In the BCS theory of superconductivity, the conduction electrons in a metal cannot be both ferromagnetically ordered and superconducting. Superconductors expel magnetic field passing through them but strong magnetic fields kill the superconductivity (SC). Even small amounts of magnetic impurities are usually enough to eliminate SC. Much work has been done both theoretically and experimentally to understand this interplay and to search for the possibility of coexistence between these two ordered states.

In the conventional theory of SC, ferromagnetism (FM) and SC compete with each other but in principle it is possible for any metal to become a SC in its non-magnetic state at a sufficiently low temperature. Even strongly ferromagnetic irons undergoes a superconducting transition at low temperature under application of sufficient pressure to bring it to its nonmagnetic state. But interest is in simultaneous existence of both of the ordered states. A. Abrikosov studied superconductivity with magnetic impurities using the RKKY interaction in which magnetic impurities interact with conduction electrons with the magnetization considered as an external parameter independent of the superconducting gap, and showed that the normal ferromagnetic state has lower energy than the SC-FM state and hence coexistence is energetically unfavorable. Fulde and Ferrell studied superconductivity with a strong exchange field produced by ferromagnetically aligned impurities and found that if the exchange field is sufficiently strong compared to the energy gap, a new type of depaired superconducting ground state will occur. Fay and Appel predicted the possibility of a p-wave superconducting state in itinerant ferromagnets where the pairing is mediated by the exchange of longitudinal spin fluctuations. They showed that when the ferromagnetic transition is approached from either the ferromagnetic or paramagnetic side, the p-wave transition temperature goes through a maximum and then falls to zero. Even if superconductivity is interpreted as arising from magnetic mediation, it was thought that the SC state will occur in the paramagnetic phase. But magnetically mediated superconductivity was never observed. Some theories predicted the possibility of s-wave SC and FM coexistence order in the paramagnetic phase but the ferromagnetic fluctuation destroys it near the Curie temperature. Consideration of s-wave superconductivity and ferromagnetism has been carried out by Suhl and Abrikosov in which the ferromagnetism is due to localized spins. There had been no experimental observations of coexistence until it was found in the ferromagnetic metal $UGe_2$. The coexistence has also been shown to exist in $ZrZn_2$ and $URhGe_2$. Experiments on these three materials show that the same electrons are responsible for both ordered states. But still the exact symmetry of the paired state and the dominant mechanism responsible for the pairing is in question. Although most authors believe there is triplet superconductivity in these materials, the possibility of s-wave superconductivity cannot be denied.

Blagoev et al. studied a weak ferromagnetic Fermi-liquid and showed that s-wave superconductivity is possible and favored on the ferromagnetic side. With similar thought, Karchev et al. developed an itinerant ferromagnetic model in a mean field approach in which the magnetic electrons are also the one responsible for the formation of the Cooper pairs. Following this model, two of the authors of this paper studied the specific heat and compared it with the experimental data of $UGe_2$. They have shown that this model exhibits the quantitative behavior of the specific heat. The phase diagram shown by them is similar to that found in $UGe_2$ experimentally. So there is enough room to believe the existence of s-wave superconductivity in the coexistence state. Motivated with these thoughts, in this paper we are going to present the results of calculations of the density of states, nuclear relaxation rate, ultrasonic attenuation and electromagnetic absorption in the model given by ref.

We refer the reader to ref. for the details, and here we look at the mean-field Hamiltonian obtained from a model Hamiltonian by the standard mean-field procedure,
\[
H_{mf} = \sum_{\vec{p}} \epsilon_{\vec{p}}(c_{\vec{p}\uparrow}^\dagger c_{\vec{p}\uparrow} + c_{\vec{p}\downarrow}^\dagger c_{\vec{p}\downarrow}) \\
+ \frac{J M}{2} \sum_{\vec{p}} (c_{\vec{p}\uparrow}^\dagger c_{\vec{p}\uparrow} - c_{\vec{p}\downarrow}^\dagger c_{\vec{p}\downarrow}) \\
- \sum_{\vec{p}} (\Delta c_{\vec{p}\uparrow}^\dagger c_{\vec{p}\downarrow}^\dagger + H.c.) + \frac{1}{2} J M^2 + \frac{\mid\Delta\mid^2}{g}. \tag{1}
\]

