Distribution of localization centers in some discrete random systems

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Abstract
As a supplement of our previous work [10], we consider the localized region of the random Schrödinger operators on $l^2(\mathbb{Z}^d)$ and study the point process composed of their eigenvalues and corresponding localization centers. For the Anderson model we show that, this point process in the natural scaling limit converges in distribution to the Poisson process on the product space of energy and space. In other models with suitable Wegner-type bounds, we can at least show that limiting point processes are infinitely divisible.

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1 Introduction
The typical model we consider is the so-called Anderson model given below.

$$(H_\omega \varphi)(x) = \sum_{|x-y|=1} \varphi(y) + \lambda V_\omega(x)\varphi(x), \quad \varphi \in l^2(\mathbb{Z}^d)$$

where $\lambda > 0$ is the coupling constant and $\{V_\omega(x)\}_{x \in \mathbb{Z}^d}$ are the independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$.

The following facts are well-known.
(1) (the spectrum of $H$) the spectrum of $H_\omega$ is deterministic almost surely

$$\sigma(H_\omega) = \Sigma := [-2d, 2d] + \lambda \supp d\nu, \quad a.s.$$
where $\nu$ is the distribution of $V_\omega(0)$ \cite{13}.

(2) (Anderson localization) There is an open interval $I \subset \Sigma$ such that with probability one, the spectrum of $H_\omega$ on $I$ is pure point with exponentially decaying eigenfunctions. $I$ can be taken (i) $I = \Sigma$ if $\lambda$ is large enough, (ii) on the band edges, and (iii) away from the spectrum of the free Laplacian if $\lambda$ sufficiently small (e.g., \cite{7,20,1,2}).

Recently, some relations between the eigenvalues and the corresponding localization centers are discussed \cite{17}. It roughly implies,

1. If $|E - E_0| \approx L^{-d}$ ($E_0 \in I$), the localization center $x(E)$ corresponding to the energy $E$ satisfies $|x(E)| \geq L$. Hence the distribution of the localization centers are “thin” in space.

2. If $|E - E'| \approx L^{-2d}$, the localization centers $x(E), x(E')$ corresponding to the energies $E, E'$ satisfies $|x(E) - x(E')| \geq L$. Hence the localization centers are repulsive if the energies get closer.

On the other hand, in \cite{10}, they study the “natural scaling limit” of the random measure in $\mathbb{R}^{d+1}$ (the product of energy and space) composed of the eigenvalues and eigenfunctions. The result there roughly implies that their distribution with eigenvalues in the order of $L^{-d}$ from the reference energy $E_0$, and with eigenfunctions in the order of $L$ from the origin, obey the Poisson law on $\mathbb{R}^{d+1}$. This work can also be regarded as an extension of the work by Minami \cite{16} who showed that the point process on $\mathbb{R}$ composed of the eigenvalues of $H$ in the finite volume approximation converges to the Poisson process on $\mathbb{R}$. To summarize, \cite{17,10} imply that the eigenfunctions whose energies are in the order of $L^{-d}$ are non-repulsive while those in the order of $L^{-2d}$ are repulsive, which are consistent with Minami’s result \cite{16}.

The aim of this paper is to supplement \cite{10} from a technical point of view: (i) to study the distribution of the localization centers which is technically different from what is done in \cite{10}, and (ii) to study what can be said for those models in which Minami’s estimate and the fractional moment bound, which are the main tool in \cite{10}, are currently not known to hold.

We set some notations.

**Notation :**

\footnote{This result follows easily from the upper bound on the density of states. So in the Lifschitz tail region, we have $|x(E)| \geq (\text{const.})e^{(\text{const.})L} \frac{n}{n+1}$ if $|E - E_0| \leq L$.}
(1) For \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d \), let \(|x| = \sum_{j=1}^d |x_j|\). \( \Lambda_L(x) := \{ y \in \mathbb{Z}^d : |x - y| \leq \frac{L}{2} \} \) is the finite box in \( \mathbb{Z}^d \) with length \( L \) centered at \( x \in \mathbb{Z}^d \). \( |\Lambda| := \#\Lambda \) is the number of sites in the box \( \Lambda \) and \( \chi_\Lambda \) is the characteristic function of \( \Lambda \).

(2) For a box \( \Lambda \), let \( \tilde{\partial}\Lambda := \{ \langle y, y' \rangle \in \Lambda \times \Lambda^c : |y - y'| = 1 \} \) be two notions of the boundary of \( \Lambda \).

(3) For a box \( \Lambda(\subset \mathbb{Z}^d) \), \( \Lambda_L := \Lambda|_x \) is the restriction of \( \Lambda \) on \( \Lambda \). We consider both Dirichlet b.c. and periodic b.c. depending on cases, to be specified in which they are defined. For \( E \notin \sigma(\Lambda_L) \), \( G_{\Lambda_L}(E; x, y) = \langle \delta_x, (\Lambda_L - E)^{-1}\delta_y \rangle_{l^2(\Lambda)} \) is the Green function of \( \Lambda_L \). \( \delta_x \in l^2(\mathbb{Z}^d) \) is defined by \( \delta_x(y) = 1(y = x), = 0(y \neq x) \) and \( \langle \cdot, \cdot \rangle_{l^2(\Lambda)} \) is the inner-product on \( l^2(\Lambda) \).

(4) Let \( \gamma > 0 \), \( E \in \mathbb{R} \). We say that the box \( \Lambda_L(x) \) is \((\gamma, E)\)-regular iff \( E \notin \sigma(\Lambda_L(x)) \) and the following estimate holds:\[ \sup_{\epsilon > 0} |G_{\Lambda_L(x)}(E + i\epsilon; x, y)| \leq e^{-\gamma L^d}, \quad \forall y \in \partial\Lambda_L(x). \]

In (1.2), (2.1) and (7.4), we will consider this condition for \( \Lambda_L(x) \) with Dirichlet b.c., while in (3.3), (5.2) with periodic b.c.\footnote{We adopt this definition to treat Lemma 3.2 and Proposition 5.2.}

(5) For \( \phi \in l^2(\mathbb{Z}^d), \phi \neq 0 \), we define the set \( X(\phi) \) of its localization centers by \( X(\phi) := \left\{ x \in \mathbb{Z}^d : |\phi(x)| = \max_{y \in \mathbb{Z}^d} |\phi(y)| \right\} \).

This definition is due to \[5\]. Since \( \phi \in l^2(\mathbb{Z}^d) \), \( X(\phi) \) is a finite set. To be free from ambiguities, we choose \( x(\phi) \in X(\phi) \) according to a certain order on \( \mathbb{Z}^d \). For a box \( \Lambda \), we say \( \phi \) is localized in \( \Lambda \) iff \( x(\phi) \in \Lambda \). If \( \{ E_j \}_j, \{ \phi_j \}_j \) are the enumerations of the eigenvalues and eigenfunctions of \( H \) counting multiplicities, we set \( X(E_j) := X(\phi_j), x(E_j) := x(\phi_j) \) and we say \( E_j \) is localized in \( \Lambda \) iff \( x(E_j) \in \Lambda \). If an eigenvalue is degenerated, we adopt any but fixed selection procedure of choosing eigenfunctions so that the quantities

\footnote{with a slight change of argument, it is possible to set periodic b.c. only in this condition.}
in concern ($\xi_k, \xi_k$ defined later) are measurable.

(6) For a Hamiltonian $H$, an interval $J(\subset \mathbb{R})$ and a box $B(\subset \mathbb{Z}^d)$, we set

$$E(H, J) := \{\text{eigenvalues of } H \text{ in } J\}$$

$$E(H, J, B) := \{\text{eigenvalues of } H \text{ in } J \text{ localized in } B\}$$

$$E_f(H, J) := \{\text{normalized eigenfunctions of } H \text{ in } J\}$$

$$E_f(H, J, B) := \{\text{normalized eigenfunctions of } H \text{ in } J \text{ localized in } B\}$$

$$N(H, J) := \# E(H, J) \text{ (counting multiplicity)}$$

$$N(H, J, B) := \# E(H, J, B)$$

(7) While our results(Theorem 1.1 and 1.2) adopt $x(\phi)$ as a definition of localization center, a more natural definition of that may be

$$\langle x \rangle_\phi := \sum_{y \in \mathbb{Z}^d} y|\phi(y)|^2 \left(\sum_{y \in \mathbb{Z}^d} |\phi(y)|^2\right)^{-1}, \quad \phi \in l^2(\mathbb{Z}^d), \; \phi \neq 0.$$  

However, it is easy to see (Lemma 4.3) that those theorems are also valid if we adopt $\langle x \rangle_\phi$ instead.

(8) For a $n$-dimensional measurable set $A(\subset \mathbb{R}^n)$, we denote by $|A|$ its Lebesgue measure. For $a \in \mathbb{R}$ and $r > 0$, $I(a, r) := \{x \in \mathbb{R} : |x - a| < r\}$ is the open interval centered at $a$ with radius $r$.

(9) Set $K = [0, 1]^d$ and let $\pi_e$ and $\pi_s$ be the canonical projections on $\mathbb{R} \times K$ onto $\mathbb{R}$ and $K$ respectively: $\pi_e(E, x) = E$, $\pi_s(E, x) = x$ for $(E, x) \in \mathbb{R} \times K$.

(10) We set

$$\xi(f) = \int_{\mathbb{R} \times K} f(x)\xi(dx)$$

for a Radon measure $\xi$ and a bounded measurable function $f$ on $\mathbb{R} \times K$. Even if $f$ is a function on $\mathbb{R}$, we write $\xi(f)$ instead of $\xi(f1_K)$ for simplicity. For a sequence $\{\xi_k\}_k$ of Radon measures, $\xi_n \rightharpoonup \xi$ means $\xi_n$ converges vaguely to $\xi$: $\xi_n(f)^{n \to \infty} \xi(f)$ for any $f \in C_c(\mathbb{R} \times K)$.

We consider the following two assumptions.

