Functional Itô Calculus, Path-dependence and
the Computation of Greeks

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Abstract

Dupire’s functional Itô calculus provides an alternative approach to the
classical Malliavin calculus for the computation of sensitivities, also called
Greeks, of path-dependent derivatives prices. In this paper, we introduce
a measure of path-dependence of functionals within the functional Itô cal-
culus framework. Namely, we consider the Lie bracket of the space and
time functional derivatives, which we use to classify functionals according
to their degree of path-dependence. We then revisit the problem of efficient
numerical computation of Greeks for path-dependent derivatives using in-
tegration by parts techniques. Special attention is paid to path-dependent
functionals with zero Lie bracket, called weakly path-dependent functionals
in our classification. We then derive the weighted-expectation formulas for
their Greeks, that was first derived using Malliavin calculus. In the more
general case of fully path-dependent functionals, we show that, equipped
with the functional Itô calculus, we are able to analyze the effect of the Lie
bracket on computation of Greeks. This was not achieved using Malliavin
calculus. Numerical examples are also provided.

1 Introduction

The theory of functional Itô calculus introduced in Dupire’s seminal paper [6]
extends the Itô stochastic calculus to functionals of the current history of a given

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process, and hence provides an excellent tool to the study of path-dependence. Further work extending this theory and its applications can be found in [1, 2, 3, 4, 7, 8, 13, 14, 16, 20, 23, 24].

We intuitively understand path-dependence of a functional as the study of its changes when we vary the history of the underlying path. Here we propose a measure of path-dependence of a functional given by the Lie bracket of the space and time functional derivatives defined in [6]. In some sense, this is a instantaneous measure of path-dependence, since we consider only path perturbation at the current time. We then classify functionals based on this measure. Moreover, we study the relation of what we called weakly path-dependent functionals and the Monte Carlo computation of Greeks in a local volatility model cf. [10].

Malliavin calculus was successfully applied to derive these Monte Carlo procedures to compute Greeks of path-dependent derivatives, see for example [9, 10, 11, 12, 17, 18]. However, the theory presented here allows us to extend these Monte Carlo procedures to a wider class of path-dependent derivatives provided that the path-dependence is not too severe. This will be made precise in Section 3. We will also see that the functional Itô calculus can be used to derive the weighted-expectation formulas shown in [10].

Furthermore, unlike the Malliavin calculus approach, we are also able to provide a formula for the Delta of functionals with more severe path-dependence, here called strongly path-dependent. In its current form, this formula enhances the understanding of the weights for different cases of path-dependence, although it is not as computationally appealing as the ones derived for weakly path-dependent functionals. It shows however the impact that the Lie bracket has on the Delta of a derivative contract.

Our main contribution is the introduction of a measure of path-dependence and the application of such measure to the computation of Greeks for path-dependent derivatives.

The paper is organized as follows. In Section 2, we give some background on functional Itô calculus. Section 3 introduces the measure of path-dependence and a classification of functionals accordingly to this measure. Finally, we present applications of this measure of path-dependence to the computation of Greeks in Section 4. Two numerical examples related to Asian options are discussed.
2 A Primer on Functional Itô Calculus

In this section we will present the functional Itô calculus first introduced in [6]. We focus on the necessary definitions and results for Sections 3 and 4.

The space of bounded càdlàg paths in $[0,t]$ will be denoted by $\Lambda_t$. We also fix a time horizon $T > 0$. The space of paths is then defined as

$$\Lambda = \bigcup_{t \in [0,T]} \Lambda_t.$$  

We will denote elements of $\Lambda$ by upper case letters and often the final time of its domain will be subscripted, e.g. $Y \in \Lambda_t \subset \Lambda$ will be denoted by $Y_t$. Note that, for any $Y \in \Lambda$, there exists only one $t$ such that $Y \in \Lambda_t$, and hence there is no confusion with this notation. The value of $Y_t$ at a specific time will be denoted by lower case letter: $y_s = Y_t(s)$, for any $s \leq t$. Moreover, if a path $Y_t$ is fixed, the path $Y_s$, for $s \leq t$, will denote the restriction of the path $Y_t$ to the interval $[0,s]$.

The following important path operations are always defined in $\Lambda$. For $Y_t \in \Lambda$ and $t \leq s \leq T$, the flat extension of $Y_t$ up to time $s \geq t$ is defined as

$$Y_{t,s-t}(u) = \begin{cases} y_u, & \text{if } 0 \leq u \leq t, \\ y_t, & \text{if } t \leq u \leq s, \end{cases}$$

see Figure 1. For $h \in \mathbb{R}$, the bumped path $Y_t^h$, see Figure 2, is defined by

$$Y_t^h(u) = \begin{cases} y_u, & \text{if } 0 \leq u < t, \\ y_t + h, & \text{if } u = t. \end{cases}$$

Figure 1: Flat extension of a path.  
Figure 2: Bumped path.

For any $Y_t, Z_s \in \Lambda$, where it is assumed without loss of generality that $s \geq t$, we define the following metric in $\Lambda$,

$$d_\Lambda(Y_t, Z_s) = \|Y_{t,s-t} - Z_s\|_\infty + |s - t|.$$
where
\[ \|Y_t\|_\infty = \sup_{u \in [0,t]} |y_u|. \]

A functional is any function \( f : \Lambda \rightarrow \mathbb{R} \) and it is said \( \Lambda \)-continuous if it is continuous with respect to the metric \( d_\Lambda \).

Moreover, for a functional \( f \) and a path \( Y_t \) with \( t < T \), if the following limit exists, the \textit{time derivative} of \( f \) at \( Y_t \) is defined as
\[ \Delta_t f(Y_t) = \lim_{\delta t \to 0^+} \frac{f(Y_{t+\delta t}) - f(Y_t)}{\delta t}. \]

The \textit{space derivative} of \( f \) at \( Y_t \) is defined as
\[ \Delta_x f(Y_t) = \lim_{h \to 0} \frac{f(Y_{t+h}) - f(Y_t)}{h}, \]
when this limit exists, and for this derivative it is allowed \( t = T \). Finally, a functional \( f : \Lambda \rightarrow \mathbb{R} \) is said to be \textit{smooth} if it is \( \Lambda \)-continuous and it has \( \Lambda \)-continuous derivatives \( \Delta_t f, \Delta_x f \) and \( \Delta_{xx} f \). We will denote the space of such functionals by \( \mathcal{C}^{1,2} \).

