Early evolution of newly born magnetars with a strong toroidal field

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ABSTRACT
We present a state-of-the-art scenario for newly born magnetars as strong sources of gravitational waves (GWs) in the early days after formation. We address several aspects of the astrophysics of rapidly rotating, ultra-magnetized neutron stars (NSs), including early cooling before transition to superfluidity, the effects of the magnetic field on the equilibrium shape of NSs, the internal dynamical state of a fully degenerate, oblique rotator and the strength of the electromagnetic torque on the newly born NS. We show that our scenario is consistent with recent studies of supernova remnant surrounding Anomalous X-ray Pulsars (AXPs) and Soft Gamma-Ray Repeaters (SGRs) in the Galaxy that constrains the electromagnetic energy input from the central NS to be ≤10⁵¹ erg. We further show that if this condition is met, then the GW signal from such sources is potentially detectable with the forthcoming generation of GW detectors up to Virgo cluster distances where an event rate ~1 yr⁻¹ can be estimated. Finally, we point out that the decay of an internal magnetic field in the 10ⁱ⁶ G range couples strongly with the NS cooling at very early stages, thus significantly slowing down both processes: the field can remain this strong for at least 10³ yr, during which the core temperature stays higher than several times 10⁸ K.

Key words: gravitational waves – magnetic fields – stars: neutron – X-rays: stars.

1 INTRODUCTION
Gravitational wave (GW) emission from spheroidal, rapidly rotating, isolated neutron stars (NSs) has long been considered in the astrophysical literature (Ostriker & Gunn 1969). A natural origin of this distortion can be the anisotropic pressure from the internal magnetic field. However, before the early 1990s, the inferred magnetic fields of NSs were ~10¹⁴ G, which implied tiny deviations from spherical symmetry and, thus, weak GW emission that would be detectable, in principle, only from the nearest sources and with years-long observations (Bonazzola & Gourgoulhon 1996).

The discovery of the Soft Gamma-Ray Repeaters (SGRs) and Anomalous X-ray Pulsars (AXPs) (cf. Mazets et al. 1979; Mereghetti & Stella 1995) led to the idea that these peculiar high-energy sources could be ultra-magnetized NSs, magnetars, with external (dipole) fields in the 10¹⁴/10¹⁵ G range and with internal fields at least one order of magnitude stronger (Duncan & Thompson 1992, hereafter DT92; Paczynsky 1992; Thompson & Duncan 1993, 1995, 1996, hereafter TD93, TD95, TD96, respectively). The core of the proto-neutron star (PNS) experiences a phase of neutrino-driven turbulent convection (during a few tens of seconds). Strong differential rotation is also present if the nascent NS is spinning at millisecond period (we indicate the initial spin, in milliseconds, as P/ms), providing a total free energy reservoir of up to ~10²⁵(P/ms)⁻² erg. The combination of these two factors can power an α – Ω dynamo that acts coherently over the whole NS core (DT92). A seed magnetic field is twisted into a mainly toroidal configuration and its intensity amplified to values ~10¹⁶ G. The total energy available from the core differential rotation corresponds to a maximum field of ~10¹³ G (TD93; Duncan 1998). Millisecond spin periods are required for differential rotation to play an important role and to achieve magnetic field coherence lengthscales comparable to the stellar radius. In more slowly rotating NSs, differential rotation is almost negligible and a convection-driven, α-type dynamo results that acts stochastically and leads to a much weaker large-scale field (DT92; TD93). In this formation scenario, magnetars represent those NSs that are born as very rapidly rotating, at ≤3 ms spin period.

As the magnetar model received increasing support from observations during the last decade, it was shortly after realized that such objects at birth represent promising sources of GWs, because of their large magnetic deformation and rapid rotation (Ioka 2001; Palomba 2001). These early suggestions assumed that a simple dipole field extended throughout the NS and its magnetosphere, thus causing NS deformation and magnetodipole spin-down. Cutler (2002) first highlighted the crucial role of the internal field structure in the GW emission efficiency of a magnetically distorted NS. He pointed out that a NS with a strong internal toroidal field has a prolate shape and would thus provide the best chance of being a strong GW emitter.

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Conversely, GW emission from a NS with an oblate distortion (such as that caused by a dipole field) would be rapidly quenched, since such an object would tend to spin around its symmetry axis (see Section 2). Based on the energetics and likely recurrence time of Giant Flares such as the Dec. 27 event from SGR 1806–20, Stella et al. (2005) derived a lower limit of $\sim 8 \times 10^{15}$ G for the strength of the core magnetic field in this object. This suggests that internal fields of magnetars may be even stronger than previously thought, reaching the $10^{16}$ G range. Lazzati, Ghirlanda & Ghisellini (2005), Nakar et al. (2006) and Popov & Stern (2006), based on the apparent lack of Giant-Flare-like events in the BATSE archive, suggest longer recurrence times for such events, thus reducing the minimum energy requirement proposed by Stella et al. (2005). However, the correlation between short GRBs and local galaxies found by Tanvir et al. (2005) leaves room for a fraction of such events to be associated with a population of young NSs. Furthermore, Kaminker et al. (2007) showed that, if the thermal emission observed from AXPs is indeed powered by the decay of a magnetar’s $B$-field, then the field strength in the NS crust must be $\sim 10^{16}$ G.

Fields $\sim 10^{16}$ G would imply remarkably large (prolate) magnetic deformations, $\epsilon_B \sim 10^{-3}$ (see Section 2.1.2), which, in turn, would make GW emission at birth from such objects strong enough to be detectable up to the Virgo cluster distance, with Advanced LIGO/Virgo class detectors. The integrated magnetar formation rate in Virgo is estimated to be $\sim 1$ per year (Stella et al. 2005), as opposed to $\sim 10^{-3}$ per year in the Galaxy alone (Gaensler, Gotthelf & Vasisht 1999), thus making newly formed magnetars the potentially most interesting sources for next-generation GW detectors, with a fairly high rate of occurrence.

As an alternative scenario, it was proposed that strongly magnetic NSs can form as a direct result of magnetic flux conservation during core collapse of a massive star with unusually strong magnetic field (Ferrario & Wickramasinghe 2006; cf. Usov 1992). In this framework, magnetars need not be born rapidly rotating: their magnetic field is stronger than in ordinary NSs because the magnetic dipole moment of their progenitor stars was particularly large. Ferrario & Wickramasinghe (2006) have shown that a population of NSs with dipole fields up to $\sim 10^{15}$ G can be obtained in this framework starting from a realistic Galactic population of high-mass, high-field stars. We note, however, that the upper end of their field distribution seems to fall short of the minimal requirement within the magnetar model. Internal fields of at least several times $10^{15}$ G are needed both to explain the overall energy output of SGRs and AXPs (TD95, Thompson & Duncan 2001, hereafter TD01) and to make the magnetic field decay time-scale comparable to, or shorter than, the estimated ages ($\sim 10^4$ yr) of these sources (TD96).

Geppert & Rheinhardt (2006) have shown that a field amplified to magnetar strength in a newly formed NS will survive an early unstable phase and (subsequent dissipation) only if the NS spin period is shorter than $\sim 5$ ms, thus reinforcing the case for fast spins at birth.

If magnetars are born with millisecond spin periods (hereafter millisecond magnetar), their initial spin energy will be $E_{\text{spin}} \sim 3 \times 10^{52} (P_i / \text{ms})^{-2}$ erg. Here and throughout, we assume a typical radius of 12 km and a mass of 1.4 $M_{\odot}$. The spin-down time-scale through magnetic dipole radiation will be very short, $\sim 1$ day for external dipole fields in the $10^{14}$ G range. Most of the initial spin energy would thus be rapidly transferred to the surrounding supernova ejecta (Allen & Horvath 2004). Recent studies have also shown that for newly formed magnetars with dipole fields in excess of $\sim (6-7) \times 10^{14}$ G and spin period of $\approx 1$ ms, strongly magnetized, relativistic winds are produced which can carry away most of the initial spin energy in a matter of minutes, thus opening the possibility of an even faster transfer of energy to the ejecta (Thompson, Chang & Quataert 2004; Bucciantini et al. 2006; Metzger, Thompson & Quataert 2007). In both cases, present-day supernova remnants (SNRs) around known magnetars should bear the signature of this larger-than-usual energy injection. For initial spin periods less than $3$ ms, the injected energy would be $\sim 3 \times 10^{51}$ erg, making these remnants significantly more energetic than those surrounding ordinary NSs.

The X-ray spectra of the SNRs surrounding known magnetar candidates (two APXs and two SGRs) have been analysed by Vink & Kuiper (2006) who found that the total energy content of the ejecta in these remnants does not appear to be different from the energy in remnants surrounding common NSs ($\approx 10^{53}$ erg). This result implies that either these NSs were not born rapidly rotating, thus challenging the $\alpha - \Omega$ dynamo scenario, or their initial spin energy must have been lost without appreciably energizing the surrounding ejecta. Dall’Osso & Stella (2007) discussed the possibility that most of the initial spin energy of millisecond magnetars is released through gravitational radiation. They concluded that, in order to account for the constraints derived by Vink & Kuiper (2006), the internal toroidal field at birth should be in the $10^{16}$ G range, consistent with the inference by Stella et al. (2005) for the field strength in SGR 1806–20 and, by extension, magnetars in general.

A similar conclusion about strong GW-driven spin-down in newly formed magnetars was reached by Arons (2003), based on a totally independent argument.

Bucciantini et al. (2007) have also studied the possibility that the huge spin energy of a millisecond magnetar is promptly released in the form of a highly collimated wind, if the external (dipole) field exceeds $10^{15}$ G. The wind breaks through the surrounding supernova ejecta and produces a GRB-like event without directly energizing the ejecta. However, the currently estimated magnetar formation rate exceeds the rate of observed GRBs by 2 orders of magnitude. Hence, this process can involve at most a tiny fraction of newly formed magnetars (Bucciantini et al. 2007).

Thus, a number of astrophysical arguments point to possible field strengths in the $10^{16}$ G range in magnetar interiors and suggest a possibly relevant role of GW emission in the early evolution of these objects. The aim of this paper is to explore the physical consequences of this hypothesis and draw a state-of-the-art scenario, in light of all approximations and uncertainties. This paper is organized as follows. In Section 2, we introduce the model for the early spin-down of a magnetar subject to both magnetodipole and GW torques and study the main factors determining the GW emission efficiency of a newly formed magnetar. In Section 3, we calculate the expected energy loss through GW emission from newly born magnetars as a function of the relevant NS parameters and re-address the problem of signal detection already discussed by Stella et al. (2005). Finally, in Section 5 we address the problem of the thermal and magnetic evolution of NSs, extending previous treatments of field decay to field strengths of $\sim 10^{16}$ G.

## 2 MAGNETAR EARLY SPIN-DOWN: ELECTROMAGNETIC VERSUS GW EMISSION

A newly formed NS with a strong toroidal magnetic field has a prolonged deformation induced by the field ($\epsilon_B$; defined in Section 2.1.2) with symmetry axis along the magnetic axis. If the magnetic and spin axes are misaligned, with tilt angle $\chi$, the angular velocity vector ($\vec{\Omega}$) precesses (in the NS frame) around the fixed angular momentum vector ($\vec{L}$). (see also Cutler 2002)
studied the precession dynamics taking also into account the effects of the centrifugal deformation of the NS. They showed that the triaxial ellipsoid behaviour is, in this case, formally equivalent to that of a biaxial rigid body, as far as the precessional motion is concerned. If $I_0$ represents the moment of inertia of the spherical NS, the eigenvalues for the distorted NS are $I_1 = I_2 = I_0 - (1/3)\Delta I_d - (1/3)\Delta I_b$ and $I_3 = I_1 + \Delta I_b$, where $\Delta I_d$ represents the centrifugal deformation. Since the latter term affects all eigenvalues in the same way, it does not enter explicitly in the precession dynamics, which are determined only by the magnetic deformation (see Appendix A). Hence, the focus here and in the following will be on the magnetic deformation only.

