A Fast Method for Array Response Adjustment with Phase-Only Constraint

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Abstract—In this paper, we propose a fast method for array response adjustment with phase-only constraint. This method can precisely and rapidly adjust the array response of a given point by only varying the entry phases of a pre-assigned weight vector. We show that phase-only array response adjustment can be formulated as a polygon construction problem, which can be solved by edge rotation in complex plain. Unlike the existing approaches, the proposed algorithm provides an analytical solution and guarantees a precise phase-only adjustment without pattern distortion. Moreover, the proposed method is suitable for an arbitrarily given weight vector and has a lower computational complexity. Representative examples are presented to demonstrate the effectiveness of the proposed algorithm.

I. INTRODUCTION

Controlling the array power response as desired plays an important role in the applications of sensor arrays. Many methods have been reported to control array response by designing an architecture including both amplitudes and phases for the given specification, see e.g., [1]–[4]. In fact, phase-only architectures which use a single power-divider network simplify the beamforming network as well as reduce cost. In this perspective, phase-only control is preferred [5]. So far, many phase-only response control and/or pattern synthesis methods have been reported. A phase-only power synthesis technique is presented for reconfigurable conformal arrays in [6], whereas for linear and planar arrays, the phase-only synthesis of minimum peak sidelobe patterns is considered in [7]. Taking advantage of advances in convex optimization [8], a class of phase-only response control algorithms have been devised. For instance, the semidefinite relaxation (SDR) [9], a class of phase-only response control algorithms have been reported to control array response by designing an architecture including both amplitudes and phases for the given specification, see e.g., [1]–[4]. In fact, phase-only architectures which use a single power-divider network simplify the beamforming network as well as reduce cost. In this perspective, phase-only control is preferred [5]. So far, many phase-only response control and/or pattern synthesis methods have been reported. A phase-only power synthesis technique is presented for reconfigurable conformal arrays in [6], whereas for linear and planar arrays, the phase-only synthesis of minimum peak sidelobe patterns is considered in [7]. Taking advantage of advances in convex optimization [8], a class of phase-only response control algorithms have been devised. For instance, the semidefinite relaxation (SDR) technique [9] has been applied in [10] and the alternating optimization algorithm is adopted in [11]. Other methods include intersection approach [12], bi-quadratic programming method [13], to name just a few.

It is worth noting that in general the aforementioned methods have to redesign the weight vector even if only a slight adjustment of the array power response is required. To this end, some accurate array response control approaches have been developed recently [14]–[16]. However, these methods design both the amplitude and phase of the weight vector. They cannot be straightforwardly applied to the phase-only design. Thus, a geometrical formulation to the problem of phase-only array response adjustment is developed in this paper. The proposed method is able to realize rapid array response adjustment with phase-only constraint for a given weight vector.

II. THE PROPOSED ALGORITHM

A. Geometrical Interpretation of the Problem

We consider an array of $N \geq 3$ elements. For a given weight vector $w_{\text{pre}}$ and a pre-assigned angle $\theta_c$, we aim to only vary the entry phases of $w_{\text{pre}}$ such that the so-obtained new weight vector $w_{\text{new}}$ adjusts the normalized array power response at $\theta_c$ to its desired level $p_c$, i.e.,

$$L_{\text{new}}(\theta_c, \theta_0) \triangleq \frac{|w_{\text{new}}^H a(\theta_c)|^2}{|w_{\text{new}}^H a(\theta_0)|^2} = p_c$$ (1)

where $\theta_0$ represents the main beam axis, $a(\theta)$ stands for the steering vector at $\theta$, and the entries of $w_{\text{new}}$ and $w_{\text{pre}}$ fulfill the condition

$$w_{\text{new},n} = |w_{\text{pre},n}| \cdot e^{j\phi_n}, \quad n = 1, \ldots, N.$$ (2)

with $\phi_n = \angle w_{\text{new},n}$. By introducing a phase parameter $\psi \in [0, 2\pi)$, we can rewrite (1) as

