Some Thoughts on the Quantum Theory of de Sitter Space

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ABSTRACT: This is a summary of two lectures I gave at the Davis Conference on Cosmic Inflation. I explain why the quantum theory of de Sitter (dS) space should have a finite number of states and explore gross aspects of the hypothetical quantum theory, which can be gleaned from semiclassical considerations. The constraints of a self-consistent measurement theory in such a finite system imply that certain mathematical features of the theory are unmeasurable, and that the theory is consequently mathematically ambiguous. There will be a universality class of mathematical theories all of whose members give the same results for local measurements, within the a priori constraints on the precision of those measurements, but make different predictions for unmeasurable quantities, such as the behavior of the system on its Poincare recurrence time scale. A toy model of dS quantum mechanics is presented.

KEYWORDS: Quantum Gravity, de Sitter Space.
1. Introduction

Observations suggest that the expansion of the universe is accelerating. The simplest explanation of this acceleration is a non-zero cosmological constant. Conventional string/M theory cannot accommodate a positive cosmological constant. At best it can have metastable positive energy density minima of an approximately defined effective potential in extreme regions of moduli space[1]. The significance of these minima is currently under study. In these talks I concentrated on the alternative hypotheses that there is an extension of string theory which defines quantum theor(ies) of gravity for any positive cosmological constant, $\Lambda$, which approach an isolated super-Poincare invariant string/M vacuum in the limit of vanishing $\Lambda$. The finite $\Lambda$ theories are conjectured[4][6] to have a finite number of physical states.

The fact that different values of the cosmological constant define different quantum systems, rather than different superselection sectors of the same system, is, for negative cosmological constant, one of the predictions of the AdS/CFT correspondence. One can view the negative cosmological constant as a parameter which controls the behavior of the high energy density of states in an asymptotically AdS universe. For positive cosmological constant the conjecture is that there is a high energy cutoff, and a finite number of states, controlled by the value of $\Lambda$. 
To understand why one would make such a conjecture, consider the description of dS space in terms of asymptotic past and future boundaries $I_{\pm}$, as advocated by Witten[2] and Strominger[3]. Naively, one might imagine a theory of correlation functions on these boundaries, which could be interpreted as defining a sort of S-matrix for dS space. In the semiclassical approximation, these would be defined by solutions of the bulk field equations, with boundary conditions on $I_{\pm}$. In asymptotically flat or AdS universes, such a definition makes sense because generic solutions with such boundary conditions exist, and have at most localized singularities\(^1\). The phase space of the system in these cases is indeed parametrized by arbitrary perturbations on the boundary.

This is not correct in asymptotically dS space. Heuristically, if we insist that each boundary probe insert a minimal finite energy density (in global coordinates), no matter how small, into the system, then the solution with some finite number of probes will have an energy density of order the Planck scale by the time the minimal radius of dS space is achieved. The solution will have a singular Big Crunch or Big Bang and will fail to be asymptotically dS in either the past or the future\(^2\). A more mathematical statement of this is that the phase space of Einstein gravity plus sensible matter with asymptotically dS boundary conditions in both past and future is compact. This is not yet a theorem, but is believed by many relativists. Preliminary results in this direction were obtained in unpublished work of Horowitz and Itzhaki[12]. It is well known that classical systems with compact phase space have a finite number of states when quantized.

A different, but related argument for a finite number of states comes from a study of classical and semiclassical physics within the causal diamond of a timelike observer in dS space. This region has a timelike Killing vector, and the corresponding classical Hamiltonian [5] has finite energy excitations corresponding to Kerr-dS black holes of various radii and angular momenta. There is a maximum energy black hole: the Nariai solution. Semiclassical arguments[13] indicate that the density matrix describing dS space is thermal, with a fixed temperature and finite entropy. When combined with an upper bound on the energy spectrum, a finite entropy thermal density matrix implies a finite number of states. This was the original argument in [4]. The two arguments

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\(^1\)If the Cosmic Censorship hypothesis is correct, every such singularity is shrouded behind a black hole horizon.