The spin energy of a rotating spheroid is minimized, at a fixed angular momentum, when its moment of inertia is maximum. For a prolate figure, this is achieved when the symmetry axis is orthogonal to the spin axis. In the presence of internal dissipative processes, the magnetic axis of a prolate NS will thus be driven towards orthogonal rotation (Mestel & Takhar 1972; Jones 1976), which maximizes the efficiency of GW emission (Cutler 2002).

For arbitrary $\chi$, GWs will be emitted at both the spin frequency and its octave, with a total rate of energy emission (cf. Cutler & Jones 2001 and references therein)

$$E_{GW} = -\frac{2}{5} \frac{G(I_\perp^2)}{c^5} \omega^6 \sin^2 \chi (1 + 15 \sin^2 \chi).$$

(1)

An orthogonal rotator emits GWs only at twice its spin frequency, at the rate given by equation (1) with $\chi = \pi/2$

$$E_{GW}(\chi = \pi/2) = -\frac{32}{5} \frac{G(I_\perp^2)}{c^5} \omega^6.$$  

(2)

Given the generation mechanism for the superstrong internal field, the magnetic axis is expected to be just slightly tilted, initially, to the spin axis and, for small $\chi$, the GW luminosity is largely suppressed (equation 1). The orthogonalization process must thus be quick enough for strong GW emission to ensue promptly, in order for it to be quickly competitive with magnetodipole radiation.

Strictly speaking, the spin-down luminosity of a magnetic dipole rotating in vacuo depends only on the magnitude of the dipole component orthogonal to the spin axis. However, according to the standard pulsar model, an aligned rotator is expected to have a spin-down luminosity comparable to that of an orthogonal rotator. Studies on the structure of force-free NS magnetospheres suggest that this is indeed the case, within a factor of the order of unity (Contopoulos, Kazanas & Fendt 1999; Gruzinov 2006; Spitkovsky 2006). Based on these results, we assume a newly formed magnetar to have the magnetodipole luminosity of an orthogonal rotator. Further discussion of these issues is delayed to Section 4.

We neglect here the effects of strongly magnetized winds from newly formed magnetars spinning at $\sim 1$ ms period and focus our attention to external dipole fields $\lesssim 5 \times 10^{14}$ G. Although such winds can be extremely efficient in carrying away angular momentum (and spin energy) from the NS in just a few minutes, their efficiency is expected to be negligible for dipole fields weaker than $(6-7) \times 10^{14}$ G (Thompson et al. 2004; Bucciantini et al. 2006; Metzger et al. 2007).

To summarize, we describe the early spin evolution of a newly born magnetar as being driven by both magnetodipole and GW torques. Hence, as the tilt angle $\chi$ increases, a sufficiently large (prolate) deformation and rapid initial rotation can make GW emission dominate angular momentum and rotational energy losses. For a given spin-down torque, $\dot{\omega} = -K_\omega\omega^2$, the corresponding spin-down time-scale is $\tau_{sd} \equiv \omega/(2\dot{\omega})$. For convenience, in the following we express the relative strength of the two torques at birth in terms of the ratio:

$$\frac{\tau_{sd}}{\tau_{GW,\nu}} = \frac{K_{GW}\omega_0^2}{K_d} = \frac{\omega_0^2}{A} \equiv x,$$

(3)

where $A \equiv K_d/K_{GW}$; GW losses are dominant for $x > 1$.

The complete spin-down equation we adopt is thus

$$\dot{\omega} = -\frac{2}{3} \frac{\mu^2}{I} \omega^3 - \frac{E_{GW}}{I_\nu} \omega^3 = -K_d\omega^3 - K_{GW}\omega^5$$

(4)

where $E_{GW}$ is given by (1), or by (2), when $\chi \approx \pi/2$.

2.1 Damping of freebody precession and orthogonalization

We consider here in some detail the very first stages when the prolate NS is formed, its symmetry axis being just slightly tilted to the spin axis. We focus in particular on those processes that might affect the orientation of the symmetry axis and thus promote (or prevent) prompt and strong GW emission.

As stated above, freebody precession of the prolate spheroid is eventually damped by internal viscous torques, which redistribute angular momentum (with no loss) inside the NS so as to minimize the spin energy. Following the analysis of Cutler (2002), Stella et al. (2005) assumed that internal (viscous) dissipation of free precession occurs on a (very short) time-scale $\tau_{\text{int}} \approx 10^5 P_{\text{prec}}$ (Alpar & Sauls 1988), where $P_{\text{prec}} (<$ a few seconds) is the free precession period. This estimate results from the crust–core coupling caused by the interaction between superfluid neutron vortices and relativistic electrons in the NS core, where only electrons follow the instantaneous rotation of the crust. The application of this prescription to a newborn NS is subject to important caveats: neutron pairing in a $1P_2$ state in the core occurs at a temperature $T_{\text{cond}} < 2 \times 10^9$ K (TD96 and references therein; Page et al. 2004), and crust formation also occurs at a temperature approximately a few $\times 10^8$ K. The coupling mechanism studied by Alpar & Sauls (1988) does not apply when $T > T_{\text{cond}}$, since there is no superfluid and, likely, not even a proper crust. The NS is more like a self-gravitating, rapidly rotating, fully degenerate fluid mass. A proper account of its early cooling is thus of crucial importance in this context.

If direct Urca processes occur in the densest parts of the core, then cooling is extremely fast due to the very large neutrino luminosity. The NS temperature drops to $10^8$ K in a matter of minutes, as opposed to $\sim 1$ yr in the case of modified Urca cooling (cf. Page, Geppert & Weber 2006 and references therein). Therefore, if newly born magnetars cool through direct Urca processes, crust formation and transition to superfluidity in the core occur very quickly. The crust–core coupling mechanism described by Alpar & Sauls (1988) could thus operate soon after NS birth and lead to the very short orthogonalization time estimated above.

In the opposite limit, when NS cooling is driven by only the modified Urca reactions, the evolution is more complex. The temperature in this case evolves as (Owen et al. 1998 and references therein; Page et al. 2006)

$$T(t) \left(\frac{10^8}{T_c}\right) = \left[1 + \left(\frac{10^8}{T_c}\right)^{1/6}\right]^{-1/6}$$

(5)

where $\tau_c \approx 1$ yr and $T_c \approx 10^8$ K are the initial temperature of the NS at the end of the $\alpha - \Omega$ dynamo phase (TD96). As we are interested in the cooling at $T > T_{\text{cond}}$, no further neutrino-emitting
reactions are considered, such as those occurring at $T \approx T_{\text{end}}$ in the so-called ‘minimal cooling scenario’ (Page et al. 2004 and references therein). According to equation (5), the NS temperature will reach a value of $2 \times 10^9 \text{K}$ in about five days after formation. During this time, the crust–core coupling mechanism introduced by Alpar & Sauls (1988) cannot operate. In the absence of other viscous processes, the prolate spheroid will thus not be orthogonalized and GW emission will remain highly suppressed during those early days. Therefore, magnetodipole radiation by a $10^{35} \text{G}$ external field will carry away most of the initial spin energy, thus spoiling the possibility of significant GW emission also at later times.

However, the prolate NS is subject to at least two further (and competing) mechanisms that can in principle alter the orientation of its symmetry axis. First, GWs will be emitted even for a small initial tilt angle $\chi$, though at a small rate. The corresponding radiation reaction torque will cause the spin and magnetic axes to align, regardless of whether the spheroid is oblate or prolate (Cutler & Jones 2001). This process always acts so as to quench the GW emission efficiency.

On the other hand, freebody precession induces internal motions in the newly formed, fluid NS, that are required by the condition of hydrostatic equilibrium (Mestel & Takhar 1972; Jones 1976). Dissipation of these internal motions through bulk viscosity is potentially able to orthogonalize the symmetry axis of the spheroid (the magnetic field axis) relative to the angular momentum vector, thus increasing the efficiency of GW emission.

The relative strength of these two mechanisms will thus determine the early evolution of the angle $\chi$ and the possibility of newly formed magnetars to become efficient GW emitters. Below, we study these two processes in more detail, showing that orthogonalization through bulk viscous damping is expected to always prevail on radiation reaction, in the parameter range of interest to our work.

### 2.1.1 Equation of state and related quantities

For our aims, the equation of state (EOS) of NS matter enters through its role in determining the global structural properties of the NS, such as the mass–radius relation and the moment of inertia. We do not consider the detailed microphysics that determine the EOS; rather we adopt a phenomenological approach, parametrizing all results in terms of the NS mass and radius.

As shown by Lattimer & Prakash (2001), most NS EOSs are well approximated by a polytrope of index $n = 1$, with $P = k \rho^n$, as far as their global properties are concerned. Here, $k$ is determined by the stellar radius (see Appendix B). We also adopt the approximate formula for the NS moment of inertia given by Lattimer & Prakash (2001): $I \approx 0.35 M R^2 \approx 1.4 \times 10^{45} (M/1.4 M_\odot)(R/12 \text{km})^2 \text{g cm}^2$. Finally, we note that, as discussed in Lattimer & Prakash (2007), internal magnetic fields weaker than $\lesssim 10^{18} \text{G}$ are not expected to sizeably affect the microphysics that determine the EOS.

For the sake of completeness, we also report here the resulting relation between the measured $P$ and $I$ of a NS and the corresponding magnetic field strength as derived by the magnetodipole spin-down formula (see equation 4)

$$B_\delta \simeq 4.4 \times 10^{19} (P P)^{1/2} \left( \frac{M}{1.4 M_\odot} \right)^{1/2} \left( \frac{R}{12 \text{km}} \right)^{-2} \text{G}$$

$$\mu_\delta \simeq 3.8 \times 10^{37} (P P)^{1/2} \left( \frac{M}{1.4 M_\odot} \right)^{1/2} \left( \frac{R}{12 \text{km}} \right) \text{G cm}^3. \quad (6)$$

### 2.1.2 The magnetically induced distortion of a neutron star

One of the key parameters of interest is the deviation from spherical symmetry $(\epsilon_B)$ of the NS that is caused by the anisotropic pressure of the internal magnetic field. This determines the frequency of freebody precession and the GW luminosity of the rapidly spinning NS. In general, $\epsilon_B$ will be determined by the volume-integrated ratio of the magnetic to gravitational binding energy densities, times a numerical coefficient accounting for field geometry and NS structure. Setting the latter factor to unity, and assuming an approximately constant density in the NS core,

$$I_3 − I_1 \equiv \Delta I_9 \equiv \frac{15 \epsilon_B}{4} = \frac{15}{4} \epsilon_B \int \frac{B^2}{8\pi} dV \quad (7)$$

as determined from general arguments based on the virial theorem (Cutler 2002 and references therein). In the above definition, the integral is extended to the whole NS volume where the magnetic field is present. We are assuming the internal field to be mostly toroidal, the poloidal component being sufficiently small to be energetically negligible, as envisaged in the millisecond magnetar formation scenario. $E_\chi$ in equation (7) is the gravitational binding energy of the NS that, for a polytrope of index $n = 1$, is $E_\chi = (3/4) GM^2/R$.

The value of $\epsilon_B$ can thus be approximated as

$$\epsilon_B \approx −1.15 \times 10^{-3} \left( \frac{E_B}{10^{50} \text{erg}} \right) \left( \frac{R}{12 \text{km}} \right) \left( \frac{M}{1.4 M_\odot} \right)^{-2} \quad (8)$$

that is negative for a prolate shape. The volume-averaged strength of the internal (toroidal) magnetic field is

$$B_t \simeq 0.93 \left( \frac{E_B}{10^{50} \text{erg}} \right)^{1/2} \left( \frac{R}{12 \text{km}} \right)^{-3/2}. \quad (9)$$

Uncertainties in the exact value of $\epsilon_B$ as a function of the (unknown) internal field distribution can be parametrized by adding a further multiplicative factor $\eta$ in the right-hand side of equation (7) where, in general, $\eta > 1$ is expected. We neglect this factor for clarity, but stress that the quantity $E_B$ used throughout can more generally be substituted with $\eta E_B$. We explicitly assume $\eta = 1$ from here onwards.

