Nonexistence of Petrov type III Space-Times
on which Weyl’s Neutrino Equation or Maxwell’s Equations
satisfy Huygens’ Principle

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Abstract. — Extending previous results we show that there are no Petrov type III
space-times on which either the Weyl neutrino equation or Maxwell’s equations satisfy
Huygens’ principle. We prove the result by using Maple’s NPspinor package to con-
vert the five-index necessary condition obtained by Alvarez and Wünsch to dyad form.
The integrability conditions of the problem lead to a system of polynomial equations.
We then apply Maple’s grobner package to show that this system has no admissible
solutions.

1 Introduction

This paper is the sixth in a series devoted to the solution of Hadamard’s problem for
the conformally invariant scalar wave equation, Weyl’s neutrino equation and source-free
Maxwell’s equations. These equations can be written respectively as

\[ \Box u + \frac{1}{6} Ru = 0, \]  
\[ \nabla^A \varphi_A = 0, \]  
\[ \nabla^{AB} \varphi_{AB} = 0. \]  

The conventions and formalism used in this paper are those of Carminati and McLe-
naghan [6] (CM in the sequel). All considerations in this paper are entirely local. Part of
the calculations presented here were performed using the NPspinor package available in
Maple [7, 8].

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Huygens’ principle is valid for eqs. (1), (2) and (3) if and only if for every Cauchy initial value problem and every point \( x_0 \) in the 4-dimensional pseudo-Riemannian space \( V_4 \), the solution depends only on the Cauchy data in an arbitrarily small neighborhood of \( S \cap C^{-}(x_0) \), where \( S \) denotes the initial surface and \( C^{-}(x_0) \) the past null conoid from \( x_0 \) \cite{12, 20, 10}. Hadamard’s problem for equations (1), (2) and (3), originally posed only for scalar equations, is that of determining all space-times for which Huygens’ principle is valid for a particular equation. As a consequence of the conformal invariance of the validity of Huygens’ principle, the determination may only be effected up to an arbitrary conformal transformation on the metric in \( V_4 \)

\[
\tilde{g}_{ab} = e^{2\varphi}g_{ab},
\]

where \( \varphi \) is an arbitrary real function.

It is known that Huygens’ principle is valid for (1), (2) and (3) on any space-time conformally related to the exact plane-wave \cite{10, 13, 21}, with metric

\[
ds^2 = 2dv \left[ du + [D(v)z^2 + \overline{D}(v)\overline{z}^2 + e(v)z\overline{z}]dv \right] - 2d\overline{z}d\overline{\sigma},
\]

in a special coordinate system where \( D \) and \( e \) are arbitrary functions. These are the only known space-times on which Huygens’ principle is valid for these equations. Furthermore, it has been shown \cite{11, 14, 21} that these are the only conformally empty space-times on which Huygens’ principle is valid.

In the non-conformally empty case several results are known. In particular for Petrov type N, the following result has been proved \cite{3, 4, 2}: Every Petrov type N space-time on which the equations (1), (2) and (3) satisfy Huygens’ principle is conformally related to an exact plane wave space-time (5). For the case of a Petrov type D the following result has been established \cite{5, 22, 17}: There exist no Petrov type D space-times on which equations (1), (2) or (3) satisfy Huygens’ principle.

In this paper we complete the analysis for Petrov type III spacetimes given by CM, in the case of equations (1) and (2). Our main result is contained in the following theorem:

**Theorem 1.** There exist no Petrov type III space-times on which Weyl’s equation (2) or Maxwell’s equations (3) satisfy Huygens’ principle.

### 2 Previous results

Let \( \Psi_{ABCD} \) denote the Weyl spinor. Petrov type III space-times are characterized by the existence of a spinor field \( o^A \) satisfying

\[
\Psi_{ABCD}o^D = 0, \quad \Psi_{ABCD}o^D \neq 0.
\]

Such spinor field is called a *repeated principal spinor* of the Weyl spinor and is determined by the latter up to an arbitrary variable complex factor. Let \( i^A \) be any spinor field satisfying

\[
o_Ai^A = 1.
\]
The ordered set \( \{ o_A, \iota_A \} \), called a dyad, defines a basis for the 1-spinor fields on \( V_4 \). The main results obtained by CM can then be stated as follows:

**Theorem 2.** The validity of Huygens’ principle for the conformally invariant scalar wave equation (1), or Maxwell’s equations (2), or Weyl’s neutrino equation (3) on any Petrov type III space-time implies that the space-time is conformally related to one in which every repeated principal spinor field \( o_A \) of the Weyl spinor is recurrent, that is

\[
\begin{align*}
o_{A;BB} &= o_A I_{BB},
\end{align*}
\]

where \( I_{BB} \) is a 2-spinor, and

\[
\begin{align*}
\Psi_{ABCD;EE} l_A l_B l_C o^D o^E o^F &= 0, \\
R &= 0, \\
\Phi_{AB\dot{A}\dot{B}} o^A o^B &= 0.
\end{align*}
\]

**Theorem 3.** If any one of the following three conditions

\[
\begin{align*}
\Psi_{ABCD;EE} l_A l_B l_C o^D o^E o^F &= 0, \\
\Phi_{AB\dot{A}\dot{B}} o^A o^B &= 0.
\end{align*}
\]

is satisfied, then there exist no Petrov type III space-times on which the conformally invariant scalar wave equation (1) or Maxwell’s equations (2), or Weyl’s neutrino equation (3) satisfies Huygens’ principle.

