Tunneling in the Electron Box in the Nonperturbative Regime

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Abstract

We study charging effects and tunneling in the single electron box. Tunneling mixes different charge states and in the nonperturbative regime the charge in the island may be strongly screened. When charge states are nearly degenerate the screening of the charge is strong even in the weak tunneling regime. Virtual tunneling processes reduce both the level splitting $\Delta$ and the tunneling strength $\alpha$. The charge on the island and the decay rates are calculated. In the strong tunneling regime also nondegenerate states are affected by tunneling. Strong-coupling scaling renormalizes the effective capacitance, a result which we confirm by Monte Carlo simulations. The tunneling strength $\alpha$ scales to smaller values into the regime where the weak-coupling scaling applies. We propose a two stage scaling procedure providing the unified picture for the problem. The scaling analysis is also extended to superconducting tunnel junctions with finite subgap conductance.

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I. INTRODUCTION

Charging effects strongly modify the transport properties of systems of small tunnel junctions. The transfer of an electron requires a typical electrostatic energy cost \( E_C = e^2/2C \), so at sufficiently low temperatures \( T \ll E_C \) charge transfer is suppressed, a phenomenon known as the Coulomb blockade \([1–3]\). Tunneling induces quantum fluctuations of the charge in the electrodes. Strong fluctuations screen the charge even at \( T = 0 \) and charging effects are weakened. Quantum fluctuations are strong if the typical tunneling resistance \( R \) is small enough \( (\alpha_0 \equiv h/2\pi eR > 1) \). Even in the weak tunneling regime \( \alpha_0 < 1 \), quantum fluctuations are strong if the lowest lying charge states are nearly degenerate.

A simple and widely studied device where these effects are manifest is the single electron box \([4]\), shown in Fig. 1. It consists of a normal metal island connected via a capacitor and a tunnel junction to a voltage source. The energy gap \( \Delta \) between the two lowest lying states can range between \( E_C \) and zero and is controlled by the external voltage (see Fig. 2). If the tunneling is strong \( (\alpha_0 > 1) \) the screening of the charge results in a reduction of the overall effective bandwidth \( E_C \). Also for weaker tunneling, \( \alpha_0 < 1 \), infrared divergent tunneling processes strongly mix nearly degenerate charge states. As a result the gap \( \Delta \) is reduced near the degeneracy points, but the overall bandwidth \( E_C \) is unaffected. The quantitative analysis of these regimes lies beyond the scope of perturbation theory. The problem has recently received much attention \([6–9]\), however several previously published results \([3,10,12]\) are in mutual disagreement. Our motivation is to provide a plausible unified description for this problem.

In section II we introduce the model. In section III we discuss the scaling in the weak tunneling regime. In section IV we review the scaling in the strong tunneling regime and propose a two stage scaling procedure to provide a unifying picture for large and small \( \alpha_0 \). We also obtained the strong coupling scaling of the effective capacitance by a Monte Carlo simulation. Then we compare with other non-perturbative techniques in the \( \alpha_0 > 1 \) regime. In section V we present results for other observable quantities, we extend our analysis to
finite temperatures and finally discuss the scaling for a superconducting junction with subgap quasiparticle tunneling.

II. THE MODEL

In the absence of tunneling the thermodynamics of the single electron box is governed by the electrostatic energy $E_0(Q) = Q^2/2C$ where $C = C_j + C_s$ (see Fig. 1) and the charge $Q = (Q_x - ne)$ is composed by the $n$ excess electrons in the island and the continuous “external charge” $Q_x = C_sV_x$ induced by the voltage source. The measurable voltage at the junction is $\langle V \rangle = \langle Q \rangle/C$. The energy spectrum of the system as a function of $Q_x$ is shown in Fig. 2. The lowest lying levels are degenerate at $Q_x = (k + 1/2)e$. At $T = 0$ a well defined $n$ is selected, which depends on $Q_x$ and changes by $\pm 1$ when $Q_x$ crosses the degeneracy points. The result is a step structure of $\langle n \rangle$ and a sawtooth shape of $\langle V \rangle$ as a function of $Q_x$ [4]. At finite temperature both are smoothened by fluctuations.