### 2.1.3 Gravitational radiation reaction

The effect of gravitational radiation reaction on damping of freebody precession was investigated by Cutler & Jones (2001), who showed that GW emission always drives the tilt angle $\chi$ to zero, independent of whether the NS is oblate or prolate. In the limit of

$2$ The exact value of $\epsilon_B$ is uncertain as it depends on the magnetic field configuration and EOS. Using a general relativistic treatment of the NS structure, Bonazzola & Gourgoulhon (1996) studied the way in which the numerical factor in equation (7) changes for different configurations of the internal magnetic field. These authors showed that this factor can be higher than $15/4$, or even much higher, for specific magnetic field geometries (e.g. the case of an internal field that is mainly concentrated in an outer shell of the NS core, or the case of an intermittent field distribution rather than a uniform one). Haskell et al. (2008) studied the same problem allowing for different EOSs and combinations of the toroidal and poloidal components of the internal field. Their results confirm that the magnetically induced ellipticity can be in general somewhat larger, or significantly larger in particular cases, than the estimate given by equations (7) and (8). Overall, for a given magnetic energy, larger ellipticities than given by equation (7) are a possibility worth further investigations.
small $\chi$, these authors express the freebody precession damping time-scale ($\tau_d$) as

$$\frac{\sin \chi}{d(\sin \chi)/dt} \approx \frac{5e^5}{2G2^2 \left( I_1 \right)} \left( \frac{E_B}{10^{38} \text{erg}} \right)^{-2} \left( \frac{P}{\text{ms}} \right)^4 \left( \frac{R}{12 \text{ km}} \right)^{-4} d. \quad (10)$$

The importance of this effect on the early evolution of a newly formed magnetar can be determined by comparing this time-scale to the bulk viscosity dissipation time-scale and to the magnetodipole spin-down time-scale.

### 2.1.4 Bulk viscosity in newly born NSs

As shown by Mestel & Takhar (1972), a field of internal motions is excited in order to maintain hydrostatic equilibrium everywhere within a fluid star undergoing freebody precession. These motions are periodic, with the same frequency as freebody precession and amplitude proportional to the (non-spherical) centrifugal deformation of the star ($\rho_0 \propto \Omega^2$) and tilt angle $\chi$ of the magnetic axis. In this section, we discuss the damping of such motions by bulk viscosity and the associated growth of the tilt angle $\chi$, within a non-superfluid NS made only of neutrons, protons and electrons (NPE matter).

The assumption of pure NPE matter corresponds to the least favourable case for damping through bulk viscosity and likely not the most realistic. At the highest densities of NS cores, muons and possibly hyperons ($\Lambda, \Sigma^-$) are expected to appear, significantly increasing the bulk viscosity coefficient (e.g. Jones 1976; Lindblom & Owen 2002 and references therein). In this respect, our calculations are conservative. Detailed models should lead to significantly shorter damping times for the excited modes.

Fluid bulk viscosity is a consequence of the finite time it takes for the fluid to react to changes in one of its thermodynamic parameters, adjusting the others to their new equilibrium values. In hot NS matter, displacement of a fluid parcel from the equilibrium position causes departure from chemical equilibrium between particle species: bulk viscosity is thus determined by the activation of $\beta$-reactions trying to restore chemical equilibrium. A general expression for the bulk viscosity coefficient in NS matter ($\zeta$) has been derived by Lindblom & Owen (2002)

$$\text{Re} \zeta = \frac{n \tau}{(\partial \rho/\partial x)n} \frac{dx/dn}{1 + (\omega \tau)^2} \geq \frac{n (\partial \rho/\partial x)n}{\omega \tau^2} dx/dn, \quad (11)$$

where $n$ is the particle density, $\rho$ is the pressure (whose derivative is calculated at constant proton fraction, $x = n_p/n_e \approx n_p/n$) and $\chi$ is the equilibrium value of $x$. The quantities in the denominator are the mode frequency ($\omega_{\text{pre}} \sim e_\rho \Omega_{\text{spin}}$) and $\beta$-reaction time-scale ($\tau_\beta$). Defining the characteristic damping time of the excited motions as $\tau_d \equiv 2E_{\text{pre}}/|E|$ one can express it as (cf. Owen et al. 1998)

$$\tau_d = \left( \frac{\int \zeta |\nabla \delta \rho|^2 dV}{2E_{\text{pre}}} \right)^{-1} = \frac{1}{\omega_{\text{pre}}} \left( \frac{\int \zeta (\delta \rho)^2 dV}{2E_{\text{pre}}} \right)^{-1}. \quad (12)$$

where $\delta \mathbf{v}$ is the velocity perturbation associated to the mode. In the last step, we have used the relation $\nabla \cdot \delta \mathbf{v} = i \delta \zeta (\Delta \rho/n)$ (Lindblom & Owen 2002) and substituted the Eulerian perturbation, $\delta \rho$, to the Lagrangian one, $\Delta \rho$. A word of caution is needed here. The analysis by Mestel & Takhar (1972) gives $\nabla \cdot \delta \mathbf{v} = 0$, i.e. $\Delta \rho = 0$ and $\delta \rho = -\zeta \cdot \nabla \rho$, a condition that would imply the absence of bulk viscosity. However, this result is a direct consequence of having considered strictly adiabatic fluid motions\(^3\) within a radiative (sub-adiabatic) layer, in a first-order perturbative analysis. As opposed to this, a polytropic NS would have an almost adiabatic gradient, in which case the calculations by Mestel & Takhar would leave $\nabla \cdot \xi$ totally unconstrained, revealing the need for a higher-order perturbation analysis. Indeed, these authors discuss the (non-)applicability of their conclusion to a convective (almost adiabatic) zone, at the end of their section 3. They discuss a number of approximations in their analysis that may well break in the case of a very rapid rotator (such as a millisecond magnetar), thus introducing non-negligible higher-order terms in the perturbations. In these cases, the internal dynamics of the oblique rotator become much more complicated and a detailed analysis is beyond our scope here.

In general, however, motions of the same order of magnitude as the one discussed by Mestel & Takhar (1972) are always expected to occur, whose rate of expansion ($\nabla \cdot \xi$) must be calculated explicitly. Here, in analogy to Reisenegger & Goldreich (1992) and to several treatments of bulk viscosity for $r$ modes (Lindblom, Owen & Morsink 1998; Owen et al. 1998), we assume that $\Delta \rho$ would be of the same order of magnitude of $\delta \rho$, leaving more detailed analyses to future work. To stress the importance of this aspect, we note that Lindblom, Mendell & Owen (1999) carried out a second-order analysis for $r$-mode damping through bulk viscosity and derived an order of magnitude longer dissipation time-scale than those based on the assumption $\Delta \rho \sim \delta \rho$.

We finally note that $\delta \mathbf{v} \sim \epsilon_B R R \sim 10^{-7} \text{ cm s}^{-1}$, while the Alfvén velocity is $v_A = B/\sqrt{4\pi \rho} \sim (10^7-10^8) \text{ cm s}^{-1}$ for the parameters of interest. Alfvén waves are thus very efficient in maintaining rigid rotation of the fluid star, despite the perturbation in principle introduced by $\xi$.

The calculation of the perturbation amplitude ($\delta \rho/\rho$) is detailed and discussed in Appendix B. We report here our result for the damping time-scale of freebody precession according to equation (12):

$$\tau_d \simeq 13.5 \frac{\cos^2 \chi}{\sin \chi} \chi \left( \frac{E_B}{10^{38} \text{erg}} \right) \left( \frac{P}{\text{ms}} \right)^2 \left( \frac{M}{1.4 M_\odot} \right)^{-1} \left( \frac{T}{10^{10} \text{ K}} \right)^{-6} s. \quad (13)$$

a fairly short time even for small initial values of the tilt angle ($\chi_i$).

Given $\tau_d$, we can eventually calculate the growth time of the tilt angle ($\tau_\chi$). As shown in Appendix A, damping of freebody precession reduces the NS spin energy, at a constant angular momentum, by changing the tilt angle $\chi$. From equation (A.9) we get

$$\tau_\chi = \frac{\cos \chi}{\sin \chi} \left( \cos^2 \chi - \sin \chi \right) \left( \frac{E_B}{10^{38} \text{erg}} \right) \left( \frac{P}{\text{ms}} \right) \left( \frac{\sin \chi}{\sin^2 \chi} \right) = \frac{\cos^2 \chi}{\sin \chi} \tau_\chi'. \quad (14)$$

Here, $\tau_\chi'$ is the time-scale to be compared to $\tau_d$ (equation 10) in order to determine the relative importance of bulk viscosity and radiation reaction in the evolution of $\chi$. In the limit of small tilt angle, for which $\cos \chi \simeq 1$, we have

$$\frac{\tau_\chi'}{\tau_d} \approx 10^{-5} \left( \frac{E_B}{10^{38} \text{erg}} \right)^3 \left( \frac{P}{\text{ms}} \right)^{-2} \left( \frac{T}{10^{10} \text{ K}} \right)^{-6} \left( \frac{R}{12 \text{ km}} \right)^{-4}. \quad (15)$$

From this, we derive the condition for bulk viscosity to largely prevail on gravitational radiation reaction, so that orthogonalization is essentially unaffected by radiation reaction. For definiteness, we

\(^3\) In fact, bulk viscosity is a result of deviations from strict adiabaticity of perturbations.
require $\tau_d^f < 0.1 \tau_i^f$, a condition that translates to
\begin{equation}
E_B \leq 2.3 \times 10^{51} \left( \frac{T}{10^{10} \text{ K}} \right)^2 \left( \frac{P}{\text{ms}} \right)^{2/3} \left( \frac{12 \text{ km}}{R} \right)^{4/3} \text{ erg} \tag{16}
\end{equation}
or $B < 9 \times 10^{16} (T/10^{10} \text{ K}) (P_{\text{ms}})^{1/3} \text{ G}.$

Soon after formation, the temperature $T$ decreases on a very short time-scale, not much different from $\tau_d^f$ itself, even for the slow cooling given by equation (5). The angle $\chi$ and $T$ thus evolve on comparable time-scales, and just considering the damping time-scale at a given temperature is not appropriate. Rather, the coupled evolution of $\chi$ and $T$ must be solved in order to derive a reliable estimate of the time it takes for $\chi$ to grow to large values.

Inserting the cooling history (5) in (13) or (14), the resulting equation for $\chi$ can be solved with initial conditions $\chi = \chi_i$ and $T_i = 10^{10} \text{ K}$. Since $\tau_d(t) = N T_{\text{em}}^i (t) \cos^2 \chi_i / \sin^2 \chi_i (1 + 3 \cos^2 \chi_i)$, the expression $T_{\text{em}}^i (t) = (t/30 + 1)$ gives
\begin{equation}
\frac{\cot \chi}{1 + 3 \cos^2 \chi} d\chi = \frac{dt}{N \left( \frac{t}{30} + 1 \right)^{240/N}},
\end{equation}
whose solution is
\begin{equation}
\frac{\sin^2 \chi}{1 + 3 \cos^2 \chi} = \frac{\sin^2 \chi_i}{1 + 3 \cos^2 \chi_i} \left( \frac{t}{30} + 1 \right)^{1440/N},
\end{equation}
From the above, we can obtain the time (in seconds) or the temperature (in units of $10^{10} \text{ K}$) at which a sufficiently large value of the angle $\chi$ is reached starting from a given, small initial tilt angle $\chi_i$.

The requirement that the damping time of free precession be much shorter than the initial spin-down time-scale of the NS allows two key constraints to be met jointly and our scenario to maintain full self-consistence. First, damping of free precession will be well described in terms of the approximation of constant angular momentum. Secondly, efficient GW emission will ensue quick enough for the NS initial spin energy to be still fully available.

