Asynchronous decentralized accelerated stochastic gradient descent

Guanghui Lan Yi Zhou

Abstract

In this work, we introduce an asynchronous decentralized accelerated stochastic gradient descent type of method for decentralized stochastic optimization, considering communication and synchronization are the major bottlenecks. We establish $O(1/\epsilon)$ (resp., $O(1/\sqrt{\epsilon})$) communication complexity and $O(1/\epsilon^2)$ (resp., $O(1/\epsilon)$) sampling complexity for solving general convex (resp., strongly convex) problems.

1 Introduction

In this paper, we consider the following decentralized optimization problem which is cooperatively solved by $m$ agents distributed over the network:

$$f^* := \min_x f(x) := \sum_{i=1}^m f_i(x)$$

s.t. $x \in X$, $X := \cap_{i=1}^m X_i$.

Here $f_i : X_i \to \mathbb{R}$ is a general convex objective function only known to agent $i$ and satisfying

$$\frac{L}{2} \|x - y\|^2 \leq f_i(x) - f_i(y) - \langle f'_i(y), x - y \rangle \leq \frac{L}{2} \|x - y\|^2 + M \|x - y\|, \ \forall x, y \in X_i,$$

for some $L, M, \mu \geq 0$ and $f'_i(y) \in \partial f_i(y)$, where $\partial f_i(y)$ denotes the subdifferential of $f_i$ at $y$, and $X_i \subseteq \mathbb{R}^d$ is a closed convex constraint set of agent $i$. (2) is a unified way of describing a wide range of problems. In particular, if $f_i$ is a general Lipschitz continuous function with constant $M_f$, then (2) holds with $L = 0, \mu = 0$ and $M = 2M_f$. If $f_i$ is a smooth and strongly convex function in $C^{1,1}_L$ (see [21 Section 1.2.2] for definition), (2) is satisfied with $M = 0$. Clearly, relation (2) also holds if $f_i$ is given as the summation of smooth and nonsmooth convex functions. Throughout the paper, we assume the feasible set $X$ is nonempty.

Decentralized optimization problems defined over complex multi-agent networks are ubiquitous in signal processing, machine learning, control, and other areas in science and engineering (see e.g. [23 [13 [24 [8]). One critical issue existing in decentralized optimization is that synchrony among network agents is usually inefficient or impractical due to processing and communication delays and the absence of a master server in the network. Note that $f_i$ and $X_i$ are private and only known to agent $i$, and all agents intend to cooperatively minimize the system objective $f$ as the sum of all local objective $f_i$’s in the absence of full knowledge about the global problem and network structure. Decentralized algorithms, therefore, require agents to communicate with their neighboring agents.

*Department of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA, 30332. (email: george.lan@isye.gatech.edu, yizhou@gatech.edu)
iteratively to propagate the distributed information in the network. Under the synchronous setting, all agents must wait for the slowest agent and/or slowest communication channel/edge in the network, and a global coordinator must be presented for synchronization, which can be extremely expensive in the large-scale decentralized network.

Following the seminal work [1], extensive research work has been conducted in recent years to design asynchronous algorithmic schemes for decentralized optimization. Asynchronous gossip-based method under the edge-based random activation setting has been proposed by [3] to solve averaging consensus problems. Later [16] extended this framework for solving (1) and established almost surely convergence to the optimal solution when $f_i$ is smooth and convex. Most recently, [30] also achieved almost surely convergence by iteratively activating a subset of agents. Besides (sub)gradient based methods, another well-known approach relies on solving the saddle point formulation of (1) (see Section 2 for the reformulation), where at each iteration a pair of primal and dual variables is updated alternatively. The distributed ADMM (e.g., [12, 28, 31, 2]) has been studied in different asynchronous setting. More specifically, [12, 2] randomly selected and updated a subset of agents iteratively where [12] assuming $f_i$ being simple convex function and [2] establishing almost surely convergence for smooth convex objectives. [28] employed the node-based random activation and achieved the $O(1/\epsilon)$ rate of convergence when $f_i$ is a simple convex function, and [31] later established the same rate of convergence by activating one agent per iteration. Most recently, [29] proposed an asynchronous parallel primal-dual type method and established almost surely convergence when $f_i$ is smooth and convex.

Asynchronous decentralized algorithms discussed above require the knowledge of exact (sub)gradients (or function values) of $f_i$, however, this requirement is not realistic when dealing with minimization of generalized risk and online (streaming) data distributed over a network. There exists limited research on asynchronous decentralized stochastic optimization (e.g., [20, 27, 6]), for which only noisy gradient information of functions $f_i$, $i = 1, \ldots, m$, can be easily computed. While asynchronous decentralized stochastic first-order methods [20, 27] established error bounds when $f_i$ is (strongly) convex, [6] achieved $O(1/\epsilon^2)$ rate of convergence for smooth and convex problems.

Recently [15] proposed a class of primal-dual type communication-efficient methods for decentralized stochastic optimization, which obtained the best-known $O(1/\epsilon)$ (resp., $O(1/\sqrt{\epsilon})$) communication complexity and the optimal $O(1/\epsilon^2)$ (resp., $O(1/\epsilon)$) sampling complexity for solving nonsmooth convex (resp., strongly convex) problems under the synchronous setting. This class of communication-efficient methods requires two rounds of communication involving all network agents per iteration, and hence may incur huge synchronous delays. Moreover, it was proposed to solve decentralized nonsmooth problems so that its convergence property is not clear when applying it to solve decentralized problems satisfying (2). Inspired by [15], we aim to propose an asynchronous decentralized algorithmic framework to solve (1) under a more general setting (2) but still maintains the complexity bounds achieved in [15]. Our main contributions in this paper can be summarized as follows. Firstly, we introduce a doubly randomized primal-dual method, namely, asynchronous decentralized primal-dual (ADPD) method, which randomly activates two agents per iteration, and hence two rounds of communication between the activated agent and its neighboring agents are performed. This proposed method can find a stochastic $\epsilon$-optimal solution in terms of both the primal optimality gap and feasibility residual in $O(1/\epsilon)$ communication rounds when the objective functions are simple convex such that the local proximal subproblems can be solved exactly.

Secondly, we present a new asynchronous stochastic decentralized primal-dual type method, called asynchronous accelerated stochastic decentralized communication sliding (AA-SDCS) method, for
solving decentralized stochastic optimization problems. It should be pointed out that AA-SDCS is a unified algorithm that can be applied to solve a wild range of problems under the general setting of (2). In particular, only $\mathcal{O}(1/\epsilon)$ (resp., $\mathcal{O}(1/\sqrt{\epsilon})$) communication rounds are required while agents perform a total of $\mathcal{O}(1/\epsilon^2)$ (resp., $\mathcal{O}(1/\epsilon)$) stochastic (sub)gradient evaluations for general convex (resp., strongly convex) functions. Moreover, the latter bounds, a.k.a. sampling complexities, of AA-SDCS can achieve a better dependence on the Lipschitz constant $L$ when the objective function contains a smooth component, i.e., $L > 0$ in (2), than other existing decentralized stochastic first-order methods. Only requiring the access to stochastic (sub)gradients at each iteration, AA-SDCS is particularly efficient for solving problems with $f_i := \mathbb{E}_{\xi_i}[F_i(x; \xi_i)]$, which provides a communication-efficient way to deal with streaming data and decentralized machine learning. We summarized the achieved communication and sampling complexities in this paper in Table 1.

Table 1: Complexity bounds for obtaining a stochastic $\epsilon$-solution under asynchronous setting

| Problem type: $f_i$ | Communication Complexity | Sampling Complexity |
|---------------------|--------------------------|---------------------|
|                     | Our results              | Existing results    |
| Simple convex        | $\mathcal{O}(1/\epsilon)$ | $\mathcal{O}(1/\epsilon)$ |
| Stochastic, convex   | $\mathcal{O}(1/\epsilon)$ | $\mathcal{O}(1/\epsilon^2)$ |
| Stochastic, strongly convex | $\mathcal{O}(1/\sqrt{\epsilon})$ | $\mathcal{O}(1/\epsilon)$ |
| Stochastic, smooth + nonsmooth | $\mathcal{O}(1/\epsilon)$ | $\mathcal{O}(1/\epsilon^2)$ |
| Stochastic, strong convex, smooth + nonsmooth | $\mathcal{O}(1/\sqrt{\epsilon})$ | $\mathcal{O}(M^2 + \sigma^2 + \sqrt{L}/\epsilon)$ |

Thirdly, we demonstrate the advantages of the proposed methods through preliminary numerical experiments for solving decentralized support vector machine (SVM) problems with real data sets. For all testing problems, AA-SDCS can significantly save CPU running time over existing state-of-the-art decentralized methods.

To the best of our knowledge, this is the first time that these asynchronous communication sliding algorithms, and the aforementioned separate complexity bounds on communication rounds and stochastic (sub)gradient evaluations under the asynchronous setting are presented in the literature.

This paper is organized as follows. In Section 2 we introduce the problem formulation and provide some preliminaries on distance generating functions and prox-functions. We present our main asynchronous centralized primal-dual framework and establish their convergence properties in Section 3. Section 4 is devoted to providing some preliminary numerical results to demonstrate the advantages of our proposed algorithms. The proofs of the main theorems in Section 3 are provided in Appendix A.

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1Here we refer to object functions satisfying the condition that $L, M > 0$ in (2).
2The proximal-gradient method proposed in [6] can only deal with the case that $f_i$ is a composite function such that it is the summation of smooth functions and a simple nonsmooth function (cf. a regularizer).
3Here we refer to object functions satisfying the condition that $\mu, L, M > 0$ in (2).
We say that \( \hat{x} \) is a solution of (4). By the method of Lagrange multipliers, problem (4) is equivalent to the following saddle point problem:

\[
\begin{align*}
\min_{x \in X^m} & \quad F(x) := \sum_{i=1}^m f_i(x_i) \\
\text{s.t.} & \quad Lx = 0,
\end{align*}
\]

where \( X^m := X_1 \times \ldots \times X_m, x = (x_1; \ldots; x_m) \in X^m, F : X^m \to \mathbb{R} \), and \( L \in \mathbb{R}^{md \times md} \). The constraint \( Lx = 0 \) is a compact way of writing \( x_i = x_j \) for all pairs \((i, j) \in \mathcal{E}\). In view of Theorem 4.2.12 in [11], \( L \) is symmetric positive semidefinite and its null space coincides with the “agreement” subspace, i.e., \( L1^T = 0^T, L = 0 \). To ensure each agent can obtain information from every other agent, we need the following assumption as a blanket assumption throughout the paper.

**Assumption 1.** The graph \( G \) is connected.

Under Assumption 1, problem (4) is equivalent to the following reformulation of (4). By the method of Lagrange multipliers, problem (4) is equivalent to the following saddle point problem:

\[
\begin{align*}
\min_{x \in X^m} & \quad F(x) + \max_{y \in \mathbb{R}^{md}} \langle Lx, y \rangle, \\
\end{align*}
\]

where \( y = (y_1; \ldots; y_m) \in \mathbb{R}^{md} \) are the Lagrange multipliers associated with the constraints \( Lx = 0 \). We assume that there exists an optimal solution \( \hat{x} \in X^m \) of (4) and that there exists \( \hat{y} \in \mathbb{R}^{md} \) such that \( (\hat{x}, \hat{y}) \) is a saddle point of (4). Finally, we define the following terminology.

**Definition 1.** A point \( \hat{x} \in X^m \) is called a stochastic \( \epsilon \)-solution of (4) if

\[
\mathbb{E}[F(\hat{x}) - F(\hat{x}^*)] \leq \epsilon \text{ and } \mathbb{E}[\|L\hat{x}\|] \leq \epsilon.
\]

We say that \( \hat{x} \) has primal residual \( \epsilon \) and feasibility residual \( \epsilon \).
Note that for problem (4), the feasibility residual \( \|L \hat{x}\| \) measures the disagreement among the local copies \( \hat{x}_i \), for \( i \in \mathcal{N} \). We will use these two criteria to evaluate the output solutions of the algorithms proposed in this paper.

2.1 Distance generating function and prox-function

Prox-function, also known as proximity control function or Bregman distance function [5], has played an important role as a substantial generalization of the Euclidean projection, since it can be flexibly tailored to the geometry of a constraint set \( U \).

For any convex set \( U \) equipped with an arbitrary norm \( \|\cdot\|_U \), we say that a function \( \omega: U \to \mathbb{R} \) is a distance generating function with modulus 1 with respect to \( \|\cdot\|_U \), if \( \omega \) is continuously differentiable and strongly convex with modulus 1 with respect to \( \|\cdot\|_U \), i.e.,

\[
\langle \nabla \omega(x) - \nabla \omega(u), x - u \rangle \geq \|x - u\|_U^2, \quad \forall x, u \in U.
\]

The prox-function induced by \( \omega \) is given by

\[
V(x, u) \equiv V_\omega(x, u) := \omega(u) - \left[ \omega(x) + \langle \nabla \omega(x), u - x \rangle \right].
\]  

We now assume that the constraint set \( X_i \) for each agent in (1) is equipped with norm \( \|\cdot\|_{X_i} \), and its associated prox-function is given by \( V_i(\cdot, \cdot) \). It then follows from the strong convexity of \( \omega \) that

\[
V_i(x_i, u_i) \geq \frac{1}{2} \|x_i - u_i\|_{X_i}^2, \quad \forall x_i, u_i \in X_i, \ i = 1, \ldots, m.
\]  

We also define the norm associated with the primal feasible set \( X^m = X_1 \times \ldots \times X_m \) of (5) as \( \|\cdot\|^2_{X^m} \equiv \|\cdot\|_{X^m} : = \sum_{i=1}^m \|x_i\|_{X_i}^2, \forall \mathbf{x} = (x_1; \ldots; x_m) \in X^m \). Therefore, the associated prox-function \( V(\cdot, \cdot) \) is defined as \( V(\mathbf{x}, \mathbf{u}) := \sum_{i=1}^m V_i(x_i, u_i), \forall \mathbf{x}, \mathbf{u} \in X^m \). In view of (8)

\[
V(\mathbf{x}, \mathbf{u}) \geq \frac{1}{2} \|\mathbf{x} - \mathbf{u}\|^2, \quad \forall \mathbf{x}, \mathbf{u} \in X^m.
\]

Throughout the paper, we endow the dual space where the multipliers \( \mathbf{y} \) of (5) reside with the standard Euclidean norm \( \|\cdot\|_2 \), since the feasible region of \( \mathbf{y} \) is unbounded. For simplicity, we often write \( \|\mathbf{y}\|_2 \) instead of \( \|\mathbf{y}\|_2 \) for a dual multiplier \( \mathbf{y} \in \mathbb{R}^{md} \).

