RESEARCH ARTICLE

Eigenvalue sensitivity analysis based on the transfer matrix method

Dieter Bestle

Abstract

For linear mechanical systems, the transfer matrix method is one of the most efficient modeling and analysis methods. However, in contrast to classical modeling strategies, the final eigenvalue problem is based on a matrix which is a highly nonlinear function of the eigenvalues. Therefore, classical strategies for sensitivity analysis of eigenvalues w.r.t. system parameters cannot be applied. The paper develops two specific strategies for this situation, a direct differentiation strategy and an adjoint variable method, where especially the latter is easy to use and applicable to arbitrarily complex chain or branched multibody systems. Like the system analysis itself, it is able to break down the sensitivity analysis of the overall system to analytically determinable derivatives of element transfer matrices and recursive formula which can be applied along the transfer path of the topology figure. Several examples of different complexity validate the proposed approach by comparing results to analytical calculations and numerical differentiation. The obtained procedure may support gradient-based optimization and robust design by delivering exact sensitivities.

KEYWORDS
adjoint variable method, direct differentiation, sensitivity analysis, transfer matrix method

1 | INTRODUCTION

Linear multibody systems are composed of simple components like bodies, springs, dampers and beams, see for example, Figure 1. The transfer matrix method utilizes the property that the kinematic and kinetic properties of all these elements can be easily and exactly described by analytical transfer matrices, see Rui et al., Abbas et al. or Appendix A. As demonstrated by some examples in Section 6, these element transfer matrices can be combined along the topology graph of a specific multibody system to finally end up with an overall transfer equation

\[ \textbf{U}_{\text{trf}}(\omega; \textbf{p})\textbf{Z}_{\text{all}} = \textbf{0}, \]  

where the overall transfer matrix \( \textbf{U}_{\text{trf}} \) depends on the system’s eigenfrequencies \( \omega \) (or in case of damped systems on complex eigenvalues \( \lambda \)) and some system parameters summarized in design vector \( \textbf{p} \). The state vector \( \textbf{Z}_{\text{all}} \) summarizes all modal displacements, rotation angles, forces and torques acting on the boundary ends of the system.

Half of these boundary states are known. For example, in case of a simply supported boundary end, displacements and torques are zero, whereas rotation angles and forces are unknown. By reducing

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

© 2021 The Authors. International Journal of Mechanical System Dynamics published by John Wiley & Sons Australia, Ltd on behalf of Nanjing University of Science and Technology.
Nontrivial solutions of Equation (2) require a singular system transfer matrix $U \in \mathbb{R}^{n \times n}$ resulting in the condition $\Delta(\omega) := \det(U(\omega; \mathbf{p})) \approx 0$ for the eigenfrequencies $\omega = \omega(\mathbf{p})$. This condition is either a polynomial or a transcendental equation depending on the type of multibody system elements being involved. Modeling steps (1)–(3) can be best seen from the simple demonstration example in Section 6.1.

The eigenfrequencies $\omega$ (and in the damped case eigenvalues $\lambda$) are an important characteristic of linear systems as they determine the system's agility and critical excitation frequencies. Thus, shifting them to desired values by proper changes of design parameters $\rho_k$ may be an important design goal, where knowledge of sensitivities $\partial \omega / \partial \rho_k$ may simplify the design task. Further, their knowledge enables the use of gradient-based optimization algorithms and provides information about the robustness of eigenvalues in case of parameter uncertainties.

Therefore, many eigenvalue sensitivity analysis studies are reported, for example, by Lancaster,1 Garg,4 or Rudisill and Chu11 for the classical eigenproblem $\mathbf{A} \mathbf{z} = \lambda \mathbf{z}$ resulting from modal transformation $\mathbf{z}(t) = \mathbf{Z} \mathbf{e}^{\mathbf{j}t}$ of a linear differential equation $\dot{\mathbf{z}} = \mathbf{A} \mathbf{z}$ in state space form. Also specific approaches are published, for example, by Fox and Kapoor6 or Choi and Kim7 for the generalized eigenproblem $\mathbf{KZ} = \omega^2 \mathbf{MZ}$ resulting from the ansatz $\mathbf{z}(t) = \mathbf{Z} \cos \omega t$ for undamped mechanical systems described by equations of motion $\mathbf{M} \ddot{\mathbf{z}} + \mathbf{Kz} = 0$. Further, damped mechanical vibration systems have been studied, for example, by Zmindak.8

A major feature of all these classical sensitivity studies is that the involved matrices $\mathbf{A}$ or $\mathbf{M}$, $\mathbf{K}$ are independent of the eigenvalue, whereas the coefficient matrix in Equation (2) is a highly nonlinear function of $\omega$. Therefore, classical approaches cannot be applied to the transfer matrix method, neither for finding the eigenfrequencies $\omega = \omega(\mathbf{p})$ nor for computing their sensitivities $\partial \omega / \partial \rho_k$. The former problem has been solved elegantly, for example, by Bestle et al.9 by transforming the root search problem into a minimization problem, the latter is the focus of the present paper.

Although classical sensitivity analysis, as presented by Murthy and Haftka,10 cannot be used directly for the transfer matrix method for multibody systems (MSTMM), at least the applied strategies of direct differentiation and the adjoint variable approach may be utilized and adapted in Sections 2 and 3 to come up with calculation procedures. Computational issues for extracting the required eigenvector information from Equation (2) are discussed in Section 4. To make these procedures applicable to practical problems, the overall sensitivity analysis is broken down to the determination of sensitivities of the analytically described element transfer matrices by differentiation and a recursive procedure in Section 5. Some simple examples in Section 6 will demonstrate how adjoints can directly be deduced from the topology graph as easily as the system transfer equations. For simplicity, all procedures will be discussed for the undamped case resulting in real quantities, which is why in Section 7 an additional example with damping will be introduced to demonstrate how the sensitivity analysis
procedures may be extended to complex eigenvalue problems. Two appendices contain a library of transfer matrices and associated sensitivities for selected elements.

