Homophyly and Randomness Resist Cascading Failure in Networks

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The universal properties of power law and small world phenomenon of networks seem unavoidably obstacles for security of networking systems. Existing models never give secure networks. We found that the essence of security is the security against cascading failures of attacks and that nature solves the security by mechanisms. We proposed a model of networks by the natural mechanisms of homophyly, randomness and preferential attachment. It was shown that homophyly creates a community structure, that homophyly and randomness introduce ordering in the networks, and that homophyly creates inclusiveness and introduces rules of infections. These principles allow us to provably guarantee the security of the networks against any attacks. Our results show that security can be achieved provably by structures, that there is a tradeoff between the roles of structures and of thresholds in security engineering, and that power law and small world property are never obstacles for
Network security has become a grand challenge in the current science and technology. We proposed a mathematical definition of network security, and a new model of networks by natural mechanisms of homophyly, randomness and preferential attachment. We found that networks of our model satisfy a serious of new topological, probabilistic and combinatorial principles, and that the new principles ensure that the networks are provably secure. Our model provides a foundation for both theoretical and practical analyses of security of networks. Generally, our model demonstrates that nature may solve security of complex systems by mechanisms, exploring a new principle for networking systems in nature, society, economics, industry and technology etc.

Many real networks satisfy the power law (1–3), and the small world phenomenon (4–6). A surprising discovery in network theory in the first 10 years after the discovery of power law in (1), is perhaps that network topology is universal in nature, society and industry (2). This universality allowed researchers from different disciplines to embrace network theory as a common paradigm. The understanding of networks is a common goal of an unprecedented array of traditional disciplines: For instance, cell biologists use networks to capture signal transduction cascade and metabolism; computer scientists are mapping the Internet and the WWW; epidemiologists follow transmission networks trough which virus spread (2).

From the second decade of network theory, security of networks has become a sharper focus and a grand challenge. We have to understand how the internet responds to attacks and traffic jams, or how the cell reacts to changes in its environments, or how the global economy responses to the current financial crisis, or even how a society reacts to a social crisis. A basic question of this issue is the security of networks.

To understand the essence of security of networks, we examine the two classic models of networks. The first is the Erdős-Rényi model (7, 8). In this model, we are given \( n \) nodes, and
a number $p$, and create an edge with probability $p$ for each pair of nodes. The second is the preferential attachment (PA, for short) model (1). In this model, for a given initial graph, $G_0$ say, and a natural number $d$, we build the network $G$ by steps. Suppose that $G_{t-1}$ is defined. At step $t$, we create a new node, linking to $d$ nodes chosen with probability proportional to the degrees of nodes in $G_{t-1}$.

Security must depend on strategies of attacks. Typical strategies are the physical attack of removal of nodes or edges, and the cascading failures of attacks.

In (9–11), it has been shown that in scale-free networks of the PA model, the overall network connectivity measured by the sizes of the giant connected components and the diameters does not change significantly under random removal of a small fraction of nodes, but is vulnerable to removal of a small fraction of the high degree nodes.

In (6, 12–14), the cascading failure model was proposed to study rumor spreading, disease spreading, voting, and advertising etc. In (15), it has been shown that in scale-free networks of the PA model even weakly virulent virus can spread.

**The Essence of Network Security**

Let $G = (V, E)$ be a network. Suppose that for each node $v \in V$, there is a threshold $\phi(v)$ associated with it. For an initial set $S \subset V$, the infection set of $S$ in $G$ is defined recursively as follows: (1) Each node $x \in S$ is called infected, and (2) A node $x \in V$ becomes infected, if it has not been infected yet, and $\phi(x)$ fraction of its neighbors have been infected. We use $\text{inf}_G(S)$ to denote the infection set of $S$ in $G$.