\(^2\)The question of whether such singular, unidirectionally asymptotically dS spaces have a finite number of states is more complicated. To answer it one must have a theory of the singularity. If one makes the hypothesis that solutions in which the singularity occurs at different values of the radius of the dS sphere should not be included in the same system, then these spaces have only finite numbers of states as well.
are related because the result of a too violent perturbation of the boundary of dS space is undoubtedly the creation of large black holes. When the black hole size exceeds the Nariai limit, we instead get a spacelike singularity covering an entire Cauchy surface of the space-time.

2. Groups, and boundaries

Isometries are diffeomorphisms and diffeomorphisms are gauge transformations. Thus, in general we do not expect an action of isometries on the physical states of a quantum system which obeys the principle of general covariance. Instead, global symmetry groups are constructed from equivalence classes of diffeomorphisms with certain action on the timelike or null boundary of a space-time. Global dS space does not have such boundaries, but the causal patch of a timelike observer does. In [8] it was suggested that the entire quantum mechanics of dS space could be understood from the point of view of any given observer, with the different descriptions related by a Cosmological Complementarity principle. This is a generalization of the idea of Black Hole Complementarity introduced by Susskind, Ughum and Thorlacius and by ’t Hooft. E. Verlinde has suggested the name Observer Complementarity for the general principle which states that observers who are classically causally disconnected, use the same Hilbert space in the quantum theory (in field theory they would use independent tensor factors of the Hilbert space), but measure mutually non-commuting observables.

Gomberoff and Teitelboim[5] have shown how to construct classical symmetry generators for general Kerr-dS black hole spacetimes. In this formalism one can treat empty dS space as the zero mass limit of a black hole. The only generators which make sense are those which preserve the causal patch and its cosmological horizon, which is the surface on which boundary conditions are imposed and the generators are defined. These form an $R \times SO(3)$ subgroup of the $SO(4, 1)$ isometry group of $dS_4$. The coset of this subgroup in the dS group is a set of gauge transformations, which map the physical Hilbert space into copies of it viewed from the point of view of different timelike observers.

This restriction to a subgroup is satisfying, because it is compatible with a quantum theory with a finite number of physical states. However, it seems to raise a puzzle about how the Poincare group will emerge in the small $\Lambda$ limit, since we usually realize it as a contraction of the full dS group. In order to understand what is going on, we should compare the boundaries of the two space-times, where the symmetry generators are defined, rather than the bulk, where the isometries act as diffeomorphisms i.e. gauge transformations.

The near horizon geometry of dS space is
\[ ds^2 = R^2 (dudv + d\Omega^2) \]  \hspace{1cm} (2.1)

where \( v \rightarrow 0 \) defines the future cosmological horizon. The static Hamiltonian is the boost generator on \( u \) and \( v \).

Future null infinity in asymptotically flat spacetime is the \( v \rightarrow 0 \) limit of

\[ ds^2 = \frac{(dudv + d\Omega^2)}{v^2} \]  \hspace{1cm} (2.2)

To remove the infinity one performs an infinite conformal transformation, and obtains a manifold with only conformal structure and coordinates \((u, \Omega)\). The asymptotic symmetry group is the semi-direct product of the conformal group of the sphere (isomorphic to the Lorentz group) with the infinite abelian group whose generators are \( f(\Omega) \partial_u \), with arbitrary \( f \). This is the Bondi-Metzner-Sachs group. If we restrict attention to classical vacuum spacetimes with no classical gravitational radiation (as would be appropriate in dimensions where an S-matrix exists), this can be reduced to the Poincare subgroup where \( f \) is restricted to be the constant function or the \( j = 1 \) spherical harmonic.

Apart from the rotations of the sphere, the static dS subgroup and the Poincare group have no generators in common. Lorentz invariance is a conformal symmetry which appears only in the limit of infinite dS radius.

Thus, in constructing a quantum theory of dS space, one is interested in finding two different Hamiltonians. One is an exact symmetry which describes time translation for a static observer at the center. The second is the Poincare Hamiltonian, which is relevant for describing scattering processes in the limiting asymptotically flat spacetime. \textit{A priori} there is no connection between these two generators, but we might expect some similarities for a subclass of localizable low energy states.

