On doubly resolving sets in graphs

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Abstract

Two vertices $u, v$ in a connected graph $G$ are doubly resolved by vertices $x, y$ of $G$ if

$$d(v, x) - d(u, x) \neq d(v, y) - d(u, y).$$

A set $W$ of vertices of the graph $G$ is a doubly resolving set for $G$ if every two distinct vertices of $G$ are doubly resolved by some two vertices of $W$. Doubly resolving number of a graph $G$, denoted by $\psi(G)$, is the minimum cardinality of a doubly resolving set for the graph $G$. The aim of this paper is to investigate doubly resolving sets in graphs. An upper bound for $\psi(G)$ is obtained in terms of order and diameter of $G$. $\psi(G)$ is computed for some graphs and all graphs $G$ of order $n$ with the property $\psi(G) = n - 1$ are determined. Also, doubly resolving sets for unicyclic graphs are studied and it is proved that the difference between the number of leaves and doubly resolving number of a unicyclic graph is at most 2.

Keywords: doubly Resolving set; doubly resolving number; resolving set; unicyclic graph.

AMS Mathematical Subject Classification [2020]: 05C12

1 Introduction

In this section, we present some definitions and known results which are necessary to prove our main results. Throughout this paper, $G$ is a simple connected graph with vertex set $V(G)$, edge set $E(G)$ and order $n(G)$. The distance between two vertices $u$ and $v$, denoted by $d_G(u, v)$, is the length of a shortest path between $u$ and $v$ in $G$. Also, $N_G(v)$ is the set of all neighbors of vertex $v$ in $G$. We write these simply $d(u, v)$ and $N(v)$, when no confusion can arise. The diameter of a graph $G$ is $\text{diam}(G) = \max_{u,v \in V(G)} d(u, v)$. A unicyclic graph is a graph with exactly one cycle. The symbols $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_n, v_1)$ represent a path of order $n$, $P_n$, and a cycle of order $n$, $C_n$, respectively.

For an ordered subset $W = \{w_1, \ldots, w_k\}$ of $V(G)$ and a vertex $v$ of $G$, the metric representation of $v$ with respect to $W$ is

$$r(v|W) = (d(v, w_1), \ldots, d(v, w_k)).$$
The set \( W \) is a *resolving set* for \( G \) if the distinct vertices of \( G \) have different metric representations, with respect to \( W \). A resolving set \( W \) for \( G \) with minimum cardinality is a *metric basis* of \( G \), and its cardinality is the *metric dimension* of \( G \), denoted by \( \text{dim}(G) \).

The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [16] and by Harary and Melter [8]. Resolving sets have several applications in diverse areas such as coin weighing problems [15], network discovery and verification [3], robot navigation [10], mastermind game [5], problems of pattern recognition and image processing [14], and combinatorial search and optimization [15]. For more results about resolving sets and metric dimension see [2, 4, 5, 7, 9].

During the study of the metric dimension of the cartesian product of graphs Caceres et al. [5] defined the concept of doubly resolving sets in graphs. Two vertices \( u, v \) in a graph \( G \) are doubly resolved by \( x, y \in V(G) \) if

\[
d(v, x) - d(u, x) \neq d(v, y) - d(u, y).
\]

A set \( W \) of vertices of the graph \( G \) is a doubly resolving set for \( G \) if every two distinct vertices of \( G \) are doubly resolved by some two vertices of \( W \). Every graph with at least two vertices has a doubly resolving set. A doubly resolving set for \( G \) with minimum cardinality is called a doubly basis of \( G \) and its cardinality is called the doubly resolving number of \( G \) and denoted by \( \psi(G) \).

Caceres et al. [5] obtained doubly resolving number of trees, cycles and complete graphs. In [11] it was proved that the problem of finding doubly bases is NP-hard. Doubly resolving number of Prism graphs and Hamming graphs is computed in [6] and [12], respectively. For more results about doubly resolving sets in graphs see [1, 5, 11, 13].

Caceres et al. [5] obtained an upper bound for the metric dimension of cartesian product of graphs \( G \) and \( H \) in terms of \( \psi(H) \) and \( \text{dim}(G) \). They also obtained a lower bound for the metric dimension of the cartesian product of a graph \( G \) with itself in terms of \( \psi(G) \). Hence computing doubly resolving number of graphs is useful for computing metric dimension of cartesian product of graphs. Moreover, studying doubly resolving sets is interesting by itself. It is clear that for each graph of order at least 2, we have \( \psi(G) \geq 2 \). Caseres et al. found the following upper bound for \( \psi(G) \).