In the presence of tunneling the single electron box can be described by the following Hamiltonian

$$H = \frac{(Q_x - ne)^2}{2C} + \sum_{p\sigma} \epsilon(p) c_{p\sigma}^+ c_{p\sigma} + \sum_{k\sigma} \epsilon(k) c_{k\sigma}^+ c_{k\sigma}, + \sum_{kp\sigma} T_{kp} c_{k\sigma}^+ c_{p\sigma} + h.c. \ . \ (1)$$

Here $n = \sum_{p\sigma} c_{p\sigma}^+ c_{p\sigma}$, $c_{p\sigma}^+$ and $c_{k\sigma}^+$ are creation operators of the electrons in the island and in the lead respectively, and $\epsilon(p)$ and $\epsilon(k)$ are their kinetic energies. We consider a wide tunnel junction where tunneling takes place through $N$ independent channels. In this case the tunneling strength is given by $\alpha_0 = N|T|^2\rho^2$, where $T$ is the tunneling amplitude for a single channel and $\rho$ is the densities of states per channel, which is assumed to be equal in both electrodes.

In order to concentrate on the quantum dynamics of the charge we integrate out the electronic degrees of freedom [3] in the partition function $Z(Q_{\phi})$. This is achieved by decoupling the quartic charging term via a Hubbard-Stratonovich transformation which introduces the phase $\varphi(\tau)$. After the integration $Z(Q_{\phi})$ can be expressed as a path integral depending on
the phase only. Expanding the effective action in powers of the tunneling amplitudes $T_{kp}$ we obtain a $1/N$ expansion, i.e. the $2n^{th}$ order term is proportional to $N|T|^{2n} \rho^{2n} = \alpha_0^n / N^{n-1}$. Since in our case the number of channels $N$ is large and the nominal conductance $\alpha_0$ is finite it is sufficient to retain only the first term of the expansion, which is given by the well known path-integral representation \[ Z(Q_x) = \sum_{m=-\infty}^{\infty} e^{2\pi i m Q_x / e} \int_{-\infty}^{\infty} d\varphi_0 \int_{\varphi_0}^{\varphi_0+2\pi m} D\varphi(\tau) e^{-S[\varphi(\tau)]} , \] with the effective action \[ S[\varphi(\tau)] = \int_0^\beta d\tau \frac{1}{4E_C} \left( \frac{d\varphi}{d\tau} \right)^2 - \int_0^\beta d\tau \int_0^\beta d\tau' \alpha(\tau - \tau') \cos[\varphi(\tau) - \varphi(\tau')] . \] The Fourier transform of the dissipative kernel is $\alpha(\omega_n) = -\pi \alpha_0 |\omega_n|$ up to a high energy cutoff $\omega_c$. The summation over winding numbers, $\varphi(\beta) = \varphi(0) + 2\pi m$, reflects the discreteness of the charge. The external charge $Q_x$ can be viewed as a gauge field and appears in the phase factor together with the winding number $m$.

It is useful to consider also the dual representation expressed in terms of charges \[ Z = \int_{Q_x-n e}^{Q_x-n e} DQ(\tau) \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^\beta d\tau_1 ... d\tau'_k \alpha(\tau_1 - \tau'_1) ... \alpha(\tau_k - \tau'_k) e^{-\int_0^\beta d\tau E_0(Q(\tau))} . \] Here $Q(\tau) \equiv Q_x - ne - e \sum_i \left[ \Theta(\tau - \tau_i) - \Theta(\tau - \tau'_i) \right]$. Equation (4) maps the tunneling problem on a gas of interacting blips and antiblips (see Fig. 3) each one representing a tunneling transition at time $\tau_i$ in which the charge changes in units of $e$. The tunneling events force a rearrangement of the other electrons in the two electrodes, which is represented by the $\alpha(\tau)$ lines.

We stress that both equations (2-3) and (4) are exact representations of the original problem in the limit of interest, $N \to \infty$ and $\alpha_0$ finite. The extension to finite values of $N$ has also been considered previously \[ \text{[10]}. \]

**III. WEAK TUNNELING REGIME**

In the weak tunneling regime $\alpha_0 < 1$ it is convenient to start from the charge representation Eq. (4). Due to the symmetries of the system we can focus on the interval
0 < Q_x < e/2. The main controlling parameters are the strength of the tunneling \( \alpha_0 \) and the energy difference between the two lowest charge states

\[
\Delta_0(Q_x) = E_0(Q_x - e) - E_0(Q_x) = E_C(1 - 2Q_x/e) \, .
\]  

Due to the infrared divergent behaviour of \( \alpha(\omega) \), perturbation theory in \( \alpha_0 \) breaks down near the degeneracy points. Indeed the leading contribution to the ground state energy, which up to singular terms is \( E_G^{(1)} = E_0(Q_x) - \alpha_0 \Delta_0 \ln(\Delta_0/\omega_c) \) shows an infrared singularity for \( \Delta_0 \to 0 \). Hence both \( \langle V \rangle = -\frac{\hbar}{\beta} \left( dE_G^{(1)}/d\Delta_0 \right) \) and \( \langle n \rangle = (Q_x - C\langle V \rangle)/e \) are logarithmically divergent.