**Assumption A**

(1) (Initial length scale estimate) Let $I(\subset \Sigma)$ be an open interval where the initial length scale estimate of the multiscale analysis holds: we can find
\( \gamma > 0 \) and \( p > 6d \) such that for sufficiently large \( L_0 \) we have

\[
P \left( \text{For any } E \in I, \Lambda_{L_0}(0) \text{ is } (\gamma, E)\text{-regular} \right) \geq 1 - L_0^{-p}.
\]

where \( H_{\Lambda_{L_0}(0)} \) has Dirichlet b.c.

(2)(Wegner’s estimate) We can find a positive constant \( C_W \) such that for any interval \( J(\subset I) \) and any box \( \Lambda \),

\[
E[|N(H_{\Lambda}, J)|] \leq C_W |\Lambda||J|.
\]

In (2), we require that both \( H_{\Lambda} \)'s with Dirichlet b.c. and those with periodic b.c. satisfy (1.1).

Assumption A is known to hold, for instance, (1) for the Anderson model when the distribution of the random potential \( \nu \) has the bounded density \( \rho \), with the allowed location of \( I \) mentioned at the beginning of this section [19, 20], (2) for the Schrödinger operators with off-diagonal disorder [6], and (3) for the Schrödinger operators on \( l^2(\mathbb{Z}^d) \) with random magnetic fluxes [14] (in (2), (3), \( I \) is on the edge of \( \Sigma \)). We need \( p > 6d \) to eliminate the contributions from the negligible events, in the proof of Proposition 2.1.

Pick \( \alpha \) with \( 1 < \alpha < \alpha_0 := \frac{2p}{p+2d}(< 2) \), and set

\[
L_{k+1} = L^\alpha_k, \quad k = 0, 1, \ldots
\]

For simplicity, we write \( \Lambda_k(x) = \Lambda_{L_k}(x) \). By the multiscale analysis [20], we have, for \( k = 1, 2, \ldots \) and for any fixed disjoint boxes \( \Lambda_k(x), \Lambda_k(y) \),

\[
P \left( \text{For any } E \in I, \text{ either } \Lambda_k(x) \text{ or } \Lambda_k(y) \text{ are } (\gamma, E)\text{-regular} \right)
\geq 1 - L_k^{-2p}.
\]

where we take Dirichlet b.c. for \( H_{\Lambda_k(x)}, H_{\Lambda_k(y)} \).

Assumption B (Minami’s estimate)

We can find a positive constant \( C_M \) such that for any finite box \( \Lambda \) and any interval \( J(\subset I) \),

\[
\sum_{k=2}^{\infty} k(k-1)P(N(H_{\Lambda}, J) = k) \leq C_M |\Lambda|^2 |J|^2
\]
where $H_\Lambda$ has periodic b.c.

Assumption B is known to be true for the Anderson model and for any interval $J(\subset \mathbb{R})$ when $\nu$ has the bounded density \cite{16}.

The integrated density of states $N(E)$ of $H$ is defined by

$$N(E) := \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} N(H_\Lambda, (-\infty, E]).$$ \hfill (1.3)

It is known that, with probability one, this limit exists for any $E \in \mathbb{R}$ and continuous \cite{3} so that its derivative $n(E)$ finitely exists a.e. which is called the density of states.

Let $M(\mathbb{R}^n)$ (resp. $M_p(\mathbb{R}^n)$) be the set of Radon measures (resp. integer-valued Radon measures) on $\mathbb{R}^n$ which is regarded as a metric space under the vague topology. Random measure (resp. point process) on $\mathbb{R}^n$ is defined to be a measurable mapping from $(\Omega, \mathcal{F}, P)$ to $M(\mathbb{R}^n)$ (resp. $M_p(\mathbb{R}^n)$). We say that a sequence $\{\xi_k\}_k$ of random measures converges in distribution to a random measure $\xi$ and write $\xi_k \overset{d}{\to} \xi$ iff the distribution of $\xi_k$ converges weakly to that of $\xi$. We state our results below.

(1) **Uniform distribution of localization centers**: We first consider localization centers corresponding to all eigenvalues in $I$. Let $H_k = H|_{\Lambda_k}$ be the restriction of $H$ on $\Lambda_k = \{1, 2, \cdots, L_k\}^d$ with periodic boundary condition. The choice of this particular boundary condition is to be free from the boundary effect which should be purely technical. Writing $\{F_j(\Lambda_k)\}_j := \mathcal{E}(H_k, I)$, we define a random measure $\bar{\xi}_k$ on $I \times K$ by

$$\bar{\xi}_k := \frac{1}{|\Lambda_k|} \sum_j \delta_{\bar{X}_j}, \quad \bar{X}_j := (F_j(\Lambda_k), L_k^{-1} x(F_j(\Lambda_k))) \in I \times K.$$

**Theorem 1.1** Assume Assumption A(1) with $p > 2d$. Then

$$\bar{\xi}_k \overset{d}{\to} \nu \otimes dx, \quad a.s.$$ 

Theorem 1.1 implies that the localization centers are uniformly distributed in the macroscopic scale. Although the proof is straightforward by using the existence of the density of states, we provide that in Section 7 for completeness. In \cite{4} Theorem 7.1, same conclusion is derived for a special
case (i.e., eq. (7.1) in Section 7 is proved for \( I = \Sigma \) and \( J = \mathbf{R}, B = K \)), and in \([10]\), almost equivalent statement is derived for all energies.

(2) Local fluctuation: To see the local fluctuation near a reference energy \( E_0 \in I \), let \( \{ E_j(\Lambda_k) \}_j \) := \( \mathcal{E}(H_k, \mathbf{R}) \), \( \{ x_j \}_j \) := \( \{ x(E_j(\Lambda_k)) \}_j \) and define a point process \( \xi_k \) on \( \mathbf{R} \times K \) as follows.

\[
\xi_k = \sum_{j=1}^{\mid \Lambda_k \mid} \delta_{x_j}, \quad X_j = (|\Lambda_k|(E_j(\Lambda_k) - E_0), L_k^{-1} x_j) \in \mathbf{R} \times K.
\]

This scaling is the same as that in \([16, 10]\) : the energies are supposed to accumulate in the order of \( L^{-d} \) around \( E_0 \) for large \( L \) if \( n(E_0) < \infty \), and if \( |E - E_0| \simeq L^{-d} \), we expect \( |x(E)| \simeq L \) \([17, \text{Theorem 1.1}]\). The main theorem of this paper is

**Theorem 1.2** Assume Assumptions A, B. If \( n(E_0) < \infty \), then \( \xi_k \overset{d}{\to} \zeta_{P, R \times K} \) as \( k \to \infty \) where \( \zeta_{P, R \times K} \) is the Poisson process on \( \mathbf{R} \times K \) with its intensity measure \( n(E_0) dE \times dx \).

If we do not assume Assumption B, we can only prove that there exists convergent subsequence and its limiting point process is infinitely divisible with absolutely continuous intensity measure (Theorem 2.4). (We say the point process \( \xi \) is infinitely divisible iff for any \( n \in \mathbf{N} \), we can find i.i.d. array of point process \( \{ \xi_{nj} \}_{j=1}^{n} \) with \( \xi \overset{d}{=} \sum_{j=1}^{n} \xi_{nj} \). Similar conclusion is proved in \([8]\) for one-dimensional random Schrödinger operator on \( \mathbf{R} \). The infinite divisibility of \( \xi \) merely implies that \( \xi \) is represented by the Poisson process on \( M_p(\mathbf{R} \times K) \) whose intensity measure is given by its canonical measure \([9, \text{Lemma 6.5, 6.6}]\). We are unable to prove Theorem 1.2 if we replace \( H_k \) by \( H \) itself (which is done in \([10]\)) because some “a priori” estimates are missing to prove Step 1 in the proof of Proposition 2.1 Lemma \([4, 4]\) and Lemma \([4, 5]\).

By “projecting” the result of Theorem 1.2 to the energy axis, we recover the result by Minami \([16]\) : the point process \( \xi_k^{(ev)} = \sum_{j} \delta_{|\Lambda_k|(E_j(\Lambda_k) - E_0)} \) converges to the Poisson process on \( \mathbf{R} \). If we project it to the space axis, we have a result on the distribution of localization centers. To be precise, pick an interval \( J(\subset \mathbf{R}) \) and let \( \{ F_j(\Lambda_k, J) \}_{j \geq 1} := \mathcal{E}(H_k, E_0 + L_k^{-d} J) \). Define a point process on \( K \) by

\[
\xi_k^{(loc)} = \xi_k(J \times \cdot) = \sum_{j \geq 1} \delta_{L_k^{-1} x_j(F_j(\Lambda_k, J))}.
\]
Corollary 1.3 Under the same assumption as in Theorem 1.2, $\xi_{k}^{(loc)} \xrightarrow{d} \zeta_{P,K}$ as $k \to \infty$ where $\zeta_{P,K}$ is the Poisson process on $K$ with intensity measure $n(E_0)|J|dx$.

In [11], they assume the Poisson distribution of localization centers and discuss the derivation of Mott’s formula on the a.c. conductivity (rigorous derivation of Mott’s formula is recently done by [12]).

The remaining sections are organized as follows. In Section 2, we prove the infinite divisibility of the limiting point process (Theorem 2.4) which is one of the main condition to apply the Poisson convergence theorem [9, Corollary 7.5] to our situation. In order to do that, we decompose $\Lambda_k$ into disjoint boxes $\{D_p\}_p$ of size $L_{k-1}$, and let $H_p = H|_{D_p}$ as is done in [16]. Since the eigenfunctions of $H_k$ corresponding to the eigenvalue $E$ in $I$ are exponentially localized, we can find a box $D_p$ such that $H_p$ has eigenvalues near $E$. By some perturbative argument, we can construct a one to one correspondence between the eigenvalues of $H_k$ and that of $\bigoplus_p H_p$, with probability close to 1. Therefore, $\xi_k$ is approximated by the sum $\eta_k = \sum_p \eta_{k,p}$ of the point process composed of the eigenvalues and localization centers of $H_p$. Wegner’s estimate ensures that $\{\eta_{k,p}\}_{k,p}$ is a null-array and relatively compact, so that $\{\eta_k\}_k$ always has the convergent subsequence whose limiting point is infinitely divisible.

