EXACT LOWER AND UPPER BOUNDS FOR SHIFTS OF
GAUSSIAN MEASURES

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Abstract. Exact upper and lower bounds on the ratio $E w(X - v)/E w(X)$ for a centered Gaussian random vector $X$ in $\mathbb{R}^n$, as well as bounds on the rate of change of $E w(X - tv)$ in $t$, where $w: \mathbb{R}^n \to [0, \infty)$ is any even unimodal function and $v$ is any vector in $\mathbb{R}^n$. As a corollary of such results, exact upper and lower bounds on the power function of statistical tests for the mean of a multivariate normal distribution are given.

1. Introduction

A classic result due to Anderson [1, Theorem 1] states the following: If $w: \mathbb{R}^n \to [0, \infty)$ is an even unimodal function and if $A$ is a symmetric convex subset of $\mathbb{R}^n$, then $\int_A w(x + tv) \, dx$ is nondecreasing in $t \geq 0$, for any vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

By a clever application of Anderson’s theorem, Marshall and Olkin [5] showed that, if a random vector $X$ in $\mathbb{R}^n$ has a Schur-concave density and if the indicator of a subset $A$ of $\mathbb{R}^n$ is Schur concave and permutation symmetric, then $P(X \in v + A)$ is Schur concave in $v$.

By using a rather different method, it was shown in [7] that for a standard Gaussian random vector $Z$ in $\mathbb{R}^n$ the probability $P(Z \in v + A)$ is Schur concave/Schur convex in $(v_1^2, \ldots, v_n^2)$ provided that the indicator of the set $A$ is so, respectively. An application of this result, also given in [7], was that, for large $n$, tests whose rejection regions are balls, centered at the origin, with respect to the $\ell_p$-norm on $\mathbb{R}^n$ with $p > 2$ will be generally preferable to the likelihood ratio test in terms of the asymptotic relative efficiency.

Here we obtain a number of results somewhat related to the just mentioned ones, including exact upper and lower bounds on the ratio $E w(X - v)/E w(X)$ for a centered Gaussian random vector $X$ in $\mathbb{R}^n$, as well as bounds on the rate of change of $E w(X - tv)$ in $t$; here, again, $w: \mathbb{R}^n \to [0, \infty)$ is any even unimodal function. As a corollary of such results, we give exact upper and lower bounds on the power function of statistical tests for the mean of a multivariate normal distribution.

The proof of cited Theorem 1 in [1] was based on the Brunn–Minkowski inequality. However (as can be seen from Remark 3.2 in the present paper), this theorem can be immediately reduced to the case when the function $w$ is log concave, and then [1, Theorem 1] can be obtained at once from the following version of the Prékopa–Leindler theorem — cf. [2, Corollary 3.5]:

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Theorem A. If a function $F: \mathbb{R}^m \times \mathbb{R}^n \to [0, \infty]$ is log concave, then the function $G: \mathbb{R}^m \to [0, \infty]$ given by the formula

$$G(x) := \int_{\mathbb{R}^n} F(x, y) \, dy$$

for $x \in \mathbb{R}^n$ is also log concave.

In turn, as shown in [2], the Prékopa–Leindler theorem follows already from the simple “one-dimensional” case of the Brunn–Minkowski inequality, for subsets of $\mathbb{R}$; concerning this “one-dimensional” case, see e.g. [4, Theorem 2.1]. It is also shown in [2, Corollary 3.4] that, vice versa, the Brunn–Minkowski inequality follows from a generalized version of the Prékopa–Leindler theorem.

Theorem A will be the main tool in the proof of the mentioned exact upper and lower bounds on the ratio $E_w(X - v)/E_w(X)$. Another ingredient, which significantly simplifies the proof, is a so-called special-case l’Hospital-type rule for monotonicity (cf. e.g. [6, Proposition 4.1]):

Theorem B. Let $-\infty \leq a < b \leq \infty$. Let $f$ and $g$ be differentiable functions defined on the interval $(a, b)$ such that $g$ and $g'$ do not take on the zero value and do not change their respective signs on $(a, b)$. Suppose also that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$. Under these conditions, if the “derivative” ratio $f'/g'$ is increasing on $(a, b)$, then the ratio $f/g$ is so as well.

General versions of this l’Hospital-type rule for monotonicity, without the assumption that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$ are also known; see again [6] and references therein.

