On $B_h[1]$-sets which are asymptotic bases of order $2h$

Sándor Z. Kiss *, Csaba Sándor †

Abstract

Let $h, k \geq 2$ be integers. A set $A$ of positive integers is called asymptotic basis of order $k$ if every large enough positive integer can be written as the sum of $k$ terms from $A$. A set of positive integers $A$ is said to be a $B_h[g]$-set if every positive integer can be written as the sum of $h$ terms from $A$ at most $g$ different ways. In this paper we prove the existence of $B_h[1]$ sets which are asymptotic bases of order $2h$ by using probabilistic methods.

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1 Introduction

Let $h, k \geq 2$ be integers. We denote the set of nonnegative integers by $\mathbb{N}$ and the set of positive integers by $\mathbb{Z}^+$. Let $A \subset \mathbb{N}$ be an infinite set and let $R_{h,A}(n)$ denote the number of solutions of the equation

\[ a_1 + a_2 + \cdots + a_h = n, \quad a_1, \ldots, a_h \in A, \quad a_1 \leq a_2 \leq \ldots \leq a_h, \quad (1) \]

where $n \in \mathbb{N}$. We say a set of positive integers $A$ forms a $B_h[g]$-set if for every $n \in \mathbb{N}$, the number of representations of $n$ as the sum of $h$ terms in the form (1) is at most $g$, that is $R_{h,A}(n) \leq g$. A set $A \subset \mathbb{N}$ is said to be an asymptotic basis of order $k$ if there exists a positive integer $n_0$ such that $R_{k,A}(n) > 0$ for $n > n_0$. In [4] and [5], P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set (i.e., a $B_2[1]$-set) which is an asymptotic basis of order 3. It is easy to see that a Sidon set cannot be an asymptotic

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*Department of Algebra, Institute of Mathematics, Budapest University of Technology and Economics, Múegyetem rkp. 3., H-1111 Budapest, Hungary; ksandor@math.bme.hu. This author was supported by the National Research, Development and Innovation Office NKFIH Grant No. K129335. This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Supported by the ÚNKP-20-5 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research Development and Innovation Fund.

†Department of Stochastics, Institute of Mathematics, Budapest University of Technology and Economics, Múegyetem rkp. 3., H-1111 Budapest, Hungary; csandor@math.bme.hu. This author was supported by the Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Múegyetem rkp. 3., H-1111 Budapest, Hungary MTA-BME Lendület Arithmetic Combinatorics Research Group, ELKH, Múegyetem rkp. 3., H-1111 Budapest, Hungary; This author was supported by the National Research, Development and Innovation Office NKFIH Grant No. K129335.
basis of order 2 because it does not have enough elements. J. M. Deshouillers and A. Plagne in [3] constructed a Sidon set which is an asymptotic basis of order at most 7. In [8] it was proved the existence of asymptotic bases of order 5 which are Sidon sets by using probabilistic methods. In [2] and [10] this result was improved on by proving the existence of an asymptotic basis of order 4 which is a Sidon set. It was also proved [2] that there exists an asymptotic basis of order 3 which is a $B_2[2]$-set. The above problem of Erdős et. al. can be formulated in a more general form. In a recent paper [11], we proved the existence of a $B_h[1]$-set which is at the same time forms an asymptotic basis of order $2h + 1$.

In this paper we continue the work in this direction. Particularly, we improve on the above result by proving the existence of a $B_h[1]$-set which is an asymptotic basis of order $2h$.

**Theorem 1.** For every $h \geq 2$ integer there exists an asymptotic basis of order $2h$ which is a $B_h[1]$-set.

Before we prove the above theorem, we propose some open problems for further research. These problems are also appear in [11]. In general, for $k, h \geq 2$ integers, one can be interested in the existence of an asymptotic basis of order $k$ which is a $B_h[1]$-set. It is easy to see that there does not exist an asymptotic basis of order $k < h$ which is a $B_h[1]$-set because it does not have enough elements. In recent years, it was proved [11] that there does not exist a $B_h[1]$-set which is an asymptotic basis of order $h$.

**Problem 1.** Determine the smallest value of $k = k(h) > h$ for which there exists a $B_h[1]$-set which is an asymptotic basis of order $k$.

In this paper, we are dealing with the case $k = 2h$ and we prove the existence of an asymptotic basis of order $2h$ which is a $B_h[1]$-set at the same time by using deeper probabilistic arguments. To prove the existence of an asymptotic basis of order $2h - 1$ which is simultaneously a $B_h[1]$-set seems to be very difficult. In the case when $k = h$, the generalization of the famous conjecture of Erdős and Turán states that there does not exist a $B_h[g]$-set which is an asymptotic basis of order $h$. This conjecture is still open and it seems to be hopeless even for $h = 2$.

It is natural question whether there exist an asymptotic basis of order $h + 1$ which is a $B_h[g]$-set for some $g = g(h)$. For $h \geq 2$, it was proved [9] the existence of an asymptotic basis of order $h + 1$ which is a $B_h[g]$-set. In [9], the order of magnitude of $g = g(h)$ was not controlled. This suggest us to study the following problem.

**Problem 2.** Determine the smallest value of $g = g(h)$ for which there exists an asymptotic basis of order $h + 1$ which is a $B_h[g]$-set.

In the following section we give a short summary of the probabilistic tools which plays the crucial role in our proof.

2 **Probabilistic tools**

In the proof of Theorem 1, we apply the probabilistic method due to Erdős and Rényi. Interested reader can find a nice summary about this method in the book of Halberstam
and Roth [6]. Let \( \Omega \) denote the set of strictly increasing sequences of positive integers. In this paper we denote the probability of an event \( \mathcal{E} \) by \( \mathbb{P}(\mathcal{E}) \), and the expectation of a random variable \( \zeta \) by \( \mathbb{E}(\zeta) \).

**Lemma 1.** Let
\[
\alpha_1, \alpha_2, \alpha_3 \ldots
\]
be real numbers satisfying
\[
0 \leq \alpha_n \leq 1 \quad (n = 1, 2, \ldots).
\]
Then there exists a probability space \((\Omega, \mathcal{X}, \mathbb{P})\) with the following two properties:

(i) For every natural number \( n \), the event \( \mathcal{E}^{(n)} = \{A: A \in \Omega, n \in \mathcal{A}\} \) is measurable, and \( \mathbb{P}(\mathcal{E}^{(n)}) = \alpha_n \).

(ii) The events \( \mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \ldots \) are independent.

See Theorem 13. in [6], p. 142.

We denote the characteristic function of the event \( \mathcal{E}^{(n)} \) by \( t_n \) or we can say the boolean random variable means that:
\[
t_n = \begin{cases} 
1, & \text{if } n \in \mathcal{A} \\
0, & \text{if } n \notin \mathcal{A}.
\end{cases}
\]
Moreover, we denote the number of solutions of the equation \( a_1 + a_2 + \ldots + a_k = n \) by \( r_{k,A}(n) \), where \( a_1 \in \mathcal{A}, a_2 \in \mathcal{A}, \ldots, a_k \in \mathcal{A}, 1 \leq a_1 < a_2 < \ldots < a_k < n \). Thus we have
\[
r_{k,A}(n) = \sum_{\eta, a_1 < \ldots < a_k < n} t_{a_1} t_{a_2} \ldots t_{a_k}.
\]

It is easy to see that \( r_{k,A}(n) \) is the sum of random variables. It is clear that for \( k > 2 \) these variables are not necessarily independent because the same \( t_n \) may appear in many terms. To handle this problem we need more advanced probabilistic tools.

Our proof is based on a method of J. H. Kim and V. H. Vu [7], [12], [13], [14]. Assume that \( t_1, t_2, \ldots, t_n \) are independent binary (i.e., all \( t_i \)'s are in \( \{0,1\} \)) random variables. Consider a polynomial \( Y = Y(t_1, t_2, \ldots, t_n) \) in \( t_1, t_2, \ldots, t_n \) with degree \( k \). A polynomial \( Y \) is said to be totally positive if it can be written in the form \( Y = \sum_i e_i \Gamma_i \), where the \( e_i \)'s are positive and \( \Gamma_i \) is a product of some \( t_j \)'s. Given any multi-index \( \overline{\eta} = (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n \), we define the partial derivative \( \partial^{(\overline{\eta})}(Y) \) of \( Y \) by
\[
\partial^{(\overline{\eta})}(Y) = \left( \frac{\partial}{\partial t_1} \right)^{\eta_1} \ldots \left( \frac{\partial}{\partial t_n} \right)^{\eta_n} Y(t_1, t_2, \ldots, t_n),
\]
and denote the order of \( \overline{\eta} \) as \( |\overline{\eta}| = \eta_1 + \ldots + \eta_n \). For any order \( d \geq 0 \), we denote \( \mathbb{E}_d(Y) = \max_{|\overline{\eta}|=d} \mathbb{E}(\partial^{(\overline{\eta})}(Y)) \). Then \( \mathbb{E}_0(Y) = \mathbb{E}(Y) \) and \( \mathbb{E}_d(Y) = 0 \) if \( d \) exceeds the degree of \( Y \). Define \( \mathbb{E}_{\geq d}(Y) = \max_{d \geq d} \mathbb{E}_d(Y) \). We will apply the following theorem proved by Kim and Vu, which informally states that when all the partial derivatives of a totally positive polynomial \( Y \) of degree \( k \) are less on average than \( Y \) itself and \( k \) is small in some sense, then \( Y \) is concentrated around its mean.
Lemma 2. (J. H. Kim - V. H. Vu) For every positive integer \( k \) and \( Y = Y(t_1, t_2, \ldots, t_n) \) totally positive polynomial of degree \( k \), where the \( t_i \)'s are independent binary random variables, and for any \( \lambda > 0 \) there exists a constant \( d_k > 0 \) depending only on \( k \) such that
\[
\mathbb{P} \left( |Y - \mathbb{E}(Y)| \geq d_k \lambda^{k - \frac{1}{2}} \sqrt{\mathbb{E}_{\geq 0}(Y) \mathbb{E}_{\geq 1}(Y)} \right) = O_k \left( e^{-\lambda/4 + (k-1) \log n} \right).
\]
See [14] for the proof. Finally, we need the Borel - Cantelli lemma which is Theorem 7. in [6], p. 135.

