Low frequency wave modes of liquid-filled flexible tubes

Yuan-Fang Chou and Tzu-Huan Peng
Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan 10617, Republic of China
E-mail: yfchou@ntu.edu.tw

Abstract. Many canals in the human body are liquid-filled thin wall flexible tubes. In general the P-wave and S-wave velocities of tube material are much slower than the sound velocity of the liquid. It is interested to study the dynamic deformation of the wall caused by pressure fluctuation of liquid. In the low frequency range, the liquid pressure is essentially axial symmetric. Therefore, axial symmetric wave propagation modes are investigated. The calculated spectrum shows there are two modes with zero frequency limit. Phase velocities of these two modes are much smaller than the sound velocity of the liquid. They are also slower than the P-wave velocity of the tube material. At very low wave number, radial displacements of both liquid particles and tube are very small compared to their axial counter parts. As the frequency goes higher, boundary waves are observed.

1. Introduction
Georg von Békésy measured the motion of basilar membrane and introduced the travelling wave theory for cochlea model [1]. He observed a slow wave with speed two orders of magnitude less than that of the water waves. Because he used a sound pressure level of 150 dB, it was questioned that there might be a contribution of broad tuning [2]. Olson developed a micro pressure sensor to measure pressure distribution inside cochlea [3, 4, 5]. She found slow and fast waves travelling in the cochlea in the sound pressure level range of 50 to 80 dB. Many researchers adopted lumped system to model acoustic waves in cochlear and confirmed that slow and fast waves do exist in the cochlea [6]. Since the lumped system model cannot completely describe pressure distribution, it is desired to employ the continuum model to study waves in cochlea. Del Grosso analysed multimode wave propagation in inviscid liquid-filled cylinders. He adopted continuum models with realistic boundary condition [7]. Lafleur and Shields employed the method developed by Del Grosso to study low-frequency mode waves in a liquid-filled elastic tube [8]. They found there are one subsonic and one supersonic modes existing in the liquid in the zero frequency limit for aluminium/water and PVC/water combinations. Baik, Jiang, and Leighton calculated waves in a PMMA tube filled with water [9]. They also found one subsonic and one supersonic modes existing in the zero frequency limit. In the studies above, the solid tube materials have P-wave and S-wave velocities higher than the acoustic wave velocity of filled-liquid. However, P-wave and S-wave velocities of the membrane materials in cochlea are slower than the liquid acoustic wave velocity. Therefore, it is interested in examining waves in liquid-filled flexible tubes to investigate dynamic behaviour of the system.

1 To whom any correspondence should be addressed.
2. Formulation

2.1. Elastic tubes

In the absence of external body force, Newton’s law gives the governing equation for the displacement $u'$ of an isotropic elastic solid as

$$ (\lambda + \mu) \nabla (\nabla \cdot u') + \mu \nabla^2 u' = \rho' \left( \partial^2 u' / \partial t^2 \right), \tag{1} $$

where $\lambda$ and $\mu$ are Lamé constants, $\rho$ is the mass density, and the superscript “$t$” indicates “tube”. The displacement can be decomposed into the scalar potential $\phi$ and vector potential $\psi$ as

$$ u' = \nabla \phi + \nabla \times \psi \quad \text{with} \quad \nabla \cdot \psi = 0. \tag{2} $$

Substituting equation (2) into equation (1) obtains two wave equations

$$ (\lambda + 2\mu) \nabla^2 \phi - \rho' \left( \partial^2 \phi / \partial t^2 \right) = 0 \quad \text{and} \quad \mu \nabla^2 \psi - \rho' \left( \partial^2 \psi / \partial t^2 \right) = 0. \tag{3} $$

Equation (3) and (4) give velocities of dilatation and shear waves respectively as

$$ c_p = \left( (\lambda + 2\mu)/\rho' \right)^{1/2} \quad \text{and} \quad c_s = \left( \mu/\rho' \right)^{1/2}. \tag{5} $$