Close to the degeneracy points, \( \Delta_0(Q_x) \ll E_C \) the low energy physics is determined by the lowest two states and involves tunneling events with energy differences smaller than \( E_C \). Thus the cutoff of the model, \( \omega_c \), is approximately given by \( E_C \). In this two-level-system (TLS) approximation only the terms \( n = 0, 1 \) of Eq. (4) are retained \( Z = \sum_{n=0,1} G_n(\beta) \) and the only trajectories \( Q(\tau) \) contributing to \( G_n(\beta) \) are such that blips and antiblips alternate. For instance we retain the diagrams (a), (b) and (c) in Fig. 3 and drop (d). The bare propagators are \( G_n(\tau) = \exp(-|\tau| E_0(Q_x - ne)) \).

The perturbative corrections are logarithmic in the regime \( \tau_c \ll \tau \). The first order term (blip-antiblip pair) in the leading logarithmic approximation reads as

\[
Z^{(1)}(Q_x, \tau) \approx e^{-E_0(Q_x)\tau} \left( 1 - \alpha_0(\Delta_0\tau + 1) \ln(\omega_c\bar{\tau}) \right) \\
+ e^{-(E_0(Q_x)+\Delta_0)\tau} \left( 1 - \alpha_0(-\Delta_0\tau + 1) \ln(\omega_c\bar{\tau}) \right) \tag{6}
\]

where \( 1/\bar{\tau} = \max[1/\tau, \Delta_0, T] \) acts as a low frequency cutoff which regularizes the infrared singularities and \( O(1) \) constants depending on the details of the high frequency cutoff procedure have been ignored.

Our model Eq. (4) is similar to the model used by Anderson et al. [14] to study the single-channel \( S = 1/2 \) Kondo problem. We could use the same renormalization group (RG) technique to treat the infrared singularities, namely progressively eliminating close blip-antiblips by increasing of the short time cutoff \( \tau_c = 1/\omega_c \) [8]. However, since
there are technical differences (e.g. here the interaction is pairwise, and the short time
cutoff procedure is chosen such as to guarantee that $\alpha(\omega_n = 0) = 0$), some care is needed
when using the scaling technique of Ref. [14]. We chose to eliminate high frequencies; so
we split the kernel $\alpha(\tau)$ into slow and fast parts, the latter containing the frequencies we
want to integrate out, $\alpha(\tau) = \alpha_s(\tau) + \alpha_f(\tau)$ . In first order in the tunneling strength we
look at all the configurations containing only one “fast” $\alpha$-line which can connect either
a close blip-antiblip pair or a pair separated by some blip-antiblip insertion. Only the
former are effective in the first step of the renormalization. Indeed the integration of the
complete (fast and slow modes) line of the latter kind does not give infrared singularities
because of phase space restrictions, thus we discard them. Interaction lines overarching
blip-antiblip insertions in higher order diagrams (eg. the “rainbow” of Fig. 3b) will enter
the renormalization in the successive steps, once the blip-antiblip insertions are eliminated.
In this procedure the running parameters will never be renormalized by diagrams containing
crossing $\alpha$-lines. Our treatment is equivalent to the “Non-Crossing Approximation” and can
be justified by direct calculation. Indeed elimination a close pair with crossing $\alpha$-lines (see
Fig. 3c) yields an interaction proportional to $\tau^{-3}$ which does not lead to any logarithmic
singularity. The bare ground state propagator $\mathcal{G}_0(\tau)$ is renormalized in lowest order by
the fast modes $\omega_c - \delta \omega_c < \omega < \omega_c$ of a single blip-antiblip close pair which generate a
contribution $-\alpha_0(1 + \Delta_0 \tau)(\delta \omega_c/\omega_c)$ . A similar expression gives the renormalization of the
propagator for the first excited state $\mathcal{G}_1(\tau)$.