**Lemma 1.1** [5] For every graph \( G \) with \( n \geq 3 \) vertices we have \( \psi(G) \leq n - 1 \).

Also, through the following three lemmas, they found the doubly resolving number of complete graphs, paths and cycles.

**Lemma 1.2** [5] For all \( n \geq 2 \) we have \( \psi(K_n) = \max\{n - 1, 2\} \).

**Lemma 1.3** [5] For each \( n \geq 2 \), \( \psi(P_n) = 2 \).

**Lemma 1.4** [5] Let \( C_n \) be a cycle of order \( n \). Then

\[
\psi(C_n) = \begin{cases} 
2 & \text{if } n \text{ is odd}, \\
3 & \text{if } n \text{ is even}.
\end{cases}
\]
In this paper we investigate doubly resolving sets for graphs. In Section 2 some properties of doubly resolving bases of graphs are presented, an upper bound for $\psi(G)$ in terms of diameter and order of $G$ is obtained and the doubly resolving number of complete bipartite graphs is computed. In Section 3 all $n$-vertex graphs $G$ with $\psi(G) = n - 1$ are determined. In Section 4 doubly resolving sets of unicyclic graphs are investigated and it is proved that the difference between the number of leaves and doubly resolving number of a unicyclic graph is at most 2.

2 Some results on doubly resolving sets

In this section, we present some results about doubly resolving sets of graphs. We obtain the doubly resolving number of some famous families of graphs. An important upper bound for doubly resolving number of graphs is obtained in terms of order and diameter of the graph.

By the following lemma, to check whether a given set $W \subseteq V(G)$ is a doubly resolving set, it does not need to consider the pair of vertices that both of them are in $W$.

Lemma 2.1 Let $W$ be a subset of size at least 2 of $V(G)$. If $x, y \in W$, then $x, y$ are doubly resolved by $x, y \in W$. Also if $a \in W$ and $b \notin W$ are two vertices in $G$ that are not doubly resolved by any pair of vertices in $W$, then for each $w \in W$ there exists a shortest path between $b$ and $w$ that contains $a$.

Proof. Let $x, y \in W$. Then $d(x, x) - d(y, x) = -d(x, y) \neq d(x, y) = d(x, y) - d(y, y)$. That is $x, y$ are doubly resolved by $x, y$. Now let $a \in W$, $b \in V(G) \setminus W$ and $a, b$ are not doubly resolved by any pair of vertices in $W$. For each $w \in W$, we have $d(a, w) - d(b, w) = d(a, a) - d(b, a)$, which implies $d(b, w) = d(b, a) + d(a, w)$. If $P$ and $Q$ are shortest paths between $b, a$ and $a, w$, respectively, then $P \cup Q$ is a path between $b, w$ with length $d(b, w)$. Therefore $P \cup Q$ is a shortest path between $b, w$ that contains $a$.

Two distinct vertices $u, v$ are said to be twins if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. It is clear that if $u, v$ are twin vertices in $G$, then for every vertex $x \in V(G)$, $d(u, x) = d(v, x)$.

Proposition 2.2 Suppose that $u, v$ are twins in a graph $G$ and $W$ is a doubly resolving set for $G$. Then at least one of the vertices $u$ and $v$ is in $W$. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ is also a doubly resolving set for $G$.

Proof. Let $u, v$ be twins, $W$ be a doubly resolving set and $u, v \notin W$. Then for each $a, b \in W$,

$$d(u, a) - d(v, a) = 0 = d(u, b) - d(v, b).$$

This contradiction implies that $W$ must contain at least one of $u$ or $v$. Now let $W$ be a doubly resolving set for $G$ such that $u \in W$ and $v \notin W$. If $(W \setminus \{u\}) \cup \{v\}$ is not a doubly resolving set for $G$, then there are vertices $x, y \in V(G)$ and $w \in W$ such that $x, y$ are doubly resolved by $u, w$ and are not doubly resolved by $v, w$. But

$$d(x, v) - d(y, v) = d(x, u) - d(y, u) \neq d(x, w) - d(y, w).$$

This means $(W \setminus \{u\}) \cup \{v\}$ is a doubly resolving set for $G$. 

3
One of the well-known families of graphs is the family of complete bipartite graphs. By the next lemma we compute the doubly resolving number of complete bipartite graphs.

