Expressive power of binary and ternary neural networks

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Abstract

We show that deep sparse ReLU networks with ternary weights and deep ReLU networks with binary weights can approximate $\beta$-Hölder functions on $[0,1]^d$. Also, for any interval $[a,b) \subset \mathbb{R}$, continuous functions on $[0,1]^d$ can be approximated by networks of depth 2 with binary activation function $1_{[a,b)}$.

Keywords: function approximation; ReLU networks; binary weights; binary activation

1 Introduction

Deep neural networks have proven to be elegant and effective computational tools for dealing with problems of classification, clustering, pattern recognition and prediction. Improvement of the accuracy of performance of DNN models usually results in significant growth of these models making them computationally expensive and memory intensive. Realizations of deep models on low-capacity devices with limited computational power can be done by restricting the possible values of network weights. Popular types of such networks are the binary and the ternary weighted neural networks with weights usually belonging to the sets $\{\pm 1\}$ and $\{0, \pm 1\}$, respectively. An effective method called BinaryConnect for training deep neural networks with binary weights was introduced in [3]. Various other learning rules and training algorithms for binary networks and their applications in natural image classification are summarized in recent comprehensive surveys [13], [16] and [19], while similar properties of ternary weighted networks are studied in [11], [5] and [10]. In a number of DNN models considered in those works, not only the weights but also the activations are quantized to the values $\{\pm 1\}$ or $\{0, \pm 1\}$. Notably, applications of binary weighted networks with ReLU activation function to molecular programming are given in [14] and [15]. Our goal is to describe the approximation properties of those networks. In particular, we show that binary weighted deep ReLU networks and sparse, ternary weighted deep ReLU networks can approximate $\beta$-Hölder functions on $[0,1]^d$. The depths of those networks have logarithmic dependence on the inverse of approximation error. Approximation properties of networks with binary valued activation functions $1_{[a,b)}$ are also considered and it is shown that for any non-empty interval $[a,b) \subset \mathbb{R}$, using only 2 hidden layers those networks can approximate continuous functions on $[0,1]^d$.

Notation. For a vector $\mathbf{v} \in \mathbb{R}^d$ and a function $f$ on $[0,1]^d$, the notations $||\mathbf{v}||_\infty$ and $||f||_\infty$ denote, respectively, the $l_\infty$ norm of $\mathbf{v}$ and the sup norm of $f$ on $[0,1]^d$, $d \in \mathbb{N}$. Also, $\mathbf{1}_S$ is the characteristic function of the set $S \subset \mathbb{R}$ and the centered dot $\cdot$ denotes the usual matrix multiplication.

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2 ReLU networks

Consider the class

\[
\mathcal{F}(L, p) := \left\{ f : [0, 1]^d \rightarrow \mathbb{R} \mid f(x) = W_L \cdot \sigma \circ W_{L-1} \cdot \sigma \circ ... \circ \sigma \circ W_0 \cdot (1, x) \right\}
\]

of feedforward ReLU networks of depth \( L \), where in each hidden layer the ReLU activation function \( \sigma(x) = \max\{0, x\} \) acts coordinate-wise on the input vectors: \( \sigma \circ (z_1, ..., z_r) = (\sigma(z_1), ..., \sigma(z_r)) \), and the coordinate 1 added to the input \( x \) allows to omit the shift vectors. The width vector \( p = (p_0, p_1, ..., p_{L+1}) \), with \( p_0 = d + 1 \), contains the sizes of the weight matrices \( W_i \in \mathbb{R}^{p_{i+1} \times p_i}, i = 0, ..., L \), and the entries of weight matrices are called the weights of the network \( f \). For a set \( A \subset \mathbb{R} \) let

\[
\mathcal{F}_A(L, p) := \{ f \in \mathcal{F}(L, p) \mid \text{all the weights of } f \text{ are in } A \}.
\]

