Right-angularity, flag complexes, asphericity

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Abstract

The “polyhedral product functor” produces a space from a simplicial complex $L$ and a collection of pairs of spaces, $\{(A(i), B(i))\}$, where $i$ ranges over the vertex set of $L$. We give necessary and sufficient conditions for the resulting space to be aspherical. There are two similar constructions, each of which starts with a space $X$ and a collection of subspaces, $\{X_i\}$, where $i \in \{0, 1, \ldots, n\}$, and then produces a new space. We give conditions for the results of these constructions to be aspherical. All three techniques can be used to produce examples of closed aspherical manifolds.

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Introduction

This paper concerns three similar methods for combining certain spaces (given as the input data) to construct a new space. These constructions

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are closely related to a standard construction of a cubical complex with link
at each vertex a specified simplicial complex (cf. [5, p. 212], [12, §1.2] or
Example [2,2] below). For each construction we prove a theorem which gives
conditions for the resulting space to be aspherical. We also give conditions
for the result to be a closed manifold.

The first result, Theorem 2.22, gives necessary and sufficient conditions
for the polyhedral product of a collection of pairs of spaces, \( \{A(i), B(i)\}_{i \in I} \),
to be aspherical. (The notion of a “polyhedral product” is defined in [2,1
as well as in [2,3].) The second result, Theorem 3.5, gives necessary and
sufficient conditions for the result of applying a reflection group trick to a
corner of spaces to be aspherical. (The notion of a “corner of spaces” is
defined in §3: roughly, it is a space \( X \) together with a collection of subspaces
\( X_i \), indexed by \( I(n) = \{0,1, \ldots, n\} \), such that all possible intersections of
the \( X_i \) are nonempty.) The third construction also starts with a corner of
spaces \( X \), but now as additional data we are given a simplicial complex \( L \)
and a coloring of its vertices \( f : L \to \Delta \) (where \( \Delta \) is the simplex on \( I(n) \)).
The third result, Theorem 3.8, gives sufficient conditions for the “pullback,”
\( f^*(X) \), to be aspherical. A common thread in all three results is that certain
auxiliary simplicial complexes need to be flag complexes.

Next, we say a few words about each construction in turn. Given a simplicial
complex \( L \) and a collection of pairs \( (A, B) = \{(A(i), B(i))_{i \in I}\} \),
indexed by the vertex set \( I \) of \( L \), the polyhedral product \( \mathcal{Z}_L(A, B) \) is defined as a
certain subspace of the product \( \prod_{i \in I} A(i) \). If each \( A(i) \) is simply connected
and each \( B(i) \) is connected, then \( \mathcal{Z}_L(A, B) \) is simply connected; however,
if either of these conditions fail, then the fundamental group of \( \mathcal{Z}_L(A, B) \)
is usually nontrivial. We compute this group in Theorem 2.18. Roughly
speaking, it is a “generalized graph product” of the fundamental groups of
the \( A(i) \). The conditions for \( \mathcal{Z}_L(A, B) \) to be aspherical are the following:

(i) each \( A(i) \) is aspherical,

(ii) for each “nonconelike vertex” \( i \in I \), each path component of \( B(i) \) is
aspherical and its fundamental group injects into \( \pi_1(A(i)) \), and

(iii) \( L \) is a flag complex.

Suppose \( X \) has a corner structure \( \mathcal{M} = \{X_i\}_{i \in I(n)} \). (The \( X_i \) are called
“mirrors.”) Suppose further that \( (W, S) \) is a spherical Coxeter system with
its set of fundamental involutions \( S \) indexed by \( I(n) \). (For example, \( W \)
could be \((C_2)^{I(n)}\) where \(C_2\) denotes the group of order two.) The “basic construction” of §1.5 yields a space with \(W\)-action, \(U(W, X)\). Necessary and sufficient conditions for \(U(W, X)\) to be aspherical are:

(i)' \(X\) is aspherical,

(ii)' for each \(i \in I(n)\), any path component of the inverse image of \(X_i\) in the universal cover \(\tilde{X}\) of \(X\) is acyclic and similarly for the inverse images of arbitrary intersections of the \(X_i\), and

(iii)' there is an “induced mirror structure” \(\tilde{M}\) on \(\tilde{X}\) and an associated simplicial complex \(N(\tilde{M})\), called its “nerve,” the condition being that \(N(\tilde{M})\) is a flag complex.

For the third construction, we are given a corner of spaces \(X\); however, in place of the Coxeter system, suppose we have a simplicial complex \(L\) with a coloring of its vertices, \(f : L \to \Delta\). The space \(f^*(X)\) is formed by pasting together copies of \(X\) in a fashion specified by \(L\). Sufficient conditions for \(f^*(X)\) to be aspherical are then conditions (i)', (ii)', (iii)', above, as well as, the condition that \(L\) be a flag complex.

A corner of spaces is a corner of manifolds if the space in question is a manifold with corners and if the mirrors are strata of codimension one. There are two good sources of corners of manifolds which satisfy conditions (i)', (ii)', (iii)' above:

- (Products of aspherical manifolds with boundary). For each \(i \in I(n)\), suppose \((M(i), \partial M(i))\) is an aspherical manifold with aspherical (and \(\pi_1\)-injective) boundary. Put \(M = \prod_{i \in I(n)} M(i)\) and let \(M_i\) be the set of points \(x \in M\) such that \(x_i \in \partial M(i)\). Then \(\{M_i\}_{i \in I(n)}\) is a corner structure on \(M\) satisfying the three conditions (cf. §3.5).

- (Borel-Serre compactifications). Suppose \(G\) is an algebraic group defined over \(\mathbb{Q}\) with real points \(G\). Let \(D\) be the associated symmetric spaces. Suppose \(\Gamma\) is an arithmetic subgroup of \(G\) and that \(\Gamma\) is torsion-free so that \(\Gamma \backslash D\) is a manifold. If the \(\mathbb{Q}\)-rank of \(G\) is greater than 0, then \(\Gamma \backslash D\) is not compact. If the \(\mathbb{Q}\) rank of \(G\) is \(n + 1\), then \(\Gamma \backslash D\) has a “Borel-Serre compactification” \(M\) which is a corner of manifolds over \(I(n)\). As we shall see in §3.5, the corner structure on \(M\) satisfies conditions (i)', (ii)', (iii)', above.
I became interested in these examples after conversations with Tam Nguyen Phan, a graduate student at the University of Chicago. She showed that one could obtain aspherical manifolds by applying the reflection group trick to a Borel-Serre compactification. This led me to start thinking about more general situations and conditions (i)', (ii)', (iii)'.

1 Basic definitions

1.1 Mirror structures

A mirror structure on a space $X$ over a set $I$ is a family of closed subspaces $\mathcal{M} = \{X_i\}_{i \in I}$. For each subset $J \subseteq I$, put

$$X_J := \bigcap_{i \in J} X_i, \quad X_J^c := \bigcup_{i \in J} X_i \quad (1.1)$$

Also, $X_\emptyset = X$ and $X^\emptyset = \emptyset$. For each $x \in X$, put

$$I(x) := \{i \in I \mid x \in X_i\}. \quad (1.2)$$

The nerve of the mirror structure is the simplicial complex $N(\mathcal{M})$ with vertex set $I$ defined by the requirement that a subset $J \subseteq I$ spans a simplex if and only if $X_J \neq \emptyset$.

Suppose that $X$ is a mirrored space over $I$, that $G = \pi_1(X)$ and that $p : \tilde{X} \to X$ is the universal covering. One can then define the “induced mirror structure on $\tilde{X}$,” as follows. For each $i \in I$, let $E_i$ be the $G$-set of path components of $p^{-1}(X_i)$. Let $E$ be the disjoint union of the $E_i$. The induced mirror structure $\tilde{\mathcal{M}} = \{\tilde{X}_e\}_{e \in E}$ on $\tilde{X}$ is defined by $\tilde{X}_e := e$.

1.2 Coxeter groups

A Coxeter matrix $(m(i, j))$ on a set $I$ is an $I \times I$ symmetric matrix with 1’s on the diagonal and with off-diagonal entries integers $\geq 2$ or the symbol $\infty$. A Coxeter matrix defines a Coxeter group $W$ with generating set $S := \{s_i\}_{i \in I}$ and with relations:

$$(s_is_j)^{m(i,j)} = 1, \quad \text{for all } (i,j) \in I \times I$$

Since each diagonal entry is equal to 1, this entails that each $s_i$ is an involution. (If $m(i,j) = \infty$, the above relation is simply omitted from the
presentation.) The pair \((W, S)\) is a Coxeter system. For a subset \(J \leq I\), let \(S_J := \{s_i \mid i \in J\}\) and let \(W_J := \langle S_J \rangle\) be the subgroup generated by \(S_J\). \(W_J\) is called a special subgroup of \(W\). The pair \((W_J, S_J)\) is a Coxeter system (cf. [12, ]). If \(W_J\) is a finite group, then \(J\) is a spherical subset of \(I\) and any conjugate of \(W_J\) is a spherical subgroup. Let \(S(W, S)\) denote the poset of spherical subsets of \(I\). There is an associated simplicial complex \(L(W, S)\) called the nerve of \((W, S)\), whose poset of simplices is \(S(W, S)\). In other words, the vertex set of \(L(W, S)\) is \(I\) and a nonempty subset of \(I\) spans a simplex if and only if it is spherical.

A Coxeter matrix is right-angled if all its off-diagonal are equal to 2 or \(\infty\). Similarly, the associated Coxeter system is a right-angled Coxeter system (a RACS) and the group is a right-angled Coxeter group (a RACG).

1.3 Buildings

A chamber system is set \(C\) (of “chambers”) together with a family of equivalence relations (“adjacency relations”) indexed by \(I\). For \(i \in I\), an \(i\)-equivalence class is called an \(i\)-panel. Two chambers are \(i\)-adjacent if they are \(i\)-equivalent and not equal. A gallery in \(C\) is a sequence \(C_0, \ldots, C_k\) of adjacent chambers. If \(C_{m-1}\) is \(i_m\)-adjacent to \(C_m\), then the type of the gallery is the word \(i_1 \cdots i_k\) in \(I\).

Suppose \((W, S)\) is a Coxeter system, where \(S = \{s_i\}_{i \in I}\) is indexed by \(I\). Given a word \(i_1 \cdots i_k\) in \(I\), its value in \(W\) is the element \(s_{i_1} \cdots s_{i_m}\) of \(W\). A building of type \((W, S)\) is a chamber system \(C\) of type \(I\) together with a Weyl distance \(\delta : C \times C \to W\) so that

- each panel has at least two elements, and
- given chambers \(C, D \in C\) with \(\delta(C, D) = w\), there is a gallery of minimal length from \(C\) to \(D\) of type \(i_1 \cdots i_n\) if and only if \(s_{i_1} \cdots s_{i_n}\) is a reduced expression for \(w\).

Given a subset \(J \leq I\), chambers \(C\) and \(D\) are said to belong to the same \(J\)-residue if they can be connected by a gallery such that all letters in its type belong to \(J\). A singleton \(\{C\}\) is a \(\emptyset\)-residue; an \(i\)-panel is the same thing as an \(\{i\}\)-residue (cf. [11, 25]).

