Invariance of the bit error rate in the ancilla-assisted homodyne detection

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We investigated the minimum achievable bit error rate of discriminating binary coherent states by homodyne measurements with the help of arbitrary ancilla states coupled via beam splitters. We took into account arbitrary pure ancillary states those were updated in real-time by those partial measurements and classical feedforward operations where the partial measurements were made by homodyne detections with a common fixed phase. It was shown that the minimum bit error rate of the system was invariant under these operations. We also discussed how generalize the homodyne detection beyond the scheme which made the bit error rate invariant.

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INTRODUCTION

Coherent states are fundamental information carriers in optical and quantum communications. They transmit in pure state intact even through a lossy channel. A primitive modulation scheme is the binary phase shift keying (BPSK) with $|\pm\alpha\rangle$. Because these two signals are not orthogonal, i.e. $\langle\alpha|\pm\alpha\rangle \neq 0$, it has been a central issue in quantum communications how to realize a quantum receiver that can discriminate them as small error as possible.

Toward approaching the minimum bound for the average error probability (the bit error rate, BER), the so-called Helstrom bound [1], optimal [2–6] and near-optimal [7–9] receivers have been studied theoretically. Some of them have been put into experimental demonstrations [8–10].

These experiments were based on photon counters with additional feedback [2, 3, 10] or coherent displacement operations [7–9]. However, due to severe limitations on their detection efficiency, the results demonstrated so far are still far from the theoretical limit without compensating the detection efficiencies. Although there is a rapid progress in the development of highly efficient photon counters [11, 12], these are still very advanced technologies and not always available in quantum optics labs.

Another important attempt is to use homodyne detectors. Homodyne detection is already an well matured technique, and is used to discriminate the BPSK signals in conventional optical communications. Some quantum receiver schemes have been proposed in which a homodyne detector is combined with additional nonlinear processes, such as Kerr medium, or higher nonlinear media [4, 6]. It is, however, still a formidable task to implement highly nonlinear unitary processes with low losses.

Instead of direct nonlinear unitary processes, measurement induced nonlinear process is another attractive possibility to combine with homodyne detectors. Such a process is made of non-classical ancillary states, linear optical circuit, homodyne detector, and classical feedforward system. It was already known that Gaussian operations and classical feedforward alone cannot overcome the homodyne BER limit [1]. Therefore non-Gaussian elements, or non-Gaussian ancillary inputs, are required essentially. Such states are currently available in the laboratory [13–17]. It would then be an interesting question whether the BER can be reduced below the homodyne BER limit by introducing non-Gaussian ancillary inputs to the homodyne-feedforward linear optical circuit.

In this paper, we investigate the achievable BER for the BPSK signals when one is available to use arbitrary pure ancillary state, beam splitter, homodyne detector, and updating the ancillary inputs based on the homodyne results. We show that under any updating of the beam splitter parameters and the ancillary inputs, the BER is invariant, being exactly the same as the homodyne limit. Our result suggests how to set up the system with homodyne detection in order to overcome the basic homodyne limit.

MULTI-ANCILLA RECOMBINATIONS

Let us show our system in FIG. 1. The received signal $|s\rangle$ ($s = \pm\alpha$, and without loss of generality $\alpha$ is real) is combined with the first ancillary state $|\psi_1\rangle$ via a beam splitter of a transmittance $\cos\theta_1$, the first homodyne detector, outputting a result $y_1$. This result is used for updating the second ancilla and the beam splitter transmittance. Any explicit expression of the updating rule is not needed to show the BER invariance.

To describe the updating, we introduce notations. Ancillary state $|\psi\rangle$ is expanded by the coherent state as

$$|\psi\rangle = \int \frac{d\alpha}{\pi} |\alpha\rangle \psi(\alpha),$$  \hspace{1cm} (1)
The received signal $|s\rangle$ is combined with the ancillary states $|\psi_1\rangle$, ..., $|\psi_N\rangle$ successively via the beam splitters with transmittance $\cos \theta_n$. Both $|\psi_n\rangle$ and $\theta_n$ are updated based on the homodyne measurement results of ancillae. The signal is finally discriminated by a homodyne detector. The parameters $x_{n-1} = u_n$ correspond to the quadrature values used for the expansion of the transformed signal state in the $n$-th step.

