Ultimate Pólya Gamma Samplers – Efficient MCMC for possibly imbalanced binary and categorical data

Gregor Zens (WU Wien)
Sylvia Frühwirth-Schnatter (WU Wien)
Helga Wagner (JKU Linz)

July 3, 2023

Abstract

Modeling binary and categorical data is one of the most commonly encountered tasks of applied statisticians and econometricians. While Bayesian methods in this context have been available for decades now, they often require a high level of familiarity with Bayesian statistics or suffer from issues such as low sampling efficiency. To contribute to the accessibility of Bayesian models for binary and categorical data, we introduce novel latent variable representations based on Pólya-Gamma random variables for a range of commonly encountered logistic regression models. From these latent variable representations, new Gibbs sampling algorithms for binary, binomial, and multinomial logit models are derived. All models allow for a conditionally Gaussian likelihood representation, rendering extensions to more complex modeling frameworks such as state space models straightforward. However, sampling efficiency may still be an issue in these data augmentation based estimation frameworks. To counteract this, novel marginal data augmentation strategies are developed and discussed in detail. The merits of our approach are illustrated through extensive simulations and real data applications.

Keywords: Bayesian, Data augmentation, Gibbs sampling, Parameter Expansion, MCMC boosting.
1 Introduction

Applied statisticians and econometricians commonly have to deal with modeling binary or categorical outcome variables. Widely used tools for analyzing such data include probit as well as binary, multinomial, and binomial logit regression models. Bayesian approaches toward inference are very useful in this context, as they allow to easily extend the standard regression framework to more complex settings such as random effects or state space models. However, as opposed to regression models with Gaussian outcomes, their implementation can be demanding from a computational viewpoint (Chopin & Ridgway 2017).

One strategy to implement sampling-based inference relies on importance sampling (Zellner & Rossi 1984) or various types of Metropolis-Hastings (MH) algorithms (Rossi et al. 2005), exploiting directly the non-Gaussian likelihood. However, these algorithms often require careful tuning and substantial experience with Bayesian computation, especially in more complex frameworks like state space models.

Routine Bayesian computation for these type of data more often relies on Markov Chain Monte Carlo (MCMC) algorithms based on data augmentation (DA, Tanner & Wong 1987). As shown by the seminal paper of Albert & Chib (1993), the binary probit model admits a latent variable representation where the latent variable equation is linear in the unknown parameters, with an error term following a standard normal distribution. As simulating the latent variables is easy when the parameters are known, the latent variable representation admits a straightforward Gibbs sampler using one level of DA, where the unknown parameters are sampled from a conditionally Gaussian model. This strategy works also for more complex models, such as probit state space or random effects models.

There is also an active literature on posterior simulation tools for probit and logit regression models that does not rely on DA. For instance, Durante (2019) introduces a framework for conjugate analysis.
However, MCMC estimation based on DA is less straightforward for a logit model which still admits a latent variable representation that is linear in the unknown parameters, but exhibits an error term that follows a logistic distribution. Related latent variable representations with non-Gaussian errors exist for multinomial logit (MNL) models (Frühwirth-Schnatter & Frühwirth 2010) and logistic regression models for binomial outcomes (Fussl et al. 2013). While the latent variables usually can be easily sampled, sampling the unknown parameters is more involved due to the non-Gaussian error terms.

A common solution relies on a scale-mixture representation of the non-Gaussian error distribution and introduces the corresponding scale parameters as a second level of DA. Conveniently, the unknown model parameters can then be sampled from a conditionally Gaussian regression model. Examples include a representation of the logistic distribution involving the Kolmogoroff-Smirnov distribution (Holmes & Held 2006) and highly accurate finite scale-mixture approximations (Frühwirth-Schnatter & Frühwirth 2007, 2010, Frühwirth-Schnatter et al. 2009). A seminal paper in this context is Polson et al. (2013) which avoids any explicit latent variable representation. They derive the Pólya-Gamma sampler that exploits a mixture representation of the non-Gaussian likelihood of the marginal model based on the Pólya-Gamma distribution and works with a single level of DA.

In the present paper, we propose a new sampling scheme involving the Pólya-Gamma distribution. Instead of working with the marginal model, we introduce a new mixture representation of the logistic distribution based on the Pólya-Gamma distribution in the of the probit model that has been generalized subsequently, see Anceschi et al. (2023) for a review. Sen et al. (2020) use a sampling framework for logistic regression based on piecewise deterministic Monte Carlo processes. We provide a discussion of these and other alternative methods in Appendix A.1.
latent variable representation of the logit model. Similar to Holmes & Held (2006) and Frühwirth-Schnatter & Frühwirth (2010), we use DA and introduce the Pólya-Gamma mixing variables as a second set of latent variables. Our new Pólya-Gamma mixture representation has the advantage that the joint posterior distribution of all augmented variables is easy to sample from, as the Pólya-Gamma mixing variable follows a tilted Pólya-Gamma distribution conditional on the latent utilities. This allows to sample the unknown model parameters from a conditionally Gaussian model, facilitating posterior simulation in complex frameworks such as state space or random effects models.

A commonly encountered challenge when working with MCMC methods based on DA is poor mixing. For binary and categorical regressions, this issue is especially pronounced for imbalanced data, where the success probability is either close to zero or one for the majority of the observations, see the excellent work of Johndrow et al. (2019). Neither the original Pólya-Gamma sampler of Polson et al. (2013) with a single level of DA, nor our new Pólya-Gamma sampler with two levels of DA, are an exception to this rule.

To resolve this issue, we introduce imbalanced marginal data augmentation (iMDA) as a boosting strategy to make our new sampler as well as the original probit sampler of Albert & Chib (1993) robust to possibly imbalanced data. This strategy is inspired by earlier work on marginal data augmentation (MDA) for binary and categorical data (Liu & Wu 1999, McCulloch et al. 2000, van Dyk & Meng 2001, Imai & van Dyk 2005). Starting from a latent variable representation of the binary model, we expand the latent variable representation with the help of two unidentified ‘working parameters’. One parameter is a global scale parameter for the latent variable, which has been shown to improve mixing considerably by Liu & Wu (1999), among others. However, this strategy alone does not resolve slow mixing when dealing with highly imbalanced data. To address this, we introduce an additional,
unknown location parameter, which improves mixing considerably in the case of imbalanced data. As iMDA only works in the context of a latent variable representation, this strategy cannot be applied to the original Pólya-Gamma sampler of Polson et al. (2013) due to the lack of such a representation. In comparison, our new Pólya-Gamma representation of the logit model is very generic and is easily combined with iMDA, not only for binary regression models, but also for more flexible models such as binary state space models. We refer to a sampling strategy combining a Pólya-Gamma mixture representation with iMDA as an ultimate Pólya-Gamma (UPG) sampler due to its efficiency.

A further contribution of the present paper is to show that such an UPG sampler can be derived for other non-Gaussian regression problems, including models for categorical and binomial data. For the MNL model, commonly a logit model based on a (partial) differenced random utility model (dRUM) representation is applied to sample the category specific parameters, see e.g. Holmes & Held (2006), Frühwirth-Schnatter & Frühwirth (2010) or Polson et al. (2013). Utilizing this partial dRUM representation, we derive a new sampler for the MNL model in the present paper. Since the latent variable equation is linear in the unknown parameters and involves a logistic error distribution, we use once more the Pólya-Gamma mixture representation of the logistic distribution and introduce the mixing variables as additional latent variables. For binomial models, a latent variable representation which did not involve a choice equation was introduced by Fussl et al. (2013). Since an explicit choice equation is needed to apply iMDA, we derive a new latent variable representation for binomial data which involves error terms that follow generalized logistic distributions. We introduce Pólya-Gamma mixture representations of these distributions and utilize the resulting auxiliary variables as an additional latent layer. Both for MNL models and for binomial models, this DA scheme leads to a conditionally Gaussian posterior
and allows to sample all unknowns through efficient block moves. Again, we apply iMDA to derive UPG samplers which mix well, also in the context of imbalanced data. Overall, we find that the various algorithms show highly competitive performance when compared to alternative DA frameworks, which we demonstrate via extensive simulation studies. In addition, we present real world data examples that further illustrate the merits of our approach. The underlying algorithms for probit regression and logistic regression models for binary, categorical and binomial outcomes have been made available in the R package UPG, which is available on CRAN (Zens et al. 2021).

The remainder of the paper is structured as follows. Section 2 introduces the UPG sampler. This sampling strategy is extended to categorical data in Section 3 and to binomial data in Section 4. In Section 5, the UPG sampler is compared to alternative DA algorithms. Section 6 applies the framework to binary state space models and discusses the utility of the approach in the context of mixture-of-experts models. Section 7 concludes.

2 Ultimate Pólya-Gamma samplers for binary data

2.1 Latent variable representations for binary data

Models for a vector of \( N \) binary observations \( y = (y_1, \ldots, y_N) \) are defined by

\[
\Pr(y_i = 1 | \lambda_i) = F_\varepsilon(\log \lambda_i),
\]

where \( \lambda_i \) depends on exogenous variables and unknown parameters \( \beta \), e.g., \( \log \lambda_i = x_i \beta \) in a standard binary regression model. Choosing the cdf \( F_\varepsilon(\varepsilon) = \Phi(\varepsilon) \) of the standard normal distribution leads to the probit model \( \Pr(y_i = 1 | \lambda_i) = \Phi(\log \lambda_i) \), whereas the cdf
\[ F_\varepsilon(\varepsilon) = e^\varepsilon / (1 + e^\varepsilon) \] of the logistic distribution leads to the logit model

\[ \Pr(y_i = 1|\lambda_i) = \lambda_i / (1 + \lambda_i). \]

A latent variable representation of model (1) involving a latent utility \( z_i \) is given by:

\[ y_i = I\{z_i > 0\}, \quad z_i = \log \lambda_i + \varepsilon_i, \quad \varepsilon_i \sim f_\varepsilon(\varepsilon_i), \]

where \( f_\varepsilon(\varepsilon) = F'_\varepsilon(\varepsilon) = \phi(\varepsilon) \) is equal to the standard normal pdf for a probit model and equal to \( f_\varepsilon(\varepsilon) = e^\varepsilon / (1 + e^\varepsilon)^2 \) for a logit model.

In Bayesian inference, the set of observed data \( y = (y_1, \ldots, y_N) \) can be augmented with the latent variables \( z = (z_1, \ldots, z_N) \) in (2) to obtain the set of complete data \((z, y)\), facilitating the implementation of MCMC algorithms. As shown by Albert & Chib (1993), this single level of DA involving \( z \) leads to a straightforward Gibbs sampler for the probit model. With \( \log \lambda_i = x_i \beta \), the following two-step sampling Scheme 1 can be set up under a Gaussian prior \( p(\beta) \):

1. **(Z)** Given \( \beta \), sample the latent variables \( z_i \) for each \( i = 1, \ldots, N \) independently from \( p(z_i|\beta, y) \) (see Appendix A.4.1);
2. **(P)** sample the unknown parameters \( \beta \) conditional on \( z \) from the Gaussian posterior \( p(\beta|z, y) \) derived from regression model (2).

Two main challenges are associated with such MCMC schemes, namely slow convergence and a lack of closed form posteriors for the unknown parameters, such as \( p(\beta|z, y) \), outside of probit models. We address both issues in the present paper.

First, to boost MCMC convergence, we rely on MDA in the spirit of Liu & Wu (1999). In that paper, the scale-based transformation \( \tilde{z}_i = \sqrt{\delta} z_i \), depending on a ‘working parameter’
\( \delta \), is used to define the expanded probit regression model

\[
y_i = I\{\tilde{z}_i > 0\}, \quad \tilde{z}_i = \sqrt{\delta}x_i\beta + \tilde{\epsilon}_i, \quad \tilde{\epsilon}_i \sim \mathcal{N}(0, \delta).
\]

(3)

In model (3), the likelihood \( p(\tilde{z} | \delta) \) of \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_N) \), marginalized w.r.t. \( \beta \), is available in closed form and yields an inverse Gamma posterior \( p(\delta | \tilde{z}) \) under a conjugate prior \( p(\delta) \).

Assuming prior independence of \( \delta \) and \( \beta \), this allows to rescale the latent variables \( z \) without involving \( \beta \). Specifically, a draw \( \tilde{\delta} \) from the working prior \( p(\delta) \) is used to ‘propose’ a scale-move \( \tilde{z}_i = \sqrt{\tilde{\delta}}z_i \) in system (3), based solely on prior information. Then, an updated value \( \delta_{\text{new}} \) is sampled from the posterior \( p(\delta | \tilde{z}) \) and the proposed scale-move is immediately ‘corrected’ (using a posteriori information) via the inverse transformation \( z_i^{\text{new}} = \tilde{z}_i / \sqrt{\delta_{\text{new}}} \), before \( \beta \) is updated conditional on \( z^{\text{new}} \). This extends Scheme 1 to Scheme 2:

(Z) Sample from \( p(z | \beta, y) \) as in Scheme 1;

(B-S) move from \( z \) to \( z^{\text{new}} \) using a scale-based expansion move under prior \( p(\delta) \);

(P) sample from \( p(\beta | z^{\text{new}}, y) \) as in Scheme 1.

The boosted Scheme 2 always provides better convergence results than Scheme 1, see van Dyk & Meng (2001) and Hobert & Marchev (2008) for further theoretical results. Indeed, as an example in Liu & Wu (1999) illustrates, Step (B-S) improves efficiency considerably in cases where the coefficient of determination in the latent regression model is large, as long as the data are balanced. However, DA schemes are in general known to be slowly mixing for imbalanced data sets where only a few cases with \( y_i = 1 \) or \( y_i = 0 \) among the \( N \) data points are observed (Johndrow et al. 2019). Indeed, sampling under Scheme 2 is still highly inefficient in such cases, as will be illustrated in Section 2.2.

A first major contribution of this paper is to protect DA algorithms for binary and categorical data against imbalanced data by using, in addition to a scale-based transformation,
a location-based expansion $\tilde{z}_i = z_i + \gamma$, depending on a ‘working parameter’ $\gamma$, to define the expanded version

$$y_i = I\{\tilde{z}_i > \gamma\}, \quad \tilde{z}_i = \gamma + \log \lambda_i + \varepsilon_i,$$

of the binary regression model (2).

As opposed to (3), the choice equation in (4) depends on $\gamma$ and defines a likelihood $p(y|\gamma, \tilde{z})$. In a probit regression model, the likelihood $p(\tilde{z}|\gamma)$ of the latent data, marginalized w.r.t. $\beta$, is available in closed form. In combination with the likelihood $p(y|\gamma, \tilde{z})$ and a Gaussian working prior $p(\gamma)$, a Gaussian posterior $p(\gamma|\tilde{z}, y)$, truncated the interval $[L, U)$ defined by, respectively, the maximum utility $L$ of the outcomes where $y_i = 0$ and the minimum utility $U$ of the outcomes where $y_i = 1$, is obtained. Assuming prior independence of $\gamma$ and $\beta$ then allows to shift the latent variables $z$ without involving $\beta$. Similar to the scale-based expansion, a location-move $\tilde{z}_i = z_i + \tilde{\gamma}$ is proposed using a draw $\tilde{\gamma}$ from the working prior $p(\gamma)$, before being immediately ‘corrected’ via the inverse transformation $z_{i}^{\text{new}} = \tilde{z}_i - \gamma_{i}^{\text{new}} = z_i + \tilde{\gamma} - \gamma_{i}^{\text{new}}$ using a draw $\gamma_{i}^{\text{new}}$ from the posterior distribution $p(\gamma|\tilde{z}, y)$, see Section 2.3 for further details. Subsequently, the regression coefficients $\beta$ are sampled conditional on $z_{i}^{\text{new}}$. We find that performing such a location-based expansion step before a scale-based transformation yields dramatic improvement compared to Scheme 1 and Scheme 2, also in cases where the data are imbalanced, see Section 2.2 and Section 5 for further illustration.

A second main contribution of the paper is to take location-based and scale-based parameter expansion beyond the probit regression model by introducing new latent variable representations for binary, binomial and multinomial logit models. For binary logit models, a second level of DA is introduced to deal with the logistic error term. For this, we apply
a new mixture representation of the logistic distribution,

\[
f_\varepsilon(\varepsilon_i) = \frac{e^{\varepsilon_i}}{1 + e^{\varepsilon_i}} = \frac{1}{4} \int e^{-\omega_i \varepsilon_i^2 / 2} p(\omega_i) d\omega_i, \tag{5}\]

where \(\omega_i \sim \mathcal{PG}(2, 0)\) follows a Pólya-Gamma distribution (Polson et al. 2013), see Appendix A.2.1 and A.2.2 for details. This representation is very convenient, as the conditional posterior \(\omega_i | \varepsilon_i \sim \mathcal{PG}(2, |\varepsilon_i|)\) of \(\omega_i\) given \(\varepsilon_i\) is a tilted Pólya-Gamma distribution which is easy to sample from, see Polson et al. (2013). For a binary logit model with \(\log \lambda_i = x_i \beta\), this new representation allows constructing a Pólya-Gamma sampler that extends Scheme 1 in the following way:

(Z) sample the latent variable \(z_i\) from \(p(z_i | \beta, y_i)\) independently for each \(i\) in the latent variable model (2) (see Algorithm 1 and Appendix A.4.1) and sample the scale parameter \(\omega_i\) conditional on \(z_i\) and \(\beta\) from \(\omega_i | z_i, \beta \sim \mathcal{PG}(2, |z_i - x_i \beta|)\);

(P) sample the unknown parameters \(\beta\) conditional on the latent variables \(z = (z_1, \ldots, z_N)\) and \(\omega = (\omega_1, \ldots, \omega_N)\) from the conditionally Gaussian posterior \(p(\beta | \omega, z, y)\).

While this scheme is easy to implement, it can be slowly mixing, like any such sampler. To deal with this issue, we additionally include the two parameter expansion steps introduced above, performing first a location-based and then a scale-based transformation. We refer to the resulting sampling scheme as Scheme 3 and provide full theoretical and computational details in Section 2.3. In later sections, we extend this strategy to logistic regression models for categorical and binomial outcomes.

While our boosting strategy is inspired by Liu & Wu (1999) and related to earlier work on MDA for binary and categorical data (McCulloch et al. 2000, van Dyk & Meng 2001, Imai & van Dyk 2005), it generalizes this literature in several aspects. Importantly, it works for any binary data model with a latent variable representation. In addition, freeing
the location of the threshold $\gamma$ in model (4) leads to an MCMC scheme that is well mixing, even in cases of extremely imbalanced data, see much of the remainder of this article for further illustration. A related strategy to improve mixing behaviour in the context of data augmentation algorithms is outlined in Duan et al. (2018). In their contribution, the authors use location and scale parameters to reparametrize augmented likelihood functions in binary and count data regression models. These calibration parameters have to be set manually, and the authors propose an involved optimization procedure based on large sample arguments and approximations to determine suitable values. The resulting algorithms show efficiency gains that are comparable to the marginal data augmentation proposed in this article when analyzing data sets with many observations and rare outcomes. A potential downside of the approach of Duan et al. (2018) is that the optimization procedure relies on the inverse of the observed Fisher information and the sampler uses Metropolis-Hastings updates. Both may result in scaling issues when many covariates are present. In such settings, a pure data augmentation approach as proposed in this article may prove more effective. Importantly, our approach is also fully automatic and does not rely on any approximations in the sense that tuning-free and exact Gibbs updates for the location and scale parameters are derived. We give details on the resulting posterior simulation scheme in Subsection 2.3. Before presenting these details, we illustrate the specific roles of the location and scale parameters $\gamma$ and $\delta$ using heuristic arguments in the next subsection.

2.2 Illustration and intuition

As a first illustration of the potential merits of the proposed iMDA scheme in imbalanced logistic regression settings, we compare estimation efficiency of the popular Pólya-Gamma sampler from Polson et al. (2013) with a plain DA sampler as in Scheme 1, a scale-based pa-
rameter expansion scheme (as in Scheme 2) and the proposed approach based on location-based and scale-based expansion (as in Scheme 3) in Figure 1. A more systematic comparison will be given in Section 5. It is clearly visible that the UPG sampler outperforms all other samplers in terms of efficiency. Notably, these efficiency gains are realized despite introducing two layers of latent auxiliary variables, which usually increases autocorrelation in the posterior draws significantly. This is counteracted by our novel iMDA strategy based on the working parameters $\gamma$ and $\delta$.

We start with the role of $\delta$, the working parameter used for scale-based expansion of the latent utility equation. Broadly speaking, this scale-based expansion will be highly effective in scenarios where the coefficient of determination in the latent utility model is high. In such settings, the current parameter draw almost perfectly determines the location of the latent utilities and vice versa. As a result, the MCMC chain is only able to move very slowly. To resolve this issue, $\delta$ artificially decreases the coefficient of determination via increasing the error variance in the latent utility equation. In turn, this decreases the dependency of the latent utilities and the regression coefficients, directly enabling larger steps of the Markov chain. In other words, $\delta$ is used to make the posterior of the latent utilities in the expanded model more diffuse than the posterior of the utilities in the original model. Similar as well as more formal arguments and further illustration of such scale-based expansion steps have been discussed for instance in Liu & Wu (1999) or Imai & van Dyk (2005).

However, a scale-based expansion alone is usually not enough to fully resolve the issue that step sizes become small relative to the range of the high posterior density region in imbalanced data settings (Johndrow et al. 2019). This can be seen from the unsatisfactory performance of the PX-DA sampler in Figure 1 and has also been discussed in Duan et al.
Figure 1: MCMC draws and corresponding autocorrelation functions of an intercept only logistic regression model fitted using plain data augmentation (DA, *Scheme 1*), the original Pólya-Gamma sampler (PSW), a MDA sampler with scale-based expansion (PX-DA, *Scheme 2*) and a DA sampler with scale- and location-based expansion (UPG, *Scheme 3*). Two out of $N = 10,000$ binary observations are non-zero.

In our approach, this issue is effectively offset through the location-based expansion of the latent utility model. In this subsection, we aim to illustrate the mechanism behind this strategy through a small numerical exercise and defer full details to Section 2.3.

