A topological approach to Neutrino masses by using exotic smoothness

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In this paper, we will consider a cosmological model with two topological transitions of the space. The smooth 4-dimensional spacetime of the model admits topological transitions of its 3-dimensional slices. The whole approach is inspired by a class of exotic smoothness structure on $S^3 \times \mathbb{R}$. In particular, this class of smoothness structures induces two topological transitions. Then, we are able to calculate the energy scales associated to these topological transitions. For the first transition we will get the value of the GUT scale and the energy of the second transition is at the electroweak scale. The topology of the exotic $S^3 \times \mathbb{R}$ determines both, the energy of the scales by certain topological invariants, and the existence of the right-handed sterile neutrino. It is the input for the seesaw mechanism. Secondly, based on this model, we are able to calculate the neutrino masses which are in a very good agreement with experiments. Finally, we will speculate, again based on topology, why there are three generations of neutrinos and an asymmetry between neutrinos and anti-neutrinos.

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1. Introduction

The problem of fermion masses is one of the most fundamental issues unresolved so far in the extensions of the standard model of elementary particles (SM, see the review [1]). To understand the neutrino masses and the mixing of the neutrinos is a particularly important ingredient of the problem. So far neutrinos are the only fermions which are neutral with respect to the electric as well color charge and they can admit masses. One of the accepted mechanisms for explaining the small (Dirac) masses of neutrinos is the seesaw mechanism (see for instance [2,3,4] and the recent paper [5] with a more complete bibliography) which postulates a heavy right-handed Majorana neutrino.

Let us briefly discuss generalities of the simplest seesaw mechanism of type I and how it fits in the proposed cosmological model based on exotic $S^3 \times \mathbb{R}$. In the
minimal SM a neutrino is represented by a Weyl spinor $$\chi$$ which belongs to the lepton isospin doublet $$L = \begin{pmatrix} \chi \\ \psi \end{pmatrix}$$. It is left-handed and massless, $$\psi$$ is the corresponding left-handed charged lepton. In SM, there are 3 generations of leptons and each of their neutrinos are represented by doublets as above.

Let now any left handed neutrino $$\chi$$ be a part of the doublet $$\begin{pmatrix} \chi \\ \eta \end{pmatrix}$$ where $$\eta$$ be a, postulated so far, right-handed neutrino (Weyl spinor). It is a singlet under weak isospin and thus does not interact weakly (sterile neutrino).

Possible mass terms are then generated by the following quadratic form

$$(\chi, \eta) \begin{pmatrix} B' & M \\ M & B \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix}.$$ 

$$B'$$ has to be set to zero because it produces a nonrenormalizable term. Thus the neutrino mass matrix reads

$$\begin{pmatrix} 0 & M \\ M & B \end{pmatrix}.$$ 

However, $$M$$ is forced to vanish by symmetries of SM. The usual way to introduce it will go through Yukawa-like interactions with the Higgs field. These interactions will generate Dirac masses and $$M$$ can be non-zero. The value of $$M$$ is thus naturally of the order of the vacuum expectation value of the Higgs field given by 246 GeV or of the Higgs field mass of order 126 GeV.

Now the value of $$B$$ is not fixed by the SM interactions since the right-handed neutrinos are sterile and uncharged under any SM gauge symmetry. Hence $$B$$ is a free parameter. Choosing $$M \ll B$$ the mass-matrix eigenvalues read

$$\lambda_1 \approx B \lambda_2 \approx -\frac{M^2}{B}$$

so that $$\lambda_1$$ is the mass of the right-handed neutrino and $$\lambda_2$$ represents the mass of the left-handed neutrino. In the unbroken phase one has $$B = 0$$. But $$B$$ becomes large around the scale of the spontaneous symmetry breaking via Yukawa interactions with the Higgs field. An increase of the scale $$B$$ will induce smaller masses of the left-handed neutrinos. Therefore one usually choose the GUT scale of order $$10^{15}\text{GeV}$$.

The problem of this approach is the existence of two independent scales, GUT and electroweak scale. The large size of $$B$$ is connected with the GUT scale which guarantees that the Dirac masses of the left-handed neutrinos are sufficiently small. But neutrinos carry no color and electric charge, i.e. the next natural scale must be the electroweak scale.