Lemma 3. (Borel - Cantelli) Let \( \{B_i\} \) be a sequence of events in a probability space. If
\[
\sum_{j=1}^{+\infty} \mathbb{P}(B_j) < \infty,
\]
then with probability 1, at most a finite number of the events \( B_j \) can occur.

In the next section we prove a lemma which plays a very important role in the proof of Theorem 1.

3 An auxiliary tool

Throughout the remaining part of the paper we use the notation \( f(x) \ll g(x) \) which means \( f(x) = O(g(x)) \). We also use the notation \( f(x) \asymp g(x) \) which means \( f(x) = \Theta(g(x)) \).

Next, we prove the following technical lemma which plays an important role in the proofs.

Lemma 4. (i) Let \( M \geq 2 \) be a positive integer, and let \( 0 < \alpha, \beta < 1 \) be arbitrary real numbers. Then
\[
\sum_{n=1}^{M-1} \frac{1}{n^\alpha} \cdot \frac{1}{(M-n)^\beta} \ll \frac{1}{M^{\alpha+\beta-1}}.
\]

(ii) Let \( M \) be an arbitrary integer, and let \( 0 < \alpha, \beta < 1 \) with \( \alpha + \beta > 1 \) be arbitrary real numbers. Then
\[
\sum_{n=1}^{\infty} \frac{1}{(|n+M|+1)^\alpha} \cdot \frac{1}{n^\beta} \ll \frac{1}{(|M|+1)^{\alpha+\beta-1}}.
\]

(iii) Let \( l \leq 2h \). Then, for every positive integer \( M \),
\[
\sum_{(z_1, \ldots, z_l) \in (\mathbb{Z}^+)^l \atop z_1 + \ldots + z_l = M} \frac{1}{(z_1 \cdots z_l)^{\frac{h-1}{4h-1}}} \ll \frac{1}{M^{1 - \frac{2l}{4h-1}}}.
\]

(iv) Let \( 0 \leq s \leq t \leq 2h \). Then, for every integer \( M \),
\[
\sum_{(z_1, \ldots, z_t) \in (\mathbb{Z}^+)^t \atop z_1 + \ldots + z_s - (z_{s+1} + \ldots + z_t) = M} \frac{1}{(z_1 \cdots z_t)^{\frac{h-1}{4h-1}}} \ll \frac{1}{(|M|+1)^{1 - \frac{2l}{4h-1}}}.
\]
3.1 Proof of Lemma 4

We estimate the sums by integral according to the well known Euler integral formula.

$$\sum_{n=1}^{M-1} \frac{1}{n^\alpha} \cdot \frac{1}{(M-n)\beta} = \sum_{n=1}^{\lfloor M/2 \rfloor} \frac{1}{n^\alpha} \cdot \frac{1}{(M-n)\beta} + \sum_{n=\lfloor M/2 \rfloor + 1}^{M-1} \frac{1}{n^\alpha} \cdot \frac{1}{(M-n)\beta}$$

$$\ll \frac{1}{M^\beta} \sum_{n=1}^{\lfloor M/2 \rfloor} \frac{1}{n^\alpha} + \frac{1}{M^\beta} \cdot \sum_{n=\lfloor M/2 \rfloor + 1}^{M-1} \frac{1}{(M-n)\beta} \ll \frac{1}{M^\beta} \int_0^M \frac{dx}{x^\alpha} + \frac{1}{M^\alpha} \int_0^M \frac{dx}{x^\beta} \ll \frac{1}{M^{\alpha+\beta-1}},$$

which proves (i).

Moreover, if $M \geq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{(\lfloor n + M \rfloor + 1)^\alpha} \cdot \frac{1}{n^\beta} = \sum_{n=1}^{M} \frac{1}{(n + M + 1)^\alpha} \cdot \frac{1}{n^\beta} + \sum_{n=M+1}^{\infty} \frac{1}{(n + M + 1)^\alpha} \cdot \frac{1}{n^\beta}$$

$$\leq \frac{1}{(M + 1)^\alpha} \sum_{n=1}^{M} \frac{1}{n^\beta} + \sum_{n=M+1}^{\infty} \frac{1}{n^\alpha+\beta}$$

$$\ll \frac{1}{(M + 1)^\alpha} \int_0^{M+1} \frac{dx}{x^\beta} + \int_{M+1}^{\infty} \frac{dx}{x^{\alpha+\beta}} \ll \frac{1}{(|M| + 1)^{\alpha+\beta-1}}.$$ 

Furthermore, if $M < 0$,

$$\sum_{n=1}^{\infty} \frac{1}{(\lfloor n + M \rfloor + 1)^\alpha} \cdot \frac{1}{n^\beta} = \sum_{n=1}^{-\lfloor M/2 \rfloor} \frac{1}{(M+n+1)^\alpha} \cdot \frac{1}{n^\beta} + \sum_{n=-\lfloor M/2 \rfloor + 1}^{-M} \frac{1}{(M+n+1)^\alpha} \cdot \frac{1}{n^\beta}$$

$$+ \sum_{n=-M}^{-2M} \frac{1}{(\lfloor M+n \rfloor + 1)^\alpha} \cdot \frac{1}{n^\beta} + \sum_{n=-2M+1}^{-\infty} \frac{1}{(M+n+1)^\alpha} \cdot \frac{1}{n^\beta}$$

$$\ll \frac{1}{|M|^\alpha} \sum_{n=1}^{-\lfloor M/2 \rfloor} \frac{1}{n^\beta} + \frac{1}{|M|^\beta} \sum_{n=-\lfloor M/2 \rfloor + 1}^{-M} \frac{1}{(M+n+1)^\alpha} + \frac{1}{|M|^\beta} \sum_{n=-M+1}^{-2M} \frac{1}{(M+n+1)^\alpha}$$

$$+ \sum_{n=-M+1}^{\infty} \frac{1}{n^{\alpha+\beta}} \ll \frac{1}{|M|^\alpha} \int_0^{-M} \frac{dx}{x^\beta} + \frac{1}{|M|^\beta} \int_0^{-M} \frac{dx}{x^\alpha} + \frac{1}{|M|^\beta} \int_0^{-M} \frac{dx}{x^\alpha} + \int_{-M}^{\infty} \frac{dx}{x^{\alpha+\beta}}$$

$$\ll \frac{1}{(|M| + 1)^{\alpha+\beta-1}},$$

which proves (ii).

Next, we prove (iii) by induction on $l$. The statement is clear for $l = 1$. Assume that it is true for $l - 1 \leq 2h - 1$. Then by (i),

$$\sum_{z_1 + \ldots + z_l = M} \frac{1}{(z_1 \cdots z_l)^{2h-\frac{1}{2}}} = \sum_{z_1 = 1}^{M-1} \frac{1}{z_1^{2h-\frac{1}{2}}} \sum_{z_1 + \ldots + z_{l-1} = M - z_1} \frac{1}{(z_1 \cdots z_{l-1})^{2h-\frac{1}{2}}}$$

5
\[ \alpha = \sum_{z_l=1}^{M-1} \frac{1}{z_l^{\frac{4h}{4k-1}}} \cdot \frac{1}{(M-z_l)^{\frac{2(2k-1)}{4k-1}}} \leq \frac{1}{M^{1-\frac{4h}{4k-1}}} \]

which gives (iii).

Finally, if either \( s = 0 \), or \( t = s \), the statement of (iv) follows immediately from (iii). Otherwise, one can assume that \( 0 < s < t \). It follows from (ii) and (iii) that

\[
\sum_{(z_1, \ldots, z_t) \in (\mathbb{Z}^+)^t} \frac{1}{(z_1 \cdots z_t)^{\frac{4h}{4k-1}}} \leq \sum_{n=1}^{\infty} \left( \sum_{(z_1, \ldots, z_t) \in (\mathbb{Z}^+)^t} \frac{1}{(z_1 \cdots z_t)^{\frac{4h}{4k-1}}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{(n+M+1)^{\frac{2(2k-1)}{4k-1}}} \leq \frac{1}{(|M|+1)^{1-\frac{2k}{4k-1}}},
\]

which proves (iv). The proof of Lemma 4 is completed.

## 4 Proof of Theorem 1

### 4.1 Outline of the proof

Let \( h \) be fixed and let \( \alpha = \frac{2}{4h-1} \). Define the sequence \( \alpha_n \) in Lemma 1 by

\[ \alpha_n = \frac{1}{n^{1-\alpha}}, \]

so that \( \mathbb{P}\{B: B \in \Omega, n \in B\} = \frac{1}{n^{1-\alpha}} \). We prove that in this probability space, almost always one can remove infinitely many elements from a set \( B \in \Omega \) such that the remaining set is both a \( B_h[1] \) set and an asymptotic basis of order \( 2h \). In the first step, we show that almost surely \( R_{2h,B}(n) \) is large, if \( n \) is large enough. Next, we delete elements from the set \( B \) to get a \( B_h[1] \)-set \( A \). Finally, we prove that with probability 1, \( R_{2h,A}(n) \) is still large, if \( n \) is large enough, which implies that the set \( A \) is suitable.

