Strings and the Gauge Theory of Spacetime Defects

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we present a new topological invariant to describe the space–time defect which is closely related to torsion tensor in Riemann–Cartan manifold. By virtue of the topological current theory and $\phi$–mapping method, we show that there must exist many strings objects generated from the zero points of $\phi$–mapping, and these strings are topological quantized and the topological quantum numbers is the Winding numbers described by the Hopf indices and the Brouwer degrees of the $\phi$–mapping.

Key words: Topological quantization, string, torsion

1. INTRODUCTION

The landscape of fundamental physics has changed substantially during the last one or two decades (Rovelli, 1998). In fact, on the one hand at a microscopical level the strong and weak interactions, while the gravitational interaction is the weakest and seems not to play any role; on the other hand, all known interactions, but gravitation, that is strong, weak and electromagnetic interactions are well described within the framework of relativistic quantum field theory in flat Minkowski space-time. So at the first sight it seems that the gravitation has no effects when we are concerned with elementary-particle physics. But today we know that is not true (De Sabbata, 1994): in fact, if we consider the quantum theory in curved instead of flat Minkowski spacetime we have some very important effects (as, for instance, neutron interferometry (De Sabbata et.al 1991)), and moreover when we go to a microphysical level, that is when we are concerned with elementary-particle physics, we

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realize that the role of gravitation becomes very important and necessary and this happens in the first place when we consider the early universe or the Planck era. In this unprecedented state of affairs, a large number of theoretical physicists from different backgrounds have begun to address the piece of the puzzle which is clearly missing: combining the two halves of the picture and understanding the quantum properties of the gravitational field, Equivalently, understanding the quantum properties of spacetime.

An exciting outcome of the interplay between particle physics and cosmology is the string theory (Hindmarsh and Kibble, 1995). Strings are linear defects, analogous to those topological defects found in some condensed matter systems such as vortex lines in liquid helium, flux tubes in type-II superconductors or disclination lines in liquid crystal, and it is closely related to the torsion tensor of the Riemann–Cartan manifold (Duan et.al, 1994). String theory is strongly believed to solve the short-distance problems of quantum gravity at the Plank scale by providing a fundamental length \( l_{\text{str}} = \sqrt{\frac{\hbar c}{T}} \), where \( T \) is the string tension, and provides a bridge between the physics of the very small and the very large.

In recent years, string theory has reached an exciting stage, where models of various types (such as Wess-Zumino-Witten model (Bakas, 1993, 1994), Ramond-Ramond charges of type II string theory (Mirjam Cvetic and Donam Youm, 1996) and supersymmetric \( SO(10) \) model (Jeannerot, 1996)) have been of much interest in differential geometry (Bakas and Stetsos, 1996), field theory (Robinson and Ziabicki, 1996), and general relativity (Larsen and Sánchez, 1996). Though all these features make string theory very attractive, but since most of them are based on some concrete models, they are not very perfect and the topological property of strings are not very clear yet.

As is well known, torsion is slight modification of the Einstein theory of relativity (proposed in the 1922-23 by E. Cartan (1922)), but is generalization that appears to be necessary when one tries to conciliate general relativity with quantum theory. The main purpose of this paper is to establish a topological theory for string through \( \phi \)-mapping method (Duan and Meng, 1993) and the theory of composed gauge potential (Duan and Lee, 1995) in 4-dimensional Riemann–Cartan manifold \( X \) without any concrete models in early universe. This theoretical framework includes three basic aspects: the generation of multistring in a 4-dimensional Riemann–Cartan manifold, the topological charges of the multistrings and
their evolution equations.

This paper is organized as follows: In section 2, we introduce a new topological invariant to describe the spacetime defect, and using the decomposition of gauge potential, we get the inner structure of torsion. In section 3, by means of the topological tensor current theory and the $\phi$–mapping method, the multistrings are generated naturally at the zero points of the vector total field $\vec{\phi}$ and the topological quantum numbers of the length of these strings are just the Hopf indices and the Brouwer degrees of $\phi$-mapping. Furthermore, using some important relations, the Lagrangian density of many strings is obtained and the corresponding evolution equations are deduced in section 4, and it is pointed out that the Lagrangian density is a generalization of that of Nielsen for strings and the evolution equations relate to the Hamornic mapping in general relativity.

