A Short Proof of the Pontryagin Maximum Principle on Manifolds

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Abstract
Applying the Tubular Neighborhood Theorem, we give a short and new proof of the Pontryagin Maximum Principle on a smooth manifold. The idea is as follows. Given a control system on a manifold $M$, we embed it into an open subset of some $\mathbb{R}^n$, and extend the control system to the open set. Then, we apply the Pontryagin Maximum Principle on $\mathbb{R}^n$ to the extended system and project the consequence to $M$.

keywords: optimal control, Pontryagin maximum principle, tubular neighborhood

1 Introduction

The classic book by Pontryagin et al. [8] gives a proof of the celebrated Pontryagin Maximum Principle (PMP) for control systems on $\mathbb{R}^n$. See also [5] for another proof of the PMP in $\mathbb{R}^n$. Although several books or journal articles have mentioned versions of the PMP for control systems on a smooth manifold, its proofs began to appear in the literature quite recently [2, 3]. In general there can be three kinds of proof of the PMP on manifolds. The first is to translate the proof in [8] into the modern differential-geometric language, [2, 3]. Although this approach gives a good geometric insight into the principle, it has the drawback that the proof becomes long since it follows the original proof in [8]. The second kind of the proof is to adapt the proof in [8] to manifolds by patching up a finite number of local charts covering an optimal trajectory without use of any modern differential-geometric machinery. A drawback of the second approach is that the proof becomes very long too, involving coordinate transformations and repeating the proof in [8]. To our knowledge, there is no literature containing the second kind of proof. The third kind of proof is the one that we present in this paper, which is a new and short proof of the PMP on manifolds, by combining the Tubular Neighborhood Theorem and the PMP on $\mathbb{R}^n$. This proof does not repeat the proof in [8], and is thus much shorter than and different from the two kinds of proofs mentioned above.

The idea in our proof is simple. Given a control system on a manifold $M$, we embed $M$ into some $\mathbb{R}^n$, take a tubular neighborhood $V$ of $M$ in $\mathbb{R}^n$, and construct a control system
on $V$ whose restriction to $M$ agrees with the original system. Since $M$ is an invariant manifold for the extended system on $\mathbb{R}^n$, we can reformulate the original optimal control problem with a point-to-point transfer on $M$ into an equivalent optimal control problem with a point-to-submanifold transfer in $\mathbb{R}^n$ where the submanifold is transversal to $M$. We apply the PMP on $\mathbb{R}^n$ to the equivalent problem, and then project (or, restrict) the result to $M$, to prove the PMP on $M$. Our proof is pedagogically meaningful in optimal control theory, and it illustrates a nice application of the Tubular Neighborhood Theorem to control theory.

2 Main Results

2.1 Review of the Pontryagin Maximum Principle on $\mathbb{R}^n$

Consider a control system on $\mathbb{R}^n$:

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in W.$$  \hspace{1cm} (1)

We want to find an optimal control $u(t)$ such that

$$\int_{t_0}^{t_1} f^0(x, u)dt$$

is minimized (2)

with free terminal time $t_1$, and

$$x(t_0) = x_0, \quad x(t_1) = x_1.$$  \hspace{1cm} (3)

For convenience, we assume that $W$ is a subset of a Euclidean space and that $f : \mathbb{R}^n \times W \rightarrow \mathbb{R}^n$ and $f^0 : \mathbb{R}^n \times W \rightarrow \mathbb{R}$ are smooth.

**Theorem 2.1** ($[8, 6]$). Suppose that $u(t), t_0 \leq t \leq t_1$ is a piecewise continuous optimal control and $x(t)$ is the corresponding optimal trajectory for (1) – (3). Then, there exists a non-vanishing continuous curve $(p_0(t), p(t)) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R} \times T^*_x\mathbb{R}^n$ such that:

1. The trajectory $(x(t), p(t))$ satisfies

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, n,$$  \hspace{1cm} (4)

i.e., it is a flow of the Hamiltonian vector field

$$X_H(x, p; p_0, u(t)) = \Omega^2dH$$  \hspace{1cm} (5)

where $\Omega$ is the canonical symplectic form on $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and the Hamiltonian $H$ is given by

$$H(x, p, p_0, u) = p_0f^0(x, u) + \langle p, f(x, u) \rangle.$$

2. $u(t) = \arg \max_{v \in W} H(x(t), p(t), p_0(t), v)$ for every $t \in [t_0, t_1]$.

