ON THE GAME INTERPRETATION OF A SHADOW PRICE PROCESS IN UTILITY MAXIMIZATION PROBLEMS UNDER TRANSACTION COSTS

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ABSTRACT. To any utility maximization problem under transaction costs one can assign a frictionless model with a price process $S^*$, lying in the bid/ask price interval $[S, S]$. Such process $S^*$ is called a shadow price if it provides the same optimal utility value as in the original model with bid-ask spread.

We call $S^*$ a generalized shadow price if the above property is true for the relaxed utility function in the frictionless model. This relaxation is defined as the lower semicontinuous envelope of the original utility, considered as a function on the set $[S, S]$, equipped with some natural weak topology. We prove the existence of a generalized shadow price under rather weak assumptions and mark its relation to a saddle point of the trader/market zero-sum game, determined by the relaxed utility function. The relation of the notion of a shadow price to its generalization is illustrated by several examples. Also, we briefly discuss the interpretation of shadow prices via Lagrange duality.

1. INTRODUCTION

A possible approach to the analysis of optimization problems under transaction costs consists in their reduction to the correspondent problems in frictionless models. The main point of this approach is to determine a frictionless price process $S^*$, called a shadow price, lying in the bid/ask price interval $[S, S]$ and ensuring the same optimal utility value. This method was successfully applied to some continuous time portfolio optimization problems in the recent papers [17], [13], [14], [12]. Previously in the same context a shadow price process with such interpretation explicitly appeared in [20].

In discrete time setting for the case of finite probability space the existence of a shadow price in an investment/consumption optimization problem was established in [18]. Inspired by this result, we consider an optimal investment problem in discrete time model over general probability space. It should be mentioned that very recently in the paper [3] the existence of a shadow price process was established in general multi-currency continuous time market models under short selling constraints.

The main feature of the present paper is the game interpretation of a shadow price process. As it was mentioned in the cited papers, a shadow price can be interpreted as a least favourable frictionless price from trader’s point of
view. So, it is natural to consider a trader/market zero-sum game determined by trader’s utility $\Psi(S, \gamma)$, regarded as a function of frictionless price process $S$ and an investment strategy $\gamma$. Moreover, one can expect that a pair $(\gamma^*, S^*)$, composed of an optimal strategy $\gamma^*$ and a shadow price $S^*$, corresponds to a saddle point of $\Psi$.

However, an application of customary minimax theorems (see [28]) is not straightforward. Firstly, usually $\Psi$ is not convex or quasiconvex in $S$. Secondly, in general it is not lower semicontinuous in a topology, ensuring the compactness of the set $[S, \bar{S}]$. To overcome at least the second difficulty, for each $\gamma$ we pass to the lower semicontinuous envelope $\hat{\Psi}$ of $\Psi$ in some natural weak topology on $[S, \bar{S}]$, and introduce the corresponding notion of a generalized shadow price process $S^*$.

The method, involving a consideration of the lower semicontinuous envelope (relaxation) of the objective functional is extensively used in analysis of variational problems [9], [2], [7]. In the present context it appears that the relaxed problem fits nicely into the framework of the intersection theorem, proved by Ha [15] (see Theorem 2 below). Applying this result, in Section 2 we establish the existence of a generalized shadow price and the minimax property of $\hat{\Psi}$ under rather weak assumptions (Theorem 1). Moreover, if there exists an optimal solution $\gamma^*$ of the original utility maximization problem, then a pair $(\gamma^*, S^*)$, where $S^*$ is a generalized shadow price, is exactly the strategic saddle point of the game, determined by the relaxed utility function $\hat{\Psi}(S, \gamma)$ (Theorem 3).

Thus, the advantage of passing to the relaxed problem is twofold: (1) the existence of a generalized shadow price process $S^*$ is guaranteed under weak assumptions, (2) the relaxed utility $\hat{\Psi}$ has nice minimax and saddle-point properties.

The relation of the notion of a shadow price to its generalization is illustrated by several examples in Section 3. If the original utility function $\Psi(S, \gamma)$ is already lower semicontinuous in $S$ in an appropriate topology, the proposed approach gives the existence of a shadow price (Examples 1 and 2). Another interesting case appears when $\hat{\Psi} \neq \Psi$ but it is still possible to give a convenient analytical description of the saddle points $(S^*, \gamma^*)$ of $\hat{\Psi}$. If $\gamma^*$ is an optimal solution of the original utility maximization problem under transaction costs, $\Psi(S^*, \gamma^*) = \hat{\Psi}(S^*, \gamma^*)$ and the optimality of $\gamma^*$ for the functions $\gamma \mapsto \Psi(S^*, \gamma)$, $\gamma \mapsto \hat{\Psi}(S^*, \gamma)$ is characterized by identical conditions, then a generalized shadow price $S^*$ is in fact a shadow price (Example 3).

Furthermore, we give an example of two-step model on a countable probability space with linear utility functional such that a generalized shadow price exists and a shadow price is not (Example 4). Independently an example of the same nature in three-step model was constructed in [3]. In spite of the nonlinearity of the objective functional, the advantage of the latter example is the use of logarithmic utility, while in Example 4 the utility contains a Banach limit. We find it interesting to test our approach on the example of [3]. It appears that the
"unsuccessful candidate" for a shadow price, mentioned in [3], is a generalized shadow price (see Example 5 of the present paper).

It is worth mentioning that usually a "shadow" or "equilibrium" resource prices are associated with an optimal solution of the Lagrange (or Fenchel) dual problem. In Section 4 we trace this connection in the problem under consideration, confining ourselves to the case of finite probability space. We show that a shadow price is equal to the relation of equilibrium prices of stock and bond. The related calculations indicate quite explicitly that the zero duality gap and the solvability of the Lagrange dual problem immediately imply the existence of a shadow price process. This point seems promising for generalizations, concerning the existence of a shadow price. However, Examples 4 and 5 show that this way is not so easy in the infinite-dimensional setting. See also the comments, concerning the papers [5], [6], in the introductory section of [3].

2. Main result

Consider a trader, who can distribute his wealth between a bond with zero interest rate (and price 1) and a risky asset (stock). As usual, he acts in random setting, described by a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with discrete-time filtration \(\mathbb{F} = (\mathcal{F}_t)_{t=-1}^T, \mathcal{F}_{-1} = \{\emptyset, \Omega\}\). The stock can be sold at the bid price \(S_t\) and purchased at the ask price \(\bar{S}_t\) at a time moment \(t\). It is assumed that \(0 < S_t \leq \bar{S}_t\) and the processes \(S, \bar{S}\) are \(\mathbb{F}\)-adapted. A trading strategy is determined by an \(\mathbb{F}\)-adapted portfolio process \((\beta_t, \gamma_t)_{t=-1}^T\), consisting of \(\beta_t\) units of bond (or cash) and \(\gamma_t\) units of stock. A trading strategy is called self-financing if any change in risky position is compensated by the cash flow:

\[
\Delta \beta_t = S_t (\Delta \gamma_t)^- - \bar{S}_t (\Delta \gamma_t)^+, \quad t = 0, \ldots, T,
\]

where \(\Delta a_t = a_t - a_{t-1}\), \(x^+ = \max\{x, 0\}\), \(x^- = \max\{-x, 0\}\). To fix the values \(\beta_{-1}, \gamma_{-1}\) we assume that the trader starts from one unit of bond: \(\beta_{-1} = 1, \gamma_{-1} = 0\). Moreover, at the terminal date \(T\) the asset holdings are converted to cash: \(\gamma_T = 0\). Hence, trader’s terminal wealth is given by

\[
X_T(\gamma) = 1 + \sum_{t=0}^T (S_t (\Delta \gamma_t)^- - \bar{S}_t (\Delta \gamma_t)^+), \quad \gamma_{-1} = 0, \gamma_T = 0. \tag{2.1}
\]

For the frictionless model \((S = S = \bar{S})\) this formula shapes to the customary form:

\[
X_T(\gamma) = 1 + (\gamma \circ S)_T := 1 + \sum_{t=1}^T \gamma_t \Delta S_t.
\]

