On symmetries of a matrix and its isospectral reduction

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Abstract

The analysis of diagonalizable matrices in terms of their so-called isospectral reduction represents a versatile approach to the underlying eigenvalue problem. Starting from a symmetry of the isospectral reduction, we show in the present work that it is possible to construct a corresponding symmetry of the original matrix.

Keywords: Isospectral reduction, Latent symmetries, Cospectrality, Eigenvalues

2010 MSC: 05C50, 15A18, 15A27

1. Introduction

The study of matrix eigenvalue problems of the form $Hx = \lambda x$ is ubiquitous in science and technology. A promising direction in analysing this eigenvalue problem is a dimensional reduction of $H$. In this work, we consider the so-called isospectral reduction (ISR) \cite{ISR}, which is defined via matrix partitioning of $H \in \mathbb{C}^{N \times N}$ as

\begin{equation}
R_S(H, \lambda) = H_{SS} + H_{S\overline{S}} \left( \lambda - H_{\overline{S}\overline{S}} \right)^{-1} H_{\overline{S}S},
\end{equation}

where the set $S \subseteq \{1, \ldots, N\}$ and its complement $\overline{S}$ are used for partitioning $H$. For example, $H_{SS}$ denotes the submatrix obtained from $H$ by taking the rows in $S$ and the columns in $S$. The ISR provides valuable insights in quantum physics, where it is referred to as an effective Hamiltonian obtained from subsystem partitioning \cite{ effective Hamiltonian 1, effective Hamiltonian 2}.

As its name suggests, the ISR preserves the spectral properties of $H$: Defining the multiset $\sigma(M)$ as the eigenvalue spectrum of a matrix $M$, it has been shown that the non-linear eigenvalue spectrum $R_S(H, \lambda)$ fulfills $\sigma(R_S(H, \lambda)) = \sigma(H) - \sigma(H_{SS})$ \cite{ISR}. Thus, whenever $H$ and $H_{SS}$ share no eigenvalues, $R_S(H, \lambda)$ preserves the eigenvalue spectrum of $H$. Building on this favourable property, the ISR has been applied \cite{ISR applications} to improve the eigenvalue approximations of Gershgorin, Brauer, and Brualdi \cite{Gershgorin, Brauer, Brualdi}, to study pseudo-spectra of graphs and matrices \cite{pseudo-spectra}, to create stability preserving transformations of networks \cite{network stability 1, network stability 2, network stability 3}, to study the survival probabilities in open dynamical systems \cite{open systems}, and very recently also to explain spectral degeneracies of physical systems \cite{spectral degeneracies}.

In this work, we concentrate on symmetries of the isospectral reduction, which we define as a normal and invertible matrix $T$ which commutes with $R_S(H, \lambda)$, that is, $[R_S(H, \lambda), T] = 0$, for

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all \( \lambda \notin \sigma(H_{SS}) \). We note that, for the special case of permutations, such symmetries of \( R_S(H, \lambda) \) have been coined “latent” or “hidden” symmetries of \( H \) [13]. In that context, \( H \) is the (weighted) adjacency matrix of a graph, and its automorphisms are described by a permutation matrix commuting with \( H \). The term “latent” then refers to the fact that the ISR of \( H \) may feature non-trivial permutation symmetries, while \( H \) features only a trivial (namely: the identity operation) permutation symmetry. Interestingly, latent symmetries have been recently connected [14] to the theory of so-called “cospectral vertices” which find applications in quantum computing [15, 16, 17]. In the remainder of this work, we will adapt the generalized notion of Ref. [3], and denote also non-permutation symmetries of \( R_S(H, \lambda) \) as “latent symmetries of \( H \)”.