In this paper we will address the question whether the existence of both scales can be uniquely determined by a cosmological model. We show that indeed it can. The method of building the model is based on the topological and geometric properties of the smooth cosmological evolution represented by the exotic $$S^3 \times \mathbb{R}$$ which inherits the smoothness from the ambient exotic $$\mathbb{R}^4$$. So thus the scales appear as
topologically supported quantities which fix the value of the parameter $B$ into the GUT energy scale. Moreover, the existence of the right-handed neutrino is also the effect of the topology change. The realistic value of the Higgs mass is assigned with the second topology change which indicates the electroweak scale. Hence electroweak and GUT scales are not independent but rather determined by topological properties of the model. Mielke\cite{6} describes also a mechanism to generate the chirality of neutrinos from topology by using the spacetime $S^2 \times S^1 \times \mathbb{R}$. Interestingly, our model is connected to this work. The 3-manifolds $Y_k$ in the infinite sequence $Y_1 \rightarrow Y_2 \rightarrow \ldots \rightarrow Y_\infty$ of topology changes (see the next section) has the same homology as $S^2 \times S^1$ (but not for the limit $Y_\infty$ which is a wildly embedded 3-sphere). Therefore the whole process represents a spacetime which is homology equivalent to $S^2 \times S^1 \times \mathbb{R}$ and the chirality of neutrinos is generated by the same mechanism like in the model of Mielke.

The main part of the paper is the description of the topological constructions showing how to translate (a part of) physics of SM into the intrinsic topology of a 4-dimensional exotic spacetime. We show that fermions, right-handed neutrinos, Higgs field and finally the GUT and electroweak energy scales all this have their natural topological counterparts in the cosmological model of our exotic $S^3 \times \mathbb{R}$. It is an amazing fact of this model that the numbers following from the topology are realistic and serve as an input for the seesaw mechanism to generate the neutrino masses.

2. Basic model

In this section we will describe the basic topological model of this paper. The whole model was described completely in\cite{7} where it was used to calculate the cosmological constant. In short, the model postulates:

- the spacetime is topologically $S^3 \times \mathbb{R}$ but not smoothly
- there two transitions

$$S^3 \xrightarrow{cork} \Sigma(2, 5, 7) \xrightarrow{gluing} P \# P.$$ 

The first transition from the 3-sphere to the Brieskorn sphere
$$\Sigma(2, 5, 7) = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^5 + z^7 = 0, \ |x|^2 + |y|^2 + |z|^2 = 1\}$$

and a second transition from $\Sigma(2, 5, 7)$ to the sum of two Poincare spheres $P \# P = (P \setminus D^3) \cup_{S^2} (P \setminus D^3)$ with
$$P = \Sigma(2, 3, 5) = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0, \ |x|^2 + |y|^2 + |z|^2 = 1\}.$$ 

The model was completely described in\cite{7} and partly in\cite{8}. The relevant properties do not depend on the particular smoothness structure of $S^3 \times \mathbb{R}$ but the so-called standard smoothness structure is ruled out (there is no topological transitions in this structure). The details of the model will be described now but the reader can also skip the rest of the section to keep only the two facts above in mind.
The distinguished feature of differential topology of manifolds in dimension 4 is the existence of open 4-manifolds carrying a plenty of non-diffeomorphic smooth structures. In the cosmological model presented here, the special role is played by the topologically simplest 4-manifold, i.e. $\mathbb{R}^4$, which carries a continuum of infinitely many different smoothness structures. Each of them except one, the standard $\mathbb{R}^4$, is called exotic $\mathbb{R}^4$. All exotic $\mathbb{R}^4$ are Riemannian smooth open 4-manifolds homeomorphic to $\mathbb{R}^4$ but non-diffeomorphic to the standard smooth $\mathbb{R}^4$. The standard smoothness is distinguished by the requirement that the topological product $\mathbb{R} \times \mathbb{R}^3$ is a smooth product. There exists only one (up to diffeomorphisms) smoothing, the standard $\mathbb{R}^4$, where the product above is smooth. There are two types of exotic $\mathbb{R}^4$: small exotic $\mathbb{R}^4$ can be embedded into the standard $S^4$ whereas large exotic $\mathbb{R}^4$ cannot. In the following, an exotic $\mathbb{R}^4$, presumably small if not stated differently, will be denoted as $\mathbb{R}^4$. In cosmology, one usually considers the topology $S^3 \times \mathbb{R}$ for the spacetime. But by using the simple topological relations $\mathbb{R}^4 \setminus D^4 = S^3 \times \mathbb{R}$ or $\mathbb{R}^4 \setminus \{0\} = S^3 \times \mathbb{R}$, one obtains also an exotic $S^3 \times \mathbb{R}$ from every exotic $\mathbb{R}^4$. In the following we will denote the exotic $S^3 \times \mathbb{R}$ by $S^3 \times \theta \mathbb{R}$ to indicate the important fact that there is no global splitting of $S^3 \times \theta \mathbb{R}$ into $S^3$-slices or it is not globally hyperbolic. This fact has a tremendous impact on cosmology and therefore we will consider our main hypothesis:

MainHypo: The spacetime, seen as smooth four-dimensional manifold, admits an exotic smoothness structure.