The cascading failure models depend on the choices of thresholds $\phi(v)$ for all $v$. We consider two natural choices of the thresholds. The first is random threshold cascading, and the second is uniform threshold cascading. We say that a cascading failure model is *random*, if for each node $v$, $\phi(v)$ is defined randomly and uniformly, that is, $\phi(v) = r/d$, where $d$ is the degree of $v$ in $G$, and $r$ is chosen randomly and uniformly from $\{1, 2, \cdots, d\}$. We say that a cascading failure
model is *uniform*, if for each node \( v \), \( \phi(v) = \phi \) for some fixed number \( \phi \).

To understand the nature and essence of security of networks, we compare the two strategies of physical attacks and the cascading failure models of attacks. For this, we introduce the notion of *injury set* of physical attacks. Let \( G = (V, E) \) be a network, and \( S \) be a subset of \( V \). The physical attacks on \( S \) is to delete all nodes in \( S \) from \( G \). We say that a node \( v \) is injured by the physical attacks on \( S \), if \( v \) is not connected to the largest connected component of the graph obtained from \( G \) by deleting all nodes in \( S \). We use \( \text{inj}_G(S) \) to denote the injury set of \( S \) in \( G \).

We depict the curves of sizes of the infection sets and the injury sets of attacks of top degree nodes of networks of the ER and PA models in Figure 1(a), and Figure 1(b) respectively. From the figures, we know that for any network, \( G \) say, generated from either the ER or the PA model, the following properties hold: (1) the infection sets are much larger than the injury sets, (2) the attacks of top degree nodes of size as small as \( O(\log n) \) may cause a constant fraction of nodes of the network to be infected under the cascading failure models of attacks, and (3) structures play a role in security of networks, by observing the difference between Figure 1(a), and Figure 1(b).

The experiments in Figures 1(a) and 1(b) show that the essence of network security is the security against any attacks of sizes polynomial in \( \log n \) under the cascading failure models.

Let \( M \) be a model of networks. We investigate the security of networks of model \( M \). We define the security of networks under both uniform and random threshold cascading failure models.

Let \( G \) be a network of \( n \) nodes constructed from model \( M \). For the random threshold cascading failure model, we say that \( G \) is *secure*, if almost surely (meaning that with probability arbitrarily close to 1 as \( n \) grows), the following holds: for any set \( S \) of size bounded by a polynomial of \( \log n \), the size of the infection set of \( S \) in \( G \) is bounded by \( o(n) \), meaning that it is negligible comparing with \( n \). For the uniform threshold cascading failure model, we say that \( G \) is *secure*, if almost surely, the following holds: for some arbitrarily small \( \phi \), i.e., \( \phi = o(1) \).
for any set $S$ of size bounded by a polynomial of $\log n$, the infection set of $S$ in $G$ with uniform threshold $\phi$ has size $o(n)$.

**Questions and Results**

By the definitions of security of networks, and by the experiments in Figures 1(a) and 1(b), we have that both the ER and the PA models never give secure networks.

Notice that randomness is the mechanism of the ER model, and is in fact the mechanism for the small world property for almost all networks (by observing all other models and real networks), and that preferential attachment is the mechanism of the PA model which guarantees the power law of the networks. The experiments in Figures 1(a) and 1(b) show that neither randomness nor preferential attachment alone is a mechanism for security of networks. This also implies that small world property and power law seem obstacles for security of networks.

The fundamental questions are thus: Are power law and small world property really obstacles for security of networks? What mechanisms and principles can guarantee security of networks? Is there an algorithm to construct secure networks? In this paper, we will answer these questions.

We found that homophyly is a new mechanism of networks, that homophyly guarantees a community structure of networks, that homophyly and randomness introduce ordering in networks and generate a degree priority principle, that homophyly creates inclusiveness and introduces infection rules in networks. These discoveries allow us to give an algorithm based on natural mechanisms of homophyly, randomness or uncertainty and preferential attachment to construct networks such that the networks are provably secure, follow a power law, have the small diameter property, and furthermore, have a navigation algorithm of time complexity $O(\log n)$.

The results show that security can be achieved by structures of networks, that there exists a tradeoff between the role of structure and the role of thresholds in security of networks, and
that neither power law nor small world property is an obstacle of security of networks.

