ON DEGENERATIONS OF PROJECTIVE VARIETIES TO COMPLEXITY-ONE T-VARIETIES

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Abstract. Let $R$ be a positively graded finitely generated $k$-domain with Krull dimension $d + 1$. We show that there is a homogeneous valuation $v : R \setminus \{0\} \to \mathbb{Z}^d$ of rank $d$ such that the associated graded $\text{gr}_v(R)$ is finitely generated. This then implies that any polarized $d$-dimensional projective variety $X$ has a flat deformation over $\mathbb{A}^1$, with reduced and irreducible fibers, to a polarized projective complexity-one $T$-variety (i.e. a variety with a faithful action of a $(d - 1)$-dimensional torus $T$). As an application we conclude that any $d$-dimensional complex smooth projective variety $X$ equipped with an integral Kähler form has a proper $(d - 1)$-dimensional Hamiltonian torus action on an open dense subset that extends continuously to all of $X$.

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Introduction

Throughout $k$ denotes an algebraically closed field of characteristic 0. In this paper we show that if $R$ is a finitely generated positively graded domain of Krull dimension $d + 1$ then there is a homogeneous valuation $v : R \setminus \{0\} \to \mathbb{Z}^d$ of rank $d$ (that is, one less than the Krull dimension) such that the corresponding associated graded

$$\text{gr}_v(R) = \bigoplus_{a \in \mathbb{Z}^d} R_{\mathbf{v} \geq a} / R_{\mathbf{v} > a}$$

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is finitely generated (Theorem 2 below). In other words, \((R, v)\) has a finite Khovanskii basis (in the sense of [KM19, Section 2]).

Taking Proj, the above then implies the following: Let \(L\) be any ample line bundle on a projective variety \(X\). Then, after going to a sufficiently high tensor power if needed, \(L\) can be deformed, in a flat family with reduced and irreducible fibers, to an equivariant ample line bundle \(L'\) on a projective complexity-one \(T\)-variety \(X'\) (see Definition \(\mathfrak{1}\) and Corollary \(\mathfrak{3}\) below). We recall that a \(d\)-dimensional variety is a complexity-one \(T\)-variety if it possesses a faithful action of the algebraic torus \(T = \mathbb{G}_m^{d-1}\) (notice that we do not require the variety to be normal).

From the results in [HK15] one then concludes that any complex \(d\)-dimensional smooth projective variety with an integral Kähler form admits a faithful \((d-1)\)-dimensional Hamiltonian torus action on a dense open subset that extends continuously to the whole variety (Theorem 4 below).

It is well-known that if \(X \subset \mathbb{P}^N \times \mathbb{A}^1 \to \mathbb{A}^1\) is a flat family of schemes embedded in a projective space \(\mathbb{P}^N\) then the Hilbert polynomials of fibers are all the same and thus, in particular, all fibers have the same degree. This illustrates why flat deformations are such useful tools in algebraic geometry, specially when dealing with intersection theoretic data. In our setting, the important and key requirement that fibers are reduced guarantees that we do not need to worry about multiplicities and embedded components at the special fiber. Whenever there is a degeneration of a variety to a variety with a torus action one can gain insight into the geometry of the variety using the torus action. The largest torus action is when the special fiber is a toric variety. Such a degeneration is usually called a toric degeneration. Toric degenerations of different special classes of varieties have been constructed and studied by many authors during the past few decades. See for example [FFL17] for a nice survey of toric degenerations of flag varieties and connections with representation theory, and [AB04, Kav15] for toric degenerations of spherical varieties.

It is an important and fundamental question whether an ample line bundle on a projective variety has a flat deformation, with reduced and irreducible fibers, to an equivariant ample line bundle on a (not necessarily normal) projective toric variety (i.e. a toric degeneration).

• The main result of the present paper (Theorem 2 and Corollary 3) shows that the answer to this question is almost affirmative in the sense that we can always degenerate to a complexity-one \(T\)-variety.

• On the other hand, we give examples of line bundles on projective curves that do not have any toric degenerations. More precisely we show the following: Let \(D\) be a very ample divisor on a smooth complex projective curve \(X\) with genus \(> 1\). Then the corresponding line bundle \(L = O(D)\) has a toric degeneration if and only if some multiple of \(D\) is equivalent to a multiple of a point (see Section 3 and Propositions 3.1 and 3.2).

In addition, we show that any toric degeneration (in the sense of Definition 1 below) comes from a valuation with a finite Khovanskii basis (Theorem 1.11). In regard to examples, in [IW18, Corollary 3.14] it is also shown that the projective coordinate ring of a very general integral rational plane curve of degree greater than 3 has no homogeneous valuation with a finite Khovanskii basis and hence has no toric degeneration by our Theorem 1.11.

Toric varieties can be classified using combinatorial and convex geometric data such as lattice points and convex polytopes. In the same spirit, there is a classification of normal
complexity-one $T$-varieties using the so-called polyhedral divisors (see [AIPSV12]). Normal complexity-one $T$-varieties over a rational curve are particularly nice and admit many degenerations to toric varieties (see [IM19]).

Let us start by giving a precise definition of the notion of degeneration that we use throughout the paper. Let $R$, $R'$ be finitely generated positively graded $k$-domains with Krull dimension $d+1$. Let $X = \text{Proj}(R)$, $X' = \text{Proj}(R')$ be the corresponding $d$-dimensional projective varieties equipped with sheaves of modules $\mathcal{O}_X(n)$ and $\mathcal{O}_{X'}(n)$, $n \in \mathbb{N}$, respectively.

**Definition 1.** By a flat degeneration (or degeneration for short) of $X$ to $X'$ we mean a $\mathbb{G}_m$-equivariant flat family $\pi : \mathcal{X} \to \mathbb{A}^1$ of varieties with reduced fibers and equipped with sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X(n)$ such that:

(a) The fibers $X_t := \pi^{-1}(t)$, $t \neq 0$, are all isomorphic to $X$ and the special fiber $X_0 := \pi^{-1}(0)$ is isomorphic to $X'$. Note that the $\mathbb{G}_m$-equivariance of the family then implies that $\mathcal{X} \setminus X_0$ is a trivial family with fiber $X$.

(b) Under the isomorphism $X_1 \cong X$, the sheaf $\mathcal{O}_X(n)|_{X_1}$ coincides with $\mathcal{O}_X(n)$, for all $n \in \mathbb{N}$.

(c) Under the isomorphism $X_0 \cong X'$, the sheaf $\mathcal{O}_X(n)|_{X_0}$ coincides with $\mathcal{O}_{X'}(n)$, for all $n \in \mathbb{N}$.

We write $X \rightsquigarrow X'$ when there is a degeneration of $X$ to $X'$. We call a degeneration as above a toric degeneration of $X$ if in addition $X'$ is a (not necessarily normal) toric variety and the $\mathcal{O}_{X'}(n)$ are torus equivariant.

We remark that our definition of a flat degeneration above is very close to the notion of test configuration from Kähler geometry.

Let $R$ be a finitely generated positively graded algebra and domain with $\dim(R) = d+1$ and $R_0 = k$. The main result of the paper which enables us to construct degenerations to complexity-one $T$-varieties is the following (Theorem 5.1):

**Theorem 2.** There exists a homogeneous rank $d$ valuation $\nu : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-1}$ such that the associated graded $\text{gr}_\nu(R)$ is finitely generated (in other words, $(R, \nu)$ has a finite Khovanskii basis). Here the first component of $\nu$ is the degree with respect to the grading of $R$, also $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-1}$ is ordered lexicographically and the ordering on $\mathbb{Z}_{\geq 0}$ is the reverse of its natural ordering.

Taking Proj and in light of [An12, Proposition 5.1] we obtain the following:

**Corollary 3.** Let $X \subset \mathbb{P}^N$ be a projective variety with homogeneous coordinate ring $R = k[X]$. Then $X$ can be degenerated to a complexity-one $T$-variety $X'$ via a $\mathbb{G}_m$-equivariant family $\pi : \mathcal{X} \to \mathbb{A}^1$ (as in Definition 1). Moreover, the family $\mathcal{X}$ can be embedded in $W \mathbb{P} \times \mathbb{A}^1$ where $W \mathbb{P}$ is a weighted projective space and the following holds:

1. The weighted projective space $W \mathbb{P} = \mathbb{P}(1, \ldots, 1, \lambda_1, \ldots, \lambda_d)$ contains $\mathbb{P}^N = \mathbb{P}(1, \ldots, 1)$.
2. $\pi : \mathcal{X} \to \mathbb{A}^1$ is given by the projection on the second factor $W \mathbb{P} \times \mathbb{A}^1 \to \mathbb{A}^1$.
3. The $\mathbb{G}_m$-action on $\mathcal{X}$ is the restriction of a linear $\mathbb{G}_m$-action on $W \mathbb{P} \times \mathbb{A}^1$.
4. The sheaves $\mathcal{O}_X(n)$ are all induced from the Serre sheaves $\mathcal{O}_{W \mathbb{P}}(n)$ on the weighted projective space via the embedding $\mathcal{X} \hookrightarrow W \mathbb{P} \times \mathbb{A}^1$.
5. The $T$-action on $X_0 \cong X'$ comes from a linear $T$-action on $W \mathbb{P} = W \mathbb{P} \times \{0\}$.

**Remark.** It seems that the proofs of Theorem 2 and Corollary 3 work over a characteristic $p$ base field as well but the authors have not checked all the details.
**Remark.** If $R$ is Cohen-Macaulay, there is a relatively simple construction of a complexity-one degeneration of $X = \text{Proj}(R)$. If moreover, $X$ is assumed to be smooth and the genus of the curve $C = V(f_1, \ldots, f_{d-1})$, where the $f_i \in R_1$ are in general position, is zero or one then we can complete our complexity-one degeneration to a toric degeneration (see Section 4).

Below we briefly mention some applications of our main result.

**Hamiltonian torus actions.** Combining the main result in [HK15] and Corollary 3 we conclude the following general result about constructing Hamiltonian torus actions on arbitrary smooth projective varieties.

**Theorem 4.** Every smooth projective variety of dimension $d$ equipped with an integral Kähler form has a proper faithful $(d - 1)$-dimensional Hamiltonian torus action on a dense open subset that extends continuously to the whole variety. Moreover, the image of the moment map is the standard simplex.

In a series of papers, Karshon and Tolman classify proper complexity-one Hamiltonian torus actions (see [KTL4]). In their terminology, the Hamiltonian action in Theorem 4 is tall.

Corollary 3 is expected to have applications in proving general results in geometric quantization theory. To this end one needs to extend some known results about toric varieties to complexity-one $T$-varieties. More specifically, building on the work [HK15], in [HHK] it is shown that given a smooth projective variety together with a quantization data as well as a toric degeneration, the “real polarization" converges to the “Kähler polarization", in a sense that is made precise in there.

