A Simpler Bit-parallel Algorithm for Swap Matching

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Abstract. The pattern matching problem with swaps is to find all occurrences of a pattern in a text while allowing the pattern to swap adjacent symbols. The goal is to design fast matching algorithm that takes advantage of the bit parallelism of bitwise machine instructions. We point out a fatal flaw in the algorithm proposed by Ahmed et al. [The swap matching problem revisited, Theor. Comp. Sci. 2014], which we describe in detail. Furthermore we devise a new swap pattern matching algorithm which is based on the same graph theoretical model as the algorithm by Ahmed et al. (thus still not based on FFT) and we prove its correctness. We also show that an approach using deterministic finite automata cannot achieve similarly efficient algorithms.

1 Introduction

In the Pattern Matching problem with Swaps (Swap Matching, for short), the goal is to find all occurrences of any swap version of a pattern $P$ in a text $T$. Where $P$ and $T$ are strings over an alphabet $\Sigma$ of length $p$ and $t$, respectively. By the swap version of a pattern $P$ we mean a string of symbols created from $P$ by swapping adjacent symbols while ensuring that each symbol is swapped at most once (see Section 2 for formal definitions). The solution of Swap Matching is a set of indices which represent where the swap matches of $P$ in $T$ begin (or, alternatively, end). Swap Matching is an intensively studied problem due to its use in practical applications such as text and music retrieval, data mining, network security and biological computing [7].

Swap Matching was introduced in 1995 as an open problem in non-standard string matching [18]. The first result was reported by Amir et al. [2] in 1997, who provided an $O(tp^3 \log p)$-time solution for alphabets of size 2, while also showing that alphabets of size exceeding 2 can be reduced to size 2 with a little overhead. Amir et al. [5] also came up with solution with $O(t \log^2 p)$ time complexity for some very restrictive cases. Several years later Amir et al. [3] showed that Swap Matching can be solved by an algorithm for the overlap matching achieving the running time of $O(t \log p \log |\Sigma|)$. This algorithm as well as all the previous ones is based on fast Fourier transformation (FFT).

In 2008 Iliopoulos and Rahman [16] came up with the first efficient solution to Swap Matching without using FFT and introduced a new graph theoretic
approach to model the problem. Their algorithm based on bit parallelism runs in $O((t+p) \log p)$ time if the pattern length is similar to the word-size of the target machine. One year later Cantone and Faro [9] presented the Cross Sampling algorithm solving Swap Matching in $O(t)$ time and $O(|\Sigma|)$ space, assuming that the pattern length is similar to the word-size in the target machine. In the same year Campanelli et al. [8] enhanced the Cross Sampling algorithm using notions from Backward directed acyclic word graph matching algorithm and named the new algorithm Backward Cross Sampling. This algorithm also assumes short pattern length. Although Backward Cross Sampling has $O(|\Sigma|)$ space and $O(tp)$ time complexity, which is worse than that of Cross Sampling, it improves the real-world performance.

In 2013 Faro [13] presented a new model to solve Swap Matching using reactive automata and also presented a new algorithm with $O(t)$ time complexity assuming short patterns. The same year Chedid [11] improved the dynamic programming solution by Cantone and Faro [9] which results in simpler algorithms and implementations. In 2014 a minor improvement by Fredriksson and Giaquinta [14] appeared, yielding slightly (at most factor $|\Sigma|$) better asymptotic time complexity (and also slightly worse space complexity) for special cases of patterns. The same year Ahmed et al. [1] took a new look on Swap Matching using ideas from the algorithm by Iliopoulos and Rahman [16]. They devised two algorithms named Smalgo-I and Smalgo-II which both run in $O(t)$ for short pattern, which utilize the so-called graph theoretical approach to Swap Matching. Unfortunately, these algorithms contain a fatal flaw, as we show in this paper.

Another remarkable effort related to Swap Matching is to actually count the number of swaps needed to match the pattern at the location [6]. This is more often studied with an extra operation of changing characters allowed [4,12,17].

**Our Contribution** We describe a fatal flaw in Smalgo-I and Smalgo-II algorithms proposed by Ahmed et al. [1]. The flaw is actually already present in the algorithm introduced by Iliopoulos and Rahman [16].

We further propose a way to employ the graph theoretical approach correctly by designing a new algorithm for Swap Matching. This algorithm has $O((\lceil \frac{p}{w} \rceil |\Sigma| + t) + p)$ time and $O((\lceil \frac{p}{w} \rceil |\Sigma|)$ space complexity where $w$ is the word-size of the machine. We would also like to stress that our solution, as based on the graph theoretic approach, does not use FFT. Therefore, it yields a much simpler non-recursive algorithm allowing bit parallelism and not suffering from the disadvantages of the convolution-based methods. Although our algorithm does not provide better asymptotic bounds than some of the previous results [9,14], we believe that its strength lies in the applications where the alphabet is small and the pattern length is at most the word-size, as it can be implemented using only $7 + |\Sigma|$ CPU registers and few machine instructions. This makes it practical for tasks like DNA sequences scanning. We have prepared an implementation of
both the SMALGO-I and our own algorithm for illustration purposes, which is available for download.\(^1\)

Finally we show that there are instances for which any deterministic finite automaton solving Swap Matching must have at least exponential number of states. This means that a possible approach to Swap Matching by designing a suitable deterministic finite automaton cannot lead to an efficient solution.

This paper is organized as follows. First we introduce all the basic definitions, recall the graph model introduced in [16] and its use for matching in Section 2. Then we describe the algorithms SMALGO-I and SMALGO-II and the flaw in detail in Section 3. We show there an input pattern and a text sequence which cause the algorithms to produce an incorrect output. In Section 4 we show our own algorithm which uses the graph model in a new way. Section 5 contains the proof that Swap Matching cannot be solved efficiently by deterministic finite automata.

2 Basic Definitions and the Graph Theoretical Model

In this section we collect the basic definitions, present the graph theoretical model and show a basic algorithm that solves Swap Matching using the model.

Notations and Basic Definitions We use the word-RAM as our computational model. That means we have access to memory cells of fixed capacity \(w\) (e.g., 64 bits). A standard set of arithmetic and bitwise instructions include And (\&), Or (\|), Left bitwise-shift (\(LShift\)) and Right bitwise-shift (\(RShift\)). Each of the standard operations on words takes single unit of time. The input is read from a read-only part of memory and the output is written to a write-only part of memory. We do not include the input and the output into the space complexity analysis.

A string \(S\) over an alphabet \(\Sigma\) is a finite sequence of symbols from \(\Sigma\) and \(|S|\) is its length. By \(S_i\) we mean the \(i\)-th symbol of \(S\) and we define a substring \(S[i,j] = S_i S_{i+1} \ldots S_j\) for \(1 \leq i \leq j \leq |S|\), and prefix \(S[1,i]\) for \(1 \leq i \leq |S|\). String \(P\) prefix matches string \(T\) \(n\) symbols on position \(i\) if \(P[1,n] = T[i,i+n-1]\).

Next we formally introduce a swapped version of a string.

Definition 1 (Campanelli et al. [8]). A swap permutation for \(S\) is a permutation \(\pi: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}\) such that:

(i) if \(\pi(i) = j\) then \(\pi(j) = i\) (symbols at positions \(i\) and \(j\) are swapped),
(ii) for all \(i, \pi(i) \in \{i-1, i, i+1\}\) (only adjacent symbols are swapped),
(iii) if \(\pi(i) \neq i\) then \(S_{\pi(i)} \neq S_i\) (identical symbols are not swapped).

