Phase Transition in Spherically Symmetric
Gravitational Collapse of a Massless Scalar Field

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Abstract

Phase transition in spherically symmetric collapse of a massless scalar field is studied in 4-d Einstein gravity. A class of exact solutions that show the evolution of a constant incoming energy flux turned on at a point in the past null infinity are constructed to serve as an explicit example. The recently discovered phase transition in this system is manifest; above a threshold value of the incoming energy flux, a black hole is dynamically formed and below that, the incoming flux is reflected into the future null infinity. The critical exponent is evaluated and discussed using the solutions.
Recent numerical investigations on gravitational collapse of a single massless scalar field in spherically symmetric Einstein gravity revealed a remarkable critical behavior near the onset of a black hole formation \[1 \] \[2 \]. A generic one-parameter class of solutions $S[p]$ decomposes into two phases depending on the magnitude of $p$ that represents the strength of self-gravitational interaction of incoming scalar field. If $p < p^*$, where $p^*$ is a threshold value, the gravitational collapse is followed by an explosion, reflecting back the incoming stress-energy flux into the future null infinity. For $p > p^*$, a black hole is formed and its mass shows a universal scaling behavior $M_{BH} \simeq |p - p^*|^\delta$ near the threshold $p = p^*$. These approaches are based on a formulation of Christodoulou where this scattering process was studied as a Cauchy problem in general relativity \[3 \]. In the context of 2-d dilaton gravity (CGHS model), a simplified theoretical model believed to capture many of the essential features of 4-d Einstein gravity, the explicit analytical understanding of the similar phenomenon is possible as the general (semi-)classical collapsing solutions are available and well understood \[4 \]. To be specific, Strominger and Thorlacius found a similar universal critical behavior when Hawking radiation effect was taken into account \[5 \]. The explicit analytical understanding of this critical behavior in 4-d Einstein gravity, however, has not yet been complete, partly due to the difficulty in solving 4-d Einstein gravity coupled with a single massless scalar field in a dynamical scattering situation. For example, the rigorous proof of the universality and the analytic calculation of the critical exponent remain to be studied.

In this Letter, we make an attempt to analytically understand this critical behavior. As a first step, we show that the constant incoming stress-energy flux of scalar particles that was turned on at some point in the past null infinity can not be reflected forward into future null infinity if the magnitude of the incoming flux exceeds a certain threshold that plays a role of $p^*$. Rather, a black hole is dynamically formed. Below this threshold, the incoming flux implodes through the origin and gets scattered forward into the future null infinity. This analysis supports the result of numerical studies, showing that this one-parameter class of solutions decomposes into the subcritical regime and the supercritical regime. The critical exponent that gives a scaling relation between the order parameter, a black hole mass in this case, and $|p - p^*|$ can be calculated for these solutions in supercritical regime. Our approach takes the classical back reaction of scalar matter fields on the space-time geometry into account exactly, since this critical phenomenon is the consequence of general relativity. Therefore, our result can not be observed in the leading order calculations of the weak field approximation. If we apply the weak field approxima-
tion taking the Minkowskian metric as a fixed background, any unbounded amount of incoming flux will be imploded through the origin and all of it bounces forward into the future.

First, we derive a class of exact dynamical solutions that can be utilized to construct an interesting physical situation. Assuming spherical symmetry, Einstein-Scalar action can be written as,

\[ I = \int d^2x \sqrt{-g} e^{-2\phi} \left( R^{(2)} + 2g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 2e^{2\phi} - \frac{1}{2} g^{\alpha\beta} \partial_\alpha f \partial_\beta f \right) \] (1)

where we integrated out the angular coordinates using Gauss-Bonnet theorem. Here our convention is \((+ - - -)\) metric signature and the 4-d metric is given by \(ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta - e^{-2\phi}(d\theta^2 + \sin^2 \theta d\phi^2)\). \(R^{(2)}\) and \(f\) represent the scalar curvature of the two dimensional longitudinal metric \(g_{\alpha\beta}\) and a massless scalar field, respectively. The gravitational constant \(G\) is \(16\pi G = 1\) in our unit. If we choose a conformal gauge for the longitudinal metric \(g_{\alpha\beta}\) as \(g_{\alpha\beta} = -e^{2\rho} \Omega dx^+ dx^-\), the action, Eq.(1), reduces to

