Smooth bang-bang shortcuts to adiabaticity for atomic transport in a moving harmonic trap

Yongcheng Ding,1,2 Tangyou Huang,1 Minjia Hao,1 and Xi Chen1,2,*

1International Center of Quantum Artificial Intelligence for Science and Technology (QuArtist) and Department of Physics, Shanghai University, 200444 Shanghai, China
2Department of Physical Chemistry, University of the Basque Country UPV/EHU, Apartado 644, 48080 Bilbao, Spain

(Dated: February 27, 2020)

Bang-bang optimization, as a shortcut to adiabaticity, provides a simple but fast protocol allowing high-fidelity shuttling of cold atoms or trapped ions, with the wide applications in interferometry, metrology, and quantum information processing. Such time-optimal control with at least two discontinuous switches requires the sudden jumps, which costs extra efforts and harms the overall performance as well. To circumvent these problems, we investigate the smooth bang-bang protocols with near-minimal time for fast atomic transport in a moving harmonic trap. Smoothening is accomplished by the Pontryagins maximal principle with the constraint that limits the first and second derivatives of control input. Moreover, the multiple shooting method is numerically presented for a smoothing procedure for the minimal-time optimization with the same constraints. By numerical examples and comparisons, we conclude that the smooth control input is capable of eliminating the energy excitation and sloshing amplitude at the cost of a slight increase in minimal time. Our smooth bang-bang protocols presented here are more practical and feasible in the relevant experiments, and can be easily extended to other classical and quantum systems where the shortcuts to adiabaticity are requested and implemented.

I. INTRODUCTION

In recent years, efficient transporting or launching of trapped ions, ultracold atoms and even quantum degenerate gases with moving macroscopic traps or chip traps are quite demanding in plenty of applications ranging from atom interferometry, metrology, holography, atom lithography to quantum information processing [1–18]. Currently, space-borne Bose-Einstein condensates (BECs) allows the reproducibility of more sophisticated protocols for transporting or shaping in low-gravity conditions [19, 20]. Motivated by this, more attention has been paid to fast and reliable transporting, by using shortcuts to adiabaticity supplemented by optimal control theory [21, 22] and machine learning [23, 24].

It goes beyond dispute that the typically adiabatic transport is the simplest way to transport the cold atoms or trapped ion without residual energy excitation [1–5], regardless of its requirements of sufficient slowness. However, this may be unacceptable for many reasons, e.g. if the repetition is requested for operation, or the accumulation of perturbation or noise will ruin the desired state. Nowadays, the techniques of “shortcuts to adiabaticity” (STA) [25–27], in addition to optimal control [9, 16] and Fourier transformation [8], are well developed, providing an alternative protocol for speeding up adiabatic progress. Among them, there are fast-forwarding scaling [28, 29] and counter-diabatic driving [30, 31], accelerating the adiabatic transport by modifying the trap trajectory or adding the auxiliary interaction directly to compensate the inertial force. Moreover, the invariant-based inverse engineering and its extension [32–36] can work as versatile toolboxes for designing the optimally fast and robust shuttling of cold atoms or trapped ions, since a family of trap trajectories designed inversely has more freedom left, with the appropriate initial and final boundary conditions. Through the Pontryagins maximal principle or other methods of optimal control, the shortcut-to-adiabaticity protocols can be further optimized with respect to transfer time [37, 38], anharmonic effect [39–41], fluctuation or noise in the trap frequency and position [42–45]. However, the time-optimal solution requires the so-called “bang-bang” control, namely, the sudden jump of controller at switching times [37, 38]. This may ruin the overall performance of STA, since the abrupt transition of control input excites more dynamical modes. Therefore, more boundary conditions and physical constraints on trap velocity/acceleration are needed [46, 47], from the implementation point of view, to prevent the final residual energy during the transport.

With this motivation, we focus on the near-minimal-time problem on the fast atomic transport in a moving harmonic trap. The basic idea is to smooth bang-bang control, by solving the time-optimal transport with the constraints that bounds the first and second derivatives of control input. The analytical solutions are discussed and compared with the numerical methods of multiple shooting method [48]. We demonstrate that the energy excitation and sloshing amplitude can be significantly eliminated by smooth bang-bang control in the cost of minimal-time increase. The smoother control input makes the experiments more feasible, and improve the overall performance of STA. Finally, we shall emphasize that our results can be easily extended to other different scenarios [49–53], although the smooth bang-bang control is applied here to atomic transport on a heuristic basis.

II. HAMILTONIAN AND MODEL

We consider the time-dependent Hamiltonian, describing the transport of cold atoms or trapped ions in a rigid harmonic trap with center \( q_0(t) \equiv q_0 \) and trap frequency \( \omega_0 \), see Fig. 1, as follows,

\[
H(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 (\hat{q} - q_0(t))^2. \tag{1}
\]
where \( \hat{p} \) and \( \hat{q} \) are momentum and position operators, \( \omega_0 \) is the trap frequency. This is a good approximation of optical dipole potential implemented by a Gaussian beam in an realistic experiment, neglecting the anharmonic terms [41, 47]. This Hamiltonian possesses a quadratic-in-momentum Lewis-Riesenfeld invariant [32, 54–56],

\[
I(t) = \frac{1}{2m}(\hat{p} - m\hat{q})^2 + \frac{1}{2m\omega_0^2}[\hat{q} - q_c(t)]^2, \tag{2}
\]

accordingly, where \( q_c(t) \equiv q_c \) satisfies an ancillary equation

\[
\ddot{q}_c + \omega_0^2(q_c - q_0) = 0, \tag{3}
\]

to guarantee self-consistency because of the invariant condition

\[
\frac{dI(t)}{dt} = \frac{d\tilde{I}(t)}{dt} + \frac{1}{i\hbar}[I(t), H(t)] = 0. \tag{4}
\]

Apparently, Eq. (3) is nothing but the Newton’s equation for classical trajectory in the forced harmonic oscillator. As well, this is the center of transport modes described as

\[
\psi_n(q, t) = e^{\frac{i\omega_0}{\hbar}t} \phi_n(q - q_c), \tag{5}
\]

where \( \phi_n \) are the eigenstates of a static harmonic oscillator. The solution of time-dependent Schrödinger equation, 

\[
i\hbar \frac{d}{dt}\Psi(q, t) = H(t)\Psi(q, t),
\]

is constructed as the superposition of “transport modes”, \( \Psi(q, t) = \sum_n c_n \exp(i\alpha_n)\psi_n(q, t) \), where \( c_n \) are time-independent coefficients and \( \psi_n(q, t) \) are the eigenstates of dynamical invariant \( I(t) \) with the constant eigenvalues \( \lambda_n \), satisfying \( I(t)\psi_n(t) = \lambda_n\psi_n(t) \), and the Lewis-Riesenfeld phase \( \alpha_n \) are calculated as

\[
\alpha_n(t) = -\frac{1}{\hbar} \int_0^t [(n + \frac{1}{2})\hbar\omega_0 + \frac{1}{2}m\dot{q}_c^2] dt', \tag{6}
\]

It is noted that all transport modes are orthogonal to each other at any time, and all centered at \( q_c(t) \).