2 | DIRECT DIFFERENTIATION METHOD

Generally, eigenvectors are not unique and may be normalized in various ways, Murthy and Haftka. In this section, it is assumed that the system has no multiple eigenvalues and that the corresponding nonunique eigenvectors are normalized such that their largest component is one, respectively, that is, \( Z_m := 1 \) where for example, \( m = \arg \max_i \lvert Z_i \rvert \). Any other eigenvector may then be deduced from this normalized vector \( Z_0 \) as \( Z = \zeta Z_0, \zeta \in \mathbb{R} \). To make the following procedure unique, the components of \( Z_0 \) are reordered by a permutation matrix \( P \), \( P^T = I \), such that the last component of the re-ordered eigenvector \( Z_0 = P \zeta Z_0 \) is one. This requires also a re-ordering of the columns of \( U \) in Equation (2) resulting in

\[ UZ = U_0 Z_0 \equiv UP \zeta Z_0 \equiv UP \zeta Z_0 = U \zeta Z_0 = 0. \]  

Since the matrix \( U(\omega) \) is singular for eigenfrequencies \( \det(\omega) = 0 \), the equations of the linear system (2) can always be re-ordered such that the last one is the dependent equation. Formally this can be achieved by pre-multiplying Equation (4) by another permutation matrix \( P, P^T = I \), on both sides. Division by \( \omega \) results in

\[ \omega \delta \delta = 0 \quad \omega \delta \delta = 0, \]  

where for \( \omega \delta \delta \) is one, respectively, and identical matrix products finally yield an eigenvalue problem

\[ U_\omega(\omega; p)Z_\omega(p) = 0. \]  

Equivalent to Equation (2) but with the specific substructure

\[ \begin{bmatrix} U_{11} & U_{12} \\ U_21 & U_{22} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = 0. \]  

or after resorting in a linear matrix equation

\[ \begin{bmatrix} U_{11} & U_{12} & U_{12} & U_{12} \\ U_21 & U_{21} & U_{22} & U_{22} \end{bmatrix} \begin{bmatrix} Z_0 & Z_0 \end{bmatrix} = 0. \]  

for the desired eigenfrequency sensitivities \( \omega_k \equiv \partial \omega / \partial p_k \) where the product rule and dependence \( \omega = \omega(p) \) have been applied, and derivatives are abbreviated as \( \cdot \equiv \partial \cdot / \partial p_k \), \( \cdot \equiv \partial \cdot / \partial \omega \).

An advantage of Equation (8) is that the coefficient matrix has to be computed only once whereas the right-hand side has to be adapted for the different parameter sensitivities \( \cdot \).

It can be shown that this procedure works in principle, however, it has several drawbacks: (i) it requires specially normalized eigenvectors and reordering of equations according to Equation (6); (ii) for each parameter sensitivity \( \omega_k \equiv \partial \omega / \partial p_k \) a complete system of equations (8) has to be set up and solved; (iii) since the system transfer matrix \( U \) results from a multiplication of several element transfer matrices, the product rule will produce multiple product terms in the derivatives \( U_\omega \) and \( U \) where identical matrix products are used several times. Thus, the procedure is computationally inefficient and not recommendable for complex models of technically relevant systems.

3 | ADJOINT VARIABLE METHOD

The adjoint variable approach is typically much more efficient than direct differentiation if only sensitivities of a scalar output variable (here \( \omega \)) are required w.r.t. multiple input parameters (here \( p_k \)). Variation of Equation (2) w.r.t. design vector \( p \) yields

\[ \delta U Z + U \delta Z = 0. \]  

Scalar product with an arbitrary vector \( \eta \in \mathbb{R}^n \) yields

\[ \eta \delta U Z + \eta^T U \delta Z = 0 \quad \forall \eta, \]  

where the second term, and thus \( \delta Z \), can be eliminated for a specific adjoint vector \( \eta \) satisfying the equation

\[ \eta^T U = 0 \quad \text{or} \quad U^T \eta = 0. \]  

This simplifies Equation (10) to

\[ \eta \delta U Z = \eta^T \left( U_\omega \delta \omega + \sum_k U_k \delta p_k \right) Z = 0. \]  

which may be solved for the variation of the eigenfrequency as

\[ \delta \omega = \sum_k - \frac{\eta^T U_k Z}{\eta^T U_\omega Z} \delta p_k. \]  

By comparing this expression with the formal variation of \( \omega = \omega(p) \) for independent design variables \( p_k \), that is,

\[ \delta \omega = \sum_k \frac{\partial \omega}{\partial p_k}, \]  

the desired sensitivities can be found as
\[
\frac{\partial \omega}{\partial \rho_k} = -\frac{\eta^T(U, Z)}{\eta^T(U, Z)}.
\]

(15)

The simple example in Section 6.1 demonstrates the validity of this scalar expression, especially by Equation (43), which is far simpler than solving vector Equation (8).

4 | COMPUTATION OF EIGENVECTORS

Besides the derivatives \( U_x \) and \( U_{\omega} \), evaluation of expression (15) requires the knowledge of right eigenvector \( Z \) of system transfer matrix \( U \) according to Equation (2) and left eigenvector \( \eta \) according to Equation (11). Both can be obtained from a singular value decomposition (SVD)

\[
U = LDR^T \quad \text{where} \quad LL^T = L^T L = I, RR^T = R^T R = I, \\
D = \text{diag} (\sigma_1, \ldots, \sigma_n, 0),
\]

(16)

for example, Golub and van Loan.\(^{11}\) The singular value 0 expresses the singularity of \( U \) for eigenfrequencies.

As already discussed in Section 2, eigenvectors are not unique. Let \( Z \) and \( \eta \) be arbitrarily selected eigenvectors; then any other equivalent eigenvectors may be expressed as \( \hat{Z} = \xi Z \) and \( \hat{\eta} = \xi \eta \) with arbitrary scaling factors. With these general eigenvectors, sensitivities (15) read as

\[
\frac{\partial \omega}{\partial \rho_k} = -\frac{\eta^T(U, Z)}{\eta^T(U, Z)} = -\frac{\xi \eta^T(U, \xi Z)}{\xi \eta^T(U, \xi Z)} = -\frac{\xi \eta^T(U, Z)}{\eta^T(U, Z)} = -\frac{\eta^T(U, Z)}{\eta^T(U, Z)}.
\]

(17)

Thus, all eigenvectors finally yield the same result for the sensitivities, which is why they can be selected freely and there is no need to normalize them.

