Extended Hamiltonian Formalism of the Pure Space-Like Axial Gauge Schwinger Model. II

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Canonical methods are not sufficient to properly quantize space-like axial gauges. In this paper, we obtain guiding principles that allow for the construction of an extended Hamiltonian formalism for pure space-like axial gauge fields. To do so, we clarify the general role that residual gauge fields play in the space-like axial gauge Schwinger model. In all the calculations, we fix the gauge using the rule $n \cdot A = 0$, where $n$ is a space-like constant vector, and we refer to its direction as $x_-$. Then, to begin with, we construct a formulation in which the quantization surface is space-like but not parallel to the direction of $n$. The quantization surface has a parameter that allows us to rotate it, but when we do so, we keep the gauge fixing direction fixed. In that formulation, we can use canonical methods. We bosonize the model to simplify the investigation. We find that the inverse differentiation, $(\partial_-)^{-1}$, is ill-defined whatever quantization coordinates we use, as long as the direction of $n$ is space-like. We find that the physical part of the dipole ghost field includes infrared divergences. However, we also find that if we introduce residual gauge fields in such a way that the dipole ghost field satisfies the canonical commutation relations, then the residual gauge fields are determined so as to regularize the infrared divergences contained in the physical part. The propagators then take the form prescribed by Mandelstam and Leibbrandt. We make use of these properties to develop guiding principles that allow us to construct consistent operator solutions in the pure space-like case, in which the quantization surface is parallel to the direction of $n$, and canonical methods do not suffice.

\section{Introduction}

In a previous paper, which is hereafter referred to as I,\textsuperscript{1)} we constructed an extended Hamiltonian formalism with which we found a family of solutions to the Schwinger model. Those solutions are of the axial or temporal gauge type. To consider the problem generally, we specified the gauge fixing direction as that of the constant vector $n^\mu = (n^0, n^3) = (\cos \theta, \sin \theta)$. We also introduced the $\pm$-coordinates $x^\mu = (x^+, x^-)$ as

\begin{equation}
  x^+ = x^0 \sin \theta + x^3 \cos \theta, \quad x^- = x^0 \cos \theta - x^3 \sin \theta.
\end{equation}

With these definitions, the gauge fixing condition,

\begin{equation}
  A_- = n \cdot A = A_0 \cos \theta - A_3 \sin \theta = 0,
\end{equation}

is that of an axial or temporal gauge. In our formulation, the temporal and axial gauges in ordinary coordinates correspond, respectively, to $\theta = 0$ and $\theta = \frac{\pi}{2}$, while
the light-front formulation corresponds to \( \theta = \frac{\pi}{4} \). We found that in the region satisfying \( 0 \leq \theta < \frac{\pi}{4} \), \( x^- \) should be taken as the evolution parameter, and we constructed the canonical temporal gauge solutions. In that case, we found that there exist residual gauge fields that depend only on \( x^+ \). These residual gauge fields are therefore static canonical variables. By continuation, we obtained an operator solution in the axial region, i.e., that in which \( \frac{\pi}{4} < \theta < \frac{\pi}{2} \), where \( x^+ \) should be taken as the evolution parameter. In that case, we find that there are infrared divergences associated with the physical degrees of freedom. These infrared divergences are regularized by the residual gauge fields. Among other results, we found that the Hamiltonian for the residual gauge fields must be calculated by integrating the divergence equation of the energy-momentum tensor over a suitable closed surface. Because the residual gauge fields do not depend on the initial value surface, \( x^- \), the \( (x^- \to \pm \infty) \) contributions from these fields have to be kept.\(^{2)\) In this way, we obtained the Hamiltonian, which includes a part obtained from integrating a density involving the residual gauge fields over a path on which \( x^- \) is constant.

In I, we found the solutions in the axial gauge region only by continuation from the temporal gauge region. In this paper, we consider the problem of finding the axial gauge solutions directly, by quantizing on the surface \( x^+ = 0 \). This axial gauge formulation involves constrained fields, and conventionally these constrained fields are eliminated in favor of physical degrees of freedom. This elimination requires that we introduce inverse differentiations, which can introduce infrared divergences.\(^{3)\) Despite extensive studies,\(^{4)\) overcoming the infrared difficulties has remained an open problem. In the present work, we find that the residual gauge fields are essential in controlling the infrared divergences. These fields may be viewed as integration constants associated with solving the constraint equations, and they are necessary to give the correct prescription for the required inverse differentiations. Quantizing the residual gauge fields is itself an interesting subject. This is because they depend on the evolution parameter in such a way that they cannot be canonical variables. A first step forward realizing this quantization was made by McCartor and Robertson\(^{5)\) in the light-front formulation of QED.

To begin with, we consider a generalization of the models considered in I. We keep the constant vector in a fixed space-like direction and take the quantization surface to also be space-like. However, we do not choose the constant vector to lie parallel to the quantization surface. In this framework, we can implement the canonical procedure. We then use the operator solutions found in such cases to clarify the dependence of the operator solutions on the quantization coordinates. We find that there are residual gauge fields allowed by the fixed gauge choice, and we can also use the operator solutions to clarify the general roles that the residual gauge fields play in these axial gauge solutions. To implement these ideas, we introduce another set of coordinates, \( x^\mu = (x^\tau, x^\sigma) \), defined by

\[
\begin{align*}
x^\tau &= x^0 \sin \phi + x^3 \cos \phi, \\
x^\sigma &= x^0 \cos \phi - x^3 \sin \phi.
\end{align*}
\tag{1.3}
\]

In these coordinates, the gauge-fixing condition and the constant vector are ex-
pressed, respectively, as

\[ A_\tau = \sin(\phi - \theta)A_\tau + \cos(\phi - \theta)A_\sigma = 0, \]
\[ n^\mu = (n^\tau, n^\sigma) = (\sin(\phi - \theta), \cos(\phi - \theta)). \]  

(1.4)  

(1.5)

To simplify our investigation, we bosonize the Schwinger model and avoid quantizing the coupled system of fermi fields and gauge fields. The solutions contain a dipole ghost field, \( X \), which consists of both physical fields and residual gauge fields. In space-like formulations in which the constant vector is not parallel to the quantization surface, we can employ \( A_\sigma \) and the dipole ghost field \( X \) as canonical variables and construct a canonical formulation without encountering any of the difficulties inherent in the pure space-like (PSL) axial gauge formulations, in which the constant vector is parallel to the quantization surface. We show that the physical part of \( X \) is determined uniquely by the gauge choice, while the residual gauge part, which exhibits manifest quantization coordinate dependence, is determined by requiring that \( X \) satisfies the canonical commutation conditions. It turns out that \((n \cdot \partial)^{-1} = (\partial_\tau)^{-1}\) is ill-defined for any quantization coordinates, as long as the gauge fixing direction is space-like (i.e., \( n^2 < 0 \)). It follows from this that the physical part of \( X \) gives rise to infrared divergences for any quantization coordinates, as long as the gauge-fixing direction is space-like. However, if we introduce residual gauge fields in such a way that \( X \) satisfies the canonical commutation conditions, then the residual gauge part automatically regularizes the infrared divergences resulting from the physical part. As a consequence, the \( x^\tau \)-time ordered propagator for \( X \) takes the form prescribed by Mandelstam (6) and by Leibbrandt (7) (the ML prescription). In this way, we see that the residual gauge degrees of freedom are necessary in formulating the space-like axial gauge Schwinger model in a way that is free from infrared divergences.

We remark here that canonical formulations in ordinary coordinates were constructed for the case \( n^2 = 0 \) by Bassetto et al. (8) and for the case \( n^0 \neq 0 \) with \( n^2 < 0 \) by Lazzizzera (9). These authors showed that to implement the ML prescription and to regularize the infrared divergences, residual gauge fields are needed.