Here are notations used in the rest of this paper:

- $X$ is a zero-mean Gaussian random vector in $\mathbb{R}^n$ with a nonsingular covariance matrix $\Sigma$;
- $Z$ is a zero-mean Gaussian random vector in $\mathbb{R}^n$ with covariance matrix $I_n$;
- $\gamma_n$ is the standard Gaussian measure over $\mathbb{R}^n$;
- unless otherwise stated, $w: \mathbb{R}^n \to [0, \infty)$ is any even unimodal function such that $E w(Z) > 0$ (and hence $E w(X) > 0$, in view of the absolute continuity of the distribution of $Z$ with respect to that of $X$); recall here that the unimodality of the function $w$ means that the set $\{ x \in \mathbb{R}^n : w(x) > c \}$ is convex for each real $c$;
- unless otherwise stated, $A$ is any symmetric convex subset of $\mathbb{R}^n$ such that $P(Z \in A) > 0$ (and hence $P(X \in A) > 0$);
- $\langle \cdot, \cdot \rangle$ denotes the standard inner product over $\mathbb{R}^n$, and $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^n$;
- $\delta^*(\cdot|A)$ is the support function of a set $A \subseteq \mathbb{R}^n$, given by the formula

$$(1.1) \quad \delta^*(v|A) := \sup\{ \langle z, v \rangle : z \in A \}$$

for $v \in \mathbb{R}^n$ (cf. e.g. [8, page 28]);
- $\Phi$ is the standard normal cumulative distribution function, and $\varphi = \Phi'$ is the standard normal density function;
• for $t \in [0, \infty)$ and $a \in [0, \infty]$,

$$r_t(a) := \begin{cases} 
 e^{-t^2/2} & \text{if } a = 0, \\
 \frac{\Phi(t+a) - \Phi(t-a)}{\Phi(a) - \Phi(-a)} & \text{if } a \in (0, \infty), \\
 1 & \text{if } a = \infty.
\end{cases}$$

(1.2)

• $I_A$ denotes the indicator function of a set $A$, and $I\{A\}$ denotes the indicator of an assertion $A$.

• $u$ denotes an arbitrary unit vector in $\mathbb{R}^n$.

2. Statements of results

2.1. Exact upper and lower bounds on the ratio $\frac{\mathbb{E}w(X - tu)}{\mathbb{E}w(X)}$.

**Theorem 2.1.** For any real $t \geq 0$

$$e^{-t^2/(u,\Sigma^{-1}u)/2} = r_t(\Sigma^{-1/2}u)(0) \leq \frac{\mathbb{E}w(X - tu)}{\mathbb{E}w(X)} \leq r_t(\Sigma^{-1/2}u)(a_{\Sigma,w,u}) \leq 1,$$

where

$$a_{\Sigma,w,u} := \delta^*(\Sigma^{-1}u | A_w) \|\Sigma^{-1/2}u\| \in [0, \infty],$$

(2.1)

$$A_w := \{ x \in \mathbb{R}^n : w(x) > 0 \}.$$  

The necessary proofs will be given in Section 3.

In the special case of a standard Gaussian random vector, the statement of Theorem 2.1 can be simplified:

**Corollary 2.2.** For any real $t \geq 0$

$$e^{-t^2/2} = r_t(0) \leq \frac{\mathbb{E}w(Z - tu)}{\mathbb{E}w(Z)} \leq r_t(a_{w,u}) \leq 1,$$

where

$$a_{w,u} := a_{I_n,w,u} = \delta^*(u | A_w).$$

(2.2)

Even though Corollary 2.2 is a special case of Theorem 2.1, it will be seen that, vice versa, Theorem 2.1 can be easily obtained from Corollary 2.2.

Letting $w = I_A$, we see that Theorem 2.1 and Corollary 2.2 immediately imply the following two corollaries.

**Corollary 2.3.** For any real $t \geq 0$

$$e^{-t^2/(u,\Sigma^{-1}u)/2} = r_t(\Sigma^{-1/2}u)(0) \leq \frac{\mathbb{P}(X \in tu + A)}{\mathbb{P}(X \in A)} \leq r_t(\Sigma^{-1/2}u)(a_{\Sigma,A,u}) \leq 1,$$

where

$$a_{\Sigma,A,u} := \delta^*(\Sigma^{-1}u | A) \|\Sigma^{-1/2}u\| \in [0, \infty].$$

(2.5)

**Corollary 2.4.** For any real $t \geq 0$

$$e^{-t^2/2} = r_t(0) \leq \frac{\gamma_n(tu + A)}{\gamma_n(A)} \leq r_t(a_{A,u}) \leq 1,$$

where

$$a_{A,u} := a_{I_n,A,u} = \delta^*(u | A).$$

(2.6)
Let us also present the following statement on the exactness of the lower and upper bounds on $Ew(X - tu)$ given in Theorem 2.1.