This hypothesis has the following consequences (see 7):

- Any $\mathbb{R}^4$ has necessarily non-vanishing Riemann curvature. Also the $S^3 \times \theta \mathbb{R}$ has a non-vanishing curvature along the $\mathbb{R}$-direction.
- Inside of $\mathbb{R}^4$, there is a compact 4-dimensional submanifold $K \subset \mathbb{R}^4$, which is not surrounded by a smoothly embedded 3-sphere. Then there is a chain of 3-submanifolds of $\mathbb{R}^4$ $Y_1 \to \cdots \to Y_\infty$ and the corresponding infinite chain of cobordisms

$$\text{End}(\mathbb{R}^4) = W(Y_1, Y_2) \cup W(Y_2, Y_3) \cup \cdots$$

where $W(Y_k, Y_{k+1})$ denotes the cobordism between $Y_k$ and $Y_{k+1}$ so that $\mathbb{R}^4 = K \cup Y_1 \text{End}(\mathbb{R}^4)$ where $\partial K = Y_1$. The cobordism $W(Y_k, Y_{k+1})$ is a 4-dimensional expression for the topological transition $Y_k \to Y_{k+1}$. Furthermore one has $\text{End}(\mathbb{R}^4) \subset S^3 \times \theta \mathbb{R}$.
- $\mathbb{R}^4$ and $S^3 \times \theta \mathbb{R}$ embeds into the standard $\mathbb{R}^4$ or $S^4$ but also in some other complicated 4-manifolds. The construction of $\mathbb{R}^4$ gives us a natural smooth embedding into the compact 4-manifold $E(2)\#\overline{CP^2}$ (with the K3 surface $E(2)$) (see 8).

But every subset $K'$, $K' \subset K \subset \mathbb{R}^4$, is surrounded by a 3-sphere. This fact is the starting point of our model. Now we choose a Planck-sized 3-sphere $S^3$ inside of the compact subset $K \subset \mathbb{R}^4$. This is the initial point where our cosmos starts to evolve. By the construction of $\mathbb{R}^4$, as mentioned above, there exists the homology 3-sphere
$\Sigma(2, 5, 7)$ inside of $\mathbb{K}$ which is the boundary of the Akbulut cork for $E(2)\#\mathbb{C}P^2$. (see chapter 9 \[9\]). If $S^3$ is the starting point of the cosmos as above, then $S^3 \subset \Sigma(2, 5, 7)$. But then we will obtain the first topological transition

$$S^3 \rightarrow \Sigma(2, 5, 7)$$

inside $R^4$. The construction of $R^4$ was based on the topological structure of $E(2)$ (the K3 surface). $E(2)$ splits topologically into a 4-manifold $|E_8 \oplus E_8|$ with intersection form $E_8 \oplus E_8$ (see \[2\]) and the sum of three copies of $S^2 \times S^2$. Here, $E_8$ denotes a $8 \times 8$ matrix of determinant 1 with integer entries. It is the Cartan matrix of the Lie algebra $\mathfrak{e}_8$ of the exceptional Lie group $E_8$. Now $E_8$ denotes the matrix of the intersection form (as integer bilinear form) of certain 4-manifold $|E_8|$ (with a boundary - see below).

In the topological splitting of $E(2)$

$$|E_8 \oplus E_8| \times (S^2 \times S^2) \times (S^2 \times S^2) \times (S^2 \times S^2)$$

the 4-manifold $|E_8 \oplus E_8|$ has a boundary which is the sum of two Poincare spheres $P\#P$. Here we used the fact that a smooth 4-manifold with intersection form $E_8$ must have a boundary (which is the Poincare sphere $P$), otherwise it would contradict Donaldson’s theorem. Then any closed version of $|E_8 \oplus E_8|$ does not exist and this fact is the reason for the existence of exotic $R^4$. To express it differently, the $R^4$ lies between this 3-manifold $\Sigma(2, 5, 7)$ and the sum of two Poincare spheres $P\#P$. We analyzed this spacetime in \[10\]. It is interesting to note that the number of the $S^2 \times S^2$ components must be three or more otherwise the corresponding spacetime is not smooth!