3. $p_0 \leq 0$ and is constant in $t$.

4. $H(x(t), p(t), p_0, u(t)) = 0$ for every $t \in [t_0, t_1]$. 

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**Theorem 2.2 ([8, 6]).** Let \( u(t), t_0 \leq t \leq t_1, \) be a piecewise continuous optimal control and \( x(t) \) the corresponding trajectory for (4), (5) and 

\[
x(t_0) \in S_0, \quad x(t_1) \in S_1
\]

where \( S_0 \) and \( S_1 \) are smooth submanifolds of \( \mathbb{R}^n \). Then, all of the conclusions in Theorem 2.1 hold, and additionally the transversality conditions

\[
\langle p(t_0), T_{x(t_0)}S_0 \rangle = 0, \quad \langle p(t_1), T_{x(t_1)}S_1 \rangle = 0
\]

are satisfied.

**Remark.** Theorems 2.1 and 2.2 will still hold for a control system on an open subset \( U \) of \( \mathbb{R}^n \) if every trajectory of the control system leaves \( U \) (positively) invariant. Since the cotangent lift of a diffeomorphism of \( U \) onto a set \( V \) is a symplectomorphism of \( (T^*U, \Omega) \) onto \( (T^*V, \Omega) \) where \( \Omega \) is the canonical symplectic form on the respective cotangent bundle (see Theorem Proposition 6.3.2. in [7]), Theorems 2.1 and 2.2 still hold for a control system on any set \( V \) that is diffeomorphic to an open subset of \( \mathbb{R}^n \). In this case, we need replace \( T^*\mathbb{R}^n \) by \( T^*V \) and use (5) preferably to (4) in Theorem 2.1.

### 2.2 Pontryagin Maximum Principle on Manifolds

We consider the optimal control problem of finding control \( u(t) \) for the control system on an \( n \)-dimensional manifold \( M \)

\[
\dot{x} = f(x, u), \quad x \in M, u \in W \tag{6}
\]

such that

\[
\int_{t_0}^{t_1} f^0(x, u)dt \text{ is minimized} \tag{7}
\]

with free terminal time \( t_1 \), and

\[
x(t_0) = x_0, \quad x(t_1) = x_1. \tag{8}
\]

For convenience, we assume that \( W \) is a subset of a Euclidean space and that \( f \) and \( f^0 \) are smooth.

**Theorem 2.3.** Suppose that \( u(t), t_0 \leq t \leq t_1 \) is a piecewise continuous optimal control and \( x(t) \) is the corresponding trajectory. Then, there exists a non-vanishing continuous curve \( (\lambda_0(t), \lambda(t)) \in \mathbb{R} \times T^*_{x(t)}M \) such that:

1. The trajectory \((x(t), \lambda(t))\) is the flow of

\[
X_H(x, \lambda; \lambda_0, u) = \Omega^t dH \tag{9}
\]

where \( \Omega \) is the canonical symplectic form on \( T^*M \) and

\[
H(x, \lambda, \lambda_0, u) = \lambda_0 f^0(x, u) + \langle \lambda, f(x, u) \rangle. \tag{10}
\]

2. \( \lambda_0 \leq 0 \) and is constant in \( t \).

3. \( u(t) = \arg \max_{v \in W} H(x(t), \lambda(t), \lambda_0(t), v) \) for every \( t \in [t_0, t_1] \).

