GROUP REPRESENTATIONS AND THE EULER CHARACTERISTIC OF ELLIPTICALLY FIBERED CALABI–YAU THREEFOLDS

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Abstract. To every elliptic Calabi–Yau threefold with a section $X$ there can be associated a Lie group $G$ and a representation $\rho$ of that group. The group is determined from the Weierstrass model, which has singularities that are generically rational double points; these double points lead to local factors of $G$ which are either the corresponding A-D-E groups or some associated non-simply laced groups. The representation $\rho$ is a sum of representations coming from the local factors of $G$, and of other representations which can be associated to the points at which the singularities are worse than generic.

This construction first arose in physics, and the requirement of anomaly cancellation in the associated physical theory makes some surprising predictions about the connection between $X$ and $\rho$. In particular, an explicit formula (in terms of $\rho$) for the Euler characteristic of $X$ is predicted. We give a purely mathematical proof of that formula in this paper, introducing along the way a new invariant of elliptic Calabi–Yau threefolds. We also verify the other geometric predictions which are consequences of anomaly cancellation, under some (mild) hypotheses about the types of singularities which occur.

As a byproduct we also discover a novel relation between the Coxeter number and the rank in the case of the simply laced groups in the “exceptional series” studied by Deligne.

It was noted by Du Val \cite{DuVal} that certain surface singularities, now known as rational double points, are classified by the Dynkin diagrams of the simply laced Lie groups of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$. Du Val pointed out that the Dynkin diagram is the dual diagram to the intersection configuration of the exceptional divisors in the minimal resolution of the singularities. Further connections between these singularities and Lie groups were subsequently discovered by Brieskorn and Grothendieck \cite{BrieskornGrothendieck}.

The resolutions of rational double points are crepant, that is, the pullback of the canonical divisor on the singular variety is the canonical divisor on the smooth

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1More precisely, Du Val recognized the combinatorial structure as occurring in the theory of finite reflection groups; the connection to Lie groups was made soon thereafter by Coxeter \cite{Coxeter} \cit{Coxeter2}.
minimal resolution. In particular, if the singular variety has trivial canonical class, so does its desingularization.

One characterization of rational double points is as quotients of $\mathbb{C}^2$ by finite subgroups of $SL(2, \mathbb{C})$ [12]. Much recent work has been done by looking at the quotient of $\mathbb{C}^3$ by a finite subgroup of $SL(3, \mathbb{C})$ (see for example [23, 19, 30]). In this paper we consider another natural generalization of the above set up.

It turns out that the singularities of the Weierstrass model of an elliptic surface are also rational double points. If the singular surface satisfies the Calabi–Yau condition, so does its resolution; the Calabi–Yau condition can be expressed as a condition on the base of the elliptic fibration and the discriminant locus. Furthermore the ranks of the A-D-E groups contribute to the rank of the Picard group of the minimal resolution of the Weierstrass model, as well as its topological Euler characteristic.

In this paper we investigate a similar situation one dimension higher, namely, elliptic Calabi–Yau threefolds which are resolutions of Weierstrass models. Here the singularities are only generically rational double points, yet it is possible to associate a group $G$ to the singularities, obtaining all the Dynkin diagrams (including the non-simply laced ones). We restrict our attention to Weierstrass models with a minimal resolution which is a flat elliptic fibration satisfying the Calabi–Yau condition—the existence of a flat resolution excludes some non-generic singularities.

The threefold also determines a specific representation of $G$ (known in the physics literature as the “matter representation”) whose irreducible summands can be described in terms of the degenerations of the general singularity; conversely, once one chooses the representations which might occur, the geometry of the Calabi–Yau is completely determined by some relations in representation theory.

As a byproduct of our analysis, we also discover a novel relation between the Coxeter number and the rank in the case of the simply laced groups in the “exceptional series” studied by Deligne [2].

This work was first motivated by the problem of verifying in this context the vanishing of an “anomaly” coming from string theory (see Section 3) and by completing a dictionary between the geometry of the Calabi–Yau and the corresponding quantities in quantum field theory. We can in fact interpret the vanishing of the anomaly as a formula for the Euler characteristic of the Calabi–Yau manifold, a formula which was quite unexpected.

We first formally define an invariant $\mathcal{R}$ (see Section 2) and show how certain representations of $G$ appear in $\mathcal{R}$ when the “general” double point degenerates to a worse singularity. We will then show how the geometry of the Calabi–Yau and its degenerations are naturally, yet surprisingly, related to the same representations occurring in $\mathcal{R}$ (see Section 9). From the string theory point of view this is explained by considering a quantum field theory associated to $X$, which suffers
from potential gauge and gravitational anomalies. Some of these anomalies can be
"cancelled" by an analogue of the Green–Schwarz mechanism, while others (which
occur as certain coefficients in a formal expression in the curvature) are required
to vanish identically. The vanishing of the latter leads to the formula for \( R \), while
the existence of the Green–Schwarz mechanism imposes the other geometric con-
straints. Note that our arguments and definitions, while inspired by the physics of
string theory, are in the realm of mathematics only; the explicit dictionary between
mathematical and field-theoretic quantities is developed in [16].

In Section 1 we discuss how we can associate a group \( G \) to an elliptic threefold
which is a resolution of a Weierstrass model; in Section 2 we introduce the invariants
\( R \) and \( H_{ch} \) and we present some first properties of \( R \). In the following Section 3
we sketch some background from physics. Section 4 shows that the results of our
Main Theorem 8.2 agree with some predictions from the physics literature.

After stating the working assumptions and some notation in Section 5, we de-
scribe an algorithm to compute the fundamental invariant \( R \) from the singularities
of the Weierstrass model (Section 6). We show that the group \( G \) determines most
of the terms occurring in \( R \) and we present these in Appendix I.

The other terms come from the degeneration of the "generic" rational double
point singularity of the Weierstrass model to a worse singularity (Sections 8.1,
8.2, 8.3). We show how such degenerations are naturally associated to certain
representations of the group \( G \) (Sections 8.2 and 8.3).

Conversely, in Section 9 we show how the assigned representations and represen-
tation-theoretic facts determine the geometry of the Calabi–Yau and the degen-
erations which can occur. We then derive the formula for \( R \), in terms of \( G \) (which
is associated to the "generic" singularity) and its representations (associated to the
"non-generic" singularities).

Only a limited set of representations occur, and only certain of the "non-generic
singularities": many others are in fact excluded by the assumption that \( \pi : X \to B \)
is a (flat) elliptic Calabi–Yau fibration.

The computation of \( R \) is slightly different in the case of the simply laced ex-
ceptional groups (including those from Deligne’s “exceptional series”): here we
obtain a novel relation between the Coxeter numbers and the rank of these groups
(Section 7.1).

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1. The group \( G \)

Definition 1.1. An elliptic Calabi–Yau threefold with section is a proper, flat
map \( \pi : X \to B \) from a nonsingular projective complex threefold \( X \) with trivial
canonical bundle to a nonsingular surface $B$, whose general fiber is an elliptic curve, and which admits a section $\sigma : B \to X$. (During certain parts of our discussion, we shall also assume that the rank of the Mordell–Weil group $\text{MW}(X/B)$ of the elliptic fibration is zero.)

Any such $X$ is a resolution of a possibly singular, Weierstrass model $\pi : W \to B$ [28, 15]. $W$ can be described (locally) by a “Weierstrass equation”

$$y^2 = x^3 + fx + g,$$

where $f$ and $g$ are sections of line bundles on the base $B$.

**Lemma 1.2.** [4, 2] In this set up we can naturally associate a reductive Lie group $G$ to the fibration as follows. Let $[E]^{\perp}$ be the orthogonal complement within $H_4(X)$ of the elliptic fiber $E$, and let $\Lambda$ be the cokernel of the natural map

$$\pi^{-1} : H_2(B) \to [E]^{\perp}.$$

Then $\Lambda$ serves as the coroot lattice of $G$, and $\Lambda \otimes U(1)$ serves as the Cartan subgroup. Moreover, to each component of the discriminant locus is associated a local factor of the group, determined by the generic Kodaira fiber along that component and by the monodromy, as follows:

| generic Kodaira fiber | $I_n$ | $I_n$ | $II$ | $III$ | $IV$ | $IV$ |
|-----------------------|-------|-------|------|-------|------|------|
| monodromy             | $\mathbb{Z}_2$ | $\{e\}$ | $\{e\}$ | $\{e\}$ | $\mathbb{Z}_2$ | $\{e\}$ |
| local group factor    | $\text{Sp}(\frac{n}{2})$ | $\text{SU}(n)$ | $\{e\}$ | $\text{SU}(2)$ | $\text{Sp}(1)$ | $\text{SU}(3)$ |

|                | $I_n^*$ | $I_n^*$ | $IV^*$ | $IV^*$ | $III^*$ | $II^*$ |
|----------------|--------|--------|-------|-------|--------|-------|
| $\mathbb{Z}_3$ or $S_3$ | $\mathbb{Z}_2$ | $\{e\}$ | $\mathbb{Z}_2$ | $\{e\}$ | $\{e\}$ | $\{e\}$ |
| $G_2$          | $\text{SO}(2n+7)$ | $\text{SO}(2n+8)$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |

**Proof.** To specify a connected reductive group, it is enough to specify a compact torus together with the collection of characters of that torus which will serve as the weights for the semisimple part of the group. The torus in turn can be described as $\Lambda \otimes U(1)$ for some lattice $\Lambda$, which is the form used in the statement of the lemma. (This choice of torus is dictated by physical considerations.)

To complete the specification, the weight spaces must be given. The Weierstrass model is singular along a (reducible) curve $C$; the general singularity over each irreducible component of $C$ is a rational double point [24]. Let us consider the intersection configuration of the exceptional curves and the exceptional divisors on $X$. In most cases, the intersection matrix is (up to a sign) the unique Cartan matrix of a Lie algebra $\mathfrak{g}$; here we also find the non-simply laced algebras, as the exceptional curves might undergo a monodromy transformation as they move in the exceptional divisors along the curve $C$. In some cases a more delicate argument is needed [2].

$\square$
Note that the group is semisimple precisely when $MW(X/B)$ has rank 0.

**Corollary 1.3.** Let $G$ be a non-simply laced group (a local factor of the entire gauge group) associated to the singularities over a curve $C$ in the discriminant, as in the above proof. Then the exceptional curves in one homology class are parameterized by a curve $C'$, a finite branched cover of $C$. The cover is of degree 2 unless $G$ is locally isomorphic to $G_2$; in the latter case, the degree of the cover is 3.

**Definition 1.4.** With the notation of the above corollary, we write $g(C') = g'$.

**Remark 1.5.** If $B$ is ruled, from $X$ and the group $G$ we can construct a K3 surface $S$ with a gauge bundle $H$, the “heterotic dual” of $X$. Many of the physics predictions stated in this paper were originally derived by analyzing this duality.

**Definition 1.6.** Let $\Sigma \subset B$ be the ramification locus of $\pi$. $\Sigma$ is a divisor. We write

\[ \Sigma = \Sigma_0 \cup_{i \geq 1} \Sigma_i, \]

where $\Sigma_0$ is the (possibly reducible) component over which the “general” singular fiber is a node (Kodaira type $I_1$), and each $\Sigma_i$ is an irreducible component of $\Sigma \setminus \Sigma_0$.

**Remark 1.7.** Since $X \to B$ is an elliptic Calabi–Yau, then:

\[ O_X \sim K_X = \pi^*(K_B + \frac{1}{12} \Sigma), \text{ and } \Sigma \in |-12K_B|. \]

The choice of the notation and of the indices in Definition 1.6 is motivated by the gauge group associated to the resolution of the general singular point of $\Sigma_i$: we denote in fact by $G_i$ this group and because $G_0$ is trivial, the relevant groups are $G_i, i \geq 1$.

2. A FIRST LOOK AT $R$: WHEN $X = W$.

Let us define the fundamental invariant:

**Definition 2.1.** $R = \frac{1}{2} \chi_{\text{top}}(X) + 30K_B^2$.

The following holds:

**Theorem 2.2.** Under the hypothesis of Definition 1.1, suppose that in addition $X = W$ is a smooth Weierstrass model. Then $R = 0$.

**Proof.** Following the algorithm provided in [34] one can in fact show that

\[ \chi_{\text{top}}(X) = -60K_B^2. \]

This statement is also buried in the proof of [28, (3.10)]; even though the author claims it only for $B = \mathbb{P}^2, F_1, \mathbb{P}^1 \times \mathbb{P}^1$. \qed
We should also point out that $\mathcal{R}=0$ is not a sufficient condition for $W$ to be smooth (see also Section 7). We will give an alternative proof of this theorem in Corollary 6.10; this proof will follow the mathematical ideas that arise from considering the “vanishing of the anomalies” in string theory. The following definition is also motivated by the physics literature:

**Definition 2.3.** $H_{ch} = \mathcal{R} + \dim(G) - \text{rk}(G)$.

Theorem 8.2 describes explicitly $H_{ch}$. We discuss in detail the motivation behind this definition and a dictionary between the geometry and the quantum field theory in [16].

3. A look from string theory:
   **Gauge theory on $X$ and vanishing of the anomalies**

Before studying other properties of the invariant $\mathcal{R}$ defined in the previous section, we look at the gauge theory interpretation of our setup. From this point of view we consider a gauge theory on $X$ (coupled to gravity), in which certain coefficients of the curvature are required to vanish: these are the “anomaly cancellations” whose geometric counterparts are a formula for $\mathcal{R}$ and certain geometric constraints. We will see that a formula for $\mathcal{R}$ is the geometric counterpart of the first anomaly cancellation (Theorem 3.1 and [16]); the second anomaly cancellation and the corresponding geometric constraints will be discussed in Section 9.

3.1. **From elliptic fibrations to gauge theory.** When one of the “type II string theories” is formulated on a ten-manifold of the form $M^{3,1} \times X$ with $X$ a Calabi–Yau threelfold and $M^{3,1}$ a flat spacetime of dimension four, the resulting theory has a low energy approximation which takes the form of a four-dimensional quantum field theory with quite realistic physical properties (depending on certain properties of the Calabi–Yau threelfold).