Denote by \(L^p(\mathcal{F}_t)\) the set of equivalence classes of \(\mathbb{P}\)-a.s. equal \(\mathcal{F}_t\)-measurable real-valued random variables. The sets \(L^p(\mathcal{F}_t), 1 \leq p < \infty\) and \(L^\infty(\mathcal{F}_t)\) consist of \(p\)-th power \(\mathbb{P}\)-integrable and \(\mathbb{P}\)-essentially bounded elements of \(L^0(\mathcal{F}_t)\) respectively. We equip \(L^0\) with the topology of convergence in probability, induced by the metric

\[
p(f, g) = \mathbb{E}\left[\frac{|f - g|}{1 + |f - g|}\right]. \tag{2.2}
\]
Unless otherwise stated, the sets $L^p$, $p \in [1, \infty)$; $L^\infty$ are considered as Banach spaces with the norms
\[ \|f\|_p = (\mathbb{E}|f|^p)^{1/p}, \quad \|f\|_\infty = \text{ess sup } |f|. \]

We consider two possible choices of spaces, containing the portfolio strategies $\gamma$: $\gamma_t \in L^s(\mathcal{F}_t)$, $s \in \{0, \infty\}$. However, in each case we equip $L^s(\mathcal{F}_t)$ with the topology $\tau_t$ of convergence in probability. Denote by $\mathfrak{F}$ the vector space $\prod_{t=0}^{T-1} L^s(\mathcal{F}_t)$ with the product topology $\tau = \prod_{t=0}^{T-1} \tau_t$ and let $\mathcal{Y}$ be a convex subset of $\mathfrak{F}$. Since the values $\gamma_{-1} = 0$, $\gamma_T = 0$ are fixed, in what follows $\gamma$ is considered as an element of $\mathfrak{F}$ (except for Section 4). We allow portfolio constraints of the form $\gamma \in \mathcal{Y}$.

Assume that $S_t, \overline{S}_t \in L^q(\mathcal{F}_t)$ for some $q \in [1, \infty]$. Put $\tau_t^w = \sigma(L^q(\mathcal{F}_t), L^p(\mathcal{F}_t))$, where $1/p + 1/q = 1$. So, $\tau_t^w$ is the weak topology of $L^q$ for $q \in [1, \infty)$ and the weak-star topology of $L^\infty$. In any case the set
\[ [S_t, \overline{S}_t] = \{ S_t \in L^q(\mathcal{F}_t) : S_t \leq S_t \leq \overline{S}_t \} \]
is $\tau_t^w$-compact. Since the closedness of $[S_t, \overline{S}_t]$ is clear, this assertion follows from the $\tau_t^w$-compactness of the unit ball for $q \in (1, \infty]$ and the uniform integrability of $[S_t, \overline{S}_t]$ for $q = 1$. Denote by $\mathfrak{E}$ the vector space $\prod_{t=0}^T L^q(\mathcal{F}_t)$ with the product topology $\tau^w = \prod_{t=0}^T \tau_t^w$ and put
\[ \mathcal{X} = [\mathfrak{S}, \mathfrak{S}] := \prod_{t=0}^T [S_t, \overline{S}_t]. \]

A functional
\[ \Phi : L^r(\mathcal{F}_T) \mapsto [-\infty, \infty], \quad r = \min\{s, q\}. \]
\[ \Phi \text{ is called \textit{monotone} if } \Phi(X) \geq \Phi(Y) \text{ whenever } X \geq Y, \ X, Y \in L^r(\mathcal{F}_T) \text{ and \textit{quasiconcave} if } \]
\[ \Phi(\alpha_1 X + \alpha_2 Y) \geq \min\{\Phi(X), \Phi(Y)\} \]
for all $X, Y \in L^r(\mathcal{F}_T)$, $\alpha_1 + \alpha_2 = 1$, $\alpha_i \geq 0$. It is easy to see that $\Phi$ is quasiconcave iff the upper level sets $\{ X \in L^r(\mathcal{F}_T) : \Phi(X) > \beta \}$ are convex for all $\beta \in \mathbb{R}$.

We admit that trader’s preferences are represented by a monotone quasiconcave and portfolio constraints, represented by $\mathcal{Y}$, is defined as follows
\[ \lambda = \sup\{ \Phi(X_T(\gamma)) : \gamma \in \mathcal{Y} \}. \quad (2.3) \]
Note that $X_T(\gamma) \in L^r(\mathcal{F}_T)$ under the above notation.

Along with (2.3) consider the optimization problem in a frictionless model, where the stock price is given by an adapted process $S \in [\mathfrak{S}, \overline{\mathfrak{S}}]$:
\[ \mu_S = \sup\{ \Phi(1 + (\gamma \circ S)_T) : \gamma \in \mathcal{Y} \}. \quad (2.4) \]
Following [18] we call an adapted process $S \in [\mathfrak{S}, \overline{\mathfrak{S}}]$ a \textit{shadow price} if $\mu_S = \lambda$. 
We are going to introduce a modification of the last notion. Put
\[ \Psi(S, \gamma) = \Phi(1 + (\gamma \circ S)_T) \]
and denote by \( \hat{\Psi}(\cdot, \gamma) \) the \( \tau^w \)-lower semicontinuous envelope (relaxation) of \( \Psi(\cdot, \gamma) \) as a function on \([S, \overline{S}]\) (see [21], Definition 2.1.13):
\[ \hat{\Psi}(S, \gamma) = \sup_{V \in \mathcal{N}(S)} \inf_{S' \in V} \Psi(S', \gamma), \]
where \( \mathcal{N}(S) \) is a local base of the topology \( \tau^w \), restricted to \([S, \overline{S}]\). As is known (see [21], Proposition 2.1.15), \( \hat{\Psi}(\cdot, \gamma) \) is the largest \( \tau^w \)-lower semicontinuous function majorized by \( \Psi(\cdot, \gamma) \). Note that
\[ \Phi(X_T(\gamma)) \leq \hat{\Psi}(S, \gamma) \leq \Psi(S, \gamma), \quad S \in [S, \overline{S}] \]
(2.5)
since \( X_T(\gamma) \leq 1 + (\gamma \circ S)_T, \overline{S} \leq S' \leq \overline{S} \) and \( \Phi \) is monotone.

Consider instead of (2.4) the optimization problem for the relaxed functional \( \hat{\Psi} \):
\[ \mu_S = \sup \{ \hat{\Psi}(S, \gamma) : \gamma \in \mathcal{Y} \}. \]
(2.6)
We call \( S \) a generalized shadow price if \( \mu_S = \lambda \).

Looking at (2.5), we immediately conclude that any shadow price \( S^* \) is a generalized shadow price:
\[ \lambda = \sup_{\gamma \in \mathcal{Y}} \Phi(X_T(\gamma)) \leq \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S^*, \gamma) = \mu_{S^*} \leq \sup_{\gamma \in \mathcal{Y}} \Psi(S^*, \gamma) = \lambda. \]

If \( \Phi \) is quasiconcave then \( \Psi(S, \cdot) \), \( \hat{\Psi}(S, \cdot) \) are quasiconcave as well. Indeed, for \( \alpha_1 + \alpha_2 = 1, \alpha_i \geq 0 \) and \( \gamma^i \in \mathcal{Y} \) we have
\[ \Psi(S, \alpha_1 \gamma^1 + \alpha_2 \gamma^2) = \Phi(\alpha_1 (1 + (\gamma^1 \circ S)_T) + \alpha_2 (1 + (\gamma^1 \circ S)_T)) \geq \min_{i=1,2} \Phi(1 + (\gamma^i \circ S)_T) = \min_{i=1,2} \Psi(S, \gamma^i). \]
Let \( \hat{\Psi}(S, \gamma^i) > \beta \). Take \( V^i \in \mathcal{N}(S) \) such that \( \inf_{S' \in V^i} \Psi(S', \gamma^i) > \beta \) and put \( V = V^1 \cap V^2 \). The inequality
\[ \hat{\Psi}(S, \alpha_1 \gamma^1 + \alpha_2 \gamma^2) \geq \inf_{S' \in V} \Psi(S', \alpha_1 \gamma^1 + \alpha_2 \gamma^2) \geq \min_{i=1,2} \inf_{S' \in V} \Psi(S', \gamma^i) > \beta \]
means that the upper level sets \( \{ \gamma \in \mathcal{Y} : \hat{\Psi}(S, \gamma) > \beta \} \) are convex.

Now we state the main result of the present paper.