For the transport problem, the instantaneous energy, \( E(t) = \langle \Psi(t)|H(t)|\Psi(t) \rangle \), reads [37],

\[
E(t) = \hbar\omega_0(n + \frac{1}{2}) + \frac{m}{2}\dot{q}_c^2 + \frac{m}{2}\omega_0^2(q_c - q_0)^2, \tag{7}
\]

where the second and third terms are the form of kinetic energy and potential energy for a classical particle in a moving harmonic trap. As a consequence, we can finally write down the time-averaged potential energy,

\[
\overline{E_p} \equiv \frac{1}{t_f} \int_0^{t_f} E_p dt = \frac{1}{t_f} \int_0^{t_f} \frac{m}{2}\omega_0^2(q_c - q_0)^2 dt, \tag{8}
\]

to characterize the energy excitation during the process. In addition, we are also interested in the sloshing amplitude \( A \),

\[
A(t_f) = \left| \int_0^{t_f} q_0(t)e^{-i\omega_0 t'} dt' \right|, \tag{9}
\]

which is the Fourier component at the trap frequency of the trap velocity trajectory. The zero final sloshing can improve the performance of STA in current realistic experiment [47].

In order to design the shortcut trajectory of harmonic trap by using inverse engineering as usual, we suppose that the harmonic trap is moving from \( q_0(0) = 0 \) to \( q_0(t_f) = d \) at finite short time \( t_f \). To avoid the final energy excitation, the boundary conditions

\[
q_c(0) = 0; \quad \dot{q}_c(0) = 0; \quad \ddot{q}_c(0) = 0, \tag{10}
\]

\[
q_c(t_f) = d; \quad \dot{q}_c(t_f) = 0; \quad \ddot{q}_c(t_f) = 0, \tag{11}
\]

are imposed along with Eq. (3). In addition, more boundary conditions

\[
\dddot{q}_c(0) = 0; \quad \dddot{q}_c(t_f) = 0, \tag{12}
\]

are required, as suggested in Ref. [47], to eliminate the sloshing amplitude \( A(t_f) \), which encapsulates the energy in the transport mode. Here we use a simple polynomial ansatz to interpolate the center of transport mode,

\[
q_c(t) = d \left[ 35s^4 - 84s^3 + 70s^2 - 20s^3 \right], \tag{13}
\]

with \( s = t/t_f \). Once \( q_c(t) \) is fixed, satisfying the eight boundary conditions listed above, the trap trajectory \( q_0(t) \) can be inversely designed through Eq. (3). However, this trajectory is far from a time-optimal solution, which will be addressed below.

### III. SMOOTH BANG-BANG CONTROL WITH NEAR-MINIMAL-TIME

In this section, we shall exploit the Pontryagin’s maximal principle to solve the near-minimal-time transport by using smooth bang-bang control input. To solve the time-optimal control problem, we define the cost functional

\[
J_T = \int_0^{t_f} 1 dt, \tag{14}
\]

and a control Hamiltonian,

\[
H_c[p(t), x(t), u] = p_0 + p_T \cdot f[x(t), u], \tag{15}
\]

that represents the dynamic system \( \dot{x} = f[x(t), u] \), governed by Eq. (3). According to the Pontryagin’s maximal principle
[57], the extremal solutions of the problem satisfy the canonical equations
\[ \mathbf{x} = \frac{\partial H_c}{\partial \mathbf{p}}, \quad \mathbf{p} = -\frac{\partial H_c}{\partial \mathbf{x}}. \] (16)
All the components of join state vector \( \mathbf{p} \) are nonzero and continuous that \( H_c \) reaches its maximum value at \( u \equiv u(t) \) for almost all \( 0 \leq t \leq t_f \).

### A. bang-bang time-optimal control

We first review the time-optimal control with the bounded relative displacement for completeness before analyzing the smooth "bang-bang" control. By introducing a new notation,
\[ x_1 = q_c, \quad x_2 = \dot{q}_c, \quad u(t) = q_c - q_0, \] (17)
we reformulate Eq. (3) the state of the system in optimal control theory as follows,
\[ \dot{x}_1 = x_2, \] (18)
\[ \dot{x}_2 = -\omega_0^2 u(t), \] (19)
where \( x_{1,2} \) are the components of state vector \( \mathbf{x} \) and \( u(t) \) is the scalar control function. Due to the anharmonicity of traps [41], the relative displacement \( u(t) \) should be bounded by \( |u(t)| \leq \delta \). With this constraint, the time-optimal problem is to minimize a cost functional \( J_f \), see Eq. (14), under the constraint \( |u(t)| \leq \delta, \) with \( u(0) = 0 \) and \( u(t_f) = 0 \), such that the transport starts at \( x_1(0) = 0, \quad x_2(0) = 0 \), ends at \( x_1(t_f) = \delta, \quad x_2(t_f) = 0 \). In this case, the control Hamiltonian (15) can be written as
\[ H_c(\mathbf{p}, \mathbf{x}, u, \dot{u}) = p_0 + p_1 x_2 - p_2 \omega_0^2 u(t), \] (20)
and the canonical equation (16) is translated into a set of costate equations
\[ \dot{p}_1 = 0, \] (21)
\[ \dot{p}_2 = -p_1. \] (22)
Once we solve the costate functions mentioned above, the time-optimal control function \( u(t) \) of bang-bang type is solved as
\[ u(t) = \begin{cases} 0, & 0 \leq t \leq t_1 \\ -\delta, & t_1 < t < t_f \\ 0, & t_f \leq t \end{cases} \] (23)
where the minimal time is obtained as
\[ t_f^{\text{min}} = \frac{2}{\omega_0} \sqrt{\frac{d}{\delta}}, \] (24)
and the switching time \( t_1 = t_f/2 \). Fig. 2 illustrates the function of controller \( u(t) \) and trajectories of mass center of atoms \( q_c(t) \) and trap center \( q_0(t) \), where the parameters are chosen to adapt to the transport experiment [8], such as \( \omega_0 = 2\pi \times 20 \text{ Hz} \), the transport distance \( d = 1 \times 10^{-2} \text{ m} \), \( m = 1.44269 \times 10^{-25} \text{ kg} \), the mass of \( ^{87}\text{Rb} \) atoms. Here the constraint on relative displacement \( \delta/d = 0.1 \) is fixed, therefore the minimal time \( t_f^{\text{min}} = 50.3 \text{ ms} \). However, there exists three sudden jumps in the control function \( u(t) \), leading to infinite relative speed of trap at switching times, which could be problematic in the experiment. To remedy the difficulty, we need to develop a smooth bang-bang trajectory by adding more constraint conditions on the relative velocity and even acceleration.