The initial time-scale for electromagnetic spin-down is $\tau_{\text{em}}^i \approx 1.1 P_{\text{ms}}^2 R_{14}^{-2} \text{ d}$, and in a time $\sim 0.1 \tau_{\text{em}}^i$ less than 10 per cent of the initial spin energy is lost to magnetic dipole radiation. Hence, we consider this as the longest time over which orthogonalization must take place for our scenario to apply. Therefore, given $\chi_i = 1^\circ$, the angle $\chi$ will grow to a sufficiently large value — say, $\chi = 60^\circ$ — in a time shorter than 0.1 $\tau_{\text{em}}^i$ if
\begin{equation}
\frac{E_B}{10^{50} \text{ erg}} < 2.1 \frac{M_4}{P_{\text{ms}}^3} \left[ \ln \left( 320 M_4 R_{12}^2 \frac{P_{\text{ms}}^2}{B_{14}^2} + 1 \right) \right]
\approx 4.2 \frac{M_4}{P_{\text{ms}}^3} \left[ 3 + \frac{P_{\text{ms}}}{B_{14}^2} + \frac{M_4}{R_{12}^3} \right]. \tag{19}
\end{equation}
Note that the numerical coefficient in the last step is $\sim 5$ if $\chi_i = 2^\circ$, and $\sim 2.8$ if $\chi_i = 0.1^\circ$.

Equation (19) is represented in Fig. 1 by two curves corresponding to initial spin periods of 0.97 and 2.58 ms, that approximately bracket the relevant range of spin periods for newly formed magnetars. Each curve is plotted for two different values of the initial tilt angle, $\chi_i = 1^\circ$ and $2^\circ$ (see caption for further details).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The region of parameter space, in the external dipole field strength ($B_d$) versus internal magnetic energy ($E_B$), where the time-scale for the angle $\chi$ to grow to a large value (60°) is less than one tenth of the initial spin-down time-scale through magnetodipole radiation. The favourable region lies on the left-hand side of the corresponding limiting curve. For each value of the initial spin period, two curves are plotted for two different values of the initial tilt angle, $\chi_i = 1^\circ$ (left-hand side) and $2^\circ$ (right-hand side). Note that further decreasing $\chi_i$ from 1° to 0.1° shifts the curves to the left-hand side by a slightly larger amount than the decrease of $\chi_i$ from 2° to 1° (cf. equation 19). This would still leave ample room in parameter space for the fastest spinning magnetars, while it would drastically reduce the available parameter space for magnetars with $P_i \geq 2.5$ ms.}
\end{figure}

The constraint shown in Fig. 1 will be used, together with independent constraints derived in later sections, to identify the region in parameter space where GW emission efficiency from newly formed magnetars is optimized. We again stress here that all our calculations were carried out by assuming pure NPE matter, a conservative assumption that minimizes the efficiency of bulk viscosity. Finally, in Appendix B we calculate the centrifugal distortion of the density profile in the slow-rotation limit, which is also likely to underestimate the deformation of a millisecond magnetar. This translates into an underestimate of $\delta \rho$ in equation (12) and, thus, of the efficiency of bulk viscosity.

We conclude that, despite having chosen a `worst case approximation’, damping of freebody precession through bulk viscosity in a newly formed, rapidly spinning and strongly magnetized NS can be very efficient in the parameter range considered here. Therefore, strong GW emission from an almost orthogonal, rapidly rotating NS is likely to ensue quickly as a consequence of a strong toroidal magnetic field and of the efficient dissipation of its freebody precession energy. The implications of this are explored further in the next section.

### 3 Amplitude and Detectability of the Emittted Signal

Stella et al. (2005) calculated the expected signal-to-noise ratio (S/N) that a putative GW signal from a newly born magnetar would have, for an optimal (matched-filter) detection, adopting the broadband design sensitivity of Advanced LIGO. At frequencies between 0.5 and 2 kHz, this is well approximated by
\begin{equation}
S_N(f) \approx S_0 f^2, \tag{20}
\end{equation}
where $S_0(f)$ represents the one-sided spectral noise distribution of the detector, $f = 2\omega$ the GW signal frequency and
\[ S_0 \approx 2.1 \times 10^{-53} \text{ Hz}^{-1} \] (Owen & Lindblom 2002; Cutler 2002 and references therein). Note that the designed sensitivity for Advanced Virgo is very similar, in this range of frequencies (Losurdo 2007), so that our calculations hold essentially for both detectors. Frequency, \( f = \omega / \pi \) for an orthogonal prolate rotator.

We re-address here this point to better qualify the role of the NS ellipticity in the detectability of the signal. We also correct a (small) numerical error in the calculated S/N curves in fig. 1 of Stella et al. (2005), whose conclusions maintain their general validity. The S/N of an optimal signal search is defined as

\[ S/N = 2 \left( \frac{\tilde{h}(f)^2}{S_0(f)} \right)^{1/2}. \] (21)

Here \( \tilde{h}(f) \) is the Fourier transform of the instantaneous signal strain \( h[f(t)] \) that, in the stationary phase approximation, is expressed as (cf. Owen & Lindblom 2002 and references therein)

\[ \tilde{h}(f) = \frac{1}{2} h^2 [f(t)] \left| \frac{df}{dt} \right|^{-1}, \] (22)

where the time derivative of \( f \) is obtained from equation (4), since \( f = \omega / \pi \). We adopt the expression for the strain amplitude averaged over source orientation — given by7 Ushomirsky, Cutler & Bildsten (2000):

\[ h_d(f) = \frac{16}{5} \left( \frac{\pi^3}{3} \right)^{1/2} \frac{GEM}{Dc^2} f^2, \] (23)

where \( D \) is the source distance. Further averaging over the detector antenna pattern, equation (21) gives the optimal S/N ratio:

\[ S/N = \sqrt{\frac{2}{5}} \left( \frac{\tilde{h}(f)^2}{S_0(f)} \right)^{1/2}, \]

\[ = 4 \sqrt{\frac{\pi GI}{6c^3 DS_b^{1/2}}} \left( \frac{K_{GW}}{K_d} \right)^{1/2} \left[ 2 \ln \frac{f_i}{f_f} - \ln \frac{a + f_i^2}{a + f_f^2} \right]^{1/2}, \] (24)

where we have set \( a = K_d/(\pi^2 K_{GW}) = A/\pi^2 \). Substituting the numerical values:

\[ S/N \approx 6 \left( \frac{E_B}{10^{50} \text{ erg}} \right) \left( \frac{B_d}{10^{14} \text{ G}} \right)^{-1} \left( \frac{R}{12 \text{ km}} \right) \left( \frac{M}{1.4 \text{ M}_\odot} \right)^{-1/2} \times \left( \frac{D}{20 \text{ Mpc}} \right)^{-1} \left[ 2 \ln \frac{f_i}{f_f} - \ln \frac{a + f_i^2}{a + f_f^2} \right]^{1/2}. \] (25)

In Fig. 2, we show the curves of constant S/N in the \( B_d \) versus \( E_B \) for selected values of the initial spin, as derived from equation (25). More details are given in the caption. According to equation (24), the maximum S/N is obtained in the limit \( a \to 0 \), which depends only on the initial spin energy of the NS and not on its oblateness. Clearly, this value, \( [S/N]_{\text{max}} \), is attained as the magnetodipole spin-down torque disappears, so that all of the initial spin energy of the NS is lost to GWs. This can also be seen directly by substituting the expression for the pure GW-driven spin-down in equation (24), which gives

\[ [S/N]_{\text{max}} \approx 4.5 \left( \frac{D}{20 \text{ Mpc}} \right)^{-1} \left( \frac{R}{12 \text{ km}} \right) \left( \frac{M}{1.4 \text{ M}_\odot} \right)^{1/2} \times \left[ \frac{f_i}{1 \text{ kHz}} \right]^{-2} \left[ \frac{f_f}{1 \text{ kHz}} \right]^{2} \right]^{1/2}. \] (26)

### 3.1 Quantifying the gravitational and electromagnetic energy output

Thus far, we have discussed the conditions under which the hypothesis that newly born magnetars be detectable sources of GWs with next-generation detectors can be true. X-ray observations of SNRs around magnetar candidates in the Galaxy give us clues on the actual viability of this hypothesis (cf. Section 1). In this section, we show that our scenario is indeed fairly consistent with such observations. We find that, if magnetars at birth were detectable sources of GWs from Virgo cluster distance, then SNRs around them would...

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7 Comparing with the ‘optimally oriented’ strain amplitude \( h_0 \) given by Abbot et al. (2007), we obtain the relation \( h_d = 4/(5 \sqrt{3} \pi) h_0 \).
likely show no significant excess of energy injection with respect to ‘ordinary’ SNRs.\footnote{Assuming GWs do not appreciably transfer energy to the supernova ejecta.}

We begin calculating the total integrated energy emitted via GWs by a newly born magnetar as

\[ \Delta E_{GW} = - \int_0^{\infty} E_{GW} dt = \int_0^{\infty} \frac{E_{GW}}{\delta} d\omega. \]  

(27)

For the spin-down model in equation (4), the above integral gives

\[ \Delta E_{GW} = I \int_0^{\infty} \frac{\omega^3}{\omega^2 + A} d\omega = I \left[ \frac{\omega^2}{2} - \frac{A}{2} \ln(\omega^2 + A) \right]_0^{\infty} \]

\[ = E_{\text{spin,}i} \left[ 1 - \frac{A}{\omega_i^2} \ln \left( 1 + \frac{\omega_i^2}{A} \right) \right]. \]  

(28)

Equation (28) expresses \( \Delta E_{GW} \) as a fraction (\( \delta \)) of the initial spin energy of the NS. The remaining energy \( \Delta E_{em} = E_{\text{spin,}i}(1 - \delta) \) is the maximum that can be transferred to the SN ejecta. Next we define the quantity \( \Delta E_{inj} = \beta \Delta E_{em} \) as the amount of the energy that is effectively transferred to the ejecta. In the absence of additional competing torques and/or a physical prescription for the transfer mechanism, we will assume it to be perfectly efficient (\( \beta = 1 \)), although leaving the explicit dependence on \( \beta \) in the calculations that follow.

Recalling the definition of \( x \) as the ratio of gravitational to magnetic dipole torque at birth (equation 3) and using equation (28) we then get

\[ \frac{\ln(1 + x)}{x} = \frac{\Delta E_{em}}{E_{\text{spin,}i}} = \frac{1}{\beta} \frac{\Delta E_{inj}}{E_{\text{spin,}i}} \leq \frac{E_{\text{inj}}^{\text{SNR}}}{E_{\text{spin,}i}}. \]  

(29)

where \( E_{\text{inj}}^{\text{SNR}} \) is the amount of energy transferred to the ejecta as determined from observations (Vink & Kuiper 2006). Observations provide an upper bound to this quantity, from which the inequality in the last step in equation (29) obtains. We derive constraints on magnetar parameters at birth by using the upper bound determined by Vink & Kuiper (2006) discussed in this paper and our GW emission scenario.

First fix a value for \( x \): equation (29) gives the maximum value of the initial spin energy or, equivalently, \( (\omega_i^2)_{\text{max}} \) compatible with the upper limit on \( E_{\text{inj}}^{\text{SNR}} \). Accordingly, the maximum value of \( A = (\omega_i^2)_{\text{max}}/x \) is constrained (cf. equation 3):

\[ A_{\text{max}} = \frac{\omega_i^2}{x} = \frac{2E_{\text{inj}}^{\text{SNR}}}{\beta I \ln(1 + x)}. \]  

(30)

Inserting equation (30) in the definition of \( A = K_d/K_{GW} \) (cf. Section 2), we get the following condition, for the electromagnetic output of a newly born, millisecond spinning magnetar should be always \( \lesssim 10^{51} \text{erg} \):

\[ E_B \geq \sqrt{\frac{G}{117.6 A_{\text{max}}^{1/2}} \frac{\text{M}_{\odot} c M^{1/2} \beta R B_3}{12 \text{km}}} \times 1.6 \times 10^{50} \sqrt{\beta I \ln(1 + x)} \left( \frac{B_3}{10^{14} \text{G}} \right) \left( \frac{M}{1.4 \text{M}_{\odot}} \right)^{3/2} \]

\[ \times \left( \frac{R}{12 \text{km}} \right) \left( \frac{E_{\text{inj}}^{\text{SNR}}}{10^{51} \text{erg}} \right)^{-1/2}. \]  

(31)

Therefore, given \( x \) one obtains \( \omega_i \) and, eventually, the appropriate curve in the \( B_3 \) versus \( E_B \) plane. Results are shown in Fig. 3, where curves in the \( B_3 \) versus \( E_B \) plane have been drawn for four different values of \( x = 10, 22, 100 \) and 150, corresponding to the initial spins indicated in each plot. Curves of given S/N from Fig. 2 are also plotted together with curves defining the region where the orthogonality time-scale is sufficiently fast (see Fig. 1).

