Distributed Generalized Nash Equilibrium Seeking for Non-Monotone Energy Sharing Games

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Abstract—With the proliferation of distributed generators and energy storage systems, traditional passive consumers in power systems have been gradually evolving into the so-called “prosumers”, i.e., proactive consumers, which can both produce and consume power. To encourage energy exchange among prosumers, energy sharing is increasingly adopted, which is usually formulated as a generalized Nash game (GNG). In this paper, a distributed approach is proposed to seek the Generalized Nash equilibrium (GNE) of the energy sharing game. To this end, we first reveal by proof that the GNG is generally not monotone. In such a situation, the pseudo-gradient of the disutilities is not monotone, which makes existing methods inapplicable. To overcome this obstacle, we convert the GNG into an equivalent optimization problem. A Krasnosel’skiĭ-Mann iteration type algorithm is thereby devised to solve the problem and consequently find the GNE in a distributed manner. We further prove the convergence of the proposed distributed algorithm based on the nonexpansive operator theory. The performance of the algorithm is validated by experiments with three prosumers, and the scalability is tested by simulations using 123 prosumers.

Index Terms—Energy sharing, prosumer, distributed algorithm, generalized Nash game, generalized Nash Equilibrium.

I. INTRODUCTION

It has been recognized that our power system is undergoing a fundamental transition due to: 1) The proliferation of distributed generations (DGs), such as wind turbines, photovoltaics, energy storage systems and electric vehicles [1]–[3]; 2) The advancement of communications and control in consumer-level via smart appliances and energy management systems [3]–[8]. Together, these changes allow traditional passive consumers to convert into the so-called “prosumers”, i.e., proactive consumers, that can actively regulate their generation and consumption [9]–[13]. Conventionally, a hierarchical structure is utilized in the power system energy management, usually in a centralized way. However, the centralized manner may face great challenges raised by the ever-increasing number of prosumers and uncertainties of renewable generations. This essentially advocates a distributed paradigm [14]–[17] in energy management. Particularly, energy sharing turns to be a promising form of the market that encourages energy trading among prosumers. As prosumers typically belong to different owners, the inherent competition may lead to strategic behaviors in energy sharing. In this situation, each prosumer intends to optimize its own profit while maintaining the power balance over the whole system. This leads to a generalized Nash game (GNG) with global constraints. In this regard, it is desirable to investigate distributed approaches to seeking the generalized Nash equilibrium (GNE) of the GNG.

There are many papers investigating distributed methods of GNE seeking, which can be roughly divided into two categories in terms of methods used: gradient-based algorithms [18]–[23], and proximal-point algorithms [24]–[27]. In the first type, the pseudo-gradient of each player’s disutility function is utilized to seek the GNE, including continuous-time algorithms [18], [19] and discrete algorithms [20]–[23]. In [18], a distributed continuous-time projection-based algorithm is proposed to seek the GNE of aggregative games with linear coupled constraints. This is extended to the case where players have nonsmooth payoff functions in [19]. In [20], two distributed primal-dual algorithms are proposed for computing a GNE in noncooperative games with shared affine constraints. The operator splitting method is utilized to prove the convergence of the algorithms. It is further improved in [21] to consider communication time delays and partial-decision information. In [22], three stochastic gradient strategies are developed to seek GNE where agents are subject to randomness in the environment of unknown statistical distribution. In [23], the relations between Nash and Wardrop equilibria of the aggregative game are investigated and two algorithms are proposed to seek the equilibrium.

In the proximal-point algorithms, the Alternating Direction Method of Multipliers (ADMM) method is widely used. An inexact-ADMM algorithm is proposed in [24]. This method is improved in [27], where each player only has partial information of their opponents and the communication graph is not necessarily the same as the cost dependency graph of each player. In [25], two double-layer preconditioned proximal-point algorithms are proposed to seek GNE with both coupled equality and inequality constraints, respectively. For the GNG with a special structure, i.e., the coupling in the cost functions of the agents is linear, a distributed proximal-point algorithm is developed in [26] for GNG with maximally monotone pseudo-gradient.

The aforementioned works have made great progress in the context of distributed GNE seeking. Generally, the pseudo-gradient of the pay-off function is assumed to be monotone [19], maximally monotone [25], [26], strictly monotone [18], strongly monotone [20]–[23], [27] or even cocoercive [24]. However, in the energy sharing game among prosumers, the
pseudo-gradient does not satisfy the above assumptions, which makes existing methods inapplicable. In this paper, we offer a different perspective for seeking the GNE in the GNG. Instead of dealing with the GNG directly, we alternatively solve an equivalent optimization problem of the original game problem. Our main contributions are as follows.

- We prove that, in the energy sharing game, the pseudo-gradient of the disutilities is generally non-monotone. It theoretically reveals that why the existing distributed algorithms for GNE seeking [13]-[27], which are built on the monotone assumption, do not apply when solving the energy sharing game problem.
- To solve such a non-monotone game problem in a distributed way, we transform the energy sharing game into an equivalent optimization problem. Then, a distributed algorithm is devised based on Krasnosel’skii-Mann iteration to solve the equivalent counterpart instead of solving the original game problem directly. By constructing a firmly nonexpansive operator, we prove that the proposed distributed algorithm converges to the GNE of the original energy sharing game.

The rest of this paper is organized as follows. In Section II we briefly introduce some necessary preliminaries. Section III formulates the energy sharing model. In Section IV key properties of the energy sharing game are analyzed including monotonicity and existence of GNE. A distributed GNE seeking algorithm is proposed in Section V. The convergence of the algorithm is proved in Section VI. The effectiveness of the algorithm is verified in Section VII by experiments and simulation studies. Section VIII concludes the paper.

II. Preliminaries

In this paper, use $\mathbb{R}^n$ to denote the $n$-dimensional Euclidean space. For a matrix $A$, $[A_{ij}]$ is the entry in the $i$-th row and $j$-th column of $A$. For vectors $x, y \in \mathbb{R}^n$, $x^T y = \langle x, y \rangle$ denotes the inner product of $x, y$. $\|x\|_2 = \sqrt{x^T x}$ denotes the Euclidean norm of $x$. Denote the inner product under a positive definite matrix $Q$ by $\langle x, y \rangle_Q = \langle Qx, y \rangle$. Similarly, the norm induced by $Q$ is $\|x\|_Q = \sqrt{(Qx)^T x}$. The following relationship holds for a $Q$-induced norm.

$$\|a - c\|_Q^2 - \|b - c\|_Q^2 = 2(a - b, a - c)_Q - \|a - b\|_Q^2$$

which can be obtained by the equation $\|a + b\|_Q^2 = \|a\|_Q^2 + 2(a, b)_Q + \|b\|_Q^2$.