2. NEW TOPOLOGICAL INVARIANT AND SPACETIME DEFECT

Using vierbein theory and the gauge potential decomposition, we will construct the Invariant formulation of the spacetime defects. The defects of space-time has been discussed from different points of view by many physicists. We will follow Duan and Zhang (1990) in which the defect of spacetime was studied from the point of view of gauge field theory. The dislocation is described by the torsion

$$T^A_{\mu\nu} = D_\mu e^A_\nu - D_\nu e^A_\mu, \quad \mu, \nu, A = 1, 2, 3, 4 \quad (1)$$

where $e^A_\mu$ is the vierbein field and its gauge covariant derivative

$$D_\mu e^A_\nu = \partial_\mu e^A_\nu - \omega^{AB}_\mu (x)e^B_\nu,$$

where $\omega^{AB}_\mu$ stands for the spin connection of the Lorentz group.

By analogy with the 't Hooft’s viewpoint ('tHooft, 1974), to establish a physical observable theory of space–time defect, we must first define a gauge invariant antisymmetrical 2–order tensor from torsion tensor with respect of a unit vector field $N^A(x)$ as follows

$$T_{\mu\nu} = T^A_{\mu\nu} N^A + e^A_\nu D_\mu N^A - e^A_\mu D_\nu N^A.$$
By making use of
\[ D_\mu N^A = \partial_\mu N^A - \omega^A_{\mu B} N^B \]
and (4), it can be rewritten as
\[ T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2} \]
where \( A_\mu = e_\mu^A N^A \) is a kind of \( U(1) \) gauge potential. This shows that the antisymmetrical tensor \( T_{\mu\nu} \) expressed in terms of \( A_\mu \) is just the \( U(1) \) like gauge field strength (i.e. the curvature on \( U(1) \) principle bundle with base manifold \( X \)), which is invariant for the \( U(1) \)-like gauge transformation
\[ A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{3} \]
where \( \Lambda(x) \) is an arbitrary function.

Now, let us investigate the integral of the two–form \( T = \frac{1}{2} T_{\mu\nu} dx^\mu \wedge dx^\nu \), which will be shown that, in topology, it is associated with the first Chern class (but not!), i.e.
\[ l = \int \frac{1}{2} T_{\mu\nu} dx^\mu \wedge dx^\nu. \]
Since the integral quantity \( l \) carries neither the coordinate index nor the group index, it must be pointed that \( l \) is invariant under general coordinate transformation as well as local Lorentz transformation. Furthermore, in \( l \) there is another \( U(1) \)-like gauge invariance for (3). In fact, \( l \) is a new topological invariant and is used to measure the size of defects of the spacetime with the dimension of length.

Very commonly, topological property of a physical system is much more important and worth investigating mediculously. It is our conviction that, in order to get a topological result, one should input the topological information from the beginning. Two useful tools—\( \phi \)-mapping method and composed gauge potential theory—just do the work. As \( A_\mu \) is a kind of \( U(1) \) gauge potential, for a section \( \Phi(x) \) of the complex line bundle \( L(X) \) with the base manifold \( X \) which is looked upon as the ordered parameter of the spacetime defects, the corresponding \( U(1) \) covariant derivative of \( \Phi(x) \) with \( A_\mu \) is
\[
D_\mu \Phi(x) = \partial_\mu \Phi(x) - i \frac{2\pi}{L_p} A_\mu \Phi(x),
\]
\[
D_\mu \Phi^*(x) = \partial_\mu \Phi^*(x) + i \frac{2\pi}{L_p} A_\mu \Phi^*(x),
\]
where $L_p$ is the Plank length introduced to make the both sides of the formula with the same dimension (Duan et.al, 1994). From the above equations, $A_\mu(x)$ can be expressed by

$$A_\mu(x) = \frac{i L_p}{4\pi \Phi \Phi^*} [(\Phi \partial_\mu \Phi^* - \Phi^* \partial_\mu \Phi) - (\Phi D_\mu \Phi^* - \Phi^* D_\mu \Phi)].$$

(4)

More deeper calculation can draw the result that

$$A_\mu(x) = \frac{i L_p}{4\pi \sqrt{\Phi \Phi^*}} (\Phi \sqrt{\Phi \Phi^*} \partial_\mu \Phi^* - \Phi^* \sqrt{\Phi \Phi^*} \partial_\mu \Phi) - \frac{i L_p}{4\pi \Phi \Phi^*} (\Phi D_\mu \Phi^* - \Phi^* D_\mu \Phi).$$