The general $n$-th order terms can be cast in a way such that the partition function
preserves the original form if the gap and the interaction scale as

$$
\frac{d(\Delta/\omega_c)}{d \ln \omega_c} = \left(2\alpha Z^2 - 1\right) \frac{\Delta}{\omega_c}, \quad \frac{d(2\alpha Z^2)}{d \ln \omega_c} = \left(2\alpha Z^2\right)^2 . \quad (7)
$$

The renormalized ground state energy is given by $E_G = E_0(Q_x) + 1/2(\Delta_0 - \Delta)$ . Here $Z$
is the wave function renormalization which enters always together with the interaction in
the combination $2\alpha Z^2$ , and the $-1$ in the first equation enters because of the dimensional
factor $\omega_c$ .
The scaling equations can be readily integrated down to a low energy scale $\omega_c$

$$\alpha(\omega_c) = \alpha_0 \left( 1 + 2\alpha_0 \ln \left( \frac{E_C}{\omega_c} \right)^{-1} \right), \quad \Delta(\omega_c) = \Delta_0 \left( 1 + 2\alpha_0 \ln \left( \frac{E_C}{\omega_c} \right)^{-1} \right). \quad (8)$$

At zero temperature the renormalized gap provides the low energy cutoff so we have to stop the scaling at $\omega_c = \Delta$. Then Eq. (8) becomes a self-consistent equation for $\Delta$. At finite temperatures, if $T > \Delta$ the RG has to be stopped at $\omega_c = \max[\Delta, T]$, where the infrared singularities in the perturbation expansion disappear.

Typical solutions are shown in Fig. 4. Notice that the renormalized gap $\Delta$ does not vanish.

**IV. STRONG COUPLING REGIME**

In the regime $\alpha_0 \gg 1$ the charge fluctuates strongly so it may be convenient to start from the phase representation Eqs. (2-3). We analyze the problem by various nonperturbative techniques. We start with the RG analysis of Refs. [5,15,16]. This treatment is restricted to the $m = 0$ sector of the partition function Eq. (2), so the discreteness of the charge is not explicitly accounted for. However, we will show that the overall conclusions of the scaling theory can be carried over to the general case, described by Eq. (2).

The scaling equations are obtained perturbatively in $1/\alpha$ [5,15,16].

$$\frac{d}{d \ln \omega_c} \left( \frac{E_C}{\omega_c} \right) = \left( \frac{1}{\bar{\alpha}} - 1 \right) \left( \frac{E_C}{\omega_c} \right); \quad \frac{d}{d \ln \omega_c} \left( \frac{1}{\bar{\alpha}} \right) = -1/\bar{\alpha}^2 , \quad (9)$$

where $\bar{\alpha} = 2\pi^2 \alpha$. In the present approach we determine the scaling of the overall bandwidth $E_C$, but not that of the gap $\Delta$, exactly the opposite of the weak coupling region. However we assume that the gap $\Delta$ scales similarly to the bandwidth. Equation (9) shows that $\bar{\alpha}$ decays in the strong coupling regime towards weak coupling; so in the case of strong tunneling a two step scaling procedure is called for which provides the desired unified picture. As the energy scale $\omega_c$ (e.g. temperature) decreases, first $\alpha$ decays from its large initial value $\tilde{\alpha}_0$ according to Eq.(9)
\[
\frac{1}{\alpha(\omega_c)} = \frac{1/\tilde{\alpha}_0}{1 - (1/\tilde{\alpha}_0) \ln(\omega_c/\omega_0)}
\]  

(10)

to \(\tilde{\alpha} \sim 1\). The bandwidth also decreases upon renormalization and for \(\tilde{\alpha} \sim 1\) reaches the value \(E^*_C \approx E^0_C \exp(-\tilde{\alpha}_0)\). At this point the cutoff reaches \(\omega_c \sim E^*_C\). From there on we use the weak coupling scaling of section III with initial values \(\omega_c \approx E^*_C\), the exponentially suppressed bandwidth, \(\Delta_0 \approx \Delta_0 \exp(-\tilde{\alpha}_0)\), and \(\tilde{\alpha}_0 \approx 1\) implying \(\alpha_0 \approx 1/2\pi^2\). The flow of \(\alpha(\omega_c)\) will be governed by Eq. (7). Again the tunneling strength decreases and eventually we stop the RG when we reach the low-energy cutoff. The final formula for the gap in the \(\alpha_0 \gg 1\) regime is

\[
\Delta(\omega_c) = \frac{\Delta_0 e^{-\tilde{\alpha}_0}}{1 + \pi^{-2} \ln(E^0_C e^{-\tilde{\alpha}_0}/\omega_c)}.
\]

(11)