### 3. States, localization, and temperature

The local rule for maximizing entropy in a gravitational system is to form a large black hole. In dS space this procedure reaches a limit at the Nariai black hole. Every black hole in dS space has both a cosmological and a black hole horizon, but even the sum of the entropies of the two horizons of the Nariai black hole is only \( 2/3 \) of the entropy of the full dS space. Indeed, the total entropy of a black hole monotonically decreases in dS space, while the entropy associated with the black hole horizon increases.

This is an indication that the black hole states are actually relatively low entropy states of the full dS spacetime. They are made by borrowing degrees from the horizon...
of empty dS space and freezing them into special configurations. We will see a rather explicit model of this below.

Another class of states in dS quantum mechanics may be termed field theoretic states. These are the states which are well approximated by the treatment of quantum field theory in curved spacetime. Their back reaction on the geometry is supposed to be negligible. At the very least, we must require that they do not collapse into black holes.

The entropy of field theory states in a given volume (with no constraint on the energy) is dominated by the ultraviolet, where the field theory is described by a fixed point. The entropy is of order $M^3 R^3$, where $M$ is a UV cutoff. The energy of these states is of order $M^4 R^3$. In order that the Schwarzchild radius of the system be less than $R$, we must have $M^4 R^2 < 1$, in Planck units. Thus, the entropy in field theoretic states is less than $R^2$. In the static patch of dS space, $R$ is at most the dS radius. Thus, field theoretic states are even less entropic than black holes.

One concludes that most of the states are viewed by the static patch observer as being associated with the horizon of empty dS space. These have classical energy 0. From the point of view of classical GR, the association of the bulk of the entropy with the horizon is quite reasonable. The static observer never sees anything fall through the horizon, but does see any object to which she is not bound, get squeezed into an infinitesimal region near the horizon. The global observer sees these objects as living in a large set of disjoint horizon volumes, and finds a natural explanation for most of the entropy being invisible to any given static observer.

Our identification of field theoretic states allows us to understand how the global point of view of quantum field theory in curved spacetime might be approximately correct. Above we have stressed Observer Complementarity: all of the physics can be described from the point of view of a given static observer. In the analogous, but different situation of black holes in asymptotically infinite spaces, none of the degrees of freedom associated to the black hole horizon by the Schwarzchild observer, commute with the complete set of asymptotic observations made by this observer. If they did, then the principle of Black Hole Complementarity would not resolve the Information Paradox. All of the information available to the Schwarzchild observer can be read by him in terms of scattering measurements which are describable by local field theory (that is, the measurement itself, not necessarily the scattering amplitude being measured, can be so described).

The static observer in dS space is, in some respects, analogous to the Schwarzchild observer, but we have identified an important difference. Only a fraction of order $R^{-\frac{3}{2}}$ of the total entropy available in the static observer’s Hilbert space refers to local field theoretic measurements. This is consistent with the possibility of making of order $R^{1/2}$
commuting copies of the field theoretic degrees of freedom in a given horizon volume. Quantum field theory in global dS space predicts that in the far future one can find an infinite number of commuting copies of the degrees of freedom in a given horizon volume. Here we see that this becomes a better approximation at large $R$, although the number of copies is never strictly infinite.

If on the other hand, we make black holes whose size scales with the horizon, then there is no similar multiplication. The full quantum theory of dS space must be able to accommodate both sorts of state. However, it is amusing to note that the field theoretic states in a maximal collection of disjoint horizon volumes seem to be more typical states of the system than those with large black holes. They can saturate the full dS entropy.

The states which a static observer associates with the horizon of empty dS space dominate the thermal density matrix at the dS temperature. This is surely true if they have strictly zero energy. A much more likely picture is that they are distributed between $E = 0$ and a cutoff $E = \Delta$, with a density $e^{-S_{dS}}$. In this case they will still dominate the thermal entropy. Higher energy states are much less entropic. As a consequence, the thermal entropy of dS space will be close (in the limit of large $R$) to the logarithm of the number of horizon states, which, in the same limit, is approximately the total number of states of the system.