**Proposition 2.5.**

(i) For any positive-definite symmetric matrix $\Sigma$, any unit vector $u \in \mathbb{R}^n$, and any real $t \geq 0$, the lower bound $e^{-t^2(u, \Sigma^{-1}u)/2}$ in (2.1) cannot be replaced by any strictly greater number.

(ii) For any positive-definite symmetric matrix $\Sigma$, any unit vector $u \in \mathbb{R}^n$, any real $t \geq 0$, and any $a \in [0, \infty]$, there is an even unimodal function $w: \mathbb{R}^n \to [0, \infty)$ such that $a_{\Sigma, w, u} = a$ and the second equality in (2.1) turns into the equality.

Letting again $w = 1_A$, we see that Proposition 2.5 follows immediately from the corresponding statement on the exactness of the lower and upper bounds on $P(X \in tu + A)$ given in Corollary 2.3.

**Proposition 2.6.**

(i) For any positive-definite symmetric matrix $\Sigma$, any unit vector $u \in \mathbb{R}^n$, and any real $t \geq 0$, the lower bound $e^{-t^2(u, \Sigma^{-1}u)/2}$ in (2.5) cannot be replaced by any strictly greater number.

(ii) For any positive-definite symmetric matrix $\Sigma$, any unit vector $u \in \mathbb{R}^n$, any real $t \geq 0$, and any $a \in [0, \infty]$, there is a symmetric convex set $A \subseteq \mathbb{R}^n$ such that $a_{\Sigma, A, u} = a$ and the second equality in (2.5) turns into the equality.

It follows that the lower and upper bounds given in Corollaries 2.2 and 2.4 for the special case $\Sigma = I_n$ are also exact, in the corresponding sense.

However, the lower bounds in Theorems 2.1 and Corollaries 2.2, 2.3, and 2.4 can be refined as shown in the following subsection.

### 2.2. Bounds on the rate of change of $Ew(X - tu)$ in $t$.

**Theorem 2.7.** If $Ew(X) < \infty$, then for any real $t \geq 0$

\begin{equation}
\frac{d}{dt} Ew(X - tu) \geq -t \langle u, \Sigma^{-1}u \rangle Ew(X - tu).
\end{equation}

**Corollary 2.8.** If $Ew(Z) < \infty$, then for any real $t \geq 0$

\begin{equation}
\frac{d}{dt} Ew(Z - tu) \geq -t Ew(Z - tu).
\end{equation}

Letting $w = 1_A$, we see that Theorem 2.7 and Corollary 2.8 immediately imply the following two corollaries.

**Corollary 2.9.** For any real $t \geq 0$

\begin{equation}
\frac{d}{dt} P(X \in tu + A) \geq -t \langle u, \Sigma^{-1}u \rangle P(X \in tu + A).
\end{equation}

**Corollary 2.10.** For any real $t \geq 0$

\begin{equation}
\frac{d}{dt} \gamma_n(tu + A) \geq -t \gamma_n(tu + A).
\end{equation}

**Remark 2.11.** The first inequality in (2.11) can be easily deduced from differential inequality (2.8). Indeed, (2.8) can be rewritten as $(\ln g)'(t) \geq -ct$ for $t \geq 0$, where $g(t) := Ew(Z - tu)$.
and $c := (u, \Sigma^{-1}u)$. Integrating the differential inequality $(\ln g)'(t) \geq -ct$, we have $g(t) \geq e^{-ct^2/2}g(0)$ for $t \geq 0$, which is indeed the first inequality in (2.1). Thus, Theorem 2.7 and Corollaries 2.8, 2.9, and 2.10 are indeed refinements of the corresponding lower bounds in Theorem 2.1 and Corollaries 2.2, 2.3, and 2.4.

For general functions $w$, which are not necessarily even or unimodal, we have

**Proposition 2.12.** Let a Borel-measurable function $w: \mathbb{R}^n \to \mathbb{R}$ be such that for some open interval $T \subseteq \mathbb{R}$ and some nonnegative Borel-measurable function $w_1: \mathbb{R}^n \to \mathbb{R}$ we have $E w_1(Z) < \infty$ and $|\langle u, Z \rangle w(Z - tu)| \leq w_1(Z)$ for all $t \in T$.