To investigate how the location-based expansion influences step sizes of the Markov chain, we consider three data sets with $N = 100$ observations each. One data set is balanced, while the others are imbalanced, with success probabilities 99% and 1%, respectively. We simulate 25,000 replications of a single MCMC iteration for a grid of starting positions of the intercept $\beta_0$, using $\mathcal{N}(0, 100)$ prior distributions for both $\beta_0$ and $\gamma$. For each starting position and for each replication, we save the absolute step size of a plain DA sampler (*Scheme 1*) and the step size of a sampler with an additional location-based expansion step, as well as the realized shift $\tilde{\gamma} - \gamma^{\text{new}}$ in the sampler including the location-based expansion.
Figure 2: Illustration of the mechanism behind the location-based expansion. (left) Log average step size of a plain DA sampler. (middle) Realized shift of utilities. (right) Log average step size of a sampler with location-based expansion. Dotted lines are the means of the posterior distribution of $\beta_0$ under a $N(0, 100)$ prior.

The results are summarized in Figure 2. The left panel shows the log average step size of the plain DA scheme. It is evident that step sizes decrease significantly when exploring posterior regions that reach far into the positive (negative) part of the real line in imbalanced scenarios with high (low) success probabilities. The purpose of the location-based expansion is to counteract this issue via shifting the utilities by $\bar{\gamma} - \gamma^\text{new}$, directly leading to larger step sizes of the Markov chain. The average shift for each data set and value of $\beta_0$ is depicted in the middle panel of Figure 2. The magnitude of the shift, $|\bar{\gamma} - \gamma^\text{new}|$, is equivalent to the increase in step size in the location-expanded sampler. While step sizes increase everywhere, the improvement is particularly large in the tails of the posterior density in imbalanced data sets, where standard DA algorithms are usually highly inefficient. In addition, the shift-move evidently acts as a ‘push into the right direction’ that
systematically leads the Markov chain back towards the highest posterior density region, effectively avoiding staying in the tails of the posterior distribution for too long. The log average step sizes of the PX-DA sampler are shown in the right panel of Figure 2. As expected from the preceding discussion, the most significant step size improvements are observed in the tail regions of the posterior distribution in the imbalanced cases.

2.3 MCMC details for binary logit regression models

The latent utility representation of the binary logit model is

\[ y_i = I\{z_i > 0\}, \quad z_i = x_i \beta + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{LO}, \quad (6) \]

where \( \mathcal{LO} \) is the logistic distribution. We assume \( \beta \sim \mathcal{N}_d(0, A_0) \) follows a multivariate Gaussian distribution a priori, where \( A_0 \) is either fixed or equipped with a hierarchical structure, e.g., to define a shrinkage prior (see e.g., Piironen & Vehtari 2017). The first block of the MCMC scheme consists of two steps that simulate the two sets of latent variables, \( z \) and \( \omega \). Given \( \beta \) and the outcome \( y_i \), we sample \( z_i \) for each \( i \) from \( p(z_i|\lambda_i, y_i) \) in the logistic model (6) where \( \log \lambda_i = x_i \beta \). Then, the Pólya-Gamma scale parameters are simulated from \( \omega_i|z_i, \beta \sim \mathcal{PG}(2, |z_i - x_i \beta|) \).

For given latent variables, a location-based parameter expansion step, based on a working prior \( p(\gamma) = \mathcal{N}(0, G_0) \), is then applied. For this, a prior draw \( \tilde{\gamma} \sim \mathcal{N}(0, G_0) \) is used to ‘propose’, for each \( i = 1, \ldots, N \), a location move \( \tilde{z}_i = z_i + \tilde{\gamma} \) in the expanded model

\[ y_i = I\{\tilde{z}_i > \gamma\}, \quad \tilde{z}_i = \gamma + x_i \beta + \varepsilon_i, \quad (7) \]

while \( \omega_i \) is unaffected. Conditional on the latent variables \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_N) \) and \( \omega = (\omega_1, \ldots, \omega_N) \), but marginally w.r.t. \( \beta \), the conditional distribution \( \gamma|\omega, \tilde{z} \sim \mathcal{N}(g_N, G_N) \) is
Gaussian where:

\[ G_N = (G_0^{-1} + \sum_{i=1}^{N} \omega_i - m_b^\top B_N m_b)^{-1}, \quad g_N = G_N(m_\gamma - m_b^\top B_N \mathbf{m}_N(\tilde{z})) \]

\[ B_N = (A_0^{-1} + \sum_{i=1}^{N} \omega_i \mathbf{x}_i^\top \mathbf{x}_i)^{-1}, \quad \mathbf{m}_N(\tilde{z}) = \sum_{i=1}^{N} \omega_i \mathbf{x}_i^\top \tilde{z}_i, \quad m_b = \sum_{i=1}^{N} \omega_i \mathbf{x}_i^\top, \quad m_\gamma = \sum_{i=1}^{N} \omega_i \tilde{z}_i, \]

as is easily shown, see Appendix A.4.2. Since the choice equation in (7) depends on \( \gamma \), \( p(\gamma|\omega, \tilde{z}) \) has to be combined with the likelihood \( p(y|\gamma, \tilde{z}) \) of the observed outcomes \( y = (y_1, \ldots, y_N) \) to define the posterior \( p(\gamma|\omega, \tilde{z}, y) \). The derivation of the likelihood \( p(y|\gamma, \tilde{z}) \) is a generic step in our sampler which does not involve the specification of \( \lambda_i \):

\[
p(y|\gamma, \tilde{z}) \propto \prod_{i:y_i=0} I\{\gamma \geq \tilde{z}_i\} \prod_{i:y_i=1} I\{\gamma < \tilde{z}_i\} \propto I\{L(\tilde{\gamma}) \leq \gamma < U(\tilde{\gamma})\},
\]

where \( I\{\cdot\} \) is the indicator function and \( L(\tilde{\gamma}) = \max_{i:y_i=0} \tilde{z}_i = \max_{i:y_i=0} z_i + \tilde{\gamma} \) and \( U(\tilde{\gamma}) = \min_{i:y_i=1} \tilde{z}_i = \min_{i:y_i=1} z_i + \tilde{\gamma} \). If no outcome \( y_i = 0 \) is observed, then \( L(\tilde{\gamma}) = -\infty \); if no outcome \( y_i = 1 \) is observed, then \( U(\tilde{\gamma}) = +\infty \). Hence, \( p(\gamma|\omega, \tilde{z}, y) \propto p(y|\gamma, \tilde{z})p(\gamma|\omega, \tilde{z}) \) is equal to a truncated version of the Gaussian posterior (8):

\[
\gamma|\omega, \tilde{z}, y \sim N(g_N, G_N) I\{L(\tilde{\gamma}) \leq \gamma < U(\tilde{\gamma})\}.
\]

An updated working parameter \( \gamma^{\text{new}} \) is sampled from (10) and the proposed location-based move is ‘corrected’ based on a posteriori information by defining the shifted utilities \( z_i^L = \tilde{z}_i - \gamma^{\text{new}} = z_i + \tilde{\gamma} - \gamma^{\text{new}} \).

This location-based move is followed by a scale-based expansion, using an inverse Gamma \( \mathcal{G}^{-1}(d_0, D_0) \) working prior \( p(\delta) \). Similar to before, \( \tilde{\delta} \) is sampled from \( p(\delta) \) and used to propose, for each \( i = 1, \ldots, N \), a scale-based move \( \tilde{z}_i = \sqrt{\tilde{\delta}} z_i^L \) in the expanded model

\[
y_i = I\{\tilde{z}_i > 0\}, \quad \tilde{z}_i = \sqrt{\delta} x_i \beta + \sqrt{\delta} \varepsilon_i.
\]
Algorithm 1 The ultimate Pólya-Gamma sampler for binary data.
Choose starting values for $\lambda = (\lambda_1, \ldots, \lambda_N)$ and repeat the following steps:

(Z) For each $i = 1, \ldots, N$, sample $z_i = \log \lambda_i + F_{\nu}^{-1}(y_i + U_i(1 - y_i - \pi_i))$ in model (2), where $U_i \sim U[0,1]$, $\pi_i = F_\nu(\log \lambda_i)$, and $F_{\nu}^{-1}(\mu) = \Phi^{-1}(\mu)$ for the probit and $F_{\nu}^{-1}(\mu) = \log p - \log(1 - p)$ for the logit model. For a logit model, sample $\omega_i | z_i, \log \lambda_i \sim PG(2, |z_i - \log \lambda_i|)$.

(B-L) Location-based parameter expansion: sample $\tilde{\gamma} \sim \mathcal{N}(0, G_0)$ and propose utilities $\tilde{z}_i = z_i + \tilde{\gamma}$ for $i = 1, \ldots, N$. Sample $\gamma_{\text{new}}$ from $\gamma | \omega, \tilde{z}, y$ and define shifted utilities $z_i^L = \tilde{z}_i - \gamma_{\text{new}}$. For a binary regression model, $p(\gamma | \omega, \tilde{z}, y)$ is given by the truncated Gaussian-posterior in (10).

(B-S) Scale-based parameter expansion: sample $\tilde{\delta} \sim \mathcal{G}^{-1}(d_0, D_0)$ and sample $\delta_{\text{new}}$ from $\delta | \tilde{\delta}, z^L, \omega$. Define rescaled utilities $z_i^{LS} = \sqrt{\tilde{\delta} / \delta_{\text{new}}} z_i^L$. For a binary regression model, $\delta | \tilde{\delta}, z^L, \omega \sim \mathcal{G}^{-1}(d_N, D_N(\tilde{\delta}))$ is an inverse Gamma distribution, with $d_N$ and $D_N(\tilde{\delta})$ given by (12).

(P) Sample the unknown parameter in log $\lambda_i$ conditional on $z^{LS}$. For a binary regression model, $\beta | \delta_{\text{new}}, \tilde{\delta}, z^L, \omega \sim \mathcal{N}\left(\sqrt{\tilde{\delta} / \delta_{\text{new}}} B_N \mathbf{m}_N(z^L), B_N\right)$ where $\mathbf{m}_N(z^L)$ and $B_N$ are given by (8).

Conditional on the Pólya-Gamma scale parameters $\omega_i$, it follows that

$$p(\tilde{z}_i | \omega_i, \delta, \beta) \propto \frac{1}{\sqrt{\delta}} \exp \left\{-\frac{\omega_i}{2} \left(\frac{\tilde{z}_i}{\sqrt{\delta}} - x_i \beta\right)^2\right\} = \frac{1}{\sqrt{\delta}} \exp \left\{-\frac{\omega_i}{2} \left(\sqrt{\tilde{\delta}} z_i^L - x_i \beta\right)^2\right\}. $$

Hence, conditional on $\delta, \tilde{\delta}$ and the shifted utilities $z^L = (z_1^L, \ldots, z_N^L)$, the posterior $\beta | \delta, \tilde{\delta}, z^L, \omega \sim \mathcal{N}\left(\sqrt{\tilde{\delta} / \delta_{\text{new}}} B_N, B_N\right)$ is Gaussian with $b_N = B_N \mathbf{m}_N(z^L)$ and $\mathbf{m}_N(z^L)$ and $B_N$, as in (8). Furthermore, conditional on $z^L$, but marginally w.r.t. $\beta$, the posterior $\delta | \tilde{\delta}, z^L, \omega \sim \mathcal{G}^{-1}(d_N, D_N(\tilde{\delta}))$ is inverse Gamma with following moments:

$$d_N = d_0 + \frac{N}{2}, \quad D_N(\tilde{\delta}) = D_0 + \frac{\tilde{\delta}}{2} \left(\sum_{i=1}^{N} \omega_i (z_i^L - x_i b_N)^2 + b_N^T A_0^{-1} b_N\right). $$

An updated working parameter $\delta_{\text{new}}$ is sampled from $\mathcal{G}^{-1}(d_N, D_N(\tilde{\delta}))$ and the proposed scale-based move is corrected by defining the rescaled utilities $z_i^{LS} = \sqrt{\tilde{\delta} / \delta_{\text{new}}} z_i^L$. This
concludes the scale-based expansion and $\beta | z^{LS}, \omega$ is sampled conditional on $z^{LS}$ or, equivalently, from the Gaussian posterior $\beta | \delta^{\text{new}}, \tilde{\delta}, z^L, \omega \sim N\left(\sqrt{\frac{\delta}{\delta^{\text{new}}}b_N, B_N}\right)$. As Algorithm 1 illustrates, many steps in this ultimate Pólya-Gamma (UPG) sampler are generic and easily extended to more complex models for binary data, as will be illustrated in Section 6.

3 Ultimate Pólya-Gamma samplers for categorical data

Let $\{y_i\}, i = 1, \ldots, N$, be a sequence of categorical data, where $y_i$ is equal to one of at least three unordered categories. The categories are labeled by $L = \{0, \ldots, m\}$, and for any $k$ the set of all categories but $k$ is denoted by $L_{-k} = L \setminus \{k\}$. We assume that the observations are mutually independent and that for each $k \in L$ the probability of $y_i$ taking the value $k$ depends on covariates $x_i$ in the following way:

$$\Pr(y_i = k | \beta_0, \ldots, \beta_m) = \frac{\exp(x_i \beta_k)}{\sum_{l=0}^{m} \exp(x_i \beta_l)}, \quad (13)$$

where $\beta_0, \ldots, \beta_m$ are category specific unknown parameters of dimension $d$. To make the model identifiable, the parameter $\beta_{k_0}$ of a baseline category $k_0$ is set equal to 0: $\beta_{k_0} = 0$.

Thus, the parameter $\beta_k$ is relative to the baseline category $k_0$ in terms of the change in log-odds. In the following, we assume without loss of generality that $k_0 = 0$. A more general version of the multinomial logit (MNL) model (again with baseline $k_0 = 0$) reads:

$$\Pr(y_i = k | \beta) = \frac{\lambda_{ki}}{1 + \sum_{l=1}^{m} \lambda_{li}}, \quad (14)$$

where $\lambda_{i1}, \ldots, \lambda_{im}$ depend on unknown parameters $\beta$, while $\lambda_{0i} = 1$. For the standard MNL regression model (13), for instance, $\log \lambda_{ki} = x_i \beta_k$ for $k = 1, \ldots, m$. 

18
Our starting point is writing the MNL model as a random utility model (RUM), see McFadden (1974):

\[ u_{ki} = \log \lambda_{ki} + \epsilon_{ki}, \quad k = 0, \ldots, m, \quad (15) \]

\[ y_i = k \iff u_{ki} = \max_{l \in L} u_{li}. \quad (16) \]

Thus the observed category is equal to the category with maximal utility. If the errors \( \epsilon_{0i}, \ldots, \epsilon_{mi} \) in (15) are i.i.d. random variables from an extreme value (EV) distribution, then the MNL model (14) results as marginal distribution of the categorical variable \( y_i \).

Conditional on \( y_i \), the posterior distribution \( p(u_{ki}|\lambda_{ki}, y_i) \) of the latent utilities \( u_{ki} = (u_{0i}, \ldots, u_{mi}) \) is of closed form and easy to sample from, see Proposition 1 which is proven in Appendix A.3.

**Proposition 1** Given \( y_i \), realisations from the distribution \( p(u_{0i}, \ldots, u_{mi}|\lambda_{1i}, \ldots, \lambda_{mi}, y_i) \) can be represented as:

\[ e^{-u_{ki}} = -\frac{\log U_i}{1 + \sum_{l=1}^{m} \lambda_{li}} - \frac{\log V_{ki}}{\lambda_{ki}} I\{y_i \neq k\}, \quad k = 0, \ldots, m, \quad (17) \]

where \( U_i \) and \( V_{0i}, \ldots, V_{mi} \) are \( m + 1 \) iid uniform random numbers.

Utilizing Proposition 1 together with a mixture approximation of the extreme value distribution to sample all unknown parameters jointly via two levels of data augmentation (Frühwirth-Schnatter & Frühwirth 2007) turned out to be inefficient. Noting that the choice equation (16) can be rewritten as a choice between any category \( k \) and all its alternatives in \( L - k \), Frühwirth-Schnatter & Frühwirth (2010) derive the partial dRUM representation of a RUM model and show that it has an explicit form if the errors \( \epsilon_{0i}, \ldots, \epsilon_{mi} \) in (15) are i.i.d. random variables from an extreme value distribution.
For a multinomial regression model this yields following well-known representation (see e.g. Holmes & Held 2006):

\[ z_{ki} = x_i \beta_k - \xi_{ki} (\beta_{-k}) + \varepsilon_{ki}, \quad \varepsilon_{ki} \sim \mathcal{LO}, \quad (18) \]

\[ y_i = \begin{cases} 
  k, & z_{ki} > 0, \\
  \neq k, & z_{ki} \leq 0.
\end{cases} \quad (19) \]

where the error term \( \varepsilon_{ki} \) follows a logistic distribution, \( z_{ki} = u_{ki} - \max_{\ell \in L \neq k} u_{\ell i} \) is the utility gap between category \( k \) and all its alternatives and the offset \( \xi_{ki} (\beta_{-k}) \) is defined as:

\[ \xi_{ki} (\beta_{-k}) = \log \left( 1 + \sum_{\ell \neq \{k,0\}} \exp(x_i \beta_{\ell}) \right). \]

While Frühwirth-Schnatter & Frühwirth (2010) use a very accurate finite mixture approximation for the logistic distribution for MCMC estimation, in the present paper we derive an ultimate Pólya-Gamma sampler based on the partial dRUM representation and proceed similarly as in Section 2. We utilize Proposition 1 to sample the utilities \( u_{0i}, \ldots, u_{mi} \) in the RUM model (15) and to define the utility gap \( z_{ki} \) between category \( k \) and all its alternatives. Given the utility gap \( z_{ki} \), we exploit the Pólya-Gamma mixture representation of the logistic distribution in (18) with category specific latent variables \( \omega_{ki} \) which are sampled from \( \omega_{ki} | \beta, z_{ki} \sim \mathcal{PG}(2, |\varepsilon_{ki}|) \), where \( \varepsilon_{ki} = z_{ki} - x_i \beta_k + \xi_{ki} (\beta_{-k}) \).

To handle imbalanced data, we apply location- and scale-based boosting as in Section 2 with category specific working parameters \( \gamma_k \) and \( \delta_k \). For instance, location-based boosting using \( \tilde{z}_{ki} = z_{ki} + \tilde{\gamma}_k \) where \( \tilde{\gamma}_k \sim \mathcal{N}(0, G_0) \), yields the following expanded model:

\[ \tilde{z}_{ki} = \gamma_k + x_i \beta_k - \xi_{ki} (\beta_{-k}) + \varepsilon_{ki}, \quad \varepsilon_{ki} \sim \mathcal{LO}, \quad (20) \]

\[ y_i = \begin{cases} 
  k, & \tilde{z}_{ki} > \gamma_k, \\
  \neq k, & \tilde{z}_{ki} \leq \gamma_k.
\end{cases} \quad (21) \]
Conditional on the latent variables $\omega_k = (\omega_{k1}, \ldots, \omega_{kN})$ and $\tilde{z}_k = (\tilde{z}_{k1}, \ldots, \tilde{z}_{kN})$, equation (20) defines a Gaussian posterior distribution $p(\gamma_k | \beta_{-k}, \omega_k, \tilde{z}_k)$, marginally w.r.t. $\beta_k$. Similarly as in Section 2 the choice equation (21) defines a likelihood function $p(y | \gamma_k, \tilde{z}_k)$ which restricts $\gamma_k$ to the interval $[L(\tilde{\gamma}_k), U(\tilde{\gamma}_k)]$, where $L(\tilde{\gamma}_k) = \max_{y_i \neq k} z_{ki} + \tilde{\gamma}_k$ and $U(\tilde{\gamma}_k) = \min_{y_i = k} z_{ki} + \tilde{\gamma}_k$. Full details on the UPG sampler for multinomial logistic regression models are provided in Appendix A.4.3.

4 Ultimate Pólya-Gamma samplers for binomial data

In this section, we consider models with binomial outcomes, i.e., models of the form

$$y_i \sim \text{BiNom}(N_i, \pi_i), \quad \text{logit} \; \pi_i = \log \lambda_i, \quad i = 1, \ldots, N,$$

with $\log \lambda_i = x_i \beta$ for a standard binomial regression model. As shown in Johndrow et al. (2019), Bayesian inference for binomial regression models based on the Pólya-Gamma sampler (Polson et al. 2013) is sensitive to imbalanced data. Similarly, the latent variable representation of binomial models of Fussl et al. (2013) is sensitive to imbalanced data, as we will show in Section 5. As for a logit model (which results for $N_i \equiv 1$), applying iMDA would be an option to improve mixing. However, Fussl et al. (2013) provide no explicit choice equation, which is needed for iMDA. The goal of this section is to define an UPG sampler which combines a new latent variable representation of binomial models, based on Pólya-Gamma mixture representations of generalized logistic distributions with iMDA to protect the algorithm against imbalanced data.
4.1 A new latent variable representation for binomial data

In Theorem 2, we introduce a new latent variable representation for binomial outcomes where two latent variable equations, both linear in log $\lambda_i$, with error terms following generalized logistic distributions are utilized. An explicit choice equation is provided which relates latent variables $w_i$ and $v_i$ to the observed binomial outcome $y_i$. We show in Theorem 3 that, conditional on $y_i$, the posterior distribution of the latent variables is of closed form and easy to sample from, see Appendix A.3. for a proof of both theorems.