It will be proved here that conditions (11) to (13) are superfluous in the cases of equations (2) and (3), i.e., they are consequences of the necessary conditions for the validity of Huygens’ principle, in particular of the five-index necessary conditions derived by Alvarez and Wünsch [11, 2].

### 3 Necessary conditions

In order to prove Theorem 1 we shall need the following necessary conditions for the validity of Huygens’ principle [9, 19, 16, 15, 20]:

\[
\begin{align*}
(III) \quad S^{kl} - \frac{1}{2} C_{ab}^l L^k_{kl} &= 0,
\end{align*}
\]

\[
\begin{align*}
(V) \quad TS \left[ k_1 C_{ab}^l l^m C_{cdl;m} + 2 k_2 C_{ab}^l l^m S_{kld} + 2(8 k_1 - k_2) S_{ab}^k S_{cd} - 2 k_2 C_{ab}^l l^m S_{cd} + k_2 C_{ab}^l l^m C_{ck} l^m_{ck} L_{dm} + 4 k_1 C_{ab}^l l^m C_{cd} l^m_{km} L_{km} \right] &= 0.
\end{align*}
\]
where
\[ C_{abcd} := R_{abcd} - 2g_{[a[d} L_{b]c]} , \]
\[ L_{ab} := -R_{ab} + \frac{1}{6} R g_{ab} , \]
\[ S_{abc} := L_{a[bkc]} . \]

In the above \( C_{abcd} \) denotes the Weyl tensor, \( R_{ab} \) the Ricci tensor and \( TS[ \ ] \) the operator which takes the trace free symmetric part of the enclosed tensor. In \( \{15\} \) \( k_1 \) takes values 3, 8, 5 and \( k_2 \) the values 4, 13, 16, depending on whether the equation under consideration is the conformally invariant scalar equation, Weyl’s equation or Maxwell’s equations respectively.

We shall also need a necessary condition involving five free indices, valid for Weyl’s equation (2) and Maxwell’s equations (3), that was found by Alvarez and Wünsch \[12\]. It can be written in the form
\[ T^{(1)}_{abce} + \sigma_1 T^{(2)}_{abce} + \sigma_2 T^{(3)}_{abce} = 0 , \]
where \( \sigma_1 \) and \( \sigma_2 \) are fixed real numbers, and

\[ T^{(1)}_{abce} := TS[4^*C^k_{ab} l C^u_{de;ukc} - 6^*C^k_{bc} l C^u_{de;uk} + 26^*C^u_{ab} k C^v_{de;vc} +\]
\[ *C^k_{ab} l n C^l_{kde;nc} + 5^*C^k_{ab} l C^m_{de;Lkn} + 4^*C^k_{ab} l C^m_{cd;kLn} + 4^*C^k_{ab} l C^m_{kcl;uLde} - 21^*C^k_{ab} l C^m_{cd;kLde} ] , \]
\[ T^{(2)}_{abce} := TS[-12^*C^k_{ab} l C^l_{kcd;nu} - 4^*C^k_{ab} l C^m_{nu;lc;u} - 4^*C^k_{ab} l C^m_{nu;lc;u} + 5^*C^k_{ab} l C^m_{nu;lc;u} + 4^*C^k_{ab} l C^m_{nu;lc;u} - 21^*C^k_{ab} l C^m_{cd;kLde} ] , \]
\[ T^{(3)}_{abce} := TS[-8^*C^k_{ab} l C^l_{kcd;nu} + 4^*C^k_{ab} l C^m_{nu;lc;u} + 4^*C^k_{ab} l C^m_{nu;lc;u} - 21^*C^k_{ab} l C^m_{cd;kLde} ] , \]

where
\[ *C^k_{abcd} := \frac{1}{2} \epsilon^{ef} C^{efk} . \]

It is worth to mention that, in our conventions, the Riemann tensor, Ricci tensor and the Ricci scalar have opposite sign to those used by Alvarez and Wünsch \[12\]. The spinor equivalents of conditions III and V are given, respectively, by \[\{17\} \]
\[ \Psi_{ABKL} K^L_A B + \Psi_{AB} KL \Phi_{KLÂB} = 0 , \]

\[ k_1 \Psi_{ABCD} K^K_{KK} \tilde{\Psi}_{ABCD} K^L_K + k_2 \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K \]
\[ + k_2 \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K - 2(8k_1 - k_2) \Psi_{ABCD} K^L_K (A|BCD) K^L_K \]
\[ - k_2 \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K ) - k_2 \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K ) + 4k_1 \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K ) + 4k_1 \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K \]
\[ + 2(k_2 - 4k_1) \Psi^K_{(ABC;D)(A|BCD)L; |K} K^L_K ) - 2(4k_1 + k_2) \Lambda \Psi_{ABCD} \tilde{\Psi}_{ABCD} = 0 . \]
4 Proof of Theorem 1

