Several applications of the moment method in random matrix theory

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Abstract. Several applications of the moment method in random matrix theory, especially, to local eigenvalue statistics at the spectral edges, are surveyed, with emphasis on a modification of the method involving orthogonal polynomials.

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1. Introduction

The goal of this article is to survey a few of the applications of the moment method (and its variants) to the study of the spectral properties of random matrices, particularly, local eigenvalue statistics at the spectral edges.

Section 2 is a brief introduction to the moment method, which we understand as the variety of ways to extract the properties of a measure $\mu$ from integrals of the form

$$\int \xi^m \, d\mu(\xi).$$

Examples, selected from the narrow part of random matrix theory in which the author feels competent, are intended to illustrate two theses. First, the moment method can be applied beyond the framework of weak convergence of a sequence of probability measures. Second, it is often convenient to replace the monomials $\xi^m$ in (1.1) with a better-conditioned sequence, such as the sequence of orthogonal polynomials with respect to a measure $\mu_\infty$ which is an approximation to $\mu$.

In Section 3 we review some applications to the local eigenvalue statistics at the spectral edges, starting from the work of Soshnikov [64]. Tracy and Widom [70] and Forrester [25] introduced the Airy point processes (see Section 3.1) and showed that they describe the limiting distribution of the largest eigenvalues for special families of large Hermitian random matrices with independent entries (the Gaussian invariant ensembles). Soshnikov [64] extended these results to Wigner

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matrices (Hermitian random matrices with independent entries and no invariance assumptions). In the terminology of Ibragimov and Linnik [37, Chapter VI], the result of [64] is a limit theorem of collective character; it is one of the instances of the ubiquity (universality) of the Airy point processes within and outside random matrix theory (as surveyed, for example, by Johansson [38], Tracy and Widom [72], and Borodin and Gorin [9]).

In Section 3.2, we consider Wigner processes, a class of matrix-valued random processes. Informally, a random matrix $H(x)$ is attached to every point $x$ of an underlying space $X$. The statistical properties of the eigenvalues of every $H(x)$ are described by the theory of Wigner matrices; the joint distribution of the eigenvalues of a tuple $(H(x_r))_{r=1}^{k}$ leads to limiting objects which depend on the geometry of $X$ (which arises from the correlations of the matrix elements of $H$) in a non-trivial way. The moment method allows to derive limit theorems of collective character (such as Theorem 3.3) pertaining to the spectral edges of $H(x)$ (the result of [64] corresponds to a singleton, $\#X = 1$).

In Section 3.3, we turn to the spectral edges of random band matrices. A random band matrix is (3.14) a random $N \times N$ Hermitian matrix with non-zero entries in a band of width $W$ about the main diagonal. When $W$ is small, a band matrix inherits the structure of the integer lattice $\mathbb{Z}$; when $W$ is large, it is similar to a Wigner matrix. The threshold at which the local eigenvalue statistics in the bulk of the spectrum exhibit a crossover is described by precise conjectures (see Fyodorov and Mirlin [33, 34], Spencer [67, 68]). The moment method allows to prove the counterpart of these conjectures for the spectral edges (the result of [64] corresponds to the special case $W = N$).

The content of Section 2 is mostly known. The modified moment method of Section 2.4 is a version of self-energy renormalisation in perturbation theory (see Spencer [67]), related to the arguments of Bai and Yin [4]. Orthogonal polynomials were explicitly used in this context in the work of Li and Solé [44], and further in [58] (where more references may be found). Some observations are incorporated from [21]. The content of Section 3.2 is an extension of [62], whereas Section 3.3 is based on [60]. The proofs of the results stated in both of these sections build on the combinatorial arguments of [21].

2. Preliminaries and generalities

The moment method is the collection of techniques inferring the properties of a measure $\mu$ on the $k$-dimensional space $\mathbb{R}^k$ from the moments

$$s(m_1, \cdots, m_k; \mu) = \int_{\mathbb{R}^k} \xi_1^{m_1} \cdots \xi_k^{m_k} d\mu(\xi) \quad (m_1, \cdots, m_k = 0, 1, 2, \cdots). \quad (2.1)$$

Introduced by Chebyshev as a means to establish Gaussian approximation for the distribution of a sum of independent random variables, the moment method achieved its first major success with the proof, given by Markov [46], of Lyapunov’s
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Central Limit Theorem and its extension to sums of weakly dependent random variables. Some of the more recent applications are surveyed by Diaconis [13].

2.1. Convergence of probability measures. In the traditional setting of the moment method, one considers a sequence of probability measures \((\mu_N)_{N \geq 1}\) on \(\mathbb{R}^k\). Suppose that the limit
\[
\lim_{N \to \infty} s(m_1, \cdots, m_k; \mu_N) = s(m_1, \cdots, m_k)
\]
exists for every \(m_1, \cdots, m_k \geq 0\). Then the sequence \((\mu_N)_{N \geq 1}\) is tight, i.e. precompact in weak topology (defined by bounded continuous functions), and every one of its limit points \(\mu\) satisfies
\[
s(m_1, \cdots, m_k; \mu) = s(m_1, \cdots, m_k) \quad (m_1, \cdots, m_k \geq 0). \tag{2.3}
\]
If, for example,
\[
s(m_1, \cdots, m_k) \leq k \prod_{r=1}^{k} (C m_r)^{m_r} \quad (m_1, \cdots, m_k \geq 0), \tag{2.4}
\]
the moment problem (2.3) is determinate, i.e. there is a unique measure \(\mu_\infty\) on \(\mathbb{R}^k\) satisfying (2.3); in this case the convergence of moments (2.2) implies that \(\mu_N \to \mu_\infty\) in weak topology; cf. Feller [23, §VIII.6].

Hardy’s sufficient condition (2.4) may be somewhat relaxed; we refer to the addenda to the second chapter of the book [1] of Akhiezer for various sufficient criteria for determinacy in the case \(k = 1\), and to the survey of Berg [5] for some extensions to \(k \geq 1\).

2.1.1. Random measures. Suppose \((\mu_N)_{N \geq 1}\) is a sequence of random measures on \(\mathbb{R}^\ell\) (i.e. random variables taking values in the space of Borel probability measures). We denote \(\xi = (\xi_1, \cdots, \xi_\ell)\), \(m = (m_1, \cdots, m_\ell)\), and \(\xi^m = \xi_1^{m_1} \cdots \xi_\ell^{m_\ell}\); thus \(\xi_{r, r}^m = \xi_{r, r, 1}^{m_r} \cdots \xi_{r, r, \ell}^{m_r}\). If
\[
\mathbb{E} \left[ \int_{\mathbb{R}^\ell} \prod_{r=1}^{k} [d\mu_N(\xi_r^m) \xi_r^{m_r}] \right] \to \mathbb{E} \left[ \int_{\mathbb{R}^\ell} \prod_{r=1}^{k} [d\mu_\infty(\xi_r) \xi_r^{m_r}] \right] \quad (k \geq 1) \tag{2.5}
\]
for some random measure \(\mu_\infty\) on \(\mathbb{R}^\ell\), and the moment problem for every moment measure \(\mathbb{E}\mu_\infty^\otimes k\) is determinate, then
\[
\mathbb{E}\mu_N^\otimes k \quad \mathbb{E}\mu_\infty^\otimes k \quad (k \geq 1). \tag{2.6}
\]
If, for every Borel set \(K \in \mathcal{B}(\mathbb{R}^\ell)\), the moment problem for the distribution of \(\mu_\infty(K)\) is determinate, then (2.6) implies that \(\mu_N \to \mu_\infty\) (weakly in distribution). Finally, if \(\mu_\infty\) is deterministic (i.e. its distribution is supported on one deterministic measure), it is sufficient to verify (2.6) for \(k = 1, 2\). See further Zessin [78].
2.2. Example: Wigner’s law. The application (going back to Chebyshev) of the moment method to sums of independent random variables is based on the identity

\[
\left[ \sum_{j=1}^{N} X_j \right]^m = \sum_{m_1 + \cdots + m_N = m} \frac{m!}{m_1! \cdots m_N!} \prod_{j=1}^{N} X_j^{m_j} \tag{2.7}
\]

expressing powers of a sum of numbers as a sum over partitions. Similarly, the application (going back to Wigner) of the moment method to random matrix theory is based on the relation

\[
\text{tr} H^m = \sum_{p=(u_0, u_1, \ldots, u_{m-1}, u_m=u_0)} \prod_{j=0}^{m-1} H(u_j, u_{j+1})
= \sum_{p} \prod_{1 \leq u \leq v \leq N} \frac{H(u, v)^\# \{(u_j, u_{j+1})=(u, v)\}}{H(u, v)^\# \{(u_j, u_{j+1})=(v, u)\}} \tag{2.8}
\]

expressing traces of powers of an Hermitian matrix \( H = (H(u, v))_{u,v=1}^{N} \) as a sum over paths.

