Connes’ Distance of One-Dimensional Lattices:

General Cases

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Abstract

Connes’ distance formula is applied to endow linear metric to three 1D lattices of different topology, with a generalization of lattice Dirac operator written down by Dimakis et al to contain a non-unitary link-variable. Geometric interpretation of this link-variable is lattice spacing and parallel transport.

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I Introduction

Lattice as a universal regulator for the non-perturbative definition of a quantum field theory works well for bosonic fields [1]. However, when fermionic fields are involved, lattice formalism encounters two well-known seemingly insurmountable
problems: implementation of grassmann number in simulations and No-Go theorem for chiral fermion on lattices \[2\]. On the other hand, lattice provides one simplest model of noncommutative geometry (NCG) \[3\]; NCG in Connes’ formulation has an intimate relation with fermion through a Hilbert space and a generalized Dirac operator \[4\]. Therefore, to explore lattice field theory in NCG context is significant for to understand those old puzzles. As the first step, because NCG endows a metric, hence a geometry, onto a space through Dirac operator, to consider this (Dirac-operator) induced metric on lattices exhibits new relation between lattice fermions and lattice geometry. In fact, the first striking nontrivial result along this line is that this distance is non-Euclidean, providing Naïve or Wilson-Dirac operator is adopted \[5\][6]. On the contrary, Dimakis and Müller-Hoissen (DM) proposed a new free Dirac operator which induces correct linear distance on a 1D lattice \[7\]. In this paper, we generalize DM’s result in case that a link-variable field is presented on this 1D lattice. We will show that the amplitude of this field modify the induced distance in the sense that its inverse provides a localized lattice spacing and that the phase of this field can play the role of a \(U(1)\)-parallel transport, hence a gauge potential.

This paper is organized as following. Connes’ distance is introduced in Sect.\[II\] and is calculated for three types of 1D lattices in Sect.\[III\] after generalized DM’s lattice Dirac operator is defined. Geometric interpretation is given in Sect.\[IV\].

II Connes’ Distance Formula

A spectral geometry in Connes’ sense, commutative or not, is defined to be a triple \((\mathcal{A}, \mathcal{H}, D)\) in which \(\mathcal{A}\) is a pre-\(C^*\) algebra being represented faithfully on Hilbert space \(\mathcal{H}\) and \(D\) is a self-adjoint operator on \(\mathcal{H}\) playing the role of Dirac operator in Classical spinor geometry \[8\]. In this paper, \(\mathcal{A}\) is taken to be the algebra of
complex functions on a lattice, $\mathcal{H}$ is the Hilbert space of fermionic fields which are not considered as grassmann-valued sections, and $\mathcal{D}$ is lattice Dirac operator to be specified. Connes’ distance is introduced by the formula

$$d_D(p,q) = \sup_f \{|f(p) - f(q)| : f \in \mathcal{A}, ||[\mathcal{D}, f]||_\mathcal{H} \leq 1\} \quad (1)$$

for all points $p,q$ of this lattice, where $||.||_\mathcal{H}$ is operator norm on $\mathcal{L}(\mathcal{H})$. Note that we do not distinguish $f$ from its imagine represented on $\mathcal{H}$ due to the faithfulness. To obtain a manipulable algorithm for Eq.(1), we define a $f$-Hamiltonian, $H(f) = [\mathcal{D}, f]^\dagger [\mathcal{D}, f]$. Then it is easy to verify that $||[\mathcal{D}, f]||_\mathcal{H}^2 = \sup_{\lambda}\{\lambda : H(f)\psi = \lambda_D(f)\psi\}$. Consequently, Eq.(1) can be expressed as

$$d_D(p,q) = \sup_f \{|f(p) - f(q)| : f \in \mathcal{A}, \forall \lambda_D(f) \leq 1\} \quad (2)$$

### III Lattice Dirac Operator and Induced Distance

We specify the term one-dimensional lattice by a discrete set $L$ together with a isomorphism $T$ acting on $L$. $\mathcal{A}(L)$ is denoted for algebra of complex functions on $L$ and Hilbert space is chosen to be $\mathcal{H} = \mathcal{A}(L) \oplus \mathcal{A}(L)$ which is a free module over $\mathcal{A}(L)$ of rank 2. $T$ induces an isomorphism of $\mathcal{A}(L)$ and an isometry of $\mathcal{H}$ to which we still write as $T$. DM’s free lattice Dirac operator can be written as

$$\mathcal{D} = T\sigma^+ + T^\dagger \sigma^-$$

where $\sigma^\pm$ are defined using Pauli matrices $\sigma^\pm = (\sigma_1 \pm i\sigma_2)/2$. We generalize it to be

$$\mathcal{D}(\omega) = \omega T\sigma^+ + T^\dagger \bar{\omega}\sigma^- \quad (3)$$

where $\omega \in \mathcal{A}(L)$. Below we consider three types of $(L,T)$ corresponding to three topologies in continuum limit.
III.1 Finite Open Lattice \(\hat{Z}_N\)

