Abstract. Weighted projective lines, introduced by Geigle and Lenzing in 1987, are one of the basic objects in representation theory. One key property is that they have tilting bundles, whose endomorphism algebras are the canonical algebras introduced by Ringel. The aim of this paper is to study their higher dimensional analogs. First, we introduce a certain class of commutative rings \( R \) graded by abelian groups \( L \) of rank 1, which we call Geigle-Lenzing complete intersections. We study their Cohen-Macaulay representations, and show that there always exists a tilting object in the stable category \( \mathsf{CM}_L R \). As an application we study when \( (R, L) \) is \( d \)-Cohen-Macaulay finite in the sense of higher dimensional Auslander-Reiten theory. Secondly, by applying the Serre construction to \( (R, L) \), we introduce the category \( \mathsf{coh}_X \) of coherent sheaves on a Geigle-Lenzing projective space \( X \). We show that there always exists a tilting bundle \( T \) on \( X \), and study the endomorphism algebra \( \text{End}_X(T) \) which we call a \( d \)-canonical algebra. Further we study when \( \mathsf{coh}_X \) is derived equivalent to a \( d \)-representation infinite algebra in the sense of higher dimensional Auslander-Reiten theory. Moreover we observe Orlov-type semiorthogonal decompositions between the stable category \( \mathsf{CM}_L R \) and the derived category \( \mathsf{D}^b(\mathsf{coh}_X) \).

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1
1. Introduction

1.1. Background. Weighted projective lines, introduced by Geigle-Lenzing in 1987 [GL], are one of the important objects in representation theory, see e.g. [CK, Ku, L1, L2, Me]. They have tilting bundles, whose endomorphism algebras are the canonical algebras introduced by Ringel [Rin], and hence we have a derived equivalence between canonical algebras and weighted projective lines. A prototype is given by the path algebra of the Kronecker quiver \[ \bullet \rightarrow \bullet \] and the projective line \( \mathbb{P}^1 \). Both weighted projective lines and canonical algebras have played important roles in representation theory. For example, by a celebrated theorem of Happel [Hap2], any hereditary abelian category with a tilting object is derived equivalent to either a weighted projective line or a path algebra of an acyclic quiver. Recently weighted projective lines play an important role in Kontsevich’s homological mirror symmetry conjecture, see e.g. [KST1, KST2].

The aim of this paper is to introduce a higher dimensional analog of weighted projective lines, then study them in the context of Auslander-Reiten theory. This provides the following new objects in representation theory:

- **GEIGLE-LENZING (GL) complete intersections \( R \).** This is a class of complete intersections in dimension \( d + 1 \) graded by an abelian group \( L \) of rank one.
- The categories \( \text{CM}^L R \) of \( L \)-graded maximal Cohen-Macaulay \( R \)-modules. This is a class of Frobenius categories satisfying Auslander-Reiten-Serre duality.
- **GEIGLE-LENZING (GL) projective spaces \( \text{coh} X \).** This is a class of abelian categories of global dimension \( d \) satisfying Auslander-Reiten-Serre duality.

In the case \( d = 1 \), these are precisely the (homogeneous coordinate rings of) weighted projective lines of Geigle-Lenzing. We will study Cohen-Macaulay representations of \( (R, L) \) and the derived category of \( \text{coh} X \) from the viewpoint of Auslander-Reiten theory and tilting theory. In particular, we show that

- the abelian category \( \text{coh} X \) has a tilting bundle \( T^{\text{ca}} \);
- the triangulated category \( \text{CM}^L R \) has a tilting object \( T^{\text{CM}} \).

Taking the endomorphism algebras of these tilting objects, we obtain

- the \( d \)-canonical algebra \( A^{\text{ca}} \) and the CM-canonical algebra \( A^{\text{CM}} \). These finite dimensional algebras of finite global dimension.

In the case \( d = 1 \), the former are precisely the canonical algebras of Ringel.

Our GL complete intersections are divided into three disjoint classes depending on the sign of their dualizing element: **Fano, Calabi-Yau** and **anti-Fano**. In the case \( d = 1 \), these correspond precisely to the famous trichotomy of weighted projective lines: **domestic, tubular** and **wild**. Our study is strongly motivated by some well-known results for the case \( d = 1 \). In this case it is the following conditions are equivalent, where we call a tilting object in a triangulated category \( n \)-tilting if its endomorphism algebra has global dimension at most \( n \).
• $(R, L)$ is domestic (or equivalently, Fano).
• $(R, L)$ is Cohen-Macaulay finite, that is, there are only finitely many isomorphism classes of indecomposable objects in $\text{CM}^{L} R$ up to degree shift.
• $X$ is vector bundle finite, that is, there are only finitely many isomorphism classes of indecomposable objects in $\text{vect} X$ up to degree shift.
• $\text{CM}^{L} R$ has a 1-tilting object.
• $\text{D}^{b}(\text{coh} X)$ has a 1-tilting bundle.

These cases bijectively corresponds to Dynkin diagrams $A_n$, $D_n$, and $E_n$. More precisely $\text{coh} X$ is derived equivalent to the path algebra of the corresponding extended Dynkin type. Note that Cohen-Macaulay finiteness and vector bundle finiteness are popular subject in representation theory [Y, LW, DG] (see also [DR, J, GK, He, A2, Ei1, Kn, BGS, EH]).

We will study generalizations of the equivalences above to GL complete intersections by replacing the classical notions of Cohen-Macaulay finiteness and vector bundle finiteness by those of $d$-Cohen-Macaulay finiteness and $d$-vector bundle finiteness, which are defined by the existence of certain $d$-cluster tilting subcategories appearing in higher dimensional Auslander-Reiten theory [H1, H2]. Our strongest hope is that the following conditions are equivalent.

• $X$ is Fano.
• $(R, L)$ is $d$-Cohen-Macaulay finite.
• $X$ is $d$-vector bundle finite.
• $\text{CM}^{L} R$ has a $d$-tilting object.
• $\text{D}^{b}(\text{coh} X)$ has a $d$-tilting bundle.

In this paper, we find much evidence for this conjecture. In particular, in the Fano case we give families of

• GL complete intersections that are $d$-Cohen-Macaulay finite,
• GL projective spaces that are $d$-vector bundle finite.

$d$-representation infinite algebras are one of the important classes of finite dimensional algebras appearing in higher dimensional Auslander-Reiten theory [HIO]. Recently Buchweitz-Hille [BrH] show that $d$-representation infinite algebras naturally appear in algebraic geometry as the endomorphism algebras of certain tilting sheaves. Some of their results can be applied to our setting, and in fact, if a GL projective space has a $d$-tilting sheaf, then its endomorphism algebra is $d$-representation infinite. For example, $d$-canonical algebras for the case $n \leq d + 1$ are always $d$-representation infinite.

We also study GL projective spaces in the context of non-commutative algebraic geometry. Recently the third author introduced the notions of ampleness of two-sided tilting complexes [Min]. Further the remarkable classes of $d$-Fano algebras and $d$-anti-Fano algebras were introduced by regarding Nakayama functors of finite dimensional algebras as analogs of canonical bundles over algebraic varieties. In joint work with Mori [MM], it was proved that $d$-Fano algebras have a strong connection with $(d+1)$-Calabi-Yau algebras. This result played an important role in our previous study of $d$-representation infinite algebras [HIO]. In this paper, we show that

• $d$-canonical algebras are $d$-Fano algebras (respectively, $d$-anti-Fano algebras) if and only if $(R, L)$ is Fano (respectively, anti-Fano).

In the next subsection, we explain our results in more detail.

1.2. Our Results on Geigle-Lenzing complete intersections. Our GL complete intersection $R$ is a commutative Noetherian $k$-algebra associated with a set $H_1, \ldots, H_n$ of hyperplaces in the projective space $P^d$ in general position and a sequence $p_1, \ldots, p_n$ of positive integers. In a natural way $R$ is graded by a certain abelian group $L$ of rank one that generally has torsion elements (see Section 3 for details). Without loss of generality, we may assume $p_i \geq 2$ for each $i$ (Observation 3.2). Our GL complete intersection $R$ is in fact a complete intersection of Krull dimension $d + 1$. 


The dualizing element (or $a$-invariant, Gorenstein parameter) of $R$ is given by

$$\vec{a} := (n - d - 1)\vec{c} - \sum_{i=1}^{n} a_i \in \mathbb{L},$$

which plays a key role in this paper. Using the degree map $\delta : \mathbb{L} \to \mathbb{Q}$, GL complete intersections are divided into the following 3 classes depending on the sign of the degree $\delta(\vec{a}) := n - d - 1 - \sum_{i=1}^{n} \frac{1}{p_i}$ of $\vec{a}$:

| $\delta(\vec{a})$ | $< 0$ | $= 0$ | $> 0$ |
|-------------------|-------|-------|-------|
| $(R, L)$          | Fano  | Calabi-Yau | anti-Fano |
| $d = 1$           | domestic | tubular | wild |

The pair $(R, \mathbb{L})$ has been widely studied in the classical case $d = 1$ in representation theory. In this case, the above 3 classes are called domestic, tubular and wild.

The first aim of this paper is to study Cohen-Macaulay representations of GL complete intersection $(R, \mathbb{L})$. We study the category $\text{CM}^{L^d}R$ of $L$-graded (maximal) Cohen-Macaulay $R$-modules. Since $R$ is a complete intersection, $\text{CM}^{L^d}R$ forms a Frobenius category and the stable category $\text{CM}^{L^d}R$ forms a triangulated category. By a classical result due to Buchweitz [Bu], we have the following basic property (see Theorem 1.2.1):

- There exists a triangle equivalence $D^b_{\text{sg}}(R) \simeq \text{CM}^{L^d}R$, where $D^b_{\text{sg}}(R) := D^b(\text{mod}^{L^d}R)/K^b(\text{proj}^{L^d}R)$ is the singular derived category of $R$ [O1]. On the other hand, we show the following basic properties of $R$ as an $L$-graded ring (see Definition 3.1.3).

**Theorem 1.1.** (Theorems 3.1.3 and 3.2.1) Any GL complete intersection $R$ is an $L$-factorial $\mathbb{L}$-domain and has $\mathbb{L}$-isolated singularities.

As an application, we have the following basic property (see Theorem 4.1.1):

- (Auslander-Reiten-Serre duality) There exists a functorial isomorphism $\text{Hom}_{\text{CM}^{L^d}R}(X, Y) \simeq D\text{Hom}_{\text{CM}^{L^d}R}(Y, X(\vec{a}))[d]$ for any $X, Y \in \text{CM}^{L^d}R$.

The notion of tilting objects (see Section 2.1) is important in representation theory since it is useful for controlling triangulated categories. It is a basic problem to study if a given triangulated category has a tilting object. Our first main result in this paper shows that this is always the case for the stable category $\text{CM}^{L^d}R$.

**Theorem 1.2.** (Theorem 4.1.7) For any GL complete intersection $(R, \mathbb{L})$, the stable category $\text{CM}^{L^d}R$ has a tilting object $T^{\text{CM}}$.

This is new even in the classical case $d = 1$. On the other hand, this is known for the hypersurface case $n = d + 2$ by Futaki-Ueda [FU] and Kussin-Meltzer-Lenzing [KLM] ($d = 1$).

We call the endomorphism algebra $A^\text{CM} := \text{End}^{L^d}(T^{CM})$ in the stable category $\text{CM}^{L^d}R$ a $CM$-canonical algebra. Then we have a triangle equivalence

$$\text{CM}^{L^d}R \simeq D^b(\text{mod} A^{CM}). \quad (1.1)$$

We give some basic properties of $A^{CM}$ including an explicit presentation of $A^{CM}$ in terms of a quiver with relations (see Theorem 4.2.1). In particular, we have the following description for the hypersurface case $n = d + 2$ (see Corollary 4.2.2):

- If $n = d + 2$, then $A^{CM} \simeq \bigoplus_{i=1}^{n} k\Lambda_{p_i-1}$, where $k\Lambda_{p_i-1}$ is the path algebra of the equioriented quiver of type $\Lambda_{p_i-1}$.

We apply these results to study when $(R, \mathbb{L})$ is Cohen-Macaulay finite (= CM finite) in the sense that there are only finitely many isomorphism classes of indecomposable objects in $\text{CM}^{L^d}R$ up to degree shift. In the classical case $d = 1$, it is well-known that $(R, \mathbb{L})$ is CM finite if and only if $(R, \mathbb{L})$ is domestic. In this paper, we give the following classification of GL complete intersections that are CM finite as an easy application of the triangle equivalence (1.1) above.
Theorem 1.3. (Theorem 4.22) A GL complete intersection \((R, \mathbb{L})\) is CM finite if and only if one of the following conditions hold.

- \(n \leq d + 1\).
- \(n = d + 2\), and \((p_1, \ldots, p_n) = (2, 2, p_3), (2, 3, 3, 3), (2, 3, 4)\) or \((2, 3, 5)\) up to permutation.

One of the interpretations of Theorem 1.3 is that there are quite few CM finite GL complete intersections since their stable categories \(\CM d R\) are triangle equivalent to the stable categories \(\CM d' R'\) for some domestic GL complete intersections \((R', \mathbb{L}')\) in \(d = 1\) by Knörrer periodicity (see Corollary 1.12).

The following observation was one of the motivations to study higher dimensional Auslander-Reiten theory: The classical notion of CM finiteness does not work so nicely in higher dimension, and we should replace it by ‘\(d\)-CM finiteness’ to have a fruitful theory. A full subcategory \(\mathcal{C}\) of \(\CM d R\) is called \(d\)-cluster tilting if \(\mathcal{C}\) is a functorially finite subcategory of \(\CM d R\) such that

\[
\mathcal{C} = \{ X \in \CM d R \mid \forall i \in \{1, 2, \ldots, d - 1\} \ Ext^i_{\mod R}(C, X) = 0 \}
\]

(see Section 2.2 for details). In this case \(\mathcal{C}\) satisfies \(\mathcal{C}(\omega) = \mathcal{C}\). We say that \((R, \mathbb{L})\) is \(d\)-Cohen-Macaulay finite (\(=\)\(d\)-CM finite) if there exists a \(d\)-cluster tilting subcategory \(\mathcal{C}\) of \(\CM d R\) such that there are only finitely many isomorphism classes of indecomposable objects in \(\mathcal{C}\) up to degree shift. In the classical case \(d = 1\), \(d\)-CM finiteness coincides with classical CM finiteness since \(\CM d R\) is a unique 1-cluster tilting subcategory of \(\CM d R\). We will study the following problem:

- When is \((R, \mathbb{L})\) \(d\)-CM finite?

We give a sufficient condition for \((R, \mathbb{L})\) to be \(d\)-CM finite in terms of tilting theory for the stable category \(\CM d R\). We say that a tilting object \(U\) in \(\CM d R\) is \(d\)-tilting if the endomorphism algebra \(\End^d_R(U)\) has global dimension at most \(d\). We have the following result.

Theorem 1.4. (Theorem 4.31) Assume that \(\CM d R\) has a \(d\)-tilting object. Then \((R, \mathbb{L})\) is Fano and \(d\)-CM finite.

Combining this with Theorem 1.2, we have the following list of GL complete intersections that are \(d\)-CM finite, though \(T^{\CM}\) itself is not necessarily \(d\)-tilting.

Theorem 1.5. (Theorem 4.31) If \(n = d + 2\) and one of the following conditions are satisfied, then \((R, \mathbb{L})\) is \(d\)-CM finite.

- \(n \geq 3\) and \((p_1, p_2, p_3) = (2, 2, p_3), (2, 3, 3), (2, 3, 4)\) or \((2, 3, 5)\).
- \(n \geq 4\) and \((p_1, p_2, p_3, p_4) = (3, 3, 3, 3)\).

We conjecture that \((R, \mathbb{L})\) is \(d\)-Fano if and only if \((R, \mathbb{L})\) is \(d\)-CM finite (see Conjecture 1.15 below).

1.3. Our Results on Geigle-Lenzing projective spaces. The second aim of this paper is to introduce the category \(\text{coh}\mathcal{X}\) of coherent sheaves on a GL projective space and study its basic properties. It is defined from the GL complete intersection \((R, \mathbb{L})\) by the Serre construction

\[
\text{coh}\mathcal{X} := \mod^d R / \mod^d R
\]

(see Section 5 for details). For the case \(n = 0\), we obtain the projective space \(\mathbb{P}^d\), and for the case \(d = 1\), we obtain the weighted projective lines of Geigle-Lenzing. We may regard \(\text{coh}\mathcal{X}\) as the category of \(G\)-equivariant coherent sheaves on the punctured spectrum \((\spec \mathcal{R}) \setminus \{R_+\}\) for \(G := \spec k[\mathbb{L}]\) (see Remark 5.11).

The following are some basic properties of \(\text{coh}\mathcal{X}\) (see Theorem 5.3 for details).

- \(\text{coh}\mathcal{X}\) is a Noetherian abelian category with global dimension \(d\).
- (Auslander-Reiten-Serre duality) There exists a functorial isomorphism \(\Ext^d_{\mathcal{X}}(X, Y) \cong D \hom_{\mathcal{X}}(Y, X(\omega))\) for any \(X, Y \in \text{coh}\mathcal{X}\).
As in the case of GL complete intersections $(R, \mathbb{L})$, we have a trichotomy of GL projective spaces: Fano, Calabi-Yau and anti-Fano given by the sign $\delta(\omega)$ of the dualizing element $\omega$. We observe (see Theorem 5.3) that these classes are characterized by ampltude of the automorphism $(-\omega)$ and $\omega$ of $\text{coh} \ X$ in the sense of Artin-Zhang [AZ] (see Definition 5.8).

We will introduce two important full subcategories of $\text{coh} \ X$: One is the category $\text{vect} \ X$ of vector bundles on $X$, and the other is its full subcategory $\text{line} \ X$ of direct sums of line bundles (see Section 5.2 for details), which play an important role in this paper.

One of the important properties of weighted projective lines is the existence of tilting bundles, whose endomorphism algebras are Ringel’s canonical algebras. We generalize this result by showing that any GL projective space $X$ has a tilting bundle.

**Theorem 1.6.** (Theorem 6.7) Any GL projective space $X$ has a tilting bundle $T^{\text{ca}} \in \text{line} \ X$.

For the case $d = 1$ this was known by Geigle-Lenzing [GL1]. Also this was known by Beilinson [Be] for $n = 0$, by Baer [Ba] for $n \leq d + 1$, and by Ishii-Ueda [I-U] for $n = d + 2$. In the context of ‘Geigle-Lenzing orders’ on $\mathbb{P}^d$, this has been shown in parallel independently by Lerner and the second author [L].

We use the tilting object $T^{\text{ca}}$ to describe the Coxeter polynomial of $\text{coh} \ X$ explicitly (see Theorem 6.20). We call the endomorphism algebra $A^{\text{ca}} := \text{End}_X(T^{\text{ca}})$ the $d$-canonical algebra, and get a derived equivalence $D^b(\text{coh} \ X) \simeq D^b(\text{mod} \ A^{\text{ca}})$.

We give the following basic properties of $A^{\text{ca}}$.

**Theorem 1.7.** (Theorem 6.7) Let $p_i \geq 2$ for all $i$. (We may assume this without loss of generality.) Then we have

$$\text{gl.dim} \ A^{\text{ca}} = \begin{cases} d & n \leq d + 1, \\ 2d & n > d + 1. \end{cases}$$

Moreover, in the first case, $A^{\text{ca}}$ is a $d$-representation infinite algebra.

We give an explicit presentation of $A^{\text{ca}}$ in terms of a quiver with relations (see Theorem 4.21 and Section 6.3). In particular, we show that, if $n \leq d + 1$, then $A^{\text{ca}}$ is a $d$-representation infinite algebra of type $\tilde{A}$ introduced in [HI] (see Theorem 6.13).

More generally, we study the endomorphism algebras $\text{End}_X(V)$ of tilting bundles $V$ on $X$ including $V = T^{\text{ca}}$ given above. For any such $V$ there is a strong relationship between the GL projective space $X$ and the endomorphism algebra $\text{End}_X(V)$. For example, we will show the following result, which uses the notions of $d$-Fano and $d$-anti-Fano algebras (see Definition 7.8) that were recently introduce by the third author [Min-MM] in non-commutative algebraic geometry.

**Theorem 1.8.** (Theorem 7.10) Let $V$ be a tilting bundle on $X$. Then $X$ is Fano (respectively, anti-Fano) if and only if $\text{End}_X(V)$ is a $d$-Fano (respectively, $d$-anti-Fano) algebra.

We study vector bundles on GL projective spaces. We say that a GL projective space $X$ is vector bundle finite ($\equiv \text{VB finite}$) if there are only finitely many isomorphism classes of indecomposable objects in $\text{vect} \ X$ up to degree shift.

**Theorem 1.9.** (Theorem 5.13) A GL projective space $X$ is VB finite if and only if $d = 1$ and $X$ is Fano (or equivalently, domestic).

Similarly to $d$-CM finiteness of GL complete intersections, we say that a GL projective space $X$ is d-vector bundle finite ($\equiv d$-VB finite) if there exists a $d$-cluster tilting subcategory $\mathcal{C}$ of $\text{vect} \ X$ (see Section 2.2 for details) such that there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$ up to degree shift. We will study the following problem as in the case of $(R, \mathbb{L})$:

- When is $X$ $d$-VB finite?

There is a close relationship between $d$-CM finiteness of $(R, \mathbb{L})$ and $d$-VB finiteness of $X$ since we always have a fully faithful functor $\text{CM}^L \ R \to \text{vect} \ X$. This is an equivalence in the classical case.
d = 1, but \( \text{vect} \mathcal{X} \) is much bigger than \( \text{CM}^{d} R \) for \( d > 1 \). In fact we have the following description of \( \text{CM}^{d} R \) inside \( \text{vect} \mathcal{X} \) (Proposition 5.13):

\[
\text{CM}^{d} R = \left\{ X \in \text{vect} \mathcal{X} \mid \forall i \in \{1, 2, \ldots, d-1\} \text{ Ext}_{\mathcal{X}}^{i}(\text{line} \mathcal{X}, X) = 0 \right\} = \left\{ X \in \text{vect} \mathcal{X} \mid \forall i \in \{1, 2, \ldots, d-1\} \text{ Ext}_{\mathcal{X}}^{i}(X, \text{line} \mathcal{X}) = 0 \right\}.
\]

Note that the objects in \( \text{CM}^{d} R \) are often called arithmetically Cohen-Macaulay bundles (e.g. \text{CH CMP}). Thanks to the similarity between the equalities above and our definition of \( d \)-cluster tilting subcategories, we obtain the following observation.

**Theorem 1.10.** (Theorem 5.13) \( d \)-cluster tilting subcategories of \( \text{CM}^{d} R \) are precisely \( d \)-cluster tilting subcategories of \( \text{vect} \mathcal{X} \) containing \( \text{line} \mathcal{X} \). In particular, if \((R, \mathcal{L})\) is \( d \)-CM finite, then \( \mathcal{X} \) is \( d \)-VB finite.

Therefore \( \mathcal{X} \) is \( d \)-VB finite for the cases listed in Theorem 1.5. Another immediate consequence of Theorem 1.10 is the following result, where the ‘if’ part generalizes Horrocks’ splitting criterion for vector bundles on projective spaces \text{[OSS 2.3.1]}.

**Corollary 1.11.** (Corollary 5.20) \( \text{line} \mathcal{X} \) is a \( d \)-cluster tilting subcategory of \( \text{vect} \mathcal{X} \) if and only if \( n \leq d + 1 \).

One of our main problems to study is the following:

- When is \( \text{coh} \mathcal{X} \) derived equivalent to a \( d \)-representation infinite algebra?

As for \( \text{CM}^{d} R \), we say that a tilting object \( V \) in \( \text{D}^{b}(\text{coh} \mathcal{X}) \) is \( d \)-\textit{tilting} if its endomorphism algebra \( \text{End}_{\text{D}^{b}(\text{coh} \mathcal{X})}(V) \) has global dimension at most \( d \). In fact this is equivalent to the global dimension being precisely \( d \) (see Proposition 7.11(a)). In this case there is the following important observation due to Buchweitz-Hille \text{[BH]} (see Proposition 7.11(e)):

- If \( V \) is a \( d \)-tilting sheaf, then \( \text{End}_{\mathcal{X}}(V) \) is \( d \)-representation infinite.

Therefore we study the following more accessible question:

- When does \( \mathcal{X} \) have a \( d \)-tilting bundle?

We will show that this is closely related to \( d \)-VB finiteness of \( \mathcal{X} \). A key role is played by the notion of a slice in a \( d \)-cluster tilting subcategory \( \mathcal{U} \) of \( \text{vect} \mathcal{X} \), which is an object \( V \in \mathcal{U} \) such that \( \text{Hom}(V, V(\ell \widetilde{\omega})) = 0 \) holds for any \( \ell > 0 \), and any indecomposable object \( X \in \mathcal{U} \) is a direct summand of \( V(\ell \widetilde{\omega}) \) for some \( \ell \in \mathbb{Z} \) (see Definition \text{[KX]} for details). The first part of the following result shows that these two notions coincide.

**Theorem 1.12.** (Theorem 7.14)

(a) (tilting-cluster tilting correspondence) \( d \)-tilting bundles on \( \mathcal{X} \) are precisely slices in \( d \)-cluster tilting subcategories of \( \text{vect} \mathcal{X} \).

(b) If \( \mathcal{X} \) has a \( d \)-tilting bundle \( V \), then \( \mathcal{X} \) is Fano, \( d \)-VB finite, and derived equivalent to \( \text{End}_{\mathcal{X}}(V) \), which is a \( d \)-representation infinite algebra.

To prove this theorem, we need the following general result on \( d \)-representation infinite algebras.

**Theorem 1.13.** (Theorem 2.11) Let \( \Lambda \) be a \( d \)-representation infinite algebra such that \( \Pi(\Lambda) \) is left graded coherent. Then the subcategory

\[
\mathcal{V}_{\Lambda} := \{ X \in \mathcal{C}_{\Lambda} \mid \forall \ell \gg 0 \nu_{d-\ell}(X) \in \text{mod} \Lambda, \nu_{d}(X) \in (\text{mod} \Lambda)[-d] \}
\]

of \( \text{D}^{b}(\text{mod} \Lambda) \) has a \( d \)-cluster tilting subcategory

\[
\mathcal{U}_{\Lambda} := \text{add} \{ \nu_{d}(\Lambda) \mid i \in \mathbb{Z} \}.
\]

Now we consider a \( d \)-tilting bundle \( V \) on \( \mathcal{X} \) which is contained in \( \text{CM}^{d} R \). In this case, \( V \) is \( d \)-cluster tilting in \( \text{CM}^{d/2d} R \) and the preprojective algebra \( \Pi(\Lambda) \) of \( \Lambda := \text{End}_{\mathcal{X}}(V) \) is isomorphic to the corresponding \( d \)-Auslander algebra \( \text{End}_{R}^{d/2d}(V) \) (see Corollary 7.20):

\[
\Lambda \quad \text{preprojective algebra} \quad \Pi(\Lambda) \simeq \text{End}_{R}^{d/2d}(V) \quad \text{d-Auslander algebra} \quad R.
\]
A similar picture already appeared in [AIR] in a different setting.

We are expecting that $d$-tilting objects in $\text{CM}^d$ always lift to $d$-tilting bundles on $X$. In fact, by using Theorems 1.12 and 1.13, we give the following result.

**Theorem 1.14.** (Theorem 7.24) If $n = d + 2$ and $p_1 = p_2 = 2$, then there exists a $d$-tilting bundle on $X$. Therefore $X$ is $d$-VB finite and derived equivalent to a $d$-representation infinite algebra.

Some of our results can be summarized as follows.

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(R, L) is d-CM finite  CM^d R has a d-tilting object  X is Fano

