New Potential-Based Bounds for the Geometric-Stopping Version of Prediction with Expert Advice

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Abstract

This work addresses the classic machine learning problem of online prediction with expert advice. A potential-based framework for the fixed horizon version of this problem was previously developed using verification arguments from optimal control theory (Kobzar, Kohn and Wang, New Potential-Based Bounds for Prediction with Expert Advice (2019)). This paper extends this framework to the random (geometric) stopping version.

Taking advantage of these ideas, we construct potentials for the geometric version of prediction with expert advice from potentials used for the fixed horizon version. This construction leads to new explicit lower and upper bounds associated with specific adversary and player strategies for the geometric problem. We identify regimes where these bounds are state of the art.

Keywords: Online Learning, Expert Advice Framework, Regret Minimization, Random (Geometric) Stopping, Upper and Lower Bounds, Potential-Based Strategies, Subsolutions and Supersolutions of Partial Differential Equations, Optimal Control, Dynamic Programming, Verification Argument, Screened Poisson Equation

1. Introduction

The problem of prediction with expert advice (the expert problem) is a classic problem in online machine learning. We will use the following representative definition of it.

Prediction with expert advice: At each period \( t \in [T] \) until the final time,

- the player determines which of the \( N \) experts to follow by selecting a discrete probability distribution \( p_t \in \Delta_N \);
- the adversary determines the allocation of losses to the experts by selecting a probability distribution \( a_t \) over the hypercube \([-1,1]^N\); and
- the expert losses \( q_t \in [-1,1]^N \) and the player’s choice of the expert \( I_t \in [N] \) are sampled from \( a_t \) and \( p_t \), respectively, and revealed to both parties.

Kobzar et al. (2019) considered the finite horizon version, where the number of periods \( T \) is fixed, the regret is \( R_T(p,a) = \mathbb{E}_{p,a} \left[ \sum_{t \in [T]} (q_t)_{I_t} - \min_i \sum_{t \in [T]} (q_t)_{i} \right] \) and the joint distributions

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$a = (a_t)_{t \in [T]}$ and $p = (p_t)_{t \in [T]}$ refer to, respectively, the adversary and player strategies or simply the adversary and player. The focus of this work is the geometric stopping version of the game, where the final time $T$ is not fixed but is rather random, chosen from the geometric distribution $G$ with mean $\frac{1}{\delta}$, and the regret is given by $R(p, a) = \mathbb{E}_G R_T(p, a)$.

The geometric stopping condition models the possibility that the prediction process may stop in any period with probability $\delta$. The effective discounting of the regret by the probability that the game will continue makes this version equivalent to the fixed horizon problem with discounted regret as $T \to \infty$.

In the geometric setting, nonasymptotic minmax optimal strategies were determined explicitly for $N = 2$ and 3 (Gravin et al., 2016), but for general $N$, optimal strategies have not been determined explicitly.

In a related line of work, strategies that are optimal asymptotically (as $\delta$ approaches zero) were determined by PDE-based methods. Drenska and Kohn (2019) showed that, for any fixed $N$, the value function associated with the scaling limit of the geometric problem is the unique solution of the associated nonlinear PDE. The last reference also gave a closed-form solution of the geometric stopping PDE for $N = 3$; Bayraktar et al. (2019b) determined the closed form solution of the geometric stopping PDE for $N = 4$, and using inverse Laplace’s transform, Bayraktar et al. (2019a) solved the finite horizon PDE for $N = 4$.

Extending the ideas of Rakhlin et al. (2012) and Rokhlin (2017), Kobzar et al. (2019) derived potential-based player and adversary strategies using sub- and supersolutions of the asymptotic PDE as potentials, and provided numerous examples (including lower as well as upper bounds) in the fixed horizon setting. This paper further extends this framework and leverages the Laplace’s transform relationship between the finite horizon and geometric problems, specifically:

1. The potential-based framework is extended to the geometric stopping setting (Theorems 1 and 5, and Remark 2).

2. Using a method based on Laplace’s transform, we construct potentials for the geometric version from potentials used for the fixed horizon version. This construction leads to explicit lower and upper bounds and the corresponding strategies for the geometric problem from the explicit potentials developed for the fixed horizon game (provided that the associated adversary fixed horizon strategies depend only on the accumulated regret, i.e., they are stationary). (Theorems 3 and 6 and Remark 4).

(a) The resulting lower bound based on the solution to the so-called screened Poisson equation is state of the art for $N \geq 5$ and relatively small but fixed $\delta$ (Sections 6 and 8).

(b) To get another family of bounds, we use the closed-form solution of a nonlinear PDE based on the largest diagonal entry of the Hessian (Section 7).

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1. This can be seen by computing the expectation with respect to $G$:

\[
R(p, a) = \sum_{T=1}^{\infty} (1 - \delta)^T \delta \mathbb{E}_{p,a} \left[ \sum_{i \in [T]} (q_i)_{t_i} - \min_{i} \sum_{t \in [T]} (q_t)_{t_i} \right]
\]

\[
= \mathbb{E}_{p,a} \left[ \sum_{t=1}^{\infty} \sum_{T=t}^{\infty} (1 - \delta)^T \delta(q_t)_{t_i} - \min_{i} \sum_{t=1}^{\infty} \sum_{T=t}^{\infty} (1 - \delta)^T \delta(q_t)_{t_i} \right] = \mathbb{E}_{p,a} \left[ \sum_{t=1}^{\infty} (1 - \delta)^T \delta(q_t)_{t_i} - \min_{i} \sum_{t=1}^{\infty} (1 - \delta)^T \delta(q_t)_{t_i} \right]
\]

where we summed the geometric series $\sum_{T=t}^{\infty} (1 - \delta)^T = \frac{(1-\delta)^T}{\delta}$ for $\delta \in (0, 1)$.
(c) The resulting upper bounds are tighter than those obtained using the exponentially weighted forecaster for $5 \leq N \lesssim 30$ and relatively small but fixed $\delta$ (Section 8).

2. Notation

We will use the following notation. For a multi-index $I$, $\partial_I$ refers to the partial derivative with respect to the spatial variable(s) in $I$. $D^3 \hat{u}[q, q, q]$ and $D^4 \hat{u}[q, q, q, q]$ denote the 3-rd and 4-th derivative in the direction of $q$ given by the linear forms $\sum_{i,j,k} \partial_{ijk} \hat{u} q_i q_j q_k$ and $\sum_{i,j,k,l} \partial_{ijkl} \hat{u} q_i q_j q_k q_l$, respectively. Whenever the region of integration is omitted, it is assumed to be $\mathbb{R}^N$.

$[T]$ denotes the set $\{1, ..., T\}$ if $T \geq 1$ or $\{T, ..., -1\}$ if $T \leq -1$. $\mathbf{1}$ is a vector in $\mathbb{R}^N$ with all components equal to 1. $\Delta_N$ refers to a discrete probability distribution over $N$ outcomes. Whenever the feasible set of $q$ is omitted, it is assumed to be $[-1, 1]^N$.

A classical solution of a partial differential equation (PDE) on a specified region is a solution such that all derivatives appearing in the statement of the PDE exist and are continuous on the specified region.

We let the vector $r_\tau = (q_\tau)_I \mathbf{1} - q_\tau$ denote the player’s losses realized in round $\tau$ relative to those of each expert (instantaneous regret) and let the vector $x = \sum_\tau r_\tau$ denote the player’s cumulative losses realized before the outcome of the current round relative to those of each expert (cumulative regret or simply the regret).

