A quantum retrograde canon: complete population inversion in $n^2$-state systems

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Abstract
We present a novel approach for analytically reducing a family of time-dependent multi-state quantum control problems to two-state systems. The presented method translates between $SU(2) \times SU(2)$ related $n^2$-state systems and two-state systems, such that the former undergo complete population inversion (CPI) if and only if the latter reach specific states. For even $n$, the method translates any two-state CPI scheme to a family of CPI schemes in $n^2$-state systems. In particular, facilitating CPI in a four-state system via real time-dependent nearest-neighbors couplings is reduced to facilitating CPI in a two-level system. Furthermore, we show that the method can be used for operator control, and provide conditions for producing several universal gates for quantum computation as an example. In addition, we indicate a basis for utilizing the method in optimal control problems.

Multi-state quantum systems are in the frontier of quantum control research, holding high potential for reducing computation complexity [1], and strengthening communication security [2] in quantum information technologies [3]. A central topic in multi-state control research is complete population inversion (CPI), where a quantum system is transferred between orthogonal states. This issue is a fundamental task in applications such as state preparation [4].

The simplest and most studied quantum control problems deal with two-state systems [5, 6]. As we move to multiple states, synthesis and analysis of control schemes become increasingly difficult. Hence, multi-state control problems are often approached by some method of reduction to one or more two-state problems [7, 8]. One such method is adiabatic elimination [9], in which the system’s dynamics are approximated by a two-state system through the elimination of irrelevant, non-resonantly coupled states. While this approach is often adequate, its applicability is conditioned on parameters’ range and its results may suffer significant inaccuracies under multiple eliminations [10]. Another reduction method [11] provides analytic solutions to $N$-level systems with $SU(2)$ dynamic symmetry in terms of the their fundamental two-state representation. While this approach is exact, it is limited to systems with $su(2)$ dynamical Lie algebra, which have only three degrees of freedom (DOF) at each moment of time.

In this paper, we present the quantum retrograde canon, a novel method for analytically reducing time-dependent multi-state quantum control problems to two-state systems. The reduction method is based on an exact translation between time-dependent $n^2$-state Hamiltonians with $su(2) \oplus su(2)$ dynamical Lie algebra and two-state systems. The principle idea underlying the translation resembles the technique of the retrograde canon in music, where a musical line is played simultaneously with another copy of it, inverted in time (see ‘Crab Canon’ by Bach [12]). Analogously, the quantum retrograde canon maps a one-qubit time-dependent Hamiltonian to a two-qubit time-dependent Hamiltonian—taking one qubit through the dynamics of the original system and the second qubit through the same dynamics, inverted in time. Given an appropriate choice of basis, the two-qubit system undergoes CPI if and only if the one-qubit system reaches a specific state. Since the
mapping is invertible, one can apply the method the other way around and translate certain two-qubit Hamiltonians, which have six DOF at each moment, to two-state Hamiltonians, which have only three DOF at each moment. By using higher dimensional representations of the two-qubit system, and an appropriate choice of basis, one gets a translation method between two-state schemes and SU(2) × SU(2) controlled $n^2$-state CPI schemes. In particular, for even $n$, the method translates any two-state CPI scheme to CPI schemes in $n^2$-state systems.

The paper is structured as follows: in section 1, we define the retrograde canon. In section 2, we illustrate a way in which the retrograde canon reduces four-level CPI problems to two-level CPI problems. In section 3, we formulate and prove the fundamental observation underlying the reduction method. In section 4, we state a more general claim regarding four-level CPI problems. In section 5, we discuss two concrete examples of the method’s application. In section 6, we indicate the use of the method for optimal and operator control problems. Lastly, in section 7, we consider generalizations of the method, including application to $n^2$-state systems through higher order representations of the SU(2) group. In section 8, we summarize. Various technical issues are discussed in the appendices.

1. Basic definitions

We begin with defining the key notions of the retrograde canon. Let $H(t)$ be a time-dependent two-state system Hamiltonian and $U(t)$ be the propagator generated by $H$, i.e., $U$ is the solution of Schrödinger’s operator equation $i\hbar U(t) = H(t)U(t)$ with the initial condition $U(0) = I_2$, where $I_2$ is the two-dimensional identity matrix. We define the Retrograde of $H$, starting from time $T > 0$, by

$$H^R(t) \equiv -H(T - t).$$

(1)

It can be verified by differentiation that $U^R$, the propagator generated by $H^R$, satisfies:

$$U^R(t) = U(T - t)U(T)^{-1}.\quad(2)$$

$U^R(t)$ takes states backwards along the path they trace when acted on by $U(t)$. That is, suppose that $|f\rangle = U(T)|i\rangle$ - i.e., $|f\rangle$ is the state to which the initial state $|i\rangle$ evolves to in the original system—then

$$U^R(t)|f\rangle = U(T - t)|i\rangle.\quad(3)$$

In what follows we will be interested in the the effect of acting simultaneously with both the original Hamiltonian and its retrograde. Thus, we define the Retrograde canon Hamiltonian, $H^{RC}$, by

$$H^{RC}(t) \equiv H^R(t) \otimes I_2 + I_2 \otimes H(t).\quad(4)$$

$H^{RC}$ acts on one qubit with the original time-dependent Hamiltonian, and on another qubit with its retrograde, starting from time $T > 0$. Clearly, $U^{RC}$, the Retrograde Canon Propagator generated by $H^{RC}$, satisfies

$$U^{RC}(t) = U^R(t) \otimes U(t).\quad(5)$$

The retrograde canon is a one-to-one mapping from a $su(2)$ two-state time-dependent Hamiltonian in the time interval $[0, T]$ and a $su(2) \otimes su(2)$ four-state time-dependent Hamiltonian in the time interval $[0, T/2]$. Accordingly, any time-dependent four-state Hamiltonian of the form $\tilde{H}(t) \equiv H^B(t) \otimes I_2 + I_2 \otimes H^F(t)$, where $H^B(t), H^F(t)$ are two-state Hamiltonians defined for $t \in [0, T/2]$, can be regarded as the retrograde canon of a unique two-state Hamiltonian, $H$, defined for $t \in [0, T]$ through

$$H(t) = \begin{cases} H^B(t) & 0 \leq t \leq T/2 \\ -H^F(T - t) & T/2 \leq t \leq T. \end{cases}\quad(6)$$

$H(t)$, being a two-state Hamiltonian, has only three DOF at each moment. On the other hand, the four-level Hamiltonian $\tilde{H}(t)$ defined above has six DOF at each moment—three coming from $H^B(t)$ and another three coming from $H^F(t)$. The one-to-one mapping of $\tilde{H}$ that has six DOF at each moment to $H$, which has only three DOF at each moment, is achieved by ‘splitting’ the DOF of $\tilde{H}(t)$ between the DOF at two different moments—i.e., for every $s \in [0, T/2]$, $\tilde{H}(s)$ is mapped into $H(s)$ and $H(T - s)$.