A building is spherical if its type is a spherical Coxeter system. A building is a right-angled building (a RAB) if its type is a RACS.
Example 1.1. (*Products of rank 1 buildings*). A rank one building (i.e., a building of type $A_1$) is just a set in which every two distinct elements are adjacent. Suppose $C_0, \ldots, C_n$ are rank one buildings. The product $C_0 \times \cdots \times C_n$ can be given the structure of a building of type $A_1 \times \cdots \times A_1$ as follows. For $i \in \{0, 1, \ldots, n\}$, elements $(c_0, \ldots, c_n)$ and $(d_0, \ldots, d_n)$ are defined to be $i$-equivalent if $c_j = d_j$ for all $j \neq i$. This gives the product the structure of a chamber system; the Weyl distance is then defined in the obvious manner. The associated Coxeter group is $C_2 \times \cdots \times C_2$; hence, the product is spherical and right-angled. (More generally one can define the product of chamber systems or of buildings in a similar fashion, cf. [25, p. 2].)

1.4 Flag complexes

Suppose $L$ is a simplicial complex with vertex set $I$. Let $S(L)$ be the poset of (vertex sets of) simplices in $L$, including the empty simplex.

**Definition 1.2.** $L$ is a flag complex if for any subset $J$ of $I$ such that every two elements of $J$ bound an edge of $L$, then $J \in S(L)$. (In other words, $L$ is flag if every nonsimplex contains a nonedge.)

Any simplicial graph $L^1$ determines a flag complex $L$ with $L^1$ as its 1-skeleton as follows: a finite subset $J \subset \text{Vert}(L^1)$ spans a simplex of $L$ if and only if it spans a complete subgraph in $L^1$. (Combinatorialists say that $L$ is the clique complex of $L^1$.)

**Example 1.3.** If $\mathcal{P}$ is a poset, then the simplicial complex Flag($\mathcal{P}$) whose simplices are all nonempty finite chains $p_0 < \cdots < p_n$ is a flag complex. This means that the barycentric subdivision of any convex cell complex is a flag complex (cf. [1, Example A.5, p. 663] or [12, Appendix A.3]).

More details about flag complexes can be found in [1 Appendix A.1.2] or [12 Appendix A.1].

The complex $K(L)$ and its mirror structure. The space $K(L)$ is defined to be the simplicial complex Flag($S(L)$) (cf. Examples 1.3 (3)). The complex $K(L)$ is isomorphic to the cone on the barycentric subdivision of $L$ (the empty set provides the cone point). There is a canonical mirror structure over $I$ on $K(L)$ defined by $K(L)_i := \text{Flag}(S(L)_{\geq\{i\}})$. In other words, $K(L)_i$ is the star of the vertex $i$ in the barycentric subdivision of $L$. It follows
from the definition in (1.1) that $K(L)_J$ is nonempty if and only if $J \in S(L)$, in which case,

$$K(L)_J = \text{Flag}(S(L)_{\geq J}). \quad (1.3)$$

**Definition 1.4.** If $L = L(W, S)$ is the nerve of a Coxeter system, then $K(W, S) := K(L(W, S))$ is its *Davis chamber*. (In other words, $K(W, S) = \text{Flag}(S(W, S))$.)

### 1.5 The basic construction

Suppose $X$ is a mirrored space with its mirrors indexed by a set $I$ and that $(W, S)$ is a Coxeter system with its fundamental set of generators $S$ indexed by the same set $I$. Define an equivalence relation $\sim$ on $W \times X$ by

$$(w, x) \sim (w', x') \iff x = x' \text{ and } wW_{I(x)} = w'W_{I(x)},$$

where $I(x)$ is defined by (1.2). The *basic construction* on this data is the $W$-space,

$$U(W, X) := (W \times X)/\sim. \quad (1.4)$$

More generally, if $\mathcal{C}$ is a building of type $(W, S)$, define $\sim$ on $\mathcal{C} \times X$ by

$$(C, x) \sim (C', x') \iff x = x' \text{ and } C, C' \text{ belong to the same } I(x)\text{-residue.}$$

As before, $U(\mathcal{C}, X) := (\mathcal{C} \times X)/\sim$.

The simplex $\Delta$ on $I$ has a mirror structure $\{\Delta_i\}_{i \in I}$ where $\Delta_i$ is defined to be the codimension-one face opposite the vertex $i$. The *classical realization* of a building $\mathcal{C}$ is defined to be $U(\mathcal{C}, \Delta)$. When $\mathcal{C}$ is the thin building $W$, then the classical realization $U(W, \Delta)$ is called the *Coxeter complex*. If $W$ is a spherical, we can identify $\Delta$ with a spherical simplex and $U(W, \Delta)$ with the round sphere. If $\mathcal{C}$ is a spherical building of type $(W, S)$, then $U(\mathcal{C}, \Delta)$ is called the *spherical realization* of $\mathcal{C}$. For example, the spherical realization of a product of rank one buildings $\mathcal{C}_0 \times \cdots \times \mathcal{C}_n$ (cf. Example 1.3 (2)) is equal to their join,

$$U(\mathcal{C}_0 \times \cdots \times \mathcal{C}_n, \Delta^n) = \mathcal{C}_0 \ast \cdots \ast \mathcal{C}_n).$$

**Definition 1.5.** The *standard realization* of a building $\mathcal{C}$ of type $(W, S)$ is $U(\mathcal{C}, K)$, where $K = K(W, S)$, the Davis chamber of $(W, S)$ (cf. Definition 1.4).
Examples 1.6. (More examples of flag complexes).

1. If \( E_0, \ldots E_n \) are discrete sets, then their join, \( E_0 \ast \cdots \ast E_n \) is an \( n \)-dimensional simplicial complex and a flag complex. (Since this is the classical realization of \( E_0 \times \cdots \times E_n \), a spherical building of type \( A_n \times \cdots \times A_1 \), the fact that it is a flag complex is a special case of (2), below.)

2. The classical realization of any building is a flag complex (cf. [1, Ex. 4.50, p. 190]). In particular, the Coxeter complex of any Coxeter system \((W, S)\) is a flag complex.

3. Let \( H \) be any simplicial arrangement of linear hyperplanes in \( \mathbb{R}^n \). Then the nonzero faces of \( H \) define a simplicial complex \( \Lambda(H) \) which is a triangulation of \( S^{n-1} \). It turns out that \( \Lambda(H) \) is a flag complex (see [6, p. 29]).

4. The Deligne complex (cf. [18] or [9]) of an Artin group of spherical type is a flag complex. (This is proved is Lemma 4.3.2 in [9], where the the proof occupies pages 623–626.) Similarly, the Deligne complex of any real simplicial arrangement is a flag complex.

5. The link of a simplex in a flag complex is easily seen to be a flag complex (cf. [5, Remarks 5.16, p. 210]).

2 Polyhedral products

2.1 Definitions and examples

Suppose we are given a family of pairs of spaces \( (A, B) = \{(A(i), B(i))\}_{i \in I} \). We shall always assume that \( A(i) \) is path connected and that \( B(i) \) is a nonempty, closed subspace of \( A(i) \). Let \( x := (x_i)_{i \in I} \) denote a point in the product \( \prod_{i \in I} A(i) \). Put

\[
\text{Supp}(x) := \{i \in I \mid x_i \in A(i) - B(i)\}.
\]

Define the polyhedral product \( Z_L(A, B) \) to be the subset of \( \prod_{i \in I} A(i) \) consisting of those \( x \) such that \( \text{Supp}(x) \in S(L) \). (This terminology comes from [1, 2]. In [19] it is called a “generalized moment-angle complex.”) If each \( (A(i), B(i)) \) is equal to the same pair, say \( (A, B) \), then we shall write \( Z_L(A, B) \) instead of \( Z_L(A, B) \). Here is another way to define the same thing. For each \( J \in S(L) \), let \( Z_J(A, B) := \{x \in \prod A(i) \mid \text{Supp}(x) \leq J\} \). In other words,

\[
Z_J(A, B) = \prod_{i \in J} A(i) \times \prod_{i \in I - J} B(i).
\]  

(2.1)
(There is a slight abuse of notation here, since the coordinates in the product do not have a natural order.) The polyhedral product is defined by

$$Z_L(A, B) = \bigcup_{J \in \mathcal{S}(L)} Z_J(A, B).$$

(2.2)

**Example 2.1.** Suppose each $A(i)$ is the unit interval $[0, 1]$ and each $B(i)$ is the base point 1. The polyhedral product

$$K(L) := Z_L([0, 1], 1)$$

is called the *chamber* associated to $L$. The geometric realization of the poset $\mathcal{S}(L)$, $\text{Flag}(\mathcal{S}(L))$, can be identified as a standard subdivision of $K(L)$ in such a fashion that a standard subdivision of each cube in the polyhedral product is a subcomplex of $\text{Flag}(\mathcal{S}(L))$ (cf. [1] Appendix A.3), [7] §4.2 [12] Ex. A.4.8, A.5.2]). If $L$ is a triangulation of $S^{n-1}$, then $K(L)$ is an $n$-disk. (For example, if $L$ is the boundary complex of a simplicial convex polytope, then $K(L)$ can be identified with the dual polytope.)

**Example 2.2.** Suppose each $(A(i), B(i))$ is $(D^1, S^0)$ where $D^1 = [-1, 1]$ and $S^0 = \{\pm 1\}$. The space $Z_L(D^1, S^0)$ is featured in [12] §1.2. Let $C_2 := \{\pm 1\}$ denote the cyclic group of order 2. Then $(C_2)^I$ acts on the product $\prod_{i \in I} [-1, 1]$ as a group generated by reflections and $Z_L(D^1, S^0)$ is a $(C_2)^I$-stable subspace. Moreover, $K(L)$ is a strict fundamental domain for the action on $Z_L(D^1, S^0)$ (because $[0, 1]$ is a strict fundamental domain for $C_2$ on $[-1, 1]$). Associated to the 1-skeleton of $L$ there is a RACG, $W_L$ (See [12] 1.2). It turns out that the universal cover of $Z_L(D^1, S^0)$ can be identified with $\mathcal{U}(W_L, K(L))$. The fundamental group of $Z_L(D^1, S^0)$ is the kernel of the natural map $W_L \to (C_2)^I$ (See Example 2.16 and 2.4 below.)

Suppose each $(A(i), B(i))$ is $(D^2, S^1)$, where $D^2$ denotes the unit disk in the complex numbers, $C$, and $S^1 = \partial D^2$. The space $Z_L(D^2, S^1)$ is called the *moment-angle complex* in [7] Ch. 6]. For $m = |I|$, the subspace $Z_L(D^2, S^1)$ of $(D^2)^m \subset C^m$ is stable under the usual $T^m$-action, where $T^m := (S^1)^m$. As in the previous example, the quotient space is $K(L) := Z_L([0, 1], 1)$. If $L$ is a triangulation of $S^{n-1}$, then $Z_L(D^2, S^1)$ is a closed $(n + m)$-manifold. (In [15] we were interested in “toric manifolds”, $M^{2n} = Z_L(D^2, S^1)/T^{m-n}$, for an appropriately chosen subgroup $T^{m-n}$ of $T^M$ acting freely on $Z_L(D^2, S^1)$. Some of these $M^{2n}$ are nonsingular toric varieties and are symplectic manifolds with a Hamiltonian $T^m$-action.)
Let $BS^1 (= \mathbb{C}P^\infty)$ be the classifying space for the Lie group $S^1$ and let $\xi$ be the canonical complex line bundle over $BS^1$. The total spaces of the associated $D^2$- and $S^1$-bundles are denoted $D(\xi)$ and $S(\xi)$, respectively. So, $S(\xi) = ES^1$, the total space of the canonical principal $S^1$-bundle over $BS^1$. Note that $D(\xi)$ is homotopy equivalent to $BS^1$ while $S(\xi)$ is contractible. The classifying space for $T^m$ is the $m$-fold product of copies of $BS^1$.