Having found the solutions to the axial gauge formulations in the cases that the constant vector is not parallel to the quantization surface, we turn to the pure space-like case. The PSL case cannot be realized by taking the limit \( \phi \rightarrow \theta \). This reflects the fact that we cannot construct a canonical formulation in the PSL case, because residual gauge fields cannot be canonical fields; only \( X \) and its conjugate remain as unconstrained canonical fields. We circumvent this difficulty by using the properties of the dipole ghost fields found in \( \S 2 \) as guiding principles. We show that operator solutions can be constructed by following these guiding principles. When these operator solutions are constructed, they are identical to those given in I.

This paper is organized as follows. In \( \S 2 \), we bosonize the space-like axial gauge Schwinger model and construct a canonical formulation in \( \tau \sigma \)-coordinates. In \( \S 3 \) we show that our canonical formulation is free from infrared difficulties. In \( \S 4 \), we carry out the quantization of the PSL case and construct the solution. Section 5 is devoted to concluding remarks.
In this paper, we keep $\phi$ in the axial region, i.e. $\frac{\pi}{4} < \phi \leq \frac{\pi}{2}$. Also, a list of conventions used here can be found in Appendix A.

§2. Bosonization of the space-like axial gauge Schwinger model

2.1. Field equation of the dipole ghost field

The space-like axial gauge Schwinger model is defined by the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - B (n \cdot A) + i \bar{\psi} \gamma^\mu (\partial_\mu + ie A_\mu) \psi,$$  \hspace{1cm} (2.1)$$

where $B$ is the Nakanishi-Lautrup field used in noncovariant formulations. From the Lagrangian, we derive the field equations

$$\partial_\mu F^{\mu\nu} = n^\nu B + J^\nu, \quad J^\nu = e \bar{\psi} \gamma^\nu \psi,$$ \hspace{1cm} (2.2)$$

and the gauge fixing condition (1.4). The field equation of $B$,

$$(n \cdot \partial) B = \partial_\tau B = (\sin(\phi - \theta) \partial_\tau + \cos(\phi - \theta) \partial_\sigma) B = 0,$$ \hspace{1cm} (2.4)$$

is obtained by operating on (2.2) with $\partial_\mu$.

We first obtain the field equation of the dipole ghost field $X$. We now know that consistent operator solutions of the Schwinger model\cite{11} can be constructed by regularizing the vector current by means of the gauge invariant point-splitting procedure.\cite{12} We therefore regularize $J^\mu$ in this manner in the present paper. With this regularization, the vector current is given by

$$J^\mu = j^\mu - m^2 A^\mu,$$ \hspace{1cm} (2.5)$$

where $m^2 = \frac{e^2}{\pi}$, and $j^\mu$ is the part given as a bilinear product of $\bar{\psi}$ and $\psi$. We now observe that Eq. (2.3) is massless; that is, $j^\mu$ satisfies $\varepsilon^{\mu\nu} \partial_\mu j_\nu = 0$. Therefore, we can define $j^\mu$ as the gradient of the dipole ghost field $X$:

$$j_\mu = m \partial_\mu X.$$ \hspace{1cm} (2.6)$$

Substituting (2.6) into (2.5) and then using current conservation, $\partial_\mu J^\mu = 0$, we obtain

$$m \Box X = m^2 \partial_\mu A_\mu.$$ \hspace{1cm} (2.7)$$

Substituting (2.5), (2.6) and (2.7) into (2.2), we get

$$\left(\Box + m^2\right) \left(A^\nu - \frac{1}{m} \partial^\nu X\right) = n^\nu B.$$ \hspace{1cm} (2.8)$$

Finally, operating with $n_\nu$ on (2.8) and using $n \cdot A = 0$, we derive the field equation of the dipole ghost field $X$,

$$\left(\Box + m^2\right) (\partial_\tau X) = -m n^2 B.$$ \hspace{1cm} (2.9)$$
2.2. Bosonization of the generalized axial gauge Schwinger model

Now we can employ Eqs. (2.7) and (2.8) as guiding principles to obtain the Lagrangian for the equivalent bosonized model. These equations are derived from

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - B(n \cdot A) + \frac{1}{2} \partial_\mu X \partial^\mu X - m \partial_\mu X A^\mu + \frac{m^2}{2} A_\mu A^\mu, \]  

(2.10)

which justifies the use of (2.10) as the Lagrangian with the present variables. From this Lagrangian, we see that in the axial region satisfying \( \pi/4 < \phi \leq \pi/2 \), where \( x^\tau \) is chosen as the evolution parameter, the fundamental fields are \( A_\sigma \) and \( X \). \( A_\tau \) is a dependent field as long as \( \phi \neq \theta \).

The canonical conjugate momenta are found from the Lagrangian to be

\[ \pi^\tau = 0, \quad \pi^\sigma = F_{\tau\sigma}, \quad \pi_B = 0, \quad \pi_X = \partial^\tau X - mA^\tau. \]  

(2.11)

Therefore, we can choose \( A_\sigma, X, \pi^\sigma \) and \( \pi_X \) as independent canonical variables and express the dependent degrees of freedom as

\[ A_\tau = \cot(\theta - \phi) A_\sigma, \quad B = (\partial_\sigma \pi^\sigma - m \pi_X)/n^\tau. \]  

(2.12)

Consequently, equal \( x^\tau \)-time canonical quantization conditions can be imposed on the independent canonical variables; the nonvanishing commutators are

\[ [A_\sigma(x), \pi^\sigma(y)] = i\delta(x^\sigma - y^\sigma), \quad [X(x), \pi_X(y)] = i\delta(x^\sigma - y^\sigma). \]  

(2.13)

For later convenience, we give here the equal \( x^\tau \)-time commutation relations for \( B \):

\[ [\pi^\sigma(x), B(y)] = [\pi_X(x), B(y)] = [B(x), B(y)] = 0, \]

\[ [X(x), B(y)] = -i \frac{m}{n^\tau} \delta(x^\sigma - y^\sigma), \quad [A_\sigma(x), B(y)] = -i \frac{n^\tau}{m^\tau} \partial_\sigma \delta(x^\sigma - y^\sigma). \]  

(2.14)

2.3. Expression of the dipole ghost field

Now that the canonical formulation is given, we proceed to solving Eq. (2.8). To obtain a particular solution, we make use of the fact that, due to (2.4), \( B \) satisfies

\[ (\Box + m^2)B = \left( m^2 - \frac{n^2 \partial_\sigma^2}{\sin^2(\phi - \theta)} \right) B = (m^2 - n^2 \partial_\perp^2)B. \]  

(2.15)

Here and in what follows, we denote, for brevity, \( -\frac{\partial}{\sin(\phi - \theta)} \) as \( \partial_\perp \) when it is applied to operators depending on only \( x^+ \). It follows immediately that a particular solution to Eq. (2.8), for the quantity \( A^\nu - \frac{1}{m} \partial^\nu X \), is

\[ A_\nu = \frac{n^\nu}{m^2 - n^2 \partial_\perp^2} B. \]

To specify the remaining homogeneous part, which satisfies the free D'Alembert equation of mass \( m \), we take account of the fact that \( F_{\tau\sigma} \) is gauge invariant and satisfies \( F_{\tau\sigma} = F_{\perp\perp} \). We see from this that \( F_{\tau\sigma} \) is independent of the quantization coordinates, and therefore it is identical to that given by the solution in I,

\[ F_{\tau\sigma} = m \tilde{\Sigma} + \frac{n^2}{m^2 - n^2 \partial_\perp^2} \partial_\perp B, \]  

(2.16)
where $\tilde{\Sigma}$ is the Schwinger field of mass $m$. We can easily see that (2.16) can be derived from the following expression for $A^\nu - \frac{1}{m} \partial^\nu X$:

$$A^\mu - \frac{1}{m} \partial^\mu X = \frac{n^\mu}{m^2 - n^2 \partial_+^2} B - \varepsilon^{\mu\nu} \partial_\nu \tilde{\Sigma}. \tag{2.17}$$

Here, $\varepsilon^{\tau\sigma} = -\varepsilon^{\sigma\tau} = 1, \varepsilon^{\tau\tau} = \varepsilon^{\sigma\sigma} = 0$.

