The motivic Adams-Novikov spectral sequence at odd primes over \( \mathbb{C} \) and \( \mathbb{R} \)

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We study the motivic Adams-Novikov spectral sequence at an odd prime \( l \) over the base fields \( \mathbb{C} \) and \( \mathbb{R} \). This spectral sequence converges to the stable motivic homotopy groups of the \( l \)-completed motivic sphere spectrum. We show that the spectral sequence is determined by the topological Adams-Novikov sequence, similar to the situation at the prime two over the base field \( \mathbb{C} \).

1 Introduction

We work in an appropriate category of motivic spectra over a base field \( k \). The aim is to compute the stable motivic homotopy groups of a completion of the motivic sphere spectrum over the base fields \( k \in \{ \mathbb{C}, \mathbb{R} \} \) at odd primes. At the prime 2 and over \( k = \mathbb{C} \), extensive calculations have been done with the motivic Adams spectral sequence (cf. [DI], [Isa]) and the motivic Adams-Novikov spectral sequence (cf. [IKO]). In this case the 2-complete Adams-Novikov spectral sequence is determined by the classical Adams-Novikov spectral sequence. The situation is entirely similar at odd primes. Moreover in topology at odd primes, the Adams-Novikov spectral sequence is a more efficient way to compute the stable homotopy groups of spheres than the Adams spectral sequence, and this holds motivically as well.

1.1 Notation

We index the motivic spheres according to the following convention: we define \( S^{1,0} \) as the suspension spectrum of the simplicial sphere and \( S^{1,1} \) as the suspension spectrum of \( \mathbb{A}^1 - 0 \). The suspension spectrum of \( \mathbb{P}^1 \) is then equivalent to \( S^{2,1} \).

This relates to the other common notation of \( S^a = S^{1,1} \) by \( S^{p,q} = S^{p-q+q\sigma} \).

Over \( k \in \{ \mathbb{C}, \mathbb{R} \} \) we have topological realization functors mapping into the stable homotopy category and the stable equivariant homotopy category:

\[
R : \mathcal{SH}_\mathbb{C} \to \mathcal{SH}_{\text{top}}
\]

and

\[
R' : \mathcal{SH}_\mathbb{R} \to \mathcal{SH}_{\mathbb{Z}/2}
\]

which map \( S^{p,q} \mapsto R S^p \) and \( S^{p,q} \mapsto R' S^{p-q+q\sigma} \). Here \( \sigma \) denotes the sign representation of \( \mathbb{Z}/2 \). An explicit review of construction and basic properties of these functors is given in e.g. [Joa, 4.3].
1.2 Coefficients of motivic cohomology and the dual motivic Steenrod algebra

Proposition 1.1. Let $l \neq 2$ be a prime

1. The coefficients $H_{\mathbb{Z}/l^{**}}$ of motivic cohomology are given as a ring by
   $$H_{\mathbb{Z}/l^{**}} \cong \mathbb{Z}/l[\tau]$$
   with $|\tau| = (0,1)$ over $k = \mathbb{C}$.

2. The coefficients $H_{\mathbb{Z}/l^{**}}$ of motivic cohomology are given as a ring by
   $$H_{\mathbb{Z}/l^{**}} \cong \mathbb{Z}/l[\theta]$$
   with $|\theta| = (0,2)$ over $k = \mathbb{R}$.

3. The map $H_{\mathbb{Z}/l^{**}} \to H_{\mathbb{Z}/l^{**}}$ is given by $\theta \mapsto \tau^2$.

4. For a fixed bidegree $p,q$ with $p \leq q$ there is a commutative square

$$
\begin{array}{ccc}
H^{p,q}_R(pt,\mathbb{Z}/l) & \cong & H^{p,q}_C(pt,\mathbb{Z}/l) \\
R & \cong & R \\
H^{p-q+q\sigma}_R(pt,\mathbb{Z}/l)^{\text{res}_{\mathbb{Z}/2}} & \cong & H^p_{\text{sing}}(pt,\mathbb{Z}/l)
\end{array}
$$

Here $H_k$ denotes motivic cohomology over the base field $k$, $H^{p-q+q\sigma}_R(pt,\mathbb{Z}/l)$ denotes Bredon cohomology (graded by the trivial representation and the sign representation $\sigma$) and $H_{\text{sing}}$ denotes singular cohomology. The top map is the one induced by Spec($\mathbb{C}$) $\to$ Spec($\mathbb{R}$), the bottom map is the restriction functor to the trivial group.

In particular, both $\tau$ and $\theta$ do not vanish under topological realization.

Proof. We know that $H_{\mathbb{Z}/l^{**}} = 0$ for $q < p$ (cf. [MVW, Theorem 3.6]). Let $q \geq p$. Then there is an isomorphism from motivic to étale cohomology:

$$H^{p,q}(\text{Spec}(k),\mathbb{Z}/l) \cong H^p_{\text{ét}}(k,\mu_l^q)$$

This isomorphism respects the product structure ([GL 1.2,4.7]).