X is d-VB finite  X has a d-tilting bundle  X is derived equivalent to a d-representation infinite algebra
```

We end this section by the following conjecture.

**Conjecture 1.15.** All conditions in the above diagram are equivalent.

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**2. Preliminaries**

In this section, we introduce basic notions which will be used throughout the paper. We refer to [ASS, ARS, Rin] for general background on representation theory of finite dimensional algebras, to [AHK, Hap] for general background on tilting theory, to [Y, LW] for general background on Cohen-Macaulay representation theory, and to [BrH] for general background on commutative ring theory.

Throughout this section, we denote by $k$ an arbitrary field, and by $D$ the $k$-dual, that is $D(-) = \text{Hom}_k(-, k)$. All modules are left modules. The composition of $f : X \to Y$ and $g : Y \to Z$ is denoted as $fg : X \to Z$. For a ring $\Lambda$, we denote by $\text{Mod}\Lambda$ the category of $\Lambda$-modules, by $\text{mod}\Lambda$ the category of finitely generated $\Lambda$-modules, by $\text{proj}\Lambda$ the category of finitely generated projective $\Lambda$-modules, For an abelian group $G$ and a $G$-graded $k$-algebra $\Lambda$, we denote by $\text{Mod}^G\Lambda$, $\text{mod}^G\Lambda$, and $\text{proj}^G\Lambda$ the $G$-graded versions.

For a class $\mathcal{X}$ of objects in an additive category $\mathcal{C}$, we denote by $\text{add}_\mathcal{C}\mathcal{X}$ or $\text{add}\mathcal{X}$ the full subcategory of $\mathcal{C}$ consisting of direct summands of finite direct sums of objects in $\mathcal{X}$.

For full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of a triangulated category $\mathcal{T}$, we denote by $\mathcal{X} \cdot \mathcal{Y}$ the full subcategory of $\mathcal{T}$ consisting of objects $Z$ such that there exists a triangle $X \to Z \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

We denote by $\mathcal{C}(-)$, $\mathcal{K}(-)$, and $\mathcal{D}(-)$ the category of complexes, the homotopy category, and the derived category, respectively. By $\mathcal{C}^b(-)$, $\mathcal{K}^b(-)$ and $\mathcal{D}^b(-)$ we mean the bounded version. We denote by $(\mathcal{D}^{-\infty}(-), \mathcal{D}^{\infty}(-))$ the standard t-structure in the derived category.

**2.1. Triangulated categories and tilting theory.** Let us start with recalling basic notions in triangulated categories. Throughout this section, let $\mathcal{T}$ be a triangulated category with a suspension functor $[1]$. 
We call a full subcategory $U$ of $\mathcal{T}$ triangulated if it is closed under cones and $[\pm 1]$. If $U$ is also closed under direct summands, we call it thick. For a subcategory $\mathcal{C}$ of $\mathcal{T}$, we denote by $\text{thick} \mathcal{C}$ or $\text{thick}_{\mathcal{T}} \mathcal{C}$ (respectively, $\text{tri} \mathcal{C}$ or $\text{tri}_{\mathcal{T}} \mathcal{C}$) the smallest thick (respectively, triangulated) subcategory of $\mathcal{T}$ containing $\mathcal{C}$.

The following observation can be checked easily.

Observation 2.1. We have $\text{thick} \mathcal{C} = \text{add} (\text{tri} \mathcal{C})$.

Tilting object. We say that an object $U \in \mathcal{T}$ is tilting if $\text{Hom}_{\mathcal{T}}(U, U[i]) = 0$ for any $i \neq 0$ and $\mathcal{T} = \text{thick} U$.

For example, for any ring $A$, the bounded homotopy category $K^b(\text{proj} A)$ of finitely generated projective $A$-modules has a tilting object $A$. Moreover a converse of this statement holds under reasonable assumptions. We call a triangulated category algebraic if it is triangle equivalent to the stable category of a Frobenius category, and idempotent-complete if any idempotent endomorphism corresponds to a direct summand. We say that a fully faithful triangle functor $F : \mathcal{T} \to \mathcal{T}'$ is an equivalence up to direct summands if, for any object $X \in \mathcal{T}'$, there exists an object $Y \in \mathcal{T}$ such that $X$ is a direct summand of $F(Y)$.

Proposition 2.2. [Hap1] Let $\mathcal{T}$ be an algebraic triangulated category with a tilting object $U$. Then there exists a triangle equivalence $F : \mathcal{T} \to K^b(\text{proj} \text{End}_{\mathcal{T}}(U))$ up to direct summands. In particular, if $\mathcal{T}$ is idempotent complete, then $F$ is a triangle equivalence.

We say that two finite dimensional algebras $\Lambda$ and $\Gamma$ are derived equivalent if one of the following equivalent conditions hold:

- There exists a triangle equivalence $K^b(\text{proj} \Lambda) \simeq K^b(\text{proj} \Gamma)$.
- There exists a triangle equivalence $D^b(\text{mod} \Lambda) \simeq D^b(\text{mod} \Gamma)$.
- There exists a triangle equivalence $D(\text{Mod} \Lambda) \simeq D(\text{Mod} \Gamma)$.
- There exists a tilting object $U$ in $K^b(\text{proj} \Lambda)$ such that $\text{End}_{K^b(\text{proj} \Lambda)}(U) \simeq \Gamma$.

The following observations are basic.

Proposition 2.3. (a) A finite dimensional $k$-algebra $\Lambda$ has finite global dimension if and only if the natural functor $K^b(\text{proj} \Lambda) \to D^b(\text{mod} \Lambda)$ is an equivalence.

(b) Assume that finite dimensional $k$-algebras $\Lambda$ and $\Gamma$ are derived equivalent. Then $\Lambda$ has finite global dimension if and only if so does $\Gamma$.

Let us recall the notion of Serre functors in triangulated categories.

Serre functor. Let $\mathcal{T}$ be a $k$-linear and Hom-finite triangulated category. A Serre functor of $\mathcal{T}$ is an autoequivalence $S : \mathcal{T} \to \mathcal{T}$ such that there exists a functorial isomorphism $\text{Hom}_{\mathcal{T}}(X, Y) \simeq D \text{Hom}_{\mathcal{T}}(Y, SX)$ for any $X, Y \in \mathcal{T}$. It is easy to show that Serre functors of $\mathcal{T}$ are unique up to isomorphism of functors.

For example, if $X$ is a smooth projective variety of dimension $d$ with a canonical sheaf $\omega$, then $D^b(\text{coh} X)$ has a Serre functor $\omega[d] \otimes_X -$. The following basic result by Happel gives another typical example of Serre functors.

Proposition 2.4. [Hap1] Let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension. Then the Nakayama functor

$$\nu := (DA) \otimes_A L : D^b(\text{mod} \Lambda) \to D^b(\text{mod} \Lambda)$$

gives a Serre functor of $D^b(\text{mod} \Lambda)$.

The following elementary observation is useful to calculate the global dimension.

Observation 2.5. Let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension. Then

$$\text{gl.dim} \Lambda = \sup \{ i \in \mathbb{Z} \mid \text{Ext}^i_\Lambda (DA, \Lambda) \neq 0 \} = \sup \{ i \in \mathbb{Z} \mid \text{Hom}_{D^b(\text{mod} \Lambda)}(\Lambda, \nu^{-1}(\Lambda)[i]) \neq 0 \}.$$
Calabi-Yau triangulated category. Let \( \mathcal{T} \) be a triangulated category with a Serre functor \( S \). We say that \( \mathcal{T} \) is fractionally Calabi-Yau of dimension \( \frac{m}{\ell} \) (or simply \( \frac{m}{\ell} \)-Calabi-Yau) for integers \( \ell \neq 0 \) and \( m \) if there exists an isomorphism \( S^\ell \cong [m] \) of functors \( \mathcal{T} \to \mathcal{T} \). Observe that \( \frac{m}{\ell} \)-Calabi-Yau implies \( \frac{m_i}{\ell_i} \)-Calabi-Yau for all integers \( i \neq 0 \).

We say that a finite dimensional \( k \)-algebra \( \Lambda \) with finite global dimension is fractionally Calabi-Yau of dimension \( \frac{m}{\ell} \) (or \( \frac{m}{\ell} \)-Calabi-Yau) if \( \mathbb{D}^b(\text{mod } \Lambda) \) is. We give a few examples.

Example 2.6. (a) [MY] Let \( kQ \) be a path algebra of Dynkin quiver. Then \( kQ \) is \( \frac{b-2}{b} \)-Calabi-Yau for the Coxeter number \( b \) of \( Q \):

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{A}_n & \text{D}_n & \text{E}_6 & \text{E}_7 & \text{E}_8 \\
\hline
n+1 & 2(n-1) & 12 & 18 & 30 \\
\hline
\end{array}
\]

(b) [HI1] Assume that \( \Lambda \) is \( \frac{m}{\ell} \)-Calabi-Yau for \( i = 1, 2 \). If \( \Lambda \otimes_k \Lambda \) has finite global dimension, then \( \Lambda \otimes_k \Lambda \) is \( \frac{m}{\ell} \)-Calabi-Yau for \( \ell := \text{l.c.m.}(\ell_1, \ell_2) \) and \( m := \frac{\ell_1m_1}{\ell_1} + \frac{\ell_2m_2}{\ell_2} \).

The following observations are easy.

Proposition 2.7. Let \( \Lambda \) be a finite dimensional \( k \)-algebra of finite global dimension \( n \).

(a) \( \nu^{-1}(\mathbb{D}^{\leq 0}(\text{mod } \Lambda)) \subset \mathbb{D}^{\leq 0}(\text{mod } \Lambda) \) and \( \nu(\mathbb{D}^{\leq 0}(\text{mod } \Lambda)) \subset \mathbb{D}^{\leq 0}(\text{mod } \Lambda) \) hold.

(b) \( \nu^{-1}(\mathbb{D}^{\leq 0}(\text{mod } \Lambda)) \subset \mathbb{D}^{\leq n}(\text{mod } \Lambda) \) and \( \nu(\mathbb{D}^{\geq 0}(\text{mod } \Lambda)) \subset \mathbb{D}^{\geq -n}(\text{mod } \Lambda) \) hold.

(c) If \( \mathbb{D}^b(\text{mod } \Lambda) \) is \( \frac{m}{\ell} \)-Calabi-Yau, then \( 0 \leq \frac{m}{\ell} \leq n \) holds. If \( n \neq 0 \), then both inequalities are strict.

Proof. (a) and (b) are elementary (see [13] 5.4(a) for (b)).

(c) We may assume \( \ell > 1 \). Since \( \nu^\ell = [m] \), we have \( \mathbb{D}^{\leq -m}(\text{mod } \Lambda) = \nu^\ell(\mathbb{D}^{\leq 0}(\text{mod } \Lambda)) \subset \mathbb{D}^{\leq 0}(\text{mod } \Lambda) \) by (a). Thus \( m \geq 0 \) and hence \( 0 \leq \frac{m}{\ell} \) holds. Moreover we have

\[ \Lambda[m] = \nu^\ell(\Lambda) = \nu^{\ell-1}(D\Lambda) \in \mathbb{D}^{\geq -(\ell-1)n}(\text{mod } \Lambda) \]

by (b). Hence \( -m \geq -(\ell-1)n \) holds, and we have \( \frac{m}{\ell} \leq \frac{n}{\ell-1} \leq n \).

In particular, if one of the equalities holds, then \( m \) has to be 0. Using \( \nu^\ell = \text{id} \) and (a), we have

\[ \mathbb{D}^{\leq 0}(\text{mod } \Lambda) = \nu^\ell(\mathbb{D}^{\leq 0}(\text{mod } \Lambda)) \subset \cdots \subset \nu(\mathbb{D}^{\geq 0}(\text{mod } \Lambda)) \subset \mathbb{D}^{\leq 0}(\text{mod } \Lambda). \]

Thus all equalities hold, and we have \( \nu^{\pm 1}(\mathbb{D}^b(\text{mod } \Lambda)) = \mathbb{D}^{\leq 0}(\text{mod } \Lambda) \). In particular, we have \( \nu^{-1}(\Lambda) \in \mathbb{D}^b(\text{mod } \Lambda) \) and \( \text{Hom}_{\mathbb{D}^b(\text{mod } \Lambda)}(\Lambda, \nu^{-1}(\Lambda)[i]) = 0 \) for any \( i > 0 \), which implies \( n = 0 \) by Observation 2.5.

Later we use the following general observation.

Proposition 2.8. Let \( \mathcal{T} \) be a triangulated category, \( S \) a thick subcategory of \( \mathcal{T} \) and \( \pi : \mathcal{T} \to \mathcal{T}/S \) the natural functor.

(a) We have a bijection between thick subcategories of \( \mathcal{T} \) containing \( S \) and thick subcategories of \( \mathcal{T}/S \) given by \( U \mapsto \text{add } \pi(U) \).

(b) For a thick subcategory \( U \) of \( \mathcal{T} \), we have \( \text{thick } \{ S, U \} = \mathcal{T} \) if and only if \( \text{thick } \mathcal{T}/S U = \mathcal{T}/S \).

Proof. (a) It is easy to check that \( \pi(U) \) is a triangulated subcategory of \( \mathcal{T}/S \). By Observation 2.1, we have that \( \text{add } \pi(U) \) is a thick subcategory of \( \mathcal{T}/S \).

For a thick subcategory \( V \) of \( \mathcal{T}/S \), it is clear that \( \pi^{-1}(V) := \{ X \in \mathcal{T} \mid \pi(X) \in V \} \) is a thick subcategory of \( \mathcal{T} \) containing \( S \). It is easy to check that these correspondences are mutually inverse.

(b) This is immediate from (a).
2.2. Higher dimensional Auslander-Reiten theory. Let us start with recalling the following basic notion.

**Functorially finiteness.** [AS] Let $A$ be an additive category and $C$ its full subcategory. For an object $A \in A$, we say that a morphism $f : C \to A$ is a right $C$-approximation of $A$ if $C \in C$ and the map $f : \text{Hom}_A(C', C) \to \text{Hom}_A(C', A)$ is surjective for any $C' \in C$. If any object in $A$ has a right $C$-approximation, then we say that $C$ is a covariantly finite subcategory of $A$. Dually we define a left $C$-approximation and a covariantly finite subcategory. We say that $C$ is functorially finite if it is contravariantly finite and covariantly finite.

The notion of a $d$-cluster tilting subcategory is central in higher dimensional Auslander-Reiten theory. Note that it is also called a maximal $(d-1)$-orthogonal subcategory.

**$d$-cluster tilting subcategory.** [I2, I3, IY] Let $A$ be an abelian category, $B$ a full extension-closed subcategory of $A$ and $C$ a full subcategory of $B$. We say that $C$ generates (respectively, cogenerates) $B$ if any object in $B$ is a subobject (respectively, factor object) of some object in $C$. We say that $C$ is a $d$-cluster tilting subcategory of $B$ if $C$ is a functorially finite subcategory of $B$ that generates and cogenerates $B$ such that

$$C = \{ X \in B | \forall i \in \{1, 2, \ldots, d-1\} \text{ Ext}^i_A(C, X) = 0 \} = \{ X \in B | \forall i \in \{1, 2, \ldots, d-1\} \text{ Ext}^i_A(X, C) = 0 \}.$$ 

Similarly, we define a $d$-cluster tilting subcategory of a triangulated category $T$ as a functorially finite subcategory $C$ satisfying

$$C = \{ X \in T | \forall i \in \{1, 2, \ldots, d-1\} \text{ Hom}_T(C, X[i]) = 0 \} = \{ X \in T | \forall i \in \{1, 2, \ldots, d-1\} \text{ Hom}_T(X, C[i]) = 0 \}.$$ 

We say that an object $C$ is $d$-cluster tilting if $\text{add} C$ is a $d$-cluster tilting subcategory.

In this paper, we apply these definitions for $B$ in the following settings:

- $B := \text{mod}^L R$ (considered as a subcategory of $A := \text{CM}^L R$) or $T := \text{CM}^L R$ for a Geigle-Lenzing complete intersection $(R, \mathbb{L})$ (see Section 3).
- $B := \text{coh} X$ (considered as a subcategory of $A := \text{vect} X$) for a Geigle-Lenzing projective space $X$ (see Section 5).
- $T := \text{D}^b(\text{mod} \Lambda)$ for a finite dimensional $k$-algebra $\Lambda$.

In the rest of this section, let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension. For an integer $d$, we define the $d$-shifted Nakayama functor by

$$\nu_d := (DA)[-d] \otimes \Lambda - : \text{D}^b(\text{mod} \Lambda) \to \text{D}^b(\text{mod} \Lambda).$$

When $\Lambda$ has global dimension at most $d$, we define the $d$-Auslander-Reiten translation by

$$\tau_d := \text{Tor}^d_A(DA, -) \simeq D \text{ Ext}^d_A(-, \Lambda) : \text{mod} \Lambda \to \text{mod} \Lambda.$$ 

Then we have

$$\tau_d \simeq H^0(\nu_d -) : \text{mod} \Lambda \to \text{mod} \Lambda.$$ 

The following classes of finite dimensional algebras are basic in this paper.

**Definition 2.9.** [I3] [I2] [HIO] Let $\Lambda$ be a finite dimensional $k$-algebra of global dimension at most $d$.

(a) We call $\Lambda$ $\tau_d$-finite if $\tau_d^i(D\Lambda) = 0$ holds for some $i \geq 0$.

(b) We call $\Lambda$ $d$-representation infinite if $\nu_d^{-1}(\Lambda) \in \text{mod} \Lambda$ for all $i \geq 0$. This is equivalent to that $\nu_d^i(D\Lambda) \in \text{mod} \Lambda$ for all $i \geq 0$ by [HIO] 2.9.
The notion of preprojective algebras plays an important role. 

**Preprojective algebra.** Let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension. The $(d + 1)$-preprojective algebra (or simply preprojective algebra) of $\Lambda$ is defined as the $\mathbb{Z}$-graded $k$-algebra

$$
\Pi = \Pi(\Lambda) := \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{D^b(\mod \Lambda)}(\Lambda, \nu^\ell_d(\Lambda)),
$$

where the multiplication is given by

$$
f \cdot g := f\nu^\ell_d(g) \in \text{Hom}_{D^b(\mod \Lambda)}(\Lambda, \nu^{\ell + m}_d(\Lambda))
$$

for any $f \in \text{Hom}_{D^b(\mod \Lambda)}(\Lambda, \nu^\ell_d(\Lambda))$ and $g \in \text{Hom}_{D^b(\mod \Lambda)}(\Lambda, \nu^m_d(\Lambda))$.

A $d$-representation infinite algebra $\Lambda$ is called $d$-representation tame if the center $Z$ of $\Pi(\Lambda)$ is a Noetherian ring and $\Pi(\Lambda)$ is a finitely generated $Z$-module.

An algebra $\Lambda$ with global dimension at most $d$ is $\tau_d$-finite if and only if $\text{dim}_k \Pi(\Lambda)$ is finite. There is a systematic construction of $d$-cluster tilting subcategories by using the subcategory

$$
U_\Lambda := \text{add}\{\nu^\ell_d(\Lambda) \mid \ell \in \mathbb{Z}\}
$$

of $D^b(\mod \Lambda)$. We will use the following result for $\tau_d$-finite algebras.

**Theorem 2.10.** [I3, 1.23] Let $\Lambda$ be a finite dimensional $k$-algebra that is $\tau_d$-finite. Then $D^b(\mod \Lambda)$ has a $d$-cluster tilting subcategory $U_\Lambda$.

Now we fix a $d$-representation infinite algebra $\Lambda$ with $\Pi = \Pi(\Lambda)$. We assume that $\Pi$ is left graded coherent, that is, finitely presented $\mathbb{Z}$-graded $\Pi$-modules are closed under kernels. We show that there exist an abelian subcategory $C_\Lambda$ in $D^b(\mod \Lambda)$ and an extension-closed subcategory $V_\Lambda$ of $C_\Lambda$ such that $U_\Lambda$ defined above is a $d$-cluster tilting subcategory of $V_\Lambda$.

We define subcategories of $D^b(\mod \Lambda)$ by

$$
C_\Lambda := \{ X \in D^b(\mod \Lambda) \mid \forall \ell \gg 0 \nu^{\ell}_d(X) \in \mod \Lambda \},
$$

$$
V_\Lambda := \{ X \in C_\Lambda \mid \forall \ell \gg 0 \nu^{\ell}_d(X) \in (\mod \Lambda)[-d] \}.
$$

We have the following main result in this section.

**Theorem 2.11.** Let $\Lambda$ be a $d$-representation infinite algebra such that $\Pi(\Lambda)$ is left graded coherent.

(a) $C_\Lambda$ is an abelian category such that $D^b(C_\Lambda) \simeq D^b(\mod \Lambda)$,

(b) $V_\Lambda$ is an extension-closed subcategory of $C_\Lambda$ and has a $d$-cluster tilting subcategory $U_\Lambda$.

To prove Theorem 2.11 we need some preparation.

Let $\mod^Z \Pi$ be the category of finitely presented $\mathbb{Z}$-graded $\Pi$-modules, and $\mod^0 Z \Pi$ the category of finite dimensional $\mathbb{Z}$-graded $\Pi$-modules. Since $\Pi$ is assumed to be left graded coherent, $\mod^Z \Pi$ is an abelian category. Let

$$
\text{qgr}^Z \Pi := \mod^Z \Pi/ \mod^0 Z \Pi
$$

be the quotient category. The third author proved [Min 3.7] that there exists a triangle equivalence

$$
D^b(\mod \Lambda) \simeq D^b(\text{qgr} \Pi)
$$

sending $\Lambda$ to $\Pi$ and inducing an equivalence $C_\Lambda \simeq \text{qgr} \Pi$. In particular, we have Theorem 2.11(a).

To prove Theorem 2.11(b), the following observation is crucial, where $\ast$ denotes the extension of categories (see Section 2.1).

**Proposition 2.12.** We have $V_\Lambda = (U_\Lambda \ast U_\Lambda[1] \ast \cdots \ast U_\Lambda[d - 1] \cap (U_\Lambda[1 - d] \ast \cdots \ast U_\Lambda[-1] \ast U_\Lambda)$.

We start by preparing the following easy observation.

**Lemma 2.13.** Let $X$ and $C^i$ be $\Lambda$-modules. If $X \subset C^n[-n] \ast \cdots \ast C^1[-1] \ast C^0$, then there exists an exact sequence $0 \to X \to C^0 \to \cdots \to C^n \to 0$ of $\Lambda$-modules.
Proof. We use the induction on \(n\). Since \(X \in C^n[-n] \ast \cdots \ast C^1[-1] \ast C^0\), there exists a triangle
\[
Y[-1] \to X \to C^0 \to Y
\]  
(2.1)
in \(D^b(\mod \Lambda)\) with \(Y \in C^n[1-n] \ast \cdots \ast C^1\). Then we have
\[
Y \in (C^n[1-n] \ast \cdots \ast C^1) \cap (C^0 \ast X[1]) \subset \mod \Lambda
\]
by looking at cohomologies. Applying \(H^0\) to the triangle (2.1), we have an exact sequence \(0 \to X \to C^0 \to Y \to 0\). On the other hand, by the induction assumption, there exists an exact sequence \(0 \to Y \to C^1 \to \cdots \to C^n \to 0\). Combining these sequences, we have the assertion. \(\square\)

Now we are ready to prove Proposition 2.12.

Proof of Proposition 2.12
(i) We prove “\(\supset\)”. Fix \(X\) in the right hand side. For any \(U_i \in \mathcal{U}_\Lambda\) and \(U' \in \mathcal{U}_\Lambda\), we have for \(\ell \gg 0\)
\[
\nu^{-\ell}_d(U_i \ast U_1[1] \ast \cdots \ast U_{d-1}[d-1]) = (\nu^{-\ell}_d(U_0) \ast \nu^{-\ell}_d(U_1)[1] \ast \cdots \ast \nu^{-\ell}_d(U_{d-1})[d-1])
\]
\subset (\mod \Lambda) \ast (\mod \Lambda)[1] \ast \cdots \ast (\mod \Lambda)[d-1],
\[
\nu^{-\ell}_d(U^1-d[1-d] \ast \cdots \ast U^{-1}[d]) \subset (\mod \Lambda)[1-d] \ast \cdots \ast (\mod \Lambda)[d-1] \ast (\mod \Lambda).
\]

Therefore we have
\[
\nu^{-\ell}_d(X) \subset ((\mod \Lambda) \ast \cdots \ast (\mod \Lambda)[d-1]) \cap ((\mod \Lambda)[1-d] \ast \cdots \ast (\mod \Lambda)) = \mod \Lambda.
\]

By a similar argument, we have \(\nu^{\ell}_d(X) \in (\mod \Lambda)[-d]\) for \(\ell \gg 0\). Therefore \(X \in \mathcal{V}_\Lambda\).

(ii) We prove “\(\subset\)”. We only prove \(\mathcal{V}_\Lambda \subset \mathcal{U}_\Lambda \ast \mathcal{U}_\Lambda[1] \ast \cdots \ast \mathcal{U}_\Lambda[d-1]\) since one can show \(\mathcal{V}_\Lambda \subset \mathcal{U}_\Lambda[1-d] \ast \cdots \ast \mathcal{U}_\Lambda[d-1] \ast \mathcal{U}_\Lambda\) dually.

Let \(X \in \mathcal{V}_\Lambda\). Then \(Y := \nu^{\ell}_d(X)[d]\) and \(Z := \nu^{-\ell}_d(X)\) belong to \(\mod \Lambda\) for \(\ell \gg 0\). Since \(\Lambda\) has global dimension \(d\), we can take an injective resolution \(0 \to Y \to I^0 \to \cdots \to I^d \to 0\) of \(Y\). Then

\[
Y \in I^d[-d] \ast \cdots \ast I^1[-1] \ast I^0
\]
holds. Now let
\[
\mathcal{P}_\Lambda := \add\{\nu^{-i}_d(\Lambda) \mid i \geq 0\} \subset \mathcal{U}_\Lambda \cap \mod \Lambda
\]
be the category of \(n\)-preprojective \(\Lambda\)-modules. Then \(P^n := \nu^{-2\ell}(I^n)[-d]\) belongs \(\mathcal{P}_\Lambda\), and we have
\[
Z = \nu^{-2\ell}_d(Y)[-d] \in P^d[-d] \ast \cdots \ast P^1[-1] \ast I^0.
\]

By Lemma 2.13 we have an exact sequence \(0 \to Z \to P^0 \to \cdots \to P^d \to 0\) of \(\Lambda\)-modules, and in particular, \(Z\) is a submodule of \(P^0 \in \mathcal{P}_\Lambda\). By applying \([HIO\ 4.28]\), we have an exact sequence
\[
0 \to P_{d-1} \to \cdots \to P_0 \to Z \to 0
\]
of \(\Lambda\)-modules with \(P_i \in \mathcal{P}_\Lambda\). Therefore
\[
Z \in \mathcal{P}_\Lambda \ast \mathcal{P}_\Lambda[1] \ast \cdots \ast \mathcal{P}_\Lambda[d-1]
\]
holds, and hence \(X = \nu^{\ell}_d(Z) \in \mathcal{U}_\Lambda \ast \mathcal{U}_\Lambda[1] \ast \cdots \ast \mathcal{U}_\Lambda[d-1]\). \(\square\)

Now we are ready to prove Theorem 2.11
Proof of Theorem 2.11
(i) It was shown in \([HIO\ 4.2]\) that \(\Hom_{D^b(\mod \Lambda)}(U_\Lambda, U_\Lambda[i]) = 0\) holds for all \(i\) with \(1 \leq i \leq d-1\).

(ii) For any \(X \in \mathcal{V}_\Lambda\), Proposition 2.12 shows that there exists a triangle
\[
U \xrightarrow{\varphi} X \xrightarrow{\varphi} Y \xrightarrow{1} U[1]
\]  
(2.2)
with \(U \in \mathcal{U}_\Lambda\) and \(Y \in \mathcal{U}_\Lambda[1] \ast \cdots \ast \mathcal{U}_\Lambda[d-1]\). Since \(\Hom_{D^b(\mod \Lambda)}(U_\Lambda, U_\Lambda[1] \ast \cdots \ast U_\Lambda[d-1]) = 0\), we have that \(g\) is a right \(U_\Lambda\)-approximation of \(X\). Thus \(U_\Lambda\) is a contravariantly finite subcategory of \(\mathcal{V}_\Lambda\).

Dually one can show that \(U_\Lambda\) is a covariantly finite subcategory of \(\mathcal{V}_\Lambda\).
We regard $S$ as an $L$-graded $k$-algebra by

$$\deg T_j := \overline{c} \quad \text{and} \quad \deg X_i := \overline{x}_i$$

(iii) Assume that $X \in V_\Lambda$ satisfies $\text{Hom}_{D^b(\text{mod } \Lambda)}(X, \mathcal{U}_\Lambda[i]) = 0$ for all $i$ with $1 \leq i \leq d - 1$. Then $f = 0$ holds in the triangle \ref{2.22}. Thus $g$ is a split epimorphism, and we have $X \in \mathcal{U}_\Lambda$.

Similarly one can show that if $X \in V_\Lambda$ satisfies $\text{Hom}_{D^b(\text{mod } \Lambda)}(\mathcal{U}_\Lambda, X[i]) = 0$ for all $i$ with $1 \leq i \leq d - 1$, then $X \in \mathcal{U}_\Lambda$. \hfill $\square$

At the end of this section, we include the following observation on a generalization of $d$-representation infinite algebras, which we will apply to our higher canonical algebras (see Theorem \ref{5.15}).

**Definition 2.14.** Let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension. We call $\Lambda$ **almost $d$-representation infinite** if $H^i(\nu^{-1}_d(\Lambda)) = 0$ holds for all $i \in \mathbb{Z}$ and all $j \in \mathbb{Z} \setminus \{0, d\}$.

Clearly any $d$-representation infinite algebra is almost $d$-representation infinite. Moreover we have the following easy observations.

**Proposition 2.15.**

(a) $d$-representation infinite algebras are precisely almost $d$-representation infinite algebras of global dimension $d$.

(b) An almost $d$-representation infinite algebra has global dimension $d$ or $2d$.

**Proof.** (a) We only have to show that any almost $d$-representation infinite algebra $\Lambda$ of global dimension $d$ is $d$-representation infinite. Since $\nu^{-1}_d(\Lambda) \in D^{\leq 0}(\text{mod } \Lambda)$ holds for any $i \geq 0$ by Proposition \ref{2.7} (b), we have $\nu^{-1}_d(\Lambda) \in \text{mod } \Lambda$. Thus the assertion follows.

(b) Since $\Lambda$ has finite global dimension, $\text{gl.dim } \Lambda = \max\{i \geq 0 \mid \text{Ext}^i_d(DA, \Lambda) \neq 0\}$ holds by Observation \ref{2.5}. Since

$$\text{Ext}^i_d(DA, \Lambda) = \text{Hom}_{D^b(\text{mod } \Lambda)}(\Lambda, \nu^{-1}_d(\Lambda)[i-d]) = H^{i-d}(\nu^{-1}_d(\Lambda))$$

vanishes except $i = d$ or $2d$, we have the assertion. \hfill $\square$

3. **Geigle-Lenzing complete intersections**

Throughout this paper we fix an arbitrary base field $k$ and a dimension $d \geq 1$. (We assume neither $k$ to have characteristic zero nor $k$ to be algebraically closed.)

3.1. **The definition and basic properties.** We start with the polynomial algebra

$$C := k[T] = k[T_0, \ldots, T_d]$$

in $d + 1$ variables and the associated projective $d$-space $\mathbb{P}^d$. We choose $n$ hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^d$ where $n \geq 0$, given as zeros of the linear forms

$$\ell_i(T) = \sum_{j=0}^{d} \lambda_{ij} T_j \in C.$$ 

Also we fix an $n$-tuple $(p_1, \ldots, p_n)$ of positive integers. Let

$$S := C[X] = k[T, X] = k[T_0, \ldots, T_d, X_1, \ldots, X_n]$$

be the polynomial algebra in $d + n + 1$ variables and

$$h_i := X_i^{p_i} - \ell_i(T) \in S.$$ 

Now we consider the factor $k$-algebra

$$R := S/(h_i \mid 1 \leq i \leq n)$$

**The grading group.** Let $L$ be an abelian group generated by $\overline{x}_1, \ldots, \overline{x}_n, \overline{c}$, modulo relations $p_i \overline{x}_i = \overline{c}$ for any $1 \leq i \leq n$:

$$L := (\overline{x}_1, \ldots, \overline{x}_n, \overline{c})/(p_i \overline{x}_i - \overline{c} \mid 1 \leq i \leq n).$$

We regard $S$ as an $L$-graded $k$-algebra by

$$\deg T_j := \overline{c} \quad \text{and} \quad \deg X_i := \overline{x}_i.$$
for any $i$ and $j$. Since $\deg h_i = c$ for any $i$, we can regard $R$ as an $\mathbb{L}$-graded $k$-algebra.

**Geigle-Lenzing complete intersection.** We call the pair $(R, L)$ a weak Geigle-Lenzing (GL) complete intersection associated with $H_1, \ldots, H_n$ and $p_1, \ldots, p_n$. This is in fact a complete intersection of dimension $d + 1$ as we will see in Proposition 3.7 below.

We call $R$ Geigle-Lenzing (GL) complete intersection if our hyperplanes $H_1, \ldots, H_n$ are in a general position in the following sense:

- Any set of at most $d + 1$ of the polynomials $\ell_i$ is linearly independent.

In the rest we assume that $R$ is a weak GL complete intersection.

Let $\mathbb{L}_+$ be the submonoid of $\mathbb{L}$ generated by all $\vec{x}_i$'s and $\vec{c}$. We equip $\mathbb{L}$ with the structure of a partially ordered set: $\vec{x} \geq \vec{y}$ if and only if $\vec{x} - \vec{y} \in \mathbb{L}_+$. Then $\mathbb{L}_+$ consists of all elements $\vec{x} \in L$ satisfying $\vec{x} \geq 0$. We denote intervals in $\mathbb{L}$ by $[\vec{x}, \vec{y}] := \{\vec{z} \in \mathbb{L} | \vec{x} \leq \vec{z} \leq \vec{y}\}$.

We collect some basic observations.

**Observation 3.1.** (a) Any element $\vec{x} \in L$ can be written uniquely as

$$\vec{x} = \sum_{i=1}^{n} a_i \vec{x}_i + a \vec{c}$$

with $0 \leq a_i < p_i$ and $a \in \mathbb{Z}$. We call it a normal form of $\vec{x}$.

(b) $L$ is an abelian group with rank one. It does not have torsion elements if and only if $p_1, \ldots, p_n$ are pairwise coprime.

(c) We have $R_0 \neq 0$ if and only if $\vec{x} \in \mathbb{L}_+$ if and only if $a \geq 0$ in the normal form in (a).

Therefore $R$ is positively graded in this sense.

**Observation 3.2 (Weights 1).** Adding a hyperplane $H_{n+1}$ given by $\ell_{n+1}$ with weight $p_{n+1} = 1$ changes neither $\mathbb{L}$ nor $R$, since the new variable $X_{n+1}$ is expressed as a linear combination of $T_i$'s by the relation $X_{n+1} = \ell_{n+1}(T)$.

Thus we may freely add or remove hyperplanes with weights 1. Therefore we can assume that $p_i \geq 2$ for any $i$ with $1 \leq i \leq n$ without loss of generality by removing all hyperplanes with weights 1.

**Observation 3.3 (Normalization).** Now we assume that $(R, \mathbb{L})$ is a GL complete intersection. Then the group $GL(d+1, k)$ acts on $\mathcal{S}$ by acting on the linear span of the variables $T_i$. Transforming coordinates in this way we may assume

$$\ell_i(T) = \begin{cases} T_{i-1} & \text{if } 1 \leq i \leq \min\{d+1, n\}, \\ \sum_{j=0}^{d} \lambda_{i,j} T_j & \text{if } \min\{d+1, n\} < i \leq n. \end{cases}$$

Then we obtain the relations $h_i = X_i^{p_i} - T_{i-1}$ for $1 \leq i \leq \min\{d+1, n\}$. Therefore the variables $T_i$ with $0 \leq i \leq \min\{d, n-1\}$ are superfluous in the presentation of $R$, and we may write

$$R = \left\{ k[X_1, \ldots, X_n, T_n, \ldots, T_d] \bigg| \begin{array}{ll} k[X_1, \ldots, X_n]/(X_i^{p_i} - \sum_{j=1}^{d+1} \lambda_{i,j-1} X_j^{p_j}) & \text{if } d + 2 \leq i \leq n \\ \{\lambda_{i,j-1} | d + 2 \leq i \leq n, 1 \leq j \leq d+1 \} & \text{if } n \geq d + 2 \end{array} \right\}.$$

In the form (3.1), our assumption that $H_1, \ldots, H_n$ are in a general position is equivalent to that all minors (including non-maximal ones) of the $(n - d - 1) \times (d + 1)$ matrix

$$|\lambda_{i,j-1}|_{d+2 \leq i \leq n, 1 \leq j \leq d+1}$$

have non-zero determinants. In the case $d = 1$, this means that the $n$ points

$$(1 : 0), (0 : 1), (\lambda_{30} : \lambda_{31}), \ldots, (\lambda_{n0} : \lambda_{n1})$$

in $\mathbb{P}^1$ are mutually distinct. If $k$ is an algebraically closed, then we can normalize the relation in (3.1) for $i = d + 2$ as

$$X_{d+2}^{p_{d+2}} = X_1^{p_1} + X_2^{p_2} + \cdots + X_{d+1}^{p_{d+1}}.$$
This presentation is widely used in $d = 1$.

Let us observe that our weak GL complete intersections can be obtained the following elementary construction, for which the name ‘root construction’ was used for stacks [AGV].

**Observation 3.4** (Root construction). Let $G$ be an abelian group, and $A$ a commutative $G$-graded ring. For a non-zero homogeneous element $\ell \in A$ of degree $g \in G$ and a positive integer $p$, let

$$G(\ell, p) := (G \oplus \mathbb{Z})/((g, -p)) \quad \text{and} \quad A(\ell, p) := A[X]/(X^p - \ell).$$

Then $A(\ell, p)$ is a $G(\ell, p)$-graded ring by $\deg X := (0, 1)$ and $\deg a := (g', 0)$ for any $a \in A_{g'}$ with $g' \in G$. We call this process to construct $(A(\ell, p), G(\ell, p))$ from $(A, G)$ a root construction. Clearly,

- $A(\ell, p)$ is a free $A$-module with a basis $\{X^i \mid 0 \leq i < p\}$.

Our weak GL complete intersection $(R, L)$ can be obtained by applying root construction iteratively to the polynomial ring $C = k[T_0, \ldots, T_2]$ with the standard $\mathbb{Z}$-grading. In fact, let

$$(R^i, L^i) : = (C, Z),$$

$$(R^i, L^i) : = (R^{i-1}(\ell, p_i), L^{i-1}(\ell, p_i))$$

for $1 \leq i \leq n$. Then one can easily check that $(R, L) = (R^n, L^n)$ holds.

The following simple observation is quite useful.

**Proposition 3.5.**

(a) $C$ is the $(\mathbb{Z}c)$-Veronese subalgebra of $R$, that is, $C = \bigoplus_{a \in \mathbb{Z}} R_{ac}$.

(b) $R$ is a free $C$-module of rank $p_1 p_2 \cdots p_n$ with a basis $\{X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n} \mid 0 \leq a_i < p_i\}$.

(c) Let $\bar{x} = \sum_{i=1}^{n} a_i \bar{x}_i + aC$ be a normal form of $\bar{x} \in L$. Then the multiplication map $X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n} : C_{ac} \to R_\bar{x}$ is bijective.

**Proof.** (b) Since $R$ is obtained form $C$ by applying root construction iteratively, the assertion is clear.

(a)(c) Immediate from (b). \qed

Let $(a_1, \ldots, a_\ell)$ be a sequence of homogeneous elements in $R$ (respectively, $S$) whose degrees are in $\mathbb{Z}_+ \setminus \{0\}$. For $M \in \text{mod}^\ell R$ (respectively, $M \in \text{mod}^\ell S$), we say that $(a_1, \ldots, a_\ell)$ is an $M$-regular sequence [BRH] if the multiplication map $a_i : M/M(a_1, \ldots, a_{i-1}) \to M/M(a_1, \ldots, a_{i-1})$ is injective for any $1 \leq i \leq \ell$. Any permutation of an $M$-regular sequence is again an $M$-regular sequence. For any positive integers $q_1, \ldots, q_\ell$, the sequence $(a_1, \ldots, a_\ell)$ is $M$-regular if and only if $(a_1^{q_1}, \ldots, a_\ell^{q_\ell})$ is $M$-regular.

We prepare the following easy observations.

**Lemma 3.6.**

(a) $(h_1, \ldots, h_n)$ is an $S$-regular sequence.

(b) Let $f_0, \ldots, f_d$ be linearly independent linear forms in $C$. Then $(h_1, \ldots, h_n, f_0, \ldots, f_d)$ is an $S$-regular sequence, and $(f_0, \ldots, f_d)$ is an $R$-regular sequence.

(c) Let $i_0, \ldots, i_s \in \{1, \ldots, n\}$ and $f_{s+1}, \ldots, f_d \in C$ be linear forms. If $\ell_{i_0}, \ldots, \ell_{i_s}, f_{s+1}, \ldots, f_d$ are linearly independent, then $(X_{i_0}, X_{i_1}, X_{i_2}, \ldots, f_d)$ is an $R$-regular sequence.

**Proof.** (b) Let $S'$ be the $k$-subalgebra of $S$ generated by $T_0, \ldots, T_d, X_1^{p_1}, \ldots, X_n^{p_n}$. Then $S'$ is a polynomial algebra with these variables. Since $h_1, \ldots, h_n, f_0, \ldots, f_d$ are linearly independent linear forms in $S'$, they form an $S'$-regular sequence. Since $S$ is a free $S'$-module of finite rank, we have the assertion.

(a) Immediate from (b).

(c) The latter assertion in (b) implies that $(\ell_{i_0} = X_{i_0}^{p_0}, \ldots, \ell_{i_s} = X_{i_s}^{p_s}, f_{s+1}, \ldots, f_d)$ is an $R$-regular sequence. Thus the assertion is immediate. \qed

Immediately we have the following observations.

**Proposition 3.7.** Let $R$ be a weak GL complete intersection.

(a) $R$ is in fact a complete intersection of dimension $d + 1$.  

Example 3.10. Assume that $R$ is a GL complete intersection and $p_i \geq 2$ for any $i$.

- $R$ is regular if and only if $n \leq d + 1$.
- $R$ is a hypersurface if and only if $n \leq d + 2$.

Proof. (a) This is immediate from Lemma 3.6(a).

(b) By Observation 3.3 the number of minimal generators of the maximal ideal $R_+ = \bigoplus_{\ell \geq 0} R_\ell$ of $R$ is $\max\{d + 1, n\}$. Thus the assertion follows. □

Dualizing element ($a$-invariant, Gorenstein parameter). By Proposition 3.7 it follows that $R$ is a Gorenstein ring. Thus

\[
\text{Ext}^i_R(k, R) = \begin{cases} 0 & i \neq d \\ k(\overrightarrow{a}) & i = d \end{cases}
\]

holds for some element $\overrightarrow{a} \in \mathbb{L}$, which is called the dualizing element (also known as $a$-invariant [BrH], Gorenstein parameter) of $(R, L)$. In this case, $\omega_R := R(\overrightarrow{a})$ is called the canonical module of $(R, L)$. We have the following explicit formula for $\overrightarrow{a}$.

Proposition 3.8. The dualizing element of $(R, L)$ is given by

\[
\overrightarrow{a} = (n - d - 1)\overrightarrow{c} + \sum_{i=1}^n \overrightarrow{x}_i \in \mathbb{L}.
\]

Proof. Since $(S, L)$ is a polynomial ring, its $a$-invariant is given by minus the sum of the degrees of all variables, i.e. $a(S) := -(d + 1)\overrightarrow{c} - \sum_{i=1}^n \overrightarrow{x}_i$. Since $h_1, \ldots, h_n$ is an $S$-regular sequence by Lemma 3.6(a), we have

\[
\overrightarrow{a} = a(S) + \sum_{i=1}^n \deg h_i = -(d + 1)\overrightarrow{c} - \sum_{i=1}^n \overrightarrow{x}_i + n\overrightarrow{c}
\]

by using standard commutative algebra [BrH, 3.6.14]. □

We define a homomorphism $\delta : \mathbb{L} \to \mathbb{Q}$ by $\delta(\overrightarrow{x}_i) = \frac{1}{p_i}$ and $\delta(\overrightarrow{c}) = 1$. Using

\[
\delta(\overrightarrow{a}) = n - d - 1 + \sum_{i=1}^n \frac{1}{p_i} \in \mathbb{Q},
\]

we have the following important trichotomy.

Trichotomy. Let $(R, L)$ be a GL complete intersection. We say that $(R, \mathbb{L})$ is Fano (respectively, Calabi-Yau, anti-Fano) if $\delta(\overrightarrow{a}) < 0$ (respectively, $\delta(\overrightarrow{a}) = 0$, $\delta(\overrightarrow{a}) > 0$).

For example, if $n \leq d + 1$, then $R$ is Fano. For the integer

\[
p := \text{l.c.m.}(p_1, p_2, \ldots, p_n),
\]

$(R, L)$ is Fano (respectively, Calabi-Yau, anti-Fano) if and only if $p\overrightarrow{a} = \ell\overrightarrow{c}$ holds for an integer $\ell < 0$ (respectively, $\ell = 0$, $\ell > 0$).

Example 3.9. Let $d = 1$. Assume that $p_i \geq 2$ for any $i$.

(a) There are 5 types for Fano case: $(p, q)$, $(2, 2, p)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$, corresponding to $\tilde{E}_{p+q-1}$, $\tilde{E}_{p+2}$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

(b) There are 4 types for Calabi-Yau case: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ and $(2, 2, 2, 2)$, corresponding to $\mathbb{E}_g^{(1,1)}$, $\mathbb{E}_7^{(1,1)}$, $\mathbb{E}_8^{(1,1)}$ and $\mathbb{D}_4^{(1,1)}$.

(c) All other cases are anti-Fano.

This is nothing but the classical trichotomy of domestic, tubular and wild types of weighted projective lines.

Example 3.10. Let $d = 2$. Assume that $p_i \geq 2$ for any $i$.

(a) There are the following cases for Fano.
Lemma 3.11. Let \( \bar{x} = \sum_{i=1}^{n} a_i \bar{x}_i + a \bar{c} \) be a normal form, that is, \( 0 \leq a_i < p_i \) and \( a \in \mathbb{Z} \). Then the following conditions are equivalent.

(a) \( \bar{x} \leq d \bar{c} \).

(b) \( a + \{ \{ i \mid a_i > 0 \} \} \leq d \).

(c) \( 0 \leq \bar{x} + \bar{\omega} \).

In particular, we have \( [0, d \bar{c}] = \{ \bar{x} \in \mathbb{L} \mid 0 \leq \bar{x}, 0 \leq \bar{x} + \bar{\omega} \} \).

Proof. (a)⇔(b) The normal form of \( d \bar{c} - \bar{x} \) is \( \sum_{a_i > 0} (p_i - a_i) \bar{x}_i + (d - a - \{ i \mid a_i > 0 \}) \bar{c} \). So \( d \bar{c} - \bar{x} \geq 0 \) if and only if \( d - a - \{ i \mid a_i > 0 \} \geq 0 \).

(b)⇔(c) The normal form of \( \bar{x} + \bar{\omega} \) is

\[
\bar{x} + \bar{\omega} = \sum_{a_i \neq 0} (a_i - 1) \bar{x}_i + \sum_{a_i = 0} (p_i - 1) \bar{x}_i + (a - d - 1 + \{ i \mid a_i > 0 \}) \bar{c}.
\]

So \( \bar{x} + \bar{\omega} \geq 0 \) if and only if \( a - d - 1 + \{ i \mid a_i > 0 \} \geq 0 \).

The quotient group \( \mathbb{L}/\mathbb{Z} \bar{\omega} \) will play an important role. We give some easy observations.

Proposition 3.12. (a) If \( n \leq d + 1 \), then the map \( [0, d \bar{c}] \to \mathbb{L}/\mathbb{Z} \bar{\omega} \) is bijective.

(b) If \((R, \mathbb{L})\) is Fano, then the map \([0, d \bar{c}] \to \mathbb{L}/\mathbb{Z} \bar{\omega} \) is surjective.

(c) If \((R, \mathbb{L})\) is not Calabi-Yau, then the cardinality of \( \mathbb{L}/\mathbb{Z} \bar{\omega} \) is equal to the absolute value of \((p_1 p_2 \cdots p_n) \delta(\bar{\omega})\).

Proof. (b) Fix \( \bar{x} \in \mathbb{L} \). Since \((R, \mathbb{L})\) is Fano, we have \( \ell \bar{\omega} < 0 \) for \( \ell \gg 0 \). Therefore there exists an integer \( \ell \) such that \( \bar{x} + \ell \bar{\omega} \geq 0 \) and \( \bar{x} + (\ell + 1) \bar{\omega} \not\geq 0 \). This is equivalent to \( \bar{x} + \ell \bar{\omega} \in [0, d \bar{c}] \) by Lemma 3.11.

(a) Since \( \bar{\omega} < 0 \) holds by \( n \leq d + 1 \), the integer \( \ell \) satisfying \( \bar{x} + \ell \bar{\omega} \geq 0 \) and \( \bar{x} + (\ell + 1) \bar{\omega} \not\geq 0 \) is unique. Therefore the assertion holds.

(c) This is equal to the absolute value of the determinant of

\[
\begin{bmatrix}
p_1 & -p_2 & 0 & \cdots & 0 & 0 \\
p_2 & -p_3 & \cdots & 0 & 0 & 0 \\
0 & p_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & p_{n-1} & -p_n & 0 \\
0 & 0 & \cdots & p_{n-1} & -p_n & 1 \\
-1 & -1 & \cdots & -1 & (n-d-1)p_{n-1} & 0
\end{bmatrix}
\]

which is equal to the desired number.

□
3.2. **R is L-factorial and has L-isolated singularities.** Let \((R, L)\) be a weak GL complete intersection associated with hyperplanes \(H_1, \ldots, H_n\) in \(\mathbb{P}^d\) and weights \(p_1, \ldots, p_n\). In this subsection, we give some ring theoretic properties of \((R, L)\). In particular, we show that, in a graded sense, \((R, L)\) is always factorial, and has isolated singularities if \((R, L)\) is a GL complete intersection.

Let us start with introducing some notions for graded rings.

**Definition 3.13.** Let \(G\) be an abelian group and \(A\) a commutative Noetherian \(G\)-graded ring.

(a) We say that \(A\) is a \(G\)-domain if a product of non-zero homogeneous elements is non-zero. We say that \(A\) is a \(G\)-field if any non-zero homogeneous element is invertible.

(b) A homogeneous ideal \(p\) of \(A\) is \(G\)-prime (respectively, \(G\)-maximal) if \(A/p\) is a \(G\)-domain (respectively, \(G\)-field). In this case we denote by \(A_p\) the localization of \(A\) with respect to the multiplicative set consisting of all homogeneous elements in \(A - p\). We denote by \(\text{Spec}^G A\) the set of all \(G\)-prime ideals of \(A\).

(c) A non-zero homogeneous element \(a \in A\) is a \(G\)-prime element if the principal ideal \(Aa\) is a \(G\)-prime ideal of \(A\). A \(G\)-domain \(A\) is \(G\)-factorial if any non-zero homogeneous element in \(A\) is a product of \(G\)-prime elements in \(A\).

(d) We say that \(A\) is \(G\)-regular if \(\text{mod}^G A\) has finite global dimension. We say that \(A\) has \(G\)-isolated singularities if \(A_{(p)}\) is \(G\)-regular for any \(p \in \text{Spec}^G A\) which is not \(G\)-maximal.

When the group is trivial \(G = \{1\}\), we recover the usual notions of domain, field etc.

These notions depend not only on the ring \(A\) but also on the group \(G\). As a simple example, let \(k\) be a field with characteristic 2 and \(A := k[x]/(1 + x^2) = k[x]/(1 + x)^2\). Then \(A\) is neither a field nor regular. On the other hand, regarding \(A\) as a \((\mathbb{Z}/2\mathbb{Z})\)-graded ring by \(\deg x = 1\), we have that \(A\) is a \((\mathbb{Z}/2\mathbb{Z})\)-field and \((\mathbb{Z}/2\mathbb{Z})\)-regular.

We start with a few easy observations.

**Observation 3.14.**

(a) Any \(G\)-field is a \(G\)-domain. Hence any \(G\)-maximal ideal is a \(G\)-prime ideal.

(b) If \(A\) is \(G\)-regular, then \(A_{(p)}\) is \(G\)-regular for any \(p \in \text{Spec}^G A\).

We show the following result, which generalizes [GL1, 1.3] for the case \(d = 1\).

**Theorem 3.15.** Any weak GL complete intersection \((R, L)\) is an \(L\)-factorial \(L\)-domain.

**Proof.** We use the following general argument due to Lenzing.

**Proposition 3.16.** For a \(G\)-domain \(A\), let \((A', G') := (A(\ell, p), G(\ell, p))\) be a root construction in Observation 3.3. Assume \(\ell \neq 0\).

(a) \(A' = A[X]/(X^p - \ell)\) is a \(G'\)-domain.

(b) Assume that \(A\) is \(G\)-factorial and \(\ell\) is a \(G\)-prime element. Then \(A'\) is \(G'\)-factorial. Moreover, \(G'\)-prime elements in \(A'\) are either \(X\) or \(G'\)-prime elements \(a \in A\) satisfying \(\text{Ann} Aa \neq \text{Ann} A\ell\).

**Proof.** Note that \(A'\) is a free \(A\)-module of rank \(p\) with a basis \(\{X^i \mid 0 \leq i < p\}\). Moreover, any non-zero homogeneous element in \(A'\) can be written uniquely as \(ax^i\) for some \(a \in A\) and \(0 \leq i < p\).

(a) Let \(ax^i\) and \(bx^j\) be homogeneous elements in \(A'\) with \(a, b \in A\) and \(0 \leq i, j < p\). Then

\[
(ax^i)(bx^j) = \begin{cases} (ab)x^{i+j} & \text{if } i + j < p, \\ (ab\ell)x^{i+j-p} & \text{if } i + j \geq p. \end{cases}
\]

Since \(A\) is a \(G\)-domain and \(\ell \neq 0\), both \(ab\) and \(ab\ell\) are non-zero. Thus \((ax^i)(bx^j) \neq 0\) holds.

(b) Since \(\ell \in A\) is \(G\)-prime, \(A'/A'X = A/\ell\ell\) is a \(G\)-domain. Hence \(X \in A'\) is \(G'\)-prime.
Let \( a \in A \) be a \( G \)-prime element such that \( Aa \neq \mathcal{A} \ell \). Since \( A/Aa \) is a \( G \)-domain and \( \ell \neq 0 \) in \( A/Aa \) by our assumption, it follows from (a) that
\[
A'/Aa = (A/Aa)[X]/(X^p - \ell) = (A/Aa)(\ell, p)
\]
is a \( G' \)-domain. Thus \( a \) is a \( G' \)-prime element in \( A' \).