where $\psi(\alpha) \equiv \langle \alpha | \psi \rangle$, and $d\alpha \equiv d\alpha_R \wedge d\alpha_I$ for the decomposition $\alpha = \alpha_R + i\alpha_I$ with $\alpha_R, \alpha_I \in \mathbb{R}$. Denoting the signal port (the horizontal line in Fig. 1) by the subscript “0”, and the $n$-th ancillary port by the subscript “$n$”, the whole $(n + 1)$-mode state up to the $n$-th step is recursively defined by

$$
|\Psi_n\rangle_{0,1,\ldots,n} \equiv \hat{B}_{0,n}(\theta_n)|\Psi_{n-1}\rangle_{0,1,\ldots,n-1} \otimes |\psi_n\rangle_n
$$

$$
= \int \frac{d\alpha}{\pi^{N/2}} |s'_n\rangle_0 \otimes |\alpha'_1\rangle_1 \psi_1(\alpha_1) \otimes \cdots \otimes |\alpha'_n\rangle_n |\psi_n(\alpha_n)\rangle. \quad (2)
$$

The action of the beam splitter $\hat{B}_{0,n}(\theta) \equiv \exp[\theta(\hat{a}_n^\dagger \hat{a}_n - \hat{a}_n^\dagger \hat{a}_n)]$ between the 0-th and $n$-th port is given by the matrix representation

$$
\begin{pmatrix}
\begin{pmatrix}
\hat{s}_n'
\end{pmatrix}'
\
\alpha_n
\end{pmatrix} = R(\theta_n)
\begin{pmatrix}
\begin{pmatrix}
\hat{s}_{n-1}'
\end{pmatrix}'
\
\alpha_n
\end{pmatrix},
$$

$$
R(\theta) \equiv \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix},
$$

where $s'_0 \equiv s$. Here we notice that the amplitude $s'_n$ depends on all the $n$ coherent amplitudes $\alpha^{(n)} \equiv (\alpha_1, \ldots, \alpha_n)$.

Now let $y_n \ (n = 1, \ldots, N)$ be the $n$-th homodyne result and $x$ be the homodyne result at port “0”. The parameter $x$ is a $N$-vector of the quadrature amplitude $\hat{x} \equiv (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$, namely $\langle x| \hat{x} = x \rangle$. The wave function is then

$$
\Psi_N(x, y^{(N)}) \equiv \langle x|_0 \otimes \langle y_1|_1 \otimes \cdots \otimes \langle y_N|_N |\Psi_N\rangle_0 \otimes \cdots \otimes |\Psi_N\rangle_N
$$

$$
= \int \frac{d^N\alpha}{\pi^N} \psi_1(\alpha_1) \cdots \psi_N \langle x|_{s_N'}y_1|\alpha_1\rangle \cdots \langle y_N|\alpha_N\rangle \quad (5)
$$

with

$$
\langle x| \alpha \rangle \equiv \frac{1}{\sqrt{\pi}} \exp \left[ -\frac{|\alpha|^2}{2} - \frac{1}{2} (x - \sqrt{2} \alpha)^2 \right]. \quad (6)
$$

The probability density of Eq. (5) is

$$
P(x, y^{(N)}|s) \equiv |\Psi_N(x, y^{(N)})|^2 = \int \frac{d^N\alpha d^N\beta}{\pi^{N+1}} \psi_1^*(\alpha_1) \psi_1(\beta_1) \cdots \psi_N^*(\alpha_N) \psi_N(\beta_N)
$$

$$
\times \exp \left[ J - (x - s_N')^2 - \sum_{n=1}^N (y_n - \alpha_n')^2 \right], \quad (7)
$$

where

$$
J \equiv \frac{1}{2} \sum_{n=1}^N \left[ (\alpha_n' \beta_n - \beta_n' \alpha_n)^2 - |\alpha_n - \beta_n|^2 \right], \quad (8)
$$

$$
\left( \begin{array}{c}
\alpha_n'' \\
\alpha_n''
\end{array} \right) \equiv \frac{1}{\sqrt{2}} \left( s_{n}^{\prime\prime} (\alpha^{(n)}) + s_{n}^{\prime\prime} (\beta^{(n)}) \right). \quad (9)
$$