The identity matrix with dimension $n$ is denoted by $I_n$. Use $\prod_{i=1}^n \Omega_i$ to denote the Cartesian product of the sets $\Omega_i$, $i = 1, \cdots, n$. Define the projection of $x$ onto a set $\Omega$ as

$$P_{\Omega}(x) = \arg \min_{\|y\|_2} \|x - y\|_2$$

Use $\text{Id}$ to denote the identity operator, i.e., $\text{Id}(x) = x$, $\forall x$. Define $N_{\Omega}(x) = \{v | \langle v, y - x \rangle \leq 0, \forall y \in \Omega \}$. We have $P_{\Omega}(x) = (\text{Id} + N_{\Omega})^{-1}(x)$ [28 Chapter 23.1].

For a single-valued operator $T : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, a point $x \in \Omega$ is a fixed point of $T$ if $T(x) = x$. The set of fixed points of $T$ is denoted by $\text{Fix}(T)$. $T$ is nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$, $\forall x, y \in \Omega$.

For $\alpha \in (0, 1)$, $T$ is called $\alpha$-averaged if there exists a nonexpansive operator $R$ such that $T = (1 - \alpha)\text{Id} + \alpha R$. Use $\mathcal{A}(\alpha)$ to denote the class of $\alpha$-averaged operators. If $T \in \mathcal{A}(\frac{1}{4})$, $T$ is called firmly nonexpansive. The graph of $T$ is $\text{gra} T = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n | u \in T(x)\}$. $T$ is monotone if $\forall (x, u), (y, v) \in \text{gra} T, \langle x - y, u - v \rangle \geq 0$. $T$ is maximally monotone if $\text{gra} T$ is not strictly contained in the graph of any other monotone operator.

III. Energy Sharing Game

In this section, the model of prosumers is introduced. Then, we formulate the energy sharing game.

A. Model of prosumers

In this paper, we consider the energy sharing problem among a set of prosumers, denoted by $\mathcal{N} = \{1, 2, \cdots, n\}$. The communication edge is denoted by $E \subseteq \mathcal{N} \times \mathcal{N}$. For a prosumer $i$, the set of its neighbors is denoted by $N_i$. If $j \in N_i$, prosumers $i$ and $j$ can communicate directly. The Laplacian matrix of the communication graph is denoted by $L$ and we have $\text{Id}^T L = 0$, where $\text{Id}$ is a vector with all of the components as $1$.

The scenario is that each prosumer has a power shortage or surplus, denoted by $D_i$. To balance its power, it can produce power itself, denoted by $p_i$, or buy power from (sell power to) the grid, denoted by $q_i$. In the sharing market, the demand function of each prosumer can be expressed by

$$q_i = a_i \mu_c + b_i$$

where $\mu_c$ is the market clearing price, $b_i$ is the willingness to buy. $a_i < 0$ implies the price elasticity. The market clears when the net quantity $\sum_i q_i = 0$ and the obtained sharing price is

$$\mu_c = -\frac{1^T b}{1^T \alpha} = -\frac{1^T b}{N \bar{\alpha}}$$

where $\bar{\alpha} = \frac{1^T \alpha}{N}$ is the average value of $a_i$. In the rest of the paper, $b_{-i} := (b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n)^T$.

B. Energy sharing game

In the energy sharing game, each prosumer intends to minimize its cost while maintaining the global power balance. The optimization problem of each prosumer is

$$\min_{p_i, b_i} f_i(p_i, b_i) = h_i(p_i) + (a_i \mu_c(b) + b_i) \mu_c(b)$$

s.t.

$$p_i + a_i \mu_c(b) + b_i = D_i, \quad \lambda_i$$

$$\sum_{i \in \mathcal{N}} (a_i \mu_c(b) + b_i) = 0, \quad \eta_i$$

$$p_{\alpha} \leq p_i \leq \bar{p}_{\alpha}$$

The disutility function consists of two parts, where $h_i(p_i)$ are the cost of prosumer $i$ to produce $p_i$ and $(a_i \mu_c(b) + b_i) \mu_c(b)$ are the cost of buying power $q_i$. $\lambda_i, \eta_i$ are

$^1$Given a collection of $z_i$ for $i$ in a certain set $Z$, $z$ denotes the column vector $z := (z_i, i \in Z)$ with a proper dimension with $z_i$ as its components.
the Lagrangian multipliers of the constraints (5b) and (5c), respectively.

Regarding the energy sharing game (5), we have following assumptions.

**Assumption 1.** The functions \( h_i(p_i) \) are convex and differentiable.

**Assumption 2.** For a given \( b_{-i} \), the Slater’s condition of the problem (5) holds [29, Chapter 5.2.3], i.e., problem (5) is feasible.

**Assumption 3.** The generation satisfies \( 1^T p < 1^T D < 1^T \overline{p} \).

The Assumption 3 implies that the energy production can strictly satisfy the load demand. There is at least one \( p_i < \overline{p}_i < \overline{p} \) in the steady state.

In summary, the energy sharing game among all prosumers is composed of the following elements:

- **Player:** all prosumers, denoted by \( N = \{1, 2, \ldots, n\} \);
- **Strategy:** generation power \( p_i \) and power interaction \( b_i \);
- **Payoff:** the disutility function \( f_i(p_i, b), \forall i \in N \).

The GNE \((p^*, b^*)\) of the game in (5) is defined as:

\[
(p^*_i, b^*_i) = \arg \min_{p_i, b_i} f_i(p_i, b_i, b^*_{-i}), \quad \text{s.t.} \quad (5b), (5b), (5c), \quad \forall i \in N
\]

**Remark 1.** The energy sharing game model (5) basically comes from [13], and is modified by additionally considering local capacity constraints (5d). Mathematically, the energy sharing game (5) is indeed a GNG by noting that the global constraint (5c) couples the strategies of players. As proved in [13], if the constraint (5d) is not taken into account, the GNG can be equivalently converted into a standard Nash game with a unique Nash equilibrium. However, when (5d) is considered, it is not trivial to eliminate the coupling constraints. This is the key difference between this paper and [13] in the modeling.

The pseudo-gradient of \( f_i(p_i, b), \forall i \in N \), denoted by \( F(p, b) \), is defined as

\[
F(p, b) = \text{col} (\nabla_{p_i, b_i} f_i(p_i, b), \ldots, \nabla_{p_N, b_N} f_N(p_N, b)) \quad (6)
\]

where \( \text{col} (x_1, \ldots, x_n) \) is the column vector stacked with column vectors \( x_1, \ldots, x_n \).

