Investigation graph isomorphism problem via entanglement entropy in strongly regular graphs

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Abstract. We investigate the quantum networks that their nodes are considered as quantum harmonic oscillators. The entanglement of the ground state can be used to quantify the amount of information one part of a network shares with the other part of the system. The networks which we studied in this paper, are called strongly regular graphs (SRG). These kinds of graphs have some special properties like they have three strata in the stratification basis. The Schur complement method is used to calculate the Schmidt numbers and entanglement entropy between two parts of graph. We could obtain analytically, all blocks of adjacency matrix in several scalable sets of strongly regular graphs. Also the entanglement entropy in the large coupling limit is considered in these graphs and the relationship between entanglement entropy and the ratio of size of boundary to size of the system is found. After that, area-law is studied to show that there are no entanglement entropy for the highest size of system.

Then, the graph isomorphism problem is considered in SRGs by using the entanglement entropies in some partitioning of graph. Two SRGs with the same parameters: \((n, \kappa, \lambda, \nu)\) are isomorphic if they can be made identical by relabeling their vertices. So the adjacency matrices of two isomorphic SRGs become identical by replacing of rows and columns. The non-isomorph SRGs could be distinguished by using the elements of blocks of adjacency matrices in the stratification basis, numerically.

Keywords: exact results, network dynamics
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References
1. Introduction

One of the important problems about networks is the graph isomorphism (GI) problem [1, 2]. Two graphs are isomorphic, if one can be transformed into the other by a relabeling of vertices (i.e. two graphs with the same number of vertices and edges are non-isomorphic, if they can not be transformed into each other by relabeling of vertices). Graph isomorphism problem has many applications in other fields of science and engineering, like as mathematical chemistry [3] (for identifying a chemical compound within a chemical database), electronic design [4] (as the basis of the layout versus schematic (LVS) circuit design step, which is a verification whether the electric circuits represented by a circuit schematic and an integrated circuit layout are the same), computer science [5, 6] (in some problems related to object-oriented Petri nets, for example in the state space generating problem and symmetry check) and biochemical tools [7].

Many graph pairs may be distinguished by a classical algorithm which runs in a time polynomial in terms of the number of vertices of the graphs, but there exist pairs which are computationally difficult to distinguish. Currently, the best general classical algorithm has a run time $O(c \sqrt{N \log N})$, where $c$ is a constant and $N$ is the number of vertices in the two graphs. It is clear that in any isomorphism, a vertex of graph $G$ with $k$ neighbors is equivalent to a vertex of graph $H$ that has $k$ neighbors. Thus, some graphs can be distinguished by degrees (the number of neighbors) of its vertices. This approach fails in the case where every vertex of $G$ and $H$ have the same number of neighbors. Graph isomorphism remains an interesting problem for strongly regular graphs because all vertices of these graphs have the same number of neighbors. Strongly regular graphs (SRGs) [8–11] are a particular class of graphs that have four dependent parameter $(n, \kappa, \lambda, \mu)$, that are difficult to distinguish classically [12]. Strongly regular graphs are important kind of graphs because they are useful in many branches of researches. Some of the most important applications of SRGs are as following: Ramsey theory in mathematics [13], having maximal energy of graphs [14] (which is in the close correspondence with the molecular orbital energy levels $\pi$-electrons in conjugated hydrocarbons), computer science and engineering such as linear codes and ternary codes [15, 16].

Many researchers have attacked to the graph isomorphism problem in strongly regular graphs. One class of algorithms that has been explored for graph isomorphism is that of quantum random walks (QRW). Shiau et al showed that the single-particle continuous-time QRW fails to distinguish pairs of SRGs with the same family parameters [17]. Gamble et al extended these results and proved that QRWs of two noninteracting particles will always fail to distinguish pairs of non-isomorphic SRGs with the same family parameters [18]. Then Rudinger et al numerically demonstrated that three-particle noninteracting walks have distinguishing power on pairs of SRGs [19, 20].

Another important tool which can be used for detecting isomorphism is entanglement. One of the operational entanglement criteria is the Schmidt decomposition [21–23]. The entanglement of a partly entangled pure state can be naturally parameterized by its entropy of entanglement, defined as the Von Neumann entropy, or equivalently as the Shannon entropy of the squares of the Schmidt coefficients [21, 23]. By computing the entanglement entropy between two parts of a network, one can quantify
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the amount of information that a part of a quantum network shares with the rest of the system. In [24] the authors considered a network of quantum harmonic oscillators and analyzed its ground state to compute the entropy of entanglement that vacuum fluctuations creates between single nodes and the rest of the network by using the entropy of entanglement.

This paper, addresses the graph isomorphism problem in strongly regular graphs by using the entanglement. To this aim, first we calculate the entanglement entropy between two parts of SRGs, then we investigate the graph isomorphism problem in them. In our model the nodes are considered as identical quantum oscillators. The ground state wave function is a gaussian state. Gaussian states have many experimental advantages. Practically, preparing the single-photon states as a qubit or two-photon entangled states as entanglement resource, deterministically are technically difficult. But gaussian states can be prepared and manipulated easily in the form of a coherent state [25, 26] (the state of light pulses from a traditional Laser) or a squeezed vacuum state [26–31] of one mode or two modes. Because of the importance of gaussian states many researchers have investigated the entanglement in gaussian states [32, 33]. They used some entanglement measures such as negativity, tangle and entanglement entropy. Here the entanglement between two parts of strongly regular graphs, in the gaussian ground state wave function is obtained by calculating the entanglement entropy. Then this measure is used for investigating the graph isomorphism problem. The ground state wave function is obtained in terms of the Laplacian \( L \) of SRGs which is related to the adjacency matrix of network. Two non-isomorph SRGs have the same energy but their ground state wave functions are different. For calculating the entanglement entropy in SRGs, first we use the stratification techniques [34–38], to rewrite the adjacency matrices of SRGs in the block form. The obtained matrix, becomes block diagonal in the stratification basis. We called it the block-diagonal adjacency matrix. The first block of obtained matrix, will be a \( 3 \times 3 \) matrix and the other blocks are \( 2 \times 2 \) or singlets. The \( 3 \times 3 \) block is related only to the parameters of the SRG and obtains analytically in terms of parameters for all SRGs. The entanglement entropy between the first stratum (which has only one vertex) and other vertices (second and third strata), can not distinguish non-isomorph pairs because it will be obtained only from \( 3 \times 3 \) block of adjacency matrix. But for calculating the entanglement between other subsets, we need the \( 3 \times 3 \) block and all of the \( 2 \times 2 \) blocks of adjacency matrix. We discuss about the elements of these \( 2 \times 2 \) blocks and give some important relations between its elements. These \( 2 \times 2 \) blocks are different for some non-isomorph SRGs, therefore the entanglement entropy will be able to distinguish many cases of non-isomorph SRGs. Also for several important sets of SRGs, such as triangular graphs, we could obtain the \( 2 \times 2 \) blocks of adjacency matrices analytically. So the entanglement entropy between all two subsets, will be obtained in these sets of SRGs analytically. For the other SRGs which their adjacency matrices are identified, we calculate the block-diagonal adjacency matrix numerically and distinguish the non-isomorph SRGs. Calculating the entanglement entropy between strata is performed by using the Schur complement method [39] and some local transformations [40]. The motivation of the paper is the investigation of graph isomorphism problem in strongly regular graphs by using the entanglement entropy. We use the entanglement entropy for detecting isomorphism for the first time. The main advantage of this research is the flexibility.
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and simplicity of the method (stratification basis and Schur complement method) in analytical calculations. Another reason for the importance of this study is the model and its ground state in the form of gaussian states which is widely applied theoretically and experimentally.

The paper is structured as follows. In the section 2, first we demonstrate the Schmidt decomposition and entanglement entropy. We give some properties of strongly regular graphs in section 2.2. Finally in section 2.3 we describe the Hamiltonian of our model and its ground state energy and wave function. In section 3, we use the stratification techniques to rewrite the adjacency matrices of SRGs in the stratification basis. In section 4 we calculate the entanglement entropy between two parts of the SRGs. By considering three strata of adjacency matrix of SRGs, the entanglement entropies are calculated for three kinds of partitioning in sections 4.1–4.3. In section 5, we construct some simple kinds of SRGs. These kinds of SRGs don’t contain non-isomorph graphs. Then in section 6 we construct four sets of SRGs that we obtain their block diagonal adjacency matrices by using the relations of section 3 and the information of graphs analytically. In section 7, the large coupling limit of entanglement entropies are considered. Also the behavior of entanglement entropy when the coupling strength tends to infinite, is investigated. The area-law for entanglement entropies are calculated in section 8. In section 9 we give some other examples of non-isomorph SRGs which can be distinguished by using their blocks of adjacency matrices numerically. The Schur complement method is in the appendix A and the stratification techniques are given in appendix B.

