Three Dimensional Differential Calculus on the Quantum Group $SU_q(2)$ and Minimal Gauge Theory

D.G. Pak *

Department of Theoretical Physics, Research Institute of Applied Physics, Tashkent State University, Vuzgorodok, 700095, Tashkent, Republic of Uzbekistan

Classification numbers: 210, 220, 240

Abstract

Three-dimensional bicovariant differential calculus on the quantum group $SU_q(2)$ is constructed using the approach based on global covariance under the action of the stabilizing subgroup $U(1)$. Explicit representations of possible $q$-deformed Lie algebras are obtained in terms of differential operators. The consistent gauge covariant differential calculus on $SU_q(2)$ is uniquely defined. A non-standard Leibnitz rule is proposed for the exterior differential. Minimal gauge theory with $SU_q(2)$ quantum group symmetry is considered.

*E-mail: dmipak@apctp.kaist.ac.kr
1 Introduction

One of the features of non-commutative geometry in the quantum group theory [1-5] is non-uniqueness in defining a differential calculus on the quantum groups and quantum spaces. The bicovariance condition determines a unique differential calculus on the linear quantum groups $GL_q(N)$ (up to symmetry corresponding to the exchange $q \rightarrow \frac{1}{q}$) [6, 7] and provides existence of the corresponding gauge covariant differential algebra [8]. Direct reducing the $GL_q(N)$-bicovariant differential calculus to a case of the special linear quantum group $SL_q(N)$ encounters difficulties connected with a loss of the centrality condition for a quantum determinant. Four-dimensional $4D_\pm$ bicovariant and three-dimensional (3D) left-covariant differential calculi on the simplest special unitary quantum group $SU_q(2)$ were considered as well using a standard Woronowicz approach [3, 6]. A full consistent construction of the 3D bicovariant differential calculus and a gauge covariant differential algebra on the $SU_q(2)$ are unknown up to now, furthermore, there are strong limitations imposed by no-go theorems [8]. A possible way to solve this problem suggests using a non-standard Leibnitz rule as it was considered in ref. [9].

In this paper possible 3D bicovariant differential calculi and gauge covariant differential algebra on the quantum group $SU_q(2)$ are considered in the framework of approach which respects a global $U(1)$-covariance. The group $U(1)$ is a stabilizing subgroup for the quantum group $SU_q(2)$ and the $U(1)$-covariant treatment allows to pass straightforward to the description of the quantum sphere $S^2_q \sim SU_q(2)/U(1)$. In Section 2 we construct explicit representations of $q$-deformed Lie algebras of left-invariant vector fields on $SU_q(2)$ in terms of differential operators. The $U(1)$-covariance constraint reduces the variety of possible covariant differential calculi on $SU_q(2)$ and leads to a unique gauge covariant differential algebra as it is shown in Section 3. We propose non-standard Leibnitz rules for the exterior differential which are compatible with the quantum group $SU_q(2)$ structure and gauge covariance. Section 4 is devoted to construction of the minimal quantum group gauge theory of $SU_q(2)$.

2 $q$-deformed Lie algebras

Following the $R$-matrix formalism [4] the main commutation relation for the generators $T_j^i$ ($i, j = 1, 2$) of the quantum group $SU_q(2)$ is defined by a standard $R$-matrix as follows

$$R_{12} T_1 T_2 = T_2 T_1 R_{12}. \quad (2.1)$$
Let us choose a covariant parametrization for the matrix $T^i_j$

$$T^i_j = \begin{pmatrix} y^1 & x^1 \\ y^2 & x^2 \end{pmatrix} \equiv (y^i x^j), \quad (2.2)$$

where $x^i, y^i$ are generators (coordinates) of the function algebra on the quantum hermitean vector space $U_q^2$ endowed with an involution $*: x^i = y_i$ and $SU_q(2)$-comodule structure. The unimodularity condition takes a simple covariant form

$$D \equiv \det_q T^i_j = x_i y^i = 1, \quad x_i = \varepsilon_{ij} x^j.$$ 

Hereafter the $SU_q(2)$ indices are raised and lowered with the invariant metric $\varepsilon_{ij}(\varepsilon_{12} = 1, \varepsilon_{21} = -\frac{1}{q}).$