Therefore we have two topological transitions resulting from the embedding into $E(2)\#\mathbb{C}P^2$

$$S^3 \xrightarrow{cork} \Sigma(2, 5, 7) \xrightarrow{gluing} P\#P.$$ 

These two topological transitions constitute the main idea of this work together with the mapping of them into two different energy scales:

- $S^3 \rightarrow \Sigma(2, 5, 7)$ which must be related to the GUT scale in between $10^4$ to $10^3$ times lower than the Planck scale, and
- $\Sigma(2, 5, 7) \rightarrow P\#P$ related to the electroweak scale $246$ GeV or to the Higgs mass $126$ GeV.

We will show in the next section that the energy scales of the transitions are consistent with this idea.
3. Energy scales and topological transitions

In this section we will show how the change of the energy scale is driven by the topological transitions

\[ S^3 \xrightarrow{\text{cork}} \Sigma(2,5,7) \xrightarrow{\text{gluing}} P#P. \]

Both transitions have different topological descriptions. The first transition \( S^3 \rightarrow \Sigma(2,5,7) \) can be realized by a smooth cobordism, i.e. by a smooth 4-manifold \( M \) with boundary \( \partial M = S^3 \sqcup \Sigma(2,5,7) \). But dimension 4 is special and one needs an infinite process to realize this transition, called a Casson handle. In \( \text{[2]} \) we described this situation extensively. In particular, we obtained a scaling formula between the length scale \( a_0 \) of the 3-sphere \( S^3 \) and the length scale \( a \) of \( \Sigma(2,5,7) \):

\[
a = a_0 \cdot \exp \left( \frac{3}{2 \cdot CS(\Sigma(2,5,7))} \right)
\]

where \( CS(\Sigma(2,5,7)) \) is the Chern-Simons invariant of this manifold. The Chern-Simons invariant \( CS(N) \) of a 3-manifold \( N \) depends on the representation of the fundamental group \( \pi_1(N) \) into \( SU(2) \) or \( SL(2,\mathbb{C}) \), see the paper \( \text{[11]} \). The minimal value for all representations is an invariant which will be exclusively considered in the following. By using the solution to the 3D Poincare conjecture, the value \( CS(N) = 0 \) is only possible for the 3-sphere \( N = S^3 \). For this value, the expression above for \( a \) would diverge but the 3-sphere cannot appear in the formula for \( a \) otherwise it contradicts the appearance of an exotic smoothness structure and a hyperbolic structure used in the derivation of this formula. In quantum gravity, the Chern-Simons invariant was used in a special class of states, the Kodama state \( \text{[12]} \) used in Loop quantum gravity. It is known to be unphysical for a variety of reasons as discussed by Witten \( \text{[13]} \), but it is still interesting to understand what Kodama state and CS invariants could correspond to in quantum gravity. The Chern-Simons term was considered also by Mielke \( \text{[14]} \) in the context of the teleparallelism theory of gravity, \( GR_{\parallel} \), which is a gauge theory of translations. Then the Chern-Simons invariant is the solution of the teleparallel Ashtekar constraints \( \text{[15]} \). In all cases, the Chern-Simons term is not a denominator of a fraction. In the presented approach, hyperbolic geometry explains how the CS term appears in the above formula \( \text{[2]} \).

In \( \text{[16]} \), we also derived this formula by relating it to the levels of the tree representing the Casson handle. By using the shortening

\[ \vartheta = \frac{3}{2 \cdot CS(\Sigma(2,5,7))} \]

we obtained

\[
a = a_0 \cdot \sum_{n=0}^{\infty} \frac{\vartheta^n}{n!}
\]

or the \( n \)th level will contribute by the term \( \frac{\vartheta^n}{n!} \) to the change of the length scale. The Casson handle is directly related to the topology change. Therefore, it is natural
from the physics point of view to identify the level number $n$ with the time coordinate. Now we will determine the smallest possible time change $\Delta t$. This change can be understood as introducing an uncertainty $\Delta t$ which results in an uncertainty of the energy $\Delta E$, see the discussion of Hilgevoord\cite{Hilgevoord} supporting our view. Then we are able to determine the corresponding energy change $\Delta E$. This is given by the well-known relation $\Delta t \sim \hbar/\Delta E$ which follows from the quantum nature of fluctuations where the smallest time change gives rise to the corresponding energy change.