(5)

From the well–known Chern–Weil homomorphism (Nash and Sen, 1983), we know that our new topological invariant is independent of the gauge potential, i.e. the last term in RHS of the equation (5) has nothing to do with the topological property in our theory. So we have many choice of gauge potential and the choice depends on the convenience of calculus. In the present work, we select $A_\mu$ as

$$A_\mu(x) = \frac{i L_p}{4\pi \sqrt{\Phi \Phi^*}} (\Phi \sqrt{\Phi \Phi^*} \partial_\mu \Phi^* - \Phi^* \sqrt{\Phi \Phi^*} \partial_\mu \Phi)$$

satisfying the relation (3) for the corresponding $U(1)$ gauge transformation $\Phi'(x) = \Lambda(x) \Phi(x)$.

As is well known, the section $\Phi(x)$ of the complex line bundle can be expressed by

$$\Phi(x) = \phi^1(x) + i \phi^2(x),$$

i.e. the section of the complex line bundle is equivalent to a 2-dimensional real vector field $\vec{\phi} = (\phi^1, \phi^2)$, and $\sqrt{\Phi \Phi^*} = || \phi || = \sqrt{\phi^a \phi^a} (a = 1, 2)$. By defining the direction of the vector field $\vec{\phi}$ as

$$n^a(x) = \frac{\phi^a(x)}{|| \phi(x) ||},$$

(6)

we can obtain the expression of $A_\mu(x)$ in terms of $n^a$ from (5)

$$A_\mu(x) = \frac{L_p}{2\pi} \epsilon_{ab} n^a(x) \partial_\mu n^b(x).$$

(7)

Obviously, $n^a(x)n^a(x) = 1$, and $n^a(x)$ is a section of the sphere bundle $S(X)$ (Duan and Meng, 1993). The zero points of $\phi^a(x)$ are just the singular points of $n^a(x)$. Thus we get the total decomposition of $U(1)$ gauge potential $A_\mu$ with the unit 2-vector field $n^a$, and because of the topological property of $n^a$, we input the topological information successfully.
3. 2-ORDER TOPOLOGICAL TENSOR CURRENT AND THE GENERATION OF STRINGS ON RIEMANN–CARTAN MANIFOLD

In recent years, the topological current theory proposed by Duan (one of the present authors) has been found to play a significant role in particle physics, field theory (especially the gauge theory) (Duan and Meng, 1993; Duan and Lee, 1995; Duan and Zhang, 1990), and the topological current theory can only be used to discussed the motion of point-like particles (or point-like singularity). In order to study the string theory, we should extend the concept, and introduce a topological tensor current of second order from torsion.

From the above discussions, we can define a dual tensor $j^{\mu\nu}$ of $T_{\lambda\rho}$ as follow

$$j^{\mu\nu} = \frac{1}{2} \frac{1}{\sqrt{g_x}} \epsilon^{\mu\nu\lambda\rho} T_{\lambda\rho}$$

$$= \frac{1}{2} \frac{1}{\sqrt{g_x}} \epsilon^{\mu\nu\lambda\rho} (\partial_\lambda A_\rho - \partial_\rho A_\lambda). \quad (8)$$

With the decomposition of $A_\mu$ in (7), $j^{\mu\nu}$ can be expressed in terms of $n^a$ by

$$j^{\mu\nu} = \frac{L_p}{2\pi} \frac{1}{\sqrt{g_x}} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \partial_\lambda n^a \partial_\rho n^b, \quad (9)$$

which shows that $j^{\mu\nu}$ is just a 2-order topological tensor current satisfying

$$j^{\mu\nu} = -j^{\nu\mu}, \quad \frac{1}{\sqrt{g_x}} \partial_\mu (\sqrt{g_x} j^{\mu\nu}) = 0,$$

i.e. $j^{\mu\nu}$ is antisymmetric and identically conserved.