In CM, Theorem 2 was proved by using conditions III and V. The explicit form of these necessary conditions is obtained by first converting the spinorial expressions to the dyad form and then contracting them with appropriate products of $\sigma^A$ and $\iota^A$ and their complex conjugates. In particular, it was shown, that there exists a dyad $\{o_A, \iota_A\}$ and a conformal transformation such that

$$\kappa = \sigma = \rho = \tau = \epsilon = 0, \quad (26)$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \quad \Psi_3 = -1, \quad (27)$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = R = 0, \quad (28)$$

$$D\alpha = D\beta = 0, \quad (29)$$

$$\delta_{11} = c, \quad (30)$$

where $c$ is a constant. By contracting condition (III) with $\iota^{AB}\bar{\sigma}^A\bar{\iota}^B$ and $\iota^A o^B \bar{\sigma}^A\bar{\iota}^B$ (where the notation $o_{A_1\cdots A_p} = o_{A_1} \cdots o_{A_p}$, etc. has been used) we get, respectively,

$$D\pi = 0, \quad (31)$$

$$\delta\beta = -\beta(\bar{\pi} + \beta). \quad (32)$$

From the Bianchi identities, using the above conditions, we obtain

$$D\Phi_{12} = 2\bar{\pi}\Phi_{11}, \quad (33)$$

$$D\Phi_{22} = -2(\beta + \bar{\beta}) + 2\Phi_{21}\bar{\pi} + 2\Phi_{12}\pi, \quad (34)$$

$$\delta\Phi_{12} = 2\bar{\pi} + 4\pi + 2\bar{\lambda}\Phi_{11} - 2\bar{\pi}\Phi_{12}, \quad (35)$$

$$\bar{\delta}\Phi_{12} = -2\beta + 2\pi\Phi_{11} - 2\bar{\beta}\Phi_{12}, \quad (36)$$

$$\delta\Phi_{22} = \Delta\Phi_{12} + 2\pi + 4\pi - 2\bar{\pi}\Phi_{11} + 2\bar{\lambda}\Phi_{21} + 2\Phi_{12}\mu - 2\Phi_{22}\beta - 2\Phi_{22}\bar{\pi} + 2\bar{\pi}\Phi_{12}. \quad (37)$$

Using Ricci identities we get

$$D\gamma = \bar{\pi}\alpha + \beta\pi + \Phi_{11}, \quad (38)$$

$$\delta\pi = D\bar{\lambda} - \pi^2 - \bar{\pi}\alpha + \pi\beta, \quad (39)$$

$$D\bar{\pi} = \Delta\pi + \bar{\pi}\mu + \bar{\lambda}\pi + \bar{\pi}\bar{\gamma} - \bar{\pi} - 1 + \Phi_{12}, \quad (40)$$

$$\delta\alpha = \bar{\lambda} + \alpha\pi + \beta\beta - 2\beta\alpha + \Phi_{11}, \quad (41)$$

$$\delta\pi = D\mu - \pi\alpha + \bar{\pi}\beta - \pi\beta. \quad (42)$$

We can obtain useful integrability conditions for the above identities by using NP commutation relations. Using (33), (34), (37), (32), (38), (39), (40) and (42) in the commutator expression $[\delta, D]\Phi_{22} - [\Delta, D]\Phi_{12}$, gives

$$\delta\beta = -2\Phi_{11} - \bar{\beta}\alpha - 4\bar{\beta}\pi - 2D\pi - \beta\beta + 2\pi\beta. \quad (43)$$
We shall consider both Maxwell and Weyl cases separately. We begin with the Maxwell equations, i.e., $k_1 = 5$ and $k_2 = 16$. By contracting condition $V$ with $\iota^{ABCD} \xi_{ABCD}$, we get
\[
62\beta\pi + 40\beta\alpha + 6\pi\alpha + 3\alpha^2 + \delta\alpha + 2\beta\pi + \delta\beta + \beta^2 = 0. \tag{44}
\]
Substituting (32) into this equation results in
\[
\delta(2\pi + \alpha) = -62\beta\pi - 39\beta\alpha - 6\pi\alpha - 3\alpha^2 . \tag{45}
\]
Now, from (41), (42) and (43) we obtain
\[
\delta(2\pi + \alpha) = 2\pi\alpha + \alpha\pi - 6\beta\pi - 3\beta\alpha - \Phi_{11} . \tag{46}
\]
Contracting condition $V$ with $\iota^{ABC} \sigma_{D}^{B} \xi_{BCD}$, using (33), (42) and the complex conjugate of (43), we get
\[
148\Phi_{11} + 152\beta\pi + 76(D\mu + D\bar{\mu}) - 8\pi\alpha - 104\beta\pi - 8\alpha\pi - 232\pi\alpha + 152\beta\pi - 8\pi\alpha = 0. \tag{47}
\]
Using (45), (46), (32), (41), (43) and (47) in $[\iota, \delta](\alpha + 2\pi) = (\alpha - \beta)(\delta\alpha + 2\pi) + (-\alpha + \beta)\delta(\alpha + 2\pi)$, we obtain one expression for $D\bar{\mu}$. Substituting it in (47) and solving for $D\mu$ we obtain
\[
D\mu = \frac{1}{152} \left( -1520\alpha\beta\pi + 208\alpha\pi\pi - 152\beta\pi\pi + 1040\beta\beta\alpha + 1968\beta\beta\pi \\
+ 1216\beta\pi\pi - 1228\pi\Phi_{11} + 1688\alpha\pi\pi - 739\alpha\Phi_{11} + 80\alpha^2\pi + 380\beta\Phi_{11} \\
+ 2496\pi\pi^2 - 2432\beta\pi^2 + 128\pi^2\pi + 80\alpha^2\pi + 760\beta\alpha\pi \right) / (-\beta + 5\alpha + 8\pi) , \tag{48}
\]
where we have assumed that $\beta - 5\alpha - 8\pi \neq 0$. By substituting this equation into (46), we find:
\[
D\bar{\mu} = \frac{1}{152} \left( 776\alpha\beta\alpha - 304\beta^2\pi + 208\beta\beta\pi + 84\beta\Phi_{11} - 1216\pi\pi^2 \\
+ 741\alpha\Phi_{11} + 1536\alpha\beta\beta - 760\alpha\pi\pi + 2744\pi\beta\pi + 1332\pi\beta\pi \\
+ 741\alpha\Phi_{11} + 1140\pi\Phi_{11} \right) / (\beta - 5\alpha - 8\pi) . \tag{49}
\]
We note that (48) and (49) have the same denominator. So, in what follows we shall use the Pfaffians $\delta\alpha$, $\delta\pi$, and $\delta\beta$, given by (11), (12) and (13), respectively, and their complex conjugates, in such a way that they have all the same denominator. This procedure simplifies the expressions to be obtained from the integrability conditions.

