ON HIERARCHY AND EQUIVALENCE OF RELATIVISTIC EQUATIONS FOR MASSIVE FIELDS

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A non-canonical correspondence of the complete set of solutions to the Dirac and Klein-Gordon free equations in Minkowski space-time is established. This allows for a novel viewpoint on the relationship of relativistic equations for different spins and on the origin of spinor transformations. In particular, starting from a solution to the Dirac equation, one obtains a chain of other solutions to both Dirac and Klein-Gordon equations. A comparison with the massless case is performed, and examples of non-trivial singular solutions are presented. A generalization to Riemannian space-time and inclusion of interactions are briefly discussed.

1. Introduction

It is generally accepted that the linear relativistic field equations are mutually independent and in a one-to-one correspondence to the finite-dimensional (tensorial or spinorial) irreducible representations of the Lorentz (Poincaré) group [1,2]. In quantum field theory, they describe particles of integer or half-integer spin, respectively. The correspondence between two observable types of elementary particles, bosons and fermions, and the representations of the invariance group of physical space-time is certainly one of the most elegant and trustworthy relationships established in physics in general.

Spinor fields, especially the Dirac spin-1/2 field, play a principal role at the microlevel being related to almost all stable elementary particles, the proton, electron and neutrino (or to quarks and leptons at a deeper level). However, in cosmology the role of spinors is generally considered to be restricted. There are a rather limited number of works dealing with cosmological scenarios that involve some (nonlinear or interacting) spinor fields invoked to avoid the primordial singularity [3], to ensure the inflationary [4] or accelerated [5] cosmological regimes etc.

However, the spinor field is in fact unable to represent the vacuum structure due to non-invariance of spinors under Lorentz transformations. Instead, scalar fields naturally perform the role of a physical vacuum both at the microlevel (Higgs fields) and in the Universe as a whole (inflaton fields etc.).

In fact, it looks an enigma why such fields, all originating from representations of the same Lorentz group, are so distinct in their manifestations in nature? It looks a mystery why there are no stable particles of zero spin described by the Klein-Gordon equation. Or, why there is no massless spin-1/2 particle (if one accepts the concept of a massive neutrino) corresponding to such a fundamental and fairly simple equation as the Weyl one?

All these and other similar questions, of both pragmatic or “methaphysical” nature, force one to return back again to the problem of a unified description of all elementary particles (irrespective of spins and masses), in particular, through some “hidden” universal structure of field equations. In the past and even in the recent times, there have been a number of attempts to bind together the formal structures of distinct relativistic equations and the sets of their solutions or, in other words, to prove in a sense their full equivalence to one another [6,7] (see also [8,9] and references therein). There are some motivations that stimulate these attempts, the most evident of them being the possibility of representing the whole set of wave (massless) equations in a universal 2-spinor form [10-13]. For example, the Weyl equation for a 2-spinor \( \{\xi_A\} \), \( A = 1, 2 \) of a massless spin-1/2 particle

\[
\partial^{AA'}\xi_A = 0, \tag{1}
\]

has a structure quite similar to that of the Maxwell equations for a (symmetric) spinor \( \varphi_{(AB)} \) of the electromagnetic field strength:

\[
\partial^{AA'}\varphi_{(AB)} = 0 \tag{2}
\]

though the latter corresponds to a particle of integer spin, the photon. Though this formal similarity does not imply any equivalence of the two equations with the respect of their solutions, in fact this is really the case. For free fields in Minkowski space-time, this has been proved, in particular, in our works [14] (see also [6]). Moreover, according to the results presented therein, any (regular or singular) solution of both Weyl and Maxwell equations may be obtained (via consecutive differentiation) from a solution of the ordinary wave equation for a one-component complex scalar field \( \phi \in \mathbb{C} \):

\[
\Box \phi = 0 \tag{3}
\]

Of course, the converse statement is also true, so that we arrive at some informal equivalence between the sets of massless equations for different (0, 1/2 and 1) spins in the above formulated sense.

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The result announced is quite nontrivial. On can note, in particular, that, owing to this equivalence, it is possible, say, to write out an analogue of the Coulomb solution for the Weyl equation \[14\], to equip this field distribution with at least two energy-like densities (the first one being the canonical indefinite density of the 2-spinor field whereas the second one is the positive-definite density inherited from the structure of the Maxwell field) etc. Numerous aspects of the equivalence obtained and powerful algebraic methods for generating solutions of wave equations in the massless case can be found in [14].