*Davis-Januszkiewicz space, $DJ(L)$,* is defined to be the Borel construction on $Z_L(D^2, S^1)$, i.e.,

$$DJ(L) := Z_L(D^2, S^1) \times_{T^m} ET^m \quad (2.3)$$

(cf. [15] and [2, 3, 7, 19]). The space $DJ(L)$ can also be written as a polyhedral product: $DJ(L) = Z_L(D(\xi), S(\xi))$ (cf. [15]). Since $D(\xi) \sim BS^1$ and $S(\xi) \sim *$ (where $\sim$ means homotopy equivalent and where $*$ is a base point), we have that $DJ(L) \sim Z_L(BS^1, *)$.

**Example 2.3.** (Real Davis-Januszkiewicz space). In a similar fashion, we can take the Borel construction on the polyhedral product in Example 2.2 to get the real Davis-Januszkiewicz space, $DJ^R(L)$,

$$DJ^R(L) := Z_L(D^1, S^0) \times_{C^m_2} EC^m_2 \quad (2.4)$$

where $BC_2 (= \mathbb{R}P^\infty)$ is the classifying space for the cyclic 2-group, $C_2$, and $EC_2$ is its universal cover. As in the previous paragraph,

$$DJ^R(L) = Z_L(D(\xi), S(\xi)) \sim Z_L(BC_2, *), \quad (2.5)$$

where now $D(\xi)$ means the canonical $D^1$-bundle over $BC_2$ and $S(\xi)$ the $S^0$-bundle.

**Example 2.4.** Let $1 \in S^1$ be the standard base point. Consider $Z_L(S^1, 1)$. It is a subcomplex of the standard torus, $T^I (= (S^1)^I)$. As we shall see in §2.4, the fundamental group of $Z_L(S^1, 1)$ is the *right-angled Artin group* (or RAAG) associated to the 1-skeleton of $L$ (in other words, it is the graph product of infinite cyclic groups with respect to the graph $L^1$ (cf. Examples 2.13 (4), below).

As we shall see in §2.4 and §2.5, $Z_L(BC_2, *)$ is the classifying space of the associated RACG associated if and only if $L$ is a flag complex. Similarly, $Z_L(S^1, 1)$ is the classifying space for the RAAG associated to $L^1$ if and only if $L$ is a flag complex (cf. [10] §3.2 or Corollary 2.23, below).
Retractions. Suppose $I'$ is a subset of $I$ and $L' \leq L$ is the full subcomplex spanned by $I'$. For each $i \in I - I'$ choose a base point $b_i$ in $B(i)$ and let $b_{I-I'}$ be the corresponding point in $\prod_{i \in I-I'} B(i)$. The inclusion $L' \hookrightarrow L$ induces an inclusion $i : Z_{L'}(A, B) \hookrightarrow Z_L(A, B)$ by taking the product with $b_{I-I'}$.

Lemma 2.5. (cf. [19, Lemma 2.2.3]) Suppose, as above, that $L'$ is a full subcomplex of $L$. Then $Z_L(A, B)$ retracts onto $Z_{L'}(A, B)$.

Proof. The retraction $r : Z_L(A, B) \to Z_{L'}(A, B)$ is induced by the projection $\prod_{i \in I} A(i) \to \prod_{i \in I'} A(i)$.

\[ \] \[ \] 2.2 Right-angled buildings

Given a discrete space $E$, let $\text{Cone}(E)$ be the cone on $E$, i.e., $\text{Cone}(E)$ is the space formed from $E \times [0, 1]$ by identifying all points with second coordinate 0. Identify $E$ with the subspace $E \times 1$. For example, $\text{Cone}(S^0) = D^1$.

Example 2.6. (Realizations of products of rank 1 buildings). As in Example 1.1, suppose $\{E_i\}_{i \in I}$ is a family of rank one buildings (i.e., each $E_i$ is a discrete set with more than one element). Let $C = \prod_{i \in I} E_i$ be the product building. The associated Coxeter group is $(C_2)^I$ and the Davis chamber is the cube, $[0, 1]^I$. The standard realization of $C$ is

$$ U(C, [0, 1]^I) = \prod_{i \in I} \text{Cone}(E_i) $$

(cf. [13]). The spherical realization of $C$ is the join of the $E_i$ (cf. Examples 2.6(1)).

Here is an important generalization of Example 2.2.

Example 2.7. (Polyhedral products of rank 1 buildings). Take notation as in the previous example. Let $L$ be a simplicial complex with vertex set $I$. Put $(\text{Cone}(E), E) := \{(\text{Cone}(E_i), E_i)\}_{i \in I}$. Then the polyhedral product of the $(\text{Cone}(E_i), E_i)$ can be identified with the $K(L)$-realization of the product building $C = \prod_{i \in I} E_i$, i.e.,

$$ U(C, K(L)) = Z_L(\text{Cone}(E), E) \quad (2.6) $$

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Example 2.8. (Universal covers). We note that if the 1-skeleton of $L$ is not a complete graph (and each $E_i$ has at least two elements), then $\mathcal{Z}_L(\text{Cone}(E), E)$ will not be simply connected. It is proved in [13] that the universal cover $\tilde{U}$ of $\mathcal{U}(\mathcal{C}, K(L))$ is the $K(L)$-realization of a building, $\mathcal{C}$. (As a set, $\mathcal{C}$ can be identified with the inverse image in $\tilde{U}$ of the central vertex of $K(L)$.) The type of this building is $(W_L, S)$, the RACS associated to the 1-skeleton of $L$.

Lemma 2.9. The fundamental group of $\mathcal{Z}_L(\text{Cone}(E), E)$ depends only on the 1-skeleton of $L$, i.e.,

$$\pi_1(\mathcal{Z}_L(\text{Cone}(E), E)) = \pi_1(\mathcal{Z}_{L^1}(\text{Cone}(E), E)).$$

Proof. First consider the case where $L$ is the 2-simplex $\Delta$ on $\{1, 2, 3\}$. Then $\mathcal{Z}_\Delta(\text{Cone}(E), E) = \text{Cone} E_1 \times \text{Cone} E_2 \times \text{Cone} E_3$, while $Z_{\partial \Delta}(\text{Cone}(E), E)$ is the join $E_1 * E_2 * E_3$, which is simply connected. So, $Z_{\Delta}(\text{Cone}(E), E)$ is the cone on a simply connected space. It follows that $Z_L(\text{Cone}(E), E)$ can be constructed from $Z_{L^1}(\text{Cone}(E), E)$ by successively adjoining cones on simply connected spaces, so the fundamental group does not change.

Alternatively, this can be proved by using (2.6) and the arguments in [12, Ch. 8, 9].

Let $\tilde{Z}_L(\text{Cone}(E), E)$ denote the universal cover of $\tilde{Z}_L(\text{Cone}(E), E)$. As a consequence of (2.6) we have the following.

Proposition 2.10. (cf. [13]). Suppose $L$ is the flag complex of a simplicial graph $L^1$. With notation as above, the standard realization of $\mathcal{C}$ is given by

$$\mathcal{U}(\tilde{\mathcal{C}}, K(L)) = \tilde{U}(\mathcal{C}, K(L)) = \tilde{Z}_L(\text{Cone}(E), E).$$

Lemma 2.11. With notation as above, $\tilde{Z}_L(\text{Cone}(E), E)$ is contractible if and only if $L$ is a flag complex.

Proof. Let $(W, S)$ be the RACS associated to $L^1$ and let $\tilde{\mathcal{C}}$ be the RAB of type $(W, S)$ defined in the Example 2.8. By Examples 2.7 and 2.8, $\tilde{Z}_L(\text{Cone}(E), E) = \mathcal{U}(\tilde{\mathcal{C}}, K(L))$, the $K(L)$-realization of $\tilde{\mathcal{C}}$. But the $K(L)$ realization of any RAB is contractible if and only if $L$ is a flag complex. (If $L$ is a flag complex, then the $K(L)$ realization of a RAB is its standard realization which is CAT(0), by [14, Thm 11.1], hence, contractible. A slightly different argument showing that the cubical complex $\tilde{Z}_L(\text{Cone}(E), E)$ is CAT(0)
goes as follows: the link of any cubical face is a flag complex since any such link is the join of a link in $L$ with a multiple join of discrete sets.) If $L$ is not flag, then any $K(L)$ realization of an apartment is not contractible ([12, Thm. 9.1.4, p. 167]) and hence, the same is true for the building since it retracts onto any apartment.

### 2.3 Graph products of groups

Suppose we are given a simplicial graph $L^1$ with vertex set $I$ and a family of discrete groups $G = \{G_i\}_{i \in I}$.

**Definition 2.12.** The graph product of the $G_i$ is the group $\Gamma$ formed by first taking the free product of the $G_i$ and then taking the quotient by the normal subgroup generated by all commutators of the form $[g_i, g_j]$ where $\{i, j\} \in \text{Edge}(L^1)$, $g_i \in G_i$ and $g_j \in G_j$.

**Examples 2.13.**

1. If the graph $L^1$ has no edges, then $\Gamma$ is the free product of the $G_i$.
2. If $L^1$ is the complete graph, then $\Gamma$ is the direct sum $\prod_{i \in I} G_i$ (i.e., when $I$ is finite it is the direct product).
3. If all $G_i = \mathbb{C}_2$, then $\Gamma$ is the RACG determined by $L^1$.
4. If all $G_i = \mathbb{Z}$, then $\Gamma$ is the right-angled Artin group (or RAAG) determined by $L^1$.

**Relative graph products.** Suppose that for each $i$ we are given a $G_i$-set $E_i$. The group $G_i$ need not be effective on $E_i$. Let $N_i$ be the normal subgroup which acts trivially. Put $(G, E) := \{(G_i, E_i)\}_{i \in I}$ and $(\text{Cone}(E), E) := \{(\text{Cone} E_i, E_i)\}_{i \in I}$. As in Example 2.7 we have the polyhedral product $\mathcal{Z}_{L^1}(\text{Cone}(E), E)$. Let $\tilde{\mathcal{Z}}_{L^1}(\text{Cone}(E), E)$ be the universal cover of the space $\mathcal{Z}_{L^1}(\text{Cone}(E), E)$. The group $G := \prod_{i \in I} G_i$ acts on $\tilde{\mathcal{Z}}_{L^1}(\text{Cone}(E), E)$ and the kernel of the action is the normal subgroup $N := \prod_{i \in I} N_i$. Let $\Gamma_L$ be the group of all lifts of the $G/N$-action to $\tilde{\mathcal{Z}}_{L}(\text{Cone}(E), E)$. In other words, there is a short exact sequence of groups,

$$1 \rightarrow \pi_1(\mathcal{Z}_{L}(\text{Cone}(E), E)) \rightarrow \Gamma_L \rightarrow G/N \rightarrow 1.$$

Using Lemma 2.9 we deduce the following.

**Lemma 2.14.** The group $\Gamma_L$ of all lifts of the $G/N$-action to $\tilde{\mathcal{Z}}_{L}(\text{Cone}(E), E)$ depends only on the 1-skeleton of $L$, i.e., $\Gamma_L = \Gamma_{L^1}$.
Put $\Gamma = \Gamma_{L^1}$. This is almost the correct definition of the “graph product of the $G_i$ relative to the $E_i$.” To make the definition correct we must put back in the subgroup $N$. Thus, the graph product of the $G_i$ relative to the $E_i$, $Z_{L^1}(G, E)$, is the extension of $\Gamma$ by $N$ defined by the pullback diagram:

$$
\begin{array}{ccc}
Z_{L^1}(G, E) & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
G & \longrightarrow & G/N
\end{array}
$$

(2.7)

Remarks 2.15. If each $G_i$ acts simply transitively on $E_i$, then $E_i \cong G_i$, $Z_{L^1}(G, E) = \Gamma$ and this group can be identified with the previously defined graph product. If each $G_i$ is only required to be transitive, then $E_i \cong G_i/H$, and $Z_{L^1}(G, E)$ coincides with the graph product of the pairs $\{G_i, H_i\}$ as defined by Januszkiewicz-Świątkowski in [22]. Discussions of “graph products of group actions,” along the above lines, can be found in [21, §4.2] and [17, §1.4].

