Role of Pressure in Quasi-Spherical Gravitational Collapse

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We study quasi-spherical Szekeres space-time (which possess no killing vectors) for perfect fluid, matter with tangential stress only and matter with anisotropic pressure respectively. In the first two cases cosmological solutions have been obtained and their asymptotic behaviour have been examined while for anisotropic pressure, gravitational collapse has been studied and the role of the pressure has been discussed.

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I. INTRODUCTION

In cosmology, solutions to Einstein’s field equations are obtained by imposing symmetries [1] on space-time. Usually, spatial homogeneity is one of the reasonable assumptions (in an average sense) while for cosmological phenomena over galactic scale or in smaller scale, inhomogeneous solutions are useful. Szekeres [2] in 1975 gave a class of inhomogeneous solutions representing irrotational dust. The space-time represented by these solutions has no killing vectors and it has invariant family of spherical hypersurfaces. Hence this space-time is referred as quasi-spherical space-time.

An extensive study of gravitational collapse [3-8] has been carried out of Tolman-Bondi-Lemaître (TBL) spherically symmetric space-times containing irrotational dust to support or disprove the cosmic censorship conjecture (CCC). A general conclusion from these studies is that a central curvature singularity forms but its local or global visibility depends on the initial data. Also over the past decades, the role of pressure within a collapsing cloud has been studied [9-13] but the actual role of pressure in determining the end state of a continual collapse is not yet clear.

On the other hand, there is very little progress in studying non-spherical collapse. Basically, the difficulty is the ambiguity of horizon formation in non-spherical geometries and the influence of gravitational radiation. Though there is hoop conjecture by Thorne [14] to characterize the formation of horizon but only few works [15-19] have been done to confirm or refute the conjecture. Recently, an extensive study of irrotational dust collapse has been done in quasi-spherical Szekeres space-time both for four [20] and higher [21] dimensions. In this paper, we have done an extensive analysis of Szekeres model with matter containing pressure and studied the collapsing procedure to examine the role of pressure in characterizing the final singularity. The paper is organized as follows: The Szekeres model [22] has been described in section II. The perfect fluid solution with asymptotic behaviour has been presented in section III. Section IV deals with the tangential stress only while section V deals with gravitational collapse with non-isotropic pressure. Finally, the paper ends with a brief discussion in section VI.
II. THE SZEKERES’ MODEL

The metric ansatz for the four dimensional Szekeres’ space-time [22] is of the form

\[ ds^2 = dt^2 - e^{2\alpha}dr^2 - e^{2\beta}(dx^2 + dy^2) \]  

(1)

where \( \alpha \) and \( \beta \) are functions of all space-time variables i.e.,

\[ \alpha = \alpha(t, r, x, y), \quad \beta = \beta(t, r, x, y). \]

Considering both radial and transverse stresses the energy momentum tensor has the following structure

\[ T_\mu^\nu = \text{diag}(\rho, -p_r, -p_T, -p_T) \]  

(2)

Hence the full set of Einstein equations are

\[ 2\dot{\alpha}\dot{\beta} + \dot{\beta}^2 - e^{-2\beta}(\alpha_x^2 + \alpha_y^2 + \alpha_{xx} + \alpha_{yy} + \beta_{xx} + \beta_{yy}) + e^{-2\alpha}(2\alpha'\beta' - 3\beta'^2 - 2\beta'') = \rho \]  

(3)

\[ 3\ddot{\beta}^2 + 2\dot{\beta} - e^{-2\alpha}\beta'^2 - e^{-2\beta}(\beta_{xx} + \beta_{yy}) = -p_r \]  

(4)

\[ \ddot{\alpha} + \dot{\alpha}^2 + \ddot{\beta} + \dot{\alpha}\dot{\beta} + e^{-2\alpha}(\alpha'\beta' - \beta'^2 - \beta'') - e^{-2\beta}(\alpha_x^2 + \alpha_y^2 - \alpha_x\beta_x + \alpha_y\beta_y) = -p_r \]  

(5)

\[ \alpha_y(-\alpha_x + \beta_x) + \alpha_x\beta_y - \alpha_{xy} = 0 \]  

(6)

\[ \dot{\alpha}\beta' - \dot{\beta}\beta' - \beta' = 0 \]  

(7)

\[ -\dot{\alpha}\alpha_x + \dot{\beta}\alpha_x - \alpha_x - \dot{\beta}_x = 0 \]  

(8)

\[ -\dot{\alpha}\alpha_y + \dot{\beta}\alpha_y - \alpha_y - \dot{\beta}_y = 0 \]  

(9)

\[ \alpha_x\beta' - \beta'_x = 0 \]  

(10)

\[ \alpha_y\beta' - \beta'_y = 0 \]  

(11)

where dot, dash and subscript stands for partial differentiation with respect to \( t, r \) and the corresponding variables respectively (e.g., \( \beta_x = \frac{\partial \beta}{\partial x} \)).

Combining the time derivatives of equations (11) and (12) with the differentiation of (8) with respect to \( x \) and \( y \) separately we have the integrability condition

\[ \beta'\beta_x = 0 = \beta'\beta_y \]  

(12)

Hence we may have three possibilities namely,
(a) $\beta' \neq 0$, $\dot{\beta}_x = 0 = \dot{\beta}_y$,

(b) $\beta' = 0$, $\dot{\beta}_x = 0 = \dot{\beta}_y$  \hspace{1cm} (14)

(c) $\beta' = 0$, $\dot{\beta}_x \neq 0$, $\dot{\beta}_y \neq 0$

Following Szekeres’ [22] the field equations are not solvable for the third case so we shall consider only the first two cases.

