Non-Abelian Gauge Configuration with a Magnetic Field Concentrated at a Point

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Abstract

A specific SU(2) gauge configuration yielding a magnetic field concentrated at a point is investigated. Its relation to the Aharonov-Bohm gauge potential and its cohomological meaning in a three dimensional space are clarified. Quantum mechanics of a spinning particle in such a gauge configuration is briefly discussed.

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In a recent paper, two of the present authors and H.-M. Zhang considered static $SU(2)$ gauge configurations of the following type:

$$ A_i(x) \equiv \frac{g}{\hbar c} a_i(x) = \frac{\lambda(r)}{r^2} \epsilon_{ijk} x_j T_k, $$

$$ i = 1, 2, 3, \quad x = r(x_1, x_2, x_3), \quad r = |x|, \quad (1) $$

where $a_i(x), g$ and $T_k$ are the gauge potential, the gauge coupling constant and a representation of the generator of $su(2)$ satisfying

$$ [T_i, T_j] = i \epsilon_{ijk} T_k, $$
$$ \text{tr}(T_i T_j) = \sigma \delta_{ij}, \quad \sigma > 0, \quad (2) $$

respectively. Under the Wu-Yang Ansatz, (1), the field equation is given by$^2$

$$ r^2 \frac{d^2}{dr^2} \lambda(r) = \lambda(r) \{ \lambda(r) - 1 \} \{ \lambda(r) - 2 \}. \quad (3) $$

Obvious solutions of (3) are $\lambda(r) = 0, 1$ and $2$. The Yang-Mills action, $S$, is maximum for $\lambda(r) = 0$ and $\lambda(r) = 2$ with $S = 0$ and minimum for $\lambda(r) = 1$ with $S = -\infty$. The case $\lambda(r) = 0$ is uninteresting. The case $\lambda(r) = 1$ corresponds to the much-discussed point-like $SU(2)$ magnetic monopole which is known as the Wu-Yang monopole.$^1$-$^3$ In this article, we concentrate on the case

$$ \lambda(r) = 2. \quad (4) $$

In this case, the gauge potential can be rewritten as a pure gauge.$^1$

$$ A_i(x) = i U(\hat{x}) \partial_i U^\dagger(\hat{x}), $$
$$ U(\hat{x}) = e^{i \pi \hat{x} \cdot T}, \quad \hat{x} = \frac{x}{r}. \quad (5) $$

It can be seen that $U(\hat{x})$ satisfies $U(\hat{x}) = U^{-1}(\hat{x}) = U^\dagger(\hat{x})$. This configuration is gauge-equivalent to the trivial configuration $A(x) = 0$ in the space $\mathbb{R}^3 \setminus \{0\}$. It is, however, nontrivial in $\mathbb{R}^3$ including the origin $0$. The field strength $F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x) - i[A_i(x), A_j(x)]$ is calculated to be

$$ F_{ij}(x) = -4 \delta(r^2) \epsilon_{ijk} T_k. \quad (6) $$

The singularity of $F_{ij}(x)$ at the origin implies the property $\partial_i \partial_j U(\hat{x})|_{x=0} \neq \partial_j \partial_i U(\hat{x})|_{x=0}$, $i \neq j$. To better understand the meaning of the magnetic field $B_i(x) \equiv \frac{1}{2} \epsilon_{ijk} F_{jk}(x) = -4 \delta(r^2) T_i$, which is nonvanishing only at the origin, we temporarily introduce a small scale parameter $s$ and replace the factor $r^{-2}$ in $A_i(x)$ by $(r^2 + s^2)^{-1}$. Then, the field strength $F_{ij}(x)$ becomes $-4s^2(r^2 + s^2)^{-2} \epsilon_{ijk} T_k$. Defining a projection of the magnetic field by $b_i(x) = \text{tr}\{ (\hat{x} \cdot T) B_i(x) \}/\sigma$, we obtain

$$ b(x) = -\text{grad}_x \int \mathbb{R}^3 d^3 x' \frac{\rho(r')}{|x - x'|}, \quad (7) $$

$$ \rho(r') = 4 \pi r'^2 |x - x'|^{-3} \delta(r - |x - x'|), \quad (8) $$
\[ \rho(r) = \frac{2s^2(r^2 - s^2)}{\pi r(r^2 + s^2)^3}. \]  

(9)

The function \( \rho(r) \) can be interpreted as the density of the projected magnetic charge. The \( \rho(r) \) is positive in the exterior region \( r > s \) and negative in the interior region \( r < s \). The total positive and negative charges in the respective regions are given by \( \int_{r>s} \rho(r)d^3x = -\int_{r<s} \rho(r)d^3x = 1 \). In the limit \( s \to 0 \), the above type of magnetic charge distribution gives rise to the magnetic field of the prescribed property. Through a similar discussion to the above, we find, in contrast with the case of the Wu-Yang monopole,\(^3\) the Bianchi identity

\[ \epsilon_{ijk}[D_i, F_{jk}(x)] = 0, \quad D_i = \partial_i - iA_i(x), \]  

(10)

is not violated.

