Sensing Quantum Nature of Primordial Gravitational Waves Using Quantum Electromagnetic Fields

F. Shojaei Arani
Department of Physics, University of Isfahan, Hezar Jerib Str., Isfahan 81746-73441, Iran;
Université de Toulouse, UPS-OMP, IRAP, F-31400 Toulouse, France and CNRS, IRAP, 14, avenue Edouard Belin, F-31400 Toulouse, France

M. Bagheri Harouni
Department of Physics, University of Isfahan, Hezar Jerib Str., Isfahan 81746-73441, Iran;
and Quantum Optics Group, Department of Physics, University of Isfahan, Hezar Jerib Str., Isfahan 81746-73441, Iran;

Brahim Lamine, and Alain Blanchard
Université de Toulouse, UPS-OMP, IRAP, F-31400 Toulouse, France and CNRS, IRAP, 14, avenue Edouard Belin, F-31400 Toulouse, France
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We establish a new formalism to describe the interaction of an optical system with a background of gravitational waves based on optical medium analogy. Besides reproducing the basic formula for the response of a LISA-type interferometer in terms of the phase shift of light, other classical observables of the electromagnetic field such as light-bending, walk-off angle and Stokes parameters are investigated. At the quantum level, this approach enables us to construct a full quantum Hamiltonian describing both electromagnetic and gravitational waves as quantum entities. It turns out that the effective coupling between the electromagnetic field and gravitational wave background resembles an opto-mechanical coupling. This points to a new strategy to evident non-classical nature of gravity within the reach of future optical experiments. As an illustrative application, we investigate the optical quadrature variance as well as the power spectrum of a laser field interacting with a background of primordial gravitational wave. The effect of the relic background of quantized gravitational waves on the optical variance is analogous to the effect of a classical mechanical oscillator, which stabilizes the optical variance. It also acts similar to a Doppler-broadening mechanism which broadens the line-width of the laser field of optical frequency by $10^{-6}$Hz. This line-width broadening and also appearance of side-bands in the spectrum of light could serve as signatures of the highly squeezed nature of primordial gravitational waves.

I. INTRODUCTION

Our findings of the early Universe cosmology are limited predominantly by the photons and neutrinos of the last scattering surface and some signatures of the inflationary scenario on the cosmic microwave background (CMB), when the Universe was opaque. However, recent observations of gravitational waves (GWs) [1] opened a new window to detect the exotic gravitational phenomena, using GWs as a new messenger capable to manifest the entire Universe, even the remnants of the big bang [2].

According to the inflationary theory, primordial quantum fluctuations constitute the origin of the structure of our Universe as well as the primordial tensor fluctuations. The existence of the primordial gravitational waves background (PGWs) is one of the crucial predictions of the inflationary scenario of the early Universe and their detection will provide us a hitherto inaccessible view of the Universe. PGWs span a full range of frequencies and may be detected by various GW detectors such as LIGO [3] and the planned laser interferometer space-based antenna (LISA) [4] or by studies on the B-mode polarization of the cosmic background radiation as is done by the LiteBIRD [5]. On top of that, it is believed that PGWs generated by the strong gravitational pumping engine of the very early Universe had evolved into the so-called squeezed states, and after that, they decoupled from the rest of matter and radiation, and freely propagated throughout the Universe [6]. Hence, detecting the non-classicality of the PGWs would not only provide us valuable information about the quantum mechanical origin of the Universe but also affirm the quantum nature of gravity.
Various theoretical schemes have been proposed to search for the non-classical nature of PGWs and in particular their squeezed essence, using either ground- and space-based gravitational wave interferometers [7, 8], or by observing their imprint on the CMB temperature fluctuations and polarizations [9, 10], or by performing a Hanbury Brown-Twiss interferometry on the PGWs [11–13]. On the experimental side, however, the search for demonstrations of the quantum nature of PGWs is still challenging.

In the present contribution, we develop a potentially powerful new theoretical framework based on optical medium analogy (OMA) [14–17], within which it’s possible to track the effect of classical and quantum features of GWs on the classical and quantum observables of the EM field. In the OMA formalism, the GW background manifests itself as a non-dispersive non-absorptive magneto-dielectric medium, possessing an anisotropic index of refraction. At the classical level, besides recovering the previously known basic formula for the response function of a LISA-like interferometer, we investigate the (i) walk-off angle of light which can be interpreted as the light-bending angle as well as (ii) the Stokes parameters of the EM signal. The remarkable aspect of the presented approach is that we can push one step further, towards a fully quantum description of the coupling of GWs with EM field, where we show that the interaction Hamiltonian resembles the opto-mechanical coupling interaction.

We perform our description in the traceless transverse (TT) gauge and keep only linear terms in $h = \mathcal{O}(|h|)$, where $h_{ij}$ is the perturbation to the flat space-time metric. Since we treat linearized gravity, the weak GWs are quantized safely. In the OMA formalism, GWs enter into play by changing the mode solutions of the optical wave equation through the varying refractive index of the medium, which is treated essentially as a quantum entity. This is similar to the way that the opto-mechanical interaction comes into play through the varying boundary conditions of the EM field inside a cavity with moving end mirror [18]. This similarity becomes more intuitive when we use the proper detector frame to describe GWs. In that frame, the geodesic deviation equation implies that the gravitational wave acts as a tidal force leading to the proper distance between two points to change with time, which can be interpreted as “changing boundary condition” for the EM field.

The opto-mechanical analogy inspires the idea of employing the equivalent full-grown field of quantum optomechanics research, to quest for the experimental demonstration of the quantum nature of PGWs, pretty similar to what was already done to expose the quantum behavior of the mesoscopic mechanical oscillators. For example, it has been shown that revivals of optical squeezing in an opto-mechanical system is a pure quantum effect which reveals the quantum nature of the mechanical oscillator [19]. However, in the gravitational analogue model the revivals of squeezing are impossible to detect, due to the insignificant coupling strength between the EM field and GWs [20]. However, we show that the amplified PGWs induce nontrivial dynamics of the optical variance that could serve as a promising tool to prove the existence of PGWs. Moreover, we will show that the background of PGWs act similar to the Doppler-broadening mechanism which reduce the coherence time of the laser field by $10^{-6}$Hz.

As a result of the quantization of GWs, one can also investigate the equations of motion for the GWs degrees of freedom to see the back-action of the optical system on them. This is the main distinctive perspective between semi-classical and quantum treatments of GWs detectors, which has been considered in a different approach recently [18]. In the present contribution we only derive the dynamics of the EM signal, albeit the dynamics of the GWs could be derived once one has the unitary evolution operator of the system.