Elliptic Calabi–Yau threelfolds with a section $\pi : X \to B$ have also been used in a different way in string theory. We can ask what happens to the type IIA theory in the limit when the Calabi–Yau metric on $X$ is varied so that the fibers of the map $\pi$ shrink to zero area. It turns out that the resulting physical theory has a low energy approximation which takes the form of a *six-dimensional* quantum field theory. This limiting theory can also be described more directly, in terms of the periods $\tau(b)$ of the elliptic curves $\pi^{-1}(b)$, regarded as a multi-valued function on $B$. The type IIB string theory is compactified on $B$ with the aid of this function, using what are known as D-branes along the discriminant locus of the map $\pi$. (This latter approach is known as “F-theory.”)

**Fact:** This six-dimensional quantum field theory includes gravity as well as a gauge field theory whose gauge group is the group $G$ defined in Section 1; in order to be a consistent quantum theory, the “anomalies” of this theory must vanish.
3.2. Curvatures, anomaly polynomial and traces. Schwarz shows [33] that these models (N=1 theories in six dimensions with a semisimple group $G$) are constrained by anomaly cancellation. The anomaly is characterized by an eight-form made from curvatures of the Levi–Civita connection and of the gauge connection. This eight-form is naturally defined on an auxiliary eight-manifold $Y$.\footnote{$Y$ is a manifold with boundary, whose boundary is the product of $S^1$ and the original six-dimensional spacetime.}

If we have a manifold $Y$ equipped with a principal $G$-bundle $\mathcal{G}$ (the “gauge bundle”), then the curvature $F$ of the gauge connection is an $\text{ad}(\mathcal{G})$-valued two-form, where each fiber $\text{ad}(\mathcal{G})_x$ of $\text{ad}(\mathcal{G})$ is isomorphic to the Lie algebra $\mathfrak{g}$ of $G$, with $\mathcal{G}_x$ acting on $\text{ad}(\mathcal{G})_x$ via the adjoint action of $G$ on $\mathfrak{g}$. Similarly, if $Y$ is equipped with a (pseudo-)Riemannian metric, then the curvature $R$ of the Levi–Civita connection is a two-form taking values in the endomorphisms of the tangent bundle.

The “anomaly polynomial” is a differential form on $Y$ which involves expressions like $\text{tr} R^k$ and $\text{Tr}_\rho F^k$, where $\rho$ is some representation of the Lie algebra. These expressions are to be interpreted as follows: the representation $\rho$ can be regarded as a homomorphism $\rho : \mathfrak{g} \to \text{End}(V)$ for some (complex) vector space $V$. As an endomorphism of $V$, $\rho(F_x)$ can be raised to the $k\text{th}$ power; the resulting endomorphism $\rho(F_x)^k$ of $V$ has a trace, which we denote as

$$\text{Tr}_\rho F^k = \text{trace}_V \rho(F)^k.$$ 

Although this expression might have depended on the choice of isomorphism to $\mathfrak{g}$, in fact it is invariant under the adjoint action of $G$ on $\mathfrak{g}$ and so is independent of choices.

Similarly, the expressions $\text{tr} R^k$ are evaluated with the help of the “vector” representation of the corresponding orthogonal group.

3.3. Vanishing of the anomalies. The first requirement is the vanishing of the coefficient of a certain curvature term, which imposes restrictions on the choice of the group and its “matter” representations which can occur. In [10] we discuss extensively the geometric realization of this formula, when F-theory is compactified on an elliptic Calabi–Yau threefold:

**Theorem 3.1.** [33, 27, 16] The anomalies are characterized by an eight-form, made from curvatures and gauge field two-forms. One requirement is the vanishing of the coefficient of the curvature term $\text{tr} R^4$, where $R$ is the curvature of the Levi–Civita connection. In our geometric set up this leads to

$$\mathcal{R} = H_{ch} - \text{dim}(G) + \text{rk}(G), \quad (3.1)$$

where $H_{ch}$ involves the dimension of certain representations of the group $G$.\footnote{$Y$ is a manifold with boundary, whose boundary is the product of $S^1$ and the original six-dimensional spacetime.}
The representations which occur in $H_{ch}$ are well defined in terms of quantum field theory; this motivates our Definition 2.3. Theorem 8.2 identifies these representations in our geometric set up. We will analyze the geometric counterpart of the following statement (“The generalized Green–Schwarz mechanism”) in Section 9:

**Theorem 3.2.** [17, 32, 33] Let us assume that $G$ is semisimple, and so in particular that $\text{rk } MW = 0$ and that $G$ is locally isomorphic to $\prod_i G_i$, where $G_i$ are simple groups. If the requirement specified in Theorem 3.1 holds then the remaining terms of the anomaly polynomial (in a suitable normalization [31]) are:

$$\frac{9 - n_T}{8} \left( \text{tr } R^2 \right)^2 + \frac{1}{6} \text{tr } R^2 \sum_i X_i^{(2)} - \frac{2}{3} \sum_i X_i^{(4)} + 4 \sum_{i<j} Y_{ij}$$

where $n_T$ denotes the number of “tensor multiplets” (which coincides with $h^{1,1}(B) - 1$ in our theories), and where

$$X_i^{(n)} = \text{Tr}_{\text{adj}} F_i^n - \sum_\rho n_\rho \text{Tr}_\rho F_i^n$$

$$Y_{ij} = \sum_{\rho, \sigma} n_{\rho\sigma} \text{Tr}_\rho F_i^2 \text{Tr}_\sigma F_j^2.$$

$\text{Tr}_{\text{adj}}$ means the trace in the adjoint representation, $\text{Tr}_\rho$ denotes the trace in the representation $\rho$ of the simple group $G_i$ (see the above subsection), $n_\rho$ is the multiplicity of the representation $\rho$ of $G_i$ in the matter representation, and $n_{i,j}$ is the multiplicity of the representation $(\rho, \sigma)$ of $G_i \times G_j$.

The Green–Schwarz cancellation mechanism (in the generalized form due to Sagnotti [32], see also Sadov [31]) says that the anomalies can be cancelled provided that (3.2) can be written in the form:

$$\left( s \text{ tr } R^2 + \sum t_i \text{ tr } F_i^2 \right) \cdot \left( u \text{ tr } R^2 + \sum v_i \text{ tr } F_i^2 \right),$$

where $s, t_i, u, v_i$ are divisors on the base $B$ (which correspond to “tensor multiplets” in the physical theory), the product is calculated using the intersection pairing on $B$, and $\text{tr } F_i^2$ is evaluated in an appropriate “fundamental” representation of $G_i$.

In the case that $G$ is not semisimple, the anomaly polynomial is also known, but it is much more complicated. In this paper, we will only consider the anomalies associated to the Green–Schwarz mechanism in the semisimple case.

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3In the physics literature one says that there are “$n_\rho$ hypermultiplets in the representation $\rho$. 
4. About $H_{ch}$: A Look from the Physics Literature.

We state some of the physics predictions on $H_{ch}$, based on Schwarz’s analysis; these predictions motivated our geometric definition of $H_{ch}$ (see Definition 2.3). (We have only described a small number of the predictions which appear in the physics literature—others can be found in [33, 1, 11, 22, 18, 6, 10].)

**Case 0:** If $W = X$ is a smooth Weierstrass model, that is $G = \{e\}$ and $\dim(G) - \text{rk}(G) = 0$, then the quantum field theory tells us that $H_{ch} = 0$ and $\mathcal{R} = 0$, as $H_{ch}$ is the (sum of) dimensions of certain irreducible representations of $G$ (see 3.1). This is in agreement with Theorem 2.2.

**Case I:** If $W$ is singular along a single, smooth curve of genus $g$ of $A_{N-1}$ singularities everywhere, we know from Section 1 that $G = \text{SU}(N)$. The authors of [22] show that under these hypothesis

$$H_{ch} = g(\dim(G) - \text{rk}(G)),$$

and also state that the same should hold for any isolated curve. In this case one would have:

$$\mathcal{R} = (g - 1)(\dim(G) - \text{rk}(G)).$$

**Case II:** If the group is non-simply laced (see Section 1) and $W$ is singular along a unique curve $C$ of genus $g$, then some of the exceptional divisors in $X$ mapping to $C$ are ruled surfaces over a curve $C'$ of genus $g'$ (see Corollary 1.3). Assume that there are $B_1$ branch points of the map $C' \to C$, and that all degenerations of the generic singular fiber occur at these branch points. The authors of [2] show that in most such cases

$$H_{ch} = g(\dim(G) - \text{rk}(G)) + (g' - g)\mathcal{R}_0,$$

where $\mathcal{R}_0$ is a constant (which corresponds to the “charged dimension” of a certain representation $\rho_0$ of $G$—see Definition 8.1). In this case one would have:

$$(4.1)\quad \mathcal{R} = (g - 1)(\dim(G) - \text{rk}(G)) + (g' - g)\mathcal{R}_0.$$

In the case of $I_{2k+1}$ with monodromy (yielding gauge group $G = \text{Sp}(k)$), this formula is modified to one which involves $B_1$ as well:

$$(4.2)\quad \mathcal{R} = (g - 1)(\dim(G) - \text{rk}(G)) + (g' - g)\mathcal{R}_0 + \frac{1}{2}B_1(2k).$$

**Case III:** If $W$ is singular along a single, smooth curve of genus $g$, the singularities are generically of type $A_{N-1}$ singularities, but they become of type $A_N$ at $B_2$ isolated points: we know from Section 1 that $G = \text{SU}(N)$. The authors of [4] and [3] show that under these hypothesis

$$H_{ch} = g(\dim(G) - \text{rk}(G)) + B_2 N.\footnote{Note in particular that [3] used anomaly cancellation—as we do—to make and verify predictions about $H_{ch}$.}$$
In this case one would have:

\[ \mathcal{R} = (g - 1)(\dim(G) - \text{rk}(G)) + B_2N. \]

In Section 8 we will prove that all of these predictions hold and give a global explanation for the above formulas; we will also derive the value of \( \mathcal{R}_0 \) (which depends on \( G \)).

5. Working assumptions and (most of the) notation

Our basic strategy for verifying the formula for \( \mathcal{R} \) is as follows. On the one hand, the Euler characteristic of \( X \) can be calculated exploiting the elliptic fibration, studying the various types of singular fibers which can occur, and assigning to each a “contribution” to the Euler characteristic. First, the generic fibers make no contributions. Second, the fibers over the curves \( \Sigma_i \) make contributions which can be accounted for in terms of the genus of \( \Sigma_i \) and of its monodromy cover as well as the type of the Kodaira fiber. This leaves the contributions from intersection points of the \( \Sigma_i \)'s, or from special points along the \( \Sigma_i \)'s at which the fiber becomes worse.

On the other hand, a parallel decomposition can be made of the representation theory. There are specific contributions to \( H_{ch} \) which are associated to the various local factors \( G_i \) of the gauge group, and depend on the genus of \( \Sigma_i \) and of its monodromy cover. If these are subtracted from our formula, what remains is a sum of contributions from the intersection points of the \( \Sigma_i \)'s, or from special points along the \( \Sigma_i \)'s at which the fiber becomes worse.

Thus, once the “generic” singularities have been matched up, the verification can be reduced to a local question—for each type of singular fiber, verify that its contribution to the Euler characteristic is compatible with the assignment of a factor in the representation to the fiber.

We will carry this out under some assumptions about the degenerations. To simplify matters and isolate the core of the problem, we will consider the case of a single non-abelian factor \( G_1 \) in the gauge group. We will also make some simplifying assumptions about which degenerate fibers are allowed. (The cases we consider can be extended to a more general set-up: see Remarks 6.11 and 8.8 as well as [16].) Our specific assumptions are as follows (see Equation (1.2)):

1. The locus of enhanced gauge symmetry is over a unique smooth curve \( \Sigma_1 \).
2. The coefficients in the Weierstrass equation are otherwise general; following [1] we assume that the local equations can be determined by the data in Table I.

**Proposition 5.1.** Under these hypothesis the group \( G \) alone determines the multiplicity \( m \) of \( \Sigma_1 \) in \( \Sigma \) (see [1] Table 2 and the Tables in Appendix I):

\[ \Sigma_0 + m\Sigma_1 = \Sigma \in \mid -12K_B \mid. \]
Equivalently

\[ \Sigma \] is defined by the equation \( s^m \sigma_0 \),

where \( \sigma_0 \) defines \( \Sigma_0 \) and \( \Sigma_1 \) is defined by \( s = 0 \).

**Definition 5.2.** We denote by \( \mu(f) \) and \( \mu(g) \) the multiplicity of \( f \) and \( g \) resp. along \( \Sigma_1 \), and by \( \mu_P(f, g) \) the intersection multiplicity of \( f/s^{\mu(f)} \) and \( g/s^{\mu(g)} \) at a point \( P \in \Sigma_1 \).

**Definition 5.3.** We denote by \( X_{\Sigma_1} \) the singular fiber of Kodaira type over the general point of \( \Sigma_1 \).

We denote by \( \{Q_1, \cdots, Q_C\} \), the singularities of \( \Sigma_0 \) away from \( \Sigma_1 \): these are cusps; \( C \) is then the number of cusps of \( \Sigma_0 \).

We denote by \( X_{\Sigma_0} \) the singular (nodal) fiber over the general point of \( \Sigma_0 \) while \( X_C \) is the singular (cuspidal) fiber over each point \( Q_j \).

If \( \Sigma_0 \) and \( \Sigma_1 \) are disjoint, all the degenerate elliptic fibers are the ones described above; if \( \Sigma_0 \cap \Sigma_1 \neq \emptyset \) there are other degenerate elliptic fibers, not necessarily of Kodaira type, over each intersection point. A complete classification of such degenerations is not available, except in the case of simple normal crossings \([26, 29]\), and the list of possibilities could be quite complicated.

These points (the \( P_\ell^i \) below) are exactly the singularities of \( \Sigma \) along \( \Sigma_1 \); the roots of \( \sigma_0 \) mod \( s \) determine the intersection of \( \Sigma_1 \) and \( \Sigma_0 \).

In particular:

**Proposition 5.4.** Our assumptions imply the following:

- (Equivalently:) \( \Sigma_0 \mod s = 0 \) splits in the product of at most two factors: \( \beta_1^r \cdot \beta_2^r \equiv \sigma_1 \mod s \). Each \( \beta_i \) is irreducible, and together with \( r_i \) is determined by the choice of the group \( G \) (see the Tables in Appendix I).