**Theorem 2 Latent variable representation of a binomial model** For $0 < y_i < N_i$, a binomial logistic model has the following random utility representation:

\[
\begin{align*}
  w_i &= \log \lambda_i + \varepsilon_{w,i}, \quad \varepsilon_{w,i} \sim \mathcal{GL}_I (k), \\
  v_i &= \log \lambda_i + \varepsilon_{v,i}, \quad \varepsilon_{v,i} \sim \mathcal{GL}_I (N_i - k), \\
  y_i &= k \iff w_i > 0, v_i \leq 0,
\end{align*}
\]

where $\mathcal{GL}_I (\nu)$ and $\mathcal{GL}_II (\nu)$ are, respectively, the generalized logistic distributions of type I and type II. For $y_i = 0$, the model reduces to

\[
  v_i = \log \lambda_i + \varepsilon_{v,i}, \quad \varepsilon_{v,i} \sim \mathcal{GL}_I (N_i), \quad y_i = 0 \iff v_i \leq 0.
\]

For $y_i = N_i$, the model reduces to

\[
  w_i = \log \lambda_i + \varepsilon_{w,i}, \quad \varepsilon_{w,i} \sim \mathcal{GL}_II (N_i), \quad y_i = N_i \iff w_i > 0.
\]

For $N_i = 1$, the logistic model results, as both $\mathcal{GL}_I (\nu)$ and $\mathcal{GL}_II (\nu)$ reduce to a logistic distribution for $\nu = 1$. For $y_i = 0$, $z_i = v_i$, whereas for $y_i = 1$, $z_i = w_i$, and the choice equation reduces to $y_i = I\{z_i > 0\}$.
Theorem 3 Sampling the utilities in the binomial RUM Given \( y_i \) and holding all model parameters in \( \lambda_i \) fixed, the latent variables \( w_i | \lambda_i, (y_i > 0) \) and \( v_i | \lambda_i, (y_i < N_i) \) are conditionally independent. The distributions of \( w_i | \lambda_i, (y_i > 0) \) and \( v_i | \lambda_i, (y_i < N_i) \) are equal in distribution to

\[
w_i = \log \left( \frac{1 + \lambda_i}{W_i^{1/y_i}} - \lambda_i \right), \quad y_i > 0, \tag{24}
\]

\[
v_i = -\log \left( \frac{1 + \lambda_i}{V_i^{1/(N_i - y_i)}} - \frac{1}{\lambda_i} \right), \quad y_i < N_i, \tag{25}
\]

where \( W_i \) and \( V_i \) are iid uniform random numbers.

4.2 Ultimate Pólya-Gamma samplers for binomial data

The two main building blocks for the UPG sampler for binomial data are a Gaussian mixture representation of the involved generalized logistic distributions based on the Pólya-Gamma distribution and the application of iMDA to handle imbalanced data.

A random variable \( \varepsilon \) following the generalized logistic distribution of type I or II can be represented as a normal mixture,

\[
f_\varepsilon(\varepsilon) = c(a, b) \frac{(e^\varepsilon)^a}{(1 + e^\varepsilon)^b} = c(a, b) \exp(\kappa \varepsilon) \int_0^\infty \exp(-\frac{\omega \varepsilon^2}{2}) p(\omega) d\omega, \tag{26}
\]

with \( \kappa = a - b/2 \) and the Pólya-Gamma distribution \( \omega \sim PG(b, 0) \), introduced by Polson et al. (2013) serving as mixing measure, see Appendix A.2.1 to A.2.3. For \( y_i > 0 \), the type II generalized logistic distribution \( \varepsilon_{w,i} \sim GL_{II}(y_i) \) in (23) has such a representation with:

\[
\kappa_{w,i} = \frac{1 - y_i}{2}, \quad \omega_{w,i} \sim PG(y_i + 1, 0),
\]

see (A.11). Similarly, for \( y_i < N_i \), the type I generalized logistic distribution \( \varepsilon_{v,i} \sim GL_{I}(y_i) \) in (23) has such a representation with:

\[
\kappa_{v,i} = \frac{y_i}{2}, \quad \omega_{v,i} \sim PG(y_i, 0),
\]
$\mathcal{GL}_i (N_i - y_i)$ in (23) has such a representation with

$$\kappa_{v,i} = \frac{N_i - y_i - 1}{2}, \quad \omega_{v,i} \sim \mathcal{PG} (N_i - y_i + 1, 0),$$

see (A.7). Note that $\kappa_{w,i} = 0$ for $y_i = 1$ and $\kappa_{v,i} = 0$ for $y_i = N_i - 1$. Hence, for $N_i = 1$, the Pólya-Gamma mixture approximation (5) of a logistic model involving $\mathcal{PG} (2, 0)$ results.

For $N_i > 1$, $\kappa_{v,i} > 0$ for $0 \leq y_i \leq N_i - 2$ and $\kappa_{w,i} < 0$ for $2 \leq y_i \leq N_i$. This leads to a slightly more challenging sampler than for binary and multinomial models.

For each $i = 1, \ldots, N$, we introduce the latent variables $z_i = (w_i, \omega_{w,i}, v_i, \omega_{v,i})$, if $0 < y_i < N_i$, $z_i = (w_i, \omega_{w,i})$, if $y_i = N_i$, and $z_i = (v_i, \omega_{v,i})$, if $y_i = 0$. Conditional on $\lambda_i$, the latent variables $w_i | \lambda_i, (y_i > 0)$ and $v_i | \lambda_i, (y_i < N_i)$ are sampled from Theorem 3 without conditioning on $\omega_{w,i}$ and $\omega_{v,i}$. Given $w_i$ and $v_i$, the parameters $\omega_{w,i} | w_i, (y_i > 0), \lambda_i$ and $\omega_{v,i} | v_i, (y_i < N_i), \lambda_i$ are independent and follow (tilted) Pólya-Gamma distributions:

$$\omega_{w,i} | w_i, y_i, \lambda_i \sim \mathcal{PG} (y_i + 1, |w_i - \log \lambda_i|), \quad y_i > 0,$$

$$\omega_{v,i} | v_i, y_i, \lambda_i \sim \mathcal{PG} (N_i - y_i + 1, |v_i - \log \lambda_i|), \quad y_i < N_i.$$

To handle imbalanced data, we apply location- and scale-based boosting as in the previous sections, based on the working parameters $\gamma$ and $\delta$. Location-based boosting, for instance, uses $\tilde{\gamma} \sim \mathcal{N} (0, G_0)$ to define $\tilde{w}_i = w_i + \tilde{\gamma}$ and $\tilde{v}_i = v_i + \tilde{\gamma}$ in the following expanded version of model (23) with an explicit choice equation involving $\gamma$:

$$\tilde{w}_i = \gamma + \log \lambda_i + \varepsilon_{w,i}, \quad y_i > 0,$$

$$\tilde{v}_i = \gamma + \log \lambda_i + \varepsilon_{v,i}, \quad y_i < N_i,$$

$$y_i = k \Leftrightarrow \begin{cases} \tilde{v}_i < \gamma < \tilde{w}_i, & 0 < k < N_i, \\ \gamma \geq \tilde{v}_i, & k = 0, \\ \gamma < \tilde{w}_i, & k = N_i. \end{cases}$$

Full details on the UPG sampler for binomial data are provided in Appendix A.4.4.
Figure 3: Sampling efficiency of intercept $\beta_0$ fitted to data sets with increasing sample size $N$ and two successes (top) and varying intercepts $\beta_0$ with $N = 1,000$ (bottom) for binary logistic regression (left), multinomial logistic regression (middle) and binomial logistic regression (right). Y-axis is on the log-scale and results are medians across 100 replications.

5 Comparison with other sampling strategies

This section compares the proposed sampling framework with other DA approaches for posterior simulation in binary and categorical regression models. Specifically, we conduct a large scale simulation study to establish the efficiency of our approach in imbalanced scenarios relative to other DA approaches. However, from a practical point of view, a number of alternative estimation algorithms that do not rely on DA are available for binary and categorical regression modeling. These algorithms can be highly efficient, and relying on them is often a reasonable choice. Hence, a thorough discussion of the unique advantages
and disadvantages of the DA strategy outlined in this article – and DA schemes in general – is warranted, and we provide such a discussion in Appendix A.1.

A set of systematic simulations is carried out to compare the efficiency of our approach to other popular Bayesian sampling schemes that involve DA. The main results are based on simulations with varying levels of imbalancedness, where imbalancedness is either induced by fixing the number of successes at two and increasing the sample size, or fixing the sample size at $N = 1,000$ and varying the intercept term in the data generating process. Each Markov chain was run for 10,000 iterations after an initial burn-in period of 2,000 iterations. To gain robustness with respect to the computed inefficiency factors, each simulation is repeated 100 times and median results across these replications are reported. The computation of the inefficiency factors is based on an estimate of the spectral density of the posterior chain evaluated at zero\(^2\).

In this section, we present results on various logistic regression models, while additional results for probit regression models and tabulated simulation results can be found in Appendix A.6.

For binary logistic regression, we compare the sampling scheme outlined in Section 2.3 (UPG), the Pólya-Gamma sampler of Polson et al. (2013) (PSW) and the auxiliary mixture DA scheme outlined in Frühwirth-Schnatter & Frühwirth (2010) (FSF). To assess sampling efficiency for the MNL model, we compare the MNL sampler proposed in Section 3 (UPG) with the sampling scheme of Polson et al. (2013) (PSW) and the partial dRUM sampler of Frühwirth-Schnatter & Frühwirth (2010) (FSF) in a setting with three categories. For the simulations with varying sample sizes, the first two categories are observed twice each and the remaining $N - 4$ observations fall into the baseline category. For the varying intercept

\(^2\)Estimating the spectral density at zero is accomplished via R package `coda` (Plummer et al. 2006) and is based on fitting an autoregressive process to the posterior draws.
simulations, the intercept of the first category is varied while the other intercepts are fixed at zero. Finally, to illustrate the efficiency gains in the case of logistic regression analysis of binomial data, we compare the approach outlined in Section 4 (UPG) to the sampling scheme of Polson et al. (2013) (PSW) and to the auxiliary mixture sampler introduced in Fussl et al. (2013) (AuxMix). For all observations, we assume $N_i = 5$ trials. In all simulations, an adaptive Metropolis-Hastings sampler (AMH) is included as a benchmark as well. Throughout all simulation settings, independent $N(0, 10)$ priors are specified on the regression parameters, and we choose $\gamma \sim N(0, 100)$ and $\delta \sim IG(2.5, 1.5)$ as working prior for the iMDA algorithms.

The results of the main simulation exercise are summarized in Figure 3. The empirical inefficiency factors confirm that standard DA techniques exhibit extremely inefficient sampling behavior when confronted with imbalanced data, as shown theoretically and empirically in Johndrow et al. (2019). The MDA strategy we propose alleviates this issue and allows for rather efficient estimation also in highly imbalanced data settings.

6 Applications to more complex models

6.1 Application to a binary state space model

Let $\{y_t\}$ be a time series of binary observations, observed for $t = 1, \ldots, T$, taking one of two possible values labelled $\{0, 1\}$. The probability that $y_t$ takes the value 1 depends on covariates $x_t$, including a constant, through time-varying parameters $\beta_t$ as follows:

$$\Pr(y_t = 1|\beta_1, \ldots, \beta_T) = \frac{\exp(x_t\beta_t)}{1 + \exp(x_t\beta_t)}. \quad (30)$$
We assume that conditional on knowing $\beta_1, \ldots, \beta_T$, the observations are mutually independent. A commonly used model for describing the time-variation of $\beta_t$ reads:

$$\beta_t = \beta_{t-1} + w_t, \quad w_t \sim N_d(0, Q),$$  \hspace{1cm} (31)

with $\beta_0 \sim N_d(0, P_0)$ and $Q = \text{Diag}(\theta_1, \ldots, \theta_d)$, where $\theta_1, \ldots, \theta_d$ are unknown variances.

MCMC estimation of binary state space models (SSM) is challenging. Single-move sampling of $\beta_t$ is potentially very inefficient \cite{ShephardPitt1997}, while blocked MH updates require suitable proposal densities in a high-dimensional space \cite{Gamerman1998}. Within the DA framework, a latent utility $z_t$ of choosing category 1 is introduced for each $y_t$:

$$y_t = 1 \iff z_t > 0, \quad z_t = x_t \beta_t + \varepsilon_t.$$  \hspace{1cm} (32)

Given $z = \{z_t\}$, this SSM is conditionally Gaussian for a probit link, but conditionally non-Gaussian for a logit link. Frühwirth-Schnatter and Frühwirth \cite{Fruhwirth-SchnatterFruhwirth2007} implemented an auxiliary mixture sampler for a binary logit SSM. Alternatively, using the Pólya-Gamma mixture representation of the logistic distribution of $\varepsilon_t$ yields a conditionally Gaussian SSM which allows multi-move sampling of the entire state process $\beta = \{\beta_0, \beta_1, \ldots, \beta_T\}$ using FFBS \cite{Fruhwirth-Schnatter1994, CarterKohn1994} in a similar fashion as for a probit SSM. To achieve robustness again imbalance, we extend the iMDA scheme introduced in Section 2 to SSMs, see Appendix A.5 for details.

To illustrate the gains in sampling efficiency for binary SSMs, we apply the UPG framework to an example data set on severe global pandemics. The data covers $T = 222$ years from 1800 to 2022 and documents disease episodes characterized by a worldwide spread and a death toll of more than 75,000. In addition, we focus on diseases that are characterized by relatively short periods of activity, hence excluding pandemics such as HIV/AIDS. This results in a total of eight pandemic events falling into the sample period, starting with a
Figure 4: Panel (a) shows the posterior of a local level model fitted to the global pandemic data (solid line: posterior mean, dashed lines: 0.05 and 0.95 posterior quantiles). Panel (b) shows the percentage gains in effective sample size when iMDA is applied, averaged across ten independent chains.

For years featuring a global pandemic, $y_t = 1$ and $y_t = 0$ otherwise. A pandemic is observed in roughly 1 out of 8 years with high state persistence, rendering the data set relatively imbalanced. We fit a logistic local level model to the data, once with and once without iMDA, using $\theta \sim \mathcal{IG}(5, 2)$ and $P_{00} = 100$ as prior settings. The Gibbs sampler is iterated 100,000 times after an initial burn-in period of 10,000 iterations. This numerical study is repeated ten times. One of the resulting posterior distributions (based on the boosted sampler) is shown in Panel (a) of Figure 4. The time-varying intercept evolves smoothly, as is typical for binary state space models. The estimated path is characterized by long periods without severe pandemics, interrupted by short pandemic episodes. In Panel (b), the percentage gains in effective sample size of the sampler with iMDA relative

\footnote{The data is sourced from \url{https://en.wikipedia.org/wiki/List_of_epidemics} and the sources therein.}
to the plain sampler are plotted for each year. The iMDA scheme described in Appendix A.5 is able to significantly improve sampling efficiency in all years. The most pronounced gains – up to 200% improvement in effective sample size – are observed during prolonged ‘imbalanced’ periods where the outcome does not change. Averaging across all periods, the inefficiency factors are roughly halved, from about 96 in the plain sampler to around 45 in the UPG sampling scheme.

6.2 Application to logistic mixture-of-experts regression models

Let \( y_i (i = 1, \ldots, N) \) be a grouped binary outcome with \( C_i = j \) denoting that observation \( i \) belongs to group \( j = 1, \ldots, J \). A logistic mixture-of-experts regression model with \( H \) \((h = 1, \ldots, H)\) components takes the form

\[
p(y_i | C_i = j, x_i, w_j) = \sum_{h=1}^{H} \eta_{jh}(w_j) \text{Ber}(\zeta_{ih}(x_i))
\]

\[
\zeta_{ih}(x_i) = \frac{\exp(x_i\beta_h)}{1 + \exp(x_i\beta_h)}
\]

\[
\eta_{jh}(w_j) = \frac{\exp(w_j\psi_h)}{\sum_{l=1}^{H}\exp(w_l\psi_l)}
\]

where \( H \) logistic regression ‘experts’ are used to model cluster-specific success probabilities \( \zeta_{ih}(x_i) \) using individual-level covariates \( x_i \) and a multinomial logistic regression plays the role of a ‘gating function’, modeling the mixture weights \( \eta_{jh}(w_j) \) based on group-level covariates \( w_j \). This model has good approximation properties (Jiang & Tanner 1999) and is popular in model-based clustering and ensemble learning. Furthermore, developing efficient inferential tools is an important research avenue (Sharma et al. 2019). A thorough treatment of mixture-of-experts models is given in Gormley & Frühwirth-Schnatter (2019).

The model in (33) naturally involves multiple layers of hierarchy, multi-modal posteriors and discrete parameter spaces, potentially rendering inference with general purpose
As a result, DA algorithms are popular tools for the estimation of mixture-of-experts models (Gormley & Frühwirth-Schnatter, 2019). However, imbalanced data and large samples may lead to convergence issues. In model (33), both the success probabilities $\zeta_{ih}$ and the mixture weights $\eta_{jh}$ may be imbalanced.

The methodology proposed in the present article is a potential remedy in such scenarios, as both the logistic regression experts and the gating function can be estimated using DA with additional location-based and scale-based parameter expansion steps. We demonstrate in a numerical exercise in Appendix A.6.3 that our iMDA scheme indeed leads to sizeable efficiency gains with respect to all involved regression parameters in simulated data. In Appendix A.7, we further illustrate logistic mixture-of-experts regression models in a large-sample real world application on maternal education and child mortality. Again, effective sample sizes increase as soon as iMDA is introduced.

7 Concluding Remarks

Due to a wide range of applications in many areas of applied science, much attention has been dedicated towards the development of estimation algorithms for generalized linear models. In the past decades, various DA algorithms have been brought forward that have steadily increased accessibility and popularity of Bayesian estimation techniques in the context of regression models for binary and categorical outcomes. In this article, we introduce new sampling algorithms based on Pólya-Gamma mixture representations for estimation of these models. The algorithms are easily implemented, intuitively appealing and allow for a conditionally Gaussian posterior distribution of the regression effects in binary, multi-
nominal and binomial logistic regression frameworks. To counteract potentially inefficient sampling behavior, we develop a novel parameter expansion strategy and apply it to the introduced sampling algorithms as well as to probit frameworks. This results in a comparative level of sampling efficiency, even in scenarios where outcomes are heavily imbalanced, as is demonstrated via extensive simulation studies and real data applications.

A number of future research avenues worth exploring come readily to mind. First, the proposed family of DA and MCMC boosting schemes could be extended to accommodate other types of limited outcomes such as ordered or count data. Second, we approached the problem of efficiency comparisons mostly empirically and left theoretical aspects largely unexplored. Extending the theoretical results of Choi & Hobert (2013) and Johndrow et al. (2019), among others, might be fruitful and assessing convergence rates of the proposed sampling schemes more formally may reveal additional insights. Finally, it is well-known that scale-based parameter expansion leads to faster convergence of expectation-maximization algorithms (Liu et al. 1998). It may be worth to investigate whether the proposed location-based expansion leads to additional efficiency gains in this context.

Acknowledgements

The authors would like to thank Darjus Hosszejni, two anonymous referees and an associate editor for helpful comments and suggestions.

References

Albert, J. H. & Chib, S. (1993), ‘Bayesian analysis of binary and polychotomous response data’, Journal of the American Statistical Association 88, 669–679.
Anceschi, N., Fasano, A., Durante, D. & Zanella, G. (2023), ‘Bayesian conjugacy in probit, tobit, multinomial probit and extensions: A review and new results’, *Journal of the American Statistical Association* (just-accepted), 1–64.

Carter, C. K. & Kohn, R. (1994), ‘On Gibbs sampling for state space models’, *Biometrika* **81**, 541–553.

Choi, H. M. & Hobert, J. P. (2013), ‘The Polya-Gamma Gibbs sampler for Bayesian logistic regression is uniformly ergodic’, *Electronic Journal of Statistics* **7**, 2054–2064.

Chopin, N. & Ridgway, J. (2017), ‘Leave Pima Indians alone: Binary regression as a benchmark for Bayesian computation’, *Statistical Science* **32**, 64–87.

URL: [https://doi.org/10.1214/16-STS581](https://doi.org/10.1214/16-STS581)

Duan, L. L., Johndrow, J. E. & Dunson, D. B. (2018), ‘Scaling up data augmentation MCMC via calibration’, *Journal of Machine Learning Research* **19**, 1–34.

Durante, D. (2019), ‘Conjugate Bayes for probit regression via unified skew-normal distributions’, *Biometrika* **106**(4), 765–779.

Frühwirth-Schnatter, S. (1994), ‘Data augmentation and dynamic linear models’, *Journal of Time Series Analysis* **15**, 183–202.

Frühwirth-Schnatter, S. & Frühwirth, R. (2007), ‘Auxiliary mixture sampling with applications to logistic models’, *Computational Statistics & Data Analysis* **51**, 3509–3528.

Frühwirth-Schnatter, S. & Frühwirth, R. (2010), Data augmentation and MCMC for binary and multinomial logit models, in T. Kneib & G. Tutz, eds, ‘Statistical Modelling
and Regression Structures – Festschrift in Honour of Ludwig Fahrmeir’, Physica-Verlag, Heidelberg, pp. 111–132.

Frühwirth-Schnatter, S., Frühwirth, R., Held, L. & Rue, H. (2009), ‘Improved auxiliary mixture sampling for hierarchical models of non-Gaussian data’, *Statistics and Computing* **19**, 479–492.

Fussl, A., Frühwirth-Schnatter, S. & Frühwirth, R. (2013), ‘Efficient MCMC for binomial logit models’, *ACM Transactions on Modeling and Computer Simulation* **23**, 3:1–3:21.

Gamerman, D. (1998), ‘Markov chain Monte Carlo for dynamic generalized linear models’, *Biometrika* **85**, 215–227.

Gormley, I. C. & Frühwirth-Schnatter, S. (2019), Mixture of experts models, in S. Frühwirth-Schnatter, G. Celeux & C. P. Robert, eds, ‘Handbook of Mixture Analysis’, CRC Press, Boca Raton, FL, chapter 12, pp. 271–307.

Hobert, J. P. & Marchev, D. (2008), ‘A theoretical comparison of the data augmentation, marginal augmentation and PX-DA algorithms’, *The Annals of Statistics* **36**, 532–554.

Holmes, C. C. & Held, L. (2006), ‘Bayesian auxiliary variable models for binary and multinomial regression’, *Bayesian Analysis* **1**, 145–168.

Imai, K. & van Dyk, D. A. (2005), ‘A Bayesian analysis of the multinomial probit model using marginal data augmentation’, *Journal of Econometrics* **124**, 311–334.

Jiang, W. & Tanner, M. A. (1999), ‘Hierarchical mixtures-of-experts for exponential family regression models: Approximation and maximum likelihood estimation’, *The Annals of Statistics* **27**(3), 987–1011.
Johndrow, J. E., Smith, A., Pillai, N. & Dunson, D. B. (2019), ‘MCMC for imbalanced categorical data’, *Journal of the American Statistical Association* **114**, 1394–1403.

