On the structure of regular $B_2$-type crystals

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1 Introduction

For simply-laced Kac-Moody algebras $\mathfrak{g}$, Stembridge [8] proposed a ‘local’ axiomatization of crystal graphs of representations of $U_q(\mathfrak{g})$. In fact, because of an important result in [3] an essential part in studying such crystals should have been carried out for the simplest nontrivial case $\mathfrak{g} = \mathfrak{sl}(3)$. Our paper [1] gives a combinatorial construction that describes the formation of any crystal of representations of $U_q(\mathfrak{sl}(3))$, a regular crystal graph of type $A_2$.

In this paper we attempt to carry out a similar programme for the algebra $\mathfrak{sp}(4)$. We follow the ideology of [1] and propose axioms of monotonicity and commutativity for edge-2-colored graphs which characterize the crystals of integrable representations of $U_q(\mathfrak{sp}(4))$, regular crystal graphs of $B_2$-type. A directed edge-colored graph which obeys our Axioms (K0)-(K5) is called an $R$-graph (for brevity), and our main result is that the regular crystals of $B_2$-type are $R$-graphs and vice versa. In particular, axiom (K5) refines Sternberg’s $B_2$-type relations by involving a certain labeling on the crystal edges. At the end of [8], Stembridge conjectured a list of relations between crystal operations for the $B_2$ case and Sternberg proved this conjecture in [7].

Further, we give a direct combinatorial construction for the crystals in question, using a general operation on graphs from [1]. On this way we introduce a new, so-called crossing model, which does not exploit Young tableaux. This combinatorial model consists of a two-component graph of a rather simple form and of a certain set of integer-valued functions on its vertices, and we show that these functions one-to-one correspond to the vertices of a regular $B_2$-crystal. In particular, due to this model, the vertices of a regular $B_2$-crystal can be located at the integer points of the union of
five 4-dimensional polyhedra in $\mathbb{R}^7$, though this union is not a convex set. (Another crossing model is developed for regular $A_n$-crystals in [2].)

The structure of this paper is the following. Section 2 is devoted to axioms. In Section 3 we formulate the main result and give a constructive characterization of the regular $B_2$-crystal. In Section 4 we introduce the crossing model for such crystals. In Section 5 we prove that certain intervals of the admissible configurations on this model are just regular $B_2$-crystals, relying on the known characterization of the latter via Littelmann’s cones. In Section 6 we prove that the graphs generated by the intervals on the crossing model are essentially the same as those constructed in Section 3.

Acknowledgements. The authors gratefully acknowledge the support of the grant 047.011.2004.017 from NWO and RFBR, and the grant 05-01-02805 from CNRSLa from CNRS and RFBR. G. Kochevoy gratefully acknowledges the support of Support Foundation of the Russian Federation and IHES for hospitality and perfect working conditions. Part of this research was done while A. Karzanov was visiting PNA1 at CWI, Amsterdam and was supported by a grant from this Center.

2 Axiomatization of $B_2$-type regular crystals

Here we consider edge-two-colored directed graphs managed by the $B_2$-type Cartan matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 2 \end{pmatrix}$. Of our interest will be those graphs that match the list of axioms below.

We consider a digraph $G = (V, A_1 \coprod A_2)$ with the vertex set $V$ and the edges set partitioned into two subsets $A_1$ and $A_2$. For convenience we refer to the edges in $A_1$ as being colored in color 1, say red, and the edges in $A_2$ as being colored in color 2, say green.

The first axiom concerns the structure of the monochromatic graphs $(V, A_1)$ and $(V, A_2)$, and states that they are constituted as the disjoint union of finite monochromatic strings. Specifically,

(K0) For $i = 1, 2$, each (weakly) connected component of $(V, A_i)$ is a finite simple (directed) path, i.e., a sequence of the form $(v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are distinct vertices and each $e_i$ is an edge going from $v_{i-1}$ to $v_i$.

In particular, each vertex $v$ has at most one outgoing 1-colored edge and at most one incoming 1-colored edge, and similarly for 2-colored edges. For brevity, we refer to a maximal monochromatic path in $K$, with color $i$ on the edges, as an $i$-string. The $i$-string passing through a given vertex $v$ (possibly
consisting of the only vertex \( v \) is denoted by \( P_i(v) \), its part from the first vertex to \( v \) by \( P_{i}^{\text{in}}(v) \), and its part from \( v \) to the last vertex by \( P_{i}^{\text{out}}(v) \). The lengths of \( P_{i}^{\text{in}}(v) \) and of \( P_{i}^{\text{out}}(v) \) (i.e., the numbers of edges in these paths) are denoted by \( t_i(v) \) and \( h_i(v) \), respectively. The next axiom tells how these lengths can change when one traverses an edge of the other color.

Consider a red edge \((x, y)\) connecting a vertex \( x \) to a vertex \( y \) (using the operator notation for Kashiwara crystals, one can write \( y = F_1 x \)), and consider the strings \( P_2(x) \) and \( P_2(y) \). For a 2-colored edge \((p, q)\) going from \( p \) to \( q \), \( q = F_2 p \), we consider the pair of strings \( P_1(p) \) and \( P_1(q) \).

(K1) a) If \( y = F_1(x) \), then \( t_2(x) - t_2(y) + h_2(y) - h_2(x) = 1 \).

b) If \( q = F_2(p) \), then \( t_1(p) - t_1(q) + h_1(q) - h_1(p) = 2 \).

Let us define the weight function \( wt : V \to \mathbb{Z}^2 \) by the rule

\[
w : x \to (h_1(x) - t_1(x), h_2(x) - t_2(x)).
\]

Then, the functions \( t_i, h_i, i = 1, 2 \), and \( wt \) define a semi-normal crystal of \( B_2 \)-type due to Kashiwara [5].

The next axioms is due to Stembridge [8].

(K2) For each 1-colored edge \( e \) going from \( x \) to \( y \), \( y = F_1(x) \), there hold \( t_2(x) - t_2(y) \geq 0 \) and \( h_2(y) - h_2(x) \geq 0 \).

For each 2-colored edge \( e' \) going from \( p \) to \( q \), \( q = F_2 p \), and there hold \( t_1(p) - t_1(q) \geq 0 \) and \( h_1(q) - h_1(p) \geq 0 \).

By this reason, we endow the edges of \( G \) with labels, and for this reason, we say that \( G \) is a decorated graph. More precisely, the red edges have labels 0 or 1 and the label being assigned to a red edge \( e = (x, y = F_1(x)) \) is equal to \( h_2(y) - h_2(x) \). The green edges have labels 0 or \( \frac{1}{2} \) or 1 and the label assigned to a green edge \( e' = (p, q) \) is equal to \( \frac{h_1(q) - h_1(p)}{2} \in \{0, \frac{1}{2}, 1\} \).

The following axioms were found experimentally and their justification is done by Theorem 1 in which we assert that regular crystal graphs of \( B_2 \)-type characterized by the whole list of axioms.