For a string \(S\) a swapped version \(\pi(S)\) is a string \(\pi(S) = S_{\pi(1)} S_{\pi(2)} \ldots S_{\pi(n)}\) where \(\pi\) is a swap permutation for \(S\).

Now we formalize the version of matching we are interested in.

\(^1\)http://users.fit.cvut.cz/blazeval/gsm.html
Definition 2. Given a text $T = T_1T_2\ldots T_t$ and a pattern $P = P_1P_2\ldots P_p$, the pattern $P$ is said to swap match $T$ at location $i$ if there exists a swapped version $\pi(P)$ of $P$ that matches $T$ at location $i$, i.e., $\pi(P) = T_{[i,i+p-1]}$.

A Graph Theoretic Model The algorithms in this paper are based on a model introduced by Iliopoulos and Rahman [16]. In this section we briefly describe this model.

For a pattern $P$ of length $p$ we construct a labeled graph $P = (V, E, \sigma)$ with vertices $V$, edges $E$, and a vertex labeling function $\sigma : V \rightarrow \Sigma$ (see Fig. 1 for an example). Let $V' = \{m_{r,c} \mid r \in \{-1, 0, 1\}, c \in \{1, 2, \ldots, p\}\}$. For $m_{r,c} \in V'$ we set $\sigma(m_{r,c}) = P_{r+c}$. The $P$-graph now represents all possible swap permutations of the pattern $P$ in the following sense. Vertices $m_{0,j}$ represent ends of prefixes of swapped version which end by a non-swapped symbol. Possible swap of symbols $P_j$ and $P_{j+1}$ is represented by vertices $m_{1,j}$ and $m_{-1,j+1}$. Edges represent symbols which can be consecutive. Each path from column 1 to column $p$ represents a swap pattern and each swap pattern is represented this way.

The following definition gives a graph model equivalent of $P$-graph for ordinary matching.

Definition 3. Let $T$ be a string. The $T$-graph of $T$ is a graph $T = (V, E, \sigma)$ where $V = \{v_i \mid 1 \leq i \leq |T|\}$, $E = \{(v_i, v_{i+1}) \mid 1 \leq i \leq |T-1|\}$ and $\sigma : V \rightarrow \Sigma$ such that $\sigma(v_i) = T_i$.

The next definition assigns to each path a string of labels of vertices along it.
Algorithm 1 The basic matching algorithm (BMA).

**Input**: Labeled directed acyclic graph \( G = (V, E, \sigma) \), set \( Q_0 \subseteq V \) of starting vertices, set \( F \subseteq V \) of accepting vertices, text \( T \), and position \( k \).

1: Let \( D_1' := Q_0 \).
2: for \( i = 1, 2, 3, \ldots, p \) do
3:   Let \( D_i := \{ x \mid x \in D_i', \sigma(x) = T_{k+i-1} \} \).
4:   if \( D_i = \emptyset \) then finish.
5:   if \( D_i \cap F \neq \emptyset \) then we have found a match and finish.
6: Define the next iteration set \( D_{i+1}' \) as vertices which are successors of \( D_i \), i.e.,
   \[ D_{i+1}' := \{ d \in V \mid (v, d) \in E \} \text{ for some } v \in D_i \} \]

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**Definition 4.** For a given \( \Sigma \)-labeled directed acyclic graph \( G = (V, E, \sigma) \) vertices \( s, e \in V \) and a directed path \( f = v_1, v_2, \ldots, v_k \) from \( v_1 = s \) to \( v_k = e \), we call \( S = \sigma(f) = \sigma(v_1)\sigma(v_2)\ldots\sigma(v_k) \in \Sigma^* \) a path string of \( f \).

**Using Graph Theoretic Model for Matching** In this section we describe an algorithm called Basic Matching Algorithm (BMA) which can determine whether there is a swap match of pattern \( P \) in text \( T \) on a position \( k \) using a graph model \( (V, E, \sigma) \), i.e., a labeled directed graph.

To use BMA, the model has to satisfy the following conditions:

- it is a directed acyclic graph,
- \( V = V_1 \cup V_2 \cup \cdots \cup V_p \) (we can divide vertices to columns) such that
- \( E = \{(u, w) \mid u \in V_i, w \in V_{i+1}, 1 \leq i < p \} \) (edges lead to next column), and
- \( Q_0 = V_1 \) are the starting vertices and \( F = V_p \) are the accepting vertices.

BMA is designed to run on any graph which satisfies these conditions. Since \( P \)-graph and \( T \)-graph satisfy these assumptions we can use BMA for \( P_P \) and \( T_T \).

The algorithm runs as follows (see also Algorithm 1). We initialize the algorithm by setting \( D_1' := Q_0 \) (Step 1). \( D_i' \) now holds information about vertices which are the end of some path \( f \) starting in \( Q_0 \) for which \( \sigma(f) \) possibly prefix matches 1 symbol of \( T_{k+k-p} \). To make sure that the path \( f \) represents a prefix match we need to check whether the label of the last vertex of the path \( f \) matches the symbol \( T_k \) (Step 3). If no prefix match is left we did not find a match (Step 4). If some prefix match is left we need to check whether we already have a complete match (Step 5). If the algorithm did not stop it means that we have some prefix match but it is not a complete match yet. Therefore we can try to extend this prefix match by one symbol (Step 6) and check whether it is a valid prefix match (Step 3). Since we extend the matched prefix in each step, we repeat these steps until the prefix match is as long as the pattern (Step 2).

Having vertices in sets is not handy for computing so we present another way to describe this algorithm. We use their characteristic vectors instead.

**Definition 5.** A Boolean labeling function \( I : V \rightarrow \{0, 1\} \) of vertices of \( P_P \) is called a prefix match signal.
Algorithm 2 BMA in terms of prefix match signals

1: Let $I^0(v) := 1$ for each $v \in Q_0$ and $I^0(v) := 0$ for each $v \notin Q_0$.
2: for $i = 0, 1, 2, 3, \ldots, p - 1$ do
3: \hspace{1cm} Filter signals by a symbol $T^k_{i+1}$.
4: \hspace{1cm} if $I(v) = 0$ for every $v \in P_F$ then finish.
5: \hspace{1cm} if $I(v) = 1$ for any $v \in F$ then we have found a match and finish.
6: \hspace{1cm} Propagate signals along the edges.

The algorithm can be easily divided into iterations according to the value of $i$ in Step 2. We denote the value of the prefix match signal in $j$-th iteration as $I^j$ and we define the following operations:

- \textit{propagate signal along the edges}, is an operation which sets $I^j(v) := 1$ if and only if there exists an edge $(u, v) \in E$ with $I^{j-1}(u) = 1$,
- \textit{filter signal by a symbol $x \in \Sigma$}, is an operation which sets $I(v) := 0$ for each $v$ where $\sigma(v) \neq x$,
- \textit{match check}, is an operation which checks whether there exists $v \in F$ such that $I(v) = 1$ and if so reports a match.