2. Preliminaries

2.1. Schmidt decomposition and entanglement entropy

Any bipartite pure state $|\psi\rangle_{AB} \in H = H_A \otimes H_B$ can be decomposed, by choosing an appropriate basis, as

$$|\psi\rangle_{AB} = \sum_{i=1}^{m} \alpha_i |a_i\rangle \otimes |b_i\rangle$$

(1)

where $1 \leq m \leq \min\{\dim(H_A); \dim(H_B)\}$, and $\alpha_i > 0$ with $\sum_{i=1}^{m} \alpha_i^2 = 1$. Here $|a_i\rangle$ ($|b_i\rangle$) form a part of an orthonormal basis in $H_A$ ($H_B$). The positive numbers $\alpha_i$ are called the Schmidt coefficients of $|\psi\rangle_{AB}$ and the number $m$ is called the Schmidt rank of $|\psi\rangle_{AB}$. The entanglement of a partly entangled pure state can be naturally parameterized by its entropy of entanglement, defined as the Von Neumann entropy of either $\rho_A$ or $\rho_B$, or equivalently as the Shannon entropy of the squares of the Schmidt coefficients.

$$E = -\text{Tr} \rho_A \log_2 \rho_A = \text{Tr} \rho_B \log_2 \rho_B = - \sum_i \alpha_i^2 \log_2 \alpha_i^2$$

(2)

2.2. Strongly regular graphs(SRG)

A graph (simple, undirected and loopless) of order $n$ is strongly regular with parameters $n$, $\kappa$, $\lambda$, $\mu$ whenever it is not complete or edgeless and
(i) each vertex is adjacent to $\kappa$ vertices,

(ii) for each pair of adjacent vertices there are $\lambda$ vertices adjacent to both,

(iii) for each pair of non-adjacent vertices there are $\mu$ vertices adjacent to both.

We assume throughout that a strongly regular graph $G$ is connected and that $G$ is not a complete graph. Consequently, $\kappa$ is an eigenvalue of the adjacency matrix of $G$ with multiplicity 1 and

$$n - 1 > \kappa \geq \mu > 0, \quad \kappa - 1 > \lambda \geq 0$$  \hspace{1cm} (3)

Counting the number of edges in $G$ connecting the vertices adjacent to a vertex $x$ and the vertices not adjacent to $x$ in two ways we obtain

$$\kappa(\kappa - \lambda - 1) = (n - \kappa - 1)\mu$$  \hspace{1cm} (4)

so the relation between these parameters is

$$\kappa^2 = (\kappa - \mu) + \mu n + (\lambda - \mu)\kappa$$  \hspace{1cm} (5)

The adjacency matrix of any SRG satisfies the particularly useful algebraic identity

$$A^2 = (\kappa - \mu)I + \mu J + (\lambda - \mu)A$$  \hspace{1cm} (6)

where $I$ is the identity and $J$ is the matrix of all ones. The eigenvalues of all SRGs are in terms of parameters of SRG as

$$x_1 = \kappa$$

$$x_2 = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(\kappa - \mu)}}{2}$$

$$x_3 = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(\kappa - \mu)}}{2}$$  \hspace{1cm} (7)

The multiplicity of $x_1 = \kappa$ is 1, and the multiplicity of two other eigenvalues are

$$m_{x_2} = -\frac{(n - 1)x_3 + \kappa}{x_2 - x_3}$$

$$m_{x_3} = -\frac{(n - 1)x_2 + \kappa}{x_2 - x_3}$$  \hspace{1cm} (8)

2.3. The model and Hamiltonian

The nodes are considered as identical quantum oscillators, interacting as dictated by the network topology encoded in the Laplacian $L$. The Laplacian of a network is defined from the adjacency matrix as $L_{ij} = k_i \delta_{ij} - A_{ij}$, where $k_i = \sum_j A_{ij}$ is the connectivity of node $i$, i.e. the number of nodes connected to $i$. The Hamiltonian of the quantum network thus reads:

$$H = \frac{1}{2}(P^T P + X^T (I + 2gL)X)$$  \hspace{1cm} (9)
here $I$ is the $N \times N$ identity matrix, $g$ is the coupling strength between connected oscillators while $p^T = (p_1, p_2, ..., p_N)$ and $x^T = (x_1, x_2, ..., x_N)$ are the operators corresponding to the momenta and positions of nodes respectively, satisfying the usual commutation relations: $[x_i, p_j] = i\hbar I$ (we set $\hbar = 1$ in the following) and the matrix $V = I + 2gL$ is the potential matrix. Then the ground state of this Hamiltonian is:

$$\psi(X) = \frac{\det(I+2gL)^{1/4}}{\pi^{N/4}} \exp \left(-\frac{1}{2}(X^T(I+2gL)X)\right)$$

(10)

where the $A_g = (\frac{\det(I+2gL)^{1/4}}{\pi^{N/4}})$ is the normalization factor for wave function. The elements of the potential matrix in terms of entries of adjacency matrix is

$$V_{ij} = (1+2g\kappa_i)\delta_{ij} - 2gA_{ij}$$

The ground state energy is in terms of the eigenvalues of potential matrix,

$$E_G = \frac{1}{2} \prod_{i=1}^{N} (1+2g\kappa_i)$$

(11)

where $\kappa_i$ are the eigenvalues of Laplacian matrix, which are written in terms of eigenvalues of adjacency matrix.

As it mentioned in equation (7), the eigenvalues of adjacency matrix in SRGs, are in terms of parameters of SRGs. So non-isomorph SRGs with the same parameters, have the same ground state energy

$$E_G(SRG) = \frac{1}{2} \prod_{i=1}^{m_{x_2}} (1+2(\kappa - x_2)g) \prod_{j=1}^{m_{x_3}} (1+2(\kappa - x_3)g)$$

(12)

where $x_2$, $x_3$, $m_{x_2}$ and $m_{x_3}$ can be calculated from equations (7) and (8). The non-isomorph SRGs, have different adjacency matrices, so their ground state wave functions are different. Therefore the entanglement entropy of ground state wave function can distinguish non-isomorph SRGs.

3. Stratification techniques in SRGs

The adjacency matrix of a strongly regular graph can be written as

$$A = \begin{pmatrix} 0 & e_k^T & 0^T \\ e_k & A_{11} & A_{12} \\ 0 & A_{12}^T & A_{22} \end{pmatrix}$$

(13)

where $e_k = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{k \times 1}$ and $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{(n-k-1) \times 1}$.
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Then

\[ A^2 = \begin{pmatrix}
  k & e_{\kappa}^T A_{11} & e_{\kappa}^T A_{12} \\
  A_{11} e_{\kappa} & e_{\kappa} e_{\kappa}^T + A_{11}^2 + A_{12} A_{12}^T & A_{11} A_{12} + A_{12} A_{22} \\
  A_{12} e_{\kappa} & A_{12}^T A_{11} + A_{22} A_{12}^T & A_{12}^T A_{12} + A_{22}^2
\end{pmatrix} \quad (14) \]

From the block (1, 3) of equations (14) and (6) we conclude that

\[ e_{\kappa}^T A_{12} = \mu e_{\kappa}' \]

\[ A_{12}^T e_{\kappa} = \mu e_{\kappa}' \quad (15) \]

So

\[ \sum_{\alpha} (A_{12})_{\alpha, j} = \mu \quad (16) \]

Also it can be written from the block (1, 2) of equation (14)

\[ \sum_{\alpha} (A_{11})_{\alpha, j} = \lambda, \quad \sum_{\alpha} (A_{11})_{j\alpha} = \lambda \quad (17) \]

From the block (2, 3) of equations (14) and (6) we have

\[ A_{11} A_{12} + A_{12} A_{22} = \mu J_{\kappa \kappa'} + (\lambda - \mu) A_{12} \quad (18) \]

Then we multiply the above equation from the left side in \( e_{\kappa}^T \) and use the equations (14) and (15) to prove that

\[ \sum_{\alpha} (A_{22})_{\alpha, j} = \kappa - \mu, \quad \sum_{\alpha} (A_{22})_{j\alpha} = \kappa - \mu \quad (19) \]

Other equations are

\[ A_{12}^T A_{12} + A_{22}^2 = (\kappa - \mu) I_{\kappa'} + \mu J_{\kappa \kappa'} + (\lambda - \mu) A_{22} \quad (20) \]

\[ A_{11}^2 + A_{12} A_{12}^T = (\kappa - \mu) I_{\kappa} + (\mu - 1) J_{\kappa \kappa} + (\lambda - \mu) A_{11} \quad (21) \]

Suppose that \( A_{12} = O_{12} D_{12} O_{22}^T \) be the singular value decomposition of \( A_{12} \), then we multiply the equation (20) from left side in \( O_{22}^T \) and from the right hand in \( O_{2} \), then by comparing the two side of relation, we see that the matrix \( A_{22} \) can be diagonal by orthogonal matrix \( O_{2} \) as

\[ A_{22} = O_{2} D_{22} O_{22}^T \]

The similar result is obtained from equation (18) for the matrix \( A_{11} \):

\[ A_{11} = O_{12} D_{12} O_{12}^T \]

By using the above result, the following transformation for adjacency matrix is obtained
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\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & O_1^T & 0 \\
0 & 0 & O_2^T
\end{bmatrix}
\begin{bmatrix}
e_\kappa^T \\
A_{11} A_{12} \\
A_{12}^T A_{22}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & O_1 & 0 \\
0 & 0 & O_2
\end{bmatrix}
= \\
\begin{bmatrix}
0 & e_\kappa^T O_1 & 0 \\
O_1^T e_\kappa & O_1^T A_{11} O_1 & O_1^T A_{12} O_2 \\
0 & O_2^T A_{12} O_1 & O_2^T A_{22} O_2
\end{bmatrix}
\equiv
\begin{bmatrix}
0 & \sqrt{k} & 0 & 0 & 0 & 0 \\
\sqrt{k} & 0 & D_{11} & D_{12} \\
\vdots & \vdots & 0 & 0 \\
0 & \vdots & D_{12} & D_{22}
\end{bmatrix}
= D
\quad (22)
\]

