Kazhdan Lusztig Cells in Infinite Coxeter Groups

Belolipetsky MV1,2* and Gunnells PE3
1 Department of Mathematical Sciences, Durham University, South Rd, Durham DH1 3LE, UK
2 Institute of Mathematics, Koptygaya 4, 630090 Novosibirsk, Russia
3 Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA

Abstract

Groups defined by presentations of the form \( \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle \) are called Coxeter groups. The exponents \( m_{ij} \in \mathbb{N} \cup \{ \infty \} \) form the Coxeter matrix, which characterizes the group up to isomorphism. The Coxeter groups that are most important for applications are the Weyl groups and affine Weyl groups. For example, the symmetric group \( S_n \) is isomorphic to the Coxeter group with presentation \( \langle s_1, \ldots, s_n \mid s_i^2 = 1 \rangle \) and is also known as the Weyl group of type \( A_{n-1} \).

Keywords: Lusztig cells; Coxeter; Hyperplane

Introduction

The notion of cells was introduced by Kazhdan and Lusztig [1,2] to study representations of Coxeter groups and their Hecke algebras. Later it was realized that cells arise in many different branches of mathematics and have many interesting properties. Some examples and references for the related results can be found in Gunnells et al. [3]. We are interested in the combinatorial structure of the cells in infinite Coxeter groups. Examples of such groups include the affine Weyl groups, as well as Weyl groups of hyperbolic Kac–Moody algebras. In contrast with the affine case, there are very few results on cells in hyperbolic Coxeter groups available so far. We refer to Bedard and Belolipetsky [4,5] for some work in this direction. Based on numerous computational experiments we introduce two conjectures that describe the structure of the cells in infinite Coxeter groups. Our conjectures use combinatorial rigidity for elements in an infinite Coxeter group ([4,5]), and is also known as the Weyl group of type \( A_{n-1} \).

We expect that this notion may be of independent interest. We were able to check the conjectures for affine groups of small rank [6] ([§5]), and to prove them in some special cases [7] ([§4]). The latter include, in particular, the right-angled Coxeter groups previously studied in Belolipetsky [5]. The proof of the conjectures in their general form remains an open problem.

Visualization of cells

Let \( W \) be a Coxeter group with a fixed system of generators \( S \). Consider a real vector space \( V \) of dimension \( |S| \) with a basis \( \{ \alpha_s \mid s \in S \} \). We define a symmetric bilinear form on \( V \) by

\[ B(\alpha_s, \alpha_t) = -\cos(\pi / m(s,t)), \quad s, t \in S. \]

Now for every \( s \in S \) we can define a linear map \( \sigma_s : V \to V \) by \( \sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s \).

This map sends \( \alpha_t \) to \(-\alpha_t\) and fixes the hyperplane \( H_t \) orthogonal to \( \alpha_t \) with respect to \( B \). Therefore, \( \sigma_t \) is a reflection of the space \( V \). One can show that the map \( s \to \sigma_s \) extends to a faithful linear representation of the group \( W \) in \( \text{GL}(V) \), called the \textit{standard geometric realization} of the group.

Next let us introduce the \textit{Tits cone} \( \mathcal{C} \subseteq V \) of \( W \). Every hyperplane \( H_t \) divides \( V \) into two halfspaces. Let \( H^+_t \) denote the closed halfspace on which the element \( \alpha_t \) is positive. The intersection of these halfspaces \( \Sigma_0 = \bigcap H^+_t \) for \( s \in S \) is a closed simplicial cone in \( V \). The closure of the union of all \( W \)-translates of \( \Sigma_0 \) is again a cone in \( V \). This cone \( \mathcal{C} \) is called the Tits cone. It is known that \( C = V \) if and only if \( W \) is finite. For infinite groups \( W \) the Tits cone is significantly smaller than the whole space and hence is more convenient for the geometric realization of the group.