The parametrization (2.2) was used in a harmonic formalism [10] of extended superfield supersymmetric theories. The coordinates $(x, y)$ parametrize the quantum sphere $S^2_q \sim SU_q(2)/U(1)$ and are just the quantum generalizations of classical harmonic functions $(u^\pm)$ (so called "harmonics")

$$x^i \equiv u^{+i}, \quad y^i \equiv u^{-i}.$$ 

The signs ($\pm$) correspond to charges ($\pm 1$) of a stabilizing subgroup $U(1)$ for the quantum group $SU_q(2)$. To simplify notations we shall not pass to the notations adopted in the harmonic formalism keeping in mind that all geometric objects (like coordinates, derivatives, differential forms etc.) have definite $U(1)$ charges.

Consider main commutation relations between the coordinates $(x, y)$ and derivatives $\partial_i \equiv \frac{\partial}{\partial x^i}, \quad \bar{\partial}^i \equiv \frac{\partial}{\partial y_i}$ on the quantum group $SU_q(2)$:

$$R_{12}(\partial_T)_1(\partial_T)_2 = (\partial_T)_2(\partial_T)_1 R_{21}, \quad (\partial_T)_j^i \equiv \begin{pmatrix} \bar{\partial}_1 & \bar{\partial}_2 \\ \partial_1 & \partial_2 \end{pmatrix},$$

$$\partial_i x^k = q^{3-2i} \delta_i^k + q Y^{nk}_m x^m \partial_n, \quad \bar{\partial}^j y_j = \delta_j^i + q y_m \bar{\partial}^n \tilde{R}_m^{nj},$$

$$\partial_i y_j = q (\tilde{R}^{-1})_{ji} y_k \partial_k, \quad \bar{\partial}^i x^j = \frac{1}{q} \tilde{R}^{ij}_{kl} x^k \bar{\partial}^l,$$ 

$$Y_{sj}^{ri} = (\tilde{R}^{-1})_{ji}^{r} q^{2(s-i)}.$$ 

The commutation relations (2.3) do not differ on principle from ones given in ref. [11]. Our choice is motivated by using manifest covariant tensor notations which are convenient in constructing explicit representations for the $q$-Lie algebras. Thus, one implies all geometric
objects with upper (lower) indices to be transformed under the quantum group co-action $\Delta$ like classical co-(contra-) variant tensors. For instance, a second rank tensor $N_{ij}$ will be transformed as follows

$$(N_{ij})' = (T^l)^i_k N^l_k$$

(2.4)

(Hereafter the signs $\otimes$ of tensor product are omitted).

Let us define the left-invariant first-order differential operators

$$D^{++} \equiv x_i \bar{\partial}^i, \quad D^{--} \equiv -y_i \bar{\partial}^i,$$

where $(\pm\pm)$ correspond to $U(1)$ charges $(\pm 2)$. The action of the operators $D^{\pm\pm}$ on the coordinates $(x, y)$ has a simple form

$$D^{++} x^i = 0, \quad D^{--} x^i = y^i,$$
$$D^{++} y_i = x_i, \quad D^{--} y_i = 0.$$

The Leibnitz rule for these differential operators may be written in a convenient form if one considers their action on functions with definite $U(1)$ charges. The functions are defined in analogy with the classical case [10] and can be decomposed in formal series

$$f^{(n \geq 0)}(x, y) = \sum_{k=1}^{\infty} C_{(i_1i_2...i_{k+n}j_1j_2...j_k)} x^{i_1} x^{i_2} ... x^{i_{k+n}} y^{j_1} y^{j_2} ... y^{j_k},$$