This paper is organized as follows. Sec. II provides a brief review on the basics of GWs and sets the required tools of the OMA. In Sec. III, we provide a classical model to study the effect of GWs on the EM field. By solving the wave equation in the presence of an arbitrary background of GWs we evaluate: the refractive index of the medium, the phase shift, the walk-off angle which is interpreted as light-bending angle and the Stokes parameters describing the polarization state of the EM field. The model is promoted to a fully quantum mechanical one in Sec. IV, where both EM and GW fields are treated quantum mechanically. We derive the unitary evolution of the field operators, based on which the optical quadrature variance and power spectrum of light are investigated in Sec. IV B. We conclude the paper in Sec. IV D.

II. PRELIMINARIES

A. Gravitational waves

The weakly perturbed flat space-time metric is described by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowsky metric with signature $(+\ldots)$ and $h_{\mu\nu}$ is the perturbation metric, $|h_{\mu\nu}| \ll 1$. In the linear regime one raises and lowers the indices with $\eta_{\mu\nu}$. The linearized perturbed Einstein equations in the weak-field limit take on the form of the familiar d’Alembert’s equation, which in the transverse-traceless (TT) gauge have the plane-wave solution,

$$h_{ij}(r, t) = \sum_{\gamma=+,-} \int \frac{d^3K}{(2\pi)^3/2} \hat{e}_{ij}^{\gamma}(\hat{K}) \left( h_{\gamma}(K) e^{-i(\Omega t - K \cdot r)} + h_{\gamma}^*(K) e^{i(\Omega t - K \cdot r)} \right),$$  \hspace{1cm} (1)$$

where $i, j = 1, 2, 3$ and $h_{\gamma}(K)$ is the amplitude (strain field) of the wave with polarization tensor $\hat{e}_{ij}^{\gamma}(\hat{K})$, and $(\Omega/c, K)$ represents the 4-momentum of GWs with $|K| = \Omega/c$ and $\hat{K} = K/|K|$. The TT gauge leaves only two independent polarization states $+$ and $\times$, which satisfy
\[ \varepsilon^i_0 = \varepsilon^j_0 K^j = 0 \text{ and } \varepsilon^i_0 \varepsilon^j_0 = 2 \] (where summation convention over repeated indices is implied). The polarization tensors are usually expressed in terms of two unit vectors \((\hat{n}, \hat{m})\) orthogonal to the propagation direction \(\hat{K}\) and to each other. In terms of the Euler’s angles \((\Theta, \Phi)\) we write
\[
\hat{K} = \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}, \quad \hat{n} = \begin{pmatrix} \cos \Theta \cos \Phi \\ \cos \Theta \sin \Phi \\ -\sin \Theta \end{pmatrix}, \quad \hat{m} = \begin{pmatrix} -\sin \Phi \\ \cos \Phi \end{pmatrix},
\]
and the polarization tensors are written as
\[
\hat{e}^i_{ij}[\hat{K}] = \hat{n}_i \hat{n}_j - \hat{m}_i \hat{m}_j, \\
\hat{e}^i_{ij}[\hat{K}] = \hat{n}_i \hat{n}_j + \hat{m}_i \hat{m}_j.
\]
For instance, a pure + polarized monochromatic GW propagating in the z-direction produces perturbations to the flat metric as
\[
ds^2 = -c^2 dt^2 + (1 + h_+) dx^2 + (1 - h_+) dy^2 + dz^2.
\]
Similarly, the cosmological GWs or PGWs are small perturbations to the metric tensor of a homogeneous isotropic expanding Universe [21]. The dynamical equations governing them imply that for sufficiently low frequencies, the mode amplitudes are subject to the super adiabatic amplification which enforces the corresponding vacuum state to evolve into a highly-correlated squeezed state [9]. As a result of amplification, the squeezing parameter grows with time to reach a typically high value, and after the end of amplification stays approximately constant. The today’s spectrum of the cosmological tensor perturbations has a specific shape, within the frequency window of the ground- and space-based interferometers [6, 22] and we will see how we can detect their non-classical essence using an EM field probe.

In this paper we consider the “small detector” condition, i.e., we set \(e^{iK\cdot r} \equiv e^{i\delta} \approx 1\) in Eq. (1), where \(\delta \equiv |K \cdot r|\). For the LIGO and LISA interferometers, one has \(\delta_{\text{LISA}} \approx (10^{-3} - 10^0)\) in frequency range \(\Omega \in (10^{-4} - 10^{-1})\) Hz, and \(\delta_{\text{LIGO}} \approx (10^{-4} - 10^{-1})\) in frequency range \(\Omega \in (10 - 10^3)\) Hz, respectively. Thus as far as \(\delta < 1\), the small detector condition reduces the complexity of the equations of motion. However the presented model can be generalized to big detectors straightforwardly.

B. Optical medium analogy

The OMA concept states that the Maxwell’s equations in a general curved space-time described by the metric \(g_{\mu\nu}\) without external charges and currents, can be replaced by the Maxwell’s equations in flat space-time but in the presence of a bi-anisotropic magneto-dielectric media [14, 23]. In other words, one can put the effect of geometry (gravity) into a certain material medium. This approach helps to better interpret and understand the gravitational phenomena.

The Cartesian coordinates of the displacement and magnetic fields \((D, H)\) are given by [14, 24]
\[
D_i(r, t) = \bar{e}_{ij} E_j(r, t) + (G \times H(r, t))_i, \\
B_i(r, t) = \bar{\mu}_{ij} H_j(r, t) + (G \times E(r, t))_i,
\]
where the permittivity tensor \(\varepsilon_{ij}\) equals the permeability tensor \(\mu_{ij}\) and is given by
\[
\varepsilon_{ij} = \bar{\mu}_{ij} = -\sqrt{g_{ij}} \frac{\omega_{gw}}{g_{00}}, \quad G_i = -\frac{g_{ii}}{g_{00}}
\]
i, j are spatial indices and the double bar on \((\varepsilon, \mu)\) indicates their tensorial nature. Eq. (6) implies that the corresponding medium is generally magneto-dielectric, inhomogeneous and anisotropic. We also see from Eq. (5) that the response is local since no convolution is involved and the medium is therefore non-dispersive and non-absorptive. The vector \(G\) represents the rotation of the reference frame [25], which vanishes for GWs in TT gauge. The general background of GWs given by Eq. (1) thus induces a correction to the vacuum permittivity tensors according to
\[
\bar{e}_{ij} = \bar{\mu}_{ij} = \delta_{ij} - h_{ij}(r, t) \\
= \delta_{ij} - \sum_{\gamma = +, \times} \int \frac{d^3K}{(2\pi)^{3/2}} \varepsilon^i_{ij}[\hat{K}] \left(\hat{h}_\gamma(K) e^{-i(\Omega t - K \cdot r)}
\right.
\left. + \hat{h}^*_\gamma(K) e^{i(\Omega t - K \cdot r)}\right).
\]
Although the medium is intrinsically inhomogeneous and anisotropic, we profit from some essential features of GWs and detectors to reduce the complexity of the equations.

