On behaviour of critical lines near ferrimagnetic phase in Higgs-Yukawa systems

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Abstract

We calculate within a mean-field approximation the slopes of the critical lines near the point of appearing the ferrimagnetic phase for the U(1) systems in the weak coupling regime. It is demonstrated that the slope of one of the critical line is continuous, while change of the slope of the other depends strongly on the number of the fermion flavours. We also find that in the ferrimagnetic phase near such a point the magnetization and the staggered magnetization align orthogonally to each other.

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1 Introduction

In our previous paper devoted to the mean-field analysis of the phase structure of the Higgs-Yukawa systems [1] we have concluded that intersection of the critical line separating the ferromagnetic (FM) and paramagnetic (PM) phases with the line separating the antiferromagnetic (AM) and PM phases always leads to appearing the ferrimagnetic (FI) phase, and that the slopes of the critical lines in the point of their intersection (denote it $B$) are continuous. These statements have been criticized recently in [2] where it was argued that the slopes are in general discontinuous and that the FI phase may not appear at all in the point $B$.

In this paper we demonstrate that in reference to the systems considered in [1] this criticism is justified only in part. Namely, considering within the mean-field approximation the weak coupling regime for the U(1) systems we show that in all the cases when the intersection of the FM-PM and AM-PM critical lines occurs, the FI phase does appear, and that the slope of the AM-PM–FI-FM line is always continuous. The reason for the latter fact is that the critical exponents of the magnetization (v.e.v. of the Higgs field) in this approximation is greater than $1/2$. The slope of the FM-PM–FI-AM line generally changes in the point $B$. However the magnitude of this change depends strongly on the number of the fermion flavours. We show that for given system (form of the lattice fermion action) there exists such number $n_0$ (generally noninteger), that for the number of the fermion flavours $n_f = n_0$ the slope of this line is continuous, too. The qualitative picture for the phase diagrams is shown in Fig. 1. By a product, we find that in the FI phase in the vicinity of the point $B$ the magnetization and the staggered magnetization align orthogonally to each other in the group space.

We want to emphasize that all the results of [1] concerning the phase structure of the systems in the regions bordering on the PM phase are not affected. The slopes of the critical lines in those regions are determined by the terms in the free energy quadratic in the mean fields. Such terms has been calculated in the ladder approximation developed in [3, 1]. For examination of the region bordering on the FI phase one needs to know the free energy up to the fourth order in the mean fields [2]. In this case the summation of the ladder diagrams becomes too complicated and the approximation loses its advantages. At the same time, unjustified neglecting the ladder diagrams may lead to obviously incorrect results: for example, in the case of the SU(2) system with the naive fermions the FI phase gets lost. Therefore we consider here the U(1) systems for which the ladder diagrams do not contribute to the free energy at least up to the terms quadratic in the mean fields. We limit ourselves to the consideration of the weak coupling regime; the examination of the strong coupling regime can be done easily in analogous way.

\[\text{Notation A has been employed in [1] for another point.}\]
2 Free energy

Our basic point is that the free energy for the U(1) systems in four dimensions up to terms quartic in the mean fields has the form (all the notations as in [1])

\[
F = \lim_{N \to \infty} N^{-4} F_{MF} = -\frac{1}{2} a h^2 - \frac{1}{2} b h_{st}^2 + \frac{1}{4} c' h_{st}^2 + \frac{1}{4} c (h \cdot h_{st})^2 \\
+ \frac{1}{4} c'' (h \times h_{st})^2 + \frac{1}{24} d h^4 + \frac{1}{24} e h_{st}^4 + \frac{1}{24} f h^4 \ln h^2, \tag{1}
\]

where coefficients \(a\) to \(f\) are functions of \(y\) and \(\kappa\), and \(h \cdot h_{st} = \sum_a h_a h_{st}^a, h \times h_{st} = \sum_{a,b} e^{ab} h_a h_{st}^a\).

