On-the-Fly Joint Feature Selection and Classification

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Abstract—Joint feature selection and classification in an online setting is essential for time-sensitive decision making. However, most existing methods treat this coupled problem independently. Specifically, online feature selection methods can handle either streaming features or data instances offline to produce a fixed set of features for classification, while online classification methods classify incoming instances using full knowledge about the feature space. Nevertheless, all existing methods utilize a set of features, common for all data instances, for classification. Instead, we propose a framework to perform joint feature selection and classification on-the-fly, so as to minimize the number of features evaluated for every data instance and maximize classification accuracy. We derive the optimum solution of the associated optimization problem and analyze its structure. Two algorithms are proposed, ETANA and F-ETANA, which are based on the optimum solution and its properties. We evaluate the performance of the proposed algorithms on several public datasets, demonstrating (i) the dominance of the proposed algorithms over the state-of-the-art, and (ii) its applicability to broad range of application domains including clinical research and natural language processing.

Index Terms—large-scale data mining, big data analytics, feature selection, classification.

1 INTRODUCTION

Feature selection is the process of selecting a subset of the most informative features from a large set of potentially redundant features with the objective of maximizing classification accuracy, alleviating the effect of the curse of dimensionality, speeding up the training process and improving interpretability [1], [2].

Most existing work on feature selection [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20] extracts a subset of discriminative features that can globally describe the data well, where the same feature subset is used to classify all instances during classification (see Fig. 1(a)). Only a handful of feature selection methods have considered the feature evaluation cost and costs associated with misclassification, which play a key role in many real-world applications [21], [22]. On the other hand, existing online classification techniques [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33] update model parameters by examining incoming data instances one at a time; such methods are not only affected by noisy and missing data, but also face scalability constraints [33].

In a departure from existing feature selection and classification methods, we study the problem of on-the-fly joint feature selection and classification (OJFC) in an online setting. Specifically, the goal of OJFC is to minimize the number of feature evaluations for classification individually for each instance, while achieving high classification accuracy across all instances. This is particularly important and necessary for real-world applications requiring time-sensitive decisions such as weather forecasting [34], transportation [35], stock markets prediction [36], clinical research [37], and natural disasters prediction [38]. Therefore, the proposed method utilizes a varying number of features to classify each data instance online using a model learned offline from all features.

Fig. 1: Using a $K \times D$ matrix, (a) existing feature selection (FS) methods extract a set $L \ll K$ of features, which is common for all data instances during classification. In contrast, (b) the proposed on-the-fly joint feature selection and classification approach utilizes a varying number of features to classify each data instance online using a model learned offline from all features.

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ously minimizes the number of features evaluated and maximizes classification accuracy. The solution to this optimization problem leads to an approach that sequentially reviews features and classifies a data instance once it determines that including additional features cannot further improve the quality of classification. However, the computational complexity of the optimum solution increases exponentially with the number of classes in multi-class classification tasks.

To improve the scalability of our approach, we propose an efficient implementation, which exploits the structure of the optimum solution. Specifically, the functions related to the optimum solution are shown to be concave, continuous and piecewise linear on the domain of a sufficient statistic. As a result, the optimum solution exhibits a threshold structure to decide between continuing the feature evaluation process and stopping. A stochastic gradient algorithm is utilized to estimate the optimal linear thresholds. Extensive experimental evaluation using seven publicly available datasets shows the superiority of the proposed approach in terms of classification accuracy, average number of features used per data instance, and time required for joint feature selection and classification compared to the state-of-the-art. Further, our evaluation results indicate that the proposed efficient implementation drastically reduces training time without a drop in accuracy as compared to the optimum solution. All proofs are included in Appendices A and B.

2 Related Work

In this section, we summarize the most relevant prior work on (i) online feature selection and (ii) online classification techniques.

