LOW-RANK APPROXIMABILITY AND ENTROPY AREA LAWS FOR GROUND STATES OF UNBOUNDED HAMILTONIANS

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ABSTRACT. We show how local bounded interactions in an unbounded Hamiltonian lead to eigenfunctions with favorable low-rank properties. To this end, we utilize ideas from quantum entanglement of multi-particle spin systems. We begin by analyzing the connection between entropy area laws and low-rank approximability. The characterization for 1D chains such as Matrix Product States (MPS) / Tensor Trains (TT) is rather extensive though incomplete. We then show that a Nearest Neighbor Interaction (NNI) Hamiltonian has eigenfunctions that are approximately separable in a certain sense. Under a further assumption on the approximand, we show that this implies a constant entropy bound.

To the best of our knowledge, this work is the first analysis of low-rank approximability for unbounded Hamiltonians. Moreover, it extends previous results on entanglement entropy area laws to unbounded operators. The assumptions include a variety of self-adjoint operators and have a physical interpretation. The weak points are the aforementioned assumption on the approximand and that the validity is limited to MPS/TT formats.

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1. Introduction

Can we represent or approximate a function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) with a complexity that does not grow exponentially in the dimension \( d \)? Assuming no specific structure for \( f \), the general answer to this question is “No” (see [34, 44]). This is commonly known as the curse of dimensionality. Nonetheless, the development of low-rank tensor methods in recent decades has shown that this curse can be broken for a variety of models (see [8, 9, 26, 35]). Can we systematically identify the necessary structures that lead to low-rank approximability?

The (approximation) theory on this topic is very scarce. We are aware of only one result in [15], where the authors considered how the inverse of a Laplace-like operator preserves low-rank structure. The original motivation for our work was a theory that describes the structure necessary for low-rank approximability. We consider a class of unbounded operators and show how local interactions in the operator structure lead to favorable approximability of eigenfunctions w.r.t. to growing \( d \).

Fortunately, we do not have to start from scratch. In physics the phenomenon of quantum entanglement has been known since as early as 1935 (see [19]). The study of multi-particle quantum systems has led to intriguing connections between the holographic principle, entropy area laws and approximability by Matrix Product States (MPS), the latter being a particular kind of a tensor format also known as the Tensor Train (TT) format. The quantum theoretical approach to approximability offers an entire set of powerful tools that we can use to answer the question posed in the beginning of this introduction. See [20] for an overview of quantum entanglement entropy area laws.

This work intertwines three things: a review of the ideas from quantum entanglement relevant to low-rank approximation; a rigorous formulation and proof of some of these ideas; an area law for a possibly exponential complexity in the dimension of \( f \).
class of Hamiltonians. The main results are an approximation estimate for the ground state projection (Theorem 4.11) and an area law for 1D continuous variable systems (Theorem 4.18). This proof illustrates essential mathematical ingredients that connect local interactions with low-rank approximability of eigenfunctions.

The proof is mainly based on ideas from [27], where the author considers NNI systems of finite bond dimension. To the best of our knowledge, this was the first proof in physics that a 1D spin system obeys an area law. Since then other proofs for discrete variable systems appeared, with sharper bounds and more general assumptions.

For instance, in [10] the authors considered the more general setting of exponentially decaying correlations and this bound was later improved in [13]. In [25] the authors considered long range interactions. In [3] the authors significantly improved the bound for NNI systems w.r.t. the spectral gap. This was also used in [24], to show that discretized NNI operators with general right hand sides lead to approximable solutions. In [4, 38] the authors showed area laws for Gaussian states in 1D and 2D, i.e., (continuous variable) harmonic oscillators.

We choose to follow the ideas from [27], since we believe these are the most natural to extend to the infinite-dimensional setting of PDEs (continuous variable systems). Of course, we do not claim that other approaches are not feasible.

The outline of this paper is as follows. In Section 2 we introduce the main notation and terminology used through out the work. In Section 3 we take a closer look at the connection between entropy scaling, approximability and discuss the issue of entropy discontinuity. In Section 4 we introduce the (NNI) Hamiltonian operator class we consider, prove some properties of the ground state and conclude with an entropy scaling bound. The latter will require an assumption on the approximate ground state (NNI) Hamiltonian operator class we consider, prove some properties of the ground state and conclude.

Moreover, for any tensor product space of the form $H$ and is typically large. In this work we focus on 1D systems (see next section for more details).

### 2. Notation

In this work we consider a separable Hilbert space which we denote by $\mathcal{H}$, with the corresponding inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$. We assume $\mathcal{H}$ is a topological tensor product of order $d$,

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_d,$$

where $d$ is (a multiple of) the number of particles in a corresponding model of a quantum system.

Note the distinction between the dimension of a tensor network, which we denote by $D$, and the order of the tensor product, which we denote by $d$. The dimension of the tensor network $D$ refers to the spatial dimension of the graph representing the network. E.g., particles ordered in a chain represent a 1D system. The corresponding tensor format is MPS or TT, which is a 1D tensor network. If particles are ordered on a lattice, the corresponding system is 2D and the corresponding tensor format is, e.g., Projected Entangled Pair States (PEPS). In all these examples $d$ is a multiple of the number of particles and is typically large. In this work we focus on 1D systems (see next section for more details).

We assume that $\mathcal{H}$ is equipped with the canonical inner product

$$\langle \psi_1 \otimes \cdots \otimes \psi_d, \phi_1 \otimes \cdots \otimes \phi_d \rangle_{\mathcal{H}} = \prod_{j=1}^{d} \langle \psi_j, \phi_j \rangle_{\mathcal{H}_j}, \quad \psi_j, \phi_j \in \mathcal{H}_j. \quad (2.1)$$

Moreover, for any tensor product space of the form $\mathcal{H}_\alpha := \bigotimes_{j \in \alpha} \mathcal{H}_j$, for $\alpha \subset \{1, \ldots, d\}$, we assume $\mathcal{H}_\alpha$ is equipped with the canonical inner product as well. This implies many nice properties for $\mathcal{H}_\alpha$ and tensor product operators on $\mathcal{H}_\alpha$ that we can take for granted. In particular, $\|\cdot\|_{\mathcal{H}_\alpha}$ is a uniform crossnorm. See [26, Chapter 4] for more details or [26, Chapter 6.7] and [12] for the case where this is not satisfied.

We consider linear operators $T : \mathcal{D}(T) \to \mathcal{H}$, where $\mathcal{D}(T)$ is some subspace of $\mathcal{H}$ (typically assumed to be dense in $\mathcal{H}$). We use $\mathcal{L}(\mathcal{H})$ to denote the space of all bounded operators. Note that w.l.o.g. we can take $\mathcal{D}(T) = \mathcal{H}$ if $\mathcal{D}(T)$ is dense in $\mathcal{H}$ and $T$ is bounded. The operator norm is

$$\|T\|_{\mathcal{L}} = \sup_{\psi \in \mathcal{H}\setminus\{0\}} \frac{\|T\psi\|_\mathcal{H}}{\|\psi\|_\mathcal{H}}.$$

Mathematically represented by a matrix acting on a finite dimensional Hilbert space.
The Hilbert adjoint is denoted by $T^*$. For self-adjoint operators we can define a partial ordering via

$$T \geq 0 \iff \langle \psi, T \psi \rangle_{\mathcal{H}} \geq 0,$$

for all $\psi \in \mathcal{D}(T)$. We refer to $T$ as being positive in this case. Consequently

$$T_1 \geq T_2 \iff T_1 - T_2 \geq 0.$$

Note that positivity already implies we assume self-adjointness.

For a complete orthonormal system $\{e_k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$, the trace is defined as

$$\text{tr}[T] := \sum_{k=1}^{\infty} \langle e_k, Te_k \rangle_{\mathcal{H}}.$$

Let $|T| := (T^*T)^{1/2}$ denote the absolute value of $T$. If $\text{tr}[|T|] < \infty$, then we say $T$ is trace class in which case $\text{tr}[T]$ is well defined and independent of the choice of $\{e_k\}_{k \in \mathbb{N}}$. We denote the space of trace class operators by $\text{Tr}(\mathcal{H})$ with the corresponding norm $\|T\|_\text{Tr} := \text{tr}[|T|]$. Since $\mathcal{H}$ is a Hilbert space, trace class operators coincide with nuclear operators. We also use the following notation

$$\text{Tr}^+(\mathcal{H}) := \{ T \in \text{Tr}(\mathcal{H}) : T \geq 0 \},$$

$$\mathcal{S}(\mathcal{H}) := \{ T \in \text{Tr}^+(\mathcal{H}) : \|T\|_\text{Tr} = \text{tr}[T] = 1 \}.$$

An important property of $\text{Tr}(\mathcal{H})$ is that it is a two sided ideal in $\mathcal{L}(\mathcal{H})$, i.e., for any $\rho \in \text{Tr}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{H})$, we have $\rho T \in \text{Tr}(\mathcal{H})$ and $T \rho \in \mathcal{L}(\mathcal{H})$ (see [11, Theorem VI.19]).

If for $T \in \mathcal{L}(\mathcal{H})$, $\text{tr}[T^*T] < \infty$, then we say $T$ is a *Hilbert Schmidt operator* and denote the corresponding space by $\text{HS}(\mathcal{H})$. This is a Hilbert space when equipped with the inner product $\langle A, B \rangle_{\text{HS}} := \text{tr}[A^*B]$ and the induced norm $\|T\|_{\text{HS}} := \sqrt{\text{tr}[T^*T]}_{\text{HS}}$. Note that the product of two Hilbert Schmidt operators is always trace class. The introduced spaces compare as follows

$$\text{Tr}(\mathcal{H}) \subset \text{HS}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}),$$

$$\|T\|_{\mathcal{L}} \leq \|T\|_{\text{HS}} \leq \|T\|_{\text{Tr}},$$

where the inclusions are strict for infinite dimensional Hilbert spaces.

If $T \in \mathcal{L}(\mathcal{H})$ is a compact operator, it can be decomposed as

$$T = \sum_{k=1}^{\infty} \sigma_k \langle \cdot, \varphi_k \rangle_{\mathcal{H}} \psi_k,$$

where $\{\varphi_k\}_{k \in \mathbb{N}}, \{\psi_k\}_{k \in \mathbb{N}}$ are orthonormal systems and $\{\sigma_k\}_{k \in \mathbb{N}}$ is a non-increasing sequence of positive numbers. This is called the *Schmidt decomposition* or the singular value decomposition (SVD) and is an important tool for low-rank approximation. The numbers $\sigma_k$ are called *singular values*. All trace class and Hilbert Schmidt operators are compact (but not vice versa).

In quantum mechanics states are modeled by so called *density matrices* $\rho \in \mathcal{S}(\mathcal{H})$. We refer to them as *density operators*, to emphasize the fact that these are, in general, not matrices in this work. A state is called *pure* if it can be written as a one dimensional projection, otherwise it is called *mixed*. I.e., for a pure state there exists $\psi \in \mathcal{H}$ with $\|\psi\|_{\mathcal{H}} = 1$ and $\rho = \langle \cdot, \psi \rangle_{\mathcal{H}} \psi$. In general states are convex combinations of one dimensional projections of the form

$$\rho = \sum_{k=1}^{\infty} \lambda_k \rho_k,$$

with positive numbers $\lambda_k$, summing to one. This is a simple consequence of the spectral decomposition. The projections $\rho_k$ can be taken to be orthogonal to each other such that $\text{tr}[\rho] = \sum_{k=1}^{\infty} \lambda_k = 1$. The numbers $\lambda_k$ have a natural interpretation as probabilities and $\rho$ as a statistical mixture of pure quantum states.

Suppose we split $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, the *partial trace* $\text{tr}_A[\cdot] : \text{Tr}(\mathcal{H}) \rightarrow \text{Tr}(\mathcal{H}_B)$ is defined as the unique trace class operator such that for any $E \in \mathcal{L}(\mathcal{H}_B)$ and any $T \in \text{Tr}(\mathcal{H})$

$$\text{tr}[\text{tr}_A[T]E] = \text{tr}[T(I_A \otimes E)].$$

This is useful in order to describe states of subsystems. I.e., if $\rho \in \mathcal{S}(\mathcal{H})$ is a density operator, then $\rho_B = \text{tr}_A[\rho] \in \mathcal{S}(\mathcal{H}_B)$ is also a density operator, describing the state of the subsystem $B$.

We use the shorthand notation $\mathcal{H}_{i,j} := \mathcal{H}_i \otimes \cdots \otimes \mathcal{H}_j$, $1 \leq i \leq j \leq d$. 

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Suppose \( \rho \in \mathcal{S}(\mathcal{H}) \) is a pure state described by \( \psi \in \mathcal{H} \). Then, applying the Hilbert Schmidt decomposition w.r.t. the bipartite cut \( \mathcal{H} = \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d} \), we can write

\[
\psi = \sum_{k=1}^{r_j} \sigma_j^k v_j^k \otimes w_j^k,
\]

where \( v_j^k \in \mathcal{H}_{1,j} \), \( w_j^k \in \mathcal{H}_{j+1,d} \) and \( r_j \in \mathbb{N} \cup \{\infty\} \). We will frequently use the notation \( \sigma_j^k \) for singular values of such a bipartite cut. The ranks \( r_j \) are the TT ranks of \( \psi \) since the chosen cuts \( \mathcal{H} = \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d} \) for \( 1 \leq j \leq d-1 \) correspond to the structure of the TT format. Approximability within MPS/TT hinges on the decay of these particular singular values \( \sigma_j^k \) or, equivalently, on the scaling of the TT ranks \( r_j \) for a fixed approximation accuracy.

**Definition 2.1** (Approximability). We say a function or a state is approximable when, for a fixed accuracy, the TT ranks grow at most polynomially in \( d \).

For the state of the subsystem on \( \mathcal{H}_{1,j} \) we have the identity

\[
\rho_{1,j} = \text{tr}_{\mathcal{H}_{j+1,d}}[\rho] = \sum_{k=1}^{\infty} (\sigma_j^k)^2 \langle v_j^k \rangle_{\mathcal{H}_{1,j}} v_j^k.
\]

Thus, we have a correspondence between the probabilities \( \lambda_k \) of the state \( \rho_{1,j} \in \mathcal{S}(\mathcal{H}_{1,j}) \) and the singular values \( \sigma_j^k \) of \( \psi \in \mathcal{H} \).

If \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \), we say an operator \( T : \mathcal{D}(T) \to \mathcal{H} \) is supported on \( \mathcal{H}_A \) or simply \( A \), if there exists an operator \( T_A : \mathcal{D}(T_A) \to \mathcal{H}_A \), such that

\[
T = T_A \otimes I_B,
\]

and we write \( \text{supp}(T) = \mathcal{H}_A \).

We evolve operators in time according to the standard Heisenberg picture: for a self-adjoint operator (system Hamiltonian) \( H : \mathcal{D}(H) \to \mathcal{H} \) and any other operator \( T \), we define the time evolution of \( T \) as

\[
T(t) := \exp[iHt]T \exp[-iHt], \quad t \in \mathbb{R}.
\]

Finally, for a state \( \rho \), the von Neumann entropy is defined as

\[
S(\rho) := -\text{tr}[\rho \log(\rho)],
\]

and the Rényi entropy is defined as

\[
S^\alpha(\rho) := \frac{1}{1-\alpha} \log_2(\text{tr}[\rho^\alpha]),
\]

for \( \alpha > 0, \alpha \neq 1 \). One can recover the von Neumann entropy from the Rényi entropy in the limit \( \alpha \to 1 \).

### 3. Entropy and Area Laws

For an overview of entanglement area laws we refer to [20]. In the following we introduce the notion of a general D-dimensional area law. For the rest of this work we focus on 1D area laws.

