LOOKING AT OSCULATING BUNDLES THROUGH THE SEMIHOLONOMY EQUALIZATIONS

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The paper glosses different forms of an introducing of higher order tangent-like functors, especially functors derived from higher order nonholonomic tangent functors. A special attention is devoted to higher order osculating bundles: their identification with higher order tangent bundles is demonstrated as the main result. Chiefly, the paper is focused on the needful unification of concepts.

1. MOTIONS AND VELOCITIES

For clearness, we recall some basic concepts in a little bit of mechanics-like language. Local differentiable maps between manifolds are differentiable maps defined on open subsets of a source manifold. Further, our source manifold are always reals \( \mathbb{R} \) (with a variable \( t \)) and our open subsets are (without loss of a generality) open real intervals. Real intervals in question will be denoted by \( I_1, I_2, \ldots \). If we take local maps from a real interval into a smooth manifold \( M \), we talk about local motions on \( M \). Especially, we deal with local motions belonging to the same germ at \( t_0 \in \mathbb{R} \), i.e. for \( f: I_1 \rightarrow M, \ t_0 \in I_1, \ g: I_2 \rightarrow M, \ t_0 \in I_2 \), the equality \( f(t_0) = g(t_0) = p \in M \) is satisfied.

Moreover, if \( Y \) is a manifold fibered over \( \mathbb{R} \), we can restrict, if it is needed, only to local sections from an open real interval to \( Y \). In such a case, local motions are called local time-transferring motions.

Two local motions \( f \) and \( g \) belonging to the same germ at \( t_0 \in \mathbb{R} \) can also belong to the same 1-jet (\( r \)-jet, respectively) at \( u \in \mathbb{R} \). We write \( j_{t_0}^1 f = j_{t_0}^1 g \) \( (j_{t_0}^r f = j_{t_0}^r g) \) and the space of all 1-jets (\( r \)-jets) at \( t_0 \in \mathbb{R} \) is denoted by \( J_{t_0}^1(\mathbb{R}, M) \) \( (J_{t_0}^r(\mathbb{R}, M)) \). For \( t_0 = 0 \in \mathbb{R} \), we talk about the same velocity (\( r \)-velocity) in \( p = f(0) = g(0) \) and \( J_0^1(\mathbb{R}, M) = TM \) yields the tangent bundle \( J_0^r(\mathbb{R}, M) = T^rM \) yields the \( r \)-th order tangent bundle). Nevertheless, we see that the choice \( t_0 = 0 \) (the choice of "true" zero) has only a formal character.

It is known that \( TM \) is also a smooth manifold. Thus, first, let us focus on velocities on the tangent bundle \( TM \). Surely, we can again take two local motions \( F, G \) on \( TM \): the same velocity is considered in \( F(0) = G(0) = (p, v) \in TM \). Forming the 1-jet, we see that we have no correspondence with the "installation" of \( v \) now. Let us consider local coordinates \( x^i : U \rightarrow \mathbb{R}^m, \ i = 1, \ldots, m, \ U \subseteq M, \)

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U \ni p, m = \dim M. Then we have the following expression in local coordinates:

\[ \mathbb{R} \ni f \mapsto p \in M \]
\[ p \xrightarrow{(x^i)} (x^i(p)) = (x^i(f(0))) \in \mathbb{R}^m \]
\[ \mathbb{R}^m \ni \left( \frac{\partial x^i}{\partial t}(0) \right) = \left( y^i(v) \right) \xrightarrow{(y^i)} v; \]

\( y^i \) are induced local coordinates; all \((p, v)\) form \( TM \);

\[ \mathbb{R} \ni I_2 \ni 0 \xrightarrow{F} (p, v) \in TM \]
\[ (p, v) \xrightarrow{(x^i, y^i)} ((x^i, y^i)(p, v)) = ((x^i, y^i)(F(0))) \in \mathbb{R}^{2m} \]
\[ \mathbb{R}^{2m} \ni \left( \left( \frac{\partial x^i}{\partial t}, \frac{\partial y^i}{\partial t} \right)(0) \right) = \left( (X^i, Y^i)(P, V) \right) \xrightarrow{(X^i, Y^i)} (P, V); \]

\( (X^i, Y^i) \) are induced local coordinates; all \((p, v, P, V)\) form \( TTM \).

So, \( TTM = J^1 \left( \mathbb{R}, J^1 \left( \mathbb{R}, M \right) \right) \).