In Section 3, under Assumption A, B, we show that $\eta_k$ converges in distribution to the Poisson process, finishing the proof of Theorem 1.2. By Minami’s estimate, $\eta_{k,p}$ has at most one atom in the corresponding region in $R \times K$ with the probability close to 1. Hence the general Poisson convergence theorem [9, Corollary 7.5] gives the result. Since the mechanism of the convergence to the Poisson process is the same as in [16], Theorem 1.2 can be regarded as the extension of that.

To construct that one to one correspondence, we used the machinery developed in [5, 13] which is reviewed in Section 4.

For the random measure studied in [10], we can show the infinite divisibility as Theorem 2.4 under Assumption A, to be mentioned in Section 5.

If we assume both Assumption A and B in the proof of Proposition 2.1, $H_p$ has at most one eigenvalues in the corresponding region with probability close to 1, so that the correspondence between eigenvalues of $H_{k+1}$ and $H_p$ becomes bijective apart from negligible contributions, which is mentioned in
Section 6. This observation allows us to construct explicitly the approximate eigenfunctions of $H$ by its finite volume operator\[18\]. In Section 7, we prove Theorem 1.1.

2 Infinite Divisibility

For simplicity, we consider $\xi_{k+1}$ instead of $\xi_k$. We first decompose $\Lambda_{k+1}$ into disjoint cubes $D_p$ of size $L_k : \Lambda_{k+1} = \bigcup_{p=1}^{N_k} D_p$, $N_k = \left(\frac{L_{k+1}}{L_k}\right)^d \left(1 + o(1)\right)$. The contribution of $D_p$'s near the boundary of $\Lambda_{k+1}$ turns out to be negligible by Lemma 2.2. We denote by $C_p$ the box obtained by eliminating the strip of width $L_{k-1}$ from the boundary of $D_p$:

$$C_p := \{x \in D_p : d(x, \partial D_p) \geq L_{k-1}\}.$$

Let $H_{k,p} := H|_{D_p}$ with periodic boundary condition. We set the following event

$$\Omega_k = \left\{\omega \in \Omega : \text{For any } E \in I, \text{ either } \Lambda_{k-1}(x) \text{ or } \Lambda_{k-1}(y) \text{ are (}\gamma, E\text{-regular)} \text{ for any disjoint pair of boxes } \Lambda_{k-1}(x), \Lambda_{k-1}(y) \subset \Lambda_{k+1} \cup \bigcup_{p=1}^{N_k} D_p \right\} \quad (2.1)$$

($H_{\Lambda_{k-1}(x)}, H_{\Lambda_{k-1}(y)}$ have Dirichlet b.c.). In (2.1), we regard $\Lambda_{k+1}$ and $D_p$'s as torus (so that now $\Lambda_{k+1} \neq \bigcup_{p=1}^{N_k} D_p$) and consider all $\Lambda_{k-1}(x)$'s contained in $\Lambda_{k+1}$ or some $D_p$. So, for $x \in \Lambda_{k+1}$ close to $\partial \Lambda_{k+1}$ (or close to some $\partial D_p$), some portion of $\Lambda_{k-1}(x)$ may appear in the opposite side to $x$ of $\partial \Lambda_{k+1}$ (or $\partial D_p$). This peculiar definition of the event $\Omega_k$ is for the proof of Lemma 4.2.

By (1.2), we have

$$P(\Omega_k) \geq 1 - (\text{const.})L_{k-1}^{-2p}L_{k+1}^{2d} = 1 - (\text{const.})L_{k-1}^{-2p+2d} \alpha^2. \quad (2.2)$$

We define the point process by

$$\eta_{k+1} = \sum_{p=1}^{N_k} \eta_{k+1,p}, \quad \eta_{k+1,p} = \sum_{j=1}^{\lvert D_p \rvert} \delta_{Y_{p,j}},$$

$$Y_{p,j} = (\lvert \Lambda_{k+1} \rvert (E_j(D_p) - E_0), L_{k+1}^{-1} y_{p,j}).$$
where \( \{E_j(D_p)\}_j := \mathcal{E}(H_{k,p}, R) \), \( y_{p,j} = x(E_j(D_p)) \). As was explained in Introduction, we expect that \( \xi_{k+1} \) can be approximated by \( \sum_p \eta_{k+1,p} \) to be shown below.

**Proposition 2.1** Under Assumption A, we have

\[
E[|\xi_{k+1}(f) - \eta_{k+1}(f)|] \to 0, \quad k \to \infty, \quad f \in C_c(R \times K).
\]

By Proposition 2.1 the Laplace transform \( L_\xi(f) = E[e^{-\xi(f)}] \) of \( \xi \) satisfies \( \lim_{k \to \infty}(L_{\xi_{k+1}}(f) - L_{\eta_{k+1}}(f)) = 0 \), for \( f \in C_c^+(R \times K) \). Hence it suffices to show \( \eta_{k+1} \Rightarrow \xi_{p,R \times K} \) to prove Theorem 1.2. By choosing \( f \) independent of the space variables, we obtain an alternative proof of [16, Step 3]. Since we use the exponential decay of eigenfunctions instead of that of Green’s function, this proof is mathematically indirect but physically direct.

**Proof. Step 1 :** We show the contribution by the event \( \Omega_k^c \) is negligible. In fact, since \( |\xi_{k+1}(f)| \leq \|f\|_\infty|\Lambda_{k+1}| \) and since \( p > 6d > \frac{3}{2}d\alpha^2 \), we have

\[
E[|\xi_{k+1}(f)|; \Omega_k^c] \leq \|f\|_\infty|\Lambda_{k+1}|L_{k-1}^{-2p+2d\alpha^2} = (\text{const.})L_{k-1}^{-2p+3d\alpha^2} = o(1)
\]

by (2.2). \( E[\sum_p \eta_{k+1,p}(f); \Omega_k^c] \) can be estimated similarly. Therefore, it suffices to show

\[
E[|\xi_{k+1}(f) - \eta_{k+1}(f)|; \Omega_k] = o(1).
\]

**Step 2 :** We show the contribution by the atoms whose localization centers are in \( \cup_p(D_p \setminus C_p) \) are negligible. We first decompose

\[
\xi_{k+1} = \xi_{k+1}^{(1)} + \xi_{k+1}^{(2)}, \quad \xi_{k+1}^{(j)} = \sum_{p=1}^{N_k} \xi_{k+1,p}^{(j)}, \quad j = 1, 2,
\]

\[
\xi_{k+1,p}^{(1)} = \sum_{x_j \in C_p} \delta X_j, \quad \xi_{k+1,p}^{(2)} = \sum_{x_j \in D_p \setminus C_p} \delta X_j.
\]

And we decompose \( \eta_{k+1,p} \) similarly. In what follows, we take any \( 0 < \gamma' < \gamma \) and let \( k \) large enough with \( k \geq k_2(\alpha, d, \gamma, \gamma') \lor k_3(\alpha, d, \gamma, \gamma') \) where \( k_2, k_3 \) are defined in Lemma 4.4 and 4.5. For simplicity, set

\[
\epsilon_{k-1} := e^{-\gamma' L_{k-1}/2}.
\]

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\[ ^4 \text{the equation "\ldots = o(1)" henceforth means "\ldots = o(1) as } k \to \infty". \]
Claim 1

\[ E[\xi_{k+1}^{(2)}(f); \Omega_k] = o(1), \quad E[\eta_{k+1}^{(2)}(f); \Omega_k] = o(1). \]

Proof of Claim 1

Let

\[ S_p = \{ x \in \Lambda_{k+1} : d(x, \partial(D_p \setminus C_p)) \leq L_{k-1} \}, \quad H'_{k,p} := H_{k+1}|_{S_p} \]

(with Dirichlet boundary condition). Pick \( a > 0 \) with \( \pi_\epsilon(\text{supp } f) \subset [-a, a] \) and set

\[ J_{k+1} := \left[ E_0 - \frac{a}{|\Lambda_{k+1}|}, E_0 + \frac{a}{|\Lambda_{k+1}|} \right] = I \left( E_0, \frac{a}{|\Lambda_{k+1}|} \right). \]

By Lemma 4.4(2) and Assumption A(2)(Wegner’s estimate), we have

\[
E[\xi_{k+1,p}^{(2)}(f); \Omega_k] \leq \|f\|_\infty E[N(H_{k+1}, J_{k+1}, D_p \setminus C_p)] \\
\leq \|f\|_\infty E[N(H'_{k,p}, J_{k+1} + I(0, \epsilon_{k-1}))] \\
\leq (\text{const.})\|f\|_\infty C_W|D_p \setminus C_p| \cdot \frac{2a}{|\Lambda_{k+1}|}.
\]

Using the inequality \( |D_p \setminus C_p| \leq (\text{const.})L_{k-1}L_k^{d-1} \) and then taking sum w.r.t. \( p \) gives

\[ E[\xi_{k+1}^{(2)}(f); \Omega_k] \leq (\text{const.})\frac{L_{k-1}}{L_k} = o(1). \]

To estimate \( \eta_{k+1}^{(2)} \), we set

\[ T_p = \{ x \in D_p : d(x, \partial(D_p \setminus C_p)) \leq L_{k-1} \}, \quad H''_{k,p} := H_{k+1}|_{T_p}. \]

Then the same argument as above with Lemma 4.4(3) gives

\[ E[\eta_{k+1}^{(2)}(f); \Omega_k] \leq (\text{const.})\frac{L_{k-1}}{L_k} = o(1) \]

and thus proves Claim 1. \( \square \)

Therefore, it suffices to show

\[ E \left[ |\xi_{k+1}^{(1)}(f) - \eta_{k+1}(f)|; \Omega_k \right] = o(1). \]

The equation \( E[\eta_{k+1}^{(2)}(f); \Omega_k] = o(1) \) in Claim 1 will be used in Step 3 below.
Step 3: We first show the following claim.