- \( \beta_i \) is smooth near \( \Sigma_1 \): \( (s, \beta_i) \) are local coordinates around each intersection point and \( r_i \) is the intersection multiplicity of \( \beta_i \) with \( \Sigma_1 \). We write:

\[ \Sigma_0 \cap \Sigma_1 = \{P_1^1, \cdots, P_1^{B_1}, P_2^1, \cdots, P_2^{B_2}\}, \]

where \( B_i \) is the number of the distinct roots of \( \beta_i \); note that if \( r_i = 1 \), the \( P_i^j \)'s are points of simple normal crossings intersections.

- (Equivalently:) \( \Sigma_0 \cdot \Sigma_1 = (-12K_B - m\Sigma_1) \cdot \Sigma_1 = r_1B_1 + r_2B_2 \).

- The degenerate elliptic fiber over each point \( P_\ell^i \) and the local equation around \( P_\ell^i \) does not depend on \( \ell \), but only on \( i = 1, 2 \): without loss of generality we write \( X_{P_\ell} = \pi^{-1}(P_\ell^i) \). We write the local equation in Table 2.

- The intersection multiplicity \( \mu_{P_\ell^i}(f, g) \) does not depend on \( \ell \), but only on \( i = 1, 2 \); we denote it by \( \mu_i(f, g) \).

**Proof.** It follows from \([4]\).
Proposition 5.5. • If $G = \text{SO}(k)$ ($k > 8$), and $m$ is as defined in Proposition 5.1 then $\text{rk}(G) + 2 = m$;

• if $G = e, \text{SU}(2), \text{SU}(3), \text{SO}(8), E_6, E_7, E_8$, and $J$ is regular, then $\text{rk}(G) + 2 = m, \mu_P(f, g) = 0$;

• if $G = \text{SU}(n)$ and $J$ has a pole along $\Sigma_1$, then $\text{rk}(G) + 1 = m, \mu(f), \mu(g) = 0$.

Note that $G = \{e\}$ corresponds here to the Kodaira fiber of type $II$.

Proof. It can be verified by inspection and explicit computations. 

We list the values of $m, r_1$, and $r_2$ in the Tables in Appendix I.

6. Deconstructing $\mathcal{R}$

In this section we set up an algorithm to compute $\mathcal{R}$, the fundamental invariant defined in 2.1.

We break up the contributions to $\chi_{\text{top}}$ as follows:

Lemma 6.1. The following lines add to the topological Euler characteristic of $X$:

\[
\frac{1}{2} \chi_{\text{top}}(\bigcup \pi^{-1}(P_i^\ell)) = \frac{1}{2} \chi_{\text{top}}(X_{P_1}) \cdot B_1 + \frac{1}{2} \chi_{\text{top}}(X_{P_2}) \cdot B_2 \\
\frac{1}{2} \chi_{\text{top}}(\pi^{-1}((\Sigma_1 \setminus \bigcup \ell,i P_i^\ell))) = \chi_{\text{top}}(X_{\Sigma_1}) \cdot [1 - g(\Sigma_1) - \frac{1}{2} B_1 - \frac{1}{2} B_2]
\]

\[
\frac{1}{2} \chi_{\text{top}}(\pi^{-1}((\Sigma_0 \setminus \bigcup Q_j \setminus \bigcup \ell,i P_i^\ell))) = \frac{1}{2} \chi_{\text{top}}(\Sigma_0 \setminus \bigcup Q_j \setminus \bigcup \ell,i P_i^\ell)
\]

\[
\frac{1}{2} \chi_{\text{top}}(\pi^{-1}(Q_j)) = \frac{1}{2} \chi_{\text{top}}(X_C) \cdot C
\]

Proof. We compute the Euler characteristic of $X$ via the structure of elliptic fibration (the Euler characteristic of the general fiber is zero) and Mayer–Vietoris’ sequence.

Now we want to effectively calculate each contribution in the above equations in terms of quantities which depend on the singularities along $C$ and the group $G$.

Note also that the singularities along $C$ are determined by the geometry of the discriminant locus on $B$; this is in turn determined by the intersections of a section of some multiple of $K_B$ with $\Sigma_1$ (see Remark 1.7).

6.1. Deconstructing $\chi_{\text{top}}(\Sigma_0 \setminus \bigcup Q_j \setminus \bigcup \ell,i P_i^\ell)$:

We start with the following definitions:

Definition 6.2. (Defining $\alpha_j$.) If $\phi_1 : B_1 \to B$ is the blow up of a point $P \in D$, with exceptional divisor $E$ and

\[
D_1 = \phi_1^*(D) - \alpha_1(P)E, \text{ its strict transform.}
\]
Corollary 6.3. With the above notation:
\[(K_B + D_1) \cdot D_1 = (K_B + D) \cdot D - \alpha_1(P) \cdot (\alpha_1(P) - 1).\]

In particular, if \(P\) is a smooth point of \(D\), \(\alpha_1(P) = 1\); if \(P\) is a cuspidal point of \(D\), \(\alpha_1(P) = 2\).

Definition 6.4. Let \(\phi_{\nu(i)}^1 \cdots \phi_{\nu}^1 \cdots \phi_1^1\) be the embedded resolution of \(\Sigma_0\) around the point \(P_1^i\) and \(\{\alpha_i^1\} := \{\alpha_v(P_1^i)\}_{v=1}^{n(i)}\) the collection of the integers as in (6.2) (\(v\) depends on \(i\), but we believe the distinction is clear.) Let us define:
\[\epsilon_j = \left[\sum \alpha^j_v(\alpha_v - 1) - \#\phi^{-1}(P_j^k)\right].\]

If \(P_1^i\) is a smooth point, \(\epsilon_1 = -1\); for the cuspidal points \(Q_j\) we have \(\epsilon = 1\).

Corollary 6.5.
\[\chi_{\text{top}}(\Sigma_0 \setminus P_j) = \chi_{\text{top}}(\tilde{\Sigma}_0 \setminus \#\phi^{-1}(P_j)) = -(K_B + \Sigma_0) \cdot \Sigma_0 + \epsilon_1 + \epsilon_2\]

Proposition 6.6. With the above notation we have:
\[\chi_{\text{top}}(\Sigma_0 \setminus \bigcup_j Q_j \setminus \bigcup_{\ell,i} P_1^\ell) + 11 \cdot 12 K_B^2 = mK_B \cdot \Sigma_1 + 2m\Sigma_1 \cdot \Sigma_0 + m^2\Sigma_1^2 + B_1 \epsilon_1 + B_2 \epsilon_2 + C.\]

\(\epsilon_1\) and \(\epsilon_2\) are defined in [6.2]; they are determined by non-generic the singularities along \(C\).

Proof. From the previous corollary we have:
\[\chi_{\text{top}}(\Sigma_0 \setminus \bigcup_j Q_j \setminus \bigcup_{\ell,i} P_1^\ell) = -(K_B + \Sigma_0) \cdot \Sigma_0 + \epsilon_1 B_1 + \epsilon_2 B_2 + 1C.\]

Note that \(\Sigma_0 \in | - 12 K_B - m \Sigma_1|\), which gives:
\[-(K_B + \Sigma_0) \cdot \Sigma_0 = -11 \cdot 12 K_B^2 + mK_B \cdot \Sigma_1 + 2m\Sigma_1 \cdot \Sigma_0 + m^2\Sigma_1^2.\]

Substituting this in the above equation we obtain the statement of the proposition.

\[\square\]

6.2. De constructing \(C\), the number of cusps:

Lemma 6.7. \(f\) and \(g\) then have
\[(-4K_B - \mu(f)\Sigma_1)(-6K_B - \mu(g)\Sigma_1) = 24K_B^2 + \{4\mu(g) + 6\mu(f)\}K_B \cdot \Sigma_1 + \mu(f)\mu(g)\Sigma_1^2\]
intersection points, counted with multiplicity (\(f \in | - 4K_B|, g \in | - 6K_B|\)).

Proposition 6.8. Then number of cusps \(C\), away from \(\Sigma_1\) is:
\[C = 24K_B^2 + \{4\mu(g) + 6\mu(f)\}K_B \cdot \Sigma_1 - \mu_1(f, g)B_1 - \mu_2(f, g)B_2 + \mu(f)\mu(g)\Sigma_1^2,\]
where \(\mu_i(f, g)\) are defined in Propositions 5.3 and 5.4.
\( \mu_i(f, g), \mu(f), \) and \( \mu(g) \) depend on the equation (I.2) and are determined by the (non-generic and generic) singularities along \( C \).

Proof. \( C \) is the number of cusps away from \( \Sigma_1 \); our assumptions in Section 5 imply that the cusps are determined by the common zeroes of the polynomials \( \{f = g = 0\} \) away from \( \Sigma_1 \) (these are ordinary vanishing, see equation (I.1)). \( f \) and \( g \) might also vanish along \( \Sigma_1 \), of orders \( \mu(f) \) and \( \mu(g) \); \( f \mod s \) and \( g \mod s \) might have a common zero along \( \Sigma_1 \). The multiplicities of these latter zeros are measured by \( \mu_i(f, g) \). (See Appendix I.)

**Proposition 6.9.** Using the formulas (6.6) and (6.8) derived above, we re-arrange the contribution to \( \chi_{\text{top}}(X) \) in 6.1 as follows:

\[
\frac{1}{2} \chi_{\text{top}}(\cup_{i, \ell} \pi^{-1}(P_{i, \ell})) = \frac{1}{2} \chi_{\text{top}}(X_{P_1}) \cdot B_1 + \frac{1}{2} \chi_{\text{top}}(X_{P_2}) \cdot B_2 \\
\frac{1}{2} \chi_{\text{top}}(\pi^{-1}(\Sigma_1 \setminus \cup_{\ell, i} P_{i, \ell})) = (g(\Sigma_1) - 1)(-m) - \frac{1}{2} m B_1 - \frac{1}{2} m B_2 \\
\frac{1}{2} \chi_{\text{top}}(\pi^{-1}(\Sigma_0 \setminus \cup_j Q_j \setminus \cup_{\ell, i} P_{i, \ell})) + 54 K_B^2 = \frac{1}{2} m K_B \cdot \Sigma_1 + \frac{1}{2} m^2 \Sigma_1^2 + m \Sigma_1 \cdot \Sigma_0 \\
+ \frac{1}{2} \xi_1 B_1 + \frac{1}{2} \xi_2 B_2 \\
+ \frac{1}{2} [4\mu(g) + 6\mu(f)] K_B \cdot \Sigma_1] - \frac{1}{2} \mu_1(f, g) B_1 \\
- \frac{1}{2} \mu_2(f, g) B_2 + \frac{1}{2} [\mu(f)\mu(g) \Sigma_1^2] \\
\frac{1}{2} \chi_{\text{top}}(\cup_j \pi^{-1}(Q_j)) = -24 K_B^2 = (4\mu(g) + 6\mu(f)) K_B \cdot \Sigma_1 - \mu_1(f, g) B_1 \\
- \mu_2(f, g) B_2 + \mu(f)\mu(g) \Sigma_1^2
\]

The entries in the left hand sides of the above equations add to \( R \); the coefficients on the right hand side are determined by the singularities, generic and non-generic, along \( C \subset W \).

**Corollary 6.10.** If \( X \) is a smooth Weierstrass model, then \( R = 0 \).

One of the aims of this paper is to show that the entries on the right hand side are a collection of dimensions of certain representations of \( G \) (Main Theorem 8.2) which are determined by the singularities, generic and non-generic along \( C \subset W \). In Section 9 we will show that also the converse is true, that is, the assigned representations determine uniquely the geometry of \( W \).

**Remark 6.11.** Note that the formula in Proposition 6.9 admits an immediate generalization to cases in which there are more simple factors in the gauge group (corresponding to additional components of the discriminant). A somewhat more
involved notation is required, to handle possibilities of singular curves $\Sigma_j$ or intersections among several components, but the same geometric principles we used above will lead to a formula of the same general type.

7. A SECOND LOOK AT $\mathcal{R}$: $\Sigma_1 \cdot \Sigma_0 = 0$ ($\Sigma_1$ IS ISOLATED).

We have considered in Section 2 the case $X = W$, we consider now the case when $\Sigma_1$ does not intersect the rest of the discriminant locus: equivalently, $W$ is singular along a single curve $C$ and the singularities are uniform along $C$. This case was also considered in the physics literature, see Section 4.

Here the computations are simpler, and we can see clearly how by using the geometry of the base we can write $\mathcal{R}$ (that is, the equation in (6.9)) as a function of the singular locus and certain representations of the group $G$. The first implication of the hypothesis is that $J : B \dasharrow \mathbb{P}^1$ is well defined around $\Sigma_1$. By analyzing the vanishing of the anomaly we find a curious relation between the Coxeter number and rank in the case of the “exceptional series” of Deligne.

Theorem 7.1. If $\Sigma_1$ does not intersect the other components of the discriminant locus then

$$\mathcal{R} = (\dim(G) - \text{rk}(G))(g - 1).$$

Note that $\dim(G) - \text{rk}(G) = \dim \text{adj}_G - \dim \text{Ker}(\text{adj}_G)$.

Proof. Case I: $J$ is regular along $\Sigma_1$ (simply laced groups in Deligne’s exceptional series.)

In this case $G = \{e\}$, $\text{SU}(2)$, $\text{SU}(3)$, $\text{SO}(8)$, $E_6$, $E_7$, $E_8$, which (except for the trivial case) are precisely the simply laced groups in Deligne’s exceptional series. Here the singular fibers are of types $II, III, IV, I_0^\ast, IV^\ast, III^\ast, II^\ast$; and $m = \chi_{top}(X_{\Sigma_1}) < 12$.