Second, the non-holonomic second order tangent bundle is constructed by the following way. The space of all 1-jets of local motions at all possible real points is denoted by \( J^1(\mathbb{R}, M) \) (i.e. \( J^1(\mathbb{R}, M) = \bigcup_{t \in \mathbb{R}} J^1_t(\mathbb{R}, M) \)). As \( J^1(\mathbb{R}, M) \) is a manifold fibered over \( \mathbb{R} \), we can take local time-transferring motions now, and, in particular, 1-jets of these time-transferring motions at 0.

\[ \mathbb{R} \ni I_2 \ni t_0 \xrightarrow{F} (0, p, v) \in J^1(\mathbb{R}, M) \]
\[ (0, p, v) \xrightarrow{(\text{id}_{\mathbb{R}}, x^i, y^i)} ((\text{id}_{\mathbb{R}}, x^i, y^i)(0, p, v)) = ((\text{id}_{\mathbb{R}}, x^i, y^i)(\sigma(0))) \in \{0\} \times \mathbb{R}^{2m} \cong \mathbb{R}^{2m} \]
\[ \mathbb{R}^{2m} \ni \{1\} \times \mathbb{R}^{2m} \ni \left( \left( \frac{\partial \text{id}_{\mathbb{R}}}{\partial t}, \frac{\partial x^i}{\partial t}, \frac{\partial y^i}{\partial t} \right)(0) \right) = \left( (1, X^i, Y^i)(P, V) \right) \xrightarrow{(1, X^i, Y^i)} (P, V); \]

\( (X^i, Y^i) \) are induced local coordinates; all \((p, v, P, V)\) form \( \tilde{T}^2 M \).

So, we find that there is a slight difference between the second iterated tangent bundle and the non-holonomic second order tangent bundle. Of course, this difference survives in general order. However, iterated tangent functor and non-holonomic tangent functor are naturally equivalent. The construction of the natural equivalence (in a general form) is noticed by Ivan Kolář in [2]; we refer also to the
paper [3] of Ivan Kolář and Raffaele Vitolo for more detailed description on pp. 4–5. We have mentioned only coordinate expressions grounds of the identification here. In the paper, we follow routine to identify iterated tangent functor and nonholonomic tangent functor, however, we prefer the iterated tangent functor in our notation.

2. PROJECTIONS IN ITERATED TANGENT BUNDLES

From here, we will denote the iterated tangent functor \( T_r \) by \( \tilde{T}^r \). This is a new notation. We believe that iterated tangent bundles deserves a special symbol. Intentionally, we will do the subsequent consideration for this bundle (and not for \( \tilde{T}^r M \)) as is perhaps more frequent and easily understandable.

Projections in \( \tilde{T}^r \) are well-known for number of decades, there were undoubtedly described already in 70’s of last century, cf. e.g. [3] and references herein. We introduce the following notation of projections in the iterated tangent bundle \( \tilde{T}^r M \). (The notation was used by author in the paper [4] and then, more precisely and with some basic properties of these projections, in [6].) So, for every \( s, 0 < s \leq r \), we denote by

\[
\pi_s : \tilde{T}^s M \to M
\]

the canonical projection to the base. Further, we denote

\[
\pi^s_b : = \pi^s_{\tilde{T}^s M} : \tilde{T}^s (\tilde{T}^b) \to \tilde{T}^b M
\]

projection with \( \tilde{T}^s M \) as the base space,

\[
\sigma^s : = \tilde{T}^a \pi^s : \tilde{T}^s (\tilde{T}^a M) \to \tilde{T}^a M
\]

induced projection originating by the posterior application of the functor \( \tilde{T}^a \), and

\[
\sigma^s_b : = \tilde{T}^a \pi^s_{\tilde{T}^a M}
\]

the general case containing applications of both previous cases. If \( a \) or \( b \) equal zero, we do not write them.

Example 1. For local coordinates \( (x^i) \) on \( M \), we obtain induced local coordinates \( (x^i, y^i) \) on \( \tilde{T}^1 M \), \( (x^i, y^i, X^i, Y^i) \) on \( \tilde{T}^2 M \) and \( (x^i, y^i, X^i, Y^i, \xi^i, \eta^i, \Xi^i, H^i) \) on \( \tilde{T}^3 M \). We have projections

\[
\pi^3 : \tilde{T}^3 M \to M, \quad \pi^3(x^i, y^i, X^i, Y^i, \xi^i, \eta^i, \Xi^i, H^i) = (x^i),
\]

\[
\pi^2 : \tilde{T}^3 M \to \tilde{T}^1 M, \quad \pi^2(x^i, y^i, X^i, Y^i, \xi^i, \eta^i, \Xi^i, H^i) = (x^i, y^i),
\]

\[
\pi^1 : \tilde{T}^3 M \to \tilde{T}^1 M, \quad \pi^1(x^i, y^i, X^i, Y^i, \xi^i, \eta^i, \Xi^i, H^i) = (x^i, \xi^i)
\]