Now need to convert (19) to the spinor form in the dyad basis. The resulting expression, obtained using Maple’s package NPspinor [5], has a considerable size, specially due to the term in (20) containing a third order derivative of the Weyl tensor, and will not be presented here. However, among the twenty one independent spinor components, we found a relatively simple one, obtained by contracting the dyad expression with $\iota^{AB} \sigma_{CDE} \xi_{ABCD}$, It has the following form:
\[
-14\alpha\delta\pi - 12\alpha^3 - \delta(\delta(\alpha + 2\pi)) - 21\beta\alpha\pi + 7\lambda\beta^2 + 14\beta^2\pi \\
-7\beta\delta(\pi + 2\pi) - 42\beta\alpha\pi - (10\pi + 6\pi)\delta\pi - 24\pi^2\pi = 0. \tag{50}
\]
We observe that the terms (21) and (22) did not contribute to this component. Using (45) to eliminate $\delta \pi$ from this equation, and solving for $\delta \pi$ we get

$$\delta \pi = 192 \beta \pi - 3 \pi^2 + 121 \beta \pi \pi$$

(51)

and

$$\delta \pi = -12 \beta \pi \beta - 80 \beta \pi - 3 \pi \alpha \pi.$$  

(52)

We have now all the Pfaffians we need to complete the proof. The integrability conditions provided by the NP commutation relations can now be used. Let us consider the NP commutator $[\overline{\delta}, \delta] \alpha$. Using the Pfaffians calculated previously, and solving for $\Phi_{11}$, we obtain

\begin{align*}
\Phi_{11} & := -8 \overline{\beta} (8 \pi + 5 \alpha)(172736 \pi^2 - 7776 \pi^2 \pi - 13294 \alpha \pi \pi + 7866 \beta \pi \pi
+ 211556 \alpha \pi \pi + 9568 \overline{\beta} \beta \pi - 22572 \beta \pi \pi + 5330 \overline{\beta} \beta \alpha + 4845 \overline{\beta} \alpha \pi \\
& - 1805 \overline{\beta}^2 \pi - 2470 \beta \overline{\beta}^2 - 13110 \alpha \pi \alpha \beta - 5290 \alpha^2 \pi \alpha + 65010 \alpha^2 \alpha^2) / \\
& (772320 \pi^2 \overline{\beta} - 2048352 \alpha \pi^2 - 1158240 \pi \alpha^2 + 4085 \overline{\alpha} \overline{\beta}^2 + 10868 \pi \overline{\beta}^2 \\
& - 335985 \alpha \overline{\beta}^2 \pi - 1020276 \alpha \pi \overline{\beta} + 214700 \alpha \rho^3 + 1191680 \pi^3).  
\end{align*}

(53)

(54)

On the other hand, from the commutator $[\overline{\delta}, \delta](\alpha + \overline{\beta})$ the following expression for $\Phi_{11}$ results:

\begin{align*}
\Phi_{11} & := -8 \overline{\beta} (-20672 \overline{\beta} \pi \pi - 12920 \pi \alpha \overline{\beta} + 8056 \overline{\beta} \pi \pi + 172736 \pi^2 \pi^2 \\
& - 7776 \pi^2 \pi + 9568 \overline{\beta} \beta \pi + 5035 \overline{\beta} \alpha \alpha - 5290 \alpha^2 \pi \alpha + 65010 \alpha^2 \alpha^2) / (-13047 \beta \alpha) \\
& + 42940 \alpha^2 + 162944 \pi \alpha + 10412 \beta \beta - 18564 \pi \overline{\beta} + 148960 \pi^2).  
\end{align*}

(55)

(56)

