Completeness of locally $k_\omega$-groups and related infinite-dimensional Lie groups

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Abstract

Recall that a topological space $X$ is said to be a $k_\omega$-space if it is the direct limit of an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact Hausdorff topological spaces. If each point in a Hausdorff space $X$ has an open neighbourhood which is a $k_\omega$-space, then $X$ is called locally $k_\omega$. We show that a topological group is complete whenever the underlying topological space is locally $k_\omega$. As a consequence, every infinite-dimensional Lie group modelled on a Silva space is complete.

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1 Introduction and statement of the results

Recall that a $k_\omega$-space is a Hausdorff topological space $X$ which carries the direct limit topology for an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets $K_n \subseteq X$ with $\bigcup_{n \in \mathbb{N}} K_n = X$ (see [8], [13]; cf. [22] and, with different terminology, [24]). A topological group is called a $k_\omega$-group if its underlying topological space is a $k_\omega$-space. Hunt and Morris [17] showed that every $k_\omega$-group is Weil complete, viz., complete in its left uniformity (cf. [29] for the case of abelian $k_\omega$-groups; see also [3] for a recent proof). The current work is devoted to generalizations and applications of this fact, with a view towards examples in infinite-dimensional Lie theory.

Following [13], a Hausdorff space $X$ is called locally $k_\omega$ if each $x \in X$ has an open neighbourhood $U \subseteq X$ which is a $k_\omega$-space in the induced topology. A topological group $G$ is called locally $k_\omega$ if its underlying topological space is locally $k_\omega$. Since every locally $k_\omega$ group has an open subgroup which is a $k_\omega$-group [13, Proposition 5.3], the Hunt-Morris Theorem implies the following:

**Proposition 1.1** Every locally $k_\omega$ topological group is Weil complete. □

We consider Lie groups modelled on arbitrary real or complex Hausdorff locally convex topological vector spaces (as in [26] and [14])\(^1\) based on the differential calculus in locally convex spaces known as Keller’s $C^\infty$-theory [18].

\(^1\)Compare also [23] (for Lie groups modelled on sequentially complete spaces).
Since the smooth maps under consideration are, in particular, continuous, the Lie groups we consider have continuous group operations. They can therefore be regarded as Hausdorff topological groups, and we can ask when they are complete. The following observation (proved in Section 2) is essential.

**Proposition 1.2** If a Lie group $G$ is modelled on a locally convex space $E$ which is a $k_\omega$-space, then $G$ is locally $k_\omega$, and its identity component $G_e$ is $k_\omega$.

Propositions 1.1 and 1.2 entail a simple completeness criterion:

**Corollary 1.3** Every Lie group modelled on a $k_\omega$-space is Weil complete. □

Recall that a locally convex space $E$ is called a Silva space if it is the locally convex direct limit $E = \varinjlim E_n$ of an ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces, such that each inclusion map $E_n \to E_{n+1}$ is a compact operator (cf. [7]). It is well known that every Silva space is a $k_\omega$-space (see, e.g., [11, Example 9.4]). Thus Corollary 1.3 entails:

**Corollary 1.4** Lie groups modelled on Silva spaces are Weil complete. □

Until recently, little was known on the completeness properties of infinite-dimensional Lie groups, except for the classical fact that Lie groups modelled on Banach spaces are Weil complete (see Proposition 1 in [2, Chapter III, §1.1]). In 2016, Weil completeness was established for many classes of infinite-dimensional Lie groups [12], but some examples modelled on Silva spaces could not be treated. The current paper closes this gap, as Corollary 1.4 establishes Weil completeness for the latter. Section 3 compiles a list of infinite-dimensional Lie groups which are modelled on Silva spaces and hence Weil complete (by Corollary 1.4). In particular, we find:

- For each compact real analytic manifold $M$, the Lie group $\text{Diff}^\omega(M)$ of all real analytic diffeomorphisms $\phi: M \to M$ is Weil complete;

- For each finite-dimensional Lie group $G$ and $M$ as before, the Lie group $C^\omega(M, G)$ of all real analytic maps $f: M \to G$ is Weil complete.

Being modelled on Silva spaces, the examples we consider are locally $k_\omega$ (by Proposition 1.2), which is sufficient to conclude Weil completeness. Yet, it is natural to ask whether the groups in question are not only locally $k_\omega$, but, actually, $k_\omega$-groups. Results in Section 4 subsume:
Proposition 1.5 For each compact real analytic manifold $M$, the Lie group $\text{Diff}^\omega(M)$ is a $k_\omega$-group. Moreover, $C^\omega(M,G)$ is a $k_\omega$-group for each $\sigma$-compact finite-dimensional Lie group $G$, and $M$ as before.