**Lemma 2.3** Let $K_{r,s}$ be a complete bipartite graph of order $n \geq 3$ and $r \leq s$. Then

$$\psi(K_{r,s}) = \begin{cases} 
  n - 1 & \text{if } r \leq 2, \\
  n - 2 & \text{if } r > 2.
\end{cases}$$

**Proof.** Suppose that $K_{r,s}$ be a complete bipartite graph with partite sets $X,Y$ such that $|X| = r$ and $|Y| = s$. Since $n \geq 3$ and $r \leq s$, we have $s \geq 2$. Let $W$ be a doubly resolving set for $K_{r,s}$. One of the following three cases can be arisen.

Case 1. $r = 1$. If $s = 2$, then by Lemma 1.3, $\psi(K_{1,2}) = 2 = n - 1$. Let $s \geq 3$. Then all vertices in $Y$ are twins and by Proposition 2.2, $|W \cap Y| \geq s - 1$. Hence $|W| \geq n - 2$. If $|W| = n - 2$, then there exists a vertex $y_0 \in Y \setminus W$. Now, for each $w \in W$, $d(y_0, w) - d(x_0, w) = 1$, where $X = \{x_0\}$. That means $W$ is not a doubly resolving set. This contradiction yields $\psi(K_{1,s}) = s = n - 1$.

Case 2. $r = 2$. In this case all vertices in $X$ are twins, also all vertices in $Y$ are twins. Hence by Proposition 2.2, $|W \cap Y| \geq s - 1$ and $|W \cap X| \geq 1$. Therefore $|W| \geq n - 2$. If $|W| = n - 2$, then $X \cap W = \{x_1\}$ and there exists a vertex $y_0 \in Y \setminus W$. Note that for each $y \in W \cap Y$, we have $d(y_0, y) - d(x_1, y) = 2 - 1 = 1$ and $d(y_0, x_1) - d(x_1, x_1) = 1 - 0 = 1$, which is impossible. Therefore, $\psi(K_{2,s}) = s + 1 = n - 1$.

Case 3. $r \geq 3$. In this case all vertices in $X$ are twins, also all vertices in $Y$ are twins. Hence by Proposition 2.2, $|W \cap Y| \geq s - 1$ and $|W \cap X| \geq r - 1$. Therefore $|W| \geq n - 2$. Let $W$ be a set of vertices of size $n - 2$ such that $|W \cap Y| = s - 1$ and $|W \cap X| = r - 1$. Consider $a,b \in V(K_{r,s})$. If $a,b \in Y$, then by Lemma 2.1 we can assume that $a \in W$ and $b \not\in W$. Since $s \geq 3$, there exists a vertex $a \neq w \in W \cap Y$ and we have $d(a,a) - d(b,a) = -2$ and $d(a,w) - d(b,w) = 0$. Hence $a,b$ are doubly resolved by $a,w$. If $a,b \in X$, by a same argument we deduce that $a,b$ are doubly resolved by a pair of vertices in $W$. Now let $a \in X$ and $b \in Y$. Since $r,s$ are at least 3, there exists vertices $x \in W \cap X \setminus \{a\}$ and $y \in W \cap Y \setminus \{b\}$. Note that

$$d(a,x) - d(b,x) = 2 - 1 \neq 1 - 2 = d(a,y) - d(b,y).$$

Therefore $W$ is a doubly resolving set for $K_{r,s}$ and $\psi(K_{r,s}) = n - 2$.

Through the following theorem we obtain an upper bound for $\psi(G)$ in terms of order and diameter of the graph $G$.

**Theorem 2.4** If $G$ is a graph with diameter $d$, then $\psi(G) \leq n - d + 1$.

**Proof.** Let $P = (v_0, v_1, \ldots, v_d)$ be a shortest path in $G$. We claim that $W = V(G) \setminus \{v_1, v_2, \ldots, v_{d-1}\}$ is a doubly resolving set for $G$. If $v_i,v_j \in V(G) \setminus W$, then

$$d(v_i, v_0) - d(v_j, v_0) = i - j \neq j - i = (d - i) - (d - j) = d(v_i, v_d) - d(v_j, v_d).$$