The coefficients \(a\) to \(c\) and \(e\) are calculated directly from the eqs. (8)–(15) of [1] and read as\(^3\)

\[
\begin{align*}
a(\kappa, y) &= -\frac{1}{2} + 4\kappa + y^2 n_f \int_p K^2(p), \\
b(\kappa, y) &= -\frac{1}{2} - 4\kappa + y^2 n_f \int_p K(p) \cdot K(p + \pi), \\
c'(\kappa, y) &= -\frac{3}{8} + \frac{1}{2} y^2 n_f \int_p \left[ K^2(p) + K(p) \cdot K(p + \pi) \right] + y^4 n_f \int_p K^2(p) K(p) \cdot K(p + \pi), \\
c''(\kappa, y) &= -\frac{3}{4} + y^2 n_f \int_p \left[ K^2(p) + K(p) \cdot K(p + \pi) \right] + \frac{1}{2} y^4 n_f \int_p K^2(p) K^2(p + \pi), \\
c''(\kappa, y) &= -\frac{1}{2} y^4 n_f \int_p K^2(p) K^2(p + \pi), \\
e(\kappa, y) &= -\frac{9}{8} - 12\kappa + 3y^2 n_f \int_p K(p) \cdot K(p + \pi) \\
&\quad + \frac{3}{2} y^4 n_f \int_p \left[ 2 (K(p) \cdot K(p + \pi))^2 - K^2(p) K^2(p + \pi) \right],
\end{align*}
\]

where \(K_\mu(p) = L_\mu(p)/L^2(p)\), so that \(\int_p K^2(p) \equiv G(0), \int_p K(p) \cdot K(p + \pi) \equiv G(\pi)\) \([1]\) (we omit the superscripts \(W\) meaning the weak coupling regime).

The coefficients \(f\) and \(d\) in (1) cannot be calculated by the term-wise integration of the standard weak coupling expansion of the logarithm of the fermion determinant: the integrals of the \(O(h^n)\) terms beginning from \(n \geq 4\) have infrared divergences, since \(K^2(p) \to 1/p^2\) at \(p^2 \to 0\). This is the reason for appearing the logarithmic term in (1). It is this term that differs our results from those of ref. [3].

To estimate the coefficient \(f\), note that the contribution of the fermion determinant to the free energy at \(h_{st} = 0\) is given by the expression

\[
I = -2n_f \int_p \ln \left[ 1 + y^2 \langle \phi \rangle^2 K^2(p) \right].
\]

The integral is finite and certainly depends on the form of the function \(K(p)\). The point, however, is that the term \(\propto h^4 \ln h^2\) comes from the infrared region \(p^2 \sim 0\) which is not

\(^3\)We do not take into account the correlations of the mean fields in the quartic terms.
sensitive to the detailed form of the function $K(p)$. Therefore, to find the coefficient $f$ we can consider the integral with the same infrared properties

$$I' = \frac{1}{(2\pi)^2} n_f \int_0^\Lambda dp \, (1 + y^2 \langle \phi \rangle^2 \frac{1}{p^2})$$

$$= \frac{1}{(4\pi)^2} y^4 n_f \langle \phi \rangle^4 \ln \langle \phi \rangle^2 + (\text{terms polynomial in } \langle \phi \rangle),$$

(4)

where $\Lambda = O(2\pi)$ is an ultraviolet cutoff. From (4), taking into account the relation $\langle \phi \rangle^2 = h^2/4 + O(h^4)$, we find

$$f(\kappa, y) = -\frac{3}{32\pi^2} y^4 n_f.$$  

(5)

Since we are interested in the domain of the phase diagram where both $h$ and $h_{st}$ tend to zero, and therefore $|h^4 \ln h^2| \gg h^4$, one can neglect the term $\propto d h^4$ in (1).

The FM-PM and AM-PM critical lines are determined by the equations $a(\kappa, y) = 0$ and $b(\kappa, y) = 0$, respectively, so that the coordinates of the point $B$ are

$$\kappa_B = -\frac{G(0) - G(\pi)}{8[G(0) + G(\pi)]}, \quad y_B^2 = \frac{1}{n_f[G(0) + G(\pi)]}.$$  

(6)

In the vicinity of the point $B$ we have (in the notations of [2])

$$a(\kappa, y) = a_\kappa(\kappa - \kappa_B) + a_y(y - y_B) + O(\Delta^2),$$

$$b(\kappa, y) = -b_\kappa(\kappa - \kappa_B) - b_y(y - y_B) + O(\Delta^2),$$

$$c^{s,v}(\kappa, y) = c^{s,v}_B + O(\Delta), \quad e(\kappa, y) = e_B + O(\Delta), \quad f(\kappa, y) = f_B + O(\Delta),$$

(7)

with

$$a_\kappa = 4, \quad a_y = 2n_f G(0) y_B, \quad b_\kappa = 4, \quad b_y = -2n_f G(\pi) y_B,$$

(8)

and $\Delta = y - y_B$ or $\kappa - \kappa_B$.

