Existence of global solutions for multi-dimensional coupled FBSDEs with diagonally quadratic generators

Yuyang Chen∗ Peng Luo †

November 16, 2021

Abstract

The present paper is devoted to study multi-dimensional coupled FBSDEs with diagonally quadratic generators. Relying on a comparison result obtained in [20], we provide conditions under which there exists a global solution. As a byproduct, we further give a comparison result for this global solution.

Key words: coupled FBSDEs; diagonally quadratic generators; BMO martingales; comparison theorem.

MSC-classification: 60H10, 60H30.

1 Introduction

In this work, we consider the following system of forward and backward stochastic differential equations

\begin{align}
X_t &= x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s)dW_s \\
Y_t &= h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]
\end{align}

where $W$ is a multi-dimensional Brownian motion on a probability space, $x$ is the initial condition, $T > 0$ is a fixed finite time horizon, and $b, \sigma, h, g$ are functions. We provide conditions which guarantee the existence of global solutions in the case where $Y$ is multi-dimensional and $g$ is diagonally quadratic in $z$.

Solvability of coupled FBSDEs with Lipschitz generators has been well studied in the literature. Antonelli [1] obtained the first solvability result of a coupled FBSDE over a small time horizon, which is further investigated in Pardoux and Tang [26]. Ma et al. [23] proposed a four step scheme method to solve coupled Markovian FBSDEs (see also [25]). Their method allows to obtain solvability result for arbitrarily large time horizon. Hu and Peng [12] introduces a continuation method, which is further developed in [30, 27]. This approach is able to provide global solvability for coupled non-Markovian FBSDEs under a so-called monotonicity condition. Combining contraction mapping method and a four step scheme method, Delarue [8] obtained the existence and uniqueness of global solutions for coupled FBSDEs with non-degenerate diffusion processes. Antonelli and Hamadène [2] investigated the solvability of coupled FBSDEs with continuous monotone coefficients. Recently, Ma et al. [24] established a unified approach to study the wellposedness of non-Markovian coupled FBSDE, where

∗School of Mathematical Sciences, Shanghai Jiao Tong University, China (cyy0032@sjtu.edu.cn)
†School of Mathematical Sciences, Shanghai Jiao Tong University, China (peng.luo@sjtu.edu.cn). Financial support from the National Natural Science Foundation of China (Grant No. 12101400) is gratefully acknowledged.
the concept of decoupling field has been introduced. This approach is further extended by Fromm and Imkeller [9].

On the other hand, BSDEs and FBSDEs with quadratic growth naturally arise in mathematical finance, see e.g. Horst et al. [11], Kramkov and Pulido [18] and Bielagke et al. [4]. One-dimensional quadratic BSDE was first studied by Kobylanski [17], where the existence, comparison and stability results are obtained. Briand and Hu [5, 6] further studied quadratic BSDEs for unbounded terminal conditions. Barrieu and El Karoui [3] considered quadratic BSDEs in terms of some general quadratic results are obtained. Briand and Hu [5, 6] considered quadratic BSDEs in terms of some general quadratic semimartingales. We further refer to Tevzadze [28], Hu and Tang [13], Jamneshan et al. [15], Luo [20] for recent developments in solvability of multi-dimensional quadratic BSDEs.

The main scope of this paper is to consider the solvability of system (1) where $Y$ is multi-dimensional and $g$ has diagonally quadratic growth in $z$. When $Y$ is one dimensional and $g$ is quadratic in $z$, Antonelli and Hamadène [2] obtained the existence of a global solution for system (1). Using the technique of decoupling field and in a Markovian case, Fromm and Imkeller [9] obtained the existence and uniqueness of a local solution and provided an extension to maximally solvable time horizon for generators which are locally Lipschitz in $z$. When $\sigma$ is independent of $X$, Luo and Tangpi [22] obtained the existence and uniqueness of a local solution for generators which can be separated into a quadratic and subquadratic part, while Kupper et al. [19] obtained global solvability for generators with arbitrary growth in $z$ in a Markovian setting. More recently, Jackson [14] established global solvability for a type of quadratic Markovian FBSDEs. Compared with all these results, we study global solution for system (1) in a multi-dimensional and non-Markovian framework. Relying on a comparison result recently established in Luo [20], we construct a sequence of processes and obtain some delicately a priori estimates. By making ample use of properties of BMO martingales, we further show the convergence of this sequence of processes in suitable spaces, which yields the existence of a global solution for system (1). Finally, we present a comparison result for this global solution. The paper is organized as follows. In the next section, we state our setups and main results. Theorem 2.1 provides the existence of global solutions of multi-dimensional FBSDEs with diagonally quadratic generators. A comparison result for this global solution is given in Theorem 2.2.
The class \( \{ M : \| M \|_{BMO} < \infty \} \) is denoted by \( BMO \). For \( (\alpha \cdot W)_t := \int_0^t \alpha_s dW_s \) in \( BMO \), the corresponding stochastic exponential is denoted by \( \mathcal{E}_t(\alpha \cdot W) \).

In this paper, we make the following assumptions. Let \( C \) be a positive constant.

\((\alpha 1)\) The function \( b := (b^1, \ldots, b^n)^* : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies that \( b(\cdot, x, y) \) is adapted for each \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \). It holds that

\[ |b(t, x, y)| \leq C(1 + |x| + |y|), \]
\[ |b(t, x, y) - b(t, \overline{x}, \overline{y})| \leq C(|x - \overline{x}| + |y - \overline{y}|) \]

for \( x, \overline{x}, y, \overline{y} \in \mathbb{R}^n \).

\((\alpha 2)\) The function \( \sigma := (\sigma^1, \ldots, \sigma^n)^* : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d} \) satisfies that \( \sigma(\cdot, x) \) is adapted for each \( x \in \mathbb{R}^n \). It holds that

\[ |\sigma(t, x)| \leq C(1 + |x|), \]
\[ |\sigma(t, x) - \sigma(t, \overline{x})| \leq C|x - \overline{x}| \]

for \( x, \overline{x} \in \mathbb{R}^n \).