Therefore $v_0$ and $v_d$ doubly resolve $v_i$ and $v_j$. If $v_i \in V(G) \setminus W$ and $a \in W$ are two vertices that are not doubly resolved by $W$, then

$$d(a,a) - d(v_i,a) = d(a,v_0) - d(v_i,v_0) \Rightarrow d(v_0,a) = d(v_i,v_0) - d(v_i,a),$$

where
Proof. By Theorem 2.4, it follows that $\text{diam}(G) \leq 2$. Suppose that $x$ is a vertex of degree $n - 1$ and $I$ is a maximum independent subset of $N(x)$. If $|I| = |N(x)|$, then $G = K_{1,n-1}$ and if $|I| = 1$, then $G = K_n$. Hence assume that $2 \leq |I| \leq n - 2$. This implies that there exists a vertex $y \in N(x) \setminus I$. We claim that $y$ is adjacent to all vertices in $I$. By definition of $I$, $y$ has a neighbour in $I$, say $a$. Since $\psi(G) = n - 1$, the set $W = V(G) \setminus \{x, y\}$ is not a doubly resolving set for $G$. If there exists a vertex $b \in I$ that is not adjacent to $y$, then

$$d(x, a) - d(y, a) = 1 - 1 \neq 1 - 2 = d(x, b) - d(y, b).$$

By Lemma 2.3 for every $n \geq 2$, $\psi(P_n) = 2 = n - \text{diam}(P_n) + 1$. Therefore the bound in Theorem 2.4 is sharp.

3 Graphs of order $n$ and doubly resolving number $n - 1$

The aim of this section is to determine all $n$-vertex graphs $G$ with $\psi(G) = n - 1$. To do this we need to compute doubly resolving number of some graphs.

Let $G$ and $H$ be two graphs with disjoint vertex sets. The join of $G$ and $H$, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. To find all graphs $G$ of order $n$ with the property $\psi(G) = n - 1$, we need to compute $\psi(K_2 \vee \overline{K_n})$.

Lemma 3.1 If $\overline{K_n}$ is the complement graph of $K_n$, then $\psi(K_2 \vee \overline{K_n}) = n + 1$.

Proof. Let $K_2 \vee \overline{K_n} = G$, $X = V(K_2)$ and $Y = V(K_n)$. If $n = 1$, then $G = P_3$ and $\psi(G) = 2 = n + 1$. Now consider $n \geq 2$. Clearly all vertices in $X$ are twins, also all vertices in $Y$ are twins. Let $W$ be a doubly resolving set for $G$. By Proposition 2.2 $|W \cap X| \geq 1$ and $|W \cap Y| \geq n - 1$. Thus $|W| \geq n$. If $|W| = n$, then $X \cap W = \{x_1\}$ and there exists a vertex $y_0 \in Y \setminus W$. Note that for each $y \in W \cap Y$, we have 

$$d(y_0, y) - d(x_1, y) = 2 - 1 = 1$$

and 

$$d(y_0, x_1) - d(x_1, x_1) = 1 - 0 = 1.$$ 

That is $y_0, x_1$ are not doubly resolved by any pair of vertices in $W$, a contradiction. Therefore, $\psi(G) = |W| = n + 1$. 

First, we find all graphs $G$ of order $n$, maximum degree $n - 1$ and $\psi(G) = n - 1$.

Proposition 3.2 Let $G$ be a graph of order $n \geq 3$ and maximum degree $\Delta$. If $\psi(G) = \Delta = n - 1$, then $G$ is $K_n, K_{1,n-1}$, or $K_2 \vee \overline{K_{n-2}}$.

Proof. By Theorem 2.4 it follows that $\text{diam}(G) \leq 2$. Suppose that $x$ is a vertex of degree $n - 1$ and $I$ is a maximum independent subset of $N(x)$. If $I = N(x)$, then $G = K_{1,n-1}$ and if $|I| = 1$, then $G = K_n$. Hence assume that $2 \leq |I| \leq n - 2$. This implies that there exists a vertex $y \in N(x) \setminus I$. We claim that $y$ is adjacent ro all vertices in $I$. By definition of $I$, $y$ has a neighbour in $I$, say $a$. Since $\psi(G) = n - 1$, the set $W = V(G) \setminus \{x, y\}$ is not a doubly resolving set for $G$. If there exists a vertex $b \in I$ that is not adjacent to $y$, then

$$d(x, a) - d(y, a) = 1 - 1 \neq 1 - 2 = d(x, b) - d(y, b).$$


This means $x, y$ are doubly resolved by $a, b \in W$. Note that $2 \leq |I| \leq |W|$, thus for each $v \in W$, there exists $v \neq u \in W$ and we have

$$d(x, u) - d(v, u) = 1 - d(v, u) \neq 1 = d(x, v) = d(x, v) - d(v, v).$$

Therefore for every $v \in W$, $x, v$ are doubly resolved by some pair of vertices in $W$. Since $W$ is not a doubly resolving set, Lemma 2.1 implies that there exists a vertex $w \in W$ such that $y, w$ are not doubly resolved by any pair of vertices in $W$. Hence,