By assumptions, $B_1 = B_2 = 0$ and $\mathcal{R}$ becomes ((6.9)):

$$\frac{1}{2} \chi_{top}(\pi^{-1}(\Sigma_1 \setminus \cup_{\ell,i} P^i_\ell)) = (g(\Sigma_1) - 1)(-m)$$
$$\frac{1}{2} \chi_{top}(\pi^{-1}(\Sigma_0 \setminus \cup_{j} Q_j \setminus \cup_{\ell,i} P^i_\ell)) + 54K_B^2 = \frac{1}{2} mK_B \cdot \Sigma_1 + \frac{1}{2} m^2 \Sigma_1^2$$
$$+ \frac{1}{2} \{4 \mu(g) + 6 \mu(f)\} K_B \cdot \Sigma_1$$
$$+ \frac{1}{2} [\mu(f) \mu(g) \Sigma_1^2]$$
$$\frac{1}{2} \chi_{top}(\cup_{j} \pi^{-1}(Q_j)) - 24K_B^2 = \{4 \mu(g) + 6 \mu(f)\} K_B \cdot \Sigma_1 + \mu(f) \mu(g) \Sigma_1^2$$
Now we use the geometry of the singularities of $W$:
\[ K_B \cdot \Sigma_1 + \Sigma_1^2 = 2g(\Sigma_1) - 2 \]
\[ \Sigma_0 \cdot \Sigma_1 = 0 \leftrightarrow 12K_B \cdot \Sigma_1 = m\Sigma_1^2 \]

By solving the system we have ($m < 12$):
\[ -K_B \cdot \Sigma_1 = \frac{m}{12 - m} 2(g - 1) \]
\[ \Sigma_1^2 = \frac{12}{12 - m} 2(g - 1) \]

By substituting the above equation in the right hand side of $\mathcal{R}$, we see that every term is a multiple of $g - 1$.

Then:
\[ \mathcal{R} = \frac{6}{12 - m} \{-2m+[m-2\mu(g)][m-3\mu(f)]+m^2\}(g-1) = \frac{6}{12 - m} m(m-2)(g-1), \]

in fact, by the definition of $m$, $[m-2\mu(g)][m-3\mu(f)] = 0$ (see Appendix I).

In this case we also have $\text{rk}(G) = m - 2$ (see Proposition 5.5) and thus:
\[ \mathcal{R} = \frac{6(\text{rk}(G) + 2)}{10 - \text{rk}(G)} \text{rk}(G)(g - 1). \]

Now, for these groups
\[ \frac{6(\text{rk}(G) + 2)}{10 - \text{rk}(G)} \text{rk}(G) = h(G) = \text{rk}(G) - \text{dim}(G), \]

where $h(G)$ is the Coxeter number (see the following Lemma 7.2).

This is in agreement with the expectations from physics (see Section 4) together with Corollary 2.3.

Case II: $J$ has a pole along $\Sigma_1$. Since $J : B \rightarrow \mathbb{P}^1$ is well defined around $\Sigma_1$, $J_\infty \cdot \Sigma_1 = K_B \cdot \Sigma_1 = 0$. This together with the assumption $0 = \Sigma_0 \cdot \Sigma_1 = (-12K_B - m\Sigma_1) \cdot \Sigma_1 = -m\Sigma_1^2$, implies $2g - 2 = 0$, that is, $g = 1$.

The substitution in (6.6) gives $\mathcal{R} = 0$. This is again consistent with the expectations in Section 4.

\[ \text{Lemma 7.2.} \quad \text{The Coxeter numbers of the simply laced groups in Deligne’s exceptional series satisfy the relation:} \]
\[ h(G) = \frac{6(\text{rk}(G) + 2)}{10 - \text{rk}(G)}. \]

\[ \text{Proof.} \quad \text{Case by case checking.} \]

This adds to the numerology of the exceptional series presented by Deligne in [9].
8. Another look at $\mathcal{R}$: the Main Theorem.

In the discussion below, we will describe the matter representation as a representation of the Lie algebra $\mathfrak{g}$; it is in fact induced from a representation of the full gauge group $G$ associated to $X$.

**Definition 8.1.** Let $\rho$ be a representation of a Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$. The charged dimension of $\rho$ is $(\dim \rho)_{ch} = \dim(\rho) - \dim(\ker \rho|_\mathfrak{h})$.

For example, if $\rho$ is the adjoint representation then

$$(\dim \text{adj})_{ch} = \dim \mathfrak{g} - \dim \mathfrak{h} = \dim G - \text{rk} G.$$

**Theorem 8.2.** Notation as in Section 5. Then:

$$\mathcal{R} = (g - 1) \dim(\text{adj})_{ch} + (g' - g) \dim(\rho_0)_{ch} + \sum_{P \in \mathcal{A}} \delta_P \dim(\rho_P)_{ch},$$

where $\mathcal{A} = \{ P \in \Sigma_1 \cap \Sigma_0 \text{ such that the fiber over } P \text{ is of Kodaira type } \}$, $g'$ is defined in Section 4, the representations $\rho_P$ all come from a small list of representations given in Table A, and the coefficient $\delta_P$ is $\frac{1}{2}$ if the representation is quaternionic and is 1 if the representation is real or complex. (The quaternionic cases are labeled with $\frac{1}{2}$ in the Table.)

In Table A, we give the Kodaira type of the general hyperplane section through the singular fibers which occur under our hypotheses. For each type of singular fiber, we either list the associated representation $\rho_j$, or (in the case of monodromy) we separate the “non-isolated part” of the representation and call it $\rho_0$, listing any residual representation as $\rho_j$. In addition, in a few cases a representation occurs with multiplicity and (for later convenience at the end of section 9) we identify an irreducible representation $\hat{\rho}$ in the Table.

**Remark 8.3.** Our assumption of a smooth, flat elliptic fibration, imposes restrictions on the type of degenerate singular fibers that might occur:

(i) If $\{ e \}$ is associated to the Kodaira type fiber $II$, there is a double point singularity in the fiber over the simple normal crossings intersection point of the two branches ($\Sigma_0$ and $\Sigma_1$). This is terminal but not canonical, leading to a smooth but *not flat* fibration and a non-minimal Calabi–Yau threefold. We assume then that such points do not occur: the curve is isolated and the theorem holds (see Theorem 7.1).

(ii) If $G = \{ e \}$ (associated to the Kodaira type fiber $I_1$) or $G = Sp(k)$ (associated to the Kodaira type fiber $I_{2k+1}$), the resolution of the generic singularities leaves a double point singularity in the fiber over the simple normal crossings intersection point of the two branches ($\Sigma_0$ and $\Sigma_1$). In fact, if the equation is otherwise generic, then no small resolution exists. We assume here for simplicity that there are no such points.
Table A. The representations which occur under our “generic” hypotheses.

| Type     | $G$       | $P_1$ | $P_2$ | $\rho_0$ | $\rho_1$ | $\rho_2$ | $\hat{\rho}$ |
|----------|-----------|-------|-------|----------|----------|----------|---------------|
| $I_1$    | $\{e\}$  | $II$  | $I_2$ |          |          |          |               |
| $I_2$    | SU(2)     | $III$ | $I_3$ |          |          |          |               |
| $I_3$    | SU(3)     | $IV$  | $I_4$ |          |          |          |               |
| $I_{2k}, k \geq 2$ | Sp($k$) | $I_{2k-4}^* I_{2k+1}$ | $\Lambda_0^2$ |          |          |          | fund          |
| $I_{2k+1}, k \geq 1$ | Sp($k$) | $I_{2k-2}^* I_{2k+2}$ | $\Lambda^2 + 2 \times \text{fund}$ | $\frac{1}{2}\text{fund}$ | NSR | fund          |
| $I_n, n \geq 4$ | SU($n$) | $I_{n-4}^* I_{n+1}$ |          |          |          | $\hat{\Lambda}^2$ | fund          |
| $II$     | $\{e\}$  | $II$  |       |          |          |          |               |
| $III$    | SU(2)     | $IV$  |       |          |          |          | $2 \times \text{fund}$ | fund          |
| $IV$     | Sp(1)     | $I_0^*$ |       | $\Lambda^2 + 2 \times \text{fund}$ | $\frac{1}{2}\text{fund}$ | fund          |
| $IV$     | SU(3)     | $I_0^*$ |       |          |          | $3 \times \text{fund}$ | fund          |
| $I_0^*$  | $G_2$     |       |       | 7        |          |          |               |
| $I_0^*$  | Spin(7)   | $I_1^*$ | $I_1^*$ | vect     |          |          | spin          |
| $I_0^*$  | Spin(8)   | $I_1^*$ | $I_1^*$ | vect     | spin_±   |          |               |
| $I_1^*$  | Spin(9)   | $I_2^*$ | $IV^*$ | vect     |          |          | spin          |
| $I_1^*$  | Spin(10)  | $I_2^*$ | $IV^*$ | vect     | spin_±   |          |               |
| $I_2^*$  | Spin(11)  | $I_3^*$ | $III^*$ | vect     |          | $\frac{1}{3}\text{spin}$ |               |
| $I_2^*$  | Spin(12)  | $I_3^*$ | $III^*$ | vect     | $\frac{1}{3}\text{spin}_±$ |               |
| $I_n^*, n \geq 3$ | SO($2n + 7$) | $I_{n+1}^*$ | NM | vect |          |          | NM           |
| $I_n^*, n \geq 3$ | SO($2n + 8$) | $I_{n+1}^*$ | NM | vect |          |          | NM           |
| $IV^*$   | $F_4$     | $III^*$ |       | 26       |          |          |               |
| $IV^*$   | $E_6$     | $III^*$ |       |          |          |          | 27            |
| $III^*$  | $E_7$     | $II^*$  |       | $\frac{1}{2}56$ |          |          |               |
| $II^*$   | $E_8$     | NM     |       |          |          |          | NM           |

(iii) If $G$ is associated to the Kodaira type fiber $II^*$, or $I_n^*, n \geq 12$, the equation of the Weierstrass model is not minimal at the non-simple normal crossing point of the two branches $\Sigma_0$ and $\Sigma_1$. In order to resolve this singularity we would need to blow up $B$ the basis of the fibration. In the resulting elliptic fibration (still flat and Calabi–Yau), the two branches of discriminant are separated. We assume then that such points do not occur.

Remark 8.4. We have used the following notation in Table A:

- (●) Cases with no small resolution are denoted “NSR”, and cases with non-minimal Weierstrass model are denoted “NM”.
- (●) A dash denotes the trivial representation, whereas a blank entry denotes a situation in which there is no representation which belongs in that location.
- (●) The classical groups SU($n$), Sp($n$) have representations on $\mathbb{C}^n$, $\mathbb{H}^n$, respectively, which are known as the fundamental representations and denoted by...
“fund”. This representation is quaternionic in the case of $\text{Sp}(n)$. The second exterior power of the fundamental representation is denoted by “$\Lambda^2$”. In the case of $\text{Sp}(n)$, $\Lambda^2$ is reducible and its irreducible “traceless” part is denoted by “$\Lambda^2_0$”. 

$\bullet$ The classical group $\text{SO}(n)$ has a representation on $\mathbb{R}^n$ called the vector representation and denoted by “vect”. Its double cover Spin($n$) has spinor representations. When $n$ is odd, there is one spinor representation, of dimension $2^{(n-1)/2}$, denoted by “spin”. When $n$ is even, there are two half-spinor representations, each of dimension $2^{(n-2)/2}$, denoted by “spin$_+$” and “spin$_-$”. Note that the spinor or half-spinor representations are real if $n \equiv 0, \pm 1 \text{ mod } 8$, complex if $n \equiv \pm 2 \text{ mod } 8$, and quaternionic if $n \equiv \pm 1, 4 \text{ mod } 8$.

$\bullet$ In the case of the exceptional groups, we label representations by their dimension (given in boldface type).

**Proof.** As we have already seen in Section 7, the intersection numbers of the various parts of the discriminant in $B$ determine the geometry of $W$ and the choice of the group $G$ and vice versa. Following Section 4, we write all the terms in $\mathcal{R}$ in Proposition 6.9, as coefficients of $g(C)$, the genus of the curve of singularities, the number of points where the singularities are non-generic, and $g(C') = g'$, when the groups are non-simply laced, and then interpret the results. The coefficients in 6.9 are determined by the group and the local geometry (the degeneration of the general rational double point) and are listed in Appendix I. We divide the proof in 3 steps.

• Step I (8.1): We show how the geometry suggests the appropriate substitutions for $\Sigma_0 \cdot \Sigma_1$, $K_B \cdot \Sigma_1$, $\Sigma_1^2$ and also $B_1$ if the group has monodromy branched at $B_1$ points.

If $B_1 = B_2 = 0$, then the substitutions are uniquely determined (see Section 6).

In section 9 we show how these substitutions are equivalent to certain representation-theoretic facts. If $G \neq \text{Sp}(k)$ or $\text{SO}(m)$, then after the substitutions we obtain the data in Table 3. That is, the resulting formula for $\mathcal{R}$ can be written as a sum of local terms, associated to various points $P$, which can be collected into a formula of the form

\begin{equation}
\mathcal{R} = (g - 1)(\dim(G) - \text{rk } G) + (g' - g)\mathcal{R}_0 + \sum_{j=1}^{2} B_j \mathcal{R}_j.
\end{equation}

The local contributions $\mathcal{R}_j$ are recorded in Table 3.

In the cases $G = \text{Sp}(k)$, $G = \text{SO}(m)$, there are choices in making the substitutions but if a careful choice is made we can again write things in the form (8.1) (see also Section 9 for a better interpretation).

As we will point out in Remark 5.8 below, the substitutions can be formulated in a very general way which allows them to be applied in cases beyond the specific ones considered here [16].
Step II (8.2): We show how we can naturally interpret the entries in Table B as charged dimensions of certain representations (multiplied by the coefficient $\delta_j$), given in Table A. That is, $R_j = \delta_j \dim(\rho_j)_{ch}$. If $p$ is not a branch point, then the (resolution of the) general elliptic surface through $P$ can be associated to a group $G'$ containing $G$, and the representation is obtained via the branching rules for the adjoint representation of $G'$.

If $G$ is non-simply laced, then we consider $G \subset G'$, $G'$ simply laced, and we use again the branching rules. (This gives the representation-theoretic interpretation of the number “$R_0$” from equations (4.1), (4.2).)

Step III (8.3). Finally we show how the number $\delta$ can be derived from the geometry of the degeneration of the general double point to the singularity over $p$.

