A REMARK ON PERIODIC POINTS ON VARIETIES
OVER A FIELD OF FINITE TYPE OVER \( \mathbb{Q} \)

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Abstract. Let \( M \) be a field of finite type over \( \mathbb{Q} \) and \( X \) a variety defined over \( M \). We study when the set \( \{ P \in X(K) \mid f^n(P) = P \text{ for some } n \geq 1 \} \) is finite for any finite extension fields \( K \) of \( M \) and for any dominant \( K \)-morphisms \( f : X \to X \) with \( \deg f \geq 2 \).

Introduction

By a variety, we mean an integral separated scheme of finite type over a ground field. Let \( M \) be a field of finite type over \( \mathbb{Q} \) and \( X \) a variety defined over \( M \). Let \( K \) be a finite extension field of \( M \) and \( f : X \to X \) a dominant morphism defined over \( K \). We say that a point \( P \in X(K) \) is periodic with respect to \( f \) if there is a positive integer \( n \) with \( f^n(P) = P \). Let \( X(K)_{\text{per}, f} \) be the set of periodic \( K \)-points with respect to \( f \). We say that \( X \) is periodically finite if \( X(K)_{\text{per}, f} \) is a finite set for any finite extension fields \( K \) of \( M \) and any dominant \( K \)-morphisms \( f : X \to X \) with \( \deg f \geq 2 \).

In this paper, we study when \( X \) is periodically finite.

In order to show the finiteness of \( X(K)_{\text{per}, f} \), we introduce the set of backward \( K \)-orbits of \( f \), denoted by \( \lim \leftarrow f X(K) \), which is defined by

\[
\lim \leftarrow f X(K) = \{(x_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} X(K) \mid f(x_{n+1}) = x_n \ (n \geq 0)\}.
\]

It is easy to see that if \( \lim \leftarrow f X(K) \) is a finite set, then so is \( X(K)_{\text{per}, f} \) and \( \# \lim \leftarrow f X(K) = \# X(K)_{\text{per}, f} \) (cf. Lemma 2.2).

We obtain the following results.

**Theorem A** (cf. Corollary 2.5 and §6). Let \( X \) be a geometrically irreducible normal projective variety defined over a field of finite type over \( \mathbb{Q} \). Assume that the Picard number of \( X \) is 1 (for example, \( X \) is \( \mathbb{P}^n \) or a geometrically irreducible normal projective curve). Then \( X \) is periodically finite.

We prove this by using Northcott’s finiteness theorem of height functions. More precisely, this result is a corollary of the fact that if there is an ample line bundle \( L \) such that \( f^*(L) \otimes L^{-1} \) is also ample, then \( \lim \leftarrow f X(K) \) is a finite set (Theorem 2.4).

We also show:

**Theorem B** (cf. Corollary 3.4 and §6). Let \( C \) be a curve defined over a field of finite type over \( \mathbb{Q} \). Then \( C \) is periodically finite.
Theorem C (cf. Theorem 4.4 and §6). Let $A$ be an abelian variety defined over a field of finite type over $\mathbb{Q}$. Then $A$ is periodically finite if and only if $A$ is simple.

For a number field case, we also show:

Theorem D (cf. Theorem 5.8). Let $X$ be a smooth projective surface with the non-negative Kodaira dimension such that $X$ is defined over a number field (for the case of a field of finite type over $\mathbb{Q}$, see §6). Then $X$ is not periodically finite if and only if $X$ is one of the following types;

(i) $X$ is an abelian surface which is not simple, or
(ii) $X$ is a hyperelliptic surface.

In order to clarify the argument, $M$ is assumed to be a number field before §6, where in §6, we deal with a field of finite type over $\mathbb{Q}$ in general.

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1. Quick review of height theory

In this section, we recall some properties of height functions. We refer to [10] for details.

Let $h : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}$ be the logarithmic height function. Namely, for a point $x \in \mathbb{P}^n(\mathbb{Q})$,

$$h(x) = \frac{1}{[K: \mathbb{Q}]} \sum_{v \in M_K} \log \left( \max_{1 \leq i \leq n} \{|x_i|_v\} \right),$$

where $x = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(K)$ is its coordinate over a sufficiently large number field $K$, and $M_K$ be the set of all places of $K$.

Now let $X$ be a projective variety defined over $\mathbb{Q}$, $\phi : X \to \mathbb{P}^n$ a morphism over $\overline{\mathbb{Q}}$. For a point $x \in X(\mathbb{Q})$, we define the height of $x$ with respect to $\phi$, denoted by $h_\phi(x)$, to be $h_\phi(x) = h(\phi(x))$.

Then the following theorem holds.

Theorem 1.1 (Height Machine). For every line bundle $L$ on a projective variety $X$ defined over $\overline{\mathbb{Q}}$, there exists a unique function $h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ modulo bounded functions with the following property;

(i) For any two line bundles $L_1, L_2$, $h_{L_1 \otimes L_2} = h_{L_1} + h_{L_2} + O(1)$.
(ii) If $f : X \to Y$ be a morphism of projective varieties over $\overline{\mathbb{Q}}$, then $h_{f^*(L)} = f^*(h_L) + O(1)$.
(iii) If $\phi : X \to \mathbb{P}^n$ a morphism over $\overline{\mathbb{Q}}$, then $h_{\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))} = h_\phi + O(1)$.

We also recall some properties of height functions.

Theorem 1.2. (i) (positiveness) If we denote $\text{Supp}(\text{Coker}(H^0(X, L) \otimes \mathcal{O}_X) \to L)$ by $\text{Bs}(L)$, then $h_L$ is bounded below on $(X \setminus \text{Bs}(L))$.

(ii) (Northcott) Assume $L$ be ample. Then for any $d \geq 1$ and $M \geq 0$,

$$\{x \in X(\overline{\mathbb{Q}}) \mid h_L(x) \leq M, \ [\mathbb{Q}(x) : \mathbb{Q}] \leq d\}$$

is a finite set.
For Theorem 3.3, we refer to [10, Theorem 3.3]. For Theorem 3.2, we refer to [10] Corollary 3.4 and Proposition 3.5. Although in [10] Theorem 1.1 (ii) is written for a morphism of smooth projective varieties, it also holds for not necessarily smooth projective varieties.

