Abstract

In this article, we use the theory of (non-abelian) exterior product of Hom-Lie algebras to prove the Hopf’s formula for these algebras. As an application, we construct an eight-term sequence in the homology of Hom-Lie algebras. We also investigate the capability property of Hom-Lie algebras via the exterior product.

Keywords: Hom-Lie algebra, non-abelian exterior product, Hom-Lie homology, capable Hom-Lie, Schur multiplier.

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1 Introduction

In [10, 11], Ellis introduced the notions of the non-abelian tensor and exterior products of Lie algebras and gave some of their basic properties. He investigated their relations to the low dimensional homology of Lie algebras. In particular, he proved that for any Lie algebra $L$ over a field $F$, $H_2(L)$, the second homology of $L$ with coefficients in the trivial $L$-module $F$, is isomorphic to the kernel of the commutator map $\lambda_L : L \wedge L \rightarrow L$. Moreover, he showed how the tensor product is related to the universal central extensions. In [22], Niroomand et al. dealt with some of the applications of exterior product to the notion of capability of Lie algebras. There is a series of papers (for instance, see [2, 9, 15, 16, 17, 21, 24, 25]) emphasizing the relevance of these notions to the development and exposition of the basic theories of the capability and the second homology of Lie algebras.

Hom-Lie algebras were originally introduced in [13], to construct deformations of the Witt algebra (which is the Lie algebra of derivations on the Laurent polynomial algebra $\mathbb{C}[x, x^{-1}]$), and the Virasoro algebra (which is a complex Lie algebra defined as the unique central extension of the Witt algebra), and used in [19] to study quantum deformations and discretisation of vector fields via twisted derivations. Hom-Lie algebras are also useful tools in studying mathematical physics. In this regard, Yau

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(30) gave some applications of these algebras to a generalization of the Yang-Barter equation and to braid group representations. Hom-Lie algebras are \( F \)-vector spaces endowed with a bilinear skew-symmetric bracket satisfying a Jacobi identity twisted by a map. When this map is identity, then the definition of Lie algebras is recovered. Hence, one is motivated to investigate some appropriate results of the Lie algebra theory in the general setting of the category of Hom-Lie algebras. Accordingly, several concepts of Lie algebras have been generalized to Hom-Lie algebras. Hom-Lie algebras on semisimple Lie algebras in [18], representation theory and (co)homology theory in [11, 8, 27, 29], universal central extensions in [6] and finally, the theory of tensor products of Hom-Lie algebras in [7], are studied. Recently, Casas and Garcia-Martinez ([5]) introduced the notions of (non-abelian) exterior product and capability of Hom-Lie algebras and extended some classical results from the Schur multiplier of Lie algebras to the multiplier of Hom-Lie algebras. In this paper, we continue the same line of research of Casas and Garcia-Martinez. In fact, we obtain further structure properties of the exterior products of Hom-Lie algebras, and use them to give several homology results, generalizing known ones for Lie algebras. In particular, we prove the Hopf’s formula for Hom-Lie algebras. Also, the relationship between the exterior product and the capability is studied.

This paper is organized as follows: In Section 2, we review some basic definitions regarding Hom-Lie algebras, with a special mention of the Hom-action and the homology of Hom-Lie algebras. Also, in this section we give some necessary results for the subsequent use in this paper. In Section 3, we recall and gather some general results related to the tensor and exterior products of Hom-Lie algebras. In addition, we provide some relations between these products and a generalized version of Whitehead’s universal quadratic functor for Hom-vector spaces. In Section 4, several key results from the theory of homology of Lie algebras are extended to Hom-Lie algebras. In particular, we describe the second homologies of Hom-Lie algebras as central subalgebras of their exterior products, and obtain an exact sequence of eight terms associated with an extension of Hom-Lie algebras in the homology. In the last section, we show that to investigate the capability of non-perfect Hom-Lie algebras, we only need to study those Hom-Lie algebras whose companion endomorphisms are surjective. We also establish the capability property for Hom-Lie algebras which are perfect or abelian. Furthermore, a relation between the tensor and exterior centers of a Hom-Lie algebra is given.

**Notations.** Throughout this paper, all vector spaces and algebras are considered over some fixed field \( F \) and linear maps are \( F \)-linear maps. We write \( \otimes \) and \( \wedge \) for the usual tensor and exterior products of vector spaces over \( F \), respectively. For any vector space (resp., Hom-Lie algebra) \( L \), a subspace (resp., an ideal) \( L' \) and \( x \in L \), we write \( \bar{x} \) to denote the coset \( x + L' \). If \( A \) and \( B \) are subspaces of a vector space \( V \) for which \( V = A + B \) and \( A \cap B = 0 \), we will write \( V = A \dot{+} B \).
2 Preliminaries on Hom-Lie algebras

In this section, we recall some basic concepts and auxiliary facts, which will be needed in the sequel.

2.1 Basic definitions

A Hom-Lie algebra \((L, \alpha_L)\) is a non-associative algebra \(L\) together with a linear map \(\alpha_L : L \rightarrow L\) satisfying the conditions

\[
\begin{align*}
(i) \quad [x, y] &= -[y, x], & \text{(skew-symmetry)} \\
(ii) \quad [\alpha_L(x), [y, z]] + [\alpha_L(y), [z, x]] + [\alpha_L(z), [x, y]] &= 0, & \text{(Hom-Jacobi identity)}
\end{align*}
\]

for all \(x, y, z \in L\), where \([\ , \ ]\) denotes the product in \(L\). In the whole paper we only deal with (the so-called multiplicative) Hom-Lie algebras \((L, \alpha_L)\) such that \(\alpha_L\) preserves the product, that is, \(\alpha_L([x, y]) = [\alpha_L(x), \alpha_L(y)]\) for all \(x, y \in L\). The Hom-Lie algebra \((L, \alpha_L)\) is said to be regular if \(\alpha_L\) is bijective. Taking \(\alpha_L = id_L\), we recover exactly Lie algebras. A Hom-vector space is a pair \((V, \alpha_V)\), where \(V\) is a vector space and \(\alpha_V : V \rightarrow V\) is a linear map. If we put \([x, y] = 0\) for all \(x, y \in V\), then \((V, \alpha_V)\) is a Hom-Lie algebra, which is called an abelian Hom-Lie algebra.

A homomorphism of Hom-Lie algebras \(\delta : (L_1, \alpha_{L_1}) \rightarrow (L_2, \alpha_{L_2})\) is an algebra homomorphism from \(L_1\) to \(L_2\) such that \(\delta \circ \alpha_{L_1} = \alpha_{L_2} \circ \delta\). The corresponding category of Hom-Lie algebras is denoted by \(\text{HomLie}\). As this category is a variety of \(\Omega\)-groups in the sense of Higgins \cite{Higgins}, it is a semi-abelian category. Therefore, the \(3 \times 3\)-Lemma and the Snake Lemma hold in this category \cite{Bourn}. We refer the reader to \cite{Bourn} for obtaining more information on this category.

Let \((L, \alpha_L)\) be a Hom-Lie algebra. The following is a list of the concepts which will be used:

(i) A (Hom-Lie) subalgebra \((H, \alpha_H)\) of \((L, \alpha_L)\) consists of a vector subspace \(H\) of \(L\), which is closed under the product and invariant under the map \(\alpha_L\), together with the linear self-map \(\alpha_H\) being the restriction of \(\alpha_L\) on \(H\). In such a case we may write \(\alpha_L|_H\) for \(\alpha_H\). A subalgebra \((H, \alpha_H)\) is an ideal if \([x, y] \in H\) for all \(x \in H, y \in L\).

(ii) The center of \((L, \alpha_L)\) is the vector space \(Z(L) = \{x \in L \mid [x, y] = 0, \text{ for all } y \in L\}\). Put \(Z_\alpha(L) = \{x \in L \mid [\alpha_L^k(x), y] = 0, \text{ for all } y \in L, k \geq 0\}\), where \(\alpha_L^0 = id_L\) and \(\alpha_L^k, k \geq 1\), denotes the composition of \(\alpha_L\) with itself \(k\) times. It is straightforward to see that \((Z_\alpha(L), \alpha_{Z_\alpha(L)})\) is the largest central ideal of \((L, \alpha_L)\). We call \(Z_\alpha(L)\) the \(\alpha\)-center of \((L, \alpha_L)\). When \(\alpha_L\) is a surjective homomorphism or \((L, \alpha_L)\) is abelian, then \(Z_\alpha(L) = Z(L)\).

(iii) If \((H, \alpha_H)\) and \((K, \alpha_K)\) are two ideals of \((L, \alpha_L)\), then the (Higgins) commutator of \((H, \alpha_H)\) and \((K, \alpha_K)\), denoted by \([H, K], \alpha_{[H,K]}\), is a Hom-Lie subalgebra spanned by the elements \([h, k], h \in H, k \in K\). Note that \([H, K], \alpha_{[H,K]}\) is an ideal of \((H, \alpha_H)\) and \((K, \alpha_K)\). Especially, \(([L, L], \alpha_{[L,L]})\) is an ideal of \((L, \alpha_L)\). The quotient \((L/\bar{L}, \alpha_{\bar{L}})\) is called the abelianisation of \((L, \alpha_L)\), and denoted by \((L^{ab}, \alpha_{L^{ab}})\). The Hom-Lie algebra \((L, \alpha_L)\) is called perfect if \(L = [L, L]\).
(iv) An exact sequence \((M, \alpha_M) \to (K, \alpha_K) \to (L, \alpha_L)\) of Hom-Lie algebras is a central extension of \((L, \alpha_L)\) if \(M \subseteq Z(K)\), or equivalently, \([M, K] = 0\).