As for the more refined case of equations with nonzero mass, it is well known that the Klein-Gordon second-order equation may be represented in the form of the Duffin-Kemmer first-order matrix equation for a 5-component wave function (composed from the initial scalar field along with its 4-gradient). However, this construction does not testify to the equivalence of these equations and, all the more, has nothing in common with the Dirac field. On the other hand, any component of the Dirac bispinor identically satisfies the Klein-Gordon equation. It is commonly accepted that, generally, the converse is not true, so that the structure of the Dirac equation is more rigid than that of, say, the Klein-Gordon equation for a “4-column” field (digressing here from their transformation properties). However, in this article we show that the situation is much more interesting and nontrivial. Namely, we demonstrate (Section 2) that the 4-component Klein-Gordon field acts as a field of potentials for any Dirac bispinor. The hidden gauge invariance of the Dirac equation with respect to special transformations of their “Klein-Gordon potentials” is also considered.

Thus we demonstrate that any solution to the Dirac equation can be obtained (by differentiation) from some solution of a 4-multiplet of the complex Klein-Gordon fields and vice versa. In this sense, these two equations may be considered to be equivalent, so that the Dirac equation is nothing but a set of four particular constraints on the derivatives of a 4-component Klein-Gordon field!

A lot of problems of interest naturally arise in this context. One of them is a curious possibility of generating a number of (in general, functionally independent) solutions to the Dirac and Klein-Gordon equations starting from an arbitrary Dirac bispinor. As a result, we arrive at a natural hierarchy of solutions to both equations. The chain of solutions resembles a similar chain of solutions to the Weyl equation related to solutions of d’Alembert and Maxwell equations. This construction is discussed in Section 3 where some remarkable examples of singular solutions to massless and massive equations are also presented.

Section 4 is devoted to the problem of origin of the spinorial transformation law for the Dirac field when the latter is obtained from a 4-component Klein-Gordon “scalar” field. Section 5 concludes the consideration.

To simplify the exposition, we do not make use of the more convenient 2-spinor formalism and operate only with the standard $\gamma$-matrix representation of the Dirac equation. Only the simplest case of free fields in the Minkowski flat background with the metric $\eta^{\mu\nu}$ (of the $+, -, -, -$ signature) is considered, and possible generalizations are briefly discussed only in the final Section 5. As usual, we accept the system of units such that $c = \hbar = 1$.

2. The Klein-Gordon “potentials” for solutions of the Dirac equation

Consider a multiplet of 4 complex fields $\phi = \{\phi_a\}$, $a = 1, 2, 3, 4$, each subject to the Klein-Gordon equation. In a 4-column representation we have

\[(\Box - m^2)\phi = 0, \quad (4)\]

\[\Box := -\eta^{\mu\nu} \partial_\mu \partial_\nu \text{ being the d’Alembert wave operator and } m \text{ the (common) mass of a “quantum” of the } \phi \text{-fields. The Klein-Gordon operator may be factorized into a product of two commuting first-order Dirac operators } D, D^* :\]

\[\Box - m^2 = D D^* = D^* D, \quad (5)\]

\[D := i\gamma^\mu \partial_\mu - m, \quad D^* := i\gamma^\mu \partial_\mu + m, \quad (6)\]

where $\gamma^\mu$ are the four canonical $4 \times 4$ Dirac matrices satisfying the commutation relations

\[\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \quad (7)\]

Let us now define another 4-component complex field $\chi$ through derivatives of the initial $\phi$ as follows:

\[\chi := D^* \phi \quad (8)\]

Then, as a consequence of (4) and (5), this field identically satisfies the Dirac equation

\[D \chi = DD^* \phi \equiv 0. \quad (9)\]

Thus any solution to the 4-component Klein-Gordon equation gives rise to a solution to the free Dirac equation. Note that the components themselves may be functionally independent or not; in particular, some of them may be identically equal to zero. Of course, we here deal with solutions differentiable in a connected domain of the Minkowski space. However, they are not necessary regular everywhere (being then a composition of plane waves); on the contrary, they may be singular at boundary points, in particular, contain poles or branching points. The corresponding examples (for the case of massless fields) may be found in [14-17] while for massive case they will be presented below (Section 3).
Conversely, let now some solution $\chi$ to the Dirac equation (9) be given. It is well known that any component of the Dirac field satisfies the Klein-Gordon equation

