Proof of Atiyah-Singer Index Theorem by Canonical Quantum Mechanics

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Abstract

We show that the Atiyah-Singer index theorem of Dirac operator can be directly proved in the canonical formulation of quantum mechanics, without using the path-integral technique. This proof takes advantage of an algebraic isomorphism between Clifford algebra and exterior algebra in small \( \tau \) (high temperature) limit, together with simple properties of quantum mechanics of harmonic oscillator. Compared to the proof given by heat kernel, we try to prove this theorem more quantum mechanically.

1 Introduction

Atiyah-Singer index theorem [1], which states that the analytical index of an elliptic complex equals to the topological index of the corresponding fiber bundle, connects analysis to topology in an insightful way. In particular, the index theorem of Dirac operator [2] is extremely important in mathematics and physics. It not only unifies the Gauss-Bonnet-Chern theorem [3], Hirzebruch-Riemann-Roch theorem [4], and Hirzebruch signature theorem [5] through vector bundle isomorphism, but also plays a central role in some novel topology-induced physical phenomena, like chiral magnetic effects in heavy-ion collisions [6], and condensed matter physics [7], as well as for the analysis of zero energy modes of the graphene sheet [8, 9] et al.

In the past several decades, mathematicians have explored various methods based on topology and analysis including cobordism [10, 11], embendding [1], heat kernel [12] to prove the Atiyah-Singer index theorem. Moreover, there are also more physical proofs based on path integral or probability [13, 14] in quantum mechanics (QM) [13, 14] with an underlined key element called supersymmetry (SUSY). As we know, QM can be equivalently formulated in different frameworks: the canonical formulation in Hilbert space, Feynman path integral, and the QM in phase space [24, 25, 26, 27] et al. In the SUSY-QM proof by path integral, one needs to take great care of the measure of both Grassmannian and ordinary variables in infinite dimensional integral. In the present work, we try to show that the index theorem of Dirac operator can be concisely proved in the canonical formulation of QM.

In our proof, the Laplace operator in Hodge’s theory [28] is treated as the Hamiltonian, and the analytical index (also Witten index) is expressed as an integration on the whole manifold. By establishing an algebraic isomorphism between the exterior algebra and the Clifford algebra [29] in small \( \tau \) limit, we can quickly simplify the integrand to be a topological characteristic. This step plays a similar role as
the localization technique [13, 23] in the path integral of the SUSY-QM proof, and is also essential for the heat kernel’s proof [32]. Moreover, the $A$ characteristic can be straightforwardly obtained by the skill of canonical transformation in QM. It is found to be closely related to the quantum harmonic oscillators. Throughout the derivation, the SUSY is naturally encoded in Hodge’s theory and Clifford algebra. Therefore, our proof can be considered as parallel to the previous SUSY-QM proofs by path integral.

2 Preliminary and notations

2.1 Analytical index of Dirac operator

We first briefly introduce a $2n$-dimension (closed) spin manifold $M$ ($n \in \mathbb{N}$) and notations. Its Riemann metric takes $\delta_{\mu\nu}e^\mu \otimes e^\nu$ ($\mu$ runs from 1 to $2n$ in Einstein summation convention) in the local unitary tangent frame $e_{\mu}(x) \in \Gamma(TM)$ and cotangent frame $e^\nu(x) \in \Omega^1(M)$. The coefficients of Levi-Civita connection are given by $\Gamma^\lambda_{\mu\nu}(x)$ which induce the curvature 2-form $R = \frac{1}{2}e^\mu \wedge e^\nu R^\rho_{\mu\nu}(x) e_\rho \otimes e^\delta$. The tangent bundle $TM$ and cotangent bundle $T^*M$ further induce the Clifford bundles $\mathcal{C}\ell(M)$ and $\mathcal{C}\ell^*(M)$, with anti-commutation relation $\{\tilde{e}_\mu(x), \tilde{e}_\nu(x)\} = -2\delta_{\mu\nu}$ and $\{\tilde{e}^\mu(x), \tilde{e}^\nu(x)\} = -2\delta^{\mu\nu}$ respectively [29]. Here a negative sign is added in front of metric $\delta_{\mu\nu}$ so that the Clifford algebra naturally represents a super complex number. (Compared with the convention $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ in physics, we have $\tilde{e}_\mu = i\gamma_\mu$.) Note there is an natural linear isomorphism $\varphi : \wedge(TM) \rightarrow \mathcal{C}\ell(M)$ that $\varphi (e_{\mu_1} \wedge \cdots \wedge e_{\mu_r}) = \tilde{e}_{\mu_1} \cdots \tilde{e}_{\mu_r}$, and $\varphi (e^{\mu_1} \wedge \cdots \wedge e^{\mu_r}) = \tilde{e}^{\mu_1} \cdots \tilde{e}^{\mu_r}$.