The gauge configuration that we are discussing should be compared with Aharonov and Bohm’s one.\(^4\) The Aharonov-Bohm gauge potential, \( A^{AB}(x) \), can be expressed by our \( A(x) \) in the following way. In \( \mathbb{R}^3 \), it is given by

\[
A^{AB}(x) = \frac{\alpha}{x_1^2 + x_2^2} \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} = \frac{\alpha}{4\sigma} \int_{-\infty}^{\infty} \frac{dZ}{|x-Z|} \text{tr}\{T_3A(x-Z)\},
\]

\( x \in \mathbb{R}^3, \quad \alpha = \text{const.}, \quad Z = \hat{z}(0, 0, Z). \)  

(11)

It is a superposition of \( A(x - Z) \) which is singular at the point \( Z \) on the \( x_3 \)-axis. In \( \mathbb{R}^2 \), the \( A^{AB}(x) \) is more simply given by

\[
A_i^{AB}(x) = \frac{\alpha}{x_1^2 + x_2^2} \epsilon_{ij}x_j, \]

\[
= \frac{\alpha}{2\sigma} \text{tr}\{T_3A_i(x)\}|_{x_3=0},
\]

\( \epsilon_{ij} = -\epsilon_{ji}, \quad \epsilon_{12} = 1, \quad x \in \mathbb{R}^2, \quad i = 1, 2. \)  

(12)

We next consider a cohomological meaning of the configuration \( A(x), x \in \mathbb{R}^3 \). Denoting the \( p \)th de Rham cohomology group of the space \( X \) by \( H^p(X) \), the existence of the Aharonov-Bohm gauge potential is due to the nontrivial \( H^1(\mathbb{R}^3 \setminus \{0\}) \). On the other hand, we know that, for the space \( M \equiv \mathbb{R}^3 \setminus \{0\} \), \( H^1(M) \) is trivial but \( H^2(M) \) is nontrivial.\(^5\) As an example, we investigate the 2-form \( \omega \) defined by

\[ \omega = \text{tr}\{U(\hat{x})A_i(x)A_j(x)\}dx_i \wedge dx_j. \]  

(13)

It is straightforward to obtain

\[ d\omega = 0, \quad x \in M. \]  

(14)

The period of \( \omega \) for a sphere surrounding the origin, however, is proportional to the integral of \( \text{tr}\{\hat{x} \cdot T\}U(\hat{x}) \) on the sphere and does not vanish. We see, through de Rham’s first theorem,\(^5\) that \( \omega \) cannot be given as an exact form. We conclude that \( \omega \) belongs to \( H^2(M) \).
and that any closed 2-form $\lambda$ on $M$ can be written as $\lambda = d\omega_1 + a\omega$ with an appropriate 1-form $\omega_1$ and a constant $a$. We thus see that $U(\hat{x})$ and $A(x)$ are convenient quantities to describe $H^2(M)$.

For our $A(x)$, a loop integral $\int_\gamma A(x) \cdot d\mathbf{x}$ for a loop $\gamma \subset M$ is in general nonvanishing. For example, for $\gamma_0 \equiv \{(r\sin \varphi \cos \varphi, r\sin \varphi \sin \varphi, r \cos \theta)|0 \leq \varphi < 2\pi, r, \theta : \text{fixed}\}$, we have $\int_{\gamma_0} A(x) \cdot d\mathbf{x} = -4\pi T_3 \sin^2 \theta$ which is not equal to a multiple of $2\pi$ in general. We see, however, that the loop variable $V(\gamma)$ defined by

$$V(\gamma) = Pe^{i\int_\gamma A(x) \cdot d\mathbf{x}}, \quad P: \text{path ordering},$$

is equal to 1:

$$V(\gamma) = 1.$$ (16)

This result is obtained with the help of the non-Abelian Stokes’ theorem, $^6$$^8$

$$V(\gamma) = P\exp \left( i \int_S d\sigma_{ij} u(x) F_{ij}(x) u^\dagger(x) \right),$$

where $P, S, d\sigma_{ij}$ and $u(x)$ are a certain 2-dimensional ordering factor, a surface with $\partial S = \gamma$, a surface element of $S$ and an $x$-dependent unitary matrix, respectively. By (7) and (17), we easily understand that $V(\gamma) = 1$ if the surface $S$ does not contain the origin. When the origin $0$ lies on $S$, we have $V(\gamma) = \exp[-4\pi i u(0)(\mathbf{n} \cdot \mathbf{T}) u^\dagger(0)]$, where $\mathbf{n}$ is the normal of $S$ at $0$. Since all the eigenvalues of $2u(0)(\mathbf{n} \cdot \mathbf{T}) u^\dagger(0)$ are integral, we conclude $V(\gamma) = 1$ in this case, too.

One more interesting property of $A(x)$ is that it yields an angular momentum satisfying a desired algebra. If we define $j$ by

$$j = U(\hat{x}) \left( x \times \frac{1}{i} \mathbf{\nabla} \right) U^\dagger(\hat{x})$$

$$= x \times \frac{1}{i} (\mathbf{\nabla} - iA)$$

$$= \frac{1}{i} x \times \nabla - \frac{2}{r^2} x \times (x \times T),$$

it satisfies the relation $[j_l, j_m] = i\epsilon_{lmn} \{j_n - 4x_n (x \cdot T) \delta(r^2)\}$. Noticing the relation $r^2 \delta(r^2) = 0$, we have

$$[j_l, j_m] = i\epsilon_{lmn} j_n,$$ (19)

even at the origin.