(2.5)

where $C_{(i_1i_2...i_{k+n}j_1j_2...j_k)}$ are $-n$-number coefficients symmetrized over all indices. Functions with negative charges are defined in a similar manner. After some calculations one can find the next Leibnitz rule for the operators $D^{\pm\pm}$:

$$D^{\pm\pm} (f^{(m)} g^{(n)}) = (D^{\pm\pm} f^{(m)}) g^{(n)} + q^{-m} f^{(m)} D^{\pm\pm} g^{(n)}.$$

(2.6)

This is a special feature of quantum group non-commutative geometry that the quantum analogue to classical $U(1)$ generator can be realized as a second-order differential operator

$$D^0 \equiv -x_i \bar{\partial}^i - q^2 y_i \bar{\partial}^i + (1 - q^2) x_i y_k \bar{\partial}^k \bar{\partial}^i.$$

(2.7)

The operator $D^0$ has eigenfunctions which are just the functions with definite $U(1)$ charges

$$D^0 f^{(n)} = \{n\}_q f^{(n)}, \quad \{n\}_q \equiv \frac{1 - q^{-2n}}{1 - q^{-2}},$$

(2.8)
where \( \{ n \}_q \) is a \( q \)-number. It is not hard to check the following Leibnitz rule for the operator \( D^0 \)

\[
D^0(f^{(m)}g^{(n)}) = (D^0 f^{(m)})g^{(n)} + q^{-2m} f^{(m)} D^0 g^{(n)}. \tag{2.9}
\]

Reducing the space of functions on \( SU_q(2) \) to the space of functions with a definite \( U(1) \) charge one obtains the covariant description of the coset \( S^2_q \sim SU_q(2)/U(1) \).

By direct calculating one can verify that the operators \( D^{\pm\pm,0} \) form the \( q \)-deformed Lie algebra of \( SU_q(2) \) [12]

\[
[D^0, D^{++}], q^{-4} = \{2\}_q D^{++}, \quad [D^0, D^{--}] q^{4} = \{-2\}_q D^{--},
\]

\[
[D^{++}, D^{--}] q^{2} = D^0, \tag{2.10}
\]

here, \([A, B]_q \equiv AB - q^s BA\). Note, that the algebra (2.10) is valid irrespective of whether one imposes the unimodularity constraint \( D = 1 \). We shall treat the algebra (2.10) as a main \( q \)-deformed Lie algebra of left-invariant vector fields on the quantum group \( SU_q(2) \).

A corresponding \( q \)-generalized Jacobi identity is available

\[
[D^0, [D^{++}, D^{--}] q^2] + [D^{++}, [D^{--}, D^0] q^{-4}] q^{-2} + q^2 [D^{--}, [D^0, D^{++}] q^{-4}] q^{-2} = 0.
\]

Let us now pass to constructing other possible \( q \)-deformed Lie algebras of left-invariant vector fields on the \( SU_q(2) \). For this purpose we consider differential operators \( \mu, \nu \) [13]

\[
\mu = 1 + (q^2 - 1) y_i \bar{\partial}^i, \quad \nu = 1 + (1 - \frac{1}{q^2}) x_i \partial^i. \tag{2.11}
\]

One can see that the operators \( \mu, \nu \) obey the simple commutation relations

\[
\mu D^{--} = q^2 D^{--}\mu, \quad \mu D^{++} = \frac{1}{q^2} D^{++}\mu,
\]

\[
\mu D^0 = D^0\mu, \quad \mu \nu = \nu \mu.
\]

Similar formulae hold for the operator \( \nu \) as well. Using these relations one can find that the operators \( D^{++}, D^{--}, D^0 \) defined by the next equations

\[
D^{++} = \mu^{-\frac{1}{2}} D^{++}, \quad D^{--} = \nu^{-\frac{1}{2}} D^{--},
\]

\[
D^0 = \frac{1}{q} \mu \nu D^0 \equiv [\partial^0]_q
\]

generate just the Drinfeld-Jimbo quantum algebra

\[
[\partial^0, D^{++}] = 2 D^{++}, \quad [\partial^0, D^{--}] = -2 D^{--},
\]

\[
[D^{++}, D^{--}] = [\partial^0]_q, \tag{2.12}
\]