3 The algorithms

In this section, we introduce an asynchronous decentralized primal-dual framework for solving (1) in the decentralized setting. Specifically, two asynchronous methods are presented, namely asynchronous decentralized primal-dual method in Subsection 3.1 and asynchronous accelerated stochastic decentralized communication sliding in Subsection 3.2, respectively. Moreover, we establish complexity bounds (number of inter-node communication rounds and/or intra-node stochastic (sub)gradient evaluations) separately in terms of primal functional optimality gap and constraint (or consistency) violation for solving (1)-(4).

3.1 Asynchronous decentralized primal-dual method

Our main goals in this subsection are to introduce the basic scheme of asynchronous decentralized primal-dual (ADPD) method, as well as establishing its complexity results. Throughout this subsection, we assume that \( f_i \) is a simple function such that we can solve the primal subproblem (15) explicitly.
We formally present the ADPD method in Algorithm 1. Each agent $i$ maintains two local sequences, namely, the primal estimates $\{x_i^k\}$ and the dual variables $\{y_i^k\}$. All primal estimates $x_i^0$ and $x_i^0$ are locally initialized from some arbitrary point in $X_i$, and each dual variable $y_i^0 = 0$. At each iteration $k \geq 1$, only one randomly selected agent (cf. activated agent) $i_k \in [m]$ updates its dual variable $y_i^k$, and then one randomly selected agent $j_k \in [m]$ updates its primal variable $x_j^k$. In particular, each agent in the activated agent’s neighborhood, i.e., agents $i \in N_{i_k}$, computes a local prediction $\tilde{x}_i^k$ using the two previous primal estimates (ref. (10)), and send it to agent $i_k$. In (11)-(12), the activated agent $i_k$ calculates its neighborhood disagreement $v_{i_k}^k$ using the receiving messages, and updates the dual variable $y_{i_k}^k$. Other agents’ dual variables remain unchanged. Then, another round of communication (14) between the activated agent $j_k$ and its neighboring agents occurs after the dual prediction step (13). Lastly, the activated agent $j_k$ solves the proximal projection subproblem (15) to update $x_{j_k}^k$, and other agents’ primal estimates remain the same as the last iteration.

It should be emphasized that each iteration $k$ only involves two communication rounds (cf. (11) and (14)) between the activated agents and its neighboring agents, which significantly reduces synchronous delays appearing in many decentralized methods (e.g., [7, 25, 26, 15]), since these methods require at least one communication round between all agents and their neighboring agents iteratively. Also note that similar to the asynchronous ADMM proposed in [28], ADPD employs node-based activation. However, while [28] requires all agents to update dual variables iteratively based on the information obtained from communication, in ADPD only the activated agent $i_k$ needs to collect neighboring information and update its dual variable (see (11) and (12)), and hence ADPD further reduces communication costs and synchronous delays comparing to [28]. Moreover, ADPD can achieve the same rate of convergence $O(1/\epsilon)$ as [28] under the assumption that (15) can be solved explicitly. We will demonstrate later that by exploiting the strong convexity, an improved $O(1/\sqrt{\tau})$ rate of convergence can be obtained.

**Algorithm 1** Asynchronous decentralized primal-dual (ADPD) update for each agent $i$

Let $x_i^0 = x_i^{-1} \in X_i$ and $y_i^0 = 0$ for $i \in [m]$, the nonnegative parameters $\{\alpha_k\}$, $\{\tau_k\}$ and $\{\eta_k\}$ be given.

for $k = 1, \ldots, N$ do

Uniformly choose $i_k, j_k \in [m]$, and update $(x_i^k, y_i^k)$ according to

$$\tilde{x}_i^k = \alpha_k (x_i^{k-1} - x_i^{k-2}) + x_i^{k-1}. \tag{10}$$

$$v_{i_k}^k = \sum_{j \in N_{i_k}} \mathcal{L}_{i_k,j} \tilde{x}_j^k \{\text{communication}\} \tag{11}$$

$$y_i^k = \begin{cases} \arg\min_{y_i \in \mathbb{R}^d} \langle -v_i^k, y_i \rangle + \frac{\tau_k}{2} \|y_i - y_i^{k-1}\|^2 = y_i^{k-1} + \frac{1}{\tau_k} v_i^k, & i = i_k, \\ y_i^{k-1}, & i \neq i_k. \end{cases} \tag{12}$$

$$\tilde{y}_i^k = m(y_i^k - y_i^{k-1}) + y_i^{k-1}. \tag{13}$$

$$w_{j_k}^k = \sum_{j \in N_{j_k}} \mathcal{L}_{j_k,j} \tilde{y}_j^k \{\text{communication}\} \tag{14}$$

$$x_i^k = \begin{cases} \arg\min_{x_i \in X_i} \langle w_i^k, x_i \rangle + f_i(x_i) + \eta_k V_i(x_i^{k-1}, x_i), & i = j_k, \\ x_i^{k-1}, & i \neq j_k. \end{cases} \tag{15}$$

end for

In the following theorem, we provide a specific selection of $\{\alpha_k\}$, $\{\tau_k\}$ and $\{\eta_k\}$, which leads to
Theorem 1. Let $\mathbf{x}^*$ be an optimal solution of (4), and $d_{\text{max}}$ be the maximum degree of graph $G$, and suppose that $\{\alpha_k\}$, $\{\tau_k\}$ and $\{\eta_k\}$ are set to

$$\alpha_k = m, \ \eta_k = 2md_{\text{max}}, \ \text{and} \ \tau_k = 2md_{\text{max}}, \forall k = 1, \ldots, N. \tag{16}$$

Then, for any $N \geq 1$, we have

$$E_{i_k,j_k}\{F(\bar{x}^N) - F(x^*)\} \leq O\left(\frac{m\Delta_0}{N+m}\right), \ E_{i_k,j_k}\{|L\bar{x}^N|\} \leq O\left(\frac{m\Delta_0}{N+m}\right), \tag{17}$$

where $\bar{x}^N = \frac{1}{N+m}\left(\sum_{k=0}^{N-1}x^k + mx^N\right)$, $\{x^k\}$ is generated by Algorithm 1 and $\Delta_{x^0} := \max\left\{C_{x^0}, |Lx^0| + md_{\text{max}}\left(|y^*| + \sqrt{\sum_{i=0}^{N}x^i + (Lx^0,y^*)}\right)\right\}$. With $C_{x^0} = F(x^0) - F(x^*) + md_{\text{max}}V(x^0, x^*)$.

Theorem 1 implies the total number of inter-node communication rounds performed by ADPD to find a stochastic $\epsilon$-solution of (4) can be bounded by

$$O\left(\frac{md_{\text{max}}\Delta_{x^0}}{\epsilon}\right). \tag{18}$$

Observed that in Algorithm 1 we assume that $f_i$'s are simple functions such that (15) can be solved explicitly. However, since $f_i$'s are possibly nonsmooth functions and/or possess composite structures, it is often difficult to solve (15) especially when $f_i$ is provided in the form of expectation. In the next subsection, we present a new asynchronous stochastic decentralized primal-dual type method, called the asynchronous accelerated stochastic decentralized communication sliding (AA-SDCS) method, for the case when (15) is not easy to solve.

### 3.2 Asynchronous accelerated stochastic decentralized communication sliding

In the subsection, we show that one can still maintain the same number of inter-node communications even when the subproblem (15) is approximately solved through an optimal stochastic approximation method, namely AC-SA proposed in [10, 9, 14], and that the total number of required stochastic (sub)gradient evaluations (or sampling complexity) is comparable to centralized mirror descent methods. Throughout this subsection, we assume that only noisy (sub)gradient information of $f_i$, $i = 1, \ldots, m$, is available or easier to compute. This situation happens when the function $f_i$'s are given either in the form of expectation or as the summation of lots of components. Moreover, we assume that the first-order information of the function $f_i$, $i = 1, \ldots, m$, can be accessed by a stochastic oracle (SO), which, given a point $u^t \in X$, outputs a vector $G_i(u^t, \xi_i^t)$ such that

$$E[G_i(u^t, \xi_i^t)] = f'_i(u^t) \in \partial f_i(u^t), \tag{19}$$

$$E[||G_i(u^t, \xi_i^t) - f'_i(u^t)||^2] \leq \sigma^2. \tag{20}$$

where $\eta_i^t$ is a random vector which models a source of uncertainty and is independent of the search point $u^t$, and the distribution $P(\xi_i)$ is not known in advance. We call $G_i(u^t, \xi_i^t)$ a stochastic (sub)gradient of $f_i$ at $u^t$. Observe that this assumption covers the case that one can access the exact (sub)gradients of $f_i$ whenever $\sigma = 0$. 

$O(1/\epsilon)$ complexity bounds for the functional optimality gap and also the feasibility residual to obtain a stochastic $\epsilon$-solution of (4).
In order to exploit the strong convexity of the prox-function $V_i$, we assume in this subsection that each prox-function $V_i(\cdot, \cdot)$ (cf. (7)) are growing quadratically with the \textit{quadratic growth constant} $C$, i.e., there exists a constant $C > 0$ such that
\begin{equation}
V_i(x_i, u_i) \leq \frac{C}{2} \|x_i - u_i\|_{X_i}, \quad \forall x_i, u_i \in X_i, \; i = 1, \ldots, m.
\end{equation}

By (8), we must have $C \geq 1$.

We now add a few comments about Algorithm 2. Firstly, similar to SDCS proposed in [15], AA-SDCS exploits two loops: the doubly randomized primal-dual scheme as outer loop and the ACS procedure as inner loop. More specifically, AA-SDCS utilizes the AC-SA method proposed in [10, 9, 14] to approximately solve the primal subproblem in (15), which provides a unified scheme for solving a general class of problems defined in [2] and leads to accelerated rate of convergence when $f_i$ possesses smooth structure. Secondly, the same dual information $w = w_{jk}^T$ (see (26)) has been used throughout the $T = T_k$ iterations of the ACS procedure, and hence no additional communication is required within the procedure. Finally, since AA-SDCS randomly selects one subproblem (15) and solved it inexactly, the outer loop also needs to be carefully designed to attain the best possible rate of convergence. In fact, the ACS procedure provides two approximate solutions of (15): one is the primal estimate of convergence. In fact, the ACS procedure provides two approximate solutions of (15): one is the primal extrapolation step of the outer loop (cf. (22)). For later convenience, we refer to the subproblem ACS solved at iteration $k$ as $\Phi^k(x_i)$, i.e.,
\begin{equation}
\arg\min_{x_i \in X_i} \left\{ \Phi^k(x_i) := \langle w_i^k, x_i \rangle + f_i(x_i) + \eta_k V_i(x_i^{k-1}, x_i) \right\}.
\end{equation}

Theorem 2 provides a specific selection of $\{\alpha_k\}$, $\{\tau_k\}$, $\{\eta_k\}$ and $\{T_k\}$ for Algorithm 2 and $\{\lambda_t\}$ and $\{\beta_t\}$ for the ACS procedure, which leads to $O(1/\epsilon)$ complexity bounds for the functional optimality gap and also the feasibility residual to obtain a stochastic $\epsilon$-solution of (4).

**Theorem 2.** Let $x^*$ be an optimal solution of (4), and $d_{\max}$ be the maximum degree of graph $G$, and suppose that the parameters $\{\lambda_t\}$ and $\{\beta_t\}$ in the ACS procedure of Algorithm 2 be set to
\begin{equation}
\lambda_t = \frac{2}{t+1}, \quad \beta_t = \frac{4(C + L)}{(t+1)}, \quad \forall t \geq 1,
\end{equation}
and $\{\alpha_k\}$, $\{\tau_k\}$, $\{\eta_k\}$ and $\{T_k\}$ are set to
\begin{align*}
\alpha_k &= 1, \quad \eta_k = 4m d_{\max}, \quad \tau_k = 2d_{\max},
\end{align*}
and
\begin{equation}
T_k = \max \left\{ \left\lfloor \frac{(M^2 + \sigma^2)N}{d_{\max} D} \right\rfloor, \left\lceil \frac{C + L}{m d_{\max}} \right\rceil \right\}, \quad \forall k = 1, \ldots, N,
\end{equation}
for some $D > 0$. Then, for any $N \geq 1$, we have
\begin{equation}
\mathbb{E}\{F(x^N) - F(x^*)\} \leq O \left\{ \frac{m\Delta_{x^0, D}}{N + m} \right\}, \quad \mathbb{E}\{\|Lx^N\|\} \leq O \left\{ \frac{m\Delta_{x^0, D}}{N + m} \right\},
\end{equation}
where $x^N = \frac{1}{N + m} \sum_{k=0}^{N-1} x^k + m x^N$, $\{x^k\}$ is generated by Algorithm 2, and $\Delta_{x^0, D} := \max \left\{ C_{x^0, D}, \|Lx^0\| \right\}$
\begin{align*}
d_{\max} &\left( \|y^*\| + \sqrt{\frac{C_{x^0, D} + (Lx^0, y^*)}{d_{\max}}} \right) \right\} \right) \text{ with } C_{x^0, D} = F(x^0) - F(x^*) + m d_{max} V(x^0, x^*) + D/m.
\end{align*}
Algorithm 2 Asynchronous Accelerated Stochastic Decentralized Communication Sliding (AA-SDCS)

Let \( x_i^0 = x_i^{-1} = \mathbf{x}_i^0 \in X_i \), \( y_i^0 = \mathbf{0} \) for \( i \in [m] \) and the nonnegative parameters \( \{\alpha_k\} \), \( \{\tau_k\} \), \( \{\eta_k\} \) and \( \{T_k\} \) be given.

For \( k = 1, \ldots, N \) do

- Uniformly choose \( i_k, j_k \in [m] \), and update \((x_i^k, y_i^k)\) according to
  \[
  \begin{aligned}
  \tilde{x}_i^k &= \alpha_k[m\tilde{x}_i^{k-1} - (m - 1)\tilde{x}_i^{k-2} - x_i^{k-2}] + x_i^{k-1}, \\
  v_{ik}^k &= \sum_{j \in N_{ik}} L_{ik,j} \tilde{x}_j^k, \text{ \{communication\}} \\
  y_i^k &= \begin{cases} y_i^{k-1} + \frac{1}{\tau_k}v_i^k, & i = i_k, \\ y_i^{k-1}, & i \neq i_k. \end{cases} \\
  \tilde{y}_i^k &= m(y_i^k - y_i^{k-1}) + y_i^{k-1}. \\
  w_{jk}^k &= \sum_{j \in N_{jk}} L_{jk,j} \tilde{y}_j^k, \text{ \{communication\}} \\
  (x_i^k, \mathbf{x}_i^k) &= \begin{cases} \text{ACS}(f_i, X_i, V_i, T_k, \eta_k, w_{ik}, x_i^1), & i = j_k, \\ (x_i^{k-1}, \mathbf{x}_i^{k-1}), & i \neq j_k. \end{cases}
  \end{aligned}
  \tag{27}
  \]

End for

The ACS (Accelerated Communication-Sliding) procedure called at \(27\) is stated as follows.