The étale cohomology groups $H^p_{\text{ét}}(k,\mu_l^q)$ can be computed as the Galois cohomology of the separable closure of the base field (in both cases the complex numbers) with coefficients in the $l$-th roots of unity. The action of the absolute Galois group $G$ is given by the trivial action if $k = \mathbb{C}$ and by complex conjugation if $k = \mathbb{R}$:

$$H^p_{\text{ét}}(k,\mu_l^q) \cong H(G,\mu_l^q(\mathbb{C}))$$
1. For $k = \mathbb{C}$, these groups all vanish for $p \neq 0$ by triviality of the Galois action, and they are $\mathbb{Z}/l$ in the degree $p = 0$ for all $q \geq 0$. The multiplicative structure is given by the tensor product of the modules.

2. For $k = \mathbb{R}$, we have the following isomorphism of $G$-modules:

$$\mu_l \otimes \mu_l \cong \mathbb{Z}/l \otimes \mathbb{Z}/l \to \mathbb{Z}/l \cong \mu_l$$

Here $G$ acts on the top left hand side by complex conjugation on each factor, on the lower left hand side by the assignment $x \mapsto -x$ on each factor, and trivial on the right hand side. Hence $\mu_l^{\otimes q}$ is isomorphic as a $G$-module to $\mu_l$ equipped with the trivial action in degrees with $q$ even, and with the Galois action in degrees with $q$ odd. Since the latter has no fix points, the description above follows additively. The multiplicative statement follows from the same reasoning as for $k = \mathbb{C}$.

3. On the level of Galois cohomology, the map induced by $\mathbb{R} \to \mathbb{C}$ corresponds to the one induced by the map of groups which embeds the trivial group (the absolute Galois group of $\mathbb{C}$) into the Galois group of $\mathbb{R}$. Since all the Galois cohomology groups are concentrated in degree 0, the map is just the inclusion of the fixed points in $\mu_l^{\otimes q}$ under the action of the Galois group of $\mathbb{R}$ into the fixed points of $\mu_l^{\otimes q}$ under the action of the trivial group, and the third statement follows.

4. The statement is a specialization of [HO, Thm. 4.18].

Remark 1.2. We have also computed $H\mathbb{Z}/l_{**} = H\mathbb{Z}/l^{-*,-}$ In an abuse of notation, we denote the elements in $H\mathbb{Z}/l_{**}$ corresponding to $\tau$ and $\theta$ by the same name, where the bidegree is the same as above multiplied by -1.

Proposition 1.3. Let $k$ be a base field of characteristic 0, and let $l$ be an odd prime. The computation of the motivic mod-l Steenrod algebra over such basefields is due to Voevodsky in [Voe]. The implications for the dual motivic Steenrod algebra are written down explicitly in the introduction of [HKO2] and many other places. In particular, the dual motivic Steenrod algebra $A_{**}$ and its Hopf algebroid structure over $k \in \{\mathbb{C}, \mathbb{R}\}$ for $l$ an odd prime can be described as follows:

$$A_{**} = H\mathbb{Z}/l_{**}[\tau_0, \tau_1, \tau_2, ..., \xi_1, \xi_2, ...]/(\tau_i^2 = 0)$$

Here $|\tau_i| = (2l^i - 1, l^i - 1)$ and $|\xi_i| = (2l^i - 2, l^i - 1)$.

The comultiplication is given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi^n_{m-i} \otimes \xi_i$$

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where \( \xi_0 := 1 \), and

\[
\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_i^{l_i} \otimes \tau_i
\]

The \( \tau_i \) are not related to the element \( \tau \) defined above. Neither the product nor the coproduct increase the number of \( \tau_i \)'s involved in any given expression in \( A_{**} \). Hence \( A_{**} \) can be graded as an \( A_{**} \)-comodule by this number. This is similar to the classical situation at odd primes.

1.3 Completions

Definition 1.4. Let \( X \) be a motivic spectrum.

1. The \( l \)-completion \( X_1^\wedge \) of \( X \) is defined as the Bousfield localization of \( X \) at the Moore spectrum \( M\mathbb{Z}/l \). Motivically this is discussed in [RO, Section 3]. In particular we have \( X_1^\wedge := L_{M\mathbb{Z}/l}X \simeq \text{holim} \ X/l^n \).

The homotopy groups of \( X \) and its \( l \)-completion are related by the short exact sequence

\[
0 \to \text{Ext}^1(\mathbb{Z}/l^\infty, \pi_{*-}X) \to \pi_{**}X_1^\wedge \to \text{Hom}(\mathbb{Z}/l^\infty, \pi_{*-1,*}X) \to 0
\]

2. In a similar manner, define the \( \eta \)-completion \( X^\wedge \) of \( X \) to be the homotopy limit

\[
\text{holim} \ X/\eta^n
\]

Definition 1.5. Let \( X \) be a motivic spectrum and \( E \) be a motivic ring spectrum. Define the \( E \)-nilpotent completion \( X_E^\wedge \) of \( X \) as the homotopy limit over the semicosimplicial object

\[
(X_E^\wedge)_n := (X \wedge E^{\wedge n})
\]

Here the coface maps are given by the unit map of \( E \).

Remark 1.6. 1. This definition of the \( E \)-nilpotent completion is slightly different to the one given in [Bau], which makes use of an alternative Adams tower. There is however an equivalence between the two by the remarks in [DI, 6.7], where the work in [Bau] is adapted to the motivic situation.