In this case, \(L\) is coordinatized by \(1, 2, \ldots, N\) and \((Tf)(i) = f(i + 1), i = 1, 2, \ldots, N - 1; (Tf)(N) = 0\) for all \(f \in \mathcal{A}(L)\), which we refer as \(\hat{Z}_N\). Notice Eq.\(\text{[3]}\), \(|D(\omega), f|\) = \(\omega^\partial^+ f T\sigma^+ + T^\dagger \bar{\omega}(-\partial^+ f)\sigma^-,\) where \((\partial^+ f)(i) = (Tf)(i) - f(i), i = 1, 2, \ldots, N - 1; (\partial^+ f)(N) = 0\). One can check that \(f\)-Hamiltonian \(H(f) = |\omega|^2|\partial^+ f|^2\sigma^+\sigma^- + |T^\dagger \omega|^2|\partial^- f|^2\sigma^-\sigma^+\) where \((\partial^- f)(i+1) = f(i) - (Tf)(i), i = 1, 2, \ldots, N - 1; (\partial^- f)(1) = 0\). Therefore, \(||D(\omega), f||_H^2 = ||(\omega \partial^+ f)||_\infty^2||_{\infty}\) in which \(||\cdot||_\infty\) is sup-norm of \(\mathcal{A}(\hat{Z}_N)\). According Eq.\(\text{[3]}\),

\[
d_{D(\omega)}(i, j) = \text{sup}_f\{|f(i) - f(j)| : f \in \mathcal{A}(\hat{Z}_N), |\omega^\partial^+ f|(k) \leq 1, k = 1, 2, \ldots, N\}
\]

for all \(i, j \in \hat{Z}_N\). If we assume \(\omega\) is non-singular, i.e. \(|\omega(k)| \neq 0\) for all \(k\), then \(d_{D(\omega)}(i, j)\) possesses an upper bound

\[
d_{D(\omega)}(i, j) \leq \sum_{k=0}^{j-i-1} |\omega(i+k)|^{-1}
\]

in which \(j\) is supposed to be larger than \(i\). Define \(f_\omega(i+1) = f_\omega(i) + |\omega(i)|^{-1}, f_\omega(1) = 0\), then \(||D(\omega), f_\omega||| \leq 1\) and \(f_\omega\) saturates the upper bound in Eq.\(\text{[3]}\). Subsequently, Eq.\(\text{[4]}\) becomes an equality, especially it holds that \(d_{D(\omega)}(i, i+1) = |\omega(i)|^{-1}\), to which a clear interpretation is that the inverse of amplitude of \(\omega(i)\) is the lattice spacing between \(i\) and \(i+1\). Note that it is obvious that the value of \(\omega\) at \(N\) makes no sense in this case.

III.2 Finite Close Lattice \(Z_N\)

Here \(L\) is labeled by \(0, 1, 2, \ldots, N - 1\) and \((Tf)(i) = f(i + 1), i = 0, 1, \ldots, N - 2; (Tf)(N - 1) = f(0)\) for all \(f \in \mathcal{A}(L)\), so addition of the argument of \(f\) makes \(L\) a finite group \(Z_N\). If we define \(\partial^+ f = Tf - f, \partial^- f = T^\dagger f - f\), then the deduction is exactly the same as that in Subsect.\(\text{II.1}\) and

\[
d_{D(\omega)}(i, j) = \text{sup}_f\{|f(i) - f(j)| : f \in \mathcal{A}(Z_N), |\omega^\partial^+ f|(k) \leq 1, k \in Z_N\}, \forall i, j \in Z_N
\]
With the non-singular assumption on $\omega$ and cyclic addition on $\mathcal{Z}_N$,

$$d_{D(\omega)}(i, j) \leq \min\{l(i, j), l(j, i)\}$$

(5)