We prove that \( B \) is an asymptotic basis of order \( 2h \) in the following stronger sense. There exists a constant \( C_h > 0 \) such that

\[ R_{2h,B}(n) > C_h n^{\frac{1}{4h-1}}, \tag{2} \]

for every \( n \) large enough, with probability 1.

To do this, we apply the following lemma proved in \([9]\) with \( S = A \), \( k = 2h \) and \( \alpha = \frac{2}{4h-1} \).
Lemma 5. Let $k \geq 2$ be a fixed integer and let $\mathbb{P}(\{A: A \in \Omega, n \in A\}) = \frac{1}{n^\alpha}$ where $\alpha > \frac{1}{k}$. Then with probability 1, $r_{k,A}(n) > cn^{k\alpha - 1}$ for every sufficiently large $n$, where $c = c(\alpha, k)$ is a suitable positive constant.

Obviously, $R_{2h,B}(n) \geq r_{2h,B}(n)$ with probability 1, so the proof of (2) is completed. To get a $B_h[1]$ set, one has to remove the elements which hurt the $B_h[1]$ property. The $B_h[1]$ property can be hurt in two different ways. First, if there exists two $h$-tuples formed by pairwise distinct elements of $B$ such that the sums of the terms in both $h$-tuples are the same i.e., there exist pairwise distinct $b_1, \ldots, b_{2h} \in B$ with

$$b_1 + \ldots + b_h = b_{h+1} + \ldots + b_{2h}.$$  

On the other hand, if there exist $b_1, \ldots, b_{2h} \in B$ with $b_1 + \ldots + b_h = b_{h+1} + \ldots + b_{2h}$, where $b_1, \ldots, b_{2h}$ are not distinct, then by subtracting the common terms from both sides and collecting the equal terms, the previous equality become of the form

$$d_1b_1 + d_2b_2 + \ldots + d_kb_k = e_1b_{k+1} + \ldots + e_ib_{k+i},$$

with positive integer weights $d_1, \ldots, d_k, e_1, \ldots, e_i \in \mathbb{Z}^+$, $k + l \leq 2h - 1$, $d_1 + \ldots + d_k = e_1 + \ldots + e_i \leq h$ and $b_1, \ldots, b_{k+i} \in B$ are already pairwise distinct.

We will prove that by removing the largest element from the above equalities, then with probability 1, the remaining set will be suitable. More formally, define the set $C$ by

$$C = \{b: b \in B, \exists b_2, \ldots, b_{2h} \in B \text{ distinct }, b_i < b, b + b_2 + \ldots + b_h = b_{h+1} + \ldots + b_{2h}\} \cup \{b: b \in B, \exists d_1, \ldots, d_k, e_1, \ldots, e_i \in \mathbb{Z}^+, k + l \leq 2h - 1, d_1 + \ldots + d_k = e_1 + \ldots + e_i \leq h, \exists b_2, \ldots, b_{k+i} \in B \text{ distinct }, b_i < b, d_1b + d_2b_2 + \ldots + d_kb_k = e_1b_{k+1} + \ldots + e_ib_{k+i}\}.$$  

We show that $A = B \setminus C$ is suitable with probability 1. To prove this, we need to show that there exists an $n_2 \in \mathbb{Z}^+$ such that

$$R_{2h,A}(n) \geq 1$$

for every $n \geq n_2$, with probability 1. We write

$$R_{2h,A}(n) = R_{2h,B,C}(n) = R_{2h,B}(n) - (R_{2h,B}(n) - R_{2h,B,C}(n)).$$

Thus we need an upper estimation to $R_{2h,B}(n) - R_{2h,B,C}(n)$. We have

$$R_{2h,B}(n) - R_{2h,B,C}(n) = |\{(b_1, \ldots, b_{2h}) : \exists 1 \leq i \leq 2h, b_i \in C\}| \leq |\{(b_1, \ldots, b_{2h}) : \exists 1 \leq i \leq 2h, b_i + \ldots + b_{2h} = n\}| + |\{(b_1, \ldots, b_{2h}) : \exists 1 < i \leq 2h, b_i + \ldots + b_{2h} = n, b_i, b_{2h} \in B\}| + |\{(b_1, \ldots, b_{2h}) : \exists 1 \leq i \leq 2h, b_i < \ldots < b_{2h}, b_i + \ldots + b_{2h} = n\}| + |\{(b_1, \ldots, b_{2h}) : \exists 1 \leq i \leq 2h, b_i < \ldots < b_{2h}, b_i + \ldots + b_{2h} = n\}|.$$
\[\exists d_1, \ldots, d_k, e_1, \ldots, e_l \in \mathbb{Z}^+, k + l \leq 2h - 1, d_1 + \ldots + d_k = e_1 + \ldots + e_l \leq h,\]

\[\exists b'_1, \ldots, b'_{k+1} \in B \text{ pairwise distinct, } b'_j < b_i, d_1 b_i + d_2 b'_2 + \ldots + d_k b'_k = e_1 b_{k+1} + \ldots + e_l b'_{k+1}\]

\[= R_B^{(1)}(n) + R_B^{(2)}(n) + R_B^{(3)}(n).\]

**Case 1.** Upper estimation for \(R_B^{(1)}(n)\). It is clear that

\[R_B^{(1)}(n) = \sum_{\substack{(f_1, \ldots, f_t) \in (\mathbb{Z}^+)^t \atop f_1 + \ldots + f_t = 2h \atop t \leq 2h - 1}} |\{(b_1, \ldots, b_t) : b_1, \ldots, b_t \in B, b_1 < \ldots < b_t, f_1 b_1 + \ldots + f_t b_t = n\}|.\]

The following lemma ensures that with probability 1, there are only bounded number of such representations.

**Lemma 6.** For positive integers \(t \leq 2h - 1\) and \(f_1, \ldots, f_t \in \mathbb{Z}^+\) with \(f_1 + \ldots + f_t \leq 2h\), the number of solutions of the equation \(f_1 x_1 + \ldots + f_t x_t = m\), where \(x_1, \ldots, x_t \in B\) are distinct, is almost always bounded for every positive integer \(m\).

It follows from Lemma 7 that with probability 1, there exists a positive integer \(C_{f_1, \ldots, f_t}(B)\) depending on the set \(B\) such that

\[R_B^{(1)}(n) \leq \sum_{\substack{(f_1, \ldots, f_t) \in (\mathbb{Z}^+)^t \atop f_1 + \ldots + f_t = 2h \atop t \leq 2h - 1}} C_{f_1, \ldots, f_t}(B) = C^{(1)}(B).\]

**Case 2.** Upper estimation for \(R_B^{(2)}(n)\). Obviously,

\[R_B^{(2)}(n) = \sum_{u=1}^{h} \sum_{v=u}^{u+h} |\{(b_1, \ldots, b_{2h}) : b_1, \ldots, b_{2h} \in B, b_1 < \ldots < b_{2h}, b_1 + \ldots + b_{2h} = n,\}
\]

\[\exists 1 \leq i \leq 2h, \exists b'_2, \ldots, b'_{2h} \in B \text{ distinct, } b'_j < b_i, 2 \leq j \leq 2h,\]

\[b_i + b'_2 + \ldots + b'_h = b'_{h+1} + \ldots + b'_{2h}, u = |\{b_i, b'_2, \ldots, b'_h\} \cap \{b_1, \ldots, b_{2h}\}|,\]

\[v - u = |\{b'_{h+1}, \ldots, b'_{2h}\} \cap \{b_1, \ldots, b_{2h}\}|\].

Define the random variable \(Y_{u,v,h,B}(n)\) by

\[Y_{u,v,h,B}(n) = |\{(b_1, \ldots, b_{4h-v}) : b_i \in B, b_1, \ldots, b_{4h-v}, \text{ distinct } b_i < b_1 \text{ if }\]

\[2 \leq i \leq v \text{ or } i > 2h, b_1 + \ldots + b_{2h} = n, b_1 + \ldots + b_u + b_{2h+1} + \ldots + b_{3h-u}\]

\[= b_{u+1} + \ldots + b_v + b_{3h-u+1} + \ldots + b_{4h-v}\}|.\]

Thus we have

\[R_B^{(2)}(n) \leq \sum_{u=1}^{h} \sum_{v=u}^{u+h} Y_{u,v,h,B}(n).\]

The following lemma gives an upper estimation for \(Y_{u,v,h,B}(n)\).
Lemma 7. With probability 1, for every $1 \leq u \leq v \leq u + h$,

$$Y_{u,v,h,B}(n) \leq \frac{n^{\frac{1}{u+h-1}}}{\log n}.$$

It follows from Lemma 7 that with probability 1, there exists a positive integer $C_{u,v,h}(B)$ such that

$$Y_{u,v,h,B}(n) \leq C_{u,v,h}(B) \cdot \frac{n^{\frac{1}{u+h-1}}}{\log n}.$$  

Let

$$C^{(2)}(B) = \sum_{u=1}^{h} \sum_{v=u}^{u+h} C_{u,v,h}(B).$$

Then with probability 1,

$$R^{(2)}_{B}(n) \leq C^{(2)}(B) \cdot \frac{n^{\frac{1}{u+h-1}}}{\log n}.$$  

Case 3. Upper estimation for $R^{(3)}_{B}(n)$. We have

$$R^{(3)}_{B}(n) = \sum_{(d_{1},\ldots,d_{k},e_{1},\ldots,e_{l}) \in (\mathbb{Z}^{+})^{k+l}} |\{(b_{1},\ldots,b_{2h}) : b_{1},\ldots,b_{2h} \in B, b_{1} < \ldots < b_{2h}, b_{1}+\ldots+b_{2h} = n, \exists 1 \leq i \leq 2h, \exists b'_{2},\ldots,b'_{k+l} \in B \text{ distinct, } b'_{j} < b_{i}, d_{1}b_{1}+d_{2}b'_{2}+\ldots+d_{k}b'_{k} = e_{1}b'_{k+1}+\ldots+e_{l}b'_{k+l} \}|$$

$$= \sum_{(d_{1},\ldots,d_{k},e_{1},\ldots,e_{l}) \in (\mathbb{Z}^{+})^{k+l}} C^{(3)}_{d_{1},\ldots,d_{k},e_{1},\ldots,e_{l}}(n).$$

For a fixed $(d_{1},\ldots,d_{k},e_{1},\ldots,e_{l})$, the following lemma gives an upper estimation for the number of such representations.