$$0 = D^*(D\chi) = (\Box - m^2)\chi \equiv 0. \quad (10)$$

However, we are interested in another thing, namely, in restoration of the generating potentials $\phi$ from the basic relation (8). This is a system of four inhomogeneous first-order PDE’s which, for any given $\chi$ in the l.h.s., can always be (locally) resolved with respect to the potentials $\phi$ (just as is the case for the field strengths and potentials of the electromagnetic field). Of course, the potentials obtained will obey the Klein-Gordon equation

$$0 = D\chi = DD^*\phi = (\Box - m^2)\phi \equiv 0. \quad (11)$$

As expected, the potentials are defined non-uniquely, up to the general solution of the “homogeneous Dirac equation” $D^*\phi = 0$. In other words, the initially specified Dirac field $\chi$ (the “Dirac field strength”) remains invariant under the following gauge transformations of its Klein-Gordon potentials (8):

$$\phi \mapsto \phi + \kappa, \quad (12)$$

where $\kappa$ is an arbitrary solution to the Dirac equation

$$D^*\kappa = 0 \quad (13)$$

Moreover, since for any $\kappa$ some Klein-Gordon potentials $\xi$ can be found, $\kappa = D\xi$, the gauge transformations (12) may be represented in the familiar gradientlike form:

$$\phi \mapsto \phi + D\xi \quad (14)$$

Now $\xi$ may be considered as an arbitrary field subject to the Klein-Gordon equation. Thus we have proved that the free Dirac field itself is a gauge invariant field resembling the Maxwell field strengths. On the other hand, the Klein-Gordon field serves as the field of potentials with respect to the Dirac field. In this sense, these both cannot be regarded as independent fields describing different kinds of particles. As in the case of electromagnetic structures, these equations describe, in fact, the same physical field system in different representations and, up to a choice of gauge for the Klein-Gordon potentials, the free Dirac and the 4-component Klein-Gordon equations are equivalent, so that arbitrary solution of the Dirac equation corresponds to some solution of the Klein-Gordon equation from the class of equivalence (14), and vice versa.

The gauge invariance of the Dirac equation, represented by the transformations (14), is defined by the gauge function $\xi(x)$ necessarily constrained by the Klein-Gordon equation. Therefore, this type of gauge invariance differs from the ordinary gauge transformations of electromagnetic potentials

$$A_\mu \mapsto A_\mu - \partial_\mu \alpha, \quad (15)$$

with $\alpha(x)$ an arbitrary smooth function of space-time coordinates $\{x_\mu\}$. However, if one requires, in addition, that the relativistic Lorentz gauge condition $\partial_\mu A^\mu = 0$ be preserved under (15), then the gauge function $\alpha$ is known to satisfy the d’Alembert wave equation and is thus no longer arbitrary. The latter situation (in the massless case) is in a full analogy with that with the gauge freedom of the “massive” Klein-Gordon potentials (14). It is also worth mentioning that a special variety of gauge structures, the so-called “weak” gauge transformations with parameters depending on coordinates only implicitly, i.e. through the initial field under transform, has been introduced in the framework of bi-quaternionic electrodynamics [16,17]. Of course, they are deeply related to the “restricted” gauge transformations (14) since if the gauge function $\xi$ depends only on the field $\xi = \xi(\phi(x))$ under transform, then it satisfies the Klein-Gordon equation, as required. Note that the “weak” gauge transformations constitute a proper subgroup of the full gauge group and are closely related to projective transformations in twistor space [17].

### 3. Hierarchy of solutions to the Dirac and Klein-Gordon equations

The above correspondence between the solutions of the Dirac and Klein-Gordon equations allows us to obtain (a chain of) derivative solutions to both equations. Let, say, a solution $\chi$ to the Dirac equation $D\chi = 0$ be given. It also identically satisfies the Klein-Gordon equation

$$0 = D^*(D\chi) = (\Box - m^2)\chi \equiv 0. \quad (16)$$

Consequently, the field $\chi_1$ defined as in (8) via derivatives of the initial field

$$\chi' = D^*\chi \quad (17)$$

again satisfies both the Dirac and Klein-Gordon equations:

$$D\chi' = D(D^*\chi) = (\Box - m^2)\chi \equiv 0, \quad (18)$$

$$D^*(\Box - m^2)\chi' = D^*D\chi' \equiv 0. \quad (19)$$

However, since $D\chi = (i\gamma^\mu \partial_\mu - m)\chi = 0$, the new solution