Let \( X \) be a graph representing a constellation of particles. For the purpose of this introduction and this chapter it is sufficient to assume \( X \subset \mathbb{Z}^D \), i.e., a quantum lattice system, and that \( X \) is finite, i.e., \#X < \infty. Each point in \( X \) represents a separable complex Hilbert space that acts as the phase space of the particle corresponding to that point. We denote the Hilbert space of the entire system by

\[
\mathcal{H}_X := \bigotimes_{\beta \in X} \mathcal{H}_\beta,
\]

where \( \beta \) are the vertices of \( X \). Throughout this work all tensor product spaces are equipped and completed w.r.t. the canonical norm (see [21]).

The interactions in this system are given by the system Hamiltonian that we can write in the general form

\[
H_X = \sum_{I \in X} \Phi(I),
\]
where $\Phi(I)$ models interactions within the subset $I$. In this general form we allow for the interactions to be trivial, i.e., either $\Phi(I) = 0$; or only one-site operations are present (no interaction): for $a, b \in X$

$$\Phi(a \cup b) = H_a \otimes H_b,$$

where $H_a$ and $H_b$ act only on sites $a$ and $b$ respectively.

Given a subset $I \subset X$, we define the set $\partial I$ to be all points in $I$ that, according to the system Hamiltonian $H_X$, have a non-trivial interaction with points in the complement $X \setminus I$. If the current state of the system is described by $\rho \in \mathcal{S}(H_X)$, then the state of a subsystem $I \subset X$ is described by the partial trace $\rho_I = \text{tr}_{X \setminus I}[\rho]$. The von Neumann entropy of any subsystem is then

$$S(\rho_I) = -\text{tr}[\rho_I \log_2(\rho_I)].$$

In principle, we could use any entropy measure to formulate area laws (see [39]).

**Definition 3.1 (Area Law).** A pure state $\rho \in \mathcal{S}(H_X)$ is said to satisfy a $D$-dimensional area law if for any $I \subset X$ the entanglement entropy scales proportional to the boundary of $I$, i.e., $S(\rho_I) \sim \#\partial I$.

This is to be contrasted with volume laws that state entropy scales as the volume of $I$, $S(\rho_I) \sim \#I$. This formulation already suggests that such laws can be formulated for continuous regions $I$ and $X$, e.g., in quantum field theory. Indeed, such laws were originally motivated by similar observations in black hole physics (see Bekenstein-Hawking area law). For our purposes non-relativistic quantum mechanics with finitely many particles will suffice.

As an illustration, see Figure 3.1. These are examples of 1D and 2D area laws. In 1D we are considering chains and thus an area law is particularly simple, $S(\rho_I) \sim 2$. On a 2D lattice, if $I$ encloses $N$ particles, an area law states $S(\rho_I) \sim \sqrt{N}$.

![Figure 3.1](image)

**Figure 3.1.** 1D and 2D area laws.

Corresponding to the structure of $X$ and interactions in $H_X$, one can tailor tensor formats in order to efficiently represent such states on $\mathcal{H}$. See [9,35] for an overview. For 1D chains the corresponding tensor format is referred to as a Matrix Product State (MPS) in the physics community, or Tensor Train (TT) in mathematics. In 1D systems there is a strong link between area laws and low-rank approximability. Due to this and the fact that, unlike in general tensor networks, the best approximation problem in TT is well posed, we focus on 1D area laws.

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3We emphasize that there is no direct relation between $d$ - the number of particles/dimensions - and $D$ - the dimension of the area law.
3.1. Entropy Scaling and Approximation. We begin by illustrating the connection between entropy scaling in $d$ and low-rank approximation in 1D systems. It can be summarized by Table 3.1.

Recall the definition of the Rényi entropy

$$S^\alpha(\rho) := (1 - \alpha)^{-1} \log_2(\text{tr}[\rho^\alpha]), \quad \alpha > 0, \alpha \neq 1.$$ 

And, since $\lim_{d \to 1} S^\alpha(\rho) = S(\rho)$, we use $S^{\alpha=1}$ to denote the von Neumann entropy.

Throughout this work we will use the shorthand notation

$$\mathcal{H}_{i,j} := \bigotimes_{k=1}^j \mathcal{H}_k,$$

where, as mentioned above, all tensor product spaces are equipped and completed w.r.t. the canonical norm. The following result is based\(^4\) upon [15] and [18].

**Proposition 3.2.** Let $\psi \in \mathcal{H}$, $\|\psi\|_1 = 1$, $\rho := \langle \cdot, \psi \rangle_{\mathcal{H}} \in S(\mathcal{H})$, $\rho_{1,j} := \text{tr}_{j+1,d}[-\rho] \in S(\mathcal{H}_{1,j})$. Suppose $S^\alpha(\rho_{1,j}) < \infty$ if $\alpha < 1$ (for $\alpha > 1$ the entropy is clearly finite since $\rho \in S(\mathcal{H})$). Let $\varepsilon_j(r)$ denote the truncation error for a bipartite cut $\mathcal{H} \cong \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d}$, $\varepsilon^2_j(r) = \sum_{k=r+1}^{\infty} (\sigma_k^j)^2$. Then,

$$S^\alpha(\rho_{1,j}) \geq \frac{\alpha}{1 - \alpha} \log_2 \left( \frac{\varepsilon^2_j(r)}{\alpha} \right) + \log_2 \left( \frac{r - 1}{1 - \alpha} \right), \quad 0 < \alpha < 1,$$

$$S^\alpha(\rho_{1,j}) \leq \frac{\alpha}{1 - \alpha} \log_2(1 - \varepsilon^2_j(r)) + \log_2(r), \quad \alpha > 1.$$

Before we proceed with the proof, we require the following lemma.

**Lemma 3.3.** Let $\alpha > 0$, $\alpha \neq 1$ and $a = \{a_k : k \in \mathbb{N}\}$, $b = \{b_k : k \in \mathbb{N}\}$ be two non-negative, non-increasing sequences such that

$$\sum_{k=1}^{\infty} a_k^\alpha < \infty, \quad \sum_{k=1}^{\infty} b_k^\alpha < \infty, \quad \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k < \infty,$$

and for any $m \in \mathbb{N},$

$$\sum_{k=1}^{m} a_k \geq \sum_{k=1}^{m} b_k.$$

Then,

$$S^\alpha(a) \leq S^\alpha(b),$$

where the entropy of a sequence is given as $S^\alpha(a) = \frac{1}{\alpha} \log_2(\sum_{k=1}^{\infty} a_k^\alpha)$.

**Proof.** The above property holds for finite sequences since entropy is Schur concave. We thus assume that there is no such $n \in \mathbb{N}$ such that $a_n = 0$ or $b_n = 0$ and reduce the above lemma to the case of finite sequences. We define truncated sequences $\tilde{a}^n$ and $\tilde{b}^n$ for any $n \in \mathbb{N}$ such that

$$\tilde{a}_i = a_i, \quad \text{for } 1 \leq i \leq n + 1, \quad \tilde{a}_{n+i} = a_{n+1}, \quad \text{for } 1 \leq i \leq m,$$

$$\tilde{a}_{n+m+1} = s - \sum_{k=1}^{n} a_k - ma_{n+1}, \quad \tilde{a}_{n+m+1+k} = 0, \quad \text{for } k \in \mathbb{N},$$

and

$$\tilde{b}_i = b_i, \quad \text{for } 1 \leq i \leq n, \quad \tilde{b}_{n+i} = b_{n+1}, \quad \text{for } 1 \leq i \leq m,$$

$$\tilde{b}_{n+m+1} = s - \sum_{k=1}^{n} b_k - mb_{n+1}, \quad \tilde{b}_{n+m+1+k} = 0, \quad \text{for } k \in \mathbb{N}.$$
where \( m \) is chosen such that \( 0 \leq \tilde{a}_{n+m+1} \leq a_{n+1} \). For \( l = l(n) \in \mathbb{N} \)
\[
\tilde{b}_i = b_i, \quad \text{for } 1 \leq i \leq l + 1, \quad \tilde{b}_{l+1} = b_{l+1}, \quad \text{for } 1 \leq i \leq p,
\]
\[
\tilde{b}_{l+p+1} = s - \sum_{k=1}^{l} b_k - pb_{l+1}, \quad \tilde{b}_{l+p+1+k} = 0, \quad \text{for } k \in \mathbb{N},
\]
where \( l(n) \) is large enough such that \( b_{l+1} \leq a_{n+1} \) and \( p \) is chosen analogously as above such that \( 0 \leq \tilde{b}_{l+p+1} \leq b_{l+1} \).
Both \( \tilde{a}^n \) and \( \tilde{b}^n \) are non-increasing sequences with finitely many non-zero terms and, by construction, \( \tilde{a}^n \) majorizes \( \tilde{b}^n \) such that we can conclude \( S^\alpha(\tilde{a}^n) \leq S^\alpha(\tilde{b}^n) \). It remains to show that
\[
\lim_{n \to \infty} S^\alpha(\tilde{a}^n) = S^\alpha(a) \quad \text{and} \quad \lim_{n \to \infty} S^\alpha(\tilde{b}^n) = S^\alpha(b).
\]
Since the \( \log_2 \)-function is continuous, it suffices to show that the argument converges. We thus estimate
\[
\left| \sum_{k=1}^{\infty} a^\alpha_k - \sum_{k=1}^{\infty} \tilde{a}^\alpha_k \right| \leq \sum_{k=n+2}^{n+m} |a^\alpha_k - a^\alpha_{n+1}| + |a^\alpha_{n+m+1} - \tilde{a}^\alpha_{n+m+1}| + \sum_{k=n+m+2}^{\infty} a^\alpha_k.
\]
The second and third term obviously converge for \( n \to \infty \). For the first term we consider \( \alpha < 1 \) (otherwise the statement is straightforward)
\[
\sum_{k=n+2}^{n+m} |a^\alpha_k - a^\alpha_{n+1}| \leq \sum_{k=n+2}^{\infty} a^\alpha_k + ma^\alpha_{n+1}.
\]
The first term converges while for the second term we obtain
\[
ma^\alpha_{n+1} = ma_{n+1}a^\alpha_{n+1} - \sum_{k=n+1}^{\infty} a^\alpha_{n+1} \leq \left( \sum_{k=n+1}^{\infty} a_k \right) a^\alpha_{n+1} \leq \sum_{k=n+1}^{\infty} a_k a^\alpha_{n+1} \leq \sum_{k=n+1}^{\infty} a_k a^\alpha_{k-1}.
\]
Similarly for \( \tilde{b}^n \) and thus the statement follows. \( \square \)

**Proof of Proposition** [7.2] We follow the arguments from [48, Lemma 2] and [45] with some adjustments. The idea is that we want to bound \( S^\alpha(\rho_{1,j}) \) by exploiting the fact that Rényi entropies are Schur concave. I.e., construct a sequence that majorizes or is majorized by the eigenvalues of \( \rho_{1,j} \) and compute the entropy for the former explicitly.

**Case** \( 0 < \alpha < 1 \). Let \( p := \epsilon_j^2(r) \). Then, \( \sum_{k=1}^{\infty} (\sigma^j_k)^2 = 1 - p \). For \( h > 0 \), consider the candidate sequence
\[
\lambda_1 = (1 - p) - (r - 1)h, \quad \lambda_i = h \quad \text{for } 2 \leq i \leq n - 1, \quad \lambda_n = p - h(n - r - 3) =: \theta, \quad \lambda_k = 0 \quad \text{for } k > n,
\]
where \( n \geq r + 1 \) is chosen such that \( 0 \leq \theta \leq h \). For \( h = 0 \), we simply set \( \lambda_1 = 1 \) and \( \lambda_k = 0 \) for \( k > 1 \). Our goal is to show that this sequence majorizes \( \{(\sigma^j_k)^2 : k \in \mathbb{N}\} \).

If \( h = 0 \), then \( \{\lambda_k : k \in \mathbb{N}\} \) is non-increasing,
\[
1 = \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} (\sigma^j_k)^2,
\]
and
\[
1 = \sum_{k=1}^{m} \lambda_k \geq \sum_{k=1}^{m} (\sigma^j_k)^2,
\]
for any \( m \in \mathbb{N} \). I.e., \( \{(\sigma^j_k)^2 : k \in \mathbb{N}\} \) is trivially majorized by \( \{\lambda_k : k \in \mathbb{N}\} \).

Thus, consider \( h > 0 \). In order to ensure the sequence is non-increasing, we require \( \lambda_1 \geq h \) and thus \( h \leq \frac{1-p}{r} \).
The entropy of the majorizing sequence can be computed as

\[ \lambda_1 = (1-p) - (r-1)h \geq (1-p) - \frac{r-1}{r}(1-p) = (1-p)\frac{1}{r} > 0. \]

For \(1 \leq k \leq n-1\)

\[ \lambda_1 - \lambda_k = \lambda_1 - h = (1-p) - rh \geq (1-p) - \frac{r}{r}(1-p) = 0. \]

And finally by the choice of \(n, 0 \leq \theta \leq h\). Thus, \(\{\lambda_k : k \in \mathbb{N}\}\) is non-increasing and by construction sums to 1.

For \(1 \leq m \leq r\),

\[ \sum_{k=1}^{m} \lambda_k + (r-m)h = 1 - p = \sum_{k=1}^{r} (\sigma_k^i)^2, \]

\[ \Leftrightarrow \sum_{k=1}^{m} \lambda_k - (\sigma_k^i)^2 = \sum_{k=m+1}^{r} (\sigma_k^i)^2 - (r-m)h \geq (r-m)((\sigma_k^j)^2 - h) = 0, \]

with equality for \(m = r\). For \(r < m < n\), we have

\[ \sum_{k=1}^{m} \lambda_k - (\sigma_k^i)^2 \geq (1-p) + (m-r)h - (1-p) - (m-r)(\sigma_k^j)^2 = 0, \]

and for \(m \geq n\),

\[ \sum_{k=1}^{m} \lambda_k - (\sigma_k^i)^2 = \sum_{k=1}^{r} \lambda_k - (\sigma_k^i)^2 + \sum_{k=r+1}^{m} \lambda_k - \sum_{k=r+1}^{m} (\sigma_k^i)^2 \]

\[ \geq \sum_{k=r+1}^{n} \lambda_k - \sum_{k=r+1}^{\infty} (\sigma_k^i)^2 = 0. \]

Thus, \(\{\lambda_k : k \in \mathbb{N}\}\) majorizes \(\{(\sigma_k^i)^2 : k \in \mathbb{N}\}\) and by Lemma 3.3 we get

\[ S^\alpha \left( \left\{(\sigma_k^i)^2 : k \in \mathbb{N}\right\} \right) \geq S^\alpha(\{\lambda_k : k \in \mathbb{N}\}). \]

The entropy of the majorizing sequence can be computed as

\[ \sum_{k=1}^{\infty} \lambda_k^\alpha = [(1-p) - (r-1)h]^\alpha + [(r-1) + \frac{p}{h}]h^\alpha + [p - h(n-r-3)]^\alpha. \]

Estimating from below

\[ \sum_{k=1}^{\infty} \lambda_k^\alpha \geq h^\alpha + (r-1)h^\alpha + (p/h - 1)h^\alpha + [p - h(n-r-3)]^\alpha \]

\[ \geq (r-1)h^\alpha + ph^{\alpha-1}. \]

We further estimate from below by minimizing the expression on the right. I.e., for

\[ f(h) = (r-1)h^\alpha + ph^{\alpha-1}, \quad f'(h) = (r-1)h^{\alpha-1} + (\alpha - 1)ph^{\alpha-2} = 0, \]

\[ \Leftrightarrow \quad h = \frac{(1-\alpha)p}{\alpha(r-1)} =: h^*. \]

Moreover,

\[ f''(h^*) = (1-\alpha)^{\alpha-1}p^\alpha(r-1)^{1-\alpha}\alpha^{-\alpha} > 0, \]

for \(0 < \alpha < 1\). Plugging in \(S^\alpha(\rho_{1,j}) \geq \frac{1}{1-\alpha} \log_2(f(h^*)), \) gives (3.2).