**Theorem 1.** Let \( \Phi \) be monotone and quasiconcave, \( S, \overline{S}, \overline{S}_t \in L^q(F_t), t = 0, \ldots, T \) for some \( q \in [1, \infty] \). Then there exists a generalized shadow price \( S^* \in [S, \overline{S}] \) and the following minimax relations hold true:
\[ \lambda = \sup_{\gamma \in \mathcal{Y}} \Phi(X_T(\gamma)) = \sup_{\gamma \in \mathcal{Y}} \inf_{S \in [S, \overline{S}]} \hat{\Psi}(S, \gamma) = \sup_{\gamma \in \mathcal{Y}} \inf_{S \in [S, \overline{S}]} \Psi(S, \gamma) \]
\[ = \inf_{S \in [S, \overline{S}]} \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S, \gamma) = \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S^*, \gamma) = \mu_{S^*}. \]
(2.7)

In fact, Theorem 1 is a direct consequence of the following intersection theorem ([15], Theorem 3). We formulate it in a slightly weaker form.
Theorem 2 (Ha, 1980). Let $\mathcal{E}$, $\mathcal{F}$ be Hausdorff topological vector spaces, $\mathcal{X} \subset \mathcal{E}$ be a convex compact set, $\mathcal{Y} \subset \mathcal{F}$ be a convex set. Let $B \subset A \subset \mathcal{X} \times \mathcal{Y}$ be subsets such that

(a) for each $y \in \mathcal{Y}$ the set \( \{x \in \mathcal{X} : (x, y) \in A\} \) is closed;
(b) for each $x \in \mathcal{X}$ the set \( \{y \in \mathcal{Y} : (x, y) \notin A\} \) is convex;
(c) $B$ is closed in $\mathcal{X} \times \mathcal{Y}$ and for each $y \in \mathcal{Y}$ the set \( \{x \in \mathcal{X} : (x, y) \in B\} \) is nonempty and convex.

Then there exists a point $x^* \in \mathcal{X}$ such that $\{x^*\} \times \mathcal{Y} \subset A$.

Proof of Theorem 2. The topological vector spaces $(\mathcal{E}, \tau^w)$, $(\mathcal{F}, \tau)$ and sets $\mathcal{X} = [\underline{\mathcal{S}}, \overline{\mathcal{S}}]$, $\mathcal{Y}$, introduced above, satisfy the conditions of Theorem 2. Put

$$A = \{(S, \gamma) \in \mathcal{X} \times \mathcal{Y} : \hat{\Psi}(S, \gamma) \leq \lambda\}.$$ 

Condition (a) of Theorem 2 is satisfied since $\hat{\Psi}(\cdot, \gamma)$ is $\tau^w$-lower semicontinuous and the validity of (b) follows from the quasiconcavity of $\hat{\Psi}(S, \cdot)$:

$$\{\gamma \in \mathcal{Y} : (S, \gamma) \notin A\} = \{\gamma \in \mathcal{Y} : \hat{\Psi}(S, \gamma) > \lambda\}.$$ 

Furthermore, consider the set-valued mapping $\hat{B}$ from $\mathcal{Y}$ to the power set of $\mathcal{X}$, defined as follows

$$\hat{B}(\gamma) = \left(\{\underline{\mathcal{S}}\}I_{\{\Delta_{\gamma} < 0\}} + \{\overline{\mathcal{S}}\}I_{\{\Delta_{\gamma} > 0\}} + [\underline{\mathcal{S}}, \overline{\mathcal{S}}]I_{\{\Delta_{\gamma} = 0\}\}\right)^T_{t=0}$$

and denote by $B$ the graph of $\hat{B}$:

$$B = \{(S, \gamma) \in \mathcal{X} \times \mathcal{Y} : S \in \hat{B}(\gamma)\}.$$ 

For $(S, \gamma) \in B$ we have

$$X_T(\gamma) = 1 + \sum_{t=0}^{T} (\underline{\mathcal{S}}I_{\{\Delta_{\gamma} < 0\}} - \overline{\mathcal{S}}I_{\{\Delta_{\gamma} > 0\}} + S_{t}\Delta_{\gamma}) = 1 - \sum_{t=0}^{T} S_{t}\Delta_{\gamma} = 1 + (\gamma \circ S)_T$$

and $\Phi(X_T(\gamma)) = \Psi(S, \gamma) \leq \lambda$. Thus, $\hat{\Psi}(S, \gamma) \leq \lambda$ and $B \subset A$.

We claim that $B$ satisfies condition (c) of Theorem 2. Clearly, the sets $\{S \in \mathcal{X} : (S, \gamma) \in B\} = \hat{B}(\gamma)$ are nonempty and convex. It remains to prove that $B$ is closed in $\mathcal{X} \times \mathcal{Y}$. Let $(S, \gamma) \in \mathcal{X} \times \mathcal{Y}$ lie in the closure of $B$ in the product topology $\tau^w \times \tau$, restricted to $\mathcal{X} \times \mathcal{Y}$. To prove that $(S, \gamma) \in B$ it is sufficient to show that

$$S_{t}I_{\{\Delta_{\gamma} \neq 0\}} = \underline{\mathcal{S}}I_{\{\Delta_{\gamma} < 0\}} + \overline{\mathcal{S}}I_{\{\Delta_{\gamma} > 0\}}, \quad t = 0, \ldots, T.$$ 

For any $t \in \{0, \ldots, T\}$, $n \in \mathbb{N}$ and $g_t \in L^\infty(\mathcal{F})$ there exist $\gamma^n \in \mathcal{Y}$ and $S^n \in [\underline{\mathcal{S}}, \overline{\mathcal{S}}]$ of the form

$$S^n = S_{t}I_{\{\Delta_{\gamma^n} < 0\}} + \overline{\mathcal{S}}I_{\{\Delta_{\gamma^n} > 0\}} + \underline{\mathcal{S}}I_{\{\Delta_{\gamma^n} = 0\},} \quad \underline{\mathcal{S}} \leq \hat{S} \leq \overline{\mathcal{S}}$$

such that $\rho(\gamma^n, \gamma_t) < 1/n$, $|\mathcal{E}(S^n_t - S_t)g_tI_{\{\Delta_{\gamma^n} \neq 0\}}| < 1/n$, where $\rho$ is defined by (2.2). Passing to subsequences (still denoted by $\gamma^n_t$, $S^n_t$), we may assume that $\gamma^n_t \to \gamma_t$ $\mathbb{P}$-a.s. Here $\gamma^n_t$, $\gamma_t$ are understood as functions, taken from the correspondent equivalence class.
On the set \( \{ \Delta \gamma_t \neq 0 \} \) we have
\[
I_{\Delta \gamma_t^o < 0} \to I_{\{ \Delta \gamma_t < 0 \}}, \quad I_{\Delta \gamma_t^o > 0} \to I_{\{ \Delta \gamma_t > 0 \}}, \quad I_{\Delta \gamma_t^o = 0} \to 0 \quad \mathbb{P}\text{-a.s.}
\]
From the dominated convergence theorem it follows that
\[
\lim_{n \to \infty} \mathbb{E} \left( g_t(S_t I_{\Delta \gamma_t^o < 0} + S_t I_{\Delta \gamma_t^o > 0}) I_{\{ \Delta \gamma_t \neq 0 \}} \right) = \mathbb{E} \left( g_t(S_t I_{\Delta \gamma_t < 0} + S_t I_{\Delta \gamma_t > 0}) \right).
\]
Hence,
\[
\mathbb{E} \left( g_t S_t I_{\{ \Delta \gamma_t \neq 0 \}} \right) = \lim_{n \to \infty} \mathbb{E} \left( g_t S_t^n I_{\{ \Delta \gamma_t \neq 0 \}} \right) = \mathbb{E} \left( g_t(S_t I_{\Delta \gamma_t < 0} + S_t I_{\Delta \gamma_t > 0}) \right)
\]
for any \( g_t \in L^\infty(F_t) \) and (2.9) is satisfied.

Now we can apply Theorem 2 and take an element \( S^* \in \mathcal{X} \) such that \( \{ S^* \} \times \mathcal{Y} \subset A \). That is, \( \hat{\Psi}(S^*, \gamma) \leq \lambda \) for all \( \gamma \in \mathcal{Y} \) and
\[
\hat{\mu}_{S^*} = \sup \{ \hat{\Psi}(S^*, \gamma) : \gamma \in \mathcal{Y} \} \leq \lambda.
\]
The reverse inequality is clear. Thus, \( S^* \) is a generalized shadow price.