The spin manifold $M$ induces a spinor bundle $\Delta(M) = \Delta^+(M) \oplus \Delta^-(M)$, where the section of $\Delta^\pm(M)$ represents a right/left-handed fermion field. The connection form and curvature form of the spinor bundle are given by $\tilde{\Gamma} = \frac{1}{4}e^\alpha \Gamma^\alpha_{\mu\nu} \tilde{e}^\nu \tilde{e}_\mu$ and $\tilde{R} = \frac{1}{8}e^\mu \wedge e^\nu R^\rho_{\mu\nu}(x) \tilde{e}_\rho \tilde{e}^\delta$, respectively. Moreover, the fermion field may have interaction through a gauge field, with gauge potential $\omega$ and strength $\Omega$. In mathematics, this is formulated by a twisted product between the spinor bundle $\Delta(M)$ and a Hermitian vector bundle $\pi : E \rightarrow M$, with connection 1-form $\omega = e^a \omega_\alpha$ and curvature 2-form $\Omega = \frac{1}{2}e^\mu \wedge e^\nu \Omega_{\mu\nu}$ of $E$. Therefore the spinor bundle becomes the twisted Dirac vector bundle $\Delta(M) \otimes E$ and its connection is given by $e^a \otimes D_\alpha$ with

$$D_\alpha = \partial_\alpha + \frac{1}{4} \Gamma^\nu_{\alpha\mu} \tilde{e}^\mu \tilde{e}_\nu + \omega_\alpha \tag{1}$$

as a differential operator on the space of section $\mathcal{H} = \Gamma(\Delta(M) \otimes E)$. Here $\partial_\alpha$ denotes the directional derivative along $e_\alpha$.

The twisted Dirac operator for the fermion field, a self adjoint first-order elliptic differential operator, is defined by $\mathcal{D} = \tilde{e}^\alpha D_\alpha : \mathcal{H} \rightarrow \mathcal{H}$. In physics it is usually expressed as $i\gamma_\alpha D_\alpha$ and arises in Dirac equation. The Dirac operator maps the right/left-handed fermion field to the left/right-handed one $\mathcal{D}^\pm = \tilde{e}^\alpha D_\alpha : \mathcal{H}^\pm \rightarrow \mathcal{H}^\mp$ with subspace $\mathcal{H}^\pm = \Gamma(\Delta^\pm(M) \otimes E)$. The analytical index of $\mathcal{D}^\pm$ is defined by

$$\text{ind}\mathcal{D}^\pm = \dim (\ker \mathcal{D}^\pm) - \dim (\text{coker} \mathcal{D}^\pm). \tag{2}$$

Since $\mathcal{D}^\pm$ is self-adjoint Fredholm operator, the analytical index also equals to $\text{ind}\mathcal{D}^\pm = \dim (\ker \mathcal{D}^\pm) - \dim (\ker \mathcal{D}^\mp)$, which can be regarded as the difference of
the degeneracy of the ground states between the right-hand and left-hand fermions, in other word, the asymmetry of the chirality of ground states. Then Atiyah-Singer index theorem for twisted Dirac operator claims that its analytical index equals to its topological index,

\[
\text{ind} \mathcal{D}^+ = \int_M \hat{A}(TM) \wedge \text{ch}(E),
\]

in which \(\hat{A}(TM)\) is the \(\hat{A}\) genus of the manifold \(M\) and \(\text{ch}(E)\) is the Chern character of vector bundle \(E\).

### 2.2 Witten index

The coboundary (nilpotent) operator \(\mathcal{D}^+\) in the analytical index can be replaced by Laplacian,

\[
\text{ind} \mathcal{D}^+ = \dim (\ker \Delta^+) - \dim (\ker \Delta^-),
\]

in which \(\Delta^\pm = \mathcal{D}^2 : \mathcal{H}^\pm \to \mathcal{H}^\pm\) is the Laplace operator on the right-handed fermion field (positive spinor) and left-handed fermion field (negative spinor), respectively. (Please notice \(\Delta\) denotes the Laplace operator while \(\Delta (M)\) denotes the spinor bundle.) From Hodge’s theory we learn: (1) the eigen value of the Laplacian \(\Delta = \mathcal{D}^2 : \mathcal{H} \to \mathcal{H}\) is non-negative; (2) \(\ker \Delta = \ker \mathcal{D}^+ \cap \ker \mathcal{D}^-\); (3) \([\mathcal{D}^\pm, \Delta] = 0\).