The diagonalization of this Hamiltonian using a Bogoliubov transformation yields,

\[
H_{MF} = E_0 + \sum_{\vec{p}} \left( E_{p\alpha}^\alpha \alpha_{\vec{p}}^\dagger \alpha_{\vec{p}} + E_{p\beta}^\beta \beta_{\vec{p}}^\dagger \beta_{\vec{p}} \right), \tag{2}
\]

where

\[
E_0 = \sum_{\vec{p}} \epsilon_{\vec{p}}^\dagger + \frac{1}{2} J M^2 + \frac{\mid\Delta\mid^2}{g},
\]

\[
\epsilon_{\vec{p}}^\dagger = \frac{p^2}{2m^*} - \mu \mp \frac{J M}{2}.
\]

The quasiparticle energy dispersion relations are,

\[
E_{\alpha_p} = \frac{J M}{2} + \sqrt{\xi_{\vec{p}}^2 + \mid\Delta\mid^2}, \tag{3}
\]

\[
E_{\beta_p} = \frac{J M}{2} - \sqrt{\xi_{\vec{p}}^2 + \mid\Delta\mid^2}. \tag{4}
\]

The final step is to minimize the free energy to produce the mean-field equations. This results in a set of two coupled equations in \(M\) and \(\Delta\) that will be solved self-consistently below. For \(M\) we find,

\[
M = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} (1 - n_{\alpha_p}^\alpha - n_{\beta_p}^\beta), \tag{5}
\]

and for \(\Delta\),

\[
\mid\Delta\mid = \frac{\midg\mid}{2} \int \frac{d^3p}{(2\pi)^3} \frac{n_{\beta_p}^\beta - n_{\alpha_p}^\alpha}{\sqrt{\xi_{\vec{p}}^2 + \mid\Delta\mid^2}}. \tag{6}
\]

The two order parameters, \(M\) and \(\Delta\), have dependences such as \(M = M(g, J, T)\) and \(\Delta = \Delta(g, J, T)\). Moreover they are coupled with each other through the distribution functions which are also functions of \(M\) and \(\Delta\). From the numerical solutions of these two coupled equations, we will get the values of \(M\) and \(\Delta\) to be used for the calculation of physical parameters we are interested in. We want to clarify that all of the future results are derived strictly from the dispersion relations and the mean-field equations only, with no other assumptions made about the coupling strength limits or small magnetization. In what follows we have studied only the case when \(J M > 2\Delta\) by which we mean that there already exists weak ferromagnetic order in which arises superconductivity.

\[
\begin{align*}
\text{FIG. 1: Energy spectrum of excited fermions. The upper curve corresponds to alpha fermions and lower one is for beta fermions. The upper horizontal line is at } JM/2 + \Delta \text{ and the lower horizontal line is at } JM/2 - \Delta. \text{ The gap is } 2\Delta. \text{ Beta fermion excitations are also associated with the zero energy excitations called gapless fermions.}
\end{align*}
\]

II. ENERGY SPECTRUM AND DENSITY OF STATES

If we compare the quasi-particle energy spectra Eqs.(3,4) with the BCS energy spectrum \(E = \sqrt{\xi_{\vec{p}}^2 + \mid\Delta\mid^2}\), we see that the modification is due to the presence of the ferromagnetic energy. The plot of the energy spectrum as a function of \(\xi_{\vec{p}} = \frac{p^2}{2m^*} - \mu\), the energy of an excited particle above the Fermi level, is shown in Fig.1. It is very clear that the gap is not at the Fermi level, rather it is pushed up. Now the energy spectrum is symmetric around \(E = \frac{J M}{2}\), contrary to the BCS case where it was symmetric around \(E = 0\). The fermions which follow the different energy dispersions have different properties. The fermions which follow the \(E_{\alpha_p}^\alpha\) energy spectrum are BCS-like excitations, and will be called alpha fermions hereafter, and have a gap in the energy. Those fermions which follow the \(E_{\beta_p}^\beta\) energy spectrum do not have a gap and will be called beta fermions or the gapless excitations hereafter. The presence of these gapless fermions will change the physical properties of the system. The maximum of the energy of beta fermions is \(JM/2 - \Delta\) and the minimum of the energy of the alpha fermions is \(JM/2 + \Delta\) and so there will be a gap of \(2\Delta\). But at \(T = 0\), the beta fermions fill only up to \(E = 0\), so the gap of alpha fermions is \(JM/2 + \Delta\).