Lemma 8. Let $k + l \leq 2h - 1$ and $d_{1},\ldots,d_{k},e_{1},\ldots,e_{l} \in \mathbb{Z}^{+}$ with $d_{1} + \ldots + d_{k} = e_{1} + \ldots + e_{l} \leq h$. Then with probability 1, there are only finitely number of solutions of the equation $d_{1}x_{1} + \ldots + d_{k}x_{k} = e_{1}x_{k+1} + \ldots + e_{l}x_{k+l}$, where $x_{1},\ldots,x_{k+l} \in B$ are distinct.

According to Lemma 8, almost surely the number of solutions of $d_{1}x_{1} + \ldots + d_{k}x_{k} = e_{1}x_{k+1} + \ldots + e_{l}x_{k+l}$, where $x_{1},\ldots,x_{k+l}$ are distinct, in $B$ is at most $C^{(3)}_{d_{1},\ldots,d_{k},e_{1},\ldots,e_{l}}(B)$. Then there are at most

$$\sum_{(d_{1},\ldots,d_{k},e_{1},\ldots,e_{l}) \in (\mathbb{Z}^{+})^{k+l}} C^{(3)}_{d_{1},\ldots,d_{k},e_{1},\ldots,e_{l}}(B) = C^{(4)}(B)$$

possibilities for $b_{i}$. By choosing $t = 2h - 1$, $f_{1} = \ldots = f_{2h-1} = 1$, it follows from Lemma 6 that the equation $x_{1} + \ldots + x_{2h-1} = n - b_{i}$, where $x_{1},\ldots,x_{2h-1}$ are distinct, has at most $C^{(5)}(B)$ solutions in the set $B$. Thus the number of representations in Case 3 is

$$R^{(3)}_{B}(n) \leq C^{(4)}(B) \cdot C^{(5)}(B)$$

with probability 1.
We know from (2) that for every \( n \geq n_1 \), \( R_{2h,B}(n) \geq C_h n^{\frac{1}{m+1}} \). Then by (4), we have

\[
R_{2h,B\backslash C}(n) \geq C_h n^{\frac{1}{m+1}} - \left( C^{(1)}(B) + C^{(2)}(B) \cdot \frac{n^{\frac{1}{m+1}}}{\log n} + C^{(4)}(B) \cdot C^{(5)}(B) \right),
\]

which gives that with probability 1, \( R_{2h,A}(n) \geq 1 \) for every large enough \( n \).

Throughout the remaining part of the paper, we prove lemmas 6-8.

### 4.2 Proof of Lemma 6

Let \( D \) be an arbitrary set of positive integers. Let

\[
R_{f_1,\ldots,f_t,D}(m) = |\{(d_1, \ldots, d_t) : d_i \in D, d_1, \ldots, d_t \text{ are distinct, } f_1d_1 + \cdots + f_td_t = m\}|
\]

Let \( R_{f_1,\ldots,f_t}^{\text{Dist}}(m) \) denote the largest nonnegative integer \( q \) such that there exist \( d_1, \ldots, d_q \in D \) distinct integers such that \( f_1d_{j_1} + \cdots + f_td_{j_t} = m \) for every \( 0 \leq j \leq q - 1 \). Let \( E_{f_1,\ldots,f_t} \) denote the event that \( R_{f_1,\ldots,f_t,D}(m) \) is not bounded. Let \( E_{f_1,\ldots,f_t}^{\text{Dist}} \) denote the event that \( R_{f_1,\ldots,f_t}^{\text{Dist}}(m) \) is not bounded. Let

\[
E = \bigcup_{(f_1,\ldots,f_t) \in (\mathbb{Z}^+)^n, f_1+\cdots+f_t \leq 2h} E_{f_1,\ldots,f_t}, \quad E^{\text{Dist}} = \bigcup_{(f_1,\ldots,f_t) \in (\mathbb{Z}^+)^n, f_1+\cdots+f_t \leq 2h} E_{f_1,\ldots,f_t}^{\text{Dist}}.
\]

Now we prove that \( E = E^{\text{Dist}} \). Obviously, \( E^{\text{Dist}} \subseteq E \). Thus we show that if \( D \) is an arbitrary subset of the set of positive integers such that \( D \subseteq E \), then \( D \in E^{\text{Dist}} \) as well. We prove that if \( 2 \leq t \leq 2h - 1 \), \( u \in \mathbb{Z}^+ \), then there exists a positive integer \( g(t,u) \) such that if there is a positive integer \( m \) with \( R_{f_1,\ldots,f_t,D}(m) \geq g(t,u) \), then there exist positive integers \( f_1, \ldots, f_t, 2 \leq t' \leq 2h - 1 \) with \( f_1 + \cdots + f_t' \leq 2h \) and a positive integer \( m' \) such that \( R_{f_1,\ldots,f_t}^{\text{Dist}}(m') \geq u \). We prove by induction on \( t \). Assume that \( t = 2 \). We show that \( g(2,u) = 3u - 1 \) is suitable. Suppose that \( R_{f_1,\ldots,f_t,D}(m) \geq 3u - 1 \). There are at most one representation of the form \( f_1d_1 + f_2d_2 = m \), such that \( d_1 \neq d_2 \in D \). It follows that there exist \( d_1^{(1)}, d_2^{(1)}, d_1^{(2)}, d_2^{(2)}, \ldots, d_1^{(3u-2)}, d_2^{(3u-2)} \in D \) positive integers such that \( d_1^{(i)} \neq d_2^{(i)} \) for every \( 1 \leq i < j \leq 3u - 2 \). Then it is easy to see that in the representation \( f_1d_1^{(i)} + f_2d_2^{(i)} = m \) the integers \( d_1^{(i)}, d_2^{(i)} \) appears at most two other representations. In particular, \( f_1d_1^{(i)} + f_2d_2^{(i)} = m \) and \( f_1d_1^{(i)} + f_2d_2^{(i)} = m \). It follows that there are at least \( u \) representations among the \( 3u - 2 \) representations which contain \( 2u \) distinct integers from \( D \).

Assume that the above statement holds for \( t - 1 \geq 1 \). Now we prove it for \( t \). We show that \( g(t,u) = t^2 u g(t-1,u) \) is suitable. Suppose that \( R_{f_1,\ldots,f_t,D}(m) \geq t^2 u g(t-1,u) \). It follows that there exist \( d_1^{(1)}, \ldots, d_1^{(t)}, d_2^{(1)}, \ldots, d_2^{(t)}, \ldots, d_1^{(v)}, \ldots, d_t^{(v)} \in D \) positive integers such that \( d_1^{(i)}, \ldots, d_t^{(i)} \) are distinct for \( 1 \leq i \leq v \) and \( f_1d_1^{(i)} + \cdots + f_td_t^{(i)} = m \) for every \( 1 \leq i \leq v \). If there exist \( d \in D \) which appears at least \( tg(t-1,u) \), then there exists a \( 1 \leq j \leq t \) such that \( d_j^{(i)} = d \) holds for at least \( g(t-1,u) \) indices \( 1 \leq i \leq v \). This implies that \( R_{f_1,\ldots,f_t,\ldots,f_t,D}(m - f_jd) \geq g(t-1,u) \). It follows from the induction hypothesis that there exist positive integers \( f_1, \ldots, f_{t'}, 2 \leq t' \leq 2h - 1 \) with \( f_1 + \cdots + f_{t'} \leq 2h \) and a positive integer \( m' \) such that \( R_{f_1,\ldots,f_{t'},D}(m') \geq u \).
If every $d \in D$ appears less than $tg(t-1, u)$ representations, then the elements from $D$ which appears in the first sum can appear in at most $t^2g(t-1, u)$ sums. Thus among the $v$ representations there is another one which contains distinct terms from the previous representations.

The elements which appear in these two representations can be at most $2t^2g(t-1, u)$ representations. Continuing this process, one can get $u$ representations such that any two representations contains distinct terms. Since every representation contains summands from $D$, we have $R_{f_1, \ldots, f_t, D}^{\text{Dist}}(m) \geq u$.