Using (8) and

$$\partial_\mu n^a = \frac{1}{\|\phi\|} \partial_\mu \phi^a + \phi^a \partial_\mu (1/\|\phi\|), \quad \frac{\partial}{\partial \phi^a} (\ln \|\phi\|) = \frac{\phi^a}{\|\phi\|^2},$$

which should be looked upon as generalized functions, $j^{\mu\nu}$ can be expressed by

$$j^{\mu\nu} = \frac{L_p}{2\pi} \frac{1}{\sqrt{g_x}} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \frac{\partial}{\partial \phi^c} \frac{\partial}{\partial \phi^a} (\ln \|\phi\|) \partial_\lambda \phi^c \partial_\rho \phi^b. \quad (10)$$

By defining the general Jacobian determinants $J^{\mu\nu}(\frac{\Phi}{x})$ as

$$\epsilon^{ab} J^{\mu\nu}(\frac{\Phi}{x}) = \epsilon^{\mu\nu\lambda\rho} \partial_\lambda \phi^a \partial_\rho \phi^b$$

for...
and making use of the Laplacian relation in $\phi$-space

$$\partial_a \partial_a \ln \|\phi\| = 2\pi \delta(\vec{\phi}), \quad \partial_a = \frac{\partial}{\partial \phi^a};$$

we obtain the $\delta$-like topological tensor current rigorously

$$j^{\mu\nu} = \frac{1}{\sqrt{g_x}} L_p \delta(\vec{\phi}) J^{\mu\nu}(\vec{x}).$$

(12)

It is obvious that $j^{\mu\nu}$ is non-zero only when $\vec{\phi} = 0$.

Suppose that for the system of equations

$$\phi^1(x) = 0, \quad \phi^2(x) = 0,$$

there are $l$ different solutions, when the solutions are regular solutions of $\phi$ at which the rank of the Jacobian matrix $[\partial_\mu \phi^a]$ is 2, the solutions of $\vec{\phi}(x) = 0$ can be expressed parameterizedly by

$$x^\mu = z_i^\mu(u^1, u^2), \quad i = 1, \cdots, l,$$

(13)

where the subscript $i$ represents the $i$-th solution and the parameters $u^I (I = 1, 2)$ span a 2-dimensional submanifold with the metric tensor $g_{IJ} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J}$ which is called the $i$-th singular submanifold $N_i$ in $X$. For each $N_i$, we can define a normal submanifold $M_i$ in $X$ which is spanned by the parameters $v^A (A = 1, 2)$ with the metric tensor $g_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial v^A} \frac{\partial x^\nu}{\partial v^B}$, and the intersection point of $M_i$ and $N_i$ is denoted by $p_i$. By virtue of the implicit function theorem, at the regular point $p_i$, it should be hold true that the Jacobian matrix $J(\phi^I/v)$ satisfies

$$J(\phi^I/v) = \frac{D(\phi^1, \phi^2)}{D(u^1, u^2)} \neq 0.$$  

(14)

As is well known (Schouten, 1951), the definition of the $\delta$ function on a submanifold $N_i\delta(N_i)$ should be satisfied the surface area relation

$$\int \delta(N_i) \sqrt{g_x} d^4x = \int_{N_i} \sqrt{g_u} d^2u,$$

where $\sqrt{g_x} d^4x$ and $\sqrt{g_u} d^2u$ ($\sqrt{g_u} = \det(g_{IJ})$)are invariant volume element of $X$ and $N_i$ respectively, and the expression of $\delta(N_i)$ is

$$\delta(N_i) = \int_{N_i} \frac{1}{\sqrt{g_x}} \delta^4(\vec{x} - \vec{z}_i(u^1, u^2)) \sqrt{g_u} d^2u.$$
Following this, by analogy with the procedure of deducing $\delta(f(x))$, since

$$
\delta(\phi) = \begin{cases} 
+\infty, & \text{for } \phi(x) = 0 \\
0, & \text{for } \phi(x) \neq 0
\end{cases}
$$

we can expand the $\delta$–function $\delta(\phi)$ as

$$
\delta(\phi) = \sum_{i=1}^{l} c_{i} \delta(N_i),
$$

where the coefficients $c_{i}$ must be positive, i.e. $c_{i} = |c_{i}|$.