$$d(y, w) = d(y, w) - d(w, w) = d(y, a) - d(w, a) = 1 - d(w, a).$$

But $d(y, w) \geq 1$ and $d(w, a) \geq 0$ imply $d(w, a) = 0$ and $w = a$. Note that $\text{diam}(G) \leq 2$, yields

$$d(a, b) - d(y, b) = 2 - 2 \neq 0 - 1 = d(a, a) - d(y, a).$$

This contradiction means that $y$ is adjacent to $b$. Therefore $y$ is adjacent to all vertices in $I$. Since $y$ is an arbitrary vertex in $N(x) \setminus I$, the above argument implies that all vertices in $I$ are adjacent to all vertices in $N(x) \setminus I$. We claim that $N(x) \setminus I = \{y\}$. Suppose on the contrary that $|N(x) \setminus I| \geq 2$. Let $W = V(G) \setminus \{a, x\}$ and $b \in I \cap W$. We have

$$d(a, b) - d(x, b) = 2 - 1 \neq 1 - 1 = d(a, y) - d(x, y),$$

and so $a, x$ are doubly resolved by $b, y \in W$. Also

$$d(x, y) - d(y, y) = 1 \neq 0 = d(x, b) - d(y, b)$$

and for each $y \neq w \in W$ we have

$$d(x, y) - d(w, y) \neq 1 = d(x, w) - d(w, w).$$

Therefore for every $w \in W$, vertices $x, w$ are doubly resolved by some vertices in $W$. Since $W$ is not a doubly resolving set, there exists a vertex $w \in W$ such that $w, a$ are not doubly resolved by any pair of vertices in $W$. If $w \in I \cap W$, then

$$d(w, y) - d(a, y) = 1 - 1 \neq -2 = d(w, w) - d(a, w),$$

which is impossible. Thus $w \in N(x) \setminus I$. Let $w \neq y' \in N(x) \setminus I$, this implies $d(a, y') - d(w, y') = 1 - d(w, y') \leq 0$ and $d(a, w) - d(w, w) = 1$. That means $a, w$ are doubly resolved by $w, y' \in W$. This contradiction leads us to $N(x) \setminus I = \{y\}$. Therefore $|I| = n - 2$ and $G = K_2 \vee \overline{K_{2,n-2}}$. ■

The next theorem determines all graphs $G$ of order $n$ and $\psi(G) = n - 1$.

**Theorem 3.3** Let $G$ be a graph of order $n \geq 3$. Then $\psi(G) = n - 1$ if and only if $G$ is $K_n, K_{1,n-1}, K_{2,n-2}$, or $K_2 \vee \overline{K_{n-2}}$.

**Proof.** By Lemmas 1.2, 2.3 and 3.1 the doubly resolving number of each of the graphs mentioned in the statement of the theorem is $n - 1$.

For the converse, assume that $G$ is a graph of order $n \geq 3$ such that $\psi(G) = n - 1$. By Theorem 2.1 it follows that $\text{diam}(G) \leq 2$. Let $\Delta$ be the maximum degree in $G$. Since $G$ is connected and has at least 3 vertices, we have $\Delta \geq 2$. 


If \( \Delta = 2 \), then \( G \) is a path \( P_n \) or a cycle \( C_n \). If \( G = P_n \), then by Lemma 1.3, \( n = 3 \) and \( G = P_3 = K_{1,2} \). In the case \( G = C_n \), Lemma 1.4 implies that \( n \in \{3,4\} \) and \( G = C_3 = K_3 \) or \( G = C_4 = K_{2,2} \).

Now let \( \Delta \geq 3 \). If \( \Delta = n - 1 \), then by Proposition 3.2, \( G \) is \( K_n, K_{1,n-1} \), or \( K_2 \vee \overline{K_{n-2}} \). If \( 3 \leq \Delta \leq n - 2 \), then let \( x \) be a vertex of degree \( \Delta \) and \( y \) be a non-adjacent vertex to \( x \). Since \( \text{diam}(G) \leq 2 \), vertices \( x, y \) have a common neighbour, say \( a \). \( \Delta \geq 3 \) implies \( x \) has at least two more neighbours, say \( b \) and \( c \). Clearly, \( W = V(G) \setminus \{a, x\} \) is not a doubly resolving set for \( G \). But \( a, x \) are doubly resolved by \( y, b \) because

\[
d(a, y) - d(x, y) = 1 - 2 = -1 \neq d(a, b) - d(x, b) = d(a, b) - 1 \geq 0.
\]