2. An application of the method

Let us present a relatively general case of reducing CPI problems from four-state systems to two-state systems through the retrograde canon. This is not the most general four-level application of the method, but it is general
enough to understand the idea. Consider the following four-level Hamiltonian:

$$\hat{H}(t) = \begin{pmatrix} 0 & A(t) & iE(t) & D(t) \\ A(t) & 0 & B(t) & iF(t) \\ -iE(t) & B(t) & 0 & C(t) \\ D(t) & -iF(t) & C(t) & 0 \end{pmatrix},$$

(7)

where $A(t), \ldots, F(t)$ are six arbitrary real functions of time. Suppose we are interested in the conditions under which $\hat{H}(t)$ facilitates CPI from $\Psi_1 = (1, 0, 0, 0)^\top$ to $\Psi_3 = (0, 0, 1, 0)^\top$ at some time $\tau > 0$, i.e., the conditions under which $1 = |\langle \Psi_3, \hat{L}(\tau) \Psi_1 \rangle|$, where $\hat{L}(t)$ is the propagator generated by $\hat{H}(t)$. Special cases of this problem are encountered in the literature [9, 13, 14] and in applications. We will show that the through the retrograde canon we reduce this question to the question of whether $\epsilon_1 = (1, 0)^\top$ evolves to $\pm \epsilon_2 = \pm (0, 1)^\top$ at time $T = 2\tau$ in a two-state system governed by the Hamiltonian $H(t) = x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z$, i.e.

$$H(t) = \begin{pmatrix} z(t) & x(t) + iy(t) \\ x(t) - iy(t) & -z(t) \end{pmatrix},$$

(8)

where

$$x(t) \equiv -\frac{1}{2} \left( C(T - t) + A(T - t) \right), \quad 0 \leq t \leq T/2,$$

$$y(t) \equiv \frac{1}{2} \left( E(t) + F(t) \right), \quad 0 \leq t \leq T/2,$$

$$z(t) \equiv \frac{1}{2} \left( B(t) + D(t) \right), \quad 0 \leq t \leq T/2,$$

$$C(T - t) - A(T - t) \quad \frac{1}{2} \left( E(T - t) - F(T - t) \right), \quad \frac{T}{2} \leq t \leq T,$$

The translation between the two-level Hamiltonian of equation (8) and the four-level Hamiltonian of equation (7) is done in two stages, illustrated schematically in figure 1: in the first stage, we change basis and define $\hat{H}(t) \equiv W^\dagger \hat{H}(t) W$, where

$$W \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

(9)

In the second stage, we apply equation (6) to $\hat{H}(t)$ and get the two-level $H(t)$. That is, we consider $\hat{H}(t)$ as the retrograde canon Hamiltonian of some $\hat{H}(t)$, and solve for $H(t)$.

The claim that $H(t)$ facilitates CPI from $\Psi_1 = (1, 0, 0, 0)^\top$ to $\Psi_3 = (0, 0, 1, 0)^\top$ at time $\tau$ if and only if $H(t)$ takes $\epsilon_1 = (1, 0)^\top$ to $\pm \epsilon_2 = \pm (0, 1)^\top$ at time $T = 2\tau$ is a special case of the claim we formulate in section 4.

### 3. The central observation

Prior to formulating our general claim for the case of four-state systems, let us state and prove the observation which underlies the use of the retrograde canon in reducing multi-state CPI problems to two-state systems:
\[ U(T)\ket{\uparrow} = \ket{\downarrow} \Leftrightarrow U^{RC}(\tau)\left( (\downarrow \downarrow) + (\uparrow \uparrow) \right) = (\uparrow \downarrow) - (\downarrow \uparrow), \]

(10)

where \( \ket{\uparrow} \equiv (1, \ 0)^T \) and \( \ket{\downarrow} \equiv e^{i\alpha}(0, \ 1)^T \) for some \( \alpha \in [0, 2\pi) \). Equation (10) states that a two-state Hamiltonian, \( H \), facilitates CPI at time \( T \) if and only if \( H^{RC} \), facilitates a transition between two specific perfectly entangled states at time \( \tau \equiv T/2 \). We emphasize that the path of \( U(t) \) up to the point \( t = T \) can be completely arbitrary. In particular, \( U(\tau) \), which appears in \( U^{RC}(\tau) = U(\tau) \otimes U^{R}(\tau) \), can be any \( SU(2) \) matrix.

To prove the \( \Rightarrow \) direction of equation (10) we first note that equation (3) tells us that for \( \tau = T/2 \),

\[ U^{R}(\tau)\ket{i} = U(\tau)\ket{i}. \]

(11)

Now, assume that indeed \( U(T)\ket{\uparrow} = \ket{\downarrow} \). Since \( U(\tau) \) is unitary, it transfers orthogonal states to orthogonal states. Thus, it is sensible to mark \( U(\tau)\ket{\downarrow} \equiv \ket{\uparrow} \) and \( U(\tau)\ket{\uparrow} \equiv \ket{\downarrow} \). From equation (11) we get that \( U^{R}(\tau)\ket{\uparrow} = \ket{\uparrow} \). Moreover, from the fact that \( U(T)^{-1} \) is a \( \pi \)-rotation in a spin-half representation, and hence \( U(T)^{-1}U(T)^{-1} = -I_{2} \), it follows that (marking \( U_{t} \equiv U(t) \))

\[ U^{R}(\tau)\ket{\uparrow} = U_{1}U_{2}\ket{\uparrow} = U_{1}U_{2}U_{1}U_{1}U_{2}\ket{\uparrow} \]

\[ = -U_{2}U_{1}\ket{\uparrow} = -U_{1}\ket{\uparrow} \]

(12)

and therefore \( U^{R}(\tau)\ket{\uparrow} = -\ket{\downarrow} \). Hence,

\[ U^{RC}(\tau)((\downarrow \downarrow) + (\uparrow \uparrow)) \]

\[ = U_{2}\ket{\downarrow} \otimes U_{1}\ket{\downarrow} + U_{2}\ket{\uparrow} \otimes U_{1}\ket{\uparrow} \]

\[ = (\uparrow \downarrow) - (\downarrow \uparrow) \]

(13)

where the last step follows from the Clebsch–Jordan fact that \( (\uparrow \downarrow) - (\downarrow \uparrow) \) is a \( SU(2) \) scalar.