Next we checked the exponential suppression of the bandwidth by a Monte Carlo simulation starting from the phase representation in the normalized form

\[
\mathcal{Z}(Q_x) = \frac{1}{\sum_m \mathcal{Z}(m)} \sum_{m=-\infty}^{\infty} \mathcal{Z}(m) e^{2\pi imQ_x/\epsilon} ; \quad \mathcal{Z}(m) = \int D\delta\varphi(\tau) e^{-S_m[\delta\varphi]}
\]

(12)

where we decomposed \(\varphi(\tau) = \delta\varphi(\tau) + \varphi_0 + 2\pi m\tau/\beta\), \(S_m[\delta\varphi(\tau)] \equiv S[\delta\varphi(\tau) + 2\pi m\tau/\beta]\), and the fluctuations satisfy \(\delta\varphi(0) = \delta\varphi(\beta) = 0\). At this stage we mention some technical points [17,18]. First, the part of the action which describes tunneling is equivalent to a one-dimensional classical XY model with long range interaction where \(\beta E^0_C\) plays the role of the system size. Then each update of the phase \(\delta\varphi(\tau_i)\) requires a summation over the whole discretized lattice. We update \(\delta\varphi(\tau_i)\) using the scheme proposed in Ref. [17] but in a standard Metropolis algorithm. Second, in order to extract \(E^*_C\) we need \(1 \ll \beta E^*_C \sim \beta E^0_C e^{-\tilde{\alpha}_0}\). Hence large values of \(\tilde{\alpha}_0\) require large \(\beta E^0_C\) i.e. a large number of lattice sites. Third, it is apparent from Eq. (12) that \(\mathcal{Z}(m)\) is proportional to the probability for the path to visit the \(m\)-th sector. Therefore we cannot perform separate simulations for each sector (we would trivially obtain \(\mathcal{Z}(m) = 1\)). On the other hand simulating continuous and discrete variables altogether is a technically difficult task. The most economical scheme we found is to calculate
sector by sector. Notice that we are interested to the marginal probability for a jump to a different $m$ sector whereas in the Monte Carlo procedure we update $(\delta \varphi, m) \to (\delta \varphi', m')$. This move can be carried out in several ways, according to how we choose the final $\delta \varphi'$ configuration. Also the relaxation of the path $\delta \varphi$ can take place by updating sequentially or randomly in the lattice. The results we present do not depend on the details described above. In each simulation we measure 300 sample points per winding number. Between two sampling points the system had time to evolve for 7 more sweeps. We measured the Gibbs energy at lower and lower temperatures, i.e. increasing the number of lattice sites for the path $\delta \varphi$ up to 1000 at the lowest temperatures. One can obtain the normalized $Z(m)$ from the results Eq. (13) by recursion relations and extract the ground state energy $E_G(Q_x)$ from finite size scaling. Further details are given in [19].

Reliable results are obtained close to $Q_x \approx 0$, and the effective bandwidth (inverse capacitance) has been extracted from the curvature of the band (see Figs. 4 and 5). However the region close to the band edges is beyond the capability of the Monte Carlo method. Indeed $O(1)$ terms in the winding number summation Eq.(12) enter with the oscillating phase factor $\exp(2\pi m Q_x/e)$. Close to the bottom of the band $Q_x \approx 0$, and the oscillations are slow, whereas for $Q_x \approx e/2$ they are fast. At the band edges one expects $Z(Q_x \sim e/2) \sim \exp(-\beta (E_C - \Delta)) \ll 1$. Hence one has to extract an extremely small number by adding many $O(1)$ terms with oscillating phase factors. This makes the numerical procedure very unstable near $Q_x = e/2$.

We stress that the same argument applies to any analytic calculation which attempts to estimate $Z(m)$ and perform the winding number summation. Examples will be discussed below.

Panyukov and Zaikin [11] studied the strong coupling regime by a non-standard instanton technique in the phase representation Eqs. (2-3). They obtained a renormalized bandwidth $E_C^* \sim E_C \tilde{\alpha}^2 \exp(-\tilde{\alpha})$, which agrees with the results of large $\alpha$ scaling, as well as the Monte
Carlo data down to surprisingly low values of $\alpha_0$. The result is shown in Fig. 5 where the curve of Panyukov et al. has been rescaled by a factor $\sim 2$, i.e. also the accuracy in the pre-exponent is remarkable.

Panyukov and Zaikin further concluded that the ground state energy completely flattens in a wide interval around $Q_x = \pm e/2$. This latter conclusion differs from our picture and below we will argue that the possible reason for this discrepancy is the lack of accuracy of the instanton calculation which becomes crucial near the degeneracy points.