8.1. Step I: The substitutions.
Proposition 8.5. Assume that resolution of the curve of singularities \( C \) leads to a non-simply laced group \( G \), as in [1, 3]. Namely, some of the exceptional divisors are ruled over a curve \( C' \), which is a finite cover of \( C \) of degree \( d = 2, 3 \) (if and only if \( G = G_2 \)), ramified at \( B_1 \) points. Write \( g' = g(C') \), then:

\[
B_1 = 2(g' - g) - (2d - 2)(g - 1)
\]

Proof. The statement follows from Hurwitz’s formula. \( \square \)

Proposition 8.6. Following the notation in Section [3], we have:

\[
\Sigma_1 \cdot \Sigma_0 = r_1 B_1 + r_2 B_2.
\]

If the group \( G \) is non-simply laced, then

\[
\Sigma_1 \cdot \Sigma_0 = r_2 B_2 + 2r_1 (g' - g) - (2d - 2)r_1 (g - 1),
\]

if there are \( B_1 \) branch points of: \( C' \to C \).

Proposition 8.7. The appropriate substitutions for \(-K_B \cdot \Sigma_1 \) and \( \Sigma_1^2 \) are the ones given in Table [C].

Proof. (a) When \( J \) is finite and there is no monodromy, i.e., cases II, III, IV, \( I_0^* \), \( IV^* \), \( III^* \), \( II^* \) corresponding to the simply laced groups \( \{e\}, SU(2), SU(3), SO(8), E_6, E_7, E_8 \), in Deligne’s exceptional series, then the local geometry is given by the following equations:

\[
K_B \cdot \Sigma_1 + \Sigma_1^2 = 2g(\Sigma_1) - 2
\]

\[
(-12K_B - m \Sigma_1) \cdot \Sigma_1 = r_1 B_1 + r_2 B_2 \quad (\leftrightarrow \Sigma_1 \cdot \Sigma_0 = r_1 B_1 + r_2 B_2),
\]

which can be solved since \( m < 12 \):

\[
-K_B \cdot \Sigma_1 = \frac{2m(g - 1) + r_1 B_1 + r_2 B_2}{12 - m}; \quad \Sigma_1^2 = \frac{24(g - 1) + r_1 B_1 + r_2 B_2}{12 - m}.
\]

(b) When \( J \) is finite and there is monodromy, i.e., cases IV, \( I_0^* \), \( I_0^* \), \( IV^* \) corresponding to groups \( Sp(1), G_2, SO(7), F_4 \) which includes the remainder of Deligne’s exceptional series, the local geometry is the same but we also use Proposition [5] to eliminate \( B_1 \) in favor of \( g' - g \):

\[
-K_B \cdot \Sigma_1 = \frac{(2m - 2r_1 (d - 1))(g - 1) + 2r_1 (g' - g) + r_2 B_2}{12 - m};
\]

\[
\Sigma_1^2 = \frac{(24 - 2r_1 (d - 1))(g - 1) + 2r_1 (g' - g) + r_2 B_2}{12 - m}.
\]

(c) If \( G = SU(n), n \geq 3 \), Table [I] in Appendix I tells us that

\[
(8.2) \quad B_1 = -K_B \cdot \Sigma_1,
\]
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Type & \(G\) & \(-K_B \cdot \Sigma_1\) & \(\Sigma_1^2\) \\
\hline
\(I_2\) & SU(2) & \(\frac{3}{2}B_1\) & \(2(g-1)+\frac{1}{2}B_1\) \\
\hline
\(I_{2k}, k \geq 2\) & Sp\((k)\) & \(-(g-1)+(g'-g)\) & \((g-1)+(g'-g)\) \\
\hline
\(I_{2k+1}, k \geq 2\) & Sp\((k)\) & \(-(g-1)+(g'+g)\) & \((g-1)+(g'-g)\) \\
\hline
\(I_n, n \geq 3\) & SU\((n)\) & \(B_1\) & \(2(g-1)+B_1\) \\
\hline
\(II\) & \{e\} & \(\frac{2}{3}(g-1)+\frac{1}{3}B_1\) & \(\frac{16}{3}(g-1)+\frac{1}{3}B_1\) \\
\hline
\(III\) & SU\((2)\) & \(\frac{2}{3}(g-1)+\frac{1}{3}B_1\) & \(\frac{8}{3}(g-1)+\frac{1}{3}B_1\) \\
\hline
\(IV\) & Sp\((1)\) & \(\frac{1}{2}(g-1)+\frac{1}{2}(g'-g)\) & \(\frac{5}{2}(g-1)+\frac{1}{2}(g'-g)\) \\
\hline
\(IV\) & SU\((3)\) & \((g-1)+\frac{1}{2}B_1\) & \(3(g-1)+\frac{1}{2}B_1\) \\
\hline
\(I_0^*\) & G\(_2\) & \(\frac{4}{3}(g-1)+\frac{1}{3}(g'-g)\) & \(\frac{16}{3}(g-1)+\frac{1}{3}(g'-g)\) \\
\hline
\(I_0^*\) & Spin\((7)\) & \(\frac{5}{3}(g-1)+\frac{1}{3}(g'-g)+\frac{1}{3}B_2\) & \(\frac{11}{3}(g-1)+\frac{1}{3}(g'-g)+\frac{1}{3}B_2\) \\
\hline
\(I_0^*\) & Spin\((8)\) & \(2(g-1)+\frac{1}{3}B_1+\frac{1}{3}B_2\) & \(4(g-1)+\frac{1}{3}B_1+\frac{1}{3}B_2\) \\
\hline
\(I_n^*, n \geq 1\) & SO\((2n+7)\) & \(2(g-1)+B_2\) & \(4(g-1)+B_2\) \\
\hline
\(I_n^*, n \geq 1\) & SO\((2n+8)\) & \(2(g-1)+B_2\) & \(4(g-1)+B_2\) \\
\hline
\(IV^*\) & F\(_4\) & \(3(g-1)+(g'-g)\) & \(5(g-1)+(g'-g)\) \\
\hline
\(IV^*\) & E\(_6\) & \(4(g-1)+B_1\) & \(6(g-1)+B_1\) \\
\hline
\(III^*\) & E\(_7\) & \(6(g-1)+B_1\) & \(8(g-1)+B_1\) \\
\hline
\(II^*\) & E\(_8\) & \(10(g-1)+B_1\) & \(12(g-1)+B_1\) \\
\hline
\end{tabular}
\caption{The substitutions.}
\end{table}

where \(B_1\) is the number of non-simple normal crossings intersections. The genus formula then says that

\[(8.3) \quad \Sigma_1^2 = 2(g-1) + B_1.\]

(The case of SU\((2)\) is similar, using \(B_1 = -2K_B \cdot \Sigma_1\).)

(d) If \(G = \text{Sp}(\left[\frac{n}{2}\right])\), \(n \geq 3\), coming from \(I_n\) with monodromy, then \(B_1 = -2K_B \cdot \Sigma_1\) so that

\[-K_B \cdot \Sigma_1 = (g'-g) - (g-1).\]

Combining this with the genus formula yields

\[\Sigma_1^2 = (g'-1) + (g-1).\]

(e) Finally, if \(G = \text{SO}(2n+7)\) or \(\text{SO}(2n+8)\) coming from \(I_n^*, n \geq 1\), then \(B_2 = -2K_B \cdot \Sigma_1 - \Sigma_1^2\) which can be combined with the genus formula and solved to give:

\[-K_B \cdot \Sigma_1 = 2(g-1) + B_2; \quad \Sigma_1^2 = 4(g-1) + B_2.\]
Step I now proceeds as follows: use the data in Tables 3 and 4 in Appendix I to evaluate the “local” contributions to the Euler characteristic, in the formula for $\mathcal{R}$ given in Proposition 6.9. Then make the substitutions given in Propositions 8.6 and 8.7 (supplementing them with Proposition 8.5 if there is monodromy) into the resulting formula; in all but a few cases (detailed below) this yields a formula of the form

$$\mathcal{R} = (g - 1)(\dim(G) - \text{rk } G) + (g' - g)\mathcal{R}_0 + \sum_{j=1}^{2} B_j \mathcal{R}_j$$

with the local contributions $\mathcal{R}_j$ recorded in Table B. (For simplicity of notation, we define $\mathcal{R}_0 = 0$ when there is no monodromy.)

The exceptional cases are $I_{2k+1}$ with monodromy, and $I_n^*$. In the case of $I_{2k+1}$ with monodromy, the formula should be written with a term $kB_1$ to which the substitution from Proposition 8.5 is not applied.

In the case of $I_n^*$, the term $m \Sigma_1 \cdot \Sigma_0$ in the formula for $\mathcal{R}$ should be broken into two parts, using the substitution from Proposition 8.6 to evaluate a term of the form $(m - 2) \Sigma_1 \cdot \Sigma_0$, but evaluating the remaining term $2 \Sigma_1 \cdot \Sigma_0$ as

$$2 \Sigma_1 \cdot \Sigma_0 = 2(-12K_B \cdot \Sigma_1 - m \Sigma_1^2) = (48 - 8m)(g - 1) + (24 - 2m)B_2$$

(using Proposition 8.7 for the last step).

The results of all of these manipulations are recorded in the coefficients given in Table B.

Remark 8.8. It is worth observing, for possible generalizations to other cases [16], that the substitutions we have used can be formulated intrinsically without reference to assumptions about the particular types of degenerate fibers which occur. This is clear for the substitutions given in Propositions 8.5 and 8.6. In the case of Proposition 8.7, when $J$ is finite the substitution only depends on the discriminant locus. If $J = \infty$ and we have type $I_n$ along $\Sigma_1$, consider the Weierstrass equation

$$y^2 = x^3 + fx + g$$

(8.4)

(which is intrinsically associated to the elliptic fibration) and note that neither $f$ nor $g$ vanishes identically along $\Sigma_1$. The location of the singularity is given by either $x = -3g/2f$ or (equivalently) $x = 2f^2/9g$. There is then a divisor $\beta$ on $\Sigma_1$ (in the class $-2\Sigma_1 \cdot B$) represented by $\text{div}_{\Sigma_1}(g) - \text{div}_{\Sigma_1}(f)$ or by $2 \text{div}_{\Sigma_1}(f) - \text{div}_{\Sigma_1}(g)$. In our case, this divisor coincides with the divisor $B_1$ (when there is monodromy) or $2B_1$ (when there is no monodromy) which we used in Proposition 8.7.

Similarly, if $J = \infty$ and we have type $I_n^*$ then neither $f/s^2$ nor $g/s^3$ vanishes identically along $\Sigma_1$. The divisor $\beta$ on $\Sigma_1$, which coincides with the divisor $B_2$.

*We are choosing to do this in order to more easily present the formula as agreeing with a calculation in representation theory; of course, the version of this formula in which all $B_1$ terms have been eliminated is also perfectly valid.*
which we used in Proposition 8.7, is represented by \( \text{div}_1(g/s^3) - \text{div}_1(f/s^2) \) or by \( 2 \text{div}_1(f/s^2) - \text{div}_1(g/s^2) \).

Note that this same computation could just as easily be carried out in the case of multiple components of the discriminant. The starting point would be a straightforward generalization of the equation in Proposition 6.9. Then for each component of the discriminant, one would use the corresponding substitution (according to the singularity type along that component) and manipulate the substituted formula precisely as above. The result is a division into “non-local” terms associated to the various factors of the gauge group (taking precisely the same form as above), and “local” terms associated to isolated points along the discriminant locus. We will explore this generalization further in [16].

8.2. **Step II: Branching rules.** In this subsection and the next, we explain how to systematically determine representations \( \rho_j \), associated to monodromy covers and to degeneration points, whose charged dimensions reproduce the numbers \( R_j \) which were calculated in Table B.

Let \( h \subset g \) be a subalgebra of a Lie algebra. Given an irreducible representation \( \rho : g \to \text{GL}(N, \mathbb{C}) \), a natural question is how \( \rho \) decomposes under \( h \). The answer can be obtained by following the “branching rules” (see for example [25]).

The representation \( \rho_0 \)

In the case of non-simply laced groups, according to [2] the representation \( \rho_0 \) is determined by the branching rules for \( g_0 \subset g \), where \( g_0 \) is the non-simply laced algebra and \( g \) is the corresponding simply laced algebra (whose Dynkin diagram covers that of \( g_0 \)). In each such case, \( g_0 \) is the fixed subalgebra of some outer automorphism of \( g \) of finite order.

**Proposition 8.9 ([25]).** The following branching rules hold (using the notation for representations established in Remark 8.4):

- \( \text{Sp}(k) \subset \text{SU}(2k) \) (involutive outer automorphism):
  \[
  \text{adj} \ \text{SU}(2k) = \text{adj} \ \text{Sp}(k) \oplus \Lambda_0^2
  \]

- \( \text{Sp}(k) \subset \text{SU}(2k + 1) \) (outer automorphism):
  \[
  \text{adj} \ \text{SU}(2k + 1) = \text{adj} \ \text{Sp}(k) \oplus \Lambda^2 \oplus \text{fund} \oplus \text{fund}
  \]

- \( \text{SO}(2k - 1) \subset \text{SO}(2k) \) (involutive outer automorphism):
  \[
  \text{adj} \ \text{SO}(2k) = \text{adj} \ \text{SO}(2k - 1) \oplus \text{vect}
  \]

- \( G_2 \subset \text{SO}(8) \) (outer automorphism):
  \[
  \text{adj} \ \text{SO}(8) = \text{adj} \ G_2 \oplus 7 \oplus 7
  \]

- \( F_4 \subset E_6 \) (involutive outer automorphism):
  \[
  \text{adj} \ E_6 = \text{adj} \ F_4 \oplus 26
  \]
In the involutive cases, we have $\rho_0$ given as the $(-1)$-eigenspace of the involution, i.e., the complement of $\text{adj} \mathfrak{g}_0$ within $\text{adj} \mathfrak{g}$. Thus $\rho_0$ coincides with $\Lambda_0$, vect, and 26 in the first, third, and fifth cases above, respectively.

In the case of $\text{Sp}(k) \subset \text{SU}(2k + 1)$, although the automorphism of $\mathfrak{g}$ has order 4, the monodromy action is only order 2, and $\rho_0$ is again given by the complement of $\text{adj} \mathfrak{g}_0$ within $\text{adj} \mathfrak{g}$, i.e., $\rho_0 = \Lambda^2 \oplus \text{fund} \oplus \text{fund}$.

