Bogolubov’s chain of equations method for temperature Wightman functions in thermodynamics of relativistic phase transition.

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Bogolubov’s chain of equations method for temperature Wightman functions is suggested for investigation of relativistic phase transition. The chain equations for the Wightman functions forming momentum–energy tensor are obtained. It is clarified that structure of the chain equations determines the basis approximation (the Hartree–Fock approximation) and corrections calculation algorithm. The basis approximation is investigated in details: renormalized equations for effective masses, order parameter and generating functional which reproduce those equations are obtained. Being considered on the solution of the gap equations for the effective masses the generating functional turns to nonequilibrium functional of free energy density, which allows to obtain phases stability conditions. Thermodynamic observables like heat capacity and sonic speed are calculated. The correction to the Hartree-Fock approximations is ascertained to be small for all temperatures excluding vicinity of the phases equilibrium point.

1 Introduction.

An idea that phase transition may take place in a quantum field system was first proposed by Kirznits [1]. First quantitative calculations were made by Kirznits and Linde [2], and later by Dolan and Jackiw [3], Weinberg [4]. Theory of relativistic phase transition (RPT) deals with nonequilibrium Landau functional (which is usually called “effective potential”). When quasiparticle approximation is used for calculation of effective potential from microscopic operational Lagrangian, a problem of quasiparticles mass spectra arises. Calculations made in Matsubara diagram technique showed, that phase transitions can’t be described in iterative perturbation theory approach — tachion pole in mass spectra of scalar particles and consequently imaginary terms in Matsubara “effective potential” appears far from the point of phase transition. Enhanced resuming schemes successfully solved the problem of tachion pole in mass spectra but not the problem of effective potential calculation.

As for calculation of nonequilibrium functional in Matsubara approach, with temperature Green functions obtained in the framework of resuming schemes, the extremum condition of such functional leads to equation for order parameter which is different from that obtained directly from equations of motion. Illusion of solving of the problem mentioned is achieved only by manipulation with high-temperature approximation of Bose-Einstein integrals.

Cornwall, Jackiw and Tomboulis suggested an “effective functional” of different kind [5]. In addition to equation for order parameter the ”CJT-functional” reproduces equations for full Green functions. At present $O(N)$—model is almost completely investigated in this approach in Hartree-Fock approximation. Nevertheless problems of renormalization of the functional,

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analytical analysis of the phases stability conditions, calculation of observables and estimation of Hartree-Fock approximation accuracy need to be discussed in details.

These problems (at least for $O(N)$—model) can be solved in alternative approach, based on Bogolubov’s chain of equations for temperature Wightman functions (WF).

In Wightman approach the equations of motion and observables of a quantum–field system are written in terms of Wightman functions. In this paper equations for all the functions forming energy–momentum tensor are presented. Field operators smoothing, used in axiomatic Wightman formalism, replaced with scheme of dimensional regularization. Self–consistency of that redefinition method is proved.

It is found out that effective masses of quasi–particles, which are formed by particles interactions with each other and vacuum, automatically arise in the Bogolubov’s chain equations. Thus, it is natural to use Hartree–Fock approximation as a basis approximation of reduction. Corrections calculation algorithm is also determined by the Bogolubov’s chain mathematical structure.

Hartree–Fock approximation was investigated in details. A functional (generating functional) which reproduce gap equations for effective masses and equation for order parameter is obtained. It is shown, that the generating functional can also be calculated from momentum–energy tensor components, which proves self–consistency of thermodynamic equations of state obtained from quantum equations of motion with those obtained from generating functional. The generating functional, being considered on solutions of the gap equations for effective masses turns into functional of nonequilibrium free energy, which is used for analytical analysis of the phases stability condition. Functional of equilibrium free energy, which is obtained by substitution of the order parameter on temperature dependence into nonequilibrium one, is used for calculation of thermodynamical observables: heat capacity and sonic speed.

An important problem of the Hartree–Fock approximation validity was clarified by calculation of correction of the first order of vanishing to this approximation.

A Lagrangian of the $O(N)$—model and equations of motions are written out in chapter 2. A definition of Wightman functions, symmetrized with regard to permutation of it’s arguments, chain’s equations for Wightman functions up to fourth rank and chain reduction algorithm are given in chapter 3. Equations for effective masses and order parameter, functionals and observables in Hartree–Fock approximation, and results of numerical calculations are given in chapter 4. Finally, chapter 5 is devoted to calculation of the first order of vanishing corrections to effective masses, order parameter and observables.

2 Lagrangian and equations of motion.

Operator equations of motions for quantum fields are obtained from Lagrangian of system of $N$ scalar fields $\phi_a$

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi_a \cdot \partial^\mu \phi_a + \mu^2 \phi_a \cdot \phi_a \right) - \frac{\lambda}{N} (\phi_a \cdot \phi_a)^2,$$

by standard variational procedure:

$$\partial_\mu \partial^\mu \phi_a - \mu^2 \phi_a + \frac{4\lambda}{N} (\phi_b \phi_b) \phi_a = 0.$$ 

For $\mu^2 < 0$, the theory is invariant under $O(N)$ transformations of the quantum fields. For $\mu^2 > 0$ and low temperatures spontaneous symmetry breaking described by nonzero vacuum expectation value of $\phi_N$,

$$\langle |\phi_N| \rangle = v$$
breaks down symmetry group to $O(N - 1)$ and leads to $N - 1$ Goldstone bosons.

$$\phi_N \equiv \varphi + v, \quad \phi_a \equiv \chi_a, \quad a = 1 \ldots N - 1$$

Equations for quantum fields $\chi_a$ and $\varphi$ look like:

$$\partial_\mu \partial^\mu \varphi - \mu^2(v + \varphi) + \frac{4\lambda}{N}(v^3 + 3v^2\varphi + 3v\varphi^2 + \varphi^3 + v\chi_a\chi_a + \varphi\chi_a\chi_a) = 0 \quad (2)$$

$$\partial_\mu \partial^\mu \chi_a - \mu^2\chi_a + \frac{4\lambda}{N}(v^2\chi_a + 2v\varphi\chi_a + \varphi^2\chi_a + \chi_b\chi_b\chi_a) = 0 \quad (3)$$

Momentum–energy tensor is obtained from Lagrangian \([\Box]\) by metric variation procedure. For homogeneous, isotropic and stationary system, momentum–energy tensor turns to:

$$\langle T^\nu_\mu \rangle = \langle \partial_\mu \varphi \partial^\nu \varphi \rangle + \langle \partial_\mu \chi_a \partial^\nu \chi_a \rangle - \delta^\nu_\mu \left[ \frac{\mu^2 v^2}{2} - \frac{\lambda v^4}{N} + \frac{\lambda}{N} \left( \langle \varphi^4 \rangle + \langle \chi_a\chi_a\chi_b\chi_b \rangle + 2\langle \varphi^2\chi_a\chi_a \rangle + 2\langle \varphi^3 \rangle + 2\langle \varphi\chi_a\chi_a \rangle \right) \right]. \quad (4)$$

### 3 Bogolubov’s chain of equations.

**Definition of symmetrical Wightman functions.** Dynamics of a system, which is determined by Lagrange equations of motions, can be described in terms of chain of equations for Wightman functions (WF) — so called Bogolubov’s chain.

Full $n$–point WF (WF of rank $n$) is an over state expectation value of $n$ quantum field operators, taken, generally speaking, at different space–time points.

$$\langle \varphi(x_1)\varphi(x_2)\cdots\varphi(x_l)\cdots\chi(x_m)\cdots\chi(x_n) \rangle.$$

It can be represented as a sum of various WF of lower rank (the sum of ranks in each product is equal to $n$) and so called correlative Wightman function of the same rank. This representation, essentially, is a definition of correlative WF. For homogeneous, isotropic space–time it is convenient to use WF symmetrized with regard to permutation of it’s arguments. Algorithm of it’s definition, which takes into account residual $O(N - 1)$ symmetry of the theory under consideration, is illustrated for two–point $G_{(x_1,x_2)}$, $D_{(x_1,x_2)}$ and four–point $G_{(x_1,x_2|x_3,x_4)}$ functions. Here left bottom index is the number of $\varphi$ fields and the right one is the number of $\chi_a$ fields which form the corresponding Wightman function.

1. An expectation value of a sum of every possible permutation of quantum–field operators is taken (symmetrization operation):

$$2G_{(x_1,x_2)} \equiv \langle \varphi(x_1)\varphi(x_2) \rangle_{\text{sym}} \equiv \langle \varphi(x_1)\varphi(x_2) \rangle + \langle \varphi(x_2)\varphi(x_1) \rangle,$$

$$2(N - 1)D_{(x_1,x_2)} \equiv \langle \chi_a(x_1)\chi_a(x_2) \rangle_{\text{sym}} \equiv \langle \chi_a(x_1)\chi_a(x_2) \rangle + \langle \chi_a(x_2)\chi_a(x_1) \rangle,$$

$$4(N - 1)G_{(x_1,x_2|x_3,x_4)} \equiv \langle \varphi(x_1)\varphi(x_2)\chi_a(x_3)\chi_a(x_4) \rangle_{\text{sym}} \equiv \langle \varphi(x_1)\varphi(x_2)\chi_a(x_3)\chi_a(x_4) \rangle + \langle \varphi(x_1)\varphi(x_2)\chi_a(x_4)\chi_a(x_3) \rangle +$$

$$+ \langle \varphi(x_1)\varphi(x_2)\chi_a(x_3)\chi_a(x_4) \rangle + \langle \varphi(x_1)\varphi(x_2)\chi_a(x_3)\chi_a(x_4) \rangle + \langle \varphi(x_2)\varphi(x_1)\chi_a(x_4)\chi_a(x_3) \rangle.$$
2. Four–point WF \( G_{22(x_1 x_2 | x_3 x_4)} \) can be represented as sum of various products of (non–symmetrized) WF of the second rank and corresponding correlative function:

\[
4(N-1)G_{22(x_1 x_2 | x_3 x_4)} = \left\langle \varphi(x_1)\varphi(x_2) \right\rangle \left\langle \chi a(x_3)\chi a(x_4) \right\rangle + \left\langle \varphi(x_1)\varphi(x_2) \right\rangle \left\langle \chi a(x_4)\chi a(x_3) \right\rangle + \\
\left\langle \varphi(x_2)\varphi(x_1) \right\rangle \left\langle \chi a(x_3)\chi a(x_4) \right\rangle + \left\langle \varphi(x_2)\varphi(x_1) \right\rangle \left\langle \chi a(x_4)\chi a(x_3) \right\rangle + 4(N-1)C_{22(x_1 x_2 | x_3 x_4)}
\]

3. The sum terms can be regrouped, which picks out symmetrized WF:

\[
G_{22(x_1 x_2 | x_3 x_4)} = G_{(x_1 x_2)}D_{(x_3 x_4)} + C_{22(x_1 x_2 | x_3 x_4)}.
\]

It should be noted that permutation of arguments separated by vertical line leave a Wightman function, in particular \( G_{22(x_1 x_2 | x_3 x_4)} \), invariant: \( G_{22(x_1 x_2 | x_3 x_4)} = G_{22(x_2 x_1 | x_3 x_4)} = G_{22(x_1 x_2 | x_4 x_3)} \).

It is convenient to divide WF which depend on odd number of arguments by expectation value \( v \).