The right eigenvector \( Z \) is contained in the orthogonal matrix \( R \) of SVD (16). Substitution of the decomposed \( U \) in Equation (2) yields

\[
UZ = LDR^T Z = 0
\]

(18)

or with abbreviation \( Z^R = R^T Z \), pre-multiplication with \( L^T \) and orthogonality property \( L^T L = I \)

\[
DZ^R = \begin{bmatrix} \sigma_1 & \cdots & \sigma_n \end{bmatrix} \begin{bmatrix} z^R_1 \\ \vdots \\ z^R_n \end{bmatrix} = 0.
\]

(19)

A possible solution is the unit vector \( Z^R = [0 \ldots 0 1]^T = e_n \). Due to \( RZ^R = RR^T Z \equiv Z \) because of orthogonality property (16), the final solution is the last column of \( R \):

\[
Z = RZ^R = R e_n = R(:, n).
\]

(20)

Analogously the left eigenvector \( \eta \) is the last column of the orthogonal matrix \( L \). Substitution of the decomposed \( U \) in Equation (11) yields

\[
U^T \eta = (LDR^T)^T \eta = RD^T \eta = 0.
\]

(21)

With abbreviation \( Z^L = L^T \eta \) and orthogonality property (16) we obtain after pre-multiplication with \( R^T \) the equation

\[
DZ^L = \begin{bmatrix} \sigma_1 & \cdots & \sigma_n \end{bmatrix} \begin{bmatrix} z^L_1 \\ \vdots \\ z^L_n \end{bmatrix} = 0
\]

(22)

with the possible solution \( Z^L = e_n \). Due to \( LZ^L = LL^T \eta \equiv \eta \), the final solution is the last column of \( L \), that is,

\[
\eta = LZ^L \equiv Le_n = L(:, n).
\]

(23)

5 | SENSITIVITY ANALYSIS BASED ON ELEMENT TRANSFER MATRICES

The remaining problem in utilizing the sensitivity Equation (15) is to derive the derivatives \( U_x \) and \( U_{\omega} \) while avoiding some of the drawbacks discussed in Section 2. A first simplification is given by the observation, that these derivatives are not required explicitly, but only as products \( U_x Z \) and \( U_{\omega} Z \) with the eigenvector \( Z \). A second simplification is given by the fact that reduction (3) is not necessary, but these sensitivity products may be obtained directly from the transfer matrix \( U_{\text{eff}} \) in Equation (1). To see this, let \( x \) be any of the quantities \( p, \omega \). Then, with relations (3) we obtain

\[
U_x Z = \frac{\partial U}{\partial x} Z = \frac{\partial (U_{\text{eff}} B)}{\partial x} Z = \frac{\partial U_{\text{eff}} B}{\partial x} Z_{\text{eff}} = \frac{\partial U_{\text{eff}}}{\partial x} Z_{\text{eff}},
\]

(24)

where \( Z_{\text{eff}} = B Z \) corresponds to and can be easily computed from the eigenvector (20).

The most significant simplification is, that the quantities (24) can be deduced directly from the topology graph by introducing element-specific adjoints. Let us consider a chain consisting of, for example, three single-input-single-output elements shown in Figure 2A, which can be described by element transfer equations

\[
Z_{ij} = U(x) Z_j.
\]

(25)

The symbol \( x \) may represent the eigenfrequency, any parameter of this specific element or any other design parameter, see Appendix A. By recursive substitution for the state vectors of the chain we get

\[
Z_4 = U_3 Z_3 = U_2 U_3 Z_2 = U_1 U_2 U_3 Z_1
\]

(26)
The expression (24) can then be computed as

\[
U_3 U_2 U_1 \begin{bmatrix} Z_1 \\ \vdots \end{bmatrix} = 0.
\]  

(27)

The expression (24) can then be computed as

\[
U_x Z = \frac{\partial U_3}{\partial x} U_{all} = \begin{bmatrix} (U_2 U_1 U_{1,1})_x & 0 \\ Z_{1,1,1} & Z_{1,1,2} & \cdots & Z_{1,1,4} \end{bmatrix}
\]

\[
= U_{3,1}(U_2 U_1) + u_1(U_2 Z_1) + u_2(U_2 U_{1,1} Z_1) + U_{3,1} U_{2,1} Z_1
\]

\[
= U_{3,1} Z_0 + U_{2,1} U_{2,1} Z_1 + U_{2,1} U_{1,1} Z_1
\]

(28)

This can be simplified by recursively assigning adjoints \(Z_{1,1,1}\) as

\[
U_x Z = U_{3,1,1} Z_0 + U_{3,2,1} Z_2 + U_{2,2,1} (U_{1,1,1} Z_1)
\]

\[
\text{where } Z_{1,1,1} = 0
\]

(29)

according to the rule

\[
Z_{0,1,1} = \begin{bmatrix} U_x Z_1 + U_{x,1} Z_1 \\ U_{x,1} Z_1 \end{bmatrix}
\]

if \( U = U(x) \), \( Z_{0,1,1} = 0 \)

if \( U = U(x) \).

(30)

Formally this rule for the output sensitivity can be found by differentiation of the transfer relation (25) w.r.t. \( x \), however, it should be pointed out that \( Z_{1,1,1} \) is not the sensitivity of eigenvector \( Z_1 \) w.r.t. parameter \( x \), which can be seen best from the formal introduction of \( Z_{1,1,1} = 0 \) in Equation (29) just to stay consistent with the general rule (30).

There is no need to firstly compute (28) with all its magic resorting, but the assignment of adjoints \( Z_{1,1,1} \) can be made directly along the transfer path according to the rule (30) starting with \( Z_{1,1,1} = 0 \) and ending with the result \( U_x Z \equiv Z_{1,1,1} \). The second part of the demonstration example in Section 6.1 shows the procedure based on the element derivative information provided in Appendix B.

