Extremal Transitions in
Heterotic String Theory

Eric Sharpe
Physics Department
Princeton University
Princeton, NJ 08544
ersharpe@puhep1.princeton.edu

In this paper we study extremal transitions between heterotic string compactifications, i.e., transitions between pairs \((M, V)\) where \(M\) is a Calabi-Yau manifold and \(V\) a gauge bundle. Bundle transitions are described using language recently espoused by Friedman, Morgan, Witten. In addition, partly as a check on our methods, we also study how small instantons are described in the same language, and also describe the sheaves corresponding to small instantons.

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1 Introduction

Historically one of the biggest challenges facing string theorists was to understand the vacuum degeneracy problem, that is, why our universe would be described by one particular string theory compactification, out of the many possibilities. It was known for some time that Calabi-Yau’s can be connected through a series of extremal transitions \[37, 12, 15\], but in the degeneration limits connecting distinct Calabi-Yaus, the conformal field theory broke down, and so such transitions were not believed to be realized physically.

This perspective was radically changed by the work of Strominger \[38\] and Greene, Morrison, Strominger \[39\] who showed in detail how nonperturbative effects would cure all such ills in type II compactifications of string theory.

However, their work did not touch on the problem of relating distinct heterotic compactifications. To compactify heterotic string theory, one must specify more than just a Calabi-Yau, one must also specify (at least) one vector bundle (or, more generally, a sheaf) which breaks \(E_8 \times E_8\) or \(\text{Spin}(32)/\mathbb{Z}_2\) to a subgroup. Although the space of Calabi-Yaus may be connected, one must also understand how the vector bundle changes during the transition.

Some amount of light was shed on this question by F-theory compactifications \[9, 10, 11\]. Compactifications of F-theory on an elliptic Calabi-Yau \(n\)-fold have a weak coupling limit in which they can be described as compactifications of heterotic string theory on a Calabi-Yau \((n-1)\)-fold, with a gauge bundle implicitly specified in the form of the elliptic fibration of the F-theory compactification. Extremal transitions between F-theory compactifications have been discussed recently in \[13, 14\].

Unfortunately F-theory compactifications are only dual to heterotic compactifications on elliptic fibrations. Heterotic string theory can, of course, be compactified on much more general Calabi-Yaus, so not all heterotic extremal transitions can be understood within F-theory. In addition, F-theory compactifications often yield non-chiral heterotic duals (in the sense that the Dirac index of the vector bundle vanishes), unless one turns on background fields (for a few comments on this issue see section 7.1 of \[1\]). As is well-known, the Dirac index is invariant under smooth deformations of the theory (modulo potentially exciting IR dynamics, as in \[3\]), so in particular it should remain invariant through an extremal transition. One would like to understand extremal transitions for arbitrary heterotic compactifications, in which the Dirac index does not necessarily vanish. Thus, in order to understand extremal transitions between non-elliptic heterotic compactifications, and to have a less cumbersome method of understanding chiral heterotic compactifications, it is not sufficient to work within F-theory.

In this paper, we develop methods to describe extremal transitions directly in heterotic string theories, using technology advanced very recently by Friedman, Morgan, Witten \[1\],
and in related work by \cite{2, 3}, and also \cite{4, 5, 6, 7}. As a result, we only consider heterotic compactifications on elliptic varieties – in a companion paper \cite{8} we will study extremal transitions using the (0,2) models of Distler, Kachru \cite{21}, which are not constrained to elliptic varieties. In addition, to better understand the relation between bundle degenerations and nonperturbative physics, we also study the sheaves associated with small instantons on $K3$, and the corresponding spectral cover degenerations.

A weakness of the present work is that we have relatively little to say about the precise nonperturbative physics occurring in the degeneration limits. For example, any potential extremal transition described in this paper, which has constant Dirac index through the transition, could be obstructed by a spacetime superpotential. In transitions which are not obstructed, we expect that in many cases asymptotically-free $N=1$ supersymmetric gauge dynamics will account for singular behavior in the degeneration limit; for an example in which such behavior is studied, see \cite{40}.

Another weakness is that we will be ignoring potential worldsheet instanton effects. Not only are we working classically in string loops, but also classically on the worldsheet.

This paper divides naturally into two parts. After a review of constraints on heterotic compactifications in section two, in section three we make a detailed study of how a gauge bundle on an elliptic three-fold with base $F_1$ transforms under an extremal transition to a three-fold with base $P^2$. After reviewing the geometry of this extremal transition, we study how the bundle transforms, then check our results by studying the transformation of the spectral cover defining the bundle. The second part of the paper is in section four, in which we study vector bundle degenerations over $K3$. We work out details of the sheaf corresponding to small instantons on $K3$, and we study the corresponding spectral curve degeneration. We conclude in section five, and have also included a pair of appendices, on the basics of ideal sheaves and homological algebra.

## 2 A Rapid Review of Heterotic Compactifications

Before studying extremal transitions between distinct heterotic compactifications, we will first review some basics of such compactifications.

For a consistent perturbative compactification of either the $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ heterotic string, in addition to specifying a Calabi-Yau $Z$ one must also specify a set of stable \cite{32}, holomorphic vector bundles (or, more generally, sheaves) $V_i$. These vector bundles must obey certain constraints. For $U(n)$ bundles\footnote{Strictly speaking, in this paper we will consider bundles whose structure group is the complexification of $U(n)$, but this should cause minimal confusion.} one constraint can be written as

$$\omega^{n-1} \cup c_1(V_i) = 0$$
where $n$ is the complex dimension of the Calabi-Yau, and $\omega$ is the Kahler form, and the other is an anomaly-cancellation condition which, if a single $V_i$ is embedded in each $E_8$, can be written as

$$\sum_i \left( c_2(V_i) - \frac{1}{2} c_1(V_i)^2 \right) = c_2(TZ)$$

It was noted [29] that the anomaly-cancellation conditions can be modified slightly by the presence of five-branes in the heterotic compactification. Let $[W]$ denote the cohomology class of the five-branes, then the second constraint above is modified to

$$\sum_i \left( c_2(V_i) - \frac{1}{2} c_1(V_i)^2 \right) + [W] = c_2(TZ)$$

Historically, for a long time the only perturbative heterotic compactifications studied were those in which one took $V = TZ$, the “standard embedding.” This was done partly because more general compactifications are more difficult to work with, and partly because it was believed more general compactifications were destabilized by worldsheet instantons [20]. For perturbative compactifications described by gauged linear sigma models, both difficulties have been overcome [21, 22].

When does one expect nonperturbative effects in heterotic string theory? For perturbative compactifications, one will get nonperturbative effects when the CFT breaks down. At least for CFTs which are low-energy limits of gauged linear sigma models [23, 24], such degenerations are controlled entirely by the vector bundle, and not at all by the Calabi-Yau base space, surprisingly enough. This would appear to contradict a naive argument from M theory: Consider the heterotic string in a strong coupling limit in which it is described as M theory compactified on $(S^1/Z_2) \times X$, where $X$ is some Calabi-Yau space. Now, in M theory compactifications on $X$, a degeneration of $X$ signals the occurrence of nonperturbative effects. Naively one would expect the same to be true for M theory on $(S^1/Z_2) \times X$. However, the massless particles in bulk are lifted by boundary effects, which is possible because the boundary theory has no BPS states.