The energy conservation equation namely $T^{\mu\nu}\rho_{\nu} = 0$ gives

\[ \dot{\rho} + \rho(\dot{\alpha} + 2\dot{\beta}) + (\dot{\alpha}p_r + 2\dot{\beta}p_r) = 0 \]  \hspace{1cm} (15)

\[ \frac{\partial}{\partial x}(p_r e^\alpha) = p_r \alpha_x e^\alpha \]  \hspace{1cm} (16)

\[ \frac{\partial}{\partial y}(p_r e^\alpha) = p_r \alpha_y e^\alpha \]  \hspace{1cm} (17)

\[ \frac{\partial}{\partial r}(p_r e^{2\beta}) = p_r \frac{\partial}{\partial r}(e^{2\beta}) \]  \hspace{1cm} (18)

Now for the first choice namely $\beta' \neq 0$, $\dot{\beta}_x = 0 = \dot{\beta}_y$ we have from the field equations the explicit form of the metric coefficients are as follows:

\[ e^\beta = R(t, r) e^{\nu(r, x, y)} \]  \hspace{1cm} (19)

\[ e^\alpha = R' + R \nu' \]  \hspace{1cm} (20)

where $R$ and $\nu$ satisfy the following differential equations

\[ 2R\dddot{R} + \dot{R}^2 + p_r R^2 = f(r), \quad (f(r) = \text{arbitrary separation function}) \]  \hspace{1cm} (21)

and

\[ e^{-2\nu}(\nu_{xx} + \nu_{yy}) = f(r) - 1 \]  \hspace{1cm} (22)

Here we have assumed $p_r = p_r(r, t)$. Equation (21) has the first integral

\[ \dot{R}^2 = f(r) + \frac{G(r)}{R} - \frac{1}{R} \int p_r R^2 dR \]  \hspace{1cm} (23)

while one of the possible solutions of equation (22) can be taken as [24]

\[ e^{-\nu} = A(r)(x^2 + y^2) + B_1(r)x + B_2(r)y + D(r) \]  \hspace{1cm} (24)

where the arbitrary functions $A(r)$, $B_1(r)$, $B_2(r)$ and $D(r)$ are related as

\[ B_1^2 + B_2^2 - 4AD = f(r) - 1 \]
and $G(r)$ is an arbitrary function.

For the choice (b) the metric coefficients are of the form

$$e^\beta = R(t)e^{\nu(x,y)}$$

(25)

$$e^\alpha = R(t)\eta(r,x,y) + \mu(t,r)$$

(26)

Then as before from the field equation (4) we have similar differential equation in $R$ and $\nu$ as

$$2R\dddot{R} + R\ddot{R}^2 + p_r R^2 = K, \quad (K \text{ is a constant})$$

(27)

$$e^{-2\nu}(\nu_{xx} + \nu_{yy}) = K$$

(28)

with $K$ an arbitrary constant. Also for $\nu$ we choose as in case (a)

$$e^{-\nu} = P(x^2 + y^2) + Q_1 x + Q_2 y + S$$

(29)

where $P$, $Q_1$, $Q_2$ and $S$ are arbitrary constants restricted by the relation

$$Q_1^2 + Q_2^2 - 4PS = K$$

Now to determine the function $\eta$ we have from the field equation (7)

$$\frac{\partial^2(e^{-\nu}\eta)}{\partial x \partial y} = 0$$

(30)

and then from the field equations (5) and (6) we have a possible solution

$$e^{-\nu}\eta = u(r)(x^2 + y^2) + v_1(r)x + v_2(r)y + w(r)$$

(31)

where $u(r)$, $v_1(r)$, $v_2(r)$ and $w(r)$ are arbitrary functions of $r$ alone.

Also the differential equation in $\mu$ is

$$R\dddot{\mu} + R\ddot{\mu} + \mu(R + p_r R) + (p_r - p_r) R^2 \eta = h(r)$$

(32)

with

$$h(r) = 2(uS + wP) - (v_1 Q_1 + v_2 Q_2).$$

Now for explicit solutions, we shall consider the following cases in the next sections:

(i) the perfect fluid model (i.e., $p_r = p_T$)

(ii) the tangential stress only (i.e., $p_r = 0$, $p_T \neq 0$)

(iii) the general case (i.e., $p_r \neq 0$, $p_T \neq 0$).

In fact, cosmological solutions are obtained (in sections III and IV) for the first two cases respectively while for the third case collapsing behaviour has been studied and the role of pressure has been examined.
III. THE PERFECT FLUID MODEL

In this case due to energy conservation equations (16)-(18) the isotropic pressure is function of $t$ only i.e., $p = p(t)$ ($p_r = p_t$). As there is no restriction on the energy density so $\rho$ is in general a function of all the 4 variables i.e., $\rho = \rho(t, r, x, y)$ and hence no equation of state is imposed.