We can regard the self-adjoint elliptic operator \(H = \frac{1}{2} \Delta\) as a quantum mechanical Hamiltonian operator which determines the statistics of the quantum state and \(\mathcal{H}\) as the corresponding Hilbert space. We immediately obtain an identity \([\mathcal{D}^\pm, H] = 0\) which is known as supersymmetry, a symmetry between fermion and boson in quantum mechanics. Therefore, the SUSY is naturally encoded in the Hodge’s theory on the spinor bundle. Assuming \(\mathcal{H}^\pm_\lambda = \{\psi | H\psi = \lambda \psi, \psi \in \mathcal{H}^\pm\}\) denotes the right/left-handed eigen subspace of \(H\) with eigen value \(\lambda\), then the isomorphic relation \(\mathcal{D}^\pm \mathcal{H}^\pm_\lambda \cong \mathcal{H}^\mp_\lambda\) can be easily established for arbitrary positive energy eigenvalue \(\lambda > 0\) by the properties (1)–(3), which leads to \(\dim \mathcal{H}^+ = \dim \mathcal{H}^-\) for \(\lambda > 0\). Evidently, the ground-state subspaces with different chirality are not isomorphic to each other, and Eq. (4) actually tells \(\dim \mathcal{H}^+ - \dim \mathcal{H}^- = \text{ind} \mathcal{D}^+\).

Then the analytical index can be reduced to Witten index \([30]\). Following Ref. [30], the fermion number operator \((-1)^F : \psi^\pm \mapsto \pm \psi^\pm, \forall \psi^\pm \in \mathcal{H}^\pm\) is defined to separate different chirality,

\[
(-1)^F = i^n \tilde{e}^1 \cdots \tilde{e}^{2n} = \varphi (\ast i^n),
\]

with \(\ast\) being the Hodge star. It is also called the complex volume element in complexified Clifford algebra \(\mathbb{C}\ell\) and the chirality operator in quantum field theory. Thus, the analytical index in Eq. (1) is reduced to Witten index

\[
\text{ind} \mathcal{D}^+ = \text{tr}_{\mathcal{H}}(-1)^F e^{-\tau H},
\]

in which the contributions of the positive energy states to the index cancel completely to each other but only those of the ground states remain. Here \(\tau\) can be an arbitrary number and we choose \(\tau > 0\) for simplicity. Moreover, the expression of the Hamiltonian in Eq. (6) is given by the Weitzenböck identity of Laplacian [31],

\[
H = \frac{1}{2} \sum_{\alpha=1}^{2n} (-D^2_\alpha + \Gamma^\nu_{\alpha \sigma} D_\nu) + \frac{\mathcal{R}}{4} + \tilde{\Omega},
\]
with $\hat{\Omega} = \varphi (\Omega) = \frac{1}{2} \varepsilon^{\mu} \varepsilon^{\nu} \Omega_{\mu \nu}$ a curvature 2-form and $\mathcal{R}$ the Ricci scalar.

## 3 Proof of index theorem

### 3.1 QM on curved manifold

The key framework of our proof is the Dirac notation of QM on curved manifold. Each point $x \in M$ correspond to a ket $|x\rangle$ which forms the basis of Hilbert space $L^2(M)$. Provided coordinate $q = (q^1, \cdots, q^{2n}) : U \to \mathbb{R}^{2n}$ on open set $U \subset M$, the ket is defined by unitary relation

$$\langle x|x' \rangle = \frac{1}{\sqrt{g(x)}} \delta [q(x) - q(x')] ,$$

in which $g = \det (g_{\mu \nu})$ is the determinant of the metric $g_{\mu \nu}(x) \, dq^\mu \otimes dq^\nu$. Since the right-hand-side of Eq. (8) is a geometric invariant, the ket defined in $L^2(M)$ is actually independent on chart $(U, q)$. It is easy to check complete relation $\int_M |x\rangle \langle x| = 1$ and the trace of operator $O$ being $\text{tr} O = \int_M \tau O |x\rangle \langle x|$ with $1 = \sqrt{g} dq^1 \wedge \cdots \wedge dq^{2n}$ being the volume element. Thus the Eq. (6) reads $\text{ind } \mathcal{D}^+ = \int_M I$ with integrand

$$I(x) = \ast \text{tr}_{Cl \otimes E_x} (-1)^F \langle x| e^{-\tau H} |x\rangle .$$

Then we need to prove $I(x)$ is a topological characteristic.