3 Extremal Transitions and Spectral Covers

In this section we will consider a prototypical example of an extremal transition which preserves an elliptic fibration structure – an extremal transition between an elliptic three-fold fibered over $F_1$, and an elliptic three-fold fibered over $P^2$. Given an $SU(n)$ vector

\[\text{For } X \text{ a Calabi-Yau three-fold, the boundary theory has N=1 supersymmetry in four dimensions, which has no BPS states. For } X \text{ a K3, the boundary theory has chiral (1,0) supersymmetry in six dimensions, and chiral superalgebras do not admit central charges.} \]
bundle $V$ over an elliptic three-fold with base $F_1$, constructed following [1], we hypothesize that after the three-fold is transformed into an elliptic fibration over $P^2$, we will get a sheaf which can be deformed into another $SU(n)$ vector bundle. We argue that the sheaf appearing in the degenerate limit is locally free away from a codimension two locus, which is naturally identified with small instantons / five-branes that necessarily appear in order to satisfy anomaly cancellation. We check that this is consistent with the description of the bundle in terms of spectral covers, by observing that the spectral cover of $V$ transforms naturally under the blowdown into the spectral cover of another $SU(n)$ bundle, modulo singular behavior at the codimension two locus just mentioned. Bundles over related spaces have also been considered in [3], though with a different perspective.

### 3.1 Review of Three-Fold Geometry

We will begin by briefly reviewing the geometry of the extremal transition between elliptic fibrations over $F_1$ and $P^2$, as has been previously discussed in [3, 11].

As is well-known, $F_1$ can be viewed as a blowup of $P^2$, so in order to send $F_1 \to P^2$, one need merely blow down the exceptional curve of self-intersection $-1$.

Naively one might think that to transform a three-fold elliptically fibered over $F_1$ into one elliptically fibered over $P^2$, one need merely blow down a divisor containing the exceptional curve in $F_1$. In the limit that the radius of the elliptic fiber is zero (the limit relevant for F theory compactifications on this three-fold), this is indeed the case. But unfortunately for nonzero Kahler modulus, the full story is somewhat more complicated.

At nonzero Kahler modulus of the fiber, the transition between three-folds is a two-step process. First, one must perform a flop in the three-fold. Before the flop, the elliptic fibration over $F_1$ is a $K3$-fibration, and the divisor containing the exceptional curve of $F_1$ is a rational elliptic surface ($P^2$ with nine blowups). The flop acts by shrinking the section of the rational elliptic surface divisor to a point, then inserting another $P^1$ at this point and orthogonal to the divisor, as sketched in Figure 1. After the flop, the rational elliptic surface becomes $P^2$ with eight blowups. The “intermediate” three-fold looks mostly like an elliptic fibration over $P^2$, with the exception of a four-cycle (the $P^2$ with eight blowups just mentioned). Finally, one blows down this four-cycle (the $P^2$ with eight blowups) to recover a three-fold that is globally an elliptic fibration over $P^2$. The $P^1$ created during the flop becomes a (singular) elliptic fiber after blowing down the four-cycle.

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³For readers less well acquainted with sheaf theory, a few definitions are in order. In this paper, by “locally free” we mean a sheaf that is associated to a well-defined vector bundle. If a sheaf is locally free everywhere except along some codimension two subvariety, then we refer to it as “torsion-free” – i.e., a torsion-free sheaf looks like a vector bundle except on some codimension two subvariety where the bundle degenerates. For more information see for example [12].
The elliptic fibration over $\mathbb{P}^2$ that one obtains via the birational transformation above is singular, and the singularity must be resolved by deformation of complex structure (as opposed to a blowup of the elliptic fiber).

3.2 General Remarks on Transforming the Bundle

To make this exercise as straightforward as possible\footnote{And for other reasons. According to [31, 2], the calculation of $c_3$ in the language of spectral covers is somewhat subtle, essentially because for a three-fold $X$, the fibered product $X \times_B \Sigma$, $\Sigma$ the spectral cover, is necessarily singular.}, we will restrict to $SU(n)$ bundles of $c_3 = 0$. Therefore the Dirac index of the bundle is automatically constrained to be constant through the transition.

We will push the locally free sheaf associated to the bundle on the three-fold over $F_1$ through the flop and the divisor blowdown, using pushforwards and pullbacks.

First, consider the first half of the flop, in which a $\mathbb{P}^1$ is shrunk to a point. For definiteness, let $E$ denote the exceptional curve in $F_1$, and let $p \in \mathbb{P}^2$ denote the point to which the exceptional curve $E$ shrinks in the blowdown $F_1 \to \mathbb{P}^2$. In the first half of the flop, the $\mathbb{P}^1$ that shrinks to a point is the image of $E$ in the section of the rational elliptic surface, as shown in Figure 1. Let $\pi_1$ denote this morphism. If $V$ is the locally free sheaf associated to a vector bundle over the elliptic three-fold with base $F_1$, then after the first half of the flop, $V \to \pi_1^*V$.

How can we compute $\pi_1^*V$? Clearly it is sufficient to restrict to the shrinking $\mathbb{P}^1$. As was shown by Grothendieck, all holomorphic vector bundles over $\mathbb{P}^1$ split, so when restricted to $\mathbb{P}^1$, $V$ will have the form

$$\mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \mathcal{O}(n_3) \oplus \cdots$$

and since $c_1(V) = 0$, $\sum n_i = 0$. If $n \geq 0$, then $\pi_1^*\mathcal{O}(n)$ is a skyscraper sheaf of rank $n + 1 = h^0(\mathbb{P}^1, \mathcal{O}(n))$. If $n < 0$, then $\pi_1^*\mathcal{O}(n)$ is an ideal sheaf, vanishing to order $-n$.

Now consider the second half of the flop. Let $\pi_2$ denote the morphism projecting the new $\mathbb{P}^1$ back down to the (singular) point we reached at the end of the first half of the flop. To
pull the sheaf through, compute $\pi_2^*\pi_1^*V$. The pullback of a rank $n$ skyscraper sheaf with support over the insertion point will be a skyscraper sheaf with support on the $\mathbb{P}^1$, and on its support will have the form $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$. The pullback of an ideal sheaf which vanishes at the insertion point is a sheaf which vanishes to the same order as at the insertion point, along the entire exceptional divisor.

So far we have formally pushed $V$ through the flop using pushforwards and pullbacks, and we have outlined how to see that the sheaf fails to be locally free along the $\mathbb{P}^1$ created during the flop. (Note, incidentally, that since the sheaf fails to be locally free at codimension two, it is always torsion-free.) In passing we should note that this procedure does not uniquely identify the sheaf that may appear after the flop – for a recent discussion of this issue, see for example [36].

Now, all that remains is to blowdown a divisor ($\mathbb{P}^2$ with eight blowups) in order to recover an elliptic fibration over $\mathbb{P}^2$. Here again, if $\pi$ denotes the blowdown morphism, then we need merely compute $\pi_2^*\pi_1^*V$.

The image of $V$ after blowing down the divisor will be locally free everywhere except along the elliptic fiber over $p \in \mathbb{P}^2$. This singular elliptic fiber is precisely the image of the $\mathbb{P}^1$ created during the flop. We will show later by a simple counting argument that in this transition, some five-branes appear, wrapped on precisely this elliptic fiber – so the failure of local freedom of the sheaf coincides with the location of the five-branes.