**Tropical geometry and existence of prime cones.** Let $A$ be a finitely generated positively graded $k$-domain presented as $A \cong k[x_1, \ldots, x_n]/I$ where $I$ is a homogeneous ideal. It is well-known that tropical variety trop($I$) of the ideal $I$ is the support of a fan in $\mathbb{R}^n$ of pure dimension $d = \dim(A)$. A cone in such a fan is called a prime cone if the corresponding initial ideal of $I$ is a prime ideal. It is a problem of great importance to understand prime cones in a presentation of $A$. In particular one would like to know whether $A$ has a presentation such that the corresponding tropical variety has a prime cone, and more strongly a prime cone of maximal dimension. This is already a very difficult question even in specific examples such as the Plücker algebra of Grassmannian $\text{Gr}(k, N)$ for $k \geq 3$ (see [MS] as well as [BLMM17]). It is known that maximal cones in the tropical variety of $\text{Gr}(2, N)$ are all prime. Prime cones give rise to degenerations of $\text{Proj}(A)$ by degenerating $I$ to its initial ideal. In particular, a maximal prime cone gives a toric degeneration.

The main result of the present paper combined with [KM19] Theorem 2 gives the following strong existence result:

**Theorem 5.** Let $\{f_1, \ldots, f_r\}$ be a set of algebra generators for a $k$-domain $A$. Then this set can be enlarged to $\{f_1, \ldots, f_s\}$ such that the ideal $I$ of relations among the $f_i$ has a prime cone of codimension at most 1 (i.e. a prime cone of dimension $d - 1$ or $d$).

**Computational algebra.** The notion of a Khovanskii basis far generalizes that of a SAGBI basis (Subalgebra Analogue of Gröbner Basis for Ideals) for subalgebras of a polynomial algebra to arbitrary domains equipped with a valuation [KM19 Section 2]. This theory allows to extend the scope of Gröbner basis methods for polynomial rings to a far larger class of algebras. While there are examples of subalgebras that do not have a finite SAGBI basis, our Theorem 2 shows that any positively graded finitely generated domain has a
valuation with rank at most one less than the Krull dimension which possesses a finite Khovanskii basis.

**Remark.** The general theory of Newton-Okounkov bodies associates a convex body $\Delta \subset \mathbb{R}^d$ (called a Newton-Okounkov body) to the algebra of sections $R$ of a line bundle $L$ on a $d$-dimensional projective variety $X$ (see [KKh12, LM09]). More generally, this construction associates convex bodies to a positively graded algebra $R$. The dimension and volume of the convex body $\Delta$ encode asymptotic information about the Hilbert function of $R$ (see [KKh12]). The construction of a Newton-Okounkov body $\Delta$ requires the extra choice of a full rank valuation $v' : k(X) \setminus \{0\} \to \mathbb{Z}^d$. The valuation $v'$ can be naturally extended to a valuation $v : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ where the first component is given by the grading of $R$. The convex body $\Delta$ is constructed from the value semigroup $S = S(R,v)$ of values of $v$ on $R \setminus \{0\}$. When this semigroup is finitely generated, the convex body $\Delta$ is a rational convex polytope.

**Remark.** We would like to mention [AKL14, Proposition 14] which gives a sufficient condition for the existence of a full rank valuation with a finitely generated value semigroup for the algebra of sections of a very ample line bundle on a smooth projective variety (which then implies the existence of a toric degeneration of this variety). This is related to our Proposition 3.1. As they point out this condition is far from necessary and often does not hold. Unlike the present paper, [AKL14] does not consider non-full rank valuations or non-full dimensional torus actions on the special fiber. For a non-full rank valuation the finite generation of the associated graded algebra is a much stronger condition than the finite generation of the value semigroup and is a source of technicalities appearing in the proof of our main theorem. Also we point out that in our main result (Corollary 3) we do not assume that the variety is smooth.

**Remark.** The recent work [IW18] among other things gives tools for checking if a complexity-one variety has a toric degeneration. In [IW18, Theorem 1.1], they show that verifying this property for a given valuation on a complexity-one variety essentially reduces to a check in the curve case. Moreover, if the complexity-one degeneration is rational, [IW18, Theorem 4.1] says one must only check this property holds for a finite number of rational curves. For this latter case see also [IW18, Theorem 3.5].

Finally we discuss the organization of the paper and outline of the proof of the main results. Section 1 covers some needed background material on filtrations, valuations, Rees algebras and symbolic powers of ideals. Section 1.4 talks about Khovanskii bases and degenerations coming from valuations. A rather new result in this section is Theorem 1.11 that shows toric degenerations necessarily come from full rank valuations.

In Section 2 we prove the main step in the proof of Theorem 2 namely we show that if $X = \text{Proj}(R)$ is a projective variety of dimension $d$ then there is a sequence of degenerations $X_0 = X \leadsto X_1 \leadsto \cdots \leadsto X_{d-1}$ that degenerates $X$ to a complexity-one $T$-variety $X' = X_{d-1}$. We call this degeneration in stages. The main idea behind degeneration in stages is the Bertini irreducibility theorem. The key technical lemma in the proof of degeneration in stages is a statement about finiteness of symbolic Rees algebras of certain height one prime ideals. This is proved in the appendix (Section 5).

In Section 3 we consider the case where $X$ is a projective curve. We give examples of line bundles on curves that do not have any toric degenerations. Section 4 discusses the particularly nice case where $R$ is Cohen-Macaulay (or in other words, $X$ is arithmetically Cohen-Macaulay). In this case, by a well-known theorem of Rees about associated graded
algebra of an ideal generated by a regular sequence (Theorem 1.1), we can construct a degeneration of $X$ to a certain complexity-one $T$-variety which is a compactification of a trivial vector bundle over a curve.

In Section 3 we show that a sequence of degenerations (as constructed in Section 2) gives rise to a rank $d$ valuation on $R$ with finitely generated associated graded algebra (in other words with a finite Khovanskii basis). This in turn gives a degeneration of $X$ to a complexity-one $T$-variety (in one step) proving Theorem 2.

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1. Preliminaries on valuations, Rees algebras and symbolic powers

Let $R$ be a finitely generated $k$-domain. In this section we discuss the Rees algebra and associated graded corresponding to a filtration, and in particular to a valuation, on $R$.

1.1. Generalities on multiplicative filtrations. A multiplicative filtration $\mathcal{F} = (F_i)_{i \in \mathbb{Z}}$ in $R$ is a descending sequence of vector subspaces $\cdots \supseteq F_{i-1} \supseteq F_i \supseteq F_{i+1} \supseteq \cdots$ in $R$ such that for all $i, j \geq 0$ we have $F_i F_j \subseteq F_{i+j}$. We also assume that $F_i = R$ for all $i \leq 0$. Given a multiplicative filtration $\mathcal{F}$ on $R$, one defines its corresponding associated graded $\operatorname{gr}_\mathcal{F}(R)$ by:

$$\operatorname{gr}_\mathcal{F}(R) = \bigoplus_i F_i / F_{i+1}.$$ 

Also one defines its Rees algebra to be the algebra:

$$A_{\mathcal{F}}(R) = \bigoplus_i F_i.$$ 

The multiplicativity assumption on $\mathcal{F}$ guarantees that these are actually algebras. We note that the Rees algebra $A_{\mathcal{F}}(R)$ is moreover a $k[t]$-module, where $k[t]$ is the polynomial algebra in one indeterminate $t$. The $k[t]$-module structure is defined as follows: for $f \in F_i$ we let $t \cdot f$ to be $f$ regarded as an element of $F_{i-1}$. One thinks of the Rees algebra $A_{\mathcal{F}}(R)$ as a family of algebras that deform the algebra $\overline{R}$ to the associated graded $\operatorname{gr}_\mathcal{F}(R)$ (see [Tei99, Proposition 2.2]).

Let $0 \neq f \in R$ and let $i \in \mathbb{Z}_{\geq 0}$ be such that $f \in F_i$ but $f \notin F_{i+1}$. We denote the image of $f$ in $F_i / F_{i+1} \subset \operatorname{gr}_\mathcal{F}(R)$ by $\bar{f}$. One verifies that $f \mapsto \bar{f}$ is a multiplicative homomorphism from $R \setminus \{0\}$ to the multiplicative set of homogeneous elements of $\operatorname{gr}_\mathcal{F}(R)$.

One can give examples of a filtration $\mathcal{F}$ on a finitely generated algebra $R$ such that the Rees algebra $A_{\mathcal{F}}(R)$ and the associated graded $\operatorname{gr}_\mathcal{F}(R)$ are not finitely generated algebras.

A main source of examples of multiplicative filtrations on an algebra are powers of ideals. Let $I \subseteq R$ be an ideal. It is clear that the sequence of subspaces $F_i = I^i$ (where $I^0 = R$ for $i \leq 0$) is a multiplicative filtration $\mathcal{F}$ on $R$. We denote the corresponding associated graded and the Rees algebra by $\operatorname{gr}_\mathcal{F}(R)$ and $A_{\mathcal{F}}(R)$ respectively. It is well-known that these algebras are finitely generated. In algebraic geometry language, $\operatorname{Spec}(\operatorname{gr}_\mathcal{F}(R))$ is the normal cone to the subscheme corresponding to $I$ in $\operatorname{Spec}(R)$. The scheme $\operatorname{Spec}(A_{\mathcal{F}}(R))$ gives the deformation of $\operatorname{Spec}(R)$ into the normal cone $\operatorname{Spec}(\operatorname{gr}_\mathcal{F}(R))$ and is related to the notion of blowup along the ideal $I$ (see [Ful98, Chapter 5]). When $R$ is a positively graded algebra
one can take Proj of these algebras to get the projective version of deformation to normal cone.

While the algebras $gr_{\mathcal{F}}(R)$ and $A_{\mathcal{F}}(R)$ are always finitely generated, it is not difficult to come up with an example in which $gr_{\mathcal{F}}(R)$ is not an integral domain (see Example 1.6). Since we are interested in degenerating a given variety to another variety with a torus action, we would like to have filtrations for which the associated graded is a domain. For this reason we consider filtrations associated to valuations.

More generally, one defines a multiplicative filtration indexed by an ordered group. By an ordered group we mean an abelian group $\Gamma$ (written additively) equipped with a total order $>$ which respects the group operation, i.e. for $a, b, c \in \Gamma$ we have $a > b$ implies that $a + c > b + c$. A multiplicative filtration indexed by $\Gamma$ is a collection $\mathcal{F} = (F_a)_{a \in \Gamma}$ of vector subspaces of $R$ such that $F_a \subset F_b$ whenever $a > b$ (i.e. the filtration is decreasing). Moreover, for $a, b \in \Gamma$ we have $F_a F_b \subset F_{a+b}$. We assume that $F_a = R$ for $a < 0$, where $0$ denotes the identity element in $\Gamma$.