We follow the usual coordinate representation in QM to define the ket basis $|q_q\rangle$ (abbreviated to $|q\rangle$) in coordinate space $q(U) \cong \mathbb{R}^{2n}$ such that

$$\langle q|q'_q\rangle = \delta (q - q') .$$

It is easy to check that $q^* |q\rangle = g^{1/4} |x\rangle$ meets the above requirement, where $q^*$ is the pull back mapping. Making use of Riemannian normal coordinate $x : U \to \mathbb{R}^{2n}$, we adopt moving frame as $q = \frac{1}{\sqrt{g}} x : U \to \mathbb{R}^{2n}$ in the following calculation. Its metric at point $x$ reads $\tau \delta_{\mu \nu} dq^\mu \otimes dq^\nu$ with $g(x) = \tau^{2n}$, which leads to $|x\rangle = \tau^{-n/2} q^* |q\rangle$. Thus through the pull-back mapping $q^* I(q) |q=0\rangle = I(x)$, the integrand Eq. (6) is expressed by coordinates,

$$I(0) = \frac{1}{\tau^n} \ast \text{tr}_{Cl \otimes E_x} (-1)^F \langle 0_q| e^{-\tau H} |0_q\rangle ,$$

in which the (scaled) Hamiltonian from Eq. (7) as well as the (scaled) connection from Eq. (11) are given by

$$\sqrt{i \tau} D_\alpha = \frac{1}{i} \frac{\partial}{\partial q^\alpha} - \frac{1}{4 \sqrt{\tau}} \Gamma^\nu_{\alpha \beta} \cdot i \tau \hat{\varepsilon}^\mu \hat{\varepsilon}_\nu - i \sqrt{\tau} \omega_\alpha ,$$

$$\tau H = \frac{1}{2} \sum_\alpha \left( \sqrt{i \tau} D_\alpha \right)^2 + \frac{\tau}{2} \hat{\Omega} + \frac{i}{2} \sum_\alpha \sqrt{\tau} \Gamma^\nu_{\alpha \beta} \cdot \frac{\sqrt{i \tau}}{i} D_\nu + \frac{\tau R}{8} .$$

Till now, the QM on $M$ has been mapped into that on $\mathbb{R}^{2n}$. 

3.2 Reduction of Clifford algebra

We notice the trace of Clifford algebra is given by $\text{tr}_{\mathbb{C}\ell}1 = 2^n$ for the zero-order algebra and $\text{tr}_{\mathbb{C}\ell}X = 0$ for the higher-order algebra, therefore, we find $(-1)^F \cdot 2^{-n}\text{tr}_{\mathbb{C}\ell}(-1)^F$ is the $2n$-order projector of $\mathbb{C}\ell$. Denoting $\mathcal{P}^r : \wedge (T^\alpha_x M) \rightarrow \wedge^r (T^\alpha_x M)$ as the $r$-form projector of the exterior algebra, one finds identity $\varphi^{-1}(-1)^F \cdot 2^{-n}\text{tr}_{\mathbb{C}\ell}(-1)^F = \mathcal{P}^{2n}\varphi^{-1}$. Relating it to $\varphi^{-1}(-1)^F = i\pi$ (see Eq. (5)), one simplifies Eq. (11) to

$$I(0) = \left(\frac{2}{i\tau}\right)^n\mathcal{P}^{2n}\varphi^{-1}\text{tr}_{E_x}(0_q)e^{-\tau H}\mid_{0_q}.$$  

Next we define a single-parameter linear isomorphism $\varphi_\epsilon : e^{\mu_1} \wedge \cdots \wedge e^{\mu_r} \mapsto e^r \tilde{e}^{\mu_1} \cdots \tilde{e}^{\mu_r}$ ($\epsilon \neq 0, \epsilon \in \mathbb{C}$) between Clifford algebra $\mathbb{C}\ell$ and complexified exterior algebra $\Lambda = \mathbb{C} \otimes \wedge (T^\alpha_x M)$. It helps distinguish the algebraic order and leads to $\frac{1}{\epsilon}\mathcal{P}^r\varphi^{-1} = \mathcal{P}^r\varphi^{-1}$. Choosing $\epsilon^2 = i\pi\tau$, one finds $\tau H = \varphi_\epsilon\Lambda^{\text{ext}}$ and $\sqrt{\tau}D_\alpha = \varphi_\epsilon D^{\text{ext}}$ with