With these definitions in hand we can describe BMA in terms of prefix match signals as Algorithm 2. See Fig. 2 for an example of use of BMA to figure out whether $P = \text{acbab}$ swap matches $T = \text{bababc}$ at a position 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig2}
\caption{BMA of $T_{[2,6]} = \text{abcab}$ on a $P$-graph of the pattern $P = \text{acbab}$. The prefix match signal propagates along the dashed edges. Index $j$ above a vertex $v$ represents that $I^j(v) = 1$, otherwise $I^j(v) = 0$.}
\end{figure}

3 Smalgo Algorithms and Why They Cannot Work

In this section we discuss how algorithms SMALGO-I and SMALGO-II introduced by Ahmed et al. [1] work, their flaw, and we also show inputs which cause false positives. However, first we describe the Shift-And algorithm for the standard pattern matching (without swaps) which forms the basis of both algorithms.
**Shift-And Algorithm** The following description is based on [10, Chapter 5] describing the Shift-Or algorithm.

For a pattern $P$ and a text $T$ of length $p$ and $t$, respectively, let $R$ be a bit array of size $p$ and $R^j$ its value after the text symbol $T_j$ has been processed. It contains information about all matches of prefixes of $P$ that end at the position $j$ in the text. For $1 \leq i \leq p$, $R_i^j = 1$ if $P_{[i,j]} = T_{[j-i+1,j]}$ and 0 otherwise. The vector $R^{i+1}$ can be computed from $R^i$ as follows. For each positive $i$ we have $R_i^{i+1} = 1$ if $R_i^j = 1$ and $P_{i+1} = T_{j+1}$, and $R_i^{i+1} = 0$ otherwise. Furthermore, $R_1^{i+1} = 1$ if $P_1 = T_{j+1}$ and 0 otherwise. If $R_p^{i+1} = 1$ then a complete match can be reported.

The transition from $R^i$ to $R^{i+1}$ can be computed very fast as follows. For each $x \in \Sigma$ let $D^x$ be a bit array of size $p$ such that for $1 \leq i \leq p$, $D_i^x = 1$ if and only if $P_i = x$. The array $D^x$ denotes the symbols of the position $x$ in the pattern $P$. Each $D^x$ can be preprocessed before the search. The computation of $R^{i+1}$ is then reduced to three bitwise operations, namely $R^{i+1} = (LShift(R^i) \mid 1) \& D^{T_{j+1}}$. When $R_p^i = 1$, the algorithm reports a match on a position $j = p + 1$.

Before we show how SMALGO-I works, we need one more definition.

**Definition 6.** A degenerate symbol $w$ over an alphabet $\Sigma$ is a nonempty set of symbols from alphabet $\Sigma$. A degenerate string $S$ is a string built over an alphabet of degenerate symbols. We say that a degenerate string $\tilde{P}$ matches a text $T$ at a position $j$ if $T_{j+i-1} \in \tilde{P}_i$ for every $1 \leq i \leq p$.

**Smalgo-I** The SMALGO-I algorithm [1] is a modification of the Shift-And algorithm for Swap Matching. The algorithm uses the graph theoretical model introduced in Section 2.

First let $\tilde{P} = \{P_1, P_2, \ldots, P_x, P_x+1, \ldots, P_{p-1}, P_p\}$ be a a degenerate version of pattern $P$. The symbol in $\tilde{P}$ on position $i$ represents the set of symbols of $P$ which can swap to that position. To accommodate the Shift-And algorithm to match degenerate patterns we need to change the way the $D^x$ masks are defined. For each $x \in \Sigma$ let $\tilde{D}^x_i$ be the bit array of size $p$ such that for $1 \leq i \leq p$, $\tilde{D}^x_i = 1$ if and only if $x \in \tilde{P}_i$.

While a match of the degenerate pattern $\tilde{P}$ is a necessary condition for a swap match of $P$, it is clearly not sufficient. The way the SMALGO algorithms try to fix this is by introducing $P$-mask $P(x_1, x_2, x_3)$ which is defined as $P(x_1, x_2, x_3)_i = 1$ if $i = 1$ or if there exist vertices $u_1, u_2,$ and $u_3$ and edges $(u_1, u_2), (u_2, u_3)$ in $\mathcal{P}_P$ for which $u_2 = m_{x_i}$ for some $r \in \{-1, 0, 1\}$ and $\sigma(u_n) = x_n$ for $1 \leq n \leq 3$, and $P(x_1, x_2, x_3)_i = 0$ otherwise. One $P$-mask called $P(x, x, x)$ is used to represent the $P$-masks for triples $(x_1, x_2, x_3)$ which only contain 1 in the first column.

Now, whenever checking whether $P$ prefix swap matches $T$ $k + 1$ symbols at position $j$ we check for a match of $\tilde{P}$ in $T$ and we also check whether $P(T_{j+k-1}, T_{j+k}, T_{j+k+1})_{k+1} = 1$. This ensures that the symbols are able to swap to respective positions and that those three symbols of the text $T$ are present in some $\pi(P)$. 
With the P-masks completed we initialize $R^1 = 1 \& \tilde{D}^T_1$. Then for every $j = 1$ to $t$ we repeat the following. We compute $R^{j+1}$ as $R^{j+1} = LSO(R^j) \& \tilde{D}^T_{j+1} \& RShift(\tilde{D}^T_{j+2}) \& P(T_j, T_{j+1}, T_{j+2})$, where $LSO$ is defined as $LSO(x) = LShift(x) \mid 1$. To check whether or not a swap match occurred we check whether $R^j_{p-1} = 1$. This is claimed to be sufficient because during the processing we are in fact considering not only the next symbol $T_{j+1}$ but also the symbol $T_{j+2}$.

**Smalgo-II** Smalgo-II algorithm is a derivative of Smalgo-I algorithm. It improves the space complexity from $O(\lceil \frac{p}{\varrho} \rceil (p + |\Sigma|^3))$ to $O(\lceil \frac{p}{\varrho} \rceil (p + |\Sigma|^2))$ for a cost of more complex algorithm. The P-masks in Smalgo-I take $O(\lceil \frac{p}{\varrho} \rceil |\Sigma|^3)$ space and are the main part of its space complexity. This can be improved by making P-masks hold information about paths of length 2 instead of 3. The change makes P-masks take space only $O(\lceil \frac{p}{\varrho} \rceil |\Sigma|^2)$.

In order to compensate for the less powerful P-masks, the algorithm introduces two new types of masks and further checks to filter out (some) invalid matches.

To explain the algorithm in more detail, we first introduce a notion of change. An upward change corresponds to (the BMA) going to vertex $m_{-1,i}$ for some $i$, a downward change corresponds to going to vertex $m_{+1,i}$, and a middle-ward change corresponds to going to vertex $m_{0,i}$.

If a downward change has occurred, then we have to check whether an upward change occurs at the next position. If an upward change has occurred, then we have to check whether a downward or middle-ward change occurs at the next position. The main problem here is how to tell whether the changes actually occur.

To this end, the authors of the algorithm introduce three new types of masks, namely up-masks $up(x,y)$, down-masks $down(x,y)$, and middle-masks $middle(x,y)$, which express whether an upward, a downward, and a middle-ward change can occur at the particular position, respectively, with the endpoints of the edge having labels $x$ and $y$.

The authors of the algorithm now claim that to perform the above checks, it is enough to save the previous down-mask and match its value with current up-mask and $R_j$, or to save the previous up-mask and match its value with current down-mask, middle-mask, and $R_j$, respectively. However, this way in both cases we only check whether the change can occur, not whether it actually occurred. This would lead not only to false positives (as shown in Section 3), but also to false negatives.