2.1 Online feature selection

In contrast to offline feature selection methods that are designed for static datasets with fixed number of features and data samples, online feature selection methods are capable of handling either streaming features or streaming data samples to choose a subset of features from a larger set of potentially redundant features [14], [19], [20]. Online feature selection methods can be generally grouped into two groups:

(a) Streaming Features: In this branch of online feature selection problems, the number of data instances is considered constant while features arrive one at a time [5], [6], [7], [8], [9], [10]. In [5], a newly arriving feature is selected if the improvement in the model is greater than a predefined threshold. [5], [6] try to extract features in the Markov blanket of the class variable using a forward algorithm, where thresholds on probability approximations to measure conditional independence (e.g. $G^2$-test [5], Fisher’s Z-test [6]) are employed. Such threshold-based methods require prior information about the feature space. [6] Recently, rough set theory based methods [7], [8], [9], [10] have been explored. Such methods do not require any domain knowledge [6]. However, methods proposed in [9], [10] are not applicable to numerical features, while methods [7], [8] are much slower in feature selection compared to the state-of-the-art streaming feature selection methods.

(b) Streaming Data: In this problem setting, the number of features is considered constant, while data instances arrive over time [11], [12], [13]. Such methods [11], [12], [13] are limited to binary classification and/or impose hard constraints on the number of non-zero elements in the model, requiring the user to define the number of features that need to be selected a priori [33].

2.2 Online Classification

Online classification methods, also referred to as online learning, use sequentially arriving data to update the function of a classifier. This is in contrast to batch learning techniques where a collection of training data is used to train a classifier offline, without further updates once training is complete [33]. Most widely used online learning methods [23], [24], [25] are either limited to binary classification [23], [24] or require solving a complex optimization problem at each iteration, and require prior information to tune parameters in the model [25]. On the other hand, traditional gradient based methods [25], [27], [28], [29] not only require to compute the gradient of a cost function, but also require to solve an optimization problem at each iteration. Cost-sensitive extensions of traditional online classification methods, which account for misclassification costs have been recently explored [25], [30], [31], [32]. Unlike our approach, [25], [30] do not optimize the misclassification cost directly [33], while [31], [32] are limited to binary classification. Last but not least, most existing methods are highly susceptible to noise and/or incomplete data [33].

3 On-the-Fly Joint Feature Selection and Classification

Consider a set $S$ of data instances, with each data instance $s \in S$ being described using an assignment of values $f = \{f_1, f_2, \ldots, f_K\}$ to a set $F = \{F_1, F_2, \ldots, F_K\}$ of $K$ features. Each data instance $s$ is drawn from some probability distribution over the feature space such that for each assignment $f$ to $F$, we have a probability $P(F = f)$. Further, each instance $s$ may belong to one of $N$ classes, with corresponding a priori probability $P(T = T_i) = p_i$ for each assignment $T_i$, $i = 1, 2, \ldots, N$, of the class variable $T$. Moreover, coefficients $c_k > 0$, $k = 1, 2, \ldots, K$, represent the cost of evaluating features $F_k$, respectively, and coefficients $M_{i,j} \geq 0$, $i, j \in \{1, \ldots, N\}$, denote the misclassification cost of selecting class $T_j$ when class $T_i$ is true.

To select one out of $N$ possible classes for each data instance $s$, our proposed approach evaluates features sequentially, where at each step it has to decide between stopping and continuing the feature evaluation process based on the accumulated information thus far and the cost of evaluating the remaining features. Herein, we introduce a pair of random variables $(R, D_R)$, where $0 \leq R < K$ (referred to as stopping time [39] in decision theory) denotes the feature at which the framework assigns $s$ to a specific class, and $D_R \in \{1, \ldots, N\}$, which depends on $R$, denotes the possibility to select among the $N$ classes. The event $\{R = k\}$ depends only on the feature set $\{F_1, F_2, \ldots, F_k\}$, whereas the event $\{D_R = j\}$ represents choosing class $T_j$ based on information accumulated up to feature $R$. The
goal is to select random variables \( R \) and \( D_R \) by solving the following optimization problem:

\[
\text{minimize } J(R, D_R),
\]

where the cost function is defined as:

\[
J(R, D_R) = \mathbb{E} \left\{ \sum_{k=1}^{R} c_k \right\} + \sum_{j=1}^{N} \sum_{i=1}^{M_j} P(D_R = j, T = T_i)
\]

in which the first term denotes the cost of evaluating features, and the second term penalizes misclassification errors.

To solve the optimization problem defined in Eq. (1), we define a sufficient statistic of accumulated information, the a posteriori probability vector \( \pi_k \), as follows:

\[
\pi_k = \begin{bmatrix} \pi_k^1, \pi_k^2, \ldots, \pi_k^N \end{bmatrix}^T,
\]

where the \( k \)th feature is evaluated to generate outcome \( f_k \), and \( \pi_k^k \) is \( P(T_i|F_k = f_k) \) subsequently. Assuming that features in set \( F \) are independent given the class variable \( T \), \( \pi_k \) can be computed recursively as in Lemma 1.