**BIT ERROR RATE WITHOUT FEEDFORWARD**

We first evaluate the BER in the case where the signal is discriminated by the $(N + 1)$ outcomes of the homodyne detectors $v^{(N)}$ and $x$ without any feedforward in Fig. 1. Putting $x_N \equiv x$, we transform the output variables $(x_n, y_n)$ into the new ones $(u_n, v_n)$ by applying

$$
\begin{pmatrix}
\begin{pmatrix}
u_n
\end{pmatrix}'
\
u_n
\end{pmatrix} = R(\theta_n)^{-1} \begin{pmatrix}
\begin{pmatrix}
u_n
\end{pmatrix}'
\
u_n
\end{pmatrix},
$$

in a descending order $n = N, N - 1, \ldots, 2, 1$, iteratively putting $x_{n-1} \equiv u_n$. The quadratic parts in Eq. (11), say $\Omega^{(N)}$, is invariant under all of the rotational operations (4), and then is equivalent to

$$
\Omega^{(N)} = (x_N - s_N'')^2 + \sum_{n=1}^N (y_n - \alpha_n'')^2 = (x_0 - \sqrt{2} s)^2 + \sum_{n=1}^N \left( v_n - \frac{\alpha_n' + \beta_n}{\sqrt{2}} \right)^2.
$$

Noting that the Jacobian of this transformation is $\prod_{n=1}^N \det R(\theta_n) = 1$, the conditional probability is now expressed in terms of the new variables $x_0, v^{(N)} = (v_1, \ldots, v_N)$ as

$$
P(x, v^{(N)}|s) \rightarrow P(x_0, v^{(N)}|s) = P(x_0|s) P(v^{(N)}|s), \quad (11)
$$

where

$$
P(x_0|s) \equiv \frac{1}{\sqrt{\pi}} \exp \left[ -(x_0 - \sqrt{2} s)^2 \right],$$

$$
P(v^{(N)}|s) \equiv \int \frac{d^N\alpha d^N\beta}{\pi^{N+1}} \psi_1^*(\alpha_1) \psi_1(\beta_1) \cdots \psi_N^*(\alpha_N) \psi_N(\beta_N)
$$

$$
\times \exp \left[ J - \sum_{n=1}^N \left( v_n - \frac{\alpha_n' + \beta_n}{\sqrt{2}} \right)^2 \right]. \quad (12)
$$

Thus it can be factorized into two parts, and the probability of obtaining the value $x_0$ is independent on those of $v^{(N)}$. 

Therefore the BER can be calculated independent of the ancillary homodyne outcomes. So, by introducing the threshold
\[ x_{th} = \frac{1}{4\sqrt{2}\alpha} \ln \left( \frac{p(-\alpha)}{p(\alpha)} \right) \] (13)
determined by \( P(x_{th}|\alpha)p(\alpha) = P(x_{th} - \alpha)p(-\alpha) \) in a priori probability \( p(\pm \alpha) \), the BER is given by
\[
BER_N = \int_{-\infty}^{\infty} d\mathbf{v}^{(N)} \left[ \int_{-\infty}^{x_{th}} dx_0 P(x_0, \mathbf{v}^{(N)}|\alpha)p(\alpha) 
+ \int_{x_{th}}^{\infty} dx_0 P(x_0, \mathbf{v}^{(N)}|\alpha)p(\alpha) \right] 
= \int_{-\infty}^{x_{th}} dx_0 P(x_0|\alpha)p(\alpha) + \int_{x_{th}}^{\infty} dx_0 P(x_0|\alpha)p(-\alpha) 
\equiv BER_0, \] (14)
which is identical with the homodyne limit. For the equivalent a priori probabilities \( p(\alpha) = p(-\alpha) = 1/2 \), we have \( BER_0 = \text{erfc}(\sqrt{2}\alpha)/2 \) as is well known.

**BIT ERROR RATE WITH FEEDFORWARD**

Next we consider generic feedforward by updating \((\psi_n, \theta_n)\) based on the homodyne outcomes \( \mathbf{y}^{(n-1)} \). Suppose the parameters \((\psi_n, \theta_n)\) are the functions of the previously measured values \( \mathbf{y}^{(n-1)} \) of the homodyne detections, and then are regarded as functions of \( v_1, \ldots, v_{n-1} \) through the rotation \((10)\). After changing variables via this rotation \((10)\), the generic functional relations become
\[
\begin{align*}
\psi_n &= \psi_n(x_0, \mathbf{v}^{(n-1)}), \\
\theta_n &= \theta_n(x_0, \mathbf{v}^{(n-1)}),
\end{align*}
\] (15)
for \( n = 2, 3, \ldots, N \). These are shown by eliminating \( \mathbf{y}^{(n-1)} \) from \((\psi_n(\mathbf{y}^{(n-1)}), \theta_n(\mathbf{y}^{(n-1)}))\) by iteratively applying
\[
\begin{align*}
y_m &= v_m \cos \theta_m + x_{m-1} \sin \theta_m, \\
x_m &= x_{m-1} \cos \theta_m - v_m \sin \theta_m,
\end{align*}
\]
for \( m = n-1, n-2, \ldots, 2, 1 \).