Recently, many papers investigate distributed algorithms of GNE seeking. In these works, a key assumption is that the pseudo-gradient of the disutility is non-monotone. However, the energy sharing game suffers from a non-monotone pseudo-gradient of the disutility, which does not satisfy this condition. Therefore, it could be solved in a distributed way by the simple application of existing methods. This will be explained in detail in the next section.

The solution approach to seeking the GNE and the relation between the associated theoretic results are illustrated in Fig. 1. The main idea is that we transform GNG to a corresponding optimization problem, and then design a distributed algorithm to solve the problem. Consequently, the GNE is obtained.

**IV. Key Properties of the Energy Sharing Game**

In this section, we first show that the pseudo-gradient of the disutility function (5a) is generally not monotone. Then, we analyze the property of the Nash equilibrium and transform the game into an equivalent optimization problem.

**A. Monotonic analysis**

Replacing \( \mu_c \) in (5) using (4), then the sharing game problem (5) can be rewritten as

\[
\begin{align*}
\min_{p_i, b_i} \quad & f_i(p_i, b) = h_i(p_i) + g_i(b_i, b_{-i}) \\
\text{s.t.} \quad & p_i + \frac{1^T a - a_i}{1^T a} b_i - \frac{a_j}{1^T a} \sum_{j \neq i} b_j = D_i, \quad \lambda_i \\
& p_i \leq p_i \leq \overline{p}_i
\end{align*}
\]

where

\[
\begin{align*}
g_i(b_i, b_{-i}) = (a_i \mu_c (b) + b_i) \mu_c (b) \\
& = - \left( \frac{1^T a - a_i}{1^T a} b_i - \frac{a_j}{1^T a} \sum_{j \neq i} b_j \right) b_i + \sum_{j \neq i} b_j \\
& + \frac{a_i}{N^2 \sigma^2} \left( \sum_{j \neq i} b_j \right)^2 \\
& = \alpha_i b_i^2 + \beta_i b_i \sum_{j \neq i} b_j + \delta_i \left( \sum_{j \neq i} b_j \right)^2
\end{align*}
\]

where

\[
\alpha_i = \frac{a_i}{N^2 \sigma^2}, \quad \beta_i = \frac{2a_i - 1^T a}{N^2 \sigma^2}, \quad \delta_i = \frac{a_i}{N^2 \sigma^2}
\]

Because \( a_i < 0 \), we have \( \alpha_i > 0, \beta_i > 0, \delta_i < 0 \). Then, \( g_i(b_i, b_{-i}) \) is strictly convex with respect to \( b_i \) given any \( b_{-i} \).

Regarding disutility function (5a), we have the following results.

**Theorem 1.** The pseudo-gradient defined in (6) is generally non-monotone.

**Proof.** For the first part of \( f_i(p_i, b) \), because \( h_i(p_i) \) is convex and only determined by prosumer \( i \), its gradient and pseudo-gradient are the same and monotone.
For the second part of \( f_i(p_i, b) \), the pseudo-gradient of \( g_i(b) \), \( i \in \mathcal{N} \) is
\[
G(b) = \text{col} \left( \nabla_{b_1} g_1(b_1, b_{-1}), \cdots, \nabla_{b_N} g_N(b_N, b_{-N}) \right)
\]
\[
= \text{col} \left( 2\alpha_i b_i + \beta_i \sum_{j \neq i} b_j \right)
\]
\[
= Ab
\]
where
\[
[A]_{ij} = \begin{cases} 2\alpha_i, & j = i \\ \beta_i, & j \neq i \end{cases}
\]
(10)

Recalling the definition of \( \alpha_i, \beta_i \) in (9), the matrix \( A \) cannot be guaranteed positively definite. Therefore, the condition \( (x - y)^T A^T (x - y) \geq 0 \) does not hold for \( \forall x \geq y \). This implies \( G(b) \) is generally non-monotone, which completes the proof. \( \square \)

Theorem 1 shows that the pseudo-gradient of \( f_i(p_i, b) \) does not satisfy the most fundamental assumption in existing works. This makes existing distributed GNE seeking methods unable to be applied, which are applicable only to monotone games [18]–[27]. In the rest of the paper, we will first transform the game (7) into an equivalent optimization problem and then propose a distributed approach to solving it.

**B. Existence of GNE**

Define the sets
\[
\Omega_i := \left\{ p_i \mid p_i \leq p_i^a \right\}, \quad \Omega = \prod_{i=1}^{n} \Omega_i
\]
and
\[
X_i^c := \left\{ (p_i, b_i) \mid p_i + \frac{1^T a - a_i}{1^T a} b_i - \frac{a_i}{1^T a} \sum_{j \neq i} b_j = D_i \right\}
\]
\[
X^c = \prod_{i=1}^{n} X_i^c, \quad X_i = \Omega_i \cap X_i^c, \quad X = \Omega \cap X^c
\]
(13)
(14)

For any given \( b \), the feasible set of prosumer \( i \), \( X_i \), is closed and convex. Then, we have the following result.

**Lemma 2.** If Assumptions 1 and 2 hold, a point \((p, b)\) is an equilibrium if and only if it is a solution of the variational inequality \( VI(X,F(p,b)) \).

**Proof.** For any given \( b \), the disutility function (7a) is continuous, differentiable and convex with respect to \((p_i, b_i)\). Moreover, the feasible set \( X_i \) is closed and convex. Thus, we have this assertion by [30] Theorem 3.3]. \( \square \)

For the existence and uniqueness of the GNE, we have the following results.

**Theorem 3.** If Assumption 1 and Assumption 2 hold, for the generalized Nash game (7), we have
1) the generalized Nash equilibrium exists;
2) if the Assumption 3 also holds, the generalized Nash equilibrium is unique.

\(^2\)The variational inequality problem \( VI(X,F(x)) \) is to find a vector \( x \in X \) such that \( (y - x)^T F(x) \geq 0 \) for all \( y \in X \) [30].