Axial symmetric modes are investigated here because they dominate most low frequency behaviour of canals in the human body. The axial symmetric displacement $u'$ of a circular tube can be expressed as

$$ u' = u'_r e_r + u'_z e_z \tag{6} $$

where subscripts “$r$” and “$z$” denote respectively radial and axial directions in cylindrical coordinate. Equations (3) and (4) are separable and the solution for time harmonic motion with frequency $\omega$ can be written as

$$ \phi = R_\phi (r) Z_\phi (z) e^{j\omega t} \quad \text{and} \quad \psi = \begin{cases} R_\psi (r) Z_\psi (z) e^{j\omega t} & (8a) \\ R_\psi (r) Z_\psi (z) e^{j\omega t} & (8b) \\ \end{cases} \psi = \begin{cases} R_\psi (r) Z_\psi (z) e^{j\omega t} & (8c) \end{cases} \quad \text{and} \quad \psi = \begin{cases} R_\psi (r) Z_\psi (z) e^{j\omega t} & (8c) \end{cases} $$

Therefore the Helmholtz equations corresponding to scalar potential $\phi$ are

$$ \left( d^2 Z_\phi / dz^2 \right) + k_c^2 Z_\phi = 0 \quad \text{and} \quad \left( d^2 R_\phi / dr^2 \right) + \left( d R_\phi / dr \right) / r + P^2 R_\phi = 0 \tag{9} \quad \text{and} \quad \left( d^2 R_\phi / dr^2 \right) + \left( d R_\phi / dr \right) / r + P^2 R_\phi = 0 \tag{10} $$

where $k_c$ is the wave number in the axial direction and $P^2 = \omega^2 / c_p^2 - k_c^2$. Combining the general solutions of equations (9) and (10) to obtain

$$ \phi = \left[ C_1 J_0 (Pr) + C_2 Y_0 (Pr) \right] e^{(j\omega - k_c z)} \quad \text{and} \quad \psi = \begin{cases} R_\psi (r) Z_\psi (z) e^{j\omega t} & (8c) \end{cases} $$

where $C_1$ and $C_2$ are constants to be determined. By the same token, solutions of $z$-variable can be found as

$$ Z_\psi (z) = Z_\psi (z) = Z_\psi (z) = e^{-j\beta z} \tag{11} $$

and the remaining Helmholtz equations for vector potential $\psi$ are

$$ \left( \partial^2 R_\psi / \partial r^2 \right) + \left( \partial R_\psi / \partial r \right) / r + (S^2 r^2 - 1) R_\psi / r^2 = 0 \quad \text{and} \quad \left( \partial^2 R_\psi / \partial r^2 \right) + \left( \partial R_\psi / \partial r \right) / r + (S^2 r^2 - 1) R_\psi / r^2 = 0 \tag{12a} \quad \text{and} \quad \left( \partial^2 R_\psi / \partial r^2 \right) + \left( \partial R_\psi / \partial r \right) / r + S^2 R_\psi = 0 \tag{12b} \quad \text{and} \quad \left( \partial^2 R_\psi / \partial r^2 \right) + \left( \partial R_\psi / \partial r \right) / r + S^2 R_\psi = 0 \tag{12c} $$

where $S^2 = (\omega/c_s)^2 - k_c^2$. The general solutions for components of vector potential are therefore
\[ \psi_r = \left[ C_3 J_1 (Sr) + C_4 Y_1 (Sr) \right] e^{j(\alpha r z)} \]  
(14a)

\[ \psi_\theta = \left[ C_3 J_1 (Sr) + C_4 Y_1 (Sr) \right] e^{j(\alpha r z)} \]  
(14b)

\[ \psi_z = \left[ C_3 J_0 (Sr) + C_4 Y_0 (Sr) \right] e^{j(\alpha r z)} \]  
(14c)

where \( C_i \)'s are constants. Substituting equations (11) and (14a, b, c) into equation (2) results in the displacement field of elastic tube

\[ u' = \left\{ -P \left[ AJ_i (Pr) + BY_i (Pr) \right] \right. 
+ \left. jk_i \left[ CJ_i (Sr) + DJ_i (Sr) \right] \right\} e^{j(\alpha r z)} e_r 
+ \left\{ S \left[ CJ_0 (Sr) + DJ_0 (Sr) \right] - jk_i \left[ AJ_0 (Pr) + BY_0 (Pr) \right] \right\} e^{j(\alpha r z)} e_z \]  
(15)

where \( A, B, C, \) and \( D \) are constants to be determined and the corresponding stress fields \( \tau_r \) and \( \tau_\theta \) can be found from the constitutive equations

\[ \tau_r = \lambda \left[ \frac{\partial (ru')}{\partial r} / r + \left( \frac{\partial u_z}{\partial z} \right) \right] + 2\mu \left( \frac{\partial u_z}{\partial r} \right) \]  
(16)

\[ \tau_\theta = \mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \]  
(17)