The equalities in the first line in (2.7) follow from (2.5) and (2.8). Furthermore, the inequalities
\[
\sup_{\gamma \in \mathcal{Y}} \inf_{S \in \mathbb{L}[S]} \hat{\Psi}(S, \gamma) \leq \inf_{S \in \mathbb{L}[S]} \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S, \gamma) \leq \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S^*, \gamma)
\]
are evident and the equality \( \lambda = \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S^*, \gamma) \) is already proved. \( \square \)

In the context of duality theory (see e.g. [25], section 1) the problems
\[
\text{maximize } f(\gamma) = \inf_{S \in \mathbb{L}[S]} \hat{\Psi}(S, \gamma) \text{ over all } \gamma \in \mathcal{Y},
\]
\[
\text{minimize } g(S) = \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S, \gamma) \text{ over all } S \in \mathbb{L}[S].
\]
are said to be dual to each other and the common value (2.7) is called the saddle-value of \( \hat{\Psi} \). The first of these problems coincides with (2.3).

The function \( g \) is lower semicontinuous as the pointwise supremum of a family of lower semicontinuous functions and the set \( \mathbb{L}[S] \) is compact. Hence, the dual problem (2.10) is solvable. The equality
\[
g(S^*) = \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S^*, \gamma) = \inf_{S \in \mathbb{L}[S]} \sup_{\gamma \in \mathcal{Y}} \hat{\Psi}(S, \gamma) = \lambda
\]
shows that generalized shadow prices are exactly the solutions of (2.10).

**Theorem 3.** A pair \((S^*, \gamma^*) \in \mathbb{L}[S] \times \mathcal{Y}\) is a saddle point of the relaxed utility function \( \hat{\Psi} \):
\[
\hat{\Psi}(S^*, \gamma) \leq \hat{\Psi}(S^*, \gamma^*) \leq \hat{\Psi}(S, \gamma^*), \quad (S, \gamma) \in \mathbb{L}[S] \times \mathcal{Y}, \quad (S^*, \gamma^*) \in \mathbb{L}[S] \times \mathcal{Y}, \quad (2.11)
\]
if and only if \( \gamma^* \) is an optimal solution of (2.3): \( \Phi(X_T(\gamma^*)) = \lambda \) and \( S^* \) is a generalized shadow price.
Proof. Condition (2.11) can be reformulated as follows:

\[
g(S^*) = \hat{\Psi}(S^*, \gamma^*) = f(\gamma^*). \tag{2.12}
\]

Since \( g(S) \geq f(\gamma), (S, \gamma) \in [\underline{S}, \overline{S}] \times \mathcal{Y} \) it follows that if \((S^*, \gamma^*)\) is a saddle point of \(\hat{\Psi}\) then \(\gamma^*\) is an optimal solution of (2.3) and \(S^*\) is an optimal solution of (2.10) (or, equivalently, a generalized shadow price). Conversely, if \(\gamma^*\) is an optimal solution of (2.3) and \(S^*\) is a generalized shadow price then

\[\lambda = g(S^*) \geq \hat{\Psi}(S^*, \gamma^*) \geq f(\gamma^*) = \lambda.\]

Thus, (2.12) holds true and \((S^*, \gamma^*)\) is a saddle point of \(\hat{\Psi}\). \(\square\)

The above arguments show that the existence of an optimal solution of (2.3) is equivalent to the existence of a saddle point of the relaxed utility function \(\hat{\Psi}\).

3. Examples

In the first two examples given below \(\Psi(\cdot, \gamma)\) is \(\tau^w\)-lower semicontinuous on \([\underline{S}, \overline{S}]\) and, hence, there exists a shadow price. Note that in these examples the topological space \(([\underline{S}, \overline{S}], \tau^w)\) is first countable and it is enough to show that

\[\Psi(S, \gamma) \leq \liminf_{n \to \infty} \Psi(S^n, \gamma), \quad S \in [\underline{S}, \overline{S}]\]

for any sequence \(S^n \in [\underline{S}, \overline{S}]\), converging to \(S\) in \(\tau^w\), to check \(\tau^w\)-lower semicontinuity of \(\Psi(\cdot, \gamma)\).

In Example 3 the relaxation \(\hat{\Psi}\) does not coincide with \(\Psi\) but \(S^*\) is a shadow price iff it is a generalized shadow price. Examples 4 and 5 (the last one is borrowed from [3]) show that even in the case of countable probability space a shadow price need not exist, while the existence of a generalized shadow price is ensured by Theorem 1.

In all examples the utility functional \(\Phi\) is concave and \(\mathcal{Y} = \mathfrak{F}\).

Example 1. Let \(\Omega\) be finite and let \(P\) be strictly positive on the atoms of \(\mathcal{F}_T\). Consider a monotone concave function \(U : \mathbb{R} \to (-\infty, \infty)\) such that \(U\) is finite (and hence continuous) on the open half-line \((0, \infty)\) and \(U(x) = -\infty, \quad x \in (-\infty, 0]\). We look for a shadow price in the optimization problem

\[
\text{maximize } \mathbb{E}U(X_T(\gamma)) \quad \text{over all } \gamma \in \prod_{t=0}^{T-1} L^0(\mathcal{F}_t)
\]

where \(X_T(\gamma)\) is defined by (2.1). The choice of \(q\) does not affect anything; \(q = 1\) for instance.

If \(\mathcal{F}_t\) is generated by the partition \((D^t_i)_{i=1}^{m_t}\) then \(L^1(\mathcal{F}_t)\) is an \(m_t\)-dimensional space. Put \(f^i_t = f_t(\omega), \omega \in D^t_i\) for \(f_t \in L^1(\mathcal{F}_t)\). All Hausdorff vector topologies on a finite dimensional space coincide ([1], Theorem 5.21). Thus, we can assume that \(\tau^w_t\) is the topology of pointwise convergence with a local base at zero generated by the sets

\[
\{f \in L^1(\mathcal{F}_t) : |f^i_t| < 1/n\}, \quad i = 1, \ldots, m_t, \quad n \in \mathbb{N}.
\]
To show that $\Psi(\cdot, \gamma)$ is lower semicontinuous in the product topology $\tau^w = \prod_{t=0}^T \tau_t^w$ it is enough to check that (3.1) is true when $S^n_t \to S_t$ pointwise.

If $1 + (\gamma \circ S)_T(\omega) > 0$ for all $t, \omega$ then the same is true for $1 + (\gamma \circ S^n)_T(\omega)$ for sufficiently large $n$. It follows that

$$\lim_{n \to \infty} \Psi(S^n, \gamma) = \lim_{n \to \infty} \mathbb{E}U(1 + (\gamma \circ S^n)_T) = \mathbb{E}U(1 + (\gamma \circ S)_T) = \Psi(S, \gamma).$$

On the other side, if $1 + (\gamma \circ S)_T(\omega) \leq 0$ for some $t, \omega$ then $\Psi(S, \gamma) = \Phi(1 + (\gamma \circ S)_T) = -\infty$.

The lower semicontinuity of $\Psi(\cdot, \gamma)$ implies the existence of a shadow price. A related result was established in [18].

**Example 2.** Let $\Omega$ be countable and let $\mathbb{P}$ be strictly positive on the atoms of $\mathcal{F}_T$. Assume that the processes $\underline{S}, \overline{S}$ are bounded: $\underline{S}_t, \overline{S}_t \in L^\infty(\mathcal{F}_t)$, and consider the optimization problem

$$\maximize \mathbb{E}U(X_T(\gamma)) \text{ over all } \gamma \in \prod_{t=0}^{T-1} L^\infty(\mathcal{F}_t)$$

with a monotone concave (and hence continuous) function $U : \mathbb{R} \to \mathbb{R}$. We put $q = 1$.

If $\mathcal{F}_t$ is generated by a partition $(D^n_i)_{i \in J_t}$, then for $f_t \in L^1(\mathcal{F}_t)$ we put $f^n_t = f_t(\omega), \omega \in D^n_i$. Consider on $L^1(\mathcal{F}_t)$ the topology $\tau_t^p$ of pointwise convergence with a local base at zero generated by the sets

$$\{ f \in L^1(\mathcal{F}_t) : |f^n_t| < 1/n \}, \quad i \in J_t, \quad n \in \mathbb{N}.$$  

The topologies $\tau_t^w = \sigma(L^1(\mathcal{F}_t), L^\infty(\mathcal{F}_t))$, $\tau_t^p$ are different on $L^1(\mathcal{F}_t)$ if the set $J_t$ is infinite, since $\tau^w$ is first countable and $\tau^p$ is not (see [1], Theorem 6.26). Clearly, $\tau_t^p \subset \tau_t^w$. It follows that they coincide on the set $[\underline{S}_t, \overline{S}_t]$ which is $\tau_t^w$-compact and $\tau_t^p$-Hausdorff (see [27], section 3.8).