Before continuing, some comments about conditional expectation in the context of paths and functionals. Until now, we have not considered any probability framework. We then fix throughout the paper a probability space \( (\Omega, \mathcal{F}, P) \). For any \( s \leq t \) in \([0,T]\), denote by \( \Lambda_{s,t} \) the space of bounded càdlàg paths in \([s,t]\). Now define the operator \((\cdot \otimes \cdot) : \Lambda_{s,t} \times \Lambda_{t,T} \rightarrow \Lambda_{s,T}\), the \textit{concatenation} of paths, by
\[ (Y \otimes Z)(u) = \begin{cases} y_u, & \text{if } s \leq u < t, \\ z_u - z_t + y_t, & \text{if } t \leq u \leq T, \end{cases} \]
which is a paste of \( Y \) and \( Z \). Consider a process \( x \) given by the Stochastic Differential Equation (SDE)
\[ dx_s = a(s,x_s)dt + b(s,x_s)dw_s, \]
with \( s \geq t \) and \( x_t = y \). The process \((w_s)_{s \in [0,T]}\) denotes a standard Brownian motion in \((\Omega, \mathcal{F}, P)\). We assume \( a \) and \( b \) satisfy the usual hypotheses to ensure existence and uniqueness of a strong solution, see for instance \([15]\). This unique solution will be denoted by \( x^y_T \) and the path solution from \( t \) to \( T \) by \( X^y_T \). Finally, we define the \textit{conditioned expectation} as
\[ \mathbb{E}[g(X_T) \mid Y_t] = \mathbb{E}[g(Y_{t} \otimes X^y_T)], \]
for any $Y_t \in \Lambda$. The path $Y_t \otimes X^y_t \in \Lambda_T$ is equal to the path $Y_t$ up to $t$ and follows the dynamics of the SDE \eqref{eq:SDE} from $t$ to $T$ with initial value equals $y_t$. The notion above can be easily extended to SDEs with functional coefficients. Moreover, if we define the filtration $\mathcal{F}_t = \sigma(x_s; s \leq t)$, one may prove
\[
\mathbb{E}[g(X_T) \mid X_t] = \mathbb{E}[g(X_T) \mid \mathcal{F}_t],
\]
after surely, where the expectation on the left-hand side is the one discussed above and the one on the right-hand side is the usual conditional expectation.

See \cite{21} for the relation of the conditioned expectation and Backward Stochastic Differential Equations (BSDE).

For the sake of completeness the functional Itô formula is stated here. The proof can be found in \cite{6}.

**Theorem 2.1** (Functional Itô Formula; \cite{6}). Let $x$ be a continuous semimartingale and $f \in C^{1,2}$. Then, for any $t \in [0, T]$,
\[
f(x_t) = f(x_0) + \int_0^t \Delta_t f(x_s)ds + \int_0^t \Delta_x f(x_s)dx_s + \frac{1}{2} \int_0^t \Delta_{xx} f(x_s)d\langle x \rangle_s.
\]

### 2.1 An Integration by Parts Formula for $\Delta_x$

In this section, we present some results from \cite{4} regarding the adjoint of $\Delta_x$. Fix a continuous square-integrable martingale $(x_t)_{t \in [0, T]}$ and the filtration generated by it, $\mathcal{F}_t = \sigma(x_s; s \leq t) = \sigma(x_t), t \in [0, T]$.

We would like to consider stochastic integrals of the form
\[
\int_0^T f(x_t)dx_t,
\]
and for this it is necessary to impose some measurability and integrability assumptions on the functional $f$. For any functional $f : \Lambda \rightarrow \mathbb{R}$, define $g_f : [0, T] \times \Lambda_T \rightarrow \mathbb{R}$ as $g_f(t, y_T) = f(y_t)$, where $Y_t$ is the restriction of $Y_T$ to $[0, t]$.

**Definition 2.2** (Progressively Measurable Functionals). For a fixed filtration $(\mathcal{A}_t)_{t \in [0, T]}$ in $\Lambda_T$, we say the functional $f$ is $\mathcal{A}_t$-progressively measurable if, for all $t \in [0, T]$, $g_f|_{[0, t] \times \Lambda_T}$ is $\mathcal{B}([0, t]) \times \mathcal{A}_t$-measurable. We denote the space of $\mathcal{A}_t$-progressively measurable functional by $\mathcal{P}(\mathcal{A}_t)$.

Following \cite{15} Section 3.5.D, we define in $\Lambda_T$ the Borel $\sigma$-algebra $\mathcal{G} = \mathcal{B}(\Lambda_T)$ (with respect to $\| \cdot \|_\infty$) and the filtration $\mathcal{G}_t = \phi_t^{-1}(\mathcal{G}), t \in [0, T]$, where
\( \phi_t : \Lambda_T \rightarrow \Lambda_T \) is given by \( \phi_t(Y_T) = Y_{t,T-t} \), for any \( Y_T \in \Lambda_T \). One can prove that if \( f \in \mathcal{P}(\mathcal{F}_t) \), then \( \psi_t = f(X_t) \) is \( \mathcal{F}_t \)-progressively measurable.

We denote the space of continuous square-integrable martingales in \([0,T]\) with respect to the filtration \((\mathcal{F}_t^x)_{t \in [0,T]}\) by \( \mathcal{M}_2^x \) and we define

\[
\mathcal{H}_x^2 = \left\{ f \in \mathcal{P}(\mathcal{F}_t^x) : \mathbb{E} \left[ \int_0^T f^2(X_t) d\langle x \rangle_t \right] < +\infty \right\},
\]

\[
\mathcal{M}_x^2 = \left\{ f : \Lambda \rightarrow \mathbb{R} : (f(X_t))_{t \in [0,T]} \in \mathcal{M}_2^x \right\},
\]

Consider now the inner products

\[
\langle f, g \rangle_{\mathcal{H}_x^2} = \mathbb{E} \left[ \int_0^T f(X_t) g(X_t) d\langle x \rangle_t \right],
\]

\[
\langle f, g \rangle_{\mathcal{M}_x^2} = \mathbb{E} [f(X_T) g(X_T)],
\]

in \( \mathcal{H}_x^2 \) and \( \mathcal{M}_x^2 \), respectively. So that (7) and (8) are proper inner products, it is necessary to suitably identify elements of these spaces as follows

\( f \sim g \Leftrightarrow f(X_t) = g(X_t) \) a.s. for all \( t \in [0,T] \).

Thus the quotient spaces \( \mathcal{H}_x^2 = \mathcal{H}_x^2 / \sim \) and \( \mathcal{M}_x^2 = \mathcal{M}_x^2 / \sim \) are both Hilbert spaces.

Define now the Itô integral operator \( \mathcal{J}_x : \mathcal{H}_x^2 \rightarrow \mathcal{M}_x^2 \) as

\[
\mathcal{J}_x(f)(t) = \int_0^t f(X_s) dx_s,
\]

which is an isometry. Indeed,

\[
\langle f, g \rangle_{\mathcal{H}_x^2} = \langle \mathcal{J}_x(f), \mathcal{J}_x(g) \rangle_{\mathcal{M}_x^2}.
\]

A test functional is an element of \( \mathcal{D}_x = \{ f \in C^{1,2} \cap \mathcal{M}_x^2 : \Delta_x f \in \mathcal{H}_x^2 \} \). The next proposition describes the integration by parts formula of the operator \( \Delta_x \) in the space \( \mathcal{D}_x \).