**Case** \(\alpha > 1\). The maximizing distribution for this case is straightforward. Pick

\[ \lambda_k = \frac{1-p}{r}, \quad 1 \leq k \leq r, \quad \lambda_k = (\sigma_k^i)^2, \quad k > r. \]
Clearly, $\sum_{k=1}^{m} \lambda_k \leq \sum_{k=1}^{m} (\sigma_k^2)^2$ for any $m \in \mathbb{N}$ with equality for $m \geq r$. Thus,

$$S^\alpha(\rho_{1,j}) \leq \frac{1}{1-\alpha} \log_2 \left( \frac{(1-p)^\alpha}{r^\alpha} + \sum_{k=r+1}^{\infty} (\sigma_k^2)^2 \right) \leq \frac{1}{1-\alpha} \log_2 \left( \frac{(1-p)^\alpha}{r^\alpha - 1} \right)$$

$$= \frac{\alpha}{1-\alpha} \log_2(1-p) + \log_2(r),$$

which shows (3.3) and completes the proof. \qed

Proposition 3.3 immediately gives.

**Corollary 3.4.** For $\psi \in \mathcal{H}$, if the Rényi entropy $S^\alpha(\rho_{1,j})$, $0 < \alpha < 1$, scales at most as some power of $\log_2(j)$, then $u$ is approximable within the TT format, i.e., TT ranks grow at most polynomially with the dimension $d$. On the other hand, if the Rényi entropy $S^\alpha(\rho_{1,j})$, $\alpha > 1$ scales as some power of $j$, then $u$ is not approximable within the TT format, i.e., TT ranks grow exponentially with $d$.

Next, we want to show that a lower bound on the von Neumann entropy implies a lower bound on rank growth. To this end, we require the ability to approximate the entropy of a given state by entropies of states with finite ranks. I.e., we require continuity of the von Neumann entropy. Unfortunately, the von Neumann entropy is continuous only on a small subset of $\text{Tr}(\mathcal{H})$ that is nowhere dense. In fact, the set of states with infinite von Neumann entropy is trace-norm dense in $\text{Tr}(\mathcal{H})$.

However, under certain “physical” assumptions, one can show continuity of $S(\cdot)$. This is will be discussed in greater detail in Section 3.2. For now we assume continuity as given. Then, we can show

**Proposition 3.5.** For $r \in \mathbb{N}$, let $\varepsilon^2(r)$ denote the best approximation error for an $r$-term truncation in the bipartite cut $\mathcal{H} \cong \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d}$, and $\rho^r_{1,j}$ the corresponding best approximation. If $S(\rho_{1,j}) \geq c \min\{j, d\}$ and $g(r) := |S(\rho^r_{1,j}) - S(\rho_{1,j})|$, then $r \geq \exp[c \min\{j, d/2\} - g(r)]$.

**Proof.** First, note that we can bound the entropy of a state of rank $r$ by $\log_2(r)$. The von Neumann Entropy is Schur concave as any entropy measure should be\(^5\), since it quantifies chaos or lack of information. Thus, the maximum entropy is attained for a uniform distribution

$$S(\rho^r_{1,j}) \leq - \sum_{k=1}^{r} \frac{1}{r} \log_2 \left( \frac{1}{r} \right) = \log_2(r).$$

See [37, Example 2.28] for more details.

Thus, following arguments from [15]

$$\log_2(r) \geq S(\rho^r_{1,j}) \geq S(\rho_{1,j}) - |S(\rho_{1,j}) - S(\rho^r_{1,j})|,$$

which completes the proof. \qed

Of course, the above estimate is useful only if $\lim_{r \to \infty} g(r) = 0$ which also assumes that $S^\alpha(\rho_{1,j}) < \infty$. For the undetermined region of Table 3.1 we refer to the examples provided in [15]. These carry over to the infinite dimensional case. We repeat one such example as a demonstration. E.g., a state that can be efficiently approximated with TT can have linearly growing Rényi entropy. To see this, assume w.l.o.g. $\mathcal{H}_j = \mathcal{H}_i =: \mathcal{H}^{id}$ for all $1 \leq i, j \leq d$ and let $\varphi_1, \varphi_2$ and $\varphi_3$ be orthonormal in $\mathcal{H}^{id}$. Then, for $0 \leq p_d \leq 1$, set

$$\psi_{2d} := \sqrt{1 - p_d(\varphi_3 \otimes \ldots \otimes \varphi_3)} + \sqrt{p_d \sum_{i_1, \ldots, i_d \in \{1, 2\}} (\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_d}) \otimes (\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_d})}.$$

Clearly $\|\psi_{2d}\|_{\mathcal{H}} = 1$. We can set $p_d := 1/d$ which, by construction, implies $\psi_{2d}$ converges to a rank-one state with growing $d$. I.e., by a simple rank-one approximation we obtain

$$\|\psi_{2d} - \sqrt{1 - p_d(\varphi_3 \otimes \ldots \otimes \varphi_3)}\|_{\mathcal{H}} \to \sqrt{p_d} \to \infty 0.$$

\(^5\)A majorizing sequence of probabilities represents less uncertainty and thus should have smaller entropy in any meaningful measure.

\(^6\)This assumption merely simplifies notation.
On the other hand, for $1 \leq j \leq d$, the density operator of the subsystem can be computed as

$$\rho_{1,j} = (1 - p_d) \rho^j + \frac{p_d}{2^j} \sum_{i_1, \ldots, i_j \in \{1, 2\}} \rho^{i_1, \ldots, i_j},$$

$$\rho^j := \left(\langle \cdot | \varphi_1 \otimes \cdots \otimes \varphi_j \rangle \right)_{S_{1,j}} \varphi_1 \otimes \cdots \otimes \varphi_j,$$

and therefore the Rényi entropy for $0 < \alpha < 1$ is

$$S^\alpha(\rho_{1,j}) = \frac{1}{1 - \alpha} \log_2 \left(1 - p_d^\alpha + 2^{1-\alpha} p_d^\alpha\right) \geq j - \frac{\alpha}{1 - \alpha} \log_2(d) = j + \frac{2\alpha}{1 - \alpha} \log_2(\varepsilon),$$

for $\varepsilon := \sqrt{p_d} = \sqrt{1/d}$. For examples of the other parts of the undetermined region of Table 3.1 we refer to [45].

We conclude this section by a brief discussion of entropy discontinuity. For $\psi \in S(\mathcal{H})$ and $\rho := \langle \cdot, \psi \rangle_{\mathcal{H}} \psi \in S(\mathcal{H})$, $S^\alpha(\rho_{1,j})$ for some $1 \leq j \leq d$ is not necessarily finite for $0 < \alpha \leq 1$. In such cases entropy is no longer a useful measure of approximability. We are thus not certain if or to what extent questions like “Does infinite entropy imply inapproximability?” or “Does approximability imply finite entropy?” make sense. For $\alpha > 1$ the Rényi entropy is always finite. In this case we can certify that a faster than logarithmic scaling implies inapproximability.

### 3.2. Entropy Convergence

In this section, we consider the question of finite entropy or entropy continuity. This is important not only for considerations in Section 3.1 but for Section 4 as well. Since one of our results is an upper bound for the von Neumann entropy in the infinite dimensional setting, we first have to consider for what states entropy makes sense in the first place.

The set of states with infinite entropy is dense in $\text{Tr}(\mathcal{H})$ (see [50, section II.D]). From a purely analytic standpoint, entropy is finite if the singular values of bipartite cuts converge fast enough. Put more precisely, given an algebraic decay of the singular values, we obtain

**Proposition 3.6.** For any $1 \leq j \leq d$, if $\sigma_k^j \lesssim k^{-s}$ for $s > \frac{1}{2\alpha}$, $0 < \alpha \leq 1$, then $S^\alpha(\rho_{1,j}) < \infty$. 

**Proof.** For the von Neumann entropy, by [6 Equation (45)], it is sufficient to show that there exists some $\delta > 0$ such that $\sum_{k=1}^{\infty} (\sigma_k^j)^2 k^\delta \lesssim \infty$. I.e., by assumption,

$$\sum_{k=1}^{\infty} (\sigma_k^j)^2 k^\delta \lesssim \sum_{k=1}^{\infty} k^{-2s+\delta} < \infty,$$

for some $0 < \delta < 2s - 1$. For $0 < \alpha < 1$, we obtain

$$S^\alpha(\rho_{1,j}) \lesssim \sum_{k=1}^{\infty} k^{-2s\alpha} < \infty,$$

since $-2s\alpha < -1$. \qed

By virtue of the fact that $\rho \in \text{Tr}(\mathcal{H})$, $S^\alpha(\rho_{1,j}) < \infty$ for any $\alpha > 1$. The requirement in Proposition 3.6 is more useful as a necessary condition: if entropy fails to be finite, it tells us how “slow” the decay rate must be. However, since we are ultimately interested in the relation between entropy scaling and approximation, it is not useful as a criteria to decide a priori the approximability of a system.

To this end, we discuss a set of conditions frequently assumed in the physics literature (see [22, 50]). There are essentially two difficulties that appear in the setting of infinite dimensional Hilbert spaces, not encountered in finite dimensional systems.

Firstly, it is possible for the expected energy described by the system Hamiltonian to be infinite in a given state. This is due to the fact that in the infinite dimensional setting Hamiltonians are generally unbounded, just like any differential operator. It can be shown that if the entropy of a given state is infinite, then so must be the expected energy of that state (see [50, section II.D]).

Secondly, in the finite dimensional case it is straightforward to determine the state of maximal entropy (maximal chaos). It is simply the state with a uniform distribution of probabilities, i.e., the density matrix is the identity operator times a normalization constant $1/d$.

This does not work anymore in infinite dimensions, since such a state is no longer normalizable. Instead, the state of maximal entropy is given by the Gibbs state, a well-known equilibrium distribution.
from statistical mechanics. This state is not unique and does not need to exist, but rather depends on
the inverse temperature (parameter \( \beta \) from Proposition 3.7) which leads to different expected energies.

Assuming the existence of the Gibbs state is equivalent to assuming the system Hamiltonian has
purely discrete spectrum and the eigenvalues diverge “fast enough” (see Thms XIII.16, XIII.67). Physically, it means that we assume the existence of a thermodynamic limit at any temperature. However, even in a physically meaningful setting, the Gibbs state does not have to exist: neither on physical grounds nor on mathematical. We will return to this issue in Section 5. For now we formulate
a model setting in which the von Neumann entropy is finite.

**Proposition 3.7.** Suppose we are given a self-adjoint PDE operator \( H : \mathcal{D}(H) \to \mathcal{H} \) such that
\[
H = H_{1,1} \otimes \mathbb{1}_{j+1,d} + R_j, \quad R_j \geq 0,
\]
for any \( 1 \leq j \leq d \), where \( H_{1,1} : \mathcal{D}(H_{1,1}) \to \mathcal{H}_{1,1} \) is self-adjoint and \( \mathbb{1}_{j+1,d} : \mathcal{H}_{j+1,d} \to \mathcal{H}_{j+1,d} \) is the
identity operator. For \( \rho \in \mathcal{S}(\mathcal{H}) \), assume that \( \text{tr}[\rho H] := E < \infty \) and \( \exp[-\beta H] \in \text{Tr}(\mathcal{H}) \) for all \( \beta > 0 \). Then, \( \mathcal{S}(\rho_{1,j}) < \infty \) for any \( 1 \leq j \leq d \).

**Proof.** The idea of the proof is as follows. Since the expected energy of the whole system is finite,
then so is the expected energy of the subsystems. For the given energy \( E \), we can choose an inverse
temperature \( \beta > 0 \), such that the corresponding Gibbs state has the same expected energy. Since the
Gibbs state has maximal entropy (for fixed \( E \), this provides the upper bound.

**I.** For any \( 1 \leq j \leq d \), it holds that
\[
\infty > E = \text{tr}[\rho H] = \text{tr}[\rho H_{1,j} \otimes \mathbb{1}_{j+1,d} + \rho R_j] \geq \text{tr}[\rho H_{1,j} \otimes \mathbb{1}_{j+1,d}] = \text{tr}[\rho_{1,j} H_{1,j}].
\]

**II.** Our assumptions imply that \( \exp[-\beta H] \) is compact and thus \( H \) must have purely discrete spectrum
which is bounded from below and has diverging eigenvalues such that
\[
\text{tr}[\exp[-\beta H]] = \sum_{k=1}^{\infty} \exp[-\beta \lambda_k] < \infty.
\]

From here on we assume the eigenvalues of \( H \), \( \{\lambda_k : k \in \mathbb{N}\} \), are ordered in non-decreasing order.

The existence of the Gibbs state for any inverse temperature \( \beta > 0 \) implies the expected that energy
is finite for any \( \beta > 0 \). To see this, note that since \( \lim_{k \to \infty} \lambda_k = \infty \), there exists an \( N \in \mathbb{N} \) such that
\[
\exp\left[\frac{\beta}{2} \lambda_k\right] \geq \lambda_k \text{ for } k \geq N. \text{ Thus,}
\]
\[
\sum_{k=N}^{\infty} \exp[-\beta \lambda_k] \lambda_k \leq \sum_{k=N}^{\infty} \exp[-\beta \lambda_k] \exp\left[\frac{\beta}{2} \lambda_k\right] = \sum_{k=N}^{\infty} \exp\left[-\frac{\beta}{2} \lambda_k\right] < \infty.
\]

By applying elementary functional calculus we compute \( \text{tr}[\exp[-\beta H]] = \sum_{k=1}^{\infty} \exp[-\beta \lambda_k] \lambda_k < \infty. \)

**III.** Thus, we have ensured the existence of the Gibbs state
\[
(3.4) \quad \rho_{\beta} := \exp[-\beta H]/Z, \quad Z := \text{tr}[\exp[-\beta H]],
\]
with finite energy
\[
E(\beta) = \text{tr}[\rho_{\beta} H] = \left(\sum_{k=1}^{\infty} \exp[-\beta \lambda_k]\right)^{-1} \sum_{k=1}^{\infty} \exp[-\beta \lambda_k] \lambda_k.
\]
The function \( E(\beta) \) is continuous for \( \beta \in (0, \infty) \). To see this, take any \( \beta > 0 \) and consider the interval
\([\beta/2, 2 \beta]\). Since the sequence \( \{\lambda_k : k \in \mathbb{N}\} \) diverges, there exists \( N_1 \in \mathbb{N} \), such that \( \exp\left[-\frac{\beta}{2} \lambda_k\right] \lambda_k \) is
monotonically decreasing for \( k \geq N_1 \) and there exists \( N_2 = N_2(\varepsilon) \in \mathbb{N} \), such that
\[
\sum_{k=N_2}^{\infty} \exp\left[-\frac{\beta}{2} \lambda_k\right] \lambda_k \leq \varepsilon.
\]
Thus, for any \( \varepsilon > 0 \), we can take \( N_2 = N_2(\varepsilon) = \max\{N_1, N_2(\varepsilon)\} \) such that for \( M \geq N \) and any
\( \beta \in [\beta/2, 2 \beta] \)
\[
\sum_{k=M}^{\infty} \exp\left[-\beta \lambda_k\right] \lambda_k \leq \sum_{k=M}^{\infty} \exp\left[-\frac{\beta}{2} \lambda_k\right] \lambda_k \leq \varepsilon.
\]
Thus, this series converges uniformly in $\beta$ on compact sets in $\mathbb{R}^+$ and is therefore continuous for any $\beta > 0$. An analogous argument shows the same for the series $\sum_{k=1}^{\infty} \exp[-\beta \lambda_k]$. Hence, $E(\beta)$ is continuous.

IV. The idea behind this part of the proof is as follows: for $\beta \to 0$, i.e., temperature $T \to \infty$, higher energies become more probable such that we anticipate for the expected energy of a Gibbs state $E(\beta) \to \infty$. On the other hand, for $\beta \to \infty$ ($T \to 0$), we expect probabilities to cluster around the ground state energy $\lambda_1$.