3. Lower bound

When the prediction process starts at a given regret $x$, the value function $\hat{v}_a$, reflecting the worst-case (smallest) regret inflicted by a given adversary $a$ over all player strategies satisfies $\hat{v}_a(x) = \delta \max_i x_i + (1 - \delta) \min_p \mathbb{E}_{a,p} \hat{v}_a(x + r)^2$. This function has an equivalent characterization

$$\hat{v}_a(x) = \min_p \mathbb{E}_{G,p,a} \max_i \left( x_i + \sum_{\tau \in [T]} (r_{\tau})_i \right)$$

To bound $\hat{v}_a$ below, we introduce a potential $\hat{u}$, and a corresponding adversary $a$. In the context of lower bounds in this paper, we will only consider those adversary strategies that assign the same probability to $q$ and $-q$ for all $q \in [-1, 1]^N$ (symmetric strategies).

| Lower-bound potential $\hat{u}$ - geometric horizon: We will use this term for a function $\hat{u} : \mathbb{R}^N \to \mathbb{R}$, such that, for every $x \in \mathbb{R}^N$, there exists some symmetric probability distribution $\alpha$ on $[-1, 1]^N$ ensuring that $\hat{u}$ is a classical solution of |

$$\begin{cases}
\hat{u}(x) \leq \max_i x_i + \frac{1 - \delta}{2\delta} \mathbb{E}_{\alpha}(D^2 \hat{u}(x) \cdot q, q) \\
\hat{u}(x + c\mathbf{1}) = \hat{u}(x) + c
\end{cases}$$

| Adversary $a$: Given $\hat{u}$ as above, the associated strategy $a$ is: in each prediction round, the adversary selects a symmetric strategy $a$ such that (2) is satisfied. |

---

2. We will denote the value functions and potentials for the geometric problem with the superscript “hat” (hat) to distinguish them from those for the fixed horizon problem (which will be denoted without that superscript).
As confirmed in Appendix A using Taylor expansion of \( \hat{u} \) with the integral form of the third-order spatial remainder, this strategy attains the following lower bound.

**Theorem 1 (Geometric l.b)** If for all \( x \), \( \hat{u}(x) - \max_i x_i \) is uniformly bounded above, \( D^2 \hat{u} \) is Lipschitz continuous and, for any \( q \) sampled from \( a \), \( \text{ess sup}_{y \in [x, x-q]} D^3 \hat{u}(y)[q, q, q] \leq K \), then \( \hat{u}(x) - \frac{1-\delta}{6\delta} K \leq \hat{v}_a(x) \).

**Remark 2 (Geometric l.b. - Lipschitz continuous higher-order derivs)** Similarly to Remark 2 in Kobzar et al. (2019) if \( \hat{u} \) has higher order Lipschitz continuous derivatives, they could be used to derive bounds on \( \hat{v}_a \). For example, if \( D^3 \hat{u} \) exists and is Lipschitz continuous for all \( x \), and for any \( q \) sampled from \( a \), \( -\text{ess inf}_{y \in [x, x-q]} D^4 \hat{u}(y)[q, q, q, q] \leq K \), then the conclusion of Theorem 1 is still holds with \( \hat{u}(x) - \frac{1-\delta}{24\delta} K \leq \hat{v}_a(x) \).

Next we establish a correspondence between a fixed horizon potential \( u \) and the geometric stopping one \( \hat{u} \). In this setting, we denote the time \( t \) by nonpositive numbers.

**Lower-bound potential \( u \) - fixed horizon:** We will use this term for a function \( u : \mathbb{R}^N \times \mathbb{R}_{\leq 0} \rightarrow \mathbb{R} \), such that, for every \( x \in \mathbb{R}^N \), there exists some symmetric probability distribution \( a \) on \([-1, 1]^N \) that depends on \( x \) only (i.e., it is stationary) ensuring that \( u \) is a classical solution of

\[
\begin{cases}
  u_t + \frac{1 - \delta}{2\delta} \mathbb{E}_a \langle D^2 u \cdot q, q \rangle \geq 0 \\
  u(x, 0) = \max_i x_i \\
  u(x + c\mathbb{1}, t) = u(x, t) + c
\end{cases}
\] (3)

Consider a potential \( \hat{u} \) given by \( \hat{u}(x) = e^{\delta} \int_{-\infty}^{-\delta} e^t u(x, t) dt - \hat{C} \) where \( \hat{C} \) is a constant that such \( u(x, -\delta) - \max_i x_i \leq \hat{C} \) for all \( x \). Integrating by parts,

\[
\hat{u}(x) = e^{\delta} \int_{-\infty}^{-\delta} \partial_t e^t u(x, t) dt - \hat{C} = u(x, -\delta) - \hat{C} - e^{\delta} \int_{-\infty}^{-\delta} e^t u_t(x, t) dt
\]

\[
\leq u(x, -\delta) - \hat{C} + e^{\delta} \int_{-\infty}^{-\delta} e^t \left( \frac{1 - \delta}{2\delta} \mathbb{E}_a \langle D^2 u \cdot q, q \rangle \right) dt \leq \max_i x_i + \frac{1 - \delta}{2\delta} \mathbb{E}_a \langle D^2 \hat{u} \cdot q, q \rangle
\]

where the last inequality holds because \( a \) is stationary. Also the linearity of \( u \) in the direction of \( \mathbb{1} \) implies the same result with respect to \( \hat{u} \). If \( \mathbb{E}_a \langle D^2 u \cdot q, q \rangle \) is bounded above uniformly in \( x \in \mathbb{R} \), \( t \leq -\delta \), and \( q \), then \( \hat{u}(x) - \max_i x \) is uniformly bounded above. Therefore \( \hat{u} \) satisfies (2) with the adversary strategy \( a \).

Finally, if for any \( q \) sampled from \( a \), \( \text{ess sup}_{y \in [x, x-q]} D^3 u(y, t)[q, q, q] \leq C_3(t) \), then

\[
\text{ess inf}_{y \in [x, x-q]} D^3 \hat{u}(y)[q, q, q] \leq e^{\delta} \int_{-\infty}^{-\delta} e^t C_3(t) dt
\]

Therefore, we obtained a geometric lower bound potential from a fixed horizon one.
Theorem 3 (Fixed horizon to geometric l.b) If for all \( x, u(x, -\delta) - \max_i x_i \leq \hat{C}, \mathbb{E}_a (D^2 u \cdot q, q) \) is bounded above uniformly in \( x, t \leq -\delta \), and \( q \), \( D^2 u \) is Lipschitz continuous and, for any \( q \) sampled from \( a \), \( \text{ess sup}_{y \in [x, x - q]} D^3 u(y, t)[q, q, q] \leq C_3(t) \), then \( \hat{u}(x) - \frac{1 - \delta}{24\delta} K \leq \hat{v}_a(x) \) where \( K = e^\delta \int_{-\delta}^{0} e^t C_3(t) dt \).