Claim 2 Let $J \subset I$ be an interval. If $\omega \in \Omega_k$, we have

$$
\sum_p N(H_{k,p}, J + I(0, \epsilon_{k-1})) \\
\leq \sum_p N(H_{k+1}, J, C_p) + \sum_p N(H_{k+1}, (J + I(0, 2\epsilon_{k-1})) \setminus J, C_p) \\
+ \sum_p N(H'_{k,p}, J + I(0, 3\epsilon_{k-1})) + \sum_p N(H''_{k,p}, J + I(0, 2\epsilon_{k-1})).
$$

Proof of Claim 2 We decompose

$$
N(H_{k,p}, J + I(0, \epsilon_{k-1})) = N(H_{k,p}, J + I(0, \epsilon_{k-1}), C_p) + N(H_{k,p}, J + I(0, \epsilon_{k-1}), D_p \setminus C_p) \\
=: I_p + II_p.
$$

By Lemma 4.4(3),

$$
II_p \leq N(H''_{k,p}, J + I(0, 2\epsilon_{k-1}))
$$

and by Lemma 4.4(2) and Lemma 4.5,

$$
\sum_p I_p \leq N(H_{k+1}, J + I(0, 2\epsilon_{k-1})) \\
= \sum_p N(H_{k+1}, J + I(0, 2\epsilon_{k-1}), C_p) + \sum_p N(H_{k+1}, J + I(0, 2\epsilon_{k-1}), D_p \setminus C_p) \\
\leq \sum_p N(H_{k+1}, J + I(0, 2\epsilon_{k-1}), C_p) + \sum_p N(H'_{k,p}, J + I(0, 3\epsilon_{k-1}))
$$

which shows Claim 2. □

For any $p = 1, 2, \ldots, N_k$, let $\{E_{p,j}\} = \mathcal{E}(H_{k+1}, J_{k+1}, C_p)$ and write $\bigcup_j I(E_{p,j}, \epsilon_{k-1})$ as the disjoint union of open intervals:

$$
\bigcup_j I(E_{p,j}, \epsilon_{k-1}) = \bigcup_i I_i.
$$

If $a_1^{(i)} < a_2^{(i)} < \cdots < a_{N_i}^{(i)}$ are the elements of $\mathcal{E}(H_{k+1}, I_i, C_p)$, then

$$
I_i = I_i' + I(0, \epsilon_{k-1}), \quad I_i' := (a_1^{(i)}, a_{N_i}^{(i)}).
$$

Letting $J = I_i'$ in Lemma 4.4(1), we have

$$
N(H_{k+1}, I_i, C_p) \leq N(H_{k,p}, I_i)
$$
and hence we have an one to one correspondence $\Phi$ from $\bigcup \sigma \mathcal{E}(H_{k+1}, I_i, C_p) =: \{E_{j,p}\}_j$ to $\bigcup \sigma \mathcal{E}(H_{k+1}, I_i)$. Since $\text{diam } (I_i) \leq L_k^d \epsilon_{k-1}$, they satisfy

$$|E_{j,p} - \Phi(E_{j,p})| \leq L_k^d \epsilon_{k-1}.$$ 

On the other hand, by letting $J = J_{k+1}$ in Lemma 4.4(1) and Claim 2 we see that, the number of elements of $\bigcup \sigma \mathcal{E}(H_{k+1}, J_{k+1} + I(0, \epsilon_{k-1}))$ which do not lie in the range of $\Phi$ is less than

$$\sum_p N(H_{k+1}, (J_{k+1} + I(0, 2\epsilon_{k-1})) \setminus J_{k+1}, C_p) + \sum_p N(H_{k+1}, J_{k+1} + I(0, 3\epsilon_{k-1}))$$

$$+ \sum_p N(H_{k+1}, J_{k+1} + I(0, 2\epsilon_{k-1})).$$

Therefore if $x_{j,p} = x(E_{j,p})$, $y_{j,p} = x(\Phi(E_{j,p}))$, we have

$$\mathbb{E} \left[ \sum_p |\xi_{k+1, p}^{(1)}(f) - \eta_{k+1, p}(f)| : \Omega_k \right]$$

$$\leq \mathbb{E} \left[ \sum_p \sum_j \left| f(|\Lambda_{k+1}|(E_{j,p} - E_0), L_{k+1}^{-1} x_{j,p}) - f(|\Lambda_{k+1}|(\Phi(E_{j,p}) - E_0), L_{k+1}^{-1} y_{j,p}) \right| : \Omega_k \right]$$

$$+ \sum_p \|f\|_{\infty} \mathbb{E} \left[ N(H_{k+1}, (J_{k+1} + I(0, 2\epsilon_{k-1})) \setminus J_{k+1}, C_p) \right]$$

$$+ \sum_p \|f\|_{\infty} \mathbb{E} \left[ N(H_{k+1}, J_{k+1} + I(0, 3\epsilon_{k-1})) \right]$$

$$+ \sum_p \|f\|_{\infty} \mathbb{E} \left[ N(H_{k+1}, J_{k+1} + I(0, 2\epsilon_{k-1})) \right]$$

$$=: I + II + III + IV.$$ 

Since $f$ is uniformly continuous, for any $\epsilon > 0$ we have $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta(\epsilon)$ with some $\delta(\epsilon) > 0$. Since

$$\left| \left( |\Lambda_{k+1}|(E_{j,p} - E_0), L_{k+1}^{-1} x_{j,p} \right) - \left( |\Lambda_{k+1}|(\Phi(E_{j,p}) - E_0), L_{k+1}^{-1} y_{j,p} \right) \right|$$

$$\leq |\Lambda_{k+1}| L_k^d \epsilon_{k-1} + L_k/L_{k+1} < \delta(\epsilon)$$

for large $k$, we have

$$I \leq \epsilon \mathbb{E} \left[ \sum_p N(H_{k+1}, J_{k+1}, C_p) \right] \leq \epsilon C_W \frac{2a}{|\Lambda_{k+1}| |\Lambda_{k+1}|} = (\text{const.}) \epsilon.$$
by Wegner’s estimate, which also gives a bound for II.

\[
II \leq \|f\|_\infty E\left[N(H_{k+1}, (J_{k+1} + I(0, 2\epsilon_{k-1})) \setminus J_{k+1}\right] \\
\leq \|f\|_\infty C_W 4e^{-\gamma L_{k-1}/2}|\Lambda_{k+1}| = o(1).
\]

III, IV can be estimated similarly as in Step 2:

\[
III, IV \leq \sum_p \|f\|_\infty C_W |D_p \setminus C_p| \left(\frac{2a}{|\Lambda_{k+1}|} + 6\epsilon_{k-1}\right) \\
\leq (\text{const.}) \|f\|_\infty \left(\frac{L_{k+1}}{L_k}\right)^d C_W L_{k-1} L_k^{d-1} L_{k+1}^{-d} \\
\leq (\text{const.}) \frac{L_{k-1}}{L_k}.
\]

The proof of Proposition 2.1 is thus completed. \[\square\]

We next show some elementary bounds of \(\xi_{k+1}, \eta_{k+1}\) to study their basic properties.

**Lemma 2.2**

(1) For a bounded interval \(A(\subset \mathbb{R} \times K)\),

\[
E[\xi_{k+1}(A)] \leq C_W |\pi_e(A)|, \quad E[\eta_{k+1,p}(A)] \leq C_W \frac{|\pi_e(A)|}{|\Lambda_{k+1}|} \cdot |\Lambda_k|
\]

for large \(k\).

(2) For \(f \in C_c(\mathbb{R} \times K)\) we have

\[
E[\sum_p |\eta_{k+1,p}(f)|] \leq (\text{const.}) C_W \|f\|_1
\]

for large \(k\).

**Proof.** (1) Since \(J_{k+1} := E_0 + |\Lambda_{k+1}|^{-1}\pi_e(A) \subset I\) for large \(k\), Wegner’s estimate gives

\[
E[\xi_{k+1}(A)] \leq E[N(H_{k+1}, J_{k+1})] \leq C_W |\pi_e(A)| \\
E[\eta_{k+1,p}(A)] \leq E[N(H_{k,p}, J_{k+1})] \leq C_W \frac{|\pi_e(A)|}{|\Lambda_{k+1}|} \cdot |\Lambda_k|.
\]
Let \( A = J \times B \) \((J \subset \mathbb{R}, B \subset K)\) be an interval. Taking \( D_p' = L_{k+1}^{-1} D_p \), we have
\[
\sum_p \eta_{k+1,p}(A) \leq \sum_{p : D_p' \cap B \neq \emptyset} N(H_{k,p}, E_0 + L_{k+1}^{-d} J).
\]
Since
\[
\# \left\{ p : D_p' \cap B \neq \emptyset \right\} \leq (\text{const.}) \left( \frac{(L_{k+1} J)^d}{L_k^d} \right) \leq (\text{const.}) \frac{|B| \cdot |\Lambda_{k+1}|}{|\Lambda_k|},
\]
we obtain, using Wegner’s estimate again,
\[
\mathbb{E} \left[ \sum_p \eta_{k+1,p}(A) \right] \leq (\text{const.}) \frac{|B| \cdot |\Lambda_{k+1}|}{|\Lambda_k|} \cdot C_W \frac{|J|}{|\Lambda_{k+1}|} |\Lambda_k| = (\text{const.}) C_W |A|.
\]
A density argument gives the result. \( \square \)

The following lemma easily follows from Lemma 2.2(1).

**Lemma 2.3**

(1) \( \{ \eta_{k+1,p} \}_{p=1}^{N_k} \) is a null-array, i.e., for any bounded interval \( A(\subset \mathbb{R} \times K) \),
\[
\lim_{k \to \infty} \sup_{1 \leq p \leq N_k} P(\eta_{k+1,p}(A) \geq 1) = 0.
\]

(2) We have the following tightness condition
\[
\lim_{t \to \infty} \limsup_{k \to \infty} P \left( \sum_p \eta_{k+1,p}(A) \geq t \right) = 0.
\]

Hence by [9, Lemma 4.5], \( \{ \sum_p \eta_{k+1,p} \}_k \) is relatively compact.

We sum up the results obtained in this section.

**Theorem 2.4** Assume Assumption A and \( n(E_0) < \infty \). Then \( \{ \xi_k \} \) has a convergent subsequence and the limiting point \( \xi \) is infinitely divisible whose intensity measure satisfies
\[
\mathbb{E}[\xi(A)] \leq n(E_0) |A|, \quad A \in \mathcal{B}(\mathbb{R} \times K).
\]

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Proof. The infinite divisibility follows from [9, Theorem 6.1], Proposition 2.1 and Lemma 2.3. The claim for the intensity measure follows from the following three considerations.