Example 2.16. Even when all the groups are trivial, the relative graph product need not be the trivial. Indeed, suppose each $G_i$ is the trivial group and each $E_i$ is $S^0$. Then $Z_{L^1}(G, E) = \pi_1(Z_{L^1}(D^1, S^0))$ (cf. Example 2.2).

2.4 The fundamental group of a polyhedral product

As before, $(A, B) = \{(A(i), B(i))\}_{i \in I}$ is a family of pairs of CW complexes and $L$ is a simplicial complex with vertex set $I$. We assume each $A(i)$ is path connected.

Let $G_i = \pi_1(A(i))$ and let $p_i : \tilde{A}(i) \rightarrow A(i)$ be the universal cover. Let $E_i$ denote the set of path components of $\tilde{B}(i)$, where $\tilde{B}(i) := p_i^{-1}(B(i))$. So, $E_i$ is naturally a $G_i$-set. The collection $(G, E) := \{(G_i, E_i)\}_{i \in I}$ is the fundamental group data associated to $(A, B)$. The natural map $\tilde{B}(i) \rightarrow E_i$ which collapses each component to a point extends to a $G_i$-equivariant map $q_i : \tilde{A}(i) \rightarrow \text{Cone } E_i$. Taking products we get a map $q : Z_L(\tilde{A}, \tilde{B}) \rightarrow Z_L(\text{Cone}(E), E)$. The universal cover $\tilde{Z}_L(A, B) \rightarrow Z_L(A, B)$ is induced from pulling back the universal cover of $Z_L(\text{Cone}(E), E)$ via $q$. Using this observation together with Lemma 2.9 we get the following.

Lemma 2.17. The fundamental group of the polyhedral product depends only on the 1-skeleton of $L$, i.e., $\pi_1(Z_L(A, B)) = \pi_1(Z_{L^1}(A, B))$.  

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Theorem 2.18. Take hypotheses as above and let \((G, E)\) be the fundamental group data associated to \((A, B)\). Then the fundamental group of the polyhedral product is the graph product of the \(G_i\) relative to the \(E_i\), i.e.,

\[ \pi_1(Z_L(A, B)) = Z_{L^1}(G, E). \]

Proof. Since \(\pi_1(Z_L(A, B))\) is the group of deck transformations of \(\tilde{Z}_L(\tilde{A}, \tilde{B}) \to Z_L(A, B)\), it acts on \(\tilde{Z}_L(\text{Cone}(E), E)\) and hence, defines a homomorphism \(\pi_1(Z_L(A, B)) \to \Gamma\) which lifts to \(\varphi : \pi_1(Z_L(A, B)) \to Z_{L^1}(G, E)\). Since the universal cover \(Z_L(\tilde{A}, \tilde{B})\) is the pullback of \(\tilde{Z}_L(\text{Cone}(E), E)\), it follows that \(\varphi\) is an isomorphism. \(\Box\)

Example 2.19. (Real Davis-Januszkiewicz space continued). This is a continuation of Example 2.3. By (2.5) \(DJ^R(L) \sim Z_L(BC_2, *)\); so, \(\pi_1(DJ^R(L))\) is the graph product of copies of \(C_2\) over \(L^1\). By Example 2.13(3), this graph product is \(W_{L^1}\), the RACG associated to \(L^1\).

Example 2.20. (Graph products of classifying spaces). More generally, given \(L^1\) and a family of groups \(G = \{G_i\}_{i \in I}\), let \((A, B) = \{(BG_i, *)\}_{i \in I}\). By Theorem 2.18 \(\pi_1(Z_{L^1}(A, B))\) is the graph product of the \(G_i\).

2.5 When is a polyhedral product aspherical?

A vertex \(v\) of a simplicial complex \(L\) is a conelike if it is joined by edges to all other vertices of \(L\).

Definition 2.21. A pair of spaces \((A, B)\) is an aspherical pair if

- \(A\) is path connected and aspherical,
- each path component of \(B\) is aspherical and the fundamental group of each such path component maps injectively into \(\pi_1(A)\).

Given a pair \((A, B)\) with \(A\) path connected, let \(\tilde{A}\) denote the universal cover of \(A\) and let \(\tilde{B}\) be the inverse image of \(B\) in \(\tilde{A}\).

Theorem 2.22. Suppose \(I\) is the vertex set of a simplicial complex \(L\) and \((A, B) = \{(A(i), B(i))\}_{i \in I}\) is a family of pairs. Then the following three conditions are necessary and sufficient for \(Z_L(A, B)\) to be aspherical.

(i) Each \(A(i)\) is aspherical.
(ii) If a vertex $i$ is not conelike, then $(A(i), B(i))$ is an aspherical pair.

(iii) $L$ is a flag complex.

Proof. For simplicity, put $Z_L \equiv Z_L(A, B)$. First we prove the necessity of these conditions. Assume $Z_L$ is aspherical. Taking $L'$ to be a single vertex $i$ in Lemma 2.5, we get that $A(i)$ is a retract of $Z_L$; hence, (i). If the vertex $i$ is not conelike then there is another vertex $j$ which is not connected to it. Consider the full subcomplex $L' = \{i, j\}$. We have

$$Z_{L'} := (A(i) \times B(j)) \cup (B(i) \times A(j)).$$

Consider the open cover of $Z_{L'}$ by the components of $A(i) \times B(j)$ and the components of $B(i) \times A(j)$. The nerve of this cover is a complete bipartite graph. The set of vertices of one type is $\pi_0(B(i))$ and of the other type, $\pi_0(B(j))$. The edges are indexed by $\pi_0(B(i)) \times \pi_0(B(j))$. The universal cover, $\tilde{Z}_{L'}$, is covered by components of $\tilde{A}(i) \times \tilde{B}(j)$ and $\tilde{B}(i) \times \tilde{A}(j)$ and it has the structure of a tree of spaces. By Lemma 2.5, $Z_{L'}$ is aspherical and hence, $\tilde{Z}_{L'}$ is contractible. Since each vertex space is a retract of $\tilde{Z}_{L'}$, we see that each component of $\tilde{B}(i) \times \tilde{A}(j)$ is contractible. Since $\tilde{A}(j)$ is contractible, this means each component of $\tilde{B}(i)$ is contractible; hence, (ii). If $i$ is not a conelike vertex, then since (ii) holds, the natural map $(\tilde{A}(i), \tilde{B}(i)) \to (\text{Cone}(E_i), E_i)$ is a homotopy equivalence. So, if no vertex is conelike, $Z_L(A, \tilde{B})$ is homotopy equivalent to $\tilde{Z}_L(\text{Cone}(E), E)$. By Lemma 2.11, $\tilde{Z}_L(\text{Cone}(E), E)$ is contractible if and only if $L$ is a flag complex. Hence, when there are no conelike vertices, if $\tilde{Z}_L(A, \tilde{B})$ is contractible, then $L$ is a flag complex. Next, suppose $i$ is a conelike vertex. Let $L'$ be the full subcomplex spanned by $I - \{i\}$. If $L \neq i \ast L'$, then there is a simplex $\sigma$ in $L'$ such that the join $i \ast \partial \sigma \subseteq L$ but the full simplex spanned by $\sigma$ and $i$ is not in $L$. Then $L'' := \sigma \cup (i \ast \partial \sigma)$ is a full subcomplex of $L$ isomorphic to the boundary of a simplex whose dimension is one more than that of $\sigma$. Note that $\tilde{Z}_{L''}(\text{Cone}(E), E)$ is the spherical realization of a product building (namely, $\mathcal{C}'' = \prod_{j \in \text{Vert}(L'')} E_j$); hence, it is homotopy equivalent to a wedge of $k$-spheres, where $k = \dim L'' + 1 \geq 2$. Since each $\tilde{A}_j$ is contractible, it is homotopy equivalent to $\text{Cone}(E_j)$. One can now show that $\tilde{Z}_{L''}(\tilde{A}, \tilde{B})$ is $(k - 1)$-connected and, by a simple spectral sequence argument, that for the first nonzero homology group we have $H_k(\tilde{Z}_{L''}(\tilde{A}, \tilde{B})) = H_k(\tilde{Z}_{L''}(\text{Cone}(E), E))$. Hence, $\pi_k(\tilde{Z}_{L''}(\tilde{A}, \tilde{B}) \neq 0$ for some $k \geq 2$, i.e., $\tilde{Z}_{L''}(\tilde{A}, \tilde{B})$ is not aspherical. But this is a retract of $Z_L(A, \tilde{B})$, contradicting the assumption that
$Z_L(A, B)$ is aspherical. Hence, $L = i \ast L'$. By induction on the number of vertices, we may suppose that $L'$ is flag and hence, so is $L$.

To prove sufficiency, suppose conditions (i), (ii) and (iii) hold. First we show that we can reduce to the case where there are no conelike vertices. Since $L$ is flag, if there are conelike vertices, then $L$ can be written as a join $L' \ast \Delta$, where $\Delta$ is the simplex spanned by all conelike vertices. Hence,

$$Z_L(A, B) = Z_L(A, B) \times Z_\Delta(A, B) = Z_{L'}(A, B) \times \prod_{i \in \text{Vert}(\Delta)} A(i)$$

Since each $A(i)$ is aspherical (by (i)), the product is aspherical and we are reduced to proving that $Z_{L'}(A, B)$ is aspherical. In other words, we can assume $L$ has no conelike vertices, i.e., that (ii) holds for all $i \in I$. This implies that each $\tilde{A}(i)$ is contractible and that each path component of $\tilde{B}(i)$ is contractible. Therefore, as before, $\tilde{Z}_L(\tilde{A}, \tilde{B})$ is homotopy equivalent to $\tilde{Z}_L(\text{Cone}(E), E)$. By Lemma 2.11, condition (iii) implies that the RAB, $\tilde{Z}_L(\text{Cone}(E), E)$, is contractible and hence, so is $\tilde{Z}_L(\tilde{A}, \tilde{B})$.

Corollary 2.23. As in Example 2.20, suppose we are given $L$ and a family of groups $G = \{G_i\}_{i \in I}$. Let $\Gamma$ be their graph product and let $(A, B) = \{(BG_i, *)\}_{i \in I}$. Then $Z_L(A, B) = B\Gamma$ if and only if $L$ is a flag complex.

3 Corners

3.1 More definitions

Nondegenerate simplicial maps and colorings. Suppose $L, L'$ are simplicial complexes with vertex sets $I, I'$, respectively.

Definition 3.1. A simplicial map $L \to L'$ is nondegenerate if its restriction to each simplex is injective. A coloring $1$ is a nondegenerate map $L \to \Delta$ onto a simplex.

Examples 3.2. (1) (Barycentric subdivisions). Suppose $L$ is the barycentric subdivision of an $n$-dimensional convex cell complex $Y$. Then each vertex $v$ of $L$ is the barycenter of some cell in $Y$. Let $d(v)$ be the dimension of this cell. Then $v \mapsto d(v)$ is a map $\text{Vert}(L) \to I(n)$, defining a coloring $d : L \to \Delta$.