Our studies are related to an open problem by K.-H. Neeb, who asked whether every Lie group modelled on a complete locally convex space is (Weil) complete (cf. [26, Problem II.9]). As long as this problem remains open, it is natural to look for classes $\Omega$ of locally convex spaces such that every Lie group modelled on a space $E \in \Omega$ is Weil complete. Banach spaces form one such class $\Omega$. By Corollary 1.3 locally convex spaces whose underlying topological space is $k_\omega$ furnish a second such class.

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2 Proof of Proposition 1.2

For each $g \in G$, there exists a homeomorphism $\phi: U \to V$ from an open neighbourhood $U \subseteq G$ of $g$ onto an open subset $V \subseteq E$. By [13, Proposition 4.2 (g)], $V$ contains an open neighbourhood $W$ of $\phi(g)$ which is a $k_\omega$-space in the induced topology. Then $\phi^{-1}(W)$ is an open neighbourhood of $g$ in $G$ and a $k_\omega$-space. Thus $G$ has an open subgroup $S$ which is $k_\omega$ and since $G_e$ is a closed subgroup of $S$, it is $k_\omega$ as well (see [13, Propositions 5.3 and 5.2 (b)]).

3 Examples of Weil complete Lie groups

Corollary 1.4 applies to many classes of infinite-dimensional Lie groups.

Example 3.1 If $M$ is a compact real analytic manifold (without boundary or corners), then the group $\text{Diff}^\omega(M)$ of all real analytic diffeomorphisms $\phi: M \to M$ is a Lie group modelled on the Silva space $\Gamma^\omega(TM)$ of all real analytic vector fields on $M$ (see [21], [19], [4]). By Corollary 1.4 $\text{Diff}^\omega(M)$ is Weil complete.

Example 3.2 Let $G$ be a finite-dimensional Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with modelling space $\mathfrak{g}$, and $K$ be a non-empty compact subset of a finite-dimensional $\mathbb{K}$-vector space $E$. Given a $\mathbb{K}$-analytic map $f: U \to G$ on an
open subset $U \subseteq E$ with $K \subseteq U$, write $[f]$ for its germ around $K$. Then the group $\operatorname{Germ}(K, G)$ of all such germs $[f]$ is a Lie group modelled on the Silva space $\operatorname{Germ}(K, g)$ (see [10]); likewise if $K \neq \emptyset$ is a compact subset of a finite-dimensional $\mathbb{K}$-analytic manifold $M$ (see [5]). By Corollary 1.3, $\operatorname{Germ}(K, G)$ is Weil complete.

**Example 3.3** If $M$ is a compact real analytic manifold, taking $\mathbb{K} := \mathbb{R}$ and $K := M$ in Example 3.2 we see that the Lie group $C^\omega(M, G)$ of $G$-valued real analytic maps on $M$ (which can be identified with $\operatorname{Germ}(M, G)$) is Weil complete, for each finite-dimensional Lie group $G$.

**Example 3.4** If $G$ is a Lie group modelled on a finite-dimensional real vector space $g$, then the group $C^\omega(\mathbb{R}, G)$ of all real analytic mappings $f: \mathbb{R} \to G$ can be made a Lie group modelled on the projective limit $C^\omega(\mathbb{R}, g) = \lim_{\leftarrow} \operatorname{Germ}([-n, n], G)$ of Silva spaces, using the strategy of [27] (cf. [5]). As $\operatorname{Germ}([-n, n], G)$ is a Weil complete Hausdorff group for each $n \in \mathbb{N}$ by Example 3.3 and $C^\omega(\mathbb{R}, G) = \lim_{\leftarrow} \operatorname{Germ}([-n, n], G)$, we see that $C^\omega(\mathbb{R}, G)$ is Weil complete.