2. If \( X \) has the structure of a motivic ring spectrum, then so has its \( E \)-nilpotent completion \( X_E^\wedge \), also by [DI, 6.7].

3. If \( X \) is a module over \( E \), then \( X \) is equivalent to its \( E \)-nilpotent completion via the canonical map \( X \to X_E^\wedge \). (See [DI, 6.9].)

4. As explained in [DI, 7.3], the preceding statement implies that \( S_{H\mathbb{Z}/l,\text{ARP}}^\wedge \cong S_{H\mathbb{Z}/l}^\wedge \). The arguments given by Dugger and Isaksen are independent of the chosen base field \( \mathbb{C} \) and their restriction to the prime 2. They work equally well if \( l \) is an odd prime and for the base field \( \mathbb{R} \).
2 The motivic Adams and Adams-Novikov spectral sequences at odd primes

2.1 Generalized motivic Adams spectral sequences

If \( E \) is an arbitrary motivic ring spectrum and \( X \) a motivic spectrum, then there is a trigraded spectral sequence \( E^s,t,u_r(E,X) \) with differentials \( d_r : E^s,t,u_r \rightarrow E^{s+r,t+t+r-1,u} \). Smashing the cofiber sequence \( \tilde{E} \rightarrow S \rightarrow E \) with \( \tilde{E}^{s} \wedge X \) yields cofiber sequences

\[
\tilde{E}^{s+1} \wedge X \rightarrow \tilde{E}^{s} \wedge X \rightarrow E^{s} \wedge \tilde{E}^{s} \wedge X
\]

The associated long exact sequences of homotopy groups form an exact couple, inducing the generalized motivic \( E \)-Adams spectral sequence.

If \( E^{\ast} \) is flat as a (left) module over the coefficients \( E^{\ast} \), one can associate a flat Hopf algebroid to \( E \) (See \[NSO, Lemma 5.1\] for the statement and \[Rav, Appendix 1\] for the definition and basic properties of Hopf algebroids). In this case the category of comodules over this Hopf Algebroid is abelian and permits homological algebra. The \( E_2 \)-page of the \( E \)-Adams spectral sequence can then be identified as:

\[
E_2^{s,t,u} = \text{Cotor}^{s,t,u}_{E^{\ast}(E)}(E^{\ast},E^{\ast}(X))
\]

Here Cotor denotes the derived functors of the cotensor product in the category of \( E^{\ast}(E) \)-comodules and can be computed as the homology of the Cobar complex \( C^\ast(E) \).

Convergence in this general situation is discussed by Bousfield in \[Bou\]; under certain assumptions, the spectral sequence converges to a filtration of the homotopy groups of the \( E \)-nilpotent completion of \( X \), which was defined in Chapter 1. Here convergence does not mean strong convergence but complete convergence, defined in \[Bou, Chapter 6\]. We will look at the following three cases:

The homological motivic Adams spectral sequence, where \( E = H\mathbb{Z}/l, X = S \); the motivic Adams spectral sequence for the computation of the coefficients of the \( l \)-completed motivic Brown-Peterson spectrum \( ABP_l^\wedge \), where \( E = H\mathbb{Z}/l, X = ABP_l^\wedge \); and the \( l \)-complete motivic Adams-Novikov spectral sequence, where \( E = ABP_l^\wedge, X = S \).

In these instances, the spectral sequence will converge strongly by virtue of a vanishing line or because the spectral sequence degenerates.

In the case \( E = H\mathbb{Z}/l \), the \( H \)-nilpotent completion has been described more explicitly by Hu, Kriz and Ormsby in \[HKO2, Theorem 1\], which we restate here for convenience:

**Proposition 2.1.** Let the base field \( k \) be of characteristic 0 and let \( X \) be a motivic cell spectrum of finite type - i.e. a motivic cell spectrum s.t. there exists a \( k \in \mathbb{N} \) with the property that \( \pi_{s,t}(X) = 0 \) for \( s - t < k \) and s.t. there exist only finitely many cells in dimensions \( (s + t, t) \) for each \( s \). Then the map \( X \rightarrow X_{H\mathbb{Z}/l}^\wedge \) is a completion at \( l \) and \( \eta \), i.e. \( X_{H\mathbb{Z}/l}^\wedge \simeq X_{l,\eta}^\wedge \).

Furthermore, if either \( l > 2 \), \(-1\) is a sum of squares in \( k \) and \( cd_l(k) < \infty \) or \( l = 2 \) and \( cd_2([k]) \) is \( \infty \), then \( X \rightarrow X_{H\mathbb{Z}/l}^\wedge \) is a completion at \( l \). Here \( cd_l(k) \) denotes the cohomological dimension of the base field at the prime \( l \).
Remark 2.2.  
1. Both the sphere spectrum $S$ and the Brown-Peterson spectrum $ABP$ (regardless of the prime $l$) are cell spectra of finite type over an arbitrary basefield.

2. The Hopf algebroid induced by $ABP$ is flat for $k = \mathbb{R}$ and $k = \mathbb{C}$.

3. Let $k = \mathbb{C}$ and $l$ be any prime. Since the condition regarding cohomological dimension obviously holds and $-1$ is a sum of squares, we have $S^\wedge_{HZ/l} \cong S^\wedge_l$ and $ABP^\wedge_{HZ/l} \cong ABP^\wedge_l$.