The notation was used by author in the paper [4] and then, more precisely and with some basic properties of these projections, in [6].
and
\[ \pi_2^1: \tilde{T}^3M \to \tilde{T}^2M, \quad \pi_1^2(x^i, y^i, X^i, Y^i, \xi^i, \eta^i, H^i) = (x^i, y^i, X^i, Y^i), \]
\[ \pi_1^1: \tilde{T}^3M \to \tilde{T}^2M, \quad \pi_1^1(x^i, y^i, X^i, Y^i, \xi^i, \eta^i, H^i) = (x^i, y^i, \xi^i, \eta^i), \]
\[ \pi_1^1: \tilde{T}^3M \to \tilde{T}^2M, \quad \pi_1^1(x^i, y^i, X^i, Y^i, \xi^i, \eta^i, H^i) = (x^i, X^i, \xi^i, H^i). \]

We can obtain further projections by compositions, e.g. we have the projection \( \pi_1 \circ \pi_1^1 = \pi_2 \circ \pi_2^1 \) here, which is not of the type \( \pi_1 \). For details, see [5].

Remark 1. In the paper of Maïdo Rahula, Petr Vašik and Nicoleta Voicu [9] is a completely different notation. The projection \( \pi_s \) there is our \( \pi_{s-1} \) and the projection \( \rho_s \) there is our \( \rho_{s-p} \). Our notation is more general.

Remark 2. Elena Pavlíková has used in [8] still another notation. Her \( j_r \) corresponds with our \( \pi_{r-s} \) and her \( p_l \) corresponds with our \( p_{r-s-p} \). Projections \( p_l \) are called (for \( p \geq 1 \)) lateral projections and some properties of these projections are derived in the cited paper.

3. THE SEMIHOLONOMY CONDITION AND THE OSCULATING BUNDLES

Let \( Z \in \tilde{T}^rM \). We say that \( Z \) is prominent, if the condition
\[ \pi_{r-1}^1(Z) = q \pi_{r-q}^1(Z) \]
for all \( q, q = 1, \ldots, r \) is satisfied. (It is clear that a condition \( \rho_1 = \cdots = \rho_r \) studied in [9] is exactly the same.) This condition borrowed from the theory of nonholonomic jets is called the semiholonomy condition.

The name "prominent" is only a working one. The prominent elements of \( \tilde{T}^rM \) form a smooth manifold with fiber bundle structure over \( M \). The obtained fibered manifold is called (see [9]) the osculating bundle of manifold \( M \) and denoted by \( \text{Osc}_{r-1} M \).

As we have seen before, the iterated tangent functor is different from the nonholonomic tangent functor, but there is a natural equivalence between them. Using this equivalence, we can identify the bundle of prominent elements as the bundle of semiholonomic 1-dimensional velocities of the order \( r \). Nevertheless, semiholonomic 1-dimensional velocities of the order \( r \) are nothing but holonomic velocities of the order \( r \). Hence the osculating bundle functor studied in [9] is nothing but the higher order tangent functor.

Thus, our result is the following.

Theorem 1. The osculating bundle functor \( \text{Osc}_{r-1} \) works on the category of manifolds as objects and smooth maps as morphisms and it is naturally equivalent to the higher order tangent functor \( T^{r-1} \).
To the completion of the proof. Almost everything concerning the proof of the theorem was already demonstrated above. We complete the proof by two comments. First, every semiholonomic jet from 1-dimensional source manifold is automatically the holonomic one. Especially, this is evident from local coordinate expressions: in general theory it was derived that nonholonomic jets which are identified after the equalizations of projections differ only by displacements of zeros in subscripts, but no symmetrisation does not come yet (in general). Nevertheless, for a 1-dimensional source manifold an additional step of a symmetrisation is unreasoning because subscripts can have only one value. Second, the turn signal for the observations in question can be a noticing dimensions of fibered manifolds; we recall that for \( r \geq 2 \) is \( \dim T^{r-1}M = rm \) (with the fiber dimension \( (r-1)m \)) and cf. [9] again.

4. FINAL REMARKS

Thus, we hope it is clearer now, why, for instance, Wolfgang Bertram has remarked in his monographical work [1] (see Introduction, page 11) that "an osculating bundle of a vector bundle introduced by F. W. Pohl" represents, "with great technical effort" . . . a construction of "a linear bundle which corresponds to \( T^kF \)."

Similarly, in projective geometry, the \( k \)-th osculating space is considered as the span of \( \gamma(0), \gamma'(0), \gamma''(0), \ldots, \gamma^{(k)}(0) \) for a smooth parameterized curve \( \gamma(t) \) (cf. the monograph of V. Ovsienko and S. Tabachnikov, [7]), which suggests exactly the same approach.

It follows that our result can be more or less known, maybe intuitively. We believe that this paper can be viewed as a contribution to the needful unification of different concepts.

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