It is useful to point out here that the right-hand side of (2.17) can be written in the divergence free form

$$\frac{n^\mu}{m^2 - n^2 \partial_+^2} B - \varepsilon^{\mu\nu} \partial_\nu \tilde{\Sigma} = -\frac{1}{m} \varepsilon^{\mu\nu} \partial_\nu \lambda, \tag{2.18}$$

$$\lambda = \tilde{\Sigma} - \frac{mn^\tau}{m^2 - n^2 \partial_+^2} \partial_\sigma^{-1} B, \tag{2.19}$$

where the operator $\partial_\sigma^{-1}$ is defined by

$$(\partial_\sigma)^{-1} f(x) = \frac{1}{2} \int_{-\infty}^{\infty} dy^\sigma \varepsilon(x^\sigma - y^\sigma) f(x^\tau, y^\sigma), \tag{2.20}$$

which imposes, in effect, the principal value regularization. It follows from (2.5), (2.6), (2.17) and (2.18) that $A^\mu$ and $J^\mu$ can be written as

$$A^\mu = \frac{1}{m} (\partial^\mu X - \varepsilon^{\mu\nu} \partial_\nu \lambda), \quad J^\mu = m \varepsilon^{\mu\nu} \partial_\nu \lambda. \tag{2.21}$$

We can now verify that $\tilde{\Sigma}$ and $\partial^\tau \tilde{\Sigma}$ satisfy the canonical equal $x^\tau$-time commutation relations

$$[\tilde{\Sigma}(x), \tilde{\Sigma}(y)] = [\partial^\tau \tilde{\Sigma}(x), \partial^\tau \tilde{\Sigma}(y)] = 0, \quad [\tilde{\Sigma}(x), \partial^\tau \tilde{\Sigma}(y)] = i \delta(x^\sigma - y^\sigma), \tag{2.22}$$

$$[B(x), \tilde{\Sigma}(y)] = [B(x), \partial^\tau \tilde{\Sigma}(y)] = 0 \tag{2.23}$$

by using their expressions in terms of the canonical variables:

$$\tilde{\Sigma} = \frac{1}{m} \left( \pi^\sigma - \frac{n^2}{m^2 - n^2 \partial_+^2} \partial_\lambda B \right), \quad \partial^\tau \tilde{\Sigma} = \partial_\sigma X - mA_\sigma + \frac{mn_\sigma}{m^2 - n^2 \partial_+^2} B. \tag{2.24}$$

We next obtain an expression for $X$. For this purpose, we multiply (2.17) by $n_\mu$ and use $n \cdot A = A_- = 0$ and $n \cdot \partial = \partial_-$. We then obtain

$$\partial_- X = -\frac{mn^2}{m^2 - n^2 \partial_+^2} B + \varepsilon^{\mu\nu} n_\mu \partial_\nu \tilde{\Sigma} = -\frac{mn^2}{m^2 - n^2 \partial_+^2} B + \partial^+ \tilde{\Sigma} \tag{2.25}$$

and see that $X$ is obtained by integrating (2.25) with respect to $x^-$. The first term has to be integrated carefully. At first glance, it seems that a linear function of $x^-$ is included, because the first term depends on only $x^+$. However, it turns out that if $X$ has such term, then the equal $x^\tau$-time canonical commutation relations for $X$ are not satisfied. We use the possibility of adding arbitrary functions of $x^+$ to write the
integral of the first term as \(-\frac{x^\tau}{n^\tau} \frac{mn^2}{m^2 - n^2 \partial_+^2} B\). To integrate the second term, we make use of the inverse differentiation \((\partial_-)^{-1}\) defined by

\[
\frac{1}{\partial_-} \tilde{\Sigma} = -\frac{n^\tau \partial^\tau + n_\sigma \partial_\sigma}{m^2 \sin^2(\phi - \theta) - n^2 \partial_\sigma^2} \tilde{\Sigma}.
\] (2.26)

We can show that (2.26) is correct by operating on both sides with \(\partial_- = n^\tau \partial_\tau + n^2 \partial_\sigma = n^\tau \partial^\tau - n_\sigma \partial_\sigma\) and using the mass shell condition \(\{(\partial^\tau)^2 - \partial_\sigma^2 - \cos 2\phi m^2\} \tilde{\Sigma} = 0\). We thus obtain the general solution, which we write in the form

\[
X = -\frac{x^\tau}{n^\tau} \frac{mn^2}{m^2 - n^2 \partial_+^2} B + \frac{\partial^+}{\partial_-} \tilde{\Sigma} + \text{integration constant}.
\]

The integration constant here is determined in the following way. To obtain the first commutation relation in the second line of (2.14), we need an operator that does not commute with \(B\); this is because \(B\) commutes with both \(\tilde{\Sigma}\) and \(\partial^\tau \tilde{\Sigma}\), as seen in (2.23), and so with \(\frac{\partial^+}{\partial_-} \tilde{\Sigma}\), which is given by

\[
\frac{\partial^+}{\partial_-} \tilde{\Sigma} = -\frac{m^2 n^\tau n_\sigma + n^2 \partial^\tau \partial_\sigma}{m^2 \sin^2(\phi - \theta) - n^2 \partial_\sigma^2} \tilde{\Sigma}.
\] (2.27)

Therefore, we must introduce another field, \(C\), which depends only on \(x^+\). To obtain the relation \([X(x), X(y)] = 0\) when \(x^\tau = y^\tau\), we need an extra term. This is because it is natural to assume that \(C\) commutes with \(\tilde{\Sigma}\) and \(\partial^\tau \tilde{\Sigma}\), and because the commutator \([\frac{\partial^+}{\partial_-} \tilde{\Sigma}(x), \frac{\partial^+}{\partial_-} \tilde{\Sigma}(y)]\) does not vanish when \(x^\tau = y^\tau\). We find that if we parameterize the integration constant in the form

\[
X = \frac{\partial^+}{\partial_-} \tilde{\Sigma} + \frac{m}{m^2 - n^2 \partial_+^2} \left( C - \frac{n^2 x^\tau}{n^\tau} B + \frac{n^2 n_\sigma}{m^2 \sin^2(\phi - \theta) - n^2 \partial_\sigma^2} \partial_\sigma B \right),
\] (2.28)

then the canonical commutation conditions yield the following equal \(x^\tau\)-time commutation relations for \(C\):

\[
[C(x), C(y)] = 0, \quad \left[ B(x), \frac{1}{m^2 - n^2 \partial_+^2} C(y) \right] = i \frac{1}{n^\tau} \delta(x^\sigma - y^\sigma), \quad [C(x), \tilde{\Sigma}(y)] = [C(x), \partial^\tau \tilde{\Sigma}(y)] = 0.
\] (2.29)

Substituting (2.28) into (2.17) then yields an explicit expression for \(A_\mu\):

\[
A_\mu = \varepsilon_{\mu \nu} n^\nu \frac{m}{\partial_-} \tilde{\Sigma} + \frac{n_\mu}{m^2 - n^2 \partial_+^2} B
\]

\[
+ \frac{\partial_\mu}{m^2 - n^2 \partial_+^2} \left( C - \frac{n^2 x^\tau}{n^\tau} B + \frac{n^2 n_\sigma}{m^2 \sin^2(\phi - \theta) - n^2 \partial_\sigma^2} \partial_\sigma B \right).
\] (2.30)

In this way, we see that the residual gauge fields are necessary to preserve the canonical commutation relations and that the residual gauge part of \(X\) must include an explicit dependence on the quantization coordinates. We close this subsection
by pointing out how infrared divergences appear in our formulation. As is seen from (2.30), the inverse of the operator \( m^2 \sin^2(\varphi - \theta) - n^2 \partial^2 \sigma \) is applied to both \( \tilde{\Sigma} \) and to the residual gauge fields. This inverse operator gives rise to infrared divergences, because \( n^2 = \cos 2\theta < 0 \) in the range \( \frac{\pi}{4} < \theta < \frac{\pi}{2} \). Therefore the operator becomes singular in our range. We show in the next section that the infrared divergences resulting from the physical field are cancelled by infrared divergences from the residual gauge part.