**Proposition 2.3.** For any \( f \in \mathcal{D}_x \) and \( g \in \mathcal{H}_x^2 \),

\[
\langle \Delta_x f, g \rangle_{\mathcal{H}_x^2} = \langle f, \mathcal{J}_x(g) \rangle_{\mathcal{M}_x^2}.
\]
Proof. Since \( \mathbb{E}[\mathcal{I}_x(g)] = 0 \) and the goal is to compute \( \Delta_x f \), it can be assumed without loss of generality that \( f(X_0) = 0 \). Then, by the Functional Itô Formula, Theorem (2.1),

\[
\mathcal{I}_x(\Delta_x f)(t) = \int_0^t \Delta_x f(X_s)dx_s
\]

(11)

\[
\quad = f(X_t) - \int_0^t \Delta_t f(X_s)ds - \frac{1}{2} \int_0^t \Delta_{xx} f(X_s)d\langle x \rangle_s,
\]

and thus, since \( f \in \mathcal{M}^2_x \), by the uniqueness of the semimartingale decomposition,

\[
\int_0^t \Delta_t f(X_s)ds + \frac{1}{2} \int_0^t \Delta_{xx} f(X_s)d\langle x \rangle_s = 0.
\]

Therefore

\[
\mathcal{I}_x(\Delta_x f)(t) = f(X_t),
\]

which implies the integration by parts formula

\[
\langle \Delta_x f, g \rangle_{\mathcal{H}^2_x} = \langle \mathcal{I}_x(\Delta_x f), \mathcal{I}_x(g) \rangle_{\mathcal{H}^2_x} = \langle f, \mathcal{I}_x(g) \rangle_{\mathcal{H}^2_x},
\]

for all \( f \in \mathcal{D}_x \) and \( g \in \mathcal{H}^2_x \), where we have used Itô Isometry (9).

\[\square\]

2.2 Path-Dependent PDE

Suppose that the dynamics of a stock price \( x \) is given by the time-homogeneous local volatility model [5],

\[
dx_t = rx_t dt + \sigma(x_t)dw_t
\]

(13)

under the risk-neutral measure. So, the no-arbitrage price of a path-dependent derivative with maturity \( T \) and payoff given by the functional \( g : \Lambda_T \rightarrow \mathbb{R} \), which will be called contract, is given by

\[
f(Y_t) = e^{-r(T-t)}\mathbb{E}[g(X_T) \mid Y_t],
\]

see Equation (4) for the exact definition of the above expectation. Finally, we state the path-dependent extension of the pricing Partial Differential Equation (PDE), which is acronymed PPDE, see for instance [7, 8, 20].
**Theorem 2.4** (Pricing PPDE; [6]). *If the price of a path-dependent derivative, denoted by the functional \( f \), is smooth, then, for any \( Y_t \) in the topological support of the process \( x \),

\[
\Delta_t f(Y_t) + \frac{1}{2} \sigma^2(y_t) \Delta_{xx} f(Y_t) + r(y_t \Delta_t f(Y_t) - f(Y_t)) = 0,
\]

with final condition \( f(Y_T) = g(Y_T) \).

**Remark 2.5.** Under suitable assumptions on \( \sigma \), the Stroock-Varadhan Support Theorem states that the topological support of \( x \) is the space of continuous paths starting in \( x_0 \), see for instance [22, Chapter 2]. So, under these assumptions, the PPDE (14) will hold for any continuous path.

### 3 Path-Dependence

The goal of this section is to analyze the commutation issue of the operators \( \Delta_x \) and \( \Delta_t \). To start, consider the following example

\[ f(Y_t) = \int_0^t y_u du. \]

A simple computation shows

\[ \Delta_t f(Y_t) = y_t \text{ and } \Delta_x f(Y_t) = 0, \]

and hence

\[ \Delta_x(\Delta_t f)(Y_t) = 1 \neq 0 = \Delta_t(\Delta_x f)(Y_t). \]

On the other hand, it is clear that the operators commute when applied to functionals of the form \( f(Y_t) = h(t, y_t) \), where \( h \) is smooth. Therefore, one could ask if the operators commute for a functional \( f \) if and only if \( f \) is of the form \( h(t, y_t) \).

The following counter-example shows that this is not true. Consider

\[ f(Y_t) = \int_0^t \int_0^s y_u du ds, \]

and then notice

\[ \Delta_t f(Y_t) = \int_0^t y_s ds \text{ and } \Delta_x f(Y_t) = 0, \]

which clearly implies that

\[ \Delta_x(\Delta_t f)(Y_t) = \Delta_t(\Delta_x f)(Y_t). \]
**Definition 3.1** (Lie Bracket). The *Lie bracket (or commutator)* of the operators $\Delta_t$ and $\Delta_x$ will play a fundamental role in what follows and it is defined as

$$\mathcal{L} f(Y_t) = [\Delta_x, \Delta_t] f(Y_t) = \Delta_{tx} f(Y_t) - \Delta_{xt} f(Y_t),$$

where $\Delta_{tx} = \Delta_x \Delta_t$ and $f$ is such that all the derivatives above exist. Abusing the nomenclature, we will call the operator $[\Delta_x, \Delta_t]$ by simply Lie bracket.

The following lemma gives an alternative and more general definition for the Lie bracket. We skip the proof because of its simplicity.

**Lemma 3.2.** *The Lie bracket of a functional $f$ is given by the following limit, when it exists,*

$$[\Delta_x, \Delta_t] f(Y_t) = \lim_{\delta t \to 0^+} \lim_{h \to 0} \frac{f((Y_t, \delta t)_h) - f((Y_t^h)_t, \delta t)}{\delta t h}.$$

This lemma gives a very interesting interpretation of the Lie bracket as well: it is a measure of the path-dependence of the functional $f$, i.e. it will be zero if, in the limit, the order of the bump and the flat extension of the path makes no difference. In Figure 3, the term $(Y_t, \delta t)_h$ is indicated in blue and the term $(Y_t^h)_t, \delta t$ in red. Lemma 3.2 also shows that the commutation issue for functionals is not just lack of smoothness as in the finite-dimensional case.

![Figure 3: Geometric Interpretation of the $[\Delta_x, \Delta_t]$.](image)

**Proposition 3.3.** *Suppose the smooth functional $f : \Lambda \to \mathbb{R}$ is given by $f(X_t) = h(t, f_1(X_t), \ldots, f_k(X_t))$, where $h : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}$ and $f_i$ are also smooth functionals. Then*

$$[\Delta_x, \Delta_t] f(X_t) = \sum_{i=1}^k \frac{\partial h}{\partial x_i}(t, f_1(X_t), \ldots, f_k(X_t)) [\Delta_x, \Delta_t] f_i(X_t)$$
Proof. This follows easily by direct computation. Notice
\[ \Delta_t f(X_t) = \frac{\partial h}{\partial t} \Delta_t f_i(X_t), \]
\[ \Delta_x f(X_t) = \frac{\partial h}{\partial x^i} \Delta_x f_i(X_t). \]
Hence, one concludes
\[ \Delta_t \Delta_x f(X_t) = \sum_{i=1}^k \frac{\partial h}{\partial x_i} \Delta_t \Delta_x f_i(X_t) + \frac{\partial^2 h}{\partial x_i \partial t} \Delta_t f_i(X_t) + \sum_{j=1}^k \frac{\partial^2 h}{\partial x_i \partial x_j} \Delta_x f_i(X_t) \Delta_t f_j(X_t), \]
\[ \Delta_x \Delta_t f(X_t) = \sum_{i=1}^k \frac{\partial^2 h}{\partial x_i \partial t} \Delta_x f_i(X_t) + \sum_{i=1}^k \left( \frac{\partial h}{\partial x_i} \Delta_x \Delta_t f_i(X_t) + \sum_{j=1}^k \frac{\partial h}{\partial x_i x_j} \Delta_x f_i(X_t) \Delta_t f_j(X_t) \right). \]

3.1 Classification of Path-Dependence of Functionals

Based on the Lie bracket of \( \Delta_t \) and \( \Delta_x \), we define several different categories of the path-dependence for functionals.