**Proof.** \( \Rightarrow 1) \) From Lemma 2 we need to find \( x^* := (p_i^*, b_i^*) \in X \) such that
\[
\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in X
\]
(15)

Now, we check the KKT condition for \( VI(X,F(p,b)) \) in [15]. In fact, \( x^* \) is a solution to \( VI(X,F(p,b)) \) if and only if \( x^* \) is the optimal solution to the following optimization problem:
\[
\min_{x \in \mathbb{R}^{2n}} \langle F(x^*), x \rangle, \quad \text{s.t.} \quad x \in X
\]
(16)

If \( x^* \) solves (16), there exists \( \lambda^* \in \mathbb{R} \) such that the following optimality conditions (KKT) are satisfied [31] Theorem 3.25:
\[
0 \in h_i^\prime(p_i^*), \quad \lambda_i^* + N_{\Omega_i}(p_i^*)
\]
(17a)
\[
0 = - (2a_i \mu_c + b_i^* + a_i \lambda_i^*) \frac{1}{1^T a} + \mu_c + \lambda_i^*
\]
(17b)
\[
0 = p_i^* + a_i \mu_c^* + b_i^* - D_i
\]
(17c)
\[
0 = 1^T a \mu_c^* + b_i^* + \sum_{j \neq i} b_j^*
\]
(17d)

From (17b), we know
\[
0 = - (2a_i \mu_c^* + b_i^* + a_i \lambda_i^*) \frac{1}{1^T a} + \mu_c^* + \lambda_i^*
\]
\[
\Rightarrow \lambda_i^* = \frac{1^T a - 2a_i}{a_i - 1^T a} \mu_c^* - \frac{1}{a_i - 1^T a} b_i^*
\]
(18)

From (17c), we know
\[
b_i^* = D_i - p_i^* - a_i \mu_c^*
\]
(19)

Combining (18) and (19), we have
\[
\lambda_i^* = \frac{1^T a - 2a_i}{a_i - 1^T a} \mu_c^* - \frac{D_i}{a_i - 1^T a}
\]
\[
= - \mu_c^* + \frac{p_i^*}{a_i - 1^T a} - \frac{D_i}{a_i - 1^T a}
\]
(20)

Then, the KKT condition (17) is
\[
0 \in h_i^\prime(p_i^*) + \frac{p_i^*}{a_i - 1^T a} - \frac{D_i}{a_i - 1^T a} - \mu_c^* + N_{\Omega_i}(p_i^*)
\]
(21a)
\[
0 = \sum_{i \in \mathcal{N}} D_i - \sum_{i \in \mathcal{N}} p_i
\]
(21b)

It is also the KKT condition of the following problem
\[
\min_p \hat{h}(p) = \sum_{i \in \mathcal{N}} \left( h_i(p_i) + \frac{p_i^2}{2(a_i - 1^T a)} - \frac{D_i}{a_i - 1^T a} p_i \right)
\]
(22a)
\[
\text{s.t.} \quad \sum_{i \in \mathcal{N}} p_i = \sum_{i \in \mathcal{N}} D_i, \quad \mu_c
\]
(22b)
\[
p_i \leq \bar{p}_i \leq p_i
\]
(22c)

where \( \mu_c \) is the Lagrangian multiplier.

Define
\[
\hat{h}_i(p_i) = h_i(p_i) + \frac{p_i^2}{2(a_i - 1^T a)} - \frac{D_i}{a_i - 1^T a} p_i
\]
(23)

We know \( \hat{h}_i(p_i) \) is strongly convex and the Slater’s condition holds by Assumption 2 Thus, \( p_i^* \) and \( \mu_c^* \) exist. By (19), \( b_i^* \)
also exists.  

⇒ 2) If Assumption 3 holds, there exists at least one prosumer \( i \) with \( p_i^* = p_i^* < p_i \). For this \( i \), we have \( \{0\} = N_{\Omega_i}(p_i^*) \). Then, by (21a), \( p_i^* \) is unique. The unique \( b_i^* \) can be obtained from (19). 

Remark 2. As explained in Section IV.A, the game (7) is difficult to solve by existing methods due to the nonmonotone pseudo-gradient. Now, we can alternatively consider the equivalent counterpart (22), which could be solved in a distributed way as long as \( I^a \) is known. In this regard, we require that the market coordinator (or a third-party platform) broadcasts the sum of \( o_i \) of all prosumers. That is needed merely when new prosumers join or existing ones quit, which can be known by the market coordinator or the third-party platform. After getting \( p_i^*, b_i^* \) can be obtained from (19). Then, the energy sharing game problem (7) can be solved.

V. DISTRIBUTED ALGORITHM FOR EQUILIBRIUM SEEKING
In this section, we first propose a distributed algorithm based on Krasnosel’skiǐ-Mann iteration to solve the problem (22), i.e., to solve the generalized game (7). Then, we use the nonexpansive operator theory to prove the convergence of the proposed algorithm.

A. Algorithm design
Before giving the algorithm, we first define a matrix and a function. Define the matrix

\[
\Theta := \begin{bmatrix}
\Gamma & 0 & -I_n \\
0 & \alpha_z^{-1}I_n & -L \\
-I_n & -L & \alpha_\mu^{-1}I_n
\end{bmatrix}
\]

where the matrix \( \Gamma = \text{diag}(\gamma_i) \). \( \gamma_i, \alpha_z \) and \( \alpha_\mu \) are constant to make \( \Theta \) positive definite.3

Define the function

\[
H_i(p_i) = \frac{\partial h_i}{\partial p_i}(p_i) + \gamma_i p_i
\]

We have the follow result.

Lemma 4. The function \( H_i(p_i) \) is strictly monotone. Moreover, its inverse function \( H_i^{-1}(p_i) \) exists and is also strictly monotone.

The proof is straightforward as \( h_i(p_i) \) is strongly convex, which is omitted here.

We propose the following algorithm, which is denoted by SGNE (Seeking the Generalized Nash Equilibrium).

Algorithm 1 SGNE
Prediction phase:
For prosumer \( i \), it computes

\[
\tilde{p}_{i,t} = p_{i,t} + \eta(p_{i,t} - \hat{p}_{i,t-1})
\]

(26a)

\[
\tilde{z}_{i,t} = z_{i,t} + \eta(z_{i,t} - \hat{z}_{i,t-1})
\]

(26b)

\[
\tilde{\mu}_{i,t} = \mu_{i,t} + \eta(\mu_{i,t} - \hat{\mu}_{i,t-1})
\]

(26c)

Update phase:
For prosumer \( i \), it computes

\[
p_{i,t+1} = \left[H^{-1}(\gamma_i \tilde{p}_{i,t} - \tilde{\mu}_{i,t})\right]_{p_i}
\]

(26d)

Communicate with its neighbors \( j \in N_i \) to get \( \tilde{\mu}_{j,t} \), and compute

\[
z_{i,t+1} = \tilde{z}_{i,t} - \sigma_z \sum_{j \in N_i} (\tilde{\mu}_{i,t} - \tilde{\mu}_{j,t})
\]

(26e)

Communicate with its neighbors \( j \in N_i \) to get \( \tilde{z}_{j,t}, \tilde{z}_{j,t+1} \), and compute

\[
\mu_{i,t+1} = \tilde{\mu}_{i,t} + \sigma_\mu \left(2p_{i,t+1} - \tilde{p}_{i,t} - D_i\right) + 2 \sum_{j \in N_i} (\tilde{z}_{i,t} - \tilde{z}_{j,t})
\]

(26f)

In the algorithm, only communications with neighbors are needed, which means that it is fully distributed.