From (2) and (5) it follows that $c_B^s > 0$ and $c_B^v < 0$. Hence, in the FI phase (if it exists) the mean fields $h$ and $h_{st}$ align in such a way that $h \cdot h_{st} = 0$ and $h \times h_{st} \neq 0$, i.e. orthogonally to each other in the group space. Therefore, in the vicinity of the point $B$ the expression (1) for the free energy is reduced to

$$F = -\frac{1}{2} [a(\kappa - \kappa_B) + a_y(y - y_B)] h^2 + \frac{1}{2} [b(\kappa - \kappa_B) + b_y(y - y_B)] h_{st}^2$$

$$+ \frac{1}{4} c_B h^2 h_{st}^2 + \frac{1}{24} e_B h_{st}^4 + \frac{1}{24} f_B h^4 \ln h^2 + o(\Delta^2),$$

(9)

where $c_B = c_B^s + c_B^v$. From (9) necessary conditions for the existence of the FI phase follow:

$$c_B^2 < \frac{1}{9} e_B f_B \ln h^2, \quad e_B > 0, \quad f_B < 0$$

(10)

(provided $a > 0$, $b > 0$). Hence, quite near the point B, where $h \to 0$, the FI phase can exist in fact at any value of the coefficient $c_B$. From (5) it follows that the third condition in (10) is satisfied always, and we shall demonstrate in the next section that the second condition for the systems considered in [3] is satisfied, too.
3 Slopes of the critical lines and the FI phase

Now, following the consideration of ref. [2] setting up the relation between the slopes of the critical lines and the critical exponents of the magnetizations, one can easily find the slopes at the point $B$.

Consider first the $h = 0$ critical line. Near the point $B$ it is determined by the equation

$$- a(\kappa, y) + \frac{1}{2} c_B h_{st}^2 = 0. \quad (11)$$

From (9) it follows that nonzero value of the staggered magnetization is determined by the equation

$$h_{st}^2 = -\frac{6}{e_B} [b_\kappa (\kappa - \kappa_B) + b_y (y - y_B)]. \quad (12)$$

Then, from (11) and (12), in full correspondence with the results of ref. [2], we have

$$\left. \frac{d\kappa}{dy} \right|_{FIM - PM} = - \frac{\partial a(\kappa, y)/\partial y}{\partial a(\kappa, y)/\partial \kappa} \bigg|_B = \frac{a_y}{a_\kappa}, \quad (13)$$

$$\left. \frac{d\kappa}{dy} \right|_{FI - AM} = - \frac{\partial a(\kappa, y)/\partial y - (1/2) c_B \partial h_{st}^2 / \partial y}{\partial a(\kappa, y)/\partial \kappa - (1/2) c_B \partial h_{st}^2 / \partial \kappa} \bigg|_B = \frac{a_y + 3b_y c_B / e_B}{a_\kappa + 3b_\kappa c_B / e_B}. \quad (14)$$

The things are different in the case of the $h_{st} = 0$ critical line. Indeed, though this line is determined by the equation similar to (11):

$$- b(\kappa, y) + \frac{1}{2} c_B h^2 = 0, \quad (15)$$

the nonzero value of the magnetization is determined by the equation ($|\ln h^2| \gg 1$)

$$h^2 \ln h^2 = \frac{6}{f_B} [a_\kappa (\kappa - \kappa_B) + a_y (y - y_B)]. \quad (16)$$

From (16) it follows that $\partial h^2 / \partial \kappa|_B = \partial h^2 / \partial y|_B = 0$, i.e. the critical exponents of the magnetization is grater than 1/2. Therefore we have in this case

$$\left. \frac{d\kappa}{dy} \right|_{AM - PM} = - \frac{\partial b(\kappa, y)/\partial y}{\partial b(\kappa, y)/\partial \kappa} \bigg|_B = \frac{b_y}{b_\kappa}, \quad (17)$$

$$\left. \frac{d\kappa}{dy} \right|_{FI - FM} = - \frac{\partial b(\kappa, y)/\partial y - (1/2) c_B \partial h^2 / \partial y}{\partial b(\kappa, y)/\partial \kappa - (1/2) c_B \partial h^2 / \partial \kappa} \bigg|_B = \frac{b_y}{b_\kappa}, \quad (18)$$

i.e. the slope of the $h_{st} = 0$ critical line does not change in the point $B$.