Then for all $t \in T$

$$
\frac{d}{dt} E w(Z - tu) = -\langle u, E Z w(Z - tu) \rangle.
$$

Letting $w = 1_A$, immediately from Proposition 2.12 we obtain the following.

**Corollary 2.13.** Let $A$ be any Borel subset of $\mathbb{R}^n$ with $\gamma_n(A) \neq 0$. Then for all real $t$ one has $P(Z \in tu + A) = \gamma_n(tu + A) > 0$ and

$$
\frac{d}{dt} \ln \frac{1}{\gamma_n(tu + A)} = \frac{d}{dt} \ln \frac{1}{P(Z \in tu + A)} = \langle u, E(Z| Z \in tu + A) \rangle;
$$

that is, the rate $\frac{d}{dt} \frac{1}{\gamma_n(tu + A)} = -\frac{d}{dt} \frac{\gamma_n(tu + A)}{\gamma_n(tu + A)}$ of the relative decrease of $\gamma_n(tu + A)$ in $t$ equals the $u$-coordinate $\langle u, E(Z| Z \in tu + A) \rangle$ of the center $E(Z| Z \in tu + A)$ of the standard Gaussian mass over the set $tu + A$.

Now Corollary 2.10 can be restated as follows:

**Corollary 2.14.** Suppose that the conditions of Corollary 2.13 hold and, in addition, the set $A$ is symmetric and convex. Then for all real $t \geq 0$

$$
\langle u, E(Z| Z \in tu + A) \rangle \leq t;
$$

that is, when the symmetric convex set $A$ is shifted by the vector $tu$, the $u$-coordinate of the center of the standard Gaussian mass over the set $tu + A$ increases by no more than $t$.

2.3. **Hypothesis testing.** Corollary 2.15 can be restated in terms of hypothesis testing:

**Corollary 2.15.** Let $Y$ be a Gaussian random vector in $\mathbb{R}^n$ with an unknown mean $\mu$ and a known nonsingular covariance matrix $\Sigma$. We test the null hypothesis $H_0: \mu = 0$ versus the alternative $H_1: \mu = \theta u$ for real $\theta > 0$, using the test $\delta(Y) := 1\{Y \notin A\}$ with a symmetric convex set $A \subseteq \mathbb{R}^n$, so that the null hypothesis is rejected if and only if $Y \notin A$, and the size of the test is $\alpha := P(X \notin A)$ (where $X$ is as in Corollary 2.9). Then for the power

$$
\beta_\delta(\theta) = P_{\theta}(Y \notin A) = P(X \notin A - \theta u) = 1 - P(X \in A - \theta u)
$$

of the test $\delta$ at any alternative $\mu = \theta u$ for a real $\theta > 0$ we have

$$
1 - e^{-\theta^2\langle u, \Sigma^{-1}u \rangle/2}(1 - \alpha) \geq \beta_\delta(\theta) \geq 1 - r_\theta\langle \Sigma^{-1/2}u \rangle(a_\Sigma, A, u)(1 - \alpha) \geq \alpha;
$$

here $P_{\theta}$ denotes the probability computed assuming that $\mu = \theta u$ is the true mean of $Y$. 

3. Proofs

Of the first four results stated in Section 2 – Theorem 2.1 and Corollaries 2.2, 2.3, and 2.4 – Corollary 2.3 is formally the least general. However, we shall prove Corollary 2.4 first. From there, it will not be hard to deduce the more general Corollary 2.2 and then in turn Theorem 2.1, which latter immediately yields Corollary 2.3 as a special case. Then a proof of Proposition 2.6 will be given.

After that, we will prove Corollary 2.8 and Theorem 2.7, in this order. Corollaries 2.9 and 2.10 will then follow immediately.

A proof of Proposition 2.12 will conclude this section.

To prove Corollary 2.4, we shall need

Lemma 3.1. The expression \( r_t(a) \), defined in (1.2), is continuous and nondecreasing in \( a \in [0, \infty] \), for each \( t \in [0, \infty) \).