In the case of $G_2$, the order 3 monodromy action leads to the representation $\rho_0$ occurring with multiplicity two in the complement of $\text{adj} \mathfrak{g}_0$. (These two copies correspond to the eigenspaces for the monodromy action with eigenvalues $e^{\pm 2\pi i/3}$.) Thus, in this case $\rho_0 = 7$.

Note that in all cases, the charged dimension of the representation $\rho_0$ agrees with the number $R_0$ calculated in Table 3.

The representations $\rho_j$

Representations associated to the points $p$ can also be determined via branching rules, using a method pioneered by Katz and Vafa [23]. If the general surface section through $p$ has a rational double point associated to $G' \supset G$, then the representation associated to $p$ is determined by the corresponding branching rule (modulo a few subtleties to be discussed in the next subsection).

**Proposition 8.10 ([23]).** The following branching rules hold (still using the notation from Remark 8.4):

- $\text{SU}(n) \subset \text{SU}(n + 1)$:
  \[
  \text{adj} \text{SU}(n + 1) = \text{adj} \text{SU}(n) \oplus \text{fund} \oplus \overline{\text{fund}} \oplus 1
  \]

- $\text{SU}(n) \subset \text{SO}(2n)$:
  \[
  \text{adj} \text{SO}(2n) = \text{adj} \text{SU}(n) \oplus \Lambda^2 \oplus \overline{\Lambda^2} \oplus 1
  \]

- $\text{SO}(2k) \subset \text{SO}(2k + 2)$:
  \[
  \text{adj} \text{SO}(2k + 2) = \text{adj} \text{SO}(2k) \oplus \text{vect} \oplus \overline{\text{vect}} \oplus 1
  \]

- $\text{Spin}(10) \subset E_6$:
  \[
  \text{adj} E_6 = \text{adj} \text{Spin}(10) \oplus \text{Spin}_+ \oplus \text{Spin}_- \oplus 1
  \]

- $\text{Spin}(12) \subset E_7$:
  \[
  \text{adj} E_7 = \text{adj} \text{Spin}(12) \oplus \text{Spin}_+ \oplus \text{Spin}_- \oplus 1 \oplus 1
  \]

- $E_6 \subset E_7$:
  \[
  \text{adj} E_7 = \text{adj} E_6 \oplus 27 \oplus \overline{27} \oplus 1
  \]

- $E_7 \subset E_8$:
  \[
  \text{adj} E_8 = \text{adj} E_7 \oplus 56 \oplus \overline{56} \oplus 1 \oplus 1 \oplus 1.
  \]

(There are also non-standard embeddings of $D_4$ into $D_5$ which lead to branching rules involving the $\text{Spin}_+$ or $\text{Spin}_-$ representations of $\text{SO}(8)$ rather than the vector representation.)
Each of these branching rules takes the form
\begin{equation}
\text{adj } \mathfrak{g} = \text{adj } \mathfrak{g}_0 \oplus \rho \oplus \overline{\rho} \oplus \mathbf{1}
\end{equation}
for some representation \(\rho\); it is \(\rho\) which determines the matter representation.

For example, when the general fiber of type \(\text{SU}(2k)\), degenerates to \(\text{SU}(2k + 1)\), then we use the branching rule corresponding to the inclusion \(\text{SU}(2k) \subset \text{SU}(2k + 1)\) to determine the correct representation “fund” appearing as \(\rho\) in the statement of the theorem.

The matter representations \(\rho_j\) for non-simply laced groups at non-branch points can be inferred by looking at the representation of the corresponding simply laced group.

The cases \(\text{SO}(12) \subset E_7\) and \(E_7 \subset E_8\) (as well as the fundamental representation of \(\text{Sp}(k)\)) lead to quaternionic representations and follow a somewhat different pattern, as we will explain in the next subsection. In all other cases, the representation \(\rho_j\) determined by these branching rules has a charged dimension which agrees with the number \(\mathcal{R}_j\) calculated in Table B.

8.3. Step III: Resolutions of non-generic singularities, deformation theory, complex and quaternionic representations.

Up to this point, we have described the “matter” representation as a complex representation of the group \(G\) (as is customary in the physics literature\(^6\)). However, the representation we need is more accurately described as a quaternionic representation, that is, a representation into \(\text{GL}(\mathbb{H}^n)\). Given a complex representation \(\rho\), the representation \(\rho \oplus \overline{\rho}\) is automatically quaternionic—this is how one passes from complex to quaternionic in many cases. However, some quaternionic representations cannot be described as the sum of a complex representation with its complex conjugate. This explains the presence of the factor \(\delta = \frac{1}{2}\) in certain terms of the formula for \(\mathcal{R}\), since in all cases we are actually counting \(1/2\) of the quaternionic dimension of the representation.

How do these complex and quaternionic representations show up in the geometry? Consider again the general elliptic surface passing through \(p\). In all the cases we are considering, this surface has a rational double point singularity, which can be associated to a simply laced group \(G'\). Deforming to a nearby surface we again find a rational double point, this time the one associated to the group \(G\).

There are three possibilities for a one-parameter family of rational double points:

1. It fails to admit a simultaneous resolution of singularities,
2. It is a base-change of a family of type (1) which admits a simultaneous resolution of singularities, or
3. It admits a simultaneous resolution of singularities, and is not the base-change of a family that failed to admit such a resolution.

When analyzed carefully, the Katz–Vafa prescription\(^{23}\) operates differently in these cases, depending on

\(^6\)In the physics literature, one refers to “hypermultiplets taking values in a complex representation” or, equivalently, “half-hypermultiplets taking values in a quaternionic representation.”
whether or not simultaneous resolution is possible. It is possible to explicitly compute whether or not this is possible in each instance, using the formulas in \[21\]. (One calculates the equation of the family after performing the base-change which ensures that simultaneous resolution is possible; the fact that a base-change has been performed can then be recognized from the dependence of all coefficients on \(t^k\) rather than \(t\), for some integer \(k\) which represents the degree of the base-change map. See \[23\], where many of these calculations have been carried out.)

Of the branching rules described in Proposition 8.10, the first one (\(\text{SU}(n) \subset \text{SU}(n + 1)\)) falls in case (3), and all others fall in cases (1) and (2) (depending on whether \(\rho_j\) is being treated as a representation of a simply laced or a non-simply laced group). There is a further distinction that can be made in case (1): making a base-change to produce a simultaneous resolution, the base-change group will act on the set of roots, and this action may or may not induce monodromy on the Dynkin diagram.

In case (1), if we perform a finite base-change, a simultaneous resolution becomes possible and the branching rules determine the representations which are involved. However, the covering group for the base-change acts on these representations, and only the invariant representation appears in the original family. In four of the branching rules from Proposition 8.10, there is monodromy on the Dynkin diagram and we have already analyzed the corresponding representations from that point of view. The representation \(\rho\) (whose weights are represented by holomorphic curves) is mapped to the representation \(\overline{\rho}\) (whose weights are represented by antiholomorphic curves) with the upshot being that each ramification point on the parameter curve is associated to \(1/2\) of the full representation. (Of course, we are not counting this as a contribution to the local representation at the branch point—this part of the representation theory is non-local, and is accounted for by the representation \(\rho_0\).)

Note that these same four branching rules also occur in the context of case (2) families, where there is no monodromy. In these cases, the entire branching rule plays a rôle, and the quaternionic representation associated to such a point is \(\rho \oplus \overline{\rho}\) (corresponding to the complex representation \(\rho\)). Note that the singularity is fully resolved in these cases, as is reflected in the Euler characteristic computations in Table 4 in Appendix I.

The remaining two types of branching rules, \(\text{SO}(12) \subset \text{E}_7\) and \(\text{E}_7 \subset \text{E}_8\), only occur in the context of case (1) in our setup, and there is no monodromy on the Dynkin diagram. In these cases, the action of the covering group similarly maps \(\rho\) to \(\overline{\rho}\), but in these cases the representation is quaternionic and \(\rho \cong \overline{\rho}\). The upshot is that the “complex representation” associated to each such point is \(1/2\) of the quaternionic representation \(\rho\). (Note that the covering group acts as \(-1\) on the “1” summands in the branching rule, so that these do not contribute as they are not

\[7\] We are grateful to Sheldon Katz for correspondence on this point.
invariant.) In both of these cases, the singularity of the surface is not fully resolved, as is reflected in the Euler characteristic computations in Table 4 in Appendix I.

The multiplicities of these points are slightly different in the cases of Kodaira fibers of types II and III, but the same representations occur. See [3], where these cases are worked out in detail.

9. Another look at the substitutions: Representation theory.

We have seen how the degenerations of the general singularity determine certain representations of the group $G$; here we show that the converse also holds: once one chooses the representations which might occur, the geometry of the Calabi–Yau is completely determined by some relations in representation theory. We will at the same time verify the additional anomaly cancellations stated in Theorem 3.2.

We will verify that the generalized Green–Schwarz anomaly cancellation mechanism works in the way that was proposed by Sadov [31]. The factored form (3.3) is taken to be

\[
\frac{1}{2} \left( \frac{1}{2} K_B \text{tr} R^2 + 2 \sum \Sigma_i \text{tr} F_i^2 \right) \cdot \left( \frac{1}{2} K_B \text{tr} R^2 + 2 \sum \Sigma_i \text{tr} F_i^2 \right).
\]

The anomaly cancellation requirements are deduced by comparing this with equation (3.2). The coefficients of $(\text{tr} R^2)^2$ agree due to the relation $9 - n_T = K_B^2$, which follows from Noether’s theorem on the surface $B$ (since $\chi(O_B) = 1$). The remaining coefficients lead to equations

\[
-6K_B \cdot \Sigma_i (\text{tr} F_i^2) = -\text{Tr}_{\text{adj}} F_i^2 + \sum_{\rho} n_{\rho} \text{Tr}_{\rho} F_i^2
\]

\[
3 \Sigma_i^2 (\text{tr} F_i^2)^2 = -\text{Tr}_{\text{adj}} F_i^4 + \sum_{\rho} n_{\rho} \text{Tr}_{\rho} F_i^4
\]

\[
\Sigma_i \cdot \Sigma_j (\text{tr} F_i^2)(\text{tr} F_j^2) = \sum_{\rho, \sigma} n_{\rho\sigma} \text{Tr}_{\rho} F_i^2 \text{Tr}_{\sigma} F_j^2
\]

which must be evaluated using the relations in the ring of $G$-invariant functions. Note that in our case there is a single local factor $G_i$ of the gauge group $G$, and we can suppress the subscript $i$ and denote its adjoint curvature by $F$.

We must also specify, for each type of group, a “fundamental representation” in which to evaluate the trace $\text{tr}$ on the left-hand side of the equations. We take $\text{tr} = \text{Tr}_{\text{fund}}$ to be the trace in the usual fundamental representation for $\text{SU}(n)$ and $\text{Sp}(k)$, we take $\text{tr} = \frac{1}{2} \text{Tr}_{\text{vect}}$ to be one-half of the trace in the vector representation for $\text{Spin}(m)$, and we take $\text{tr}$ to be the trace in the smallest representation of the group in the case of the exceptional groups.

\footnote{We have corrected some minor numerical errors in [31].}
Note that if we were to replace \( \text{tr} \) by some multiple of it, say \( \lambda \text{tr} \), then we would multiply \( -6K_B \cdot \Sigma_1 \) by \( \lambda \) and \( 3\Sigma_2^2 \) by \( \lambda^2 \). Making the geometry match the representation theory completely constrains our choice of \( \lambda \), and we express everything below in terms of the “correct” trace for each group.

Having specified the fundamental representation, \( \text{tr} F^2 \) will correspond to a basis of Casimir operators of second order, and \( (\text{tr} F^2)^2 \) will be one of the basis elements for Casimir operators of the fourth order; when there is a second independent fourth-order Casimir, the second basis element can be taken to be \( \text{tr} F^4 \). Traces taken in other representations can be expressed in terms of these. We have collected the data of this sort that we need (mostly taken from Erler [13]) in Table D (in which we use the notation spin\(^*\) to denote either spin or spin\(_\pm\)).

| \( G \) | \( \rho \) | \( \text{Tr}_\rho F^2 \) | \( \text{Tr}_\rho F^4 \) |
|---|---|---|---|
| SU(2) | adj | \( 4 \text{tr} F^2 \) | \( 8(\text{tr} F^2)^2 \) |
|  | fund | \( \text{tr} F^2 \) | \( \frac{1}{2}(\text{tr} F^2)^2 \) |
| SU(3) | adj | \( 6 \text{tr} F^2 \) | \( 9(\text{tr} F^2)^2 \) |
|  | fund | \( \text{tr} F^2 \) | \( \frac{1}{3}(\text{tr} F^2)^2 \) |
| SU(\( n \)), \( n \geq 4 \) | adj | \( 2n \text{tr} F^2 \) | \( 6(\text{tr} F^2)^2 + 2n \text{tr} F^4 \) |
| \( \Lambda^2 \) | fund | \( \text{tr} F^2 \) | \( 0(\text{tr} F^2)^2 + 3 \text{tr} F^4 \) |
|  | \( n-2 \text{tr} F^2 \) | \( 0(\text{tr} F^2)^2 + 3 \text{tr} F^4 \) |
| Sp(\( k \)), \( k \geq 2 \) | adj | \( (2k+2) \text{tr} F^2 \) | \( 3(\text{tr} F^2)^2 + (2k+8) \text{tr} F^4 \) |
| \( \Lambda^2 \) | fund | \( \text{tr} F^2 \) | \( 0(\text{tr} F^2)^2 + 3 \text{tr} F^4 \) |
|  | \( (2k-2) \text{tr} F^2 \) | \( 0(\text{tr} F^2)^2 + 3 \text{tr} F^4 \) |
| Spin(\( m \)), \( m \geq 7 \) | adj | \( (2m-4) \text{tr} F^2 \) | \( 12(\text{tr} F^2)^2 + (2m-16) \text{tr} F^4 \) |
| \( \text{vect} \) | | \( 2 \text{tr} F^2 \) | \( 0(\text{tr} F^2)^2 + 2 \text{tr} F^4 \) |
| \( \text{spin}^* \) | | \( \dim(\text{spin}^*)(\frac{1}{4} \text{tr} F^2) \) | \( \dim(\text{spin}^*)(\frac{3}{8}(\text{tr} F^2)^2 - \frac{1}{8} \text{tr} F^4) \) |
| \( E_6 \) | adj | \( 24 \text{tr} F^2 \) | \( 18(\text{tr} F^2)^2 \) |
|  | 27 | \( 6 \text{tr} F^2 \) | \( 3(\text{tr} F^2)^2 \) |
| \( E_7 \) | adj | \( 36 \text{tr} F^2 \) | \( 24(\text{tr} F^2)^2 \) |
|  | 56 | \( 12 \text{tr} F^2 \) | \( 6(\text{tr} F^2)^2 \) |
| \( E_8 \) | adj | \( 60 \text{tr} F^2 \) | \( 36(\text{tr} F^2)^2 \) |
| \( F_4 \) | adj | \( 18 \text{tr} F^2 \) | \( 15(\text{tr} F^2)^2 \) |
|  | 26 | \( 6 \text{tr} F^2 \) | \( 3(\text{tr} F^2)^2 \) |
| \( G_2 \) | adj | \( 8 \text{tr} F^2 \) | \( 10(\text{tr} F^2)^2 \) |
|  | 7 | \( 2 \text{tr} F^2 \) | \( (\text{tr} F^2)^2 \) |

Table D.