Under some additional assumptions we can define the action of \( W \) on a section of the Tits cone. In particular, this way we can describe the action of affine or hyperbolic Coxeter groups of rank 3 on a Euclidean or Lobachevsky plane, respectively. For example, consider the affine group \( W \) of type \( \tilde{A}_2 \). This group is generated by three involutions \( s_1, s_2, s_3 \) satisfying the relations \( (s_i s_j)^{m_{ij}} = 1 \) for all \( i \neq j \). The space \( V \) is isomorphic to \( \mathbb{R}^3 \), and via this the Tits cone \( \mathcal{C} \) can be identified with the upper halfspace \( \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0 \} \). It is easy to check that the action of \( W \) preserves the affine plane \( M = \{ z = 1 \} \), and that the images of \( \Sigma_0 \) intersect \( M \) in equilateral triangles (Figure 1). This construction implies that the group \( W \) of type \( \tilde{A}_2 \) can be realized as the discrete subgroup of the affine isometries of a plane generated by reflections in the sides of an equilateral triangle.

A generalization of this construction leads to \textit{triangle groups}. Let

*Corresponding author: Belolipetsky MV, Department of Mathematical Sciences, Durham University, South Rd, Durham DH1 3LE, UK, Institute of Mathematics, Koptygaya 4, 630090 Novosibirsk, Russia, Tel: +44 191 334 2000; E-mail: mikhail.belolipetsky@durham.ac.uk

Received July 21, 2015; Accepted August 03, 2015; Published August 31, 2015

Citation: Belolipetsky MV, Gunnells PE (2015) Kazhdan Lusztig Cells in Infinite Coxeter Groups. J Generalized Lie Theory Appl S1: 002. doi:10.4172/1736-4337.S1-002

Copyright: © 2015 Belolipetsky MV, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
Δ be a triangle with angles \( \pi/p, \pi/q, \pi/r \), where \( p, q, r \in \mathbb{N} \cup \{\infty\} \). The triangle Δ lives on a sphere, or in an affine or hyperbolic plane, depending on whether \( 1/p + 1/q + 1/r \) is \( > 1 \), \( = 1 \), or \( < 1 \). Then the group \( W_\infty \) of isometries of the corresponding space generated by reflections in the sides of Δ is respectively a finite, affine, or hyperbolic Coxeter group. For example \( W_\infty \) is the affine group of type \( A_2 \). In Figure 2 we show the tessellations of the hyperbolic plane corresponding to \( W_\infty \) (the Hurwitz group, Figure 2a and \( W_\infty \) (the modular group, Figure 2b). The coloring of the triangles indicates the partition of \( W \) into cells, which we will define in the next section.

**Main Definitions**

Consider a Coxeter group \( W \) with a system of generators \( S \). Any element \( w \in W \) can be written as a product, or word, in the generators: \( w = s_1 \ldots s_k \in S \). Such an expression is called **reduced** if we cannot use the relations in \( W \) to produce a shorter expression for \( w \).

An element can have different reduced expressions but it is not hard to check that all of them have the same length. Therefore, we can define the **length function** \( l: W \to \mathbb{N} \cup \{0\} \), which assigns to an element \( w \in W \) the length of a reduced expression with respect to the generators \( S \) [1]. Another important notion which can be defined using the reduced expressions is the **partial order** \( \leq \) of Chevalley–Bruhat. Let \( s_1 \ldots s_k \) be a word in the generators. We define a **subexpression** as to be any (possibly empty) product of the form \( s_{i_1} \ldots s_{i_m} \), where \( 1 \leq i_1 \leq \ldots \leq i_m \leq M \). We say that \( y \leq w \) if an expression for \( y \) appears as a subexpression of a reduced expression for \( w \). It can be shown that the relation \( \leq \) is a partial order on the group \( W \) [1].