Since the sequence $\{\lambda_k : k \in \mathbb{N}\}$ diverges, for any $C > 0$, there exists $N = N(C) \in \mathbb{N}$ such that $\sum_{k=1}^{N} \lambda_k \geq C$ and, clearly, $N \to \infty$ as $C \to \infty$.

As was shown in III, both the nominator and denominator in $E(\beta)$ are series that converge uniformly in $\beta$. Thus, for any $\varepsilon > 0$, there exists $\tilde{\beta} > 0$, such that for any $\beta \leq \tilde{\beta}$

$$E(\beta) \geq \frac{\sum_{k=1}^{\infty} \lambda_k - \varepsilon}{\sum_{k=1}^{N} \lambda_k + \varepsilon} = \frac{1}{N + \varepsilon} \sum_{k=1}^{N} \lambda_k - \varepsilon \geq \frac{N - \varepsilon}{N + \varepsilon} C - \frac{\varepsilon}{N + \varepsilon}.$$  

Since this is possible for any $\varepsilon$ and any $C > 0$, we have $\lim_{\beta \to 0} E(\beta) = \infty$.

On the other hand, let $p_k := \exp[-\beta \lambda_k]/Z(\beta)$. Then,

$$|E(\beta) - \lambda_1| = \sum_{k=2}^{\infty} p_k (\lambda_k - \lambda_1).$$

Since this series converges, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\sum_{k=N}^{\infty} p_k (\lambda_k - \lambda_1) \leq \varepsilon$. We can choose $N$ independently of $\beta$, since for $\beta \to \infty$ the series converges faster and thus $N$ gets smaller. Due to the convergence of $\{p_k(\beta), \ldots, p_{N-1}(\beta)\}$ for $\beta \to \infty$, there exists $\beta > 0$ such that for $\beta \geq \tilde{\beta}$, $p_k(\beta)(\lambda_k - \lambda_1) \leq \frac{\varepsilon}{N - 2}$ for $2 \leq k \leq N - 2$. Thus,

$$\sum_{k=2}^{\infty} p_k (\lambda_k - \lambda_1) = \sum_{k=2}^{N-1} p_k (\lambda_k - \lambda_1) + \sum_{k=N}^{\infty} p_k (\lambda_k - \lambda_1) \leq 2\varepsilon.$$  

Since this is possible for any $\varepsilon > 0$, we conclude $\lim_{\beta \to \infty} E(\beta) = \lambda_1$.

V. Finally, take any $1 \leq j \leq d$. By I, $E_j := \text{tr}[\rho_1 H_{1,j}] \leq E = \text{tr}[\rho H] < \infty$. By III-IV, there exists $\beta > 0$ such that $\text{tr}[\rho_\beta H_{1,j}] = E_j$.

In [50] section I.B.5, inequality (1.41)) it is shown that if $\rho$ is a state with $E = \text{tr}[\rho H]$ and $\rho_\beta$ is defined as in [54] with $E = \text{tr}[\rho_\beta H] = \text{tr}[\beta H]$, then $S(\rho) \leq S(\rho_\beta)$. Applying this to our problem we get $S(\rho_{1,j}) \leq S(\rho_\beta) = \beta E_j + \log_2(Z) < \infty$. This completes the proof. \hfill \square

4. Ground State Approximability

We turn to the main result of this work. We illustrate how local interactions in a Hamiltonian operator imply that the ground state can be approximated by operators which have a small overlap in the support. Under an additional assumption on the approximand (see Assumption 4.16), we show that the entropy of the ground state does not scale with the dimension. Consequently, by the considerations in Section 3.1 such eigenfunctions enjoy favorable separability properties within the TT format. To stay consistent with the typical notation in physics, we will slightly adapt indices in this Section by shifting all summations to start from 0 such that the first eigenvalue will be denoted by $\lambda_0$.

4.1. Ground State Density Operator. Before we can proceed with the entropy bound, we require some preparations. The key ingredient will be Theorem 4.11 which essentially states that the ground state density operator can be approximated by a product of 3 local operators with overlapping support, where the error converges exponentially in the length of the overlap. Indeed, Theorem 4.11 is interesting in its own right and we consider it to be the main contribution of this work. This approximation is possible if the Hamiltonian operator satisfies the following properties.

Assumption 4.1. Let $H : \mathcal{D}(H) \to \mathcal{H}$ be a densely defined self-adjoint (possibly unbounded) operator.  

(1) (Locality). We assume $H$ can be decomposed as

$$H = \sum_{j=1}^{d-1} H_{j,j+1},$$
Remark 4.2. Assumption (1) means we only consider local 2-site interactions. Our results would remain unchanged for N-site interactions, for a fixed N. The point is that the complexity of approximating an eigenfunction scales exponentially with N and not d. Moreover, we expect similar results could be obtained for long range interactions that decay sufficiently fast.

We require Assumption (5) since the proof heavily relies on the spectral decomposition. One could possibly generalize the proofs presented here to sectorial operators. We are not certain to what extent approximability actually depends on the form of the resolvent/spectrum of the operator in \( \mathbb{C} \).

Assumption (2) is necessary for an area law to hold. Systems with degenerate ground states are at a quantum critical point and have been observed to exhibit divergent entanglement entropies (see [12, 20, 49]). However, uniqueness of the ground state is not necessary for the main estimate in Theorem 4.11.

Assumptions (3) and (1) are required for the application of Lieb-Robinson bounds, i.e., finite speed information propagation. There are essentially two difficulties when considering information propagation for dynamics prescribed by an unbounded operator.

First, unlike with classic Lieb-Robinson bounds (see [30]), bounded local operators do not have to remain local when evolved via the unitary operator \( \exp[\iota H t] \) (see [21]). This can be remedied as in [7, 33] by, e.g., assuming the interactions in \( H \) are of a certain type, such as bounded. Hence, we require Assumption (3).

Second, when applying time dynamics to an unbounded local operator, it is not clear in which sense the operator remains approximately local. Thus, Assumption (1) ensures that the non-local part is bounded.

However, we essentially require only an application of Lieb-Robinson. Although Assumptions (3) and (1) are certainly sufficient, they are perhaps not necessary.

Example 4.3 (Nearest Neighbor Interaction (NNI)). We provide an example of how the general structure of such an NNI Hamiltonian might look like. Perhaps the most famous example of an NNI Hamiltonian is the Ising model (see [11]).

In this work we consider infinite dimensional Hilbert spaces and unbounded Hamiltonians. A typical example to keep in mind is \( \mathcal{H} = \otimes_{j=1}^{d} \mathcal{H}_{j} = \otimes_{j=1}^{d} L^{2}(\mathbb{R}^{n}, \mathbb{C}) \), where \( n \in \{1, 2, 3\} \) if \( H \) is to model a physical phenomenon.

Let the Hamiltonian operator be given as

\[
H = -\Delta + V.
\]

Footnotes:

1. That models interactions between particles \( j \) and \( j + 1 \).
2. Or can be uniquely extended to bounded operators.
The Laplacian $\Delta$ is the one-site unbounded operator where $H_j = -\frac{\partial^2}{\partial x_j^2}$. The potential $V$ contains the bounded interaction operators. E.g., $V = \sum_{j=1}^{d-1} \Phi_{j,j+1}$, where $\Phi_{j,j+1} : \mathcal{H} \to \mathcal{H}$ is a bounded operator such as

$$(\Phi_{j,j+1}\psi)(x) = c(x_j, x_{j+1})\psi(x), \quad \text{or}$$

$$(\Phi_{j,j+1}\psi)(x) = \int_{\mathbb{R}^n} \kappa(x_j, x_{j+1}, y_j, y_{j+1})\psi(x_1, \ldots, y_j, y_{j+1}, \ldots, x_d) \, d(y_j, y_{j+1}),$$

where $c(\cdot)$ is a bounded coefficient function and $\kappa(\cdot)$ is an integral kernel.

We would have to check that a given Hamiltonian has a gap above the ground state and if the ground state is unique. Note that the gap property is much more important than uniqueness, since the latter is only necessary for the area law in Theorem 2.18 and not for the approximation in Theorem 4.11.

Spectral properties and uniqueness of ground states have been extensively studied before and we refer to, e.g., [43, Chapter XIII] for more details.

We begin with a lemma that shows how we can approximately express the ground state projector through the Hamiltonian operator. This will provide the necessary link between the local operator structure and the local structure of the density operator.

**Lemma 4.4.** Let Assumption 4.1 hold. Assume w.l.o.g. that $\lambda_0 = 0$ and let $\rho^0$ denote the corresponding density operator (see (4.12)). Then, for any $q > 0$ and

$$\rho^q := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp[iHt] \exp\left[-\frac{t^2}{2q}\right] \, dt,$$

we have

$$\|\rho^q - \rho^0\|_\mathcal{L} \leq \exp\left[-\frac{1}{2}(\Delta E)^2 q\right],$$

with $\Delta E$ from (4.1).

**Proof.** The operator $U(t) := \exp[iHt] \exp\left[-\frac{t^2}{2q}\right]$ is strongly continuous for all $t \in \mathbb{R}$. Thus, a finite integral of $U(t)$ is well defined. For any $\psi \in \mathcal{H}$

$$\lim_{c \to \infty} \left\| \frac{1}{\sqrt{2\pi q}} \int_{-c}^{c} U(t)\psi \, dt \right\|_{\mathcal{H}} \leq \lim_{c \to \infty} \|\psi\|_{\mathcal{H}} \frac{1}{\sqrt{2\pi q}} \int_{-c}^{c} \exp\left[-\frac{t^2}{2q}\right] \, dt = \|\psi\|_{\mathcal{H}}.$$

Thus, the integral (4.4) is well defined.

Since $H$ is self-adjoint, we have the spectral decomposition

$$H = \int_{\sigma(H)} \lambda \, dP(\lambda),$$

where $P : \sigma(H) \to \mathcal{L}(\mathcal{H})$ is a projection valued measure. Due to the gap assumption, we get that $\rho^0 = P(\lambda_0)$.

Applying functional calculus for self-adjoint operators

$$\exp[iHt] = \int_{\sigma(H)} \exp[i\lambda t] \, dP(\lambda).$$

Equation (4.1) is to be interpreted as the unique operator such that for any $\psi \in \mathcal{H}$

$$\langle \psi, \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} U(t)\psi \, dt \rangle_{\mathcal{H}} = \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp\left[-\frac{t^2}{2q}\right] \langle \psi, U(t)\psi \rangle_{\mathcal{H}} \, dt$$

$$= \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \exp\left[-\frac{t^2}{2q}\right] \int_{\sigma(H)} \exp[i\lambda t] \, dP(\lambda) \, dt,$$

where $P_\psi(\cdot) = \langle \psi, P(\cdot)\psi \rangle_{\mathcal{H}}$ and the equality follows from the linearity and continuity of the $\mathcal{H}$-inner product. For the last integral we can apply Fubini’s Theorem for general product measures. This allows
Lemma 4.5. Suppose Assumption 4.1 (1-4) holds. 
For a fixed $l \in \mathbb{N}$ and a fixed $1 + l \leq j \leq d - 2 - l$, 
\[
H_L := \sum_{k \leq j - l - 2} H_{k,k+1}, \quad H_B := \sum_{j-l-1 \leq k \leq j+l+1} H_{k,k+1}, \quad H_R := \sum_{k \geq j+l+2} H_{k,k+1}.
\]

W.l.o.g. let $\langle \psi_0, H_L \psi_0 \rangle 1 \mathcal{H} = \langle \psi_0, H_B \psi_0 \rangle 1 \mathcal{H} = \langle \psi_0, H_R \psi_0 \rangle 1 \mathcal{H} = 0$ and $\lambda_0 = 0$. Then, for any $q > 0$ and 
\[
\tilde{H}_L := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_L(t) \exp \left[ -\frac{t^2}{2q} \right] dt,
\]
\[
\tilde{H}_B := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_B(t) \exp \left[ -\frac{t^2}{2q} \right] dt,
\]
\[
\tilde{H}_R := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_R(t) \exp \left[ -\frac{t^2}{2q} \right] dt,
\]
where $H_{\ldots}(t)$ is given as in (2.2) by, e.g., 
\[
H_L(t) := \exp[i H t] H_L \exp[-i H t],
\]
we have 
\[
\| \tilde{H}_L \psi_0 \|_{1 \mathcal{H}} \leq 3J^2 (\Delta E)^{-1} \exp \left[ -\frac{1}{2} (\Delta E)^2 q \right],
\]
\[
\| \tilde{H}_B \psi_0 \|_{1 \mathcal{H}} \leq 3J^2 (\Delta E)^{-1} \exp \left[ -\frac{1}{2} (\Delta E)^2 q \right],
\]
\[
\| \tilde{H}_R \psi_0 \|_{1 \mathcal{H}} \leq 3J^2 (\Delta E)^{-1} \exp \left[ -\frac{1}{2} (\Delta E)^2 q \right].
\]

The constant $J$ is the interaction strength from (4.3).

Proof. By the same arguments as in Lemma 4.4, the integrals are well defined. Next, application to the ground state yields 
\[
H_L(t) \psi_0 = \exp[i H t] H_L \int_{\sigma(H)} \exp[-i \lambda t] dP(\lambda) \psi_0, \quad H_B \psi_0 = \int_{\sigma(H)} \exp[i H t] H_B \psi_0, \quad H_R \psi_0 = \int_{\sigma(H)} \exp[i \lambda t] dP(\lambda) H_R \psi_0.
\]

The authors considered this technique in order to describe states that belong to the same phase.

Next, we want to approximate $H$ by a sum of three local operators, where each operator approximately annihilates the ground state. To this end, we apply Hasting’s quasi-adiabatic continuation technique (see [27][28]), which was also studied in [7] in the infinite dimensional setting.

By the same arguments as in Lemma 4.4, the integrals are well defined. Next, application to the ground state yields 
\[
H_L(t) \psi_0 = \exp[i H t] H_L \int_{\sigma(H)} \exp[-i \lambda t] dP(\lambda) \psi_0, \quad H_B \psi_0 = \int_{\sigma(H)} \exp[i H t] H_B \psi_0, \quad H_R \psi_0 = \int_{\sigma(H)} \exp[i \lambda t] dP(\lambda) H_R \psi_0.
\]
Thus,
\[
\hat{H}_L \psi_0 = \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} H_L(t) \psi_0 \exp\left[-\frac{t^2}{2q}\right] \, dt \\
= \int_{\sigma(H)\setminus\{\lambda_0\}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[i \lambda t] \exp\left[-\frac{t^2}{2q}\right] \, dt \, dP(\lambda) H_L \psi_0 \\
= \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] \, dP(\lambda) H_L \psi_0.
\]

On the other hand,
\[
H \hat{H}_L \psi_0 = \int_{\sigma(H)\setminus\{\lambda_0\}} \lambda \exp\left[-\frac{1}{2} \lambda^2 q\right] \, dP(\lambda) H_L \psi_0.
\]

Hence,
\[
(4.5) \quad \|H \hat{H}_L \psi_0\|_{\mathcal{H}} \geq \Delta E \left\| \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] \, dP(\lambda) H_L \psi_0 \right\|_{\mathcal{H}} = \Delta E \|\hat{H}_L \psi_0\|_{\mathcal{H}}.
\]

Next, since \( H \psi_0 = 0 \),
\[
H \hat{H}_L \psi_0 = (H \hat{H}_L - \hat{H}_L H) \psi_0 = \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] \, dP(\lambda)[H, H_L] \psi_0 \\
= \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] \, dP(\lambda)[H_{j-l-1,j-l}, H_{j-l-2,j-l-1}] \psi_0 \\
= \int_{\sigma(H)\setminus\{\lambda_0\}} \exp\left[-\frac{1}{2} \lambda^2 q\right] \, dP(\lambda) \left\{ [H_{j-l-1}, (\Phi_{j-l-2,j-l-1} - \Phi_{j-l-1,j-l})] \\
+ [\Phi_{j-l-1,j-l}, \Phi_{j-l-2,j-l-1}] \right\} \psi_0,
\]

where the last equality follows from Assumption 3. And thus
\[
\|H \hat{H}_L \psi_0\|_{\mathcal{H}} \leq 3J^2 \exp\left[-\frac{1}{2} (\Delta E)^2 q\right].
\]

Together with (4.5)
\[
\|\hat{H}_L \psi_0\|_{\mathcal{H}} \leq 3J^2 (\Delta E)^{-1} \exp\left[-\frac{1}{2} (\Delta E)^2 q\right],
\]

and analogously for \( \hat{H}_B, \hat{H}_R \). This completes the proof.

\[\square\]

**Remark 4.6.** Note that the above bound depends explicitly only on \( q, \Delta E \) and \( J \). In fact, more precisely, the bound depends on an estimate for
\[
\|\|[H, H_L]\|_{\mathcal{L}},
\]
and the latter was assumed in (4.3) to be uniformly bounded, i.e., in particular it is independent of \( j \) or \( l \).