Approximations of \( \beta \)-Hölder functions belonging to the ball

\[
C_\beta^d(K) := \left\{ f : [0, 1]^d \rightarrow \mathbb{R} : \sum_{0 \leq |\alpha| < |\beta|} \| \partial^\alpha f \|_\infty + \sum_{|\alpha| = |\beta|} \sup_{x, y \in [0, 1]^d} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^{\beta - |\beta|}} \leq K \right\}
\]

with sparse networks from \( \mathcal{F}_{[-1, 1]}(L, p) \) are established in [11]. Using that result and substituting the interval \([-1, 1]\) by the finite set of weights \( \{0, \pm \frac{1}{2}, \pm 1, 2\} \), the following theorem is given in [2]:

**Theorem 2.1.** For any function \( f \in C_\beta^d(K) \) and any integers \( m \geq 1 \) and \( N \geq (\beta + 1)^d \lor (K + 1)e^d \), there exists a ReLU network \( \tilde{f} \in \mathcal{F}_A(L, p) \) with \( A = \{0, \pm \frac{1}{2}, \pm 1, 2\} \),

\[
L = 16 + 2(m + 5)(1 + \lceil \log_2(d \lor \beta) \rceil) + 8 \log_2(N^{\beta + d}Ke^d),
\]

\[
|p|_\infty = \left[ 2 \left( 1 + d + (2\beta)dN + 2 \log_2(N^{\beta + d}Ke^d) \right) \right] \lor \left[ 2^d 6(d + \lceil \beta \rceil)N \right],
\]

having at most \( 141(d + \beta + 1)^{3+d}L|p|_\infty \) nonzero weights, such that

\[
\| \tilde{f} - f \|_\infty \leq C(N^{2-m} + N^{-\frac{\beta}{2}}),
\]

where \( C = C(\beta, d, K) \) is some constant depending only on \( \beta, d \) and \( K \).

Let us restate the above theorem for ternary and binary weighted ReLU networks. First, let us show that any network from \( \mathcal{F}_A(L, p, s) \) with weights in \( A = \{0, \pm \frac{1}{2}, \pm 1, 2\} \) can be implemented by a network from \( \mathcal{F}_{[0, \pm \frac{1}{2}]}(L + 2, p', s') \) with

\[
|p'|_\infty = 4|p|_\infty
\]

and with \( s' \leq 16s + 20(d + 1) \). Indeed, using the weights 0 and \( 1/2 \) and 2 hidden layers we can compute the vector

\[
(1, x) \mapsto (1, ..., 1, x_1, ..., x_1, ..., x_d, ..., x_d).
\]

This computation requires \( 20(d + 1) \) nonzero weights. As each of \( 0, \pm \frac{1}{2}, \pm x \) and \( 2x, x \in \mathbb{R} \), can be represented in the form \( \sum_{i=1}^4 w_i x \) for some \( w_i \in \{0, \pm 1/2\}, i = 1, ..., 4 \), the desired inclusion

\[
\mathcal{F}_A(L, p, s) \subset \mathcal{F}_{[0, \pm 1/2]}(L + 2, p', 16s + 20(d + 1))
\]