**Security Model**

How can we construct secure networks? Networks are proved universal in a wide range of disciplines in both nature and society. This suggests that natural mechanisms of the evolution of complex systems in nature and society maybe helpful for us to construct secure networks.

Let us consider a mental experiment in evolution of networking systems in nature. Assume that $H$ is the current network. Suppose that a new individual $v$ is born. Then $v$ has its own characteristic from the very beginning of its birth either as a remarkable element or a normal element. If $v$ is born as a remarkable element, then it develops some links to individuals in $H$ by the preferential attachment scheme in the whole $H$, and some links to remarkable elements in $H$ by chance, and $v$ will develop its own community. If $v$ is born as a normal individual, then it is very likely that $v$ joins randomly some group of individuals, in which case, $v$ links to some individuals in that group by a preferential attachment scheme.

Based on this mental experiment, we propose a new model of networks, the security model below.

The security model proceeds as follows: (1) Given a homophyly exponent $a$ and a natural number $d$, let $G_d$ be an initial $d$-regular graph. Each node of $G_d$ is associated with a distinct color and called a seed. For $i > d$, let $G_{i-1}$ be the graph constructed at the end of step $i - 1$. At step $i$, set $p_i = 1/(\log i)^a$. (2) At time step $i$, create a new node $v$. (3) With probability $p_i$, $v$ chooses a new color, $c$ say, in which case: (a) we say that $v$ is a seed node, (b) (PA scheme) add one edge $(v, u)$ such that $u$ is chosen with probability proportional to the degrees among all nodes in $G_{i-1}$, and (c) (Randomness) add $d - 1$ edges $(v, u_j)$ for $j = 1, 2, \cdots, d - 1$, where $u_j$ is chosen randomly and uniformly among all seed nodes in $G_{i-1}$. (4) Otherwise, then $v$ chooses an old color, in which case: (a) (Randomness) $v$ chooses randomly and uniformly an old color, and (b) (Homophyly and PA scheme) create $d$ edges from $v$ to nodes of the same color as $v$ chosen
with probability proportional to the degrees of the nodes in $G_{i-1}$.

Obviously the model is dynamic. The mechanisms of the model are homophyly, randomness (or uncertainty) and preferential attachment. Clearly, each of the three mechanisms is a natural mechanism in evolution of networking systems in nature and society.

**Mathematical Principles**

We will show that networks generated from the security model are secure against any attacks of small-scales under both uniform and random threshold cascading failure models.

The authors have shown that networks of the security model satisfy four groups of topological, probabilistic and combinatorial principles (A. Li, Y. Pan and W. Zhang, Provable security of networks).

Let $a > 1$ be the homophyly exponent, and $d \geq 4$ be a natural number. Let $G = (V, E)$ be a network constructed by our model. Then with probability $1 - o(1)$, $G$ satisfies the following four principles each of which consists of a number of interesting properties.

The first is a fundamental principle, consisting of a number of topological and probabilistic properties: (1) (Basic properties): (i) The number of seed nodes is bounded in the interval $\left[ \frac{n}{2 \log n}, \frac{2n}{\log n} \right]$, and (ii) Each homochromatic set has a size bounded by $O(\log^{a+1} n)$; (2) For degree distributions, we have: (i) The degrees of the induced subgraph of a homochromatic set follow a power law, (ii) The degrees of nodes of a homochromatic set follow a power law, and (iii) (Power law) Degrees of nodes in $V$ follow a power law; (3) For node-to-node distances, we have: (i) The induced subgraph of a homochromatic set has a diameter bounded by $O(\log \log n)$, (ii) (Small world phenomenon) The average node to node distance of $G$ is bounded by $O(\log n)$, and (iii) (Local algorithm for navigating) There is an algorithm to find a short path between arbitrarily given two nodes in time complexity $O(\log n)$; and (4) (Small community phenomenon) There are $1 - o(1)$ fraction of nodes of $G$ each of which belongs to a homochromatic set, $W$, say, such that the conductance of $W$, $\Phi(W)$, is bounded by $O\left(\frac{1}{|W|}\right)$ for $\beta = \frac{a-1}{4(a+1)}$. 