In the following, we will discuss the dynamic form of the tensor current $j^{\mu\nu}$ and study the topological quantization of strings through the Winding numbers (Guillemin and Pollack, 1974) $W_i$ of $\phi$ on $M_i$ at $p_i$

$$
W_i = \frac{1}{2\pi} \int_{\partial\Sigma_i} d\arctan(\phi^2/\phi^1),
$$

where $\partial\Sigma_i$ is the boundary of a neighborhood $\Sigma_i$ of $p_i$ on $M_i$ with $p_i \notin \partial\Sigma_i$. It is well-known that the Winding numbers $W_i$ are corresponding to the first homotopy group $\pi[S^1] = Z$ (the set of integers). By making use of (6), it can be precisely proved that

$$
W_i = \frac{1}{2\pi} \int_{\partial\Sigma_i} n^* (\epsilon_{ab} n^a d n^b),
$$

where $n^*$ is the pull back of map $n$. This is another definition of $W_i$ by the Gauss map (Dubrosin et.al, 1985) $n : \partial\Sigma_i \rightarrow S^1$. In topology it means that, when the point $v = (v^1, v^2)$ covers $\partial\Sigma_i$ once, the unit vector $n^a$ will cover $S^1$ $W_i$ times, which is a topological invariant and is also called the degree of Gauss map. Using the Stokes’ theorem in the exterior differential form and (17), one can deduce that

$$
W_i = \frac{1}{2\pi} \int_{\Sigma_i} \epsilon_{ab} \partial_A n^a \partial_B n^b dv^A \wedge dv^B
$$

$$
= \frac{1}{2\pi} \int_{\Sigma_i} \epsilon^{AB} \epsilon_{ab} \partial_A n^a \partial_B n^b d^2v.
$$

Then, by duplicating the above process, we have

$$
W_i = \int_{\Sigma_i} \delta(\phi) J(\phi/v) d^2v,
$$

substituting (16) into (18), and considering that only one $p_i \in \Sigma_i$, we can get

$$
W_i = \int_{\Sigma_i} c_i \delta(N_i) J(\phi/v) d^2v
$$
\[
\int_{N_i} c_i \frac{1}{\sqrt{g_x \sqrt{g_v}}} \delta^4(\vec{x} - \vec{z}_i(u^1, u^2)) J(\frac{\phi}{v}) \sqrt{g_u} d^2 u \sqrt{g_v} d^2 v,
\]
where \(\sqrt{g_v} = \det(g_{AB})\). Because \(\sqrt{g_u} \sqrt{g_v} d^2 u d^2 v\) is the invariant volume element of the Product manifold \(M_i \times N_i\), so it can be rewritten as \(\sqrt{g_x} d^4 x\). Thus, by calculating the integral and with positivity of \(c_i\), we get

\[
c_i = \frac{\beta_i \sqrt{g_v}}{|J(\frac{\phi}{v})|_{p_i}} = \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\phi}{v})|_{p_i}},
\]
where \(\beta_i = |W_i|\) is a positive integer called the Hopf index (Milnor, 1965) of \(\phi\)-mapping on \(M_i\), it means that when the point \(v\) covers the neighborhood of the zero point \(p_i\) once, the function \(\tilde{\phi}\) covers the corresponding region in \(\tilde{\phi}\)-space \(\beta_i\) times, and \(\eta_i = \text{sign} J(\frac{\phi}{v})|_{p_i} = \pm 1\) is the Brouwer degree (Milnor, 1965) of \(\phi\)-mapping. Substituting this expression of \(c_i\) and (16) in (12), we gain the total expansion of the string current

\[
j^{\mu\nu} = \frac{L_p}{\sqrt{g_x}} \sum_{i=1}^{l} \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\phi}{v})|_{p_i}} \delta(N_i) J^{\mu\nu}(\tilde{\phi})
\]

From the above equation, we conclude that the inner structure of \(j^{\mu\nu}\) is labelled by the total expansion of \(\delta(\tilde{\phi})\), which includes the topological information \(\beta_i\) and \(\eta_i\).