It is now a straightforward matter to verify the remaining anomaly cancellations. We illustrate the procedure in the case of \( G = \text{SU}(n), n \geq 4 \), with a matter
representation in which the adjoint representation has multiplicity \( g \), the fundamental representation has multiplicity \( B_2 \), and \( \Lambda^2 \) has multiplicity \( B_1 \) (as specified in Theorem 8.2).

From Table D, we read off the facts which must hold in order for the gauge and mixed anomalies to cancel:

\[
-6K_B \cdot \Sigma_1 = 2n(g - 1) + B_2 + (n - 2)B_1 \\
3\Sigma_1^2 = 6(g - 1) + 0B_2 + 3B_1 \\
0 = 2n(g - 1) + B_2 + (n - 8)B_1.
\]

(Note that there are two equations coming from the quartic anomaly, since there are two independent fourth order Casimirs.)

To verify these, we use the geometric relations which characterize \( g \), \( B_1 \), and \( B_2 \), namely

\[
B_1 = -K_B \cdot \Sigma_1 \\
B_2 = (-8K_B - n\Sigma_1) \cdot \Sigma_1 \\
g = \left(\frac{1}{2}K_B + \frac{1}{2}\Sigma_1\right) \cdot \Sigma_1.
\]

When these are substituted into the right-hand side of the proposed anomaly relations,

\[
2n\left(\frac{1}{2}K_B + \frac{1}{2}\Sigma_1\right) + (-8K_B - n\Sigma_1) + (n - 2)(-K_B) = -6K_B \\
6\left(\frac{1}{2}K_B + \frac{1}{2}\Sigma_1\right) + 0(-8K_B - n\Sigma_1) + 3(-K_B) = 3\Sigma_1 \\
2n\left(\frac{1}{2}K_B + \frac{1}{2}\Sigma_1\right) + (-8K_B - n\Sigma_1) + (n - 8)(-K_B) = 0,
\]

the relations are verified.

A similar verification can be carried out in all cases. It is convenient to supplement the geometric formulas for \( g \) and \( B_j \)’s with a formula for for \( g' - g \) in the case of monodromy, and to compute a quantity \( \hat{B} \) in a few cases (in order to match the representation \( \hat{\rho} \) in the representation theory, as determined in Theorem 8.2). We summarize the data in Table E. (We have omitted the relation \( g = \left(\frac{1}{2}K_B + \frac{1}{2}\Sigma_1\right) \cdot \Sigma_1 \), which always holds.) Carrying out the verification is then a simple exercise in combining Tables D and E, as we have done in the case of SU(\( n \)) above.

Remark 9.1. As in Remark 8.8, we can express this part of the verification of the anomaly cancellation in terms which are somewhat more intrinsic. We defer the details of this to [16], but observe here how this can be carried out in the case of SU(\( n \)), \( n \geq 4 \).
The intrinsic geometric quantities we need are the divisor $\Sigma_0 \cdot \Sigma_1$, the arithmetic genus $p_a(\Sigma_1)$, and the divisor $\beta$ from Remark 8.8 (which is the intrinsic version of $2B_1$). We derive from these an intrinsic version of $B_2$, represented as $\Sigma_0 \cdot \Sigma_1 - 2\beta$. 

**Table E.** The relations.
Then in the anomaly cancellation requirements, we can represent the coefficient of $\text{tr} F_i^2$ as

$$2n(p_a(\Sigma_1) - 1) + (\Sigma_0 \cdot \Sigma_1 - 2\beta) + \frac{n - 2}{2}\beta$$

and the coefficient of $(\text{tr} F_i^2)^2$ as

$$6(p_a(\Sigma_1) - 1) + \frac{3}{2}\beta.$$
Appendix I: How to compute \( R \) (the coefficients in Proposition 6.9 and other things)

In this section we study the local equations and the geometric data for each group and their generic degenerations.

Following [4] we analyze the local equations in Tables 1 and 2. In Tables 3 and 4 we list, for each group, the coefficients of the right hand side of the equation defining \( R \), in Proposition 6.9. The entries of Table 1 are taken from [4], those of Table 3 are well known; to compute the others we need the affine equations of (I.2) and (I.1). We will work out the details for the case \( G = SU(2k) \) in Appendix II.

We need to use a more general form of the Weierstrass equation (1.1), namely

\[
    y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. 
\]

Since \( W \) is assumed to be a Calabi–Yau a \( j_k \in | -jK_B| \).

**Definition I.1.** It is convenient to use the following:

\[
    b_2 = a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6, \\
    b_8 = (a_1^2 + a_2)a_6 - a_4^2.
\]

The coefficients in (1.1) are now:

\[
    f = -\frac{1}{48}(b_2^2 - 24b_4), \quad g = -\frac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6).
\]

If \( a_j \) (resp. \( b_j \)) vanish along \( \Sigma_1 \) of order \( k \), then we write

\[
    a_{j,k} = \frac{a_j}{s^k} \quad (\text{resp.} \quad b_{j,k} = \frac{b_j}{s^k}).
\]

Table 1 is mostly taken from [4]: the first two columns list the Kodaira fiber and the associated group (see Section 1); in the middle columns we write the order of vanishing of each \( a_i \) along \( \Sigma_1 \). Recall that our hypothesis (a flat Calabi–Yau fibration) imposes some restriction on the self-intersection of the ramification divisor (see the Remark after the Main Theorem 8.2). In the last column, we exhibit how the equation for \( \Sigma_0 \) mod \( s \) breaks into factors; the power \( r_j \) which gives the multiplicity of the factor \( \beta_j \) is indicated in the factorization in each case.

We have incorporated some necessary corrections to the Table from [4]. First, the entry for \( I_{2k+1}, \ k \geq 1 \), with gauge group \( SU(2k+1) \) corresponds to the Weierstrass equation

\[
    y^2 + a_1 xy + a_{3,ks^k} y = x^3 + a_{2,1} sx^2 + a_{4,k+1}s^{k+1}x + a_{6,2k+1}s^{2k+1},
\]

which has discriminant

\[
    -\frac{1}{16} a_1^4 (a_{1}^2 a_{6,2k+1} - a_1 a_{3,k} a_{4,k+1} + a_{3,k}^2 a_{2,1}) s^{2k+1} - \frac{1}{16} a_1^3 a_{3,k}^3 s^k + O(s^{2k+2}).
\]
Thus, the correct leading term in the local equation of $\Sigma_0$ in this case (the “residual discriminant”) takes the form
\[ a_1^3(a_1 b_{8,2k+1} - a_{3,k}^3), \quad \text{if } k = 1, \]
and
\[ a_1^4 b_{8,2k+1}, \quad \text{if } k > 1 \]
(not $a_1^6 a_{6,2k+1}$ as was written in [4]).

Second, the residual discriminant in the case $IV$ (with gauge group $SU(3)$) should read $-27a_{3,1}^4$ rather than $-27a_{3,2}^4$.

| Type   | $G$        | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_6$ | $(\beta_1)^3(\beta_2)^2$ |
|-------|------------|-------|-------|-------|-------|-------|------------------------|
| $I_1$ | $\{e\}$   | 0     | 0     | 1     | 1     | 1     | $(b_2)^4(a_{6,1})^4$    |
| $I_2$ | SU(2)     | 0     | 0     | 1     | 1     | 2     | $(b_2)^2(b_{8,2})^4$    |
| $I_3$ | SU(3)     | 0     | 1     | 1     | 2     | 3     | $(a_1)^3(a_{1,8,3} - a_{3,1})^4$ |
| $I_{2k}, k \geq 2$ | Sp(k) | 0     | 0     | $k$   | $k$   | $2k$  | $b_2^4(b_{8,2k})^4$    |
| $I_{2k+1}, k \geq 1$ | Sp(k) | 0     | 0     | $k+1$ | $k+1$ | $2k+1$ | $(b_2)^4(a_{6,2k+1})^4$ |
| $I_n, n \geq 4$ | SU(n) | 0     | 1     | \left[ \frac{n}{2} \right] | \left[ \frac{n+1}{2} \right] | n  | $(a_1)^4(b_{8,n})^4$    |
| $II$  | $\{e\}$   | 1     | 1     | 1     | 1     | 1     | $(a_{6,1})^2$          |
| $III$ | SU(2)     | 1     | 1     | 1     | 1     | 2     | $(a_{4,1})^3$          |
| $IV$  | SU(3)     | 1     | 1     | 1     | 2     | 3     | $(a_{3,1})^4$          |
| $I_0^*$ | $G_2$  | 1     | 1     | 2     | 2     | 3     | $(\Delta_{12,6})^4$    |
| $I_0^*$ | Spin(7) | 1     | 1     | 2     | 2     | 4     | $(a_{2,1} - a_{4,2})^4(a_{4,2})^2$ |
| $I_0^*$ | Spin(8) | 1     | 1     | 2     | 2     | 4     | $(\sqrt{a_{2,1}^2 - a_{4,2}^2})^2(a_{4,2})^2$ |
| $I_0^*$ | Spin(9) | 1     | 1     | 2     | 3     | 4     | $(a_{6,4})^4(a_{2,1})^3$ |
| $I_0^*$ | Spin(10) | 1     | 1     | 2     | 3     | 5     | $(a_{3,2})^4(a_{2,1})^3$ |
| $I_0^*$ | Spin(11) | 1     | 1     | 3     | 3     | 5     | $(a_{2,1}^3 - 4a_{2,1}a_{6,5})^4(a_{2,1})^2$ |
| $I_0^*$ | Spin(12) | 1     | 1     | 3     | 3     | 5     | $(\sqrt{a_{4,3}^2 - 4a_{2,1}a_{6,5}})^2(a_{2,1})^2$ |
| $I_{2k-3}, k \geq 3$ | SO(4k+1) | 1     | 1     | $k$   | $k+1$ | 2$k$  | $(b_{6,2k})^4(a_{2,1})^3$ |
| $I_{2k-3}, k \geq 3$ | SO(4k+2) | 1     | 1     | $k$   | $k+1$ | 2$k+1$ | $(3,2)^4(a_{2,1})^3$ |
| $I_{2k-2}, k \geq 3$ | SO(4k+3) | 1     | 1     | $k+1$ | $k+1$ | 2$k+1$ | $(a_{2,k+1}^2 - 4a_{2,1}a_{6,2k+1})^4(a_{2,1})^2$ |
| $I_{2k-2}, k \geq 3$ | SO(4k+4) | 1     | 1     | $k+1$ | $k+1$ | 2$k+1$ | $(\sqrt{a_{4,2,k+1}^2 - 4a_{2,1}a_{6,2k+1}})^2(a_{2,1})^2$ |
| $IV^*$ | $F_4$     | 1     | 2     | 2     | 3     | 4     | $(b_{6,4})^2$          |
| $IV^*$ | $E_6$     | 1     | 2     | 2     | 3     | 5     | $(a_{3,2})^3$          |
| $III^*$ | $E_7$    | 1     | 2     | 3     | 3     | 5     | $(a_{4,3})^3$          |
| $II^*$ | $E_8$     | 1     | 2     | 3     | 4     | 5     | $(a_{6,5})^2$          |

Table 1.
Remark I.2. Following \[4\] (see also Proposition 5.4) we see that if
\[
\Sigma_0 \cap \Sigma_1 = \{P_1^1, \cdots P_1^{B_1}, P_2^1, \cdots P_2^{B_2}\},
\]
then the local equation of \(\Sigma_0\) around \(P_\ell\) does not depend on \(\ell\), but only on \(i = 1, 2\).

In Table 2 we list the local equation (l.e.) of \(\Sigma_0\) around \(P_1\) and \(P_2\). As usual, we denote by \(s = 0\) the divisor \(\Sigma_1\); \(t\) is a convenient coordinate vanishing at \(P_i\) and \(\gamma_i\) is a suitable invertible function near \(\{s = t = 0\}\).