\((\alpha 3)\) The function \( h := (h^1, \ldots, h^n)^* : \Omega \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies that \( h \) is \( \mathcal{F}_t \)-measurable. It holds that

\[ |h(x)| \leq C, \]
\[ |h(x) - h(\overline{x})| \leq C|x - \overline{x}| \]

for \( x, \overline{x} \in \mathbb{R}^n \).

\((\alpha 4)\) For \( i = 1, \ldots, n \), the function \( g^i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R} \) satisfies that \( g^i(\cdot, x, y, z^i) \) is adapted for each \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^n \) and \( z^i \in \mathbb{R}^d \). It holds that

\[ |g^i(t, x, y, z^i)| \leq C(1 + |y| + |z^i|^2), \]
\[ |g^i(t, x, y, z^i) - g^i(t, \overline{x}, \overline{y}, \overline{z}^i)| \leq C|x - \overline{x}| + C|y - \overline{y}| + C(1 + |z^i| + |\overline{z}^i|)|z^i - \overline{z}^i| \]

for \( x, \overline{x}, y, \overline{y} \in \mathbb{R}^n \), \( z^i, \overline{z}^i \in \mathbb{R}^d \).

\((\alpha 5)\) For \( t \in [0, T] \), \( i = 1, \ldots, n \), it holds that \( b^i(t, x, y) \leq \overline{b}^i(t, \overline{x}, \overline{y}) \) for any \( x, \overline{x}, y, \overline{y} \) satisfying

\[ x^i = \overline{x}^i, x^j \leq \overline{x}^j, j \neq i, y \leq \overline{y}, \]

\((\alpha 6)\) For \( t \in [0, T] \), it holds that \( h(x) \leq h(\overline{x}) \) for any \( x, \overline{x} \) satisfying \( x \leq \overline{x} \).

\((\alpha 7)\) For \( t \in [0, T] \), \( i = 1, \ldots, n \), it holds that \( g^i(t, x, y, z^i) \leq g^i(t, \overline{x}, \overline{y}, \overline{z}^i) \) for any \( x, \overline{x}, y, \overline{y} \) satisfying

\[ y^i = \overline{y}^i, y^j \leq \overline{y}^j, j \neq i, x \leq \overline{x}. \]
Theorem 2.1 Let assumptions (A1) – (A7) be satisfied, then there exists a solution \((X, Y, Z)\) of (2) such that \((X, Y, Z, W) \in \mathbb{S}^p(\mathbb{R}^n) \times \mathbb{S}^\infty(\mathbb{R}^n) \times \text{BMO}\) for any \(p \geq 2\). Besides this solution is the minimal one, in the sense that if \((X', Y', Z')\) is another solution of (2), for any \(t \in [0, T]\), we have
\[ X_t \leq X'_t, Y_t \leq Y'_t. \]

**Proof.** We divide the proof into three steps:

**Step 1:** For any \(t \in [0, T]\), \(i = 1, \ldots, n\), we consider the following BSDE and SDE:

\[
U^i_t = C + \int_t^T C(1 + |U^i_s| + |V^i_s|^2)\,ds - \int_t^T V^i_s\,dW_s, \tag{3}
\]
\[
S^i_t = x^i_0 + \int_0^t C(1 + |S^i_s| + |U^i_s|)\,ds + \int_0^t \sigma^i(s, S^i_s)\,dW_s. \tag{4}
\]

There exists a unique solution \((U, V)\) in (2) such that \((U, V) \in \mathbb{S}^\infty(\mathbb{R}^n) \times \text{BMO}\) from [13, Theorem 2.3]. Moreover,
\[
\|U\|_{\mathbb{S}^\infty(\mathbb{R}^n)} + \|V \cdot W\|_{\text{BMO}} \leq K
\]
where \(K\) is a positive constant only depending on \(C\) and \(T\). Also, the SDE (3) has a solution \(S \in \mathbb{S}^p(\mathbb{R}^n)\) due to the SDE theory.

Then we consider \((Y^{(0)}, Z^{(0)})\) is the solution of
\[
Y^{i, (0)}_t = -C - \int_t^T C(1 + |Y^{(0)}_s| + |Z^{i, (0)}_s|^2)\,ds - \int_t^T Z^{i, (0)}_s\,dW_s. \tag{5}
\]

Noting the assumption (A4), we can obtain \(Y^{i, (0)}_t \leq U^{i, (0)}_t\) for any \(t \in [0, T]\) from [20, Theorem 2.2]. Without loss of generality, assume \(\|Y^{(0)}\|_{\mathbb{S}^\infty(\mathbb{R}^n)} + \|Z^{(0)} \cdot W\|_{\text{BMO}} \leq K\).

Similarly, we consider \(X^{(0)}\) is the solution of
\[
X^{i, (0)}_t = x^i_0 + \int_0^t b^i(s, X^{(0)}_s)\,ds + \int_0^t \sigma^i(s, X^{(0)}_s)\,dW_s. \tag{6}
\]

Noting the assumption (A1), we can obtain \(X^{i, (0)}_t \leq S^i_t\) for any \(t \in [0, T]\) from [10, Theorem 1.1].