The third axiom is

(K3) For any \( i \) and any \( i \)-string \( P_i \), the labels on consecutive edges \( e_1 \) and \( e_2 \) (where \( e_2 \) follows \( e_1 \)) do not decrease.

The next axiom presents important commutation between special pairs of green and red edges.
(K4) Suppose a red edge enters a vertex $v$ and has label $a$ and let a green edge leave $v$ and have label $b$. Then there holds $b \neq a$, moreover, if $a < b$, then $a = 0$, $b = 1$ and the commutative diagram takes place:

Similarly, suppose a green edge enters a vertex $q$ and has label $b$ and a red edge leaves $q$ and has label $a$. Then there holds $b \neq a$, and if $a > b$, then $a = 1$, $b = 0$, and the commutative diagram takes place:

From these axioms we have the following property of graphs which satisfy axioms $K1 - K4$:

A green edge labeled 1 is at the top of the corresponding commutative square in Axiom 4; a green edge labeled 0 is at the bottom the corresponding commutative square in Axiom 4. A green edge labeled $\frac{1}{2}$ is always a part of the crystal graph of a fundamental representation of $U_q(sp(4))$.

In fact, because of axiom (K2), there is an ingoing red edge to the starting point of a green edge labeled $\frac{1}{2}$, and there is an outgoing red edges from the ending point of the green edge. Due to axiom K4, the ingoing red edge can not be labeled 0 and the outgoing can not be labeled 1. Thus they are labeled 1 and 0 respectively.

**Corollary.** In an $S$-graph any green string has at most one edge labeled $\frac{1}{2}$.

**Proof.** Suppose two green edges, both labeled $\frac{1}{2}$, belong to a green string. Then, because of Axiom K3, these edges have a common vertex. Then due
to above property of green edges labeled \( \frac{1}{2} \), the ingoing red edge to this vertex is labeled 1 and the outgoing edge is labeled 0. That contradicts to the monotonicity Axiom K3.

Because to Axiom K3 each red string has a unique vertex (possibly the beginning or ending one) at which labels switch from 0 to 1, we call this vertex critical. Because of Axiom 3 and Corollary each green string has either a critical vertex or a critical edge labeled \( \frac{1}{2} \).

Because of the above property of S-graphs, of interest are the green edges labeled 1 which are not bottom edges of commutative squares (of course, each such an edge is at the top a commutative square) and the green edges labeled 0 which are not top edges of the corresponding commutative squares. Possible configurations ”above” and ”below” such edges, respectively, are managed by the following axiom.

(K5) The implications illustrated on Pictures 1, 2 and dual to these implications have to hold.

On Pictures 1 and 2 we demonstrate four possible configurations which involve a green edge labeled 1, which does not located at the bottom of a commutative square. Possible configurations involving a green edge labeled 0, which does not located at the top of a commutative square, are obtained dually to that indicated at Pictures 1 and 2, that is one has to reverse directions of edges and replace a label \( a \) to \( 1 - a \).
Picture 1.
Remark. The relation on Picture 2 is the Verma $B_2$-type relation of degree 7 (if we ignore labels), though depicted in the form somewhat different from that of the custom Verma $B_2$-type relation. This form of the Verma relation can be interpreted as commuting of crystals of the fundamental representations of $U_q(sp(4))$, the paths between $A$ and $B$, and $B$ and $C$, and the paths between $A$ and $D$, and $D$ and $C$, respectively.

The custom Verma relation is obtained if one draws two commutative diagrams which follow from Axiom K4.

If the labels are ignored, on Pictures 1a) and 1b) one gets the degree 4 relations due to Sternberg [7]. The relation on Picture 1c) is of degree 4, but due to this relation we have to go twice against the orientation (this relation was indicated by [8]). The relations 1b) and 1c) imply the relation on Picture 3, the degree 5 relation due to Sternberg [7] (if the labels are ignored). One can easy check that the degree 5 relation and the relation 1b) imply the relation 1c). (The dual degree 5 relation is depicted at Picture 4.)
In that follows it will be of use the following two implications for $B_2$-type Verma relations depicted on Picture 2a, which follows from implications on Pictures 2, 3 and 4.

Let us prove that the bottom implication, that is the graph $ABCDEF G$ implies the $B_2$-type Verma relation, and we leave to the reader to prove the upper implication. Because the edge $CD$ is labeled 0, there is a green edge emanating from $C$ (because of Axiom K3, this edge labelled 1). This implies that there is a green edge emanating from $A$. Moreover, this edge labeled 1. In fact if the label would be 0, then, because of Axiom K4, there would be a commutative diagram, and hence the label of the edge $BC$ has to be 0, that is not the case. If we assume that there is the 5 term relation as on the Picture 3, then the label on the edge $FG$ has to be 0, that is not the case. Hence the $B_2$-type Verma relation takes place for this graph.

**Definition.** A edge-2-colored graph for which Axioms (K0)-(K5) hold true is called an *R-graph* of $B_2$-type.
Our aim is to prove the following

**Theorem 1.** A connected $R$-graph of $B_2$-type is a crystal graph of an integrable irreducible representation of $U_q(sp(4))$, that is a regular $B_2$-type crystal, and vice versa.

### 3 Construction of $R$-graphs of $B_2$-type

Here we give a direct construction of such graphs. Specifically, we give a direct constructions of the 3-dimensional "view from the sky" on a 4-dimensional $R$-graph $G$.

**Definition.** Given an $R$-graph $G$, we define another labeled digraph $\hat{G} = (\hat{V}, \hat{E}, (X, Y) : \hat{V} \to \mathbb{Z}_4)$ which we call the *view from the sky* (along red strings) on $G$:  

- the set of vertices, $\hat{V}$, of $\hat{G}$ is constituted from red strings of $G$, that is a whole red string in the graph $G$ becomes a vertex in $\hat{G}$; to each vertex $\hat{v}$ is attributed a label being a pair of numbers $(X(\hat{v}), Y(\hat{v})) = (t_1(\hat{v}), h_1(\hat{v}))$, where we let $\ast$ to denote the critical point on the red string $P_1$ corresponding to $\hat{v}$;
- the set of edges, $\hat{E}$, is formed by the following rule: we join vertices $\hat{v}$ and $\hat{v}'$ by an edge going from $\hat{v}$ to $\hat{v}'$ and put on it a label either 1, or $\frac{1}{2}$, or 0, if there exists a green edge labeled 1, or $\frac{1}{2}$, or 0, respectively, which joins a pair of vertices (in $G$) located on the red strings which correspond to $\hat{v}$ and $\hat{v}'$, respectively.

**Lemma 1.** Let $G$ be an $R$-graph. Then any vertex $\hat{v}$ of $\hat{G}$ has at most three ingoing edges and at most three outgoing edges; the ingoing edges have different labels and the outgoing edges have different labels; there are no parallel edges in $\hat{G}$.

For a proof of this Lemma we use the following properties, which follow from the "local commutative diagrams" illustrated on Pictures 1 and 2 (and dual to them) and the commutative squares figured in Axiom K4.