To construct other possible $q$-Lie algebras one introduces another differential operators $\Delta^{++}, \Delta^{--}, \Delta^0$ as follows

$$
\Delta^{++} = D^{++}, \quad \Delta^{--} = \frac{q^2 - 1}{q^{2p}} \hat{Z}^{1-p} D^{--},
$$

$$
\Delta^0 = 1 - \frac{\hat{Z}^s}{1 - q^{2s}},
\hat{Z} \equiv (\mu \nu)^{-\frac{1}{2}}, \quad \hat{Z} f^{(n)} = q^n f^{(n)}.
$$

(2.13)

The operators $\Delta^{\pm\pm,0}$ generate the next $q$-deformed Lie algebra:

$$
\Delta^{++} \Delta^{--} - q^{2p} \Delta^{--} \Delta^{++} = \frac{\hat{Z}^2 - 1}{\hat{Z}^{1+p}},
\Delta^0 \Delta^{++} - q^{2s} \Delta^{++} \Delta^0 = \Delta^{++},
\Delta^0 \Delta^{--} - q^{-2s} \Delta^{--} \Delta^0 = -q^{-2s} \Delta^{--},
$$

where $s, p$ – arbitrary integers. The equation (2.13) allows to express the operator $\hat{Z}$ in terms of $\Delta^0$, then the arbitrariness in the choice of parameters $s, p$ can be reduced by considering only quadratic in $\Delta^{\pm\pm,0}$ $q$-commutators.

3 Gauge covariant differential algebra

In this section we give description of possible $SU_q(2)$ bicovariant differential algebras with $U(1)$ conserved charge. The gauge covariance condition leads to a unique differential algebra of $SU_q(2)$. At the same time a Leibnitz rule for the exterior differential is not fixed yet. To find the differentiation rules one needs to choose a corresponding $q$-Lie algebra of left-invariant vector fields.

Consider the left-invariant Cartan 1-forms $\Omega$ on the quantum group $SU_q(2)$

$$
\Omega = dT^\dagger T \equiv \left( \begin{array}{cc} \omega^0 & \omega^{++} \\ \omega^{--} & -q^2 \omega^0 \end{array} \right),
$$

where $\omega^0, \omega^{++}, \omega^{--}$ are the basic left-invariant differential 1-forms with corresponding $U(1)$ charges $(0, +2, -2)$. One defines gauge transformations as follows

$$
T^g = \tilde{T} T,
\Omega^g = \Omega - T^\dagger \tilde{\Omega} T,
\tilde{\Omega} \equiv d\tilde{T}^{-1} \tilde{T},
$$

(3.1)

where the matrix $\tilde{T}$ commutes with the matrices $T, dT$ and satisfies the same equation (2.1) as for the matrix $T$. In the case of gauge symmetry the matrices $\tilde{T}, T$ depend on
the coordinates of a base space-time and the connection 1-form $A$ has the same transformation and commutation properties as the right invariant 1-form $dT_T^\dagger$. It turns out that the requirement of global $U(1)$-covariance and the consistency with the quantum group structure determine uniquely all commutation relations between the differential 1-forms $\omega$ and the coordinates $(x, y)$. As a result we have

$$\begin{align*}
\omega^{++}x &= qx^{++}, \\
\omega^{--}x &= \frac{1}{q}x^{--} + \frac{1}{q} - q^4 y\omega^0, \\
\omega^{++}y &= \frac{1}{q}y^{++}, \\
\omega^{--}y &= qy^{--}, \\
\omega^0x &= x\omega^0 + \left(1 - \frac{1}{q^2}\right)y\omega^{++}, \\
\omega^0y &= y\omega^0.
\end{align*}$$

(3.2)