B. Algorithm reformulation
From (26d), we know

\[
H^{-1}(\gamma_i \tilde{p}_{i,t} - \tilde{\mu}_{i,t}) \begin{cases} 
\leq p_{i,t+1}, \quad p_{i,t+1} = p_i \\
= p_{i,t+1}, \quad p_i < p_{i,t+1} < p_i \\
\geq p_{i,t+1}, \quad p_{i,t+1} = p_i
\end{cases}
\]

(27)

Because \( H_i^{-1}(p_i) \) is monotone, we have

\[
H(p_{i,t+1}) - \gamma_i \tilde{p}_{i,t} + \tilde{\mu}_{i,t} \begin{cases} 
\geq 0, \quad p_{i,t+1} = p_i \\
= 0, \quad p_i < p_{i,t+1} < p_i \\
\leq 0, \quad p_{i,t+1} = p_i
\end{cases}
\]

(28)

It is equivalent to

\[
p_{i,t+1} = P_{\Omega_i} \left(p_{i,t+1} - \alpha_\mu (\nabla h_i(p_{i,t+1}) + \gamma_i (p_{i,t+1} - \tilde{p}_{i,t}) + \tilde{\mu}_{i,t})\right)
\]

(29)

Recalling \( P_{\Omega_i}(x) = (I + N_{\Omega_i})^{-1}(x) \), we have

\[
\gamma_i (\tilde{p}_{i,t} - p_{i,t+1}) + \mu_{i,t+1} \in N_{\Omega_i}(p_{i,t+1})
\]

(30)

From (26e), we have

\[
\sigma_z^{-1}(\tilde{z}_{i,t} - z_{i,t+1}) - \sum_{j \in N_i} (\tilde{\mu}_{i,t} - \tilde{\mu}_{j,t})
\]

(31)

\[
+ \sum_{j \in N_i} (\mu_{i,t+1} - \mu_{j,t+1}) = \sum_{j \in N_i} (\mu_{i,t+1} - \mu_{j,t+1})
\]

(31)
From (26), we have
\[
\sigma^-_\mu^{-1}(\tilde{\mu}_{i,t} - \mu_{i,t+1}) + \sum_{j \in N_i} (z_{i,t+1} - z_{j,t+1})
- \sum_{j \in N_i} (\tilde{z}_{i,t} - \tilde{z}_{j,t}) + p_{i,t+1} - \tilde{p}_{i,t}
= -p_{i,t+1} + D_t - \sum_{j \in N_i} (z_{i,t+1} - z_{j,t+1})
\] (32)

The compact form of the algorithm is
\[
\begin{align*}
\tilde{p}_t &= p_t + \eta(p_t - p_{t-1}) \\
\tilde{z}_t &= z_t + \eta(z_t - z_{t-1}) \\
\tilde{\mu}_t &= \mu_t + \eta(\mu_t - \mu_{t-1}) \\
\Gamma(\tilde{p}_t - p_{t+1}) - (\tilde{\mu}_t - \mu_{t+1}) &\in N_\Omega (p_{t+1}) + \nabla \tilde{h}(p_{t+1}) + \mu_{t+1}
\end{align*}
\] (33)

Theorem 20.25. Thus, \( U_1 \) is maximally monotone.

Theorem 6. Suppose assumptions \( 1, 2 \) and \( 3 \) hold. At the equilibrium of the algorithm SGNE, we have \( \mu^*_i = \mu^*_j = \mu_0, \forall i, j \) is the clearing price and \( (p^*, b^*) \) is the GNE of the game, where \( \mu_0 \) is constant.

Proof. From Definition \( 1 \) and \( 33 \), we have
\[
\begin{align*}
0 &= N_\Omega (p^*) + \nabla \tilde{h}(p^*) + \mu^* \\
0 &= L\mu^* \\
0 &= -p^* + D - Lz^*
\end{align*}
\] (38)

From (38b), we have
\[
\mu^*_i = \mu^*_j = \mu_0, \forall i, j
\] (39)

From (38c), we have
\[
0 = -1^T p^* + 1^T D - 1^T Lz^* = -1^T p^* + 1^T D
\] (40)

The equations (38a), (39) and (40) are the KKT condition (21). This completes the proof. \( \square \)

VI. NASH EQUILIBRIUM SEEKING AND CONVERGENCE

In this section, we address the optimality of the equilibrium point and the convergence of the algorithm SGNE, i.e., the discrete dynamic system (20).

A. Nash Equilibrium

First, we define the equilibrium of the algorithm SGNE.

Definition 1. A point \( w^* = (w^*_i, i \in N) = (p^*_i, z^*_i, \mu^*_i) \) is an equilibrium point of (26) if \( \lim_{t \to +\infty} w_{i,t} = w^*_i \) holds for all \( i \).

Recalling the (19), and we have the following result.

Theorem 6. Suppose assumptions \( 1, 2 \) and \( 3 \) hold. At the equilibrium of the algorithm SGNE, we have \( \mu^*_i = \mu^*_j = \mu_0, \forall i, j \) is the clearing price and \( (p^*, b^*) \) is the GNE of the game, where \( \mu_0 \) is constant.

Proof. From Definition \( 1 \) and \( 33 \), we have
\[
\begin{align*}
0 &= N_\Omega (p^*) + \nabla \tilde{h}(p^*) + \mu^* \\
0 &= L\mu^* \\
0 &= -p^* + D - Lz^*
\end{align*}
\] (38)

From (38b), we have
\[
\mu^*_i = \mu^*_j = \mu_0, \forall i, j
\] (39)

From (38c), we have
\[
0 = -1^T p^* + 1^T D - 1^T Lz^* = -1^T p^* + 1^T D
\] (40)

The equations (38a), (39) and (40) are the KKT condition (21). This completes the proof. \( \square \)

B. Convergence

In this subsection, we analyze the convergence of the algorithm SGNE based on the compact form (37). First, we give the following result.