Next, we derived the expressions for the density of states for the corresponding fermions. We use the usual relation,

\[
N_i(E) = \frac{1}{2\pi^3} \int d\vec{p} \delta(E - E_p^i). \tag{7}
\]
where \( i \) refers to alpha and beta fermions. Using the property of the delta function and solving the equation, we get,

\[
\frac{N^\alpha(E)}{N(0)} = \frac{1}{4\pi^2} \left[ \frac{p^2_F + 2m^* (E - JM / 2)^2 - \Delta^2}{\sqrt{\frac{1}{2m^*} \sqrt{\frac{2\mu - E}{2\mu - E}}} - \Delta^2} \right].
\]

Both of the expressions converge to the density of normal fermions at the Fermi level, \( N(0) = \frac{mv_F^2}{2\pi^2} \), in the corresponding limit. And, for \( M = 0 \), the expressions converge to the density of states of the BCS model, \( N(E)/N(0) = E/\sqrt{E^2 - \Delta^2} \). Both of the expressions are the same mathematically, the only difference lies in the fact that the energy for gapless fermions ranges from \( -\infty \) to \( JM/2 - \Delta \), including zero obviously and that for alpha fermions ranges from \( JM/2 + \Delta \) to \( +\infty \). The result of the calculation of normalized density of states with respect to density of states of normal state fermions at the Fermi level is as shown in Fig.2.

As we have mentioned above, both expressions for the density of states are identical. If we plot both the expressions from \( -\infty \) to \( +\infty \), the plot will be same. So we can just use any one of the density of states expressions wherever needed, preserving the limit of energy ranges suitable for the corresponding fermions. In the above plot, the left most curve corresponds to the beta fermion densities and the right most is for the alpha fermions. The gap is pushed up (compared to the BCS gap) entirely in the positive energy side giving rise to a finite density for gapless fermions at the original Fermi level. The density of states is enhanced at \( JM/2 + \Delta \) for beta fermions and at \( JM/2 - \Delta \) for alpha fermions and has no density in the gap of \( 2\Delta \). Looking at the energy spectrum and the density of states, we can guess that the physical properties of the system will be dominated by normal metal like behavior due to the presence of gapless fermions at the Fermi energy.

In order to get a good feeling about the type of gapless fermions we are dealing with, it is better to compare this with gapless fermions discussed in the context of a superconductor with a magnetic impurity. The origin of gapless fermions in FS (our model) and a magnetic impurity superconductor is different. In magnetic impurity superconductors for zero impurity concentration, there will be full fermions condensation, full gap and diverging density of states at the edge of gap energy. But an increase in impurity concentration brings de-pairing effects in Cooper pairs. The interaction of fermions with the impurity triggers a flip of the spin of the Cooper pair giving rise to what is called a pair breaking effect. The fermions resulting from the pair breaking mechanism will start to fill in the gap. The diverging density of states starts to smear out resulting in a finite density of states for all the energy. At certain impurity density the gap will be fully populated giving rise to gapless fermions. The zero energy gap does not necessarily mean that there is no superconductivity because the order parameter will not be zero together with the gap for that impurity concentration. But the transition temperature will be reduced indicating a suppressed superconductivity. The superconductivity will be completely suppressed at a certain critical impurity density which is bigger than the density at which the gap will be zero. So the gaplessness is due to the de-pairing effect of the Cooper pair. But in our model of FS we see that the presence of gapless fermions is the inherent property of FS. Neither the gap should be zero nor should the density of states be smeared out for the gapless fermions to exist. There is always Fermi surface where gapless excitations are present no matter what the value of the gap is. Their existence is independent of each other.

Even though the origin of gapless fermions is different in these two different cases, their presence has a common effect in reducing the transition temperature. In our model of FS the presence of gapless fermions is necessary for the magnetic property of the system and hence will reduce the available fermions to pair up. This reduces the transition temperature. Mathematically we can un-
nderstand how the transition temperature is reduced due to the presence of magnetism. Let us compare the equation for the gap parameter in the BCS and FS cases. For the BCS case,
\[
\frac{1}{gN(0)} = \int_0^\beta \frac{d\epsilon}{\sqrt{\epsilon^2 + |\Delta|^2}}
\] (10)
where \(\lambda\) is the cutoff value. In the weak coupling limit, the equation can be solved to get an expression for the gap which looks like,
\[
\Delta = 2\lambda \exp\left(\frac{1}{\sqrt{\epsilon_0(\epsilon_0^2 - \beta^2)}}\right)
\] (11)
For our model of FS, referring to Eq. (14), the equation (10) above looks like,
\[
\frac{1}{gN(0)} = \int_0^{\beta \epsilon_0^2 - \Delta^2} \frac{d\epsilon}{\sqrt{\epsilon^2 + |\Delta|^2}}
\] (12)
and in the weak coupling limit, the gap equation reads as,
\[
\Delta = 2\lambda \exp\left(-\frac{1}{\sqrt{\epsilon_0(\epsilon_0^2 + \Delta^2)}}\right)
\] (13)
Analyzing these expressions we clearly see that the presence of the magnetization reduces the volume of phase space that is available for the Cooper pair. This leads to a decrease in the gap at \(T = 0\) and hence \(T_c\) will also be reduced compared to the BCS value for given values of other relevant parameters. So the increase in magnetization pushes the gap away from the Fermi surface and reduces its magnitude also.