To prove the \( \Leftarrow \) direction of equation (10) we utilize a slightly different perspective, which will also be useful for the generalizations of the retrograde canon presented below. We assume that \( U^{RC}(\tau)((\downarrow \downarrow) + (\uparrow \uparrow)) = (\uparrow \downarrow) - (\downarrow \uparrow) \) and need to prove that \( U(T)\ket{\uparrow} = \ket{\downarrow} \). For the proof we shall use two simple facts: the first is that for any three 2-by-2 matrices, \( A, B, m \),

\[ (A \otimes B)F(m) = F(ABm^T), \]

(14)

where \( F \) simply flattens matrices, i.e., \( F \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = (a, b, c, d)^T \). Equation (14) may be verified by direct calculation. The second, is that for \( Y \equiv e^{i\alpha \sigma} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) (\( \sigma \) are the Pauli matrices) and any \( u \in SU(2) \),

\[ uYu^T = Y. \]

(15)

Equation (15) is in fact equivalent to the claim that \( (\downarrow \uparrow) - (\downarrow \downarrow) \) is a \( SU(2) \) scalar—the equivalence can be seen through equation (14) and the observation that \( (\uparrow \downarrow) - (\downarrow \uparrow) \rightarrow e^{i\alpha}F(Y) \) in the basis \( (\uparrow \downarrow), (\downarrow \downarrow), (\uparrow \uparrow), (\downarrow \uparrow) \).

To proceed, we define \( R \equiv e^{-i\alpha \sigma} \), and note that in the above basis

\[ (\downarrow \downarrow) + (\uparrow \uparrow) \rightarrow F \left( \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \right) = e^{i\alpha}F(R). \]

Now, it follows that the assumption is equivalent to

\[ Y = F^{-1}(U^{RC} \otimes U^{R})F(R) = U_{2}^{R}R^{T}U_{1}^{T} = U_{2}^{R}RY^{-1}U_{1}^{-1}U_{1}^{T} = U_{2}^{R}RY^{-1}U_{1}^{-1}, \]

(16)

where we used equation (14) in the second equality, inserted \( I_{2} = Y^{-1}U_{1}^{-1}U_{1} \) in the third equality, and used equation (15) in the fourth equality. Multiplying equation (16) by \( Y^{-1} \) we get that \( U_{2}^{R}RY^{-1}U_{1}^{-1} = I_{2} \) and therefore \( U_{2}^{R} = R^{-1}U_{1}^{-1} \). So finally we reach our goal:

\[ U(T)\ket{\uparrow} = (U_{2}^{R})^{-1}U_{1}\ket{\uparrow} = -RY^{-1}U_{1}^{-1}U_{1}\ket{\uparrow} = -RY\ket{\uparrow} = R(0,1)^T = e^{i\alpha}(0, \ 1)^T = \ket{\uparrow}, \]

where the first equality can be easily derived from equation (2).

### 4. CPI in four-state systems

Equation (10) describes a correspondence between two-state CPI and the four-state transition of:

\[ (\downarrow \downarrow) + (\uparrow \uparrow) \rightarrow (\uparrow \downarrow) - (\downarrow \uparrow) \] (17). It takes just a few small steps to connect this transition to four-state CPI between standard basis vectors.

Marking \( w_{1} = \frac{1}{\sqrt{2}}((\downarrow \downarrow) + (\uparrow \uparrow)) \) and \( w_{2} = \frac{1}{\sqrt{2}}((\downarrow \uparrow) - (\downarrow \uparrow)) \), we first note that

\[ U(T)\ket{\uparrow} = \pm\ket{\downarrow} \Leftrightarrow U^{RC}(\tau)w_{1} = \pm w_{3} \]

is a trivial generalization of equation (10). Next, we note that in the fundamental representation of \( SU(2) \times SU(2) \) (and in higher even dimensional representations), \( e^{i\alpha}w_{1} \) and \( e^{i\beta}w_{2} \) are orthonormal for any \( \phi_{1}, \phi_{2} \in [0, 2\pi) \). They can therefore be completed to an orthonormal basis of \( \mathbb{C}^{4} \) with some vectors \( w_{3}, w_{4} \in \mathbb{C}^{4} \). This basis determines a unitary matrix \( W \equiv (e^{i\alpha}w_{1}, w_{2}, e^{i\beta}w_{3}, w_{4}) \in M_{4}[\mathbb{C}] \) which we call the **conjugating matrix**
H(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}

5. Examples

We now present two concrete examples of the method’s application. To better see what is going on in these examples we shall visualize the evolution of the propagators as a path in the unit 3-ball, \( B = \{ x \in \mathbb{R}^3 \mid ||x|| \leq 1 \} \). We quickly review how we do that: any element \( u \in SU(2) \) corresponds to a rotation by an angle \( \phi \in [0, 4\pi) \) around an axis \( \hat{r} \in \mathbb{R}^3 \) \( (||\hat{r}|| = 1) \) according to \( u = \hat{R}_\theta(\phi) \equiv e^{i\phi \hat{r}} \), where \( \hat{r}^{(2)} \equiv (\hat{r}_1, \hat{r}_2, \hat{r}_3) \equiv \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z) \). A path in \( SU(2) \) can thus be projected to the unit 3-ball using the two-to-one map \( \eta \), defined by

\[
\eta(\hat{R}_\theta(\phi)) \equiv \sin \left( \frac{\phi}{2} \right) \hat{r}.
\]
where $p, q$ are odd integers. A Hamiltonian of the above form facilitates a $p\pi$-rotation around the the $\hat{x}$-axis, followed by a $q\pi$-rotation around the $\hat{z}$-axis. Such sequences are equivalent to a $\pm \pi$-rotation around the $\hat{y}$-axis. Hence, these Hamiltonians facilitate CPI and satisfy the left hand side of 19 with $|\tilde{\eta}\rangle = \pm (0, 1)^T$ for time $T \equiv 2\pi$. We shall translate $H(t)$ to a four-level system, using the the retrograde canon and the following conjugating matrix $W$

$$W \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sin(\theta) & 0 & \cos(\theta) \\ \cos(\theta) & 1 & -\sin(\theta) & 0 \\ \cos(\theta) & -1 & -\sin(\theta) & 0 \\ 1 & -\sin(\theta) & 0 & -\cos(\theta) \end{pmatrix},$$

(22)

where $\theta = -\tan^{-1}\left(\frac{p}{q}\right)$. The result, $\mathcal{H}^{\text{CRC}}$, is a constant coupling four-state Hamiltonian which facilitates CPI from $\Psi_1$ to $\Psi_3$ at time $\tau = T/2 = \pi$, where

$$\mathcal{H}^{\text{CRC}} = n(p, q) \cdot \begin{pmatrix} 0 & C(p, q) & 0 & 0 \\ \cdots & 0 & B(p, q) & 0 \\ \cdots & \cdots & 0 & A(p, q) \\ \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

(23)

where $n(p, q) = \frac{1}{\sqrt{p^2 + q^2}}$ and $A(p, q), B(p, q), C(p, q)$ are time-constant integers, depending on $p$ and $q$, which satisfy the Pythagorean relation $A^2(p, q) + B^2(p, q) = C^2(p, q)$. The fact that such Hamiltonians perform CPI was derived and utilized in previous works [13, 14]. Figure 3 presents the dynamics of the original two-state system and the resulting retrograde canon system.