### 3.3 Analysis of the Spectral Cover

Having made these general observations, we now turn to a detailed analysis of the spectral cover. As the spectral cover is defined over precisely the base of the elliptic fibration, we will be able to largely ignore the complexities of the three-fold geometry of the transition. We will discover that the spectral cover degenerates at $p \in \mathbb{P}^2$, in precise agreement with our general remarks above on failure of local freedom of the sheaf along the elliptic fiber over this point. We will also discover that the spectral cover of an $SU(n)$ bundle transforms into a (singular) spectral cover of another $SU(n)$ bundle, which can be deformed to describe a rank $n$ vector bundle, i.e., the rank is unchanged by the extremal transition.

We begin by describing bundles on three-folds with base $F_1$. (Then, we will blowdown $F_1 \to \mathbb{P}^2$ and describe what happens to the bundle.) The Hirzebruch surface can be described by a GIT quotient, i.e., homogeneous coordinates and $\mathbb{C}^*$ actions, as

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5Given a morphism of varieties $f : Y \to X$ and a sheaf $V$ over $X$, there are other ways to pull the sheaf $V$ back to a sheaf over $Y$ than $f^*V$ – for example, one could compute $f^{-1}V$. However, $f^{-1}$ of a locally free sheaf is not necessarily locally free, so we will only consider $f^*V$.

6Geometric Invariant Theory. For more information, see [30]. In this paper we will be sloppy and usually ignore the fact that, in addition to specifying $\mathbb{C}^*$ actions on a set of homogeneous coordinates, one must
In other words, just as a projective space can be described in terms of homogeneous coordinates identified under single $\mathbb{C}^\times$ action, here we are describing $F_1$ in terms of homogeneous coordinates $u, v, w, s$, identified under two $\mathbb{C}^\times$ actions $\lambda, \mu$ with weights as shown above.

In this description we can recover $\mathbb{P}^2$ (the blowdown limit) by using the $\mathbb{C}^\times$ action $\mu$ to fix $s$ to a nonzero value, then $u, v, w$ are homogeneous coordinates on $\mathbb{P}^2$, and the blowup is located at $u = 0, v = 0$ (the point $p \in \mathbb{P}^2$).

We will assume there are two unitary gauge bundles, $V_1, V_2$, each of $c_1 = 0$ and of ranks $n_1, n_2$, respectively. Let $Z$ denote an elliptic three-fold fibered over $F_1$, i.e., $\pi : Z \to F_1$. Following closely the notations and conventions of [1], we will only demand $\pi_*c_2(V_1) + \pi_*c_2(V_2) = \pi_*c_2(TZ)$, both before and after the transition $F_1 \to \mathbb{P}^2$, rather than the full condition for perturbative compactifications $c_2(V_1) + c_2(V_2) = c_2(TZ)$, so the background will contain five-branes, wrapped on the elliptic fiber.

The spectral cover of an $SU(n)$ bundle is specified by functions $a_0, a_2, a_3, \ldots$, which are sections of the bundles $\mathcal{N}, \mathcal{N} \otimes \mathcal{L}^{-2}, \mathcal{N} \otimes \mathcal{L}^{-3}, \ldots$, where $\mathcal{L} = K_{F_1}^{-1}$, and $\mathcal{N} \to F_1$ is a line bundle. Also, recall from [1] that $\pi_*c_2(TZ) = 12c_1(F_1)$. Finally, letting $\mathcal{N}_i$ denote the bundle associated with the spectral data of $V_i$, for reasons of technical convenience we will impose a pair of additional conditions on the bundle:

\[
\begin{align*}
n_i &\equiv 0 \text{ mod } 2, \text{ for all } i \\
c_1(\mathcal{N}_i) &\equiv c_1(\mathcal{L}) \text{ mod } 2, \text{ for all } i
\end{align*}
\]

and for illustration we will demand $c_1(\mathcal{N}_1) = (6+t)c_1(\mathcal{L}), c_1(\mathcal{N}_2) = (6-t)c_1(\mathcal{L})$, for some odd integer $t$.

Before we go on, we will introduce a little notation. Let $D_u$ denote both the divisor $\{u = 0\}$ and its Poincare-dual element of $H^2(F_1, \mathbb{Z})$, in somewhat sloppy notation. (So in particular if we view $F_1$ as a $\mathbb{P}^1$ fibration, $D_u$ is the fiber and $D_s$ is the isolated section.) Let $\mathcal{O}(m,n)$ denote the line bundle over $F_1$ such that $c_1 = mD_u + nD_s$, so in particular $c_1(F_1) = 3D_u + 2D_s$.

Putting this all together, we find

\[
\begin{align*}
\mathcal{L} &= \mathcal{O}(3,2) \\
\mathcal{N}_1 &= \mathcal{L}^{6+t}
\end{align*}
\]

also specify a set of points to be omitted before quotienting.

These conditions impose a particular $\mathbb{Z}_2$ symmetry on the bundle. See equation (7.49) of [1].
\[ N_2 = \mathcal{L}^{6-t} \]
\[ a_0(1) \in \Gamma(N_1) = \Gamma(O(18 + 3t, 12 + 2t)) \]
\[ a_2(1) \in \Gamma(N_1 \otimes \mathcal{L}^{-2}) = \Gamma(O(12 + 3t, 8 + 2t)) \]
\[ a_3(1) \in \Gamma(N_1 \otimes \mathcal{L}^{-3}) = \Gamma(O(9 + 3t, 6 + 2t)) \]
\[ a_4(1) \in \Gamma(N_1 \otimes \mathcal{L}^{-4}) = \Gamma(O(6 + 3t, 4 + 2t)) \]

and so forth.

We can expand out the \(a_k\) in terms of the homogeneous coordinates on \(F_1\) as follows:

\[ a_0(1) = \sum_{i,j} a_{0,i,j} u^i v^j w^{18+3t-i-j} s^{i+j-6-t} \]
\[ a_2(1) = \sum_{i,j} a_{2,i,j} u^i v^j w^{12+3t-i-j} s^{i+j-4-t} \]
\[ a_3(1) = \sum_{i,j} a_{3,i,j} u^i v^j w^{9+3t-i-j} s^{i+j-3-t} \]
\[ a_4(1) = \sum_{i,j} a_{4,i,j} u^i v^j w^{6+3t-i-j} s^{i+j-2-t} \]

and so forth.

Were we on \(P^2\) rather than \(F_1\), the expansions of the \(a_k\) in terms of homogeneous coordinates would be identical, except for the omission of the \(s\) factor.

Note that the \(s\) factor constrains the \(a_k\) on \(F_1\) more than they would be on \(P^2\). To be specific, consider \(a_0(1)\). On \(P^2\), the sum over \(i, j\) would run over \(0 \leq i + j \leq 18 + 3t\), whereas on \(F_1\) because of the \(s\) factor the sum is restricted to \(6 + t \leq i + j \leq 18 + 3t\). In particular this means that each of the \(a_k(1)\) (for \(k \leq 5 + t\)) vanishes over the point \(p \in P^2\) \((u = v = 0)\), so in the blowdown limit of \(F_1\) the vector bundle becomes some torsion-free sheaf. Note that this is consistent with the remarks made earlier, that after pulling \(V\) through the birational transformation it fails to be locally free on the (singular) elliptic fiber over \(p \in P^2\).

The idea in the last paragraph will appear many more times in this paper and so is worth repeating. Given a section of some line bundle over \(\tilde{X}\), a blowup of \(X\), in the blowdown limit the section will often have zeroes at the location of the blowup on \(X\). By expanding out sections of line bundles explicitly in terms of homogeneous coordinates, we are able to pick off this behavior directly.