Now we construct some FBSDEs. For any \(t \in [0, T]\), \(k \geq 1\), \(i = 1, \ldots, n\),
\[
Y^{i, (k)}_t = h^i(X^{(k-1)}_T) + \int_t^T g^i(s, X^{(k-1)}_s, Y^{i, (k)}_s, Z^{i, (k)}_s)\,ds - \int_t^T Z^{i, (k)}_s\,dW_s, \tag{7}
\]
\[
X^{i, (k)}_t = x^i_0 + \int_0^t b^i(s, X^{(k)}_s, Y^{i, (k)}_s)\,ds + \int_0^t \sigma^i(s, X^{(k)}_s)\,dW_s. \tag{8}
\]

If it holds for any \(t \in [0, T]\) that
\[
X^{(k-1)}_t \leq X^{(k)}_t \leq S_t,
\]
we can immediately get
\[
h(X^{(k-1)}_T) \leq h(X^{(k)}_T) \leq C,
\]
\[
g^i(t, X^{(k-1)}_t, y, z^i) \leq g^i(t, X^{(k)}_t, y, z^i) \leq C(1 + |y| + |z^i|^2)
\]
due to the assumptions $\mathcal{A}6(\mathcal{A}7)$. It follows from [20] Theorem 2.2 that

$$Y_t^{(k)} \leq Y_t^{(k+1)} \leq U_t, \quad t \in [0, T].$$

Similarly, when

$$Y_t^{(k)} \leq Y_t^{(k+1)} \leq U_t$$

holds for any $t \in [0, T]$, we can obtain

$$b(t, x, Y_t^{(k)}) \leq b(t, x, Y_t^{(k+1)}) \leq C(1 + |x| + |U_t|)$$

due to the assumption $\mathcal{A}5$. Using [10] Theorem 1.1, we will get

$$X_t^{(k)} \leq X_t^{(k+1)} \leq S_t, \quad t \in [0, T].$$

Since it is easy to check that

$$X_t^{(0)} \leq X_t^{(1)} \leq S_t, \quad Y_t^{(0)} \leq Y_t^{(1)} \leq U_t$$

are satisfied for any $t \in [0, T]$, then for any $k \geq 0$, we will have

$$X_t^{(k)} \leq X_t^{(k+1)} \leq S_t, \quad Y_t^{(k)} \leq Y_t^{(k+1)} \leq U_t, \quad t \in [0, T]$$

by induction. By the monotonic convergence theorem, we note that there exists two adapted processes $X$ and $Y$ such that $\|X^{(k)} - X\|_{\mathcal{H}^p(\mathbb{R}^n)} \to 0$, $\|Y^{(k)} - Y\|_{\mathcal{H}^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$ for any $p \geq 2$. What’s more,

$$E \left[ \int_0^T |X_t^{(k)} - X_t|^p \, dt \right] \to 0, \quad E \left[ \int_0^T |Y_t^{(k)} - Y_t|^p \, dt \right] \to 0$$

as $k \to \infty$ for any $p \geq 2$.

**Step 2:** We prove that $(X^{(k)}, Y^{(k)}, Z^{(k)})$ converges to $(X, Y, Z)$ in $\mathcal{S}^p(\mathbb{R}^n) \times \mathcal{S}^p(\mathbb{R}^n) \times \mathcal{H}^p(\mathbb{R}^{n \times d})$ as $k \to \infty$ for any $p \geq 2$.

Since for any $p \geq 2$, $k, h \geq 0$ it holds that

$$E \left[ \left( \sup_{0 \leq s \leq t} \int_0^s |b(s, X_s^{(k)}, Y_s^{(k)}) - b(s, X_s^{(h)}, Y_s^{(h)})| \, ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}}$$

$$\leq E \left[ \left( \sup_{0 \leq s \leq t} \int_0^s \frac{|b(s, X_s^{(k)}, Y_s^{(k)}) - b(s, X_s^{(h)}, Y_s^{(h)})|^2 \, ds}{2} \right)^{\frac{p}{2}} \right]^{\frac{2}{p}}$$

$$\leq E \left[ \left( \int_0^t (C|X_s^{(k)} - X_s^{(h)}| + C|Y_s^{(k)} - Y_s^{(h)}|)^2 \, ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}}$$

$$\leq CE \left[ \left( \int_0^t |X_s^{(k)} - X_s^{(h)}|^2 \, ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} + CE \left[ \left( \int_0^t |Y_s^{(k)} - Y_s^{(h)}|^2 \, ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}}$$
where $C_p$ is the coefficient in the Burkholder-Davis-Gundy inequality, which implies $\|X^{(k)} - X\|_{\mathcal{F}(\mathbb{R}^n)} \to 0$ as $k \to \infty$ for any $p \geq 2$. Let $k \to \infty$ in (8), we have

$$X_t^i = x_0^i + \int_0^t b^i(s, X_s, Y_s) \, ds + \int_0^t \sigma^i(s, X_s) \, dW_s.$$  

Then we discuss the convergence of $Y^{(k)}$ and $Z^{(k)}$. Because of the assumption (A4), we can denote

$$g^i(t, X_t^{(k-1)}, Y_t^{(k)}, Z_t^{i,(k)}) - g^i(t, X_t^{(h-1)}, Y_t^{(h)}, Z_t^{i,(h)}) = \alpha^i_t(k, h)(X_t^{(k-1)} - X_t^{(h-1)}) + \beta^i_t(k, h)(Y_t^{(k)} - Y_t^{(h)}) + \gamma_t^i(k, h)(Z_t^{i,(k)} - Z_t^{i,(h)})$$

and

$$|\alpha^i_t(k, h)| \leq C, \quad |\beta^i_t(k, h)| \leq C, \quad |\gamma_t^i(k, h)| \leq C(1 + |Z_t^{i,(k)}| + |Z_t^{i,(h)}|)$$

for any $k, h \geq 1$, $i = 1, \cdots, n$. We note that $W_t(i, k, h) := W_t - \int_0^t \gamma_t^i(k, h) \, ds$ is a Brownian motion under the equivalent probability measure $P_{i,k,h}$ defined by

$$dP_{i,k,h} := E(\gamma^i(k, h) \cdot W_t^T) \, dP.$$  

Since $\int_0^t \gamma_t^i(k, h) \, dW_s$ is a BMO martingale, we can deduce that there exists $p_1, q_1 > 1$ depending on $\gamma^i(k, h)$, such that