Take a sequence $S^n \in [\underline{S}, \overline{S}]$, converging to $S$ in the product topology $\tau^p = \prod_{t=0}^T \tau_t^p$. This amounts to the pointwise convergence $S^n_t \to S_t$. The correspondent sequence $U(1 + (\gamma \circ S^n)_T)$ is uniformly bounded and $\Psi(\cdot, \gamma)$ is $\tau^p$-continuous on $[\underline{S}, \overline{S}]$:

$$\lim_{n \to \infty} \Psi(S^n, \gamma) = \lim_{n \to \infty} \mathbb{E}U(1 + (\gamma \circ S^n)_T) = \mathbb{E}U(1 + (\gamma \circ S)_T) = \Psi(S, \gamma)$$

due to the dominated convergence theorem. This implies the existence of a shadow price.

A counterexample, given in [3] (see Example 5 below), indicates that the assumptions on boundedness of $\underline{S}_t, \overline{S}_t$ and finiteness of $U$ cannot be dropped simultaneously.

**Example 3.** Let $T = 1, \Omega = [0, 1], \mathcal{F}_0 = \{\emptyset, \Omega\}$, and let $\mathcal{F}_1$ be the Borel $\sigma$-algebra of $[0, 1]$ with the Lebesgue measure $\mathbb{P}(d\omega) = d\omega$. Assume that $\underline{S}_0 = \overline{S}_0 = S_0$, $\underline{S}_1, \overline{S}_1 \in L^\infty(\mathcal{F}_1)$ and

$\overline{S}_1 - \underline{S}_1 \geq \alpha > 0$
for some real number $\alpha > 0$. Consider the optimization problem
\[
\text{maximize } E U(X_1(\gamma)) \text{ over all } \gamma_0 \in \mathbb{R} \tag{3.2}
\]
with a monotone concave function $U: \mathbb{R} \mapsto \mathbb{R}$. From (2.1) we get
\[
X_1(\gamma) = 1 + \gamma_0(S_1 - S_0) - \gamma_0(S_1 - S_0)
\]
Put $s = \infty$, $q = 1$. Thus, $[S_1, \overline{S}_1]$ is considered as a set in $L^1(F_1)$ with the weak topology $\tau_{L^1}$ of $L^1(F_1)$. We look for the lower semicontinuous envelope of the functional
\[
\Psi(S, \gamma) = E U(1 + \gamma_0(S_1 - S_0))
\]
declared on the set $\{S_0\} \times [S_1, \overline{S}_1]$. This problem reduces to relaxation of the integral functional
\[
F(S_1) = \int_0^1 \left[ U(1 + \gamma_0(S_1 - S_0)) + \delta(S_1|[S_1, \overline{S}_1]) \right] d\omega, \quad S_1 \in L^1(F_1),
\]
where $\delta(x|A) = 0$, $x \in A$; $\delta(x|A) = +\infty$, $x \not\in A$ since
\[
F(S_1) = \begin{cases} \Psi(S, \gamma), & S_1 \in [S_1, \overline{S}_1] \text{ a.s.,} \\ +\infty, & \text{otherwise}. \end{cases}
\]
Furthermore, the function
\[
f(\omega, x) = U(1 + \gamma_0(x - S_0)) + \delta(x|[S_1(\omega), \overline{S}_1(\omega)])
\]
is Borel on $[0, 1] \times \mathbb{R}$ and lower semicontinuous in $x$ for each $\omega$. Hence, $f$ is a normal integrand (see [9], Chapter VIII, Definition 1.1), uniformly bounded from below. The relaxation of $F$ is given by the formula
\[
\hat{F}(S_1) = \int_0^1 \hat{f}(\omega, S_1(\omega)) d\omega,
\]
where $\hat{f}(\omega, \cdot)$ is the largest convex lower semicontinuous minorant of $f(\omega, \cdot)$ for each $\omega$: see [9], Chapter IX, Propositions 1.2 and 2.3 or [26], [16] (chapter 2, section 9) for more general results of this sort. Using the concavity of $f(\omega, \cdot)$ on $[S_1(\omega), \overline{S}_1(\omega)]$, we conclude that $\hat{f}(\omega, \cdot)$ is linear on this interval:
\[
\hat{f}(\omega, S_1) = f(\omega, \overline{S}_1) \frac{S_1 - S_}\overline{S}_1 + f(\omega, S_1) \frac{\overline{S}_1 - S_1}{S_1 - \overline{S}_1} + \delta(S_1|[S_1, \overline{S}_1]).
\]
Thus, for $S_1 \in [S_1, \overline{S}_1]$ we have
\[
\hat{\Psi}(S, \gamma) = \hat{F}(S_1) = E \left( U(1 + \gamma_0(S_1 - S_0)) \frac{S_1 - S_1}{S_1 - \overline{S}_1} \right.
\]
\[
+ \left. U(1 + \gamma_0(S_1 - S_0)) \frac{\overline{S}_1 - S_1}{S_1 - \overline{S}_1} \right).
\]
Now assume that the function $U$ is strictly increasing and differentiable and there exists an optimal solution $\gamma_0^*$ of (3.2). From Theorem 3 it follows that
$S^*$ is a generalized shadow price iff $(S^*, \gamma^*)$ is a saddle point of $\hat{\Psi}$. From the representation

$$
\hat{\Psi}(S, \gamma) = \mathbb{E}\left( \frac{U(1 + \gamma_0(S_1 - S_0)) - U(1 + \gamma_0(S_1 - S_0))}{S_1 - S_1} \right) \left( S_1 - S_1 \right) + \mathbb{E}\left( \frac{S_1U(1 + \gamma_0(S_1 - S_0)) - S_1U(1 + \gamma_0(S_1 - S_0))}{S_1 - S_1} \right).
$$

it is clear that the inequality $\hat{\Psi}(S^*, \gamma^*) \leq \hat{\Psi}(S, \gamma^*)$, $S_1 \in [\underline{S}_1, \overline{S}_1]$ is equivalent to the condition

$$
S_1^*I_{(\gamma_0^* \neq 0)} = S_1^*I_{(\gamma_0^* > 0)} + \overline{S}_1^*I_{(\gamma_0^* < 0)}.
$$

Furthermore, the inequality $\hat{\Psi}(S^*, \gamma^*) \leq \hat{\Psi}(S^*, \gamma^*)$, $\gamma_0 \in \mathbb{R}$ reduces to the condition

$$
\frac{\partial \hat{\Psi}}{\partial \gamma_0}(S^*, \gamma^*) = 0
$$
due to the concavity of $\hat{\Psi}(S^*, \gamma)$. After elementary calculations we get

$$
\frac{\partial \hat{\Psi}}{\partial \gamma_0}(S^*, \gamma^*) = \mathbb{E}\left( (S_1^* - S_0)U'(1 + \gamma_0^*(S_1^* - S_0)) \right) = 0 \text{ for } \gamma_0^* > 0,
$$

$$
\frac{\partial \hat{\Psi}}{\partial \gamma_0}(S^*, \gamma^*) = \mathbb{E}\left( (S_1^* - S_0)U'(1 + \gamma_0^*(S_1^* - S_0)) \right) = 0 \text{ for } \gamma_0^* < 0,
$$

$$
\frac{\partial \hat{\Psi}}{\partial \gamma_0}(S^*, \gamma^*) = U'(1)\mathbb{E}(S_1^* - S_0) = 0 \text{ for } \gamma_0^* = 0.
$$

Taking into account (3.3), we conclude that the last three equalities are equivalent to the following one:

$$
\frac{\partial \hat{\Psi}}{\partial \gamma_0}(S^*, \gamma^*) = \mathbb{E}\left( (S_1^* - S_0)U'(1 + \gamma_0^*(S_1^* - S_0)) \right) = 0.
$$

Thus, $(S^*, \gamma^*)$ is a saddle point of $\hat{\Psi}$ iff the relations (3.3), (3.4) hold true.