(I) First block of adjacency matrix in the stratification basis
In this section we want to calculate the adjacency matrix in the basis of first strata, so the equations (18), (20) and (21) should be rewritten for the nonzero eigenvalue of matrix \(J\), so

\[
\sqrt{\lambda_{12}} (\lambda_1 + \lambda_2) = \mu \sqrt{\kappa (n - \kappa - 1)} + (\lambda - \mu) \sqrt{\lambda_{12}} \\
\lambda_{12} + \lambda_2^2 = (\kappa - \mu) + \mu (n - \kappa - 1) + (\lambda - \mu) \lambda_2 \\
\lambda_{12} + \lambda_1^2 = (\kappa - \mu) + (\mu - 1) k + (\lambda - \mu) \lambda_1
\]

(23)

Where \(\lambda_1\) (\(\lambda_2\)) is the first eigenvalue of the block \(A_{11}\) (\(A_{22}\)) and \(\lambda_{12}\) is the first singular value decomposition of block \(A_{12}\), \(D^2\) from equation (22) must be calculate

\[
D^2 \equiv (\kappa - \mu) I + \mu J + (\lambda - \mu) D
\quad (24)
\]

From above equation

\[
\sqrt{\kappa} \sqrt{\lambda_{12}} = \mu \sqrt{n - k - 1} \\
\sqrt{k} \lambda_1 = \lambda \sqrt{\kappa}
\]

(25)

So by substituting these results into equation (23), we can calculate the parameters \(\lambda_1\), \(\lambda_2\) and \(\lambda_{12}\) as

\[
\lambda_1 = \lambda \\
\lambda_2 = \kappa - \mu \\
\lambda_{12} = \frac{\mu^2 (n - \kappa - 1)}{\kappa} = \sqrt{\mu} \sqrt{\kappa - \lambda - 1}
\]

(26)

So the \(3 \times 3\) block of adjacency matrix in the stratification basis is

\[
\begin{bmatrix}
0 & \sqrt{\kappa} & 0 \\
\sqrt{\kappa} & \lambda & \sqrt{\mu} \sqrt{\kappa - \lambda - 1} \\
0 & \sqrt{\mu} \sqrt{\kappa - \lambda - 1} & \kappa - \mu
\end{bmatrix}
\]

(27)
(II) $2 \times 2$ blocks and singlets of adjacency matrix in the stratification basis:

The equations (18), (20) and (21) for the other zero eigenvalues of matrix $J$ become

$$\sqrt{\lambda_{12}} (\lambda_1 + \lambda_2) = (\lambda - \mu) \sqrt{\lambda_{12}}$$

$$\lambda_{12} + \lambda_2^2 = (\kappa - \mu) + (\lambda - \mu) \lambda_2$$

$$\lambda_{12} + \lambda_1^2 = (\kappa - \mu) + (\lambda - \mu) \lambda_1$$

where $\lambda_1$s ($\lambda_2$s) are the eigenvalues (except the first eigenvalue) of the block $A_{11}$ ($A_{22}$), and $\lambda_1$'s are the singular value decomposition (except the first one) of block $A_{12}$. The solution for these equations is

$$\begin{cases}
\lambda_1 + \lambda_2 = \lambda - \mu & \lambda_{12} \neq 0 \\
\lambda_{12}^2 = (\kappa - \mu) + (\lambda - \mu) \lambda_{12} = 0
\end{cases} \quad (29)$$

The $2 \times 2$ matrix for other strata is obtained by

$$\begin{pmatrix} \lambda_1 & \sqrt{\lambda_{12}} \\ \sqrt{\lambda_{12}} & \lambda_2 \end{pmatrix} \quad (30)$$

The eigenvalues of SRGs are $x_{1,2} = \frac{1}{2} \left( \lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(\kappa - \mu)} \right)$ [20, 21]. So, by using this fact that the sum of two eigenvalues is $\lambda - \mu$ and the multiply of two eigenvalues is $\mu - \kappa$, we have

$$\lambda_{12} - \lambda_1 \lambda_2 = \kappa - \mu$$. 

So if anyone have one of the $\lambda_1$, $\lambda_2$ or $\lambda_{12}$ for $2 \times 2$ blocks of adjacency matrix, one can calculate the two other parameters from the equations (29) and (31).

For example the Petersen graph is strongly regular graph with parameters $(n, \kappa, \lambda, \mu) = (10, 3, 0, 1)$ (see figure 1(a)). The adjacency matrix of Petersen graph is

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

It’s clear that the blocks $A_{11}$, $A_{12}$ and $A_{22}$ are

$$A_{11} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad A_{12} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad A_{22} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}$$
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Figure 1. (a) The Petersen graph. (b) The three strata of Petersen graph, first stratum with one vertex, the second stratum with three vertices and the last stratum with six vertices.

In figure 1(b) the Petersen graph is shown in the stratification form. We choose one of the vertices as reference vertex. It will be the first stratum, then all vertices which are connected to the reference vertex are in the second stratum, and the other vertices are in the third(last) stratum. The block diagonal adjacency matrix of Petersen graph in the stratification basis from equations (27) and (30) is

\[
A = \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{2} & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
\end{pmatrix}
\]

4. Entropy of entanglement in the ground state of quantum harmonic oscillators

In order to calculate the entanglement entropy between two parts of SRGs (partitioning is between strata in SRGs, for example the vertex of first stratum and the vertices of second and third strata) in the graph, we introduce the following process: first one divide the potential matrix of the graph into three part (according to the the first $3 \times 3$ block of adjacency matrix)

\[
V = I + 2gL = \begin{pmatrix}
V_{11} & V_{12} & 0 \\
V_{21} & V_{22} & V_{23} \\
0 & V_{32} & V_{33} \\
\end{pmatrix}
\]

Then by using the generalized Schur complement method (for more details see appendix B), the potential matrix can be write
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\[
\begin{pmatrix}
V_{11} & V_{12} & 0 \\
V_{21} & V_{22} & V_{23} \\
0 & V_{32} & V_{33}
\end{pmatrix}
\]

\[
\begin{pmatrix}
I_1 & 0 & 0 \\
0 & I_2 & V_{23}V_{33}^{-1} \\
0 & 0 & I_3
\end{pmatrix}
\begin{pmatrix}
V_{11} & V_{12} & 0 \\
V_{12}^T & V_{22} - V_{23}V_{33}^{-1}V_{32} & 0 \\
0 & 0 & V_{33}^{-1}V_{32}I_3
\end{pmatrix}
\begin{pmatrix}
I_1 & 0 & 0 \\
0 & I_2 & 0 \\
0 & V_{33}^{-1}V_{32}I_3
\end{pmatrix}
\]

(33)

In the transformed matrix the blocks are scalar. So for calculating the entanglement between two subsets, it is sufficient to use the \(2 \times 2\) matrix as

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{12}^T & a_{22}
\end{pmatrix}
= \begin{pmatrix}
V_{11} & V_{12} \\
V_{12}^T & V_{22} - V_{23}V_{33}^{-1}V_{32}
\end{pmatrix}
\]

(34)

The wave function in this stage is

\[
\psi(x, y) = A_y \exp\left(-\frac{1}{2}(x, y)\begin{pmatrix}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{pmatrix}\begin{pmatrix}
x \\
y
\end{pmatrix}\right)
\]

(35)

By rescaling the variables \(x\) and \(y\):

\[
x = a_{11}^{-1/2}x
\]

\[
y = a_{22}^{-1/2}y
\]

(36)

The ground state wave function is transformed to

\[
\psi(\bar{x}, \bar{y}) = A_y \exp\left(-\frac{1}{2}(\bar{x}, \bar{y})\begin{pmatrix}
1 & d \\
d & 1
\end{pmatrix}\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix}\right)
\]

(37)

where

\[
d = a_{11}^{-1/2}a_{12}a_{22}^{-1/2}
\]

(38)

So the ground state wave function is

\[
\psi(\bar{x}, \bar{y}) = A_y e^{-\frac{\bar{x}^2}{2} - \frac{\bar{y}^2}{2} - d\bar{x}\bar{y}}
\]

(39)

From above equation, it’s clear that the node \(\bar{x}\) is just entangled with \(\bar{y}\), so one can use following identity to calculate the Schmidt number of this wave function,

\[
\frac{1}{\pi^{1/2}} \exp\left(-\frac{1 + t^2}{2(1 - t^2)}(\bar{x}^2 + (\bar{y})^2) + \frac{2t}{1 - t^2}\bar{x}\bar{y} \right) = \left(1 - t^2\right)^{1/2} \sum_n t^n \psi_n(\bar{x}) \psi_n(\bar{y})
\]

(40)

In order to calculating the entropy, we apply a change of variable as

\[
1 - t^2 = \frac{2}{\gamma + 1}
\]
\[ t^2 = \frac{\gamma - 1}{\gamma + 1} \]