This picture of the spectrum of the dS Hamiltonian, suggests a self consistent explanation for the origin of the dS temperature. Namely, if we postulate this dense set of low energy states, and assume that their Hamiltonian is a random matrix, so that dynamics in this subspace is chaotic (as chaotic as a finite quantum system can ever be), then perhaps we need only postulate a weak coupling between these states and any other states of the system in order to explain why the localized systems experience thermal fluctuations. The dS temperature would then be determined by the cutoff $\Delta$ on the low energy spectrum. My student, Lorenzo Mannelli, is trying to prove this.

4. Measurement theory in dS space

Theoretical physics was invented to describe the result of outside measurement on an isolated system. With the advent of quantum mechanics, we have had to pay a little more attention to what we really mean by a measuring process. Initial discussions of this had to assume the existence of a separate classical world of measuring equipment. More modern discussions view this as an approximate description of a self consistent process of measurement of one quantum system by another. The discussion in this section of measurement theory in dS space is based on the paper[9].
There has been much discussion in the recent measurement theory literature of “environmental decoherence”: The effects of random interactions between the measurement apparatus and a large unmeasured “environment”. While not denying the existence of such effects for all realistic measurements, I would like to believe that they are not logically necessary to the existence of a sensible theory of measurement. If we are to take the step of extending the formalism of quantum theory to describe the entire universe, we must give up the crutch of unmeasured environments.

I believe that a reasonable measurement theory exists, without postulating environmental decoherence. All of measurement theory rests on Von Neumann’s observation that ordinary unitary evolution can take an uncorrelated state of a system plus a measuring apparatus into an entangled state in which each eigenstate of a complete set of commuting observables of the system, is correlated with a different “pointer” state of the apparatus.

\[
\sum a_n |n > \rightarrow |A > \rightarrow \sum a_n |n > |A_n >
\] (4.1)

In the theory of environmental decoherence, it is assumed properties of the pointer states’ interaction with the random environment that enable one to claim that further measurements on the system will not be sensitive to the relative phases of the \(a_n\). An alternative explanation of decoherence is illustrated by a simple model. Suppose we are trying to measure a single spin, and we model our measuring apparatus by a cutoff quantum field theory with two degenerate minima, \(\phi_±\) in a volume \(V\) which is large in cutoff units. Postulate a nonlocal coupling of the spin to the field theory which correlates the state \(\sigma_3 = 1\) with \(\phi_+\) and \(\sigma_3 = -1\) with \(\phi_-\). This is a cartoon of the amplification that is necessary to get a microscopic phenomenon to register on a macroscopic apparatus.

What do we mean by this correlation? \(\phi_±\) are not really single states but labels for whole ensembles of states in which the field takes values very close to \(\phi_±\) in most of the volume \(V\). Now consider any operator \(O_{loc}\) which is localized in a volume much less than \(V\). The matrix elements of \(O_{loc}\) between any pair of states from the two different ensembles, is of order \(e^{-V}\). I now claim that our correlated state is one in which we can say that a measurement has been made. Further local perturbations of the apparatus will not change the fact that the states where the spin is positive and negative can communicate only by amounts of order \(e^{-V}\). Expectation values of system operators in the correlated state and all states it evolves into under local perturbations over times short compared to \(e^V\) will follow the rules of classical probability.

Over times of order \(e^V\), tunnelling between the two would be superselection sectors will occur, and the measurement will lose its coherence. But in ordinary quantum
mechanics we can imagine taking $V$ as large as we like. Thus we can approximate Copenhagen measurements of quantum systems as well as we like, and thus give operational meaning to the mathematically precise formulae of the quantum theory.

In theories of quantum gravity this argument must be rethought. The large, almost classical, measuring devices will gravitate and have potentially large effects on the system they are supposed to be measuring. The only way to avoid this, is to place the measuring devices further and further away from the system, as we try to make them larger in order to make the measurements more precise and more robust against quantum fluctuations in the apparatus. This is why the only mathematically precise observables in good theories of quantum gravity are S-matrix elements (and their analogs in other infinite geometries).

We can see that when we come to dS space we are in a bind. If we are trying to measure the results of an experiment, which is traveling along a particular timelike geodesic, the best we can do is to measure its influence on a freely falling detector that is practically at the cosmological horizon of the experimental system. This is the closest analog of a scattering matrix that can be achieved in dS space. The detector can be made very large without significant effect on the experiment.