\textbf{procedure: } \((x, \mathbf{x}) = \text{ACS}(\phi, U, V, T, \eta, w, x)\)

- Let \( u^0 = u_i^0 = x \) and the parameters \( \{\beta_t\} \) and \( \{\lambda_t\} \) be given.

For \( t = 1, \ldots, T \) do

\[
\begin{aligned}
\hat{u}^t &= \frac{(1 - \lambda_t)(\mu + \eta + \beta_t)}{\beta_t + (1 - \lambda_t)(\mu + \eta)} u^{t-1} + \frac{\lambda_t[(1 - \lambda_t)(\mu + \eta) + \beta_t]}{\beta_t + (1 - \lambda_t)(\mu + \eta)} u^t. \\
G^t &= G(\hat{u}^t, \xi^t), \text{ \{Call the SO\}} \\
u^t &= \arg \min_{u \in U} \left\{ \lambda_u [(w + G^t + \eta(\nabla w(\hat{u}^t) - \nabla w(x))) u] + (\mu + \eta) V(\hat{u}^t, u) \right. \\
&\quad \left. + \left[(1 - \lambda_t)(\mu + \eta) + \beta_t\right] V(u^{t-1}, u) \right\}. \\
u^t &= (1 - \lambda_t)u^{t-1} + \lambda_t u^t. \tag{31}
\end{aligned}
\]

End for

Set \( x = u^T \) and \( \mathbf{x} = u^T \).

\textbf{end procedure}
In view of Theorem 2, letting $\mathcal{D} = \mathcal{O}(m^2 d_{\text{max}})$, we can see that the total number of inter-node communication rounds and intra-node (sub)gradient evaluations required by AA-SDCS for finding a stochastic $\epsilon$-solution of (4) can be bounded by
\[
\mathcal{O}\left\{ \frac{m d_{\text{max}} \Delta x^0, D, \mu}{\epsilon} \right\} \quad \text{and} \quad \mathcal{O}\left\{ \frac{(M^2 + \sigma^2) \Delta x^0, D}{\epsilon^2 d_{\text{max}}^2} + \sqrt{\frac{m(C + L) \Delta x^0, D}{\epsilon}} \right\},
\]
respectively. It also needs to be emphasized that the sampling complexity (second bound in (36)) only sublinearly depends on the Lipschitz constant $L$.

Now consider the case when $f_i$’s are strongly convex (i.e., $\mu > 0$ in (2)). The following theorem instantiates Algorithm 2 by providing a selection of $\{\alpha_k\}$, $\{\tau_k\}$, $\{\tilde{\eta}_k\}$ and $\{T_k\}$, which leads to an improved $\mathcal{O}(1/\sqrt{\epsilon})$ complexity bound for the functional optimality gap and also the feasibility residual to obtain a stochastic $\epsilon$-solution of (4).

**Theorem 3.** Let $x^*$ be an optimal solution of (4), and $d_{\text{max}}$ be the maximum degree of graph $G$, and suppose that the parameters $\{\lambda_i\}$ and $\{\beta_i\}$ in the ACS procedure of Algorithm 2 be set to (33), and $\{\alpha_k\}$, $\{\tau_k\}$, $\{\tilde{\eta}_k\}$ and $\{T_k\}$ are set to
\[
\alpha_k = \frac{k + 3m - 1}{k + 3m}, \quad \tilde{\eta}_k = \frac{(k + 3m - 1)\mu}{2} - \frac{C + L}{T_k(T_k + 1)}, \quad \tau_k = \frac{32md_{\text{max}}^2}{(k + 3m)\mu - 1},
\]
and $T_k = \max\left\{ \frac{6m(M^2 + \sigma^2)N}{D\mu^2}, \sqrt{\frac{4(C + L)}{(k + 3m)^2\mu)}} \right\}$, $\forall k = 1, \ldots, N$. (37)

Then, for any $N \geq 1$, we have
\[
\mathbb{E}\{F(\mathbf{x}^N) - F(x^*)\} \leq \mathcal{O}\left\{ \frac{m^2 \Delta x^0, D, \mu}{m^2 + N} \right\}, \quad \mathbb{E}\{\|L\mathbf{x}^N\|\} \leq \mathcal{O}\left\{ \frac{m^2 \Delta x^0, D, \mu}{m^2 + N} \right\},
\]
where $\mathbf{x}^N = \frac{2}{6m^2 + N(N + 6m + 1)} \sum_{k=0}^{N-1} (k + 2m + 1)\mathbf{x}^k + m(N + 3m)\mathbf{x}^N$, $\{\mathbf{x}^k\}$ is generated by Algorithm 2, and $\Delta x^0, D, \mu := \max\left\{ C x^0, D, \mu \|L\mathbf{x}^0\| + \frac{d_{\text{max}} \|y^*\|}{\mu} + d_{\text{max}} \sqrt{\frac{C x^0, D, \mu + (L \mathbf{x}^0, y^*)}{\mu}} \right\}$ with $C x^0, D, \mu = F(x^0) - F(x^*) + m\mu \mathbf{V}(x^0, x^*) + \frac{D\mu}{m^2}$.

As a consequence of Theorem 3, letting $\mathcal{D} = \mathcal{O}(m^3)$, we can see that the total number of inter-node communication rounds and intra-node (sub)gradient evaluations required by AA-SDCS for finding a stochastic $\epsilon$-solution of (4), respectively, can be bounded by
\[
\mathcal{O}\left\{ md_{\text{max}} \sqrt{\frac{\Delta x^0, D, \mu}{\epsilon}} \right\}, \quad \text{and} \quad \mathcal{O}\left\{ \frac{(M^2 + \sigma^2) \Delta x^0, D, \mu}{\mu^2 \epsilon} + \sqrt{\frac{m(C + L)}{\mu} \left( \frac{\Delta x^0, D, \mu}{\epsilon} \right)^{1/4}} \right\}.
\]

4 Numerical experiments

We demonstrate the advantages of our proposed AA-SDCS method over the state-of-art synchronous algorithm, stochastic decentralized communication sliding (SDCS) method, proposed in [15] through some preliminary numerical experiments.

Let us consider the decentralized linear Support Vector Machines (SVM) model with the following hinge loss function
\[
\max\{0, 1 - y\langle x, u \rangle\},
\]

(40)
where \((v, u) \in \mathbb{R} \times \mathbb{R}^d\) is the pair of class label and feature vector, and \(x \in \mathbb{R}^d\) denotes the weight vector. We consider two types of stochastic decentralized linear SVM problems in this paper. For the convex case, we study 1-norm SVM problem \([32, 4]\) defined in (41), while for the strongly convex case, we study 2-norm SVM model defined in (42). Moreover, we use the Erhos-Renyi algorithm to generate the underlying decentralized network. Note that nodes with different degrees are drawn in different colors (cf. Figure 1). We also used the real dataset named “ijcnn1” from LIBSVM and drew 40,000 samples from this dataset as our problem instance data to train the decentralized linear SVM model. These samples are evenly split over the network agents. For example, if we have \(m = 8\) nodes (or agents) in the decentralized network (see Figure 1), each network agent has 5,000 samples.

![Figure 1: The 8-agent decentralized network randomly generated by Erhos-Renyi algorithm.](image)

With the same initial points \(x^0 = 0\) and \(y^0 = 0\), we compare the performances of our algorithms with the SDCS method \([15]\) for solving (1)-(4) by reporting the progresses of objective function values and feasibility residuals \(\|Lx\|\) versus the elapsed CPU running time (in seconds) for solving the aforementioned two different types of problems. In all problem instances, we use \(\|\cdot\|_2\) norm in both the primal and dual spaces, and hence in the parameter settings of SDCS \(\|L\|\) refers to the maximum eigenvalue of the Laplacian matrix \(L\). Moreover, all algorithms are implemented in MATLAB R2016a and run in the computer environment of with 32-core (Intel(R) Xeon(R) CPU E5-2673 v3 2.40GHz) virtual machine on Microsoft Azure. Since the underlying network has 8 agents, we utilized the parallel toolbox in MATLAB to simulate the synchronous setting for SDCS. However, inter-node communication is instant and no delay is simulated in all experiments. In fact, such simulation setup is in favor of the synchronous methods, since these methods can be heavily slowed down by different processing speeds of the agents (cores) and inter-node communication speeds.

**Convex case: decentralized 1-norm SVM \([32, 4]\)** Consider a stochastic decentralized linear SVM problem defined over the \(m\)-agent decentralized network as

\[
\min_x \sum_{i=1}^m \left[ f_i(x_i) := E_{(v_i, u_i)} \left[ \max\{0, 1 - v_i \langle x_i, u_i \rangle\} \right] + \frac{1}{\|S_i\|} \|x_i\|_1 \right]
\]

(41)

s.t. \(Lx = 0\),

where \((v_i, u_i)\) represents a uniform random variable with support \(S_i\) and \(S_i\) denotes the dataset.

---

1. We implemented the Erhos-Renyi algorithm based on a MATLAB function written by Pablo Blider, which can be found in [this link](https://www.mathworks.com/matlabcentral/fileexchange/4206).
2. This real dataset can be downloaded from [this link](https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/).
belonging to node $i$. We compare the performances of AA-SDCS with SDCS for the decentralized network setups, $m = 8$ (cf. R. Figure 1). For all problem instances, we choose the parameters of AA-SDCS as in Theorem 2. For SDCS, we choose parameters as suggested in [15].

Figure 2: 1-norm SVM defined over the 8-agent decentralized network (cf. Figure 1), and we report the progresses of the objective function values on the left and the feasibility residuals on the right versus the elapsed CPU running time in seconds.

In Figure 2, the vertical-axis of the left subgraph represents the objective function values, the vertical-axis of the right subgraph represents the feasibility measure $\|\mathbf{Lx}\|$, and the horizontal-axis is the elapsed CPU running time in seconds. These numerical results are consistent with our theoretical analysis. We also need to emphasize that AA-SDCS can significantly save CPU running time over SDCS in terms of both objective function values and feasibility residuals as shown in Figure 2 even when each agent (Core) has the same processing speed.

**Strongly convex case: decentralized 2-norm SVM** Consider a decentralized linear SVM problem with $l_2$ regularizer defined over the $m$-agent decentralized network as the following

$$
\min_{\mathbf{x}} \sum_{i=1}^{m} \left[ f_i(x_i) := E(v_i, u_i) \max\{0, 1 - v_i\langle x_i, u_i \rangle\} \right] + \frac{1}{2|S_i|} \|x_i\|_2^2
$$

s.t. $\mathbf{Lx} = 0$.  \hspace{1cm} (42)

We compare the performances of AA-SDCS with SDCS for the decentralized network setups, $m = 8$ (cf. R. Figure 1). For all problem instances, we choose the parameters of AA-SDCS as in Theorem 3. For SDCS, we choose parameters as suggested in [15].
The above figures clearly show that AA-SDCS can significantly save CPU running time over SDCS in terms of both objective function values and feasibility residuals. Moreover, comparing Figure 3 with Figure 2, we can find out AA-SDCS obtains more improvements over SDCS for solving decentralized 2-norm SVM problems than decentralized 1-norm SVM problems. In fact, the decentralized 2-norm SVM problem defined in (42) has a composite objective structure that consists of a nonsmooth hinge loss function and a smooth strongly convex $l_2$-regularizer, and the convergence results of AA-SDCS has a better dependence on the Lipschitz constant $L$, which indicates that it can obtain a faster convergence speed than SDCS for solving decentralized 2-norm SVM problems.

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A Convergence analysis

In this section, we provide detailed convergence analysis of ADPD (cf. Algorithm 1) and AA-SDCS (cf. Algorithm 2) presented in Section 3.

A.1 Some basic tools: gap functions, termination criteria and technical results

Given a pair of feasible solutions \( z = (x, y) \) and \( \bar{z} = (\bar{x}, \bar{y}) \) of (5), we define the primal-dual gap function \( Q(z; \bar{z}) \) by

\[
Q(z; \bar{z}) := F(x) + \langle Lx, \bar{y} \rangle - [F(\bar{x}) + \langle L\bar{x}, y \rangle].
\] (43)

Sometimes we also use the notations \( Q(z; \bar{z}) := Q(x, y; \bar{x}, \bar{y}) \) or \( Q(z; \bar{z}) := Q(z; \bar{z}) \).

One can easily see that \( Q(z; \bar{z}) \leq 0 \) and \( Q(z; \bar{z}) \geq 0 \) for all \( z \in X^m \times \mathbb{R}^{md} \), where \( z^* = (x^*, y^*) \) is a saddle point of (5). For compact sets \( X^m \subseteq \mathbb{R}^{md}, Y \subseteq \mathbb{R}^{md} \), the gap function

\[
\sup_{z \in X^m \times Y} Q(z; \bar{z})
\] (44)

measures the accuracy of the approximate solution \( z \) to the saddle point problem (5).

However, the saddle point formulation (5) of our problem of interest (1) may have an unbounded feasible set. We adopt the perturbation-based termination criterion by Monteiro and Svaiter [17, 18, 19] and propose a modified version of the gap function in (44). More specifically, we define

\[
g_Y(s, z) := \sup_{y \in Y} Q(z; x^*, y) - \langle s, y \rangle,
\] (45)

for any closed set \( Y \subseteq \mathbb{R}^{md}, z \in X^m \times \mathbb{R}^{md} \) and \( s \in \mathbb{R}^{md} \). If \( Y = \mathbb{R}^{md} \), we omit the subscript \( Y \) and simply use the notation \( g(s, z) \). This perturbed gap function allows us to bound the objective function value and the feasibility separately.

In the following proposition, we adopt a result from [22, Proposition 2.1] to describe the relationship between the perturbed gap function (45) and the approximate solutions (see Definition 1) to problem (4).

**Proposition 4.** For any \( Y \subseteq \mathbb{R}^{md} \) such that \( 0 \in Y \), if \( g_Y(Lx, z) \leq \epsilon < \infty \) and \( ||Lx|| \leq \delta \), where \( z = (x, y) \in X^m \times \mathbb{R}^{md} \), then \( x \) is an \((\epsilon, \delta)\)-solution of (4). In particular, when \( Y = \mathbb{R}^{md} \), for any \( s \) such that \( g(s, z) \leq \epsilon < \infty \) and \( ||s|| \leq \delta \), we always have \( s = Lx \).