Unfortunately, no more details are available about the algorithm in the original paper. The pseudocode of Smalgo-II (which contains numerous errors) performs something different and we include its analysis in the appendix for completeness. Nevertheless, the example presented in the Section 3 still makes the pseudocode (with the small errors corrected) report a false positive.
Table 1. ˜D-masks and P-masks for \( P = abab \). A column \( xyz \) contains values \( P(x, y, z) \).

| i | \( P \) | ˜D\(^a\) | ˜D\(^b\) | aab | aba | baa | abb | bab | bba | bbb |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | \( ab \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | \( ba \) | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 3 | \( ab \) | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 4 | \( ba \) | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2. SMLGO-I algorithm execution for \( P = abab \) and \( T = aaba \). The column \( RD^x \) denotes the values of \( RShift(\tilde{D}^x) \).

| i | \( R^3 \) | LSO(\( R^3 \)) | \( D^a \) | \( RD^a \) | \( P(a, a, b) \) | \( R^2 \) | LSO(\( R^2 \)) | \( D^b \) | \( RD^b \) | \( P(a, b, a) \) | \( R^1 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

The Flaw in the SMLGO Algorithms We shall see that for a pattern \( P = abab \) and a text \( T = aaba \) both algorithms SMLGO-I and SMLGO-II give false positives.

The concept of SMLGO-I and SMLGO-II is based on the assumption that we can find a path in \( \mathcal{P}_P \) by searching for consecutive paths of length 3 (triplets), where each two consecutive share two columns and can partially overlap. However, this only works if the consecutive triplets actually share the two vertices in the common columns. If the assumption is not true then the found substring of the text might not match any swap version of \( P \).

The above input gives such a configuration and therefore the assumption is false. In Tables 1 and 2 we can see the step by step execution of SMLGO-I algorithm on pattern \( P = abab \) and text \( T = aaba \). In Table 2 we see that \( R^3 \) has 1 in the 3-rd row which means that the algorithm reports a pattern match on a position 1. This is a false positive, because it is not possible to swap match the pattern with two \( b \) symbols in the text with only one \( b \) symbol.

The reason behind the false positive match is as follows. The algorithm checks whether the first triplet of symbols \( (a, a, b) \) matches. It can match the swap pattern \( aabb \). Next it checks the second triplet of symbols \( (a, b, a) \), which can match \( baba \). We know that \( baba \) is not possible since it did not appear in the previous check, but the algorithm cannot distinguish them since it checks only the triplets and nothing more. Since each step gave us a positive match the algorithm reports a swap match of the pattern in the text.

In the Fig. 3 we see the two triplets which SMLGO-I assumes have two vertices in common. Even though SMLGO-II checks level changes, it does so just to simulate how SMLGO-I works. Therefore the flaw is also contained in SMLGO-II.
4 Our Algorithm

In this section we will show an algorithm which solves Swap Matching and prove correctness of the new algorithm. We call the algorithm GSM (Graph Swap Matching). GSM uses the graph theoretic model presented in Section 2 and is also based on the Shift-And algorithm from Section 3.

The basic idea of the GSM algorithm is to represent prefix match signals (see Definition 5) from the basic matching algorithm (Section 2) over $P$ in bit vectors. The GSM algorithm represents all signals $I$ in the bitmaps $RX$ formed by three vectors, one for each row. Each time GSM processes a symbol of $T$, it first propagates the signal along the edges, then filters the signal and finally checks for matches. All these operations can be done very quickly thanks to bitwise parallelism.

First, we make the concept of GSM more familiar by presenting a way to interpret the Shift-And algorithm by means of the $T$-graph model and the basic matching algorithm (BMA) from Section 2 to solve the (ordinary) Pattern Matching problem. Then we expand this idea to Swap Matching by using the graph theoretic model.

**Graph Theoretic View of the Shift-And Algorithm** Let $T$ and $P$ be a text and a pattern of lengths $t$ and $p$, respectively. We create the $T$-graph $T_P = (V, E, \sigma)$ of the pattern $P$. We know that the $T$-graph is directed acyclic graph which can be divided into columns $V_i, 1 \leq i \leq p$ (each of them containing one vertex $v_i$) such that the edges lead from $V_j$ to $V_{j+1}$. This means that the $T$-graph satisfies all assumptions of BMA. We apply BMA to $T_P$ to figure out whether $P$ matches $T$ at a position $j$. We get a correct result because for each $i \in \{1, \ldots, p\}$ we check whether $T_{j+i-1} = \sigma(v_i) = P_i$.

To find every occurrence of $P$ in $T$ we would have to run BMA for each position separately. This is basically the naive approach to solve the pattern matching. We can improve the algorithm significantly when we parallelize the computations of $p$ runs of BMA in the following way.

The algorithm processes one symbol at a time starting from $T_1$. We say that the algorithm is in the $j$-th step when a symbol $T_j$ has been processed. BMA represents a prefix match as a prefix match signal $I : V \rightarrow \{0, 1\}$. Its value in the $j$-th step is denoted $I^j$. Since one run of the BMA uses only one column of
the $T$-graph at any time we can use other vertices to represent different runs of the BMA. We represent all prefix match indicators in one vector so that we can manipulate them easily. To do that we prepare a bit vector $R$. Its value in $j$-th step is denoted $R^j$ and defined as $R^j = I^j(v_i)$.

First operation which is used in BMA (propagate signal along the edges) can be done easily by setting the signal of $v_i$ to value of the signal of its predecessor $v_{i-1}$ in the previous step. I.e., for $i \in \{1, \ldots, p\}$ we set $I(v_i) = 1$ if $i = 1$ and $I^j(v_i) = I^{j-1}(v_{i-1})$ otherwise. In terms of $R^j$ this means just $R^j = LSO(R^{j-1})$.

We also need a way to set $I(v_i) := 0$ for each $v_i$ for which $\sigma(v_i) \not= T_{j+1}$ which is another basic BMA operation (filter signal by a symbol). We can do this using the bit vector $D^x$ from Section 3 and taking $R \& D^x$. I.e., the algorithm computes $R^j$ as $R^j = LSO(R^{j-1}) \& D^{j+1}$.

The last BMA operation we have to define is the match detection. We do this by checking whether $R^j = 1$ and if this is the case then a match starting at position $j - p + 1$ occurred.

Our Algorithm for Swap Matching Using the Graph Theoretic Model

Now we are ready to describe the GSM algorithm.

We again let $P_P = (V, E, \sigma)$ be the $P$-graph of the pattern $P$, apply BMA to $P_P$ to figure out whether $P$ matches $T$ at a position $j$, and parallelize $p$ runs of BMA on $P_P$.

Again, the algorithm processes one symbol at a time and it is in the $j$-th step when a symbol $T_j$ is being processed. We again denote the value of the prefix match signal $I : V \rightarrow \{0, 1\}$ of BMA in the $j$-th step by $I^j$. I.e., the semantic meaning of $I^j(m_{r,c})$ is that $I^j(m_{r,c}) = 1$ if there exists a swap permutation $\pi$ such that $\pi(c) = c + r$ and $\pi(P)_{[1, c]} = T_{[i-c+1, j]}$. Otherwise $I^j(m_{r,c}) = 0$.