$$w_{\text{new}}^H (a(\theta_c) - \sqrt{\rho_c e^{j\psi}} a(\theta_0)) = \sum_{n=1}^{N} h_n |w_{\text{pre},n}| e^{-j\phi_n} = 0$$ (3)

where $h_n$ is the $n$th element of $h(\theta_c, \theta_0, \rho_c, \psi_c)$. Given $\theta_c$, $\theta_0$, $\rho_c$ and $\psi_c$, our concern is finding appropriate $\phi_n$, $n = 1, \ldots, N$, to satisfy (3).

For notational simplicity, let us define

$$v_n \triangleq h_n \cdot |w_{\text{pre},n}|, \quad n = 1, \ldots, N$$ (4)

and hence re-express (3) as $\sum_{n=1}^{N} v_n e^{-j\phi_n} = 0$. In a view of complex plane, $v_n e^{-j\phi_n}$ corresponds to a vector, denoted as $v_n e^{-j\phi_n}$, whose coordinate is given by $(\Re(v_n e^{-j\phi_n}), \Im(v_n e^{-j\phi_n})))$. With this geometrical concept, one can denote (3) as

$$\sum_{n=1}^{N} v_n e^{-j\phi_n} = \sum_{n=1}^{N} |v_n| e^{j(\theta_n - \phi_n)} = 0$$ (5)

where $\phi_n = \angle v_n = \angle h_n$, $n = 1, \ldots, N$. The problem of solving (5) with respect to $\phi_n$ becomes how to rotate the vectors $v_n$, $n = 1, \ldots, N$, in complex plane to sum them up to a zero vector.
B. A Geometric Solution via Triangle Construction

We now derive a solution to the above problem via triangle construction [17]. To begin with, let us sort the entries of \( [v_1, \cdots, v_N] \) in a descending order as

\[
[d_1, \cdots, d_N]^T = J [v_1, \cdots, v_N]^T \tag{6}
\]

where \( d_1 \geq d_2 \geq \cdots \geq d_N > 0 \) and \( J \) denotes a certain permutation matrix. As a result, solving the problem (5) with \( w \) it is experimentally found that the resulting beampattern of \( \phi \) differs from that of the piecewise summation function construction [17]. To begin with, let us sort the entries of \( B \). A Geometric Solution via Triangle Construction

\[
Q(k, l) \triangleq \sum_{i=k}^{l} d_i, \quad 1 \leq k \leq l \leq N. \tag{9}
\]

After solving this problem with respect to \( \phi_n, n = 1, \cdots, N \), the phase \( \phi_n \) can be recovered as

\[
[\phi_1, \cdots, \phi_N]^T = [\vartheta_1, \cdots, \vartheta_N]^T - J^{-1} [\varphi_1, \cdots, \varphi_N]^T. \tag{8}
\]

Before proceeding, a useful lemma [17] is given.

Lemma 1: Assume \( d_1 \geq d_2 \geq \cdots \geq d_N > 0 \), define a piecewise summation function \( Q(\cdot) \) as

\[
Q(i, N) \triangleq \sum_{i=1}^{N} d_i, \quad 1 \leq i \leq N. \tag{10}
\]

On the basis of Lemma 1, the following important corollary can be obtained.

Corollary 1: If \( d_1 \leq Q(2, N) \), the non-linear Eqn. (7) has the following solution as

\[
\varphi_i = \begin{cases} 
\pi, & \text{if } i = 1 \\
\alpha_1, & \text{if } 2 \leq i \leq m \\
\alpha_1 + \alpha_2 + \pi, & \text{if } m + 1 \leq i \leq N
\end{cases} \tag{11}
\]

where \( m \) is the index satisfying

\[
m \triangleq \arg \min_{i \in \{2, \cdots, N-1\}} |Q(2, i) - Q(i + 1, N)|. \tag{12}
\]