So the above identity becomes

\[ \frac{1}{\pi^{1/2}} \exp\left(-\frac{\gamma}{2}((\bar{x})^2 + (\bar{y})^2) + (\gamma^2 - 1)^{1/2} \bar{x} \bar{y}\right) = \left(\frac{2}{\gamma + 1}\right)^{1/2} \sum_n \left(\frac{\gamma - 1}{\gamma + 1}\right)^{n/2} \psi_n(\bar{x}) \psi_n(\bar{y}) \]

and the reduced density matrix is

\[ \rho = \frac{2}{\gamma + 1} \sum_n \left(\frac{\gamma - 1}{\gamma + 1}\right)^n |n\rangle\langle n| \]

and the entropy is

\[ S(\rho) = \frac{\gamma + 1}{2} \log\left(\frac{\gamma + 1}{2}\right) - \frac{\gamma - 1}{2} \log\left(\frac{\gamma - 1}{2}\right) \]

By definition the scale \( \mu^2 \), we obtain

\[ \gamma = 1 \times \mu^2 \]

\[ (\gamma^2 - 1)^{1/2} = -d \times \mu^2 \]

After some straightforward calculation

\[ \gamma = \left(\frac{1}{1 - d^2}\right)^{1/2}. \]

4.1. Entanglement entropy between parts: (first stratum) and (second and third strata)

In this case only the first \( 3 \times 3 \) block of adjacency matrix plays role, which is given in equation (27). The potential matrix of this \( 3 \times 3 \) block of adjacency matrix is

\[
\begin{pmatrix}
1 + 2g\kappa & -2g\kappa & 0 \\
-2g\kappa & 1 + 2g(\kappa - \lambda) & -2g\mu \frac{\sqrt{n - \kappa - 1}}{\sqrt{\kappa}} \\
0 & -2g\mu \frac{\sqrt{n - \kappa - 1}}{\sqrt{\kappa}} & 1 + 2g\mu
\end{pmatrix}
\]

(45)

Then by substituting the elements of above potential matrix in equation (34), the parameter \( d_{1,(2,3)} \) is obtained from equation (38) as

\[ d_{1,(2,3)} = \frac{2\sqrt{\pi}\sqrt{1 + 2g\mu}}{\sqrt{1 + 2g\kappa \sqrt{(1 + 2g\mu)(1 + 2g(\kappa - \lambda)) - 4g^2\mu(\kappa - \lambda - 1)}}} \]

(46)

Therefore the entanglement entropy can be calculated from equations (44) and (43).
4.2. Entanglement entropy between parts: (first and second strata) and (third stratum)

Now we want to investigate the bipartite entanglement entropy in SRGs in the case that the vertices of first and second strata are in the first subset and the other vertices are in the second subset.

By applying the generalized Schur complement similar to equation (33), the transformed potential matrix is

$$
\begin{pmatrix}
V_{11} & 0 & 0 \\
0 & V_{22} - V_{12}^T V_{11}^{-1} V_{12} & V_{23} \\
0 & V_{23}^T & V_{33}
\end{pmatrix}
$$

Then for calculating the entanglement between the vertices of two first strata and third stratum, we can use the following $2 \times 2$ matrix:

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{12}^T & a_{22}
\end{pmatrix} = \begin{pmatrix}
V_{22} - V_{12}^T V_{11}^{-1} V_{12} & V_{23} \\
V_{23}^T & V_{33}
\end{pmatrix}
$$

The parameter $d_{(12,3)}^{(1)}$ in this case is calculated from equation (38) as

$$
d_{(12,3)}^{(1)} = \frac{2\mu \sqrt{n - \kappa - \lambda_1} - 1}{\sqrt{\kappa} \sqrt{1 + 2g\mu} \sqrt{1 + 2g\kappa}}
$$

where the index (1) shows that this parameter is obtained from the first block of adjacency matrix. The potential matrix of other $2 \times 2$ blocks of adjacency matrix can be calculated by using equation (30).

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{12}^T & a_{22}
\end{pmatrix} = \begin{pmatrix}
1 + 2g(\kappa - \lambda_1^i) & -2g\lambda_{12}^i \\
-2g\lambda_{12}^i & 1 + 2g(\kappa - \lambda_2^i)
\end{pmatrix}
$$

The parameter $d_{i}^{(2,3)}$, $i = 2, 3, \ldots, m$ ($m$ is the number of $2 \times 2$ blocks in adjacency matrix) again is calculated from equation (38):

$$
d_{i}^{(2,3)} = \frac{2g \sqrt{\lambda_{12}^i}}{\sqrt{1 + 2g(\lambda_{12}^i - \lambda_{1}^i)} \sqrt{1 + 2g(\lambda_{12}^i - \lambda_{2}^i)}}
$$

Therefore the entanglement entropy can be calculated from equations (44) and (43). The total entanglement entropy in this case is

$$
S(\rho)_T = S(\rho)_{d_{(12,3)}^{(1)}} + \sum_{i=2}^{m} S(\rho)_{d_{i}^{(2,3)}}
$$

4.3. Entanglement entropy between parts: (second stratum) and (first and third strata)

The last case is that the vertices of second stratum is in the first subset and the other vertices are in the second subset. So, we apply a permutation $1 \rightleftharpoons 2$ to the equation (45). Then we rewrite the potential matrix in this form

\[ \text{doi:10.1088/1742-5468/2015/08/P08013} \]
\begin{equation}
V = \begin{pmatrix}
\hat{V}_{11} & \hat{V}_{12} \\
\hat{V}_{12}^T & \hat{V}_{22}
\end{pmatrix}
\end{equation}

where

\[
\hat{V}_{11} = 1 + 2g(\kappa - \lambda), \quad \hat{V}_{12} = \left\{-2g\sqrt{\kappa} - 2g\mu\frac{\sqrt{n - \kappa - 1}}{\sqrt{\kappa}}\right\}, \quad \hat{V}_{22} = \begin{pmatrix}
1 + 2g\kappa & 0 \\
0 & 1 + 2g\mu
\end{pmatrix}
\]

By applying a change of variable similar to equation (36), the above elements of potential matrix will be transformed as following

\[
\hat{V}_{11} = 1, \quad \hat{V}_{12} = \hat{V}_{11}^{-\frac{1}{2}}\hat{V}_{12}\hat{V}_{22}^{-\frac{1}{2}} = \frac{1}{\sqrt{1 + 2g(\kappa - \lambda)}}\begin{pmatrix}\end{pmatrix}
\]

The parameter \(d^{(1)}_{(13,2)}\) is obtained by calculating the singular value decomposition of above matrix \(\hat{V}_{12}\), as following

\[
d^{(1)}_{(13,2)} = 2g\sqrt{\frac{\kappa}{(1 + 2g\kappa)(1 + 2g(\kappa - \lambda))} + \frac{\mu(\kappa - \lambda - 1)}{(1 + 2g\mu)(1 + 2g(\kappa - \lambda))}}
\]

The parameter \(d^{(2)}_{(2,3)}\) can be calculated from equation (50) again and the entanglement entropy can be calculated from equations (44) and (43).

Similar to the previous case, the total entanglement entropy is

\[
S(\rho)_T = S(\rho)_{d^{(1)}_{(1,3)}} + \sum_{i=2}^{m} S(\rho)_{d^{(i)}_{(2,3)}}
\]

5. Constructing some important kinds of SRG classes which don’t contain nonisomorph SRGs

In this section we want to study some kinds of SRGs. We identify their adjacency matrices in the stratification basis. For these kinds of SRGs, there are not any nonisomorph SRG, because all blocks of their adjacency matrices are in terms of the parameters of SRG: \((n, \kappa, \lambda, \mu)\)

5.1. Normal subgroup graph \((2m, m, 0, m)\)

Let \(G\) be a finite group, and \(P = P_0, P_1, ..., P_d\) be a blueprint of it. we assume that the sets \(P_i\) are so numbered that the identity element \(e\) of \(G\) belongs to \(P_0\), if \(P_0 = e\), the \(P\) is called homogeneous. Let \(R_0, R_1, ..., R_d\) be the set of relations \(R_i = (\alpha, \beta) \in G \otimes G|\alpha^{-1}\beta \in P_i\) on \(G\). Now, we define a blueprint for group \(G\) which form a strongly regular graph. If \(H\) is a subgroup of \(G\), we define the blueprints by
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\[ P_0 = e, P_1 = G - H, p_2 = H - e \] (55)

This blueprint form a strongly regular graph with parameters \((n, \kappa, \lambda, \mu) = (|G|, |G|-|H|, |G|-2|H|, |G|-|H|)\). As an example, we consider \(G = D_{2m}(m = \text{odd})\)

\[ H = e, a, a^{-1}, \ldots, a^{(m-1)/2}, a^{-(m-1)/2} \] (56)

Therefore the blueprints are given by

\[ P_0 = e, P_1 = b, ab, a^2b, \ldots, a_{m-1}b, P_2 = e, a, a^{-1}, \ldots, a^{(m-1)/2}, a^{-(m-1)/2} \] (57)

which form a strongly regular graphs with parameters \((2m, m, 0, m)\). The first \(3 \times 3\) block of adjacency matrix from equation (27) is

\[
\begin{pmatrix}
0 & \sqrt{m} & 0 \\
\sqrt{m} & 0 & \sqrt{m(m-1)} \\
0 & \sqrt{m(m-1)} & 0
\end{pmatrix}
\] (58)

From equation (17), we conclude that the block \(A_{11}\) for this kind of SRG, is the zero matrix. So all of it’s eigenvalues are zero, i.e. \(\lambda_j = 0\) for all \((i = 1, 2, \ldots, \kappa)\). Then by substituting this in equations (29) and (31), we conclude that \(\lambda_1^{i=1} = -m\) and \(\lambda_2^{i=1} = 0\). So these kinds of SRG, don’t have the \(2 \times 2\) blocks of adjacency matrices and they have only singlets: \(-m\). So these kinds of SRGs are not nonisomorph with any other SRG with the same parameters. The total entanglement entropies for each kinds of explained partitioning, are obtained from equations (46), (48) and (53).