Let \( \mathcal{H} \) denote the Hecke algebra of \( W \) over the ring \( \mathbb{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}] \) of Laurent polynomials in \( q^{1/2} \). This algebra is a free \( \mathbb{A} \)-module with a basis \( T_w \) \( w \in W \) and with multiplication defined by \( T_w T_{w'} = T_w \) if \( l(ww') = l(w) + l(w') \), and \( T_q^2 = q + (q - 1)T_q \) for \( s \in S \). Together with the basis \( \{T_w\}_{w \in W} \), we can define in \( \mathcal{H} \) another basis \( \{C_w\}_{w \in W} \). This new basis, introduced by Kazhdan and Lusztig [2], has a number of important properties and has proven to be very convenient for describing the representations of \( W \) and \( \mathcal{H} \). The elements \( C_w \) can be expressed in terms of \( T_w \) by the formulae

\[
C_w = \sum_{y \in w} (-1)^{l(y)}q^{l(y)/2 - l(y)} P_{J_w}(q^{-1})T_y,
\]

where the \( P_{J_w}(t) \in \mathbb{A} \) are the Kazhdan–Lusztig polynomials. The polynomials \( P_{J_w}(t) \) are nonzero exactly when \( y, w \in W \) satisfy \( y \leq w \), equal 1 when \( y = w \), and otherwise have degree \( \deg(P_{J_w}) \) at most \( d(y, w) := (l(w) - l(y) - 1)/2 \). If \( \deg(P_{J_w}) \) we denote the leading coefficient by \( \mu(y, w) = \mu(w, y) \), and in all other cases (including when \( y \) and \( w \) are not comparable in the partial order) we put \( \mu(y, w) = \mu(w, y) = 0 \). We indicate that \( \mu(y, w) \neq 0 \) by \( y \perp w \).

Using the polynomials \( P_{J_w} \) we can define the partial orders \( \leq_M \) and \( \leq_L \) on \( W \). First, for \( w \in W \) we define the left and right descent sets:

\[
L(w) = \{ s \in S | sw < w \}, \quad R(w) = \{ s \in S | ws < w \}.
\]

Next, we say that \( y \leq_L w \) if there exists a sequence

\[
y = y_0, y_1, \ldots, y_n = w \quad \text{in} \quad W \quad \text{such that} \quad y_{i-1} - y_i \quad \text{and} \quad L(y_i) \subsetneq L(y_{i+1})
\]

for all \( 0 \leq i \leq n \). The relation \( \leq_{R} \) can be defined using \( \leq_{L} \); we put \( y \leq_{R} w \) if \( y^{-1} \leq_{L} w^{-1} \). Finally, \( y \leq_{LR} w \) means that there exists a sequence

\[
y = y_0, y_1, \ldots, y_n = w \quad \text{such that} \quad \forall i \in n \quad \text{we have either} \quad y_i \leq_L y_{i+1} \quad \text{or} \quad y_i \geq_R y_{i+1}.
\]

It is easy to check that \( \leq_L \) and \( \leq_{LR} \) are partial orders on \( W \); we denote the corresponding equivalence relations by \( \sim_L \) and \( \sim_{LR} \). The equivalence classes of \( \sim_L \) (respectively, \( \sim_{LR} \)) are called the left cells (resp. right cells, two-sided cells) of \( W \). It follows from the definitions that the left and right cells have very similar properties, thus we will mostly deal with only one of these two types of cells.

The colors in Figure 2 indicate the left cells of \( W_{\text{aff}} \) and the 2-sided cells of \( W_{\infty} \). Furthermore, the left cells of \( W_{\text{aff}} \) can be seen in Figure 2b: also, they are the connected unions of triangles of a given color. Note that both of these groups apparently have infinitely many left cells. Now consider the multiplication of the \( C \)-basis element in \( \mathcal{H} \). We can write

\[
C_y C_z = \sum h_{x,y,z} C_x, \quad h_{x,y,z} \in A.
\]

Let \( a(z) \) be the smallest integer such that \( q^{a(z)2}h_{x,y,z} \in A^+ \) for all \( x, y \in W \), where \( A^+ = \mathbb{Z}[q^{1/2}] \). Denote by \( D_0 \) the set \( \{ z \in W | l(z) = a(z) = 2(\delta(z) - 1) \} \), where \( \delta(z) \) is the degree of the polynomial \( P_{J_{\infty}} \) and \( \delta(z) \) is the length function on \( W \), and \( a(z) \) is defined as above. The set \( D_0 \) consists of distinguished involutions of \( W \) introduced by Lusztig [8].