$$H^{\text{ext}} = \frac{1}{2}\sum_\alpha \left(\frac{i}{\epsilon}D^{\text{ext}}_\alpha\right)^2 + \frac{i}{\epsilon}\sum_\alpha \sqrt{\tau}\Gamma_{\alpha\alpha}^{\nu} \frac{1}{i}D^{\text{ext}}_\nu - i\frac{\tau}{2\epsilon} \Omega + i\frac{\tau \mathcal{R}}{8},$$

$$\frac{1}{\epsilon}D^{\text{ext}}_\alpha = \frac{1}{i} \frac{\partial}{\partial q^\alpha} - \frac{1}{4\epsilon\sqrt{\tau}}\Gamma_{\alpha\alpha}^{\nu} \cdot e^\nu \wedge e_\nu - i\epsilon\sqrt{\tau}\omega_\alpha.$$  

Thus the integrand is reduced to

$$I(0) = (2\pi)^n\mathcal{P}^{2n}\varphi^{-1}\text{tr}_{E_x}(0_q)e^{-\varphi_\epsilon H^{\text{ext}}}\mid_{0_q}.$$  

Since $\lim_{|\epsilon| \rightarrow 0}\{\varphi_\epsilon(e^{\mu}) : \varphi_\epsilon(e^{\nu})\} = 0 \in \mathbb{C}\ell$ displays the anti-commutation relation of exterior algebra, then $\varphi_\epsilon : \Lambda \rightarrow \mathbb{C}\ell$ becomes an algebraic isomorphism when $|\epsilon| \rightarrow 0$, i.e., $\lim_{|\epsilon| \rightarrow 0}\varphi^{-1}_\epsilon[\varphi_\epsilon(\xi) \cdot \varphi_\epsilon(\eta)] = \xi \wedge \eta$ for all $\xi, \eta \in \Lambda$. Therefore, we obtain

$$\lim_{\tau \rightarrow 0} I(0) = (2\pi)^n\mathcal{P}^{2n}\text{tr}_{E_x}0_qe^{-H^{\text{ext}}}\mid_{0_q},$$

in which the Clifford algebra in the integrand has been reduced to the differential form.

3.3 Derivation of topological invariant

We clearly see that the terms $\sqrt{\tau}\Gamma_{\alpha\alpha}^{\nu}, \sqrt{\tau}\omega_\alpha$ and $\tau \mathcal{R}$ in Eq. (15) and Eq. (16) vanish in the limit $\tau \rightarrow 0$, so that we just need to calculate the limit of $\Gamma_{\beta\mu}^{\nu}/\sqrt{\tau}$. Returning back to the Riemannian normal coordinate $x = \sqrt{\tau} q$, one finds

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} \Gamma_{\beta\mu}^{\nu}(\sqrt{\tau} q) = q^\alpha \frac{\partial \Gamma_{\beta\mu}^{\nu}}{\partial x^\alpha} \mid_{x = 0}. $$

With the help of the definition $R_{\mu\alpha\beta}^{\nu} = \frac{\partial}{\partial x^\nu} \Gamma_{\beta\mu}^{\nu} - \frac{\partial}{\partial x^\mu} \Gamma_{\beta\alpha}^{\nu} - \frac{\partial}{\partial x^\alpha} \Gamma_{\beta\mu}^{\nu}$ and identity $\frac{\partial}{\partial x^\alpha} \Gamma_{\beta\mu}^{\nu} + \frac{\partial}{\partial x^\mu} \Gamma_{\beta\alpha}^{\nu} + \frac{\partial}{\partial x^\nu} \Gamma_{\beta\alpha}^{\nu} = 0$ at $x = 0$, one obtains the expansion of the connection

$$\frac{\partial}{\partial x^\alpha} \Gamma_{\beta\mu}^{\nu} = -\frac{1}{3} (R_{\mu\beta\alpha}^{\nu} + R_{\nu\beta\mu}^{\nu}) \mid_{x = 0}. $$
From the first Bianchi identity as well as the symmetric properties of the Riemann tensor, one gets \((2R_{\gamma\beta\mu\nu} - R_{\alpha\beta\mu\nu}) e^\mu \wedge e^\nu = 0\). Combining this equation with Eq. (20), we thus obtain

\[
\frac{\partial}{\partial x^\alpha} \Gamma^\nu_{\beta\mu} \cdot e^\alpha \wedge e^\nu = -\frac{1}{2} R_{\alpha\beta\mu\nu} e^\mu \wedge e^\nu. \tag{21}
\]