Definition 3.4. A functional \( f : \Lambda \rightarrow \mathbb{R} \) is called

- weakly path-dependent if \( [\Delta_x, \Delta_t] f = 0 \);
- path-independent if there exists \( h : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(Y_t) = h(t, y_t) \);
- discretely monitored if there exist \( 0 < t_1 < \cdots < t_n \leq T \) and, for each \( t \in [0, T] \), \( \phi_t : \mathbb{R}^{i(t)} \rightarrow \mathbb{R} \) such that
\[
(15) \quad f(Y_t) = \phi_t(y_{t_1}, \ldots, y_{i(t)}, y_t),
\]
where \( i(t) \) is the maximum \( i \in \{1, \ldots, n\} \) such that \( t_i \leq t \);
- \( t_1 \)-delayed path-dependent if \( [\Delta_x, \Delta_t] f(Y_t) = 0 \), \( \forall t < t_1 \). Moreover, a functional \( f \) is said to be delayed path-dependent if there exists \( t_1 > 0 \) such that \( f \) is \( t_1 \)-delayed path-dependent;
• strongly path-dependent if \( \forall [s,t] \subset [0,T], \exists u \in [s,t], [\Delta_x, \Delta_t]f(Y_u) \neq 0. \)

The next proposition analyzes the Lie-bracket of discretely monitored functionals.

**Proposition 3.5.** If \( f \) is a discretely monitored functional, then \([\Delta_x, \Delta_t]f(Y_t) = 0\) but for \( t_1, \ldots, t_n \).

**Proof.** Take \( t \in (t_i, t_{i+1}) \). So, for sufficiently small \( \delta t > 0 \) such that \( t + \delta t \in (t_i, t_{i+1}) \), we have \( f((Y_t, \delta t)^h) = \phi_{t+\delta t}(y_{t_1}, \ldots, y_{t_{i(h)}}, y_t + h) = f((Y_t^h)_t, \delta t) \). Hence, \([\Delta_x, \Delta_t]f(Y_t) = 0\). \( \square \)

## 4 Greeks for Path-Dependent Derivatives

### 4.1 Introduction

In [10], the authors presented a computationally efficient way to calculate Greeks for some path-dependent derivatives using tools of the Malliavin calculus. More specifically, under the local volatility model (13), for contracts of the form

\[
g(Y_T) = \phi(y_{t_1}, \ldots, y_{t_n}),
\]

where \( 0 < t_1 < \cdots < t_n \leq T \) are fixed times and \( \phi : \mathbb{R}^n \to \mathbb{R} \) is such that \( g(X_T) \in L^2(\Omega, \mathcal{F}, P) \), it is shown that

\[
\Delta_x f(X_0) = \mathbb{E} \left[ \phi(x_{t_1}, \ldots, x_{t_n}) \int_0^T \frac{a(t)z_t}{\sigma(x_t)} dw_t \bigg| X_0 \right],
\]

where \( z_t \) is the tangent process described by the SDE

\[
dz_t = rz_t dt + \sigma'(x_t)z_t dw_t
\]

with \( z_0 = 1 \), and

\[
a \in \Gamma = \left\{ a \in L^2[0,T] : \int_0^{t_i} a(t) dt = 1, \forall i = 1, \ldots, n \right\}.
\]

If we define the weight

\[
\pi = \int_0^T \frac{a(t)z_t}{\sigma(x_t)} dw_t,
\]
which does not depend on the derivative contract $g$, we may restate the result above as:

$$\Delta_t f(X_0) = E[\phi(x_{t_1}, \ldots, x_{t_n}) \pi | X_0].$$

The goal of this section is twofold: show how this result can be achieved using functional Itô calculus and then provide a better understanding of assumption (16). In short, this assumption implies that contracts of the form (16) generate derivatives prices that are discretely monitored functionals, see Definition 3.4. The main feature of these functionals is that they exhibit weakly path-dependence but for the finite set of times $\{t_1, \ldots, t_n\}$, see Proposition 3.5.

In each section that follows we comment also on the comparison of our method with the results of [10].

For simplicity, we shall restrict ourselves to the local volatility model (13) with $r = 0$,

$$dx_t = \sigma(x_t)dw_t, \tag{20}$$

with $x_0 = x$. We suppose $\sigma : \mathbb{R} \rightarrow \mathbb{R}_+$ is $C^1(\mathbb{R})$ with bounded derivative and growth at most linearly in order to guarantee existence and uniqueness of the solution of (20) and of the tangent process (17). We also assume that $\sigma$ is bounded from below: $\sigma(x) \geq a > 0$. This assures us the topological support of the process $x$ is all the continuous function in $[0, T]$ starting at $x_0$, see Remark 2.5.

We are constraining ourselves to one-dimensional processes in order to make the exposition clearer, although the extension to multi-dimensional processes is straightforward.

### 4.2 Delta

The Delta of a derivative contract is the sensitivity of its price with respect to the current value of the underlying asset. Hence, if $f(X_t)$ denotes the price of the aforesaid derivative at time $t$, the Delta of $f$ is given by $\Delta_t f(X_t)$. Consider a path-dependent derivative with maturity $T$ and contract $g : \Lambda_T \rightarrow \mathbb{R}$. The price of this derivative is given by the functional $f : \Lambda \rightarrow \mathbb{R}$:

$$f(Y_t) = E[g(X_T) | Y_t],$$

for any $Y_t \in \Lambda$. In what follows we will perform some formal computations and hence we assume that $f$ is as smooth as necessary for such calculations. By the Pricing PPDE, Theorem 2.4, we know

$$\Delta_t f(Y_t) + \frac{1}{2} \sigma^2(Y_t) \Delta_{xx} f(Y_t) = 0,$$
for any continuous path $Y_t$. Now, define the tangent process

$$dz_t = \sigma'(x_t)z_t dw_t,$$

with $z_0 = 1$, which can be written as

(21) \[ z_t = \exp\left\{ -\frac{1}{2} \int_0^t \sigma'(x_s)^2 ds + \int_0^t \sigma'(x_s) dw_s \right\}. \]

The main idea is to apply the Functional Itô Formula, Theorem 2.1, to $\Delta_x f(X_t)z_t$. First, notice that applying $\Delta_x$ to the PPDE gives

(22) \[ \Delta_x f(Y_t) + \sigma(y_t)\sigma'(y_t)\Delta_{xx} f(Y_t) + \frac{1}{2} \sigma^2(y_t)\Delta_{xxx} f(Y_t) = 0. \]