Liu, C., Rubin, D. B. & Wu, Y. N. (1998), ‘Parameter expansion to accelerate EM: the PX-EM algorithm’, *Biometrika* **85**, 755–770.

Liu, J. S. & Wu, Y. N. (1999), ‘Parameter expansion for data augmentation’, *Journal of the American Statistical Association* **94**, 1264–1274.

McCulloch, R. E., Polson, N. G. & Rossi, P. E. (2000), ‘A Bayesian analysis of the multinomial probit model with fully identified parameters’, *Journal of Econometrics* **99**, 173–193.

McFadden, D. (1974), Conditional logit analysis of qualitative choice behaviour, in P. Zarembka, ed., ‘Frontiers of Econometrics’, Academic, New York, pp. 105–142.

Piironen, J. & Vehtari, A. (2017), ‘Sparsity information and regularization in the horseshoe and other shrinkage priors’, *Electronic Journal of Statistics* **11**, 5018–5051.

Plummer, M., Best, N., Cowles, K. & Vines, K. (2006), ‘CODA: Convergence diagnosis and output analysis for MCMC’, *R News* **6**(1), 7–11.

Polson, N. G., Scott, J. G. & Windle, J. (2013), ‘Bayesian inference for logistic models using Pólya-Gamma latent variables’, *Journal of the American Statistical Association* **108**, 1339–49.

Rossi, P. E., Allenby, G. M. & McCulloch, R. (2005), *Bayesian Statistics and Marketing*, Wiley, Chichester.

Sen, D., Sachs, M., Lu, J. & Dunson, D. B. (2020), ‘Efficient posterior sampling for high-dimensional imbalanced logistic regression’, *Biometrika* **107**(4), 1005–1012.
Sharma, A., Saxena, S. & Rai, P. (2019), ‘A flexible probabilistic framework for large-margin mixture of experts’, *Machine Learning* **108**(8), 1369–1393.

Shephard, N. & Pitt, M. K. (1997), ‘Likelihood analysis of non-Gaussian measurement time series’, *Biometrika* **84**, 653–667.

Tanner, M. A. & Wong, W. H. (1987), ‘The calculation of posterior distributions by data augmentation’, *Journal of the American Statistical Association* **82**, 528–540.

van Dyk, D. & Meng, X.-L. (2001), ‘The art of data augmentation’, *Journal of Computational and Graphical Statistics* **10**, 1–50.

Zellner, A. & Rossi, P. E. (1984), ‘Bayesian analysis of dichotomous quantal response models’, *Journal of Econometrics* **25**, 365–393.

Zens, G., Frühwirth-Schnatter, S. & Wagner, H. (2021), ‘Efficient Bayesian modeling of binary and categorical data in R: The UPG package’, *arXiv preprint arXiv:2101.02506*.
Online supplementary material

Ultimate Pólya Gamma Samplers – Efficient MCMC for possibly imbalanced binary and categorical data

Gregor Zens, Sylvia Frühwirth-Schnatter, Helga Wagner

May 2023
A Appendix

A.1 Discussion: Advantages and disadvantages compared to approaches without data augmentation

In the context of the probit model, Durante (2019) discusses conjugate analysis under unified skew-normal priors. This approach allows to sample from the resulting posterior distribution extremely efficiently and works well in the context of small $N$, large $P$ settings. However, even for moderately large $N$, the computational burden of the approach makes posterior simulation infeasible. In addition, deviating from the standard probit regression setup and introducing modifications such as time-varying parameters is a non-trivial task in this framework. Finally, this sampling strategy is restricted to the probit link function. In comparison, the approach outlined in this article is much more general, scales better to larger data sets and extensions to more complex setups are often trivial to achieve due to the conditionally Gaussian representation.

A number of recent contributions have established piecewise deterministic Markov processes (Vanetti et al. 2017; Fearnhead et al. 2018) as a successful tool for posterior simulation. One particularly useful approach arising from this literature is the so-called Zig-Zag sampler (Bierkens et al. 2019). The advantages of this approach in the context of logistic regression with imbalanced data have been pointed out by Sen et al. (2020). However, methods based on piecewise deterministic Markov processes are rather involved, both from a computational and a mathematical perspective. This makes them relatively inaccessible to applied researchers and renders extensions to customized, complex modeling tasks difficult.

Compared to that, gradient-based posterior simulation techniques, such as the Metropolis-
adjusted Langevin algorithm (MALA) or Hamiltonian Monte Carlo (HMC) are commonly encountered in practice. These methods have become popular due to readily available software implementations such as Stan (Carpenter et al. 2017). When applied to simple models with a small to moderate parameter dimension, these approaches are likely to produce posterior samples that are nearly independent of each other, conditional on having access to a well-chosen set of tuning parameters. If tuning parameters are chosen suboptimally, gradient-based methods may fail in scenarios with ill-conditioned likelihoods. A particularly relevant example are highly imbalanced logistic regression problems, see for instance Hird et al. (2020). In comparison, one very convenient property of DA approaches is the absence of tuning parameters. Nonetheless, the issue of searching for good tuning parameters can be facilitated via automatic tuning approaches such as the No-U-turn sampler outlined in Hoffman & Gelman (2014) or via methods that are more robust to tuning parameters, such as the Barker proposal (Livingstone & Zanella 2022). However, even when good tuning parameters can be automatically obtained, standard gradient-based methods may encounter issues when confronted with complex, hierarchical frameworks involving multi-modal or discrete (i.e., discontinuous) posterior distributions, where specialized solutions have to be employed (Mangoubi et al. 2018; Nishimura et al. 2020).

Compared to that, DA is an easily applicable out-of-the-box tool that often is one of the few available approaches that is easy to implement and achieves convergence, even in more complex scenarios. As a result, DA is still one of the standard tools used by practitioners in many modeling frameworks. Examples include mixture and mixture-of-experts models, where multi-modal posteriors, discrete-valued parameters and imbalanced data are commonly encountered. Expanding Section 6.2, we discuss such mixture-of-experts frameworks in more detail in Appendix A.6.3 and in Appendix A.7. Another typical application
of DA algorithms are state space models, due to the potentially high-dimensional parameter space. In Section 6.1, we extended our iMDA approach to logistic state space models and provide further details in Appendix A.5.

A.2 Mixture representations

A.2.1 The Pólya-Gamma mixture representation

For all latent variable representations derived in this paper for binary, binomial or categorical data, the error term in the latent equations arises from a distribution \( \varepsilon \sim F_\varepsilon(\varepsilon) \), for which the density \( f_\varepsilon(\varepsilon) \) can be represented as a mixture of normals using the Pólya-Gamma distribution as mixing measure:

\[
f_\varepsilon(\varepsilon) = c(a, b) \frac{(e^\varepsilon)^a}{(1 + e^\varepsilon)^b} e^{\kappa \varepsilon} \int e^{-\frac{\omega^2}{2}} p(\omega) d\omega, \tag{A.1}
\]

where \( \kappa = a - b/2 \) and \( \omega \sim \mathcal{PG}(b, 0) \) follows the Pólya-Gamma distribution introduced by Polson et al. (2013) with parameter \( b \).

This new representation is very convenient, as the conditional posterior of \( \omega|\varepsilon \) can be derived from following (tilted) Pólya-Gamma distribution with the same parameter \( b \):

\[
\omega|\varepsilon \sim \mathcal{PG}(b, |\varepsilon|). \tag{A.2}
\]

On the other hand, conditional on \( \omega \), the likelihood contribution of \( \varepsilon \) is proportional to that of a \( \mathcal{N}(\kappa/\omega, 1/\omega) \) observation.\footnote{Based on rewriting}

\[
\kappa \varepsilon - \frac{\omega \varepsilon^2}{2} = -\frac{\omega(\varepsilon - \kappa/\omega)^2}{2} + d, \tag{A.3}
\]

where \( d \) is a constant not depending on \( \varepsilon \).
To simulate from a (tilted) Pólya-Gamma $\mathcal{PG}(q, c)$ distribution, the following convolution property is exploited:

$$X_1 \sim \mathcal{PG}(q_1, c), \quad X_2 \sim \mathcal{PG}(q_2, c) \Rightarrow X_1 + X_2 \sim \mathcal{PG}(q_1 + q_2, c).$$

where $X_1$ and $X_2$ are independent. Hence, to simulate from $Y \sim \mathcal{PG}(q, c)$, use $Y = \sum_{j=1}^{q} X_j$, where $X_j \sim \mathcal{PG}(1, c)$ are $q$ independent draws from the $\mathcal{PG}(1, c)$ distribution.

A.2.2 The logistic and the type I generalized logistic distribution

For the type I generalized logistic distribution $\varepsilon \sim \mathcal{GL}_1(\nu)$ with parameter $\nu > 0$, the density reads

$$f_\varepsilon(\varepsilon) = \frac{\nu e^{-\varepsilon}}{(1 + e^{-\varepsilon})^{\nu+1}} = \frac{\nu e^{\nu \varepsilon}}{(1 + e^{\varepsilon})^{\nu+1}}. \quad (A.4)$$

$\mathcal{GL}_1(\nu)$ reduces to the logistic distribution for $\nu = 1$. The c.d.f. of a type I generalized logistic distribution takes a simple form:

$$F_\varepsilon(\varepsilon) = \frac{1}{(1 + e^{-\varepsilon})^\nu} = \frac{e^{\nu \varepsilon}}{(1 + e^{\varepsilon})^\nu}. \quad (A.5)$$

Hence, the quantiles are available in closed form:

$$\varepsilon_p = F_\varepsilon^{-1}(p) = -\log \left( \frac{1}{p^{1/\nu}} - 1 \right). \quad (A.6)$$

The type I generalized logistic distribution $\varepsilon \sim \mathcal{GL}_1(\nu)$ can be represented as a mixture of normals with a Pólya-Gamma distribution serving as mixing measure, where

$$\omega \sim \mathcal{PG}(\nu + 1, 0), \quad \kappa = \frac{\nu - 1}{2}, \quad (A.7)$$

see Appendix A.2.1. For the logistic distribution, $\omega \sim \mathcal{PG}(2, 0)$ and $\kappa = 0$. 

4
A.2.3 The type II generalized logistic distribution

For the type II generalized logistic distribution \( \varepsilon \sim \mathcal{GL}_{\text{II}}(\nu) \) with parameter \( \nu > 0 \), the density reads

\[
f_\varepsilon(\varepsilon) = \frac{\nu e^{-\nu \varepsilon}}{(1 + e^{-\varepsilon})^{\nu+1}} = \frac{\nu e^\varepsilon}{(1 + e^\varepsilon)^{\nu+1}}. \tag{A.8}
\]

Also \( \mathcal{GL}_{\text{II}}(\nu) \) reduces to the logistic distribution for \( \nu = 1 \).

The c.d.f. of a type II generalized logistic distribution takes a simple form:

\[
F_\varepsilon(\varepsilon) = 1 - \frac{1}{(1 + e^\varepsilon)^\nu} = 1 - \frac{e^{-\nu \varepsilon}}{(1 + e^{-\varepsilon})^\nu}. \tag{A.9}
\]

Hence, the quantiles are available in closed form:

\[
\varepsilon_p = F_\varepsilon^{-1}(p) = \log \left( \frac{1}{(1-p)^{1/\nu}} - 1 \right). \tag{A.10}
\]

The type II generalized logistic distribution \( \varepsilon \sim \mathcal{GL}_{\text{II}}(\nu) \) can be represented as a mixture of normals with a Pólya-Gamma distribution serving as mixing measure, where

\[
\omega \sim \mathcal{PG}(\nu + 1, 0), \quad \kappa = \frac{1 - \nu}{2}, \tag{A.11}
\]

see Appendix A.2.1. Again, for the logistic distribution, \( \omega \sim \mathcal{PG}(2, 0) \) and \( \kappa = 0 \) results.

A.3 Proofs

Proof of Proposition 1

The proof of Proposition 1 is straightforward. Depending on the observed category \( y_i \), the corresponding utility \( u_{y_i,i} \) is the maximum among all latent utilities. Equivalently, given that \( y_i = k \), \( e^{-u_{ki}} \) attains the minimum among all random variables \( e^{-u_{0i}}, \ldots, e^{-u_{mi}} \). Since
$e^{-u_{ki}} \sim \mathcal{E}(1)$ for $\ell = 0, \ldots, m$ are iid standard exponential a priori, we obtain that a posteriori (given that $y_i = k$) the minimum $e^{-u_{ki}}$ follows an exponential distribution:

$$e^{-u_{ki}}|y_i = k \sim \mathcal{E}(\lambda_i^*) ,$$

where $\lambda_i^* = 1 + \sum_{l=1}^{m} \lambda_{li}$. Furthermore, given the minimum $e^{-u_{ki}}$ all remaining random variables $e^{-u_{li}}$ with $\ell \neq k$ are conditionally independent with following distributions:

$$e^{-u_{0i}} = e^{-u_{ki}} + \mathcal{E}(1),$$

$$e^{-u_{\ell i}} = e^{-u_{ki}} + \mathcal{E}(\lambda_{li}), \quad \forall \ell \in L-(k,0).$$

For efficient joint sampling of all utilities for all $i = 1, \ldots, N$, this can be rewritten as in (17).

**Proof of Theorem 2**

A binomial observation $y_i$ can be regarded as the aggregated number of successes among $N_i$ independent binary outcomes $z_{1i}, \ldots, z_{N_i;i}$, labelled $\{0,1\}$, and each following the binary logit model $\Pr(z_{ni} = 1|\pi_i) = \pi_i$. For each individual binary observation $z_{ni}$, the logit model can be written as a RUM [McFadden 1974]:

$$u_{ni} = \log \lambda_i + \epsilon_{ni}, \quad \epsilon_{ni} \sim \mathcal{LO},$$

$$z_{ni} = I\{u_{ni} > 0\},$$

involving a latent variable $u_{ni}$, where $\epsilon_{ni}$ are i.i.d. errors following a logistic distribution. Among the $N_i$ binary experiment, $y_i$ outcomes $z_{ni}$ choose the category 1, whereas the remaining $N_i - y_i$ outcomes $z_{ni}$ choose the category 0. The challenge is to aggregate the latent variables $u_{ni}$ to a few latent variables in such a way that an explicit choice equation
is available. As it turns out, such an aggregation can be based on the order statistics 
\( u_{(1),i} < \cdots < u_{(N_i),i} \) of \( u_{1i}, \ldots, u_{N_i,i} \).

Consider first the case that \( y_i = 0 \). Such an outcome is observed, iff \( z_{ni} = 0 \) or, equivalently, the latent utility is negative (\( u_{ni} \leq 0 \)) for all \( n = 1, \ldots, N_i \). Hence, a necessary and sufficient condition for \( y_i = 0 \) is that the maximum of all utilities is negative, or equivalently,

\[
y_i = 0 \iff u_{(N_i),i} \leq 0.
\] (A.12)

Next, consider the case that \( y_i = N_i \). Such an outcome is observed, iff \( z_{ni} = 1 \) or, equivalently, the latent utility is positive (\( u_{ni} > 0 \)) for all \( n = 1, \ldots, N_i \). Hence, a necessary and sufficient condition for \( y_i = N_i \) is that the minimum of all utilities is positive, or equivalently,

\[
y_i = N_i \iff u_{(1),i} > 0.
\] (A.13)

Also for outcomes \( 0 < y_i < N_i \), the order statistics \( u_{(N_i−k),i} \) and \( u_{(N_i−k+1),i} \) provide necessary and sufficient conditions:

\[
y_i = k, \ 0 < k < N_i \iff u_{(N_i−k),i} \leq 0, u_{(N_i−k+1),i} > 0.
\] (A.14)

Note that (A.12) – (A.14) are choice equations involving either a single or two order statistics. Hence, we introduce the corresponding order statistics as aggregated latent variables. Given \( 0 < y_i = k < N_i \), we define \( v_i = u_{(N_i−k),i} \) and \( w_i = u_{(N_i−k+1),i} \). The choice equation then follows from (A.14):

\[
y_i = k \iff v_i \leq 0, \ w_i > 0,
\]

with obvious modifications for \( y_i = 0 \) and \( y_i = N_i \).
It remains to prove that the latent variables can be represented as in the aggregated model (23):

\[ w_i = \log \lambda_i + \varepsilon_{w,i}, \quad \varepsilon_{w,i} \sim \mathcal{GL}_{II}(y_i), \quad y_i > 0, \quad (A.15) \]

\[ v_i = \log \lambda_i + \varepsilon_{v,i}, \quad \varepsilon_{v,i} \sim \mathcal{GL}_1(N_i - y_i), \quad y_i < N_i. \]

Note that the order statistics \( u_{(1),i}, \ldots, u_{(N_i),i} \) can be represented for \( j = 1, \ldots, N_i \) as \( u_{(j),i} = \log \lambda_i + \varepsilon_{(j),i} \), involving the order statistics \( \varepsilon_{(1),i}, \ldots, \varepsilon_{(N_i),i} \) are of \( N_i \) iid realisations \( \varepsilon_{1,i}, \ldots, \varepsilon_{N_i,i} \) of a logistic distribution. Their distribution can be derived from the order statistics \( X_{(1),i}, \ldots, X_{(N_i),i} \) of \( N_i \) uniform random numbers \( X_1,i, \ldots, X_{N_i,i} \) using:

\[ \varepsilon_{(j),i} = F^{-1}(X_{(j),i}) = \log \frac{X_{(j),i}}{1 - X_{(j),i}} \iff X_{(j),i} = F(\varepsilon_{(j),i}), \quad (A.16) \]

where \( F \) is the cdf of the logistic distribution.

First, for the special cases where \( y_i = 0 \) or \( y_i = N_i \), we use that \( X_{(j),i} \sim B(j, N_i - j + 1) \).

Using (A.16), we can derive the density of \( \varepsilon_{(N_i),i} \):

\[ p(\varepsilon_{(N_i),i}) = N_i F(\varepsilon_{(N_i),i})^{N_i-1} f(\varepsilon_{(N_i),i}) = \frac{N_i \exp(\varepsilon_{(N_i),i})^{N_i}}{(1 + \exp(\varepsilon_{(N_i),i}))^{N_i+1}}, \]

which is the density of a \( \mathcal{GL}_1(N_i) \) distribution, see (A.4). Hence, for \( y_i = 0 \),

\[ v_i = \log \lambda_i + \varepsilon_{v,i}, \quad \varepsilon_{v,i} = \varepsilon_{(N_i),i} \sim \mathcal{GL}_1(N_i). \]

Using (A.16), we can derive the density of \( \varepsilon_{(1),i} \):

\[ p(\varepsilon_{(1),i}) = N_i (1 - F(\varepsilon_{(1),i}))^{N_i-1} f(\varepsilon_{(1),i}) = \frac{N_i \exp(\varepsilon_{(1),i})}{(1 + \exp(\varepsilon_{(1),i}))^{N_i+1}}, \]

which is the density of a \( \mathcal{GL}_{II}(N_i) \) distribution, see (A.8). Hence, for \( y_i = N_i \):

\[ w_i = \log \lambda_i + \varepsilon_{w,i}, \quad \varepsilon_{w,i} = \varepsilon_{(1),i} \sim \mathcal{GL}_{II}(N_i). \]
Second, for any $0 < y_i = k < N_i$ we need the joint distribution of $(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i})^\top$, where \(\varepsilon_{(N_i-k+1),i} = \varepsilon_{(N_i-k),i} + \Delta \varepsilon_i\) with \(\Delta \varepsilon_i > 0\). Using that \((X_{(j),i}, X_{(j+1),i} - X_{(j),i}, 1 - X_{(j+1),i}) \sim \mathcal{D}(j, 1, N_i - j)\) follow a Dirichlet distribution, see e.g. Robert & Casella (1999), we obtain that \((X_{(N_i-k),i}, 1 - X_{(N_i-k+1),i}, X_{(N_i-k+1),i} - X_{(N_i-k),i}) \sim \mathcal{D}(N_i - k, k, 1)\). To derive \(p(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i})\), we consider the transformations

\[
\varepsilon_{(N_i-k),i} = F^{-1}(X_{(N_i-k),i}), \quad \varepsilon_{(N_i-k+1),i} = F^{-1}(X_{(N_i-k+1),i}),
\]

and their inverse, \(X_{(N_i-k),i} = F(\varepsilon_{(N_i-k),i})\) and \(X_{(N_i-k+1),i} = F(\varepsilon_{(N_i-k+1),i})\). We determine

\[
\begin{vmatrix}
\frac{\partial(X_{(N_i-k),i}, X_{(N_i-k+1),i})}{\partial(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i})}
\end{vmatrix} = 
\begin{vmatrix}
 f(\varepsilon_{(N_i-k),i}) & 0 \\
 0 & f(\varepsilon_{(N_i-k+1),i})
\end{vmatrix} = f(\varepsilon_{(N_i-k),i}) f(\varepsilon_{(N_i-k+1),i}),
\]

where \(f\) is the pdf of the logistic distribution. Since

\[
p(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i}) = 
\frac{\Gamma(N_i + 1)}{\Gamma(k) \Gamma(N_i - k)} \frac{\exp(\varepsilon_{(N_i-k),i})^{N_i-k-1} (1 - F(\varepsilon_{(N_i-k+1),i}))^{k-1}}{(1 + \exp(\varepsilon_{(N_i-k),i}))^{N_i-k+1}} \cdot \frac{\exp(\varepsilon_{(N_i-k+1),i})}{(1 + \exp(\varepsilon_{(N_i-k+1),i}))^{k+1}}
\]

whenever \(\varepsilon_{(N_i-k),i} > \varepsilon_{(N_i-k),i}\), the density \(p(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i})\) can be expressed as

\[
p(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i}) = p(\varepsilon_{(N_i-k),i}) p(\varepsilon_{(N_i-k+1),i}) \cdot C \cdot I\{\varepsilon_{(N_i-k+1),i} > \varepsilon_{(N_i-k),i}\},
\]

where

\[
p(\varepsilon_{(N_i-k),i}) = \frac{(N_i - k) \exp(\varepsilon_{(N_i-k),i})^{N_i-k}}{(1 + \exp(\varepsilon_{(N_i-k),i}))^{N_i-k+1}}
\]

is the density of a \(\mathcal{GL}_1(N_i - k)\) distribution, see (A.4),

\[
p(\varepsilon_{(N_i-k+1),i}) = \frac{k \exp(\varepsilon_{(N_i-k+1),i})}{(1 + \exp(\varepsilon_{(N_i-k+1),i}))^{k+1}}
\]

(A.18)
is the density of a $\mathcal{GL}_II(k)$ distribution, see (A.8), and

$$C = \frac{\Gamma(N_i + 1)}{\Gamma(k + 1)\Gamma(N_i - k + 1)}$$

is a normalising constant. It is possible to verify that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i}) d\varepsilon_{(N_i-k),i} d\varepsilon_{(N_i-k+1),i} = 1.$$ 

Defining $(\varepsilon_{v,i}, \varepsilon_{w,i})^\top = (\varepsilon_{(N_i-k),i}, \varepsilon_{(N_i-k+1),i})^\top$, yields (A.15).

