ROKHLIN DIMENSION AND $C^*$-DYNAMICS

ILAN HIRSHBERG, WILHELM WINTER, AND JOACHIM ZACHARIAS

Abstract. We develop the concept of Rokhlin dimension for integer and for finite group actions on $C^*$-algebras. Our notion generalizes the so-called Rokhlin property, which can be thought of as Rokhlin dimension 0. We show that finite Rokhlin dimension is prevalent and appears in cases in which the Rokhlin property cannot be expected: the property of having finite Rokhlin dimension is generic for automorphisms of $\mathbb{Z}$-stable $C^*$-algebras, where $\mathbb{Z}$ denotes the Jiang-Su algebra. Moreover, crossed products by automorphisms with finite Rokhlin dimension preserve the property of having finite nuclear dimension, and under a mild additional hypothesis also preserve $\mathbb{Z}$-stability. Finally, we prove a topological version of the classical Rokhlin lemma: automorphisms arising from minimal homeomorphisms of finite dimensional compact metrizable spaces always have finite Rokhlin dimension.

0. Introduction

An action of a countable discrete group $G$ on a (possibly noncommutative) space $X$ has a Rokhlin property if there are systems of subsets of $X$, indexed by large subsets of $G$,

(i) which are (approximately) pairwise disjoint,
(ii) which (approximately) cover $X$ and
(iii) on which the group action is (approximately) compatible with translation on the index set.

In a sense, Rokhlin properties create a ‘reflection’ of the acting group in the underlying space which often allows for a dynamic viewpoint on properties of the latter. The concrete interpretation of conditions (i), (ii) and (iii) above allows for a certain amount of freedom, and may vary with the applications one has in mind. For a general group it is usually not so clear how to synchronize indexing by large subsets and compatibility of the group action with translation on these subsets. The situation is much less ambiguous if $G$ is finitely generated, in particular for $\mathbb{Z}$ (or $\mathbb{Z}^d$) or for finite groups.

In this paper we will be mostly interested in the noncommutative case, more specifically in actions on $C^*$-algebras. The study of such actions, and their associated crossed products, has always been a central theme in operator algebras, beginning with Murray and von Neumann’s group measure space construction which associates a von Neumann algebra to an action of a group on a measure space. Since then the crossed product construction has been generalized to actions on noncommutative von Neumann and $C^*$-algebras, providing an inexhaustible source of highly nontrivial
examples. It combines in an intricate way dynamical properties of the action and properties of the algebra of coefficients.

A basic, yet important early result in the dynamics of group actions on measure spaces is the Rokhlin Lemma saying that a measure preserving aperiodic (i.e. almost everywhere non-periodic) action of $\mathbb{Z}$ can be approximated by cyclic shifts, in the sense that there exists a finite partition of the space (Rokhlin tower) which is almost cyclically permuted. It can be reformulated as a result about strongly outer automorphisms of commutative von Neumann algebras involving partitions of unity by orthogonal projections (towers of projections) and this formulation has led to the various versions of $C^*$-algebraic Rokhlin properties. (We recall the version which is now most commonly used in Definition 2.1).

As it turns out there are many important examples of actions of finite groups and $\mathbb{Z}$ possessing Rokhlin properties. See, for instance, [15, 8] and references therein for actions of $\mathbb{Z}$, and [4, 9, 10, 20], for the finite group case. In fact, Rokhlin properties, especially for the single automorphism case, are quite prevalent, and indeed as a byproduct we establish in this paper that they are generic for automorphisms of unital $C^*$-algebras which absorb a UHF algebra of infinite type. For related work establishing genericity of Rokhlin type conditions, see [23].

Most currently used $C^*$-algebraic Rokhlin properties involve towers consisting of projections. However, unlike in the case of von Neumann algebras, requiring the existence of projections poses severe restrictions on the coefficient algebra. In particular, by lack of projections, automorphisms of the Jiang-Su algebra $\mathcal{Z}$ (the smallest possible $C^*$-algebraic analogue of the hyperfinite $II_1$ factor $\mathcal{R}$) or automorphisms arising from homeomorphisms of connected spaces will not satisfy most of the current definitions of the Rokhlin property. Moreover, even if there are sufficiently many projections (e.g. in the real rank zero case) Rokhlin properties do not always hold. In the $C^*$-setting they become regularity properties of actions which may be regarded as a strong form of outerness (producing mostly simple $C^*$-algebras).

The main purpose of this paper is to develop generalizations of Rokhlin properties motivated by the idea of covering dimension. Roughly speaking, instead of requiring towers consisting of projections we allow for partitions of unity involving several towers of positive elements, where elements from different towers are no longer required to be orthogonal, but the number of different towers is restricted. (A further requirement, which we need for some results and which is automatic in many important cases, is that these towers commute.) This is analogous to the definition of covering dimension using partitions of unity with functions allowing for controlled overlaps. We like to think of the number of towers as the coloring number, or dimension, of the cover of the dynamical system, the usual Rokhlin property corresponding to the zero dimensional case. It turns out that this generalization provides a much more flexible concept, which applies to a much wider range of examples than the currently used Rokhlin properties. We can even show that in the $\mathcal{Z}$-stable case finite Rokhlin dimension is topologically generic. A very nice and explicit example for which we can show finiteness of our Rokhlin dimension and where the usual Rokhlin property clearly fails are irrational rotation automorphisms of $C(\mathbb{T})$. Our main result in this direction is that automorphisms corresponding to minimal homeomorphisms of compact spaces of finite covering dimension always have finite Rokhlin dimension. This result might be regarded as a topological version of the classical Rokhlin Lemma.
Covering dimension for topological spaces was generalized to the context of C*-algebras in [14] and [34], in which the related concepts of decomposition rank and nuclear dimension were introduced. These are based on placing a uniform bound on the decomposability of c.p. approximations of the identity map (cf. [5]). Simple C*-algebras with finite decomposition rank or finite nuclear dimension have been shown to have significant regularity properties, in particular they are Z-stable ([32, 33]). Whilst these classes have good permanence properties, in particular those with finite nuclear dimension, there are still no general results known involving crossed products. Our original motivation was to find bounds on the decomposition rank and/or nuclear dimension of crossed products, which is indeed possible for Rokhlin actions of finite groups. Moreover it turns out that similar bounds can be established for automorphisms satisfying higher dimensional Rokhlin properties, at least for the nuclear dimension. Combining these results with the finiteness of Rokhlin dimension for minimal systems on compact spaces of finite covering dimension we obtain finiteness of the nuclear dimension of crossed products of such systems. This provides a different proof of the nuclear dimension result in [29].

We note that other generalizations of the Rokhlin property have been considered which are based on the idea of leaving out a remainder which is small in the tracial sense (see [19, 25, 16, 22]).

We also wish to point out that finite Rokhlin dimension can be interpreted as a dynamic version of Kirchberg’s covering number, introduced in [12] (also cf. [13]).

Hiroki Matui and Yasuhiko Sato have informed us that they have a proof that all strongly outer automorphisms of separable, simple, unital, monotracial and Z-stable C*-algebras have Rokhlin dimension at most 1, thus partially generalizing our Theorem 3.4. Their argument is based on the techniques developed in [18].

As a further variation of our Rokhlin dimension one might in addition ask the Rokhlin systems to approximate the underlying algebra; one might also compare the necessary number of colors with the possible lengths of the Rokhlin towers of such approximations (instead of asking for a global bound on the number of colors). This idea leads to a version of ‘slow dimension growth’ which in turn is closely related to the notion of mean dimension introduced by Lindenstrauss and Weiss in [17].

Finally, we mention that Bartels, Lück and Reich have studied a concept closely related to ours in [2]; this idea plays a key role in their proof of the Farrell-Jones conjecture for hyperbolic groups in [3].

These connections will be studied in subsequent work.

The organization of the paper is as follows. We first consider the case of finite group actions in Section 1 which in some respects is simpler than the case of actions of Z and serves as a good motivation displaying some essential ideas. In Section 2 we introduce various versions of Rokhlin dimension for Z-actions and study their interplay. In Section 3 we show that finite Rokhlin dimension is generic in the Z-stable case. A similar argument shows that the Rokhlin property is generic in the UHF-stable case. In Section 4 we show that finite nuclear dimension is preserved under forming crossed products by automorphisms with finite Rokhlin dimension, and in Section 5 we show that Z-stability is preserved if we assume a further commutativity condition in the definition of Rokhlin dimension (which holds in all cases we are aware of); the latter result holds for actions of finite groups and of Z.

The remainder of the paper is devoted to showing that minimal Z-actions on finite dimensional compact Hausdorff spaces always have finite Rokhlin dimension. As the
proof is quite technical, we give a (different) short and direct argument for irrational rotations of the circle in Section 6. The proof of the general result is carried out in Sections 7 through 12. The argument closely follows that of [31], computing the decomposition rank for orbit breaking large subalgebras of transformation group $C^*$-algebras while at the same time carefully keeping track of the underlying dynamics.

In an appendix we recall the crossed product construction as well as the notions of nuclear dimension and of $\mathcal{Z}$-stability for the reader’s convenience.

CONTENTS

0. Introduction 1
1. Rokhlin dimension for actions of finite groups 5
2. Rokhlin dimension for actions of $\mathbb{Z}$ 8
3. Genericity in the $\mathcal{Z}$-stable case 15
4. Permanence of finite nuclear dimension 18
5. Permanence of $\mathcal{Z}$-stability 21
6. Irrational rotations 27
7. Cyclic vs. non-cyclic shifts 29
8. Minimal dynamics, first return times and compatible approximations 31
9. Relative barycentric subdivision 36
10. Elementary polynomials 48
11. Existence of compatible approximations 49
12. Compatible approximations and Rokhlin dimension 67
Appendix: Crossed products, nuclear dimension and $\mathcal{Z}$-stability 70
References 72
1. Rokhlin dimension for actions of finite groups

In this section we define the concept of Rokhlin dimension for finite group actions on $C^*$-algebras and show how this notion can be used to prove permanence of finite nuclear dimension under taking crossed products.

**Definition 1.1.** Let $G$ be a finite group, $\mathcal{A}$ a unital $C^*$-algebra and

$$\alpha : G \to \text{Aut}(\mathcal{A})$$

an action of $G$ on $\mathcal{A}$. We say that $\alpha$ has Rokhlin dimension $d$ if $d$ is the least integer such that the following holds: for any $\varepsilon > 0$ and every finite subset $F \subset \mathcal{A}$ there are positive contractions

$$\left( f_g^{(l)} \right)_{l=0,\ldots,d;\, g \in G} \subset \mathcal{A}$$

satisfying

1. for any $l$, $\left\| f_g^{(l)} f_h^{(l)} \right\| < \varepsilon$, whenever $g \neq h$ in $G$,
2. $\left\| \sum_{l=0}^d \sum_{g \in G} f_g^{(l)} - 1 \right\| < \varepsilon$,
3. $\left\| \alpha_h \left( f_g^{(l)} \right) - f_h^{(l)} \right\| < \varepsilon$ for all $l \in \{0, \ldots, d\}$ and $g \in G$,
4. $\left\| f_g^{(l)} a \right\| < \varepsilon$ for all $l \in \{0, \ldots, d\}$, $g \in G$ and $a \in F$.

We will refer to the family $(f_g^{(l)})_{l=0,\ldots,d;\, g \in G}$ as a multiple tower and to the $d + 1$ families $(f_g^{(0)})_{g \in G}, \ldots, (f_g^{(d)})_{g \in G}$ as towers of color $0, \ldots, d$, respectively. Notice that if $d = 0$ then we obtain the usual Rokhlin property for actions of finite groups (see [10]).

A possible variant of the above definition would be to weaken (2) to

(2') $f := \sum_{l=0}^d \sum_{g \in G} f_g^{(l)} \geq (1 - \varepsilon) \cdot 1_{\mathcal{A}}$.

If the towers can be chosen to commute or approximately commute it’s not hard to see that if (2') is true then, upon replacing $f_g^{(l)}$ by $f^{-1/2} f_g^{(l)} f^{-1/2}$ and $\varepsilon$ by a sufficiently small fraction of $\varepsilon$, we can also obtain (2), so that (2) and (2') are equivalent in this case.

**Remark 1.2.** By approximating the function $t \mapsto t^{1/2}$ by polynomials on $[0, 1]$ we may assume in Definition 1.1 above that for $h, g \in G$ and $a \in F$ we have

$$\left\| f_g^{(l)} f_h^{(l)} \right\| < \varepsilon \quad \text{if} \quad g \neq h, \quad \left\| \left[ f_g^{(l)} f_h^{(l)} , a \right] \right\| < \varepsilon \quad \text{and} \quad \left\| \alpha_h \left( f_g^{(l)} \right) - f_h^{(l)} \right\| < \varepsilon$$

for all $l \in \{0, \ldots, d\}$. Also, we may replace the finite set $F$ in the definition by a norm compact set.

**Theorem 1.3.** Let $G$ be a finite group, $\mathcal{A}$ a unital $C^*$-algebra with finite decomposition rank and $\alpha : G \to \text{Aut}(\mathcal{A})$ be an action with Rokhlin dimension $d$.

Then the crossed product $\mathcal{A} \rtimes_\alpha G$ has finite decomposition rank as well - in fact,

$$\text{dr}(\mathcal{A} \rtimes_\alpha G) \leq (\text{dr}(\mathcal{A}) + 1)(d + 1) - 1.$$ 

The respective statement is true for nuclear dimension.

**Proof.** We will only give a proof for the decomposition rank; the proof for nuclear dimension is similar. Set $n = |G|$ and recall that $\text{dr}(M_n(\mathcal{A})) = \text{dr}(\mathcal{A})$ and denote this number by $N$. 

Let \( F \subset \mathcal{A} \rtimes_{\alpha} G \subset M_n(\mathcal{A}) \) finite and \( \varepsilon > 0 \) be given; we may assume that \( F \) consists of contractions. Choose an \( N \)-decomposable c.p.c. approximation (for \( M_n(\mathcal{A}) \)) consisting of \( F = F^{(0)} + \cdots + F^{(N)} \), \( \psi : M_n(\mathcal{A}) \to F \), \( \varphi : F \to M_n(\mathcal{A}) \), \( \varphi = \varphi^{(0)} + \cdots + \varphi^{(N)} \) for \( F \) to within \( \varepsilon/6 \).

We will construct c.p. approximations (for \( \mathcal{A} \rtimes_{\alpha} G \)) as below, and then use stability of order zero maps to replace the maps \( \rho \varphi \) by ones that are decomposable into order zero maps:

Let \( K_0 = \bigcup_{g \in G} \bigcup_{j=0}^{N}(\text{id}_{M_n} \otimes \alpha_g)(\varphi(j)(B_{F})) \), where \( B_{F} \) is the closed unit ball in \( F \), and let \( K \subseteq \mathcal{A} \) be the collection of all matrix entries of elements in \( K_0 \). Notice that \( K_0, K \) are compact subsets of \( M_n(\mathcal{A}) \) and \( \mathcal{A} \) respectively. Using stability of order zero maps, we choose \( \eta > 0 \) such that \( \eta < \varepsilon/6 \) and such that if \( \theta : F \to \mathcal{A} \rtimes_{\alpha} G \) is a c.p.c. map which satisfies \( \|\theta(a)\theta(b)\| \leq (n + 4)^3\eta \) for all positive orthogonal contractions \( a, b \in F \), then there exists a c.p.c. order zero map \( \theta' : F \to \mathcal{A} \rtimes_{\alpha} G \) such that \( \|\theta' - \theta\| < \varepsilon/(6(d + 1)(N + 1)) \). Finally we also require \( \eta \) to be such that \( (2(d + 1)n + 1)\eta \) will be used later.

Every element \( x \in F \subseteq \mathcal{A} \rtimes_{\alpha} G \) can be written in the form \( x = \sum_{g \in G} x(g)u_g \), where \( x(g) \in \mathcal{A} \) are uniquely determined contractions. Let \( \tilde{F} \) be the finite set \( \tilde{F} = \{x(g) \mid x \in F, g \in G\} \subseteq \mathcal{A} \).

For the particular \( \eta > 0 \) we have chosen and the compact set \( K \cup \tilde{F} \), choose a multiple tower \( \left( f_g^{(l)} \right)_{l=0,\ldots,d} \) such that (1)–(4) of Definition 1.1 hold true for \( \varepsilon \) replaced by \( \eta \) and \( a \in K \cup \tilde{F} \) and also \( \left\| f_g^{(l)1/2} f_h^{(l)1/2} \right\| < \eta \) \( (g, h \in G \) different), \( \left\| \alpha_h(f_g^{(l)1/2}) - f_h^{(l)1/2} \right\| < \eta \) and \( \left\| f_g^{(l)1/2}, a \right\| < \eta \) for all \( l \in \{0, \ldots, d\}, a \in K \cup \tilde{F}, g, h \in G \).

Now, define \( \rho^{(l)} : M_n(\mathcal{A}) \to \mathcal{A} \rtimes_{\alpha} G \) by

\[
\rho^{(l)}(e_g \otimes a) = f_g^{(l)1/2} u_g^* a u_h f_h^{(l)1/2}.
\]

We note that \( \rho^{(l)} \) is c.p. since it is the composition of the canonical inclusion \( M_n \otimes \mathcal{A} \to M_n \otimes \mathcal{A} \rtimes_{\alpha} G \) with the map \( M_n(\mathcal{A} \rtimes_{\alpha} G) \to \mathcal{A} \rtimes_{\alpha} G \) given by \( a \mapsto v_lau_l^* \), where \( v_l \in M_{1,n}(\mathcal{A} \rtimes_{\alpha} G) \) is defined by \( v_l = \left( f_g^{(l)1/2} u_g \right)_{g \in G} \) (as a row vector whose entries are indexed by the elements of \( G \)). Thus the sum \( \rho = \sum_{l=0}^{d} \rho^{(l)} \) is also c.p.
For \( a \in K \cup \tilde{F}, \ g \in G, \ l \in \{0, \ldots, d\} \) we have

\[
\rho^{(l)}(au_g) = \rho^{(l)} \left( \sum_{h \in G} \epsilon_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) \right)
\]

\[
= \sum_{h \in G} f_h^{(l) \frac{1}{2}} u_h \alpha_{h^{-1}}(a) u_{h^{-1}g} f_{g^{-1}h}(l)^{\frac{1}{2}}
\]

\[
= \sum_{h \in G} f_h^{(l) \frac{1}{2}} au_g f_{g^{-1}h}(l)^{\frac{1}{2}}
\]

\[
= n\eta \sum_{h \in G} f_h^{(l) \frac{1}{2}} au_g.
\]

Thus \( \|\rho(au_g) - au_g\| < (2(d + 1)n + 1)\eta \) by assumption (note that \( \|a\| \leq 1 \)) and so \( \|\rho(x) - x\| < (2(d + 1)n + 1)\eta < \varepsilon/6 \) for \( x \in F \).

Note that \( \rho(1) = \sum_{l=0}^{d} \rho^{(l)}(1) = \sum_{l=0}^{d} \sum_{g \in G} f_g^{(l)} \), therefore \( \|\rho\| < 1 + \varepsilon/6 \) and so after possible normalization (dividing by its norm) we have \( \|\rho\| = 1 \) and \( \|\rho(x) - x\| < \varepsilon/3 \) for all \( x \in F \).

Define \( f_r^{(l)} := \sum_{g \in G} f^{(l)} \), where \( l \in \{0, \ldots, d\} \). One checks that for all \( a, b \in K \cup \tilde{F} \), and \( g_1, g_2, h_1, h_2 \in G \), we have

\[ \|\rho^{(l)}(a \otimes e_{g_1,h_1}) \rho^{(l)}(b \otimes e_{g_2,h_2}) - f^{(l)} \rho((a \otimes e_{g_1,h_1}) \cdot (b \otimes e_{g_2,h_2}))\| < (n + 4)\eta. \]

Therefore, if \( a, b \) are elements of \( K_0 \subset M_n(A) \), then

\[ \|\rho^{(l)}(a) \rho^{(l)}(b) - f^{(l)} \rho^{(l)}(ab)\| \leq (n + 4)n^3\eta. \]

Consequently, for any \( j = 0, \ldots, d \), we have that

\[ \|\rho^{(l)}(\varphi^{(j)}(a)) \rho^{(l)}(\varphi^{(j)}(b)) - f^{(l)} \rho^{(l)}(\varphi^{(j)}(a) \varphi^{(j)}(b))\| < (n + 4)n^3\eta \]

for all \( a, b \in B_F \). In particular, if \( a, b \in B_F \) are positive and orthogonal, then \( \|\rho^{(l)}(\varphi^{(j)}(a)) \rho^{(l)}(\varphi^{(j)}(b))\| \leq (n + 4)n^3\eta \), and \( f^{(l)} \circ \varphi^{(j)} \) is a c.p.c. map. By the choice of \( \eta \), there are c.p.c. order zero maps

\[ \sigma^{(l,j)} : F \to A \rtimes_G \alpha \]

such that

\[ \|\sigma^{(l,j)} - \rho^{(l)} \circ \varphi^{(j)}\| < \varepsilon/[6(d + 1)(N + 1)]. \]

Set

\[ \sigma^{(l)} := \sum_{j=0}^{N} \sigma^{(l,j)} : F \to A \rtimes_G \alpha \]

and

\[ \sigma = \sum_{l=0}^{d} \sigma^{(l)} : \bigoplus_{l=0}^{d} F \to A \rtimes_G \alpha. \]

If \( \|\sigma\| > 1 \), then we normalize \( \sigma \); since \( \|\sigma\| < 1 + \varepsilon/6 \), normalizing will again introduce an error of at most \( \varepsilon/6 \).
Thus, possibly after normalizing, we have that \( \| \sigma - \rho \circ \varphi \| < \varepsilon/3 \), and putting everything together, the triple \((\bigoplus_{l=0}^{d} F, \psi, \sigma)\) is a \((d+1)(N+1)-1\)-decomposable approximation for \( F \) within \( \varepsilon \), as required. \( \square \)

**Remark 1.4.** We will see in Theorem 5.9 that \( \mathcal{Z} \)-stability also passes to crossed products by finite group actions with finite Rokhlin dimension, provided one in addition assumes that the Rokhlin towers commute.

### 2. Rokhlin Dimension for Actions of \( \mathbb{Z} \)

In this section we introduce the concept of Rokhlin dimension for \( \mathbb{Z} \)-actions on \( C^* \)-algebras. There is a certain amount of freedom in the definition, just as for the original Rokhlin property. For the latter, several versions have been studied (cf. [15], [8] and the references therein). Correspondingly, we introduce several versions of Rokhlin dimension and compare these. As it turns out, in general one such version is finite if and only if the other is – roughly, they bound one another by a factor 2.

Since we are mostly interested in when the Rokhlin dimension is finite (as opposed to the precise value), for our purposes it will not be too important which variant we choose. The different versions of Rokhlin dimension are designed so that they are just the zero dimensional instances of the various Rokhlin properties; in this regard, our concept bridges the gaps between the latter. For our exposition we will give slight preference to the version generalizing the definition below.

Let us first recall the Rokhlin property which is now most commonly used.

**Definition 2.1.** Let \( A \) be a unital \( C^* \)-algebra, and let \( \alpha : \mathbb{Z} \to \text{Aut}(A) \) be an action of the integers. We say that \( \alpha \) has the Rokhlin property if for any finite set \( F \subseteq A \), any \( \varepsilon > 0 \) and any \( p \in \mathbb{N} \) there are projections

\[
e_{0,0}, \ldots, e_{0,p-1}, e_{1,0}, \ldots, e_{1,p} \in A
\]

such that

1. \( \sum_{r=0}^{1} \sum_{j=0}^{p+r-1} e_{r,j} = 1 \),
2. \( \| \alpha_1(e_{r,j}) - e_{r,j+1} \| < \varepsilon \) for all \( j = 0, 1, \ldots, p - 2 + r \),
3. \( \| [e_{r,j}, a] \| < \varepsilon \) for all \( r, j \) and all \( a \in F \).

We shall refer to the projections in the definition as Rokhlin projections and to the set of those projections as Rokhlin towers.

**Remarks 2.2.** (i) It follows immediately from the definition that the Rokhlin projections are pairwise orthogonal, and that \( \| \alpha_1(e_{0,p-1} + e_{1,p}) - (e_{0,0} + e_{1,0}) \| < \varepsilon \). In the generalization we consider, we will need to add such assumptions explicitly.

(ii) It could be that the \( e_{1,j} \)'s (or the \( e_{0,j} \)'s) are all zero. If this can always be arranged then we say that \( \alpha \) has the single tower Rokhlin property. The single tower Rokhlin property is stronger than the Rokhlin property - it implies immediately that the identity can be decomposed into a sum of projections which are equivalent via an automorphism, which amounts to a nontrivial divisibility requirement. For instance, there can be no automorphism on the UHF algebra of type \( 2^n \) which has the single tower Rokhlin property as stated here, whereas we shall see that the Rokhlin property is generic for automorphisms of this algebra (for this \( C^* \)-algebra, there are many automorphisms with single Rokhlin towers if one restricts to towers of a height which is a power of 2 – in fact, basically the same proof shows that such automorphisms are generic).
We now turn to our main definition.

**Definition 2.3.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra and \( \alpha : \mathbb{Z} \to \text{Aut}(\mathcal{A}) \) an action of the integers.

a) We say \( \alpha \) has Rokhlin dimension \( d \),

\[
\dim_{\text{Rok}}(\mathcal{A}, \alpha) = d,
\]

if for any finite subset \( F \subset \mathcal{A} \), any \( p \in \mathbb{N} \) and any \( \varepsilon > 0 \), there are positive elements

\[
f^{(l)}_{0,0}, \ldots, f^{(l)}_{0,p-1}, f^{(l)}_{1,0}, \ldots, f^{(l)}_{1,p}, l \in \{0, \ldots, d\},
\]

satisfying

1. for any \( l \in \{0, \ldots, d\} \), \( \|f^{(l)}_{q,i} f^{(l)}_{r,j}\| < \varepsilon \), whenever \( (q, i) \neq (r, j) \),
2. \( \left\| \sum_{l=0}^{d} \sum_{r=0}^{1} \sum_{j=0}^{p-1+r} f^{(l)}_{r,j} - 1 \right\| < \varepsilon \),
3. \( \|\alpha_1(f^{(l)}_{r,j}) - f^{(l)}_{r,j+1}\| < \varepsilon \) for all \( r \in \{0,1\}, j \in \{0,1,\ldots, p-2+r\} \) and all \( l \in \{0, \ldots, d\} \),
4. \( \|\alpha_1(f^{(l)}_{0,p-1} + f^{(l)}_{1,0}) - (f^{(l)}_{0,0} + f^{(l)}_{1,0})\| < \varepsilon \) for all \( l \),
5. \( \|[f^{(l)}_{r,j}, a]\| < \varepsilon \) for all \( r, j, l \) and \( a \in F \).

If there is no such \( d \) then we say that \( \alpha \) has infinite Rokhlin dimension and write \( \dim_{\text{Rok}}(\mathcal{A}, \alpha) = \infty \).

b) If, moreover, for all towers we can arrange \( \|[f^{(l)}_{q,i}, f^{(m)}_{r,j}]\| < \varepsilon \) for all \( q, i, l, r, j, m \), then we say that \( \alpha \) has Rokhlin dimension \( d \) with commuting towers,

\[
\dim_{\text{Rok}}^c(\mathcal{A}, \alpha) = d.
\]

c) If one can obtain this property such that for all \( l \) one of the towers \( (f^{(l)}_{0,j})_j \) or \( (f^{(l)}_{1,j})_j \) vanishes, then we shall say that \( \alpha \) has Rokhlin dimension \( d \) with single towers,

\[
\dim_{\text{Rok}}^s(\mathcal{A}, \alpha) = d.
\]

d) We write

\[
\overline{\dim}_{\text{Rok}}(\mathcal{A}, \alpha) = d
\]

if \( \alpha \) satisfies the conditions above except that (2) is replaced by the weaker condition

\[
(2') \quad \sum_{l=0}^{d} \sum_{r=0}^{1} \sum_{j=0}^{p-1+r} f^{(l)}_{r,j} \geq (1 - \varepsilon) \cdot 1_{\mathcal{A}}.
\]

e) We define \( \overline{\dim}_{\text{Rok}}(\mathcal{A}, \alpha) \) and \( \overline{\dim}_{\text{Rok}}(\mathcal{A}, \alpha) \) in the respective manner, replacing property (2) by (2’) in b) and c).

We shall refer to each sequence \( (f^{(l)}_{r,j})_{j \in \{0, \ldots, p-1+r\}} \) as a tower, to the length of the sequence as the height of the tower, and to the pair of towers \( (f^{(l)}_{r,j})_{j \in \{0, \ldots, p-1+r\}} \), \( r \in \{0, 1\} \), as a double tower. The superscript \( l \) will sometimes be referred to as the color of the tower.

**Remarks 2.4.**

(i) Obviously \( \dim_{\text{Rok}}(\mathcal{A}, \alpha) \) and \( \dim_{\text{Rok}}^c(\mathcal{A}, \alpha) \) coincide if \( \mathcal{A} \) is commutative and we will show below (Proposition 2.7) that in this case we also have \( \overline{\dim}_{\text{Rok}}(\mathcal{A}, \alpha) = \dim_{\text{Rok}}(\mathcal{A}, \alpha) \) and \( \overline{\dim}_{\text{Rok}}^s(\mathcal{A}, \alpha) = \dim_{\text{Rok}}^s(\mathcal{A}, \alpha) \).

(ii) Note that in the single tower version defined in c) above, property (4) of 2.3(a) forces property (3) to hold cyclically, i.e., for \( j \in \{0, 1, \ldots, p-1+r\} \).

(iii) It is clear that in general \( \overline{\dim}_{\text{Rok}}(\mathcal{A}, \alpha) \leq \dim_{\text{Rok}}(\mathcal{A}, \alpha) \). However in all cases in which we establish that a \( \mathbb{Z} \)-action has Rokhlin dimension \( d \) the towers are...
commuting, and we do not know if there are any examples for which there is a strict inequality. The additional assumption concerning commuting towers is needed as a hypothesis for some results but not others.