Similar to filtrations indexed by $\mathbb{Z}$, we define the associated graded of $\mathcal{F}$ by $gr_{\mathcal{F}}(R) = \bigoplus_{a \in \Gamma} F_a/F_{>a}$. Here $F_{>a} = \sum_{b > a} F_b$. We also define its Rees algebra to be $A_{\mathcal{F}}(R) = \bigoplus_{a \in \Gamma} F_a$. It follows from the definition that the algebras $gr_{\mathcal{F}}(R)$ and $A_{\mathcal{F}}(R)$ are graded by the group $\Gamma$.

Suppose for any $0 \neq f \in R$ the set $\{b \mid f \in F_b\}$ has a maximum $a$. Then $f \in F_a$ but $f \notin F_{>a}$. Similar to before, we denote the image of $f$ in $F_a/F_{>a} \subset gr_{\mathcal{F}}(R)$ by $\bar{f}$. Again $f \mapsto \bar{f}$ is a multiplicative homomorphism from $R \setminus \{0\}$ to the multiplicative set of homogeneous elements of $gr_{\mathcal{F}}(R)$.

1.2. Valuations. We recall some generalities about valuations and their associated filtrations. Let $R$ be a $k$-algebra and domain, and let $\Gamma$ be an ordered group. Throughout the paper, the ordered groups we work with are $\mathbb{Z}$ or $\mathbb{Q}$ (with their standard ordering) or $\mathbb{Z}^r, \mathbb{Q}^r$ (for some $r > 0$) and equipped with some total order (often a lexicographic order).

**Definition 1.1.** A function $\nu : R \setminus \{0\} \to \Gamma$ is a quasivaluation on $R$ if the following hold:

1. For all $0 \neq f, g, f + g \in R$ we have $\nu(f + g) \geq \min(\nu(f), \nu(g))$.
2. For all $0 \neq f, g \in R$ we have $\nu(fg) \geq \nu(f) + \nu(g)$.
3. For all $0 \neq f \in R$ and $0 \neq c \in k$ we have $\nu(cf) = \nu(f)$.

If for all $0 \neq f, g \in R$ we have $\nu(fg) = \nu(f) + \nu(g)$ then $\nu$ is called a valuation.

It is sometimes useful to define a quasivaluation to be a map $\nu : R \to \Gamma \cup \{\infty\}$ satisfying the above axioms, where $\infty$ is greater than all elements in $\Gamma$.

In the case that $\nu$ is a valuation, the image of $R \setminus \{0\}$ under $\nu$ is a subsemigroup of the value group $\Gamma$; we call this the value semigroup of $(R, \nu)$, and we denote it by $S(R, \nu)$.

In this paper we only work with discrete valuations, that is, the image of the valuation is a discrete subset of $\mathbb{Q}^r$. The rank (also called the rational rank) of a valuation is the rank of the subgroup generated by the image of the valuation. One shows that the rank of a valuation $\nu : R \setminus \{0\} \to \mathbb{Q}^r$ is less than or equal to $\dim(R)$, the Krull dimension of $R$ (see [SZ60], Chapter VI). We call $\nu$ a full rank valuation if the rank of $\nu$ is equal to $\dim(R)$.

Let $\mathcal{F} = (F_a)_{a \in \Gamma}$ be a multiplicative filtration. Define the function $\nu_{\mathcal{F}} : R \to \Gamma \cup \{\infty\}$ as follows: for $0 \neq f \in R$,

1. $\nu_{\mathcal{F}}(f) = \max\{a \mid f \in F_a\}$.

If the maximum is not attained we let $\nu_{\mathcal{F}}(f) = \infty$, in particular, $\nu(0) = \infty$. One verifies that $\nu_{\mathcal{F}}$ is a quasivaluation on the algebra $R$. Conversely, if $\nu : R \to \Gamma \cup \{\infty\}$ is a quasivaluation,
one can define a filtration $F_v = (F_{v \geq a})_{a \in \Gamma}$ as follows. For each $a$, let:

$$F_{v \geq a} = \{0 \neq f \in R \mid v(f) \geq a\} \cup \{0\}.$$ 

One checks that $F_v$ is indeed a multiplicative filtration, and moreover, the operations of $\mathcal{F} \mapsto \mathcal{F}_v$ and $v \mapsto F_v$ are inverse of each other.

**Example 1.2.** Suppose $R = \bigoplus_{a \in \Lambda} R_a$ is an algebra graded by an ordered abelian group $\Lambda$. Suppose the support semigroup $S(R, \Gamma) = \{a \in \Gamma \mid R_a \neq \{0\}\}$ is maximum well-ordered. For $f = \sum_a f_a \in R$ define $v(f) = \min\{a \mid f_a \neq 0\}$. Then $v : R \setminus \{0\} \to \Gamma$ is a valuation with value semigroup $S(R, \Gamma)$. A common example of this situation is when $R$ is a positively graded algebra.

Letting $R = \bigoplus_{a \in \Lambda} R_a$ be an algebra graded by an abelian group $\Lambda$, we can consider any homomorphism $\pi : \Lambda \to \Gamma$ of abelian groups, where $\Gamma$ is an ordered group. The function $v_\pi : R \setminus \{0\} \to \Gamma$ which sends $f = \sum_a f_a$ to $\min\{\pi(a) \mid f_a \neq 0\}$ is easily checked to define a valuation on $R$. We call functions which arise in this way grading functions.

More generally, let $R = \bigoplus_{a \in \Lambda} R_a$ be an algebra graded by an abelian group $\Lambda$ and let $v : R \setminus \{0\} \to \Gamma$ be a valuation. We say that $v$ is homogeneous with respect to the $\Lambda$-grading if for any $f = \sum_a f_a \in R$ we have:

$$v(f) = \min\{v(f_a) \mid f_a \neq 0\}.$$ 

Given a quasivaluation $v$ on $R$ we denote the Rees algebra and the associated graded algebra corresponding to the filtration $F_v$ by $A_v(R)$ and $\text{gr}_v(R)$ respectively. The following propositions are straightforward to verify from the definitions.

**Proposition 1.3.** Let $\mathcal{F}$ be a multiplicative filtration on a domain $R$ with corresponding quasivaluation $v_\mathcal{F}$ (given by (1)). Then $\text{gr}_\mathcal{F}(R)$ is a domain if and only if $v_\mathcal{F}$ is a valuation, that is, for any $0 \neq f, g \in R$ we have $v_\mathcal{F}(fg) = v_\mathcal{F}(f) + v_\mathcal{F}(g)$.

**Proposition 1.4.** Suppose $R$ is graded by an abelian group $\Lambda$ as above, and $v$ is homogeneous with respect to this grading. Then the algebras $A_v(R)$ and $\text{gr}_v(R)$ are graded by $\Lambda$, and the support semigroups (see Example 1.2: $S(R, \Lambda)$, $S(A_v(R), \Lambda)$ and $S(\text{gr}_v(R), \Lambda)$ all coincide. Furthermore, if $v$ is a grading function, the algebras $\text{gr}_v(R)$ and $R$ are seen to be isomorphic as $\Lambda$-graded algebras.

1.3. **Symbolic powers and symbolic normal cone.** It is natural here to mention the notion of symbolic powers of an ideal. Let $p \subset R$ be a prime ideal. For $n \geq 0$, the $n$-th *symbolic power* $p^{(n)}$ is by definition the smallest $p$-primary ideal that contains $p^n$, also for $n \leq 0$ we put $p^{(n)} = R$ (see [Ei04, Section 3.9] for more about the notion of symbolic power). One sees that the symbolic powers $(p^{(n)})_{n \in \mathbb{Z}}$ form a multiplicative filtration. We denote the corresponding Rees algebra and associated graded by $A_{p^{(\cdot)}}(R)$ and $\text{gr}_{p^{(\cdot)}}(R)$ and call them the *symbolic Rees algebra* and *symbolic associated graded of $p$* respectively. In analogy with the usual normal cone, we call $\text{Spec}(\text{gr}_{p^{(\cdot)}}(R))$, the *symbolic normal cone* to the subscheme defined by $p$ in $\text{Spec}(R)$. Now, suppose that the local ring $R_p$ is a discrete valuation ring and let $v$ be the corresponding valuation. One shows that the $n$-th symbolic power is given by the $n$-th subspace in the filtration $\mathcal{F}_v$, that is, $p^{(n)} = F_{v,n} = \{f \in R \mid v(f) \geq n\}$.

It is important to point out that the algebra $A_{p^{(\cdot)}}(R)$ could be non-finitely generated. Some interesting examples of this situation can be found in [Cut88, Lemma 6.1] shows finite generation of $A_{p^{(\cdot)}}(R)$ for certain prime ideals of height 1.

To illustrate the above concepts, below we give an example of the associated graded with respect to powers of a prime ideal versus the associated graded with respect to its symbolic
powers. To facilitate the computation we state a lemma first. Let $\mathcal{F}$ be a filtration on $R$. Take $f \in R$ and let $n = \nu_{\mathcal{F}}(f)$, that is, $f \in F_n$ but $f \notin F_{n+1}$. Provided that $n \neq \infty$, we denote the image of $f$ in $F_n/F_{n+1} \subset \text{gr}_{\mathcal{F}}(R)$ by $\bar{f}$ and call it the initial form of $f$ with respect to the filtration $\mathcal{F}$. The next lemma is well-known and straightforward to prove (see [Ei01, Exercise 5.3]).

**Lemma 1.5.** Let $R$ be an algebra and $J \subset I$ ideals in $R$. Then the associated graded $\text{gr}_{I/J}(A/J)$ is naturally isomorphic to the quotient algebra $\text{gr}_I(A)/\text{in}(J)$, where $\text{in}(J)$ is the ideal generated by the initial forms of elements of $J$ with respect to the filtration given by the powers of $I$.