Also for each \( w \in W \), vertices \( x, w \) are doubly resolved by some pair of vertices in \( W \). To see this we must consider two cases \( w = c \) or \( w \neq c \). If \( w = c \), then \( d(c, c) - d(x, c) = -d(x, c) = -1 \) and \( d(c, b) - d(x, b) = d(c, b) - 1 \geq 0 \). If \( w \neq c \), then \( d(w, w) - d(x, w) = -d(x, w) < 0 \) and \( d(w, c) - d(x, c) = d(w, c) - 1 \geq 0 \). Since \( W \) is not a doubly resolving set for \( G \), there exists a vertex \( w \in W \) such that \( a, w \) are not doubly resolved by any pair of vertices in \( W \). Therefore

\[
1 \leq d(a, w) = d(a, w) - d(w, w) = d(a, y) - d(w, y) = 1 - d(w, y).
\]

This means \( w = y \). Note that \( 1 = d(a, y) - d(y, y) = d(a, c) - d(y, c) \). Hence \( d(a, c) = 2 \) and \( d(y, c) = 1 \). Since \( c \) is an arbitrary neighbour of \( x \), we deduce that \( N(y) = N(x) \) and \( N(x) \) is an independent set. We claim that \( V(G) = N(x) \cup \{x, y\} \). Suppose on the contrary that there exists a vertex \( y' \in V(G) \setminus (N(x) \cup \{x, y\}) \). By similar to the above argument we have \( N(y') = N(x) \). Since \( \psi(G) = n - 1 \), the set \( W = V(G) \setminus \{y', c\} \) can not be a doubly resolving set for \( G \). But \( y', c \) are doubly resolved by \( y, b \). If \( w \in W \cap N(x) \), then

\[
d(c, w) - d(w, w) = 2 \neq 0 = d(c, x) - d(w, x).
\]

For \( w \in W \setminus N(x) \), let \( w \neq w' \in W \setminus N(x) \). Hence

\[
d(c, w') - d(w, w') = 1 - 2 \neq 1 = d(c, w) - d(w, w).
\]

Therefore for every \( w \in W \), vertices \( c, w \) are doubly resolved by some pair of vertices in \( W \). By a same way for every \( w \in W \), vertices \( y', w \) are doubly resolved by some pair of vertices in \( W \). This means \( W \) is a doubly resolving set for \( G \), which is a contradiction. Thus \( V(G) = N(x) \cup \{x, y\} \), \( N(x) = N(y) \) is an independent set, and \( x \) is not adjacent to \( y \). Therefore \( G = K_{2,n-2} \).

## 4 Doubly resolving number of unicyclic graphs

In this section we investigate doubly resolving sets in unicyclic graph. A unicyclic graph is a graph that have exactly one cycle. If \( G \) is a unicyclic graph \( C(G) \) indicates the unique cycle of \( G \). For each vertex \( x \in V(C(G)) \) of degree at least 3 we define

\[
V(x) = \{v \in V(G) \setminus V(C(G)) \mid \forall y \in V(C(G)), d(v, x) < d(v, y)\},
\]

and \( T(x) \) as the induced subgraph \( \langle \{x\} \cup V(x) \rangle \) of \( G \). Clearly \( T(x) \) is a tree. A leaf in a graph is a vertex of degree 1. We use the notations \( L(G) \) and \( l(G) \) for the set of all leaves in the graph \( G \) and its cardinality, respectively. Caceres et al. [5] proved the following lemma for doubly bases of trees.
Lemma 4.1 The set of leaves is the unique doubly resolving basis for a tree.

By the same proof we can see that each resolving set of a graph contains all leaves. The following lemma prepares a lower bound for doubly resolving number in terms of the number of leaves in a graph.

Lemma 4.2 Let $v$ be a vertex of degree 1 in a graph $G$. Then $v$ belongs to all doubly basis of $G$, and

$$\psi(G) \geq l(G).$$

Proof. Let $B$ be a doubly basis of $G$ and $u$ be the neighbour of $v$. If $v \notin B$, then for each $x \in B$, $d(v, x) - d(u, x) = 1$ which is impossible. Therefore $v \in B$ and $\psi(G) \geq l(G)$.

Proposition 4.3 Let $G$ be a unicyclic graph and $W$ be a doubly resolving set for $C(G)$. If every vertex of $W$ is of degree at least 3, then $L(G)$ is a doubly basis of $G$.

Proof. Let $r, s \in V(G)$, we need to consider the following three cases.