Now for explicit solution according to Szafron [25] and Szafron and Wainwright [26] we choose

$$p(t) = p_c t^{-s}$$  \hfill (33)

($p_c$ and $s$ are positive constants) and we have the general solution for $R$ as [23]

$$R^2 = \begin{cases} \sqrt{T} \left( C_1 J_\xi \left[ \frac{2\sqrt{\lambda}}{s-2} t^{-\frac{s-2}{2}} \right] + C_2 Y_\xi \left[ \frac{2\sqrt{\lambda}}{s-2} t^{-\frac{s-2}{2}} \right] \right) \\ \sqrt{T} \left( C_1 J_{-\xi} \left[ \frac{2\sqrt{\lambda}}{s-2} t^{-\frac{s-2}{2}} \right] + C_2 J_{-\xi} \left[ \frac{2\sqrt{\lambda}}{s-2} t^{-\frac{s-2}{2}} \right] \right) \\ C_1 t^{q_1} + C_2 t^{1-q_1} \end{cases}$$ \hfill (34)

according as $\xi$ is an integer, non-integer and $s = 2$. Here $C_1$ and $C_2$ are arbitrary functions of $r$ and we have chosen

$$\xi = \frac{1}{s-2}, \quad \lambda = \frac{3p_c}{4}, \quad q_1 = \frac{1}{2}(1 + \sqrt{1 - 3p_c}).$$

It is to be noted that to derive the above solution we have chosen $f(r) = 0$. Further, if we consider the dust model (i.e., $p = 0$) then the above solution simplifies to $R \propto t^{2/3}$ which is the form of the scale factor in the usual Friedman model.

Now, the physical and kinematical parameters have the following expressions

$$\rho = \frac{G'(r) + 3Gv'}{R'(R' + R\nu') - \frac{p_c}{t^s}}$$ \hfill (35)

$$\theta = \frac{R\dot{R}' + 3R\dot{\nu}' + 2\ddot{R}R'}{R(R' + R\nu')},$$ \hfill (36)

$$\sigma^2 = \frac{1}{12} \left[ \frac{R(Rf' - 2R'f) + (RG'(r) - 3R'G)}{RR^2(R' + R\nu')} \right]^2.$$ \hfill (37)

The above solution is for the choice (a) (see eq.(13)). For the choice (b) (i.e., eq.(14)) the explicit form for $R$ is same as in equation (34) except here $C_1$ and $C_2$ are arbitrary constants and $K = 0$. But we note that the differential equation (30) is not solvable for the above explicit solutions for $R$. The physical and kinematical parameters have the expressions as

$$\rho = \frac{2(\ddot{R} - K)}{R^2} - \frac{2(\ddot{\mu} + \eta \ddot{R})}{\mu + \eta R} - \frac{p_c}{t^s}$$ \hfill (38)

$$\theta = \frac{R\ddot{\mu} + 2\mu\ddot{R} + 3R\ddot{\eta}}{R(\mu + \eta R)}.$$ \hfill (39)
\[ \sigma^2 = \frac{1}{3} \left[ \frac{R\dot{\mu} - \dot{R}\mu}{R(\mu + \eta R)} \right]^2 \]  

(40)

Asymptotic Behaviour

We shall now discuss the asymptotic behaviour of the above solutions. The co-ordinates vary over the range: \( t_0 < t < \infty; \ -\infty < r < \infty; \ -\infty < x, y < \infty \). For both the choices (a) and (b) as \( p \geq 0, \ p \neq 0 \) so we must have \( \frac{1}{2} < q_1 < 1 \). So for \( s = 2 \) for large \( t \),

\[
R \sim t^{\frac{2p}{q_1}} \\
\rho \sim t^{-2} \\
p \sim t^{-2} \\
\theta \sim t^{-1} \\
\sigma^2 \sim t^{-2}
\]

(41)

Hence as \( t \to \infty \), \((p, \rho)\) fall off faster compare to \((\theta, \sigma)\), while the scale factor \( R \) gradually increases with time. So the model approaches isotropy along fluid world line as \( t \to \infty \).

IV. MODEL WITH TANGENTIAL STRESSES ONLY

For this model we have from the conservation equation (18) \( \beta' = 0 \) i.e. choice (b) is only possible here. Also from the other conservation equations namely equation (16) and (17) we have

\[ p_T = A(r, t)e^{-\alpha} \]

(42)

where \( A(r, t) \) is an arbitrary function of \( r \) and \( t \). But for the consistency of the differential eq.(30) \( p_T \) (as stated earlier) must be a function of \( r \) and \( t \) and hence \( \alpha \) should be independent of \( x \) and \( y \). As a consequence, in the solution (26) for \( \alpha, \eta \) must be independent of \( x \) and \( y \). Thus for the solution (32) for \( \eta \) we should have

\[ u(r) = P\eta_0(r), \ v_1(r) = Q_1\eta_0(r), \ v_2(r) = Q_2\eta_0(r) \] and \( w(r) = S\eta_0(r) \)

Hence we have \( \eta = \eta_0(r) \), an arbitrary function of \( r \) alone and \( h(r) = -K\eta_0(r) \).