Although the proposition was originally developed for deterministic cases, the extension of this to stochastic cases is straightforward. In fact, if we define \( g_Y(s, z) \) as follows

\[
g_Y(s, z) := \sup_{y \in Y} \mathbb{E}[Q(z; x^*, y) - \langle s, y \rangle],
\]

from the definition of \( Q \), we have \( g_Y(s, z) = \sup_{y \in Y} \mathbb{E}[F(x) - F(x^*) - \langle Lx - s, y \rangle] \) due to \( Lx^* = 0 \). Therefore, when \( Y = \mathbb{R}^{md} \), the results in Proposition 4 holds, since \( g_Y(s, z) \) is bounded for any \( y \in Y \).

We also define some auxiliary notations which play important roles in the convergence analysis. Let \( \bar{x}^k, \hat{y}^k, \bar{x}_t^k \) and \( \hat{x}^k \) be defined as follows, \( \forall t = 1, \ldots, k \)

\[
\bar{x}^k = \arg \min_{x \in X^m} \{ L\bar{y}^k, x \} + F(x) + \eta_t V(x^{k-1}, x),
\] (46)

\[
\hat{y}^k = y^{k-1} + \frac{1}{\tau_k} L\bar{x}^k,
\] (47)

\[
(\bar{x}_t^k, \hat{x}^k) = ACS(F, X^m, V, T_k, \eta_k, L\bar{y}^k, x^{k-1}),
\] (48)

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and \( \hat{x}^2 = \hat{x}^3 = \hat{x}^4 = \hat{x}^5 = \hat{x}^6 = \hat{x}^7 = \hat{x}^8 = \hat{x}^9 = 0 \). Note that some notations may be abused in the above definitions, since \( x^k, \hat{y}^k, y^k, \hat{x}^k \) can be generated by both Algorithm 1 and Algorithm 2. However, these definitions become clear when we refer to them in the convergence analysis of certain algorithm. For example, when we refer to \( \hat{x}^k \) in the convergence analysis of Algorithm 1, notations \( \hat{y}^k \) and \( x^{k-1} \) in its definition clearly refer to (13) and (15) in Algorithm 1.

The following lemma below characterizes the solution of the primal and dual projection steps (13), (12), (24) (also (46), (47)) as well as the projection in inner loop (31). The proof of this result can be found in Lemma 2 of [9].

**Lemma 5.** Let the convex function \( q : U \to \mathbb{R} \), the points \( \bar{x}, \bar{y} \in U \) and the scalars \( \mu_1, \mu_2 \in \mathbb{R} \) be given. Let \( \omega : U \to \mathbb{R} \) be a differentiable convex function and \( V(x, z) \) be defined in (7). If

\[
\sum_{k=0} \theta_k \geq 0, \quad \sum_{k=0} \theta_k = 1,
\]

then for any \( u \in U \), we have

\[
q(u^*) + \mu_1 V(\bar{x}, u^*) + \mu_2 V(\bar{y}, u^*) \leq q(u) + \mu_1 V(\bar{x}, u) + \mu_2 V(\bar{y}, u) - (\mu_1 + \mu_2)V(u^*, u).
\]

For any given weight sequence \( \{\theta_k\} \) such that \( \theta_k \geq 0, \quad \sum_{k=0} N \theta_k = 1 \), let \( \theta_k \) be defined as

\[
\theta_k = \begin{cases} 
\theta_0 - (m - 1)\theta_1, & k = 0, \\
\theta_{k-1} - (m - 1)\theta_{k+1}, & k = 1, \ldots, N - 1, \\
\theta_N, & k = N.
\end{cases}
\]

Therefore, \( \sum_{k=0} N \theta_k = \sum_{k=0} N \theta_k = 1 \). In the following lemma, we provide some important relations that will be used later in the convergence analysis.

**Lemma 6.** For weight sequence \( \{\theta_k\} \) defined as in (49) and any \( \mathbf{x} \in X^m, \mathbf{y} \in \mathbb{R}^{md} \), we have

\[
\mathbb{E}_{[i_k, j_k]} \left\{ \sum_{k=0} N \theta_k [F(\mathbf{x}^k) - F(\mathbf{x}) + \langle \mathbf{Lx}^k, \mathbf{y} \rangle] \right\} = \mathbb{E}_{[i_k, j_k]} \left\{ \sum_{k=1} N \theta_k [F(\hat{x}^k) - F(\mathbf{x}) + \langle \mathbf{Lx}^k, \mathbf{y} \rangle] \right\},
\]

\[
\mathbb{E}_{[i_k, j_k]} \left\{ \sum_{k=0} N \theta_k [F(\hat{x}^k) - F(\mathbf{x}) + \langle \mathbf{Lx}^k, \mathbf{y} \rangle] \right\} = \mathbb{E}_{[i_k, j_k]} \left\{ \sum_{k=1} N \theta_k [F(\hat{x}^k) - F(\mathbf{x}) + \langle \mathbf{Lx}^k, \mathbf{y} \rangle] \right\},
\]

\[
\mathbb{E}_{[i_k, j_k]} \{ V(\hat{x}^k, \mathbf{x}) \} = \mathbb{E}_{[i_k, j_k]} \{ m V(\hat{x}^k, \mathbf{x}) - (m - 1) V(\mathbf{x}^{k-1}, \mathbf{x}) \},
\]

\[
\mathbb{E}_{[i_k, j_k]} \{ V(\hat{x}^k, \mathbf{x}^{k-1}) \} = \mathbb{E}_{[i_k, j_k]} \{ m V(\hat{x}^k, \mathbf{x}^{k-1}) \},
\]

where \( \mathbb{E}_{[i_k, j_k]} \) represents taking expectation over \( i_1, j_1, \ldots, i_k, j_k, \) and \( \mathbf{x}^k, \hat{x}^k, \mathbf{x}^k \) and \( \hat{x}^k \) are defined in (15), (46), (27), (48) respectively.

**Proof.** Note that by (46) and the fact that \( j_k \) is chosen uniformly from \{1, \ldots, m\}, we have

\[
\mathbb{E}_{j_k} \{ F(\mathbf{x}^k) - F(\mathbf{x}) + \langle \mathbf{Lx}^k, \mathbf{y} \rangle \}
\]

\[
= (1 - \frac{1}{m}) [F(\mathbf{x}^{k-1}) - F(\mathbf{x}) + \langle \mathbf{Lx}^{k-1}, \mathbf{y} \rangle] + \frac{1}{m} [F(\hat{x}^{k-1}) - F(\mathbf{x}) + \langle \mathbf{Lx}^{k-1}, \mathbf{y} \rangle].
\]
Therefore, by (49) we obtain
\[
\mathbb{E}_{[i_k,j_k]} \left\{ \sum_{k=0}^{N} \theta_k [F(x^k) - F(x) + \langle Lx^k, y \rangle] \right\} \\
= \mathbb{E}_{[i_k]} \mathbb{E}_{[j_k]} \left\{ (\hat{\theta}_0 - (m-1)\hat{\theta}_1) [F(x^0) - F(x) + \langle Lx^0, y \rangle] [i_k] \right\} \\
+ \mathbb{E}_{[i_k]} \mathbb{E}_{[j_k]} \left\{ \sum_{k=1}^{N} (m\hat{\theta}_k - (m-1)\hat{\theta}_{k+1}) [F(x^k) - F(x) + \langle Lx^k, y \rangle] [i_k] \right\} \\
+ \mathbb{E}_{[i_k]} \mathbb{E}_{[j_k]} \left\{ m\hat{\theta}_N [F(x^N) - F(x) + \langle Lx^N, y \rangle] [i_k] \right\} \\
= \mathbb{E}_{[i_k,j_k]} \left\{ \hat{\theta}_0 [F(\hat{x}^0) - F(x) + \langle L\hat{x}^0, y \rangle] \right\} \\
+ \mathbb{E}_{[i_k,j_k]} \left\{ \sum_{k=1}^{N} m\hat{\theta}_k [F(x^k) - F(x) + \langle Lx^k, y \rangle] [i_k] \right\} \\
- \mathbb{E}_{[i_k,j_k]} \left\{ \sum_{k=1}^{N} (m-1)\hat{\theta}_k [F(x^{k-1}) - F(x) + \langle Lx^{k-1}, y \rangle] [i_k] \right\} \\
= \mathbb{E}_{[i_k,j_k]} \left\{ \sum_{k=0}^{N} \hat{\theta}_k [F(x^k) - F(x) + \langle L\hat{x}^k, y \rangle] \right\},
\]
where the last equality is obtained by applying (50) and rearranging the terms. Similarly, in view of (48), we have
\[
\mathbb{E}_{j_k} \{ F(x^k) - F(x) + \langle Lx^k, y \rangle \} \\
= (1 - \frac{1}{m}) [F(x^{k-1}) - F(x) + \langle Lx^{k-1}, y \rangle] + \frac{1}{m} [F(\hat{x}^{k-1}) - F(x) + \langle L\hat{x}^{k-1}, y \rangle],
\]
and hence the second identity follows from the same argument. Moreover, for any \( x \in \mathbb{R}^m, k \geq 1 \), we have
\[
\mathbb{E}_{[i_k,j_k]} \{ V(x^k, x) \} = \mathbb{E}_{[i_k,j_k]} \left\{ \sum_{j=1}^{m} V_j(x_j^k, x_j) \right\} \\
= \mathbb{E}_{[i_k,j_k]} \left\{ \frac{1}{m} V(\hat{x}^k, x) + (1 - \frac{1}{m}) V(x^{k-1}, x) \right\},
\]
where the first equality follows from the definition of \( V(\cdot, \cdot) \), and the last equality follows by taking expectation on \( j_k \). Similarly, we can obtain the last relation of the lemma. \( \square \)

We define \( \hat{y}^k \) (see (47)) and \( \hat{y}_k^k \) (see (48)) in a similar way as \( \hat{x}_k \), and hence, following the same technique as in the above lemma, we can conclude
\[
\mathbb{E}_{[i_k,j_k]} \{ \hat{y}^k \} = \mathbb{E}_{[i_k,j_k]} \{ \hat{y}^k \}, \quad \mathbb{E}_{[i_k,j_k]} \{ ||y - \hat{y}^k||^2 \} = \mathbb{E}_{[i_k,j_k]} \{ m ||y - y^k||^2 - (m-1)||y - y^{k-1}||^2 \}, \quad \mathbb{E}_{[i_k,j_k]} \{ ||y^{k-1} - \hat{y}^k||^2 \} = \mathbb{E}_{[i_k,j_k]} \{ m ||y^{k-1} - y^k||^2 \},
\]
and
\[
\mathbb{E}_{[i_k,j_k]} \{ V(\hat{x}_k^k, x) \} = \mathbb{E}_{[i_k,j_k]} \{ m V(x^k, x) - (m-1) V(x^{k-1}, x) \},
\]
where \( x^k \) (cf. (27)) is generated by Algorithm 2.
A.2 Convergence properties of Algorithm 1 (A-DPD)

We now provide an important recursion relation of Algorithm 1 in the following lemma.

**Lemma 7.** Let the gap function $Q$ be defined as in (43), and $ar{z}^N := (\bar{x}^N, \bar{y}^N) = \sum_{k=0}^{N}(\theta_k x^k, \hat{\theta}_k y^k)$, where $\{\theta_k\}$ is a nonnegative sequence that satisfies (49). Also let $x^k$ and $y^k$ be defined in (15) and (12), respectively. Then for any $k \geq 1$, we have

$$
\mathbb{E}_{[i_k, j_k]}\{Q(\bar{z}^N; z)\} \leq \hat{\theta}_0 Q_0(x, y) + \mathbb{E}_{[i_k, j_k]}\left\{\sum_{k=0}^{N} \hat{\theta}_k\langle L(mx^k - (m - 1)x^{k-1} - \hat{x}^k), y - \hat{y}^k \rangle\right\} + \mathbb{E}_{[i_k, j_k]}\left\{\sum_{k=1}^{N} m\hat{\theta}_k \eta_k [V(x^{k-1}, x) - V(x^k, x) - V_j(x_{j_k}^{k-1}, x_{j_k}^k)]\right\}
$$

where $Q_0(x, y)$ is defined as

$$
Q_0(x, y) := F(x^0) - F(x) + \langle Lx^0, y \rangle.
$$

**Proof.** By the definitions of $Q(\cdot, \cdot)$ in (43) and $\bar{z}^N$, we have

$$
Q(\bar{z}^N; z) = F(\bar{x}^N) - F(x) + \langle L\bar{x}^N, y \rangle - \langle Lx, \tilde{y}^N \rangle
$$

$$
\leq \sum_{k=0}^{N} \theta_k [F(x^k) - F(x) + \langle Lx^k, y \rangle] - \sum_{k=0}^{N} \hat{\theta}_k \langle Lx, \tilde{y}^k \rangle,
$$

where the inequality follows from the convexity of $F(\cdot).$ By taking expectation over $i_1, j_1, \ldots, i_k, j_k$ and applying Lemma 6, we obtain

$$
\mathbb{E}_{[i_k, j_k]}\{Q(\bar{z}^N; z)\} \leq \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=0}^{N} \hat{\theta}_k \langle F(x^k) - F(x) + \langle Lx^k, y \rangle - \langle Lx, \tilde{y}^k \rangle \rangle\right\}.
$$

Note that by applying Lemma 5 to (46) and (47), we have

$$
\langle L\tilde{y}^k, \hat{x}^k - x \rangle + F(\hat{x}^k) - F(x) \leq \eta_k [V(x^{k-1}, x) - V(\bar{x}^k, x) - V(x^{k-1}, \hat{x}^k)],
$$

$$
\langle L\tilde{y}^k, y - \tilde{y}^k \rangle \leq \frac{n}{2} \left[\|y - y^{k-1}\|^2 - \|y - \tilde{y}^k\|^2 - \|y^{k-1} - \tilde{y}^k\|^2\right].
$$

Combining the above three inequalities and in view of $\tilde{y}^0 = y^0 = 0$, we can conclude that

$$
\mathbb{E}_{[i_k, j_k]}\{Q(\bar{z}^N; z)\} \leq \hat{\theta}_0 Q_0(x, y) + \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=1}^{N} \hat{\theta}_k \langle L\tilde{y}^k, \hat{x}^k - x \rangle + \langle L(\bar{x}^k - \hat{x}^k), y \rangle + \langle L(\tilde{x}^k - x), \tilde{y}^k \rangle\right\}
$$

$$
+ \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=1}^{N} m\hat{\theta}_k \eta_k [V(x^{k-1}, x) - V(\bar{x}^k, x) - V_j(x_{j_k}^{k-1}, x_{j_k}^k)]\right\}
$$

$$
+ \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=1}^{N} \frac{m\hat{\theta}_k \eta_k}{2} \left[\|y - y^{k-1}\|^2 - \|y - \tilde{y}^k\|^2 - \|y^{k-1} - \tilde{y}^k\|^2\right]\right\}
$$

$$
\leq \hat{\theta}_0 Q_0 + \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=1}^{N} \hat{\theta}_k \langle L(\bar{x}^k - \hat{x}^k), y - \tilde{y}^k\rangle\right\}
$$

$$
+ \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=1}^{N} m\hat{\theta}_k \eta_k [V(x^{k-1}, x) - V(x^k, x) - V_j(x_{j_k}^{k-1}, x_{j_k}^k)]\right\}
$$

$$
+ \mathbb{E}_{[i_k, j_k]} \left\{\sum_{k=1}^{N} \frac{m\hat{\theta}_k \eta_k}{2} \left[\|y - y^{k-1}\|^2 - \|y - y^k\|^2 - \|y^{k-1} - y^k\|^2\right]\right\},
$$

where the second inequality follows from Lemma 6 and the result in (51) immediately follows from taking expectation on $j_k$. \qed
The following proposition establishes the main convergence property of the A-DPD method stated in Algorithm 1.