**Example 3.5** For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and a non-empty compact subset $K$ of a finite-dimensional $\mathbb{K}$-vector space $E$, consider the group $\operatorname{GermDiff}(K)$ of germs $[f]$ around $K$ of $\mathbb{K}$-analytic diffeomorphisms $f: U \to V$ between open subsets $U, V \subseteq E$ with $K \subseteq U \cap V$, such that $f|_K = \text{id}_K$. Then $\operatorname{GermDiff}(K)$ is a Lie group modelled on the Silva space $\operatorname{Germ}(K, E)_0$ of all germs of $E$-valued $\mathbb{K}$-analytic maps around $K \subseteq E$ which vanish on $K$ (see [11]). By Corollary 1.3, $\operatorname{GermDiff}(K)$ is Weil complete. As a special case, taking $\mathbb{K} := \mathbb{C}$, $E := \mathbb{C}^n$ and $K := \{0\}$, we see that the Lie groups $G_{h_n}(\mathbb{C}) = \operatorname{GermDiff}(\{0\})$ studied by Pisanelli [28] are Weil complete. We also mention that certain Lie groups of real analytic diffeomorphisms considered by Leitenberger (see [20]) are closed subgroups of $\operatorname{GermDiff}(\{0\})$ with $\mathbb{K} := \mathbb{R}, E := \mathbb{R}^n$, and $K := \{0\}$, whence they are Weil complete as well.

**Example 3.6** The tame Butcher group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (see [1]) is an infinite-dimensional Lie group related to integration methods for ordinary differential equations in numerical analysis. As the tame Butcher group is globally diffeomorphic to a Silva space (see [1] Lemma 1.17), it is a $k_\omega$-group and hence Weil complete, by the Hunt-Morris Theorem.
Example 3.7 Let $M$ be a compact smooth manifold, $G$ be a Lie group modelled on a finite-dimensional real vector space $\mathfrak{g}$, and $s$ be a real number such that $s \geq \dim(M)/2$. Then a Lie group $H^{s}(M,G)$ can be defined which is modelled on the Silva space $\lim_{\to} H^{s+\frac{1}{2}}(M,\mathfrak{g})$ of all $\mathfrak{g}$-valued functions on $M$ which are of Sobolev class $t$ for some $t > s$ (see [15]). By Lemma 1.4, $H^{s}(M,G)$ is Weil complete.

Example 3.8 Consider an ascending sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional real Lie groups such that all inclusion maps $G_n \to G_{n+1}$ are smooth group homomorphisms. Then $G := \bigcup_{n \in \mathbb{N}} G_n$ can be made a Lie group modelled on the Silva space $\lim_{\to} L(G_n)$ (see [9], or also [25], [19] for special cases). By Corollary 1.4, $G$ is Weil complete.

4 Proof of Proposition 1.5

In this section, we prove Proposition 1.5 and obtain results concerning separability and the $k_\omega$-property also for some further examples of Lie groups.

Recall that a topological space is called separable if it has a countable, dense subset. The following observation will help us to prove Proposition 1.5:

Lemma 4.1 If a topological group $G$ is separable and locally $k_\omega$, then $G$ is a $k_\omega$-group.

Proof. Let $D \subseteq G$ be a dense countable subset. Since $G$ is locally $k_\omega$, it has an open subgroup $U$ which is a $k_\omega$-group (see [13] Proposition 5.3)). Then $G = DU$, enabling us to choose a subset $R \subseteq D$ of representatives for the left cosets of $U$. Thus $G = RU$ and $rU \cap sU = \emptyset$ for all $r \neq s$ in $R$. As each coset is open, the disjoint union $G = \bigcup_{r \in R} rU$ is a topological sum. Since countable topological sums of $k_\omega$-spaces are $k_\omega$-spaces (see [13] Proposition 4.2(e))), we see that $G$ is a $k_\omega$-space. □

4.2 Every subset of a separable metric space is separable, as is well known.

The next lemma compiles further well-known elementary facts.

Lemma 4.3 (a) If a topological space $X$ is separable, then every open subset of $X$ is separable.
(b) If \((X_j)_{j \in J}\) is a family of separable topological spaces with countable index set \(J \neq \emptyset\), then \(\prod_{j \in J} X_j\) is separable when endowed with the product topology.

(c) If \((X_j)_{j \in J}\) is a family of separable topological spaces with countable index set \(J \neq \emptyset\), then also the topological sum \(\bigoplus_{j \in J} X_j\) is separable.

(d) If \(f: X \to Y\) is a surjective continuous map between topological spaces and \(X\) is separable, then \(Y\) is separable.

(e) If \(X\) is a topological space which has a separable dense subset, then \(X\) is separable.