With the discovery of an explicit four-particle amplitude that combines the narrow-resonance approximation with Regge behavior and crossing symmetry, some physicists began to study the dual resonance models, i.e. string model, which can be generated from our topological tensor current theory. It is obvious that, in (13), when \(u^1\) and \(u^2\) are taken to be time-like evolution parameter and space-like string parameter, respectively, the inner structure of \(j^{\mu\nu}\) just represents \(l\) strings moving in the 4-dimensional Riemann-Cartan manifold \(X\). The 2-dimensional singular submanifolds \(N_i\) \((i = 1, \cdots, l)\) are their world sheets. Here we see that the strings are generated from where \(\tilde{\phi} = 0\) and does not tie on any concrete models. Furthermore, we see that the Hopf indices \(\beta_i\) and Brouwer degree \(\eta_i\) classify these strings. In detail, the Hopf indices \(\beta_i\) characterize the absolute values of the topological quantization and the Brouwer degrees \(\eta_i = 1\) correspond to strings while \(\eta_i = -1\) to antistrings.

4. THE EVOLUTION EQUATIONS OF STRINGS

At the beginning of this section, we firstly give some useful relations to study many
strings theory. On the \( i \)-th singular submanifold \( N_i \) we have

\[ \phi^a(x)|_{N_i} = \phi^a(z^1_i(u), \ldots, z^4_i(u)) \equiv 0, \]

which leads to

\[ \partial_\mu \phi^a \partial x^\mu \bigg|_{u^I_i} = 0, \quad I = 1, 2. \]

Using this relation and the expression of the Jacobian matrix \( J(\frac{\hat{\phi}}{v}) \), we can obtain

\[ J^{\mu \nu}(\frac{\hat{\phi}}{x})|_{\phi=0} = \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} \epsilon_{ab} \partial \phi^a \partial x^\lambda \partial \phi^b \partial x^\rho \]

\[ = \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} \epsilon_{AB} \partial v^A \partial x^\lambda \partial v^B \partial x^\rho \]

\[ = \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} \epsilon_{AB} J(v^A \partial x^\lambda, v^B \partial x^\rho), \]

then from this expression, the rank–two tensor current can be expressed by

\[ j^{\mu \nu} = \frac{L_p}{2 \sqrt{g_x}} \sum_{i=1}^{l} \beta_i \eta_i \sqrt{g_v} \delta(N_i) \epsilon^{\mu \nu \lambda \rho} \epsilon_{AB} \partial v^A \partial x^\lambda \partial v^B \partial x^\rho. \]  

\[ \text{(20)} \]

Corresponding to the rank–two topological tensor currents \( j^{\mu \nu} \), it is easy to see that the Lagrangian of many strings is just

\[ L = \frac{1}{L_p} \sqrt{-\frac{1}{2} g_{\mu \lambda} g_{\nu \rho} j^{\mu \nu} j^{\lambda \rho}} = \delta(\hat{\phi}) \]

which includes the total information of strings in \( X \) and is the generalization of Nielsen’s Lagrangian for string (Nielsen and Olesen, 1973). The action in \( X \) is expressed by

\[ S = \int_X L \sqrt{g_x} d^4 x = \sum_{i=1}^{l} \beta_i \eta_i \int_{N_i} \sqrt{g_v} d^2 u = \sum_{i=1}^{l} \beta_i \eta_i S_i \]

where \( S_i \) is the area of the singular manifold \( N_i \). It must be pointed out here that the Nambu–Goto action (Nambu, 1970; Forster, 1974; Orland, 1994), which is the basis of many works on string theory, is derived naturally from our theory. From the principle of least action, we obtain the evolution equations of many strings

\[ g^{IJ} \frac{\partial g_{\mu \lambda}}{\partial x^\nu} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\lambda}{\partial u^J} - 2 \frac{1}{\sqrt{g_u}} \frac{\partial}{\partial u^I} \left( \sqrt{g_v} g^{IJ} g_{\mu \nu} \frac{\partial x^\nu}{\partial u^J} \right) = 0, \quad I, J = 1, 2. \]  

\[ \text{(22)} \]

As a matter of fact, this is just the equation of harmonic map (Duan et.al, 1992).
5. CONCLUSION

In summary, we have studied the topological quantization of the strings in Riemann–Cartan space–time by making use of the composed gauge theory and the $\phi$–mapping topological current theory. As a result, the strings are generated from the zero points of $\phi$–mapping and the topological quantum numbers of these strings are the Winding numbers which are determined by the Hopf indices and the Brouwer degrees of $\phi$–mapping, the singular manifolds of $\vec{\phi}$ are just the evolution surfaces of these strings. The whole theory in this paper is not only covariant under general coordinate transformations but also gauge invariant.

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