(1) If \( \xi_k \xrightarrow{d} \xi \), then \( \xi_k f \xrightarrow{d} \xi f \) for \( f \in C_c(\mathbb{R} \times K), f \geq 0 \) [9, Lemma 4.4]. Hence

\[
\mathbb{E}[\xi(f)] \leq \liminf_{k \to \infty} \mathbb{E}[\xi_k(f)].
\]

(2) By a density argument using Lemma 2.2(2), we deduce from (3.6) (Assumption B is not used to derive (3.6))

\[
\mathbb{E}\left[ \sum_p \eta_{k+1,p}(f) \right] \to n(E_0)n ||f||_1, \quad f \in C_c(\mathbb{R} \times K).
\]

(3) By Proposition 2.1

\[
\mathbb{E}\left[ \sum_p \eta_{k+1,p}(f) \right] - \mathbb{E}[\xi_k(f)] \to 0, \quad f \in C_c(\mathbb{R} \times K).
\]

\[\square\]

3 Poisson Limit Theorem

In this section, we show that \( \{\xi_k\} \) converges in distribution to the Poisson process, under Assumption A, B. Two conditions (3.1), (3.2) in Proposition 3.1 below are sufficient to prove that.

Proposition 3.1 Under Assumption A, B, we have for a bounded interval \( A(\subset \mathbb{R} \times K) \),

(1) \( \sum_p \mathbb{P}(\eta_{k+1,p}(A) \geq 2) \to 0 \),

(2) \( \sum_p \mathbb{P}(\eta_{k+1,p}(A) \geq 1) \to n(E_0)|A| \).

Proposition 3.1 together with [9, Corollary 7.5], Proposition 2.1 and Lemma 2.3 proves Theorem 1.2. For its proof, a preparation is necessary.

Lemma 3.2 Assume Assumption A. For an interval \( J(\subset \mathbb{R}) \), we have

\[
\sum_{p=1}^{N_k} \mathbb{E}[\eta_{k+1,p}(J \times K)] \to n(E_0)|J|.
\]
Proof. Since \( \mathbb{E} [ (\xi_{k+1} (J \times K) - \sum_p \eta_{k+1,p} (J \times K) ) ] \to 0 \) by Proposition 2.1 and Lemma 2.2(1), it suffices to show
\[
\mathbb{E} [\xi_{k+1} (J \times K)] \to n(E_0) |J|.
\]
As is done in [16], it is further sufficient to show the above equation for the following function instead of \( 1_J \)
\[
f_\zeta(x) = \frac{\tau}{(x - \sigma)^2 + \tau^2}, \quad \zeta = \sigma + i\tau \in \mathbb{C}_+,
\]
because the set
\[
\mathcal{A} := \left\{ \sum_{j=1}^n a_j f_{\zeta_j}(x) : a_j \geq 0, \ \zeta_j \in \mathbb{C}_+ \right\}
\]
of the finite linear combinations of \( f_\zeta \) with positive coefficients is dense in \( L^1_+(\mathbb{R}) \) [16, Lemma 1], and Lemma 4.6 then enables us to carry out the density argument. Hence it suffices to show
\[
\mathbb{E}[\xi_{k+1}(f_\zeta)] \to \pi n(E_0), \quad \zeta \in \mathbb{C}_+.
\]
For any \( x \in \Lambda_{k+1} \), we have
\[
\mathbb{E}[\xi_{k+1}(f_\zeta)] = \frac{1}{|\Lambda_{k+1}|} \mathbb{E}[\text{Tr } \Im G_{k+1}(E_0 + \frac{\zeta}{|\Lambda_{k+1}|}; x, x)],
\]
since \( H_{k+1} \) has periodic b.c. Let \( G(z) = (H - z)^{-1}, G_{k+1}(z) = (H_{k+1} - z)^{-1} \) be Green’s function of \( H, H_{k+1} \) respectively. Let \( x \) be the center of \( \Lambda_{k+1} \) and let \( z_{k+1} = E_0 + \frac{\zeta}{|\Lambda_{k+1}|} \). Then by the resolvent equation,
\[
|G_{k+1}(z_{k+1}; x, x) - G(z_{k+1}; x, x)| \leq \sum_{\langle y, y' \rangle \in \partial \Lambda_{k+1} \cup \partial \Lambda_{k+1}} |G_{k+1}(z_{k+1}; x, y)G(z_{k+1}; y', x)|
\]
where \( \langle y, y' \rangle \in \partial \Lambda_{k+1} \) means that \( y' \in \partial \Lambda_{k+1} \) is connected to \( y \in \partial \Lambda_{k+1} \) if we regard \( \Lambda_{k+1} \) as a torus. By the multiscale analysis, the event
\[
\mathcal{G}_{k+1}(E) := \{ \omega \in \Omega : \Lambda_{k+1} \text{ is } (\gamma_0, E)\text{-regular} \}
\]
satisfies
\[
\mathbb{P}(\mathcal{G}_{k+1}(E)) \geq 1 - L_{k+1}^{-p}
\]
for any $E \in I$ and $0 < \gamma_0 < \gamma$. Although $H_{k+1}$ has periodic b.c., the procedure of the multiscale analysis requires no essential modifications to prove (3.4). Take $k$ large enough and let $E_{k+1} = \Re z_{k+1}$. We decompose

$$ [E[G_{k+1}(z_{k+1}; x, x)] - E[G(z_{k+1}; x, x)] ] $$

$$ \leq \sum_{(y, y') \in \partial \Lambda_{k+1} \cup \partial \Lambda_{k+1}} E[|G_{k+1}(z_{k+1}; x, y)| \cdot |G(z_{k+1}; y', x)|; G_{k+1}(E_{k+1})] $$

$$ + \sum_{(y, y') \in \partial \Lambda_{k+1} \cup \partial \Lambda_{k+1}} E[|G_{k+1}(z_{k+1}; x, y)| \cdot |G(z_{k+1}; y', x)|; G_{k+1}(E_{k+1})^c] $$

$$ =: I + II. $$

By (3.4), we have

$$ I \leq c d L^{d-1} e^{-\gamma_0 L_{k+1}^{d-1}/2} L_{k+1}^{d} = o(1), \quad II \leq c d L^{d-1} L_{k+1}^{2d} L_{k+1}^{-p}, $$

so that $p > 3d - 1$ is required to have $II = o(1)$, which is guaranteed by Assumption A(1). Therefore

$$ E[\xi_{k+1}(f_c)] = E[\Im G(z_{k+1}; x, x)] + o(1) = \pi n(E_0) + o(1). $$

as $k \to \infty$. $\square$

**Proof of Proposition 3.1** Let $A(\subset \mathbb{R} \times K)$ be a bounded interval. As is discussed in [16], it suffices to show the following equations to prove Proposition 3.1

\begin{align}
(1) & \quad \sum_p \sum_{j \geq 2} P(\eta_{k+1, p}(A) \geq j) \to 0, \quad (3.5) \\
(2) & \quad \sum_p E[\eta_{k+1, p}(A)] \to n(E_0)|A|. \quad (3.6)
\end{align}

In fact, (3.5) trivially implies (3.1), and (3.2) follows from

$$ \sum_p P(\eta_{k+1, p}(A) \geq 1) = \sum_p E[\eta_{k+1, p}(A)] - \sum_p \sum_{j \geq 2} P(\eta_{k+1, p}(A) \geq j) \to n(E_0)|A|. $$

(3.5) in turn follows from Assumption B.Minami’s estimate:

$$ \sum_p \sum_{j \geq 2} P(\eta_{k+1, p}(A) \geq j) \leq \sum_p j(j - 1) P(\eta_{k+1, p}(\pi_c(A) \times K) = j) $$

$$ \leq C M N_k |\Lambda_k|^2 |\Lambda_{k+1}|^2 \to 0 $$

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which is the only (and fundamental) step to use Minami’s estimate.

To prove (3.6), let $J = \pi_e(A), B = \pi_s(A)$ and let $D'_p = L_{k+1} D_p$. We then have

$$
\sum_p \mathbb{E}[\eta_{k+1,p}(A)] = \sum_{D'_p \subset B} \mathbb{E}[\eta_{k+1,p}(A)] + \sum_{B \cap D'_p \neq \emptyset, B \cap D'_c \neq \emptyset} \mathbb{E}[\eta_{k+1,p}(A)] =: I + II.
$$

(3.7)

By Lemma 2.2(1) and by the inequality $\sharp\{p : D'_p \cap B \neq \emptyset, D'_p \cap B^c \neq \emptyset\} \leq (\text{const.}) \left(\frac{L_{k+1}}{L_k}\right)^{d-1}$, we have

$$
II \leq (\text{const.}) \left(\frac{L_{k+1}}{L_k}\right)^{d-1} \cdot \frac{|\Lambda_k|}{|\Lambda_{k+1}|} = (\text{const.}) \frac{L_k}{L_{k+1}}.
$$

(3.8)

To compute $I$, we note $I = \sharp\{p : D'_p \subset B\} \mathbb{E}[\eta_{k+1,p}(J \times K)]$. Substituting $N_k \mathbb{E}[\eta_{k+1,p}(J \times K)] = n(E_0)|J| + o(1)$, which follows from Lemma 3.2 we have

$$
I = \frac{\sharp\{p : D'_p \subset B\}}{N_k} (n(E_0)|J| + o(1)) = n(E_0)|B| \cdot |J| + o(1)
$$

(3.9)

as $k \to \infty$. By (3.7), (3.8) and (3.9), we obtain (3.6). \[\square\]

4 Appendix 1

4.1 Basic properties of localization centers

We review some basic properties of localization centers [5, 13].

Lemma 4.1 [13] Let $H \phi = E \phi, \phi \in l^2(\mathbb{Z}^d)$ ($H = H_{k+1}$ or $H = H_{k,p}$). Then we can find $L_0(d, \gamma)$ such that for $L \geq L_0$, $\Lambda_L(x(\phi))$ (with Dirichlet b.c.) is ($\gamma, E$)-singular.