2. Finiteness

Let $X$ be a variety defined over a number field $M$. Let $K$ be a finite extension of $M$ and $f : X \to X$ a dominant morphism defined over $K$.

We say that a point $P \in X(K)$ is periodic with respect to $f$ if there is a positive integer $n$ with $f^n(P) = P$. Let $X(K)_{\text{per}, f}$ be the set of periodic $K$-points with respect to $f$.

We also define the set of backward $K$-orbits of $f$, denoted by $\varprojlim_f X(K)$, to be

$$\varprojlim_f X(K) = \left\{(x_n)_{n=0}^\infty \in \prod_{n=0}^\infty X(K) \mid f(x_{n+1}) = x_n \ (n \geq 0)\right\}.$$

We say that $X$ is periodically finite if for any finite extension $K$ of $M$ and for any dominant $K$-morphism $f : X \to X$ with $\deg f \geq 2$, $X(K)_{\text{per}, f}$ is a finite set.

In this paper, we would like to study what kind of $X$ is periodically finite.

We first remark elementary properties of $X(K)_{\text{per}, f}$ and $\varprojlim_f X(K)$.

Lemma 2.1. Let $S \subset X(K)$ be a finite set and $(x_n)_{n=0}^\infty \in \varprojlim_f X(K)$. Assume that there is a subsequence $(x_n)_{n=0}^\infty$ consisting of elements in $S$. Then $(x_n)_{n=0}^\infty$ is periodic, i.e., there is a positive integer $p$ with $x_{n+p} = x_n$ for $n \geq 0$. Moreover, $(x_n)_{n=0}^\infty$ is uniquely determined by $x_0$.

Proof. Since $S$ is a finite set, there is an element $s \in X(K)$ such that, for infinitely many $n$, $x_n$ equals to $s$. Let $(x_n)_{n=0}^\infty$ be the subsequence of $(x_n)_{n=0}^\infty$ with $x_n = s$ for $j \geq 0$. Let us set $p = n_1 - n_0$. We show that $n_2 - n_1 = p$. Indeed, since $f^s(x_{n_2}) = x_{n_1}$, if we set $q = n_2 - n_1$, then we have $f^q(s) = s$. If we assume $q > p$, then $n_2 > n_2 - p > n_1$ and $x_{n_2} = x_{n_2 - p} = x_{n_1} = s$. This is a contradiction. If we assume $p > q$, then we similarly have a contradiction. Thus $n_2 - n_1 = n_1 - n_0 = p$. In the same way, $n_{j+1} - n_j = p$ for any $j \geq 0$. Now let us take any $n \geq 0$. We fix an $n_j$ with $n_j > n$ and set $r = n_j - n$. Then $n_j + p = n_{j+1}$ and $n_{j+1} - (n + p) = r$. Therefore, we get

$$x_{n+p} = f^r(x_{n_j+1}) = f^r(s) = f^r(x_{n_j}) = x_n.$$

This shows that $(x_n)_{n=0}^\infty$ is periodic. Moreover if we divide $n$ by $p$ and write $n = qp + l$ with $0 \leq l \leq p - 1$, then it is easy to see that $x_n = f^{(p-l)}(x_0)$. This shows the latter assertion of the lemma.

The next lemma gives the relationship between $\varprojlim_f X(K)$ and $X(K)_{\text{per}, f}$.

Lemma 2.2. (i) If $P$ is a $K$-periodic point, then there is an element $(x_n)_{n=0}^\infty \in \varprojlim_f X(K)$ such that $P = x_0$. By this correspondence, $X(K)_{\text{per}, f}$ can be seen as a subset of $\varprojlim_f X(K)$. We say an element of $\varprojlim_f X(K)$ which lies in the image of $X(K)_{\text{per}, f}$ to be periodic.

(ii) If $X(K)_{\text{per}, f} \subseteq \varprojlim_f X(K)$ in the above correspondence, then $\varprojlim_f X(K)$ is an infinite set.
(iii) If \( \lim_f X(K) \) is a finite set, then \( X(K)_{\text{per},f} = \lim_f X(K) \) in the above correspondence.

In particular, \( X(K)_{\text{per},f} \) is also a finite set.

Proof. (i) Let \( f^{\circ p}(P) = P \). For any \( n \geq 0 \), we divide \( n \) by \( p \) and write \( n = qp + l \) with \( 0 \leq l \leq p - 1 \). Then if we put \( x_n = f^{\circ (p-1)}(P) \), \( (x_n)_{n=0}^{\infty} \) is an element of \( \lim_f X(K) \).

(ii) Suppose \( (x_n)_{n=0}^{\infty} \in \lim_f X(K) \) is not periodic. By Lemma 2.1, for any fixed \( m \), there are only finitely many \( k \) with \( x_k = x_m \). Then \( \{(x_n)_{n=m}^{\infty} \mid m \geq 0\} \subset \lim_f X(K) \) is an infinite set.

(iii) If \( \lim_f X(K) \) is a finite set, then every \( (x_n)_{n=0}^{\infty} \in \lim_f X(K) \) is periodic by (ii). In particular, \( x_0 \) is periodic. Therefore, the correspondence of (i) becomes bijective. \( \square \)

Next lemma shows that finiteness still holds if we change \( f \) to some powers of \( f \).

Lemma 2.3. Let \( k \) be a positive integer.

(i) \( X(K)_{\text{per},f^k} \) is a finite set if and only if \( X(K)_{\text{per},f} \) is a finite set.

(ii) \( \lim_{f^k} X(K) \) is a finite set if and only if \( \lim_f X(K) \) is a finite set.

Proof. (i) Suppose \( P \) satisfies \( f^{\circ m}(P) = P \). Then \( P \) satisfies \( (f^{\circ k})^{\circ m}(P) = P \). This shows that \( X(K)_{\text{per},f} = X(K)_{\text{per},f^k} \).