From this observation it follows that any generalized shadow price is a shadow price. Indeed, condition (3.4) ensures that $\gamma^*$ is an optimal solution in the frictionless model with the price process $S^*$:

$$
\mathbb{E}U(1 + \gamma_0(S_1^* - S_0)) \leq \mathbb{E}U(1 + \gamma_0^*(S_1^* - S_0^*)).
$$

Moreover, in view of (3.3) we have

$$
\mathbb{E}U(1 + \gamma_0^*(S_1^* - S_0^*)) = \mathbb{E}U(1 + \gamma_0^*(S_1^* - S_0^*)) = \mathbb{E}U(1 + \gamma_0^*(S_1^* - S_0^*)) = \mathbb{E}U(1 + \gamma_0^*(S_1^* - S_0^*)) = \hat{\Psi}(S^*, \gamma^*) = \lambda.
$$

Hence, although

$$
\hat{\Psi}(S, \gamma) \neq \Psi(S, \gamma) = \mathbb{E}U(1 + \gamma_0(S_1 - S_0))
$$

(if, e.g., $U$ is strictly concave), in this example a process $S^*$ is a generalized shadow price iff it is a shadow price.
Example 4. Let $\Omega = \mathbb{N}$, $T = 1$, $\mathcal{F}_0$ is generated by the atoms $D_n = \{2n - 1, 2n\}$, $n \in \mathbb{N}$ and $\mathcal{F}_1$ coincides with the power set of $\mathbb{N}$. The probability measure is defined by $P(\{n\}) = 2^{-n}$. Put $S_0 = 1$, $S_0 = 4$, $S_1 = \infty \sum_{n=1}^{\infty} (4I_{(2n)} + I_{(2n-1)})$.

Since $S_t$ is bounded we can put $s = q = r = \infty$. From the definitions of $X_1(\gamma)$ and $S_t$ it follows that

$$X_1(\gamma) = 1 + \gamma^+_0(S_1 - S_0) - \gamma^-_0(S_1 - S_0) \leq 1.$$ 

and $\gamma_0 = 0$ is an optimal trading strategy for any monotone functional $\Phi$ on $L^\infty(\mathcal{F}_1)$.

Denote by $\text{LIM} : L^\infty(\mathcal{F}_1) \mapsto \mathbb{R}$ a Banach limit (see e.g. [8], Chapter II, Exercise 22) and put

$$\Phi(X) = E(X) + \text{LIM}(X).$$

Clearly, $\Phi$ is a linear monotone functional on $L^\infty(\mathcal{F}_1)$. We have

$$\lambda = \sup \{ \Phi(X_1(\gamma)) : \gamma_0 \in L^\infty(\mathcal{F}_0) \} = 1.$$

We show that there is no shadow price in this model. Assume first that $S_0$ is a shadow price which is not equal to the conditional expectation

$$E(S_1|\mathcal{F}_0) = \sum_{n=1}^{\infty} \frac{E(S_1I_{D_n})}{P(D_n)} I_{D_n} = \sum_{n=1}^{\infty} \frac{4P(2n) + P(2n - 1)}{P(2n) + P(2n - 1)} I_{D_n} = 2.$$

If $S_0 \neq 2$ on $D_n$, then putting $\gamma_0(i) = 0$, $i \notin D_n$; $\gamma_0(i) = \delta$, $i \in D_n$ we get

$$\Phi(1 + (\gamma \circ S)_1) = E(1 + \gamma_0 \Delta S_1) = 1 + E(\gamma_0(2 - S_0)) = 1 + \delta(2 - S_0) P(D_n).$$

It follows that

$$\mu_S = \sup \{ \Phi (1 + (\gamma \circ S)_1) : \gamma_0 \in L^\infty(\mathcal{F}_0) \} = +\infty.$$

Now assume that $S_0 = E(S_1|\mathcal{F}_0) = 2$. For $\gamma_0 = 1$ we have

$$\Phi(1 + (\gamma \circ S)_1) = \text{LIM}(1 + \Delta S_1) = \frac{3}{2}.$$

For computing the value of the Banach limit in the last equality we have used its shift-invariance property:

$$2\text{LIM}(\Delta S_1) = \text{LIM} \sum_{n=1}^{\infty} (2I_{(2n)} - I_{(2n-1)}) + \text{LIM} \sum_{n=1}^{\infty} (2I_{(2n-1)} - I_{(2n)}) = \text{LIM}(1) = 1.$$

Thus, $\mu_S \geq 3/2 > \lambda = 1$ and $S_0 = 2$ is not a shadow price.

The existence of a generalized shadow price $S^*$ is guaranteed by Theorem [1].

Let us show that $S^*_0 = 2$. For $S_0 = 2$ we have

$$\Psi(S, \gamma) = \Phi(1 + (\gamma \circ S)_1) = \text{LIM}(1 + \gamma_0(S_1 - S_0)).$$
Consider a neighbourhood $U$ of $S_0$ in the topology $\tau_w = \sigma(L^\infty(\mathcal{F}_0), L^1(\mathcal{F}_0))$, restricted to $[S_n, S_0]$:

$$U = \{S'_0 \in [S_0, S_n] : |Eg_i(S'_0 - S_0)| < \varepsilon, \ i = 1, \ldots, m\}, \ g_i \in L^1(\mathcal{F}_0), \ \varepsilon > 0.$$ 

The set

$$U_n = \left\{ S'_0 \in [S_0, S_n] : S'_0 = S_0 \text{ on } \bigcup_{j=1}^n D_j \right\}$$

is contained in $U$ for sufficiently large $n$. Indeed, for $S'_0 \in U_n$ we have

$$|Eg_i(S'_0 - S_0)| \leq \sum_{j=n+1}^\infty |g^j_i||S'_0 - S_0|P(D_j),$$

where $g^j_i = g_i(\omega, \omega \in D_j$ and $S'_0, S^j_i$ are defined similarly. The right-hand side of the last inequality can be made arbitrary small by an appropriate choice of $n$.

Take $S'_0 \in U_n \subset U$ such that

$$1 + \gamma_0 \Delta S'_1 = 1 + \gamma_0^+(S_1 - S'_0) - \gamma_0^-(S_1 - S'_0) = X_1(\gamma) \text{ on } \bigcup_{j \geq n+1} D_j.$$ 

Clearly,

$$\Psi(S', \gamma) = 1 + \text{LIM } (\gamma_0 \Delta S'_1) = \text{LIM } (X_1(\gamma)).$$

It follows that $\hat{\Psi}(S, \gamma) \leq \text{LIM } (X_1(\gamma)) \leq 1$ and $S^*_0 = 2$ determines a generalized shadow price.

**Example 5.** Let us reproduce the counterexample of [3]. Put $\Omega = \mathbb{N}, T = 2, \mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{F}_1$ be generated generated by the atoms

$$D_k = \{2k + 1, 2k + 2\}, \ k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N},$$

and let $\mathcal{F}_2$ be the power set of $\mathbb{N}$.

Assume that the stock bid prices are falling deterministically: $S_0 = 3, S_1 = 2$ $S_2 = 1$ and the ask prices are defined as follows: $S_0 = 3,$

$$S_1 = 2 + k \text{ on } D_k, \ k \in \mathbb{N}_0,$$

$$S_2(\omega) = 1 \ \text{ for } \omega = 2k + 1, \ k \in \mathbb{N}_0;$$

$$S_2(\omega) = 3 + k \ \text{ for } \omega = 2k + 2, \ k \in \mathbb{N}_0.$$ 

The probability measure is defined as follows:

$$P(D_0) = 1 - 2^{-n}, \ P(D_k) = 2^{-n-k},$$

$$P(\{2k + 1\}) = (1 - 2^{-n-k})P(D_k), \ P(\{2k + 2\}) = 2^{-n-k}P(D_k), \ k \in \mathbb{N}_0,$$

where $n \in \mathbb{N}$ is fixed sufficiently large to make $E(S_2 - S_1|\mathcal{F}_1) < 0$.

The problem cosists in maximization of the logarithmic utility $\Phi(X_2(\gamma)) = \mathbb{E}\ln(X_2(\gamma))$ (we put $\ln x = -\infty$ for $x \leq 0$). Expectation is defined by the formula

$$Ef = \lim_{M \to +\infty} \mathbb{E}(f \wedge M).$$

for any measurable function $f$ with values in the extended real line $\mathbb{R} \cup \{\pm \infty\}$. Particularly, $Ef = -\infty$ if $Ef^- = +\infty$. 