**Proof of Theorem 3**

Knowing that $y_i = k$, with $0 < k < N_i$, $w_i | \lambda_i, y_i$ and $v_i | \lambda_i, y_i$ are conditionally independent and the following holds:

$$w_i = \log \lambda_i + \varepsilon_{w,i}, \quad v_i = \log \lambda_i + \varepsilon_{v,i},$$

where $\varepsilon_{w,i} | y_i = k \sim \mathcal{GL}_II(k)$ is truncated to $(-\log \lambda_i, +\infty)$, since $w_i > 0$ and $\varepsilon_{v,i} | y_i = k \sim \mathcal{GL}_I(N_i - k)$ is truncated to $(-\infty, -\log \lambda_i]$, since $v_i \leq 0$. For $y_i = 0$, only $v_i | \lambda_i, y_i$ is sampled using that $\varepsilon_{v,i} | y_i = 0 \sim \mathcal{GL}_I(N_i)$, truncated to $(-\infty, -\log \lambda_i]$. For $y_i = N_i$, only $w_i | \lambda_i, y_i$ is sampled using that $\varepsilon_{w,i} | y_i = N_i \sim \mathcal{GL}_II(N_i)$, truncated to $(-\log \lambda_i, +\infty)$.

Since both $F_\varepsilon$ and $F_\varepsilon^{-1}$ are available in closed form for both types of generalized logistic distributions, we obtain:

$$\varepsilon_{w,i} = \log \left( \frac{1}{(1-p)^{1/y_i} - 1} \right), \quad 1 - p = W_i(1 - F_\varepsilon(-\log \lambda_i)) = W_i \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{y_i},$$

where $W_i$ is a uniform random number, see (A.10). This proves equation (24):

$$\varepsilon_{w,i} = \log \left( \frac{1 + \lambda_i}{\lambda_i} \frac{1}{W_i^{1/y_i} - 1} \right) \Rightarrow w_i = \log \lambda_i + \varepsilon_{w,i} = \log \left( 1 + \lambda_i \frac{1}{W_i^{1/y_i} - \lambda_i} \right).$$
Furthermore,
\[ \varepsilon_{v,i} = -\log \left( \frac{1}{p^{1/(N_i-y_i)}} - 1 \right), \quad p = V_i F_\varepsilon(-\log \lambda_i) = V_i \frac{1}{(1 + \lambda_i)^{N_i-y_i}}, \]

where \( V_i \) is a uniform random number, see (A.6). This proves equation (25):
\[ \varepsilon_{v,i} = -\log \left( \frac{1 + \lambda_i}{V_i^{1/(N_i-y_i)}} - 1 \right) \Rightarrow v_i = \log \lambda_i + \varepsilon_{v,i} = -\log \left( \frac{1 + \lambda_i}{\lambda_i} \frac{1}{V_i^{1/(N_i-y_i)}} - \frac{1}{\lambda_i} \right). \]

It is easy to verify that indeed \( w_i > 0 \) and \( v_i \leq 0 \).

### A.4 Computational details

#### A.4.1 Sampling the utilities in a binary model

Consider the latent variable representation of a binary model
\[ \Pr(y_i = 1|\lambda_i) = F_\varepsilon(\log \lambda_i), \quad (A.20) \]

involving the latent variables \( z_i \):
\[ y_i = I\{z_i > 0\}, \quad z_i = \log \lambda_i + \varepsilon_i, \quad \varepsilon_i \sim f_\varepsilon(\varepsilon_i), \quad (A.21) \]

where \( f_\varepsilon(\varepsilon) \) is the pdf of the cdf \( F_\varepsilon(\varepsilon) \). \( f_\varepsilon(\varepsilon) = \phi(\varepsilon) \) is equal to the standard normal pdf for a probit model and equal to \( f_\varepsilon(\varepsilon) = e^{\varepsilon}/(1 + e^{\varepsilon})^2 \) for a logit model. Given \( \lambda_1, \ldots, \lambda_N \) and \( y = (y_1, \ldots, y_N) \), the latent variables \( z_1, \ldots, z_N \) in the latent variable representation (A.21) are conditionally independent with conditional posterior
\[ p(z_i|y_i, \lambda_i) \propto p(y_i|z_i)f_\varepsilon(z_i - \log \lambda_i). \]

The posterior of \( z_i \) is \( f_\varepsilon(z_i - \log \lambda_i) \) truncated to \( (-\infty, 0] \), if \( y_i = 0 \), and truncated to \( (0, \infty) \), if \( y_i = 1 \), hence, \( z_i = \log \lambda_i + \varepsilon_i \), where \( \varepsilon_i \sim f_\varepsilon(\varepsilon_i)I\{\varepsilon_i > -\log \lambda_i\} \), if \( y_i = 1 \),
and \( \varepsilon_i \sim f_\varepsilon(\varepsilon_i)I\{\varepsilon_i \leq -\log \lambda_i\} \), if \( y_i = 0 \). Since the quantile function \( F_\varepsilon^{-1}(p) \) is available in closed form, it is easy to sample \( z_i \) from the posterior density \( z_i|\lambda_i, y_i \). Since \( f_\varepsilon(\varepsilon) \) is symmetric around 0,

\[
z_i = \log \lambda_i + F_\varepsilon^{-1}(y_i + U_i(1 - y_i - \pi_i)),
\]

where \( U_i \sim U[0, 1] \) and \( \pi_i = \Pr(y_i = 1|\lambda_i) = F_\varepsilon(\log \lambda_i) \), where \( F_\varepsilon^{-1}(p) = \Phi^{-1}(p) \) for the probit model and \( F_\varepsilon^{-1}(p) = \log p - \log(1 - p) \) for the logit model.

### A.4.2 Proof of (8)

From the Pólya-Gamma mixture representation (5), it follows that

\[
p(\tilde{z}_i|\omega_i, \gamma, \beta) \propto \exp \left\{ -\frac{\omega_i}{2} (\tilde{z}_i - \gamma - x_i\beta)^2 \right\},
\]

see also Appendix [A.2.1](#).

Hence, conditional on \( \gamma \), the posterior \( \beta|\gamma, \tilde{z}, \omega \sim \mathcal{N}(B_N(m_N(\tilde{z}) - m_b\gamma), B_N) \) is Gaussian with moments as in (8). Using a well-known result, \( p(\tilde{z}|\omega, \gamma) \) can be expressed as

\[
p(\tilde{z}|\omega, \gamma) = \frac{\prod_{i=1}^N p(\tilde{z}_i|\omega_i, \gamma, \beta)p(\beta)}{p(\beta|\gamma, \tilde{z}, \omega)}. \tag{A.23}
\]

Evaluating the right hand side of (A.23) at \( \beta = 0 \) yields, in combination with the Gaussian working prior \( p(\gamma) \), the conditional posterior \( p(\gamma|\omega, \tilde{z}) \) given in (8).

### A.4.3 Details on the UPG sampler for MNL regression models using the partial dRUM representation

In the standard MNL model ([13](#)), we define independent Gaussian priors, \( \beta_k \sim \mathcal{N}_d(0, A_k) \), for the category specific regression parameters which can be equipped with a hierarchical

---

6To simulate \( \varepsilon \) from a distribution \( F_\varepsilon(\varepsilon) \) truncated to \([a,b] \) we simulate a uniform random number \( U \) and define either \( \varepsilon = F_\varepsilon^{-1}(F_\varepsilon(a) + U(F_\varepsilon(b) - F_\varepsilon(a))) \) or \( \varepsilon = F_\varepsilon^{-1}(F_\varepsilon(b) - U(F_\varepsilon(b) - F_\varepsilon(a))) \).
structure on the prior covariance matrices $A_k$. Algorithm 1 can be extended in a fairly straightforward manner to MNL models. The ultimate Pólya-Gamma sampler for categorical data is summarized in Algorithm 2.

To implement category specific boosting, we proceed in the following way. The priors for the working parameters $\gamma_k$ and $\delta_k$ are chosen similarly as for the binary model, namely

$$\gamma_k \sim \mathcal{N}(0, G_0), \quad \delta_k \sim \mathcal{G}^{-1}(d_0, D_0). \tag{A.24}$$

In location-based boosting, we sample $\tilde{\gamma}_k \sim \mathcal{N}(0, G_0)$, and define $\tilde{z}_{ki} = z_{ki} + \tilde{\gamma}_k$. This leads to the expanded model

$$\tilde{z}_{ki} = \gamma_k + x_i \beta_k - \xi_{ki}(\beta_{-k}) + \varepsilon_{ki}, \tag{A.25}$$

$$y_i = \begin{cases} k, & \tilde{z}_{ki} > \gamma_k, \\ \neq k, & \tilde{z}_{ki} \leq \gamma_k. \end{cases} \tag{A.26}$$

where the error term $\varepsilon_{ki}$ follows a logistic distribution. Based on this expanded model, we derive the posterior $\gamma_k | \beta_{-k}, \omega_k, \tilde{z}_k$ marginalized w.r.t. the regression parameter $\beta_k$, sample a new location parameter $\gamma_k^{\text{new}}$ from this posterior and define the shifted utility gap $z_{ki}^L = \tilde{z}_{ki} - \gamma_k^{\text{new}}$, or equivalently, $z_{ki}^L = z_{ki} + (\tilde{\gamma}_k - \gamma_k^{\text{new}})$.

Given the outcomes $y = (y_1, \ldots, y_N)$, the choice equation (A.26) implies the constraint

$L(\tilde{\gamma}_k) \leq \gamma_k < U(\tilde{\gamma}_k)$, conditional on $z_k$, where

$$L(\tilde{\gamma}_k) = \max_{i: y_i \neq k} \tilde{z}_{ki} = \max_{i: y_i \neq k} z_{ki} + \tilde{\gamma}_k, \tag{A.27}$$

$$U(\tilde{\gamma}_k) = \min_{i: y_i = k} \tilde{z}_{ki} = \min_{i: y_i = k} z_{ki} + \tilde{\gamma}_k.$$

If $\#\{y_i = k\} = 0$, then $U(\tilde{\gamma}_k) = +\infty$ and the constraint reads $\gamma_k > L(\tilde{\gamma}_k)$; if $\#\{y_i \neq k\} = 0$, then $L(\tilde{\gamma}_k) = -\infty$ and the constraint reads $\gamma_k < U(\tilde{\gamma}_k)$. It can be shown that the posterior
\[ \gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k, y \text{ takes the following form,} \]
\[ \gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k, y \sim \mathcal{N}(g_k, G_k) I\{L(\gamma_k) \leq \gamma_k < U(\gamma_k)\} \tag{A.28} \]
where \( L(\gamma_k) \) and \( U(\gamma_k) \) are the boundaries defined in \( (A.27) \) and \( g_k \) and \( G_k \) are defined in \( (A.29) \). To derive \( (A.28) \), the likelihood function \( p(y|\gamma_k, \tilde{z}_k) \) of the location shift parameter \( \gamma_k \) is combined with the conditional distribution \( \gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k \sim \mathcal{N}(g_k, G_k) \), marginalized w.r.t. \( \beta_k \). The moments of this distribution are given by:

\[ G_k = \left( G_0^{-1} + \sum_{i=1}^{N} \omega_{ki} - m_{kb} B_k m_{kb} \right)^{-1}, \]
\[ g_k = G_k \left( m_k \gamma - m_{kb} B_k m_k (\tilde{z}_k) \right), \]
\[ B_k = (A_k^{-1} + \sum_{i=1}^{N} \omega_{ki} x_i^T x_i)^{-1}, \]
\[ m_k(\tilde{z}_k) = \sum_{i=1}^{N} x_i^T \omega_{ki}(\tilde{z}_k + \xi_k(\beta_{-k})), \]
\[ m_{kb} = \sum_{i=1}^{N} \omega_{ki} x_i^T, \quad m_k = \sum_{i=1}^{N} \omega_{ki} (\tilde{z}_k + \xi_k(\beta_{-k})). \]

\( (A.29) \) is derived from the latent equation \( (A.25) \) under the Gaussian working prior \( \gamma_k \sim \mathcal{N}(0, G_0) \) similarly as for the logit model. Conditional on \( \gamma_k \), the posterior \( p(\beta_k|\beta_{-k}, \gamma_k, \tilde{z}_k, \omega_k) \) is Gaussian,

\[ \beta_k|\beta_{-k}, \gamma_k, \tilde{z}_k, \omega_k \sim \mathcal{N}(B_k (m_k(\tilde{z}_k) - m_{kb} \gamma_k), B_k) \]
with moments as in \( (A.29) \). \( p(\tilde{z}_k|\beta_{-k}, \omega_k, \gamma_k) \) can be expressed as

\[ p(\tilde{z}_k|\beta_{-k}, \omega_k, \gamma_k) = \prod_{i=1}^{N} p(\tilde{z}_{ki}|\omega_{ki}, \gamma_k, \beta_{-k}, \beta_k) p(\beta_k) \frac{p(\beta_k|\beta_{-k}, \gamma_k, \tilde{z}_k, \omega_k)}{p(\beta_k|\beta_{-k}, \gamma_k)}. \tag{A.30} \]

Evaluating the right hand side of \( (A.30) \) at \( \beta_k = 0 \) yields, in combination with the Gaussian prior \( p(\gamma_k) \), the conditional distribution \( p(\gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k) \sim \mathcal{N}(g_k, G_k) \) with moments given in \( (A.29) \). Taking the choice equation \( (A.26) \) into consideration, the posterior
\( p(\gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k, y) \propto p(y|\gamma_k, \tilde{z}_k)p(\gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k) \) given the outcomes \( y \) is a truncated version of the Gaussian distribution \( p(\gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k) \sim \mathcal{N}(g_k, G_k) \) and yields the posterior \( p(\gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k, y) \) given in (A.28). An updated working parameter \( \gamma_k^{\text{new}} \) is sampled from \( \mathcal{N}(g_k, G_k) \) and the proposed location-based move is corrected by defining the shifted utility gap \( z_{ki}^L = \tilde{z}_{ki} - \gamma_k^{\text{new}} = z_{ki} + \tilde{z}_k - \gamma_k^{\text{new}} \).

This location-based move is followed by a scale-based move using a scale parameter \( \delta_k \) following the inverse Gamma prior defined in (A.24). More specifically, we move to

\[
z_{ki}^{LS} = \sqrt{\frac{\tilde{\delta}_k}{\delta_k}} z_{ki}^L,
\]

for all \( i = 1, \ldots, N \). To perform the move from \( z_{ki}^L \) to \( z_{ki}^{LS} \), first \( \tilde{\delta}_k \) is sampled from the prior \( p(\delta_k) \), which is the \( \mathcal{G}^{-1}(d_0, D_0) \)-distribution, and the scale move is then corrected by sampling \( \delta_k \) from the posterior \( p(\delta_k|\beta_{-k}, \omega_k, \tilde{z}_k) \) where \( \tilde{z}_k = (\tilde{z}_{k1}, \ldots, \tilde{z}_{kN}) \) and \( \tilde{z}_{ki} = \sqrt{\frac{\tilde{\delta}_k}{\delta_k}} z_{ki}^L \) for all \( i = 1, \ldots, N \). This leads to the expanded model

\[
\tilde{z}_{ki} = \sqrt{\delta_k} (x_i \beta_k - \xi_{ki}(\beta_{-k})) + \sqrt{\delta_k} \varepsilon_{ki},
\]

\[
y_i = \begin{cases} 
    k, & \tilde{z}_{ki} > 0, \\
    \neq k, & \tilde{z}_{ki} \leq 0,
\end{cases}
\]

where the error term \( \varepsilon_{ki} \) follows a logistic distribution. Given the outcomes \( y = (y_1, \ldots, y_N) \), the choice equation in (A.31) does not impose any constraint on \( \delta_k \). The likelihood of \( \tilde{z}_{ki} \) conditional on the scale parameter \( \delta_k \) is given for all \( i = 1, \ldots, N \) as:

\[
p(\tilde{z}_{ki}|\omega_{ki}, \delta_k, \beta_k, \beta_{-k}) \propto \]

\[
\propto \frac{1}{\sqrt{\delta_k}} \exp \left\{ -\frac{\omega_{ki}}{2} \left( \sqrt{\frac{\delta_k}{\delta_k}} z_{ki}^L - x_i \beta_k + \xi_{ki}(\beta_{-k}) \right)^2 \right\}.
\]
Hence, conditional on $\delta_k$, $\tilde{\delta}_k$ and all latent variables the vector of regression coefficients $\beta_k|\beta_{-k}, \delta_k, \tilde{\delta}_k, z_L^k, \omega_k \sim \mathcal{N}(b_k, B_k)$ follows a Gaussian distribution with parameters $b_k = B_k m_k(z_L^k)$ where $B_k$ is given in (A.29) and $m_k(z_L^k)$ is given as:

$$m_k(z_L^k) = m_{bk} + \sqrt{\tilde{\delta}_k \delta_k} m_{ak},$$ \hspace{1cm} (A.32)

We sample $\delta_k$ conditional on $z_L^k$, $\tilde{\delta}_k$ and $\beta_{-k}$, but marginally w.r.t. $\beta_k$, from the posterior $p(\delta_k|\beta_{-k}, \omega_k, \tilde{\delta}_k, \delta_k)$. To determine the likelihood $p(z_L^k|\beta_{-k}, \omega_k, \tilde{\delta}_k, \delta_k)$ marginally w.r.t. $\beta_k$, we evaluate the right hand side of following ratio at $\beta_k = 0$

$$p(z_L^k|\beta_{-k}, \omega_k, \tilde{\delta}_k, \delta_k) \propto \frac{p(\beta_k) \prod_{i=1}^N p(z_{ki}^L|\beta_{-k}, \tilde{\delta}_k, \omega_{ki}, \delta_k, \beta_k) p(\omega_{ki})}{p(\beta_k|\beta_{-k}, \tilde{\delta}_k, \delta_k, z_L^k, \omega_k)} = \frac{A(\delta_k)}{B(\delta_k)}.$$

We have

$$\log A(\delta_k) = -0.5 \left( \frac{\tilde{\delta}_k}{\delta_k} \sum_{i=1}^N \omega_{ki} (z_{ki}^L)^2 + 2 \sqrt{\tilde{\delta}_k} \sum_{i=1}^N \omega_{ki} z_{ki}^L \xi_{ki} (\beta_{-k}) \right) - N/2 \log(\delta_k) + c_A,$$

and

$$\log B(\delta_k) = -0.5 m_k(z_L^k)^\top B_k m_k(z_L^k) =$$

$$= -0.5 \left( (m_{bk} + \sqrt{\tilde{\delta}_k \delta_k} m_{ak})^\top B_k (m_{bk} + \sqrt{\tilde{\delta}_k \delta_k} m_{ak}) \right) + c_B =$$

$$= -0.5 \frac{\tilde{\delta}_k}{\delta_k} m_{ak}^\top B_k m_{ak} - \sqrt{\tilde{\delta}_k \delta_k} m_{ak}^\top B_k m_{bk} + c_C,$$
where \( m_{ak} \) and \( m_{bk} \) have been defined in (A.32) and \( c_A, c_B \) and \( c_C \) are terms independent of \( \delta_k \). Due to the presence of the off-set \( \xi_{ki}(\beta_{-k}) \), the likelihood \( p(z^L_k|\beta_{-k}, \omega_k, \tilde{\delta}_k, \delta_k) \) does not admit a conjugate analysis as opposed to a binary model. This would be possible iff \( m = 1 \) and the MNL reduces to a binary logit model in which case all \( \xi_{ki}(\beta_{-k})s \) are zero. Combined with the prior \( \delta_k \sim G^{-1}(d_0, D_0) \), the posterior for \( \delta_k|\beta_{-k}, \tilde{\delta}_k, z^L_k, \omega_k \) is given as

\[
p(\delta_k|\beta_{-k}, \tilde{\delta}_k, z^L_k, \omega_k) \propto \left( \frac{1}{\delta_k} \right)^{d_k + 1} \exp \left( -\frac{D_k}{\delta_k} \right) \exp \left( B_k \sqrt{\frac{\delta_k}{\tilde{\delta}_k}} \right),
\]

with parameters

\[
d_k = d_0 + \frac{N}{2},
\]

\[
D_k = D_0 + \frac{\tilde{\delta}_k}{2} \left( \sum_{i=1}^{N} \omega_{ki}(z^L_{ki})^2 - m_{ak}^\top B_k m_{ak} \right),
\]

\[
B_k = \sqrt{\tilde{\delta}_k} \left( -\sum_{i=1}^{N} \omega_{ki}z^L_{ki} \xi_{ki}(\beta_{-k}) + m_{ak}^\top B_k m_{ak} \right).
\]

Details how to sample \( \tilde{\delta}_k^{\text{new}} \) from such a distribution are provided in Appendix [A.4.3].