The technique presented in [11] is unusual in two senses: i) each instanton possesses the standard zero mode related to the location, but also a non-standard one related to their width. Thus instantons of all lengths enter $Z(Q_x)$ with similar weight. ii) A summation over winding numbers $m$ connected with oscillating phase factors has to be performed (cf. Eq. (12)). Hence, near the band edges innocent looking approximations can profoundly alter the result. We reinvestigated the accuracy and consistency of the method used in Ref. [11].

In particular a dilute instanton approximation has been used which in standard fixed size instantons calculations is justified when the $N$-instanton configurations playing the main role have finite $N$. Then for fixed size instantons the dilute limit is always reached for $\beta \to \infty$. As the size of the instantons is not limited here, we have to calculate the average width $\langle \sigma \rangle_N$ of an instanton in an $N$-instanton configuration and the dilute limit is reached only if $N\langle \sigma \rangle_N \ll \beta$ for the relevant values of $N$. In the partition function for $N$-instantons $Z(N)$ we first separate the zero modes (the locations $\tau_j$ and the width $\sigma_j$ of each instanton) and integrate the remaining fluctuation determinant to get

$$
\langle \sigma_i \rangle_N = \frac{N! \left(4T E_C^* \cos (2\pi Q_x / e) \right)^N}{Z(N)} \int \prod_{j=1}^N d\tau_j \int \prod_{j=1}^N d\sigma_j \sigma_i = \frac{\beta}{2N + 1} .
$$

As in Ref. [11] we constrain the integration to non-overlapping instanton configurations ($\tau_j + \sigma_j/2 < \tau_{j+1} - \sigma_{j+1}/2$). Hence the result (14) is a lower limit to the average size and $N\langle \sigma \rangle_N$ is never small compared to $\beta$. We conclude that the instanton gas is never dilute. Thus the contributions from overlapping instanton configurations are comparable to the non-overlapping ones considered in [11].
We calculated also the bare interaction between an instanton and an anti-instanton
\[ S(i, j) = -16\tilde{\alpha}\sigma_i\sigma_j/[(\sigma_i + \sigma_j)^2 + (\tau_i - \tau_j)^2] \] which then attract each other. Instantons of the same “sign” do not interact. Close pairs of instantons are favoured, and this will amplify the deviations from the dilute, non-interacting instanton gas picture. Since close to the degeneracy points extreme accuracy is required we consider the result at the edges of the bands derived in Ref. [11], which are based on considerable approximations, not conclusive. However close to the bottom of the band results are less sensitive to approximations, which explains the excellent agreement between the instanton calculation and our Monte Carlo results.

The exponential suppression of the bandwidth is also found in the straight-line approximation, introduced in [12]. We make use of the decomposition of \(\varphi(\tau)\), introduced for the Monte Carlo simulation, and expand the action in terms of the fluctuations \(\delta\varphi(\tau)\) around the straight lines \(2\pi m\tau/\beta + \varphi_0\)

\[ S_m(\delta\varphi) \approx \tilde{\alpha}|m| + T\sum_{\nu>0} \left[ \frac{\omega_\nu^2}{4E_C} + \frac{\tilde{\alpha}}{2\pi} (|\omega_\nu - \omega_m| + |\omega_\nu + \omega_m| - 2|\omega_m|) \right] |\delta\varphi_\nu|^2. \quad (15) \]

This action gives rise to soft modes for \(|\omega_\nu| < |\omega_m|\). The fluctuation determinant relative to the \(m = 0\) term is

\[ \frac{det(m)}{det(0)} = a^{-m}m! \prod_{\nu=1}^{m} \frac{1}{1 + \nu/a} \prod_{\nu=m+1}^{\infty} [1 - \frac{m}{\nu(1 + \nu/a)}] \quad (16) \]

where \(a \equiv \frac{2E_C\tilde{\alpha}}{\pi^2T}\). In the \(a \gg 1\) limit, we obtain the partition function

\[ Z(Q_x) \approx \sum_{m < a} \frac{a^m}{m!}e^{-\tilde{\alpha}|m|}e^{i2\pi m Q_x/e} \quad (17) \]