Lusztig proved that in affine groups there is a bijection between the set \( D_0 \) and the (or right) cells of \( W \) [8]. This deep result has many important corollaries and applications. One of the main goals of our work is to make this correspondence between distinguished involutions and cells explicit, so that one can describe the structure of the cells using the distinguished involutions of the group. Another goal is to find an algorithm that produces distinguished involutions of a given group. In the next section we will formulate two conjectures that answer these questions and present some results to support the conjectures.

**Conjectures and Results**

We will need some more notations and definitions. Given \( w \in W \), we write \( w = x \cdot y \) if \( w = xy \) and \( l(w) = l(x) + l(y) \). Denote by \( Z(w) \) the set of all \( v \in W \) such that \( w = x \cdot y \cdot z \) for some \( x, y \in W \) and \( v \in W' \) for some \( I \subset S \) with \( W' \) finite. (We recall that for a subset \( I \subset S \), \( W' \) denotes the standard parabolic subgroup of \( W \) generated by \( I \).) We call \( v \in Z(w) \) **maximal** in \( w \) if it is not a proper sub word of any other \( v' \in Z(w) \) such that \( w = x' \cdot y' \cdot z' \) with \( x' \subseteq x \) and \( y' \subseteq y \).

Let \( Z = Z(W) \) be the union of \( Z(w) \) over all \( w \in W \). \( D_0' = D_0 \cap Z \) be the set of distinguished involutions of the finite standard parabolic subgroups of \( W \) and \( D_0' = D_0' \setminus (S \cup \{l\}) \). Note that each of the sets \( Z(W), D_0 \) and \( D_0' \) is finite.

**References**
We call $w = x_0y$ rigid at $v$ if (i) $v \in \mathcal{D}_v$, (ii) $v$ is maximal in $w$, and (iii) for every reduced expression $w = x'_0y'$ with $a(v') \geq a(v)$, we have $I(x') = I(x')$ and $I(y') = I(y')$. This notion of combinatorial rigidity plays an important role in our considerations. Figure 3 helps to understand its meaning using the Cayley graph of the group $W$. Maximal distinguished involutions of the finite parabolic subgroups correspond to the "long cycles" in the graph. Combinatorial rigidity means that such a cycle cannot be shifted along the presentation of $w$ in any direction. For example, in the triangle group $\Delta_{3,3,3}$, the element $w = s_3s_2s_3s_2s_3s_2$ is rigid at $v = s_3s_2s_3$, but $w' = s_3s_2s_3s_2s_1s_2$ is not rigid at $v$ (Figure 3).

Our conjectures can be formulated as follows:

**Conjecture 1:** ("distinguished involutions") Let $v = x_0y \in \mathcal{D}$ with $v_1 \in \mathcal{D}_v$ and $a(v) = a(v_1)$, and let $v' = s_0v, s_1v, s_2v, s_3v, s_4v$, and $v'' = s_6v$, rigid at $v$, then $y' = v_1y$. Then if $s v_0 = v_1$, we have $v' \in \mathcal{D}$.

**Conjecture 2:** ("basic equivalences") Let $w = y v_1y \in \mathcal{D}$ with $v_0$ maximal in $w$.

(a) Let $u = x_0y, x_0y' \in \mathcal{D}$ satisfies $a(u) = a(v_0)$ and $w' = wu$ is reduced and has $a(w') = a(w)$. Then there exists $v_0'$ such that $v_0' \in \mathcal{D}_v$ and $l(v_0') = l(v_0')$, is rigid at $v$, for every $v_0'$, such that $v_0' \in \mathcal{D}_v$, is rigid at $v_1$.