In this setting, we can also take advantage of the continuity of the higher-order spatial derivatives. For example, if for all \( x \) and \( t < 0 \), \( D^3 u(\cdot, t) \) exists and is Lipschitz continuous, and for any \( q \) sampled from \( a \), \( \text{ess inf}_{y \in [x, x - q]} D^4 u(y, t)[q, q, q, q] dt \leq C_4(t) \),

\[
\text{ess inf}_{y \in [x, x - q]} D^4 \hat{u}(y)[q, q, q] \leq e^\delta \int_{-\delta}^{-} e^t C_4(t) dt
\]

Remark 4 (Fixed to geometric l.b - Lipschitz continuous higher-order derivs.) Similarly to Remark 1, if \( u \) has higher order Lipschitz continuous derivatives, they could be used to derive a lower bound for \( \hat{v}_a \). For example, if for all \( x \) and \( t < 0 \), \( D^3 u(\cdot, t) \) exists and is Lipschitz continuous, and for any \( q \) sampled from \( a \), \( \text{ess inf}_{y \in [x, x - q]} D^4 u(y, t)[q, q, q, q] dt \leq C_4(t) \), then the conclusion of Theorem 3 still holds with \( \hat{u}(x) - \frac{1 - \delta}{24\delta} K \leq \hat{v}_a(x) \) where \( K = e^\delta \int_{-\delta}^{0} e^t C_3(t) dt \).

4. Upper bound

The value function \( \hat{v}_p \), reflecting the largest regret possible for a given player \( p \) over all adversary strategies satisfies \( \hat{v}_p(x) = \delta \max_i x_i + (1 - \delta) \max_i \mathbb{E}_{a, b} \hat{v}_p(x + r) \). This function also has an equivalent characterization:

\[
\hat{v}_p(x) = \max_a \mathbb{E}_{G, p, a} \max_i \left( x_i + \sum_{r \in [T]} (r_r)_i \right)
\]

To bound the regret above, we introduce a potential \( \hat{w} \) and a corresponding player strategy \( p \).

**Upper-bound potential \( \hat{w} \) - geometric stopping:** We will use this term for a function \( \hat{w} : \mathbb{R}^N \rightarrow \mathbb{R} \), which is nondecreasing as a function of each \( x_i \), and which, for all \( x \in \mathbb{R}^N \), \( q \in [-1, 1]^N \), and \( c \in \mathbb{R} \), is a classical solution of

\[
\begin{align*}
\hat{w}(x) &\geq \max_i x_i + \frac{1 - \delta}{24\delta} (D^2 \hat{w}(x) \cdot q, q) \\
\hat{w}(x + c\mathbb{1}) &= \hat{w}(x) + c
\end{align*}
\]

(4)

**Player p:** Given \( \hat{w} \) as above, the associated player strategy \( p \) is: In each period, the player selects \( p = \nabla \hat{w}(x) \).

Since \( \hat{w} \) is nondecreasing as a function of each \( x_i \) and \( \sum_i \partial_i \hat{w}(x) = 1 \) by linearity of \( \hat{w}(x) \) in the direction of \( \mathbb{1} \), \( p \in \Delta_N \). In Appendix B, using Taylor expansion of \( \hat{w} \) with the integral form of the third-order spatial remainder, we confirm that \( p \) guarantees the following upper bound.

Theorem 5 (Geometric horizon u.b) If, for all \( x \), \( \hat{w}(x) - \max_i x_i \) is uniformly bounded below, \( D^2 \hat{w} \) is Lipschitz continuous and, for any \( q \), \( \text{ess inf}_{y \in [x - q]} D^3 \hat{w}(y)[q, q, q] \leq K \), then \( \hat{v}_p(x) \leq \hat{w}(x) + \frac{1 - \delta}{6\delta} K \).
We can also construct a geometric stopping lower bound potentials from a fixed horizon one.

**Upper-bound potential w · fixed horizon:** We use this term for a function \( w : \mathbb{R}^N \times \mathbb{R}_{\leq 0} \to \mathbb{R} \), which is nondecreasing as a function of each \( x_i \), and which is, for all \( x \in \mathbb{R}^N, t < 0 \), \( q \in [-1, 1]^N \), and \( c \in \mathbb{R} \), a classical solution of

\[
\begin{align*}
& w_t + \frac{1 - \delta}{2\delta} \langle D^2 w \cdot q, q \rangle \leq 0 \\
& w(x, 0) \geq \max_i x_i \\
& w(x + c\mathbb{1}, t) = w(x, t) + c
\end{align*}
\tag{5}
\]

Consider a potential \( \dot{w}(x) = e^\delta \int_{-\infty}^{-\delta} e^t w(x, t) dt + \hat{C} \) where \( \hat{C} \) is a constant that such \( w(x, -\delta) - \max_i x_i \geq -\hat{C} \) for all \( x \). Integrating by parts,

\[
\dot{w}(x) = e^\delta \int_{-\infty}^{-\delta} \partial_t e^t w(x, t) dt + \hat{C} = w(x, -\delta) + \hat{C} - e^\delta \int_{-\infty}^{-\delta} e^t w_t(x, t) dt \\
\geq w(x, -\delta) + \hat{C} + e^\delta \int_{-\infty}^{-\delta} e^t \left( \frac{1 - \delta}{2\delta} \langle D^2 w \cdot q, q \rangle \right) dt
\tag{6}
\]

When \( q = q(x) \) is independent of \( t \), we have \( \langle D^2 \dot{w} \cdot q, q \rangle = e^\delta \int_{-\infty}^{-\delta} e^t \langle D^2 w \cdot q, q \rangle dt \). Therefore, (6) is bounded below by \( \max_i x_i + \frac{1 - \delta}{2\delta} \langle D^2 \dot{w} \cdot q, q \rangle \).

Also the linearity of \( w \) in the direction of \( \mathbb{1} \) implies the same result with respect to \( \dot{w} \). If \( \langle D^2 w \cdot q, q \rangle \) is bounded below uniformly in \( q \in [-1, 1]^N \), \( x \in \mathbb{R} \) and \( t \leq -\delta \), then \( \dot{w}(x) - \max_i x \) is uniformly bounded below. Finally, if \( w \) is nondecreasing with respect to each \( x_i \), then so is \( \dot{w} \). Therefore \( \dot{w} \) satisfies (2) with the adversary strategy \( a \).

Finally, if for any \( q \), \( -\text{ess inf}_{y \in [x, x-q]} D^3 w(y, t)[q, q, q] \leq C_3(t) \), then

\[
-\text{ess inf}_{y \in [x, x-q]} D^3 \dot{w}(y)[q, q, q] \leq e^\delta \int_{-\infty}^{-\delta} e^t C_3(t) dt
\]

Therefore, we obtain the following correspondence.

**Theorem 6 (Fixed horizon to geometric u.b.)** If for all \( x, u(x, -\delta) - \max_i x_i \geq -\hat{C}, \langle D^2 u \cdot q, q \rangle \) is bounded below uniformly in \( q, x \) and \( t \leq -\delta \), \( D^2 u \) is Lipschitz continuous and, for any \( q \), \( -\text{ess inf}_{y \in [x, x-q]} D^3 u(y, t)[q, q, q] \leq C_3(t) \), then \( \check{v}_p(x) \leq \check{u}(x) + \frac{1 - \delta}{2\delta} K \) where \( K = e^\delta \int_{-\infty}^{-\delta} e^t C_3(t) dt \).

The analysis becomes simpler if \( w_t \) is a constant. Then \( w_t + \frac{1 - \delta}{2\delta} \langle D^2 w \cdot r, r \rangle \leq 0 \) holds for all \( q \) even if \( w_t \) and \( \frac{1 - \delta}{2\delta} \langle D^2 w \cdot q, q \rangle \) are evaluated at different points. This allows us to use Taylor’s expansion of \( \dot{w} \) with the mean value form of the second-order spatial remainder, which eliminates the discretization error \( K \). (Also in such case we do not need to ensure convergence of the integral for the error term \( K \), and therefore we can take also \( \delta = 0 \).) We will use this approach below in the context of the exponentially weighted average player.