Proof. The case \( t = 0 \) is trivial. Fix now any \( t \in (0, \infty) \). That \( r_t(a) \) is continuous in \( a \) at \( a = \infty \) is obvious. It is also obvious that \( r_t(a) \) is continuous in \( a \) at each point \( a \in (0, \infty) \). That \( r_t(a) \) is continuous in \( a \) at \( a = 0 \) follows by the l’Hospital rule.

It remains to show that \( r_t(a) \) is increasing in \( a \in (0, \infty) \). For such \( a \), we have

\[
 r_t(a) = \frac{\psi_t(a)}{\psi_0(a)},
\]

where \( \psi_t(a) := \Phi(t + a) - \Phi(t - a) \). Note that \( \psi_t(0+) = \psi_0(0+) = 0 \) and the “derivative ratio”

\[
\frac{\psi_t'(a)}{\psi_0'(a)} = \frac{\varphi(t + a) + \varphi(t - a)}{2\varphi(a)} = e^{-t^2/2} \cosh ta
\]

is increasing in \( a \in (0, \infty) \). So, by Theorem B (stated in Section 1), \( r_t(a) \) is increasing in \( a \in (0, \infty) \). The proof of Lemma 3.1 is complete. \( \square \)

Proof of Corollary 2.4. In view of (2.7) and because the set \( A \) is symmetric,

\[
 a_{A,u} = \delta^*(u | A) = \sup \{ \langle z, u \rangle : z \in A \} = \sup \{ ||z, u|| : z \in A \}.
\]

By the spherical symmetry of the standard Gaussian measure \( \gamma_n \), without loss of generality \( u \) equals \( e_1 \), the first vector of the standard basis of \( \mathbb{R}^n \). So, denoting by \( \varphi_k \) the density of the standard Gaussian measure \( \gamma_k \) over \( \mathbb{R}^k \) (with respect to the Lebesgue measure over \( \mathbb{R}^k \)), we have

\[
 g_{A,u}(t) := \gamma_n(tu + A) = \int_{tu + A} dz \varphi_n(z)
\]

(3.2)

\[
 = \int_A dx \varphi_n(x + tu)
\]

(3.3)

\[
 = \int_{-a_{A,u}}^{a_{A,u}} dx \varphi(x + t) h_{A,u}(x)
\]

(3.4)

where

\[
 h_{A,u}(x) := \int_{\mathbb{R}^{n-1}} dy \varphi_{n-1}(y) l_A(xu + y)
\]
and the orthogonal complement \( \{ y \in \mathbb{R}^n : \langle y, u \rangle = 0 \} \) of the vector \( u = e_1 \) is identified with \( \mathbb{R}^{n-1} \); equality \((3.4)\) holds because, if \( |x| > a_{A,u} \), then for all \( y \in \mathbb{R}^{n-1} \) we have \( |\langle xu + y, u \rangle| = |x| > a_{A,u} \); so, by \((3.1)\), \( xu + y \not\in A \) for all \( y \in \mathbb{R}^{n-1} \), whence \( h_{A,u}(x) = 0 \).

The functions \( \mathbb{R} \times \mathbb{R}^{n-1} \ni (x, y) \mapsto I_A(xu + y) \in [0, \infty) \) and \( \varphi_{n-1} \) are even and log concave, and hence so is the function

\[
\mathbb{R} \times \mathbb{R}^{n-1} \ni (x, y) \mapsto \varphi_{n-1}(y)I_A(xu + y) \in [0, \infty).
\]

Therefore, in view of Theorem A, the function \( h_{A,u} : \mathbb{R} \to [0, \infty) \) is also even and log concave, and hence unimodal; it also follows that \( h_{A,u} \) is continuous on the interval \((-a_{A,u}, a_{A,u})\). So, there is a (unique, nonnegative, finite) Borel measure \( \mu_{A,u} \) over the interval \((0, a_{A,u})\) such that \( h_{A,u}\big((x, a_{A,u})\big) = h_{A,u}(x) \) for all \( x \in (0, a_{A,u}) \), and then for \( x \in (-a_{A,u}, a_{A,u}) \) we have

\[
h_{A,u}(x) = h_{A,u}(|x|) = \mu_{A,u}\left(|x|, a_{A,u}\right) = \int_{(0, a_{A,u})} \mu_{A,u}(da) \mathbf{1}\{a > |x|\}
\]

where

\[
h_a(x) := \mathbf{1}\{|x| < a\}.
\]