We conclude that there exists a wide region of the parameter space where all constraints for efficient GW emission from newly born, millisecond spinning magnetars are met jointly.

4 ELECTROMAGNETIC SPIN-DOWN AT BIRTH

The ideal magnetodipole spin-down formula (cf. equation 4) implies that a sizeable fraction of the initial spin energy of a millisecond spinning magnetar would be lost to electromagnetic emission, for \( B_3 \geq 3 \times 10^{14} \text{G} \).

SGRs, and two out of seven AXPs, have spin periods and period derivatives leading to estimated dipole fields of a few times \( 10^{14} \text{G} \), up to \( \sim 10^{15} \) (cf. Woods & Thompson 2006 and references therein), some of which are apparently out of the optimal range for GW emission at birth, if they maintained their dipole field since then. Although relation (4) is certainly true to a good degree of approximation, the ideal magnetodipole formula does not appear to hold exactly in the few isolated NSs for which sufficiently accurate timing measurements exist.

Observations show that the spin-down of pulsars, assumed to be wholly due to an electromagnetic torque, might be a somewhat weaker function of the spin frequency than expected in the ideal case. Indeed, in the few pulsars where the second time derivative of the spin frequency (\( \ddot{\nu} \)) could be measured, a ‘braking index’ \( n = (\ddot{\nu}/\dot{\nu})^2 < 3 \) is typically derived. In particular, the Crab pulsar has a measured \( n = 2.5 \) (Lyne, Pritchard & Smith 1988), and PSR J1119–6127 has \( n = 2.9 \) (Camilo et al. 2000). Livingstone et al. (2007) have recently measured braking indices of three relatively young X-ray pulsars. Again, all results give \( n < 3 \) (2.14 for PSR B0540–69, 2.84 for PSR B1509–58 and 2.65 for PSR J1846–0258). The Vela pulsar has the largest known deviation from the ideal case, with \( n = 1.4 \) (Lyne et al. 1996), although subtraction of the frequent glitches from its spin history presents difficulties and that result should be taken with some caution.

Similar (or even greater) uncertainties are associated to the putative \( n \approx 6 \) determined by Marshall et al. (2004) for PSR J0537–6910 (the ‘Big Glitcher’). This source is found to exhibit very frequent glitches; post-glitch ‘recoveries’ are known to significantly affect the second time derivative of the spin frequency. For this reason, the measured values of \( \ddot{\nu} \) from such a frequent glitcher are probably unsuitable to determine reliably the ‘true’ secular spin-down trend. Middlelich et al. (2006) discuss in detail several aspects concerning the timing properties of this source, arguing for a much smaller value of \( n \).

The examples of \( 10^{13}–10^{14} \) yr old pulsars with braking indices lower than 3 suggest considering the same possibility for magnetar candidates, which indeed have similar ages and similarly high glitch activity (Dall’Osso et al. 2003; Israel et al. 2007; Dib, Kaspi & Gavriil 2008). No measurements of \( n \) have yet been obtained for AXPs and SGRs. It would be interesting to know from the data whether these sources share this same spin-down property of other isolated NSs.

Several mechanisms able to produce a braking index smaller than 3 have been proposed in the literature, since the early measurements of \( n \) in the Crab pulsar. Blandford & Romani (1988) discuss in general the effects of evolving the parameters of the magnetodipole.
torque. They identify a secular increase in the surface magnetic field, at least over the first $10^5-10^6$ yr, as a most promising explanation for $n < 3$ in young NSs. Amplification of crustal fields through a thermomagnetic battery over this early phase had been previously discussed (Blandford, Applegate & Hernquist 1983 and references therein), which could find application also in this context.

Alternatively, several suggestions have been made for a secular increase of the tilt angle between the magnetic dipole and the spin axis, based on a wide variety of physical mechanisms (Goldreich 1970; Ruderman 1991; Link, Epstein & Baym 1992; Ruderman, Zhu & Chen 1998). More recently, the possibility that the zone of closed field lines in the magnetosphere (the corotating region) does not reach the light cylinder has been widely discussed, both in the general context of studies of pulsar magnetospheres (Contopoulos et al. 1999; Contopoulos & Spitkovsky 2006; Gruzinov 2006; Spitkovsky 2006) and in relation to the magnetospheric structure of newly born magnetars (Bucciantini et al. 2006; Metzger et al. 2007).

Simulations by Spitkovsky (2006) have shown that the closed field line region reaches the light cylinder in a matter of one (at most) rotation period, as a model NS is set into rotation. However, it is not obvious whether the region of closed field lines can subsequently track the expansion of the light cylinder as the NS spins down or, rather, lags behind it at an increasing distance, which would naturally lead to $n < 3$ (Contopoulos & Spitkovsky 2006 and references therein).

Without indicating any particular mechanism, we discuss the consequences that an electromagnetic braking index $n < 3$ would have, at a purely phenomenological level.

Considering a generic model for electromagnetic spin-down with braking index $n$, one has

$$\dot{\omega}^{(0)} = -K_4^{(0)} \dot{\omega}^n,$$  \hspace{1cm} (32)

where the constant $K_4^{(0)} = (\dot{\omega}_0/\dot{\omega}_0^0)$ and the subscript '0' refer to present-day values of the parameters. According to equation (32), given measured timing parameters ($\dot{\omega}$, $\dot{\omega}$) of a source the expression for its spin-down at birth in the cases $n \neq 3$ and $n = 3$ is related through

$$\dot{\omega}_i = -K_4^{(0)} \dot{\omega}_i^0 = -K_4^{(3)} \dot{\omega}_0^3 \left( \frac{\dot{\omega}_i}{\dot{\omega}_0} \right)^n = \dot{\omega}_i^{(3)} \left( \frac{\dot{\omega}_i}{\dot{\omega}_0} \right)^{n-3},$$  \hspace{1cm} (33)

where the subscript 'i' indicates quantities at birth.

Figure 3. Curves defining the region of parameter space – in the $B_3$ versus $E_3$ plane – where all constraints for efficient emission and detection of GWs are fulfilled, for four initial spin periods. The region on the left-hand side of (and below) the solid curve labelled $\tau < 0.1 \tau_{\text{em}}$ is where the tilt angle $\chi$ becomes $\geq 60^\circ$ in less than one fifth of the initial magnetodipole spin-down time. The region on the right-hand side of (and below) the straight, solid curve is where the energy emitted electromagnetically by the newly formed magnetar is $\leq 10^{31}$ erg. It is obtained from equation (31) with four different values of $x = 150, 100, 22, 10$ from top to bottom and right- to left-hand side, corresponding to the initial spins indicated in the figures. The region on the right-hand side of the dashed (or dot–dashed) curves is where a match-filtered search with Advanced LIGO/Virgo would give a $S/N$ greater than indicated in the figures, with the source at the distance of the Virgo cluster (20 Mpc).
Note that the quantity in parentheses is smaller than unity for $n < 3$. It is therefore natural to ask what value of $n$ would be required in magnetar candidates (AXPs/SGRs), given their measured $\omega_i$ and $\dot{\omega}_i$, for their spin-down at birth to have been dominated by GW emission rather than by the magnetic dipole radiation.

In order to answer this, we define the maximum allowed strength of the electromagnetic spin-down at birth ($\dot{\omega}_i(\text{max})$). Given the results of the previous section, this maximum value will equal the ideal magnetodipole formula ($n = 3$) would give for $B_d = B_d(\text{max}) = 2 \times 10^{14} \text{ G}$ (a value that guarantees strong GW emission at birth; cf. Section 3.1). Therefore, we write $\dot{\omega}_i(\text{max}) = K_{d,\text{max}} \omega_i$ and the last step of equation (33) must be smaller than $\dot{\omega}_i(\text{max})$, which gives

$$K_{d}^{(3)} \left( \frac{\omega_i}{\omega_0} \right)^{n-3} \leq K_{d}^{(3)}(\text{max}).$$

From this, since the ratio of the torque functions corresponds to the ratio of the magnetic dipole fields:

$$(3 - n) \log \left( \frac{\omega_i}{\omega_0} \right) \geq 2 \log \left( \frac{B_d}{B_d(\text{max})} \right)$$

or

$$n \leq 3 - \frac{2}{\log B_d / B_d(\text{max})} \left( \frac{\log P_d - \log P_{ms}}{3} \right).$$

Among SGRs and AXPs, SGR 1806–20 has the strongest inferred dipole field ($\sim 1.1 \times 10^{15} \text{ G}$ with $R = 12 \text{ km}$ and $M = 1.4 \text{ M}_{\odot}$; cf. Woods & Thompson 2006), thus requiring the largest deviation from $n = 3$ in the hypothesis discussed here. Even assuming a (relatively) slow initial spin period for this source, $P = 3 \text{ ms}$, equation (35) gives $n \leq 2.6$, a wholly plausible value compared to other isolated NSs. Note that the constraint is slightly weaker for other SGRs and/or considering a spin period at birth shorter than 3 ms. For AXPs, the limit on $n$ ranges from 2.7 to 2.85, for an (unfavourable) initial spin of 3 ms. This speculative argument would require direct measurements of braking indices in magnetar candidates. However, our aim here was to emphasize the dependence (strong, in some cases) of the calculations of previous sections on a number of poorly constrained physical parameters, and the importance of further studies on all of the above aspects.

5 THE DECAY OF CORE FIELDS IN THE 10^{16} G RANGE

The secular evolution (and dissipation) of the magnetic field in NS cores was studied in detail by Goldreich & Reisenegger (1992) (hereafter GR92) and their analysis was extended by TD96 to the specific case of magnetar fields in the $10^{16}$ G range. In magnetars, large-scale field instabilities leading to fast dissipation events were studied in detail as well (TD95; TD01; Lyutikov 2003), in order to interpret the powerful bursts and flares of SGRs.

In GR92, three separate processes for secular field evolution were identified; two of which, ohmic dissipation and ambipolar diffusion, are dissipative while the third, the Hall drift, conserves magnetic energy. In particular, ambipolar diffusion was found to be more sensitive to the field intensity (GR92) which in fact implies that, while this process is not very important in normal NSs, it is the main mechanism for direct field decay in magnetars (TD96).

The Hall drift can indirectly affect the evolution and dissipation of magnetic fields in NS interiors, on longer time-scales than those characteristic of ambipolar diffusion (TD96; Arras, Cumming & Thompson 2004). As suggested in GR92 (and recently studied in detail by Cumming, Arras & Zweibel 2004), excited Hall modes of field diffusion could decay (or cascade) to shorter wavelengths, that are subject to enhanced ohmic dissipation. This has particular relevance for accelerating field decay in a magnetar’s crust, as recent studies pointed out (Pons et al. 2007; Pons & Geppert 2007). Furthermore, the Hall diffusion can drive – yet conserving the total energy – an initially stable magnetohydrodynamics configuration close to a new equilibrium configuration with smaller total energy. A point can be reached where the field suddenly relaxes to the new equilibrium, if (sufficiently fast) fluid motions are allowed within the stably stratified NS interior.

As long as the early (ages much less than $\sim 10^4 \text{ yr}$) evolution of magnetars is concerned, however, ambipolar diffusion in the NS core is expected to be the dominant mode of field decay.

Ambipolar diffusion drives a slow motion of charged particles with respect to background neutrons, which is opposed by both particle friction and chemical potential gradients in the stably stratified NS medium. GR92 identified two separate modes of ambipolar diffusion, differing by their effect on chemical composition. The solenoidal mode does not perturb chemical equilibrium and thus is counteracted only by particle friction. The irrotational mode, on the other hand, does perturb chemical equilibrium and cannot evolve on time-scales shorter than the $\beta$-reaction time-scale.