**Proposition 8.** Let the iterates \( (x^k, y^k) \), \( k = 1, \ldots, N \), be generated by Algorithm 1 and be defined as in [47], respectively, and let \( z^N := (\sum_{k=0}^N \theta_k x^k, \sum_{k=0}^N \hat{\theta}_k y^k) \). Assume that the parameters \( \{\alpha_k\} \), \( \{\tau_k\} \), and \( \{\eta_k\} \) in Algorithm 1 satisfy

\[
\hat{\theta}_k \tau_k = \hat{\theta}_{k-1} \tau_{k-1}, \quad k = 2, \ldots, N, \tag{55}
\]

\[
\hat{\theta}_k \eta_k \leq \hat{\theta}_{k-1} \eta_{k-1}, \quad k = 2, \ldots, N, \tag{56}
\]

\[
\alpha_k \hat{\theta}_k = m \hat{\theta}_{k-1}, \quad k = 2, \ldots, N + 1, \tag{57}
\]

\[
4m \alpha_k d_{max}^2 \leq \eta_{k-1} \tau_k, \quad k = 2, \ldots, N, \tag{58}
\]

\[
4(m - 1)^2 d_{max}^2 \leq \eta_k \tau_k, \quad k = 1, \ldots, N. \tag{59}
\]

where \( \{\hat{\theta}_k\} \) is some given weight sequence and \( d_{max} \) is the maximum degree of graph \( G \). Then, for any \( z := (x, y) \in X^m \times \mathbb{R}^{md} \), we have

\[
\mathbb{E}_{[i_k, j_k]} \{Q(z^N; z)\} \leq \hat{\theta}_0 (F(x^0) - F(x)) + m \hat{\theta}_1 \eta_1 V(x^0, x) + \mathbb{E}_{[i_k, j_k]} \{s, y\}, \tag{60}
\]

where \( Q \) is defined in [43], and \( s \) is defined as

\[
s := \theta_0 Lx^0 + m \theta_1 L(x^N - x^{N-1}) + m \hat{\theta}_1 \tau_1 y^N. \tag{61}
\]

Furthermore, for any saddle point \( (x^*, y^*) \) of [5], we have

\[
\frac{m \hat{\theta}_0}{2} \left( 1 - \frac{4d_{max}}{\eta N \tau N} \right) \max \left\{ \frac{\eta N}{2} \mathbb{E}_{[i_k, j_k]} \|x_{i_k} - x_{j_k}^N \|^2, \tau N \mathbb{E}_{[i_k, j_k]} \|y^* - y^N \|^2 \right\}
\]

\[
\leq \hat{\theta}_0 (F(x^0) - F(x^*) + \langle Lx^0, y^* \rangle) + m \hat{\theta}_1 \eta_1 V(x^0, x^*) + \frac{m \hat{\theta}_1 \tau_1}{2} \|y^*\|^2. \tag{62}
\]

**Proof.** In view of Lemma 7 we have

\[
\mathbb{E}_{[i_k, j_k]} \{Q(z^N; z)\} \leq \hat{\theta}_0 Q_0(x, y) + \mathbb{E}_{[i_k, j_k]} \left\{ \sum_{k=1}^N \hat{\theta}_k \Delta_k \right\}, \tag{63}
\]

where

\[
\Delta_k := \langle L(mx^k - (m - 1)x^{k-1} - \hat{x}^k), y - \hat{y}^k \rangle + m \eta_k [V(x^{k-1}, x) - V(x^k, x) - V_{jk}(x_{jk}^{k-1}, x_{jk}^k)]
\]

\[
+ \frac{m \eta}{2} [\|y - y^{k-1}\|^2 - \|y - y^k\|^2 - \|y_{jk}^{k-1} - y_{jk}^k\|^2]. \tag{64}
\]

Now we will provide a bound for \( \sum_{k=1}^N \hat{\theta}_k \Delta_k \). Observe that by [10], we obtain

\[
\sum_{k=1}^N \hat{\theta}_k \langle L(mx^k - (m - 1)x^{k-1} - \hat{x}^k), y - \hat{y}^k \rangle
\]

\[
= \sum_{k=1}^N \hat{\theta}_k \langle L(m(x^k - x^{k-1}) - \alpha_k(x^{k-1} - x^{k-2})), y - \hat{y}^k \rangle
\]

\[
= \sum_{k=1}^N \left[ m \hat{\theta}_k \langle L(x^k - x^{k-1}), y - \hat{y}^k \rangle - \hat{\theta}_k \alpha_k \langle L(x^{k-1} - x^k), y - \hat{y}^k \rangle \right]
\]

\[
+ \sum_{k=1}^N \hat{\theta}_k \alpha_k \langle L(x^{k-1} - x^k), \hat{y}^k - \hat{y}^k \rangle
\]

\[
= m \hat{\theta}_N \langle L(x^N - x^{N-1}), y - y^N - (m - 1)(y^N - y^{N-1}) \rangle
\]

\[
+ \sum_{k=2}^N \hat{\theta}_k \alpha_k \langle L(x^{k-1} - x^k), m(y^k - y^{k-1}) - (m - 1)(y^{k-1} - y^{k-2}) \rangle
\]

\[
= m \hat{\theta}_N \langle L(x^N - x^{N-1}), y - y^N \rangle + \sum_{k=2}^N m \hat{\theta}_k \alpha_k \langle L(x^{k-1} - x^k), y^k - y^{k-1} \rangle
\]

\[
- \sum_{k=1}^N (m - 1) \hat{\theta}_{k+1} \alpha_{k+1} \langle L(x^k - x^{k-1}), y^k - y^{k-1} \rangle,
\]

20
where the third equality follows from (57), (13) and the fact that $x^{-1} = x^0$, and the last equality follows from (57) and rearranging the terms. Also note that

$$\sum_{k=1}^{N} m\hat{\theta}_k \eta_k \langle V(x^{k-1}, x) - V(x^k, x) - V_j(x_{jk}^{-1}, x_{jk}) \rangle$$

$$= m\hat{\theta}_1 \eta_1 V(x^0, x) + \sum_{k=2}^{N} (m\hat{\theta}_k \eta_k - m\hat{\theta}_{k-1} \eta_{k-1}) V(x^{k-1}, x) - m\hat{\theta}_N \eta_N V(x^N, x)$$

$$- \sum_{k=1}^{N} m\hat{\theta}_k \eta_k V_j(x_{jk}^{-1}, x_{jk})$$

$$\leq m\hat{\theta}_1 \eta_1 V(x^0, x) - m\hat{\theta}_N \eta_N V(x^N, x) - \sum_{k=1}^{N} m\hat{\theta}_k \eta_k V_j(x_{jk}^{-1}, x_{jk}),$$

where the last inequality follows from (56). Similarly, by (55) we have

$$\sum_{k=1}^{N} \frac{m\hat{\theta}_k \tau_k}{2} (\|y - y^{k-1}\|^2 - \|y - y^k\|^2)$$

$$\leq \frac{m\hat{\theta}_N}{2} (\|y - y^0\|^2 - \|y - y^N\|^2) - \sum_{k=1}^{N} \frac{m\hat{\theta}_k \tau_k}{2} \|y_{ik} - y_{ik}^k\|^2.$$  

Combining the above three results, we conclude that

$$\sum_{k=1}^{N} \hat{\theta}_k \Delta_k \leq m\hat{\theta}_N \langle L(x^N - x^{N-1}), y - y^N \rangle + \sum_{k=2}^{N} m\hat{\theta}_k \alpha_k \langle L(x^{k-1} - x^k), y^k - y^{k-1} \rangle$$

$$- \sum_{k=1}^{N} (m - 1) \hat{\theta}_{k+1} \alpha_{k+1} \langle L(x^{k} - x^{k-1}), y^k - y^{k-1} \rangle$$

$$+ m\hat{\theta}_1 \eta_1 V(x^0, x) - m\hat{\theta}_N \eta_N V(x^N, x) - \sum_{k=1}^{N} m\hat{\theta}_k \eta_k V_j(x_{jk}^{-1}, x_{jk})$$

$$+ \frac{m\hat{\theta}_N}{2} (\|y - y^0\|^2 - \|y - y^N\|^2) - \sum_{k=1}^{N} \frac{m\hat{\theta}_k \tau_k}{2} \|y_{ik} - y_{ik}^k\|^2$$

$$\leq \sum_{k=1}^{N} \left\{ m\hat{\theta}_k \alpha_k L_{ik,jk-1} \langle x_{jk-1}^{k-1} - x_{jk-1}^k, y_{ik}^k - y_{ik}^{k-1} \rangle - \frac{m\hat{\theta}_k \eta_k}{4} \|y_{ik}^{k-1} - y_{ik}^k\|^2 \right\}$$

$$+ \sum_{k=1}^{N} \left\{ (m - 1) \hat{\theta}_{k+1} \alpha_{k+1} L_{ik,jk} \langle x_{jk}^k - x_{jk}^{k-1}, y_{ik}^k - y_{ik}^{k-1} \rangle - \frac{m\hat{\theta}_k \eta_k}{4} \|x_{jk}^{k-1} - x_{jk}^k\|^2 \right\}$$

$$+ \frac{m\hat{\theta}_N}{2} \left\{ (\|y - y^0\|^2 - \|y - y^N\|^2) - \sum_{k=1}^{N} \frac{m\hat{\theta}_k \tau_k}{2} \|y_{ik}^{k-1} - y_{ik}^k\|^2 \right\}.$$  

Note that by (58) and the fact that $b(u, v) - a\|v\|^2/2 \leq b^2\|u\|^2/(2a), \forall a > 0$, for all $k \geq 2$, we have

$$m\hat{\theta}_k \alpha_k L_{ik,jk-1} \langle x_{jk-1}^{k-1} - x_{jk-1}^k, y_{ik}^k - y_{ik}^{k-1} \rangle - \frac{m\hat{\theta}_k \eta_k}{4} \|y_{ik}^{k-1} - y_{ik}^k\|^2$$

$$\leq m \left( \frac{\hat{\theta}_k \alpha_k}{\hat{\theta}_{k-1} \alpha_{k-1}} \right) \|y_{ik}^{k-1} - y_{ik}^k\|^2 \leq 0.$$  