Now we show that \( A' \) is \( G' \)-factorial. Fix a non-zero homogeneous element \( aX^i \in A' \) with \( a \in A \) and \( 0 \leq i < p \). Since \( A \) is \( G \)-factorial, we can write \( a = a_1 \cdots a_\ell \ell^t \) for \( G \)-prime elements \( a_j \) satisfying \( Aa_j \neq \mathcal{A} \ell \) and \( t \geq 0 \). Then \( aX^i \) is a product
\[
aX^i = a_1 \cdots a_\ell X^p i
\]
of \( G' \)-prime elements in \( A' \). Thus the assertion follows. \( \square \)

Now Theorem 3.15 follows immediately from Observation 3.4 and Proposition 3.16. \( \square \)

By Theorem 3.15, the zero ideal \((0)\) of \( R \) is an \( L \)-prime ideal. Therefore the localization of \( R_{(0)} \) of \( R \) is an \( L \)-field, and its degree 0 part \((R_{(0)})_0\) is a field.

**Rank function.** For \( X \in \text{mod}^L R \), we define the rank of \( X \) by
\[
\text{rank } X := \dim_{(R_{(0)})_0}(X_{(0)})_0.
\]

We need the following observations, where \( K_0(\text{mod}^L R) \) is the Grothendieck group of \( \text{mod}^L R \).

**Proposition 3.17.**
(a) \( \text{rank } R(I) = 1 \) for any \( I \in \mathbb{L} \).
(b) \( \text{rank } \) extends to a morphism \( K_0(\text{mod}^L R) \to \mathbb{Z} \) of abelian groups.
(c) We have an equivalence \((-)_0 : \text{mod}^L R_{(0)} \simeq \text{mod}(R_{(0)})_0 \).

**Proof.** (a) For any \( I \in \mathbb{L} \), there exist monomials \( r, s \in R \) such that \( I = \deg r - \deg s \). Thus we have an isomorphism \( rs^{-1} : (R_{(0)})_0 \simeq (R(I))_0 \) of \((R_{(0)})_0\)-modules.

(c) \( R_{(0)} \) is strongly graded in the sense that \((R_{(0)})_x \cdot (R_{(0)})_{-x} = (R_{(0)})_0 \) for any \( x \in L \). Thus the assertion follows from an \( \mathbb{L} \)-graded analog of [NV] I.3.4.

(b) This is clear since we have a morphism \((-)_0 : K_0(\text{mod}^L R) \to K_0(\text{mod}(R_{(0)})_0) \simeq \mathbb{Z} \) by (c).

In the rest of this subsection, we will show that GL complete intersections have \( L \)-isolated singularities. We need the assumption that our hyperplanes are in a general position.

A crucial role is played by the following non-commutative ring, which already appeared in Gabriel’s classical covering theory [Ga].

**Definition 3.18.** Let \( A \) be a \( G \)-graded ring and \( H \) a subgroup of \( G \) with finite index. We fix a complete set \( I \subset G \) of representatives of \( G/H \). We define an \( H \)-graded ring \( A[H] \) called the covering [Ga] [LL] (or quasi-Veronese subalgebra [MM]) of \( A \) as
\[
A[H] := \bigoplus_{h \in H} B_h, \quad B_h := (A_{x-y+h})_{x,y \in I}
\]
where the multiplication \( B_h \times B_{h'} \to B_{h+h'} \) for \( h, h' \in H \) is given by
\[
(a_{x,y})_{x,y \in I} \cdot (a'_{x,y})_{x,y \in I} := \left( \sum_{z \in I} a_{x,z} \cdot a'_{z,y} \right)_{x,y \in I}.
\]

Note that the (ungraded) \( k \)-algebra structure of \( A[H] \) does not depend on the choice of \( I \).

We know from [LL 3.1] that we have an equivalence of categories
\[
F : \text{mod}^G A \simeq \text{mod}^H A[H],
\]
which is given as follows: For \( M \in \text{mod}^G A \), define \( FM \in \text{mod}^H A[H] \) by
\[
FM := \bigoplus_{h \in H} (FM)_h \quad \text{where} \quad (FM)_h := (M_{x+h})_{x \in I}.
\]
Now we apply this general observation to GL complete intersections. For the subgroup $\mathbb{Z}\vec{c}$ of $L$ generated by $\vec{c}$, we take the complete set

$$I := \{ \sum_{i=1}^{n} a_i \vec{x}_i \mid 0 \leq a_i \leq p_i - 1 \ (1 \leq i \leq n) \} \quad (3.4)$$

of representatives of $\mathbb{L}/\mathbb{Z}\vec{c}$. Then we have the corresponding covering $R^{[\mathbb{Z}]}$ of $R$, which is a $\mathbb{Z}$-graded $k$-algebra. Applying (3.3), we have the following observation.

**Proposition 3.19.** We have an equivalence of categories

$$\text{mod}^2 R \simeq \text{mod}^2 R^{[\mathbb{Z}]}.$$

We are ready to prove our main result in this subsection. Note that our $R$ has a unique $L$-maximal ideal

$$R_+ := \bigoplus_{\vec{x} \in \mathbb{L}\setminus\{0\}} R_{\vec{x}}.$$ 

We denote by $R_{T_j}$ (respectively, $C_{T_j}$) is the localization of $R$ (respectively, $C$) with respect to the multiplicative set $\{ T_j^\ell \mid \ell \in \mathbb{Z} \}$ for any $j$ with $0 \leq j \leq d$.

**Theorem 3.20.** Let $(R, L)$ be a GL complete intersection over an arbitrary field $k$.

(a) $R$ has $L$-isolated singularities.

(b) $R_{T_j}$ is $L$-regular for any $0 \leq j \leq d$.

For the subgroup $\mathbb{Z}\vec{c}$ of $\mathbb{L}$ generated by $\vec{c}$, we take the complete set $I$ in (3.4) of representatives of $\mathbb{L}/\mathbb{Z}\vec{c}$. We define a $\mathbb{Z}$-graded algebra $T(j)$ by

$$T(j) := (R_{T_j})^{[\mathbb{Z}]}$$

(see Definition 3.18). Then we have the following observation.

**Proposition 3.21.** We have equivalences

$$\text{mod}^2 R_{T_j} \simeq \text{mod}^2 T(j) \simeq \text{mod} T(j)_0.$$

**Proof.** The first equivalence follows from (3.3). Since $T_j$ is an invertible element of degree 1 in $T(j)$, we have $T(j)_\ell = T(j)_0$ for any $j \in \mathbb{Z}$. Thus $T(j)$ is strongly graded in the sense that

$$T(j)_\ell \cdot T(j)_{-\ell} = T(j)_0$$

holds for any $\ell \in \mathbb{Z}$. By [NV, I.3.4], we have the second equivalence. \qed

Now we give a description of $T(j)_0$ in terms of a tensor product. For $p > 0$ and a ring $A$ with an element $a \in A$, we define a subring of the full matrix ring $M_p(A)$ by

$$T_p(A, a) := \begin{bmatrix} A \ (a) & \cdots & (a) & (a) \\ A \ A \ \cdots \ (a) & (a) \\ \vdots & \vdots & \ddots & \vdots \\ A \ A \ \cdots \ A & (a) \\ A \ A \ \cdots \ A & A & \cdots & A \end{bmatrix}.$$ 

We have the following explicit description of $T(j)_0$, which is a analog of a description of $R^{[\mathbb{Z}]}$ given in [LL] 3.6.

**Lemma 3.22.** We have $T(j)_0 \simeq T_{p_1}((C_{T_j})_0, \ell_1/T_j) \otimes \cdots \otimes T_{p_n}((C_{T_j})_0, \ell_n/T_j)$, where the tensor products are over $(C_{T_j})_0$.

**Proof.** If $\vec{x} = \sum_{i=1}^{n} a_i \vec{x}_i + a \vec{c}$ is in normal form, then we have $(R_{T_j})_{\vec{x}} = (\prod_{i=1}^{n} X_i^{a_i}) T_j^{a}(C_{T_j})_0$.

For $\vec{x} = \sum_{i=1}^{n} a_i \vec{x}_i$ and $\vec{y} = \sum_{i=1}^{n} b_i \vec{x}_i$ in $I$, let $\epsilon_i := 0$ if $a_i \geq b_i$ and $\epsilon_i := 1$ otherwise. Then $\vec{x} - \vec{y}$ has the normal form $\sum_{i=1}^{n} (a_i - b_i + \epsilon_i p_i) \vec{x}_i - (\sum_{i=1}^{n} \epsilon_i) \vec{c}$, and we have

$$(R_{T_j})_{\vec{x} - \vec{y}} = \left( \prod_{i=1}^{n} (X_i^{a_i-b_i+\epsilon_i p_i}/T_j^{\epsilon_i}) \right) (C_{T_j})_0$$

$$= ((X_1^{a_1-b_1+\epsilon_1 p_1}/T_j^{\epsilon_1})(C_{T_j})_0) \otimes \cdots \otimes ((X_n^{a_n-b_n+\epsilon_n p_n}/T_j^{\epsilon_n})(C_{T_j})_0),$$
Therefore we have an isomorphism

\[
T(j)_0 = \bigotimes_{i=1}^{n} \begin{pmatrix}
(C_T)_0 & (X_i^{p_i-1}/T_i)(C_T)_0 & \cdots & (X_i^2/T_i)(C_T)_0 & (X_i/T_i)(C_T)_0 \\
X_i(C_T)_0 & (C_T)_0 & \cdots & (X_i^2/T_i)(C_T)_0 & (X_i^3/T_i)(C_T)_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_i^{p_i-2}(C_T)_0 & X_i^{p_i-3}(C_T)_0 & \cdots & (C_T)_0 & (X_i^{p_i-1}/T_i)(C_T)_0 \\
X_i^{p_i-1}(C_T)_0 & X_i^{p_i-2}(C_T)_0 & \cdots & X_i(C_T)_0 & (C_T)_0
\end{pmatrix}
\approx \bigotimes_{i=1}^{n} T_{p_i}((X_i^{p_i}/T_i) \otimes \cdots \otimes T_{p_n}((C_T)_0, X_n^{p_n}/T_n))
\]

of \( k \)-algebras.

**Lemma 3.23.** \( T(j)_0 \) has global dimension \( d \).

**Proof.** \( T(j)_0 \) is a \((C_T)_0\)-algebra which is a free \((C_T)_0\)-module of finite rank. It suffices to show that for any maximal ideal \( \mathfrak{m} \) of \((C_T)_0\), the global dimension of \((T(j)_0)_0\) at \( \mathfrak{m} \) is \( d \). Let \( x \) be the closed point in \( \mathbb{P}^d \setminus H_j \) defined by \( \mathfrak{m} \). Let \( I_x := \{ i \mid 1 \leq i \leq n, x \in H_i \} \). For any \( i \notin I_x \), we have that \( \ell_i/T_j \) is a unit in \((C_T)_0)_0\) and hence \( T_{p_i}((C_T)_0)_0, \ell_i/T_j) = M_{p_i}((C_T)_0)_0\) holds. Thus we have

\[
(T(j)_0)_0 \cong \bigotimes_{i=1}^{n} T_{p_i}(((C_T)_0)_0, \ell_i/T_j) \cong M_{p_i}((C_T)_0)_0, \ell_i/T_j)
\]

for \( p := \prod_{i \notin I_x} p_i \). Since \( H_1, \ldots, H_n \) are in a general position, \((\ell_i/T_j)_{i \in I_x} \) is a regular sequence of \((C_T)_0)_0\). Thus the global dimension of \( \bigotimes_{i \in I_x} T_{p_i}((C_T)_0)_0, \ell_i/T_j) \) is \( d \) by [IL, 2.14], and we have the assertion.

Now we are ready to prove Theorem 3.20.

**Proof of Theorem 3.20** (b) The statement follows from Propositions 3.21 and 3.23.

(a) Fix \( p \in (\text{Spec}^L R) \setminus \{ R_+ \} \). We will show that \( R(p) \) is \( \mathbb{L} \)-regular. Since \( p \neq R_+ \), there exists \( j \) such that \( T_j \notin p \). Clearly \( R(T_j) \) is an \( \mathbb{L} \)-prime ideal of \( R(T_j) \) such that \( R(p) = (R(T_j))(R(T_j), p) \). Since \( R(T_j) \) is \( \mathbb{L} \)-regular by (b), we have that \( R(p) = (R(T_j))(R(T_j), p) \) is \( \mathbb{L} \)-regular by Observation 3.11. Thus the assertion follows.

If we forget the \( \mathbb{L} \)-grading, then our GL complete intersection does not necessarily have isolated singularities, e.g. \( R = k[X_1, X_2, X_3]/(X_1^2 + X_2^2 + X_3^2) \) and \( k \) has characteristic \( p \).

We end this subsection with the following easy observation when \( k \) has characteristic zero.

**Proposition 3.24.** Let \((R, L)\) be a GL complete intersection over a field \( k \) of characteristic zero. Then \( R_p \) is a regular local ring for any \( p \in (\text{Spec} R) \setminus \{ R_+ \} \). In particular \( R \) has isolated singularities.

**Proof.** If \( n \leq d + 1 \), then the assertion follows from Proposition 3.21(b). Assume \( n \geq d + 2 \). By Observation 3.13, we have \( R = S'/(X_i^{p_i} - \sum_{j=1}^{d+1} \lambda_{i,j-1} X_j^{p_j} \mid d + 2 \leq i \leq n) \), where all maximal minors of the \((n - d - 1) \times n \) matrix

\[
L := [\lambda_{i,j-1} - \mathbb{I}_{n-d-1} | d + 2 \leq i \leq n, 1 \leq j \leq d+1]
\]

have non-zero determinants. The Jacobian matrix is given by

\[
M := L \cdot \text{diag}(p_1 X_1^{p_1-1}, \ldots, p_n X_n^{p_n-1}).
\]

By Jacobian criterion [E12, 16.20], the singular locus of \( R \) is given by \( V(J) \cap \text{Spec} R \), where \( J \) is the ideal of \( S' \) generated by all maximal minors of \( M \) and \( V(J) := \{ p \in \text{Spec} S' \mid p \supset J \} \). Since \( p_i \neq 0 \) in \( k \) for any \( i \) and all maximal minors of \( L \) have non-zero determinants, we have

\[
J = \left\{ \prod_{i \in I} X_i^{p_i-1} \mid I \subset \{1, \ldots, n\}, |I| = n - d - 1 \right\}.
\]
Therefore it is easy to check that \( p \in \text{Spec } S' \) contains \( J \) if and only if \( p \) contains at least \( d + 2 \) elements from \( \{X_1, \ldots, X_n\} \). Thus

\[
V(J) = \bigcup_{I \subseteq \{1, \ldots, n\}, \ |I| = d + 2} V((X_i \mid i \in I))
\]

holds. On the other hand, for any subset \( I \) of \( \{1, \ldots, n\} \) with \( |I| \geq d + 1 \), we have \( R/(X_i \mid i \in I) \in \text{mod}^d R \) by Lemma 3.6(c), and hence \( V((X_i \mid i \in I)) = \{R_+\} \) holds. In particular, the singular locus of \( R \) is contained in \( \{R_+\} \).

\[\Box\]

4. Cohen-Macaulay representations on Geigle-Lenzing complete intersections

Let \((R, L)\) be a Geigle-Lenzing (GL) complete intersection ring associated with hyperplanes \( H_1, \ldots, H_n \) in \( \mathbb{P}^d \) and weights \( p_1, \ldots, p_n \) over an arbitrary field \( k \). For an integer \( i \) with \( 0 \leq i \leq d + 1 \), the objects in the category

\[
\text{CM}^i L: = \{X \in \text{mod}^d R \mid \text{Ext}_{\text{mod}^d R}^j(X, R) = 0 \text{ for all } j \neq d + 1 - i\}
\]

are called (L-graded) Cohen-Macaulay \( R \)-modules of dimension \( i \). In particular the objects in the category

\[
\text{CM}^i L: = \text{CM}^i_{d+1} R
\]

are simply called (L-graded maximal) Cohen-Macaulay \( R \)-modules.

4.1. Basic properties. We give basic properties of the category \( \text{CM}^i L \). The stable category \([ABr]\) defined as follows is fundamental in representation theory.

**Stable category.** We denote by \( \text{mod}^i L \) the stable category of \( \text{mod}^d R \) \([ABr]\). Thus \( \text{mod}^i L \) has the same objects as \( \text{mod}^d R \), and the morphism set is given by

\[
\text{Hom}_{\text{mod}^i L}(X, Y) = \text{Hom}_{\text{mod}^d R}(X, Y)/\text{Hom}_{\text{mod}^d R}(X, P(X, Y))
\]

for any \( X, Y \in \text{mod}^d R \), where \( P(X, Y) \) is the submodule of \( \text{Hom}_{\text{mod}^d R}(X, Y) \) consisting of morphisms that factor through objects in \( \text{proj}^d L \). The full subcategory \( \text{CM}^i L \) of \( \text{mod}^i L \) corresponding to the full subcategory \( \text{CM}^i L \) of \( \text{mod}^i L \) plays an important role in this paper.

We give some of the basic properties in Cohen-Macaulay representation theory:

**Theorem 4.1.** (a) (Auslander-Reiten-Serre duality) We have a functorial isomorphism for any \( X, Y \in \text{CM}^i L \):

\[
\text{Hom}_{\text{mod}^i L}(X, Y) \cong D \text{Ext}_{\text{mod}^d R}^d(Y, X(\omega)).
\]

(b) The category \( \text{CM}^i L \) has almost split sequences.

(c) (Auslander-Buchweitz approximation) \( \text{CM}^i L \) is a functorially finite subcategory of \( \text{mod}^i L \).

**Proof.** (a)(b) \( R \) is an L-isolated singularity by Theorem 3.20. Thus the assertions follow from a general result in \([AB]\) (see also \([IT]\)).

(c) The argument in \([AB]\) works in L-graded setting, and we have that \( \text{CM}^i L \) is a contravariantly finite subcategory of \( \text{mod}^d R \). This implies covariantly finiteness of \( \text{CM}^i L \) as follows: For any \( X \in \text{mod}^i R \), let \( X^* = \text{Hom}_R(R, X) \in \text{mod}^d R \). Let \( a: Y \to X^* \) be a right \( \text{CM}^i L \)-approximation of \( X^* \). It is easily checked that the composition

\[
X \xrightarrow{\epsilon_X} X^{**} \xrightarrow{a^*} Y^{**}
\]

of the evaluation map \( \epsilon_X \) and \( a^* \) gives a left \( \text{CM}^i L \)-approximation of \( X \).

\[\Box\]

Let us recall basic results on the structure of the stable category \( \text{CM}^i L \) as a triangulated category. We call the quotient category

\[
D^i_{\text{sg}}(R) := D^b(\text{mod}^i L)/K^b(\text{proj}^i L)
\]
the singular derived category of $R$ \cite{BuO1}. We have the following results due to Happel, Auslander and Reiten, Buchweitz and Eisenbud.

**Theorem 4.2.**  
(a) $\text{CM}^R$ is a Frobenius category whose projective objects are $\text{proj}^R$, and $\text{CM}^R$ is a triangulated category.  
(b) $\text{CM}^R$ has a Serre functor $S := (\bar{\omega})[d]$.  
(c) There is a triangle functor $\rho : \mathcal{D}^b(\text{mod}^R) \to \text{CM}^R$ which induces a triangle equivalence $\mathcal{D}^b(R) \simeq \text{CM}^R$.  
(d) If $n = d + 2$, then we have an isomorphism $[2] \simeq (\bar{c})$ of functors $\text{CM}^R \to \text{CM}^R$.  

**Proof.** (a) Since $R$ is Gorenstein, $\text{CM}^R$ is a Frobenius category. Therefore its stable category $\text{CM}^R$ is a triangulated category by a general result by Happel \cite{Hap}.  
(b) This is immediate from Auslander-Reiten-Serre duality in Theorem 4.1(a).  
(c) This is a classical result by Buchweitz \cite{Bu}.  
(d) $R$ is a hypersurface for $n = d + 2$ by Proposition 3.7(b). Therefore this is a well-known result for matrix factorizations \cite{Hap}. \hfill $\square$

**Proposition 4.3.** Assume that $p_i \geq 2$ for any $i$. Then $\text{CM}^R$ $\text{proj}^R$ holds if and only if $\text{CM}^R$ $= 0$ holds if and only if $n \leq d + 1$.  

**Proof.** The first equivalence is clear. Moreover $\text{CM}^R = 0$ holds if and only if $\mathcal{D}^b(\text{mod}^R)$ holds by Theorem 4.1(c). This is clearly equivalent to that $\text{mod}^R$ has finite global dimension. Thus we have the second equivalence. \hfill $\square$

Now we have the following characterization of when the stable category $\text{CM}^R$ is fractionally Calabi-Yau (see Section 2.1).

**Corollary 4.4.** Assume that $p_i \geq 2$ for any $i$. Then the triangulated category $\text{CM}^R$ is fractionally Calabi-Yau if and only if one of the following conditions holds, where $p := \text{l.c.m.}(p_1, \ldots, p_n)$. 

- $n \leq d + 1$ holds. In this case $\text{CM}^R = 0$.  
- $n = d + 2$ holds. In this case $\text{CM}^R$ is $\frac{\text{p}(d+2\bar{c}(\bar{\omega}))}{p}$-Calabi-Yau.  
- $(R, \mathbb{L})$ is Calabi-Yau. In this case $\text{CM}^R$ is $\frac{\text{p}(\bar{c})}{p}$-Calabi-Yau. 

In particular, if $n = d + 2$, then $(R, \mathbb{L})$ is Fano (respectively, Calabi-Yau, anti-Fano) if and only if the fractional Calabi-Yau dimension of $\text{CM}^R$ is less than (respectively, equal to, more than) $d$.  

**Proof.** We need the following Tate’s result.

**Proposition 4.5.** \cite{Tat, Y} If there is an upper bound for Betti series of $k$, then $R$ is a hypersurface.  

Now we are ready to prove Corollary 4.4.  

By Theorem 4.2(b), the Serre functor of $\text{CM}^R$ is given by $S := (\bar{\omega})[d]$. Clearly $p\bar{\omega} = p\bar{\delta}(\bar{\omega})\bar{c}$ holds. First we show the ‘if’ part. If $n \leq d + 1$, then $\text{CM}^R = 0$ by Proposition 3.8. If $n = d + 2$, then $S^p = (p\bar{\omega})[pd] = (p\bar{\delta}(\bar{\omega})\bar{c})[pd] = [p(d + 2\bar{\delta}(\bar{\omega}))]$ holds, where we used $(\bar{c}) = [2]$ from Theorem 4.2(d). Thus $\text{CM}^R$ is $\frac{p(d+2\bar{c}(\bar{\omega}))}{p}$-Calabi-Yau. If $(R, \mathbb{L})$ is Calabi-Yau, then we have $p\bar{\omega} = 0$ and $S^p = (p\bar{\omega})[dp] = [dp]$. Thus $\text{CM}^R$ is $\frac{dp}{p}$-Calabi-Yau.

Next we show the ‘only if’ part. Assume that $\text{CM}^R$ is fractionally Calabi-Yau. If $\bar{\omega} \in \mathbb{L}$ is a torsion element, then $(R, \mathbb{L})$ is Calabi-Yau. Now we assume that $\bar{\omega}$ is not torsion. Then $(p\bar{\omega}) = (p\bar{\delta}(\bar{\omega})\bar{c}) = [2p\delta(\bar{\omega})]$ holds. By Proposition 4.5, we have that $R$ is a hypersurface. Therefore $n \leq d + 2$ holds by Proposition 3.7. \hfill $\square$
For $X \in \mod^L R$, we consider the support $\text{Supp}^L X := \{ p \in \text{Spec}^L R \mid X(p) \neq 0 \}$. The following observation is elementary.

**Lemma 4.6.** $X \in \mod^L R$ belongs to $\mod^L_0 R$ if and only if $\text{Supp}^L X \subset \{ R_+ \}$ if and only if $X_{T_j} = 0$ for any $j$ with $0 \leq j \leq d$.

**Proof.** We only show the first equivalence since the second one is clear.

Note that, for $p \in \text{Spec}^L R$, the $R$-module $R/p$ belongs to $\mod^L_0 R$ if and only if $p = R_+$. By a similar argument to that in the ungraded setting, we have a filtration $X_0 = 0 \subset X_1 \subset \cdots \subset X_\ell = X$ such that $X_i/X_{i-1} \simeq (R/p_i)(\tilde{a}_i)$ for $p_i \in \text{Spec}^L R$ and $\tilde{a}_i \in L$ in $\mod^L R$ for any $i$ with $1 \leq i \leq \ell$. In this case, we have $\text{Supp}^L R = \bigcup_{i=1}^\ell V(p_i)$ for $V(p_i) := \{ q \in \text{Spec}^L R \mid p_i \subset q \}$.

Then $X \in \mod^L_0 R$ holds if and only if $R/p_i \in \mod^L_0 R$ holds for any $1 \leq i \leq \ell$ if and only if $p_i = R_+$ for any $1 \leq i \leq \ell$ if and only if $\text{Supp}^L R \subset \{ R_+ \}$.

□

The following notion is basic in Auslander-Reiten theory for Cohen-Macaulay modules [A1, Y].

**Definition-Proposition 4.7.** We say that $M \in \mod^L R$ is locally free on the punctured spectrum if the following equivalent conditions are satisfied.

(a) For any $p \in (\text{Spec}^L R) \setminus \{ R_+ \}$, we have $M(p) \in \text{proj}^L R(p)$.

(b) For any $j$ with $0 \leq j \leq d$, we have $M_{T_j} \in \text{proj}^L R_{T_j}$.

(c) $\text{Ext}^i_R(M, R) \in \mod^L_0 R$ for any $i > 0$.

(d) For any $X \in \mod^L R$, the $R$-modules $(\text{Hom}^R(M, X)$ and $\text{Ext}^i_R(M, X)$ for any $i > 0$ belong to $\mod^L_0 R$.

**Proof.** (d)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(b) Since $R_{T_j}$ is $L$-regular by Theorem 3.20, the $R_{T_j}$-module $M_{T_j}$ has finite projective dimension. On the other hand, by Lemma 4.6, we have $\text{Ext}^i_R(M_{T_j}, R_{T_j}) \simeq \text{Ext}^i_R(M, R)_{T_j} = 0$ for any $i > 0$. Thus $M_{T_j}$ must be a projective $R_{T_j}$-module.

(b)$\Rightarrow$(a) Clear.

(a)$\Rightarrow$(d) We only show $\text{Ext}^i_R(M, X) \in \mod^L_0 R$ since the other assertion can be shown similarly. For any $p \in (\text{Spec}^L R) \setminus \{ R_+ \}$, we have $M(p) \in \text{proj}^L R$ by (a). Thus $\text{Ext}^i_R(M, X)(p) = \text{Ext}^i_R(M(p), X(p)) = 0$ holds. Hence $\text{Supp}^L R \subset \{ R_+ \}$ holds, and the assertion follows from Lemma 4.6.

□

Since $R$ has $L$-isolated singularities, we have the following useful property of $L$-graded Cohen-Macaulay $R$-modules.

**Proposition 4.8.** Any object in $\text{CM}^L R$ is locally free on the punctured spectrum.

**Proof.** Since $M \in \mod^L R$, we have $\text{Ext}^i_R(M, R) = 0$ for any $i > 0$. Thus the assertion follows from Lemma 4.7(c)$\Rightarrow$(a)(d).

□

At the end of this subsection, we note the following property. We need the following result, which is an analog of [KMV] A.2 [O2] A.2 [I] 2.4.[Tak1 2.4].

**Theorem 4.9.** Let $R$ be a GL complete intersection over an arbitrary field $k$. Then we have

$$D^b(\mod^L R) = \text{thick}\{ \text{proj}^L R, \mod^L_0 R \}$$

and

$$\text{CM}^L R = \text{thick}(\rho(\mod^L_0 R)),$$

where $\rho : D^b(\mod^L R) \to \text{CM}^L R$ is the triangle functor in Theorem 4.3(c).

We give a simple proof following Takahashi’s method [Tak2 3.4, 4.1].

**Proof.** For $X \in D^b(\mod^L R)$, let

$$\Lambda := \text{End}_R(\rho(X)) = \bigoplus_{\bar{x} \in L} \text{Hom}_R^\Lambda(\rho(X), \rho(X)(\bar{x})).$$
By Proposition 4.8(b), we have \( A \in \text{mod}_0^L R \). Let \( I \) be the annihilator of the \( R \)-module \( A \), then we have \( R/I \in \text{mod}_0^L R \). Let \( r_1 \in R_{a_1}, \ldots, r_t \in R_{a_t} \) be a set of homogeneous generators of \( I \). Let \( K_i \) be the cone of the morphism \( r_i : R(-a_i) \to R \) in \( D^b(\text{mod}^L R) \), and \( K := K_1 \otimes_R \cdots \otimes_R K_t \in D^b(\text{mod}^L R) \) the Koszul complex. Then all homologies of both \( K \) and \( K \otimes_R X \) are annihilated by \( I \). In particular we have \( K \otimes_R X \in D^b(\text{mod}_0^L R) \).

On the other hand, we have a triangle
\[
X(-a_i) \to X \to K_i \otimes_R X
\]
Since \( \rho(r_i) = 0 \) holds, we have that \( \rho(f_i) \) is a split monomorphism in \( \text{CM}_R \). Hence \( \rho(X) \in \text{add} \rho(K_i \otimes_R X) \) holds for any \( i \). Using this inductively we have \( \rho(X) \in \text{add} \rho(K \otimes_R X) \). Therefore we have \( \text{CM}_R = \text{thick}(\rho(\text{mod}_0^L R)) \).

Applying Proposition 4.8(b), we have \( \text{D}^b(\text{mod}_0^L R) = \text{thick}(\text{proj}^L R, \text{mod}_0^L R) \). \( \square \)

4.2. Tilting theory in the stable categories and \( I \)-canonical algebras. Let \((R, \mathbb{L})\) be a Geigle-Lenzing complete intersection associated with hyperplanes \( H_1, \ldots, H_n \) and weights \( p_1, \ldots, p_n \).

Recall from Theorem 4.2(c) that we have a triangle functor \( \rho : \text{D}^b(\text{mod}^L R) \to \text{CM}_R \)
which induces a triangle equivalence \( \text{D}^b_{\text{gp}}(R) = \text{D}^b(\text{mod}^L R)/\text{K}^b(\text{proj}^L R) \simeq \text{CM}_R \). In this section, we show that certain subcategories of \( \text{D}^b(\text{mod}^L R) \) are triangle equivalent to \( \text{CM}_R \) through \( \rho \).

As an application, we obtain a triangle equivalence \( \text{D}^b(\text{mod} A_{\text{CM}}) \simeq \text{CM}_R \) for a certain finite dimensional \( k \)-algebra \( A_{\text{CM}} \) defined below.

We say that a subset \( I \) of \( \mathbb{L} \) is \textit{convex} if for any \( \bar{x}, \bar{y}, \bar{z} \in \mathbb{L} \) such that \( \bar{x} \leq \bar{y} \leq \bar{z} \) and \( \bar{x}, \bar{z} \in I \), we have \( \bar{y} \in I \). We say that a subset \( I \) of \( \mathbb{L} \) is a poset ideal if \( I + \mathbb{L} \subset I \) holds. Moreover we say that a poset ideal is non-trivial if it is neither \( \mathbb{L} \) nor \( \emptyset \).

\( I \)-canonical algebra. For a finite subset \( I \) of \( \mathbb{L} \), we define a \( k \)-algebra
\[
A^I := \langle R_{x - \bar{y}} \rangle_{\bar{x}, \bar{y} \in I}
\]
in a similar way to Definition 3.18. Namely the multiplication of \( A^I \) is given by
\[
(r_{\bar{x}, \bar{y}})_{\bar{x}, \bar{y} \in I} \cdot (r'_{\bar{x}, \bar{y}})_{\bar{x}, \bar{y} \in I} := \left( \sum_{\bar{x} \in I} r_{\bar{x}, \bar{y}} \cdot r'_{\bar{x}, \bar{z}} \right)_{\bar{x}, \bar{y} \in I}.
\]
We call \( A^I \) the \( I \)-canonical algebra. Moreover we call
\[
\bar{\delta} := d\bar{\epsilon} + 2\bar{\omega} \in \mathbb{L}
\]
the dominant element (cf. [KLM]), and
\[
A_{\text{CM}} := A^{[0, \bar{\delta}]}
\]
the \( CM \)-canonical algebra, which plays an important role in this subsection.

We give the first properties of \( I \)-canonical algebras.

Proposition 4.10. Let \( I \) be a finite subset of \( \mathbb{L} \).

(a) The \( k \)-algebra \( A^I \) has finite global dimension. In particular, we have \( \text{K}^b(\text{proj} A^I) \simeq \text{D}^b(\text{mod} A^I) \).

(b) We have isomorphisms of \( k \)-algebras \( A^{-I} \simeq (A^I)^{\text{op}} \) and \( A^{I+\bar{z}} \simeq A^I \) for any \( \bar{z} \in \mathbb{L} \).

(c) For any \( \bar{x}, \bar{y} \in \mathbb{L} \), we have an isomorphism \( A^{[\bar{x}, \bar{y}]} \simeq (A^{[\bar{x}, \bar{y}]})^{\text{op}} \) of \( k \)-algebras.

Proof. (a) All the diagonal entries of \( A^I \) are \( R_{x - \bar{x}} = R_0 = k \). Moreover, since \( \mathbb{L} \) is a partially ordered set, either \( \bar{x} \not\geq \bar{y} \) or \( \bar{x} \not\leq \bar{y} \) holds. Therefore either \( R_{\bar{x} - \bar{y}} = 0 \) or \( R_{\bar{y} - \bar{x}} = 0 \) holds. These observations imply that \( A^I \) has finite global dimension.

(b) These are clear.

(c) This follows from (b) since \( -[\bar{x}, \bar{y}] + \bar{x} + \bar{y} = [\bar{x}, \bar{y}] \) holds. \( \square \)
Proposition 4.11. Let $I$ be a finite subset of $\mathbb{L}$. Then

(a) We have an equivalence $\text{proj}^I R \simeq \text{proj} A^I$.

(b) If $I$ is convex, then we have an equivalence $\text{mod}^I R \simeq \text{mod} A^I$.

Proof. (a) We have a triangle equivalence

$$P \simeq \text{proj}^I R \simeq \text{proj} A^I.$$ 

(b) This is an analog of (3.3). The equivalence is given by $M = \bigoplus_{\bar{x} \in I} M_{\bar{x}} \mapsto (M_{\bar{x}})_{\bar{x} \in I}$. □

To simplify notations, let

$$D := D^b(\text{mod}^I R), \quad S := D^b(\text{mod}^0_0 R), \quad \mathcal{P} := K^b(\text{proj}^I R),$$

$$D^I := D^b(\text{mod}^I R), \quad S^I := D^b(\text{mod}^0_0 R), \quad \mathcal{P}^I := K^b(\text{proj}^I R).$$

By the following observations, we can regard $\mathcal{P}^I$ as a thick subcategory of $D$, and when $I$ is convex, we can regard $D^I$ and $S^I$ as thick subcategories of $D$.

Observation 4.12. Let $I$ be a subset of $\mathbb{L}$.

(a) $\mathcal{P}^I$ is triangle equivalent to the subcategory $\text{thick}(R(-\bar{x}) \mid \bar{x} \in I)$ of $D$.

(b) If $I$ is convex, then $D^I$ (respectively, $S^I$) are triangle equivalent to the subcategory $D^b_{\text{mod}^I R}(\text{mod}^I R)$ (respectively, $\text{thick}(k(-\bar{x}) \mid \bar{x} \in I) = D^b_{\text{mod}^0_0 R}(\text{mod}^I R)$) of $D$.

When $I$ is a convex subset, we have a functor

$$(-)^I : \text{mod}^I R \to \text{mod}^I R$$

given by $X_I := \bigoplus_{\bar{x} \in I} X_{\bar{x}}$.

Note that $X_I$ has a natural structure of an $\mathbb{L}$-graded $R$-module: Let $\bar{T}$ be the smallest poset ideal of $\mathbb{L}$ containing $I$. Then $X_I = \bigoplus_{\bar{x} \in \bar{T}} X_{\bar{x}}$ and $X_{\bar{T} \cup I} = \bigoplus_{\bar{x} \in \bar{T} \cup I} X_{\bar{x}}$ are subobjects of $X$ in $\text{mod}^I R$, and we regard $X_I$ as the quotient object $X_{\bar{T}}/X_{\bar{T} \cup I}$ in $\text{mod}^I R$.

The following easy observations play an important role in this paper.

Theorem 4.13. Let $I$ be a finite subset of $\mathbb{L}$.

(a) We have a triangle equivalence $D^b(\text{mod} A^I) \simeq \mathcal{P}^I$. Moreover $\mathcal{P}^I$ has a tilting object $T^I := \bigoplus_{\bar{x} \in I} R(-\bar{x}) \in \text{proj}^I R$ such that $\text{End}_{\mathcal{P}^I}(T^I) \simeq A^I$.

(b) Assume that $I$ is convex. Then we have a triangle equivalence $D^b(\text{mod} A^I) \simeq S^I$. Moreover $S^I$ has a tilting object $U^I := \bigoplus_{\bar{x} \in I} R(-\bar{x})I \in \text{mod}^0_0 R$ such that $\text{End}_{S^I}(U^I) \simeq A^I$.

Proof. (a) We have an equivalence $\text{proj}^I R \simeq \text{proj} A^I$ by Proposition 4.11(a). Thus we have a triangle equivalence

$$\mathcal{P}^I = K^b(\text{proj}^I R) \simeq K^b(\text{proj} A^I)$$

sending $T^I$ to $A^I$, which shows that $T^I$ is a tilting object in $\mathcal{P}^I$. The isomorphism $\text{End}_{\mathcal{P}^I}(T^I) \simeq A^I$ follows from $\text{Hom}_{\mathcal{P}^I}(R(-\bar{x}), R(-\bar{y})) \simeq R_{\bar{x}-\bar{y}}$ and $\text{End}_{\mathcal{P}^I}(T^I) \simeq (R_{\bar{x}-\bar{y}})_{\bar{x}, \bar{y} \in I} = A^I$. By Proposition 4.11(a), we have triangle equivalences $\mathcal{P}^I \simeq K^b(\text{proj} A^I) \simeq D^b(\text{mod} A^I)$. 

For a subset $I$ of $\mathbb{L}$, let

\[
\begin{align*}
\text{mod}^I R &:= \{ X = \bigoplus_{\bar{x} \in I} X_{\bar{x}} \in \text{mod}^I R \mid \forall \bar{x} \in \mathbb{L} \setminus I, \ X_{\bar{x}} = 0 \}, \\
\text{mod}^0_0 R &:= \text{mod}^I R \cap \text{mod}^0_0 R, \\
\text{proj}^I R &:= \text{add}\{ R(-\bar{x}) \mid \bar{x} \in I \}.
\end{align*}
\]

If $I$ is finite, then $\text{mod}^I R = \text{mod}^0_0 R$ holds clearly. We have the following elementary properties.
Theorem 4.14. For any non-trivial poset ideal \( CM \) of \( I \), the composition

\[
\mathcal{D}^{\mathcal{I}} \cap (\mathcal{D}^{\mathcal{I}})^{\ast} \subset \mathcal{D}^{b}(\mathcal{M}^{\mathcal{I}} R) \to \mathcal{M}^{\mathcal{I}} R
\]

is a triangle equivalence.

We start with some easy observations. Recall from Section 2 that \( (\mathcal{X} \ast \mathcal{Y}) \) denotes the category of extensions of subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \mathcal{D} \). If \( \mathcal{D}(\mathcal{X}, \mathcal{Y}) = 0 \) holds, we write

\[
\mathcal{X} \perp \mathcal{Y} := \mathcal{X} \ast \mathcal{Y}.
\]

A semiorthogonal decomposition (or stable t-structure) of \( \mathcal{D} \) is a pair of thick subcategories of \( \mathcal{D} \) satisfying \( \mathcal{D} = \mathcal{X} \perp \mathcal{Y} \). In this case we have triangle equivalences \( \mathcal{X} \simeq \mathcal{D}/\mathcal{Y} \) and \( \mathcal{Y} \simeq \mathcal{D}/\mathcal{X} \).

The following distributive law is clear.

Lemma 4.15. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be thick subcategories of \( \mathcal{D} \) such that \( \mathcal{X} \ast \mathcal{Y} \) is a thick subcategory of \( \mathcal{D} \). Then \( (\mathcal{X} \ast \mathcal{Y}) \cap \mathcal{Z} \) (respectively, \( \mathcal{X} \cap \mathcal{Z} \)) is a thick subcategory of \( \mathcal{D} \). In this case we have triangle equivalences \( \mathcal{X} \simeq \mathcal{C}/\mathcal{Y} \) and \( \mathcal{Y} \simeq \mathcal{C}/\mathcal{X} \).

We have the following elementary observation, which is an \( L \)-graded version of [O1, 2.3].

Lemma 4.16. Let \( \mathcal{I} \) be a non-zero poset ideal of \( L \).

(a) We have \( K(\mathcal{P}^{\mathcal{I}}) = K(\mathcal{P}^{\mathcal{I}}) \perp K(\mathcal{P}^{\mathcal{I}}) \), \( \mathcal{P} = \mathcal{P}^{\mathcal{I}} \perp \mathcal{I} \) and \( \mathcal{D} = \mathcal{P}^{\mathcal{I}} \perp \mathcal{D}^{\mathcal{I}} \). More generally, for a poset ideal \( I \) containing \( \mathcal{I} \) we have \( \mathcal{P} = \mathcal{P}^{\mathcal{I}} \perp \mathcal{I} \) and \( \mathcal{D} = \mathcal{P}^{\mathcal{I}} \perp \mathcal{D}^{\mathcal{I}} \).

(b) We have a triangle equivalence \( \mathcal{D}^{\mathcal{I}} / \mathcal{P}^{\mathcal{I}} \simeq \mathcal{M}^{\mathcal{I}} R \).

Proof. (a) Clearly \( K(\mathcal{P}^{\mathcal{I}}) = K(\mathcal{P}^{\mathcal{I}}) \perp K(\mathcal{P}^{\mathcal{I}}) \) holds. For any \( P \in \mathcal{P}^{\mathcal{I}} \), we denote by \( P^{\mathcal{I}} \) the sub-\( R \)-module of \( P \) generated by the subspace \( \bigoplus_{\mathcal{I} \in \mathcal{I}^{\ast}} P_{\mathcal{I}} \), and \( P^{\mathcal{I}} := P/P^{\mathcal{I}} \). We give functors \( (-)^{\mathcal{I}} : \mathcal{P}^{\mathcal{I}} \to \mathcal{P}^{\mathcal{I}} \), \( (-)^{\mathcal{I}} : \mathcal{P}^{\mathcal{I}} \to \mathcal{P}^{\mathcal{I}} \) and \( \mathcal{D}^{\mathcal{I}} \) a and sequence

\[
0 \to (-)^{\mathcal{I}} \to \mathcal{P}^{\mathcal{I}} \to (-)^{\mathcal{I}} \to 0
\]

of natural transformations which is objectwise split exact. Therefore we have induced triangle functors \( (-)^{\mathcal{I}} : \mathcal{K}(\mathcal{P}^{\mathcal{I}}) \to \mathcal{K}(\mathcal{P}^{\mathcal{I}}) \), \( (-)^{\mathcal{I}} : \mathcal{K}(\mathcal{P}^{\mathcal{I}}) \to \mathcal{K}(\mathcal{P}^{\mathcal{I}}) \) and a functorial triangle \( Q^{\mathcal{I}} \to Q \to Q^{\mathcal{I}} \to Q^{\mathcal{I}}[1] \) for any \( Q \in \mathcal{K}(\mathcal{P}^{\mathcal{I}}) \). Thus we have the first equality. The second equality \( \mathcal{P} = \mathcal{P}^{\mathcal{I}} \perp \mathcal{D}^{\mathcal{I}} \) follows immediately. The third equality \( \mathcal{D} = \mathcal{P}^{\mathcal{I}} \perp \mathcal{D}^{\mathcal{I}} \) follows from an equivalence between \( \mathcal{D} \) and the homotopy category \( K_{-\mathcal{B}}(\mathcal{P}^{\mathcal{I}}) \) of complexes bounded above with bounded cohomologies.

The remaining equalities can be shown similarly.

(b) By (a), we have triangle equivalences \( \mathcal{P}^{\mathcal{I}} \simeq \mathcal{P}/\mathcal{P}^{\mathcal{I}} \) and \( \mathcal{D}^{\mathcal{I}} \simeq \mathcal{D}/\mathcal{D}^{\mathcal{I}} \). Therefore we have

\[
\mathcal{M}^{\mathcal{I}} R \simeq \mathcal{D}/\mathcal{P} \simeq (\mathcal{D}/\mathcal{P}^{\mathcal{I}})/(\mathcal{P}/\mathcal{P}^{\mathcal{I}}) \simeq \mathcal{D}^{\mathcal{I}}/\mathcal{P}^{\mathcal{I}}.
\]

□
Proof of Theorem 4.14. Applying Lemma 4.11(a) to the poset ideal \( -I^c \), we have \( D = P^\top \perp D^{\perp} \). Applying \((-)^+\), we have \( D = D^* = (D^{\perp})^* \perp P^i \). Taking the intersections of \( D^i \) with both sides and applying Lemma 4.15 we have \( D^i = (D^i \cap (D^*)^*) \perp P^i \). Therefore \( D^i \cap (D^*)^* \simeq D^i / P^i \simeq \CM^{k,i} \) holds by Lemma 4.16(b). \( \square \)

Now we are ready to prove the following main result in this section.

**Theorem 4.17.** Let \((R, L)\) be a Geigle-Lenzing complete intersection.

(a) The following composition is a triangle equivalence:

\[
S^{[0, \vec{\delta}]} \subseteq D^h(\mod^{d^\top} R) \xrightarrow{\rho} \CM^{k^\top} R.
\]

(b) We have triangle equivalences

\[
D^h(\mod A^{CM}) \simeq S^{[0, \vec{\delta}]} \simeq \CM^{k^\top} R \text{ such that } A^{CM} \mapsto U^{[0, \vec{\delta}]} \mapsto T^{CM} := \rho(U^{[0, \vec{\delta}]}).
\]

In particular \( \CM^{k^\top} R \) has a tilting object \( T^{CM} \).

(c) We have \( S^{[0, \vec{\delta}]} = D^{L_+ + (D^{-L_+})*} \).

For the hypersurface case \( n = d + 2 \), this result was shown by Futaki-Ueda [FU] and Kussin-Lenzing-Meltzer [KLM] \((d = 1)\) using quite different methods. For the non-hypersurface case, Theorem 4.17 is new even for the case \( d = 1 \).

**Proof.** We only have to prove (c). In fact, (a) follows from (c) and Theorem 4.14 and (b) follows from (a) and Theorem 4.13(b).

In the rest, we prove the statement (c). By Lemma 3.11 we have \( -\vec{\omega} \nsubseteq \vec{x} \) if and only if \( \vec{x} \leq d\vec{c} \). Thus we have \( \LL_+ \cap (L_+ + \vec{\omega}) = [0, \vec{\delta}] \).

As a consequence, we have

\[
D^{L_+ + (D^{-L_+})*} \supseteq S^{L_+ + (S^{-L_+})*} \overset{\text{(4.3)}}{=} S^{L_+ + \vec{\omega}} = S^{L_+ \cap (L_+ + \vec{\omega})} \simeq S^{[0, \vec{\delta}]}.
\]

To show the reverse inclusion, it is enough to show that the composition \( S^{[0, \vec{\delta}]} \subseteq D \xrightarrow{\rho} \CM^{k^\top} R \) is dense. This is equivalent to \( D = \thick(S^{[0, \vec{\delta}], P}) \) by Proposition 2.8. Since we have \( D = \thick(S, P) \) by Theorem 4.10, it is enough to show the following statement.

**Proposition 4.18.** \( S \subseteq \thick(S^{[0, \vec{\delta}], P}) \).

**Proof.** We prepare the following simple observation.

**Lemma 4.19.** Let \( J \) be a subset of \( \{1, \ldots, n\} \) with \( |J| = n - d - 1 \) and \( J^c := \{1, \ldots, n\} \backslash J \).

(a) The \( R \)-module \( F := R/(X_i^{p_i} \mid i \in J^c) \) is finite dimensional and belongs to \( P \).

(b) \( \soc F = k(-\sum_{i=1}^n (p_i - 1)\vec{z}_i) \).

(c) \( F / \soc F \) belongs to \( \thick(k(-\vec{x}))_{0 \leq \vec{x} < \sum_{i=1}^n (p_i - 1)\vec{z}_i} \).

**Proof.** Since \( \ell_i (i \in J^c) \) are linear independent, it follows from Lemma 3.6(c) that \( X_i^{p_i} (i \in J^c) \) forms an \( R \)-regular sequence. Thus the assertion (a) follows.

Moreover \( F \) is a finite dimensional Gorenstein algebra whose \( a \)-invariant is given by

\[
\vec{\omega} + (d + 1)\vec{c} = \sum_{i=1}^n (p_i - 1)\vec{z}_i.
\]

This equals the degree of the socle of \( F \), and the assertion (b) follows. Since the degree 0 part of \( F \) is \( k \), its degree \( \sum_{i=1}^n (p_i - 1)\vec{z}_i \) part has to be one dimensional. Thus the assertion (c) follows. \( \square \)
Now we prove Proposition 4.18.

(i) We show that \( k(-\vec{x}) \in \text{thick}\{S^{(0,\delta)}_R, \mathcal{P}\} \) holds for any \( \vec{x} \in \mathbb{L}_+ \) by using induction with respect to the partial order on \( \mathbb{L}_+ \). We write \( \vec{x} \in \mathbb{L}_+ \) in a normal form \( \vec{x} = \sum_{i=1}^n a_i \vec{x}_i + a \vec{c} \) (see Observation 3.1 (a)). Let \( J := \{i \mid a_i = p_i - 1\} \).

Assume \(|J| + a \leq n - d - 2\). Then we have

\[
0 \leq \vec{x} \leq (n - d - 2)\vec{c} + \sum_{i=1}^n (p_i - 2)\vec{x}_i = (2n - d - 2)\vec{c} - 2 \sum_{i=1}^n \vec{x}_i = \vec{d},
\]

and hence \( k(-\vec{x}) \in S^{(0,\delta)}_R \).

Assume \(|J| + a \geq n - d - 1\). Then there exists a subset \( J' \) of \( J \) such that \(|J'| = n - d - 1\). Then \( \vec{y} := \vec{x} - \sum_{i \in J'} (p_i - 1)\vec{x}_i \) belongs to \( \mathbb{L}_+ \). By Lemma 4.19, we have an exact sequence

\[
0 \rightarrow k(-\vec{x}) \rightarrow F(-\vec{y}) \rightarrow (F/\text{soc} F)(-\vec{y}) \rightarrow 0
\]

in \( \text{mod}^\delta R \) with \( F(-\vec{y}) \in \mathcal{P} \) and \( (F/\text{soc} F)(-\vec{y}) \in \text{thick}\{S^{(0,\delta)}_R, \mathcal{P}\} \). By the induction hypothesis, we have \( (F/\text{soc} F)(-\vec{y}) \in \text{thick}\{S^{(0,\delta)}_R, \mathcal{P}\} \) and therefore \( k(-\vec{x}) \in \text{thick}\{S^{(0,\delta)}_R, \mathcal{P}\} \).

(ii) Similarly, by using induction with respect to the reverse of the partial order on \( \mathbb{L}_+ \), we can show that \( k(-\vec{x}) \in \text{thick}\{S^{(0,\delta)}_R, \mathcal{P}\} \) holds for any \( \vec{x} \in \mathbb{L}_+ \).

We have finished proving Theorem 4.17 (c), which implies all other statements as we observed.

In the rest of this subsection, we will give a quiver presentation of \( I \)-canonical algebras. To give presentations in a uniform way, we should start with a presentation of our GL complete intersection \((R, L)\) satisfying the following condition.

**Assumption 4.20.** Applying Observation 3.2 to \((R, \mathbb{L})\), we may assume that one of the following conditions holds.

- \( n = d + 1 \).
- \( n > d + 1 \) and \( p_i \geq 2 \) for all \( 1 \leq i \leq n \).

Furthermore, by Observation 3.3 we may assume in both cases that

\[
R = k[X_1, \ldots, X_n]/(X_{i_1}^{p_1} - \sum_{j=1}^{d+1} \lambda_{i,j-1} X_{j}^{p_j} \mid d + 2 \leq i \leq n).
\]

Now we have the following quiver presentations of \( I \)-canonical algebras.

**Theorem 4.21.** Let \( I \) be a finite convex subset of \( \mathbb{L} \). Under Assumption 4.20, the \( I \)-canonical algebra \( A_I \) is presented by the quiver \( Q_I \) defined by

- \( Q_0^I = I \),
- \( Q_1^I = \{i : x \rightarrow \vec{x} \mid 1 \leq i \leq n, \vec{x} \in I \cap (I - \vec{x})\} \)

with the following relations:

- \( x_{i,j} x_{j,i} : \vec{x} \rightarrow \vec{x} + \vec{x}_i \mid 1 \leq i < j \leq n \) and \( \vec{x} \in I \cap (I - \vec{x}_i - \vec{x}_j) \),
- \( x_{i}^{p_i} - \sum_{j=1}^{d+1} \lambda_{i,j-1} x_j^{p_j} : \vec{x} \rightarrow \vec{x} + \vec{c} \mid d + 2 \leq i \leq n \) and \( \vec{x} \in I \cap (I - \vec{c}) \).

**Proof.** The vertices of \( Q_I \) naturally corresponds to the primitive idempotents of \( A_I \). The arrows \( x_{1}, \ldots, x_{n} \) of \( Q_I \) corresponds to the generators \( X_1, \ldots, X_n \) of \( R \). Thus we have a morphism \( kQ_I \rightarrow A_I \) of \( k \)-algebras, which is surjective since \( I \) is convex. Clearly the commutativity relations \( x_{i,j} x_{j,i} = x_{j,i} x_{i,j} \) are satisfied in \( A_I \). Also the relations \( X_i^{p_i} = \sum_{j=1}^{d+1} \lambda_{i,j-1} X_j^{p_j} \) in \( R \) correspond to the relations \( x_i^{p_i} = \sum_{j=1}^{d+1} \lambda_{i,j-1} x_j^{p_j} \) in \( A_I \). Thus we have a surjective morphism \( B_I \rightarrow A_I \) of \( k \)-algebras, where \( B_I \) is the factor algebra of \( kQ_I \) by these relations. This is an isomorphism since it clearly induces an isomorphism \( B_I e_{\vec{x}} \simeq A_I e_{\vec{x}} = \bigoplus_{\vec{y} \in I \cap (I + L_+)} R_{\vec{y}} \) for any \( \vec{x} \in Q_0^I \). Therefore we have the assertion.

\( \square \)
Now we concentrate on our CM-canonical algebra $A^\text{CM}$. We work under Assumption 4.20. It follows from Proposition 4.10(a) that $A^\text{CM}$ has finite global dimension. We give more properties below, where the statement (b) was shown in [KLM, 6.1] (cf. [FU, 1.2]). We refer to [HM] for more information on the tensor product of the path algebras of type $A$.

**Corollary 4.22.** Let $(R, L)$ be a Geigle-Lenzing complete intersection in dimension $d + 1$ with weights $p_1, \ldots, p_n$, and $A^\text{CM}$ the corresponding CM-canonical algebra.

(a) $A^\text{CM} = 0$ if and only if $n = d + 1$.

If $n = d + 2$, then $\vec{\delta} = \sum_{i=1}^{n} (p_i - 2) \vec{x}_i$ and the following assertions hold.

(b) We have $A^\text{CM} \simeq \bigotimes_{i=1}^{n} k \mathcal{A}_{p_i-1}$, where $k \mathcal{A}_{p_i-1}$ is the path algebra of the equioriented quiver of type $A_{p_i-1}$. In particular, $\text{CM}^L R$ is independent of the choice of hyperplanes.

(c) The global dimension of $A^\text{CM}$ is equal to $|\{i \mid p_i \geq 2\}|$.

(d) The Grothendieck group $K_0(\text{CM}^R)$ is a free abelian group of rank $\prod_{i=1}^{n} (p_i - 1)$.

(e) (Knörrer periodicity) Let $(R^\prime, L')$ be a Geigle-Lenzing complete intersection in dimension $d + 2$ with weights $2, p_1, \ldots, p_n$. Then we have a triangle equivalence $\text{CM}^L R \simeq \text{CM}^{L'} R'$.

**Proof.** (a) Clearly $A^\text{CM} \neq 0$ if and only if $0 \leq \vec{\delta}$. This is equivalent to $n \geq d + 2$ since $\vec{\delta}$ has a normal form $\vec{\delta} = \sum_{i=1}^{n} (p_i - 2) \vec{x}_i + (n - d - 2) \vec{c}$.

(b) Clearly $\vec{\delta} = \sum_{i=1}^{n} (p_i - 2) \vec{x}_i$ holds. It is easy to check that the quiver $Q^{[0, \vec{\delta}]}$ coincides with the quiver of $\bigotimes_{i=1}^{n} k Q_{p_i-1}$. Moreover $A^{[0, \vec{\delta}]}$ has only commutativity relations since $[0, \vec{\delta}] \cap ([0, \vec{\delta}] - \vec{c}) = \emptyset$ holds by $0 \leq \vec{\delta} - \vec{c}$. Hence the assertion follows.

(c) This follows from (b).

(d) The assertion follows from the triangle equivalence $D^b(\text{mod } A^\text{CM}) \simeq \text{CM}^L R$ in Theorem 4.17 since the Grothendieck group of $\bigotimes_{i=1}^{n} k \mathcal{A}_{p_i-1}$ is a free abelian group of rank $\prod_{i=1}^{n} (p_i - 1)$.

(e) By (b), the CM-canonical algebras of $R$ and $R'$ are isomorphic. Thus we have the desired triangle equivalence.

We give a few examples of CM-canonical algebras.

**Example 4.23.** (a) Let $d = 1$, $n = 3$ and all $(p_1, p_2, p_3) = (3, 3, 3)$. Then the quiver of $A^\text{CM}$ is