The picture and clear economical argumentation, given in [3], show that it is optimal not to trade at step 0 ($\gamma_0^* = 0$) and to go short at step 1 ($\gamma_1^* < 0$). To be a bit more formal consider

$$X_2(\gamma) = 1 + \sum_{t=0}^{2} (S_t(\Delta\gamma_t)^- - S_t(\Delta\gamma_t)^+)$$

$$= 1 - S_0\gamma_0 + S_1(\gamma_1 - \gamma_0)^- - S_1(\gamma_1 - \gamma_0)^+ + S_2\gamma_2^+ - S_2\gamma_2^-, \quad (3.5)$$

where $S_0 = S_0' = S_0$. It is easy to see that $\gamma_0 < 0$ leads to a negative value of $X_2(\gamma)$ for some $\omega = 2k$. Assuming that $\gamma_0 \geq 0$, it is not optimal to posses a positive amount of stock at step 1:

$$X_2(\gamma) = 1 + (S_1 - S_0)\gamma_0 \quad \text{for } \gamma_1 = 0; \quad$$

$$X_2(\gamma) = 1 + (S_1 - S_0)\gamma_0 + (S_2 - S_1)\gamma_1 \leq 1 + (S_1 - S_0)\gamma_0 \quad \text{for } \gamma_1 \in (0, \gamma_0); \quad$$

$$X_2(\gamma) = 1 + (S_1 - S_0)\gamma_0 + (S_2 - S_1)\gamma_1 \leq 1 + (S_1 - S_0)\gamma_0 \quad \text{for } \gamma_1 \geq \gamma_0,$$

since $(S_2 - S_1)\gamma_1 \leq (S_1 - S_0)\gamma_1 \leq (S_1 - S_0)\gamma_0$ for $\gamma_1 \geq \gamma_0$.

Under the assumptions $\gamma_0 \geq 0, \gamma_1 \leq 0$ the expression $(3.5)$ reduces to

$$X_2(\gamma) = 1 + (S_1 - S_0)\gamma_0 + (S_2 - S_1)\gamma_1 \leq 1 + (S_2 - S_1)\gamma_1.$$

It follows that the maximization of $E \ln(X_2(\gamma))$ can be carried over the set $\{\gamma_0 = 0, \gamma_1 \leq 0\}$:

$$\lambda = \sup\{E \ln(X_2(\gamma)) : \gamma_1 \in L^0(\mathcal{F}_t), \ t = 0, 1\} \quad$$

$$= \sup\{E \ln(1 + (S_2 - S_1)\gamma_1) : \gamma_1 \in L^0(\mathcal{F}_1, -\mathbb{R}_+)\}. \quad (3.6)$$

Moreover, since $E(S_2 - S_1|\mathcal{F}_1) < 0$, it is not optimal to do nothing. Denote by $\gamma_1^k$ the value of $\gamma_1$ on $D_k$. We have

$$E \ln(1 + (S_2 - S_1)\gamma_1) = \sum_{k=0}^{\infty} \left( (1 - 2^{-n-k}) \ln(1 - \gamma_1^k) \right. \quad$$

$$\left. + 2^{-n-k} \ln \left( 1 + (1 + k)\gamma_1^k \right) \right) P(D_k). \quad (3.7)$$

The optimal portfolio $\gamma_1^{*,k} < 0$ can be obtained by maximizing each term in this sum. Taking into account that $\gamma_1^{*,k} \in (-1 + k)^{-1}, 0)$, we get an estimate

$$\left( 1 - 2^{-n-k} \right) \ln(1 - \gamma_1^k) + 2^{-n-k} \ln \left( 1 + (1 + k)\gamma_1^k \right) \leq \left( 1 - 2^{-n-k} \right)(-\gamma_1^k) \leq \frac{1 - 2^{-n-k}}{k + 1}$$

which shows that the optimal utility value $\lambda$ is finite.

Since the optimal strategy $\gamma_1^* < 0$, $\gamma_2^* = -\gamma_1^*$ is active, shadow prices $S_1^*, S_2^*$ should coincide with $S_1, S_2$. Otherwise, the same strategy would give strictly higher utility value in the frictionless market with stock price $S^*$. But in this frictionless market the optimal utility value $\mu_{S^*}$ is infinite since

$$1 + \gamma_0 \Delta S_1^* = 1 - \gamma_0 \rightarrow +\infty, \quad \gamma_0 \rightarrow -\infty.$$
Thus, there is no shadow price in this model. However, as we will see shortly, the process \( S^* = (S_0, S_1, S_2) \) is a generalized shadow price. The point is that in the relaxed problem short selling at step 0 is automatically prohibited. Put \( s = 0, q = 1 \) in the notation of Section 2. By the same reasons as in Example 2, the topology \( \tau_i^w \) of pointwise convergence coincides with the weak topology \( \tau_i^w = \sigma(L^1(F_i), L^\infty(F_i)) \) on the set \([S_0, S_1] \). For any \( \prod_{i=0}^n \tau_i^w \)-neighbourhood \( U \) of \( S^* \) there exist sufficiently large \( k \) and \( S' \in U \) such that \( S'_1 = S_1, S'_2 = S_2 \) on \( D_k \). We have

\[
1 + (\gamma \circ S')_2 = 1 + (S_1 - S_0)\gamma_0 + (S_2 - S_1)\gamma_1 \\
= (k - 1)\gamma_0 + (I_{k+2} - (k + 1)I_{k+1})\gamma_1^k \quad \text{on } D_k.
\]

If \( \gamma_0 < 0 \), then \( 1 + (\gamma \circ S')_2(2k + 2) < 0 \) for large \( k \). Thus,

\[
\Psi(S', \gamma) = E \ln(1 + \gamma \circ S')_2 = -\infty
\]

and \( \hat{\Psi}(S^*, \gamma) = -\infty \) for any \( \gamma_0 < 0 \).

Furthermore,

\[
1 + (\gamma \circ S^*)_2 = 1 + (S_1 - S_0)\gamma_0 + (S_2 - S_1)\gamma_1 \leq 1 + (S_2 - S_1)\gamma_1 \quad \text{for } \gamma_0 \geq 0.
\]

It follows that \( \hat{\Psi}(S^*, (\gamma_0, \gamma_1)) \leq \hat{\Psi}(S^*, (0, \gamma_1)) \) and one can assume \( \gamma_0 = 0 \) in the relaxed utility maximization problem (2.10):

\[
\hat{\mu}_{S^*} = \sup\{\hat{\Psi}(S^*, \gamma) : \gamma_0 = 0, \gamma_1 \in L^0(F_1)\}.
\]

To prove that \( \mu_{S^*} = \lambda \), we go back to the "unrelaxed" frictionless problem:

\[
\hat{\mu}_{S^*} \leq \sup\{\Psi(S^*, \gamma) : \gamma_0 = 0, \gamma_1 \in L^0(F_1)\} = \sup\{E \ln (1 + (S_2 - S_1)\gamma_1) : \gamma_1 \in L^0(F_1)\}.
\]

Looking again at (3.7), we conclude that optimal values \( \gamma_1^{*, k} \) are negative. Comparing the last expression with (3.6), we obtain the inequality \( \hat{\mu}_{S^*} \leq \lambda \). The reverse inequality is evident.

4. Shadow prices via Lagrange duality

As is known, in mathematical economics shadow resource prices are associated with the optimal solution of the dual problem: see e.g. [19], [10] (Chapter 5). To avoid conflicts with the terminology of the present paper we, following [24], use the term "equilibrium prices" instead. These prices are introduced along the following lines. Let \( x = (x_1, \ldots, x_n) \) represent activities of a firm and let \( f(x) \) be the cost of the corresponding operation. The activities are subject to the resource constraints \( g_i(x) \leq 0, i = 1, \ldots, m \). Put

\[
\varphi(u) = \inf\{f(x) : g_i(x) \leq u_i, i = 1, \ldots, m\}.
\]

The components of a vector \( \lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \) are called equilibrium resource prices if the firm cannot reduce the optimal cost of the operation by buying or
selling resources at these prices:
\[ \varphi(u) + \sum_{i=1}^{m} \lambda_i^* u_i \geq \varphi(0), \quad u \in \mathbb{R}^m. \]

For a convex problem vectors \( \lambda^* \) of equilibrium prices are exactly the optimal solutions of the dual problem (see [24], Theorem 28.2 and Corollary 28.4.1 for the precise statement).

In the problem under consideration the trader has two resources at his disposal: bonds and stocks. It is natural to expect that the equilibrium price \( s \) of these resources are related to the shadow price process introduced above.