(b) Let $w = v_0v_1$ with $v_0 \in \mathcal{D}_v$ not maximal in $w'$ and $a(w') = a(v_0)$. Then we can write $v = y_0v_0y_0v_0y_0$, so that $y_0v_0$ is maximal in $w'$, $R(w_0) = R(w)$, and $R(w, w_0) \neq 0$, which implies $w' = w_0w_0w_0w_0w_0w_0w_0$. Conjecture 1 can be used to inductively construct distinguished involutions in an infinite Coxeter group $W$ starting from the involutions of its finite standard parabolic subgroups. Conjecture 2, in turn, allows one to obtain equivalences in the group using its distinguished involutions. Let us note that the usual method for obtaining results of this kind is based on computing Kazhdan-Lusztig polynomials. This requires many computations and, moreover, does not give any priori information about the elements that are distinguished involutions. Let us note that the usual method for obtaining results of this kind is based on computing Kazhdan-Lusztig polynomials. This requires many computations and, moreover, does not give any priori information about the elements that are distinguished involutions. Let us note that the usual method for obtaining results of this kind is based on computing Kazhdan-Lusztig polynomials. This requires many computations and, moreover, does not give any priori information about the elements that are distinguished involutions.

(2) The relations described in Conjecture 2 determine the partition of $W$ into right cells.

(3) The relations described in Conjecture 2 together with its $\sim_{\tau}$-analogue determine the partition of $W$ into two-sided cells.

The conjectures of Lusztig to which we refer in the theorem are so-called "positivity conjecture" and a conjecture about a combinatorial description of the function $a(v)$. The positivity conjecture is now proved for a wide class of infinite Coxeter groups that includes affine Weyl groups. The precise statements of the conjectures and the references for the known results can be found in Belolipetsky et al. [7].

We were able to prove Conjectures 1 and 2 under certain additional assumptions. The results are given in the following two theorems.

**Theorem 2:** [BG1] Let $v = x_0y, x_0y' \in \mathcal{D}$ with $v_1 \in \mathcal{D}_v$, $a(v) = a(v_1)$, and $L(v) \setminus R(v) \neq \emptyset$. Then if $v' = s_0v, s_1v, s_2v, s_3v, s_4v$, rigid at $v$, then $v' \in \mathcal{D}$.

**Theorem 3:** Let $w = x_0y, x_0y' \in \mathcal{D}$ with $v_0 \in \mathcal{D}_v$ such that $a(u) = a(v_0)$, and $w' = w_0w_0w_0w_0w_0w_0w_0$ with $v_0 = s_0v_0, s_1v_0, s_2v_0, s_3v_0, s_4v_0$ has $a(w') = a(w)$ and $R(w') \subseteq R(w)$ [7].

Assume that

(1) For any $v_j = t_j, t_j v_0, t_j v_0, j = 0, \ldots, n, -1 = t_j + 1$ or $t = t_j + 1$ if $t_j - 1 \in R(v_0)$, we have $a(v_j) = a(v_0)$, $L(v_0) \setminus R(v_0) \neq \emptyset$ and $v_0v_0$ is rigid at $v_0$.

(2) For any $v_j = t_j, t_j v_0, t_j v_0, j = 1, \ldots, n - 1$ with $u_0 = u$, we have $a(u_j) = a(u_0)$, $L(u_0) \setminus R(u_0) \neq \emptyset$ and $u_0u_0u_0$ is rigid at $u_0$.

Then $\mu(w, w') = 0v'$ and $w - w'$.