In case of branched multibody systems, dummy bodies with two inputs \( I_1 \) and \( I_2 \) may be used. Abbas et al. According to Figure 2B, such massless dummies are described by a transfer equation

\[
Z_0 = U_2^2 Z_{I_1} + U_2^2 Z_{I_2}
\]

(31)

and a consistency equation

\[
h = -H_2^2 Z_{I_1} + H_2^2 Z_{I_2} = 0.
\]

(32)

where transfer matrices \( U_1^2 \), \( U_2^2 \) as well as extraction matrices \( H_2^2 \), \( H_2^2 \) are constant, see Appendix A. Therefore, all their derivatives are zero and formal differentiation of Equations (31) and (32) w.r.t. some quantity \( x \) yields the rules

\[
\begin{align*}
Z_{0,x} & = U_2^2 Z_{I_1} + U_2^2 Z_{I_1}, \\
h_x & = -H_2^2 Z_{I_2} + H_2^2 Z_{I_2}
\end{align*}
\]

(33)

(34)

for adjoint assignment. The use of these quantities becomes clear from the example in Section 6.3.

6 | APPLICATION EXAMPLES

In the following, three examples with different complexity and intentions will be provided. The first is a simple spring-mass vibrator which allows demonstrating the whole concept analytically. The second one shall validate the derivatives of the beam transfer equation, and by attaching an additional spring support, the concept for branched multibody system can be shown in the third example.

6.1 | Spring-mass vibrator

The system in Figure 1A combining spring and lumped mass is considered only in the horizontal \( y \)-direction. From its topology graph we can immediately see

FIGURE 2 Transition of state vectors and adjoints for (A) single-input-single-output element and (B) dummy connection element.
\[
Z_3 = U^T Z_1 = U^T U^S Z_1, \tag{35}
\]

and thus, by using element transfer Equations (75) and (77), find the overall system transfer Equation (1) as

\[
U_{\text{eff}} Z_{\text{eff}} = \left[ U^T U^S - I \right] Z_3 = \left[ \begin{array}{ccc} 1 & -1/k & 0 \\ m o^2 & 1 - m o^2/k & 0 \\ 0 & -1 & 0 \end{array} \right] \left[ \begin{array}{c} Y_1 \\ Q_{r,1} \\ Y_5 \end{array} \right] = 0.
\tag{36}
\]

The boundary conditions are \( Y_1 = 0 \) due to the pin joint and \( Q_{r,3} = 0 \) for the free end, which may be expressed by Equation (3) as

\[
Z_{\text{eff}} = \left[ \begin{array}{c} Q_{r,1} \\ Y_5 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{array} \right]. \tag{37}
\]

This leads to the reduced state transfer Equation (2)

\[
U(\omega; m, k) Z = \left[ \begin{array}{c} -1/k \\ 1 - m o^2/k \end{array} \right] \left[ \begin{array}{c} Q_{r,1} \\ Y_5 \end{array} \right] = 0. \tag{38}
\]

In this very simple case, the eigenfrequency \( \omega \) can be found analytically from \( \det U = 1 - m o^2/k = 0 \). Differentiation of \( \omega \) w.r.t. to parameters \( m \) and \( k \) yields its sensitivities which will serve as reference values later:

\[
\frac{\partial \omega}{\partial m} = \frac{k}{m}, \quad \frac{\partial \omega}{\partial k} = 1 - m o^2/k = 0.
\tag{39}
\]

To validate sensitivity Equation (15), we first may compute the eigenvectors. By substituting the eigenfrequency (39) in Equation (38), the (right) eigenvector may be obtained:

\[
U(\omega^2 = k/m) Z = \left[ \begin{array}{c} -1/k \\ 0 \\ 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} Q_{r,1} \\ Y_5 \end{array} \right] = -1/k Z_3 = 0. \tag{40}
\]

Similarly, the left eigenvector results from Equation (11) as

\[
U^T(\omega^2 = k/m) \eta = \left[ \begin{array}{c} -1/k \\ 0 \\ 0 \end{array} \right] \eta = 0. \tag{41}
\]

With these eigenvectors and the derivatives of system transfer matrix (38) the required terms for feeding Equation (15) may be computed:

\[
\frac{\partial \omega}{\partial m} = \frac{\eta^T U_{\text{eff}} Z_{\text{eff}}}{\eta^T U_{\text{eff}} Z_{\text{eff}}} = \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = -1/k Z_3 = 0. \tag{42}
\]

This results in sensitivities

\[
\frac{\partial \omega}{\partial m} = \frac{\eta^T U_{\text{eff}} Z_{\text{eff}}}{\eta^T U_{\text{eff}} Z_{\text{eff}}} = \frac{\omega^2}{m}, \quad \frac{\partial \omega}{\partial k} = \frac{\omega}{2k}. \tag{43}
\]

which is identical with the analytical results (39) obtained by direct differentiation, and thus validates Equation (15).

This calculation based on the overall system transfer matrix \( U \) is for demonstration only, because for more complex technical problems \( U \) cannot be calculated symbolically. Therefore, the recommended concept is based on eigenvector (40), derivatives (82) and (83) of element transfer matrices and adjoints drawn from the topology graph in Figure 1A according to rule (30):

\[
Z_3 = \left[ \begin{array}{c} Y_1 \\ Q_{r,1} \\ Y_5 \end{array} \right] = 0 \rightarrow Z_3 = U^S Z_3 = \left[ \begin{array}{c} 1 \\ -1/k \\ 0 \end{array} \right]. \tag{44}
\]

Since \( U_{\text{eff}} Z \equiv Z_{\text{eff}} \) analogously to Equation (29), the sensitivities (15) can be computed from left eigenvector (41) and (44) as

\[
\frac{\partial \omega}{\partial m} = \frac{\eta^T Z_{\text{eff}}}{\eta^T Z_{\text{eff}}} = \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = -\frac{\omega^2}{m}, \quad \frac{\partial \omega}{\partial k} = \frac{\omega}{2k}. \tag{45}
\]

which is identical with analytical solutions (39) and result (43).
In technical applications, symbolic computations (44) and (45)
would be performed numerically, which makes computation much
easier. Further, in Equation (44) it can be clearly observed that
adjoints $Z_{ix}$ are not identical with derivatives $\partial Z_i/\partial x$, but only arti-
ficial quantities introduced in Equation (29) after resorting the
derivative terms properly.