4. \( H(x(t), \lambda(t), \lambda_0, u(t)) = 0 \) for every \( t \in [t_0, t_1] \).
Proof. By the Whitney Embedding Theorem [3], we may assume that $M$ is an embedded submanifold and a closed subset of $\mathbb{R}^N$ for some $N \in \mathbb{N}$. By the Tubular Neighborhood Theorem [5], there is an open neighborhood $V$ of $M$ in $\mathbb{R}^N$ with a smooth retraction $\pi_V$ of $V$ onto $M$. Since $M$ is a closed subset of $V$, by Proposition 2.26 in [5] there is a smooth bump function $\rho : \mathbb{R}^N \to [0, 1]$ such that $\text{supp } \rho \subset V$, and $\rho(z) = 1$ for every $z \in M$. Define a control vector field $F : V \times W \to TV$ by

$$F(z, u) = \rho(z)f(\pi_V(z), u) \in \mathbb{R}^N. \quad (11)$$

It is straightforward to verify that the restriction of $F$ to $M$ agrees with $f$ and that both $V$ and $M$ are invariant under the flow of $F$. Hence, the optimal control problem in (6) – (8) is equivalent to the problem of finding control $u(t)$, $t_0 \leq t \leq t_1$ for

$$\dot{z} = F(z, u), \quad z \in V, u \in W \quad (12)$$

such that

$$\int_{t_0}^{t_1} f^0(\pi_V(z), u)dt \text{ is minimized} \quad (13)$$

with free terminal time $t_1$, and

$$z(t_0) = x_0 \in M \hookrightarrow V, \quad z(t_1) \in \pi_V^{-1}(x_1) \quad (14)$$

where $\pi_V^{-1}(x_1)$ is an $(N - n)$-dimensional submanifold of $V$, and $\pi_V^{-1}(x_1) \cap V = \{x_1\}$.

Let $u(t)$, $t_0 \leq t \leq t_1$, be a piecewise continuous optimal control for (6) – (8), and $x(t)$ be its optimal trajectory in $M$. By the equivalence of the two optimal control problems (3) – (8) and (12) – (14) and by the invariance of $M$, the trajectory

$$z(t) := x(t) \in M \hookrightarrow V \quad (15)$$

is the optimal trajectory associated with the control $u(t)$ for (12) – (14). By the remark below Theorem 2.22 we can apply Theorem 2.22 to (12) – (14). Hence, there exists a non-vanishing continuous curve $(p_0, p(t)) \in \mathbb{R} \times T_{z(t)}^*V$ such that with the Hamiltonian $\tilde{H} : T^*V \times \mathbb{R} \times W \to \mathbb{R}$ defined by

$$\tilde{H}(z, p; p_0, u) = p_0 f^0(\pi_V(z), u) + \langle p, F(z, u) \rangle,$$

statements 1–4 in Theorem 2.21 and the transversality condition

$$\langle p(t_1), T_{x_1}(\pi_V^{-1}(x_1)) \rangle = 0 \quad (16)$$

are satisfied where

$$T_{x_1}(\pi_V^{-1}(x_1)) \oplus T_{x_1}V = T_{z_1}V = \mathbb{R}^N. \quad (17)$$

Define a function $H : T^*M \times \mathbb{R} \times W \to \mathbb{R}$ by

$$H(x, \lambda; \lambda_0, u) = \lambda_0 f^0(x, u) + \langle \lambda, f(x, u) \rangle. \quad (18)$$

Let $T^*V|_M$ be the restriction of $T^*V$ to $M$. Let $i : T^*V|_M \hookrightarrow T^*V$ be the canonical inclusion and $\pi : T^*V|_M \to T_*M$ the canonical projection defined by restricting $\alpha \in T_z^*V$ for $z \in M$ to $T_zM$. Then, it is easy to show