(ii) We have only to prove the “only if” part. If \( \lim_{f^k} X(K) \) is a finite set, its elements are all periodic by Lemma 2.2(ii). Thus if we set

\[
S = \{ x \in X(K) \mid \text{there is an} \ (x_n)_{n=0}^{\infty} \in \lim_{f^k} X(K) \ \text{and an} \ m \ \text{such that} \ x = x_m \},
\]

then \( S \) is a finite set. Now the finiteness of \( \lim_f X(K) \) follows from Lemma 2.1. \( \square \)

Now we prove the following theorem.

Theorem 2.4. Let \( X \) be a projective variety defined over a number field \( K \) and \( f: X \to X \) a surjective morphism defined over \( K \). Assume that there is an ample line bundle \( L \) such that \( f^*(L) \otimes L^{-1} \) is ample. Then \( \lim_f X(K) \) is a finite set. In particular, \( X(K)_{\text{per},f} \) is also a finite set and \( \# \lim_f X(K) = \# X(K)_{\text{per},f} \).

Proof. If we take a positive rational number \( \epsilon' \) which is sufficiently small, then \( f^*(L) \otimes L^{-1+\epsilon'} \) is still ample as a \( \mathbb{Q} \)-line bundle. Then by Theorem 2.2(i), and by the fact that \( h_{f^*(L)}(P) - h_L(f(P)) \) is a bounded function, we have a constant \( C \) such that

\[
h_L(f(P)) - (1 + \epsilon')h_L(P) \geq C.
\]

for all \( P \in X(K) \). Let us take an \( \epsilon \) with \( 0 < \epsilon < \epsilon' \). Then there is a constant \( M \) such that if \( h_L(P) > M \), then

\[
h_L(f(P)) - (1 + \epsilon)h_L(P) > 0.
\]

Now let us define a set \( S \) to be

\[
S = \{ x \in X(K) \mid h_L(x) \leq M \}.
\]

Since \( L \) is ample, \( S \) is a finite set by Northcott.

In the following we show that, if \( (x_n)_{n=0}^{\infty} \in \lim_f X(K) \), then there is a subsequence \( (x_{n_i})_{i=0}^{\infty} \) consisting of elements in \( S \). In fact, suppose on the contrary that there is an \( m \) such that, for any \( n \geq m \), \( x_n \) does not belong to \( S \). Since \( h_L(x_n) > M \) for \( n \geq m \), we have

\[
\cdots < (1 + \epsilon)^2h_L(x_{m+2}) < (1 + \epsilon)h_L(x_{m+1}) < h_L(x_m)
\]
This is a contradiction because
\[ h_L(x_n) < \frac{1}{(1 + \epsilon)^n} h_L(x_m) \to 0 \quad (n \to \infty) \]

Now by applying Lemma 2.1, we get that \((x_n)_{n=0}^\infty\) is periodic and uniquely determined by \(x_0\). We also get that the number of \(\lim fX(K)\) does not exceed the number if \(S\). This proves the first assertion. The second assertion follows from Lemma 2.2. \(\square\)

As a corollary, we obtain the finiteness for a certain class of varieties.

**Corollary 2.5.** Let \(X\) be a geometrically irreducible normal projective variety defined over a number field \(M\). Assume that the Picard number of \(X\) is 1 (for example, \(X\) is \(\mathbb{P}^n\) or a geometrically irreducible normal projective curve). Then \(X\) is periodically finite.

**Proof.** Let \(K\) be a finite extension field of \(M\) and \(f : X \to X\) be a surjective \(K\)-morphism of \(\deg f \geq 2\). We take an arbitrary ample line bundle \(L\) on \(X\). Then by our hypothesis, there is an integer \(d \geq 2\) such that \(f^*(L)\) is numerically equivalent to \(L^\otimes d\). In particular, \(f^*(L) \otimes L^{-1}\) is ample. \(\square\)

Let us keep the notation of Theorem 2.4. Assume here that \(f^*(L)\) is linearly equivalent to \(L^\otimes d\). In this case, due to Tate, there exists a unique height function \(h_{L,f}\) such that \(h_{L,f} = h_L + O(1)\) and that \(h_{L,f}(f(P)) = dh_L(P)\) (cf. [3, Chap 4, Proposition 1.9]). Then for any periodic points with respect to \(f\), their height must be 0 with respect to \(h_{L,f}\). For example:

**Corollary 2.6.** Let \(K\) be a number field, \(A\) an Abelian variety defined over \(K\) and \([m] : A \to A\) the \(m\)-plication map with \(m \geq 2\). Then \(\lim [m]A(K)\) is a finite set and the number of \(\lim [m]A(K)\) does not exceed the number of torsion \(K\)-points.

**Proof.** Extending \(K\) if necessary, we may assume that there is an ample symmetric line bundle \(L\) on \(A\). Then \(f^*L \simeq L^\otimes m^2\) and we can apply the theorem. In this case, if \(x\) is a periodic point, then \(x\) is a torsion point. \(\square\)

We finish this section by giving examples such that \(X(K)_{per,f}\) is infinite.

**Example 2.7.** We give an example such that \(X(K)_{per,f}\) (and thus \(\lim fX(K)\)) is infinite. Let \(E\) be an elliptic curve defined over a number field \(K\) such that \(E(K)\) is an infinite set. Let \(X\) be \(E \times E\) and \(f : X \to X\) map \((P,Q)\) to \((P,[2](Q))\). Then \(f\) is finite of degree 4 and the points of the form \((P,0)\) are all periodic points.

**Example 2.8.** We give an example such that \(X(K)_{per,f}\) is finite but \(\lim fX(K)\) is infinite. Let \(E\) be an elliptic curve defined over a number field \(K\) for which \(E(K)\) contains non-torsion points. Let \(P_0 \in E(K)\) be a non-torsion point. Let \(X\) be \(E \times E\) and \(f : X \to X\) map \((P,Q)\) to \((P + P_0, [2](Q))\). Then \(f\) is finite of degree 4 and contains a sequence \((x_n)_{n=0}^\infty \in \lim fX(K)\) with \(x_n = (-[n](P_0),0)\). Thus by Lemma 2.2, \(\lim fX(K)\) is not finite. On the other hand, there are no periodic points.