We want to represent all prefix match indicators in vectors so that we can manipulate them easily. We can do this by mapping the values of $I$ for rows $r \in \{-1, 0, 1\}$ of the $P$-graph to vectors $RU, RM$, and $RD$, respectively. We denote value of the vector $RX \in \{RU, RM, RD\}$ in $j$-th step as $R^x_j$. We define values of the vectors as $RU^j_i = I^j(m_{-1, i})$, $RM^j_i = I^j(m_{0, i})$, and $RD^j_i = I^j(m_{1, i})$, where the value of $I^j(v) = 0$ for every $v \not\in V$.

We define BMA propagate signal along the edges as setting the signal of $m_{r,c}$ to 1 if at least one of its predecessors have signal set to 1. I.e., we set $I^{j+1}(m_{-1, i}) := I^j(m_{-1, i-1})$, $I^{j+1}(m_{0, i}) := I^j(m_{0, i-1}) \& I^j(m_{0, i-1})$, $I^{j+1}(m_{1, i}) := I^j(m_{1, i-1}) \& I^j(m_{1, i-1})$, and $I^{j+1}(m_{1, 1}) := 1$. We can perform the above operation using the operation $LSO(R)$. We obtain the operation propagate signal along the edges for our algorithm in the form $RU^{j+1} := LSO(RU^j)$, $RM^{j+1} := LSO(RM^j \& RU^j)$, and $RD^{j+1} := LSO(RM^j \& RU^j)$.

The operation filter signal by a symbol can be done by first constructing a bit vector $D^x$ for each $x \in \Sigma$ as $D_x^j = 1$ if $x = P_i$ and $D_x^j = 0$ otherwise, and letting $DU^x = LShift(D^x)$, $DM^x = D^x$, and $DD^x = RShift(D^x)$. Then we use these vectors to filter signal by a symbol $x$ by taking $RU^j := RU^j \& LShift(D^x)$, $RM^j := RM^j \& D^x$, and $RD^j := RD^j \& RShift(D^x)$.
Lemma 1. For every $x \in \Sigma$ we have $D_x := 0^x$. Similarly, we define $D_x$ for all $x \in \Sigma$. Proof.

The last operation we define is the match detection. We do this by checking whether $RU_{p}^j = 1$ or $RM_{p}^j = 1$ and if this is the case, then a match starting at a position $j - p + 1$ occurred.

The final GSM algorithm (Algorithm 3) first prepares the D-masks $D_x$ for every $x \in \Sigma$ and initializes $RU^0 := RM^0 := RD^0 := 0$ (Steps 1–3). Then the algorithm computes the value of vectors $RU^j$, $RM^j$, and $RD^j$ for $j \in \{1, \ldots, t\}$ by first using the above formula for signal propagation (Steps 5 and 6) and then the formula for signal filtering (Step 7) and checks whether $RU_{p}^j = 1$ or $RM_{p}^j = 1$ and if this is the case the algorithm reports a match (Step 8).

Correctness of Our Algorithm To ease the notation let us define $R^j(m_{r,c})$ to be $RU_{c}^j$ if $r = -1$, $RM_{c}^j$ if $r = 0$, and $RD_{c}^j$ if $r = 1$. We define $R^j(m_{r,c})$ analogously. Similarly, we define $D_x(m_{r,c})$ as $(LShift(D_x))^c = D_{c-1}^x$ if $r = -1$, $D_{c}^x$ if $r = 0$, and $(RShift(D_x))^c = D_{c+1}^x$ if $r = 1$. By the way the masks $D_x$ are computed on lines 2 and 3 of Algorithm 3, we get the following observation.

Observation 1 For every $m_{r,i} \in V$ and every $j \in \{i, \ldots, t\}$ we have $D_{j}^T(m_{r,i}) = 1$ if and only if $T_j = P_{j+i}$.

Lemma 1. For every $m_{r,i} \in V$ and every $j \in \{i, \ldots, t\}$ if there exists a swap permutation $\pi$ such that $\pi(P)_{[1,i]} = T_{[j-i+1,j]}$ and $\pi(i) = i + r$, then $R^j(m_{r,i}) = 1$.

Proof. We prove the claim by induction on $i$. If $i = 1$ and there is a swap permutation $\pi$ such that $\pi(1) = 1 + r$ and $P_{1+r} = T_j$, then the algorithm sets $R^j(m_{r,1})$ to 1 on line 5 or 6 (recall the definition of LSO). As $P_{1+r} = T_j$, we have $D_{j}^T(m_{r,1}) = 1$ by Observation 1 and, therefore, by line 7, also $R^j(m_{r,1})$.

Now assume that $i > 1$ and that the claim is true for every smaller $i$. Assume that there exists a swap permutation $\pi$ such that $\pi(P)_{[1,i]} = T_{[j-i+1,j]}$ and $\pi(i) = i + r$. By induction hypothesis we have that $R^{j-1}(m_{r',i-1}) = 1$, where $r' = i - 1 - \pi(i-1)$. Since $r$ equals $-1$ if and only if $r'$ equals $+1$ by Definition 1, we have $(r, r') \in \{(-1, 1), (0, -1), (0, 0), (1, -1), (1, 0)\}$. Therefore the algorithm sets
we must have $D_j^i(m_{r,i})$ to 1 on line 5 or 6. Moreover, since $P_{i+r} = T_j$, we have $D^T_i(m_{r,i}) = 1$ by Observation 1 and the algorithm sets $R^i(m_{r,i})$ to 1 on line 7.

**Lemma 2.** For every $m_{r,i} \in V$ and every $j \in \{i, \ldots, t\}$ if $R^j(m_{r,i}) = 1$, then there exists a swap permutation $\pi$ for $P$ such that $\pi(P)[1, i] = T[j - i + 1, j]$ and $\pi(i) = i + r$.

**Proof.** We prove the claim by induction on $i$. If $i = 1$ and $R^j(m_{r,i}) = 1$, then we must have $D^T_i(m_{r,1}) = 1$ and, by Observation 1, also $P_{1+r} = T_j$. We obtain $\pi$ by setting $\pi(1) = 1 + r$, $\pi(2) = 2 - r$ and $\pi(i') = i'$ for every $i' \in \{2, \ldots, p\}$. It is easy to verify that this is a swap permutation for $P$ and has the desired properties.

Now assume that $i > 1$ and that the claim is true for every smaller $i$. Assume that $R^j(m_{r,i}) = 1$. Then, due to line 7 we must have $D^T_i(m_{r,i}) = 1$ and, hence, by Observation 1, also $P_{i+r} = T_j$. Moreover, we must have $R^j(m_{r,i}) = 1$ and, hence, by lines 5 and 6 of the algorithm also $R^{j-1}(m_{r',i-1}) = 1$ for some $r'$ with $(r, r') \in \{(-1, 1), (0, -1), (0, 0), (1, -1), (1, 0)\}$. By induction hypothesis there exists a swap permutation $\pi'$ for $P$ such that $\pi'(P)[1, i-1] = T[j - i + 1, j-1]$ and $\pi'(i - 1) = i - 1 + r'$. If $\pi'(i) = i + r$, then setting $\pi = \pi'$ finishes the proof. Otherwise we have either $r = 0$ or $r = 1$ and $i < p$. In the former case we let $\pi(i') = i'$ for every $i' \in \{i, \ldots, p\}$ and in the later case we let $\pi(i) = i + 1$, $\pi(i + 1) = i$ and $\pi(i') = i'$ for every $i' \in \{i + 2, \ldots, p\}$. In both cases we let $\pi(i') = \pi'(i')$ for every $i' \in \{1, \ldots, i - 1\}$. It is again easy to verify that $\pi$ is a swap permutation for $P$ with the desired properties.