$$E_{\tau} \left( \frac{dP_{i,k,h}}{dP} \right)^{p_1} < \infty, \quad E_{\tau}^{i,k,h} \left( \frac{dP}{dP_{i,k,h}} \right)^{p_2} < \infty$$

from [10] Theorem 3.1, where $E_{\tau}$ is a conditional expectation for an arbitrary stopping time $\tau$. We let $q_1, q_2 > 1$ satisfying

$$\frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$  

Denote the conditional expectation with respect to $P$ by $E_t$ and $P_{i,k,h}$ by $E_{i,k,h}^t$ and use Itô’s formula, for any $p \geq 2$, $i = 1, \cdots, n$, we have

$$|Y_t^{i,(k)} - Y_t^{i,(h)}|^p + \frac{1}{2}(p - 1)E_t^{i,k,h} \int_0^T |Y_s^{i,(k)} - Y_s^{i,(h)}|^{p-2}|Z_s^{i,(k)} - Z_s^{i,(h)}|^2 \, ds$$

$$= E_t^{i,k,h} |h^i(X_t^{(k-1)} - X_t^{(h-1)})|^p$$

$$+ E_t^{i,k,h} \int_0^T |Y_s^{i,(k)} - Y_s^{i,(h)}|^{p-2}(Y_s^{i,(k)} - Y_s^{i,(h)})|\alpha_t^i(k, h)(X_s^{(k-1)} - X_s^{(h-1)}) + \beta_t^i(k, h)(X_s^{(k}) - X_s^{(h)})| \, ds.$$  

6
First, we calculate

$$E_t^{i,k,h} \int_t^T p|Y_s^{i,(k)} - Y_s^{i,(h)}|^{p-2}(Y_s^{i,(k)} - Y_s^{i,(h)})\alpha_s^{i}(k,h)(X_s^{(k-1)} - X_s^{(h-1)})ds$$

$$\leq pCE_t^{i,k,h} \int_t^T |Y_s^{(k)} - Y_s^{(h)}|^{p-1}|X_s^{(k-1)} - X_s^{(h-1)}|ds$$

$$\leq (p-1)CE_t^{i,k,h} \int_t^T |Y_s^{(k)} - Y_s^{(h)}|^{p}ds + CE_t^{i,k,h} \int_t^T |X_s^{(k-1)} - X_s^{(h-1)}|^{p}ds$$

and

$$E_t^{i,k,h} \int_t^T p|Y_s^{i,(k)} - Y_s^{i,(h)}|^{p-2}(Y_s^{i,(k)} - Y_s^{i,(h)})\beta_s^{i}(k,h)(Y_s^{i,(k)} - Y_s^{(h)})ds$$

$$\leq pCE_t^{i,k,h} \int_t^T |Y_s^{(k)} - Y_s^{(h)}|^{p}ds$$

for any $p \geq 2$, $k,h \geq 1$, $i = 1,\ldots,n$, which implies

$$|Y_s^{i,(k)} - Y_s^{i,(h)}|^p \leq C^p E_t^{i,k,h} |X_s^{(k-1)} - X_s^{(h-1)}|^p + (2p-1)CE_t^{i,k,h} \int_t^T |Y_s^{(k)} - Y_s^{(h)}|^{p}ds$$

$$+ CE_t^{i,k,h} \int_t^T |X_s^{(k-1)} - X_s^{(h-1)}|^{p}ds.$$ 

Then we show that for any $p \geq 2$, $k,h \geq 1$, $i = 1,\ldots,n$,

$$E_t^{i,k,h} \left[ \sup_{t \in [0,T]} E_t^{i,k,h} \int_t^T |Y_s^{(k)} - Y_s^{(h)}|^{p}ds \right] \leq 2E_t^{i,k,h} \left[ \left( \int_0^T |Y_s^{(k)} - Y_s^{(h)}|^{p}ds \right)^2 \right]^{{\frac{1}{2}}}$$

$$= 2E \left[ \left( \frac{dP^{i,k,h}}{dP} \right) \left( \int_0^T |Y_s^{(k)} - Y_s^{(h)}|^{p}ds \right)^2 \right]^{{\frac{1}{2}}}$$

$$\leq 2E \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{P_1} \right]^{\frac{1}{2 P_1}} E \left[ \left( \int_0^T |Y_s^{(k)} - Y_s^{(h)}|^{p}^{2q_1}ds \right)^{q_1} \right]^{\frac{1}{q_1}}$$

$$\leq 2E \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{P_1} \right]^{\frac{1}{2 P_1}} E \left[ \left( \int_0^T |Y_s^{(k)} - Y_s^{(h)}|^{2q_1}ds \right) \right]^{\frac{1}{q_1}}$$

holds due to Doob’s $L^p$-inequality and H"older’s inequality.

Similarly, we have

$$E_t^{i,k,h} \left[ \sup_{t \in [0,T]} E_t^{i,k,h} \int_t^T |X_s^{(k-1)} - X_s^{(h-1)}|^{p}ds \right] \leq 2E \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{P_1} \right]^{\frac{1}{2 P_1}} E \left[ \left( \int_0^T |X_s^{(k-1)} - X_s^{(h-1)}|^{2q_1}ds \right)^{q_1} \right]^{\frac{1}{q_1}},$$

$$E_t^{i,k,h} \left[ \sup_{t \in [0,T]} E_t^{i,k,h} \left| X_T^{(k-1)} - X_T^{(h-1)} \right|^p \right] \leq 2E \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{P_1} \right]^{\frac{1}{2 P_1}} E \left[ \left| X_T^{(k-1)} - X_T^{(h-1)} \right|^{2q_1} \right]^{\frac{1}{q_1}}.$$
So it holds that for any $p \geq 2$, $k, h \geq 1$, $i = 1, \ldots, n$,