**Example 1.6** (Normal cone and symbolic normal cone for an elliptic curve). Let $S = k[x, y, z]$ and consider the algebra $R = S/(f)$ where

$$f(x, y, z) = y^2z - x^3 + xz^2.$$ 

The algebra $R$ is the homogeneous coordinate ring of the elliptic curve $X \hookrightarrow \mathbb{P}^2$ defined by the homogeneous polynomial $f$. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the images of $x, y, z$ in $R$. Consider the prime ideal $\mathfrak{p} = (\tilde{z}) = (\tilde{x}, \tilde{z}) \subset R$. Note that $\text{gr}_{(x, z)}(S)$ is naturally isomorphic to $S$. We have $f \equiv 0 \mod (x, z)$ and $f \equiv y^2z \mod (x, z)^2$. This implies that the initial form $\tilde{f}$ of $f$, with respect to the ideal $\mathfrak{p}$, is $y^2z$. By Lemma 1.5, we have:

$$\text{gr}_{\mathfrak{p}}(R) \cong S/(y^2z).$$

Let $\text{gr}_{\mathfrak{p}^{(1)}}(R) = \bigoplus_{n=0}^{\infty} \mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)}$ be the symbolic associated graded of $\mathfrak{p}$. It is $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0})$-graded with the first $\mathbb{Z}_{\geq 0}$-grading inherited from $R$ and the second one given by the direct sum. We have $y^2\tilde{z} = \tilde{x}^3 - \tilde{x}\tilde{z}^2 \in \mathfrak{p}^3$ and so $\tilde{z} \in \mathfrak{p}^{(3)}$. One observes that $\tilde{z} \notin \mathfrak{p}^{(4)}$ (because $\tilde{z}$ vanishes of order 3 at the point $(0 : 1 : 0)$ on $X$) and so with respect to the $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0})$-grading, the bidegree of $\tilde{z}$ is $(1, 3)$. Similarly, the bidegrees of $\tilde{x}$ and $\tilde{y}$ generate the algebra $\text{gr}_{\mathfrak{p}^{(1)}}(R)$. Let $R' \subset \text{gr}_{\mathfrak{p}^{(1)}}(R)$ be the subalgebra generated by the images of $\tilde{x}, \tilde{y}, \tilde{z}$. On the one hand, for $n \geq 0$, we have:

$$\dim_k(R'_n) = |\{n_1(1, 3) + n_2(1, 1) + n_3(1, 0) \mid n_1 + n_2 + n_3 = n, n_i \geq 0\}| \quad = 3n.$$ 

On the other hand, using the Riemann-Roch theorem, one computes that $\dim_k(\text{gr}_{\mathfrak{p}^{(1)}}(R))_n = \dim_k R_n = 3n$. This shows that $R' = \text{gr}_{\mathfrak{p}^{(1)}}(R)$ as claimed. Hence we have:

$$\text{gr}_{\mathfrak{p}^{(1)}}(R) \cong S/(y^2z - x^3).$$

Geometrically, $X$ degenerates to the cuspidal cubic curve $y^2z = x^3$ (cf. An12, Example 4.2).

**Remark 1.7** (Balanced normal cone). Although not needed here in this paper, we would like to mention another variant of the normal cone, namely the balanced normal cone (see [Kui05]). Let $R$ be a Noetherian ring and $I \subset R$ an ideal. Let $v : R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be the quasivaluation associated to the filtration by powers of $I$, that is, for $f \in R$, $v(f) = n$ if $f \in I^n \setminus I^{n+1}$. If there is no such $n$, $v(f) = \infty$. Then define $\overline{v}$ by

$$\overline{v}(f) = \lim_{n \to \infty} \frac{v(f^n)}{n}.$$ 

Samuel proved that the above limit exists and thus $\overline{v}(f)$ is well-defined. It is also known that $\overline{v}$ has values in $\mathbb{Q}$. One sees that $\overline{v}$ is a quasivaluation on $R$. Hence one can form the
associated graded ring:

$$\text{gr}_I(R) = \bigoplus_{n \in \mathbb{Q}} \{ x \in R \mid v(f) \geq n \} / \{ f \in R \mid v(f) > n \}. $$

Nagata showed that $$\sup_{f \in R} |v(f) - v(f)| < \infty.$$ From this and the inequality $$v(f) \geq v(f),$$ $$\forall f \in R,$$ it follows that $$\text{gr}_I(R)$$ is a Noetherian ring. The scheme $$\text{Spec}(\text{gr}_I(R))$$ is called the balanced normal cone for the subscheme of $$\text{Spec}(R)$$ defined by $$I.$$ Two important properties of $$\text{gr}_I(R)$$ are the following: (1) It is a reduced ring. (2) The kernel of the natural homomorphism $$\text{gr}_I(R) \to \text{gr}_I(R)$$ coincides with the nilradical of $$\text{gr}_I(R),$$ and moreover, when the kernel vanishes, we have $$\text{gr}_I(R) = \text{gr}_I(R).$$

1.4. Khovanskii bases and flat degenerations. Let $$R$$ be a finitely generated positively graded domain and let $$v : R \setminus \{0\} \to \mathbb{Z}^+$$ be a homogeneous valuation.

**Definition 1.8** (Khovanskii basis [KM19]). A subset $$\mathcal{B} \subset R$$ is called a Khovanskii basis for $$(R,v)$$ if the image of $$\mathcal{B}$$ in $$\text{gr}_v(R)$$ forms a set of algebra generators.

One can show that if $$v$$ has full rank and the base field is algebraically closed, $$\mathcal{B}$$ is a Khovanskii basis if $$v(\mathcal{B}) = \{ v(f) \mid f \in \mathcal{B} \}$$ generates $$S(R,v)$$ as a semigroup.

Given a Khovanskii basis one can do algebra operations in $$R$$ algorithmically, in particular one can represent any element of $$R$$ as a polynomial in the Khovanskii basis elements using a simple algorithm known as the subduction algorithm. In [KM19] a theory of Khovanskii bases is developed. Moreover, it is shown that a finite set of algebra generators $$\mathcal{B}$$ is a Khovanskii basis, with respect to some valuation, if and only if the tropical variety of the ideal of relations among the elements of $$\mathcal{B}$$ contains a “prime cone”.

Whenever we have a Khovanskii basis, we can construct a deformation of $$R$$ to the associated graded $$\text{gr}_v(R).$$ More precisely we have the following ([An12 Proposition 5.1] and [Tei99 Proposition 2.2]).

**Theorem 1.9.** Suppose $$(R,v)$$ has a finite Khovanskii basis, then there is a finitely generated, positively graded, flat $$k[t]$$-subalgebra $$A \subset R[t],$$ such that:

(a) $$A[t^{-1}] \cong R[t,t^{-1}]$$ as $$k[t,t^{-1}].$$

(b) $$A / (t) \cong \text{gr}_v(R).$$

We have the following geometric interpretation of Theorem 1.9.

**Corollary 1.10.** There is a degeneration of $$X = \text{Proj}(R)$$ to the variety $$\text{Proj}(\text{gr}_v(R))$$ (in the sense of Definition 3).

When $$v$$ is full rank and the base field $$k$$ is algebraically closed, the associated graded $$\text{gr}_v(R)$$ is isomorphic to the semigroup algebra $$k[S(R,v)]$$ and we get a toric degeneration of $$\text{Proj}(R)$$ to the toric variety $$\text{Proj}(k[S(R,v)]).$$

Below we prove a converse to Theorem 1.9. Roughly speaking, it says that any toric degeneration must come from a full rank valuation. Its proof relies on [KM19 Theorem 4].

**Theorem 1.11.** Let $$R$$ be a positively graded domain and let $$A$$ be a finitely generated positively graded $$k[t]$$-module and domain with the following properties:

(a) $$A[t^{-1}] \cong R[t,t^{-1}]$$ as $$k[t].$$

(b) The algebra $$R' = A / (t)$$ is a graded semigroup algebra $$k[S]$$ where $$S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$$ is a finitely generated semigroup.

(c) The standard $$\mathbb{G}_m$$-action on $$k[t]$$ extends to an action on $$A$$ respecting its grading. Moreover, this $$k[t]$$-action acts through $$\mathbb{G}_m$$ acting on the semigroup algebra $$R'.$$
Then there is a full rank valuation $v : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ such that $R' \cong k[S(R, v)]$.

Proof. Let $A = \bigoplus_{n \in \mathbb{Z}} F_n$ be the isotypical decomposition of $A$ with respect to the $\mathbb{G}_m$-action. The parameter $t$ is a non-zero divisor, so we must have $tF_i \subset F_{i+1}$. By setting $t = 1$ we obtain the isomorphism of algebras $R \cong A/(t - 1)$, furthermore both of these algebras are identified with $\bigcup_{n \in \mathbb{Z}} F_n$ as vector spaces. In this way, each $F_n$ is mapped isomorphically to a subspace of $R$, and these spaces form a homogeneous filtration $\mathcal{F}$ with $R' = \text{gr}_\mathcal{F}(R)$.

We select generators $u_1, \ldots, u_k \in R'$ which are homogeneous with respect to the $S$-grading on $R$; this in turn implies they are homogeneous with respect to both the positive grading on $R'$ inherited from $A$ and the action of $\mathbb{G}_m$. If $u_i \in F_j/F_{j+1} \subset R'$ we select a homogeneous lift $y_i \in F_j$; we claim the $y_j$ and $t$ generate $A$. Let $F_{n,d} \subset F_n$ be the homogeneous degree $d$ part of $F_n$. We must have $F_n = \bigoplus_d F_{n,d}$, and $u_i$ define a vector space filtration of the $d$-graded component of $R$. It suffices to show that every $f \in F_{n,d}$ can be written as a polynomial in the $y_j$. For this we use a variant of the subduction algorithm (see [KM19, Algorithm 2.11]). Since the $u_j$ generate $R' = \bigoplus_n F_n/F_{<n}$, we can find a monomial term $m = C_s y^\alpha$ such that $f - m \in tF_{n-1,d}$. Now we can repeat this procedure with $f - m$, noting that it must terminate eventually as $A$ is positively graded.

It now follows that the images $z_j \in R$ of the $y_j$ generate $R \cong A/(t - 1)$. We consider a monomial weighting $r = (r_1, \ldots, r_k)$ of these generators, where $z_j$ is given the weight $r_j$ such that $u_i \in F_{r_j}/F_{r_j-1}$. Let $J$ be the homogeneous ideal which vanishes on the $z_j$. We consider $r$ as a point in the Gröbner fan of $J$. Choose a Gröbner basis $G \subset J$ accordingly, and let $J_r \subset k[x_1, \ldots, x_k, t]$ be the ideal which cuts out the Gröbner degeneration of $J$ corresponding to $r$. Observe that each polynomial in $J_r$ must vanish on the $y_k$ and $t$ in $A$. But $k[x_1, \ldots, x_k, t]/J_r$ has the same Krull dimension as $A$, and both algebras are domains, so it must follow that $A = k[x_1, \ldots, x_k, t]/J_r$.

Let $I$ be the binomial ideal which vanishes on the $u_j \in R'$. We have $\text{in}_r(J) = I$, and that $r$ is in the lineality space of $I$. Now a standard argument from Gröbner theory shows that if $B$ is a sufficiently small relatively open ball in the lineality space of $I$, we must have $\text{in}_{r+s}(J) = \text{in}_r(J) = I$, for any $s \in B$. Since $I$ is prime, this implies that the Gröbner fan of $J$ contains a prime cone $C$ of dimension $d$. Now [KM19, Theorem 4] implies that there is a full rank valuation $v$ on $R$ with $R' = k[S(R, v)]$. \hfill \Box

2. Degeneration in stages

In this section we prove the following which is one of the main results of the paper.