Now we show that if $D \in E$, then $D \in E^{\text{Dist}}$. If $D \in E$, then there exist $f_1, \ldots, f_t$, $2 \leq t \leq 2h - 1$ positive integers with $f_1 + \ldots + f_t \leq 2h$ such that $D \in E_{f_1, \ldots, f_t}$. It follows that for every positive integer $u$, there exists a positive integer $m$ such that $R_{f_1, \ldots, f_t, D}(m) \geq g(t, u)$. Thus for every positive integer $u$, there exist positive integers $f'_1, \ldots, f'_t$, $2 \leq t' \leq 2h - 1$ with $f'_1 + \ldots + f'_t \leq 2h$ and a positive integer $m'$ such that $R_{f'_1, \ldots, f'_t, D}^{\text{Dist}}(m') \geq u$. Since there are only finitely many possibilities for $(f'_1, \ldots, f'_t)$, there exist positive integers $f'_1, \ldots, f'_t$, $2 \leq t' \leq 2h - 1$ with $f'_1 + \ldots + f'_t \leq 2h$ such that for infinitely many positive integer $u$, there exist a positive integer $m'_u$ such that $R_{f'_1, \ldots, f'_t, D}^{\text{Dist}}(m'_u) \geq u$. Thus we have $D \in E_{f'_1, \ldots, f'_t}^{\text{Dist}}$ and then $D \in E^{\text{Dist}}$.

To prove Lemma 6 we need to show that $P(B \in E_{f_1, \ldots, f_t}) = 1$, i.e., $P(B \in E_{f_1, \ldots, f_t}) = 0$. We prove that $P(B \in E_{f_1, \ldots, f_t}) = 0$. Let

$$p_n = P(R_{f_1, \ldots, f_t, D}^{\text{Dist}}(n) \geq 4h).$$

By $t \leq 2h - 1$ and (iii) from Lemma 4,

$$p_n \leq \left( \sum_{(x_1, \ldots, x_t) \in (\mathbb{Z}^+)^t} \frac{1}{(x_1 \cdot \ldots \cdot x_t)^{4h - 1}} \right)^{4h} \leq \left( \sum_{(x_1, \ldots, x_t) \in (\mathbb{Z}^+)^t} \frac{1}{((f_1 x_1) \cdot \ldots \cdot (f_t x_t))^{4h - 1}} \right)^{4h} \leq \left( \sum_{(z_1, \ldots, z_t) \in (\mathbb{Z}^+)^t} \frac{1}{(z_1 \cdot \ldots \cdot z_t)^{4h - 1}} \right)^{4h} \leq \left( \frac{1}{n^{1 - \frac{4h}{4h - 1}}} \right)^{4h} \leq \frac{1}{n^{4h - 1}}.$$ 

Since $\frac{4h}{4h - 1} > 1$, then by the Borel-Cantelli lemma we get that $R_{f_1, \ldots, f_t, D}(n) \leq 4h$ for every large enough $n$, with probability 1. Thus we have $P(B \in E_{f_1, \ldots, f_t}^{\text{Dist}}) = 0$ and so $P(B \in E^{\text{Dist}}) = 0$. Since $E = E^{\text{Dist}}$, we get that $P(B \in E) = 0$ and so $P(B \in E_{f_1, \ldots, f_t}) = 0$ i.e., $P(B \in E_{f_1, \ldots, f_t}) = 1$. The proof of Lemma 6 is completed.
4.3 Proof of Lemma 8

Let \(d_1, \ldots, d_k, e_1, \ldots, e_l, k+l \leq 2h-1\) be positive integers with \(d_1 + \ldots + d_k = e_1 + \ldots + e_l \leq h\). Let \(n\) be a positive integer. Let

\[ q_n = P(\exists b_1, \ldots, b_{k+l} \in B \text{ distinct }, d_1 b_1 + \ldots + d_k b_k = e_1 b_{k+1} + \ldots + e_l b_{k+l} = n). \]

By (iii) in Lemma 4,

\[ q_n \leq \sum_{(x_1, \ldots, x_{k+l}) \in (\mathbb{Z}^+)^{k+l}} \frac{1}{(x_1 \cdots x_{k+l})^{\frac{2h-3}{4h-4}}} \]

\[ \leq \sum_{(d_1 x_1) \in (\mathbb{Z}^+)^k} \frac{1}{(d_1 x_1 \cdots d_k x_k)^{\frac{2h-3}{4h-4}}} \cdot \sum_{(e_1 x_{k+1}) \in (\mathbb{Z}^+)^l} \frac{1}{(e_1 x_{k+1} \cdots e_l x_{k+l})^{\frac{2h-3}{4h-4}}} \]

\[ \leq \sum_{(z_1, \ldots, z_k) \in (\mathbb{Z}^+)^k} \frac{1}{(z_1 \cdots z_k)^{\frac{2h-3}{4h-4}}} \cdot \sum_{(z_{k+1}, \ldots, z_{k+l}) \in (\mathbb{Z}^+)^l} \frac{1}{(z_{k+1} \cdots z_{k+l})^{\frac{2h-3}{4h-4}}} \]

\[ \leq \frac{1}{n^{1 - \frac{2l}{4h-4}}} \cdot \frac{1}{n^{1 - \frac{2l}{4h-4}}} = \frac{1}{n^{2 - \frac{2(k+l)}{4h-4}}}. \]

Since \(k+l \leq 2h-1\), then \(2 - \frac{2(k+l)}{4h-4} > 1\). It follows that the infinite series \(\sum_{n=1}^{\infty} q_n\) is convergent, thus by the Borel-Cantelli lemma almost surely there exist only finitely many positive integer \(n\) for which there exist distinct positive integers \(b_1, \ldots, b_{k+l} \in B\) such that \(d_1 b_1 + \ldots + d_k b_k = e_1 b_{k+1} + \ldots + e_l b_{k+l}\). The proof of Lemma 8 is completed.

4.4 Proof of Lemma 7

Let \(\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)\), \(\alpha_i \in \mathbb{N}, \alpha_i \geq 0\). We will prove Lemma 7 by using Lemma 2. Recall

\[ t_n = \begin{cases} 1, & \text{if } n \in A \\ 0, & \text{if } n \notin A \end{cases} \]
The random variable $Y_{u,v,h,B}(n)$ is clearly

$$
Y_{u,v,h,B}(n) = \sum_{(x_1,\ldots,x_u) \in (\mathbb{Z}^+)^u} t_{x_1} \cdots t_{x_u} \sum_{(x_{u+1},\ldots,x_v) \in (\mathbb{Z}^+)^{v-u}} t_{x_{u+1}} \cdots t_{x_v}
$$

$$
\sum_{(x_{v+1},\ldots,x_{2h}) \in (\mathbb{Z}^+)^{2h-v}} t_{x_{v+1}} \cdots t_{x_{2h}} \sum_{(x_{2h+1},\ldots,x_{3h}) \in (\mathbb{Z}^+)^{h-u}} t_{x_{2h+1}} \cdots t_{x_{3h-u}}
$$

$$
\sum_{(x_{3h-u+1},\ldots,x_{4h}) \in (\mathbb{Z}^+)^{h-v}} t_{x_{3h-u+1}} \cdots t_{x_{4h-v}}
$$

Let us denote $\partial^{(\pi)} Y_{u,v,h,B}(n) = Y_{u,v,h,B}(n)^{\pi}$. We show that

$$
\mathbb{E}(Y_{u,v,h,B}(n)^{\pi}) = O \left( \frac{n^{\frac{1}{4h-1}}}{(\log n)^{4h}} \right), \quad (5)
$$

where the constant depends only on $h$. It follows from (5) that

$$
\mathbb{E}(Y_{u,v,h,B}(n)) = O \left( \frac{n^{\frac{1}{4h-1}}}{(\log n)^{4h}} \right),
$$

$$
\mathbb{E}_{\geq 0}(Y_{u,v,h,B}(n)) = O \left( \frac{n^{\frac{1}{4h-1}}}{(\log n)^{4h}} \right),
$$

$$
\mathbb{E}_{\geq 1}(Y_{u,v,h,B}(n)) = O \left( \frac{n^{\frac{1}{4h-1}}}{(\log n)^{4h}} \right).
$$

Applying Lemma 2 with $k = 4h - v \leq 4h - 1$ and $\lambda = 16h \log n$, we get that

$$
P \left( |Y_{u,v,h,B}(n) - \mathbb{E}(Y_{u,v,h,B}(n))| \geq C_{4h-v}(16h \log n)^{4h-v-\frac{1}{2}} \sqrt{\mathbb{E}_{\geq 0}(Y_{u,v,h,B}(n)) \cdot \mathbb{E}_{\geq 1}(Y_{u,v,h,B}(n))} \right)
$$

$$
= O \left( e^{-16h \log n - (4h-2) \log n} \right) = O \left( \frac{1}{n^2} \right).
$$