```
0 \rightarrow \vec{x}_1 \rightarrow \vec{x}_1 + \vec{x}_2
0 \rightarrow \vec{x}_2 \rightarrow \vec{x}_1 + \vec{x}_3
0 \rightarrow \vec{x}_3 \rightarrow \vec{x}_2 + \vec{x}_3
```

In fact this is a well-known tubular algebra of type $(3, 3, 3)$.

(b) Let $d = 1$, $n = 4$ and $(p_1, p_2, p_3, p_4) = (2, 2, 2, 3)$. Then the quiver of $A^\text{CM}$ is

```
0 \rightarrow \vec{x}_1 \rightarrow \vec{x}_1 + \vec{x}_4
0 \rightarrow \vec{x}_2 \rightarrow \vec{x}_2 + \vec{x}_4
0 \rightarrow \vec{x}_3 \rightarrow \vec{x}_3 + \vec{x}_4
0 \rightarrow \vec{x}_4 \rightarrow 2 \vec{x}_1 + \vec{c}
```

We end this subsection with the following question.

**Question 4.24.** What is $\text{gl.dim } A^\text{CM}$ in general?
4.3. Cohen-Macaulay finiteness and $d$-Cohen-Macaulay finiteness. Let $(R, L)$ be a Geigle-Lenzing complete intersection associated with hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^d$ and weights $p_1, \ldots, p_n$.

Cohen-Macaulay finiteness. We say that a GL complete intersection $(R, L)$ is Cohen-Macaulay finite (= CM finite) if there are only finitely many isomorphism classes of indecomposable objects in $\text{CM}^\geq R$ up to degree shift.

As an application of results in previous section, we have the following classification of CM finite GL complete intersections.

**Theorem 4.25.** Let $(R, L)$ be GL complete intersection. Assume that $p_i \geq 2$ for any $i$. Then $(R, L)$ is CM finite if and only if one of the following conditions hold.

- $n \leq d + 1$.
- $n = d + 2$, and $(p_1, \ldots, p_n) = (2, \ldots, 2, p_1), (2, \ldots, 2, 3, 3), (2, \ldots, 2, 3, 4)$ or $(2, \ldots, 2, 3, 5)$ up to permutation.

**Proof.** If $n \leq d + 1$, then $(R, L)$ is CM finite since $\text{CM}^\geq R = \text{proj}^\geq R$ holds. If $(R, L)$ is CM finite, then $n \leq d + 2$ holds by Proposition 4.25. Therefore we assume $n = d + 2$ in the rest of proof.

**Lemma 4.26.** Assume $n = d + 2$. Then $(R, L)$ is CM finite if and only if there are only finitely many isomorphism classes of indecomposable objects in $\text{D}^b(\text{mod} A)$ up to suspension.

**Proof.** We have a triangle equivalence $\text{D}^b(\text{mod} A) \simeq \text{CM}^\geq R$ by Theorem 4.17. Moreover $[2] = (\ell)$ holds by Theorem 4.2(d). Thus the assertion holds. \qed

Now let us recall the following well-known result.

**Lemma 4.27.** (a) For $\ell = 2, 3, 4$, the algebra $kA_2 \otimes_k kA_\ell$ is derived equivalent to $kA_4$ if $\ell = 2$, $kE_6$ if $\ell = 3$, and $kE_8$ if $\ell = 4$.

(b) $kA_2 \otimes_k kA_5$ is derived equivalent to the canonical algebra of type $(2, 3, 6)$.

(c) $kA_3 \otimes_k kA_3$ is derived equivalent to the canonical algebra of type $(2, 4, 4)$.

(d) $kA_2 \otimes_k kA_2 \otimes_k kA_2$ is derived equivalent to the canonical algebra of type $(3, 3, 3)$.

In particular, for the algebra $A$ in (b), (c) or (d), there are infinitely many indecomposable objects in $\text{D}^b(\text{mod} A)$ up to degree shift.

**Proof.** For (a), we refer to [KLM] §8. For (b), (c) and (d), we refer to [KLM] 5.6. \qed

For the four cases with $n = d + 2$ listed in Theorem 4.26, it follows from Lemma 4.27 that $A^\text{CM} \simeq \bigotimes_{i=1}^{d+2} kA_{p_i-1}$ is derived equivalent to a path algebra of a Dynkin quiver. In particular there are only finitely many isomorphism classes of indecomposable objects in $\text{D}^b(\text{mod} A^\text{CM})$ up to shift. Hence $(R, L)$ is CM finite by Lemma 4.26.

Conversely, assume that the weights are not one of $(2, \ldots, 2, p_n), (2, \ldots, 2, 3, 3), (2, \ldots, 2, 3, 4)$ or $(2, \ldots, 2, 3, 5)$. It is easy to check that one of the following conditions hold up to permutation:

- $p_1 \geq 3$ and $p_2 \geq 6$.
- $p_1 \geq 4$ and $p_2 \geq 4$.
- $p_1 \geq 3, p_2 \geq 3$ and $p_3 \geq 3$.

Thus there exists an idempotent $e$ in $A^\text{CM}$ such that $eA^\text{CM}e$ is isomorphic to $kA_2 \otimes_k kA_5, kA_3 \otimes_k kA_3$ or $kA_2 \otimes_k kA_2 \otimes_k kA_2$. In each case, there are infinitely many isomorphism classes of indecomposable objects in $\text{D}^b(\text{mod} eA^\text{CM}e)$ (and hence in $\text{D}^b(\text{mod} A^\text{CM})$) up to degree shift by Lemma 4.27. By Lemma 4.26 we have that $(R, L)$ is not CM finite. \qed

Theorem 4.26 tells us that there are only a few CM finite GL complete intersections. In higher dimensional Auslander-Reiten theory, we introduce the notion of ‘$d$-CM finiteness’ as a proper substitute of CM finite finiteness. Before defining them, we prepare the following easy observations.

**Lemma 4.28.** Let $\bar{a} \in L$ be an element which is not a torsion.
(a) For any $X, Y \in \text{mod}^d R$ and $i \geq 0$, we have

\[ \text{Ext}^i_{\text{mod}^d/R}(X, Y) = \bigoplus_{\ell \in \mathbb{Z}} \text{Ext}^i_{\text{mod}^d/R}(X, Y((\ell))) \]

(b) For any $M \in \text{mod}^d R$, the subcategory $\mathcal{C} := \text{add}\{M(\ell \tilde{a}) \mid \ell \in \mathbb{Z}\}$ (respectively, $\mathcal{C}^+ := \text{add}\{M(\ell \tilde{a}) \mid \ell \geq 0\}$, $\mathcal{C}^- := \text{add}\{M(\ell \tilde{a}) \mid \ell \leq 0\}$) is functorially finite in $\text{mod}^d R$.

Proof. (a) This is clear.

(b) We only show that $\mathcal{C}$ is covariantly finite in $\text{mod}^d R$ since other assertions can be shown similarly. The $(\mathbb{Z}\tilde{a})$-Veronese subalgebra $R^{(\mathbb{Z}\tilde{a})} = \bigoplus_{\ell \in \mathbb{Z}} R_{\tilde{a}}$ of $R$ is noetherian. For any $X \in \text{mod}^d R$, the $R^{(\mathbb{Z}\tilde{a})}$-module $\text{Hom}_{R^{(\mathbb{Z}\tilde{a})}}^{i/2\mathbb{Z}}(X, M) = \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{R_{\tilde{a}}}(X, M(\ell \tilde{a}))$ is finitely generated. Let $f_1, \ldots, f_m$ with $f_i \in \text{Hom}_{R_{\tilde{a}}}(X, M(\ell_i \tilde{a}))$ be homogeneous generators. It is easy to check that $f := (f_1, \ldots, f_m) : X \to \bigoplus_{i=1}^m M(\ell_i \tilde{a})$ is a left $\mathcal{C}$-approximation. \qed

Recall from Section 2.2 that a full subcategory $\mathcal{C}$ of $\text{CM}^d R$ is called a $d$-cluster tilting if it is a functorially finite subcategory of $\text{CM}^d R$ such that

\[ \mathcal{C} = \{X \in \text{CM}^d R \mid \forall i \in \{1, 2, \ldots, d-1\} \quad \text{Ext}^i_{\text{mod}^d R}(\mathcal{C}, X) = 0 \} \quad \text{and} \quad \mathcal{C} = \{X \in \text{CM}^d R \mid \forall i \in \{1, 2, \ldots, d-1\} \quad \text{Ext}^i_{\text{mod}^d R}(X, \mathcal{C}) = 0 \} \]

Note that one of the equalities above implies the other [11.2.2.2]. In this case $\mathcal{C}$ generates and cogenerates $\text{CM}^d R$ since it contains $\text{proj}^d R$. Moreover $\mathcal{C} = (\mathbb{Z}\tilde{a})$ holds by Auslander-Reiten-Serre duality given in Theorem 4.1 (a).

$d$-Cohen-Macaulay finiteness. We say that a GL complete intersection $(R, L)$ is $d$-Cohen-Macaulay finite ($=d$-CM finite) if there exists a $d$-cluster tilting subcategory $\mathcal{C}$ of $\text{CM}^d R$ (see Section 2.2) such that there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$ up to degree shift.

In the case $d = 1$, $d$-CM finiteness coincides with classical CM finiteness since $\text{CM}^1 R$ is the unique 1-cluster tilting subcategory of $\text{CM}^d R$.

When $(R, L)$ is not Calabi-Yau, we have the following equivalent conditions.

**Lemma 4.29.** Assume that $(R, L)$ is not Calabi-Yau. Then the following conditions are equivalent.

(a) $(R, L)$ is $d$-CM finite.

(b) There exists $M \in \text{CM}^d R$ satisfying

\[ \text{add}_{\text{CM}^d R}\{M(\ell \tilde{a}) \mid \ell \in \mathbb{Z}\} = \{X \in \text{CM}^d R \mid \forall i \in \{1, 2, \ldots, d-1\} \quad \text{Ext}^i_{\text{mod}^d \text{Z}/2\mathbb{Z}}(M, X) = 0\}. \]

(\text{or} \quad \text{add}_{\text{CM}^d R}\{M(\ell \tilde{a}) \mid \ell \in \mathbb{Z}\} = \{X \in \text{CM}^d R \mid \forall i \in \{1, 2, \ldots, d-1\} \quad \text{Ext}^i_{\text{mod}^d \text{Z}/2\mathbb{Z}}(X, M) = 0\}).

**Proof.** (b)⇒(a) If $M \in \text{CM}^d R$ satisfies the equality above, then $\mathcal{C} := \text{add}\{M(\ell \tilde{a}) \mid \ell \in \mathbb{Z}\}$ is a functorially finite subcategory of $\text{CM}^d R$ by Lemma 1.28 (b). Therefore $\mathcal{C}$ is a $d$-cluster tilting subcategory of $\text{CM}^d R$ by Lemma 1.28 (a). Since $\mathcal{C}$ clearly has only finitely many isomorphism classes of indecomposable objects up to degree shift, $(R, L)$ is $d$-CM finite.

(a)⇒(b) Let $\mathcal{C}$ be a $d$-cluster tilting subcategory of $\text{CM}^d R$ with only finitely many isomorphism classes of indecomposable objects up to degree shift. Since $\mathcal{C} = (\mathbb{Z}\tilde{a})$, there exists $M \in \text{CM}^d R$ such that $\mathcal{C} = \text{add}\{M(\ell \tilde{a}) \mid \ell \in \mathbb{Z}\}$. It follows from Lemma 1.28 (a) that $M$ satisfies the desired equality. \qed

We will give a sufficient condition for $d$-CM finiteness in terms of tilting theory in $\text{CM}^d R$. We start with the following basic observations.

**Lemma 4.30.** Let $U$ be a tilting object in $\text{CM}^d R$, and $\Lambda := \text{End}^d_R(U)$.

(a) $\Lambda$ has finite global dimension for any tilting object $U$ in $\text{CM}^d R$. 


(b) We have a triangle equivalence $F : \text{D}^b(\text{mod } \Lambda) \simeq \text{CM}^L_R$ such that the diagram
\[
\begin{array}{ccc}
\text{D}^b(\text{mod } \Lambda) & \xrightarrow{F} & \text{CM}^L_R \\
\nu_d \downarrow & & \downarrow (\omega) \\
\text{D}^b(\text{mod } \Lambda) & \xrightarrow{F} & \text{CM}^L_R
\end{array}
\]
commutes, where $\nu_d = (DA)[−d] \otimes \Lambda$ is the $d$-shifted Nakayama functor.

**Proof.** By Proposition 2.2, we have a triangle equivalence $F : \text{D}^b(\text{mod } \Lambda) \simeq \text{CM}^L_R$. 
(a) By Theorem 4.17(b), $\Lambda$ is derived equivalent to $A^\text{CM}$, which has finite global dimension by Proposition 4.10(a). Thus the assertion follows from Proposition 2.3(b).
(b) Both $\nu$ and $(\omega)[d]$ are Serre functors of $\text{D}^b(\text{mod } \Lambda)$ and $\text{CM}^L_R$ respectively by Proposition 2.4 and Theorem 4.2. Since the Serre functor is unique up to isomorphism, the diagram commutes. □

**d-tilting object.** We say that a tilting object $U$ in $\text{CM}^L_R$ is $d$-tilting if $\text{End}_{\text{L}}(U)$ has global dimension at most $d$.

For example, if $n = d + 2$, then $T^\text{CM}$ given in Theorem 4.17 is $d$-tilting if and only if $\left| \{ i \mid p_i \geq 2 \} \right| \leq d$ by Corollary 4.22(c).

Our main result in this subsection is the following.

**Theorem 4.31.** Let $(R, L)$ be a GL complete intersection. If $\text{CM}^L_R$ has a $d$-tilting object $U$, then we have the following.

(a) $(R, L)$ is $d$-CM finite and $\text{CM}^L_R$ has a $d$-cluster tilting subcategory

$$
U := \text{add}\{ U(\ell \omega), R(\bar{x}) \mid \ell \in \mathbb{Z}, \bar{x} \in \mathbb{L} \}.
$$

(b) $(R, L)$ is Fano.

(c) $\Lambda := \text{End}_{\text{R}}^L(U)$ is $\tau_d$-finite (see Definition 2.9).

We start with an easy observation. For $i \geq 0$, the $i$-th syzygy of $X \in \text{mod}^L_R$ is defined as $\Omega^i X := \text{Ker} f_i$, where

\[
\cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \rightarrow 0
\]
is a minimal projective resolution of $X$ in $\text{mod}^L_R$.

**Lemma 4.32.** For any $X, Y \in \text{mod}^L_R$, there exists $\bar{a} \in \mathbb{L}$ such that $\text{Hom}_{\text{R}}^L(X, \Omega^i Y(\bar{x})) = 0$ for any $i \geq 0$ and $\bar{x} \in \mathbb{L}$ satisfying $\bar{x} \leq \bar{a}$.

**Proof.** Assume that the $R$-module $X$ (respectively, $Y$) is generated by homogeneous elements of degrees $\bar{a}_1, \ldots, \bar{a}_\ell$ (respectively, $\bar{b}_1, \ldots, \bar{b}_m$). There exists $\bar{a} \in \mathbb{L}$ such that

$$
\bar{a}_1, \ldots, \bar{a}_\ell \notin \bigcup_{j=1}^m (\bar{b}_j - \bar{a} + \mathbb{L}_+).
$$

Fix $i \geq 0$ and $\bar{x} \leq \bar{a}$. Then the $R$-module $\Omega^i Y(\bar{x})$ is generated by homogeneous elements whose degrees are in $\bigcup_{j=1}^m (\bar{b}_j - \bar{a} + \mathbb{L}_+)$. Hence we have

$$(\Omega^i Y(\bar{x}))_{\bar{y}} = 0 \text{ for any } \bar{y} \notin \bigcup_{j=1}^m (\bar{b}_j - \bar{a} + \mathbb{L}_+),$$

and $(\Omega^i Y(\bar{x}))_{\bar{a}_j} = 0$ for any $1 \leq j \leq \ell$ by our choice (4.4) of $\bar{a}$. Therefore $\text{Hom}_{\text{R}}^L(X, \Omega^i Y(\bar{x})) = 0$ holds. □
Now we are ready to prove Theorem 4.31. Note that $K^b(\text{proj}\, \Lambda) = D^b(\text{mod}\, \Lambda)$ holds since $\Lambda$ has finite global dimension by Lemma 4.30 (a).

(b) Assume that $(R, L)$ is Calabi-Yau. Then the triangulated category $CM^{-1}R$ is fractionally derived Calabi-Yau by Corollary 4.3. Hence $\Lambda$ has global dimension strictly bigger than $d$ by Proposition 2.7, a contradiction.

Now we assume that $(R, L)$ is anti-Fano. Since $\Lambda$ has global dimension at most $d$, we have $\nu_d^\ell(\Lambda) \in D^{\leq 0}(\text{mod}\, \Lambda)$ for $\ell \geq 0$ by Proposition 2.7. Hence $\text{Hom}_{D^b(\text{mod}\, \Lambda)}(\Lambda, \nu_d^\ell(\Lambda)[-i]) = 0$ holds for any $\ell \geq 0$ and $i > 0$. On the other hand, for $\ell \gg 0$, the element $-\ell\omega \in L$ becomes sufficiently small since $(R, L)$ is anti-Fano. Therefore

$$\text{Hom}_{D^b(\text{mod}\, \Lambda)}(\Lambda, \nu_d^\ell(\Lambda)[-i]) = \text{Hom}_{R}(U, \Omega^U(-\ell\omega)) = 0$$

holds for any $i > 0$ by Lemma 4.32. Consequently, we have $\nu_d^\ell(\Lambda) = 0$ for $\ell \gg 0$. This is a contradiction since $\nu_d$ is an autoequivalence.

Therefore $(R, L)$ must be Fano.

(c) We have $\text{gl.dim} \, \Lambda \leq d$ by our assumption. Moreover

$$H^0(\nu_d^\ell(\text{DA})) = \text{Hom}_{D^b(\text{mod}\, \Lambda)}(\Lambda, \nu_d^\ell(\text{DA})) = \text{Hom}_{D^b(\text{mod}\, \Lambda)}(\Lambda, \nu_d^\ell(\Lambda)[d])$$

$$\simeq \text{Hom}_{R}(U, \Omega^U((1 - \ell\omega)[d]) = \text{Ext}^d_{\text{mod-}R}(U, U)((1 - \ell\omega) = 0$$

holds for $\ell > 0$ since $\text{Ext}^d_{\text{mod-}R}(U, U) \in D^b_{\text{mod}} R$ holds by Proposition 4.8. Therefore $\Lambda$ is $\tau_R$-finite.

(a) Since $\Lambda$ is $\tau_R$-finite, we have a $d$-cluster tilting subcategory $U_{\Lambda} := \text{add}\{\nu_d^i(\Lambda) \mid i \in \mathbb{Z}\}$ of $D^b(\text{mod}\, \Lambda)$ by Theorem 2.10. By Proposition 4.30 (b), we have that

$$F(U_{\Lambda}) = \text{add}\{U(i\omega) \mid i \in \mathbb{Z}\}$$

is a $d$-cluster tilting subcategory of $CM^{-1}R$. Therefore $\text{add}\{U(i\omega), \, R(\vec{x}) \mid i \in \mathbb{Z}, \, \vec{x} \in \mathbb{L}\}$ is a $d$-cluster tilting subcategory of $CM^{-1}R$, and $(R, L)$ is $d$-CM finite.

We end this subsection by posing the following conjecture.

**Conjecture 4.33.** The following conditions are equivalent.

(a) $(R, L)$ is Fano.

(b) $(R, L)$ is $d$-CM finite.

(c) $CM^{-1}R$ has a $d$-tilting object.

(d) $A^{CM}_{d}$ is derived equivalent to an algebra of global dimension at most $d$.

We know that (c) is equivalent to (d) by Theorem 4.17 (b). The statement (c)$\Rightarrow$(a)(b) was shown in Theorem 4.31. On the other hand, we give the following partial answer to the statement (a)$\Rightarrow$(b)(c).

**Theorem 4.34.** If $n = d + 2$ and one of the following conditions are satisfied, then $CM^{-1}R$ has a $d$-tilting object $U$ and $(R, L)$ is $d$-CM finite.

- $n \geq 3$ and $(p_1, p_2, p_3) = (2, 2, 3), \, (2, 3, 3), \, (2, 3, 4)$ or $(2, 3, 5)$.
- $n \geq 4$ and $(p_1, p_2, p_3, p_4) = (3, 3, 3)$.

Proof. Since we have a triangle equivalence $CM^{-1}R \simeq D^b(\text{mod}\, A^{CM})$ by Theorem 4.17, it suffices to show that $A^{CM}$ is derived equivalent to an algebra of global dimension at most $d$.

In the case $n \geq 3$ and $(p_1, p_2) = (2, 2)$, the algebra $A^{CM}$ has global dimension at most $d$ by Corollary 4.22 (c).

In the case $n \geq 3$ and $(p_1, p_2, p_3) = (2, 3, 3)$ (respectively, $(2, 3, 4), \, (2, 3, 5)$), the algebra $\bigotimes_{i=1}^n kA_{p_i-1}$ is derived equivalent to a path algebra $kQ$ of type $D_4$ (respectively, $E_6, \, E_8$) by Lemma 4.27. Hence $\bigotimes_{i=1}^n kA_{p_i-1}$ is derived equivalent to $kQ \otimes_k (\bigotimes_{i=4}^n kA_{p_i-1})$ which has global dimension at most $d$.

In the case $n \geq 4$ and $(p_1, p_2, p_3, p_4) = (3, 3, 3, 3)$, the algebra $\bigotimes_{i=1}^n kA_{p_i-1}$ is derived equivalent to $kD_4 \otimes_k kD_4$ by Lemma 4.27. Hence $\bigotimes_{i=1}^n kA_{p_i-1}$ is derived equivalent to $kD_4 \otimes_k kD_4 \otimes_k (\bigotimes_{i=5}^n kA_{p_i-1})$ which has global dimension at most $d$. \qed
To study Conjecture 4.33 it is important to know global dimension of algebras which are derived equivalent to $\bigotimes_{i=1}^n kA_{p_i-1}$.

**Proposition 4.35.** Let $n \geq 3$ and $p_1, \ldots, p_n \geq 2$ be integers. Then (a) $\Rightarrow$ (b) holds for the following conditions.

(a) $\bigotimes_{i=1}^n kA_{p_i-1}$ is derived equivalent to an algebra $\Lambda$ with global dimension at most $n - 2$.

(b) $\sum_{i=1}^n \frac{1}{p_i} > 1$.

**Proof.** The algebra $\bigotimes_{i=1}^n kA_{p_i-1}$ is fractionally Calabi-Yau of dimension $\sum_{i=1}^n \frac{p_i-2}{p_i} = n-2 \sum_{i=1}^n \frac{1}{p_i}$ by Example 2.6(a)(b). Therefore by Proposition 2.7(c), we have

$$n - 2 \sum_{i=1}^n \frac{1}{p_i} < \text{gl.dim} \Lambda \leq n - 2.$$ 

or $\Lambda$ is semisimple. In the first case we immediately have $\sum_{i=1}^n \frac{1}{p_i} > 1$. In the second case it follows that $p_1 = \cdots = p_n = 2$ and hence $\sum_{i=1}^n \frac{1}{p_i} = \frac{n}{2} > 1$. □

Now we pose the following, which implies Conjecture 4.33 (a)$\Rightarrow$(b)(c)(d).

**Conjecture 4.36.** The conditions (a) and (b) in Proposition 4.35 are equivalent.

4.4. Matrix factorizations for the hypersurface case $n = d + 2$. Let $(R, \mathfrak{m})$ be a Geigle-Lenzing complete intersection with $n = d + 2$. By Observation 3.3 it is given by

$$R = k[X_1, \ldots, X_n]/(\sum_{i=1}^n \lambda_i X_i^{p_i}).$$

The aim of this subsection is to give an explicit description of a tilting object in $\mathcal{CM}^{L,R}R$ as a Cohen-Macaulay $R$-module, which will be used in Section 7.2. We have $\tilde{s} = \sum_{i=1}^n (p_i - 2)\bar{x}_i$ in this case. We consider an interval

$$\bar{s} + [0, \bar{\delta}] = \{\bar{\ell} = \sum_{i=1}^n \ell_i \bar{x}_i \mid \forall i \ 1 \leq \ell_i \leq p_i - 1\} \text{ for } \bar{s} := \sum_{i=1}^n \bar{x}_i.$$ 

We describe the following variant $U^{\mathcal{CM}}$ of the tilting object $T^{\mathcal{CM}}$ given in Theorem 4.17.

**Proposition 4.37.** Let

$$U^{\mathcal{CM}} := \bigoplus_{\bar{\ell} \in \bar{s} + [0, \bar{\delta}]} \rho(F^{\bar{\ell}}) \in \mathcal{CM}^{L,R}, \text{ where }$$

$$F^{\bar{\ell}} := R/(X_i^{\ell_i} | 1 \leq i \leq n) \in \text{mod}_{0,\bar{\delta}}^R.$$ 

Then $U^{\mathcal{CM}} = T^{\mathcal{CM}([\bar{s}])}|d|$ holds. In particular $U^{\mathcal{CM}}$ is a tilting object in $\mathcal{CM}^{L,R}$.

**Proof.** The equivalence $\text{mod} A^{\mathcal{CM}} \simeq \text{mod}_{0,\bar{\delta}}^R$ in Proposition 4.11 sends $DA^{\mathcal{CM}}$ to $\bigoplus_{\bar{\ell} \in \bar{s} + [0, \bar{\delta}]} F^{\bar{\ell}}$. Therefore the triangle equivalence $\text{D}^b(\text{mod} A^{\mathcal{CM}}) \simeq \mathcal{CM}^{L,R}$ in Theorem 4.17(b) sends $A^{\mathcal{CM}}$ to $T^{\mathcal{CM}}$, and $DA^{\mathcal{CM}}$ to $U^{\mathcal{CM}}$. Since $DA^{\mathcal{CM}} = \nu(A^{\mathcal{CM}})$, we have the assertion by Theorem 4.2(b). □

For each $\bar{\ell} \in \bar{s} + [0, \bar{\delta}]$, we construct two matrices over the polynomial ring $S = k[X_1, \ldots, X_n]$:

Let $[1, n] := \{1, \ldots, n\}$, and let $M^{\bar{\ell}}$ be the matrix whose rows are indexed by odd subsets of $[1, n]$ (i.e., subsets with an odd number of elements), and whose rows are indexed by even subsets of $[1, n]$, with entries

$$M^{\bar{\ell}}_{I,J} = \begin{cases} (-1)^{p(I,J)} X_i^{\ell_i} & \text{if } I = J \cup \{i\}, \\ (-1)^{p(J,J)} \lambda_j X_j^{p_i - \ell_i} & \text{if } J = I \cup \{j\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\lambda_j$ is the matrix whose rows are indexed by even subsets of $[1, n]$.
where \( p(i, I) \) denotes the position of \( i \) in \( I \), that is
\[
p(i, I) = |\{ j \in I \mid j \leq i \}|.
\]

Let \( N^\vec{\ell} \) be the matrix with the exact same description of entries, but where the columns are indexed by even and the rows by odd subsets of \([1, n]\). For \( a \in \mathbb{Z} \), we define \( \mathbb{L} \)-graded free \( R \)-modules by
\[
p^\vec{\ell}, a = \begin{cases} \bigoplus_{I \subseteq [1, n]} R \left( \frac{|I| + a}{2} \right) \vec{c} - \sum_{i \in I} \vec{b}_i & \text{if } a \text{ is even}, \\ \bigoplus_{I \subseteq [1, n]} R \left( \frac{|I| + a}{2} \right) \vec{c} - \sum_{i \in I} \vec{b}_i & \text{if } a \text{ is odd}. \end{cases}
\]

The following is the main result in this subsection.

**Theorem 4.38.** For \( \vec{\ell} \in \vec{s} + [0, \vec{\delta}] \), let \( E^\vec{\ell} := \text{Cok} \left( M^\vec{\ell} : P^\vec{\ell}, -1 \rightarrow P^\vec{\ell}, 0 \right) \).

(a) \((M^\vec{\ell}, N^\vec{\ell})\) gives a matrix factorization of \( \sum_{i=1}^n \lambda_i X_i^{p_i} \), that is,
\[
M^\vec{\ell} N^\vec{\ell} = \sum_{i=1}^n \lambda_i X_i^{p_i} = N^\vec{\ell} M^\vec{\ell}.
\]

(b) \( E^\vec{\ell} \) belongs to \( \text{CM}^L R \) and has a complete resolution
\[
\ldots \rightarrow P^\vec{\ell}, -2 \xrightarrow{N^\vec{\ell}} P^\vec{\ell}, -1 \rightarrow M^\vec{\ell} \rightarrow P^\vec{\ell}, 0 \xrightarrow{N^\vec{\ell}} P^\vec{\ell}, 1 \rightarrow \ldots.
\]

(c) \( E^\vec{\ell} \simeq \rho(F^\vec{\ell}) \) holds in \( \text{CM}^L R \). Therefore \( \bigoplus_{i \in [0, \vec{\delta}]} E^\vec{\ell} \simeq U^{CM} \) is a tilting object in \( \text{CM}^L R \).

(d) \( \text{rank } E^\vec{\ell} = 2^d \) and \( \text{rank } P^\vec{\ell}, a = 2^{d+1} \), where rank is defined in Section 3.2.

Before giving a proof, we give examples.

**Example 4.39.**

(a) Let \( d = 1 \) and \( n = 3 \). Then we have
\[
p^\vec{\ell}, 0 = R \oplus R(\vec{c} - \ell_1 \vec{x}_1 - \ell_2 \vec{x}_2) \oplus R(\vec{c} - \ell_1 \vec{x}_1 - \ell_3 \vec{x}_3) \oplus R(\vec{c} - \ell_2 \vec{x}_2 - \ell_3 \vec{x}_3),
\]
\[
p^\vec{\ell}, -1 = R(-\ell_1 \vec{x}_1) \oplus R(\vec{c} - \ell_1 \vec{x}_1) \oplus R(-\ell_2 \vec{x}_2) \oplus R(\vec{c} - \ell_1 \vec{x}_1 - \ell_2 \vec{x}_2 - \ell_3 \vec{x}_3),
\]
\[
M^\vec{\ell} = \begin{bmatrix} -X_1^{p_1-\ell_1} & \lambda_2 X_2^{p_2-\ell_2} & \lambda_3 X_3^{p_3-\ell_3} & 0 \\ -X_2^{p_2-\ell_2} & \lambda_1 X_1^{p_1-\ell_1} & 0 & \lambda_3 X_3^{p_3-\ell_3} \\ -X_3^{p_3-\ell_3} & 0 & -\lambda_1 X_1^{p_1-\ell_1} & -\lambda_2 X_2^{p_2-\ell_2} \\ 0 & -X_3^{\ell_3} & X_2^{\ell_2} & -X_1^{\ell_1} \end{bmatrix},
\]
\[
N^\vec{\ell} = \begin{bmatrix} -\lambda_1 X_1^{p_1-\ell_1} & -\lambda_2 X_2^{p_2-\ell_2} & -\lambda_3 X_3^{p_3-\ell_3} & 0 \\ X_1^{\ell_1} & -X_1^{\ell_1} & 0 & -\lambda_3 X_3^{p_3-\ell_3} \\ X_2^{\ell_2} & 0 & -X_1^{\ell_1} & \lambda_2 X_2^{p_2-\ell_2} \\ 0 & X_3^{\ell_3} & -X_2^{\ell_2} & -\lambda_1 X_1^{p_1-\ell_1} \end{bmatrix}.
\]

(b) Let \( d = 2 \) and \( n = 4 \). Then we have
\[
p^\vec{\ell}, 0 = R \oplus R(\vec{c} - \ell_1 \vec{x}_1 - \ell_2 \vec{x}_2) \oplus R(\vec{c} - \ell_1 \vec{x}_1 - \ell_3 \vec{x}_3) \oplus R(\vec{c} - \ell_1 \vec{x}_1 - \ell_4 \vec{x}_4) \oplus R(\vec{c} - \ell_2 \vec{x}_2 - \ell_3 \vec{x}_3) \oplus R(\vec{c} - \ell_2 \vec{x}_2 - \ell_4 \vec{x}_4) \oplus R(\vec{c} - \ell_3 \vec{x}_3 - \ell_4 \vec{x}_4) \oplus R(\vec{c} - \sum_{i=1}^4 \ell_i \vec{x}_i),
\]
\[
p^\vec{\ell}, -1 = R(-\ell_1 \vec{x}_1) \oplus R(-\ell_1 \vec{x}_1) \oplus R(-\ell_2 \vec{x}_2) \oplus R(-\ell_3 \vec{x}_3) \oplus R(-\ell_4 \vec{x}_4) \oplus R(\vec{c} - \sum_{i=1,2,3} \ell_i \vec{x}_i) \oplus R(\vec{c} - \sum_{i=1,2,4} \ell_i \vec{x}_i),
\]
Moreover, for $Y_i := X_i^{\ell_i}$ and $Y_i := \lambda_i X_i^{p_i - \ell_i}$, we have

$$
M^\ell = 
\begin{bmatrix}
-Y_1 & Z_2 & Z_3 & Z_4 & 0 & 0 & 0 & 0 \\
-Y_2 & -Z_1 & 0 & 0 & Z_3 & Z_4 & 0 & 0 \\
-Y_3 & 0 & -Z_1 & 0 & -Z_2 & 0 & Z_4 & 0 \\
-Y_4 & 0 & 0 & -Z_1 & 0 & -Z_2 & -Z_3 & 0 \\
0 & Y_3 & Y_2 & 0 & -Y_1 & 0 & 0 & Z_4 \\
0 & -Y_4 & 0 & Y_2 & 0 & -Y_1 & 0 & -Z_3 \\
0 & 0 & -Y_4 & Y_3 & 0 & 0 & -Y_1 & Z_2 \\
0 & 0 & 0 & 0 & -Y_4 & Y_3 & -Y_2 & -Z_1 \\
\end{bmatrix},
$$

$$
N^\ell = 
\begin{bmatrix}
-Z_1 & -Z_2 & -Z_3 & -Z_4 & 0 & 0 & 0 & 0 \\
Y_2 & -Y_1 & 0 & 0 & -Z_3 & -Z_4 & 0 & 0 \\
Y_3 & 0 & -Y_1 & 0 & Z_2 & 0 & -Z_4 & 0 \\
Y_4 & 0 & 0 & -Y_1 & 0 & Z_2 & Z_3 & 0 \\
0 & Y_3 & -Y_2 & 0 & -Z_1 & 0 & 0 & Z_3 \\
0 & Y_4 & 0 & -Y_2 & 0 & -Z_1 & 0 & Z_3 \\
0 & 0 & Y_4 & -Y_3 & 0 & 0 & -Z_1 & -Z_2 \\
0 & 0 & 0 & Y_4 & -Y_3 & Y_2 & -Y_1 & \end{bmatrix}.
$$

Proof of Theorem 4.38 (a) We only have to show $M^\ell N^\ell = \sum_{i=1}^{n} \lambda_i X_i^{p_i}$. Let $I$ and $J$ be even subsets of $[1, n]$. We need to check that $(M^\ell N^\ell)_{I,J}$ is $\sum_{i=1}^{n} \lambda_i X_i^{p_i}$ if $I = J$ and 0 otherwise.

For $I$ and $J$ differing by precisely two elements, let us consider the case $I = J \cup \{i_1, i_2\}$. Then

$$(M^\ell N^\ell)_{I,J} = M^\ell_{I, \cup \{i_1\}} N^\ell_{I, \cup \{i_1\}, J} + M^\ell_{I, \cup \{i_2\}} N^\ell_{I, \cup \{i_2\}, J},$$

$$
= (-1)^{p(i_2, J)} X_{i_2}^{\ell_{i_2}} (-1)^{p(i_1, J \cup \{i_1\})} X_{i_1}^{\ell_{i_1}} + (-1)^{p(i_1, J)} X_{i_1}^{\ell_{i_1}} (-1)^{p(i_2, J \cup \{i_2\})} X_{i_2}^{\ell_{i_2}}
$$

$$
= 0,
$$

holds, where we used the equation $(-1)^{p(i_2, J)} (-1)^{p(i_1, J \cup \{i_1\})} + (-1)^{p(i_1, J) (-1)^{p(i_2, J \cup \{i_2\})} = 0$.

A similar argument applies if $J$ contains two additional elements compared to $I$.

For $I$ and $J$ satisfying $I = (I \cap J) \cup \{i\}$ and $J = (I \cap J) \cup \{j\}$ with $i \neq j$, we obtain

$$(M^\ell N^\ell)_{I,J} = M^\ell_{I, \cup \{i\}} N^\ell_{I, \cup \{i\}, J} + M^\ell_{I, \cup \{j\}} N^\ell_{I, \cup \{j\}, J},$$

$$
= (-1)^{p(i, J)} X_i^{\ell_i} (-1)^{p(j, J \cup \{i\})} \lambda_j X_j^{p_j - \ell_i} + (-1)^{p(j, J \cup \{i\})} \lambda_j X_j^{p_j - \ell_i} (-1)^{p(i, J \cup \{j\})} X_i^{\ell_i}
$$

$$
= 0,
$$

where we used the equation $(-1)^{p(i, J)} (-1)^{p(j, J \cup \{i\})} + (-1)^{p(j, J \cup \{i\})} (-1)^{p(i, J \cup \{j\})} = 0$.

For $I = J$ we obtain

$$(M^\ell N^\ell)_{I,I} = \sum_{i \in I} M^\ell_{I, \cup \{i\}} M^\ell_{I, \cup \{i\}, I} + \sum_{j \notin I} M^\ell_{I, \cup \{j\}} M^\ell_{I, \cup \{j\}, I},$$

$$
= \sum_{i \in I} (-1)^{p(i, I)} X_i^{\ell_i} (-1)^{p(i, I)} \lambda_i X_i^{p_i - \ell_i} + \sum_{j \notin I} (-1)^{p(j, I \cup \{i\})} \lambda_j X_j^{p_j - \ell_i} (-1)^{p(i, J \cup \{j\})} X_i^{\ell_i}
$$

$$
= \sum_{i=1}^{n} \lambda_i X_i^{p_i}.
$$

It is clear that $(M^\ell N^\ell)_{I,J} = 0$ holds for all other cases. Thus the assertion follows.

(b) It follows from general facts on matrix factorizations [11] that $\{M^\ell\}$ is an acyclic complex.

(c) Let $\ell \in \mathcal{S} + [0, \delta]$. We only have to show that a sufficiently left part of $\{M^\ell\}$ coincides with a minimal projective resolution of the $\mathbb{L}$-graded $R$-module $F^\ell$. 

For $a \in \mathbb{Z}$, we denote by $Q^{\ell,-a}$ the direct summand of $P_{\ell,-a}$ corresponding to subsets $I \subset [1, n]$ satisfying $|I| \leq a$, i.e.

$$Q^{\ell,-a} := \bigoplus_{I \subset [1, n], a-|I| \in \mathbb{Z}_{\geq 0}} R \left( \frac{|I|-a}{2} \varepsilon - \sum_{i \in I} \xi_i \right).$$

Then $Q^{\ell,-a} = 0$ for all $a < 0$, and $Q^{\ell,-a} = P_{\ell,-a}$ for all $a \geq n$. Moreover we have a commutative diagram

$$
\begin{array}{cccccc}
... & M^\ell & \to & P_{\ell,-2} & \to & P_{\ell,-1} & \to & M^\ell \\
... & L^\ell \to & Q_{\ell,-2} & \to & L^\ell_{-1} & \to & Q_{\ell,-1} & \to & 0 \\
\end{array}
$$

(4.6)
in $\text{mod}^\ell R$, where vertical maps are natural projections and $L_{\ell,-a} : Q^{\ell,-a} \to Q^{\ell,1-a}$ is the corresponding minors of $M^\ell$ if $a$ is odd (respectively, $N^\ell$ if $a$ is even).

**Lemma 4.40.** The lower sequence in (13) gives a minimal projective resolution of $F^\ell$ in $\text{mod}^\ell R$.

**Proof.** Clearly $\text{Cok} L_{\ell,-1} = F^\ell$ holds. We need to show that the lower sequence is exact except at $Q^{\ell,0}$. We define an $L$-graded free $S$-modules by

$$Q^{\ell,-a} := \bigoplus_{I \subset [1, n], a-|I| \in \mathbb{Z}_{\geq 0}} S \left( \frac{|I|-a}{2} \varepsilon - \sum_{i \in I} \xi_i \right).$$

This gives a lift of $Q^{\ell,-a}$, that is $Q^{\ell,-a} \otimes_S R = \tilde{Q}^{\ell,-a}$. Hence we have a commutative diagram:

$$
\begin{array}{cccccc}
\tilde{Q}^{\ell,-1-a} & \to & \tilde{Q}^{\ell,-a} & \to & \tilde{Q}^{\ell,1-a} \\
\tilde{Q}^{\ell,-1-a} & \to & \tilde{Q}^{\ell,-a} & \to & \tilde{Q}^{\ell,1-a} \\
\end{array}
$$

The matrix $L^{\ell,-a}$ gives a morphism $\tilde{Q}^{\ell,-a} \to \tilde{Q}^{\ell,1-a}$ in $\text{mod}^\ell S$. It follows easily from (a) that

$$(L^{\ell,-a}L^{\ell,1-a})_{I,J} = \delta_{I,J} \sum_{i=1}^n \lambda_i X_i^{p_i}$$

(4.7)
holds for any subsets $I$ and $J$ of $[1, n]$ corresponding to summands of $\tilde{Q}^{\ell,-a}$ and $\tilde{Q}^{\ell,2-a}$ respectively.

Fix any $x \in \text{Ker} \left( L^{\ell,-a} : Q^{\ell,-a} \to Q^{\ell,1-a} \right)$. Take any lift $x \in \tilde{Q}^{\ell,-a}$ of $x$. Then we have

$$(x)L^{\ell,-a} = y \sum_{i=1}^n \lambda_i X_i^{p_i}$$

for some $y \in \tilde{Q}^{\ell,1-a}$. Now regard $y$ as an element in $\tilde{Q}^{\ell,1-a}$, which contains $\tilde{Q}^{\ell,1-a}$ as a direct summand. Then $x' := x - (y)L^{\ell,-1-a} \in \tilde{Q}^{\ell,-a}$ satisfies $(x')L^{\ell,-a} = 0$ by (4.7). Since $(x')L^{\ell,-a}L^{\ell,1-a} = 0$, the equality (4.7) shows that $x'_I = 0$ holds for all $I \subset [1, n]$ satisfying $|I| \leq a-2$. We have a commutative diagram

$$
\begin{array}{cccccc}
\bigoplus_{|I|=a+1} S \left( -\sum_{i \in I} \xi_i \right) & \to & \bigoplus_{|I|=a} S \left( -\sum_{i \in I} \xi_i \right) & \to & \bigoplus_{|I|=a-1} S \left( -\sum_{i \in I} \xi_i \right) \\
\tilde{Q}^{\ell,-1-a} & \to & \tilde{Q}^{\ell,-a} & \to & \tilde{Q}^{\ell,1-a} \\
\end{array}
$$

where the upper sequence is the Koszul complex with respect to the $S$-regular sequence $X_1^{p_1}, \ldots, X_n^{p_n}$, and the vertical maps are the natural inclusions. Since the above observation shows that $x'$ belongs to $\text{Ker} f^{-a}$ and the upper sequence is exact, there exists $z$ such that $x' = (z)f^{-a}$. Regarding
z as an element in $Q^s_{-1} \cdot L^{\tilde{s}^{1-a}}$. Consequently, $x = (y + z)L^{\tilde{s}^{1-a}} \in \text{Im} \left( L^{\tilde{s}^{1-a}} : Q^s_{-1} \cdot Q^{\tilde{s}^{1-a}} \right)$ holds.

It remains to prove (d). We need the following observation, which will be used later again.

**Lemma 4.41.** For $i = 0, 1$, let $\tilde{T}^{0,i}$ be the direct summands of $P^{\tilde{s},i}$ corresponding to subsets $I$ of $[1, n]$ satisfying $n \notin I$, and $\tilde{M}$ and $\tilde{N}$ the corresponding minors of $M^{\tilde{s}}$ of $N^{\tilde{s}}$ respectively.

(a) rank $E^{\tilde{s}} = 2^d$.

(b) Ker($E^{\tilde{s}} \rightarrow P^{\tilde{s},1} \rightarrow \tilde{T}^{0,1}$) = 0 and Cok($\tilde{T}^{0,0} \rightarrow P^{\tilde{s},0} \rightarrow E^{\tilde{s}}$) has rank 0.

**Proof.** The composition $\tilde{T}^{0,0} \rightarrow P^{\tilde{s},0} \rightarrow \tilde{T}^{0,1} \rightarrow \tilde{T}^{1,1}$ is given by $\tilde{N}$, and we have a matrix factorization $\tilde{M} \tilde{N} = \sum_{i=1}^{n} \lambda_i X_i^{p_i} = \tilde{N} \tilde{M}$ of the hypersurface $\sum_{i=1}^{n} \lambda_i X_i^{p_i}$ with one fewer variables. Since it does not have a common factor with our hypersurface $\sum_{i=1}^{n} \lambda_i X_i^{p_i}$, the determinant of $\tilde{N}$ is non-zero in $R$. Thus the morphism $\tilde{N} : (\tilde{T}^{0,0})_0 \rightarrow (\tilde{T}^{1,1})_0 = (\tilde{T}^{0,1})_0$ between $(R_{(0)})_0$-vector spaces of dimension $2^d$ is an isomorphism. Since this factors through $(E^{\tilde{s}}(0))_0$, we have rank $E^{\tilde{s}} \geq 2^d$.

Applying the same argument to the morphism $M^{\tilde{s}} : P^{\tilde{s},1} \rightarrow P^{\tilde{s},0}$, we have rank($\Omega E^{\tilde{s}}$) $\geq 2^d$. Since rank $E^{\tilde{s}} + \text{rank}(\Omega E^{\tilde{s}}) = \text{rank} P^{\tilde{s},0} = 2^d$, we have rank $E^{\tilde{s}} = \text{rank}(\Omega E^{\tilde{s}}) = 2^d$. Thus the assertion (a) follows. Since $(E^{\tilde{s}}(0))_0 \rightarrow (\tilde{T}^{1,1})_0$ and $(\tilde{T}^{0,0})_0 \rightarrow (E^{\tilde{s}}(0))_0$ are isomorphisms, we have the assertion (b).

We have completed the proof of Theorem 4.38.

In the rest of this section, we give a direct proof of the following statement, which was shown in Proposition 4.38 by using heavy machinery in the derived category $D^b(\text{mod}^L R)$.