Similarly, by (59) for all $k \geq 1$, we have

$$(m - 1) \hat{\theta}_{k+1} \alpha_{k+1} L_{ik,jk} \langle x_{jk}^k - x_{jk}^{k-1}, y_{ik}^k - y_{ik}^{k-1} \rangle - \frac{m\hat{\theta}_k \eta_k}{4} \|x_{jk}^{k-1} - x_{jk}^k\|^2 - \frac{m\hat{\theta}_k \tau_k}{4} \|y_{ik}^{k-1} - y_{ik}^k\|^2 \leq 0.
Hence, combining the above three inequalities, we conclude that
\[
\sum_{k=1}^{N} \hat{\theta}_k \Delta_k \leq m \hat{\theta}_N \langle L(x^N - x^{N-1}), y - y^N \rangle - \frac{m \hat{\theta}_N \eta N}{4} \| x_{jN}^{N-1} - x_{jN}^N \|^2 \\
+ m \hat{\theta}_1 \eta_1 \langle V(0, x), x \rangle + \frac{m \hat{\theta}_N \tau_1}{2} \{ \| y - y^0 \|^2 - \| y - y^N \|^2 \} \\
\leq m \hat{\theta}_N \langle L(x^{N-1} - x^N), y^N \rangle - \frac{m \hat{\theta}_N \eta N}{4} \| x_{jN}^{N-1} - x_{jN}^N \|^2 - \frac{m \hat{\theta}_N \tau N}{2} \| y^N \|^2 \\
+ m \hat{\theta}_1 \eta_1 \langle V(0, x), x \rangle + \frac{\hat{\theta}_1 \tau_1}{2} \| y^0 \|^2 + m \langle \hat{\theta}_N L(x^N - x^{N-1}) + \hat{\theta}_1 \tau_1 (y^N - y^0), y \rangle \\
\leq m \hat{\theta}_N \sum_{i=1}^{m} \left( \frac{\hat{C}_i^2}{\eta N} - \frac{\tau N}{2} \right) \| y_i^N \|^2 + m \langle \hat{\theta}_N L(x^N - x^{N-1}) + \hat{\theta}_1 \tau_1 y_N^N, y \rangle \\
+ m \hat{\theta}_1 \eta_1 \langle V(0, x), x \rangle,
\] (65)
where the second inequality follows from (8) and the fact that \( b(u, v) - a \| v \|^2 / 2 \leq b^2 \| u \|^2 / (2a), \forall a > 0 \), and the last inequality also follows from the fact and \( y^0 = 0 \). In view of (59) and (63), we obtain
\[
E_{[i_k, j_k]} \{ Q(z^N, z) \} \leq \hat{\theta}_0 Q_0 (x, y) + m \hat{\theta}_1 \eta_1 \langle V(0, x), x \rangle \\
+ E_{[i_k, j_k]} \left\{ m \langle \hat{\theta}_N L(x^N - x^{N-1}) + \hat{\theta}_1 \tau_1 y^N, y \rangle \right\} \\
= \hat{\theta}_0 \langle F(0^0) - F(x), x \rangle + m \hat{\theta}_1 \eta_1 \langle V(0, x), x \rangle \\
+ E_{[i_k, j_k]} \left\{ \langle \hat{\theta}_0 L x^0 + m \hat{\theta}_N L(x^N - x^{N-1}) + m \hat{\theta}_1 \tau_1 y_N^N, y \rangle \right\},
\] (66)
where the last equality follows from the definition of \( Q_0 \) in (52). The result in (60) immediately follows from the above relation. Furthermore, from (63), (65), (55) and the facts that \( Q(z^N, z^*) \geq 0, y^0 = 0 \), we have
\[
0 \leq E_{[i_k, j_k]} \{ Q(z^N, z^*) \} \leq \hat{\theta}_0 Q_0 (x^*, y^*) + m \hat{\theta}_1 \eta_1 \langle V(0, x^*), x^* \rangle + \frac{m \hat{\theta}_N \tau_1}{2} \| y^* \|^2 \\
+ E_{[i_k, j_k]} \left\{ m \hat{\theta}_N \langle L(x^N - x^{N-1}), y^* - y^N \rangle - \frac{m \hat{\theta}_N \eta N}{4} \| x_{jN}^{N-1} - x_{jN}^N \|^2 \right\} \\
- E_{[i_k, j_k]} \frac{m \hat{\theta}_N \tau N}{2} \{ \| y^* - y^N \|^2 \} \\
\leq \hat{\theta}_0 Q_0 (x^*, y^*) + m \hat{\theta}_1 \eta_1 \langle V(0, x^*), x^* \rangle + \frac{m \hat{\theta}_N \tau_1}{2} \| y^* \|^2 \\
+ E_{[i_k, j_k]} \left\{ \sum_{i=1}^{m} m \hat{\theta}_N E_{i, j_N} (x_{jN}^N - x_{jN}^{N-1}, y_i^N - y_i^0) \right\} \\
- E_{[i_k, j_k]} \left\{ \frac{m \hat{\theta}_N \tau N}{2} \sum_{i=1}^{m} \| y_i^N - y_i^0 \|^2 + \frac{m \hat{\theta}_N \eta N}{4} \| x_{jN}^{N-1} - x_{jN}^N \|^2 \right\},
\]
which together with (52) and the fact that \( b(u, v) - a \| v \|^2 / 2 \leq b^2 \| u \|^2 / (2a), \forall a > 0 \) imply that
\[
\frac{m \hat{\theta}_N \eta N}{4} E_{[i_k, j_k]} \| x_{jN}^{N-1} - x_{jN}^N \|^2 \leq \hat{\theta}_0 \langle F(0^0) - F(x^*), \langle L x^0, y^* \rangle \rangle + m \hat{\theta}_1 \eta_1 \langle V(0, x^*), x^* \rangle + \frac{m \hat{\theta}_N \tau_1}{2} \| y^* \|^2 \\
+ E_{[i_k, j_k]} \left\{ \frac{m \hat{\theta}_N \eta N^2}{\tau N \max} \| x_{jN}^{N-1} - x_{jN}^N \|^2 \right\},
\]
where the last inequality follows from the definition of \( L \) in (3). Similarly, we obtain
\[
\frac{m \hat{\theta}_N \tau N}{2} E_{[i_k, j_k]} \| y^* - y^N \|^2 \leq \hat{\theta}_0 \langle F(0^0) - F(x^*), \langle L x^0, y^* \rangle \rangle + m \hat{\theta}_1 \eta_1 \langle V(0, x^*), x^* \rangle + \frac{m \hat{\theta}_N \tau_1}{2} \| y^* \|^2 \\
+ E_{[i_k, j_k]} \left\{ \frac{m \hat{\theta}_N \eta N^2}{\tau N \max} \| y^* - y^N \|^2 \right\},
\]
which implies the result in (62). \( \Box \)
Proof of Theorem 1. Let us set \( \hat{\theta}_k \) as follow
\[
\hat{\theta}_k = \begin{cases} 
\frac{m}{N+m}, & k = 0, \\
\frac{1}{N+m}, & k = 1, \ldots, N.
\end{cases}
\] (67)
Therefore, it is easy to check that (16) satisfies conditions (56)-(59). Also note that by (49), we have
\[
\theta_k = \begin{cases} 
\frac{1}{N+m}, & k = 1, \ldots, N-1, \\
\frac{m}{N+m}, & k = N,
\end{cases}
\] (68)
which implies that \( \bar{x}^N = \frac{1}{N+m}(\sum_{k=0}^{N-1} x^k + m x^N) \). By plugging the parameter setting in (60), we have
\[
\mathbb{E}_{[ik,jk]}\{Q(\bar{x}^N; x^*, y)\} \leq \frac{m}{N+m} \left[ F(x^0) - F(x^*) + 2md_{max} V(x^0, x^*) \right] + \mathbb{E}_{[ik,jk]}\{(s, y)\}. \tag{69}
\]
Observe that from (61) and (16),
\[
\mathbb{E}_{[ik,jk]}\{\|s\|\} \leq \frac{m}{N+m} \mathbb{E}_{[ik,jk]}\left[ \|Lx^0\| + 2d_{max}\|x_{jn}^N - x_{jn}^{N-1}\| + 2md_{max}(\|y^* - y^N\| + \|y^*\|) \right].
\]
By (62), (16) and Jensen’s inequality, we have
\[
(\mathbb{E}\{\|x_{jn}^N - x_{jn}^{N-1}\|\})^2 \leq \mathbb{E}\{\|x_{jn}^N - x_{jn}^{N-1}\|^2\} \leq 4 \left[ \frac{F(x^0) - F(x^*) + (Lx^0, y^*)}{md_{max}} + 2V(x^0, x^*) + \|y^*\|^2 \right],
\]
\[
(\mathbb{E}\{\|y^* - y^N\|\})^2 \leq \mathbb{E}\{\|y^* - y^N\|^2\} \leq 2 \left[ \frac{F(x^0) - F(x^*) + (Lx^0, y^*)}{md_{max}} + 2V(x^0, x^*) + \|y^*\|^2 \right].
\]
Hence, in view of the above three inequalities, we conclude that
\[
\mathbb{E}_{[ik,jk]}\{\|s\|\} \leq \frac{m}{N+m} \left\{ \|Lx^0\| + 2md_{max}\|y^*\| \\
+ 7md_{max} \sqrt{\frac{F(x^0) - F(x^*) + (Lx^0, y^*)}{md_{max}} + 2V(x^0, x^*) + \|y^*\|^2} \right\}
= O \left\{ \frac{m}{N+m} \left[ \|Lx^0\| + md_{max}\|y^*\| + md_{max} \sqrt{\frac{F(x^0) - F(x^*) + (Lx^0, y^*)}{md_{max}} + V(x^0, x^*)} \right] \right\}.
\]
Furthermore, by (69) we have
\[
\mathbb{E}_{[ik,jk]}\{g(s, \bar{x}^N)\} \leq \frac{m}{N+m} \left[ F(x^0) - F(x^*) + 2md_{max} V(x^0, x^*) \right].
\]
The results in (17) immediately follow from Proposition 4 and the above two inequalities. \(\square\)

### A.3 Convergence properties of Algorithm 2

Before we provide the proof for Theorem 2 which establishes the main convergence results for AA-SDCS, we state in the following proposition a general result for the ACS procedure. For notation convenience, we use the notations defined the in ACS procedure (cf. Algorithm 2) and let
\[
\Lambda_t := \begin{cases} 
1, & t = 1, \\
(1 - \lambda_t)\Lambda_t, & t \geq 2.
\end{cases}
\] (70)
Proposition 9. If \{\beta_t\} and \{\lambda_t\} in the ACS procedure satisfy
\begin{align}
\lambda_1 &= 1, \\
\mu + \eta + \beta_t &> (C + L)^2, \quad t = 1, \ldots, T, \\
\frac{\beta_t}{\lambda_t} &= \frac{\beta_{t-1}}{\lambda_{t-1}}, \quad t = 1, \ldots, T,
\end{align}
then, under assumptions (19) and (20), for \( u \in U \),
\begin{equation}
\mathbb{E}[\xi] \Phi(u^T) - \Phi(u) \leq \Lambda_T \beta_1 V(u^0, u) - (\Lambda_T \beta_1 + \mu + \eta) \mathbb{E}[\xi] V(u^T, u) + \Lambda_T \sum_{t=1}^{T} \frac{2(M^2 + \sigma^2)\lambda_t^2}{\lambda_t},
\end{equation}
where \( \mathbb{E}[\xi] \) represents taking the expectation over \{\xi^1, \ldots, \xi^T\} and \( \Phi \) is defined as
\begin{equation}
\Phi(u) := \langle w, u \rangle + \phi(u) + \eta V(x, u).
\end{equation}

Proof. Note that in view of (7), (8) and (21), we have
\begin{equation}
\frac{1}{2} \|u_1 - u_2\|^2 \leq V(x, u_1) - V(x, u_2) - \langle \nabla V(x, u_2), u_1 - u_2 \rangle = V(u_2, u_1) \leq \frac{C}{2} \|u_1 - u_2\|^2, \quad \forall u_1, u_2 \in U,
\end{equation}
where \( \nabla V(x, u_2) \) denotes the gradient of \( V(x, \cdot) \) w.r.t. \( u_2 \) for a given \( x \), and the above result together with (2) imply \( \phi(\cdot) \) satisfies
\begin{equation}
\frac{\mu + 2}{2} \|u_1 - u_2\|^2 \leq \Phi(u_1) - \Phi(u_2) - \langle \nabla \Phi(u_2), u_1 - u_2 \rangle \leq \frac{C + L}{2} \|u_1 - u_2\|^2 + M \|u_1 - u_2\|, \quad \forall u_1, u_2 \in U.
\end{equation}
Hence, by the proof of Theorem 1 in [9], we can conclude that
\begin{equation}
\mathbb{E}[\xi] \Phi(u^T) - \Phi(u) \leq \Lambda_T \beta_1 V(u^0, u) - (\Lambda_T \beta_1 + \mu + \eta) \mathbb{E}[\xi] V(u^T, u) + \Lambda_T \sum_{t=1}^{T} \frac{2\lambda_t^2(M^2 + \sigma^2)}{\lambda_t}.
\end{equation}
\hfill \Box

We are now ready to present the main convergence property of the AA-SDCS method stated in Algorithm 2 when the objective functions \( f_i, i = 1, \ldots, m \), are general convex.

Proposition 10. Let the iterates \((x^k, \tilde{x}^k)\) and \(y^k, k = 1, \ldots, N\), be generated by Algorithm 2 and be defined as in (47), respectively, and let \( \tilde{z}^N := (\sum_{k=0}^{N} \theta_k \tilde{x}^k, \sum_{k=0}^{N} \tilde{\gamma} \tilde{x}^k) \). Assume that the objective \( f_i, i = 1, \ldots, m \), are general convex functions, i.e., \( \mu = 0, L, M \geq 0 \) in [2]. Let the parameters \{\alpha_k\}, \{\tau_k\}, and \{\eta_k\} in Algorithm 2 satisfy (55) and
\begin{align}
\hat{\theta}_k \left( \frac{c + L}{T_k(T_{k+1})} + \eta_k \right) &\leq \hat{\theta}_{k-1} \left( \frac{c + L}{T_{k-1}(T_{k-1}+1)} + \eta_{k-1} \right), \quad k = 2, \ldots, N, \\
\alpha_k \hat{\theta}_k &= \hat{\theta}_{k-1}, \quad k = 2, \ldots, N, \\
8m \alpha_k \beta_{max}^2 &\leq \eta_k \tau_k, \quad k = 2, \ldots, N, \\
8(m - 1)^2 \beta_{max}^2 &\leq m \eta_k \tau_k, \quad k = 1, \ldots, N,
\end{align}
where \{\hat{\theta}_k\} is some given weight sequence. Let the parameters \{\lambda_t\} and \{\beta_t\} in the ACS procedure of Algorithm 2 be set to (33). Then, for any \( z := (x, y) \in X^m \times \mathbb{R}^{md} \), we have
\begin{equation}
\mathbb{E}\{Q(\tilde{z}^N; z)\} \leq \hat{\theta}_0 (F(x^0) - F(x)) + m \hat{\theta}_1 \left( \frac{4(c + L)}{T_{1}(T_{1}+1)} + \eta_1 \right) V(x^0, x) + \mathbb{E}\{(s, y)\} + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{\eta_k (T_k+1)},
\end{equation}
where \( \beta_t = \beta_t \).
where $\mathbb{E}$ represents taking the expectation over all random variables, $Q$ is defined in (43) and $s$ are defined as
\begin{align}
s := \hat{\theta}_0 L x^0 + \hat{\theta}_N L (\hat{x}^N - x^{N-1}) + m \hat{\theta}_1 \tau_1 y^N.
\end{align}

Furthermore, for any saddle point $(x^*, y^*)$ of [5], we have
\begin{align}
\frac{\hat{\theta}_N}{4} \left( 1 - \frac{2|L|^2}{m \tilde{\eta}_N \tau_N} \right) \max \left\{ \eta_N \mathbb{E} \|\hat{x}^N - x^{N-1}\|^2, 2m \tau_N \mathbb{E} \|y^* - y^N\|^2 \right\} \\
\leq \hat{\theta}_0 (F(x^0) - F(x^*)) + \langle L x^0, y^* \rangle + m \hat{\theta}_1 \left( \frac{4(C+L)}{\tau(T+1)} + \eta \right) V(x^0, x^*) \\
+ \frac{m \hat{\theta}_1 \tau_1}{2} \|y^*\|^2 + \sum_{k=1}^N \frac{8m(M^2 + \sigma^2) \hat{\theta}_k}{(t_k + 1) \eta_k}.
\end{align}

Proof. Since $f_i$’s are general convex function, we have $\mu = 0$ and $L, M \geq 0$ (cf. (2)). Also note that $\lambda_t$ and $\beta_t$ defined in (33) satisfy condition (71)-(73). Therefore, substituting $\phi := f_i$, and $\lambda_t$ and $\beta_t$, relation (74) can be rewritten as the following:\footnote{We added the subscript $i$ to emphasize that this inequality holds for any agent $i \in N$ with $\phi = f_i$. More specifically, $\Phi_i(u_i) := \langle w_i, u_i \rangle + f_i(u_i) + \eta V_i(x_i, u_i)$.}

\begin{align}
\mathbb{E}_{[\xi]} \Phi_{i}(u_i^T) - \Phi_{i}(u_i) \leq \Lambda_T \beta_1 V_i(\hat{x}_i^+, \hat{y}_i) + (\Lambda_T \beta_1 + \eta) \mathbb{E}_{[\xi]} V_i(\hat{x}_i^+, \hat{y}_i) + \Lambda_T \sum_{t=1}^T \frac{2(M^2 + \sigma^2) \lambda_t^2}{(t_k + 1) \eta_k},
\end{align}

