1. INTRODUCTION

A significant approach, known as the expansion of the small parameter, is used to investigate nonlinear oscillatory systems, which is erected upon the perturbation theory. The perturbation theory can be defined as mathematical methods by which an approximate solution to a mathematical problem is figured out. This theory was used as a basis for one of the widely known methods called the Krylov-Bogoliubov and Mitropolskii (KBM) method which is applied to study nonlinear oscillatory and non-oscillatory differential systems with small nonlinearities.

KBM [4] developed a perturbation method to determine an approximate solution of a second order nonlinear differential system described by

\[ \ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \]  

(1.1)

Then the method was amplified and justified by Bogoliubov and Mitropolskii [1]. Mitropolskii [5] has extended the method to nonlinear differential system with slowly varying coefficients

\[ \ddot{x} + \omega_0^2 (\tau) x = -\varepsilon f(x, \dot{x}, \tau), \quad \tau = \varepsilon t \]  

(1.2)

Following the extended Krylov-Bogoliubov-Mitropolskii (KBM) [1, 4, 5, 8] method, Bojadziev and Edwards [2] studied some damped oscillatory and non-oscillatory systems modeled by

\[ \ddot{x} + c(\tau) \dot{x} + \omega^2(\tau) x = -\varepsilon f(x, \dot{x}, \tau), \]  

(1.3)

where \( c(\tau) \) and \( \omega(\tau) \) are positive. An extension of the Krylov-Bogoliubov-Mitropolskii method and Harmonic Balance method applied to systems of hyperbolic partial differential equations modeling physical processes with damping and slowly varying parameters. The applicability of this technique or the study of similar systems involving retardation effects is shown. In the beginning, this method was developed by Krylov and Bogoliubov to find out the periodic solutions of second order nonlinear differential systems with small nonlinearities. Shortly after that, Popov extended it to damped oscillatory processes. Roy and Shamsul [9] found an asymptotic solution of a differential system in which the coefficient changes in an exponential order of slowly varying time. In another recent article Pinakke Dey et al. [7] has presented an extended KBM for solving nonlinear problem in which the coefficients change slowly and periodically with time. Akbar et al. [3] have investigated a technique for obtaining over-damped solutions of \( n \)-th order nonlinear differential equation. The aim of this article is to find an approximate solution for
several damping effects and strong nonlinearity based on the extended KBM method and general HB method.

2. METHODOLOGY

Let us consider the nonlinear differential system

$$\dot{x} + 2k(\tau)\dot{x} + (c_1 + c_2 \cos \tau)x = -\varepsilon f(x, \dot{x}, \tau),$$

$$\tau = \varepsilon t$$  \hspace{1cm} (2.1)

$\varepsilon$ is a small parameter, $c_1$ and $c_2$ are constants, $c_2 = O(\varepsilon)$, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \geq 0$, $f$ is a given nonlinear function. Setting $\omega(\tau) = (c_1 + c_2 \cos \tau)$, $\omega(\tau)$ is known as frequency.

Setting $\varepsilon = 0$ and $\tau = \tau_0 = \text{constant}$, in Eq.(2.1), we obtain the unperturbed solution of (2.1) in the form

$$x(t, 0) = a_1 e^{\lambda_1(t_0)\tau} + a_2 e^{\lambda_2(t_0)\tau},$$

(2.2)

Let Eq.(2.1) has two eigenvalues, $\lambda_j(\tau_0)$, $j = 1, 2$, where $\lambda_j(\tau_0)$ are constants, but when $\varepsilon \neq 0$, $\lambda_j(\tau)$ vary slowly with time.

When $\varepsilon \neq 0$ we seek a solution in accordance with the KBM method, of the form

$$x(t, \varepsilon) = a_1(t, \tau) + a_2(t, \tau) + a_3(t, \tau, t, \tau) + \varepsilon^2 ..., $$

(2.3)

where $a_1$ and $a_2$ satisfy the differential equations

$$\dot{a}_1 = \lambda_1(a_1 + \omega A_1 a_1 + \varepsilon^2 ..., $$

$$\dot{a}_2 = \lambda_2(a_2 + \omega A_2 a_2 + \varepsilon^2 ..., $$

(2.4)

Confining our attention to the first few term, 1, 2, ..., $m$ in the series expansion of (2.3) and (2.4), we evaluate functions $u_1, u_2, ...$, such that $a_1$ and $a_2$ appearing in (2.3) and (2.4) satisfy (2.1) with an accuracy of $\varepsilon^{m+1}$.