where $l(i, j) = |\omega(i)|^{-1} + |\omega(i + 1)|^{-1} + \ldots + |\omega(j - 1)|^{-1}$. Now we design a function to saturate this upper bound. Without losing generality, let $l(i, j) \leq l(j, i)$ and define $f_\omega(i) = 0, f_\omega(i + 1) = |\omega(i)|^{-1}, f_\omega(i + 2) = f_\omega(i + 1) + |\omega(i + 1)|^{-1}, \ldots, f_\omega(j) = f_\omega(j - 1) + |\omega(j - 1)|^{-1}, f_\omega(j + 1) = f_\omega(j) - |\omega(j)|^{-1}l(i, j)l(j, i)^{-1}, f_\omega(j + 2) = f_\omega(j + 1) - |\omega(j + 1)|^{-1}l(i, j)l(j, i)^{-1}, \ldots, f_\omega(i - 1) = f_\omega(i - 2) - |\omega(i - 2)|^{-1}l(i, j)l(j, i)^{-1}$. It is easy to check that $||[D(\omega), f_\omega]|| \leq 1$ and that $f_\omega$ saturates the upper bound in Eq.(5). If $\omega$ satisfies triangle-inequalities $|\omega(i)|^{-1} \leq \sum_{k=1}^{N-1} |\omega(i + k)|^{-1}, \forall i \in \mathcal{Z}_N$, then $|\omega(i)|^{-1}$ is able to be interpreted as lattice spacing between $i$ and $i + 1$.

### III.3 Infinite Lattice $\mathcal{Z}$

$L$ is parametrized by integer $\mathcal{Z}$ in this case and $(T f)(i) = f(i + 1), \forall i \in \mathcal{Z}, f \in \mathcal{A}(L)$. However to guarantee convergency, we must consider $\mathcal{H} = l^2(\mathcal{A}(L) \oplus \mathcal{A}(L))$ and

$\mathcal{A}(\mathcal{Z}) = \{f \in \mathcal{A}(L) : ||[D(\omega), f]||_{\mathcal{H}} < \infty\}$

here. Still define $\partial^+ f = T f - f, \partial^- f = T^t f - f$, then deduction is the same as that in Subsect.III.1 and it follows that

$$d_{D(\omega)}(i, j) = \sup_f \{|f(i) - f(j)| : f \in \mathcal{A}(\mathcal{Z}), |\omega \partial^+ f|(k) \leq 1, k \in \mathcal{Z}\}, \forall i, j \in \mathcal{Z}$$

With non-singular $\omega$ and that $i < j$,

$$d_{D(\omega)}(i, j) \leq \sum_{k=0}^{j-i-1} |\omega(i + k)|^{-1}$$

(6)

Let $f_\omega(0) = 0, f_\omega(k) = f_\omega(k - 1) + |\omega(k - 1)|^{-1}, f_\omega(-k) = f_\omega(-k + 1) - |\omega(-k)|^{-1}, k = 1, 2, \ldots$, then $||[D(\omega), f_\omega]|| \leq 1$ and $f_\omega$ saturates the upper bound in Eq.(5). Since $d_{D(\omega)}(i, i + 1) = |\omega(i)|^{-1}, |\omega(i)|^{-1}$ is the lattice spacing between $i$ and $i + 1$.

Notice that non-singular $\omega$ can be polarized as $a_+^{-1} e^{i a_+ A}$ with two real functions $a_+, A$, we conclude that $d_{D(\omega)}$ is determined entirely by lattice spacing function $a_+$ and that $d_{D(\omega)}$ is still linear distance in the sense of additivity.
IV Discussions

We claim that $e^{ia+A}$ in the above decomposition plays the role of unitary link-variable in lattice gauge theory, or equivalently parallel transport in mathematical literature. In fact, a local $U(1)$-gauge transformation on $\mathcal{H}$ is defined to be $\psi \mapsto u\psi, \forall \psi \in \mathcal{H}$ where $u$ is a unitary in $A(L)$ and a $U(1)$-parallel transport $U$ on $L$ is a link-variable satisfying $U \rightarrow uU\bar{u}$, $<U\psi,U\psi> = <U\bar{T}\psi,U\bar{T}\psi>$ in which $<,>$ is hermitian-structure on $\mathcal{H}$. If $e^{ia+A} \rightarrow ue^{ia+A}(T\bar{u}), a_+ \rightarrow a_+$, then $e^{ia+A}T$ is a parallel transport and $(\psi, D_\omega \psi)$ is gauge-invariant where $(,)$ is inner product of $\mathcal{H}$. Therefore geometric interpretation of $\omega$ is clear: $\omega$ is a link-variable not necessarily unitary, whose amplitude provides a vierbein and phase is the usual integrated $U(1)$-connection.

Non-unitary link-variable has been noticed in the work of Majid and Raineri [9] discussing field theory on permutation group $S_3$ and ours [10]. Nevertheless, its geometric picture is the clearest on 1D lattices.

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