\[
E^{i,k,h} \left[ \sup_{t \in [0,T]} \left| Y^{i,(k)}_t - Y^{i,(h)}_t \right|^p \right] \leq 2C^p E \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{p_1} \right]^{\frac{1}{2p_1}} E \left[ \left| X^{i,(k)}_T - X^{i,(h)}_T \right|^{2pq} \right]^{\frac{1}{2pq}} + 2(2p - 1)CE \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{p_1} \right]^{\frac{1}{2p_1}} E \left[ \left( \int_0^T \left| Y^{i,(k)}_s - Y^{i,(h)}_s \right|^{2pq} ds \right) \right]^{\frac{1}{2pq}} + 2CE \left[ \left( \frac{dP^{i,k,h}}{dP} \right)^{p_1} \right]^{\frac{1}{2p_1}} E \left[ \left( \int_0^T \left| X^{i,(k)}_s - X^{i,(h)}_s \right|^{2pq} ds \right) \right]^{\frac{1}{2pq}}.
\]

Since we have that

\[
E \left[ \sup_{t \in [0,T]} \left| Y^{i,(k)}_t - Y^{i,(h)}_t \right|^p \right] = E^{i,k,h} \left[ \left( \frac{dP}{dP^{i,k,h}} \right)^{p_2} \right]^{\frac{1}{2p_2}} E^{i,k,h} \left[ \sup_{t \in [0,T]} \left| Y^{i,(k)}_t - Y^{i,(h)}_t \right|^p \right]
\]

and

\[
E \left( \frac{dP^{i,k,h}}{dP} \right)^{p_1} < \infty, \quad E^{i,k,h} \left( \frac{dP}{dP^{i,k,h}} \right)^{p_2} < \infty,
\]

\[
\lim_{k,h \to \infty} E \left[ \int_0^T \left| X^{i,(k)}_t - X^{i,(h)}_t \right|^p dt \right] = 0, \quad \lim_{k,h \to \infty} E \left[ \int_0^T \left| Y^{i,(k)}_t - Y^{i,(h)}_t \right|^p dt \right] = 0
\]

for any $p \geq 2$, we can obtain

\[
\lim_{k,h \to \infty} E \left[ \sup_{t \in [0,T]} \left| Y^{i,(k)}_t - Y^{i,(h)}_t \right|^p \right] = 0
\]

for any $p \geq 2$. So $\|Y^{(k)} - Y\|_{S^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$ for any $p \geq 2$.

We show that $(Y^{(k)}, Z^{(k)}, W) \in S^{\infty}(\mathbb{R}^n) \times BMO$ for any $k \geq 0$.

Since we have

\[
Y^{(0)}_t \leq Y^{(k)}_t \leq U_t, \quad t \in [0,T]
\]

and $Y^{(k)}$ is continuous in $S^p(\mathbb{R}^n)$ for any $p \geq 2$, $k \geq 0$, it holds that

\[
\|Y^{(k)}\|_{S^\infty(\mathbb{R}^n)} \leq \max\{\|Y^{(0)}\|_{S^\infty(\mathbb{R}^n)}, \|U\|_{S^\infty(\mathbb{R}^n)}\} \leq K,
\]

which implies $Y^{(k)}$ is in $S^{\infty}(\mathbb{R}^n)$.

Besides, define

\[
\phi(x) = \frac{e^{2C|x|} - 2C|x| - 1}{4C^2} \geq 0
\]