Theorem 7. Suppose Assumption \( 1 \) and Assumption \( 2 \) hold. Given a parameter \( \eta \) satisfying \( 0 < \eta < \frac{1}{2} \) and the step sizes \( \Gamma, \alpha_z \) and \( \alpha_p \) such that \( \Theta \) is positive definite. Then with SGNE, \( w_t \) converges to a primal-dual optimal solution \( \omega^* \) of the problem (22). Then, \( (p^*, b^*) \) is the GNE of the problem (17).

Proof. First, we prove that \( \lim_{t \to +\infty} (\omega_{t+1} - \omega_t) = 0 \). Given any equilibrium point \( \omega^* \), use the equation (1), and we have
\[
\|\omega_t - \omega^*\|^2_\Theta - \|\omega_{t+1} - \omega^*\|^2_\Theta
\]
\[\begin{align*}
  &= 2(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2 \\
  &= 2(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta + \|\omega_t - \omega_{t+1}\|_\Theta^2 \\
  &= \|\omega_t - \omega_{t+1}\|_\Theta^2 + 2(\tilde{\omega}_t - \omega_t, \omega_{t+1} - \omega^*)_\Theta \\
  &- 2\eta(\omega_t - \omega_{t-1}, \omega_{t+1} - \omega^*)_\Theta
  \end{align*}\]

where the last equation is derived by integrating (37a).

By (35b), we have

\[0 \in \mathcal{U}(\omega^*)\]

Since \(\mathcal{U}\) is maximally monotone, we have

\[\langle \Theta(\omega_t - \omega_{t+1}), \omega_{t+1} - \omega^* \rangle \geq 0\]

Thus, we have

\[\|\omega_{t+1} - \omega^*\|_\Theta^2 - \|\omega_t - \omega^*\|_\Theta^2 \leq 2\eta(\omega_t - \omega_{t-1}, \omega_{t+1} - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2\]

Moreover,

\[\eta(\|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t-1} - \omega^*\|_\Theta^2) = \eta(2(\omega_t - \omega_{t-1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t-1}\|_\Theta^2)\]

From (44) and (45), we have

\[\begin{align*}
\|\omega_{t+1} - \omega^*\|_\Theta^2 - \|\omega_t - \omega^*\|_\Theta^2
- \eta(\|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t-1} - \omega^*\|_\Theta^2)
\leq & 2\eta(\omega_t - \omega_{t-1}, \omega_{t+1} - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2 \\
- & \eta(2(\omega_t - \omega_{t-1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t-1}\|_\Theta^2) \\
= & - \|\omega_t - \omega_{t+1}\|_\Theta^2 + 2\eta(\omega_t - \omega_{t-1}, \omega_{t+1} - \omega_t)_\Theta \\
& + \eta \|\omega_t - \omega_{t-1}\|_\Theta^2 \\
\leq & (\eta - 1) \|\omega_t - \omega_{t+1}\|_\Theta^2 + 2\eta \|\omega_t - \omega_{t-1}\|_\Theta^2
\end{align*}\]

Define a sequence \(s_k = \|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t-1} - \omega^*\|_\Theta^2 + 2\eta \|\omega_t - \omega_{t-1}\|_\Theta^2\). Then

\[s_{k+1} - s_k = \|\omega_{t+1} - \omega^*\|_\Theta^2 - \|\omega_t - \omega^*\|_\Theta^2 \leq 2\eta \|\omega_t - \omega_{t+1}\|_\Theta^2 + 2\eta \|\omega_{t+1} - \omega_t\|_\Theta^2 \\
- 2\eta \|\omega_t - \omega_{t-1}\|_\Theta^2 \leq (3\eta - 1) \|\omega_{t+1} - \omega_t\|_\Theta^2\]

From (47), we have \((1 - 3\eta)\sum_{t=1}^{k+1} \|\omega_{t+1} - \omega_t\|_\Theta^2 \leq s_1 - s_{k+1} \leq s_1 + \eta \|\omega_t - \omega^*\|_\Theta^2\]. As \(0 < \eta < \frac{1}{3}\), we have \(s_{k+1} \leq s_1 \leq s_1 + \eta \|\omega_t - \omega^*\|_\Theta^2\]. Then, \(\|\omega_t - \omega^*\|_\Theta^2 \leq \eta(\|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2) \leq s_1 < s_1\). Thus, \(\|\omega - \omega^*\|_\Theta^2 \leq \eta^k(\|\omega - \omega^*\|_\Theta^2 - \frac{\mu}{1 - \alpha}) + \frac{\mu_s}{1 - \alpha}\). Therefore, \(\omega_t\) is bounded. Thus, \(\|\omega_t - \omega_{t-1}\|_\Theta^2 \leq s_1 + \eta^{k+1}(\|\omega_t - \omega^*\|_\Theta^2 - \frac{\mu_s}{1 - \alpha}) + \frac{\mu_s}{1 - \alpha}\). As \(0 < \eta < \frac{1}{3}\), we have

\[\sum_{t=1}^{\infty} \|\omega_{t+1} - \omega_t\|_\Theta^2 < \infty\]

Thus, \(\lim_{t \to \infty} (\omega_{t+1} - \omega_t) = 0\). Then, we prove that \(\|\omega_t - \omega^*\|_\Theta^2\] also converges by the similar analysis in [20, Theorem 6.3]. We also write it here for completeness.

Denote \(\varphi_t = \max \left\{0, \|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2\right\}\) and \(\zeta_t = 2\eta \|\omega_t - \omega_{t-1}\|_\Theta^2\). We know that \(\varphi_t\) is lower bounded. Recall (46), and we have \(\varphi_{t+1} \leq \eta \varphi_t + \zeta_t\). Apply this relationship recursively, and we have

\[\varphi_{t+1} \leq \eta^t \varphi_1 + \sum_{i=0}^{t-1} \eta^i \zeta_i\]

Adding both sides of (49) from \(t = 1 \to \infty\), we have

\[\sum_{t=1}^{\infty} \varphi_t \leq \frac{\varphi_1}{1 - \eta} + \frac{1}{1 - \eta} \sum_{t=1}^{\infty} \zeta_t\]

From (48), we know \(\sum_{t=1}^{\infty} \zeta_t < \infty\). This implies that \(\sum_{t=1}^{\infty} \varphi_t\) is bounded, non-decreasing and converges.