So, by \((3.4)\) and the Fubini theorem,

\[
g_{A,u}(t) = \int_{(0, a_{A,u})} \mu_{A,u}(da) \int_{-a_{A,u}}^{a_{A,u}} dx \varphi(x + t)h_a(x) = \int_{(0, a_{A,u})} \mu_{A,u}(da) g_a(t),
\]

where, for \( a \in (0, a_{A,u}) \),

\[
g_a(t) := \int_{-a}^{a} dx \varphi(x + t) = r_t(a)g_a(0),
\]

in view of \((1.2)\). So, by Lemma 3.1,

\[
r_t(0)g_a(0) \leq g_a(t) \leq r_t(a_{A,u})g_a(0)
\]

for \( a \in (0, a_{A,u}) \), whence, by \((3.5)\),

\[
r_t(0)g_{A,u}(0) \leq g_{A,u}(t) \leq r_t(a_{A,u})g_{A,u}(0).
\]

In view of \((3.2)\), inequalities \((3.7)\) are the same as the first two inequalities in \((2.6)\). The equality and the third inequality in \((2.6)\) follow by \((1.2)\) and Lemma 3.1 Corollary \((2.3)\) is now proved.

\[\square\]

The following remark will be used in the proof of Corollary \((2.2)\)

**Remark 3.2.** We have

\[
w(x) = \int_{0}^{\infty} w_c(x) dc
\]

for all \( x \in \mathbb{R}^n \), where

\[
w_c := 1_{A_{w,c}}, \quad A_{w,c} := \{ x \in \mathbb{R}^n : w(x) > c \}.
\]

Note also that the functions \( w_c \) are log concave. Thus, the unimodal function \( w \) is a mixture of log concave functions \( w_c \); moreover, the functions \( w_c \) are even whenever the function \( w \) is even.
Proof of Corollary 2.7. For any $c \geq 0$, by (2.9) and (2.2), $A_{w,c} \subseteq A_{w,0} = A_w$, whence, by (2.4) and (2.1), $a_{A_{w,c},u} \leq a_{A_w,u} = a_{w,u}$. Now Lemma 3.1 yields $r_t(a_{A_{w,c},u}) \leq r_t(a_{w,u})$. Hence, by Remark 3.2 and the second inequality in (2.0),

$$E w(Z - tu) = \int_0^\infty E w_c(Z - tu) dc$$

$$= \int_0^\infty P(Z \in tu + A_{w,c}) dc$$

$$= \int_0^\infty \gamma_n(tu + A_{w,c}) dc$$

$$\leq \int_0^\infty r_t(a_{A_{w,c},u}) \gamma_n(A_{w,c}) dc$$

$$\leq r_t(a_{w,u}) \int_0^\infty \gamma_n(A_{w,c}) dc = r_t(a_{w,u}) E w(Z),$$

which proves the second inequality in (2.3). The proofs of the other inequalities in (2.3) and of the equality there are similar and even somewhat simpler.  

Proof of Theorem 2.1. Given $X, \Sigma, w, u$, and $t$ as in the statement of Theorem 2.1, define $\tilde{X}, \tilde{w}, \tilde{u}$, and $\tilde{t}$ as follows: $\tilde{X} := \Sigma^{-1/2}X$, $\tilde{w}(\tilde{x}) := w(\Sigma^{1/2}\tilde{x})$ for $\tilde{x} \in \mathbb{R}^n$, $\tilde{u} := \Sigma^{-1/2}u/\|\Sigma^{-1/2}u\|$, and $\tilde{t} := t\|\Sigma^{-1/2}u\|$. Applying now Corollary 2.2 with, respectively, $\tilde{X}$, $\tilde{w}$, $\tilde{u}$, and $\tilde{t}$ in place of $Z$, $w$, $u$, and $t$ there, we obtain Theorem 2.1.

Proof of Proposition 2.6. As in the proof of Theorems 2.1 the consideration can be easily reduced to the case $\Sigma = I_n$, so that $X = Z$, a standard Gaussian random vector. Take then indeed any $a \in [0, \infty]$ and let

$$A := \{z \in \mathbb{R}^n : |\langle z, u \rangle| \leq a\}.$$

Then

$$a_{\Sigma, A,u} = a_{I_n, A,u} = \delta^u(A) = a$$

by (1.1). Also, for $a \in [0, \infty)$,

$$P(Z \in tu + A) = P(|\langle Z - tu, u \rangle| \leq a)$$

$$= P(t - a \leq \langle Z, u \rangle \leq t + a)$$

$$= \Phi(t + a) - \Phi(t - a) = r_t(a) P(Z \in A)$$

by (1.2), so that the second equality in (2.5) (with $I_n$ and $Z$ in place of $\Sigma$ and $X$) turns into the equality; the case $a = \infty$ is even simpler than this. This proves part (ii) of Proposition 2.6.