Let \( G_N = (V_N, E_N)_{N \geq 1} \) be a sequence of graphs, so that \( G_N \) is \( \kappa_N \)-regular (meaning that every vertex is adjacent to exactly \( \kappa_N \) edges), and the connectivity \( \kappa_N \) tends to infinity:

\[
\lim_{N \to \infty} \#V_N = \infty, \quad \lim_{N \to \infty} \kappa_N = \infty. \tag{2.8}
\]

For every \( N \), consider a random matrix \( H^{(N)} = (H(u, v))_{u,v \in V_N} \) with rows and columns indexed by the elements of \( V_N \), so that \( (H(u, v))_{u,v \in V_N} \) are independent up to the constraint \( H(v, u) = H(u, v) \); the diagonal entries \( \{H(u, u)\}_u \) are sampled from a distribution \( \mathcal{L}_{\text{diag}} \) on \( \mathbb{R} \) satisfying

\[
\mathbb{E}H(u, u) = 0, \quad \mathbb{E}|H(u, u)|^2 < \infty; \tag{2.9}
\]

the off-diagonal entries \( \{H(u, v)\}_{(u,v) \in E_N} \) are sampled from a distribution \( \mathcal{L}_{\text{off-diag}} \) on \( \mathbb{C} \) satisfying

\[
\mathbb{E}H(u, v) = 0, \quad \mathbb{E}|H(u, v)|^2 = 1; \tag{2.10}
\]

and all the other entries \( H(u, v) \) are set to zero.

Let \( \xi_1^{(N)} \geq \xi_2^{(N)} \geq \cdots \xi_{\#V_N}^{(N)} \) be the eigenvalues of \( H^{(N)} \), and let

\[
\mu_N = \frac{1}{\#V_N} \sum_{j=1}^{\#V_N} \delta \left( \xi - \frac{\xi_j^{(N)}}{2 \sqrt{\kappa_N} - 1} \right).
\]

(The scaling is natural since, for instance, the \( \ell_2 \) norm of every column of the \( N \times N \) matrix is of order \( \sqrt{\kappa_N} \).)
Theorem (Wigner’s law). In the setting of this paragraph (i.e. assuming (2.8), (2.9), (2.10)), the sequence of random measures \( (\mu_N)_N \) converges (weakly, in distribution) as \( N \to \infty \) to the (deterministic) semicircle measure \( \sigma_{\text{Wig}} \) with density
\[
\frac{d\sigma_{\text{Wig}}}{d\xi} = \frac{2}{\pi} \sqrt{1 - \xi^2} + .
\] (2.11)

Wigner considered [75–77] the case when \( G_N \) is the complete graph on \( N \) vertices (Wigner matrices), and the entries satisfy some additional assumptions, the important of them being that all moments are finite. Wigner’s argument is based on the moment method.

Bogachev, Molchanov, and Pastur [6] observed (in the context of random band matrices) that a similar argument can be applied as long as (2.8) is satisfied. The first argument for Wigner matrices without additional restrictions on the distribution of the entries was given by Pastur [51], using the Stieltjes transform method introduced by Marchenko and Pastur [47, 48] (see Pastur [52] and the book of Pastur and Shcherbina [53] for some of the further applications of the method). Khorunzhiy, Molchanov, and Pastur [39] applied the Stieltjes transform method to prove Wigner’s law for random band matrices; their argument is applicable in the setting described here.

Let us outline a proof of Wigner’s law in the form stated above, following [6] (and incorporating Markov’s truncation argument [46]). We refer for details to the book of Anderson, Guionnet, and Zeitouni [3, Chapter 2.1], where similar arguments are also applied to questions such as the Central Limit Theorem for linear statistics
\[
\phi(\xi_1^{(N)}) + \cdots + \phi(\xi_{\#V_N}^{(N)}).
\]

Proof of Wigner’s law. Due to (2.8), (2.9) and (2.10) one can find a sequence \( \delta_N \to +0 \) so that
\[
\mathbb{E}|H(u, v)|^2 \mathbb{1}_{|H(u, v)| \geq \delta_N \sqrt{\kappa_N} \leq \delta_N .
\]
Consider the matrix \( H^{(N)}_\wedge \) with truncated matrix elements
\[
H^{(N)}_\wedge(u, v) = \begin{cases} H(u, v), & |H(u, v)| \leq \sqrt{\kappa_N} \\ 0, & |H(u, v)| > \sqrt{\kappa_N} \end{cases}.
\]
Then
\[
\mathbb{P}\{H^{(N)}_\wedge(u, v) \neq H(u, v)\} \leq \delta_N \kappa_N^{-1} ,
\]
whence, bounding rank by the number of non-zero matrix elements and applying the Chebyshev inequality,
\[
\mathbb{P}\{\text{rank}(H^{(N)}_\wedge - H^{(N)}) \geq \delta^{1/2}_N \#V_N\} \leq \delta^{1/2}_N .
\]
For any \( \xi \in \mathbb{R} \), the interlacing property of rank-one perturbation yields
\[
|\mu_N(-\infty, \xi) - \mu_\wedge, N(-\infty, \xi)| \leq \#V_N^{-1} \text{rank}(H^{(N)}_\wedge - H^{(N)}) ,
\] (2.12)
therefore it is sufficient to establish the result for $H^{(N)}_\omega$ in place of $H^{(N)}$. For large $N$, the elements of $H^{(N)}_\omega$ enjoy the following estimates:

$$|EH(u,v)| \leq \delta N^{\kappa - \frac{1}{2}};$$  \hspace{1cm} (2.13)  

$$|\mathbb{E}H(u,v)|^2 - 1 \leq \delta N \quad ((u,v) \in E_N): \quad \mathbb{E}|H_\omega(u,v)|^2 \leq \text{const};$$  \hspace{1cm} (2.14)  

$$\mathbb{E}|H_\omega(u,v)|^k \leq 2\delta N^{\kappa - \frac{k-2}{2}} (k \geq 3).$$  \hspace{1cm} (2.15)  

Next, consider the expansion

$$s(m_1, \cdots, m_k; \mathbb{E}\mu^{\otimes k}_\omega,N) = \mathbb{E} \prod_{r=1}^k \int\xi^{m_r} d\mu_{\omega,N}(\xi)$$

$$= \mathbb{E} \frac{1}{(#V_N)^k} \prod_{r=1}^k \text{tr} \left( \frac{H^{(N)}_\omega}{2\sqrt{\kappa_N} - 1} \right)^{m_r}$$

$$= \sum \frac{1}{(#V_N)^k} \mathbb{E} \prod_{r=1}^k \prod_{j=0}^{m_r-1} \frac{H_\omega(u_{r,j}, u_{r,j+1})}{2\sqrt{\kappa_N} - 1},$$

where the sum is over $k$-tuples of closed paths

$$\left\{ u_{1,0}, u_{1,1}, \cdots, u_{1,m_1-1}, u_{1,m_1} \right\}$$

$$\left\{ u_{2,0}, u_{2,1}, \cdots, u_{2,m_2-1}, u_{2,m_2} \right\}$$

$$\cdots$$

$$\left\{ u_{k,0}, u_{k,1}, \cdots, u_{k,m_k-1}, u_{k,m_k} \right\}$$

in the augmented (multi-)graph $G^+_N = (V_N, E^+_N)$, $E^+_N = E_N \cup \{ (u,u) \mid u \in V_N \}$. Two such $k$-tuples are called isomorphic if one is obtained from one another by a permutation of the vertices $V_N$. For example, the pair $(1 3 1, 2 1 4 2)$ is isomorphic to $(7 4 7, 1 7 2 1)$. According to (2.13), (2.14) and (2.15), the contribution of an isomorphism class consisting of $k$-tuples spanning a graph $g$ with $v$ vertices and $e$ edges, of which $e_2$ are traversed exactly twice, is bounded by $\kappa_N^{-k}\epsilon(e_2/\kappa_N/\text{const})^{-\epsilon}$.