(f) Let \(X\) be a topological space, \((X_j)_{j \in J}\) be a family of separable topological spaces with countable index set \(J \neq \emptyset\), and \((f_j)_{j \in J}\) be a family of continuous maps \(f_j: X_j \to X\) with \(\bigcup_{j \in J} f_j(X_j) = X\). Then \(X\) is separable. \(\square\)

If \(X\) and \(Y\) Hausdorff topological spaces, we write \(C(X,Y)\) for the set of all continuous maps \(f: X \to Y\). We shall always endow \(C(X,Y)\) with the compact-open topology (as discussed in [6], [30], or also [14, Appendix A.5]). If \(K\) is a compact Hausdorff topological space, we write \(\text{Homeo}(K)\) for the group of all homeomorphisms \(\phi: K \to K\). We endow \(\text{Homeo}(K)\) with the topology induced by \(C(K,K)\). Some folklore facts concerning the compact-open topology will be used, as compiled in the next lemma.

**Lemma 4.4**

(a) For each compact subset \(K \subseteq \mathbb{R}^m\) and \(n \in \mathbb{N}\), the locally convex space \(C(K,\mathbb{R}^n)\) is separable.

(b) For each \(K\) as before and \(\sigma\)-compact finite-dimensional Lie group \(G\), the topological group \(C(K,G)\) is separable.

(c) If \(K\) is a compact smooth manifold (or a compact subset of \(\mathbb{R}^n\) for some \(n \in \mathbb{N}\)), then \(\text{Homeo}(K)\) is a separable topological group.

**Proof.**

(a) By the Stone-Weierstraß Theorem, the algebra of real-valued polynomial functions is dense in \(C(K,\mathbb{R})\), and hence also its dense countable subset of polynomials with rational coefficients. Thus \(C(K,\mathbb{R}^m) \cong C(K,\mathbb{R})^m\) is separable.

(b) It is well known that \(C(K,G)\) is a topological group (see, e.g., [30, Theorem 11.5]). By the Whitney Embedding Theorem (see, e.g., [16]), for
some $n \in \mathbb{N}$ there exists a $C^\infty$-diffeomorphism $j: G \to N$ from $G$ onto a $C^\infty$-submanifold $N \subseteq \mathbb{R}^n$. Now $C(K,G) \to C(K,\mathbb{R}^n)$, $f \mapsto j \circ f$ is a topological embedding. Hence $C(K,G)$ is separable, by (a) and 4.2.

(c) By Whitney’s Embedding Theorem, every compact smooth manifold is $C^\infty$-diffeomorphic to a smooth submanifold of some $\mathbb{R}^n$ (see, e.g., [16]). It therefore suffices to consider a compact subset $K \subseteq \mathbb{R}^n$. As the metrizable space $C(K,\mathbb{R}^n)$ is separable by (a), also its subset $\text{Homeo}(K)$ is separable, by 4.2. It is well known that $\text{Homeo}(K)$ is a topological group: See, e.g., [30, Lemma 9.4 (c)] for continuity of multiplication. As the evaluation map $\varepsilon: \text{Homeo}(K) \times K \to K$, $(\phi, x) \mapsto \phi(x)$ is continuous (see [30, Lemma 9.8]), it has closed graph, whence also $h: \text{Homeo}(K) \times K \to K$, $(\phi, x) \mapsto \phi^{-1}(x)$ has closed graph. Since $K$ is compact, continuity of $h$ follows (see, e.g., [30, Theorem 1.21 (b)]) and thus also continuity of $h^\vee$: $\text{Homeo}(K) \to \text{Homeo}(K)$, $\psi \mapsto h(\psi, \cdot) = \psi^{-1}$ (cf., e.g., [6, Theorem 3.4.1 and p. 110]).

We shall recognize $k_\omega$-groups using the following lemma:

**Lemma 4.5** Let $G$ be a topological group. Assume that there exists a continuous homomorphism $\alpha: G \to H$ to a separable metrizable topological group $H$ and an open identity neighbourhood $U \subseteq H$ such that $\alpha^{-1}(U)$ is separable in the topology induced by $G$. Then $G$ is separable.