2.4. Fermion field operator

Now that we have an explicit expression for \( A_\mu \), we can construct the fermion field operators in the same way as in I. From the expression for \( A_\mu \) in (2.21), we see that the fermion operators are formally given by

\[
\psi_\alpha(x) = \frac{Z_\alpha}{\sqrt{(\gamma^0 \gamma^\tau)_{\alpha\alpha}}} \exp[-i\sqrt{\pi} \Lambda_\alpha(x)], \quad (\alpha = 1, 2)
\]

where \( Z_\alpha \) is a normalization constant and

\[
\Lambda_\alpha(x) = X(x) + (-1)^\alpha \lambda(x).
\]

We need to rewrite the formal solution as a normal ordered product. However, if we simply normally order the exponential and then calculate the canonical anticommutation relations, we find another infrared divergence inherent in two-dimensional massless scalar fields. In our formulation this divergence results from the singular operator, \( \partial^{-1}_\sigma B \), in \( \lambda \) given in (2.19). We overcome this difficulty by not rewriting the infrared parts of the singular operator and its conjugate operator into normal ordered form. In what follows, we keep the condition \( \phi > \theta \), and to incorporate the ML prescription, we employ the following representations of \( B \) and \( C \):

\[
B(x) = \frac{m}{n^2 \sqrt{2\pi}} \int_{-\infty}^{\infty} dk_\sigma \theta(-k_\sigma) \sqrt{|k_\sigma|} \{B(k_\sigma)e^{-ik_\sigma x} + B^*(k_\sigma)e^{ik_\sigma x}\},
\]

\[
\frac{m}{m^2 - n^2 \partial^2 \sigma} C(x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk_\sigma}{\sqrt{|k_\sigma|}} \theta(-k_\sigma) \{C(k_\sigma)e^{-ik_\sigma x} - C^*(k_\sigma)e^{ik_\sigma x}\}.
\]

Here, \( k_\tau = \cot(\theta - \phi)k_\sigma \), creation and annihilation operators satisfy

\[
[B(k_\sigma), C^*(q_\sigma)] = [C(k_\sigma), B^*(q_\sigma)] = -\delta(k_\sigma - q_\sigma),
\]

and all other commutators are zero. These relations allow us to define the physical subspace \( V \) as

\[
V = \{ |\text{phys}\rangle \mid B(k_\sigma)|\text{phys}\rangle = 0 \}
\]

and to define the infrared part, \( A^{(0)}_\alpha \), of \( A_\alpha \) by

\[
A^{(0)}_\alpha = \frac{i}{\sqrt{2\pi}} \int_{-\kappa}^{0} \frac{dk_\sigma}{\sqrt{|k_\sigma|}} \{C(k_\sigma) - C^*(k_\sigma) + (-1)^\alpha (B(k_\sigma) - B^*(k_\sigma))\},
\]

where \( \kappa \) is a small positive constant.
Now we can define the fermion field operators to be
\[
\psi_\alpha(x) = \frac{Z_\alpha}{\sqrt{(\gamma^{0}\gamma^{r})_{\alpha\alpha}}} \exp[-i\sqrt{\pi}A^{(-)}_{\alpha r}(x)]\sigma_\alpha \exp[-i\sqrt{\pi}A^{(+)}_{\alpha r}(x)],
\]
where \( A^{(-)}_{\alpha r} \) and \( A^{(+)}_{\alpha r} \) are the creation and annihilation operator parts of \( A_{\alpha r} \equiv A_\alpha - A^{(0)}_\alpha \), and
\[
\sigma_\alpha = \exp\left[-i\sqrt{\pi}\left(A^{(0)}_\alpha - (-1)^\alpha \frac{Q}{2m}\right)\right].
\]

Here, we have \( Q = -n^\tau \int_{-\infty}^{\infty} dx^\sigma B(x) \). (Note that \( Q \) in \( \sigma_\alpha \) constitutes a Klein transformation.) We refer to \( \sigma_\alpha \) as the spurion operator.\(^{13}\)

We now enumerate the properties of \( \psi_\alpha \), which show that the bosonized model is actually equivalent to the original model defined by the Lagrangian (2.1). We note that the symmetric energy-momentum tensor given by (2.41)–(2.43) below follows directly from the Lagrangian (2.10).

1. The following Dirac equation is satisfied:
\[
i\gamma^\mu(\partial_\mu + ieA_\mu)\psi = 0.
\]

2. The canonical commutation relations with \( A_\sigma \) and \( \pi^\sigma \) and the anticommutation relations are satisfied.

3. By applying the gauge invariant point-splitting procedure to \( e\bar{\psi}\gamma^\mu\psi \), we obtain the vector current \( J^\mu = m\partial^\mu X - m^2 A^\mu = me^{\mu\nu}(\partial_\nu + \Lambda) \). This result verifies that \( j^\mu \) is given by \( j^\mu = m\partial^\mu X \), so that it satisfies \( \epsilon^{\mu\nu}(\partial_\mu j_\nu) = 0 \). The charge operator, \( Q \), is given by
\[
Q = \int_{-\infty}^{\infty} dx^\sigma J^\tau(x) = -n^\tau \int_{-\infty}^{\infty} dx^\sigma B(x),
\]
where the derivative terms vanish in the integration.

4. Applying the gauge invariant point-splitting procedure to the Fermi products in the symmetric energy-momentum tensor and subtracting a divergent \( c \)-number (we denote this procedure by \( R \), we obtain
\[
\Theta^\sigma_\tau = iR(\bar{\psi}\gamma^\sigma\partial_\tau\psi) - A_\tau J^\sigma - n^\sigma A_\tau B = \partial_\tau \lambda \partial^\sigma \lambda - n^\sigma A_\tau B,
\]
\[
\Theta^\tau_\tau = -iR(\bar{\psi}\gamma^\tau\partial_\tau\psi) + A_\tau J^\tau + \frac{1}{2}(F_{\tau\sigma})^2 - n^\tau B A_\tau
\]
\[
= \cos^2 \frac{\phi}{2} \left\{(\partial_\tau \lambda)^2 + (\partial_\sigma \lambda)^2\right\} + \frac{1}{2}(F_{\tau\sigma})^2 - n^\tau B A_\tau,
\]
\[
\Theta^\sigma_\sigma = iR(\bar{\psi}\gamma^\sigma\partial_\sigma\psi) - A_\sigma J^\sigma + \frac{1}{2}(F_{\tau\sigma})^2 - n^\sigma B A_\sigma
\]
\[
= \cos^2 \frac{\phi}{2} \left\{(\partial_\tau \lambda)^2 + (\partial_\sigma \lambda)^2\right\} + \frac{1}{2}(F_{\tau\sigma})^2 - n^\sigma B A_\sigma,
\]
\[
\Theta^\tau_\sigma = iR(\bar{\psi}\gamma^\tau\partial_\sigma\psi) - A_\sigma J^\tau - n^\tau B A_\sigma = \partial_\sigma \lambda \partial^\tau \lambda - n^\tau B A_\sigma.
\]