Finally, after the scale boosting step, we sample the regression parameter \( \beta_k \) from following Gaussian distribution: \( \beta_k|\beta_{-k}, z^L_k, \tilde{\delta}_k^{\text{new}}, \delta_k, \omega_k \sim N_\theta(B_k m^S_k, B_k) \), where \( B_k \) is given by (A.29) and

\[
m^S_k = m_{bk} + \sqrt{\frac{\tilde{\delta}_k}{\delta_k^{\text{new}}}} m_{ak},
\]

where \( m_{bk} \) and \( m_{ak} \) are given by (A.32). This is equivalent to sampling \( \beta_k \) from the partial dRUM (18) based on the boosted utility gap

\[
z^L_{ki} = \sqrt{\frac{\tilde{\delta}_k}{\delta_k^{\text{new}}}} z^L_{ki} = \sqrt{\frac{\tilde{\delta}_k}{\delta_k^{\text{new}}}} (z_{ki} + \tilde{\gamma}_k - \gamma_k^{\text{new}}).
\]

The ultimate Pólya-Gamma sampler for categorical data based on the partial dRUM representation is summarized in Algorithm 2.
Algorithm 2 The ultimate Pólya-Gamma sampler for categorical data based on the partial dRUM model.

Choose starting values for $\beta = (\beta_1, \ldots, \beta_m)$. For each MCMC sweep, loop over the categories $k = 1, \ldots, m$ and perform the following steps:

(Z) For each $i = 1, \ldots, N$, sample the latent variables $u_i = (u_{0i}, u_{1i}, \ldots, u_{mi}) \sim p(u_i|\beta, y_i)$, using $m + 1$ independent uniform random numbers $U_i$ and $V_{1i}, \ldots, V_{mi}$, i.e. for all $\ell = 0, \ldots, m$, sample $u_{\ell i} = -\log(-\log(U_i)) + \sum_{k=1}^{m} \lambda_{ki} - \log(V_{\ell i}) \lambda_{\ell i} I\{y_i \neq \ell\}$, where $\log \lambda_{ki} = x_i \beta_k$ and $\lambda_{0i} = 1$. Define $z_{ki} = u_{ki} - \max_{k' \neq k} u_{k' i}$, and define $\xi_{ki}(\beta_{-k}) = \log \left( 1 + \sum_{\ell \neq \{k, 0\}} \exp(x_i \beta_\ell) \right)$. Given $\varepsilon_{ki} = z_{ki} - x_i \beta_k + \xi_{ki}(\beta_{-k})$, sample $\omega_{ki}|\beta, z_{ki} \sim \mathcal{PG}(2,|\varepsilon_{ki}|)$.

(B-L) Location-based parameter expansion: sample $\tilde{\gamma}_k \sim \mathcal{N}(0, G_0)$ and propose $\tilde{z}_{ki} = z_{ki} + \tilde{\gamma}_k$ for $i = 1, \ldots, N$, while all other latent variables remain unchanged. Sample $\gamma_{k, new}$ from the truncated Gaussian-posterior $p(\gamma_k|\beta_{-k}, \omega_k, \tilde{z}_k, y)$ given by (A.28) and define the shifted utility gap $z_{ki}^L = \tilde{z}_{ki} - \gamma_{k, new}$, for $i = 1, \ldots, N$, and $z^L_k = (z^L_{k1}, \ldots, z^L_{kN})$.

(B-S) Scale-based parameter expansion: sample $\tilde{\delta}_k \sim \mathcal{G}^{-1}(d_0, D_0)$ and sample $\delta_{k, new}$ from the posterior $p(\delta_k|\beta_{-k}, \tilde{\delta}_k, z^L_k, \omega_k)$ given in (A.33) using the resampling technique described in Appendix A.4.5.

(P) Sample $\beta_k|\beta_{-k}, z^L_k, \delta_{k, new}, \tilde{\delta}_k, \omega_k \sim \mathcal{N}_d(B_k m^S_k, B_k)$ where $B_k$ is given by (A.29) and $m^S_k = m_{6k} + \sqrt{\delta_k/\delta_{k, new}} m_{ak}$, where $m_{6k}$ and $m_{ak}$ are given by (A.32).

A.4.4 Details on the UPG sampler for binomial logistic regression models

In the standard binomial regression model, the prior $\beta \sim \mathcal{N}_d(0, A_0)$ is assumed, where $A_0$ can be equipped with a hierarchical structure. The working priors are the same as in a logit model, namely $\gamma \sim \mathcal{N}(0, G_0)$ and $\delta \sim \mathcal{G}^{-1}(d_0, D_0)$. The ultimate Pólya-Gamma sampler for binomial regression models is summarized in Algorithm 3.

Using DA, it follows from the Pólya-Gamma mixture representation (26) and Ap-
Choose starting values for $\lambda = (\lambda_1, \ldots, \lambda_N)$ and repeat the following steps:

(Z) For each $i = 1, \ldots, N$, sample $\omega_{w,i} | \lambda_i, (y_i > 0)$ and $\omega_{v,i} | \lambda_i, (y_i < N_i)$ using Theorem 3 and sample $\omega_{w,i} | w_i, \lambda_i, (y_i > 0)$ and $\omega_{v,i} | v_i, \lambda_i, (y_i < N_i)$ using (27).

(B-L) Location-based parameter expansion: sample $\hat{\gamma} \sim \mathcal{N}(0, G_0)$ and propose $\hat{w}_i = w_i + \hat{\gamma}$ and $\hat{v}_i = v_i + \hat{\gamma}$, for $i = 1, \ldots, N$. Sample $\gamma_{\text{new}}$ from $\gamma | \hat{z}, y$, conditional on $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_N)$, where $\hat{z}_i = (\hat{w}_i, \omega_{w,i}, \hat{v}_i, \omega_{v,i})$ and define shifted utilities $w_i^L = \hat{w}_i - \gamma_{\text{new}}$ and $v_i^L = \hat{v}_i - \gamma_{\text{new}}$. For a standard binomial regression model, $p(\gamma | \hat{z}, y)$ is a truncated Gaussian posterior, given in (A.37).

(B-S) Scale-based parameter expansion: sample $\delta \sim \mathcal{G}^{-1}(d_0, D_0)$ and sample $\delta_{\text{new}}$ from $\delta | \delta, z^L$. Define rescaled utilities $w_i^{LS} = \sqrt{\delta/\delta_{\text{new}}} w_i^L$ and $v_i^{LS} = \sqrt{\delta/\delta_{\text{new}}} v_i^L$. For a standard binomial regression model, $p(\delta | \delta, z^L)$ is Gaussian with moments given in (A.37), and $m_N = \sqrt{\delta/\delta_{\text{new}}} m_a - m_b$, with $m_a$ and $m_b$ being defined in (A.40).

(P) Sample the unknown parameter in $\lambda_i$ conditional on $z^{LS}$. For a standard binomial regression model, this is equivalent to sampling from $\beta | \delta_{\text{new}}, \delta, z^L \sim \mathcal{N}(B_N m_N, B_N)$ where $B_N$ is defined in (A.35) and $m_N = \sqrt{\delta/\delta_{\text{new}}} m_a - m_b$, with $m_a$ and $m_b$ being defined in (A.40).

Appendix A.2.1 that for all $i$ with $y_i > 0$,

$$p(\tilde{w}_i | \omega_{w,i}, \gamma, \beta) \propto \exp \left\{ -\frac{\omega_{w,i}}{2} (\tilde{w}_i - \kappa_{w,i} )^2 \right\},$$

while for all $i$ with $y_i < N_i$,

$$p(\tilde{v}_i | \omega_{v,i}, \gamma, \beta) \propto \exp \left\{ -\frac{\omega_{v,i}}{2} (\tilde{v}_i - \kappa_{v,i} )^2 \right\}.$$

Conditional on $\gamma$ and the latent variables $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_N)$, where $\hat{z}_i = (\hat{w}_i, \omega_{w,i}, \hat{v}_i, \omega_{v,i})$, the posterior $\beta | \gamma, \hat{z} \sim \mathcal{N}(B_N (m_N (\hat{z}) - m_b \gamma), B_N)$ is Gaussian with moments given in (A.35). Evaluating the right hand side of following ratio at $\beta = 0$,

$$p(\hat{z} | \gamma) \propto \frac{p(\beta) \prod_{i:y_i>0} p(\tilde{w}_i | \omega_{w,i}, \gamma, \beta) p(\omega_{w,i}) \prod_{i:y_i<N_i} p(\tilde{v}_i | \omega_{v,i}, \gamma, \beta) p(\omega_{v,i})}{p(\beta | \gamma, \hat{z})},$$

19
yields, in combination with the Gaussian prior $p(\gamma)$, the conditionally Gaussian distribution
\[ \gamma | \tilde{z} \sim \mathcal{N}(g_N, G_N), \]
marginalized w.r.t. $\beta$, where
\[ G_N = (G_0^{-1} + \sum_{i=1}^{N} M_i - m_b^i B_N m_b)^{-1}, \quad g_N = G_N(m_\gamma - m_b^i B_N m_N(\tilde{z})), \] (A.35)
\[ B_N = (A_0^{-1} + \sum_{i=1}^{N} M_i x_i^\top x_i)^{-1}, \quad m_N(\tilde{z}) = \sum_{i=1}^{N} m_i(\tilde{z}_i) x_i^\top, \]
\[ M_i = \{y_i > 0\} \omega_{w,i} + \{y_i < N_i\} \omega_{v,i}, \quad m_b = \sum_{i=1}^{N} M_i x_i^\top, \quad m_\gamma = \sum_{i=1}^{N} m_i(\tilde{z}_i), \]
\[ m_i(\tilde{z}_i) = \{y_i > 0\} (\tilde{w}_i \omega_{w,i} - \kappa_{w,i}) + \{y_i < N_i\} (\tilde{v}_i \omega_{v,i} - \kappa_{v,i}). \]

Given the observed choices $y = (y_1, \ldots, y_N)$, the choice equation (29) implies the constraint
\[ L(\tilde{\gamma}) \leq \gamma < U(\tilde{\gamma}) \] for $\gamma$, where conditional on $\tilde{z}$:
\[ L(\tilde{\gamma}) = \max_{i=1,\ldots,N} \tilde{v}_i = \max_{i=1,\ldots,N} v_i + \tilde{\gamma}, \]
\[ U(\tilde{\gamma}) = \min_{i=1,\ldots,N} \tilde{w}_i = \min_{i=1,\ldots,N} w_i + \tilde{\gamma}. \] (A.36)

Hence, $p(\gamma | \tilde{z}, y) \propto p(y | \gamma, \tilde{z})p(\gamma | \tilde{z})$ is a truncated version of the Gaussian posterior (A.35):
\[ \gamma | \tilde{z}, y \sim \mathcal{N}(g_N, G_N) I\{L(\tilde{\gamma}) \leq \gamma < U(\tilde{\gamma})\}. \] (A.37)

An updated working parameter $\gamma^{\text{new}}$ is sampled from (A.37) and the proposed location-based move is corrected by defining the shifted utilities $w_i^L = \tilde{w}_i - \gamma^{\text{new}} = w_i + \tilde{\gamma} - \gamma^{\text{new}}$ and $v_i^L = \tilde{v}_i - \gamma^{\text{new}} = v_i + \tilde{\gamma} - \gamma^{\text{new}}$ and, correspondingly, $z_i^L = (z_i^L, \ldots, z_N^L)$, where $z_i^L = (w_i^L, \omega_{w,i}, v_i^L, \omega_{v,i})$.

This location-based move is followed by a scale-based expansion. $\tilde{\delta} \sim \mathcal{G}^{-1}(d_0, D_0)$ is sampled from $p(\delta)$ to propose, for each $i = 1, \ldots, N$, the scale moves $\tilde{w}_i = \sqrt{\tilde{\delta}} w_i^L$ and $\tilde{v}_i = \sqrt{\tilde{\delta}} v_i^L$ in the expanded model
\[ \tilde{w}_i = \sqrt{\tilde{\delta}} x_i^L \beta + \sqrt{\tilde{\delta}} \varepsilon_{w,i}, \quad y_i > 0, \] (A.38)
\[ \tilde{v}_i = \sqrt{\tilde{\delta}} x_i^L \beta + \sqrt{\tilde{\delta}} \varepsilon_{v,i}, \quad y_i < N_i, \]
where the choice equation is independent of $\delta$. Using similar arguments as above, it follows from the Pólya-Gamma mixture representation (26) that for all $i$ with $y_i > 0$,

$$
p(\tilde{w}_i|\omega_{w,i}, \delta, \beta) \propto \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{\omega_{w,i}}{2} \left( \frac{\tilde{w}_i}{\sqrt{\delta}} - \frac{\kappa_{w,i}}{\omega_{w,i}} - x_i\beta \right)^2 \right\},
$$

while for all $i$ with $y_i < N_i$,

$$
p(\tilde{v}_i|\omega_{v,i}, \delta, \beta) \propto \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{\omega_{v,i}}{2} \left( \frac{\tilde{v}_i}{\sqrt{\delta}} - \frac{\kappa_{v,i}}{\omega_{v,i}} - x_i\beta \right)^2 \right\}.
$$

Completing the squares yields that $\beta|\delta, \tilde{\delta}, \tilde{z} \sim \mathcal{N}(B_N m_N, B_N)$ is Gaussian with $B_N$ as in (A.35) and $m_N = \sqrt{\delta/\tilde{\delta}} m_a - m_b$, where $m_a$ and $m_b$ are defined in (A.40).

Evaluating the right hand side of following ratio at $\beta = 0$ yields a closed form expression for the likelihood $p(\tilde{z}|\delta)$:

$$
p(\tilde{z}|\delta) \propto \frac{p(\beta) \prod_{i:y_i > 0} p(\tilde{w}_i|\omega_{w,i}, \delta, \beta)p(\omega_{w,i}) \prod_{i:y_i < N_i} p(\tilde{v}_i|\omega_{v,i}, \delta, \beta)p(\omega_{v,i})}{p(\beta|\delta, \tilde{z})}
\propto \left( \frac{1}{\delta} \right)^{d_L} \exp \left( -\frac{D_L}{\delta} \right) \exp \left( \frac{B_L}{\sqrt{\delta}} \right),
$$

(A.39)

where

$$d_L = \frac{1}{2} \sum_{i=1}^{N} \left( I\{y_i > 0\} + I\{y_i < N_i\} \right), \quad (A.40)$$

$$D_L = \frac{\tilde{\delta}}{2} \left( \sum_{i=1}^{N} \left( I\{y_i > 0\}(w^L_i)^2 \omega_{w,i} + I\{y_i < N_i\}(v^L_i)^2 \omega_{v,i} \right) - m_a^\top B_N m_a \right),$$

$$B_L = \sqrt{\delta} \left( \sum_{i=1}^{N} \left( I\{y_i > 0\} w^L_i \kappa_{w,i} + I\{y_i < N_i\} v^L_i \kappa_{v,i} \right) - m_a^\top B_N m_b \right),$$

$$m_a = \sum_{i=1}^{N} \left( I\{y_i > 0\} w^L_i \omega_{w,i} + I\{y_i < N_i\} v^L_i \omega_{v,i} \right) x_i^\top,$$

$$m_b = \sum_{i=1}^{N} \left( I\{y_i > 0\} \kappa_{w,i} + I\{y_i < N_i\} \kappa_{v,i} \right) x_i^\top,$$

21
and where $B_N$ is the same as in (A.35). However, due to the presence of the (fixed) location parameters $\kappa_{w,i}$ and $\kappa_{v,i}$, the likelihood $p(\tilde{z}|\delta)$ does not take a conjugate form as opposed to a binary logistic model. Combined with the inverse gamma prior $\delta \sim G^{-1}(d_0, D_0)$, the posterior $p(\delta|\tilde{\delta}, z^L)$ belongs for a binomial regression model to the same generalized distribution family as the posterior $p(\delta_k|\tilde{\delta}_k, \beta_{-k}, z^L_k, \omega_k)$ arising for a MNL model, with density
\[
p(\delta|\tilde{\delta}, z^L) \propto \left(\frac{1}{\delta}\right)^{d_I+1} e^{-D_I/\delta} e^{B_I/\sqrt{\delta}},
\]
where $d_I = d_0 + d_L$ and $D_I = D_0 + D_L$. $\delta^{\text{new}}$ is sampled from $\delta|\tilde{\delta}, z^L$ to define rescaled utilities $w^{LS}_i = \sqrt{\tilde{\delta}/\delta^{\text{new}}} w^L_i$ and $v^{LS}_i = \sqrt{\tilde{\delta}/\delta^{\text{new}}} v^L_i$. The posterior $p(\delta|\tilde{\delta}, z^L)$ reduces to the inverse gamma distribution $G^{-1}(d_I, D_I)$, iff $N_i = 1$ for all $i = 1, \ldots, N$ and the binomial model reduces to a logit model in which case all $\kappa_{w,i}$s and $\kappa_{v,i}$s are zero and $B_I = 0$. It can be shown that in any case $D_I > 0$, provided that $D_0 > 0$. Details how to sample from such a distribution are provided in Appendix A.4.5.

This concludes the boosting step and $\beta|z^{LS}$ is sampled conditional on $z^{LS}$, or equivalently from the Gaussian posterior $\beta|\delta^{\text{new}}, \tilde{\delta}, z^L \sim N(B_N m_N, B_N)$ where $B_N$ is defined in (A.35) and $m_N = \sqrt{\tilde{\delta}/\delta^{\text{new}}} m_a - m_b$, with $m_a$ and $m_b$ being defined in (A.40).

A.4.5 Sampling the scale parameter in scale boosting for binomial and MNL models

In general, the marginal density for the scale parameter arising in scale boosting for binomial and MNL models does not belong to a well-known distribution family unless these models reduce to a logistic model.
Figure A.1: Approximating \( p(\delta | \cdot) \) by the inverse Gamma density \( \delta \sim G^{-1}(d^*_I, D^*_I) \) for a case where \( B_I < 0 \) (left) and \( B_I > 0 \) (right). The posterior densities are covered by the auxiliary prior densities.

The following resampling technique is used to sample in Algorithm 3 from

\[
p(\delta | \tilde{\delta}, z^L) \propto \left( \frac{1}{\delta} \right)^{d_I+1} e^{-D_I/\delta} e^{B_I/\sqrt{\delta}}
\]

with obvious modifications to sample in Algorithm 2 from the posterior

\[
p(\delta_k | \tilde{\delta}_k, \beta_{-k}, z^L_k, \omega_k) \propto \left( \frac{1}{\delta_k} \right)^{d_k+1} \exp \left( -\frac{D_k}{\delta_k} \right) \exp \left( \frac{B_k}{\sqrt{\delta_k}} \right).
\]

Choose an ‘auxiliary prior’ \( \pi(\delta) \) for resampling such that the mode and the curvature of \( \pi(\delta) \) coincide with the mode \( \delta_M \) and the curvature \( I_p \) of the posterior \( p(\delta | \tilde{\delta}, z^L) \) which are given by:

\[
\delta_M = \frac{16D_I^2}{B_I + \sqrt{B_I^2 + 16D_I(d_I + 1)}} \quad I_p = -\frac{\sqrt{B_I^2 + 16D_I(d_I + 1)}}{4 \cdot (\delta_M)^2}.
\]

Resampling works as follows. \( L \) draws \( \delta^{(l)} \sim \pi(\delta) \), \( l = 1, \ldots, L \), from the auxiliary prior are resampled using weights proportional to the ‘auxiliary likelihood’ \( \ell(\delta) = p(\delta | \tilde{\delta}, z^L)/\pi(\delta), \)
given by:

\[
\log \ell(\delta) \propto \log p(\delta | \tilde{z}, y) - \log \pi(\delta) \\
\propto -(d_I + 1) \log \delta - \frac{D_I}{\delta} + \frac{B_I}{\sqrt{\delta}} - \log \pi(\delta).
\]

The desired draw from \( p(\delta | \cdot) \) is given by \( \delta^{(l^*)} \), where \( l^* \sim \text{MulNom}(1; w_1, \ldots, w_L) \) and the weights \( w_l \propto \ell(\delta^{(l)}) \) are normalized to 1.

The auxiliary likelihood \( \ell(\delta) \) is expected to be rather flat over the support of \( \pi(\delta) \), see Figure A.1 for illustration. Hence, \( w_1, \ldots, w_L \) is expected to be close to a uniform distribution and \( L \) can be pretty small (\( L = 5 \) or \( L = 10 \) should be enough).