Therefore, due to the quantum nature of the fluctuations we are enforced to determine the shortest time change. But this change has natural topological interpretation via the minimal number of levels in the tree of the Casson handle where the topology change appears. How many levels are necessary to get such topology change? In \cite{Freedman} Freedman answered this question: three levels are needed to embed entire CH in the 3-tower of a Casson handle! Then if we assume that the shortest time scale of one level is given by the Planck time $t_{\text{Planck}}$ then we will get for the smallest time change

$$\Delta t = \left(1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6}\right)t_{\text{Planck}} \quad (3)$$

so that we obtain for the energy scale

$$\Delta E = \frac{E_{\text{Planck}}}{1 + \vartheta + \frac{\vartheta^2}{2} + \frac{\vartheta^3}{6}} \quad . \quad (4)$$

For the transition $S^3 \to \Sigma(2, 5, 7)$ we have the values

$$CS(\Sigma(2, 5, 7)) = \frac{9}{280}, \quad \vartheta = \frac{140}{3}, \quad \Delta E_{\text{GUT}} \approx 0.67 \cdot 10^{15} \text{GeV}$$

and thus the first energy scale is of an order

$$B \approx 10^{15} \text{GeV}$$

of the GUT energy scale.

In the formula \cite{Freedman} above, we obtained a relation between time and the Chern-Simons invariant. This relation has some similarities with a proposal of Smolin and coworker\cite{Smolin} which has been discussed recently\cite{Freedman}. In contrast to our formula, the time is directly given by the Chern-Simons invariant (or its imaginary part). The difference to our proposal can be explained by the essential role played by hyperbolic geometry in the derivation of \cite{Freedman} along with a (dynamically induced) time foliation which is fixed by rigidity in hyperbolic geometry.

As described in \cite{Freedman} for the second transition $\Sigma(2, 5, 7) \to P \# P$, we need a different argumentation. As shown in \cite{Freedman}, this transition cannot be represented by a smooth cobordism (a smooth 4-manifold with boundary $P \# P \sqcup \Sigma(2, 5, 7)$). But we are able to get a smooth 4-manifold by adding three hyperbolic 3-manifolds (see \cite{Freedman} for an explanation). Here we will need entire tower with all infinite many levels of Casson handles to realize the transition, which stays in contrast to the 3-stages
of the previous transition. As argued in [21] these hyperbolic 3-manifolds can be physically interpreted as matter.

But adding of three hyperbolic 3-manifolds means that we need now three different Casson handles for the transition. To express it differently: From the physics point of view, we have three channels for the change \( \Sigma(2,5,7) \rightarrow P\#P \). The time change \( \Delta t \) of this change will be determined by one channel, in contrast to the scaling of the length where one needs the whole expression (see [7]). Therefore we have to modify the usual exponent by the number of channels

\[
\frac{1}{3} \left( \frac{3}{2 \cdot CS(P\#P)} \right)
\]

to obtain the change in the time scale

\[
\Delta t = \tilde{\Delta} t \cdot \exp \left( \frac{1}{2 \cdot CS(P\#P)} \right)
\]

where \( \tilde{\Delta} t \) represents the time change for one level in the Casson handle, see above.

Then with the value

\[ CS(P\#P) = \frac{1}{60} \]

we will get for the time change of both transitions

\[
\Delta t = \frac{t_{\text{Planck}} \cdot \exp \left( -\frac{1}{2 \cdot CS(P\#P)} \right)}{1 + \vartheta + \frac{a^2}{T} + \frac{a^3}{6}}
\]

where we used [3]. Then we obtain for the corresponding energy scale

\[
M = \Delta E = \frac{E_{\text{Planck}} \cdot \exp \left( -\frac{1}{2 \cdot CS(P\#P)} \right)}{1 + \vartheta + \frac{a^2}{T} + \frac{a^3}{6}} \approx 63\text{GeV}
\] (5)

which is exactly half of the Higgs mass and we will identify this value with the second energy scale. Now we have fixed the two energy scales for the seesaw mechanism. But one ingredient is missing: which structures describes the usual left-handed neutrino in our topological scheme and how did we get the very massive right-handed neutrino (sterile neutrino). This question will be addressed in the next section.