Let \((M, \alpha_M)\) and \((N, \alpha_N)\) be two Hom-Lie algebras. By a Hom-action of \((M, \alpha_M)\) on \((N, \alpha_N)\), we mean a \(\mathbb{F}\)-bilinear map \(M \times N \to N, (m, n) \mapsto m n\), satisfying the axioms
\[
(a) \quad [m, m'] \alpha_N(n) = \alpha_M(m)(m'n) - \alpha_M(m')(mn), \\
(b) \quad \alpha_M(m)[n, n'] = [mn, \alpha_N(n')] + [\alpha_N(n), m'n], \\
(c) \quad \alpha_N(m'n) = \alpha_M(m)\alpha_N(n),
\]
for all \(m, m' \in M, n, n' \in N\). The action is called trivial if \(mn = 0\), for all \(m \in M, n \in N\). Also, if \((N, \alpha_N)\) is an abelian Hom-Lie algebra enriched with a Hom-action of \((M, \alpha_M)\), then \((N, \alpha_N)\) is said to be a Hom-\(M\)-module (see [29]). Clearly, if \((M, \alpha_M)\) is a subalgebra of some Hom-Lie algebra \((L, \alpha_L)\) and \((N, \alpha_N)\) is an ideal of \((L, \alpha_L)\), then the product in \(L\) induces a Hom-action of \((M, \alpha_M)\) on \((N, \alpha_N)\) given by \(m'n = [m, n]\). In particular, there is a Hom-action of \((L, \alpha_L)\) on itself given by the product in \(L\).

Let \((M, \alpha_M)\) and \((L, \alpha_L)\) be Hom-Lie algebras together with a Hom-action of \((L, \alpha_L)\) on \((M, \alpha_M)\). Their semi-direct product \((M \rtimes L, \alpha_x)\) is the Hom-Lie algebra with the underlying vector space \(M \bowtie L\), the endomorphism \(\alpha_x : M \times L \to M \times L\) given by \(\alpha_x(m, l) = (\alpha_M(m), \alpha_L(l))\), and the product
\[
[(m_1, l_1), (m_2, l_2)] = ([m_1, m_2] + \alpha_L(l_1)m_2 + \alpha_L(l_2)m_1, [l_1, l_2]).
\]
When \((L, \alpha_L)\) acts trivially on \((M, \alpha_M)\), we get the direct sum structure of Hom-Lie algebras.

Let \((M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\zeta} (L, \alpha_L)\) be a split short exact sequence of Hom-Lie algebras, that is, there exists a homomorphism \(\eta : (L, \alpha_L) \to (K, \alpha_K)\) such that \(\zeta \circ \eta = \text{id}_L\). Then we can find a Hom-action of \((L, \alpha_L)\) on \((M, \alpha_M)\) defined by \(i^* = i^{-1}[\eta(l), i(m)]\) for all \(m \in M, l \in L\). Furthermore, we have the following commutative diagram with exact rows:
\[
\begin{array}{ccc}
(M, \alpha_M) & \xrightarrow{i} & (M \rtimes L, \alpha_x) & \xrightarrow{\rho} & (L, \alpha_L) \\
\downarrow{id_M} & & \downarrow{\xi} & & \downarrow{id_L} \\
(K, \alpha_K) & \xrightarrow{\zeta} & (L, \alpha_L) & & \\
\end{array}
\]
where \(i(m) = (m, 0), \rho(m, l) = l, \) and \(\xi\) is an isomorphism defined by \(\xi(m, l) = m + \eta(l)\). In particular, if we put \(T = \eta(L)\), then \((T, \alpha_T)\) is a subalgebra of \((K, \alpha_K)\) such that \(K = M \bowtie T\).

A crossed module is a homomorphism \(\partial : (M, \alpha_M) \to (L, \alpha_L)\) of Hom-Lie algebras together with a Hom-action of \((L, \alpha_L)\) on \((M, \alpha_M)\) such that \(\partial^*(m) = [l, \partial(m)]\) and \(\partial(m)m' = [m, m']\) for all \(m, m' \in M, l \in L\). If \((M, \alpha_M)\) is an ideal of \((L, \alpha_L)\), then the inclusion map \((M, \alpha_M) \hookrightarrow (L, \alpha_L)\) is a crossed module. It is worth noting that for any crossed module \(\partial : (M, \alpha_M) \to (L, \alpha_L)\), \((\text{Im}(\partial), \alpha_{M_L})\) is an ideal of \((L, \alpha_L)\) and \((\ker(\partial), \alpha_M)\) is a central subalgebra of \((M, \alpha_M)\).
2.2 The Schur multiplier and the homology of Hom-Lie algebras

Consider the functors $\mathcal{U} : \text{HomLie} \rightarrow \text{HomSet}$, from the category of Hom-Lie algebras to the category of Hom-sets, that assigns to any Hom-Lie algebra $(B, \alpha_B)$ the Hom-set obtained by forgetting the operations, and $\mathfrak{S}_r : \text{HomSet} \rightarrow \text{HomLie}$, that assigns to any Hom-set $(X, \alpha_X)$ the free Hom-Lie algebra $\mathfrak{S}_r(X, \alpha_X) = (A_X/I, \tilde{\alpha}_A)$ (here, $(A_X, \alpha_A)$ is a non-associative Hom-algebra such that $A_X$ is the $\mathbb{F}$-algebra generated by the free magma $M_X$ and $\alpha_A$ is the endomorphism induced by $\alpha_X$; and $I$ is the two-side ideal of $(A_X, \alpha_A)$ spanned by the elements of the forms

$$ab + ba \quad \text{and} \quad \alpha_A(a)(bc) + \alpha_A(b)(ca) + \alpha_A(c)(ab),$$

where $a, b, c \in A_X$). It is proved in [4] that the functor $\mathfrak{S}_r$ is the left adjoint to the functor $\mathcal{U}$. As an immediate consequence, we conclude that every Hom-Lie algebra admits at least one free presentation.

Let $(L, \alpha_L)$ be an arbitrary Hom-Lie algebra with a free presentation $(R, \alpha_R) \xrightarrow{\rho} (F, \alpha_F) \xrightarrow{\theta} (L, \alpha_L)$. Then the Schur multiplier of $(L, \alpha_L)$ is defined in [5] to be the abelian Hom-Lie algebra

$$\mathcal{M}(L, \alpha_L) = \frac{\langle R, \alpha_R \rangle \cap ([F, F], \alpha_{[F,F]})}{([F, F], \alpha_{[F,F]})}.$$

As the category of Hom-Lie algebras is semi-abelian, Theorem 6.9 of [12] indicates that the Schur multiplier of $(L, \alpha_L)$ is independent of the choice of the free presentation of $(L, \alpha_L)$.

The homology of Hom-Lie algebras, which is a generalization of the Chevalley-Eilenberg homology of Lie algebras, is constructed as follows: Let $(L, \alpha_L)$ be a Hom-Lie algebra and $(M, \alpha_M)$ be a Hom-$L$-module. The homology of $(L, \alpha_L)$ with coefficients in $(M, \alpha_M)$, denoted by $H^\alpha_n(L, M)$, is the homology of the Hom-chain complex $(C^\alpha_n(L, M), d_\ast)$, where $C^\alpha_n(L, M) = (M \otimes L^\wedge n, \alpha_M \otimes \alpha_L^\wedge n), n \geq 0$ ($L^\wedge n$ denotes the $n$-th exterior power of $L$, with $L^\wedge 0 = \mathbb{F}$), and the boundary map $d_n : C^\alpha_n(L, M) \rightarrow C^\alpha_{n-1}(L, M), n \geq 1$, is a homomorphism of Hom-vector spaces defined by

$$d_n(m \otimes x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^n (-1)^i (x_i m \otimes \alpha_L(x_1) \wedge \cdots \wedge \alpha_L(x_i) \wedge \cdots \wedge \alpha_L(x_n))$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} \alpha_M(m) \otimes [x_i, x_j] \wedge \alpha_L(x_1) \wedge \cdots \wedge \alpha_L(x_i) \wedge \cdots \wedge \alpha_L(x_j) \wedge \cdots \wedge \alpha_L(x_n),$$

where the notation $\alpha_L(x_i)$ means that the variable $\alpha_L(x_i)$ is omitted. It is obvious that $H^\alpha_n(L, M) = (\ker(d_n)/\text{Im}(d_{n-1}), \alpha_M \otimes \alpha_L^\wedge n)$ has a Hom-vector space structure for all $n \geq 1$ (note that $H^\alpha_n(L, M)$ is defined in [29] to be the vector space $\ker(d_n)/\text{Im}(d_{n-1})$, while we here take it as a Hom-vector space).