Now, the differential equation for \( R \) has the simple form

\[ \dot{R}^2 = a_1 + \frac{a_2}{R}, \quad (a_1, a_2 \text{ are constants}) \]

(43)

which has a parametric solution of the form

\[
R = \frac{a_2}{2(a_1)^{2/3}}(1 - \cos \phi) \\
t_c - t = \frac{a_2}{2(a_1)^{1/3}}(\phi - \sin \phi)
\]

for \( a_1 < 0 \) \((0 < \phi < 2\pi)\)

(44)
\[
\begin{align*}
R &= \frac{a_2}{2a_1}(\cosh \phi - 1) \\
t_c - t &= \frac{a_2}{2a_1'}(\sinh \phi - \phi)
\end{align*}
\]
for \( a_1 > 0 \ (\phi > 0) \)

and

\[
R = \left(\frac{a_2}{a_1}\right)^{1/3} (t_c - t)^{2/3}
\]
for \( a_1 = 0 \)

Here \( t_c \) is an integration constant that corresponds to the time of arrival of each shell to the central singularity.

Choosing the power law solution (i.e. \( a_1 = 0 \)) for \( R \) (i.e., if \( R \propto T^{2/3} \) then equation (27) implies that \( K = 0 \) and assuming \( p_T = p_{\varpi T}/T^2, (T = t_c - t) \) (i.e., a function of \( T \) alone), it is possible to have a solution for \( \mu \) (from eq.(32)) as

\[
\mu(r, t) = \begin{cases} 
C_1(r) T^{n_1} + C_2(r) T^{n_2}, & p_{\varpi_T} < 1/4 \\
C_1(r) T^{1/6} \cos(k \ln T) + C_2(r) T^{1/6} \sin(k \ln T), & p_{\varpi_T} > 1/4 \\
C_1(r) T^{1/6} + C_2(r) T^{1/6} \ln T, & p_{\varpi_T} = 1/4
\end{cases}
\]

where \( C_1 \) and \( C_2 \) are arbitrary functions of \( r \), \( n_1 = \frac{1}{6} + \frac{1}{2} \sqrt{1 - 4p_{\varpi_T}}, \ n_2 = \frac{1}{6} - \frac{1}{2} \sqrt{1 - 4p_{\varpi_T}}, \ k = \frac{1}{2} \sqrt{4p_{\varpi_T} - 1} \) and \( R_0 = \left(\frac{a_2}{a_1}\right)^{1/3} \)

(Here we have set \( \eta = 0 \), which has no effect on the metric).

Further, the physical and kinematical parameters have the expressions

\[
\rho = \frac{2(\dot{R}^2 - K)}{R^2} - \frac{2(\dot{\mu} + \eta \dot{R})}{\mu + \eta R} - 2p_T
\]

\[
\theta = \frac{R \dot{\mu} + 2 \mu \dot{R} + 3 \eta R \ddot{R}}{R(\mu + \eta R)}
\]

\[
\sigma^2 = \frac{1}{3} \left[ \frac{\dot{\mu} R - \mu \dot{R}}{R(\mu + \eta R)} \right]^2
\]

It is to be noted that the solution for \( R \) does not depend on \( p_T \) so \( R \) has same expression for dust model. But for the solution of \( \mu \) we have only the power law form \( T^{2/3} \) (or \( T^{-1/3} \)) when \( R \) has Friedmann like behaviour (i.e. \( R \sim T^{2/3} \)). The difference comes in the matter density. For dust model \( \rho \) is a function of all the four co-ordinate variables while in the presence of tangential stress \( \rho \) is a function of \( t \) and \( r \) only. Finally, the asymptotic behaviour for both the model will be very similar.

V. ROLE OF PRESSURE IN GRAVITATIONAL COLLAPSE

In the general case when both radial and tangential pressures are non-zero and distinct then from the Einstein equations they can be obtained in compact form as

\[
\rho = \frac{\rho'}{\zeta \zeta'}
\]

\[
p_r = -\frac{\dot{\rho}}{\zeta \zeta'}
\]

\[
p_T = p_r + \frac{\zeta' \zeta}{2 \zeta'}
\]
where $F(r, t) = R e^{3\nu(\dot{R}^2 - f(r))}$ and $\zeta = e^\beta$.

Since $p_r$ is regular initially at the centre and blows up at the singularity so we can choose $p_r$ to be of the form:

$$p_r = \frac{g(r)}{R^n}$$ (50)

where $g(r)$ is an arbitrary function such that $g(r) \sim r^n$ near $r = 0$ to make initial matter density non-zero at the centre $r = 0$ and $n$ is any constant. As a consequence, the expressions for matter density and tangential stress become

$$\rho = \frac{H' + 3H\nu'}{R^2(R' + \nu')}$$ (51)

$$p_r = \frac{g(r)}{R^n} \left[ 1 - \frac{nR'}{2(R' + \nu')} \right] + \frac{g'(r)}{2R^{n-1}(R' + \nu')}, \quad (n \neq 3)$$ (52)

where $H(R, t) = C(r) - \frac{g(r)}{3-n} R^{3-n}$ and $C(r)$ is an arbitrary integration function. Also the radial velocity of collapsing shells at a distance $r$ from the centre is given by

$$\dot{R}^2 = f(r) + \frac{H(R, t)}{R}$$ (53)

Now if we choose $R = r$ initially then at the beginning of the collapse the density and the tangential stress have the initial values

$$\rho_i(r, x, y) = \rho_i(r, t_i, x, y) = \frac{c' + 3\nu'}{r^2(1 + r\nu')}$$ (54)

$$p_{ri} = p_r(t = t_i) = \frac{g(r)}{r^n} \left[ 1 - \frac{n}{2(1 + r\nu')} \right] + \frac{g'(r)}{2r^{n-1}(1 + r\nu')}$$ (55)

where $c(r) = H(r, t_i) = C(r) - \frac{g(r)}{3-n} r^{3-n}$.