Similar consideration of commutation relations for the basic differential 1-forms $\omega^{\pm\pm,0}$ leads to covariant algebras parametrized by a real number $\sigma$:

$$\begin{align*}
\omega^{++}\omega^{++} &= \omega^{--}\omega^{--} = 0, \quad \omega^{\pm\pm,0} + q^{\pm2}\omega^0\omega^{\pm\pm} = 0, \\
\omega^{++}\omega^{--} + q^\sigma\omega^{--}\omega^{++} + \frac{q^2(1 - q^\sigma)(1 + q^2)}{q^2 - 1}\omega^0\omega^0 &= 0, \\
\omega^0\omega^0 &= \frac{1 - q^2}{q^2(1 + q^2)}\omega^{++}\omega^{--}.
\end{align*}$$

(3.3-3.5)

It should be noted that the algebra defined by eqs. (3.3-3.5) is bicovariant irrespective of whether one considers the last relation (3.5). Requiring the covariance under the gauge transformations and using the additional commutation constraint

$$\tilde{\Omega}\Omega = -q^2\Omega\tilde{\Omega}$$

one finds a unique gauge covariant differential algebra at $\sigma = 4$:

$$\begin{align*}
\omega^{++}\omega^{++} &= \omega^{--}\omega^{--} = 0, \\
\omega^{\pm\pm,0} + q^{\pm2}\omega^0\omega^{\pm\pm} &= 0, \\
(1 + q^2)^2\omega^0\omega^0 &= \frac{1}{q^2}\omega^{++}\omega^{--} + q^2\omega^{--}\omega^{++}.
\end{align*}$$

(3.6)

The equation (3.5) is not gauge covariant and should be omitted. So defined gauge covariant differential algebra differs from one considered in refs. [6,9]. Our treatment does not contain the condition of vanishing for the central element $C_2 \equiv tr_q(\Omega^2)$, which is not gauge covariant. Here we have used the notion of the $q$-deformed covariant trace [4, 7].

One can rewrite the commutation relations for the gauge covariant differential algebra in terms of the $R$-matrix. Direct checking leads to the next formulae

$$\begin{align*}
R_{12}dT_1T_2 &= T_2dT_1R_{12}, \\
R_{12}\Omega_2R_{12}^{-1}\Omega_1 + \frac{1}{q^2}\Omega_1R_{12}\Omega_2R_{12}^{-1} - \frac{q^2}{1 + q^2 + q^4(E_{12} - (1 + q^2)A_{21})tr_q\Omega^2} &= 0.
\end{align*}$$

(3.7-3.8)
here $E_{kl}^{ij} = \delta_k^i \delta_l^j$ and $A_{21}$ is the quantum antisymmetrizer [11]. Note, that the first relation in (3.7) had been obtained earlier in ref. [9].

To construct an exterior differential it is convenient to use the definition based on the dualism between the exterior algebra of differential forms and the $q$-Lie algebra of vector fields. In this way the Leibnitz rule is followed straightforwardly and it depends only on a special choice of the $q$-Lie algebra.

Let us start from a general 3D $q$-Lie algebra of left-invariant vector fields $D^a = (D^{++}, D^{--}, D^0)$ on the quantum group $SU_q(2)$ with a Lie bracket

$$[D^a, D^b]_B \equiv D^a D^b - B^{abcd} D^c D^d = C^{abc} D^c.$$

We consider the matrix $B^{abcd}$ to be unitary, so that it generates a representation of the permutation group. Thus, one can easily define the alteration rules for the tensor algebra of vector fields. Moreover, a generalized Jacobi identity will be available as well.

The basic left-invariant differential 1-forms $\omega^a$ are defined as dual objects by means of the scalar product $\omega^a(D^b) = \delta^{ab}$. The action of the exterior differential on arbitrary functions $f$ and differential 1-forms $u$ is defined in analogy with the classical case [14]

$$df(D^a) = D^a f,$$

$$du(D^a, D^b) = -\frac{1}{2} (D^a u(D^b) - B^{abcd} D^c u(D^d) - u([D^a, D^b]_B)),$$

$$du(D^a, D^b) = -B^{abcd} du(D^c, D^d).$$

Rules for the exterior differentiation of the differential $(n > 1)$-forms can be generalized in a similar fashion. The Cartan-Maurer equations have a standard form