For the second example, we take a two-state time-dependent Hamiltonian that performs CPI through adiabatic following [15] by means of a Landau–Zener scheme [16, 17]. We define $H(t) \equiv \tilde{\mathcal{H}}(t) \cdot J^{(2)}$ with

$$\tilde{\mathcal{H}}(t) = \left(\Omega_0, 0, B\left(1 - \frac{2t}{T}\right)\right),$$

(24)

satisfying $B \gg \Omega_0^2$ and $\Omega_0^2 \gg B/T$. We define the retrograde Hamiltonian with $T = \tilde{T}$ as the moment of two-state CPI. For $t \approx T$, $U(t)(1, 0) \approx e^{\frac{2i}{T}}(0, 1)^T$, with $\alpha(t)$ changing rapidly. Applying the method requires using $\alpha(T)$ in the definition of $W$. Alternatively, we can rotate $H(t)$ around the $\hat{z}$-axis with $H(t) \rightarrow R_z(-\alpha(T))H(t)R_z(\alpha(T))$, and use $W$ defined in equation (22), to get the following time-dependent Hamiltonian:

$$\mathcal{H}^{\text{CRC}}(t) = \begin{pmatrix} 0 & h_z(t)\sin(\theta) & \sin(\alpha) & h_z(t)\cos(\theta) \\ \cdots & 0 & -\cos(\alpha)\sin(\theta) & 0 \\ \cdots & \cdots & 0 & \cos(\alpha)\cos(\theta) \\ \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

(25)
Figure 4 presents the resulting dynamics. It can be observed in the plot of the original propagator’s dynamics (bottom left), that for \( t \ll T/2 \) the two-state propagator is approximately a rotation by some angle around \( \hat{z} \)-axis, while for \( t \gg T/2 \) it is approximately a rotation by an angle \( \pi \) around some axis in the \( \hat{x} - \hat{y} \) plane (see equation (20)). While this explains the robustness of the two-state dynamics starting at \((1, 0)^T\), it also explains the transient nature of the CPI in the retrograde canon system: Using equations (2), (5) and (14) we see that if \( t \) is sufficiently far from \( T/2 \), then \( \mathcal{U}^{CRC}(t) \text{w}_1 \approx F(R_2(\beta)R_3(\pi R_{\text{cos}(\gamma)})^{\dagger}R_{\text{sin}(\gamma)})^T) = F(R_2(\delta)) \) with \( \beta, \delta \in [0, 4\pi] \), \( \gamma \in [0, 2\pi] \). The low occupancy of the third state for such \( t \) is then explained by \( 0 = \langle F(R_2(\delta)), F(R_3(\pi)) \rangle \forall \delta \in [0, 4\pi] \)—where \( \langle \cdot, \cdot \rangle \) is the standard inner product.

6. Optimal and operator control

Next, we indicate ways in which the retrograde canon can be used for optimal control and operator control problems. We begin with optimal control. It is typical for such problems to contain optimization criteria or constraints relating to some norm defined through the parameters of the Hamiltonian \([18]\). We therefore note that a simple relation between natural norms of the four-state \( \mathcal{H}^{CRC} \) and the two-state \( H \), holds under the suggested translation method, namely

\[
\frac{1}{\sqrt{2}} \| \mathcal{H}^{CRC}(t) \|_F = \sqrt{\| \hat{h}(t) \|_2^2 + \| \hat{h}(2\tau - t) \|_2^2},
\]

where \( \hat{h}(t) \) is the two-state Hamiltonian through \( H(t) = \hat{h}(t) \cdot t^{(2)} \), \( \| \cdot \|_2 \) is the Euclidean norm and \( \| \cdot \|_F \) is the Frobenius norm defined by \( \| \mathcal{H} \|_F = \sqrt{\sum |\mathcal{H}_{ij}|^2} \). It follows from equation (26) that

\[
\frac{1}{2} \int_0^\tau \| \hat{h}^{CRC}(t) \|_F^2 \, dt = \int_0^{2\tau} \| \hat{h}(t) \|_2^2 \, dt.
\]

Relations such as equations (26) and (27) provide a basis for identifying certain multi-state optimal control problems with two-state optimal control problems.

Up to now we have only discussed the retrograde canon in the context of the CPI state-control problem, yet the retrograde canon method may also shed light on operator control problems. Indeed, knowing that \( \mathcal{U}^{CRC} \) performs CPI at time \( \tau \) holds only partial information on \( \mathcal{U}^{CRC}(\tau) \). However, further information regarding \( \mathcal{U}^{CRC}(\tau) \) can be related to information on \( \mathcal{U}(\tau) \)—i.e. the point where the curve of \( \mathcal{U}(t) \) and \( \mathcal{U}(T - t) \) meet. Consider, for example, four-state propagator operators of the following form

\[ R = |eg\rangle \langle e| \pm |ce\rangle \langle eg| \mp |ge\rangle \langle ge| \pm |gg\rangle \langle gg| \text{ where } |g\rangle, |e\rangle \text{ form a basis of a two-state system and } |gg\rangle \equiv |g\rangle \otimes |g\rangle. \]

Operators such as \( R \) are universal operators for quantum computation, since they maximally entangle separable states (\( R(|e + g\rangle \otimes \psi) = |ce\rangle \pm |gg\rangle \)) and thus satisfy a criterion of being a universal gate \([19]\). If we identify \( |\Psi_h, \Psi_s, |\Psi_q\rangle \) with \(( \uparrow, \uparrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow \), and use the following conjugating matrix

\[
W \equiv \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & i \sin(\theta) & i \cos(\theta)
0 & -1 & i \cos(\theta) & -i \sin(\theta)
0 & 1 & i \cos(\theta) & -i \sin(\theta)
1 & 0 & -i \sin(\theta) & -i \cos(\theta)
\end{pmatrix}
\]

then, for \( \theta = 0 \), the condition for getting \( \mathcal{U}^{CRC}(\tau) = R \) is that \( \eta(\mathcal{U}(\tau)) \) is either \((0, \pm 1/\sqrt{2}, 0) \) or \((\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2}) \). Changing \( \theta \) relates the above points to other operators—for example, for \( \theta = \pi/4 \), the
meeting points \((\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2})\) correspond to a ‘double-rail’ operator, making the transitions \(e_1 \leftrightarrow e_2\) and \(e_3 \leftrightarrow e_4\).