(iv) Similarly, \( \dim_{\text{Rok}}(\mathcal{A}, \alpha) \leq \dim_{\text{Rok}}(\mathcal{A}, \alpha) \) in general and it is straightforward to verify that \( \dim_{\text{Rok}}(\mathcal{A}, \alpha) = 0 \) if and only if \( \alpha \) has the Rokhlin property (if and only if \( \dim_{\text{cRok}}(\mathcal{A}, \alpha) = 0 \)).

(v) As is the case for the regular Rokhlin property, one could have defined a notion of a Rokhlin dimension using more than two towers for each color. That is, require that for any given \( N \) one could find numbers \( R_N, p_0, \ldots, p_{R_N} \) such that \( p_r > N \) and positive elements \( f_{r,j}^{(l)}, l = 0, 1, \ldots, d, r = 0, 1, \ldots, R_N, \) for all \( r \) and \( j = 0, 1, \ldots, p_r \) such that the analogous definition holds (of course, if one allows for arbitrary tower lengths and numbers of towers, then \( \varepsilon \) in conditions (1)–(v) has to be replaced by something like \( \varepsilon/(p_0 + \ldots + p_{R_N}) \)). However, this can be reduced to the case of two towers in a standard way, which we quickly outline. For a given \( p \), we find an \( N \) such that any \( n > N \) can be written as \( a(p - 1) + bp \) for some given non-negative integers \( a, b \). Using this \( N \), one finds \( f_{r,j}^{(l)} \) as above. We fix \( a_r, b_r \) such that \( a_r(p - 1) + b_r p = p_r + 1 \), and then define

\[
\tilde{f}_{0,j}^{(l)} = \sum_{r=0}^{R_N} a_r^{-1} \sum_{m=0}^{p_r-1} f_{r,m(p-1)+j}^{(l)} \\
\tilde{f}_{1,j}^{(l)} = \sum_{r=0}^{R_N} b_r^{-1} \sum_{m=0}^{p_r-1} f_{r,a_r(p-1)+mp+j}^{(l)}
\]

and this reduces the case of an arbitrary number of towers of each color, each in an arbitrary length of at least \( N \) to our notion of Rokhlin dimension.

(vi) The definition of Rokhlin dimension (and Rokhlin dimension with commuting towers) is equivalent to the following apriori weaker formulation: instead of asking for double towers of arbitrary height \( p \), one can require that there are such towers for arbitrarily large \( p \) (independently of \( \varepsilon \)) but not necessarily for any \( p \). This can be shown by using a rearrangement of tower terms as explained above, and is useful in certain examples (e.g. if it is easier to produce towers which are of prime height, or of height which is a power of 2). We note, however, that in the single tower case this indeed gives a different formulation. For example (cf. Remark 2.2(ii)), the infinite tensor shift on the CAR algebra has the Rokhlin property and one can obtain single towers of height \( 2^n \), but if one requires heights which are not powers of 2 then one needs two towers (this follows from \( K \)-theoretic considerations).

**Notation 2.5.** We like to think of finite Rokhlin dimension as a property of a group action rather than of an automorphism, even if the group is cyclic. However, when it is understood that the group is \( \mathbb{Z} \) we will sometimes slightly abuse notation and write \( \dim_{\text{Rok}}(\mathcal{A}, \alpha) \) even when \( \alpha \in \text{Aut}(\mathcal{A}) \) denotes the automorphism inducing the \( \mathbb{Z} \)-action rather than the action itself. In this situation we also say that the automorphism has finite Rokhlin dimension.

There is a useful and straightforward reformulation of the definition of Rokhlin dimension which will be used later on:

**Proposition 2.6.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra and \( \alpha : \mathbb{Z} \to \text{Aut}(\mathcal{A}) \) a \( \mathbb{Z} \)-action.

a) Then \( \dim_{\text{Rok}}(\mathcal{A}, \alpha) \leq d \) if and only if the following holds:
For every $p \in \mathbb{N}$, $F \subset \mathcal{A}$ finite and $\varepsilon > 0$ there are c.p.c. maps
\[
\zeta^{(i)} : \mathbb{C}^p \oplus \mathbb{C}^{p+1} \to \mathcal{A}, \ l \in \{0, \ldots, d\},
\]
such that

\begin{enumerate}[(1)]
\item $\|\zeta^{(i)}(e)\zeta^{(i)}(e')\| \leq \varepsilon \|e\|\|e'\|$ whenever $e, e' \in \mathbb{C}^p \oplus \mathbb{C}^{p+1}$ are orthogonal positive elements,
\item $\|\sum_l (\zeta^{(i)}(1_{C^p} \oplus 1_{C^{p+1}}) - 1_A)\| \leq \varepsilon$,
\item $\|\alpha_1(\zeta^{(i)}(e)) - \zeta^{(i)}(\tilde{\sigma}(e))\| \leq \varepsilon \|e\|$ for all elements $0 \leq e \in \mathbb{C}^p \oplus \mathbb{C}^{p+1}$ satisfying $e \perp (e_p, f_{p+1})$, where $\tilde{\sigma} \in \text{Aut}(\mathbb{C}^p \oplus \mathbb{C}^{p+1})$ is the cyclic shift on each summand and $e, f_j$ denote the canonical generators of $\mathbb{C}^p$ and $\mathbb{C}^{p+1}$, respectively,
\item $\|\alpha_1(\zeta^{(i)}((e_p, f_{p+1}))) - \zeta^{(i)}((e_1, f_1))\| \leq \varepsilon$,
\item $\|\zeta^{(i)}(e, a)\| \leq \varepsilon \|e\|$ for all $0 \leq e \in \mathbb{C}^p \oplus \mathbb{C}^{p+1}$ and $a \in F$.
\end{enumerate}

b) We have $\dim_{\text{Rok}}^R(\mathcal{A}, \alpha) \leq d$ if for every $p$, $F$ and $\varepsilon$ as above there are $\zeta^{(0)}, \ldots, \zeta^{(d)}$ as above such that each $\zeta^{(i)}$ vanishes on one of the summands $\mathbb{C}^p$ or $\mathbb{C}^{p+1}$.

c) We have $\dim_{\text{Rok}}^R(\mathcal{A}, \alpha) \leq d$ if for every $p$, $F$ and $\varepsilon$ as above there are $\zeta^{(0)}, \ldots, \zeta^{(d)}$ as above with $\|\|\zeta^{(i)}(e, \zeta^{(i)}(e'))\| \leq \varepsilon \|e\|\|e'\|$ for all $l, l' \in \{0, \ldots, d\}$ and $e, e' \in \mathbb{C}^p \oplus \mathbb{C}^{p+1}$.

d) We have $\overline{\dim}_{\text{Rok}}^R(\mathcal{A}, \alpha) \leq d$ if for every $p$, $F$ and $\varepsilon$ as in a) there are $\zeta^{(0)}, \ldots, \zeta^{(d)}$ as in a) satisfying
\[
(2') \sum_l \zeta^{(i)}(1_{\mathbb{C}^p} \oplus 1_{\mathbb{C}^{p+1}}) \geq (1 - \varepsilon) \cdot 1_A
\]
instead of (2).

A combination of b), c) and d) yields the respective statements for $\overline{\dim}_{\text{Rok}}^R(\mathcal{A}, \alpha)$ and $\overline{\dim}_{\text{Rok}}(\mathcal{A}, \alpha)$.

When the underlying $C^*$-algebra is commutative, then clearly each version of Rokhlin dimension agrees with its commuting tower counterpart – but in this case, we even have that $\dim_{\text{Rok}}^R$ agrees with $\overline{\dim}_{\text{Rok}}$.

**Proposition 2.7.** Let $(T, h)$ be a dynamical system with $T$ compact and metrizable; let $\alpha$ be the induced action on $C(T)$. Then,

\[
\overline{\dim}_{\text{Rok}}^R(C(T), \alpha) = \dim_{\text{Rok}}(C(T), \alpha)
\]
and
\[
\overline{\dim}_{\text{Rok}}^R(C(T), \alpha) = \dim_{\text{Rok}}^R(C(T), \alpha).
\]

**Proof.** We clearly have $\overline{\dim}_{\text{Rok}}(C(T), \alpha) \leq \dim_{\text{Rok}}(C(T), \alpha)$. For the reverse inequality, assume
\[
\overline{\dim}_{\text{Rok}}^R(C(T), \alpha) \leq d
\]
and suppose we are given $p \in \mathbb{N}$, $0 < \bar{\varepsilon} < 1$ and $F \subset C(T)$ finite (in fact, the subset $F$ is irrelevant here since $C(T)$ is commutative).

Define $f \in C([0, d + 1])$ by
\[
f(t) := \begin{cases}
t^{-1} & \text{if } t \geq 1/2 \\
0 & \text{if } t = 0 \\
\text{linear} & \text{else}.
\end{cases}
\]

By approximating $f$ uniformly by polynomials we can find
\[
0 < \delta \leq \frac{\bar{\varepsilon}}{4}
\]
such that, for any positive $a$ with $\|a\| < d + 1$, if $\|\alpha_1(a) - a\| < \delta$ then $\|\alpha_1(f(a)) - f(a)\| < \frac{\varepsilon}{2}$.

Set
\[
\varepsilon := \frac{\delta}{d + 1};
\]
since $\dim_{\text{Rok}}(C(T), \alpha) \leq d$, there are c.p.c. maps
\[
\zeta^{(l)} : \mathbb{C}^p \oplus \mathbb{C}^{p+1} \to C(T)(= \mathcal{A})
\]
satisfying (2.6), (2'), (3), (4) and (5). Note that by (2.6) and since the $\zeta^{(l)}$ are c.p.c. we have
\[
\frac{1}{2} \cdot 1_{C(T)} \leq \sum_{l=0}^{d} \zeta^{(l)}(1) \leq (d + 1) \cdot 1_{C(T)},
\]
whence
\[
f \left( \sum_{l=0}^{d} \zeta^{(l)}(1) \right) = \left( \sum_{l=0}^{d} \zeta^{(l)}(1) \right)^{-1}.
\]
From (2.6) we see that
\[
\left\| \alpha_1 \left( \sum_{l=0}^{d} \zeta^{(l)}(1) \right) - \sum_{l=0}^{d} \zeta^{(l)}(1) \right\| \leq (d + 1) \varepsilon = \delta.
\]
By the choice of $\delta$ and by (2) this entails
\[
\left\| \alpha_1 \left( \sum_{l=0}^{d} \zeta^{(l)}(1) \right)^{-1} - \left( \sum_{l=0}^{d} \zeta^{(l)}(1) \right)^{-1} \right\| \leq \frac{\varepsilon}{2}.
\]
Define maps
\[
\bar{\zeta}^{(l)} : \mathbb{C}^p \oplus \mathbb{C}^{p+1} \to C(T)
\]
by
\[
\bar{\zeta}^{(l)}(.) := \left( \sum_{k=0}^{d} \zeta^{(k)}(1) \right)^{-1} \zeta^{(l)}(.)
\]
for $l = 0, \ldots, d$; it is clear that the $\bar{\zeta}^{(l)}$ are c.p.c. and that
\[
\sum_{l=0}^{d} \bar{\zeta}^{(l)}(1) = 1_{C(T)},
\]
whence (2.6) holds for the $\bar{\zeta}^{(l)}$ (in fact, with 0 in place of $\varepsilon$).
By (2.6), (2') and since $C(T)$ is commutative, we have
\[
\|\bar{\zeta}^{(l)}(e)\bar{\zeta}^{(l)}(e')\| \leq 4\varepsilon\|e\|\|e'\| \leq \frac{\varepsilon}{d + 1}\|e\|\|e'\|
\]
whenever $e, e' \in \mathbb{C}^p \oplus \mathbb{C}^{p+1}$ are positive orthogonal elements; it follows that (2.6) holds for the $\bar{\zeta}^{(l)}$ and $\varepsilon$ in place of $\zeta^{(l)}$ and $\varepsilon$.
(2.6) holds automatically for the $\bar{\zeta}^{(l)}$ (again with 0 in place of $\varepsilon$) since $\mathcal{F}$ and the range of the $\bar{\zeta}^{(l)}$ are in $C(T)$. 
Finally, we estimate for each \( l \in \{0, \ldots, d\} \) and \( 0 \leq e \in \mathbb{C}^p \oplus \mathbb{C}^{p+1} \)
\[
\|\alpha_1(\bar{\zeta}^{(l)}(e)) - \bar{\zeta}^{(l)}(\tilde{\sigma}(e))\| \\
= \left\| \alpha_1 \left( \left( \sum_{k=0}^{d} \zeta^{(k)}(1) \right)^{-1} \bar{\zeta}^{(l)}(e) \right) - \left( \sum_{k=0}^{d} \zeta^{(k)}(1) \right)^{-1} \zeta^{(l)}(\tilde{\sigma}(e)) \right\| \\
\leq \left\| \left( \sum_{k=0}^{d} \zeta^{(k)}(1) \right)^{-1} \left( \alpha_1(\zeta^{(l)}(e)) - \zeta^{(l)}(\tilde{\sigma}(e)) \right) \right\| + \bar{\varepsilon} \|e\| \\
\leq \left( 2\varepsilon + \frac{\bar{\varepsilon}}{2} \right) \|e\| \\
\leq \bar{\varepsilon} \|e\|,
\]
so \((2.6.3)\) holds for \( \bar{\zeta}^{(l)} \) and \( \bar{\varepsilon} \) in place of \( \zeta^{(l)} \) and \( \varepsilon \).

We have now shown \((1)\). Literally the same proof also yields the single tower version. \(\square\)

Rokhlin dimension with single towers in general does not coincide with Rokhlin dimension. That is already true in the case of the regular Rokhlin property: there are automorphisms with the Rokhlin property but in which requiring double-towers is essential. One way to see this is via \(K\)-theoretic considerations (cf. Remarks 2.2(ii) and 2.4(vi)): for example, if \( \alpha \) is an approximately inner automorphism which has a Rokhlin tower then all Rokhlin projections are equivalent in \(K_0\), and therefore if \( \alpha \) has the Rokhlin property with single towers then \(K_0(A)\) must be divisible – and it is well known that there are Rokhlin automorphisms on \(C^*\)-algebras which do not have divisible \(K_0\), e.g. the CAR algebra. However, if we allow for higher Rokhlin dimensions, then we see that the distinction is relatively mild:

**Proposition 2.8.** Let \( A \) be a unital \(C^*\)-algebra and \( \alpha: \mathbb{Z} \to \text{Aut}(A) \) an action. Then

\[(4) \quad \text{dim}_{\text{Rok}}(\alpha) \leq \text{dim}^8_{\text{Rok}}(\alpha) \leq 2 \text{dim}_{\text{Rok}}(\alpha) + 1\]

and

\[(5) \quad \overline{\text{dim}}_{\text{Rok}}(\alpha) \leq \overline{\text{dim}^8_{\text{Rok}}}(\alpha) \leq 2 \overline{\text{dim}}_{\text{Rok}}(\alpha) + 1.\]

**Proof.** As in 2.7, we only show \((4)\); the same argument will also yield \((5)\).

The left hand inequality is trivial. As for the right hand one, we may assume that \( \alpha \) has finite Rokhlin dimension. Let \( f_{r,j}^{(l)} \), \( l = 0, 1, \ldots, d \), \( r = 0, 1, \ldots, p-1+r \) be Rokhlin tower elements for \( \alpha \), with respect to a given \( \varepsilon > 0 \), height \( p \) and finite set \( F \subseteq A \). Let us first assume that \( p \) is large enough so that \( \varepsilon > \frac{1}{p-1/2} \). We shall construct single towers \( (g_{j}^{(l)}) \) for \( l = 0, 1, \ldots, 2d \), the same finite set \( F \) and with \( 2\varepsilon \) instead of \( \varepsilon \).

Define decay factors \( \mu_r: \{0, 1, \ldots, p-1+r\} \to \mathbb{R} \) by
\[
\mu_r(n) = 1 - \frac{|(p-1+r)/2 - n|}{(p-1+r)/2},
\]

so 2.6(3) holds for \( \bar{\zeta}^{(l)} \) and \( \bar{\varepsilon} \) in place of \( \zeta^{(l)} \) and \( \varepsilon \).
where we denote $\mu_0(p) = 0$. We use this decay factor to merge each double tower into two separate (cyclic) towers. Schematically, we use the decay factor $\mu_r$ which is large in the middle and small in the ends to bunch up the tower terms as follows

\[
\begin{pmatrix}
  f_{0,0} & f_{0,1} & f_{0,2} & \cdots & f_{0,p-1} \\
f_{1,0} & f_{1,1} & f_{1,2} & \cdots & f_{1,p-1} \\
\end{pmatrix}
\]

and we use $1 - \mu_r$, which is small at the center, to bunch up the tower terms like this (treat $p$ as even for the purpose of the diagram):

\[
\begin{pmatrix}
  f_{0,0} & \cdots & f_{0,\frac{p}{2}-1} & f_{0,\frac{p}{2}} & \cdots & f_{0,p-1} \\
f_{1,0} & \cdots & f_{1,\frac{p}{2}-1} & f_{1,\frac{p}{2}} & f_{1,\frac{p}{2}+1} & \cdots & f_{1,p} \\
\end{pmatrix}
\]

For $l = 0, 1, \ldots, d$, and $j = 0, 1, \ldots, p$, define tower elements as follows, where in the formulas below $f_{r,j}^{(l)}$ for $j > p - 1 + r$ is meant to be read modulo $p + r$:

\[
g_j^{(l)} = \mu_0(j)f_{0,j}^{(l)} + \mu_1(j)f_{1,j}^{(l)} \quad j = 0, 1, \ldots, p
\]

and for $l = 1, \ldots, d$

\[
g_j^{(l+d)} = (1 - \mu_0(j))f_{0,j}^{(l)} + (1 - \mu_1(j))f_{1,j}^{(l)} \quad j = 0, 1, \ldots, [p/2] - 1
\]

\[
g_j^{(l+d)} = (1 - \mu_1(j))f_{1,j}^{(l)} \quad \text{for } j = [p/2]
\]

\[
g_j^{(l+d)} = (1 - \mu_0(j-1))f_{0,j-1}^{(l)} + (1 - \mu_1(j))f_{1,j}^{(l)} \quad j = [p/2] + 1, \ldots, p.
\]

One now readily checks that $g_j^{(l)}$ satisfy the requirements.

If $p$ is not sufficiently large, we choose a $k$ such that $\varepsilon > \frac{1}{kp-3/2}$, choose Rokhlin towers $(f_{r,j}^{(l)})$ for $j = 1, \ldots, kp - 1 + r$ and $l = 0, \ldots, d$ with tolerance $\varepsilon/k$ instead of $\varepsilon$, and repeat the previous procedure to obtain Rokhlin towers $(g_j^{(l)})$ for $j = 0, 1, \ldots, kp - 1$ and $l = 0, \ldots, 2d$. Defining $g_j^{(l)} = \sum_{n=0}^{k-1} g_{j+np}^{(l)}$ for $j = 0, 1, \ldots, p - 1$ and $l = 0, \ldots, 2d$ gives us Rokhlin towers of height $p$ as required. \(\square\)

**Remark 2.9.** It follows from the preceding proof that in fact all the single towers can be chosen to be of the same height (rather than allowing for some to be of height $p$ and others of height $p + 1$).
3. Genericity in the $\mathcal{Z}$-stable case

We recall that the automorphism group of a $C^*$-algebra $A$ can be endowed with the topology of pointwise convergence generated by the sets

$$U_{\alpha,a} = \{ \beta \mid \|\beta(a) - \alpha(a)\| + \|\beta^{-1}(a) - \alpha^{-1}(a)\| < 1 \},$$

where $\alpha$ runs over all $\alpha \in \text{Aut}(A)$ and $a$ runs over all elements of $A$. If $A$ is separable, then this topology is Polish.

In this section we want to show that the property of having finite Rokhlin dimension is generic, more precisely the automorphisms a $\mathcal{Z}$-stable $C^*$-algebra which have Rokhlin dimension at most 1 form a dense $G_δ$ set in the topology of pointwise norm convergence.

Recall that for $p \in \mathbb{N}$ the dimension drop interval $\mathcal{Z}_{p,p+1}$ is given as

$$\mathcal{Z}_{p,p+1} = \{ f \in C([0,1], M_p \otimes M_{p+1}) \mid f(0) \in M_p \otimes 1, f(1) \in 1 \otimes M_{p+1} \}$$

and that the Jiang-Su algebra $\mathcal{Z}$ is an inductive limit of such dimension drop intervals.

**Lemma 3.1.** There exists a unitary $u \in \mathcal{Z}_{p,p+1}$ and positive elements

$$f_0, \ldots, f_{p-1}, g_0, \ldots, g_p \in \mathcal{Z}_{p,p+1}$$

such that

1. $f_if_j = 0, g_ig_j = 0$ and $[f_i, g_j] = 0$ for all $i \neq j$.
2. $f_0 + \ldots + f_{p-1} + g_0 + \ldots + g_p = 1$.
3. $uf_ju^* = f_{j+1}$ and $ug_ju^* = g_{j+1}$ for all $j$, where addition is modulo $p$ or $p+1$, respectively.

**Proof.** Let $h \in C([0,1])$ be defined by:

Let $v_x \in M_p$ be a continuous path of unitary matrices such that $v_x$ is a fixed cyclic permutation matrix of order $p$ for all $x \in [0,2/3]$, and $v_1 = 1$. Similarly, let $w_x \in M_{p+1}$ be a continuous path of unitaries, which is a fixed cyclic permutation matrix of order $p+1$ for all $x \in [1/3,1]$ and $w_0 = 1$. Let $u_x = v_x \otimes w_x$, then $u = (u_x)_{x \in [0,1]}$ is a unitary in $\mathcal{Z}_{p,p+1}$. Now, choose projections $a_0, \ldots, a_{p-1} \in M_p$ which add up to 1, such that $v_0a_jv_0^* = a_{j+1}$ (modulo $p$), and similarly choose projections $b_0, \ldots, b_p \in M_{p+1}$ which add up to 1 such that $v_1b_jv_1^* = b_{j+1}$ modulo $p+1$.

Setting $f_j(x) = h(x) \cdot a_x \otimes 1, g_j = (1-h(x)) \cdot 1 \otimes b_x$ gives us elements as required. □

**Lemma 3.2.** Let $A$ be a separable $C^*$-algebra. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, set $c = \sum_{j=1}^{n} a_j \otimes \max b_i \in A \otimes_{\max} A$, and let $p(x, x^*)$ be a noncommutative polynomial. For any $\varepsilon > \|p(c, c^*)\|$ there are a $\delta > 0$ and a finite subset $W \subseteq A$ such that the following holds.

For any $C^*$-algebra $B$ and any two $*$-homomorphisms $\varphi, \psi : A \to B$ which satisfy $\|\varphi(x), \psi(y)\| < \delta$ for all $x, y \in W$, if we set $d = \sum_{i=1}^{n} \varphi(a_i)\psi(b_i)$ then we have that $\|p(d, d^*)\| < \varepsilon$. 


Proof. Suppose not. Let $W_m$ be an increasing sequence of finite sets with dense union, then for any $m > 0$ we can find a $C^*$-algebra $B_m$ and a pair of homomorphisms $\varphi_m, \psi_m : A \to B_m$ such that $\|[(\varphi_m(x), \psi_m(y))]\| < 1/m$ for all $x, y \in W_m$ but if we set $d_m = \sum_{i=1}^n \varphi_m(a_i)\psi_m(b_i)$ then $\|p(d_m, d_m^*)\| \geq \varepsilon$. Let $\bar{\varphi} = (\varphi_1, \varphi_2, \ldots)$, $\bar{\psi} = (\psi_1, \psi_2, \ldots)$ be the induced homomorphisms $\bar{\varphi}, \bar{\psi} : A \to \prod B_n/\bigoplus B_n$. Notice that

$$\bar{\varphi}(x), \bar{\psi}(y) = 0$$

for any $x, y$, and therefore we have a well-defined homomorphism

$$\bar{\varphi} \otimes \bar{\psi} : A \otimes_{\max} A \to \prod B_n/\bigoplus B_n.$$

But then it follows that $\|p(\bar{\varphi}(c), \bar{\psi}(c)^*)\| \geq \varepsilon$ - contradiction.

We recall the following simple application of functional calculus:

**Lemma 3.3.** For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any unital $C^*$-algebra $A$, if $a \in A$ satisfies $\max\{|1 - a^*a|, |1 - aa^*|\} < \delta$ then there is a unitary $u \in A$ such that $\|u - a\| < \varepsilon$.

**Theorem 3.4.** Let $A$ be a unital separable $\mathbb{Z}$-stable $C^*$-algebra, then a dense $G_\delta$ set of automorphisms of $A$ has Rokhlin dimension $\leq 1$ with commuting towers (and in fact with single Rokhlin towers). In particular, Rokhlin dimension at most 1 is generic.

If furthermore $A \cong A \otimes D$ for $D$ a UHF algebra of infinite type then the Rokhlin property (i.e. Rokhlin dimension 0) is generic.

**Proof.** We shall give a proof for the $\mathbb{Z}$-stable case. The UHF-stable case is similar and will be omitted. Given $p \in \mathbb{N}$, a finite set $F \subset A$ and $\varepsilon > 0$, we shall say that an automorphism $\alpha \in \text{Aut}(A)$ has the $(p, F, \varepsilon)$-approximate 1-dimensional Rokhlin property if there are positive elements $f_0, f_{p-1}, g_0, \ldots, g_p \in A$ such that

1. $\|f_if_j\| < \varepsilon$ for all $i \neq j$, and $\|g_i g_j\| < \varepsilon$ for all $i \neq j$,
2. $\|f_0 + \ldots + f_{p-1} + g_0 + \ldots + g_p - 1\| < \varepsilon$,
3. $\|\alpha(f_j) - f_{j+1}\| < \varepsilon$ and $\|\alpha(g_j) - g_{j+1}\| < \varepsilon$ for all $j$, where addition is modulo $p$ or $p + 1$, respectively,
4. $\|[f_j, a]\| < \varepsilon$ and $\|[g_j, a]\| < \varepsilon$ for all $j$ and all $a \in F$,
5. $\|[f_i, g_j]\| < \varepsilon$ for all $i, j$.

We denote by $V_{p, F, \varepsilon}$ the set of all automorphisms $\alpha \in \text{Aut}(A)$ which satisfy the $(p, F, \varepsilon)$-approximate 1-dimensional Rokhlin property. It is clear that $V_{p, F, \varepsilon}$ is open.

Now, if we choose an increasing sequence of finite sets $F_n \subset A$ with dense union, then any

$$\alpha \in \bigcap_{p \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} V_{p, F_n, \frac{1}{n}}$$

has Rokhlin dimension at most 1. It thus suffices to prove that $V_{p, F, \varepsilon}$ is dense for any $p, F, \varepsilon$. We thus fix an automorphism $\alpha$ and a triple $(p, F, \varepsilon)$. For any finite set $F_0 \subset A$ and $\gamma > 0$ we need to find $\beta \in V_{p, F, \varepsilon}$ such that

$$\max\{|\beta(a) - \alpha(a)|, |\beta^{-1}(a) - \alpha^{-1}(a)|\} < \gamma$$

for all $a \in F_0$. Since enlarging $F$ simply imposes additional conditions, we may assume without loss of generality that $F \supseteq F_0$, for notational convenience, and
we furthermore assume that all elements of $F$ have norm at most 1. We may furthermore assume that $\gamma < \varepsilon$.

Fix $\gamma / 10 > \eta > 0$ as in Lemma 3.3 such that for any unital $C^*$-algebra $A$, if $a \in A$ satisfies $\text{max}\{\|1 - a^*a\|,\|1 - aa^*\|\} < \eta$ then there is a unitary $u \in A$ such that $\|u - a\| < \gamma / 10$.

Choose a unitary $u \in Z_{p,p+1} \subset Z$ and elements $f_0, \ldots, f_{p-1}, g_0, \ldots, g_p \in Z_{p,p+1} \subset Z$ as in Lemma 3.1. Let $G = \{u, f_0, \ldots, f_{p-1}, g_0, \ldots, g_p\}$.

Choose $w \in Z \otimes Z$ such that $\|w(x \otimes 1)w^* - 1 \otimes x\| < \eta$ for all $x \in G$. Find contractions $a_1, \ldots, a_n, b_1, \ldots, b_n \in Z$ such that $\|w - \sum_{i=1}^n a_i \otimes b_i\| < \eta$. Notice that

$$\left\| 1 - \sum_{i,j=1}^n a_i a_j^* \otimes b_j b_j^* \right\|, \left\| 1 - \sum_{i,j=1}^n a_i^* a_j \otimes b_j^* b_j \right\| < 2\eta$$

and furthermore

$$\left\| \sum_{i=1}^n a_i x a_i^* \otimes b_j b_j^* - 1 \otimes x \right\|, \left\| \sum_{i=1}^n a_i^* a_j \otimes b_j^* x b_j - x \otimes 1 \right\| < 3\eta$$

Choose $W$ and $\delta$ as in Lemma 3.2 for the four expressions above and with $2\eta$, $3\eta$ instead of $\varepsilon$. We may assume without loss of generality that $\delta < \eta / 3n$ and that $W$ contains $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Choose a unital embedding $\varphi : Z \to A$ such that $\|\varphi(x), y\| < \delta$ for all $x \in G \cup W$ and $y \in \alpha(F) \cup F \cup \alpha^{-1}(F)$. Notice that $\|\varphi(x), y\| < \delta$ for all $x \in G \cup W$ and $y \in F$.

Now, choose a unital embedding $\psi : Z \to A$ such that $\|\psi(x), y\| < \delta$ for all $x \in W$, $y \in F \cup \alpha^{-1}(F) \cup \varphi(G \cup W) \cup \alpha \circ \varphi(G \cup W)$.