### B. smooth bang-bang control with constrained relative velocity/acceleration

Motivated by the problem arised from bang-bang time-optimal control, we introduce more constraints to fix the sudden jumps while fitting the experimental requests as well. A new component \( x_3 \) is added into the state vector \( \mathbf{x} \), with the relations between two nearby components to be
\[ x_1 = q_c, \quad x_2 = \dot{q}_c, \quad x_3 = -\omega_0^2 u(t), \quad u(t) = q_c - q_0, \] (25)
and after considering the relative speed limitation, Eq. (3) can be rewritten into the form for solving the time-optimal control...
problem as
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -\omega_0^2 \dot{u}(t).
\end{align*}
\] (26) (27) (28)

The new control Hamiltonian \( H_c \) (15) can be updated with the cost functional \( J_f \) in Eq. (14),
\[
H_c(p, x, u, \dot{u}) = p_0 + p_1 x_2 + p_2 x_3 + p_3 \omega_0^2 \dot{u}(t),
\] (29)

and the canonical equations (16) add a new equation into the original costate equations as
\[
\begin{align*}
\dot{p}_1 &= 0, \\
\dot{p}_2 &= -p_1, \\
\dot{p}_3 &= -p_2.
\end{align*}
\] (30) (31) (32)

which can be solved easily as \( p_1 = c_1 \), \( p_2 = -c_1 t + c_2 \) and \( p_3 = -c_1 t^2/2 + c_2 t + c_3 \) with the constants \( c_1, c_2 \) and \( c_3 \). Based on the Pontryagin’s maximum principle [57], the time-optimal control \( u(t) \) maximizes the control Hamiltonian (29) with the new constraint on the relative velocity, \( |u(t)| \leq \epsilon \). In order to smooth the function of controller \( u(t) \), \( \dot{u}(t) \) can be given by
\[
\dot{u}(t) = \begin{cases} 
-\epsilon, & 0 \leq t < t_1 \\
\epsilon, & t_2 < t < t_3 \\
-\epsilon, & t_4 < t \leq t_f \\
\end{cases},
\] (33)

By combining the previous constraint on the relative displacement, \( |u(t)| \leq \delta \), the “sudden-jump-free” control function \( u(t) \) is
\[
\begin{align*}
\dot{u}(t) &= \begin{cases} 
-\epsilon t + c_1, & 0 \leq t < t_1 \\
\epsilon t + c_3, & t_2 < t < t_3 \\
-\epsilon t + c_5, & t_4 < t \leq t_f
\end{cases},
\end{align*}
\] (34)

where \( c_2 = -c_4 = -\delta \), \( c_1 = 0 \), \( c_3 = -\delta (t \leq t_2) \) and \( c_5 = \epsilon t_f \). According to boundary conditions, the symmetry and continuity conditions, one can find four switching times \( t_1, t_2, t_3 \) and \( t_4 \) with the values \( \delta / \epsilon, t_f/2 - \delta / \epsilon, t_f/2 + \delta / \epsilon \) and \( t_f - \delta / \epsilon \), respectively. Substituting \( u(t) \) into Eq. (3), and with boundary conditions, see Eqs. (10) and (11), we find the solution of \( q_c(t) \) in different time intervals as follows
\[
q_c(t) = \begin{cases} 
\frac{1}{6} \omega_0^2 \epsilon \delta^3 t^3 & \text{for } 0 \leq t < t_1 \\
\frac{1}{6} \omega_0^2 \delta \left( t^3 - \frac{2}{3} t + \frac{\delta^2}{\epsilon} \right) & \text{for } t_1 < t < t_2 \\
-\frac{1}{6} \omega_0^2 \epsilon \delta (t - \frac{\delta}{\epsilon})^3 - \frac{1}{2} \omega_0^2 \delta t \left( \frac{2}{3} \epsilon \delta - \frac{1}{2} \right) - \frac{1}{2} \omega_0^2 \delta t \left( \frac{2}{3} \epsilon \delta - \frac{1}{2} \right) & \text{for } t_2 < t < t_3 \\
-\frac{1}{6} \omega_0^2 \delta \left( t - \frac{\delta}{\epsilon} \right)^3 - \frac{1}{2} \omega_0^2 \delta \left( \frac{2}{3} \epsilon \delta - \frac{1}{2} \right) & \text{for } t_3 < t < t_4 \\
\frac{1}{6} \omega_0^2 \delta \left( t - \frac{\delta}{\epsilon} \right)^3 & \text{for } t_4 < t \leq t_f
\end{cases}
\] (35)

from which the trajectory of the trap center \( q_0(t) \) can be easily obtained through Eq. (3), but the lengthy expression is omitted for convenience. After straightforward calculation, we obtain the near-minimal time as follows,
\[
t_f = \frac{\delta}{\epsilon} + \frac{2}{\omega_0} \sqrt{\frac{\delta^3}{\epsilon^2} + \frac{\omega_0^2 \delta^2}{4 \epsilon^2}},
\] (36)

which tends to the minimal time in Eq. (24), when there is no limit to the relative velocity, namely, \( \epsilon \rightarrow \infty \). Fig. 2 demonstrates the trajectories of trap center and the mass center of atoms with the smoother controller \( u(t) \) at switching times, when the relative velocity is bounded. Apparently, the constraint in relative velocity prolongs the process of time-optimal transport, as shown also in Fig. 3, where the trajectory becomes smoother, by taking the relative velocity constraint into consideration. In details, the transport time increases from \( t_f = 58.9 \) ms to \( t_f = 68.7 \) ms, when the constraint on the relative velocity decreases from \( \epsilon/(\omega_0) = 0.1 \) to \( \epsilon/(\omega_0) = 0.05 \), with the same constraint on the relative displacement, \( \delta/d = 0.1 \).

Next, we shall find the near-minimal-time protocol with an extra constraint condition on the relative acceleration, namely \( |\dot{u}(t)| \leq \zeta \), where there exists discontinuity of speed which leads to infinite acceleration in previous protocols. To set the limit to the second derivative of controller, the new notation \( x_4 = \dot{x}_3 = -\omega_0^2 \dot{u}(t) \) is added to equations for defining the control Hamiltonian as
\[
H_c(p, x, u, \dot{u}, \ddot{u}) = p_0 + p_1 x_2 + p_2 x_3 + p_3 x_4 + p_4 \omega_0^2 \dot{u},
\] (37)

from which we use canonical equation (16) to obtain the following costate functions:
\[
\begin{align*}
\dot{p}_1 &= 0, \\
\dot{p}_2 &= -p_1, \\
\dot{p}_3 &= -p_2, \\
\dot{p}_4 &= -p_3.
\end{align*}
\] (38) (39) (40) (41)

With the same method mentioned above, the components of the new \( p \) can be calculated as well. According to the Pontryagin’s maximum principle, the time-optimal control \( u(t) \) maximizes the control Hamiltonian in Eq. (37). The optimal control that maximizes \( H_c \) is determined by the sign of \( p_4 \),