Residual global and discrete \( (\chi a \rightarrow -\chi a) \) symmetries of the theory restrict number of nontrivial WF. In particular, there exists only two nontrivial three–point

\[
C_{03(x_1 x_2 x_3)} = \frac{1}{3!} v \left\langle \varphi(x_1)\varphi(x_2)\varphi(x_3) \right\rangle_{sym}
\]

\[
C_{21(x_1 x_2 | x_3)} = \frac{1}{2!} \left(\frac{1}{N-1}\right) \left\langle \chi a(x_i)\chi a(x_j)\varphi(x_3) \right\rangle_{sym}
\]

and three nontrivial four–point correlative WF: \( C_{22(x_1 x_2 | x_3 x_4)} \) and

\[
C_{04(x_1 x_2 x_3 x_4)} = \frac{1}{4!} \left(\frac{1}{N-1}\right) \left\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \right\rangle_{sym} - G_{(x_1 x_2)}G_{(x_3 x_4)}
\]

\[
- G_{(x_1 x_3)}G_{(x_2 x_4)} - G_{(x_1 x_4)}G_{(x_2 x_3)}
\]

\[
C_{40(x_1 x_2 | x_3 x_4)} = \frac{1}{4!} \left(\frac{1}{N-1}\right)^2 \left\langle \chi a(x_1)\chi a(x_2)\chi a(x_3)\chi a(x_4) \right\rangle_{sym} - \\
- \frac{1}{N-1} \left[ (N-1)D_{(x_1 x_2)}D_{(x_3 x_4)} + D_{(x_1 x_3)}D_{(x_2 x_4)} + D_{(x_1 x_4)}D_{(x_2 x_3)} \right]
\]

Correlative functions of higher rank are defined in the same way. Strictly speaking, written above Wightman functions are divergent and should be redefined by procedures of regularization and renormalization. \[1\] In this paper the scheme of dimensional regularization is used. This technique lets retain exact thermodynamic relations between equations of state, functionals and observables after renormalization.

In terms of Wightman functions momentum–energy tensor \( \mathbb{T}^{\nu} \) looks like:

\[
\left\langle T^{\nu}_{\mu} \right\rangle = \lim_{x \rightarrow x_1} \left( \partial_{\mu}(x)\partial_{\nu}(x_1)G_{(xx)} + (N-1)\partial_{\mu}(x)\partial_{\nu}(x_1)D_{(xx)} + \right) - \delta_{\mu\nu} \left( \frac{\mu^2 v^2}{2} - \frac{\lambda v^4}{N} \right) + \\
+ \frac{\lambda}{N} \left[ 3G_{(xx)}G_{(xx)} + \left( N^2 - 1 \right)D_{(xx)}D_{(xx)} + 2\left( N-1 \right)G_{(xx)}D_{(xx)} + \right] + \frac{\lambda}{N} \left[ C_{04(xx)} + \right. \\
+ \left. (N-1)^2 C_{40(xx)} + 2(N-1)C_{22(xx)} + 2v^2 C_{03(xx)} + 2v^2(N-1)C_{21(x|x)} \right]
\]

\[1\] In discussed formalism these operations replace quantum–field operators product smoothing, used in axiomatic quantum–field theory.
Derivation of the chain equations. First of all equations for WF forming momentum–energy tensor should be obtained.

Equation of state for vacuum expectation value is obtained by averaging (2):

\[ v \left[ -\mu^2 + \frac{4\lambda}{N} \left( v^2 + 3G_{xx} + C_{03(xx)} + (N - 1)D_{xx} + (N - 1)C_{21(xx|x)} \right) \right] = 0. \] (6)

Averaging of (3) gives

\[ 2v\langle \varphi \chi_a \rangle + \langle \varphi^2 \chi_a \rangle + \langle \chi_b \chi_b \chi_a \rangle = 0, \]

which is identity owing to mentioned above symmetry \( \chi_a \to -\chi_a \).

Averaging equations (3) and (4) multiplied by one or several operators, one obtain equations of the Bogolubov’s chain. After symmetrization, which is required in account of symmetrical definition of WF, commutators arise:

\[ \Delta \varphi(x_1) \equiv \varphi(x)\varphi(x_1) - \varphi(x_1)\varphi(x), \quad \Delta \chi(x_1) \equiv \frac{1}{N - 1} \left( \chi_a(x)\chi_a(x_1) - \chi_a(x_1)\chi_a(x) \right). \]

Equations for two–point WF look like:

\[ \partial_\mu \partial_\nu G(x_1) \equiv \left[ -\mu^2 + \frac{4\lambda}{N} \left( 3v^2 + 3G_{xx} + (N - 1)D_{xx} \right) \right] G(x_1) + \\
\quad + \frac{4\lambda}{N} \left[ \left( C_{04(xxx|x_1)} + (N - 1)C_{22(xx|x_1)} \right) + v^2 \left( 3C_{03(xxx)} + (N - 1)C_{21(xx|x_1)} \right) \right] = 0 \] (7)

\[ \partial_\mu \partial_\nu D(x_1) \equiv \left[ -\mu^2 + \frac{4\lambda}{N} \left( v^2 + G_{xx} + (N + 1)D_{xx} \right) \right] D(x_1) + \\
\quad + \frac{4\lambda}{N} \left[ (N - 1)C_{40(xx|x_1)} + C_{22(xx|x_1)} + 2v^2C_{21(xx|x_1)} \right] = 0 \] (8)

Owing to homogeneity, isotropy and stationarity of the system under consideration \( G(x_1) \) and \( D(x_1) \) depend only on module of two arguments difference \( |x - x_1| \); consequently, when \( x = x_1 \) their value are determined only by internal system parameters. Thus, emerge in equations (11), (12), all equations for WF and equations for commutators values

\[ m_1^2 \equiv -\mu^2 + \frac{4\lambda}{N} \left( 3v^2 + 3G_{xx} + (N - 1)D_{xx} \right), \] (9)

\[ m_2^2 \equiv -\mu^2 + \frac{4\lambda}{N} \left( v^2 + G_{xx} + (N + 1)D_{xx} \right), \] (10)

depend on the system state but not the 4–coordinates. Hence these values play role of parameters when chain equations are solved.

Taking into account the designations introduced, equations for two–point WF can be rewritten as follows:

\[ \partial_\mu \partial_\nu G(x_1) + m_1^2 G(x_1) + \frac{4\lambda}{N} \left[ \left( C_{04(xxx|x_1)} + (N - 1)C_{22(xx|x_1)} \right) + v^2 \left( 3C_{03(xxx)} + (N - 1)C_{21(xx|x_1)} \right) \right] = 0, \] (11)

\[ \partial_\mu \partial_\nu D(x_1) + m_2^2 D(x_1) + \frac{4\lambda}{N} \left[ (N - 1)C_{40(xx|x_1)} + C_{22(xx|x_1)} + 2v^2C_{21(xx|x_1)} \right] = 0. \] (12)
Equation for WF $C_{03(x_1 x_2)}$ looks like:

$$\partial_{\mu} \partial^{\mu} C_{03(x_1 x_2)} + m^2 C_{03(x_1 x_2)} +$$

$$+ \frac{4\lambda}{N} \left[ G(x_{x_1}) \left( 3C_{03(x_1 x_2)} + (N-1)C_{21(x_1 | x_2)} \right) + G(x_{x_2}) \left( 3C_{03(x_1 x_2)} + 
+ (N-1)C_{21(x_1 | x_1)} \right) + (N-1)C_{22(x_1 | x_2)} + 3C_{04(x_1 x_1 x_2)} + C_{05(x_1 x_1 x_2)} + 
+ (N-1)C_{23(x_1 | x_1 x_2)} \right] = -\frac{24\lambda}{N} \left[ G(x_{x_1})G(x_{x_2}) + \frac{1}{12}\Delta \varphi(x_{x_1}) \Delta \varphi(x_{x_2}) \right].$$

Since $C_{03(x_1 x_2)}$ is invariant under transformations $x \leftrightarrow x_1$, $x \leftrightarrow x_2$ and $x_1 \leftrightarrow x_2$, solution for this WF is a sum of solution of (13) and solutions obtained from that by permutations $x \leftrightarrow x_1$ and $x \leftrightarrow x_2$.

For $C_{21(x_1 | x_2)}$, which depends on $\varphi$ and $\chi_0$ quantum fields, one should derive two equations, which describe contributions of $\varphi$ and $\chi_0$ respectively.

The first one is derived from (3),

$$\partial_{\mu} \partial^{\mu} C_{21(x_1 | x_2)} + m^2 C_{21(x_1 | x_2)} +$$

$$+ \frac{4\lambda}{N} \left[ 2C_{22(x_1 | x_1 x_2)} + C_{23(x_1 | x_1 x_2)} + D(x_{x_1}) C_{03(x_1 x_2)} + 2G(x_{x_2}) C_{21(x_1 | x_2)} + 
+ (N-1)C_{41(x_1 | x_1 x_2)} + (N+1)D(x_{x_1}) C_{21(x_1 | x_2)} \right] = -\frac{8\lambda}{N} D(x_{x_1}) G(x_{x_2}),$$

and the second one from (2),

$$\partial_{\mu} \partial^{\mu} C_{21(x_1 | x_2)} + m^2 C_{21(x_1 | x_2)} +$$

$$+ \frac{4\lambda}{N} \left[ 3C_{22(x_1 | x_2 x_2)} + C_{23(x_1 | x_2 x_2)} + (N-1)C_{41(x_1 | x_2 x_2)} + 
+ (N-1)C_{40(x_1 | x_2 x_2)} + 2\lambda D(x_{x_2}) C_{21(x_1 x_2 | x_2)} + 2\lambda G(x_{x_2}) C_{21(x_1 x_2 | x_2)} \right] =$$

$$= -\frac{8\lambda}{N} \left[ D(x_{x_2}) D(x_{x_2}) + \frac{1}{12}\Delta \chi(x_{x_2}) \Delta \chi(x_{x_2}) \right].$$

Since $C_{21(x_1 | x_2)}$ is invariant under transformation $x \leftrightarrow x_1$ solution for this WF is a sum of solutions of (13) and (14), and solution of (14) after permutation $x \leftrightarrow x_1$. The same assertions are valid for all the other WF.

Equation for $C_{04(x_1 x_1 x_2 x_3)}$ is derived from (2):}
\[(N - 1)C_{24}(x_1 x_3 | x_1 x_2 x_3) \] = \(-\frac{\lambda}{N} \left[ 24G(x_1)G(x_2)G(x_3) + \\
+ \Delta \varphi(x_1) \Delta \varphi(x_2) G(x_3) + \Delta \varphi(x_2) \Delta \varphi(x_3) G(x_1) + \Delta \varphi(x_3) \Delta \varphi(x_1) G(x_2) \right] \]

Since the chain equations are quite bulky, a simple criterion of it’s propriety is very useful. For example WF \( C_{04}(x_1 x_2 x_3) \) is invariant of permutation of its arguments, so that equation (1) should be invariant under transformations \( x_1 \leftrightarrow x_2 \), \( x_1 \leftrightarrow x_3 \), \( x_2 \leftrightarrow x_3 \) (but, due to differential operator \( \partial_\mu \partial_\nu \), not the permutations involve \( x \)). Easy to check that the equation, written above, satisfy this criterion.