7. Generalizations

Last, we present the general version of the method, which allows translating two-level schemes to a wide family of \(SU(2) \times SU(2)\) controlled \(n^2\)-level systems. The generalization is based on relinquishing three assumptions made above which concern: (a) the pace of movement along the dynamical path traced by the original one-qubit Hamiltonian; (b) the dimension of representation of the original Hamiltonian; and (c) the final state of the original system. For a discussion of these assumptions and a proof of the following general translation claim see appendix B.

To carry out the generalization we revise our definitions. We begin by taking a higher order representation of the original Hamiltonian: given a two-state Hamiltonian, \(H(t)\), we define \(H^{(n)}(t)\) to be its image in an \(n\)-dimensional representation. That is,

\[
H_n(t) \equiv \pi_n(H(t)),
\]

(29)

where \(\pi_n\) is a \(n\)-dimensional irreducible representation of \(su(2)\), fixed by satisfying \(\pi_n(U^{(2)}) = H^{(n)}_i\) for \(i = 1, 2, 3\) where \(J^{(n)}_i \equiv (J^{(n)}_x, J^{(n)}_y, J^{(n)}_z)\) satisfies the commutation relation [\(J^{(n)}_x, J^{(n)}_y]\] = \(i\hbar J^{(n)}_z\) with a real \(J^{(n)}_0\) and a diagonal \(J^{(n)}_0\). Next, we define a \(n\)-dimensional retrograde canon Hamiltonian that goes back along the original trajectory in a non-constant pace. That is, we define

\[
H^R_n(t) \equiv -\dot{\tau}(t)H^{(n)}(T - \tau(t)),
\]

(30)

where \(0 \leq \tau(t) \leq T\) is a general differentiable function of time. It can be verified by differentiation that \(U_n^R(t) = U_n(t - \tau(t))U_n(T)\) is the propagator generated by \(H^R_n\), where \(U_n\) is the \(n\)-dimensional propagator generated by \(H_n\). We continue be defining the \(n^2\)-state retrograde canon Hamiltonian, \(H^R_{n^2}\) by

\[
H^R_{n^2}(t) \equiv H_n^R(t) \otimes I_2 + I_2 \otimes H_n(t).
\]

(31)

Clearly \(U_n^R\), the propagator generated by \(H_n^R\), satisfies \(U_n^R(t) = U_n^R(t) \otimes U_n(t)\).

We will also to generalize the definition of \(W\), the conjugating matrix defined above in section 3: given \(n \in \mathbb{N}\), \(k \in [1, \ldots, n - 1]\), and a unit vector \(\hat{r} \in \mathbb{R}^2\) \((|\hat{r}| = 1)\), we shall designate as a \(n^2\)-conjugating matrix of \(k\) and \(\hat{r}\) any \(n^2\)-dimensional unitary matrix \(W(k, \hat{r}) = (w_{11}, w_{22}, \ldots, w_{n^2})\) satisfying

\[
w_1 \propto F_n(R^{(n)}_k(\omega_k) \cdot R^{(n)}_y(\pi)); \quad w_2 \propto F_n(R^{(n)}_y(\pi)),
\]

(32)

where \(\omega_k \equiv 2\kappa e^{\frac{2\pi i}{n}} R^{(n)}_y(\phi) \equiv e^{i\omega} R^{(n)}_y\) and \(F_n\) is the \(n\)-dimensional analog of the flattening function \(F\) encountered in equation (14), i.e., it takes a \(n\)-by-\(n\) matrix and returns a \(n^2\) column vector defined by \(F(m)_{(n-1)i+j} \equiv m_{ij}\).

Now, we can define the \(n^2\)-state conjugated retrograde canon Hamiltonian:

\[
H^R_{n^2}(t) \equiv W_n(k, \hat{r}) H^R_{n^2}(t) W_n(k, \hat{r}),
\]

(33)

Its corresponding propagator, \(U^R_{n^2}\), satisfies \(U^R_{n^2} = W_n(k, \hat{r}) U^R_{n^2} W_n(k, \hat{r})\).

Now we can finally formulate the general translation claim: Let there be \(k, \hat{r}, \tau, a n^2\)-state conjugating matrix \(W_n(k, \hat{r})\), a differentiable pace function \(\tau : [0, T) \rightarrow \mathbb{R}\) and a two-state system Hamiltonian \(H : [0, T) \rightarrow su(2)\), whose propagator is \(U(t)\). Then, \(\forall \tau \in (0, T)\), for which \(U(T - \tau) = U(\tau)\), the following holds

\[
U(T) = \pm R_\tau(\omega) \Rightarrow 1 = \{e_i, U^R_{n^2}(\tau) e_i\},
\]

(34)

where \(e_i \in \mathbb{C}^{n^2}(i = 1, 2)\) are standard basis vectors. In particular, for any time \(\tau\), for which \(T - \tau(\tau) = \tau\), \(H^R_{n^2}\) facilitates the transition \(e_1 \rightarrow e_2\) at time \(\tau \in (0, T)\) if and only if \(U(T) = R_\tau(\omega)\). Note that for even \(n\) we could take \(k = n/2 \in \mathbb{N}\) and \(\hat{r} = \hat{y}\) to get \(U(T) = R^{(n)}_\tau(\pi)\) and \(w_1 = R^{(n)}_y(\pi)\) for equations (34) and (32) respectively—such a choice would be a straight forward generalization of the four-state translation method, one which converts two-state CPI schemes to \(n^2\)-state CPI schemes.

8. Conclusion

In conclusion, we introduced a novel method for translating multi-state control problems to two-state systems. The method provides a new framework for importing the knowledge, tools and intuition related to two-state systems, into multi-state research. In particular, the method offers an exact reduction into two-state systems of multi-state CPI problems that cannot be reduced by other currently available methods. The idea of the retrograde canon in control theory can be further explored in future research: e.g., in the application of the retrograde canon to other groups that have appropriate properties, such as the symplectic group.
$Sp(n) \rightarrow Sp(n) \times Sp(n)$ instead of $SU(2) \rightarrow SU(2) \times SU(2)$ or in using $k$ two-states CPI schemes together to generate a CPI scheme in a $SU(2) \times SU(2) \times \ldots \times SU(2)$ controlled system. It may also be interesting to examine the possibility of applying the method in the context of mixed states. We hope that the analytical reduction of multi-state control problems to two-state systems offered by the quantum retrograde canon will be helpful in deepening the understanding of multi-state dynamics, and in simplifying the analysis in certain cases of particular interest.