**Proposition 4.42.** We have $\text{Hom}^i_R(U^{CM}, U^{CM}[i]) = 0$ for any $i \neq 0$.

**Proof.** It suffices to check that $\text{Hom}^i_R(\Omega^i E^{\tilde{s}}, F^{\tilde{m}}) = 0$ for all non-zero $i$ and all $\tilde{x}, \tilde{m} \in \tilde{s} + [0, \tilde{s}]$. By Auslander-Buchweitz approximation theory [ABu], there exists an exact sequence $0 \rightarrow P \rightarrow E^{\tilde{m}} \rightarrow F^{\tilde{m}} \rightarrow 0$ in $\text{mod}^L R$ with an $R$-module $P$ of finite projective dimension. Applying $\text{Hom}^i_R(\Omega^i E^{\tilde{s}}, -)$, we have $\text{Hom}^i_R(\Omega^i E^{\tilde{s}}, F^{\tilde{m}}) \simeq \text{Hom}^i_R(\Omega^i E^{\tilde{s}}, F^{\tilde{m}})$ which is $H^i(\text{Hom}_R^i(C_\bullet, F^{\tilde{m}}))$ for the complete resolution $C_\bullet$ form [15]. Since $\text{Hom}^i_R_R(R(\tilde{x}), F^{\tilde{m}}) = \begin{cases} k & \text{if } \tilde{x} \in \left[ \sum_{i=1}^{n} (m_i - 1) \tilde{r}_i, 0 \right] \\
0 & \text{otherwise.} \end{cases}$ holds, the complex $\text{Hom}^i_R(C_\bullet, F^{\tilde{m}})$ is isomorphic to $\cdots \rightarrow k \rightarrow k[[m_i, \tilde{r}_i]] \rightarrow k[[ (1^{i + m_i} \tilde{r}_i) ] ] \rightarrow k[[ (1^{i + m_i} \tilde{r}_i) ]] \rightarrow \cdots$. 

If \( \{i \mid m_i > \ell_i\} \) is non-empty, then this is a Koszul complex and hence acyclic. Otherwise, this has the only non-zero term at degree 0. Therefore we have the assertion.

**Example 4.43.** Let \( d = 2 \), \( n = 4 \) and \( p_1 = p_2 = p_3 = p_4 = 2 \). Then \( \text{CM}^2 R \) has the Auslander-Reiten quiver:

\[
\begin{array}{c}
\cdots \rightarrow E_1 \rightarrow E_2 \rightarrow E_1(-\overline{w}) \rightarrow E_2(-2\overline{w}) \rightarrow \cdots \\
\vdots \rightarrow \overline{x}_4 \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \overline{x}_4 - \overline{c} \rightarrow \overline{x}_2 + \overline{x}_3 + \overline{x}_4 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \cdots \\
\end{array}
\]

where \( E_1 := \rho(F) \) and \( E_2 := E_1(\overline{x}_1) \).

Moreover \( \text{CM}^2 R \) has two 2-cluster tilting subcategories

\[
\mathcal{U}_i := \text{add}\{E_i(\ell\overline{w}), R(\overline{x}) \mid \ell \in \mathbb{Z}, \overline{x} \in \mathbb{L}\}
\]

for \( i = 1, 2 \). The quiver of \( \mathcal{U}_2 \) is the following:

\[
\begin{array}{c}
\cdots \rightarrow E_2 \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \overline{x}_4 - \overline{c} \rightarrow \overline{x}_2 + \overline{x}_3 + \overline{x}_4 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \cdots \\
\vdots \rightarrow \overline{x}_4 \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_2 + \overline{x}_3 + \overline{x}_4 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \overline{x}_1 + \overline{x}_2 + \overline{x}_3 - \overline{c} \rightarrow \cdots \\
\end{array}
\]

5. **Geigle-Lenzing projective spaces**

Let \( R \) be a Geigle-Lenzing complete intersection over a field \( k \) associated with hyperplanes \( H_1, \ldots, H_n \) in \( \mathbb{P}^d \) and weights \( p_1, \ldots, p_n \). Let \( \text{mod}^L R \) be the category of \( \mathbb{L} \)-graded finitely generated \( R \)-modules, and let \( \text{mod}^L_0 R \) be the full subcategory of \( \text{mod}^L R \) consisting of finite dimensional modules.

**Geigle-Lenzing projective space.** In the setup above, the category of coherent sheaves on Geigle-Lenzing (GL) projective space \( X \) is defined as the quotient category

\[
\text{coh} X = \text{qgr}^L R := \text{mod}^L R / \text{mod}^L_0 R
\]
of \( \text{mod}^R \) by its Serre subcategory \( \text{mod}_R \).

We denote by \( \pi : \text{mod}^R \to \text{coh} \) the natural functor. The object
\[ \mathcal{O} := \pi(R) \]

is called the structure sheaf of \( \mathbb{X} \). We have a triangle equivalence (e.g. [MiY 3.2])
\[ D^b(\text{mod}^R) / D_{\text{mod}^R}(\text{mod}^R) \simeq D^b(\text{coh} \mathbb{X}), \]
and we denote by \( \pi : D^b(\text{mod}^R) \to D^b(\text{coh} \mathbb{X}) \) the natural functor.

5.1. Basic properties. In this section, we give some basic properties of the categories \( \text{coh} \mathbb{X} \) and \( D^b(\text{coh} \mathbb{X}) \). Let us start with recalling the notion of local cohomology [BrH] which relates the categories \( \text{mod}^L R \) and \( \text{coh} \mathbb{X} \).

Local cohomology. For any \( X \in \text{mod}^R, i \geq 0 \) and \( \bar{x} \in L \), let
\[ H^i_m(X)_{\bar{x}} := \lim_{\to \bar{x}} \text{Ext}^{i+1}_R(R_{\bar{x}}m, X(\bar{x})), \]
where \( R_{\bar{x}}m = R/R_{\bar{x}} \) is the quotient of \( R \) by its subobject \( R_{\bar{x}} = \bigoplus_{x \geq \bar{x}} R_x \) in \( \text{mod}^R \). Let \( H^i_m(X) := \bigoplus_{\bar{x} \in L} H^i_m(X)_{\bar{x}} \).

The following result is fundamental, where \( D = \text{Hom}_R(\cdot, k) \) is the \( k \)-dual.

Proposition 5.1. [BrH 3.6.19] (local duality) For any \( X \in \text{mod}^R \) and \( i \geq 0 \), we have an isomorphism in \( \text{mod}^L R \):
\[ \text{Ext}^{d+1-i}_R(X, R(\bar{a})) \simeq \bigoplus_{\bar{x} \in L} D(H^i_m(X)_{\bar{x}}). \]

Evaluating for \( X := R \), we have
\[ H^i_m(R)_{\bar{x}} \simeq \begin{cases} D(R_{\bar{x}}) & \text{if } i = d + 1, \\ 0 & \text{otherwise}. \end{cases} \tag{5.1} \]

Another immediate consequence is the following description of \( \text{CM}_R \) :
\[ \text{CM}_R = \{ X \in \text{mod}^R | \forall j \neq i, H^i_m(X) = 0 \}. \tag{5.2} \]

The following exact sequence is basic.

Proposition 5.2. [BV 4.1.5] For any \( X \in \text{mod}^R \), we have an exact sequence
\[ 0 \to H^i_m(X)_{\bar{x}} \to X_{\bar{x}} \to \text{Hom}_R(\mathcal{O}, X(\bar{x})) \to H^i_m(X)_{\bar{x}} \to 0 \]
and an isomorphism \( \text{Ext}^i_{\mathbb{X}}(\mathcal{O}, X(\bar{x})) \to H^{i+1}_m(X)_{\bar{x}} \) for any \( i \geq 1 \).

We have the following useful description of extension spaces between line bundles.

Proposition 5.3. For any \( \bar{x}, \bar{y} \in L \) and \( i \in \mathbb{Z} \), we have
\[ \text{Ext}^i_{\mathbb{X}}(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y})) = \begin{cases} R_{\bar{y}-\bar{x}} & \text{if } i = 0, \\ D(R_{\bar{y}-\bar{x}}) & \text{if } i = d, \\ 0 & \text{otherwise}. \end{cases} \]

Proof. For \( i > 0 \), we have \( \text{Ext}^i_{\mathbb{X}}(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y})) = H^{i+1}_m(R)_{\bar{y}-\bar{x}} \) by Proposition 5.2. Thus the assertion follows from [1.1].

For \( i = 0 \), we have \( \text{Hom}_{\mathbb{X}}(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y})) = R_{\bar{y}-\bar{x}} \) by Proposition 5.2 and [1.1]. \( \square \)

Now we give a list of fundamental properties of our category \( \text{coh} \mathbb{X} \).

Theorem 5.4.
(a) \( \text{coh} \mathbb{X} \) is a Noetherian abelian category.
(b) \( \text{coh} \mathbb{X} \) has global dimension \( d \).
(c) \( \text{Ext}^i(X, Y) \) is a finite dimensional \( k \)-vector space for any \( i \geq 0 \) and \( X, Y \in \text{coh} \mathbb{X} \).
(d) \( \text{Hom}_{D^b(\text{coh} \mathbb{X})}(X, Y) \) is a finite dimensional \( k \)-vector space for any \( X, Y \in D^b(\text{coh} \mathbb{X}) \).
(e) We have $D^b({\text{coh}} X) = \text{thick}\{ \mathcal{O}(\bar{x}) \mid \bar{x} \in \mathbb{L}\}$.

(f) (Auslander-Reiten-Serre duality) We have a functorial isomorphism for any $X, Y \in D^b({\text{coh}} X)$:

$$\text{Hom}_{D^b({\text{coh}} X)}(X, Y) \simeq D\text{Hom}_{D^b({\text{coh}} X)}(Y, X)([d]).$$

In other words, $D^b({\text{coh}} X)$ has a Serre functor $(\bar{\omega})[d]$.

Proof. (a) See e.g. [P, 5.8.3].

(c)(d) It is enough to prove (d). By (e), it is enough to show that $\text{Ext}_X^d(\mathcal{O}(\bar{x}), \mathcal{O}(y))$ is finite dimensional for any $\bar{x}, \bar{y} \in \mathbb{L}$ and $i \geq 0$. This was shown in Proposition [5.3].

(f) We will give a complete proof in Section 5.3.

(b) For all $i > d$ and $X, Y \in \text{coh } X$ we have, by (f), that

$$\text{Ext}_X^d(Y, X)(\bar{\omega}) = 0.$$

We have the following basic results.

**Theorem 5.5.**

(a) For any $X \in \text{coh } X$, there exists an epimorphism $Y \to X$ in $\text{coh } X$ with $Y \in \text{add}(\mathcal{O}(-\bar{x}) \mid \bar{x} \in \mathbb{L}_+)$. 

(b) (Serre vanishing) For any $X \in \text{coh } X$, there exists $\bar{a} \in \mathbb{L}$ such that $\text{Ext}_X^d(\mathcal{O}, X(\bar{x})) = 0$ holds for any $i > 0$ and any $\bar{x} \in \mathbb{L}$ satisfying $\bar{x} \geq \bar{a}$.

Proof. (a) For $X \in \text{mod } d^+$, let $X_{\mathbb{L}_+} := \bigoplus_{\bar{x} \in \mathbb{L}_+} X_{\bar{x}}$ be a subobject of $X$ in $\text{mod } d^+$. Since $X_{\mathbb{L}_+}$ is finitely generated, there exists a surjection $f : P \to X_{\mathbb{L}_+}$ in $\text{mod } d^+$ with $P \in \text{proj}^{d+}_1$. Then $\pi(f) : \pi(P) \to \pi(X)$ is an epimorphism in $\text{coh } X$ since $\text{Cok } f = X/X_{\mathbb{L}_+}$ belongs to $\text{mod } i_0$. 

(b) If $X = \mathcal{O}(\bar{y})$ for some $\bar{y} \in \mathbb{L}$, then the assertion follows from Proposition [5.3].

For general $X \in \text{coh } X$, applying (a) repeatedly, we have an exact sequence

$$\cdots \to \mathcal{O} \to L_1 \to \mathcal{O} \to L_0 \to X \to 0$$

in $\text{coh } X$, where each $L_i$ is a finite direct sum of the degree shifts of $\mathcal{O}$. We take $\bar{a} \in \mathbb{L}$ such that $\text{Ext}_X^d(\mathcal{O}, \bigoplus_{i=0}^{d-1} L_i(\bar{x})) = 0$ for any $i > 0$ and any $\bar{x} \in \mathbb{L}$ satisfying $\bar{x} \geq \bar{a}$. Applying $\text{Hom}_X(\mathcal{O}, -)$ with $\bar{x} \geq \bar{a}$ to (5.3), we have

$$\text{Ext}_X^d(\mathcal{O}, X(\bar{x})) \simeq \text{Ext}_X^{d+1}(\mathcal{O}, \text{Im } f_1(\bar{x})) \simeq \cdots \simeq \text{Ext}_X^{d+i}(\mathcal{O}, \text{Im } f_d(\bar{x})) = 0$$

since $\text{coh } X$ has global dimension $d$ by Theorem [5.3](b). □

We note the following easy property, which will be used later. Recall that $C$ is the polynomial algebra $k[T_0, \ldots, T_d]$ in $d + 1$ variables.

**Lemma 5.6.** For $X \in \text{mod } d^+$ and $\ell \geq 0$, let $f_\ell = (t)_\ell : \bigoplus_i X \to X(\ell \bar{c})$ be the morphism in $\text{mod } d^+$, where $t$ runs over all monomials on $T_0, \ldots, T_d$ of degree $\ell$.

(a) The cokernel of $f_\ell : \bigoplus_i X \to X(\ell \bar{c})$ belongs to $\text{mod } d^+$.

(b) $\pi(f_\ell) : \bigoplus_i \pi(X) \to \pi(X)(\ell \bar{c})$ is an epimorphism in $\text{coh } X$.

Proof. (a) Since the cokernel is annihilated by all monomials on $T_0, \ldots, T_d$ in degree $\ell$, it is a finitely generated module over the finite dimensional $k$-algebra $C/(T_0, \ldots, T_d)^\ell$. Thus the assertion follows.

(b) Immediate from (a). □

The full subcategory

$$\text{mod } d^+)^{i+1} := \{ Y \in \text{mod } d^+ \mid \text{Ext}_R^i(Y, X) = 0 \text{ for any } Y \in \text{mod } d^+ \text{ and } i = 0, 1 \}$$

of $\text{mod } d^+$ is called the perpendicular category [GL2] of $\text{mod } d^+$. The following basic observation will be used later.

**Lemma 5.7.** [GL2] 2.1 The functor $(\text{mod } d^+)^{i+1} \to \text{coh } X$ is fully faithful.
In the rest of this subsection, we study the following trichotomy of GL projective spaces.

**Trichotomy.** We say that $X$ is *Fano* (respectively, *Calabi-Yau, anti-Fano*) if so is $(R,L)$, that is, $\delta(\bar{\omega}) < 0$ (respectively, $\delta(\bar{\omega}) = 0$, $\delta(\bar{\omega}) > 0$) holds, where $\delta(\bar{\omega})$ was given in (3.2).

We will characterize these three types using the following categorical ampleness due to Artin-Zhang [AZ].

**Definition 5.8.** Let $\mathcal{A}$ be an abelian category. We say that an automorphism $\alpha$ of $\mathcal{A}$ *ample* if there exists an object $V \in \mathcal{A}$ satisfying the following conditions.

- For any $X \in \mathcal{A}$, there exists an epimorphism $Y \to X$ in $\mathcal{A}$ with $Y \in \text{add}\{\alpha^{-\ell}(V) \mid \ell \geq 0\}$.
- For any epimorphism $f : X \to Y$ in $\mathcal{A}$, there exists an integer $\ell_0$ such that for every $\ell \geq \ell_0$ the map $f : \text{Hom}_\mathcal{A}(\alpha^{-\ell}(V), X) \to \text{Hom}_\mathcal{A}(\alpha^{-\ell}(V), Y)$ is surjective.

We have the following interpretation of our trichotomy in terms of ampleness.

**Theorem 5.9.** Let $\text{coh} X$ be a GL projective space.

(a) $X$ is Fano if and only if the automorphism $(-\bar{\omega})$ of $\text{coh} X$ is ample.

(b) $X$ is anti-Fano if and only if the automorphism $(\bar{\omega})$ of $\text{coh} X$ is ample.

(c) $X$ is Calabi-Yau if and only if $D^b(\text{coh} X)$ is a fractionally Calabi-Yau triangulated category.

**Proof.** Let $p := \text{l.c.m.}(p_1, \ldots, p_n)$.

(c) Since $D^b(\text{coh} X)$ has a Serre functor $[\bar{\omega}]$ by Theorem 5.5(f), it is fractionally Calabi-Yau if and only if $\bar{\omega}$ is a torsion element in $L$. This means that $X$ is Calabi-Yau.

(a) Assume that $X$ is Fano. Then there exists a finite subset $S$ of $L$ such that $S + \mathbb{Z}_{\leq 0}\bar{\omega} \supset \mathbb{L}_+$. We show that $V := \bigoplus_{\bar{x} \in S} \mathcal{O}(\bar{x})$ satisfies the two conditions in Definition 5.8. Since

$$\text{add}\{V(-\ell\bar{\omega}) \mid \ell \geq 0\} \supset \{\mathcal{O}(\bar{x}) \mid \bar{x} \in \mathbb{L}_+\},$$

the first condition is satisfied by Theorem 5.5(a). For an epimorphism $f : X \to Y$ in $\text{coh} X$, we have $\text{Ext}_X^2(V,(\ker f)(-\ell\bar{\omega})) = 0$ for $\ell \gg 0$ by Theorem 5.5(b). Thus the second condition follows.

On the other hand, assume that $(-\bar{\omega})$ is ample, but $X$ is not Fano. Let $V \in \text{coh} X$ be an object satisfying the conditions in Definition 5.8. By Theorem 5.5(a), there exists a finite subset $S$ of $L$ such that there exists an epimorphism $L \to V$ in $\text{coh} X$ with $L \in \text{add}\{\mathcal{O}(\bar{x}) \mid \bar{x} \in S\}$. Then any object in $\text{coh} X$ is a quotient of an object in $C := \text{add}\{\mathcal{O}(\bar{x} + \ell\bar{\omega}) \mid \bar{x} \in S, \ell \geq 0\}$. On the other hand, since $X$ is not Fano, there exists an element $\bar{a} \in L$ which is smaller than all elements in $-S + \mathbb{Z}_{\geq 0}\bar{\omega}$. Then any morphism from an object in $C$ to $\mathcal{O}(\bar{a})$ is zero, a contradiction. Therefore $X$ has to be Fano.

(b) The proof is parallel to that of (a). \qed

**Remark 5.10.** In [11], Lerner and the second author introduce an order $A$ on the projective space $\mathbb{P}^d$ called a *Geigle-Lenzing order* associated with hyperplanes $H_1, \ldots, H_n$ and weights $p_1, \ldots, p_n$. They prove that there exists an equivalence

$$\text{coh} X \simeq \text{mod} A.$$  

This gives another approach to GL projective spaces, which will not be used in this paper.

**Remark 5.11.** Although our GL projective space $\text{coh} X$ is defined purely categorically, it also has a geometric interpretation, that is, we have an equivalence

$$\text{coh} X \simeq \text{coh}^G Y$$

with the category of $G$-equivariant coherent sheaves on $Y := (\text{Spec } R) \setminus \{R_+\}$ for $G := \text{Spec } k[\mathbb{L}_+]$. In particular, in stack theoretic language, it also can be interpreted as coherent sheaves on the quotient stack $[Y/G]$:

$$\text{coh} X \simeq \text{coh}[Y/G].$$
5.2. Vector bundles. Recall that the canonical module $\omega_R$ of $R$ is defined as $\omega_R := R(\varpi)$. Since $R$ is Gorenstein, we have a duality

$$(-)^\vee := R\text{Hom}_R(-, \omega_R) : \mathcal{D}^b(\text{mod}^R_0) \to \mathcal{D}^b(\text{mod}^R_0)$$

which induces dualities $(-)^\vee : \mathcal{D}^b(\text{mod}^R_0) \to \mathcal{D}^b(\text{mod}^R_0)$ and

$$(-)^\vee : \mathcal{D}^b(\text{coh} \mathcal{X}) \to \mathcal{D}^b(\text{coh} \mathcal{X}).$$

Cohen-Macaulay sheaves For each $i$ with $0 \leq i \leq d$, we define the category of Cohen-Macaulay sheaves of dimension $i$ by

$$\text{CM}_i \mathcal{X} := \text{coh} \mathcal{X} \cap (\text{coh} \mathcal{X}[i - d])^\vee.$$

Cohen-Macaulay sheaves of dimension $d$ are called vector bundles:

$$\text{vect} \mathcal{X} := \text{CM}_d \mathcal{X}.$$

Clearly $O(\vec{x}) \in \text{vect} \mathcal{X}$ for any $\vec{x} \in \mathbb{L}$. Let

$$\text{line} \mathcal{X} := \text{add}\{O(\vec{x}) \mid \vec{x} \in \mathbb{L}\} \subset \text{vect} \mathcal{X}.$$

Immediately we have a duality

$$(−)^\vee[d − i] : \text{CM}_i \mathcal{X} \to \text{CM}_i \mathcal{X}$$

for any $0 \leq i \leq d$. The following observation characterizes objects in the category $\text{CM}_i \mathcal{X}$ in terms of $\mathbb{L}$-graded $R$-modules. Let $\pi : \text{mod}^R_0 R \to \text{coh} \mathcal{X}$ be the natural functor.

**Proposition 5.12.** Let $0 \leq i \leq d$.

(a) We have

$$\pi^{-1}(\text{CM}_i \mathcal{X}) = \{X \in \text{mod}^R_0 R \mid \forall j \neq d - i \text{ Ext}^j_{R}(X, R) \in \text{mod}^R_0 R\}.$$ 

(b) $\pi : \text{mod}^R_0 R \to \text{coh} \mathcal{X}$ restricts to a functor $\text{CM}^R_{i+1} R \to \text{CM}_i \mathcal{X}$.

(c) We have

$$\pi^{-1}(\text{vect} \mathcal{X}) = \{X \in \text{mod}^R_0 R \mid X \text{ is locally free on the punctured spectrum (Definition 4.7)}\}.$$ 

**Proof.** (a) Let $X \in \text{mod}^R_0 R$. Then $\pi(X^\vee)$ belongs to $(\text{coh} \mathcal{X})[i - d]$ if and only if $\pi(X^\vee)) = 0$ for all $j \neq d - i$ if and only if $H^j(X^\vee) = \text{Ext}^j_{R}(X, R)$ belongs to $\text{mod}^R_0 R$ for any $j \neq d - i$. Thus the first assertion follows.

(b) Since $\text{CM}^R_{i+1} R = \{X \in \text{mod}^R_0 R \mid \forall j \neq d - i \text{ Ext}^j_{R}(X, R) = 0\}$, the assertion follows immediately from (a).

(c) The assertion is immediate from (a) and Definition-Proposition 4.7. 

**Vector bundle finiteness.** We say that a GL projective space $\mathcal{X}$ is vector bundle finite (= VB finite) if there are only finitely many isomorphism classes of indecomposable objects in $\text{vect} \mathcal{X}$ up to degree shift.

The following result gives a classification of VB finite GL projective spaces.

**Theorem 5.13.** A GL projective space $\mathcal{X}$ is VB finite if and only if $d = 1$ and $\mathcal{X}$ is Fano (or equivalently, domestic).

**Proof.** For the case $d = 1$, it is classical that $\mathcal{X}$ is VB finite if and only if $\mathcal{X}$ is domestic [GL1].

We show that, if $d \geq 2$, then $\mathcal{X}$ is never VB finite. For any $X \in \text{mod}^R_0 R$, we consider an exact sequence

$$0 \to \Omega^2(X) \xrightarrow{g} P_1 \xrightarrow{f} P_0 \to X \to 0$$

of $\mathbb{L}$-graded $R$-modules with $P_0, P_1 \in \text{proj} \mathcal{X}$. Since $R$ is a Gorenstein ring of dimension $d + 1 \geq 3$, we have that $\text{Ext}^i_R(X, R) = 0$ for any $i \leq 2$. Therefore $g$ above is a left $(\text{proj}^R_1 R)$-approximation of $\Omega^2(X)$, and $f$ above gives a left $(\text{proj}^R_1 R)$-approximation of $\Omega(X)$. Hence the correspondence $X \mapsto \Omega^2(X)$ preserves indecomposability and respects isomorphism classes.
On the other hand, \( \Omega^2(X) \) is locally free on the punctured spectrum, and hence \( \pi(\Omega^2(X)) \) belongs to \( \text{vect} \, X \) by Proposition 5.12(c). Moreover \( \Omega^2(X) \) belongs to \( (\text{mod}^\perp R)^{+0,1} \). Therefore by Lemma 5.7 the functor
\[
\text{mod}^\perp R \to \text{vect} \, X, \quad X \mapsto \pi(\Omega^2(X))
\]
preserves indecomposables and respects isomorphism classes. Since there are infinitely many indecomposable objects in \( \text{mod}^\perp R \) up to degree shift, we have the assertion. \( \square \)

We have the following easy property.

**Lemma 5.14.** Any object in \( \text{vect} \, X \) is isomorphic to \( \pi(X) \) for some \( X \in \text{mod}^\perp R \) such that there exists an exact sequence \( 0 \to X \to P^0 \to P^1 \) in \( \text{mod}^\perp R \) with \( P^0, P^1 \in \text{proj}^\perp R \). In particular, \( \text{vect} \, X \subset \pi((\text{mod}^\perp R)^{+0,1}) \).

**Proof.** Let \( V \in \text{vect} \, X \). By (5.4), there exists \( Y \in \pi^{-1}(\text{vect} \, X) \) such that \( V \simeq \pi(Y) \). Taking a projective resolution \( P_1 \to P_0 \to Y \to 0 \) in \( \text{mod}^\perp R \) and applying \( \text{Hom}_{\text{R}}(\cdot, \omega_{R}) \), we have the desired exact sequence.

It is clear from the exact sequence that \( X \) belongs to \( (\text{mod}^\perp R)^{+0,1} \).

As a special case of Proposition 5.12(b), we have a functor
\[
\text{CM}^\perp R \to \text{vect} \, X.
\]

The statement (a) below shows that this is fully faithful. Therefore \( \text{CM}^\perp R \) has two exact structures, one is the restriction of the exact structure on \( \text{mod}^\perp R \) and the other is the restriction of that on \( \text{coh} \, X \). These are certainly different (e.g. \( R \) is projective in \( \text{mod}^\perp R \), but not in \( \text{coh} \, X \)), but the statement (c) below shows that they are still very close.

**Proposition 5.15.**

(a) \( \pi : \text{mod}^\perp R \to \text{coh} \, X \) restricts to a fully faithful functor \( \text{CM}^\perp R \to \text{vect} \, X \) and an equivalence \( \text{proj}^\perp R \to \text{line} \, X \).

(b) We have
\[
\pi(\text{CM}^\perp R) = \{ X \in \text{vect} \, X \mid \forall i \in \{1, 2, \ldots, d-1\} \, \text{Ext}^i_{\text{X}}(\text{line} \, X, X) = 0 \} = \{ X \in \text{vect} \, X \mid \forall i \in \{1, 2, \ldots, d-1\} \, \text{Ext}^i_{\text{X}}(X, \text{line} \, X) = 0 \}.
\]

(c) For any \( i \) with \( 0 \leq i \leq d-1 \), we have a functorial isomorphism
\[
\text{Ext}^i_{\text{mod}^\perp R}(X, Y) \cong \text{Ext}^i_{\text{X}}(X, Y)
\]
for any \( X \in \text{mod}^\perp R \) and \( Y \in \text{CM}^\perp R \).

Note that, by (b) above, the equality \( \pi(\text{CM}^\perp R) = \text{vect} \, X \) holds for \( d = 1 \), which is classical [GL1, 5.1, GL2, 8.3]. On the other hand, for \( d \geq 2 \), the category \( \text{vect} \, X \) is much bigger than \( \pi(\text{CM}^\perp R) \). The objects in \( \pi(\text{CM}^\perp R) \) are often called \emph{arithmetically Cohen-Macaulay bundles} (e.g. [CH, CMP]).

**Proof.** (a) Since \( \text{CM}^\perp R \subset (\text{mod}^\perp R)^{+0,1} \), the former assertion follows from Lemma 5.7. The latter assertion is an immediate consequence.

(b) By Lemma 5.14, any object in \( \text{vect} \, X \) can be written as \( \pi(X) \) with \( X \in (\text{mod}^\perp R)^{+0,1} \).

Since \( H^i_{\text{m}}(X) = 0 \) holds for \( i = 0, 1 \), it follows from (5.2) that \( X \) belongs to \( \text{CM}^\perp R \) if and only if \( H^i_{\text{m}}(X) = 0 \) for any \( i \) with \( 2 \leq i \leq d \). By Proposition 5.2 this is equivalent to \( \text{Ext}^i_{\text{X}}(\text{line} \, X, X) = 0 \) for any \( i \) with \( 1 \leq i \leq d-1 \). Thus the first equality follows. The second one is a consequence of Auslander-Reiten-Serre duality.

(c) Let \( \cdots \to P_1 \to P_0 \to X \to 0 \) be a projective resolution of \( X \) in \( \text{mod}^\perp R \). Applying \( \text{Hom}^i_{\text{R}}(\cdot, Y) \), we have a complex
\[
0 \to \text{Hom}^i_{\text{R}}(P_0, Y) \to \text{Hom}^i_{\text{R}}(P_1, Y) \to \text{Hom}^i_{\text{R}}(P_2, Y) \to \cdots
\]
whose homology at \( \text{Hom}^i_{\text{R}}(P_i, Y) \) is \( \text{Ext}^i_{\text{mod}^\perp R}(X, Y) \).
On the other hand, applying \( \text{Hom}_X(-, Y) \) to an exact sequence \( \cdots \to P_1 \to P_0 \to X \to 0 \) in \( \text{coh}_X \), we have a complex
\[
0 \to \text{Hom}_X(P_0, Y) \to \text{Hom}_X(P_1, Y) \to \text{Hom}_X(P_2, Y) \to \cdots.
\]
(5.6)
Since we have \( \text{Ext}^i_X(P_j, Y) = 0 \) for all \( i \) and \( j \) with \( 1 \leq j \leq d-1 \) by (b), it is easily checked that the homology of \( (5.6) \) at \( \text{Hom}_X(P_1, Y) \) is \( \text{Ext}^1_X(X, Y) \) for any \( i \) with \( 1 \leq i \leq d-1 \).

Since the complexes \( (5.5) \) and \( (5.6) \) are isomorphic by (a), we have \( \text{Ext}^i_{\text{mod}^+ R}(X, Y) \simeq \text{Ext}^i_X(X, Y) \) for all \( i \) with \( 0 \leq i \leq d-1 \). \( \square \)

We have Serre vanishing for vector bundles, which is a generalization of Theorem 5.5.

**Theorem 5.16.** Let \( V \in \text{vect} X \) be non-zero.

(a) For any \( X \in \text{co} \text{h}_X \), there exists an epimorphism \( Y \to X \) in \( \text{co} \text{h}_X \) with \( Y \in \text{add} \{V(-\bar{x}) \mid \bar{x} \in \mathbb{L}_+\} \).

(b) (Serre vanishing) For any \( X \in \text{co} \text{h}_X \), there exists \( \bar{a} \in L \) such that \( \text{Ext}^i_X(X(\bar{x})) = 0 \) holds for any \( i > 0 \) and any \( \bar{x} \in \mathbb{L} \) satisfying \( \bar{x} \geq \bar{a} \).

**Proof.** Take a lift \( V \in \text{mod}^+ R \) of \( V \in \text{vect} X \). Then \( V \) is locally free on the punctured spectrum by Proposition 5.12(c).

(a) For \( X \in \text{mod}^+ R \), let \( \text{Hom}_R(V, X)_{\mathbb{L}_+} := \bigoplus_{\bar{x} \in \mathbb{L}_+} \text{Hom}^i_R(V, X(\bar{x})) \). The natural morphism
\[
f : V \otimes_R \text{Hom}_R(V, X)_{\mathbb{L}_+} \to X
\]
has a cokernel in \( \text{mod}^+ R \) since for any \( p \in (\text{Spec}^+ R) \setminus \{R_+\} \), we have \( \left( \text{Hom}_R(V, X)_{\mathbb{L}_+} \right)_{(p)} = \text{Hom}_{R(p)}(V_{(p)}, X_{(p)}) \) and
\[
f(p) : V_{(p)} \otimes_R \text{Hom}_{R(p)}(V_{(p)}, X_{(p)}) \to X_{(p)}
\]
is an epimorphism since \( V_{(p)} \) is a non-zero free \( R_{(p)} \)-module.

Let \( g_i : V(-\bar{x}_i) \to X \) with \( 1 \leq i \leq m \) and \( \bar{x}_i \in \mathbb{L}_+ \) be homogeneous generators of the \( R \)-module \( \text{Hom}_R(V, X)_{\mathbb{L}_+} \). Then the morphism
\[
g := (g_1, \ldots, g_m)^t : V(-\bar{x}_1) \oplus \cdots \oplus V(-\bar{x}_m) \to X
\]
in \( \text{mod}^+ R \) has a cokernel in \( \text{mod}^+ R \). Therefore \( \pi(g) \) is an epimorphism in \( \text{co} \text{h}_X \).

(b) By the argument of Theorem 5.5, we only have to consider the case \( X = O \).

First we consider the case \( 0 < i < d \). By Proposition 5.15(c), we have an isomorphism
\[
\text{Ext}^i_X(O, \bar{x}(\bar{x})) \simeq \text{Ext}^i_{\text{mod}^+ R}(V, R(\bar{x})).
\]
By Proposition 4.7, we have \( \text{Ext}^i_q(V, R) \in \text{mod}^0 R \). Thus \( \text{Ext}^i_{\text{mod}^+ R}(V, R(\bar{x})) = 0 \) holds for all but finitely many \( \bar{x} \in \mathbb{L} \), and the assertion follows.

Now we consider the case \( i = d \). By Auslander-Reiten-Serre duality and Proposition 5.15(c), we have isomorphisms
\[
\text{Ext}^d_X(V, O(\bar{x})) \simeq D \text{Hom}_X(O, V(\bar{x} - \bar{x})) \simeq D \text{Hom}_{R(p)}(R, V(\bar{x} - \bar{x})) = D(V_{\bar{a} - \bar{x}}).
\]
This is zero for sufficiently large \( \bar{x} \). \( \square \)

Recall from Section 2.22 that a full subcategory \( C \) of \( \text{vect} X \) is called \( d \)-cluster tilting if \( C \) is a generating and cogenerating functorially finite subcategory of \( \text{vect} X \) such that
\[
C = \{ X \in \text{vect} X \mid \forall i \in \{1, 2, \ldots, d-1\} \text{ Ext}^1_X(C, X) = 0 \} \text{ and } C = \{ X \in \text{vect} X \mid \forall i \in \{1, 2, \ldots, d-1\} \text{ Ext}^i_X(X, C) = 0 \}.
\]
Note that one of the equalities above implies the other as in the case of \( d \)-cluster tilting subcategories of \( \text{CM}^+ R \) [11 2.2.2]. Now we give some basic properties of \( d \)-cluster tilting subcategories, which will be used later, where we need our assumption that \( C \) generates and cogenerates \( \text{vect} X \).

**Theorem 5.17.** For a \( d \)-cluster tilting subcategory \( C \) of \( \text{vect} X \), the following assertions hold.

(a) We have \( C(\bar{a}) = C \).

 Proposition 5.19. \( \pi(\cdot) \) is a functorially finite subcategory of \( \text{vect} \mathbb{K} \).

(b) For any \( X \in \text{vect} \mathbb{K} \), there exist exact sequences
\[
0 \to C_{d-1} \to \cdots \to C_0 \to X \to 0 \quad \text{and} \quad 0 \to X \to C^0 \to \cdots \to C^{d-1} \to 0
\]
in \( \text{coh} \mathbb{K} \) with \( C_i \in \mathcal{C} \) for any \( 0 \leq i \leq d-1 \).

(c) For any indecomposable object \( X \in \mathcal{C} \), there exists an exact sequence (called a \( d \)-almost split sequence)
\[
0 \to X(\mathcal{J}) \to C_{d-1} \to \cdots \to C_1 \to C_0 \to X \to 0
\]
such that the following sequences are exact:
\[
0 \to \text{Hom}_\mathcal{C}(-, X(\mathcal{J})) \to \text{Hom}_\mathcal{C}(-, C_{d-1}) \to \cdots \to \text{Hom}_\mathcal{C}(-, C_0) \to \text{rad}_\mathcal{C}(-, X) \to 0,
\]
\[
0 \to \text{Hom}_\mathcal{C}(X, -) \to \text{Hom}_\mathcal{C}(C_0, -) \to \cdots \to \text{Hom}_\mathcal{C}(C_{d-1}, -) \to \text{rad}_\mathcal{C}(X(\mathcal{J}), -) \to 0.
\]

Proof. (a) This is immediate from Auslander-Reiten-Serre duality given in Theorem 5.14(f).

(b) is shown in [11, Theorem 3.3.1], and (c) is shown in [11, Theorem 3.4.4].

**d-vector bundle finiteness.** We say that a GL projective space \( \mathbb{K} \) is \( d \)-vector bundle finite (=\( d \)-VB finite) if there exists a \( d \)-cluster tilting subcategory \( \mathcal{C} \) of \( \text{vect} \mathbb{K} \) (see Section 2.2) such that there are only finitely many isomorphism classes of indecomposable objects in \( \mathcal{C} \) up to degree shift.

Now it is easy to prove the following key result in this subsection.

**Theorem 5.18.** The correspondence \( \mathcal{C} \mapsto \pi(\mathcal{C}) \) gives a bijection between the following objects.

- \( d \)-cluster tilting subcategories of \( \text{CM}^L \mathbb{K} \).
- \( d \)-cluster tilting subcategories of \( \text{vect} \mathbb{K} \) containing \( \pi \mathbb{K} \).

In particular, if \( (\mathcal{R}, \mathcal{L}) \) is \( d \)-CM finite, then \( \mathcal{X} \) is \( d \)-VB finite.

In particular, \( \mathcal{X} \) is \( d \)-VB finite in the cases given in Theorem 5.14.

To prove Theorem 5.18, we need the following observation.

**Proposition 5.19.** \( \pi(\text{CM}^L \mathbb{K}) \) is a functorially finite subcategory of \( \text{vect} \mathbb{K} \).

Proof. It follows from Theorem 5.11(c) that \( \text{CM}^L \mathbb{K} \) is a functorially finite subcategory of \( \text{mod}_L \mathbb{K} \), and hence of \( (\text{mod}_0^L \mathbb{K})^{\perp_0} \). By Lemma 5.14 we have that \( \pi(\text{CM}^L \mathbb{K}) \) is a functorially finite subcategory of \( \pi((\text{mod}_0^L \mathbb{K})^{\perp_0}) \). Since \( \text{vect} \mathbb{K} \) is contained in \( \pi((\text{mod}_0^L \mathbb{K})^{\perp_{0,1}}) \) by Lemma 5.14 we have the assertion.

Now we are ready to prove Theorem 5.18.

Proof of Theorem 5.18. For a full subcategory \( \mathcal{C} \) of \( \text{CM}^L \mathbb{K} \), it follows from Proposition 5.19 that \( \mathcal{C} \) is functorially finite in \( \text{CM}^L \mathbb{K} \) if and only if \( \pi(\mathcal{C}) \) is functorially finite in \( \text{vect} \mathbb{K} \).

Let \( \mathcal{C} \) be a \( d \)-cluster tilting subcategory of \( \text{CM}^L \mathbb{K} \). Since \( \mathcal{C} \) contains \( \text{proj}^L \mathbb{K} \), it follows that \( \pi(\mathcal{C}) \) contains \( \pi \mathbb{K} \). Then we have
\[
\pi(\mathcal{C}) = \pi \left( \left\{ X \in \text{CM}^L \mathbb{K} \mid \forall i \in \{1, 2, \ldots, d-1\} \ Ext_{\text{mod}^L \mathbb{K}}^{i+1}(\mathcal{C}, X) = 0 \right\} \right)
\]
\[
= \left\{ Y \in \text{vect} \mathbb{K} \mid \forall i \in \{1, 2, \ldots, d-1\} \ Ext_{\mathcal{X}}^{i}(\pi(\mathcal{C}), Y) = 0 \right\},
\]
where the second equality follows from Proposition 5.15(b)(c). Dually we have \( \pi(\mathcal{C}) = \left\{ Y \in \text{vect} \mathbb{K} \mid \forall i \in \{1, 2, \ldots, d-1\} \ Ext_{\mathcal{X}}^{i}(Y, \pi(\mathcal{C})) = 0 \right\} \). Therefore \( \pi(\mathcal{C}) \) is a \( d \)-cluster tilting subcategory of \( \text{vect} \mathbb{K} \).

Conversely, any \( d \)-cluster tilting subcategory of \( \text{vect} \mathbb{K} \) containing \( \pi \mathbb{K} \) is contained in \( \pi(\text{CM}^L \mathbb{K}) \) by Proposition 5.15(c), and hence can be written as \( \pi(\mathcal{C}) \) for a subcategory \( \mathcal{C} \) of \( \text{CM}^L \mathbb{K} \). By a similar argument as above, one can check that \( \mathcal{C} \) is a \( d \)-cluster tilting subcategory of \( \text{CM}^L \mathbb{K} \).

As an immediate consequence of Theorem 5.18 we have the following result.

**Corollary 5.20.** Assume that \( p_i \geq 2 \) for all \( i \). Then the following conditions are equivalent.

- \( n \leq d + 1 \) (or equivalently, \( \mathcal{R} \) is regular).
• line $X$ is a $d$-cluster tilting subcategory of $\text{vect} X$.

Proof. The first condition is equivalent to $\text{CM}^d R = \text{proj}^d R$ by Proposition 4.3. On the other hand, it is clear from the definition of $d$-cluster tilting subcategories that $\text{CM}^d R = \text{proj}^d R$ holds if and only if $\text{proj}^d R$ is a $d$-cluster tilting subcategory of $\text{CM}^d R$. This is equivalent to the second condition by Theorem 5.18. \qed