The key question now is how large it can be. If we require that the detector’s workings can be understood with “current technology”, then according to the above discussion, the detector must be built from what we have called field theoretic states in the static patch. In that case, an extremely conservative lower bound on the tunneling amplitudes between pointer states of the detector, is of order $e^{-bR^3/2}$, with $b$ a constant (much) less than one and $R$ the radius of dS space. This can only be achieved with detectors whose size is a finite fraction of the cosmological horizon.

There is a hypothetical possibility for the construction of more robust detectors. For field theoretic detectors, tunneling amplitudes, are of order $N^{-p}$, where $p$ can be a number of order 1, and $N$ is the number of states of the detector. Black holes whose size scales like the cosmological horizon have a much larger number of states than any field theoretic system. In principle they could provide the mechanism for more robust detectors. However, in order for that to work, one must be able to construct pointer states for the black hole. In field theoretic models the robustness of pointer states depends on the concept of superselection sector, which is itself a consequence of locality. Such considerations do not apply to the states on a black hole horizon. Indeed, the existence of an elaborate set of pointer states of a black hole, which would enable us to make precise and robust measurements of a multitude of observables external to the black hole, would seem to contradict the no hair theorem and the thermal nature of black hole physics. Nonetheless, since we cannot rule out the possibility rigorously, the use of black holes as detectors must be considered. The tunneling amplitudes between
pointer states of such monstrous detectors would be bounded from below by something of order $e^{-cR^2}$. Again we would expect $c$ to be much less than one.

These considerations imply that there is a fundamental limit, both to the precision of any measurement in dS space and to the amount of time for which any actual physical object in dS space can play the role of an idealized Copenhagen measuring device. *This time scale is always much less than the Poincare recurrence time*, even if we accept the bizarre possibility of detectors constructed from the microstates of black holes.

Historically, mathematical formulae for observable quantities in theoretical physics were presumed (to the extent they were presumed exactly correct) to be precise results to which actual measurements could approximate with any required degree of precision. Once we accept the rules of quantum mechanics, and the hypothesis that the entire universe has a finite number of physical states this can no longer be correct. Considerations of gravitational interactions and the geometry of dS space give us a more refined estimate of the fundamental limits on the precision of measurements in such a situation. It seems absolutely clear that there will then be *many* Hamiltonian descriptions of the physics of dS space, that will fit all conceivable experiments within the fundamental limits on their precision. It also seems clear where the modifications that do not affect ordinary measurements will come from. Most of the states on the horizon do not affect measurements in the interior, apart from providing the thermal bath at the dS temperature, and perhaps renormalizing the effective local field theory Lagrangian describing field theory states in the interior[7]. We have already suggested the idea that the horizon states could be described by a random Hamiltonian with an appropriate spectral cutoff related to the dS temperature.

It seems likely to me that the proper mathematical description of this situation will utilize the concept of universality classes from the theory of phase transitions. dS space is a finite system. The vanishing cosmological constant limit is a critical limit in which the number of states goes to infinity. There will be a universality class of Hamiltonians which describe dS space in this limit, and give the same answers for all observables with the fundamental limits on precision that we have outlined. Our considerations suggest that the predictions of these different mathematical theories will be the same, over reasonable periods of time, to all orders in powers of the cosmological constant. The imprecisions we have identified vanish like the exponential of a power of the cosmological constant. This means that for all practical purposes, the mathematical formulation of dS space will be predictive.

However, when it comes to questions of what happens to the system over a Poincare recurrence time [10] the different Hamiltonians will give different results. The Poincare recurrence time is the inverse of the level splitting we have hypothesized between states on the cosmological horizon. Thus, if the Hamiltonian ambiguity is indeed mainly
associated with the description of the horizon states, we expect all of the physics on
the recurrence time scale to be completely unpredictable. Different Hamiltonians in the
universality class will give different results. Since, in principle, no actual observations of
this physics can be made, this should not bother us. Rather, we might want to view it
as a sort of gauge ambiguity in the description of dS space, which affects mathematical
aspects of the formalism, without affecting the predictions for observable physics.