In order to conclude the above, the following result is needed: if $f(Y_t) = 0$, for all continuous paths $Y$, and $f$ is smooth, then $\Delta_x f(Y_t) = 0$, for all continuous paths $Y$. The proof of this can be found in [4, Corollary 4.4]. Hence

$$d(\Delta_x f(X_t)z_t) = z_t d(\Delta_x f(X_t)) + \Delta_x f(X_t)dz_t + d(\Delta_x f(X_t))dz_t$$

$$= \left( \Delta_{xx} f(X_t) + \frac{1}{2} \sigma^2(x_t)\Delta_{xxx} f(X_t) + \sigma(x_t)\sigma'(x_t)\Delta_{xx} f(X_t) \right) z_t dt$$

$$+ (\sigma'(x_t)\Delta_x f(X_t) + \sigma(x_t)\Delta_{xx} f(X_t)) z_t dw_t,$$

and therefore, if we define the martingale

$$m_t = \int_0^t (\sigma'(x_s)\Delta_x f(X_s) + \sigma(x_s)\Delta_{xx} f(X_s)) z_s dw_s,$$

with $m_0 = 0$, and use Equation (22), we are able to derive the important formula

(23) \[ d(\Delta_x f(X_t)z_t) = (\Delta_{xx} f(X_t) - \Delta_{xx} f(X_t)) z_t dt + dm_t \]

$$= -[\Delta_x, \Delta_t] f(X_t) z_t dt + dm_t.$$

In the following we will derive formulas for the Delta of a derivative contract assuming each path-dependence structure presented in Definition 3.4.

### 4.2.1 Weakly Path-Dependent Functionals

**Theorem 4.1.** Consider a path-dependent derivative with maturity $T$ and contract $g : \Lambda_T \to \mathbb{R}$. So, if the price of this derivative is weakly path-dependent, then $\Delta_x f(X_t)z_t$ is a martingale and the following formula for the Delta is valid

$$\Delta_x f(X_0) = \mathbb{E} \left[ g(X_T) \frac{1}{T} \int_0^T \frac{z_t}{\sigma(x_t)} dw_t \right].$$
Proof. From Equation (23) and by assumption $[\Delta_x, \Delta_t]f = 0$, we have that
\[ \Delta_t f(X_t)z_t = \Delta_t f(X_0) + m_t, \]
and then $\Delta_x f(X_t)z_t$ is clearly a martingale. Now, integrating with respect to $t$, we get
\[ \int_0^T \Delta_x f(X_t)z_t dt = \Delta_x f(X_0)T + \int_0^T m_t dt. \]
Then taking expectations and noticing that $m_t$ is a martingale with $m_0 = 0$, we get
\[ \mathbb{E} \left[ \int_0^T \Delta_x f(X_t)z_t dt \right] = \Delta_x f(X_0)T, \]
which implies
\begin{align*}
\Delta_x f(X_0) &= \mathbb{E} \left[ \int_0^T \Delta_x f(X_t) \frac{1}{T} \frac{z_t}{\sigma^2(x_t)} \sigma^2(x_t) dt \right] \\
&= \left\langle \Delta_x f(X), \frac{1}{T} \frac{z}{\sigma^2(x)} \right\rangle_{\mathcal{H}_2^x}. \quad (24)
\end{align*}
Finally, since $f(X_t)$ and $x_t$ are martingales, by the integration by parts formula (10),
\[ \Delta_x f(X_0) = \left\langle f(X_t), I_x \left( \frac{1}{T} \frac{z_t}{\sigma^2(x_t)} \right) \right\rangle_{\mathcal{H}_2^x} = \mathbb{E} \left[ g(X_T) \frac{1}{T} \int_0^T \frac{z_t}{\sigma^2(x_t)} dw_t \right]. \quad (25) \]

Remark 4.2. In the Black–Scholes model (i.e. $\sigma(x_t) = \sigma x_t$), we find the same result as in [10]
\[ \Delta_x f(X_0) = \mathbb{E} \left[ g(x_T) \frac{w_T}{x_0 \sigma T} \right]. \]

Remark 4.3. Theorem 4.1 states that, for weakly path-dependent functionals, the weight can take the form
\[ \pi = \frac{1}{T} \int_0^T \frac{z_t}{\sigma(x_t)} dw_t, \quad (26) \]
cf. Equation (19).
Remark 4.4. Theorem 4.1 also enlightens the question when the Delta is martingale. The theorem affirms that the lost of martingality of the Delta comes from two factors: the stock price model through its tangent process \( z_t \) and the path-dependence of the derivative in question.

For instance, let us consider a call option. It is a well-know fact that, under the Black-Scholes model, the Delta is not a martingale. Although the price of a call option is weakly path-dependent (actually it is path-independent), the tangent process in this model is given by \( z_t = x_t / x_0 \). On the other hand, under the Bachelier model, the Delta is indeed a martingale, since \( z_t = 1 \) in this case.

Corollary 4.5. Under the same hypotheses as before, for any \( s \in [0, T] \), one has

\[
\Delta_x f(x_s) = \frac{1}{(T-s)z_s} \mathbb{E} \left[ g(X_T) \int_s^T \frac{z_t}{\sigma(x_t)} dw_t \bigg| X_s \right].
\]

Proof. The same argument is applied with some minor differences. Notice the study of the integration by parts formula for \( \Delta_x \) can be easily extended to handle the conditional expectation. \( \square \)

Example 4.6. In order to exemplify the method above, consider the following contract

\[
g(X_T) = \left( x_T - \frac{2}{T^2} \int_0^T \int_0^t x_u du dt \right)^+,
\]

which is a modification of the floating-strike Asian call option. We will call it here floating-strike double Asian call option. One can easily deduce that \( f(Y_t) = \mathbb{E}[g(X_T) | Y_t] \) is a weakly path-dependent functional by Proposition 3.3. Consider now the Black–Scholes model with \( r = 0 \),

\[dx_t = \sigma x_t dw_t,\]

where \( \sigma > 0 \) is the volatility. Given a realization of this process, that will be denoted by \( Y_T \), we will compute the Delta on this path. So, we apply Theorem 4.1 (see also Remark 4.2 and Corollary 4.5) to find

\[
\Delta_x f(Y_s) = \mathbb{E} \left[ g(X_T) \frac{w_T - w_s}{x_s \sigma(T-s)} \bigg| Y_s \right].
\]

Before proceeding, if we denote by \( h \) the functional

\[h(X_T) = \int_0^T \int_0^t x_u du dt,\]
we then notice

\[ h(Y_s \otimes X_T^y) = \int_0^s \int_0^t y_u du dt + \int_s^T \left( \int_0^s y_u du + \int_s^t x_u^y du \right) dt. \]

The formula above is used to compute \( \Delta_x f(Y_s) \). Hence, for parameters given in Table 1, we present the results in Table 2 and in Figure 6. Moreover, we used \( 10^5 \) Monte Carlo paths to compute the expectation and we discretized \([0, T]\) in 500 points.

| Parameter | Value |
|-----------|-------|
| \( X_0 \) | 100   |
| \( \sigma \) | 0.4   |
| \( T \)    | 1     |

Table 1: Parameters.

|                        | Mean     | Standard Error |
|------------------------|----------|----------------|
| \( f(X_0) \)          | 11.2891  | 0.0693         |
| \( \Delta_x f(X_0) \) | 0.3960   | 0.0037         |

Table 2: Monte Carlo Estimation of the Price and Delta.
Figure 4: Realized path $Y_T$ of the process $x$.

Figure 5: Price of the double Asian option on the realized path.