III. CLASSICAL MODEL

A. Solution to wave equation

The Maxwell’s equation in the presence of the medium specified by Eq. (7) are
\[
\nabla \cdot B = 0, \\
\nabla \cdot (\bar{e}(r, t) E(r, t)) = 0, \\
\nabla \times (\bar{\mu}^{-1}(r, t) B(r, t)) = \partial_t (\bar{e}(r, t) E(r, t)), \\
\n\nabla \times E(r, t) = -\partial_t B.
\]
In terms of the scalar and vector potentials one has \(E = -\nabla \phi - \partial_t A\) and \(B = \nabla \times A\). It can be shown that in order to exclude the extra degree of freedom, i.e., to have \(\phi = 0\), it is enough to fix the gauge by imposing the condition \(\nabla \cdot [\bar{e}(r, t) A(r, t)] = 0\) which gives rise to the following equation of motion for the vector potential
\[
\nabla \times [\bar{\mu}^{-1}(r, t) \nabla \times A] = -\bar{e}(r, t) \partial_t^2 A.
\]
In this equation, we have neglected the term proportional to \(\delta A\) in the adiabatic approximation, where \(\Omega_{gw} \ll \omega_{em}\). This approximation is valid for the LIGO and LISA with
operating laser at frequency \( \sim 10^{14} \text{Hz} \). In order to solve Eq. (9), let us expand the field \( \mathbf{A}(\mathbf{r}, t) \) over a complete set of mode functions \( \mathbf{F}_k(\mathbf{r}, t) \) as \[26, 27\]

\[
\mathbf{A}(\mathbf{r}, t) = \sum_k \sqrt{\hbar} \frac{\alpha_k \nu_k(t)}{\omega_k} \mathbf{F}_k(\mathbf{r}, t) + c.c., \tag{10}
\]

where \( k = (\lambda, \mathbf{k}) \) denotes different modes of the EM field with wave vector \( \mathbf{k} \) and polarization \( \lambda \). Here the quantum prefactor \( \hbar \) is included intentionally, but at the classical level one can substitute \( \alpha_k \sqrt{\hbar} \to A_k \). The time-harmonic behaviour of \( \mathbf{A} \) is included in \( \nu_k(t) \) which recasts to \( e^{\pm i \omega_k t} \) in the vacuum case. The mode-functions \( \mathbf{F}_k(\mathbf{r}, t) \) contain the spatial factor \( e^{i \mathbf{k} \cdot \mathbf{r}} \), and we let it to have a slowly varying envelope as well. The slowly varying envelope approximation (SVEA) implies that \[24\]

\[
\left| \frac{\tilde{\mathbf{F}}_k}{\mathbf{F}_k} \right| \ll \omega_k, \tag{11}
\]

By substituting the expansion Eq. (10) into Eq. (9) we get

\[
\nabla \times \left[ \vec{\mu}^{-1}(t) \nabla \times \mathbf{F}_k \right] = - \left( \frac{\dot{\nu}_k(t)}{\nu_k(t)} \right) \tilde{\varepsilon}(t) \mathbf{F}_k, \tag{12}
\]

where the spatial dependency of the permittivity tensors is dropped in the small detector approximation. By defining \( \dot{\nu}_k(\nu_k) \equiv -\omega_k^2(t) \), Eq. (12) becomes an eigenvalue equation for the mode functions \( \mathbf{F}_k(\mathbf{r}, t) \) which satisfy the following orthogonality relation \[26, 27\]

\[
\int d^3 r \tilde{\varepsilon}_{ij}(t) F^*_{ki} F_{kj} = \delta_{kk}. \tag{13}
\]

Due to the local response Eq. (5) the (temporal) Fourier transformation of the permittivity tensors Eq. (7) is proportional to delta function, \( \delta(Q' \pm Q) \), which means that the medium is not dispersive and absorptive in the optical range. One can easily check that the following mode functions

\[
\mathbf{f}_k(\mathbf{r}, t) = \frac{\mathbf{u} e^{i \mathbf{k} \cdot \mathbf{r}}}{\sqrt{m(t)}}, \tag{14}
\]

satisfy the orthogonality condition Eq. (13) as well as the SVEA condition \[28, 29\]. Here \( \mathbf{u} \) is the unit polarization vector and \( m(t) \equiv \tilde{\varepsilon}_{ii}(t) u_i u_j \) is a slowly varying function. The gauge condition \( \nabla \cdot \left[ \tilde{\varepsilon}(\mathbf{r}, t) \mathbf{A}(\mathbf{r}, t) \right] = 0 \) now implies that \( \mathbf{k} \cdot (\tilde{\varepsilon}(t) \mathbf{u}) = 0 \). Plugging the solutions Eq. (14) in Eq. (12) leads to

\[
\mathbf{k} \times \left[ \tilde{\mu}^{-1}(t) \mathbf{k} \times \mathbf{u} \right] = -\omega_k^2(t) \tilde{\varepsilon}(t) \mathbf{u}, \tag{15}
\]

For sake of simplicity and without loss of generality we assume that the wave vector of light lays in the \( x-z \) plane and makes angle \( \theta \) with the \( z \)-axis, \( \mathbf{k} \equiv (\sin \theta, 0, \cos \theta) \). By inserting Eq. (7) into Eq. (15), one finds that the temporal mode functions \( \nu_k(t) \) obey the time-dependent frequency equation

\[
\ddot{\nu}_k(t) = -\omega_k^2(t) \nu_k(t), \tag{16}
\]

where

\[
\omega_k^2(t) = \omega_0^2 \left( 1 + h_{11}(t) \cos^2 \theta + h_{22}(t) + h_{33}(t) \sin^2 \theta - h_{13}(t) \sin 2\theta \right) \equiv \omega_0^2/\eta^2(t). \tag{17}
\]

Here \( \omega_0 = c |\mathbf{k}| \) and \( h_{ij}(t) \) are given by Eq. (1) by setting \( e^{i \mathbf{k} \cdot \mathbf{r}} = 1 \). The function \( \eta(t) \) defined in Eq. (17) is the refractive index of the medium and can be found using the Fresnel equation \[25\],

\[
n(t) = \sqrt{\det(\tilde{\varepsilon})/\varepsilon_{ij} k_i k_j} = 1 - \frac{1}{2} \left( h_{11}(t) \cos^2 \theta + h_{22}(t) + h_{33}(t) \sin^2 \theta - h_{13}(t) \sin 2\theta \right). \tag{18}
\]