**Theorem 2.** The GSM algorithm is correct.

**Proof.** By Lemma 1 if there is a swap match of $P$ on position $j - p + 1$ in $T$, then $R^j(m_{p,-1}) = 1$ or $R^j(m_{p,0}) = 1$. In either case the algorithm reports a match on position $j - p + 1$.

On the other hand, if the algorithm reports a match on position $j - p + 1$, then $R^j(m_{p,-1}) = 1$ or $R^j(m_{p,0}) = 1$. But then, by Lemma 2 there is a swap match of $P$ on position $j - p + 1$ in $T$.

and express the complexity of the algorithm.

**Theorem 3.** The GSM algorithm runs in $O\left(\left\lceil \frac{p}{w} \right\rceil |\Sigma| + t \right) + p$ time and uses $O\left(\left\lceil \frac{p}{w} \right\rceil |\Sigma|\right)$ memory cells (not counting the input and output cells), where $t$ is the length of the input text, $p$ length of the input pattern, $w$ is the word-size of the machine, and $|\Sigma|$ size of the alphabet.$^2$

**Proof (of Theorem 3).** The initialization of $RX$ and $D_x$ masks (lines 1 and 2) takes $O\left(\left\lceil \frac{p}{w} \right\rceil |\Sigma|\right)$ time. The bits in $D_x$ masks are set according to the pattern in $O(p)$ time (line 3). The main cycle of the algorithm (lines 4–8) makes $t$ iterations. Each iteration consists of computing values of $RX$ in 13 bitwise operations, i.e.,

$^2$ To simplify the analysis, we assume that log $t < w$, i.e., the iteration counter fits into one memory cell.
in $O\left(\lceil \frac{p}{w} \rceil \right)$ machine operations, and checking for the result in $O(1)$ time. This gives $O\left(\lceil \frac{p}{w} \rceil (|\Sigma| + t) + p \right)$ time in total. The algorithm saves 3 $RX$ masks (using the same space for all $j$ and also for $RX'$ masks), $|\Sigma|$ $D^x$ masks, and constant number of variables for other uses (iteration counters, temporary variable, etc.). Thus, in total the GSM algorithm needs $O\left(\lceil \frac{p}{w} \rceil |\Sigma| \right)$ memory cells.

**Corollary 1.** If $p = cw$ for some constant $c$, then the GSM algorithm runs in $O(|\Sigma| + p + t)$ time and has $O(|\Sigma|)$ space complexity. Moreover, if $p \leq w$, then the GSM algorithm can be implemented using only $7 + |\Sigma|$ memory cells.

**Proof (of Corollary 1).** The first part follows directly from Theorem 3. Let us show the second part. We need $|\Sigma|$ cells for all $D$-masks, 3 cells for $R$ vectors (reusing the space also for $R'$ vectors), one pointer to the text, one iteration counter, one constant for the match check and one temporary variable for the computation of the more complex parts of the algorithm. Altogether, we need only $7 + |\Sigma|$ memory cells to run the GSM algorithm.

From the space complexity analysis we see that for some sufficiently small alphabets (e.g. DNA sequences) the GSM algorithm can be implemented in practice using solely CPU registers with the exception of text which has to be loaded from the RAM.

## 5 Limitations of the Finite Deterministic Automata Approach

It is easy to construct a non-deterministic finite automaton that solves Swap Matching. An alternative approach would thus be to determinize and execute this automaton. The drawback is that the determinization process may lead to an exponential number of states. We show that in some cases it actually does, contradicting the conjecture of Holub [15], stating that the number of states of this determinized automaton is $O(p)$.

**Theorem 4.** There is an infinite family $F$ of patterns such that any deterministic finite automaton $A_P$ accepting the language $L_S(P) = \{ u\pi(P) \mid u \in \Sigma^*, \pi \text{ is a swap permutation for } P \}$ for $P \in F$ has $2^{\Omega(|P|)}$ states.

**Table 3.** An example of the construction from proof of Theorem 4 for $k = 2$. Suppose we have $i = 0$ and $j = 1$. Then $T_0 = acabcabc$, $T_1 = acabcbac$, $T_0' = acabcabcabc$, $T_1' = acabacbacabc$ and the suffixes are $X = bacabcabc$ and $Y = acabcabc$. 

$$
\begin{array}{c|c|c|c}
P & T_0 & T_1 & T_0' \\
\hline
acabcabc & acabcbac & acabcabcabc \\
acabcbac & acabcbacabc \\
acabacbc & acabacbcabc \\
acabacbc & acabacbcabc \\
\end{array}
$$
Proof. For any integer $k$ we define the pattern $P_k := ac(abc)^k$. Note that the length of $P_k$ is $\Theta(k)$. Suppose that the automaton $A_P$ recognizing language $L(P)$ has $s$ states such that $s < 2^k$. We define a set of strings $T_0, \ldots , T_{2^k-1}$ where $T_i$ is defined as follows. Let $b_{k-1}, b_{k-2} \ldots b_0$ be the binary representation of the number $i$. Let $B_j^i = abc$ if $b_j^i = 0$ and let $B_j^i = bac$ if $b_j^i = 1$. Then, let $T_i := acB_{k-1}b_{k-2} \ldots b_0$. See Table 3 for an example. Since $s < 2^k$, there exist $0 \leq i < j \leq 2^k - 1$ such that both $T_i$ and $T_j$ are accepted by the same accepting state of the automaton $A$. Let $m$ be the minimum number such that $b_{k-1-m}^i \neq b_{k-1-m}^j$. Note that $b_m^i = 0$ and $b_m^j = 1$. Now we define $T_i' = T_i(abc)^{(m+1)}$ and $T_j' = T_j(abc)^{(m+1)}$. Let $X = (T_i'[3(m+1)+1,3(m+1+k)+2]$ and $Y = (T_j'[3(m+1)+1,3(m+1+k)+2]$ be the suffixes of the strings $T_i'$ and $T_j'$ both of length $3k + 2$. Note that $X$ begins with $bc \ldots$ and $Y$ begins with $ac \ldots$ and that block $abc$ or $bac$ repeats for $k$ times in both. Therefore pattern $P$ swap matches $Y$ and does not swap match $X$. Since for the last symbol of both $T_i$ and $T_j$ the automaton is in the same state $q$, the computation for $T_i'$ and $T_j'$ must end in the same state $q'$. However as $X$ should not be accepted and $Y$ should be accepted we obtain contradiction with the correctness of the automaton $A$. Hence, we may define the family $F$ as $F = \{P_1, P_2, \ldots \}$, concluding the proof.

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A Appendix: Analysis of the Pseudocode of SMALGO-II

In this section we analyze the pseudocode of the SMALGO-II algorithm as given by Ahmed et al. in [1] in order to understand the meaning of the checks the pseudocode actually performs.

The original pseudocode is as follows.