4. Let $k = \mathbb{R}$ and $l$ be an odd prime. By comment 1 on this theorem in [HKO2], the $HZ/l$-nilpotent completion of the sphere spectrum $S$ is a completion at $l$ and $\eta$, and this completion is not equivalent to the completion at $l$ alone.

Proposition 2.3. Let $k$ be any field. Then $ABP_{\eta,l}^\wedge \cong ABP_l^\wedge$.

Proof. Since $MGL \wedge \eta = 0$ (for a proof, see e.g. [Joa, Lemma 7.1.1]), we also have $ABP \wedge \eta = 0$ and $ABP^\wedge_l \wedge \eta = 0$. Hence all maps in the homotopy limit

$$\text{holim} ABP^\wedge_l/\eta^a$$

are equivalences, and this homotopy limit models the completion at $l$ and $\eta$. \hfill \Box

2.2 The coefficients of the motivic Brown-Peterson spectrum

If $E = ABP^\wedge_l$ and $X = S$, the $E$-Adams spectral sequence is called the $l$-completed motivic Adams Novikov spectral sequence (considered for example in [OO 3],[HKO]), abbreviated to MANSS. By the previous remarks, this agrees with the $HZ/l$-complete MANSS considered in [DI].

For $l$-low dimensional fields (a base field with exponential characteristic $p \neq l$ s.t. $cd_l(k) \leq 2$) the $E_2$-page of the $l$-primary MANSS has been described in terms of the $E_2$-page of the classical ANSS and $HZ_{l^{**}}$ by Ormsby and Østvær in [OO] in order to calculate $\pi_{1,**}$ over these fields. To use this, we first need to compute the coefficients $ABP_{l,**}(ABP_{l,**}^\wedge)$.

Let $k \subset \mathbb{C}$ and $l$ be an arbitrary prime. Let $MGL$ denote the algebraic cobordism spectrum of Voevodsky and $MU$ the topological cobordism spectrum. We follow the definition of the motivic Brown-Peterson as in e.g. [Joa 6.3.1], i.e. we define the spectrum $ABP$ as the quotient of $MGL_{(l)}$ by a particular regular sequence of elements generating the vanishing ideal of the $l$-typical formal group law. As remarked in [Hoy, Remark 6.20], this is equivalent to the definition by either the Quillen idempotent or by the motivic Landweber exact functor theorem.

Precisely, we choose elements $a_{i}^{top} \in MU_{2i}$ satisfying the conditions in [Hoy 6.1, last section] and write $a_i$ for their image in $MGL_{2i}$ under the canonical map $L \cong MU_{2*} \rightarrow MGL_{2**,}$. Then $R(a_i) = a_{i}^{top}$, and we can define

$$ABP := MGL_{(l)}/(a_{i} | i \neq l^j - 1)$$
Additionally, define $v_n^{\text{top}} := a_{n-1}^{\text{top}}$ and $v_n := a_{n-1}$. Over $\mathbb{C}$ we have $R(ABP) = BP$.

We summarise a few properties of $ABP$ that we will use later:

**Remark 2.4.** 1. If $k = \mathbb{C}$, we have $R(v_n) = v_n^{\text{top}}$ ( Mentioned in [Joa, 6.3.2] and clear by construction)

2. By [Hoy, Thm. 6.11] we have an isomomorphism $HZ/l^{* *}(ABP) \cong P_{* *}$ as $A_{* *}$-comodules.

**Proposition 2.5.** If $k = \mathbb{C}$ and $l$ an odd prime, then

$$
\pi_{* *}(ABP^{l \wedge}_l) = \mathbb{Z}/l[\tau, v_1, v_2, ...]
$$

If $k = \mathbb{R}$ and $l$ an odd prime, then

$$
\pi_{* *}(ABP^{l \wedge}_l) = \mathbb{Z}[\theta, v_1, v_2, ...]
$$

Here the elements $v_i$ have bidegree $(2l^i - 2, l^i - 1)$.

**Proof.** Let $k = \mathbb{C}$. Consider the motivic Adams spectral sequence for $ABP$. Since $ABP$ is cellular it converges to $\pi_{* *}(MGL^{l \wedge}_l)$. The $E_2$-page has the form $E_2^{s,t,u} = \mathbb{Z}/l[\tau, Q_0, Q_1, ...]$, where $Q_n$ lives in degree $(1, 2l^n - 2, l^n - 1)$. Apart from the grading by weight this agrees with the classical ASS for $BP$, and the spectral sequence collapses at the $E_2$-page. The extension problem is solved by considering topological realization over $\mathbb{C}$.

**Remark 2.6.** More generally, the coefficients of the $l$-completed motivic cobordism spectrum $MGL^{l \wedge}_l$ have been computed over a $l$-low dimensional base field by Ormsby and Østvær using the slice spectral sequence, cf. [OO, Corollary 2.6].