In the special case, if $(M, \alpha_M) = (\mathbb{F}, \text{id}_\mathbb{F})$ is a trivial Hom-$L$-module, then $H^\alpha_n(L, \mathbb{F})$ is said to the $n$-th homology of $(L, \alpha_L)$ and denoted by $H^\alpha_n(L)$. It is easily checked that there is an isomorphism of Hom-vector spaces $H^\alpha_1(L) \cong (L^{ab}, \alpha_{L,ab})$. Also, it is established in [5, Theorem 3.14] that $H^\alpha_2(L) \cong \mathcal{M}(L, \alpha_L)$ for any perfect Hom-Lie algebra $(L, \alpha_L)$. In Section 4, we extend this result for any arbitrary Hom-Lie algebra.
3 The tensor and the exterior products of Hom-Lie algebras

This section is devoted to the study of the properties of the (non-abelian) tensor and exterior products of Hom-Lie algebras. We begin by recalling these concepts.

Let $\partial_1 : (M,\alpha_M) \rightarrow (L,\alpha_L)$ and $\partial_2 : (N,\alpha_N) \rightarrow (L,\alpha_L)$ be two crossed modules of Hom-Lie algebras. There are Hom-actions of $(M,\alpha_M)$ on $(N,\alpha_N)$ and of $(N,\alpha_N)$ on $(M,\alpha_M)$ given by $m_n = \partial_1(m)n$ and $n_m = \partial_2(n)m$. We take $(M,\alpha_M)$ (and $(N,\alpha_N)$) to act on itself by the product. Then the (Hom-Lie) tensor product $(M \star N, \alpha_*)$ is defined in [7] as the Hom-Lie algebra generated by the symbols $m \star n$ (for $m \in M, n \in N$) subject to the relations

$$
(A1) \quad c(m \star n) = cm \star n = m \star cn,
(A2) \quad (m + m') \star n = m \star n + m' \star n,
(A3) \quad m \star (n + n') = m \star n + m \star n',
(A4) \quad [m, m'] \star \alpha_N(n) = \alpha_M(m) \star (\alpha_N(n)),
(A5) \quad \alpha_M(m) \star [n, n'] = (\alpha_M(n)) \star \alpha_N(n),
(A6) \quad [(m \star n), (m' \star n')] = - (\alpha_M(m)) \star (\alpha_N(n'))
$$

Note that the identity homomorphism $id_L : (L,\alpha_L) \rightarrow (L,\alpha_L)$ is a crossed module with $(L,\alpha_L)$ acting on itself by the product, so we can always form the tensor products $(L \star M, \alpha_*)$, $(L \star N, \alpha_*)$ and $(L \star L, \alpha_*)$. Also, if $\alpha_M = id_M$, $\alpha_N = id_N$ and $\alpha_L = id_L$, then $M \star N$ coincides with the tensor product of Lie algebras given in [11].

The following proposition gives some useful information on Hom-Lie tensor products, the proof of which are left to the reader (see also [7]).

**Proposition 3.1.** With the above assumptions and notations, we have :

(i) The maps

$$
\lambda : (M \star N, \alpha_*) \rightarrow (L,\alpha_L), \quad m \star n \rightarrow [\partial_1(m), \partial_2(n)]
$$

$$
\lambda_M : (M \star N, \alpha_*) \rightarrow (M,\alpha_M), \quad m \star n \rightarrow -n_m
$$

$$
\lambda_N : (M \star N, \alpha_*) \rightarrow (N,\alpha_N), \quad m \star n \rightarrow m_n
$$

are homomorphisms of Hom-Lie algebras with the kernels contained in the center of $(M \star N, \alpha_*)$.

(ii) There is a Hom-action of $(L,\alpha_L)$ on $(M \star N, \alpha_*)$ given by

$$
l^r(m \star n) = l^r m \star \alpha_N(n) + \alpha_M(m) \star l^r n,
$$

and then $(M,\alpha_M)$ and $(N,\alpha_N)$ act on $(M \star N, \alpha_*)$ via $\partial_1$ and $\partial_2$. Moreover,

$$
\lambda(l^r x) = [\alpha_L(l), \lambda(x)] \quad \text{and} \quad \lambda^x(x') = [\alpha_N(x), x']
$$

for all $x, x' \in M \star N$, $l \in L$, and the relations similar to (1) are valid for $\lambda_M$ and $\lambda_N$.

(iii) If $(M,\alpha_M)$ and $(N,\alpha_N)$ act trivially on each other and both maps $\alpha_M$ and $\alpha_N$ are surjective, then there is an isomorphism of abelian Hom-Lie algebras $(M \star N, \alpha_*) \cong (M^{ab} \otimes N^{ab}, \alpha_{\otimes})$, where $\alpha_{\otimes}$ is induced by $\alpha_M$ and $\alpha_N$. 

(iv) Let \( (M, \alpha_M) \mapsto (K, \alpha_K) \mapsto (L, \alpha_L) \) and \( (M', \alpha_{M'}) \mapsto (K', \alpha_{K'}) \mapsto (L', \alpha_{L'}) \) be short exact sequences of Hom-Lie algebras, where \( (M', \alpha_{M'}) \) and \( (K', \alpha_{K'}) \) are ideals of \( (K, \alpha_K) \), and \( (L', \alpha_{L'}) \) is an ideal of \( (L, \alpha_L) \). Then there exists an exact sequence of Hom-Lie algebras

\[
((M \star K') \times (K \star M'), \alpha_\otimes) \longrightarrow (K \star K', \alpha_\star) \longrightarrow (L \star L', \alpha_\star),
\]

where the Hom-action of \( (K \star M', \alpha_\star) \) on \( (M \star K', \alpha_\star) \) is induced by the homomorphism \( \lambda_{M'} : (K \star M', \alpha_\star) \longrightarrow (M', \alpha_{M'}) \).

In view of part (i) of the above proposition, for any Hom-Lie algebra \( (L, \alpha_L) \), the Hom-map \( \lambda_L : (L \star L, \alpha_\star) \longrightarrow (L, \alpha_L) \), \( l_1 \star l_2 \rightsquigarrow [l_1, l_2] \), is a homomorphism (which is called the commutator Hom-map). We define \( J_2^\alpha(L) = (\ker \lambda_L, \alpha_\star) \).

Let \( (M \boxtimes N, \alpha_\boxtimes) \) be the Hom-vector subspace of \( (M \star N, \alpha_\star) \), where \( M \boxtimes N \) is the vector subspace of \( M \star N \) spanned by the elements of form \( m \star n \) with \( \partial_1(m) = \partial_2(n) \) and \( \alpha_\boxtimes \) is the restriction of \( \alpha_\star \) to \( M \boxtimes N \). Then \( M \boxtimes N \subseteq Z(M \star N) \) and so \( (M \boxtimes N, \alpha_\boxtimes) \) is an ideal of \( (M \star N, \alpha_\star) \). Following [5], the (Hom-Lie) exterior product \( (M \wedge N, \alpha_\wedge) \) is defined to be the quotient \( (\ker (M \star N)/(M \boxtimes N), \tilde{\alpha}_\star) \). We write \( m \wedge n \) to denote the image in \( M \wedge N \) of the generator \( m \star n \).

It is readily seen that the parts (i) and (ii) of Proposition 3.1 hold with \( \star \) replaced by \( \wedge \). We can also conclude from Proposition 3.1(iv) that every short exact sequence \( e : (M, \alpha_M) \mapsto (K, \alpha_K) \mapsto (L, \alpha_L) \) of Hom-Lie algebras induces an exact sequence

\[
(M \wedge K, \alpha_\wedge) \longrightarrow (K \wedge K, \alpha_\wedge) \longrightarrow (L \wedge L, \alpha_\wedge). \tag{2}
\]

Note that the functorial homomorphisms \( (K \wedge M, \alpha_\wedge) \longrightarrow (K \wedge K, \alpha_\wedge) \) and \( (M \wedge K, \alpha_\wedge) \longrightarrow (K \wedge K, \alpha_\wedge) \) have the same images. We can say more if the extension \( e \) is split.

Lemma 3.2. A split extension \( (M, \alpha_M) \mapsto (K, \alpha_K) \mapsto (L, \alpha_L) \) of Hom-Lie algebras induces a short exact sequence of Hom-Lie algebras \( (M \wedge K, \alpha_\wedge) \overset{i}{\longrightarrow} (K \wedge K, \alpha_\wedge) \overset{\text{j}}{\longrightarrow} (L \wedge L, \alpha_\wedge) \).

Proof. We only require to prove that the homomorphism \( i \) is injective. To do this, it suffices to construct a homomorphism of Hom-Lie algebras \( \theta : (K \wedge K, \alpha_\wedge) \longrightarrow ((M \wedge K) \wedge (L \wedge L), \alpha_\wedge) \) such that the composite map \( \theta \circ i \) is the canonical inclusion. Here the Hom-action of \( (L \wedge L, \alpha_\wedge) \) on \( (M \wedge K, \alpha_\wedge) \) is defined as follows:

\[
x(m \wedge k) = [\lambda_L(x), m] \wedge \alpha_K(k) + \alpha_M(m) \wedge [\lambda_L(x), k],
\]

for all \( x \in L \wedge L, m \in M, k \in K \), in which \( \lambda_L : (L \wedge L, \alpha_\wedge) \longrightarrow (L, \alpha_L) \), \( l_1 \wedge l_2 \rightsquigarrow [l_1, l_2] \), is the commutator Hom-map and \( (L, \alpha_L) \) is considered as a subalgebra of \( (K, \alpha_K) \). Since \( (K, \alpha_K) \cong (M \wedge L, \alpha_\wedge) \), we can define

\[
\theta : ((M \wedge L) \wedge (M \wedge L), \alpha_\wedge) \longrightarrow ((M \wedge (M \wedge L)) \wedge (L \wedge L), \alpha_\wedge).
\]

\[
(m_1, l_1) \wedge (m_2, l_2) \rightsquigarrow (m_1 \wedge (m_2, l_2) - m_2 \wedge (0, l_1), l_1 \wedge l_2)
\]
It is routine to check that \( \theta \) preserves the defining relations of the Hom-Lie exterior product with \( \theta \circ \alpha_\lambda = \alpha_A \circ \theta \), and is therefore the required homomorphism.