### 6.2 Simply supported beam

The next example shown in Figure 1B is chosen for validating
beam sensitivity equations (84)- (86). The beam parameters are
chosen as $m = 0.78$kg/m, $L = 1$m and $EI = 166.67$Nm². The analy-
tical result for the first eigenfrequency of the simply supported
beam is well known from engineering mechanics textbooks as

$$\omega = \pi^2 \sqrt{EI/m} L^2 = 144.2717 \text{rad/s}$$

resulting in sensitivities

$$\nabla \omega = \begin{bmatrix} \frac{\partial \omega}{\partial m} \\ \frac{\partial \omega}{\partial L} \\ \frac{\partial \omega}{\partial EI} \end{bmatrix} = \begin{bmatrix} -2\omega/2m \\ -2\omega/L \\ \omega/2EI \end{bmatrix} = \begin{bmatrix} -92.48186 \\ -288.5434 \\ 0.4328065 \end{bmatrix}$$

Exactly the same values are obtained by evaluating the equations

$$Z_{ix} := 0 \rightarrow Z_{2x} = U^x_i Z_1 + U^x_i Z_{1x} = U^x_i Z_{1x}, \quad x \in \{\omega, m, L, EI\},$$

with $U^x_i$ from (78) and $U^i_x$ from (86). The eigenvectors
matrix $Z = \begin{bmatrix} \Theta_{x1}, \Theta_{x2}, \Theta_{x3} \end{bmatrix}^T = \begin{bmatrix} 4.3 \cdot 10^{-4}, -0.7071, -4.3 \cdot 10^{-4}, 0.7071 \end{bmatrix}^T$ and $n = [-1, 0, 6.1 \cdot 10^{-4}, 0]^T$ are obtained according to Section 4 from a SVD of the reduced system transfer matrix

$$U = \begin{bmatrix} T/\beta & V/\beta E^2 & 0 & 0 \\ S & E^2 & -1 & 0 \\ \beta E V & T/\beta & 0 & 0 \\ \beta^2 E U & S & 0 & -1 \end{bmatrix}$$

with abbreviations (79). The eigenvector $Z_1 = \begin{bmatrix} 0 & \Theta_{x1} & 0 & \Theta_{x3} \end{bmatrix}^T$

is extracted from $Z$.

### 6.3 Simply supported beam with elastic support

The third example in Figure 1C comes already closer to
practical problems, since the branching enforces a typical substructure
of the overall transfer equation (1). By recursive substitution of element
transfer relations (25) and (31) from right to left in the topology graph of
Figure 1C, we find for the transfer equation

$$Z_x = U_x (U_x^T U_x Z_1 + U_x^T U_x Z_{1x}).$$

Additionally the scalar consistency Equation (32), respectively (81),
that is,

$$h = -H^2 U_x Z_1 + H^2 U_x Z_{1x} = 0$$

has to be formulated for the dummy element. Both together may be
summarized as

$$\begin{bmatrix} T_1 & T_2 & -1 \\ G_1 & G_2 & 0 \\ u_{al} & z_{al} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = 0,$$

where $T_1 = U_x U_x^T U_x$, $T_2 = U_x U_x^T U_x$, $G_1 = -H^2 U_x$, $G_2 = H^2 U_x$.

According to Equation (24) we need for sensitivity analysis the
derivatives

$$U_x Z = \frac{\partial U_{al}}{\partial x} = \begin{bmatrix} T_1 & T_2 & 0 \\ G_1 & G_2 & 0 \\ u_{al} & z_{al} \end{bmatrix} = \begin{bmatrix} \frac{T_{1x} Z_1 + T_{2x} Z_{1x}}{u_{al}} \end{bmatrix} = \begin{bmatrix} h_x \end{bmatrix}$$

w.r.t. any of the interesting design variables or eigenfrequency
$x \in \{m, L, EI, \omega\}$ consisting of two parts $u_x$ and $h_x$. By reordering the
products of the upper part, we get

$$u_x = \left(U_x^0 \left(U_x^0 U_x^0 Z_1 + U_x^0 U_x^0 Z_{1x}\right) + \left(U_x^0 U_x^0 U_x^0 + U_x^0 U_x^0 U_x^0\right) Z_0\right) Z_0$$

This resorting allows to recursively introduce adjoints $Z_{4x}$ according to
rules (30) and (33) for each intermediate state vector $Z_i$ along the
transfer path in Figure 1C as

$$u_x = \begin{bmatrix} Z_2 & Z_3 \end{bmatrix} \begin{bmatrix} Z_2 & Z_3 \end{bmatrix} \begin{bmatrix} Z_2 & Z_3 \end{bmatrix} = \begin{bmatrix} Z_5 & Z_6 & Z_7 & Z_8 \end{bmatrix}$$

where we have to set $Z_{1x} = 0$ and $Z_{3x} = 0$ for the formally missing
terms in $Z_{2x}$ and $Z_{4x}$. With these adjoints, the lower part of Equation
(54) yields

$$h_x = -H^2 U_x Z_1 + H^2 U_x Z_{1x},$$

which is consistent with the rule (34) for dummy elements.
Both equations can be directly deduced from the topology graph in Figure 1C and the eigenvector $Z = \begin{bmatrix} \Theta_{x,1} & \Theta_{x,2} \end{bmatrix}$ to deliver quantities (56) and (57) for being used as substitutes (54) in sensitivity formula (15):

$$Z = \begin{bmatrix} 0 & 0 \end{bmatrix}^{T}, \quad Z_{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{T}, \quad Z_{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{T}.$$

For beam parameters $m = 0.78 \text{kg/m}$, $L = 0.5 \text{m}$, $EI = 166.67 \text{Nm}^2$ and spring stiffness $k = 10E/2L = 166.67 \text{N/m}$ we obtain the first eigenfrequency $\omega = 158.35 \text{rad/s}$ from calling the root search function $\text{om} = \text{FMin1D}(1, 400, 150, 0, 0.0001, [\cdot], \text{absdetU}, p)$ published by Bestle et al., where $\text{absdetU}$ computes the absolute value $|\Delta(\omega)| = |\text{det}(U(\omega); p)|$ of the determinant of the reduced system transfer matrix to be minimized. The eigenvector $Z = [-0.0004, 0.6909, 0.2126, 0.0004]^{T}$ is obtained from the SVD-based procedure described in Section 4 resulting in the sensitivities shown in Table 1. This again makes clear that the adjoints are artificial and no differentials of associated quantities.