Here \( \varepsilon_{ij} \) is given by Eq. (7) and \( k_i \) is the \( i \)-th component of the EM wave vector. That the refractive index of the medium is anisotropic (depends on the EM propagation direction \( \hat{k} \) reflects the fact that the medium is anisotropic and consequently the EM field experiences different refractive indexes depending on it’s propagation direction. When deriving Eq. (17) the identity \( \kappa \cdot (\tilde{\varepsilon}(t) \mathbf{u}) = 0 \) has been used, which implies that

\[
\frac{u_z}{u_x} = -\tan \theta \left( 1 - h_{11} + h_{33} - h_{13} (\cot \theta - \tan \theta) \right). \tag{19}
\]

Here the polarization state of light varies with time, as will be seen later in Sec. III C (yet \( |u_x|^2 + |u_z|^2 = 1 \)). The time dependency of the frequency in Eq. (16) have to be considered with care. This points to the particle production by the gravitational field and in the corresponding quantum field theory, leads to the ambiguity in defining the vacuum state of the EM field \[30\]. However, we can show that for a slowly varying GW background the corresponding Bogoliubov coefficient is zero and the particle production is negligible. Hence the WKB prescription gives rise to the adiabatic vacuum given by \[30\]

\[
\nu_k(t) = \frac{1}{\sqrt{\omega_k(t)}} e^{i \int_0^t \omega_k(t') dt'}. \tag{20}
\]

As far as related to the phase of the EM field, Eq. (20) implies that the effect of GWs on the electromagnetic field is to change its frequency, which was expected in the TT gauge.

In order to clarify the link between the present approach with the previously known results of a LISA-type interferometer, let us write the phase of the wave as \( \Psi \equiv \mathbf{k} \cdot \mathbf{r} - \int_0^t \omega(t') dt' \), the time derivative of which results in the phase velocity of light according to

\[
\frac{dr}{dt} = \frac{\omega(t)}{k} = \frac{c}{\eta(t)}, \tag{21}
\]

which shows the rate of change of positions of constant phase. Eq. (21) could be derived alternatively following the geometrical approach in general relativity; by specifying the light-like geodesics of the perturbed flat space-time metric in the TT gauge \[31, 32\].
B. Walk-off angle and light-bending

Within the OMA framework, the propagation direction of the ray is determined by the Poynting vector, \( \mathbf{S} = \mathbf{E} \times \mathbf{H} \). The Maxwell’s equations Eq. (8) imply that (i) \( \mathbf{k} \perp \mathbf{D} \) and (ii) \( \mathbf{k} \times \mathbf{H} = -\omega_0/n\mathbf{D} \) (in small-detector condition). As is shown in Fig. 1, at the initial time \( \mathbf{E} \parallel \mathbf{D} \). As time elapses, \( \mathbf{D} \) stays perpendicular to the propagation direction \( \mathbf{k} \), but \( \mathbf{E} \) starts to rotate. It turns out the angle between \( \mathbf{E} \) and \( \mathbf{D} \), represented by \( \psi(t) \), is the same as that of \( \mathbf{S} \) and \( \mathbf{k} \), which is known as the “walk-off” angle in birefringence media [33]. Thus the wave vector \( \mathbf{k} \) points perpendicular to the wave fronts while the Poynting vector \( \mathbf{S} \) shows the direction of energy flux. To find the walk-off angle \( \psi(t) \), one can rotate the \( x - z \) plane by angle \( \theta \) and rewrite the polarization vector in the new coordinate system as

\[
\mathbf{u}' = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} u_x (\cos \theta - \sin \theta u_z / u_x) \\ u_y \\ u_x (\sin \theta + \cos \theta u_z / u_x) \end{pmatrix}.
\]

By defining the walk-off angle in the \( x' \)-direction as \( \psi(t) \approx |\tan \psi(t)| = u_{z'}/u_{z'} \), and using Eq. (19) one can show that

\[
\psi(t) \approx (h_{11}(t) - h_{33}(t)) \sin \theta \cos \theta + h_{13}(t) \cos 2\theta.
\]

Similarly, we can define the corresponding walk-off angle in the \( y \)-direction as \( \psi_y \approx \tan \psi_y \equiv u_{z'}/(u_{x'}^2 + u_{z'}^2)^{1/2} \) which would be zero if one sets \( u_y = 0 \) at initial time.

Thus the GWs background acts as a birefringence medium which deflects the propagation direction of energy flux. However, as we have seen, the wave vector \( \mathbf{k} \) is a constant vector in space time. Equivalently, one could use the perturbed geodesic equation for light rays and show that in the small-detector condition that the gradient of metric perturbations vanishes, there would be no deflection of light, i.e., \( \mathbf{k} \) is a constant vector.

C. Stokes parameters of the EM field

As a useful treatment to determine the polarization state of light we investigate the Stokes parameters [34], since they are quadratic in the field strength and can be determined through intensity measurements only. Their measurement completely determine the polarization state of the wave. Using the rotated basis \((\hat{x}', \hat{y}', \hat{z}')\), one can define the Stokes parameters as [34]

\[
\begin{align*}
    s_0 &= |\hat{x}' \cdot \mathbf{E}|^2 + |\hat{y}' \cdot \mathbf{E}|^2 = a_1^2 + a_2^2, \\
    s_1 &= |\hat{x}' \cdot \mathbf{E}|^2 - |\hat{y}' \cdot \mathbf{E}|^2 = a_1^2 - a_2^2, \\
    s_2 &= 2 \Re((\hat{x}' \cdot \mathbf{E})^* (\hat{y}' \cdot \mathbf{E})) = 2a_1a_2 \cos(\delta_2 - \delta_1), \\
    s_3 &= 2 \Im((\hat{x}' \cdot \mathbf{E})^* (\hat{y}' \cdot \mathbf{E})) = 2a_1a_2 \sin(\delta_2 - \delta_1).
\end{align*}
\]

where we have defined \( \hat{x}' \cdot \mathbf{E} = a_1 e^{i\delta_1}, \hat{y}' \cdot \mathbf{E} = a_2 e^{i\delta_2} \) and \( a_1, a_2, \delta_1, \delta_2 \) are real numbers. With the help of Eq. (19) and assuming \( a_y = 0 \) without loss of generality, one gets

\[
\begin{align*}
    s_0 &\approx s_1 \approx |u_{z'}|^2 = u_x^2 \left(1 + 2(h_{33}(t) - h_{11}(t)) \sin^2 \theta + 2h_{13}(t)(\tan \theta - \sin 2\theta)\right) \\
    s_2 &= s_3 = 0.
\end{align*}
\]

where we have used the previous notation for \( h_{ij}(t) \). The parameter \( s_0 \) measures the intensity of the wave, the parameter \( s_1 \) gives the preponderance of \( x' \)-linear polarization over \( y \)-linear polarization and \( s_2 \) and \( s_3 \) give phase information [34]. Thus Eq. (25) implies that a \( x' \)-linear polarized light stays linear (since the \( y \)-linear contribution is zero). However, as the medium is birefringence, the polarization vector rotates in the \( x' - z' \) plane by angle \( \psi(t) \) given by Eq. (23) (see Fig. 1).