$$\tilde{H} \circ i = H \circ \pi \quad (19)$$

on $T^*V|_M$. Since $z(t) \in M$, $(z(t), p(t)) \in T^*V|_M$ for every $t \in [t_0, t_1]$. Let us assume that both $T^*V$ and $T^*M$ are equipped with the canonical symplectic forms. We need the following lemma:
Lemma 2.4. Suppose that $M$ is an $n$-dimensional embedded submanifold of an $N$-dimensional manifold $L$. Let $T^*L|_M$ be the restriction of $T^*L$ to $M$. Let $i : T^*L|_M \to T^*L$ the canonical inclusion and $\pi : T^*L|_M \to T^*M$ the canonical projection, where both $T^*L$ and $T^*M$ are equipped with the canonical symplectic forms. If $\tilde{H} : T^*L \to \mathbb{R}$ and $H : T^*M \to \mathbb{R}$ are functions such that
\[
\tilde{H} \circ i = H \circ \pi
\] (20)
on $T^*L|_M$, then the Hamiltonian flow $\tilde{\varphi}_t$ with the Hamiltonian $\tilde{H}$ leaves $T^*L|_M$ invariant, so it induces canonically a flow $\varphi_t$ on $T^*M$ satisfying $\pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi$ on $T^*L|_M$, and the flow $\varphi_t$ is a Hamiltonian flow on $T^*M$ for the Hamiltonian $H$.

Proof. This lemma is a simple corollary to standard results in geometric mechanics [1, 7], but we could not find a reference that explicitly states this lemma, so we give a quick proof.

Let us choose a set of local coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$ for $L$ such that $y = 0$ corresponds to $M$. Let $((x, y), (\alpha, \beta))$ be the corresponding cotangent bundle coordinates for $T^*L$. Then, $(x, \alpha, \beta)$ are local coordinates for $T^*L|_M$ and $(x, \alpha)$ for $T^*M$. In these coordinates, (20) becomes $\tilde{H}((x, 0), (\alpha, \beta)) = H(x, \alpha)$ for every $((x, 0), (\alpha, \beta)) \in T^*L|_M$.

Then, for any $((x, 0), (\alpha, \beta)) \in T^*L|_M$, the Hamiltonian vector field $X_{\tilde{H}}(z)$ is given by

\[
\begin{align*}
\dot{x}^i &= \frac{\partial \tilde{H}}{\partial \alpha_i}((x, 0), (\alpha, \beta)) = \frac{\partial H}{\partial \alpha_i}(x, \alpha), \\
\dot{y}^a &= \frac{\partial \tilde{H}}{\partial \beta_a}((x, 0), (\alpha, \beta)) - \frac{\partial H}{\partial \beta_a}(x, \alpha) = 0, \\
\dot{\alpha}_i &= \frac{\partial \tilde{H}}{\partial x^i}((x, 0), (\alpha, \beta)) = \frac{\partial H}{\partial x^i}(x, \alpha), \\
\dot{\beta}_a &= -\frac{\partial \tilde{H}}{\partial y^a}((x, 0), (\alpha, \beta)).
\end{align*}
\] (21) and (22)

Since $\dot{y}^a = 0$, $X_{\tilde{H}}(z)$ is tangent to $T^*L|_M$ at every $z \in T^*L|_M$, so its flow $\tilde{\varphi}_t$ leaves $T^*L|_M$ invariant. Then, it is not hard to see that we have a globally well-defined flow $\varphi_t$ on $T^*M$ such that $\pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi$ on $T^*L|_M$. Equations (21) and (22) imply that $\varphi_t$ is a Hamiltonian flow of the Hamiltonian $H$. \qed

From Lemma 2.4 with $L = V$, it follows that $(x(t), \lambda(t)) := \pi(z(t), p(t))$ is a flow of the Hamiltonian vector field

\[
X_H(\cdot; p_0, u(t)) = \Omega^B dH(\cdot; p_0, u(t))
\] (23)
on $T^*M$, where $\Omega$ is the canonical symplectic form on $T^*M$. Notice that $x(t)$ is the optimal trajectory we began with in [15] and that $\lambda(t)$ is the restriction of $p(t)$ to $T_{x(t)}M$, i.e.,

\[
\lambda(t) = p(t)|_{T_{x(t)}M}.
\] (24)

Setting $\lambda_0 = p_0$, we see that statement 1 in Theorem 2.3 holds. By statement 2 in Theorem 2.1, $\lambda_0 = p_0$ is a non-positive constant, so statement 2 in Theorem 2.3 holds. Since $(z(t), p(t)) \in T^*V|_M$, by (19)

\[
\tilde{H}(z(t), p(t); p_0, v) = H(z(t), \lambda(t); \lambda_0, v)
\] (25)
for every $v \in W$. By statement 3 in Theorem 2.1, for every $t \in [t_0, t_1]$

\[
\begin{align*}
u(t) &= \arg \max_{v \in W} \tilde{H}(z(t), p(t); p_0, v) \\
&= \arg \max_{v \in W} H(x(t), \lambda(t); \lambda_0, v),
\end{align*}
\]
which implies that statement 3 in Theorem 2.3 holds. By (25) and statement 4 in Theorem 2.4, statement 4 in Theorem 2.3 holds.