Set

$$v_1 = \sum_{i=1}^n \varphi(a_i) \psi(b_i), \quad v_2 = \sum_{i=1}^n \alpha(\varphi(a_i)) \psi(b_i).$$

Now, for any $x \in F$ we have that

$$\|v_1 \varphi(x) v_1^* - \psi(x)\| < 3\eta$$

and

$$\|v_2^* \psi(x) v_2 - \alpha(\varphi(x))\| < 3\eta.$$
It remains to check that $\beta \in V_{p,F,\varepsilon}$. We claim that the elements $\varphi(f_i), \varphi(g_i)$ have the required properties. The facts that they are orthogonal to each other, add up to 1, $\varepsilon$-commute with the elements of $F$ and with each other follow immediately from the requirements we imposed on $\varphi$. It thus remains to check that they are almost permuted by $\beta$. Indeed,

$$\beta(\varphi(f_i)) = z\alpha(\varphi(u)\varphi(f_i)\varphi(u^*))z^* = z\alpha(\varphi(f_{i+1}))z^*$$

and therefore

$$\|\beta(\varphi(f_i)) - \varphi(f_{i+1})\| = \|z\alpha(\varphi(f_{i+1}))z^* - \varphi(f_{i+1})\| \leq 4\gamma/10 + 6\eta < \varepsilon$$

as required.

**Remark 3.5.** A modification of this argument can be used to show the following. Let $D$ be a strongly self-absorbing $C^*$-algebra, and let $A$ be a unital $D$-stable $C^*$-algebra, then for a dense $G_\delta$ set of automorphisms of $A$ we have that $A \rtimes_\alpha \mathbb{Z}$ is $D$-stable as well. This can be done by showing that for any given $\varepsilon > 0$ and finite sets $F \subseteq A$, $G \subseteq D$, the set of automorphisms $\alpha$ such that there exists unital homomorphism $\varphi : D \to A$ such that $\|\varphi(x), a\| < \varepsilon$ for all $x \in G$, $a \in F$ and furthermore $\|\alpha(\varphi(x)) - \varphi(x)\| < \varepsilon$ for all $x \in G$ is a dense open set. We omit the proof.

4. Permanence of finite nuclear dimension

In this section we show that forming a crossed product by an automorphism with finite Rokhlin dimension preserves finiteness of nuclear dimension.

**Theorem 4.1.** Let $A$ be a separable unital $C^*$-algebra of finite nuclear dimension and $\alpha \in \text{Aut}(A)$ an automorphism with finite Rokhlin dimension. Then $A \rtimes_\alpha \mathbb{Z}$ has finite nuclear dimension with

$$\dim_{\text{nuc}}(A \rtimes_\alpha \mathbb{Z}) \leq 4(\dim_{\text{Rok}}(\alpha) + 1)(\dim_{\text{nuc}}(A) + 1) - 1$$

**Proof.** Denote the nuclear dimension of $A$ by $N$ and denote $d = 2\dim_{\text{Rok}}(\alpha) + 1$. By Proposition 2.8 and the subsequent remark, we have that $\dim_{\text{Rok}}^*(\alpha) \leq d$ and furthermore the towers can all be chosen to be of the same height. (The fact that the towers can be chosen to be of the same height is not important for the proof, but simplifies notation a bit.) Let $F \subseteq A \rtimes_\alpha \mathbb{Z}$ be a given finite set. We need to construct a piecewise contractive c.p. approximation $(F, \Phi, \Psi)$ which is $2(d+1)(N+1) - 1$-decomposable and of tolerance $\varepsilon$ on $F$.

Recall that $A \rtimes_\alpha \mathbb{Z} \to B(\ell^2(\mathbb{Z}, H))$ is generated by a copy of $A$ and a unitary $u$ acting on $\ell^2(\mathbb{Z}, H)$, cf. the appendix. We may and shall assume that $F$ consists of contractions all lying in the algebraic crossed product i.e. there exists $q \in \mathbb{N}$ such that all elements of $F$ are of the form $x = \sum_{i=-q}^q x(i)u^i$, where $x(i) \in A$ are coefficients uniquely determined by $x$. Let $\bar{F} \subseteq A$ be the finite set of all such coefficients of the elements in $F$. Let $p \in \mathbb{N}$ be a positive integer (much) larger than $q$ to be specified later. We shall furthermore require that $p$ is even to slightly simplify notation.

Let $Q$ be the projection onto the subspace $\ell^2(\{0,\ldots, p-1\}, H) \subseteq \ell^2(\mathbb{Z}, H)$ so that $x \mapsto QxQ$ is a u.c.p. map from $A \rtimes_\alpha \mathbb{Z}$ to $M_p(A)$ indexed by $0,\ldots, p-1$.

Define decay factors

$$d_n = 1 - \frac{|(p-1)/2 - n|}{(p-1)/2}, \quad n = 0,\ldots, p-1,$$
We observe that $|\sqrt{d_i} - \sqrt{d_j}| \leq \frac{2|i-j|}{p-1}$. Define $D \in M_p$ to be diag$(d_0, d_1, \ldots, d_{p-1})$. Define $\mu : A \rtimes_{\alpha} \mathbb{Z} \to M_p(A)$ to be the c.p.c. map given by

$$\mu(x) = \sqrt{D}QxQ\sqrt{D}.$$ 

Notice that

$$\|[\sqrt{D}, Qau^mQ]\| = \left\| \sum_{i=m}^{p-1} (\sqrt{d_i} - \sqrt{d_{i-m}})e_{i,i-m} \otimes \alpha^{-i}(a) \right\| \leq \max_i(|\sqrt{d_i} - \sqrt{d_{i-m}}|) \|a\| \leq \frac{2|m|}{p-1} \cdot \|a\|,$$

where $a \in A$ and $|m| \leq q$. In fact these estimates hold for all bounded operator matrices $x = [x_{i,j}]$ of width $q$ (i.e. $x_{i,j} \neq 0$ implies $|i-j| \leq q$). It follows that for $|m| \leq q$

$$\|\mu(au^m) - DQau^mQ\| \leq \sqrt{\frac{2q}{p-1}} \cdot \|a\|.$$ 

Next let $\tilde{F}_1 := \tilde{F} \cup \alpha(\tilde{F}) \cup \ldots \cup \alpha^p(\tilde{F})$ and let $(\mathcal{F}, \psi, \phi)$ be a piecewise contractive $N$-decomposable c.p. approximation,

$$\psi : A \to \mathcal{F} = \mathcal{F}^{(0)} \oplus \ldots \oplus \mathcal{F}^{(N)},$$

$$\phi : \mathcal{F} = \mathcal{F}^{(0)} \oplus \ldots \oplus \mathcal{F}^{(N)} \to A,$$

where $\phi^{(i)} = \phi|\mathcal{F}^{(i)}$ is an order zero contraction for every $i$ and $\|\phi(\psi(x)) - x\| < \varepsilon$ for all $x \in \tilde{F}_1$. Consider

$$\tilde{\psi}^{(i)} := \text{id}_{M_p} \otimes \psi : M_p(A) \to M_p(\mathcal{F}^{(0)}) \oplus \ldots \oplus M_p(\mathcal{F}^{(N)}) = M_p(\mathcal{F})$$

$$\tilde{\phi}^{(i)} := \text{id}_{M_p} \otimes \phi : M_p(\mathcal{F}) = M_p(\mathcal{F}^{(0)}) \oplus \ldots \oplus M_p(\mathcal{F}^{(N)}) \to M_p(A)$$

be defined by $\tilde{\psi}^{(i)} = \text{id}_{M_p} \otimes \psi^{(i)}$, $\tilde{\phi}^{(i)} = \text{id}_{M_p} \otimes \phi^{(i)}$. Then one checks that $\tilde{\phi}^{(i)} : M_p(\mathcal{F}^{(i)}) \to M_p(A)$ are order zero contractions so that $(\tilde{\psi}, \tilde{\phi})$ is a piecewise contractive $N$-decomposable approximations for $M_p(A)$ of tolerance $p^2 \varepsilon$ on $M_p(\tilde{F}_1)$.

Let now $B_{\mathcal{F}}$ be the closed norm compact unit ball in $\mathcal{F}$ and define the norm compact subset $K = \bigcup_{j=0}^{p} \bigcup_{i=0}^{N} \alpha^{-j}(\mathcal{F}^{(i)}(B_{\mathcal{F}}))$ of $A$.

Let $(f^{(l)}_i)_{i=0,\ldots,p-1}$ be single Rokhlin towers with respect to the given $\varepsilon$ and the compact set $K$ (we note that one may obviously replace the finite set in the definition by a compact set). For $j > p-1$ we understand $f^{(l)}_j$ to mean $f^{(l)}_{j(\mod p)}$. Define $\rho_0^{(l)}, \rho_1^{(l)} : M_p(A) \to A \rtimes_{\alpha} \mathbb{Z}$ by

$$\rho_0^{(l)}(e_{i,j} \otimes a) = f_{i,j}^{(l)(1/2)} u^i au^{-j} f_j^{(l)(1/2)}, \quad \rho_1^{(l)}(e_{i,j} \otimes a) = (f_{p/2+i}^{(l)})^{1/2} u^i au^{-j}(f_{p/2+i}^{(l)})^{1/2}.$$
One checks that $\rho_0^l$ is approximately order zero i.e. denoting $f^l = \sum_i f_i^l$ we have:

$$\|\rho_0^l(e_{i,j} \otimes a)\rho_0^l(e_{k,m} \otimes b) - f^l \rho_0^l((e_{i,j} \otimes a)(e_{k,m} \otimes b))\| < c p \varepsilon$$

for a constant $c > 0$ which depends only on $\varepsilon$ and for all $a, b \in K$. The same applies for $\rho_1^l$. This is easily verified by approximating the square root function by a polynomial vanishing at 0, and we omit the calculation. Similarly, we see that there is a constant $c'$ such that

$$\|\rho_0^l(e_{i,j} \otimes a) - f_i^l u^i a u^{-j}\| < c' \varepsilon$$

for all $a \in \tilde{F}_1$, and a similar estimate holds for $\rho_1^l$.

Let $x = a u^m$ for $|m| \leq q$, $a \in \tilde{F}$. Denote $y = Q x Q = \sum_{i=m}^{p-1} e_{i,i-m} \otimes \alpha^{-i}(a)$. Then we have

$$\rho_0^l(Dy) + \rho_1^l(Dy) = \sum_{i=m}^{p-1} d_i f_i^{l/2} u^i \alpha^{-i}(a) u^{m-i} f_i^{l/2} + \sum_{i=m}^{p-1} d_i f_i^{l/2} u^i \alpha^{-i}(a) u^{m-i} f_i^{l/2} = 2pc \varepsilon$$

$$\sum_{i=m}^{p-1} d_i (f_i^{l/2} + f_i^{l/2}) u^m$$

Now, notice that $\sum_{i=0}^{p-1} d_i (f_i^{l/2} + f_i^{l/2}) = f^l$, and

$$\left| \sum_{i=0}^{m-1} d_i (f_i^{l/2} + f_i^{l/2}) \right| \leq 2 \sum_{i=0}^{q-1} d_i = \frac{q(q-1)}{(p-1)/2}.$$

It follows that $\|\rho_0^l(Dy) + \rho_1^l(Dy) - f^l a u^m\| \leq 2pc \varepsilon + \frac{q(q-1)}{(p-1)/2}$, and therefore

$$\|\rho_0^l \circ \mu(y) + \rho_1^l \circ \mu(y) - f^l a u^m\| \leq 2pc \varepsilon + \frac{q(q-1)}{(p-1)/2} + 2 \sqrt{\frac{2q}{p-1}}.$$

Thus, given $\eta > 0$ we could first choose $p$ large enough and then $\varepsilon > 0$ small enough so that

$$\|\rho_0^l \circ \mu(a u^m) + \rho_1^l \circ \mu(a u^m) - f^l a u^m\| \leq \eta$$

for all $a \in F$ and all $|m| \leq q$. Consider the maps

$$\tilde{\psi} : M_p(A) \to \bigoplus_{j=0}^{2d+1} M_p(\mathcal{F})$$

given by $\tilde{\psi}(x) = \psi^{(p)}(x) \oplus \psi^{(p)}(x) \oplus \ldots \psi^{(p)}(x)$ and

$$\tilde{\varphi} : \bigoplus_{j=0}^{2d+1} M_p(\mathcal{F}) \to \bigoplus_{j=0}^{2d+1} M_p(A)$$

given by $\tilde{\varphi} = \bigoplus_{j=0}^{2d+1} \varphi^{(p)}$.

Now define $\rho : \bigoplus_{j=0}^{2d+1} M_p(A) \to A \cong \mathbb{Z}$ by

$$\rho(x_0, x_1, \ldots, x_{2d+1}) = \sum_{j=0}^{d-1} \sum_{i=0}^{1} \rho_i^j(x_{2j+i})$$
With $a, m$ and $\eta$ as above, we have that
\[
\left\| \rho(\mu(au^m), \mu(au^m), \ldots, \mu(au^m)) - \sum_{l=0}^{d} f^{(l)} au^m \right\| \leq d\eta
\]
and therefore
\[
\|\rho(\mu(au^m), \mu(au^m), \ldots, \mu(au^m)) - au^m\| \leq d\eta + \varepsilon.
\]

To summarize, we constructed the following maps.

\[
\begin{array}{c}
A \rtimes_{\alpha} Z \\
\mu \\
M_p(A) \\
\Phi
\end{array}
\quad
\begin{array}{c}
A \rtimes_{\alpha} Z \\
\rho \\
\bigoplus_{j=0}^{2d+1} M_p(A) \\
\phi
\end{array}
\quad
\begin{array}{c}
\bigoplus_{j=0}^{2d+1} M_p(F) \\
\psi
\end{array}
\]\n
We notice that $\rho^{(j)}_i \circ \varphi^{(p)}|_{M_p(F^{(i)})}$ is approximately order zero, i.e. for any $a, b \in M_p(F^{(i)})$ of norm at most 1, we have that
\[
\|\rho^{(j)}_i \circ \varphi^{(p)}(a) \rho^{(j)}_i \circ \varphi^{(p)}(b) - f^{(j)} \rho^{(j)}_i \circ \varphi^{(p)}(ab)\| < c\varepsilon.
\]
By stability of order zero maps, given $\eta > 0$ one can choose $\varepsilon > 0$ small enough so that for this choice of $\varepsilon$, there exists a contractive order zero map $\zeta : M_p(F^{(i)}) \to A \rtimes_{\alpha} Z$ such that $\|\zeta - \rho^{(j)}_i \circ \varphi^{(p)}\| < \eta$. Putting together such approximating order zero maps $\zeta$ for the various maps $\rho^{(j)}_i \circ \varphi^{(p)}$, we see that there is a $2(d+1)(N+1)-1$-decomposable map $\Phi' : \bigoplus_{j=0}^{2d+1} M_p(F) \to A \rtimes_{\alpha} Z$ such that $\|\Phi' - \Phi\| \leq 2(d+1)(N+1)\eta$.

One obtains that
\[
\begin{array}{c}
A \rtimes_{\alpha} Z \\
\psi \\
\bigoplus_{j=0}^{2d+1} M_p(F) \\
\Phi'
\end{array}
\quad
\begin{array}{c}
A \rtimes_{\alpha} Z \\
\Phi \\
\bigoplus_{j=0}^{2d+1} M_p(F) \\
\phi
\end{array}
\]
is a $2(d+1)(N+1)-1$-decomposable approximation for $F$ to within tolerance $C\eta$ for some constant $C$ which depends only on $d$ and $N$. This shows that $\dim_{\text{huc}}(A \rtimes_{\alpha} Z) \leq 2(d+1)(N+1) - 1$, as required.

5. Permanence of $\mathcal{Z}$-stability

The purpose of this section is to show that if $A$ is a unital separable $\mathcal{Z}$-stable $C^*$-algebra, then so is any crossed product by an automorphism which has finite Rokhlin dimension with commuting towers. As shown above, this property is generic and we do not know whether there is an automorphism which has finite Rokhlin dimension but without commuting towers. Our preferred criterion for $\mathcal{Z}$-stability will be Proposition A.5 of the appendix.

We begin with a simple preliminary lemma, whose proof is straightforward and left to the reader.
Lemma 5.1. Let $B$ be a $C^*$-algebra. Suppose $B$ is generated by a compact set $Y \subseteq B$. For any $\varepsilon > 0$ and any compact subset $F \subseteq B$ there is an $\varepsilon' > 0$ which satisfies the following. For any $C^*$-algebra $A$, any automorphism $\alpha$ of $A$ and any homomorphism $\pi : B \to A$, if $\|\alpha(\pi(y)) - \pi(y)\| < \varepsilon'$ for any $y \in Y$ then $\|\alpha(\pi(x)) - \pi(x)\| < \varepsilon$ for any $x \in F$.

The following lemma is also easy to verify and we leave its proof to the reader. We denote by $A^+$ the unitization of $A$ such that $A$ is a codimension 1 ideal in $A^+$ (that is, if $A$ is unital then $A^+ \cong A \oplus \mathbb{C}$).

Lemma 5.2. Let $A, B$ be two $C^*$-algebras. Let $D$ be the kernel of the canonical map $A^+ \otimes_{\text{max}} B^+ \to \mathbb{C}$, and let us denote by $\iota_A, \iota_B$ the following homomorphisms.

\[ \iota_A(a) = a \otimes 1 \quad \iota_B(b) = 1 \otimes b \]

Then $D$ has the following universal property. For any $C^*$-algebra $E$ and any two homomorphisms $\gamma_A : A \to E$, $\gamma_B : B \to E$ with commuting images there is a unique homomorphism $\theta : D \to E$ such that the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_A} & D \\
\downarrow{\gamma_A} & & \downarrow{\theta} \\
B & \xleftarrow{\iota_B} & E
\end{array}
\]

Lemma 5.3. Let $D_n^{(k)} = \ker((CM_n^+)^{\otimes k} \to \mathbb{C})$. For any positive contraction $h \in Z(D_n^{(k)}) \cong C_0([0, 1]^k \setminus \{(0, 0, \ldots, 0)\})$ there exists an order zero map $\theta : M_n \to D_n^{(k)}$ such that $\theta(1)(x) = h(x)$ for all $x \in [0, 1]^k$.

Proof. For $t \in [0, 1]$, let

\[ E_t = \left\{ \begin{array}{ll} M_n & | t > 0 \\ \mathbb{C} \cdot 1 \subseteq M_n & | t = 0 \end{array} \right. \]

For $\vec{t} = (t_1, \ldots, t_k) \in [0, 1]^k$, we let

\[ E_{\vec{t}} = E_{t_1} \otimes E_{t_2} \otimes \ldots \otimes E_{t_k} \]

We observe that

\[ D_n^{(k)} \cong \left\{ f \in C_0([0, 1]^k \setminus \{(0, 0, \ldots, 0)\}, M_n^{\otimes k}) \mid f(\vec{t}) \in E_{\vec{t}} \right\} \]

It is easy to construct a unital homomorphism $\pi : M_n \to M(D_n^{(k)})$, and it is readily seen that $\theta(x)(\vec{t}) = h(\vec{t}) \cdot \theta(x)(\vec{t})$ satisfies the desired properties. \hfill $\square$

The following is a simple modification of Lemma 2.4 from [7]. While this lemma can be generalized to actions of non-discrete groups in a straightforward way (as in the said lemma), we state it here just for actions of discrete groups to avoid notation which we do not need in this paper. We denote $A_\infty = \ell^\infty(N, A)/c_0(N, A)$, with $A$ embedded as the subalgebra of constant sequences in $A_\infty$. We use a similar idea to one which was used in Proposition 2.2 from [27].

If $\alpha : G \to \text{Aut}(A)$ is an action then we have naturally induced actions of $G$ on $A_\infty$ and $A_\infty \cap A$. We denote those actions by $\bar{\alpha}$.
Lemma 5.4. Let $A, B$ be a unital separable $C^*$-algebras. Let $G$ be a discrete countable group with an action $\alpha : G \to \text{Aut}(A)$. Suppose that $B_n$ is a sequence of unital nuclear subalgebras (with a common unit) of $B$ with dense union. Suppose that for any $n$, any finite subset $F \subseteq B_n$, any $\varepsilon > 0$ and any finite set $G_0 \subseteq G$ there is a unital homomorphism $\gamma : B_n \to A_\infty \cap A'$ such that
\[ \|\gamma_g(\gamma(a)) - \gamma(a)\| < \varepsilon \]
for all $a \in F$ and all $g \in G_0$, then there is a unital homomorphism from $B$ into the fixed point subalgebra of $A_\infty \cap A'$. If $B_n$ is finitely generated, then it suffices to check this for a generating set $F$.

If $B$ is furthermore strongly-self absorbing then it follows that the maximal (hence any) crossed product absorbs $B$ tensorially.

Proof. To verify the properties of Lemma 2.4 from [7], it suffices to show that for any finite set $F \subseteq B$, any finite subset $G_0 \subseteq G$ and any $\varepsilon > 0$ there is a unital c.p. map $\varphi : B \to A_\infty \cap A'$ such that
\begin{enumerate}
  \item $\|\varphi_g(\varphi(b)) - \varphi(b)\| < \varepsilon$ for all $g \in G_0$, $b \in F$.
  \item $\|\varphi(x)\varphi(y) - \varphi(xy)\| < \varepsilon$ for all $x, y \in F$.
\end{enumerate}
Doing a small perturbation of the elements of $F$ if need be, we may assume without loss of generality that $F \subseteq B_n$ for a sufficiently large $n$, and that all the elements of $F$ have norm at most 1, and that $1 \in F$ (so that $F \cdot F \supseteq F$ - this is just for notational convenience). Fix such an $n$. Since $B_n$ is nuclear, one can find a finite dimensional algebra $E$ and unital c.p. maps
\[ B_n \xrightarrow{\psi} E \xrightarrow{\theta} B_n \]
such that $\|\theta \circ \psi(x) - x\| < \varepsilon/6$ for all $x \in F \cdot F$. It follows that
\[ \|\theta \circ \psi(x) \cdot \theta \circ \psi(y) - \theta \circ \psi(xy)\| < \varepsilon/2 \]
for all $x, y \in F$. By the Arveson extension theorem, we can extend $\psi$ to a unital c.p. map $\tilde{\psi} : B \to \tilde{E}$. Choose a homomorphism $\gamma$ as in the statement of the lemma, with $\varepsilon/2$ instead of $\varepsilon$. Define
\[ \varphi = \gamma \circ \theta \circ \tilde{\psi} \]
and $\varphi$ satisfies the required conditions. It is straightforward to check that if one can find such a $\varphi$ for a generating set then one could find it for any other finite set.

If $B$ is strongly self-absorbing then the fact that $A \rtimes_{\alpha} G$ absorbs $B$ tensorially now follows from the results in [7].

The previous lemma, together with the characterization of prime dimension drop algebras from [24], immediately gives the following.

Corollary 5.5. Let $A$ be a unital separable $C^*$-algebra. Let $G$ be a discrete countable group with an action $\alpha : G \to \text{Aut}(A)$. Suppose that for any positive integer $n$, any $\varepsilon > 0$ and any finite subset $G_0 \subseteq G$ there are order zero maps $\theta : M_n \to A_\infty \cap A'$, $\eta : M_{n+1} \to A_\infty \cap A'$ with commuting ranges such that
\[ \theta(1) + \eta(1) = 1 \]
and such that
\[ \|\alpha_g(\theta(x)) - \theta(x)\| < \varepsilon \]
and
\[ \|\alpha_g(\eta(x)) - \eta(x)\| < \varepsilon \]
for any $x$ in the unit ball of $M_n$ or $M_{n+1}$, respectively, and any $g \in G_0$, then $A \rtimes_\alpha G$ is $Z$-stable.

**Remark 5.6.** In the previous lemma and corollary, if $G$ is generated by a subset $\Gamma$ then it is sufficient to consider finite subsets $G_0 \subseteq \Gamma$ rather than all finite subsets in $G$. In particular, for actions of $\mathbb{Z}$ we will consider a single generator.

**Lemma 5.7.** Let $A$ be a unital separable $C^*$-algebra. Let $G$ be a discrete countable group with an action $\alpha : G \to \text{Aut}(A)$. Let $d$ be a non-negative integer. Suppose that for any positive integer $n$, any $\varepsilon > 0$ and any finite subset $G_0 \subseteq G$ there are order zero maps $\theta_0, \ldots, \theta_d : M_n \to A_\infty \cap A'$, $\eta_0, \ldots, \eta_d : M_{n+1} \to A_\infty \cap A'$ with commuting ranges such that for all $g \in G_0$, $k = 0, 1, \ldots, d$ and all $x$ in the unit ball of $M_n$ or $M_{n+1}$ respectively we have that

$$\|\bar{\alpha}_g(\theta_k(x)) - \theta_k(x)\| < \varepsilon,$$

$$\|\bar{\alpha}_g(\eta_k(x)) - \eta_k(x)\| < \varepsilon$$

and

$$\sum_{k=0}^{d} \theta_k(1) + \eta_k(1) = 1$$

then $A \rtimes_\alpha G$ is $Z$-stable.

**Proof.** Let $D_n^{(d+1)}$, $D_{n+1}^{(d+1)}$ be as in Lemma 5.3.

Denote by $\zeta_j : CM_n \to D_n^{(d+1)}$ the $j$'th coordinate embedding map

$$\zeta_j(x) = 1 \otimes 1 \otimes \ldots \otimes x \otimes \ldots \otimes 1$$

Let $\beta : M_n \to CM_n$ be the canonical order zero map given by

$$\beta(a)(t) = t \cdot a, \quad a \in M_n, t \in [0, 1]$$

Denote $h = \sum_{k=0}^{d} \zeta_k(\beta(1))$. Note that $h$ is a strictly positive element of the center of $D_n^{(d+1)}$. Denote by $M_n^1$ the closed unit ball of $M_n$, and set

$$Y = \bigcup_{k=0}^{d} \zeta_k(\beta(M_n^1))$$

Notice that $Y$ is compact and generates $D_n^{(d+1)}$.

Use Lemma 5.3 to find an order zero map $\mu : M_n \to D_n^{(d+1)}$ such that $\mu(1) = h$. (Lemma 5.3 was only formulated for contractions, which $h$ is not – but we can simply replace $h$ by $||h||^{-1}h$, then apply the lemma, and multiply the resulting order zero map by $||h||$.) Denote $F = \mu(M_n^1)$.

We repeat the same procedure for $D_n^{(d+1)}$, denoting the resulting maps, sets and elements by $\zeta'_j$, $\beta'$, $h'$, $Y'$, $\mu'$ and $F'$.

Choose $\varepsilon'$ as in Lemma 5.1 with respect to the compact set of generators $Y$ and the compact set $F$ and with respect to the compact set of generators $Y'$ and the compact set $F'$ (take the least of the two).

Let

$$\pi : D_n^{(d+1)} \to A_\infty \cap A'$$
be the unique homomorphism which satisfies that $\pi \circ \zeta_k \circ \beta = \theta_k$ for $k = 0, 1, \ldots, d$. Notice that

$$\pi(h) = \sum_{k=0}^{d} \theta_k(1)$$

(and that $\pi(h)$ in fact is a contraction). We have then that $\pi \circ \mu : M_n \to \mathcal{A}_\infty \cap \mathcal{A}'$ is an order zero map, that

$$\pi \circ \mu(1) = \sum_{k=0}^{d} \pi \circ \zeta_k \circ \beta(1)$$

and that for any $x \in M_n^1$ and any $g \in G_0$ we have

$$\|\bar{\alpha}_g(\pi \circ \mu(x)) - \pi \circ \mu(x)\| < \varepsilon$$

We similarly obtain a homomorphism

$$\pi' : D_{d+1} \to \mathcal{A}_\infty \cap \mathcal{A}'$$

whose range commutes with that of $\pi$, such that

$$\pi'(h') = \sum_{k=0}^{d} \eta_k(1)$$

and such that $\pi' \circ \mu'$ satisfies the analogous properties to that of $\pi \circ \mu$.

Thus, $\pi \circ \mu(1) + \pi' \circ \mu'(1) = 1$, and those two order zero maps satisfy the conditions of Corollary 5.5. Therefore $\mathcal{A} \rtimes_\alpha G$ is $\mathcal{Z}$-stable, as required. \qed

The main theorem of this section is a partial generalization of Theorem 4.4 from [7] (that theorem is for the Rokhlin property, but it applies to absorption of general strongly self-absorbing $C^*$-algebras and not just the Jiang-Su algebra).

For the proof, we shall implicitly use the following immediate observation: by choosing a sequence of Rokhlin tower elements, one can view them as sitting in the central sequence algebra $\mathcal{A}_\infty \cap \mathcal{A}'$, in which case the approximate properties in the definition of Rokhlin dimension hold exactly.

**Theorem 5.8.** Let $\mathcal{A}$ be a separable unital $\mathcal{Z}$-stable $C^*$-algebra, and let $\alpha$ be an automorphism of $\mathcal{A}$ with $\text{dim}_{\text{Rok}}(\alpha) = d < \infty$, then the crossed product $\mathcal{A} \rtimes_\alpha \mathcal{Z}$ is $\mathcal{Z}$-stable as well.

**Proof.** We establish that the conditions of Lemma 5.7 hold.

Let $r$ be a given positive integer. Fix two order zero maps $\theta : M_r \to \mathcal{Z}$, $\eta : M_{r+1} \to \mathcal{Z}$ with commuting ranges such that $\theta(1) + \eta(1) = 1$. Let $K$ be the union of the images of the unit balls of $M_r, M_{r+1}$ under those maps.

We define unital homomorphisms $\iota_0, \ldots, \iota_d, \mu_0, \ldots, \mu_d : \mathcal{Z} \to \mathcal{A}_\infty \cap \mathcal{A}'$ as follows. First fix $\iota_0 : \mathcal{Z} \to \mathcal{A}_\infty \cap \mathcal{A}'$. Proceeding inductively, we choose $\iota_k : \mathcal{Z} \to \mathcal{A}_\infty \cap \mathcal{A}'$ such that its image further commutes with $\bar{\alpha}^j(\iota_i(\mathcal{Z}))$ for all $j \in \mathbb{Z}$ and $i < k$ (this can be done by Lemma 4.5 of [7]). We define $\mu_0, \ldots, \mu_d$ in a similar way, such that the image of $\mu_k$ commutes with $\bar{\alpha}^j(\iota_i(\mathcal{Z}))$ for all $j \in \mathbb{Z}$ and $i < k$ as well as $\bar{\alpha}^j(\iota_i(\mathcal{Z}))$ for all $j \in \mathbb{Z}$ and all $i = 0, 1, \ldots, d$.