The graph $g$ has at most $k$ connected components, whence

$$v - e \leq k,$$  \hspace{1cm} (2.17)  

with equality for graphs which are vertex-disjoint unions of $k$ trees. For fixed $m_1, \cdots, m_k$, the number of isomorphism classes remains bounded as $N \to \infty$, therefore the limit of (2.16) is given by the contribution of vertex-disjoint $k$-tuples of paths corresponding to graphs with

$$v - e = k,$$  \hspace{1cm} (2.18)  

Every path in such a $k$-tuple is tree-like (see Figure 2.1 left); each isomorphism class contributes $2^{-\sum \epsilon_r m_r}$ (due to (2.14)), and the number of classes is given by a
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![Figure 2.1](image)

Figure 2.1. The tree-like path 1 2 3 4 5 6 7 8 9 10 8 4 2 1 with v = 10 and e = e_2 = 9 (left) and the non-backtracking path 1 2 3 4 5 6 7 8 4 3 2 1 with v = 8 and e = e_2 = 8 (right). Among the two, only the first one contributes to the semi-circle limit.

The product of Catalan numbers:

\[ \prod_{r=1}^{k} \left\{ \begin{array}{ll} \frac{m_r}{m_r+2}, & m_r \text{ is even} \\ 0, & m_r \text{ is odd} \end{array} \right\} = \prod_{r=1}^{k} \left[ 2^{m_r} s(m_r; \sigma_{Wig}) \right]. \]

Thus

\[ \lim_{N \to \infty} s(m_1, \ldots, m_k; \mu_{\cap N}) = s(m_1, \ldots, m_k; \sigma_{\cap N}). \] (2.19)

Applying the relation (2.19) with \( k = 1, 2 \), we conclude (cf. Section 2.1.1) that \( \mu_{\cap N} \) converge to \( \sigma_{Wig} \) weakly in distribution, and thus (by (2.12)) so do \( \mu_N \).

2.3. Some quantitative aspects. Whenever the moment convergence (2.2) is a consequence of the stronger property

\[ s(m_1, \ldots, m_k; \mu_N) = s(m_1, \ldots, m_k; \mu_\infty) \quad (N \geq N_0(m_1, \ldots, m_k)), \] (2.20)

the arguments quoted in Section 2.1 can be recast in quantitative form. This is illustrated by the following inequality due to Sonin [63]. Let \( \gamma \) be the Gaussian measure,

\[ \frac{d\gamma}{d\xi} = \frac{1}{\sqrt{2\pi}} \exp(-\xi^2/2) \quad (\xi \in \mathbb{R}), \]

and assume that

\[ s(m; \mu_N) = s(m; \gamma) \left[ \begin{array}{l} 0, \\ \left( \frac{m}{(2)^{\frac{m}{2}}} \right), & \text{m is odd} \\ \left( \frac{m}{(2)^{\frac{m}{2}}} \right), & \text{m is even} \end{array} \right] \quad (N \geq N_0(m)). \] (2.21)

Then

\[ \sup_{\xi \in \mathbb{R}} |\mu_N(\xi) - \gamma(\xi)| \leq \frac{\sqrt{\pi}}{m-1} \quad (N \geq \max_{m' \leq m} N_0(m')). \] (2.22)

Measures \( \mu_N \) of random matrix origin for which (2.21) holds may be found in the survey of Diaconis [13]. Inequalities of the form (2.22) may be also derived for other measures \( \mu_\infty \) (see Akhiezer [11] Section II.5.4) for the general framework of Chebyshev–Markov–Stieltjes inequalities, and Krawtchouk [40] for additional examples).
Similar inequalities can be derived for $k > 1$. On the other hand, already in the setting of the Central Limit Theorem for sums of independent random variables, (2.21) is not valid (unless the addends are Gaussian themselves); the correct relation $s(m; \mu_N) \approx s(m; \gamma)$, even with the optimal dependence of the error term on $m$ and $N$, yields a poor bound on the rate of convergence of $\mu_N$ to $\gamma$ (the sharp Berry–Esseen bound, see Feller [23, §XVI.5], was proved by the Fourier-analytic approach). The reason is that monomials form an ill-conditioned basis; see Gautschi [35] for a discussion of computational aspects (and of remedies similar to the one discussed in the next section).

2.4. A modification of the moment method. The following modification makes the moment method better conditioned. Let $(\mu_N)_{N \geq 1}$ be a sequence of probability measures on $\mathbb{R}$, and suppose $\mu_\infty$ is a candidate for the weak limit of the sequence $(\mu_N)_{N \geq 1}$. Let $P_n(\xi)$ ($n = 0, 1, 2, \cdots$) be the orthogonal polynomials with respect to $\mu_\infty$:

$$\deg P_n = n, \quad \int P_n(\xi)P_{n'}(\xi)d\mu_\infty(\xi) = \delta_{nn'} .$$

Also set

$$\tilde{s}(n; \mu; \mu_\infty) = \int_{-\infty}^{\infty} P_n(\xi)d\mu(\xi) .$$

Then the convergence of moments

$$\lim_{N \to \infty} s(m; \mu_N) = s(m; \mu_\infty) \quad (m \geq 0)$$

is equivalent to

$$\lim_{N \to \infty} \tilde{s}(n; \mu_N; \mu_\infty) = \delta_{n0} \quad (n \geq 0) .$$

Thus (2.25) implies that $\mu_N \to \mu_\infty$, provided that the moment problem for $\mu_\infty$ is determinate.

While the modification of the moment method advertised here seems to have no general counterpart in dimension $k > 1$, in the special case when $\mu_\infty$ is the $k$-th power of a one-dimensional measure with orthogonal polynomials $P_n$ we define:

$$\tilde{s}(n_1, \cdots, n_k; \mu; \mu_\infty) = \int_{\mathbb{R}^k} \prod_{r=1}^{k} P_{n_r}(\xi_r) d\mu(\xi) .$$

2.4.1. A random matrix example. If $X_1, \cdots, X_n$ are independent random variables with zero mean, unit variance, and finite moments, one may give a combinatorial interpretation to

$$\mathbb{E} \frac{1}{\sqrt{n!}} \text{He}_n \left[ \frac{X_1 + \cdots + X_N}{\sqrt{N}} \right] ,$$

where

$$\text{He}_n(\xi) = (-1)^ne^{\xi^2/2} \frac{d^n}{d\xi^n}e^{-\xi^2/2}$$
are the Hermite polynomials; the three-term recurrent relation
\[ \text{He}_{n+1}(\xi) = \xi \text{He}_n(\xi) - n \text{He}_{n-1}(\xi) \]
eliminates the asymptotically leading terms of the moments (2.7) of \( X_1 + \cdots + X_N \). Here we focus on a different example, pertaining to random matrices of the form considered in Section 2.2.