Algorithm 4 SMALGO-II

| Require: Text T, up-mask up, down-mask down, middle-mask middle. |
| P-mask pmask, D-mask D for given pattern p |
| 1: \( R_0 \leftarrow 2^{\text{pattern length} - 1} \) |
| 2: checkup \( \leftarrow \) checkdown \( \leftarrow 0 \) |
| 3: \( R_0 \leftarrow R_0 \& D_T_0 \) |
| 4: \( R_1 \leftarrow R_0 >> 1 \) |
| 5: for \( j = 0 \) to \( (n - 2) \) do |
| 6: \( R_j \leftarrow R_j \& \ p\text{mask}(T_j,T_{j+1}) \& D_{T_{j+1}} \) |
| 7: checkup \( \leftarrow \ \text{prevcheckup} >> 1 \) |
| 8: checkup \( \leftarrow \ \text{checkup} \ | \ \text{up}(T_j,T_{j+1}) \) |
| 9: checkup \( \leftarrow \ \text{checkup} \ & \sim \ \text{down}(T_j,T_{j+1}) \ & \sim \ \text{middle}(T_j,T_{j+1}) \) |
| 10: prevcheckup \( \leftarrow \ \text{checkup} \) |
| 11: \( R_j \leftarrow \sim (\text{temp} \& \text{checkup}) \& R_j \) |
| 12: temp \( \leftarrow \ \text{prevcheckdown} >> 1 \) |
| 13: checkdown \( \leftarrow \ \text{checkdown} \ | \ \text{down}(T_j,T_{j+1}) \) |
| 14: checkdown \( \leftarrow \ \text{checkdown} \ & \sim \ \text{up}(T_j,T_{j+1}) \) |
| 15: prevcheckdown \( \leftarrow \ \text{checkdown} \) |
| 16: \( R_j \leftarrow \sim (\text{temp} \& \text{checkdown}) \& R_j \) |
| 17: if \( (R_j \& 1) = 1 \) then |
| 18: Match found ending at position \( (j - 1) \) |
| 19: \( R_{j+1} \leftarrow R_j >> 1 \) |
| 20: checkup \( \leftarrow \ \text{checkup} >> 1 \) |
| 21: checkdown \( \leftarrow \ \text{checkdown} >> 1 \) |

The pseudocode has several problems. First, in the first iteration of the cycle, the algorithm uses the value of the variable \( \text{prevcheckup} \) which was never initialized. Second, the algorithm never adds new ones to the variable \( R \) and, hence, can never report a match after position \( \text{pattern length} \) of the text. Third, if the text is of the same length as the pattern, the algorithm only applies the shift \( \text{pattern length} - 2 \) times to the original value of \( 2^{\text{pattern length} - 1} \) (note that in the first iteration it uses \( R_0 \) and overwrites the value of \( R_1 \)) before the last match check. Therefore, at the last check, the value could only drop to \( 2^{\text{pattern length} - 1 - \text{pattern length} + 2} = 2^1 = 2 \) and the match check cannot be successful. Also the reported position of the match does not make much sense.

Let us first correct all this easy problems.
Algorithm 5 SMALGO-II

1: $R_0 \leftarrow 2^{\text{pattern length}} - 1$
2: prevcheckup $\leftarrow$ prevcheckdown $\leftarrow$ checkup $\leftarrow$ checkdown $\leftarrow 0$
3: $R_0 \leftarrow R_0 \& D_{T_0}$
4: $R_1 \leftarrow R_0 >> 1$
5: for $j = 0$ to $(n - 2)$ do
6: $R_{j+1} \leftarrow R_{j+1} \& \text{pmask}(T_j, T_{j+1}) \& D_{T_{j+1}}$
7: temp $\leftarrow$ prevcheckup $\gg 1$
8: checkup $\leftarrow$ checkup $\uparrow \text{up}(T_j, T_{j+1})$
9: checkup $\leftarrow$ checkup $\& \sim \text{down}(T_j, T_{j+1})$ $\& \sim \text{middle}(T_j, T_{j+1})$
10: prevcheckup $\leftarrow$ checkup
11: $R_{j+1} \leftarrow \sim (\text{temp} \& \text{checkup}) \& R_{j+1}$
12: temp $\leftarrow$ precheckdown $\gg 1$
13: checkdown $\leftarrow$ checkdown $\downarrow \text{down}(T_j, T_{j+1})$
14: checkdown $\leftarrow$ checkdown $\& \sim \text{up}(T_j, T_{j+1})$
15: prevcheckdown $\leftarrow$ checkdown
16: $R_{j+1} \leftarrow \sim (\text{temp} \& \text{checkdown}) \& R_{j+1}$
17: if $(R_{j+1} \& 1) = 1$ then
18: Match found ending at position $(j + 1)$
19: $R_{j+2} \leftarrow (R_{j+1} >> 1) | 2^{\text{pattern length}} - 1$
20: checkup $\leftarrow$ checkup $\gg 1$
21: checkdown $\leftarrow$ checkdown $\gg 1$

If we now move the line setting prevcheckup to checkup after the line where the check with the temp variable is performed and similarly with prevcheckdown, we do not need the temp variable anymore. We also move the shifts of checkup and checkdown closer to where this variables are used. We only show the important part of the algorithm.

Algorithm 6 SMALGO-II

5: for $j = 0$ to $(n - 2)$ do
6: $R_{j+1} \leftarrow R_{j+1} \& \text{pmask}(T_j, T_{j+1}) \& D_{T_{j+1}}$
7: checkup $\leftarrow$ checkup $\uparrow \text{up}(T_j, T_{j+1})$
8: checkup $\leftarrow$ checkup $\& \sim \text{down}(T_j, T_{j+1})$ $\& \sim \text{middle}(T_j, T_{j+1})$
9: $R_{j+1} \leftarrow \sim (\text{prevcheckup} \gg 1 \& \text{checkup}) \& R_{j+1}$
10: prevcheckup $\leftarrow$ checkup
11: checkup $\leftarrow$ checkup $\gg 1$
12: checkdown $\leftarrow$ checkdown $\downarrow \text{down}(T_j, T_{j+1})$
13: checkdown $\leftarrow$ checkdown $\& \sim \text{up}(T_j, T_{j+1})$
14: $R_{j+1} \leftarrow \sim (\text{prevcheckdown} \gg 1 \& \text{checkdown}) \& R_{j+1}$
15: prevcheckdown $\leftarrow$ checkdown
16: checkdown $\leftarrow$ checkdown $\gg 1$
17: if $(R_{j+1} \& 1) = 1$ then
18: Match found ending at position $(j + 1)$
19: $R_{j+2} \leftarrow (R_{j+1} >> 1) | 2^{\text{pattern length}} - 1$
Now we swap the order of setting \texttt{prevcheckup} to \texttt{checkup} and the shift of \texttt{checkup}. As this makes \texttt{prevcheckup} shifted by one, we remove the additional shift in the check. Similarly for \texttt{checkdown}.

\textbf{Algorithm 7} SMALGO-II  

\begin{verbatim}
7: \texttt{checkup} ← \texttt{checkup} | \texttt{up}(T_j, T_{j+1})
8: \texttt{checkup} ← \texttt{checkup} & \sim \texttt{down}(T_j, T_{j+1}) & \sim \texttt{middle}(T_j, T_{j+1})
9: R_{j+1} ← (\texttt{prevcheckup} & \texttt{checkup}) & R_{j+1}
10: \texttt{checkup} ← \texttt{checkup} >> 1
11: \texttt{prevcheckup} ← \texttt{checkup}
12: \texttt{checkdown} ← \texttt{checkdown} | \texttt{down}(T_j, T_{j+1})
13: \texttt{checkdown} ← \texttt{checkdown} & \sim \texttt{up}(T_j, T_{j+1})
14: R_{j+1} ← (\texttt{prevcheckdown} & \texttt{checkdown}) & R_{j+1}
15: \texttt{checkdown} ← \texttt{checkdown} >> 1
16: \texttt{prevcheckdown} ← \texttt{checkdown}
\end{verbatim}

Now we institute \texttt{checkup} into the check and move its computation after the check.