To check the result, we may use forward differences based on eigenfrequency evaluations at design point and after disturbing the interesting design parameter by some small parameter variation $\epsilon$. The result in Table 1 for $\epsilon = 10^{-5}$, however, is rather disappointing for sensitivities w.r.t. $EI$ and $k$ questioning the procedure. Therefore, additionally central differences are evaluated for $\epsilon = 5 \cdot 10^{-5}$ resulting in sensitivities now consistent with those calculated with the adjoint method, and thus validating the procedure (58).

This reveals one of the major drawbacks of numerical differentiation often used in combination with gradient-based optimization algorithms. If the parameter perturbation $\epsilon$ is too small, numerical errors in $\omega$-evaluation will be amplified too much; if $\epsilon$ is too large, approximation errors of numerical differences come into play. Bestle, in Figure 3, showing deviations of numerical differences (59) and (60) from sensitivities (58), both effects can be seen rather clearly for forward differences (o) associated with sensitivities $\partial \omega / \partial \delta m$ and $\partial \omega / \partial \delta l$. Errors in $\partial \omega / \partial k$ are so large for (59) that the curve lies outside the display window. The central differences (x) are obviously less prone to these effects and coincide with the results of the adjoint method (–) in a rather broad $\epsilon$-range. However, they require twice as much computational effort as forward differences since eigenfrequency computations have to be performed for two parameter perturbations $\pm \epsilon$, respectively. Generally, also for central differences the parameter perturbations $\epsilon$ have to be chosen carefully in order not to mislead optimization algorithms by wrong gradients, whereas the adjoint method is both efficient and robust without the need of any $\epsilon$-decisions.

To get a feeling for the efficiency of the proposed approach, some numbers about the required CPU time on an Intel Core i5-8400 CPU @ 2.80 GHz may be given. With MATLAB, the root search takes about 6.7 ms, whereas the proposed sensitivity analysis including the setup of the reduced transfer matrix, SVD for eigenvector computation and evaluation of sensitivity formula (58) takes only 0.1 ms. For comparison, the four additional eigenvalue searches for the forward differences (59) take 37 ms and the central differences with 74 ms twice as much CPU time. Thus, the proposed method has not only the advantage of being exact, but is also several orders of magnitude faster than numerical difference approximations.

### Table 1 Sensitivities for beam with elastic support

| Transfer matrix method | Forward differences | Central differences |
|------------------------|---------------------|---------------------|
| $\epsilon$             | $10^{-4}$           | $5 \cdot 10^{-5}$   |
| $\omega$               |                     |                     |
| $\partial \omega / \partial k$ | -101.5068       | -101.5068           |
| $\partial \omega / \partial \delta m$ | -101.5068       | -101.317            |
| $\partial \omega / \partial \delta l$ | -552.9303        | -552.5788           |
| $\partial \omega / \partial \delta \Theta I$ | 0.3945719        | 0.5746097           |
| $\partial \omega / \partial \delta \Theta I$ | 0.00804702       | 0.1880848           |
| $\partial \omega / \partial \delta \Theta I$ |          |                     |

7 | Extension to Damped Systems

The concept in Section 3 is also applicable to damped systems where the overall transfer matrix is complex and the real eigenfrequency has to be substituted by the complex eigenvalue $\lambda = \sigma \pm i \omega \in \mathbb{C}$, $i = \sqrt{-1}$. For showing this, the spring in Figure 4A is substituted by an element with spring and damper in parallel, which is governed by the force law $a_{q,0} = a_{q,1} = k(y_{1} - y_{0}) + d(y_{1} - y_{0})$ with spring stiffness $k$ and damping coefficient $d$. With modal transformation $z(t) = Z e^{\lambda t}$ this results in $Q_{q,0} = Q_{q,1} = (k + d\lambda)(y - Y_{0})$ or element transfer equation.
The spring-damper transfer matrix and their derivatives are then

\[
\frac{\partial \omega}{\partial m} = \begin{bmatrix}
1 & 0 \\
-1/(k + d\lambda) & 1
\end{bmatrix}, \quad \frac{\partial \omega}{\partial L} = \begin{bmatrix}
\frac{d}{(k + d\lambda)^2} \\
0
\end{bmatrix}.
\]

(61)

For damped vibrations, also the transfer relations (75) and (82) of the lumped mass need to be adapted to the changed modal transformation resulting in

\[
U^m(\lambda; m) = \begin{bmatrix}
1 & 0 \\
-\frac{1}{m\lambda^2} & 1
\end{bmatrix}, \quad U^m(\lambda; d) = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad U^m(\lambda; k) = \begin{bmatrix}
0 & 1 \\
-\lambda^2 & 0
\end{bmatrix}.
\]

(62)

Analogously to Equation (36) we find the overall system transfer equation

\[
U_{all} Z_{all} = \begin{bmatrix}
U^mU^d - I & Z_0
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
1 & -1/(k + d\lambda) & -1 \\
-1/(k + d\lambda) & 1 + m\lambda^2/(k + d\lambda) & 0 & -1
\end{bmatrix} Y_1 & Q_{r,1} \\
Y_3 & Q_{r,3}
\end{bmatrix} = 0.
\]

(64)

Taking into account the boundary conditions \( Y_1 = 0 \) and \( Q_{r,3} = 0 \) by Equation (37), the reduced state transfer Equation (2) reads as

\[
U(\lambda; m, d, k)Z = \begin{bmatrix}
-1/(k + d\lambda) & -1 \\
1 + m\lambda^2/(k + d\lambda) & 0 & -1
\end{bmatrix} Y_3 = 0
\]