There is a more invariant method to describe this result. In general, suppose \(\pi: \tilde{X} \to X\) is a blowup of \(X\), and \(\pi^*(L) \otimes O(-n)\) some line bundle over \(\tilde{X}\), where \(c_1(O(-n)) = -n\) Judging from the codimension of the singularities in the bundle. In general, the fact that the \(a_k\) all vanish at a point does not necessarily imply the bundle degenerates over that point – this will be discussed in section \[42\].
times the dual to the exceptional divisor. Then the direct image sheaf \( \pi_\ast(\pi^\ast(L) \otimes \mathcal{O}(-n)) = L \otimes \mathcal{I}_n \), where \( \mathcal{I}_n \) is an ideal sheaf on \( X \) which vanishes to order \( n \) at the location of the blowup. In the present case, consider for example \( a_0(1) \). Since \( a_0(1) \in \Gamma(N_1) \) and \( c_1(N_1) = 3(6 + t)u_0 - (6 + t)u_1 \) (in conventions where \( K_{F_1} = -3u_0 + u_1 \), so \( u_1 \) is dual to the exceptional divisor of the blowup of \( \mathbb{P}^2 \)), we have \( \pi_\ast(N_1) = K_{\mathbb{P}^2}^{(6+t)} \otimes \mathcal{I}_{6+t} \), and indeed we showed in the last paragraph that in the blowdown limit, \( a_0(1) \) vanishes to order \( 6 + t \) at \( p \in \mathbb{P}^2 \).

At this point we will take a moment to describe how the complex structure of the elliptic fibration \( Z \to F_1 \) degenerates, as another example of the concept above. \( Z \) is given as a hypersurface in the total space of \( \mathbb{P}((\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)) \to F_1 \), with homogeneous coordinates \((z, x, y)\) (respectively) on the \( \mathbb{P}^2 \) fiber. The hypersurface is of the form \( y^2z = x^3 + fxz^2 + gz^3 \), where \( f \in \Gamma(\mathcal{L}^4) \), \( g \in \Gamma(\mathcal{L}^6) \). Since \( c_1(\mathcal{L}) = 3u_0 - u_1 \), we can read off that in the blowdown limit, \( f \) will have an order 4 zero, \( g \) an order 6 zero, at \( p \in \mathbb{P}^2 \). Clearly we can resolve this singularity in \( Z \to \mathbb{P}^2 \) by deforming the complex structure. One can often also resolve singularities by blowups of the elliptic fiber; however, in the present case, the singularity in \( Z \) is too severe to be blown up fiber-wise into another Calabi-Yau. The bundle degeneration can be resolved simply by deforming the \( a_k \) to more generic sections.

Conversely, consider starting with an \( SU(n) \) bundle on \( Z \to \mathbb{P}^2 \). In order to blow up the point \( p \in \mathbb{P}^2 \), we must first adjust both the sections defining the Weierstrass fibration as well as the \( a_k \) defining the bundle. In particular, it is not sufficient to arrange for only the base to be singular – in order to be able to blow up the base with bundle consistently, one must also adjust the \( a_k \) to be singular. (This is in accordance with the observation in the introduction that, for perturbative heterotic compactifications, the conformal field theory degeneration is controlled by the vector bundle – so deforming only the base to the singular locus is insufficient for the conformal field theory to break down, and make an extremal transition possible.)

So far we have described how in a transition between elliptic three-folds over \( F_1 \) and \( \mathbb{P}^2 \), the \( a_k \) defining the spectral cover for a pair of \( SU(n) \) bundles changes. To completely specify the bundles, one must in addition specify a line bundle on each spectral cover. We assumed at the beginning that each bundle was invariant under a \( \mathbb{Z}_2 \) symmetry (called \( \tau \) in \([1]\)), in which case the line bundle on the spectral cover is trivial. In particular, since the bundles on either side of the transition are taken to be \( \tau \) invariant, the line bundles on the spectral covers are trivial throughout the transition.

How many five-branes appear in the transition? Recall that if \([W]\) denotes the cohomology class of the five-branes, then

\[
[W] = c_2(TZ) - \sum_i c_2(V_i)
\]

\[
= 11c_1(\mathcal{L})^2 + c_2(B) - \frac{1}{24}c_1(\mathcal{L})^2 \left[ n_1^3 - n_1 + n_2^3 - n_2 \right]
\]

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where \( B \) is the base of the elliptic three-fold (either \( F_1 \) or \( P^2 \)). Despite appearances, it can be shown using the mod 2 conditions mentioned earlier that \([W]\) is an element of integral cohomology, that is, that the number of five-branes is an integer. To insure supersymmetry is unbroken one must check in general that the number of five-branes is nonnegative. In any event, it is clear that in general the number of five-branes present (wrapped on the elliptic fiber) is different on either side of the transition. Recall from our discussion above that in the blowdown \( F_1 \to P^2 \), the bundle becomes singular along the singular elliptic fiber over \( p \in P^2 \); clearly this codimension two locus should be interpreted as the locations of five-branes.

To review, given a pair of \( \tau \)-invariant bundles on an elliptic three-fold fibered over \( F_1 \), we have explicitly worked out how the bundles transform under the blowdown \( F_1 \to P^2 \), in terms of the spectral data defining them. In the degenerate limit the bundle has a singularity over a codimension two locus, which is interpreted as due to the presence of five-branes. In principle the same idea should apply much more generally. (Although not all rational surfaces are toric, we have outlined how to attack other cases, in terms of direct image sheaves and ideal sheaves).

We have not attempted to determine whether the spacetime superpotential obstructs this transition, though in principle it might be possible to work this out following [24, 25].

### 3.4 Splitting-Type Extremal Transitions

Lest the reader get the impression that all bundle degenerations are reflected by a spectral cover degeneration, in this section we will give a counterexample. We will consider a splitting-type transition \([12, 13]\), between an elliptic three-fold with base \( P^2 \) and another elliptic three-fold with the same base. It will turn out that at the transition point, singularities in the elliptic three-fold will lie along the section of the elliptic fibration. We will not have anything to say about how bundles behave through such a transition, nevertheless we felt it appropriate to include this discussion.

First we will describe an elliptic three-fold with base \( P^2 \). The Calabi-Yau is described as a hypersurface in an ambient space which is obtained by fibering \( P^2 \) over the base \( P^2 \). Let \( u, v, w \) be homogeneous coordinates on the base and \( x, y, z \) homogeneous coordinates on the fiber, then we have \( \mathbb{C}^\times \) actions as

\[
\begin{array}{c|ccc|ccc}
\lambda & u & v & w & x & y & z \\
1 & 1 & 1 & 2 \alpha & 3 \alpha & 0 \\
\nu & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]
with hypersurface defined by
\[ y^2z = x^3 + f(u, v, w)xz^2 + g(u, v, w)z^3 \]

For this hypersurface to be Calabi-Yau, we demand \( \alpha = 3 \). Also, \( f \) has degree \( 4\alpha = 12 \) under \( \lambda \), and \( g \) has degree \( 6\alpha = 18 \) under \( \lambda \).