(*) If a vertex $v$ is located at least two red edges above a critical vertex on $P_1(v)$ or below the critical vertex on $P_1(v)$, that is $t_1(v) \geq X + 2$ or $t_1(v) \leq X - 2$, respectively. Then a pair of the red and green edges ingoing in $v$ form a commutative square, and a pair of the red and green edges outgoing from $v$ form a commutative square.

If a vertex $v$ is such that $t_1(v) = X + 1$ (one edge above the critical), then the pair of ingoing red and green edges form a commutative square.
If a vertex $v$ is such that $t_1(v) = X - 1$ (one edge below the critical), then the pair of outgoing red and green edges form a commutative square.

(**) For each red string $P$, there can be at most two green strings having critical edge $e$ (i.e., labeled $\frac{1}{2}$) such that $e$ enters or leaves a vertex in $P$, and if this is the case, then the picture is as follows

![Diagram](image)

That is, either $e$ enters $P$ one red edge before the critical point or leaves $P$ one red edge after this point.

**Proof of Lemma 1.** Due to the properties (*) and (**), in an R-graph, any red string can be joined with at most six other red strings through ingoing and outgoing green edges. The claim that the labels are different and that there are no parallel edges also follow from these properties. Q.E.D.

For each R-graph $G$, it will be useful to color edges in the “view from the sky” graph $\tilde{G}$ in two colors, green and blue. The edges labeled $\frac{1}{2}$ remain green. An edge $\tilde{e}$ labeled 0 (or 1) is colored in blue if $|X(\tilde{v}) - X(\tilde{v}')| = 1$ holds, and $\tilde{e}$ remains green if either $|X(\tilde{v}) - X(\tilde{v}')| = 2$ or $|X(\tilde{v}) - X(\tilde{v}')| = 0$ hold.

On the following pictures we illustrate ”views from the sky” along a ‘typical’ red strings in an R-graph.
Picture 6a.

Picture 6b.
3.1 The ”view from the sky” on relations in Axiom K5

On the next pictures we illustrate the “view from the sky” on the relations in R-graphs of $B_2$-type.

Picture 6c.

Picture 7a
On Picture 7a we illustrated the relation on Picture 1a) and the dual relation. Let us note that from Axiom K5 also follows the following relation: two consequent blue edges labeled 0 and 1, respectively, the roof of the right hand side, also imply the right-hand side relation on Picture 7a.

On the next picture we present the "view from the sky" on the relation on Picture 1b) and its dual (the dual is depicted at the north-west corner of Picture 7b).

![Diagram](image)

**Picture 7b**

On the next picture we illustrated the "view from the sky" on the relation at Picture 1c) and its dual.
And finally, the view from the sky on Pictures 2 and 2a is presented as follows: each of the left-hand side combinations of the green and blue edges implies the Verma relation, that takes the form of commuting "views from the sky" on the crystals of the fundamental integrable modules of $U_q(sp(4))$. 
3.2 Graphs $K(H, A)$

The above-mentioned graph $K(H, 0)$ is an R-graph of $B_2$-type which has a red string of the form $(0, H)$ (with $H$ red edges labeled 1 and no edges labeled 0) outgoing from the minimal vertex and no green edge outgoing from this vertex. These graphs have the “view from the sky” (depending on the parity of $H$) for $H = 1, 2, 3, 4$ as depicted below. One can realize how to construct such a view for an arbitrary $H$: the building blocks are the graph $\hat{K}(2, 0)$ (the view from the sky on the right-hand side relation from Picture 1b (see Picture 7b) and two rhombuses from the graph $\hat{K}(3, 0)$, these rhombuses are view from the sky the relations from Picture 3 and 4 (see Pictures 7c). The rule for travelling along the red edges follows from the commutativity Axiom K3. We define the set of distinguished vertices in $\hat{K}(H, 0)$ to consist of the vertices at the ground floor, or ‘zero etage’, in the corresponding triangles and trapezoids; by an analogy with $\square$ we call this set the diagonal. Note that the red paths at the diagonal are of the form $(l, H - l)$, $l = 0, \ldots, H$, and the red paths at the etage $k$ are of the form $(l, H - 2k - l)$, $l = 0, \ldots, H - 2k$. One can see that $\hat{K}(H, 0)$ has no blue edges.

Picture 7d.

Picture 8.
In its turn, the R-graph of $B_2$-type $K(0, A)$ is an R-graph which has no red edge outgoing from the minimal vertex, and the green string beginning at it consists of $A$ edges labeled 1. The graph $\hat{K}(0, A)$ has no green edges. Therefore the structure of such a graph is forced by the $A_2$-type Verma relation presented at Picture 7a. This graph has the form of a triangular grid, and we depict two examples with $A = 1$ and $A = 2$ in Picture 9. The distinguished vertex subset in the graph $\hat{K}(0, A)$, the diagonal, is defined to be constituted by the unique vertices of the degenerated red paths from the top etage.

Now we are going to give a combinatorial characterization of the "view from the sky" on an R-graph. A significance of this characterization is due to that any R-graph is determined by its "view from the sky".

The following operation introduced in [1] will be of use.

Consider arbitrary graphs or digraphs or labeled digraphs $G = (V, E)$ and $H = (V', E')$. Let $S$ be a distinguished subset of vertices of $G$, and $T$ a distinguished subset of vertices of $H$. Take $|T|$ disjoint copies of $G$, denoted as $G_t$ ($t \in T$), and $|S|$ disjoint copies of $H$, denoted as $H_s$ ($s \in S$). We glue these copies together in the following way: for each $s \in S$ and each $t \in T$, the vertex $s$ in $G_t$ is identified with the vertex $t$ in $H_s$. The resulting graph consisting of $|V||T| + |V'||S| - |S||T|$ vertices and $|E||T| + |E'||S|$ edges is denoted by $(G, S) \bowtie (H, T)$.

In our case the role of $G$ and $H$ is played by 2-colored digraphs $\hat{K}(H, 0)$ and $\hat{K}(0, A)$ depending on parameters $H, A \in \mathbb{Z}_+$ (it will be clear later that these digraphs are the "views from the sky" on the crystals of irreducible
representations of $U_q(sp(4))$ with the highest weight $H\lambda_1$ and $A\lambda_2$, respectively, where $\lambda_1$ and $\lambda_2$ denote the fundamental weights for the $B_2$ case).

Specifically, let us consider a labeled digraph $\hat{K}(H, A) = \hat{K}(H, 0) \bowtie \hat{K}(0, A)$ (the distinguished subsets in the graphs $\hat{K}(H, 0)$ and $\hat{K}(0, A)$ are the diagonals as described above). Then we set labels at the vertices of $\hat{K}(H, A)$ by the following rule: if a vertex $\hat{v}$ belongs to a copy of $\hat{K}(H, 0)$, then we set $(\tilde{X}(\hat{v}), \tilde{Y}(\hat{v})) = (X(\hat{v}), Y(\hat{v}))$, where $(X(\hat{v}), Y(\hat{v}))$ is the label on $\hat{v}$ in $\hat{K}(H, 0)$; if $\hat{v}$ belongs to a copy of $\hat{K}(0, A)$ being attached to the $l$-th distinguished point of $\hat{K}(H, 0)$, then $(\tilde{X}(\hat{v}), \tilde{Y}(\hat{v})) = (X(\hat{v}) + l, Y(\hat{v}) + H - l)$, where $(X(\hat{v}), Y(\hat{v}))$ is the label on $\hat{v}$ in $\hat{K}(0, A)$.