$$d\omega^d(D^a, D^b) = \frac{1}{2} C^{abc} \omega^d(D^c).$$

As a concrete example we consider the $q$-Lie algebra (2.10) which is consistent with the gauge covariant algebra of left-invariant differential 1-forms (3.7). In that case differentiation rules (3.9) can be rewritten in a more familiar form after using the explicit tensor representation for the exterior products of 2-forms $\omega^a \wedge \omega^b$. After some calculations one finds

$$df = \omega^a D^a f,$$

$$d(\omega^{++} f) = d\omega^{++} f + \beta \omega^0 \omega^{++} D^0 f - d\omega^0 D^{++} f,$$

$$d(\omega^{--} f) = d\omega^{--} f + \beta q^2 \omega^0 \omega^{--} D^0 f + q^2 d\omega^0 D^{--} f,$$

$$d(\omega^0 f) = d\omega^0 f + \beta q^2 \omega^{++} \omega^0 D^{--} f + \beta \omega^{--} \omega^0 D^{++} f,$$

$$\beta \equiv \frac{1 + q^4}{q^2(1 + q^2)}.$$
It should be noted that the formulae (3.11) involve just three independent basis differential 2-forms $\omega^0\omega^{++},\omega^0\omega^{--},d\omega^0$ in the space of exterior 2-forms in correspondence with the classical case. The fourth linearly independent basis 2-form $\sigma^0$ can be defined as follows

$$\sigma^0 = \frac{1}{1 + q^2}(\omega^{++}\omega^{--} + q^2\omega^{--}\omega^{++}) ,$$

The form $\sigma^0$ takes a non-zero value only for the symmetrical tensor product $D^0 \otimes D^0$:

$$\sigma^0(D^0, D^0) = \rho,$$

where the number $\rho$ vanishes in the classical limit $q \to 1$. Due to this property the form $\sigma^0$ does not appear in eqs. (3.11).

To construct the differentiation rules for the $(n > 1)$-forms it is convenient to use the specific structure of the exterior algebra of $SU_q(2)$. It is easy to check that the invariant 2-form

$$C \equiv \frac{\beta q^4}{1 + q^2 + q^4} C_2 = \frac{\beta q^2}{1 + q^2}(\omega^{++}\omega^{--} + \omega^{--}\omega^{++})$$ (3.12)

is a central element. We put the natural constraint

$$d(C\omega^{(n)} f) = C d(\omega^{(n)} f).$$ (3.13)

Using this constraint and starting from the most general form for differentiation rules one finds

$$d(\omega^0\omega^{++} f) = \pm q^4 v D^{++} f,$$

$$d(\omega^{++}\omega^{--} f) = \frac{1}{\beta} C df + v D^0 f,$$

$$d(\omega^{--}\omega^{++} f) = \frac{1}{\beta q^2} C df - v D^0 f,$$

$$d(\omega^0\omega^{++}\omega^{--} f) = \frac{1}{\beta} C d \omega^0 f - q^4 C \omega^0 \omega^{++} D^{--} f - \frac{1}{q^2} C \omega^0 \omega^{--} D^{++} f ,$$

$$d(v f) = 0 ,$$

$$v \equiv \frac{1}{2}(\omega^0\omega^{++}\omega^{--} - q^2\omega^0\omega^{--}\omega^{++}) ,$$ (3.14)

where $v$ is a volume 3-form on the $SU_q(2)$.

All basis differential forms of order $n > 3$ can be obtained from lower order forms multiplied by the invariant $C$ in an appropriate degree. So that the relations (3.11, 3.14) complete the differential rules for the differential algebra of $SU_q(2)$.

Having carried out some calculations one can also find the explicit expressions for the Cartan-Maurer equations (3.10)

$$d\Omega = \Omega^2 - \frac{q^2}{1 + q^2} I \text{tr}_q \Omega^2 .$$

The right hand side of the equation contains only the traceless part of $\Omega^2$. 