Then by the Borel-Cantelli lemma, we get that with probability 1,

$$
|Y_{u,v,h,B}(n) - \mathbb{E}(Y_{u,v,h,B}(n))| < C_{4h-v}(16h \log n)^{4h-v-\frac{1}{2}} \sqrt{\mathbb{E}_{\geq 0}(Y_{u,v,h,B}(n)) \cdot \mathbb{E}_{\geq 1}(Y_{u,v,h,B}(n))}
$$

holds for every $n$ large enough. Then with probability 1,

$$
Y_{u,v,h,B}(n) = O \left( (16h \log n)^{4h-v-\frac{1}{2}} \cdot \frac{n^{\frac{1}{4h-1}}}{(\log n)^{4h}} \right) = O \left( \frac{n^{\frac{1}{2h-1}}}{(\log n)^h} \right).
$$
Thus it is enough to prove (5). Obviously,

\[
Y_{u,v,h,B}^{(\alpha)}(n) = \frac{\partial}{\partial \alpha} \left( \sum_{(x_1, \ldots, x_u) \in (\mathbb{Z}^+)^u} \sum_{x_i < x_j, \text{ if } 2 \leq i \leq u} t_{x_1} \cdots t_{x_u} \right) \left( \sum_{(x_{u+1}, \ldots, x_v) \in (\mathbb{Z}^+)^{v-u}} \sum_{x_i < x_j, \text{ if } u+1 \leq i \leq v} t_{x_{u+1}} \cdots t_{x_v} \right)
\]

\[
= \sum_{(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, \bar{\beta}, \bar{\sigma}, \bar{\mu}, \bar{\nu})} \left( \sum_{(x_1, \ldots, x_u) \in (\mathbb{Z}^+)^u} \sum_{x_i < x_j, \text{ if } 2 \leq i \leq u} \left( \frac{\partial}{\partial \alpha} \right) t_{x_1} \cdots t_{x_u} \right) \left( \sum_{(x_{u+1}, \ldots, x_v) \in (\mathbb{Z}^+)^{v-u}} \sum_{x_i < x_j, \text{ if } u+1 \leq i \leq v} \left( \frac{\partial}{\partial \beta} \right) t_{x_{u+1}} \cdots t_{x_v} \right)
\]

\[
= \sum_{(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, \bar{\beta}, \bar{\sigma}, \bar{\mu}, \bar{\nu})} \left( \sum_{(x_1, \ldots, x_{2h}) \in (\mathbb{Z}^+)^{2h-v}} \sum_{x_i + \cdots + x_{2h} = n} \left( \frac{\partial}{\partial \alpha} \right) t_{x_1} \cdots t_{x_{2h}} \right) \left( \sum_{(x_{2h+1}, \ldots, x_{3h-u}) \in (\mathbb{Z}^+)^{h-u}} \sum_{x_i < x_j, \text{ if } 2h+1 \leq i \leq 3h-u} \left( \frac{\partial}{\partial \beta} \right) t_{x_{2h+1}} \cdots t_{x_{3h-u}} \right)
\]

\[
= \sum_{(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, \bar{\beta}, \bar{\sigma}, \bar{\mu}, \bar{\nu})} Y_{u,v,h,B}^{(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{\alpha}, \bar{\beta}, \bar{\sigma}, \bar{\mu}, \bar{\nu})}(n).
\]
It is clear that there are at most $5^{5h-v}$ possibilities for the 5-tuples $(\beta, \gamma, \delta, \mu, \nu)$. It is enough to prove that for a fixed 5-tuples $(\beta, \gamma, \delta, \mu, \nu)$,

$$
\mathbb{E}(Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n)) = O\left(\frac{n^{\frac{3h}{h-v}}}{(\log n)^{4h}}\right).
$$

Assume that the nonzero $\beta_1, \gamma_1, \delta_1, \mu_1, \nu_1$'s have the indices $i_1, \ldots, i_{|\beta|}$ and $i_{|\beta|+1}, \ldots, i_{|\beta|+|\gamma|}$ and $i_{|\beta|+|\gamma|+1}, \ldots, i_{|\beta|+|\gamma|+|\delta|}$ and $i_{|\beta|+|\gamma|+|\delta|+1}, \ldots, i_{|\beta|+|\gamma|+|\delta|+|\mu|}$ and $i_{|\beta|+|\gamma|+|\delta|+|\mu|+1}, \ldots, i_{|\beta|+|\gamma|+|\delta|+|\mu|+|\nu|}$ respectively. If $i_1 + \ldots + i_{|\beta|+|\gamma|+|\delta|} > n$, then $Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n) = 0$. If $i_1 + \ldots + i_{|\beta|+|\gamma|+|\delta|} = n$, then we can assume that $|\beta| = u, |\gamma| = v - u$ and $|\delta| = 2h - v$, otherwise $Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n) = 0$. Then we have

$$
Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n) \leq \left|\{(w_1, \ldots, w_{2h-v}) : w_i \in B, 1 \leq w_i \leq n, w_1, \ldots, w_{2h-v} \text{ are distinct}, w_1 + \ldots + w_{h-u} - (w_{h-u+1} + \ldots + w_{2h-v}) = i_{u+1} + \ldots + i_v - (i_1 + \ldots + i_u) = M\}\right|.
$$

It follows from (iv) in Lemma 4 that

$$
\mathbb{E}(Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n)) \leq \sum_{\substack{(w_1, \ldots, w_{2h-v}) \in (\mathbb{Z}^{2h-v}) \setminus (w_1 + \ldots + w_{h-u} - (w_{h-u+1} + \ldots + w_{2h-v}) = i_{u+1} + \ldots + i_v - (i_1 + \ldots + i_u) = M}} \frac{1}{(w_1 \cdots w_{2h-v})^{\frac{2h-v}{h-v}}} \leq \frac{1}{(|M| + 1)^{1 - \frac{2(h-v)}{4h-v}}} = O\left(\frac{n^{\frac{3h}{h-v}}}{(\log n)^{4h}}\right).
$$

Now, we assume that $i_1 + \ldots + i_{|\beta|+|\gamma|+|\delta|} < n$. We denote the following conditions by cond$_3$ and cond$_4$, respectively:

$$
\text{cond}_3 : \quad z_1 + \ldots + z_{2h-|\beta|-|\gamma|-|\delta|} = n - (i_1 + \ldots + i_{|\beta|+|\gamma|+|\delta|}),
$$

$$
\text{cond}_4 : \quad z_1 + \ldots + z_{u-|\beta|} + z_{2h-|\beta|-|\gamma|-|\delta|+1} + \ldots + z_{3h-u-|\beta|-|\gamma|-|\delta|} - (z_{u-|\beta|+1} + \ldots + z_{v-|\gamma|-|\delta|} + z_{2h-u-|\beta|-|\gamma|-|\delta|+1} + \ldots + z_{4h-v-|\beta|-|\gamma|-|\delta|}) = i_{|\beta|+1} + \ldots + i_{|\gamma|+1} + i_{|\gamma|+|\delta|+1} + \ldots + i_{|\gamma|+|\delta|+|\mu|+1} + i_{|\gamma|+|\delta|+|\mu|+|\nu|} - (i_1 + \ldots + i_{|\beta|+|\gamma|+|\delta|} + i_{|\beta|+|\gamma|+|\delta|+1} + \ldots + i_{|\beta|+|\gamma|+|\delta|+|\mu|+|\nu|}) = M.
$$

Then we have

$$
Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n) = \left|\{(z_1, \ldots, z_{4h-v-|\beta|-|\gamma|-|\delta|}) : z_i \in B, z_j \neq i_t \text{ for } 1 \leq j \leq 4h-v-|\beta|-|\gamma|-|\delta|, 1 \leq t \leq |\beta| + |\gamma| + |\delta| + |\mu| + |\nu|, \text{ cond}_3 \text{ and cond}_4 \text{ hold}\}\right|.
$$

It follows that

$$
\mathbb{E}(Y_{u,v,h,B}^{(\beta, \gamma, \delta, \mu, \nu)}(n)) \leq \frac{n^{\frac{3h}{h-v}}}{(\log n)^{4h}}.
$$
\[
\sum_{\text{cond}_3 \text{ holds}} \left( \frac{1}{(z_1 \cdots z_{2h-|\beta|-|\gamma|-|\delta|})^{\frac{4h-3}{4h-1}}} \right)
\cdot \frac{1}{(\sum_{\text{cond}_4 \text{ holds}}) \left( z_{2h-|\beta|-|\gamma|-|\delta|} \right)^{\frac{4h-3}{4h-1}}}
\]

By (iv) of Lemma 4, the inner sum

\[
\sum_{\text{cond}_4 \text{ holds}} \frac{1}{(z_{2h-|\beta|-|\gamma|-|\delta|})^{\frac{4h-3}{4h-1}}}
\]

\[\leq \frac{1}{(1 + |M|)^{1 - \frac{2(2h-v+1)}{4h-1}}} \leq \frac{1}{(1 + |M|)^{1 - \frac{2h-1}{4h-1}}} = O(1).\]

On the other hand, by (iii) of Lemma 4,

\[
\sum_{\text{cond}_3 \text{ holds}} \frac{1}{(z_1 \cdots z_{2h-|\beta|-|\gamma|-|\delta|})^{\frac{4h-3}{4h-1}}}
\]

\[= \frac{1}{(n - (i_1 + \ldots + i_{|\beta|+|\gamma|+|\delta|}))^{\frac{2(|\beta|+|\gamma|+|\delta|)-1}{4h-1}}}.\]

Thus if \(|\beta| + |\gamma| + |\delta| \geq 1\), then \(E(Y_{u,v,h,B}^\alpha(n)) = O(1)\). Now we can assume that \(|\beta| = |\gamma| = |\delta| = 0\), i.e., \(\beta = \gamma = \delta = 0\). Then the equation in \text{cond}_4 can be written in the form

\[
z_1 + \ldots + z_u + z_{2h+1} + \ldots + z_{3h-u-|\beta|} - (z_{u+1} + \ldots + z_v + z_{3h-u-|\beta|+1} + \ldots + z_{4h-v-|\beta|-|\gamma|})
\]

\[= i_{|\beta|+1} + \ldots + i_{|\gamma|+|\delta|} - (i_1 + \ldots + i_{|\beta|}),\]

that is, \text{cond}_4 is equivalent to

\[
\text{cond}_5 : \quad z_{2h+1} + \ldots + z_{3h-u-|\beta|} - (z_{h-u-|\beta|+1} + \ldots + z_{4h-v-|\beta|-|\gamma|}) = z_{u+1} + \ldots + z_v - (z_1 + \ldots + z_u)
\]

\[+ i_{|\beta|+1} + \ldots + i_{|\gamma|+|\delta|} - (i_1 + \ldots + i_{|\beta|}).\]