The additional assumption $L(v) \setminus R(v) \neq \emptyset$ in Theorem 2 may seem minor, but unfortunately this is not the case. In particular, conditions (1) and (2) in Theorem 3 appear as a consequence of this assumption. The proof of the theorems 2 and 3 in Belolipetsky et al. [7] essentially uses the results from two unpublished letters of Springer and Lusztig [9]. A possible approach to the proof of our conjectures in general requires developing further the ideas of this correspondence.

Although the conditions of Theorems 2 and 3 are not always met for all $W$, the theorems can still be used to produce interesting results. We will give some examples in the next section, other applications of the theorems are considered in Belolipetsky et al. [7].

**Cells in Affine Groups of Rank 3**

Affine Weyl groups of rank 3 have type $A$, $B(=C)$ or $G$. The cells in these groups were first described by Lusztig [10]. In this section we will show how the same results can be relatively easily obtained using conjectures from §4.

**Type $\tilde{A}_2$:** The group $W$ is generated by involutions $s_1, s_2, s_3$, with relations $(s_1)^2 = (s_2)^2 = (s_3)^2 = 1$. We have $\mathcal{D}_v = \langle s_1s_2s_1, s_2s_3s_2, s_3s_1s_3 \rangle$, $\mathcal{D}_v' = \langle s_3s_1s_3, s_2s_3s_2, s_3s_1s_3 \rangle$.

Applying Conjecture 1 with $v = v_1, s_3s_2s_3$, we get $v' = s_2s_3s_2 \in \mathcal{D}_v'$. Note that after this the inductive procedure terminates as the elements $s_2s_3$ and $s_3s_2$ which would come out on the next step are both non-rigid at $v$. We can apply the same procedure to the other two involutions from $\mathcal{D}_v'$. As a result we get

![Figure 3: (a) Rigid and (b) Non-rigid expressions in the Cayley graph.](image)
The application of Conjecture 1 in this case already requires some effort because of a large number of possible variants. In order to generate the list of distinguished involutions we used a computer. Our algorithms and their application to other affine Weyl groups are described in Belolipetsky et al. [6]. As a result of these computations, we obtain

\[ D = \{ 1, s_1, s_2, s_3, s_1s_2s_1, s_1s_2s_3, s_1s_2s_3s_1, s_2s_3s_1s_2, s_1s_2s_3s_1s_2s_3s_1 \}. \]

Therefore, the group \( W \) of type \( \tilde{A}_2 \) has 10 left (right) cells. The interested reader can check that Conjecture 2 allows us to obtain the partition of \( W \) into cells which is shown on Figure 4c. Note that for this case the conditions of neither Theorem 2 nor 3 are satisfied. For instance, we have

\[ L(s_1s_2s_3s_1s_2s_3s_1) \setminus R(s_1s_2s_3s_1s_2s_3s_1) = \emptyset. \]

Thus our results here rely on unproved instances of the conjectures, but nevertheless the results agree with Lusztig et al. [10].

References

1. Humphreys J (1990) Reflection groups and Coxeter groups. Cambridge studies in advanced mathematics 29, Cambridge University Press, Cambridge.
2. Kazhdan D, Lusztig G (1979) Representations of coxeter groups and hecke algebras. Invent Math 53: 165-184.
3. Gunnells PE (2006) Cells in coxeter groups. Notices of the AMS 53: 528-535.
4. Bedard R (1989) Left V-cells for hyperbolic coxeter groups. Comm Algebra: 17: 2971-2997.
5. Belolipetsky M (2004) Cells and representations of right-angled coxeter groups. Selecta Math NS 10: 329-339.
6. Belolipetsky M, Gunnells PE (2014) Cells in coxeter groups II in preparation.
7. Belolipetsky M, Gunnells PE (2013) Cells in Coxeter groups. J Algebra 385: 134-144.
8. Lusztig G (1987) Cells in affine Weyl groups II. J Algebra 109: 536-548.
9. Lusztig G, Springer TA Correspondence.
10. Lusztig G (1985) Cells in affine Weyl groups, Algebraic Groups and Related Topics. Adv Stud Pure Math 40: 255-287.