Consider another sequence \(\left\{\|\omega_t - \omega^*\|_\Theta^2 - \sum_{i=1}^{t} \varphi_i\right\}\), which is lower bounded. Moreover,

\[\|\omega_{t+1} - \omega^*\|_\Theta^2 - \sum_{i=1}^{k+1} \varphi_i = \|\omega_{t+1} - \omega^*\|_\Theta^2 - \sum_{i=1}^{t} \varphi_i\]

\[\leq \|\omega_{t+1} - \omega^*\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2 + \|\omega_t - \omega^*\|_\Theta^2 - \sum_{i=1}^{t} \varphi_i\]

\[= \|\omega_t - \omega^*\|_\Theta^2 - \sum_{i=1}^{t} \varphi_i\]

This implies that \(\left\{\|\omega_t - \omega^*\|_\Theta^2 - \sum_{i=1}^{t} \varphi_i\right\}\) is a non-increasing sequence and also converges. \(\|\omega_t - \omega^*\|_\Theta^2\) is the sum of two convergent sequences, and also converges.

Since \(\omega_t\) is bounded, it has a convergent subsequence \(\omega_{n_t}\) converging to a point \(\tilde{\omega}^*\). From \(\lim (\omega_{t+1} - \omega_t) = 0\), we have \(\lim (\omega_{n_t+1} - \omega_{n_t}) = 0\) and \(\lim (\omega_{n_t} - \omega_{n_t+1}) = 0\). Due to the continuity of the right hand side of (37), we have \(\tilde{\omega}^* = (I_d + \Theta^{-1}U)\tilde{\omega}^*\). Thus, \(\omega^*\) is an equilibrium point of the sequence \(\omega_t\). This also implies that \(\|\omega_t - \tilde{\omega}^*\|_\Theta^2\) converges. Because \(\|\omega_{n_t} - \tilde{\omega}^*\|_\Theta^2\) converges to zero, \(\|\omega_t - \tilde{\omega}^*\|_\Theta^2\) also converges to zero.

Based on Theorem 6, we have \((p^*, b^*)\) is the GNE of (7). This completes the proof.

Invoking Theorem 5 if Assumption 5 also holds, the algorithm will converge to the unique GNE of the generalized Nash game (7), equivalently the energy sharing game (5).

VII. EXPERIMENTS AND SIMULATION STUDIES

In this section, experiments and numerical simulations are introduced to verify the effectiveness of the proposed method. First, experiments with three prosumers are carried out to illustrate the basic properties of the algorithm. Then, a case with 123 prosumers is investigated to test the scalability, where the communication topology is identical to the topology of the IEEE 123-bus system [32].

A. Experimental results

The proposed method is verified on an experimental platform based on the dSPACE RTI 1202 controller, which is also presented in Fig.2. It is composed of three inverters, one dSPACE RTI 1202 controller, two switchable loads, and one host computer. Each inverter represents a prosumer, which can both produce and consume power. The system topology is given in Fig.3. In the experiments, the breaker B0 is open,
i.e., the system operates in an isolated mode. One load is connected at the bus of Prosumer1 and the other is at the bus of Prosumer2. Three prosumers are connected through impedances. The communication topology is Prosumer1 ↔ Prosumer2 ↔ Prosumer3 ↔ Prosumer1. The disutility function is \( h_i(p_i) = \frac{1}{2}c_i p_i^2 + d_i p_i \) with \( c_1 = 0.00075, c_2 = 0.0006, c_3 = 0.001, d_i = 0 \). The price elasticity of each prosumer is set as \( a_i = -1000 \). The load demand of each prosumer is \( D_1 = 730W, D_2 = 365W, D_3 = 0 \).

The simulation scenario is: 1) At \( t = 10s \), two loads are connected; 2) at \( t = 30s \), load 2 is disconnected. Then, each DG regulates its generation to balance the power difference. The frequency dynamics are illustrated in Fig.4. When loads are connected, the frequency drops to about 49.2Hz and recovers to the nominal value in four seconds. On the contrary, when load 2 is switched off, the frequency increases and recovers in 3 seconds.

The GNE of the first stage is \( (p_1^* = 505.4W, b_1^* = 103.3W), (p_2^* = 408.4W, b_2^* = 355.0W), (p_3^* = 177.8W, b_3^* = 310.0W) \). Dynamics of seeking the GNE is illustrated in Fig.5 where the left part is the generation \( p_i \) of each prosumer and the right part is the purchase willingness \( b_i \). They vary slightly around the equilibrium. In this stage, Prosumer1 buys power from Prosumer2 and Prosumer3. The GNE of the second stage is \( (p_1^* = 440.6W, b_1^* = -41.7W), (p_2^* = 168.9W, b_2^* = -310.0W), (p_3^* = 123.9W, b_3^* = -444.2W) \). Compared with the results obtained from the centralized method, the steady state generations are optimal to the problem [22]. This shows that the GNE is obtained. Moreover, real power can always trace the reference value. This verifies that the proposed method can get the correct results in the experiment. Dynamics of the payoff functions are given in Fig.6 which also converges in five seconds. In the first stage, Prosumer3 earns profit by selling power to Prosumer1 and Prosumer2. In the second stage, there is only Prosumer1 buying power, while others selling. The cost of Prosumer2 changes from positive to negative, which implies that it earns profit by selling power to Prosumer1.

B. Scalability

In this subsection, the scalability of proposed method is illustrated. The 123 prosumers is numbered with the same fundamental assumption of existing distributed methods for the Nash equilibrium seeking of GNG. Then, we transform the GNG to an equivalent optimization problem, which is solved by the proposed distributed algorithm. Experimental results with three prosumers show that the GNE can be obtained at a fast speed. Simulations on 123 prosumers verify the scalability of the proposed method.

VIII. CONCLUSION

In this work, we investigate the distributed energy sharing game among prosumers. We first prove that this game does not satisfy the monotone condition, which serves as the fundamental assumption of existing distributed methods for the Nash equilibrium seeking of GNG. Then, we transform the GNG to an equivalent optimization problem, which is solved by the proposed distributed algorithm. Experimental results with three prosumers show that the GNE can be obtained at a fast speed. Simulations on 123 prosumers verify the scalability of the proposed method.
This paper offers a different perspective for seeking the GNE in the GNG. If the monotone condition is not satisfied, it is possible to transform the GNG to an equivalent optimization problem. In our future work, it is interesting to investigate the conditions of the equivalent transformation for a more general GNG.