In the temperature regime \(E_C\tilde{\alpha}e^{-\tilde{\alpha}} \ll T \ll E_C\) we can confine ourselves to \(m = 0, \pm 1\). This yields \((1/\beta) \ln Z(Q_x) \approx \text{const} + E_C^* \cos(2\pi Q_x/e)\) where \(E_C^* = \frac{4E_C\tilde{\alpha}}{\pi^2} e^{-\tilde{\alpha}}\) in agreement with the previous results. However at lower temperatures the approximation breaks down and the numerical summation over \(m\) gives a negative partition function. This breakdown is understandable since it happens at the crossover temperature where according to the scaling analysis the effective \(\alpha\) decreases below 1, and quantum fluctuations of \(\delta\varphi(\tau)\) become large.
V. OTHER RESULTS AND DISCUSSION

A scaling analysis, very similar to the one we present in section III, was performed in Ref. [5]. The gap renormalization of Eq. (7a) had been found there, but the possibility of wave function or $\alpha$ renormalization was not considered. As a result a phase transition between a finite gap and a zero gap region at $\alpha_0 = 1/2$ was predicted. The main consequence of the additional scaling of $\alpha$ of Eq.(7b) is that the gap remains finite. The transition is smeared, leaving only a strong crossover around $\alpha_0 \sim 1/2$ (see Fig. 4).

Two studies addressed directly the weak coupling regime, one performing a poor man’s scaling analysis [10], and one solving a Dyson equation [6]. Our results agree with those in the leading logarithmic approximation. Difference arises because in our formalism the gap appears as an explicit low energy cutoff, yielding a self-consistent equation for $\Delta$. This will be important when $\alpha_0$ is not very small.

The ground state energy $E_G$, the voltage at the junction $C\langle V \rangle = dE_G(Q_x)/dQ_x$ and the average number $\langle n \rangle$ of electrons on the island are shown in Figs. 6, 7 and 8. As Ref. [11] found a complete flattening of the band around the degeneracy points, it was suggested that in the middle of the vertical part of the original sawtooth pattern of $\langle V \rangle$ a new $S$-shape develops. In the light of the above analysis there is no support for the complete flattening. As no reliable treatment of the gap is available in the large $\alpha$ regime, we used the scaling formalism in the $\alpha_0 \sim 1$ regime (see Fig. 7). The logarithmic corrections of Eq.(7) modify the vertical part in a weak manner. The clearest consequence is the strong suppression of the amplitude of the sawtooth oscillations already at moderate values of $\alpha_0$.

Another observable feature is the possible broadening of the excited states, caused by the appearance of a finite lifetime $\Gamma^{-1}$. We adopt the method of Ref. [4] to determine $\Gamma$ close to the band edges, in the $\alpha_0 < 1$ regime. Recall that for $Q_x < e/2$ the ground state energy close to the degeneracy point is given by $E_G(Q_x) \approx E_C/4 - \Delta(Q_x)/2$. We can determine the energy of the first excited state $E_1(Q_x)$ by analytic continuation of $E_G(Q_x)$ to $Q_x > e/2$. The argument of the logarithm in $\Delta$, turns negative when passing $Q_x = e/2$
and an imaginary part develops in $E_1(Q_x)$. The inverse lifetime is then determined by
\[ \Gamma = \Im m(E_1(Q_x)) \approx \frac{4\pi\alpha_0\Delta_0}{\left(1 + 2\alpha_0 \ln(E_C/\Delta_0)\right)^2 + (4\pi\alpha_0)^2}. \quad (18) \]
As $\Gamma \ll \Delta$, the excited levels remain well defined \[21\]. The primary experimental consequence of $\Gamma > 0$ is that the I-V curves become smoothed proportional to $\Gamma/\Delta$ around the onset of the Coulomb-blockade.

At finite temperatures also thermal fluctuations have to be considered. If $T \ll E_C$ and for $\alpha_0 < 1$ the physics near the band edges still involves only the two lowest charge states and the Gibbs energy is given by
\[ G(T, Q_x) \approx E_0(Q_x) + \frac{\Delta_0}{2} - \frac{1}{\beta} \ln \left[ 2 \cosh \left( \frac{\beta\Delta}{2} \right) \right] \quad (19) \]
The renormalized gap is then calculated using Eq.(8) with $\omega_c = T$ for $T > \Delta(Q_x)$ and with $\omega_c = \Delta$ for $T < \Delta(Q_x)$. All the quantities of interest can be calculated and in particular the normalized $\langle V \rangle$ vs. $Q_x$ has now a finite slope at $Q_x = e/2$, given by $(\frac{1}{2}\beta E_C)/[1 + 2\alpha_0 \ln(\beta E_C)]^2$. This result has also been found by other methods \[21\]. Notice that a finite tunneling strength is very effective in suppressing the slope when $2\alpha_0 \ln(\beta E_C) \sim 1$. In the experiments of the Saclay group \[4\], $2\alpha_0 \ln(\beta E_C) \sim 10^{-2}$, so quantum fluctuations of the charge do not explain the observed suppression of the slope at $Q_x = e/2$. In this case the screening of the charge is probably due to the fact that thermal noise coming from the electromagnetic environment can excite tunneling \[22\].