Case 1. $r, s \in V(C(G))$. Since $W$ is a doubly resolving set for $C(G)$, there are vertices $x, y \in W$ that $r, s$ are doubly resolved by $x, y$. Let $x_1 \in V(x)$ and $y_1 \in V(y)$ be leaves. Then

$$d(r, x_1) - d(s, x_1) = d(r, x) + d(x_1) - (d(s, x) + d(x_1)) =$$

$$= d(r, x) - d(s, x) \neq d(r, y) - d(s, y) =$$

$$= d(r, y) + d(y_1) - (d(s, y) + d(y_1)) =$$

$$= d(r, y_1) - d(s, y_1).$$

Therefore $r, s$ are doubly resolved by leaves $x_1, y_1$.

Case 2. $r \in V(C(G))$ and $s \notin V(C(G))$. Let $s \in V(t)$ for some $t \in V(C(G))$. Hence there are vertices $x, y \in W$ such that $r, t$ are doubly resolved by $x, y$. There are two possibilities, $t \in \{x, y\}$ or $t \notin \{x, y\}$. If $t \in \{x, y\}$, say $t = x$, then let $y_1$ be a leaf in $V(y)$ and $x_1$ be a leaf in $V(x)$ such that $s$ is a vertex in the shortest path between $x$ and $x_1$. Hence

$$d(x, x_1) = d(x, s) + d(s, x_1).$$

If $r, s$ are not resolved by $x_1, y_1$, then

$$d(r, x) + d(x, s) = (d(r, x) + d(x, s) + d(s, x_1)) - d(s, x_1) = d(r, x_1) - d(s, x_1) =$$

$$= d(r, y_1) - d(s, y_1) = d(r, y) + d(y_1) - (d(s, y) + d(y_1)) =$$

$$= d(r, y) - (d(s, y) = d(r, y) - (d(s, x) + d(x, y)).$$

Hence $d(r, x) + d(x, y) = d(r, y) - 2d(x, s)$. But we know that $d(r, y) \leq d(r, x) + d(x, y)$ satisfies for all three vertices $x, y, r \in V(G)$. Therefore

$$d(r, y) \leq d(r, x) + d(x, y) = d(r, y) - 2d(s, x) \leq d(r, y) - 2,$$

which is impossible. Thus $r, s$ are doubly resolved by $x_1, y_1$. If $t \notin \{x, y\}$, let $x_1$ be a leaf in $V(x)$ and $y_1$ be a leaf in $V(y)$. Then

$$d(r, x_1) - d(s, x_1) = d(r, x) + d(x_1) - (d(s, t) + d(t, x) + d(x_1)) =$$

$$= d(r, x) - (d(s, t) + d(t, x)) \neq d(r, y) - (d(s, t) + d(t, y)) =$$

$$= d(r, y) + d(y_1) - (d(s, t) + d(t, y) + d(y_1)) =$$

$$= d(r, y_1) - d(s, y_1).$$
Therefore $r, s$ are doubly resolved by leaves $x_1, y_1$.

Case 3. $r, s \notin V(C(G))$. Let $r \in V(y)$ and $s \in V(x)$ for some $x, y \in V(C(G))$. If $x \neq y$, let $x_1 \in V(x)$ be a leaf such that $s$ is in the shortest path between $x$ and $x_1$ and $y_1 \in V(y)$ be a leaf such that $r$ is in the shortest path between $y$ and $y_1$. Thus

$$d(r, x_1) - d(s, x_1) = d(r, y) + d(y, x) + d(x, s) + d(s, x_1) - d(s, x_1) > 0,$$
and

$$d(r, y_1) - d(s, y_1) = d(r, y_1) - (d(s, x) + d(x, y) + d(y, r) + d(r, y_1)) < 0.$$  
Therefore $r, s$ are doubly resolved by $x_1, y_1$. If $x = y$, then by Lemma 4.1 leaves of $T(x)$ is a basis for $T(x)$. Let $u, t$ be two leaves of $T(x)$ that doubly resolve $r, s$. If $x \notin \{u, t\}$, then $u, t$ are leaves of $G$. If $x \in \{u, t\}$, say $x = t$, then let $y_1 \in V(G) \setminus V(x)$ be a leaf of $G$. Thus

$$d(r, y_1) - d(s, y_1) = d(r, x) + d(x, y_1) - (d(s, x) + d(x, y_1)) = d(r, x) - d(s, x) \neq d(r, t) - d(s, t).$$
Therefore $r, s$ are doubly resolved by $x, t$ and the result follows. \hfill \blacksquare

**Corollary 4.4** Let $G$ be a unicyclic graph and $W$ be a doubly resolving set for $C(G)$. If $U$ is the set of all vertices of degree 2 in $W$, then $L(G) \cup U$ is a doubly resolving set for $G$.