**Proof.** Since $H$ is separable and metrizable, $\alpha(G)$ is separable in the topology induced by $H$ (see 4.2). We therefore find a countable subset $C \subseteq G$ such that $\alpha(C)$ is dense in $\alpha(G)$. Let $D$ be a dense countable subset of $\alpha^{-1}(U)$. Then $C^{-1}D$ is a countable subset of $G$. For $g \in G$, we find $c \in C$ such that $\alpha(c) \in U \alpha(g^{-1})$ and thus $cg \in \alpha^{-1}(U)$. Now $d_j \to cg$ for a net $(d_j)_{j \in J}$ in $D$. Then $c^{-1}d_j \in C^{-1}D$ and $c^{-1}d_j \to g$, whence $C^{-1}D$ is dense in $G$. $\Box$

The next lemma yields separability of relevant modelling spaces.

**Lemma 4.6** Every Silva space is separable.

**Proof.** Let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of Banach spaces, such that all inclusion maps $E_n \to E_{n+1}$ are compact operators. Consider the locally convex direct limit $E := \bigcup_{n \in \mathbb{N}} E_n$. Let $B_n$ be the unit ball in $E_n$ and $K_n$ be its closure in $E_{n+1}$. Then $K_n$ is compact and metrizable as a subset of $E_{n+1}$ (hence also in $E$), and thus separable. As a consequence,

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2Note that graph($h$) is obtained from graph($\varepsilon$) $\subseteq \text{Homeo}(K) \times K \times K$ by swapping the last and penultimate components. 

7
\[ E = \bigcup_{n,m \in \mathbb{N}} mK_n \] is separable, being a countable union of subsets which are separable in the induced topology (see Lemma 4.3(f)).

In the next proposition, \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \).

**Proposition 4.7** Let \( M \) be a finite-dimensional \( \mathbb{K} \)-analytic manifold, \( G \) be a finite-dimensional \( \mathbb{K} \)-analytic Lie group which is \( \sigma \)-compact and \( K \subseteq M \) be a compact, non-empty subset. Then \( \text{Germ}(K, G) \) (as in Example 3.2) is separable and a \( k_\omega \)-group.

**Proof.** Let \( \phi: U \to V \) be a \( \mathbb{K} \)-analytic diffeomorphism from an open identity neighbourhood \( U \subseteq G \) onto an open subset \( V \subseteq g \) such that \( \phi(e) = 0 \), where \( g \) (\( \cong \mathbb{K}^n \)) is the modelling space of \( G \). Then

\[ \text{Germ}(K, V) := \{ [f] \in \text{Germ}(K, g) : f(K) \subseteq V \} \]

is an open 0-neighbourhood in the Silva space \( \text{Germ}(K, g) \) and hence separable, by Lemmas 4.6 and 4.3(a). After shrinking \( U \) if necessary,

\[ \text{Germ}(K, U) := \{ [f] \in \text{Germ}(K, G) : f(K) \subseteq U \} \]

is an open identity neighbourhood and the map \( \text{Germ}(K, U) \to \text{Germ}(K, V) \), \( [f] \mapsto [\phi \circ f] \) is a homeomorphism, whence \( \text{Germ}(K, U) \) is separable. Recall from Lemma 4.4(b) that \( C(K, G) \) is a separable topological group. Now

\[ \alpha: \text{Germ}(K, G) \to C(K, G), \quad [f] \mapsto f|_K \]

is a continuous homomorphism and \( C(K, U) \) an open identity neighbourhood in \( C(K, G) \) such that \( \alpha^{-1}(C(K, U)) = \text{Germ}(K, U) \) is separable. Thus \( \text{Germ}(K, G) \) is separable and \( k_\omega \), by Lemma 4.5. \( \square \)

**Proof of Proposition 1.5.** Let \( M \) be a compact real analytic manifold. If \( G \) is a \( \sigma \)-compact, finite-dimensional Lie group, taking \( K := M \) and \( \mathbb{K} := \mathbb{R} \) in Proposition 4.7, we see that \( C^\omega(M, G) = \text{Germ}(M, G) \) is a \( k_\omega \)-group.

The Lie group \( \text{Diff}^\omega(M) \) is an open subset of the set \( C^\omega(M, M) \) of all real analytic self-maps of \( M \), endowed with its natural smooth manifold structure (and topology); see [19, Theorem 43.3] or [4, Proposition 1.9]. It is useful to recall an aspect of the construction of the manifold structure on \( C^\omega(M, M) \), as described in [19] and [4]. Write \( 0_x \) for the zero vector in \( T_xM \) for \( x \in M \), and \( \pi_{TM}: TM \to M \) for the bundle projection taking \( v \in T_xM \) to \( x \). Let
Σ: Ω \to M be a real analytic local addition; thus Ω is an open neighbourhood of \{0_x: x \in M\} in TM, we have Σ(0_x) = x for all x \in M, and