If \(K = i_1 + \ldots + i_{|\beta|} - (i_{|\beta|+1} + \ldots + i_{|\gamma|+|\delta|})\), then, by replacing \text{cond}_4 to \text{cond}_5 and by (iv) of Lemma 4, the inner sum in (6) can be estimated in the following way.

\[
\sum_{\text{cond}_5 \text{ holds}} \frac{1}{(z_{2h+1} \cdots z_{4h-v-|\beta|-|\gamma|})^{\frac{4h-3}{4h-1}}}
\]

\[\leq \frac{1}{(|z_1 + \ldots + z_u - (z_{u+1} + \ldots + z_v) + K| + 1)^{\frac{4h-3}{4h-1}}},\]
and then
\[
\mathbb{E}(Y_{u,v,h,B}(n)) \ll 
\]
\[
\sum_{(z_1,\ldots,z_{2h}) \in (\mathbb{Z}^+)_{2h \leq n}} \frac{1}{(z_1 \cdots z_{2h})^{\frac{2h-1}{2h+1}}} \cdot \frac{1}{(|z_1 + \ldots + z_u - (z_{u+1} + \ldots + z_v) + K| + 1)^{\frac{2h-1}{2h+1}}}.
\]

We will prove that
\[
\sum_{(z_1,\ldots,z_{2h}) \in (\mathbb{Z}^+)_{2h \leq n}} \frac{1}{(z_1 \cdots z_{2h})^{\frac{2h-1}{2h+1}}} \cdot \frac{1}{(|z_1 + \ldots + z_u - (z_{u+1} + \ldots + z_v) + K| + 1)^{\frac{2h-1}{2h+1}}} = O\left(\frac{n^{-\frac{1}{4h-1}}}{(\log n)^{4h}}\right).
\]

We have three cases.

**Case 1.** \(v = 2h\). Then \(u = h\) and \(\overline{u} = \overline{v} = 0\), and so
\[
Y_{u,v,h,B}(n) = |\{(z_1,\ldots,z_{2h}) : z_i \in B, z_1 + \ldots + z_{2h} = n, z_1 + \ldots + z_h = z_{h+1} + \ldots + z_{2h}, z_i'\text{ are distinct, } z_i < z_1, \text{ if } 2 \leq i \leq 2h\}|.
\]

It follows from (iii) of Lemma 4 that
\[
\mathbb{E}(Y_{u,v,h,B}(n)) \ll \left(\sum_{(z_1,\ldots,z_h) \in (\mathbb{Z}^+)_h} \frac{1}{(z_1 \cdots z_h)^{\frac{h-1}{h+1}}} \right)^2 \ll \left(\frac{1}{n^{1-\frac{2h}{2h+1}}}\right)^2 = O(1).
\]

**Case 2.** \(u = v\). Then \(u = v \leq h\) and so
\[
\mathbb{E}(Y_{u,v,h,B}(n)) \ll \]
\[
\sum_{(z_1,\ldots,z_{2h}) \in (\mathbb{Z}^+)_{2h \leq n}} \frac{1}{(z_1 \cdots z_{2h})^{\frac{2h-1}{2h+1}}} \cdot \frac{1}{(|z_1 + \ldots + z_u - (z_{u+1} + \ldots + z_v) + K| + 1)^{\frac{2h-1}{2h+1}}}.
\]

\[
= \sum_{m=1}^{n-1} \left( \sum_{(z_1,\ldots,z_u) \in (\mathbb{Z}^+)_u} \left( \frac{1}{(z_1 \cdots z_u)^{\frac{h-1}{h+1}}} \cdot \frac{1}{(|z_1 + \ldots + z_u + K| + 1)^{\frac{h-1}{h+1}}} \right) \right) \cdot \left( \sum_{(z_{u+1},\ldots,z_{2h}) \in (\mathbb{Z}^+)_h} \frac{1}{(z_{u+1} \cdots z_{2h})^{\frac{h-1}{h+1}}} \right).
\]
It follows from (iii) of Lemma 4 that

\[
\mathbb{E}(Y_{u,v,h,B}^{(\gamma,\gamma,\gamma,\beta)}) (n) \ll \sum_{m=1}^{n-1} \left( \sum_{(z_1, \ldots, z_u) \in \mathbb{Z}^u} \frac{1}{(z_1 \cdots z_u)^{2n-1}} \right) \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
\ll \sum_{m=1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}.
\]

We have three subcases.

**Subcase 2.1.** $K \geq 0$. It follows from (i) of Lemma 4 that

\[
\mathbb{E}(Y_{u,v,h,B}^{(\gamma,\gamma,\gamma,\beta)}) (n) \ll \sum_{m=1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
= \sum_{m=1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
\ll \frac{1}{m^{2u-1}} = O(1).
\]

**Subcase 2.2.** $-\frac{n}{3} < K < 0$. Then we have

\[
\mathbb{E}(Y_{u,v,h,B}^{(\gamma,\gamma,\gamma,\beta)}) (n) \ll \sum_{m=1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
+ \sum_{m=-[K/2]+1}^{-K} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
+ \sum_{m=-K+1}^{-2K+1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
+ \sum_{m=-2K+1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
\ll \frac{1}{|K|^{1-2u-1}} \cdot \frac{1}{n^{2u-1}} \sum_{m=1}^{n-1} \frac{1}{m^{1-2u-1}} + \frac{1}{|K|^{1-2u-1}} \cdot \frac{1}{n^{2u-1}} \sum_{m=-[K/2]+1}^{-K} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}}
\]

\[
+ \frac{1}{|K|^{1-2u-1}} \cdot \frac{1}{n^{2u-1}} \sum_{m=-K+1}^{-2K+1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} + \sum_{m=-2K+1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(|m + K| + 1)^{2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]

\[
\ll \frac{1}{|K|^{1-2u-1}} \cdot \frac{1}{n^{2u-1}} \int_0^{|K|} \frac{dx}{x^{1-2u-1}} + \frac{1}{|K|^{1-2u-1}} \cdot \frac{1}{n^{2u-1}} \int_0^{|K|} \frac{dx}{x^{2u-1}}
\]

\[
+ \frac{1}{|K|^{1-2u-1}} \cdot \frac{1}{n^{2u-1}} \int_0^{|K|} \frac{dx}{x^{2u-1}} + \sum_{m=1}^{n-1} \frac{1}{m^{1-2u-1}} \cdot \frac{1}{(n-m)^{2u-1}}
\]
\[
\frac{[K]^{2\mu}}{2n^{4k-1}} + \frac{[K]^{1-2\nu} - [K]^{1-2\mu}}{2n^{4k-1} \cdot n^{2\mu-1}} + \frac{[K]^{1-2\nu}}{2n^{4k-1} \cdot n^{2\nu-1}} + \frac{1}{n^{2\nu-2} n^{1-2\mu}} = O(1).
\]

**Subcase 2.3.** \(K < -\frac{n}{3}\) Then we have
\[
\mathbb{E}(Y^{(7,7,5,5,6,6,6)}_{u,v,B})(n) \ll \sum_{m=1}^{-\lfloor K/2 \rfloor} \frac{1}{m^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{n^{2\nu-1}} \cdot \frac{1}{(n - m)^{2\nu-1}} + \sum_{m=-\lfloor K/2 \rfloor + 1}^{n-1} \frac{1}{m^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{n^{2\nu-1} \cdot (n - m)^{2\nu-1}} = O(1).
\]

If \(1 \leq m < -K/2\), then \(|m + K| \geq |K/2| \geq \frac{n}{6}\). It follows from (i) of Lemma 4 that
\[
\sum_{m=1}^{-\lfloor K/2 \rfloor} \frac{1}{m^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{n^{2\nu-1}} \cdot \frac{1}{(n - m)^{2\nu-1}} \ll \frac{1}{n^{2\nu-1}} \cdot \sum_{m=1}^{-\lfloor K/2 \rfloor} \frac{1}{m^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{n^{2\nu-1} \cdot (n - m)^{2\nu-1}} = O(1).
\]

On the other hand, by \(-K/2 \geq \frac{n}{6}\),
\[
\sum_{m=-\lfloor K/2 \rfloor + 1}^{n-1} \frac{1}{m^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{n^{2\nu-1} \cdot (n - m)^{2\nu-1}} \ll \frac{1}{n^{2\nu-1}} \cdot \sum_{m=-\lfloor K/2 \rfloor}^{n-1} \frac{1}{m^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{n^{2\nu-1} \cdot (n - m)^{2\nu-1}}.
\]

We know that \(|K| \leq nh\), thus in the above sum, \(1 \leq |m + K| + 1 \leq (1 + h)n\) and \(1 \leq n - m \leq n - 1\). Thus a positive integer among the integers \(n - m\) and \(|m + K| + 1\) appear, as the non-largest, at most three times. Then,
\[
\frac{1}{n^{1-2\nu} \cdot (|m + K| + 1)^{2\nu}} \cdot \frac{1}{(n - m)^{2\nu-1}} \cdot \frac{1}{k^{2\nu-1}} \cdot \sum_{k=1}^{(h+1)n} \frac{1}{k^{1-2\mu}} \cdot \frac{1}{k^{2\mu-1}} \ll \frac{1}{n^{1-2\nu} \cdot \int_0^{(h+1)n} \frac{dx}{x^{4\nu-1}} \cdot \frac{n^{1-4\nu-2}}{n^{1-2\mu}} = O(1).
\]