**Proof.** Let $U = \{u_1, u_2, \ldots, u_t\}$. We construct the graph $G'$ by adding leaves $v_i; 1 \leq i \leq t$, to the graph $G$ such that for each $i; 1 \leq i \leq t$, $v_i$ is adjacent to $u_i$. By Proposition 4.3 $L(G')$ is a doubly basis of $G'$. Note that $L(G') = L(G) \cup \{v_1, v_2, \ldots, v_t\}$. Let $x, y$ be two vertices in $G$. Then for every $i$, $d(x, v_i) - d(y, v_i) = d(x, u_i) - d(y, u_i)$. Therefore $L(G) \cup U$ is a doubly resolving set for $G$. \hfill \blacksquare

The next theorem obtains an upper bound for the doubly resolving number of unicyclic graphs.

**Theorem 4.5** Let $G$ be a unicyclic graph that is not a cycle. If $C(G)$ is of order $m$, then

$$l(G) \leq \psi(G) \leq \begin{cases} l(G) + 1 & \text{if } m \text{ is odd}, \\ l(G) + 2 & \text{if } m \text{ is even}. \end{cases}$$

**Proof.** Let $C(G) = (v_1, v_2, \ldots, v_m, v_1)$. If $m$ is odd by lemma 4.4, $\psi(C(G)) = 2$. Let $\{v_i, v_j\}$ be a doubly basis of $C(G)$. Since $G$ is not a cycle at least one of the vertices in $C(G)$ is of degree 3. By relabeling vertices of $C(G)$, we can assume that $v_i$ is of degree at least 3. Hence by Corollary 4.4 $W = L(G) \cup \{v_j\}$ is a doubly resolving set for $G$. Therefore $\psi(G) \leq l(G) + 1$. If $m$ is even by lemma 4.4, $\psi(C(G)) = 3$. The same argument implies that $\psi(G) \leq l(G) + 2$. \hfill \blacksquare

**Corollary 4.6** Let $G$ be a unicyclic graph that is not a cycle and $B$ be a doubly basis of $C(G)$. If all vertices of $B$ are of degree at least 3, then $\psi(G) = l(G)$. 

9
References

[1] A. Ahmad, M. Baca and S. Sultan, Minimal doubly resolving sets of necklace graph, Math. Reports. 20(70) (2018) 123-129.

[2] R.F. Bailey and P.J. Cameron, Base size, metric dimension and other invariants of groups and graphs, Bull. London Math. Soc. 43 (2011) 209-242.

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalak and L. S. Ram, Network discovery and verification, IEEE Journal On Selected Areas in Communications 24(12) (2006) 2168-2181.

[4] P.S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, On k-dimensional graphs and their bases, Periodica Mathematica Hungarica 46(1) (2003) 9-15.

[5] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, and D.R. Wood, On the metric dimension of cartesian products of graphs, SIAM Journal Discrete Mathematics 21(2) (2007) 423-441.

[6] M. Cangalovic, J. Kratica, V. Kovacevic-Vujcic, and M. Stojanovic, Minimal doubly resolving sets of Prism graphs, Optimization 62 (2013) 1037-1043.

[7] G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Ollermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (2000) 99-113.

[8] F. Harary and R.A Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191-195.

[9] C. Hernando, M. Mora, I.M. Pelayo, C. Seara, and D.R. Wood, Extremal Graph Theory for Metric Dimension and Diameter, The Electronic Journal of Combinatorics 17 (2010) #R30.

[10] S. Khuller, B. Raghavachari, and A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70(3) (1996) 217-229.

[11] J. Kratica, M. Cangalovic, and V. Kovacevic-Vujcic, Computing minimal doubly resolving sets of graphs, Computers & Operations Research 36 (2009) 2149-2159.

[12] J. Kratica, V. Kovacevic-Vujcic, M. Cangalovic, and M. Stojanovic, Minimal doubly resolving sets and the strong metric dimension of Hamming graphs, Applicable Analysis and Discrete Mathematics 6(1) (2012) 63-71.

[13] J.B. Liu, A. Zafari, and H. Zarei, Metric dimension, minimal doubly resolving sets, and the strong metric dimension for jellyfish graph and cocktail party graph, Complexity 2020 (2020) 7 pages, Article ID 9407456.

[14] R. A. Melter and I. Tomescu, Metric bases in digital geometry, Computer Vision Graphics and Image Processing 25 (1984) 113-121.

[15] A. Sebo and E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29(2) (2004) 383-393.

[16] P.J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549-559.