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**Appendix A. The general form of the conjugating matrix**

We shall now present and explain the general form of the four-state conjugating matrix $W$, and general form of $H^{CRC}(t)$ derived from it. Remember that $w_3, w_4 \in \mathbb{C}^4$ (discussed between equations 17 and 18) can be chosen to be any two vectors that complete $e^{i\phi}w_1, e^{i\phi}w_3$ to an orthonormal basis of $\mathbb{C}^4$. This gives the choice of $W$ six DOF: the angle $\alpha$ in the definition of $\{ 1\}$, the four phases of the four vectors, and the orientation of $w_3$ and $w_4$ in the subspace orthogonal to $\text{span}\{w_1, w_2\}$. Since one degree of freedom can be regarded as a global phase which changes nothing in the shape of $H(t)$, we effectively have only five DOF. To slightly simplify the presentation we set $\alpha = 0$ in what follows and place $w_3$ in the second column of $W$. We can organize these DOF by first defining a basic alternative $\tilde{W} = \{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4\}$ with

\[
\begin{align*}
\tilde{w}_1 &\equiv \frac{1}{\sqrt{2}} F_2(I_2) = \frac{1}{\sqrt{2}} (1, 0, 0, 1) \\
\tilde{w}_2 &\equiv -\frac{1}{\sqrt{2}} F_2(R_\pi) = \frac{1}{\sqrt{2}} (0, 1, -1, 0) \\
\tilde{w}_3 &\equiv -i F_2(R_\pi) = \frac{1}{\sqrt{2}} (0, 1, 1, 0) \\
\tilde{w}_4 &\equiv -i F_2(R_\pi) = \frac{1}{\sqrt{2}} (1, 0, 0, -1).
\end{align*}
\]

Note that $\tilde{W}$ is just a permutation of the columns of $W$ defined in (9). Next, we define the general form of the conjugating matrix $W \equiv \{w_1, w_3, w_2, w_4\}$, using four variables, $\phi_2, \phi_3, \phi_4, \theta \in [0, 2\pi]$ to be

\[
\begin{align*}
w_1 &\equiv \tilde{w}_1 \\
w_3 &\equiv e^{i\phi_3} \tilde{w}_3 \\
w_2 &\equiv e^{i\phi_3}(\cos(\theta) \tilde{w}_3 + \sin(\theta) \tilde{w}_4) \\
w_4 &\equiv e^{i\phi_3}(\cos(\theta) \tilde{w}_3 - \sin(\theta) \tilde{w}_4)
\end{align*}
\]

to finally get a general form for $W$

\[
W \equiv \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 & e^{i\phi_3} \sin(\theta) & e^{i\phi_3} \cos(\theta) \\
0 & e^{i\phi_2} & e^{i\phi_2} \cos(\theta) & -e^{i\phi_2} \sin(\theta) \\
0 & e^{-i\phi_2} & e^{-i\phi_2} \cos(\theta) & -e^{-i\phi_2} \sin(\theta) \\
1 & 0 & -e^{i\phi_3} \sin(\theta) & -e^{i\phi_3} \cos(\theta)
\end{bmatrix}.
\]

We proceed to present the form of $H^{CRC}$, the form of the Hamiltonian resulting from a general two state system Hamiltonian and a choice of $W$. The general form of $H^{CRC}$ shows which four-state Hamiltonians can be translated by the retrograde canon method to a two-level system. We parameterize the two-state Hamiltonian by writing $H(t) = \frac{1}{2}\tilde{H}(t) \cdot \tilde{\sigma}$ where $\tilde{H}(t) \equiv (h_x(t), h_y(t), h_z(t)) \in \mathbb{R}^3$. Then, in order to simplify the form of $H^{CRC}(t)$ we introduce the vectors $\tilde{A}^\theta(t) \equiv (a^x(t), b^x(t), c^x(t))$ and $\tilde{A}^\phi(t) \equiv (a^y(t), b^y(t), c^y(t))$ which are defined from $\tilde{A}^\phi(t)$ through a rotation of $\theta$ around $\tilde{y}$-axis, i.e., $a^y(t) = \cos(\theta)a^x(t) + \sin(\theta)c^x(t)$ and $c^y(t) = \cos(\theta)c^x(t) - \sin(\theta)a^x(t)$. We finally get the general matrix form of $H^{CRC}(t)$, presented in terms of the parameters of $H(t)$ and the DOF inherited from $W$: 

The under-diagonal entries in equation (A2) follow from Hermiticity. A permutation of the conjugating matrix columns (see equation (A1)) would shuffle the places of the ‘a, b, c’ letters and ‘+/-’ signs in equation (A2).

We note that with an appropriate choice of $\phi_2, \phi_3, \phi_4$ all six above-diagonal entries of $\mathcal{H}^{RC}(t)$ in equation (A2) can be set imaginary, yet no more than four can be real. There are three possible ways to get four real above-diagonal entries—in each, either the couple $(a_0, A_0)$ or $(b_0, B_0)$ or $(c_0, C_0)$ will have imaginary coefficients. Figure 2 presents an example of such couplings. Suppose we wonder whether $\mathcal{H}(t)$, a Hamiltonian of the form in equation (A2) performs CPI at time $\tau$. Such questions can be reduced to questions regarding a two-state Hamiltonian $H(t)$ by inverting equation (18)—while taking suitable phase parameters $\phi_i (i = 2, 3, 4)$ and choosing $\theta \in [0, 2\pi]$ at will—followed by applying equation (6). The resulting two-state Hamiltonian $H(t)$ would facilitate CPI from $|\uparrow\rangle$ to $|\downarrow\rangle$ at time $2\tau$ if and only if the four-state Hamiltonian $\mathcal{H}(t)$ would evolve the state $e_1$ to $\pm e_2$ at time $\tau$.

Appendix B. The general translation claim—discussion and proof

The general version of the retrograde canon method allows translating two-level schemes to a wide family of $SU(2) \times SU(2)$ controlled $n^2$-level systems. The definitions of the general method and its corresponding general translation claim appear in equations (31)–(34) in the main text. The first part of this appendix is concerned with explaining the rational behind the generalization, while the second part contains a proof of the general translation claim.

B.1. A discussion of the generalizations

We recall that the method’s general version is based on relinquishing three assumptions that underlie the fundamental version—assumptions which concern: (a) the pace of movement along the path of the original propagator; (b) the dimension of representation of the original Hamiltonian; and (c) the final state of the original system. We'll explain the rational of dropping these three assumptions.

We begin with (a), the pace of movement along the path of the original propagator. The definition of $H^{RC}$, given in equation (30), is such that $U(t)$ and $U^{RC}(t)$ move at the same constant pace (albeit in opposite directions). This, however, is not a necessary condition for the method to work. The propagator, $U^R$ can move at practically any pace, $\dot{t}(t)$, so long as there’s a moment $\tau$ for which $U(T - \tau) = U(\tau)$—which always happens for a time $\tau \in [0, T]$ such that $T - \tau = \tau$. Thus, loosening the pace assumption allows creating a wide family of significantly different multi-state Hamiltonians, even if the pace remains constant (i.e. $\dot{t}(t) = at$ for some $a > 0$). Changing the pace can be used to move $U(\tau)$—the meeting point of $U(t)$ and $U(T - t)$—which defines the operator $\mathcal{H}^{RC}(\tau)$, to any point along the path of $U(t)$.