**Remark.** If we regard $\hat{K}(H, A)$ as edge green-blue colored graph, and will follow the rule of changing of the labels attached to vertices along the edges in according to the color of the edge and the label attached to the edge, then the labels at the vertices are determined if we set the label $(0, H)$ to the source vertex of $\hat{K}(H, A)$.

**Definition.** We define $\hat{K}(H, A)$ to be the R-graph for which the ”view from the sky” is of the form $\hat{K}(H, A) = \hat{K}(H, 0) \bowtie \hat{K}(0, A)$ with the above defined labels at the vertices.

**Proposition 1.** Any connected R-graph of $B_2$-type takes the form $K(H, A)$ with $H$ and $A$ being the lengths of the red and green strings, respectively, emanating from the source vertex.

**Proof.** In the beginning, we place the list of forbidden configurations in graphs being the views from the sky on R-graphs on the next Picture.

![Picture](image-url)
The cases c) and d) easy follow since in these cases there have to be two outgoing (in c)) or ingoing (in d)) edges for the vertex in $G$ from which the corresponding green edge labeled $\frac{1}{2}$ emanates or terminates, that is not the case due to Axiom K0. The case a) follows from the $A_2$-type Verma relation depicted on Picture 1a. The case b) follows the relation depicted on Picture 1b.

Now let $G$ be an R-graph and let us consider a green edge labeled $\frac{1}{2}$. Let us consider the "view from the sky" on the graph and consider the corresponding edge. Then, because of the "view from the sky" on the relations on Pictures 7b) and 7c), we get that such an edge has to be an edge in the graph of the form $\tilde{K}(H,0)$, for some $H$, we call it a sail of the upper type. Now consider a vertex of the bottom etage of this sail. Then, because of the above list of the forbidden diagrams (Picture 10), it can be either an ingoing blue edge with the label 0 in such a vertex or an outgoing blue edge with the label 1, or the both blue edges, or none. In the latter case we are done. For other cases, because of the Verma relation and its "view from the sky" we can translate such a blue edge across all the bottom vertices of the upper sail and to translate the sail across such an edge. Now with the help of the $A_2$-type Verma relation (Picture 7a), we get that the bottom sail constituted from the blue edges attached to a bottom vertex takes the "view from the sky" of the form $\tilde{K}(0,A)$. The point is than any blue edge attached to a vertex from the below etage of $\tilde{K}(H,0)$, due to Verma $B_2$-type relation (its view from the sky) implies that a configuration of the form $\tilde{K}(0,1)$ is also attached and it contains such a blue edge. Then applying $A_2$-type Verma relation, we get the whole bottom sail. Thus, any connected R-graph $G$ has the "view from the sky" of the form $\tilde{G} = \tilde{K}(H,0) \bowtie \tilde{K}(0,A)$. It is clear that $H$ and $A$ are the lengths of the red and green strings passing through the minimal vertex of $G$, and the source vertex has the label $(0,H)$.

Vice versa, given a graph $\tilde{G} = \tilde{K}(H,0) \bowtie \tilde{K}(0,A)$ we reconstruct $G$ using the Pictures 6a, 6b, 6c.

The rest of the paper is devoted to a proof that the graph $K(H,A)$ is the crystal graph of the irreducible representation of $U_q(sp(4))$ with the highest weight $H \lambda_1 + A \lambda_2$. To prove this, we will use a new model related to $B_2$-type crystals.

4 Crossings model for $B_2$-type crystals

Here we present a model of the free regular crystal $K^\infty$ of $B_2$-type. That is an edge-2-colored graph with infinite monochromatic strings passing through each vertex, and we will show that each $B_2$-type crystal for irreducible representation can be obtained as an interval of $K^\infty$.

The vertices of $K^\infty$ are the functions on the vertex-set the diagram
depicted on Picture 11, for which the inequalities indicated by arrows are imposed, namely: \( f(a) \geq f(b) \geq f(c) \) and \( f(x) \geq f(y) \geq f(z) \geq f(w) \). Also it is required that the following parity conditions hold: \( f(y) + f(z) \in 2\mathbb{Z} \) and \( f(a), f(c) \in 2\mathbb{Z} \), and that at least three of the above inequalities turn into equalities. The possible combinations of equalities are given in Picture 12. (One may consider an equivalent model, with odd \( f(a), f(c) \in 2\mathbb{Z} + 1 \) and half-integer weights.)

Such functions are called \textit{admissible configurations}.

\begin{center}
\includegraphics[width=0.5\textwidth]{picture11}
\end{center}

Picture 11.

\begin{center}
\includegraphics[width=0.9\textwidth]{picture12}
\end{center}

Picture 12.
Now we define the structure of a crystal on this set of admissible configurations $f$ by explaining to which admissible configuration $f$ goes under the action of the operations that we denote by $F_1$ and $F_2$ as before. That is, the red edges of the crystal take the form $(f, F_1 f)$ and the green edges take the form $(f, F_2 f)$.

The operation $F_1$ “moves” $f$ to an admissible configuration $f'$ which can differ from $f$ at exactly one of the points $x$, $b$, $w$. Specifically,

if $f(w) < f(z)$, then $F_1$ increases $f$ by one at the vertex $w$: $f'(w) = f(w) + 1$;

if $f(w) = f(z)$ and $f(b) < f(a)$, then $F_1$ increases $f$ by one at $b$: $f'(b) = f(b) + 1$;

and if $f(w) = f(z)$ and $f(b) = f(a)$, then $F_1$ increases $f$ by one at $x$: $f'(x) = f(x) + 1$.

In its turn, $f' = F_2 f$ differs from $f$ at one or two points among $a$, $y$, $z$, $c$, and $F_2$ either increases $f$ by 2 at one point, or increases $f$ by 1 at $y$ and at $z$. More precisely,

if $f(a) - f(b) < f(a) - f(c)$, then $f'(c) = f(c) + 2$;
if $f(a) - f(b) = f(a) - f(c)$ and $f(y) + 2 < f(x)$, then $f'(y) = y + 2$;

if $f(a) - f(b) > f(a) - f(c)$ and $f(y) + 1 = f(x)$, then $f'(y) = f(y) + 1$ and $f'(z) = f(z) + 1$;

and if $f(a) = f(b) > f(a) - f(c)$, $f(y) = f(x)$, and $f(z) < f(y)$, then $f'(z) = f(z) + 2$;

and if $f(a) = f(b) ≥ f(a) - f(c)$, $f(y) = f(x)$, and $f(z) = f(y)$, then $f'(a) = f(a) + 2$.

It is easy to see that these operations preserve the admissibility.