\[ I = \int dx^+ dx^- \left( 4\Omega \dot{\rho} + 4\Omega^{1/2} \right) \] (2)

along with two gauge constraints

\[ \partial_\pm^2 \Omega - 2\partial_\pm \Omega \partial_\pm \rho + \frac{1}{2} \Omega (\partial_\pm f)^2 = 0 \]

where \(\Omega = e^{-2\phi}\). The general static solutions of the field equations obtained from the action (2) under the gauge constraints can be readily found as follows; we can consistently reduce those partial differential equations into the coupled second order ordinary differential equations (ODE’s) by assuming all functions depends on a single space-like coordinate \(x = x^+ x^-\). The resulting ODE’s can also be derived from an effective action

\[ I = \int dx(x \dot{\Omega} \dot{\rho} + \frac{1}{4} e^{2\rho} \Omega^{1/2} - \frac{1}{4} \Omega x f^2) \] (3)

and the original gauge constraints become

\[ \ddot{\Omega} - 2\dot{\rho} \dot{\Omega} + \frac{1}{2} \Omega f^2 = 0, \]

where the dot represents taking a derivative with respect to \(x\). The complete solutions to this effective action represent the general solutions of the static Einstein-Scalar fields under a particular choice of the conformal coordinates. To obtain these, we observe that the action (3) has three rigid continuous symmetries that allow us to
construct three corresponding Noether charges, reducing the order of ODE’s by one. The first symmetry is \( f \rightarrow f + \alpha \), which is clear as \( f \) field appears only through its derivative. The second symmetry is \( x \rightarrow x e^\alpha \) and \( \rho \rightarrow \rho - \alpha/2 \), which corresponds to the translation of the asymptotically flat spatial coordinate. The last symmetry transformation, \( x \rightarrow x^{1+\alpha} \), \( \rho \rightarrow \rho - \alpha/2 \), \( \Omega \rightarrow \Omega(1+\alpha) \), changes the action by a total derivative. This symmetry corresponds to the rescaling of the asymptotically flat spatial coordinate. The Noether charges for these symmetries are constructed to be

\[
 f_0 = x\Omega \dot{f},
\]

\[
 c_0 = x^2 \rho \dot{\Omega} + \frac{1}{2} x \dot{\Omega} - \frac{1}{4} \Omega x^2 \dot{f}^2 - \frac{1}{4} x e^{2\rho} \Omega^{1/2},
\]

and

\[
 x \dot{\rho} \dot{\Omega} - \frac{1}{4} x \dot{\Omega} + \frac{1}{2} \Omega = c_0 \ln x + c_1.
\]

As these charges are conserved, \( f_0, c_0 \) and \( c_1 \) are constants of integration independent of \( x \). The gauge constraint, when combined with equations of motion from Eq.(3), reduces to a condition \( c_0 = 0 \). We note that Eq.(5) can be readily solved for \( \rho \) and by putting this into Eq.(3) we get a decoupled ODE. Once \( \Omega \) is solved in terms of \( x, \rho \) and \( f \) are easily determined as a function of \( x \). If we take \( f_0 = 0 \), we recover static black hole solutions in Kruskal-Szekers coordinates with \( c_1 = 4M^2 \) where \( M \) is the mass of a black hole. For arbitrary value of \( f_0, c_1 > 0 \) case solutions are identical to the solutions obtained by Janis et.al. using a technique to generate some Einstein-Scalar solutions from the vacuum solutions of the Einstein equations [6]. Since \( c_1 = 0 \) case contains no black hole as long as the static analysis is concerned and since the scalar field diverges logarithmically near the horizon in \( c_1 > 0 \) cases, our calculation here is effectively a proof of no-scalar-hair theorem. The details of this consideration can be found in [6] where complete static solutions of the more general action than Eq.(1) are obtained.