FIG. 3. Phase diagram of smooth bang-bang control of fast transport with different relative velocity constraints, where the relative displacement is bounded by \( \delta/d = 0.1 \), while the constraint on the relative velocity is variable. The trajectories with \( \epsilon/(\omega_0) = 0.05 \) (solid red) and \( \epsilon/(\omega_0) = 0.1 \) (dotted black) become much smoother with larger time \( t_f \) of 68.7 ms and \( t_f \) of 58.9 ms, calculated by Eq. (36), respectively, as compared to the case of bang-bang control \( t_f = 50.3 \) ms in Fig. 2. Other parameters are the same as those in Fig. 2.
when \( \ddot{u}(t) \) is bounded by \( |\ddot{u}(t)| \leq \zeta \). Here we apply three constraints simultaneously, with the other two to be \( |u(t)| \leq \delta \) and \( |\dot{u}(t)| \leq \epsilon \). The near-minimal-time protocol meets the limitation to the relative acceleration, velocity, and displacement together. As a consequence, the appropriate controller, \( \ddot{u}(t) \), has the bang-bang type of

\[
\ddot{u}(t) = \begin{cases} 
-\zeta, & 0 \leq t < t_1 \\
0, & t_1 \leq t < t_2 \\
\zeta(t-t_2), & t_2 \leq t < t_3 \\
0, & t_3 \leq t < t_4 \\
\zeta, & t_4 \leq t < t_5 \\
0, & t_5 \leq t < t_6 \\
-\zeta, & t_6 \leq t < t_7 \\
0, & t_7 \leq t < t_8 \\
-\zeta, & t_8 \leq t < t_9 \\
0, & t_9 \leq t \leq t_{10} \\
\zeta, & t_{10} \leq t < t_f 
\end{cases}
\]

(42)

With boundary conditions at switching times, after a simple integration, \( \ddot{u}(t) \) can be given by

\[
\ddot{u}(t) = \begin{cases} 
-\zeta t, & 0 \leq t < t_1 \\
-\epsilon, & t_1 \leq t < t_2 \\
\zeta(t-t_2) - \epsilon, & t_2 \leq t < t_3 \\
0, & t_3 \leq t < t_4 \\
\zeta(t-t_4), & t_4 \leq t < t_5 \\
\epsilon, & t_5 \leq t < t_6 \\
-\zeta(t-t_6) + \epsilon, & t_6 \leq t < t_7 \\
0, & t_7 \leq t < t_8 \\
-\zeta(t-t_8), & t_8 \leq t < t_9 \\
-\epsilon, & t_9 \leq t < t_{10} \\
\zeta(t-t_{10}) - \epsilon, & t_{10} \leq t \leq t_f 
\end{cases}
\]

(43)

from which the switching times can be calculated, using the symmetry, boundary conditions and continuity conditions of \( \dot{u}(t) \) and \( \ddot{u}(t) \), as

\[ t_1 = \epsilon/\zeta, \quad t_2 = \delta/\epsilon + \epsilon/\zeta, \quad t_3 = t_1 + t_2 = \epsilon/\zeta + \zeta/\epsilon, \quad t_4 = t_3 + t_5 = \epsilon/\zeta + \zeta/\epsilon + \delta, \quad t_5 = t_4 + t_5 = \epsilon/\zeta + \zeta/\epsilon + 2\delta, \quad \text{and} \quad t_{10} = t_f - t_1. \]

Once we have the switching times, the control function, see Fig. 4, is finally expressed by

\[
u(t) = \begin{cases} 
-\frac{1}{2}\zeta^2 + \epsilon(t - \frac{\delta}{\epsilon}) - \frac{\zeta^2}{\epsilon^2}, & 0 \leq t < t_1 \\
-\frac{1}{2}\zeta^2 + \epsilon(t - \frac{\delta}{\epsilon}) - \frac{\zeta^2}{\epsilon^2} + \frac{4\delta^2}{3\epsilon^2} + \frac{\zeta^2}{\epsilon^2}, & t_1 \leq t < t_2 \\
-\frac{1}{2}\zeta^2 + \epsilon(t - \frac{\delta}{\epsilon}) - \frac{\zeta^2}{\epsilon^2} + \frac{4\delta^2}{3\epsilon^2} + \frac{\zeta^2}{\epsilon^2} + \frac{(\epsilon(t - \frac{\delta}{\epsilon}) - 2\delta)^2}{8\epsilon^2\zeta}, & t_2 \leq t < t_3 \\
-\delta + \frac{(\epsilon^2 + \epsilon\zeta + \epsilon(t-t_3)^2)}{\epsilon}, & t_3 \leq t < t_4 \\
-\delta + \frac{(\epsilon^2 + \epsilon\zeta + \epsilon(t-t_4)^2)}{\epsilon}, & t_4 \leq t < t_5 \\
-\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_5)^2}{\epsilon}, & t_5 \leq t < t_6 \\
-\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_6)^2}{\epsilon}, & t_6 \leq t < t_7 \\
-\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_7)^2}{\epsilon}, & t_7 \leq t < t_8 \\
-\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_8)^2}{\epsilon}, & t_8 \leq t < t_9 \\
-\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_9)^2}{\epsilon}, & t_9 \leq t < t_{10} \\
\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_{10})^2}{2\epsilon}, & t_{10} \leq t < t_f \\
\frac{\epsilon^2 + \epsilon\zeta + \epsilon(t-t_f)^2}{2\epsilon}, & t_f \leq t \leq t_f 
\end{cases}
\]

(44)

Here we omit the lengthy expression, but the trajectories of trap center \( q_0(t) \) and mass center of cold atoms \( \bar{q}_c(t) \) can be easily calculated through Eq. (3), see Fig. 4. Obviously, this shows a feasible way to realize smooth transport, only taking a little more time as cost, than the bang-bang case and smooth bang-bang case with the bounded velocity. The minimum time for transport \( t_f \) can be calculated from \( q_0(t) \) if the atom is transported by a distance \( d \). As a result, the final expression of near-minimal time in this case is given by

\[
t_f = \delta + \frac{\delta \epsilon}{\zeta} + \frac{2}{\omega_0} \sqrt{\frac{d}{\delta} + \frac{\omega_0^2 \delta^2}{4\epsilon^2}}.
\]

(45)

The minimal time given here is just increase a bit by \( \delta \epsilon / \zeta \), which is the exact price for smooth bang-bang control by setting the constrain on the relative acceleration. For instance, we choose the different constraints in Fig. 4, where \( \epsilon/(\omega_0 d) = 0.1, \zeta/(\omega_0 d) = 0.5 \) (solid red), \( \epsilon/(\omega_0 d) = 0.2, \zeta/(\omega_0 d) = 1 \) (dashed blue), \( \epsilon/(\omega_0 d) = 0.5, \zeta/(\omega_0 d) = 2 \) (dotted black), and other parameters are the same as those in Fig. 2. In those cases, the near-minimal times are \( t_f = 60.5 \) ms, \( t_f = 56.1 \) ms, and \( t_f = 53.9 \) ms, accordingly, given by Eq. (45). (b,c) The corresponding trajectories of mass center \( q_0(t) \) of cold atoms and trap center \( q_0(t) \) under different constraints.