We note that we can give examples similar to the above two examples by using \(\mathbb{P}^1\).
3. Curves

By a curve, we mean an integral separated scheme of finite type over a ground field. In this section, we prove that a curve is periodically finite. Since there is no surjective morphism $f : C \rightarrow C$ with $\deg f \geq 2$ if $C$ is a smooth projective curve of genus $\geq 2$, we are mainly concerned with a curve $C$ such that $C \otimes \mathbb{Q}$ is a reduced scheme consisting of rational curves and elliptic curves. First we prove two lemmas.

**Lemma 3.1.** Let $C$ be a curve defined over $\overline{\mathbb{Q}}$, and $f : C \rightarrow C$ a morphism over $\overline{\mathbb{Q}}$. Then there is a completion $\overline{C}$ of $C$ and a morphism $\overline{f} : \overline{C} \rightarrow \overline{C}$ which is an extension of $f$.

**Proof.** Let us take an arbitrary complete curve $\overline{C}$ which is a completion of $C$ and set $T' = \overline{C} \setminus C(\overline{\mathbb{Q}})$. If $t \in T'$ is a singular point of $\overline{C}$, then we blow it up. Iterating this procedure, we get a completion $\overline{C}$ such that every point in $T = \overline{C} \setminus C(\overline{\mathbb{Q}})$ is a smooth point of $\overline{C}$. Now $f$ defines a rational map $\overline{f} : \overline{C} \cdots \rightarrow \overline{C}$. Since it is defined over $\overline{T}$ and $\overline{C}$, $\overline{f}$ is actually a morphism. \hfill \Box

**Lemma 3.2.** Let $C$ be a curve defined over a number field $M$ which is geometrically irreducible. Then $C$ is periodically finite.

**Proof.** Let $K$ be a finite extension of $M$ and $f : X \rightarrow X$ a surjective morphism defined over $K$ with $\deg f \geq 2$. By taking a finite extension of $K$ if necessary, Lemma 2.1 indicates that there is a completion $\overline{C}$ of $C$ and a extension $\overline{f}$ of $f$ which are defined over $K$. Then $\lim \overline{f}C(K)$ can be seen as a subset of $\lim \overline{f}C(K)$. For a general point $P \in \overline{C}(\overline{\mathbb{Q}})$, let $L = \mathcal{O}_{\overline{C}}(P)$. Then, since $\deg \overline{f} \geq 2$, $f^*(L) \otimes L^{-1}$ is ample. Thus, by Theorem 2.4, $\lim \overline{f}C(K)$ is a finite set. This proves the lemma. \hfill \Box

Now we prove the following proposition.

**Proposition 3.3.** Let $C$ be a reduced scheme which is a chain of geometrically irreducible curves over $\overline{\mathbb{Q}}$. Let $f : C \rightarrow C$ be a surjective morphism such that, for every irreducible component $C_i$ of $C$, $f|_{C_i} : C_i \rightarrow f(C_i)$ has degree $\geq 2$. Then for a number field $K \subset \overline{\mathbb{Q}}$ such that $C$ and $f$ are defined over $K$, $\lim f^*C(K)$ is a finite set.

**Proof.** If $K'$ is a extension field of $K$, then the finiteness of $\lim f^*C(K')$ implies the finiteness of $\lim f^*C(K)$. Thus to prove the proposition, we may take a finite extension of $K$ if necessary. Let $C_1, C_2, \ldots, C_l$ be the irreducible components of $C$. Since $f$ is surjective, the dimension of $f(C_\alpha)$ is $1$ for every $\alpha$. Thus $f$ is seen to induce a transposition of the set $C_1, C_2, \ldots, C_l$. Then $f^\text{red}$ maps $C_\alpha$ to $C_\beta$ for $1 \leq i \leq l$. Let us set $S = (\cup_{\alpha \neq \beta} C_\alpha \cap C_\beta)_{\text{red}}$. By Lemma 2.3, we have only to show that $\lim f^\text{red}C(K)$ is a finite set. We may take a sufficiently large $K$, so that $C_\alpha$’s and $S$ are all defined over $K$. Now let $(x_n)_{n=0}^\infty \in \lim f^\text{red}X(K)$.

Case 1 Suppose that there exists a subsequence $(x_n)_{i=0}^\infty$ consisting of elements in $S$. Then by Lemma 2.1, the number of $(x_n)_{n=0}^\infty$ in this case is finite.

Case 2 Suppose that there is no subsequence $(x_n)_{n=0}^\infty$ consisting of elements in $S$. Then there is an $\alpha$ such that every $x_n$ belongs to $C_\alpha$. By Lemma 3.1, $\lim f^\text{red}C_\alpha(K)$ is a finite set. Thus the number of $(x_n)_{n=0}^\infty$ in this case is also finite. \hfill \Box

As a corollary, we get
Corollary 3.4. Let $C$ be a curve defined over a number field $M$. Then $C$ is periodically finite.

Proof. Let $K$ be a finite extension of $M$ and $f : C \to C$ be a surjective $K$-morphism with $\deg f \geq 2$. Let us consider $C_{\overline{K}}$ and let $C_1, C_2, \ldots, C_t$ be its irreducible components. By abbreviation, $f$ also denotes the induced morphism $C_{\overline{K}} \to C_{\overline{K}}$. Since $C_1, C_2, \ldots, C_t$ are all conjugate to each other, the degree of $f_{\mid C_{\overline{K}}}$ is greater or equal to 2 for each $1 \leq \alpha \leq t$. Now the assertion follows from Proposition 3.3. \qed

4. Abelian varieties

Let $A$ be an abelian variety defined over a number field $M$. Recall that $A$ is said to be simple if $\text{End}(A)_{\overline{Q}}$ is simple. In this section, we show that $A$ is periodically finite if and only if $A$ is simple.

First we show that if an abelian variety is simple, then it is periodically finite.

Proposition 4.1. Let $A$ a simple abelian variety defined over a number field $M$. Then $A$ is periodically finite.

Proof. Let $K$ be a finite extension field of $M$ and $f : X \to X$ a finite $K$-morphism with $\deg f \geq 2$. Let us set $B_n = \{ P \in A(K) \mid f^{\circ n}(P) = P \}$. We prove the finiteness of $A(K)_{\text{per}, f}$ in two steps.