The \( c_1 = 0 \) cases are of particular importance as we can generalize it to dynamical situations we are interested in. The static solutions in this case are calculated to be

\[
 \Omega = \frac{e^{-2\rho_0}}{4}(e^{4\rho_0}(\ln(x/x_0))^2 - 4f_0^2),
\]

\[
 \rho = \frac{1}{4} \ln \Omega - \frac{1}{2} \ln x + \rho_0,
\]

\[
 f = \ln(\sqrt{e^{2\rho_0}\Omega + f_0^2} - f_0) + f_1 = \ln(\frac{e^{2\rho_0} \ln(x/x_0) - 2f_0}{e^{2\rho_0} \ln(x/x_0) + 2f_0}) + f_1,
\]

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where $\rho_0$, $f_1$ and $x_0$ are arbitrary constants. For simplicity, we take $x_0 = 1, \rho_0 = 0$ and $f_1 = 0$ for further discussions. Then, as $\Omega \to \infty$, the behavior of $f$ asymptotically approaches to $-2f_0/r$ where the geometric radius $r$ is defined to be $\sqrt{\Omega}$. In this limit, we find $r \to (\ln x^+ + \ln x^-)/2$ and the 4-d metric becomes 
\[ ds^2 \to -dx^+dx^-/(x^+x^-) - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \]
After a conformal transformation $\ln x^\pm \to x^\pm$, we find that the asymptotic space-time is a flat Minkowskian. Additionally, taking $G \to 0$ limit reproduces the same results. This limit is the same as the weak field approximation taking a Minkowskian metric as a fixed background metric. The wave equation for $f$ in s-wave sector under this fixed background geometry has general solutions of the form $f = -2f_0/r + constant$. Thus, Eqs.(7) are the relativistic generalization of the point scalar charge solution. In a sense, $c_1 = 0$ limit is similar to taking an extremal limit of Reissner-Nordström black hole. This consideration suggests that the scalar charge $f_0$, in some dynamic situations, can be a chiral field instead of being a strict constant, for $f_0$ is indeed an arbitrary chiral field in the weak field approximation. In general relativistic case, however, this simple generalization is not possible in general, since there can be a non-trivial corrections to Eqs.(8) of the order of $\partial_{\pm} f_0$. In case of asymptotically steady incoming and outgoing stress-energy flux, though, that simple generalization is possible.

Forgetting about global boundary conditions, the result of this extension is

\[ \Omega = \frac{1}{4}(\ln x)^2 - (k_+ \ln x^+ - k_- \ln x^- + q_0)^2; \]
\[ \rho = \frac{1}{4} \ln \Omega - \frac{1}{2} \ln x + \frac{1}{2} \ln(1 + 4k_+k_-), \]
\[ f = \ln\left(\frac{\ln x - 2(k_+ \ln x^+ - k_- \ln x^- + q_0)}{\ln x + 2(k_+ \ln x^+ - k_- \ln x^- + q_0)}\right), \]

where $k_\pm$ and $q_0$ are constants. We can straightforwardly verify that these solutions satisfy field equations derived from Eq.(6) and the corresponding gauge constraints.

The additional constant term in the expression for $\rho$ is the correction term originating from $\partial_{\pm} f$. This term turns out to be rather simple in this case where the charge $f_0$ has terms only up to linear terms in $\ln x^{\pm}$. The asymptotic stress-energy tensor averaged over the transversal sphere in a conformal coordinate system that becomes asymptotically flat near the past or future infinity is calculated to be $T_{\pm\pm} = 4k_{\pm}^2$. Thus, $4k_{\pm}^2$ are interpreted to be an incoming and an outgoing energy flux, respectively. The similar asymptotic analysis shows that $q_0$ represents the background component of the scalar charge. Recently, $q_0 = 0$ and $k_+k_- = 0$ case of Eqs.(8) was...
reported in mathematics literature [8] and named scale invariant solutions. For scattering situations in our consideration where incoming and outgoing flux can coexist, a slightly generalized version [8] proves to be useful. Additionally, the presence of \( q_0 \) term enables us to consider the time evolution of the multiple square-type incoming energy pulses by successively gluing our solutions, unless a black hole is formed in an intermediate stage. To name a few other applications possible with our result, we can construct various scattering solutions, cosmological solutions and point particle solutions with time-varying charge at the origin, depending on the boundary conditions and initial conditions.