In the above, we have considered a gauge configuration yielding a magnetic field at one point. Of course, any configuration of the form (5) with $U$ singular at a point will exhibit similar properties to (7) and (19). If we consider $A^\nu(x) = iU^\nu(\hat{x}) \{\nabla U^{\nu\dagger}(\hat{x})\}$ with $U^\nu(\hat{x}) = e^{i\nu \hat{x} \cdot T}, 0 \leq \nu < 2\pi$, we are led to $A^\nu(x) = [(\hat{x} \times T)(1 - \cos \nu) - (\hat{x} \times (\hat{x} \times T))] \sin \nu r^{-1}$. In contrast with our $A(x)$, the $A^\nu(x)$, $\nu \neq 0, 1$, does not satisfy the Wu-Yang Ansatz and is not a solution of the field equation. If we do not restrict ourselves to the solutions of the Yang-Mills field equation, we can think of many interesting configurations. For instance, it is clear that the $A_i^{(N)}(x)$ defined by

$$A_i^{(N)}(x) = iU^{(N)} \partial_i U^{(N)\dagger},$$ (20)
\[ U^{(N)} = U_1 U_2 \cdots U_N, \quad U_n = \exp \left\{ i \pi \frac{x - r_n}{|x - r_n|} \cdot T \right\}, \]

causes a magnetic field located at the points \( r_1, r_2, \ldots, r_N \). An appropriate limit of the gauge potential of this type yields the Aharonov-Bohm potential as is seen in (11).

We now turn to a brief discussion of quantum mechanics of a spinning particle put in the gauge configuration given by (1) and (4). The Hamiltonian of the system is

\[ \mathcal{H} = -\frac{\hbar^2}{2m} \{ \sigma \cdot (\nabla - i A(x)) \}^2 + V(r) \]

\[ = -\frac{\hbar^2}{2m} \left\{ (\nabla - i A)^2 + \sigma \cdot B(x) \right\} + V(r), \]  

where \( m \) and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) denote the mass of the particle and the Pauli matrices, respectively. The potential term \( V(r) \) is assumed to be independent of the spin \( \frac{1}{2} \hbar \sigma \) and the isospin \( T \) and dependent only on \( r \). The nontriviality of this example is manifest in the term \( \sigma \cdot B(x) = -4(\sigma \cdot T)\delta(r^2) \). Since the \( \mathcal{H} \) can be written as

\[ \mathcal{H} = U(\hat{x})\mathcal{H}_0 U^\dagger(\hat{x}), \]

\[ \mathcal{H}_0 = -\frac{\hbar^2}{2m} (\sigma \cdot \nabla)^2 + V(r), \]  

its eigenfunction \( \psi(x) \) takes the form

\[ \psi(x) = U(\hat{x}) \varphi(x) v. \]  

In (24), \( \varphi(x) \) is an eigenfunction of \(-\frac{\hbar^2}{2m} \nabla^2 + V(r) \) and is regular at the origin, \( v \) describing the spin and the isospin degrees of freedom. Note that \( \varphi(x) v \) satisfies the relation \( (\sigma \cdot \nabla)^2 \{ \varphi(x) v \} = \{ \nabla^2 \varphi(x) \} v \). The angular momentum \( \hbar J \) of the system is given by

\[ \hbar J = \hbar \left( j + \frac{\sigma}{2} \right) \]  

with \( j \) defined by (18). The commutativity of \( J \) with \( \mathcal{H} \) is assured not on \( \varphi(x) v \) but on \( \psi(x) \):

\[ [\mathcal{H}, J] \psi(x) = 0. \]  

If we respect this property, we are forced to make use of \( \psi(x) \) which is not single-valued unless \( \varphi(x) \) vanishes at the origin. \( \text{i} \)From (19) and (25), the components of \( \hbar J \) satisfy the desired algebra of angular momentum. Because of the afore-mentioned property \( \partial_i \partial_j U(\hat{x})|_{x=0} \neq \partial_i \partial_j U(\hat{x})|_{x=0}, \text{ } i \neq j \), we see that the canonical commutation relation \([p_i, p_j] = 0, \text{ } p_i = (\hbar/i)\partial_i \) is violated unless \( \varphi(x) \) vanishes at the origin:

\[ [p_i, p_j] \psi(x)|_{x=0} \neq 0 \text{ } (i \neq j) \text{ if } \varphi(0) \neq 0. \]  

On the other hand, if the \( \varphi(x) \) with \( \varphi(0) \neq 0 \) is excluded, the completeness of the set \( \{ \psi(x) \} \) and the self-adjointness of \( \mathcal{H} \) will be lost.

In a future communication, we will discuss whether the multi-valuedness of \( \psi(x) \) and the violation of the canonical commutation relation at the origin cause physical effects or not.
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