\( \alpha_1 \) and \( \alpha_2 \) are given by

\[
\begin{align*}
\alpha_1 &= \text{acos} \left( \frac{d_1^2 + Q^2(2, m) - Q^2(m + 1, N)}{2d_1 Q(2, m)} \right) \tag{13a} \\
\alpha_2 &= \text{acos} \left( \frac{Q^2(2, m) + Q^2(m + 1, N) - d_1^2}{2Q(2, m) Q(m + 1, N)} \right). \tag{13b}
\end{align*}
\]

Proof: Given the \( m \) in (13), if \( d_1 \leq Q(2, N) \), we can obtain from Lemma 1 that \( d_1 \geq |Q(2, m) - Q(1, m + 1, N)| \). Thus, the three edges, i.e., \( d_1, Q(2, m) \) and \( Q(m + 1, N) \), can construct a triangle as shown in Fig. 1. In a geometrical manner, we have \( d_1 e^{j\pi} + Q(2, m) \alpha_1 + Q(1, m + 1, N) \vartheta = 0 \), which indicates that there exists a solution to the problem (7) as specified in (10). This completes the proof.

Note that although Corollary 1 provides a closed-form solution to the problem (7) with the aid of triangle construction, it is experimentally found that the resulting beampattern of \( w \) may lead to large pattern variations in the uncontrolled region, compared to the previous beampattern.

C. Solution Analysis via Polygon Construction

In this subsection, the above-obtained solution is analyzed with polygon construction. Before further discussion, we first give the following lemma, which has also been reported and proofed in [17].

Lemma 2: Given \( d_1 \geq d_2 \geq \cdots \geq d_N > 0 \), there exists a solution to \( \sum_{i=1}^{N} d_i e^{j\varphi_i} = 0 \) (or all the edges \( d_i \)'s can form a polygon), if and only if the condition \( d_1 \leq Q(2, N) \) is satisfied.

According to the derivation of Corollary 1, we can construct a triangle using \( d_1, Q(2, m) \) and \( Q(m + 1, N) \). To make this clear, let us introduce an auxiliary vector \( x_2 e^{j\gamma_2} \) (with modulus \( x_2 \) and phase \( \gamma_2 \)) pointing from \( 0 \) to \( d_1 e^{j\pi} + d_2 e^{j\varphi_2} \). We know that all the edges \( d_i \)'s \( (i = 1, \cdots, N) \) can form a polygon if and only if:

1) The edges \( d_1, d_2 \) and \( x_2 \) can form a triangle.
2) The edges \( x_2, d_3, \cdots, d_N \) can form a polygon.

Recalling Lemma 2, we can then calculate the range of feasible \( x_2 \) and further determine the set of feasible \( \varphi_2 \). Once \( \varphi_2 \) has been determined as \( \varphi_2^* \) (discussed in the next subsection), the resulting \( \sum_{i=1}^{N} d_i e^{j\varphi_i} = 0 \) (or all the edges \( d_i \)'s can form a polygon).

In a general sense, if \( x_{i-1} e^{j\gamma_{i-1}} \) has been determined as \( x_{i-1} e^{j\gamma_{i-1}} \), we can then calculate the set of feasible \( \varphi_i \), by introducing an auxiliary vector \( x_i e^{j\gamma_i} \) pointing from \( 0 \) to \( x_{i-1} e^{j\gamma_{i-1}} + d_i e^{j\varphi_i}, i = 2, \cdots, N-2 \), where \( x_1 \) and \( \gamma_1 \) are defined, respectively, as \( x_1 \triangleq d_1, \gamma_1, \varphi_1 \triangleq \pi \). To satisfy (7), the edges \( x_1, d_2, d_3, \cdots, d_N \) should form a polygon. According to Lemma 2, we can obtain the set of \( x_i \), denoted as \( \mathbb{X}_i = [x_{i_{\min}}, x_{i_{\max}}], 2 \leq i \leq N-2 \), with