Figure 6: Delta of the double Asian option on the realized path.
4.2.2 Discretely Monitored Functionals

In this section we consider a simple modification of the method described in Section 4.2.1 to handle discretely monitored functionals as studied in [10], see Equation (16).

**Theorem 4.7.** Consider a contract \( g(X_T) = \phi(x_{t_1}, \ldots, x_{t_n}) \), where \( 0 < t_1 < \cdots < t_n \leq T \) are fixed times and \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \). The no-arbitrage price \( f \) of this contract is a discretely monitored functional and, if we assume \( f \) is smooth, we find the same representation for the Delta as in [10]:

\[
\Delta_x f(X_0) = \mathbb{E} \left[ g(X_T) \int_0^T \frac{a(t)z_t}{\sigma(x_t)} \, dw_t \right],
\]

for any \( a \in \Gamma \), see Equation (18).

**Proof.** First of all, notice that \( f \) is clearly a discretely monitored functional as one can simply deduce from

\[
f(Y_t) = \mathbb{E}[\phi(x_{t_1}, \ldots, x_{t_n}) \mid Y_t],
\]

and from Definition 3.4.

To focus on the essential arguments of the proof, we consider the case where \( g(X_T) = \phi(x_{t_1}, x_{t_2}) \). This setting let us introduce all the elements of the proof without the burden of heavy notations. A similar reasoning could be applied to the general case \( g(X_T) = \phi(x_{t_1}, \ldots, x_{t_n}) \).

As we have seen in (23),

\[
d(\Delta_x f(X_t) z_t) = -[\Delta_t, \Delta_x] f(X_t) z_t \, dt + dm_t,
\]

with \((m_t)_{t \in [0, T]} \) being a martingale. As seen in Proposition 3.5 we have \([\Delta_t, \Delta_x] f(X_t) = 0 \) for all \( t \in [0, t_1) \cup (t_1, T] \). Fix \( \varepsilon > 0 \) and for \( t \in [t_1, T] \), we integrate the SDE (30) over the interval \([t_1 + \varepsilon, t] \), we get

\[
\Delta_x f(X_t) z_t = \Delta_x f(X_{t_1} + \varepsilon) z_{t_1 + \varepsilon} + m_t - m_{t_1 + \varepsilon}.
\]

So, multiplying by any \( a \in \Gamma \) (which then satisfies \( \int_{t_1}^T a(t) \, dt = 0 \)) and integrating with respect to \( t \), we have

\[
\int_{t_1 + \varepsilon}^T \Delta_x f(X_t) z_t a(t) \, dt = \int_{t_1 + \varepsilon}^T \Delta_x f(X_{t_1 + \varepsilon}) z_{t_1 + \varepsilon} a(t) \, dt
\]

\[
+ \int_{t_1 + \varepsilon}^T (m_t - m_{t_1 + \varepsilon}) a(t) \, dt = \int_{t_1 + \varepsilon}^T (m_t - m_{t_1 + \varepsilon}) a(t) \, dt
\]
For \( t \in [0, t_1 - \varepsilon] \), integrating again Equation (30) now over the interval \([0, t]\), we get

\[
\Delta_x f(X_t) z_t = \Delta_x f(X_0) + m_t - m_0.
\]

Multiplying by \( a \) (that satisfies \( \int_0^{t_1 - \varepsilon} a(t) dt = 1 \)) and integrating with respect to \( t \) give us

\[
\int_0^{t_1 - \varepsilon} \Delta_x f(X_t) z_t a(t) dt = \int_0^{t_1 - \varepsilon} \Delta_x f(X_0) a(t) dt
\]

(33)

\[
+ \int_0^{t_1 - \varepsilon} (m_t - m_0) a(t) dt = \Delta_x f(X_0) + \int_0^{t_1 - \varepsilon} (m_t - m_0) a(t) dt
\]

(34)

Summing the two Equations (31) and (33), taking the expectation and using the fact \( m_t \) is a martingale, we find

\[
\mathbb{E} \left[ \left( \int_0^{t_1 - \varepsilon} + \int_{t_1 + \varepsilon}^T \right) \Delta_x f(X_t) z_t a(t) dt \right] = \Delta_x f(X_0)
\]

\[
+ \mathbb{E} \left[ \int_0^{t_1 - \varepsilon} (m_t - m_0) a(t) dt \right] + \mathbb{E} \left[ \int_{t_1 + \varepsilon}^T (m_t - m_{t_1 + \varepsilon}) a(t) dt \right] = \Delta_x f(X_0).
\]

Therefore, letting \( \varepsilon \to 0^+ \) and a simple application of the integration by parts formula give the result.

\[\square\]

**Remark 4.8.** Comparing with Equation (19), we conclude that Theorem 4.7 finds the same weight as in [10].

We would like to conclude this section observing that we were able to derive, using the techniques of the functional Itô calculus, the same results of [10], in which Malliavin calculus was used. Furthermore, the method implemented here enlightens the assumption that the derivative price needs to be a discretely monitored functional to employ Theorem 4.7. Indeed, the main feature of such functionals is that it is weakly path-dependent in the interval \((t_i, t_{i+1})\) allowing us to apply the integration by parts formula in the intervals \((t_i, t_{i+1})\).

### 4.2.3 Delayed Path-Dependent Functionals

The argument presented in the proof of Theorem 4.7 can be generalized to the delayed path-dependent functionals. The next proposition states precisely the result.

Define

\[
\Gamma_s = \left\{ a \in L^2([0, T]) : \int_0^s a(t) dt = 1 \text{ and } a(t) = 0, \text{ for } t \geq s \right\}.
\]
Proposition 4.9. Fix a $t_1$-delayed path-dependent functional $f$ and consider $a \in \Gamma_{t_1}$. Thus,

\[ \Delta_x f(X_0) = \mathbb{E} \left[ g(X_T) \int_0^{t_1} \frac{a(t) z_t}{\sigma(x_t)} \, dw_t \right]. \]  
(35)

Proof. As before, by Equation (37),

\[ m_t = \Delta_x f(X_t) z_t - \Delta_x f(X_0) + \int_0^t \left[ \Delta_x, \Delta_t \right] f(X_s) z_s ds. \]

Multiplying by any $a \in \Gamma_{t_1}$, integrating with respect to $t$ and taking expectation,

\[ \Delta_x f(X_0) = \mathbb{E} \left[ \int_0^T a(t) \Delta_x f(X_t) z_t dt \right] + \mathbb{E} \left[ \int_0^T a(t) \int_0^t \left[ \Delta_x, \Delta_t \right] f(X_s) z_s ds dt \right] \]
\[ = \mathbb{E} \left[ \int_0^{t_1} a(t) \Delta_x f(X_t) z_t dt \right]. \]

Therefore a simple application of the integration by parts formula yields the result.

Remark 4.10. Therefore, in the case of delayed path-dependent derivative, we have found the weight

\[ \pi = \int_0^{t_1} a(t) \frac{z_t}{\sigma(x_t)} \, dw_t. \]

One should compare this formula with (19).

Remark 4.11. In the case when the Lie bracket is zero in $[u, s] \subset [0, T]$, when can adapt the proof above to find a similar expression of (35) for the Delta at time $u$, $\Delta_x f(X_u)$.