It is tantalizing to try to associate this ambiguity with the gauge invariance of
general relativity under change of time coordinate, the famous Problem of Time. Inde-
deer, in spaces without asymptotic boundary, a generally covariant theory does not
give any definite prescription for what the time evolution operator is. Wheeler-deWitt
quantization suggests instead that a system may have many non-commuting time evo-
lution operators associated with different semiclassical clocks. The mutual quantum
incompatibility between different semiclassical clocks is at the root of the principle of
observer Complementarity. At the classical level, we have tried to remove this ambi-
guity for dS space by choosing the proper time of a given timelike observer to define
the Hamiltonian. However, a fixed timelike observer is a classical concept. Perhaps the
inevitable imprecisions we have discovered in the quantum mechanics of dS space can
be related to a quantum version of the Problem of Time.

There is one final note about measurement theory in dS space, which connects
this discussion to our previous remarks about symmetry generators. The freely falling
devices we have been thinking about up to this point do not really correspond to mea-
surements made by an observer bound to the experiment which defines the particular
static coordinate system that our quantum formalism refers to. Rather, they are the
best dS approximation to “S-matrix meters”. They measure amplitudes which will
become the scattering matrix in the $\Lambda \to 0$ limit.

Actual measurements done by a static observer are of necessity less precise than
these S-matrix measurements. Since he remains bound to the experiment, the size of
device that he can build without gravitationally interacting with the experiment and
changing its result, is much more limited. In order to read the results of the S-matrix
meters he must send devices out to their position, which must then accelerate back to
him. These devices will be affected by the very high temperature radiation that an
accelerated observer experiences near the horizon.

The approximate Poincare generators will have a natural action on the states mea-
sured by the freely falling S-matrix meters. On the other hand, the measurements made
by the bound observer will be naturally described in terms of the static dS Hamiltonian.
5. A toy model of dS quantum mechanics

This model has been constructed in collaboration with B. Fiol. It is definitely work in progress, and a lot more progress needs to be made. There are several basic principles that we used. The first was to realize the spherical geometry of the cosmological horizon, in a way that was compatible with having a finite number of states. This motivates the introduction of fuzzy spheres (M. Li[11] has utilized fuzzy spheres for a hypothetical description of dS quantum mechanics.). For the moment our considerations are restricted to four spacetime dimensions. The corresponding fuzzy sphere is two dimensional and this is the only case where a complete technology exists. The restriction to four dimensions may be only technical, but it may have a deeper significance. If the $\Lambda \to 0$ limit of the theory is supersymmetric, it must be four dimensional. Only minimal four dimensional SUGRA admits a dS deformation.

The second principle that we use is the approximation of Asymptotic Darkness. That is, we attempt to describe a quantum theory with stable black holes, and account for the entropy and energy of these black holes. The idea is to find a description of the high energy spectrum, where Hawking decay is negligible. This should make sense for asymptotically small $\Lambda$. In dS space, in contrast to asymptotically infinite spaces, one must, even in the asymptotic darkness approximation, take into account the huge reservoir of dS vacuum states. The asymptotic darkness approximation also neglects the splittings between black hole eigenstates, as well as those between vacuum eigenstates.

There is a peculiar feature of the asymptotic darkness approximation in dS space. In AdS space, large black holes are stable. In asymptotically flat space, they are unstable but correspond to long lived resonances in scattering amplitudes. The black hole mass thus has significance even when corrections to the asymptotic darkness approximation are taken into account. By contrast, in dS space a black hole decays into objects which fall through its cosmological horizon\(^3\). Thus, in the full theory, a black hole must be viewed as a state which can be written as a superposition of vacuum eigenstates. It’s energy cannot be much above the dS temperature. Thus, corrections to the asymptotic darkness approximation are large.