It is seen now that the FI phase appears always, provided the conditions (10) are fulfilled. The change of the slope of the $h = 0$ critical line in the point $B$ is determined mainly by the value of $c_B$: if $c_B = 0$ the slope does not change; if $c_B > 0$ the domain with the FI phase
gets narrow and in the limit \( c_B \to \infty \) it shrinks; on the contrary, if \( c_B < 0 \) the domain with the FI phase becomes wider and in the limit \( c_B \to -\infty \) it occupies near the point \( B \) all the region between the FI-FM and AM-PM lines (Fig. 1).

We shall demonstrate now that all the three variants: \( c_B = 0 \), \( c_B > 0 \), and \( c_B < 0 \) are realised for the systems considered in [1]. The FM-PM and AM-PM lines intersect in three of those systems: in the systems with the SLAC, Weyl, and the mirror fermion actions. In these cases we have: \( I_1 \equiv \int_p K^2(p) K^2(p + \pi) \approx 8.23 \times 10^{-3} \), \( I_2 \equiv \int_p (K(p) \cdot K(p + \pi))^2 \approx 3.73 \times 10^{-5} \), \( I_3 \equiv \int_p K^2(p) K(p) \cdot K(p + \pi) \approx -7.83 \times 10^{-3} \) for the SLAC action; \( I_1 \approx 4.59 \times 10^{-4} \), \( I_2 \approx 1.86 \times 10^{-4} \), \( I_3 \approx -6.09 \times 10^{-4} \) for the Weyl action; \( I_1 \approx 5.92 \times 10^{-5} \), \( I_2 \approx 5.92 \times 10^{-5} \), \( I_3 \approx -2.70 \times 10^{-5} \) for the mirror fermion action (the values of \( G(0) \) and \( G(\pi) \) are given in [1]).

Although \( e_B \) depends on the number of flavours \( n_f \), in all these cases \( e_B > 0 \) for any \( n_f \) and therefore the FI phase appears in all of these systems. The sign and the value of \( c_B \) however depends strongly on \( n_f \): this is due to the fact that the \( O(y^2_B) \) and \( O(y^4_B) \) terms in \( c_B \) are independent of \( n_f \), while the \( O(y^0_B) \) terms are \( \propto 1/n_f \). Moreover, for given system there exists such number \( n_0 \), that \( c_B = 0 \) if \( n_f = n_0 \), \( c_B > 0 \) if \( n_f > n_0 \), and \( c_B < 0 \) if \( n_f < n_0 \). For our systems we find:

\[
\begin{align*}
n_0 &\approx 32 \quad \text{for the SLAC action}, \\
&\approx 4.7 \quad \text{for the Weyl action}, \\
&\approx 1.3 \quad \text{for the mirror fermion action}. \\
\end{align*}
\]

The corresponding phase diagrams are shown qualitatively in Fig. 1.

4 Summary

Thus, we have demonstrated that in the U(1) Higgs-Yukawa systems with the SLAC, Weyl, or mirror fermion actions the FI phase does appear in the weak coupling regime in the point \( B \). In the FI phase near this point, the magnetization \( h \) and the staggered magnetization \( h_{st} \) align orthogonally to each other in the group space. Due to the fact that the free energy involves the logarithmic term, that, in turn, leads to the critical exponents of the magnetization \( h \) grater than 1/2, the \( h_{st} = 0 \) critical line (the AM-PM–FI-FM line) is continuous together with its first derivative in the point \( B \). The derivative of the \( h = 0 \) critical line (FM-PM–FI-AM line) is generally discontinuous in the point \( B \). However the discontinuity depends strongly on the number of the fermion flavours \( n_f \), so that in each system there exist such number \( n_0 \), that for \( n_f = n_0 \) the line is smooth, too.

To conclude, note that for the systems in which one has \( K(p + \pi) = -K(p) \) (as it is, for example, for the naive fermions), the logarithmic terms in the free energy appear also for the staggered magnetization. This means that the critical exponents for both \( h \) and \( h_{st} \) are grater than 1/2. Therefore, in the cases when the FM-PM and AM-PM lines intersect in such systems, the slopes of both critical lines are continuous in the point \( B \). This may occur, for example, in the case of the SU(2) system with the naive fermions. We admit, however, that to make definite conclusion on the existence of the FI phase in the SU(2) systems a special investigation is necessary.
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Figure captions

Fig. 1. Qualitative picture of the phase diagrams: (1) for $n_f < n_0$, (2) for $n_f = n_0$, (3) for $n_f > n_0$. 