Using the fact that $\delta(\Phi_{11}) = 0$, we obtain, from (55), a third expression for $\Phi_{11}$:

\begin{align*}
\Phi_{11} & := 8 \overline{\beta} (-48191081692160 \pi^3 \pi^2 + 180931104170496 \pi^3 \alpha^2 \\
& + 7968511840 \overline{\beta}^2 \pi \alpha - 265004094784 \overline{\beta} \beta \pi^2 - 5268590200832 \pi \pi \pi \overline{\beta} \\
& + 24894675520 \overline{\beta} \beta \beta \pi + 221453789400 \pi \alpha \overline{\beta} \overline{\beta} - 20739582848 \overline{\beta} \overline{\beta} \pi \pi \\
& + 4980319900 \overline{\beta} \pi \pi - 30330200378072 \pi \pi \pi \alpha \alpha - 1208861900450 \alpha^5 \alpha \\
& + 522592181050 \pi \alpha \alpha + 724772046800 \overline{\beta} \pi \pi \alpha \\
& - 3675701240760 \overline{\beta} \pi \pi \alpha^2 - 6145239989312 \overline{\beta} \pi \pi^2 \alpha \\
& - 7649757648780 \overline{\beta} \pi \pi \alpha^3 - 16323677160464 \overline{\beta} \pi \pi \pi \alpha^2 \\
& - 15274850502912 \overline{\beta} \pi \pi \pi \pi \alpha - 26601021440 \overline{\beta} \pi \pi \alpha \\
& - 1040399202440 \overline{\beta} \pi \pi \pi \pi \alpha \alpha - 136570135136 \overline{\beta} \pi \pi \pi \pi \pi \alpha \\
& + 5632482563850 \overline{\beta} \pi \pi \pi \pi \pi \alpha^3 + 13162176133400 \overline{\beta} \pi \pi \pi \pi^2 \alpha \alpha \\
& + 13687783600768 \pi \pi \pi \pi \pi \alpha - 329130147840 \overline{\beta} \pi \pi \pi \pi \pi \alpha \\
& + 265058659320 \overline{\beta} \pi \beta \pi^2 \alpha^2 + 469041329536 \overline{\beta} \pi \beta \pi^2 \alpha \alpha \\
& + 1952913512680 \overline{\beta} \pi \beta \pi^2 \alpha^3 + 4375379783424 \overline{\beta} \pi \beta \pi^2 \alpha \alpha \\
& + 4338917366784 \overline{\beta} \pi \beta \pi^2 \alpha^3 - 102469601500 \overline{\beta} \beta \alpha^2 + 49609754350 \overline{\beta} \beta \alpha^3 \\
& + 325649974650 \overline{\beta} \beta \alpha^4 + 15559172200 \overline{\beta} \beta \alpha^4 - 38360907652096 \pi \pi \alpha \alpha.
\end{align*}

(57)

(58)
\[
+ 39628423187260 \pi \alpha^4 \pi - 1329967209650 \pi \alpha^4 \beta - 858758431040 \pi^2 \alpha^3 \pi
- 20447502080 \beta^2 \pi^2 \pi - 8524238850 \beta^3 \alpha^2 \pi - 12779688800 \pi \alpha \beta^4
+ 905003032125 \pi \alpha^4 \pi + 136081885849600 \pi \alpha^4 \alpha
+ 11991507351888 \pi \alpha^2 \alpha^3 - 95640595390 \alpha^4 \alpha \pi
+ 534491133400 \pi \alpha \beta - 12237656133632 \pi^2 \beta \pi + 1606267826176 \beta \pi^4
+ 4081870077952 \pi \alpha^5 - 730927683900 \pi \alpha^3 \beta^2 - 220969445675 \beta^2 \alpha^3 \pi
+ 274949282816 \beta^5 \beta \pi^3 + 592713574016 \beta^3 \pi \pi^2 - 341693169632 \beta^3 \pi \pi^3) /\\[(−13047 \beta \alpha + 42940 \alpha^2 + 162944 \pi a + 10412 \beta^2 - 18564 \pi \beta + 148960 \pi^2)(102596352 \pi^3 - 7684576 \pi^2 \beta + 194465152 \alpha \pi^2 + 24320 \pi \beta^2 - 9460852 \alpha \pi \beta
+ 123050612 \pi \alpha^2 + 25995895 \alpha^3 - 133000 \alpha \beta^2 - 2915745 \alpha^2 \beta)].
\]

We now suppose that the denominators in these three expressions for \( \Phi_{11} \) are different from zero. Later we consider the cases in which each of them is different from zero. We also suppose that spin coefficients \( \alpha, \beta, \pi \) are different from zero. It was shown in CM that if one of these spin coefficients is equal to zero, the others must be zero too, and Huygens’ principle is violated.

The next step consists in proving that (58), (59), and (60) imply that \( \alpha, \beta \) and \( \pi \) are proportional to each other. In order to get a system of with only two complex variables, instead of three, new variables are defined as follows:

\[
x_1 := \frac{\alpha}{\pi}, \quad x_2 := \frac{\beta}{\pi}.
\]

By subtracting (58) from (59), taking the numerator and dividing by \((8-x_2-5x_1)(5776x_2^2\pi \pi)\), we get

\[
N_1 := 178100 x_1 x_2 x_2^2 + 284960 x_2 x_2^2 + 208240 x_2^2 \pi \pi + 130150 x_2 x_2^2 x_1
+ 109825 x_2 x_2^2 x_2^2 + 252850 x_2 x_2^2 x_2^2 + 523744 x_2 x_2^2 + 3451480 x_1 x_2^2
+ 341900 x_2 x_1 x_1 + 265888 x_2 x_1 + 735800 x_2 x_2 + 2915264 x_2
+ 1018400 x_2 x_2^2 - 879008 x_2^2 + 18263488 - 408050 x_2 x_2^2 + 4864450 x_2^3
+ 35335248 x_1 + 22731900 x_2^2 - 2101032 x_1 x_1 - 1622550 x_2^2 x_2 = 0.
\]