IV. QUANTUM MECHANICAL MODEL

A. Hamiltonian formalism

So far we have considered the classical description of light propagation through a background of GWs. As we aim at detecting quantum nature of GWs, we need to promote our classical treatment to a fully quantum mechanical description where both EM field and GWs (in linearized theory) are quantized properly.

In this section we derive the Hamiltonian of the total system by considering EM and GW fields as two subsystems coupled to each other using OMA framework.
We start by the total Lagrangian density $\mathcal{L} \equiv \mathcal{L}_{gw} + \mathcal{L}_{em}$ where $L = \int d^4 r \mathcal{L}$ gives the total Lagrangian of the system,

$$
\mathcal{L}_{gw}(r, t) = \frac{c^4}{64\pi G} \left( \frac{1}{c^2} (\dot{h}_{ij})^2 - (\partial h_{ij})^2 \right), 
$$

(26)
describes free dynamics of GWs in the TT gauge [35] and

$$
\mathcal{L}_{em}(r, t) = \frac{1}{2} \left( \bar{E}_{ij}(r, t) E^i E^j - \mu_{ij}^1(r, t) B^i B^j \right),
$$

(27)
results in the classical Maxwell’s equations in the presence of GW background, Eq. (8) [27]. Tensors $(\bar{E}, \bar{B})$ are given by Eq. (7) and consist of two terms including the effect of vacuum as well as the GW background: $\mathcal{L}_{em} = \mathcal{L}_{free} + \mathcal{L}_{int}$ where $\mathcal{L}_{free}$ describes free EM field in flat spacetime and $\mathcal{L}_{int}$ shows the coupling between EM and GW fields.

In the following, we define the corresponding conjugate variable momenta of EM and GW fields, and find the quantized form of the total Hamiltonian. Before doing that, we remind that (i) $E = -\nabla \phi - \partial_t A$, $B = \nabla \times A$ and the suitable gauge condition $\nabla \cdot [\bar{E}(r, t)\bar{A}(r, t)] = 0$ excluid $\phi$. Therefore the dynamical freedom of the EM field is the vector potential $A$, and (ii) as mentioned before, the analogue optical medium is non-absorptive and non-dispersive. Consequently, contrary to the usual procedures of quantization of absorptive-dispersive magneto-dielectric media [36–38], here we do not treat the medium as a heat bath made of continuum bosonic modes. Instead, we apply the standard method of quantum field theory to the linearized GWs described by tensor field of rank 2.

1. Quantization of linearized gravitational waves

Quantization of linearized GWs is usually done within the framework of quantum field theory (QFT), where the GW field is simply treated as a rank-2 tensor possessing 2 different polarization states [39–42]. With the help of the classical Lagrangian Eq. (26) we define the canonical conjugate momentum of $h_{ij}$, as $\Xi_{ij} \equiv \partial \mathcal{L}_{gw} / \partial \dot{h}_{ij} = c^2/(32\pi G)\dot{h}_{ij}$. In quantum description, one promotes $h_{ij}$ and $\Xi_{ij}$ to operators and conventionally expands them as (compare to Eq. (1) for classical description)

$$
\hat{h}_{ij}(r, t) = A \sum_{\gamma = +, \times} \int \frac{d^3 K}{(2\pi)^3/2} \frac{\tilde{\epsilon}_{ij}[\hat{K}]}{\sqrt{2\Omega_K}} \left( \hat{b}_{K, \gamma} e^{-i(\Omega t - \mathbf{K} \cdot \mathbf{r})} + \hat{b}_{K, \gamma}^\dagger e^{i(\Omega t - \mathbf{K} \cdot \mathbf{r})} \right),
$$

(28)
and

$$
\hat{\Xi}_{ij}(r, t) = (-i) \frac{c^2 A}{32\pi G} \sum_{\gamma = +, \times} \int \frac{d^3 K}{(2\pi)^3/2} \sqrt{\frac{\Omega_K}{2}} \tilde{\epsilon}_{ij}[\hat{K}] \times \left( \hat{b}_{K, \gamma} e^{-i(\Omega t - \mathbf{K} \cdot \mathbf{r})} - \hat{b}_{K, \gamma}^\dagger e^{i(\Omega t - \mathbf{K} \cdot \mathbf{r})} \right),
$$

(29)
in which the constant $A$ has to be determined, $\Omega_K = cK$ and $\gamma$ refers to the polarization state. $(\hat{b}_{K, \gamma}, \hat{b}_{K, \gamma}^\dagger)$ are the ladder operators of mode $(K, \gamma)$. Following the standard QFT approach, we impose the following equal time commutation relation between $h_{ij}$ and it’s conjugate momentum,

$$
\left[ \hat{h}_{ij}(r, t), \hat{\Xi}_{ij}(r', t) \right] = i\hbar \delta^{(3)}(\mathbf{r} - \mathbf{r}').
$$

(30)
By plugging Eqs. (28, 29) into Eq. (30) one gets $A = \sqrt{16\pi \ell_p^2}$ [43], where $\ell_p = \sqrt{\hbar G/c^3}$ is the Planck length, and the bosonic commutation relation $[\hat{b}_{K, \gamma}, \hat{b}_{K', \gamma'}^\dagger] = \delta_{\gamma, \gamma'} \delta^{(3)}(\mathbf{K} - \mathbf{K}')$ outcomes. The free Hamiltonian of GWs turns out to be

$$
\hat{H}_{gw} = \frac{1}{2} \int d^3 r \left( \frac{32\pi G}{c^2} \frac{\hbar^2}{2} + \frac{c^4}{32\pi G} (\partial h_{ij})^2 \right),
$$

(31)
With substituting Eqs. (28, 29) in Eq. (31) one gets

$$
\hat{H}_{gw} = \sum_{\gamma = +, \times} \int d^3 K \hbar \Omega_K \hat{b}_{K, \gamma}^\dagger \hat{b}_{K, \gamma}
$$

(32)
which describes free evolution of GWs.