1Sometimes we have used the word “folding” for this concept.
(2) (Finite Coxeter complexes). Suppose \((W, S)\) is a spherical Coxeter system of rank \(n + 1\). A fundamental simplex for \(W\) on \(\mathbb{S}^n\) can be identified with \(\Delta\) (where the mirror structure on \(\Delta\) is defined by letting \(\Delta_i\) be the codimension one face opposite to the vertex \(i\). The Coxeter complex, \(U(W, \Delta)\), is homeomorphic to \(\mathbb{S}^n\) and the natural projection \(U(W, \Delta) \to \Delta\) is a coloring. A special case of this is where \(W = (C_2)^{I(n)}\). In this case the Coxeter complex can be identified with the boundary complex of an \((n + 1)\)-dimensional octahedron.

(3) (Spherical buildings). More generally, if \(\mathcal{C}\) is a spherical building of type \((W, S)\), then its simplicial realization, \(U(\mathcal{C}, \Delta)\), is a simplicial complex and the natural projection \(U(\mathcal{C}, \Delta) \to \Delta\) is a coloring.

(4) (Simplices of groups). Suppose \(G\) is a simplex of groups as defined in \(\S 3.4\) below and that \(G\) is its fundamental group (= \(\pi_1(\mathcal{G})\)). Let \(U(G, \Delta)\) be its universal cover as defined in \(\S 3.4\). If \(U(G, \Delta)\) is a simplicial complex, then the projection \(U(G, \Delta) \to \Delta\) is a coloring. For example, this shows that the Deligne complex of an Artin group of spherical type is colorable (see \([9]\), \([18]\) or Examples \(3.11(3)\), below, for the definition of “Deligne complex”).

Properties for mirror structures. Suppose \(\mathcal{M}\) is a mirror structure on a space \(X\) over a set \(I\). The mirror structure is flag if its nerve \(N(\mathcal{M})\) is a flag complex. Equivalently, \(\mathcal{M}\) is flag if for any nonempty subset \(J \subseteq I\), the intersection \(X_J\) is nonempty whenever all pairwise intersections \(X_i \cap X_j\) are nonempty for all \(i, j \in J\).

A space \(X\) is \(m\)-acyclic (resp. acyclic) if its reduced homology groups, \(H_i(X)\), vanish for all \(i \leq m\) (resp. for all \(i\)). A mirror structure \(\mathcal{M}\) on a connected space \(X\) is 0-acyclic if each mirror is path connected as is each nonempty intersection of mirrors. The mirror structure is acyclic if \(X\) is acyclic as is each nonempty intersection of mirrors.

Next, we define the “induced mirror structure” on the universal cover \(\tilde{X}\) of \(X\). Let \(p : \tilde{X} \to X\) be the covering projection. For each \(i \in I\), let \(E_i\) denote the set of path components of \(p^{-1}(X_i)\). Let \(E\) denote the disjoint union of the \(E_i\). Let \(i : E \to I\) be the natural projection which sends \(E_i\) to \(i\). There is an induced mirror structure \(\tilde{\mathcal{M}} = \{\tilde{X}_e\}_{e \in E}\) defined by \(\tilde{X}_e := e\). Extend \(i\) to a natural projection \(i : N(\tilde{\mathcal{M}}) \to N(\mathcal{M})\), defined by \(J \mapsto i(J)\).

Corners of spaces. Suppose \(I(n) := \{0, 1, \ldots, n\}\) and \(P(I(n))\) is its power set. A corner of spaces is a path connected space \(X\) with a mirror structure

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\[ \mathcal{M} = \{ X_i \}_{i \in I(n)} \] such that \( X_J \neq \emptyset \) for all \( J \in \mathcal{P}(I(n)) \). So, the nerve of this mirror structure is the \( n \)-simplex, \( \Delta \). If \( \tilde{\mathcal{M}} \) is the induced mirror structure on \( \tilde{X} \), then \( \iota : N(\tilde{\mathcal{M}}) \to \Delta \) is a coloring.

**Example 3.3.** (*Products*). Suppose we are given the data for a polyhedral product over \( \Delta \), i.e., a family of pairs \( (A, B) = \{(A(i), B(i))\}_{i \in I(n)} \). Put \( X = Z_\Delta(A, B) = \prod_{i \in I(n)} A(i) \). Define a mirror structure on \( X \) over \( I(n) \) by letting \( X_i \) be the set of those points \( x \) such that \( x_i \in B(i) \), i.e., \[ X_i \cong B(i) \times \prod_{j \neq i} A(j). \]

Note that for \( J \leq I(n) \),
\[ X_J = \bigcap_{i \in J} X_i \cong \prod_{i \in J} B(i) \times \prod_{j \notin J} A(j). \]

**Example 3.4.** (*An \((n + 1)\)-cube*). Suppose \( \square \) is the \((n + 1)\)-dimensional cube \([0, 1]^{n+1}\). Define the mirror \( \square_i \) by \( x_i = 1 \). This is the special case of the previous example where \( (A(i), B(i)) = ([0, 1], 1) \). It is also a special case of Example 2.1.

### 3.2 Reflection groups and asphericity

Suppose that \( (m(i', j')) \) is a Coxeter matrix on a set \( I' \), that the nerve of the corresponding Coxeter system is \( L' \) and that \( f : L \to L' \) is nondegenerate simplicial map. Define a new Coxeter matrix \( (m(i, j)) \) on \( I \) (= \text{Vert}(L)) by
\[
m(i, j) = \begin{cases} 
1, & \text{if } i = j; \\
m(f(i), f(j)), & \text{if } \{i, j\} \text{ is an edge of } L; \\
\infty, & \text{otherwise.}
\end{cases}
\] (3.1)

The new Coxeter system corresponding to \( (m(i, j)) \) is induced from the old one via \( f \) (cf. [11]).

Suppose \( (W, S) \) is a spherical Coxeter system where \( S \) is indexed by \( I(n) \) and that \( X \) is a corner of spaces over \( I(n) \). When \( U(W, X) \) is aspherical?

**Theorem 3.5.** Suppose \( \mathcal{M} \) is a mirror structure on \( X \) giving it the structure of a corner of spaces over \( I(n) \) and that \( (W, S) \) is a spherical Coxeter system where \( S \) is also indexed by \( I(n) \). Then \( U(W, X) \) is aspherical if and only if the following three conditions hold.

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(i)’ $X$ is aspherical.

(ii)’ The induced mirror structure $\tilde{M}$ on the universal cover is acyclic.

(iii)’ $\tilde{M}$ is flag.

Proof. Let $\hat{W}$ be the Coxeter group induced from $W$ via $\iota : E \to I(n)$, as in (3.1). Let $S_E = \{s_e\}_{e \in E}$ be the corresponding fundamental system of generators for $W$. Let $L(\hat{W}, S_E)$ denote the nerve of this Coxeter system. According to [12, p. 168], the natural map $U(\hat{W}, \hat{X}) \to U(W, X)$ defined by $[\tilde{w}, \tilde{x}] \mapsto [\varphi(\tilde{w}), p(\tilde{x})]$ is the universal cover (where $\varphi : \hat{W} \to W$ is the natural epimorphism). By definition, $U(W, X)$ is aspherical if and only if $U(\hat{W}, \hat{X})$ is contractible. By [12, Thm. 9.1.5, p. 167] and [12, Cor. 8.2.8], this is the case precisely when the following two conditions hold.

(i)’ $\hat{X}$ is contractible.

(ii)’ For each nonempty spherical subset $J \leq E$, the subcomplex $\hat{X}_J$ is acyclic.

It follows from the assumption that $W$ is spherical, that $L(\hat{W}, S_E)$ is a flag complex. Since the property of being acyclic implies the property of being nonempty, condition (ii)’ entails $N(\hat{M}) = L(\hat{W}, \hat{X})$. So, we must have that

(iii)’ $N(\hat{M})$ is also a flag complex.

\[\square\]

Definition 3.6. A corner structure on $X$ is \textit{aspherical} if

- $X$ is aspherical,

- for all $J \in \mathcal{P}(I(n))$, each path component of $X_J$ is aspherical and the inclusion of any such component into $X$ induces a monomorphism on fundamental groups.

Note that if $X$ is an aspherical corner of spaces, then the induced mirror structure on $\hat{X}$ satisfies conditions (i)’ and (ii)’ of Theorem 3.5 (since for each spherical subset $J \leq E$, the space $\hat{X}_J$ is contractible and hence, a \textit{a fortiori} acyclic). Condition (iii)’ is problematical. However, there are two nice examples when (iii)’ is satisfied. The first is in the case of products as in Example 3.3 and the second is the case of Borel-Serre compactifications (as discussed in §3.5 below).
3.3 Pullbacks and asphericity

For any subset \( J \leq I(n) \), let \( J^c \) denote its complement:

\[
J^c = I(n) - J
\]  

(3.2)

Suppose we are given a corner structure \( M = \{ X_i \}_{i \in I(n)} \) on a space \( X \) and a coloring \( f : L \to \Delta^n \). This induces a map, also denoted by \( f \), from \( S(L) \) to \( \mathcal{P}(I(n)) \) (sending \( J \) to \( f(J) \)). For each \( J \in S(L) \), define a certain subspace \( Q(J) \) of \( S(L) \times X \) (where \( S(L) \) has the discrete topology) by

\[
Q(J) := (J, X_{f(J)}),
\]  

(3.3)

and let \( Q = \bigcup_{J \in S(L)} Q(J) \) be the disjoint union of the \( Q(J) \). Define an equivalence relation \( \sim \) on \( Q \) by identifying \( J' \times X_{f(J')} \) with the corresponding subspace of \( J \times X_{f(J)} \) whenever \( J' \leq J \). By definition, the pullback, \( f^*(X) \), is the quotient space \( Q/\sim \).

Example 3.7. Suppose, as in Example 3.4, that \( X \) is the cube \([0, 1]^{n+1}\). Then \( f^*(L) \) can be identified with \( K(L) \), the chamber of \( L \) defined in Example 2.1.

Theorem 3.8. Suppose the corner structure \( M \) on \( X \) satisfies the following conditions.

(i)’ \( X \) is aspherical.

(ii)’ The induced mirror structure \( \tilde{M} \) on the universal cover \( \tilde{X} \) is acyclic.

(iii)’ \( \tilde{M} \) is flag.

If \( L \) is also a flag complex, then \( f^*(X) \) is aspherical.

The induced mirror structure on \( \tilde{X} \) is indexed by \( E \). Using \( \iota : E \to I(n) \) we get a mirror structure over \( I(n) \) on \( \tilde{X} \) defined by

\[
\tilde{X}_i = \coprod_{e \in \iota^{-1}(i)} \tilde{X}_e.
\]  

(3.4)

Since \( f^*(\tilde{X}) \) is clearly a covering space of \( f^*(X) \), it is equivalent to prove that \( f^*(\tilde{X}) \) is aspherical. We will prove this by constructing an auxiliary cubical complex \( Y \), homotopy equivalent to \( f^*(\tilde{X}) \), and then use the theory of polyhedral nonpositive curvature to show that \( Y \) is aspherical.
To simplify notation let $N$ be $N(\tilde{\mathcal{M}})$, the nerve of $\tilde{\mathcal{M}}$, and put $K = K(N)$. $K$ also has a mirror structure over $I(n)$ defined as in (3.4) by $K_i = \bigsqcup e \in e^{-1}(i) K_e$. Our complex $Y$ is $f^*(K)$. As explained in Example 2.1, the space $K = Z_N([0,1],1)$ is a cubical complex. Since $f^*(K)$ is constructed by pasting together copies of $K$ along cubical subcomplexes, we see that $Y$ is also a cubical complex.

**Lemma 3.9.** The link of any cubical face of $Y$ is isomorphic to the join of a link of a simplex in $L$ with the link of a simplex in $N$ (in both cases the empty simplex is allowed).