Differentiating $x(t, \varepsilon)$ twice with respect to $t$, substituting for the derivatives $\ddot{x}$ and $x$ in the original equation (2.1) and equating the coefficient of $\varepsilon$, we obtain

$$\lambda_1 a_1 + \lambda_2 a_2 - \lambda_1 A_1 - \lambda_2 A_2 + \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(A_1 + A_2 \right) $$

$$+ \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(A_1 + A_2 \right) $$

$$= f^{(0)}(a_1, a_2, \tau),$$

where

$$f^{(0)}(a_1, a_2, \tau) = f(x_0, \dot{x}_0, \tau)$$

and the function $f^{(0)}$ becomes

$$f^{(0)} = -(a_1^2 + 3a_1 a_2^2 + 3a_1 a_2^2).$$

(3.1)

We substitute in (2.5) and separate it into two parts as

$$\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(A_1 + A_2 \right) $$

$$= - (3a_1^2 a_2 + 3a_3 a_2^2)$$

(3.2)

and

$$\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(A_1 + A_2 \right) $$

$$= -(a_1^3 + a_2^3)$$

(3.3)

It is assumed that both $f^{(0)}$ can be expanded in Taylor’s series (2.3-2.4)

$$f^{(0)} = \sum_{\eta=0}^{\infty} F_{\eta, r} (a_1^\eta, a_2^r) ,$$

(2.6)

Therefore, equation (2.5) can be separated into three equations for unknown functions $u_1$ and $A_1, A_2$. We obtain

$$\left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) A_1 + \lambda_1 a_1 = \sum_{\eta=0, r=0}^{\infty} F_{\eta, r} (a_1^\eta, a_2^r) , \text{ if } r_1 = r_2 + 1 $$

(2.7)

$$\left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) A_2 + \lambda_2 a_2 = \sum_{\eta=0, r=0}^{\infty} F_{\eta, r} (a_1^\eta, a_2^r) , \text{ if } r_2 = r_1 + 1 $$

(2.8)

and

$$\left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) u_1 = \sum_{\eta=0, r=0}^{\infty} F_{\eta, r} (a_1^\eta, a_2^r) ,$$

(2.9)

where $\sum_{\eta=0, r=0}^{\infty} F_{\eta, r} (a_1^\eta, a_2^r)$ exclude those terms for

$$r_1 = r_2 \pm 1.$$

Thus the particular solutions of (2.7) (2.9) give the values of the unknown functions $A_1, A_2$ and $u_1$ which completes the determination of the solution of non-linear problem (2.1).

3. RELATED EXAMPLE SOLVED BY PROPOSED METHOD

We consider a second order nonlinear system with slowly varying coefficients

$$\ddot{x} + 2k(\tau)\dot{x} + (c_1 + c_2 \cos \tau)x = -\varepsilon^3 f(x, \dot{x}, \tau),$$

(3.1)

and the function $f^{(0)}$ becomes

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2^2 + 3a_1 a_2^2 + a_1^2).$$

(3.2)

We substitute in (2.5) and separate it into two parts as

$$\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(A_1 + A_2 \right) $$

$$= - (3a_1^2 a_2 + 3a_3 a_2^2)$$

(3.3)

and

$$\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_2 a_2\right) \left(A_1 + A_2 \right) $$

$$= -(a_1^3 + a_2^3)$$

(3.4)
The particular solution of (3.16) is
\[ u_i = -\frac{a_i}{2\lambda_i(3\lambda_i - \lambda_i^2)} - \frac{a_i^2}{2\lambda_i(3\lambda_i - \lambda_i^2)} \]  
(3.5)

Now we have to solve (3.3) for two functions \( A_1 \) and \( A_2 \). According with the unified KBM method \( A_1 \) contains the term \( 3a_i^2a_i \) and \( A_2 \) contains the term \( 3a_i^2a_i^2 \) (Shamsul [10]) and thus we obtain the following equations
\[
\left( \lambda_i a_i + \lambda_i a_2 \frac{\delta}{\delta a_i} - \lambda_i \right) A_1 + \lambda_i a_i = -3a_i^2a_i^2 \]  
(3.6)

and
\[
\left( \lambda_i a_i + \lambda_i a_2 \frac{\delta}{\delta a_i} - \lambda_i \right) A_2 + \lambda_i a_i = -3a_i^2a_i^2 \]  
(3.7)

The particular solutions of (3.6) and (3.7) are
\[ A_1 = -\frac{\lambda_i a_i}{\lambda_i - \lambda_i^2} - \frac{3a_i^2a_i}{2\lambda_i} \]  
(3.8)

and
\[ A_2 = \frac{\lambda_i a_i}{\lambda_i - \lambda_i^2} - \frac{3a_i^2a_i}{2\lambda_i} \]  
(3.9)

Substituting the functional values of \( A_1, A_2 \), and the perturbation method, one compares the solutions determined by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact), we have the following equations
\[
\lambda_i a_i + \lambda_i a_2 \frac{\delta}{\delta a_i} - \lambda_i A_1 + \lambda_i a_i = -3a_i^2a_i^2 \]  
(3.10)

and
\[
\lambda_i a_i + \lambda_i a_2 \frac{\delta}{\delta a_i} - \lambda_i A_2 + \lambda_i a_i = -3a_i^2a_i^2 \]  
(3.11)