However, the subsequent lemmas will employ Lieb-Robinson bounds that depend explicitly on the parameter \( l \). Thus, we will eventually use the above lemma and choose the constant \( q \) depending on the spectral gap \( \Delta E \) and the parameter \( l \).

Next, we show that \( \hat{H}_L, \hat{H}_B \) and \( \hat{H}_R \) are approximately local. This is mainly due to Lieb-Robinson type estimates.

**Lemma 4.7.** Under Assumption 4.3 (4.3), there exist local bounded operators \( \Theta_L, \Theta_B \) and \( \Theta_R \) supported on \( \mathcal{H}_{j-2l-2,j}, \mathcal{H}_{j-2l-2,j+2l+3} \) and \( \mathcal{H}_{j+1,j+2l+3}, \) respectively, such that for
\[
M_L := H_L + \Theta_L, \quad M_B := H_B + \Theta_B, \quad M_R := H_R + \Theta_R.
\]

\[\text{In the sense specified by the following lemma.}\]
there exist constants $c_1 > 0$, $C_1 > 0$ such that
\[
\left\| \hat{H}_L - M_L \right\|_\mathcal{L} \leq C_1 J^2 \max\left\{ q^{1/2}, q^{3/2} \right\} \exp[-c_1 t], \\
\left\| \hat{H}_B - M_B \right\|_\mathcal{L} \leq C_1 J^2 \max\left\{ q^{1/2}, q^{3/2} \right\} \exp[-c_1 t], \\
\left\| \hat{H}_R - M_R \right\|_\mathcal{L} \leq C_1 J^2 \max\left\{ q^{1/2}, q^{3/2} \right\} \exp[-c_1 t],
\]
where $q$, $l$ are the parameters from Lemma 4.2 and $J$ is the interaction strength from (4.3).

Proof. First, note that we can differentiate $H_L(t)$ to obtain
\[
\frac{d}{dt} H_L(t) = \frac{d}{dt} \exp[iHt] H_L \exp[-iHt] \\
= \exp[iHt] iH L \exp[-iHt] - \exp[iHt] iH L H \exp[-iHt] \\
= \exp[iHt] i[H, L] \exp[-iHt] =: i[H, L](t),
\]
and $H_L(0) = H_L$. Thus, we can write
\[(4.6) \quad H_L(t) = H_L + \int_0^t i[H, H_L](\tau) \, d\tau.\]
By Assumption, $[H, H_L]$ is bounded and supported on $\mathcal{H}_{j-l-2,j-1}$. Consequently, we can write
\[
\hat{H}_L = H_L + \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \int_0^t i[H, H_L](\tau) \, d\tau \exp\left[\frac{t^2}{2q}\right] \, dt.
\]
Since the commutator is bounded and local, and the interactions in $H$ are bounded, by [33 Corollary 2.2], we know a Lieb-Robinson bound applies to $[H, H_L]$. I.e., there exists a constant (velocity) $v \geq 0$ and constants $C > 0$, $a > 0$, such that
\[(4.7) \quad \|[[[H, H_L](\tau), B]\|_{\mathcal{L}} \leq C \|H, H_L\|_{\mathcal{L}} \|B\|_{\mathcal{L}} \exp[-a \{ \text{dist}([H, H_L], B) - v|\tau| \}],
\]
for all bounded and local $B$. Thus, by [4] Lemma 3.2, there exists a map $\Pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ such that $\Pi(A)$ is supported on $\mathcal{H}_{j-2l-2,j}$ and, for any $A \in \mathcal{L}(\mathcal{H})$ satisfying (4.7) with $\text{dist}(A, B) \geq l$, we have
\[
\|A - \Pi(A)\|_{\mathcal{L}} \leq 2C \|A\|_{\mathcal{L}} \exp[-a \{l - v|\tau| \}].
\]
Then, using [4] Theorem 3.4, we integrate over time and further estimate
\[
\left\| \int_0^t [H, H_L](\tau) \, d\tau - \int_0^t \Pi([H, H_L](\tau)) \, d\tau \right\|_{\mathcal{L}} \leq |t| C J^2 \exp[-a \{l - v|t| \}].
\]
We define the operator
\[
\Theta_L := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \int_0^t \Pi(i[H, H_L](\tau)) \, d\tau \exp\left[\frac{t^2}{2q}\right] \, dt.
\]
By all of the above, this operator is bounded and supported on $\mathcal{H}_{j-2l-2,j}$. Analogously, we define $\Theta_B$ and $\Theta_R$ with supports in $\mathcal{H}_{j-2l-2,j+2l+3}$ and $\mathcal{H}_{j+1,j+2l+3}$, respectively.

What remains is to truncate the tails of the integral to obtain an overall error of the same order as the Lieb-Robinson bound. Let $T = \frac{4}{2e}$. Then,
\[
\left\| \hat{H}_L - M_L \right\|_{\mathcal{L}} = \\
= \left\| \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \int_0^t (i[H, H_L](\tau) - \Pi(i[H, H_L](\tau))) \, d\tau \exp\left[\frac{t^2}{2q}\right] \, dt \right\|_{\mathcal{L}} \\
= \left\| \frac{1}{\sqrt{2\pi q}} \left( \int_{|t| \leq T} + \int_{|t| > T} \right) \right\|_{\mathcal{L}}
\]
(4.8)
\[
\leq C J^2 \exp[-al] \left( \exp[avT] \frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} |t| \exp\left[\frac{t^2}{2q}\right] \, dt + \frac{1}{\sqrt{2\pi q}} \int_{|t| > T} |t| \exp\left[av|t| - \frac{t^2}{2q}\right] \, dt \right).
\]
For the first term
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} |t| \exp \left[ -\frac{t^2}{2q} \right] \, dt \leq \sqrt{\frac{q}{2\pi}}.
\]

For the second
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| > T} |t| \exp \left[ a\sqrt{\frac{1}{2}} \left( t^2 - \frac{t^2}{2q} \right) \right] \, dt = \frac{1}{\sqrt{2\pi q}} \int_{T}^{\infty} t \exp \left[ a\sqrt{\frac{1}{2}} \left( t^2 - \frac{t^2}{2q} \right) \right] \, dt \\
= q \exp \left[ a\sqrt{\frac{1}{2}} \right] (\sqrt{\frac{q}{2}} T - \frac{T^2}{2q}) \\
+ aq \int_{T}^{\infty} \exp \left[ a\sqrt{\frac{1}{2}} \left( t^2 - \frac{t^2}{2q} \right) \right] \, dt.
\]

For the latter term
\[
\int_{T}^{\infty} \exp \left[ a\sqrt{\frac{1}{2}} \left( t^2 - \frac{t^2}{2q} \right) \right] \, dt = \int_{T}^{\infty} \left( \frac{\exp[\sqrt{\frac{1}{2}} t] - \sqrt{\frac{q}{2}} t}{\sqrt{\frac{q}{2}}} \right) \, dt \\
= \sqrt{2q} \int_{\sqrt{\frac{q}{2}} T - a\sqrt{\frac{1}{2}}}^{\infty} \exp[-\tau^2] \, d\tau \leq \sqrt{\frac{4\pi}{2}} \exp \left[ a\sqrt{\frac{1}{2}} \right] (\sqrt{\frac{q}{2}} T - \frac{T^2}{2q}).
\]

And thus
\[
\| \tilde{H}_L - M_L \|_\mathcal{L} \leq C J^2 \exp[-a\sqrt{\frac{1}{2}}] \times \\
\left( \exp[\sqrt{\frac{1}{2}} t] + a\exp[\sqrt{\frac{1}{2}}] \exp \left[ \sqrt{\frac{q}{2}} t \right] \right) \leq C J^2 \exp \left[ -\frac{a}{2} \right] \max \left\{ \sqrt{\frac{q}{2}}, q, a\sqrt{\frac{1}{2}} \right\} \left( 1 + 2 \exp \left[ -\frac{t^2}{8a^2} \right] \right)
\]
with $C_1$ and $c_1$ defined in an obvious way as above. This completes the proof. \qed

We can conclude the existence of the first two operators that we will need to approximate $\rho^0$.

**Lemma 4.8.** Under Assumption 4.1, there exist local, bounded and self-adjoint (projection) operators $O_L = O_L(l)$, $O_R = O_R(l)$ with the property
\[
\| (O_L - I) \psi \|_\mathcal{H} \leq \exp[-c_1 l/2], \quad \| (O_R - I) \psi \|_\mathcal{H} \leq \exp[-c_1 l/2].
\]

The operators $O_L$ and $O_R$ have the same support as $M_L$ and $M_R$, respectively, and $\| O_L \|_\mathcal{L} = \| O_R \|_\mathcal{L} = 1$.

**Proof.** Recall from Lemma 4.7 that we applied Lieb-Robinson to operators such as $i[H, H_L](\tau)$. By Assumption 4.1 since $H$ and $H_L$ are self-adjoint, the commutators are bounded and one has $i[H, H_L] = -i(H_LH - HH_L) = i[H, H_L] \Rightarrow$ we can conclude that the commutator is self-adjoint. Applying Lieb-Robinson as in [7] Lemma 3.2], one can construct the local approximation to preserve self-adjointness. We briefly elaborate.

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. For the construction of $\Pi : \mathcal{H} \to \mathcal{H}$ in [7] Lemma 3.2], the authors use an arbitrary state $\rho \in S(\mathcal{H}_2)$, though the resulting bound does not depend on the choice of $\rho$. Then, for this state, applying the spectral decomposition, we have
\[
\rho = \sum_{k=1}^{\infty} \lambda_k \langle \cdot, \psi_k \rangle_{\mathcal{H}_2} \psi_k,
\]
for $\{ \psi_k : k \in \mathbb{N} \}$ orthonormal in $\mathcal{H}_2$. Next, for each $\psi_k$, the authors define the map $A_k$ as
\[
\langle v, A_k w \rangle_{\mathcal{H}_2} = \langle v \otimes \psi_k, A w \otimes \psi_k \rangle_{\mathcal{H}}, \quad \forall v, w \in \mathcal{H}_2.
\]

Finally, the map $\Pi(A)$ is defined as
\[
\Pi(A) := \left( \sum_{k=1}^{\infty} \lambda_k A_k \right) \otimes I.
\]
Note that each $A_k$ is self-adjoint if $A$ is self-adjoint. Therefore, $\Pi(A)$ is self-adjoint.

Thus, $M_L$, $M_B$ and $M_R$ can be chosen self-adjoint. By Lemmas 4.7 and 4.8 picking $q = c_1 \frac{2}{(A_L^2)^{1/4}}$, we get $\|M_L \psi_0\|_{\mathcal{H}} \leq C_1 J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1 t]$. Moreover, since $M_L$ is self-adjoint, there exists a projection valued measure $P(\cdot)$ such that

$$\|M_L \psi_0\|_{\mathcal{H}}^2 = \langle M_L \psi_0, M_L \psi_0 \rangle_{\mathcal{H}} = \langle \psi_0, M_L^2 \psi_0 \rangle_{\mathcal{H}} = \int_{\sigma(M_L)} \lambda^2 \, dP_{\psi_0}(\lambda).$$

We split the spectrum of $M_L$ as

$$\sigma_1(M_L) := \left\{ \lambda \in \sigma(M_L) : |\lambda| \leq C_1 J^2 \max\{q^{1/2}, q^{3/2}\} \exp[-c_1 t/2] \right\},$$

$$\sigma_2(M_L) := \sigma(M_L) \setminus \sigma_1(M_L).$$

Define $O_L$ as

$$O_L := \int_{\sigma_1(M_L)} dP(\lambda).$$

Clearly, $O_L$ is a bounded self-adjoint operator with $\|O_L\|_{\mathcal{L}} = 1$ and the same support as $M_L$. Moreover, by orthogonality of the spectral subspaces

$$C_1^2 J^4 \max\{q, q^2\} \exp[-c_1 2t] \geq \|M_L \psi_0\|_{\mathcal{H}} = \int_{\sigma_1(M_L)} \lambda^2 \, dP_{\psi_0}(\lambda) \psi_0 + \int_{\sigma_2(M_L)} \lambda^2 \, dP_{\psi_0}(\lambda) \psi_0 \geq C_1^2 J^4 \max\{q, q^2\} \exp[-c_1 t] \int_{\sigma_2(M_L)} dP_{\psi_0}(\lambda).$$

Thus, $\int_{\sigma_2(M_L)} dP_{\psi_0}(\lambda) \leq \exp[-c_1 t]$. Finally, this gives

$$\| (O_L - I) \psi_0 \|_{\mathcal{H}} = \left\| \int_{\sigma_2(M_L)} dP(\lambda) \psi_0 \right\|_{\mathcal{H}} = \left( \int_{\sigma_2(M_L)} dP_{\psi_0}(\lambda) \right)^{1/2} \leq \exp[-c_1 t/2].$$

Analogously for $O_R$. This completes the proof.

What remains is a step by step approximation of $\rho^g$ as a product of three local operators.

**Lemma 4.9.** Under Assumption \[4.4\], we can further approximate $\rho^0$ as

$$\tilde{\rho}^g := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \mathcal{T} \left( \exp \left[ \frac{\int_0^t A(\tau) \, d\tau}{2q} \right] \right)^* \exp \left[ -\frac{t^2}{2q} \right] O_L O_R \, dt,$$

$$A(t) := \exp[i(M_L + M_R)t]iM_B \exp[-i(M_L + M_R)t],$$

where $\mathcal{T} \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^*$ is the negative time-ordered exponential and $q = c_1 \frac{2}{(A_L^2)^{1/4}}$. We have

$$\|\tilde{\rho}^g - \rho^0\|_{\mathcal{L}} \leq C_2 \max\{q, q^2\} \left( \exp[-c_1 t/2] + \exp[-c_1 t] \right),$$

for some constant $C_2 > 0$.

**Proof.** Note that $H = \tilde{H}_L + \tilde{H}_B + \tilde{H}_R$. By utilizing the estimates from Lemma 4.7, we have

$$\left\| \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \left\{ \exp \left[ i(\tilde{H}_L + \tilde{H}_B + \tilde{H}_R)t \right] - \exp[i(M_L + MB + M_R)t] \right\} \exp \left[ -\frac{t^2}{2q} \right] \right\|_{\mathcal{L}}$$

$$\leq 3C_1 J^2 \max\{q^{1/2}, q^{3/2}\} \int_{-\infty}^{\infty} |t| \exp \left[ -\frac{t^2}{2q} \right] \, dt$$

$$= 3(2\pi)^{-1/2} C_1 J^2 \max\{q, q^2\} \exp[-c_1 t],$$

where the inequality can be shown using \[29\] Chapter 9, Theorem 2.12, Equation (2.22)].