\[
x_{i_{\min}} = \max \left\{ x_i - d_i, d_{i+1} + \sum_{k=i+2}^{N} d_k \right\} \tag{14}
\]

\[
x_{i_{\max}} = \min \left\{ x_i + d_i, \sum_{k=i+1}^{N} d_k \right\}. \tag{15}
\]
Moreover, if
\[ \delta \leq \delta_{i,\text{min}} \] with
\[ \delta_{i,\text{min}} = \frac{\pi}{2} \left( x_{i-1}^2 + d_i^2 - 2x_{i-1}d_i \right) \]
\[ \delta_{i,\text{max}} = \frac{\pi}{2} \left( x_{i-1}^2 + d_i^2 - 2x_{i-1}d_i \right) \].

Furthermore, we can obtain the set of feasible \( \varphi_i \) as
\[ \varphi_i \in \Psi_i \triangleq \left[ \gamma_{i-1} + \pi - \delta_{i,\text{max}}, \gamma_{i-1} + \pi - \delta_{i,\text{min}} \right] \cup \left[ \gamma_{i-1} + \pi + \delta_{i,\text{min}}, \gamma_{i-1} + \pi + \delta_{i,\text{max}} \right]. \] (17)

Assume that \( \varphi_i \) is determined as \( \varphi_i^* \) (see next subsection), the ultimate \( x_i e^{j\varphi_i^*}, i = 2, \ldots, N - 2 \), can be expressed as
\[ x_i e^{j\varphi_i^*} = x_{i-1} e^{j\gamma_{i-1}^*} + d_i e^{j\varphi_{i-1}^*}. \] (18)

Note that if \( i = N - 2 \) is applied, the resulting \( x_{N-2} \) can form a triangle with the other two edges \( d_{N-1} \) and \( d_N \). In this case, it’s not hard to learn that there are two candidates at most for \( \varphi_{N-1} \) as
\[ \varphi_{N-1} \in \Psi_{N-1} \triangleq \left\{ \gamma_{N-2}^* + \pi - \delta_{N-1}^* - \delta_{N-1}^* \right\}, \]

where \( \delta_{N-1} = \frac{\pi}{2} \left( x_{N-2}^2 + d_{N-1}^2 - 2x_{N-2}d_{N-1} \right) \).

Moreover, if \( \varphi_{N-1} \) is selected as \( \varphi_{N-1}^* \) (discussed in the next subsection), there would be only one choice for the ultimate \( \varphi_N \) (denoted as \( \varphi_{N}^* \)), which can be expressed accordingly as
\[ \varphi_N^* = \angle \left( x_{N-2} e^{j\gamma_{N-2}^*} + d_{N-1} e^{j\varphi_{N-1}^*} \right). \] (19)

D. Determination of the Phase Parameter \( \varphi_i \)

In this subsection, we consider the determination of \( \varphi_{i*} \), \( i = 2, \ldots, N \), and complete the proposed algorithm by finding the ultimate \( \varphi_n, n = 1, \ldots, N \).

To begin with, for any given \( i \in \{1, \ldots, N\} \), we note from (8) that there exists a unique index \( n \in \{1, \ldots, N\} \) such that \( J(i, n) = 1 \), and we denote its resulting value for clarity as \( n = c(i) \). With the above notation, one obtains from (6) that
\[ \phi(i, n) = \varphi_{i}(i) - \varphi_i, \quad i = 1, \ldots, N. \] (20)

Since the beampattern is invariant to a fixed phase shift, we set the ultimate selection of \( \phi_{i}(i) \) straightforwardly as \( \phi_{i}(i) = \varphi_{i}(i) - \pi \).