**Lemma 1.** The a posteriori probability vector \( \pi_k \in [0,1]^N \) can be recursively computed as:

\[
\pi_k = \frac{\text{diag}(\Delta_k(F_k)) \pi_{k-1}}{\Delta_k^T(F_k) \pi_{k-1}},
\]

where \( \Delta_k(F_k) = [P(F_k|T_1), P(F_k|T_2), \ldots, P(F_k|T_N)]^T \), \( \text{diag}(A) \) denotes a diagonal matrix with diagonal elements being the elements in vector \( A \), and \( \pi_0 = [p_1, p_2, \ldots, p_N]^T \).

Next, we simplify the probability \( P(D_R = j, T = T_i) \) exploiting the definition of the a posteriori probability \( \pi_k^i \).

**Lemma 2.** Based on the fact that \( x = \sum_{k=0}^{K} x_k \mathbb{I}_{(R=k)} \) for any sequence of random variables \( \{x_k\} \), where \( \mathbb{I}_A \) is the indicator function for event \( A \) (i.e., \( \mathbb{I}_A = 1 \) when \( A \) occurs, and \( \mathbb{I}_A = 0 \) otherwise), the probability \( P(D_R = j, T = T_i) \) can be written as follows:

\[
P(D_R = j, T_i) = \mathbb{E} \left\{ \frac{\pi_k^j \mathbb{I}_{(D_R=j)}}{\pi_k^i \mathbb{I}_{(D_R=j)}} \right\}.
\]

Using Lemma 2, the average cost in Eq. (2) can be written compactly as:

\[
J(R, D_R) = \mathbb{E} \left\{ \sum_{k=1}^{R} c_k + \sum_{j=1}^{N} \left( \sum_{i=1}^{M_j} \pi_k^i \right) \mathbb{I}_{(D_R=j)} \right\}
\]

which in turn can be rewritten as follows:

\[
J(R, D_R) = \mathbb{E} \left\{ \sum_{k=1}^{R} c_k + \sum_{j=1}^{N} \pi_k^j \mathbb{I}_{(D_R=j)} \right\}
\]

where \( M_j = [M_{1,j}, M_{2,j}, \ldots, M_{N,j}] \).

To obtain the optimum stopping time \( R^* \), we must first obtain the optimum decision rule \( D_R^* \) for any given \( R \). In the process of finding the optimum decision rule, we need to find a lower bound (independent of \( D_R \)) for the second term inside the expectation in Eq. (7), which is the part of the equation that depends on \( D_R \). Theorem 1 provides such bound.

**Theorem 1.** For any classification rule \( D_R \) given stopping time \( R \), \( \sum_{j=1}^{N} M_j^T \pi_k \mathbb{I}_{(D_R=j)} \geq g(\pi_k) \), where \( g(\pi_k) \) is the optimum decision rule defined as follows:

\[
D_R^\text{optimum} = \arg \min_{1 \leq j \leq N} \left[ M_j^T \pi_k \right].
\]

From Theorem 1, we conclude that:

\[
J(R, D_R) \geq J(R, D_R^\text{optimum}),
\]

where \( J(R, D_R^\text{optimum}) = \min_{D_R} J(R, D_R) \).

Thus, we can reduce the cost function in Eq. (7) to one which depends only on the stopping time \( R \) as follows:

\[
J(R) = \mathbb{E} \left\{ \sum_{k=1}^{R} c_k + g(\pi_k) \right\}.
\]

To optimize the cost function in Eq. (10) with respect to \( R \), we need to solve the following optimization problem:

\[
\min_{R \geq 0} J(R) = \min_{R \geq 0} \mathbb{E} \left\{ \sum_{k=1}^{R} c_k + g(\pi_k) \right\}.
\]

Since \( R \in \{0, 1, \ldots, K\} \), the optimum strategy consists of a maximum of \( K + 1 \) stages, where the optimum solution must minimize the corresponding average cost going from stages 0 to \( K \). The solution can be obtained using dynamic programming.