**Proof.** Given a closed cubical face $F$ of $K = Z_N([0,1],1)$, put
\[
Z(F) = \{e \in E \mid x_e|_F = 0\}
\]
\[
D(F) = \{e \in E \mid x_e|_F = 1\}
\]
\[
\text{Supp}(F) = E - D(F) \in \mathcal{S}(N).
\]
A cubical face $F'$ of $Y$ is a pair $(J, F)$ where $J \in \mathcal{S}(L)$ and $F$ is a cubical face of $K$ with $f(J) = \iota(Z(F)) \in \mathcal{P}(I(n))$. Moreover,
\[
\text{Lk}(F', Y) = \text{Lk}(J, L) \ast \text{Lk}(\text{Supp}(F), N).
\]

**Proof of Theorem 3.8.** The proof consists of two parts. First of all, we show that the cubical complex $Y$ is locally CAT(0); hence, aspherical. Secondly, we show that a natural map $\varphi : Y \to f^*(\tilde{X})$ is a homotopy equivalence.

By Gromov’s Lemma (cf. [5, Thm. 5.18, p. 211] or [12, Appendix I.6]), the piecewise Euclidean metric on a cubical complex is locally CAT(0) if and only if the link of each vertex is a flag complex. It is easy to see that the link of any simplex in a flag complex is again a flag complex. So, by Lemma 3.9, $Y$ is locally CAT(0). (However, it is generally not simply connected since the intersection of various copies of $K$ in $Y$ need not be connected.)

Since, for each $J \in \mathcal{S}(N)$, $K_J$ is a cone, there is a natural map $\varphi : \tilde{X} \to K$, sending $\tilde{X}_J$ to $K_J$. By condition (ii)', for each $J \in \mathcal{S}(N)$, the intersection of mirrors $\tilde{X}_J$ is acyclic. It follows that, for each $J \in \mathcal{S}(N)$, the union $X^J$ is acyclic. In particular, each such $X^J$ is path connected. Since $f^*(K)$ and $f^*(\tilde{X})$ are both covered by contractible pieces homeomorphic to $K$ or $\tilde{X}$ respectively, and since these pieces are glued together in the same
combinatorial fashion along the same number of components, it follows from van Kampen’s Theorem that \( \pi_1(f^*(\tilde{X})) \cong \pi_1^*(f^*(K)) \quad (= \pi_1^*(Y)) \). Let \( U \) denote the universal cover of \( f^*(\tilde{X}) \) (this is also the universal of \( f^*(X) \)) and let \( \tilde{Y} \) denote the universal cover of \( Y \). Let \( \tilde{\varphi} : U \to \tilde{Y} \) be the map covering \( \varphi \). Since \( U \) and \( \tilde{Y} \) are both unions of contractible pieces which are glued together along acyclic subcomplexes in the same combinaorial pattern, it follows that \( \tilde{\varphi} \) is a homotopy equivalence. By the previous paragraph \( \tilde{Y} \) is contractible; hence, so is \( U \).

**Remark 3.10.** Suppose that \( L \) is the Coxeter complex of a spherical Coxeter system \((W, S)\) where \( S \) is indexed by \( I(n) \) and that \( X \) is a corner of spaces over \( I(n) \). By Examples 3.2(2), we can choose a coloring \( f : L \to \Delta \). Then \( f^*(X) \cong U(W, X) \). Hence, the reflection group trick of §3.2 is a special case of the pullback construction.

### 3.4 Simplices of groups and corners of groups

A poset \( P \) can be regarded as a category where there is a morphism from \( p \) to \( q \) if and only if \( p \leq q \). A poset of spaces over \( P \) is a functor from \( P \) to the category of spaces and inclusions. (We require the inclusions to be cofibrations.) Similarly, a poset of groups is a functor from \( P \) to the category of groups and monomorphisms.

The colimit of a poset of spaces \( \{X(p)\}_{p \in P} \) is the union,

\[
X := \bigcup_{p \in P} X(p)
\]

Similarly, given a poset of groups \( \{G(p)\} \) we can form the colimit, \( \lim_{\to} G(p) \). Of course, if \( P \) has a final element \( q \), then \( X = X(q) \) and \( G = G(q) \).

A (simple) simplex of groups is a poset of groups over \( P(I(n))_{<I(n)} \). Thus, \( \mathcal{G} \) associates to each proper subset \( J < I(n) \) a group \( G_J \) and to each pair \((J', J)\) with \( J' \leq J \) a monomorphism \( G_{J'} \hookrightarrow G_J \). The colimit of these groups is the fundamental group of \( \mathcal{G} \), denoted \( \pi_1(\mathcal{G}) \). (See Chapter III.C of \([5]\).) \( \mathcal{G} \) is developable if, for each \( J < I(n) \), the natural homomorphism \( G_J \to \pi_1(\mathcal{G}) \) is injective. One way in which a developable simplex of groups can arise is from a cell-preserving action of a group \( G \) on a simplicial cell complex with strict fundamental domain an \( n \)-simplex \( \Delta \) with its codimension one faces indexed by elements of \( I(n) \). (For us, a “simplicial cell complex” will not necessarily be a simplicial complex – it means a space formed by gluing together
simplices along unions of faces; however, the intersection of two simplices is not required to be a common face.) The group $G_J$ is then the isotropy subgroup at $\Delta_J$. Conversely, as we shall explain in the next paragraph, given any developable simplex of groups $\mathcal{G}$, there is an action of $G = \pi_1(\mathcal{G})$ on a simply connected simplicial cell complex $U$ with strict fundamental domain $\Delta$ so that the isotropy group at $\Delta_J$ is $G_J$. $U$ is called the universal cover of $\mathcal{G}$. (Even when $\mathcal{G}$ is not developable a universal cover can still be defined; however, the isotropy group at $\Delta_J$ might only be a quotient of $G_J$.)

Similarly, a corner of groups $\mathcal{G}$ is a functor from $\mathcal{P}(I(n))$ to the category of groups and monomorphisms. Since $I(n)$ is a terminal object of $\mathcal{P}(I(n))$, the colimit of the $G_J$ is $G_{I(n)} (= \pi_1(\mathcal{G}))$. So, a corner of groups is automatically developable. If $\mathcal{G}$ is any corner of groups, then we get a developable simplex of groups, $G_0$, by deleting the terminal object. Moreover, $G_{I(n)}$ is a quotient of $\pi_1(\mathcal{G}_0)$. Given a corner of groups $\mathcal{G}$, put $G = G_{I(n)}$. We can use a version of the basic construction to get cubical complex, $U(G/G_0, \Box)$ as follows. As in Example 3.4, let $\Box$ denote the $(n+1)$-dimensional cube. As in (1.2), given $x \in \Box$, put $I(x) := \{ i \in I(n) \mid x_i = 1 \}$. Define an equivalence relation $\sim$ on $G \times \Box$ by

$$(gG_0, x) \sim (g'G_0, x') \iff x = x' \text{ and } gG_{I(x)} = g'G_{I(x)},$$

and as in (1.4), put $U(G, \Box) = (G \times \Box)/\sim$. It is the universal cover of $\mathcal{G}$ as a complex of groups. If $\Delta$ denotes the $n$-dimensional simplex, then $U(G, \Delta)$ can be similarly defined. Call it link of $\mathcal{G}$. The universal cover of $U(G, \Delta)$ as a space is the universal cover of $G_0$ as a complex of groups.

Examples 3.11. Suppose $(W, S)$ is a spherical Coxeter system with $S$ indexed by $I(n)$. This gives rise to various corners of groups.

1. First, it gives the data for corner of groups $\mathcal{W}$ which assigns $W_J$ to $J$. Its link is the Coxeter complex $U(W, \Delta)$. (This is the universal cover of associated simplex of groups $\mathcal{W}_0$ when $n > 1$.) The Coxeter complex is a triangulation of $S^n$.

2. Suppose $G$ is a chamber-transitive automorphism group of a spherical building $\mathcal{C}$ of type $(W, S)$. Fix a chamber $C \in \mathcal{C}$ and let $G_J$ denote the stabilizer of $J$-residue containing $C$. This gives us a corner of groups $\mathcal{G}$ with $G_{I(n)} = G$. The link of $\mathcal{G}$ is the classical realization, $U(\mathcal{C}, \Delta)$.

3. Let $A (= A_{I(n)}$) be the Artin group of spherical type associated to $(W, S)$. In a similar fashion to (1), $A$ gives rise to the corner of Artin groups.
\(\mathcal{A}\) which associates to a subset \(J \leq I(n)\) the corresponding Artin group \(A_J\). In this case, the link of \(\mathcal{A}\) is the Deligne complex.

Actually in all three examples the assumption that \(W\) is spherical is unnecessary.

**Nonexample 3.12.** Suppose that \((W,S)\) is a finite Coxeter system of rank \(n + 1 \geq 3\). Let \(\Delta\) be a fundamental simplex so that the Coxeter complex, \(U(W,\Delta)\), is a triangulation of \(\mathbb{S}^n\). Let \(W^+\) be the orientation-preserving subgroup. Color the \(n\)-simplices which are translates of \(\Delta\) by an element of \(W^+\) white and color the others black. Call the union of the white simplices \(\Lambda\). (It is the complement in \(\mathbb{S}^n\) of the interiors of the black simplices.) The group \(W^+\) acts on \(\Lambda\) with \(\Delta\) a strict fundamental domain. It acts on the Cone \(\Lambda\) as well. Hence we get a corner of groups with link \(\Lambda\). Since it has “missing simplices”, \(\Lambda\) is obviously not a flag complex. If \(n + 1 > 3\), then \(\Lambda\) is simply connected and hence, is the universal cover of the corresponding simplex of groups. So, universal covers of simplices of groups need not be flag complexes.

**Example 3.13.** *(The associated corner of groups).* Next suppose \(\mathcal{M} = \{X_i\}_{i \in I(n)}\) is a corner structure on \(X\) and that \(\mathcal{M}\) is 0-acyclic. Put \(G_{I(n)} = \pi_1(X)\) and let \(G_J\) be the image of \(\pi_1(X_J)\) in \(G_{I(n)}\), where \(J^*\) is defined in [3.2]. (Here we want to choose the base point in \(X_{I(n)}\).) This defines a corner of groups \(\mathcal{G}(\mathcal{M})\), called the associated corner of groups. Consider the induced mirror structure \(\tilde{\mathcal{M}} = \{\tilde{X}_e\}_{e \in \mathcal{E}}\) on the universal cover \(\tilde{X}\) and let \(N(\tilde{\mathcal{M}})\) be its nerve. The group \(G_{I(n)}\) acts on \(N(\tilde{\mathcal{M}})\). If \(\tilde{\mathcal{M}}\) is also assumed to be 0-acyclic, then any \(n\)-simplex of \(N(\tilde{\mathcal{M}})\) is a strict fundamental domain. It follows that \(N(\tilde{\mathcal{M}}) = U(G/G_\emptyset, \Delta)\), the link of the associated corner of groups.

Next, we need to define a corner structure on the \((n + 1)\)-cube, which is different from the one defined in Example 3.4 and used above. To emphasize the difference we will regard the new one as giving the cube the structure of a poset of spaces over \(\mathcal{P}(I(n))^{\text{mathrm{op}}}\).

**Example 3.14.** *(The cube again)* For each \(J \leq I(n)\) let \(\Box(J)\) be the face of \(\Box\) defined by

\[\Box(J) := \{x \in \Box \mid x_i = 0 \text{ for all } i \in J\}.\]  

(3.5)

(The difference from Example 3.4 is that a mirror is defined by \(x_i = 0\) rather than \(x_i = 1\).) Note that \(\text{Flag}(\mathcal{P}(I(n)))\) is a standard subdivision of \(\Box\) and
Flag(\(\mathcal{P}(I(n))\)) is a standard subdivision of \(\Box(J)\), cf. [11, Appendix A.3], [7, §4.2], or [12, Ex. A.2.5].