Equation for WF \( C_{40}(x_1 x_2 x_3) \) is derived from (3):

\[ \partial_\mu \partial_\nu C_{40}(x_1 x_2 x_3) + m_2^2 C_{40}(x_1 x_2 x_3) + \\
+ \frac{4\lambda}{N} \left(D(x_1)\left( (N + 1)C_{40}(x_1 x_2 x_3) + C_{22}(x_2 x_3 | x_1) \right) + 2\nu^2 C_{21}(x_1 x_2 | x_1) \right) + D(x_2)\left( C_{40}(x_1 x_1 x_3) + \\
+ 2C_{40}(x_1 x_1 x_3) + \frac{1}{N - 1} C_{22}(x_1 x_3 | x_1) \right) + 2\nu^2 \left[ C_{21}(x_1 x_2 | x_1)C_{21}(x_2 x_1 | x_1) + \\
+ C_{21}(x_2 x_1 | x_3)C_{21}(x_1 x_1 | x_3) + (N - 1)C_{21}(x_1 x_1 | x_3)C_{21}(x_2 x_1 | x_3) \right] + 2\nu^2 C_{41}(x_1 x_2 x_3 | x_1) + \\
+ C_{42}(x_1 x_2 x_3 | x_1) + (N - 1)C_{60}(x_1 x_2 x_3 | x_1) \right] = -\frac{8\lambda(N + 1)}{N(N - 1)} D(x_1)D(x_2)D(x_3) - \\
- \frac{\lambda}{N(N - 1)} \left[ \frac{(5N - 7)}{3} \Delta \chi(x_2) \Delta \chi(x_3)D(x_1) + \Delta \chi(x_1) \Delta \chi(x_2)D(x_3) + \\
+ \Delta \chi(x_1) \Delta \chi(x_3)D(x_2) \right] \]

For WF \( C_{22}(x_2 x_3 | x_1) \) one should obtain two independent equations. The first one is derived from (2) by multiplying by \( \chi_a(x_2) \chi_a(x_3) \varphi(x_1) \) and \( \chi_a(x_3) \chi_a(x_2) \varphi(x_1) \):

\[ \partial_\mu \partial_\nu C_{22}(x_2 x_3 | x_1) + m_1^2 C_{22}(x_2 x_3 | x_1) + \\
+ \frac{4\lambda}{N} \left[G(x_1) \left( (N - 1)C_{40}(x_1 x_2 x_3) + 3C_{22}(x_2 x_3 | x_2) \right) \right] + 2 \left[ D(x_2)C_{22}(x_2 x_3 | x_1) + \\
+ 3\nu^2 C_{21}(x_2 x_3 | x_1)C_{03}(x_3 x_1) + \frac{1}{N}C_{21}(x_1 x_2 | x_1)C_{21}(x_2 x_1 | x_1) + 2C_{21}(x_2 x_1 | x_1)C_{21}(x_2 x_2 | x_1) + \\
+ (N - 1)C_{42}(x_2 x_3 x_3 | x_1) \right] = -\frac{8\lambda}{N} \left[ D(x_2)D(x_3)G(x_1) + \frac{1}{12} \Delta \chi(x_2) \Delta \chi(x_3)G(x_1) \right] ; \]

and the second one from (3) by transformation \( x \to x_2 \) and multiplying by \( \chi(x_3) \varphi(x_3) \varphi(x_1) \) and \( \chi(x_3) \varphi(x_1) \varphi(x) \):

\[ \partial_\mu \partial_\nu C_{22}(x_2 x_3 | x_1) + m_2^2 C_{22}(x_2 x_3 | x_1) + \\
+ \frac{4\lambda}{N} \left[D(x_2)C_{40}(x_3 x_1 x_2) + (N + 1)C_{22}(x_3 x_3 | x_3) \right] + 2 \left[ D(x_2)C_{22}(x_2 x_3 | x_1) + \\
+ 3\nu^2 C_{21}(x_2 x_3 | x_1)C_{03}(x_3 x_1) + \frac{1}{N}C_{21}(x_1 x_2 | x_1)C_{21}(x_2 x_1 | x_1) + 2C_{21}(x_2 x_1 | x_1)C_{21}(x_2 x_2 | x_1) + \\
+ (N - 1)C_{42}(x_2 x_3 x_3 | x_1) \right] = -\frac{8\lambda}{N} \left[ D(x_2)D(x_3)G(x_1) + \frac{1}{12} \Delta \chi(x_2) \Delta \chi(x_3)G(x_1) \right] ; \]
contain small constant of self–interaction, correlative WF $C$ functions, which describe weak effects of many–particles interaction, in considered iterative approximation is dictated by the chain mathematical structure.

Energy spectrum, which, in turn, can be obtained only after renormalization.

To necessity of WF redefinition: in order to renormalize divergent integrals one should know to an algorithm specified. Use of this approach in the considered formalism is difficult due to the sources.

First order of vanishing in compare with $G$ rank Wightman functions are evaluated as a certain combination of lower–rank WF according to

\[ G(x_1 x_2) C_{22}(x_2 x_3 | x_2 x_3) + 2v^2 \left[ D(x_2 x_3) C_{03}(x_1 x_2) + G(x_1 x_2) C_{21}(x_2 x_3 | x) \right] + \]

\[ + G(x_2 x_3) C_{21}(x_2 x_3 | x_1) + v^2 \left[ 2C_{21}(x_2 x_3 | x_2) C_{03}(x_1 x_2) + C_{21}(x_2 x_3 | x_1) C_{03}(x_2 x_2) \right] + \]

\[ + C_{21}(x_2 x_3 | x) C_{03}(x_1 x_2 x_2) \right] + v^2(N + 1) \left[ C_{21}(x_2 x_2 | x_1) C_{21}(x_2 x_3 | x_1) + C_{21}(x_2 x_2 | x_1) C_{21}(x_2 x_3 | x) \right] + \]

\[ + (N - 1) C_{21}(x_2 x_2 x_2 | x_1) + C_{24}(x_2 x_3 | x_1 x_2 x_2) + 2v^2 C_{23}(x_2 x_3 | x_1 x_2) \right] = \]

\[ = - \frac{8\lambda}{N} \left[ D(x_2 x_3) G(x_2 x_2) G(x_1 x_2) + \frac{1}{12} \Delta \varphi(x_2 x_2) \Delta \varphi(x_2 x_1) D(x_2 x_1) \right] \]

Equations for commutators, which present almost in all chain equations, look like:

\[ \partial_\mu \partial_\nu \Delta \varphi(x_1 x_1) + m_1^2 \Delta \varphi(x_1 x_1) = 0, \]  

(17)

\[ \partial_\mu \partial_\nu \Delta \chi(x_1 x_1) + m_2^2 \Delta \chi(x_1 x_1) = 0. \]  

(18)

In high symmetry phase, where vacuum expectation value is equal to zero, symmetry with regard to $\varphi \to -\varphi$ transformation is restored, consequently all WF which depend on odd number of $\varphi$ quantum–field operators are identically equal to zero. The chain equations for this phase are easily derived from those for low–symmetry phase.

The chain reduction. Two approaches to the chain reduction are known: model–approximative and iterative.

In model–approximative approach a few first chain equations are solved, and the higher rank Wightman functions are evaluated as a certain combination of lower–rank WF according to an algorithm specified. Use of this approach in the considered formalism is difficult due to necessity of WF redefinition: in order to renormalize divergent integrals one should know energy spectrum, which, in turn, can be obtained only after renormalization.

In iterative approach corrections to a basis approximation are calculated. Choice of basis approximation is dictated by the chain mathematical structure.

A chain equation is a generalized D’Alamber equation

\[ \partial_\mu \partial_\nu C_{nm}(x ... x_m) + m^2 C_{nm}(x ... x_m) = f (x ... x_m) \]

with combinations of correlative functions and field commutators as sources, which solution is a sum of general solution of the corresponding homogeneous equation and partial solution concerned with the sources.

Solution of homogeneous equation, considered in chapter 4, should be taken into account only for two–point WF $G(x_1 x_2)$ and $D(x_1 x_2)$, which form effective masses $m_1$, $m_2$. For correlative functions, which describe weak effects of many–particles interaction, in considered iterative approach one should take into account only solutions concerned with the sources. Since sources contain small constant of self–interaction, correlative WF $C_{03}(x_1 x_2 x_3)$, $C_{21}(x_1 x_2 | x_3)$ are of the first order of vanishing in compare with $G(x_1 x_2)$ and $D(x_1 x_2)$.

Independently of choice of basis approximation the following algorithm of calculations can be formulated:

I.1. Derive solutions of the chain equations (considered in a certain approximation) keeping $m_1$, $m_2$ and $v$ as parameters. As a result one obtain WF as functions of coordinates, effective masses, order parameter and temperature:

\[ W_{(x_1 ... x_n x_m)} = W(x x_1 ... x_n x_m, m_1, m_2, v, T) \]
I.2. Calculate WF at coincident points. Owing to homogeneity, isotropy and stationarity of the system under consideration, the result depends on temperature, \( m_1, m_2 \) and \( v \), but not 4–coordinates:

\[
W_{(x \ldots x)} = W(m_1, m_2, v, T)
\]

I.3. Substitute Wightman functions, calculated at coincident points, in (9), (10) and (6) and solve this system of nonlinear equations for \( v, m_1, m_2 \). After substitution of solutions

\[
m_1 = m_1(T), m_2 = m_2(T), v = v(T)
\]

one obtain WF as functions of coordinates and temperature.

I.4. By known dependence WF on temperature calculate thermodynamic observables using momentum–energy tensor (5).

The chain reduction basis approximation corresponds to allowing only for two–point \( G_{(x x_1)} \) and \( D_{(x x_1)} \) Wightman functions in equations (11) and (12), equation of state for order parameter (6) and momentum–energy tensor (5).

When calculating corrections to the basis approximation, which assumes calculation of higher rank WF, already known two–point Wightman functions are used:

II.1. Substitute \( G_0(x_1 x_2) \) and \( D_0(x_1 x_2) \), calculated in basis approximation, in equations for correlative WF, in particular \( C_{03(x_1 x_2 x_3)} \) and \( C_{21(x_1 x_2 | x_3)} \), and obtain partial solution, concerned with sources.

II.2. Substitute obtained solutions for correlative WF into the chain equations for \( G_{(x_1 x_2)} \), \( D_{(x_1 x_2)} \), obtain their solutions and so on...

II.3. Calculate \( v, m_1, m_2 \) and thermodynamic observables on temperature dependence (according to points I.2 — I.4 of algorithm).

It is easy to see, that corrections are of \( O(N^{-1}) \) order of vanishing, thus at the limit \( N \to \infty \) the basis approximation becomes exact solution of the chain equations.

4 Hartree–Fock approximation.

The Bogolubov’s chain basis reduction approximation is called Hartree–Fock approximation (HFA).

The chain equations in Hartree–Fock approximation. According to algorithm, stated above, at first step equations for two–point functions, with \( m_1 \) and \( m_2 \) considered as parameters, are written out:

\[
\partial_\mu \partial_\nu G_{(x x_1)} + m_1^2 G_{(x x_1)} = 0, \quad \partial_\mu \partial_\nu D_{(x x_1)} + m_2^2 D_{(x x_1)} = 0.
\]

These equations are similar to those for noninteracting fields, thus it is natural to represent \( \varphi \) and \( \chi_a \) as follows:

\[
\varphi = \sum_p \frac{1}{\sqrt{2\epsilon_1 p}}(a_+ p e^{-i\epsilon_1 p} e^{ix p} + a_{-} e^{i\epsilon_1 p} e^{ix p}), \quad \epsilon_1^2 p = p^2 + m_1^2,
\]

\[
\chi_a = \sum_p \frac{1}{\sqrt{2\epsilon_2 p}}(b_{p(a)} e^{-i\epsilon_2 p} e^{ix p} + b_{p(a)} e^{i\epsilon_2 p} e^{ix p}), \quad \epsilon_2^2 p = p^2 + m_2^2,
\]

Which is often mentioned as “mean field approximation”, “Hartree approximation”.  

\[\text{Which is often mentioned as “mean field approximation”, “Hartree approximation”}.
\]
with usual commutators for $a_\mathbf{p}, b_{\mathbf{p}(a)}, a_\mathbf{p}^+, b_{\mathbf{p}(a)}^+$ operators. This representation corresponds to description of the system as an ideal gas of particles with masses dependent on temperature.