Denote
\[ P_n^{(\kappa)}(\xi) = U_n(\xi) - \frac{1}{\kappa - 1} U_{n-2}(\xi), \]
where
\[ U_n(\cos \theta) = \frac{\sin((n+1)\theta) - \sin \theta}{n} \]
are the Chebyshev polynomials of the second kind (orthogonal with respect to \( \sigma_{\text{Wig}} \)), and \( U_{-1} \equiv U_{-2} \equiv 0 \). Let \( G = (V, E) \) be a regular graph of connectivity \( \kappa \), and let \( H \) be an \( \#V \times \#V \) Hermitian matrix, such that
\[ |H(u, v)| = \mathbb{1}_{(u, v) \in E}, \quad (u, v \in V). \quad (2.27) \]
The three-term recurrent relation
\[ P_{n+1}^{(\kappa)}(\xi) = 2\xi P_n^{(\kappa)}(\xi) - (1 + (\kappa - 1)^{-1} \mathbb{1}_{n=1}) P_{n-1}^{(\kappa)}(\xi) \]
for \( P_n^{(\kappa)} \) leads to

**Proposition 2.1** (cf. [58, Lemma 2.7], [21, Claim II.1.2]). For any Hermitian matrix \( H \) satisfying (2.27),
\[ P_n^{(\kappa)} \left[ \frac{H}{2\sqrt{\kappa - 1}} \right] (u, v) = \sum_{j=1}^n \prod_{j=1}^n \frac{H(u_j, u_{j+1})}{\sqrt{\kappa - 1}}, \quad (2.28) \]
where the sum is over paths \( u_0, u_1, \cdots, u_{n-1}, u_n \) in \( G \) from \( u_0 = u \) to \( u_n = v \) which satisfy the non-backtracking condition \( u_j \neq u_{j+2} \) \( (0 \leq j \leq n - 2) \).

Consider a sequence of random matrices \( H^{(N)} \) associated to a sequence of graphs \( G_N \) with \( \kappa_N \to \infty \) as in Section 2.2; let us assume that the entries of \( H \) satisfy the unimodality assumptions (2.27). A non-backtracking path can not be tree-like (see Figure 2.1), therefore the modified moments tend to zero; this provides an alternative proof to Wigner’s law in the form of Section 2.2 under the additional assumptions (2.27).

The generalisation of Propostion 2.1 to matrices which do not satisfy (2.27) is somewhat technical, and we do not present it here. In the context of Wigner (and sample covariance) matrices, it is described in [21, Part III]; for the (more involved) case of band matrices we refer to the work of Erdős and Knowles [17].
2.4.2. Advantages of modified moments. Although the convergence of modified moments (2.25) is equivalent to the convergence of moments (2.24), quantitative forms of the former yield better estimates on the rate of convergence $\mu_N \to \mu_\infty$. As an illustration, we recall a variant of the Erdős–Turán inequality [20] proved in [22]. Consider again the semi-circle measure $\sigma_{\text{Wig}}$ with density (2.11).

Proposition 2.2 ([22, Proposition 5]). Let $\mu$ be a probability measure on $\mathbb{R}$. Then, for any $\xi \in \mathbb{R}$ and any $n_0 \geq 1$,

$$
|\mu(-\infty, \xi] - \sigma_{\text{Wig}}(-\infty, \xi]| \leq C \left\{ \frac{\rho(\xi; n_0)}{n_0} + \sqrt{\frac{\rho(\xi; n_0)}{n}} \sum_{n=1}^{n_0} \left| \tilde{s}(n; \mu; \sigma_{\text{Wig}}) \right| \right\},
$$

where $C > 0$ is a numerical constant, and $\rho(\xi; n_0) = \max(1 - |\xi|, n_0^{-2})$.

The original Erdős–Turán inequality provides a bound of similar structure for the measure with density $d\mu_\infty/d\xi = \pi^{-1} ((1 - \xi^2)_+)^{-1/2}$ (in this case, $\rho(\xi; n_0)$ should be replaced with 1).

2.5. Convergence of rescaled probability measures. The rescaling $R_{\xi_0}^\eta [\mu]$ of a measure $\mu$ on $\mathbb{R}^k$ about $\xi_0 \in \mathbb{R}^k$ by $\eta > 0$ is defined by

$$
R_{\xi_0}^\eta [\mu](K) = \mu(\eta(K - \xi_0)) \quad (K \in \mathcal{B}(\mathbb{R}^k)).
$$

In a class of questions outside the narrow framework of Section 2.1, one is interested in vague limits (weak limits with respect to the topology defined by compactly supported continuous functions) of

$$
(\epsilon_N^{-1} R_{\xi_0}^\eta [\mu_N])_{N \geq 1},
$$

where $\mu_N$ is a sequence of probability measures on $\mathbb{R}^k$, $\xi_0 \in \mathbb{R}^k$, and two sequences $\epsilon_N, \eta_N \to +0$ determine the scaling of $\mu_N$ on the value ($\uparrow$) and variable ($\leftrightarrow$) axes, respectively.

2.5.1. Edges (corners) of the support. Moments allow to study the rescaling of $\mu_N$ about a point $\xi_0$ which is close to the corners of the cube supporting $\mu_N$. Variants of this observation were used, for example, by Sinai and Soshnikov [50, 57].

Assume that we are given a sequence $(\mu_N)_{N \geq 1}$ of probability measures on $\mathbb{R}^k$, two sequences $\epsilon_N, \eta_N \to +0$ which determine the scaling (2.30), and $2^k$ continuous functions $\phi_\varepsilon : (\alpha_0, \infty)^{2k} \to \mathbb{R}_+$ ($\varepsilon \in \{-1, 1\}^k$) which will describe the limiting Laplace transform at the $2^k$ corners of the cube.

Proposition 2.3. Suppose

$$
\epsilon_N^{-k} s(m_1, N, \ldots, m_k, N; \mu_N) - \sum_{\varepsilon \in \{-1, 1\}^k} \prod_{r=1}^k \epsilon_r^{m_r} \phi_\varepsilon(\alpha_1, \ldots, \alpha_k) \to 0 \quad (N \to \infty)
$$

A similar inequality for the Gaussian measure, combined with a careful estimate of the modified moments [22, 20], could perhaps yield a proof of the Berry–Esseen theorem along the lines suggested by Chebyshev.
for any sequence \((m_1,N, \cdots, m_k,N)_{N \geq 1}\) for which
\[
\lim_{N \to \infty} \eta_N m_r,N = \alpha_r > \alpha_0 \quad (1 \leq r \leq k).
\]

Then, for any \(\varepsilon \in \{-1,1\}^k\), the sequence \((\varepsilon_N^{-1} R_N \mu_N)_{N \geq 1}\) converges vaguely to a measure \(\nu^\varepsilon\) which is uniquely determined by the equations
\[
\int \exp(\alpha_1 \lambda_1 + \cdots + \alpha_k \lambda_k) d\nu^\varepsilon (\varepsilon_1 \lambda_1, \cdots, \varepsilon_k \lambda_k) = \phi_\varepsilon (\alpha_1, \cdots, \alpha_k) \quad (\alpha \in (\alpha_0, \infty)^k).
\]

**Remark 2.4.** Convergence actually holds in the stronger topology defined by continuous functions supported (for some \(R > 0\)) in
\[
\prod_{r=1}^k \left\{ \begin{array}{ll}
(-R, \infty), & \varepsilon_r = 1 \\
(-\infty, R), & \varepsilon_r = -1
\end{array} \right.
\]

The counterparts of Proposition 2.3 for modified moments depend on the structure of the limiting measure \(\mu_\infty\). For the case \(\mu_\infty = \sigma_{\text{Wig}}^k\) such a statement was proved in [60, Section 6]. It is somewhat technical, and we do not reproduce it here; instead of the Laplace transform, the limiting measures \(\nu^\varepsilon\) are characterised in terms of the transform
\[
\int_{\mathbb{R}^k} \prod_{r=1}^k \frac{\sin \alpha_r \sqrt{-\lambda_r}}{\sqrt{-\lambda_r}} d\nu^\varepsilon (\varepsilon_1 \lambda_1, \cdots, \varepsilon_k \lambda_k) \tag{2.31}
\]
(which becomes convergent after a certain regularisation). The system of functions \(\lambda \mapsto \frac{\sin \alpha \sqrt{-\lambda}}{\sqrt{-\lambda}}\) forms a continuous analogue of orthogonal polynomials (as introduced by Krein, see Denisov [12]) with respect to the measure \(\frac{2\sqrt{2}}{\pi} \frac{\sqrt{-\lambda}}{\sqrt{-\lambda}}\) (obtained by rescaling \(\sigma_{\text{Wig}}\) about \(\xi_0 = 1\)).