Theorem 2.1. Let $R$ be a finitely generated positively graded $k$-domain. Let $X = \text{Proj}(R)$ and $d = \dim(X)$. Then there exists a sequence of $k$-domains $R_0 = R, \ldots, R_{d-1}$ and degenerations $X_i = \text{Proj}(R_i) \rightsquigarrow X_{i+1} = \text{Proj}(R_{i+1})$, $i = 0, \ldots, d - 2$, such that $X_{d-1}$ is a complexity-one $T$-variety for an action of torus $T = \mathbb{G}_m^{d-1}$.

The theorem is a corollary of the following lemma. Its proof in turn relies on Lemma 6.1 proved in the appendix, using some technical commutative algebra.

Lemma 2.2. Let $A$ be a finitely generated $(\mathbb{Z}_{\geq 0} \times \mathbb{Z})$-graded $k$-domain and let $X = \text{Proj}(A)$ (where $\text{Proj}$ is taken with respect to the $\mathbb{Z}_{\geq 0}$-grading). The $\mathbb{Z}^r$-grading on $A$ induces an action of the torus $T = \mathbb{G}_m^r$ on $X$. We assume that $d = \dim(X) \geq 2$, $0 \leq r \leq d - 2$ and the following hold:

1. $T$-stabilizer of a general point in $X$ is finite.
2. $\dim(X_{\text{us}}) \leq r$ where $X_{\text{us}}$ denotes the unstable locus for the $T$-action on $X$.
3. $\dim(A^T) \geq d + 1 - r$. 


Then $X$ can be degenerated to $X' = \text{Proj}(A')$ (in the sense of Definition 7) where $A'$ is a finitely generated $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}^{r+1})$-graded $k$-domain with the following properties: Consider the $T' = \mathbb{G}_m^{r+1}$-action on $X'$ induced from the $\mathbb{Z}^{r+1}$-grading, then (1)-(3) above hold for $(A', X', T', r + 1)$ in place of $(A, X, T, r)$ respectively.

Proof. We construct $A'$ as the symbolic associated graded of a height one prime ideal in $A$. Let $Y = X//T = \text{Proj}(A^T)$ be the GIT quotient of $X$ with $\pi : X_{ss} \to Y$ the GIT quotient map. Let $k > 0$ be sufficiently large so that the Veronese subalgebra $(A^T)[k] = \bigoplus_{i \geq 0} A^T_{ki}$ is generated in degree 1. Take a finite set $\{f_0, \ldots, f_s\}$ of homogeneous degree 1 algebra generators for $(A^T)[k]$ to get an embedding $Y \hookrightarrow \mathbb{P}^s$ and consider $\pi : X_{ss} \to Y \to \mathbb{P}^s$. For $0 \neq f \in (A^T)_k$, let $H_f \subset Y$ be the hyperplane section defined by $f$. Since $\dim(A^T) \geq d + 1 - r$ we have $\dim(Y) \geq d - r \geq 2$. Thus we can apply the Bertini irreducibility theorem (see [Jou83, Theorem 0.1]) to conclude that if $f$ is in general position then $\pi^{-1}(H_f) \subset X^s$ is an irreducible subvariety. Moreover, $\pi^{-1}(H_f)$ intersects the open subset $U \subset X^s$ of points with finite $T$-stabilizer. By assumption $\dim(X \setminus X^s) = \dim(X^u) \leq r < d - 1$. That is, $\text{codim}(X^u) \geq 2$. It follows that the subvariety in $X$ defined by the principal ideal $(f)$ is irreducible and hence coincides with $Z = \pi^{-1}(H_f)$. We note that $\pi^{-1}(H_f)$ and hence $Z$ are $T$-invariant subvarieties. This then implies that the radical ideal $p = \sqrt{(f)}$, which is the ideal of the subvariety $Z$, is a homogeneous $T$-invariant prime ideal in $A$. Moreover, since $f \in (A^T)_k$ is in general position, we can assume that:

- The local ring $A_p$ is a discrete valuation ring.
- $Z \cap U \neq \emptyset$

Now let $A' = \text{gr}_{p^\infty} A$ be the symbolic associated graded of $p$. It is a $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}^r \times \mathbb{Z})$-graded algebra where the last $\mathbb{Z}$-grading is the natural $\mathbb{Z}_{\geq 0}$-grading on $A' = \bigoplus_{i \geq 0} p^{(i)} / p^{(i+1)}$. Let $X' = \text{Proj}(A')$. Let $T' = \mathbb{G}_m^{r+1}$, then the $\mathbb{Z}^{r+1}$-grading on $A'$ induces a $T'$-action on $X'$. Also the torus $T' = T \times \mathbb{G}_m$ acts on $Z \times \mathbb{A}^1$ where $T$ acts on $Z$ and $\mathbb{G}_m$ acts on $\mathbb{A}^1$ in the standard way.

By Lemma 6.1 we know that there is a natural embedding of the polynomial ring $(A/p)[u]$ into $A'$ and $A'$ is a finite module over $(A/p)[u]$. Moreover, this embedding induces a finite morphism $\phi : \text{Proj}(A') \to \text{Proj}((A/p)[u])$. One verifies that the embedding $(A/p)[u] \to A'$ preserves the $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}^{r+1})$-gradings and hence $\phi$ is $T'$-equivariant.

The variety $\text{Proj}((A/p)[u])$ is the projectivization of $\tilde{Z} \times \mathbb{A}^1$, where $\tilde{Z}$ is the affine cone over $Z$. It is the product $Z \times \mathbb{A}^1$ with a point $\infty$ added. The point $\infty$ is in the closure of each fiber $\mathbb{A}^1$. It is easy to see that the dimension of unstable locus for the $T'$-action is at most dimension of $T$-unstable locus on $Z$ plus 1 which in turn is less than or equal to $r + 1$.

Since $Z \cap U \neq \emptyset$ we know that the generic $T'$-stabilizer in $Z \times \mathbb{A}^1$ is finite. From $T'$-equivariance of $\phi$ it follows that the generic $T'$-stabilizer in $X'$ is also finite.

Now by [MFK93, Chap I, Section 5] we know that inverse image of the $T'$-unstable locus of $\text{Proj}((A/p)[u])$ under the finite morphism $\phi$ contains the $T'$-unstable locus of $X'$. It follows that the $T'$-unstable locus of $X'$ has dimension $\leq r + 1$. To finish the proof we need to verify that $\dim(A^T) \geq d + 1 - (r + 1)$. We note that $A^T = A^T/p^T$ with $p^T = p \cap A^T$. Since $p = \sqrt{TA}$ and $f$ is $T$-invariant, we have $p \cap A^T = \sqrt{TA}$. Thus, $p^T = p \cap A^T$ is a height 1 prime in $A^T$ and $\dim(A^T/p^T) = \dim(A^T) - 1 \geq d + 1 - r - 1$ as required. \hfill \qed

Proof of Theorem 7. Starting from $r = 0$ repeatedly apply Lemma 7 until $r = \dim(X) - 2$. We arrive at a sequence of degenerations of $X$. \hfill \qed
Remark 2.3. Observe that the Bertini theorem breaks down in dimension 1, so our method above cannot produce a toric degeneration in general.

3. Curve case

In this section we consider the case where the variety $X$ is a projective curve. We give an example of the homogeneous coordinate ring of a smooth projective curve such that for any choice of a homogeneous full rank valuation the corresponding associated graded (equivalently the value semigroup) is non-finitely generated. Also, as a corollary of Lemma 6.1 we see that given a (not necessarily smooth) projective curve $X$, we can find a very ample line bundle $L$ and a valuation $v$ (corresponding to a smooth point on $X$) such that the associated graded of the homogeneous coordinate ring of $(X, L)$ with respect to $v$ is finitely generated and hence $(X, L)$ has a toric degeneration.

For a smooth point $p \in X$ let us consider the homogeneous valuation $v_p : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ as follows. For every $f \in R$ put:

$$v_p(f) = (\deg(f), \ord_p(f)).$$

(The first factor $\mathbb{Z}_{\geq 0}$ is equipped with the reverse ordering of integers).

Proposition 3.1 (Finitely generated semigroup case). Let $X$ be a (not necessarily smooth) projective curve. Let $L$ be a very ample line bundle on $X$ such that the divisor class of $L$ is a multiple of a smooth point $p \in X$. Then the algebra of sections $R = R(X, L) = \bigoplus H^0(X, L^\delta)$ has a finite Khovanskii basis with respect to the valuation $v_p$, i.e. the value semigroup $S(R, v_p)$ is finitely generated.

Proof. From Lemma 6.1 it follows that $\text{gr}_{v_p} R$ is finitely generated which is what the proposition claims.

A variant of Proposition 3.1 is observed in [IW18, Remark 3.17], and an example of the situation in Proposition 3.1 is Example 1.6. Also see the discussion after Corollary 17 in [AKL14].

Proposition 3.2 (Non-finitely generated semigroup case). Suppose the base field $k$ is uncountable. Let $X$ be a smooth projective curve with genus > 1 and let $L$ be a very ample line bundle on $X$ such that no tensor power of $L$ has a divisor which is a multiple of a point. Then for any choice of $p \in X$ the value semigroup $S(R, v_p)$ is not finitely generated and hence $(R, v_p)$ does not have a finite Khovanskii basis (and hence no toric degeneration for any choice of a homogeneous valuation).

Proof. Take a point $p \in X$ with the corresponding valuation $v_p$. We know that the cone of the value semigroup $S(R, v_p)$ is the cone $C = \{(k, x) \mid 0 \leq x \leq k\delta\}$ where $\delta = \deg(L)$ is the degree of the line bundle $L$. By contradiction suppose that the value semigroup $S(R, v_p)$ is indeed finitely generated. Then for some $m$ there should exists $f \in R_m$ such that $\ord_p(f) = m\delta$. This means that the divisor of the section $f$ is $m\delta p$ which contradicts the assumption, namely the line bundle $L^m$ does not have a divisor which is a multiple of a single point.

In fact, it is also possible to find rational curves with a projective coordinate ring which cannot have a finite Khovanskii basis. In [IW18, Corollary 3.14] this is shown for the projective coordinate ring of a very general integral rational plane curve of degree $d > 3$. 
Throughout this section $R$ is the homogeneous coordinate ring of a $d$-dimensional projective variety $X$. When the ring $R$ is Cohen-Macaulay (in other words, $X$ is arithmetically Cohen-Macaulay or ACM for short), there is a simple construction of a degeneration of $X$ to a complexity-one $T$-variety.

Recall that a sequence $f_1, \ldots, f_r$, $f_i \in R$, is called a regular sequence if $f_1$ is not a zero divisor and for every $i = 2, \ldots, r$, the image of $f_i$ in $R/(f_1, \ldots, f_{i-1})$ is not a zero divisor. If $R$ is an algebra over a field $k$, it follows from the definition of a regular sequence that $f_1, \ldots, f_r$ are linearly independent over $k$.