Remark 4.12. Clearly, a discretely monitored functional is also delayed path-dependent, but it could be computationally advantageous to consider $a \in \Gamma$ instead of $a \in \Gamma_{t_1}$.

Example 4.13. Consider the following contract

\[ g(X_T) = \left( x_T - \frac{1}{T-t_1} \int_{t_1}^T x_u du \right)^+, \]

where $0 < t_1 < T$. This derivative is called forward-start floating-strike Asian call option, see [11] for more details. One can easily deduce that, for $t < t_1$, 

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\( f(Y_t) = \mathbb{E}[g(X_T) \mid Y_t] \) depends only of \( Y_t \). Therefore, \( f \) is a delayed path-dependent functional.

Assuming the Black–Scholes model with \( r = 0 \),
\[
    dx_t = \sigma x_t dw_t,
\]
where \( \sigma > 0 \), and applying Proposition \ref{prop}, we find
\[
\Delta_x f(X_0) = \mathbb{E}\left[ g(X_T) \int_0^{t_1} \frac{a(t)z_t}{\sigma x_t} dw_t \right].
\]

Consider then the weight
\[
\pi = \int_0^{t_1} \frac{a(t)z_t}{\sigma x_t} dw_t,
\]
and further notice that in this model the tangent process satisfies \( z_t = x_t / x_0 \). Hence,
\[
\pi = \frac{1}{\sigma x_0} \int_0^{t_1} a(t)dw_t \sim N\left(0, \frac{1}{\sigma^2 x_0} \int_0^{t_1} a^2(t)dt\right).
\]
One can show that the choice \( a \equiv 1/t_1 \) attains minimum variance for \( \pi \) over \( \Gamma_{t_1} \). Then,
\[
\pi = \frac{w_{t_1}}{t_1 \sigma x_0}.
\]

Considering the parameters given in Table \ref{table3}, we find the results presented in Table \ref{table4} and in Figure \ref{figure7}.

| Parameter | Value |
|-----------|-------|
| \( X_0 \) | 100   |
| \( \sigma \) | 0.4   |
| \( t_1 \) | 0.2   |
| \( T \) | 1     |

Table 3: Parameters

4.2.4 Strongly Path-Dependent Functionals

How would these formulas change if \( f \) is strongly path-dependent? The integral form of Equation \ref{eq23} is
\[
\Delta_x f(X_0) = \Delta_x f(X_t)z_t + \int_0^t [\Delta_x, \Delta_t] f(X_s)z_s ds + m_t.
\]
Table 4: Price and Delta

|                | Mean | Standard Error |
|----------------|------|----------------|
| $f(X_0)$       | 7.6536 | 0.0475         |
| $\Delta_x f(X_0)$ | 0.0755 | 0.0032         |

Figure 7: Convergence Plot of Monte Carlo Method to Compute $\Delta_x f(X_0)$.

Integrating with respect to $t$ and taking expectation, we get

$$
\Delta_x f(X_0) = \mathbb{E} \left[ \frac{1}{T} \int_0^T \Delta_x f(X_t) z_t dt \right] + \mathbb{E} \left[ \frac{1}{T} \int_0^T \int_0^t [\Delta_x, \Delta_t] f(X_s) z_s ds dt \right].
$$

Now, for the first expectation, we use the same argument as in Theorem 4.1 to conclude

$$
\mathbb{E} \left[ \frac{1}{T} \int_0^T \Delta_x f(X_t) z_t dt \right] = \mathbb{E} \left[ g(X_T) \frac{1}{T} \int_0^T z_t \frac{\partial}{\partial x_t} dw_t \right].
$$
Finally, we get

\[ \Delta_x f(X_0) = E \left[ g(X_T) \frac{1}{T} \int_0^T \frac{z_t}{\sigma(x_t)} \, dw_t \right] + E \left[ \frac{1}{T} \int_0^T \int_0^t [\Delta_x, \Delta_t] f(X_s) z_s \, ds \, dt \right]. \tag{40} \]

For the second term, one should study the adjoint and/or an integration by parts for \( \Delta_t \) and \( \Delta_x \) in \( \mathcal{H}_x^2 \). An integration by parts formula for \( \Delta_x \) in \( \mathcal{H}_x^2 \) is presented in [6, Section 3].

Furthermore, an important interpretation of the second term of the right-hand side of Equation (40) is as a path-dependent correction to the weakly path-dependent “Delta” of Equation (39), which does not take into consideration the strong path-dependence structure of the derivative contract. This is one of the most important achievements of the functional Itô calculus framework: it allows us to quantify how the path-dependence of the functional influences the Delta of this contract.

In the next section we provide formulas for the Gamma and the Vega of a path-dependent derivative contract. For both cases we assume that the contract is weakly path-dependent. Similar formulas and proofs for the different classifications of path-dependence of Definition 3.4 can be derived following akin arguments.

### 4.3 Gamma

The Gamma of a derivative is the sensitivity of the Delta of the derivative price with respect to the current value of the underlying asset, i.e. \( \Delta_{xx} f(X_t) \). Here we will derive a similar formula to (27) for the Gamma.

**Theorem 4.14.** Under the hypotheses of Theorem 4.1 and assuming \([\Delta_x, \Delta_t](\Delta_x f) = [\Delta_t, \Delta_{xx}] f = 0\),

\[ \Delta_{xx} f(X_s) = E[g(X_T) \xi_{s,T} | X_s], \]

where

\[ \eta_s = \int_0^s \frac{z_t}{\sigma(x_t)} \, dw_t, \]

\[ \xi_{s,T} = \frac{(\eta_T - \eta_s)^2}{(T-s)^2 z_s^2} - \frac{\sigma'(x_s) \eta_T - \eta_s}{\sigma(x_s) (T-s) z_s} - \frac{1}{(T-s) \sigma^2(x_s)}. \]

23
Proof. First, using the formula \([21]\) for the tangent process \(z_s\) and the generalization of the Functional Itô Formula shown in \([19]\), one has

\[
\Delta_t \left( \int_0^t \frac{z_s}{\sigma(x_s)} \, dw_s \right) = \frac{z_t}{\sigma^2(x_t)} \text{ and } \Delta_t z_s = \frac{\sigma^\prime(x_s)}{\sigma(x_s)} z_s,
\]

\[
\Delta_t \left( \int_0^t \frac{z_s}{\sigma(x_s)} \, dw_s \right) = 0 \text{ and } \Delta_t z_s = 0.
\]

Remember the following formula given in Corollary 4.5:

\[(T - s)z_s \Delta_t f(X_s) + f(X_s) \eta_s = \mathbb{E} \left[ g(X_T) \eta_T | X_s \right].\]

Define now \(\tilde{g}(X_T) = g(X_T) \eta_T\) and \(\tilde{f}(X_s) = \mathbb{E}[\tilde{g}(X_T) | X_s]\). It is easy to show that

\[\tilde{f}(X_s) = (T - s)z_s \Delta_t f(X_s) + f(X_s) \eta_s\]