Next we study some interesting properties of the coexistent system such as nuclear relaxation, ultrasonic attenuation and electromagnetic absorption following the calculation done by many authors for BCS model.\[15,17,18,19\].

### III. TRANSITION PROBABILITIES

To study transition probabilities we need to find out the expressions for the coherence factors appropriate for different cases. For that we use the coefficients of the Bogoliubov transformations, namely \(u\) and \(v\),
\[
|v_k|^2 = 1 - |u_k|^2 = \frac{1}{2}(1 - \frac{\xi}{\sqrt{\xi^2 + |\Delta|^2}})
\] (14)
Then the expressions for the coherence factors for the beta fermions and alpha fermions are
\[
F^{\beta,\alpha} = (uv' \mp uv')^2 = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{\sqrt{\xi^2 + |\Delta|^2}}\sqrt{\xi^2 + |\Delta|^2})
\] (15)
for the scattering process, and
\[
F^{\beta,\alpha} = (vu' \pm vu')^2 = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{\sqrt{\xi^2 + |\Delta|^2}}\sqrt{\xi^2 + |\Delta|^2})
\] (16)
for the creation or annihilation process. If we express in terms of quasi-particles energy explicitly, these equations reduce to
\[
F^{\beta\beta}(\Delta, E, E') = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{(\frac{J-M}{2} - E)(\frac{J-M}{2} - E')})
\] (17)
\[
F^{\alpha\alpha}(\Delta, E, E') = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{(E - \frac{JM}{2})(E' - \frac{JM}{2})})
\] (18)
\[
F^{\beta\alpha}(\Delta, E, E') = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{(\frac{JM}{2} - E)(E' - \frac{JM}{2})})
\] (19)
for the scattering process, and
\[
F^{\beta\beta}(\Delta, E, E') = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{(\frac{JM}{2} - E)(\frac{JM}{2} - E')})
\] (20)
\[
F^{\alpha\alpha}(\Delta, E, E') = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{(E - \frac{JM}{2})(E' - \frac{JM}{2})})
\] (21)
\[
F^{\beta\alpha}(\Delta, E, E') = \frac{1}{2}(1 \mp \frac{|\Delta|^2}{(\frac{JM}{2} - E)(E' - \frac{JM}{2})})
\] (22)
for the pair creation or annihilation process. It is clear now that \(F\) is like a matrix with diagonal components for intra band transitions and off diagonal components for inter band transitions. The scattering process which prefers the upper sign is called case I and the process which prefers the lower sign is called case II. In the low frequency limit these corresponds to ultrasonic attenuation and nuclear relaxation respectively.

The effect of the coherence factors will be discussed with respect to the properties studied here. We are now ready to have an expression for the net transition rate \(\frac{1}{T_1}\), for the coexistent state, between energy levels \(E\) and \(E' = E + \hbar\omega\), and is expressed below as,
\[
\frac{1}{T_1} = |M|^2 \int F(\Delta, E, E')N_{\alpha}(E)N_{\alpha}(E')(f(E) - f(E'))dE
\] (23)
where \(M\) is the magnitude of a one-electron matrix element. Since we are interested in the ratio to the normal state scattering rate, we do not need to know more about the actual value of \(M\). The limit of integration is from
\(-\infty\) to \(+\infty\) with an obvious understanding that the limits of the integral have a gap of \(2\Delta\), from \(JM/2 - \Delta\) to \(JM/2 + \Delta\), if we are working with finite temperature but the maximum limit of the integral will be only zero if we are working at zero temperature since \(f(E)\) and \(f(E')\) both will be zero for \(E > 0\). The coherence factors will be chosen appropriate to the band involved in the transition. \(f(E)\) is the usual probability distribution function. We will study the ratio of this transition rate with respect to that of the normal state fermions at the Fermi level (\(\frac{\Delta}{\pi} = |M|^2 N^2(0) h\omega\)).

In the limit of \(h\omega \to 0\) there will be transitions due only to scattering and we do not need to consider inter-band transitions. This applies to nuclear relaxation and ultrasonic attenuation but for electromagnetic absorption we can vary the frequency to higher values so that the inter-band transition will also contribute to the scattering.