Let

$$\mathcal{B}_k := C^*(\mu_k(\mathcal{Z}) \cup \bigcup_{j=-\infty}^{\infty} \bar{\alpha}^j(\iota_k(\mathcal{Z})))$$

note that $\mathcal{B}_k \cong \mathcal{B}_k \otimes \mathcal{Z}$, and that the elements of $\mathcal{B}_k$ commute with those of $\mathcal{B}_m$ for $k \neq m$. 

Choose a unitary \( w \in U(Z \otimes Z) \) such that
\[
\| w(x \otimes 1)w^* - 1 \otimes x \| < \frac{\varepsilon}{4}
\]
for all \( x \in K \). Notice that the unitary group of the Jiang-Su algebra is connected. Thus, \( w \) can be connected to 1 via a rectifiable path. Let \( L \) be the length of such a path. Choose \( n \) such that \( L\|x\|/n < \varepsilon/8 \) for all \( x \in F \).

Define homomorphisms
\[
\rho_k, \rho'_k : Z \otimes Z \to B_k \quad k = 0, 1, \ldots, d
\]
by
\[
\rho_k(x \otimes y) = \iota_k(x)\mu_k(y), \quad \rho'_k(x \otimes y) = \bar{\alpha}^n(\iota_k(x))\mu_k(y).
\]
Pick unitaries \( 1 = w_0, w_1, \ldots, w_n = w \in U_0(Z \otimes Z) \) such that \( \|w_j - w_{j+1}\| \leq L/n \) for \( j = 0, \ldots, n - 1 \). Now, for \( k = 0, \ldots, d \), let
\[
u^{(k)}_j = \rho_k(w_j)\rho'_k(w_j)
\]
Note that
\[
\|\nu^{(k)}_j - \nu^{(k)}_{j+1}\| \leq \frac{2L}{n}
\]
for \( j = 0, \ldots, n - 1 \), that \( \nu^{(k)}_0 = 1 \), and that
\[
\|\nu^{(k)}_n\bar{\alpha}^n(\iota_k(x))\nu^{(k)*}_n - \iota_k(x)\| < \frac{\varepsilon}{2}
\]
for all \( x \in K \).

Similarly, we choose unitaries \( 1 = v^{(k)}_0, v^{(k)}_1, \ldots, v^{(k)}_{n+1} \in B_k \) such that
\[
\|v^{(k)}_j - v^{(k)}_{j+1}\| \leq \frac{2L}{n+1}
\]
for \( j = 0, \ldots, n \) and
\[
\|v^{(k)}_{n+1}\bar{\alpha}^{n+1}(\iota_k(x))v^{(k)*}_{n+1} - \iota_k(x)\| < \frac{\varepsilon}{2}
\]
for all \( x \in K \).

Let \( \{f^{(l)}_0, \ldots, f^{(l)}_{0,n-1}, f^{(l)}_{1,0}, \ldots, f^{(l)}_{1,n} \mid l = 0, \ldots, d \} \in \mathcal{A}_\infty \cap \mathcal{A}' \) commuting Rokhlin elements in \( \mathcal{A}_\infty \cap \mathcal{A}' \) which furthermore commute with \( B_0, B_1, \ldots, B_d \).

Now, set
\[
\theta_k(x) = \sum_{j=0}^{n-1} f^{(k)}_{0,j} \bar{\alpha}^j(\iota_k \circ \theta(x))\bar{\alpha}^j(\iota_k \circ \theta(x)) - u^{(k)*}_j + \sum_{j=0}^{n} f^{(k)}_{1,j} \bar{\alpha}^j(\iota_k \circ \theta(x))\bar{\alpha}^j(\iota_k \circ \theta(x)) - u^{(k)*}_j
\]
and
\[
\eta_k(x) = \sum_{j=0}^{n-1} f^{(k)}_{0,j} \bar{\alpha}^j(\iota_k \circ \eta(x))\bar{\alpha}^j(\iota_k \circ \eta(x)) - u^{(k)*}_j + \sum_{j=0}^{n} f^{(k)}_{1,j} \bar{\alpha}^j(\iota_k \circ \eta(x))\bar{\alpha}^j(\iota_k \circ \eta(x)) - u^{(k)*}_j.
\]

We can check that
\[
\|\bar{\alpha}(\theta_k(x)) - \theta_k(x)\| < \varepsilon
\]
and

\[ \| \bar{\alpha}(\eta_k(x)) - \eta_k(x) \| < \varepsilon \]

for all \( x \) in the unit balls of \( M_r, M_{r+1} \), respectively. Furthermore,

\[ \theta_k(1) + \eta_k(1) = \sum_{j=0}^{n-1} f_{0,j}^{(k)} + \sum_{j=0}^{n} f_{1,j}^{(k)} \]

Therefore those maps satisfy the conditions of Corollary 5.5. □

We can also obtain in a similar way the analogous result for actions of finite groups with finite Rokhlin dimension and commuting towers.

**Theorem 5.9.** Let \( A \) be a separable unital \( \mathbb{Z} \)-stable \( C^* \)-algebra, let \( G \) be a finite group and let \( \alpha : G \to \text{Aut}(A) \) be an action with Rokhlin dimension \( d < \infty \), such that furthermore the Rokhlin elements from Definition 1.1 can be chosen to commute with each other. Then the crossed product \( A \rtimes_{\alpha} G \) is \( \mathbb{Z} \)-stable as well.

**Proof.** The proof is a simpler version of the proof of Theorem 5.8 - here we do not need the choice of correcting unitaries. We again establish that the conditions of Lemma 5.7 hold, and we start in a similar way.

Let \( r \) be a given positive integer. Fix two order zero maps \( \theta : M_r \to \mathbb{Z} \), \( \eta : M_{r+1} \to \mathbb{Z} \) with commuting ranges such that \( \theta(1) + \eta(1) = 1 \). We define unital homomorphisms \( \iota_0, \ldots, \iota_d, \mu_0, \ldots, \mu_d : \mathbb{Z} \to A_\infty \cap A' \) as follows. First fix \( \iota_0 : \mathbb{Z} \to A_\infty \cap A' \). Proceeding inductively, we choose \( \iota_k : \mathbb{Z} \to A_\infty \cap A' \) such that its image furthermore commutes with \( \bar{\alpha}_g(\iota_i(\mathbb{Z})) \) for all \( g \in G \) and \( i < k \). We define \( \mu_0, \ldots, \mu_d \) in a similar way, such that the image of \( \mu_k \) commutes with \( \bar{\alpha}_g(\iota_i(\mathbb{Z})) \) for all \( g \in G \) and \( i < k \) as well as \( \bar{\alpha}_j(\iota_i(\mathbb{Z})) \) for all \( j \in \mathbb{Z} \) and all \( i = 0, 1, \ldots d \).

Let

\[ B_k := C^* \left( \mu_k(\mathbb{Z}) \cup \bigcup_{g \in G} \bar{\alpha}_g(\iota_k(\mathbb{Z})) \right) ; \]

note that \( B_k \cong B_k \otimes \mathbb{Z} \), and that the elements of \( B_k \) commute with those of \( B_m \) for \( k \neq m \).

Let \( \left( f_g^{(l)} \right)_{l=0,\ldots,d; g \in G} \subseteq A_\infty \cap A' \) be Rokhlin elements, and as in the statement of the theorem we assume that they all commute with each other, and which are furthermore chosen to commute with \( B_0, B_1, \ldots, B_d \).

Define

\[ \theta_k(x) = \sum_{g \in G} f_g^{(k)} \bar{\alpha}_g(\iota_k(\theta(x))) , \quad \eta_k(x) = \sum_{g \in G} f_g^{(k)} \bar{\alpha}_g(\iota_k(\eta(x))) . \]

One checks that the images of those maps are fixed by the action of \( G \) on \( A_\infty \cap A' \), and satisfy the conditions of Corollary 5.5. □

**6. Irrational rotations**

In this section we first give a direct proof of the fact that irrational rotations have Rokhlin dimension 1. In the subsequent sections we consider much more generally minimal actions on finite dimensional compact spaces and show that these must always have finite Rokhlin dimension.

**Theorem 6.1.** Let \( \alpha \) be the irrational rotation by \( \theta \) on \( C(\mathbb{T}) \), then \( \dim_{\text{Rok}}^s(\alpha) = 1 \).
Remark 6.2. It follows immediately that if \((X, h)\) is a dynamical system which has an irrational rotation as a factor, that is there is a commuting diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X \\
\pi & & \pi \\
\downarrow & & \downarrow \\
T & \xrightarrow{\rho} & T \\
\end{array}
\]

then \(\dim_{\text{Rok}}^\alpha(C(X), \alpha) \leq 1\) as well.

Proof of Theorem 6.2. Given a prime number \(p\) and \(\varepsilon > 0\), we will exhibit positive \(\{f_i, g_i\}_{i=0,1,\ldots,p-1}\) such that

1. \(\|\alpha(f_j) - \tilde{f}_{j+1} \text{ (mod) } p\|, \|\alpha(g_j) - \tilde{g}_{j+1} \text{ (mod) } p\| < \varepsilon\)
2. The \(f_j\)’s are pairwise orthogonal and the \(g_j\)’s are pairwise orthogonal.
3. \(\tilde{f}_0 + \tilde{f}_1 + \ldots + \tilde{f}_{p-1} + \tilde{g}_0 + \tilde{g}_1 + \ldots + \tilde{g}_{p-1} = 1\)

and that shows that \(\alpha\) has Rokhlin dimension 1 with single towers. We note that since \(C(\mathbb{T})\) has no nontrivial projections, \(\alpha\) clearly cannot have Rokhlin dimension 0.

We first recall some basic facts about continued fractions (see [26], Chapter 1). If \(t\) is an irrational number and if \(m_j/n_j\) is the sequence of approximants for \(t\), then \(|t - m_j/n_j| < \frac{1}{n_j^2}\) for all \(j\). Furthermore, the \(m_j\)’s satisfy a recursive formula of the form \(m_{j+1} = a_j m_j + m_{j-1}\). It follows by induction that gcd\((m_j, m_{j+1}) = \text{gcd}(m_0, m_1) = 1\), and thus any two successive \(m_j\)’s are coprime. In particular, given a prime number \(p\) and an irrational number \(t\), there are infinitely many integers \(m, n\) such that \(p \nmid m\) and \(|t - m/n| < \frac{1}{n^2}\).

Fix a prime number \(p\). We have infinitely many coprime numbers \(m, n\) such that \(|p\theta - m/n| < \frac{1}{n^2}\), and \(p \nmid m\), i.e. \(|\theta - m/n| < \frac{1}{pn^2}\) and \((m, pn) = 1\).

Identifying the circle with the reals mod 1, we set

\[
f_0(x) = \begin{cases} 
2np & \text{if } 0 \leq x < \frac{1}{2np} \\
2 - 2np & \text{if } \frac{1}{2np} \leq x < \frac{1}{np} \\
0 & \text{if } \frac{1}{np} \leq x \leq 1
\end{cases}
\]

and we set \(g_0(x) = f_0(x - 1/2np)\). Let \(\gamma(f) = f(x - m/np)\) (rational rotation by \(m/np\)). Notice that \(\gamma\) is \(np\)-periodic (and does not have a smaller period). Set \(f_j(x) = \gamma^j(f_0), g_j(x) = \gamma^j(g_0)\) for \(j = 1, 2, \ldots, np - 1\). One easily verifies that \(f_0 + f_1 + \ldots + f_{np-1} + g_0 + g_1 + \ldots + g_{np-1} = 1\) and that \(f_j f_k = g_j g_k = 0\) for \(j \neq k\).

Notice furthermore that each \(f_j\) and \(g_j\) are Lipschitz with Lipschitz constant \(2np\). For \(j = 0, 1, \ldots, p - 1\), set

\[
\tilde{f}_j = \sum_{k=0}^{n-1} f_{kp+j}, \quad \tilde{g}_j = \sum_{k=0}^{n-1} g_{kp+j}
\]

Notice that again, the \(\tilde{f}_j\)’s are pairwise orthogonal, the \(\tilde{g}_j\)’s are pairwise orthogonal, we have

\[
\tilde{f}_0 + \tilde{f}_1 + \ldots + \tilde{f}_{p-1} + \tilde{g}_0 + \tilde{g}_1 + \ldots + \tilde{g}_{p-1} = 1,
\]

those functions all have Lipschitz constant \(2np\), and

\[
\gamma(\tilde{f}_j) = \tilde{f}_{j+1} \text{ (mod) } p, \quad \gamma(\tilde{g}_j) = \tilde{g}_{j+1} \text{ (mod) } p
\]
Now, we note that for all $x$ we have
\[
\left| \gamma(\tilde{f}_j)(x) - \alpha(\tilde{f}_j)(x) \right| = \left| \tilde{f}_j \left( x - \frac{m}{pn} \right) - \tilde{f}_j(x - \theta) \right| \leq 2np \cdot \left| \theta - \frac{m}{pn} \right| \leq 2np \frac{pn}{n} = 2
\]
and the same estimate holds for the $\tilde{g}_j$'s. By choosing $n$ sufficiently large, we have the required almost-permutation estimate. The other requirements in the definition of Rokhlin dimension 1 are immediate. (And clearly, no action on a connected space can have Rokhlin dimension 0.)

The remainder of the paper is devoted to our general version of [6.1], Theorem [12.1] below. The argument closely follows that of [31], computing the decomposition rank for (approximately subhomogeneous) orbit breaking subalgebras of transformation group $C^*$-algebras. In this computation we need to carefully keep track of the underlying dynamics, which causes a considerable amount of technical difficulty. The role of the orbit breaking subalgebras is that of a book-keeping device, which keeps track of first return times for points of closed subsets with nonempty interiors.

7. Cyclic vs. non-cyclic shifts

Below we fix notation and some elementary facts on diagonal subalgebras of matrix algebras. Proposition [7.2] may be thought of as a splicing principle, which allows to compare a truncated shift to two cyclic shifts. We have already seen a very similar phenomenon in the proof of Proposition [2.8].

**Notation 7.1.** For $r \in \mathbb{N}$ we denote the set of diagonal elements of $M_r$ by $D_r$; we call diagonal elements with vanishing $(r, r)$-entry shiftable and denote the shiftable diagonal elements of $M_r$ by $\overline{D}_r^r$. We denote by $\bar{\sigma}_r$ the truncated shift
\[
\bar{\sigma}_r : D_r \to D_r,
\]
\[
\bar{\sigma}_r(e_{i, i}) := \begin{cases} 
 e_{i+1, i+1} & \text{if } i < r, \\
 0 & \text{if } i = r.
\end{cases}
\]
Set
\[
S_r := \begin{pmatrix} 
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]
and note that
\[
\bar{\sigma}_r(e_{i, i}) = S_r e_{i, i} S_r^* \tag{6}
\]
and that $\bar{\sigma}_r$ is injective on $\overline{D}_r^r$. Define a partial inverse
\[
\bar{\sigma}_r^{-} : D_r \to D_r
\]
of $\bar{\sigma}_r$ by

$$\bar{\sigma}_r^-(e) := S_r^* e S_r.$$ 

When there is no ambiguity, we will omit the subscript and just write $D, D^\sigma, \bar{\sigma}, \bar{\sigma}^-$ and $S$.

If

$$F = M_{r_1} \oplus \ldots \oplus M_{r_s}$$

is a finite dimensional $C^*$-algebra, we will usually denote by

$$D := D_{r_1} \oplus \ldots \oplus D_{r_s}$$

the diagonal elements of $F$; we denote by

$$D^\sigma := D_{r_1}^{\bar{\sigma}_{r_1}} \oplus \ldots \oplus D_{r_s}^{\bar{\sigma}_{r_s}}$$

the shiftable diagonal elements of $F$ and write

$$\bar{\sigma} : D \to D$$

for the sum of the truncated shifts on each component; we do similar for $\bar{\sigma}^-$ and $S$.

Note that this notation relies on the particular identification (7) of the finite dimensional $C^*$-algebra $F$ with a sum of matrix algebras.

**Proposition 7.2.** Given $k \in \mathbb{N}$ and $\delta > 0$ there is $s \in \mathbb{N}$ such that the following holds:

For any natural number $r \geq 4s$ there is a c.p.c. order zero map

$$\mu : \mathbb{C}^k \to D^\sigma \subset M_r$$

such that

$$\|\bar{\sigma}(\mu(e)) - \mu(\bar{\sigma}(e))\| \leq \delta\|e\|$$

for all $0 \leq e \in \mathbb{C}^k$, and such that

$$(\bar{\sigma}^-)^s(\mu(1_{\mathbb{C}^k})) + \bar{\sigma}^s(\mu(1_{\mathbb{C}^k})) \geq 1_{M_r},$$

where $\bar{\sigma}$ is the truncated shift on $D^\sigma$, $\bar{\sigma}^-$ is its partial inverse and $\bar{\sigma}$ is the cyclic shift on $\mathbb{C}^k$, cf. 7.1.

**Proof.** Choose a natural number

$$L \geq \frac{1}{\delta}$$

and set

$$s := kL.$$  

Now suppose $r \geq 4s$ is given. Choose $L \in \mathbb{N}$ such that

$$r - 2s \leq kL < r - 2s + k;$$

note that

$$2s \leq kL.$$  

Define a linear map

$$\mu : \mathbb{C}^k \to M_r$$

by

$$\mu(e_i) := \sum_{j=1}^{L-1} \frac{j}{L} \cdot f_{kj+i} + \sum_{j=L}^{L+L-1} f_{kj+i} + \sum_{j=L+L}^{L+2L-2} \left(1 - \frac{j - L - L + 1}{L}\right) \cdot f_{kj+i}$$

for all $0 \leq e_i \in \mathbb{C}^k$. 


for $i \in \{1, \ldots, k\}$, where $e_1, \ldots, e_k \in \mathcal{C}_k$ and $f_1, \ldots, f_r \in \mathcal{D}$ are the respective canonical generators.

It is straightforward to check that $\mu$ indeed is a c.p.c. order zero map; since

$$k(L + 2\bar{L} - 2) + k \leq kL + 2s - k < r,$$

it follows that

$$\mu(\mathcal{C}_k) \subset \mathcal{D} \subset M_r.$$

For $i \in \{1, \ldots, k-1\}$ we have

$$\tilde{\sigma}(\mu(e_i)) = \mu(e_{i+1}) = \mu(\tilde{\sigma}(e_i)),$$

and one checks that

$$\|\tilde{\sigma}(\mu(e_k)) - \mu(\tilde{\sigma}(e_k))\| = \|\tilde{\sigma}(\mu(e_k)) - \mu(e_1)\| \leq \frac{1}{L} \leq \delta. \quad (13)$$

Furthermore, it is straightforward to verify that

$$(\tilde{\sigma}(\mu(e_i)) - \mu(\tilde{\sigma}(e_i))) \perp (\tilde{\sigma}(\mu(e_i')) - \mu(\tilde{\sigma}(e_i')))$$

if $i \neq i' \in \{1, \ldots, k\}$; in combination with (14) this yields (8).

Finally, we have

$$(\tilde{\sigma}^{-})^\ast(\mu(1_{\mathcal{C}_k})) + \bar{\sigma}^\ast(\mu(1_{\mathcal{C}_k})) \geq (\tilde{\sigma}^{-})^\ast\left(\sum_{i=1}^{k} \sum_{j=L}^{L+L-1} f_{kj+i}\right) + \bar{\sigma}^\ast\left(\sum_{i=1}^{k} \sum_{j=L}^{L+L-1} f_{kj+i}\right)$$

$$= (\tilde{\sigma}^{-})^\ast\left(\sum_{l=kL+1}^{k(L+L)} f_l\right) + \bar{\sigma}^\ast\left(\sum_{l=kL+1}^{k(L+L)} f_l\right)$$

$$= (\tilde{\sigma}^{-})^\ast\left(\sum_{l=s+1}^{kL+s} f_l\right) + \bar{\sigma}^\ast\left(\sum_{l=s+1}^{kL+s} f_l\right)$$

$$\geq \sum_{l=1}^{r} f_l + \sum_{l=1}^{r} f_l$$

$$\geq \sum_{l=1}^{r} f_l = 1_{M_r}. \quad (12)$$

8. Minimal dynamics, first return times and compatible approximations

In this section we fix notation on transformation group $C^*$-algebras and certain recursive subhomogeneous subalgebras of these. We then define what it means for approximations of those subalgebras to be compatible with the underlying dynamics.
Notation 8.1. (Cf. [16].) Let \((T, h)\) be a minimal dynamical system with \(T\) compact and metrizable. We let
\[
\mathcal{E} := C(T) \rtimes \mathbb{Z} = C^*(C(T), u)
\]
be the crossed product \(C^*\)-algebra, where \(u\) is the unitary which implements the action by
\[
u f(. ) u^* = f \circ h^{-1}(. ).
\]
If \(Z \subset T\) is closed with nonempty interior, consider
\[
C(T) \subset \mathcal{E}_Z := C^*(C(T), uC_0(T \setminus Z)) \subset \mathcal{E}.
\]
Let
\[
m_1 < m_2 < \ldots < m_L \in \mathbb{N}
\]
denote the first return times for \(Z\) with associated pairwise disjoint locally compact subsets
\[
Z_1, \ldots, Z_L \subset Z.
\]
Note that
\[
Z_1 \cup \ldots \cup Z_l \subset Z
\]
is closed for each \(l\), so that
\[
\overline{Z_{l+1}} \setminus Z_{l+1} \subset (Z_1 \cup \ldots \cup Z_l).
\]
Define \(\ast\)-homomorphisms
\[
\sigma_l : \mathcal{E}_Z \to C(\overline{Z_l}) \otimes M_{m_l}
\]
by
\[
\sigma_l(f) := \begin{pmatrix}
f \circ h|_{\overline{Z_l}} & 0 \\
0 & f \circ h^{m_l}|_{\overline{Z_l}}
\end{pmatrix}
\]
for \(f \in C(T)\) and
\[
\sigma_l(uf) := \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]
(16) \(\sigma_l(f) = S\sigma_l(f)\).
for $f \in C_0(T \setminus Z)$ (these are easily checked to be well-defined). Define
\begin{equation}
\rho_l := \sigma_1 \oplus \ldots \oplus \sigma_l : E_Z \to (C(Z_l) \otimes M_{m_1}) \oplus \ldots \oplus (C(Z_l) \otimes M_{m_l})
\end{equation}
and set
\begin{equation}
B_l := \rho_l(E_Z) \subset (C(Z_1) \otimes M_{m_1}) \oplus \ldots \oplus (C(Z_l) \otimes M_{m_l}).
\end{equation}
We now have a diagram
\[ \begin{array}{ccc}
E_Z & \xrightarrow{\rho_l} & B_l \\
\downarrow & & \downarrow \\
B_l & \xrightarrow{\rho_{l-1}} & B_{l-1} \\
\end{array} \]
where the horizontal maps are the canonical projections. Note that the map
\begin{equation}
\rho_L : E_Z \to B_L
\end{equation}
is an isomorphism.

Let
\[ \tilde{\sigma}_l : E_Z \to C(Z_l \setminus Z_l) \otimes M_{m_l} \]
denote the composition of $\sigma_l$ with the restriction map. If $t \in \overline{Z_{l+1} \setminus Z_{l+1}}$, then it is straightforward to check that the kernel of $e_{l+1} \circ \tilde{\sigma}_{l+1}$ contains the kernel of $\rho_l$; from this one concludes that $\tilde{\sigma}_{l+1}$ factorizes through $\sigma_l$, i.e., there is a $*$-homomorphism
\begin{equation}
\pi_l : B_l \to C(Z_l \setminus Z_l) \otimes M_{m_l}
\end{equation}
such that
\[ \tilde{\sigma}_{l+1} = \pi_l \circ \rho_l. \]
This in turn implies that $B_{l+1}$ can be regarded as a pullback, i.e.,
\begin{equation}
B_{l+1} = B_l \oplus \pi_l(Z_l \setminus Z_{l+1}) \oplus (C(Z_{l+1}) \otimes M_{m_{l+1}})
\end{equation}
\begin{equation}
\subset (C(Z_1) \otimes M_{m_1}) \oplus \ldots \oplus (C(Z_l) \otimes M_{m_l}) \oplus (C(Z_{l+1}) \otimes M_{m_{l+1}}).
\end{equation}
Next, let
\[ D^{\tilde{\sigma}_l}_{m_l} \subset D_{m_l} \subset M_{m_l} \]
denote the (shiftable) diagonal elements of $M_{m_l}$. We claim that
\begin{equation}
C_0(Z_l) \otimes D_{m_l} \subset \sigma_l(C(T)) \subset C(Z_l) \otimes D_{m_l}
\end{equation}
and
\begin{equation}
C_0(Z_l) \otimes D^{\tilde{\sigma}_l}_{m_l} \subset \pi_l(C_0(T \setminus Z_l)) \subset C(Z_l) \otimes D^{\tilde{\sigma}_l}_{m_l}.
\end{equation}
In each line, the second inclusions follow directly from (15); the first inclusions are a little less obvious but follow easily from (15) together with the fact that the sets $h(Z_l), \ldots, h^m(Z_l)$ are pairwise disjoint.

**Definition 8.2.** Let $A, F$ be $C^*$-algebras, $F = M_{r_1} \oplus \ldots \oplus M_{r_s}$, and $\varphi : F \to A$ a c.p. map. If there is an order $\prec$ on $\{1, \ldots, s\}$ s.t.
\[ [\varphi(1_{M_{r_j}}), \varphi(M_{r_i})] = 0 \]
for $i \prec j$, we say $\varphi$ is piecewise commuting (p.c.) with respect to $\prec$. 

8.3. With notation as in 8.1, let
\[ C(T) \subset \mathcal{E}_Z = C^*(C(T), uC_0(T \setminus Z)) \subset \mathcal{E} = C^*(C(T), u) \]
be given, and let
\[ \rho : \mathcal{E}_Z \rightarrow \mathcal{B} \]
be a surjection onto some \( C^* \)-algebra \( \mathcal{B} \). Let
\[ \mathcal{F} = M_{r_1} \oplus \ldots \oplus M_{r_s} \]
be a finite dimensional \( C^* \)-algebra and let
\[ \mathcal{D}^\# \subset \mathcal{D} \subset \mathcal{F} \]
denote the (shiftable) diagonal elements of \( \mathcal{F} \). Let
\[ \varphi : \mathcal{F} \rightarrow \mathcal{B} \]
be c.p.c., p.c. and \( n \)-decomposable; let \( \eta \geq 0 \).
We say \( \varphi \) is \( (\rho, \eta) \)-compatible, if there is a c.p.c. map
\[ \varphi^\#: \mathcal{D} \rightarrow C(T) \]
such that
(i) \( \| \varphi(e) - \rho \varphi^\#(e) \| \leq \eta \| e \| \) for \( e \in \mathcal{D}_+ \)
(ii) \( \varphi^\#(\mathcal{D}^\#) \subset C_0(T \setminus Z) \subset \mathcal{E}_Z \)
(iii) \( \| \rho(u \varphi^\#(e)) - \varphi(Se) \| \leq \eta \| e \| \) for \( e \in \mathcal{D}^\#_+ \), where \( S = \bigoplus_{i=1}^s S_i \) is as in 7.1.
We call \( \varphi^\# \) an \( \eta \)-compatible approximate c.p.c. lift for \( \varphi \).

8.4. For every \( n \in \mathbb{N} \) and \( \eta > 0 \) there is \( \bar{\eta} > 0 \) such that the following holds:

Let \( (T, h) \) be a minimal dynamical system with \( T \) compact and metrizable; suppose that \( \dim T \leq n < \infty \) and that \( Z \subset T \) is a closed subset with nonempty interior. Let
\[ C(T) \subset \mathcal{E}_Z = C^*(C(T), uC_0(T \setminus Z)) \subset \mathcal{E} = C^*(C(T), u) \]
be as in 8.1, let
\[ \rho_L : \mathcal{E}_Z \rightarrow \mathcal{B}_L \]
be the isomorphism of 18 and let
\[ \mathcal{F} = M_{r_1} \oplus \ldots \oplus M_{r_s} \]
be a finite dimensional \( C^* \)-algebra with (shiftable) diagonal elements
\[ \mathcal{D}^\# \subset \mathcal{D} \subset \mathcal{F} \]
as in 7.1. Whenever
\[ \varphi : \mathcal{F} \rightarrow \mathcal{B}_L \]
is c.p.c., p.c., \( n \)-decomposable and \( (\rho_L, \bar{\eta}) \)-compatible, then
\[ \| u \rho_L^{-1}(\varphi(e)) u^* - \rho_L^{-1} \varphi(\bar{\sigma}(e)) \| \leq \eta \| e \| \]
for \( e \in \mathcal{D}^\#_+ \).
Proof. Let \( n \in \mathbb{N} \) and \( \eta > 0 \) be given; we may clearly assume that \( \eta \leq 1 \).

Choose

\[
0 < \gamma \leq \frac{\eta}{n+1},
\]

then

\[
0 < \beta \leq \frac{\gamma^2}{64},
\]

then

\[
0 < \tilde{\eta} \leq \frac{\gamma \beta}{8}.
\]

Now suppose \( T, h, Z, \rho_L, \mathcal{F} \) and \( \varphi \) are as in the proposition. Suppose \( \varphi \) is decomposable with respect to

\[
\mathcal{F} = \mathcal{F}^{(0)} \oplus \ldots \oplus \mathcal{F}^{(n)};
\]

denote the respective components of \( \varphi \) by \( \varphi^{(i)} \), similar for \( \sigma^{(i)}, S^{(i)}, D^{(i)} \) and \( (D^{(i)})^\circ \).