Now, in order to apply the same argument as in the proof of the Theorem 4.1, it is necessary to prove \([\Delta_t, \Delta_t] \tilde{f}(X_s) = 0:\)

\[
\Delta_t \tilde{f}(X_s) = (T - s) \frac{\sigma^\prime(x_s)}{\sigma(x_s)} z_s \Delta_t f(X_s) + (T - s) z_s \Delta_t \Delta_t f(X_s)
\]

\[+ \Delta_t f(X_s) \eta_s + f(X_s) \frac{z_s}{\sigma^2(x_s)};\]

\[
\Delta_t \tilde{f}(X_s) = -z_s \Delta_t f(X_s) + (T - s) z_s \Delta_t \Delta_t f(X_s) + \Delta_t f(X_s) \eta_s,
\]

\[
\Delta_{tt} \tilde{f}(X_s) = -\frac{\sigma^\prime(x_s)}{\sigma(x_s)} z_s \Delta_t f(X_s) + (T - s) \frac{\sigma^\prime(x_s)}{\sigma(x_s)} z_s \Delta_t \Delta_t f(X_s) - z_s \Delta_{tt} f(X_s)
\]

\[+ (T - s) z_s \Delta_{tt} \Delta_t f(X_s) + \Delta_{tt} f(X_s) \eta_s + \Delta_t f(X_s) \frac{z_s}{\sigma^2(x_s)};\]

\[
\Delta_{tx} \tilde{f}(X_s) = -\frac{\sigma^\prime(x_s)}{\sigma(x_s)} z_s \Delta_t f(X_s) - z_s \Delta_{tx} f(X_s) + (T - s) \frac{\sigma^\prime(x_s)}{\sigma(x_s)} z_s \Delta_{tt} f(X_s)
\]

\[+ (T - s) z_s \Delta_{tx} \Delta_t f(X_s) + \Delta_{tt} f(X_s) \eta_s + \Delta_t f(X_s) \frac{z_s}{\sigma^2(x_s)}.
\]

Finally, since \([\Delta_t, \Delta_t] \tilde{f}(X_s) = 0 = [\Delta_t, \Delta_t] (\Delta_t f)(X_s)\), we find

\[ [\Delta_t, \Delta_t] \tilde{f}(X_s) = 0.\]

Hence, by Theorem 4.1 \(\Delta_t \tilde{f}(X_s) z_s\) is a martingale and therefore

\[(T - s)z_s \Delta_t \tilde{f}(X_s) + \tilde{f}(X_s) \int_0^s \frac{z_t}{\sigma(x_t)} \, dw_t = \mathbb{E} \left[ \tilde{g}(X_T) \int_0^T \frac{z_t}{\sigma(x_t)} \, dw_t | X_s \right].\]
Going from $\tilde{f}$ to $f$, we find
\[
\Delta_x \tilde{f}(X_s) = (T - s) z_s \frac{\sigma'(x_s)}{\sigma(x_s)} \Delta_x f(X_s) + (T - s) z_s \Delta_{xx} f(X_s)
\]
\[+ \Delta_x f(X_s) \int_0^s \frac{z_t}{\sigma(x_t)} dw_t + f(X_s) \frac{z_s}{\sigma^2(x_s)}.
\]
Lastly, the result can be easily derived from the equation above.

\[\blacksquare\]

**Corollary 4.15.** At $s = 0$,
\[
\Delta_{xx} f(X_0) = \mathbb{E}[g(X_T) \xi],
\]
where
\[
\xi = \xi_{0,T} = \frac{\eta_T^2}{T^2} - \frac{\sigma'(x)}{\sigma(x)} \frac{\eta_T}{T} - \frac{1}{T \sigma^2(x)}.
\]

**Remark 4.16.** In the Black-Scholes model, we find the same result as in [10]:
\[
\Delta_{xx} f(X_0) = \mathbb{E}\left[g(x_T) \frac{1}{x_0} \left( \frac{w_T^2}{\sigma T} - w_T^2 - \frac{1}{\sigma} \right) \right].
\]
However, in [10] the Gamma is derived only in the Black–Scholes model and for path-independent derivative with contract of the form $g(X_T) = \phi(x_T)$.

One should also notice that, making the proper adaptations, a similar result to Theorem 4.14 holds true for discretely monitored functionals, since their Deltas are also discretely monitored functionals.

### 4.4 Vega

Consistently to [6], under the local volatility model, we define the Vega of $f(X_t)$ as the Fréchet derivative of $f(X_t)$ with respect to $\nu = \sigma^2$. Using the result presented in [6 Section 4, Example 1], we know that the Vega of $f(X_t)$ in the direction of $u$ is given by
\[
\langle \nabla_\nu f, u \rangle = \int_0^T \int_\mathbb{R} u(t,x) m(t,x) dx dt,
\]
where
\[
m(t,x) = \frac{1}{2} \mathbb{E}_0 [\Delta_{xx} f(X_t) \mid x_t = x] \phi_0^{v_0}(t,x).
\]
Here, \( E_{v_0} \) is the expectation under the local volatility model (13) with \( v_0 = \sigma^2 \).

This equation can be rewritten as

\[
\langle \nabla_v f, u \rangle = \frac{1}{2} \int_0^T E_{v_0} [u(t,x_t) \Delta_{xx} f(X_t)] dt.
\]

Assuming that the conditions of Theorem 4.14 are satisfied, we can write

\[
\Delta_{xx} f(X_t) = E_{v_0} [g(X_T) \xi_{t,T} | X_t],
\]

and then the following is true

\[
\langle \nabla_v f, u \rangle = \frac{1}{2} \int_0^T E_{v_0} [u(t,x_t) E_{v_0} [g(X_T) \xi_{t,T} | X_t]] dt
\]

\[
= \frac{1}{2} \int_0^T E_{v_0} [u(t,x_t) g(X_T) \xi_{t,T}] dt
\]

\[
= E_{v_0} \left[ g(X_T) \frac{1}{2} \int_0^T u(t,x_t) \xi_{t,T} dt \right],
\]

with

\[
\xi_{t,T} = \frac{(\eta_T - \eta_t)^2}{(T-t)^2} \frac{1}{\nu_0(x_t)} \frac{\eta_T - \eta_t}{(T-t)\nu_0(x_t)} - \frac{1}{(T-t)\nu_0(x_t)}
\]

and

\[
\eta_t = \int_0^t \frac{z_s}{\sqrt{\nu_0(x_s)}} dw_s.
\]

**Remark 4.17.** Comparing this result with the one presented in [10], we notice that our formula (43) avoids the necessity to compute Skorohod integrals. Actually, one can show that the formula for the Vega in [10] can be simplified to (43) when \( g(X_T) = \phi(x_T) \).

Again, we should note that, making the proper adaptations, we could derive the equivalent of formula (43) for discretely monitored functionals.

## 5 Conclusion

We have introduced a measure of path-dependence of a functional using the functional Itô calculus first introduced in Dupire’s influential work [6]. This measure is defined as the Lie bracket of the time and space functional derivatives. We then
proposed a classification of functionals by their degree of path-dependence. Furthermore, for functional with less severe path-dependence structures, we studied the weighted-expectation formulas for the Delta, Gamma and Vega. In the case of a strong path-dependent functional, we were able to understand the impact of the Lie bracket on its Delta. Numerical examples of the theory were also presented.

Further research will be conducted to analyze the case of strong path-dependence. In particular, it is really important to explicitly describe the adjoint of the functional time derivative $\Delta_t$.

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