We define a community of $G$ to be the induced subgraph of a homochromatic set. By the fundamental principle, we know that all the communities are small, that $G$ has both a power law local structure and a power law global structure, that $G$ has not only a short diameter, but also a local algorithm of time complexity $O(\log n)$ to find a short path between arbitrarily given two nodes, and that $G$ has a remarkable community structure.

The second is a degree priority principle, consisting of some properties of the degree priority of vertices of $G$.

Let $v$ be a node of $G$. We consider the homochromatic sets of all the neighbors of $v$. We define the length of degrees of $v$ to be the number of colors of the neighbors of $v$, written by $l(v)$. For each $j \in \{1, 2, \cdots, l(v)\}$, let $X_j$ be the $j$-th largest homochromatic set of all the neighbors of $v$ (break ties arbitrarily). We define the $j$-th degree of $v$ to be the size of $X_j$.

Then we have the following degree priority principle:

For a randomly chosen node $v$, with probability $1 - o(1)$, the following properties hold: (1) The length of degrees of $v$ is bounded by $O(\log n)$, (2) The first degree of $v$ is the number of $v$'s neighbors that share the same color as $v$, (3) The second degree of $v$ is bounded by $O(1)$, so that for any possible $j > 1$, the $j$-th degree of $v$ is $O(1)$, and (4) The first degree of a seed node is at least $\Omega(\log^{\frac{a+1}{4}} n)$.

The third one is an infection-inclusion principle, created by homophyly and randomness of the model.

Let $x$ and $y$ be two nodes of $G$. We say that $x$ injures $y$, if the infection of $x$ contributes to the probability that $y$ becomes infected. Otherwise, we say that $x$ fails to injure $y$.

Let $X$ and $Y$ be two homochromatic sets. Suppose that $G_X$ and $G_Y$ are the induced communities by $X$ and $Y$ respectively. Let $x_0$, and $y_0$ be the seed nodes of $X$, and $Y$ respectively. Suppose that $x_0$ and $y_0$ are created at time step $s$ and $t$ respectively. The infection-inclusion principle ensures that with probability $1 - o(1)$, the following properties hold: (1) If $s < t$, then:
(i) community \( G_X \) fails to injure any non-seed node in community \( G_Y \), and (ii) the number of neighbors of the seed node \( y_0 \) that are in \( X \) is bounded by a constant \( O(1) \); and (2) If \( s > t \), then:

(i) all the non-seed nodes in \( G_X \) fail to injure any node in community \( G_Y \), (ii) the number of neighbors of the seed node \( y_0 \) that are in \( X \) is bounded by 1, and (iii) the injury of a non-seed node in \( Y \) from the seed node of \( X \) follows only the edge created by step (3) (b) of the definition of the model.

The infection-inclusion principle shows that homophily creates some inclusiveness among the non-seed nodes, and that a community protects its non-seed members from being arbitrarily injured by the collection of their neighbor communities.

The fourth one is an infection priority tree principle. We define the infection priority tree \( T \) of \( G \) as follows: (i) for each edge \( e = (u, v) \) in \( G \), if \( u \) and \( v \) were created at time steps \( s > t \) respectively, then we interpret the edge \( e = (u, v) \) as a directed edge from \( u \) to \( v \), (ii) let \( H \) be the graph obtained from \( G \) by deleting all edges created by (3) (c) of definition of our model, and (iii) let \( T \) be the graph obtained from \( H \) by merging each of the homochromatic set into a single node, and at the same time, keeping all the directed edges.

We have the following infection priority tree principle: (1) \( T \) is a tree on which the injury directions always going to the early created nodes, and (2) with probability \( 1 - o(1) \), the infection priority tree \( T \) has a height bounded by \( O(\log n) \).

Note that the direction in \( T \) is determined by the injury of a non-seed node from a seed node of a neighbor community as shown in the infection-inclusion principle.

**Proofs of Security**

By combining the four principles together, we are able to prove some security results.