which give rise to

\[ \tau_r = \left\{ A \left[ -\lambda \left( P^2 + k_z^2 \right) J_0 (Pr) - 2\mu P^2 J_0 (Pr) + 2\mu PJ_1 (Pr) \right] / r \right\} + \right. 
\left. B \left[ -\lambda \left( P^2 + k_z^2 \right) Y_0 (Pr) - 2\mu P^2 Y_0 (Pr) + 2\mu PY_1 (Pr) \right] / r \right\} + 
\right. 
\left. j2\mu C \left[ Sk_x J_0 (Sr) - k_x J_0 (Sr) / r \right] + j2\mu D \left[ Sk_y Y_0 (Sr) - k_y Y_1 (Sr) / r \right] \right\} e^{j(\alpha r z)} \]  
(18)

and

\[ \tau_\theta = \left\{ j2\mu P k_x \left[ AJ_1 (Pr) + BY_1 (Pr) \right] + \mu \left( k_z^2 - S^2 \right) \left[ CJ_1 (Sr) + DJ_1 (Sr) \right] \right\} e^{j(\alpha r z)} . \]  
(19)

2.2. Fluid inside tube

The acoustic wave in fluid is the pressure fluctuation \( p' \) caused by fluid compressibility that is measured by the bulk modulus \( B \). The fluid is assumed to be lossless so there are no dissipative effects arising from viscosity or heat conduction. The analysis will be limited to waves of relatively small amplitude, so the density change of the medium will be small compared with its equilibrium value \( \rho_f \). Therefore a corresponding linear, lossless displacement field \( u' \) can be expressed with a wave equation

\[ c_f^2 \nabla \cdot \nabla u' = \partial^2 u' / \partial r^2 \]  
(20)

where \( c_f \) is the sound velocity. The particle velocity field corresponding to a small amplitude acoustic processes is irrotational. This means that the displacement field can be expressed as the gradient of a scalar potential \( \chi \)

\[ u' = \nabla \chi \]  
(21)

and equation (26) becomes

\[ c_f^2 \nabla \cdot \nabla \chi = \partial^2 \chi / \partial r^2 \]  
(22)

or

\[ \partial \left[ r (\partial \chi / \partial r) \right] / \partial r + r \left( \partial^2 \chi / \partial z^2 \right) - r \left( \partial^2 \chi / \partial r^2 \right) / c_f^2 = 0 \]  
(23)

in cylindrical coordinate for axial symmetric case. The solution of equation (30) is similar to equation (11) and can be expressed as

\[ \chi = \left[ D_1 J_0 (Fr) + D_2 Y_0 (Fr) \right] e^{j(\alpha r z)} \]  
(24)

where \( D_1 \) and \( D_2 \) are constants and \( F^2 = \left( \alpha / c_f \right)^2 - k_z^2 \). Due to the restriction of finite value at \( r = 0 \), \( D_2 \) must vanish and the solution becomes

\[ \chi = D_1 J_0 (Fr) e^{j(\alpha r z)} \]  
(25)
Substituting equation (25) into equation (21) results in the displacement field of fluid

\[ \mathbf{u}^f = E \left[ FJ_1(Fr) \mathbf{e}_r + jk \mathcal{J}_0(Fr) \mathbf{e}_z \right] e^{j(\omega t - kz)} . \]  

(26)

and the corresponding acoustic pressure

\[ p^f = -E \rho^f \omega^2 \mathcal{J}_0(Fr) e^{j(\omega t - kz)} . \]  

(27)

The constants \( A, B, C, D, \) and \( E \) in equations (15) and (27) are determined by satisfying tube’s boundary conditions on the outer surface and the interface conditions on tube’s inner surface.

2.3. Boundary and interface conditions

The illustration of a fluid filled elastic tube is shown in figure 1 where \( R_o \) and \( R_i \) respectively are the inner and outer radius of the tube. The outside surface of the tube is traction free as shown in figure 2. Therefore the corresponding stress components must vanish at \( r = R_o \), i.e.,

\[ \tau^r_r (R_o) = 0 \quad \text{and} \quad \tau^r_z (R_o) = 0 . \]  

(28)

(29)

The solid-fluid interface is located on the inner surface of the tube. Because the inviscid fluid cannot resist shear, the traction on the inner surface of the tube has no shear components. Therefore the stress components in the tube have to satisfy

\[ \tau^r_z (R_i) = 0 \quad \text{and} \quad \tau^z_z (R_i) = -p^f (R_i) . \]  