**Remark 7 (Fixed to geometric u.b. - constant first-order time deriv.)** If \( w_t \) is constant, then the conclusion of Theorem 6 still holds with \( \check{v}_p(x) \leq \check{w}(x) \).
5. Exponentially weighted average

In Appendix C, we confirm that \( \langle D^2 \Phi \cdot r, r \rangle = \langle D^2 \Phi \cdot q, q \rangle \leq \eta \) for the potential \( \Phi(x) = \frac{1}{\eta} \log(\sum_{k=1}^{N} e^{\eta x_k}), r = \langle p, q \rangle \mathbf{1} + q \) and all \( x, t, p \) and \( q \). Also note that \( \Phi(x + c \mathbf{1}) = \Phi(x) + c \).

Accordingly, the upper bound potential \( w^e \) given by \( w^e(x, t) = \Phi(x) - \frac{1-\delta}{2\delta} \eta t \) satisfies (5). Let \( \hat{w}^e \)
be given by

\[
\hat{w}^e(x) = \int_{-\infty}^{0} e^t w^e(x, t) dt = \Phi(x) + \frac{1-\delta}{2\delta} \eta
\]

Note that \( w^e(x, 0) - \max_i x_i \geq 0 \). Therefore, \( \hat{w}^e \) satisfies (4), and by Remark 7, we obtain

**Example 1 (Exp u.b. - geometric)** The Exp player \( p^e \) attains the following upper bound \( \hat{v}^e(x) \leq \hat{w}^e(x) \) where \( \hat{v}^e \) is the value function for this player.

Note that

\[
\hat{w}^e(0) = \log \frac{N}{\eta} + \frac{1-\delta}{2\delta} \eta
\]

Taking \( \eta = \sqrt{\frac{2\Delta \log N}{1-\delta}} \), leads to the following bound

\[
\max_a R(a, p^e) = \hat{v}^e(0) \leq \sqrt{\frac{2(1-\delta) \log N}{\delta}}
\]

6. Screened Poisson equation-based potential

In this section, let \( u \) be given by

\[
u(x, t) = \alpha \int e^{-\frac{\|y\|^2}{2\sigma^2}} \max_k (x_k - y_k) dy
\]

where \( \alpha = (2\pi\sigma^2)^{-\frac{N}{2}} \) and \( \sigma^2 = -2\kappa t \) and \( t < 0 \). This \( u \) is the classical solution, on \( \mathbb{R}^N \times \mathbb{R}_{<0} \), of the following linear heat equation

\[
\begin{cases}
u_t + \kappa \Delta \nu = 0 \\
u(x, 0) = \max_i x
\end{cases}
\]

We consider a potential \( \hat{u} \) given by

\[
\hat{u}(x) = e^\delta \int_{-\infty}^{-\delta} e^t u(x, t) dt - \hat{C}
\]

3. Rokhlin (2017) proposed a fixed-horizon potential corresponding to the exponentially weighted average (Exp) player \( p^e \) providing a PDE perspective on the bound in Corollary 2.2 of Cesa-Bianchi and Lugosi (2006). Our PDE-based approach gives the best known upper bound for the geometric stopping problem and it is straightforward to extend this approach to the fixed horizon problem and provide a PDE perspective on the best known bound in Theorem 2.2 of Cesa-Bianchi and Lugosi (2006). Since the focus of this paper is the geometric stopping problem, we will not revisit the fixed horizon problem here.
where $\hat{C} = \sqrt{2N\delta} \mathbb{E} \max_i G_i$, and $G$ is an $N$-dimensional Gaussian $N(0, I)$. Integrating by parts,

$$
\hat{u}(x) = e^{\delta} \int_{-\infty}^{x} \partial_{x} e^{\tau} u(x, t) dt - \hat{C} = u(x, -\delta) - e^{\delta} \int_{-\infty}^{-\delta} e^{\tau} u_t(x, t) dt
$$

$$
\hat{u}(x) = u(x, -\delta) - \hat{C} + \kappa \Delta \hat{u}(x)
$$
on $\mathbb{R}^N$. (Partial differential equations of the form $u = \phi + \kappa \Delta u$ are called screened Poisson equations with a source $\phi$). By Section 5 of Kobzar et al. (2019), this $u$ with the diffusion factor

$$
\kappa_s = \begin{cases} 
\frac{1 - \delta}{\delta} & \text{if } N = 2 \\
\frac{1}{2} + \frac{1}{2N} & \text{if } N \text{ is odd} \\
\frac{1}{2} + \frac{1}{2N} & \text{otherwise}
\end{cases}
$$

satisfies (3) when the adversary uses the following strategy.$^4$

**Screened Poisson-based adversary $a^s$:** Adversary $a^s$ samples $q$ from a uniform distribution on

$$
S = \begin{cases} 
\{ q \in \{-1, 1\}^N | \sum_{i=1}^{N} q_i = \pm 1 \} & \text{for } N \text{ odd} \\
\{ q \in \{-1, 1\}^N | \sum_{i=1}^{N} q_i = 0 \} & \text{for } N \text{ even}
\end{cases}
$$

Note that the $a^s$ strategy is symmetric and stationary. Next, as confirmed in Appendix D, $|u(x, -\delta) - \max_i x_i| \leq \hat{C}$, for all $x \in \mathbb{R}^N$. Also since

$$
\max_q \langle D^2 u \cdot q, q \rangle \leq 2 \Delta u = - \sum_i \frac{2c_N}{\sigma^2} \int e^{-\frac{\|y\|^2}{\sigma^2}} y_i \mathbb{1}_{|x_i - y_i| > \max_j y_j, x_j - y_j} dy = \frac{2}{\sigma} \mathbb{E}_G \max |G_i|
$$

max_q $\langle D^2 \hat{u} \cdot q, q \rangle$ is uniformly bounded above. Therefore $a^s$ with the potential $\hat{u}^s$ given by (8) with the diffusion factor $\kappa_s$ satisfies (2).

Finally, by Appendix E in Kobzar et al. (2019), $-\text{ess inf}_{y \in [x, x-q]} D^4 u(y, t)[q, q, q, q] \leq \frac{C_4}{(-t)^{\frac{3}{2}}}$ for $q \sim a^s$ with $C_4 = O(\kappa_s^{-3/2} N \sqrt{N})$. Therefore, since $\int \frac{e^{t}}{(-t)^{\frac{3}{2}}} dt = 2\sqrt{\pi} \text{erf}(\sqrt{-t}) + \frac{2e^{t}}{\sqrt{\pi t}} + \text{const}$

$$
-\text{ess inf}_{y \in [x, x-q]} D^4 \hat{u}(y)[q, q, q, q] \leq e^{\delta} C_4 \int_{-\infty}^{-\delta} \frac{e^{t}}{(-t)^{\frac{3}{2}}} dt \leq C_4 \left( \frac{2}{\sqrt{\delta}} - 2e^{\delta} \sqrt{\pi} \text{erfc}(\sqrt{\delta}) \right) \leq C'_4
$$

where $C'_4 = O \left( \delta N \sqrt{N} \right)$.