$$\chi' = D^*\chi = (i\gamma^\mu \partial_\mu + m)\chi = 2m\chi \quad (20)$$

is proportional to the old one. In order to obtain actually a functionally independent solution, one should make use of the internal symmetry group $\phi \mapsto P\phi$, $P \in SL(4, \mathbb{C})$ of the solutions $\phi$ to the Klein-Gordon equation (this symmetry relates also to the Lorentz invariance, see Section 4). Let us thus, instead of (17), take

$$\chi_1 = D^*P_1\chi \quad (21)$$
with some arbitrary matrix \( P_1 \in SL(4, \mathbb{C}) \). Since \((\Box - m^2) P_1 \chi = P_1 (\Box - m^2) \chi = 0\), the anew defined solution (21) will satisfy the Dirac equation,

\[
D \chi = D^* P_1 \chi = 0.
\] (22)

In the case when the matrix \( P_1 \) is not proportional to the \( \gamma^5 \) (in this case the solution \( \chi_1 \) is identically zero) or to the unit matrix, solution (21) will be functionally independent from the initial one. Continuing the procedure, one arrives at an (infinite) chain of solutions to the Dirac free equation of the following form:

\[
\chi_N = D^* P_N \cdots D^* P_2 D^* P_1 \chi,
\] (23)

where, generally, all the matrices \( P_i \) could be different.

In order to imagine better the above procedure, one can first consider an analogous construction of hierarchy of solutions in the massless case, that is, to the Weyl equation. In the spinor coordinates

\[
u = t + z, \quad v = t - z, \quad w = x - iy, \quad p = x + iy
\] (24)

\(u, v\) being real and \(w, p\) complex conjugated) the latter has the form of a set of two equations

\[
\partial_u \alpha = \partial_p \beta \quad (:= \gamma), \quad \partial_w \alpha = \partial_v \beta \quad (:= \delta)
\] (25)

for the two components \(\{\alpha, \beta\}\) of the Weyl 2-spinor. The d’Alembert equation manifests itself here as the compatibility condition for the Weyl system (25):

\[
\Box \alpha = \frac{1}{4} (\partial_u \partial_v - \partial_w \partial_p) \alpha \equiv 0,
\] (26)

and analogously for the \(\beta\)-component. From (26) it directly follows that the derivative spinor \(\{\delta, \gamma\}\) satisfies the “reflected” Weyl equation

\[
\partial_u \delta = \partial_w \gamma \quad (:= \pi), \quad \partial_p \delta = \partial_v \gamma \quad (:= \rho),
\] (27)

(wheras the Weyl equation of initial type holds for the complex conjugated 2-spinor \(\{\delta^*, \gamma^*\}\)). On the other hand, for any solution to the Weyl equation there exist “2-spinor potentials” which not only satisfy the d’Alembert equation but are themselves a new solution to the Weyl equation. Indeed, the system (25) is solved locally by a new 2-spinor \(\{\mu, \nu\}\) such that

\[
\alpha = \partial_p \mu, \quad \beta = \partial_w \mu;
\] (28)

\[
\alpha = \partial_v \nu, \quad \beta = \partial_w \nu,
\] (29)

so that the potential 2-spinor satisfies again the (reflected) Weyl equation

\[
\partial_u \mu = \partial_w \nu, \quad \partial_p \mu = \partial_v \nu
\] (30)

Such procedure can be repeated and leads to a chain of solutions to the Weyl equation, to the d’Alembert equation and to the associated equations for complex Maxwell field (for details see [14]). On the other hand, if one resolves only, say, the first of the Weyl equations (25) by means of the potential \(\mu\) as in (28), then the second equation requires that the potential \(\mu\) should satisfy the d’Alembert equation

\[
\Box \mu = \frac{1}{4} (\partial_u \partial_v - \partial_w \partial_p) \mu = 0
\] (31)

Thus any Weyl 2-spinor can be obtained by derivation from a d’Alembert potential, i.e. from a solution to a one-component wave equation.