The nonlinear eigenvalues of \( R_S(H, \lambda) \) correspond to eigenvectors fulfilling
\[
R_S(H, \lambda)y = \lambda y,
\]
given by the projection \( y = x_S \) of the eigenvector \( x \) of \( H \) to \( S \) [18]. This allows to derive the profound impact of latent symmetries on the eigenvectors of \( H \): Whenever the symmetry \( T \) has only simple eigenvalues, and additionally \( H \) and \( H_{SS} \) share no eigenvalues, then all eigenvectors of \( H \) fulfill
\[
Tx_S = tx_S,
\]
with \( t \) being an eigenvalue of \( T \). However, when \( H \) and \( H_{SS} \) share eigenvalues, the behavior of the corresponding eigenvectors is still an open issue for general \( T \). Recently, an interesting first step in solving this problem has been made for the special case of \( T = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) [14, 19]: It has been shown that \( [R_S(H, \lambda), P] = 0 \) corresponds to the existence of an orthogonal block-diagonal matrix \( Q = P \oplus \overline{Q} \) fulfilling \( [Q, H] = 0 \). Thus, \( H \) and \( Q \) can be simultaneously diagonalized, and—assuming no degeneracies of \( H \)—it follows that all eigenvectors either fulfill Eq. (2) or vanish on \( S \), that is, \( x_S = 0 \). We generalize this result in the following.

**Theorem 1.** Let \( R_S(H, \lambda) \) be the isospectral reduction of a self-adjoint matrix
\[
H = \begin{pmatrix} H_{SS} & H_{SS}^T \\ H_{SS}^T & H_{SS} \end{pmatrix}
\]
and \( T \) be a normal and invertible \(|S| \times |S| \) matrix. Then the following are equivalent

(i) \([R_S(H, \lambda), T] = 0 \forall \lambda \notin \sigma(H_{SS})\).

(ii) \([H^k]_{SS}, T] = 0 \forall k \in \mathbb{N} \).

(iii) There exists a normal matrix \( Q = T \oplus \overline{Q} \) fulfilling \([Q, H] = 0\).

**Proof.** The equivalence of (i) and (ii) has already been proven in Ref. [3]. Proving (iii) \(\Rightarrow\) (ii) is trivial, since
\[
[H, Q] = 0 \Rightarrow [H^k, Q] = 0,
\]
and writing the \( SS \)-block of this commutator gives \([H^k]_{SS}, Q_{SS} = [H^k]_{SS}, T] = 0\).

We now prove the remaining step of (ii) \(\Rightarrow\) (iii): Since \( T \) is normal and invertible, it can be spectrally decomposed as \( T = \sum_{i=1}^{n} \sum_{j=1}^{d_i} t_i \phi_{i,j} \phi_{i,j}^\dagger \) with each of its nonzero eigenvalues \( t_i \).
corresponding to a set of \(d_i\) orthonormal eigenvectors \(\phi_{i,j}\) with degeneracy index \(j = 1, \ldots, d_i\), and with \(\dagger\) denoting the conjugate transpose.

Let now \(\Phi_{i,j}\) be the \(N\)-dimensional vector obtained from \(\phi_{i,j}\) by padding it with zeros such that 
\[
(\Phi_{i,j})_S = \phi_{i,j} \quad \text{and} \quad (\Phi_{i,j})_\overline{S} = 0, \quad \text{with} \quad N \text{ denoting the dimension of } H.
\]
We denote by \(K_{i,j} = \text{span} \left( \Phi_{i,j}, H_{i,j}, \ldots, H^{N-1}_{i,j} \right)\) the Krylov subspace generated by \(\Phi_{i,j}\). As we now prove, when \(l \neq m\), \(K_{i,j} \perp K_{m,j'}\) for all \(j'\). Equivalently, the hermitian inner product \(\langle H^{k_1}_{i,j}, H^{k_2}_{m,j'} \rangle = 0\) for all \(k_1, k_2\): Since \(H = H\dagger\) is self-adjoint, this inner product can be written as
\[
\Phi_{i,j} H^k \Phi_{m,j'} = (a) \phi_{i,j}^\dagger H^k S \phi_{m,j'} = \phi_{i,j}^\dagger T^{-1} H^k S T \phi_{m,j'}
\]
with \(k = k_1 + k_2\) and where we have used that (a) both \(\Phi_{i,j}\) and \(\Phi_{m,j'}\) vanish on \(S\) and (b) that \(t_i \neq t_m\).