For convenience, let \( M \) denote the ambient space described above. Consider a splitting-type transition in which the hypersurface \( W = 0 \) in \( M \) transforms into a complete intersection \( W_1 = W_2 = 0 \) in \( \mathbb{P}^1 \times M \), and the degrees of the hypersurfaces are
\[
\begin{array}{c|cc}
\text{P}^1 & 1 & 1 \\
\lambda & 6 & 12 \\
\nu & 1 & 2 \\
\end{array}
\]

Note that the complete intersection above has base \( \mathbb{P}^2 \), the same as previously, and fiber
\[
\begin{array}{c|cc}
\text{P}^1 & 1 & 1 \\
\nu & 1 & 2 \\
\end{array}
\]

which is manifestly an elliptic curve. So, in other words, this particular splitting-type transition takes place entirely within the elliptic fiber. In addition, on both sides of the transition the elliptic fibration has a section: the threefold in \( M \) has elliptic section \( \{ x = z = 0, y = 1 \} \), and the threefold in \( M \times \mathbb{P}^1 \) has section \( \{ x = z = 0, y = 1, t_0 = 0, t_1 = 1 \} \) where \( t_0, t_1 \) are homogeneous coordinates on the \( \mathbb{P}^1 \).

The fact that the base is invariant under this transformation somewhat simplifies the analysis of vector bundles over the three-fold in the language of [1]. Suppose, for definiteness, we have an \( SU(n) \) bundle, whose spectral cover is specified by \( a_k \) (sections of line bundles over the base). Since the base of the elliptic fibration is unchanged by the transition, and the \( a_k \) are sections of bundles over the base, the \( a_k \) are invariant through the transition.

Although the \( a_k \) are invariant through the transition, the description of the bundle does break down at the transition point, because all of the conifold singularities are located along the section of the elliptic fibration. This is relatively straightforward to see. Let \( W_1, W_2 \) be the hypersurfaces in \( M \times \mathbb{P}^1 \) whose complete intersection is the Calabi-Yau. Write
\[
\begin{align*}
W_1 &= t_0 P + t_1 Q \\
W_2 &= t_0 R + t_1 S
\end{align*}
\]
then in the blowdown limit, this complete intersection becomes the hypersurface \( PS - QR = 0 \), with (conifold) singularities at \( P = Q = R = S = 0 \). By expanding out \( P, Q, R, \) and \( S \), it is easy to see they all vanish at \( x = z = 0, y = 1 \), which is precisely the section of the elliptic fibration.
Clearly the bundle degeneration that occurs in this transition is quite different from that in the previous section. There, the bundle singularity was signaled by the fact that the $a_k$ all vanished over some point on the base, at the transition point. Here, by contrast, the $a_k$ are unaffected by the transition!

4 Vector Bundle Degenerations on Surfaces

In this section we examine vector bundle degenerations over $K3$. We begin by analyzing the case of a single small instanton, by conjecturing the form of the sheaf describing such a degeneration and then by using F theory to understand the precise spectral cover behavior. We also examine spectral cover degenerations corresponding to multiple small instantons.

4.1 Small Instantons and Sheaves on $K3$

In this subsection we conjecture that the precise sheaf corresponding to a small instanton of $SU(2)$ is of the form $\mathcal{O} \oplus J$ in a neighborhood of the small instanton, where $J$ is an appropriate ideal sheaf. We motivate this conjecture by closely examining a well-known \cite{26, 27} construction of $SU(2)$ bundles on $K3$ from an unordered set of points on $K3$. The arguments are necessarily rather technical in nature; readers not familiar with sheaf-theoretic homological algebra are encouraged to skip to the next subsection.

First we shall review how to associate some number of unordered points with an $SU(2)$ bundle $E$ on $K3$ of $c_2(E) = k$, closely following \cite{26}. First, find a line bundle $L$ on $K3$ such that $\chi(E \otimes L^{-1}) = 1$, then generically $E \otimes L^{-1}$ will have a unique (up to scalar multiple) section $s$. The section $s$ will have $c_2(E \otimes L^{-1}) = 2k - 3$ isolated zeroes. The zeroes of this section are precisely the unordered points that we associate with the bundle $V$.

Now, in order to make our conjecture regarding small instanton sheaves, we shall closely examine the converse: given a set of $2k - 3$ points on $K3$, we will construct a bundle $E$. We construct $E$ as a nontrivial extension \cite{33, 28}

\[ 0 \to L \to E \to L^{-1} \otimes J \to 0 \]

where $L$ is an invertible sheaf and $J$ is an ideal sheaf, vanishing at the $2k - 3$ points on $K3$. These extensions are of course classified by the group global $\text{Ext}^1(L^{-1} \otimes J, L)$.

The extensions $E$ are not necessarily locally free – to recover a bundle as an extension, one must impose additional constraints. Now, global $\text{Ext}$ is defined by a spectral sequence, and in particular

\[ 0 \to H^0(K3, \text{Ext}^1(L^{-1} \otimes J, L)) \to \text{Ext}^1(L^{-1} \otimes J, L) \to H^1(K3, \text{Hom}(L^{-1} \otimes J, L)) \]
\[ \rightarrow H^0(K3, Ext^2(L^{-1} \otimes J, L)) \]

is an exact sequence. (We are assuming the $K3$ has nonzero Picard number, and that $J$ vanishes at isolated points with multiplicity one.) Thus, elements of $Ext^1(L^{-1} \otimes J, L)$ are partly determined by elements of $H^0(K3, Ext^1(L^{-1} \otimes J, L)) = H^0(K3, \mathcal{O}/J)$. It can be shown that the sheaf $E$ is locally free precisely when each element of $H^0(K3, \mathcal{O}/J)$ is a unit.

Thus, if we want to find non-locally-free sheaves that are in the same $S$-equivalence classes as stable bundles on the moduli space, all we need to do is pick extensions such that some element of $H^0(K3, \mathcal{O}/J)$ is not a unit. For example, if $\mathcal{O}/J$ has support over a point $x$ in $K3$, choose an extension corresponding to a section with value 0 over $x$. Then, locally, the corresponding extension will have the form $\mathcal{O} \oplus J$ rather than $\mathcal{O} \oplus \mathcal{O}$.

This, then, is a possibility for the sheaf corresponding to small instantons. Simply, in a neighborhood in which the sheaf fails to be locally free, it takes the form $\mathcal{O} \oplus J$ (though is not globally of this form) for some ideal sheaf $J$. Deforming the sheaf to be locally free everywhere (by deforming the sections in a neighborhood of $x$) would correspond to peeling a five-brane off the end-of-the-world, in the language of M-theory compactified on $S^1/\mathbb{Z}_2$. (As a check, note that in such a deformation, $c_2$ drops by one.)

The reader may wonder why we did not mention another possibility: that when an instanton becomes small, the sheaf takes the form $V \otimes I$, where $V$ is locally free and $I$ is an ideal sheaf vanishing at the location of the small instanton. A quick Chern class computation will convince the reader that it is not possible to describe only one small instanton.

4.2 Spectral Cover Degenerations on $K3$

In this subsection we will work out the precise spectral cover degeneration corresponding to a single small instanton on $K3$, more precisely, a small $E_8$ instanton. We will find that the spectral cover becomes reducible, with one component being precisely the spectral cover of a bundle with one less instanton.

We will use the duality between heterotic strings on $K3$ and F theory on elliptic threefolds. F theory on an elliptic three-fold (with section) whose base is $F_n$ is dual to a heterotic compactification on $K3$ with instanton numbers $(12 + n, 12 - n)$ embedded in either $E_8$. Blowups of the base of the three-fold correspond to instantons degenerating into five-branes. Physically one can imagine five-branes propagating between the two ends-of-the-world in the M theory description, which corresponds to blowups, blowdowns transforming $F_n$ into $F_{n+1}$. (To avoid certain technical issues, we will assume $n \leq 6$.)

\[ ^9 \text{Though in principle such a sheaf might describe multiple small instantons.} \]
The transformation \( F_n \to F_{n\pm 1} \) has an elegant mathematical understanding as an elementary transformation on the ruled surface \( F_n \) \(^{[33]}\). Viewed as a \( P^1 \) fibration, \( F_n \) has sections, of self-intersection \( \pm n \). The transformation proceeds by first blowing up a point on one of the sections, then blowing down the strict transform of the fiber\(^{[10]}\).