2. Electromagnetic field Quantization

In Sec. III A We have specified the mode expansion of the vector potential. We now rewrite the quantized form of Eq. (10) as

$$
\hat{A}(r, t) = \sum_k \sqrt{\frac{\hbar}{2}} \left( \hat{a}_k \nu_k(t) \mathbf{F}_k(r, t) + \hat{a}_k^\dagger \nu_k^*(t) \mathbf{F}_k(r, t) \right),
$$

(33)
where the coefficients $\alpha_k$ have been promoted to operators. The canonical conjugate momentum of $\hat{A}$ is defined as $\hat{\Pi}_i(r, t) \equiv \partial \mathcal{L} / \partial \dot{A}_i = \bar{E}_{ij}(r, t) \hat{A}_j(r, t)$,

$$
\hat{\Pi}_i(r, t) = \bar{E}_{ij} \sum_k \sqrt{\frac{\hbar}{2}} \left( \hat{a}_k \nu_k(t) \mathbf{F}_{kj} + \hat{a}_k^\dagger \nu_k^*(t) \mathbf{F}_{kj}^* \right).
$$

(34)
Now one can construct the Hamiltonian density by performing the Legendre transform on the Lagrangian density $\mathcal{L}_{em}$. The resulting Hamiltonian is

$$
\hat{H}_{em} = \frac{1}{2} \int d^3 r \left[ \bar{E}_{ij}(r, t) \hat{\Pi}_j - \bar{E}_{ij}^{-1}(r, t) \hat{\Pi}_j \hat{\Pi}_i \right] \right) \hat{\Xi}_{ij} \right),
$$

(35)
where we have used the identity $\bar{E}_{ik} \bar{E}_{k}^{-1} = \delta_{ii}$. The standard canonical quantization is done by imposing the following equal time commutation relation between the Cartesian components of the conjugate variables $\hat{A}$ and $\hat{\Pi}$ as

$$
\left[ \hat{A}_i(r, t), \hat{\Pi}_j(r', t) \right] = i\hbar P_{ij}^r(r, r'),
$$

(36)
Here $P_{\perp}^T$ is the generalized transverse delta-distribution [26, 27]. By inserting Eqs. (33, 34) into the canonical commutation relation Eq. (36) one gets the bosonic commutation relation $[\hat{a}_k, \hat{a}_k^\dagger] = \delta_{kk'}$ and the other commutators vanish. The resulting Hamiltonian is

$$\hat{H}_{em} = \hbar \sum_k \left( \hat{a}_k \hat{a}_{-k} \left[ \hat{\nu}_k^2 + \omega_k^2(t) \nu_k^2 \right] + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \left[ \hat{\nu}_k^2 + \omega_k^2(t) \nu_k^2 \right] + \left( 2\hat{a}_k^\dagger \hat{a}_k + \delta^{(3)}(0) \right) \left[ |\hat{\nu}_k|^2 + \omega_k^2(t) |\nu_k|^2 \right] \right),$$

The Hamiltonian depends explicitly on time and thus doesn’t posses time-independent eigenvectors. Nevertheless, we can still define the instantaneous vacuum $|\langle \nu_0| 0 \rangle_t\rangle$ to be the lowest energy state of the Hamiltonian $\hat{H}(t_0)$ at time $t_0$. By minimizing the expectation value $\langle \langle \nu_0| \hat{H}(t_0) |\nu_0\rangle \rangle$ with respect to the mode functions $\nu_k(t)$ one can show that the mode functions

$$\nu_k(t_0) = \frac{1}{\sqrt{\omega_k(t_0)}} e^{i\eta_k(t_0)}, \quad \hat{\nu}_k(t_0) = i\omega_k(t_0) \nu_k(t_0),$$

are the preferred mode functions which determine the vacuum at a particular moment of time $t_0$ [30]. Here $\eta_k(t_0)$ is a function of time which in the adiabatic limit coincides with the WKB solution given by Eq. (20). By plugging the mode functions Eq. (38) in the Hamiltonian Eq. (37) we obtain the diagonalized Hamiltonian at a given moment of time as

$$\hat{H}_{em}(t) = \sum_k \hbar \omega_k(t) \hat{a}_k^\dagger \hat{a}_k = \sum_k \frac{\hbar \omega_k}{n(t)} \hat{a}_k^\dagger \hat{a}_k.$$

where $n(t)$ is given by Eq. (18), and the zero-point energy is excluded. Hence, we find the general form of the interaction Hamiltonian between EM and GWs fields, which is often used in the literature as an ansatz [20].

Despite the fact that the Hamiltonian is explicitly time-dependent and the vacuum state is not unique in general, in case of a slowly varying GW background the WKB approximation indicates that the mode functions Eq. (20) diagonalize the Hamiltonian at all times and the particle creation by the GW background is negligible, as previously noted by Kip Thorn et al [23]. Using Eq. (18) for $n(t)$, we now split the Hamiltonian Eq. (39) into two parts. The resulting interaction Hamiltonian turns out to be

$$\hat{H}_{int} = \frac{-1}{2} \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k \left( \hat{h}_{13} \sin 2\theta - \hat{h}_{11} \cos^2 \theta \right) - \hat{h}_{22} - \hat{h}_{33} \sin^2 \theta \right).$$

where $\hat{h}_{ij}(t)$ is given by Eq. (28) by setting $e^{iK \cdot r} = 1$. $\hat{H}_{int}$ represents the interaction between EM and GW fields through the intensity dependent coupling (proportional to $\hat{a}_k^\dagger \hat{a}_k$) of EM field with GWs. This leads to either possible nonlinear interactions $\hat{b}_K \hat{a}_k^\dagger \hat{a}_k$ or $\hat{b}_K^\dagger \hat{a}_k \hat{a}_k^\dagger$, analogous to the three wave mixing process in nonlinear optics and reminiscent of the opto-mechanical coupling in the cavity opto-mechanics [44]. This coupling can also be viewed virtually as an intensity dependent “displacement” of GWs, similar to the mechanical motion of a massive mirror induced by the radiation pressure in a cavity opto-mechanical system. The opto-mechanical analogy becomes more evident with the association GW strain field $\leftrightarrow$ mechanical oscillator displacement field. In a similar manner, we call this interaction “opto-gravitational” coupling. As a result, the mutual interaction between EM and GWs not only induces EM field dynamics, but also the radiation pressure of the light induces the strain field dynamics as discussed in a different approach recently [18].