**Definition.** An admissible configuration is called a fat vertex if $f(a) = f(b) = f(c) ∈ 2\mathbb{Z}$ and $f(x) = f(y) = f(z) = f(w)$.

The subset $B(H, A) ⊂ K^∞$ of configurations which satisfy the restrictions $f(c) ≥ 0$, $f(w) ≥ 0$, $f(a) ≤ A$, $f(x) ≤ H$ is called the interval of weight $A/2\lambda_2 + H\lambda_1$ ($A ∈ 2\mathbb{Z}$).

**Remark.** Note that the interval ‘join’ the fat vertex $0$ and the fat vertex $f(a) = f(b) = f(c) = A$ and $f(x) = f(y) = f(z) = f(w) = H$, and the whole rectangle of the fat vertices $0 ≤ f(a) = f(b) = f(c) ≤ A$ and $0 ≤ f(x) = f(y) = f(z) = f(w) ≤ H$ belong to this interval. Also note that the interval joining a fat vertex $f(a) = f(b) = f(c) = A'$ and $f(x) = f(y) = f(z) = f(w) = H'$ and the fat vertex $f(a) = f(b) = f(c) = A + A'$ and $f(x) = f(y) = f(z) = f(w) = A + H'$, $A ∈ 2\mathbb{Z}$, $H, H' ∈ \mathbb{Z}$, $A, H ≥ 0$, is isomorphic to the interval of weight $A/2\lambda_2 + H\lambda_1$.
We define the operations $F_1$ and $F_2$ on the interval $B(H,A)$ as they are defined in $K^\infty$ with the following modification at the final cases: if in the last case of the definition of $F_1$, one has $f(x) = H$, then we say that $F_1$ is not defined at $f$; and if in the last case of the definition of $F_2$, one has $f(a) = A$, then we say that $F_2$ is not defined at $f$. Accordingly, the reverse operation $F_2^{-1}$ does not act if it would result in the value on $c$ below zero, and similarly for $F_1^{-1}$ and $w$.

One can check that this model gives the inclusion

$$B(H,A) \supset B(H,0) \bowtie B(0,A),$$

where the distinguished subsets in $B(H,0)$ and $B(0,A)$ are constituted by the fat points.

In the next section we will prove, using Littelmann’s path model \cite{Littelmann}, that the interval $B(H,A) \subset K^\infty$ is a crystal graph of the integrable irreducible representation of $U_q(sp(4))$ with weight $\frac{A}{2}\lambda_2 + H\lambda_1$.

**Remark.** On the free regular crystal $K^\infty$, there is the following involution which reverse the direction of edges

$$* : K^\infty \to K^\infty,$$

$$*f(a) = -f(c), *f(b) = -f(b), *f(c) = -f(a), *f(x) = -f(w), *f(y) = -f(z), *f(z) = -f(y), *f(w) = -f(x).$$

One can see that $*f$ is an admissible configuration, moreover, if $f$ is the type $(a)$, then $*f$ is also of type $(a)$; if $f$ is of type $(bi), i = 1,2$, then $*f$ is of type $(b(3-i))$; and if $f$ is of type $(ci), i = 1,2$, then $*f$ is of type $(c(3-i))$. This involution sends each interval $B(H,A)$ to an interval $*B(H,A)$. The interval $*B(H,A)$ is isomorphic to the interval $B(H,A)$ (see Remark above), and the composition $B(H,A) \to *B(H,A) \cong B(H,A)$ is the Kashiwara involution \cite{Kashiwara} indeed.

## 5 Littelmann cones and admissible configurations

### 5.1 Canonical coordinates for configurations

It is easy to check that for any configuration $f \in B(H,A)$ (where $f(c) \geq 0, f(w) \geq 0, f(a) \leq A$ and $f(x) \leq H$), we can reach the sink (or the maximal vertex, that is, the fat vertex with $A = f(a) = f(b) = f(c)$ and $H = f(x) = f(y) = f(z) = f(w)$) by applying a word of the form $F_1^n F_2^m F_1^p F_2^q$ or $F_2^q F_1^p F_2^m F_1^r$ for appropriate $n,m,p,q$, and reach the source (or the minimal vertex, that is, the fat vertex with $0 = f(a) = f(b) = f(c)$ and $0 = f(x) = f(y) = f(z) = f(w)$) by applying a words of the same form but substituting $F_i$ on $E_i, i = 1,2$. These words are called canonical words for $f$,
and the corresponding \((n, m, p, q)\) and \((q', p', m', n')\) are called the canonical coordinates of \(f\).

We can explicitly calculate these coordinates for an admissible configuration \(f\). Indeed, there are 5 cases, depending on the type of the equalities diagram. The answers are as follows:

(a) Let \(\alpha = f(a), \beta = f(b), \gamma = f(c), \) and \(h = f(x) = f(y) = f(z) = f(w)\). Then the following canonical words move \(f\) to the sink. The first word can be of two types, relative to \(\beta\) and \(\frac{\alpha + \gamma}{2}\). Specifically, if \(f(a) - f(b) \geq f(b) - f(a)\), then the word is

\[
1H-h 2\frac{\alpha - \alpha}{2} + H-h 1H-h+A-\beta 2\frac{\gamma + \gamma}{2},
\]

that is \(q = \frac{\frac{\alpha - \alpha}{2}}{2}, p = H - h + A - \beta, m = \frac{\frac{\alpha + \gamma}{2}}{2} + H - h, n = H - h;\) and if \(f(a) - f(b) < f(b) - f(c)\), then the word is

\[
1H-h 2\frac{\alpha - \alpha}{2} + H-h 1H-h+A-\beta 2\frac{\alpha + \gamma}{2},
\]

that is \(q = \frac{\frac{\alpha - \alpha}{2}}{2} + \beta - \frac{\alpha + \gamma}{2}, p = H - h + A - \beta, m = \frac{\frac{\alpha - \alpha}{2}}{2} - \beta + H - h, n = H - h.\)

The second word is

\[
2\frac{\alpha - \alpha}{2} 1H-h+A-\alpha 2\frac{\alpha - \alpha}{2} + H-h 1\alpha-\beta+H-h,
\]

that is \(n' = \alpha - \beta + H - h, m' = \frac{\frac{\alpha + \gamma}{2}}{2} + H - h, q' = H - h + A - \alpha, p' = \frac{\frac{\alpha - \alpha}{2}}{2}.\)

(b1) Let \(\alpha = f(a) = f(b) = f(c), h = f(x), g = f(y), e = f(z) = f(w)\). Then the first and second words are (respectively)

\[
1H-h 2H-h+\frac{\alpha - \alpha}{2} 1h-e+A-\alpha+H-h 2h-\frac{\alpha + \gamma}{2} + \frac{\alpha - \alpha}{2} =
\]

\[
1H-h 2H-h+\frac{\alpha - \alpha}{2} 1H-e+A-\alpha 2h-\frac{\alpha + \gamma}{2} + \frac{\alpha - \alpha}{2};
\]

\[
2\frac{\alpha - \alpha}{2} 1H-e+A-\alpha 2H-\frac{\alpha + \gamma}{2} + \frac{\alpha - \alpha}{2} 1H-h.
\]