5.2. \(\kappa = \mu\) which leads to: \((2k - \lambda, k, \lambda, k)\)

For the case \(\kappa = \mu\) by using the equation (5), we conclude that

\[ n = 2k - \lambda \] (59)

So, the parameters are \((2k - \lambda, k, \lambda, k)\) and from equation (19),

\[ A_{22} = 0 \]

also from equation (20),

\[ A_{12}^T A_{12} = \kappa J_{k'k'} \]

So, \(A_{12} = J_{k'k'}\). So by considering the singular value decomposition of \(A_{12}\), we conclude that \(\lambda_{12}^{(1)} = \sqrt{k} \sqrt{n - \kappa - 1}\) and \(\lambda_{12}^{(i=1)} = 0\) Therefore this case don’t have the \(2 \times 2\) blocks of adjacency matrix. It has singlets in the form of \(\lambda - \kappa\).

Therefore we conclude that the Schmidt number and entanglement entropy is obtained from the first \(3 \times 3\) block of adjacency matrix.

\[
\begin{pmatrix}
0 & \sqrt{\kappa} & 0 \\
\sqrt{\kappa} & \lambda & \sqrt{\kappa(\kappa - \lambda - 1)} \\
0 & \sqrt{\kappa(\kappa - \lambda - 1)} & 0
\end{pmatrix}
\] (60)
It’s clear that the entanglement entropy can not distinguish two non-isomorphic graphs of these kinds.

5.3. $\lambda = 0$ which leads to $\left( \frac{k(k-1)}{\mu} + k + 1, k, 0, \mu \right)$

In this case, from equation (17), we find that

$$A_{11} = 0$$

The first $3 \times 3$ block of adjacency matrix from equation (27) is

$$\begin{pmatrix}
0 & \sqrt{\kappa} & 0 \\
\sqrt{\kappa} & 0 & \sqrt{\kappa(\kappa-1)} \\
0 & \sqrt{\kappa(\kappa-1)} & \kappa - \mu
\end{pmatrix}$$

(61)

So by substituting $\lambda_{i}^{i} = 0, (i = 1, 2, \ldots, \kappa)$ in equation (31)

$$\lambda_{12}^{(i=1)} = \kappa - \mu$$

(62)

We have explained the case $\kappa = \mu$ in the previous example, so we suppose that $\kappa \neq \mu$, therefore $\lambda_{12} \neq 0$. So from equation (29) we find that

$$\lambda_{2} = \lambda - \mu = -\mu$$

In this case also the Schmidt number is related to parameters of SRG, so it can not distinguish non-isomorph graphs similar to previous example.

The parameter $d_{(2,3)}^{(i=1)}$ for these kind of graphs become

$$d_{(2,3)}^{(i=1)} = \frac{2g\sqrt{\kappa - \mu}}{\sqrt{1 + 2g\kappa}\sqrt{1 + 2g(\kappa - \mu)}}$$

(63)

5.4. $A_{12}A_{12}^{T} = \kappa J_{\kappa\kappa}$

Now we want to investigate the SRG graphs which their $A_{12}$ is $\kappa \times 1$ dimensional complete graph. So

$$A_{12}^{T}A_{12} = \kappa$$

(64)

In this case $A_{22} = 0$ is a $1 \times 1$ scalar, so $\lambda_{2}^{(1)} = 0, n - \kappa - 1 = 1$ and from equation (19), $\kappa - \mu = 0$. Therefore we conclude that

$$n = \kappa + 2$$

$$\kappa = \mu$$

Then from equation (5):

$$\lambda = \kappa - 2$$

(65)
Therefore the parameters of this case will be:
\[(\kappa + 2, \kappa, \kappa - 2, \kappa)\]

We know that it is possible to write the matrices \(A_{11}\) and \(A_{22}\) in terms of the matrix representations of permutation group, so suppose
\[A_{11} = J - I - \pi\]
\[A_{11}^2 = (\kappa - 4)J + I + \pi^2 + 2\pi\]

Also from equation (21), we find that
\[A_{11}^2 = (\kappa - 2)J - 2A_{11}\]

After comparing the two above equation
\[\pi^2 = I\]

We conclude that \(\pi\) is an element of cycle group with order two, therefore the parameter \(\kappa\) can not be odd. By substituting the parameters of these kinds of SRG into equation (6) we find
\[A^2 = \kappa J - 2A\]

By comparing this relation for \(A^2\) with relation for \(A_{11}^2\) it can be concluded that the matrix \(A_{11}\) of this graph for the case with degree \(\kappa\), is the adjacency matrix of these kinds of graphs with degree \(\kappa - 2\). These kinds of graphs don’t contain the non-isomorph graphs.

6. Some important scalable sets of SRGs which contain nonisomorph SRGs

The eigenvalues of adjacency matrices of strongly regular graphs only depend on the parameters of graph. So the non-isomorph SRGs with the same parameters have the same eigenvalues, i.e. they are cospectral. In fact their energies of ground states are identical, but their ground states wave functions are different, because the potential matrices of graphs are obtained from adjacency matrices of graphs and the adjacency matrices are different for non-isomorph graphs. For some important scalable kinds of SRGs, we identify their adjacency matrices in the stratification basis analytically. Then we investigate the graph isomorphism problem by using the entanglement entropies of sections 4.2 and 4.3. In the entanglement entropies of these sections, the \(2 \times 2\) blocks of adjacency matrices play an important role. The eigenvalues of \(A_{12}\), i.e. \(\lambda_{12}\) are different for some non-isomorph SRGs. The entanglement entropies which are obtained from parameters \(d_{12,3}^{(1)}, d_{2,13}^{(1)}\), are identical for non-isomorph SRGs. But the entanglement entropies obtained from \(d_{2,3}^{(1,\neq 1)}\), are different for non-isomorph SRGs. Therefore the total entanglement entropy of sections 4.2 and 4.3 can distinguish non-isomorph SRGs.
6.1. Triangular (Johnson) graph \( \left( \frac{\nu(\nu - 1)}{2}, 2(\nu - 2), \nu - 2, 4 \right) \)

For positive integer \( \nu \) the triangular graph \( T_n \) is strongly regular graph. As the construction is completely symmetric, we may begin by considering any vertex, say the one labeled by the set \((1, 2)\). Every vertex labeled by a set of form \((1, i)\) or \((2, i)\), for \( i \geq 3 \), will be connected to this set. So, this vertex, and every vertex, has degree \( 2(\nu - 2) \). For any neighbor of \((1, 2)\), say \((1, 3)\), every other vertex of form \((1, i)\) for \( i \geq 4 \) will be a neighbor of both of these, as will the set of \((2, 3)\). Carrying this out in general, we find that \( \lambda = \nu - 2 \). Finally any non-neighbor of \((1, 2)\), say \((3, 4)\), will have 4 common neighbors with \((1, 2)\). So, \( \mu = 4 \) and \( n = \binom{\nu}{2} \).

In triangular graph the \( A_{11} \) is defined as following form:
\[
A_{11} = I_2 \otimes (J_{\nu - 2} - I_{\nu - 2}) + X \otimes I_{\nu - 2} 
\]  
where \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

And the eigenvalues of \( A_{11} \) is
\[
\lambda_1^{(i)} = \nu - 2, \nu - 4, \frac{\nu - 3}{0}, -2
\]
\( \nu - 2 \) is the biggest eigenvalue. From equation (27), we can calculate the \( 3 \times 3 \) block of adjacency matrix as
\[
\begin{pmatrix}
0 & \sqrt{2(\nu - 2)} & 0 \\
\sqrt{2(\nu - 2)} & \nu - 2 & 2\sqrt{(\nu - 3)} \\
0 & 2\sqrt{(\nu - 3)} & 2(\nu - 4)
\end{pmatrix}
\]  
(67)

By substituting the values of \( \lambda_1^{(i)} \) in the equations (29) and (31), we can calculate other blocks of adjacency matrix of triangular graph as \( \lambda_1 = 0 \).
\[
\begin{pmatrix}
0 & \sqrt{2(\nu - 4)} \\
\sqrt{2(\nu - 4)} & \nu - 6
\end{pmatrix}
\]  
(68)

The eigenvalues \( \nu - 4 \) and \(-2\) are singlets. The parameters \( d_{12,3}^{(1)}, d_{2,13}^{(1)} \) and \( d_{2,3}^{(i=1)} \) can be calculated from equations(48), (50) and (53). Then the entanglement entropy can be obtained from equations (43) and (44).

The strongly regular graph with parameters \((28, 12, 6, 4)\) have 4 non-isomorphic graphs that one of them is Triangular graph.