Step 1 We assume here that $f$ is a homomorphism. Let us denote by $A(K)_{\text{tor}}$ the set of $K$-valued torsion points on $A$. It is well known that $A(K)_{\text{tor}}$ is a finite set (cf. Corollary 2.3). Since $A$ is simple and $f^{\circ n} \neq 1$, $B_n = \text{Ker}(f^{\circ n} - 1)(K)$ is a finite abelian group. In particular, $B_n \subset A(K)_{\text{tor}}$. Thus $A(K)_{\text{per}, f} = \cup_{n=1}^{\infty} B_n \subset A(K)_{\text{tor}}$ is a finite set.

Step 2 Here we treat a general $f$. If $B_n = \emptyset$ for $n \geq 1$, then we have nothing to prove. Thus we assume that there is an $k$ with $B_k \neq \emptyset$ and we shall prove $A(K)_{\text{per}, f}$ is a finite set. Since $A(K)_{\text{per}, f}^{\circ k} = A(K)_{\text{per}, f}$ by Lemma 2.3, we may assume that $B_1 \neq \emptyset$. We take $x_0 \in B_n$, i.e., $f(x_0) = x_0$. We give $A$ another group structure such that the identity is $x_0$. We denote this abelian variety by $A'$. Since $f$ maps $x_0$ to itself, $f$ is a homomorphism of $A'$. Therefore, $A'(K)_{\text{per}, f}$ is a finite set by Step 1. Since $A$ and $A'$ are identical as a set and thus $A(K)_{\text{per}, f} = A'(K)_{\text{per}, f}$, we are done. \qed

Next we show that if $A$ is not simple, then $A$ is not periodically finite. First we prove the following lemma.

Lemma 4.2. Let $A$ be an abelian variety defined over a number field $M$. Then there exists a finite extension field $K$ of $M$ such that $A(K)$ is an infinite set.

Proof. By Bertini, there is a curve $C$ of genus $\geq 2$ on $A_{\overline{Q}}$. By Raynaud’s theorem (Manin-Mumford conjecture), $C(\overline{Q}) \cap A(\overline{Q})_{\text{tor}}$ is a finite set. Take a finite extension field $K$ of $M$ such that $C(K)$ contains a non-torsion point $P$. Then since $A(K)$ contains $P$, the rank of $A(K)$ is positive. (The author does not know easier proofs of this lemma.) \qed

Proposition 4.3. Let $A$ be an abelian variety defined over a number field $M$. If $A$ is not simple, then $A$ is not periodically finite.
Proof. Since \(A\) is not simple, there is an \(\overline{\mathbb{Q}}\)-isogeny \(g: B \times C \to X\), where \(B\) and \(C\) are positive-dimensional abelian varieties. Let us set \(D = \text{Ker} \, g\), which is a finite group of order \(d = \#D\).

We consider a morphism
\[
[d+1] \times [1]: B \times C \longrightarrow B \times C.
\]
Since, for a point \((b, c) \in D\), \([d]b, [d]c = 0\), we get \([k+1] \times [1](b, c) = (b, c)\) for any \((b, c) \in D\). In particular, \([d+1] \times [1]\) induces a morphism
\[f: A \longrightarrow A.\]

By the snake lemma, \(\text{Ker}([d+1] \times [1]) = \text{Ker} \, f\), thus \(f\) is a surjective morphism with \(\deg f \geq 2\). Now we take a finite extension field \(K\) of \(M\) such that \(B\) and \(C\) are defined over \(K\) and that \(C(K)\) is an infinite set. Then the infinite set
\[g(\{(0, Q) \in B(K) \times C(K)\})\]
is contained in \(A(K)_{\text{per}, f}\).

Combining Proposition \ref{prop:1.1} and Proposition \ref{prop:1.3}, we get:

**Theorem 4.4.** Let \(A\) be an abelian variety defined over a number field. Then \(A\) is periodically finite if and only if \(A\) is simple.

5. surfaces with non-negative Kodaira dimensions

In this section we consider smooth projective surfaces with non-negative Kodaira dimensions.

E. Sato and Y. Fujimoto \cite{2} \cite{3} determined smooth projective varieties of \(\dim = 3\) with the non-negative Kodaira dimensions which has a non-trivial surjective endomorphism.

As a test case, they consider the surface case, which is as in the following.

**Theorem 5.1** (E. Sato and Y. Fujimoto). If a smooth projective surface \(X\) has a surjective endomorphism \(f: X \to X\) with \(\deg f \geq 2\), then \(X\) must be minimal and is one of the following types;

(i) \(X\) is an abelian surface,

(ii) \(X\) is a hyperelliptic surface, or

(iii) The Kodaira dimension of \(X\) is 1 and \(X\) carries an elliptic fibration \(\pi: X \to B\) whose singular fibers are at most multiple of the type \(mI_0\) in the sense of Kodaira, where \(B\) is a smooth projective curve.

**Proof.** For the reader’s sake, we give a brief sketch of a proof.

Since \(X\) has non-negative Kodaira dimension, \(f: X \to X\) must be étale (cf. \cite{3} Theorem 11.7]). Suppose there is an exceptional curve \(C\) on \(X\). Then the equality
\[f^*(C) \cdot K_X = f^*(C) \cdot f^*K_X = -(\deg f)\]
shows that there are at least two exceptional curves on \(X\). Iterating this procedure, we get a contradiction.

We note that since \(f\) is étale, \(\chi_{\text{top}}(X) = (\deg f)\chi_{\text{top}}(X)\). Then \(\deg f \geq 2\) implies \(\chi_{\text{top}}(X) = 0\). In the same way, we get \(\chi(O_X) = 0\).
If the Kodaira dimension of $X$ is 2, there are no surjective morphisms $f : X \to X$ with $\deg f \geq 2$ (cf. [6, Proposition 10.10]).

If the Kodaira dimension of $X$ is 1, then $\chi_{top}(X) = 0$ indicates that $X$ has possibly only multiple singular fibers of type $mI_0$.

If the Kodaira dimension of $X$ is 0, then $\chi(O_X) = 0$ indicates that $X$ cannot be a K3 surface nor an Enriques surface.

We determined in the previous section when an abelian surface is periodically finite. Now we study whether a surface of the case (ii) or (iii) is periodically finite.