We now show that the continuous curve \((\lambda_0, \lambda(t))\) never vanishes. Suppose that there is a \(\bar{t} \in [t_0, t_1]\) such that \((\lambda_0, \lambda(\bar{t})) = (0, 0)\). By \(\lambda_0\) being constant and (23), the curve \((\lambda_0, \lambda(t))\) satisfies

\[
\dot{\lambda}_0 = 0; \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x^i}, i = 1, \ldots, n
\]

in a local chart containing \((x(\bar{t}), \lambda(\bar{t}))\), which is a (non-autonomous) linear ordinary differential equation for \((\lambda_0, \lambda)\) since \(H\) in (18) is linear in \((\lambda_0, \lambda)\). By the uniqueness of ODE solutions, \((\lambda_0, \lambda(t)) \equiv (0, 0)\) as long as \((x(t), \lambda(t))\) stays in the local chart. By patching up a finite number of local charts, we get \((\lambda_0, \lambda(t_1)) = (0, 0)\), which, together with (16), (17) and (24), implies that \((p_0, p(t_1)) = (0, 0)\), contradicting the non-vanishing assumption on \((p_0, p(t))\). Therefore, \((\lambda_0, \lambda(t))\) never vanishes. This completes the proof of Theorem 2.4. ❄

**Remark.** Notice that using \(z(t_1) \in \pi^{-1}_V(x_1)\) in (14) instead of \(z(t_1) = x_1\) is crucial in the proof since the condition \(z(t_1) = x_1\) would not guarantee the non-vanishing property of \((\lambda_0, \lambda(t))\) in the last part of the proof.

Consider the optimal control problem in (6) – (8) where the fixed endpoint condition in (8) is replaced by

\[
x(t_0) \in S_0, \quad x(t_1) \in S_1
\]

where \(S_0\) and \(S_1\) are smooth submanifolds of \(M\). Then, the following theorem holds:

**Theorem 2.5.** Let \(u(t), t_0 \leq t \leq t_1\), be a piecewise continuous optimal control and \(x(t)\) the corresponding trajectory for (6), (7) and (20). Then, it is necessary that there exists a nonzero continuous \((\lambda_0, \lambda(t)) \in \mathbb{R} \times T_{x(t)}M\), which satisfies the conclusions in Theorem 2.4 and, in addition, the transversality conditions

\[
\langle \lambda(t_0), T_{x(t_0)}S_0 \rangle = 0, \quad \langle \lambda(t_1), T_{x(t_1)}S_1 \rangle = 0.
\]

**Proof.** The same arguments in the proof of Theorem 2.4 apply here except that we use

\[
z(t_0) \in S_0 \leftrightarrow V, \quad z(t_1) \in \pi^{-1}_V(S_1)
\]

instead of (14). ❄

**Example.** Consider a time optimal control problem for the following system on the unit 2-sphere \(S^2\) that is embedded in \(\mathbb{R}^3\):

\[
\dot{x} = \begin{pmatrix}
0 & u_1 & 0 \\
-u_1 & 0 & u_2 \\
0 & -u_2 & 0
\end{pmatrix} x, \quad x \in \mathbb{R}^3
\]

where \(|u_1| \leq 1, |u_2| \leq 1, x(0) = (1, 0, 0),\) and \(x(t_1) = (0, 0, 1)\). As in the proof of Theorem 2.3 we can replace the terminal condition \(x(t_1) = (0, 0, 1)\) by

\[
x(t_1) \in \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2 = 0, \ 0 < x_3 < 2 \}
\]

and apply the PMP on \(\mathbb{R}^3\). This time optimal control problem is solved from this viewpoint in [4].
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