Let \( G \) be a network constructed by our model. By the fundamental principle, all the communities are small, and the number of seed nodes is large. Let \( X \) be a homochromatic set with seed \( x_0 \). Suppose that there is no node in \( X \) which has been targeted. By the degree priority
principle, the first and second degrees of \( x_0 \) is at least \( \Omega(\log^{a+1} n) \), and at most \( O(1) \) respectively. Therefore, the seed node \( x_0 \) of \( G_X \) is hard to be infected by a single neighbor community \( G_Y \), if any. Furthermore, by the same principle, the length of degrees of \( x_0 \) is at most \( O(\log n) \), therefore, for properly chosen \( a \), the seed node \( x_0 \) of \( G_X \) is hard to be infected by the collection of all its neighbor communities.

We say that a community \( G_X \) is strong, if the seed \( x_0 \) of \( X \) can not be infected by the collection of all its neighbor communities alone, and vulnerable, otherwise. The degree priority principle ensures that for properly chosen \( a \), almost all communities are strong.

By definition of the infection priority tree \( T \), and by the infection-inclusion principle, infections among strong communities from its neighbor communities must be triggered by an edge in the infection priority tree \( T \) of \( G \). We further explain this as follows.

Suppose that \( G_X, G_Y \) and \( G_Z \) are strong communities. Let \( x_0, y_0 \) and \( z_0 \) be the seed nodes of \( X, Y \) and \( Z \) respectively. Suppose that \( x_0, y_0 \) and \( z_0 \) are created at time step \( t_1, t_2 \) and \( t_3 \) respectively.

Then it is possible that \( x_0 \) infects a non-seed node \( y_1 \in Y \), \( y_1 \) infects the seed node \( y_0 \) of \( Y \) and \( y_0 \) infects a non-seed node \( z_1 \in Z \). By the infection-inclusion principle, we have that \( t_1 > t_2 > t_3 \), and that the edges \((x_0, y_1)\) and \((y_0, z_1)\) are created by step (3) (b) of the construction of the network so that the edges are embedded in the infection priority tree \( T \) of \( G \).

By the infection priority tree principle, \( T \) is directed with direction always going towards the early created nodes, and \( T \) has height bounded by \( O(\log n) \), with probability \( 1 - o(1) \). Therefore whenever a strong community triggers an infection in the priority tree, it generates at most \( O(\log n) \) many strong communities to be infected, where a community is infected, if at least one node of the community has been infected.

By the fundamental principle, each community has size at most \( O(\log^{a+1} n) \). Therefore an infected community contributes at most \( O(\log^{a+1} n) \) many infected nodes.
By using the ideas above, we estimate the number of infected nodes by an attack of small scales. By the degree priority principle, for properly chosen \( a \), almost all communities are strong. Let \( k \) be the number of vulnerable communities. Then \( k \) must be negligible.

Suppose that \( S \) is a set of nodes of size \( \log^c n \) for some constant \( c \). We attack all the nodes in \( S \). Then there are at most \( |S| + k \) communities each of which triggers an infection in the infection priority tree \( T \). By the infection priority tree principle, the total number of infected communities is at most \( O((|S| + k) \cdot \log n) \). Therefore even if all the nodes of an infected community are infected, the total number of nodes that are infected by attacks on \( S \) is at most \( O((|S| + k) \log^{a+2} n) \), which could be \( o(n) \).

Now the only problem is to estimate the number of vulnerable communities \( k \) which is some probabilistic arguments. In fact, we have shown that: (1) For the uniform threshold cascading model, for \( a > 4 \), and \( d \geq 4 \), let \( G \) be a network constructed by our security model, then with probability \( 1 - o(1) \), the following event occurs: For some \( \phi = o(1) \), for any constant \( c \), and any set \( S \) of vertices of \( G \), if \( S \) has size bounded by \( \log^c n \), then the infection set of \( S \) in \( G \) with uniform threshold \( \phi \) has size \( o(n) \), where \( n \) is the number of vertices of \( G \); and (2) For the random threshold cascading model, for \( a > 6 \), and \( d \geq 4 \), let \( G \) be a network of the security model, then with probability \( 1 - o(1) \), the following event occurs: For any constant \( c \), any set \( S \) of vertices of \( G \), if the size of \( S \) is bounded by \( \log^c n \), then the infection set of \( S \) in \( G \) has size \( o(n) \).