To show how the exterior product is related to universal central extensions of Hom-Lie algebras, we need the following

**Lemma 3.3.** If \( e : (M,\alpha_M) \rightarrow (K,\alpha_K) \) is a central extension of Hom-Lie algebras, then there exists a homomorphism of Hom-Lie algebras \( \psi : (L \triangleleft L,\alpha_\lambda) \rightarrow (K,\alpha_K) \) such that \( \phi \circ \psi = id_L \circ \lambda_L \). Moreover, if \((L,\alpha_L)\) is perfect then \( \psi \) is unique.

**Proof.** We define the Hom-map \( \psi \) on generators by \( \psi(l_1 \triangleleft l_2) = [k_1,k_2] \), where \( \phi(k_i) = l_i \), \( i = 1,2 \). Due to the centrality of the extension \( e \), \( \psi \) preserves the relations of the exterior product with \( \psi \circ \alpha_\lambda = \alpha_K \circ \psi \), and is thus the required homomorphism. If \( \psi, \psi' : (L \triangleleft L,\alpha_\lambda) \rightarrow (K,\alpha_K) \) are two homomorphisms with \( \phi \circ \psi = \phi \circ \psi' \), then \( \psi - \psi' = \iota \circ \eta \), where \( \eta : (L \triangleleft L,\alpha_\lambda) \rightarrow (M,\alpha_M) \) is a homomorphism such that \( [L \triangleleft L,L \triangleleft L] \) is contained in \( \ker \eta \). By the relation (A6), if \((L,\alpha_L)\) is perfect, then so is \((L \triangleleft L,\alpha_\lambda)\), forcing the uniqueness of \( \psi \), as desired.

The above lemma, together with \cite{7} Theorem 3.4, leads us to the following result.

**Proposition 3.4.** For any perfect Hom-Lie algebra \((L,\alpha_L)\), the extension

\[
(\ker(\lambda_L),\alpha_\lambda) \rightarrow (L \triangleleft L,\alpha_\lambda) \xrightarrow{\lambda_L} (L,\alpha_L)
\]

is the universal central extension of \((L,\alpha_L)\) and so, there is an isomorphism of Hom-Lie algebras \((L \triangleleft L,\alpha_\lambda) \cong (L \star L,\alpha_\ast)\). In particular, \(H^0_\ast(L) \cong (\ker(\lambda_L),\alpha_\lambda)\).

We end this section by analyzing the kernel of the natural homomorphism \((M \star N,\alpha_\ast) \rightarrow (M \triangleleft N,\alpha_\lambda)\). To do this, we require the following definition, which is a generalized version of Whitehead’s quadratic functor in the context of Hom-vector spaces.

**Definition.** The universal quadratic Hom-functor \( \Gamma \) is defined for any Hom-vector space \((A,\alpha_A)\) to be the Hom-vector space \((\Gamma(A),\alpha_\Gamma)\), where \(\Gamma(A)\) is the vector space generated by the symbols \( \gamma(a) \) with \( a \in A \), subject to the relations

\[
\lambda^2 \gamma(a) = \gamma(\lambda a),
\]

\[
\gamma(\lambda a + b) + \lambda \gamma(a) + \lambda \gamma(b) = \lambda \gamma(a + b) + \gamma(\lambda a) + \gamma(b),
\]

\[
\gamma(a + b + c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a + b) + \gamma(a + c) + \gamma(b + c),
\]

for all \( \lambda \in \mathbb{F} \), \( a, b, c \in A \), and the linear map \( \alpha_\Gamma : \Gamma(A) \rightarrow \Gamma(A) \) is given by \( \alpha_\Gamma(\gamma(a)) = \gamma(\alpha_A(a)) \).

Under the assumptions at the beginning of this section, we set

\[
M \times_L N = \{(m,n) \in M \oplus N \mid \partial_1(m) = \partial_2(n)\}
\]

and \( (M,N) = \{(\lambda_M(x),\lambda_N(x)) \mid x \in M \star N\} \).
Then \((M \times_L N, \alpha\oplus)\) is a subalgebra of the direct sum \((M \oplus N, \alpha\oplus)\), \((\langle M, N \rangle, \alpha\oplus)\) is an ideal of \((M \times_L N, \alpha\oplus)\), and the quotient \((M \times_L N/\langle M, N \rangle, \alpha\oplus)\) is abelian. By arguments similar to those used in [11] Proposition 14], we can obtain the following natural exact sequence of Hom-Lie algebras
\[
\begin{align*}
(\Gamma(\frac{M \times_L N}{\langle M, N \rangle}), \alpha) & \xrightarrow{\psi} (M \star N, \alpha) \to (M \wedge N, \alpha),
\end{align*}
\]
where \(\psi(\gamma((m, n))) = m \star n\) for \((m, n)\) the coset of \(\langle M, N \rangle\) represented by \((m, n) \in M \times_L N\). In particular, if \((M, \alpha_M)\) and \((N, \alpha_N)\) are ideals of a Hom-Lie algebra, then there is an exact sequence
\[
\begin{align*}
(\Gamma(\frac{M \cap N}{\langle M, N \rangle}), \alpha) & \xrightarrow{\psi} (M \star N, \alpha) \to (M \wedge N, \alpha), \quad (3)
\end{align*}
\]
In next result, we show that in the case of \((M, \alpha_M) = (N, \alpha_N)\), the homomorphism \(\psi\) in (3) is injective.

**Proposition 3.5.** For any Hom-Lie algebra \((L, \alpha_L)\) with surjective endomorphism \(\alpha_L\), there is an exact sequence of Hom-Lie algebras \((\Gamma(L^{ab}), \alpha) \xrightarrow{\psi} (L \star L, \alpha) \to (L \wedge L, \alpha)\).

**Proof.** In view of the exact sequence (3), it suffices to prove that \(\psi\) is injective. Owing to Proposition 3.1(iii), \((L^{ab} \star L^{ab}, \alpha) \cong (L^{ab} \otimes L^{ab}, \alpha\oplus)\). Taking into account [28] Proposition 8.6, one easily sees that the composite homomorphism
\[
\varphi : (\Gamma(L^{ab}), \alpha) \xrightarrow{\psi} (L \Box L, \alpha) \xrightarrow{\subseteq} (L \star L, \alpha) \xrightarrow{\text{nat}} (L^{ab} \star L^{ab}, \alpha) \xrightarrow{=} (L^{ab} \otimes L^{ab}, \alpha),
\]
maps a basis of \(\Gamma(L^{ab})\) injectively into a set of linearly independent elements. Therefore \(\varphi\) and then \(\psi\) are injective. \(\square\)

We deduce from Proposition 3.5 the following consequence, which is used in Section 5.

**Corollary 3.6.** With the assumptions of Proposition 3.5, there is an isomorphism of Hom-Lie algebras \(\rho : (L \Box L, \alpha) \longrightarrow (L^{ab} \Box L^{ab}, \alpha\oplus)\), given by \((x \star y) \mapsto (\bar{x} \star \bar{y})\).

**Proof.** It is sufficient to note that \(\rho\) is equal to the composite homomorphism
\[
(L \Box L, \alpha) \xrightarrow{=} (\Gamma(L^{ab}), \alpha) \xrightarrow{=} (L^{ab} \Box L^{ab}, \alpha)\).
\(\square\)

### 4 The Hopf’s formula for Hom-Lie algebras

In [10], Ellis proves that, for any Lie algebra \(K\), the second homology of \(K\), \(H_2(K)\), is isomorphic to the kernel of the commutator map \(K \wedge K \xrightarrow{\lbrack-\rbrack} K\) (here \(\wedge\) denotes the non-abelian exterior product of Lie algebras). Using this result, he determines in [11] the behavior of the functor \(H_2(-)\) with respect to the direct sum of Lie algebras and also, applying topological techniques, gets an eight-term exact sequence in homology of Lie algebras
\[
H_3(K) \to H_3(L) \to H_2(K, M) \to H_2(K) \to H_2(L) \to M/[M,K] \to H_1(K) \to H_1(L). \quad (4)
\]
from a short exact sequence of Lie algebras \( M \to K \to L \) (here, \( H_2(K, M) \) denotes the second relative Chevalley-Eilenberg homology of the pair \( (K, M) \), which is isomorphic to \( \ker(M \wedge K \to K) \)). In particular, he obtains the Hopf’s formula for Lie algebras and, moreover, shows that if \( F/R \) is a free presentation of \( K \) and \( S/R \) is the induced presentation of \( M \) for some ideal \( S \) of \( F \), then \( H_3(K) \cong \ker(R \wedge F \to F) \) and \( H_3(L) \cong \ker(S \wedge F \to F) \). In this section, we generalize these results to Hom-Lie algebras. We start with the following theorem.