### 2.3 The motivic Adams-Novikov spectral sequence

By a result of Naumann, Spitzweck and Østvær ([NSO, Lemma 9.1]) $ABP^{l \wedge}_{l, * *}(ABP^{l \wedge}_l)$ and the structure of the Hopf algebroid induced by $ABP^{l \wedge}_l$ are known:

$$
ABP^{l \wedge}_{l, * *}(ABP^{l \wedge}_l) \cong ABP^{l \wedge}_{l, * *} \otimes_{\pi_*(BP)} BP_*(BP)
$$

Over $\mathbb{C}$ and $\mathbb{R}$ the Hopf algebroid structure is given by tensoring the Hopf algebroid of the Brown Peterson spectrum $BP$ with $\mathbb{Z}[\tau]$ resp. $\mathbb{Z}[\theta]$ over $\mathbb{Z}_l$.

This implies a similar decomposition for the cobar complex by its explicite definition:

$$
C^*(ABP^{l \wedge}_l) \cong C^*(BP) \otimes_{\pi_*(BP)} \pi_{* *}(ABP^{l \wedge}_l)
$$

If $k \in \{\mathbb{C}, \mathbb{R}\}$ and $l$ an odd prime, we can now use the computations of $\pi_{* *}(ABP^{l \wedge}_l)$:

$$
C^*(ABP^{l \wedge}_l) \cong C^*(BP) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\tau]
$$
over $\mathbb{C}$ and
\[
C^*(ABP^\wedge_l) \cong C^*(BP) \otimes \mathbb{Z}_l[\theta]
\]
over $\mathbb{R}$.

The map induced by $\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})$ is given by $\theta \mapsto \tau^2$ and by the identity on the other generators.

The universal coefficient theorem of homological algebra, applied to the cobar complex, then yields the following short exact sequence for the $E_2$-term of the $l$-complete MANSS, where $E_2^{\text{top}}$ denotes the $E_2$-term of the classical topological ANSS:

\[
0 \to E_{2,\text{top}}^{s,t} \otimes \mathbb{Z}_l\tau^n \to E_2^{s,t,\frac{1}{l}-n} \to \text{Tor}_1^Z(E_{2,\text{top}}^{s-1,t}, \mathbb{Z}_l\tau^n) \to 0
\]

over $\mathbb{C}$, and

\[
0 \to E_{2,\text{top}}^{s,t} \otimes \mathbb{Z}_l\theta^n \to E_2^{s,t,\frac{1}{l}-n} \to \text{Tor}_1^Z(E_{2,\text{top}}^{s-1,t}, \mathbb{Z}_l\theta^n) \to 0
\]

over $\mathbb{R}$.

**Remark 2.7.** We recall some facts about the topological ANSS associated to the spectrum $BP$ for reference.

1. There is a vanishing line $E_2^{s,t} = \text{Cotor}_{BP,BP}(BP_*BP_*) = 0$ if $t < 2s$. If we translate this in the Adams grading $(s,s') := (s,t-s)$, the condition reads as $s' < s$. (cf. [Rav, 5.1.23 (a)])

2. $E_2^{0,0} = \mathbb{Z}_l$ and $E_2^{0,t} = 0$ for $t \neq 0$. (cf. [Rav, 5.2.1 (b)])

3. If $(s,t) \neq (0,0)$, then $\text{Cotor}_{BP,BP}(BP_*BP_*)$ is a finite $l$-group. (cf. [Rav, 5.2.1 (a)] together with the previous statement). In particular, if we construct the generalized Adams spectral sequence for $BP^\wedge_l$, then this spectral sequence agrees with the ANSS in all bidegrees away from $(0,0)$, and in this bidegree there is a single copy of $\mathbb{Z}_l$.

4. ("Sparseness"): $E_2^{s,t} \neq 0$ only if $t$ is divisible by $q = 2l - 2$. This means that a differential $d_r$ can be nontrivial only if $r \equiv 1 \pmod{q}$. Hence $E_{nq+2} \cong E_{nq+3} \cong \cdots \cong E_{(n+1)q+1}$. [Rav, 4.4.2]

The following results take the same form as in the case $l = 2$, discussed in [Isa] and [HKO].

**Proposition 2.8.** Assume that we know the $E_2$-page of the classical ANSS (associated to either $BP$ or $BP^\wedge_l$) in a certain range. The $E_2$ page of the odd-primary MANSS over $\mathbb{C}$ can then be constructed from this information as follows:

1. $E_2^{0,0,u} = 0$ if $u > 0$ and $E_2^{0,0,u} \cong \mathbb{Z}_l\tau^u$ if $u \leq 0$. By the multiplicative structure of the MANSS, each $E_2^{s,t,*}$ is a $\mathbb{Z}_l[\tau]$-module, so we can speak of $\tau$-torsion.
2. Let \((s, t) \neq (0, 0)\): For each group \(\mathbb{Z}/l^n\) of the ANSS, there is a group \(\mathbb{Z}/l^n\tau\) in \(E^{s,t}_2\) of the MANSS, and its generator as a \(\mathbb{Z}_l[\tau]\)-module has weight \(\frac{r}{2}\). There are no other groups in \(E^{s,t,u}_2\) of the MANSS.

3. The vanishing line in the classical ANSS carries over to the MANSS, so the MANSS converges strongly.

4. \(E^{s,t,u}_2 \neq 0\) only if \(t\) is divisible by \(q = 2l - 2\). This means that a differential \(d_r\) can be nontrivial only if \(r \equiv 1 \pmod{q}\). Hence \(E_{nq+2} \cong E_{nq+3} \cong \ldots \cong E_{(n+1)q+1}\).