**Theorem 2.** For \( k = K - 1, \ldots, 0 \), function \( \hat{J}_k(\pi_k) \) is related to \( J_{k+1}(\pi_{k+1}) \) through the equation:

\[
\hat{J}_k(\pi_k) = \min \left\{ g(\pi_k), c_{k+1} + \sum_{F_{k+1}} \frac{\Delta_{k+1}(F_{k+1}) \pi_k}{\Delta_{k+1}^T(F_{k+1}) \pi_k} \right\},
\]

where \( J_K(\pi_K) = g(\pi_K) \).

The optimum stopping strategy derived from Eq. (12) has a very intuitive structure. Specifically, it stops at stage \( k \), where the cost of stopping (the first expression in the minimization) is no greater than the expected cost of continuing given all information accumulated at the current stage \( k \) (the second expression in the minimization). Equivalently, at each stage \( k \), our method faces two options given \( \pi_k \): (i) stop evaluating features and select optimally between the \( N \) classes, or (ii) continue with the next feature. The cost of stopping is \( g(\pi_k) \), whereas the cost of continuing is \( c_{k+1} + \sum_{F_{k+1}} \frac{\Delta_{k+1}(F_{k+1}) \pi_k}{\Delta_{k+1}^T(F_{k+1}) \pi_k} \).

Based on Lemma 1 and Theorems 1 and 2, we present ETANA, an on-the-fly Feature selection and classification Algorithm. Initially, the posterior probability vector \( \pi_0 \) is set to \( [p_1, p_2, \ldots, p_N] \), and the two terms in Eq. (12) are compared. If the first term is less than or equal to the second term, ETANA classifies the instance under examination to the appropriate class, based on the optimum rule in Eq. (8). Otherwise, the first feature is evaluated. ETANA repeats these steps until either it decides to classify the instance using \( < K \) features, or using all \( K \) features.

To implement ETANA, we first need to solve the dynamic programming recursion in Eq. (12). This can be achieved by quantizing the interval \([0,1]\) over \( N \) values such that
Lemma 4 summarizes the key properties enjoyed by this function.

Lemma 4. The functions \( \bar{J}_k(\varpi) \), \( k = 0, \ldots, K-1 \), are concave, continuous, and piecewise linear.

The fact that \( g(\varpi) \) and \( \bar{J}_k(\varpi) \) are concave and piecewise linear allows for a compact representation of these functions. Recall that according to Theorem 2 we stop at stage \( k \) whenever \( g(\varpi) \leq \bar{C}_{k+1}(\varpi) \), where \( \bar{C}_{k+1}(\varpi) \) is the optimum cost-to-go at stage \( k \) given by

\[
\bar{C}_{k+1}(\varpi) = c_{k+1} + \sum_{k+1} \Delta_{k+1}^T(F_{k+1})\varpi \times \bar{J}_{k+1}(\text{diag}(\Delta_{k+1}(F_{k+1}))\varpi) \Delta_{k+1}(F_{k+1})\varpi).
\]

In particular, to decide between continuing and stopping, it is sufficient to keep track of the thresholds at the intersections of \( g(\varpi) \) with every \( \bar{C}_{k+1}(\varpi) \) as stated in Theorem 5 below.

Theorem 3. At every stage \( k \), there exists at most \( N \) threshold curves that separate the unit simplex into regions which alternatively switch between continuation to the next stage and stopping. In particular, the region starting from every corner of the \( N-1 \) dimensional unit simplex always corresponds to stopping the feature evaluation process.

Theorem 3 turns out to be very important. Specifically, the region where the \textit{a posteriori} probability vector \( \pi_R \) falls into will help decide between continuing to the next stage or stopping. This provides an alternative fast implementation of the optimum solution using thresholds. Fig. 3 shows a visualization of Theorem 3; both sub–figures contain maximum number of threshold curves (i.e., 3 since \( N = 3 \)).

4.1 Stochastic Gradient Algorithm for Estimating Optimum Linear Thresholds

We propose a stochastic gradient algorithm to estimate the threshold curves described in Theorem 3. For ease of implementation, we restrict the approximation to linear threshold curves of the form given in Eq. (15).