**Theorem 4.1.** Let \((L, \alpha_L)\) be any Hom-Lie algebra. Then there is an isomorphism of Hom-vector spaces \( H^3_2(L) \cong (\ker(\lambda_L), \alpha_{\ker(\lambda_L)}) \), where \( \lambda_L : (L \wedge L, \alpha_L) \to (L, \alpha_L) \) is the commutator Hom-map.

**Proof.** We recall that the linear Hom-maps \( d_2 : C^2_2(L, F) \to C^3_2(L, F) \) and \( d_3 : C^3_3(L, F) \to C^3_2(L, F) \) are defined by \( d_2(x \wedge y) = [x, y] \) and \( d_3(x \wedge y \wedge z) = -[x, y] \wedge \alpha_L(z) + \alpha_L(x) \wedge [y, z] + [x, z] \wedge \alpha_L(y) \), respectively. Hence \( \text{Im}(d_3) = \text{span}\{[x, y] \wedge \alpha_L(z) + \alpha_L(x) \wedge [y, z] + [x, z] \wedge \alpha_L(y) \ | \ x, y, z \in L\} \). Obviously, the linear Hom-map \( \psi : ((L \wedge L)/\text{Im}(d_3), \alpha_L^2) \to (L \wedge L, \alpha_L) \) given by \( \psi(x \wedge y + \text{Im}(d_3)) = x \wedge y \) is surjective. We now have the following diagram of Hom-vector spaces, in which the rows are exact:

\[
\begin{array}{c}
\bigtriangledown \phantom{\bigtriangledown} H^3_2(L) \phantom{\bigtriangledown} \\
\downarrow^{\phi} \downarrow^{\psi} \\
(\ker(\lambda_L), \alpha_{\ker(\lambda_L)}) \phantom{\bigtriangledown} \xrightarrow{\lambda_L} \phantom{\bigtriangledown} ([L, L], \alpha_{[L, L]}) \phantom{\bigtriangledown} \xrightarrow{\alpha_L^2} \phantom{\bigtriangledown} ([L, L], \alpha_{[L, L]})
\end{array}
\]

Note that we here consider \((L \wedge L, \alpha_L)\) as a Hom-vector space. By comparing the relation (A4) and the generators of \( \text{Im}(d_3) \), it follows that \( \psi \) has an inverse \( \psi' \) that sends \( x \wedge y \) to \( x \wedge y + \text{Im}(d_3) \). We therefore deduce that \( \phi \) is an isomorphism, as required. \( \square \)

The following corollary is a direct consequence of the above theorem and Proposition 3.5.

**Corollary 4.2.** For any Hom-Lie algebra \((L, \alpha_L)\) with surjective endomorphism \( \alpha_L \), there is an exact sequence of Hom-vector spaces \((\Gamma(L^a), \alpha_\Gamma) \to J_2^3(L) \to H^3_2(L)\). In particular, if \( L \) is of finite dimension, then \( \dim(J_2^3(L)) = \dim(H^3_2(L)) + \dim(\Gamma(L^a)) \) (as vector spaces).

**Theorem 4.3.** Let \((F, \alpha_F)\) be any free Hom-Lie algebra. Then there is an isomorphism of Hom-Lie algebras \((F \wedge F, \alpha_\wedge) \cong ([F, F], \alpha_{[F, F]})\).

**Proof.** We have only to prove that \( \lambda_F : (F \wedge F, \alpha_\wedge) \to ([F, F], \alpha_{[F, F]}) \) is injective. Using the same notations as in Subsection 2.2, suppose \((F, \alpha_F) = (X/I, \alpha_A)\) and \((Y, \alpha_Y)\) is a subalgebra of \((X, \alpha_A)\), where \( Y \) is the \( \alpha_A \)-invariant subspace of \( X \) generated by the set \( \{ab \ | \ a, b \in X\} \). Note that each element \( x \in Y \) is written as a unique finite sum \( \sum_{i=1}^{n} a_i b_i, \ a_i, b_i \in X \). Therefore, \( \phi : (Y, \alpha_Y) \to \)
\((F \land F, \alpha_\lambda)\), given by \(ab \mapsto \bar{a} \land \bar{b}\), is a well-defined linear Hom-map (where \(\bar{a}\) denotes the coset \(a + I \in A_X/I\)). For any \(a, b, c \in A_X\), we have

\[
\phi(ab + ba) = \bar{a} \land \bar{b} + \bar{b} \land \bar{a} = 0,
\]

\[
\phi(\alpha_A(a)bc + \alpha_A(b)ca + \alpha_A(c)ab) = \bar{\alpha}_A(\bar{a}) \land [\bar{b}, \bar{c}] + \bar{\alpha}_A(\bar{b}) \land [\bar{c}, \bar{a}] + \bar{\alpha}_A(\bar{c}) \land [\bar{a}, \bar{b}]) = 0.
\]

Hence \(\phi\) induces a linear Hom-map \(\tilde{\phi} : ([F, F] = Y/I, \alpha_{[F, F]}) \rightarrow (F \land F, \alpha_\lambda)\). Furthermore, \(\lambda_F \circ \tilde{\phi} = id_{[F, F]}\) and \(\tilde{\phi} \circ \lambda_F = id_{F \land F}\). This completes the proof. \(\square\)

From the above theorem, we have the following corollary.

**Corollary 4.4.** Let \((R, \alpha_R) \xrightarrow{\pi} (F, \alpha_F) \xrightarrow{\lambda_F} (L, \alpha_L)\) be a free presentation of the Hom-Lie algebra \((L, \alpha_L)\). Then there is an isomorphism of Hom-Lie algebras \((L \land L, \alpha_\lambda) \cong ([F, F]/[R, F], \bar{\alpha}_{[F, F]}).\) In particular, \(M(L, \alpha_L) \cong (\ker(\lambda_L), \alpha_{\ker(\lambda_L)})\).

**Proof.** Consider the following commutative diagram of Hom-Lie algebras with exacts rows:

\[
\begin{array}{ccc}
(F \land R, \alpha_\lambda) & \xrightarrow{\theta} & (F \land F, \alpha_\lambda) \\
\downarrow \lambda_F & & \downarrow \lambda_F \\
(\ker(\pi), \alpha_{\ker(\pi)}) & & ([F, F]/[R, F], \bar{\alpha}_{[F, F]}) \\
\end{array}
\]

Evidently, \(\lambda_F\) maps the subalgebra \((\text{Im}(\theta), \alpha_{\text{Im}(\theta)})\) isomorphically onto \(([R, F], \alpha_{[R, F]})\). We consequently conclude from Theorem 4.3 that

\[
(L \land L, \alpha_\lambda) \cong (F \land F, \alpha_\lambda) \cong ([F, F]/[R, F], \bar{\alpha}_{[F, F]}),
\]

and the proof is complete. \(\square\)

Combining the above corollary with Theorem 4.1, we have the main result of this section.

**Corollary 4.5** (Hopf’s formula for Hom-Lie algebras). For any Hom-Lie algebra \((L, \alpha_L)\), there is an isomorphism of Hom-vector spaces \(H^0_2(L) \cong M(L, \alpha_L)\).

As an immediate consequence of Corollary 4.5, we conclude that the second homology of any free Hom-Lie algebra is trivial.

Using Theorem 4.1, we generalize the exact sequence (4) for Hom-Lie algebras.

**Theorem 4.6.** Let \(e : (M, \alpha_M) \rightarrow (K, \alpha_K) \rightarrow (L, \alpha_L)\) be an extension of Hom-Lie algebras. Let \((R, \alpha_R) \rightarrow (F, \alpha_F) \rightarrow (K, \alpha_K)\) be a free presentation of \((K, \alpha_K)\) and \((M, \alpha_M) \cong (S/R, \bar{\alpha}_S)\) for some ideal \((S, \alpha_S)\) of \((F, \alpha_F)\). Then there is an exact sequence of Hom-vector spaces

\[
\ker(R \land F \rightarrow F, \alpha_\lambda) \rightarrow \ker(S \land F \rightarrow F, \alpha_\lambda) \rightarrow \ker(M \land K \rightarrow K, \alpha_\lambda) \rightarrow H^0_2(K)
\]
\[ H_2^\alpha(L) \rightarrow (\frac{M}{[M,K]}, \bar{\alpha}_M) \rightarrow H_1^\alpha(K) \rightarrow H_1^\alpha(L). \] (5)

Moreover, if the extension \( e \) is split, then the sequence (5) induces a short exact sequence

\[ (\ker(M \oplus K \rightarrow K), \alpha_\perp) \rightarrow H_2^\alpha(K) \rightarrow H_2^\alpha(L). \] (6)