(65)

with the well-known characteristic equation \( \det U = 1 + m\lambda^2/(k + d\lambda) = 0 \). Resorting it to

\[
m\lambda^2 + d\lambda + k = 0
\]

(66)

yields the eigenvalues

\[
\lambda = \sigma \pm i\omega \quad \text{where} \quad \sigma = -\frac{d}{2m}, \quad \omega = \sqrt{\frac{k}{m} - \sigma^2}.
\]

(67)
Reference values for the sensitivities may be best found from the variation
\[ \delta m \lambda^2 + 2m \lambda \delta \lambda + \delta m \delta \lambda + d \delta \lambda + \delta k = 0 \]
\[ \Rightarrow \delta \lambda = -\frac{\delta m \lambda^2 + \delta d \lambda + \delta k}{2m \lambda + d} \]  
(68)
of characteristic Equation \((66)\). Comparison with the formal variation \(\delta \lambda = \lambda_m \delta m + \lambda_d \delta d + \lambda_k \delta k \) of \(\lambda = \lambda(m, d, k)\) finally yields
\[ \nabla \lambda = \begin{bmatrix} \lambda_m \\ \lambda_d \\ \lambda_k \end{bmatrix} \begin{bmatrix} \lambda_m \\ \lambda_d \\ \lambda_k \end{bmatrix} = -\frac{1}{2m \lambda + d} \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}. \]  
(69)

To demonstrate the concept of Sections 3 and 5, we first need to compute the eigenvectors of \( U \). Substitution of the eigenvalues \((67)\) in Equation \((65)\) yields the right eigenvector as
\[ \begin{bmatrix} -1/(k + d) \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} Q_{1,1} \\ Y_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} Q_{1,1} \\ Y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  
(70)
and the left eigenvector from Equation \((11)\) as
\[ \begin{bmatrix} -1/(k + d) \\ -1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  
(71)

By following the transfer path of a topology graph similar to that in Figure 1A, we get with eigenvectors \((70)\) and \((71)\), matrices \((62)\) and \((63)\), and rule \((30)\) where \( U^f = U^f(m) \), \( U^k = U^k(d, k) \):
\[ Z_1 = \begin{bmatrix} Y_1 \\ Q_{1,1} \end{bmatrix}, \]
\[ Z_2 = 0 \\ -1 \]
\[ \begin{bmatrix} Z_{1,m} = Z_{1,d} = Z_{1,k} = 0 \end{bmatrix}, \]
\[ Z_3 = U^f Z_1 \rightarrow Z_3, Z_3, Z_3 = U^f Z_1, Z_3, Z_3 = U^f Z_1, \]
\[ Z_5 = U^k Z_2 \rightarrow Z_5, Z_5, Z_5 = U^k Z_2, Z_5, Z_5 = U^k Z_2, \]
\[ Z_{5,d} = U^k Z_{5,d}, Z_{5,k} = U^k Z_{5,k}, \]
\[ \Rightarrow \nabla \lambda = -\frac{1}{\eta Z_{1,m}} \begin{bmatrix} \eta Z_{1,m} \\ \eta Z_{1,d} \\ \eta Z_{1,k} \end{bmatrix}. \]  
(72)

For parameter values \( m = 1 \) kg, \( d = 2 \) Ns/m and \( k = 5 \) N/m, both calculus \((69)\) and adjoint method \((72)\) deliver the same result \( \lambda = -1 \pm 2i \) and
\[ \nabla \lambda = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} \pm \begin{bmatrix} -0.75 \\ -0.25 \\ 0.25 \end{bmatrix}. \]  
(73)
which validates the applicability of the concept also for damped systems.

8 | CONCLUSIONS

The paper develops two different strategies for computing eigenvalue sensitivities of multibody systems modeled with the transfer matrix method, where only the adjoint method is efficient enough to be applied to complex mechanical systems. Simple rules are found for assigning adjoints to each state vector along the transfer path of the topology graph, which are then combined with the left eigenvector of the reduced system transfer matrix in a simple, explicit formula for the desired eigenvalue sensitivities. The rules can be deduced element-wise and require only derivatives of the analytically given element transfer matrices. The procedure is easy to use, exact and computationally efficient by using vector relations instead of matrix multiplications. Although demonstration examples in the paper are restricted to selected multibody system elements only, the library of element transfer matrices and their derivatives can be easily extended to bodies and spatial systems.

ACKNOWLEDGMENTS

The paper is dedicated to late Laith Abbas who was professor at the Nanjing University of Science and Technology (NUST) and challenged the author by many interesting discussions regarding the transfer matrix method. This study of eigenvalue sensitivities was initialized by Prof. Xiaoting Rui during a research stay at his Institute of Launch Dynamics supported by NUST in 2019, which is greatly appreciated.

CONFLICT OF INTEREST

The author declares that there are no conflict of interest.

DATA AVAILABILITY STATEMENT

Data are available from the corresponding author upon reasonable request.

ORCID

Dieter Bestle http://orcid.org/0000-0002-7636-376X

ENDNOTE

1 Although the value is rather small, enforcing \( h_x = 0 \) would slightly distort the sensitivity results.

REFERENCES

1. Rui X, Wang G, Zhang J. Transfer Matrix Method for Multibody Systems: Theory and Applications. Wiley; 2018.
2. Abbas L, Zhou Q, Bestle D, Rui X. A unified approach for treating multibody systems involving flexible beams. Mech Mach Theory. 2017;107:197-209.
3. Lancaster P. On eigenvalues of matrices dependent on a parameter. Num Mathematik. 1964;6:377-387.
4. Garg S. Derivatives of eigensolutions for a general matrix. AIAA J. 1973;11:1192-1194.
5. Rudisill CS, Chu Y-Y. Numerical methods for evaluating the derivatives of eigenvalues and eigenvectors. AIAA J. 1975;13:834-837.
The translational spring in Figure A1B with stiffness $k$ is considered as massless, which is why input and output forces $q_{y,0} = q_{y,i}$ are equal and given by displacement difference $q_{y,i} = k(y - y_0)$. Above modal transformation yields