5. The translational generators consist of those of the constituent fields:
\[
P_\tau = \int_{-\infty}^{\infty} dx^\sigma : \Theta^\tau_\tau := \int_{-\infty}^{\infty} dx^\sigma : -\frac{\cos^2 \phi}{2} \left\{(\partial_\tau \hat{\Sigma})^2 + (\partial_\sigma \hat{\Sigma})^2\right\} + \frac{m^2}{2}(\hat{\Sigma})^2.
\]
\[ P_{\sigma} = \int_{-\infty}^{\infty} dx^{\sigma} : \Theta_{\sigma}^{\tau} := \int_{-\infty}^{\infty} dx^{\sigma} : \left\{ \partial_{\sigma} \bar{\Sigma} \partial^{\tau} \bar{\Sigma} - B \frac{n^{\tau}}{m^{2} - n^{2} \partial_{\tau}^{2}} \partial_{\sigma} C \right\} : . \]  

\section{3. Cancellation of infrared divergences resulting from $\partial^{-1}$}

We begin this section by pointing out that $X$ incorporates Higgs phenomena and that the possible infrared singularities in $A_{\mu}$ are in $X$. Therefore, we confine ourselves to showing that $X$ is free from infrared divergences. More precisely, we show that the commutator function and the propagator for $X$ are free from infrared divergences. For this purpose we represent $\bar{\Sigma}$ as

\[ \bar{\Sigma}(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \frac{dp_{\sigma}}{p^{\tau}} \{ a(p_{\sigma}) e^{-ip^{\tau}x} + a^{*}(p_{\sigma}) e^{ip^{\tau}x} \}. \]  

Here, $p^{\tau} = \sqrt{p_{\sigma}^{2} + m_{0}^{2}}$, with $m_{0}^{2} = -\cos 2\phi m^{2}$, $p_{\tau} = -\sin 2\phi^{\sigma} + p^{\tau}$, and

\[ [a(p_{\sigma}), a(q_{\sigma})] = 0, \quad [a(p_{\sigma}), a^{*}(q_{\sigma})] = \delta(p_{\sigma} - q_{\sigma}). \]  

We first show that the commutator function of $X$ includes the commutator function, $E(x)$, characteristic of a dipole ghost field. From (2.28), (2.32) and (3.1), we obtain

\[ [X(x), X(y)] = i \{ \Delta(x - y; m^{2}) + n^{2}m^{2}E(x - y) \}, \]  

where $\Delta(x; m^{2})$ is the commutator function of the free boson field of mass $m$, and

\[ E(x) = \frac{1}{\partial_{-}^{2}} \Delta(x; m^{2}) - \frac{x^{\tau}}{m^{2}(n^{\tau})^{2} - n^{2} \partial_{\sigma}^{2}} \delta(x^{\sigma} - \cot(\theta - \phi)x^{\tau}) \]

\[ + \frac{2n^{\tau}n_{\sigma}}{(m^{2}(n^{\tau})^{2} - n^{2} \partial_{\tau}^{2})^{2}} \partial_{\sigma} \delta(x^{\sigma} - \cot(\theta - \phi)x^{\tau}). \]

When $x^{\tau} = y^{\tau}$, the commutator $[X(x), X(y)]$ vanishes. This can be seen as follows. The first term in (3.3) vanishes trivially. The second term of $E(x - y)$, which is proportional to $x^{\tau} - y^{\tau}$, also vanishes trivially. If we evaluate the first term in (3.4) using $\partial^{-1}$ as defined in (2.26), then we obtain a nonvanishing term. However, this term is cancelled by the third term in (3.4).

The commutator function $E(x)$ satisfies the following:

\[ (\Box + m^{2})E(x) = -\frac{x^{\tau}}{(n^{\tau})^{2}} \delta(x^{\sigma} - \cot(\theta - \phi)x^{\tau}), \quad \partial_{-}^{2}E(x) = \Delta(x; m^{2}), \]  

\[ E(x)|_{x^{\tau}=0} = 0, \quad \partial_{-}E(x)|_{x^{\tau}=0} = 0, \quad \partial^{2}E(x)|_{x^{\tau}=0} = 0, \]  

\[ (\Box + m^{2})E(x)|_{x^{\tau}=0} = 0, \quad (\Box + m^{2})\partial_{-}E(x)|_{x^{\tau}=0} = -\frac{1}{n^{\tau}} \delta(x^{\sigma}). \]  

Next, we show that the vacuum expectation value $\langle 0|X(x)X(y)|0\rangle$ does not diverge when $x^{\tau} = y^{\tau}$. We need to use $\langle 0|X(x)X(y)|0\rangle$ evaluated at $x^{\tau} = y^{\tau}$ to
calculate the equal $x^\tau$-time anticommutation relations of the fermion field operators. It is straightforward to obtain

$$
\langle 0|X(x)X(y)|0\rangle = \Delta^{(+)}(x-y; m^2) + m^2n^2E^{(+)}(x-y),
$$

(3.8)

where $\Delta^{(+)}(x; m^2)$ is the positive frequency part of $i\Delta(x; m^2)$, and

$$
E^{(+)}(x) = \frac{1}{2\pi} \Delta^{(+)}(x; m^2) - \frac{ix^\tau}{2\pi} \int_{-\infty}^{0} \frac{dk_\sigma}{m^2(n^\tau)^2 + n^2k_\sigma^2} e^{-ik_\sigma x^\tau} + \frac{1}{2\pi} \int_{-\infty}^{0} \frac{dk_\sigma}{(m^2(n^\tau)^2 + n^2k_\sigma^2)^2} e^{-ik_\sigma x^\tau}.
$$

(3.9)

A logarithmic divergence appears in the second term here but we regularize it using the principal value prescription. In addition, linear divergences appear in the first and third terms when $x^\tau = 0$. We set $x^\tau = 0$ and divide the integration region of the first term into a region in which the integration variable is positive and a region in which the integration variable is negative. We then combine the integration in the negative region with the third term and obtain

$$
\frac{-1}{4\pi} \int_{-\infty}^{0} \frac{dp_\sigma}{p^\tau} \frac{1}{p^\tau} e^{-ip_\sigma x^\tau} + \frac{1}{2\pi} \int_{-\infty}^{0} \frac{dk_\sigma}{(m^2(n^\tau)^2 + n^2k_\sigma^2)^2} e^{-ik_\sigma x^\tau} = \frac{-1}{4\pi} \int_{-\infty}^{0} \frac{dp_\sigma}{p^\tau} \frac{(n^\tau p^\tau - n_\sigma p_\sigma)^2}{(m^2(n^\tau)^2 + n^2p_\sigma^2)^2} e^{-ip_\sigma x^\tau} = \frac{-1}{4\pi} \int_{-\infty}^{0} \frac{dp_\sigma}{p^\tau} \frac{(-\cos 2\phi)^2}{(n^\tau p^\tau + n_\sigma p_\sigma)^2} e^{-ip_\sigma x^\tau}.
$$

(3.10)

It is useful to recall here that $\phi$ and $\theta$ lie in the regions satisfying $\frac{\pi}{4} < \theta < \phi \leq \frac{\pi}{2}$, and therefore $n^\tau = \sin(\phi - \theta) > 0$ and $n_\sigma = \cos(\phi + \theta) < 0$. As a result, no infrared divergences appear from (3.10). Furthermore, changing the integration variable from $p_\sigma$ to $-p_\sigma$ verifies that (3.10) is equal to the positive integration part of the first term of $E^{(+)}(x)$. It follows that $E^{(+)}(x-y)$ is well defined at $x^\tau = y^\tau$, which implies that we can incorporate the equal $x^\tau$-time anticommutation relations of the fermion field operators in the same way as in §3 of I.