In the next example we show that any corner of groups can occur as \(N(\mathcal{M})\) for a corner structure on some space \(X\).

**Example 3.15.** Suppose \(G\) is a corner of groups and \(G = G_{I(n)}\). Let \(Y\) be any space with \(\pi_1(Y) = G\) and with universal cover \(\tilde{Y}\). Put

\[
X := \tilde{Y} \times_G \mathcal{U}(G/G_\emptyset, \Box).
\]

Since \(\mathcal{U}(G/G_\emptyset, \Box)\) is contractible (it is a cone), by taking projection onto the first factor, we see that \(X\) is homotopy equivalent to \(Y\). In particular, if \(Y = BG\), then \(X\) is also a model for the classifying space of \(G\). Projection onto the second factor induces a map \(p : X \to \mathcal{U}(G/G_\emptyset, \Box)/G = \Box\). Put

\[
X_J := p^{-1}(\Box(J)) \quad \text{and} \quad X_i := X_{(i)},
\]

so that \(\mathcal{M} = \{X_i\}_{i \in I(n)}\) is a corner structure on \(X\). Since \(\Box(J)\) is the union of simplices in \(\text{Flag}(\mathcal{P}(I(n))_{\leq J})\) with maximum vertex \(J\); \(\mathcal{U}(G/G_\emptyset, \Box(J))\) is homotopy equivalent to \(G/G_J\). So, \(p^{-1}(\Box(J))\) is homotopy equivalent to \(\tilde{Y} \times_G (G/G_J) = \tilde{Y}/G_J\). Thus, \(\pi_1(X_J) = G_J\).

As a further example we could let \(Y = BG\) where, as in Examples 3.11, \(G\) is either a spherical Coxeter group \(W\) or a spherical Artin group of rank \(n + 1\). Then

\[
BW \sim EW \times_W \mathcal{U}(W, \Box) \quad \text{(3.6)}
\]
\[
BA \sim EA \times_A \mathcal{U}(A, \Box). \quad \text{(3.7)}
\]

These two formulas play a role in [9].

**Corners of \(G\)-sets.** The hypothesis in Example 3.13 that the corner structure on \(X\) is 0-acyclic seems unnatural. To get around this we need to talk about corners of \(G\)-sets rather than corners of groups. Suppose \(G\) is a group. A **poset of \(G\)-sets** is a cofunctor from a poset \(\mathcal{P}\) to the category of \(G\)-sets and \(G\)-equivariant maps. For example, given a poset of groups \(\{G(p)\}_{p \in \mathcal{P}}\) with a final object \(G = G(q)\), we get a poset of \(G\)-sets \(\{G/G(p)\}_{p \in \mathcal{P}}\). A **corner of \(G\)-sets** is a cofunctor from \(\mathcal{P}(I(n))\) to \(G\)-sets. Suppose \(\mathcal{E}\) is a corner of \(G\)-sets. In other words, for each \(J \leq I(n)\) we are given a \(G\)-set \(E_J\) and whenever
$J \leq J'$ a $G$-equivariant map $\varphi_{J,J'} : E_{J'} \to E_J$. Put $\varphi_J = \varphi_{J,I(n)} : E_{I(n)} \to E_J$. As before, put $\mathcal{U}(\mathcal{E}, \square) = (E_{I(n)} \times \square)/\sim$, where $\sim$ is defined by:

$$(e, x) \sim (e', x') \iff x = x' \text{ and } \varphi_I(e) = \varphi_I(e').$$

The space $\mathcal{U}(\mathcal{E}, \square)$ is the universal cover of the corner of $G$-sets $\mathcal{E}$. The simplicial cell complex $\mathcal{U}(\mathcal{E}, \Delta)$ is defined similarly. Associated to any corner structure $\mathcal{M}$ on $X$, we have a corner of $G$-sets $\mathcal{E}(\mathcal{M})$ as in Example 3.13 and as in that example we get the following.

**Proposition 3.16.** (cf. Example 3.13). Suppose $\mathcal{M}$ is a mirror structure on $X$ and $\tilde{\mathcal{M}}$ is the induced structure on the universal cover. Then the nerve of $\tilde{\mathcal{M}}$ satisfies $N(\tilde{\mathcal{M}}) = \mathcal{U}(\mathcal{E}(\mathcal{M}), \Delta)$. Hence, $K(N(\tilde{\mathcal{M}})) = \mathcal{U}(\mathcal{E}, \square)$.

### 3.5 Examples of aspherical corners satisfying the flag condition

When is $N(\tilde{\mathcal{M}})$ a flag complex? For us, the most common reason for this to be true will be that it is a spherical building (cf. Example 3.6(2)).

**Products.** Suppose $(A, B) = \{(A(i), B(i))\}_{i \in I(n)}$ and as in Example 3.3, $X := Z_\Delta(A, B) = \prod_{i \in I(n)} A(i)$ is the corresponding corner, where $X_i$ is the set of points with $i$-th coordinate lying in $B(i)$. Put $G(i) = \pi_1(A(i))$ and $G = \pi_1(X) = G(0) \times \cdots \times G(n)$. Let $p : \tilde{X} \to X$ be the universal cover and $E_i$ the $G$-set of path components of $p^{-1}(X_i)$. Let $p(i) : \tilde{A}(i) \to A(i)$ be the universal cover and let $E(i)$ be the $G(i)$-set of path components of $p(i)^{-1}(B(i))$. Since each $A(j)$ is connected, the $G$-set $E_i$ is identified with the $G(i)$-set $E(i)$, equivariantly with respect to the projection $G \to G(i)$. (For example, if each $B(i)$ is connected and $H(i) := \pi_1(B(i))$, then $G_i := \pi_1(X_i) = G(0) \times \cdots \times H(i) \times \cdots G(n)$ and

$$E_i = G / G_i \cong G(i) / H(i) = E(i).$$

As at the end of the previous subsection, we get a corner of $G$-sets $\mathcal{E}$, with $E_J \cong \prod_{i \in J} E(i)$, for $J \in \mathcal{P}(I(n))$. It follows that the simplicial cell complex $\mathcal{U}(\mathcal{E}, \Delta)$ is the join $E(0) * \cdots * E(n)$. In other words, $\mathcal{U}(\mathcal{E}, \Delta)$ is the spherical realization of a product of rank one buildings, $E(0) \times \cdots \times E(n)$; hence, a flag complex (cf. Examples 3.6(1), (2)). So, by Proposition 3.16 $\mathcal{M}$ is
flag. Suppose each pair \((A(i), B(i))\) is an aspherical pair (cf. Definition 2.21). Then since each component of \(X_J\) is aspherical, each component of its inverse image in \(\widetilde{X}\) is contractible. Thus, \(N(\mathcal{M})\) is acyclic. So, we have proved the following.

**Theorem 3.17.** Suppose \(\{(A(i), B(i))\}_{i \in I(n)}\) is a family of aspherical pairs and that \(X = A(0) \times \cdots \times A(n)\) has corner structure \(\mathcal{M}\) as above. Then \(X\) and \(\mathcal{M}\) satisfy conditions (i)', (ii)', (iii)' of Theorems 3.5 and 3.8.

**Borel-Serre compactifications.** Suppose \(G\) is a semisimple linear algebraic group defined over \(\mathbb{Q}\). Let \(G(\mathbb{R})\) and \(G(\mathbb{Q})\) denote its real and rational points, respectively. Suppose \(G\) has \(\mathbb{Q}\) rank \(n + 1\). Let \(K\) be the maximal compact subgroup of \(G(\mathbb{R})\) and \(D\) denote the symmetric space \(K \setminus G(\mathbb{R})\). In [4] Borel and Serre introduced a bordification \(\overline{D}\) of \(D\) such that

- \(\overline{D}\) is a manifold with corners (see §4.1 below).
- \(\overline{D}\) is \(G(\mathbb{Q})\)-stable.
- If \(\Gamma < G(\mathbb{Q})\) is any arithmetic lattice in \(G(\mathbb{R})\), then \(\overline{D}/\Gamma\) is compact.

If \(\Gamma\) is torsion-free, then \(M := K \setminus G(\mathbb{R})/\Gamma\) is a smooth manifold with corners, called the Borel-Serre compactification of \(D/\Gamma\). Associated to a \(\mathbb{Q}\)-split torus there is a spherical Coxeter system \((W', S')\) of rank \(n + 1\). Index \(S'\) by \(I(n)\). The strata of \(M\) (or of \(\overline{D}\)) correspond to subsets of \(I(n)\). Let \(M_J\) denote the stratum corresponding to \((W'_J, S'_J)\), where \(J\) is defined by (3.2). The codimension one strata are the \(M_i (= M_{\{i\}})\). This gives \(M\) a corner structure, \(\mathcal{M} = \{M_i\}_{i \in I(n)}\). (In fact, \(M\) is a “corner of manifolds” as in Definition 4.1 below.) For us, the most salient feature of this mirror structure is the following

- The the nerve of induced mirror structure \(\widetilde{\mathcal{M}}\) on \(\overline{D}\) can be identified with the spherical building for \(G(\mathbb{Q})\).

So, \(N(\widetilde{\mathcal{M}})\) is a flag complex (cf. Examples 1.6(2)). The strata have the following structure:

- Each \(M_J\) is a manifold with corners.
- The interior of \(M_J\) is a fiber bundle over a locally symmetric space with fiber a nilmanifold.

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In particular, each $M_J$ is aspherical and the map $\pi_1(M_J) \to \pi_1(M) = \Gamma$ is injective.

(See [1, pp. 637–640], [4], [20], [23].) So, we have proved the following.

**Theorem 3.18.** Suppose $M$ is the Borel-Serre compactification of $D/\Gamma$, where $\Gamma$ is a torsion-free arithmetic group of $\mathbb{Q}$-rank $n + 1$ and where $M$ has corner structure $\mathcal{M}$ as above. Then $M$ and $\mathcal{M}$ satisfy conditions (i)', (ii)', (iii)' of Theorems 3.7 and 3.8.

In the case of the Borel-Serre compactification, the fact that the reflection group trick yields an aspherical manifold $U(W, M)$ (i.e., that the conclusion of Theorem 3.5 holds) has been proved previously by Phan [24].

**Remark 3.19.** Suppose that, for $i \in I(n)$, $\Gamma(i)$ is a torsion-free arithmetic group of rank 1, that $D(i)/\Gamma(i)$ is the corresponding locally symmetric space and that $M(i)$ is its Borel-Serre compactification. Then $\Gamma := \Gamma(0) \times \cdots \times \Gamma(n)$ is an arithmetic group of rank $n + 1$ and $M = M(0) \times \cdots \times M(n)$ is the Borel-Serre compactification of its locally symmetric space. Moreover, $(M(i), \partial M(i))$ is an aspherical manifold with $\pi_1$-injective, aspherical boundary. So, in this case, the Borel-Serre compactification is a special case of the products discussed at the beginning of this subsection.

**More examples.** In Examples 1.6 and 3.2 we discussed some more examples of colorable flag complexes, e.g.,

- triangulations of $S^n$ arising from simplicial arrangements in $\mathbb{R}^{n+1}$ (Examples 1.6 (3)) and Examples 3.20 (1) below,

- barycentric subdivisions, i.e., flag complexes of the poset of cells in a cell complex (Examples 3.2 (1)).

**Examples 3.20.** *(Some simple polytopes).* We discuss three types of simple polytopes, which are dual to colorable flag complexes.