**Proof.** In \(\mathbb{R}\) the torsion term \(\text{Tor}^{Z_2}_1\) vanishes (the second argument is free over the ground ring). This proves the first two statements. The last two statements can either be proven directly by examining the cobar complex or they can be seen as a corollary of the first two.

**Remark 2.9.** Since \(A_{ss} \cong A^s_{top} \otimes \mathbb{Z}/l[\tau]\) over \(\mathbb{C}\) and \(A_{ss} \cong A^s_{top} \otimes \mathbb{Z}/l[\theta]\) over \(\mathbb{R}\), we can also apply the universal coefficient theorem for the MASS. Since the Tor-term vanishes, it follows that the \(E_2\)-term of the MASS is the classical ASS \(E_2\)-term (with weights dictated by the cobar representatives of the generators) tensored with the appropriate coefficient ring.

Let now \(k = \mathbb{C}\). The topological realization functor \(R : SH_\mathbb{C} \to SH_{\text{top}}\) induces a map of towers and hence a map of spectral sequences \(\Psi\) from the \(l\)-completed MANSS to the \(l\)-completed topological ANSS. Away from the bidegree \((0, 0)\) this is just the classical ANSS. Since \(\Psi\) is a map of spectral sequences and commutes with differentials, it follows that \(d_{r-1}(x) = 0 \implies d_{r-1}(\Psi(x)) = 0\). Similar to the case \(l = 2\) ([HKO, Lemma 16]) the converse is also true. As \(\tau\)-torsion vanishes under topological realization, it hence cannot support nontrivial differentials in the MANSS.

**Proposition 2.10.** Let \(q = 2l - 2\) as above and \(k = \mathbb{C}\).

1. In each bidegree \((s, t)\) the entry \(E^{s,t}_r\) of the \(E_r\)-page of the \(l\)-completed MANSS contains \(E^s_{r,\text{top}} \otimes \mathbb{Z}_l[\tau]\). Here the elements with the highest weight have weight \(\frac{r}{2}\).

2. For any \(x \in E_r\), we have \(d_r(\Psi(x)) = 0 \implies d_r(x) = 0\). In particular, if an element is \(\tau\)-torsion, it does not support a nontrivial differential.

3. For each \(E^{s,t}_\infty\) of the classical ANSS, there is a group \(E^{s,t}_\infty[\tau]\) in the \(E_\infty\)-page of the MANSS. The torsion submodule of the \(E_\infty\)-page is generated by the groups above.
Proof. In weight 0, topological realization induces an isomorphism already on the level of exact couples, so \( \Psi \) is an isomorphism of spectral sequences at weight 0 for all \( r \).

For the other claims we proceed by induction over \( r \). For \( r = 2 \) there is no torsion and \( \Psi \) is an isomorphism in weight 0, so all claims follow by the previous proposition. For the induction step, we can restrict to the case \( r = nq + 1 \) by sparsity. Assume all the claims are true for \( r = nq + 1 \), and we wish to show that they also hold for \( r + 1 \).

Every nontrivial differential \( d_r : E^{s,t}_{r,\text{top}} \to E^{s+r,t+r-1}_{r,\text{top}} \) in the ANSS lifts to a nontrivial differential \( d_r : E^{s,t}_{s,t+r} \to E^{s+r,t+r-1}_{s,t+r} \) in the MANSS. Generators in the target have weight \( \frac{2t+1}{2} \), where \( t+r-1 = t+nq \) is divisible by two since \( t \) and \( q \) are. Hence a \( \tau \) torsion group of order \( \frac{t-1}{2} \) appears at the \( E_{r+1} \) page of the MANSS.

Consider a nontrivial differential \( d_r : E^{s,t}_{s,t} \to E^{s+r,t+r-1}_{s,t} \) in the MANSS. By the induction assumption there can be no \( \tau \) torsion in the target at weight \( \frac{t}{2} \) anymore, and multiplication by \( \tau^2 \) is an isomorphism between weight \( \frac{t}{2} \) and weight 0. In weight 0, \( \Psi \) is an isomorphism of spectral sequences, so \( d_r(\Psi(x)) = 0 \implies d_r(x) = 0 \) follows. \( \square \)

Proposition 2.11. Consider \( k = \mathbb{R} \). The results of the previous proposition apply if we replace \( \tau \) with \( \theta \).

Proof. The map of MANSSs induced by \( \mathbb{R} \to \mathbb{C} \) is the inclusion of the terms in even weight on each page. \( \square \)

Let now again be \( k = \mathbb{C} \). It is known by a result of Morel \((\text{Mor})\) that \( \pi_{s',u}(S) = 0 \) if \( u > s' \), but it can also be deduced from the vanishing line of the ANSS:

Proposition 2.12. \( \pi_{s',u}(S_{\mathbb{C}}) = 0 \) if \( u > s' \).