FIG. 4. (a) Control function \( u(t) \) with different constraints, where \( \epsilon/(\omega d) = 0.1, \zeta/(\omega d) = 0.5 \) (solid red), \( \epsilon/(\omega d) = 0.2, \zeta/(\omega d) = 1 \) (dashed blue), \( \epsilon/(\omega d) = 0.5, \zeta/(\omega d) = 2 \) (dotted black), and other parameters are the same as those in Fig. 2. (b,c) The corresponding trajectories of mass center \( q_0(t) \) of cold atoms and trap center \( q_0(t) \) under different constraints.
Fig. 2. It is obvious that the larger constraint we have, the control function $u(t)$ is more likely to the original bang-bang control in Fig. 2, which is harder to be implemented experimentally. Meanwhile, the near-minimal times, $t_f = 60.5$ ms, $t_f = 56.1$ ms and $t_f = 53.9$ ms, accordingly, become closer to the minimal time, $t_f^{\text{min}} = 50.3$ ms, given by Eq. (24). Here we should emphasize that one can further smooth by adding more constraints on high-order derivative of controller $u(t)$, since the second derivative of controller, $\dot{u}(t)$ still keeps the bang-bang form.

Figure 5 illustrates how much price that we pay to smooth the bang-bang control by restricting the relative velocity and acceleration. In general, the influence of constraint on the relative velocity is more pronounced, as compared to the constraint on the relative acceleration. When the constrains are slacked, the operation time approaches the minimal time $t_f^{\text{min}}$ from bang-bang control. From Fig. 5, it is concluded that we can smooth bang-bang time-optimal solution with sudden changes of control input by setting more constraints on the first and second derivatives of controller, but the price that we paid is to increase the transport time a bit more.

Moreover, we present the potential energy (8) and sloshing amplitude (9) for the smooth bang-bang protocols in Fig. 6. The energy excitation during the shortcuts is suppressed by smoothing the control input, since the time-averaged potential energy (8) is proportional to the square of $u(t)$, see Fig. 6, where the time-averaged potential energy $\bar{E}_p$ is rescaled by $\bar{E}_p^0$, the time-averaged potential energy for time-optimal bang-bang control. For additional comparison, we calculate $\mathcal{A}(t_f) = 4 \times 10^{-3}$ for the time-optimal bang-bang control, while $\mathcal{A}(t_f) \approx 10^{-12}$ for the polynomial ansatz (13) with $t_f \in (1.1 t_f^{\text{min}}, 1.24 t_f^{\text{min}})$. Apparently, the sloshing amplitude $\mathcal{A}(t_f)$ is completely nullified by adding more physical constraints on the relative velocity and acceleration, as compared to that for time-optimal bang-bang control. This implies that the smooth bang-bang protocol can improve the overall performance of STA in a realistic experiment [47], in the manner that the energy excitation modes can be significantly avoided by stopping the trap smoothly at the final time, with more strict constraints on relative velocity/acceleration.

IV. NUMERICAL MULTIPLE SHOOTING ALGORITHM

In this section, we shall present the numerical method to solve such time-optimal control with the two-fold reasons. On one hand, the solvable and analytical expressions become complicated, when high-order derivative of controller is considered. Thus, the numerical algorithm is required to calculate automatically the switching times and minimum time with different constraints for double-check and simplicity. On the other hand, when a shooting method is applied, one may encounter numerical difficulty for solving the optimal control because the shooting function is not smooth when the control is bang-bang [48]. In what follows, we shall formulate the boundary-value problem, and solve the smoothing proce-
dure by using multiple shooting method. The detailed steps of our multiple shooting algorithm are shown as follows:

(i) We get the expression of in (second) converges to smooth bang-bang solution of time-optimal transport, with the constraint conditions . The trajectories of trap center and one. After that, calculate the norm of new .

(ii) Then we can write a column vector . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(iii) A Jacobian matrix is introduced to calculate the modifications of switching times.

(iv) We set a column vector to be . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(v) Calculate the values of with . Modification of column vector Del will be given by . In this way, the modified will be . Here, is a constant deciding the speed of convergence between zero and one. After that, calculate the norm of new .

(vi) Repeat Step.v until the norm of is smaller than an acceptable fixed tolerance value.

To demonstrate the algorithm, we give an example of smooth transport calculated with multiple shooting method and plot the norm of vector , showing how the transport protocol converges to time-optimal solution with Fig. 7, where the constraint conditions are .

Besides time-optimal control, the relevant energy minimization can also be dealt with the numerical algorithm mentioned above. In fact, the time-averaged potential energy during the process characterizes the energy excitation, that is, the energy cost of shortcuts. Particularly, when the anharmonic effect is considered, high potential energy during the process will make the atom escape from the trap easily, harming the performance of STA. Therefore, one has to minimize the time-averaged potential energy . From Eq. (8), the cost functional reads

leading to costate equations in Eq. (30) and (31). Thus, by using the Pontryagin’s maximal principle, the “unbounded control”, i.e., without any constraints on , gives the lowest bound

with linear time-varying controller

Obviously, the controller is not zero at and , thus requiring the sudden transition at time edges. The case of bounded can be calculated as well in Ref. [37]. However, the analytical method is too complicated to obtain the smooth controller at initial and final times, when higher-order derivative of controller is requested. Here, we use multiple shooting method to find control trajectories with arbitrary dimension of states and constraints. Transport interval is partitioned by grid points, where control functional consists of subintervals with length of . For finding solution of boundary value problems, we define a state and its derivative , initializing it by . We define a timestep as a criterion. The norm of this column vector should be zero when all the switching times are optimized.

(i) We get the expression of in (second) converges to smooth bang-bang solution of time-optimal transport, with the constraint conditions . The trajectories of trap center and mass center of cold atoms are presented by red solid and dashed blue lines, respectively. Other parameters are the same as those in Fig. 2.

(ii) Then we can write a column vector . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(iii) A Jacobian matrix is introduced to calculate the modifications of switching times.

(iv) We set a column vector to be . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(v) Calculate the values of with . Modification of column vector Del will be given by . In this way, the modified will be . Here, is a constant deciding the speed of convergence between zero and one. After that, calculate the norm of new .

(vi) Repeat Step.v until the norm of is smaller than an acceptable fixed tolerance value.

To demonstrate the algorithm, we give an example of smooth transport calculated with multiple shooting method and plot the norm of vector , showing how the transport protocol converges to time-optimal solution with Fig. 7, where the constraint conditions are .

Besides time-optimal control, the relevant energy minimization can also be dealt with the numerical algorithm mentioned above. In fact, the time-averaged potential energy during the process characterizes the energy excitation, that is, the energy cost of shortcuts. Particularly, when the anharmonic effect is considered, high potential energy during the process will make the atom escape from the trap easily, harming the performance of STA. Therefore, one has to minimize the time-averaged potential energy . From Eq. (8), the cost functional reads

leading to costate equations in Eq. (30) and (31). Thus, by using the Pontryagin’s maximal principle, the “unbounded control”, i.e., without any constraints on , gives the lowest bound

with linear time-varying controller

Obviously, the controller is not zero at and , thus requiring the sudden transition at time edges. The case of bounded can be calculated as well in Ref. [37]. However, the analytical method is too complicated to obtain the smooth controller at initial and final times, when higher-order derivative of controller is requested. Here, we use multiple shooting method to find control trajectories with arbitrary dimension of states and constraints. Transport interval is partitioned by grid points, where control functional consists of subintervals with length of . For finding solution of boundary value problems, we define a state and its derivative , initializing it by . We define a timestep as a criterion. The norm of this column vector should be zero when all the switching times are optimized.