Below we present some examples of the generating procedure in the massless and massive cases. Note that we do not deal here with rather a trivial case of regular waveline solutions but, instead, consider solutions with a complicated structure of singularities. Let us first take the potential \(\mu\) in (28) of the form

\[
\mu = \frac{p}{z + r} = \frac{x + iy}{z + r} = \tan \frac{\theta}{2} \exp(i \varphi),
\] (32)

which corresponds to the stereographic projection \(S^2 \mapsto \mathbb{C}\). It is easy to verify that this function satisfies the d’Alembert equation (2). Then the 2-spinor \(\{\alpha, \beta\}\) derived from (32) in accordance with (28),

\[
\alpha = \partial_p \mu = \frac{1}{2r}, \quad \beta = \partial_w \mu = -\frac{\mu}{2r},
\] (33)

identically satisfies the Weyl equation (25), and for the derivative spinor \(\{\delta, \gamma\}\) one gets from the latter:

\[
\delta = -\frac{p}{4r^3}, \quad \gamma = -\frac{z}{4r^3}
\] (34)

The above spinor also satisfies the (reflected) Weyl equation (24), and the corresponding derivatives determine the next 2-spinor \(\{\pi, \rho\}\) in the chain of solutions to the Weyl equation:

\[
\pi = \frac{3z p}{8 r^5}, \quad \rho = \frac{r^2 - 3z^2}{8 r^5}
\] (35)

It is not difficult to guess now that, in fact, this chain is directly related to the multipole expansion of the (complexified) electromagnetic potentials, the spinors \(\{\alpha, \beta\}, \{\delta, \gamma\}\) and \(\{\pi, \rho\}\) representing the monopole, dipole and quadrupole terms, respectively. In more detail this correspondence will be presented elsewhere.

Let us now return to the massive case and take, say, the following stationary solution to the Klein-Gordon free equation:

\[
\Phi = \frac{p}{r(z + r)} e^{-i m t} = \frac{1}{r} \tan \frac{\theta}{2} \exp (i \varphi - i m t)
\] (36)

where the “frequency” is necessarily equal to the Compton one: \(\omega = m\). For the 4-component column of the Klein-Gordon potentials we trivially take

\[
\phi^T = (\Phi, 0, 0, 0).
\] (37)

Then, making use of the above-described procedure based on the expression (38) and allowing for generating...
the solutions to Dirac equation from the Klein-Gordon potentials, we easily obtain the following Dirac field $\chi$:

$$\chi^T = \left( \frac{2mp}{r(z + r)}, 0, -\frac{ip}{r}, -\frac{ip^2(z + 2r)}{r^2(z + r)^2} \right) e^{-imt}. \quad (38)$$

It is easy to check now that the solution defined through (38) as in (17) is in fact proportional to (38). If, however, one makes use of the recipe (21) and takes an auxiliary matrix $P_1 = \gamma^3$ (this choice evidently preserves the Z-axial symmetry) then the following Ansatz results:

$$\chi_1^T = (D^*\gamma^3\chi)^T = \left( \frac{2mp}{r^3}, 0, \frac{6pz}{r^5}, \frac{6p^2}{r^5} \right) e^{-imt}. \quad (39)$$

One can see that the above field is indeed functionally independent from the initial one (38) and obeys the Dirac free equation. Taking again, in accord with general prescription (23),

$$\chi_2 = D^*\gamma^3\chi_1, \quad \chi_3 = D^*\gamma^3\chi_2, \ldots, \quad (40)$$

and so on, one can obtain other axisymmetrical solutions to the Dirac free equation and thus construct an hierarchy of these starting, in fact, from a single solution (38) to the Klein-Gordon free equation.

4. Spinor transformations from scalar potentials

Motivated by the above procedure, there naturally arises the problem of correspondence between the scalar nature of the Klein-Gordon potentials and the (bi)spinor type of transformations of the induced Dirac field. Resolution of this problem is based on the existence of two independent symmetry groups of the 4-component Klein-Gordon equation, namely, of the Lorentz (Poincaré) space-time group and the internal symmetry group $G$ by 360

where the matrix $S \in SL(4, \mathbb{C})$ is now determined (up to a sign, $\pm S$) by the Lorentz transformations parameters that define the matrix $\{L_p^e\}$. The above representation (13) is a direct consequence of a well known commutation property of the Dirac operator

$$(D^*)_yS = SD^*$$

responsible for the relativistic invariance of the Dirac equation in the generally accepted sense.