\textbf{Algorithm 8} SMALGO-II  

\begin{verbatim}
6: R_{j+1} ← R_{j+1} & \texttt{pmask}(T_j, T_{j+1}) & D_{T_{j+1}}
7: R_{j+1} ← (\texttt{prevcheckup} & (\texttt{checkup} | \texttt{up}(T_j, T_{j+1})) & \sim \texttt{down}(T_j, T_{j+1}) & \sim \texttt{middle}(T_j, T_{j+1}) & R_{j+1}
8: \texttt{checkup} ← (\texttt{checkup} | \texttt{up}(T_j, T_{j+1})) & \sim \texttt{down}(T_j, T_{j+1}) & \sim \texttt{middle}(T_j, T_{j+1})
9: \texttt{checkup} ← \texttt{checkup} >> 1
10: \texttt{prevcheckup} ← \texttt{checkup}
11: R_{j+1} ← (\texttt{prevcheckdown} & (\texttt{checkdown} | \texttt{down}(T_j, T_{j+1})) & \sim \texttt{up}(T_j, T_{j+1}) & R_{j+1}
12: \texttt{checkdown} ← (\texttt{checkdown} | \texttt{down}(T_j, T_{j+1}) & \sim \texttt{up}(T_j, T_{j+1})
13: \texttt{checkdown} ← \texttt{checkdown} >> 1
14: \texttt{prevcheckdown} ← \texttt{checkdown}
\end{verbatim}

Now note that during the check, the content of \texttt{prevcheckup} is exactly the same as the content of \texttt{checkup}, so we can remove \texttt{prevcheckup} completely.
Algorithm 9 SMALGO-II

1: \( R_0 \leftarrow 2^{\text{pattern length} - 1} \)
2: checkup \( \leftarrow \) checkdown \( \leftarrow 0 \)
3: \( R_0 \leftarrow R_0 \& D_T_0 \)
4: \( R_1 \leftarrow R_0 >> 1 \)
5: \( \text{for } j = 0 \text{ to } (n-2) \text{ do} \)
6: \( R_{j+1} \leftarrow R_{j+1} \& \text{pmask}_{(T_j, T_{j+1})} \& D_{T_{j+1}} \)
7: \( R_{j+1} \leftarrow \lnot (\text{checkup} \& (\text{up}_{(T_j, T_{j+1})} \& \text{down}_{(T_j, T_{j+1})} \& \text{middle}_{(T_j, T_{j+1})}) \& R_{j+1} \)
8: \( \text{checkup} \leftarrow (\text{checkup} \& \text{up}_{(T_j, T_{j+1})}) \& \text{down}_{(T_j, T_{j+1})} \& \text{middle}_{(T_j, T_{j+1})} \)
9: \( \text{checkup} \leftarrow \text{checkup} >> 1 \)
10: \( R_{j+1} \leftarrow \text{checkdown} \& (\text{checkdown} \& \text{down}_{(T_j, T_{j+1})}) \& \text{up}_{(T_j, T_{j+1})} \& R_{j+1} \)
11: \( \text{checkdown} \leftarrow (\text{checkdown} \& \text{down}_{(T_j, T_{j+1})} \& \text{up}_{(T_j, T_{j+1})} \)
12: \( \text{checkdown} \leftarrow \text{checkdown} >> 1 \)
13: \( \text{if } (R_{j+1} \& 1) = 1 \text{ then} \)
14: \( \text{Match found ending at position } (j + 1) \)
15: \( R_{j+2} \leftarrow (R_{j+1} >> 1) \mid 2^{\text{pattern length} - 1} \)

Now we modify the expressions by laws of logic to arrive at the following formulation.

Algorithm 10 SMALGO-II

7: \( R_{j+1} \leftarrow R_{j+1} \& (\lnot \text{checkup} \mid \text{down}_{(T_j, T_{j+1})} \& \text{middle}_{(T_j, T_{j+1})}) \)
8: \( \text{checkup} \leftarrow (\text{checkup} \& \text{down}_{(T_j, T_{j+1})} \& \text{middle}_{(T_j, T_{j+1})}) \& (\text{up}_{(T_j, T_{j+1})} \& \text{down}_{(T_j, T_{j+1})} \& \text{middle}_{(T_j, T_{j+1})}) \)
9: \( \text{checkup} \leftarrow \text{checkup} >> 1 \)
10: \( R_{j+1} \leftarrow R_{j+1} \& (\lnot \text{checkdown} \mid \text{up}_{(T_j, T_{j+1})}) \)
11: \( \text{checkdown} \leftarrow (\text{checkdown} \& \text{up}_{(T_j, T_{j+1})}) \mid (\text{down}_{(T_j, T_{j+1})} \& \text{up}_{(T_j, T_{j+1})}) \)
12: \( \text{checkdown} \leftarrow \text{checkdown} >> 1 \)

Now, if the first subexpression in the logical or setting the new value of checkup is true, then the appropriate bit of \( R_{j+1} \) was just set to 0 on the previous line and filtrating this bit again in future is useless. Hence, we can omit this part of the expression. We arrive at the following resulting pseudocode.
Algorithm 11 SMALGO-II

1: \( R_0 \leftarrow 2^{\text{pattern length} - 1} \)
2: \( \text{checkup} \leftarrow \text{checkdown} \leftarrow 0 \)
3: \( R_0 \leftarrow R_0 \& D_T_0 \)
4: \( R_1 \leftarrow R_0 \gg 1 \)
5: \( \text{for } j = 0 \text{ to } (n - 2) \text{ do} \)
6: \( R_{j+1} \leftarrow R_{j+1} \& \text{pmask}(T_j, T_{j+1}) \& D_T_{j+1} \)
7: \( R_{j+1} \leftarrow R_{j+1} \& (\sim \text{checkup} \mid \text{down}(T_j, T_{j+1}) \mid \text{middle}(T_j, T_{j+1})) \)
8: \( \text{checkup} \leftarrow \text{up}(T_j, T_{j+1}) \& \sim \text{down}(T_j, T_{j+1}) \& \sim \text{middle}(T_j, T_{j+1}) \)
9: \( \text{checkup} \leftarrow \text{checkup} \gg 1 \)
10: \( R_{j+1} \leftarrow R_{j+1} \& (\sim \text{checkdown} \mid \text{up}(T_j, T_{j+1})) \)
11: \( \text{checkdown} \leftarrow \text{down}(T_j, T_{j+1}) \& \sim \text{up}(T_j, T_{j+1}) \)
12: \( \text{checkdown} \leftarrow \text{checkdown} \gg 1 \)
13: \( \text{if } (R_{j+1} \& 1) = 1 \text{ then} \)
14: \( \text{Match found ending at position } (j + 1) \)
15: \( R_{j+2} \leftarrow (R_{j+1} \gg 1) \mid 2^{\text{pattern length} - 1} \)

Now it is easy to see, that \text{checkup} stores the information on whether an \text{upward-change} must have occurred in the previous step (provided that there was a prefix match) and this is compared with the information whether \text{downward-change} or \text{middle-change} can occur. Similarly for the \text{downward-change}. This is not sufficient to avoid false positives since sometimes both \text{upward-change} and \text{downward-change} can occur (e.g., as in our counterexample), in which case no filtration is performed at all.