(i) We get the expression of in (second) converges to smooth bang-bang solution of time-optimal transport, with the constraint conditions . The trajectories of trap center and mass center of cold atoms are presented by red solid and dashed blue lines, respectively. Other parameters are the same as those in Fig. 2.

(ii) Then we can write a column vector . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(iii) A Jacobian matrix is introduced to calculate the modifications of switching times.

(iv) We set a column vector to be . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(v) Calculate the values of with . Modification of column vector Del will be given by . In this way, the modified will be . Here, is a constant deciding the speed of convergence between zero and one. After that, calculate the norm of new .

(vi) Repeat Step.v until the norm of is smaller than an acceptable fixed tolerance value.

To demonstrate the algorithm, we give an example of smooth transport calculated with multiple shooting method and plot the norm of vector , showing how the transport protocol converges to time-optimal solution with Fig. 7, where the constraint conditions are .

Besides time-optimal control, the relevant energy minimization can also be dealt with the numerical algorithm mentioned above. In fact, the time-averaged potential energy during the process characterizes the energy excitation, that is, the energy cost of shortcuts. Particularly, when the anharmonic effect is considered, high potential energy during the process will make the atom escape from the trap easily, harming the performance of STA. Therefore, one has to minimize the time-averaged potential energy . From Eq. (8), the cost functional reads

leading to costate equations in Eq. (30) and (31). Thus, by using the Pontryagin’s maximal principle, the “unbounded control”, i.e., without any constraints on , gives the lowest bound

with linear time-varying controller

Obviously, the controller is not zero at and , thus requiring the sudden transition at time edges. The case of bounded can be calculated as well in Ref. [37]. However, the analytical method is too complicated to obtain the smooth controller at initial and final times, when higher-order derivative of controller is requested. Here, we use multiple shooting method to find control trajectories with arbitrary dimension of states and constraints. Transport interval is partitioned by grid points, where control functional consists of subintervals with length of . For finding solution of boundary value problems, we define a state and its derivative , initializing it by . We define a timestep as a criterion. The norm of this column vector should be zero when all the switching times are optimized.

(i) We get the expression of in (second) converges to smooth bang-bang solution of time-optimal transport, with the constraint conditions . The trajectories of trap center and mass center of cold atoms are presented by red solid and dashed blue lines, respectively. Other parameters are the same as those in Fig. 2.

(ii) Then we can write a column vector . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(iii) A Jacobian matrix is introduced to calculate the modifications of switching times.

(iv) We set a column vector to be . All the elements’ initial values are our assumptions of switching times and minimum transport time which will be modified by the algorithm automatically.

(v) Calculate the values of with . Modification of column vector Del will be given by . In this way, the modified will be . Here, is a constant deciding the speed of convergence between zero and one. After that, calculate the norm of new .

(vi) Repeat Step.v until the norm of is smaller than an acceptable fixed tolerance value.

To demonstrate the algorithm, we give an example of smooth transport calculated with multiple shooting method and plot the norm of vector , showing how the transport protocol converges to time-optimal solution with Fig. 7, where the constraint conditions are .

Besides time-optimal control, the relevant energy minimization can also be dealt with the numerical algorithm mentioned above. In fact, the time-averaged potential energy during the process characterizes the energy excitation, that is, the energy cost of shortcuts. Particularly, when the anharmonic effect is considered, high potential energy during the process will make the atom escape from the trap easily, harming the performance of STA. Therefore, one has to minimize the time-averaged potential energy . From Eq. (8), the cost functional reads

leading to costate equations in Eq. (30) and (31). Thus, by using the Pontryagin’s maximal principle, the “unbounded control”, i.e., without any constraints on , gives the lowest bound

with linear time-varying controller

Obviously, the controller is not zero at and , thus requiring the sudden transition at time edges. The case of bounded can be calculated as well in Ref. [37]. However, the analytical method is too complicated to obtain the smooth controller at initial and final times, when higher-order derivative of controller is requested. Here, we use multiple shooting method to find control trajectories with arbitrary dimension of states and constraints. Transport interval is partitioned by grid points, where control functional consists of subintervals with length of . For finding solution of boundary value problems, we define a state and its derivative , initializing it by . We define a timestep as a criterion. The norm of this column vector should be zero when all the switching times are optimized.
FIG. 8. Controller $u(t)$ that minimizes the time-averaged potential energy $E_p$ within a fixed time $t_f = 60$ ms. Controller $u(t)$ is bounded by $\delta/d = 0.1$ (solid red); $\delta/d = 0.1$ and $\epsilon/(d\omega_0) = 0.1$ (dashed blue); and $\delta/d = 0.1$, $\epsilon/(d\omega_0) = 0.1$ and $\zeta/(d\omega_0^2) = 0.5$ (dotted black), respectively. The corresponding time-averaged potential energies $E_p/E_p^{\text{min}}$ are 1.0002, 1.4918 and 1.6099, which are larger than the lowest bound for the potential energy in unbounded control. We let $N = 100$, $M = 10$, and other parameters to be the same as those in Fig. 2.

FIG. 9. Phase diagram of smooth bang-bang control of fast transport, minimizing the time-averaged potential energy $E_p$ with numerical algorithms. Constraint conditions and parameters are the same as those in Fig. 8.

In summary, we present analytical and numerical methods for smooth bang-bang transport of cold atoms in a moving trap, achieving the near-minimal-time shortcuts to adiabaticity. Since the time-optimal bang-bang control requires the sudden jump of control input, resulting in higher residual energy or difficulty in practical implementation, we propose the smooth bang-bang control, corresponding to the near-minimal time, by setting more constraints on the first and second derivatives of controller. Comparison demonstrates the energy excitation and sloshing amplitude are significantly suppressed for smooth bang-bang control while the time is slightly increased at cost. To make our results more applicable, the numerical multiple shooting algorithm is developed for time minimization and energy minimization, to avoid the sudden transition of control input. Of course, the different ansatz with smooth polynomial or triangular functions can be compared and used to approach (sub-)optimal solutions.

Finally, we should emphasize that the analytical and numerical methods proposed for smooth bang-bang control are useful in the field of quantum control, since the time-optimal bang-bang control are ubiquitous, for instance, in the atom cooling [49–51] and ground-state preparation [52, 53]. Moreover, our results can be supplemented by machine learning [23, 24, 59] for the rapid transport with a harmonic trap [43–45] and optical lattice [60–62] in presence of noise or spin-orbit coupling [63], and will be extended to other problems, including the load manipulation by cranes in classical system [64], and Brownian motion in statistical physics as well [65].

ACKNOWLEDGEMENT

This work is partially supported from National Natural Science Foundation of China (NSFC) (11474193), STCSM (18010500400 and 18ZR1415500), and Program for Excellent Scholar. XC also acknowledges Ramón y Cajal program of the Spanish MCIU (RYC-2017-22482), QMiCS (820505) and OpenSuperQ (820363) of the EU Flagship on Quantum Technologies, Spanish Government PGC2018-095113-B-I00 (MCIU/AEI/FEDER, UE), Basque Government IT986-16, as well as the and EU FET Open Grant Quromorphic.