B. Quantum dynamics

In a similar manner to the cavity opto-mechanical interaction, the opto-gravitational interaction Eq. (40) gives rise to EM field amplitude-phase noise correlation, and one may expect that this acts as a new source of ponderomotive squeezing (PS) of light, similar to the mechanically-induced PS [45]. Although the opto-mechanical squeezing of light can not be an evidence of non-classicality of mechanical oscillator, i.e. one can still get squeezing of light without quantizing the mechanical motion [46], it can be shown that “revival of squeezing” is a pure quantum mechanical effect, in the sense that it exists only if both EM field and mechanical oscillator are quantum entities [19]. Following this argument, one may conclude that the same happens also for the GWs and observing the gravitational-induced revivals of light squeezing could serve as verification of quantum nature of GWs.

In this section we compute the optical variance for a continuum of highly-squeezed PGWs and explicitly show that as a result of a very small coupling strength between GWs and EM field, it’s unfortunately impossible to experimentally see the revivals of optical squeezing and the above argument fails. However, it turns out that the background of squeezed PGWs induces non-trivial dynamics for the optical variance. Moreover, it acts as a mechanism of light line-width-broadening. The other foot-print of PGWs is the emergence of side-bands in the spectrum of light, which directly comes from the amplified nature of squeezed PGWs. The light-broadening and side-bands could in principle be tested by the spectrometric analysis in a GW detector.

In the following, we consider the interaction of a single mode of EM field with frequency $\omega_0$ with a continuum of quantized GWs and find the unitary evolution of the total system, governed by the Hamiltonian Eq. (40). Using
Eq. (28) we rewrite \( \hat{H}_{\text{int}} \) as
\[
\hat{H}_{\text{int}}/\hbar = -\frac{1}{2} \frac{A\omega_0}{(2\pi)^{3/2}} \sum_{\gamma} \int \frac{d^3 K}{\sqrt{2\pi K}} F_\gamma(K, \theta) \left( \hat{b}_{K, \gamma} e^{-i\Omega t} + \hat{b}_{K, \gamma}^\dagger e^{i\Omega t} \right) \hat{a}^\dagger \hat{a},
\]
where we reminded that \( A = \sqrt{16\pi c\ell_P} \) and the function \( F_\gamma(K, \theta) \) depends on the configuration of the EM probe (the interferometer) with respect to the incoming GWs polarization,
\[
F_\gamma(K, \theta) = e_{11}^\gamma[K] \sin 2\theta - e_{12}^\gamma[K] \cos 2\theta - e_{22}^\gamma[K] - e_{33}^\gamma[K] \sin^2 2\theta.
\]
The unitary evolution operator of the total system turns out to be
\[
\hat{U} = e^{iE(t)(\hat{a}^\dagger \hat{a})} e^{-i\omega \hat{a}^\dagger \hat{a} - i \int d^3K \kappa (\Omega K) (\hat{b}_{K}^\dagger \eta_K(t) - \hat{b}_K \eta_K(t))}
\]
where \( \eta_K(t) = 1 - e^{i\Omega_K t} \) and
\[
E(t) = \int d^3K \kappa^2(\Omega_K)(\Omega_K t - \sin \Omega_K t),
\]
shows the effect of the vacuum of the GWs field. Moreover,
\[
\kappa(\Omega_K) = \frac{A\omega_0}{2(4\pi \Omega_K)^{3/2}} F_\gamma(K, \theta).
\]

\[
\Delta \chi_0(t) = 1 + 2\tilde{n} \left\{ e^{-\tilde{n}(1 - \cos 4E(t)) - 2B(t)} \cos \left[ -2\theta + 4E(t) + \tilde{n}\sin(4E) \right] - e^{-2\tilde{n}(1 - \cos 2E(t)) - B(t)} \cos \left[ -2\theta + 2E(t) + \tilde{n}\sin(2E) \right] + 1 - e^{-2\tilde{n}(1 - \cos 2E(t)) - B(t)} \right\},
\]

where
\[
B(t) = \int d^3K \frac{\kappa^2(\Omega_K)}{2} \kappa^2(\Omega_K) \left\{ 4\sin^2(\Omega_K t/2) + \cos \theta_\zeta - 2\cos(\Omega_K t - \theta_\zeta) + \cos(2\Omega_K t - \theta_\zeta) \right\}.
\]
GWs, one could set $\zeta_K = 0$ to obtain result from [20]. Here we used the notation $\zeta_K \equiv r_Ke^{i\theta}$ for each squeezed mode of the GWs, where $\theta$ is the squeezing angle and $\hat{n}_K \sim e^{2r_K}$ takes into account the highly squeezed parameter. Since the order of magnitude of the coupling strength $\kappa(\Omega_K)$ is extremely small ($\kappa^2 \sim \mathcal{O}(10^{-61})$), it takes a very long time (more than the age of the Universe) to observe effect of the gravitational vacuum, such as the revival of squeezing [20]. Nevertheless, the highly-squeezed nature of PGWs leaves a non-trivial effect on the EM field behaviour. The time dependence of the optical variance Eq. (47) is shown in Fig. (2)(a) for $\tilde{n} = 1$, $\omega = 10^{14}$Hz and $\vartheta = 0$ (for numerical calculations see Sec. IV C). The variance starts from the reference level ($\Delta \chi = 1$) and increases to a value $1 + 2\tilde{n}$ after a period $10^4$s. The increase to and stabilization at $1 + 2\tilde{n}$ is similar to the effect of a mechanical oscillator on the EM field, in the opto-mechanical analogue model [19]. This arises from a phase mismatch between different Fock state components making up the initial coherent state, due to the fact that we add up the contribution of different GW modes incoherently. As discussed in [19], the typical time scale needed to observe the recurrence is $t_{obs} \sim T/\kappa^2$, where $T = 2\pi/\Omega$ is the typical mechanical (or GWs) period time. For the set of parameters related to the quantized GWs, $\kappa^2 \sim 10^{-61}$ which results in an irrationally large observation time to see the revivals. Despite the fact that it is impossible to verify the recurrence behaviour of the variance, Eq. (47) shows that any violation from $\Delta \chi = 1$ would directly imply the existence of a highly-squeezed PGWs background. That is because if we do not consider the high factor $e^{2r_K}$ in Eq. (48), both $E(t)$ and $B(t)$ would be approximately zero and the variance would not be affected by the GWs background.