Subtracting (59) from (60), taking the numerator and dividing by \((8-x_2-5x_1)(152x_2^2\pi)\), gives

\[
N_2 := -11651821200 x_1^3 - 26531539120 x_1^3 - 10132263424 - 26800629441 x_1
+ 242619584 x_2^2 + 593671488 x_2^2 + 2256829184 x_2^2 - 35155250 x_2^2 x_2 + 2 x_1^2
+ 21550100 x_2^2 x_1 - 12858950 x_2^2 x_1^3 + 11036720 x_2^2 x_1^3 x_1
+ 21755825 x_2^2 x_1 x_1^3 + 130839250 x_1^4 x_1^3 - 216631750 x_2 x_1 x_1^3
- 17700400 x_2^3 x_1 + 372958500 x_1^4 x_1^3 + 22060070 x_2^2 x_2^2 + 34480160 x_2 x_2^2
- 1915591500 x_1^3 - 28326040 x_2^3 x_1 + 6897950 x_2^3 x_1 x_1^3 + 724005800 x_2 x_2^2 x_2^2 x_2
- 112640320 x_1 x_2 x_2^2 - 90878112 x_2 x_2^2 + 59839936 x_2^2 x_2^2 x_2
+ 7220928 x_1 x_2 x_2^2 x_2 - 1002142990 x_1 x_2 x_1 x_2^2 - 585508600 x_2 x_2^2 x_1^2
- 448045312 x_2 x_2^2 + 632690016 x_1 x_2 - 1546809184 x_2 x_1 x_1^3 - 796702240 x_1 x_2 x_1
- 887976960 x_2 x_2 x_1 + 599836460 x_2 x_1^2 + 2328634300 x_1 x_2
+ 5728848860 x_1 x_2 + 5470002280 x_1 x_2 = 0.
\]
At this point we shall consider $x_1$, $x_2$, $x_3$, $x_4$ as independent variables, and use the package \texttt{groebner} in Maple, for the polynomial system formed by the polynomials $N_1$, $N_2$. This package computes a collection of reduced (lexicographic) Gröbner bases corresponding to a set of polynomials. The result is a list of reduced subsystems whose roots are those of the original system, but whose variables have been successively eliminated and separated as far as possible. In the present case we obtain four subsystems given by

$$G_1 := \{ 42 - 3x_4 + 65x_2x_3, 8 + 5x_1 \},$$

$$G_2 := \{ 205049562510x_1x_2 - 2072817918600x_2x_3 + 529175067720x_2 \\
+ 500150007328x_3^2 - 3239213905470x_3 + 3029503111800x_3^2 \\
- 26163100475032, 23707187714600x_2x_3^2 - 1207011345240x_2x_3^2 \\
- 15004500219849x_3 - 34648966669800x_3^2 + 11975391986580x_3^3 \\
+ 41386627076564, -1113092 - 431311x_1 + 1909780x_2 + 954890x_2x_1, \\
205x_1 + 368 \},$$

$$G_3 := \{ 2175607695656460086570x_1x_2 - 24442906094419171242925x_2x_3x_1 \\
- 36201645633753432617920x_2x_3 - 25492004395136420363950x_2x_1 \\
- 3377755223900242460240x_2 - 352210319977170626297190x_1x_1^3 \\
- 527515033185400238012371x_1x_1^3 - 372609773697867989940085x_1x_1^3 \\
- 56875826635800959992694x_1 + 35724247368766804124275x_1x_1^3 \\
+ 529100974647178863056960x_2x_1 + 241165196163010851629650x_1 \\
+ 613523694569903050334320x_2x_1^2 \\
- 372108356841000x_2x_1^2 - 595373379045600x_2x_2 \\
+ 202992871981785x_1x_1 - 309188233840256x_1x_1^2 - 108770135076600x_2x_2 \\
+ 408707348737250x_1x_1^2 + 731690446259960x_2x_2^2 - 363026773505180x_1 \\
- 558465136160528, \\
139740x_2x_1^2 - 497365x_1x_1 - 799324x_1x_1 - 279480x_2x_2 - 1187280x_1 - 1879248, \\
43975x_1^2 + 137900x_1 + 107824 \},$$

$$G_4 := \{ 15138500x_1x_2x_3 - 200682625x_2x_1^2x_4 - 589775940x_2x_1x_1 \\
- 425769864x_4x_2 + 30277000x_2x_2^2 - 677199250x_2x_1^2 - 2061417280x_2x_1 \\
- 1553711328x_2 - 34073270x_2x_1^2 - 2202762525x_1x_1^2 - 6761383290x_1x_1^2 \\
- 5193018500x_1x_1^2 - 13923818776 - 6178746050x_1^2 - 18555219840x_1 \\
- 180544080x_1x_2 - 17865753288x_1x_1 - 158418580x_2x_2 - 23445716600x_1x_1x_1 \\
- 7696469075x_1x_1^3x_4, 38482345375x_1x_1^3 + 309893790250x_1^3 \\
+ 11013812625x_1x_1^2x_4 + 1003413125x_2x_1x_1^2 + 3385962650x_2x_1^3 \\
+ 51211999525x_1x_1^2 - 16053417900x_2x_1^2 + 4651759450x_2x_1^2x_1x_1 \\
+ 177479507850x_1x_1^3 + 140245780600x_1^3 + 21214048880x_1 \\
+ 713335840x_2x_1x_1x_1 + 272934958520x_2x_1x_1x_1 + 25260582880x_2x_1x_1 \\
+ 79415365840x_1x_1^3x_4 + 106697298856 + 13183919424x_2 + 41078949112x_1x_1^2 \}.$
By applying Grobner to (64), (65) and their complex conjugates (in this case, two new equations), we obtain a system of polynomials for which all solutions have \( x_1 \) and \( x_2 \) constant.