[1] W. H¨ansel, P. Hommelhoff, T. W. H¨ansch, and J. Reichel, Bose-Einstein condensation on a microelectronic chip. Nature 413, 498501 (2001).
[2] W. H¨ansel, J. Reichel, P. Hommelhoff, and T. W. H¨ansch, Magnetic Conveyor Belt for Transporting and Merging Trapped Atom Clouds. Phys. Rev. Lett. 86, 608 (2001).
[3] T. L. Gustavson, A. P. Chikkatur, A. E. Leanhardt, A. G¨orlitz, S. Gupta, D. E. Pritchard, and W. Ketterle, Transport of Bose-Einstein Condensates with Optical Tweezers. Phys. Rev. Lett. 88, 020401 (2001).
[4] A. Ben-Kish, B. DeMarco, D. Leibfried, V. Meyer, J. Beall, J. Britton, J. Hughes, W. M. Itano, B. Jelenkovic, C. Langer, T.
Rosenband, and D. J. Wineland, Transport of quantum states and separation of ions in a dual RF ion trap. Quant. Inf. Comp. 4, 257 (2002).

[5] R. Reichle, D. Leibfried, R. B. Blakestad, J. Britton, J. D. Jost, E. Knill, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland, Transport dynamics of single ions in segmented microstructured Paul trap arrays. Fortschr. Phys. 54, 666-685 (2006).

[6] S. Schmid, G. Thallhammer, K. Winkler, F. Lang, and J. H. Denschlag, Long distance transport of ultracold atoms using a 1D optical lattice. New J. Phys. 8, 159 (2006).

[7] T. Lahaye, G. Reinaudi, Z. Wang, A. Couvert, and D. Guéry-Odelin, Transport of atom packets in a train of Ioffe-Pritchard traps, Phys. Rev. A 74, 033622 (2006).

[8] A. Couvert, T. Kawalec, G. Reinaudi, and D. Guéry-Odelin, Optimal transport of ultracold atoms in the non-adiabatic regime. Europhys. Lett. 83, 13001 (2008).

[9] M. Murphy, L. Jiang, N. Khaneja, and T. Calarco, High-fidelity fast quantum transport with imperfect controls, Phys. Rev. A 79, 020301(R) (2009).

[10] D. Chen, H. Zhang, X. Xu, T. Li, and Y. Wang, Nonadiabatic transport of cold atoms in a magnetic quadrupole potential. Appl. Phys. Lett. 96, 134103 (2010).

[11] H.-K. Lau and D. F. V. James, Decoherence and dephasing errors caused by the dc Stark effect in rapid ion transport. Phys. Rev. A 83, 062330 (2011).

[12] R. Bowler, J. Gaebler, Y. Lin, T. R. Tan, D. Hanneke, J. D. Jost, J. P. Home, D. Leibfried, and D. J. Wineland, Coherent Diabatic Ion Transport and Separation in a Multizone Trap Array. Phys. Rev. Lett. 109, 080502 (2012).

[13] A. Walther, F. Ziesel, T. Ruster, S. T. Dawkins, K. Ott, M. Hettich, K. Singer, F. Schmidt-Kaler, and U. Poschinger, Controlling Fast Transport of Cold Trapped Ions. Phys. Rev. Lett. 109, 080501 (2012).

[14] T. Middelmann, S. Falke, C. Lisdat, and U. Sterr, Long-range transport of ultracold atoms in a far-detuned one-dimensional optical lattice. New J. Phys. 14, 073020 (2012).

[15] J. Alonso, F. M. Leupold, B. C. Keitch, and J. P. Home, Quantum control of the motional states of trapped ions through fast switching of trapping potentials. New J. Phys. 15, 023001 (2013).

[16] A. Negretti, A. Benseny, J. Mompert, and T. Calarco, Speeding up the spatial adiabatic passage of matter waves in optical microtraps by optimal control. Quantum Inf. Process 12, 1439 (2013).

[17] M. Dupont-Nivet, C. I. Westbrook, and S. Schwartz, Contrast and phase-shift of a trapped atom interferometer using a thermal ensemble with internal state labelling. New J. Phys. 18, 113012 (2016).

[18] S. Pandey, H. Mas, G. Drougakis, P. Thekkeppatt, V. Bolpasi, G. Vasilakis, K. Poulion, and W. von Klitzing, Hypersonic Bose-Einstein condensates in accelerator rings. Nature 570, 205209 (2019).

[19] D. Becker, M. D. Lachmann, S. T. Seidel et al., Space-borne Bose-Einstein condensation for precision interferometry. Nature 562, 391395 (2018).

[20] R. Corgier, S. Amri, W. Herr, H. Ahlers, J. Rudolph, D. Guéry-Odelin, E. M. Rasel, E. Charron, and N. Glaoual, Fast manipulation of Bose-Einstein condensates with an atom chip, New J. Phys. 20, 055002 (2018).

[21] R. Roy, P. C. Condylis, V. Prakash, D. Sahagun, and B. Hessmo, A minimalistic and optimized conveyor belt for neutral atoms. Sci. Rep. 7, 1-8 (2017).

[22] S. Amri, R. Corgier, D. Sugny, E. M. Rasel, N. Glaoul, and E. Charron, Optimal control of the transport of Bose-Einstein condensates with atom chips. Sci. Rep. 9, 5346 (2019).

[23] J. J. W. H. Sørensen, M. K. Pedersen et al., Exploring the quantum speed limit with computer games, Nature 532, 210213 (2016).

[24] D. Sels, Stochastic gradient ascent outperforms gamers in the Quantum Moves game, Phys. Rev. A 97, 040302(R) (2018).

[25] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin, and J. G. Muga, Fast Optimal Frictionless Atom Cooling in Harmonic Traps: Shortcut to Adiabaticity. Phys. Rev. Lett. 104, 063002 (2010).

[26] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, et al., Shortcuts to adiabaticity, Advances in atomic, molecular, and optical physics. Academic Press, 62, 117-169 (2013).

[27] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga, Shortcuts to adiabaticity: Concepts, methods, and applications. Rev. Mod. Phys. 91, 045001 (2019).

[28] S. Masuda and K. Nakamura, Fast-forward of adiabaticity in quantum mechanics. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 466, 1135-1154 (2010).

[29] S. Masuda, Acceleration of adiabatic transport of interacting particles and rapid manipulations of a dilute Bose gas in the ground state. Phys. Rev. A 86, 063624 (2012).

[30] S. Defner, C. Jarzynski, and A. del Campo, Classical and Quantum Shortcuts to Adiabaticity for Scale-Invariant Driving. Phys. Rev. X 4, 021013 (2014).

[31] S. An, D. Lv, A. del Campo, and K. Kim Shortcuts to adiabaticity by counterdiabatic driving for trapped-ion displacement in phase space. Nat. Commun. 7, 12999 (2016).

[32] E. Torrontegui, S. Ibáñez, X. Chen, A. Ruschhaupt, D. Guéry-Odelin, and J. G. Muga, Fast atomic transport without vibrational heating. Phys. Rev. A 83, 013415 (2011).