Finally, we show that the factors $(m^2(n^\tau)^2 + n^2p_\sigma^2)^{-1}$ relevant to the infrared divergences drop out completely from the propagator for $X$, which is given by

$$
\langle 0|T(X(x)X(y))|0\rangle = \Delta_F(x-y; m^2) + m^2n^2E_F(x-y),
$$

(3.11)

where $\Delta_F(x-y; m^2)$ is the propagator for a free boson field of mass $m$ and $E_F(x-y)$, defined by

$$
E_F(x-y) = \theta(x^\tau - y^\tau)E^{(+)}(x-y) + \theta(y^\tau - x^\tau)E^{(+)}(y-x).
$$

(3.12)

Substituting the expression in (3.9) into (3.12) and Fourier transforming it yields

$$
E_F(q) = -\frac{i}{q^2 - m^2 + i\varepsilon} \frac{(n^\tau)^2(q_\sigma^2 - \cos 2\phi m^2) + n_\sigma^2 q_\sigma^2 + 2n^\tau n_\sigma q^\tau q_\sigma}{(m^2(n^\tau)^2 + n^2q_\sigma^2)^2} + \frac{i}{q_- + i\varepsilon \text{sgn}(q_+)} \frac{2(n^\tau)^2 n_\sigma q_\sigma}{(m^2(n^\tau)^2 + n^2q_\sigma^2)^2},
$$

(3.13)
where \( q_+ \equiv -\frac{q_\tau}{n^\tau} \). The term in the first line comes from the physical degrees of freedom, whereas the terms in the second line come from the residual gauge degrees of freedom. It is noteworthy that if we combine them into one term, then we obtain
\[
E_F(q) = \frac{1}{q^2 - m^2 + i\epsilon} \times \frac{i}{(q_- + i\epsilon \text{sgn}(q_+))^2},
\]
and hence
\[
\langle 0 | T(X(x)X(y)) | 0 \rangle = \frac{1}{(2\pi)^2} \int d^2q \frac{i}{q^2 - m^2 + i\epsilon} \left( 1 - \frac{n^2m^2}{(q_- + i\epsilon \text{sgn}(q_+))^2} \right) e^{-iq\cdot(x-y)}. \quad (3.14)
\]
In this way, the infrared divergences are eliminated, and the singularity associated with the gauge fixing is prescribed in such a way that causality is preserved in complex \( q_\tau \) coordinates. It can be shown that the same is true of the propagator for \( A_{\mu} \), and we obtain
\[
\int d^2x \langle 0 | T(A_{\mu}(x)A_{\nu}(0)) | 0 \rangle e^{iq\cdot x} = \frac{iP_{\mu\nu}}{q^2 - m^2 + i\epsilon}, \quad (3.15)
\]
\[
P_{\mu\nu} = -g_{\mu\nu} + \frac{n_\mu q_\nu + n_\nu q_\mu}{q_- + i\epsilon \text{sgn}(q_+)} - n^2 \left( \frac{q_\mu q_\nu}{(q_- + i\epsilon \text{sgn}(q_+))^2} \right). \quad (3.16)
\]

§4. Pure space-like case

We begin by noting that the limit \( \phi \to \theta \) of the residual gauge part of the operator solution given in §2, which has factors that depend on quantization coordinates, is not well-defined. We see from this that an operator solution in the (PSL) case \( \phi = \theta \) is not constructed in the same manner as that given in §2.

The Lagrangian and the equations of motion for \( A_{\mu} \) and \( X \) in the PSL case are obtained, respectively, by transforming (2.1), (2.7) and (2.8) into those in \( \pm \)-coordinates. Then, we obtain two new constraints,
\[
\pi^- + \partial_+ A_+ = 0, \quad \partial_- \pi^- - m\pi_X = 0, \quad (4.1)
\]
in addition to the gauge fixing condition \( A_- = 0 \). As a result, only \( X \) and \( \pi_X \) are left as independent canonical variables. This reflects the fact that the residual gauge fields depend on only \( x^+ \), so they cannot be canonical variables. Therefore we cannot obtain their quantization conditions from the Dirac procedure. Instead, we employ the following items as guiding principles to introduce them in the PSL case:

1. \( X \) and \( \pi_X \) satisfy the canonical commutation conditions.
2. The residual gauge fields commute with the massive field.
3. \( B \) satisfies \([B(x), X(y)] = im\delta(x^+ - y^+)\), and so generates c-number residual gauge transformations.
4. The infrared divergences that come from the physical part of \( X \) are regularized by infrared divergences from the residual gauge fields.

We start constructing an operator solution by solving Eq. (2.8) and obtain an expression similar to (2.17). The massive field obtained will be identified below as \( \tilde{\Sigma} \). Therefore, because \( A_- = 0 \), we can write
\[
\partial_- X = \partial^+ \tilde{\Sigma} - \frac{mn^2}{m^2 - n^2\partial^2_+} B. \quad (4.2)
\]
We see from this that $X$ is given by

$$X = \frac{\partial^+}{\partial_-} \tilde{\Sigma} - \frac{m n x^-}{m^2 - n^2 \partial^+_+} B + \text{integration constant}.$$  

The massive, physical part of $X$ is now known, and $\pi_X$ can be written as

$$\pi_X = \partial^+ X - mA^+ = \partial_- \lambda = \partial_- \tilde{\Sigma}. \quad (4.3)$$

From this, we see that if we impose the canonical commutation conditions on $X$ and $\pi_X$, the following equal $x^+$-time commutation relations are implied:

$$[\tilde{\Sigma}(x), \tilde{\Sigma}(y)] = 0, \quad [\tilde{\Sigma}(x), \partial^+ \tilde{\Sigma}(y)] = i \delta(x^- - y^-), \quad [\partial^+ \tilde{\Sigma}(x), \partial^+ \tilde{\Sigma}(y)] = 0. \quad (4.4)$$

The integration constant is determined in the following way. To implement the residual gauge transformation, we add $\frac{m}{m^2 - n^2 \partial^+_+} C$ to $X$, where $C$ is conjugate to $B$ and satisfies the following commutation relations:

$$[C(x), C(y)] = 0, \quad \left[B(x), \frac{1}{m^2 - n^2 \partial^+_+} C(y)\right] = i \delta(x^+ - y^+). \quad (4.5)$$

Furthermore, we require that the infrared divergence resulting from $\frac{\partial^+}{\partial_-} \tilde{\Sigma}$ be cancelled through the mechanism worked out in §2. If we consider a surface of constant $x^-$, then we can write

$$\frac{\partial^+}{\partial_-} \tilde{\Sigma} = \frac{n_+ m^2 + n_- \partial_+ \partial_-}{m^2 - n^2 \partial^+_+} \tilde{\Sigma} \quad (4.6)$$

and see that the equal $x^-$-time commutator, $[\frac{\partial^+}{\partial_-} \tilde{\Sigma}(x), \frac{\partial^+}{\partial_-} \tilde{\Sigma}(y)]$, does not vanish. To correct for this, we must add $\frac{m n_+ n_-}{m^2 - n^2 \partial^+_+} \partial_+ B$ to $X$. Summing all the terms, we obtain

$$X = \frac{\partial^+}{\partial_-} \tilde{\Sigma} + \frac{m}{m^2 - n^2 \partial^+_+} \left( C - n^2 x^- B + \frac{n_+ n_-}{m^2 - n^2 \partial^+_+} \partial_+ B \right). \quad (4.7)$$

This is exactly the solution given in I. Therefore, we need not repeat the construction of the fermion field operators or the description of their properties.