Uniqueness theorems for the transform (2.31) were proved (in dimension \(k = 1\)) in the 1950-s by Levitan [12], Levitan and Meiman [13], and Vul [73] (listed in order of increasing generality); the argument in [60] builds on [12].

One advantage of the approach based on modified moments is that, for a measure supported on several intervals, it allows to consider the rescaling about edges (corners) which are not maximally distant from the origin, and even internal edges. In the context of random matrices, this was exploited in [21].

**2.5.2. Interior points of the support.** If \(\xi_0\) is an interior point of the support of \(\mu_\infty\), it seems impossible to extract any information regarding the measures \(\varepsilon_N^{-1} R_N \mu_N\) from the asymptotics of the moments of \(\mu_N\). The modified moments \(\tilde{s}\) carry such information. For example, Proposition 2.2 shows that if one can find a sequence \((n_0(N))_{N \geq 1}\) so that
\[
\lim_{N \to \infty} \epsilon_N n_0(N) = +\infty, \quad \lim_{N \to \infty} n_0(N) \sum_{n=1}^{n_0(N)} \frac{\tilde{s}(n; \mu_N; \sigma_{\text{Wig}})}{n} = 0, \tag{2.32}
\]
then
\[ \epsilon_N^{-1} R_{\epsilon N}^N [\mu_N] \xrightarrow{\text{vague}} \frac{1}{N} \sqrt{1 - \xi_0^2} \text{mes} (-1 < \xi_0 < 1) \] (2.33)

(where mes is the Lebesgue measure on the real line).

Let us briefly comment on the shorter scales \( \epsilon_N \), for which (2.32) fails. The challenge is to give meaning to the expansion
\[ \mu_N [\xi', \xi''] \sim \sum_{n \geq 0} \tilde{s}(n; \mu_N; \mu_\infty) \int_{\xi'}^{\xi''} P_n(\xi) d\mu_\infty(\xi) \] (2.34)
when \( |\xi' - \xi''| \) is small. For \( \mu_\infty = \sigma_{\text{Wig}} \), a regularisation procedure suggested in [61] allows to establish (2.33) (and even to determine the subleading asymptotic terms) in the cases when (2.32) is violated due to divergent contribution to
\[ \tilde{s}(n; \mu_N; \sigma_{\text{Wig}}) = \int_{-\infty}^{\infty} U_n(\xi) d\mu_N(\xi) \]
coming from the neighbourhood of \( \xi = \pm 1 \). It would be interesting to find a way to consider even shorter scales \( \epsilon_N \), for which the limit of \( \epsilon_N^{-1} R_{\epsilon N}^N [\mu_N] \) is distinct from that of \( \epsilon_N^{-1} R_{\epsilon N}^N [\mu_\infty] \). In the random matrix applications, such a method would allow to study the local eigenvalue statistics in the bulk of the spectrum via modified moments (in particular, in problems where alternative methods are not currently available).

3. Spectral edges of random matrices

3.1. Wigner matrices. The application of the moment method to local eigenvalues statistics originates in the work of Soshnikov [64] on universality for Wigner matrices. Let us recall the result of [64], after some preliminaries.

As before, we consider a sequence \( (H^{(N)})_{N \geq 1} \) of Wigner matrices, i.e. random Hermitian matrices such that the diagonal entries of every \( H^{(N)} \) are sampled from a probability distribution \( \mathcal{L}_{\text{diag}} \) satisfying (2.9), and the off-diagonal entries are sampled from a probability distribution \( \mathcal{L}_{\text{off-diag}} \) satisfying (2.10); the eigenvalues of \( H^{(N)} \) are denoted
\[ \xi_1^{(N)} \geq \xi_2^{(N)} \geq \cdots \geq \xi_N^{(N)} . \]

Consider the random point process (i.e. a random collection or points, or, equivalently, a random integer-valued measure)
\[ \Lambda^{(N)} = \sum_{j=1}^{N} \delta \left( \lambda - N^{1/6} \left[ \xi_j^{(N)} - 2 \sqrt{N} \right] \right) \] (3.1)
(the scaling is natural in view of the square-root singularity of \( \sigma_{\text{Wig}} \) at 1).

Two special cases, the Gaussian Orthogonal Ensemble (GOE), and the Gaussian Unitary Ensemble (GUE) [as well as the Gaussian Symplectic Ensemble (GSE, not
discussed here], enjoy an invariance property which allows to apply the method of orthogonal polynomials (see Mehta \[49\]). The limits of $Λ^{(N)}$ for GOE and GUE, called the Airy$_1$ ($\mathfrak{A}_1$) and the Airy$_2$ ($\mathfrak{A}_2$) point processes, respectively, were found by Tracy and Widom \[70, 71\] and Forrester \[25\]. The correlation functions, which are (by definition) the densities $ρ_{β,k}(λ_1, \cdots, λ_k) = \frac{d}{d mes_k} E Ai^{⊗k} | λ_1 < \cdots < λ_k$ of the off-diagonal parts of the moment measures $E Ai^{⊗k}$, are expressed via determinants involving the Airy function $Ai$:

$$ρ_{2,k}(λ_1, \cdots, λ_k) = \det_{k×k} (A(λ_p, λ_r))_{p,r=1}^{k},$$

$$ρ_{1,k}(λ_1, \cdots, λ_k) = \sqrt{\det_{2k×2k} (A_1(λ_p, λ_r))_{p,r=1}^{k}},$$

where

$$A(λ, λ') = \int_0^∞ Ai(λ + u) Ai(λ' + u) du, \quad A_1(λ, λ') = \begin{pmatrix} A(λ, λ') & DA(λ, λ') \\ JA(λ, λ') & A(λ, λ') \end{pmatrix},$$

$$DA(λ, λ') = \frac{∂}{∂λ'} A(λ, λ'), \quad JA(λ, λ') = -\int_∞^λ A(λ'', λ') dλ'' - \frac{1}{2} \text{sign}(λ - λ').$$

**Theorem** (Soshnikov \[64\]). Let $(H^{(N)})_{N≥1}$ be a sequence of Wigner matrices satisfying the additional assumptions

$$H(u, v) \xrightarrow{distr} -H(u, v); \quad (\text{symmetry})$$

$$E|H(u, v)|^{2k} ≤ (Ck)^k \quad (\text{subgaussian tails})$$

on $L_{diag}$ and $L_{off-diag}$. If $L_{off-diag}$ is supported on the real line, the point processes $Λ^{(N)}$ converge (in the topology of Remark \[4.4\]) to $\mathfrak{A}_1$; otherwise, $Λ^{(N)} → \mathfrak{A}_2$.

**Remark 3.1.** Lee and Yin \[41\] have shown that the theorem remains valid if \(3.4\) and \(3.5\) are replaced with the assumption

$$\lim_{R→∞} R^4 P \{ |H(1, 2)| ≥ R \} = 0,$$

which they have shown to be necessary and sufficient. Their argument makes use of the methods developed in the works of Erdős, Bourgade, Knowles, Schlein, Yau, and Yin on universality in the bulk for Wigner matrices, cf. Erdős \[15\].

**Remark 3.2.** The work of Soshnikov was followed by numerous other applications of the moment method to local eigenvalue statistics in random matrix theory (see Soshnikov \[65\], Péché \[54\]) as well as outside it (see Okounkov \[50\]).