9
4 Construction of the minimal $SU_q(2)$ gauge theory

A minimal gauge theory corresponding to the quantum algebra $SU_q(2)$ was proposed in ref. [15], where the initial gauge transformations contain by definition the antipodal map. One can try to formulate a gauge Yang-Mills theory for the quantum group $SU_q(2)$ in analogy with a covariant $GL_q(N)$ version proposed in ref. [7]. For this purpose one should define the algebra of main operators (matter fields and gauge potential) and a corresponding comodule structure.

In the case of the quantum group $SU_q(2)$ it is convenient to put the matter field $\phi^i(\bar{\phi}^i)$ into one matrix $\Phi = (\bar{\phi}^i \phi^i)$ . The operators $(\Phi, d\Phi)$ generate the $Z_2$-graded algebra $Z$ with the same commutation relations as for the differential algebra of $SU(2)$ . The algebra $Z$ is a left $SU_q(2)$-comodule with the following co-action:

$$\Phi \to \Phi^g = T\Phi, \hspace{1cm} \text{(4.1)}$$

$$d\Phi \to (d\Phi)^g = (dT)\Phi + T(d\Phi). \hspace{1cm} \text{(4.2)}$$

All axioms for the comodule are fulfilled.

We introduce also an operator $A$ (gauge potential 1-form) satisfying the same commutation relations as for the right-invariant Cartan 1-forms on the $SU_q(2)$ . One can consider the quantum analogue to the gauge transformation for the $A$:

$$A \to A^g = TAT^\dagger + (dT)T^\dagger. \hspace{1cm} \text{(4.3)}$$

A covariant differential $\Delta$ acting on the matter field is defined as

$$\Delta \Phi = (d - A)\Phi. \hspace{1cm} \text{(4.4)}$$

The curvature 2-form $F$ is introduced as follows

$$F = dA - A^2 + \frac{1}{1 + q^2}tr_q A^2 \hspace{1cm} \text{(4.5)}$$

and it contains only traceless part.

To define the commutation relations in the algebra $G$ generated by the operators $(\Phi, d\Phi, A, dA)$ we will take into account the compatibility conditions with the gauge transformations and the centrality property for the quantum determinant $\det_q \Phi = 1$. The commutation relations for the operators $\Phi, A$ are uniquely defined from the formula (3.7), since the gauge potential $A$ in a pure gauge limit is just the right-invariant Cartan form on the quantum group $SU_q(2)$ , so we have

$$\Phi_1A_2 = R_{21}A_2R_{21}^{-1}\Phi_1. \hspace{1cm} \text{(4.6)}$$
We demand the covariant combination \((F \Phi)\) to have the same commutation relations with itself and \(A\) as for the operator \(\Phi\)

\[
(F \Phi)_1 A_2 = R_{21} A_2 R_{21}^{-1} (F \Phi)_1, \tag{4.7}
\]

\[
R_{12} (F \Phi)_1 (F \Phi)_2 = (F \Phi)_2 (F \Phi)_1 R_{12}. \tag{4.8}
\]

Using these formulae one can easily derive the next commutation relations:

\[
F_1 R_{21} A_2 R_{21}^{-1} = R_{21} A_2 R_{21}^{-1} F_1, F_1 R_{21} \tilde{r}_2 R_{21}^{-1} = R_{21} \tilde{r}_2 R_{21}^{-1} F_1, \tag{4.9}
\]

where \(\tilde{r} = (d \Phi) \Phi^\dagger\).