t follows.
Let us now show that networks from \( \mathcal{F}_{(\pm 1/2)}(L + 2, p') \) can be implemented by binary weighted networks from \( \mathcal{F}_{(\pm 1/4)}(L + 5, p'') \) with
\[
|p''|_\infty = 8|p'|_\infty.
\]
It suffices to show that using the weights \( \pm 1/4 \) and 3 hidden layers we can compute the vector
\[
(1, x) \mapsto (1, 1, x_1, \ldots, x_d, x_d).
\]
The inclusion
\[
\mathcal{F}_{(0, \pm 1/2)}(L + 2, p') \subset \mathcal{F}_{(\pm 1/4)}(L + 5, p'')
\]
would then follow as each of \( 0, \pm 1/4 \) can be represented in the form \( w_1 x + w_2 x \) for some \( w_1, w_2 \in \{\pm 1/4\} \). Denote
\[
y_0 := \frac{1}{4}(1 + \sum_{i=1}^{d} x_i) \quad \text{and} \quad y_k := \frac{1}{4}(1 - x_k + \sum_{i \neq k} x_i),
\]
\( k = 1, \ldots, d \). Then, using the weights \( \pm 1/4 \) and one hidden layer, we can compute the vector
\[
(1, x) \mapsto (y_0, \ldots, y_0, y_1, \ldots, y_1, \ldots, y_d, \ldots, y_d) := y.
\]
Note that as \( x = (x_1, \ldots, x_d) \in [0, 1]^d \), then \( y_i \geq 0 \), and, therefore, the action of the ReLU activation function does not change the value of \( y_i \), \( i = 0, \ldots, d \). Also, since each coordinate of the output vector \( y \) is repeated 8 times, then, in the following layer, if needed, each of them can be eliminated by multiplying 4 equal coordinates by 1/4 and the other 4 coordinates by \(-1/4\). Hence, as
\[
\frac{1}{4}8y_0 = 2y_0 \quad \text{and} \quad \frac{1}{4}8y_0 - \frac{1}{4}8y_k = x_k, \quad k = 1, \ldots, d,
\]
then, using the weights \( \pm 1/4 \), in the following hidden layer we can compute the vector
\[
y \mapsto \left( \frac{2y_0, \ldots, 2y_0, x_1, \ldots, x_1, \ldots, x_d, \ldots, x_d}{8} \right).
\]
Finally, as
\[
\frac{1}{4}(16y_0 - 4 \sum_{i=1}^{d} x_i) = 1,
\]
then, using the weights \( \pm 1/4 \) and one more hidden layer, we can compute
\[
\left( \frac{2y_0, \ldots, 2y_0, x_1, \ldots, x_1, \ldots, x_d, \ldots, x_d}{8} \right) \mapsto (1, 1, x_1, \ldots, x_d, x_d)
\]
and the inclusion \( 4 \) follows.

Thus, combining \( 1 \)-\( 4 \), we get the following corollary of Theorem 2.1.

**Theorem 2.2.** For any function \( f \in C^\beta_d(K) \) and any integers \( m \geq 1 \) and \( N \geq (\beta + 1)^d \vee (K + 1)d^d \), there exist networks \( f_1 \in \mathcal{F}_{(0, \pm 1/2)}(L + 2, p', s') \) and \( f_2 \in \mathcal{F}_{(\pm 1/4)}(L + 5, p'') \) with
\[
|p'|_\infty = 4|p|_\infty, \quad |p''|_\infty = 32|p|_\infty \quad \text{and} \quad s' \leq 16s + 20(d + 1),
\]
such that
\[
\|f_i - f\|_\infty \leq C(N2^{-m} + N^{\frac{-\beta}{d}}), \quad i = 1, 2,
\]
where \( L, p \) and \( s \) are same as in Theorem 2.1 and \( C = C(\beta, d, K) \) is some constant.

**Remark 2.1.** Note that as the ReLU function \( \sigma \) is positive homogeneous (\( \sigma(ax) = a\sigma(x) \) for any \( a > 0 \)), then \( \mathcal{F}_{(0, \pm 1/2)}(L + 2, p', s') = 1/2^{L+1} \mathcal{F}_{(0, \pm 1)}(L + 2, p', s') \) and \( \mathcal{F}_{(\pm 1/4)}(L + 5, p'') = 1/4^{L+6} \mathcal{F}_{(\pm 1)}(L + 5, p'') \). Also, to avoid the scaling factors, the above theorem can be restated for networks with weights \( \{0, \pm 1\} \) and \( \{\pm 1\} \) and with activation functions \( \sigma/2 \) and \( \sigma/4 \), respectively.
3 Networks with activation function \(1_{[a,b]}\)