For \( i \in \{0, \ldots, n\} \) and \( e \in (D^{(i)})^\circ \), we estimate

\[
\lVert u(\rho_L^{-1} \circ \varphi^{(i)})(e) - \rho_L^{-1} \circ \varphi^{(i)}(S(e)) + \tilde{\eta} \lVert \leq \frac{\beta}{2} \lVert e \lVert.
\]

Define a function \( f_\beta \in C([0, 1]) \) by

\[
f_\beta(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \beta/2 \\ t^{-1/2} & \text{if } t \geq \beta \\ \text{linear} & \text{else} \end{cases}
\]

and observe that

\[
\lVert f_\beta \rVert = \beta^{-1/2}
\]

and

\[
\lVert \text{id}_{[0, 1]} \circ f_\beta - (t \mapsto t^{1/2}) \lVert \leq \beta^{1/2}.
\]

We apply functional calculus for the order zero maps \( \rho_L^{-1} \varphi^{(i)} \) (as introduced in \( \mathcal{F}^{(i)} \)) to obtain

\[
\lVert (\rho_L^{-1} \varphi^{(i)})(a)f_\beta(\rho_L^{-1} \varphi^{(i)})(b) - (\rho_L^{-1} \varphi^{(i)})^{1/2}(ab) \lVert \leq \beta^{1/2} \lVert a \lVert \lVert b \lVert
\]

for all \( a, b \in \mathcal{F}^{(i)} \).
For $e \in (\mathcal{D}^{\varphi(i)})_+^*$ we now estimate
\[
\|u(\rho_L^{-1}\varphi(i))(e)u^* - (\rho_L^{-1}\varphi(i))(\bar{\varphi}(i))(e)\|
\]
\[
\leq \|u(\rho_L^{-1}\varphi(i))(e)u^* - (\rho_L^{-1}\varphi(i))(\bar{\varphi}(i))(S^{(i)}e(S^{(i)})^*)\|
\]
\[
= \|u(\rho_L^{-1}\varphi(i))^{1/2}(e)(\rho_L^{-1}\varphi(i))^{1/2}(1_{(D^{(i)})^*})u^*
\]
\[
- (\rho_L^{-1}\varphi(i))^{1/2}(S^{(i)}e)(\rho_L^{-1}\varphi(i))^{1/2}(1_{(D^{(i)})^*})(S^{(i)})^*)
\]
\[
+ 4\beta^{1/2}\|e\|
\]
\[
\leq (4\tilde{\eta}\beta^{-1} + 4\beta^{1/2})\|e\|
\]
\[
\leq \frac{\gamma}{2} + \frac{\gamma}{2}
\]
\[
(30) = \gamma.
\]

When $e \in \mathcal{D}^\varphi_0$, write $e = \oplus e^{(i)}$ with $e^{(i)} \in (\mathcal{D}^{\varphi(i)})_+^*$, then the desired estimate \cite{31} follows from \cite{30} and \cite{24}.

9. Relative barycentric subdivision

Below we recall the notion of relative barycentric subdivision from \cite{31} Section 5 and prove a perturbation version of \cite{31} Proposition 5.3.

**Proposition 9.1.** As in \cite{31} 5.3, let $K$ be a simplicial complex with vertex set
\[
V(K) = \Sigma^+ = \Sigma^{(1)} \cup \Sigma^{(2)} \cup \{\ast\};
\]
let $J$ be the subcomplex generated by
\[
V(J) = \Sigma^{(1)},
\]
and let $\text{Sd}_JK$ denote the relative barycentric subdivision with vertex set
\[
V(\text{Sd}_JK) = \Gamma,
\]
as constructed in \cite{31} 5.7. Let $\text{Sd}K$ be the (full) barycentric subdivision with vertex set
\[
V(\text{Sd}K) = \Delta,
\]
and let
\[
\tilde{h} : C^{\Sigma^+} \to C(|K|),
\]
\[
\tilde{k} : C^{\Gamma} \to C(|\text{Sd}_JK|),
\]
\[
\tilde{j} : C^{\Delta} \to C(|\text{Sd}K|)
\]
denote the u.c.p. coordinate maps; let
\[
S : C(|\text{Sd}K|) \xrightarrow{\cong} C(|K|)
\]
and
\[
S_J : C(|\text{Sd}_JK|) \xrightarrow{\cong} C(|K|)
\]
denote the induced isomorphisms, cf. \cite{31} 5.4 and 5.5.
Then, there are canonical u.c.p. maps
\[ h^\times : C^{\Sigma^+} \to C^\Gamma \]
and
\[ k^\times : C^\Gamma \to C^\Delta \]
such that the diagram
\[
\begin{array}{ccc}
C^{\Sigma^+} & \xrightarrow{h} & C(|K|) \\
\downarrow h^\times & \cong & \downarrow S_J^{-1} \\
C^\Gamma & \xrightarrow{k} & C(|Sd_JK|) \\
\downarrow k^\times & \cong & \downarrow S^{-1}s_J \\
C^{\Delta} & \xrightarrow{j} & C(|SdK|)
\end{array}
\]
commutes.

**Proof.** By construction, the relative barycentric subdivision SdJ K of K is obtained by inductively subdividing certain faces of K with the help of [31 Proposition 5.5], also cf. [31 5.6]. At each step one obtains a linear homeomorphism, called \( \tilde{\beta} \) in [31 Proposition 5.5], between the geometric realizations. The composition of all these is our map \( S_J \). If we keep subdividing SdJ K inductively at the remaining original faces of K, eventually we arrive at the (full) barycentric subdivision SdK. This yields the factorization
\[
C(|K|) \xrightarrow{S_J^{-1}} C(|Sd_JK|) \xrightarrow{S^{-1}S_J} C(|SdK|).
\]
The map
\[ h^\times : C^{\Sigma^+} \to C^\Gamma \]
is then just given by
\[ h^\times := \bigoplus_{\gamma \in \Gamma} ev_{\gamma}(S_J^{-1}(\tilde{h})), \]
where
\[ ev_{\gamma} : C(|Sd_JK|) \to \mathbb{C} \]
denotes evaluation at the vertex \( \gamma \) (regarded as a point in the geometric realization of SdJ K).

Similarly, one defines
\[ k^\times := \bigoplus_{\delta \in \Delta} ev_{\delta}(S^{-1}S_J(\tilde{k})). \]

With these maps it is straightforward to check that the diagram (31) indeed commutes, since the maps between the geometric realizations are linear homeomorphisms (given explicitly in the proof of [31 Proposition 5.5]).

**Proposition 9.2.** Given \( L \in \mathbb{N} \) and \( \theta > 0 \), there is \( \bar{\theta} > 0 \) such that the following holds:

Suppose \( K \) is a simplicial complex with at most \( L \) vertices; denote by \( \Sigma \) its vertex set and by
\[ \bar{h} : C^\Sigma \to C(|K|) \]
the u.c.p. coordinate map.

Denote by SdK the barycentric subdivision of K, \( \Delta \) its vertex set,
\[ \bar{j} : C^\Delta \to C(|SdK|) \]
the u.c.p. coordinate map and

\[ S : C(|\text{Sd}K|) \xrightarrow{\cong} C(|K|) \]

the canonical isomorphism.

Let \( \mathcal{D} \) be a \( C^* \)-algebra and let

\[ \tilde{h}_1, \tilde{h}_2 : C(|K|) \to \mathcal{D} \]

be two \( * \)-homomorphisms satisfying

(32) \[ \| \tilde{h}_1 \tilde{h} - \tilde{h}_2 \tilde{h} \| < \tilde{\theta}. \]

Then, the c.p.c. maps

(33) \[ \tilde{j}_i := \tilde{h}_i \mathbb{S}j : C^\Delta \to \mathcal{D}, \quad i = 1, 2, \]

satisfy

(34) \[ \| \tilde{j}_1 - \tilde{j}_2 \| < \theta. \]

Proof. Let \( K^o \) denote the full simplex with \( L \) vertices, \( \Sigma^o = \{1, \ldots, L\} \) its vertex set,

\[ \tilde{h}^o : C(\Sigma^o) \to C(|K^o|) \]

the u.c.p. coordinate map, and let \( \text{Sd}K^o, \Delta^o, \tilde{j}^o \) and \( S^o \) denote the respective data for the barycentric subdivision.

Approximate each coordinate function

\[ \tilde{j}^o(e_\delta) \in C(|\text{Sd}K^o|), \quad \delta \in \Delta^o \]

uniformly by a polynomial in \( L \) commuting variables,

\[ p_\delta(x_1, \ldots, x_L), \]

such that

(35) \[ \| \tilde{j}^o(e_\delta) - p_\delta((S^o)^{-1} \tilde{h}^o(e_1), \ldots, (S^o)^{-1} \tilde{h}^o(e_L)) \| < \frac{\theta}{3 \cdot \text{card}(\Delta^o)} \]

for each \( \delta \in \Delta^o \); this is possible since \( C(|K^o|) \cong C(|\text{Sd}K^o|) \) is generated by the image of \( \tilde{h}^o \).

Choose \( \theta > 0 \) such that, whenever \( a_1, \ldots, a_L, b_1, \ldots, b_L \) are positive elements of norm at most 1 in some \( C^* \)-algebra satisfying

\[ \| a_i - b_i \| < \tilde{\theta} \]

for each \( i \), then

(36) \[ \| p_\delta(a_1, \ldots, a_L) - p_\delta(b_1, \ldots, b_L) \| < \frac{\theta}{3 \cdot \text{card}(\Delta^o)} \]

for each \( \delta \in \Delta^o \).

Now suppose \( K, \mathcal{D} \) and \( \tilde{h}_1, \tilde{h}_2 \) are as in the Proposition; choose an inclusion

\[ \Sigma \hookrightarrow \Sigma^o \]

and let

\[ S_K : C(|K^o|) \to C(|K|) \]
be the induced surjection. Note that we have a commutative diagram

\[ C^{\Sigma^1} \xrightarrow{\tilde{h}} C([K^0]) \]

\[ C^{\Sigma} \xrightarrow{\tilde{h}} C([K]) \xrightarrow{\tilde{h}_i} D \]

\[ C^{\Delta^1} \xrightarrow{\tilde{j}} C([SdK^1]) \]

\[ C^{\Delta} \xrightarrow{\tilde{j}} C([SdK]). \]

For \( \delta \in \Delta \), choose \( \delta' \in \Delta^1 \) such that

\[ T^{\Delta}(e_{\delta'}) = e_{\delta} \in C^{\Delta}. \]

We then estimate

\[ \|j_1(e_{\delta}) - j_2(e_{\delta})\| \]

\[ \leq \|\tilde{h}_1S\tilde{j}(e_{\delta}) - \tilde{h}_2S\tilde{j}(e_{\delta})\| \]

\[ \leq \|\tilde{h}_1S\tilde{j}S\tilde{p}_\delta((S^0)^{-1}\tilde{h}_0(e_1), \ldots, (S^0)^{-1}\tilde{h}_0(e_L)) \]

\[ -\tilde{h}_2S\tilde{j}S\tilde{p}_\delta((S^0)^{-1}\tilde{h}_0(e_1), \ldots, (S^0)^{-1}\tilde{h}_0(e_L))\| \]

\[ + \frac{2\theta}{3 \cdot \text{card}(\Delta^0)} \]

\[ = \|p_\delta(\tilde{h}_1S\tilde{h}_K\tilde{h}_0(e_1), \ldots, \tilde{h}_1S\tilde{h}_K\tilde{h}_0(e_L)) \]

\[ -p_\delta(\tilde{h}_2S\tilde{h}_K\tilde{h}_0(e_1), \ldots, \tilde{h}_2S\tilde{h}_K\tilde{h}_0(e_L))\| \]

\[ + \frac{2\theta}{3 \cdot \text{card}(\Delta^0)} \]

\[ \theta \]

\[ \frac{\text{card}(\Delta^0)}{\text{card}(\Delta^0)}, \]

since

\[ \|\tilde{h}_1\tilde{h}T_{\Sigma}(e_\sigma) - \tilde{h}_2\tilde{h}T_{\Sigma}(e_\sigma)\| < \theta \]

for each \( \sigma \in \{1, \ldots, L\} \) by (32).

From this it follows immediately that

\[ \|j_1 - j_2\| < \theta, \]

since \( \text{card}\Delta \leq \text{card}\Delta^0 \).
Before we proceed, let us fix notation for some continuous functions as follows; this will be useful for the present and the subsequent sections.

**Definition 9.3.** If $0 \leq \alpha < \beta \leq 1$, define $f_{\alpha, \beta}, g_{\alpha, \beta}, d_{\alpha, \beta} \in C([0, 1])$ by

$$f_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } t \leq \alpha \\ t & \text{if } t \geq \beta \\ \text{linear} & \text{if } \alpha \leq t \leq \beta, \end{cases}$$

$$g_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } t \leq \alpha \\ 1 & \text{if } t \geq \beta \\ \text{linear} & \text{if } \alpha \leq t \leq \beta, \end{cases}$$

and

$$d_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } 0 < t \leq \alpha \\ 1/t & \text{if } \beta \leq t \leq 1 \\ \text{linear} & \text{if } \alpha \leq t \leq \beta. \end{cases}$$

**Proposition 9.4.** Let $n, R \in \mathbb{N}$ be given. Then, for any $\theta > 0$ there is $\tilde{\theta} > 0$ such that the following holds:

As in \cite[5.3]{31}, let $K$ be a simplicial complex with vertex set

$$V(K) = \Sigma^+ = \Sigma^{(1)} \cup \Sigma^{(2)} \cup \{*\};$$

let $J$ be the subcomplex generated by $V(J) = \Sigma^{(1)}$.

Let $B$ be an $R$-subhomogeneous $C^*$-algebra and let

$$h : C^{\Sigma^+} \to C := C^*(h(C^{\Sigma^+})) \subset B$$

be a u.c.p. map (with $C$ commutative) as in \cite[5.3]{31}, in particular with $h|_{C^{\Sigma^+}}$ $n$-decomposable and such that $h$ factorizes through $C(|K|)$ (see \cite[50]{50}). Suppose that

$$h^\dagger : C^{\Sigma^+} \to C^\dagger := C^*(h^\dagger(C^{\Sigma^+})) \subset B$$

is another u.c.p. map with commutative range satisfying

$$\|h - h^\dagger\| < \tilde{\theta}.$$  \hspace{1cm} (40)

Then, there is a u.c.p. map

$$k^\dagger : C^\Gamma \to C^\dagger$$

such that $k^\dagger|_{C^\Gamma \setminus \{*\}}$ is $n$-decomposable and such that

$$\|k - k^\dagger\| < \theta,$$ \hspace{1cm} (41)

where $\Gamma$ and

$$k : C^\Gamma \to C$$

are as in \cite[5.3]{31}. We may choose $k^\dagger|_{C^\Gamma \setminus \{*\}}$ to be $n$-decomposable with respect to the same decomposition as $k|_{C^\Gamma \setminus \{*\}}$.

Upon extending the map

$$\nu : \Gamma \setminus (\Sigma^{(1)})^+ \to \Sigma^{(2)}$$

(cf. \cite[5.4]{31}) to a map (also denoted)

$$\nu : \Gamma \setminus \{*\} \to \Sigma^{(1)} \cup \Sigma^{(2)}$$

by

$$\nu(\sigma) := \sigma \text{ for } \sigma \in \Sigma^{(1)},$$

(42)
we may furthermore assume that
\( \|g_{\tilde{\theta}/2}(h^\dagger(e_{\nu(\gamma)}))k^\dagger(e_\gamma) - k^\dagger(e_\gamma)\| < \theta \)
for each \( \gamma \in \Gamma \setminus \{\ast\} \) and that
\( k^\dagger(e_\gamma) \in J(h^\dagger(e_{\nu(\gamma)})) \)
for \( \gamma \in \Gamma \setminus \{\ast\} \).

**Proof.** Given \( \theta, n \) and \( R \), set
\[ L := 2R(n + 2); \]
we may clearly assume \( \theta \leq 1 \). Obtain
\( \bar{\theta} < \frac{\theta}{4(n + 2)} \)
from Proposition 9.2 choose
\( \bar{\theta} < \frac{\bar{\theta}}{4(n + 2)} \).
Take
\( 0 < \bar{\theta} \leq \frac{\bar{\theta}}{12R(n + 2)} \)
such that, if \( a, b \) are positive elements in some \( C^* \)-algebra which satisfy
\( \|a - b\| \leq 3(R(n + 2) + 1)\bar{\theta} \),
then
\( \|g_{\bar{\theta}/2}(a) - g_{\bar{\theta}/2}(b)\| \leq \frac{\theta}{4} \).
this is possible by a simple functional calculus argument upon approximating \( g_{\bar{\theta}/2} \)
uniformly by polynomials.
Finally, choose
\( \bar{\theta} < \frac{\bar{\theta}^2}{2\bar{\theta}} \).

Now let \( B, K, J, C, \) as in the proposition be given; note that \( h \) factorizes as
\( h : C^{\Sigma^+} \rightarrow C(|K|) \rightarrow C \).
Define a u.c.p. map
\( h^\circ : C^{\Sigma^+} \rightarrow C^\dagger \)
by
\( h^\circ(e_{\sigma}) := \left( \sum_{\sigma' \in \Sigma^+} f_{\bar{\theta}/2}(h^\dagger(e_{\sigma'})) \right)^{-1} \cdot f_{\bar{\theta}/2}(h^\dagger(e_\sigma)) \),
for \( \sigma \in \Sigma^+ \); we will check below that the inverse exists.
Let
\( \hat{\pi} : B \rightarrow M_r, \ r \leq R, \)
be an irreducible representation. Since \( \bar{h}_{|C^{\Sigma^+}} \) is \( n \)-decomposable, \( K \) is at most \( (n + 1) \)-dimensional. Now \( \hat{\pi}h \) is a sum of at most \( R \) characters of \( C(|K|) \), and each of the
corresponding points of $|K|$ sits in some face with at most $n + 2$ vertices. Therefore, there is a subset

$$\Sigma' \subset \Sigma^+$$

with at most $R(n + 2)$ elements such that

$$\hat{\pi} h \left( \sum_{\sigma' \in \Sigma'} e_{\sigma'} \right) = \hat{\pi} h \left( \sum_{\sigma' \in \Sigma'} e_{\sigma'} \right),$$

whence

$$\hat{\pi} h \left( \sum_{\sigma' \in \Sigma' \setminus \Sigma'} e_{\sigma'} \right) = 0.$$

It follows that

$$\left\| \hat{\pi} h^\dagger \left( \sum_{\sigma' \in \Sigma' \setminus \Sigma'} e_{\sigma'} \right) \right\| = \left\| \hat{\pi} h^\dagger \left( \sum_{\sigma' \in \Sigma'} e_{\sigma'} \right) - \hat{\pi} h \left( \sum_{\sigma' \in \Sigma' \setminus \Sigma'} e_{\sigma'} \right) \right\| < \tilde{\theta},$$

and we estimate

$$\left\| \hat{\pi} \left( \sum_{\sigma' \in \Sigma'} h^\dagger(e_{\sigma'}) - \sum_{\sigma' \in \Sigma^+} f_{\tilde{\theta},2\tilde{\theta}}(h^\dagger(e_{\sigma'})) \right) \right\| \leq \left\| \hat{\pi} \left( \sum_{\sigma' \in \Sigma'} (h^\dagger(e_{\sigma'}) - f_{\tilde{\theta},2\tilde{\theta}}(h^\dagger(e_{\sigma'}))) \right) \right\| + \left\| \hat{\pi} \left( \sum_{\sigma' \in \Sigma' \setminus \Sigma'} (h^\dagger(e_{\sigma'}) - f_{\tilde{\theta},2\tilde{\theta}}(h^\dagger(e_{\sigma'}))) \right) \right\| < R(n + 2)\tilde{\theta} + \tilde{\theta}$$

(52) $< 1.$

Since $\hat{\pi}$ was arbitrary and

$$\sum_{\sigma' \in \Sigma^+} h^\dagger(e_{\sigma'}) = 1_{C_1},$$

it follows that

$$\sum_{\sigma' \in \Sigma^+} f_{\tilde{\theta},2\tilde{\theta}}(h^\dagger(e_{\sigma'}))$$

is in fact invertible with

$$\left( \sum_{\sigma' \in \Sigma^+} f_{\tilde{\theta},2\tilde{\theta}}(h^\dagger(e_{\sigma'})) \right)^{-1} - 1_{C_1} \left\| < 2(R(n + 2)\tilde{\theta} + \tilde{\theta}), \right.$$  

whence $h^\circ$ is well-defined.

A similar reasoning shows that

$$\left\| \sum_{\sigma \in \Sigma^+} \lambda_\sigma \cdot h^\dagger(e_{\sigma}) - \sum_{\sigma \in \Sigma^+} \lambda_\sigma \cdot f_{\tilde{\theta},2\tilde{\theta}}(h^\dagger(e_{\sigma})) \right\| < R(n + 2)\tilde{\theta} + \tilde{\theta}$$

(54).
for any normalized \((\lambda_{\sigma})_{\sigma \in \Sigma^+} \in \mathbb{C}^{\Sigma^+}\). We then obtain
\[
\|h^0 - h^1\| < 3(R(n+2)\bar{\theta} + \bar{\theta}) \leq 3(R(n + 2) + 1)\bar{\theta} \leq \frac{\bar{\theta}}{2}.
\]

Next, suppose that for some \(\sigma, \sigma' \in \Sigma^+\) we have
\[h^0(e_{\sigma}) h^0(e_{\sigma'}) \neq 0.
\]
Then,
\[
\|h^1(e_{\sigma}) h^1(e_{\sigma'})\| \geq \bar{\theta}^2,
\]
and
\[
\|h(e_{\sigma}) h(e_{\sigma'})\| \geq \|h^1(e_{\sigma}) h^1(e_{\sigma'})\| - 2\bar{\theta}
\]
\[
\geq \bar{\theta}^2 - 2\bar{\theta}
\]
\[
> 0.
\]
From this and (50) it follows in particular that \(h^0\) also factorizes through \(C(|K|)\), say via a u.c.p. map \(\tilde{h}^0\),
\[
h^0 : \mathbb{C}^{\Sigma^+} \xrightarrow{\tilde{k}} C(|K|) \xrightarrow{\tilde{k}^0} \mathbb{C}^\dagger,
\]
and that \(h^0|_{\Sigma^+}\) is \(n\)-decomposable with respect to the same decomposition as \(h\).

Let \(\Delta\) denote the vertex set of the barycentric subdivision \(\text{Sd}K\) of \(K\), and consider the commutative diagrams
\[
\begin{array}{ccc}
\text{h} : \mathbb{C}^{\Sigma^+} & \xrightarrow{k} & C(|K|) \xrightarrow{h} \mathbb{C} \\
\downarrow & & \downarrow \\
\text{j} : \mathbb{C}^\Delta & \xrightarrow{j} & C(|\text{Sd}K|) \xrightarrow{j} \mathbb{C}
\end{array}
\]
and
\[
\begin{array}{ccc}
\text{h}^0 : \mathbb{C}^{\Sigma^+} & \xrightarrow{k} & C(|K|) \xrightarrow{\tilde{h}^0} \mathbb{C}^\dagger \\
\downarrow & & \downarrow \\
\text{j}^\dagger : \mathbb{C}^\Delta & \xrightarrow{j^\dagger} & C(|\text{Sd}K|) \xrightarrow{j^\dagger} \mathbb{C}^\dagger.
\end{array}
\]

To show that
\[
\|j - j^\dagger\| < \theta,
\]
let again
\[
\hat{\pi} : \mathcal{B} \to M_r, r \leq R,
\]
be some irreducible representation; we set
\[
\pi := \hat{\pi}|_\mathcal{C} : \mathcal{C} \subset \mathcal{B} \to M_r
\]
and
\[
\pi^\dagger := \hat{\pi}|_{\mathcal{C}^\dagger} : \mathcal{C}^\dagger \subset \mathcal{B} \to M_r.
\]
Both $\pi\hat{h}$ and $\pi^\dagger\hat{h}^\circ$ can be written as sums of at most $R$ characters of $C(|K|)$. Again each of the corresponding points in $|K|$ sits in some face with at most $n + 2$ vertices. Let

\begin{equation}
\tilde{\Sigma} \subset \Sigma^+
\end{equation}

denote the set of these at most $2R(n + 2)$ vertices, and let $\tilde{K}$ denote the subcomplex of $K$ generated by $\tilde{\Sigma}$. Then, $\pi\hat{h}$ and $\pi^\dagger\hat{h}^\circ$ both factorize through $C(|\tilde{K}|)$, i.e. we have commuting diagrams

\begin{equation}
\begin{array}{ccc}
\pi \hat{h} : C^{\Sigma^+} \xrightarrow{\hat{h}} C(|K|) \xrightarrow{\tilde{h}} C \xrightarrow{\pi} M_r \\
\tilde{h} : C^{\tilde{\Sigma}} \xrightarrow{\tilde{h}} C(|\tilde{K}|) \xrightarrow{\tilde{h}} M_r
\end{array}
\end{equation}

and

\begin{equation}
\begin{array}{ccc}
\pi^\dagger h^\circ : C^{\Sigma^+} \xrightarrow{\hat{h}^\circ} C(|K|) \xrightarrow{\tilde{h}^\circ} C^{\dagger} \xrightarrow{\pi^\dagger} M_r \\
\hat{h}^\circ : C^{\tilde{\Sigma}} \xrightarrow{\tilde{h}^\circ} C(|\tilde{K}|) \xrightarrow{\tilde{h}^\circ} M_r
\end{array}
\end{equation}

where the first vertical maps are compatible with (61).

It now follows that

\begin{equation}
\|\hat{h} - \hat{h}^\circ\| \leq \|\hat{\pi}h\|_{C^{\Sigma}} - \|\hat{\pi}h^\circ\|_{C^{\Sigma}} \leq \|\hat{\pi}h - \hat{\pi}h^\circ\| \leq \|\hat{h} - \hat{h}^\circ\| \leq \|h - h^\circ\| + \|h^\dagger - h^\circ\| \leq \tilde{\theta}.
\end{equation}

Denote by $\text{Sd}\tilde{K}$ the barycentric subdivision of $\tilde{K}$, $\tilde{\Delta}$ its vertex set,

\[ \tilde{j} : C^{\tilde{\Delta}} \to C(|\text{Sd}\tilde{K}|) \]

the u.c.p. coordinate map and

\[ \tilde{S} : C(|\text{Sd}\tilde{K}|) \xrightarrow{\tilde{\Sigma}} C(|\tilde{K}|) \]

the canonical isomorphism. Note that

\[ \tilde{\Delta} \subset \Delta, \]

that

\begin{equation}
\pi\tilde{j}j_{C\Delta\setminus\tilde{\Delta}} = \pi^\dagger h^\circ \tilde{j}_{C\Delta\setminus\Delta} = 0
\end{equation}
and that we have commuting diagrams

\[
\begin{array}{cccccc}
C^\Sigma_+ & \xrightarrow{\bar{h}} & C(|K|) & \xrightarrow{\bar{h}} & C & \xrightarrow{\pi} M_r \\
\downarrow & & \downarrow & & \downarrow & \\
C^\Sigma & \xrightarrow{\bar{k}} & C(|\bar{K}|) & \xrightarrow{\bar{k}} & C & \xrightarrow{\pi} M_r \\
\downarrow S^{-1} & & \downarrow S^{-1} & & \downarrow & \\
C^\Delta & \xrightarrow{j} & C(|SdK|) & \xrightarrow{j} & C & \xrightarrow{\pi} M_r \\
\downarrow & & \downarrow & & \downarrow & \\
C^\Delta & \xrightarrow{j} & C(|Sd\bar{K}|) & \xrightarrow{j} & C & \xrightarrow{\pi} M_r \\
\end{array}
\]

and

\[
\begin{array}{cccccc}
C^\Sigma_+ & \xrightarrow{\bar{h}} & C(|K|) & \xrightarrow{\bar{h}^\sigma} & C^\dag & \xrightarrow{\bar{h}^\sigma} \pi r \\
\downarrow & & \downarrow & & \downarrow & \\
C^\Sigma & \xrightarrow{\bar{k}} & C(|\bar{K}|) & \xrightarrow{\bar{k}^\sigma} & C^\dag & \xrightarrow{\bar{k}^\sigma} \pi r \\
\downarrow S^{-1} & & \downarrow S^{-1} & & \downarrow & \\
C^\Delta & \xrightarrow{j} & C(|SdK|) & \xrightarrow{j^\sigma} & C^\dag & \xrightarrow{j^\sigma} \pi r \\
\downarrow & & \downarrow & & \downarrow & \\
C^\Delta & \xrightarrow{j} & C(|Sd\bar{K}|) & \xrightarrow{j} & C & \xrightarrow{\pi} M_r \\
\end{array}
\]

Apply Proposition 9.2 with \(\tilde{K}\) in place of \(K\), \(M_r\) in place of \(D\) and with

\[\tilde{h}_1, \tilde{h}_2 : C(|\tilde{K}|) \to M_r\]

in place of \(\tilde{h}_1, \tilde{h}_2\); observe that

\[
\|\tilde{h}_1 \bar{h} - \tilde{h}_2 \bar{h}\| = \|\tilde{h} \bar{h} - \tilde{h}^\sigma \bar{h}\| \leq \tilde{\theta}
\]

\[
\|\tilde{h} - \tilde{h}^\sigma\| < \tilde{\theta}
\]

to conclude from (33) and (34) that the maps

\[
\text{(66)} \quad \tilde{j} := \tilde{h} \tilde{S} \tilde{j} = \tilde{j} \tilde{j}
\]

and

\[
\text{(67)} \quad \tilde{j}^\sigma := \tilde{h}^\sigma \tilde{S} \tilde{j} = \tilde{j}^\sigma \tilde{j}
\]

satisfy

\[
\|\tilde{j} - \tilde{j}^\sigma\| < \theta.
\]
As a consequence, we have

\[
\|\hat{\pi}_j - \tilde{\pi}_j\| \leq \|\tilde{\pi}_j\| < \theta, \tag{58}
\]

since \(\hat{\pi}\) was an arbitrary irreducible representation of \(\mathcal{B}\), it follows that in fact

\[
\|j - j^\sharp\| < \theta,
\]

so (58) holds.