Let us consider now the fourth and fifth equations in set \( G_4 \) given, respectively, by

\[
31185310 \, \overline{x_2} \, x_1 + 52924196 \, \overline{x_2} \, x_2 - 6814654 \, \overline{x_2} \, \overline{x_1} - 716849880 \, x_1^2 \\
- 440552505 \, x_1^2 \, \overline{x_1} - 1352366658 \, x_1 \, \overline{x_1} - 2156806048 \, x_1 - 45578530 \, x_1 \, \overline{x_2} \\
- 1038603700 \, \overline{x_1} - 1620399064 - 90980056 \, \overline{x_2} = 0
\]  

(64)

and

\[
505750 \, x_1^2 + 149875 \, x_1^2 \, \overline{x_1} + 440460 \, x_1 \, \overline{x_1} + 116450 \, \overline{x_2} \, x_1 + 1160352 \, x_1 + 317976 \, \overline{x_1} + 372640 \, \overline{x_2} + 186320 \, \overline{x_2} \, \overline{x_1} = 0
\]  

(65)

By applying Grobner to \( (64) \), \( (65) \) and their complex conjugates (in this case, two new equations), we obtain a system of polynomials for which all solutions have \( x_1 \) and \( x_2 \) constant.

Let us consider now the special cases in which the denominators in the previous expressions for \( \Phi_{11} \) are zero. The denominators of \( (53) \), \( (55) \), and \( (57) \) are given, respectively, by

\[
d_1 := 10868 \, \overline{x_2}^2 + 772320 \, \overline{x_2} + 1020276 \, x_1 \, \overline{x_2} + 4085 \, \overline{x_2}^2 \, x_1 \\
- 1158240 \, x_1^3 - 2048352 \, x_1 - 214700 \, x_1^3 + 335985 \, \overline{x_2} \, x_2^1 - 1191680,
\]  

(66)

\[
d_2 := 148960 + 42940 \, x_1^2 + 10412 \, \overline{x_2}^2 - 18564 \, \overline{x_2} - 13047 \, x_1 \, \overline{x_2} + 162944 \, x_1,
\]  

(67)

\[
d_3 := (148960 + 42940 \, x_1^2 + 10412 \, \overline{x_2}^2 - 18564 \, \overline{x_2} - 13047 \, x_1 \, \overline{x_2} + 162944 \, x_1) \\
(-24320 \, \overline{x_2}^2 + 7684576 \, \overline{x_2} + 9460852 \, x_1 \, \overline{x_2} + 2915745 \, \overline{x_2} \, x_2 - 194465152 \, x_1 \\
- 123050612 \, x_1^2 - 102596352 - 25995895 \, x_1^3 + 133000 \, \overline{x_2} \, x_2)
\]  

(68)

Applying \( \mathfrak{f} \) to \( (69) \) we obtain

\[
63729588 \, \overline{x_2} \, x_1^2 - 670120 \, x_1^2 \, \overline{x_2}^2 + 12371205 \, x_1^3 \, \overline{x_2} - 617652 \, \overline{x_2}^2 \, x_1^2 \\
+ 108263616 \, x_1 \, \overline{x_2} + 10424160 \, x_1^3 + 10725120 \, x_1 - 1048608 \, \overline{x_2}^2 + 60746496 \, \overline{x_2} \\
- 383325 \, \overline{x_2} \, x_1 - 1749232 \, \overline{x_2}^2 \, x_1 + 1932300 \, x_1^4 = 0.
\]  

(69)
Applying \( \delta \) again on (69), gives

\[
- 71849032 x_2^3 x_1 - 2917197592 x_1^3 x_2^2 + 2126429184 x_2 x_1^2 \\
+ 3683108352 x_1 x_2 + 357185400 x_1^3 x_2 - 128701440 x_1^2 - 125089920 x_1^4 \\
- 67408896 x_2^3 - 18590290 x_1^3 x_2^2 + 6696360 x_2^2 x_1 + 4179810 x_1 x_2^3 \\
- 221222016 x_1^3 - 2418547200 x_2 x_1^2 + 2059223040 x_2 - 22411650 x_1 x_2 \\
- 614629870 x_1 x_2^2 - 23187600 x_1^5 - 4605932928 x_2 x_1 x_2^3 x_1 = 0. 
\] (70)

Using \texttt{grobner} package on (66), (69) and (70) gives the empty set solution. Applying \( \delta \) to (75) and (76) we obtain again the empty solution set.