**IV. ULTRASONIC ATTENUATION**

The relevant matrix elements for treating the attenuation of longitudinal sound waves have case I scattering coherence factors. So in the limit of \(h\omega \to 0\) the ratio of the transition rate \((T_n/T_1)\) can be expressed as,

\[
\frac{T_n}{T_1} = \int_{-\infty}^{JM/2-\Delta} (1 - \frac{\Delta^2}{(\frac{JM}{2} - E)^2}) \left( \frac{N^\beta(E)}{N(0)} \right)^2 \left( \frac{\partial f(E)}{\partial E} \right) dE \\
+ \int_{JM/2+\Delta}^{\infty} (1 - \frac{\Delta^2}{(E - \frac{JM}{2})^2}) \left( \frac{N^\alpha(E)}{N(0)} \right)^2 \left( \frac{\partial f(E)}{\partial E} \right) dE.
\]

(24)

We did the numerical calculations of this expression. The variation of this transition rate ratio as a function of temperature has been presented in Fig.3.

There is a significant difference in this graph as compared to the result based on BCS theory. At \(T \approx 0\), there is a finite contribution in the scattering rate whereas in the BCS case it is zero. The alpha fermion does not contribute around this temperature because there will be no fermions available to contribute for scattering since all fermions will be frozen in the low energy region. Mathematically it is due to the delta function behavior of the differential of the distribution function at energy equal to zero so that for integration all the higher energy states will be irrelevant. So the finite contribution is from the gapless fermions only. For the gapless fermion, at \(E \approx 0\), even though the density of states is comparable to, but slightly greater than the density of states of normal fermions at fermi level, the scattering rate ratio is close to but less than one. This effect is due to the coherence factor which will be less than one and dominates the effect of the density of states ratio.

As the temperature is increased the scattering rate is decreased. It is definitely due to the effect of the coherence factors. The logic is the following: With the increase in temperature, higher energy states which are close to the gap will also be relevant and we know there the effect of the coherence factors will be bigger. Even if the ratio of the density of states is very high in this energy range, the product of the density ratio and differential of the distribution function will not be enough to overcome the decreasing effect of the coherence factors. If the temperature is still increased, higher energy states will also be excited. The delta function then broadens out and the coherence factors increase slowly up to 1 for energy less than zero but decrease to zero for energy greater than zero. The density of states ratio becomes almost constant for energy less than zero but enhances for positive energy, and the alpha fermion starts to contribute but very weakly. For the alpha fermion the coherence factor has the most negative effect at the diverging edge of the density of states. The future of the scattering rate is determined by the competition between all of these effects. The result is that the rate ratio decreases until the temperature is close to \(0.8T_c\) and starts to increase until it is one at \(T_c\). From the numerical calculation we see that the alpha fermion contribution is responsible for the increase in the scattering rate. The curve shows similar behavior as in BCS at around \(T_c\) but as we lowered the temperature it does not go to zero as in the BCS case since in the BCS case no fermion would be available for scattering at low temperature. Here, however, the beta fermion contribution is comparable to normal metal scattering. So if we reduced the temperature from \(T_c\) we see that the initial reduction on the scattering rate is due to the alpha fermions which are again BCS-like and later, the increase is due to the beta fermions which are gapless.

![Graph showing temperature dependence of low-frequency absorption process obeying case I scattering coherence factor.](image)
tor has maximum positive effect at which can contribute to scattering. The coherence fac-

ture. Only the beta fermions contribute at low temper-

most equal but greater than one at around zero temper-

FIG. 4: Temperature dependence of the low-frequency ab-
sorption process obeying case II scattering coherence factors. The increasing peaks correspond to increase in the magnitude of the gap. The dotted curve corresponds to the BCS case.

V. NUCLEAR RELAXATION

The matrix elements for nuclear-spin relaxation by interaction with quasi-particles have the case II coherence factors, which corresponds to the constructive interference in the relevant low-energy scattering process. The scattering coherence factor will have the lower sign. In the limit of \( \hbar \omega \to 0 \) the ratio of transition rate \( (T_n/T_1) \) can be expressed as,

\[
\frac{T_n}{T_1} = \int_{-\infty}^{\frac{J_M}{2} - \Delta} (1 + \frac{\Delta^2}{(\frac{J_M}{2} - E)^2}) \left( \frac{N^{\beta}(E)}{N(0)} \right)^2 \left( \frac{-\partial f(E)}{\partial E} \right) dE + \int_{\frac{J_M}{2} + \Delta}^{\infty} (1 + \frac{\Delta^2}{(E - \frac{J_M}{2})^2}) \left( \frac{N^{\alpha}(E)}{N(0)} \right)^2 \left( \frac{-\partial f(E)}{\partial E} \right) dE. \tag{25}
\]

The variation of this transition rate ratio as a function of temperature has been presented in fig.4.