Lemma 4.2 For any $0 < \gamma_m < \gamma$ we can find $k_1 = k_1(\alpha, d, \gamma, \gamma_m)$ with the following properties. Suppose $\omega \in \Omega_k$ and $\phi \in E f(H_{k+1}, I, C_p)$ for some $p = 1, 2, \cdots, N_k$. Then if $k \geq k_1$ we have

$$
\| (1 - \chi_{D_p}) \phi \|_{L^2(\Lambda_{k+1})} \leq e^{-\gamma_m \frac{L_{k-1}}{2}}.
$$

$D_p, C_p$ are defined in Section 2.
Proof. Take $k_1$ large enough with $L_{k_1} \geq L_0(d, \gamma)$. Since $\omega \in \Omega_k$ and since $\Lambda_{k-1}(x(\phi))$ is $(\gamma, E)$-singular by Lemma 4.1, $\Lambda_{k-1}(x)$ (with Dirichlet b.c.) is $(\gamma, E)$-regular for $x \notin D_p$. Here, as in (2.1), we regard $\Lambda_{k+1}$ as a torus and $\Lambda_{k-1}(x) \subset \Lambda_{k+1}$. Therefore, using $|\phi(y)| \leq 1$ we have

$$|\phi(x)| \leq \sum_{(y,y') \in \partial \Lambda_{k-1}(x)} |G_{\Lambda_{k-1}(x)}(E; x, y)||\phi(y')| \leq c_dL_{k-1}^{-1}e^{-\gamma L_{k-1}/2}.$$  

Taking $k_1(\alpha, d, \gamma, \gamma_m)$ large enough with $L_{k_1}^d c_2 L_{k-1}^{2(d-1)} e^{-\gamma L_{k-1}} \leq e^{-\gamma_m L_{k-1}}$ gives the result. \[\square\]

The following lemma says two localization centers $x(\phi), \langle x \rangle_\phi$ are close in the scale of $L_k$.

**Lemma 4.3** Let $\omega \in \Omega_k, \phi \in \mathcal{E}f(H_{k+1}, I)$. Then for $k \geq k_1(\alpha, d, \gamma, \gamma_m)$, we have

$$|\langle x \rangle_\phi - x(\phi)| \leq L_{k-1} + (\text{const.})e^{-\gamma'' L_{k-1}}, \quad 0 < \gamma'' < \gamma_m.$$  

**Proof.** Set $A_k := \{x \in \Lambda_k : d(x(\phi), x) \leq L_{k-1}\}$. Since by Lemma 4.2, $\sum_{x \in A_k} ||x\rangle \phi(x)||^2 \leq (\text{const.})e^{-\gamma'' L_{k-1}}, 0 < \gamma'' < \gamma_m$, we have

$$|\langle x \rangle_\phi - x(\phi)| \leq \sum_{x \in A_k} ||x-x\rangle\phi(x)||^2 \leq \sum_{x \in A_k} ||x-x\rangle\phi(x)||^2 \leq L_{k-1} + (\text{const.})e^{-\gamma'' L_{k-1}}.$$  

\[\square\]

### 4.2 Comparison of eigenvalues of big and small boxes

In Section 2, we need to show that eigenvalues of $H_{k+1}$ localized in $C_p$ produce those of $H_{k,p}$. The following lemma is an elementary extension of [13, Lemma 1].

**Lemma 4.4** For any $0 < \gamma' < \gamma$, we can find $k_2(\alpha, d, \gamma, \gamma')$ with the following properties. Let $J(\subset I)$ be an interval, $\omega \in \Omega_k$ and $k \geq k_2$. Then

1. $N(H_{k+1}, J, C_p) \leq N(H_{k,p}, J + I(0, \epsilon_{k-1}))$
2. $N(H_{k+1}, J, D_p \setminus C_p) \leq N(H_{k,p}', J + I(0, \epsilon_{k-1}))$
3. $N(H_{k,p}, J, D_p \setminus C_p) \leq N(H_{k,p}', J + I(0, \epsilon_{k-1}))$
where \( \epsilon_{k-1} := e^{-\gamma' L_{k-1}/2} \). \( D_p, C_p, H_{k,p}, H'_{k,p} \) and \( H''_{k,p} \) are defined in Section 2.

**Proof.** It is sufficient to show (1). Let \( \{ \phi_j \}_{j=1}^{M_p} := \mathcal{E}_f(H_{k+1}, J, C_p) \), \( M_p := N(H_{k+1}, J, C_p) \) and set \( \psi_j = \chi_{D_p} \phi_j \). Letting \( \gamma_m = \frac{\gamma + \gamma'}{2} \), we have by Lemma 4.2

\[
\|\psi_j\|_{l^2(D_p)}^2 \geq 1 - e^{-\gamma_m L_{k-1}},
\]

\[
|\langle \psi_i, \psi_j \rangle_{l^2(D_p)}| \leq e^{-\gamma_m L_{k-1}}, \quad i, j = 1, 2, \ldots, M_p, \; i \neq j
\]

for \( k \geq k_1(\alpha, d, \gamma, \gamma_m) \). By (4.1) and (4.2), it is straightforward to prove the following Claim.

**Claim**

1. We can find \( k'(\alpha, d, \gamma_m) \) such that if \( k \geq k' \), \( \psi_1, \ldots, \psi_{M_p} \) are linearly independent.
2. \( \|(H_{k,p} - E_j)\psi_j\|_{l^2(D_p)}^2 \leq \sqrt{2} e^{-\gamma_m L_{k-1}/2}, \; j = 1, 2, \ldots, M_p. \)

Let \( J' := J + I(0, \epsilon_{k-1}) \), let \( P \) be the spectral projection of \( H_{k,p} \) corresponding to \( J' \) and let \( Q = I - P \). Since \( \|(H_{k,p} - E_j)Q\psi_j\|_{l^2(D_p)}^2 \geq \epsilon_{k-1}^2 \|Q\psi_j\|_{l^2(D_p)}^2 \) by the spectral theorem, we have

\[
\|Q\psi_j\|_{l^2(D_p)} \leq \sqrt{2} e^{-(\gamma - \gamma') L_{k-1}/2}, \; j = 1, 2, \ldots, M_p
\]

by Claim (2). Let \( V := \text{Span} \{ \psi_1, \ldots, \psi_{M_p} \} \) and take \( \psi \in V, \|\psi\|_{l^2(D_p)} = 1. \) Writing \( \psi = \sum_j a_j \psi_j \), we have

\[
1 = \|\psi\|_{l^2(D_p)}^2 = \sum_j |a_j|^2 \|\psi_j\|_{l^2(D_p)}^2 + \sum_{i \neq j} a_i \overline{a_j} \langle \psi_i, \psi_j \rangle_{l^2(D_p)},
\]

(4.3)

By inequalities (4.1), (4.2) and

\[
|\text{2nd term of (4.3)}| \leq e^{-\gamma_m L_{k-1}} \sum_{i \neq j} |a_i||a_j| \leq e^{-\gamma_m L_{k-1}} (M_p - 1) \sum_i |a_i|^2,
\]

we have \( \sum_j |a_j|^2 \leq (1 - M_p e^{-\gamma_m L_{k-1}})^{-1} \) and hence

\[
\|Q\psi\|_{l^2(D_p)}^2 \leq \sum_j |a_j|^2 \cdot \sum_j \|Q\psi_j\|_{l^2(D_p)}^2 \leq \frac{2 |\Lambda_{k+1}| e^{-(\gamma - \gamma') L_{k-1}}}{1 - |\Lambda_{k+1}| e^{-\gamma_m L_{k-1}}}.
\]

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Taking $k \geq k_2(\alpha, d, \gamma, \gamma')$ such that \[ \frac{2|\Lambda_k+1|e^{-\gamma m - \gamma'}L_{k-1}}{1-|\Lambda_k+1|e^{-\gamma m L_{k-1}}} < \frac{1}{2}, \] we have \[ \|Q\psi\|_{L^2(D_p)}^2 < \frac{1}{2}\|\psi\|_{L^2(D_p)}^2 \] and hence
\[ \|P\psi\|_{L^2(D_p)}^2 > \frac{1}{2}\|\psi\|_{L^2(D_p)}^2, \]
which implies $P$ is injective on $V$. Therefore \( \dim \text{Ran } P \geq \dim PV = M_p. \)

We next do the converse: we show that an eigenvalues of $H_{k,p}$ localized in $C_p$ produce those of $H_{k+1}$. Since the proofs are similar to those of Lemma 4.2 and 4.4, we state the result only.

**Lemma 4.5** For any $0 < \gamma' < \gamma$, we can find $k_3 = k_3(\alpha, d, \gamma, \gamma')$ with the following property. Suppose $\omega \in \Omega_k, J(\subset I)$ is an interval and $k \geq k_3$, then
\[ \sum_p N(H_{k,p}, J, C_p) \leq N(H_{k+1}, J + I(0, \epsilon_{k-1}). \]

**4.3 A priori estimate**

We show a priori estimate for $E[|\xi_{k+1}(g)|]$ for $g(x) = O(|x|^{-2})$ as $|x| \to \infty$.

**Lemma 4.6** Suppose $g$ is bounded and measurable on $\mathbb{R}$, satisfying
\[ |g(x)| \leq \frac{C_R}{x^2}, \quad |x| \geq R \]
for some $R > 0$ and $C_R > 0$. Let $r := d(E_0, I^c) > 0$. If $r|\Lambda_{k+1}| \geq R$, we have
\[ E[|\xi_{k+1}(g)|] \leq C_W \int_{|\lambda|<r|\Lambda_{k+1}|} |g(\lambda)|d\lambda + \frac{C_R}{r^2|\Lambda_{k+1}|}. \]

**Proof.** We decompose
\[ \xi_{k+1}(g) = \sum_{E_j \in I} g(|\Lambda_{k+1}|(E_j(\Lambda_{k+1}) - E_0)) + \sum_{E_j \in I^c} g(|\Lambda_{k+1}|(E_j(\Lambda_{k+1}) - E_0)) \]
\[ = I + II. \]

$II$ is estimated by using the assumption on $g$.
\[ |II| \leq |\Lambda_{k+1}| \cdot \frac{C_R}{r^2|\Lambda_{k+1}|^2}. \] (4.4)
To estimate $I$, we note $I = \xi_{k+1}(g1_{\{|x|<r|\Lambda_{k+1}|\}})$. If $g = 1_J$ for some interval $J \subset \{x \in \mathbb{R} : |x| < r|\Lambda_{k+1}|\}$, we have by Wegner’s estimate,

$$\mathbb{E}[\xi_{k+1}(g1_{\{|x|<r|\Lambda_{k+1}|\}})] = \mathbb{E}[N(H_{k+1}, E_0 + \frac{J}{|\Lambda_{k+1}|})] \leq C_W |J| = C_W \int_{\{|\lambda|<r|\Lambda_{k+1}|\}} |g(\lambda)|d\lambda. \quad (4.5)$$

A density argument proves (4.5) for $g$ bounded and measurable. Together with (4.4), we arrive at the conclusion.