In the rest of this section, we give a geometric characterization of Cohen-Macaulay sheaves on $X$ in terms of the projective space $\mathbb{P}^d$. For each $i$ with $0 \leq i \leq d$, let

$$\text{CM}_i \mathbb{P}^d := \{ X \in \text{coh} \mathbb{P}^d \mid \forall \text{ closed point } x \in \mathbb{P}^d, \ X_x \in \text{CM}_i(\mathcal{O}_{\mathbb{P}^d,x}) \}$$

be the category of Cohen-Macaulay sheaves of dimension $i$ on $\mathbb{P}^d$. In particular

$$\text{vect} \mathbb{P}^d := \text{CM}_d \mathbb{P}^d$$

is the category of vector bundles. We identify $\text{coh} \mathbb{P}^d$ with $\text{mod}^p C/ \text{mod}^p C$ for the $(2\mathbb{C})$-Veronese subalgebra $C = k[T_0, \ldots, T_d]$ of $R$. We have an exact functor

$$f_* : \text{mod}^p R \to \text{mod}^p R_{[2\mathbb{C}]} \to \text{mod}^p C,$$

where the first functor is given in Proposition 4.13 and the second one is the restriction with respect to the inclusion $C \to R_{[2\mathbb{C}]}$. Since $f_* (\text{mod}^p C) \subset \text{mod}^p C$, we have an induced exact functor

$$f_* : \text{coh} X = \text{mod}^p R/ \text{mod}^p C \to \text{mod}^p C/ \text{mod}^p C = \text{coh} \mathbb{P}^d.$$

We have the following reasonable description of $\text{CM}_i X$ in terms of $\text{CM}_i \mathbb{P}^d$.

Theorem 5.21. For $0 \leq i \leq d$, we have

$$\text{CM}_i X = \{ X \in \text{coh} X \mid f_* X \in \text{CM}_i \mathbb{P}^d \},$$

$$\text{vect} X = \{ X \in \text{coh} X \mid f_* X \in \text{vect} \mathbb{P}^d \}.$$
In particular, we have an isomorphism $\mathbf{R}\text{Hom}_{\mathbb{P}^d}(-, \omega_{\mathbb{P}^d}) \simeq \mathbf{R}\text{Hom}_{C}(-, \omega_{C})$ of functors $D^b(\text{coh } \mathbb{P}^d) \rightarrow D^b(\text{coh } \mathbb{P}^d)$.

Proof. Consider a diagram

$$\begin{array}{ccc}
\text{mod}^2 C & \xrightarrow{\pi} & \text{mod}^2 C \\
\downarrow f & & \downarrow \pi \\
\text{coh } \mathbb{P}^d & \xrightarrow{G := \mathbf{R}\text{Hom}_{\mathbb{P}^d}(-, \omega_{\mathbb{P}^d})} & \text{coh } \mathbb{P}^d
\end{array}$$

By [Gr 2.5.13], there exists a natural isomorphism $\pi \circ F \sim G \circ \pi$, which induces a natural isomorphism $\mathbf{R}(\pi \circ F) \sim \mathbf{R}(G \circ \pi)$ of functors $D^b(\text{mod}^2 C) \rightarrow D^b(\text{coh } \mathbb{P}^d)$.

Since $\pi : \text{mod}^2 C \rightarrow \text{coh } \mathbb{P}^d$ is an exact functor, by [Har 1.5.4.(b)] we have $\mathbf{R}(\pi \circ F) \sim \pi \circ \mathbf{R}F$. Note that we can compute the derived functor $\mathbf{R}G$ by using locally free resolutions. Since the image of projective resolutions in $D^b(\text{mod}^2 C)$ by $\pi$ give locally free resolutions in $D^b(\text{coh } \mathbb{P}^d)$, by [Har 1.5.4.(b)] we have $\mathbf{R}(G \circ \pi) \sim \mathbf{R}G \circ \pi$. Combining all of this, we have $\mathbf{R}G \circ \pi \simeq \pi \circ \mathbf{R}F$ as desired.

Our exact functor $f_* : \text{coh } X \rightarrow \text{coh } \mathbb{P}^d$ induces a triangle functor $f_* : D^b(\text{coh } X) \rightarrow D^b(\text{coh } \mathbb{P}^d)$ which makes the diagram

$$\begin{array}{ccc}
D^b(\text{coh } X) & \xrightarrow{H^i} & \text{coh } X \\
\downarrow f_* & & \downarrow f_* \\
D^b(\text{coh } \mathbb{P}^d) & \xrightarrow{H^i} & \text{coh } \mathbb{P}^d
\end{array} \ (5.7)$$

commutative for any $i \in \mathbb{Z}$. The following observation is easy.

Lemma 5.24. We have the following commutative diagrams

$$\begin{array}{ccc}
D^b(\text{coh } X) & \xrightarrow{(\ )^\vee = \mathbf{R}\text{Hom}_{\mathbb{P}^d}(-, \omega_R)} & D^b(\text{coh } \mathbb{P}^d) \\
\downarrow f_* & & \downarrow f_* \\
D^b(\text{coh } \mathbb{P}^d) & \xrightarrow{(\ )^\vee = \mathbf{R}\text{Hom}_{C}(-, \omega_C)} & D^b(\text{coh } \mathbb{P}^d)
\end{array}$$

Proof. We have $\omega_R = \mathbf{R}\text{Hom}_{C}(R, \omega_C)$ (e.g. [BrH 3.3.7(b)]). Thus

$$\mathbf{R}\text{Hom}_{\mathbb{P}^d}(-, \omega_R) = \mathbf{R}\text{Hom}_{R}(-, \mathbf{R}\text{Hom}_{C}(R, \omega_C)) = \mathbf{R}\text{Hom}_{C}(-, \omega_C)$$

holds, and we have the assertion.

Now we are ready to prove Theorem 5.21.

Proof of Theorem 5.21 Let $X \in \text{coh } X$. By Proposition 5.22, $f_* X \in \mathcal{C}M, \mathbb{P}^d$ if and only if $f_*(X^\vee) = (f_* X)^\vee \in (\text{coh } \mathbb{P}^d)[i - d]$, where the equality holds by Lemma 5.24. This is equivalent to $X^\vee \in (\text{coh } \mathbb{X})[i - d]$ by the commutative diagram (5.7). This means $X \in \mathcal{C}M, \mathbb{X}$ by definition.

5.3. Proof of Auslander-Reiten-Serre duality. In the rest of this section, we give a complete proof of Theorem 5.21 following idea of proof of [DV Theorem A.4]. We refer to [P] Chapter 4 for background on quotients of abelian categories.

Let $\text{Mod}^\mathbb{L} R$ be the category of $\mathbb{L}$-graded $R$-modules, and $\text{Mod}^\mathbb{L} \mathbb{P}^d R$ be the localizing subcategory of $\mathbb{L}$-graded modules obtained as a colimit of finite dimensional modules. We set

$$Q\text{coh } X := \text{Mod}^\mathbb{L} R/ \text{Mod}^\mathbb{L} \mathbb{P}^d R.$$
Then the quotient functor \( \pi : \text{Mod}^d R \to \text{Qcoh} \mathcal{X} \) has the section functor \( \varpi : \text{Qcoh} \mathcal{X} \to \text{Mod}^d R \), that is, the right adjoint of \( \pi \) such that \( \pi \circ \varpi \cong \text{id}_{\text{Qcoh} \mathcal{X}} \). We set \( Q := \varpi \circ \pi \) to be the localization functor. Since \( \pi \circ \omega \) is identity, it follows \( Q^2 = Q \). The torsion functor \( \Gamma_m \) associate an \( L \)-graded \( R \)-module \( M \) with its largest torsion submodule \( \Gamma_m^* (M) \), which is the kernel of the unit morphism \( u_M : M \to Q(M) \). Recall that \( \Gamma_m(M) \) coincides with the 0-th local cohomology group \( H^0_m(M) \), and that \( i \)-th local cohomology group \( H^i_m(M) \) is the \( i \)-th derived functor \( R^i \Gamma_m(M) \).

The following is well-known.

**Lemma 5.25.** We have an exact triangle
\[
R^i \Gamma_m(M) \to M \to RQ(M) \to 
\]
for \( M \in \text{D}(\text{Mod}^d R) \).

**Proof.** By [AZ, Proposition 7.1 (5)] and [P, Lemma 5.1] every injective object \( I \) is a direct sum \( I_t \oplus I_f \) where \( I_f \) is an injective object such that \( \Gamma_m(I_t) = I_t, Q(I_t) = 0 \) and \( I_t \) is an injective object such that \( \Gamma_m(I_t) = 0, Q(I_t) = I_t \). Therefore the exact sequence \( 0 \to \Gamma_m(I) \to I \to Q(I) \) is isomorphic to the split exact sequence \( 0 \to I_t \to I \to I_f \to 0 \).

Since \( \Gamma_m \) and \( Q \) are left exact, we can compute the derived functors by using K-injective resolution. Since each term of a K-injective complex is injective, we have the assertion. \( \square \)

We denote by \( \text{Ext}^i_R(M,N) \) the graded extension group. Namely
\[
\text{Ext}^i_R(M,N) := \bigoplus_{\overline{x} \in \mathcal{X}} \text{Ext}^{i,\text{Mod}^d R}(M,N(\overline{x})).
\]

**Lemma 5.26.**
(a) We have an isomorphism of \( L \)-graded \( R \)-modules \( \text{Ext}^i_R(DR, R(\overline{z})) \cong R \).

In particular \( \text{Ext}^i_R(\text{DR} R(\overline{z})) \cong k \)
(b) We have an isomorphism \( R^i Q(\text{DR} Q(R)) \cong D R Q(R) \).

**Proof.**
(a) By local duality and Proposition 3.53 for any finite dimensional \( L \)-graded \( R \)-module \( M \), we have \( \text{Ext}^{d+1}_R(M, R(\overline{z})) \cong DM \) and \( \text{Ext}^i_R(M, R(\overline{z})) = 0 \) for \( i \neq d \).

Let \( \phi_m : R_{\leq m \overline{z}} \to R_{\leq (m-1) \overline{z}} \) be the canonical projection. Since \( DR \) is the colimit of the following linear diagram
\[
D(R_0) \xrightarrow{\phi_1} D(R_{\leq 1 \overline{z}}) \xrightarrow{\phi_2} D(R_{\leq 2 \overline{z}}) \xrightarrow{\phi_3} \cdots,
\]

it fits into the exact sequence in \( \text{Mod}^d R \)
\[
0 \to \bigoplus_{m \geq 0} D(R_{\leq m \overline{z}}) \xrightarrow{\Psi} \bigoplus_{m \geq 0} D(R_{\leq m \overline{z}}) \to DR \to 0
\]

where the components of \( \Psi \) is \( (\text{id}, -D(\phi_{m+1})) : D(R_{\leq m \overline{z}}) \to D(R_{\leq m \overline{z}}) \oplus D(R_{\leq (m+1) \overline{z}}) \). Since the contravariant functor \( \text{Ext}^i_R(-, R(\overline{z})) \) sends coproducts to products, considering the Ext long exact sequence of the above exact sequence, we obtain the exact sequence
\[
0 \to \text{Ext}^{d+1}_R(DR, R(\overline{z})) \to \prod_{m \geq 0} R_{\leq m \overline{z}} \xrightarrow{\Phi} \prod_{m \geq 0} R_{\leq m \overline{z}},
\]

such that the components of \( \Phi \) is \( (-\phi_m, \text{id}) : R_{\leq m \overline{z}} \to R_{\leq (m-1) \overline{z}} \oplus R_{\leq m \overline{z}} \). Since \( R \) is the limit of the following linear diagram in \( \text{Mod}^d R \)
\[
\cdots \phi_3 R_{\leq 2 \overline{z}} \xrightarrow{\phi_2} R_{\leq \overline{z}} \xrightarrow{\phi_1} R_0,
\]

we obtain the desired result.

(b) Applying the graded \( k \)-duality \( D \) to the exact triangle \( RH_m(R) \to R \to RQ(R) \to \), we obtain the exact triangle \( DR \to DH_m(R) \to D(Q(R))[1] \to \). By proposition 3.53 we have isomorphisms
DR ≅ RH_m(R)(\omega)[d + 1] and DH_m(R) ≅ R(\omega)[d + 1]. Since Ext^{d+1}_R(DR, R)^{\omega} ≅ k, we obtain the following commutative diagram both rows of which are exact triangles

\[
\begin{array}{c}
DR \\
\rightarrow
\end{array}
\begin{array}{c}
DR_m(R) \\
\rightarrow
\end{array}
\begin{array}{c}
D(RQ(R))[1] \\
\rightarrow
\end{array}
\begin{array}{c}
H_m(R)(\omega)[d + 1] \\
\rightarrow
\end{array}
\begin{array}{c}
R(\omega)[d + 1] \\
\rightarrow
\end{array}
\begin{array}{c}
RQ(R)(\omega)[d + 1] \\
\rightarrow
\end{array}
\tag{5.8}
\]

Therefore we conclude \(D(RQ(R)) \cong RQ(R)(\omega)[d - 1]\). Since the localization functor \(RQ\) is idempotent, we have the assertion. \(\square\)

Now we are ready to prove Theorem 5.4(f).

**Proof of Theorem 5.4(f).** Let \(\mathcal{P} := \text{thick}\{R(\omega) \mid \omega \in \mathcal{L}\}\). By Theorem 4.9, any object in \(D(\text{coh} \mathcal{X})\) is a direct summand of \(\pi(M)\) for some object \(M\) of \(\mathcal{P}\). Thus it is enough to show that there exists a functorial isomorphism \(\text{Hom}_{\text{coh} \mathcal{X}}(\pi N, \pi M) \cong \text{Hom}_{\text{coh} \mathcal{X}}(\pi M, \pi(N(\omega)[d]))\) for any \(M, N \in \mathcal{P}\).

Let \(D\) denote the graded \(k\)-dual. For complexes \(M, N, S\) of \(\mathcal{L}\)-graded \(R\)-modules, we obtain the following diagram

\[
\text{Hom}(M, N \otimes DS) \rightarrow \text{Hom}(M, D \text{Hom}(N, S)) \rightarrow D(M \otimes \text{Hom}(N, S)) \leftarrow D \text{Hom}(N, M \otimes S)
\]

which is natural in \(M, N, S\) by combining the natural morphisms

\[
N \otimes DS \rightarrow D \text{Hom}(N, S), \text{Hom}(M, DT) \rightarrow D(M \otimes T), M \otimes \text{Hom}(N, S) \rightarrow \text{Hom}(N, M \otimes S)
\]

where we put \(T := \text{Hom}(N, S)\). If the complexes \(M, N\) are \(K\)-projective, then the above diagram gives the diagram in the derived category

\[
\text{R Hom}(M, N \otimes^L DS) \rightarrow \text{R Hom}(M, D \text{RHom}(N, S)) \rightarrow D(M \otimes^L \text{R Hom}(N, S)) \leftarrow \text{DR Hom}(N, M \otimes^L S).
\tag{5.9}
\]

Note that if \(M\) and \(N\) belong to \(\mathcal{P}\), then the above morphisms are isomorphisms.

We claim that if an object \(M \in D(\text{Mod}^+ R)\) belongs to \(\mathcal{P}\), then the canonical morphism \(M \otimes^L \text{RQ} R \rightarrow \text{RQ} M\) is an isomorphism. Indeed let \(R \rightarrow I\) be an injective resolution of \(R\). If \(M\) is a bounded complex of finitely generated graded projective modules, then the complex \(M \otimes I\) is a left bounded complex of injective \(R\)-modules which is quasi-isomorphic to \(M\). Hence we have

\[
\text{RQ} M \cong Q M \otimes I \cong M \otimes Q I \cong M \otimes^L \text{RQ} R.
\]

Therefore if we substitute \(S\) with \(\text{RQ} R\) in the diagram (5.9), then we obtain an isomorphism

\[
\text{Hom}(M, N \otimes^L D \text{RQ} R) \cong D \text{Hom}(N, \text{RQ} M)
\]

for \(M, N \in \mathcal{P}\).

Since \(\pi(H_m(R)(\omega)[d+1]) \cong \pi(DR) = 0\), we have \(\pi(N \otimes^L D \text{RQ} R) \cong \pi(N(\omega)[d])\) by the bottom exact triangle in the diagram (5.8).

Combining all, we obtain natural isomorphisms for \(M, N \in \mathcal{P}\).

\[
\text{Hom}_{\text{coh} \mathcal{X}}(\pi M, \pi(N(\omega)[d])) \cong \text{Hom}_{\text{coh} \mathcal{X}}(\pi M, \pi(N \otimes^L D \text{RQ} R))
\]

\[
\cong \text{Hom}_{\text{Mod}^+ R}(M, \text{RQ}(N \otimes^L D \text{RQ} R))
\]

\[
\cong \text{Hom}_{\text{Mod}^+ R}(M, N \otimes D \text{RQ} R)
\]

\[
\cong D \text{Hom}_{\text{Mod}^+ R}(N, \text{RQ} M)
\]

\[
\cong D \text{Hom}_{\text{coh} \mathcal{X}}(\pi N, \pi M).
\]

This finishes the proof. \(\square\)
6. \textit{d}-canonical algebras

Let $\mathbb{X}$ be a Geigle-Lenzing projective space over a field $k$ associated with hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^d$ and weights $p_1, \ldots, p_n$. In this section we show that $\mathbb{X}$ has a tilting bundle, and in particular the category of coherent sheaves is derived equivalent to a certain finite dimensional algebra $A^{\text{ca}}$ which we call a $d$-canonical algebra. Then we study basic properties of $d$-canonical algebras. In particular we show that the global dimension of $A^{\text{ca}}$ is $d$ if $n \leq d + 1$ and $2d$ otherwise. In the former case, $A^{\text{ca}}$ belongs to a special class of algebras called ‘$d$-representation infinite algebras of type $\mathcal{A}$’ studied in [HIO].

6.1. Basic properties. Our $d$-canonical algebras are a special class of $I$-canonical algebras introduced in Section 3.2.

\textit{d}-canonical algebra. The $d$-\textit{canonical algebra} of $\mathbb{X}$ (or $(R, L)$) is defined as

$$A^{\text{ca}} := A^{[0, d\vec{x}] = (R_{\vec{x}} - \vec{y})_{\vec{x}, \vec{y} \in [0, d\vec{x}]}.$$ 

The multiplication of $A^{\text{ca}}$ is given by

$$(r_{\vec{x}} \vec{y})_{\vec{x}, \vec{y} \in [0, d\vec{x}]} \cdot (r'_{\vec{x}} \vec{y})_{\vec{x}, \vec{y} \in [0, d\vec{x}]} = \left( \sum_{\vec{x} \in [0, d\vec{x}]} r_{\vec{x}} \cdot r'_{\vec{x}} \right)_{\vec{x}, \vec{y} \in [0, d\vec{x}]}.$$

Our main result in this subsection is the following.

\textbf{Theorem 6.1.} The object

$$T^{\text{ca}} := \bigoplus_{\vec{x} \in [0, d\vec{x}]} O(\vec{x})$$

is tilting in $D^b(\text{coh } \mathbb{X})$ such that $\text{End}_X(T^{\text{ca}}) \simeq A^{\text{ca}}$.

The following special cases are known.

- Let $d = 1$. Then $T^{\text{ca}}$ is a tilting bundle on a weighted projective line due to Geigle-Lenzing [GL1], and $A^{\text{ca}}$ is the canonical algebra due to Ringel [Rin] (see Example 6.9).
- The case $n = 0$ is due to Bellinson [Be] (see Example 6.10), the case $n \leq d + 1$ is due to Baer [Ba] (see Theorem 6.13), and the case $n = d + 2$ is due to Ishii-Ueda [IU].
- In terms of the Geigle-Lenzing order (see Remark 5.10), Theorem 6.1 is independently given by Lerner and the second author [IL].

We give two different proofs: One is to show directly that $T^{\text{ca}}$ is rigid in Proposition 6.2 and that $T^{\text{ca}}$ generates the derived category of $\text{coh } \mathbb{X}$ in Proposition 6.3. The other proof will be given in the next subsection, which is parallel to the proof of Theorem 4.17.

\textbf{Proposition 6.2.} We have $\text{Ext}^i_{\mathbb{X}}(T^{\text{ca}}, T^{\text{ca}}) = 0$ for all $i > 0$.

\textbf{Proof.} Since $\text{coh } \mathbb{X}$ has global dimension $d$ by Theorem 5.4(b), we only have to consider $i$ with $1 \leq i \leq d$. By Proposition 5.3 we have $\text{Ext}^i_{\mathbb{X}}(T^{\text{ca}}, T^{\text{ca}}) = 0$ for all $i$ with $1 \leq i \leq d - 1$.

Let $\vec{x}, \vec{y} \in [0, d\vec{x}]$. By Proposition 5.3 we have $\text{Ext}^i_{\mathbb{X}}(O(\vec{x}), O(\vec{y})) = D(R_{\vec{x} - \vec{y} + i})$. By Lemma 5.1 we have $\vec{x} + \vec{y} \neq 0$. Therefore $\vec{x} - \vec{y} + i \neq 0$ holds, and so $R_{\vec{x} - \vec{y} + i} = 0$ by Observation 5.1(c). Thus we have $\text{Ext}^i_{\mathbb{X}}(T^{\text{ca}}, T^{\text{ca}}) = 0$. \hfill $\square$

In the rest, we show that $T^{\text{ca}}$ generates the derived category of $\text{coh } \mathbb{X}$.

\textbf{Proposition 6.3.} We have $\text{thick } T^{\text{ca}} = D^b(\text{coh } \mathbb{X})$.

\textbf{Proof.} Let $L' := \{ \vec{x} \in L \mid O(\vec{x}) \in \text{thick } T^{\text{ca}} \}$. Since $D^b(\text{coh } \mathbb{X}) = \text{thick } (\text{line } X)$ holds by Theorem 5.4(c), it is enough to prove that $L' = L$. Clearly $[0, d\vec{x}] \subset L'$ holds.

The key observation is the following.

\textbf{Lemma 6.4.} Let $\vec{x} \in L$. If there exists a subset $I$ of $\{1, \ldots, n\}$ satisfying the two conditions below, then $\vec{x} \in L'$.

- $|I| = d + 1$. 
-
• For any non-empty subset $I'$ of $I$, we have $\bar{x} - \sum_{i \in I'} \bar{x}_i \in \mathbb{L}'$.

Proof. By Lemma 3.6(c), $(X_i)_{i \in I}$ is an $R$-regular sequence. Hence the corresponding Koszul complex

$$0 \rightarrow \mathcal{R}(\bar{x} - \sum_{i \in I} \bar{x}_i) \rightarrow \bigoplus_{i,j \in I, i \neq j} \mathcal{R}(\bar{x}_i - \bar{x}_j) \rightarrow \bigoplus_{i \in I} \mathcal{R}(\bar{x}_i) \rightarrow \mathcal{R}(\bar{x}) \rightarrow 0. \quad (6.1)$$

of $R$ is exact except in the rightmost position whose the homology is $(\mathcal{R}/(X_i \mid i \in I))(\bar{x})$. Since this belongs to $\mathcal{mod} R$, the image of $(6.1)$ in $\mathcal{coh} \mathbb{X}$ is exact. Since all the terms except $\mathcal{R}(\bar{x})$ belongs to thick $\mathcal{T}^\mathcal{ca}$ by our assumption, we have $\mathcal{R}(\bar{x}) \in \text{thick } \mathcal{T}^\mathcal{ca}$.

We continue our proof of Proposition 6.3 by showing that $\mathbb{L}_+ \subset \mathbb{L}'$. We use induction with respect to the partial order on $\mathbb{L}_+$. Let $\bar{x} \in \mathbb{L}_+$ and assume that any $y \in [0, \bar{x}]$ with $y \neq \bar{x}$ belongs to $\mathbb{L}'$. If $\bar{x} \in [0, d\bar{c}]$, then $\bar{x} \in \mathbb{L}'$ holds. Otherwise let

$$\bar{x} = \sum_{i=1}^{n} a_i \bar{x}_i + a\bar{c}$$

be the normal form of $\bar{x}$ with $0 \leq a_i < p_i$. Since $\bar{x} \notin d\bar{c}$, we have $a + |\{i \mid a_i > 0\}| \geq d + 1$ by Lemma 3.11. Thus there exists a subset $I$ of $\{1, \ldots, n\}$ with $|I| = d + 1$ such that $\bar{x} - \sum_{i \in I} \bar{x}_i \in \mathbb{L}_+$. Thus for any non-empty subset $I'$ of $I$, we have $\bar{x} - \sum_{i \in I'} \bar{x}_i \in \mathbb{L}'$ by the induction assumption, and we have $\bar{x} \in \mathbb{L}'$ by Lemma 6.4.

Now, the fact that $\mathbb{L}' = \mathbb{L}$ is shown by the induction with respect to the opposite of the partial order on $\mathbb{L}$ by using $\mathbb{L}_+ \subset \mathbb{L}'$ and the dual of Lemma 6.4.

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. $\mathcal{T}^\mathcal{ca}$ is a tilting object in $\mathcal{D}^\mathcal{b}(\mathcal{coh} \mathbb{X})$ by Propositions 6.2 and 6.3.

It remains to show $\text{End}_\mathcal{X}(\mathcal{T}^\mathcal{ca}) \simeq A^\mathcal{ca}$. Since $\text{Hom}_\mathcal{X}(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y})) = \mathcal{R}_{\bar{y}-\bar{x}}$ holds, we have

$$\text{End}_\mathcal{X}(\mathcal{T}^\mathcal{ca}) = (\mathcal{R}_{\bar{y}-\bar{x}})_{\mathcal{X}, \bar{y} \in [0, d\bar{c}]} = (A^\mathcal{ca})^{\mathcal{op}},$$

which is isomorphic to $A^\mathcal{ca}$ by Proposition 4.10(c).

We give a list of basic properties of the $d$-canonical algebras, where $\nu_d := (DA)[-d]^\mathcal{L} \otimes \Lambda - : \mathcal{D}^\mathcal{b}(\mathcal{mod } \Lambda) \rightarrow \mathcal{D}^\mathcal{b}(\mathcal{mod } \Lambda)$ is the $d$-shifted Nakayama functor.

Proposition 6.5. (a) $A^\mathcal{ca}$ has finite global dimension.

(b) $A^\mathcal{ca}$ is isomorphic to the opposite algebra $(A^\mathcal{ca})^{\mathcal{op}}$.

(c) We have a triangle equivalence $\mathcal{T}^\mathcal{ca} \otimes A^\mathcal{ca} : \mathcal{D}^\mathcal{b}(\mathcal{mod } A^\mathcal{ca}) \rightarrow \mathcal{D}^\mathcal{b}(\mathcal{coh } \mathbb{X})$ which makes the following diagram commutative:

$$\begin{array}{ccc}
\mathcal{D}^\mathcal{b}(\mathcal{mod } A^\mathcal{ca}) & \xrightarrow{T^\mathcal{ca} \otimes A^\mathcal{ca}} & \mathcal{D}^\mathcal{b}(\mathcal{coh } \mathbb{X}) \\
\downarrow{\nu_d} & & \downarrow{(2)} \\
\mathcal{D}^\mathcal{b}(\mathcal{mod } A^\mathcal{ca}) & \xrightarrow{T^\mathcal{ca} \otimes A^\mathcal{ca}} & \mathcal{D}^\mathcal{b}(\mathcal{coh } \mathbb{X}).
\end{array}$$

Proof. (a)(b) These are shown in Proposition 4.10.

(c) Since the triangle functor $T^\mathcal{ca} \otimes A^\mathcal{ca} - : \mathcal{D}^\mathcal{b}(\mathcal{mod } A^\mathcal{ca}) \rightarrow \mathcal{D}^\mathcal{b}(\mathcal{coh } \mathbb{X})$ sends a tilting object $A^\mathcal{ca}$ to a tilting object $T^\mathcal{ca}$, it is a triangle equivalence (cf. Proposition 2.2). The diagram commutes by uniqueness of Serre functors.

The above commutative diagram is useful to study further properties of $d$-canonical algebras. In particular, we apply it to study $d$-canonical algebras in the context of (almost) $d$-representation infinite algebras (see Definitions 2.9 and 2.12).

Our $d$-canonical algebras have the following property.
Theorem 6.6. Without loss of generality, we assume that \( p_i \geq 2 \) for all \( i \).

(a) We have
\[
\text{gl.dim } A^{ca} = \begin{cases} 
\ d & n \leq d + 1, \\
2d & n > d + 1.
\end{cases}
\]

(b) \( A^{ca} \) is an almost \( d \)-representation infinite algebra.

(c) \( n \leq d + 1 \) if and only if \( A^{ca} \) is a \( d \)-representation infinite algebra.

\textbf{Proof.} By Proposition 6.5(a)(c), the \( d \)-canonical algebra \( A^{ca} \) has finite global dimension, and
\[
\text{Hom}_{D^b(\text{mod } A^{ca})}(A^{ca}, \nu_{i}^{d}(A^{ca})[i]) = \text{Ext}^{i}_{X}(T^{ca}, T^{ca}(\omega)),
\]
which is zero except \( i = 0 \) or \( i = d \) by Proposition 5.3. Thus \( A^{ca} \) is almost \( d \)-representation infinite, and hence it has global dimension \( d \) or \( 2d \) by Proposition 5.3. Again by Proposition 5.3, we have
\[
\text{Ext}^{2d}_{A^{ca}}(DA^{ca}, A^{ca}) = \text{Hom}_{D^b(\text{mod } A^{ca})}(A^{ca}, \nu_{d}^{-1}(A^{ca})[d]) = \text{Ext}^{d}_{X}(T^{ca}, T^{ca}(-\omega))
\]
\[
= \bigoplus_{\vec{x}, \vec{y} \in [0,d\vec{c}]} DR(\vec{x}, \vec{y} + \vec{2\omega}).
\]

By Observation 2.5, it remains to show that \( \text{Ext}^{2d}_{A^{ca}}(DA^{ca}, A^{ca}) = 0 \) if and only if \( n \leq d + 1 \).

For \( n \leq d + 1 \), we have \( \vec{\omega} < 0 \). For any \( \vec{x}, \vec{y} \in L \), we have \( \vec{x} - \vec{y} + 2\vec{c} \not\geq -\vec{y} + \vec{\omega} \) by Lemma 5.11. In particular \( \vec{x} - \vec{y} + 2\vec{c} \not\geq 0 \) holds since \( -\vec{y} + \vec{\omega} < 0 \). Hence \( R_{\vec{x} - \vec{y} + \vec{2\omega}} = 0 \) holds by Observation 3.1(c), and we have \( \text{Ext}^{d}_{A^{ca}}(DA^{ca}, A^{ca}) = 0 \).

For \( n > d + 1 \), let \( \vec{x} = d\vec{c} \) and \( \vec{y} = 0 \). then
\[
\vec{x} - \vec{y} + 2\vec{c} = (2n - d - 2)\vec{c} - 2 \sum_{i=1}^{n} \vec{c} + \vec{2c} \geq 0,
\]
so we have \( R_{\vec{x} - \vec{y} + \vec{2\omega}} \neq 0 \) holds by Observation 3.1(c), and we have \( \text{Ext}^{2d}_{A^{ca}}(DA^{ca}, A^{ca}) \neq 0 \). \( \square \)

We give an explicit description of the preprojective algebra (see Section 2.2) of the \( d \)-canonical algebra under the assumption that \( X \) is not \( d \)-Calabi-Yau.

\textbf{Proposition 6.7.} Let \( X \) be a GL projective space which is not Calabi-Yau. Let \( R^{[\vec{c}, \vec{e}]} \) be the covering of \( R \) (Definition 3.18), and \( e \) the idempotent of \( R^{[\vec{c}, \vec{e}]} \) corresponding to the image of the map \( [0,d\vec{c}] \to L/\mathcal{Z} \).

(a) The preprojective algebra \( \Pi(A^{ca}) \) of \( A^{ca} \) is Morita equivalent to \( eR^{[\vec{c}, \vec{e}]}e \).

(b) Let \( S \subset [-d\vec{c}, 0] \) be a complete set of representatives of \( \{[-d\vec{c}, 0] + \mathcal{Z}\}/\mathcal{Z} \). For \( e_{S} := \sum_{\vec{x} \in S} e_{-\vec{x}} \in A^{ca}, \) there is an isomorphism \( e_{S}\Pi(A^{ca})e_{S} \simeq eR^{[\vec{c}, \vec{e}]}e \) of \( \mathbb{Z} \)-graded \( k \)-algebras.

\textbf{Proof.} (b) For any \( \ell \in \mathbb{Z} \), the degree \( \ell \) part of \( e_{S}\Pi(A^{ca})e_{S} \) is given by
\[
(e_{S}\Pi(A^{ca})e_{S})_{\ell} = \text{Hom}_{D^b(\text{mod } A^{ca})}(A^{ca}e_{S}, \nu_{d}^{\ell}(A^{ca}e_{S})) \simeq \text{Hom}_{X}(T^{ca}e_{S}, T^{ca}e_{S}(\ell\vec{\omega}))
\]
\[
= (\text{Hom}_{X}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{-y} + \ell\vec{\omega})))_{\vec{x}, \vec{y} \in S} = (R_{\vec{x} - \vec{y} + \ell\vec{\omega}})_{\vec{x}, \vec{y} \in S} = (eR^{[\vec{c}, \vec{e}]}e)_{\ell}.
\]

Thus the assertion follows.

(a) It suffices to show that \( e_{S}\Pi(A^{ca})e_{S} \) is Morita equivalent to \( \Pi(A^{ca}) \).

For any \( \vec{x} \in [-d\vec{c}, 0] \), there exists \( \vec{y} \in S \) such that \( \vec{x} - \vec{y} \in \mathcal{Z} \). We have an isomorphism
\[
\Pi(A^{ca})e_{-\vec{x}} \simeq \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{X}(T^{ca}, \mathcal{O}(\vec{-x} + \ell\vec{\omega})) \simeq \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{X}(T^{ca}, \mathcal{O}(\vec{-y} + \ell\vec{\omega})) \simeq \Pi(A^{ca})e_{-\vec{y}}
\]
of projective \( \Pi(A^{ca}) \)-modules. Thus \( \Pi(A^{ca})e_{S} \) is a progenerator of \( \Pi(A^{ca}) \), and we have the assertion. \( \square \)

As an immediate consequence, we have the following result.

\textbf{Theorem 6.8.} Let \( X \) be a GL projective space which is not Calabi-Yau, \( A^{ca} \) the \( d \)-canonical algebra and \( \Pi(A^{ca}) \) the preprojective algebra of \( A^{ca} \).
(a) The center of $\Pi(A^{ca})$ is the $\mathbb{Z}[\vec{\omega}]$-Veronese subalgebra $R^{\mathbb{Z}[\vec{\omega}]}$ of $R$, and $\Pi(A^{ca})$ is a finitely generated $R^{\mathbb{Z}[\vec{\omega}]}$-module. In particular $\Pi(A^{ca})$ is a Noetherian algebra.

(b) If $n \leq d + 1$, then we have an isomorphism $\Pi(A^{ca}) \simeq R^{\mathbb{Z}[\vec{\omega}]}$ of $\mathbb{Z}$-graded $k$-algebras. In particular, $A^{ca}$ is a $d$-representation tame algebra.

(c) If $X$ is Fano, then $\Pi(A^{ca})$ is Morita equivalent to $R[\mathbb{Z}[\vec{\omega}]]$.

Proof. (a) By Proposition 6.7(a), the center of $\Pi(A^{ca})$ is isomorphic to that of $eR[\mathbb{Z}[\vec{\omega}]]e$, which is clearly the diagonal $R^{\mathbb{Z}[\vec{\omega}]}$ of $eR[\mathbb{Z}[\vec{\omega}]]e$. Since $R$ is a finitely generated $R^{\mathbb{Z}[\vec{\omega}]}$-module, the remaining assertion follows immediately.

(b) By Proposition 3.12(a), we have that $S := [-d\vec{c}, 0]$ itself gives a complete set of representatives and hence $e_S = 1$ and $e = 1$. Thus the assertion follows from Proposition 6.7(b).

(c) By Proposition 3.12(b), we have $e = 1$. Thus the assertion follows from Proposition 6.7(a). □

In the rest of this subsection, we give examples of $d$-canonical algebras using the quiver presentations given in Theorem 4.21. Hence we work under Assumption 4.20.

Example 6.9. For $d = 1$ we obtain the classical canonical algebras $[\text{Rin}]$, $[\text{GL}]$. More explicitly, the 1-canonical algebra of type $(p_1, \ldots, p_n)$ has the quiver

with relations $x_i^{p_i} = \lambda_{i0}x_1^{p_1} + \lambda_{1i}x_2^{p_2}$ for any $i$ with $3 \leq i \leq n$.

Example 6.10. The $d$-canonical algebra of weight type $(1, \ldots, 1)$ (where $n = d + 1$) is isomorphic to the $d$-Beilinson algebra $[\text{Be}]$ and has the quiver

with relations $x_i x_j = x_j x_i$ for any $i$ and $j$ with $1 \leq i < j \leq d + 1$.

The $d$-canonical algebras with $d + 1$ weights (or less) will be treated in detail in the end of this section. The smallest example with more than $d + 1$ weights is considered in the next example.
Example 6.11. The 2-canonical algebra of type $(2,2,2,2)$ has the quiver

![Quiver Diagram]

with relations $\sum_{i=1}^{4} \lambda_i x_i^4 = 0$ and $x_i x_j = x_j x_i$ for any $i$ and $j$ with $1 \leq i < j \leq 4$.

For any two element subset $\{i, j\} \subset \{1, 2, 3, 4\}$ we get a full subquiver of the quiver in Example 6.11 by identifying the two vertices labeled $\vec{c}$ in the following quiver

\[
\begin{array}{c}
\vec{c} \rightarrow \vec{x}_i + \vec{c} \rightarrow 2\vec{c} \\
\uparrow \quad \uparrow \\
\vec{x}_j \rightarrow \vec{x}_i + \vec{x}_j \rightarrow \vec{x}_j + \vec{c} \\
\uparrow \quad \uparrow \\
0 \rightarrow \vec{x}_i \\
\end{array}
\]

In fact, the whole quiver is the union of these subquivers.

Observation 6.12. (a) The quiver of any 2-canonical algebra of weight type $(p_1, \ldots, p_n)$ is the union of the full subquivers parametrized by two element subsets $\{i, j\} \subset \{1, \ldots, n\}$, that are obtained from

\[
\begin{array}{c}
\vec{c} \rightarrow \vec{x}_i + \vec{c} \rightarrow \cdots \rightarrow 2\vec{c} \\
\uparrow \quad \uparrow \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
\uparrow \quad \uparrow \\
\vec{x}_j \rightarrow \vec{x}_i + \vec{x}_j \rightarrow \cdots \rightarrow \vec{x}_j + \vec{c} \\
\uparrow \quad \uparrow \\
0 \rightarrow \vec{x}_i \\
\end{array}
\]

by identifying the two vertices labeled $\vec{c}$.

(b) For $d > 2$ a similar construction can be carried out replacing the above rectangles by $d$ dimensional parallelepipeds.

In the rest of this subsection we treat the case $n \leq d + 1$ in detail, which is precisely the case studied by Baer [Ba]. In this case $A^{\alpha}$ is $d$-representation infinite by Theorem 6.6. A class of $d$-representation infinite algebras called type $\hat{A}$ was introduced in [HIO]. We will show that $d$-canonical algebras for the case $n \leq d + 1$ are $d$-representation infinite algebras of type $\hat{A}$.

$d$-representation infinite algebras of type $\hat{A}$. Let $L$ be the root lattice of the root system

\[\{e_i - e_j \mid 1 \leq i \neq j \leq d + 1\}\]
of type $\Lambda_d$ in \{v ∈ \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} v_i = 0\}. The abelian group $L$ is freely generated by the simple roots $\alpha_i = e_i - e_{i-1}$, where $2 ≤ i ≤ d + 1$. We further define $\alpha_1 = e_1 - e_{d+1}$ and obtain the relation $\sum_{i=1}^{d+1} \alpha_i = 0$. Let $\tilde{Q}$ be the quiver defined by

- $\tilde{Q}_0 := L$,
- $\tilde{Q}_1 := \{ \alpha_i : v \mapsto (v + \alpha_i) \mid v ∈ L, 1 ≤ i ≤ d + 1 \}$.

The group $L$ acts on $\tilde{Q}_0$ by translations, which induces a unique $L$-action on $\tilde{Q}$.

Let $B$ be a subgroup of $L$ such that $L/B$ is finite. Denote by $\tilde{Q}/B$, the $B$-orbit quiver of $\tilde{Q}$. We denote the $B$-orbit of a vertex or arrow $x$ by $\overline{x}$. Let

$$A_B := k(\tilde{Q}/B)/(\overline{\pi_i} \overline{\pi_j} - \overline{\pi_j} \overline{\pi_i} : \overline{\pi} \mapsto \overline{\pi_1} + \overline{\pi_j} \mid \overline{\pi} ∈ (\tilde{Q}/B)_0, 1 ≤ i < j ≤ d + 1).$$

A set of arrows $C$ in $\tilde{Q}$ is called a cut if it contains exactly one arrow from each cycle of length $d + 1$ in $\tilde{Q}$. A cut $C$ is called $B$-acyclic if $BC = C$ and the quiver $\tilde{Q}/C$ is acyclic. In this case, we call the factor algebra

$$\Lambda_{B,C} := A_B/(C/B)$$

a $d$-representation infinite algebra of type $\hat{\mathbb{k}}$ associated to $(B, C)$. In fact it is $d$-representation infinite by [HIO, 5.6]. Observe that $\Lambda_{B,C}$ is presented by the quiver $(\tilde{Q}/C)/B$ and all commutativity relations of the form $\overline{\pi} \overline{\pi_j} - \overline{\pi_j} \overline{\pi}$, that appear in this quiver.

We will prove the following observation.

**Theorem 6.13.** Let $\mathbb{k}$ be a GL projective space with weights $(p_1, \ldots, p_n)$ with $n ≤ d + 1$. Then the $d$-canonical algebra $A^\mathbb{k}$ is isomorphic to a $d$-representation infinite algebra $\Lambda_{B,C}$ of type $\hat{\mathbb{k}}$ for $B := \langle p_i \alpha_i - p_j \alpha_j \mid 1 ≤ i, j ≤ n \rangle$ and some $B$-acyclic cut $C$.

**Proof.** By Observation 6.2, we assume without loss of generality that $n = d + 1$ by adding hyperplanes with weights 1. In Theorem 4.21 the $d$-canonical algebra $A^\mathbb{k} = A^{[0, d\mathfrak{c}]}$ was presented by the quiver $Q^{[0, d\mathfrak{c}]}$ with commutativity relations $x_i x_j = x_j x_i$.

By our choice of $B$, we have an epimorphism of abelian groups

$$\phi : L \rightarrow L/B$$