6.2. Lattice graphs \((\nu^2, 2(\nu - 1), \nu - 2, 2)\)

For positive integer \( \nu \), the lattice graph \( L_n \) is the graph with vertex set \( 1, \ldots, \nu^2 \) in which vertex \((a, b)\) is connected to vertex \((c, d)\) if \( a = c \) or \( b = d \). Thus the vertices may be
In investigating graph isomorphism problem via entanglement entropy in strongly regular graphs, graphs are arranged at the points in an \( \nu \)-by-\( \nu \) grid, with vertices being connected if they lie in the same row or column. It is routine to see that the parameters of this graph are:

\[
\kappa = 2(\nu - 1), \quad \lambda = \nu - 2, \quad \mu = 2
\]

(69)

In lattice graph adjacency matrix is

\[
A = I_{\nu} \otimes (J_{\nu} - I_{\nu}) + (J_{\nu} - I_{\nu}) \otimes I_{\nu}
\]

(70)

The \( A_{11} \) is written as following

\[
A_{11} = I_{2} \otimes (J_{\nu-1} - I_{\nu-1})
\]

(71)

and the eigenvalues of \( A_{11} \) are

\[
\lambda_{i} = \nu - 2, \quad \frac{2(\nu - 2)}{-1}
\]

\( \nu - 2 \) is the biggest eigenvalue. From equation (27), we can calculate the \( 3 \times 3 \) block of adjacency matrix as

\[
\begin{pmatrix}
0 & \sqrt{2(\nu - 1)} & 0 \\
\sqrt{2(\nu - 1)} & \nu - 2 & \sqrt{2(\nu - 1)} \\
0 & \sqrt{2(\nu - 1)} & 2(\nu - 2)
\end{pmatrix}
\]

(72)

By substituting \( \lambda_{1} = -1 \) in the equations (29) and (31), we can calculate other blocks of adjacency matrix of Lattice graph

\[
\begin{pmatrix}
-1 & \sqrt{(\nu - 1)} \\
\sqrt{(\nu - 1)} & \nu - 3
\end{pmatrix}
\]

(73)

The parameters \( d_{12,3}^{(1)}, d_{2,13}^{(1)} \) and \( d_{2,3}^{1 \times 1} \) can be calculated from equations (48), (50) and (53). Then the entanglement entropy can be obtained from equations (43) and (44).

The strongly regular graphs with parameters \( (16, 6, 2, 2) \) have 2 non-isomorphic graphs that one of them is lattice graph.

### 6.3. Latin square graphs \( (\nu^2, 3(\nu - 1), \nu, 6) \)

A Latin Square is an \( \nu \)-by-\( \nu \) grid, each entry of which is a number between 1 and \( \nu \), such that no number appears twice in any row or column. So, it will have \( \nu^2 \) nodes, one for each cell in the square. Two nodes are joined by an edge if

1. They are in the same row,
2. They are in the same column, or
3. They hold the same number.
So, such a graph has degree $\kappa = 3(\nu - 1)$. Any two nodes in the same row will both be neighbors with every other pair of nodes in their row. They will have two more common neighbors: The nodes in their columns holding the other’s number. So, they have $\nu$ common neighbors. The same obviously holds for columns, and is easy to see for nodes that have the same number. Therefore, every pair of nodes that are neighbors have exactly $\lambda = \nu$ common neighbors. On the other hand, consider two vertices that are not neighbors, they lie in different rows, lie in different columns, and hold different numbers. So, $\mu = 6$.

In Latin square graph adjacency matrix is

$$A = I_\nu \otimes (J_\nu - I_\nu) + (J_\nu - I_\nu) \otimes I_\nu + \sum_{k=1}^{\nu} S^k \otimes S^{n-k}$$  \hspace{1cm} (74)

where $S$ is shift operator. The $A_{11}$ is in the following form

$$A_{11} = I_3 \otimes (J_{3-1} - I_{3-1}) + (J_3 - I_3) \otimes f_{\nu -1}$$  \hspace{1cm} (75)

where $f_\nu = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix}$ is off-diagonal matrix. If $\nu = 2l$, i.e.($\nu$ is even), the eigenvalues of $A_{11}$ is

$$\lambda_1 = \nu, \nu - 3, 1, 0, -2, -3$$  \hspace{1cm} (76)

$\nu$ is the biggest eigenvalue. So the $3 \times 3$ block of adjacency matrix is

$$\begin{pmatrix} 0 & \sqrt{3(\nu - 1)} & 0 \\ \sqrt{3(\nu - 1)} & \nu & 2\sqrt{3(\nu - 2)} \\ 0 & 2\sqrt{3(\nu - 2)} & 3(\nu - 3) \end{pmatrix}$$  \hspace{1cm} (76)

By using the equation (31), we see that $\lambda_1 = \nu - 3, -3$ are singlets. So, we can calculate other $2 \times 2$ blocks of Latin square graph by considering $\lambda_1 = 1, 0, -2$

$$\begin{pmatrix} 1 & \sqrt{4(\nu - 4)} \\ \sqrt{4(\nu - 4)} & \nu - 7 \end{pmatrix}$$  \hspace{1cm} (77)

and

$$\begin{pmatrix} 0 & \sqrt{3(\nu - 3)} \\ \sqrt{3(\nu - 3)} & \nu - 6 \end{pmatrix}$$  \hspace{1cm} (78)

and

$$\begin{pmatrix} -2 \sqrt{(\nu - 1)} \\ \sqrt{(\nu - 1)} & \nu - 4 \end{pmatrix}$$  \hspace{1cm} (79)
If $\nu = 2l + 1$ i.e. ($\nu$ is odd), the eigenvalues of $A_{11}$ are

$$\lambda_i = \nu, \nu - 3, \frac{\nu - 3}{1}, \frac{\nu - 1}{-2}, \frac{i}{0}, \frac{i}{-3}$$

The blocks of this case ($\nu$ is odd) are the same as even $\nu$.

The strongly regular graphs with parameters $(16, 9, 4, 6)$ have 2 non-isomorphic graphs and $(25, 12, 5, 6)$ have 15 non-isomorphic graphs and $(49, 18, 7, 6)$ have 147 non-isomorphic graphs that one of them is Latin square graph.

6.4. Generalized quadrangle $GQ(s, t)$, $((st + 1)(s + 1), s(t + 1), s - 1, t + 1)$

A generalized quadrangle $GQ(s, t)$ is an incidence structure of points and lines with the following properties.

1. Every line has $s + 1$ points and every point is on $t + 1$ lines.

2. Any two distinct points are incident with at most one line.

3. Given a line $L$ and a point $p$ not on $L$, there is a unique point on $L$ collinear with $p$ (two points are said to be collinear if there is a line incident with both).

Its strongly regular graph’s parameter set is $((st + 1)(s + 1), s(t + 1), s - 1, t + 1)$. Necessary conditions for existence of a $GQ(s, t)$ are $1 \leq t \leq s^2$ if $s > 1$, and $s + t$ divides $st(s + 1)(t + 1)$. The $A_{11}$ is

$$A_{11} = I_{t+1} \otimes (J_s - L_s)$$

$$\lambda_i = \frac{s - 1}{-1}, \frac{s}{-1}$$

$s - 1$ is the biggest eigenvalue. So the $3 \times 3$ block of adjacency matrix is

$$\begin{pmatrix}
0 & \sqrt{s(t + 1)} & 0 \\
\sqrt{s(t + 1)} & s - 1 & \sqrt{st(t + 1)} \\
0 & \sqrt{st(t + 1)} & (s - 1)(t + 1)
\end{pmatrix}$$

By equation (31), we can calculate other blocks of generalized quadrangle graph. The other blocks are

$$\begin{pmatrix}
-1 & \sqrt{st} \\
\sqrt{st} & s - t - 1
\end{pmatrix}$$

The parameters $d_{12,3}^{(1)}, d_{2,13}^{(1)}$ and $d_{2,3}^{(1)}$ can be calculated from equations (48), (50) and (53). Then the entanglement entropy can be obtained from equations (43) and (44).

The strongly regular graphs with parameters $(40, 12, 2, 4)$ have 28 non-isomorphic graphs and $(45, 12, 3, 3)$ have 78 non-isomorphic graphs and $(64, 18, 2, 6)$ have 167 non-isomorphic graphs that one of them is Generalized Quadrangle graph.
7. Entanglement entropy in the large coupling limit

In this section, our derivation is based on the entanglement entropy for large coupling strength. We can rewrite the \(d_{(1,23)}\) from equation (46) as following

\[
d_{(1,23)}^{(1)} = \frac{1}{\sqrt{1 + \frac{1}{2g\kappa} \left( \frac{1}{2g} + \kappa - \frac{2g\kappa(n - \kappa - 1)}{\kappa(1 + 2g\kappa)} \right)}}
\]

where \(\frac{2g\kappa(n - \kappa - 1)}{\kappa(1 + 2g\kappa)} \sim \frac{n(n - \kappa - 1)}{\kappa} \left( 1 - \frac{1}{2g\kappa} \right)\)

Therefore, by using equation (4) we have

\[
d_{(1,23)}^{(1)} \simeq \frac{1}{\sqrt{1 + \frac{1}{2g\kappa} \left( \frac{1}{2g} + \frac{n - 1}{2g\kappa} \right)}} \simeq 1 - \frac{1}{2} \varepsilon
\]

and \(\varepsilon = \frac{1}{2g\kappa} + \frac{n - 1}{2g\kappa} = \frac{n}{2g\kappa}\)