(30)

(31)

Finally, the radial displacement component of fluid is equal to its counterpart of tube at the interface

\[ u^f_r (R_i) = u^t_r (R_i) . \]  

(32)

These five conditions can be expressed with general solutions in tube and fluid as

\[ \begin{align*}
& \left[ \mathcal{Q} \mathcal{J}_0 \left( PR_o \right) + \mathcal{P} \mathcal{J}_1 \left( PR_o \right) / R_o \right] A + \left[ \mathcal{Q} \mathcal{Y}_0 \left( PR_o \right) + \mathcal{P} \mathcal{Y}_1 \left( PR_o \right) / R_o \right] B \\
& + jk \left[ \mathcal{S} \mathcal{J}_0 \left( SR_o \right) - \mathcal{J}_1 \left( SR_o \right) / R_o \right] C + jk \left[ \mathcal{S} \mathcal{Y}_0 \left( SR_o \right) - \mathcal{Y}_1 \left( SR_o \right) / R_o \right] D = 0 \\
& jk \mathcal{J}_1 \left( PR_o \right) A + jk \mathcal{Y}_1 \left( PR_o \right) B + \mathcal{Q} \mathcal{J}_1 \left( SR_o \right) C + \mathcal{Q} \mathcal{Y}_1 \left( SR_o \right) D = 0 \\
& \mathcal{J} \mathcal{J}_0 \left( PR_i \right) A + \mathcal{Q} \mathcal{J}_1 \left( PR_i \right) B + \mathcal{J} \mathcal{Y}_0 \left( SR_i \right) C + \mathcal{Q} \mathcal{Y}_1 \left( SR_i \right) D = 0 \\
& -\mathcal{P} \mathcal{J}_1 \left( PR_i \right) A + \mathcal{P} \mathcal{Y}_1 \left( PR_i \right) B + jk \mathcal{J}_1 \left( SR_i \right) C + jk \mathcal{Y}_1 \left( SR_i \right) D - F \mathcal{J}_1 \left( FR_i \right) E = 0 \\
\end{align*} \]

(33)

(34)

(35)

(36)

(37)

The parameter \( Q \) in these equations are defined as \( Q = k_o^2 \left( \omega^2 \rho^f / 2 \mu \right) \).

---

![Figure 1. A fluid filled tube.](image1)

![Figure 2. Boundary and interface conditions.](image2)
The system of homogeneous equations (33), (34), (35), (36), and (37) forms an eigenvalue problem. Pairs of \((k_z, \omega)\) offering nontrivial solutions are dispersive relations and the corresponding eigenvector \(\{A, B, C, D, E\}\) gives mode shape of waves.

3. Example and discussion

An elastic tube with inner radius \(R_i=1.5\ mm\) and outer radius \(R_o=1.6\ mm\) are chosen for study. Since the material properties of membranes in cochlea are similar to that of cartilage [10], Lamé constants \(\lambda=0.167\ MPa\) and \(\mu=0.25\ MPa\) and the density \(\rho'=1000\ kg/m^3\) are adopted. The filled fluid has a density \(\rho_f=1000\ kg/m^3\) and an acoustic wave velocity \(c_f=1500\ m/sec\). Dilatation and shear wave velocities are respectively 25.82 \(m/sec\) and 15.81 \(m/sec\) for tube material. These velocities are much slower than the acoustic wave velocity of the filled fluid. The system has two dispersion curves without cut-off frequency as shown in figure 3. Figure 4 gives phase velocities of these two low-frequency modes and they are respectively 4.366 \(m/sec\) and 24.98 \(m/sec\) as wave number approaches zero. Characteristics of waves that wavelength is much longer than tube’s diameter, in the same order of tube’s diameter, and much smaller than tube’s diameter are studied as follows.

3.1. Low wave number ranges

Mode shapes corresponding to a specific point on dispersion curve is obtained from the eigenvector

![Figure 3. The lowest two dispersion curves.](image1)

![Figure 4. Phase velocities for lowest two modes.](image2)

![Figure 5. Mode shapes at \(k_z=25\). (a) First mode. (b) Second mode.](image3)
\{A, B, C, D, E\} corresponding to the \((k, \omega)\) pair. Figure 5 shows the mode shape of these two modes at the wave number of 25 rad/m that has a wave length of 251.3 mm. The real parts of displacement fields are indicated with blue solid lines while the imaginary parts are drawn with green dashed lines. Figure 5(a) presents the displacement field of first mode at 17.36 Hz. The radial displacement is zero and the axial displacement field is mainly happened in fluid. This planelike wave has a phase velocity of 4.364 m/sec that is much slower than the acoustic velocity of 1500 m/sec for the fluid. The displacement field of second mode at 99.38 Hz is shown in figure 5(b). It is a longitudinal plane wave of the tube with axial component only. The phase velocity is 24.98 m/sec, very close to the solid P-wave velocity of 25.82 m/sec.