To make this section accessible to a larger number of readers, we will use the fact that Hirzebruch surfaces and some of their blowups are toric varieties\(^{[11]}\). The fan describing the surface relevant here as a toric variety, a (smooth toric) blowup of \( F_n \), is shown in Figure 2.

The fan has edges along \((0, 1), (1, 0), (−1, n), (−1, n + 1), \) and \((0, −1)\). Using standard methods, the toric variety can be described as a GIT quotient. If we also fiber \( P^2 \) over the base, then we have homogeneous coordinates and \( C^\times \) actions defining the total space as follows:

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & s & t & u & v & w & x & y & z \\
\hline
\lambda & 1 & 1 & n & 0 & 0 & 2\alpha & 3\alpha & 0 \\
\mu & 0 & 0 & 1 & 1 & 0 & 2\beta & 3\beta & 0 \\
\tau & 1 & 0 & n + 1 & 0 & 1 & 2\gamma & 3\gamma & 0 \\
\nu & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hline
\end{array}
\]

At this point a few words of explanation are in order. The homogeneous coordinates \( x, y, z \) are coordinates on the fiber \( P^2 \). The coordinates \( u, v \) can be understood as homogeneous coordinates on the \( P^1 \) fiber of the Hirzebruch surface that exists in either of the blowdown limits to be described next.

To recover \( F_n \), we blowdown the toric divisor described by the edge \((−1, n + 1)\) in the fan, which corresponds to using the \( \tau \) action to fix the value of \( w \) and taking \( s, t \) to be

\(^{[10]}\)Note if \( S \) is the section and \( F \) the fiber, then the strict transform of the fiber \( \tilde{F} = \pi^*F - E \) has self-intersection \(-1\), as \( E^2 = −1, E \cdot \pi^*F = 0, \) and \( F^2 = 0 \), so by Castelnuovo’s contractibility criterion and the fact \( \tilde{F} \cong P^1 \), one can blowdown \( \tilde{F} \) to recover a smooth surface.

\(^{[11]}\)For introductions to toric varieties see \([14, 15, 18, 19]\).
homogeneous coordinates on the $\mathbb{P}^1$ base of $F_n$. The exceptional divisor is inserted at the point $v = t = 0$.

To recover $F_{n+1}$, we blowdown the toric divisor described by the edge $(-1, n)$ in the fan, which corresponds to using the $\lambda$ action to fix the value of $t$ and taking $s, w$ to be the homogeneous coordinates on the $\mathbb{P}^1$ base of $F_{n+1}$.

The Calabi-Yau three-fold is described as a hypersurface in the toric four-fold above:

$$y^2z = x^3 + f(s, t, u, v, w)xz^2 + g(s, t, u, v, w)z^3$$

For this to describe a Calabi-Yau, we must demand $\alpha = n + 2$, $\beta = 2$, $\gamma = n + 3$, and that $f, g$ have degrees under the $\mathbb{C}^\times$ actions as below:

|   | degree $f$ | degree $g$ |
|---|-----------|-----------|
| $\lambda$ | $4\alpha$ | $6\alpha$ |
| $\mu$ | $4\beta$ | $6\beta$ |
| $\tau$ | $4\gamma$ | $6\gamma$ |

so we can expand $f, g$ as

$$f(s, t, u, v, w) = \sum_{i=0}^{8} u^iv^{8-i} \left[ \sum_j f_{i,j} s^{8+n(4-i)-j} t^j w^{4-i+j} \right]$$

$$g(s, t, u, v, w) = \sum_{i=0}^{12} u^iv^{12-i} \left[ \sum_j g_{i,j} s^{12+n(6-i)-j} t^j w^{6-i+j} \right]$$

Now assume that we have a section of $E_6$ singularities at $v = 0$, which corresponds to an $SU(3)$ bundle in one of the $E_6$s. Following [10, 11] this means that in a neighborhood of $v = 0$, in terms of the affine coordinate $w_1 = v/u$, we have

$$f(w_1, s, t, w) = w_1^3 f'_{8-n}(s, t, w)$$

$$g(w_1, s, t, w) = w_1^4 \left[ q'_{6-n}(s, t, w) \right]^2 + w_1^5 g'_{12-n}(s, t, w)$$

where

$$f'_{8-n}(s, t, w) = \sum_{j=1}^{8-n} f'_{8-n,j} s^{8-n-j} t^j w^{j-1}$$

$$q'_{6-n}(s, t, w) = \sum_{j=1}^{6-n} q'_{6-n,j} s^{6-n-j} t^j w^{j-1}$$

$$g'_{12-n}(s, t, w) = \sum_{j=1}^{12-n} g'_{12-n,j} s^{12-n-j} t^j w^{j-1}$$

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As explained in [1], in either blowdown limit, $f'$, $q'$, and $g'$ above define a spectral curve over the $\mathbb{P}^1$ which is the base of the elliptic $K3$ (with section) of the heterotic compactification.

Consider the blowdown limit to $F_n$. Were it not for the homogeneous coordinate $w$ above, $f'$, $q'$, and $g'$ could each have an additional term, with coefficients $f'_{8-n,0}$, $q'_{6-n,0}$, and $g'_{12-n,0}$. However, in the blowdown limit $f'$, $q'$, and $g'$ all have a common factor of $t$, i.e., $f'$, $q'$, and $g'$ all vanish at $t = 0$. In other words, in the blowdown limit in which we recover $F_n$ as the base of the elliptic fibration in F theory, the spectral curve (partially) defining the vector bundle on $K3$ in the heterotic dual becomes reducible, and now describes a sheaf, not a vector bundle. This vector bundle degeneration coincides with the appearance of tensionless strings in the compactified six-dimensional theory.

It is possible to streamline the derivation above. Let $\pi$ denote the blowdown morphism to $F_n$, and $\pi' : F_n \to \mathbb{P}^1$ the projection morphism, then if $\tilde{\mathcal{L}}$ denotes the anticanonical bundle of the blowup of $F_n$, $\pi_* \tilde{\mathcal{L}}^4 = \mathcal{L}^4 \otimes \mathcal{I}$, where $\mathcal{I}$ is an ideal sheaf vanishing at the point $t = v = 0$, and

$$\pi'_* \pi_* \tilde{\mathcal{L}}^4 = \bigoplus_{i=0}^{3} \mathcal{O}(8 + n(4 - i)) \bigoplus_{i=5}^{8} \left[ \mathcal{O}(8 + (n + 1)(4 - i)) \otimes \mathcal{I}'_{i-4} \right]$$

where $\mathcal{I}'$ is an ideal sheaf vanishing over $t = 0$. (As a check, note that $h^0(\mathbb{P}^1, \mathcal{O}(8)) = 9 = \text{rank } \pi'_* \pi_* \tilde{\mathcal{L}}^4$.) In this fashion we can read off the spectral data defining each bundle as well as the data defining the heterotic Weierstrass fibration.