By equation (44), we can write

\[
\gamma = \frac{1}{\sqrt{1 - \left(1 - \frac{1}{2}\varepsilon\right)^2}} \simeq \frac{1}{\sqrt{\varepsilon}} = \sqrt{\frac{2g\kappa}{n}} \quad (83)
\]

Then from equation (43)

\[
S(\rho) = \frac{\gamma}{2} \left(1 + \frac{1}{\gamma}\right) \log \frac{\gamma}{2} \left(1 + \frac{1}{\gamma}\right) - \frac{\gamma}{2} \left(1 - \frac{1}{\gamma}\right) \log \frac{\gamma}{2} \left(1 - \frac{1}{\gamma}\right)
\]

\[
= \frac{1}{2} \left( (\gamma + 1) \left(\log \frac{\gamma}{2} + \frac{1}{\gamma}\right) - \frac{1}{2} (\gamma - 1) \left(\log \frac{\gamma}{2} - \frac{1}{\gamma}\right) \right) \quad (84)
\]

So

\[
S(\rho)_{1,23} = \log \frac{\gamma}{2} + 1 = \frac{1}{2} \log \frac{g\kappa}{2n} + 1 \quad (85)
\]

where \(\kappa\) is the size of the boundary between the first and the second subsets. So, we see that the entanglement entropy has a logarithmic relation with the ratio of size of boundary to the size of the system. Also we can rewrite the \(d_{(12,3)}^{(1)}\) from equation (48) by using equation (4), as following

\[
d_{(12,3)}^{(1)} = \frac{1}{\sqrt{1 + \frac{1}{2g\kappa} \left( \frac{1}{2g(\kappa - \lambda - 1)} + \frac{\kappa - \lambda}{\kappa - \lambda - 1} \frac{2g\kappa}{\kappa - \lambda - 1 + 2g\kappa} \right)}}
\]

where \(\frac{2g\kappa}{(\kappa - \lambda - 1)(1 + 2g\kappa)} \simeq \frac{1}{(\kappa - \lambda - 1) \left(1 - \frac{1}{2g\kappa}\right)}\)
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So

\[ d_{(12,3)}^{(1)} \simeq \frac{1}{\sqrt{1 + \frac{1}{2\mu} \sqrt{1 + \frac{\kappa + 1}{2\mu(n - \kappa - 1)}}}} = 1 - \frac{1}{2} \varepsilon \]

and \( \varepsilon = \frac{1}{2\mu} + \frac{\kappa + 1}{2\mu(n - \kappa - 1)} = \frac{n}{2\mu(n - \kappa - 1)} \)

Similar to the case of \( d_{1,23} \), by using equations (84) and (85) we have

\[ S(\rho)_{12,3} = \log \frac{\gamma}{2} + 1 = \frac{1}{2} \log \frac{\mu g(n - \kappa - 1)}{2n} + 1 \quad (86) \]

where \( \mu(n - \kappa - 1) \) is the size of boundary.

Also it’s interesting to study the behavior of entanglement entropy of two non-isomorph graphs when coupling strength tends to infinite. We know that the first \( 3 \times 3 \) block of adjacency matrices are the same for non-isomorph graphs, because it is in terms of parameters of SRG. But the other \( 2 \times 2 \) blocks of adjacency matrices are different for non-isomorph graphs. By some simple calculations, it can be shown that \( \lim_{g \to \infty} d_{1,23} = \lim_{g \to \infty} d_{(12,3)} = \lim_{g \to \infty} d_{(2,13)} = 1 \). So the entanglement entropy from equations (44) and (43) tends to infinite. But \( \lim_{g \to \infty} d_{2,3}^{(1)} < 1 \). So the entanglement entropy of other \( 2 \times 2 \) blocks of adjacency matrix will be finite but the total entanglement entropy for each of explained partitioning in sections 4.1–4.3 tend to infinite. Therefore the behavior of entanglement entropies of non-isomorph SRGs for \( g \to \infty \) are the same. It shows that when the coupling strength tends to infinite, the entanglement entropy can not distinguish non-isomorph SRGs.

8. Area-law

Entanglement entropy is a quantitative measure of the quantum entanglement. A natural problem then, is to divide the system into two regions and study the entanglement entropy between them. In general, short-range correlations, which are non-universal, give a contribution proportional to the area of the boundary between the two regions to the entanglement entropy. This is often called as the area-law contribution. In one dimension, the area-law contribution is constant with respect to the system size or to the size of the regions. For a regular lattice, the size of the boundary of an element is given by twice its dimensionality thus, in analogy, for a node in a complex network its boundary is given by its connectivity.

In our system, the area law is studied in bipartite systems. Two cases will be choose

Case I: \( \mu \) is finite and \( \lambda, \kappa \) are infinite.

When \( \lambda, \kappa \) are infinite, it means that the size of the system is infinite. The parameter \( \gamma \) from equation (44) can be written as
By finite $\mu$, in highest connectivity, the statement \( \frac{4\kappa^2(1 + 2g_1^2)}{1 + 4\kappa^2\mu + 2g_1(\kappa - \lambda + \mu)} \) tends to zero and parameter $\gamma$ tends to one and the entanglement entropy $S(\rho_{(1,23)}) \to 0$. In the large size of the system, there is no entanglement between strata.

**Case II**: $\lambda$ is finite and $\kappa = \mu$ is infinite

In this case, the parameter $\gamma$ from equation (44) can be written as

\[
\gamma_{(1,23)} = \sqrt{\frac{1 + 2g_1\kappa}{1 + 2g_1\kappa - \frac{4g_1^2(1 + 2g_1\kappa)}{1 + 4g_1^2\mu + 2g_1(\kappa - \lambda + \mu)}}}
\]  

Also, in this case the parameter $\gamma$ tends to one and the entanglement entropy $S(\rho_{(1,23)}) \to 0$. So, there is no entanglement between strata.

**9. Numerical investigation of graph isomorphism problem in SRGs**

Two graphs will be isomorphic, when those are related to each other by a relabeling of vertices. Here, we want to investigate the graph isomorphism problem by using different eigenvalues of the matrix $A_{12}$ or the total entanglement entropies of 4.2 and 4.3. Our method can distinguish non-isomorphic graphs with simple method.

There are some non-isomorphic SRGs with the same parameters, which their $\lambda_{12}$s are different. Our numerical results are in tables 1 and 2.

**10. Conclusion**

For calculating the entanglement entropy and investigating the graph isomorphism problem in SRGs, we rewrote the adjacency matrices of strongly regular graphs in the stratification basis. In this basis the adjacency matrix became block diagonal. Rewriting the adjacency matrix in the stratification basis simplified the calculating of entanglement entropy and investigating GI problem. Then the Schur complement method was used for calculating the Schmidt number and entanglement entropy between two parts (various partitioning of strata) of strongly regular graphs. In the new basis, the adjacency matrices of SRGs were decomposed to a $3 \times 3$ block and some $2 \times 2$ blocks and some singlets. The $3 \times 3$ blocks of all SRGs were obtained in terms of parameters of SRGs, so they didn’t play any role in distinguishing non-isomorph pairs of SRGs. we
could obtain some important relations for the elements of other blocks of adjacency matrices. These blocks are useful in distinguishing non-isomorph pairs of SRGs because the elements of $2 \times 2$ blocks of adjacency matrices in the stratification basis are different for two non-isomorph SRGs with the same parameter, so the entanglement entropies

Table 1. Some non-isomorph SRGs with their singular value decomposition of $A_{12}$.

| SRG family $(n, \kappa, \lambda, \mu)$ | No. of different eigenvalues of $A_{12}$ | Singular value decomposition of $A_{12}$ |
|--------------------------------------|----------------------------------------|----------------------------------------|
| (25, 12, 5, 6)                      | 6                                      | $\lambda_{12}(1) = 6, 2.4495(3), 2.3268, 2.1753, 2, 1.6080, 1.1260, 0(3)$ |
|                                      |                                        | $\lambda_{12}(2) = 6, 2.4495(4), 1.7321(4), 0(3)$ |
|                                      |                                        | $\lambda_{12}(3) = 6, 2.4495(4), 2.1753(2), 1.1260(2), 0(3)$ |
|                                      |                                        | $\lambda_{12}(4) = 6, 2.4953(2), 2.2770(2), 2(3), 0.7672(2), 0(2)$ |
|                                      |                                        | $\lambda_{12}(5) = 6, 2.4495(4), 2(3), 0(4)$ |
|                                      |                                        | $\lambda_{12}(6) = 6.2(9), 0(2)$ |
| (26, 10, 3, 4)                      | 5                                      | $\lambda_{12}(1) = 4.8990, 2.4972, 2.3073, 2.2361(4), 1.3556, 1.3281, 0.5645$ |
|                                      |                                        | $\lambda_{12}(2) = 4.8990, 2.495(2), 2(6), 0$ |
|                                      |                                        | $\lambda_{12}(3) = 4.8990, 2.494(2), 2.4812, 2.1342(2), 1.7883(2), 1.1701, 0.6889$ |
|                                      |                                        | $\lambda_{12}(4) = 4.8990, 2.4953(2), 2.2770(2), 2(3), 0.7672(2)$ |
|                                      |                                        | $\lambda_{12}(5) = 4.8990, 2.4495(4), 2(3), 0(2)$ |
| (28, 12, 6, 4)                      | 4                                      | $\lambda_{12}(1) = \sqrt{20}, 2.9356(2), 2.5263(2), 2.2361(2), 0(5)$ |
|                                      |                                        | $\lambda_{12}(2) = \sqrt{20}, \sqrt{8}(5), 0(6)$ |
|                                      |                                        | $\lambda_{12}(3) = \sqrt{20}, \sqrt{8}(4), 2(2), 0(5)$ |
|                                      |                                        | $\lambda_{12}(4) = \sqrt{20}, \sqrt{8}(2), 2.4495(4), 0(5)$ |
| (36, 14, 4, 6)                      | 3                                      | $\lambda_{12}(1) = 7.3485, 2.9849, 2.9832, 2.9713, 2.9244, 2.8810, 2.7777, 2.6722, 2.2143, 2.1213, 1.7809, 0.7695, 0.6985, 0(4)$ |
|                                      |                                        | $\lambda_{12}(2) = 7.3485, 2.9713(4), 2.8284(3), 1.7809(4), 0(2)$ |
|                                      |                                        | $\lambda_{12}(3) = 7.3485, 2.9863, 2.9785, 2.9356, 2.9173, 2.7501, 2.5354, 2.5263, 2.3189, 2.2998, 2.0165, 1.2072, 0.7204, 0$ |
| (40, 12, 2, 4)                      | 3                                      | $\lambda_{12}(1) = 6.3(4), 2.8284(2), 2.2361(4), 0$ |
|                                      |                                        | $\lambda_{12}(2) = 6.3(6), 2.8284, 2.2361(2), 0(2)$ |
|                                      |                                        | $\lambda_{12}(3) = 6.3(8), 0(3)$ |