Note that the vertical maps of (56) and (57) factorize through the relative barycentric subdivision, cf. Proposition 9.1, so that we have commuting diagrams

\[
(68) \quad h : \mathbb{C}^\Sigma^+ \xrightarrow{\bar{h}} \mathbb{C}([K]) \xrightarrow{j} \mathbb{C} \quad \quad k : \mathbb{C}^\Gamma \xrightarrow{\bar{k}} \mathbb{C}([SdJ\bar{K}]) \xrightarrow{j} \mathbb{C} \quad \quad j : \mathbb{C}^\Delta \xrightarrow{\bar{j}} \mathbb{C}([SdK]) \xrightarrow{j} \mathbb{C}
\]

and

\[
(69) \quad h^\circ : \mathbb{C}^\Sigma^+ \xrightarrow{\bar{k}} \mathbb{C}([K]) \xrightarrow{\tilde{j}} \mathbb{C}^\sharp \quad \quad k : \mathbb{C}^\Gamma \xrightarrow{\bar{k}} \mathbb{C}([SdJ\bar{K}]) \xrightarrow{j} \mathbb{C}^\sharp \quad \quad j : \mathbb{C}^\Delta \xrightarrow{\bar{j}} \mathbb{C}([SdK]) \xrightarrow{j} \mathbb{C}^\sharp.
\]

We then clearly have

\[
\|k - k^\sharp\| = \|jk^\times - j^\sharp k^\times\| \leq \|j - j^\sharp\| < \theta,
\]

so (69) holds.

The decomposition for \(k|_{C^\Gamma \setminus \{\ast\}}\) only depends on a coloring for the covering of \(|SdJ\bar{K}|\) by open stars around vertices, cf. [31, 5.7]; therefore, \(k^\sharp|_{C^\Gamma \setminus \{\ast\}}\) can be decomposed in the same way.

Next, observe that

\[
\|g_{\tilde{\theta}, 2\tilde{\theta}}(h^\circ(e_{\nu(\gamma)})|k^\sharp(e_{\gamma}) - k^\sharp(e_{\gamma}))\| \leq \|g_{\tilde{\theta}, 2\tilde{\theta}}(S_{j}^{-1} \bar{h}(e_{\nu(\gamma)})\bar{k}(e_{\gamma}) - \bar{k}(e_{\gamma}))\| \leq (n + 2) 2\tilde{\theta} < \frac{\theta}{4}, \tag{70}
\]

where the second estimate follows by functional calculus and since the simplex \(SdK\) is at most \((n + 2)\)-dimensional, also cf. Remark 9.5 below.
Moreover, we have

\[(71) \quad \|g_{\theta,2\delta}^\star(h^\circ(e_{\nu(\gamma)})) \| \leq \frac{\theta}{4}\]

for \(\gamma \in \Gamma \setminus \{\ast\}\) by (48) and (55).

We obtain for \(\gamma \in \Gamma \setminus \{\ast\}\)

\[(70) \quad \|g_{\theta/2,\delta}(h^\circ(e_{\nu(\gamma)})) k^\ddagger(e_\gamma) - k^\ddagger(e_\gamma)\| \leq \frac{\theta}{2}\]

\[(72) \quad \|g_{\theta/2,\delta}(h^\circ(e_{\nu(\gamma)})) k^\ddagger(e_\gamma) - k^\ddagger(e_\gamma)\| \leq \frac{3\theta}{4}\]

Finally, we have

\[(73) \quad k^\ddagger(e_\gamma) \in J(h^\circ(e_{\nu(\gamma)})) \subset J(h^\ddagger(e_{\nu(\gamma)})) \]

by the choice of \(\nu\) (cf. [31, 5.4]) and (51). □

**Remark 9.5.** In the situation of Proposition 9.4, we always have

\[(72) \quad \|g_{\delta,2\delta}(h^\circ(e_{\nu(\gamma)})) k(e_\gamma) - k(e_\gamma)\| \leq (n + 1)2\delta\]

and

\[(73) \quad k(e_\gamma) \leq (n + 1)h(e_{\nu(\gamma)})\]

for any \(\delta > 0\) and \(\gamma \in \Gamma \setminus \{\ast\}\).

**Proof.** The subdivision \(S_dJ K\), and hence the map

\[\bar{k} : C^\Gamma \rightarrow C(\text{span}J K)\]

come from a repeated application of [31 Proposition 5.5].

Let \(K^-\) denote the subcomplex of \(K\) generated by \(\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)}\), then \(K^-\) is at most \(n\)-dimensional.

By [31 5.3(i) and 5.4] each \(\gamma \in \Gamma \setminus \{\ast\}\) is some face of \(K^-\) which contains \(\nu(\gamma)\) and which has dimension at most \(n\). Let \(t_\gamma \in |K^-|\) denote the barycenter of this face and observe that

\[(74) \quad \frac{1}{n+1} \cdot S\bar{k}(e_\gamma)(t_\gamma) \leq h(e_{\nu(\gamma)})(t_\gamma);\]
the same estimate holds for each $t \in K$, since $\tilde{h}(e_{\gamma})$ and $\tilde{h}(e_{\nu(t)})$ are linear coordinate functions. The second estimate (73) now follows after applying $\tilde{h}$ to (74); the first estimate (72) is then straightforward using functional calculus. □

10. Elementary polynomials

The notion of elementary polynomials introduced below is used frequently in Section 11.

Definition 10.1. Let $r \in \mathbb{N}$ and let $e$ be a polynomial in $r$ commuting variables $x_1, \ldots, x_r$. We say $e$ is elementary, if $e(x_1, \ldots, x_r)$ is of the form

$$\sum_{k=1}^{r} \prod_{l=1}^{r} y^{(k,l)},$$

where each $y^{(k,l)}$ is either $x_j$ or $1 - x_j$ for some $j \in \{1, \ldots, r\}$.

Remark 10.2. Given $r \in \mathbb{N}$, there are no more than $(2r)^r$ elementary polynomials in $r$ commuting variables $x_1, \ldots, x_r$.

Proposition 10.3. Let $r \in \mathbb{N}$ be given. If $q_1, \ldots, q_s \in M_r$ are pairwise commuting projections for some $s \in \mathbb{N}$, then for every projection $p \in C^*(q_j \mid j = 1, \ldots, s) \subset M_r$ there are an elementary polynomial $e$ in $r$ commuting variables and $j_1, \ldots, j_r \in \{1, \ldots, s\}$ such that

$$p = e(q_{j_1}, \ldots, q_{j_r}).$$

Proof. It will suffice to show that, for any projection $e$ which is minimal in $C^*(q_j \mid j = 1, \ldots, s)$, we have

$$e = h_1 \cdots h_r,$$

where each $h_i$ is either $q_{j_i}$ or $1 - q_{j_i}$ for some $j_i \in \{1, \ldots, s\}$. The result will then follow since each projection in $C^*(q_j \mid j = 1, \ldots, s)$ is a sum of at most $r$ minimal ones (and since 0 can be written in the form $\prod_{l=1}^{r} y^{(l)}$ with $y^{(l)}$ either $q_j$ or $1 - q_j$).

So let $e \in C^*(q_j \mid j = 1, \ldots, s)$ be minimal, then

$$e \leq q_{j_1}$$

for some $j_1 \in \{1, \ldots, s\}$. Set

$$d_1 := q_{j_1}.$$

If $d_1$ is minimal in $C^*(q_j \mid j = 1, \ldots, s)$, then

$$e = d_1 = (q_{j_1})^{r-1} d_1$$

is of the desired form. If $d_1$ is not minimal, there is $q_{j_2}$ such that either

$$e \leq d_1 q_{j_2} \leq q_{j_1}$$

or

$$e \leq d_1 (1 - q_{j_2}) \leq q_{j_1}.$$

In the first case, set

$$d_2 := d_1 q_{j_2};$$

in the second case, set

$$d_2 := d_1 (1 - q_{j_2}).$$

If $d_2$ is minimal in $C^*(q_j \mid j = 1, \ldots, s)$, then

$$e = d_2 = (q_{j_1})^{r-2} d_2$$
is of the desired form. Otherwise, keep repeating the construction inductively to obtain
\[ e \leq d_l \leq \ldots \leq d_1. \]
Since \( d_1 \) has rank (as an element of \( M_r \)) at most \( r \) and the ranks of the \( d_j \) are reduced at each step, the construction will terminate for some \( l \leq r \), in which case \( d_l \) is minimal and
\[ e = d_l = (q_j)_{l}^{-1}d_l \]
is of the desired form. \[\square\]

11. Existence of compatible approximations

The purpose of this section is it to show that compatible approximations indeed exist. To this end, we carefully revisit the proof of [31, Theorem 6.1] in the case of recursive subhomogeneous subalgebras of transformation group \( C^*\)-algebras.

**Lemma 11.1.** Let \((T, h)\) be a minimal dynamical system with \( T \) compact and metrizable and such that \( \dim T \leq n < \infty \). Let \( Z \subset T \) be closed with nonempty interior and let
\[ C(T) \subset E_Z = C^*(C(T), uC_0(T \setminus Z)) \subset E = C^*(C(T), u), \]
and
\[ B_{l+1} = B_l \oplus \mathbb{Z}_{l+1} \mathbb{Z}_{l+1} (C(Z_{l+1}) \otimes M_{m_{l+1}}) \]
be as in [8, 7].

Then, for each \( l \in \{1, \ldots, L\} \) and \( \eta > 0 \) there is a system of c.p.c., p.c., \( n \)-decomposable and \((\rho_l, \eta)\)-compatible approximations for \( B_l \); the approximating finite dimensional \( C^*\)-algebras may be chosen so that all their irreducible representations have rank at least \( m_1 \).

**Proof.** Let us first prove the assertion of the lemma for \( l = 1 \). Note that \( Z_1 \subset Z \) is closed with \( \dim Z_1 \leq n \), and that
\[ B_1 = C(Z_1) \otimes M_{m_1}. \]
It follows from [15, 21, 16] and [22] that
\[ \rho_1(C(T)) = C(Z_1) \otimes \mathcal{D}_{m_1} \]
and that
\[ \rho_1(C_0(T \setminus Z)) = C(Z_1) \otimes \mathcal{D}_{m_1}^\phi. \]
Let
\[ \left( C(Z_1) \xrightarrow{\varphi_\lambda} C^k \xrightarrow{\varphi_\lambda} C(Z_1) \right)_{\lambda \in \Lambda} \]
be a system of c.p.c., \( n \)-decomposable approximations for \( C(Z_1) \), then
\[ \left( C(Z_1) \otimes M_{m_1} \xrightarrow{\psi_\lambda \otimes \text{id}_{M_{m_1}}} C^k \otimes M_{m_1} \xrightarrow{\varphi_\lambda \otimes \text{id}_{M_{m_1}}} C(Z_1) \otimes M_{m_1} \right)_{\lambda \in \Lambda} \]
is a system of c.p.c., p.c., n-decomposable approximations for $C(Z_1) \otimes M_{m_1}$. Upon identifying $C^{k_\lambda} \otimes M_{m_1}$ with $\bigoplus_{k_\lambda} M_{m_1}$, we write $\bar{\psi}_\lambda$ and $\bar{\varphi}_\lambda$ for $\psi_\lambda \otimes \text{id}_{M_{m_1}}$ and $\varphi_\lambda \otimes \text{id}_{M_{m_1}}$, respectively. We check that each $\bar{\varphi}_\lambda$ is $(\rho_1, \eta)$-compatible. Let

$$D_k^\lambda \subset D_\lambda \subset \bigoplus_{k_\lambda} M_{m_1}$$

denote the (shiftable) diagonal elements, and observe that

$$\varphi_\lambda^*(D_\lambda) \subset C(Z_1) \otimes D_{m_1} \supseteq \rho_1(C(T))$$

and

$$\varphi_\lambda(D_k^\lambda) \subset C(Z_1) \otimes D_{m_1} \supseteq \rho_1(C_0(T \setminus Z)).$$

It is straightforward to construct a c.p.c. lift

$$\varphi_\lambda^*: D_\lambda \to C(T) \subset E_Z$$

for $\varphi_\lambda|_{D_\lambda}$ along $\rho_1$ satisfying

$$\varphi_\lambda^*(D_\lambda) \subset C_0(T \setminus Z) \subset E_Z,$$

so that we have properties 8.3(i) and (ii). Property 8.3(iii) follows directly from (16), so that indeed each $\bar{\varphi}_\lambda$ is $(\rho_1, \eta)$-compatible. Note that in fact our argument shows each that $\bar{\varphi}_\lambda$ is $(\rho_1, 0)$-compatible, since $\varphi_\lambda^*$ is an exact lift of $\varphi_\lambda|_{D_\lambda}$; for higher values of $l$ we will only be able to produce approximate lifts, which will complicate matters significantly.

Now suppose the lemma has been established for some $l \in \{1, \ldots, L - 1\}$. To verify the statement for $l + 1$, it will suffice to show that, for any $0 < \varepsilon, \eta < 1$ and positive contractions $a_1, \ldots, a_k \in B_{l+1}$ (where we may assume $a_1 = 1_{B_{l+1}}$), there is a c.p.c., p.c., $n$-decomposable approximation $(\mathcal{F}, \psi, \varphi)$ (for $B_{l+1}$) of $\{a_1, \ldots, a_k\}$ within $\varepsilon$ such that $\varphi$ is $(\rho_{l+1}, \eta)$-compatible. (We also need to make sure the ranks of the irreducible representations of $\mathcal{F}$ are at least $m_1$ — but this will turn out automatically, since only $m_1, \ldots, m_{l+1}$ occur as possible ranks.)

Our construction will be a variation of the proof of [31, Theorem 6.1]. Let us first adjust our notation. Set

$$\mathcal{B} := \mathcal{B}_l, \mathcal{A} := \mathcal{B}_{l+1}, r := m_{l+1}, \Omega := Z_{l+1} \text{ and } X := Z_{l+1} \setminus Z_{l+1}.$$

Note that $X \subset \Omega$ is closed and that

$$\mathcal{A} = \mathcal{B} \oplus_{\pi, X} (C(\Omega) \otimes M_r)$$

(cf. [20]), where

$$\pi: \mathcal{B} \to C(X) \otimes M_r$$

is the map $\pi_l$ from [19]. Note that $\mathcal{A}$ is $r$-subhomogeneous of topological dimension at most $n$, cf. [21], hence satisfies the hypotheses of [31, Theorem 6.1].

We will run the proof of [31, Theorem 6.1] almost verbatim to obtain a c.p.c., p.c., $n$-decomposable approximation for $\mathcal{A}$. In a few places along the way we will have to be more careful about certain choices to make the approximations $(\rho_{l+1}, \eta)$-compatible.

First, we choose numbers

$$(77) \quad \frac{1}{4(n + 1)} = \mu_0 < \mu_1 < \ldots < \mu_r < \frac{1}{2(n + 1)}$$
and define functions
\[ c_1, \ldots, c_r, d_1, \ldots, d_r \in C_0([0,1])_+ \]
by
\[ c_l := g_{\mu_{l-1}, \mu_l} \quad \text{and} \quad d_l := d_{\mu_{l-1}, \mu_l}, \]
where \( g_{\alpha, \beta} \) and \( d_{\alpha, \beta} \) are as in \[9.3\].

Choose
\[ 0 < \theta^{(11)} < \frac{\eta}{6(n+1)r}, \]
\[ 0 < \theta^{(10)} < \frac{\theta^{(11)}}{5(4(n+1))^2}, \]
\[ 0 < \theta^{(9)} < \left( \frac{\theta^{(10)}}{7(n+1)r} \right)^8, \]
\[ 0 < \theta^{(8)} < \frac{\theta^{(9)}}{6(n+1)r} \left( \leq \frac{\theta^{(10)} \theta^{(11)}}{5} \right), \]
\[ 0 < \theta^{(7)} < \frac{\theta^{(8)}}{24(n+1)} \]
and
\[ 0 < \theta^{(6)} < \frac{\theta^{(7)} \theta^{(8)}}{32(n+1)} \]
Apply Proposition \[9.4\] with \( \theta^{(6)} \) in place of \( \theta \) and \( r \) in place of \( R \) to obtain \( \tilde{\theta} > 0 \);
choose
\[ 0 < \theta^{(5)} < \frac{\tilde{\theta}}{2}, \theta^{(6)}. \]
Choose
\[ 0 < \theta^{(4)} < \frac{\theta^{(5)}}{16(n+1)r}. \]
Choose
\[ 0 < \theta^{(3)} < \theta^{(4)} \]
such that, if \( a, b \) are positive contractions in some \( C^* \)-algebra which satisfy \[ \|a - b\| \leq \theta^{(3)}, \]
then
\[ \|d_{\theta^{(7)/2}, \theta^{(7)}}(a) - d_{\theta^{(7)/2}, \theta^{(7)}}(b)\| \leq \theta^{(6)}; \]
this is possible by approximating the function \( d_{\theta^{(7)/2}, \theta^{(7)}} \) uniformly by polynomials on \([0,1] \).
Choose
\[ 0 < \theta^{(2)} < \frac{\theta^{(3)}}{12(n+1)}. \]
Choose
\begin{equation}
0 < \theta^{(1)} < \theta^{(2)}
\end{equation}
such that, if \(a_1, \ldots, a_r\) and \(b_1, \ldots, b_r\) are positive contractions in some \(C^*\)-algebra so that the \(a_i\) commute mutually and the \(b_i\) commute mutually, and so that
\[
\|a_i - b_i\| \leq \theta^{(1)},
\]
then
\begin{equation}
\|e(a_1, \ldots, a_r) - e(b_1, \ldots, b_r)\| \leq \theta^{(2)}
\end{equation}
for any elementary polynomial \(e\) in \(r\) commuting variables. (Note that \(\theta^{(1)}\) exists since there are at most \((2r)^r\) such elementary polynomials, cf. Remark 10.2.)

Choose
\begin{equation}
0 < \eta' < \frac{\theta^{(3)}}{36(n+1)}
\end{equation}
such that, if \(a, b\) are positive contractions in some \(C^*\)-algebra, and if
\[
\|a - b\| \leq 3\eta',
\]
then
\begin{equation}
\|c_l(a) - c_l(b)\| \leq \theta^{(1)}
\end{equation}
and
\begin{equation}
\|d_l(a) - d_l(b)\| \leq \theta^{(1)}
\end{equation}
for \(l = 1, \ldots, r\), cf. [78].

We now take
\[
\alpha > 0
\]
and a c.p.c., p.c., \(n\)-decomposable approximation
\[
(F' = \bigoplus_{i=1}^s M_{r_i}, \psi', \varphi')
\]
(of \(B\)) as in Step 1 of the proof of [31, 6.1]. We may in addition assume that
\begin{equation}
\alpha < \eta';
\end{equation}
much more important, by our induction hypothesis we may even assume that \(\varphi'\) is \((\rho, \eta')\)-compatible.

As in [31, 6.1], we note that by [31, 4.5] there are a closed neighborhood \(Y' \subset \Omega\) of \(X\) and a c.p.c., p.c., \(n\)-decomposable map
\[
\hat{\varphi} : F' \rightarrow B \oplus_{\pi,X} C(Y') \otimes M_{r}
\]
satisfying (6.1) and (6.2) of [31].

As we claim that, upon making \(Y'\) smaller if necessary, we may in addition assume that \(\hat{\varphi}\) is \((\rho_{Y'}, 3\eta')\)-compatible, where
\begin{equation}
\rho_{Y'} : \mathcal{E}_Z \xrightarrow{\rho_{Y'}^1} \mathcal{A} \xrightarrow{\pi_{Y'}} B \oplus_{\pi,X} (C(Y') \otimes M_{r})
\end{equation}
is the natural surjection.

To verify this, note that by (6.1) of [31] we have
\begin{equation}
\|\beta \circ \hat{\varphi}(x) - \varphi'(x)\| \leq \alpha\|x\| \leq \eta'\|x\|
\end{equation}
for \( x \in F_+ \), where (as in \[31\] (6.1)) we use \( \beta \) to denote both the projection maps

\[
\mathcal{A} = \mathcal{B} \oplus_{\pi,X} (C(\Omega) \otimes M_r) \to \mathcal{B}
\]

and

\[
\mathcal{B} \oplus_{\pi,X} (C(Y') \otimes M_r) \to \mathcal{B}
\]

(this will cause no ambiguity). Let

\[
(D')^\beta \subset D' \subset F'
\]

de note the (shiftable) diagonal elements of \( F' \); now any \((\beta \rho, \eta')\)-compatible c.p.c. approximate lift

\[
\mathcal{D}' \to C(T) \subset \mathcal{E}_Z
\]

for \( \varphi' \) will be a \((\beta \rho, 2\eta')\)-compatible c.p.c. approximate lift for \( \beta \circ \hat{\varphi} \) by (96); upon making \( Y' \) smaller if necessary, it will then also be a \( 3\eta' \)-compatible c.p.c. approximate lift for \( \hat{\varphi} \).

We now run Step 2 of the proof of \[31\] Theorem 6.1 to obtain

\[
V, W \subset_{\text{open}} \Omega
\]

and

\[
Y \subset_{\text{closed}} \Omega
\]

such that

\[
X \subset W \subset Y \subset V \subset Y'
\]

and a collection

\[
(U_\lambda)_{\lambda \in \Lambda}
\]

of open subsets of \( \Omega \setminus X \) satisfying properties (i)–(vi) of the proof of \[31\] Theorem 6.1, in particular with

\[
U_\lambda \subset Y' \quad \text{for all } \lambda \in \Lambda',
\]

where

\[
\Lambda' = \{ \lambda \in \Lambda \mid U_\lambda \cap Y \neq \emptyset \}
\]

(cf. property (v) of \[31\] 6.1, Step 2).

It is straightforward to show that, upon making the \( U_\lambda \) smaller if necessary, we may in addition assume that the following are satisfied:

(vii) there is a map

\[
\kappa : \Lambda' \to \{1, \ldots, r\}
\]

such that

\[
\mu(\lambda) = \mu_{\kappa(\lambda)}
\]

(cf. property (vi) and (77)), such that

\[
c_{\kappa(\lambda)}(\hat{\varphi}(1_{M_{ri}})|_{U_\lambda}) = q(\lambda, i)
\]

and such that

\[
d_{\kappa(\lambda)}(\hat{\varphi}(1_{M_{ri}})|_{U_\lambda}) = (q(\lambda, i)\hat{\varphi}(1_{M_{ri}})|_{U_\lambda})^{-1}
\]

for \( \lambda \in \Lambda', i \in \{1, \ldots, s\} \), where

\[
q(\lambda, i) := \chi_{\mu(\lambda)}(\hat{\varphi}(1_{M_{ri}})|_{U_\lambda}) \in C^*(\hat{\varphi}(1_{M_{ri}})|_{U_\lambda}) \subset C_b(U_\lambda) \otimes M_r
\]

is a projection for each \( \lambda \in \Lambda' \) and \( i \in \{1, \ldots, s\} \), cf. property (vi);
(viii) for each \(\lambda \in A'\) and each \(t \in U_\lambda\), the map
\[
ev_t : C^*(q(\lambda, j) \mid j = 1, \ldots, s) \to C^*(q(\lambda, j)(t) \mid j = 1, \ldots, s)
\]
is an isomorphism of finite dimensional commutative \(C^*\)-algebras with dimension at most \(r\).

Just as in the proof of [31, 6.1], we complete Step 2 by choosing functions
\[(101)\]
\[g_\lambda \in C_0(U_\lambda), \; \lambda \in A;\]
we set
\[(102)\]
\[g := \sum_{\lambda \in A} g_\lambda.\]

We note that in the proof of [31, 6.1] each \(C_0(U_\lambda)\) was identified with \(C_0(U_\lambda) \otimes 1_{M_r}(\subset Z(A) \subset A)\) (where \(Z(A)\) denotes the center of \(A\)); it follows from (17) that in our setting and with this identification we have
\[(103)\]
\[C_0(U_\lambda) \otimes 1_{M_r} \subset \rho_{l+1}(C(T)).\]

From here on, the construction runs exactly as in the proof of [31, 6.1]. Step 3 yields c.p.c. maps
\[(104)\]
\[\tilde{\varphi}^{(1)} : \mathcal{F}^{(1)} \,(= \mathcal{F}' = \bigoplus_{i=1}^s M_{r_i}) \to A\]
and
\[(105)\]
\[\tilde{\varphi}^{(2)} : \mathcal{F}^{(2)} \,(= \bigoplus_{A(2)} M_r) \to A\]
which add up to a c.p.c. map
\[(106)\]
\[\varphi = (\tilde{\varphi}^{(1)}, \tilde{\varphi}^{(2)}) : \mathcal{F} = \mathcal{F}^{(1)} \oplus \mathcal{F}^{(2)} \to A\]
and are given by
\[\varphi_i(x) = \tilde{\varphi}_i^{(1)}(x) \]
\[(107)\]
\[\varphi_\lambda(x) = \tilde{\varphi}_\lambda^{(2)}(x) = g_\lambda \cdot x\]
for \(\lambda \in A(2)\) and \(x \in M_r\).

Here, the projections \(p(\lambda, i)\) (for \(\lambda \in A'\) and \(i \in \{1, \ldots, s\}\)) are obtained from [31, Lemma 4.2] and satisfy (6.4) of [31, 6.1] as well as
\[(108)\]
\[\sum_{j=1}^s p(\lambda, j) = 1_{U_\lambda},\]
and
\[\{p(\lambda, i), q(\lambda, j)p(M_{r_j})\} = 0 \text{ for } j \in \{1, \ldots, s\},\]
and
\[p(\lambda, i)q(\lambda, i) = p(\lambda, i)\]
This is essentially (6.3) of [31, 6.1], except that we have corrected a small quibble stemming from the statement of [31, Lemma 4.2]; with this correction, neither our
proof nor that of [31, 6.1] are affected. We are indebted to A. Tikuisis for pointing out this issue.

We also need to observe an additional fact:
For each \( \lambda \) and \( i \), we have
\[
p(\lambda, i) \in C^*(q(\lambda, j) \mid j = 1, \ldots, s)
\]
by [31, Lemma 4.2]. By (viii) above, for each \( t \in U_\lambda \) we have
\[
evt : C^*(q(\lambda, j) \mid j = 1, \ldots, s) \xrightarrow{\cong} C^*(q(\lambda, j)(t) \mid j = 1, \ldots, s) \subset M_r;
\]
for each \( i \) we may thus apply Proposition 10.3 to obtain
\[
\text{(109)}
\]
and an elementary polynomial
\[
\text{(110)}
\]
in \( r \) commuting variables such that
\[
p(\lambda, i)(t) = e^{(\lambda, i)}(q(\lambda, j_1(t), \ldots, q(\lambda, j_r(t)));
\]
as \( ev_t \) is an isomorphism, we may apply its inverse to obtain
\[
\text{(111)}
\]
Step 4 now involves the relative barycentric subdivision method from [31, 5.3 and 5.4]; it yields an index set
\[
\text{(112)}
\]
a u.c.p. map
\[
\text{(113)}
\]
and a map
\[
\text{(114)}
\]
which in particular satisfies
\[
\text{(115)}
\]
We then construct
\[
\mathcal{F} = \mathcal{F}^{(1)} \oplus \mathcal{F}^{(2)} \oplus \mathcal{F}^{(3)}
\]
where
\[
\text{(116)}
\]
a c.p.c. map
\[
\psi : A \to \mathcal{F}
\]
and a c.p.c. map
\[
\phi := (\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) : \mathcal{F} \to A
\]
with
\[
\text{(117)}
\]
for \( i = 1, \ldots, s \) and \( x \in M_{r_i} \) (with \((\hat{\varphi}_i(1_{M_{r_i}}))^{-1}\varphi_i(\cdot)) being the supporting *-homomorphism for \( \hat{\varphi}_i(\cdot) \), cf. [30, Lemma 1.1.3])
\[
\text{(118)}
\]
for $\gamma \in \Gamma'$, $i \in \{1, \ldots, s\}$ and $x \in M_{r_i}$,
\begin{equation}
\varphi^{(3)}_{\gamma}(x) := k_\gamma \cdot x
\end{equation}
for $\gamma \in \Gamma''$ and $x \in M_r$.

Step 5 of our proof of [31, 6.1] now applies verbatim to show that $(\mathcal{F}, \psi, \varphi)$ is a c.p.c., p.c., $n$-decomposable approximation (for $\mathcal{A}$) of $a_1, \ldots, a_k$ within $\varepsilon$.

In the remainder of the proof we will show that $\varphi$ indeed is $(\rho_{l+1}, \eta)$-compatible. To this end, we need to construct a c.p.c. map
\[ \varphi^\sharp : \mathcal{D} \to C(T) \]
as in Definition 8.3, where
\[ \mathcal{D}^\vartheta \subset \mathcal{D} \subset \mathcal{F} \]
denotes the (shiftable) diagonal elements of $\mathcal{F}$, and similar for
\[ (\mathcal{D}^{(1)})^\vartheta \subset \mathcal{D}^{(1)} \subset \mathcal{F}^{(1)}, (\mathcal{D}^{(2)})^\vartheta \subset \mathcal{D}^{(2)} \subset \mathcal{F}^{(2)}, (\mathcal{D}^{(3)})^\vartheta \subset \mathcal{D}^{(3)} \subset \mathcal{F}^{(3)}. \]

Recall from Step 1 that
\[ \hat{\varphi} : \mathcal{F}' \to \mathcal{B} \oplus_{\pi, X} (C(\gamma') \otimes M_r) \]
is $(\rho_{\gamma'}, 3\eta')$-compatible, so that we may choose a $3\eta'$-compatible approximate c.p.c. lift
\[ \hat{\varphi}^\sharp : \mathcal{D}' \to C(T) \subset \mathcal{E}_Z. \]

This will satisfy
\begin{enumerate}
\item \[ \|\hat{\varphi}(e) - \rho_{\gamma'\varphi^\sharp(e)}\| \leq 3\eta\|e\| \text{ for } e \in \mathcal{D}'_+ \]
\item \[ \hat{\varphi}^\sharp((\mathcal{D}')^\vartheta) \subset C_0(T \setminus Z) \]
\item \[ \|\rho_{\gamma'}(\hat{\varphi}^\sharp(e)) - \hat{\varphi}(Se)\| \leq 3\eta\|e\| \text{ for } e \in (\mathcal{D}')^\vartheta \]
\end{enumerate}
(\text{cf. Definition 8.3(i), (ii), (iii)}).