Reasoning similarly as above, the scattering rate is almost equal but greater than one at around zero temperature. Only the beta fermions contribute at low temperature since there will be no fermions in the upper band which can contribute to scattering. The coherence factor has maximum positive effect at \( \frac{J_M}{2} - \Delta \) for the beta fermion and \( \frac{J_M}{2} + \Delta \) for the alpha fermion. The initial rise in the ratio of the scattering rate, at a temperature close to but less than the critical temperature, is due to BCS-like alpha fermions. By around 0.87\( T_c \) the rate is maximum and starts to decrease, due to the freezing out of fermions in the beta fermion band. The finite scattering ratio at lower energy is due to the gapless nature of the beta fermions. We solved the coupled equations [5, 6] for \( M \) and \( \Delta \) for different values of the interaction parameters \( J \) and \( g \). We refer the reader to ref.5 for the details. In fig.4 the scattering rate corresponds to the pairs of \( J \) and \( g \) for which \( M \) is small and remains almost constant, and \( \Delta \) increases faster. And for those values of \( M \) and \( \Delta \), we observed that the increase in relaxation rate ratio from around 1.35 to around 1.55 is due to the increase in the value of the gap. We see that the peak is reduced considerably compared to the BCS case and it is due to the presence of the finite magnetization. The increased magnetization reduces the density of \( \alpha \)-fermions, the magnitude of the coherence factors and the probability distribution function. We will show the effect of high magnetization explicitly in what follows. But here we want to draw attention to the fact that we observed the coherence peak which is a signature of an s-wave pairing mechanism.

In fig.5, we have presented the effect of magnetization on the nuclear relaxation rate. We again solved the coupled Eqs.(5, 6) for \( M \) and \( \Delta \) for different values of \( J \) and \( g \). Fig.5 corresponds to the scattering rate for pairs of \( J \) and \( g \) for which \( M \) is large and increases faster with the chosen values of \( J \) and \( g \).

From the graph we can easily infer that even for a finite magnetization we can get a finite peak in the relaxation rate. For low magnetization the peak is strong and hence we can argue that the superconductivity is dominant. More fermions are forming pairs than aligning in certain direction to give strong magnetization. But with the increase in magnetization the peak has been reduced. This can be understood by arguing that when the magnetization is increased, the fraction of fermions which are giving rise to ferromagnetism will increase with the ultimate effect of decreasing the fraction of fermions that form Cooper pairs. We can increase the temperature to
get more fermions to form more pairs, but that will still hurt the pairing since the thermal energy will overcome the binding energy of the pairs. Since the coherence peak is the inherent property of BCS pairing, once the pairing is less effective the peak will be reduced. Nevertheless, there are always a finite number of fermions which are forming the pairs. So we want to put forward the idea that it is natural to have a reduced relaxation due to the intrinsic magnetization, but there should in s-wave superconductors always survive Cooper pairs to give an enhanced relaxation rate, no matter how small it may be. In addition, for the time being we believe that the mechanism of superconductivity is s-wave and are optimistic about observing the coherence peak in a ferromagnetic superconductor, as will be seen in a later section.

VI. ELECTROMAGNETIC ABSORPTION

Unlike the case of nuclear relaxation, it is now possible to utilize large enough frequencies to allow the quasiparticles to absorb energy, but the absorption process will now be different compared to the case of BCS. Now there are fermions to absorb any finite energy of electromagnetic wave which was already revealed in the calculation of the nuclear relaxation and ultrasonic attenuation where the whole curve corresponds to low energy absorption.

The electromagnetic absorption is proportional to the real part of the complex conductivity which we can directly compute from Eq. (19), with the Fermi function being either zero or one at zero temperature. The coherence factor will be the one for the pair creation or annihilation process. We will take the corresponding component of coherence factors appropriate for the involved band. For different values of incident frequency, the limit of integration and the expression to evaluate the absorption will be different. For two cases, namely for $\bar{h}\omega$ to 0, where as the limit of integration for the second case is $-\bar{h}\omega$ to $-\bar{h}\omega + \frac{J M}{2} - \Delta$. These two integrals correspond to intra band transition in the beta fermion band. In the second case the incident radiation will encounter the effect of the gap. Still, since $\bar{h}\omega \leq \frac{J M}{2} + \Delta$, the alpha fermion does not contribute to absorption. For