Let \( \theta_{D,R} = [\theta_{D,R}^1, \theta_{D,R}^2, \ldots, \theta_{D,R}^N] \) denote the parameters of a linear hyperplane, where \( R \) is the number of features evaluated so far, and \( D,R = j, R \in \{0, 1, \ldots, K\}, j \in \{1, \ldots, N\} \) represents a decision choice. Then, the decision \( Z_{\theta_D} \) to “stop” or “continue” at each stage \( R \) under the decision choice \( D,R = j \), as function of \( \varpi \), is defined as follows:

\[
Z_{\theta_D}(\varpi) = \begin{cases} 
\text{stop}, & \text{if } \theta_{D,R}^T \varpi \leq 0 \\
\text{continue}, & \text{otherwise}
\end{cases}
\] (15)

Decision \( Z_{\theta_D} \) is indexed by \( \theta_{D,R}^T \) to show the explicit dependency of the parameters on the decision, where \( \theta_{D,R} = [\theta_{D,R}^1, \theta_{D,R}^2, \ldots, \theta_{D,R}^N] \) is the concatenation of \( \theta_{D,R} \) vectors, one for each stage \( R \). Now, recall the cost function in Eq. (10). Since we are interested in finding linear thresholds for each decision choice \( D,R = j \) independently, we use a modified version of the cost function in Eq. (10) as follows:

\[
\bar{H}(\theta_{D,R}) = \mathbb{E}_{Z_{\theta_D}} \left\{ \sum_{k=1}^{R} c_k + M_{D,R} \pi_R \right\}.
\] (16)

Algorithm 1 generates a sequence of estimates \( \theta_{D,t} \) by computing the gradient \( \nabla_{\theta_{D}} \bar{H}(\theta) \). Here, \( \theta_{D,t} \) denotes the
direction iteration literature [43]. SPSA algorithm estimates the gradient at each computed using a simulation–based gradient estimator. For where dependency of \( \tilde{\theta} \) is intractable due to the non–linear estimate of \( \theta \) of iterations reached.

Output: \( \tilde{\theta} \)

Require: Initial parameters \( \theta_D,0 \)

Output: Optimal parameters \( \theta_D, opt \)

1: for iterations \( t = 0, 1, 2, \ldots \) do
2: Evaluate \( \hat{H}(\theta_D,t + \beta_t \alpha_t) \) and \( \hat{H}(\theta_D,t - \beta_t \alpha_t) \) using \( \nabla \hat{\theta} \hat{H}(\theta_D,t) \) using \( \nabla \theta D \theta H(\theta_D,t-1) \) using \( \nabla \theta D \theta H(\theta_D,t) \)
3: Estimate \( \nabla \theta D \theta H(\theta_D,t) \) using Eq. (16)
4: Update \( \theta_D, t \) to \( \theta_D, t+1 \) using Eq. (17)
5: Stop if \( \| \nabla \theta D \theta H(\theta_D,t) \|_2 \leq \varepsilon \) or maximum number \( t_{max} \) of iterations reached
6: end for
7: return \( \theta_D, opt \)

estimate of \( \theta_D \) at iteration \( t \). Although evaluating the gradient in closed form is intractable due to the non–linear dependency of \( \hat{H}(\theta_D) \) and \( \theta_D \), estimate \( \nabla \theta D \theta H(\theta_D) \) can be computed using a simulation–based gradient estimator. For simplicity, we opted for the SPSA algorithm [42], among the several simulation–based gradient estimators in the literature [43]. SPSA algorithm estimates the gradient at each iteration \( t \) using a finite difference method and a random direction \( \alpha_t \), as follows:

\[
\nabla \theta D \theta H(\theta_D,t) = \frac{\hat{H}(\theta_D,t + \beta_t \alpha_t) - \hat{H}(\theta_D,t - \beta_t \alpha_t)}{2\beta_t}, (17)
\]

where \( \alpha_t = \begin{cases} -1, & \text{with probability 0.5} \\ +1, & \text{with probability 0.5} \end{cases} \).

Using the gradient estimate in Eq. (17), parameter \( \theta_D, t \) is updated as follows:

\[
\theta_D, t+1 = \theta_D, t + \alpha_t \nabla \theta D \theta H(\theta_D,t), \quad (18)
\]

where \( \alpha_t \) and \( \beta_t \) are typically chosen as in [42]:

\[
a_t = \varepsilon(t + 1 + \varsigma)^{-\kappa}, \quad 0.5 < \kappa < 1, \quad \varepsilon, \varsigma > 0, \\
\beta_t = \mu(t + 1)^{-v}, \quad 0.5 < v < 1, \quad \mu > 0.
\]

Algorithm 1 is guaranteed to converge to a local minimum with probability one [42]. We consider the following stopping criteria: \( \| \nabla \theta D \theta H(\theta_D,t) \|_2 \leq \varepsilon \), or algorithm stops when it reaches a user–defined maximum number of iterations. Finally, \( \hat{H}(\cdot) \) in Eq. (16) is estimated using Function 2.