For $\lambda \in \Lambda'$ and $i \in \{1, \ldots, s\}$ we define
\begin{equation}
q(\lambda, i)^\dagger := e_{\kappa(\lambda)}(\rho_{l+1}\varphi^\sharp(1_{M_{r_i}})) \in \rho_{l+1}(C(T))
\end{equation}
and
\begin{equation}
p(\lambda, i)^\dagger := e(\lambda, i)(q(\lambda, j_1^{(i)})^\dagger, \ldots, q(\lambda, j_r^{(i)})^\dagger) \in \rho_{l+1}(C(T)),
\end{equation}
where $\kappa(\lambda)$, $e(\lambda, i)$ and $j_r^{(i)}$ come from (98), (78), (110) and (109), respectively. Note that the $q(\lambda, i)^\dagger$ and $p(\lambda, i)^\dagger$ are positive contractions, but not necessarily projections.

By (a) above we have that
\begin{equation}
\|\hat{\varphi}(1_{M_{r_i}})|_{U_{\lambda}} - \rho_{\gamma'}(\hat{\varphi}^\sharp(1_{M_{r_i}}))|_{U_{\lambda}}\| \leq 3\eta
\end{equation}
for each $\lambda \in \Lambda'$ and $i \in \{1, \ldots, s\}$.

Since
\[ \pi_{\gamma'}(q(\lambda, i)^\dagger)|_{U_{\lambda}} \overset{[120]}{=} c_{\kappa(\lambda)}(\rho_{\gamma'}(\hat{\varphi}^\sharp(1_{M_{r_i}}))|_{U_{\lambda}}), \]
we see from (122), (99) and (92) that
\begin{equation}
\|\pi_{\gamma'}(q(\lambda, i)^\dagger)|_{U_{\lambda}} - q(\lambda, i)\| \leq \theta^{(1)}
\end{equation}
for each $\lambda \in \Lambda'$ and $i \in \{1, \ldots, s\}$. Similarly we see from (123) and (90) that
\begin{equation}
\|\pi_{\gamma'}(p(\lambda, i)^\dagger)|_{U_{\lambda}} - p(\lambda, i)\| \leq \theta^{(2)}.
\end{equation}
As before, we denote by
\[
\overline{D} \subset \mathcal{D} \subset \mathcal{F}, \quad (\overline{D}(1))^{\sigma} \subset \mathcal{D}(1) \subset \mathcal{F}(1) \quad \text{and} \quad (\overline{D}(2))^{\sigma} \subset \mathcal{D}(2) \subset \mathcal{F}(2)
\]
the (shiftable) diagonal elements of \(\mathcal{F}, \mathcal{F}(1)\) and \(\mathcal{F}(2)\), respectively. Recall that (see (104)) \(\mathcal{F}(1) = \mathcal{F}'\), so
\[
\overline{D}(1) = \mathcal{D}'.
\]
Define a c.p. map
\[
\varphi^{(1)}_{\downarrow}: \overline{D}(1) \to \rho_{l+1}(C(T))
\]
by
\[
\varphi^{(1)}_{\downarrow}(e) := (1_A - g) \cdot \rho_{l+1}\hat{\varphi}^{1}_i(e) + \sum_{\lambda \in \Lambda^{(1)}} g_{\lambda} \cdot p(\lambda, i) \hat{d}_{\kappa}(\lambda)(\rho_{l+1}\hat{\varphi}^{1}_i(1_{M_{r_i}}))\rho_{l+1}\hat{\varphi}^{1}_i(e)
\]
for \(e \in \overline{D}(1) \cap M_{r_i}\), cf. (116), (101), (102) and (103).

For notational consistency, let us define another c.p. map
\[
\varphi^{(2)}_{\downarrow}: \overline{D}(2) \to \rho_{l+1}(C(T))
\]
by
\[
\varphi^{(2)}_{\downarrow} := \varphi^{(2)}|_{\mathcal{D}(2)};
\]
\(\varphi^{(2)}_{\downarrow}\) is well defined with the right target algebra by (107) and (103).

The maps \(\varphi^{(1)}_{\downarrow}\) and \(\varphi^{(2)}_{\downarrow}\) add up to a c.p. map
\[
\varphi_{\downarrow}: \overline{D} \to \rho_{l+1}(C(T)).
\]

With
\[
\Sigma = \Sigma^{(1)} \cup \Sigma^{(2)} = \{1, \ldots, s\} \cup \Lambda^{(2)}, \quad \Sigma^{+} = \Sigma \cup \{\ast\}
\]
(cf. (112), (114) and (105)) and
\[
\iota: \mathbb{C}^\Sigma \to \mathcal{F}
\]
being the canonical unital embedding, in Step 4 of the proof of [31, 6.1] the u.c.p. map
\[
h: \mathbb{C}^{\Sigma^+} \to C^\ast(\bar{\varphi}_i(\mathbb{C}), 1_A) \subset \mathcal{A}
\]
was defined by
\[
h|_{\mathbb{C}^\Sigma} := \varphi_i,
\]
so that (cf. (113))
\[
\mathcal{C} = C^\ast(h(\mathbb{C}^{\Sigma^+})).
\]

We define unital and linear maps
\[
h^\downarrow, h^\downarrow: \mathbb{C}^{\Sigma^+} \to \rho_{l+1}(C(T)) \subset \mathcal{A}
\]
by
\[
h^\downarrow(e_i) := \left(\varphi^{(1)}_{i}(1_{M_{r_i}}) - \theta^{(4)}\right)_+
\]
for \(i \in \{1, \ldots, s\},\)
\[
h^\downarrow(e_\lambda) := \left(\varphi^{(2)}_{\lambda}(1_{M_{r}}) - \theta^{(4)}\right)_+
\]
for \( \lambda \in \Lambda^{(2)} \) and

\[(131) \quad h^\dagger|_{C^\lambda} := \frac{1}{1 + \theta(5) \cdot h^\dagger|_{C^\lambda}}.\]

Set

\[C^\dagger := C^*(h^\dagger(C^{n^+})) \subset \rho_{t+1}(C(T)) \subset \mathcal{A}.\]

Before proceeding with our construction, we check that \( h^\dagger \) is u.c.p. and that

\[\|h(e) - h^\dagger(e)\| \leq 2\theta(5)\|e\|\]

for \( 0 \leq e \in C^{n^+} \).

For \( i \in \{1, \ldots, s\} \) and \( 0 \leq e \in \mathcal{D}^{(1)} \cap M_{r_i} \) we have

\[(132) \quad \|\hat{\varphi}_i(e) - \rho_Y\hat{\varphi}_i(e)\| \leq 3\eta'\|e\|\]

and, for \( \lambda \in \Lambda^{(1)} \subset \Lambda' \),

\[(133) \quad \|\varphi_i - \rho_Y\varphi_i\| \leq \theta(3)\|e\|.\]

We may now use the fact that \( \mathcal{A} - g \) vanishes outside \( \mathcal{Y}' \) and that each \( g_i \) vanishes outside \( U_{\lambda} \) to conclude from (126), (106), (132) and (133) that

\[(134) \quad \|\varphi_i^{(1)}(e) - \varphi_i^{(1)}(e)\| \leq \theta(3)\|e\|\]

for any \( 0 \leq e \in \mathcal{D}^{(1)} \cap M_{r_i}, i = 1, \ldots, s.\)

If \( \zeta \) is an irreducible representation of \( \mathcal{A} \), then by (134), (129) and (86) we have the implication

\[(135) \quad \zeta(h^\dagger(e_i)) \neq 0 \implies \zeta(\varphi_i^{(1)}(1_{M_{r_i}})) \neq 0.\]

Since \( \varphi^{(1)} \) is \( n \)-decomposable, so is \( \zeta \circ \varphi^{(1)} \); since the image of \( \zeta \) is a matrix algebra of rank at most \( r \), we see that \( \zeta(\varphi_i^{(1)}(1_{M_{r_i}})) \) is nonzero for at most \( (n + 1) \cdot r \) values of \( i \); by (135), \( \zeta(h^\dagger(e_i)) \) is nonzero for at most \( (n + 1) \cdot r \) values of \( i \) as well.
Now let
\[ 0 \leq e \leq \|e\| \cdot \sum_{i \in \{1, \ldots, s\}} e_i \in C^{\Sigma(1)} \subset C^{\Sigma^+}, \]
then
\[ \|\zeta(h(e)) - \zeta(h^{++}(e))\| \leq \sum_{i \in \{1, \ldots, s\}} \|\zeta(h(e_i)) - \zeta(h^{++}(e_i))\| \cdot \|e\| \]
\[ \leq \frac{(n + 1)r(\theta(3) + \theta(4))}{\theta(5)} \cdot \|e\|. \]
Similarly, one checks that
\[ \|\zeta(h(e)) - \zeta(h^{++}(e))\| \leq \frac{\theta(5)}{2} \cdot \|e\| \]
for \( 0 \leq e \in C^{\Sigma(2)} \). Since \( \zeta \) was an arbitrary irreducible representation of \( A \), we obtain
\[ \|h(e) - h^{++}(e)\| \leq \theta(5) \cdot \|e\| \]
for \( 0 \leq e \in C^{\Sigma} \). It is then clear from (131) that
\[ \|h(e) - h^{\dagger}(e)\| \leq 2\theta(5) \cdot \|e\| \]
for \( e \in C^{\Sigma^+} \) and that
\[ h^{\dagger}(1_{C^{\Sigma}}) \leq 1_A, \]
whence \( h^{\dagger} \) (being the unital linear extension of \( h^{\dagger}|_{C^{\Sigma}} \)) indeed is a u.c.p. map. We can now continue our construction.

With
\[ \Gamma = \{\ast\} \cup \Sigma(1) \cup \Gamma' \cup \Gamma'' \]
(cf. (112)) like in the proof of [31, 6.1] (as obtained from [31, 5.4]), by (136) Proposition 9.4 yields a u.c.p. map
\[ k^{\dagger}: C^\Gamma \rightarrow C^{\dagger} \subset \rho_{l+1}(C(T)) \subset A \]
such that \( k^{\dagger}|_{C^\Gamma \setminus \{\ast\}} \) is \( n \)-decomposable with respect to the same decomposition as \( k|_{C^\Gamma \setminus \{\ast\}} \), with
\[ \|k - k^{\dagger}\| < \theta(6) \]
and such that
\[ \|g_{\theta(5)/2, \theta(5)}(h^{k}(e_{\nu(\gamma)}))k^{\dagger}(e_{\gamma}) - k^{\dagger}(e_{\gamma})\| < \theta(6) \]
for each \( \gamma \in \Gamma \setminus \{\ast\} \). We will denote the components of \( k^{\dagger}|_{C^\Gamma \setminus \{\ast\}} \) by \( k^{\dagger}_{\gamma} \) and \( k^{\dagger}_{\gamma} \), respectively.

We now define a c.p. map
\[ \varphi^{\dagger}_{\gamma}: D \rightarrow \rho_{l+1}(C(T)) \subset A \]
by
\[ \varphi^{\dagger}_{\gamma}(e) := k^{\dagger}_{\gamma} \cdot d_{\theta(7)/2, \theta(7)}(\varphi^{\dagger}(1_{M_{\gamma}})) \cdot \varphi^{\dagger}_{\gamma}(e) \]
for $i = 1, \ldots, s$ and $e \in \mathcal{D}^{(1)} \cap M_{r_{i}}$,  
(139) \[ \varphi_{\gamma,i}^\dagger(e) := k_{\gamma}^\dagger \cdot p(\nu(\gamma), i)^{\dagger} d_{\nu(\gamma)}(\rho_{i+1}(\varphi_{\gamma}(1_{M_{r_{i}}}))\rho_{i+1}(\varphi_{\gamma}(e)) \]

for $\gamma \in \Gamma'$, $i \in \{1, \ldots, s\}$ and $e \in \mathcal{D}^{(2)} \cap M_{r_{i}}$ and

(140) \[ \varphi_{\gamma}^\dagger(e) := k_{\gamma}^\dagger \cdot e \]

for $\gamma \in \Gamma''$ and $e \in \mathcal{D}^{(3)} \cap M_{r}$.

Each of the maps $\varphi_{1}^\dagger$, $\varphi_{\gamma,i}^\dagger$ and $\varphi_{\gamma}^\dagger$ are c.p.c. order zero; we moreover claim that

(141) \[ \|\varphi_{1}^\dagger(e) - \varphi_{1}^{(1)}(e)\| \leq \theta^{(8)} \|e\| \]

for $i = 1, \ldots, s$ and $0 \leq e \in \mathcal{D}^{(1)} \cap M_{r_{i}}$,

(142) \[ \|\varphi_{\gamma,i}^\dagger(e) - \varphi_{\gamma,i}^{(2)}(e)\| \leq \theta^{(8)} \|e\| \]

for $\gamma \in \Gamma'$, $i \in \{1, \ldots, s\}$ and $0 \leq e \in \mathcal{D}^{(2)} \cap M_{r_{i}}$ and

(143) \[ \|\varphi_{\gamma}^\dagger(e) - \varphi_{\gamma}^{(3)}(e)\| \leq \theta^{(8)} \|e\| \]

for $\gamma \in \Gamma''$ and $0 \leq e \in \mathcal{D}^{(3)} \cap M_{r}$.

For $i = 1, \ldots, s$ and $0 \leq e \in \mathcal{D}^{(1)} \cap M_{r_{i}}$, we have

\[ \|\varphi_{1}^{(1)}(e) - \varphi_{1}^\dagger(e)\| \]

\[ \leq \|k_{1} \cdot (\varphi_{1}(1_{M_{r_{i}}}))^{-1}\varphi_{1}(e) - \varphi_{1}^\dagger(e)\| \]

\[ \leq \|k_{1} \cdot g_{\theta(1/2, \theta(1)}(\varphi_{1}(1_{M_{r_{i}}}))^{-1}\varphi_{1}(e) - \varphi_{1}^\dagger(e)\| \]

\[ + 2(n + 1)\theta^{(7)} \|e\| \]

\[ \leq \|k_{1} \cdot g_{\theta(1/2, \theta(1)}(\varphi_{1}(1_{M_{r_{i}}}))\varphi_{1}(e) - \varphi_{1}^\dagger(e)\| \]

\[ + 2(n + 1)\theta^{(7)} \|e\| \]

\[ \leq \|k_{1} \cdot d_{\theta(1/2, \theta(1)}(\varphi_{1}(1_{M_{r_{i}}}))\varphi_{1}(e) - \varphi_{1}^\dagger(e)\| \]

\[ + 4(n + 1)\theta^{(7)} \|e\| \]

\[ \leq \|k_{1} \cdot d_{\theta(1/2, \theta(1)}(\varphi_{1}^\dagger(1_{M_{r_{i}}}))\varphi_{1}^\dagger(e) - \varphi_{1}^\dagger(e)\| \]

\[ + 4(n + 1)\theta^{(7)} \|e\| \]

\[ \leq \|k_{1} \cdot d_{\theta(1/2, \theta(1)}(\varphi_{1}^\dagger(1_{M_{r_{i}}}))\varphi_{1}^\dagger(e) - \varphi_{1}^\dagger(e)\| \]

\[ + 4(n + 1)\theta^{(7)} \|e\| \]

\[ \leq \left( 4(n + 1)\theta^{(7)} + \frac{2\theta^{(6)}}{\theta^{(7)}} + \frac{\theta^{(3)}}{\theta^{(7)}} \right) \|e\| \]

\[ \leq \theta^{(8)} \|e\|. \]
For $\gamma \in \Gamma'$, $i \in \{1, \ldots, s\}$ and $0 \leq e \in D^{(2)} \cap M_{r_i}$ we estimate

\[ \|\pi^\gamma_Y(\varphi^{(2)}_{\gamma,i}(e)) - \pi^\gamma_Y(k_{\gamma}^* \cdot p(\nu(\gamma), i)^{\frac{1}{2}} d_{\kappa(\nu(\gamma))}(\rho_Y(\hat{\varphi}_{\gamma}^i(1_{M_{r_i}})))\rho_Y(\hat{\varphi}_{\gamma}^i(e))\| \]

\[ \leq (\theta^{(6)})(4(n + 1) + \theta^{(2)}4(n + 1) + \theta^{(1)} + 3\eta'4(n + 1)||e||, \]

where for the last estimate we have used \(78\), \(137\), \(90\), \(111\), \(121\), \(93\) and (a).

Since $k_{\gamma}$ is 0 outside $Y'$ and $\|k_{\gamma}^* - k_{\gamma}\| \leq \theta^{(6)}$ (see \(137\)) we obtain

\[ \|\varphi^{(2)}_{\gamma,i}(e) - \varphi^{\dagger}_{\gamma,i}(e)\| \]

\[ \leq (\theta^{(6)})(8(n + 1) + \theta^{(2)}4(n + 1) + \theta^{(1)} + 3\eta'4(n + 1)||e|| \]

We have

\[ \|\varphi^{(3)}_{\gamma}(e) - \varphi^{\dagger}_{\gamma}(e)\| \leq \theta^{(8)}||e|| \]

for $\gamma \in \Gamma''$ and $0 \leq e \in D^{(3)} \cap M_r$ since

\[ \|k_{\gamma} - k_{\gamma}^*\| \leq \theta^{(6)} \leq \theta^{(8)}, \]

cf. \(140\), \(119\) and \(137\). We have thus confirmed \(141\), \(142\) and \(143\).

Let \(\tilde{c} \in C_0((0, 1])_+\) be the function satisfying

\[ (145) \quad \tilde{c} \cdot \text{id}_{[0, 1]} = (\text{id}_{[0, 1]} - \theta^{(9)})_+. \]

Define a c.p. map

\[ \varphi^b : D \to \rho_{l+1}(C(T)) \subset \mathcal{A} \]

by

\[ (146) \quad \varphi_{\gamma}^b(e) \quad \tilde{c}(\varphi_{\gamma,i}^i(1_{M_{r_i}}))\varphi_{\gamma,i}^i(e) \]

for $i = 1, \ldots, s$ and $e \in D^{(1)} \cap M_{r_i}$,

\[ (147) \quad \varphi_{\gamma,i}^b(e) \quad \tilde{c}(\varphi_{\gamma,i}^i(1_{M_{r_i}}))\varphi_{\gamma,i}^i(e) \]

for $\gamma \in \Gamma'$, $i = 1, \ldots, s$ and $e \in D^{(2)} \cap M_{r_i}$, and

\[ (148) \quad \varphi_{\gamma}^b(e) \quad \tilde{c}(\varphi_{\gamma}^i(1_{M_r}))\varphi_{\gamma}^i(e) \]

for $\gamma \in \Gamma''$ and $e \in D^{(3)} \cap M_r$. 

ROKHLIN DIMENSION AND $C^*$-DYNAMICS
For the c.p.c. order zero components of \( \varphi^\dagger \) we have
\[
(149) \quad \varphi^\dagger_{\gamma}(1_{M_r}) = (\varphi_{\gamma}(1_{M_r}) - \theta^{(9)})_+ 
\]
for \( i \in \{1, \ldots, s\} \),
\[
(150) \quad \varphi^\dagger_{\gamma,i}(1_{M_r}) = (\varphi_{\gamma,i}(1_{M_r}) - \theta^{(9)})_+ 
\]
for \( \gamma \in \Gamma', i \in \{1, \ldots, s\} \) and
\[
(151) \quad \varphi^\dagger_{\gamma}(1_{M_r}) = (\varphi_{\gamma}(1_{M_r}) - \theta^{(9)})_+ 
\]
for \( \gamma \in \Gamma'' \).

Note that it follows from the definition that if a component of \( \varphi^\dagger \) is nonzero, then so is the respective component of \( \varphi^\dagger \).

Now if \( \zeta \) is an irreducible representation of \( \mathcal{A} \), and \( \zeta(\varphi^{(1)\dagger}(1_{M_r})) \neq 0 \) for \( i \in \{1, \ldots, s\} \), then by (158) we have \( \zeta(g_{\nu(\gamma)/2,\theta}(\varphi^{(1)\dagger}(1_{M_r}))) \neq 0 \), which in turn implies \( \zeta(h^{(\dagger)}(e_i)) \neq 0 \) by (129). But we have already seen after (135) that this can happen for at most \( (n+1) \cdot r \) values of \( i \).

Similarly, if \( \zeta(\varphi^{(2)\dagger}(1_{M_r})) \) is nonzero for some \( \gamma \in \Gamma' \) and \( i \in \{1, \ldots, s\} \), then \( \zeta(d_{\nu(\gamma)}(\rho_{i+1}(\varphi^{\dagger}_{\gamma,i}(1_{M_r})))) \) and \( \zeta(k_{\gamma,i}^\dagger) \) must both be nonzero.

The latter implies \( \zeta(h^{(\dagger)}(e_{\nu(\gamma)})) \neq 0 \) by (44), which in turn yields \( \zeta(\varphi^{(2)\dagger}(1_{M_r})) \neq 0 \) by (114), (131) and (130). From (127) and (107) we see that \( \zeta(g_{\nu(\gamma)}) \neq 0 \). Since \( \gamma \in \Gamma'' \), we have \( \nu(\gamma) \in \Lambda' \) (cf. the second paragraph of 31 6.1, Step 4), hence \( U_{\nu(\gamma)} \subset Y' \) (by property (v) of 31 6.1, Step 2], cf. (97)), and so \( \zeta \) factorizes through \( \pi_{Y'} \), so that in fact \( \zeta(d_{\nu(\gamma)}(\rho_{\nu(\gamma)}(\varphi^{\dagger}_{\gamma,i}(1_{M_r})))) \neq 0 \), whence \( \|\zeta(\rho_{\nu(\gamma)}(\varphi^{\dagger}_{\gamma,i}(1_{M_r}))\| \geq \frac{1}{(n+1)} \) (see (78)) and (by (a) above) \( \zeta(\varphi_{\gamma,i}(1_{M_r})) \neq 0 \). Once again, this can happen for at most \( (n+1) \cdot r \) values of \( i \), regardless of \( \gamma \).

Finally, if \( \zeta(\varphi_{\gamma,i}(1_{M_r})) \neq 0 \) for some \( \gamma \in \Gamma'' \), then \( \zeta(k_{\gamma,i}^\dagger) = \zeta(\varphi_{\gamma,i}(1_{M_r})) \neq 0 \), see (140). But since \( k_{\gamma,i}^\dagger \in \mathcal{C} \setminus \{\cdot\} \) is \( n \)-decomposable, \( \zeta(k_{\gamma,i}^\dagger) \) can be nonzero for at most \( (n+1) \cdot r \) values of \( \gamma \).

For \( j = 1, 2, 3 \) and \( e^{(j)} \in \mathcal{D}_r^{(j)} \) we now conclude from (141), (142) and (143) together with the preceding remarks and the fact that \( \varphi \) is \( n \)-decomposable that
\[
\|\varphi^{(j)\dagger}(e^{(j)}) - \varphi^{(j)}(e^{(j)})\| = \sup_\zeta \|\zeta(\varphi^{(j)\dagger}(e^{(j)})) - \zeta(\varphi^{(j)}(e^{(j)}))\| 
\]
\[
\leq 2(n+1)r\theta^{(9)}\|e^{(j)}\|, 
\]
where the supremum runs over all irreducible representations \( \zeta \) of \( \mathcal{A} \). For \( e \in \mathcal{D}_+ \) this yields
\[
(152) \quad \|\varphi^\dagger(e) - \varphi(e)\| \leq 3 \cdot 2(n+1)r\theta^{(9)}\|e\| \leq 82 \theta^{(9)}\|e\|. 
\]

By (149), (150) and (151) we have
\[
\varphi^\dagger(1_D) \leq \frac{1}{1 + \theta^{(9)}} \cdot \varphi^\dagger(1_D) 
\]
\[
\leq \frac{1}{1 + \theta^{(9)}} \cdot (\varphi(1_D) + \theta^{(9)} \cdot 1_A) 
\]
\[
\leq 1_A, 
\]
whence \( \varphi^\dagger \) indeed is a c.p.c. map.
We next observe that
\[(153) \; \hat{g} := (1 - (\theta^{(9)})^{\frac{1}{2}}) \cdot g_{\theta^{(9)},(\theta^{(9)})^{\frac{1}{2}}} \leq \hat{c} \leq 1_{[0,1]};\]
for \(i \in \{1, \ldots, s\}\) and \(e \in D^{(1)} \cap M_{\tilde{r}_i}\) positive and normalized we estimate
\[
\| \varphi^\dagger_{\gamma,i}(e) - \hat{g}(\varphi^\dagger_{\gamma,i}(1M_{\tilde{r}_i}))\varphi^\dagger_{\gamma,i}(e) \| \\
\leq \| (1A - \hat{g}(\varphi^\dagger_{\gamma,i}(1M_{\tilde{r}_i})))\varphi^\dagger_{\gamma,i}(e)(1A - \hat{g}(\varphi^\dagger_{\gamma,i}(1M_{\tilde{r}_i}))) \|^\frac{1}{2} \\
\leq \| (1A - \hat{g}(\varphi^\dagger_{\gamma,i}(1M_{\tilde{r}_i})))\varphi^\dagger_{\gamma,i}(1M_{\tilde{r}_i})(1A - \hat{g}(\varphi^\dagger_{\gamma,i}(1M_{\tilde{r}_i}))) \|^\frac{1}{2} \\
\leq \theta^{(9)}\frac{1}{4}.
\]
We combine this with (146) and (153) to obtain
\[(154) \; \| \varphi^\dagger_{\gamma,i}(e) - \varphi^\dagger_{\gamma,i}(e) \| \leq (\theta^{(9)})^{\frac{1}{2}}\| e \|\]
for any \(0 \leq e \in D^{(1)} \cap M_{\tilde{r}_i}\).

In the same manner we see that
\[(155) \; \| \varphi^\dagger_{\gamma,i}(e) - \varphi^\dagger_{\gamma,i}(e) \| \leq (\theta^{(9)})^{\frac{1}{2}}\| e \|\]
for \(\gamma \in \Gamma',\; i \in \{1, \ldots, s\}\) and \(0 \leq e \in D^{(2)} \cap M_{\tilde{r}_i}\), and
\[(156) \; \| \varphi^\dagger_{\gamma,i}(e) - \varphi^\dagger_{\gamma,i}(e) \| \leq (\theta^{(9)})^{\frac{1}{2}}\| e \|\]
for \(\gamma \in \Gamma''\) and \(0 \leq e \in D^{(3)} \cap M_{\tilde{r}}\).

Now with essentially the same reasoning as for (152) (and using (148), (155) and (156)), we obtain for \(e \in D_{+}\)
\[
\| \varphi^\dagger(e) - \varphi^\dagger(e) \| \leq 3 \cdot 2(n + 1)r(\theta^{(9)})^{\frac{1}{2}}\| e \|. \\
\]
By combining this with (152) we get
\[
\| \varphi^\dagger(e) - \varphi(e) \| \leq 7(n + 1)r(\theta^{(9)})^{\frac{1}{2}}\| e \| \leq \theta^{(10)}\| e \| \leq \eta\| e \|(157)
\]
for any \(e \in D_{+}\).

For \(e \in (D^{(1)})^\theta \cap M_{\tilde{r}_i}\), by (116) and (b) above we have
\[(158) \; \rho_{t+1}(\varphi^\dagger_{\gamma,i}(e)) \in \rho_{t+1}(C_0(T \setminus Z)),\]
whence
\[
\varphi^\dagger_{\gamma,i}(e) \in \rho_{t+1}(C_0(T \setminus Z)) \\
\]
by (126) and therefore
\[
\varphi^\dagger_{\gamma,i}(e), \varphi^\dagger_{\gamma,i}(e) \in \rho_{t+1}(C_0(T \setminus Z)) \\
\]
by (138) and (146).
This argument also yields (using (147) and (139))
\[
\varphi^\dagger_{\gamma,i}(e), \varphi^\dagger_{\gamma,i}(e) \in \rho_{t+1}(C_0(T \setminus Z))
\]
for \( \gamma \in \Gamma', i \in \{1, \ldots, s\} \) and \( e \in (D(2))^{\delta} \cap M_{r_i} \). It follows from \( 148, 140 \) and \( 22 \) that 
\[
\phi_{\gamma}^{h}(e), \phi_{\gamma}^{d}(e) \in \rho_{i+1}(C_{0}(T \setminus Z))
\]
for \( \gamma \in \Gamma'' \) and \( e \in (D(3))^{\eta} \cap M_{r}. \) We have thus confirmed 
\[
\phi^{h}(D^{\delta}) \in \rho_{i+1}(C_{0}(T \setminus Z)).
\]
Let 
\[
\phi^{\sharp} : D \to C(T)
\]
be a c.p.c. lift for 
\[
\phi^{\delta} : D \to \rho_{i+1}(C(T));
\]
It is not hard to arrange for \( \phi^{\sharp} \) to satisfy 
\[
\phi^{\sharp}(D^{\delta}) \subset C_{0}(T \setminus Z).
\]

We have now confirmed properties \( 8.3(i) \) (by \( 157 \), with \( \rho_{i+1} \) in place of \( \rho \) and \( 8.3(ii) \) (by \( 160 \)). It remains to verify \( 8.3(iii) \).

For each \( i \in \{1, \ldots, s\} \) choose 
\[
h_{i}^{\sharp} \in C_{0}(T \setminus Z)
\]
positive and normalized such that 
\[
\|h_{i}^{\sharp} \phi_{i}^{\sharp}(e) - \phi_{i}^{\sharp}(e)\| \leq \theta^{(8)} \|e\|
\]
and 
\[
\|h_{i}^{\sharp} \phi_{i}^{\sharp}(e) - \phi_{i}^{\sharp}(e)\| \leq \theta^{(8)} \|e\|
\]
for any \( 0 \leq e \in (D(1))^{\delta} \cap M_{r_i}; \) this is possible by \( 160 \) and by \( 116 \) and (b) above.