**Proof.** Consider the following commutative diagrams of Hom-Lie algebras:

\[
\begin{array}{cccccc}
(M \cdot K, \alpha_K) & (K \cdot K, \alpha_K) & (L \cdot L, \alpha_L) & (R \cdot F, \alpha_L) & (S \cdot F, \alpha_L) & (M \cdot L, \alpha_L) \\
\lambda_M & \lambda_K & \lambda_L & \lambda_R & \lambda_S & \lambda_M \\
(M, \alpha_M) & (K, \alpha_K) & (L, \alpha_L) & (R, \alpha_R) & (S, \alpha_S) & (M, \alpha_M)
\end{array}
\]

where, the rows are exact. Applying the Snake Lemma to these diagrams, we have the exact sequences

\[
\begin{align*}
(\ker(\lambda_M), \alpha_\perp) & \xrightarrow{\delta} (\ker(\lambda_K), \alpha_\perp) \rightarrow (\ker(\lambda_L), \alpha_\perp) \xrightarrow{\Delta_1} (\frac{M}{[M,K]}, \bar{\alpha}_M) \rightarrow H_1^\alpha(K) \rightarrow H_1^\alpha(L), \\
(\ker(\lambda_R), \alpha_\perp) & \rightarrow (\ker(\lambda_S), \alpha_\perp) \rightarrow (\ker(\lambda_M), \alpha_\perp) \xrightarrow{\Delta_2} (\frac{R}{[R,F]}, \bar{\alpha}_R),
\end{align*}
\]

where \( \Delta_i, i=1,2 \), is the connecting homomorphism of Hom-vector spaces. It now remains to show that \( \ker \Delta_2 = \ker \delta \). But this immediately follows from the commutative diagram

\[
\begin{array}{ccc}
(\ker(\lambda_M), \alpha_\perp) & \xrightarrow{\Delta_2} & (\frac{R}{[R,F]}, \bar{\alpha}_R) \\
\downarrow & & \downarrow \beta \subseteq H_2^\alpha(K), \\
(\ker(\lambda_K), \alpha_\perp) & & \beta
\end{array}
\]

where \( \beta \) is the isomorphism obtained in Theorem 4.1. Thus, the sequence (5) is exact.

We now verify the exactness of (6). According to the points mentioned at the end of Subsection 2.1, we can assume that \( (L, \alpha_L) \) is a subalgebra of \( (K, \alpha_K) \) such that \( K = M \cdot L \). Then

\[ [K, K] \cap M = ([L, L] + [M, K]) \cap M = ([L, L] \cap M) + [M, K] = [M, K]. \]

Also, invoking Lemma 3.2, we may consider \( (M \cdot K, \alpha_K) \) as a subalgebra of \( (K \cdot K, \alpha_K) \), which implies that \( (M \cdot K) \cap \ker(\lambda_K) = \ker(\lambda_M) \). So, we obtain the following commutative diagram of Hom-Lie algebras:

\[
\begin{array}{cccc}
(\ker(\lambda_M), \alpha_\perp) & \rightarrow & H_2^\alpha(K) & \rightarrow H_2^\alpha(L) \\
\downarrow & & \downarrow \vartheta & \downarrow \\
(M \cdot K, \alpha_K) & \rightarrow & (K \cdot K, \alpha_K) & \rightarrow (L \cdot L, \alpha_L) \\
\downarrow & & \downarrow & \downarrow \\
([M, K], \alpha_{[M,K]}) & \rightarrow & ([K, K], \alpha_{[K,K]}) & \rightarrow ([L, L], \alpha_{[L,L]}),
\end{array}
\]

where the columns and rows are exact. One easily sees from the above diagram that \( \vartheta \) is surjective.

\[ \square \]
Using the above theorem and the explanations at the beginning of this section, we have the following conjecture.

**Conjecture.** With the assumptions of Theorem 4.6, $H^3_2(F) = 0$ and there is an isomorphism of Hom-vector spaces $H^3_2(L) \cong (\ker(S \wedge F \to F), \alpha_L)$.

Finally, we close this section by examining the behavior of functors $J_2^a(-)$ and $H_2^a(-)$ to the direct sum of Hom-Lie algebras.

**Theorem 4.7.** Let $(L_1, \alpha_{L_1})$ and $(L_2, \alpha_{L_2})$ be two Hom-Lie algebras with surjective endomorphisms $\alpha_{L_1}$ and $\alpha_{L_2}$. Then there are isomorphisms of Hom-vector spaces

$$J_2^a(L_1 \oplus L_2) \cong J_2^a(L_1) \oplus J_2^a(L_2) \oplus (L_1^b \otimes L_2^b, \alpha_\otimes) \oplus (L_2^b \otimes L_1^b, \alpha_\otimes),$$

$$H_2^a(L_1 \oplus L_2) \cong H_2^a(L_1) \oplus H_2^a(L_2) \oplus (L_1^b \otimes L_2^b, \alpha_\otimes).$$

**Proof.** We only need to prove that

$$((L_1 \oplus L_2) \star (L_1 \oplus L_2), \alpha_*) \cong (L_1 \star L_1, \alpha_*) \oplus (L_2 \star L_2, \alpha_*) \oplus (L_1^b \otimes L_2^b, \alpha_\otimes) \oplus (L_2^b \otimes L_1^b, \alpha_\otimes), \quad (7)$$

$$((L_1 \oplus L_2) \wedge (L_1 \oplus L_2), \alpha_\wedge) \cong (L_1 \wedge L_1, \alpha_\wedge) \oplus (L_2 \wedge L_2, \alpha_\wedge) \oplus (L_1^b \otimes L_2^b, \alpha_\otimes). \quad (8)$$

Put $(L, \alpha_L) = (L_1 \oplus L_2, \alpha_\otimes)$, and identify $(L_1, \alpha_{L_1})$ and $(L_2, \alpha_{L_2})$ with their images in $(L, \alpha_L)$. Then $[L_1, L_2] = 0$ and so, invoking Proposition 3.1(iii), $(L_1 \star L_2, \alpha_*) \cong (L_1^b \otimes L_2^b, \alpha_\otimes)$ and $(L_2 \star L_1, \alpha_*) \cong (L_2^b \otimes L_1^b, \alpha_\otimes)$. We claim that for any ideal $(K, \alpha_K)$ of $(L, \alpha_L)$, there is an isomorphism of Hom-Lie algebras

$$(K \star L, \alpha_*) \cong ((K \star L_1) \oplus (K \star L_2), \alpha_\otimes). \quad (9)$$

Define the Hom-map $\varphi : (K \star L, \alpha_*) \to ((K \star L_1) \oplus (K \star L_2), \alpha_\otimes)$, $(k \star (l_1, l_2)) \mapsto ((k \star l_1), (k \star l_2))$. A page of routine calculations shows that $\varphi$ preserves the defining relations of the exterior product and also, $\varphi \circ \alpha_* = \alpha_\otimes \circ \varphi$. On the other hand, the inclusions of $L_1$ and $L_2$ into $L$ yield linear Hom-maps $(K \star L_1, \alpha_*) \to (K \star L, \alpha_*)$ and $(K \star L_2, \alpha_*) \to (K \star L, \alpha_*)$, which combine to give an inverse $\psi$ of $\varphi$. It therefore follows that $\varphi$ is an isomorphism of Hom-Lie algebras, as claimed. We now get the isomorphism (7) by applying the isomorphism (9) twice.

For the isomorphism (8), it is enough to note that for the isomorphism $\varphi$ obtained in proving (7), we have $\varphi((L_1 \oplus L_2) \Box (L_1 \oplus L_2)) = (L_1 \Box L_1) \oplus (L_2 \Box L_2)$, which deduces the result.

The following corollary generalizes a result due to Simson and Tye [28].

**Corollary 4.8.** Let $(V_1, \alpha_{V_1})$ and $(V_2, \alpha_{V_2})$ be two Hom-vector spaces with surjective endomorphisms $\alpha_{V_1}$ and $\alpha_{V_2}$. Then there is an isomorphism of Hom-vector spaces

$$(\Gamma(V_1 \oplus V_2), \alpha_\Gamma) \cong (\Gamma(V_1) \oplus \Gamma(V_2) \oplus (V_1 \otimes V_2), \alpha_\otimes).$$

**Proof.** It follows immediately from Theorem 4.7 and Proposition 3.5. \qed
5 The capability of Hom-Lie algebras

In this section, we investigate some of the applications of exterior product for developing the theory of capability of Hom-Lie algebras. We first recall from [5] that:

- A Hom-Lie algebra \((L, \alpha_L)\) is said to be capable if there exists a Hom-Lie algebra \((K, \alpha_K)\) such that \((L, \alpha_L) \cong (K/Z_\alpha(K), \tilde{\alpha}_K)\). When \(\alpha_L = Id_L\), this definition recovers the notion of capable Lie algebra in [26].

- Let \((L, \alpha_L)\) be any Hom-Lie algebra.
  
  (a) The tensor centre \(Z^*_\alpha(L)\) of \((L, \alpha_L)\) is defined to be the vector space \(Z^*_\alpha(L) = \{x \in L \mid \alpha^k_L(x) \ast l = 0_{L \ast L}, \text{ for all } l \in L, k \geq 0\}\).

  (b) The exterior centre \(Z^0_\alpha(L)\) of \((L, \alpha_L)\) is defined to be the vector space \(Z^0_\alpha(L) = \{x \in L \mid \alpha^k_L(x) \ast l = 0_{L \ast L}, \text{ for all } l \in L, k \geq 0\}\).

Plainly, \(Z^0_\alpha(L) \subseteq Z^*_\alpha(L)\), and the equality holds whenever \((L, \alpha_L)\) is perfect, by Proposition 3.4.