Function 2 \( \hat{H}(\cdot) \)

Require: parameter \( \theta_D \) and \( \pi_0 \)

Output: \( \hat{H}(\theta_D) \)

Initialization : \( k = 0 \) and \( \hat{H} = 0 \)
1: while \( \theta_T D, k \pi_k \geq 0 \) do
2: \( k = k + 1 \)
3: Obtain a new feature \( F_k \)
4: Update \( \pi_k \) using Eq. (4)
5: \( \hat{H} = \hat{H} + c_k \)
6: end while
7: \( \hat{H} = \hat{H} + M_T D, k \pi_k \)
8: return \( \hat{H} \)

5 Experimental Results

In this section, we conduct an extensive set of experiments to evaluate the performance of ETANA and F–ETANA using seven benchmark datasets: 4 DNA Microarray Datasets (Lung Cancer, Lung2, MLL, Car) [41], 2 NIPS 2003 feature selection challenge datasets (Dexter, Madelon) [44], and 1 high dimensional dataset (News20) [45]. Table 1 summarises these datasets. For Madelon, Dexter and MLL datasets, we use the originally provided training and validation sets,
5.1 Practical Considerations

Here, we discuss some practical considerations. We use a smoothed maximum likelihood estimator to estimate $p(F_k|T_i), k = 1, \ldots, K, i = 1, \ldots, N$, after quantizing the feature space. Specifically, $\hat{p}(F_k|T_i) = \frac{S_{k,i}}{T_i}$, where $S_{k,i}$ denotes the number of samples that satisfy $F_k = f_k$ and belong to class $T_i$, $S_i$ denotes the total number of samples belonging to class $T_i$, and $V$ is the number of bins considered. The effect of the number $V$ of bins on the performance of our algorithm is studied in Section 5.2. We estimate the a priori probabilities as $P(T_i) = \frac{S_i}{\sum_{i=1}^{N} S_i}, i = 1, \ldots, N$.

Feature ordering is crucial for early stopping. Different features can hinder or facilitate the quick identification of the class of which an instance may belong to. Consider an example of classifying fruits as either ‘Apple’ or ‘Orange’ using two features $F_1$ and $F_2$, where $F_1$ is the color of the fruit, and $F_2$ is the weight of the fruit. Intuitively, the color of the fruit can potentially simplify the classification process as compared to the weight of the fruit. As a result, if feature $F_2$ was to be examined first, it would be very probable for feature $F_1$ to be examined as well to improve the chances of accurate classification. Instead, if $F_1$ was to be evaluated first, a decision could be made using one feature only. To avoid the computational complexity of evaluating all $K!$ possible feature orderings, we sort features in increasing order of the sum of type I and II errors (considering the true class as the positive class and all the rest classes as a single negative class), scaled by the cost coefficient of the $n$th feature to promote low cost features that at the same time are expected to result in few errors. Finally, for F–ETANA, we set $\epsilon = 10^{-5}$, and $t_{\text{max}} = 10^5$ as the stopping criteria in Algorithm 1.

5.2 Effect of Feature Space Quantization

In Section 5.1, we estimated the conditional probabilities of features given the class using a data binning technique (i.e., $\hat{p}(F_k|T_i) = \frac{N_{k+1}}{N_i+V}$). In this subsection, we analyze the effect of the number $V$ of bins on ETANA using four datasets (Lung, Dexter, Madelon, MLL).

In Fig. 4, we plot the variation of the accuracy, the average number of features used for classification, and the training time as a function of $V$. ETANA’s accuracy and the average number of features used for classification is relatively robust to the number of bins (see Fig. 4(a) and Fig. 4(b)). However, increasing the number of bins from 50 to 100 results in a drop in accuracy for the MLL dataset. Most probably, this is due to overfitting as a result of increasing the resolution of the feature space to a very high value. The linear relationship between training time and the number of bins (see Fig. 4(c)) is due to Eq. (12). Without any loss of generality, in the rest of the experiments we set $V$ to the number of class variables (i.e., $N$) in the evaluating dataset.