Here it is to be noted that for regular initial data $C(r)$ and $g(r)$ to be $C^\infty$ functions and hence we have the following series expansions

$$g(r) = \sum_{j=0}^{\infty} g_j r^{n+j}$$

$$C(r) = \sum_{j=0}^{\infty} C_j r^{3+j}$$

$$\rho_i(r) = \sum_{j=0}^{\infty} \rho_j r^j$$ (56)

$$\nu' = \sum_{j=-1}^{\infty} \nu_j r^j$$

$$p_{ri} = \sum_{j=0}^{\infty} p_j r^j$$

where $\nu_{-1} \geq -1$.

In these series expansions the coefficients $g_j$’s and $C_j$’s are constants while $\rho_j$’s, $\nu_j$’s and $p_j$’s are functions of $x$ and $y$. These coefficients are related among themselves through the relations (54) and (55) as follows:
\[ p_0 = g_0, \quad p_1 = g_1 \left( 1 + \frac{1}{2(1+p^{-1})} \right), \quad p_2 = g_2 \left( 1 + \frac{1}{(1+p^{-1})} \right) - \frac{g_1 g_0}{2(1+p^{-1})^2}, \quad \ldots \quad (57) \]

\[ \rho_0 = 3c_0, \quad \rho_1 = \frac{4+3\rho^{-1}}{1+p^{-1}} c_1, \quad \rho_2 = \frac{5+3\rho^{-1}}{1+p^{-1}} c_2 - \frac{c_1}{(1+p^{-1})^2}, \quad \ldots \quad (58) \]

or

\[ p_0 = g_0 + \frac{g_1}{2v_0}, \quad p_1 = g_1 \left( 1 - \frac{1}{2v_0} \right) + \frac{g_2}{v_0}, \quad p_2 = g_2 \left( 1 - \frac{1}{v_0} \right) + \frac{g_1^2 - g_0 g_2}{2v_0^2} g_1 + \frac{3g_2}{2v_0}, \quad \ldots \quad (58) \]

\[ \rho_0 = 3c_0 + \frac{c_1}{v_0}, \quad \rho_1 = \frac{3c_0}{v_0} + c_1 \left( 3 - \frac{4}{v_0^2} \right), \quad \rho_2 = \frac{3c_0}{v_0} + c_2 \left( 3 - \frac{4}{v_0^2} \right) + c_1 \frac{(v_1^2 - v_0^2)}{v_0^2}, \quad \ldots \quad (58) \]

according as \( \nu_{-1} > -1 \) or \( \nu_{-1} = -1 \) and \( c_i = C_i - \frac{g_i}{3-n}, \quad i = 0, 1, 2, \ldots \)

The hypersurface \( t = t_s(r) \) describing the shell focusing singularity is characterized by

\[ R(t_s(r), r) = 0 \quad (59) \]

As the differential equation in \( R \) (i.e., eq.(53)) is not solvable so we shall consider only the marginally bound case (i.e., \( f(r) = 0 \)). Hence in this case, the singularity hypersurface can be written in explicit form as

\[ t_s(r) - t_i = \frac{2r^{3/2}}{3\sqrt{C(r)}} \, _2F_1 \left[ \frac{1}{2}, b + 1, \frac{g(r)r^{3-n}}{C(r)(3-n)} \right] \quad (60) \]

where \( _2F_1 \) is the usual hypergeometric function with \( b = \frac{3}{2(3-n)} \).

### A. Formation of Trapped Surfaces

The event horizon of observers at infinity plays an important role in the nature of the singularity. As formation of event horizon depends greatly on the computation of null geodesics whose computation are almost impracticable for the present space-time geometry, so we consider closely related concept of a trapped surface (namely a compact space-like 2-surface whose normals on both sides are future pointing converging null geodesic families). In fact, if the 2-surface \( S_{r,t} \) \( (r=\text{constant}, \quad t=\text{constant}) \) is a trapped surface then it and its entire future development lie behind the event horizon provided the density falls off fast enough at infinity. So mathematically, if \( K^\mu \) denotes the tangent vector field to the null geodesics which is normal to \( S_{r,t} \) then we have

\[ K_\mu K^\mu = 0, \quad K^\mu_\mu, K^\nu = 0. \]

Now the convergence or divergence of the null geodesics is characterized by the sign of the invariant \( K^\mu_\mu \) evaluated on the surface \( S_{r,t} = 0 \) (in fact, \( K^\mu_\mu < 0 \) indicates convergence while \( K^\mu_\mu > 0 \) stands for divergence). It can be shown that the inward geodesics converges initially and throughout the collapsing process while the outward geodesics diverges initially but becomes convergent after a time \( t_{ah}(r) \) (time of formation of apparent horizon) given by

\[ \dot{R}(t_{ah}(r), r) = -\sqrt{1 + f(r)} \]