It remains to be shown that infrared divergences do not appear when we evaluate $\langle 0 | X(x) X(y) | 0 \rangle$ at $x^+ = y^+$, or when we calculate the $x^+$-time ordered propagator for $X$. By using the representations for the constituent operators given in I, we obtain the vacuum expectation value

$$\langle 0 | X(x) X(y) | 0 \rangle = \Delta^{(+)}(x - y; m^2) + n^2 m^2 \Pi_{PSL}^{(+)}(x - y), \quad (4.8)$$

where

$$\Pi_{PSL}^{(+)}(x) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} dp_- e^{-ip_- x} \int_{0}^{\infty} dp^+ p^+ \quad (4.9)$$

$$- \frac{i x^-}{2\pi} \int_{0}^{\infty} dk_+ \frac{e^{-i k_+ x^+}}{m^2 + n^2 k_+^2} + \frac{1}{2\pi} \int_{0}^{\infty} dk_+ \frac{2 n_+ k_+ e^{-i k_+ x^+}}{(m^2 + n^2 k_+^2)^2}.$$
Here, $p^+$ and $p_+$ are defined, respectively, as $p^+ = \sqrt{p_-^2 + m_0^2}$ with $m_0^2 = -n^2 m^2$ and $p_+ = \frac{p_+^{n_+} p_-^{n_-}}{-n_-}$. The integral in the first line comes from $\Sigma$, while the integrals in the second line come from the residual gauge fields. The value of $E_{PSL}^{(\pm)}(x)$ at $x^+ = 0$ is formally given by

$$E_{PSL}^{(\pm)}(x)|_{x^+ = 0} = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} e^{-ip_- x^-} - \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} \frac{1}{p^+}$$

$$+ \int_{0}^{\infty} dk_+ \frac{1}{m^2 + n^2 k_+^2} + \frac{1}{2\pi} \int_{0}^{\infty} dk_+ \frac{2n_+ k_+}{(m^2 + n^2 k_+^2)^2}, \quad (4.10)$$

where we have divided the first term into a finite term and a divergent term. It should be noted here that $p_-$ is conjugate to the spatial variable $x^-$, while $k_+$ is conjugate to the temporal variable $x^+$. To make the infrared divergence cancellation mechanism work as in §3, both integration variables have to be either spatial or temporal. Therefore, we change the integration variable from the spatial $p_-$ to the temporal $p_+ = \sqrt{p_-^2 + m_0^2}$. Also, we denote $k_+$ by $p_+$. If we take account of the fact that $p_+$ is a two-valued function of $p_-$, then we can rewrite the divergent integral in the form

$$\int_{-\infty}^{\infty} \frac{dp_-}{p^+} \frac{1}{p^+} = \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p_+^2 - m_0^2}} \left( \frac{-n_-}{n_+ p_+ - \sqrt{p_+^2 - m_0^2}} \right)^2$$

$$+ \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p_+^2 - m_0^2}} \left( \frac{-n_-}{n_+ p_+ + \sqrt{p_+^2 - m_0^2}} \right)^2, \quad (4.11)$$

where the first term diverges, but the second term is finite. Now we see that if we combine the first integral in (4-11) with the third one in the second line of (4-10), we obtain the following finite integrals:

$$- \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p_+^2 - m_0^2}} \left( \frac{-n_-}{n_+ p_+ - \sqrt{p_+^2 - m_0^2}} \right)^2$$

$$+ \int_{0}^{\infty} dp_+ \frac{4n_+ p_+}{(m^2 + n^2 p_+^2)^2}$$

$$= - \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p_+^2 - m_0^2}} \left( \frac{n_+ p_+ + \sqrt{p_+^2 - m_0^2}}{m^2 + n^2 p_+^2} \right)^2$$

$$+ \int_{0}^{\infty} dp_+ \frac{4n_+ p_+}{(m^2 + n^2 p_+^2)^2}$$

$$= - \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p_+^2 - m_0^2}} \left( \frac{n_+ p_+ - \sqrt{p_+^2 - m_0^2}}{m^2 + n^2 p_+^2} \right)^2$$

$$+ \int_{0}^{m_0} dp_+ \frac{4n_+ p_+}{(m^2 + n^2 p_+^2)^2}$$

$$= - \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p_+^2 - m_0^2}} \left( \frac{-n_-}{n_+ p_+ + \sqrt{p_+^2 - m_0^2}} \right)^2$$

$$+ \int_{0}^{m_0} dp_+ \frac{4n_+ p_+}{(m^2 + n^2 p_+^2)^2}. \quad (4.12)$$
After tedious but straightforward calculations, we finally obtain

$$E_{PSL}^{(+)}(x)|_{x^+=0} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{dp_-}{p_-} \frac{1 - \cos(p_-x^-)}{p_-^2} + \frac{1}{2\pi m^2} \frac{-n_-}{1 + n_+}. \quad (4.13)$$

Finally, without demonstration, we give the $x^+$-time ordered propagator for $X$. Its demonstration can be carried out in parallel with that given in Appendix A of I. We find the following:

$$\langle 0| T(X(x)X(y))|0 \rangle = \frac{1}{(2\pi)^2} \int d^2 q \frac{i}{q^2 - m^2 + i\varepsilon} \left(1 - \frac{n^2m^2}{(q_+ + i\varepsilon \text{sgn}(q_+))^2}\right) e^{-iq\cdot(x-y)}. \quad (4.14)$$

It is noteworthy that as a consequence of the fact that all factors depending on the quantization coordinates drop out, we have the same propagator that we obtained in (3.14).

§5. Concluding remarks

In this paper, the framework used in I has been generalized by introducing the $\tau\sigma$-coordinates and at the same time simplified by bosonizing the model. The new framework has allowed us to investigate the way in which operator solutions develop a dependence on the quantization coordinates. In the new framework, we can take the dipole ghost field, $X$, and the component of the gauge field, $A_\sigma$, as canonical variables. We have given special attention to the determination of $X$, because we know that it cannot be a manifest Lorentz scalar, as it develops an explicit dependence on the quantization coordinates. We have found that the physical part of $X$ is determined uniquely by the gauge choice, while the residual gauge part, which contains the manifest dependence on the quantization coordinates, is determined by requiring that $X$ satisfy the canonical commutation conditions. The main findings of this paper are the following:

1. The residual gauge fields are necessary ingredients of the space-like axial gauge Schwinger model.
2. $(n\cdot\partial)^{-1} = (\partial_-)^{-1}$ is ill-defined for any quantization coordinates, as long as the gauge fixing direction is space-like (i.e. $n^2 < 0$).
3. As a consequence, the physical part of $X$ includes infrared divergences irrespective of the quantization coordinates, as long as the gauge fixing direction is space-like.
4. If we introduce the residual gauge fields in such a way that $X$ satisfies the canonical commutation relations, then the residual gauge part is determined so as to regularize the infrared divergences resulting from the physical part.

In the PSL case, the residual gauge fields cannot be canonical variables, due to the fact that they depend only on the evolution parameter $x^+$. Therefore the operator solution for this case cannot be constructed purely by canonical methods. We have overcome this difficulty by employing the items described in §4 as guiding principles supplementing canonical methods. The operator solution we obtain with the extended methodology is satisfactory in every respect. In particular, all ill-defined factors drop out of the $x^+$-time ordered propagators for $X$ and $A_\mu$, and therefore we
have the same ML form for the propagators for any quantization coordinates.

The light-cone gauge, $\theta = \frac{\pi}{4}$, is exceptional. In that case, $n^2$ is zero, and hence the manifest dependence on the quantization coordinates disappears whatever coordinates we may have. Therefore, the light-cone axial gauge formulation is obtained by simply setting $\theta = \phi = \frac{\pi}{4}$, and we obtain

$$X = \tilde{\Sigma} + m^{-1}C, \quad A_+ = 2\frac{\partial_+ \tilde{\Sigma}}{m} + \frac{\partial_+}{m^2}(C + \partial_+^{-1}B).$$

Comparing these operators with the corresponding operators we obtained in a previous paper,\textsuperscript{16} we find that $B$ and $C$ are related to $\eta$ and $\phi$ as $C = m(\eta + \phi)$ and $B = m\partial_+(\eta - \phi)$.