Up to now, the limit \(\tau \to 0\) in Eq. (15) is completely worked out,

\[
I(0) = \mathcal{P}^{2n} (2\pi)^n \langle 0_q | e^{-\frac{1}{2} \sum_{\beta=1}^{2n} \left( \frac{\partial}{\partial q^\beta} + \frac{i}{\hbar} q^\beta R_{\alpha\beta\mu\nu} e^\mu \wedge e^\nu \right)^2} | 0_q \rangle \text{tr}_E e^{\frac{i}{\hbar} \Omega}. \tag{22}
\]

To see the structure of Eq. (22) clearly, we treat \(p_\beta = \frac{\partial}{\partial q^\beta}\) and \(q^\alpha\) as the momentum and spatial displacement operators in the QM on coordinate space \(q(U) \cong \mathbb{R}^n\). We also denote the eigen state of an operator \(\hat{O}\) by \(|\lambda \rangle\) for eigen value \(\lambda\) (with abbreviation \(|\bar{O}\rangle\) to \(|O\rangle\)), so that \(|0_q\rangle\) is denoted by \(|0_q^0\rangle \otimes \cdots \otimes |0_q^n\rangle\). Furthermore, we apply the splitting principle of the characteristic \(\frac{1}{\hbar} \bar{R}_{2l-1, 2l, \mu\nu} e^\mu \wedge e^\nu = -y_l \in \Omega^2(M)\) \((l = 1, \cdots, n)\). Thus Eq. (22) is decomposed to

\[
I(0) = \mathcal{P}^{2n} \prod_{l=1}^n A_l \wedge \text{ch}(E) |_{q=0}. \tag{23}
\]

where \(\text{tr}_E e^{\frac{i}{\hbar} \Omega} = \text{ch}(E)\) is the Chern character of the vector bundle \(E\), and factor \(A_l\) is given by

\[
A_l = 2\pi \langle 0_{q_{2l-1}} 0_{q_{2l}} | e^{-\frac{1}{2} \left[ \left( p_{2l-1} - \frac{\hbar}{2} y_{2l-1} \right)^2 + \left( p_{2l} + \frac{\hbar}{2} y_{2l} \right)^2 \right] } | 0_{q_{2l-1}} 0_{q_{2l}} \rangle, \tag{24}
\]

We find that \(\prod_{l=1}^n A_l\) is independent on the choice of the unitary frame \(e^\mu(x)\), so that the integrand \(I\) is actually an \(SO(2n)\)-invariant polynomial regarding to the curvature forms. According to the Chern-Weil homomorphism, the integrand \(I\) is a cohomology class which is the characteristic of the twisted spinor bundle. Therefore, the local Atiyah-Singer index theorem is proved.

### 3.4 Calculation of \(\hat{A}\) characteristic

The calculation of characteristic Eq. (24), power series of 2-form \(y_l\), becomes rather straightforward. Its value can be extracted by \(A_l = A(y_l)\) with the generating function

\[
A(y) = 2\pi \langle 0_{\hat{q}_1} 0_{\hat{q}_2} | e^{-\frac{1}{2} \left[ \left( \hat{p}_{2} - \frac{\hbar}{2} \hat{q}_{2} \right)^2 + \left( \hat{p}_1 + \frac{\hbar}{2} \hat{q}_1 \right)^2 \right] } | 0_{\hat{q}_1} 0_{\hat{q}_2} \rangle, y \in \mathbb{R}, \tag{25}
\]

where \(\hat{q}_{1,2}\) and \(\hat{p}_{1,2}\) are the usual displacement and momentum operators for QM on \(\mathbb{R}^2\). Since \(A(y)\) is an even function, we just consider the case of \(y \geq 0\). After making canonical transformation

\[
\hat{q}_1 = \frac{1}{\sqrt{y}} \hat{p}_2 + \frac{\sqrt{y}}{2} \hat{q}_1, \hat{p}_1 = \frac{1}{\sqrt{y}} \hat{p}_1 - \frac{\sqrt{y}}{2} \hat{q}_2, \tag{26}
\]

\[
\hat{q}_2 = -\frac{1}{\sqrt{y}} \hat{p}_2 + \frac{\sqrt{y}}{2} \hat{q}_2, \hat{p}_2 = \frac{1}{\sqrt{y}} \hat{p}_1 + \frac{\sqrt{y}}{2} \hat{q}_2, \tag{27}
\]
with \( \left[ \hat{Q}_1, \hat{P}_1 \right] = \left[ \hat{Q}_2, \hat{P}_2 \right] = i \), we eliminate one degree of freedom in Eq. (25) and arrive at

\[
A (y) = 2\pi \langle 0_q, 0_{q_2} | e^{-\frac{y}{2} (\hat{Q}_2^2 + \hat{P}_2^2)} | 0_q, 0_{q_2} \rangle.
\]

(28)

Here \( \frac{y}{2} (\hat{Q}_2^2 + \hat{P}_2^2) \) represents the Hamiltonian of a quantum harmonic oscillator (QHO).