Now using equations (23) and (50) we have
where in evaluating the limit we have used the series form of \( g(r) \) and \( C(r) \) (from eq. (56)). Now if we restrict \( n < 3 \) then we have a comparative expression between \( t_{ah}(r) \) and \( t_0 \) as

\[
t_{ah}(r) - t_0 = \left[ -C_0^{-3/2} C_1 2 F_1 \left( \frac{1}{2}, b, b + 1, z \right) + \frac{(C_0 g_1 - C_1 g_0)}{(3 - n)(9 - 2n)} C_0^{-5/2} 2 F_1 \left( \frac{3}{2}, b + 1, b + 2, z \right) \right] r
\]

\[+ O(r^2) - \frac{C_0^{3-n} g_0}{(3 - n)(9 - 2n)} 2 F_1 \left( \frac{1}{2}, b, b + 1, C_0^{3-n} z \right) r^{9-2n} + \ldots \ldots , \quad (n < 3) \quad (64)
\]

Note that here \( t_0 \) is the time of formation of singularity at \( r = 0 \) while \( t_{ah}(r) \) is the epoch at which a trapped surface is formed at a distance \( r \). Thus if trapped surface is formed at a later instant than \( t_0 \) then it is possible that any light signal from the singularity can reach an observer. As the geometry of the present model is a class of spherical space-time having different centres, so the condition for formation of NS (or BH) will be same as TBL model. Therefore, \( t_{ah}(r) > t_0 \) is the necessary condition for formation of naked singularity, while to form black hole the sufficient condition is \( t_{ah}(r) \leq t_0 \). It should be mentioned that this criterion for naked singularity is purely local.

Due to complicated form of equation (64) it is very difficult to make a comparative study between \( t_{ah} \) and \( t_0 \). Hence for simplicity we choose \( n = 3/2 \). Then the difference between \( t_{ah} \) and \( t_0 \) has the form

\[
t_{ah}(r) - t_0 = \frac{2 \left( C_0 g_1 - C_1 g_0 - g_1 \sqrt{C_0} \sqrt{C_0 - \frac{2}{3} g_0} \right)}{3 g_0 \sqrt{C_0} \left( \sqrt{C_0} + \sqrt{C_0 - \frac{2}{3} g_0} \right)} r + O(r^2) \quad (65)
\]

Hence in the present problem it is possible to have local naked singularity or a black hole form under the conditions shown in the following table (see Table I):
Figs. 1 and 2 show variation of $t_{\text{ah}} - t_0$ of eq.(65) for the variation of $k_0 (= g_0/C_0)$ and $k_1 (= g_1/C_1)$. Fig.1 corresponds to $C_1 > 0$ while Fig.2 corresponds to $C_1 < 0$.

**TABLE-I**

| Choice of the parameters | Naked Singularity | Black hole |
|--------------------------|-------------------|------------|
| (i) $g_1 > 0, C_1 < 0$   | Always possible   | Not possible |
| (ii) $g_1 < 0, C_1 > 0$  | Not possible      | Always possible |
| (iii) $g_1 > 0, C_1 > 0$ | $\frac{g_1}{C_1} > \frac{5}{2} \left(1 + \sqrt{1 - \frac{2}{3}k_0}\right)$ | $\frac{g_1}{C_1} < \frac{5}{2} \left(1 + \sqrt{1 - \frac{2}{3}k_0}\right)$ |
| (iv) $g_1 < 0, C_1 < 0$  | $\frac{|g_1|}{|C_1|} > \frac{5}{2} \left(1 + \sqrt{1 - \frac{2}{3}k_0}\right)$ | $\frac{|g_1|}{|C_1|} < \frac{5}{2} \left(1 + \sqrt{1 - \frac{2}{3}k_0}\right)$ |

Here we note that for initial density gradient to be negative at the centre (i.e., $\rho_1 < 0$) we must have $(C_1 - \frac{2g_0}{3}) < 0$ (for $\nu_1 > -1$). In the first case (i.e., $g_1 > 0, C_1 < 0$) we have negative definite $\rho_1$ and there is always naked singularity as in the dust model. Similarly in the second case (i.e., $g_1 < 0, C_1 > 0$), $\rho_1$ is positive definite and we always get black hole same as dust model. For the third and fourth cases (when $g_1$ and $C_1$ are of same sign) both naked singularity (NS) and black hole (BH) are possible for the restrictions given in the table I. When both $g_1$ and $C_1$ are positive (third case) or negative (fourth case) then for formation of NS $\rho_1$ is negative but for BH case as there is no lower limit (or upper limit) of $\frac{g_1}{C_1}$ (or $\frac{|g_1|}{|C_1|}$) so $\rho_1 > 0$ or $\rho_1 < 0$ are possible. Further for $g_1 = C_1 = 0$ we have $\rho_1 = 0$ then we have similar behaviour for the parameters $(g_2, C_2)$. A diagrammatic representation of $t_{\text{ah}} - t_0$ for variation of $k_0 (= g_0/C_0)$ and $k_1 (= g_1/C_1)$ has been shown in figures 1 and 2 for positive and negative $C_1$ respectively. In both the figures the vertical positive region corresponds to NS while the negative region stands for BH solution. Finally, we see that if the initial density or pressure has opposite behaviour (i.e., one increases and other decreases and vice-versa) near the centre $r = 0$ then we have similar character of the singularity as in dust model i.e., pressure has no significant effect on the singularity formation. On the other hand, if initial density and pressure increase or decrease simultaneously near the centre then even for negative density gradient at the centre it is possible to have a BH formation at $r = 0$, which is a distinct result in compare to dust model. Therefore, we may conclude that pressure tries to resist formation of NS.
B. Study of Geodesics