So far we have only studied the bundle moduli of the heterotic compactification, in the limit that a tensionless string should appear (by duality with F theory). What about the moduli of the elliptic fibration in the heterotic compactification, the $K3$ moduli? The polynomials defining the heterotic Weierstrass fibration also appear in the F theory Weierstrass polynomials: if we expand

$$f(s, t, u, v, w) = \sum_{i=0}^{8} u^i v^{8-i} f_i(s, t, w)$$
$$g(s, t, u, v, w) = \sum_{i=0}^{12} u^i v^{12-i} g_i(s, t, w)$$

then the polynomials appearing in the heterotic Weierstrass fibration are precisely $f_4(s, t, w)$ and $g_6(s, t, w)$. Now,

$$f_4(s, t, w) = \sum_{j} f_{4,j} t^j s^{8-j} w^j$$
$$g_6(s, t, w) = \sum_{j} g_{6,j} t^j s^{12-j} w^j$$

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so in the blowdown limit there are no additional constraints on these polynomials. In other words, in the limit in which a tensionless string should appear, the bundle becomes singular, but the elliptic $K3$ need not. This is completely consistent with our understanding of the physics, as we expect tensionless strings to appear when an instanton shrinks to zero size – $K3$ moduli should be irrelevant.

What is the dimension of the subvariety of $K3$ along which the bundle degenerates? Since the spectral data $a_k$ all vanish along a codimension one locus, it is naively tempting to speculate that the bundle must degenerate along some curve in the $K3$. However, there is an important subtlety – it will turn out that the bundle does not degenerate along an entire curve in $K3$, but at most at points.

What is this subtlety? The fact that the $a_k$ all vanish along some codimension one subvariety means that the $a_k$ are not really sections of $\mathcal{N} \otimes \mathcal{L}^{-k}$, but rather sections of $\mathcal{N} \otimes \mathcal{L}^{-k} \otimes \mathcal{M}$, for some line bundle $\mathcal{M}$ (determined by the divisor $\{t = 0\}$ in this case). We can recover manifestly well-defined spectral data simply by making some redefinitions: $a_0 = g'_{12-n} \to g'_{12-n}/t$, and so forth. The new spectral data $a_k$ are sections of $\mathcal{N} \otimes \mathcal{L}^{-k}$. In this particular case, the spectral cover is reducible, with one component being the spectral cover of a bundle with $c_2 = 11 - n = 12 - n - 1$, reflecting the fact that a single instanton has become small.

To repeat, we have observed that if the spectral data $a_k$ vanishes along some codimension one locus on the base of the elliptic fibration, it should not be interpreted as a bundle degeneration along a codimension one locus, but rather as a poor interpretation of the spectral data. Any naive codimension one bundle degenerations should really be interpreted as, at most, codimension two. Note that the procedure outlined above only applies if the $a_k$ vanish at codimension one, not at codimension two.

The fact that the bundle does not degenerate at codimension one on $K3$, but only at codimension two, is perfectly consistent with the physical interpretation of the bundle degeneration as due to the presence of a five-brane, localized at a point on $K3$.

The spectral curve degeneration occurs at complex codimension three in the moduli space of spectral curves, as we have lost one monomial in each of $f'$, $q'$, and $g'$. What is the codimension of the vector bundle degeneration (in the moduli space of vector bundles on $K3$)? Recall that the vector bundle on $K3$ is defined by both a spectral cover of the base of the elliptic $K3$ as well as a line bundle on the spectral cover. In this case, since the base is $\mathbb{P}^1$, the spectral cover is a branched cover of $\mathbb{P}^1$, i.e., a Riemann surface, and the space of line bundles of fixed degree is of course the Jacobian of the Riemann surface. If the Riemann surface degenerates at complex codimension three, then surely the Jacobian will also degenerate at codimension three, so the degenerate vector bundle moduli space lies.

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12For every cycle in the Riemann surface that shrinks, the space of flat line bundles loses precisely one dimension.
along a complex codimension six subvariety, which is precisely correct to describe one less
SU(3) instanton on $K3$.

This degeneration of $f(s, t, u, v, w)$ and $g(s, t, u, v, w)$ also yields a singular elliptic fiber (in
the F theory compactification) over the point $t = 0$. The singularity is type $II$ in Kodaira’s
classification, which means that although the fiber is singular, the total space is not singular
in a neighborhood of this point, and so there is no (nonperturbative) contribution to the
six-dimensional gauge symmetry, precisely as expected.

So far, we have discussed the degeneration of the SU(3) bundle in the $E_8$ located over
$v = 0$. Proceeding similarly, one can show the bundle in the other $E_8$, located over $u = 0$,
does not degenerate. This is completely consistent with the fact that we have blown up $F_n$
over the point $v = 0$, $t = 0$, so the bundle over the blown-up point degenerates, and the
spectral curve degenerates over the same point on the base.

We can also study the limit in Kahler moduli space in which we recover $F_{n+1}$. This limit
corresponds to a blow-up of $F_{n+1}$ over the point $u = 0$, $w = 0$. The bundle over $u = 0$
degenerates, and the spectral curve degenerates over the point $w = 0$ on the base. Note that
all of this is completely consistent with the standard interpretation of such blowups: they
should correspond to an instanton in one $E_8$ shrinking and becoming a five-brane, travelling
to the second $E_8$, and reverting to an instanton.

### 4.3 Multiple Small Instantons

In this subsection we will study spectral curve degeneration in a case in which two instantons
shrink to become five-branes, whose F-theory dual is an elliptic three-fold over $F_n$ with two
of blowups, as shown in Figure 3. (Note that the second blowup is located on the exceptional
divisor of the first blowup.) Closely related results have been obtained in [4]. The details
proceed in a very similar manner to the last section, so we will simply outline the relevant
results.
If we fiber $\mathbb{P}^2$ over this base, then the complete description in terms of homogeneous coordinates and $\mathbb{C}^\times$ actions is as follows:

| $s$ | $t$ | $u$ | $v$ | $w$ | $r$ | $x$ | $y$ | $z$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\lambda$ | 1 | 1 | $n$ | 0 | 0 | 0 | $2\alpha$ | $3\alpha$ | 0 |
| $\mu$ | 0 | 0 | 1 | 1 | 0 | 0 | $2\beta$ | $3\beta$ | 0 |
| $\tau$ | 1 | 0 | $n+1$ | 0 | 1 | 0 | $2\gamma$ | $3\gamma$ | 0 |
| $\rho$ | 1 | 0 | $n+2$ | 0 | 0 | 1 | $2\delta$ | $3\delta$ | 0 |
| $\nu$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |

As before, we will describe our elliptic three-fold by the hypersurface

$$y^2z = x^3 + f(s, t, u, v, w, r) x z^2 + g(s, t, u, v, w, r) z^3$$

and for this to be a Calabi-Yau, we demand $\alpha = n+2$, $\beta = 2$, $\gamma = n+3$, and $\delta = n+4$.

Suppose we have an $SU(3)$ bundle over $v = 0$, then the Weierstrass fibration is of the form

$$f(w_1, s, t, w, r) = w_1^3 f'_{8-n}(s, t, w, r)$$
$$g(w_1, s, t, w, r) = w_1^4 \left[ q'_{6-n}(s, t, w, r) \right]^2 + w_1^5 g'_{12-n}(s, t, w, r)$$

and we can expand each of the terms as

$$f'_{8-n}(s, t, w, r) = \sum_i f'_{8-n,i} s^{8-n-i} t^i w^{-1} r^{i-2}$$
$$g'_{6-n}(s, t, w, r) = \sum_i g'_{6-n,i} s^{6-n-i} t^i w^{-1} r^{i-2}$$
$$g'_{12-n}(s, t, w, r) = \sum_i g'_{12-n,i} s^{12-n-i} t^i w^{-1} r^{i-2}$$

Thus, in the blowdown limit in which we recover $F_n$, we find that each of $f'$, $q'$, $g'$ is proportional to $t^2$, not just $t$. A dimension count just like the one in the last section reveals that this is perfect to describe the shrinking of two $SU(3)$ instantons on $K3$.