Using the results obtained so far, we construct the one-parameter class of exact solutions that reduce to \( f = (2k \ln x^+ H(\ln x^+ ) - 2k \ln x^- H(- \ln x^- ))/r \) as we take the leading order weak field approximation, i.e., taking \( G \to 0 \) limit. Here, \( H(x) \) denotes the usual step function. This approximate solution in a fixed Minkowskian background has a property that any unbounded amount of the incoming stress-energy flux can be totally reflected off from the origin into the future null infinity. However, this picture qualitatively changes as we consider the exact solutions focusing on the space-time geometry change due to the stress-energy of the scalar field. The physical region of space-time in our consideration is specified by the requirement \( \Omega \geq 0 \), since the angular coordinates should not have time-like signature. Thus, the natural boundary is \( \Omega = 0 \). The asymptotic incoming wave section of the weak field solutions is chosen as initial data and it represents the turn-on of the constant incoming energy flux at a point in past null infinity. Under these conditions, the class of solutions parameterized by a constant \( k \geq 0 \) are found to be

\[
\begin{align*}
\Omega &= \frac{1}{4}(u + v)^2 & \text{I} \\
\Omega &= \frac{1}{4}(u + v)^2 - k^2 v^2 & \text{II} \\
\Omega &= \frac{1}{4}(u + v)^2 - (kv - k_- (k) u)^2 & \text{III}
\end{align*}
\]

where we introduced \( v = \ln x^+ \) and \( u = \ln x^- \). Other functions can be read off from Eqs.(8). The specific form of \( k_- (k) \) depends on the boundary condition at \( \Omega = 0 \) in region III. In our case, imposing a covariant version of the reflecting boundary condition yields \( k_- (k) = (\sqrt{1 + 8k - 16k^2} - 1)/4 \). The region I, bounded by the past null infinity, \( v = 0 \) and the origin \( u = -v \), represents the Minkowski space before the turn-on of the constant incoming flux. The region II, bounded by the future null infinity, \( u = 0 \) and the past null infinity, represents the propagation of the incoming particles before any of them hits the boundary at the origin. Note
that our coordinate choice sets the turn-on time of the incoming energy flux as $v = 0$. Finally, the region III, bounded by $u = 0$, the future null infinity and the origin $u = -v(1-2k)/(1+2k\_-(k))$, contains the scattering of the incoming particles off the origin and their further propagation toward the future null infinity. As the matter particles hit the origin, they linearly tilt it, adding some space-like component to its tangent line. If $k < 1/2$, the path of the origin remains time-like and the incoming particles are scattered forward into the future null infinity, just as in the subcritical regime of the numerical simulations. Especially, if $k \ll 1/2$, the weak field approximation results are recovered and the geometry is almost Minkowskian.

At $k = 1/2$, which can be interpreted as a phase transition point, i.e., the black hole formation threshold, the path of the origin, $u = 0$, becomes light-like for $v > 0$. If $k > 1/2$, the path of the origin becomes space-like and form a trapped region in space-time. In this case, the region III disappears and our solutions here become exactly the same as the one obtained in [8]. As explained in detail in [8], the resulting space-time is a black hole with indefinitely increasing mass, for we do not turn off the constant incoming flux. Thus, this corresponds to the supercritical phase of this scattering system. As the numerical studies and the above considerations suggest, the order parameter of this system is the black hole mass. Then, the important physical quantity to compute, given our exact one-parameter class of solutions in supercritical regime, is the critical exponent. The geometric radius $r = r_H(v)$ of the apparent horizon of the dynamic black hole in supercritical case, determined by $\partial_v r = 0$, is calculated to be

$$r_H(v) = 2(k - \frac{1}{2})^{1/2}(k + \frac{1}{2})^{1/2}kv.$$  (10)

The $1/(2G) = 8\pi$ times the value of this corresponds to the apparent mass $M_A$ of the dynamic black hole. The linear dependence on $v$ is understandable as it is the time duration between the turn-on of the incoming flux and the reference time $v$.