For simplicity of calculation here we shall consider as before the marginally bound case 
\( f(r) = 0 \) with \( n = 3/2 \). Then \( R(t, r) \) has the explicit solution (choosing the initial time \( t_i = 0 \)) which can be written in convenient form as

\[
t(r) = \frac{2}{g(r)} \left[ \sqrt{C(r) - \frac{2}{3}g(r)R^{3/2}} - \sqrt{C(r) - \frac{2}{3}g(r)r^{3/2}} \right]
\]

To examine whether the singularity at \( t = t_0, r = 0 \) is naked or not, we investigate whether there exist one or more outgoing null geodesics which terminate in the past at the central singularity. In particular, we shall concentrate to radial null geodesics only.

First we assume that it is possible to have one or more such geodesics and we choose the equation of the outgoing radial null geodesic (ORNG) which passes through the central singularity in the past as (near \( r = 0 \))

\[
t_{\text{ORNG}} = t_0 + a \ r^\xi
\]

to leading order in the \((t, r)\) plane with \( a > 0, \xi > 0 \).

Now the expression for the singularity time (characterized by \( R(t_s(r), r) = 0 \)) from (66) is

\[
t_s(r) = \frac{2}{g(r)} \left[ \sqrt{C(r)} - \sqrt{C(r) - \frac{2}{3}g(r)r^{3/2}} \right]
\]

and hence the time for central singularity is

\[
t_0 = \frac{2}{g_0} \left( \sqrt{C_0} - \sqrt{C_0 - \frac{2}{3}g_0} \right)
\]

Here we choose for \( C(r) \) and \( g(r) \) as

\[
C(r) = C_0 r^3 + C_k r^{k+3}
\]
\[
g(r) = g_0 \ r^{3/2} + g_l \ r^{l+3/2}
\]

where \( C_0, g_0 \) are constants and \( C_k (< 0) \) and \( g_l (< 0) \) are the first non-vanishing term beyond \( C_0 \) and \( g_0 \) respectively. As a consequence the expression for \( t_s(r) \) becomes

\[
t_s(r) = t_0 + \frac{C_k}{g_0} \left( \frac{1}{\sqrt{C_0}} - \frac{1}{\sqrt{C_0 - \frac{2}{3}g_0}} \right) r^k + \frac{2g_l}{g_0} \left( \frac{1}{3\sqrt{C_0 - \frac{2}{3}g_0}} - \frac{\sqrt{C_0} - \sqrt{C_0 - \frac{2}{3}g_0}}{g_0} \right) r^l + ....
\]

we shall now study the following possibilities:

(i) \( k < l \),  \( ii \) \( k > l \)

Case I: \( k < l \)

Here for \( t_s(r) \) we write

\[
t_s(r) = t_0 - \frac{C_k}{g_0} \left( \frac{1}{\sqrt{C_0}} - \frac{1}{\sqrt{C_0}} \right) r^k, \quad (C_k < 0)
\]
Now comparing with the geodesic equation (67) we get the relations

\[(a) \ \xi > k \text{ or } (b) \ \xi = k \text{ and } a < -\frac{C_k}{g_0} \left( \frac{1}{\sqrt{C_0 - \frac{2}{3} g_0}} - \frac{1}{\sqrt{C_0}} \right) \] (73)

When \( \xi > k \) then near \( r = 0 \) the solution for \( R \) simplifies to

\[ R = r \left[ 1 - \frac{3}{8} g_0 t^2 - \frac{3}{2} t \left( \sqrt{C_0 - \frac{2}{3} g_0} + \frac{C_k t^k}{2 \sqrt{C_0 - \frac{2}{3} g_0}} \right) \right]^{2/3} \] (74)

Further for the given metric an ORNG should satisfy

\[ \frac{dt}{dr} = R' + R \nu' \] (75)

To examine the feasibility of the null geodesic starting from the singularity, we combine equations (67) and (74) in equation (75) and we get (upto leading order in \( r \))

\[ a\xi^{\xi-1} = \left( 1 + \nu_{-1} + \frac{2k}{3} \right) \left[ -\frac{3C_k t_0}{4\sqrt{C_0 - \frac{2}{3} g_0}} \right]^{2/3} r^{2/3}, \quad (\nu_{-1} \neq 0) \] (76)

which implies

\[ \xi = 1 + \frac{2k}{3} \text{ and } a = \frac{1}{\xi} \left( 1 + \nu_{-1} + \frac{2k}{3} \right) \left[ -\frac{3C_k t_0}{4\sqrt{C_0 - \frac{2}{3} g_0}} \right]^{2/3} \] (77)

As \( \xi > k \), so from (77) \( k < 3 \). Since \( k \) is an integer, we could have

\[ k = 1, \ \xi = \frac{5}{3} \]

or

\[ k = 2, \ \xi = \frac{7}{3} \]