Example 5.1. (i) Let \((L, \alpha_L)\) be an abelian Hom-Lie algebra with linear basis \(\{e_i \mid i \in I\}\), where \(I\) is a well-ordered set. Consider the Hom-Lie algebra \((K, \alpha_K)\) with linear basis \(\{e_i, e_{jk} \mid i, j, k \in I, j < k\}\), product given by \([e_j, e_k] = e_{jk}\) for all \(j < k\), and zero elsewhere, and the endomorphism \(\alpha_K\) is defined as \(\alpha_K(e_i) = \alpha_L(e_i)\) and \(\alpha_K(e_{jk}) = [\alpha_K(e_j), \alpha_K(e_k)]\). Since all triple products of elements in \(K\) are zero, one sees that \((K, \alpha_K)\) satisfies the conditions of a Hom-Lie algebra. Furthermore, \(Z_\alpha(K) = \text{span}\{e_j \mid j < k\}\), and \((Z_\alpha(K), \alpha_{Z_\alpha(K)}) \rightarrow (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L)\) is an extension of Hom-Lie algebras, where \(\pi\) is defined by \(\pi(e_i) = e_i\) and \(\pi(e_{jk}) = 0\). Hence \((L, \alpha_L)\) is capable.

(ii) Let \((L, \alpha_L)\) be a Hom-Lie algebra with \(\alpha_L = 0\). We claim that \((L, \alpha_L)\) is capable. The result is clear when \(Z_\alpha(L) = 0\) or \((L, \alpha_L)\) is abelian. Choose a linear basis \(\{e_i \mid i \in I\}\) for \(Z_\alpha(L)\) and extend it to a linear basis \(\{e_i, f_j \mid i \in I, j \in J\}\) for \(L\), where \(I\) and \(J\) are non-empty well-defined ordered sets. Take the vector space \(K\) with a linear basis \(\{e_i, t_i, f_j \mid i \in I, j \in J\}\) (in fact, \(K\) is a vector space direct sum of \(L\) and the space generated by the set \(\{t_i \mid i \in I\}\)), together with the following product: \([e_i, f_{j_0}] = t_i\) for some fixed \(j_0 \in J\) and all \(i \in I\), \([f_j, f_k]\) is the same in \(L\) for all \(j, k \in J\), and zero elsewhere. Then \(K\) with companion endomorphism \(\alpha_K = 0\) is a Hom-Lie algebra such that \(Z_\alpha(K) = \text{span}\{t_i \mid i \in I\}\) and \((K/Z_\alpha(K), \tilde{\alpha}_K) \cong (L, \alpha_L)\), as claimed.

(iii) (iii) Consider the three-dimensional Hom-Lie algebra \((L, \alpha_L)\) with linear basis \(\{e_1, e_2, e_3\}\), the product given by \([e_1, e_2] = e_3\) (unwritten products are equal to zero), and the non-surjective endomorphism \(\alpha_L\) defined by \(\alpha_L(e_1) = e_3, \alpha_L(e_2) = e_2, \alpha_L(e_3) = 0\). Evidently, \(Z_\alpha(L) = L^2 = \text{span}\{e_3\}\). Let \((K, \alpha_K)\) be the four-dimensional Hom-Lie algebra with linear basis \(\{f_1, f_2, f_3, f_4\}\), the product given by \([f_1, f_2] = f_3, [f_3, f_1] = f_4\), and the endomorphism defined by \(\alpha_K(f_1) = f_3, \alpha_K(f_2) = f_2, \alpha_K(f_3) = \alpha_K(f_4) = 0\). Then \(Z_\alpha(K) = \text{span}\{f_4\}\) and \((K/Z_\alpha(K), \tilde{\alpha}_K) \cong (L, \alpha_L)\), that is, \((L, \alpha_L)\) is capable.
The following proposition plays an important role in obtaining the results of this section.

**Proposition 5.2.** Let \((L, \alpha_L)\) be a Hom-Lie algebra. Then:

(i) Both \((Z^*_\alpha(L), \alpha_{Z^*_\alpha(L)})\) and \((Z^\alpha(L), \alpha_{Z^\alpha(L)})\) are central ideals of \((L, \alpha_L)\).

(ii) \((L, \alpha_L)\) is capable if and only if \(Z^*_\alpha(L) = 0\).

(iii) If \((R, \alpha_R) \rightarrow (F, \alpha_F) \xrightarrow{\pi} (L, \alpha_L)\) is a free presentation of \((L, \alpha_L)\), then \(Z^*_\alpha(L) = \bar{\pi}(Z\alpha(F/[R, F]))\), where \(\bar{\pi} : (F/[R, F], \bar{\alpha}_F) \rightarrow (L, \alpha_L)\) is the homomorphism induced by \(\pi\).

(iv) If \(\alpha_L\) is a surjective endomorphism, then \(Z^*_\alpha(L)\) is contained in \([L, L]\).

(v) \((Z^\alpha(L), \alpha_{Z^\alpha(L)})\) is the smallest central subalgebra containing all central subalgebras \((N, \alpha_N)\) for which the canonical homomorphism \(H^\alpha_2(L) \rightarrow H^\alpha_2(L/N)\) is a monomorphism (or equivalently, for which the canonical homomorphism \((L \wedge L, \alpha_L) \rightarrow (L/N \wedge L/N, \alpha_L)\) is an isomorphism).

(vi) \((Z^\alpha(L), \alpha_{Z^\alpha(L)})\) is the smallest central subalgebra containing all central subalgebras \((N, \alpha_N)\) for which the canonical homomorphism \(J^\alpha_2(L) \rightarrow J^\alpha_2(L/N)\) is a monomorphism (or equivalently, for which the canonical homomorphism \((L \ast L, \alpha_*) \rightarrow (L/N \ast L/N, \alpha_*)\) is an isomorphism).

**Proof.** The parts (i)-(iii) are found in [5].

(iv) Consider the composite homomorphism \((L \ast L, \alpha_*) \xrightarrow{\rho} (L^ab \ast L^ab, \bar{\alpha}_*) \xrightarrow{\phi} (L^ab \otimes L^ab, \bar{\alpha}_\otimes)\), where \(\rho\) is the natural surjective homomorphism and \(\phi\) is the isomorphism given in Proposition 3.1(iii). If there is some \(x \in L - [L, L]\), then \(\bar{x} \otimes \bar{y} \neq 0\) for some \(y \in L\), implying that the pre-image of this element under \(\phi \circ \rho\) can not be vanished, that is, \(x \ast y \neq 0\) in \(L \ast L\). This means that \(Z^*_\alpha(L) \subseteq [L, L]\).

(v) The result immediately follows from the following commutative diagram of Hom-Lie algebras

\[
\begin{array}{c}
(N \wedge L, \alpha_L) \xrightarrow{=} H^\alpha_2(L) \xrightarrow{H^\alpha_2(N/L)} H^\alpha_2(L)\\
(N \wedge L, \wedge_L) \xrightarrow{(L \wedge L, \wedge_L)} (L \wedge L, \wedge_L) \xrightarrow{\phi} (L \wedge L, \wedge_L) \\
([L, L], \wedge_{L,L}) \xrightarrow{=} (L \wedge L, \wedge_{L,L}) \\
\end{array}
\]

where, by the sequence (2) and Theorem 4.1, the rows and columns are exact.

The proof of (vi) is similar to that of (v). \qed

We have the following consequences, which are of interest in their own account.

**Corollary 5.3.** Let \((N, \alpha_N)\) be a central subalgebra of a finite dimensional Hom-Lie algebra \((L, \alpha_L)\). Then \(N \subseteq Z^\alpha_\alpha(L)\) if and only if \(\dim(H^\alpha_2(L/N)) = \dim(H^\alpha_2(L)) + \dim(N \cap [L, L])\) (as vector spaces).

**Proof.** The centrality of \((N, \alpha_N)\) together with Theorem 4.6 implies an exact sequence

\[
(N \wedge K, \alpha_L) \rightarrow H^\alpha_2(N/L) \rightarrow H^\alpha_2(L) \rightarrow (N \cap [L, L], \alpha_N|_{[L, L]}).
\]

The result now follows from Proposition 5.2(v). \qed
Corollary 5.4. For any Hom-Lie algebra \((L, \alpha_L)\) with surjective endomorphism \(\alpha_L\),
\[
Z^*_\alpha(L) = Z^\lambda_\alpha(L) \cap [L, L].
\]

Proof. Let \(x \in Z^\lambda_\alpha(L) \cap [L, L]\). Then \(\alpha^k(x) \in [L, L]\) and \(\alpha^k(x) \star l \in L \square L\) for all \(l \in L\) and \(k \geq 0\), implying from Corollary 3.6 that \(\alpha^k(x) \star l = 0\). Hence \(x \in Z^*_\alpha(L)\). The inverse containment follows from Proposition 5.2(iv) and the fact that \(Z^*_\alpha(L) \subseteq Z^\lambda_\alpha(L)\).

If \(\alpha_L = id_L\), Corollaries 5.3 and 5.4 reduce to [20] Theorem 4.4 and [20] Corollary 2.7, respectively.

Corollary 5.5. Let \((L, \alpha_L)\) be a perfect Hom-Lie algebra with surjective endomorphism \(\alpha_L\). Then \((L, \alpha_L)\) is capable if and only if \(Z(L) = 0\).