In the previous section we considered binary and ternary weighted neural networks with ReLU activation function. Although the ReLU function is one of the most commonly used activation functions in deep learning models, various other activation functions have been considered in the literature for constructing neural networks that approximate functions of given smoothness. In particular, networks with piece-wise linear, RePU and hyperbolic tangent activation functions as well as networks with activations belonging to the families \{\sin, \arcsin\} and \{\lfloor \cdot \rfloor, 2^x, 1_{x\geq 0}\} have been studied in the works \[4, 7, 9, 12\] and \[18\]. Particular choice of the (family of) activation function(s) may be caused, for example, by its computational simplicity, representational sparsity, smoothness or (super)expressivity. Also, in \[6\] it is shown that shallow networks with arbitrary squashing activation function \(\Psi(x)\) (i.e., \(\Psi : [0, 1] \to \mathbb{R}\) is non-decreasing, \(\lim_{x \to -\infty} \Psi(x) = 0\) and \(\lim_{x \to +\infty} \Psi(x) = 1\)) can approximate Borel measurable functions. In particular, the Heaviside step function is an example of binary valued squashing activation function. Approximations of continuous functions by networks of depth 2 with squashing activation functions are given in \[8\]. The proofs presented in \[8\] are based on Kolmogorov’s representation theorem and allow to estimate the number of hidden units needed to attain given accuracy of approximation of continuous functions with given rate of increase. Approximating networks can also be constructed by neural network approximation/implementation of polynomials \([9, 17]\) or by mapping small continuous functions with given rate of increase. Approximating networks can also be constructed to estimate the number of hidden units needed to attain given accuracy of approximation of continuous functions from the set

\[
\mathcal{H}_d^\beta(F, K) := \left\{ f : [0, 1]^d \to [-F, F]; \ |f(x) - f(y)| \leq K|x - y|^\beta \right\}
\]

with \(\beta \in (0, 1]\) and \(F, K \in \mathbb{R}_+\).

**Theorem 3.1.** For any \(\varepsilon > 0\) and for any natural number \(M \geq \left(\frac{F}{\varepsilon}\right)^{1/\beta}\) there are matrices \(W \in \mathbb{R}^{(dM+1)\times(d+1)}\) and \(V \in \mathbb{R}^{(M+1)^d \times (dM+1)}\) such that for any \(f \in \mathcal{H}_d^\beta(F, K)\) there is a vector \(U_f \in \mathbb{R}^{(M+1)^d}\) with

\[
\|U_f \cdot 1_{[0,1]} \circ V \cdot 1_{[0,1]} \circ W \cdot (1, x) - f(x)\|_\infty \leq \varepsilon.
\]

**Proof.** Consider the collection \(I_{d,M}\) of \((M + 1)^d\) hypercubes of the form

\[
\left[\frac{m_1}{M}, \frac{m_1 + 1}{M}\right) \times \cdots \times \left[\frac{m_d}{M}, \frac{m_d + 1}{M}\right),
\]

where \((m_1, \ldots, m_d) \in [0, M]^d \cap \mathbb{N}_0^d\). Note that those hypercubes are pairwise disjoint and their union covers \([0, 1]^d\). To enumerate the sets from \(I_{d,M}\), denote the hypercube \((5)\) by \(I_k\), where \(k = \sum_{i=1}^d m_i (M + 1)^{i-1}\). We thus map each hypercube from \(I_{d,M}\) to a unique integer index \(k \in [0, (M + 1)^d - 1]\). Choose the matrix \(W \in \mathbb{R}^{(dM+1)\times(d+1)}\) such that

\[
W \cdot (1, x) = \left(0, x_1 - \frac{1}{M + 1}, \ldots, x_1 - \frac{M}{M + 1}, \ldots, x_d - \frac{1}{M + 1}, \ldots, x_d - \frac{M}{M + 1}\right).
\]