Let $p = (f_1, \ldots, f_r)$ be the ideal generated by the $f_i$. The following theorem is due to Rees. The idea of proof goes back to Macaulay and is based on a double induction (see [Rees57, Theorem 2.1]).

**Theorem 4.1.** Let $\bar{f}_i$ be the image of $f_i$ in $p/p^2$. Then the map $(R/p)[t_1, \ldots, t_r] \to \text{gr}_p(R)$ which is identity on $R/p$ and sends $t_i$ to $\bar{f}_i$, $i = 1, \ldots, r$, gives an isomorphism between $(R/p)[t_1, \ldots, t_r]$ and $\text{gr}_p(R)$.

Let $X$ be a projective variety of dimension $d \geq 2$. Fix an embedding of $X$ in a projective space $\mathbb{P}^N$ and let $R = k[X]$ be its homogeneous coordinate ring. We write $R = \bigoplus_{i \geq 0} R_i$ where $R_i$ is the $i$-th homogeneous piece of $R$ with respect to its natural grading. We also let $\bar{X} \subset \mathbb{A}^{N+1}$ denote the affine cone over $X$.

The following is a corollary of the Bertini irreducibility theorem and unmixedness of Cohen-Macaulay rings.

**Proposition 4.2.** Suppose $R$ is Cohen-Macaulay. Then there exists a Zariski open subset $U \subset R_1 \times \cdots \times R_1$ ($d - 1$ times) such that for every $(f_1, \ldots, f_{d-1}) \in U$ and any subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, d - 1\}$ the ideal generated by $f_{i_1}, \ldots, f_{i_k}$ is prime. In particular, $(f_1, \ldots, f_{d-1})$ is a regular sequence in $R$ and the ideal $p = (f_1, \ldots, f_{d-1})$ is prime.

Now fix a regular sequence $f_1, \ldots, f_r \in R_1$, $r \leq d$, such that $p = (f_1, \ldots, f_r)$ is prime. Let $Y \subset X$ be the subvariety defined by the homogeneous prime ideal $p$ in $X$. Also let $\bar{Y} \subset \bar{X}$ be the affine cone over $Y$ and let $\bar{X}_0 = C_Y \bar{X}$ denote the normal cone of $\bar{Y}$ in $\bar{X}$. Let $\pi : \bar{X} \to \mathbb{A}^1$ be the deformation to normal cone family which deforms $\bar{X}$ to $\bar{X}_0$ and let $\bar{x}$ be the projectivization of $\bar{X}$, that is, $\bar{x} = \text{Proj}(\bigoplus_{t \geq 0} p^t)$. The normal cone $\bar{X}_0$ is the Spec of the associated graded algebra $\text{gr}_p(R)$. Since each graded piece of $\text{gr}_p(R)$ is a finite dimensional $k$-vector space, $X_0 = \text{Proj}(\text{gr}_p(R))$ is a projective variety.

Let us give a precise description of the projective variety $X_0$. Since $\{f_1, \ldots, f_r\} \subset R_1$ is linearly independent over $k$ we can extend it to a basis $\{f_1, \ldots, f_N\}$ for $R_1$. We observe that $\text{gr}_p(R) = \bigoplus_{t \geq 0} p^t/p^{t+1}$ is generated as an algebra by $\{f_1, \ldots, f_N\}$ where $\bar{f}_i$ is the image of $f_i$ in $\text{gr}_p(R)$. This choice of generators gives rise to an embedding of $X_0 = \text{Proj}(\text{gr}_p(R))$ into $\mathbb{P}^N$. Consider the rational map $p : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-r}$ given by:

$$(x_1 : \cdots : x_{N+1}) \mapsto (x_{r+1} : \cdots : x_{N+1}).$$

This rational map is regular outside the locus $Z$ where $x_{r+1} = \cdots = x_{N+1} = 0$. Also $p : \mathbb{P}^N \setminus Z \to \mathbb{P}^{N-r}$ is a vector bundle and fiber over each point can be identified with $\mathbb{A}^r$. Moreover, $p : X_0 \setminus Z \to Y$ is also a vector bundle with fibers isomorphic to $\mathbb{A}^r$ (Theorem 4.1). We thus obtain the following:

**Proposition 4.3.** The variety $X_0 = \text{Proj}(\text{gr}_p(R))$ is the disjoint union of a vector bundle of rank $r$ over $Y$ and a copy $P$ of the projective space $\mathbb{P}^r$. Moreover, each fiber of the vector
bundle union with $P \cong \mathbb{P}^r$ is isomorphic to $\mathbb{P}^{r+1}$. Also $X_0$ has a natural action of torus $T = G^r_n$. It acts on the fibers of the vector bundle and on the projective space $P$ in the natural way. Thus when $r = d - 1$, $X_0$ is a complexity-one $T$-variety.

**Remark 4.4.** With notation as above, let $v : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-1}$ be the homogeneous valuation corresponding to the regular sequence $(f_1, \ldots, f_{d-1})$. We obtain a Khovanskii basis for $(R, v)$ by augmenting any generating set of $R$ with the regular sequence $f_1, \ldots, f_{d-1}$.

Now we assume that $R$ is both Cohen-Macaulay and normal. A general principle in algebraic geometry dictates that the complete intersection $Y$ isomorphic to a complexity-one variety. We write $h_R(n)$ as a sum of binomials:

\begin{equation}
(2) \quad h_R(n) = \sum_{i=0}^{d} c_i \binom{n+i}{i}, \quad c_0, \ldots, c_d \in \mathbb{Z}.
\end{equation}

The graded ring $R/p$ is obtained from $R$ by a complete intersection of $d - 1$ linear forms, as a consequence we have:

\begin{equation}
(3) \quad h_{R/p}(n) = c_d(n+1) + c_{d-1},
\end{equation}

and the genus of $Y$ is $1 - c_d - c_{d-1}$.

**Corollary 4.6.** Let $R$ and $h_R(n)$ be as above, then if $c_d + c_{d-1}$ equals 0 or 1, the algebra $R$ can be degenerated to a finitely generated affine semigroup algebra (in the sense of Theorem 4.1).

**Proof.** If the genus of $Y$ is 0 or 1 then the divisor class of $L$ is a multiple of a point, now we use Proposition 1.3.

If $c_d + c_{d-1} = 1$ then $Y \cong \mathbb{P}^1$ and $L \cong \mathcal{O}(\ell)$ for some $\ell \in \mathbb{Z}_{>0}$. It is easy to see from \[3\] that $\ell = c_d$, the degree of $X \subset \mathbb{P}^N$. It follows that $R/p$ is the $c_d$-th Veronese subring of the polynomial ring $k[x, y]$. Let $\Delta(\bar{1}, \ell)$ be the simplex with vertices $(0, \ldots, 0)$, $(1, \ldots, 0)$, $\ldots$, $(0, \ldots, 1, 0)$, $\ldots$, $(0, 0, \ldots, 0)$, and let $S(\bar{1}, \ell)$ be the graded affine semigroup algebra associated to $\Delta(\bar{1}, \ell)$. From Theorem 4.1 we can conclude the following.

**Corollary 4.7.** Let $R$ be a normal Cohen-Macaulay domain as above, then the following are equivalent:

1. $R$ has a flat degeneration to $S(\bar{1}, c_d)$ (in the sense of Theorem 1.20).
2. A complete intersection in $X$ by $d - 1$ generic hyperplanes is isomorphic to $\mathbb{P}^1$.
3. $c_d + c_{d-1} = 1$. 
The algebra $S(\bar{1}, c_d)$ is a quotient of the polynomial ring $k[x_0, \ldots, x_{cd}, y_1, \ldots, y_{d-1}]$ by the $2 \times 2$ minors of the matrix:

\[
\begin{bmatrix}
x_0 & x_1 & \cdots & x_{cd-1} \\
x_1 & x_2 & \cdots & x_{cd}
\end{bmatrix}
\]

These forms are a quadratic square-free Gröbner basis. It follows that $R$ can likewise be presented as a quotient of $k[x_0, \ldots, x_{cd}, y_1, \ldots, y_d]$ by a quadratic square-free Gröbner basis. The algebra $S(\bar{1}, c_d)$ is Gorenstein if and only if $c_d = 2$, this is also the case when $R$ can be presented as coordinate ring of a hypersurface.

**Remark 4.8.** Let $R$ be Cohen-Macaulay and normal, with $X$ and $X_0$ as above. We choose a point $p \in Y$ and consider the ample divisor $D_p \subset X_0$ defined by the copy of $\mathbb{P}^{r+1}$ obtained from the fiber over $p$ as in Proposition 4.3. The projective coordinate ring of $D_p$ is a polynomial ring over the projective coordinate ring of the divisor defined by $p \in \bar{Y}$. By Proposition 4.5, $X_0$ can be degenerated to a toric variety using this alternative embedding. It follows that any normal projective variety with an arithmetically Cohen-Macaulay embedding can be degenerated in multiple steps to a toric variety.

## 5. Existence of a finite Khovanskii basis and degeneration in one step

In this section, using results from Section 2, we show that a finitely generated positively graded algebra and domain with Krull dimension $d + 1$ has a rank $d$ valuation with finitely generated associated graded algebra. This then implies that the projective variety corresponding to $R$ can be degenerated (in one step) to a complexity-one $T$-variety (in the sense of Definition 1).

More precisely, we have the following:

**Theorem 5.1.** Let $R$ be a finitely generated positively graded algebra and domain with $\text{dim}(R) = d + 1$ and $R_0 = k$. Then there exists a homogeneous rank $d$ valuation $v : R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{d-1}$ such that the associated graded $\text{gr}_v(R)$ is finitely generated (in other words, $(R, v)$ has a finite Khovanskii basis).

**Remark 5.2.** Note that the finite generation of $\text{gr}_v(R)$ as an algebra implies the finite generation of $S(R, v)$ as a semigroup but not the other way around. Since the valuation $v$ is not full rank, the finite generation of $\text{gr}_v(R)$ is stronger than the value semigroup be finitely generated.

**Corollary 5.3.** With notation as above, the projective variety $X = \text{Proj}(R)$ has a degeneration to the complexity-one $T$-variety $\text{Proj}(\text{gr}_v(R))$ (in the sense of Definition 7).