and we can calculate that

\[
\phi'(x) = \frac{e^{2C|x|} - 1}{2C} sgn(x), \quad \phi''(x) = e^{2C|x|}, \quad \phi'''(x) = 2C|\phi'(x)| = 1.
\]
By Itô’s formula, it holds that for any \( k \geq 1, \ i = 1, \cdots, n, \)
\[
\phi(Y_t^{i,(k)}) + \frac{1}{2} E_t \int_t^T \phi''(Y_s^{i,(k)})|Z_s^{i,(k)}|^2 ds = E_t \phi(Y_T^{i,(k)}) + E_t \int_t^T \phi'(Y_s^{i,(k)})g'(X_s, Y_s^{i,(k)}, Z_s^{i,(k)}) ds
\]
\[
\leq \phi(\|Y_T^{i,(k)}\|_\infty) + E_t \int_t^T \phi'(Y_s^{i,(k)})C(1 + |Y_s^{i,(k)}| + |Z_s^{i,(k)}|^2) ds,
\]
and then
\[
\phi(Y_t^{i,(k)}) + \frac{1}{2} E_t \int_t^T |Z_s^{i,(k)}|^2 ds \leq \phi(\|Y_T^{i,(k)}\|_\infty) + E_t \int_t^T \phi'(Y_s^{i,(k)})C(1 + |Y_s^{i,(k)}|) ds
\]
\[
\leq \phi(\|Y_T^{i,(k)}\|_\infty) + C(T-t)\phi'(|Y^{(k)}|_{S^\infty(\mathbb{R}^n)})(1 + \|Y\|_{S^\infty(\mathbb{R}^n)})
\]
\[
\leq \phi(K) + CT\phi'(K)(1 + K),
\]
which implies
\[
\|Z^{(k)} \cdot W\|_{BMO} < \infty.
\]
Then use Itô’s formula for \( |Y_t^{i,(k)} - Y_t^{i,(h)}|^2 \), we have
\[
|Y_t^{i,(k)} - Y_t^{i,(h)}|^2 + \int_t^T |Z_s^{i,(k)} - Z_s^{i,(h)}|^2 ds
\]
\[
= |h^i(X_T^{(k-1)} - X_T^{(h-1)})|^2 - 2 \int_t^T (Y_s^{i,(k)} - Y_s^{i,(h)})(Z_s^{i,(k)} - Z_s^{i,(h)})dW_s
\]
\[
+ 2 \int_t^T (Y_s^{i,(k)} - Y_s^{i,(h)})(\alpha_s(k,h)(X_s^{(k-1)} - X_s^{(h-1)}) + \beta_s(k,h)(Y_s^{(k)} - Y_s^{(h)}) + \gamma_s(k,h)(Z_s^{i,(k)} - Z_s^{i,(h)})) ds.
\]
We can get
\[
\int_0^T |Z_t^{i,(k)} - Z_t^{i,(h)}|^2 dt \leq C^2 |X_T^{(k-1)} - X_T^{(h-1)}|^2 + 2 \int_0^T |(Y_t^{i,(k)} - Y_t^{i,(h)})(Z_t^{i,(k)} - Z_t^{i,(h)})| dW_t
\]
\[
+ 2CK \int_0^T |X_T^{(k-1)} - X_T^{(h-1)}| ds + 2CT \sup_{t \in [0,T]} |Y_t^{(k)} - Y_t^{(h)}|^2
\]
\[
+ 6C \sup_{t \in [0,T]} |Y_t^{(k)} - Y_t^{(h)}| \int_0^T (1 + |Z_t^{i,(k)}|^2 + |Z_t^{i,(h)}|^2) dt.
\]
With \( C^r \) inequality and Hölder’s inequality, for any \( p \geq 2 \), it holds that
\[
E\left[\int_0^T |Z_s^{i,(k)} - Z_s^{i,(h)}|^2 ds\right]^{\frac{2}{p}} \leq C_p \left\{ C^p E \left[|X_T^{(k-1)} - X_T^{(h-1)}|^p\right] + 2\tilde{p} E \left[\left(\int_0^T |(Y_s^{i,(k)} - Y_s^{i,(h)})(Z_s^{i,(k)} - Z_s^{i,(h)})|^2 ds\right)^{\frac{\tilde{p}}{2}}\right]\right.
\]
\[
+ (2CK)^{\tilde{p}} E \left[\left(\int_0^T |X_s^{(k-1)} - X_s^{(h-1)}| ds\right)^{\tilde{p}}\right] + (2CT)^{\tilde{p}} E \left[\left(\sup_{t \in [0,T]} |Y_t^{(k)} - Y_t^{(h)}|^2\right)^{\frac{\tilde{p}}{2}}\right]\right.
\]
\[
+ (6C)^{\tilde{p}} E \left[\left(\sup_{t \in [0,T]} |Y_t^{(k)} - Y_t^{(h)}|\right)^{\frac{\tilde{p}}{2}} \left(\int_0^T (1 + |Z_t^{i,(k)}|^2 + |Z_t^{i,(h)}|^2) ds\right)^{\frac{\tilde{p}}{2}}\right]\right\}
\]
\[ \leq C_p' \left\{ E \left| X_T^{(k-1)} - X_T^{(h-1)} \right|^p + E \left[ \sup_{t \in [0,T]} \left| Y_t^{(k)} - Y_t^{(h)} \right|^p \right]^{\frac{1}{p}} E \left[ \left( \int_0^T (|Z_s^{i,(k)}|^2 + |Z_s^{i,(h)}|^2) ds \right)^{\frac{2}{p}} \right] \right\} \]

\[ + E \left[ \left( \int_0^T |X_s^{(k-1)} - X_s^{(h-1)}|^2 ds \right)^{\frac{2}{p}} \right] + E \left[ \left( \sup_{t \in [0,T]} \left| Y_t^{(k)} - Y_t^{(h)} \right|^2 \right)^{\frac{2}{p}} \right] \]

\[ + E \left[ \sup_{t \in [0,T]} \left| Y_t^{(k)} - Y_t^{(h)} \right|^p \right]^{\frac{1}{p}} E \left[ \left( \int_0^T (1 + |Z_s^{i,(k)}|^2 + |Z_s^{i,(h)}|^2) ds \right)^{\frac{2}{p}} \right] \}

Since for any \( p \geq 2, k, h \geq 0 \), we have

\[ E \left( \int_0^T \left| Z_s^{i,(k)} \right|^2 ds \right)^{\frac{2}{p}} < \infty \]

and

\[ \lim_{k,h \to \infty} E \left[ \sup_{t \in [0,T]} \left| X_t^{(k)} - X_t^{(h)} \right|^p \right] = 0, \quad \lim_{k,h \to \infty} E \left[ \sup_{t \in [0,T]} \left| Y_t^{(k)} - Y_t^{(h)} \right|^p \right] = 0, \]

then \( Z^{(k)} \) is convergent in \( H^p(\mathbb{R}^{n+d}) \). So there exists \( Z \in H^p(\mathbb{R}^{n+d}) \), such that \( \| Z^{(k)} - Z \|_{H^p(\mathbb{R}^{n+d})} \to 0 \) as \( k \to \infty \) for any \( p \geq 2 \).

Therefore let \( k \to \infty \) in (7), we have

\[ Y_t = h(X_t) + \int_0^T g^i(s, X_s, Y_s, Z_s^i) ds - \int_0^T Z_s^i dW_s. \]

To conclude, \((X, Y, Z)\) is a solution for the FBSDE (2) and use the same method in (9) and (10) we have \((Y, Z \cdot W) \in S^\infty(\mathbb{R}^n) \times BMO \).

**Step 3:** We show the minimality of the solution that we construct above. Let us assume that \((\tilde{X}, \tilde{Y}, \tilde{Z})\) is another solution of (2) such that \((\tilde{X}, \tilde{Y}, \tilde{Z} \cdot W) \in S^p(\mathbb{R}^n) \times S^\infty(\mathbb{R}^n) \times BMO \) for any \( p \geq 2 \).