Under the transformations, \( a_i = ae^{i\omega t}/2 \) and \( a_2 = ae^{-i\omega t}/2 \) together with \( \lambda_1 = -k + i\omega \), \( \lambda_2 = -k - i\omega \) equations (3.10) and (3.11) reduce to
\[ \dot{u} = \epsilon^2 A_i(a) + \epsilon^2 \ldots \]  
(3.12)

and
\[ \dot{\phi} = \omega + \epsilon^2 B_i(a) + \epsilon^2 \ldots \]  
(3.13)

We shall obtain the variational equations of \( \bar{u} \) and \( \varphi \) in the real form \((\bar{u} \) and \( \varphi \) are known as amplitude and phase respectively) which transform (3.12) to
\[ \dot{\bar{u}} = -ka - \frac{3a_i^2k}{2\omega} + \frac{3a_i^2k}{8(k^2 + \omega^2)} \]  
(3.13)

and
\[ \dot{\varphi} = \omega - \frac{3a_i^2k}{2\omega} + \frac{3a_i^2k}{8(k^2 + \omega^2)} \]  
(3.14)

where \( \omega = \sqrt{c_1 + c_2\cos\tau} \)

The variational equations (3.13) and (3.14) are in the form of the KBM [1, 4, 5, 8] solution.

Therefore, the improved solution of the equation (3.1) is
\[ x(t, \epsilon) = a \cos \phi + \epsilon u_1 + \cdots, \]  
(3.15)

where \( \phi = \omega t + \varphi \), and \( a \) and \( \phi \) are the solutions of the equations (3.13) and (3.14) respectively.

4. Discussion of Results

Based on the extended KBM [1, 4, 5, 8] method and HB method an asymptotic solution of second order damped nonlinear systems has been found. With a view to check the exactness of an approximate solution determined by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). We have compared the perturbation solutions (3.15) of Duffing’s equations (3.1) to those determined by Runge-Kutta fourth-order procedure.

Figure 4.1(a). Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions \( a = 0.6000, \omega = -0.005433 \) \( x(0) = 0.60000, \dot{x}(0) = 0.00000 \) for \( \epsilon = 0.7, \omega_0 = 1, h = 0.05 \), with damping coefficient is \( k = 0.1\cos\tau \).

Figure 4.1(b). Unified KBM solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions \( x(0) = 0.60000, \dot{x}(0) = 0.00000 \) for \( \epsilon = 0.7, \omega_0 = 1, h = 0.05 \), with damping coefficient is \( k = 0.1\cos\tau \).
Fig 4.2(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 0.6000, \varphi = -0.010866 \quad [x(0) = 0.60000, \dot{x}(0) = 0.00000]$ for $e = 7, \omega_b = 1, h = .05$, with damping coefficient is $k = 0.02\sqrt{\cos}$

Fig 4.2(b): Unified KBM solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 0.6000, \varphi = -0.016548 \quad [x(0) = 1.00000, \dot{x}(0) = 0.00000]$ for $e = 7, \omega_b = 1, h = .05$, with damping coefficient is $k = 0.02\sqrt{\cos}$

Fig 4.3(a): Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 0.6000, \varphi = -0.027172 \quad [x(0) = 0.60000, \dot{x}(0) = 0.00000]$ for $e = 7, \omega_b = 1, h = .05$, with damping coefficient is $k = 0.05\sqrt{\cos}$

First of all, $x$ is calculated by (3.6) with initial conditions $[x(0) = 0.6000, \dot{x}(0) = 0.00000]$ or $a = 0.6000, \varphi = -0.00543$ for $e = 7, \omega_b = \omega_0\sqrt{(c_1 + c_2 \cos \tau)}, k = 0.01$. Then we can use the numerical solutions which is also obtained by Runge-Kutta fourth-order method. The result is shown in Fig. 4.1(a).

Also Fig 4.1(b) represents Unified KBM solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0) = 0.60000, \dot{x}(0) = 0.00000]$ for $e = 7, \omega_b = 1, h = .05$, with damping coefficient is $k = 0.01\sqrt{\cos}$. We see that in Fig. 4.1(a) represents the perturbation solution well observe to the numerical solution, but in this conditions subsisting perturbation solution (unified method) in Fig. 4.1(b) does not observe. From Fig. 4.2(a), Fig. 4.3(a), Fig. 4.4(a) and Fig 4.5(a) we notice that the approximate solutions opinion at numerical outcome well even if $k \geq 0.1$ and $e \geq 7$, but in Fig. 4.2(b), Fig. 4.3(b), Fig. 4.4(b), Fig 4.5(b) do not opinion and the solution fails to give aimed result. The KBM method was developed for the systems in which the linear restoring forces must present and the case where only nonlinear restoring force exists. The methods give more realistic solutions that converge very rapidly in physical problems.
In this article we find the approximate solution of a nonlinear differential system with slowly varying coefficients under the action of several damping forces. The solution is simpler than classical KBM method. Here it is found that if the damping force is significant, the solution is stable.

5. CONCLUSIONS

In this article we find the approximate solution of a nonlinear differential system with slowly varying coefficients under the action of several damping forces. The solution is simpler than classical KBM method. Here it is found that if the damping force is significant, the solution is stable.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest related to the publication of this article.

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