We also note that the angular averaged incoming energy flux is $4k^2$. Defining the transition point $p^* = 1/2$ and $p = k$, we find that the critical exponent in this case is $\delta = 1/2$ in a scaling relation $M_A \simeq (p - p^*)^\delta$.

In the aforementioned numerical study [1], the incoming flux is a pulse type and because of this, the asymptotic out-region is a static black hole with asymptotically flat geometry. However, in our case, since we put infinite mass into the black hole, the whole universe collapses to a future singularity, barring the existence of any realistic out-region. As a result, we should consider turning off the incoming flux at a finite time $v = v_1 > 0$. Gluing a space-time with purely outgoing particles
above $v = v_1$ is straightforward in subcritical case using Eqs.(8). In supercritical case, however, the gluing is highly non-trivial, for we can not directly glue the static Schwarzschild solution at $v = v_1$. For example, in 4-d Einstein gravity, gluing the Minkowski space to the Schwarzschild geometry requires an impulsive shock-wave type injection of matter particles [9]. Thus, after we turn off the source, the geometry goes through a brief transient period to asymptotically settle down into a quasi-Schwarzschild geometry. During this process, the apparent horizon that was initially space-like at the turn-off time will settle down to a future null direction, slightly changing its geometric radius. The numerically obtained $M_{BH} \simeq (p - p^*)^\delta$ with $\delta \simeq 0.37$ is very difficult to calculate, as the scaling relation between $M_{BH}$ and $M_A$ gets complicated through this process. The similar situation in the CGHS model is not as difficult as ours. The complicated transient behavior is absent in this case, due to the simplified dynamics of the model. Therefore, it would be possible to directly glue a static dilatonic black hole at $v = v_1$ and thereby getting a relation $M_A \simeq M_{BH}$ if we had considered CGHS model from the outset. An interesting observation in this regard is that our $\delta = 0.5$ is the same as the critical exponent obtained by Strominger and Thorlacius in CGHS model [5]. There is a reason for this connection as suggested in [10]. The pure gravity sector of CGHS model, other than being a target space effective action from string theory, can be considered as a leading order theory in the $1/d$-expansion of the spherically symmetric $d$-dimensional Einstein gravity. We can, therefore, adopt $1/d$-expansion and consider the leading order behavior in the description of the complex transient process in 4-d Einstein gravity. As a zeroth order approximation, we glue a CGHS black hole directly to our solutions to deduce the approximate scaling relation $M_A \simeq M_{BH}$. Thus, the leading order approximation of the exact critical exponent for finite pulse-type incoming energy flux is now calculated to be 0.5. Since the next order correction to the critical exponent is expected to be an order of $1/d = 0.25$ and the numerically calculated value is about 0.37, our leading order value, 0.5, seems plausible. Generalizing this consideration, it is conceivable that the exact critical exponent for finite pulse-type incoming energy flux in $d$-dimensional spherically symmetric Einstein gravity will interpolate $\delta(4) \simeq 0.37$ in 4-d Einstein gravity and $\delta(\infty) = 0.5$ in CGHS model as a continuous function $\delta(d)$ of the space-time dimensionality $d$. In many other cases of phase transitions, the critical exponent rapidly saturates into a limiting value at $d = \infty$. It will be an interesting exercise to verify this conjecture and, additionally, develop a systematic perturbation theory with a dimensionless expansion parameter $1/d$ to tackle other difficult problems in 4-dimensional gravity.
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