**Proposition 5.2.** Let $X$ be a hyperelliptic surface defined over a number field $M$. Then $X$ is not periodically finite.

**Proof.** Let $E$, $F$ be arbitrary elliptic curves, $G$ a group of translations of $E$ which operates on $F$. According to the Bagnera-De Franchis list ([5, Liste VI.20]), all the hyperelliptic curves are one of the following types:

(i) $X \cong (E \times F)/G$, $G = \mathbb{Z}/2$ operating on $F$ by $x \mapsto -x$,
(ii) $X \cong (E \times F)/G$, $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ operating on $F$ by $x \mapsto -x$, $x \mapsto x + \epsilon$ ($\epsilon \in F_2$),
(iii) $X \cong (E \times F_1)/G$, $G = \mathbb{Z}/4$ operating on $F$ by $x \mapsto i x$, where $F_1 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$,
(iv) $X \cong (E \times F_2)/G$, $G = \mathbb{Z}/4$ operating on $F$ by $x \mapsto i x$,
(v) $X \cong (E \times F_3)/G$, $G = \mathbb{Z}/3$ operating on $F$ by $x \mapsto \rho x$, where $\rho = \frac{-1 + \sqrt{3}}{2}$ and $F_3 = \mathbb{C}/\mathbb{Z} + \rho\mathbb{Z}$.
(vi) $X \cong (E \times F_4)/G$, $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ operating on $F$ by $x \mapsto \rho x$, $x \mapsto X + \frac{\rho - 1}{3}$
(vii) $X \cong (E \times F_5)/G$, $G = \mathbb{Z}/6$ operating on $F$ by $x \mapsto -\rho x$.

Now we consider the case (i). In this case,

$$[3] \times [1] : E \times F \longrightarrow E \times F$$

induces a surjective morphism

$$f : X \to X$$

with $\deg f \geq 2$. If we take a sufficiently large finite extension field $K$ of $M$, Then the infinite set $\{(0, Q) \mid Q \in F(K)\}$ is contained in $(E \times F)(K)_{\text{per},[3] \times [1]}$. Thus $X(K)_{\text{per},f}$ is also an infinite set. The other cases can be treated in similar ways. In lieu of $[3] \times [1]$, we have only to consider $[g + 1] \times [1]$ where $g = \#G$. □

Next we treat a case of an elliptic surface. We first consider an elliptic surface such that the genus of the base curve is greater or equal to 2, and then one such that the genus of the base curve is 0 or 1.

**Proposition 5.3.** Let $M$ be a number field. Let $X$ be a smooth projective surface defined over $M$ with the Kodaira dimension 1. We assume that $X$ carries an elliptic fibration $\pi : X \to B$ with at most multiple singular fibers of the type $mI_0$ in the sense of Kodaira, where $B$ is a smooth projective curve of genus $g(B) \geq 2$. Then $X$ is periodically finite.

**Proof.** Let $f : X \to X$ be a surjective morphism with $\deg f \geq 2$. Since $X$ has a unique structure of an elliptic fibration up to isomorphisms, there is an automorphism $g : B \to B$ with $\pi \circ f = g \circ \pi$. Moreover there is a positive integer $k$ such that $g^{\circ k}$ is the identity morphism. By Lemma 2.3, we may assume by interchanging $f$ with $f^{\circ k}$ that $g$ is the identity morphism.
Then $f$ is a finite morphism preserving fibers. Let $K$ be a sufficiently large number field such that $X, B, f, \pi, g$ are all defined over $K$.

Now let $b$ be a point of $B$ and consider $f|_{(X_b)_{\text{red}}} : (X_b)_{\text{red}} \to (X_b)_{\text{red}}$. Since $f$ is an étale morphism (cf. [3, Theorem 11.7]), $f|_{(X_b)_{\text{red}}} : (X_b)_{\text{red}} \to (X_b)_{\text{red}}$ is a surjective morphism with $\deg f|_{(X_b)_{\text{red}}} \geq 2$.

Now we prove the finiteness of $X(K)_{\text{per}, f}$ by showing the finiteness of $\varprojlim fX(K)$ (cf. Lemma 2.2). Let $(x_n)^n_{n=0} \in \varprojlim fX(K)$. Since $f$ preserves fibers, $x_n$ are all contained in the fiber $X_{\pi(x_n)}$. On the other hand, by Mordell-Faltings’ theorem, $B(K)$ is a finite set. Since $\pi(x_0) \in B(K)$, the number of $b$ such that $b = \pi(x_0)$ for some $(x_n)^n_{n=0} \in \varprojlim fX(K)$ is finite. Since $\varprojlim f|_{(X_b)_{\text{red}}} (X_b)_{\text{red}}(K)$ is a finite set for each such $b$ by Lemma 3.2 we get the assertion.

Next we consider an elliptic surface such that the genus of the base curve is 0 or 1. We prove the following two lemmas in advance.

Lemma 5.4. Let $\pi : X \to B$ be an elliptic surface with the Kodaira dimension 1. Then the geometric genus of every multi-section of $\pi$ is greater or equal to 2.

Proof. Suppose there is a multi-section $C$ on $X$ such that the geometric genus of $C$ is 0 or 1. Then there is an elliptic curve $B'$ with a surjection $u : B' \to C$. Let us set $v = u \circ \pi : B' \to B$. Now we consider the following Cartesian product,

$$
\begin{array}{ccc}
X' & \xrightarrow{\pi'} & X \\
\downarrow & & \downarrow \pi \\
B' & \xrightarrow{v} & B.
\end{array}
$$

Since the singular fibers of $\pi'$ are at most multiple fibers of type $mI_0$ and since $\pi' : X' \to B'$ has a section, $\pi'$ must be a smooth morphism. Then there is an elliptic curve $B''$ and an étale covering $B'' \to B'$ such that its pull-back $\pi'' : X'' = X' \times_B B''$ is trivial, i.e., $X''$ is a product of elliptic curves (cf. [4, Proposition VI.8]). On the other hand, since there is a surjective morphism $X'' \to X$, the Kodaira dimension of $X''$ must be greater or equal to 1. This is a contradiction.