One can get an intuitive idea for why this might be so by considering moving black holes in the asymptotic darkness approximation. Consider a pair of black holes, which are not bound by their mutual gravitational attraction, as viewed from the static frame defined by one of them. The second black hole will fall into the cosmological horizon of

\(^3\)Even if the black hole leaves behind a stable remnant, its mass will be much smaller than that of the hole. In this case some of the statements below will be modified, but only by replacing vacuum state by stable remnant state in appropriate places.
the first and therefore has (approximately) zero energy as measured by the Hamiltonian in the static frame. This is true even for black holes which are for some finite range of time, very close to the central one. One concludes that there must be superpositions of vacuum eigenstates whose spacetime description is arbitrarily close to that of the static frame black hole. The number of these states is much larger than the number of static black hole states. It is easy to imagine that when we split the Hamiltonian as $H = H_{AD} + V$, in the asymptotic darkness approximation, that the perturbation $V$ will have order one matrix elements between the static black hole state and superpositions of vacuum states which represent close by, moving, black holes. These can lead to a significant lowering of the actual black hole eigenvalue.

Given this remark, one may question the utility of the asymptotic darkness approximation for studying dS space. Recall however that there are two interesting Hamiltonians to construct in dS space with small $\Lambda$. The other one is the approximate Poincare Hamiltonian. The splitting of the Hilbert space into black hole states and vacuum states will definitely be useful for the Poincare generator. Although I will not discuss the Poincare generator here, this is the best we can do at present, so let us proceed.

Our fundamental variable will by a complex $N \times N + 1$ matrix, $\Psi_i^A$. We view it as a bimodule over the fuzzy sphere by allowing the appropriate irreducible representation of $SU(2)$ to act on it both on the left and the right. It is clear that $\Psi$ transforms in a half integral spin representation. In the limit $N \to \infty$ it will be a section of the spinor bundle on the sphere. We will quantize $\Psi$ as a fermion, consistent with the spin statistics theorem,

$$[\Psi_i^A, (\Psi^\dagger)_B]_+ = \delta_i^j \delta_A^B.$$

The Fock space formed by these fermionic operators has dimension $2^{N(N+1)}$. Recalling that the radius of the fuzzy sphere scales like $N$, we see an entropy that scales like the area, at least for the completely uncertain density matrix on this space.

In the asymptotic darkness approximation we expect the entire Hilbert space to decompose into eigenspaces of an approximate Hamiltonian, corresponding to the vacuum, and to black holes of various masses. We will take the Hamiltonian in this approximation to commute with the total fermion number (we do not expect such a quantum number in the exact theory). It is then natural to choose the vacuum density

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4This discussion is valid for black holes whose size does not scale to infinity with the dS radius. The concept of multiple black holes moving with respect to each other probably does not make sense close to the Nariai limit.
matrix to be the projection operator on half-filled states, relative to the Fock vacuum of $\Psi_A$.

A corresponding guess for the black hole states is to write $N = N_+ + N_-$, with $N_+ \geq N_-$. The black hole density matrix is then the projection on states, where we only allow filling by creation operators from either the first $N_+$ rows and $N_+ + 1$ columns, or the last $N_-$ rows and last $N_- + 1$ columns to act on the vacuum, and consider both subsystems to be at half filling.

The microcanonical entropy of this state is $\frac{N_+^2}{2} + \frac{N_-^2}{2}$ (for large values of the two integers). We identify the two terms in this formula with the entropies of the cosmological and black hole horizons of the Schwarzschild-deSitter black holes. They coincide for the maximal black hole horizon area, which occurs at $N_+ = N_-$. The total entropy is then equal to one half what we have identified as the entropy of empty dS space.

Although this is qualitatively the behavior we expect from the semiclassical thermodynamics of dS space, one would like to do better and get the relative coefficient in the entropy on the nose (the absolute coefficient will just be the identification of Newton’s constant in this system). The cosmological and black hole horizons satisfy the classical relations

$$R_+^2 + R_-^2 + R_+ R_- = R^2$$

which gives a factor of $2/3$ between the total Nariai black hole entropy and the empty dS entropy. Our calculation is clearly missing an entropy of order $R_+ R_-$. 

There are two possible explanations for this discrepancy. First, we should really be calculating a thermal entropy in the canonical ensemble at the dS temperature. We have defined black hole states by forbidding the excitation of the off diagonal operators entirely. Perhaps instead they should be allowed, but with Boltzmann suppression. Since the log of the number of these states is of order $N_+ N_-$ one can hope to make up our entropy deficit by including them. In order to get a finite fraction of the state counting entropy, the energy we assign to these states has to be of order the dS temperature.