The strategy of \[64\] is to compute the asymptotics of moments and to show, using a version of Proposition \[2.3\] that the limit of $Λ^{(N)}$ exists and does not depend on the distribution of the entries. Thus the theorem is reduced to its special case appertaining to the Gaussian invariant ensembles.
3.1.1. An argument based on modified moments. In [21], modified moments were used to re-prove Soshnikov’s theorem quoted above (the method was also applied to sample covariance matrices, to re-prove the results of Soshnikov [65] and Péché [54] on the largest eigenvalues, and to prove a new result on the smallest ones). Let us outline the argument of [21] (incorporating modifications from [60]), which serves as the basis for the extensions described later in this section.

Let us assume that the diagonal entries $H(u,u)$ are identically zero, and that the off-diagonal entries $H(u,v)$ are randomly chosen signs $\pm 1$. Then Proposition 2.1 identifies
\[
E \prod_{r=1}^{k} \text{tr} P_{n_r}^{(N-1)} \left( \frac{H(N)}{2\sqrt{N-2}} \right)
\]
with (3.7) as $(N-2)^{-\sum n_r/2}$ times the number of $k$-tuples of closed non-backtracking paths in the complete graph on $N$ vertices, in which every edge is traversed an even number of times (in total). Such $k$-tuples are divided in topological equivalence classes ($k$-diagrams of Section 3.2.1 below). For $n_r \approx N^{1/3}$, the contribution of every equivalence class can be asymptotically evaluated.

In the regime $n_r \approx N^{1/3}$, (3.7) captures the asymptotics of the transform (2.31) of the moment measures of $\Lambda(N)$. The combinatorial classification yields a convergent series for this transform. This allows to describe the vague limit of $\Lambda(N)$. A more general argument making use of an extension of Proposition 2.1 allows to show that the same limit appears for any sequence of matrices satisfying the assumptions of Soshnikov’s theorem (with $L_{\text{off-diag}}$ supported on the real line) in particular, for the GOE for which the answer is already identified as the Airy$_1$ point process $\mathfrak{Ai}_1$.

In the remainder of this section we describe (without proofs) two results which may be seen as generalisations of [64].

3.2. Wigner processes. Instead of a single Wigner matrix $H(N)$, let us consider a family $H(N)(x) = (H(x;u,v))_{1 \leq u \leq v \leq N}$ of Wigner matrices depending on a parameter $x \in X$; then we are interested in the eigenvalues $\xi_1^{(N)}(x) \geq \xi_2^{(N)}(x) \geq \cdots \geq \xi_N^{(N)}(x)$ as a random process on $X$.

Let us assume that $(x \mapsto H(x;u,u))_{1 \leq u \leq N}$ are independent copies of a random process $\text{diag} : X \to \mathbb{R}$,
\[
E \text{diag}(x) = 0 , \quad E \text{diag}(x)^2 < \infty ,
\]
and that $(x \mapsto H(x;u,v))_{1 \leq u < v \leq N}$ are independent copies of $\text{off-\text{-diag}} : X \to \mathbb{C}$,
\[
E \text{off-\text{-diag}}(x) = 0 , \quad E |\text{off-\text{-diag}}(x)|^2 = 1 .
\]
The process $\text{off-\text{-diag}}(x)$ equips $X$ with the $L_2$ metric
\[
\rho(x,x') = \sqrt{\frac{1}{2} E |\text{off-\text{-diag}}(x) - \text{off-\text{-diag}}(x')|^2} .
\]
The local properties of the eigenvalues rescaled about $x_0 \in X$ depend on the behaviour of $\rho$ near $x_0$, which may be captured by the tangent cone $T_{x_0}X$ to $X$ at $x_0$ (the tangent cone to a metric space was introduced by Gromov [36, Section 7]).

Moment-based methods allow to obtain rigorous results at the spectral edges. Here we focus on the special case in which $X = \mathbb{R}^d$ and

$$\rho(x,x')^2 = ||x - x'||_p + o(||x||_p + ||x'||_p) \quad (x, x' \to 0) \quad (3.8)$$

for some $1 \leq p \leq 2$, which includes, for example, the Ornstein–Uhlenbeck sheet. In this case the tangent cone at the origin is the space $X_p^d = (\mathbb{R}^d, \sqrt{||\cdot||_p})$.

**Theorem 3.3.** Let $d \in \mathbb{N}$ and $1 \leq p \leq 2$. Let $X = \mathbb{R}^d$, and suppose the processes $\text{diag}(x)$ and off–diag$(x)$ have symmetric distribution (3.2) and subgaussian tails (3.3) at every point $x \in \mathbb{R}^d$, and that the covariance of off–diag$(x)$ has the asymptotics (3.3) near the origin. Then the processes

$$\Lambda^{(N)}(x) = \sum_{j=1}^{N} \delta \left( \lambda - N^{1/6} \left[ \xi_j^{(N)}(xN^{1/3}) - 2\sqrt{N} \right] \right)$$

converge (as $N \to \infty$, in the sense of finite-dimensional distributions) to a limiting process

$$\text{AD}_\beta[X_p^d(x)] = \sum_{j=1}^{\infty} \delta(\lambda - \lambda_j(x)) \quad (x \in \mathbb{R}^d)$$

taking values in sequences $\lambda_1(x) \geq \lambda_2(x) \geq \cdots$, where $\beta = 1$ if off–diag$(0)$ is real-valued, and $\beta = 2$ — otherwise.

The level of generality chosen here is motivated in particular by the following result proved in [82]: the process $\text{AD}_\beta[X_p^d]$ also describes the edge scaling limit of corners of time-dependent random matrices (for a discussion of these, see Borodin [7,8]).

**Proposition 3.4.** For $\beta \in \{1, 2\}$, the process $\text{AD}_\beta[X_p^d]$ boasts the properties:

1. There exists a modification of $\text{AD}_\beta[X_p^d(x)]$ in which every $\lambda_j(x)$ is a continuous function of $x \in \mathbb{R}^d$.

2. At a fixed $x \in \mathbb{R}^d$, $\text{AD}_\beta[X_p^d(x)]$ is equal in distribution to the Airy$_\beta$ point process $\mathfrak{A}_\beta$.

3. The distribution of $\text{AD}_\beta[X_p^d(x)]$ at a $k$-tuple of points $(x_q)_{q=1}^k$ in $\mathbb{R}^d$ depends only on $\beta$ and on the distances $||x_q - x_r||_p$ ($1 < q < r < k$).

The last item implies that the distribution of the restriction of $\text{AD}_\beta[X_p^d]$ to a geodesic in $\mathbb{R}^d$ does not depend on the choice of geodesic (and neither on $p$ and $d$), and thus coincides in distribution with $\text{AD}_\beta[X^1]$.

We note that $\text{AD}_2[X^1]$ admits a concise determinantal description. Indeed, the $\beta = 2$ Dyson Brownian motion satisfies the assumptions of Theorem 3.3 thus its
edge scaling limit is described by the process $A D_2[X^1]$. On the other hand, Macêdo [15] and Forrester, Nagao, and Honner [20] (see further Forrester [26, 7.1.5]) found this limit directly. This process, the moment measures of which are given by determinants, appeared again in the work of Prähofer and Spohn [55] on models of random growth (see the lecture notes of Johansson [38] for further limit theorems in which it appears); Corwin and Hammond [11] studied its properties, and coined the term ‘Airy line ensemble’. Thus the distribution of the restriction of $AD_2[X^d_p]$ to any geodesic in $\ell^d_p$ is given by the Airy line ensemble.

3.2.1. Construction of the processes $AD_\beta$. With the exception of the case $\beta = 2, d = 1$, the process $X = AD_\beta[X^d_p]$ does not seem to be described by determinantal formulæ. The construction presented here is motivated by the combinatorial arguments of Soshnikov [64] and further by the work of Okounkov [50], and makes use of the results of [21].