At the second step two–point WF are calculated in coincident points:

$$
\lambda G_{o(x)} = \lambda \sum_{\mathbf{p}} \frac{N_1 \mathbf{p} + 1/2}{V \varepsilon_1 \mathbf{p}} = \lambda \sum_{\mathbf{p}} \frac{1}{V \varepsilon_1 \mathbf{p}} \cdot \frac{1}{\exp \frac{\mathbf{p} \cdot \mathbf{T}}{T} - 1} + \frac{1}{2} \left( \lambda \sum_{\mathbf{p}} \frac{1}{V \varepsilon_1 \mathbf{p}} \right)_{\text{ren}},
$$

$$
\lambda D_{o(x)} = \lambda \sum_{\mathbf{p}} \frac{N_2 \mathbf{p} + 1/2}{V \varepsilon_2 \mathbf{p}} = \lambda \sum_{\mathbf{p}} \frac{1}{V \varepsilon_2 \mathbf{p}} \cdot \frac{1}{\exp \frac{\mathbf{p} \cdot \mathbf{T}}{T} - 1} + \frac{1}{2} \left( \lambda \sum_{\mathbf{p}} \frac{1}{V \varepsilon_2 \mathbf{p}} \right)_{\text{ren}}.
$$

As it was already mentioned, divergent terms were renormalized in the framework of dimensional regularization method. Essential part of the method is that a theory is initially formulated in non-integer dimension space and transmutation parameters present in Lagrangian as factor multiplying dimensionless constants of interaction. Indeed, in Lagrangian (1), written for space–time of dimension $D = 1 + n = 1 + 3 - 2\varepsilon$, number of dimension of self–interaction constant $\lambda$ is $L^{n-3}$; dimensional transmutation parameter is defined as $\lambda_n = \lambda^{n-3}$. Analytic continuation to integer dimension space $\varepsilon \to 0$, $D \to 1 + 3$ is made only after subtraction of $\varepsilon^{-1}$ poles (this operation is equivalent to redefinition of inoculating constants of the model). After renormalization of divergent terms (20) one obtain:

$$
\left( \lambda \sum_{\mathbf{p}} \frac{1}{2V \varepsilon \mathbf{p}} \right)_{\text{ren}} = \lambda \frac{m^2}{16\pi^2} \ln \left( \frac{m^2}{\Lambda^2} \right), \quad \Lambda^2 = \frac{4\pi l^{-2}}{e^{c-1}},
$$

where $C = 0.5772157$ is Euler constant.

At third step renormalized WF are substituted to (9), (10) and (6).

It should be noted, that independently of renormalization method used, equation of state for order parameter (8) in Hartree–Fock approximation is written as:

$$
v \left[ m_1^2 - \frac{8\lambda}{N} v^2 \right] = 0
$$

Gap equations for effective masses (9) and (10) for further purposes should be rewritten as follows:

$$
\left[ J_1(m_1, T) + \frac{m_1^2}{16\pi^2} \ln \left( \frac{m_1^2}{\Lambda^2} \right) \right] - \frac{N}{8\lambda(N + 2)} \left( (N + 1)m_1^2 - (N - 1)m_2^2 + 2\mu^2 \right) + v^2 = 0,
$$

$$
(N - 1) \left[ J_1(m_2, T) + \frac{m_2^2}{16\pi^2} \ln \left( \frac{m_2^2}{\Lambda^2} \right) \right] - \frac{N(N - 1)}{8\lambda(N + 2)} \left[ 3m_2^2 - m_1^2 + 2\mu^2 \right] = 0,
$$

where following designations for integrals over Bose–Einstein distribution are introduced:

$$
J_n(m, T) = \frac{1}{2\pi^2} \int_0^\infty \frac{p^{2n} dp}{\sqrt{p^2 + m^2}} \cdot \frac{1}{\exp \frac{\sqrt{p^2 + m^2}}{T} - 1}, \quad n = 0, 1, 2,
$$

Temperature integrals satisfy recurrent relations:

$$
\frac{\partial J_n(m, T)}{\partial m} = -(2n - 1)m J_{n-1}(m, T), \quad \frac{\partial J_n(m, T)}{\partial T} = \frac{2n}{T} J_n(m, T) + (2n - 1) \frac{m^2}{T} J_{n-1}(m, T)
$$

Gap equation for effective mass at high–symmetry phase is obtained from (22) by zero filling of order parameter. It is easy to see, that at high–symmetry phase $m_1 = m_2 \equiv m$. 

10
It follows from (21) and (22) that in Hartree–Fock approximation the following effects are taken into account:

1) vacuum influence on quasi–particles properties (order parameter \( v \) in definition of effective masses).

2) whole particles influence on condensate and properties of each quasi–particle (temperature integrals in equations for condensate and effective masses).

3) zero-point oscillations of quantum fields influence on condensate and properties of each quasi–particle (renormalized vacuum integrals in equations for \( v \) and effective masses).

Generating functional. Lagrangian (1) is a “generating functional” for operator equations of state, cause it reproduce these equations on it’s extremals. Since Bogolubov’s chain is equivalent to set of operator equations, a “generating functional” which reproduce chain equations must exist. The same assertion is valid for the chain reduced in any approximation.

In self–consistent approximation this functional should reproduce gap equations for effective masses and equation of state for order parameter, or their linear combination.

By equations (21) and (22) generating functional, which satisfy requirements

\[
\left( \frac{\partial F}{\partial m_1} \right)_{m_2,v} = 0, \quad \left( \frac{\partial F}{\partial m_2} \right)_{m_1,v} = 0, \quad \left( \frac{\partial F}{\partial v} \right)_{m_1,m_2} = 0.
\] (23)

is reconstructed up to an arbitrary constant. It’s explicit form:

\[
F(T, m_1, m_2, v) = -\frac{1}{3} \left[ J_2(m_1, T) + (N - 1)J_2(m_2, T) \right] + U(m_1, m_2, v),
\]

\[
U(m_1, m_2, v) = \frac{m_1^4}{64\pi^2} \ln \left( \frac{m_1^2}{\sqrt{\epsilon}\Lambda^2} \right) + (N - 1)\frac{m_2^4}{64\pi^2} \ln \left( \frac{m_2^2}{\sqrt{\epsilon}\Lambda^2} \right) - \frac{N}{8\lambda(N + 2)} \left[ (N + 1)m_1^4 + 3(N - 1)m_2^4 \right] - \frac{N - 1}{2} m_1^2 m_2^2 + \\
\mu^2(m_1^2 + (N - 1)m_2^2) + \frac{m_1^2 v^2}{2} - \frac{2\lambda v^4}{N} - \frac{N^2\mu^4}{16\lambda(N + 2)}.
\] (24)

Gibbs potential. The goal of the theory of phase transitions is a calculation of transition temperature, phase co-existence region, and thermodynamic observables. The last can be obtained from momentum–energy tensor components (3), which for homogeneous and isotropic system is diagonal \( \langle T_{\nu}^\mu \rangle = \text{diag}(\epsilon, -p, -p, -p) \). Since state of the system under consideration is defined by temperature and volume, one should use Gibbs potential for thermodynamic description. Due to chemical potential equality to zero, potential of free energy coincides with \( \Omega \)-potential, which is easily calculated via momentum–energy tensor spatial components:

\[
\Omega = -\lim_{n \to 3} p_n V_n = \lim_{n \to 3} \frac{\langle T_{i}^{i} \rangle V_{n}}{n}.
\] (25)

As (25) is divergent, it should be renormalized. In (3) two–point WF \( G_{(x,x)} \) and \( D_{(x,x)} \) are expressed through effective masses and, consequently, are not divergent. “Kinetic” momentum–energy tensor terms

\[
\lim_{n \to 3} \frac{V_{n}}{n} \cdot \left( \langle \partial_i \varphi \, \partial^i \varphi \rangle + \langle \partial_i \chi_a \, \partial^i \chi_a \rangle \right)
\]
except directly dependent on temperature items,
\[-\frac{1}{3} \sum \frac{p^2 N_1 p}{\varepsilon_1 p} - \frac{N-1}{3} \sum \frac{p^2 N_2 p}{\varepsilon_2 p}\]
contain divergent vacuum terms,
\[\lim_{n \to 3} \frac{V_n}{n} \cdot \left( -\sum \frac{p^2}{2V_n \varepsilon_1 p} - (N-1) \sum \frac{p^2}{2V_n \varepsilon_2 p} \right)\]
which should be renormalized. The parameter of dimensional regularization method is introduced via volume \(V_n\) in non–integer dimension space, which is connected with integer–dimension space volume by simple relation:
\[V_n = N l^n = N l^3 \cdot l^{-2\varepsilon} = V l^{-2\varepsilon}.\]

Renormalization of divergent terms leads to the following result:
\[\lim_{n \to 3} \frac{V_n}{n} \cdot \left( -\sum \frac{p^2}{2V_n \varepsilon_1 p} \right) = V \cdot \frac{m^4}{64\pi^2} \ln \left( \frac{m^2}{\sqrt{e}\Lambda^2} \right)\]

Substitution of results of renormalization of divergent terms and two–point WF, expressed via effective masses and order parameter, to (26) gives:
\[F(T, m_1(T), m_2(T), v(T)) = \frac{\mathcal{F}(T, V)}{V} \quad (27)\]

Free energy density (27) is distinguished into two parts: the first one describes contribution of ideal gas (temperature integrals over Bose–Einstein distribution), and the second one contribution of so called “self–consistent field” (indirectly dependent on temperature).

It follows from (23) and (27), that generating functional, being considered on solution of gap equations for effective masses and equation of state for order parameter, turns into equilibrium free energy density.
\[F(T, m_1(T), m_2(T), v(T)) = \frac{\mathcal{F}(T, V)}{V} \]

For this reason arbitrary constant in (23) was chosen to be equal to \[-\frac{N^2 \mu^4}{16\lambda(N+2)}.\]

**Thermodynamic observables.** Specific heat capacity, entropy and sonic speed are obtained from equilibrium free energy by usual thermodynamic relations.
\[s = \frac{S}{V} = -\frac{1}{V} \left( \frac{\partial \mathcal{F}}{\partial T} \right)_V, \quad c_V = \frac{C_V}{V} = -\frac{T}{V} \left( \frac{\partial^2 \mathcal{F}(T)}{\partial T^2} \right)_V, \quad u^2 = \frac{s}{c_V}.\]