Applying \( \delta \) again on (72), gives

\[
- 128832 x_2^3 - 101344 x_2^2 x_1 - 2591852 x_2 x_1^2 - 8248048 x_1 x_2 - 257640 x_1^3 \\
- 6550592 x_2 - 977664 x_1^2 - 20824 x_2^3 - 893760 x_1 = 0. 
\] (72)

Applying \( \delta \) again on (72), gives

\[
62472 x_2^3 - 60094064 x_2^2 x_1 + 35714228 x_2 x_1^2 + 2832764 x_1 x_2^3 \\
- 2838720 x_2^3 + 8043840 x_1^2 - 210698752 x_1 x_2 + 8798976 x_1^3 + 86775744 x_2^2 \\
- 171601920 x_2 - 1690904 x_2^3 x_1 + 111204048 x_2 x_1 + 2318760 x_1^4 = 0. 
\] (73)

Using \texttt{grobner} package on (67), (72) and (73) we obtain the empty set solution. We observe now that one of the factors in \( d_3 \) is \( d_2 \). Thus, if \( d_3 = 0 \), we need to consider just the expression

\[
133000 x_2^2 x_1 - 24320 x_2^2 + 7684576 x_2 + 9460852 x_1 x_2 + 2915745 x_2 x_1^2 \\
- 19446152 x_1 - 123050612 x_1^2 - 102596352 - 25995895 x_1^3 = 0. 
\] (74)

Applying \( \delta \) twice on this equation, results in

\[
1923742456 x_2 x_1^2 - 54838615 x_2^2 + 387128860 x_2 x_1^3 + 28673280 x_2^3 \\
+ 1750186368 x_1^2 + 3181762656 x_1 x_2 + 1107455508 x_1^3 + 923367168 x_1 \\
- 143083296 x_2^2 + 1751900928 x_2 + 17772600 x_2^3 x_1 - 175990444 x_2^2 x_1 \\
+ 233963055 x_1^4 = 0. 
\] (75)

and

\[
3071183072 x_2^2 x_1 - 65381238064 x_1 x_2^2 + 363466389312 x_2 x_1^2 \\
+ 414477092352 x_1 x_2 + 141775882688 x_1^3 x_2 - 11080406016 x_1^2 \\
- 13289466096 x_1^4 + 2839158528 x_2^3 + 811332320 x_1^2 x_2^3 - 315187200 x_2^4 \\
- 196695600 x_1 x_2^3 - 21002236416 x_1^3 - 58327724544 x_2^2 + 177286496256 x_2 \\
+ 20770389380 x_1 x_2^2 - 13320040240 x_1^2 x_2^2 - 2807556660 x_1^5 \\
- 106950649152 x_2 x_1 x_2^3 x_1 = 0. 
\] (76)

Using \texttt{grobner} package to (75) and (76) we obtain again the empty solution set.
We need now to study the case in which \( x_1 \) and \( x_2 \) are constants. From \( \delta x_1 = \delta (\alpha/\pi) = 0 \) and \( \delta x_2 = \delta (\beta/\pi) = 0 \) we get, respectively,

\[
31 x_1 + 10 x_1^2 + 2 = 0 \tag{77}
\]

and

\[
63 x_2 + 40 x_2 \overline{x_1} + \overline{x_1} = 0 \tag{78}
\]

The above equations have two solutions, given by \( x_1 = -3/2, \) \( x_2 = 1/2 \) and \( x_1 = x_2 = -8/5. \) The first one satisfies \( 5x_1 + \overline{x_2} - 8 = 0, \) which will be considered next. The second case is impossible, since these values don’t make \( N_1 \) equal to zero.

Let us now consider the case:

\[
\beta - 5\pi - 8\pi = 0, \tag{79}
\]

or, using variables \( x_1 \) and \( x_2, \) and dividing by \( \pi, \)

\[
8 - x_2 - 5x_1 = 0. \tag{80}
\]

Applying \( \delta \) to this equation, using \( (32), (51) \) and \( (52), \) we get

\[
34x_2\overline{x_1} - x_2^2 + 56x_2 + 15\overline{x_1}^2 + 24\overline{x_1} = 0. \tag{81}
\]

The only solution for both equations is given by

\[
x_1 = -3/2, \quad x_2 = 1/2. \tag{82}
\]

Since the numerator on the right side of \( (48) \) must be zero, we obtain, solving for \( \Phi_{11}, \)

\[
\Phi_{11} = -\frac{86}{141}\pi\overline{\pi}. \tag{83}
\]

Applying \( \delta \) on this equation, we obtain

\[
\delta\pi = -\pi\overline{\pi}. \tag{84}
\]

Equations \( (39) \) and \( (52) \) give

\[
\overline{\delta\pi} = \pi^2. \tag{85}
\]

Computation of the commutator \( [\delta, \delta]\pi, \) using \( (84) \) and \( (85) \) gives \( \pi = 0. \)

Finally, we consider the case \( \Phi_{11} = 0. \) Here both denominators of \( (55), \) and of \( (57), \) must be equal to zero. Applying Buchberger-Gröbner algorithm to them and their complex conjugates and using the variables defined in \( (60), \) we can verify that the only possible solutions again fall in cases we have studied before, i.e., either \( x_1 \) or \( x_2 \) are zero, or both \( x_1 \) and \( x_2 \) are constants.
5 Conclusions

Theorem 1 was proved for the case of Maxwell equations, i.e., there are no Petrov type III space-times on which Maxwell’s equations satisfy Huygens’ principle. For the neutrino case, $k_1 = 8, k_2 = 13$, the proof is similar [18]. The use of the package $\texttt{NPsinor}$ in Maple was essential for the conversion of the Alvarez-Wünsch five-index necessary condition from the tensorial to dyad form.

The polynomial system obtained from the integrability conditions was simplified using Maple’s package $\texttt{grobner}$. Since a direct application of the algorithm is apparently impossible due to the large size of the polynomial system, a “divide and conquer” approach was applied with success, showing that the necessary conditions III and V for the validity of Huygens’ principle cannot be simultaneously satisfied for Maxwell’s equations in Petrov type III space-times.

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