The parameters \((\psi_1, \theta_1)\) are given prior to any ancillary states. The \( BER_N \) will be calculated with the help of the relation \((15)\) in the feedforward control case. Note that although the probability density of \( y_n \) depends on the value of signal \( s \), there is no functional relation with the parameters \((\psi_n, \theta_n)\) to the value of signal \( s \). The signal \( s \) affects the value of \( y_n \) only through changing its probability distribution. The functional relations \((15)\) show that ancillary states do not depend directly on the input signal \( s \), but depend only through probability densities of \( y_n \)'s. This fact helps us to prove the invariance of the BER for the updating operation considered here. Because the explicit dependence on the parameter \( s \) is confined in the probability distribution \( P(x_0|s) = \exp[-(x_0 - \sqrt{2}s)^2]/\sqrt{\pi} \), it is not the variables \( v_1, \ldots, v_N \) but \( x_0 \) that is affected by the value of the signal \( s = \pm \alpha \).

By using the maximum likelihood method, a \( N \)-dimensional boundary hyper-surface on the \((N + 1)\)-dimensional parameter space \( \{(x_0, \mathbf{v}^{(N)}) \in \mathbb{R}^{N+1}\} \) that divides the likelihood regions belonging to the values \( s = \pm \alpha \in \mathbb{R} \) is determined by
\[
P(x_0, \mathbf{v}^{(N)}|\alpha)p(\alpha) = P(x_0, \mathbf{v}^{(N)}|\alpha)p(-\alpha).
\] (16)
Here we note that the input signal \( s = \pm \alpha \) appears in the probability \( P(x_0, \mathbf{v}^{(N)}|s) \) only through the overall factor \( \exp[-(x_0 - \sqrt{2}s)^2] \), which is the same as Eq. \((12)\). Then, the boundary condition \((10)\) does not affect the variables \( \mathbf{v}^{(N)} \), although does determine the threshold point of \( x_0 \) as \( x_0 = x_{th} \), which is the same value as that in the simplest homodyne case.

Now, let us calculate the bit error rate \( BER_N \) in the feedforward control. The definition
\[
BER_N' \equiv \int_{-\infty}^{\infty} d\mathbf{v}^{(N)} \left[ \int_{-\infty}^{x_{th}} dx_0 P(x_0, \mathbf{v}^{(N)}|\alpha)p(\alpha) 
+ \int_{x_{th}}^{\infty} dx_0 P(x_0, \mathbf{v}^{(N)}|\alpha)p(-\alpha) \right] 
\] (17)
is the same as that of \( BER_N \) except the coefficient \( \psi_n \) of the probability density is a function of \( x_0 \) and \( \mathbf{v}^{(n-1)} \) for \( n = 2, \ldots, N \), as seen from Eqs. \((15)\). We perform the integrations from \( v_N \) to \( v_1 \) in this ordering, using the normalization conditions \( \langle \psi_n|\psi_n \rangle = 1 \) for \( n = N, \ldots, 1 \), and then integrate \( x_0 \) with the threshold point \((13)\).

More concretely, let us consider the first integral \( \int d\mathbf{v}_N \). From Eq. \((15)\) the coefficients \( \psi_2, \ldots, \psi_N \) does not depend on the variable \( v_N \), although it does depend on the variables \( v_1, \ldots, v_{N-1} \). Then, we carry out the integration \( \int dx_0 \) by using the normalization condition \( \langle \psi_N|\psi_N \rangle = 1 \), and we eliminate the coefficient \( \psi_N \). The same step is repeated for \( v_{N-1}, \ldots, v_1 \), and then we eliminate the coefficients \( \psi_{N-1}, \ldots, \psi_1 \). Finally, there remains one integral \( \int dx_0 \). Now, this integration becomes the same one as that without feedforward and that without ancillary states (the simplest homodyne case). Finally, we obtain the same formula \((14)\).