Next, consider assumption (b), concerning the dimension of representation of the original Hamiltonian: we shall see in the proof of the general translation claim below that in the definition of the retrograde canon Hamiltonian we need not restrict ourselves to using the fundamental representation of the original Hamiltonian. Correspondingly, the output system does not have to be a four-level system. Rather, it can be a $n^2$-state system for every $n \in \mathbb{N}$. The change of dimension of the retrograde canon Hamiltonian has to be accompanied by a non-trivial revision of the definition of the conjugating matrix $W$. One natural way of revising $W$—which is appropriate only for even $n$—is defining the $n^2$-state conjugating matrix as any $n^2$-dimensional unitary matrix $W_0 = \{ w_1, w_2, \ldots, w_{n^2} \}$ such that

\[ w_1 \propto E_0(I_n); \quad w_2 \propto E_0(R^{(n)}_f(\pi)), \]

where $E_0$ is the flattening function presented in (32). Note, that while $I_n$ has $1$s on the diagonal, and $0$s everywhere else, the matrix $R^{(n)}_f(\pi)$ has $1$s and $(-1)$s alternately on the anti-diagonal, and $0$s everywhere else. Therefore, since for even $n$ the anti-diagonal and the diagonal of a $n$-dimensional matrix have no common entry, $w_1$ and $w_2$ will indeed be orthogonal for every even $n$. On the other hand, for odd $n$, the anti-diagonal and the diagonal of a $n$-dimensional matrix do have a common entry, and therefore $F_0(I_n)$ and $E_0(R^{(n)}_f(\pi))$ will not be orthogonal and cannot be columns of the same unitary matrix.

The generalization of the third assumption (c), regarding the final state of the original system $U(T)$, is designed to solve the above mentioned problem of odd $n$ representations. In the process, it opens up the method to a wider range of conjugating matrices and two-state schemes, thus enabling the production of a wider variety
of \( n^2 \)-state CPI schemes—useful also for even \( n > 2 \). The generalization with regards to the final state of the original system, \( U(T) \), comes from the insight that the method and the translation claim essentially rely on just three basic conditions (assuming for the moment that \( \alpha = 0 \)). To present these conditions we mark the first two columns of the required \( n^2 \)-state conjugating matrix as \( w_1 \equiv F_n(m_1) \) and \( w_2 \equiv F_n(m_2) \).

The first condition is that

\[ U_n(T)^{-1} m_1 = m_2. \]  

(B1)

The second condition is that

\[ m_2 = R^{(w_1)}_n(\pi). \]  

(B2)

The third condition is that

\[ F_n(m_1) \perp F_n(m_2). \]  

(B3)

The last condition simply ensures that \( w_1 \) and \( w_2 \) are orthogonal and can therefore be two columns in the same unitary matrix. The role of the first two conditions shall be clarified in the proof below. If these conditions are satisfied we can define as the \( n^2 \)-state conjugating matrix any unitary matrix whose first two columns are \( F_n(m_1) \) and \( F_n(m_2) \), and formulate a general translation claim for two-state systems whose final states satisfy (B1), i.e., \( U_n(T) = m_2 m_1^{-1} \).

Interestingly, assuming equations (B1) and (B2), all that is needed to satisfy equation (B3) is that \( U(T) \), the original propagator at time \( T \), should satisfy

\[ \chi_n(U(T)) = 0, \]  

(B4)

where \( \chi_n : SU(2) \rightarrow \mathbb{C} \) is the character of the \( n \)-dimensional irreducible representation of \( SU(2) \), i.e., the function which for every element of \( SU(2) \) returns the trace of its image in the \( n \)-dimensional irreducible representation. Hence, (B4) is equivalent to

\[ \text{trace}(U_n(T)) = 0. \]  

(B5)

We shall now prove that indeed, if equations (B1) and (B2) are satisfied then equation (B3) follows from (B4). We mark \( Y_n = R^{(w_1)}_n(\pi) \). Now, from (B1) and (B2) we get \( m_2 \equiv \pm Y_n \) and \( m_1 \equiv U_n(T) Y_n \). Therefore,

\[ \langle F_n(m_1), F_n(m_2) \rangle = \langle F_n(U_n(T) Y_n), F_n(Y_n) \rangle. \]  

(B6)

Note that for every \( A, B \in M_n(\mathbb{C}) \) the following identity holds

\[ \langle F_n(A), F_n(B) \rangle = \langle F_n(A Y_n), F_n(B Y_n) \rangle. \]  

(B7)

Equation (B7) can be understood as following from the form of \( Y_n \), whose only non-zero entries are 1s and \((-1)\)s on the anti-diagonal. Therefore, multiplication by \( Y_n \) from the right simply permutes the columns of the multiplied matrix while providing factors of \( \pm 1 \). Hence, applying \( Y_n \) to the matrices on both sides of the inner product only changes the order of summation and not the result. Using (B7), and noting the fact that \( Y_n Y_n = -I_n \), we see that under assumptions (B1) and (B2), equation (B3) is indeed equivalent to (B4), since

\[ 0 = \langle F_n(m_1), F_n(m_2) \rangle = \langle F_n(U_n(T) Y_n), F_n(Y_n) \rangle \]
\[ = \langle F_n(U_n(T)), F_n(Y_n) \rangle = \text{trace}(U_n(T)) = \chi_n(U(T)). \]  

(B8)

We shall now present and prove a criterion for a matrix \( U(T) \in SU(2) \) to satisfy equation (B4): for every unit vector \( \hat{r} \in \mathbb{R}^2 \) the following equivalence holds

\[ \chi_n(R_{\hat{r}}(\omega)) = 0 \Leftrightarrow \omega = 2k\pi/n. \]  

(B9)

\( k \in \mathbb{Z}, k \neq 0 \). To prove (B9) we note that for every unit vector \( \hat{r} \in \mathbb{R}^3 \) there exists, \( \hat{r}_n : SU(2) \rightarrow M_n[\mathbb{C}] \), an irreducible \( n \)-dimensional representation of \( SU(2) \), for which \( \hat{R}_n^{(\hat{r})}(\omega) \equiv \hat{r}_n(R_{\hat{r}}(\omega)) \) is diagonal. In such a representation we have

\[ \hat{R}_n^{(\hat{r})}(\omega) = \begin{pmatrix} e^{i\omega} & e^{i(j-1)\omega} & \cdots & e^{-i\omega} \\ e^{i(j-2)\omega} & e^{i(j-3)\omega} & \cdots & e^{-2i\omega} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i(1-j)\omega} & e^{i(2-j)\omega} & \cdots & e^{-\omega} \end{pmatrix}, \]  