\[(\pi_{TM}, \Sigma): \Omega \to M \times M\]
is a \(C^\omega\)-diffeomorphism onto an open subset of \(M \times M\). According to [4, Theorem 1.6 (a)], the set

\[U_{id_M} := \{\psi \in C^\omega(M, M) : (id_M, \psi)(M) \subseteq (\pi_{TM}, \Sigma)(\Omega)\}\]
is open in \(C^\omega(M, M)\) and the map

\[\Phi_{id_M}: U_{id_M} \to \Gamma^\omega(TM), \quad \psi \mapsto (\pi_{TM}, \Sigma)^{-1} \circ (id_M, \psi)\]
is a homeomorphism onto an open subset of the Silva space \(\Gamma^\omega(TM)\) of real analytic vector fields on \(M\). Note that

\[
\{(\phi, \psi) \in C(M, M \times M) : (\phi, \psi)(M) \subseteq (\pi_{TM}, \Sigma)(\Omega)\}
\]
is open in \(C(M, M \times M)\). As \(C(M, M) \to C(M, M \times M), \psi \mapsto (id_M, \psi)\) is a continuous map, we deduce that

\[W := \{\psi \in \text{Homeo}(M) : (id_M, \psi)(M) \subseteq (\pi_{TM}, \Sigma)(\Omega)\}\]
is an open identity neighbourhood in \(\text{Homeo}(M)\). The inclusion map \(\alpha: \text{Diff}^\omega(M) \to \text{Homeo}(M), \phi \mapsto \phi\), is a continuous homomorphism. Now

\[\alpha^{-1}(W) = W \cap \text{Diff}^\omega(M) = \{\psi \in \text{Diff}^\omega(M) : (id_M, \psi)(M) \subseteq (\pi_{TM}, \Sigma)(\Omega)\}\]
is an open \(id_M\)-neighbourhood in \(\text{Homeo}(M)\). The Proposition 4.8

**Proposition 4.8** For each \(n \in \mathbb{N}\) and non-empty compact set \(K \subseteq \mathbb{K}^n =: E\), the Lie group \(\text{GermDiff}(K)\) is separable and a \(k_\omega\)-group.

**Proof.** As shown in [11], \(\text{Germ}(K, E)_0 := \{[f] \in \text{Germ}(K, E) : f|_K = 0\}\) is a Silva space, \(\Omega := \{[f] \in \text{Germ}(K, E)_0 : [id_E + f] \in \text{GermDiff}(K)\}\) is open in \(\text{Germ}(K, E)_0\) and the bijection \(\Omega \to \text{GermDiff}(K), [f] \mapsto [id_E + f]\) is a homeomorphism for the Lie group structure on \(\text{GermDiff}(K)\). Since \(\Omega\) is separable by Lemma 4.6 and 4.3(a) that \(\alpha^{-1}(W)\) is separable. Since \(\text{Homeo}(M)\) is separable and metrizable by Lemma 4.3(c), we deduce with Lemma 4.5 that \(\text{Diff}^\omega(M)\) is separable and hence \(k_\omega\) by Lemma 4.1. \(\square\)

In the next proposition, \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\).

**Proposition 4.8** For each \(n \in \mathbb{N}\) and non-empty compact set \(K \subseteq \mathbb{K}^n =: E\), the Lie group \(\text{GermDiff}(K)\) is separable and a \(k_\omega\)-group.

**Proof.** As shown in [11], \(\text{Germ}(K, E)_0 := \{[f] \in \text{Germ}(K, E) : f|_K = 0\}\) is a Silva space, \(\Omega := \{[f] \in \text{Germ}(K, E)_0 : [id_E + f] \in \text{GermDiff}(K)\}\) is open in \(\text{Germ}(K, E)_0\) and the bijection \(\Omega \to \text{GermDiff}(K), [f] \mapsto [id_E + f]\) is a homeomorphism for the Lie group structure on \(\text{GermDiff}(K)\). Since \(\Omega\) is separable by Lemma 4.6 and 4.3(a) that \(\alpha^{-1}(W)\) is separable. Since \(\text{Homeo}(M)\) is separable and metrizable by Lemma 4.3(c), we deduce with Lemma 4.5 that \(\text{Diff}^\omega(M)\) is separable and hence \(k_\omega\) by Proposition 4.1. \(\square\)
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