[33] E. Torrontegui, X. Chen, M. Modugno, S. Schmidt, A. Ruschhaupt, and J. G. Muga, Fast transport of Bose-Einstein condensates. New J. Phys. 14, 013031 (2012).

[34] M. Palmero, E. Torrontegui, D. Guéry-Odelin, and J. G. Muga, Fast transport of two ions in an anharmonic trap. Phys. Rev. A 88, 053423 (2013).

[35] M. Palmero, R. Bowler, J. P. Gaebler, D. Leibfried, and J. G. Muga, Fast transport of mixed-species ion chains within a Paul trap. Phys. Rev. A 90, 053408 (2014).

[36] A. Tobalina, M. Palmero, S. Martínez-Garaot, and J. G. Muga, Fast atom transport and launching in a nonrigid trap. Sci. Rep. 7, 5753 (2017).

[37] X. Chen, E. Torrontegui, D. Stefanatos, J.-S. Li, and J. G. Muga, Optimal trajectories for efficient atomic transport without final excitation. Phys. Rev. A 84, 043415 (2011).

[38] H. A. Fürst, M. H. Goerz, U. G. Poschinger, M. Murphy, S. Montangero, T. Calarco, F. Schmidt-Kaler, K. Singer, and C. P. Koch, Controlling the transport of an ion: classical and quantum mechanical solutions. New J. Phys. 16, 075007 (2014).

[39] J. Li, Q. Zhang, and X. Chen, Trigonometric protocols for shortcuts to adiabatic transport of cold atoms in anharmonic traps. Physics Letters A 381, 3272 (2017).

[40] Q. Zhang, X. Chen, and D. Guéry-Odelin, Fast and optimal transport of atoms with nonharmonic traps. Phys. Rev. A 92, 043410 (2015).

[41] Q. Zhang, J. G. Muga, D. Guéry-Odelin, and X. Chen, Optimal shortcuts for atomic transport in anharmonic traps. J. Phys. B: At. Mol. Opt. Phys. 49, 125503 (2016).

[42] D. Guéry-Odelin and J. G. Muga, Transport in a harmonic trap: Shortcuts to adiabaticity and robust protocols. Phys. Rev. A 90, 063425 (2014).
[43] X.-J. Lu, J. G. Muga, X. Chen, U. G. Poschingher, F. Schmidt-Kaler, and A. Ruschhaupt, Fast shuttling of a trapped ion in the presence of noise. Phys. Rev. A 89, 063414 (2014).
[44] X.-J. Lu, M. Palmero, A. Ruschhaupt, X. Chen, and J. Gonzalo Muga, Optimal transport of two ions under slow spring-constant drifts. Phys. Scr. 90, 074038 (2015).
[45] X.-J. Lu, A. Ruschhaupt, and J. G. Muga, Fast shuttling of a particle under weak spring-constant noise of the moving trap. Phys. Rev. A 97, 053402 (2018).
[46] D. Stefanatos and J.-S. Li, Minimum-Time Quantum Transport With Bounded Trap Velocity. IEEE Trans. Automat. Contr. 59, 733 (2014).
[47] G. Ness, C. Shkedrov, Y. Florshaim, and Y. Sagi, Realistic shortcuts to adiabaticity in optical transfer. New J. Phys. 20, 095002 (2018).
[48] B. A. Albassam, Optimal near-minimum-time control design for flexible structures. Journal of guidance, control, and dynamics 25, 618 (2002).
[49] D. Stefanatos, J. Ruths, and J.-S. Li, Frictionless atom cooling in harmonic traps: A time-optimal approach. Phys. Rev. A 82, 063422 (2010).
[50] K. H. Hoffmann, P. Salamon, Y. Rezek, and R. Kosloff, Time-optimal controls for frictionless in harmonic traps, EPL 96, 60015 (2011).
[51] X.-J. Lu, X. Chen, J. Alonso, and J. G. Muga, Fast transitionless expansions of Gaussian anharmonic traps for cold atoms: Bang-singular-bang control. Phys. Rev. A 89, 023627 (2014).
[52] S. Balasubramanian, S. Han, B. T. Yoshimura, and J. K. Freericks, Bang-bang shortcut to adiabaticity in trapped-ion quantum simulators. Phys. Rev. A 97, 022313 (2018).
[53] J. Cohn, A Safavi-Naini, R. J. Lewis-Swan, J. G. Bohnet, M Gärttner, K. A. Gilmore, J. E. Jordan, A. M. Rey, J. J. Bollinger, and J. K. Freericks, Bang-bang shortcut to adiabaticity in the Dicke model as realized in a Penning trap experiment. New J. Phys. 20, 055013 (2018).
[54] H. R. Lewis and W. B. Riesenfeld, An Exact Quantum Theory of the TimeDependent Harmonic Oscillator and of a Charged Particle in a TimeDependent Electromagnetic Field. J. Math. Phys. 10, 1458 (1969).
[55] H. R. Lewis and P. G. Leach, A direct approach to finding exact invariants for onedimensional time-dependent classical Hamiltonians. J. Math. Phys. 23, 2371 (1982).
[56] A. K. Dhara and S. W. Lawande, Feynman propagator for time-dependent Lagrangians possessing an invariant quadratic in momentum. J. Phys. A 17, 2324 (1984).
[57] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishechenko, The Mathematical Theory of Optimal Processes (Interscience, New York, 1962).
[58] V. Martikyan, D. Guéry-Odelin, and D. Sugny, Comparison between optimal control and shortcut to adiabaticity protocols in a linear control system. Phys. Rev. A 101, 013423 (2020).
[59] A. J. Barker, H. Style, K. Luksch, S. Sunami, D. Garrick, F. Hill, C. J. Foot, and E. Bentine, Applying machine learning optimization methods to the production of a quantum gas. Mach. Learn.: Sci. Technol. 1, 015007 (2020).
[60] C. Robens, S. Brakhane, W. Alt, D. Meschede, J. Zopes, and A. Alberti, Fast, High-Precision Optical Polarization Synthesizer for Ultracold-Atom Experiments. Phys. Rev. Applied 9, 034016 (2018).
[61] T. Dowdall and A. Ruschhaupt, Transport of atoms across an optical lattice using an external harmonic potential. arXiv:2002.05976.
[62] X.-J. Lu, A. Ruschhaupt, S. Martínez-Garaot, and J. G. Muga, Noise sensitivities for an atom shuttled by a moving optical lattice via shortcuts to adiabaticity. arXiv:2002.03951.
[63] X. Chen, R.-L. Jiang, J. Li, Y. Ban, and E. Ya. Sherman, Inverse engineering for fast transport and spin control of spin-orbit-coupled Bose-Einstein condensates in moving harmonic traps. Phys. Rev. A 97, 013631 (2018).
[64] S. González-Resines, D. Guéry-Odelin, A. Tobalina, I. Lizuain, E. Torrontegui, and J. G. Muga, Invariant-Based Inverse Engineering of Crane Control Parameters. Phys. Rev. Applied 8, 054008 (2017).
[65] C. A. Plata, D. Guéry-Odelin, E. Trizac, and A. Prados, Optimal work in a harmonic trap with bounded stiffness. Phys. Rev. E 99, 012140 (2019).