5 Appendix 2

In this section, we consider the random measure $\xi$ studied in [10], and examine its natural scaling limit under Assumption A. $\xi$ is defined by

$$\xi(J \times B) := \text{Tr} \left( 1_B(x) P_J(H) \right)$$

for an interval $J \times B$ ($J \subset \mathbb{R}$, $B \subset \mathbb{R}^d$), and its scaling $\xi_L$ is given by

$$\xi_L(J \times B) := \text{Tr} \left( 1_{LB}(x) P_{E_0+L^{-d}J}(H) \right), \quad L > 0$$

which is done in the same spirit of $\xi_k$. $P_J(H)$ is the spectral projection of $H$ corresponding to $J$. We then have

**Theorem 5.1** Suppose Assumption A (with $p > 8d - 2$) and $n(E_0) < \infty$. Then we can find a convergent subsequence $\{L_k\}_{k=1}^\infty$ such that $\xi_{L_k}$ converges in distribution to a infinitely divisible point process $\xi$ on $\mathbb{R}^{d+1}$ with its intensity measure satisfying

$$\mathbb{E}\xi(dE \times dx) \leq n(E_0)dE \times dx.$$ 

For its proof, we take $l_L = O(L^\beta)$ for some $0 < \beta < 1$ and consider

$$B_p(L) := \{x \in \mathbb{Z}^d : p_j l_L \leq x_j < (p_j + 1)l_L, \quad j = 1, \cdots, d\}, \quad p \in \mathbb{Z}^d$$

$$H_{L,p} := H|_{B_p(L)}, \quad H_L := \oplus_p H_{L,p}$$

as is done in [10], with the periodic boundary condition. Let $\tilde{\eta}_{L,p}$ be the random measure defined by

$$\tilde{\eta}_{L,p}(J \times B) := \text{Tr} \left( 1_{LB}(x) P_{E_0+L^{-d}J}(H_{L,p}) \right).$$

We then have
Proposition 5.2 Suppose Assumption A with \( p > 8d - 2 \). Then for \( f \in C_c(\mathbb{R}^{d+1}) \),
\[
\mathbb{E}[|\xi_L(f) - \sum_p \tilde{\eta}_{L,p}(f)|] \to 0. \tag{5.1}
\]

Sketch of proof of Proposition 5.2

Step 1: We first show (5.1) for \( f(E,x) = 1_B(x) f_\zeta(E) \) for a box \( B \subset \mathbb{Z}^d \) and \( \zeta \in C_+ \). \( f_\zeta \) is defined in Section 3. In order to do that, we use the resolvent equation, decompose the expectation into good and bad events, and use the following estimate given by the multiscale analysis:
\[
\mathbb{P} \left( \sup_{\epsilon > 0} |G_\Lambda(E + i\epsilon; x, y)| \leq e^{-\frac{2}{\epsilon} |x-y|} \right) \geq 1 - C |\Lambda| |x-y|^{-p/2} \tag{5.2}
\]
for any \( E \in I \), any box \( \Lambda \) (\( H_\Lambda \) has periodic b.c.) and any \( x, y \in \Lambda \) with \( |x-y| \geq C \) for some \( C \).

Step 2: We prove a simple estimate
\[
\mathbb{E} \left[ \left| \int 1_B(x) g(E) d\xi_L \right| \right] \leq C_W (1 + o(1)) |B| \|g\|_1 + \frac{(\text{const.})}{L^d} \tag{5.3}
\]
for \( g \) bounded and measurable with \( |g(x)| \leq \frac{C_R}{x} \), \( |x| \geq R \) for some \( R > 0 \) and \( C_R > 0 \). The estimate (5.3) can be proved similarly as Lemma 3.2 and Lemma 4.6. By a density argument using (5.3), we can show (5.1) for \( f(E,x) = 1_B(x) g(E) \) for a box \( B \subset \mathbb{Z}^d \) and \( g \in C_c(\mathbb{R}) \). Then we can further extend (5.1) to arbitrary \( f \in C_c(\mathbb{R}^{d+1}) \) by using some a priori estimates stated below: for any \( C > 0 \) we can find \( L_0(C) \) with
\[
(1) \quad \mathbb{E} \left[ \left| \int f(E,x) d\xi_L \right| \right] \leq 2n(E_0) \|f\|_1 \tag{5.4}
\]
(2) \[ \mathbb{E} \left[ \sum_p \left| \int f(E,x) d\tilde{\eta}_{L,p} \right| \right] \leq C_W \|f\|_1. \]

for \( \text{supp} \ f \subset \{(E,x) | \leq C \} \) and \( L \geq L_0(C) \). An alternative way to prove Proposition 5.2 is to use the almost analytic extensions which also applies to the continuum analog of this statement.

The facts that the sequence \( \{\xi_L\} \) is a null-array and relatively compact follow from (5.4), and then Proposition 5.2 proves the infinite divisibility of the limiting random measure \( \xi \). The infinite divisibility of \( \xi \) as a point process and the estimate for its intensity measure \( \mathbb{E} \xi(dE \times dx) \) follow similarly as in [10], completing the proof of Theorem 5.1.
Remark 5.3 Let $B \subset \mathbb{Z}^d$ be a finite box and let $H_{LB} := H\big|_{LB}$ be a restriction of $H$ on $LB$ with some boundary condition. Define a random measure $\xi_{L,B}$ on $\mathbb{R} \times B$ by

$$
\xi_{L,B}(J \times C) = \text{Tr} \left( 1_{\mathcal{L}_C}(x) P_{E_0 + L^{-d} J(H_{LB})} \right), \quad J \subset \mathbb{R}, \ C \subset B.
$$

Then for $f \in C_c(\mathbb{R} \times B)$, the proof of Proposition 5.2 tells us that $\xi_L(f) - \xi_{L,B}(f) \overset{a.s.}{\to} 0$. Therefore the eigenvalues and the eigenfunctions on $H_{LB}$ and those of $H$ localized in $LB$ has the same behavior in this sense.

6 Appendix 3

In this section we assume both Assumption A and B, and present another presentation of Step 3 in the proof of Proposition 2.1: we show the following equation for $f \in C_c(\mathbb{R} \times K)$.

$$
\sum_p E \left[ |\xi_{k+1,p}^{(1)}(f) - \eta_{k+1,p}(f)|; \Omega_k \right] = o(1). \quad (6.1)
$$

For simplicity, let

$$
J_{k+1}' := J_{k+1} + I(0, \epsilon_{k-1}).
$$

and decompose the LHS of (6.1) as

LHS of (6.1) = \sum_p E \left[ |\xi_{k+1,p}^{(1)}(f) - \eta_{k+1,p}(f)|; \Omega_k \cap \{N(H_{k,p}, J_{k+1}'; \Omega_k) = 1\} \right]

+ \sum_p E \left[ |\xi_{k+1,p}^{(1)}(f) - \eta_{k+1,p}(f)|; \Omega_k \cap \{N(H_{k,p}, J_{k+1}'; \Omega_k) \geq 2\} \right]

=: A + B.

Claim 1 : $B = o(1)$.

Proof of Claim 1 We write $B = \sum_p B_p$. By Lemma 4.4(1) and by Minami’s estimate, we have

$$
B_p \leq 2\|f\|_\infty E \left[ N(H_{k,p}, J_{k+1}'; \Omega_k) \cap \{N(H_{k,p}, J_{k+1}') \geq 2\} \right]

\leq 2\|f\|_\infty \sum_{j \geq 2} j(j-1) \mathcal{P} \left( N(H_{k,p}, J_{k+1}') = j \right)

\leq 2\|f\|_\infty C_M \left( \frac{2\alpha}{|\Lambda_{k+1}|} + 2\epsilon_{k-1} \right)^2 \cdot |\Lambda_k|^2
$$
which shows \( B \leq (\text{const.}) \frac{|A_k|}{|\Lambda_{k+1}|} \) and thus proves Claim 1. \( \square \)

To estimate \( A \), we further decompose \( A = A_1 + A_2 \) with
\[
A_1 = \sum_p \mathbb{E} \left[ \eta_{k+1,p}(f) - \xi^{(1)}_{k+1,p}(f); \Omega_k \cap \{ N(H_{k,p}, J'_{k+1}) = 1, N(H_{k+1}, J_{k+1}, C_p) = 1 \} \right]
\]
\[
A_2 = \sum_p \mathbb{E} \left[ \eta_{k+1,p}(f); \Omega_k \cap \{ N(H_{k,p}, J'_{k+1}) = 1, N(H_{k+1}, J_{k+1}, C_p) = 0 \} \right].
\]

Here we note that \( |\xi^{(1)}_{k+1,p}(f) - \eta_{k+1,p}(f)| = 0 \) if \( N(H_{k,p}, J'_{k+1}) = 0 \) by Lemma 4.4(1).