given by $\phi(\overline{x}_i) = \alpha_i + B$ such that the kernel is generated by $\overline{\omega} = -\sum_{i=1}^n \overline{x}_i$. Since the map $[0, d\mathfrak{c}] → \mathbb{L}/\mathbb{Z}\mathfrak{J}$ is a bijection by Proposition 5.12(a), the induced map

$$\phi : [0, d\mathfrak{c}] → L/B$$

is bijective. Thus $\phi$ is a bijection between the sets of vertices of the quivers $Q^{[0, d\mathfrak{c}]}$ and $\tilde{Q}/B$. In fact $\phi$ extends to a morphism

$$\phi : Q^{[0, d\mathfrak{c}]} \rightarrow \tilde{Q}/B$$

of quivers sending each arrow $x_i$ to the corresponding arrow $\overline{x}_i$. Moreover $\phi$ induces an isomorphism

$$\phi : Q^{[0, d\mathfrak{c}]} \cong (\tilde{Q}/C)/B$$

of quivers, where $C$ is the union of all $B$-orbits of arrows in $\tilde{Q}$ not in the image of $\phi$.

We will show that $C$ is a cut. Any cycle of length $d + 1$ in $\tilde{Q}$ is of the form

$$v_0 \xrightarrow{\alpha_{\sigma(1)}} v_1 \xrightarrow{\alpha_{\sigma(2)}} \cdots \xrightarrow{\alpha_{\sigma(d+1)}} v_{d+1} = v_0 \quad (6.2)$$

for some $v_0 ∈ L$ and permutation $\sigma$ of $\{1, \ldots, d + 1\}$, where $v_i = v_0 + \sum_{j=1}^i \alpha_{\sigma(j)}$. We need to show that all arrows in (6.2) except one belongs to $\phi(Q^{[0, d\mathfrak{c}]})$. Now let $\overline{y}_0$ be the unique element in $[0, d\mathfrak{c}]$ such that $\phi(\overline{y}_0) = \overline{\pi}_0$. Setting $\overline{y}_i := \overline{y}_{i-1} + \overline{x}_{\sigma(i)}$ for each $1 ≤ i ≤ d + 1$, we have a sequence

$$0 ≤ \overline{y}_0 < \overline{y}_1 < \cdots < \overline{y}_{d+1} = \overline{y}_0 - \overline{\omega}$$

in $L$ satisfying $\phi(\overline{y}_i) = \overline{\pi}_i$. Since $\overline{y}_{d+1} = \overline{y}_0 - \overline{\omega} ≥ -\overline{\omega}$, we have $\overline{y}_{d+1} ≥ d\mathfrak{c}$ by Lemma 3.11. Thus there is a unique $1 ≤ i ≤ d + 1$ such that $\overline{y}_i ≤ d\mathfrak{c}$ and $\overline{y}_{i-1} ≥ d\mathfrak{c}$. Hence we have a sequence

$$0 ≤ \overline{y}_i + \overline{\omega} < \overline{y}_{i+1} + \overline{\omega} < \cdots < \overline{y}_{d+1} + \overline{\omega} = \overline{y}_0 < \overline{y}_1 < \cdots < \overline{y}_{i-1} ≤ d\mathfrak{c};$$
in $[0, dc]$, and so there is a path
\[
\bar{y}_t + \bar{x} \xrightarrow{x(t+1)} \bar{y}_{t+1} + \bar{x} \xrightarrow{x(t+2)} \ldots \xrightarrow{x(d+1)} \bar{y}_{d+1} + \bar{x} = \bar{y}_0 \xrightarrow{x(t)} \bar{y}_1 \xrightarrow{x(2)} \ldots \xrightarrow{x(t-1)} \bar{y}_{t-1}
\]
in the quiver $Q^{[0,dc]}$. The image of the arrows in this path under $\varphi$ consists of all orbits of arrows in the cycle $c$ except $a_x(t) : v_{t-1} \rightarrow v_t$. Moreover, there is no arrow labeled $x(t)$ starting at $\bar{y}_{t-1}$, since $\bar{y}_{t-1} + x(t) = \bar{y}_t \not\in [0, dc]$. Hence $a_x(t) : v_{t-1} \rightarrow v_t$ is the unique arrow in $\{0,2\}$ that is not contained in $\phi(Q^{[0,dc]})$. We conclude that $C$ is a cut.

The cut $C$ is $B$-acyclic since $BC = C$ holds clearly and $(\bar{Q} \setminus C) / B \simeq Q^{[0,dc]}$ is acyclic. Finally, the quiver isomorphism $Q^{[0,dc]} \simeq (\bar{Q} \setminus C) / B$ induces an isomorphism
\[
A^{[0,dc]} \rightarrow \Lambda_{B,C}
\]
of $k$-algebras since both algebras are defined by all the commutativity relations in their quivers. □

We illustrate Theorem 6.13 for $d = 2$ in the following example.

**Example 6.14.** Here is the quiver $Q^{[0,dc]}$ of the 2-canonical algebra $A^{ca}$ of weights $(2, 3, 4)$, where for each $i = 1, 2, 3$, the full subquivers with vertices $\{a\bar{c}_i + \bar{c} \mid 0 \leq a \leq p_i\}$ that appear twice should be identified.

For comparison we give the corresponding quiver $\bar{Q}$ below, with cut $C$ indicated in bold. The vertices in the sublattice $B$ are labeled 0. The vertices whose $B$-orbits are in the image of $\bar{c}$ and $2\bar{c}$ under $\phi$ are also labelled accordingly.
6.2. Orlov-type semiorthogonal decompositions. In this section, we give an alternative proof of Theorem 6.1.1 by constructing an embedding of $\mathbb{D}^b(\text{coh } \mathcal{X})$ to $\mathbb{D}^b(\text{mod}^L R)$, which is parallel to the proof of Theorem 4.14. Then we will give Orlov-type semiorthogonal decompositions for $\mathbb{D}^b(\text{coh } \mathcal{X})$ and $\mathbb{D}^b R$. We use the notation from Section 4.2.

We start with the following analog of Theorem 4.14.

**Theorem 6.15.** For any non-trivial poset ideal $I$ of $\mathbb{L}$, the composition $$\mathcal{D}^I \cap (\mathcal{D}^{2-I})^* \subset \mathbb{D}^b(\text{mod}^L R) \xrightarrow{\pi} \mathbb{D}^b(\text{coh } \mathcal{X})$$ is a triangle equivalence.

**Lemma 6.16.** Let $I$ be a non-zero poset ideal of $\mathbb{L}$.

(a) We have $\mathcal{S} = \mathcal{S}^I \perp \mathcal{S}^{2-I}$ and $\mathcal{D} = \mathcal{D}^I \perp \mathcal{D}^{2-I}$. More generally, for a poset ideal $J$ containing $I$, we have $\mathcal{S}^J = \mathcal{S}^I \perp \mathcal{S}^{J \setminus I}$ and $\mathcal{D}^J = \mathcal{D}^I \perp \mathcal{D}^{J \setminus I}$.

(b) We have a triangle equivalence $\mathcal{D}^I/\mathcal{S}^I \simeq \mathbb{D}^b(\text{coh } \mathcal{X})$.

**Proof.** (a) Clearly $\text{Hom}_{\mathcal{D}}(\mathcal{D}^I, \mathcal{S}^{2-I}) = 0$ holds. We have functors $(-)^I : \text{mod}^L R \to \text{mod}^I R$, $(-)^{2-I} : \text{mod}^L R \to \text{mod}^{2-I} R$ and a sequence $$0 \to (-)^I \to \text{id} \to (-)^{2-I} \to 0$$ of natural transformations which is objectwise exact. Therefore we have induced triangle functors $(-)^I : \mathcal{D} \to \mathcal{D}^I$, $(-)^{2-I} : \mathcal{D} \to \mathcal{S}^{2-I}$ and a functorial triangle $X^I \to X \to X^{2-I} \to X^I[1]$ for any $X \in \mathcal{D}$. Thus we have the first two equalities.

The remaining equalities are similar.

(b) By (a), we have triangle equivalences $\mathcal{S}^I \simeq \mathcal{S}/\mathcal{S}^{2-I}$ and $\mathcal{D}^I \simeq \mathcal{D}/\mathcal{S}^{2-I}$. Therefore we have $\mathbb{D}^b(\text{coh } \mathcal{X}) \simeq \mathcal{D}/\mathcal{S} \simeq (\mathcal{D}/\mathcal{S}^{2-I})/(\mathcal{S}/\mathcal{S}^{2-I}) \simeq \mathcal{D}^I/\mathcal{S}^I$. □

**Proof of Theorem 6.15.** Applying Lemma 6.16(a) to the poset ideal $\mathcal{J} - I^c$, we have $\mathcal{D} = \mathcal{D}^{\mathcal{J} - I^c} \perp \mathcal{S}^{\mathcal{J} - 1^c}$. Applying $(-)^*$ and using (4.2), we have $\mathcal{D} = \mathcal{D}^* = \mathcal{S}^I \perp (\mathcal{D}^{2-I})^*$. Taking the intersections of $\mathcal{D}^I$ with both sides and applying Lemma 6.15 we have $\mathcal{D}^I = \mathcal{S}^I \perp (\mathcal{D}^I \cap (\mathcal{D}^{2-I})^*)$. Thus we have $\mathcal{D}^I \cap (\mathcal{D}^{2-I})^* \simeq \mathcal{D}^I/\mathcal{S}^I \simeq \mathbb{D}^b(\text{coh } \mathcal{X})$ by Lemma 6.16(b). □

Using Theorem 6.15, we are able to give an alternative proof of Theorem 6.1. The following analog of Theorem 4.14 is a main result in this subsection.

**Theorem 6.17.** Let $\mathcal{X}$ be a Geigle-Lenzing projective space.

(a) The following composition is a triangle equivalence: $$\mathcal{P}^{[0, d\mathcal{C}]} \subset \mathbb{D}^b(\text{mod}^L R) \xrightarrow{\pi} \mathbb{D}^b(\text{coh } \mathcal{X}).$$

(b) We have triangle equivalences $$\mathbb{D}^b(\text{mod}^L A^c) \simeq \mathcal{P}^{[0, d\mathcal{C}]} \simeq \mathbb{D}^b(\text{coh } \mathcal{X})$$ and $\mathbb{D}^b(\text{coh } \mathcal{X})$ has a tilting object $\pi(T^{[0, d\mathcal{C}]}), \text{which is isomorphic to } T^{[0, d\mathcal{C}]}$.

(c) We have $\mathcal{P}^{[0, d\mathcal{C}]} = \mathcal{L}^{+}_{\mathcal{C}} \cap (\mathcal{D}^{2-L^{+}_{\mathcal{C}}})^*.$

**Proof.** We only have to prove (c). In fact, (a) follows from (c) and Theorem 6.15 and (b) follows from (a) and Theorem 4.13(a).

In the rest, we prove the statement (c). By Lemma 6.11 $-\mathcal{J} \notin \mathcal{J}$ if and only if $\mathcal{J} \leq d\mathcal{C}$. Thus we have $$\mathcal{L}^{+}_{\mathcal{C}} \cap (L^{+}_{\mathcal{C}} - \mathcal{J}) = [0, d\mathcal{C}].$$ (6.3)

As a consequence, we have $$\mathcal{D}^{L^{+}_{\mathcal{C}}} \cap (\mathcal{D}^{2-L^{+}_{\mathcal{C}}})^* \supset \mathcal{D}^{L^{+}_{\mathcal{C}}} \cap (\mathcal{D}^{2-L^{+}_{\mathcal{C}}})^* \supset \mathcal{D}^{L^{+}_{\mathcal{C}}} \cap (\mathcal{D}^{2-L^{+}_{\mathcal{C}}})^* \supset P^{[0, d\mathcal{C}]}.$$ (6.3)

To show the converse, it is enough to show that the composition $\mathcal{P}^{[0, d\mathcal{C}]} \subset \mathcal{D} \xrightarrow{\pi} \mathbb{D}^b(\text{coh } \mathcal{X})$ is dense. This follows from Proposition 6.3. □
In the rest of this section, we show that there is a close connection between \( \text{CM}_{\ell}R \) and \( \text{D}^b(\text{coh } X) \) given in terms of the following Orlov-type semiorthogonal decompositions \([\text{KLM} 2.5]\). [KLM] C.4.

**Theorem 6.18.** There exist embeddings \( \text{D}^b(\text{coh } X) \to \text{D}^b(\text{mod}^L R) \) and \( \text{CM}_{\ell}R \to \text{D}^b(\text{mod}^L R) \) such that we have semiorthogonal decompositions

\[
\begin{align*}
\text{D}^b(\text{coh } X) &\cong \text{CM}_{\ell}R \perp \text{thick} \{ R(-\vec{x}) \}_{\vec{x} \in \delta(\vec{L}) < -d} & \text{if } X \text{ is Fano}, \\
\text{D}^b(\text{coh } X) &\cong \text{CM}_{\ell}R & \text{if } X \text{ is Calabi-Yau}, \\
\text{D}^b(\text{coh } X) &\cong \text{CM}_{\ell}R \perp \text{thick} \{ k(-\vec{x}) \}_{\vec{x} \in \delta(\vec{L}) < \delta(\vec{\omega})} & \text{if } X \text{ is anti-Fano}.
\end{align*}
\]

In particular, we have the equality:

\[
\text{rank} K_0(\text{coh } X) - \text{rank} K_0(\text{CM}_{\ell}R) = \left\{ \begin{array}{ll} |L/\mathbb{Z}\omega| & \text{if } X \text{ is Fano}, \\ 0 & \text{if } X \text{ is Calabi-Yau}, \\ -|L/\mathbb{Z}\omega| & \text{if } X \text{ is anti-Fano}. \end{array} \right.
\]

**Proof.** Using the map \( \delta : L \to \mathbb{Q} \) defined by \( \delta(\vec{x}) = 1/p_i \), we define a non-trivial poset ideal \( I := \{ \vec{x} \in L \mid \delta(\vec{x}) \geq 0 \} \). Applying Theorems [6.15] and [H.14] to \( I \), we have identifications

\[
\text{D}^b(\text{coh } X) = D^I \cap (D^{\text{op}})^* \quad \text{and} \quad \text{CM}_{\ell}R = D^I \cap (D^{\text{op}})^*.
\]

Note that we have \( \vec{\omega} - I^c = \{ \vec{x} \in L \mid \delta(\vec{x}) > 0 \} \). Using the map \( \delta : L \to \mathbb{Q} \) defined by \( \delta(\vec{x}) = 1/p_i \), we define a non-trivial poset ideal \( I := \{ \vec{x} \in L \mid \delta(\vec{x}) > 0 \} \).

(i) Assume that \( X \) is Calabi-Yau. Then \( \vec{\omega} - I^c = -I^c \), and we have the assertion immediately from (6.4).

(ii) Assume that \( X \) is Fano. Then \( \vec{\omega} - I^c \supset -I^c \). We have \( D^{\text{op}} \cap S(-I^c) \perp D^{-I^c} \) by Lemma [6.16] (a). Applying \((-)^*\), we have \( (D^{\text{op}}) \perp (D^{-I^c})^\perp \perp S(-\vec{\omega} + I^c) \cap I^c \). Taking the intersection of \( D^I \) with both sides and applying Lemma [6.15] we have

\[
D^I \cap (D^{\text{op}}) = (D^I \cap (D^{-I^c})^*) \perp S(-\vec{\omega} + I^c) \cap I^c.
\]

Since \( S(-\vec{\omega} + I^c) \cap I^c = \text{thick} \{ k(-\vec{x}) \}_{\vec{x} \in \delta(\vec{L}) < \delta(\vec{\omega})} \subset D^I \), we have the desired assertion from (6.4).

(iii) Assume that \( X \) is anti-Fano. Then \( \vec{\omega} - I^c \subset -I^c \). We have \( D^{\text{op}} \perp S(-I^c) \cap I^c = D^{-I^c} \) by Lemma [6.16] (a). Applying \((-)^*\), we have \( S(-\vec{\omega} + I^c) \cap I^c \cap (D^{-I^c})^* = (D^{-I^c})^* \). Taking the intersection of \( D^I \) with both sides and applying Lemma [6.15] we have

\[
D^I \cap (D^{\text{op}}) \perp S(-\vec{\omega} + I^c) \cap I^c = (D^I \cap (D^{-I^c})^*) \cap I^c.
\]

Since \( S(-\vec{\omega} + I^c) \cap I^c \) \( \subset D^I \), we have the desired assertion from (6.4).

As a consequence of Theorem 6.18 we have semiorthogonal decompositions between the derived categories of the d-canonical algebra \( A^{ca} \) and the CM-canonical algebra \( A^{CM} \).

We often have a more direct connection between \( A^{ca} \) and \( A^{CM} \). The following is such an example.

**Example 6.19.** Assume that \( n = 2d + 2 \) holds. Then we have \( \vec{\delta} = d\vec{e} + \sum_{i=1}^n (p_i - 2)\vec{e}_i \), and in particular, \([0, d\vec{e}] \subset [0, \vec{\delta}] \) holds. Therefore there exists an idempotent \( e \) of \( A^{CM} \) such that

\[
A^{ca} = eA^{CM}e.
\]

If moreover all \( p_i = 2 \), then \([0, d\vec{e}] \subset [0, \vec{\delta}] \) holds and we have \( A^{ca} = A^{CM} \). In this case \( X \) is Calabi-Yau, and a derived equivalence between \( A^{ca} \) and \( A^{CM} \) can be obtained from Theorem 6.18.

6.3. **Coxeter polynomials.** The aim of this section is to determine the Coxeter polynomials of Geigle-Lenzing projective spaces. Recall that the Coxeter polynomial is the characteristic polynomial of any matrix representing the action of the d-th suspension of the Serre functor, that is of \( (\vec{\omega}) \), on the Grothendieck group.

For \( X \in \text{coh } X \) we denote by \([X]\) the corresponding element in the Grothendieck group. By abuse of notation we denote by \( (\vec{x}) \) the action of the shift by \( \vec{x} \) on the Grothendieck group, that is \([X](\vec{x}) = [X(\vec{x})]\).
**Definition 6.20.** Let $0 \leq e \leq d$, and let $I \subseteq \{1, \ldots, n\}$ have cardinality at most $d - e$.

Choose $d - e - |I|$ homogeneous linear polynomials $f_1, \ldots, f_{d - e - |I|}$, such that these polynomials and the $\ell_i$ with $i \not\in I$ are linearly independent.

We denote by $G^e_I$ the element of the Grothendieck group of the coherent sheaf corresponding to the graded $R$-module

$$\left(\mathcal{O}_{X_i} \cap f_j(T) \mid i \in I, j \in \{0, \ldots, d - e - |I|\}\right).$$

Note that a priori this depends on our choice of the $f_i$. However Lemma 6.21 below shows that in fact any choice gives the same element in the Grothendieck group.

Vaguely the interpretation of this module is that it corresponds to the structure sheaf on the intersection of the $e$ dimensional subspace formed by the intersection of the $d - e - |I|$ “generic” hyperplanes $f_j$, and the $|I|$ special hyperplanes $\ell_i$.

**Lemma 6.21.** We have

$$G^e_I = \sum_{I' \subseteq I} \sum_{a=0}^{d - e - |I|} (-1)^{a + |I|} \binom{d - e - |I|}{a} \mathcal{O}(\bar{\alpha} - \bar{x}_j).$$

In particular $G^e_I$ is independent of the choice of hyperplanes $f_j$.

**Proof.** Since $(X_i, f_j(T))$ form a regular sequence by Lemma 3.6(c), we may use the associated Koszul complex to compute the dimension vector of $R/(X_i, f_j(T))$ in terms of dimension vectors of shifts of projective modules. The formula follows. \[\Box\]

We collect the following immediate consequences of the definition, which will allow us to compute Coxeter polynomials.

**Proposition 6.22.** For $I$ and $e$ as above we have the following

- If $j \not\in I$ then $G^e_I - G^e_I(-\bar{x}_j) = \begin{cases} 0 & \text{if } e = 0 \\ G^e_{I \cup \{j\}} & \text{otherwise.} \end{cases}$

- $G^e_I - G^e_I(-\bar{\alpha}) = \begin{cases} 0 & \text{if } e = 0 \\ G^e_I & \text{otherwise.} \end{cases}$

- For $i \in I$ we have $\sum_{a=0}^{p_i-1} G^e_I(-a\bar{x}_i) = G^e_{I \setminus \{i\}}$.

**Proof.** For the first point note that, in the notation of Lemma 6.21 we may choose the $f_k$ linearly independent to $\ell_i$ with $i \in I \cup \{j\}$. Consider the sequence

$$0 \to R/(X_i, f_k \mid i \in I)(-\bar{x}_j) \to R/(X_i, f_k \mid i \in I) \to R/(X_i, f_k \mid i \in I \cup \{j\}) \to 0,$$

which is short exact by Lemma 3.6(c). The claim follows from the definition of Grothendieck groups.

The second point can be seen similarly.

For the final point, note that $R/(X_i, \ell_i, f_k \mid j \in I \setminus \{i\})$ is filtered by $(R/(X_j, f_k \mid j \in I))(a\bar{x}_i)$ for $a = 0, \ldots, p_i-1$. Again the claim follows immediately. \[\Box\]

**Proposition 6.23.** The set

$$\{G^e_I(\bar{x}) \mid I \subseteq \{1, \ldots, n\}, 0 \leq e \leq d - |I|, \bar{x} = \sum_{i \in I} a_i \bar{x}_i \text{ for some } 0 < a_i < p_i\}$$

is a basis of the Grothendieck group.

**Proof.** We check that the subgroup of the Grothendieck group generated by the above set contains all $G^e_I(\bar{x})$ (for arbitrary $\bar{x} \in \mathbb{L}$). Inductively we may assume this claim to hold for

$G^e_{i'}(\bar{x})$ for $e' < e$, and any $I' \subseteq I$. Hence we have

$$G^e_I(\bar{x}) = \sum_{I' \subseteq I} \sum_{a=0}^{d - e - |I|} (-1)^{a + |I|} \binom{d - e - |I|}{a} \mathcal{O}(\bar{x} - \sum_{i \in I} a_i \bar{x}_i).$$
Now the three points of Proposition 6.22 show that the set of shifts of $G^i_j$ we obtain is closed under addition (and subtraction) of $x_j$ ($j \not\in I$), $c$, and $x_i$ ($i \in I$), respectively. Indeed, in all cases the term on the right side of the equalities is already in what is generated by our set inductively, and so the left side tells us that we can do these additions and subtractions without leaving what is generated by the set of the proposition. (For the last point note that we already have $p_i - 1$ adjacent shifts inside the set.)

In particular we have now seen that the subgroup generated by the above set contains all $G^i_j(A) = [O(A)]$ and thus is the entire Grothendieck group by Proposition 6.3.

Finally note that $G^i_j(A) \Rightarrow c\bar{c} + A$ defines a bijection of the above set to the interval $[0, d\bar{c}^i]$, and hence to the indecomposable summands of the tilting module $T$ (see Theorem 6.1). Thus the generating set has cardinality equal to the rank of the Grothendieck group, so it is a basis. □

Example 6.24. Consider the usual projective line $\mathbb{P}^1$. Then the basis of the Grothendieck group given in Proposition 6.23 consists of $G^0_0 = [O]$ and $G^0_i = [S]$ for any simple sheaf $S$.

In particular we may note that this basis does not arise as the dimension vectors of a tilting object.

For the calculation of the Coxeter polynomials, we prepare the following combinatorially defined polynomials.

Construction 6.25. For $a_1, \ldots, a_s \in \mathbb{N}_{>0}$, denote by $B_{a_1,...,a_s}$ the abelian group

$$ \mathbb{Z}/(a_1) \times \cdots \times \mathbb{Z}/(a_s) = \mathbb{Z}/(a_1) \times \cdots \times \mathbb{Z}/(a_s), $$

and consider the action on it given by adding $(1, \ldots, 1)$. One observes that under this action, $B_{a_1,...,a_s}$ decomposes into orbits of length $\text{lcm}(a_1, \ldots, a_s)$.

Clearly this extends to a linear map $\Xi$ on the group ring $\mathbb{Z}B_{a_1,...,a_s}$. It follows from the above observation on the orbits on $B_{a_1,...,a_s}$ that the characteristic polynomial of this endomorphism is

$$ \chi_{\mathbb{Z}B_{a_1,...,a_s}} = (1 - \frac{1}{\text{lcm}(a_1, \ldots, a_s)}) \prod_{i=1}^{a_1} (\sum_{j=0}^{a_i-1} x). $$

We denote by

$$ \Delta_i = \sum_{x \in \mathbb{Z}/(a_i)} (1, \ldots, 1, x, 1, \ldots, 1) $$

the sum over all elements in the cyclic group in the $i$-th factor of $B_{a_1,...,a_s}$. Then we have a filtration of $\mathbb{Z}B_{a_1,...,a_s}$

$$ I_s \subseteq I_{s-1} \subseteq \cdots \subseteq I_1 \subseteq I_0 = \mathbb{Z}B_{a_1,...,a_s} $$

where $I_j$ is the ideal generated by all products of $j$ distinct $\Delta_i$, that is

$$ I_1 = (\Delta_1, \ldots, \Delta_s), \quad I_2 = (\Delta_1 \Delta_2, \Delta_1 \Delta_3, \ldots, \Delta_{s-1} \Delta_s), \ldots, I_s = (\Delta_1 \Delta_2 \cdots \Delta_n). $$

It is immediate that $\Xi$ restrict to these ideals. The polynomials which we will be interested in are

$$ \phi_{a_1,...,a_s} := \chi_{\mathbb{Z}B_{a_1,...,a_s}}/I_1 $$

that is the characteristic polynomial of the action of $\Xi$ of the top quotient of this filtration.

One may observe that the characteristic polynomials of the action of $\Xi$ on the other subquotients are of the same form, and thus

$$ (1 - \frac{1}{\text{lcm}(a_1, \ldots, a_s)}) \prod_{i=1}^{a_i-1} (\sum_{j=0}^{a_i} x_{a_i}/I_i) = \prod_{i=0}^{s} \chi_{I_i}/I_{i+1} = \prod_{i=0}^{s} \prod_{I \subseteq \{1, \ldots, s\}} \phi_{a_i,j \in I} \prod_{I \subseteq \{1, \ldots, s\}} \phi_{a_j,j \in I} $$

This formula allows us to calculate the $\phi_{a_1,...,a_s}$ inductively, starting with $\phi$ with fewer indices.

For the first few we obtain

$$ \phi_1 = 1 - t, \quad \phi_0 = \frac{1}{1-t^a}, \quad \phi_{a,b} = \frac{(1-t^{\text{lcm}(a,b)}) \prod_{j \in \{1, \ldots, s\}, a \neq j} (1-t^a)}{(1-t^a)^2 (1-t^b)/(1-t^c)), \ldots
We are now ready to compute the Coxeter polynomial of a \(d\)-dimensional Geigle-Lenzing projective space.

**Theorem 6.26.** The Coxeter polynomial of a \(d\)-dimensional Geigle-Lenzing projective space with weights \(p_1, \ldots, p_n\) is

\[
\chi = \prod_{I \subseteq \{1, \ldots, n\}, |I| \leq d} \phi_{\{p_i : i \in I\}}^{d + 1 - |I|}.
\]

**Proof.** By Theorem 5.4(b) the Serre functor is given by \((\vec{\omega})[d]\).

We use the basis of the Grothendieck group given in Proposition 6.23. By the proof of that proposition, the action of \((\vec{\omega})\) with respect to that basis has a block diagonal shape, and thus the Coxeter polynomial is the product

\[
\chi = \prod_{I \subseteq \{1, \ldots, n\}, 0 \leq e \leq d - |I|} \chi^e_I,
\]

where \(\chi^e_I\) is the characteristic polynomial of the action of \((\vec{\omega})\) on

\[
\left\langle G_I^e(\vec{x}) | \vec{x} = \sum_{i \in I} a_i \vec{x}_i \text{ for some } 0 < a_i < p_i \right\rangle + \left\langle G_I^e(\vec{x}) | I' \subseteq I, \vec{x} \text{ arbitrary} \right\rangle
\]

By Proposition 6.22 the action of \((\vec{\omega})\) on such a subquotient is precisely the same as the action of \(\Xi\) on \(ZB_{\{p_i : i \in I\}} / I_1\) in Construction 6.25 above. Therefore

\[
\chi^e_I = \phi_{\{p_i : i \in I\}}.
\]

The claim follows. \(\square\)

As an easy consequence, we have the following information on Grothendieck groups.

**Corollary 6.27.** The rank of the Grothendieck group of \(K_0(\text{coh} X)\) and \([0, d\vec{c}]\) are equal to

\[
\sum_{I \subseteq \{1, \ldots, n\}, |I| \leq d} (d + 1 - |I|) \prod_{i \in I} (p_i - 1).
\]

**Example 6.28.** Let \(X\) be a 2-dimensional Geigle-Lenzing projective space with weights 2, 3. Then the Coxeter polynomial of \(X\) is given by

\[
\chi = \phi_1^3 \phi_2^3 \phi_3^3 \phi_{2,3} = (1 - t)^3 \left[ \frac{1 - t^2}{1 - t} \right]^2 \left[ \frac{1 - t^3}{1 - t} \right]^2 \left[ \frac{(1 - t^6)(1 - t)}{(1 - t^2)(1 - t^3)} \right] = (1 - t)^3 (1 + t)^2 (2 + t + t^2)^2 (2 - t - t^2),
\]

and the rank of the Grothendieck group is 11.

We end this section by posing the following question.

**Question 6.29.** What is the Coxeter polynomial of \(\text{CM}_{\mathbb{R}}^d R\)?

For the hypersurface case \(n = d + 2\), an answer was given by Hille-Müller [HM].

### 7. Tilting Theory on Geigle-Lenzing Projective Spaces

Let \(X\) be a Geigle-Lenzing projective space over a field \(k\) associated with hyperplanes \(H_1, \ldots, H_n\) on \(\mathbb{P}^d\) and weights \(p_1, \ldots, p_n\). In this section, we study tilting objects \(V\) in \(\text{D}^b(\text{coh} X)\) that belong to \(\text{vect} X\) (respectively, \(\text{coh} X\)). We call such a \(V\) a tilting bundle (respectively, tilting sheaf) on \(X\). We study the endomorphism algebras \(\text{End}_X(V)\) of tilting bundles. A typical example of a tilting bundle is \(T^c_a\) given in Theorem 6.3. In this case, the endomorphism algebra is the \(d\)-canonical algebra \(A^c_a\) studied in the previous section.
7.1. Basic properties of tilting sheaves. Throughout this section, let \( V \) be a tilting bundle on \( X \) with \( \Lambda := \text{End}_X(V) \). Then we have triangle equivalences

\[
V \otimes_{\Lambda} - : \mathcal{D}^b(\text{mod } \Lambda) \to \mathcal{D}^b(\text{coh } X) \quad \text{and} \quad \mathbf{R}\text{Hom}_X(V, -) : \mathcal{D}^b(\text{coh } X) \to \mathcal{D}^b(\text{mod } \Lambda)
\]

which are mutually quasi-inverse and make the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{D}^b(\text{mod } \Lambda) & \sim & \mathcal{D}^b(\text{coh } X) \\
\downarrow^{\nu_d} & & \downarrow^{(2)} \\
\mathcal{D}^b(\text{mod } \Lambda) & \sim & \mathcal{D}^b(\text{coh } X).
\end{array}
\]

In the rest, we identify \( \mathcal{D}^b(\text{mod } \Lambda) \) and \( \mathcal{D}^b(\text{coh } X) \) by these triangle equivalences.

Let \( 0 \leq j \leq d \). Recall that \( \mathcal{R}_{T_j} \) is the localization of \( R \) with respect to the multiplicative set \( \{T_j^\ell \mid \ell \in \mathbb{Z}\} \). Since \((\text{mod }^j R)_{T_j} = 0\), holds, the natural functor \((-)_{T_j} : \mathcal{D}^b(\text{mod }^j R) \to \mathcal{D}^b(\text{mod }^j R_{T_j})\) factors as

\[
\mathcal{D}^b(\text{mod }^j R) \xrightarrow{s} \mathcal{D}^b(\text{coh } X) \to \mathcal{D}^b(\text{mod }^j R_{T_j})
\]

by universality. The following observation shows that tilting bundles on \( X \) give generators in \( \text{mod }^j R_{T_j} \).

**Lemma 7.1.** Assume that \( V \in \text{mod }^j R \) gives a tilting bundle on \( X \). Then for any \( j \) with \( 0 \leq j \leq d \), we have \( \text{proj}^j \mathcal{R}_{T_j} = \text{add } \mathcal{V}_{T_j} \).

**Proof.** It follow from Proposition 5.12(c) that \( \mathcal{V}_{T_j} \in \text{proj}^j \mathcal{R}_{T_j} \). Since the functor \( \mathcal{D}^b(\text{coh } X) = \text{thick } V \to \mathcal{D}^b(\text{mod }^j R_{T_j}) \) is dense, we have \( \mathcal{D}^b(\text{mod }^j R_{T_j}) = \text{thick } \mathcal{V}_{T_j} \). In particular, \( \mathcal{V}_{T_j} \) has to be a generator in \( \text{mod }^j R_{T_j} \).

The following useful result strengthens Theorem 5.16(a) for tilting bundles.

**Theorem 7.2.** Let \( V \) be a tilting bundle on \( X \). Then for any \( X \in \text{coh } X \), there exists \( \bar{a} \in L \) such that for any \( \bar{x} \in L \) satisfying \( \bar{x} \geq \bar{a} \), there exists an epimorphism \( V' \to X(\bar{x}) \) in \( \text{coh } X \) with \( V' \in \text{add } V \).

**Proof.** It suffices to show that for any \( W \in \text{mod }^j R \), there exists \( \bar{a} \in L \) such that for any \( \bar{x} \in L \) satisfying \( \bar{x} \geq \bar{a} \), there exists a morphism \( V' \to W(\bar{x}) \) in \( \text{mod }^j R \) with \( V' \in \text{add } V \) which has a cokernel in \( \text{mod }^j R_{T_j} \).

(i) Fix \( X \in \text{mod }^j R \). We show that for \( \ell \gg 0 \), there exists a morphism \( V' \to X(\ell \bar{c}) \) in \( \text{mod }^j R \) with \( V' \in \text{add } V \) which has a cokernel in \( \text{mod }^j R_{T_j} \).

Fix \( j = 0, \ldots, d \). Since \( \text{proj}^j R_{T_j} = \text{add } \mathcal{V}_{T_j} \) holds by Lemma 7.1, there exists an epimorphism \( f_j : \mathcal{V}^j_{T_j} \to X_{T_j} \) in \( \text{mod }^j R \) with \( V' \in \text{add } V \). Since

\[
\text{Hom}^j_{R_{T_j}}(\mathcal{V}^j_{T_j}, X_{T_j}) = (\text{Hom}_R(V^j, X)_{T_j})_0 = \sum_{a \geq 0} \text{Hom}^j_R(V^j, X(a\bar{c}))T_j^{-a},
\]

we can write \( f_j = g_{jT_j}^{-a} \) with \( g_j \in \text{Hom}^j_R(V^j, X(a\bar{c})) \). Then \( \text{Cok } g_j \in \text{mod }^j R \) satisfies \( \text{Cok } g_j(\mathcal{R}_{T_j}) = 0 \).

For \( a := \max\{a_0, \ldots, a_d\} \), let \( e_j := g_{jT_j}^{a^{-a}} \in \text{Hom}^j_R(V^j, X(a\bar{c})) \) and

\[
e := (e_0, \ldots, e_d)^t : V^0 \oplus \cdots \oplus V^d \to X(a\bar{c}).
\]

Then there exists an epimorphism \( \text{Cok } e_j \to \text{Cok } e \). Since \( \text{Cok } e_j(\mathcal{R}_{T_j}) = 0 \) holds, we have \( \text{Cok } e_j(\mathcal{R}_{T_j}) = 0 \) for any \( j \) with \( 0 \leq j \leq d \). Thus \( \text{Cok } e \) belongs to \( \text{mod }^j R \).

Using Lemma 5.1(a), we have the assertion.
(ii) Let \( I := \{ \sum_{i=1}^{n} a_i \bar{x}_i \mid 0 \leq a_i < \bar{p}_i \} \) be the complete set of representatives in \( \mathbb{L}/\mathbb{Z}\bar{c} \). Applying (i) to \( X := W(\bar{x}) \) for each \( \bar{x} \in I \), we have the assertion. \( \square \)

We have the following description of \( \text{coh} X \) in terms of \( \Lambda \).

**Theorem 7.3.** Let \( V \) be a tilting bundle on \( X \), and \( \Lambda := \text{End}_X(V) \).

(a) For any \( \bar{a} \in \mathbb{L} \) satisfying \( \delta(\bar{a}) > 0 \), we have
\[
\text{coh} X = \{ X \in D^b(\text{coh} X) \mid \forall \ell \gg 0 \ X(\ell \bar{a}) \in \text{mod} \Lambda \}.
\]

(b) If \( X \) is Fano, then \( \text{coh} X = \{ X \in D^b(\text{coh} X) \mid \forall \ell \gg 0 \ X(-\ell \bar{a}) \in \text{mod} \Lambda \} \).

(c) If \( X \) is anti-Fano, then \( \text{coh} X = \{ X \in D^b(\text{coh} X) \mid \forall \ell \gg 0 \ X(\ell \bar{a}) \in \text{mod} \Lambda \} \).

More strongly, we describe the standard t-structure of \( D^b(\text{coh} X) \) in terms of \( \Lambda \):

**Proposition 7.4.** Let \( V \) be a tilting bundle on \( X \). Then for any \( \bar{a} \in \mathbb{L} \) satisfying \( \delta(\bar{a}) > 0 \), we have equalities
\[
D^{\leq 0}(\text{coh} X) = \{ X \in D^b(\text{coh} X) \mid \forall \ell \gg 0 \ X(\ell \bar{a}) \in D^{\leq 0}(\text{mod} \Lambda) \},
\]
\[
D^{> 0}(\text{coh} X) = \{ X \in D^b(\text{coh} X) \mid \forall \ell \gg 0 \ X(\ell \bar{a}) \in D^{> 0}(\text{mod} \Lambda) \}.
\]

**Proof.** We only show the first equality since the second one can be shown similarly.

Fix \( X \in D^b(\text{coh} X) \). Then \( X \) belongs to \( D^{\leq 0}(\text{coh} X) \) if and only if the following condition holds:

(i) \( H^i(X) = 0 \) for any \( i > 0 \).

By Theorem 7.3, this is equivalent to the following condition (since \( H^i(X) \neq 0 \) for almost all \( i \)):

(ii) For \( \ell \gg 0 \), we have \( \text{Hom}_X(V,H^i(X)(\ell \bar{a})) = 0 \) for any \( i > 0 \).

By Serre vanishing Theorem 5.16(b), for \( \ell \gg 0 \), we have \( \text{Ext}^j_X(V,X^i(\ell \bar{a})) = 0 \) for any \( i \in \mathbb{Z} \) and \( j > 0 \) (since \( X^i \neq 0 \) for almost all \( i \)). Therefore for \( \ell \gg 0 \), we have
\[
\text{Hom}_X(V,H^i(X)(\ell \bar{a})) \simeq \text{Hom}_D^{\bar{b}}(\text{coh} X)(V,X(\ell \bar{a})[i])
\]
for any \( i \in \mathbb{Z} \). Therefore (ii) is equivalent to the following condition:

(iii) For \( \ell \gg 0 \), we have \( \text{Hom}^{D^b(\text{coh} X)}(V,X(\ell \bar{a})[i]) = 0 \) for any \( i > 0 \).

This is equivalent to the following condition since \( V \) corresponds to \( \Lambda \) under the identification \( D^b(\text{coh} X) = D^b(\text{mod} \Lambda) \):

(iv) For \( \ell \gg 0 \), we have \( X(\ell \bar{a}) \in D^{\leq 0}(\text{mod} \Lambda) \).

Thus the first equality follows. \( \square \)

Now we are ready to prove Theorem 7.3.

**Proof of Theorem 7.3.** (a) Since \( \text{coh} X = D^{\leq 0}(\text{coh} X) \cap D^{> 0}(\text{coh} X) \), the assertion follows from Proposition 7.4.

(b)(c) Immediate from (a). \( \square \)

Now we describe the duality \((-)^\vee = R\text{Hom}_R(-,\omega_R) : D^b(\text{coh} X) \to D^b(\text{coh} X) \) in terms of \( \Lambda \).

**Proposition 7.5.**

(a) \( V^\vee \) is also a tilting bundle on \( X \).

(b) The following diagram is commutative:
\[
\begin{array}{c}
D^b(\text{coh} X) \xrightarrow{R\text{Hom}_X(V,-)} D^b(\text{mod} \Lambda) \\
\downarrow{(-)^\vee[d]} \quad \downarrow{D=\text{Hom}_X(-,\omega_X)} \\
D^b(\text{coh} X) \xrightarrow{R\text{Hom}_X(V(\underline{\omega})^\vee,-)} D^b(\text{mod} \Lambda^{op}).
\end{array}
\]

(c) For any \( \bar{a} \in \mathbb{L} \) satisfying \( \delta(\bar{a}) > 0 \), we have
\[
(\text{coh} X)^\vee = \{ X \in D^b(\text{coh} X) \mid \forall \ell \gg 0 \ X(-\ell \bar{a}) \in (\text{mod} \Lambda)[-d] \}.
\]
Proof. (a) This is clear since $(-)^\vee : D^b(\text{coh } \mathcal{X}) \rightarrow D^b(\text{coh } \mathcal{X})$ is a duality.
(b) Using Auslander-Reiten-Serre duality, we have $\text{RHom}_X(V, V(\tilde{\omega})) = DA[-d]$. Thus we have isomorphisms of functors:

$\text{RHom}_X(V, -)[d] = \text{RHom}_\Lambda(\text{RHom}_X(V, -), DA[-d])$

$= \text{RHom}_\Lambda(\text{RHom}_X(V, -), \text{RHom}_X(V, \tilde{\omega}))$

$= \text{RHom}_\Lambda(V \otimes \Lambda \text{RHom}_X(V, -), \text{RHom}_X(V, \tilde{\omega}))$

$= \text{RHom}_X(\Lambda, \text{RHom}_X(V, \tilde{\omega}))$

Thus the assertion follows.

(c) Let $X \in D^b(\text{coh } \mathcal{X})$. Applying Theorem 7.3 to $V(\tilde{\omega})^{\vee}$, we have that $X^{\vee} \in \text{coh } \mathcal{X}$ if and only if $X(-\ell \tilde{\alpha})^{\vee} = X^{\vee}(\ell \tilde{\alpha}) \in \text{mod } \Lambda$ for $\ell \gg 0$. Using the commutative diagram in (b), this is equivalent to $X(-\ell \tilde{\alpha}) \in (\text{mod } \Lambda)[-d]$ for $\ell \gg 0$. Thus the assertion follows. \qed

Identifying $D^b(\text{coh } \mathcal{X})$ with $D^b(\text{mod } \Lambda)$, we have the following description of $\text{CM}_i \mathcal{X}$.