On the other hand for \( \xi = k \), as before we get \( k = 3 \) and

\[ a = \frac{1}{3} \left[ \frac{3}{4} \left( a g_0 t_0 + 2a \sqrt{C_0 - \frac{2}{3} g_0} + \frac{C_k t_0}{\sqrt{C_0 - \frac{2}{3} g_0}} \right) \right]^{-1/3} \times \]

\[ \left[ -\frac{3}{4} \left( 1 + \nu_{-1} \right) \left( a g_0 t_0 + 2a \sqrt{C_0 - \frac{2}{3} g_0} + \frac{2 + \nu_{-1} + \frac{4k}{3} C_k t_0}{\sqrt{C_0 - \frac{2}{3} g_0}} \right) \right] \] (79)
Case II: \( k > l \)

In this case

\[
t_s(r) = t_0 - \frac{2g_l}{g_0} \left( \sqrt{C_0 + \frac{g_0 - 3C_0}{3 \sqrt{C_0 - \frac{2}{3}g_0}}} \right) r^l
\]  

(80)

Now matching the geodesic equation as above we get

\[
(a) \quad \xi > l \quad \text{or} \quad (b) \quad \xi = l \quad \text{and} \quad a < -\frac{2g_l}{g_0} \left( \sqrt{C_0 + \frac{g_0 - 3C_0}{3 \sqrt{C_0 - \frac{2}{3}g_0}}} \right)
\]

(81)

Then for \( \xi > l \) we get

\[
l = 1, \quad \xi = \frac{5}{3} \quad \text{or} \quad l = 2, \quad \xi = \frac{7}{3} : \quad a = \frac{1}{\xi} \left( 1 + \nu_{-1} + \frac{2l}{3} \right) \left[ \frac{3g_l t_0}{8} \left( t_0 - \frac{4}{3 \sqrt{C_0 - \frac{2}{3}g_0}} \right) \right]^{2/3}
\]

(82)

and for \( \xi = l \)

\[
l = 3, \quad a = \frac{1}{3} \left[ -\frac{3}{8} \left( g_0 t_0^2 + 2a t_0 g_0 \right) - \frac{3}{2} a \sqrt{C_0 - \frac{2}{3}g_0} + \frac{g_1 t_0}{2 \sqrt{C_0 - \frac{2}{3}g_0}} \right]^{-1/3} \times \]

\[
\left[ \frac{3}{8} \left( 1 + \nu_{-1} + \frac{2l}{3} \right) g_0 t_0^2 + 2a t_0 (1 + \nu_{-1}) g_0 \right] - \frac{3}{2} a (1 + \nu_{-1}) \sqrt{C_0 - \frac{2}{3}g_0} + \frac{(1 + \nu_{-1} + \frac{2l}{3}) g_1 t_0}{2 \sqrt{C_0 - \frac{2}{3}g_0}}
\]

(83)

We note that the expressions for ‘a’ is very complicated both in equations (79) and (83). So no definite conclusion is possible on the role of pressure in determining in the final state of collapse by ORNG.

VI. DISCUSSIONS AND CONCLUDING REMARKS

An extensive analysis of the four dimensional Szekeres model has been done for the matter containing pressure. When matter is in the form of perfect fluid then the isotropic pressure turns out to be a function of time only while the matter density is a function of all the four space-time variables. In this case, assuming a polynomial form for pressure, cosmological solutions have been obtained and their asymptotic behaviour have been studied. Both in quasi-spherical and quasi-cylindrical model the solution approaches isotropy along fluid world line as \( t \to \infty \).

Secondly, for the matter with tangential stress only, solutions are possible for quasi-cylindrical model. Here both the tangential stress and the matter density turns out to be a function of \( t \) and \( r \) only. The scale factor \( R \) has parametric solution as for dust model and does not depend on the tangential stress. However, choosing the parameter \( C_1 = 0 \), \( R \) has a power law solution and it is possible to have a complete solution if we assume the
tangential stress proportional to $t^{-2}$.

Lastly, gravitational collapse has been studied in details for anisotropic pressure (i.e., both radial and tangential pressures are non-zero and distinct) in quasi-spherical model. Here we have to assume the radial pressure as a function of $r$ and $t$ of the form (see eq. (50)) $p_r = g(r)/R^n$. Also to solve the differential equation in $R$ (see eq. (53)) we consider only the marginally bound case (i.e., $f = 0$) only. Then equation (64) shows a comparative study between the time of formation of trapped surface and the time of formation of central singularity. To simplified further, we choose $n = 3/2$ and detailed analysis has been done using equation (65). Table I shows all possibilities for the parameters involved in the expression. If the initial density gradient at the centre is positive definite (or negative definite) then as in dust case we have definitely a black hole (or naked singularity) as the final state of collapse. But when $\rho_1$ has no definite sign (as in third and fourth cases) then for black hole solution it is possible to have negative density gradient at the centre initially. In fact, near the singularity if the initial density and pressure has identical behaviour (i.e., increase or decrease simultaneously) then even with negative density gradient (initially at the centre) we can have black hole as the end state but if the initial density and pressure has opposite behaviour (i.e., one decrease while other increases and vice versa) then we have identical character as in dust case. So we conclude that pressure tries to resist the formation of naked singularity. Finally we have studied the geodesics to examine whether it is possible to have any future directed non space-like geodesic terminating in the past at the singularity. For simplicity, we have considered only radial null geodesic and it is found that the end state of collapse is characterized by the coefficients of the series expansion of initial density and pressure (radial). Due to complicated expressions we can not definitely characterize the role of pressure. Therefore, in the context of local visibility, we say that pressure tries to cover the singularity.

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