Summing up the above inequality from $i \in [m]$, and using the definitions of $\hat{x}_i^+$ and $\hat{x}_i^-$ in (48), we obtain
\begin{align}
\mathbb{E}_{[\xi]} \Phi^k(\hat{x}_i^+ - \hat{x}_i^- (x) \leq \Lambda_T \beta_1 V(x^{k-1}, x) + (\Lambda_T \beta_1 + \eta) \mathbb{E}_{[\xi]} V(\hat{x}_i^+, x) + \Lambda_T \sum_{t=1}^T \frac{2(M^2 + \sigma^2) \lambda_t^2}{(t_k + 1) \eta_k},
\end{align}

where $\Phi^k(x) = \langle L x, \hat{y}_i^- \rangle + F(x) + \eta_k V(x^{k-1}, x)$. By plugging into the above relation the values of $\lambda_t$ and $\beta_t$ in (33), together with the definition of $\Phi^k(x)$ and rearranging the terms, we have $\forall x \in X^m$

\begin{align}
\mathbb{E}_{[\xi]} \left\{ \langle L (\hat{x}_i^+ - x), \hat{y}_i^+ \rangle + F(\hat{x}_i^+ - F(x) \right\} &\leq \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta \right) \mathbb{E}_{[\xi]} \left[ V(x^{k-1}, x) - V(\hat{x}_i^+, x) \right] \\
&- \eta_k \mathbb{E}_{[\xi]} \left\{ V(x^{k-1}, \hat{x}_i^-) \right\} + \frac{8m(M^2 + \sigma^2)}{(T_k + 1) \eta_k}.
\end{align}

By the definitions of $Q$ in (43) and $z^N$, and the convexity of $F(\cdot)$, we have

\begin{align}
Q(z^N, z) \leq \sum_{k=0}^N \theta_k [F(x^k) - F(x) + \langle L x^k, y \rangle - \sum_{k=0}^N \hat{\theta}_k (L x, \hat{y}_i)].
\end{align}

Taking expectation over $i_1, i_2, \ldots, i_k, j_k$ and applying Lemma 6 we obtain

\begin{align}
\mathbb{E}_{[\xi, \eta]} \left\{ Q(z^N, z) \right\} \leq \mathbb{E}_{[\xi, \eta]} \left\{ \sum_{k=0}^N \hat{\theta}_k [F(\hat{x}_i^+ - F(x) + \langle L x^k, y \rangle - \langle L x, \hat{y}_i \rangle] \right\}.
\end{align}

Moreover, if we replace (53) by (83) in Lemma 7 we can conclude the following result similar to (54)

\begin{align}
\mathbb{E} \{ Q(z^N, z) \} \leq \hat{\theta}_0 Q_0(x, y) + \sum_{k=1}^N \frac{8m(M^2 + \sigma^2) \hat{\theta}_k}{(T_k + 1) \eta_k} \\
+ \sum_{k=1}^N \left\{ \hat{\theta}_k [F(\hat{x}_i^+ - \hat{x}_i^-) - \hat{\theta}_k \eta_k V(x^{k-1}, \hat{x}_i^-)] \right\} \\
+ \mathbb{E} \left\{ \sum_{k=1}^N m \hat{\theta}_k \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta \right) \left[ V(x^{k-1}, x) - V(x^k, x) \right] \right\} \\
+ \mathbb{E} \left\{ \sum_{k=1}^N \frac{m \hat{\theta}_k \tau_1}{2} \|y - y^{k-1}\|^2 - \|y - y^k\|^2 - \|y_i^k - y_i^k\|^2 \right\}.
\end{align}
where $\mathbb{E}$ represents taking the expectation over all random variables. Therefore, we have

$$
\mathbb{E}\{Q(z^N; z)\} \leq \hat{\theta}_0 Q_0(x, y) + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2) \hat{\theta}_k}{(T_k + 1) \eta_k} + \mathbb{E} \left\{ \sum_{k=1}^{N} \hat{\theta}_k \tilde{\Delta}_k \right\},
$$

where

$$
\tilde{\Delta}_k := \langle L(\hat{x}^k - \hat{x}^k), y - \hat{y}^k \rangle + m \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta_k \right) \mathbb{V}(x^{k-1}, x) - \mathbb{V}(x^k, x) - \eta_k \mathbb{V}(x^{k-1}, \hat{x}^k) + \frac{m \sigma^2}{2} \|y - y^{k-1}\|^2 - \|y - y^k\|^2 - \|y_{i_k^k}^k - y_{i_k^k}\|^2.
$$

We now provide a bound for $\mathbb{E}\{\sum_{k=1}^{N} \hat{\theta}_k \tilde{\Delta}_k\}$. Observe that $\tilde{\Delta}_k$ is different from $\Delta_k$ defined in (64) in first three terms, however, they can be bounded via the same technique. Note that by (22), we obtain

$$
\mathbb{E} \left\{ \sum_{k=1}^{N} \hat{\theta}_k \langle L(\hat{x}^k - \hat{x}^k), y - \hat{y}^k \rangle \right\} = \mathbb{E} \left\{ \sum_{k=1}^{N} \hat{\theta}_k \langle L((\hat{x}^k - x^{k-1}) - \alpha_k (m\hat{x}^k - (m-1)\hat{x}^{k-2} - x^{k-2}), y - \hat{y}^k) \rangle \right\} = \mathbb{E} \left\{ \sum_{k=1}^{N} \left[ \hat{\theta}_k \langle L(\hat{x}^k - x^{k-1}), y - \hat{y}^k \rangle - \hat{\theta}_k \alpha_k \langle L(\hat{x}^{k-1} - x^{k-2}), y - \hat{y}^{k-1} \rangle \right] \right\} + \mathbb{E} \left\{ \sum_{k=1}^{N} \hat{\theta}_k \alpha_k \langle L(\hat{x}^{k-1} - x^{k-2}), \hat{y}^k - \hat{y}^{k-1} \rangle \right\}
$$

which together with (76) and (55) imply that

$$
\mathbb{E}\{\sum_{k=1}^{N} \hat{\theta}_k \tilde{\Delta}_k\} \leq \mathbb{E} \left\{ \hat{\theta}_N \langle L(\hat{x}^N - x^{N-1}), y - y^N \rangle + \sum_{k=2}^{N} m \hat{\theta}_k \alpha_k \langle L(\hat{x}^{k-1} - x^{k-2}), y^k - y^{k-1} \rangle \right\} - \mathbb{E} \left\{ \sum_{k=1}^{N} (m-1) \hat{\theta}_k \alpha_{k+1} \langle L(\hat{x}^k - x^{k-1}), y^k - y^{k-1} \rangle + \sum_{k=1}^{N} \hat{\theta}_k \eta_k \mathbb{V}(x^{k-1}, \hat{x}^k) \right\} + \mathbb{E} \left\{ m \hat{\theta}_1 \left( \frac{4(C+L)}{T_1(T_1 + 1)} + \eta_1 \right) \mathbb{V}(x^0, x) - m \hat{\theta}_N \left( \frac{4(C+L)}{T_N(T_N + 1)} + \eta_N \right) \mathbb{V}(x^N, x) \right\} + \mathbb{E} \left\{ \frac{m \hat{\theta}_N}{2} \|y - y^0\|^2 - \frac{m \hat{\theta}_N}{2} \|y - y^N\|^2 - \sum_{k=1}^{N} \frac{m \hat{\theta}_k}{2} \|y_{i_k^k} - y_{i_k^k}\|^2 \right\}
$$

Noting that by the fact that $b(u, v) - a\|v\|^2/2 \leq b^2\|u\|^2/(2a), \forall a > 0$ and (77) and (78), for all $k \geq 2$, we have

$$
\sum_{j=1}^{m} \left\{ m \hat{\theta}_k \alpha_k \mathcal{L}_{i,j}(\hat{x}^k - x^k, y^k, y^{k-1}) - \frac{\hat{\theta}_{k-1} \eta_{k-1}}{4} \|z_j^k - x_j^{k-2}\|^2 - \frac{\hat{\theta}_{k-1} \eta_{k-1}}{4} \|z_j^k - x_j^{k-2}\|^2 \right\} - \frac{m \hat{\theta}_k}{4} \|y_{i_k}^k - y_{i_k}^k\|^2 \leq 0.
$$
Similarly, by (79) for all $k \geq 1$, we have

$$\sum_{j=1}^{m} \left( (m - 1)\dot{\theta}_{k+1} \alpha_{k+1} L_{i,j} \langle \dot{x}_{j}^{k}, y_{j}^{k} - y_{ik}^{-1} - \frac{\dot{\theta}_{k}}{4} \|x_{j}^{k} - \dot{x}_{j}^{k}\|^2 \right) - \frac{m\dot{\theta}_{k}}{4} \|y_{ik}^{k} - y_{ik}^{-1}\|^2 \leq 0.$$  

Hence, in view of the above three results, we obtain

$$\mathbb{E} \left\{ \sum_{k=1}^{N} \dot{\theta}_{k} \dot{\Delta}_{k} \right\} \leq \mathbb{E} \left\{ \langle L(\dot{x}_{k}^{N} - x^{N-1}), y - y^{N} \rangle \right\} \leq \frac{\dot{\theta}_{N,N}}{4} \mathbb{E} \left\{ \|\dot{x}_{k}^{N} - x^{N-1}\|^2 \right\} + m\dot{\theta}_{1} \left( \frac{4(C+L)}{T_{1}(T_{1}+1)} + \eta_{1} \right) \mathbb{E} \left\{ \|y^{0}\|^2 - \|y - y^{N}\|^2 \right\}.$$  

(86)

Following the same procedure as we used in Proposition 8 (cf. (66)), and using the above result and (84), we can conclude that

$$\mathbb{E} \left\{ Q(\dot{z}_{N}, z) \right\} \leq \mathbb{E} \left\{ \dot{\theta}_{0} (F(x^{N}) - F(x^{*})) + m\dot{\theta}_{1} \left( \frac{4(C+L)}{T_{1}(T_{1}+1)} + \eta_{1} \right) \mathbb{E} \left\{ \|x^{0}, x\|^2 \|y^{0}\|^2 \right\} + \mathbb{E} \left\{ \langle \dot{\theta}_{0} Lx^{0} + \dot{\theta}_{N} L(\dot{x}_{N}^{N} - x^{N-1}) + m\dot{\theta}_{1} \tau_{N} y^{0}, y \rangle \right\},$$  

which implies the result in (80). Furthermore, from (84), (86), (55), and the fact that $Q(\dot{z}^{N}, z^{*}) \geq 0$, $y^{0} = 0$, we have

$$0 \leq \mathbb{E} \left\{ Q(\dot{z}_{N}, z^{*}) \right\} \leq \dot{\theta}_{0} Q_{0}(x^{*}, y^{*}) + m\dot{\theta}_{1} \left( \frac{4(C+L)}{T_{1}(T_{1}+1)} + \eta_{1} \right) \mathbb{E} \left\{ \|x^{0}, x^{*}\|^2 \|y^{0}\|^2 \right\} + \mathbb{E} \left\{ \langle \dot{\theta}_{0} Lx^{0} + \dot{\theta}_{N} L(\dot{x}^{N} - x^{N-1}) + m\dot{\theta}_{1} \tau_{N} y^{0}, y \rangle \right\},$$  

which together with the fact that $b(u, v) = a\|u\|^2 / 2 \leq b\|u\|^2 / (2a), \forall a > 0$ and (82) imply that

$$\frac{\dot{\theta}_{N,N}}{4} \mathbb{E} \left\{ \|\dot{x}^{N-1} - x^{N}\|^2 \leq \dot{\theta}_{0} (F(x^{0}) - F(x^{*}) + \langle Lx^{0}, y^{*} \rangle) + m\dot{\theta}_{1} \left( \frac{4(C+L)}{T_{1}(T_{1}+1)} + \eta_{1} \right) \mathbb{E} \left\{ \|x^{0}, x^{*}\|^2 \|y^{0}\|^2 \right\} + \mathbb{E} \left\{ \langle \dot{\theta}_{0} Lx^{0} + \dot{\theta}_{N} L(\dot{x}^{N} - x^{N-1}) + m\dot{\theta}_{1} \tau_{N} y^{0}, y \rangle \right\},$$  

(87)

Similarly, we can obtain

$$\frac{m\dot{\theta}_{N,N}}{2} \mathbb{E} \left\{ \|y^{*} - y^{N}\|^2 \right\} \leq \dot{\theta}_{0} (F(x^{0}) - F(x^{*}) + \langle Lx^{0}, y^{*} \rangle) + m\dot{\theta}_{1} \left( \frac{4(C+L)}{T_{1}(T_{1}+1)} + \eta_{1} \right) \mathbb{E} \left\{ \|x^{0}, x^{*}\|^2 \|y^{0}\|^2 \right\} + \mathbb{E} \left\{ \langle \dot{\theta}_{0} Lx^{0} + \dot{\theta}_{N} L(\dot{x}^{N} - x^{N-1}) + m\dot{\theta}_{1} \tau_{N} y^{0}, y \rangle \right\},$$  

which implies the result in (82).  

\[ \square \]

**Proof of Theorem 2** Let us set $\{\dot{\theta}_{k}\}$ as (67). Therefore, it is easy to check that parameter settings (33) and (34) satisfies conditions (71) - (73), (55), and (76) - (79). Also note that by (49), $\{\dot{\theta}_{k}\}$ is given by (68), which implies that $\dot{x}_{k}^{N} = \frac{1}{N+m}(\sum_{k=0}^{N-1} \dot{x}^{k} + m\dot{x}^{N}).$ By plugging the parameter setting in (80), we have

$$\mathbb{E} \left\{ Q(\dot{z}_{N}^{N}, x^{*}, y) \right\} \leq \frac{m}{N+m} \left[ F(x^{0}) - F(x^{*}) + 8m\max_{V} \mathbb{E} \left\{ \|x^{0}, x^{*}\|^2 + \frac{2d}{m} \right\} + \mathbb{E} \left\{ \langle s, y \rangle \right\}. \right.$$  

(88)
Observe that from (81) and (34)
\[
E\{\|s\|\} \leq \frac{m}{N+m} E\left[\|Ls_0\| + \frac{\|L\|}{m} \|\hat{x}^N - x^{N-1}\| + 2d_{\max}(\|y^* - y^N\| + \|y^*\|)\right].
\]

By (82), (34), and Jensen’s inequality, we have
\[
(\mathbb{E}\{\|\hat{x}^N - x^{N-1}\|\})^2 \leq \mathbb{E}\{\|\hat{x}^N - x^{N-1}\|^2\} \leq \frac{2(F(x^0) - F(x^*) + \langle Lx_0, y^* \rangle)}{d_{\max}} + 16m \mathbf{V}(x^0, x^*) + 2\|y^*\|^2 + \frac{4D}{md_{\max}},
\]
\[
(\mathbb{E}\{\|y^* - y^N\|\})^2 \leq \mathbb{E}\{\|y^* - y^N\|^2\} \leq \frac{2(F(x^0) - F(x^*) + \langle Lx_0, y^* \rangle)}{d_{\max}} + 16m \mathbf{V}(x^0, x^*) + 2\|y^*\|^2 + \frac{4D}{md_{\max}}.
\]

Hence, in view of the above three inequalities, we obtain
\[
\mathbb{E}\{\|s\|\} \leq \frac{m}{N+m} \left\{\|Ls_0\| + 2d_{\max}\|y^*\| + 3d_{\max} \sqrt{\frac{2(F(x^0) - F(x^*) + \langle Lx_0, y^* \rangle)}{d_{\max}} + 16m \mathbf{V}(x^0, x^*) + 2\|y^*\|^2 + \frac{4D}{md_{\max}}}\right\} = \mathcal{O}\left\{\frac{m}{N+m} \left[\|Ls_0\| + d_{\max}\|y^*\| + d_{\max} \sqrt{\frac{F(x^0) - F(x^*) + \langle Lx_0, y^* \rangle}{d_{\max}}} + m \mathbf{V}(x^0, x^*) + \frac{D}{md_{\max}}\right]\right\}.
\]

Furthermore, by (88) we have
\[
E\{g(s, \hat{z}^N)\} \leq \frac{m}{N+m} \left[F(x^0) - F(x^*) + 8md_{\max} \mathbf{V}(x^0, x^*) + \frac{2D}{m}\right].
\]

The results in (35) immediately follow from applying Proposition 4 to the above two inequalities.