Note: The numbers inside the parentheses show the degeneracy of the singular value.
between two kinds of partitioning of strata, are different for those non-isomorph SRGs. By this method, we could distinguish some non-isomorphic pairs of SRGs, easily. Also in four important scalable sets of SRGs (Triangular graph, Lattice graph, Latin Square graph and generalized quadrangle graph), all blocks of adjacency matrices could be found analytically. More, the relationship between size of the boundary of strata and entanglement entropy is obtained in the limit of large coupling.

In our model the vertices of graphs are considered as quantum harmonic oscillators. So the ground state of wave function was a gaussian state. one expects that our methods (stratification basis and the generalized Schur complement method) can be used for calculating entanglement entropy in the excited states of quantum harmonic oscillator and other quantum models.

The other aim is that the considered techniques, be generalized to other kinds of graphs such as association schemes. It is under investigation for some distance regular graphs.

| SRG family \( (n, \kappa, \lambda, \mu) \) | No. of different \( \lambda \) | Singular value decomposition of \( A_{12} \) |
|---|---|---|
| 50, 21, 8, 9 10 | \( \lambda_{12}(1) = 10.3923, 3.4971, 3.4681, 3.4568, 3.4514, 3.3753, 3.2582, 3.1566, 3.1279, 2.9672, 2.9551, 2.8718, 2.6479, 2.4173, 2.1405, 1.9254, 1.7811, 1.2576, 1.2507, 0.9902, 0.1757 \) | 2.1026, 1.9616, 1.7264, 1.6747, 1.1559, 0.8272, 0.2970 |
| | \( \lambda_{12}(2) = 10.3923, 3.4998, 3.4873, 3.4097, 3.4095, 3.3353, 3.3019, 3.2827, 3.2455, 2.9175, 2.8205, 2.7827, 2.6663, 2.3496 \) | |
| | \( \lambda_{12}(3) = 10.3923, 3.5, 3.4826, 3.4790, 3.4612, 3.3755, 3.2339, 3.1841, 3.1585, 3.0605, 3.0311, 2.5686, 2.5446, 2.4968, 2.3267, 2.2466, 1.3636, 1.3571, 0.9672, 0.7667, 0.1583 \) | |
| | \( \lambda_{12}(4) = 10.3923, 3.4963, 3.4945, 3.4877, 3.47, 3.4433, 3.2566, 3.1966, 3.1858, 2.9511, 2.9340, 2.5904, 2.5879, 2.3506, 1.9055, 1.9489, 1.8770, 1.7911, 1.1268, 0.8170, 0.0947 \) | |
| | \( \lambda_{12}(5) = 10.3923, 3.4757, 3.4589, 3.4482, 3.3634, 3.3024, 3.2533, 3.1747, 3.0016, 2.9364, 2.7631, 2.7481, 2.6822, 2.6359, 2.4051, 1.9295, 1.8550, 1.7033, 1.0740, 1.0527, 0.3333 \) | |
| | \( \lambda_{12}(6) = 10.3923, 3.4978, 3.4826, 3.4741, 3.3731, 3.2942, 3.2315, 3.2165, 3.1663, 2.9801, 2.9188, 2.5423, 2.5208, 2.2940, 2.2563, 2.0890, 1.8087, 1.7151, 1.1861, 1.1438, 0.8130 \) | |
| | \( \lambda_{12}(7) = 10.3923, 4.0749, 3.4989, 3.4975, 3.4216, 3.3640, 3.2608, 3.2068, 3.0944, 2.8368, 2.7623, 2.6790, 2.6558, 2.1554, 2.0768, 1.9787, 1.7463, 1.2244, 0.9112, 0.6396, 0 \) | |
| | \( \lambda_{12}(8) = 10.3923, 3.4641(4), 3.1623(6), 2.4495(6), 0(4) \) | |
| | \( \lambda_{12}(9) = 10.3923, 3.4998, 3.4835(2), 3.3621, 3.3535(2), 3.1446(2) \) | |
Appendix A. Schur complement method

Let $M$ be an $n \times n$ matrix written as a 2×2 block matrix

$$
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

where $A$ is a $p \times p$ matrix and $D$ is a $q \times q$ matrix, with $n = p + q$ (so, $B$ is a $p \times q$ matrix and $C$ is a $q \times p$ matrix). We can try to solve the linear system

$$
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}
$$

that is

$$
Ax + By = c \\
Cx + Dy = d
$$

by mimicking gaussian elimination, that is, assuming that $D$ is invertible, we first solve for $y$ getting

$$
y = D^{-1}(d - Cx)
$$

and after substituting this expression for $y$ in the first equation, we get

$$
Ax + B(D^{-1}(d - Cx)) = c
$$

and

$$
(A - BD^{-1}C)x = c - BD^{-1}d
$$

The invertible matrix, $A - BD^{-1}C$, is called the Schur complement of $D$ in $M$.

Appendix B. Stratification

For an underlying network $\Gamma$, let $W = \mathbb{C}^n$ (with $n = |V|$) be the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by vertex set $V$ of $\Gamma$, and whose entries are in $\mathbb{C}$. For all $\beta \in V$, let $|\beta|$ denotes the element of $W$ with a 1 in the $\beta$ coordinate and 0 in all other coordinates. We observe $\{|\beta|: \beta \in V\}$ is an orthonormal basis for $W$, but in this basis, $W$ is reducible and can be reduced to irreducible subspaces $W_i$, $i = 0, 1, ..., d$, i.e.

$$
W = W_0 \oplus W_1 \oplus ... \oplus W_d,
$$

where, $d$ is diameter of the corresponding association scheme. If we define $\Gamma_i(o) = \{\beta \in V: (o, \beta) \in R_i\}$ for an arbitrary chosen vertex $o \in V$ (called reference vertex), then, the vertex set $V$ can be written as disjoint union of $\Gamma_i(o)$, i.e.

$$
V = \bigcup_{i=0}^{d} \Gamma_i(o).
$$
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In fact, the relation (B.2) stratifies the network into a disjoint union of strata (associate classes) \( \Gamma_i(o) \). With each stratum \( \Gamma_i(o) \) one can associate a unit vector \( |\phi_i\rangle \) in \( W \) (called unit vector of \( i\)-th stratum) defined by

\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle,
\]

where, \(|\alpha\rangle\) denotes the eigenket of \( \alpha \)-th vertex at the associate class \( \Gamma_i(o) \) and \( \kappa_i = |\Gamma_i(o)| \) is called the \( i\)-th valency of the network \((\kappa_i := p_i^0 = |\{\gamma : (o, \gamma) \in R_i\}| = |\Gamma_i(o)|)\). For \( 0 \leq i \leq d \), the unit vectors \( |\phi_i\rangle \) of equation (B.3) form a basis for irreducible submodule of \( W \) with maximal dimension denoted by \( W_0 \). Since \( \{ |\phi_i\rangle \}_{i=0}^d \) becomes a complete orthonormal basis of \( W_0 \), we often write \([15]\)

\[
W_0 = \bigoplus_{i=0}^d \mathbb{C} |\phi_i\rangle.
\]

Let \( A_i \) be the adjacency matrix of the underlying network \( \Gamma \). From the action of \( A_i \) on reference state \( |\phi_0\rangle \) (\( |\phi_0\rangle = |o\rangle \), with \( o \in V \) as reference vertex), we have

\[
A_i |\phi_0\rangle = \sum_{\beta \in \Gamma_i(o)} |\beta\rangle.
\]

Then by using (B.3) and (B.5), we obtain

\[
A_i |\phi_0\rangle = \sqrt{\kappa_i} |\phi_i\rangle.
\]

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