$$\frac{\sigma_1}{\sigma_n} = \frac{1}{\bar{h}\omega} \int \langle 1 \pm \frac{\Delta^2}{(\frac{J M}{2} - E)(\frac{J M}{2} - E')} \rangle \frac{N^\beta(E) N^\beta(E')}{N(0)^2} dE,$$

where the limit of integration for the first case is from $-\bar{h}\omega$ to 0, where as the limit of integration for the second case is $-\bar{h}\omega$ to $-\bar{h}\omega + \frac{J M}{2} - \Delta$. These two integrals correspond to intra band transition in the beta fermion band. In the second case the incident radiation will encounter the effect of the gap. Still, since $\bar{h}\omega \leq \frac{J M}{2} + \Delta$, the alpha fermion does not contribute to absorption. For

$$\frac{\sigma_1}{\sigma_n} = \frac{1}{\bar{h}\omega} \int \langle 1 \pm \frac{\Delta^2}{(\frac{J M}{2} - E)(\frac{J M}{2} - E')} \rangle \frac{N^\beta(E) N^\beta(E')}{N(0)^2} dE,$$

where the limit of integration for the first case is from $-\bar{h}\omega$ to 0, where as the limit of integration for the second case is $-\bar{h}\omega$ to $-\bar{h}\omega + \frac{J M}{2} - \Delta$. These two integrals correspond to intra band transition in the beta fermion band. In the second case the incident radiation will encounter the effect of the gap. Still, since $\bar{h}\omega \leq \frac{J M}{2} + \Delta$, the alpha fermion does not contribute to absorption. For

FIG. 6: Frequency dependence of absorption process obeying case I coherence factor.

$$\bar{h}\omega \geq \frac{J M}{2} + \Delta,$$

$$\frac{\sigma_1}{\sigma_n} = \frac{1}{\bar{h}\omega} \int \langle 1 \pm \frac{\Delta^2}{(\frac{J M}{2} - E)(\frac{J M}{2} - E')} \rangle \frac{N^\beta(E) N^\beta(E')}{N(0)^2} dE,$$

$$\frac{\sigma_1}{\sigma_n} = \frac{1}{\bar{h}\omega} \int \langle 1 \pm \frac{\Delta^2}{(\frac{J M}{2} - E)(\frac{J M}{2} - E')} \rangle \frac{N^\beta(E) N^\beta(E')}{N(0)^2} dE,$$

where $\frac{\sigma_1}{\sigma_n}$ is the total conductivity and is the sum of conductivity due to transition from beta to beta $\frac{\sigma_{1\beta}}{\sigma_n}$ band and from beta to alpha band $\frac{\sigma_{1\alpha}}{\sigma_n}$. The limit of integration for the first integral is from $-\bar{h}\omega$ to $-\bar{h}\omega + \frac{J M}{2} - \Delta$ and for the second integral is $-\bar{h}\omega + \frac{J M}{2} - \Delta$ to 0. The upper sign is for case I and the lower sign is for case II. The second integral is the contribution of the alpha fermion which reduces to a BCS like expression if M goes to zero. The first term still gives intra-band transitions and the second term gives inter band transitions. It is very easy to see that the true gap is only $2\Delta$. We evaluated the ratio of conductivity for both the cases. The variation of this transition rate ratio as a function of temperature at fixed value of $J, g$ has been presented in Fig.(6) and Fig.(7) for case I and case II respectively.

Both the cases show almost similar behavior except for some peculiarity in each case. For both the cases, the absorption rate is never zero because of the presence of gapless fermions. It increases proportionally to the increase.
Interestingly enough, both the cases satisfy the optical properties of the beta fermion band. For energy higher than the minimum possible value of the alpha fermion energy, the effect of the gap will come into play and the absorption rate drops sharply. But this will not go to zero since still there is the presence of intra band transitions in the beta fermion bands. For energy higher than the minimum possible value of the alpha fermion energy and less than minimum value of the beta fermion energy, the absorption rate will start to increase again and reach the value of the normal fermion absorption rate in the high energy limit. To be precise, in this energy limit the ratio of the rate for case I will be higher and saturate out faster than that of case II. If we compare with the BCS case, the behavior is similar. The only difference in the BCS case is that for case I the ratio has a discontinuous jump when the energy of the electromagnetic radiation is bigger than the gap energy and has the absorption rate much bigger than that of normal fermions. For the case II the ratio starts to increase from zero with a faster rate in the BCS case, but here it starts at a finite value and rises more slowly. Here, both the inter band and intra band transitions will contribute for absorption. Another important point, unlike the BCS case, there will be no delta function-like behavior of the conductivity corresponding to zero frequency (DC conductivity). In the BCS case the lower band will be at an energy which is less than zero, hence occupied, and has infinite density of states which contribute an infinite conductivity but here at zero temperature, the system will never have infinite density since an enhanced density corresponds to positive energy and will be unoccupied at zero temperature.