Due to mentioned property of generating functional (24) and in view of (23) are identities on solution of gap equations and equation of state
\[s = -\frac{1}{V} \left( \frac{\partial \mathcal{F}(T, V)}{\partial T} \right)_V = -\left( \frac{\partial F}{\partial T} \right)_{v,m_1,m_2}, \quad c_V = -\frac{T}{V} \left( \frac{\partial^2 \mathcal{F}(T, V)}{\partial T^2} \right)_V = -T \left[ \left( \frac{\partial^2 F}{\partial T^2} \right)_{v,m_1,m_2} + \frac{d v}{d T} \left( \frac{\partial^2 F}{\partial T \partial v} \right)_{m_1,m_2} + \left( \frac{\partial^2 F}{\partial m_1 \partial T} \right)_{v,m_2} \frac{dm_1}{d T} + \left( \frac{\partial^2 F}{\partial T \partial m_2} \right)_{v,m_1} \frac{dm_2}{d T} \right].\]
Effective masses and condensate order parameter derivatives are obtained by derivation of (21) and (22), considered as identities. Result of calculation is given below. Specific heat capacity in low–symmetry phase:

\[
c_V = \frac{1}{T} \left[ 4 [J_2(m_1, T) + (N - 1)J_2(m_2, T)] + 5 \left[ m_1^2 J_1(m_1, T) + (N - 1)m_2^2 J_1(m_2, T) \right] + \\
+ \left[ m_1^4 J_0(m_1, T) + (N - 1)m_2^4 J_0(m_2, T) \right] + \frac{4\lambda(N + 2)}{\delta \cdot N} \left[ f_2^2(m_1, T) \left[ 3 - \frac{4\lambda(N + 2)}{N} f_1(m_2, T) \right] - \\
f_2^2(m_2, T)(N - 1) \left[ \frac{4\lambda(N + 2)}{N} f_1(m_1, T) + 1 \right] \right] + 2(N - 1)f_2(m_1, T)f_2(m_2, T) \right].
\]

Here following designations are introduced:

\[
f_1(m, T) \equiv \frac{1}{8\pi^2} \ln \left( \frac{m^2e}{\Lambda^2} \right) - J_0(m, T), \quad f_2(m, T) \equiv 2J_1(m, T) + m^2 J_0(m, T),
\]

\[
\delta \equiv N - 1 - \left( \frac{4\lambda(N + 2)}{N} f_1(m_2, T) - 3 \right) \left( \frac{4\lambda(N + 2)}{N} f_1(m_1, T) + 1 \right).
\]

Specific heat capacity in high–symmetry phase:

\[
c_V = \frac{N}{T} \left( 4J_2(m, T) + 5m^2 J_1(m, T) + m^4 J_0(m, T) - \frac{f_2^2(m, T)}{f_1(m, T)} - \frac{N}{2\lambda(N + 2)} \right).
\]

Specific entropy in high–symmetry phase is easily obtained from that in low–symmetry phase:

\[
s = \frac{1}{T} \left( \frac{4}{3} [J_2(m_1, T) + (N - 1)J_2(m_2, T)] + \left[ m_1^2 J_1(m_1, T) + (N - 1)J_1(m_2, T) \right] \right)
\]

**Phases stability conditions.** Free energy density (26), obtained earlier by clearly thermodynamic relation (25), can be calculated via statistical sum:

\[
\mathcal{F} = -T \ln Z.
\]  

(28)

The system energy spectrum, which is needed for the calculation, is determined by infill numbers, effective masses and condensate values:

\[
E = \sum_{\mathbf{p}} \varepsilon_{1\mathbf{p}} N_{1\mathbf{p}} + (N - 1) \sum_{\mathbf{k}} \varepsilon_{2\mathbf{k}} N_{2\mathbf{k}} + V \cdot U(m_1, m_2, v),
\]

where \(N_{1\mathbf{p}}\) and \(N_{2\mathbf{p}}\) are numbers of \(\varphi\) and \(\chi_a\) particles respectively. For the theory to be self–consistent in approximation used, one should sum over infill numbers \(N_{1\mathbf{p}}\) and \(N_{2\mathbf{p}}\), but not the effective masses and order parameter, considering \(m_1\), \(m_2\) and \(v\) as constants (as it was stated in [4, §30]). After summation statistical sum is represented as a product of two terms: the first one corresponds to ideal gas and the second one to “self–consistent” field. Taking the logarithm and summing over impulses gives a function which coincide with generating functional. Consideration of gap equations for effective masses (22) and equation of state for order parameter (21) leads to the specific equilibrium free energy density (26).

Keeping symmetrical parameters as constants, when calculating statistical sum, corresponds to the algorithm of calculation of non–equilibrium Landau statistical sum \(Z_L\) [8], which allows
to obtain non-equilibrium free energy functional $F_L$. Since the only symmetrical parameter in our case is condensate $v$, to obtain non-equilibrium Landau functional, one should substitute effective masses on order parameter dependence into generating functional.

The functional obtained $F(T, v) = F(T, m_1(T, v), m_2(T, v), v)$ reproduce equation of state for order parameter:

$$\frac{d}{dv} F(T, v) = \left( \frac{\partial F}{\partial m_1} \right)_{T, m_2, v} + \left( \frac{\partial F}{\partial m_2} \right)_{T, m_1, v} + \left( \frac{\partial F}{\partial v} \right)_{T, m_1, m_2},$$

and on solution of that equation $v = v(T)$ it turns to free energy density, i.e. it possess all properties of non-equilibrium functional.

A phase stability condition is order parameter Landau functional second derivation positivity:

$$\frac{d^2 F}{dv^2} = m_1^2 - \frac{24\lambda}{N} v^2 + 2m_1 v \frac{dm_1}{dv}.$$ \hspace{1cm} (29)

Inclusion of formal external classical source, interacting with condensate, in Lagrangian changes equation of state for

$$\mathcal{L} \longrightarrow \mathcal{L} + \rho_a \phi_a, \quad \rho_a = \delta_{a, N} \cdot \eta$$

order parameter, which now depends on external source $v = v(\eta)$. In Hartree–Fock approximation:

$$v \left[ m_1^2 - \frac{8\lambda}{N} v^2 \right] = \eta.$$  

Generalized sensitivity at $\eta \to 0$ turns out to be in inverse proportion with stability condition (29):

$$\left( \frac{dv}{d\eta} \right)_{\eta \to 0} = \left( \frac{d^2 F}{dv^2} \right)^{-1}.$$  

Being usual in Landau phase transitions theory, this result proves correctness of non-equilibrium free energy and stability condition (29).

Appearing in (29) condensate derivative of mass is obtained by order parameter differentiating of gap equations for effective masses (22), considered as identities. In particular, low–symmetry phase stability condition looks like:

$$\frac{d^2 F}{dv^2} = 2m_1^2 \left( 4\lambda(N+2) \frac{f_1(m_2, T)}{N} - 3 \right) \left( 4\lambda(N+2) f_1(m_1, T) + 1 \right).$$

In the vicinity of critical temperature $T_{c_2}$ stability condition denominator is a slow variable function. Numerator temperature derivative, which designation was introduced earlier, $\delta'_T \sim \delta^{-1}$, this implies that $\delta \sim \sqrt{T_{c_2} - T}$ at the vicinity of $T_{c_2}$.

Specific heat capacity of low–symmetry phase contains $\delta$ in inverse proportion, thus it is formally divergent at $T_{c_2}$ temperature. Sonic speed, which is inverse proportion with heat capacity, tends to zero.

Stability condition of high–symmetry phase proportional to effective mass squared

$$\frac{d^2 F}{dv^2} = m_1^2,$$

which allows to obtain value of critical temperature $T_{c_1}$ analytically:

$$T_{c_1} = \mu \sqrt{\frac{3N}{\lambda(N+2)}}.$$
Numerical calculations in Hartree–Fock approximation. Results of numerical calculations of effective masses, order parameter and thermodynamical observables on temperature dependence for $N = 2$, $N = 4$ and $N = 9$ and $\lambda = 0.01 \cdot 2\pi^2$ are given below.

If parameter $\Lambda$ is chosen to be $\Lambda = m_1 \ vac$, $m_1 \ vac \equiv m_1(T = 0)$, gap equations for effective masses and equation for order parameter posses a solution $m_1^2 = 2\mu^2$, $m_2^2 = 0$, so that quanta of $\chi_a$ fields turn to Goldstone bosons.

It is convenient to normalize effective masses, order parameter and observables to $m_1 \ vac$. Graphs of their temperature dependence are given in mentioned normalization ($T \Rightarrow \frac{T}{m_1 \ vac}$ and so forth.).

![Graphs of effective masses, order parameter, sonic speed and specific heat capacity](image)

Fig.1 Temperature dependence of effective masses, order parameter, sonic speed and specific heat capacity in Hartree–Fock approximation.

Equations (22) and (21) in low–symmetry phase admit thermodynamically stable in temperature region $0 \ .. \ T_{c_2}$ (solid lines at figures 1.(a) and 1.(b)) and in temperature region $T_{c_1} \ .. \ T_{c_2}$ thermodynamically unstable (not shown) branches of solutions. In high–symmetry phase only one, stable, branch of solutions exists (dashed line).

Below temperature $T_{c_1}$ in high–symmetry phase and above $T_{c_2}$ in low–symmetry phase, where stability conditions of the phases turn to zero, equations for effective masses (22) and order parameter (21) admit no solutions. When calculating correction to Hartree–Fock approximation, this criterion allows to easily find phases stability thresholds.

As it follows from order parameter on temperature dependence, at phase equilibrium point $T_{c_1} < T_c < T_{c_2}$, where equality of free energies of both phases is achieved, order parameter is small but not zero, consequently for finite $N$ first type (close to second type) phase transition
takes place. Decreasing of condensate value at equilibrium point \( T_c \) and decreasing of phase coexistence region to \( T_c \) ratio \( \xi \equiv \Delta T / T_c = T_{c_2} - T_{c_1} / T_c \) with increase \( N \) (\( \xi |_{N=2} \approx 0.021, \xi |_{N=4} \approx 0.011, \xi |_{N=9} \approx 0.006 \)) indicates that at the limit \( N \rightarrow \infty \) phase transition of second type takes place. Due to zero \( \chi_a \) fields quanta masses, sonic speed is not zero at zero temperature.

Numerical calculations confirm mentioned above conclusion of heat capacity divergence at vicinity of \( T_{c_2} \). However, it is well known, that heavy fluctuations in this region “blur” jump of heat capacity and other thermodynamic observables.

Corrections to Hartree–Fock approximation turn out large at phases equilibrium point, which confirms inadaptability of the approximation at this temperature and, consequently, formality of made above conclusion about heat capacity divergence.

5 Corrections to Hartree–Fock approximation.

Calculation of corrections to Hartree–Fock approximation, which implies calculation of higher–rank WF, is important not only for estimation of the Hartree–Fock approximation temperature region of adaptability, but also for proof of correctness of mentioned in 4 renormalization method.

Algorithm of calculations. Since equation for order parameter (3), which can be rewritten as follows

\[
v \left[ m_1^2 - \frac{8v^2}{N} + \frac{4\lambda}{N} \left( C_{03(x x x)} + (N - 1)C_{21(x x |x)} \right) \right] = 0, \quad (30)
\]

contains three–point WF, at first step one should derive partial solution of equation for WF \( C_{03(x_1 x_2 x_3)} \) and \( C_{21} \) concerned with sources. Proximate system of equations for three-point WF is derived from (31), (32) and (33) by neglecting of higher–rank WF:

\[
\partial_\mu \phi^\mu_{03} C_{03(x_1 x_2)} + m_1^2 C_{03(x_1 x_2)} \approx -\frac{24\lambda}{N} \left[ G(x_1 x_2) + \frac{1}{12} \Delta \phi(x_1) \Delta \phi(x_2) \right], \quad (31)
\]

\[
\partial_\mu \phi^\mu_{21} C_{21(x_1 |x_2)} + m_1^2 C_{21(x_1 |x_2)} \approx -\frac{8\lambda}{N} \left[ D(x_2 x_1) + \frac{1}{12} \Delta \chi(x_2) \Delta \chi(x_2 x_1) \right], \quad (32)
\]

\[
\partial_\mu \phi^\mu_{21} C_{21(x_1 |x_2)} + m_1^2 C_{21(x_1 |x_2)} \approx -\frac{8\lambda}{N} D(x_1) G(x_2). \quad (33)
\]

In the framework of the iterative procedure used, for solving equations (31) and (32), (33) two–point WF \( G_0(x_1 x) \) and \( D_0(x_1 x) \) calculated in Hartree–Fock approximation are used.