As shown in TD96, $\beta$-reactions are very efficient at erasing chemical equilibrium imbalance when $T > T_u \approx 5.73 \times 10^8 (\rho_{15}/0.7)^{1/2} \text{ K}$. At these high temperatures, both modes of ambipolar diffusion are effectively opposed by neutron–proton friction only. Field decay occurs on the same time-scale in both modes (GR92)

$$t_d^{(\text{early})} = \frac{4\pi m_e^2}{k_B^2} \left( \frac{L}{a} \right)^2 \approx 2.2 \times 10^7 \left( \frac{T_u}{10^8 \text{ K}} \right)^2 \left( \frac{\rho_{15}}{0.7} \right)^{1/2} \left( \frac{B}{10^{16} \text{ G}} \right)^{-2} \text{ yr},$$

where $L$ and $a$ are the characteristic scale of variation of the Lorentz force and chemical potential, respectively, $(L/a) \approx 9.16 T_u^{-1/2} (L/2 \text{ km})$ (GR92). The latter is $\gg 1$ at high temperatures, given the high efficiency of $\beta$-reactions, while it becomes $< 1$ as the temperature drops and the efficiency of $\beta$-reactions decreases. At $T > 10^9 \text{ K}$, the time-scale (36) is much longer than the NS age or its cooling time-scale. Therefore, field decay is negligible as long as the temperature is this high.

As NS cooling proceeds, the point is reached (at $T \lesssim T_u$) where chemical equilibrium imbalance becomes the main obstacle against which magnetic stresses must work to drive particle diffusion. This affects only the irrotational mode, while the solenoidal mode still decays on the time-scale of equation (36). Hence, the two modes grow at different rates with the irrotational mode evolving on a longer time-scale\(^9\) (TD96; GR92)

$$t_d^{(\text{late})} = \frac{4\pi m_e^2}{k_B^2} = \left( \frac{L}{a} \right)^2 t_d^{(\text{early})} \approx 7 \times 10^3 \left( \frac{T_u}{T_u} \right)^{-6} \left( \frac{\rho_{15}}{0.7} \right)^{5/6} \left( \frac{L}{2 \text{ km}} \right)^{3/2} \left( \frac{B}{10^{16} \text{ G}} \right)^{-2} \text{ yr}. \quad (37)$$

TD96 considered in detail this lower-temperature regime. Based on stability arguments, these authors suggested that the solenoidal

\(^9\)The two time-scales are formally equal at $T = T_u$, but the irrotational mode becomes much more slowly below $T = T_u$. For example, a 20 per cent decrease of $T$ below $T_u$ gives a six times longer decay time-scale for the irrotational mode.
mode is expected to carry just a small fraction of the magnetic energy, most of it being tapped by the irrotational mode. The main conclusion of this scenario is that only a tiny fraction of the magnetic energy reservoir in the NS core is lost either in the high-$T$ regime or via field decay through the solenoidal mode at lower temperatures. Most of the magnetic energy dissipation occurs via the slow decay of the irrotational mode, at $T \leq T_{tr}$.

The above scenario holds for fields in the $10^{15}$ G range. According to equation (36), field decay at higher temperatures is significantly faster for stronger magnetic fields. Further, dissipation of even a small fraction of the magnetic energy reservoir may in principle affect NS cooling, if the reservoir is sufficiently large. In fact, field decay at $T > 10^{10}$ K is not frozen if $B$ is larger than $10^{16}$ G. In analogy to the treatment of TD96, we check here whether an equilibrium condition between heating and cooling in the high-$T$ regime can apply as well, with field decay described by equation (36). For uniformity with that work we adopt the same normalizations for the parameters used by TD96.

We can write the heating rate per unit volume through field decay

$$\frac{dU^+}{dt} = \frac{B^2}{4\pi \tau_d^{(\text{early})}} \approx 3.69 \times 10^{19} \frac{B_16^2}{T_{27}^{1/2}} \rho_{15} \text{erg cm}^{-3} \text{s}^{-1}, \quad (38)$$

while the cooling rate per unit volume through modified Urca reactions is

$$\frac{dU^-}{dt} \approx 9.6 \times 10^{20} T_9^{2/3} \rho_{15} \text{erg cm}^{-3} \text{s}^{-1}. \quad (39)$$

Equating the two rates gives the equilibrium temperature

$$T_{eq} \approx 6.6 \times 10^{6} \left( \frac{B}{10^{16} \text{ G}} \right)^{2/5} \left( \frac{\rho_{15}}{0.7} \right)^{-2/15} \left( \frac{L}{2 \text{ km}} \right)^{-1/5} \text{K}. \quad (40)$$

Note that $T_{eq}$ is higher than $T_{tr}$ if $B \geq 7 \times 10^{15}$ G ($\rho_{15}/0.7)^{1.5/2} (L/2 \text{ km})^{-1/8}$ G. Fields larger than that would thus be able to dissipate enough energy and balance neutrino cooling even in the early phase when the solenoidal and irrotational modes are still degenerate. This conclusion describes a regime that was not considered in TD96: very strong magnetic fields ($\sim 10^{16}$ G) decaying and heating a NS core at very high temperatures ($\sim 10^{9}$ K) and at very young ages (years to centuries). The resulting evolution of magnetic field and temperature is coupled, as already shown by TD96. Our solution (equation 40) joins smoothly the one found by TD96 (their equation 31) in the sense that both give the same value of the magnetic field strength ($B \approx 7 \times 10^{15}$ G) when calculated at $T_{tr}$. The two regimes are, in this sense, complementary, forming a continuous evolutionary sequence through $T_{tr}$ for an arbitrarily large magnetic field whose decay is driven by ambipolar diffusion.

In order to better illustrate this, we calculate here the joint evolution of the magnetic field strength and the equilibrium temperature, according to the equilibrium conditions discussed above. Consider the rate of magnetic energy dissipation per unit volume

$$\frac{B}{4\pi} \frac{dB}{dt} = -\frac{B^2}{4\pi \tau_d^{(\text{early})}}. \quad (41)$$

Inserting equations (36) and (40) in (41) gives

$$\frac{dB_{16}}{dt} \approx -3.12 \times 10^{-12} \left( \frac{\rho_{15}}{0.7} \right)^{-2/5} B_{16}^{11/5} \left( \frac{L}{2 \text{ km}} \right)^{-8/5}, \quad (42)$$

whose solution is

$$B_{16}(t) = \left[ 1.12 \times 10^{-4} \left( \frac{\rho_{15}}{0.7} \right)^{-2/5} \frac{t}{\text{yr}} + \left( \frac{1}{B_{16}} \right)^{6/5} \right]^{-5/6} \quad (43)$$

with $B_{16}$ the strength of the magnetic field at the initial time $t$. As an illustrative example, in Fig. 4 we show the evolution of the core magnetic field according to equation (43) for three different initial values. The corresponding equilibrium temperature evolution is

$$T_{eq} \approx 6.6 \left[ 1.1 \times 10^{-4} \left( \frac{\rho_{15}}{0.7} \right)^{-2/5} \frac{t}{\text{yr}} + \left( \frac{1}{B_{16}} \right)^{6/5} \right]^{-1/3} \quad (44)$$

We caution that the equilibrium temperature as a function of density is in principle different from the actual temperature profile throughout the NS core. Finding this would require solving the heat flux problem self-consistently, which includes account for the strong suppression of heat transport across magnetic field lines in a superstrong magnetic field.

With the above expressions for the equilibrium regime, the decay time-scales of the two modes of ambipolar diffusion can be evaluated self-consistently – given an initial magnetic core field $B_j$ – and their values compared. In Fig. 5 we show, for illustration, the evolution of the two time-scales for an initial (uniform) core magnetic field $B_{16} = 5 \times 10^{16}$ G and at a given value of the density $\rho_{15} = 0.7$. Clearly, field evolution is always determined by the longest decay time: as long as particle friction dominates, both modes decay on the time-scale $t_{fr}^{(\text{early})}$. Once chemical equilibrium imbalance overtakes particle friction (at $t \approx 10^7$ yr in our example), the two modes split: the solenoidal mode is unaffected and continues to evolve on the (now shorter) time $t_{fr}^{(\text{early})}$, while the irrotational mode is now subject to a slower evolution, determined by the efficiency of $\beta$-reactions.

We show in Fig. 4 the evolution of the core magnetic field in the regime described here, for three different initial values. More details are given in the captions.

We can also estimate the time $\Delta t$ after which the temperature, once equilibrium between heating and cooling holds, reaches the
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Figure 5. Evolution of the ambipolar diffusion decay time-scales determined by each of the two friction mechanisms, separately, calculated at a particular density $\rho = 2.5 \rho_{\text{puc}}$ and for a given initial field strength $B_i = 5 \times 10^{16} \text{ G}$. The time-scales evolve according to their dependence on the temperature $T$ and magnetic field $B$. The evolution of $T(t)$ and $B(t)$ is given by equations (44) and (43) and is thus determined by np-friction alone, initially. However, the $\beta$-reaction time-scale is a much stronger function of time than the np-friction time-scale, in this regime. Curves are dotted where the two time-scales become comparable, within a factor of a few, and the approximation under which time-scales are calculated — through equations (43) and (44) — does not hold anymore. At this point, the temperature is close to the transition temperature $T_u$ and the magnetic field close to the value $\approx 7 \times 10^{15} \text{ G}$ (see text).

transition value $^{10} T_u$

$$
\Delta t \approx 1.36 \times 10^4 \left( \frac{\rho_{15}}{0.7} \right)^{-1/4} \left( \frac{L}{2 \text{ km}} \right)^{7/4} \left[ 1 - 0.657 \left( \frac{1}{B_{16}} \right)^{4/5} \left( \frac{\rho_{15}}{0.7} \right)^{4/5} \left( \frac{L}{2 \text{ km}} \right)^{32/35} \right].
$$

This gives $\Delta t \sim 4500$ yr for $B_i = 10^{16} \text{ G}$ and has an asymptotic value $\Delta t_{\text{max}} \approx 1.36 \times 10^4$ yr for very large $B$. For $B_i = 3 \times 10^{16} \text{ G}$, $\Delta t \approx 1 \times 10^4$ yr, $\sim 80$ per cent of the asymptotic value. The comparison with a NS standard cooling scenario, without any heat sources, is striking, since in this case $T_u$ is reached in somewhat less than 10 yr.

Once the field has decayed to the limiting value $\approx 7 \times 10^{15} \text{ G}$, the temperature equals $T_u$ and the results obtained by TD96 hold. We conclude that the early thermal evolution of the core can be significantly altered (slowed down) by ultra-strong field decay. A detailed analysis of the consequences of this conclusion is beyond our scope here and will be the subject of future study.

5.1 Comparison with recent studies of core heating in magnetars

Surface temperatures inferred from the X-ray spectra of SGRs/AXPs ($>0.35$ keV or $>4 \times 10^8$ K) are a factor of $\approx 2-3$ higher than expected from cooling calculations of NSs with their estimated ages, $\sim 10^4$ yr (Kaminker et al. 2007). The heat source internal to the NS need thus not only be strong enough to provide the excess energy but it also need to be efficiently converted to surface heating.

Kaminker et al. (2007) considered how the surface temperature of a cooling NS would be affected when a (slowly decaying) heating source in its interior were included, in order to account for the heating provided by the decaying magnetic field. These authors suggest that, if heat is released in the core or deep crustal layers, only a minor fraction of it is transported to the surface and radiated thereby as X-ray photons. Rather, most of the released heat produces an enhancement in neutrino emission — which is strongly temperature dependent — and is thus radiated locally (via neutrinos). In fact, in their calculations the local temperature in the core is affected only slightly by internal heating. Core temperatures never reach $10^9$ K and the surface cannot ever be kept as warm as AXPs/SGRs surface emission indicates. Kaminker et al. (2007) conclude that heat is most likely released in the outer regions of the crust, where neutrino-emitting processes are much less efficient and heat is thus most efficiently transported to the surface, with only minor losses.

Their results are therefore at variance with ours, that envisage an efficient heating of the core up to very high temperatures (as a function of the magnetic field strength). The difference between the two conclusions is readily found in the parametrization of the heating source chosen by Kaminker et al. (2007), which corresponds to a different physical scenario.