For the upper bound, Appendix D in Kobzar et al. (2019) provides that when $u$ is defined using $\kappa = \frac{1 - \delta}{\delta}$, it satisfies (5). Therefore, $\hat{w}(x) = e^{\delta} \int_{-\infty}^{-\delta} e^{\tau} u(x, t) dt + \hat{C}$ with that choice of $\kappa$ satisfies (4) leading to

$^4$ As more fully discussed in the reference cited in the text accompanying this footnote, $\langle D^2 u \cdot q, q \rangle$ is equivalent to a cut of an undirected graph with the Laplacian matrix given by $D^2 u$. From this perspective, the set $S$ represents balanced cuts.
**Screened Poisson-based player** $p^s$: In each period, the player selects $p^s = \nabla \hat{w}^s(x)$.

By Appendix E in Kobzar et al. (2019), $-\text{ess inf}_{y \in [x, x-q]} D^3 u(y)[q, q, q] dt \leq -\frac{C_3}{t}$ for all $q$. Therefore, using a standard bound of the exponential integral

$$-\text{ess inf}_{y \in [x, x-q]} D^3 \hat{u}(y)[q, q, q] \leq C_3 e^\delta \int_{-\delta}^- \frac{e^t}{t} dt \leq C_3 \left(1 + \log \frac{1}{\delta}\right)$$

where $C_3 = O(\kappa^{-1} \sqrt{N}) = O(\frac{1}{N} \sqrt{N})$.

Applying Theorems 3 and 6, yields the following upper and lower bounds on the relevant value functions. Since the solution to the heat equation is smooth, for the lower bound we control $E_{a^*}$ uniformly in $\delta$ using Remark 4.

**Example 2 (Screened Poisson equation-based bounds - geometric)**

(l.b.) the adversary strategy $a^*$ attains the following lower bound $\hat{u}^s(x) - E_{a^*} \leq \hat{v}_{a^*}(x)$ where $\hat{v}_{a^*}$ is the geometric stopping value function of $a^*$ and $E_{a^*}$ is bounded uniformly in $\delta$ and $E_{a^*} = O(\sqrt{N})$.

(u.b.) the player strategy $p^s$ attains the following upper bound $\hat{v}_{p^*}(x) \leq \hat{w}^s(x) + E_{p^*} \leq \hat{v}_{p^*}$ is the geometric stopping value function of $p^s$ and $E_{p^*} = O(\sqrt{N} \log \frac{1}{\delta})$.

In this example, we choose to use the bound in Remark 4. Theorem 3 is also available, and provides $E_{a^*} = O(\sqrt{N} \log \frac{1}{\delta})$, which gives a smaller discretization error in a regime where $\delta$ is fixed and $N$ increases. Therefore, $E_{a^*}$ is in fact the lesser of $O\left(N \sqrt{N}\right)$ and $O\left(\sqrt{N} \log \frac{1}{\delta}\right)$, which is reflected in our plots in the Figures below.

Note that

$$\hat{u}(0) = \sqrt{2\kappa \xi_G} \max \{G_i \left( e^\delta \int_{-\delta}^0 e^t \sqrt{-tdt} - \sqrt{\delta} \right) \} = \sqrt{2\kappa \xi_G} \max \{G_i \left( e^\delta \frac{1}{2} \sqrt{\pi \text{erfc}(\sqrt{\delta})} \right) \}$$

where $G$ is a Gaussian $N$-dimensional vector $N(0, I)$. Therefore, the bounds on the value function lead to the following bounds on the regret

$$\sqrt{2\kappa \xi_G} \max \{G_i \left( e^\delta \frac{1}{2} \sqrt{\pi \text{erfc}(\sqrt{\delta})} \right) \} \leq \hat{v}_{a^*}(0) = \min_p R(a^*, p)$$

and

$$\max_d R(a, p^s) = v_{p^s}(0) \leq \sqrt{\frac{2(1-\delta)}{\delta} \xi_G} \max \{G_i \left( e^\delta \frac{1}{2} \sqrt{\pi \text{erfc}(\sqrt{\delta})} \right) \} + O\left(\sqrt{N} \log \frac{1}{\delta}\right)$$

**7. Max-based potential**

In this section, let $u$ be given by the solution of

$$\begin{cases} u_t + \kappa \max_i \partial_i^2 u = 0 \\ u(x, 0) = \max_i x \end{cases} \quad (10)$$
For all \( x \in \mathbb{R}^N \), we will denote by \( \{x(i)\}_{i=1,...,N} \) the ranked coordinates of \( x \), such that \( x(1) \geq x(2) \geq ... \geq x(N) \). As shown in Kobzar et al. (2019), the classical solution of (10) on \( \mathbb{R}^N \times \mathbb{R}_{\geq 0} \) is given by

\[
u(x, t) = \frac{1}{N} \sum_i x(i) + \sqrt{-2Nt} \sum_{l=1}^{N-1} c_l f(z_l) \tag{11}\]

where \( z_l = \frac{1}{\sqrt{-2Nt}} \left( \left( \sum_{n=1}^l x(n) \right) - lx(l+1) \right) , f(z) = \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} + \operatorname{erf} \left( \frac{z}{\sqrt{2}} \right) , \operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-s^2} ds \) and \( c_l \equiv \frac{1}{l(l+1)} \). We consider a potential \( \hat{u} \) given by

\[
\hat{u}(x) = e^\delta \int_{-\infty}^{-\delta} e^t \nu(x, t) dt - \hat{C} \tag{12}
\]

where \( \hat{C} = 2 \sqrt{\frac{\delta N}{N-1}} \). (We shall call this the max potential.) Integrating by parts,

\[
\hat{u}'(x) = e^\delta \int_{-\infty}^{-\delta} \partial_t e^t \nu(x, t) dt - \hat{C} = u(x, -\delta) - \hat{C} - e^\delta \int_{-\infty}^{-\delta} e^t \nu_t(x, t) dt \\
= u(x, -\delta) - \hat{C} + \kappa \max \partial^2_i \hat{u}(x)
\]

Also, as confirmed in Appendix E below, \( u(x, -\delta) - \max_i x_i \leq \hat{C} \) for all \( x \in \mathbb{R}^N \). Appendix F in Kobzar et al. (2019) confirms that \( \max_i \partial^2_i u(x, t) \) is uniformly bounded above for all \( x \) and \( t \leq -\delta \).

Section 6 of Kobzar et al. (2019) confirms \( u \) with \( \kappa = \frac{2(1-\delta)}{\delta} \) satisfies (3). Therefore, \( \hat{u}^m \) given by (12) using that choice of \( u \) and \( \kappa \) satisfies (2) for the following adversary \( a^m \).

**max adversary \( a^m \):** In each period, the adversary selects the distribution \( a^m \) by assigning probability \( \frac{1}{2} \) to each of \( q^m \) and \( -q^m \) where the entry of \( q^m \) corresponding to the largest component of \( x \) is set to 1 and the remaining entries are set to -1.

By Appendix H.3 in Kobzar et al. (2019), for \( q \sim a^m \),

\[
\text{ess sup}_{y \in [x, x-q]} D^3 \hat{u}(y)[q, q, q] = \text{ess sup}_{y \in [x, x-q]} \int_{-\infty}^{-\delta} e^t D^3 \hat{u}(y, t)[q, q, q] dt \leq C_3 \int_{-\infty}^{-\delta} e^t \frac{dt}{t} \leq C_3 e^{-\delta} \left( 1 + \log \frac{1}{\delta} \right)
\]

where \( C_3 = O \left( \frac{N}{\delta} \right) = O \left( \frac{N^\delta}{1-\delta} \right) \).