Proof. The surjectivity of \(\alpha_L\) yields that \(Z_\alpha(L) = Z(L)\). We have \(Z(L) \star L, \alpha_* \cong (Z(L) \otimes L^{ab}, \alpha_{\otimes}) = 0\), thanks to Proposition 3.1(iii). Consequently \((Z(L) \lambda L, \alpha_\lambda) = 0\), forcing the natural homomorphism \(H_2^\alpha(L) \rightarrow H_2^\alpha(L/Z(L))\) is injective. Therefore, the parts (ii) and (v) of Proposition 5.2 imply the result.

In the following, we try to omit the condition of surjectivity in Corollary 5.5.

Proposition 5.6. A perfect Hom-Lie algebra \((L, \alpha_L)\) is capable if and only if \(Z_\alpha(L) \subseteq \ker(\alpha_L)\).

Proof. We can assume \(Z_\alpha(L) \neq 0\). If there is \(x \in Z_\alpha(L)\) such that \(\alpha_L(x) \neq 0\), then \(\alpha_L^k(x) \star [a, b] = 0\) for all \(a, b \in L\) and \(k \geq 1\). The perfectness of \((L, \alpha_L)\) yields that \(\alpha_L(x) \in Z^*_\alpha(L)\), and then \((L, \alpha_L)\) is not capable, thanks to Proposition 5.2(ii). We now suppose \(\alpha_L(Z_\alpha(L)) = 0\) and consider the algebra \(K\) introduced in Example 5.1(ii). We define the endomorphism \(\alpha_K\) such that \(\alpha_K|_L = \alpha_L\) and \(\alpha_K|_{\text{span}\{t_i \mid i \in I\}} = 0\). It is straightforward to check that \((K, \alpha_K)\) is a Hom-Lie algebra with \(Z_\alpha(L) = \text{span}\{t_i \mid i \in I\}\) and \((K/Z_\alpha(K), \alpha_K) \cong (L, \alpha_L)\).

The following theorem shows that for deciding on the property of capability, we may restrict ourselves to Hom-Lie algebras with surjective endomorphisms.

Theorem 5.7. Let \((L, \alpha_L)\) be a non-perfect Hom-Lie algebra with non-surjective endomorphism. Then \((L, \alpha_L)\) is capable.

Proof. Owing to Example 5.1(i), we can assume that \((L, \alpha_L)\) is non-abelian. If \(L - [L, L] \subseteq \text{Im}(\alpha_L)\), then \(L = [L, L] \cup \text{Im}(\alpha_L)\). But this implies that \((L, \alpha_L)\) is perfect or \(\alpha_L\) is surjective, contradicting the assumptions. So, there is an element \(x \in L - ([L, L] \cup \text{Im}(\alpha_L))\) with \(x \notin Z_\alpha(L)\). Choose a linear basis \(\{e_i \mid i \in I\}\) for \(Z_\alpha(L)\) and extend it to a linear basis \(\{e_i, f_j \mid i \in I, j \in J\}\) for \(L\) containing \(x\), where \(I\) and \(J\) are non-empty sets. Take the algebra \(K = L + T\), where \(T\) is generated by the set \(\{t_i \mid i \in I\}\), together with the following product: \(e_i, x = t_i\) for all \(i \in I\), \([f_j, f_k]\) is the same in \(L\) for all \(j, k \in J\),
and zero elsewhere. Define $\alpha_K$ such that $\alpha_K|_L = \alpha_L$ and $\alpha_K(t_i) = 0$ for $i \in I$. Since $x \notin \text{Im}(\alpha_K)$, $x$ does not appear in the brackets of the form $[\alpha_K(a), b]$, where $a \in K$, $b \in [K, K]$. Hence, for all $i \in I$, the elements $t_i$ are never vanished by force the Hom-Jacobi identity. On the other hand, we have $[K, t_i] = 0$ for any $i \in I$. Therefore, $(K, \alpha_K)$ is a Hom-Lie algebra, $Z_\alpha(K) = \text{span}\{t_i \mid i \in I\}$ and $(K/Z_\alpha(K), \alpha_K) \cong (L, \alpha_L)$.

It is obvious that if the Hom-Lie algebras $(L_1, \alpha_{L_1})$ and $(L_2, \alpha_{L_2})$ are capable, then $(L_1 \oplus L_2, \alpha_{\oplus})$ is capable, because $Z_\alpha^L(L_1 \oplus L_2) \subseteq Z_\alpha^L(L_1) \oplus Z_\alpha^L(L_2)$. In the following, we prove the converse of this result, under some conditions.

**Theorem 5.8.** Let $(L, \alpha_L)$ be a finite dimensional capable regular Hom-Lie algebra such that $(L, \alpha_L) = (L_1 \oplus L_2, \alpha_{\oplus})$. Then $(L_1, \alpha_{L_1})$ and $(L_2, \alpha_{L_2})$ are capable.

**Proof.** We only need to prove that $Z_\alpha^L(L_1 \oplus L_2) = Z_\alpha^L(L_1) \oplus Z_\alpha^L(L_2)$. By Example 5.1(i), we can assume that $(L, \alpha_L)$ is non-abelian. For convenience, we divide the rest of the proof into three steps.

**Step 1.** Here we prove that if $(L_i, \alpha_{L_i})$ is non-abelian, then $(L_i, \alpha_{L_i}) = (T_i \oplus A_i, \alpha_{\oplus})$ such that $(A_i, \alpha_{A_i})$ is an abelian Hom-Lie algebra and $Z_\alpha^{L_i}(L_i) = Z_\alpha^{T_i}(T_i)$, for $i = 1, 2$.

It follows from [23, Theorem 3.8] that $(L_i, \alpha_{L_i}) = (T_i \oplus A_i, \alpha_{\oplus})$ in which $(A_i, \alpha_{A_i})$ is abelian and $[L_i, L_i] \cap Z(L_i) = Z(T_i) \subseteq [T_i, T_i]$. By Example 5.1(i), $(A_i, \alpha_{A_i})$ is capable and then $Z_\alpha^{A_i}(A_i) = 0$, implying that $Z_\alpha^{L_i}(L_i) \subseteq Z_\alpha^{T_i}(T_i)$. We now claim that $Z_\alpha^{T_i}(T_i) \subseteq Z_\alpha^{L_i}(L_i)$. By virtue of Theorem 4.7 and Corollary 5.3, we have

$$
dim(H_2^\alpha(L_i)) = \dim(H_2^\alpha(T_i)) + \dim(H_2^\alpha(A_i)) + \dim((T_i^{ab} \otimes A_i) \oplus (A_i \otimes T_i^{ab}))$$

$$= \dim(H_2^\alpha(T_i)) - \dim(Z_\alpha^A(T_i)) + \dim(H_2^\alpha(A_i))$$

$$+ \dim((T_i^{ab} \otimes A_i) \oplus (A_i \otimes T_i^{ab}))$$

$$= \dim(H_2^\alpha(T_i)) - \dim(Z_\alpha^A(T_i))$$

which, using again Corollary 5.3, gives the required result.

**Step 2.** Here we prove that if the Hom-Leibniz algebras $(L_i, \alpha_{L_i})$, $i = 1, 2$, are both non-abelian and $Z(L_i) \subseteq [L_i, L_i]$, then $Z_\alpha^{L_i}(L_i) = Z_\alpha^{L_1}(L_1) \oplus Z_\alpha^{L_2}(L_2)$.

It is enough to verify that $Z_\alpha^{L_i}(L_i) \subseteq Z_\alpha^{L_i}(L)$. Using Theorem 4.7 and an argument similar to Step 1, we deduce that

$$\dim(H_2^\alpha(L_i)) = \dim(H_2^\alpha(L)) - \dim(Z_\alpha^A(L_i))$$

from which we have $Z_\alpha^{L_i}(L_i) \subseteq Z_\alpha^{L_i}(L)$.

**Step 3.** The completion of the proof.

If the one of Hom-Lie algebras $(L_1, \alpha_{L_1})$ or $(L_2, \alpha_{L_2})$ is abelian, the result obtains from Step 1. So suppose that both are non-abelian. Then, using again Step 1, there are non-abelian-subalgebras
\((T_i, \alpha_{T_i})\) of \((L_i, \alpha_{L_i})\), \(i = 1, 2\), such that \((L, \alpha_L) = (T_1 \oplus T_2 \oplus A, \alpha_{\oplus})\), \(Z_{\alpha}(L) = Z_{\alpha}(T_1) \oplus Z_{\alpha}(T_2)\), \(Z_{\alpha}(L_i) = Z_{\alpha}(T_i)\) and \(Z(T_i) \subseteq [T_1, T_2]\). But by Step 2, \(Z_{\alpha}(T_1 \oplus T_2) = Z_{\alpha}(T_1) \oplus Z_{\alpha}(T_2)\). We therefore conclude that \(Z_{\alpha}(L_1 \oplus L_2) = Z_{\alpha}(L_1) \oplus Z_{\alpha}(L_2)\), as desired. \(\square\)

The following example shows that the regular condition is essential in Theorem 5.8.

**Example 5.9.** Let \(H(m)\) be the Heisenberg algebra of dimension \(2m + 1\). Then by [22, Theorem 3.4], the Hom-Lie algebra \((H(m), id_{H(m)})\) is not capable for any \(m \geq 2\). But if we define \(K = \text{span}\{t\}\) and \(\alpha_K(t) = 0\), then thanks to Theorem 5.7, we infer that \((H(m), id_{H(m)}) \oplus (K, \alpha_K)\) is capable.

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