For the index \( i \in \{2, \ldots, N-1\} \), we propose to optimize \( \phi_{i}(i) \) \( (i = 2, \ldots, N-1) \) as
\[ \text{minimize}_{\phi_{i}(i)} | \exp(j\phi_{i}(i)) - \exp(j\varpi_{i}(i)) | \] (21a)

subject to \( \phi_{i}(i) \in \Phi_{i}(i) \) (21b)

where \( \Phi_{i}(i) = \{ \phi_{i}(i) | \phi_{i}(i) = \varphi_{i}(i) - \varphi_i, \; \varphi_i \in \Psi_i \} \). In (21), \( \varpi_{i}(i) \) is the \( i \)-th element of a pre-designed weight vector \( \mathbf{w} \) that can result a satisfactory beampattern. Note that the phase-only constraint (2) may not be satisfied by \( \mathbf{w} \). In this paper, we construct \( \mathbf{w} \) using the weight vector orthogonal decomposition (WORD) algorithm [16]. Giving a weight vector \( \mathbf{w}_{\text{pre}} \), WORD can precisely and analytically adjust array response level at one pre-assigned angle \( \theta_c \) as the desired level \( \rho_c \). When \( i = N \) is applied, there is only one candidate for \( \varphi_N \), and hence, one can calculate the corresponding \( \phi_{i}(N) \) as \( \phi_{i}(N) = \varphi_{i}(N) - \varphi_N \).

Consequently, the new weight vector \( \mathbf{w}_{\text{new}} \) can be obtained as
\[ \mathbf{w}_{\text{new}} = [w_{\text{pre}}, \ldots, w_{\text{pre}}, \mathbf{e}_{j\varphi_{N}}^{T} \oplus \mathbf{e}_{j\varphi_{N}}^{T}]^{T} \] (22)

where \( \oplus \) denotes elementwise product.

III. Numerical Results

In this section, simulations are presented to demonstrate the effectiveness of the proposed algorithm. For comparison purpose, the results of SDR method in [10] and convex relaxation (CR) method in [11] will also be presented.

1) Phase-Only Array Response Adjustment for a Non-uniformly Spaced Linear Array: In the first example, we consider a non-uniformly spaced linear array with 11 isotropic elements, see \( x_n \) in Table I for the specifications of sensor positions. The main beam axis is set as \( \theta_0 = -30^\circ \). The controlled angle and its desired level are taken as \( \theta_c = 52^\circ \) and \( \rho_c = -30 \text{dB} \), respectively. In this case, the previous
The consequence is the semidefinite relaxation operator, which

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shape a deep notch at $\theta_w$ and the setting of $\theta_w$ doesn’t realize a complete phase-only control. The resulting beampattern is listed in Table I, and the weight of the proposed algorithm is shown in Fig. 3.

configuration. We can see that all the three methods adjust the corresponding beampattern brings small pattern variations at the uncontrolled region as shown in Fig. 3.

2) Phase-Only Nulling: Following the array configuration and the setting of $w_{\text{pre}}$ in the preceding example, we keep $\theta_0 = -30^\circ$ and attempt to form a null at $\theta_c = 35^\circ$. The resulting beampatterns of different methods are depicted in Fig. 4. One can clearly see that all the three methods shape a deep notch at $\theta_c$. Moreover, the CR method and the proposed one result small pattern variations at the uncontrolled region. On the contrary, the ultimate pattern of SDR method is distorted seriously. The possible reason for this adverse consequence is the semidefinite relaxation operator, which reformulates a convex problem that is not equivalent to the original one. In this case, the proposed algorithm can realize a phase-only control, which may not be always guaranteed by the CR method. In addition, the execution time of the proposed algorithm is great shorter than those of SDR and CR.

IV. Conclusion

In this paper, we have presented a geometrical approach to fast array response adjustment with phase-only constraint. The devised method can precisely and rapidly adjust the response of a given point with the phase-only constraint, staring from a pre-assigned weight vector. In the proposed algorithm, the single-point phase-only array response adjustment is recast as a polygon construction problem, which can be solved by edge rotation. Representative simulations have been carried out to verify the effectiveness and superiority of the proposed algorithm.

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