4. Fermions and knot complements

In this section we will discuss the topological reasons for the existence of fermions. In our previous paper [21] we obtained a relation between an embedded 3-manifold and a spinor in the spacetime. The main idea can be simply described by the following line of argumentation. Let \( \iota : \Sigma \hookrightarrow M \) be an embedding of the 3-manifold \( \Sigma \) into the 4-manifold \( M \) with the normal vector \( \vec{N} \). A small neighborhood \( U_\varepsilon \) of \( \iota(\Sigma) \subset M \) looks like \( U_\varepsilon = \iota(\Sigma) \times [0,\varepsilon] \). Furthermore we identify \( \Sigma \) and \( \iota(\Sigma) \) (\( \iota \) is an embedding). Every 3-manifold admits a spin structure with a spin bundle, i.e. a principal \( Spin(3) = SU(2) \) bundle (spin bundle) as a lift of the frame bundle
(principal $SO(3)$ bundle associated to the tangent bundle). There is a (complex) vector bundle associated to the spin bundle (by a representation of the spin group), called spinor bundle $S_{\Sigma}$; see Trautman [22] for a careful definition of spinors. A section in the spinor bundle is called a spinor field (or a spinor). In case of a 4-manifold, we have to assume the existence of a spin structure. But for a manifold like $M = S^3 \times \mathbb{R}$, there is no restriction, i.e. there is always a spin structure and a spinor bundle $S_M$. In general, the unitary representation of the spin group in $D$ dimensions is $2^{[D/2]}$-dimensional. From the representational point of view, a spinor in 4 dimensions is a pair of spinors in dimension 3. Therefore, the spinor bundle $S_M$ of the 4-manifold splits into two sub-bundles $S^+_M$ where one sub-bundle, say $S^+_M$, can be related to the spinor bundle $S_{\Sigma}$ of the 3-manifold. Then the spinor bundles are related by $S_{\Sigma} = \iota^* S^+_M$ with the same relation $\phi = \iota_* \Phi$ for the spinors ($\phi \in \Gamma(S_{\Sigma})$ and $\Phi \in \Gamma(S^+_M)$). Let $\nabla_X^M, \nabla_X^\Sigma$ be the covariant derivatives in the spinor bundles along a vector field $X$ as section of the bundle $T\Sigma$. Then we have the formula

$$\nabla_X^M(\Phi) = \nabla_X^\Sigma \phi - \frac{1}{2} (\nabla_X \vec{N}) \cdot \vec{N} \cdot \phi$$

with the embedding $\phi \mapsto \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \Phi$ of the spinor spaces from the relation $\phi = \iota_* \Phi$. There are, certainly, two possible embeddings. For later use we will choose the left-handed version. The expression $\nabla_X \vec{N}$ is the second fundamental form of the embedding where the trace $tr(\nabla_X \vec{N}) = 2H$ is related to the mean curvature $H$. Then from (6) one obtains the following relation between the corresponding Dirac operators

$$D^M \Phi = D^\Sigma \phi - H \phi$$

with the Dirac operator $D^\Sigma$ on the 3-manifold $\Sigma$. This relation (as well as (9)) is only true for the small neighborhood $U_\epsilon$ where the normal vector points is parallel to the vector defined by the coordinates of the interval $[0, \epsilon]$ in $U_\epsilon$. In [23] we extend the spinor representation of an immersed surface into the 3-space to the immersion of a 3-manifold into a 4-manifold according to the work in [24]. Then the spinor $\phi$ defines directly the embedding (via an integral representation) of the 3-manifold. Then the restricted spinor $\Phi|_{\Sigma} = \phi$ is parallel transported along the normal vector and $\Phi$ is constant along the normal direction (reflecting the product structure of $U_\epsilon$). But then the spinor $\Phi$ has to fulfill

$$D^M \Phi = 0$$

in $U_\epsilon$ i.e. $\Phi$ is a parallel spinor. Finally we get

$$D^\Sigma \phi = H \phi$$

with the extra condition $|\phi|^2 = const.$ (see [24] for the explicit construction of the spinor with $|\phi|^2 = const.$ from the restriction of $\Phi$). The idea of the paper [21] was to use the Einstein-Hilbert action for a spacetime with boundary $\Sigma$. The boundary
term is the integral of the mean curvature for the boundary, see [25, 26]. Then by the relation (9) we will obtain
\[ \int_{\Sigma} H \sqrt{\hat{h}} \, d^3 x = \int_{\Sigma} \bar{\phi} D^\Sigma \phi \sqrt{\hat{h}} \, d^3 x \] (10)
using \(|\phi|^2 = \text{const.}\). As shown in [21], the extension of the spinor \( \phi \) to the 4-dimensional spinor \( \Phi \) by using the embedding
\[ \Phi = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \] (11)
can be seen as embedding, if (and only if) the 4-dimensional Dirac equation
\[ D^M \Phi = 0 \] (12)
on \( M \) is fulfilled (using relation (7)). This Dirac equation is obtained by varying the action
\[ \delta \int_{M} \bar{\Phi} D^M \Phi \sqrt{g} \, d^4 x = 0 \] (13)
In [21] we went a step further and discussed the topology of the 3-manifold leading to a fermion. On general grounds, one can show that a fermion is given by a knot complement admitting a hyperbolic structure. The connection between the knot and the particle properties is currently under investigation. But first calculations seem to imply that the particular knot is only important for the dynamical state (like the energy or momentum) but not for charges, flavors etc. But some properties can be derived from the approach above:

• The construction of the 4-spinor \( \Phi \) by (11) gives us a fermion of fixed chirality. Currently we know only one such particle, the neutrino.
• A Dirac operator over \( \Sigma \) is strongly related to this manifold, i.e. a local diffeomorphism cannot change the operator. But there are diffeomorphisms which are not connected to the identity or the diffeomorphism group is not connected. The mapping class group, the group of connection components for the diffeomorphism group, labels the different Dirac operators on \( \Sigma \) (see also [23]).

Even the last point has a strong impact on the model. Now we will consider the case that the diffeomorphism group \( \text{Diff}(\Sigma) \) has two disjoint components, i.e. \( \pi_0(\text{Diff}(\Sigma)) = \mathbb{Z}_2 \). The group \( \pi_0(\text{Diff}(\Sigma)) \) is known as mapping class group. These two different components lead to two different Dirac operators on the 3-manifold \( \Sigma \) which can be combined in a single 4-dimensional Dirac operator. In case of the neutrino, one would obtain a left-handed and a right-handed neutrino. Keeping this fact in mind, we will look at the first transition
\[ S^3 \to \Sigma(2, 5, 7) \]
and the mapping class is known to be

$$\pi_0(\text{Diff}(\Sigma(2, 5, 7))) = \mathbb{Z}_2. \quad (14)$$

Interestingly, the second transition

$$\Sigma(2, 5, 7) \rightarrow P\#P$$

leads to $P\#P$ having a trivial mapping class group

$$\pi_0(\text{Diff}(P)) = 1. \quad (15)$$

Here, we recommend the papers [27,28] for the relevant informations about mapping class groups of 3-manifolds. Therefore we have the following picture:

- At the end of the first transition $S^3 \rightarrow \Sigma(2, 5, 7)$, there is a left-handed neutrino and a right-handed neutrino.
- The right-handed neutrino is caused by the first transition. Therefore the mass of the right-handed neutrino must be proportional to the energy scale of the first transition which is the GUT scale (see [4]).
- The left-handed neutrino is not connected with any special transition. Then the corresponding mass must be (much) smaller.

For the calculation of this small mass we will use the seesaw mechanism as described in the next section.

5. Seesaw mechanism

Now it is time to join the topological approach with the seesaw mechanism to generate the neutrino mass. So, let us present an overview of this approach

- Fermions are given by hyperbolic knot complements represented by a 3D spinor. The corresponding 4D spinor is chiral and given by the embedding [11]. The corresponding particle, the neutrino, must be left-handed.
- In the first transition, $S^3 \rightarrow \Sigma(2, 5, 7)$, a right-handed neutrino is generated because of a nontrivial mapping class group [14]. It is strongly connected with this transition. The energy scale is given by [4] which gives $B \approx 0.67 \cdot 10^{15}\text{GeV}$.
- In the second transition, $\Sigma(2, 5, 7) \rightarrow P\#P$, we have only the left-handed neutrino. The energy scale can be expressed by [5] giving approximately $M \approx 63\text{GeV}$.

Now we have all ingredients to calculate the neutrino mass by the seesaw mechanism. We can start with the non-diagonal mass matrix like in the Introduction

$$\begin{pmatrix} 0 & M \\ M & B \end{pmatrix}$$
with two mass scales $B$ and $M$ fulfilling $M \ll B$. This matrix has eigenvalues

$$\lambda_1 \approx B \quad \lambda_2 \approx -\frac{M^2}{B}$$

so that $\lambda_1$ is the mass of the right-handed neutrino and $\lambda_2$ represents the mass of the left-handed neutrino. Above we fix the scales to the values (4) and (5)

$$B \approx 6.7 \cdot 10^{15} \text{GeV}, \quad M \approx 63 \text{GeV}$$

and we will obtain for the neutrino mass

$$m = \frac{M^2}{B} \approx 0.006 \text{eV}$$

The sum of the three neutrino masses is constrained by the results of the PLANCK mission (29,30). If we assume three identical masses we will get for the sum $0.018 \text{eV}$. The PLANCK mission determined only an upper value of $0.3 \text{eV}$ but the Baryon Acoustic Oscillations (BAO) lower this value to $0.12 - 0.17 \text{eV}$ for the sum in good agreement with our result. There are many experimental results (31,32) for the neutrino masses obtained with different methods (CMB, BAO, weak lensing). The smallest upper bound of $0.06 \text{eV}$ can be found in the work (33). All experimental results give only upper bounds and one has to wait for further results showing how small the neutrino masses could be. The model presented here demonstrates how to generate small masses in a natural way.