Their model heat source is independent of the magnetic field strength and the NS internal temperature. The maximum initial heating rate they consider ($Q_0 = 3 \times 10^{20} \text{ erg s}^{-1}$) effectively corresponds to our expression (38) with $B \approx 1.6 \times 10^{16} \text{ G}$. The heating rate in their model decreases exponentially in time, with a fixed time constant comparable to, but somewhat longer than, the source estimated age. As opposed to this, we have (i) allowed for initially stronger fields and, thus, larger heating rates and (ii) considered a time-varying decay time-scale, since the ambipolar diffusion-driven field decay is a function of the NS internal temperature and field strength (equation 38). In particular, the values of $T$ and $B$ at each epoch are determined self-consistently by the equilibrium condition between cooling and heating.

As a consequence, in the model by Kaminker et al. (2007) the released heat does not affect the value of $Q$ while it enhances neutrino emission. The latter being a strong function of the temperature, the net result is a slight enhancement of the interior temperature, to make neutrinos able to carry away almost all the excess heat.

In ambipolar diffusion-driven field decay, on the other hand, heating has a feedback on both cooling and heating itself. As long as the temperature is very high the NS cools, with heating providing just a minor perturbation to the dominant process of $\nu$-cooling. As the core temperature drops, however, the field decay rate grows (equation 38) while neutrino emission drops. Eventually, the two rates become almost equal and at this stage their equilibration plays a key role, as stressed by TD96. Near equilibrium, temperature variations have a strong feedback on both heating and cooling — and with opposite effects. This forces the temperature towards $T_{\text{eq}}$ (equation 40): as the field dissipates the temperature drops slightly and a new equilibrium between heating and cooling is reached, at a slightly smaller temperature and field strength. The equilibrium temperature within the NS core, as a function of density, obtained through our equation (40) is shown in Fig. 6 for four different values of the average core magnetic field strength. In Fig. 7, we show, for a given initial magnetic field strength, the equilibrium temperature throughout the magnetar core at four different epochs of its evolution according to equation (41). More details are given in the caption.
if the decaying field is around a few days after formation. This results if there exists at least a tiny misalignment (angle $\chi$) between the rotation axis and the symmetry axis of the magnetic field, at birth. The newly born NS is distorted to a prolate shape by the toroidal field, and is freely precessing because of $\chi \neq 0$. Under these circumstances, internal viscous dissipation of the precessional motion will drive the magnetic symmetry axis orthogonal to the spin axis, the most favourable geometry for GW emission.

We discussed current uncertainties in various aspects of the model and introduced simple approximations to treat each of them. We developed a simple analytical model describing the early rotational evolution of newly formed magnetars, that includes an ideal magnetic dipole torque plus the GW torque acting on an orthogonal, prolate rotator (see equation 4). We then estimated the magnitude of GW emission from magnetars as a function of their initial spin, internal magnetic energy and external (dipole) magnetic field (as well as their mass and radius).

Our main conclusions can be summarized as follows.

(i) If magnetars are born with spin period less than 3 ms, internal toroidal fields $> 3 \times 10^{15}$ G and external dipole fields $\lesssim 2 \times 10^{14}$ G, then the expected GW signal would be strong enough to be detectable with Advanced LIGO/Virgo class detectors out to the Virgo cluster, where their formation rate may be $\sim$1 per year.

(ii) The estimated optimal S/N ratios for match-filtered signal searches (with one detector only) are very encouraging. However, as already noted by Stella et al. (2005), optimal signal searches have unaffordable computational costs. The development of sub-optimal signal search strategies is required; a task that is currently under way.

(iii) If our scenario holds, the rotational energy $\approx 3 \times 10^{52} (P \text{ ms}^{-1})^2$ erg of the newly formed magnetar will be emitted mostly as GWs, in the first few days after formation. As a consequence, SNRs surrounding evolved magnetars would not be expected to bear the signature of an excess energy injection ($> 10^{51}$ erg) soon after formation, since GWs do not interact appreciably with the expanding shell.

In particular, the condition that a newly formed magnetar be detectable as a GW source from Virgo cluster distances turns out to be almost equivalent to the condition that it radiate less than $10^{51}$ erg through magnetic dipole radiation. The two requirements are met for nearly overlapping regions of the internal and external magnetic fields parameter space. Stated differently, if the Galactic magnetar candidates studied by Vink & Kuiper (2006) were born with millisecond spin periods, $B_1 > 3 \times 10^{15} \text{ G}$ and $B_2 \leq 2 \times 10^{16}$, they would have lost most of their rotational energy through GW emission. Their SNRs would thus not show any excess energy compared to other SNRs - as observed, indeed - and the GW signal they emitted could have been detected out to $\sim$20 Mpc with Advanced LIGO/Virgo class interferometers.

Finally, we considered the evolution of an internal field $> 10^{16}$ G as a result of ambipolar diffusion, as already envisaged by GR92 and TD96. Our aim here was twofold: first, we investigate this high $B$-field regime, for which the calculations by TD96 are not appropriate. Secondly, we showed that even fields this strong have (at least) a slow decay mode through ambipolar diffusion, that is active soon after formation. This process can prevent the cooling

Therefore, the internal heating source considered by Kaminker et al. (2007) differs from ambipolar diffusion-driven field decay. When ambipolar diffusion in the core – and the associated heating – is taken into account, the NS core can remain at fairly high temperatures for a long time if the decaying field is around $\sim 10^{16}$ G.

6 CONCLUSIONS

In this paper, we have investigated some implications of one of the key ansatz of the magnetar model; namely that magnetars do form with millisecond spin periods and a (mainly) toroidal magnetic field, generated through the strong differential rotation of the collapsing PNS (DT92 and TD93).

Building on our earlier work (Stella et al. 2005; Dall’Osso & Stella 2007), we showed that one major implication of this scenario is that such objects can become strong sources of GWs in the first few days after formation. Under these circumstances, internal viscous dissipation of the precessional motion will drive the magnetic symmetry axis orthogonal to the spin axis, the most favourable geometry for GW emission.

We discussed current uncertainties in various aspects of the model and introduced simple approximations to treat each of them. We developed a simple analytical model describing the early rotational evolution of newly formed magnetars, that includes an ideal magnetic dipole torque plus the GW torque acting on an orthogonal, prolate rotator (see equation 4). We then estimated the magnitude of GW emission from magnetars as a function of their initial spin, internal magnetic energy and external (dipole) magnetic field (as well as their mass and radius).

Our main conclusions can be summarized as follows.

(i) If magnetars are born with spin period less than 3 ms, internal toroidal fields $\geq 3 \times 10^{15}$ G and external dipole fields $\leq 2 \times 10^{14}$ G, then the expected GW signal would be strong enough to be detectable with Advanced LIGO/Virgo class detectors out to the Virgo cluster, where their formation rate may be $\sim$1 per year.

(ii) The estimated optimal S/N ratios for match-filtered signal searches (with one detector only) are very encouraging. However, as already noted by Stella et al. (2005), optimal signal searches have unaffordable computational costs. The development of sub-optimal signal search strategies is required; a task that is currently under way.

(iii) If our scenario holds, the rotational energy $\approx 3 \times 10^{52} (P \text{ ms}^{-1})^2$ erg of the newly formed magnetar will be emitted mostly as GWs, in the first few days after formation. As a consequence, SNRs surrounding evolved magnetars would not be expected to bear the signature of an excess energy injection ($> 10^{51}$ erg) soon after formation, since GWs do not interact appreciably with the expanding shell.

In particular, the condition that a newly formed magnetar be detectable as a GW source from Virgo cluster distances turns out to be almost equivalent to the condition that it radiate less than $10^{51}$ erg through magnetic dipole radiation. The two requirements are met for nearly overlapping regions of the internal and external magnetic fields parameter space. Stated differently, if the Galactic magnetar candidates studied by Vink & Kuiper (2006) were born with millisecond spin periods, $B_1 > 3 \times 10^{15} \text{ G}$ and $B_2 \leq 2 \times 10^{16}$, they would have lost most of their rotational energy through GW emission. Their SNRs would thus not show any excess energy compared to other SNRs - as observed, indeed - and the GW signal they emitted could have been detected out to $\sim$20 Mpc with Advanced LIGO/Virgo class interferometers.

Finally, we considered the evolution of an internal field $> 10^{16}$ G as a result of ambipolar diffusion, as already envisaged by GR92 and TD96. Our aim here was twofold: first, we investigate this high $B$-field regime, for which the calculations by TD96 are not appropriate. Secondly, we showed that even fields this strong have (at least) a slow decay mode through ambipolar diffusion, that is active soon after formation. This process can prevent the cooling

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**Figure 6.** Value of the equilibrium temperature (equation 44) as a function of density throughout the NS core, for four selected values of the magnetic field strength, as indicated in the figure.

**Figure 7.** The evolution of the temperature profile within a magnetar core ($10^{14} \text{ g cm}^{-3} < \rho \leq 10^{15} \text{ g cm}^{-3}$) for a specific initial value of the toroidal field strength ($B_t = 4 \times 10^{16}$ G), corresponding to a total magnetic energy $E_B \approx 4.6 \times 10^{50}$ erg or an ellipticity $\epsilon_t \approx -4.6 \times 10^{-3}$. The four curves describe the temperature profile at four different epochs, the thick solid curve at the bottom defines the temperature at (and below) which chemical equilibrium imbalance becomes the major limiting factor for ambipolar diffusion of the irrotational mode (the regime described in TD96). Above the thick line, the irrotational and solenoidal modes are degenerate and the regime described in the previous section holds.

In particular, the condition that a newly formed magnetar be detectable as a GW source from Virgo cluster distances turns out to be almost equivalent to the condition that it radiate less than $10^{51}$ erg through magnetic dipole radiation. The two requirements are met for nearly overlapping regions of the internal and external magnetic fields parameter space. Stated differently, if the Galactic magnetar candidates studied by Vink & Kuiper (2006) were born with millisecond spin periods, $B_1 > 3 \times 10^{15} \text{ G}$ and $B_2 \leq 2 \times 10^{16}$, they would have lost most of their rotational energy through GW emission. Their SNRs would thus not show any excess energy compared to other SNRs - as observed, indeed - and the GW signal they emitted could have been detected out to $\sim$20 Mpc with Advanced LIGO/Virgo class interferometers.

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of the magnetar core below a temperature of ~10^9 K for hundreds to thousands years. This conclusion is expected to have significant implications on our understanding of AXPs/SGRs.

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APPENDIX A: SPIN ENERGY AND FREEBODY PRECESSION ENERGY OF A ROTATING ELLIPSOID

We give here a quick derivation of the expression for the energy of freebody precession of a fluid star subject to both centrifugal and magnetic deformations. A more general discussion, in the context of ‘ordinary’ NSs, is found in Jones & Andersson (2001).

First of all, we write down the frequency of the freebody precession mode derived by Mestel & Takhar (1972):

\[ \omega_{\text{pre}} = \frac{I_1 - I_0}{I_0} \cos \psi = \epsilon_\theta \Omega \cos \psi, \]  

(A1)

where angles throughout this section are those defined in Fig. A1.