We end this paper by pointing out that in axial gauge quantizations in 4 dimensions, the same infrared divergence cancellation mechanism acts. We already have some work on this topic.\textsuperscript{17} We hope to report more in subsequent studies.

### Appendix A

---

**Conventions**

In this appendix we list the conventions we use. The space-like constant vector $n$ is specified in ordinary coordinates as

$$n^\mu = (n^0, n^3) = (\cos \theta, -\sin \theta), \quad (A.1)$$

which yields $n^2 = \cos 2\theta$. The direction of $n$ is $x_-$ in the $\pm$-coordinates — $x^\mu = (x^+, x^-)$ — defined by (1.1). In addition, we define the $\tau\sigma$ coordinates — $x^\mu = (x^\tau, x^\sigma)$ — by (1.3), in order to allow for simple reference to the quantization surface, which is space-like but not parallel to $n$. The coordinates $x^+$ and $x^-$ are expressed in terms of $x^\tau$ and $x^\sigma$ as

$$x^+ = \cos(\phi - \theta)x^\tau - \sin(\phi - \theta)x^\sigma, \quad x^- = \sin(\phi - \theta)x^\tau + \cos(\phi - \theta)x^\sigma. \quad (A.2)$$

The derivative with respect to $x^\mu$ is defined by $\frac{\partial}{\partial x^\mu} = \partial_\mu$. Consequently, $\partial_+$ and $\partial_-$ are expressed in terms of $\partial_\tau$ and $\partial_\sigma$ as

$$\partial_+ = \cos(\phi - \theta)\partial_\tau - \sin(\phi - \theta)\partial_\sigma, \quad \partial_- = \sin(\phi - \theta)\partial_\tau + \cos(\phi - \theta)\partial_\sigma. \quad (A.3)$$

Inverting these expressions yields

$$\partial_\tau = \cos(\phi - \theta)\partial_+ - \sin(\phi - \theta)\partial_-, \quad \partial_\sigma = -\sin(\phi - \theta)\partial_+ + \cos(\phi - \theta)\partial_. \quad (A.4)$$

(1) Substituting (A.1) into (1.3) gives us the expression for $n$ in the $\tau\sigma$ coordinates:

$$n^\mu = (n^\tau, n^\sigma) = (\sin(\phi - \theta), \cos(\phi - \theta)). \quad (A.5)$$

In $\tau\sigma$ coordinates, the metric tensor is given by

$$g_{\tau\tau} = -\cos 2\phi, \quad g_{\sigma\tau} = g_{\tau\sigma} = \sin 2\phi, \quad g_{\sigma\sigma} = \cos 2\phi, \quad (A.6)$$

$$g^{\tau\tau} = -\cos 2\phi, \quad g^{\sigma\tau} = g^{\tau\sigma} = \sin 2\phi, \quad g^{\sigma\sigma} = \cos 2\phi. \quad (A.7)$$
and therefore the lower components of $n$ are given by

$$n_\mu = (n_\tau, n_\sigma) = (\sin(\phi + \theta), \cos(\phi + \theta)). \quad (A.8)$$

Because the mass of $\tilde{\Sigma}$ is $m$, the mass shell condition is given by

$$0 = p_\mu p^\mu - m^2 = -\cos 2\phi p_\tau^2 + 2\sin 2\phi p_\tau p_\sigma + \cos 2\phi p_\sigma^2 - m^2. \quad (A.9)$$

Solving this equation, we obtain

$$p_\tau = -\sin 2\phi p_\sigma + \sqrt{p_\sigma^2 - \cos 2\phi m^2}. \quad (A.10)$$

By comparing (A.10) with the relation $p^\tau = -\cos 2\phi p_\tau + \sin 2\phi p_\sigma$, we see that $p^\tau$ is given by

$$p^\tau = \sqrt{p_\sigma^2 - \cos 2\phi m^2}, \quad (A.11)$$

which indicates that $p^\tau$ is the energy of the Schwinger particle of momentum $p_\sigma$. Note, in particular, that (A.10) enables us to obtain $\partial_-$, $\partial_-^{-1}$ and $\partial^+$, when they are applied to $\tilde{\Sigma}$, as

$$\begin{align*}
\partial_+ &= n^\tau \partial_\tau + n^\sigma \partial_\sigma = \frac{n^\tau \partial^\tau - n_\sigma \partial_\sigma}{-\cos 2\phi}, \quad \frac{1}{\partial_-} = -\frac{n^\tau \partial^\tau + n_\sigma \partial_\sigma}{m^2(n^\tau)^2 - n^2 \partial_\sigma^2}, \\
\partial_- &= n^\tau \partial_\tau + n^\sigma \partial_\sigma = \frac{-n_\sigma \partial^\tau + n^\tau \partial_\sigma}{-\cos 2\phi}, \quad \partial_+ = -\frac{m^2 n^\tau n_\sigma + n_\sigma \partial_\sigma}{m^2(n^\tau)^2 - n^2 \partial_\sigma^2}. \quad (A.12)
\end{align*}$$

(2) The pure space-like case is realized by setting $\phi = \theta$ and thus, in $\pm$-coordinates, we have

$$(n^+, n^-) = (0, 1), \quad (n_+, n_-) = (\sin 2\theta, \cos 2\theta), \quad (A.14)$$

$$p_+ = \frac{p^+ - n_+ p_-}{n_-}, \quad p^+ = \sqrt{p_-^2 - \cos 2\theta m^2}. \quad (A.15)$$

The quantity $p_+$ is a two-valued function of $p_-$, and it takes its minimum value, $\sqrt{-n_- m \equiv m_0}$, at $p_- = \frac{n_+}{n_-} m_0$. Therefore, when we change the integration variable from $p_-$ to $p_+$, we use $p_- = \frac{n_+ p_+ - \sqrt{p_+^2 - m_0^2}}{-n_-}$ in order to allow the region $m_0 \leq p_+ < \infty$ to correspond to $-\infty < p_- \leq \frac{n_+}{n_-} m_0$, whereas we use $p_- = \frac{n_+ p_+ + \sqrt{p_+^2 - m_0^2}}{-n_-}$ in order to allow the region $m_0 \leq p_+ < \infty$ to correspond to $\frac{n_+}{n_-} m_0 \leq p_- < \infty$. In the case that $p_- = \frac{n_+ p_+ + \sqrt{p_+^2 - m_0^2}}{-n_-}$, we have

$$p^+ = \frac{p_+ - n_+ \sqrt{p_+^2 - m_0^2}}{-n_-}, \quad p^- = \sqrt{p_+^2 - m_0^2}, \quad (A.16)$$

whereas in the case that $p_- = \frac{n_+ p_+ + \sqrt{p_+^2 - m_0^2}}{-n_-}$, we have

$$p^+ = \frac{p_+ + n_+ \sqrt{p_+^2 - m_0^2}}{-n_-}, \quad p^- = -\sqrt{p_+^2 - m_0^2}. \quad (A.17)$$
The gamma matrices $\gamma^\tau$ and $\gamma^\sigma$ are defined by

$$\gamma^\tau = \gamma^0 \sin \phi + \gamma^3 \cos \phi, \quad \gamma^\sigma = \gamma^0 \cos \phi - \gamma^3 \sin \phi,$$

(A.18)

where

$$\gamma^0 = \sigma_1, \quad \gamma^3 = i\sigma_2, \quad \gamma^5 = -\sigma_3.$$

(A.19)

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