**Case 3.** \(0 < u < v < 2h\). Then,
\[
\mathbb{E}(Y^{(7,7,5,5,6,6,6)}_{u,v,B})(n) \ll
\]
It follows from (iv) of Lemma 4 that

\[ \mathbb{E}(\mathcal{Y}_{u,v,h,B}(n)) \ll \]

\[ \ll \sum_{(m_1,m_2) \in (\mathbb{Z}^+)^2, m_1+m_2<n} \left( \sum_{(z_1,\ldots,z_u) \in (\mathbb{Z}^+)^u, z_1+\ldots+z_u=m_1} \left( \sum_{(z_u+1,\ldots,z_v) \in (\mathbb{Z}^+)^v-u, z_u+1+\ldots+z_v=m_2} \left( \frac{1}{(z_1 \cdots z_u) \frac{4u-2}{4u-1}} \cdot \frac{1}{(z_u+1 \cdots z_v) \frac{4u-v}{4u-1}} \cdot \frac{1}{(n-m_1-m_2) \frac{2u-1}{4u-1}} \right) \right) \right) \]

\[ \ll \sum_{(m_1,m_2) \in (\mathbb{Z}^+)^2, m_1+m_2<n} \left( \sum_{(z_1,\ldots,z_u) \in (\mathbb{Z}^+)^u, z_1+\ldots+z_u=m_1} \left( \frac{1}{(z_1 \cdots z_u) \frac{4u-2}{4u-1}} \cdot \frac{1}{m_2} \cdot \frac{1}{n-m_1-m_2} \right) \right) \]

\[ = \sum_{(m_1,m_2) \in (\mathbb{Z}^+)^2, m_1+m_2<n} \frac{1}{m_1} \cdot \frac{1}{m_2} \cdot \frac{1}{(n-m_1-m_2) \frac{2u-1}{4u-1}} \]

Now, it is enough to prove that this sum is \( O\left( \frac{1}{(\log n)^{15}} \right) \). It follows from (i) of Lemma 4 that

\[ \sum_{(m_1,m_2) \in (\mathbb{Z}^+)^2, m_1+m_2<n} \frac{1}{m_1} \cdot \frac{1}{m_2} \cdot \frac{1}{(n-m_1-m_2) \frac{2u-1}{4u-1}} \]

\[ \leq \sum_{m_1=1}^{n-1} \frac{1}{m_1} \cdot \sum_{m_2=1}^{n-m_1-1} \frac{1}{m_2} \cdot \frac{1}{(n-m_1-m_2) \frac{2u-1}{4u-1}} \]

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We have three subcases.

Subcase 3.1 \(|K| < 10(\log n)^{64h^3}\). Then

\[
\sum_{\substack{(m_1, m_2) \in \mathbb{Z}^+^2 \\ m_1 + m_2 < n \\ |m_1 - m_2 + K| \leq (\log n)^{16h^2}}} \frac{1}{m_1^{1 - \frac{2n}{4h-1}}} \cdot \frac{1}{m_2^{1 - \frac{2n}{4h-1}}} \cdot \frac{1}{(|m_1 - m_2 + K| + 1)^{\frac{2n}{4h-1}}} \cdot \frac{1}{n - m_1 - m_2^{\frac{2n}{4h-1}}} \leq \frac{n^{\frac{1}{4h}}}{(\log n)^{4h}}
\]

Thus it is enough to show that

\[
S = \sum_{\substack{(m_1, m_2) \in \mathbb{Z}^+^2 \\ m_1 + m_2 < n \\ |m_1 - m_2 + K| \leq (\log n)^{16h^2}}} \frac{1}{m_1^{1 - \frac{2n}{4h-1}}} \cdot \frac{1}{m_2^{1 - \frac{2n}{4h-1}}} \cdot \frac{1}{(|m_1 - m_2 + K| + 1)^{\frac{2n}{4h-1}}} \cdot \frac{1}{n - m_1 - m_2^{\frac{2n}{4h-1}}}
\]

\[
= S_1 + S_2.
\]

In \(S_1\), there are \(O((\log n)^{64h^3})\) possibilities for \(m_1\), and for a fixed \(m_1\), there are \(O((\log n)^{16h^2})\) possibilities for \(m_2\). Since every term in \(S_1\) is bounded, thus

\[
S_1 = O((\log n)^{64h^3 + 16h^2}) = O \left( \frac{n^{\frac{1}{4h}}}{(\log n)^{4h}} \right).
\]
If $m_1 \geq 100(\log n)^{64h^3}$, then $|K| \leq 10(\log n)^{64h^3}$ and $|m_1 - m_2 + K| \leq (\log n)^{16h^2}$, which implies that $m_1 \asymp m_2$, and so

$$S_2 \ll \sum_{m_1=1}^{n-1} \frac{1}{m_1} \cdot \frac{1}{m_1^{2(\log n-1)}} \cdot (\log n)^{16h^2}$$

because for a fixed $m_1$, there are $O((\log n)^{16h^2})$ possibilities for $m_2$ and

$$\frac{1}{(m_1 - m_2 + K) + 1} \cdot \frac{1}{(n - m_1 - m_2)^{2(\log n-1)}} = O(1).$$

Then

$$S_2 \ll \sum_{m_1=1}^{n-1} \frac{1}{m_1} \cdot (\log n)^{16h^2} \ll (\log n)^{16h^2} = O \left( \frac{n^{4hn-1}}{(\log n)^{4h}} \right)$$

because for a fixed $m_1$, there are $O((\log n)^{16h^2})$ possibilities for $m_2$ and $v \leq 2h - 1$.

**Subcase 3.2** $K \geq 10(\log n)^{64h^3}$. Then by $|m_1 - m_2 + K| \leq (\log n)^{16h^2}$, $m_2 \geq m_1 + K - (\log n)^{16h^2} \geq m_1$. Thus,

$$S_2 \ll \sum_{m_1=1}^{n-1} \frac{1}{m_1} \cdot \frac{1}{m_1^{2(\log n-1)}} \cdot (\log n)^{16h^2} = (\log n)^{16h^2} \sum_{m_1=1}^{\infty} \frac{1}{m_1} \cdot \frac{1}{2^{2(\log n-1)}}$$

$$= O((\log n)^{16h^2}) = O \left( \frac{n^{4hn-1}}{(\log n)^{4h}} \right)$$

because for a fixed $m_1$, there are $O((\log n)^{16h^2})$ possibilities for $m_2$ and $v \leq 2h - 1$.

**Subcase 3.3** $K \leq -10(\log n)^{64h^3}$. Then by $|m_1 - m_2 + K| \leq (\log n)^{16h^2}$, $m_1 \geq m_2 - K - (\log n)^{16h^2} \geq m_2$. Then

$$S_2 \ll \sum_{m_2=1}^{n-1} \frac{1}{m_2} \cdot \frac{1}{m_2^{2(\log n-1)}} \cdot (\log n)^{16h^2} \ll (\log n)^{16h^2} \sum_{m_2=1}^{\infty} \frac{1}{m_2} \cdot \frac{1}{2^{2(\log n-1)}}$$

$$= O((\log n)^{16h^2}) = O \left( \frac{n^{4hn-1}}{(\log n)^{4h}} \right)$$

because for a fixed $m_2$, there are $O((\log n)^{16h^2})$ possibilities for $m_1$ and $v \leq 2h - 1$. The proof is completed.

**References**

[1] G. Cao-Labora, J. Rué, C. Spiegel. *An Erdős-Fuchs Theorem for Ordered Representation Functions*, The Ramanujan Journal. **56** (2021), 183-201.

[2] J. Cilleruelo. *On Sidon sets and asymptotic bases*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 5, 1206-1230.
[3] J. M. Deshouillers, A. Plagne. A Sidon basis, Acta Math. Hungar. 123 (2009), no. 3, 233-238.

[4] P. Erdős, A. Sárközy, V. T. Sós. On additive properties of general sequences, Discrete Math. 136 (1994), no. 1-3, 75-99.

[5] P. Erdős, A. Sárközy, V. T. Sós. On sum sets of Sidon sets I, J. Number Theory 47 (1994), no. 3, 329-347.

[6] H. Halberstam, K. F. Roth. Sequences, 2nd ed. Springer - Verlag, New York-Berlin, 1983.

[7] J. H. Kim, V. H. Vu. Concentration of multivariate polynomials and its applications, Combinatorica, 20 (2000), 417-434.

[8] S. Z. Kiss. On Sidon sets which are asymptotic bases, Acta Math. Hungar. 128 (2010), no. 1-2, 46-58.

[9] S. Z. Kiss. On generalized Sidon sets which are asymptotic bases, Annales Univ. Sci. Budapest. Eötvös 57 (2014), 149-160.

[10] S. Z. Kiss, E. Rozgonyi, Cs. Sándor. On Sidon sets which are asymptotic bases of order 4, Func. Approx. Comment. Math. 51 (2014), no. 2, 393-413.

[11] S. Z. Kiss, Cs. Sándor. Generalized asymptotic Sidon basis, Discrete Math., 344 (2021), 112208, 5pp.

[12] T. Tao, V. H. Vu. Additive Combinatorics, Cambridge University Press, 2006.

[13] V. H. Vu. Chernoff type bounds for sum of dependent random variables and applications in additive number theory, Number theory for the millennium, III (Urbana, IL, 2000), 341-356, A K Peters, Natick, MA, 2002.

[14] V. H. Vu. On the concentration of multivariate polynomials with small expectation, Random Structures and Algorithms, 16 (2000), 344-363.