We also need to express \( |0_q, 0_{q_2} \rangle \) by the eigen states for new canonical variables. Actually \( |q_1 p_2 \rangle \) is the eigen state of operators \( \hat{Q}_1 \) and \( \hat{Q}_2 \) as seen in Eqs. (26-27), which means \( |q_1 p_2 \rangle = C |Q_1 Q_2 \rangle \) with \( C \in \mathbb{C} \). After choosing the following convention of representations,

\[
\langle q | q' \rangle = \delta (q - q') , \langle p | p' \rangle = \delta (p - p'), \quad (29)
\]

\[
|p\rangle = \int dq \frac{1}{\sqrt{2\pi}} e^{iqp} |q\rangle , |q\rangle = \int dp \frac{1}{\sqrt{2\pi}} e^{-iqp} |p\rangle ,
\]

we find \( C = 1 \) up to a phase factor. Then the relation between the old and new eigen states is established,

\[
|0_q, 0_{q_2} \rangle = \int dq_1 \frac{dp_2}{\sqrt{2\pi}} \delta (q_1) e^{-ip_2 0} |q_1 p_2 \rangle
\]

\[
= \frac{1}{\sqrt{2\pi}} \int dQ_1 dQ_2 \cdot \delta \left( \frac{Q_1 + Q_2}{\sqrt{y}} \right) |Q_1 Q_2 \rangle
\]

\[
= \sqrt{\frac{y}{2\pi}} \int dQ_2 \cdot \left| \left( -Q_2 \right) \hat{Q}_1 \right\rangle \otimes |Q_2 \rangle .
\]

(31)

Substituting it to Eq. (28), one obtains

\[
A (y) = y \int dQ_2 \langle Q_2 | e^{-\frac{y}{2} \left( \hat{Q}_2^2 + \hat{P}_2^2 \right)} | Q_2 \rangle = y \cdot \text{tr} e^{-\frac{y}{2} (\hat{Q}_2^2 + \hat{P}_2^2)}.
\]

(32)

We see that \( A (y) \) is closely related to the partition function of QHO. Since the eigenvalue of \( \frac{1}{2} \left( \hat{Q}_2^2 + \hat{P}_2^2 \right) \) reads \( m + \frac{1}{2} \) with non-negative integer \( m \), we reach

\[
A (y) = ye^{-y/2} \sum_{m=0}^{\infty} e^{-my} = \frac{y/2}{\sinh (y/2)}.
\]

(33)

Therefore, factor \( \prod_{l=1}^{n} A_l = \prod_{l=1}^{n} \frac{y/2}{\sinh (y/2)} \) is exactly the \( \hat{A} \) characteristic for the manifold \( M \). Relating it to Eq. (23), we obtain \( I (x) = \mathcal{P}^{2n} \hat{A} (TM) \wedge \text{ch} (E) \).

Finally, we make integration and prove the Atiyah-Singer index theorem

\[
\text{ind} \mathcal{B}^+ = \int_M \hat{A} (TM) \wedge \text{ch} (E) .
\]

(34)

4 Summary

The profound Atiyah-Singer theorem of Dirac operator have given many inspirations to physicists and mathematician to understand the geometry, topology and quantum behaviour of particles and fields in microscopic world. In mathematics, it relates
analysis to topology, and in physics, it relates high temperature physics to low temperature physics. We illustrate how this theorem can be concisely proved in the canonical formulation of QM. Firstly, the analytical index is expressed as the Witten index in the QM with the Hamiltonian given by the Weitzenböck identity. In the derivation, the SUSY is naturally encoded in the properties of Hodge’s theory and Clifford algebra. Secondly, in the canonical formulation of QM, the Clifford algebra is reduced to the exterior differential form through their algebraic isomorphism in small $\tau$ limit, leading to the emergence of topological characteristics. This step plays a similar role as the Localization technique in the path integral formulation for the SUSY-QM models. Finally, the expression of $\hat{A}$ characteristic is obtained with the help of the property of quantum harmonic oscillator, while the Chern character naturally appears in the topological index. Therefore, our proof can be regarded as an independent version to the SUSY-QM proof by using path integral.