To determine an upper bound, from Appendix G of Kobzar et al. (2019), \( u \) with the diffusion factor

\[
\kappa_m = \frac{1 - \delta}{\delta} \left( \frac{N^2}{2(N-1)} \right) \quad \text{for } N \text{ even}
\]

\[
\kappa_m = \frac{1 - \delta}{\delta} \left( \frac{N+1}{2} \right) \quad \text{for } N \text{ odd}
\]

(13)
satisfies (5) Also confirmed in Appendix E below, \( u(x, -\delta) - \max_i x_i \geq \hat{0} \), and Appendix F in Kobzar et al. (2019) confirms that \( \max_i \partial^2_t u(x, t) \) is uniformly bounded below for all \( x \) and \( t \leq -\delta \). Therefore, an upper bound potential \( \hat{w}^m \) given by

\[
\hat{w}_m(x) = e^{\delta} \int_{-\infty}^{-\delta} e^t u(x, t) dt
\]

(14)

with the diffusion factor \( \kappa_m \) satisfies (4) for all \( \mathbf{q} \) and this yields the player strategy \( p^m \).

### max-potential player \( p^m \): In each period, the player selects \( p = \nabla \hat{w}^m(x) \).

By Appendix H.4 in Kobzar et al. (2019), for all \( \mathbf{q} \in [-1, 1]^N \),

\[
-\text{ess inf}_{y \in [x, x - \mathbf{q}]} D^3 \hat{u}(y)[\mathbf{q}, \mathbf{q}, \mathbf{q}] = -\text{ess inf}_{y \in [x, x - \mathbf{q}]} \int_{-\delta}^{-\delta} e^t D^3 \hat{u}(y, t)[\mathbf{q}, \mathbf{q}, \mathbf{q}] dt \leq C_3 \int_{-\infty}^{-\delta} \frac{e^t}{t} dt \leq C_3 e^{-\delta} \left( 1 + \log \frac{1}{\delta} \right)
\]

where \( C_3 = O \left( \frac{N^2}{\kappa_m^2} \right) = O \left( \frac{N}{\delta} \right) \).

Applying Theorems 3 and 6, we obtain

### Example 3 (max-based bounds)

(l.b.) \( \hat{v}_m(x) - E \leq \hat{v}_m(x) \) where \( \hat{v}_m(x) \) is the value function of \( a^m \); and

(u.b.) \( \hat{v}_m(x) \leq \hat{v}_p(x) + E \) where \( \hat{v}_p(x) \) is the value function of \( p^m \)

and, in each case, \( E = O \left( N \log \frac{1}{\delta} \right) \).

Since \( u(0, t) = \frac{2(N-1)}{N} \sqrt{-\frac{\kappa t}{\pi}} \),

\[
\hat{u}^m(0) = \frac{2(N-1)}{N} \sqrt{\frac{\kappa}{\pi}} \left( e^{\delta} \int_{-\infty}^{-\delta} e^t \sqrt{-\delta t} dt - \sqrt{\delta} \right) = \frac{(N-1)}{N} \sqrt{\frac{2(1 - \delta)}{\delta}} e^{\delta} \text{erfc}(\sqrt{\delta})
\]

and

\[
\hat{w}^m(0) = \frac{2(N-1)}{N} \sqrt{-\frac{\kappa_m}{\pi}} \left( e^{\delta} \int_{-\infty}^{-\delta} e^t \sqrt{\delta t} dt \right) = \frac{(N-1)}{N} \sqrt{\kappa_m} \left( e^{\delta} \text{erfc}(\sqrt{\delta}) + \sqrt{\delta} \right)
\]

Therefore, the bounds on the value function lead to the following bounds on the regret

\[
\frac{(N-1)}{N} \sqrt{\frac{2(1 - \delta)}{\delta}} \left( e^{\delta} \text{erfc}(\sqrt{\delta}) \right) - O \left( N \log \frac{1}{\delta} \right) \leq \hat{v}_m(0) = \min_p R(a^m, p)
\]

and

\[
\max_a R(a, p^m) = \hat{v}_p(0) \leq \frac{(N-1)}{N} \sqrt{\frac{\kappa_m}{\pi}} \left( e^{\delta} \sqrt{\pi} \text{erfc}(\sqrt{\delta}) + \sqrt{\delta} \right) + O \left( N \log \frac{1}{\delta} \right)
\]
8. Related work

Our regret upper bound $\sqrt{\frac{2(1-\delta)}{\delta} \log N}$ for the Exp player $p^e$ improves in the nonasymptotic setting, the upper bound $\sqrt{\frac{2}{\delta} \log N}$ determined in Gravin et al. (2017) (as rescaled for our $[-1, 1]^N$ losses).

Also since $\lim_{N \to \infty} \max_i G_i \sqrt{2 \log N} = 1$, we have

$$\lim_{N \to \infty} \frac{1}{\sqrt{2 \log N}} \lim_{\delta \to 0} \sqrt{\delta} \left[ \hat{u}_s(0) - E_{a^s} \right] = \frac{\sqrt{\pi}}{2} \approx .89$$

(15)

This shows that in the limit where $\delta \to 0$ first and then $N \to \infty$, the lower bound for the screened Poisson-based adversary $a^s$ is tighter than the lower bound $\sqrt{\frac{\log N}{2\delta}}$ determined in Gravin et al. (2017) with respect to the Exp player only (for the latter bound, (15) is equal to $\frac{1}{\sqrt{2}}$). Also our lower bound guarantee is given with respect to an arbitrary player strategy.

We now consider the nonasymptotic setting. In the fixed horizon setting there exists several known lower bounds on $\min_p R(p, a)$. Gravin et al. (2017) noted that the only lower bound for the geometric problem is one that could be inferred from the fixed horizon lower bound given in Chapter 7 of György et al. and showed that the lower bound when the player is using Exp is $\sqrt{\frac{\log N}{2\delta}} \geq \max_a R(p^e, a)$ as $\delta \to 0$, which is tighter than any so inferred lower bound.

Our nonasymptotic lower bound for $\min_p R(p, a^s)$ based on the screened Poisson potential is even tighter (for sufficiently small but fixed $\delta$) and applies to all player strategies $p$. See Figure 1. In particular, since $\max_a R(p^e, a) \geq \min_p R(p, a^p)$, we also improve the lower bound with respect to Exp.

Furthermore, when $5 \leq N \lesssim 30$ and $\delta$ is relatively small, as illustrated by Figure 2, the max-based player $p^m$ and the screened-Poisson based player $p^s$ improve the upper bounds guarantee given with respect to the Exp (the Poisson-based adversary $a^s$ also remains tighter than the corresponding lower bound in Gravin et al. (2017)).

The max potential was known from Drenska and Kohn (2019) to solve the PDE associated with the scaling limit of the optimal value function for $N = 3$ in the geometric case. Therefore, the upper and lower bounds we obtained have to match at the leading order, and in fact they do for $N = 2$ and 3 experts. We note that in Gravin et al. (2016) the exact nonasymptotic value function was determined for two and three experts and any fixed $\delta$. In Appendix F, we check directly that our lower and upper bounds for $N = 2$ and 3 coalesce at the leading order as expected.

9. Conclusions

In this work, we extended the potential-based framework and strategies for the expert problem from the fixed horizon setting to the geometric stopping one.