Let $X(x) (x \in \mathbb{R}^d)$ be a random process which takes values in point configurations on the line (i.e. locally finite sums of $\delta$-functions). That is, for every $x \in \mathbb{R}^d$ the random variable $X(x)$ is a point process on $\mathbb{R}^d$. Denote

$$\tilde{\rho}_{X,k}(x_1, \ldots, x_k) = \mathbb{E} \prod_{r=1}^k X(x_r) \ (x_1, \ldots, x_k \in \mathbb{R}^d)$$

be the moment measures of $X$, and

$$\tilde{R}_{X,k}(x_1, \ldots, x_k; \alpha_1, \ldots, \alpha_k) = \int \prod_{r=1}^k \frac{\sin \alpha_r \sqrt{-\lambda_r}}{\sqrt{-\lambda_r}} d\tilde{\rho}_{X,k}(x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_k)$$

(in our case, the divergent integral can be regularised, cf. [60, Section 6]; the transform appears from the asymptotics of orthogonal polynomials, cf. (2.31)). Then, let

$$\tilde{R}_{X,k}^\#(x; \alpha) = \sum_{I \subset \{1, \ldots, k\}} \tilde{R}_{X,k}\#_{I}(x|_I, \alpha|_I) \tilde{R}_{X,k}\#_{I^c}(x|_{I^c}, \alpha|_{I^c}) ,$$

where $x = (x_1, \ldots, x_k) \in (\mathbb{R}^d)^k$, and $x|_I = (x_r)_{r \in I}$. (The sum over partitions has to do with the contribution of the two spectral edges to the asymptotics.) We define $AD_\beta$ via a formula (3.12) for $\tilde{R}_{AD_\beta[X^d_p],k}^\#$ (uniqueness follows from the considerations of [60, Section 6]).

Let us consider the collection of $k$-tuples of non-backtracking walks for which every edge of the spanned graph $g$ (cf. Section 2.2) is traversed exactly twice, and every vertex in $g$ has degree at most three. Such $k$-tuples can be divided into topological equivalence classes ($k$-diagrams). For example, for $k = 1$, the paths 1 2 3 4 5 6 7 8 4 5 6 7 8 4 3 2 1 on Figure 2.1 (right) and the path 8 7 2 3 9 2 3 9 2 7 8 belongs to the same equivalence class, schematically depicted on Figure 3.1 (left). The formal definition is given in [21, 60].
Several applications of the moment method in random matrix theory

Figure 3.1. Three 1-diagrams. The diagram on the left corresponds to the projective plane with \( s = 1 \) (left). The two diagrams in the centre and on the right correspond to surfaces with \( s = 2 \); the one in the centre is the torus.

Different \( k \)-diagrams correspond to homotopically distinct ways to glue \( k \) disks with a marked point on the boundary. The result of such a gluing is a two-dimensional manifold. Thus to every \( k \)-diagram one can associate a number \( s \), which is related to the Euler characteristic \( \chi \) of the manifold by the formula \( s = 2k - \chi \); for \( k = 1 \) the number \( s \) is the non-oriented genus. The multi-graph associated to a diagram with a certain value of \( s \) (see Figures 3.1 and 3.2) has \( 2s \) vertices and \( 3s - k \) edges. The number \( D_k(s) \) of \( k \)-diagrams with a given value of \( s \) satisfies the estimates ([21, Proposition II.3.3])

\[
\frac{(s/C)^{s+k-1}}{(k-1)!} \leq D_k(s) \leq \frac{(Cs)^{s+k-1}}{(k-1)!} ;
\]

the upper bound guarantees that the series (3.12) which we derive below converges.

Figure 3.2. Three 2-diagrams: \( s = 2 \) (left), \( s = 3 \) (centre, right). The leftmost diagram, corresponding to a sphere glued from two disks, is often responsible for fluctuations of linear eigenvalue statistics on global and mesoscopic scales.

Next, we associate to a \( k \)-diagram \( D \) and to \( \alpha \in (0, \infty)^k \) a \((3s-2k)\)-dimensional polytope \( \Delta_D(\alpha) \) in \( \mathbb{R}^{3s-k} \), as follows. The variables \( w(e) \) are labeled by the edges \( e \) of \( D \); the polytope is defined by the inequalities

\[
\begin{align*}
  w(e) & \geq 0 \quad (e \in \text{Edges}(D)) \\
  \sum_e c_r(e)w(e) & = \alpha_r \quad (1 \leq r \leq k)
\end{align*}
\]

where \( c_r(e) \in \{0,1,2\} \) is the number of times the edge \( e \) is traversed by the \( p \)-th path in the diagram. For example, the polytope associated with the rightmost
2-diagram of Figure 3.2 is given by

\[
\begin{align*}
\{ w(I), w(II), \cdots, w(VII) \} & \geq 0 \\
2w(I) + 2w(II) + 2w(III) + 2w(IV) + w(V) + w(VI) & = \alpha_1 \\
w(V) + w(VI) + 2w(VII) & = \alpha_2.
\end{align*}
\]

Let \(\mathcal{D}_1(k)\) be the collection of all \(k\)-diagrams, and let \(\mathcal{D}_2(k) \subset \mathcal{D}_1(k)\) be the sub-collection of diagrams in which every edge is traversed once in one direction and once in another one (such as Figure 3.1 centre, and Figure 3.2 left; these diagrams correspond to gluings preserving orientation). Now we can finally write the series for \(\tilde{R}^\#\):

\[
\tilde{R}^\#_{\mathcal{D}_2, k}(x; \alpha) = \sum_{\mathcal{D} \in \mathcal{D}_2} \int_{\Delta_{\mathcal{D}}(\alpha)} \exp \left\{- \sum_{e \in \text{Edges}(\mathcal{D})} \| x_{r_+(e)} - x_{r_-(e)} \| w(e) \right\} d \text{mes}_{3s-2k}(w),
\]

where \(k \geq r_+(e) \geq r_-(e) \geq 1\) are the indices of the two paths traversing \(e\) in \(\mathcal{D}\).

For example, when \(X\) is a singleton, all the terms in the exponent vanish, and (3.12) yields an expression for the Airy point process in terms of volumes of the polytopes \(\Delta_{\mathcal{D}}(\alpha)\), which may be compared to the one given by Okounkov [50, §2.5.4].

3.3. Band matrices. In this section, we discuss an extension of Soshnikov’s theorem to a class of matrices of the form considered in Section 2.2. First, we recall a conjecture, based on the Thouless criterion [69]. Then we discuss a particular case, the spectral edges of random band matrices, in which the conjecture can be proved. Finally, we comment on mesoscopic scales.

3.3.1. Thouless criterion. The Thouless criterion [69], originally introduced in the context of Anderson localisation, can be applied to predict the behaviour of local eigenvalue statistics; cf. Fyodorov and Mirlin [33, 34]. Consider a sequence of matrices \(H(N)\) associated with a sequence of graphs \(G_N = (V_N, E_N)\) as in Section 2.2 Then the measures

\[
\mu_N = \frac{1}{\#V_N} \sum_{j=1}^{N} \delta \left( \xi - \frac{\xi_j^{(N)}}{2\sqrt{2W_N}} \right)
\]

converge to the semi-circle measure \(\sigma_{\text{Wig}}\).