(b2) Let \(\alpha = f(a) = f(b) = f(c), h = f(x) = f(y), g = f(z), e = f(w)\). Then the words are

\[
1H-h 2H-h+\frac{\alpha - \alpha}{2} 1h-e+A-\alpha+H-h 2h-\frac{\alpha + \gamma}{2} + \frac{\alpha - \alpha}{2};
\]

\[
2\frac{\alpha - \alpha}{2} 1H-g+A-\alpha 2H-\frac{\alpha + \gamma}{2} + \frac{\alpha - \alpha}{2} 1g-e+H-h.
\]

(c1) Let \(\alpha = f(a), \beta = f(b) = f(c), h = f(x) = f(y) = f(z), e = f(w)\). Then the words are

\[
1H-h 2\frac{\alpha - \alpha}{2} + H-h 1h-e+A-\beta+H-h 2\frac{\alpha - \alpha}{2};
\]

\[
2\frac{\alpha - \alpha}{2} 1h-e+H-h 2\frac{\alpha - \alpha}{2}.
\]
\[ 2^{\frac{\alpha - \beta}{2}} H^{-h+\alpha - H - h} + \frac{(\alpha - \beta)\alpha}{2} + H - h + \frac{\alpha - \beta}{2} 1^{h-e+\alpha - \beta + H - h}. \]

(c2) Let \( \alpha = f(a) = f(b), \gamma = f(c), h = f(x), g = f(y) = f(z) = f(w). \) Then the words are
\[ 1^{H-h} 2^{\frac{\alpha - \beta}{2} + h-g + A-\alpha + H-h} 2^{\alpha - \beta + h-g + \frac{\alpha - \beta}{2}}; \]
\[ 2^{\frac{\alpha - \beta}{2} 1^{H-g+A-\alpha} 2^{\alpha - \beta + H-g + \frac{\alpha - \beta}{2}} 1^{H-h}}. \]

5.2 The cones of canonical words

According to [6], the canonical words of \( B_2 \)-type constituted the cones of the following form:
\[ C_1 = \{(n', m', p', q') : 2m' \geq p' \geq 2q' \} \text{ and } C_2 = \{(q, p, m, n) : p \geq m \geq n \}. \]

Let us check that the canonical coordinates of admissible configurations belong to these cones. We have to check gradually all 5 cases:

(a) \( C_1: \) \[ 2^{\frac{\alpha - \beta}{2} + H - h} \geq H - h + A - \alpha \geq 2^{\frac{\alpha - \beta}{2}}; \]
\( C_2: \) if \( 2 \beta \leq \alpha + \gamma, \) then there holds \( H - h + A - \beta \geq H - h + \frac{\alpha - \beta}{2} \geq H - h, \)
since \( 2 \beta \leq \alpha + \gamma. \) If \( 2 \beta > \alpha + \gamma, \) then there holds \( H - h + A - \beta \geq H - h + \frac{\alpha - \beta}{2} \geq H - h. \)

(b1) \( C_1: \) \[ 2H - g - e + A - \alpha \geq H - e + A - \alpha \geq 2^{\frac{\alpha - \beta}{2}}; \]
\( C_2: \) \[ H - e + A - \alpha \geq H - h + \frac{\alpha - \beta}{2} \geq H - h. \]

(b2) \( C_1: \) \[ 2H - g - h + A - \alpha \geq H - g + A - \alpha \geq 2^{\frac{\alpha - \beta}{2}}; \]
\( C_2: \) \[ H - e + A - \alpha \geq H - h + \frac{\alpha - \beta}{2} \geq H - h. \]

(c1) \( C_1: \) \[ 2(H - h) + A - \alpha + 2^{\frac{\alpha - \beta}{2}} \geq H - h + A - \alpha \geq A - \beta \geq [\frac{\alpha - \beta}{2}]; \]
\( C_2: \) \[ H - h + A - \beta + h - e \geq H - h + \frac{\alpha - \beta}{2} \geq H - h. \]

(c2) \( C_1: \) \[ 2(H - g) + A - \gamma \geq H - g + A - \alpha \geq A - \alpha; \]
\( C_2: \) \[ H - h + A - \beta + h - e \geq H - h + \frac{\alpha - \beta}{2} \geq H - h. \]

Thus, the coordinates of the admissible configurations belong to the cones. Moreover, one can see that configurations of \( B(H, A) \) cover the ‘intervals’ \( q \leq \frac{\alpha - \beta}{2}, p \leq A + H, m \leq H + \frac{\alpha - \beta}{2}, n \leq H \) and \( n' \leq H, m' \leq H + \frac{\alpha - \beta}{2}, \)
\( p' \leq A + H, q' \leq \frac{\alpha - \beta}{2}, \) respectively.
In order to get the equivalence of our model and the Littelmann path model, we have to check that the piece-wise linear transformations on the canonical coordinates of admissible configurations are the same as for the Littelmann cones. That is, we have to check the validity of the following relations:

\[ q = \max(q', p' - m', m' - n') \]
\[ p = \max(p', n' + 2p' - 2m', 2q' + n') \]
\[ m = \min(m', 2m' - p' + q', n' + q') \]
\[ n = \min(n', 2m' - p', p' - 2q') \]

Note that since \( n + p = n' + p' \) and \( m + q = m' + q' \), it suffices to check the first two equalities.

The most involved case to be checked is the case of the configurations with the equalities of type (a), and here are the calculations for this case (other cases are much easier and we leave them for the reader).

So, in the case of equalities of type (a):

If \( \beta \leq \frac{\alpha + \gamma}{2} \), then we have \( q = \frac{A - \alpha}{2} \), \( q' = \frac{A - \alpha}{2} \), \( p' - m' = A - \alpha - \frac{A - \gamma}{2} \), and \( m' - n' = \frac{A - \gamma}{2} - (\alpha - \beta) \). So we obtain \( q' \geq m' - n' \), whence \( q = q' \);

If \( \beta \leq \frac{\alpha + \gamma}{2} \), then \( q = \frac{A - \gamma}{2} - (\alpha - \beta) \), \( q' = \frac{A - \alpha}{2} \), \( p' - m' = A - \alpha - \frac{A - \gamma}{2} \), and \( m' - n' = \frac{A - \gamma}{2} - (\alpha - \beta) \). So we obtain \( q' \leq m' - n' \), whence \( q = m' - n' \).

Thus, the relation \( q = \max(q', p' - m', m' - n') \) is valid.

The equality \( p = \max(p', n' + 2p' - 2m', 2q' + n') \) holds, since in the case under consideration, we have \( p = n' + 2q' = \max(p', n' + 2p' - 2m', 2q' + n') \).

Thus, we can conclude with the following

**Theorem 2.** For any \( A \in 2\mathbb{Z}_+ \) and \( H \geq 0 \), the interval \( B(H, A) \) is the crystal graph of an integrable irreducible representation of the rank 2 algebra of \( B_2 \)-type and vice versa.