As accurate within first order of vanishing terms \( \lambda v^2 = \frac{N}{8} m_1^2 \), corrections to WF \( G_1(x_1 x) \) and \( D_1(x_1 x) \) are of the first order of vanishing and should be taken into account along with \( C_{03(x_1 x_2 x_3)} \) and \( C_{21(x_1 x_2 |x_3)} \). Thus, at second step equations for \( G_1(x_1 x) \) and \( D_1(x_1 x) \), which derived from (31) and (32) by substitution \( \lambda v^2 \), expressed through \( m_1^2 \), substitution of three–point WF and neglecting of the second order of vanishing sources (in particular four–point WF, multiplied by \( \lambda \)),

\[
\partial_\mu \phi^\mu_{21} G_1(x_1 x) + m_1^2 G_1(x_1 x) \approx -\frac{m_1^2}{2} \left[ 3C_{03(x x x_1)} + (N - 1)C_{21(x x |x_1)} \right], \quad (34)
\]

\[
\partial_\mu \phi^\mu_{21} D_1(x_1 x) + m_2^2 D_1(x_1 x) \approx -m_2^2 C_{21(x x |x_1)}. \quad (35)
\]

Taking into account tree-point WF and corrections to two–point WF in equation for order parameter and equations for effective masses leads to new temperature dependence of these
values in compare with Hartree–Fock approximation. New dependencies are found at third step.

Preliminary conclusion of corrections to \(m_1, m_2\) and \(v\) magnitudes one could made without numerical analysis. Assuming the corrections small, introduce effective masses and order parameter as follows:

\[
m_1^2 = m_{10}^2 + \delta m_1^2, \quad m_2^2 = m_{20}^2 + \delta m_2^2, \quad v^2 = v_0^2 + \delta v^2,
\]

Here \(m_{10}, m_{20}\) and \(v_0\) corresponds to self consistent field approximation. Substitution of (36) in (30) and (3), (10) and separation by orders of vanishing gives the following expressions for corrections:

\[
\delta m_2^2 = -\frac{4\lambda(N+2)}{\delta \cdot N} \left[ 2(N-1)D_1(x,x) - \left( \frac{4\lambda(N+2)}{N} f_1(m_{20}, T) - 3 \right) \cdot \left( 2G_1(x,x) + C_{03}(x,x) + (N-1)C_{21}(x,x|x) \right) \right],
\]

\[
\delta m_2^2 = -\frac{4\lambda(N+2)}{\delta \cdot N} \left[ -2D_1(x,x) \left( \frac{4\lambda(N+2)}{N} f_1(m_{10}, T) + 1 \right) + 2G_1(x,x) + C_{03}(x,x) + (N-1)C_{21}(x,x|x) \right],
\]

\[
\delta v^2 = \frac{\delta m_1^2 \cdot N}{8\lambda} + \frac{1}{2} \left( C_{03}(x,x) + (N-1)C_{21}(x,x|x) \right).
\]

Designations \(f_1(m, T)\), which up to a coefficient is a squared mass derivative of two-point WF considered in Hartree–Fock approximation,

\[
f_1(m_{10}, T) = 2\frac{\partial G_{0(x,x)}^1}{\partial m_{10}^2}, \quad f_1(m_{20}, T) = 2\frac{\partial G_{0(x,x)}^2}{\partial m_{20}^2}
\]

and \(\delta\) were introduced in chapter [3]. As long as at the vicinity of \(T_{c2}\) \(\delta \sim \sqrt{T_{c2} - T}\), corrections, calculated by (37), (38) and (39), are of high magnitude in this region.

At fourth step corrections to thermodynamical observables are calculated. Neglecting of four-point Wightman functions, which contributions are of second order of vanishing, and substitution of two- and three-point WF, expressed through effective masses and condensate value, in “potential” terms of momentum–energy tensor gives:

\[
\langle T^\mu_\nu \rangle = \lim_{x \to x_1} \left( \partial_{\mu(x)} \partial^\nu_{(x_1)} G(x_{x_1}) + (N-1)\partial_{\mu(x)} \partial^\nu_{(x_1)} D(x_{x_1}) \right) - \\
- \delta^\mu_\nu \left( \frac{N}{8\lambda(N+2)} \left[ \frac{(N+1)m_1^4 + 3(N-1)m_2^4}{4} - \frac{N-1}{2} m_1^2 m_2^2 + \mu^2(m_1^2 + (N-1)m_2^2) + \frac{N\mu^4}{2} \right] - m_1^2 v^2 + \frac{6\lambda v^4}{N} \right).
\]

Since in high–symmetry phase three-point WF are identically equal to zero, corrections are of the second order of vanishing, and should not be considered in approximation which takes into account only effects of zero and first order of vanishing.

**Calculation of three-point WF.** In Hartree–Fock approximation two-point WF \(G_{0(x,x)}\) and \(D_{0(x,x_1)}\), and commutators look like:

\[
G_{0(x,x_1)} = \sum_p \frac{n_p + \frac{1}{2}}{2\varepsilon_p \sqrt{p^2}} \left( e^{ip(x-x_1)} + e^{ip(x_1-x)} \right), \quad D_{0(x,x_1)} = \sum_k \frac{n_k + \frac{1}{2}}{2\varepsilon_k \sqrt{k^2}} \left( e^{ik(x-x_1)} + e^{ik(x_1-x)} \right),
\]
\[ \Delta \varphi_0(x_1) = \sum_{p} \frac{1}{2\varepsilon p V} \left( e^{ip(x_1 - x)} - e^{ip(x_1 - x)} \right), \quad \Delta \chi_0(x_1) = \sum_{k} \frac{1}{2\varepsilon k V} \left( e^{ik(x_1 - x)} - e^{ik(x_1 - x)} \right), \]

so that sources in equations (33), (34), (35) and (36) are sum of exponents. Since exponent is an eigen-function of differentiation operator, action of D’Alambert operator at each terms reduce to multiplying by correspondent function of masses and impulses \( f(q, m) \). Hence, solutions of the equations are functions which differ from sources by factors \( f(q, m)^{-1} \) of exponents.

Use of this algorithm and consideration of symmetrical properties of Wightman functions \( C_{03}(x_1, x_2) \) gives:

\[ \lambda C_{03}(x_1, x_2) = \lim_{n \to 3} \frac{-12\lambda^2 n}{N(2\pi)^{2n}} \int d^np \, d^np' \left[ \left( \frac{1}{\exp \frac{p}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{p'}{T} - 1} + \frac{1}{2} \right) \times \right. \\
\left. \Re \left( e^{ip(x-x_1)} e^{ip'(x-x_2)} + e^{ip(x_1-x)} e^{ip'(x_1-x_2)} + e^{ip(x-x_2)} e^{ip'(x_2-x_1)} \right) \right] \]

Here \( p = (p^0, p^i), \, p' = (p^0', p'^i) = (\sqrt{p'^2 + m^2}, p') \). To obtain (41) \( G_0(x_1), \)

\[ D_0(x_1), \Delta \varphi_0(x_1) \) and \( \Delta \chi_0(x_1) \) one should turn from summation to integration over non–integer dimension space.

In solutions of (32) and (33) terms \( \frac{1}{m_2^2 - (p-k)^2} \) and \( \frac{1}{m_1^2 - (k+k')^2} \) respectively arise. Denominators of these terms turn to zero for some impulses \( p, \, k \) if condition \( m_1 \geq 2m_2 \) is satisfied. Physically satisfaction of this condition corresponds to possibility of \( \varphi \) to pair \( \chi_0 \) decay. In tree approximation decay width is:

\[ \Gamma = \frac{N - 1}{N^2} \cdot \frac{2\lambda^2 v^2}{\pi m_1} \sqrt{1 - \left( \frac{2m_2}{m_1} \right)^2}. \]

Let us now introduce helper function \( C_{21}(x_1 | x_2) \), which satisfy to equations obtained from (32) and (33) respectively by substitutions

\[ \partial^\mu \partial^\mu_x + m_1^2 \Rightarrow \partial^\mu \partial^\mu_x + m_1^2 - im_1 \Gamma, \quad \partial^\mu \partial^\mu_x + m_2^2 \Rightarrow \partial^\mu \partial^\mu_x + m_2^2 - im_2 \Gamma. \]

Since WF are real, by definition \( C_{21}(x_1 | x_2) = \frac{1}{2} \left[ C_{21}^{au}(x_1 | x_2) + \bar{C}_{21}^{au*}(x_1 | x_2) \right] \).

After consideration of \( C_{21}(x_1 | x_2) \) symmetries with regard to arguments permutation:

\[ \lambda C_{21}(x_1 | x_2) = \lim_{n \to 3} \frac{-4\lambda^2 n}{N(2\pi)^{2n}} \int d^np \, d^nk \left[ \left( \frac{1}{\exp \frac{p}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{k}{T} - 1} + \frac{1}{2} \right) \right. \]

\[ \left. \Re \left( e^{ip(x-x_1)} e^{ik(x-x_2)} + e^{ip(x_1-x)} e^{ik(x_1-x_2)} \right) \right] + \left. \Re \left( e^{ip(x-x_1)} e^{ik(x-x_2)} + e^{ip(x_1-x)} e^{ik(x_2-x_1)} \right) \right) \]

(42)
lead not only to divergent by both integration variables, analytical continuation into four regions where values of these parameters are greater (less) than one. In each region an expansion valid in the region is used.

The integrals are well defined in space of dimension $1 < n < 2$. Since "vacuum integrals" are divergent by both integration variables, analytical continuation into $n = 3$ dimension space lead not only to $\varepsilon^{-1}$ pole (which is responsible for $\ln \left( \frac{m^2}{\Lambda^2} \right)$ terms in renormalized integrals), but also $\varepsilon^{-2}$ pole (which is responsible for $\ln^2 \left( \frac{m^2}{\Lambda^2} \right)$ terms).

As "mixed integrals" are divergent only by one integration variable, at the limit $n \to 3$ only $\varepsilon^{-1}$ pole arise. Coefficient at $\ln \left( \frac{m^2}{\Lambda^2} \right)$, formed by Bose–Einstein distribution, is a function of particle mass and temperature.

After poles subtraction one obtain the following expressions for three–point Wightman functions:

\[
C_{03(x\,x\,x)} \rightarrow \frac{-4\Lambda^2}{(2\pi)^2 N} \int \frac{d^n k}{\varepsilon_k \varepsilon_{k'}} \left[ \left( \frac{1}{\exp \frac{k}{\Lambda} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{k'}{\Lambda} - 1} + \frac{1}{2} \right) \times \right. \]

\[
\operatorname{Re} \left( \frac{\exp \left( i k (x_2 - x_1) \right)}{m_1^2 - (k + k')^2 - im_1 \Gamma} \right) + \operatorname{Re} \left( \frac{\exp \left( i k (x_2 - x_1) \right)}{m_1^2 - (k - k')^2 - im_1 \Gamma} \right) + \]

\[
+ \frac{1}{12} \times \left[ \frac{\exp \left( i k (x_2 - x_1) \right)}{m_1^2 - (k + k')^2 - im_1 \Gamma} - \frac{\exp \left( i k (x_2 - x_1) \right)}{m_1^2 - (k - k')^2 - im_1 \Gamma} \right].
\]

Here $k = (k_1, k) = (\sqrt{k^2 + m_1^2}, k)$, $k' = (k_0', k') = (\sqrt{k'^2 + m_1^2}, k')$. Solutions, written above, are symmetrical of permutations of arguments. Taken in coincide points, they depend on inner system parameters, but not 4–coordinates.