$$\begin{bmatrix} Y_0 \\ Q_{y,i} \end{bmatrix} \cos \omega t = \begin{bmatrix} 1 & -1/k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_i \\ Q_{y,i} \end{bmatrix} \cos \omega t$$

or transfer Equation (25) where

$$Z = \begin{bmatrix} Y \\ Q_i \\ \Theta \end{bmatrix}, \quad U = U^i(\beta, \beta L) = \begin{bmatrix} S & T/\beta & U/EI_2 & V/EI_b^2 & V/EI_b^3 \\ \beta V & S & T/EI_b & U/EI_2 \\ \beta^2 EI_u & EI_b V & S & T/\beta \end{bmatrix}$$

where

$$\beta = \sqrt{\frac{m \omega^2}{EI}}, \quad S = \frac{ch + c}{2}, \quad T = \frac{sh + s}{2}, \quad U = \frac{ch - c}{2}, \quad V = \frac{sh - s}{2},$$
$$ch = \cosh \beta L, \quad sh = \sinh \beta L, \quad c = \cos \beta L, \quad s = \sin \beta L.$$
A dummy body connecting a 2D beam (78) with a 1D element like (75) or (75) may be described by transfer Equation (31) summing up the input forces \( Q_{y,0} + Q_{y,0} \) through

\[
\begin{bmatrix}
Y \\
\dot{Q}_2 \\
\dot{M}_2 \\
Q_{y,0}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{Y} \\
\dot{Q}_2 \\
\dot{M}_2 \\
Q_{y,0}
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{Y} \\
\dot{Q}_2 \\
\dot{M}_2 \\
Q_{y,0}
\end{bmatrix}.
\tag{80}
\]

while transmitting the other quantities, see Abbas et al.\(^2\) Additionally geometric consistency Equation (32) has to be taken into account which equates displacements \( Y_{\alpha} = Y_0 \) of both inputs:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Y \\
\dot{Q}_2 \\
\dot{M}_2 \\
Q_{y,0}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Y \\
\dot{Q}_2 \\
\dot{M}_2 \\
Q_{y,0}
\end{bmatrix}.
\tag{81}
\]

APPENDIX B: SENSITIVITIES OF ELEMENT TRANSFER MATRICES

The transfer matrix (75) of a lumped mass depends on mass parameter \( m \) and eigenfrequency \( \omega \). The corresponding derivatives are

\[
U^m_{\omega} = \frac{\partial U^m}{\partial \omega} = \begin{bmatrix} 0 & 0 \\ 2m & 0 \end{bmatrix}, \quad U^m_{m} = \frac{\partial U^m}{\partial m} = \begin{bmatrix} 0 & 0 \end{bmatrix}.
\tag{82}
\]

The transfer Equation (77) of a translational spring depends only on stiffness \( k \), which is why the derivative w.r.t. eigenfrequency \( \omega \) vanishes:

\[
U^k_{\omega} = \frac{\partial U^k}{\partial \omega} = 0, \quad U^k_{k} = \frac{\partial U^k}{\partial k} = \begin{bmatrix} 0 & 1/k^2 \end{bmatrix}.
\tag{83}
\]

For the Euler-Bernoulli beam (78) we may first compute the derivatives w.r.t. the secondary parameters (EI), \( \beta \) and \( \beta L \):

\[
U^\beta = \frac{\partial U^\beta}{\partial \beta} = \begin{bmatrix} 0 & 0 & -U/EI \beta^2 & -V/EI \beta^3 \\ 0 & 0 & -T/EI \beta & -U/EI \beta^2 \end{bmatrix},
\]

\[
U^\beta_{\beta} = \frac{\partial U^\beta}{\partial \beta} = \begin{bmatrix} 0 & -T/\beta^2 & -2U/EI \beta^3 & -3V/EI \beta^4 \\ V & 0 & -T/EI \beta^2 & -2U/EI \beta^3 \end{bmatrix},
\]

\[
U^\beta_{\beta L} = \frac{\partial U^\beta}{\partial \beta} = \begin{bmatrix} 0 & 0 & 0 & S/EI \beta^3 \beta L \\ 0 & 0 & 0 & T/EI \beta^2 \beta L \\ 3E \beta \beta L & 2E \beta L^2 & V & 0 \end{bmatrix}.
\tag{84}
\]

For the finally required derivatives w.r.t. the primary parameters we have to use the chain rule with the \( \beta \)-derivatives

\[
\beta_{\omega} = \frac{1}{4} \left( \frac{m \omega^2}{EI} \right)^{-\frac{3}{2}} \cdot \frac{2}{m} = \frac{1}{2 \omega} \left( \frac{m \omega^2}{EI} \right)^{-\frac{1}{2}} = \frac{\beta}{2 \omega},
\]

\[
\beta_m = \frac{\beta}{4m} \cdot \beta_{EL} = -\frac{\beta}{4m E I},
\tag{85}
\]

resulting from the \( \beta = \beta(\omega; m, EI) \) dependence (79). Thus, the derivatives w.r.t. to \( \omega \) and the primary beam parameters \( m, L \) and \( EI \) are obtained as

\[
U^\beta_{\omega} = \frac{\partial U^\beta}{\partial \omega} = \beta U^\beta_{\omega} + U^\beta_{\beta \omega} - \frac{\beta}{2 \omega} \left( U^\beta_{\omega} + L U^\beta_{\beta} \right),
\]

\[
U^\beta_{m} = \frac{\partial U^\beta}{\partial m} = \beta U^\beta_{m} + U^\beta_{\beta m} - \frac{\beta}{4m E I} \left( U^\beta_{\beta m} + L U^\beta_{\beta} \right),
\]

\[
U^\beta_{L} = \frac{\partial U^\beta}{\partial L} = \beta U^\beta_{L} + U^\beta_{\beta L} - \frac{\beta}{4m E I} \left( U^\beta_{\beta L} + L U^\beta_{\beta} \right),
\]

\[
U^\beta_{EI} = \frac{\partial U^\beta}{\partial EI} = \beta U^\beta_{EI} + U^\beta_{\beta EI} - \frac{\beta}{4m E I} \left( U^\beta_{\beta EI} + L U^\beta_{\beta} \right).
\tag{86}
\]