Proof. If we index the MANSS according to the usual Adams grading \((s, s') := (s, t-s)\), the column \( s' \) converges against \( \pi_{s,s}(S_{\mathbb{C}}) \). Recall the vanishing line on the \( E_2 \) page of the MANSS: \( E_{2,t,u}^{s,t,u} = 0 \) if \( t < 2s \) or equivalently \( s' < s \). The weight of a generator in \( E_{2,s,s'}^{s,s} \) is \( \frac{s'}{2} \), and the weight of arbitrary elements is lesser or equal. As the weight grows with \( s \), for fixed \( s' \) the weight of a nonzero element in the column \( s' \) can be at most \( s' \), if the element lies in the maximal nonzero filtration degree \( s = s' \).

Proof. \( \square \)

Proposition 2.13. \( \tau \in E_{2}^{0,0,-1} \) is a permanent cycle in the MANSS and defines an element \( \tau \in \pi_{0,1}(S_{\mathbb{C}}) \). On \( \pi_{s',u}(S_{\mathbb{C}}) \) multiplication by \( \tau \) is an isomorphism if \( u \leq \frac{s'}{2} + 1 \) and \( s' > 0 \), or if \( s' = 0 \) and \( u \leq 0 \). Since the groups \( \pi_{s',0}(S_{\mathbb{C}}) \) in weight 0 are known to be the (completed) classical groups by \((\text{Lev})\) this determines the motivic groups in the given range.

Proof. It is clear by its position in the MANSS that \( \tau \) is a permanent cycle.

At the \( E_2 \) page of the MANSS, the multiplication \( E_2^{s,s',u} \xrightarrow{\tau} E_2^{s,s',u-1} \) is obviously an isomorphism. Differentials are \( \tau \)-linear and do not map into \( \tau \)-torsion, so we know \( d_r(\tau x) = \tau y \implies d_r(x) = y \). In particular outgoing differentials lift to the highest weight \( u = \frac{s'}{2} \) at each fixed bidegree \((s, t)\), so multiplication by \( \tau \) can only fail to be an isomorphism at a later page if there is a nontrivial incoming differential.
Then the statement follows for \( s' = 0 \), so let \( s' > 0 \). If there is a nontrivial differential of this form, there is an \( x \in E_r^{s-r,s+1,u-1} \) such that \( dr(x) \neq 0 \) and \( \tau \not| x \). Then \( x \) has weight \( w = \frac{s' + 1 + s - r}{2} \). Since the line \( s = 0 \) is concentrated at \( s' = 0 \), the lowest possible weight \( x \) can assume is the weight \( w = \frac{s' + 2}{2} \) if the differential maps out of the 1-line (\( s - r = 1 \)).

**Example 2.14.** 1. The first nontrivial differential in the classical ANSS is \( d_{2-1}(\beta_{l/1}) = a\alpha_1\beta_1^l \), cf. [Rav, Thm.4.4.22], where \( a \) is an unknown constant. If we index the MANSS in the classical Adams grading \( s, t - s, u \), these elements live in \( \alpha_1 \in E_2^{2,l-3,l-1} ; \beta_1 \in E_2^{2,2l^2-2l-2,l^2-l}, \) and \( \beta_{l/1} \in E_2^{2,2l^3-2l^2-2l^3-l} \). This gives us an example of a \( \tau \)-torsion group of order \( l - 1 \) in the stem \( t = s = 2l^3 - 2l^2 - 3 \). For \( l = 3 \) this yields a permanent cycle in the MANSS in degree \( s = 7, t - s = 33, u = 20 \) that is annihilated by \( \tau^2 \).

2. Consider the classical ANSS and the elements representing the Hopf maps. The motivic counterpart of the Hopf maps have been defined in [DI3]. \( l = 3 \), the element \( \alpha_1 \) lives in \( s = 1, t - s = 3 \) and is not involved in any differentials. It represents the Hopf map \( \nu \). Its motivic counterpart has the same coordinates and weight 2. Similarly, the element \( \alpha_2 \) has coordinates \( s = 1, t - s = 7 \). It is a permanent cycle and represents the Hopf map \( \sigma \). Its motivic counterpart has weight 4.

   For \( l = 5 \) the element \( \alpha_1 \) in the degree \( s = 1, t - s = 7 \) similarly is a permanent cycle, representing the Hopf map \( \sigma \). Its motivic counterpart also has weight 4.

   Since at odd primes, products of the \( \alpha \)-elements all vanish in the classic ANSS, this is true motivically as well. Hence all three elements are nilpotent in the MANSS.

3. Any classical differential of the form \( Z/l^2 \xrightarrow{\partial} Z/l^2 \) in the ANSS would imply an entry of the form \( Z/l[\tau]/(l\tau^n) \) in the MANSS. However, I do not know an example of such a differential.

## 3 The relation of MANSS and MASS

Assume \( k \in \{C, R\} \) and \( l \neq 2 \). Define \( P_{\ast \ast} \) as the polynomial sub-Hopf algebra of the dual motivic Steenrod Algebra and \( E_{\ast \ast} \) as the exterior part, i.e.

\[
P_{\ast \ast} = H\mathbb{Z}/l_{\ast \ast}[\xi_1, \xi_2, ...]
\]

\[
E_{\ast \ast} = H\mathbb{Z}/l_{\ast \ast}[\tau_0, \tau_1, \tau_2, ...]/(\tau_1^2 = 0)
\]

There is a square of spectral sequences relating the \( l \)-complete ANSS and the ASS, discussed classically in [Rav, Theorem 4.4.3, Theorem 4.4.4] and motivically in [AM].
Here the Cartan-Eilenberg spectral sequence, short CESS is induced by the Hopf algebra extension $P \to A \to E$ and the algebraic Novikov spectral sequence, short AANNS, is induced by a filtration of the cobar complex by an invariant ideal.