**DISCUSSION**

The invariance proof relies on the transformation property of homodyne measurement base, i.e., the eigenstates of the quadrature amplitude and phase. The tensor product of those bases are transformed into a separable state
through the beam splitter

\[ |x\rangle_a \otimes |y\rangle_b = \hat{B}_{ab}(\theta)|x\rangle_a \otimes |y\rangle_b = |x \cos \theta + y \sin \theta \rangle_a \otimes |y \cos \theta - x \sin \theta \rangle_b, \tag{18} \]

where \( \hat{B}_{ab}(\theta) = \exp[i\theta(x_b p_a - p_a x_b)] \). This is sharply contrasted to the beam splitter transformation property of the number state bases. For example,

\[ \hat{B}_{ab}|1\rangle_a \otimes |0\rangle_b = \cos \theta |1\rangle_a \otimes |0\rangle_b + \sin \theta |0\rangle_a \otimes |1\rangle_b, \tag{19} \]

and the output states are entangled. By the very fact of Eq. (15), the homodyne distributions of the \((N + 1)\) mode systems of the signal and generic ancillae can be reduced to the same distribution of the homodyne detection without any ancilla. This shows that the homodyne detectors fail in causing any useful quantum interference to reduce the BER below the homodyne limit. Thus, the present setup does not have any necessary non-linearity to improve the BER performance.

The future topic remained is to take into account an adaptive updating of the local oscillator phase at each homodyne measurement. Let us briefly look at it at a glance. Varying the local oscillator phase in our model is equivalently realized by generalizing the beam splitting operation from the \( SO(2) \) group to the \( U(2) \) group. One of the most general parametrization of the \( U(2) \) group element is

\[ U = e^{i\delta} e^{i\phi_1 \sigma_x} e^{i\phi_2 \sigma_y} e^{i\chi \sigma_z} = e^{i\delta} \left( e^{i(\alpha + \chi)} \cos \theta \ -e^{-i(\alpha - \chi)} \sin \theta \right), \tag{20} \]

where \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices. In the two port case of the simplest example in which we have only one ancilla and one signal, let us adopt the most general beam splitter \( \hat{B}(U) \), the wave function becomes

\[ |\Psi\rangle = \hat{B}(U)|\Psi\rangle = \int \frac{d\alpha}{\pi} |s\rangle \otimes |\alpha\rangle \psi(\alpha), \]

\[ \left( \begin{array}{c} s' \\ \alpha' \end{array} \right) = U \left( \begin{array}{c} s \\ \alpha \end{array} \right). \tag{21} \]

Next, let us consider the arbitrary phase in the each homodyne measurement of the signal and the ancilla. The phase rotated coordinate base of the homodyne measurement is given as \( \langle x(\phi) | \psi_\phi = x(\phi'|) \rangle \) where \( \psi_\phi = q \cos \phi + p \sin \phi \). Then, we obtain

\[ P(x(\phi_0), y(\phi_1)|s) = \left| \langle x(\phi_0), y(\phi_1)|\Psi\rangle \right|^2 = \int \frac{d\alpha d\beta}{\pi^2} \psi(\alpha)^* \psi(\beta) e^{-K-(x-s')^2-(y-a')^2}, \tag{22} \]

where

\[ K = \frac{1}{2} (\alpha^* \beta - \alpha \beta^*) - \frac{1}{2} (\alpha - \beta)^2 \]

\[ s'' = \frac{s'(\alpha)^* e^{-i\phi_0} + s'(\beta) e^{i\phi_0}}{\sqrt{2}} \]

\[ a'' = \frac{\alpha'^* e^{-i\phi_1} + \beta' e^{i\phi_1}}{\sqrt{2}}. \]

Notice that the phase dependence of a local oscillator, say \( \phi \), is translated in terms of the phase of the coherent state as \( \langle x(\phi) | \alpha \rangle = \langle x | e^{i\phi} \rangle \). The rotation of the homodyne outcomes \( x, y \) and the reparametrization of the phases of the homodyne measurements forms a group \( SO(2) \times U(1)^2 \). If we could reparametrize \( x, y, \phi_0, \phi_1 \) to eliminate the beam splitter parameter \( U \) in the two port probability [22], the methodology used in this letter would be directly applicable and thus would obtain the same conclusion; namely, the BER is invariant. However, it is not the case in actual, that is, we cannot absorb all of these parameters in Eq. (22). Precisely, the unabsorbed parameters is in the coset group \( U(2)/(SO(2) \times U(1)^2) \). The remained question is to find a way of treating them to clarify whether this parameter space helps us to reduce the bit error rate further than that of a simple homodyne detection.

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