VII. PREDICTION AND DISCUSSION FOR UGe$_2$

Two of the present authors have studied some physical properties of UGe$_2$ using the same model as we have used here. In the study of the specific heat of this system, they found that their calculation fits well the data observed experimentally. The interaction parameters used in that calculation are $J = 1.001$ and $g = 1.1$. We did the calculation of the relaxation rate ratio with respect to the normal state contribution with the same interaction parameter. The graph has been presented in Fig. 8. It is clear from the figure that the enhancement in the relaxation rate ratio is very small, only around 7% over the normal metal value. We think it might be hard to see this small enhancement experimentally.

To give a quantitative feeling about the predicted values and the experimental values of some properties we have presented a table which shows the maximum percentage variation of these physical parameters with respect to the normal metal values for $J = 1.001$ and $g = 1.1$.

| property      | predicted | experimental |
|---------------|-----------|--------------|
| specific heat | $(\Delta C_V/C_V) = 20\%$ | $(\Delta C_V/C_V) = 20\%$ |
| nuclear relax. | $T_n/T_1 = 7\%$ | $T_n/T_1 \leq 15\%$ |
| ultras.atten.  | $T_n/T_1 = 1.5\%$ | $T_n/T_1 = ?$ |
| electrom. abs. | $\sigma_1/\sigma_n = 4\%$ | $\sigma_1/\sigma_n = ?$ |
It is clear from the data presented in the table that the specific heat jump is greatly reduced compared to the BCS case with a maximum enhancement of only 20% over the normal value. We assume that the maximum enhancement in the coherence peak height is less than 15%, explained below, which is very small compared to the more than 100% rise over the normal value in the BCS case. The theoretical study of other properties like ultrasonic attenuation and electromagnetic absorption also show very feeble variation from the normal state value.

Experimental data for $1/T_1$ as a function of $T$ at pressures around $13Kbar$ is available in Ref.\textsuperscript{20}. To compare our prediction with those data, we evaluate our expression and present the result in Fig.9.

We can see that the coherence peak is very small using the interaction parameters that have been used for all of the calculations, including the specific heat. In the data of Ref.\textsuperscript{20}, the peak is unseen. However, the plot there is on a log-log scale, which could reduce its appearance based on a reproduction of that data that we have done. The point is that the evidence is not clear that there is not a coherence peak in the experiments that have been carried out so far, and there needs to be a more detailed exploration with smaller temperature steps around the superconducting transition. We believe there is a very small, yet noticeable upturn in the relaxation rate.

Furthermore, in the experiments by the same group on the s-wave superconductor $MgB_2$ for $1/T_1$,\textsuperscript{22} the plot looks incredibly similar to the one for $UGe_2$. The peak is slightly more noticeable for this well-studied material, but there is no background ferromagnetism present to suppress it as much as in the case of $UGe_2$.

Another interesting point of our model with respect to experiment can be seen in a paper soon to be published\textsuperscript{23} where $1/T_1T$ for $UGe_2$ is studied well below the superconducting transition temperature. It is seen that this value holds very constant, independent of pressure and temperature. One possible reason surmised for this behavior is the likely presence of low lying gapless excitations, which is an inherent property of the coexistent phase of our model.

**VIII. CONCLUSION**

We closely studied the density of states, ultrasonic attenuation, nuclear relaxation and electromagnetic absorption for a ferromagnetic superconductor using a mean field theoretical approach where superconductivity is due to s-wave pairing and the magnetism is due to spontaneously broken spin rotation symmetry. Due to the finite density of states of gapless fermions, the low temperature behavior of the properties we studied is dominated by normal metal like behavior. At temperature less than the superconducting transition temperature, our study supports the presence of superconductivity of an s-wave nature. Namely, there is the presence of a Hebel-Slichter peak. The peak is reduced due to the presence of a ferromagnetic background. There is a slight decrease in the ultrasonic attenuation rate ratio when the temperature is less than the transition temperature. The electromagnetic absorption shows not much difference in the conductivity for either case. Nonetheless, it shows some interesting behavior compared to the result in the BCS case. The optical sum rule is followed by both cases in a more general way. We are partially successful in describing the experimental observation of the temperature dependence of the relaxation rate in superconducting $UGe_2$.

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