One can write down the square for not completed $BP$ as well if one does not care about the exact filtration on $\pi^{*\ast}(S)$.

In topology (and motivically over $\mathbb{C}$ as well) at the prime $l = 2$, computations in the ANSS and the ASS supplement each other. In contrast to this, at odd primes the ANSS is sparser than the ASS. More precisely, the CESS degenerates at the $E_1$-page, so the reindexed $E_2$ term of the ASS is the $E_1$ term of a spectral sequence converging to the $E_2$ term of the ANSS. This degeneration of the CESS at odd primes is true motivically as well over $\mathbb{C}$ and $\mathbb{R}$, i.e. we have a a square of convergent spectral sequences:

Furthermore, the CESS degenerates at the $E_1$-page.

### 3.1 The motivic Cartan Eilenberg spectral sequence

**Definition 3.1.** If we tensor the classical extension of Hopf algebras $P \to A \to E$ with $HZ/l_{ss}$, which by our computation above is just a polynomial ring in one generator, we obtain an extension of Hopf algebras $P^{*\ast} \to A^{*\ast} \to E^{*\ast}$ in the sense of [Rav, A1.1.15].

Associated to such an extension we immediately get a motivic counterpart to [Rav, Theorem 4.4.3]:

**Proposition 3.2.**
1. $\text{Cotor}_{E^{*\ast}}(HZ/l_{ss}, HZ/l_{ss}) = P(a_0, a_1, \ldots)$, where $P(\ldots)$ refers to the polynomial algebra over the ground ring $HZ/l_{ss}$ and $a_i \in \text{Cotor}_{E^{*\ast}}^{1,2l_i-1,1,l_i-1}$ is represented by the cobar cycle $[\tau_i]$.

2. There is a motivic Cartan Eilenberg spectral sequence converging to $\text{Cotor}_{A^{*\ast}}(HZ/l_{ss}, HZ/l_{ss})$ (the reindexed $E_2$-term of the motivic Adams spectral sequence) with the following $E_2$-page:

$$E_2^{s_1,s_2,t,u} = \text{Cotor}_{P^{*\ast}}^{s_1,t,u}(HZ/l_{ss}, \text{Cotor}_{E^{*\ast}}^{s_2,t,u}(HZ/l_{ss}, HZ/l_{ss}))$$

and differential $d_r : E_r^{s_1,s_2,t,u} \to E_r^{s_1+r,s_2-r+1,t,u}$. 

3.1 The motivic Cartan Eilenberg spectral sequence
3. The $P_{**}$-coaction on $\text{Cotor}_{E_{**}}(H_{**},H_{**})$ is given by
$$\psi(a_n) = \sum_i \xi_n^i \otimes a_i$$

4. The motivic Cartan Eilenberg SS collapses from the $E_2$ page with no nontrivial extensions.

3.2 The motivic algebraic Novikov spectral sequence

**Remark 3.3.** The ideal $I = (l,v_1,v_2,\ldots) \subset \text{ABP}_{l,**}^\wedge = \mathbb{Z}[[\tau,v_1,v_2,\ldots]]$ defines a decreasing filtration by its powers on $\text{ABP}_{l,**}^\wedge$. The associated graded of $\text{ABP}_{l,**}^\wedge$ is $E_0\text{ABP}_{l,**}^\wedge \cong \mathbb{Z}/l[q_0,q_1,q_2,\ldots]$. Here $q_0$ is the class of $l$ and $q_i$ is the class of $v_i$ for $i \geq 1$, so the grading of $q_i$ is $(1,2(l^i - 1))$.

One can check by the same arguments as for $I = (l,v_1,v_2,\ldots) \subset \text{BP}_*^\wedge$ that $I$ is an invariant ideal in regard to the structure maps of the Hopf Algebroid $\text{ABP}_{l,**}^\wedge(\text{ABP}_{l,**}^\wedge)$. Since $I$ is invariant, the filtration on the Hopf algebroid defined by its powers induces a Hopf algebroid structure on the associated graded, and the induced filtration on comodules induces a comodule structure on the associated graded of the comodules. In particular we obtain a filtration of the cobar complex as a differential graded comodule.

**Proposition 3.4.** There is a spectral sequence called the motivic algebraic Novikov spectral sequence converging to $\text{Cotor}_{\text{ABP}_{l,**}^\wedge}(\text{ABP}_{l,**}^\wedge,\text{ABP}_{l,**}^\wedge)$. The $E_1$-page is given by $E_1 = \text{Cotor}_{P_**}(HZ/l_{**},E_0\text{ABP}_{l,**}^\wedge) = \text{Cotor}_{P_**}(HZ/l_{**},\text{Cotor}_{E_{**}^\wedge}(HZ/l_{**},HZ/l_{**})).$

**Proof.** This works as the classical proof (See [Rav], 4.4.4). □

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