**Claim 2 :** \( A_2 = o(1). \)

**Proof of Claim 2** Lemma 6.1 and the argument in the proof of Claim 1 gives
\[
\mathbb{E}[N(H_{k+1}, J_{k+1})] = \sum_p \mathbb{E} \left[ N(H_{k+1}, J_{k+1}, C_p); \Omega_k \cap \{ N(H_{k,p}, J'_{k+1}) = 1 \} \right] + o(1) \quad (6.2)
\]
\[
\mathbb{E}[N(H_{k+1}, J_{k+1})] = \sum_p \mathbb{E}[N(H_{k,p}, J'_{k+1})] + o(1)
\]
\[
= \sum_p \mathbb{E} \left[ N(H_{k,p}, J'_{k+1}); \Omega_k \cap \{ N(H_{k,p}, J'_{k+1}) = 1 \} \right] + o(1). \quad (6.3)
\]

By Lemma 4.4(1), (6.2) and (6.3), we have
\[
0 \leq \sum_p \mathbb{E} \left[ N(H_{k,p}, J'_{k+1}) - N(H_{k+1}, J_{k+1}, C_p); \Omega_k \cap \{ N(H_{k,p}, J'_{k+1}) = 1 \} \right] = o(1). \quad (6.4)
\]

Since we have
\[
|\eta_{k+1,p}(f)| \leq \|f\|_\infty \left( N(H_{k,p}, J'_{k+1}) - N(H_{k+1}, J_{k+1}, C_p) \right)
\]
on the event in which \( A_2 \) is computed, (6.4) implies \( A_2 = o(1) \) and thus proves Claim 2. \( \square \)

On the event in which \( A_1 \) is computed, it is easy to construct bijective correspondence between \( \mathcal{E}(H_{k+1}, J_{k+1}, C_p) \) and \( \mathcal{E}(H_{k,p}, J'_{k+1}) \) which proves Proposition 2.1. It remains to show the following lemma.
Lemma 6.1 If $p > 12d$ in Assumption A(1), we have

$$\mathbb{E}[N(H_{k+1}, J_{k+1})] = \sum_p \mathbb{E}[N(H_{k,p}, J'_{k+1})] + o(1).$$

Proof. By Wegner’s estimate, it suffices to show $\mathbb{E}[N(H_{k+1}, J_{k+1})] = \sum_p \mathbb{E}[N(H_{k,p}, J_{k+1})] + o(1)$. By Lemma 2.2, it is further reduced to $\mathbb{E} \left[ |\xi_{k+1}(f) - \sum_p \eta_{k+1,p}(f) | \right] = o(1)$ for any $f \in C_c(\mathbb{R})$. By the density of $A$ in $L^1_+(\mathbb{R})$, it is sufficient to take $f = f_{\zeta}$, $\zeta \in C_+ \text{ in which case the proof can be done by using the resolvent equation and the exponential decay of Green’s functions } (5.2)$. \[\square\]

7 Appendix 4: Proof of Theorem 1.1

To prove Theorem 1.1 it suffices to show

$$\tilde{\xi}_k(J \times B) \rightarrow \nu(J)|B|, \text{ a.s.} \quad (7.1)$$

for intervals $J \subset I, B \subset K$ with rational endpoints. Let $B'_k := (L_k B) \cap \mathbb{Z}^d$. Then $|B'_k| = |B|L^d_k(1 + o(1))$ for large $k$ and

$$\tilde{\xi}_k(J \times B) = \frac{1}{|\Lambda_k|} N(H_k, J, B'_k). \quad (7.2)$$

We also consider a box $D_k$ by eliminating a strip of width $2L_{k-1}$ from the boundary of $B'_k$ and further consider boxes $B''_k$ (resp. $B'''_k$) obtained by adding a strip of width $L_{k-1}$ in both sides of the strip $B'_k \setminus D_k$ in $\Lambda_k$ (resp. in $B'_k$):

- $D_k := \{x \in B'_k : d(x, \partial B'_k) \geq 2L_{k-1} \}$
- $B''_k = \{x \in \Lambda_k : d(x, \partial(B'_k \setminus D_k)) \leq L_{k-1} \}$,
- $B'''_k = \{x \in B'_k : d(x, \partial(B'_k \setminus D_k)) \leq L_{k-1} \}$.

We take any $0 < \gamma' < \gamma$ and let

- $H'_k := H|_{B'_k}$ (periodic b.c.)
- $H''_k := H|_{B''_k}$, $H'''_k := H'|_{B'''_k}$ (Dirichlet b.c.)
- $\epsilon_{k-1} := e^{-\gamma'L_{k-1}/2}$.
We first decompose

\[ N(H_k, J, B'_k) = N(H_k, J, D_k) + N(H_k, J, B'_k \setminus D_k). \]  \hspace{1cm} (7.3)

To estimate the second term, we consider the following event

\[ \Omega'_k := \left\{ \omega \in \Omega : \text{For all } E \in I, \text{ either } \Lambda_{L_k - 1}(x) \text{ or } \Lambda_{L_k - 1}(y) \text{ is } (\gamma, E)-\text{regular} \right\} \]  \hspace{1cm} (7.4)

where \( H_{\Lambda_{L_k - 1}(x)}, H_{\Lambda_{L_k - 1}(y)} \) have Dirichlet b.c. As in (2.1), we regard \( \Lambda_k, B'_k \) as torus. Since

\[ P(\Omega'_k) \leq (const.) L_{2d - 2p}^{-2p}, \quad p > 2d, \quad \omega \in \Omega'_0 := \liminf_{k \to \infty} \Omega'_k \]

satisfies \( P(\Omega'_0) = 1 \), and for \( \omega \in \Omega'_0 \) we can find \( k'_0(\omega) \) with \( \omega \in \Omega'_k \) if \( k \geq k'_0 \).

The following lemma is proved similarly as Lemma 4.4.

**Lemma 7.1** We can find \( k_4(\alpha, d, \gamma, \gamma') \) such that, if \( k \geq k_4(\alpha, d, \gamma, \gamma') \) and \( \omega \in \Omega'_k \), we have

\( 1 \) \hspace{0.5cm} \( N(H_k, J, D_k) \leq N(H'_k, J + I(0, \epsilon_{k - 1})) \)

\( 2 \) \hspace{0.5cm} \( N(H_k, J, B'_k \setminus D_k) \leq N(H''_k, J + I(0, \epsilon_{k - 1})) \)

\( 3 \) \hspace{0.5cm} \( N(H'_k, J, B'_k \setminus D_k) \leq N(H'''_k, J + I(0, \epsilon_{k - 1})) \).

The following lemma is similar to Lemma 4.5 but additionally has a control on the location of localization centers of the big box.

**Lemma 7.2** We can find \( k_5(\alpha, d, \gamma, \gamma') \) such that, if \( k \geq k_5(\alpha, d, \gamma, \gamma') \) and \( \omega \in \Omega'_k \), we have

\[ N(H'_k, J, D_k) \leq N(H_k, J + I(0, \epsilon_{k - 1}), B'_k). \]

**Idea of proof of Lemma 7.2**

Let

\[ C_k := \left\{ x \in B'_k : d(x, \partial B'_k) \geq L_{k - 1} \right\}, \]

let \( M := N(H'_k, J, D_k) \) and let \( P \) be the spectral projection of \( H_k \) corresponding to \( J + I(0, \epsilon_{k - 1}) \). Since \( \phi_1, \ldots, \phi_M \in \mathcal{E} f(H'_k, J, D_k) \) decay exponentially on \( C'_k \), so are \( P\phi_1, \ldots, P\phi_M \). We can write

\[ P\phi_1 = \psi_1 + \psi'_1, \ldots, P\phi_M = \psi_M + \psi'_M, \]
where \( \{ \psi_j \} \subset \text{Span } \mathcal{E} f(H_k, J + I(0, \epsilon_{k-1}), B_k') \), \( \{ \psi'_j \} \subset \text{Span } \mathcal{E} f(H_k, J + I(0, \epsilon_{k-1})), (B_k')^c \). Since \( \{ P \phi_l \} \) are ONS on \( l^2(C_k) \) modulo exponential error, and since \( \psi'_j \) decays exponentially on \( C_k \), \( \psi_1, \ldots, \psi_M \) are linearly independent so that \( N(H_k, J + I(0, \epsilon_{k-1}), B_k') \geq M. \)

We further take \( k \geq k_4(\alpha, d, \gamma, \gamma') \vee k_5(\alpha, d, \gamma, \gamma') \). Since by Lemma 7.1(2), \( N(H_k, J, B_k' \setminus D_k) \leq N(H''_k, J + I(0, \epsilon_{k-1})) \leq (\text{const.})L_{k-1}L_{d-1}^d \), we have

\[
\frac{1}{|\Lambda_k|} N(H_k, J, B_k' \setminus D_k) \leq \frac{L_{k-1}}{L_k} = o(1). \tag{7.5}
\]

In what follows, we assume that the origin is the lower-left endpoint of \( B \). By (1.3) and by Lemma 7.1(1) it follows that, for any \( \epsilon > 0 \)

\[
N(H_k, J, D_k) \leq N(H'_k, J + I(0, \epsilon_{k-1})) \leq |B||\Lambda_k|(|\nu(J) + \epsilon|
\]

for large \( k \). Together with (7.2), (7.3) and (7.5), we have

\[
\limsup_{k \to \infty} \frac{1}{|\Lambda_k|} \xi_k(J \times B) \leq |B|\nu(J). \tag{7.6}
\]

On the other hand, by Lemma 7.1(2)

\[
N(H'_k, J + I(0, \epsilon_{k-1})) = N(H'_k, J + I(0, \epsilon_{k-1}), D_k) + N(H'_k, J + I(0, \epsilon_{k-1}), B'_k \setminus D_k) \leq N(H_k, J + I(0, 2\epsilon_{k-1}), B'_k) + N(H''_k, J + I(0, \epsilon_{k-1})) \leq N(H_k, J + I(0, 2\epsilon_{k-1}), B'_k) + (\text{const.})L_{k-1}L_{k-1}^d \]

Hence for large \( k \),

\[
N(H'_k, J - I(0, \epsilon)) \leq N(H_k, J, B_k') + (\text{const.})L_{k-1}L_{k-1}^d \]

for any \( \epsilon > 0 \). Dividing by \( |\Lambda_k| \) and letting \( k \to \infty \), we have

\[
|B|(\nu(J) - \epsilon) \leq \liminf_{k \to \infty} \frac{1}{|\Lambda_k|} \xi_k(J \times B). \tag{7.7}
\]

(7.6), (7.7) prove (7.1) if the origin is the lower-left endpoint of \( B \). For general \( B \), (7.1) follows from a subtraction argument.

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