The other quantity of interest is the power spectrum of light, defined as [48]

$$S(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty d\tau e^{i\omega\tau} G^{(1)}(r, r; \tau),$$

where $G^{(1)}(r, r; \tau)$ is the first order degree of coherency of light. For a linearly polarized single mode EM field we have

$$G^{(1)}(r, r; \tau) = \langle \hat{a}^\dagger(0)\hat{a}^{\dagger}(\tau) \rangle = e^{-B(\tau)}e^{-i\omega_0\tau},$$

where in the last line we have used Eq. (46) and $B(\tau)$ is given by Eq. (48). In the absence of GWs, $G^{(1)}(\tau) \sim e^{-i\omega_0\tau}$. Fig. (2)(b) shows the time behaviour of the absolute value of the coherence function in the presence of PGWs. The effect of PGWs background appears as the modulating factor $e^{-B(\tau)}$. This means that the PGWs background act as a broadening mechanism with a characteristic gravitational “coherence time”, $\tau_c \sim 2 \times 10^8$s. Thus $1/\tau_c \sim 10^{-6}$Hz, which is the inverse of the coherence time of the light field, is a measure of the frequency broadening of light. Fig. (3) shows the power spectrum of light, with a broadened pick centered at $\omega_0$ and emergence of side-bands at $\omega_0 \pm m10^{-6}$Hz. A physical understanding of this behavior is described in the following.

### C. Results and discussion

When estimating the function $B(t)$, we used results of [22, 49] for the squeezing parameter $e^{2r_K}$, which leads to

$$B(t) = \left[ \int_{10^{-20}}^{10^{-18}} d\Omega \left( \frac{10^{12.28}}{\Omega^2} \right) + \int_{10^{-18}}^{10^{-16}} d\Omega \left( \frac{10^{-59}}{\Omega^2} \right) + \int_{10^{-16}}^{10^{-4}} d\Omega \left( \frac{10^{-27}}{\Omega^2} \right) + \int_{10^{-16}}^{10^{10}} d\Omega \left( \frac{10^{-52.8}}{\Omega^3} \right) \right] \sin^4(\Omega t/2) / \Omega$$

(51)

Although the amplified PGWs span a full range of the frequencies, $10^{-20}$Hz $\leq \Omega \leq 10^{10}$Hz, and the integrands in Eq. (51) are fast oscillating as time increases, it can be shown numerically that the first three terms in Eq. (51) constitute the main contribution in $B(t)$. Therefore the

“small detector” approximation would not be destroyed since $\delta_{\text{visa}} \leq 10^{-3}$ for frequency range $10^{-20}$Hz $\leq \Omega \leq 10^{-4}$Hz. Moreover, the angular pre-factor in the integration, $\int d(\cos \Theta) d\Phi f_K^2[\dot{K}, \theta]$ is of order 1. The resulting time behaviour of the optical variance is shown in

![FIG. 3. The spectrum of the optical field. $S(\omega)$, based on numerical calculations with considering $G^{(1)}(\tau) \propto e^{-\langle e^{i\omega_0\tau} \rangle}$. The laser line-width is about $10^{-6}$Hz and the side-bands appear at $\omega_0 \pm m10^{-6}$Hz.](image-url)
Fig. (2)(a) where it can be seen that it stabilize after about one year ($t \sim 2 \times 10^6$ s). The first order degree of coherence is plotted in Fig. (2)(b). Numerical calculations show that the well fitted curve to the function $B(t)$ given by Eq. (51) would be a forth-order polynomial,

$$B(t) \approx \left[ \int_{10^{-18}}^{10^{-16}} d\Omega \left( \frac{10^{12.28}}{\Omega^2} \right) + \int_{10^{-18}}^{10^{-16}} d\Omega \left( \frac{10^{-59}}{\Omega^6} \right) \right] + \int_{10^{-4}}^{10^{-16}} d\Omega \left( \frac{10^{-27}}{\Omega^4} \right) \left( \frac{\Omega t}{2} \right)^4 = (10^{-6} t)^4 \equiv (\Omega_{\text{eff}} t)^4. \quad (52)$$

In the last line we have introduced an effective frequency $\Omega_{\text{eff}}$ with numerical value $\Omega_{\text{eff}} \sim 10^{-6}$ Hz. Eq. (52) results in $G^{(1)}(\tau) \propto e^{-\left(\tau/10^6\right)^4}$, which means that the PGWs background act similar to a Doppler-broadening mechanism and $\Omega_{\text{eff}}$ is a measure of the light line-width-broadening, as is shown in Fig. (3).

The appearance of the side-bands at $\omega_0 \pm m\Omega_{\text{eff}}$ (where $m \in \mathbb{N}^*$) is also a direct consequence of the highly squeezed nature of PGWs. This can be justified by using the opto-mechanical cavity opto-analogy. In an opto-mechanical cavity, the cavity exhibit - owning to its harmonic motion by frequency $\Omega_m$- an absorption spectrum which possesses both a carrier at the cavity resonance frequency and a series of side-bands at $\omega_0 \pm m\Omega_m$ [44, 50].

The effect of GWs on light, if we use the proper observer frame instead of the TT gauge, would be changing boundary condition of light due to the variations in the arm-length of the interferometer. Each mode in the GWs background acts independently, and the collective effect would be the sum of the individual displacements owning to different modes with corresponding amplitudes $h_K$ and frequencies $\Omega_K$. The cumulative effect of the squeezed background of PGWs on light is similar to the effect of different velocities of emitters in a gas, on the emitted light, i.e., the Doppler broadening effect. On the other hand, the collective displacement of the interferometer arm-length is the sum of individual displacements owning to different GWs modes, which leads to the variation of laser frequency and phase modulation of light, as what happens in an opto-mechanical cavity [50]. This can be viewed as if there is a single mechanical oscillator with frequency $\Omega_m \equiv \Omega_{\text{eff}} = 10^{-6}$ Hz that results in the appearance of the side-bands.

### D. Conclusions

In conclusion, a new framework to describe the interaction between GWs background with an EM field was established based on optical medium analogy. By deriving the quantized Hamiltonian of the total system, we obtain the optical variance and the spectrum of a laser light interacting with a squeezed background of PGWs. Our results show that: (1) the optical variance stabilizes to a value of $1 + n$ after about one year, (2) PGWs background acts as a new source of line-width broadening, the numerical value of which is about $10^{-6}$ Hz for the set of parameters reported in [22, 49], and (3) as a result of interaction with GWs, the appearance of side-bands occurs at $\omega_0 \pm m\Omega_{\text{eff}}$ in the light spectrum.

It has to be noted that observation of the line-width broadening or non-trivial behaviour of optical variance is a direct consequence of the existence of the highly-squeezed PGWs, since without large amplification of the PGWs there is no observable effect.

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