Another possibility is that the factor of $4/3$ between the two answers should be viewed as an artifact of the asymptotic darkness approximation, analogous to the factor that occurs in the free field calculation of the entropy of near extremal black three branes. In this view, only the fully interacting theory will get the coefficient correct. The fully interacting theory will however have to deal with the fact that the black holes are unstable. In such a calculation, the entropy of empty dS space should be counted as the thermal entropy of the entire Hilbert space, including the sectors that are black hole eigenstates in the asymptotic darkness approximation. It is not clear
to me whether the semiclassical calculation of Gibbons and Hawking refers to the full entropy, or the entropy of the empty dS vacuum in the approximation in which black holes are stable eigenstates, orthogonal to the vacuum states.

As noted in the beginning of this section, the explicit model of dS quantum mechanics is as yet in a very primitive stage. Nonetheless, it gives a hint about the way in which a consistent quantum theory could reproduce the semiclassical thermodynamics of dS space.

6. Conclusions

Semiclassical analysis leads to the conclusion that a quantum theory of dS space should have a finite number of states. This is implied both by (as yet non-rigorous) arguments that the phase space of quantum gravity with past and future asymptotically dS boundary conditions, is compact, and by the combination of the finiteness of the Gibbons-Hawking entropy and the cutoff on static energies implied by the existence of a maximal mass black hole.

The dS entropy is thermal, but analysis of states in dS space leads to the conclusion that it must primarily represent a very dense spectrum of levels of the static Hamiltonian at energies below the dS temperature. The entropy is then, approximately the logarithm of the number of these states. The entropy of states that can be described by local field theory in a given horizon volume is bounded by something of order $R^{3/2}$. This is consistent with a dual description of the full set of states in terms of $R^{1/2}$ commuting copies of the field theoretic degrees of freedom. I argued that this is approximately the same as the description of (cutoff )local field theory in global coordinates, except that the latter formalism implies an infinite number of copies of the static patch degrees of freedom. From the point of view of the static observer, the states corresponding to local excitations outside his horizon are viewed as very low energy states on the horizon. The global field theoretic picture breaks down drastically when processes which create horizon scale black holes in a single static patch are considered. These put the system into a low entropy state in which the dynamics outside the horizon is frozen.

The existence of this dense spectrum of levels suggests a mechanism for understanding the temperature of dS space. It is simply the result of interaction of the localizable states with these low energy horizon degrees of freedom. The temperature is an indication of the energy cutoff on the horizon states. Calculations to verify this conjecture and understand the precise relation between temperature and cutoff are in progress.

The finiteness of the number of states and the paucity of states that can be described by field theory inside the cosmological horizon, puts fundamental limits on
measurements in dS space. In particular, no self consistent measuring device can be constructed in the theory, which will retain its classical character over times comparable to the Poincare recurrence time.

In my view this represents a fundamental ambiguity in the mathematical description of dS space. Many mathematical theories will give the same results for all measurable quantities within the limits set by the unavoidable lack of precision of measurement in this system. The ambiguities are smaller than any power of the cosmological constant in Planck units, and have little practical significance, but they are conceptually important. To someone in a pretentious frame of mind, they represent the fundamental limit on the basic assumption of theoretical physics, that the observer can be separated from the object it observes. It is likely that most of the ambiguity refers to the dynamics of the horizon states. I would conjecture that they can be described by a more or less random Hamiltonian, subject to a few constraints.

The above discussion was relevant to the observations made by a timelike observer in dS space. The quantum mechanics of such an observer uses the static dS Hamiltonian. I showed that in the limit of vanishing cosmological constant, we should expect the system to exhibit a new symmetry group, the Poincare group\(^5\) (most of) which is not related to the dS generators, and in particular, not to the static Hamiltonian. The complicated horizon states completely decouple from this limiting dynamics. It describes observations made by freely falling detectors, near the cosmological horizon. In the limit, the horizon becomes null infinity and we find the dynamics of an asymptotically flat space-time.

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