Next, for the exponential term we can write

$$\exp[i(M_L + MB + M_R)t] = \exp[i(M_L + M_B + M_R)t] \exp[-i(M_L + M_R)t] \exp[i(M_L + M_R)t],$$
and we define \( U(t) := \exp[i(M_L + M_B + M_R)t] \exp[-i(M_L + M_R)t] \). In \( U(t) \) the term in the exponent commutes for different \( t \), since the time-dependence is a simple multiplication by \( t \). We thus compute

\[
\frac{d}{dt} U(t) = \exp[i(M_L + M_B + M_R)t]i(M_L + M_B + M_R) \exp[-i(M_L + M_R)t]
- \exp[i(M_L + M_B + M_R)t]i(M_L + M_R) \exp[-i(M_L + M_R)t]
= \exp[i(M_L + M_B + M_R)t]iM_B \exp[-i(M_L + M_R)t]
= \exp[i(M_L + M_B + M_R)t] \exp[-i(M_L + M_R)t]iM_B \exp[-i(M_L + M_R)t]
= U(t) \exp[i(M_L + M_R)t]iM_B \exp[-i(M_L + M_R)t]
\]

and \( U(0) = 1 \). We abbreviate

\[
M_B(t) := \exp[i(M_L + M_R)t]M_B \exp[-i(M_L + M_R)t].
\]

Due to the simple form of \( iM_B(t) \), the solution to this initial value problem exists, is unique and is given by the (negative) time-ordered exponential of \( \exp[iM_B(t)] \), see [12] Chapter X.12 (interaction representation) or [23] for ordered exponentials of more general unbounded time dependent Hamiltonians.

Thus, our approximation so far is

\[
\rho^0 = \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^\infty T(\exp\left[\int_0^t A(\tau) \, d\tau\right])^* \exp[i(M_L + M_R)t] \exp\left[\frac{-t^2}{2\sigma}\right] dt.
\]

By multiplying \( O_L, O_R \) from the right we obtain

\[
\| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L} = \| (\rho^0 - \rho^0) O_L O_R - \rho^0 \|_\mathcal{L} = \| (\rho^0 - \rho^0) O_L O_R + \rho^0 O_L O_R - \rho^0 \|_\mathcal{L}
\leq \| \rho^0 - \rho^0 \|_\mathcal{L} + \| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L}.
\]

For the latter term we use Lemma 116 set \( q = c_1 \frac{2}{\sqrt{2\pi \sigma}} \) and obtain

\[
\| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L} \leq 3 \| (O_L - I + I) + (O_R - I + I) - I \|_\mathcal{L} = 3 \| (O_L - I) \rho^0 \|_\mathcal{L} + \| (O_R - I) \rho^0 \|_\mathcal{L} \leq 4 \exp[-c_1 l/2].
\]

since \( \| \rho^0 \|_\mathcal{L} = \| O_L \|_\mathcal{L} = 1 \) and \( \rho^0, O_L \) and \( O_R \) are self-adjoint. Thus, overall

\[
\| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L} \leq 3(2\pi)^{-1/2} C_1 J^2 \max\{q, q^2\} \exp[-c_1 l/2] + 4 \exp[-c_1 l/2].
\]

Finally, by definition, \( O_L \) and \( O_R \) project onto the spectral subspaces of \( M_L \) and \( M_R \) corresponding to small eigenvalues. I.e.,

\[
\| \exp[i(M_L + M_R)t] - I \|_{O_L O_R} \leq 2 \| C_1 J^2 \max\{q, q^2\} \exp[-c_1 l/2],
\]

hence

\[
\| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L} \leq \left\| \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^\infty T(\exp\left[\int_0^t A(\tau) \, d\tau\right])^* \rho^0 O_L O_R - \rho^0 \right\|_{\mathcal{L}}
\leq 2 C_1 J^2 \max\{q, q^2\} \exp[-c_1 l/2] \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^\infty |t| \| \exp\left[\frac{-t^2}{2\sigma}\right] dt
\]

= \left\| C_1 J^2 (2\pi)^{-1/2} \max\{q, q^2\} \exp[-c_1 l/2] \right\|_{\mathcal{L}}.
\]

Overall we obtain

\[
\| \rho^0 - \rho^0 \|_\mathcal{L} \leq \| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L} + \| \rho^0 O_L O_R - \rho^0 \|_\mathcal{L}
\leq C_2 J^2 \max\{q, q^2\} \left( \exp[-c_1 l/2] + \exp[-c_1 l] \right),
\]

for an appropriately chosen constant \( C_2 > 0 \). This completes the proof.

It remains to show how we can obtain a local operator \( O_B \), maintaining the same approximation order. This follows once more from a Lieb-Robinson bound.
Lemma 4.10. Consider the operator
\[ \tilde{O}_B := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} T \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* \exp \left[ -\frac{t^2}{2q} \right] \, dt, \]
with \( A(t) \) as above
\[ A(t) := \exp[i(M_L + M_R)t]iM_B \exp[-i(M_L + M_R)t]. \]
Then, there exists a local bounded operator \( O_B \) supported on \( \mathcal{H}_{j-3\ell-2,j+3\ell+3} \), with \( \|O_B\|_\mathcal{L} \leq 1 \) such that
\[ \|\tilde{O}_B - O_B\|_\mathcal{L} \leq C_3 J_2^2 \max\{q^{1/2}, q\} \exp[-c_3 t], \]
for some constants \( C_3 > 0, c_3 > 0. \)

Proof. We use the same trick as in Lemma 4.7 to show that the non-local part of \( M_B(t) \) is bounded.
\[ M_B(t) := \exp[i(M_L + M_R)t]M_B \exp[i(M_L + M_R)t], \]
\[ \frac{d}{dt}M_B(t) = \exp[i(M_L + M_R)t][M_L + M_R, M_B] \exp[-i(M_L + M_R)t], \]
\[ [M_L + M_R, M_B] = [H_L, H_B] + [H_L, \Theta_B] + [\Theta_L, H_B] + [\Theta_L, \Theta_B] + [H_R, H_B] + [H_R, \Theta_B] + [\Theta_R, H_B] + [\Theta_R, \Theta_B]. \]
Since \( \Theta_L \) and \( \Theta_R \) are supported on a superset of the supports of \([H, H_L]\) and \([H, H_R]\), we obtain
\[ \text{supp}(M_L + M_R, M_B) \subset \text{supp}(\Theta_L) \cup \text{supp}(\Theta_R) = \text{supp}(M_B). \]
Thus, as in Lemma 4.7, we can approximate \( M_B(t) \) by a local operator \( \tilde{M}_B(t) \) supported on \( \mathcal{H}_{j-3\ell-2,j+3\ell+3} \) such that
\[ \|M_B(t) - \tilde{M}_B(t)\|_\mathcal{L} \leq C J_2^2 |t| \exp[-a(t-v|t|)]. \]
We utilize this estimate in a similar way as in [4.9]. To this end, we write the time-ordered exponential as a product integral and use a step function approximation to the integral (see [16] Chapter 3.6), namely
\[ T \left( \exp \left[ \int_0^t A(\tau) \, d\tau \right] \right)^* = \prod_{0}^{t} \exp[A(\tau)] \, d\tau = \lim_{N \to \infty} (\exp[A(t_N)\Delta t] \cdots \exp[A(t_0)\Delta t]), \]
where \( t_i = i\Delta t, \Delta t = t/N \) and the convergence is meant in the strong sense. This is possible due to the simple form of \( A(t) \), i.e., \( \exp[A(t)] \) is bounded with norm 1.
For \( N = 1 \), we obtain as in [4.9]
\[ \| \exp[i M_B(t)\Delta t] - \exp[i \tilde{M}_B(t)\Delta t] \|_\mathcal{L} \leq |\Delta t| \| M_B(t) - \tilde{M}_B(t) \|_\mathcal{L}, \]
with \( |\Delta t| = |t| \). For \( N \to \infty \), by induction
\[ \| \exp[i M_B(t_N)\Delta t] \cdots \exp[i M_B(t_0)\Delta t] - \exp[i \tilde{M}_B(t_N)\Delta t] \cdots \exp[i \tilde{M}_B(t_0)\Delta t] \|_\mathcal{L} \]
\[ \leq \| \exp[i M_B(t_N)\Delta t] - \exp[i \tilde{M}_B(t_N)\Delta t] \|_\mathcal{L} \| \exp[i M_B(t_N)\Delta t] \cdots \exp[i M_B(t_{N-1})\Delta t] - \exp[i \tilde{M}_B(t_N)\Delta t] \cdots \exp[i \tilde{M}_B(t_0)\Delta t] \|_\mathcal{L} \]
\[ + \| \exp[i \tilde{M}_B(t_N)\Delta t] \times \]
\[ \| \exp[i M_B(t_{N-1})\Delta t] \cdots \exp[i M_B(t_0)\Delta t] - \exp[i \tilde{M}_B(t_{N-1})\Delta t] \cdots \exp[i \tilde{M}_B(t_0)\Delta t] \|_\mathcal{L} \]
\[ \leq |\Delta t| N \| M_B(t) - \tilde{M}_B(t) \|_\mathcal{L}, \]
where by definition \( |\Delta t| N = |t| \). Thus, we can estimate for the ordered exponential
\[ \| T \left( \exp \left[ \int_0^t i M_B(\tau) \, d\tau \right] \right)^* - T \left( \exp \left[ \int_0^t i \tilde{M}_B(\tau) \, d\tau \right] \right)^* \|_\mathcal{L} \]
\[ \leq |t| \| M_B(t) - \tilde{M}_B(t) \|_\mathcal{L} \leq C J_2^2 \exp[-a(t/3 - v|t|)]. \]
We define the local operator $O_B$ as
\[
O_B := \frac{1}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \mathcal{T} \left( \exp \left[ i \int_0^t \dot{M}_B(\tau) \, d\tau \right] \right)^* \exp \left[ -\frac{t^2}{2q} \right] \, dt.
\]
Estimating as in (4.8) for $T = \frac{4}{6e}$, we obtain
\[
\left\| \hat{O}_B - O_B \right\| \leq \left\| \int_{|t| \leq T \ldots} + \int_{|t| > T \ldots} \right\| \leq C J^2 \exp[-a l/3]
\]
\[
\times \left( \exp[a v T] \frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} t^2 \exp \left[ -\frac{t^2}{2q} \right] \, dt + \frac{1}{\sqrt{2\pi q}} \int_{|t| > T} t^2 \exp \left[ a v t - \frac{t^2}{2q} \right] \, dt \right).
\]
In analogy to (4.8), for the first term we obtain
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| \leq T} t^2 \exp \left[ -\frac{t^2}{2q} \right] \, dt \leq \frac{q}{2}.
\]
For the second term, applying integration by parts, we obtain
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| > T} t^2 \exp \left[ a v t - \frac{t^2}{2q} \right] \, dt = \sqrt{\frac{2}{\pi q}} \int_{|t| > T} t^2 \exp \left[ a v t - \frac{t^2}{2q} \right] \, dt
\]
\[
= \sqrt{\frac{2}{\pi q}} \left( T q \exp \left[ a v T - \frac{T^2}{2q} \right] + q \int_{|t| > T} \exp \left[ a v t - \frac{t^2}{2q} \right] \, dt \right) + a v q \int_{|t| > T} t \exp \left[ a v t - \frac{t^2}{2q} \right] \, dt.
\]
And hence
\[
\frac{1}{\sqrt{2\pi q}} \int_{|t| > T} t^2 \exp \left[ a v t - \frac{t^2}{2q} \right] \, dt \leq \exp \left[ a v - \frac{T^2}{2q} \right] \sqrt{\frac{2}{\pi q}} \left( \frac{\sqrt{q\pi^2} + T}{1 - a v q} \right).
\]
The final estimate is thus
\[
\left\| \hat{O}_B - O_B \right\| \leq C_3 J^2 \max \left\{ q^{1/2}, q \right\} \exp[-3c_3 l],
\]
for appropriate constants $C_3 > 0$, $c_3 > 0$. This completes the proof. 

We are now ready to state the main result of this subsection.

**Theorem 4.11.** Under Assumption 4.1, there exist local, bounded and self-adjoint operators $O_L = O_L(j, l)$, $O_B = O_B(j, l)$, $O_R = O_R(j, l)$ with norms bounded by 1, such that for some constants $C_4 > 0$, $c_4 > 0$
\[
(4.10) \quad \left\| O_BO_LO_R - \rho^0 \right\| \leq C_4 J^2 \exp[-c_4 l].
\]
The respective supports are $\mathcal{H}_{1,j}$, $\mathcal{H}_{j-3l-2,j+3l+3}$ and $\mathcal{H}_{j+1,d}$. The operator $O_B$ can be chosen w.l.o.g. to be positive.

**Proof.** The operators $O_L$ and $O_R$ were defined in Lemma 4.8 and their properties follow therefrom. The operator $O_B$ was defined in Lemma 4.10. W.l.o.g. we can assume it is positive, otherwise the same arguments as in [27] Lemma 4] apply.

By Lemmas 4.9, 4.10 and since $\|O_LO_R\| \leq 1$, we obtain an error bound with asymptotic dependence on $l$ of the form $l^2 \exp[-c l]$. Hence, we can pick constants $C_4 > 0$, $c_4 > 0$ to satisfy (4.10). This completes the proof. □

4.2. Relative Entropy Bounds. The key idea for the proof of the area law is that we can bound the relative entropy from below by a non-vanishing term that grows with $l$. The precise asymptotics of this lower bound will imply a constant upper bound on the value of entropy.

**Lemma 4.12.** Suppose Assumption 4.1 holds. Then, for
\[
\mathbb{E}_B := \text{tr} \left[ O_B(\rho_{1,j}^0 \otimes \rho_{j+1,d}^0) \right],
\]
and $\varepsilon(l) := C_4 J^2 \exp[-c_4 l]$, we have the lower bound
\[
S(\rho_{j-1-2,j}^0) + S(\rho_{j+1,j+l+3}^0) - S(\rho_{j-1-2,j+l+3}^0) \geq (1 - 2\varepsilon(l)) \log_2 \left[ \frac{1 - 2\varepsilon(l)}{\mathbb{E}_B} \right] + 2\varepsilon(l) \log_2 \left[ \frac{2\varepsilon(l)}{1 - \mathbb{E}_B} \right].
\]
Lemma 4.13. The estimated quantity is commonly referred to as quantum mutual information. We briefly show that it is equal to a specific expression of relative entropy. By definition of relative entropy

$$S(\rho_{AB} || \rho_A \otimes \rho_B) = -S(\rho_{AB}) - \text{tr}[\rho_{AB} \log_2(\rho_A \otimes \rho_B)].$$

For the latter term we compute

$$\text{tr}[\rho_{AB} \log_2((\rho_A \otimes \rho_B)(I_A \otimes \rho_B))] = -\text{tr}[\rho_{AB} \log_2(\rho_A \otimes \rho_B) + \log_2(I_A \otimes \rho_B)]$$

$$= -\text{tr}[\rho_{AB} \log_2(\rho_A)] - \text{tr}[\rho_{AB} \log_2(\rho_B)] = S(\rho_A) + S(\rho_B).$$

From Theorem 4.11 it follows

$$\text{tr}[\rho^0 O_B O_L O_R] = \text{tr}[\rho^0 (\rho^0 - \rho^0 + O_B O_L O_R)] = \text{tr}[(\rho^0)^2] + \text{tr}[\rho^0 (O_B O_L O_R - \rho^0)]$$

$$\geq 1 - \text{tr}[\rho^0] \|O_B O_L O_R \rho^0 - \rho^0\|_1 \geq 1 - \varepsilon(l).$$

Thus, applying the Cauchy-Schwarz inequality we can estimate

$$\text{(4.12)} \quad \text{tr}[\rho^0 O_B] = \text{tr}[\rho_{j-l-2j+i+3} O_B] \geq 1 - 2\varepsilon(l).$$

Next, define the map (quantum channel) $E : \text{Tr}(\mathcal{H}) \rightarrow \text{Tr}(C^2)$ by

$$\text{(4.13)} \quad E(\rho) := \text{tr}[\rho O_B](\cdot, (1, 0))_{C^2} + \text{tr}[\rho(\| - O_B)](\cdot, (0, 1))_{C^2}.$$ 

By Theorem 4.11 $O_B$ is a bounded, positive operator with $\|O_B\|_1 \leq 1$. Thus, one easily checks that $E$ is a positive trace preserving map. By [32] Theorem 1] the relative entropy is monotone under $E$ and we get

$$S(\rho^0_j - \rho^0_{j+1}) + S(\rho^0_{j+1} + \rho^0_{j+1}) = S(\rho^0_j - \rho^0_{j+1} \otimes \rho^0_{j+1})$$

$$\geq S(\rho^0_j - \rho^0_{j+1} \otimes \rho^0_{j+1}) + (1 - \text{tr}[\rho^0_j - \rho^0_{j+1} \otimes \rho^0_{j+1} + \rho^0_{j+1}]) \log_2[1 - \text{tr}[\rho^0_j - \rho^0_{j+1} + \rho^0_{j+1}]]$$

$$\geq 2\varepsilon(l) \log_2 \left[ \frac{1 - 2\varepsilon(l)}{E_B} \right] + 2\varepsilon(l) \log_2 \left[ \frac{2\varepsilon(l)}{1 - E_B} \right]$$

where $(*)$ is due to (4.12) and since the first term is positive while the second is negative. This completes the proof.