In the last section, when one instanton shrank, there was no nonperturbative enhanced gauge symmetry – the singularity in the fiber was Kodaira type $II$. Here, however, the singularity is Kodaira type $IV$, which corresponds to an $A_2$ singularity in the total space of the F-theory compactification, so we expect to recover a nonperturbative enhanced $SU(3)$ gauge symmetry in six dimensions.
5 Conclusions

In this paper we developed technology to describe a class of extremal transitions directly in heterotic string theory. We have also studied the sheaves associated with small instantons on $K3$, and the corresponding spectral cover degenerations in each case.

The work described here leaves many questions answered. Perhaps foremost among these questions concerns the nonperturbative physics at the transitions, about which we have had very little to say. Each of the transitions between four-dimensional compactifications discussed in this paper could conceivably be obstructed by a superpotential, an issue we have not been able to address at all. Without having a detailed understanding of the physics occurring at these transitions, we have only been able to make some preliminary checks of whether they might be allowed – by checking that the Dirac index is constant through the transition, and that the number of five-branes present on both sides of the transition is positive.

However, in spite of not understanding the superpotential, we have found that understanding classical heterotic extremal transitions is within reach of current technology.

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A Ideal Sheaves

An ideal sheaf is simply a subsheaf of the trivial rank one sheaf (the structure sheaf) with the property that all sections of the ideal sheaf vanish along some subvariety.

More precisely, in a local coordinate neighborhood an ideal sheaf is defined by some polynomial ideal, in the sense that all local sections are elements of the ideal. For example, consider a local coordinate neighborhood on some variety containing coordinates $u, v$, among others. An ideal sheaf that vanishes to first order at the point $u = v = 0$ is defined by the ideal generated by $(u, v)$, i.e., local sections are all of the form $uf + vg$ for holomorphic functions $f, g$.

\(^{13}\)Experts will recognize this is a ham-handed treatment of a simple idea. Let $(\text{Spec } A, \mathcal{O})$ be an affine scheme, and $I$ an ideal of the ring $A$. Then the stalk of the ideal sheaf $\hat{I}$ defined by $I$ over a prime ideal $p$ in $\text{Spec } A$ is the localization $I_p$, and $\Gamma(\text{Spec } A, \hat{I}) = I$. 

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In particular, an ideal sheaf that vanishes along a codimension one locus is associated to a line bundle.

Chern classes can be defined for ideal sheaves. To do so, one needs a locally free resolution of the ideal sheaf. For an ideal generated by a regular sequence \([4, 2, 43]\) (that is, a locally complete intersection), we can use the Koszul resolution

\[
\cdots \rightarrow \oplus \mathcal{O}(-D_i - D_j) \rightarrow \oplus \mathcal{O}(-D_i) \rightarrow I \rightarrow 0
\]

where \(I\) is the ideal sheaf, associated with the ideal generated by the regular sequence \((f_1, f_2, \ldots)\), and \(D_i\) are the divisors \(D_i = \{f_i = 0\}\). Then, given this resolution, we can compute the total Chern class as

\[
c(I) = c(\oplus \mathcal{O}(-D_i)) c(\oplus \mathcal{O}(-D_i - D_j))^{-1} \cdots
\]

When an ideal sheaf is associated with an ideal not generated by a regular sequence, one must work harder, as the naive Koszul resolution does not yield an exact sequence.

Note that an ideal sheaf is not uniquely defined by the subvariety along which it vanishes and the order to which it vanishes along this subvariety. To return to the previous example, consider the ideals generated by \((u^n, v^n)\) and by \((u^n, u^{n-1}v, u^{n-2}v^2, \ldots, v^n)\). Both of these vanish to order \(n\) at the subvariety \(u = v = 0\). However, the first is a regular sequence, the second is not, so in general the ideal sheaves generated by either will have distinct Chern classes.

\section{Homological Algebra}

In this appendix we will give an extremely schematic outline of some homological algebra used in this paper. For a basic introduction to the subject, see [34, 42], and for information on the sheaf-theoretic version, see [32, 33, 43].

Let \(M, N\) be modules over some ring \(A\), then a set of groups labelled \(\text{Ext}^n(M, N)\) can be associated with the pair \((M, N)\), and classify isomorphism classes of exact sequences

\[
0 \rightarrow N \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow M \rightarrow 0
\]

Rather than explain the technical definition, we will merely state some relevant properties:

1) If the ring \(A\) is a principal ideal domain, then \(\text{Ext}^n(M, N) = 0\) for all \(n\) and for all \(N\) precisely when \(M\) is freely generated.

2) \(\text{Ext}^0(M, N) = \text{Hom}(M, N)\)
3) If the following sequences are exact,
\[
\begin{align*}
0 & \to M' \to M \to M'' \to 0 \\
0 & \to N' \to N \to N'' \to 0
\end{align*}
\]
then we have the long exact sequences
\[
\begin{align*}
\cdots & \to \text{Ext}^n(M, N) \to \text{Ext}^n(M', N) \to \text{Ext}^{n+1}(M'', N) \to \cdots \\
\cdots & \to \text{Ext}^n(M, N) \to \text{Ext}^n(M, N'') \to \text{Ext}^{n+1}(M, N') \to \cdots
\end{align*}
\]
i.e., as a functor, \(\text{Ext}^n(-, -)\) is contravariant in the first variable and covariant in the second.

So far we have described \(\text{Ext}\) for modules. It is also possible to define \(\text{Ext}\) for sheaves, and in fact one recovers two distinct possibilities, called local \(\text{Ext}\) and global \(\text{Ext}\).

Local \(\text{Ext}\) is a sheaf derived from a pair of coherent sheaves, say \(\mathcal{M}\) and \(\mathcal{N}\). It can be derived as \(\text{Ext}\) for modules acting on stalks of the sheaves \(\mathcal{M}, \mathcal{N}\), and has the same properties as for modules:

1) If \(\mathcal{M}\) is locally free, i.e., associated to a vector bundle, then \(\text{Ext}^n(\mathcal{M}, -) = 0\) for all \(n > 0\).

2) \(\text{Ext}^0(\mathcal{M}, \mathcal{N}) = \text{Hom}(\mathcal{M}, \mathcal{N})\)

3) And one gets long exact sequences of \(\text{Ext}\) sheaves from short exact sequences of sheaves as above.

In addition to the sheaf local \(\text{Ext}\), it is also possible to define a group, global \(\text{Ext}\). This group is defined as the limit of either of two spectral sequences, with second level terms
\[
\begin{align*}
E_2^{p,q} & = H^p(\text{Ext}^q(\mathcal{M}, \mathcal{N})) \\
E_2'^{p,q} & = H^q(\text{Ext}^p(\mathcal{M}, \mathcal{N}))
\end{align*}
\]

Isomorphism classes of exact sequences of sheaves
\[
0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{M} \to 0
\]
are classified by elements of global \(\text{Ext}^1(\mathcal{M}, \mathcal{N})\). In particular, if \(\mathcal{M}\) is locally free, then \(\text{Ext}^1(\mathcal{M}, \mathcal{N}) = H^1(\text{Hom}(\mathcal{M}, \mathcal{N}))\), a result oft-mentioned in \([1]\).

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