The resulting lower bound based on the solution to the screened Poisson equation is the best known for $N \geq 5$. This bound is obtained by a simple randomized strategy that is independent of the accumulated regret or any other history.
Figure 1: This figure illustrates that the screened Poisson-based adversary $a^\delta$ is advantageous when $\delta$ is relatively small but fixed. $C_N$ is such that for a given adversary $a$, $R(a, p) \geq C_N \sqrt{\frac{1}{\delta}}$ for all $p$.

Also the resulting upper bounds based on the screened Poisson potential and a new max-based potential are tighter for small $N$ than those guaranteed by the exponentially weighted average forecaster.

Since the discounting of the regret by the probability that the game will continue makes the geometric stopping problem similar to a discounted regret problem, we expect that our potential-based framework could be extended to a suitable class of problems involving discounted regret.

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Figure 2: This figure illustrates that the max-based player $p^m$ and the screened Poisson-based player $p^s$ are advantageous when $N$ is small and $\delta$ is small but fixed (the screened Poisson-based adversary $a^s$ also remains advantageous in this setting). $C_N$ is such that for an adversary $a$, $R(a, p) \geq C_N \sqrt{\frac{1}{\delta}}$ for all players $p$, and for a given player $p$, $R_T(a, p) \leq C_N \sqrt{\frac{1}{\delta}}$ for all adversaries $a$. 

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Nick Gravin, Yuval Peres, and Balasubramanian Sivan. Tight Lower Bounds for Multiplicative Weights Algorithmic Families. In 44th International Colloquium on Automata, Languages, and Programming (ICALP 2017), volume 80, pages 48:1–48:14, Warsaw, Poland, 2017.
Appendix A. Proof of Theorem 1

We show that \( \hat{u} \) bounds \( \hat{v}_a \) below in two steps:

1. Define a new problem, which is the same as the geometric stopping problem except that it starts at a given time \( t \leq -1 \), and ends at time 0 (if it does not end sooner in accordance with the geometric stopping condition) and show that the value of the new problem approaches the value of the original problem as \( t \to -\infty \); and

2. Show that \( \hat{u} \) bounds below the regret in the new problem for all \( t \).

A.1. Convergence of the value of the new problem to the original one

The value function \( g \) of the new problem reflecting the worst-case (lowest) value is constructed by the following DP:

\[
g(x, 0) = \max_i x_i \\
g(x, t) = \delta \max_i x_i + (1 - \delta) \min_p \mathbb{E}_{a_t, p_t} g(x + r_t, t + 1)
\]

It has an equivalent definition given by \( g(x, t) = \min_p \mathbb{E}_{G, p, a} \max_i \left( x_i + \sum_{\tau \in [\max(t,T)]} (r_{\tau})_i \right) \).

For these purposes, the geometric random variable \( G \) is the same as the standard geometric distribution except that it is multiplied by -1 so that to ensure that its outcomes are negative numbers consistently with our time counting convention. Since in each period the regret can decrease by at most 2, from the definition of \( \hat{v} \) in (1), we obtain \( g(x, t) - s(t) \leq \hat{v}(x) \) for any \( t \leq 0 \) where

\[
s(t) = \mathbb{E}_G \left[ \mathbb{I}_{T < t} \sum_{T < t} 2 \right] = 2 \sum_{\tau = -t + 1}^{\infty} (1 - \delta)^{-1} \delta (\tau + t) = \frac{2(1 - \delta)^{-t}}{\delta}
\]

Since \( \lim_{t \to -\infty} s(t) = 0 \), it suffices to bound \( g \) below.
A.2. Lower bound on the value of the new problem

To bound $g$ below, we show that (i) $\hat{u}(x) - E(t) \leq g(x,t)$ for all $x \in \mathbb{R}$ and $t \leq 0$, and (ii) $\lim_{t \to -\infty} E(t) = \frac{1 - \delta}{\delta} K$. Since $g$ is characterized by a DP, we can use induction to show (i). The initial step $\hat{u}(x) + E(0) \leq \max_i x_i$ follows from the uniform upper bound on $\hat{u}(x) - \max_i x_i$.

To prove the inductive step, as a preliminary result, we bound below the difference $\min_{p_t} \mathbb{E}_{a_t,p_t} [\hat{u}(x + r_t)] - \hat{u}(x)$ in terms of $K$. Since $\hat{u}$ is $C^2$ with Lipschitz continuous second-order derivatives, we can use Taylor’s theorem with the integral remainder:

$$
\min_{p_t} \mathbb{E}_{a_t,p_t} [\hat{u}(x + r_t)] - \hat{u}(x) = \min_{p_t} \mathbb{E}_{a_t,p_t} [\hat{u}(x - q_t)] + (q_t)_t - \hat{u}(x)
\geq \frac{\delta}{1 - \delta} (\hat{u}(x) - \max_i x_i) - K
$$

We eliminated the dependence on $p$ using the fact that $\hat{u}(x + r_t, t + 1) = \hat{u}(x - q_t, t + 1) + (q_t)_t$, in the first equality and the fact that the expectation of $(q_t)_t$ is zero by the symmetry of $a_t$ in the second equality. We also used the condition on the potential

$$
u(x) - \max_i x_i - \frac{1 - \delta}{2\delta} \mathbb{E}_{a_t} \langle D^2 \hat{u} \cdot q,q \rangle \leq 0
$$

Thus,

$$(1 - \delta)(\min_{p_t} \mathbb{E}_{a_t,p_t} [\hat{u}(x + r_t)] - \hat{u}(x)) - \delta(\hat{u}(x) - \max_i x_i) + (1 - \delta)K \geq 0
$$

From this and the inductive hypothesis $u(x + r) - E(t + 1) \leq g(x + r, t + 1)$, we obtain

$$
\hat{u}(x) - E(t) \leq \hat{u}(x) - \delta(\hat{u}(x) - \max_i x_i) + (1 - \delta) \min_{p_t} \mathbb{E}_{a_t,p_t} [\hat{u}(x) + r] - \hat{u}(x)] + (1 - \delta)K - E(t)
\leq \delta \max_i x_i + (1 - \delta) \min_{p_t} \mathbb{E}_{a_t,p_t} [\hat{u}(x) + r] + (1 - \delta)K - E(t)
\leq \delta \max_i x_i + (1 - \delta) \min_{p_t} \mathbb{E}_{a_t,p_t} [g(x + r, t + 1) + E(t + 1)] + (1 - \delta)K - E(t)
= g(x, t)
$$

Finally, $E(t) = (1 - \delta)^{-t} E(0) + \sum_{\tau \in \mathbb{N}} (1 - \delta)^\tau K$ for $t \leq -1$ satisfies the recursion $E(t) = (1 - \delta)(E(t + 1) + K)$ necessary for the last equality to hold. Also $\lim_{t \to -\infty} E(t) = \frac{1 - \delta}{\delta} K$ as desired.

The proof of Remark 2 is the same except that we expand $\hat{u}(x - q_t)$ up to fourth order derivatives and use the fact that $\mathbb{E}_{a_t} D^3 \hat{u}(x)[q,q,q] = 0$ by symmetry ($q$ and $-q$ have the same probability).
Appendix B. Proof of Theorem 5

The proof of Theorem 5 is similar to that of Theorem 1 except that we eliminated the first-order spatial derivative from the Taylor expansion

\[ \max_a \mathbb{E}_{a,p} [\hat{w}(x + r) - \hat{w}(x)] = \max_a \mathbb{E}_{a,p} \left[ \frac{1}{2} \langle D^2 \hat{u}(x)q, q \rangle - \int_0^1 D^3 \hat{u}(x - \mu q, x) \frac{(1 - \mu)^2}{2} d\mu \right] \]

\[ \leq \frac{\delta}{1 - \delta} (\hat{w}(x) - \max_i x_i) + K \]

since \( p = \nabla \hat{w}(x) \), and therefore \( \nabla \hat{w}(x) \cdot r = \nabla \hat{w}(x) \cdot (\sum_i \partial_i \hat{w}(x)q_i) \mathbb{1} - q) = 0 \) for all \( q \).