From the assumption (A4), we have

\[-K(1 + |y| + |z|^2) \leq g^i(t, \tilde{X}, y, z), \quad \forall y \in \mathbb{R}^n, \quad z^i \in \mathbb{R}^d, \quad i = 1, \ldots, n, \quad t \in [0, T] \]

and we may conclude immediately that

\[ Y_t^{(0)} \leq \tilde{Y}_t, \quad t \in [0, T] \]

from [20, Theorem 2.2]. On the other hand, since \( b \) is nondecreasing in \( y \), from our construction we have

\[ b(t, x, Y_t^{(0)}) \leq b(t, x, \tilde{Y}_t), \quad t \in [0, T]. \]

So by [10, Theorem 1.1] we have

\[ X_t^{(0)} \leq \tilde{X}_t, \quad t \in [0, T] \]
and then
\[ h^i(X_t^{(0)}) \leq h^i(\bar{X}_t), \]
\[ g^i(t, X_t^{(0)}, y, z) \leq g^i(t, \bar{X}_t, y, z), \quad i = 1, \ldots, n, \quad t \in [0, T]. \]

Again by [20, Theorem 2.2], we conclude that \( Y_t^{(1)} \leq \bar{Y}_t, \quad t \in [0, T]. \)

Iterating the procedure we get
\[ Y_t^{(0)} \leq Y_t^{(1)} \leq \cdots \leq Y_t^{(k)} \leq \cdots \leq \bar{Y}_t, \]
\[ X_t^{(0)} \leq X_t^{(1)} \leq \cdots \leq X_t^{(k)} \leq \cdots \leq \bar{X}_t, \]
\[ k \in \mathbb{N}, \quad t \in [0, T]. \]

Therefore, \( Y \leq \bar{Y}, \quad X \leq \bar{X}, \) that is the minimality of the solution \((X, Y)\).

**Theorem 2.2** Assume \((x_0, b, \sigma, h, g)\) and \((\bar{x}_0, \bar{b}, \bar{\sigma}, \bar{h}, \bar{g})\) satisfy \((\mathcal{A}1) - (\mathcal{A}7)\). Let \((X, Y, Z)\) (resp. \((\bar{X}, \bar{Y}, \bar{Z})\)) be the minimal solution of FBSDE (2) associated to \((x_0, b, \sigma, h, g)\) (resp. \((\bar{x}_0, \bar{b}, \bar{\sigma}, \bar{h}, \bar{g})\)). If it holds that \(x_0 \leq \bar{x}_0\) and for any \(i = 1, \ldots, n, \quad t \in [0, T],\)
\[ b^i(t, x, y) \leq \bar{b}^i(t, x, \bar{y}), \]
for any \(x, \bar{x}, y, \bar{y} \in \mathbb{R}^n\) satisfying \(x^i = \bar{x}^i, x^j \leq \bar{x}^j, j \neq i, y \leq \bar{y}\) and
\[ h(x) \leq \bar{h}(x), \]
\[ g^i(t, x, y, z^i) \leq \bar{g}^i(t, x, \bar{y}, z^i), \]
for any \(z^i \in \mathbb{R}^i\) and \(x, \bar{x}, y, \bar{y} \in \mathbb{R}^n\) satisfying \(y^i = \bar{y}^i, y^j \leq \bar{y}^j, j \neq i, x \leq \bar{x}\), we have
\[ X_t \leq \bar{X}_t, \quad Y_t \leq \bar{Y}_t, \quad t \in [0, T]. \]

**Proof.** Since the solution constructed in Theorem 2.1 is the minimal one, we consider \((Y^{(0)}, Z^{(0)}) = (\bar{Y}^{(0)}, \bar{Z}^{(0)})\) satisfying
\[ Y_t^{(0)} = -C - \int_t^T (1 + |Y_s^{(0)}| + |Z_s^{(0)}|^2) ds - \int_t^T Z_s^{(0)} dW_s, \quad i = 1, \ldots, n, \quad t \in [0, T]. \]

Then we consider
\[ X_t^{(0)} = x_0 + \int_0^t b(s, X_s^{(0)}, Y_s^{(0)}) ds + \int_0^t \sigma(s, X_s^{(0)}) dW_s, \]
\[ \bar{X}_t^{(0)} = \bar{x}_0 + \int_0^t \bar{b}(s, \bar{X}_s^{(0)}, \bar{Y}_s^{(0)}) ds + \int_0^t \bar{\sigma}(s, \bar{X}_s^{(0)}) dW_s. \]

By our assumptions, it holds that
\[ x_0^i \leq \bar{x}_0^i, \quad b^i(t, x, Y_t^{(0)}) \leq \bar{b}^i(t, x, \bar{Y}_t^{(0)}), \quad t \in [0, T] \]
for any \(x, \bar{x} \in \mathbb{R}^n\) satisfying \(x^i = \bar{x}^i, x \leq \bar{x}, j \neq i\), which implies
\[ X_t^{(0)} \leq \bar{X}_t^{(0)}, \quad t \in [0, T] \]

due to the [10] Theorem 1.1. Now we introduce \((Y^{(1)}, Z^{(1)})\) and \((\overline{Y}^{(1)}, \overline{Z}^{(1)})\) as follows

\[
Y_t^{(1)} = h^i(X_t^{(0)}) + \int_t^T g^i(s, X_s^{(0)}, Y_s^{(1)}, Z_s^{(1)})ds - \int_t^T Z_s^{(1)}dW_s,
\]

\[
\overline{Y}_t^{(1)} = \overline{h}^i(X_t^{(0)}) + \int_t^T \overline{g}^i(s, X_s^{(0)}, \overline{Y}_s^{(1)}, \overline{Z}_s^{(1)})ds - \int_t^T \overline{Z}_s^{(1)}dW_s.
\]

We can deduce

\[
Y_t^{(1)} \leq \overline{Y}_t^{(1)}, \quad t \in [0, T]
\]

from

\[
h^i(X_t^{(0)}) \leq \overline{h}^i(X_t^{(0)}), \quad g^i(t, X_t^{(0)}, y, z) \leq \overline{g}^i(t, X_t^{(0)}, \overline{y}, \overline{z}), \quad i = 1, \cdots, n, \quad t \in [0, T]
\]

for any \(z^i \in \mathbb{R}, \overline{y}, \overline{y} \in \mathbb{R}^n\) satifying \(y^i = \overline{y}^i, y^j \leq \overline{y}^j, j \neq i\) and \([20]\) Theorem 2.2].