**Remark.** We want to stress the following aspect of the crossing model. From this model one can see that a crystal graph of an integrable representation of \( B_2 \)-type is located on a union of 5 polyhedra, and the latter set is not a polyhedron itself. There are two projections of this union of polyhedra to the Littelmann cones, and these cones have to be related via the specific piece-wise linear transformation. Thus, the crossing model captures the global non-convex structure of regular crystals, and the language of the Littelmann cones describes the projections of this non-convex structure.
In view of Theorem 2, Theorem 1 would follow from the equality $B(H, A) = K(H, A/2)$.

In the beginning, we establish the required bijection for the case $H = 0$ and for the case $A = 0$.

**Claim.** $B(H, 0) = K(H, 0)$.

**Proof.** Note that the admissible configurations in the interval $B(H, 0)$ have equalities of types (b1) or (b2).

Take a vertex $v$ in $K(H, 0)$. The corresponding admissible configuration is either of type (b1) or of type (b2). Consider the red-colored path passing through this vertex. Then this vertex is located either below the critical vertex (Case 1) or strictly above it (Case 2). Now consider the corresponding vertex for this path in the graph being the “view from the sky” for $K(H, 0)$. Let this vertex be located on the $k$-th etage and at $l$ steps to the right (that is, the corresponding red path is of the form $(l, H - 2k - l)$, $l$ edges with label 0 and $H - 2k - l$ with label 1).

**Case 1.** Suppose $l = 2t$. We consider the following route from the source to $v$: first we traverse $2(k + t)$ red edges (these edges have label 1), then $k + t$ green edges with label 0 and $t$ green edges with label 1, and finally, $l' \leq l = 2t$ red edges with label 0 until we reach the vertex $v$.

The configuration corresponding to $v$ is of type (b2), and it is represented as follows: $f(a) = f(b) = f(c) = 0$, $f(x) = 2(k + t)$ ($= \text{the number of red edges with label 1 on the route}$); $f(y) = 2(k + t)$ ($= \text{twice the number of green edges with label 0 on the route}$); $f(z) = 2t + 0$ ($= \text{twice the number of green edges with label 1 plus 0 edges with label } \frac{1}{2}$); and $f(w) = l' \leq 2t$ ($= \text{the number of final red edges with label 0}$). Conversely, by a configuration of (b2)-type with $f(z) \in 2\mathbb{Z}$, we reconstruct the vertex $v$ (in general case, lying in the sail $f(a) = f(b) = f(c) = \alpha$) as follows: $k = \frac{f(x) - f(z)}{2}$, $l = f(z)$, and $f(w)$ determines the position of $v$ in the red path $P_1(v)$ (that is $f(w)$ equals the number of edges to go from the beginning of the path $P_1(v)$ to $v$).

Now let $l = 2t + 1$. We start from the source and traverse one red edge with label 1, then one green edge with label $\frac{1}{2}$, then $2(k + t)$ red edges (all but one have label 1), then $(k + t)$ green edges with label 0 and $t$ green edges with label 1, and finally, $l' \leq 2t + 1$ red edges until we reach $v$.

The corresponding configuration is again of type (b2) and represented as: $f(a) = f(b) = f(c) = 0$; $f(x) = 1 + 2(k + t) - 1 = 2(k + t)$ ($= \text{the number}$.}
of red edges with the label 1); \( f(y) = 2(k + t) \) (= the number of green edges with label 0); \( f(z) = 2t + 1 \) (= twice the number of green edges with label 1 plus the number of green edges with label \( \frac{1}{2} \)); \( f(w) = l' \leq l = 2t + 1 \) (= the number of red edges with label 0 on the final part of the route). Conversely, a configuration of (b2)-type with \( f(z) \in 2Z + 1 \) reconstructs the vertex \( v \) as follows:

\[
k = \frac{f(x) - f(z) - 1}{2}, \quad l = f(z),
\]

and \( f(w) \) determines the position of \( v \) in the red path \( P_1(v) \).

Case 2. Suppose \( v \) is located on \( P_1(v) \) above the critical point, and \( l = 2t \). Then, starting from the source, we traverse \( 2(k - 1) + t \) red edges with label 1, then green \( k + t - 1 \) edges with label 0 and \( t \) green edges with label 1, then move along the red path until we reach the tail of the green edge with label 0 that enters \( v \) (on this part of the route we have passed through \( 2t \) edges with label 0 and \( s \geq 1 \) edges with label 1), and finally, traverse this green edge to reach \( v \).

The corresponding configuration is of type (b1) for which: \( f(a) = f(b) = f(c) = 0 \); \( f(x) = 2(k + t) - 2 + (s + 2) \) (= the number of red edges with label 1 (note: on the red path from which we turn to the \( v \) through the green edge, the number of edges with label 1 is greater by two; see the corresponding relation on Picture 3, see also Picture 10a)); \( f(y) = 2(k + t - 1 + 1) \) (= the number of green edges with label 0); \( f(z) = f(w) = 2t \) (= twice the number of green edges with label 1). Conversely, a configuration of (b1)-type with \( f(z) \in 2Z \) gives

\[
l = f(z), \quad k = \frac{f(y) - f(z)}{2},
\]

and \( f(w) + f(x) - f(y) \) determines the position of \( v \) on the path \( P_1(v) \).

Now let \( l = 2t + 1 \). Then, like the corresponding situation in Case 1, we first traverse one red edge with label 1, then one green edge with label \( \frac{1}{2} \), and then follow the route which is ‘parallel’ to the route in the case with \( l' = l - 1 \). This gives us only one change compared with the configuration in the above case, namely: \( f(z) := 2t + 1 = f(w) \) (= twice the number of green edges with label 1 plus the number of green edges with label \( \frac{1}{2} \)). Conversely, a configuration of (b1)-type with \( f(z) \in 2Z + 1 \) gives

\[
l = f(z), \quad k = \frac{f(y) - f(z) - 1}{2},
\]

and \( f(w) + f(x) - f(y) \) determines the position of \( v \) on the path \( P_1(v) \).

Now we have to check that the action of \( F_1 \) and \( F_2 \) on the configurations in \( B(H, 0) \) agrees with the edges in \( K(H, 0) \). Let a red edge emanate from \( v \) and end in \( v' \). If the label on this edge is 0, then we are in Case 1, and
hence $f'(w) = f(w) + 1$, that is the case. And if the label is 1, then we are in Case 2, and hence $f'(x) = f(x) + 1$, that is the case.

Let $v$ and $v'$ be connected by a green edge. Then three cases are possible.

(i) The label of $(v, v')$ is 1. Then the coordinates of $v$ and $v'$ in $K(H, 0)$ are $(k + 1, l - 2, m)$ and $(k, l, m)$, respectively, and we get $f'(z) = f(z) + 2$, and there are no other changes, that is the case.