Appendix C. Fixed horizon exponential potential

Note that

\[ \partial_{ij} w(x, t) = \begin{cases} \psi''(y) \phi'(x_i) \phi'(x_j) & \text{if } i \neq j \\ \psi''(y) \phi(x_i)^2 + \psi'(y) \phi''(x_i) & \text{if } i = j \end{cases} \]

where

\[ y = \sum_{k=1}^N \phi(x_k) \]

\[ \psi(y) = \frac{1}{\eta} \log(y), \quad \psi'(y) = \frac{1}{\eta y}, \quad \psi''(y) = -\frac{1}{\eta y^2} \]

\[ \phi(x_k) = \exp(\eta x_k), \quad \phi'(x_k) = \eta \exp(\eta x_k) \quad \text{and} \quad \phi''(x_k) = \eta^2 \exp(\eta x_k) \]

and observe that \( D^2 w(x + a \mathbb{1}) = D^2 w(x) \), and \( \sum_j \partial_j w = 0 \) (and thus, \( D^2 w \cdot \mathbb{1} = 0 \)) by linearity of \( w \) in the direction of \( \mathbb{1} \). Using these result and

\[ \langle D^2 w \cdot q, q \rangle = -\eta \left( \sum_{k=1}^N \exp(\eta x_k) \right)^{-2} \sum_{i,j} e^{\eta x_i} e^{\eta x_j} q_i q_j + \eta = \eta - \eta \langle p^c, q \rangle^2 \leq \eta \]

we obtain

\[ \langle D^2 w(x, t) \cdot r, r \rangle = \langle D^2 w(x, t) \cdot q, q \rangle \leq \eta \]

for \( r = \langle p, q \rangle \mathbb{1} + q \) and all \( x, t, p \) and \( q \). Therefore, \( w_t + \frac{(1 - \delta)}{25} \max_q \langle D^2 w \cdot q, q \rangle \leq 0 \).

Appendix D. Final time step (screened Poisson potential)

Since \(- \max_i (x - y)_i \geq - \max_i x_i + \min_i y_i \), we have

\[ u(x, 0) - u(x, -\delta) = \alpha \int e^{-\frac{||y||^2}{2\sigma^2}} \max_i (x_{-1})_i - \max_i ((-1)_i - y_i) dy \]

\[ \geq \alpha \int e^{-\frac{||y||^2}{2\sigma^2}} \min_i y_i dy = -\sigma \max_i G_i \]

where \( \sigma = \sqrt{-2\kappa \delta} \).

Similarly, since \(- \max_i (x - y)_i \leq - \max_i x_i + \max_i y_i \), we obtain

\[ u(x, 0) - u(x, -\delta) \leq \sqrt{-2\kappa \delta} \max_i G_i \]
Appendix E. Final time step (max potential)

We have for any \( x \)

\[
    u(x, 0) - u(x, -\delta) = x(1) - \frac{1}{N} \sum_{l=1}^{N} x(l) - \sqrt{2} \kappa \delta \sum_{l=1}^{N-1} c_l f(z_l) = \sqrt{2} \kappa \delta \sum_{l=1}^{N-1} c_l z_l - \sqrt{2} \kappa \delta \sum_{l=1}^{N-1} c_l f(z_l)
\]

Since \(-\sqrt{2}\pi \leq z - f(z) \leq 0\) for \( z \geq 0 \),

\[
    -2\sqrt{\frac{\kappa \delta}{\pi}} \frac{N-1}{N} \leq u(x, 0) - u(x, -\delta) \leq 0
\]

Appendix F. Relationship to the known exact minmax value functions

F.1. Two experts

Since \(1 - O(\sqrt{\delta}) \leq e^\delta \text{erfc}(\sqrt{\delta}) \leq 1\), for the screened Poisson-based potentials, we obtain:

\[
    \hat{u}^s(0) = \sqrt{2} \kappa_h \max_{G_i} \sqrt{\frac{\pi}{2}} \left(1 - O(\sqrt{\delta})\right) = \sqrt{\kappa_h E_G} \max_{G_i} \left(\frac{\sqrt{\pi}(1 - \delta)}{\sqrt{1 - (1 - \delta)^2}} - O(1)\right)
\]

\[
    \hat{w}^s(0) \leq \sqrt{\frac{1 - \delta}{\delta}} \max_{G_i} \sqrt{\frac{\pi}{2}} = \max_{G_i} \left(\frac{\sqrt{\pi}(1 - \delta)}{\sqrt{1 - (1 - \delta)^2}}\right)
\]

where

\[
    \kappa_h = \begin{cases} 
    \frac{1}{2} + \frac{1}{2N} & \text{if } N \text{ is odd} \\
    \frac{1}{2} + \frac{1}{2N-2} & \text{otherwise}
\end{cases}
\]

Thus, for \( N = 2 \), \( E_G \max_{G_i} = \frac{1}{\sqrt{\pi}} \), and

\[
    \frac{(1 - \delta)}{\sqrt{1 - (1 - \delta)^2}} - O(1) \leq \min_p R(a^s, p)
\]

and

\[
    \max_a R(a, p^s) \leq \frac{(1 - \delta)}{\sqrt{1 - (1 - \delta)^2}} + O\left(\log \frac{1}{\delta}\right)
\]

Therefore, to the leading order, the lower and upper bounds match the optimal regret \( \frac{1 - \delta}{\sqrt{1 - (1 - \delta)^2}} \rightarrow \frac{1}{\sqrt{2}\delta} \) as \( \delta \rightarrow 0 \), as determined in Theorem 4.1 of Gravin et al. (2016) (as rescaled for our loss function).
F.2. Three experts

For $N = 3$, the bounds on the value functions $\hat{v}_{am}$ and $\hat{v}_{pm}$ lead to the following bounds on the regret

$$\frac{2}{3} \sqrt{\frac{2(1 - \delta)}{\delta}} \left( 1 - O(\sqrt{\delta}) \right) - O \left( \log \frac{1}{\delta} \right) = \frac{4}{3} \sqrt{\frac{1 - \delta}{1 - (1 - \delta)^2}} - O \left( \log \frac{1}{\delta} + 1 \right) \leq \min_p R(a^m, p)$$

and

$$\max_a R(a, p^m) \leq \frac{2}{3} \sqrt{2} \left( 1 + \frac{\sqrt{\delta}}{\pi} \right) + O \left( \log \frac{1}{\delta} \right) = \frac{4}{3} \sqrt{\frac{1 - \delta}{1 - (1 - \delta)^2}} + O \left( \log \frac{1}{\delta} + 1 \right)$$

Therefore, to the leading order, the lower and upper bounds match the optimal regret $\frac{4}{3} \sqrt{\frac{1 - \delta}{(1 - \delta)^2}} \to \frac{4}{3} \frac{1}{\sqrt{2\pi}}$ as $\delta \to 0$, as determined in Theorem 4.2 of Gravin et al. (2016) (again, as rescaled for our loss function).