We can claim that after iterating several times in the same way,

\[
X_t^{(k)} \leq \overline{X}_t^{(k)}, \quad Y_t^{(k)} \leq \overline{Y}_t^{(k)}
\]

holds for any \(k \geq 0, t \in [0, T]\).

Therefore we have

\[
X_t = \lim_{k \to \infty} X_t^{(k)} \leq \lim_{k \to \infty} \overline{X}_t^{(k)} = \overline{X}_t,
\]

\[
Y_t = \lim_{k \to \infty} Y_t^{(k)} \leq \lim_{k \to \infty} \overline{Y}_t^{(k)} = \overline{Y}_t, \quad t \in [0, T].
\]

\[\blacksquare\]

References

[1] F. Antonelli, Backward-forward stochastic differential equations, Ann. Appl. Probab. 3 (1993): 777-793.
[2] F. Antonelli, S. Hamadène, Existence of the solutions of backward-forward SDE’s with continuous monotone coefficients, Statistics and Probability Letters. 76 (2006) 1559-1569.
[3] P. Barrieu, N. El Karoui, Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs, The Annals of Probability. 41(3B) (2013) 1831-1863.
[4] J. Bielagk, A. Lionnet, G. dos Reis, Equilibrium pricing under relative performance concerns, SIAM Journal on Financial Mathematics. 8(1) (2017) 435–482.
[5] P. Briand and Y. Hu, BSDE with quadratic growth and unbounded terminal value, Probab. Theory Related Fields. 136 (2006) 604-618.
[6] P. Briand and Y. Hu, Quadratic BSDEs with convex generators and unbounded terminal conditions, Probab. Theory Related Fields. 141 (2008) 543-567.
[7] P. Cheridito and K. Nam, Multidimensional quadratic and subquadratic BSDEs with special structure, Stochastics 87(5) (2014) 1257-1285.
[8] F. Delarue, On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case, Stochastic Processes and their Applications. 99 (2002) 209-286.
[9] A. Fromm, P. Imkeller, Existence, uniqueness and regularity of decoupling fields to multidimensional fully coupled FBSDEs, Preprint. (2013).

[10] C. Geiß, R. Manthey, Comparison theorems for stochastic differential equations in finite and infinite dimensions, Stochastic Processes and their Applications. 53 (1994) 23-35.

[11] U. Horst, Y. Hu, P. Imkeller, A. Réveillac, J. Zhang, Forward backward systems for expected utility maximization, Stochastic Processes and their Applications. 124 (5) (2014) 1813–1848.

[12] Y. Hu, S. Peng, Solution of forward-backward stochastic differential equations, Probab Theory RelFields. 103 (1995) 273–283.

[13] Y. Hu, S. Tang, Multi-dimensional backward stochastic differential equations of diagonally quadratic generators, Stochastic Processes and their Applications. 126 (2016) 1066-1086.

[14] J. Jackson, Global existence for quadratic FBSDE systems and application to stochastic differential games, arXiv:2110.01588 (2021).

[15] A. Jamneshan, M. Kupper and P. Luo, Multidimensional quadratic BSDEs with separated generators, Electron. Comm. Probab. 22(58) (2017) 1-10.

[16] N. Kazamaki, Continuous Exponential Martingales and BMO, Springer-Verlag, Berlin, 1994.

[17] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth, The Annals of Probability. 28(2) (2000) 588-602.

[18] D. Kramkov, S. Pulido, Stability and analytic expansions of local solutions of systems of quadratic BSDEs with applications to a price impact model, SIAM Journal on Financial Mathematics. 7(1) (2016) 567–587.

[19] M. Kupper, P. Luo, L. Tangpi, Multidimensional Markovian FBSDEs with super-quadratic growth, Stochastic Processes and their Applications. 129 (2019) 902-923.

[20] P. Luo, Comparison theorem for diagonally quadratic BSDEs, Discrete and Continuous Dynamical Systems. 41(6) (2021) 2543-2557.

[21] P. Luo, A type of globally solvable BSDEs with trianulary quadratic generators, Electron. J. Probab. 25(112) (2020) 1-23.

[22] P. Luo, L. Tangpi, Solvability of coupled FBSDEs with diagonally quadratic generators, Stochastics and Dynamics. 17(6) (2017) 1750043.

[23] J. Ma, P. Protter, J. Yong, Solving forward-backward stochastic differential equations explicitly-a four step scheme, Probability Theory and Related Fields. 98 (1994) 339-359.

[24] J. Ma, Z. Wu, D. Zhang and J. Zhang, On wellposedness of forward-backward SDE - a unified approach. Ann. Appl. Probab. 25(4) (2015) 2168-2214.

[25] J. Ma, J. Yong, Y. Zhao, Four step scheme for general Markovian forward-backward SDEs, Journal of Systems Science and Complexity. 23 (2010) 546–571.

[26] E. Pardoux and S. Tang, Forward-backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Relat. Fields. 114 (1999) 123-150.

[27] S. Peng, Z. Wu, Fully Coupled Forward-Backward Stochastic Differential Equations and Applications to Optimal Control, SIAM Journal on Control and Optimization. 37(3) (1999) 825–843.
[28] R. Tevzadze, Solvability of backward stochastic differential equations with quadratic growth, Stochastic Process. Appl. 118 (2008) 503-515.

[29] H. Xing and G. Žitković, A class of globally solvable Markovian quadratic BSDE systems and applications, Ann. Probab. 46(1) (2018) 491-550.

[30] J. Yong, Linear Forward-Backward Stochastic Differential Equations, Applied Mathematics and Optimization. 39 (1999) 93–119.