Let \(\xi_0 \in \mathbb{R}\), and let \(\eta_N > 0\) be chosen so that the sequence of (random) measures

\[
\#V_N R^{\xi_0}_{\eta_N} [E \mu_N] = \#V_N \sum_{j=1}^{N} \delta \left( \lambda - \frac{1}{\eta_N} \left[ \frac{\xi_j^{(N)}}{2\sqrt{2W_N}} - 1 - \xi_0 \right] \right)
\]

will have a non-trivial vague limit (cf. Section 2.5). Thus chosen, $\eta_N$ measures the mean spacing between eigenvalues, whereas

$$(\text{spacing/DOS})(\xi_0) = \eta_N^2 \#V_N$$

measures the mean spacing in units of the density of states. Let us compare the inverse of this quantity with the mixing time $T_{mix}^{\text{G}}$ of the random walk on $G_N$. In many cases the following seems to be correct: the eigenvalue statistics of $H^{(N)}$ near $\xi_0$ are described by random matrix theory if and only if

$$T_{mix}^{\text{G}}(G_N) \ll \frac{1}{(\text{spacing/DOS})(1)}.$$  \hspace{1cm} (3.13)

This interpretation of the Thouless criterion is based on the assumption that the semi-classical approximation is valid up to the scales governing the local eigenvalue statistics; we refer to the reviews of Spencer [66, 67] for a discussion of various aspects of Thouless scaling and its mathematical justification, and to the work of Spencer and Wang [74] for some rigorous results. Here we focus our attention on the particular case of

### 3.3.2. Random band matrices.

Denote $\| u - v \|_N = \min_{\ell \in \mathbb{Z}} | u - v - \ell N |$. A (one-dimensional) random band matrix of bandwidth $W$ is for us a random Hermitian $N \times N$ matrix $H^{(N)} = (H(u, v))_{1 \leq u, v \leq N}$ such that

$$
\begin{cases}
H(u, v) = 0, & \| u - v \|_N > W, \\
H(u, v) \sim L_{\text{off-diag}}, & 1 \leq \| u - v \|_N \leq W, \\
H(u, u) \sim L_{\text{diag}}, &
\end{cases}
$$

(3.14)

where $L_{\text{diag}}$ and $L_{\text{off-diag}}$ satisfy the normalisation conditions (2.9) and (2.10), respectively. In the setting of Section 2.2 it corresponds to the graph $G_N = (V_N, E_N)$,

$$V_N = \{ 1, \cdots, N \}, \quad (u, v) \in E_N \iff 1 \leq \| u - v \|_N \leq W_N .$$

(3.15)

More general band matrices are discussed, for example, in [17, 39, 67].

For $-1 < \xi_0 < 1$ (the bulk of the spectrum),

$$(\text{spacing/DOS})(\xi_0) \asymp \frac{1}{N}, \quad T_{mix} \asymp \frac{N^2}{W^2},$$

therefore the criterion (3.13) suggests the following: the eigenvalue statistics of $H^{(N)}$ near $\xi_0$ are described by random matrix theory if and only if $W \gg \sqrt{N}$. This prediction is supported by the detailed super-symmetric analysis performed by Fyodorov and Mirlin [33, 34]. Mathematical justification remains a major challenge, cf. Spencer [67, 68] and references therein.

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2equivalently, the ratio of the mixing time and the density of states, which is interpreted as the energy-dependent mixing time, is compared to the usual inverse eigenvalue spacing $\eta_N$. 

3.3.3. Spectral edges. The (modified) moment method allows to confirm the criterion (3.13) at the spectral edges of random band matrices.

**Theorem 3.5** (cf. [60, Theorem 1.1]). Let \((H^{(N)})_{N \geq 1}\) be a sequence of random band matrices satisfying the unimodality assumptions (2.27). If the bandwidth \(W_N\) of \(H^{(N)}\) satisfies

\[
\lim_{N \to \infty} \frac{W_N}{N^{5/6}} = \infty,
\]

then

\[
\sum_{j=1}^{N} \delta \left( \lambda - \frac{N^{2/3}}{\sqrt{2}W_N} \left( \xi_j^{(N)} - 2\sqrt{2W_N} \right) \right) \to \Re \beta,
\]

where \(\beta = 1\) if \(\text{supp} \, \mathcal{L}_{\text{off-diag}} \subset \mathbb{R}\), and \(\beta = 2\) otherwise.

The threshold \(N^{5/6}\) in (3.16) is sharp, see [60, Theorem 1.2]. The same [60, Theorem 1.2] implies that \(\eta_N \simeq \min(W_N^{2/5} N^{-1}, N^{-2/3})\), therefore

\[
(\text{spacing/DOS})(1) \simeq \min(W_N^{2/5} N^{-1}, N^{-1/3}),
\]

and (3.16) is consistent with (3.13).

The unimodality conditions (2.27) simplify the analysis (cf. Proposition 2.1); we expect that they can be relaxed using the methods of [21, Part III] and [17].

3.3.4. Mesoscopic scales. On mesoscopic scales \(1 \gg \epsilon_N \gg 1/\#V_N\), the following counterpart of the Thouless criterion goes back to the (physical) work of Altshuler and Shklovskii [2]. Let \(\eta_N\) be such that the sequence \((\epsilon_N^{-1} R_{\alpha N} [E \mu_N])_N\) has a non-trivial vague limit. If

\[
\epsilon_N \eta_N^{-2} \gg T_{\text{mix}},\]

the fluctuations of linear eigenvalue statistics should be described by a log-correlated Gaussian field, whereas when (3.17) is violated, one expects a more regular field depending on the geometry of the underlying lattice. We refer to the works of Fyodorov, Le Doussal, and Rosso [32] and of Fyodorov and Keating [30] for a discussion of the significance of log-correlated fields within and outside random matrix theory, and to the work of Fyodorov, Khoruzhenko, and Simm [31] for results pertaining to the Gaussian Unitary Ensemble.

Erdős and Knowles proved a series of results pertaining to mesoscopic statistics for a wide class of \(d\)-dimensional band matrices. In the works [16, 17], they developed a moment-based approach which allowed them to control the quantum dynamics associated for time scales \(t \leq W_N^{d/3-\delta}\). In [18, 19], they gave mathematical justification to the criterion (3.17) in the range \(\epsilon_N \geq W_N^{-d/3+\delta}\). It would be interesting to extend the results of [16, 17] and [18, 19] to the full mesoscopic range.
4. Some further questions

Other limiting measures The spectral measures in this article converge to the semicircle distribution $\mu_\infty = \sigma_{\text{Wig}}$. The modified moment method described here has been also applied to the Kesten–McKay measure (the orthogonality measure for $P_n^{(c)}$), the Godsil–Mohar measure (its bipartite analogue), and the Marchenko–Pastur measure (the infinite connectivity limit of the Godsil–Mohar measure); see e.g. [58, 59]. It would be interesting to adapt the method to situations in which the recurrent relation has less explicit form.

$\beta$-ensembles The (convincing, although so far unrigorous) ghost and shadows formalism introduced by Edelman [14] strongly suggests that the construction (3.12) should have an extension to general $\beta > 0$. See Forrester [27] and [26] for background on $\beta$-ensembles, and Borodin and Gorin [10] for a recent result pertaining to the spectral statistics of submatrices of $\beta$-Jacobi random matrices.

Time-dependent invariant ensembles It seems plausible that, for general (non-Gaussian) invariant ensembles undergoing Dyson-type evolution, the spectral statistics near a soft edge should be described by the processes $A\Delta \beta$ of Section 3.2.1. Currently, there seem to be no proved results of this form (even for the case $\beta = 2$ in which determinantal formulae for finite matrix size are given by the Eynard–Mehta theorem [39, Chapter 23]).

Beyond random matrices Motivated by the proof of the Baik–Deift–Johansson conjecture given by Okounkov [50], one may look for the appearance of (3.12) outside random matrix theory, particularly, in the context of random growth models, for a discussion of the subtle connection between which and random matrix theory we refer to the lecture notes of Ferrari [24].

Bulk of the spectrum We are not aware of any derivation (rigorous or not) of the local eigenvalue statistics in the bulk of the spectrum using any version of the moment method. Even for the test case of the Gaussian Unitary Ensemble (tractable by other means), perturbative methods such as Chebyshev expansions have not been of use beyond the scales $\epsilon_N \gg N^{-1+\delta}$. For random band matrices the expansion (2.54) has been only regularised for $\epsilon_N \gg W^{-1+\delta}$ (see [61]).

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