We need to replace $E_B$ by an expectation value that is independent of the approximation operator $O_B$.

Lemma 4.13. For $E := \text{tr}[\rho^0_j \rho^0_{j+1} \otimes \rho^0_{j+1}]$ we have the bound

$$\text{(4.14)} \quad E_B \leq \frac{E - \sqrt{2E_B \varepsilon(l) + 2\varepsilon(l)}}{1 - 2\varepsilon(l)}.$$ 

Proof. Define $E_{LR} := \text{tr}[O_L O_R(\rho^0_j \rho^0_{j+1})]$. As in (4.12), $E_{LR} \geq 1 - 2\varepsilon(l)$. Applying the same arguments as in [27] Equation (24), i.e., by applying Cauchy-Schwarz to the co-variance of operators and since $E_B \leq 1$, we obtain

$$E \geq \text{tr}[O_L O_R(\rho^0_j \rho^0_{j+1})] - \varepsilon(l) \geq E_B E_{LR} - \sqrt{E_B - E_B^2} \sqrt{E_{LR} - E_{LR}^2} - \varepsilon(l)$$

$$\geq E_B(1 - 2\varepsilon(l)) - \sqrt{E_B} \sqrt{2\varepsilon(l)} + \varepsilon(l).$$

Hence,

$$E_B \leq \frac{E - \sqrt{2E_B \varepsilon(l) + 2\varepsilon(l)}}{1 - 2\varepsilon(l)}.$$ 

This completes the proof.
We want to use Lemma 4.12 to estimate entropy asymptotics w.r.t. the length of a chain \( l \). To this end, we need to quantify the worst possible entropy scaling. Of course, to avoid a tautology, this estimate has to include the worst case of exponential scaling in ranks, i.e., linearly growing entropy.

In Section 3.2, we discussed assumptions under which the entropy is finite for any subsystem. However, unlike in the finite dimensional case, we do not know exactly how the entropy bounds differ from site to site. Since a detailed investigation of this goes beyond the scope of this work, for now we require the following assumption.

**Assumption 4.14.** We assume that there exists a constant \( S_{\text{max}} > 0 \) such that

\[
S\left( \rho_{j,j}^{0} \right) \leq S_{\text{max}} \quad \text{for any } 1 \leq j \leq d,
\]

i.e., the single site entropies are bounded by \( S_{\text{max}} \). Moreover, we assume for any \( 1 \leq j \leq d \) and \( 0 \leq l \leq d - j \) that we have

\[
S\left( \rho_{j,j+l}^{0} \right) \leq (1 + l)S_{\text{max}},
\]

i.e., entropy grows at most linearly in \( S_{\text{max}} \) (which still includes exponential scaling in ranks for a given approximation accuracy).

**Lemma 4.15.** Let

\[
S_{l} := \max \{ S(\rho_{j,j+l-1}^{0}) \colon 1 \leq j \leq d, (j, j + l - 1) \subset [1, d] \}.
\]

Then, under Assumptions 4.1 and 4.14 there exists a constant \( C_{5} > 0 \) such that

\[
S_{l} \leq C_{5}(S_{\text{max}} + 1)l - \log_{2}(l) \log \left[ \frac{1}{E + 2\varepsilon(l)} \right]\ 
\]

(4.15)

**Proof.** The 2nd term in (4.11) can be neglected since it vanishes rapidly for large \( l \). For the first term we use (4.14) to obtain

\[
S\left( \rho_{j,j+l-1}^{0} \right) + S\left( \rho_{j+1,j+l+3}^{0} \right) - S\left( \rho_{j,j+2}^{0} \right) \geq (1 - 2\varepsilon(l)) \log_{2} \left[ \frac{1}{E + 2\varepsilon(l)} \right] - C_{5}.
\]

For an appropriately chosen \( C_{5} > 0 \). Thus, we can estimate

\[
S_{2l} \leq 2S_{l} + C_{5} - (1 - 2\varepsilon(l)) \log_{2} \left[ \frac{1}{E + 2\varepsilon(l)} \right].
\]

Now we simply iterate this inequality as in [27, Equation (10)] and use Assumption 4.14 to get

\[
S_{l} \leq lS_{\text{max}} + lC_{5} + \sum_{k=0}^{\infty} 2\varepsilon(2^{k}) \log_{2} \left[ \frac{1}{E + 2\varepsilon(2^{k})} \right] - \log_{2}(l) \log_{2} \left[ \frac{1}{E + 2\varepsilon(l)} \right].
\]

The log term in the series diverges at most linearly, thus the series can be bounded by a constant. Adjusting \( C_{5} > 0 \) we obtain the final statement

\[
S_{l} \leq C_{5}(S_{\text{max}} + 1)l - \log_{2}(l) \log \left[ \frac{1}{E + 2\varepsilon(l)} \right].
\]

This completes the proof. \( \square \)

**4.3. Expectation Value Bounds.** Equation (4.15) is the key for the final argument. This in turn depends on a bound for

\[
E = \text{tr}\left[ \rho^{0}[\rho_{1,j}^{0} \otimes \rho_{j+1,d}^{0}] \right].
\]

In order to derive such a bound, we want to use the technique from [27, Lemma 2] and first replace \( \rho^{0} \) by the approximation from Theorem 4.11 \( \rho^{0} \approx O_{B}O_{L}O_{R} \).

To this end, consider the mapping

\[
T_{m} = O_{B}(m)O_{L}(m)O_{R}(m)[\rho_{1,j}^{0} \otimes \rho_{j+1,d}^{0}]O_{R}(m)O_{L}(m)O_{B}(m),
\]

where we used \( m \) to indicate the support length. It is trace class since \( \rho_{1,j}^{0} \otimes \rho_{j+1,d}^{0} \in S(\mathcal{H}) \) and the trace class is a two sided ideal in \( \mathcal{L}(\mathcal{H}) \). Thus, we can apply the Schmidt decomposition to \( T_{m} \) w.r.t. the bipartite cut \( \mathcal{H} = \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d} \).

Our bound requires some knowledge about the low-rank approximation properties of \( T_{m} \), i.e., convergence w.r.t. the Schmidt rank \( r \). Note that even though \( O_{L}O_{R}[\rho_{1,j}^{0} \otimes \rho_{j+1,d}^{0}]O_{L}O_{R} \) has rank 1, the operator \( T_{m} \) may have infinite rank. The operator \( O_{B} \) is a unitary transformation with local support. However, beyond this, it is not clear how exactly \( O_{B} \) influences low-rank approximability. Since a deeper investigation of this is rather intricate, for this work we focus on three cases. These cases are
prototypical for known approximation rates based on assumptions of smoothness for the operator kernels, i.e., in our case smoothness of eigenfunctions of $H$ (see [4]). Note that in all three cases ranks scale exponentially for a given approximation accuracy.

**Assumption 4.16.** Define the normalized map

$$T_m := O_B(m)O_L(m)O_R(m)[\rho_{1,j} \otimes \rho_{j+1,d}^0]O_R(m)O_L(m)O_B(m) \in \text{Tr}^+(\mathcal{H}),$$

$$\rho_m := T_m/\text{tr}[T_m] \in S(\mathcal{H}).$$

Note that the support and thus effective dimensionality of this map is $2m + 6$. Let $\rho_m^*$ denote the best rank-$r$ approximation w.r.t. the bipartite cut $\mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d}$. For some rate $s > 0$ and constant $C_6 > 0$, the three cases we consider are

\begin{align*}
(4.16) \quad & \|\rho_m - \rho_m^*\|_\mathcal{L} \leq C_6 r^{- \frac{2m+6}{s}}, \\
(4.17) \quad & \|\rho_m - \rho_m^*\|_\mathcal{L} \leq C_6 r^{- s} \log_2(r) \|\rho_m^*\|_\mathcal{L}^{s(2m+6)}, \\
(4.18) \quad & \|\rho_m - \rho_m^*\|_\mathcal{L} \leq C_6 r^{m+6} r^{-s}.
\end{align*}

**Lemma 4.17.** Let Assumption 4.1 hold. Then, for the three rates in Cases 4.16, we get the respective bounds

\begin{align*}
(4.19) \quad & S(\rho_{1,j}) \leq C_7 \left(1 + \log_2 \left[ \frac{C_8}{E} \right] \right), \\
(4.20) \quad & S(\rho_{1,j}) \leq C_7 \left(1 + \log_2 \left[ \frac{C_8}{E} \right] \log_2 \left\{ \log_2 \left[ \frac{C_8}{E} \right] \right\} \right), \\
(4.21) \quad & S(\rho_{1,j}) \leq C_7 \left(1 + \log_2 \left[ \frac{C_8}{E} \right] \right),
\end{align*}

for some constants $C_7 > 0, C_8 > 0$.

**Proof.** We only prove the statement for the first case, the others are analogous. The space $\text{Tr}(\mathcal{H})$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$. Moreover, we can interpolate the Hilbert Schmidt norm as follows

\[\|\rho A\|_\mathcal{H}^2 = \|A\rho\|_\mathcal{H}^2 = \text{tr}[A^* \rho A] \leq \|\rho A\|_\mathcal{H} \|\rho A\|_\mathcal{L}.\]

And so we arrive at an estimate as in [24] Equation (17). For

\[A := O_B(m)O_L(m)O_R(m) \in \mathcal{L}(\mathcal{H}), \quad \rho := \rho_{1,j} \otimes \rho_{j+1,d}^0 \in S(\mathcal{H}),\]

we obtain

\[\text{tr}[\rho^0 T_m] = \text{tr}[\rho^0 A \rho A^*] = \langle u_0 , A \rho A^* u_0 \rangle_{\mathcal{H}} = \langle u_0 , A \rho A^* u_0 \rangle_{\mathcal{H}},\]

\[= \text{tr}[\rho^0 A \rho A^* \rho^0] = \|\rho^0 A \sqrt{\rho}\|_\mathcal{H}^2 = \|\rho^0 (A - \rho^0) \sqrt{\rho}\|_\mathcal{H}^2 \geq \left(\|\sqrt{\rho}\|_\mathcal{H} - \|\rho (A - \rho^0)\|_\mathcal{H}\right)^2 \geq \left(\sqrt{\rho_{1,j}} - \frac{\rho_{1,j}}{\sqrt{\rho_{1,j}}}\right)^2 \geq \left(\sqrt{E} - \varepsilon(m)\right)^2,\]

where we used the identity

\[E = \text{tr}[\rho^0 \rho] = \text{tr}[\rho^0 \rho \rho^0] = \|\rho^0 \sqrt{\rho}\|_\mathcal{H}^2.\]

Next, note the identity

\[A - \rho^0 A = A - (\rho^0)^2 + (\rho^0)^2 - \rho^0 A = (A - \rho^0) + \rho^0 (\rho^0 - A).\]

Moreover,

\[\text{tr}[\mathbf{1} - \rho^0]T_m(\mathbf{1} - \rho^0) = \text{tr}[T_m - T_m \rho^0 - \rho^0 T_m + \rho^0 T_m \rho^0] = \text{tr}[T_m(\mathbf{1} - \rho^0)] = \text{tr}[\mathbf{1} - \rho^0]T_m.\]

And thus

\[\text{tr}[\mathbf{1} - \rho^0]T_m = \|A - \rho^0 A\sqrt{\rho}\|_\mathcal{H}^2 \leq 4\varepsilon^2(m).\]

Combining both estimates

\[\text{tr}[\rho^0 \rho_m] \geq \frac{\text{tr}[\rho^0 T_m]}{\text{tr}[\rho^0 T_m] + 4\varepsilon^2(m)} = 1 - \frac{4\varepsilon^2(m)}{\text{tr}[\rho^0 T_m] + 4\varepsilon^2(m)} \geq 1 - \frac{4\varepsilon^2(m)}{5\varepsilon^2(m) + \varepsilon - 2\varepsilon(m)\sqrt{E}} \geq 1 - \frac{8\varepsilon^2(m)}{E}.\]
We chose \( r \) in (4.16) such that
\[
\| \rho_m - \rho^r_m \|_\mathcal{L} \leq C_4^2 J^4 \exp[-2c_4m]/\mathbb{E}.
\]
Then,
\[
\text{tr}[\rho^0 \rho^r_m] = \text{tr}[\rho^0(\rho^r_m - \rho_m + \rho_m)] \geq \text{tr}[\rho^0 \rho^r_m] - \text{tr}[\rho^0] \| \rho_m - \rho^r_m \|_\mathcal{L} \geq 1 - 9C_4^2 J^4 \exp[-2c_4m]/\mathbb{E}.
\]
Let \( \rho^0 \) denote the best rank \( r \) approximation to the ground state projection w.r.t. the same bipartite cut \( \mathcal{H}_{1,j} \otimes \mathcal{H}_{j+1,d} \). Then, since clearly \( \mathbb{I} \geq \rho^r_m \geq 0 \), we obtain
\[
\text{tr}[\rho^0 \rho^r_m] = \text{tr}[\rho^0 - \rho^0(\mathbb{I} - \rho^r_m)] \leq 1 - \inf_{\text{rank}(\rho^r_m) \leq r, 0 \leq v_r \leq \mathbb{I}} \text{tr}[\rho^0(\mathbb{I} - v_r)] = 1 - \text{tr}[\rho^0(\mathbb{I} - \rho^0)] = \sum_{k=1}^r (\sigma_k)^2.
\]
And thus
\[
\sum_{k=r+1}^\infty (\sigma_k)^2 = 1 - \sum_{k=1}^r (\sigma_k)^2 \leq 1 - \text{tr}[\rho^0 \rho^r_m] \leq 9C_4^2 J^4 \exp[-2c_4m]/\mathbb{E}.
\]
The above inequality readily provides enough information about the decay of the singular values to derive (4.19).

Note that the above rank \( r \) depends on \( m \), i.e., \( r = r(m) \). Let \( m' \) be minimal such that
\[
9C_4^2 J^4 \exp[-2c_4m']/\mathbb{E} \leq 1.
\]
Then, for any \( m > m' \),
\[
\sum_{k=r(m)+1}^\infty (\sigma_k)^2 \leq 9C_4^2 J^4 \exp[-2c_4m]/\mathbb{E} = \exp[-2c_4(m - m')] (9C_4^2 J^4 \exp[-2c_4m']/\mathbb{E}) \leq \exp[-2c_4(m - m')].
\]
We can now compute the maximal possible entropy satisfying this decay condition. As discussed in Section 3.1 since entropy is Schur concave, it is maximized for a uniform-like distribution, subject to the constraint (4.22).