Lemma 5.5. Let $A$ be a simple abelian variety defined over a algebraically closed field $k$ and $f : A \to A$ a surjective morphism with $\deg f \geq 2$. Assume that there is a positive integer $l$ such that $f^l(0) = 0$. Then $f$ maps a torsion point to a torsion point.

Proof. Let us set $f(0) = a$.

We first show that $a$ is a torsion point. For this purpose, we may assume that $l \geq 2$. Let us set $h = f - a : A \to A$. Then $h$ is a homomorphism with $\deg h \geq 2$. The equality $f^l(0) = 0$ indicates that

$$
\begin{align*}
h^l(a) + h^{l-2}(a) + \cdots + h(a) + a = 0.
\end{align*}
$$

Let us set

$$
h' = h^l + h^{l-2} + \cdots + h + id.
$$
We claim that $h' : A \to A$ is a surjective homomorphism of $\deg h' \geq 2$. Indeed, suppose $h'$ is not surjective. Then, since $A$ is simple, $h'$ must be the zero map. Then we get

$$-id = h \circ \left( h^{\circ(l-2)} + h^{\circ(l-2)} + \cdots + id \right),$$

which contradicts with $\deg h \geq 2$. Thus $h'$ is surjective. Moreover, since $h'$ maps a non-zero element $a$ to 0, $h'$ is not an isomorphism.

Then, since $A$ is simple, $\ker h'$ must be a finite group. In particular, $a \in \ker h'$ is a torsion point.

Now let $b$ be any torsion point of $A$. We take a positive integer $n$ such that $na = 0$ and $nb = 0$. Then

$$nf(b) = n(h(b) + a) = h(nb) + na = 0.$$ 

Thus $f$ maps a torsion point to a torsion point.

\[ \square \]

**Remark 5.6.** The above lemma does not hold in general for abelian varieties. For example, let $A$ be an abelian variety and $a$ a non-torsion point. If we set $f : A \times A \to A \times A$ by $f(x, y) = (2x, -y + a)$, then $f^{\circ 2}(0, 0) = (0, 0)$. However, $f(0, 0) = (0, a)$ is not a torsion point.

**Proposition 5.7.** Let $M$ be a number field. Let $X$ be a smooth projective surface defined over $M$ with the Kodaira dimension 1. We assume that $X$ carries an elliptic fibration $\pi : X \to B$ with at most multiple singular fibers of the type $mI_0$ in the sense of Kodaira, where $B$ is a smooth projective curve of genus 0 or 1. Then $X$ is periodically finite.

**Proof.** Let $f : X \to X$ be a surjective morphism with $\deg f \geq 2$. Since $X$ has a unique structure of an elliptic fibration up to isomorphisms, there is an automorphism $g : B \to B$ with $\pi \circ f = g \circ \pi$. Let $K$ be a sufficiently large number field such that $X, B, f, \pi, g$ are all defined over $K$.

**Case 1** Suppose that for any $k \geq 1$, $g^{\circ k}$ is not the identity morphism. Let us set

$$S = \{ b \in B(\bar{\mathbb{Q}}) \mid g^{\circ k}(b) = b \text{ for some } k \geq 1 \}.$$

We claim that $S$ consists at most two points. Indeed, suppose $S$ contains three points $b_1, b_2, b_3 \in B(\bar{\mathbb{Q}})$ such that $g^{\circ k_i}(b_i) = b_i$ for $i = 1, 2, 3$. Then for $k = k_1k_2k_3$ we get $g^{\circ k}(b_i) = b_i$ for $i = 1, 2, 3$. Since $B$ is $\mathbb{P}^1$ or an elliptic curve, this shows that $g^{\circ k}$ is the identity morphism, which contradicts our assumption of Case 1.

We take $l$ such that $g^{\circ l}(b) = b$ for any $b \in S$. Now we prove the finiteness of $X(K)_{\per,f}$ by showing the finiteness of $\lim f^{\circ l}X(K)$ (cf. Lemma 2.2 and Lemma 2.3). Let $(x_n)_{n=0}^{\infty}$ be an element of $\lim f^{\circ l}X(K)$. Since $\pi(x_n)$ belongs to $S$, $x_n$ are all contained in the fiber $X_{\pi(x_0)}$. Since $f^{\circ l}$ is an étale morphism (cf. [4, Theorem 11.7]), $\lim f^{\circ l}(X_{b_{\text{red}}}(X_{b_{\text{red}}})(K)$ is a finite set for $b \in S$ by Lemma 3.2. Using the finiteness of $S$, we obtain the finiteness of $\lim f^{\circ l}X(K)$.

**Case 2** Suppose that there is a $k \geq 1$ such that $g^{\circ k}$ is the identity morphism. To prove the finiteness of $X(K)_{\per,f}$, we may (and will) assume by interchanging $f$ with $f^{\circ k}$ that $g$ is the identity morphism (cf. Lemma 2.3).
To prove the theorem, we first recall the Merel theorem (cf. [7]) : There is a positive integer \(m_0 = m_0(K)\) which depends only on \(K\) such that for any elliptic curve \(E\) defined over \(K\),

\[
\#E(K)_{tor} \leq m_0(K).
\]

**Claim 5.7.1.** There is a positive integer \(m = m(K)\) which depends only on \(K\) such that if \(x\) belongs to \(X(K)_{per,f}\), then \(f^m(x) = x\).