Assume that \( \Omega \) is finite, \( \mathcal{F}_T \) coincide with the power set of \( \Omega \) and \( \mathbb{P}(\omega) > 0, \omega \in \Omega. \) First of all we rewrite the self-financing condition, separating the "resource constraints":
\[
(\Delta \beta_t - L_t \bar{S}_t + M_t \bar{S}_t)(\omega) \leq 0, \quad t \in 0, \ldots, T, \quad \omega \in \Omega; \quad (4.1)
\]
\[
(\Delta \gamma_t + L_t - M_t)(\omega) \leq 0, \quad t \in 0, \ldots, T, \quad \omega \in \Omega; \quad (4.2)
\]
\[
- L_t(\omega) \leq 0, \quad - M_t(\omega) \leq 0, \quad t \in 0, \ldots, T, \quad \omega \in \Omega. \quad (4.3)
\]

Here, as above, \( \beta_{-1} = 1, \gamma_{-1} = 0. \) By \( L_t \) (respectively, \( M_t \)) we denote the number of stocks sold (respectively, purchased) at time \( t \) at price \( \bar{S}_t \) (respectively, \( \bar{S}_t \)). Clearly, passing to the inequality constraints (corresponding to the possibility of consumption) and allowing the simultaneous transfers from bonds to stocks and back: \( L_t M_t \neq 0 \) do not increase trader’s monotone utility. We should also take into account the "boundary condition":
\[
\gamma_T(\omega) = 0, \quad \omega \in \Omega \quad (4.4)
\]
and the "information constraints":
\[
(\beta_t, \gamma_t, L_t, M_t) \in L^0(\mathcal{F}_t, \mathbb{R}^4), \quad t \in 0, \ldots, T. \quad (4.5)
\]

Consider a concave utility function \( U \) as in Example 1: \( U \) is finite on \((0, \infty)\) and \( U(x) = -\infty, x \leq 0 \) and denote by \( C \) the set of processes \( (\beta, \gamma, L, M) \), satisfying (4.5) and such that \( \beta_T > 0. \) The problem is to minimize
\[
- \mathbb{E}U(\beta_T) \quad (4.6)
\]
over the set \( C \) under the constraints (4.1) – (4.4). Formally, this is an ordinary convex optimization program ([24], Section 28).

Consider the Lagrange function
\[
\mathcal{L} = - \mathbb{E}U(\beta_T) + \sum_{t=0}^{T} \mathbb{E}(Z_t^1(\Delta \beta_t - L_t \bar{S}_t + M_t \bar{S}_t)) + \sum_{t=0}^{T} \mathbb{E}(Z_t^2(\Delta \gamma_t + L_t - M_t))
\]
\[
- \sum_{t=0}^{T} \mathbb{E}(Z_t^4 L_t) - \sum_{t=0}^{T} \mathbb{E}(Z_t^4 M_t) + \mathbb{E}(\nu_T \gamma_T) \quad \text{for } (\beta, \gamma, L, M) \in C. \quad (4.7)
\]

The Lagrange multipliers are represented by a process \( Z_t = (Z_t^1, Z_t^2, Z_t^3, Z_t^4) \) with non-negative components: \( Z_t \in L^0(\mathcal{F}_t, \mathbb{R}_+^4), t = 0, \ldots, T \) and \( \nu_T \in L^0(\mathcal{F}_T). \) Note that the process \( Z \) may be assumed adapted since for adapted processes
\( (\beta, \gamma, L, M) \in C \) the number of constraints in (4.1) – (4.3) for fixed \( t \) coincides with the number of atoms of \( F_t \) and \( Z_t \) can be taken constant on these atoms.

To complete the definition of \( \mathcal{L} \), we put in accordance to the general scheme of [24] (Section 28)

\[
\mathcal{L} = +\infty, \quad \text{if} \ (\beta, \gamma, L, M) \notin C; \\
\mathcal{L} = -\infty, \quad \text{if} \ (\beta, \gamma, L, M) \in C, \quad Z_t \notin L^0(\mathcal{F}_t, \mathbb{R}_+^4) \text{ for some } t.
\]

Collecting terms, containing the same elements \( \beta_t, \gamma_t, L_t, M_t \), we rewrite (4.7) in the following way:

\[
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,
\]

\[
\mathcal{L}_1 = \mathbb{E} \left( -U(\beta_T) + Z^1_T \beta_T \right) - \sum_{t=0}^{T-1} \mathbb{E} \beta_t \Delta Z^1_{t+1} - \mathbb{E} Z^1_0,
\]

\[
\mathcal{L}_2 = \mathbb{E} \gamma_T (Z^2_T + \nu_T) - \sum_{t=0}^{T-1} \mathbb{E} \gamma_t \Delta Z^2_{t+1},
\]

\[
\mathcal{L}_3 = \sum_{t=0}^{T} \mathbb{E} L_t (Z^2_t - Z^1_t S_t - Z^3_t) + \sum_{t=0}^{T} \mathbb{E} M_t (-Z^2_t + Z^1_t S_t - Z^4_t).
\]

The objective function of the dual problem is given by

\[
g(Z, \nu_T) = \inf \{ \mathcal{L} : (\beta, \gamma, L, M) \in C \}.
\]

Put \( V(x) = \inf_y (-U(y) + x y) \). After simple calculations we get

\[
g(Z, \nu_T) = \mathbb{E} V(Z^1_T) - \mathbb{E} Z^1_0
\]

if \( Z_t \in L^0(\mathcal{F}_t, \mathbb{R}_+^4), \ t = 0, \ldots, T \) and the following conditions hold true

\[
\mathbb{E}(\Delta Z^1_{t+1}|\mathcal{F}_t) = 0, \quad \mathbb{E}(\Delta Z^2_{t+1}|\mathcal{F}_t) = 0, \quad t = 0, \ldots, T - 1; \\
Z^2_t - Z^1_t S_t = Z^3_t, \quad -Z^2_t + Z^1_t S_t = Z^4_t, \quad t = 0, \ldots, T; \quad \nu_T = -Z^2_T.
\]

Otherwise, \( g(Z, \nu_T) = -\infty \).

It readily follows that the optimal value of the dual problem can be represented as

\[
\sup \{ \mathbb{E} (V(Z^1_T) - Z^1_0) : Z \in D \},
\]

\[
D = \{ Z^1 \in \mathcal{M}_+ : Z^1 S_t \leq Z^2_t \leq Z^1 S_t \text{ for some } Z^2 \in \mathcal{M}_+ \},
\]

where \( \mathcal{M}_+ \) is the set of non-negative \( \mathbb{P} \)-martingales. The representations of this sort are well known: see [5] for continuous time case and [22, 23] for generalizations in discrete time.

The objective function (4.6) of the primal problem is finite on \( C \) and the point \( (\beta, \gamma) \), where \( \beta_t = 1, \gamma_t = 0, \ t = 0, \ldots, T \), belongs to the relative interior of \( C \) and satisfies the constraints (4.11) – (4.3), which are affine. If the optimal value \(-\lambda\) of the primal problem is finite then there is no duality gap and the dual problem is solvable ([24], Theorem 28.2 and Corollary 28.4.1). That is,

\[
-\lambda = \sup \{ \mathbb{E} (V(\hat{Z}^1_T) - \hat{Z}^1_0) : Z \in D \} = \mathbb{E} (V(\hat{Z}^1_T) - \hat{Z}^1_0)
\]

for some \( \hat{Z}^1 \in D \).
Let us introduce an adapted process \( S^*_t \in [S_t, \overline{S}_t] \) such that
\[
S^*_t \overline{Z}^1_t = \overline{Z}^2_t.
\] (4.9)

On the atoms of \( \mathcal{F}_t \) with \( \overline{Z}^1_t = 0, \overline{Z}^2_t = 0 \) the values \( S^*_t \in [S_t, \overline{S}_t] \) are chosen arbitrary. Put
\[
-\mu_{S^*} = \sup \{ E(V(Z_t^1) - Z_0^1) : Z \in D(S^*) \},
\] (4.10)
\[
D(S^*) = \{ Z^1 \in \mathcal{M}_+ : Z^2 = Z^1 S^* \in \mathcal{M}_+ \}.
\]
The maximization in (4.10) is carried over smaller set as compared to (4.8), and the objective functions are the same. Hence, \(-\lambda \geq -\mu_{S^*}\). On the other hand, the optimal solution \( \overline{Z} \) of (4.8) is feasible for (4.10): \( \overline{Z}^1 \in D(S^*) \) since \( \overline{Z}^2 = S^*_t \overline{Z}^1_t \in \mathcal{M}_+ \). It follows that \( \lambda = \mu_{S^*} \). But (4.10) is the dual to the frictionless optimization problem with the price process \( S^* \). This means that \( S^* \) is a shadow price in the sense of the definition of Section 2.

In fact, we have obtained the same result as in Example 1 (and in [18]). But formula (4.9) reveals one more interpretation of a shadow price process: it is the equilibrium bond/stock exchange rate, that is, the relation of stock and bond equilibrium prices.

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