Thus, for the sequence \( \lambda_k := (\sigma_k)^2 \), we maximize the von Neumann entropy under the conditions
\[
\sum_{k=1}^{r(m'+1)} \lambda_k = 1, \quad \sum_{k=1}^{r(m+1)} \lambda_k = 1 - \exp[-2c_4],
\]
\[
\sum_{k=r(m)+1} \lambda_k = (1 - \exp[-2c_4]) \exp[-2c_4(m - m')], \quad m > m'.
\]
Computing the upper bound for the entropy
\[
S(\rho_{1,j}^0) \leq -(1 - \exp[-2c_4]) \log_2 \left[ \frac{1 - \exp[-2c_4]}{r(m'+1)} \right] - (1 - \exp[-2c_4]) \sum_{n=1}^\infty \exp[-2c_4n] \log_2 \left[ \frac{(1 - \exp[-2c_4]) \exp[-2c_4n]}{r(m'+n+1) - r(m'+n)+1} \right] = \log_2[r(m'+1)] - \log_2[1 - \exp[-2c_4]] - \exp[-2c_4] \log_2[r(m'+1)] + [1 - \exp[-2c_4]] \sum_{n=1}^\infty \exp[-2c_4n]2c_4n + [1 - \exp[-2c_4]] \sum_{n=1}^\infty \exp[-2c_4n](\log_2[r(m'+n+1) - r(m'+n)+1]).
\]
We express \( r(m) \) explicitly through \( m \) using Assumption 4.16 as
\[
C_6 r^{-\frac{2m+6}{2}} \leq C_4^2 J^4 \exp[-2c_4m]/\mathbb{E} \Rightarrow r(m) \geq \left( \frac{C_4^2}{C_6} \right)^{-\frac{2m+6}{2}} \mathbb{E}^{-\frac{2m+6}{2}} \exp\left[ \frac{c_4^2}{8} [4m^2 + 12m] \right].
\]
We get similar asymptotic bounds for \( r(m' + n + 1) - r(m' + n) \). Taking logarithms and choosing an appropriate constant \( C_7 > 0 \), we obtain

\[
S(\rho_{1,j}^0) \leq C_7(1 + (m')^2 - \log_2(\varepsilon)).
\]

Finally, from the requirement on \( m' \) we compute

\[
3C_4^2 J^4 \exp\left[-2C_4 m'\right]/\varepsilon \leq 1, \quad \Rightarrow \quad m' \leq \frac{1}{2C_4} \log_2 \left[ \frac{9C_4^2 J^4}{\varepsilon} \right] + 1.
\]

Thus, again choosing an appropriate constant \( C_8 > 0 \) (and possibly adjusting \( C_7 \)), we get

\[
S(\rho_{1,j}^0) \leq C_7 \left( 1 + \log_2 \left[ \frac{C_8}{\varepsilon} \right] \right).
\]

The proof for the other cases is analogous. \( \Box \)

4.4. Area Law. We are finally able to prove an area law for the ground state: the entropy of a chain is bounded by a constant and thus does not increase with the dimension. We are only able to show this for the third case (4.18), since for the other two the bound on \( \varepsilon \) decays too slow to apply (4.15) successfully. However, we conjecture that Theorem 4.11 is sufficient to show approximability directly, without going through entropy. A further investigation of this goes beyond the scope of this work.

**Theorem 4.18 (Area Law).** Under Assumptions 4.1, 4.14 and provided case three (4.18) is valid, i.e., there exists a constant \( 0 < C_{\text{Area}} < \infty \), independent of \( j_0 \) or \( d \), such that for any \( 1 \leq j_0 \leq d \)

\[
S(\rho_{1,j_0}^0) \leq C_{\text{Area}}.
\]

The constant \( C_{\text{Area}} \) depends on the physical properties of \( H \), such as gap size, interaction length, interaction strength and Lieb-Robinson velocity.

**Proof.** For any pure state of a tripartite system \( \rho \in \mathcal{S}(\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \), we have by sub-additivity of entropy

\[
S(\rho_A) = S(\rho_{BC}) \leq S(\rho_B) + S(\rho_C) = S(\rho_B) + S(\rho_{AB}).
\]

Hence, \( S(\rho_{AB}) \geq S(\rho_A) - S(\rho_B) \). Moreover, partial traces can be “chained”, in the sense that \( \text{tr}_{BC}[\cdot] = \text{tr}_A[\cdot] \circ \text{tr}_B[\cdot] = \text{tr}_B[\cdot] \circ \text{tr}_A[\cdot] \). To see this, let \( T_A \in \mathcal{L}(\mathcal{H}_A) \) and \( T_{AB} \in \mathcal{L}(\mathcal{H}_{AB}) \). Then, by definition of the partial trace, for \( \rho \in \mathcal{S}(\mathcal{H}) \)

\[
\text{tr}[\rho_{BC}[\rho]T_A] = \text{tr}[\rho T_A \otimes I_{BC}],
\]

\[
\text{tr}[\rho C[\rho]T_{AB}] = \text{tr}[\rho T_{AB} \otimes I_C],
\]

\[
\text{tr}[\rho_B[\rho]T_{BC}] = \text{tr}[\rho T_A \otimes I_B \otimes I_C] = \text{tr}[\rho T_A \otimes I_{BC}].
\]

Thus, for \( k > j_0 \) and by applying Assumption 4.14

\[
S(\rho_{1,k}^0) \geq S(\rho_{1,j_0}^0) - (k - j_0) S_{\text{max}}.
\]

Hence, for \( j_0 \leq k \leq j_0 + l_0 \) and some \( \theta \in (0, 1) \), where \( l_0 \leq \frac{1 - \theta}{S_{\text{max}}} S(\rho_{1,j_0}^0) \), \( l_0 + 1 \geq \frac{1 - \theta}{S_{\text{max}}} S(\rho_{1,j_0}^0) \), we have

\[
S(\rho_{1,k}^0) \geq \theta S(\rho_{1,j_0}^0) =: S_{\text{Cut}}.
\]

Applying (4.21),

\[
\varepsilon \leq C_8 \exp \left[ -\frac{S_{\text{Cut}}}{C_7} + 1 \right],
\]

where the inequality is valid for any \( j \in [j_0, j_0 + l_0] \). With this bound, we can pick the constants \( \theta \) and \( S_{\text{max}} \) such that for \( l \leq l_0 \), either \( \varepsilon \leq \varepsilon(l) \) or \( S_{\text{Cut}} \) is bounded by a constant. The latter automatically bounds \( S(\rho_{1,j_0}^0) \), so we only consider the former. Then, equation (4.15) becomes

\[
S_l \leq C_5 (S_{\text{max}} + 1) l + \log_2(l) \log \left[ 3 l_0 \right],
\]

for any \( 0 \leq l \leq l_0 \). The positive term scales linearly in \( l \) while the negative term scales log-linearly in \( l \). Thus, since entropy is always positive, \( l_0 \) can not be arbitrarily large and therefore, by definition of \( l_0, S(\rho_{1,j_0}^0) \) must be bounded. This completes the proof. \( \Box \)
5. Summary

In this work, we investigated the properties of a Hamiltonian that allow for low-rank approximability. To this end, we have exploited the vast knowledge and experience available in the literature on quantum entanglement. In Section 4, we have shown that approximability is linked to entropy scaling. Though this characterization is not complete, it is rather extensive for 1D systems (TT format). We also discussed the issue of entropy continuity in infinite dimensions and considered a common model setting.

In Section 3 we have shown how local interactions in a Hamiltonian lead to eigenfunctions, whose projectors can be well approximated by local operators. We have also shown that, under further assumptions on the approximand, this implies an area law for the von Neumann entropy.

While we demonstrated the essential mathematical techniques and benefits of the entropy approach, many issues remain. In the following we discuss some of these questions. Since the proofs presented in this chapter are rather lengthy and technical, we begin by reviewing the main steps.

Key Ingredients. The starting point to derive an entropy estimate is the lower bound on relative entropy in (4.11). The fact that we could derive (4.11) relies on (4.10), i.e., the eigenfunction projection $\rho^0$ is essentially a product of three local operators. This, in turn, is directly implied by the local structure of $H$, see Assumptions 4.1. Note that it is not essential that we considered approximating $\rho^0$: any part of the spectrum of $H$ would do, as long as it is separated from the rest of the spectrum.

The derivation of (4.10) essentially relies on the spectral decomposition of $H$ and the fact that we can express parts of $H$ through commutators. At this point the locality of $H$ comes in, since these commutators reduce to local operators with small support. The approximating arguments rely on Lieb-Robinson type bounds, i.e., support/information has a finite propagation speed depending solely on $H$.

Having derived the bound on relative entropy in (4.11), the last step is to show that this lower bound scales sufficiently fast, such that (4.15) can not be valid for values of entropy that are too large. The scaling of the lower bound is then determined by the expectation value $\mathbb{E} = \text{tr} \left[ \rho^0 | \rho^0_0 \otimes \rho^0_j_{+1,d} \right]$, which was bounded in Lemma 4.17. Note that since $\rho^0 = \rho^0_{0,0}$ is a pure state, it is factorized if and only if $\psi_0$ is rank one. Thus, $\mathbb{E}$ can be seen as a measure of entanglement for $\rho^0$ w.r.t. the bipartite cut $\mathcal{H} = \mathcal{H}_{0,j} \otimes \mathcal{H}_{j_{+1,d}}$.

Although all three bounds seem to have similar asymptotic behavior, only the last one yields an area law. The reason is that in (4.15) we consider $C_1 l - C_2 \log_2(l) l^p$ and argue that this becomes negative. If we would have $C_1 l - C_2 \log_2(l) l^p$ for any $p < 1$, the argument fails. This delicate balance, thus, restricts us to (4.21).

Assumptions on the Hamiltonian. We have already discussed Assumption 4.1 in Remark 4.2. Local interactions and a gap are necessary. We could have considered any part of the spectrum, not necessarily the ground state, as long as we have a gap above and possibly below. For non-local interactions, one could derive similar estimates, given sufficient decay of interaction strength for long-range interactions.

Self-adjointness is a technical assumption, necessary for the spectral decomposition. This can be perhaps generalized.

Finite interaction strength is necessary for a finite support propagation speed. However, since we only require the application of Lieb-Robinson bounds, any interaction that admits such bounds would work. It is known that this is not possible for any unbounded interaction (see [21]). Nonetheless, some unbounded interactions can be controlled (see [33][40]) such that the finite interaction strength can be generalized.

The weakest point is, however, the requirement made in Assumption 4.1. In essence, $O_B$ was constructed by approximating the spectrum of $H$. However, we were not explicit in the construction such that it is not clear to us what effect $O_B$ has on the low-rank structure of $\rho_m$. An explicit construction of $O_B$ may be possible due to [7, Lemma 3.2]. Although we believe that (4.11) is a necessary step to bound the entropy, we are not certain if Lemma 4.17 could be entirely avoided.

The result of Theorem 4.11 is interesting in its own right as it is already a statement about approximability and avoids many seemingly artificial assumptions required for the final area law in Theorem 4.18. Theorem 4.11 essentially states

$$\rho^0 \approx O_B O_L O_R,$$
where $O_L O_R$ is rank-one, $O_B$ has an overlapping support of size $l$ and the error bound depends exponentially on $l$ only, not on the full dimension $d$. For the final low-rank approximation, we would have to approximate $O_B$ with low-rank. Intuitively, since $O_B$ has support size $l$, a low-rank approximation to $O_B$ has a complexity scaling at worst exponentially in $l$, not $d$. Indeed, if the underlying Hilbert spaces are finite-dimensional, this is trivially satisfied. However, it is not clear to us how to express this correctly if the Hilbert spaces are infinite dimensional — hence the need for Assumption 4.10.

Electronic Schrödinger Equation. Finally, we mention limitations concerning the well known electronic Schrödinger equation:

\begin{equation}
H = K + V = -\frac{1}{2} \sum_{k=1}^{N} \Delta_k - \sum_{k=1}^{N} \sum_{\nu=1}^{M} \frac{Z_\nu}{|x_k - a_\nu|} + \frac{1}{2} \sum_{k,j=1}^{N} \frac{1}{|x_k - x_j|},
\end{equation}

where $Z_\nu$ and $a_\nu$ are the nuclei charges and positions, respectively, $x_1, \ldots, x_N \in \mathbb{R}^3$. Clearly, the last term in the potential is not local. Moreover, due to the singularity in the potential, the interactions are only relatively bounded (see [29, Chapter 4.1.1]). Thus, Assumption 4.1 does not apply.

We may still obtain a bound as in (4.10), if the interactions are sufficiently small for large $r_{kj}$, where $r_{kj}$ measures the distance between the sites $k$ and $j$. The different electrons in (5.1) interact equally,\footnote{By the nature of the described physical phenomenon, there is no underlying lattice-like structure.} for any $|k-j|$ so that we do not see how to extend the results to this case. We refer to [25] for area laws with long range interactions. On the other hand, Lieb-Robinson bounds could be perhaps extended to relatively bounded potentials. We mention [33, 40] for Lieb-Robinson bounds for unbounded interactions.

Entropy Measures. When switching to the infinite-dimensional regime, the von Neumann entropy is no longer necessarily continuous or finite. In fact, it is discontinuous for ‘most’ states in $\text{Tr}(\mathcal{H})$.

In Proposition 3.7 we have seen that entropy is finite if the expected energy is finite and the Gibbs state exists for any inverse temperature. For eigenstates the energy is obviously finite. The existence of a Gibbs state is more restrictive, though certainly there are many examples where this is true. This is the case if the ionization threshold diverges. The spectrum is discrete in the presence of a diverging (confinement) potential or if the domain is bounded (e.g., infinite potential well).

However, for instance, the spectrum of (5.1) is not purely discrete (see [52, Thm. 5.16]), since the ionization threshold remains bounded and thus the Gibbs state does not exist in this case. This suggests that, at the very least, other entropy measures are worth a consideration for PDEs. Indeed, the question of possible alternative entropy measures, particularly for infinite dimensions, has been previously addressed. We refer to, e.g., [5, 17, 18, 22, 38, 39, 40, 46, 47] for more details.

General Right-Hand-Side. In this work we considered the low-rank structure of eigenfunctions. In general, both in application and approximation theory, one would be interested in general right-hand-sides. Put precisely, given that the right-hand-side is low-rank, does the same hold for the solution? We are only aware of one work that addressed this question for PDEs in [15]. In [24] the authors successfully utilized area laws for spin systems as in [3], to derive low-rank approximability estimates for discretized PDEs with a general right hand side.

In [15] the authors considered a Laplace-like PDE operator. In particular, no interactions are involved. For such an operator the eigenfunctions are rank one tensor products. The authors use an explicit representation for the inverse operator and an exponential sum approximation for the inverse eigenvalues. The trick is then to show that this approximation does not significantly increase ranks, in the sense that the solution is in a slightly worse approximation class than the right hand side.

In the spirit of [24], one could try to extend results from spectrum approximability to approximability of general solutions. Possibly utilizing ideas as in [13], i.e., the eigenfunctions are no longer rank-one but are TT-approximable. However, this is a far from trivial task and will probably require stronger restrictions on the operator structure.

Beyond the TT Format. We analyzed approximability of 1D systems within the MPS/TT format. It is generally known that the approximation format should fit the PDE structure if one is to obtain good approximation results. By now an entire variety of higher-dimensional tensor structures is available, see, e.g., [35] for a recent overview. A natural question is thus: to what extent does the above apply to multi-dimensional systems?
Even though multi-dimensional formats can be very successful for tailored applications, a general theory of approximability seems elusive. Firstly, general tensor networks with loops are not closed (see [31]).

Secondly, on one hand, the holographic principle seems robust w.r.t. the dimension: area laws hold in a thermal equilibrium in any dimension [51]. On the other hand, area laws are insufficient to describe approximability in higher dimensions. In, e.g., [14] multi-dimensional bosonic systems in quantum critical states were shown to satisfy area laws.

Thus, it seems an approximation theory of multi-dimensional systems would require physical and mathematical ideas fundamentally different from the ones applied in this work...

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