The construction of the valuation $v$ relies on the following basic construction which we call concatenation of valuations. Let $R$ be as above and let $v : R \setminus \{0\} \rightarrow \Gamma_1$ be a valuation with values in an ordered group $\Gamma_1$. Recall that for $0 \neq f \in R$ with $v(f) = a$ we let $\bar{f}$ denote the image of $f$ in $R_{e\geq a}/R_{e\geq a} \subset \text{gr}_v(R)$. The map $f \mapsto \bar{f}$ is a multiplicative homomorphism from $R \setminus \{0\}$ to the multiplicative set of homogeneous elements of $\text{gr}_v(R)$. Also let $w : \text{gr}_v(R) \setminus \{0\} \rightarrow \Gamma_2$ be a valuation on the associated graded of $v$ and with values in an ordered group $\Gamma_2$. We define a function $u : R \setminus \{0\} \rightarrow \Gamma_1 \times \Gamma_2$ as follows: for $0 \neq f \in R$ put:

\[(4) \quad u(f) = (v(f), w(\bar{f})).\]
Let us equip $\Gamma_1 \times \Gamma_2$ with a lexicographic order. More precisely, for $(a_1, a_2), (b_1, b_2) \in \Gamma_1 \times \Gamma_2$ we have $(a_1, a_2) > (b_1, b_2)$ if $a_1 > b_1$, or $a_1 = b_1$ and $a_2 > b_2$. The following is straightforward to verify.

**Proposition 5.4.** With notation as above, the function $u : R \setminus \{0\} \to \Gamma_1 \times \Gamma_2$ is a valuation.

We call the valuation $u$ the concatenation of $v$ and $w$ and denote it by $u = v \circ w$.

The following proposition relates the associated graded algebra of the concatenation $v \circ w$ with those of $v$ and $w$.

**Proposition 5.5.** With notation as above, let $v : R \setminus \{0\} \to \Gamma_1$ be a valuation and $w : \text{gr}_v(R) \setminus \{0\} \to \Gamma_2$ be a homogeneous valuation with respect to the $\Gamma_1$-grading on $\text{gr}_v(R)$.

Then we have an isomorphism of graded $(\Gamma_1 \times \Gamma_2)$-algebras:

$$\text{gr}_{v \circ w}(R) \cong \text{gr}_w(\text{gr}_v(R)).$$

**Proof.** Let $F = F_v$, $G = F_w$ be the filtrations on $R$ and $\text{gr}_v(R)$ induced by $v$ and $w$ respectively. Also let $H = F_{v \circ w}$ be the filtration on $R$ induced by the concatenation $v \circ w$. As $w$ is assumed to be homogeneous, the algebras $\text{gr}_{v \circ w}(R)$ and $\text{gr}_w(\text{gr}_v(R))$ are both $(\Gamma_1 \times \Gamma_2)$-graded algebras, and $\text{gr}_w(\text{gr}_v(R))$ is the direct sum of the spaces $(G_{\geq s} \cap F_{\geq r}/F_{\geq r})/(G_{\geq s} \cap F_{\geq r}/F_{\geq r})$. We show that that the graded components of these algebras are isomorphic and that their homogeneous multiplication operations coincide.

First we construct a map $\phi : (G_{\geq s} \cap F_{\geq r}/F_{\geq r}) \to H_{\geq r(s)} / H_{\geq r(s)}$. Any $f \in F_{\geq r}/F_{\geq r}$ is represented by some $f \in F_{\geq r}$, we let $\phi(f)$ for $f \in (G_{\geq s} \cap F_{\geq r}/F_{\geq r})$ be the equivalence class of $f$ in $H_{\geq r(s)} / H_{\geq r(s)}$. If we choose some other representative $f' \in F_{\geq r}$ then the difference $f - f'$ must lie in $F_{\leq r}$, and therefore $H_{\geq r(s)}$, so this map is well-defined. Any $g \in H_{\geq r(s)}$ must have $v(g) \geq r$ and $w(g) \geq s$ for $g \in F_{\geq r}/F_{\geq r}$ by definition, it follows that $\phi$ is also onto.

We claim that the kernel of $\phi$ is $(G_{\geq s} \cap F_{\geq r}/F_{\geq r})$. Clearly this space lies in the kernel of $\phi$, and if $\phi(f) = 0$ it must be the case that a lift $f \in F_{\geq r}$ is in $H_{\geq r(s)}$. If $f \notin F_{\geq r}$ (i.e. $\tilde{f} \neq 0$), then $w(\tilde{f})$ will be strictly less than $s$. It follows that the induced map on the quotient space $\phi : (G_{\geq s} \cap F_{\geq r}/F_{\geq r})/(G_{\geq s} \cap F_{\geq r}/F_{\geq r}) \to H_{\geq r(s)} / H_{\geq r(s)}$ is an isomorphism.

Now let $f_1, f_2 \in \text{gr}_w(\text{gr}_v(R))$ be homogeneous. The product $f_1 f_2$ is represented by the product $f_1 f_2$, where $f_i$ is a lift of $f_i$ in the definition of $\phi$ above. The construction above shows that multiplicative map $R \to \text{gr}_{v \circ w}(R)$ is the composition of the maps $R \to \text{gr}_v(R)$ and $\text{gr}_v(R) \to \text{gr}_w(\text{gr}_v(R)) \cong \text{gr}_{v \circ w}(R)$, the latter being isomorphism as graded vector space, so it follows that $\phi(f_1 f_2) = \phi(f_1) \phi(f_2)$. \hfill \Box

**Remark 5.6.** For any valuation $w$ on $\text{gr}_v(R)$, not necessarily homogeneous, there is an associated homogeneous valuation $\tilde{w}$ defined as follows:

$$\tilde{w}(\sum_{\gamma} f_\gamma) = \min\{w(f_\gamma) \mid f_\gamma \neq 0\}. $$

**Proposition 5.7.** With notation as above, let $v : R \setminus \{0\} \to \Gamma$ have rank $r$, and let $w : \text{gr}_v(R) \to \mathbb{Q}$ be homogeneous with respect to the $\Gamma$-grading on $\text{gr}_v(R)$. If $w$ is not a grading function then $v \circ w : R \setminus \{0\} \to \Gamma \times \mathbb{Q}$ has rank $r + 1$. 

Proof. By definition, the rank of \( v \circ w \) is greater than or equal to \( r \). Suppose by contradiction that it is equal to \( r \). Then \( S(R, v \circ w) \) lies in a hyperplane in \( \Gamma \times \mathbb{Q} \) and projection onto the first factor in \( \Gamma \times \mathbb{Q} \) gives an isomorphism from \( S(R, v \circ w) \) to \( S(R, v) \). Let \( \pi : S(R, v) \to \mathbb{Q} \) be the linear function given by the projection onto the second factor in \( \Gamma \times \mathbb{Q} \). We see that if \( f \in \text{gr}_v(R) \) is homogeneous of degree \( a \) then \( w(f) = \pi(a) \). But this implies that \( w \) is a grading function. \( \square \)

Proof of Theorem 5.7. Following the proof of Lemma 2.2 we can construct a sequence of algebras \( R = R_0, R_1, \ldots, R_{d-1} \), and a sequence of valuations \( v_i : R \setminus \{0\} \to \mathbb{Z} \) such that \( \text{gr}_{v_i}(R_i) = R_{i+1} \). Here \( v_0 \) is the grading function given by the positive grading on \( R \). Each \( v_i \) is constructed by the symbolic normal cone method, and is homogeneous with respect to a \( \mathbb{G}_m^i \)-action on \( R_i \). Inductively, the \( \mathbb{G}_m^i \)-action on \( R_i = \text{gr}_{v_i}(R_{i-1}) \) is constructed from the \( \mathbb{G}_m^{i-1} \)-action on \( R_{i-1} \), and the \( \mathbb{G}_m \)-action on \( \text{gr}_{v_i}(R_{i-1}) \) coming from its natural \( \mathbb{Z}_{\geq 0} \)-grading.

The construction in the proof of Lemma 2.2 also implies that \( v_i \) is not a grading function with respect to the \( \mathbb{G}_m^i \)-action. Repeated application of Proposition 5.7 now implies that the concatenation \( v = v_0 \circ \cdots \circ v_{d-1} : R \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d-1} \) has rank \( d \). Furthermore, by Proposition 5.3 we have \( \text{gr}_v(R) \cong R_{d-1} \). Since \( R_{d-1} \) is finitely generated by construction, it follows that \( v \) has a finite Khovanskiǐ basis. \( \square \)

We finish this section by showing that any valuation, whose value group is \( \mathbb{Q}^r \) (for some \( r > 0 \)) and ordered lexicographically, can be obtained as a concatenation of rank 1 valuations in the fashion described above. This is a consequence of the following proposition.

Proposition 5.8. Let \( \mathbb{Q}^r \) and \( \mathbb{Q}^s \) be equipped with group orderings and order \( \mathbb{Q}^r \times \mathbb{Q}^s \) lexicographically such that the \( \mathbb{Q}^r \) component is preferred to the \( \mathbb{Q}^s \) component. Let \( v : R \setminus \{0\} \to \mathbb{Q}^r \times \mathbb{Q}^s \) be a valuation. Then there are valuations \( v : R \setminus \{0\} \to \mathbb{Q}^r \) and \( w : \text{gr}_v(R) \setminus \{0\} \to \mathbb{Q}^s \) such that \( w \) is homogeneous with respect to the grading on \( \text{gr}_v(R) \), and \( v = v \circ w \).

Proof. First observe that the axioms of valuations implies that the projection \( \pi_1 \circ v = v \)
onto the first \( r \) components is itself a valuation. For \( f \in R \) let \( \mathbf{v}(f) = (a, b) \), we claim that \( b \) only depends on the equivalence class \( f \in \text{gr}_v(R) \). If \( v(f) = v(f') < v(f - C f') \) for some \( C \in k \), then in turn we must have \( \mathbf{v}(f - C f') > \mathbf{v}(f), \mathbf{v}(f') \), but this implies that \( \mathbf{v}(f) = \mathbf{v}(f') \). Therefore, the projection \( \pi_2 \circ v = w \) is a well-defined function on homogeneous elements of \( \text{gr}_v(R) \); we extend this function to all of \( \text{gr}_v(R) \) using the min convention. It is straightforward to verify that this function is super-additive. We take two inhomogeneous elements \( \sum f \) and \( \sum h \). We have \( w(f h) = w(f h) = w(f) + w(h) \). If we fix a grading degree \( a \) and consider the sum \( \sum f_i h_i \) of all products with degree \( a \), we must have \( w(\sum f_i h_i) \geq \min \{ w(f_i) + w(h_i) \} \). It follows that without loss of generality we may assume that all the elements in \( \sum f \) have the same \( w \)-value \( b_1 \), and likewise it can be assumed that all of the summands of \( \sum h \) have \( w \)-value \( b_2 \). Now we consider the Newton polytopes of \( \sum f \) and \( \sum h \) in \( \mathbb{Q}^r \). Let \( f_c \) and \( h_c \) be components respectively at extremal vertices of these polytopes. Then then \( f_c h_c \) define an extremal vertex of the product \( (\sum f)(\sum h) \), and the value \( w(f_c h_c) \) must be \( b_1 + b_2 = w(\sum f) + w(\sum h) \). \( \square \)

6. APPENDIX: A LEMMA ON FINITE GENERATION OF SYMBOLIC REES ALGEBRA

In this appendix we prove the key lemma used in the proof of degeneration in stages (Lemma 2.2).