(ii) The label of $(v, v')$ is $\frac{1}{2}$. Then the coordinates of $v$ and $v'$ in $K(H, 0)$ are $(k, l - 1, m)$ and $(k, l, m)$, but $v$ concerns Case 2 and $v'$ concerns Case 1 (see Picture 5), and we get $f(x) = f(y) + 1$, $f(z) = f(w)$, $f'(y) = f(y) + 1 = f(x)$ and $f'(z) = f(z) + 1$, which is the case.

(iii) The label of $(v, v')$ is 0. Then the coordinates of $v$ and $v'$ are $(k - 1, l, m + 2)$ and $(k, l, m)$, that yield $f'(y) = f(y) + 2$, which is the case.

Thus, we established the isomorphism between $K(H, 0)$ and $B(H, 0)$.

QED

**Claim.** $K(0, A) = B(0, \frac{A}{2})$.

The configurations corresponding to such intervals are of type (a).

Note that each red path at $k$-th level below the distinguished diagonal-set (which is located at the 0-th level) is constituted from $k$ edges with label 0 and $k$ edges with label 1.

**Case 1:** a vertex $v$ is located above the critical point in the red path $P_1(v)$ and the “view from the sky” of $v$ is located at the level $k$ and at the position $l$ to the right from the left “border”, $l \leq A + 1 - k$. Thus, each vertex of $K(0, A)$ is characterized by a triple $(k, l, k' + k)$, $k' \leq k$, $0 \leq l \leq A - k$ (if $l = 0$, then the vertex is on the left-hand side border).

On the next picture we illustrate the “view from the sky” of the point with the coordinates $(1, 2, 1)$ in $K(0, 4)$

```
  ❅❅❅❅ butterfly
  ❅❅❅❅ source
```

The route from the source to $v$ is of the following form: first it traverses $k + l$ green edges with label 1, then $k + 2l$ red edges, then $l$ green edges, and finally, $k'$ red edges to enter $v$. 
The corresponding admissible configuration is: \( f(x) = f(y) = f(z) = f(w) = 0, f(a) = 2(k + l) \) (= twice the number of green (or green-blue) edges with label 1); \( f(c) = 2l \) (= twice the number of green edges with label 0); \( f(b) = k + 2l + k' \) (= the number of red edges on the route (obviously, one holds \( f(a) \geq f(b) \geq f(c) \)). Conversely, we have

\[
    k = \frac{f(a) - f(c)}{2}, \quad l = \frac{f(c)}{2}, \quad k' = \frac{f(a) + f(c)}{2},
\]

and \( f(b) - f(c) \) is equal to the position of \( v \) in \( P_1(v) \).

Case 2: a vertex \( v \) is located below the critical point in the red path \( P_1(v) \). Let \((k, l, k')\) be the coordinates of \( v \). Then our route is as follows: it starts at the source and first goes to the vertex with the coordinates \( v' = (0, l - k, 0) \) if \( l \geq k \), and to the vertex \((0, l, 0)\) if \( l \leq k \), then it goes from \( v' \) using green edges with label 1 to the beginning of the string \( P_1(v) \), and then makes \( k' \) steps along \( P_1(v) \).

As a result of this route, we will traverse \((l - k)\) green edges with label 1, then \(2(l - k)\) red edges, then \((l - k)\) green edges with label 0 (coming in \( v' \)), then \( k \) green edges with label 1, and then \( k' \) steps to \( v \).

So we obtain: \( f(a) = 2(l - k) + k \) (= twice the number of green (green-blue) edges with label 1); \( f(c) = 2(l - k) \) (= the number of green (green-blue) edges with label 0); \( f(b) = k' + 2(l - k) \) (= the number of red edges on the route). Like Case 1, the converse is true too.

A simple verification shows that the action of \( F_1 \) and \( F_2 \) in \( B(0, A) \) agrees with the edges in \( K(0, \frac{A}{2}) \). The Claim is proven.

Finally, the graph \( K(H, A) \) contains translated sails of the types \( K(0, A) \) and \( K(H, 0) \).

The sail \( K(H, 0) \) attached to the \( l \)-th distinguished point of \( K(0, A) \), \( l \leq A \), is described by a copy of the configurations \( K(H, 0) \) with the following unimportant modification: \( f(a) = f(b) = f(c) = l \).

As to the sail \( K(0, A) \), the situation is a bit tricky. Specifically, such a sail, attached to the distinguished point of \( K(H, 0) \) through the red path \((l, H - l)\), gets the following modification: modulo the coordinates in the red vertices the “view from the sky” of \( K(l, H - l)(0, A) \) is, in fact, the same as of \( K(0, A) \), and concerning to the coordinates, all coordinates of \( K(l, H - l)(0, A) \) have to be change by adding the vector \((l, H - l)\), that is, by adding \( l \) edges from below to the ‘original edges’ in \( K(0, A) \) (with label 0) and \( H - l \) edges from above (with label 1).

The additional vertices in such a sail are managed by the admissible configurations of type (c1) or (c2), and the ‘original’ ones are managed by configuration of type (a) with the only modification \( f(x) = f(y) = f(z) = f(w) = l \).
Specifically, if a vertex of $K_{(l,H-l)}(0,A)$ is located below the 'original vertices' in $K(0,A)$, then $f(x) = f(y) = f(z) = l$, and $f(w) = l'$, $0 \leq l' \leq l$, and $f(b) = f(c)$ (the (c1)-type configuration) till we reach original vertices. If a vertex of $K_{(l,H-l)}(0,A)$ is above the 'original vertices' in $K(0,A)$, then $f(y) = f(z) = f(w) = l$, and $f(x) = l + l'$ for $0 \leq l' \leq H - l$, and $f(a) = f(b)$ (the (c2)-type configuration) after the moment we leave original vertices. A verification of the coincidence of the crystal operations in $B(H,A/2)$ and $K(H,A)$ in these additional cases is by direct calculations.

Thus, the isomorphism $B(H,A) = K(H,A/2)$ is shown, and this completes the proof of Theorem 1.

References

[1] V.Danilov, A.Karzanov, and G.Koshevoy, Combinatorics of $A_2$-crystals, J. of Algebra (to appear), e-print arXiv:math.RT/0604333.
[2] V.I. Danilov, A.V. Karzanov, and G.A. Koshevoy, The crossing model for regular $A_n$-crystals, in preparation.
[3] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, Affine crystals and vertex models, Adv. Ser. Math. Phys. 16 (World Sci. Publ., River Edge, NJ, 1992).
[4] M. Kashiwara, The crystal base and Littelmann’s refined Demazure character formula, Duke Math. J. 71 (3) (1993) 839–857.
[5] M. Kashiwara, On crystal bases, in: Representations of Groups (Banff, AB,1994), CMS Conf. Proc. 16 (2) (1995) 155–197.
[6] P.Littelmann, Cones, crystals, and patterns. Transform. Groups 3 (1998), no. 2, 145–179.
[7] P.Sternberg, On the local structure of doubly laced crystals, arXiv:math.RT/0603547
[8] J.R. Stembridge, A local characterization of simply-laced crystals, Transactions of the Amer. Math. Soc. 355 (12) (2003) 4807–4823.