(B10)

where \( j = (n - 1)/2 \). Therefore

\[ \text{trace}(\hat{R}_n^{(\hat{r})}(\omega)) = \sum_{m=1}^{n-1} e^{im\omega/2} = \left\{ \begin{array}{ll} \sin(n\omega/2) & \omega \neq 0 \\ n & \omega = 0 \end{array} \right. \]  

(B11)

from which (B9) follows. To summarize the discussion of assumption (c), regarding the final state of the original system, we see that conditions (B1)–(B3) entail that \( U_n(T) = (R_{\hat{r}}(2k\pi/n)) \) for \( k \in \mathbb{Z}, k \neq 0 \), and ensure that the definition of the \( n^2 \)-state conjugating matrix given in equation (32) can be satisfied, since the first two
columns are orthogonal. The fact that under definition (32), the general translation claim, given in equation (34), follows, is what we shall now prove.

B.2. A proof of the general translation claim

We need some more preparation before presenting the proof. We shall use the fact that a high order representation of a propagator is a propagator of the high order Hamiltonian, i.e., that for all \( t \in [0, T] \) we have

\[
\Pi_n(U(t)) = U_n(t),
\]

where \( \Pi_n : SU(2) \to M_n(\mathbb{C}) \) is the lie group representation of SU(2) which for every \( R_\ell(\phi) \in SU(2) \) satisfies

\[
\Pi_n(R_\ell(\phi)) = R_\ell^{(n)}(\phi).
\]

Equation (B12) follows directly from the fact that \( H_n(t) \equiv \pi_n[H(t)] \) where \( \pi_n : su(2) \to M_n(\mathbb{C}) \), is the \( n \)-dimensional lie algebra linear representation of \( su(2) \), fixed by \( \pi_n(U_i^{(1)}) = f_i^{(n)} \) for \( i = 1, 2, 3 \). This can be proved by writing \( \Pi_n(t) = R_{\delta(t)}(\phi_n(t)) \) where \( \phi_n : [0, T] \to \mathbb{R} \) and \( \delta : [0, T] \to S^1 \subset \mathbb{R}^2 \), and showing that for every \( n \geq 2, \phi_n = \phi \) and \( \delta_n = \delta \). From (B12) and (B13) it follows that the left side of equation (34) means that \( U_n(T) \) satisfies

\[
U_n(T) = R_\ell^{(n)}(\omega_k).
\]

Another important fact for the proof is that the analog of equations (14) and (15) also holds for higher dimensional irreducible representations of SU(2)—i.e., for every \( u_n \in \Pi_n(SU(2)) \) the following equation holds

\[
Y_n = u_n Y_n u_n^\dagger.
\]

We shall prove (B15) using the defining property of group representations, which is that the multiplication of images of group elements under the representation equals the image of the multiplication of the group elements. Hence, together with equations (15), (B13), and the fact that there exists \( u \in SU(2) \) such that \( u_n = \Pi_n(u) \) we get

\[
Y_n = \Pi_n(Y) = \Pi_n(u Y u^\dagger) = \Pi_n(u) \Pi_n(Y) \Pi_n(u^\dagger) = u_n Y_n u_n^\dagger.
\]

From equation (B12) it follows that

\[
U_n(T - \tau(\tau)) = U_n(\tau).
\]

We will proceed to show that

\[
w_2 = \pm U_n^{RC}(\tau) w_1
\]

from which the conclusion follows. From equations (B14) and (B17) we see that

\[
U_n^{RC}(\tau) w_1 = (U_n(\tau) R_\ell^{(n)}(\omega_k)^{-1} \otimes U_n(\tau)) w_1
\]

from which, using equations (32), the generalization of (14) and (B15), we get

\[
U_n^{RC}(\tau) w_1 = E_n(U_n(\tau) R_\ell^{(n)}(\omega_k)^{-1} R_\ell^{(n)}(\omega_k) Y_n U_n(\tau)^\dagger) = \pm E_n(U_n(\tau) Y_n U_n(\tau)^\dagger) = \pm E_n(Y_n) = \pm w_2.
\]

For the \( \Leftarrow \) direction of equation (34) we shall assume that \( U(T - \tau(\tau)) = U(\tau) \) and that \( 1 = |\langle e_2, \ U_n^{CRC}(\tau) e_1 \rangle| \) to conclude that \( U(T) = R_\ell(\omega_k) \). From the assumption it follows that

\[
\hat{e}_2 = U_n^{CRC}(\tau) e_1
\]

for some \( \alpha \in [0, 2\pi] \). Or equivalently,

\[
\hat{e}_2 w_2 = U_n^{RC}(\tau) w_1 = (U_n^{RC}(\tau) \otimes U_n(\tau)) w_1,
\]

Hence, marking \( \hat{R} \equiv R_\ell^{(n)}(\omega_k) Y_n \) and \( V \equiv U_n^{RC}(\tau) \hat{R} U_n^{-1}(\tau) \) and noting that \( Y_n^{-1} = \pm Y_n \) (where the sign depends on whether \( n \) is even) we get that

\[
\hat{e}_2 Y_n = F^{-1}((U_n^{RC}(\tau) \otimes U_n(\tau)) F(\hat{R})) = U_n^{RC}(\tau) \hat{R} U_n^{-1}(\tau) = \pm V Y_n(\tau) Y_n U_n^{-1}(\tau) = \pm V Y_n,
\]

where \( \alpha \in [0, 2\pi] \). Hence

\[
\hat{e}_2 Y_n = \pm V Y_n.
\]
Hence $V = \pm e^{i\varphi}I_n$. Since $V \in \Pi_n(SU(2)) \subset SU(n)$ it follows that $1 = |V| = |\pm e^{i\varphi}I_n| = (\pm 1)^{|\varphi|} \Rightarrow \varphi = 2\pi k/n$ with $k \in \{0, 1, \ldots, n-1\}$. Hence $V = (\pm 1)^{|\varphi|} e^{i2\pi k/n}I_n$. Yet an irreducible representations of $SU(2)$ may contain only $I_n$ and possibly $-I_n$. Hence, we finally get $V = \pm I_n$ and $U_n^{R}(\tau) = U_n(\tau)Y_nR^{-1}$. From which it follows that

$$U(T) = (U_n^{R}(\tau))^{-1}U_n(\tau) = \pm \bar{R}Y_n(U_n(\tau))^{-1}U_n(\tau) = \pm \bar{R}Y_n = \pm R^{(n)}(\omega_k).$$

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