Our assumption on the existence of a smooth Calabi–Yau resolution imposes of \(\Sigma_1\) and \(\Sigma_0\) We write “NM” or “NSR” if the intersection type, as stated in Table 1, is not compatible with our hypothesis due to the singularities being non-minimal or having no small resolution.

| Type \(\Sigma_0\) \(\cap\) \(\Sigma_1\) | \(G\) | L.e. at \(P_1\) | L.e. at \(P_2\) |
|---|---|---|---|
| \(I_1\) \(\{e\}\) | \(\gamma_1 t^4 + \gamma_2 s = 0\) | NSR |
| \(I_2\) SU(2) | \(\gamma_0 t^2 + \gamma_1 s = 0\) | transversal |
| \(I_3\) SU(3) | \(\gamma_0 t^2 + \gamma_1 s = 0\) | transversal |
| \(I_{2k}, k \geq 2\) Sp\((k)\) | \(t^2 - \gamma s^k = 0\) | transversal |
| \(I_{2k+1}, k \geq 2\) Sp\((k)\) | \(t^2(\gamma_0 t + \gamma_1 s) + \gamma_2 t s^{k+1} + \gamma_3 s^{k+2} = 0\) | NSR |
| \(I_n, n \geq 4\) SU\((n)\) | \(s^n + \gamma t^2 = 0\) | transversal |

| Type \(\Sigma_0\) \(\cap\) \(\Sigma_1\) | \(G\) | L.e. at \(P_1\) | L.e. at \(P_2\) |
|---|---|---|---|
| \(II\) \(\{e\}\) | | NSR |
| \(III\) SU(2) | \(\gamma_1 t^4 = 0\) | |
| \(IV\) Sp\((1)\) | \(\gamma t^2 + s = 0\) | |
| \(IV\) SU(3) | \(t^4 + \gamma_0 s^2 + \gamma_1 st^2 = 0\) | |
| \(I_0\) G\(_2\) | | transversal |
| \(I_0\) Spin\((7)\) | | transversal |
| \(I_0\) Spin\((8)\) | \(\gamma_0 s + \gamma_1 t^2 = 0\) | \(\gamma_0 s + \gamma_1 t^2\) |
| \(I_0\) Spin\((9)\) | | transversal |
| \(I_0\) Spin\((10)\) | \(\gamma t^2 + s = 0\) | \(\gamma t^2 + s = 0\) |
| \(I_0\) Spin\((11)\) | | transversal |
| \(I_0\) Spin\((12)\) | \(\gamma t^2 + s = 0\) | \(\gamma t^2 + s = 0\) |
| \(I^*_{n, n \geq 3}\) SO\((2n + 7)\) | | transversal |
| \(I^*_{n, n \geq 3}\) SO\((2n + 8)\) | \(\gamma t^2 + s = 0\) | \(\gamma t^2 + s = 0\) |
| \(IV^*\) E\(_4\) | \(\gamma s + t^2 = 0\) | |
| \(IV^*\) E\(_6\) | \(\gamma s + t^2 = 0\) | |
| \(III^*\) E\(_7\) | \(\gamma_0 s + \gamma_1 t^4 = 0\) | |
| \(III^*\) E\(_8\) | | NM |

Table 2.

In Table 3, \(h\) denotes the Coxeter number of the group \(G\), \(m\) the multiplicity of \(\Sigma_1\) in the discriminant, and \(\mu(f)\) (resp. \(\mu(g)\)) the vanishing of \(f\) (resp. \(g\)) in equation (1.1) along \(\Sigma_1\) (see also Section 3).
In Table 4 we write, for each Kodaira type fiber and associated group, the coefficients needed to compute \( \mathcal{R} \), as in Proposition 6.9. The general Kodaira type fiber over \( \Sigma_1 \) degenerates over both \( P_i \) at the intersection with \( \Sigma_0 \). As in Table 2 we write “NM” or “NSR” if the intersection type, as stated in Table 1, is not compatible with our hypothesis. We describe the degenerate singular fibers: if they are of Kodaira type we use Kodaira’s notation. Note that these are not necessarily the Kodaira type of the general Weierstrass surface passing through the degenerate fiber; for example in the case of \( G = E_7 \) (III\(^*\)), the degenerate fiber is again of type III\(^*\), but the general Weierstrass surface has a II\(^*\) singularity (see also Section 8.3). These distinctions are important in computing \( \mathcal{R} \) as in Theorem 8.2.

| Type  | \( G \)   | \( h \) | \( \text{rk} \) | \( m \) | \( \mu(f) \) | \( \mu(g) \) |
|-------|--------|------|-------|-----|------|------|
| \( I_1 \) | \{e\} | –    | –    | 1   | 0    | 0    |
| \( I_2 \) | SU(2) | 2    | 1    | 2   | 0    | 0    |
| \( I_3 \) | SU(3) | 3    | 2    | 3   | 0    | 0    |
| \( I_{2k}, k \geq 2 \) | \( \text{Sp}(k) \) | \( 2k \) | \( k \) | \( 2k \) | 0    | 0    |
| \( I_{2k+1}, k \geq 1 \) | \( \text{Sp}(k) \) | \( 2k \) | \( k \) | \( 2k+1 \) | 0    | 0    |
| \( I_{n,n \geq 4} \) | SU(\( n \)) | \( n \) | \( n-1 \) | \( n \) | 0    | 0    |
| \( II \) | \{e\} | –    | –    | 2   | 1    | 1    |
| \( III \) | SU(2) | 2    | 1    | 3   | 1    | 2    |
| \( IV \) | \( \text{Sp}(1) \) | 2    | 1    | 4   | 2    | 2    |
| \( IV \) | SU(3) | 3    | 2    | 4   | 2    | 2    |
| \( I_0^* \) | \( G_2 \) | 6    | 2    | 6   | 2    | 3    |
| \( I_n^*, n \geq 0 \) | SO(\( 2n+7 \)) | \( 2n+6 \) | \( n+3 \) | \( n+6 \) | 2    | 3    |
| \( I_n^*, n \geq 0 \) | SO(\( 2n+8 \)) | \( 2n+6 \) | \( n+4 \) | \( n+6 \) | 2    | 3    |
| \( IV^* \) | \( F_4 \) | 12   | 4    | 8   | 3    | 4    |
| \( IV^* \) | \( E_6 \) | 12   | 6    | 8   | 3    | 4    |
| \( III^* \) | \( E_7 \) | 18   | 7    | 9   | 3    | 5    |
| \( II^* \) | \( E_8 \) | 30   | 8    | 10  | 4    | 5    |

Table 3.
The fibers of non-Kodaira type are the branch points of an outer automorphism of the group; we denote these with “br.”.

| Type      | $G$       | $\mu_1(f, g)$ | $\mu_2(f, g)$ | $\epsilon_1$ | $\epsilon_2$ | $\chi_{top}(X_{P_1})$ | $\chi_{top}(X_{P_2})$ |
|-----------|-----------|---------------|---------------|--------------|--------------|------------------------|------------------------|
| $I_1$     | $\{e\}$  | 2             | 0             | -1           | -1           | 2 ($II$)               | NSR                    |
| $I_2$     | SU(2)     | 3             | 0             | -1           | -1           | 3 ($III$)              | 3 ($I_3$)              |
| $I_3$     | SU(3)     | 8             | 0             | -1           | -1           | 4 ($IV$)               | 4 ($I_4$)              |
| $I_{2k}, k \geq 2$ | Sp($k$)  | 3$k$          | 0             | $k - 2$      | -1           | $k + 2$ (br.)         | $2k + 1$ ($I_{2k+1}$) |
| $I_{2k+1}, k \geq 1$ | Sp($k$)  | 3$k + 3$      | 0             | $k + 2$      | -1           | $k + 2$ (br.)         | NSR                    |
| $I_n, n \geq 4$ | SU($n$) | 3$n$          | 0             | $n - 2$      | -1           | $n + 2$ ($D_n$)        | $n + 1$ ($I_{n+1}$)    |
| $II$      | $\{e\}$  | 0             |               | -1           |              | 4 ($IV$)               |                        |
| $III$     | SU(2)     | 0             |               | -1           |              | 3 ($I_3$)              |                        |
| $IV$      | Sp(1)     | 0             |               | -1           |              | 6 ($I_0$)              |                        |
| $IV$      | SU(3)     | 0             |               | 2            |              | 6 ($I_0^*$)            |                        |
| $I_0^*$   | $G_2$     | 0             |               | -1           |              | 5 (br.)                |                        |
| $I_0^*$   | Spin(7)   | 0             | 0             | -1           | -1           | 7 ($I_1^*$)            | 7 ($I_1^*$)            |
| $I_0^*$   | Spin(8)   | 0             | 0             | -1           | -1           | 7 ($I_1^*$)            | 7 ($I_1^*$)            |
| $I_1^*$   | Spin(9)   | 0             | 2             | -1           | -1           | 6 (br.)                | 8 ($IV^*$)             |
| $I_1^*$   | Spin(10)  | 0             | 2             | -1           | -1           | 8 ($I_2^*$)            | 8 ($IV^*$)             |
| $I_2^*$   | Spin(11)  | 0             | 3             | -1           | -1           | 7 (br.)                | 8 ($I_2^*$)            |
| $I_2^*$   | Spin(12)  | 0             | 3             | -1           | -1           | 9 ($I_3^*$)            | 8 ($I_2^*$)            |
| $I_n*, n \geq 3$ | SO(2$n + 7$) | 0  | NM           | -1           | NM           | $n + 5$ (br.)         | NM                    |
| $I_n*, n \geq 3$ | SO(2$n + 8$) | 0  | NM           | -1           | NM           | $n + 7$ ($I_{n+1}^*$) | NM                    |
| $IV^*$    | $F_4$     | 0             | -1           |              |              | 6 (br.)                |                        |
| $IV^*$    | $E_6$     | 0             | -1           |              |              | 9 ($III^*$)            |                        |
| $III^*$   | $E_7$     | 0             | -1           |              |              | 9 ($III^*$)            |                        |
| $II^*$    | $E_8$     | NM            | NM           |              |              | NM                    |                        |

Table 4.
APPENDIX II: THE ENTRIES IN THE ABOVE TABLES FOR $G = SU(2k), \ k \geq 2$
AND $I_{2k}$ FIBER TYPE.

We illustrate the pattern of computations needed to compile the Tables in Appendix I with the specific example $G = SU(2k)$.

The generalized Weierstrass equation has the form:

$$y^2 + a_1xy = x^3 + a_2sx^2 + a_4sx^kx + a_6s^{2k}.$$

$$b_2 = a_1^2 + 4a_2s, \ b_4 = 2a_4s^k, \ b_6 = 4a_6s^{2k},$$

$$b_8 = [(a_1^2 + 4a_2s)a_6 - a_4^2]s^{2k}, \ b_{8,2k} = (a_1^2 + 4a_2s)a_6 - a_4^2$$

$$f = \frac{1}{48}(b_2^2 - 24b_4), \ g = \frac{1}{864}(-b_2^3 + 36b_2^2s - 216b_6).$$

$$\Sigma: \ s^{2k}\{-a_1^4 + 16a_2^2s^2 + 8a_1^2a_2s\}[a_1^2 + 4a_2s]+$$

$$-8s^k[8a_1^4 + 27 \cdot 2a_6^2s^k - 9a_4a_6(a_1^2 + 4a_2s)],$$

$$\Sigma_0: \ -a_1^4 + 16a_2^2s^2 + 8a_1^2a_2s\}[a_1^2 + 4a_2s]+$$

$$-8s^k[8a_1^4 + 27 \cdot 2a_6^2s^k - 9a_4a_6(a_1^2 + 4a_2s)].$$

At the points of intersections of $\Sigma_0$ and $\Sigma_1$, either $a_1 = 0 (P^1_1)$ or $b_{8,2k} = 0 (P^1_2)$.

(There the notation of Section $\overline{3}$, $a_1 = \beta_1$.)

Remark II.1. $r_1 = 4$ and $r_2 = 1$; there are $B_1 = -K_B \cdot \Sigma_1$ points of $P_1$ type, and $(-8K_B - 2k\Sigma_1) \cdot \Sigma_1 = B_2$ points of $P_2$ type. The second condition follows from the first one, as $\Sigma_1 \cdot \Sigma_0 = 4B_1 + B_1$.

II.1. Computing $\epsilon_1$: Let $t := a_1, s$ be the local coordinates around a point $P_1^1$.

(In the notation of Section $\overline{3}$, $a_1 = \beta_1$.)

Then

$$\Sigma_0: \gamma_0t^4 + \gamma_1s^2 + \gamma_2t^2s + \gamma_3s^k,$$

where $\gamma_1$ is invertible at $s = t = 0$.

We can write

$$\Sigma_0: \gamma_0t^{2k} + \gamma_1s^2 = 0$$

which defines an $A_{2k-1}$ curve singularity. Since the blowup of an $A_{2k-1}$ curve singularity yields an $A_{2k-3}$ singularity, we have

$$#\phi^{-1}(P_1); \{\alpha^1_\phi\}) = (2; 2, \cdots 2), \ (k \text{ times}); \ \text{then} \ \epsilon_1 = 2k - 2.$$

II.2. Computing $\epsilon_2$: Since $\Sigma_0$ is smooth around each point $P_2$

$$#\phi^{-1}(P_2); \{\alpha^2_\phi\}) = (1; 1); \epsilon_2 = -1.$$
II.3. Computing $\mu(f, g)$: From the equations we see that $f$ and $g$ have a common zero along $\Sigma_1$ when $b_2 = 0$, and there $-2K_B \cdot \Sigma_1$ such points. Now set

$$g' = \frac{b_2}{18} f + g = \frac{1}{72}(-b_2 b_4) - \frac{1}{12} b_6.$$  

Then $\mu(f, g) = \mu(f, g')$ [14, Section 1].

From the equation above we see that $P \in \Sigma_1$ is a common zero of $f$ and $g$ if and only if $a_1 = 0$. As in [11] we take $t := a_1, s$ as the local coordinates around $P$. 

$$\mu(f, g) = \dim C\left[[s, t]/(f, g')\right],$$  

for suitable invertible functions $\gamma_i$ (around $s = t = 0$). Then [14, Ex. 1.2.5] 

$$\mu(f, g) = 6k.$$  

II.4. Computing $\chi_{\text{top}}(X_{P_1})$. After $\ell$ blowups the Weierstrass equation becomes: 

$$y^2 + a_1 xy = x^3 s^\ell + a_2 s x^2 + a_4 x s^{k-\ell} + a_6 s^{2k-2\ell},$$  

and there are isolated singular points (nodes) on the fiber at $P_1 = 0$. These points can be blown up with small resolutions: the fiber over the points $P_1$ is of Kodaira type $D_{2k}$ and $\chi_{\text{top}}(X_{P_1}) = 2k + 2$.

II.5. Computing $\chi_{\text{top}}(X_{P_2})$. $\Sigma_0$ and $\Sigma_1$ intersect transversally at $P_2$, and it is easy to see that the corresponding fiber $X_{P_2}$ is of type $I_{2k+1}$ and $\chi_{\text{top}}(X_{P_2}) = 2k + 1$.

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