**Proof.** Let \(x\) be an element of \(X(K)_{per,f}\). Let us set \(\pi(x) = b \in B(K)\). We introduce a group structure on \(X_b\) by letting \(x\) be the origin. Then \(X_b\) is an elliptic curve defined over \(K\). Since \(x\) is a periodic point, there is a positive integer \(l\) such that \(f^{ol}(x) = x\). Then by Lemma 5.3, \(f\) maps the set \(X_b(K)_{tor}\) to itself. On the other hand, by the Merel theorem,

\[
\#X_b(K)_{tor} \leq m_0.
\]

Thus, if we set \(m = m_0(K)!\), then \(m\) depends only on \(K\) and \(f^m(x) = x\). \(\square\)

Going back to the proof of Proposition 5.7, we define a reduced subscheme \(C\) of \(X\) by

\[
C = \{x \in X \mid f^m(x) = x\}.
\]

By the above claim, \(X(K)_{per,f} \subset C(K)\). On the other hand, let

\[
C = C_1 \cup \cdots \cup C_\alpha
\]

be the irreducible decomposition of \(C\). Since \(f\) is étale with \(\text{deg } f \geq 2\), \(C\) does not contain a fibral curve. If \(C_i\) is a horizontal curve, then \(C_i(K)\) is a finite set by Lemma 5.4 and the Mordell-Faltings theorem. Therefore \(C(K)\) and thus \(X(K)_{per,f}\) is a finite set. \(\square\)

Combining all the results of this section, we get:

**Theorem 5.8.** Let \(X\) be a smooth projective surface with the non-negative Kodaira dimension such that \(X\) is defined over a number field. Then \(X\) is not periodically finite if and only if \(X\) is one of the following types;

(i) \(X\) is an abelian surface which is not simple, or
(ii) \(X\) is a hyperelliptic surface.

6. Fields of finite type over \(\mathbb{Q}\)

In this section, we work over a field of finite type over \(\mathbb{Q}\). A. Moriwaki has recently constructed the theory of height functions over a field of finite type over \(\mathbb{Q}\). We first recall a part of his theory. We refer to [8] for details.

Let \(K\) be a field of finite type over \(\mathbb{Q}\) with \(\text{tr.deg}_{\mathbb{Q}}(K) = d\). Let \(B\) be a normal variety which is projective and flat over \(\mathbb{Z}\) such that the field of rational functions of \(B\) is \(K\). Let \(\mathcal{H} = (H, h_H)\) be a nef \(C^\infty\)-hermitian line bundle on \(B\), i.e., \(H\) is a line bundle on \(B\) and \(h_H\) is a \(C^\infty\)-hermitian line bundle such that for any curve on \(C\) on \(B\), \(\widehat{\deg \left(c_1(\mathcal{H}|_C)\right)} \geq 0\) (in the sense of the Arakelov geometry) and that the Chern form \(c_1(\mathcal{H})\) is semi-positive. There exist many such \(\mathcal{H} = (B, \mathcal{H})\). We pick up a \(\mathcal{H}\) and fix it in the following.
Now, for a point \( x \in \mathbb{P}^n(K) \), let us define \( h^\mathcal{P}(x) \) to be

\[
h^\mathcal{P}(x) = \sum_\Gamma \log \left( \max_{1 \leq i \leq n} \{- \operatorname{ord}_\Gamma(x_i)\} \deg(c_1(\mathcal{P}|\Gamma)^d) \right) + \int_{B(\mathbb{C})} \log \left( \max_{1 \leq i \leq n} \{|x_i|\} \right) c_1(\mathcal{P})^d,
\]

where \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(K') \) is its coordinate over a sufficiently large extension field \( K' \) of \( K \), and \( \Gamma \) runs through all prime divisors on \( B \). This gives rise to a function \( h^\mathcal{P} : \mathbb{P}^n(K) \to \mathbb{R} \).

Now let \( X \) be a projective variety defined over \( K \), \( \phi : X \to \mathbb{P}^n \) a morphism over \( K \). For a point \( x \in X(\overline{K}) \), we define the height of \( x \) with respect to \( \phi \), denoted by \( h^\phi(x) \), to be \( h^\phi(x) = h(\phi(x)) \).

Then the following theorem holds as is the number field case (cf. [8, §3 - §4]).

**Theorem 6.1.** For every line bundle \( L \) on a projective variety \( X \) defined over \( K \), there exists a unique function \( h^\mathcal{P} : X(\overline{K}) \to \mathbb{R} \) modulo bounded functions with the following property;

(i) For any two line bundles \( L_1, L_2 \), \( h^\mathcal{P}_{L_1 \otimes L_2} = h^\mathcal{P}_{L_1} + h^\mathcal{P}_{L_2} + O(1) \).
(ii) If \( f : X \to Y \) be a morphism of projective varieties over \( K \), then \( h^\mathcal{P}_{f^*(L)} = f^*(h^\mathcal{P}_L) + O(1) \).
(iii) If \( \phi : X \to \mathbb{P}^n \) a morphism over \( K \), then \( h^\phi_{\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))} = h^\phi + O(1) \).

Moreover the following properties hold.

(a) (positiveness) If we denote \( \operatorname{Supp}(\operatorname{Coker}(H^0(X, L) \otimes \mathcal{O}_X) \to L) \) by \( \operatorname{Bs}(L) \), then \( h^\mathcal{P}_L \) is bounded below on \( (X \setminus \operatorname{Bs}(L)) \).
(b) (Northcott) Assume \( L \) is ample. Then for any \( e \geq 1 \) and \( M \geq 0 \),

\[
\{ x \in X(\overline{K}) \mid h^\mathcal{P}_L(x) \leq M, \quad [K(x) : K] \leq e \}
\]

is a finite set.

Aside from the Northcott finite theorem, we used three big theorems.

The first one is the Mordell-Faltings theorem (cf. Proposition 5.3). It is known that this is also true for a finitely generated field over \( \mathbb{Q} \) (cf. [1, Chapter VI]).

The next one is the Raynaud theorem (cf. Lemma 4.2). This is actually proven for a field of finite type over \( \mathbb{Q} \).

The last one is the Merel theorem (cf. Proposition 5.7). Unfortunately this is not known for a field of finite type over \( \mathbb{Q} \). Thus Theorem 5.8 must be replaced by the following weaker theorem for a field of finite type over \( \mathbb{Q} \).

**Theorem 6.2.** Let \( X \) be a smooth projective surface with the non-negative Kodaira dimension such that \( X \) is defined over a field of finite type over \( \mathbb{Q} \). Assume that \( X \) does not carry an elliptic fibration \( X \to B \) with \( g(B) \leq 1 \), where \( g(B) \) denotes the genus of the base curve \( B \). Then \( X \) is not periodically finite if and only if \( X \) is one of the following types;

(i) \( X \) is an abelian surface which is not simple, or
(ii) \( X \) is a hyperelliptic surface.

Aside from Proposition 5.7 and Theorem 6.2, all the other results before this section also hold for a field of finite type over \( \mathbb{Q} \).
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