The factorization of the posterior suggests two distribution families as auxiliary prior \( \pi(\delta) \). First, the inverse Gamma prior \( \delta \sim \mathcal{G}^{-1}(d^*_I, D^*_I) \) with mode \( \delta_{IG} \) and curvature \( I_{IG} \) of the pdf given by:

\[
\delta_{IG} = \frac{D^*_I}{d^*_I + 1}, \quad I_{IG} = -\frac{(d^*_I + 1)^3}{(D^*_I)^2}.
\]

Matching the mode, i.e. \( \delta_{IG} = \delta_M \), and the curvature, i.e. \( I_{IG} = I_p \), to the posterior, i.e.,

\[
I_{IG} = -\frac{(d^*_I + 1)^3}{(D^*_I)^2} = -\frac{d^*_I + 1}{(\delta_{IG})^2} = -\frac{d^*_I + 1}{(\delta_M)^2} = I_p,
\]

\[
\delta_{IG} = \frac{D^*_I}{d^*_I + 1} = \delta_M,
\]

yields following optimal choice for the parameters \( (d^*_I, D^*_I) \):

\[
d^*_I = -I_p \cdot (\delta_M)^2 - 1, \quad D^*_I = \delta_M(d^*_I + 1).
\]

The log likelihood ratio reads:

\[
\log \ell(\delta) \propto -(d_I - d^*_I) \log \delta - \frac{D_I - D^*_I}{\delta} + \frac{B_I}{\sqrt{\delta}}.
\]

24
Second, provided that $B^*_t > 0$, the translated inverse Gamma prior $\sqrt{\delta} \sim G^{-1}(2b^*_t, B^*_t)$ with mode $\delta_{SQ}$ and the curvature $I_{SQ}$ of the pdf given by:

$$\delta_{SQ} = \frac{(B^*_t)^2}{4(b^*_t + 1)^2}, \quad I_{SQ} = -\frac{8(b^*_t + 1)^5}{(B^*_t)^4}. $$

Matching the mode, i.e. $\delta_{SQ} = \delta_M$, and the curvature, i.e. $I_{SQ} = I_p$, to the posterior, i.e.,

$$I_{SQ} = -\frac{8(b^*_t + 1)^5}{(B^*_t)^4} = -\frac{b^*_t + 1}{2(\delta_{SQ})^2} = -\frac{b^*_t + 1}{2(\delta_M)^2} = I_p, $$

$$\delta_{SQ} = \frac{(B^*_t)^2}{4(b^*_t + 1)^2} = \delta_M,$$

yields following optimal choice for the parameters $(b^*_t, b^*_t)$:

$$b^*_t = -2I_p \cdot (\delta_M)^2 - 1, \quad B^*_t = 2(b^*_t + 1)\sqrt{\delta_M}. $$

The log likelihood ratio reads:

$$\log \ell(\delta) \propto -(d_I - b^*_t) \log \delta - \frac{D_I}{\delta} + \frac{B_t + B_t^*}{\sqrt{\delta}}. $$

### A.5 The UPG sampler for binary state space models

First, location-based boosting based on a $\mathcal{N}(0, G_0)$ working prior is applied. $\tilde{\gamma} \sim \mathcal{N}(0, G_0)$ is sampled from this prior to propose, for each $t = 1, \ldots, T$, a location move $\tilde{z}_t = z_t + \tilde{\gamma}$ in an expanded state space model, where the observation equation is affected in the following

As follows from the law of transformation of densities, a random variable $\delta$, where the transformed variable $\sqrt{\delta} \sim G^{-1}(2a, b)$, has density

$$p_{SQ}(\delta; 2a, b) = \frac{b^{2a}}{2\Gamma(2a)} \left(\frac{1}{\delta}\right)^{a+1} e^{-\frac{1}{\delta}}. \quad (A.41)$$

To sample from such a density, we sample $X \sim G^{-1}(2a, b)$ and take the square, i.e. $\delta = X^2$. 25
while the transition equation (31) and the initial distribution $\beta_0 \sim \mathcal{N}_d(0, P_{0|0})$ remain the same. Conditional on the latent variables $\omega = (\omega_1, \ldots, \omega_T)$, where $\omega_t \sim \mathcal{P}\mathcal{G}(2, 0)$, it follows from the Pólya-Gamma mixture representation (5) that the observation density $p(\tilde{z}_t|\omega_t, \gamma, \beta_t)$ takes the form

$$p(\tilde{z}_t|\omega_t, \gamma, \beta_t) \propto \exp \left\{ -\frac{\omega_t}{2} (\tilde{z}_t - x_t\beta_t - \gamma)^2 \right\}.$$ 

Hence, in combination with the transition density (31) and the initial distribution $\beta_0 \sim \mathcal{N}_d(0, P_{0|0})$, a conditionally Gaussian state space model is obtained and the Kalman filter can be applied to determine the moments of the filtering density conditional on $\gamma$, given $\tilde{z}^t = (\tilde{z}_1, \ldots, \tilde{z}_t)$. These moments can be expressed as:

$$\beta_t|\tilde{z}^t, \omega, \gamma, \theta_1, \ldots, \theta_d \sim \mathcal{N}_d \left( \hat{\beta}_{t|t} + g_t\gamma, P_{t|t} \right),$$

where $g_t$ will be defined below in (A.46) and $\hat{\beta}_{t|t}$ and $P_{t|t}$ are the moments of the filtering density $\beta_t|\gamma = 0, \tilde{z}^t, \omega, \theta_1, \ldots, \theta_d \sim \mathcal{N}_d \left( \hat{\beta}_{t|t}, P_{t|t} \right)$ for the specific state space model where $\gamma = 0$:

$$\beta_t = \beta_{t-1} + w_t, \quad w_t \sim \mathcal{N}_d(0, Q), \quad \tilde{z}_t = x_t\beta_t + \varepsilon_t.$$ 

Starting with $\hat{\beta}_{0|0} = 0$, these moments are given for $t = 1, \ldots, T$ by the Kalman filter:

$$P_{t|t-1} = P_{t-1|t-1} + \text{Diag} \left( \theta_1, \ldots, \theta_d \right),$$

$$\tilde{z}_t|\tilde{z}^{t-1}, \omega, \theta_1, \ldots, \theta_d \sim \mathcal{N} \left( \tilde{z}_{t|t-1}, S_{t|t-1} \right),$$

$$\tilde{z}_{t|t-1} = x_t\hat{\beta}_{t-1|t-1}, \quad S_{t|t-1} = x_tP_{t-1|t-1}x_t^\top + 1/\omega_t,$$

$$\hat{\beta}_{t|t} = \hat{\beta}_{t-1|t-1} + K_t(\tilde{z}_t - \tilde{z}_{t|t-1}),$$

$$P_{t|t} = (I - K_tx_t)P_{t|t-1}, \quad K_t = P_{t|t-1}x_t^\top S_{t|t-1}^{-1}. $$
In addition, the weight $g_t$ in (A.43) satisfies the following recursion for $t = 1, \ldots, T$ with $g_0 = 0$:

$$g_t = (I - K_t x_t)g_{t-1} - K_t. \tag{A.46}$$

The representation (A.43) is easy to prove. Based on the Kalman filter for a model with arbitrary $\gamma$, we obtain for $t = 1$:

$$\beta_{1|1} \mid \tilde{z}, \omega, \gamma, \theta_1, \ldots, \theta_d \sim N_d \left( \hat{\beta}_{1|1}(\gamma), P_{1|1} \right),$$

$$\hat{\beta}_{1|1}(\gamma) = \hat{\beta}_{0|0} + K_1(\tilde{z}_t - \gamma - x_1\hat{\beta}_{0|0}) = K_1\tilde{z}_t - K_1\gamma = \hat{\beta}_{1|1} + g_1\gamma,$$

where $\hat{\beta}_{1|1}$ and $P_{1|1}$ are the same as in (A.45) and $g_1 = -K_1$. Assuming that (A.43) holds up to $t - 1$ and based on the Kalman filter for a model with arbitrary $\gamma$, we obtain at time-point $t$:

$$\beta_t \mid \tilde{z}', \omega, \gamma, \theta_1, \ldots, \theta_d \sim N_d \left( \hat{\beta}_{t|t}(\gamma), P_{t|t} \right),$$

$$\hat{\beta}_{t|t}(\gamma) = \hat{\beta}_{t-1|t-1}(\gamma) + K_t(\tilde{z}_t - \gamma - x_t\hat{\beta}_{t-1|t-1}(\gamma)) = (I - K_t x_t)\hat{\beta}_{t-1|t-1}(\gamma) + K_t\tilde{z}_t - K_t\gamma =$$

$$= (I - K_t x_t)\hat{\beta}_{t-1|t-1} + K_t\tilde{z}_t + (I - K_t x_t)g_{t-1}\gamma - K_t\gamma = \hat{\beta}_{t|t} + g_t\gamma,$$

where $\hat{\beta}_{t|t}$ and $P_{t|t}$ are the same as in (A.45) and $g_t$ satisfies recursion (A.46).

To derive the likelihood $p(\tilde{z} \mid \omega, \gamma, \theta_1, \ldots, \theta_d)$, we exploit the well-known representation of the likelihood as a product of the one-step-ahead predictive densities resulting from Kalman filtering:

$$p(\tilde{z} \mid \omega, \gamma, \theta_1, \ldots, \theta_d) = \prod_{t=1}^T p(\tilde{z}_t \mid \tilde{z}_{t-1}, \omega, \gamma, \theta_1, \ldots, \theta_d) \propto \exp \left\{ -\frac{1}{2} \sum_{t=1}^T \frac{(\tilde{z}_t - \hat{\tilde{z}}_{t|t-1}(\gamma))^2}{S_{t|t-1}} \right\}.$$

Since the mean of the one-step-ahead predictive distribution is given by

$$\hat{\tilde{z}}_{t|t-1}(\gamma) = x_t\hat{\beta}_{t-1|t-1}(\gamma) = \hat{\tilde{z}}_{t-1} + x_t g_{t-1}\gamma,$$
while the variance is the same as for $\gamma = 0$, we obtain:

$$p(\tilde{z} | \omega, \gamma, \theta_1, \ldots, \theta_d) \propto \exp \left\{ -\frac{1}{2} \sum_{t=1}^{T} \frac{(\tilde{z}_t - \hat{z}_{t|t-1} - x_t g_t^\gamma)^2}{S_{t|t-1}} \right\},$$

where $\hat{z}_{t|t-1}$ and $S_{t|t-1}$ are defined in (A.45). Combining this likelihood with the Gaussian working prior $p(\gamma)$, yields the conditional Gaussian posterior $\gamma | \omega, \tilde{z}, \theta_1, \ldots, \theta_d \sim \mathcal{N}(G_T m_{\gamma}, G_T)$ where:

$$G_T = \left( G_0^{-1} + \sum_{t=1}^{T} \frac{(x_t g_t^\gamma)^2}{S_{t|t-1}} \right)^{-1}, \quad m_{\gamma} = \sum_{t=1}^{T} \frac{(\tilde{z}_t - \hat{z}_{t|t-1}) x_t g_t^\gamma}{S_{t|t-1}}.$$

Since the choice equation in (A.42) depends on $\gamma$, $p(\gamma | \omega, \tilde{z}, \theta_1, \ldots, \theta_d)$ has to be combined with the likelihood $p(y | \gamma, \tilde{z})$ of the observed outcomes $y = (y_1, \ldots, y_T)$ as before to define the posterior $p(\gamma | \omega, \tilde{z}, \theta_1, \ldots, \theta_d, y)$:

$$\gamma | \omega, \tilde{z}, \theta_1, \ldots, \theta_d, y \sim \mathcal{N}(G_T m_{\gamma}, G_T) I\{L(\tilde{\gamma}) \leq \gamma < U(\tilde{\gamma})\}, \quad (A.47)$$

where $L(\tilde{\gamma}) = \max_{t:y_t=0} \tilde{z}_t = \max_{t:y_t=0} z_t + \tilde{\gamma}$ and $U(\tilde{\gamma}) = \min_{t:y_t=1} \tilde{z}_t = \min_{t:y_t=1} z_t + \tilde{\gamma}$. An updated working parameter $\gamma^{\text{new}}$ is sampled from (A.47) and the proposed location-based move is corrected by defining the shifted utilities $z_t^L = \tilde{z}_t - \gamma^{\text{new}} = z_t + \tilde{\gamma} - \gamma^{\text{new}}$.

This location-based move is followed by a scale-based expansion, using an inverse gamma distribution, $\mathcal{G}^{-1}(d_0, D_0)$, as working prior $p(\delta)$. $\tilde{\delta}$ is sampled from $p(\delta)$ to propose, for each $t = 1, \ldots, T$, a scale move $\tilde{z}_t = \sqrt{\delta} z_t^L$ in the expanded state space model

$$y_t = 1 \Leftrightarrow \tilde{z}_t > 0, \quad \tilde{z}_t = \sqrt{\delta} x_t \beta_t + \sqrt{\delta} \varepsilon_t,$$

while the transition equation (31) and the initial distribution $\beta_0 \sim \mathcal{N}_d(0, P_{00})$ remain the same. From the Pólya-Gamma mixture representation of the error terms $\varepsilon_t$, it follows that
the observation equation takes the form

\[ p(\tilde{z}_t | \omega_t, \delta, \beta_t) \propto \frac{1}{\sqrt{\delta}} \exp \left\{ -\frac{\omega_t}{2} \left( \frac{\tilde{z}_t}{\sqrt{\delta}} - x_t \beta_t \right)^2 \right\}. \]

Again, in combination with the transition density (31) and the initial distribution \( \beta_0 \sim N_d(0, P_{0|0}) \), a conditionally Gaussian state space model is obtained and the Kalman filter can be applied to determine the moments of the filtering density conditional on \( \delta \), given \( \tilde{z}^t \). These moments can be expressed as:

\[ \beta_t | \tilde{z}^t, \omega, \delta, \theta_1, \ldots, \theta_d \sim N_d \left( \frac{1}{\sqrt{\delta}} \hat{\beta}_{t|t}, P_{t|t} \right), \tag{A.48} \]

where \( \hat{\beta}_{t|t} \) and \( P_{t|t} \) are the moments of the filtering density \( \beta_t | \delta = 1, \tilde{z}^t, \omega, \theta_1, \ldots, \theta_d \sim N_d \left( \hat{\beta}_{t|t}, P_{t|t} \right) \) for the specific state space model where \( \delta = 1 \). This model takes the same form as in (A.44), however with a different outcome variable \( \tilde{z}_t \) than before. Its moments are given by the Kalman filter outlined in (A.45), where all (co)variances, i.e. \( P_{t|t-1}, P_{t|t}, \) and \( S_{t|t-1} \), and the Kalman gain \( K_t \) are the same as for the location boost, whereas \( \hat{\beta}_{t|t} \) and \( \hat{z}_{t|t-1} \) depend on \( \tilde{z}_t \) and have to be recomputed.

The representation (A.48) is easy to prove. Based on the Kalman filter for a model with arbitrary \( \delta \), we obtain for \( t = 1 \):

\[ \beta_1 | \tilde{z}^1, \omega, \delta, \theta_1, \ldots, \theta_d \sim N_d \left( \hat{\beta}_{1|1}(\delta), P_{1|1} \right), \]

\[ \hat{\beta}_{1|1}(\delta) = \hat{\beta}_{0|0} + K_1(\tilde{z}_1/\sqrt{\delta} - x_1 \hat{\beta}_{0|0}) = K_1 \tilde{z}_1/\sqrt{\delta} = \frac{1}{\sqrt{\delta}} \hat{\beta}_{1|1}, \]

where \( \hat{\beta}_{1|1} \) and \( P_{1|1} \) are the moments for \( \delta = 1 \). Assuming that (A.48) holds up to \( t - 1 \)
and based on the Kalman filter for a model with arbitrary $\delta$, we obtain for time $t$:

$$
\beta_t | \tilde{z}^t, \omega, \delta, \theta_1, \ldots, \theta_d \sim \mathcal{N}_d \left( \hat{\beta}_{t|t}(\delta), P_{t|t} \right),
$$

$$
\hat{\beta}_{t|t}(\delta) = \hat{\beta}_{t-1|t-1}(\delta) + K_t (\tilde{z}_t / \sqrt{\delta} - x_t \hat{\beta}_{t-1|t-1}(\delta))
$$

$$
= \frac{1}{\sqrt{\delta}} \left( \hat{\beta}_{t-1|t-1} + K_t (\tilde{z}_t - x_t \hat{\beta}_{t-1|t-1}) \right) = \frac{1}{\sqrt{\delta}} \hat{\beta}_{t|t}.
$$

To derive the likelihood $p(\tilde{z} | \omega, \delta, \theta_1, \ldots, \theta_d)$, we use once more the product of the one-step-ahead predictive densities resulting from Kalman filtering:

$$
p(\tilde{z} | \omega, \delta, \theta_1, \ldots, \theta_d) = \prod_{t=1}^{T} p(\tilde{z}_t | \tilde{z}_{t-1}, \omega, \delta, \theta_1, \ldots, \theta_d).
$$

Since the one-step-ahead predictive distribution is given by:

$$
\tilde{z}_t | \tilde{z}_{t-1}, \omega, \delta, \theta_1, \ldots, \theta_d \sim \mathcal{N} \left( \hat{z}_{t|t-1}(\delta), S_{t|t-1}(\delta) \right),
$$

where

$$
\hat{z}_{t|t-1}(\delta) = \sqrt{\delta} x_t \hat{\beta}_{t-1|t-1}(\delta) = x_t \hat{\beta}_{t-1|t-1} = \hat{z}_{t|t-1}, \quad S_{t|t-1}(\delta) = \delta S_{t|t-1},
$$

we obtain:

$$
p(\tilde{z} | \omega, \delta, \theta_1, \ldots, \theta_d) \propto \left( \frac{1}{\delta} \right)^{T/2} \exp \left\{ -\frac{1}{2\delta} \sum_{t=1}^{T} \frac{(\tilde{z}_t - \hat{z}_{t|t-1})^2}{S_{t|t-1}} \right\}.
$$

Combining this likelihood with the inverse gamma working prior, the posterior $\delta | \tilde{z}, \omega, \theta_1, \ldots, \theta_d \sim \mathcal{G}^{-1}(d_T, D_T)$ with the following moments results:

$$
d_T = d_0 + \frac{T}{2}, \quad D_T = D_0 + \frac{1}{2} \sum_{t=1}^{T} \frac{(\tilde{z}_t - \hat{z}_{t|t-1})^2}{S_{t|t-1}}. \quad \text{(A.49)}
$$
An updated working parameter $\delta_{\text{new}}$ is sampled from (A.49) and the proposed scale-based move is corrected by defining the rescaled utilities $z_t^{LS} = \tilde{z}_t / \sqrt{\delta_{\text{new}}} = \sqrt{\delta / \delta_{\text{new}}} \tilde{z}_t$ for all $t$. This concludes the boosting step.

The state process $\beta_0, \ldots, \beta_T$ is then sampled conditional on $z^{LS}$ from the smoothing density $p(\beta_0, \ldots, \beta_T | z^{LS}, \omega, \theta_1, \ldots, \theta_d)$ of the boosted state space model

$$
\beta_t = \beta_{t-1} + w_t, \quad w_t \sim N_d(0, Q), \quad z_t^{LS} = x_t \beta_t + \varepsilon_t \quad (A.50)
$$

using FFBS (Frühwirth-Schnatter 1994). It is easy to verify that the moments of the filtering density in the boosted state space model (A.50) are identical to

$$
\beta_t | z_t^{LS,t}, \omega, \theta_1, \ldots, \theta_d \sim N_d \left( \frac{1}{\sqrt{\delta_{\text{new}}}} \hat{\beta}_{t|t}, P_{t|t} \right), \quad (A.51)
$$

where $z_t^{LS,t} = (z_1^{LS}, \ldots, z_t^{LS})$. Indeed, $\hat{\beta}_{t|t}(\delta_{\text{new}}) = K_1 \tilde{z}_1 / \sqrt{\delta_{\text{new}}} = K_1 z_1^{LS}$ is equal to the mean of the filtering density at $t = 1$. Assuming this identity holds up to $t - 1$, we obtain that

$$
\hat{\beta}_{t|t}(\delta_{\text{new}}) = \hat{\beta}_{t-1|t-1}(\delta_{\text{new}}) + K_t (\tilde{z}_t / \sqrt{\delta_{\text{new}}} - x_t \hat{\beta}_{t-1|t-1}(\delta_{\text{new}})) \]
$$

$$
= \hat{\beta}_{t-1|t-1}(\delta_{\text{new}}) + K_t (z_t^{LS} - x_t \hat{\beta}_{t-1|t-1}(\delta_{\text{new}}))
$$

is equal to the mean of the filtering density at time $t$.

Hence, the moments $\hat{\beta}_{t|t}$ and $P_{t|t}$ of the Kalman filter (A.48) underlying the scale-based parameter expansion can be recycled in the backward sampling step of FFBS. Starting with a draw $\beta_T$ from the filter density

$$
\beta_T | z_T^{LS,T}, \omega, \theta_1, \ldots, \theta_d \sim N_d \left( \frac{1}{\sqrt{\delta_{\text{new}}}} \hat{\beta}_{T|T}, P_{T|T} \right), \quad (A.52)
$$

31
the state $\beta_t$ is sampled backwards in time for $t = T - 1, \ldots, 0$ using\footnote{The conditional density $p(\beta_t | \beta_{t+1}, \ldots, \beta_T, z^{LS}, \omega, \theta)$ is proportional to}

$$\beta_t | \beta_{t+1}, \ldots, \beta_T, z^{LS}, \omega, \theta \sim N_d \left( \hat{\beta}_{t|T} (\beta_{t+1}), P_{t|T} \right),$$

(A.53)

$$\hat{\beta}_{t|T} (\beta_{t+1}) = \frac{1}{\sqrt{\delta_{new}}} (I - B_{t+1}) \hat{\beta}_{t|t} + B_{t+1} \beta_{t+1},$$

$$P_{t|T} = (I - B_{t+1}) P_{t|t}, \quad B_{t+1} = P_{t|t} (P_{t|t} + \text{Diag} (\theta_1, \ldots, \theta_d))^{-1}.$$

Finally, the unknown variances $\theta_1, \ldots, \theta_d$ are updated conditional on the state process $\beta_0, \ldots, \beta_T$ and $z^{LS}$. This requires the choice of a prior $p(\theta_j)$ and the inverse gamma prior $\theta_j \sim \mathcal{G}^{-1}(c_0, C_0)$ is used for illustration. However, this prior is easily substituted by variance selection priors, such as the triple gamma prior (Cadonna et al. 2020).

The ultimate Pólya-Gamma sampler for binary SSMs is summarized in Algorithm 4.

A.6 Additional Simulation Results

This section provides some additional simulation results for probit, logit, multinomial and binomial regression models. In addition, a numerical study on mixture-of-experts models is presented. The results for the probit model are discussed in detail in Appendix A.6.1.

These moments can be expressed as in (A.53).
Algorithm 4 The ultimate Pólya-Gamma sampler for binary state space models.

Choose starting values for $(\theta_1, \ldots, \theta_d)$, and $\lambda_1, \ldots, \lambda_T$, where $\lambda_t = x_t \beta_t$ and repeat the following steps:

(Z) For each $t = 1, \ldots, T$, sample $z_t = \log \lambda_t + F^{-1}_c(y_t + U_t(1 - y_t - \pi_t))$ in the SSM (34), where $U_t \sim U[0,1]$ and $\pi_t = F_z(\log \lambda_t)$. Sample $\omega_t | z_t, \lambda_t, y_t \sim \mathcal{PG}(2, |z_t - \log \lambda_t|)$ for the logit SSM.

(B-L) Location-based parameter expansion: sample $\tilde{\gamma} \sim N(0, G_0)$ and propose $\tilde{z}_t = z_t + \tilde{\gamma}$ for $t = 1, \ldots, T$. Sample $\gamma^{\text{new}}$ from $\gamma | \omega, \tilde{z}, \theta_1, \ldots, \theta_d, y$ given in (A.47) and define the shifted utilities $z^L_t = \tilde{z}_t - \gamma^{\text{new}} = z_t + \tilde{\gamma} - \gamma^{\text{new}}$.