Thus the transformed solution to the Dirac equation in a 4-rotated frame takes the form

$$\chi' = (SD^*S^{-1})\phi, \quad D'\chi' \equiv 0. \quad (45)$$

However, these transformations, though linear in $\chi$, contain the derivatives of the fields, so that generally the new Dirac field cannot be represented in the form of a “spintensorial” transformation of the initial field $\chi$,

$$\chi' \neq M\chi = M(D^*\phi)$$

with some “representation matrix” $M \in SL(4, \mathbb{C})$. Nonetheless, we have managed to generate transformed solutions to the Dirac equations in an arbitrary 4-rotated frame, which are in general different from the fields obtained by the canonical bispinor transformations. Remarkably, in the above representation, we never meet any sort of two-valuedness of the transformed Dirac field. In particular, it is easy to see from (45) that the field $\chi'$ evolves uniquely and continuously under the 3D rotations of coordinate frame, changes its sign under a rotation by 180° and returns to its initial form (without change in sign!) under the complete turn by 360°.

It is now necessary to understand how one can restore the canonical bispinor transformations of the Dirac field. To do so, one should recall that, even in a fixed frame, the Dirac fields $\chi$ are defined non-uniquely since their Klein-Gordon potentials $\phi$ may be subject to transformations from the symmetry group:

$$\phi \mapsto P\phi, \quad \chi \mapsto D^*(P\phi)$$

with an arbitrary matrix $P \in SL(4, \mathbb{C})$. From a generic viewpoint, all these Dirac fields should be regarded as (physically) equivalent!

Let us combine now the Dirac field transformations related to Lorentz rotations (with invariant potentials) (45) with those related to alteration of potentials (17) in accordance with their internal symmetry group. Then we arrive at the equivalence class of the Lorentz transformed Dirac fields

$$\chi_p = (SD^*S^{-1})(P\phi). \quad (48)$$

The last step one has to make, in order to obtain an explicit (though giving rise to the two-valuedness!) transformations law for the Dirac fields, is to identify two

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4Indeed, if the Dirac field undergoes the ordinary (bi)spinor transformations, then one requires $(D^*)_y\psi = (D^*)_yS\psi = SD^*\psi = 0$ to ensure the Dirac equation be form-invariant.
initially independent matrices: that of a space-time rotation \( S \) and that of an internal intermingling of the potential components \( P \), that is, to set \( P \equiv S \). Then the transformed Dirac field (48) gets the familiar form
\[
\chi' = SD^\ast \phi \equiv S\chi.
\] (49)
This is the canonical bispinor linear transformations law for the Dirac fields in terms of themselves. It does not contain any derivation and does not appeal to generating Klein-Gordon potentials. Nonetheless, one should not forget that this law is, in fact, not uniquely possible and even not the best one as compared, say, with the single-valued transformed Dirac field [15].

5. Conclusion

We have no opportunity here to discuss other peculiar problems related to the concept of Klein-Gordon potentials for the Dirac field. In particular, we postpone the discussion of a number of independent conservative energies, momenta, angular momenta and charges that may be ascribed to any solution of the associated Dirac and Klein-Gordon fields. The existence of such an ambiguity can, in principle, force one to reconsider the canonical quantization scheme for relativistic free fields based, in its considerable part, on the indefiniteness of energy density for the Dirac field and on its positive-definiteness for the Klein-Gordon field.

We also put off a generalization of the above procedure to a Riemannian space-time background or to the case when external, say, electromagnetic fields are present. Since in these cases the ordinary Klein-Gordon operator can no longer factorized into a product of Dirac-like operators, many features of the above connections between these fields are lost in the presence of external fields or on a curved background. However, the squared Dirac equation can then be used instead of the Klein-Gordon equation for potentials, and, from its solutions, one is able, as before, to derive a whole chain of solutions to the canonical Dirac equation with interactions. We are going to present the corresponding examples in a forthcoming publication.

In any case, the obtained correspondence between the solutions and transformation properties of “scalar” and “spinor” fields looks rather unexpected and may shed new light on the classification of really independent fields and on possible ways of a unified description of bosons and fermions, alternative to those based on the supersymmetry hypothesis.

In our construction, the Klein-Gordon and Dirac fields do not exist as two independent ones but manifest themselves as one and the same field in different representations. Specifically, the Klein-Gordon fields serve as potentials for the Dirac one. As for the question of what kind of particles actually corresponds to this field (if any!), an answer lies in the analysis of particular solutions and dynamical quantities that may be ascribed to them. In this respect, it seems intriguing that solutions to the free Dirac and Klein-Gordon equations are not at all exhausted by the regular plane wave solutions [8] but can possibly have a complicated structure of (point-like or extended) singularities. The simplest examples of such solution are represented by formulae (48) and (49), for the Klein-Gordon and Dirac equations, respectively.

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