The coordinate-wise application of the activation function \(1_{[0,1]}\) to the obtained vector gives

\[
1_{[0,1]} \circ W \cdot (1, x) = \left(1, x_1^{M, M}, \ldots, x_1^{M, M}, \ldots, x_d^{M, M}, \ldots, x_d^{M, M}\right),
\]
where \( x_i^{j,M} = 1 \) if \( x_i \geq j/(M+1) \) and otherwise \( x_i^{j,M} = 0, \ i = 1, \ldots, d; \ j = 1, \ldots, M \). Note that

\[
\Sigma_{j=1}^{M} x_i^{j,M} = m \iff x_i \in \left[ \frac{m}{M}, \frac{m+1}{M} \right], \ i = 1, \ldots, d,
\]

that is,

\[
\Sigma_{i=1}^{d} \left( \Sigma_{j=1}^{M} x_i^{j,M} \right) (M+1)^{-i} = k \iff x \in I_k,
\]

where \( k = 0, \ldots, (M + 1)^d - 1 \). Thus, there is a matrix \( V \in \mathbb{R}^{(M+1)^d \times (dM+1)} \) with entries from \((- (M+1)^d, (M+1)^d) \cap \mathbb{Z}\) such that for \( x \in I_k \)

\[
V \cdot 1_{[0,1)} \circ W \cdot (1, x) = (0 - k; 1 - k, \ldots, (M + 1)^d - 1 - k).
\]

Hence, for \( x \in I_k \)

\[
1_{[0,1)} \circ V \cdot 1_{[0,1)} \circ W \cdot (1, x) = (0, 0, 0, \ldots, 0),
\]

where the \((k+1)\)-th coordinate of the last vector is 1 and all other coordinates are equal to 0, \( k = 0, \ldots, (M + 1)^d - 1 \).

Let us now take any function \( f \in \mathcal{R}_d^2(F, K) \). Denote \( x_k := (m_1/M, \ldots, m_d/M) \) to be the vector from \( I_k, k = \Sigma_{i=1}^{d} m_i (M+1)^{-i-1} \), with smallest coordinates. Then, taking \( U_f = (f(x_0), \ldots, f(x_{(M+1)^d-1}) \in \mathbb{R}^{(M+1)^d} \), for \( x \in I_k \) we have

\[
|U_f \cdot 1_{[0,1)} \circ V \cdot 1_{[0,1)} \circ W \cdot (1, x) - f(x)| = |f(x_k) - f(x)| \leq \frac{K}{M^\beta} \leq \varepsilon.
\]

As \([0,1]^d \subseteq \cup_{k=1}^{(M+1)^d-1} I_k\), then

\[
\|U_f \cdot 1_{[0,1)} \circ V \cdot 1_{[0,1)} \circ W \cdot (1, x) - f(x)\|_{\infty} \leq \varepsilon.
\]

\[\square\]

**Corollary 3.1.** For any continuous function \( f \in C([0,1]^d) \) and for any \( \varepsilon > 0 \) there is a network \( \tilde{f} \) of depth 2 with activation function \( 1_{[0,1)} \) such that \( \|\tilde{f} - f\|_{\infty} \leq \varepsilon \).

**Proof.** Follows from uniform continuity of \( f \) and the proof of previous theorem. \[\square\]

**Remark 3.1.** As for any interval \([a,b) \subset \mathbb{R}\) we have \( 1_{[0,1)}(x) = 1_{[a,b)}((b-a)x+a) \), then the above results also hold for networks with activation function \( 1_{[a,b)} \).

### 4 Conclusion

Expressivities of binary and ternary weighted deep ReLU networks are considered. While the ReLU function is seemingly the most popular choice of activation function in deep neural networks, the restriction of possible choice of network weights to binary and ternary values allows to implement those networks on devices with limited processing power and small memory capacity. Importantly, the sizes of described binary and ternary weighted ReLU networks that attain a given rate of approximation of \( \beta \)-Hölder functions on \([0,1]^d \) differ only by constant and logarithmic factors from the sizes of corresponding approximating networks with continuous weights. We also show that networks of depth 2 with binary activation function \( 1_{[a,b)} \) can approximate continuous functions on \([0,1]^d \), where \([a,b) \subset \mathbb{R}\) is an arbitrary non-empty interval.
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