**Experiments**

Our theoretical results require \( a > 4 \) and \( a > 6 \) for the uniform and random threshold cascading models respectively. The reason is the degree priority principle. We know that the lower bound of the first degree of a seed node is \( \Omega(\log^{\frac{a+1}{2}} n) \), which depends on \( a \), and that in the worst case, the degree of the seed node contributed by all its neighbor communities is \( O(\log n) \). To make sure that a community is strong in the case that the threshold of the seed node is
sufficiently small, we have to choose $a$ to be appropriately large. Therefore, if $a$ is too small, then the number of strong communities will be relatively small, in which case, the network of the model will be less secure. However we will show that networks of the security model with small $a$ have much better security than that of the other models.

Our experiments below show that even if just for $a > 1$, the networks of our model are much more secure than that of the ER and PA models. From Figure 2, we have that for any size $n$, for a network of either the PA model or the ER model, the attacks of size only $\log n$ would probably generate a global cascading failure of the network. In sharp contrast to this, for any $n$, any attacks of $\log n$ size are unlikely to generate a global cascading failure of the networks generated from the security model. For the uniform threshold cascading failure model, we compare the security thresholds of networks of the three models. From Figure 3, we have that the curve of the security thresholds of networks of the security model is the lowest, much better than that of the ER and PA models.

In summary, both theoretical analysis and experiments show that networks of our model resist cascading failure of attacks, for which homophyly, randomness and preferential attachment are the underlying mechanisms. Our fundamental principle also shows that networks of the security model follow a power law and satisfy the small world property, and more importantly, allowing a navigation algorithm of time complexity $O(\log n)$. This shows that power law and small world property are never obstacles of security of networks.

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Additional information

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Figure 1: (a), (b) are the curves of cascading failure and injured nodes by nodes removal of networks for $n = 10,000$ and $d = 10$ of the ER and PA models respectively. The red curves are fractions of injury sets of attacks on the top degree nodes of size up to $5 \cdot \log n$, and the curves colored blue are the fractions of the largest infection sets among 100 times of attacks of the top degree nodes of size less than $5 \cdot \log n$ under the random threshold cascading failure model.
Figure 2: Security curves of networks of the Erdős-Rényi model, the preferential attachment model and the security model. In this figure, we consider the case of random thresholds. It depicts the curves of the greatest size of the cascading failure sets of attacks of 100 times of size $\log n$ for each $n$ less than or equal to 10,000. The curves describe the greatest sizes of the final cascading failure sets among 100 times attacks of the random thresholds. The sizes of the initial attacks are always $\log n$, where $n$ is the number of nodes of the networks.

Figure 3: Security curves for initial size $\log n$, $a = 1.5$ and $d = 5$. We consider networks of nodes up to 100,000, each of which has average number of edges $d = 5$. For the security model, we set the homophly exponent $a = 1.5$ for all the networks. This describes the curves of security thresholds of the networks generated from the 3 models of nodes up to 100,000 and of average number of edges $d$ for $d = 5$ respectively. In this experiment, the initial set of attacks are the top degree nodes of size $\log n$. The homophly exponent $a$ is chosen as 1.5 for the security model.
security curves ($|S| = \log(N), a = 1.5, d = 10$)
Degree distribution (N=10000, a=1.5, d=10)

\[ G_1 \]
\[ G_2 \]
\[ H = \mathcal{R}(G_2) \]
cascading vs node attack (ER model: $N = 10000$, $d = 15$)
cascading vs node attack (PA model: $N=10000$, $d=15$)
cascading curves ($N=10000$, $a=1.5$, $d=10$)
cascading curves \((N=10000, a=1.5, d=5)\)