Following Jones & Andersson 2001 (and references therein), we define the moment of inertia tensor of the fluid NS as a linear combination of three contributions, namely a (spherical) gravitational part, plus two axisymmetric perturbations provided, in our case, by the centrifugal and magnetic fields, respectively. Hence,

\[ I = I_0 \hat{a} + \Delta I_0 (\tilde{a} \tilde{a}_B - \delta B / 3) + \Delta I_B (\tilde{a} \tilde{a}_B - \delta B / 3), \]  

(A2)

where \( \hat{a} \) is the identity, \( \tilde{a}_B \) represent the unit vectors along the spin and magnetic field axis and \( \Delta I_0, \Delta I_B \) are the magnitudes of the

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corresponding perturbations of the inertia tensor. The eigenvalues of the latter are (Jones & Andersson 2001)

\[
\begin{align*}
I_1 &= I_0 + 2 \frac{2}{3} \Delta I_0 - 2 \frac{1}{3} \Delta I_B \\
I_2 &= I_0 + 2 \frac{2}{3} \Delta I_0 - 2 \frac{1}{3} \Delta I_B \\
I_3 &= I_0 + 2 \frac{2}{3} \Delta I_0 + 2 \frac{1}{3} \Delta I_B = I_1 + I_2 + 2 I_B.
\end{align*}
\]  

(A3)

These expressions show that the centrifugal deformation modifies all eigenvalues in the same way, as if it was effectively a spherically symmetric, additive term. The magnetic deformation, on the other hand, does introduce an asymmetry between the first two eigenvalues and the third one, inducing the magnetic ellipticity \( \epsilon_B \equiv (I_3 - I_2)/I_1 \). The angle \( \phi \) between \( \Omega \) and the angular momentum axis is always smaller than the tilt angle \( \chi \) of the magnetic symmetry axis (Jones & Andersson 2001). In formulae, to first order in \( \epsilon_B \) it can be shown that (by using equation A2)

\[
\sin \phi \approx \phi \approx \epsilon_B \sin \psi \cos \psi \ll \psi,
\]

(A4)

from which \( \psi = \chi - \phi \approx \chi \).

Following the argument by Cutler & Jones (2001), the NS angular momentum can be written as \( L = I \Omega \) and its kinetic energy as \( E_k = (1/2)I \Omega^2 \), from which (to first order in \( \epsilon_B \))

\[
L = I_1 \Omega (1 + 2 \epsilon_B \cos^2 \psi)^{1/2} \approx I_1 \Omega (1 + 2 \epsilon_B \cos^2 \chi)^{1/2}
\]

\[
E_k = \frac{1}{2} I_1 \Omega^2 (1 + \epsilon_B \cos^2 \psi)^{1/2} \approx \frac{1}{2} I_1 \Omega^2 (1 + \epsilon_B \cos^2 \chi).
\]

(A5)

From the latter equation, we can obtain the energy of freebody precession by taking the difference between the second equation in (A5) and the spin energy of the same ellipsoid that, with the same total angular momentum \( L \), spins around the axis having the greatest moment of inertia. The latter configuration is indeed the one that minimizes the energy, at constant angular momentum. It is thus the one towards which a freely precessing spheroid will evolve, given an internal dissipative process (Mestel & Takhar 1972; Jones

\[
\frac{\Omega_c}{\Omega} = \left( \frac{I_1}{I_{\text{max}}} \right)^2 (1 + 2 \epsilon_B \cos^2 \chi).
\]

(A6)

Writing the minimum spin energy as \( E_{\text{min}} = \frac{1}{2} I_{\text{max}} \Omega_c^2 \), the freebody precession energy is

\[
E_{\text{prec}} = E_k - \frac{1}{2} I_{\text{max}} \Omega_c^2
\]

\[
= \frac{1}{2} I_1 \Omega^2 \left( 1 + \epsilon_B \cos^2 \chi - \frac{I_1}{I_{\text{max}}} (1 + 2 \epsilon_B) \right).
\]

(A7)

This general expression can be specialized to the case of an oblate \( (I_{\text{max}} = I_1) \) or a prolate \( (I_{\text{max}} = I_1 = I_2) \) ellipsoid giving, respectively

\[
E_{\text{prec}} \approx \frac{1}{2} I_1 \Omega^2 \epsilon_B \sin^2 \chi \text{ oblate ellipsoid} \quad (\epsilon_B > 0)
\]

\[
E_{\text{prec}} \approx -\frac{1}{2} I_1 \Omega^2 \epsilon_B \cos^2 \chi \text{ prolate ellipsoid} \quad (\epsilon_B < 0)
\]

(A8)

to the first order in \( \epsilon_B \).

Self consistence of the above is warranted by \( E_{\text{prec}} \to 0 \) for \( \chi \to 0 \) in the oblate case, and for \( \chi \to \pi/2 \) in the prolate case. From the above formulae, we can eventually relate the time derivative of the freebody precession energy to the time derivative of the tilt angle \( \chi \), in the case of a prolate ellipsoid. Since conservation of angular momentum is required, \( \Omega \) changes only as a consequence of changes in \( \chi \) (cf. equation A6), which makes the time derivative of \( E_{\text{prec}} \) a function of \( \chi \) only. Taking the time derivatives of the first of equation (A5) and the second of equation (A8) and requiring angular momentum conservation, we obtain (to first order in \( \epsilon_B \))

\[
\frac{dE_{\text{prec}}}{dt} \approx I_1 \epsilon_B \Omega^2 \chi \cos \chi \sin \chi = -2E_{\text{prec}} \tau_d \chi^2.
\]

(A9)

where \( \tau_d \) was defined in Section 2.1.4.

**APPENDIX B: CALCULATION OF THE DAMPING TIME OF FREEBODY PRECESSION THROUGH BULK VISCOITY**

In this Appendix, we describe the calculation that leads to the estimated time-scale for dissipation of the free precessional motion through bulk viscosity (equation 13). According to the definition of \( \tau_d \) (equation 12), we need an expression for both the bulk viscosity coefficient, \( \xi \), and the precession-induced density perturbation, \( \delta \).

For the bulk viscosity coefficient, we recall here the general expression (11)

\[
\text{Re}(\xi) = \frac{\nu \tau}{\omega^2 \tau} \frac{\partial \rho / \partial \chi}{\partial x / \partial \tau} \approx \frac{n \tau}{\omega^2 \tau} \frac{\partial \rho / \partial \chi}{\partial x / \partial \tau}.
\]

(B1)

For NPE matter, as we have assumed throughout, the time-scale for \( \beta \)-reactions (\( \tau_{\beta} \)) is (cf. Reisenegger & Goldreich 1992)

\[
\tau_{\beta} = \frac{3 \pi^2}{\lambda_{\beta} E_F} \approx \frac{0.23}{T_9} \left( \frac{\rho}{\rho_{\text{uac}}} \right)^{2/3} \text{yr}.
\]

(B2)

Here, \( \chi \) is the equilibrium fraction of charged particles (see below), \( E_F = (h^2/2m_n)(3\pi^2 n_B)^{1/3} \) is the neutron Fermi energy and \( \lambda_{\beta} \approx \frac{3 \pi^2}{\lambda_{\beta} E_F} \approx \frac{0.23}{T_9} \left( \frac{\rho}{\rho_{\text{uac}}} \right)^{2/3} \text{yr}.

Note that the precession energy is positive in both cases, as it must be.
where $\theta_0$ is the unperturbed density profile, solution to the Lane–Emden equation for a non-rotating polytrope. The parameter $\alpha$ above is related to the polytropic EOS

$$\alpha = \left[\frac{(n+1)k}{4\pi G} \rho_0^{-1+1/n}\right]^{1/2} \left(\frac{(n+1)}{k} \frac{2}{\pi G}\right)^{1/2}. \quad (B10)$$

The problem is thus reduced to finding an appropriate expression for the function $\psi(\xi)$. In particular, for polytropes of index $n$ the following expansion holds:

$$\Theta(\xi) = \theta_0 + v [\psi_0(\xi) + A_2\psi_2(\xi)\Psi_2(\mu)], \quad (B11)$$

where $\psi_0(\xi)$ is the spherical part of the centrifugal deformation and the non-spherical part of the deformation corresponds to the second term in square brackets. The coefficient $A_2$ is

$$A_2 = \frac{5}{6} \frac{\xi_k^2}{3\psi_3(\xi_k)} + \xi_k \psi_3'(\xi_k). \quad (B12)$$

which formally completes the problem at hand.

As it has been shown by numerical integration of the equilibrium equations, a first-order expansion in $v$ provides a relatively good approximation to slowly rotating polytropes; in the case $n = 1$, $v \leq 0.075$ gives approximately the rotation rate at which deviations from the first-order perturbation theory become non-negligible (Tassoul 1978). In particular, direct integration of the equilibrium equations reveals a systematically larger deformation of rapid rotators compared to the results obtained through the first-order perturbative approximation. Therefore, our general expectation would be that the latter underestimates the real deformation of a rapidly rotating NS ($\delta \rho$), thus underestimating the efficiency of bulk viscous dissipation.

The polytropic index $n = 1$ greatly simplifies the analytical treatment and allows one to derive quite straightforward formulae. The adimensional radius of the polytropic star will be $\xi_k = \pi$ and the numerical values of $\psi_3(\xi_k)$ and $\psi_3'(\xi_k)$ have been tabulated by Chandrasekhar (1933), allowing to derive $A_2(n = 1) \simeq -0.54833$. Finally, $\alpha$ is determined by the linear scale for the NS radius so that

$$\alpha \simeq 3.82 \times 10^3 \left(\frac{R}{12 \text{ km}}\right) \text{ cm} \quad (B14)$$

and the EOS thus becomes $p = 6.1416 \times 10^4 (R/12\text{ km})^2\rho^2$. The density perturbation of equation (B5), the one that determines the precession-induced internal motions damped by bulk viscosity, amounts to only the non-spherical part of the centrifugal deformation.

The spherical part does not cause any radial periodic motions and, thus, does not perturb the local chemical potential equilibrium of NS matter. We are therefore left with

$$f(r(\xi)) = \rho_0 v A_2 \psi_2(\xi), \quad (B15)$$

where the angular part corresponds to the function $K(\chi, \theta, \lambda, \Omega)$. This must be inserted in equation (B5) to obtain the full expression for the non-spherical density perturbation. Recalling equation (12), we thus get the full expression for the bulk viscosity damping rate:

$$E_{\text{diss}} = 60 T_0^6 \rho^2 \int \text{d}V \left(\frac{\delta \rho}{\rho}\right)^2 \frac{\rho^2}{\rho^2} = 60 T_0^6 \int \text{d}V (\delta \rho)^2 \quad (B16)$$
where
\[
[\delta \rho(r(\xi))]^2 = \frac{1}{4} f^2(r(\xi)) K^2(\chi, \theta, \lambda, \Omega)
\]
\[
= \frac{1}{4} (\rho c v)^2 A_2^2 \psi_2^2(\xi) \hat{K}^2(\chi, \theta, \lambda, \Omega)
\]  
(B17)
so that, eventually, the full expression for \( \dot{E}_{\text{diss}} \) can be written as
\[
\dot{E}_{\text{diss}} \approx 2.6 \times 10^{13} T_10^6 \Omega^2 \alpha^3 \int \psi_2^2(\xi) \xi^2 d\xi \int \hat{K}^2 \sin \theta d\theta d\lambda
\]
\[
\approx 2.2 \times 10^{45} \left( \frac{T}{10^{10} \text{K}} \right)^6 \left( \frac{\text{ms}}{P} \right)^4 \left( \frac{R}{12 \text{km}} \right)^3
\]
\[
\times \int \psi_2^2(\xi) \xi^2 d\xi \int \hat{K}^2 \sin \theta d\theta d\lambda.
\]  
(B18)
The radial integral, solved from \( \xi = 0 \) to \( \xi_R = \pi \), gives 138.04, while the angular part gives \((24\pi/5) \sin^2 \chi(1 + 3 \cos^2 \chi)\). Eventually,

\[
\dot{E}_{\text{diss}} \approx 4.7 \times 10^{48} \sin^2 \chi(1 + 3 \cos^2 \chi)
\]
\[
\times \left( \frac{T}{10^{10} \text{K}} \right)^6 \left( \frac{\text{ms}}{P} \right)^4 \left( \frac{R}{12 \text{km}} \right)^3 \text{ergs}^{-1}.
\]  
(B19)

As a last step, given the energy of the freebody precession mode derived in Appendix A, we obtain the dissipation time-scale (see equation 13)

\[
\tau_d = \frac{2 E_{\text{pre}}}{\dot{E}_{\text{diss}}} \approx 13.5 \cos^2 \chi \frac{E_B}{10^{50} \text{erg}} \left( \frac{P}{\text{ms}} \right)^2 \left( \frac{T}{10^{10} \text{K}} \right)^{-6} \left( \frac{1.4 M_\odot}{M} \right) \left( \frac{R}{12 \text{km}} \right)^3 \text{s}.
\]  
(B20)

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