Finally we reconsider the case of S-S junctions with finite subgap quasiparticle tunneling. Here the effect of a Josephson coupling between the electrodes has to be considered as well. For large $E_J$ a Kosterlitz-Thouless type transition was found in Ref. \[5\]: for $\alpha_0 > 1/4$ the Josephson coupling $E_J$ scales towards larger values. Here we discuss the small $E_J$ limit, where $E_J$ does not renormalize (see below). The flow diagram in the $E_J/E_C-\alpha$ is shown in Fig. 9. plane. In the strong tunneling regime the flow lines of are given by $E_J/E_C \propto \exp(-\tilde{\alpha}_0)$ \[9\] which is due to the exponential suppression of $E_C$ discussed in section IV. One of these lines is the separatrix between the two phases which ends at $\alpha = 1/4$.
for $E_J \to \infty$. In the regime $\alpha < 1$ (i.e. $\tilde{\alpha} < 2\pi^2$) we can study at the charge representation Eq. (4), modified by the effect of the Josephson tunneling [2,3]. The typical configurations differ from those shown in Fig. 3 because $2e$ blips and antiblips are present, due to the transfer of Cooper pairs. However, no infrared process is connected with the transfer of Cooper pairs and hence $E_J$ does not renormalize.

We thus arrive at the following picture for the the small $E_J$ regime: for large $\alpha$ the system flows towards larger $E_J/E_C$ until it reaches $\tilde{\alpha} < 1$ ($\alpha < 1/2\pi^2$) where $E_C$ itself does not scale appreciably anymore, and the flow lines flatten. In general $\alpha$ will scale towards smaller (cf. Eq.(8)) but finite values, as discussed in section III. Thus the flow lines in Fig. 9 do not reach the $\alpha = 0$ axis. The actual final value of $\alpha$ depends on the single particle gap $\Delta_0(Q_x)$. Also the detailed behaviour of the separatrix in the intermediate $\alpha$ and $E_J$ regimes may depend on $\Delta_0$.

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Figure Captions :

Fig. 1 : The single electron box. It shows the interplay between the discrete charge \(-ne\) on the island and the continuous charge \(Q_x = C_s V_x\) controlled by the voltage source.

Fig. 2 : The band structure for \(\alpha_0 = 0\). The energy gap \(\Delta_0\) can be tuned between \(E_C\) and zero by varying the external voltage.

Fig. 3 : Some diagrams contributing to \(Z\). Full lines represent the allowed trajectories in the charge space, wiggly lines are associated with \(\alpha(\tau_j - \tau'_j)\). Repeated blip-antiblip pairs (a), rainbows (b) and crossings (c) are retained in for the TLS approximation. The diagram in (d) involves four charge states.

Fig. 4 : The effective gap \(\Delta\) and tunneling strength \(\alpha\) obtained by integrating the scaling equations for different initial values \((\Delta_0/E_C = 0.01, 0.05, 0.1)\), compared with the values inferred from Monte Carlo simulations (diamonds).

Fig. 5 : Monte Carlo results (diamonds) for the effective capacitance renormalization in the weak tunneling regime and in the (non-perturbative) intermediate tunneling regime. Comparison is made with the results from perturbation theory, from Ref. [11] and from Ref. [5,15].

Fig. 6 : The renormalized energy bands close to the edges for various \(\alpha_0\) in the weak coupling limit.

Fig. 7 : The normalized voltage at the junction \(C\langle V\rangle/e = \langle Q\rangle/e\) close to the band edges, at \(T = 0\), for various \(\alpha_0\) in the weak coupling limit.

Fig. 8 : The expectation value of the number of excess electrons in the island close to the band edges, at \(T = 0\), for various \(\alpha_0\) in the weak coupling limit. In the absence of charging
effects the “ohmic” linear dependence is found.

Fig. 9: Flow diagram in the $\frac{E_i}{E_C}$ – $\alpha$ plane for S-S tunnel junctions with finite subgap conductance $\alpha_0$. 