3.2. Mid wave number range
Mode shapes of the first and second modes at the wave number of 2000 rad/m are shown in Figure 6. The wave length is 3.142 mm that is about the same as the tube diameter. Figure 6(a) presents the displacement field of first mode at 961.8 Hz. This wave has a phase velocity of 3.022 m/sec that is even slower than its low wavenumber counterparts. The radial and axial components of the displacement have the same order in amplitude but they hold a phase difference of \(\pi/2\). Therefore the fluid particles move clockwise in elliptical loci while the radial component dominates the motion of tube particles. The displacement field of second mode at 7946 Hz is shown in figure 6(b). It is a longitudinal plane wave of the tube which particles move in axial direction. The phase velocity is 24.96 m/sec, very close to the P-wave velocity of 25.82 m/sec. The fluid is stay in rest.

3.3. High wave number range
Figure 7 shows the mode shape of these two modes at the wave number of 60000 rad/m that has a wave length of 0.1047 mm. It shows significant coupling between fluid and tube. Figure 7(a) presents the displacement field of first mode at 112554 Hz. This wave has a phase velocity of 11.79 m/sec. Most of the fluid particles do not move and only those close to the interface circle around clockwise in z-r plane. The displacement of solid particles on the tube surface is greater than that of interior particles. All solid particles move elliptically. The wave of second mode at 142143 Hz has a phase velocity of 14.89 m/sec. The displacement field is shown in figure 7(b) where maximum displacement occurred at the outer surface of the tube and particles move counterclockwise. Similar to the first mode, large clockwise motion is also observed for particles in the neighborhood of solid-fluid interface. The motion decays rapidly toward the center so that most of the fluid particles stay still.

![Figure 6](image.png)

**Figure 6.** Mode shapes at \(k=2000\). (a) First mode. (b) Second mode.
4. Conclusions
A liquid-filled thin wall flexible tube which P-wave and S-wave velocities are much slower than the sound velocity of the filled-liquid is studied in this paper. Two lowest axial symmetric modes with zero frequency limit are investigated. Their phase velocities are much smaller than both the sound velocity of the liquid. For very long wave length, the radial displacement fields in liquid and tube are much smaller than their axial counter parts. The first mode is a planelike wave of fluid and the second mode is a longitudinal plane wave of the tube. Displacement fields of these two waves are dominated by axial component. When the wave length is comparable to the tube diameter, the first mode has fluid particles move clockwise in elliptical loci while the radial component dominates the motion of tube particles. The displacement field of second mode is a longitudinal plane wave in the tube which particles move in axial direction and the fluid is stay in rest. As the frequency goes higher, boundary waves can be observed in very short wave length range. The slow wave observed by Von Békésy and Olsen is verified in this study.

References
[1] Von Békésy G 1960 Experiments in Hearing McGraw-Hill New York
[2] Johnstone B M, Patuzzi R, and Yates G K 1986 Hearing Research, 22 147-53
[3] Olson E S 1998 J. Acous. Soc. Am. 103 3445-63
[4] Olson E S 1999 Nature 402 526-9
[5] Olson E S 2001 J. Acous. Soc. Am. 110 349-67
[6] Duifhuis H 2012 Cochlear Mechanics- Introduction to a Time Domain Analysis of the Nonlinear Cochlea, Springer, New York
[7] Del Grosso V A 1971 Acustica 24 299-311
[8] Lafleur L D and Shields F D 1995 J. Acous. Soc. Am. 97 1435-45
[9] Baik K, Jiang J, and Leighton T G 2009 Theoretical Investigation of Phase Velocity, Group Velocity and Attenuation of Acoustic Waves in a Liquid-Filled Cylinder, ISVR Technical Report 329, University of Southampton
[10] Mansour J M, 2003 Biomechanics of cartilage, In Kinesiology: the Mechanics and Pathomechanics of Human Movement Lippincott Williams & Wilkins, Philadelphia