(B-S) Scale-based parameter expansion: sample $\tilde{\delta} \sim G^{-1}(d_0, D_0)$ and propose $\tilde{z}_t = \sqrt{\tilde{\delta}} z^L_t$ for all $t$. Sample $\delta^{\text{new}}$ from the inverse Gamma density $\delta | \tilde{z}, \omega, \theta_1, \ldots, \theta_d$ given in (A.49). Define rescaled utilities $z^{LS}_t = \tilde{z}_t / \sqrt{\delta^{\text{new}}} = \sqrt{\tilde{\delta} / \delta^{\text{new}}} z^L_t$ for all $t$.

(F) Sample the state process $\beta_0, \ldots, \beta_T | z^{LS}, \omega, \theta_1, \ldots, \theta_d$ using backward-sampling, based on the Kalman filter from step (B-S), see (A.52) and (A.53).

(P) Sample $\theta_j | \{\beta_{jt}\} \sim G^{-1}(c_0 + T/2, C_j)$, for $j = 1, \ldots, d$, where

$$C_j = C_0 + \frac{1}{2} \sum_{t=1}^T (\beta_{jt} - \beta_{jt-1})^2.$$
Figure A.2: Log median inefficiency factors across 100 simulation runs for binary probit models.

(a) Varying sample sizes with two successes.

(b) Varying Intercepts with \( N = 1,000 \).

Selected tabulated simulation results for all models are given in Appendix A.6.2. The mixture-of-experts results are given in Appendix A.6.3.

### A.6.1 Probit Results

For binary probit models, we compare the iMDA approach outlined in this article (UPG) to an adaptive Metropolis-Hastings sampler (AMH), the MDA scheme from Liu & Wu (1999) (PX-DA), as well as the original probit Gibbs sampler outlined in Albert & Chib (1993) (A&C). The results for are summarized visually in Figure A.2 and numerical results are provided in Table A.1. It is obvious that the commonly used standard DA approach from A&C has severe efficiency problems when increasing the level of imbalancedness of the data. In these cases, increasing inefficiency can also be observed for the PX-DA algorithm. On the other hand, constantly high levels of efficiency are characterizing both UPG and
the AMH algorithm.

A.6.2 Tabulated Simulation Results

This subsection provides some additional tables summarizing results from various simulation studies. Table A.1 to Table A.4 give detailed results on sampling efficiency for probit as well as binary, multinomial and binomial logistic regression models in scenarios with changing sample size $N$. For the probit case, we choose the PX-DA algorithm of Liu & Wu (1999) as benchmark, while the algorithms of Polson et al. (2013) serve as baseline for the logistic regression algorithms. The results of all other competing models are reported relative to these benchmarks for ease of interpretation.
Table A.1: Varying Sample Sizes for Binary Probit Models

| Binary Probit | N = 10 | N = 100 | N = 1,000 | N = 5,000 | N = 10,000 | N = 50,000 |
|---------------|-------|---------|-----------|-----------|-----------|-----------|
| PX-DA         | ESS^B | 4451.285| 2120.910  | 502.611   | 164.747   | 98.851    | 29.174    |
|               | IE^B  | 2.247   | 4.715     | 19.896    | 60.699    | 101.162   | 342.791   |
| UPG           | ESS / ESS^B | 1.546 | 1.735     | 3.341     | 7.638     | 11.693    | 36.785    |
|               | IE / IE^B | 0.647 | 0.576     | 0.299     | 0.131     | 0.086     | 0.027     |
| AMH           | ESS / ESS^B | 0.501 | 1.006     | 4.088     | 12.355    | 20.265    | 68.240    |
|               | IE / IE^B | 1.998 | 0.994     | 0.245     | 0.081     | 0.049     | 0.015     |
| A&C           | ESS / ESS^B | 0.707 | 0.278     | 0.193     | 0.164     | 0.153     | 0.217     |
|               | IE / IE^B | 1.414 | 3.599     | 5.185     | 6.083     | 6.534     | 4.608     |

*Note:* Values are medians across 100 simulation runs. Every simulated dependent variable has two successes. IE = inefficiency factors, ESS = effective sample size. IE^B and ESS^B correspond to the simulation results of the benchmark sampler PX-DA of Liu & Wu (1999). Results for all other samplers are reported relative to this benchmark. Values of, respectively, ESS / ESS^B > 1 and IE / IE^B < 1 indicate superiority compared to the benchmark.
Table A.2: Varying Sample Sizes for Binary Logit Models

| Binary Logit | N = 10     | N = 100    | N = 1,000  | N = 5,000  | N = 10,000 | N = 50,000 |
|--------------|------------|------------|------------|------------|------------|------------|
| PSW          | ESS<sup>B</sup> | 5851.204   | 895.380    | 146.321    | 39.673     | 24.457     | 8.027      |
|              | IE<sup>B</sup> | 1.709      | 11.168     | 68.343     | 252.062    | 408.884    | 1245.794   |
| UPG          | ESS / ESS<sup>B</sup> | 0.901      | 2.802      | 9.649      | 29.521     | 46.139     | 132.403    |
|              | IE / IE<sup>B</sup> | 1.110      | 0.357      | 0.104      | 0.034      | 0.022      | 0.008      |
| AMH          | ESS / ESS<sup>B</sup> | 0.266      | 1.687      | 10.304     | 37.911     | 61.811     | 187.009    |
|              | IE / IE<sup>B</sup> | 3.763      | 0.593      | 0.097      | 0.026      | 0.016      | 0.005      |
| FSF          | ESS / ESS<sup>B</sup> | 0.367      | 0.270      | 0.209      | 0.228      | 0.231      | 0.378      |
|              | IE / IE<sup>B</sup> | 2.723      | 3.707      | 4.782      | 4.378      | 4.322      | 2.643      |

*Note:* Values are medians across 100 simulation runs. Every simulated dependent variable has two successes. IE = inefficiency factors, ESS = effective sample size. IE<sup>B</sup> and ESS<sup>B</sup> correspond to the simulation results of the benchmark sampler PSW. Results for all other samplers are reported relative to this benchmark. Values of, respectively, ESS / ESS<sup>B</sup> > 1 and IE / IE<sup>B</sup> < 1 indicate superiority compared to the benchmark.
| Multinomial Logit | N = 10 | N = 100 | N = 1,000 | N = 5,000 | N = 10,000 | N = 50,000 |
|------------------|--------|---------|-----------|-----------|-----------|-----------|
| PSW              | ESS\textsuperscript{B} | 5148.712 | 898.435   | 148.772   | 40.204    | 24.609    | 7.654     |
|                  | IE\textsuperscript{B}  | 1.943    | 11.154    | 67.690    | 252.732   | 424.122   | 1414.031  |
| UPG              | ESS / ESS\textsuperscript{B} | 0.905    | 2.779     | 9.422     | 28.961    | 45.219    | 138.636   |
|                  | IE / IE\textsuperscript{B} | 1.105    | 0.359     | 0.106     | 0.034     | 0.021     | 0.007     |
| AMH              | ESS / ESS\textsuperscript{B} | 0.286    | 1.734     | 10.454    | 39.047    | 63.901    | 206.778   |
|                  | IE / IE\textsuperscript{B} | 3.503    | 0.576     | 0.095     | 0.025     | 0.015     | 0.004     |
| FSF              | ESS / ESS\textsuperscript{B} | 0.372    | 0.277     | 0.211     | 0.218     | 0.248     | 0.417     |
|                  | IE / IE\textsuperscript{B} | 2.686    | 3.638     | 4.947     | 4.912     | 4.224     | 2.435     |

Note: Values are medians across 100 simulation runs. Every simulated dependent variable has two successes. IE = inefficiency factors, ESS = effective sample size. IE\textsuperscript{B} and ESS\textsuperscript{B} correspond to the simulation results of the benchmark sampler PSW. Results for all other samplers are reported relative to this benchmark. Values of, respectively, ESS / ESS\textsuperscript{B} > 1 and IE / IE\textsuperscript{B} < 1 indicate superiority compared to the benchmark.
### Table A.4: Varying Sample Sizes for Binomial Logit Models

| Binomial Logit | N = 10       | N = 100      | N = 1,000    | N = 5,000    | N = 10,000   |
|---------------|--------------|--------------|--------------|--------------|--------------|
| PSW           | ESS\(^B\)   | 1556.512     | 248.392      | 40.558       | 11.797       | 8.808        |
|               | IE\(^B\)    | 6.425        | 40.259       | 246.568      | 847.677      | 1135.303     |
| UPG           | ESS / ESS\(^B\) | 2.314        | 8.137        | 32.121       | 96.350       | 123.117      |
|               | IE / IE\(^B\) | 0.432        | 0.123        | 0.031        | 0.010        | 0.008        |
| AMH           | ESS / ESS\(^B\) | 0.988        | 6.076        | 37.598       | 128.136      | 172.595      |
|               | IE / IE\(^B\) | 1.012        | 0.165        | 0.027        | 0.008        | 0.006        |
| AuxMix        | ESS / ESS\(^B\) | 0.260        | 0.198        | 0.202        | 0.330        | 0.342        |
|               | IE / IE\(^B\) | 3.853        | 5.054        | 4.961        | 3.031        | 2.923        |

*Note: Values are medians across 100 simulation runs. Every simulated dependent variable has two successes. IE = inefficiency factors, ESS = effective sample size. IE\(^B\) and ESS\(^B\) correspond to the simulation results of the benchmark sampler PSW. Results for all other samplers are reported relative to this benchmark. Values of, respectively, ESS / ESS\(^B\) > 1 and IE / IE\(^B\) < 1 indicate superiority compared to the benchmark.*
Experts Gating Function Covariate Intercept Covariate Intercept
2.5
3.0
3.5
4.0
4.5
1.4
1.6
1.8
2.0
2.2
\text{Ln(Inefficiency Factors)}

Figure A.3: Log inefficiency factors for simulated mixture-of-experts data. \textit{UPG} is a data augmentation scheme with location-based and scale-based parameter expansion steps.

A.6.3 Mixture-of-Experts Simulations

To evaluate whether the proposed MDA scheme can be expected to increase efficiency in complex settings such as mixture-of-experts models, we conduct a numerical experiment as follows. As in the application presented in Section 6.2, we consider logistic regression experts and a multinomial logistic gating function. We simulate $H = 3$ mixture components and $N = 1,000$ units that are observed $N_i = 20$ times each. Both for the gating function and the logistic regression experts, an intercept term and a $\mathcal{N}(0,1)$ covariate are used to simulate the respective outcomes. The regression coefficients of the gating function are $\psi_1 = (-4,2)^\prime$, $\psi_2 = (-4,-2)^\prime$ and $\psi_3 = (0,0)^\prime$. The regression parameters in the logistic regression experts are set to $\beta_1 = (-4,2.5)^\prime$, $\beta_2 = (4,2.5)^\prime$ and $\beta_3 = (0,0)^\prime$. This corresponds to a setup where both the classification and the success probabilities within the logistic regression components are relatively imbalanced. Specifically, the class membership probabilities that the gating function produces are on average between 6% and 7% for the first two categories and 87% for the baseline category. Within the logistic experts, the first
category has an average success probability of around 10%, the second one of around 90% and the baseline category is balanced with 50% success probability.

We simulate 50 replicate data sets using these parameters and estimate the model once with and once without iMDA, collecting inefficiency factors in each simulation run. Throughout the simulation runs, we assume independent \( \mathcal{N}(0, 4) \) priors for all elements of \( \psi_1, \psi_2, \beta_1, \beta_2 \) and \( \beta_3 \). Figure A.3 summarizes the results of these 50 runs and shows that the proposed MDA scheme offers significant performance gains when compared to a plain DA sampler. These gains are expected to become larger as the data becomes more imbalanced.

Finally, it is worth to note that mixture-of-experts models with logistic regression experts potentially suffer from inefficiencies stemming from two sources. First, the degree of dependency of the parameters during sampling is potentially high due to class membership being dependent on the parameters in the gating function and vice versa. Second, there may be inefficiencies due to imbalanced outcomes, either in the gating function or the logistic experts or both. The iMDA scheme introduced in the present article is concerned with the latter type of inefficiency.

A.7 Mixture-of-experts illustration using child mortality data

The goal of this illustration is to examine heterogeneity and non-linearities in the relationship of child mortality and maternal education in developing countries. For this, we construct a large data set on \( N = 99,641 \) births in eight countries in sub-Saharan Africa using household survey data from the Demographic and Health Survey (DHS) program. For each birth, the survey data indicates whether the child died before its fifth birthday. To measure maternal education, a categorical indicator on whether a child’s mother has
no formal education or achieved a primary, secondary or tertiary level of education is extracted. The data itself has been collected at $J = 5,558$ distinct geographical locations. At an average location, around 18 children are observed, and a location can be thought of as, e.g., a small village or a neighborhood within a city. The data set contains information on malaria incidence and urban/rural status for all $J$ survey clusters. All data is publicly available from IPUMS-DHS [Heger Boyle et al. 2020].

We denote the mortality status of child $i = 1, \ldots, N$ as $y_i$ and use $C_i = j$ to indicate that child $i$ has been observed at location $j$. To examine potential non-linearities in the relationship of child mortality and maternal education, we assume that $y_i$ can be modeled using a $H$-component mixture ($h = 1, \ldots, H$) of logistic regression models

$$y_i \mid C_i = j \sim \sum_{h=1}^{H} \eta_{jh}(w_j)\text{Ber}(\zeta_{ih}(x_i)). \quad (A.54)$$

In model (A.54), where $\sum_{h=1}^{H} \eta_{jh}(w_j) = 1$, $\eta_{jh}(w_j)$ is the probability that location $j$ is a member of mixture component $h$. The corresponding categorical class membership indicator $S_j$ takes on values $h = 1, \ldots, H$ and is modeled as the outcome of a multinomial logistic regression model such that

$$\Pr(S_j = h \mid w_j) = \eta_{jh}(w_j) = \frac{\exp(w_j \psi_h)}{\sum_{i=1}^{H} \exp(w_j \psi_i)}, \quad (A.55)$$

where the regression parameters $\psi_h$ of one category are set to zero for identification purposes. The covariate vector $w_j$ contains an intercept, the malaria incidence at the survey location in the closest available year to the survey year, as well as a binary variable indicating whether a survey cluster is located in an urban area. In addition, to let country-specific factors such as the effectiveness of public health care systems influence class membership, binary vectors indicating whether the cluster is located in Ethiopia, Kenya, Malawi,
Mozambique, Rwanda, Tanzania or Zimbabwe are included, with the DR Congo serving as baseline. The country-specific intercepts also control for the fact that the surveys in the countries have been conducted in different years.

For each component $h$ in model (A.54), $\zeta_{ih}(x_i)$ is the mortality rate, i.e., the probability of a child $i$ dying before the age of five, conditional on survey location $j$ being a member of mixture component $h$ (i.e. $S_j = h$). $\zeta_{ih}(x_i)$ is modeled as a function of maternal education using the logistic link

$$\zeta_{ih}(x_i) = \Pr(y_i = 1 \mid C_i = j, S_j = h, x_i) = \frac{\exp(\beta_{0h} + \sum_{s=1}^{3} \beta_{sh}EDUC_s)}{1 + \exp(\beta_{0h} + \sum_{s=1}^{3} \beta_{sh}EDUC_s)}, \tag{A.56}$$

where $x_i = (EDUC_1^i, EDUC_2^i, EDUC_3^i)$ are binary indicators for primary, secondary and tertiary education level of the mother of child $i$ and no formal education serves as reference category.

Both the component-specific logistic regressions and the multinomial logistic regression that serves as class membership prior are estimated using the methodology introduced in this paper, with $\mathcal{N}(0, 4)$ priors on all regression parameters. We examine models of order $H = 1, \ldots, 7$ and use the full mixture likelihood to determine the appropriate number of components. The BIC selects $H = 2$ as the most suitable number of components, and we discuss results for this case in more detail below. Posterior estimates are based on 25,000 posterior draws after a burn-in period of 10,000 iterations using a random permutation sampler to deal with label switching. Identification of the mixture parameters is achieved via k-means clustering of the posterior draws in the point process representation, see for instance Malsiner Walli et al. (2016) for more details. In terms of sampling efficiency, the effective sample sizes of the parameters of the logistic experts and the gating function

---

9For a formal discussion on model selection in mixture models, refer to Celeux et al. (2019).
Figure A.4: (left) Component-specific child mortality estimates by maternal education groups. (right) Estimated gating function coefficients. Baseline category is the low mortality component. Uncertainty bounds correspond to 95% credible intervals, and the dotted line in the left panel is the sample average mortality rate.

Panel (a) in Figure A.4 shows the estimated relationship of maternal education and under five mortality rates in the two mixture components. The 'low mortality' component exhibits relatively low mortality rates that do not systematically decrease with maternal education. The 'high mortality' component is characterized by significantly higher mortality rates and a pronounced, negative relationship of maternal education and child mortality. Panel (b) in Figure A.4 shows posterior summaries of the gating function coefficients $\psi_h$ that determine the mixture weights $\eta_{jh}(w_j)$. The baseline category corresponds to the high mortality component. The most striking observation is that a high malaria incidence at the
survey cluster location strongly decreases the odds of being a member of the low mortality component. This is in line with the fact that malaria is particularly dangerous for children under five, who for instance accounted for 67% of malaria deaths worldwide in 2019 ([World Health Organization][2020]). A potential channel behind the implied significantly negative relationship of maternal education and mortality in the high mortality component is that maternal education is a critical factor in preventing malaria infections in young children ([Njau et al., 2014](#)).

**Supplementary References**

Albert, J. H. & Chib, S. (1993), ‘Bayesian analysis of binary and polychotomous response data’, *Journal of the American Statistical Association* **88**, 669–679.

Bierkens, J., Fearnhead, P. & Roberts, G. (2019), ‘The zig-zag process and super-efficient sampling for Bayesian analysis of big data’, *The Annals of Statistics* **47**(3), 1288–1320.

Cadonna, A., Frühwirth-Schnatter, S. & Knaus, P. (2020), ‘Triple the gamma – A unifying shrinkage prior for variance and variable selection in sparse state space and TVP models’, *Econometrics* **8**, 20.

Carpenter, B., Gelman, A., Hoffman, M. D., Lee, D., Goodrich, B., Betancourt, M., Brubaker, M., Guo, J., Li, P. & Riddell, A. (2017), ‘Stan: A probabilistic programming language’, *Journal of Statistical Software* **76**, 1–32.

Celeux, G., Frühwirth-Schnatter, S. & Robert, C. P. (2019), Model selection for mixture models – perspectives and strategies, in S. Frühwirth-Schnatter, G. Celeux & C. P.
Robert, eds, ‘Handbook of Mixture Analysis’, CRC Press, Boca Raton, FL, chapter 7, pp. 117–154.

Durante, D. (2019), ‘Conjugate Bayes for probit regression via unified skew-normal distributions’, *Biometrika* 106(4), 765–779.

Fearnhead, P., Bierkens, J., Pollock, M. & Roberts, G. O. (2018), ‘Piecewise deterministic Markov processes for continuous-time Monte Carlo’, *Statistical Science* 33(3), 386–412.

Frühwirth-Schnatter, S. (1994), ‘Data augmentation and dynamic linear models’, *Journal of Time Series Analysis* 15, 183–202.

Heger Boyle, E., King, M. & Sobek, M. (2020), ‘IPUMS Demographic and Health Surveys. Version 8 [dataset]’, Minneapolis, MN: IPUMS and ICF.

Hird, M., Livingstone, S. & Zanella, G. (2020), ‘A fresh take on’Barker dynamics’ for MCMC’, *arXiv preprint arXiv:2012.09731*.

Hoffman, M. D. & Gelman, A. (2014), ‘The No-U-Turn sampler: adaptively setting path lengths in Hamiltonian Monte Carlo.’, *J. Mach. Learn. Res.* 15(1), 1593–1623.

Liu, J. S. & Wu, Y. N. (1999), ‘Parameter expansion for data augmentation’, *Journal of the American Statistical Association* 94, 1264–1274.

Livingstone, S. & Zanella, G. (2022), ‘The Barker proposal: Combining robustness and efficiency in gradient-based MCMC’, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 84(2), 496–523.

Malsiner Walli, G., Frühwirth-Schnatter, S. & Grün, B. (2016), ‘Model-based clustering based on sparse finite Gaussian mixtures’, *Statistics and Computing* 26, 303–324.
Mangoubi, O., Pillai, N. S. & Smith, A. (2018), ‘Does Hamiltonian Monte Carlo mix faster than a random walk on multimodal densities?’, arXiv preprint arXiv:1808.03230.

McFadden, D. (1974), Conditional logit analysis of qualitative choice behaviour, in P. Zarembka, ed., ‘Frontiers of Econometrics’, Academic, New York, pp. 105–142.

Nishimura, A., Dunson, D. B. & Lu, J. (2020), ‘Discontinuous Hamiltonian Monte Carlo for discrete parameters and discontinuous likelihoods’, Biometrika 107(2), 365–380.

Njau, J. D., Stephenson, R., Menon, M. P., Kachur, S. P. & McFarland, D. A. (2014), ‘Investigating the important correlates of maternal education and childhood malaria infections’, The American Journal of Tropical Medicine and Hygiene 91(3), 509–519.

Polson, N. G., Scott, J. G. & Windle, J. (2013), ‘Bayesian inference for logistic models using Pólya-Gamma latent variables’, Journal of the American Statistical Association 108, 1339–49.

Robert, C. P. & Casella, G. (1999), Monte Carlo Statistical Methods, Springer Series in Statistics, Springer, New York/Berlin/Heidelberg.

Sen, D., Sachs, M., Lu, J. & Dunson, D. B. (2020), ‘Efficient posterior sampling for high-dimensional imbalanced logistic regression’, Biometrika 107(4), 1005–1012.

Vanetti, P., Bouchard-Côté, A., Deligiannidis, G. & Doucet, A. (2017), ‘Piecewise-deterministic Markov chain Monte Carlo’, arXiv preprint arXiv:1707.05296.

World Health Organization (2020), ‘World Malaria Report 2020: 20 Years of Global Progress and Challenges’, Licence: CC BY-NC-SA 3.0 IGO.