REFERENCES

[1] Y. M. Chiang, “Building a better battery,” Science, vol. 330, no. 6010, pp. 1485–1486, 2010.
[2] P. Mani, J. Lee, K. Kang, and Y. H. Joo, “Digital controller design via lmis for direct-driven surface mounted pm-sig-based wind energy conversion system,” IEEE Transactions on Cybernetics, pp. 1–12, 2019.
[3] Z. Wang, Y. Chen, S. Mei, S. Huang, and Y. Xu, “Optimal expansion planning of isolated microgrid with renewable energy resources and controllable loads,” IET Renewable Power Generation, vol. 11, no. 7, pp. 931–940, 2017.
[4] M. Tran, D. Banister, J. D. K. Bishop, and M. D. Mcculloch, “Realizing the electric-vehicle revolution,” Nature Climate Change, vol. 2, no. 5, pp. 328–333, 2012.
[5] Y. Zhang, L. Chu, Y. Ou, C. Guo, Y. Liu, and X. Tang, “A cyber-physical system-based velocity-profile prediction method and case study of application in plug-in hybrid electric vehicle,” IEEE Transactions on Cybernetics, pp. 1–12, 2019.
[6] D. M. Han and J. H. Lim, “Smart home energy management system using ieee 802.15.4 and zigbee,” IEEE Transactions on Consumer Electronics, vol. 56, no. 3, pp. 1403–1410, 2010.
[7] S. Weng, D. Yue, C. Dou, J. Shi, and C. Huang, “Distributed event-triggered cooperative control for frequency and voltage stability and power sharing in isolated inverter-based microgrid,” IEEE transactions on cybernetics, no. 99, pp. 1–13, 2018.
[8] H. Zhang, D. Yue, C. Dou, K. Li, and X. Xie, “Event-triggered multiagent optimization for two-layered model of hybrid energy system with price bidding-based demand response,” IEEE Transactions on Cybernetics, pp. 1–12, 2019.

[9] A. Dimeas, S. Drenkard, N. Hatziargyriou, S. Karnouskos, K. Kok, J. Ringelstein, and A. Weidlich, “Smart houses in the smart grid: Developing an interactive network,” IEEE Electrification Magazine, vol. 2, no. 1, pp. 81–93, 2014.
[10] Y. Parag and B. K. Sovacool, “Electricity market design for the prosumer era,” Nature Energy, vol. 1, no. 4, pp. 1–6, 2016.
[11] T. Morstyn, N. Farrell, S. J. Darby, and M. D. Mcculloch, “Using peer-to-peer energy-trading platforms to incentivize prosumers to form federated power plants,” Nature Energy, vol. 3, no. 2, pp. 94–101, 2018.
[12] N. Liu, X. Yu, W. Cheng, C. Li, and J. Lei, “An energy sharing model with price-based demand response for microgrids of peer-to-peer prosumers,” IEEE Transactions on Power Systems, vol. 32, no. 5, pp. 3569–3583, 2017.
[13] Y. Chen, S. Mei, F. Zhou, S. H. Low, and F. Liu, “An energy sharing game in prosumers based on generalized demand bidding: Model and properties,” IEEE Transactions on Smart Grid, in press, 2019.
[14] Z. Wang, F. Liu, S. H. Low, C. Zhao, and S. Mei, “Distributed frequency control with operational constraints, part ii: Network power balance,” IEEE Trans. Smart Grid, vol. 10, no. 1, pp. 53–64, Jan 2019.
[15] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, “Breaking the hierarchy: Distributed control and economic optimality in microgrids,” IEEE Trans. Control Network Syst., vol. 3, no. 3, pp. 241–253, 2016.
[16] Z. Wang, F. Liu, Y. Chen, S. H. Low, and S. Mei, “Unified distributed control of stand-alone dc microgrids,” IEEE Trans. Smart Grid, vol. 10, no. 1, pp. 1013–1024, Jan 2019.
[17] Z. Wang, F. Liu, S. H. Low, P. Yang, and S. Mei, “Distributed load-side control: Coping with variation of renewable generations,” Automatica, in press, 2019.
[18] S. Liang, P. Yi, and Y. Hong, “Distributed nash equilibrium seeking for aggregative games with coupled constraints,” Automatica, vol. 85, pp. 179–185, 2017.
[19] X. Zeng, J. Chen, S. Liang, and Y. Hong, “Generalized nash equilibrium seeking strategy for distributed nonsmooth multi-cluster game,” Automatica, vol. 103, pp. 20 – 26, 2019.
[20] P. Yi and L. Pavel, “An operator splitting approach for distributed generalized nash equilibria computation,” Automatica, vol. 102, pp. 111–121, 2019.
[21] P. Yi and L. Pavel, “Asynchronous distributed algorithms for seeking generalized nash equilibria under full and partial-decision information,” IEEE transactions on cybernetics, in press, 2019.
[22] C. Yu, M. van der Schaar, and A, H. Sayed, “Distributed learning for stochastic generalized nash equilibrium problems,” IEEE Transactions on Signal Processing, vol. 65, no. 15, pp. 3893–3908, Aug 2017.
[23] D. Paccagnan, B. Gentile, F. Parise, M. Kamgarpoor, and J. Lygeros, “Nash and wardrop equilibria in aggregative games with coupling constraints,” IEEE Transactions on Automatic Control, vol. 64, no. 4, pp. 1373–1388, April 2019.
[24] F. Salehsadaghiani and L. Pavel, “Distributed nash equilibrium seeking via the alternating direction method of multipliers,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 6166 – 6171, 2017.
[25] P. Yi and L. Pavel, “Distributed generalized nash equilibria computation of monotone games via double-layer preconditioned proximal-point algorithms,” IEEE Transactions on Control of Network Systems, vol. 6, no. 1, pp. 299–311, 2018.
[26] G. Beligioso and S. Grammatico, “A distributed proximal-point algorithm for nash equilibrium seeking of generalized potential games with linearly coupled cost functions,” in 2019 18th European Control Conference (ECC), June 2019, pp. 1–6.
[27] F. Salehsadaghiani, W. Shi, and L. Pavel, “Distributed nash equilibrium seeking via the alternating direction method of multipliers,” Automatica, vol. 103, pp. 27 – 35, 2019.
[28] H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, 2017.
[29] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge university press, 2004.
[30] F. Facchinei and C. Kanzow, “Generalized nash equilibrium problems,” Annals of Operations Research, vol. 175, no. 1, pp. 177–211, 2010.
[31] A. P. Ruszczyński and A. Ruszczyński, Nonlinear optimization. Princeton university press, 2006, vol. 13.
[32] W. H. Kersting, “Radial distribution test feeders,” IEEE Transactions on Power Systems, vol. 6, no. 3, pp. 975–985, 1991.