We first use the fact that \( \hat{\phi}_{i}(x) \) and \( \pi_{Y^{\gamma}} \phi_{i}^{(1)}(1_{M_{r_i}}) \) commute (see \( 31 \) Steps 4 and 5)) to compute 
\[
\|\pi_{Y^{\gamma}}(\phi_{i}^{(1)}(x)) - \hat{\phi}_{i}(x)\pi_{Y^{\gamma}}(\phi_{i}^{(1)}(1_{M_{r_i}}))d_{\theta(10)/2,\theta(10)}(\hat{\phi}_{i}(1_{M_{r_i}}))\|
\]
\[
\leq 117 \|\pi_{Y^{\gamma}}(k_{i} \cdot (\hat{\phi}_{i}(1_{M_{r_i}}))^{-1}\phi_{i}(x)) - \pi_{Y^{\gamma}}(\phi_{i}^{(1)}(1_{M_{r_i}}))\phi_{i}(x)d_{\theta(10)/2,\theta(10)}(\hat{\phi}_{i}(1_{M_{r_i}}))\|
\]
\[
\leq 117 \|\pi_{Y^{\gamma}}(k_{i} \cdot (\hat{\phi}_{i}(1_{M_{r_i}}))^{-1}\phi_{i}(x)) - \pi_{Y^{\gamma}}(k_{i} \cdot (\hat{\phi}_{i}(1_{M_{r_i}}))^{-1}\phi_{i}(1_{M_{r_i}}))\phi_{i}(x)d_{\theta(10)/2,\theta(10)}(\hat{\phi}_{i}(1_{M_{r_i}}))\|
\]
\[
\leq 106 \|\pi_{Y^{\gamma}}(k_{i} \cdot (\hat{\phi}_{i}(1_{M_{r_i}}))^{-1}\phi_{i}(x)) - \pi_{Y^{\gamma}}(k_{i} \cdot (\hat{\phi}_{i}(1_{M_{r_i}}))^{-1}\phi_{i}(1_{M_{r_i}}))\phi_{i}(x)d_{\theta(10)/2,\theta(10)}(\hat{\phi}_{i}(1_{M_{r_i}}))\|
\]
\[
\leq 106 \leq 4(n+1)2(n+1)\theta(10)\|x\|
\]
for \( i \in \{1, \ldots, s\} \) and \( x \in M_{r_i}. \)
We then estimate for \( i \in \{1, \ldots, s\} \) and \( 0 \leq e \in (\mathcal{D}^{(1)})^\vartheta \cap M_{r_i} \)
\[
\|\rho_{t+1}(u \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| \\
\overset{\text{(161)}}{\leq} \|\rho_{t+1}(u h^t_i \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| + \theta^{(8)}\|e\| \\
\overset{\text{(159)}}{\leq} \|\rho_{t+1}(u h^t_i \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| + \theta^{(8)}\|e\| \\
\overset{\text{(157)}}{\leq} \|\rho_{t+1}(u h^t_i \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| + (\theta^{(8)} + \theta^{(10)})\|e\| \\
\overset{\text{(160)}}{\leq} \|\rho_{t+1}(u h^t_i \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| + (\theta^{(8)} + \theta^{(10)})\|e\| \\
\overset{\text{(59)}}{\leq} \|\rho_{t+1}(u h^t_i \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| \cdot \frac{1}{\theta^{(10)}} \\
+ (\theta^{(8)} + \theta^{(10)} + (4(n + 1))^2\theta^{(10)})\|e\| \\
\overset{\text{(a)}}{\leq} \|\rho_{t+1}(u \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| \cdot \frac{1}{\theta^{(10)}} \\
+ \left(\theta^{(8)} + \theta^{(10)} + (4(n + 1))^2\theta^{(10)} + \frac{3\eta'}{\theta^{(10)}}\right)\|e\| \\
\overset{\text{(162)}}{\leq} \|\rho_{t+1}(u \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| \cdot \frac{1}{\theta^{(10)}} \\
+ \left(\theta^{(8)} + \theta^{(10)} + (4(n + 1))^2\theta^{(10)} + \frac{3\eta'}{\theta^{(10)}} + \frac{\theta^{(8)}}{\theta^{(10)}}\right)\|e\| \\
\overset{\text{(c)}}{\leq} \left(\theta^{(8)} + \theta^{(10)} + (4(n + 1))^2\theta^{(10)} + \frac{3\eta'}{\theta^{(10)}} + \frac{\theta^{(8)}}{\theta^{(10)}} + \frac{3\eta'}{\theta^{(10)}}\right)\|e\| \\
\overset{\text{(80), (82)}}{\leq} \theta^{(11)}\|e\|.
\]

For any irreducible representation \( \zeta \) of \( \mathcal{A} \), we have
\[
\zeta \rho_{t+1} \varphi^{(1)}(1)_{M_{r_i}} \overset{\text{(159)}}{=} \zeta \varphi^{(1)}(1)_{M_{r_i}} \overset{\text{(149)}}{=} \zeta (\varphi^{(1)}(1)_{M_{r_i}} - \theta^{(9)})_+,
\]
and (after \( \text{(151)} \)) we have already seen that at most \((n+1)r\) components of \( \zeta \rho_{t+1} \varphi^{(1)}_{\varphi}(e) \) are nonzero; the same holds for \( \zeta \varphi^{(1)} \), since \( \varphi \) is \( n \)-decomposable. Together with the preceding estimate this yields
\[
\|\rho_{t+1}(u \varphi^{(1)}(e)) - \varphi^{(1)}(Se)\| \overset{\text{(79)}}{\leq} \frac{n}{3}\|e\| \\
\overset{\text{(164)}}{\leq} 2(n + 1)r \theta^{(11)}\|e\|
\]
for any \( 0 \leq e \in (\mathcal{D}^{(1)})^\vartheta \).

Next, choose
\[
h^t_{r,i} \in C_0(T \setminus Z)
\]
for each \( \gamma \in \Gamma' \) and \( i \in \{1, \ldots, s\} \) such that

\[
\|h_{\gamma,i}^2 \varphi^{(2)}_{\gamma,i}(e) - \varphi^{(2)}_{\gamma,i}(e)\| \leq \theta^{(8)}\|e\|
\]

and

\[
\|h_{\gamma,i}^2 \hat{\varphi}_{\gamma,i}^r(e) - \hat{\varphi}_{\gamma,i}^r(e)\| \leq \theta^{(8)}\|e\|
\]

for any \( 0 \leq e \in (\mathcal{D}(2))^\sigma \cap M_{\gamma} \); again this is possible by (160), (116) and (b) above. We now estimate

\[
\|\rho_{l+1}(u\varphi^{(2)}_{\gamma,i}(e)) - \varphi^{(2)}_{\gamma,i}(Se)\|
\]

for \( \gamma \in \Gamma', i \in \{1, \ldots, s\} \) and \( 0 \leq e \in (\mathcal{D}(2))^\sigma \cap M_{\gamma} \).

By the same argument as for \( \varphi^{(1)} \) and \( \varphi^{(1)}_{\gamma,i} \) we see that for any irreducible representation \( \zeta \) of \( \mathcal{A} \) at most \( (n+1)r \) components of \( \zeta \varphi^{(2)} \) and at most \( (n+1)r \) components of \( \zeta \rho_{l+1} \varphi^{(2)}_{\gamma,i} \) are nonzero. For \( 0 \leq e \in (\mathcal{D}(2))^\sigma \) we obtain

\[
\|\rho_{l+1}(u\varphi^{(2)}_{\gamma,i}(e)) - \varphi^{(2)}_{\gamma,i}(Se)\| \leq 2(n+1)r\eta'^{(11)}\|e\|
\]

(166)

Finally, we note that for \( \gamma \in \Gamma'' \) and \( 0 \leq e \in (\mathcal{D}(2))^\sigma \cap M_{\gamma} \) we may choose an (isometric) lift

\[
x_e \in C_0(T \setminus Z)
\]

of \( k_{\gamma} \cdot e \); this will automatically satisfy

\[
\rho_{l+1}(ux_e) = k_{\gamma} \cdot Se,
\]

cf. (16) and (17).

Choose \( h_{\gamma,i}^2 \in C_0(T \setminus Z) \) positive and normalized such that

\[
\|h_{\gamma,i}^2 \varphi^{(3)}_{\gamma,i}(e) - \varphi^{(3)}_{\gamma,i}(e)\| \leq \theta^{(8)}\|e\|
\]

(168)
and
\[(169) \quad \| h^k_x x_e - x_e \| \leq \theta(e) \| e \|; \]
this is possible by (167) and (115).

We now estimate
\[
\| \rho_{t+1}(u \varphi^{(3)}(e)) - \varphi^{(3)}(Se) \| \leq \| \rho_{t+1}(uh^k_x \varphi^{(3)}(e)) - \varphi^{(3)}(Se) \| + \theta(e) \| e \|
\]

(8) \[ \leq \| \rho_{t+1}(uh^k_x \varphi^{(3)}(e)) - \varphi^{(3)}(Se) \| + \theta(e) \| e \|
\]

(11) \[ \leq \| \rho_{t+1}(uh^k_x) - \varphi^{(3)}(Se) \| + \theta(e) \| e \|
\]

and
\[
\| \rho_{t+1}(uh^k_x x_e) - k \cdot (Se) \| + \theta(e) \| e \|
\]

Obtain this is possible by (167) and (115). We are finally prepared to show that actions coming from minimal homeomorphisms for any $0 < \delta < \varepsilon$

Proof. Let $T$ be a minimal dynamical system with $T$ compact and metrizable; suppose that $\dim T \leq n < \infty$ and let $\alpha$ be the induced action on $C(T)$. Then,

\[
\dim_{\text{Rok}}(C(T), \alpha) \leq 2n + 1.
\]

We are finally prepared to show that actions coming from minimal homeomorphisms on finite dimensional spaces indeed have finite Rokhlin dimension.

12. COMPATIBLE APPROXIMATIONS AND ROKHLIN DIMENSION

We are finally prepared to show that actions coming from minimal homeomorphisms on finite dimensional spaces indeed have finite Rokhlin dimension.

**Theorem 12.1.** Let $(T, h)$ be a minimal dynamical system with $T$ compact and metrizable; suppose that $\dim T \leq n < \infty$ and let $\alpha$ be the induced action on $C(T)$. Then,

\[
\dim_{\text{Rok}}(C(T), \alpha) \leq 2n + 1.
\]

**Proof.** Given $k \in \mathbb{N}$ and $\varepsilon > 0$, choose
\[
0 < \delta < \varepsilon \quad \text{and} \quad 0 < \eta < \frac{\varepsilon}{4(n + 1)(s + 1)}.
\]

Obtain $0 < \tilde{\eta}$ from Proposition 8.4; we may assume that $\tilde{\eta} < \eta$. 

and choose a closed subset $Z \subset T$ with nonempty interior such that, with the notation of 8.1, $m_1 \geq 4s$ (using minimality of $h$).

Use Lemma 11.1 to find an $n$-decomposable and $(\rho_L, \tilde{\eta})$-compatible c.p.c. approximation 

$$(\mathcal{F} = M_{r_1} \oplus \ldots \oplus M_{r_s}, \psi, \varphi)$$

for $B_L$ with

$$(174) \quad \|\varphi(1_F) - 1_{B_L}\| \leq \frac{\varepsilon}{2}$$

and such that, for each $i$,

$$r_i \geq m_1 \geq 4s.$$ 

Now apply Proposition 7.2 to each summand of $F$; the resulting maps add up to a c.p.c. order zero map

$$\mu : C^k \to D^\tilde{\sigma} \subset F$$

such that

$$(175) \quad \|\tilde{\sigma}(\mu(e)) - \mu(\tilde{\sigma}(e))\| \leq \delta\|e\|$$

for all $0 \leq e \in C^k$ (where as in 7.1 $\tilde{\sigma}$ denotes the truncated shift on $D \subset F$, and $\tilde{\sigma}$ is the cyclic shift on $C^k$), and such that

$$(\tilde{\sigma})^s(\mu(1_{C^k})) + \tilde{\sigma}^s(\mu(1_{C^k})) \geq 1_F.$$ 

Suppose $\varphi$ is $n$-decomposable with respect to the decomposition

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)};$$

for $j = 0, \ldots, n$ let

$$\varphi^{(j)} : F^{(j)} \to B_L$$

denote the order zero components of $\varphi$ and let

$$\mu^{(j)} : C^k \to (D^{(j)})^\tilde{\sigma} \subset F^{(j)}$$

denote the respective components of $\mu$.

For $j \in \{0, \ldots, n\}$ and $l \in \{-1, 1\}$ define maps

$$\tilde{\zeta}^{(j,l)} : C^k \to E$$

by

$$(176) \quad \tilde{\zeta}^{(j,l)}(e) := u^{ls}(\rho_L^{-1}\varphi^{(j)}(\mu^{(j)}(e))u^{-ls}.$$ 

It is clear that these are c.p.c. order zero (the $\mu^{(j)}$ and the $\varphi^{(j)}$ are).

Let

$$\varphi^\sharp : D \to C(T)$$

be an $\tilde{\eta}$-compatible approximate c.p.c. lift for $\varphi$ with components

$$(\varphi^\sharp)^{(j)} : D^{(j)} \to C(T).$$

Define c.p.c. maps

$$\zeta^{(j,l)} : C^k \to C(T) \subset E$$

by

$$(177) \quad \|\zeta^{(j,l)}(e) - \tilde{\zeta}^{(j,l)}(e)\| \leq \tilde{\eta}\|e\|$$

for $j \in \{0, \ldots, n\}$ and $l \in \{-1, 1\}$.
for any $0 \leq e \in \mathbb{C}^k$; since $\tilde{\zeta}^{(j,l)}$ has order zero, we see that whenever $e, e' \in \mathbb{C}^k$ are orthogonal positive elements, then

\begin{equation}
\|\zeta^{(j,l)}(e)\zeta^{(j,l)}(e')\| \leq 2\tilde{\eta}\|e\|\|e'\| \leq \varepsilon\|e\|\|e'\|.
\end{equation}

Next, we estimate for $j \in \{0, \ldots, n\}$, $l \in \{-1, 1\}$ and $0 \leq e \in \mathbb{C}^k$

\begin{equation}
\begin{aligned}
\|\zeta^{(j,l)}(\sigma(e)) - u\zeta^{(j,l)}(e)u^*\| & \leq \|\tilde{\zeta}^{(j,l)}(\sigma) - u\tilde{\zeta}^{(j,l)}(e)u^*\| + 2\tilde{\eta}\|e\| \\
\|\sigma^{-1}(\varphi^{(j,l)}(\sigma)) - u(\sigma^{-1}(\varphi^{(j,l)}(e)))u^*\| & \leq \|\sigma^{-1}(\varphi^{(j,l)}(\sigma)) - (\sigma^{-1}(\varphi^{(j,l)}(e)))u^*\| + 2\tilde{\eta}\|e\| \\
& \leq (2\tilde{\eta} + \delta + \eta)\|e\|
\end{aligned}
\end{equation}

We furthermore have

\begin{equation}
\zeta^{(j,1)}(1_{\mathbb{C}^k}) \geq \tilde{\zeta}^{(j,1)}(1_{\mathbb{C}^k}) - \tilde{\eta} \cdot 1_{\mathbb{C}^k}
\end{equation}

\begin{equation}
\begin{aligned}
u^s(\sigma^{-1}(\varphi^{(j,l)}(1_{\mathbb{C}^k}))) - \tilde{\eta} \cdot 1_{\mathbb{C}^k} & \geq u^s(\sigma^{-1}(\varphi^{(j,l)}((1_{\mathbb{C}^k}))u^s - \tilde{\eta} \cdot 1_{\mathbb{C}^k} \\
& \geq \rho_{L}^{-1}(\varphi^{(j,l)}(\sigma^{-1}(\varphi^{(j,l)}(\sigma^{-1}(\varphi^{(j,l)}(1_{\mathbb{C}^k})))) - (s\eta + \tilde{\eta}) \cdot 1_{\mathbb{C}^k} \\
& = \rho_{L}^{-1}(\varphi^{(j,l)}(\sigma^{-1}(\varphi^{(j,l)}(1_{\mathbb{C}^k})))) - (s\eta + \tilde{\eta}) \cdot 1_{\mathbb{C}^k}
\end{aligned}
\end{equation}

for $j \in \{0, \ldots, n\}$; in a similar manner one checks

\begin{equation}
\zeta^{(j,-1)}(1_{\mathbb{C}^k}) \geq \rho_{L}^{-1}(\varphi^{(j,l)}((1_{\mathbb{C}^k}) - (s\eta + \tilde{\eta}) \cdot 1_{\mathbb{C}^k}.
\end{equation}

This in turn yields

\begin{equation}
\begin{aligned}
\sum_{j=0,\ldots,n} \sum_{l=1,-1} \zeta^{(j,l)}(1_{\mathbb{C}^k}) & \geq \rho_{L}^{-1}(\varphi^{(j,l)}((1_{\mathbb{C}^k}) - (s\eta + \tilde{\eta}) \cdot 1_{\mathbb{C}^k} \\
& \geq \rho_{L}^{-1}(\varphi^{1_{\mathbb{C}^k}}) - (s\eta + \tilde{\eta}) \cdot 1_{\mathbb{C}^k} \\
& \geq (1 - \varepsilon/2 - 2(n + 1)(s\eta + \tilde{\eta})) \cdot 1_{\mathbb{C}^k}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\geq (1 - \varepsilon) \cdot 1_{\mathbb{C}^k}.
\end{aligned}
\end{equation}

Upon relabeling the $\zeta^{(j,l)}$ as $\zeta^{(j)}$, $j = 0, \ldots, 2n + 1$, we see from (178), (180) and (179) that indeed

\begin{equation}
\overline{\dim}_{\text{Rok}}(C(T), \alpha) \leq 2n + 1,
\end{equation}

cf. Definition 2.3 and Proposition 2.6 respectively (the $\mathbb{C}^{k+1}$ component of $\zeta^{(j)}$ is just zero for each $j$). It follows from Proposition 2.7 that

\begin{equation}
\dim_{\text{Rok}}(C(T), \alpha) \leq 2n + 1.
\end{equation}

$\square$
Appendix: Crossed Products, Nuclear Dimension and $\mathcal{Z}$-stability

Notation A.1. To fix notation we recall the construction of the reduced crossed product. Let $G$ be a discrete group, $A \to B(H)$ a $C^*$-algebra acting faithfully on the Hilbert space $H$ and $\alpha : G \to \text{Aut}(A)$ an action of $G$ on $A$. Define $\pi : A \to B(\ell^2(G) \otimes H)$ by

$$\pi(a)(e_g \otimes \xi) = e_g \otimes \alpha_{g^{-1}}(a)\xi$$

and

$$\lambda_h(e_g \otimes \xi) = e_{hg} \otimes \xi,$$

where $g, h \in G$, $a \in A$ and $\xi \in H$. Then $\lambda_g \pi(a) = \pi(\alpha_g(a))\lambda_g$ i.e. $(\pi, \lambda)$ is a covariant representation. The reduced crossed product $A \rtimes_{\alpha, r} G$ is the $C^*$-subalgebra of $B(\ell^2(G) \otimes H)$ generated by $\{\pi(a)\lambda_g \mid a \in A, g \in G\}$. This algebra does not depend on the choice of the faithful representation of $G$. The groups we consider in this paper are either finite or equal to $\mathbb{Z}$, hence amenable. For amenable groups the reduced crossed product $A \rtimes_{\alpha, r} G$ coincides with the universal crossed product $A \rtimes_{\alpha, r} G$. Using matrix units, $\pi(a)$ can also be written as

$$\pi(a) = \sum_{g \in G} e_{g,g} \otimes \alpha_{g^{-1}}(a)$$

and

$$\pi(a)\lambda_h = \sum_{g \in G} e_{g,h^{-1}g} \otimes \alpha_{g^{-1}}(a).$$

Thus if $G$ is finite with $n$ elements then we may regard $A \rtimes_{\alpha, r} G$ as a subalgebra of $M_n(\mathbb{C})$ in a natural way and if $G = \mathbb{Z}$ we will regard $A \rtimes_{\alpha, r} G$ as a subalgebra of $B(\ell^2(\mathbb{Z}) \otimes H)$. We will usually drop $\pi$ from the notation and denote $\lambda_g$ by $u_g$.

We recall that a c.p. contraction $\varphi : A \to B$ is said to be an order zero map if whenever $x, y$ are positive elements in $A$ such that $xy = 0$ then $\varphi(x)\varphi(y) = 0$.

Order zero maps play a central role in the definition of decomposition rank and nuclear dimension which we recall for the reader’s convenience. (Cf. [14] and [34].)

Definition A.2. Let $A$ be a $C^*$-algebra, $F$ a finite-dimensional $C^*$-algebra and $n \in \mathbb{N}$.

1. A c.p. map $\varphi : F \to A$ is $n$-decomposable if there is a decomposition

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

such that the restriction $\varphi^{(i)}$ of $\varphi$ to $F^{(i)}$ has order zero for each $i \in \{0, \ldots, n\}$.

2. $A$ has decomposition rank $n$, $\text{dr} A = n$, if $n$ is the least integer such that the following holds: For any finite subset $F \subset A$ and $\varepsilon > 0$, there is a finite-dimensional c.p.c. approximation $(F, \psi, \varphi)$ for $F$ with tolerance $\varepsilon$ (i.e., $F$ is finite-dimensional, $\psi : A \to F$ and $\varphi : F \to A$ are c.p.c. and $\|\varphi\psi(b) - b\| < \varepsilon \forall b \in F$) such that $\varphi$ is $n$-decomposable. If no such $n$ exists, we write $\text{dr} A = \infty$.

3. $A$ has nuclear dimension $n$, $\text{dim}_{\text{nuc}} A = n$, if $n$ is the least integer such that the following holds: For any finite subset $F \subset A$ and $\varepsilon > 0$, there is a finite-dimensional c.p.c. approximation $(F, \psi, \varphi)$ for $F$ to within $\varepsilon$ (i.e., $F$ is finite-dimensional, $\psi : A \to F$ and $\varphi : F \to A$ are c.p.c. and $\|\varphi\psi(b) - b\| < \varepsilon \forall b \in F$) such that $\psi$ is c.p.c., and $\varphi$ is $n$-decomposable with c.p.c. order zero components $\varphi^{(i)}$. If no such $n$ exists, we write $\text{dim}_{\text{nuc}} A = \infty$. 
Note that in (3) we may assume $\varphi \circ \psi$ to be contractive by \cite[Remark 2.2(iv)]{34}.

**Definition A.3.** Let $\mathcal{F}$ be a finite dimensional $C^*$-algebra. Let $\mathcal{A}$ be a $C^*$-algebra, and let $\delta > 0$. A c.p. contraction $\varphi : \mathcal{F} \to \mathcal{A}$ is a $\delta$-order zero map if for any positive contractions $x, y \in \mathcal{F}$ such that $xy = 0$ we have that $\|\varphi(x)\varphi(y)\| \leq \delta$.

We recall that order zero maps from finite dimensional $C^*$-algebras have the following stability property (\cite[Proposition 2.5]{14}). Let $\mathcal{F}$ be a finite dimensional $C^*$-algebra, then for any $\varepsilon > 0$ there is a $\delta > 0$ (depending on $\mathcal{F}$ and $\varepsilon$) such that if $\mathcal{A}$ is any $C^*$-algebra and $\varphi : \mathcal{F} \to \mathcal{A}$ is a $\delta$-order zero map then there is an order zero map $\varphi' : \mathcal{F} \to \mathcal{A}$ such that $\|\varphi - \varphi'\| < \varepsilon$. Using this, we easily obtain the following technical Lemma.

**Lemma A.4.** Let $\mathcal{A}$ be a $C^*$-algebra. $\mathcal{A}$ has nuclear dimension at most $n$ if and only if for any finite set $F \subseteq \mathcal{A}$ and any $\varepsilon > 0$ there exists a finite dimensional $C^*$-algebra $\mathcal{F} = \mathcal{F}(0) \oplus \ldots \oplus \mathcal{F}(n)$ such that for any $\delta > 0$ there exist c.p. maps $\psi : \mathcal{A} \to \mathcal{F}$, $\varphi : \mathcal{F} \to \mathcal{A}$ such that $\psi$ is contractive, $\|\varphi \circ \psi(a) - a\| < \varepsilon$ for all $a \in \mathcal{F}$ and $\varphi|_{\mathcal{F}(j)}$ is a c.p.c. $\delta$-order zero map for any $j = 0, 1, \ldots, n$.

$\mathcal{A}$ has decomposition rank at most $n$ if the same holds where furthermore $\varphi$ is assumed to be a contraction.

Recall from \cite{11} and \cite{28} that the Jiang-Su algebra $\mathcal{Z}$ can be written as an inductive limit of prime dimension drop intervals of the form

$$\mathcal{Z}_{p,p+1} = \{f \in C([0,1] \times M_p \otimes M_{p+1}) \mid f(0) \in M_p \otimes 1, f(1) \in 1 \otimes M_{p+1}\}$$

and that a separable unital $C^*$-algebra $\mathcal{A}$ is $\mathcal{Z}$-stable if and only if it admits almost central unital embeddings of $\mathcal{Z}_{p,p+1}$ for some $p \geq 2$. With the aid of \cite{24} one can give the following useful alternative formulation, cf. \cite{32}.

**Proposition A.5.** Let $\mathcal{A}$ be a separable unital $C^*$-algebra. $\mathcal{A}$ is $\mathcal{Z}$-stable if and only if for some $p \geq 2$ there are c.p.c. order zero maps

$$\Phi : M_p \to A_\infty \cap A' \quad \text{and} \quad \Psi : M_2 \to A_\infty \cap A'$$

satisfying

$$\Psi(e_{11}) = 1_{A_\infty} - \Phi(1_{M_p})$$

and

$$\Phi(e_{11})\Psi(e_{22}) = \Psi(e_{22})\Phi(e_{11}) = \Psi(e_{22}).$$
References

1. Dawn E. Archey, Crossed product $C^*$-algebras by finite group actions with a generalized tracial Rokhlin property, Ph.D. thesis, University of Oregon, 2008.
2. Arthur Bartels, Wolfgang Lück, and Holger Reich, Equivariant covers for hyperbolic groups, Geom. Topol. 12 (2008), no. 3, 1799–1882.
3. , The K-theoretic Farrell-Jones conjecture for hyperbolic groups, Invent. Math. 172 (2008), no. 1, 29–70.
4. Richard H. Herman and Vaughan F. R. Jones, Period two automorphisms of UHF $C^*$-algebras, J. Funct. Anal. 45 (1982), no. 2, 169–176.
5. Ilan Hirshberg, Eberhard Kirchberg, and Stuart A. White, Decomposable approximations of nuclear $C^*$-algebras, Adv. Math. 230 (2012), 1029–1039.
6. Ilan Hirshberg and Joav Orovitz, Tracially Z-absorbing $C^*$-algebras, arXiv preprint math.OA/12082444, 2012.
7. Ilan Hirshberg and Wilhelm Winter, Rokhlin actions and self-absorbing $C^*$-algebras, Pacific J. Math. 233 (2007), no. 1, 125–143.
8. Masaki Izumi, The Rohlin property for automorphisms of $C^*$-algebras, Mathematical physics in mathematics and physics (Siena, 2000), Fields Inst. Commun., vol. 30, Amer. Math. Soc., Providence, RI, 2001, pp. 191–206.
9. Masaki Izumi, Finite group actions on $C^*$-algebras with the Rohlin property. I, Duke Math. J. 122 (2004), no. 2, 233–280.
10. Masaki Izumi, Finite group actions on $C^*$-algebras with the Rohlin property. II, Adv. Math. 184 (2004), no. 1, 119–160.
11. Xinhui Jiang and Hongbing Su, On a simple unital projectionless $C^*$-algebra, Amer. J. Math. 121 (1999), no. 2, 359–413.
12. Eberhard Kirchberg, Central sequences in $C^*$-algebras and strongly purely infinite $C^*$-algebras, Abel Symposia 1 (2006), 175–231.
13. Eberhard Kirchberg and Mikael Rørdam, Non-simple purely infinite $C^*$-algebras, Amer. J. Math. 122 (2000), 637–666.
14. Eberhard Kirchberg and Wilhelm Winter, Covering dimension and quasidiagonality, Internat. J. Math. 15 (2004), no. 1, 63–85.
15. Akitaka Kishimoto, The Rohlin property for automorphisms of UHF algebras, J. Reine Angew. Math. 465 (1995), 183–196.
16. Huaxin Lin and N. Christopher Phillips, Crossed products by minimal homeomorphisms, J. Reine Angew. Math. 641 (2010), 95–122.
17. Elon Lindenstrauss and Benjamin Weiss, Mean topological dimension, Israel J. Math. 115 (2000), 1–24.
18. Hiroki Matui and Yasuhiko Sato, Strict comparison and $Z$-absorption of nuclear $C^*$-algebras, arXiv preprint math.OA/1111.1637; to appear in Acta Math., 2011.
19. Hiroki Matui and Yasuhiko Sato, The Rohlin property for automorphisms of the Jiang-Su algebra, J. Funct. Anal. 259 (2010), no. 2, 453–476.
20. Andrew S. Toms and Wilhelm Winter, Z-stable ASH algebras, Canad. J. Math. 60 (2008), no. 3, 703–720.
28. ______, Z-stable ASH algebras, Canad. J. Math. 60 (2008), no. 3, 703–720.
29. ______, Minimal dynamics and the classification of C*-algebras, Proc. Natl. Acad. Sci. USA 106 (2009), no. 40, 16942–16943.
30. Wilhelm Winter, Covering dimension for nuclear C*-algebras, J. Funct. Anal. 199 (2003), no. 2, 535–556.
31. ______, Decomposition rank of subhomogeneous C*-algebras, Proc. London Math. Soc. 89 (2004), 427–456.
32. ______, Decomposition rank and Z-stability, Invent. Math. 179 (2010), no. 2, 229–301.
33. ______, Nuclear dimension and Z-stability of pure C*-algebras, Invent. Math. 187 (2012), no. 2, 259–342.
34. Wilhelm Winter and Joachim Zacharias, The nuclear dimension of C*-algebras, Adv. Math. 224 (2010), no. 2, 461–498.

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE’ER SHEVA 84105, ISRAEL
E-mail address: ilan@math.bgu.ac.il

MATHEMATISCHES INSTITUT, WESTFALISCHE WILHELMUS-UNIVERSITÄT, 48149 MÜNSTER, GERMANY
E-mail address: wwinter@uni-muenster.de

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GLASGOW, UNIVERSITY GARDENS, GLASGOW Q12 8QW, SCOTLAND
E-mail address: Joachim.Zacharias@glasgow.ac.uk