Wightman functions $C_{03(x\,x\,x)}$ and $C_{21(x\,x\,x)}$ automatically separate into items of three types. Items of the first type contain product of Bose–Einstein distributions under the integral and are finite at the limit $n \to 3$. Items of the second type ("mixed integrals") which contain under integral Bose–Einstein distribution for one integration variable and items of the third type ("vacuum integrals") which don’t contain Bose–Einstein distribution under integral are divergent at the limit $n \to 3$.

Asymptotic expansion of integrands is used for regularization of divergent terms of three–point WF in the framework of dimensional regularization. Three–dimensional impulse $p$ ($k$) to effective mass $m_1$ ($m_2$) quotient serves as expansion parameter, so that region of integration is divided into four regions where values of these parameters are greater (less) than one. In each region an expansion valid in the region is used.

The integrals are well defined in space of dimension $1 < n < 2$. Since "vacuum integrals" are divergent by both integration variables, analytical continuation into $n = 3$ dimension space lead not only to $\varepsilon^{-1}$ pole (which is responsible for $\ln \left( \frac{m^2}{\Lambda^2} \right)$ terms in renormalized integrals), but also $\varepsilon^{-2}$ pole (which is responsible for $\ln^2 \left( \frac{m^2}{\Lambda^2} \right)$ terms).

As "mixed integrals" are divergent only by one integration variable, at the limit $n \to 3$ only $\varepsilon^{-1}$ pole arise. Coefficient at $\ln \left( \frac{m^2}{\Lambda^2} \right)$, formed by Bose–Einstein distribution, is a function of particle mass and temperature.

After poles subtraction one obtain the following expressions for three–point Wightman functions:

\[
C_{03(x\,x\,x)} \rightarrow \frac{-18\lambda}{(2\pi)^2 N} \left[ I_0^0(m_1, m_1, m_1, T) + J_1(m_1, T) \left( C_1 - 2 \ln \left( \frac{m^2}{\Lambda^2} \right) \right) + \right.
\]

\[
+ K_0^0(m_1, m_1, m_1, T) + C_2 m_1^2 \ln^2 \left( \frac{m^2}{\Lambda^2} \right) + C_3 m_1^2 \ln \left( \frac{m^2}{\Lambda^2} \right) + C_4 m_1^2,
\]

\[
C_{21(x\,x\,x)} \rightarrow \frac{-4\lambda}{(2\pi)^2 N} \left[ I_0^0(m_1, m_2, m_2, T) + \frac{1}{2} \left( J_1(m_1, T) + m_1 m_2 J_1(m_2, T) \right) \left( C_1 - 2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) \right) + \right.
\]

\[
+ 2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) + \frac{1}{2} \left( K_0^0(m_1, m_2, m_2, T) + C_1 m_1 m_2 \right) + \frac{1}{2} \left( K_1^0(m_2, m_1, m_2, T) - m_1 m_2 J_1(m_2, T) \right) \left( C_1 - \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) \right) + \right.
\]

\[
+ C_3 m_1^2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) + C_5 m_1^2 + L_0^0(m_1, m_2, m_2, 1) + \frac{1}{2} \frac{m_1 m_2}{m_2^2} J_1(m_2, T) \left( C_1 - 2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) \right) - C_5 m_1^2 + C_6 m_2^2 +
\]
\[ + C_2 (2m_2^2 - m_1^2) \ln^2 \left( \frac{m_2^2}{\Lambda^2} \right) + C_3 (2m_2^2 - m_1^2) \ln \left( \frac{m_2^2}{\Lambda^2} \right) + L_1^0(m_2, m_2, m_1, 4/3) \].

Coefficients \( C_i \) were calculated approximately: \( C_1 \approx -1.386, C_2 \approx 0.044, C_3 \approx 0.013, C_4 \approx -0.031, C_5 \approx -0.012, C_6 \approx -0.031, C_6 \approx -0.089 \). Introduced above functions \( I_n(x, y, \alpha, T) \), \( K_n(x, y, \alpha, T) \) and \( L_n(x, y, \alpha, \beta, T) \) are defined as follows:

\[
I_n^m(x, y, \alpha, \beta) = \frac{2}{(2\pi)^4} \int \frac{d^3p}{\varepsilon_p - k} \frac{1}{\varepsilon_p - k} \frac{1}{\varepsilon_p - k, \beta} \frac{\beta(p + k)^{2m}}{(\alpha^2 - (p + k)^2 - i\alpha\Gamma)^n} + \frac{\beta(p + k)^{2m}}{(\alpha^2 - (p - k)^2 - i\alpha\Gamma)^n},
\]

\[
K_n^m(x, y, \alpha, \beta) = \frac{2}{(2\pi)^4} \int \frac{d^3p}{\varepsilon_p - k} \frac{1}{\varepsilon_p - k} \frac{1}{\varepsilon_p - k, \beta} \frac{(p + k)^{2m}}{(\alpha^2 - (p + k)^2 - i\alpha\Gamma)^n} + \frac{(p + k)^{2m}}{(\alpha^2 - (p - k)^2 - i\alpha\Gamma)^n},
\]

\[
L_n^m(x, y, \alpha, \beta) = \frac{1}{2(2\pi)^4} \int \frac{d^3p}{\varepsilon_p - k} \frac{1}{\varepsilon_p - k} \frac{1}{\varepsilon_p - k, \beta} \frac{\beta(p + k)^{2m}}{(\alpha^2 - (p + k)^2 - i\alpha\Gamma)^n} + \frac{\beta(p + k)^{2m}}{(\alpha^2 - (p - k)^2 - i\alpha\Gamma)^n}.
\]

In (13), (14) and (15) 4–impulses \( p \) and \( k \) depend on arguments \( x \) and \( y \) as: \( p = (p^0, p) = (\varepsilon_p, k) = (\sqrt{p^2 + x^2}, k) \), \( k = (k^0, k) = (\sqrt{k^2 + x^2}, k) \).

Proportional to introduced in chapter 4 temperature integral \( J_1(m, T) \) terms are renormalized “mixed integrals”. Renormalized “vacuum integrals” are not explicitly temperature dependent.

Wightman function \( C_{21(x|x)}^{(x)} \), which contain terms \( m_1^2 \ln \left( \frac{m_1^2}{\Lambda^2} \right) \) and \( m_1^2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) \) is logarithmically divergent at the limit \( T \to 0 \). This divergence, caused by long-range action, is described by diagrams of massless \( \chi_0 \) quanta exchange. Since at the most interesting region of phase transition \( m_2 \) is not zero, this problem was not investigated in details.

**Calculation of corrections to two–point Wightman functions.** Taking into account (13), (14) and (15) it is convenient to rewrite equations (14) and (15) as follows:

\[
\left[ \partial^\mu_{x_1} + \partial^\mu_{x_2} \right] \left[ \partial^\mu_{x_1} + \partial^\mu_{x_2} \right] G_{1(x_1)} = \frac{36\lambda m_1^2}{N} \left[ G_{0(x_1)}G_{0(x_1)} + \frac{1}{12}\Delta \varphi_{0(x_1)}\Delta \varphi_{0(x_1)} \right] + \frac{4\lambda m_1^2(N - 1)}{N} \left[ D_{0(x_1)}D_{0(x_1)} + \frac{1}{12}\Delta \chi_{0(x_1)}\Delta \chi_{0(x_1)} \right],
\]

\[
\left[ \partial^\mu_{x_1} + \partial^\mu_{x_2} \right] \left[ \partial^\mu_{x_1} + \partial^\mu_{x_2} \right] D_{1(x_1)} = \frac{8\lambda m_1^2}{N} D_{0(x_1)}G_{0(x_1)}.
\]

Since, like in the case of three–point WF, divergent for \( m_1 > 2m_2 \) and some impulses terms \( \frac{1}{(m_2^2 - (p - k)^2)^2} \) and \( \frac{1}{(m_1^2 - (k + k')^2)^2} \) arise in solutions, equations for two–point WF should be redefined. Use of the same way of redefinition as was used for \( C_{21(x_1|x_2)} \) leads to:

\[
\lambda G_{1(x_1)} = \lim_{n \to 3} \frac{2\lambda m_1^2}{N(2\pi)^2} \left[ \frac{1}{\varepsilon_p - k} + \frac{1}{\varepsilon_p - k, \beta} \right] \left( \frac{1}{\exp \frac{\varepsilon_p}{\beta} - 1 + \frac{1}{2}} \right) \times
\]
energy tensor components by relation (25), considering correct ion to Hartree–Fock approxima-

tion.

Differentiation of (5) and (49) gives respectively:

\[
\frac{1}{12} \cdot \text{Re} \left( \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} \right) +
\]
\[
\left( N - 1 \right) \int \frac{d^n k \cdot d^n k'}{\varepsilon_k \varepsilon_{k'}} \left[ \left( \frac{1}{\exp \frac{\varepsilon_k}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{\varepsilon_{k'}}{T} - 1} + \frac{1}{2} \right) \times \right.
\]
\[
\text{Re} \left( \frac{\text{Re}(e^{i(k + k')^2 - i m_1 \Gamma)^2}}{m_1^2 - (k + k')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(k + k')^2 - i m_1 \Gamma)^2}}{m_1^2 - (k + k')^2 - i m_1 \Gamma^2} \right) +
\]
\[
\left. \frac{1}{12} \cdot \text{Re} \left( \frac{\text{Re}(e^{i(k + k')^2 - i m_1 \Gamma)^2}}{m_1^2 - (k + k')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(k + k')^2 - i m_1 \Gamma)^2}}{m_1^2 - (k + k')^2 - i m_1 \Gamma^2} \right) \right]
\]
\[
\approx - \frac{9 \lambda}{N(2\pi)^2} \int \frac{d^n p \cdot d^n k}{\varepsilon_p \varepsilon_k} \left[ \left( \frac{1}{\exp \frac{\varepsilon_p}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{\varepsilon_k}{T} - 1} + \frac{1}{2} \right) \times \right.
\]
\[
\text{Re} \left( \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} \right) +
\]

After renormalization of corrections to two–point WF, considered in coincide points, in the framework of dimensional regularization one obtain:

\[
G_{1(x)}_{\text{ren}} = \frac{9 \lambda}{N(2\pi)^2} \left[ m_1^2 I_2^0(m_1, m_1, 1, T) + J_1(m_1, T) \left( C_1 - 2 \ln \left( \frac{m_1^2}{A^2} \right) \right) +
\]
\[
+ m_1^2 K_2^0(m_1, m_1, T) \right. + C_2 m_1^2 \ln^2 \left( \frac{m_1^2}{A^2} \right) \right. + C_7 m_1^2 \ln \left( \frac{m_1^2}{A^2} \right) + C_8 m_1^2 \right. +
\]
\[
\left. \left( N - 1 \right) \lambda \right. \int \frac{d^n p \cdot d^n k}{\varepsilon_p \varepsilon_k} \left[ \frac{m_1^2}{m_2^2} J_1(m_2, T) \left( C_1 - 2 \ln \left( \frac{m_2^2}{A^2} \right) \right) + C_9 m_1^2 + C_{10} m_2^2 +
\]
\[
+ m_1^2 K_2^0(m_2, m_2, T) \right. + C_2 m_1^2 \ln^2 \left( \frac{m_2^2}{A^2} \right) \right. + C_7 m_1^2 \ln \left( \frac{m_2^2}{A^2} \right) + m_1^2 L_2^0(m_2, m_2, T) \right]
\]
\[
\lambda D_{1(x,x)} = \lim_{n \to 3} \frac{4 \lambda n^2}{N(2\pi)^2} \int \frac{d^n p \cdot d^n k}{\varepsilon_p \varepsilon_k} \left[ \left( \frac{1}{\exp \frac{\varepsilon_p}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{\varepsilon_k}{T} - 1} + \frac{1}{2} \right) \times \right.
\]
\[
\text{Re} \left( \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} \right) \right]
\]

\[
\lambda D_{1(x,x)} = \lim_{n \to 3} \frac{4 \lambda n^2}{N(2\pi)^2} \int \frac{d^n p \cdot d^n k}{\varepsilon_p \varepsilon_k} \left[ \left( \frac{1}{\exp \frac{\varepsilon_p}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{\varepsilon_k}{T} - 1} + \frac{1}{2} \right) \times \right.
\]
\[
\text{Re} \left( \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} \right) \right]
\]

Here \( C_7 \approx -0.568, C_8 \approx -0.929, C_9 \approx -0.111, C_{10} \approx -0.818, C_{11} \approx -1.042. \)

In order to calculate (specific) equilibrium free energy, which is expressed via momentum–energy tensor components by relation (23), considering correction to Hartree–Fock approximation of the first order of vanishing, one should derive and renormalize derivatives of corrections to two–point WF.