1. *(Zonotopes).* Suppose $\Lambda$ is the boundary complex of the simplicial polytope associated to some simplicial hyperplane arrangement in $\mathbb{R}^{n+1}$. Its dual polytope $P$ is a *zonotope* (a simple zonotope since $\Lambda$ is a simplicial complex). Let $\mathcal{F}(P)$ be the set of codimension one faces of $P$. It can be seen that $\Lambda$ is colorable (cf. [16, Lemma 4.2.6]). A coloring $f: \Lambda \to \Delta$ induces
a function also denoted by \( f \) from \( F \) to \( I(n) \). Define a mirror structure \( \{P_i\}_{i \in I(n)} \) on \( P \) by

\[
P_i = \bigcup_{F \in f^{-1}(i)} F.
\]

For example, if we start with the coordinate hyperplane arrangement, then \( \Lambda \) is the boundary complex of a \((n + 1)\) dimensional octahedron and \( P \) is the cube \( \square = \mathcal{Z}_\Delta(D^1, S^0) \) where \( \square_i \) is defined by \( x_i = \pm 1 \).

(2) (Duals of barycentric subdivisions). Suppose \( \Lambda \) is the barycentric subdivision of the boundary complex of a convex polytope in \( \mathbb{R}^{n+1} \) and that \( P \) is the dual polytope to \( \Lambda \). The coloring \( d \) of Examples 3.2(1) induces \( d: F(P) \to I(n) \) and we define \( P_i \) as before.

(3) (Coxeter zonotopes). Suppose \((W, S_{I(n)})\) is a spherical Coxeter system of rank \((n + 1)\). Then \( W \) has a representation as an orthogonal reflection group on \( \mathbb{R}^{n+1} \) and the corresponding triangulation \( \Lambda \) of \( S^n \) is the Coxeter complex, \( U(W, \Delta) \). The dual zonotope \( P \) is called the Coxeter polytope in [12]. We have that \( P \cong U(W, \square) \), which is the universal cover for the corner of groups \( W \) discussed in Examples 3.11(1). So, as in (3.6), we have the aspherical corner \( EW \times_W P \).

In all three cases, conditions (i)', (ii)', (iii)' of Theorems 3.5 and 3.8 are satisfied. The aspherical manifolds and spaces which result from applying these theorems were discussed previously in [11].

Example 3.21. (Artin groups of spherical type). Suppose, as in Examples 3.20 (3), that \((W, S)\) is a spherical Coxeter group of rank \( n + 1 \). Let \( A \) be the associated Artin group. Salvetti defined a cell complex \( X' \) which was homotopy equivalent to the complement of the reflection hyperplane arrangement in \( \mathbb{C}^{n+1} \). (The fundamental group of \( X' \) is the associated pure Artin group.) The quotient \( X := X'/W \) is called the Salvetti complex. Its fundamental group is \( A \). The CW complex \( X \) can be formed by identifying faces of the same type in the Coxeter zonotope \( P \) associated to \((W, S)\) (cf. [10]). (When \( W \) is right-angled and \( L \) is the associated flag complex, \( X \) is the polyhedral product \( \mathcal{Z}_L(S^1, 1) \) of Example 2.4. This defines a mirror structure \( M = \{X_i\}_{i \in I(n)} \), where \( X_i \) is the image of \( P_i \) in \( X \) (cf. Example 3.20 (1)). The associated corner of groups is the corner of groups \( A \) defined in Example 3.11(3). Its link is the Deligne complex \( U(A, \Delta) \). As mentioned previously in Examples 1.6 (4), it was proved in [9] that \( U(A, \Delta) \) is a flag complex. The CW complexes \( X \) and \( EA \times_A U(A, \square) \) are homotopy equivalent.
(as corners of spaces). It follows that $X$ is an aspherical corner, satisfying conditions (i)$'$, (ii)$'$, (iii)$'$ of Theorems 3.5 and 3.8.

4 Closed aspherical manifolds

4.1 Corners of manifolds

An $n$-dimensional smooth manifold with corners is a second countable Hausdorff space $M$ which is differentiably locally modeled on $[0, \infty)^n$. If $\varphi : U \rightarrow [0, \infty)^n$ is any coordinate chart, then the number of coordinates of $\varphi(x)$ which are equal to 0 is independent of the chart and is denoted by $c(x)$. An (open) stratum of codimension $k$ is a component of $\{x \in M \mid c(x) = k\}$. A stratum is the closure of such a component. As in [12, p. 180] one can define a topological manifold with corners in a similar fashion.

For example, a simple polytope is a manifold with corners.

Definition 4.1. Suppose $M$ is a manifold with corners. A corner structure $\{M_i\}_{i \in I(n)}$ on $M$ is a corner of manifolds if each $M_i$ is a union of closed strata of codimension $|J|$.

Examples 4.2. (Corners of manifolds).

(1) (Products of manifolds with boundary). Suppose $\{(M(i), \partial M(i))\}_{i \in I(n)}$ is a collection of manifolds with boundary. Then $M := \prod_{i \in I(n)} M(i)$ is a manifold with corners. As in Example 3.3 a corner structure on $M$ is defined by

$$M_i = M(0) \times \cdots \times \partial M(i) \times \cdots \times M(n) \cong \partial M(i) \times \prod_{j \neq i} M(j).$$

This gives $M$ the structure of a corner of manifolds. A point $x \in M$ lies in an (open) stratum of codimension $k$ if $\{i \mid x_i \in \partial M(i)\}$ has precisely $k$ elements.

(2) (Sequence of fiber bundles with manifolds with boundary as fibers). This is a generalization of the previous example. Suppose we have a sequence of fiber bundles $\pi_i : X(i) \rightarrow X(i - 1)$, with $-1 \leq i \leq n$, where $X(-1)$ is a point and where the fiber of $\pi_i$ is a manifold with boundary, $M(i)$. Then the points of $X(i)$ lying in the boundaries of the fibers form a subbundle with fiber $\partial M(i)$, which we denote $\partial_i X(i) \rightarrow X(i - 1)$. Put $\partial_0 X_0 = \partial M(0)$. Suppose $i > 0$ and suppose by induction that we have defined $\partial_j X(i - 1)$ for all $j \leq i - 1$. Put $\partial_i(X(i)) := \pi_i^{-1}(\partial_j X(i - 1))$. It is clear that $X(i)$ is
a manifold with corners locally modeled on $M(0) \times \cdots \times M(i)$. Moreover, if we set $X = X(n)$ and define a mirror structure $\{X_i\}_{i \in I(n)}$ by $X_i = \partial_i X$, then $X$ is a corner of manifolds.

(3) (Borel-Serre compactifications). See §3.5.

(4) (Certain hyperbolic manifolds with corners). These examples are related to the “strict hyperbolization procedures” of [3]. According to [3] Cor. 6.2, for each $n$, one can construct a hyperbolic $(n+1)$-manifold with corners $X$ with corner structure $\mathcal{M} = \{X_i\}_{i \in I(n)}$ such that

- Each $X_i$ has two components, each of which is a codimension one stratum. (Moreover, if we regard each $X_i$ as consisting of two mirrors, then the nerve of the resulting mirror structure is the boundary complex of an $(n+1)$-octahedron.)

- Each component of $X_i$ is totally geodesic and these components intersect orthogonally.

Let $\tilde{X}$ be the universal cover. We claim that the nerve $N = N(\tilde{M})$ of the induced mirror structure on $\tilde{X}$ is a flag complex. The argument is somewhat indirect. Apply the right-angled reflection group trick to $X$ to get a closed hyperbolic manifold $U(\mathbb{C}_2(I(n)), X)$. Its universal cover $U(W_N, X)$ can be identified with hyperbolic $(n+1)$-space, which is contractible. (Here $W_N$ is the RACG associated to the 1-skeleton of $N$.) As we pointed out in the proof of Theorem 3.5, a necessary condition for $U(W_N, X)$ to be acyclic is that $N$ be a flag complex (cf. [12, Cor. 8.2.8]). Hence, $N$ is a flag complex. So, $X$ and $\mathcal{M}$ satisfy conditions (i)', (ii)', (iii)' of Theorems 3.5 and 3.8. If $f : L \to \Delta$ is a coloring of a flag complex $L$ and we apply Theorem 3.8, then $f^*(X)$ is a space constructed previously in [3]: the strict hyperbolization of the cubical complex $f^*(\square)$.

### 4.2 Closed manifolds

**Proposition 4.3.** Suppose $L$ is a simplicial complex with vertex set $I$ and $(M, \partial M) = \{(M(i), \partial M(i))\}_{i \in I}$ is a collection of manifolds with boundary. If $L$ is a triangulation of a sphere, then $Z_L(M, \partial M)$ is a closed manifold.

**Proposition 4.4.** Suppose $\{(M_i, \partial M_i)\}_{i \in I(n)}$ is a corner of manifolds structure on $M$.  

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(a) Suppose $W$ is a finite Coxeter group of rank $n + 1$. Then $U(W, M)$ is a closed manifold.

(b) Suppose $L$ is a triangulation of a sphere and $f : L \to \Delta$ is a coloring (so that the pullback can be defined). Then $f^*(M)$ is a closed manifold.

As we pointed out in Remark 3.10, part (a) of Proposition 4.4 is a special case of part (b). Proposition 4.3 is also essentially a special case of part (b).

So, we only prove part (b).

Proof of Proposition 4.4 (b). It follows from (3.3) in the definition of a pullback that $f^*(M)$ is a union of pieces of the form $J \times M_f(J)$, where $J \in S(L)$ and where $M_f(J)$ denotes the relative interior of $M_f(J)$. A neighborhood of $\emptyset \times M_f(n)$ in $f^*(L)$ has the form $\text{Cone}(L) \times M_f(n)$ which is a manifold since $\text{Cone}(L)$ is a disk. Similarly, $J \times M_f(J)$ has a neighborhood of the form $\text{Cone}(\text{Lk}(J, L)) \times M_f(J)$ and this is a manifold since $\text{Cone}(\text{Lk}(J, L))$ is a homology manifold and becomes locally Euclidean once we take its product with a Euclidean space of dimension at least $|J|$. (Note: $\dim M_f(J) = |J| + \dim M_f(n) \geq |J|$.)

We can combine Theorem 2.22 and Proposition 4.3 as follows.

Proposition 4.5. Suppose a flag complex $L$ is a triangulation of a sphere, that $(M, \partial M) = \{(M(i), \partial M(i))\}_{i \in I}$ is a family of manifolds with boundary indexed by the vertex set $I$ of $L$ and, as in Definition 2.27, that $(M(i), \partial M(i))$ is an aspherical pair. Then the polyhedral product $Z_L(M, \partial M)$ is a closed aspherical manifold.

For the remainder of this subsection suppose $M$ is a corner of manifolds with set of codimension one faces, $M = \{M_i\}_{i \in I(n)}$, satisfying the following conditions (cf. 3.5).

(i)' $M$ is aspherical.

(ii)' The induced mirror structure $\tilde{\mathcal{M}}$ on the universal cover is acyclic.

(iii)' $\tilde{\mathcal{M}}$ is flag.

We can combine Theorem 3.5 and Proposition 4.4 (a) as follows.
Proposition 4.6. With hypotheses as above, suppose that \((W, S)\) is a spherical Coxeter system of rank \(n + 1\). Then \(U(W, M)\) is a closed aspherical manifold.

Finally, we can combine Theorem 3.8 and Proposition 4.4(b) to get the following proposition.

Proposition 4.7. With hypotheses as above, suppose \(L\) is a flag triangulation of a sphere and that \(f : L \to \Delta\) is a coloring. Then \(f^*(M)\) is a closed aspherical manifold.

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