Part (i) of it now follows because

$$\frac{P(Z \in tu + A)}{P(Z \in A)} = r_t(a) \underset{a \to 0}{\longrightarrow} r_t(0) = e^{-t^2/2},$$

by Lemma 3.1.

To prove Corollary 2.8 we shall need

Lemma 3.3. Recall the definition of $g_a(t)$ in (2.6). We have

$$0 \geq g_a(t) \geq -tg_a(t)$$

for all $a \in [0, \infty]$ and $t \in [0, \infty)$. 

then the second inequality in (3.11) can be rewritten as
\[ \lambda \quad \text{for all} \quad t, \]
since \( \tanh 0 = 0 \) and \( \tanh' = 1/\cosh^2 \leq 1 \). So, for \( t > 0 \), we do have \( \lambda_a(t) \geq 0 \), which completes the proof of the second inequality in (3.11). Lemma 3.3 is now proved. \( \blacksquare \)

Proof of Corollary 2.8 By (3.10), (3.2), and (3.5),
\[ Ew(Z - tu) = \int_0^\infty dc \gamma_n(tu + Aw, c) = \int_0^\infty dc g_{Aw, c}(t) = \int_0^\infty dc \int_{(0, Aw, c, u]} \mu_{Aw, c, u}(da) g_a(t). \]
By Lemma 3.3 for all \( a \in [0, \infty] \) and \( t \in [0, \infty) \) we have
\[ 0 \geq g_a'(t) \geq -tg_a(t) \geq -tg_a(0), \]
whence
\[ \int_0^\infty dc \int_{(0, Aw, c, u]} \mu_{Aw, c, u}(da) |g_a'(t)| \leq t \int_0^\infty dc \int_{(0, Aw, c, u]} \mu_{Aw, c, u}(da) g_a(0) \]
\[ = tEw(Z) < \infty. \]
So, using (3.13) and (3.14) again, together with the standard rule of the differentiation of an integral with respect to a parameter – see e.g. [3, Theorem (2.27)(b)], we see that for all real \( t \geq 0 \)
\[ \frac{d}{dt} Ew(Z - tu) = \int_0^\infty dc \int_{(0, Aw, c, u]} \mu_{Aw, c, u}(da) g_a'(t) \]
\[ \geq -t \int_0^\infty dc \int_{(0, Aw, c, u]} \mu_{Aw, c, u}(da) g_a(t) = -tEw(Z - tu), \]
which completes the proof of Corollary 2.8. \( \blacksquare \)

Proof of Theorem 2.7 Given \( X, \Sigma, w, u, \) and \( t \) as in the statement of Theorem 2.7, define \( \tilde{X}, \tilde{w}, \tilde{u}, \) and \( \tilde{t} \) as follows: \( \tilde{X} := \Sigma^{-1/2}X, \tilde{w}(\tilde{x}) := w(\Sigma^{1/2}x) \) for \( \tilde{x} \in \mathbb{R}^n, \tilde{u} := \Sigma^{-1/2}u/\|\Sigma^{-1/2}u\|, \) and \( \tilde{t} := t\|\Sigma^{-1/2}u\| \). Applying now Corollary 2.8 with, respectively, \( \tilde{X}, \tilde{w}, \tilde{u}, \) and \( \tilde{t} \) in place of \( Z, w, u, \) and \( t \) there, we obtain Theorem 2.7. \( \blacksquare \)
Proof of Proposition 2.12. Using the domination condition involving the function $w_1$ and (again) the rule of the differentiation of an integral with respect to a parameter, for all $t \in T$ we have

$$\frac{d}{dt} E w(Z - tu) = \frac{d}{dt} \int_{\mathbb{R}^n} dz \varphi_n(z) w(z - tu)$$

$$= \int_{\mathbb{R}^n} dx \frac{d}{dt} \varphi_n(x + tu) w(x)$$

$$= -\int_{\mathbb{R}^n} dx \varphi_n(x + tu) \langle x + tu, u \rangle w(x)$$

$$= -\langle u, E Z w(Z - tu) \rangle,$$

as claimed. \qed

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