Differentiation of (3) and (49) gives respectively:

\[
\lim_{n \to 3} \frac{V_{n,1}}{n} \cdot \partial_{(x_1, y_1)} G_{1(x_1)} =
\]
\[
= - \lim_{n \to 3} \frac{4 \lambda n^2}{N(2\pi)^2} \int \frac{d^n p \cdot d^n k}{\varepsilon_p \varepsilon_k} \left[ \left( \frac{1}{\exp \frac{\varepsilon_p}{T} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{\varepsilon_k}{T} - 1} + \frac{1}{2} \right) \times \right.
\]
\[
\text{Re} \left( \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} + \frac{\text{Re}(e^{i(p_1 + p')^2 - i m_1 \Gamma)^2}}{m_1^2 - (p + p')^2 - i m_1 \Gamma^2} \right) \right]
\]
After renormalization one obtain:

\[
\text{Re} \left( \frac{(p + p')^2 \cdot \text{Re}(e^{ip(x_1 - x)} e^{ip'(x_1 - x)})}{(m^2_1 - (p + p')^2 - im_1 \Gamma)^2} + \frac{(p - p')^2 \cdot \text{Re}(e^{ip(x_1 - x)} e^{ip'(x - x_1)})}{(m^2_1 - (p - p')^2 - im_1 \Gamma)^2} \right) + \\
\frac{1}{12} \cdot \text{Re} \left( \frac{(p + p')^2 \cdot \text{Re}(e^{ip(x_1 - x)} e^{ip'(x_1 - x)})}{(m^2_1 - (p + p')^2 - im_1 \Gamma)^2} - \frac{(p - p')^2 \cdot \text{Re}(e^{ip(x_1 - x)} e^{ip'(x - x_1)})}{(m^2_1 - (p - p')^2 - im_1 \Gamma)^2} \right) + \\
+ (N - 1) \int \frac{d^m k \, d^m k'}{\varepsilon \, \varepsilon'} \left[ \left( \frac{1}{\exp \frac{k'}{\lambda} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{k}{\lambda} - 1} + \frac{1}{2} \right) \times \right. \\
\text{Re} \left( \frac{(k + k')^2 \cdot \text{Re}(e^{ik(x_1 - x)} e^{ik'(x_1 - x)})}{(m^2_1 - (k + k')^2 - im_1 \Gamma)^2} + \frac{(k - k')^2 \cdot \text{Re}(e^{ik(x_1 - x)} e^{ik'(x - x_1)})}{(m^2_1 - (k - k')^2 - im_1 \Gamma)^2} \right) + \\
+ \frac{1}{12} \cdot \text{Re} \left( \frac{(k + k')^2 \cdot \text{Re}(e^{ik(x_1 - x)} e^{ik'(x_1 - x)})}{(m^2_1 - (k + k')^2 - im_1 \Gamma)^2} + \frac{(k - k')^2 \cdot \text{Re}(e^{ik(x_1 - x)} e^{ik'(x - x_1)})}{(m^2_1 - (k - k')^2 - im_1 \Gamma)^2} \right) \right] \\
\lim_{n \to \infty} V_n \frac{1}{n} \cdot \partial_i(x) \partial^j(x_1) G_1(x_1) = \\
- \lim_{n \to \infty} \frac{1}{n} \cdot \frac{4 \lambda m_1^2}{N(2\pi)^2} \int d^m p \, d^m k \left( \frac{1}{\exp \frac{p}{\lambda} - 1} + \frac{1}{2} \right) \left( \frac{1}{\exp \frac{k}{\lambda} - 1} + \frac{1}{2} \right) \times \\
\text{Re} \left( \frac{(p + k)^2 \cdot \text{Re}(e^{ip(x_1 - x)} e^{ik(x_1 - x)})}{(m^2_1 - (p + k)^2 - im_1 \Gamma)^2} + \frac{(p - k)^2 \cdot \text{Re}(e^{ip(x_1 - x)} e^{ik(x_1 - x)})}{(m^2_1 - (p - k)^2 - im_1 \Gamma)^2} \right) \right).
\]

After renormalization one obtain:

\[
\lim_{x_1 \to x} C''_{1(x_1), \text{ren}} \equiv \left( \lim_{x_1 \to x} \frac{1}{n} \cdot \partial_i(x) \partial^j(x_1) G_1(x_1) \right)_{\text{ren}} = -\frac{1}{3} \cdot \frac{\lambda m_1^2}{N(2\pi)^2} \left[ \text{Re} \left( I^2_2(m_1, m_1, m_1, T) + \\
+ K^2_1(m_1, m_1, m_1, T) - \frac{1}{2} J_1(m_1, T) + \frac{1}{2m_2^2} J_2(m_1, T) \left( C_{12} - 2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) \right) + \\
+ C_{13} \, m_1^2 \ln^2 \left( \frac{m_2^2}{\Lambda^2} \right) + C_{14} \, m_1^2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) + C_{15} \, m_1^2 \right) + (N - 1) \left( I^2_2(m_2, m_2, m_2, T) + \\
+ K^2_2(m_2, m_2, m_1, T) - \frac{1}{2} J_1(m_2, T) + \frac{1}{2} J_2(m_2, T) \left( C_{12} - 2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) \right) + \\
+ C_{16} \, m_1^2 \ln^2 \left( \frac{m_2^2}{\Lambda^2} \right) + C_{17} \, m_2^2 \ln^2 \left( \frac{m_2^2}{\Lambda^2} \right) + C_{18} \, m_2^2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) + C_{19} \, m_2^2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) + \\
+ C_{20} \, m_2^2 + C_{21} \, m_2^2 + L^2_2(m_2, m_2, 4/3) \right) \right) \right] \\
\lim_{x_1 \to x} D''_{1(x_1), \text{ren}} \equiv \left( \frac{1}{n} \cdot \partial_i(x) \partial^j(x_1) D_1(x_1) \right)_{\text{ren}} = -\frac{1}{3} \cdot \frac{2 \lambda m_1^2}{N(2\pi)^2} \left[ \text{Re} \left( I^2_2(m_1, m_1, m_2, T) + \\
+ \frac{1}{2} K(m_1, m_2, m_2, T) + \frac{1}{2} K(m_2, m_1, m_2, T) - \frac{1}{4} m_1 J_1(m_1, T) - \frac{1}{4} m_2 J_1(m_2, T) + \\
+ \frac{1}{4} m_1^2 J_2(m_1, T) \left( C_{12} - 2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) \right) + \frac{1}{4} m_2^2 J_2(m_2, T) \left( C_{12} - 2 \ln \left( \frac{m_2^2}{\Lambda^2} \right) \right) + \\
+ C_{22} \, m_1^2 \ln^2 \left( \frac{m_1 m_2}{\Lambda^2} \right) + C_{23} \, m_1 m_2 \ln^2 \left( \frac{m_1 m_2}{\Lambda^2} \right) + C_{24} \, m_2^2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) + \\
+ C_{25} \, m_1 m_2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) + C_{26} \, m_1^2 + C_{27} \, m_1 m_2 + L^2_2(m_1, m_2, m_2) \right) \right].
\]

Numerical values of the coefficients $C_i$ are: $C_{12} \approx 0.719$, $C_{13} = 0.011$, $C_{14} \approx 0.177$, $C_{15} \approx -0.019$, $C_{16} \approx 0.065$, $C_{17} \approx -0.054$, $C_{18} \approx -0.021$, $C_{19} \approx 0.198$, $C_{20} \approx 0.026$, $C_{21} \approx -0.046$, $C_{22} \approx -0.526$, $C_{23} \approx 0.617$, $C_{24} \approx 0.168$, $C_{25} \approx 1.341$, $C_{26} \approx 0.396$, $C_{27} \approx -0.636$. 

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Thus specific free energy density calculated accurate to first order of vanishing terms looks like:

\[ \tilde{F} = \lim_{x \to x_1} G''(x_1)_{ren} + (N - 1) \lim_{x \to x_2} D''(x_2)_{ren} + \frac{m_1^2 v^2}{2} - \frac{4\lambda v^4}{N} + F(T, m_1(T), m_2(T), v(T)), \]

where (24) is evaluated on solution of (3), (10) and (30).

**Numerical calculations and conclusions.** Taking into account correction to Hartree–Fock approximation does not bring to appearance of a new branch of solutions of equations for effective masses and condensate. These values on temperature dependencies are depicted at Fig. 2 (solid lines at Fig. 2.a, Fig. 2.b, Fig. 2.c). For easier comparison, by dashed line these values on temperature dependencies in Hartree–Fock approximation are depicted.

Fig. 2 Effective masses, order parameter, equilibrium free energy and specific heat capacity on temperature dependencies calculated accurate to corrections of the first order of vanishing.

From Fig. 2 it follows that in low-symmetry phase only one branch of solutions remain thermodynamically stable in temperature region \( 0 \leq T \leq T_{c2} \).

Significant deviation of \( m_2 \) on temperature dependence in compare with that in Hartree–Fock approximation concerns with mentioned above logarithmically divergent terms \( m_1^2 \ln \left( \frac{m_2}{\Lambda^2} \right) \) and \( m_1^2 \ln \left( \frac{m_1 m_2}{\Lambda^2} \right) \) in three-point WF and corrections to two-point WF.
Critical temperature $T_{c_2}$ decreases in comparison with that calculated in Hartree–Fock approximation. With increase of $N$, relative magnitude of corrections, estimated as quotient $\gamma \equiv \frac{T_{c_2,SP} - T_{c_2}}{T_{c_2}} \cdot 100\%$ decreases $(\gamma|_{N=2} \approx 3.26\%, \gamma|_{N=4} \approx 3.21\%, \gamma|_{N=9} \approx 2.61\%)$ and at the limit $N \to \infty$ tends to zero, which agrees with conclusion made in chapter 3.

Though $m_2$ on temperature dependence significantly changes, at low temperatures thermodynamical observables (equilibrium free energy and heat capacity) values are close to those in Hartree–Fock approximation, as expected from general conclusions. The closer to $T_{c_2}$, the greater deviation of observables from that in Hartree–Fock approximation, in particular heat capacity, which agrees with made above conclusion of formality of heat capacity divergence at $T_{c_2(0)}$ vicinity.

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