Lemma 6.1. Let \( R \) be a finitely generated \( k \)-domain. Suppose \( 0 \neq x \in R \) is such that:
Consider the homomorphism $u$ in one indeterminate $x$, given by $u(r) = r$, for $r \in R/p$ and $u(x) = \bar{x}$, the image of $x$ in $\text{gr}_p(R)$.

Lemma 6.2. Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded integral domain. If for some integer $m > 0$, the $m$-th Veronese subring $A^{[m]} = \bigoplus_{i \geq 0} A_{mi}$ is a Noetherian ring, then $A$ is finite over $A^{[m]}$.

Proof. Let us write $A = \bigoplus_{j=0}^{m-1} M_j$ where $M_j = \bigoplus_{k=0}^{\infty} A_{km+j}$. If $M_j \neq \{0\}$, then it contains a nonzero homogeneous element $r$. Then the multiplication by $r^{m-1}$ map $M_j \to A^{[m]}$ is well-defined and injective since $A$ is an integral domain. It is also a homomorphism of $A^{[m]}$-modules. It follows that $A$ is a finitely generated $A^{[m]}$-module as required.

Proof of Lemma 6.1. (1) Let $n > 0$ be such that $x \in p^{(n)} \setminus p^{(n+1)}$. First we show that for all $k > 0$ we have:

$$p^{(kn)} = \langle x^k \rangle,$$

where $\langle x^k \rangle$ denotes the integral closure of the principal ideal $(x^k)$ in the field of fractions of $R$. Firstly, $\sqrt{\langle x^k \rangle} = \sqrt{(x^k)} = p$. Thus $p$ is an associated prime of $\langle x^k \rangle$. By [HS06, Corollary 5.4.2] we know that $\langle x^k \rangle$ is unmixed. It follows that in its primary decomposition only primary ideals corresponding to minimal primes appear. Thus $\langle x^k \rangle = q$ where $q$ is a $p$-primary ideal. Moreover, $q = (x^k)A_p \cap A$. Now $A_p$ is a DVR so there exists $n > 0$ such that

$$\langle x^k \rangle A_p = (x^k)A_p = p^{kn}A_p.$$

Here we are using the fact that any principal ideal in a normal ring is integrally closed. Thus

$$q = p^{kn}A_p \cap A = p^{(kn)}$$

which proves the claim. We note that $S' = \bigoplus_k (x^k) = \bigoplus_k (x^k)$, the integral closure of the ring $S = \bigoplus_k (x^k)$ in $\bigoplus_k R$ (see [HS06, Proposition 5.2.1]). By the finiteness of integral closure we know that $S'$ is finite over $S$. Now, the finite generation of $\bigoplus_k p^{(k)}$ follows from Lemma 6.2.

(2) By definition $\psi$ preserves the natural $\mathbb{Z}_{\geq 0}$-gradings on $(R/p)[u]$ and $\text{gr}_p(R)$. So ker $\psi$ is a homogeneous ideal in $(R/p)[u]$. Let $(a + p)u \in \text{ker} \psi$, then $\bar{a}(\bar{x})^k = 0$ where $\bar{a}$, $\bar{x}$ are the images of $a$, $x$ in $\text{gr}_p(R)$ respectively. But since $\text{gr}_p(R)$ is a domain this is not possible unless ker $\psi = \{0\}$. 
Intersection theory

Convergence of polarizations, toric degenerations, and Okounkov bodies

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• (3) Recall from proof of (1) that $S' = \bigoplus_k (x^k) = \bigoplus_k p^{(kn)}$ is finite over $S = \bigoplus_k (x^k)$.

Let $\{f_1, \ldots, f_s\} \subset S'$ be a finite set of homogeneous $S$-module generators with $f_i \in (x^k)$. For $k > 0$ take $f \in (x^k) = p^{(kn)}$ such that $f \notin p^{(kn+1)}$, that is, $0 \neq \bar{f} \in \bar{p}^{(kn)} / p^{(kn+1)}$. There exists $r_i \in R$ such that

$$f = \sum_i r_ix^{k-k_i}f_i.$$ 

We recall that $x \in p^{(n)}$. Going modulo $p^{(kn+1)}$ we have:

$$f = \sum_j r_j x^{k-k_j}f_j \pmod{p^{(kn+1)}},$$

where the sum is over all $j$ such that $r_j \notin p$ and $f_j \notin p^{(k+j+1)}$. Now by definition $\bar{f}$ (respectively $\bar{f}_j$) is the image of $f$ in $p^{(kn)} / p^{(kn+1)}$ (respectively $p^{(k+j+1)} / p^{(k+j+1)}$). It follows from (7) that $f$ is a linear combination of the $\bar{f}_j$ with coefficients $\psi(r_j u^{k-k_j}) = (r_j + p)\bar{x}^{k-k_j}$ respectively. This shows that the $\bar{f}_j$ are a finite set of module generators for $\bigoplus_k p^{(kn)} / p^{(kn+1)}$ over $(R/p)[u]$. As before, the claim follows from Lemma 6.2.

(4) It follows from the construction of $\psi$. (5) We show the following more general fact.

Let $A$, $B$ be positively graded domains and $\psi : A \to B$ an injective finite homomorphism. Then $\psi$ induces a morphism $\phi : \text{Proj}(B) \to \text{Proj}(A)$. For this, consider the morphism $\psi^* : \text{Spec}(B) \to \text{Spec}(A)$. Finiteness of $\psi$ implies that the inverse image (under $\psi^*$) of any point is a finite set and hence $\psi^*$ cannot send a $\mathbb{G}_m$-invariant subvariety of $\text{Spec}(B)$ of positive dimension to the vertex $V(A_+)$ where $A_+$ is the irrelevant ideal, generated by all the elements in $A$ of positive degree. It follows that the inverse image of the vertex of $\text{Spec}(A)$ is the vertex of $\text{Spec}(B)$ and hence $\psi^*$ induces a morphism $\phi : \text{Proj}(B) \to \text{Proj}(A)$. The finiteness of $\phi$ follows from the finiteness of $\psi$. 

□

References

[AB04] Alexeev, V.; Brion, M. Toric degenerations of spherical varieties. Selecta Math. (N.S.) 10 (2004), no. 4, 453–478.

[AIPSV12] Altmann, K.; Ilten, N.; Petersen, L.; Süss, H.; Vollmert, R. The geometry of $T$-varieties. In Contributions to algebraic geometry, EMS Ser. Congr. Rep., pages 17–69. Eur. Math. Soc., Zürich, 2012.

[An12] Anderson, D. Okounkov bodies and toric degeneration. Math. Ann. 356 (2013), no. 3, 1183–1202.

[AKL14] Anderson, D.; K"{u}ronya, A.; Lozovanu, V. Okounkov bodies of finitely generated divisors. Int. Math. Res. Not. IMRN (2014), no. 9, 2343–2355.

[BLMM17] Bossinger, L.; Lamboglia, S.; Mincheva, K.; Mohammad, K. Computing toric degenerations of flag varieties. In Combinatorial algebraic geometry, volume 80 of Fields Inst. Commun., pages 247–281. Fields Inst. Res. Math. Sci., Toronto, ON, 2017.

[Cut88] Cutkosky, S. Weil divisors and symbolic algebras. Duke Math. J. 57 (1988), no. 1, 175–183.

[Ei04] Eisenbud, D. Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[FFL17] Fang, X.; Fourier, G.; Littelmann, P. On toric degenerations of flag varieties. Representation theory current trends and perspectives, 187–232, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017.

[FOV] Flenner, H., O’Carroll, L., and Vogel, W. Joints and Intersections. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1999.

[Ful98] Fulton, W. Intersection theory. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2. Springer-Verlag, Berlin, 1998.

[HHK] Hamilton, M.; Harada, M.; Kaveh, K. Convergence of polarizations, toric degenerations, and Newton-Okounkov bodies. To appear in Communications in Analysis and Geometry.

[HK15] Harada, M.; Kaveh, K. Integrable systems, toric degenerations and Okounkov bodies. Invent. Math. 202 (2015), no. 3, 927–985.
[HS06] Huneke, C.; Swanson, I. Integral closure of ideals, rings, and modules. London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.

[IM19] Ilten, N.; Manon, C. Rational Complexity-One T-Varieties are Well-Poised. Int. Math. Res. Not. IMRN (2019), no. 13, 4198–4232.

[IW18] Ilten, N.; Wrobel, M. Khovanskii-Finite Valuations, Rational Curves, and Torus Actions. arXiv:1807.08780 [math.AG]

[Jou83] Jouanolou, J. P. Théorèmes de Bertini et applications. (French) [Bertini theorems and applications]. Progress in Mathematics, 42. Birkhäuser Boston, Inc., Boston, MA, 1983.

[KT14] Karshon, Y.; Tolman, S. Classification of Hamiltonian torus actions with two-dimensional quotients. Geom. Topol. 18 (2014), no. 2, 669–716.

[Kav15] Kaveh, K. Crystal bases and Newton-Okounkov bodies. Duke Math. J. 164 (2015), no. 13, 2461–2506.

[KKh12] Kaveh, K.; Khovanskii, A. G. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. Ann. of Math. (2) 176 (2012), no. 2, 925–978.

[KM19] Kaveh, K.; Manon, C. Khovanskii bases, higher rank valuations and tropical geometry. SIAM J. Appl. Algebra Geom. 3 (2019), no. 2, 292–330.

[Kn05] Knutson, A. Balanced normal cones and Fulton-MacPherson’s intersection theory. Pure Appl. Math. Q. 2 (2006), no. 4, Special Issue: In honor of Robert D. MacPherson. Part 2, 1103–1130.

[MS] Mohammadi, F.; Shaw, K. Toric degenerations of Grassmannians from matching fields. arXiv:1809.01020

[SZ60] Samuel, P.; Zariski, O. Commutative algebra. Vol. II. Reprint of the 1960 edition. Graduate Texts in Mathematics, Vol. 29. Springer-Verlag, New York-Heidelberg, 1975.

[LM09] Lazarsfeld, R.; Mustaţa, M. Convex bodies associated to linear series. Ann. Sci. Éc. Norm. Super. (4) 42 (2009), no. 5, 783–835.

[MFK93] Mumford, D.; Fogarty, J.; Kirwan, F. Geometric invariant theory. Third ed., Springer, 1993.

[Rees57] Rees, D. The grade of an ideal or module. Proc. Camb. Phil. Soc. 53, pp. 28–42 (1957).

[Tei99] Teissier, B. Valuations, deformations, and toric geometry. In Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), 361–459, Fields Inst. Commun., 33, Amer. Math. Soc., Providence, RI, 2003.

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