References

[1] M. F. Atiyah and I. M. Singer, The index of elliptic operators, Ann. Math. 87 (1968) 484, 531, 546; 93 (1971) 119, 139.

[2] T. Friedrich, Dirac Operators in Riemannian Geometry, Graduate Studies in Mathematics 25, AMS, Providence, USA (2000).

[3] S. S. Chern, On the curvatura integra in a Riemannian manifold, Ann. Math. 46 (1945) 674.

[4] F. Hirzebruch, Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties, Proc. Natl. Acad. Sci. U.S.A. 40, (1954) 110.

[5] F. Hirzebruch, Topological methods in algebraic geometry, Springer, third edition, (1978).

[6] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Chiral magnetic effect, Phys. Rev. D 78 (2008) 074033.

[7] Q. Li, et al, Chiral magnetic effect in ZrTe$_5$, Nature Phys. 12 (2016) 550.

[8] J. K. Pachos and M. Stone, An index theorem for graphene, Int. J. Mod. Phys. B 21 (2007) 5113.

[9] B. Dietz, et al, Fullerene C$_{60}$ simulated with a superconducting microwave resonator and test of the Atiyah-Singer index theorem, Phys. Rev. Lett. 115 (2015) 026801.

[10] M. F. Atiyah, The index of elliptic operators on compact manifolds, Sem. Bourbaki, Exp. 253 (1963).

[11] R. Palais, Seminar on the Atiyah-Singer Index Theorem, Ann. of Math. Studies 57, Princeton University Press, Princeton, (1965).
[12] M. F. Atiyah and I. M. Singer, The index of elliptic operators on compact manifolds, Bull. Am. Math. Soc. 69 (1963) 422.

[13] L. Alvarez-Gaumé, Supersymmetry and the Atiyah-Singer index theorem, Commun. Math. Phys. 90 (1983) 161.

[14] D. Friedan and P. Windey, Supersymmetry and index theorems, Physica D15 (1985) 71.

[15] G. Parisi and N. Sourlas, Supersymmetric field theories and stochastic differential equations, Nucl. Phys. B206 (1982) 321.

[16] P. Salomonson and J. W. van Holten, Fermionic coordinates and supersymmetry in quantum mechanics, Nucl. Phys. B196 (1982) 509.

[17] C. A. Blockley and G. A. Stedman, Simple supersymmetry: I. Basic examples, Eur. J. Phys. 6 (1985) 218.

[18] L. F. Urrutia and E. Hernandez, Long-range behavior of nuclear forces as a manifestation of supersymmetry in nature, Phys. Rev. Lett. 51 (1983) 755.

[19] A. Khare and J.Maharana, Supersymmetry quantum mechanics in one, two and three dimensions, Nucl. Phys. B244 (1984) 409.

[20] A. B. Balantekin, Accidental degeneracies and supersymmetric quantum mechanics, Ann. Phys. (N.Y.) 164 (1985) 277.

[21] V. A. Kostelecký and M. M. Nieto, Evidence for a phenomenological supersymmetry in atomic physics, Phys. Rev. Lett. 53 (1984) 2285.

[22] A. R. P. Rau, Comments on "Evidence for a phenomenological supersymmetry in atomic physics", Phys. Rev. Lett. 56 (1986) 95.

[23] A. Schwarz, O. Zaboronsky, Supersymmetry and localization, Commun. Math. Phys. 183 (1997) 463.

[24] E. Wigner, On the quantum correction for thermodynamics equilibrium, Phys. Rev. 40 (1932) 749.

[25] J. E. Moyal, Quantum mechanics as a statistical theory, Proc. Camb. Phil. Soc. 45 (1949) 99.

[26] H. J. Groenewold, On the principles of elementary quantum mechanics, Physica 12 (1946) 405.

[27] R. J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. 378 (2003) 207.

[28] M. Nakahara, Geometry, Topology and Physics, 2nd Edition, Graduate Student Series in Physics, Taylor & Francis, (2003).
[29] H. B. Lawson, and M. L. Michelsohn, Spin Geometry, Princeton University Press (1989).

[30] E. Witten, Supersymmetry and morse theory, J. Diff. Geometry 17 (1982) 661.

[31] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley, New York, (1978).

[32] E. Getzler, A Short Proof of the Local Atiyah-Singer Index Theorem, Topology 25, (1985) 111.