The bicovariance condition, the traceless property for the \(F\) and the centrality of the quantum determinant \(\text{det}_q \Phi\) lead to the following commutation relation for the operators \((\Phi, F)\):

\[
\Phi_1 F_2 = R_{21} F_2 R_{21}^{-1} \Phi_1. \tag{4.10}
\]

From this equation taking into account the eqn. (34) one finds

\[
R_{12} F_1 R_{21} F_2 = F_2 R_{12} F_1 R_{21}. \tag{4.11}
\]

Note, that the last equation coincides with the corresponding relation in \(GL_q(N)\) gauge theory [7]. It is not difficult to check the next simple relations as well:

\[
R_{12} \Delta \Phi_1 \Phi_2 = \Phi_2 \Delta \Phi_1 R_{12}, \tag{4.12}
\]

\[
\Phi_1 dA_2 = R_{21} dA_2 R_{21}^{-1} \Phi_1. \tag{4.13}
\]

The commutation relations obtained above are the main ones which imply all other commutations in the algebra \(G\). To construct a formal expression for the Lagrangian of the gauge theory one needs to specify the underlying space-time and define the dual \(*\)-operation. The simplest variant correspond to the choice of the space-time isomorphic to the quantum space of \(SU_q(2)\). Another problem is the construction of Leibnitz rules for the exterior differential which have essentially non-standard form. Differentiation rules for the algebra of matter fields \((\Phi^i, d \Phi)\) are defined by similar formulae (23, 26) in full analogy with the case of the differential algebra of \(SU_q(2)\). For instance, one has the following equations for the quadratic combinations of matter fields

\[
d(\phi^i \phi^j) = d\phi^i \phi^j + \frac{1}{q^2} \phi^i d\phi^j + \frac{q^2 - 1}{q^3} \epsilon^{ijk} d\phi_k \phi^k,
\]

\[
d(\bar{\phi}^i \bar{\phi}^j) = d\bar{\phi}^i \bar{\phi}^j + q^2 \bar{\phi}^i d\bar{\phi}^j,
\]

\[
d(\bar{\phi}^i \phi^j) = d\bar{\phi}^i \phi^j + q^2 \bar{\phi}^i d\phi^j + (q^2 - 1) \bar{\phi}^i \bar{\phi}^j d\phi_k \phi^k.
\]
Another possible way towards a consistent minimal quantum group gauge Yang-Mills theory corresponds to the differential calculus with a $q$-Lie algebra differed from one defined by eqs. (2.10).

**Acknowledgments**

The author would like to acknowledge Professor M. Arik and the members of the Organizing Committee for kind hospitality and financial support. Author thanks F. Mueller-Hoissen, V.D. Gershun and M. Nomura for useful discussions.

**References**

[1] V.G. Drinfeld *Quantum Groups*, in: Proc. Int. Cong. Math. **1** (Berkeley, CA, USA, 1986), 793.

[2] M. Jimbo, Int. J. Mod. Phys. **A4** (1989) 3759.

[3] S.L. Woronowicz, Publ. Res. Inst. Math. Sci., Kyoto University **23** (1987) 117.

[4] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Algebra i Analiz. **1** (1989) 178.

[5] S.L. Woronowicz, Comm. Math. Phys. **122** (1989) 125.

[6] A.P. Isaev and P.N. Pyatov, Phys. Lett. **A179** (1993) 81.

[7] A.P. Isaev and Z. Popowicz, Phys.Lett. **B307** (1993) 353.

[8] I.Ya.Aref’eva and G.E.Antyunov, ”On *-representations of the $Z_2$-graded extension of the quantum group $U_q(2)$” Trudi MIAN, v.203 (1994)

[9] L.D. Faddeev and P.N. Pyatov, *The Differential Calculus on Quantum Linear groups*. Preprint [hep-th/9402070](http://arxiv.org/abs/hep-th/9402070) (1994).

[10] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984) 469.

[11] J. Wess and B. Zumino, Nucl. Phys. (Proc. Suppl.) **B18** (1990) 302.

[12] A. Schirrmacher, J. Wess and B. Zumino, Zeit. Ph. **C49** (1991) 317.

[13] O. Ogievetsky, Lett. Math. Phys. **24** (1992) 245.
[14] S. Kobayashi and K. Nomizu, Vol.1. *Foundations of Differential Geometry* (Interscience Publishers, N.Y., London, 1963).

[15] T. Sudbery Phys.Lett. B375 (1996) 75.