We then proceed by defining the \(H\)-invariant subspaces \(\widetilde{K}_i = \bigoplus_j K_{i,j}\). From the above, it is clear that \(\widetilde{K}_i \perp \widetilde{K}_m\) when \(l \neq m\). We now construct an orthonormal basis of each \(\widetilde{K}_i\) as follows: As the first \(d_i\) basis vectors, we choose the generating vectors \(\Phi_{i,1}, \ldots, \Phi_{i,d_i}\) of the Krylov spaces \(K_{i,1}, \ldots, K_{i,d_i}\). These vectors are already pairwise orthonormal and are necessarily contained in \(\widetilde{K}_i\). Denoting the dimension of \(\widetilde{K}_i\) by \(\tilde{d}_i\), the remaining \(\tilde{d}_i - d_i = r_i \geq 0\) basis vectors \(\overline{\Phi}_{i,1}, \ldots, \overline{\Phi}_{i,r_i}\) can be shown to vanish on \(S\): First, being a basis vector of \(\widetilde{K}_i\), each \(\overline{\Phi}_{i,j}\) must be orthogonal to all other basis vectors of this space, and in particular to \(\Phi_{i,j'}\) for all \(j'\). Second, since \(\widetilde{K}_i \perp \widetilde{K}_{i'}\) with \(i \neq i'\), each \(\overline{\Phi}_{i,j}\) must be orthogonal to each basis vector of \(\widetilde{K}_{i'}\), and in particular to \(\Phi_{i',j'}\) for all \(j'\). Thus, \(\overline{\Phi}_{i,j}\) is orthogonal to \(\Phi_{i',j'}\) for all \(i', j'\). Now, since the set \(\{\Phi_{i,j}\}\) forms an orthogonal basis for any vector that vanishes on \(S\), and since each element of this set vanishes on \(S\), it follows that each \(\overline{\Phi}_{i,j}\) must vanish on \(S\).

The above insights allow us to finally construct
\[
Q = \sum_{i=1}^n t_i \left[ \sum_{j=1}^{d_i} \Phi_{i,j} \Phi_{i,j}^\dagger + \sum_{j=1}^{r_i} \overline{\Phi}_{i,j} \overline{\Phi}_{i,j}^\dagger \right]. \tag{7}
\]
By construction, \(Q\) is a normal matrix. We now prove that \([H, Q] = 0\). To this end, let \(x_i \in \widetilde{K}_i\).
It follows that \(H x_i \in \widetilde{K}_i\) as well, since \(\widetilde{K}_i\) is by construction an \(H\)-invariant subspace. Then, by Eq. (7), all vectors in \(\widetilde{K}_i\) are eigenvectors of \(Q\) with identical eigenvalue \(t_i\). We thus have \(Q H x_i = t_i H x_i\), and also \(H Q x_i = H (t_i x_i)\). Let now \(V\) denote the orthogonal complement of \(\bigoplus_i \widetilde{K}_i\). It is obvious that for \(v \in V\) we have \(Q v = 0\) and thus also \(H Q v = 0\). Being the orthogonal complement of \(H\)-invariant subspaces, \(V\) is also \(H\)-invariant, and we also get \(H v \in V\) implying \(Q H v = 0\). In summary,
\[
Q H x = H Q x \tag{8}
\]
for any \(x\). Thus, \([Q, H] = 0\) as claimed.

We proceed by showing that \(Q_{SS} = T\). To this end, we define the vector \(e_i\) having a 1 on component \(i\) and with all other components vanishing. Then, for \(s, s' \in S\), the matrix element
\[ Q_{s,s'} = e^\dagger_s Q e_{s'}, \text{ and since each basis vector } \Phi_{i,j} \text{ vanishes on } S, \text{ we have} \]

\[ Q_{s,s'} = \sum_{i=1}^{n} \sum_{j=1}^{d_i} t_i e^\dagger_s \Phi_{i,j} \Phi^\dagger_{i,j} e_{s'} = \sum_{i=1}^{n} \sum_{j=1}^{d_i} t_i \left( \Phi_{i,j} \Phi^\dagger_{i,j} \right)_{s,s'} = \sum_{i=1}^{n} \sum_{j=1}^{d_i} t_i \left( \phi_{i,j} \phi_{i,j}^\dagger \right)_{s,s'} = T_{s,s'} \]

where in the last step we recognized the spectral decomposition of \( T \).

To see that \( Q_{s,s} = 0 \) and also \( Q_{s,s} = 0 \), it suffices to note that, due to Eq. (7), \( e^\dagger_s Q e_s = 0 \) for any \( s \in S, s' \in S \).

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