**Proposition 7.6.** Let $0 \leq i \leq d$.
(a) For any $\tilde{\alpha} \in L$ satisfying $\delta(\tilde{\alpha}) > 0$, we have

$\text{CM}_i \mathcal{X} = \{ X \in \text{coh } \mathcal{X} \mid \forall \ell \gg 0 \ X(\ell \tilde{\alpha}) \in \text{mod } \Lambda, \ X(-\ell \tilde{\alpha}) \in (\text{mod } \Lambda)[-i] \}$.

(b) If $X$ is Fano, then $\text{CM}_i \mathcal{X} = \{ X \in \text{coh } \mathcal{X} \mid \forall \ell \gg 0 \ X(\ell \tilde{\alpha}) \in \text{mod } \Lambda, \ X(-\ell \tilde{\alpha}) \in (\text{mod } \Lambda)[-i] \}$.
(c) If $X$ is anti-Fano, then $\text{CM}_i \mathcal{X} = \{ X \in \text{coh } \mathcal{X} \mid \forall \ell \ll 0 \ X(\ell \tilde{\alpha}) \in \text{mod } \Lambda, \ X(-\ell \tilde{\alpha}) \in (\text{mod } \Lambda)[-i] \}$.

**Proof.** (a) By Theorem 7.3 an object $X \in D^b(\text{coh } \mathcal{X})$ belongs to $\text{coh } \mathcal{X}$ if and only if $X(\ell \tilde{\alpha}) \in \text{mod } \Lambda$ holds for $\ell \gg 0$. Now we fix $X \in \text{coh } \mathcal{X}$. By definition, $X \in \text{CM}_i \mathcal{X}$ if and only if $X[i - d]^{\vee} \in \text{coh } \mathcal{X}$. By Proposition 7.5(c), this is equivalent to that $X(-\ell \tilde{\alpha}) \in (\text{mod } \Lambda)[-i]$ holds for $\ell \gg 0$. Thus the assertion follows.

(b)(c) Immediate from (a). \qed

Next we give some properties of $\Lambda = \text{End}_\mathcal{X}(V)$.

In general, let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension. Let $U$ be a two-sided tilting complex of $\Lambda$. For $i \geq 0$, we simply write

$U^\ell := U \otimes^\Lambda \cdots \otimes^\Lambda U$.

We have an autoequivalence $U^\ell \otimes^\Lambda \cdots \otimes^\Lambda U$ of $D^b(\text{mod } \Lambda)$, which we simply denote by $U^\ell$. The following notion was introduced by the third author [Min].

**Definition 7.7.** (a) We say that $U$ is quasi-ample if $U^\ell \in \text{mod } \Lambda$ for $\ell \gg 0$.
(b) We say that $U$ is ample if it is quasi-ample and $(D^{U \leq 0}, D^{U \geq 0})$ is a $t$-structure in $D^b(\text{mod } \Lambda)$, where

$D^{U \leq 0} := \{ X \in D^b(\text{mod } \Lambda) \mid \forall \ell \gg 0 \ U^\ell(X) \in D^{\leq 0}(\text{mod } \Lambda) \}$,

$D^{U \geq 0} := \{ X \in D^b(\text{mod } \Lambda) \mid \forall \ell \gg 0 \ U^\ell(X) \in D^{\geq 0}(\text{mod } \Lambda) \}$.

Note that $\omega_\Lambda := DA[-d]$ and $\omega_\Lambda^{-1} := \text{RHom}_\Lambda(\omega_\Lambda, \Lambda)$ are 2-sided tilting complexes of $\Lambda$.

**Definition 7.8.** We say that $\Lambda$ is quasi $d$-Fano (respectively, quasi $d$-anti-Fano) if the two-sided tilting complex $\omega_\Lambda^{-1}$ (respectively, $\omega_\Lambda$) is quasi-ample. More strongly, we say that $\Lambda$ is $d$-Fano (respectively, $d$-anti-Fano) if the two-sided tilting complex $\omega_\Lambda^{-1}$ (respectively, $\omega_\Lambda$) is ample.
Remark 7.9. It is clear from definition any $d$-representation infinite algebra is quasi $d$-Fano. But the converse is not true. For example, if $X$ is Fano with $n \geq d + 2$ and $p_i \geq 2$ for any $i$, then the $d$-canonical algebra $A^{ca}$ is a $d$-Fano algebra which is not $d$-representation infinite by Theorems 7.10 and 6.6.

Note that $d$-Fano algebras are not necessarily almost $d$-representation infinite. For example let $A^{ca}_1$ be a Kronecker algebra and $A^{ca}_2$ a 1-canonical algebra with $n \geq 3$ and $p_i \geq 2$ for any $i$. Then $\Lambda := A^{ca}_1 \otimes_k A^{ca}_2$ is a 2-Fano algebra. But $\text{gl.dim} \Lambda = \text{gl.dim} A^{ca}_1 + \text{gl.dim} A^{ca}_2 = 3$ holds, which is not possible for almost 2-representation infinite algebras by Proposition 2.15.

Next we will show the following trichotomy, which generalizes the case $d = 1$ [Min]:

Theorem 7.10. Let $V$ be a tilting bundle on $X$, and $\Lambda := \text{End}_X(V)$.

(a) $X$ is Fano if and only if $\Lambda$ is a $d$-Fano algebra.

(b) $X$ is anti-Fano if and only if $\Lambda$ is a $d$-anti-Fano algebra.

(c) $X$ is Calabi-Yau if and only if $\text{D}^b(\text{mod} \Lambda)$ is a fractionally Calabi-Yau triangulated category.

Proof. Since $d$-Fano algebras, fractionally Calabi-Yau algebras and $d$-anti-Fano algebras are disjoint classes, we only have to show the ‘if’ part of all statements.

(c) This is clear from the diagram (7.1).

(a) Assume that $X$ is Fano. By (7.1), we have

$$H^i(\omega_X^\ell) = \text{Hom}_{\text{D}^b(\text{mod} \Lambda)}(\Lambda, \nu^\ell_0(\Lambda)[i]) = \text{Hom}_{\text{D}^b(\text{coh} X)}(V, V(-\ell \omega)[i]).$$ (7.2)

This is clearly zero for $i < 0$. Assume $i > 0$. Since $X$ is Fano, the element $-\ell \omega$ is sufficiently large for $\ell \gg 0$. Therefore (7.2) is zero for $\ell \gg 0$ by Serre vanishing Theorem 5.16. Therefore $\omega_X^\ell$ is quasi-ample.

On the other hand, Proposition 7.14 shows that $D^{<0} = D^{<0}(\text{coh} X)$ and $D^{>0} = D^{>0}(\text{coh} X)$ hold. In particular $(D^{<0}, D^{>0})$ is a t-structure in $D^b(\text{mod} \Lambda)$. Thus $\Lambda$ is a $d$-Fano algebra.

(b) Assume that $X$ is anti-Fano. One can show that $\Lambda$ is a $d$-anti-Fano algebra by a parallel argument as in (a) above. □

7.2. Tilting-cluster tilting correspondence. The aim of this section is to study when $X$ is derived equivalent to a $d$-representation infinite algebra. We show that this is closely related to $d$-VB finiteness of $X$. As in the case of $\text{CM}^d R$, the following notion plays an important role.

$d$-tilting objects. As in $\text{CM}^d R$, a tilting object $V$ in $D^b(\text{coh} X)$ is called $d$-tilting if $\text{End}_{D^b(\text{coh} X)}(V)$ has global dimension at most $d$. By Proposition 7.11(a) below, this is equivalent to the global dimension being precisely $d$.

We give some basic properties of the endomorphism algebras of tilting objects. The result (a) below shows that $d$ gives a lower bound for $\text{gl.dim} \Lambda$. Moreover the result (e) below due to Buchweitz-Hille [BuH] explains the importance of $d$-tilting sheaves.

Proposition 7.11. Let $V \in D^b(\text{coh} X)$ be a tilting object and $\Lambda = \text{End}_{D^b(\text{coh} X)}(V)$.

(a) $\text{gl.dim} \Lambda \geq d$ holds.

(b) If $V \simeq \pi(U)$ for $U \in (\text{mod}^d R)^{1,0,1}$, then $\Pi(\Lambda) \simeq \text{End}_{R^Z(\pi \omega)}(U)$ as $Z$-graded $k$-algebras.

Assume that $V$ is a $d$-tilting object. Then:

(c) $\text{Hom}_X(V, V(\ell \omega)) = 0$ for any $\ell > 0$.

(d) $X$ is Fano.

(e) [BuH] If $V$ belongs to $\text{coh} X$, then $\Lambda$ is $d$-representation infinite.

First we prepare the following general observation.

Lemma 7.12. Let $p := \text{l.c.m.}(p_1, \ldots, p_n)$. If there exists a non-zero object $X \in D^b(\text{coh} X)$ satisfying $\text{Hom}_{D^b(\text{coh} X)}(X, X(p \omega)) = 0$, then $X$ is Fano.
Moreover, if Λ is Fano, then pd,q = q e holds for some q ≥ 0. Consider a morphism
\( f_\ell := (t_\ell) : \bigoplus_\ell X \to X(q_\ell) = X(pd_\ell) \) where \( t_\ell \) runs over all monomials on \( T_0, \ldots, T_d \) of degree q. For any \( i \leq 0 \), the morphism \( H^i(f_\ell) : \bigoplus_\ell H^i(X) \to H^i(X(q_\ell)) \) is an epimorphism in \( \text{coh} \mathbb{X} \) by Lemma 5.20(b). In particular, \( f_\ell \) is non-zero in \( D^b({\text{coh}} \mathbb{X}) \), a contradiction. Thus \( X \) is Fano.

Now we are ready to prove Proposition 7.11.

Proof of Proposition 7.11
(a) Assume \(\text{gl.dim } \Lambda < d\). Then for any \( X, Y \in D^b(\text{mod } \Lambda) \), we have \( \text{Hom}_{D^b(\text{mod } \Lambda)}(X, \nu_d^\ell(Y)) = 0 \) for almost all \( \ell \in \mathbb{Z} \). On the other hand, by 7.14 and Lemma 5.6(b), we have

\[ \text{Hom}_{D^b(\text{mod } \Lambda)}(\Lambda, \nu_d^\ell(\Lambda)) \cong \text{Hom}_{D^b(\text{coh } \mathbb{X})}(V, V(\ell \vec{\omega})) \neq 0 \]

for infinitely many \( \ell \in \mathbb{Z} \), a contradiction.

(b) By 7.14 and Lemma 5.24 we have

\[ \Pi(\ell) = \text{Hom}_{D^b(\text{mod } \Lambda)}(\Lambda, \nu_d^{-\ell}(\Lambda)) \cong \text{Hom}_{D^b(\text{coh } \mathbb{X})}(V, V(-\ell \vec{\omega})) = \text{Hom}_{R^I}(U, U(-\ell \vec{\omega})) \]

for any \( \ell \in \mathbb{Z} \). Therefore we have \( \Pi(\ell) = \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{R^I}(U, U(-\ell \vec{\omega})) = \text{End}_{R^I}^{1/2 \vec{\omega}}(U) \).

(c) Since \( \Lambda \) has global dimension at most \( d \), we have \( \nu_d^{\ell-1}(DA) \in D^{\geq 0}(\text{mod } \Lambda) \) for any \( \ell > 0 \) by Proposition 2.7. Hence the commutative diagram 7.11 shows

\[ \text{Hom}_{\mathbb{X}}(V, V(\ell \vec{\omega})) \cong \text{Hom}_{D^b(\text{mod } \Lambda)}(\Lambda, \nu_d^\ell(\Lambda)) = H^0(\nu_d^{\ell-1}(DA)[-d]) = 0. \]

(d) The assertion follows from (c) and Lemma 7.12.

(e) For a convenience of the reader, we include a proof. We only have to show that \( H^i(\nu_d^{-\ell}(\Lambda)) = 0 \) holds for any \( i \neq 0 \) and \( \ell \geq 0 \). This is clear for \( i < 0 \) since \( V \in \text{coh } \mathbb{X} \) implies

\[ H^i(\nu_d^{-\ell}(\Lambda)) = \text{Hom}_{D^b(\text{mod } \Lambda)}(\Lambda, \nu_d^{-\ell}(\Lambda)[i]) = \text{Hom}_{D^b(\text{coh } \mathbb{X})}(V, V(-\ell \vec{\omega})[i]) = 0 \]

by 7.11. On the other hand, for \( \ell \geq 0 \), since \( \nu_d^\ell(\Lambda) \in D^{\leq 0}(\text{mod } \Lambda) \) holds by Proposition 2.7, we have \( H^i(\nu_d^{-\ell}(\Lambda)) = 0 \) for any \( i > 0 \). Thus the assertion follows.

The following plays a key role in relating \( d \)-VB finiteness of \( \mathbb{X} \) to the existence of \( d \)-tilting bundle on \( \mathbb{X} \).

Definition 7.13. Let \( \mathcal{U} \) be a \( d \)-cluster tilting subcategory of \( \text{vect } \mathbb{X} \) (respectively, \( \text{CM}^R_{\mathbb{X}} \), \( \text{CM}^R_{\mathbb{X}} \)). We call an object \( V \in \mathcal{U} \) slice if the following conditions are satisfied.

(a) For any indecomposable object \( X \in \mathcal{U} \), there exists \( \ell \in \mathbb{Z} \) such that \( X(\ell \vec{\omega}) \in \text{add } V \).

(b) \( \text{Hom}_{\mathcal{U}}(V, V(\ell \vec{\omega})) = 0 \) for any \( \ell > 0 \).

In this case, any \( \langle \vec{\omega} \rangle \)-orbit of indecomposable objects in \( \mathcal{U} \) contains exactly one element in \( \text{add } V \).

The following is a main result in this section.

Theorem 7.14. (a) (tilting-cluster tilting correspondence) \( d \)-tilting bundles on \( \mathbb{X} \) are precisely slices in \( d \)-cluster tilting subcategories of \( \text{vect } \mathbb{X} \).

(b) If \( \mathbb{X} \) has a \( d \)-tilting bundle \( V \), then \( \mathbb{X} \) is Fano, \( d \)-VB finite, and derived equivalent to \( \text{End}_{\mathbb{X}}(V) \), which is a \( d \)-representation infinite algebra.

Proof of Theorem 7.14. Part 1. Let \( V \) be a \( d \)-tilting bundle of \( \mathbb{X} \), and \( \Lambda := \text{End}_{\mathbb{X}}(V) \). Then we have a derived equivalence \( D^b(\text{coh } \mathbb{X}) \to D^b(\text{mod } \Lambda) \) which makes the diagram 7.11 commutative. Moreover, \( \Lambda \) is \( d \)-representation infinite by Proposition 7.11(e). In particular, Theorem 2.11 shows that

\[ V_\Lambda := \{ X \in D^b(\text{mod } \Lambda) \mid \forall \ell \gg 0, \nu_d^{-\ell}(X) \in \text{mod } \Lambda, \nu_d^\ell(X) \in (\text{mod } \Lambda)[-d] \}. \]

has a \( d \)-cluster tilting subcategory

\[ \mathcal{U}_\Lambda := \text{add } \nu_d^\ell(\Lambda) \mid \ell \in \mathbb{Z} \].
On the other hand, X is Fano by Proposition 7.11(d). By Proposition 7.8(b), the equivalence $\text{D}^b(\text{mod} \Lambda) \rightarrow \text{D}^b(\text{coh} X)$ restricts to an equivalence

\[ \nu_\Lambda \rightarrow \text{vect} X, \]

\[ \mathcal{U}_\Lambda \rightarrow \mathcal{U} := \text{add}\{V(\ell \bar{\omega}) \mid \ell \in \mathbb{Z}\}. \]

Therefore $\mathcal{U}$ is a $d$-cluster tilting subcategory of $\text{vect} X$, and we have that $X$ is $d$-VB finite.

It follows from Proposition 7.11(c) that $V$ is a slice in $\mathcal{U}$. We have shown that the statement (b) holds.

**Proof of Theorem 7.14** Part II. Let $V$ be a slice in a $d$-cluster tilting subcategory $\mathcal{U}$ of $\text{vect} X$ and $\Lambda := \text{End}_X(V)$. Since $\text{Hom}_X(V, V(\ell \bar{\omega})) = 0$ holds for any $\ell > 0$ by our assumption, $X$ is Fano by Lemma 7.12. Now we show that $V$ satisfies one of the conditions for being a tilting object.

**Lemma 7.15.** $\text{Ext}^2_\Lambda(V, V) = 0$ holds for any $i \neq 0$.

**Proof.** Since $V$ is an object in a $d$-cluster tilting subcategory $\mathcal{U}$, we have $\text{Ext}^2_\Lambda(V, V) = 0$ for any $i$ with $1 \leq i \leq d - 1$. On the other hand, by Auslander-Reiten-Serre duality, we have $\text{Ext}^2_\Lambda(V, V) = D \text{Hom}_X(V, V(\bar{\omega}))$, which is zero since $V$ is a slice. Thus the assertion follows.

Next we show the following easy properties of a slice.

**Lemma 7.16.** Let $X$ and $Y$ be indecomposable objects in $\mathcal{U}$.

(a) There exists a unique integer $\ell = \ell(X)$ satisfying $X \in \text{add} V(\ell)$.

(b) If there exist indecomposable objects $X_i \in \mathcal{U}$ for $1 \leq i \leq m$ with $X = X_1$ and $Y = X_m$ and non-zero morphisms $f_i : X_i \rightarrow X_{i+1}$ in $\mathcal{U}$ for $1 \leq i \leq m - 1$, then $\ell(X) \geq \ell(Y)$.

(c) If $\ell(X) \geq 0$, then the $\Lambda$-module $\text{Hom}_X(V, X)$ is projective.

(d) Let

\[ 0 \rightarrow X(\bar{\omega}) \rightarrow C_{d-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow X \rightarrow 0 \]

be a $d$-almost split sequence from Theorem 5.17(c). If $Y$ is a direct summand of $C_i$ for some $0 \leq i \leq d - 1$. Then either $\ell(Y) = \ell(X)$ or $\ell(Y) = \ell(X) + 1$ holds.

**Proof.** (a)(b) Both assertions are clear from our definition of slice.

(c) If $\ell(X) > 0$, then $\text{Hom}_X(V, X) = 0$. If $\ell(X) = 0$, then the assertion holds since $X \in \text{add} V$.

(d) Since there exists a chain of non-zero morphisms from $Y$ to $X$, we have $\ell(Y) \geq \ell(X)$ by (b). Since there exists a chain of non-zero morphisms from $X(\bar{\omega})$ to $Y$, we have $\ell(X) + 1 \geq \ell(Y)$ by (b). Thus the assertion follows.

Now we are ready to prove the following observation.

**Proposition 7.17.** $\Lambda$ has global dimension $d$.

**Proof.** It suffices to show that any simple $\Lambda$-module $S$ has projective dimension at most $d$. We regard $S$ as an $\mathcal{U}$-module naturally. Then there exists an indecomposable object $X \in \text{add} V$ such that $S = \text{Hom}_\mathcal{U}(\bar{\omega}, X)/\text{rad}_\mathcal{U}(\bar{\omega}, X)$. Let

\[ 0 \rightarrow X(\bar{\omega}) \rightarrow C_d \rightarrow \cdots \rightarrow C_1 \rightarrow X \rightarrow 0 \]

be a $d$-almost split sequence in $\mathcal{U}$ given in Theorem 5.17(c). Since $V$ is a slice and $X \in \text{add} V$, we have $\text{Hom}_\mathcal{U}(V, X(\bar{\omega})) = 0$. Hence we have an exact sequence

\[ 0 \rightarrow \text{Hom}_\mathcal{U}(V, C_d) \rightarrow \cdots \rightarrow \text{Hom}_\mathcal{U}(V, C_1) \rightarrow \text{Hom}_\mathcal{U}(V, X) \rightarrow S \rightarrow 0 \]

(7.3)

of $\Lambda$-modules. On the other hand, let $Y$ be an indecomposable summand of $C_1$. Then $\ell(Y) \geq 0$ holds by Lemma 7.10(d). Hence $\text{Hom}_\mathcal{U}(V, Y)$ is a projective $\Lambda$-module by Lemma 7.10(c). Therefore the sequence (7.3) gives a projective resolution of the simple $\Lambda$-module $S$, and we have the assertion.

We also need the following observation.
Lemma 7.18. There exist exact sequences
\[ 0 \to U_d \to \cdots \to U_0 \to V(-\omega) \to 0 \quad \text{and} \quad 0 \to V(\omega) \to U^0 \to \cdots \to U^d \to 0 \]
in \( \text{coh} \mathbb{X} \) with \( U_i, U^i \in \text{add} V \) for any \( 0 \leq i \leq d \).

Proof. We only construct the first sequence since the second one can be constructed in a similar way. Let \( \mathcal{U}^+ := \text{add} \{ V(\ell \omega) \mid \ell \geq 0 \} \). This is a functorially finite subcategory of \( \text{vect} \mathbb{X} \) by Lemmas 4.28(b) and 5.7. Let \( f : U_0 \to V(-\omega) \) be a right \( \mathcal{U}^+ \)-approximation of \( V(-\omega) \). Then \( f \) is surjective by Theorem 7.2. Since \( \mathcal{U} \) is a \( d \)-cluster tilting subcategory of \( \text{vect} \mathbb{X} \), there exists an exact sequence
\[ 0 \to U_d \to \cdots \to U_1 \to \text{Ker} f \to 0 \]
in \( \text{vect} \mathbb{X} \) with \( U_i \in \mathcal{U} \) by Theorem 5.17(b). It suffices to show that \( U_i \) belongs to \( \text{add} V \) for any \( i \) with \( 0 \leq i \leq d \). By Lemma 7.16(b), we have
\[ U_i \in \mathcal{U}^+ \quad (7.4) \]
for any \( i \) with \( 0 \leq i \leq d \). Now we define an \( \mathcal{U} \)-module \( F \) by an exact sequence
\[ \text{Hom}_\mathcal{U}(\mathcal{U}, U_0) \xrightarrow{f} \text{Hom}_\mathcal{U}(\mathcal{U}, V(\omega)) \to F \to 0. \]
Since \( V \) is a slice, we have \( F(V(\ell \omega)) = 0 \) for any \( \ell < -1 \). Since \( f \) is a right \( \mathcal{U}^+ \)-approximation, we have \( F(\mathcal{U}^+) = 0 \). Hence the support of \( F \) is contained in \( \text{add} V \), and therefore \( F \) has a finite length as an \( \mathcal{U} \)-module. In particular, \( F \) has a finite filtration by simple \( \mathcal{U} \)-modules of the form \( S_X := \text{Hom}_\mathcal{U}(\mathcal{U}, X) / \text{rad}_\mathcal{U}(\mathcal{U}, X) \) for indecomposable direct summands \( X \) of \( V(\omega) \).

On the other hand, a minimal projective resolution of \( S_X \) is given by a \( d \)-almost split sequence whose terms belong to \( \text{add}(V(\omega) \oplus V) \) by Lemma 7.16(d). Applying Horseshoe Lemma repeatedly, we have that each \( U_i \) belongs to \( \text{add}(V(\omega) \oplus V) \). By (7.4), we have that \( U_i \in \mathcal{U}^+ \cap \text{add}(V(\omega) \oplus V) = \text{add} V \) for any \( i \) with \( 0 \leq i \leq d \). Thus the assertion follows.

Now we show that \( V \) satisfies the remaining condition for being a tilting object.

Lemma 7.19. We have thick \( V = \mathcal{D}^b(\text{coh} \mathbb{X}) \).

Proof. Using Lemma 7.18 repeatedly, we have \( V(\ell \omega) \in \text{thick} V \) for any \( \ell \in \mathbb{Z} \). Thus \( \mathcal{U} \subset \text{thick} V \). Since \( \mathcal{U} \) is a \( d \)-cluster tilting subcategory of \( \text{vect} \mathbb{X} \), we have \( \text{vect} \mathbb{X} \subset \text{thick} \mathcal{U} \subset \text{thick} V \) by Theorem 5.17(b). Therefore \( \mathcal{D}^b(\text{coh} \mathbb{X}) = \text{thick} V \) holds by Theorem 5.4(e). Consequently \( V \) is a \( d \)-tilting bundle on \( \mathbb{X} \). We have finished proving Theorem 7.14.

Now we consider a special class of \( d \)-tilting bundles which are contained in \( \text{CM}^{d+} R \). In this case, Theorem 7.14 gives the following result.

Corollary 7.20. (a) \( d \)-tilting bundles on \( \mathbb{X} \) contained in \( \text{CM}^{d+} R \) are precisely slices in \( d \)-cluster tilting subcategories of \( \text{CM}^{d+} R \).

(b) Let \( V \in \text{CM}^{d+} R \) be a \( d \)-tilting bundle on \( \mathbb{X} \) and \( \Lambda := \text{End}_\mathbb{X}(V) \). Then \( V \) is a \( d \)-cluster tilting object in \( \text{CM}^{d+} \mathbb{Z} \), and the preprojective algebra \( \Pi(\Lambda) \) is isomorphic to the corresponding \( d \)-Auslander algebra \( \text{End}^{d+}_{\text{CM}^{d+}}(V) \).

Proof. (a) This is immediate from Theorems 5.18 and 7.14.

(b) This is immediate from Proposition 7.11(b).

Example 7.21. Let \( \mathbb{X} \) be a GL projective space with \( d = 1 \) which is Fano. Then there exists a tilting bundle \( V \) on \( \mathbb{X} \) such that \( \text{End}_\mathbb{X}(V) \) is isomorphic to the path algebra \( kQ \) of an extended Dynkin quiver \( Q \), and \( \Pi(kQ) \) is the corresponding classical preprojective algebra. On the other hand, \( (R, L) \) is (1-)CM finite and we have \( \text{CM}^{-} R = \text{add} \{ V(\ell \omega) \mid \ell \in \mathbb{Z} \} \). It is classical that the (1-)Auslander algebra of \( (R, L) \) is isomorphic to the preprojective algebra \( \Pi(kQ) \).
Remark 7.22. The following diagram shows connections between different kinds of slices.
\[
\begin{align*}
\{ \text{d-tilting bundles on } \mathcal{X} \} & \cup \{ \text{slices in d-cluster tilting subcat. of } \text{vect } \mathcal{X} \} \\
\{ \text{d-tilting bundles on } \mathcal{X} \text{ contained in } \text{CM}^L R \} & \cup \{ \text{slices in d-cluster tilting subcat. of } \text{CM}^L R \} \\
\{ \text{d-tilting objects in } \mathcal{CM}^L R \} & \subset \{ \text{slices in d-cluster tilting subcat. of } \text{CM}^L R \}
\end{align*}
\]
We do not know if the bottom inclusion \( \subset \) is strict or not. Also we do not know if the right map \( \downarrow \) is surjective or not.

Now we pose the following.

Conjecture 7.23. The following conditions are equivalent.

(a) \( \mathcal{X} \) is Fano.
(b) \( \mathcal{X} \) is d-VB finite.
(c) \( \mathcal{X} \) has a d-tilting bundle.
(d) \( \mathcal{X} \) is derived equivalent to a d-representation infinite algebra.

The statement (c) \( \Rightarrow \) (a)(b)(d) was shown in Theorem 7.14.

We give the following partial answer.

Theorem 7.24. Let \( \mathcal{X} \) be a GL projective space with \( n = d + 2 \) and \( p_1 = p_2 = 2 \). Then \( \mathcal{X} \) has a d-tilting bundle
\[
V := \pi(U_{\text{CM}}) \oplus (\oplus_{\mathcal{L} \in S} \mathcal{O}(\mathcal{L})),
\]
where \( U_{\text{CM}} \in \text{CM}^L R \) is given in Theorem 4.38 and \( S \) is the subset of \( \mathcal{L} \) given by
\[
S := \left\{ \bigcup_{i=1,2} \left[ \left\{ \left[ -\frac{d+1}{d+2} \mathcal{X} + \mathcal{X} \right] \bigcup \left[ -\frac{d+2}{d+1} \mathcal{X} + \mathcal{X} \right] \right\} \bigcup \left[ -\frac{d+2}{d+1} \mathcal{X} + \mathcal{X} \right] \right\} \right\}.
\]
In particular, \( \mathcal{X} \) is d-VB finite and derived equivalent to a d-representation infinite algebra.

Proof. Since \( p_1 = p_2 = 2 \), the algebra \( \text{End}_{\mathcal{L}}(U_{\text{CM}}) = \bigotimes_{\ell=1}^n kA_{p_\ell - 1} \) has global dimension at most \( n - 2 = d \), Theorem 4.38 shows that \( \text{CM}^L R \) has a d-cluster tilting subcategory
\[
\mathcal{U} := \text{add}\{U_{\text{CM}}(\ell\mathcal{X}), R(\mathcal{X}) | \ell \in \mathbb{Z}, \mathcal{X} \in \mathcal{L} \}.
\]
By Theorem 7.14 it suffices to show that \( V \) is a slice in \( \mathcal{U} \). We start with proving the following observations.

Lemma 7.25. (a) \( S \) is a complete set of representatives of \( \mathcal{L}/\mathbb{Z}\mathcal{X} \).
(b) We have \( \text{Hom}_{\mathcal{L}}(\mathcal{O}(\mathcal{X}), \mathcal{O}(\mathcal{Y} + \ell\mathcal{X})) = 0 \) for any \( \mathcal{X}, \mathcal{Y} \in S \) and \( \ell > 0 \).
(c) For any \( i \) with \( 1 \leq i \leq n \), let \( \ell_i \) be an integer satisfying \( 1 \leq \ell_i \leq p_\ell - 1 \). Then \( \frac{1}{2}\sum_{i=1}^{\ell_i} \ell_i \mathcal{X} \) (respectively, \( \frac{1}{2}\sum_{i=1}^{\ell_i} \ell_i \mathcal{X} \) \( - \sum_{i=1}^{\ell_i} \ell_i \mathcal{X} \)) belongs to \( S \) for any even (respectively, odd) subset \( I \) of \( \{1, \ldots, n\} \) which does not contain \( n \).

Proof. (a) Consider the abelian group
\[
\mathcal{L} := \langle \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n \rangle/\langle 2\mathcal{Y}_i - p_i\mathcal{Y}_i | 3 \leq i \leq n \rangle.
\]
Then we have an exact sequence
\[
0 \to \langle \mathcal{X}_1 - \mathcal{X}_2 \rangle \to \mathcal{L} \xrightarrow{q} \mathcal{L} \to 0
\]
given by \( q(\mathcal{X}_i) := \mathcal{Y}_i \), where \( \langle \mathcal{X}_1 - \mathcal{X}_2 \rangle \) is a subgroup of \( \mathcal{L} \) of order 2 by \( p_1 = p_2 = 2 \).

Let \( \mathcal{L} \) be a submonoid of \( \mathcal{L} \) generated by all \( \mathcal{Y}_i \)'s. We regard \( \mathcal{L} \) as a partially ordered set: \( \mathcal{X} \leq \mathcal{Y} \) if and only if \( \mathcal{Y} - \mathcal{X} \in \mathcal{L} \). The map \( q : \mathcal{L} \to \mathcal{L} \) is a morphism of partially ordered sets. Then \( q^{-1}(\mathcal{X}) \) consists of two elements which give the lower (respectively, upper) bounds of the intervals defining \( S \). On the other hand, for \( \mathcal{X}, \mathcal{Y} \in \mathcal{L} \), it is easy to check that \( q(\mathcal{X}) \leq q(\mathcal{Y}) \) holds if and only if either \( \mathcal{X} \leq \mathcal{Y} \) or \( \mathcal{X} \leq \mathcal{Y} + \mathcal{X} - 2 \mathcal{X} \) holds. These observations imply
\[
S = q^{-1}(\langle 1 - \mathcal{Y} \rangle, d\mathcal{Y}_1 \rangle).
\]
By using a similar argument as in the proof of Proposition 5.12(a), one can easily check that the interval \([0, (2d - 1)\bar{y}_1]\) in \(\mathbb{L}\) gives a complete set of representatives of \(\mathbb{L}/\mathbb{Z}\bar{v}\) for
\[
\bar{v} := -\sum_{i=3}^n \bar{y}_i.
\]
Shifting by \((1 - d)\bar{y}_1\), the interval \([(1 - d)\bar{y}_1, d\bar{y}_1]\) also gives a complete set of representatives of \(\mathbb{L}/\mathbb{Z}\bar{v}\). Since \(q(\bar{w}) = \bar{v}\) holds, we have an exact sequence
\[
0 \to \langle \bar{x}_1 - \bar{x}_2 \rangle \to \mathbb{L}/\mathbb{Z}\bar{w} \xrightarrow{\bar{y}} \mathbb{L}/\mathbb{Z}\bar{v} \to 0.
\]
Therefore \((\ref{7.25})\) implies that \(S\) gives a complete set of representatives of \(\mathbb{L}/\mathbb{Z}\bar{w}\).

(b) We have \(\text{Hom}_X(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y} + \ell\bar{w})) = R_{\bar{y}, \ell, \bar{w}, \bar{x}}\) by Proposition 5.3. If \(\bar{y} + \ell\bar{w} - \bar{x} \geq 0\) for some \(\ell > 0\), then we have
\[
0 \leq q(\bar{y} + \ell\bar{w} - \bar{x}) = d\bar{y}_1 + \ell\bar{v} - (1 - d)\bar{y}_1 = -\bar{y}_1 + \sum_{i=3}^n (p_i - \ell)\bar{y}_i,
\]
in \(\mathbb{L}\), a contradiction. Thus \(R_{\bar{y}, \ell, \bar{w}, \bar{x}} = 0\) holds by Observation 3.1(c).

(c) Let \(J := \{1, 2\}\). Then \(|J| \leq d - 1\) holds since \(n \notin I\). Since \(\ell_1 = \ell_2 = 1\), we have
\[
q \left( \frac{|I| + a}{2} \bar{v} - \sum_{i \in I} \ell_i \bar{x}_i \right) = (|I| + a)\bar{y}_1 - \sum_{i \in I} \ell_i \bar{y}_i = (|J| + a)\bar{y}_1 - \sum_{i \in J} \ell_i \bar{x}_i,
\]
where \(a = 0\) if \(I\) is even, and \(a = 1\) if \(I\) is odd. Therefore it is enough to show
\[
(1 - d)\bar{y}_1 \leq (|J| + a)\bar{y}_1 - \sum_{i \in J} \ell_i \bar{y}_i \leq d\bar{y}_1.
\]
The right inequality is clear since \(|J| \leq d - 1\). The left inequality is equivalent to \((|J| - a + 1 - d)\bar{y}_1 \leq \sum_{i \in J} (p_i - \ell_i)\bar{y}_i\), which holds since \(|J| \leq d - 1\).

Lemma 7.26. There exist exact sequences
\[
L \to \pi(U^{\text{CM}}) \to C \to 0 \quad \text{and} \quad 0 \to \pi(U^{\text{CM}}) \to L'
\]
in \(\text{coh} \mathbb{X}\) such that \(L, L' \in \text{add} \bigoplus_{E \in S} \mathcal{O}(\bar{x})\) and \(\text{Hom}_X(C, \text{CM}L) = 0\).

Proof. We use the notations in Theorem 4.35. Recall that \(U^{\text{CM}} = \bigoplus_{E \in \mathcal{E} + [0, \bar{d}]} E^\bar{x}\) holds, where \(E^\bar{x}\) is the image of the morphism \(N^\bar{x} : P^\bar{x,0} \to P^\bar{x,1}\). It suffices to construct
- a morphism \(L \to \pi(E^\bar{x})\) in \(\text{coh} \mathbb{X}\) whose cokernel has rank 0,
- a monomorphism \(\pi(E^\bar{x}) \to L'\) in \(\text{coh} \mathbb{X}\)

with \(L, L' \in \text{add} \bigoplus_{E \in S} \mathcal{O}(\bar{x})\).

Let \(\text{add} \bigoplus_{E \in S} \mathcal{O}(\bar{x})\), the compositions
\[
L := \pi(P^\bar{x,0}) \to \pi(P^\bar{x,0}) \to \pi(E^\bar{x}) \quad \text{and} \quad \pi(E^\bar{x}) \to \pi(P^\bar{x,1}) \to L' := \pi(P^\bar{x,1})
\]
satisfy the desired conditions since \(\pi(P^\bar{x,1}) \in \text{add} \bigoplus_{E \in S} \mathcal{O}(\bar{x})\) holds by Lemma 7.25(c).

Now we are ready to prove Theorem 7.24. It is enough to show that \(V = \pi(U^{\text{CM}}) \oplus (\bigoplus_{E \in S} \mathcal{O}(\bar{x}))\) is a slice in \(\mathcal{U}\). By Lemma 7.25(a), we have \(\mathcal{U} = \text{add} \{V(\bar{w}) \mid \bar{w} \in \mathbb{Z}\}\).

It remains to show \(\text{Hom}_X(V, V(\bar{w})) = 0\) for any \(\ell > 0\). Assume that there exists a non-zero morphism \(V \to V(\bar{w})\) for some \(\ell > 0\). By Lemma 7.26 there exists a non-zero morphism \(L \to L'(\bar{w})\) for some \(L, L' \in \text{add} \bigoplus_{E \in S} \mathcal{O}(\bar{x})\). This contradicts Lemma 7.25(b).
Example 7.27. Let $X$ be a GL projective space with $d = 2$ and $n = 4$ weights $(2, 2, 3, 4)$. Then $V$ given in Theorem 7.24 is the following, where $\vec{t} := \vec{x}_1 - \vec{x}_2$.
[IY] O. Iyama, Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, Invent. Math. 172 (2008), no. 1, 117–168.

[J] H. Jacobinski, Sur les ordres commutatifs avec un nombre fini de sous-modules indecomposables, Acta Math. 118 1967 1–31.

[KST1] H. Kajiura, K. Saito, A. Takahashi, Matrix factorization and representations of quivers. II. Type ADE case, Adv. Math. 211 (2007), no. 1, 327–362.

[KST2] H. Kajiura, K. Saito, A. Takahashi, Triangulated categories of matrix factorizations for regular systems of weights with \( e = -1 \), Adv. Math. 220 (2009), no. 5, 1602–1654.

[Ke] B. Keller, Deriving DG categories, Ann. Sci. Ecole Norm. Sup. (4) 27 (1994), no. 1, 63–102.

[KM] V. Khoroshkin, S. Marmaridis, On two examples by Iyama and Yoshino, Compos. Math. 147 (2011), no. 2, 591–612.

[Kn] H. Knörrer, Cohen-Macaulay modules on hypersurface singularities. I, Invent. Math. 88 (1987), no. 1, 153–164.

[Ke] B. Keller, Deriving DG categories, Ann. Sci. Ecole Norm. Sup. (4) 27 (1994), no. 1, 63–102.

[KMV] B. Keller, D. Murfet, M. Van den Bergh, On two examples by Iyama and Yoshino, Compos. Math. 147 (2011), no. 2, 591–612.

[Kn] H. Knörrer, Cohen-Macaulay modules on hypersurface singularities. I, Invent. Math. 88 (1987), no. 1, 153–164.

[Ku] D. Kussin, Noncommutative curves of genus zero: related to finite dimensional algebras, Mem. Amer. Math. Soc. 201 (2009), no. 942.

[KLM] D. Kussin, H. Lenzing, H. Meltzer, Triangle singularities, ADE-chains, and weighted projective lines, Adv. Math. 237 (2013), 194–251.

[L1] H. Lenzing, Rings of singularities, Advanced School and Conference on Homological and Geometrical Methods in Representation Theory.

[L2] H. Lenzing, Weighted projective lines and applications, Representations of algebras and related topics, 153–187, EMS Ser. Congr. Rep., Eur. Math. Soc., Zurich, 2011.

[LO] B. Lerner, S. Oppermann, A recollement approach to Geigle-Lenzing weighted projective varieties, in preparation.

[LW] G. Leuschke, R. Wiegand, Cohen-Macaulay representations, Mathematical Surveys and Monographs, 181. American Mathematical Society, Providence, RI, 2012.

[Me] H. Meltzer, Exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines, Mem. Amer. Math. Soc. 171 (2004), no. 808.

[Min] H. Minamoto, Ampleness of two-sided tilting complexes, Int. Math. Res. Not. IMRN 2012, no. 1, 67–101.

[MM] H. Minamoto, I. Mori, Structures of AS-regular algebras, Adv. Math. 226 (2011) 4061–4095.

[Miy] J. Miyachi, Localization of triangulated categories and derived categories, J. Algebra 141 (1991), no. 2, 463–483.

[MY] J. Miyachi, A. Yekutieli, Derived Picard groups of finite-dimensional hereditary algebras, Compositio Math. 129 (2001), no. 3, 341–368.

[NV] C. Nastasescu, F. van Oystaeyen, Graded ring theory, North-Holland Mathematical Library, 28. North-Holland Publishing Co., Amsterdam-New York, 1982.

[O1] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 503–531, Progr. Math., 270, Birkhauser Boston, Inc., Boston, MA, 2009.

[O2] D. Orlov, Formal completions and idempotent completions of triangulated categories of singularities, Adv. Math. 226 (2011), no. 1, 206–217.

[P] N. Popescu, Abelian categories with applications to rings and modules, London Mathematical Society Monographs, No. 3. London-New York: Academic Press, 1973.

[Q] R. Quirós, Derived category and stable equivalence, J. Pure Appl. Algebra 61 (1989), no. 3, 303–317.

[Rin] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Mathematics, 1099, Springer-Verlag, Berlin, 1984.

[S] H. Schoutens, Projective dimension and the singular locus, Comm. Algebra 31 (2003), no. 1, 217–239.

[Tak1] R. Takahashi, Classifying thick subcategories of the stable category of Cohen-Macaulay modules, Adv. Math. 225 (2010), no. 4, 2076–2116.

[Tak2] R. Takahashi, Reconstruction from Koszul homology and applications to module and derived categories, to appear in Pacific J. Math. [arXiv:1210.1982]

[Tat] J. Tate, Homology of Noetherian rings and local rings, Illinois J. Math. 1 (1957), 14–27.

[Y] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990.