6. Conclusion

We were able to get the seesaw mechanism for generating neutrino masses in a topological model for the cosmic evolution based on two topological transitions. This model describes the cosmology of the evolving universe and is based on the exotic smoothness of $S^3 \times \Theta \subset R^4$. There are two topology changes within the 3-dimensional slice of $S^3 \times \Theta \subset R^4$, i.e. $S^3 \rightarrow \Sigma(2, 5, 7)$ related to the GUT scale, and $\Sigma(2, 5, 7) \rightarrow P \#P$ related to the electroweak scale. Both scales can be calculated by using the topological numbers of the transitions rather than just assigned it from the outside. The basic ingredient of the seesaw mechanism, i.e. the right-handed neutrino, is also topologically supported in the model. It is connected with the nontrivial mapping class group $\pi_0(\text{Diff}(\Sigma(2, 5, 7))) = \mathbb{Z}_2$ of the first transition. On the other hand the mapping class $\pi_0(\text{Diff}(P)) = 1$ of the second transition is trivial. These topological results determine much of the physics behind the seesaw mechanism which was demonstrated in our model. Again, we have to state that the results depend strongly on the usual incorporation of the Chern-Simons invariant. In this paper we used the expression (2) with the Chern-Simons invariant at the denominator. Usually the Chern-Simons invariant appeared in the numerator like in the Kodama state (12) in teleparallel gravity as the solution of the constraints (15) or to define time (19,20). But the expression (2) is crucial to get the correct results for the energy scales and neutrino masses.
We can speculate about farther consequences of the model. As explained above, the splitting (1) has a tremendous impact on the spacetime, the exotic $S^3 \times \theta \mathbb{R}$. The appearance of three $S^2 \times S^2$ has an influence on the energy scale (see section 3). In section 4 we constructed the fermion as embedded 3-manifolds (knot complements). As shown in (21), the particular Casson handle determines the fermion. Therefore the three $S^2 \times S^2$ in (1) give three Casson handles with three kinds of fermions. To express it differently, there are three generations. Currently we have no idea about the difference between the generations but as discussed above and in (10), there must exist at least three generations. Otherwise the spacetime cannot be smooth.

It is interesting to note that the three $S^2 \times S^2$ cannot be distinguished at the level of spacetime. Therefore, the whole approach must be invariant with respect to the symmetric group $S_3$ isomorphic to the dihedral group $D_3 = \langle r, s | r^3 = e = s^2, srs = r^{-1} \rangle$. We conjecture that this symmetry should appear somehow in the mixing matrix of the neutrinos (Pontecorvo-Maki-Nakagawa-Sakata matrix).

Finally, the topology of the spacetime opens also a way to produce an asymmetry between neutrinos and anti-neutrinos. In section 4 we described the appearance of the fermion. We constructed a Dirac operator $D$ on the spacetime. The kernel of this operator $\ker D$ are the solutions of the Dirac equation which correspond to the neutrino. The kernel $\ker D^\dagger$ of the conjugated operator $D^\dagger$ is associated to the anti-neutrino. Thanks to the Atiyah-Singer index theorem (see (34, 35)) there is a relation between the difference $\dim \ker D - \dim \ker D^\dagger$ and the topology of the underlying manifold or this difference is given by a topological invariant (so-called A-roof genus). In our case, one obtains

$$\dim \ker D - \dim \ker D^\dagger = 2$$

and one has an asymmetry between the neutrinos and anti-neutrinos, because the number of possible solutions are different. Interestingly, the difference is given by the number of $E_8$ factors in the splitting (1) of $E(2)\#\mathbb{CP}^2$. Currently we have no idea to quantify this difference to get in contact with current measurements.

As we have already mentioned many details must be worked out. It is ongoing work on this topic. The entire approach relating topology in dimension 3 and 4 and physics of SM is by no means exhausted, it is rather at its initial stage. It is surprising indeed how direct and specific connections of topology and physics can be worked out and how many connections are presumably hidden.

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