In the following proposition, we provide the main convergence property of the AA-SDCS method stated in Algorithm 2 when the objective functions \(f_i, i = 1, \ldots, m\), are strongly convex.

**Proposition 11.** Let the iterates \((\hat{x}^k, \hat{y}^k)\) and \(\tilde{y}^k, k = 1, \ldots, N\), be generated by Algorithm 2 and be defined as in (47) respectively, and let \(\hat{z}^N := \left(\sum_{k=0}^{N} \theta_k x^k, \sum_{k=0}^{N} \tilde{\theta}_k \tilde{y}^k\right)\). Assume that the objective function \(f_i, i = 1, \ldots, m\), are strongly convex functions, i.e., \(\mu > 0\), \(L, M \geq 0\) in (2). Let the parameters \(\{\alpha_k\}\), \(\{\tau_k\}\), and \(\{\eta_k\}\) in Algorithm 2 satisfy (55), (77) - (79),
\[
\hat{\theta}_k \left(\frac{C + L}{\tau_k(T_k + 1)} + \eta_k\right) \leq \theta_{k-1} \left(\frac{C + L}{\tau_{k-1}(T_{k-1} + 1)} + \eta_{k-1} + \mu\right), \quad k = 2, \ldots, N,
\]
where \(\{\hat{\theta}_k\}\) is some given weight sequence. Let the parameters \(\{\lambda_t\}\) and \(\{\beta_t\}\) in the ACS procedure of Algorithm 2 be set to (33). Then, for any \(z := (x, y) \in \mathbb{R}^m \times \mathbb{R}^{md}\), we have
\[
E\{Q(\hat{z}^N; z)\} \leq \hat{\theta}_0(F(x^0) - F(x)) + m\bar{\theta}_1 \left(\frac{4(C + L)}{T_1(T_1 + 1)} + \eta_1\right) \mathbf{V}(x^0, x) + E\{s, y\} + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{\eta_k + \mu(T_k + 1)},
\]
where \(E\) represents the taking expectation over all random variables, \(Q\) and \(s\) are defined in (43) and (81) respectively. Furthermore, for any saddle point \((x^*, y^*)\) of (5), we have
\[
\hat{\theta}_N \left(1 - \frac{2\|L\|^2}{m\eta_N\tau_N}\right) \max \left\{\eta_N \mathbb{E}\|\hat{x}^N - x^{N-1}\|^2, 2m\tau_N \mathbb{E}\|y^* - y^N\|^2\right\} \leq \hat{\theta}_0(F(x^0) - F(x^*) + \langle Lx_0, y^* \rangle) + m\bar{\theta}_1 \left(\frac{4(C + L)}{T_1(T_1 + 1)} + \eta_1\right) \mathbf{V}(x^0, x^*) + \frac{m\bar{\theta}_1}{2}\|y^*\|^2 + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{\eta_k + \mu(T_k + 1)}.
\]
Proof. Since $f_i$’s are strongly convex function, we have $\mu > 0$ and $L, M \geq 0$ (cf. (2)). Observe that $\lambda_t$ and $\beta_t$ defined in (33) satisfy conditions (71)–(73). Therefore, following similar procedure in Proposition 10, in view of Proposition 9, and the definition of $\hat{x}_k^*$ and $\hat{x}_k$ in (18), we can obtain

$$
\mathbb{E}[ \Phi^k(\hat{x}_k) - \Phi^k(x) \leq \Delta_k \beta_1 \mathbf{V}(x^{-1}, x) - (\Delta_k \beta_1 + \mu + \eta_k) \mathbb{E}[ \mathbf{V}(x^{-1}, x) + \Delta_k \sum_{t=1}^{T_k} \frac{2m(M^2 + \sigma^2)\lambda_t^2}{(\mu + \eta_t - (C + L)\lambda_t)}],
$$

where $\Phi^k(x) = \langle \mathbf{L}x, \hat{y}^k \rangle + F(x) + \eta_k \mathbf{V}(x^{-1}, x)$. By plugging into the above relation the values of $\lambda_t$ and $\beta_t$ in (33), together with the definition of $\Phi^k(x)$ and rearranging the terms, we have $\forall x \in X^m$

$$
\mathbb{E}[\phi_k] \left( \langle \mathbf{L}(x^k - x), \hat{y}^k \rangle + F(\hat{x}^k) - F(x) \right) \leq \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta_k \right) \mathbb{E}[ \mathbf{V}(x^{-1}, x) ]
$$

Observe that if we replace (83) by (92) in Proposition 10, we can conclude the following result similar to (84)

$$
\mathbb{E}[Q(z^N, z)] \leq \hat{\theta}_0 Q_0(x, y) + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{(T_k + 1)(\eta_k + \mu)} + \mathbb{E}\left\{ \sum_{k=1}^{N} \hat{\theta}_k \Delta_k \right\},
$$

where $\mathbb{E}$ represents taking the expectation over all random variables and

$$
\Delta_k := \langle \mathbf{L}(x_k^* - \hat{x}_k), y - \hat{y}^k \rangle + m \left[ \frac{4(C+L)}{T_k(T_k + 1)} + \eta_k \right] \mathbf{V}(x^{-1}, x) - \left[ \frac{4(C+L)}{T_k(T_k + 1)} + \eta_k + \mu \right] \mathbf{V}(x, x)
$$

$$
- \eta_k \mathbf{V}(x^{-1}, \hat{x}_k) + \frac{m\tau_k}{2} \left[ \|y - y^{-1}\|^2 - \|y - y^k\|^2 - \|y_k - y^k\|^2 \right].
$$

Since $\Delta_k$ defined above shares a similar structure with $\bar{\Delta}_k$ in (85), we can follow a similar procedure as in Proposition 10 to obtain a bound for $\mathbb{E}[Q(z^N, z)]$. Note that the only difference between (94) and (85) exists in the coefficient of the term $\mathbf{V}(x^{-1}, x)$ and $\mathbf{V}(x, x)$. Hence, by using condition (89) in place of (76), we obtain

$$
\mathbb{E}\left\{ Q(z^N, z) \right\} \leq \hat{\theta}_0 (F(x^0) - F(x)) + m\hat{\theta}_1 \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta_1 \right) \mathbf{V}(x^0, x) + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{(T_k + 1)(\eta_k + \mu)}
$$

$$
+ \mathbb{E}\left\{ \langle \hat{\theta}_0 \mathbf{L}x^0 + \hat{\theta}_N \mathbf{L}(\hat{x}_N - x_{N^{-1}}) + m\hat{\theta}_1 \tau_1 y^N, y \rangle \right\}.
$$

Our result in (90) immediately follows. Following the same procedure as we obtain (87), for any saddle point $z^* = (x^*, y^*)$ of (5), we have

$$
\frac{\hat{\theta}_N}{4} \mathbb{E}[z^N - x^N]^2 \leq \hat{\theta}_0 (F(x^0) - F(x)) + \langle \mathbf{L}x^0, y^* \rangle + m\hat{\theta}_1 \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta_1 \right) \mathbf{V}(x^0, x^*)
$$

$$
+ \frac{m\tau_1}{2} \|y^*\|^2 + \mathbb{E}\left\{ \frac{\hat{\theta}_N}{2m}\|y^*\|^2 \right\} + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{(T_k + 1)(\eta_k + \mu)},
$$

$$
\frac{m\hat{\theta}_N}{2} \mathbb{E}[y^* - y^N]^2 \leq \hat{\theta}_0 (F(x^0) - F(x^*)) + \langle \mathbf{L}x^0, y^* \rangle + m\hat{\theta}_1 \left( \frac{4(C+L)}{T_k(T_k + 1)} + \eta_1 \right) \mathbf{V}(x^0, x^*)
$$

$$
+ \frac{m\tau_1}{2} \|y^*\|^2 + \mathbb{E}\left\{ \frac{m\hat{\theta}_N}{\eta_N}\|y^* - y^N\|^2 \right\} + \sum_{k=1}^{N} \frac{8m(M^2 + \sigma^2)\hat{\theta}_k}{(T_k + 1)(\eta_k + \mu)},
$$

from which the result in (91) follows. □
**Proof of Theorem 3** Let us set

\[
\hat{\theta}_k = \begin{cases} 
\frac{6m^2}{6m^2 + N(N + 6m + 1)}, & k = 0, \\
\frac{2(k+3m)}{6m^2 + N(N + 6m + 1)}, & k = 1, \ldots, N.
\end{cases}
\]  

(95)

Observe from (37) that

\[
\eta_k = \frac{(k+3m-1)\mu}{2} - \frac{C + L}{f_k(T_k + 1)} \geq \frac{(k+3m-1)\mu}{2} - \frac{(k+3m-1)\mu}{4} = \frac{(k+3m+1)\mu}{4}.
\]

Therefore, it is easy to check that parameter settings (33) and (37) satisfy conditions (71) - (73), (55), (77) - (79), and (89). Also by (49), we have

\[
\theta_k = \begin{cases} 
\frac{2(k+2m+1)}{6m^2 + N(N + 6m + 1)}, & k = 1, \ldots, N - 1, \\
\frac{2m(N + 3m)}{6m^2 + N(N + 6m + 1)}, & k = N,
\end{cases}
\]

which implies that \( \tilde{x}^N = \frac{2}{6m^2 + N(N + 6m + 1)}(\sum_{k=0}^{N-1} (k + 2m + 1)x^k + m(N + 3m)x^N) \). By plugging the parameter setting in (90), we have

\[
\mathbb{E}\{Q(z^N; x^*, y)\} \leq \frac{6m^2}{6m^2 + N(N + 6m + 1)} \left[ F(x^0) - F(x^*) + \frac{(3m+1)\mu}{2} \mathbf{V}(x^0, x^*) + \frac{D\mu}{6m^2} \right] + \mathbb{E}\{\langle s, y \rangle\}. 
\]  

(96)

Observe that from (37) and (90),

\[
\mathbb{E}\{||s||\} \leq \frac{2m^2}{6m^2 + N(N + 6m + 1)} \mathbb{E}\left[ 3||Lx^0|| + \frac{(N+3m)||L||}{m^2} ||\tilde{x}^N - x^{N-1}|| + \frac{32d_{max}^2}{\mu} (||y^* - y^N|| + ||y^*||) \right].
\]

In view of (91) and (37), we have

\[
\mathbb{E}\{||\tilde{x}^N - x^{N-1}||^2\} \leq \frac{8}{\theta_N \eta_N} \frac{6m^2}{6m^2 + N(N + 6m + 1)} \left[ F(x^0) - F(x^*) + \langle Lx^0, y^* \rangle + \frac{(3m+1)\mu}{2} \mathbf{V}(x^0, x^*) + \frac{32d_{max}^2}{\mu} \right]
\]

\[
\mathbb{E}\{||y^* - y^N||^2\} \leq \frac{3\mu}{4d_{max}^2} \left[ \frac{3(F(x^0) - F(x^*) + \langle Lx^0, y^* \rangle)}{\mu} + \frac{(3m+1)}{2} \mathbf{V}(x^0, x^*) + \frac{32d_{max}^2}{\mu^2} ||y^*||^2 + \frac{D}{6m^2} \right].
\]

Hence, in view of the above three inequalities and Jensen’s inequality, we obtain

\[
\mathbb{E}\{||s||\} \leq \frac{2m^2}{6m^2 + N(N + 6m + 1)} \left[ 3||Lx^0|| + \frac{32d_{max}^2}{\mu} ||y^*|| \right] + 24d_{max} \left[ \frac{3(F(x^0) - F(x^*) + \langle Lx^0, y^* \rangle)}{\mu} + \frac{(3m+1)}{2} \mathbf{V}(x^0, x^*) + \frac{32d_{max}^2}{\mu^2} ||y^*||^2 + \frac{D}{2m^2} \right]
\]

\[
= O \left\{ \frac{m^2}{m^2 + N^2} \left[ ||Lx^0|| + \frac{d_{max}^2}{\mu} ||y^*|| + d_{max} \left[ \frac{3(F(x^0) - F(x^*) + \langle Lx^0, y^* \rangle)}{\mu} + \frac{(3m+1)}{2} \mathbf{V}(x^0, x^*) + \frac{D}{m^2} \right] \right] \right\}.
\]

Furthermore, by (96) we have

\[
\mathbb{E}\{g(s, \tilde{x}^N)\} \leq \frac{6m^2}{6m^2 + N(N + 6m + 1)} \left[ F(x^0) - F(x^*) + \frac{(3m+1)\mu}{2} \mathbf{V}(x^0, x^*) + \frac{D\mu}{6m^2} \right].
\]

The results in (38) immediately follow from applying Proposition 4 to the above two inequalities. \(\square\)