5.3 Accuracy as a Function of the Average Number of Features Used

To study the behavior of ETANA for varying values of feature evaluation cost $c$, when all features incur same cost (i.e., $c_k = c$), we measured accuracy for constant misclassification costs (i.e., $M_{ij}, j = 1 \forall i \neq j, M_{j,j} = 0, i,j \in \{1, \ldots, N\}$) and $c = \{0.1, 0.08, 0.06, 0.04, 0.02, 0.01, 0.001, 0\}$. Different $c$ values result in different number of features and levels of accuracy. Intuitively, using a small potion of the total feature set leads to low accuracy, whereas when the average number of features used increases, the performance improves dramatically. From here onwards, unless specified, we report results for $c = 0.01$.

![Fig. 4: Variation of (a) accuracy, (b) average number of features, and (c) training time (sec) as a function of the number of bins using Lung Cancer, Dexter, Madelon and MLL datasets.](Image)

### TABLE 1: Datasets used in our experiments.

| Dataset    | # Instances | # Features | # classes |
|------------|-------------|------------|-----------|
| Madelon    | 2,000       | 500        | 2         |
| Lung       | 181         | 12,533     | 2         |
| MLL        | 72          | 5,848      | 3         |
| Dexter     | 300         | 20,000     | 2         |
| Car        | 174         | 9,182      | 11        |
| Lung2      | 203         | 3,312      | 5         |
| News20     | 19,996      | 1,955,191  | 2         |
TABLE 2: Comparison of accuracy. The highest accuracy is bolded and gray–shaded. The second highest value is also gray shaded. Cells are marked with ‘−’ if the corresponding method was unable to generate results within a cutoff time of 12 days.

| Dataset   | ETANA | F–ETANA | OPS–Density | OPS–A3M | SAOLA | Fast–OSFS | OSFS | Alpha–Investing |
|-----------|-------|---------|-------------|---------|-------|-----------|------|-----------------|
| Madelon   | 0.6233| 0.555   | 0.5117      | 0.5117  | 0.5817| 0.5417    | 0.5817| 0.6050          |
| Lung C.   | 0.9673| 0.9835  | 0.9779      | 0.9557  | 0.9890| 0.9890    | 0.9724| 0.9613          |
| MLL       | 1.00  | 1.00    | 0.9333      | 0.8667  | 0.8667| 0.8000    | 0.8000| 0.9333          |
| Dexter    | 0.84  | 0.84    | 0.8167      | 0.7800  | 0.7800| 0.7800    | 0.7800| 0.7600          |
| Car       | 0.9217| 0.9217  | 0.5874      | 0.7574  | 0.7882| 0.5809    | 0.5000| 0.5429          |
| Lung2     | 0.8720| 0.8818  | 0.8968      | 0.8865  | 0.8817| 0.8420    | 0.8471| 0.8820          |
| News20    | 0.7352| 0.6346  | 0.7846      | 0.7846  | 0.7846| 0.7846    | 0.7846| 0.7846          |

TABLE 3: Comparison of average number of features used. Values corresponding to highest and second highest accuracy are bolded and gray–shaded, and gray–shaded accordingly. Cells are marked with ‘−’ if the corresponding method was unable to generate results within a cutoff time of 12 days.

| Dataset   | ETANA | F–ETANA | OPS–Density | OPS–A3M | SAOLA | Fast–OSFS | OSFS | Alpha–Investing |
|-----------|-------|---------|-------------|---------|-------|-----------|------|-----------------|
| Madelon   | 4.09  | 211.21  | 2           | 2       | 3     | 3         | 3    | 4               |
| Lung C.   | 2.78  | 8.48    | 25.40       | 8.20    | 52    | 6.8       | 4.0  | 4.6             |
| MLL       | 5.07  | 10.8    | 12          | 9       | 28    | 5         | 3    | 7               |
| Dexter    | 6.32  | 7.04    | 10          | 98      | 21    | 9         | 6    | 1               |
| Car       | 13.85 | 389.38  | 34.8        | 36.6    | 41.40 | 8.4       | 4.4  | 24.4            |
| Lung2     | 13.77 | 71.66   | 14.2        | 18.4    | 28.2  | 9.4       | 5.8  | 34.4            |
| News20    | 81.70 | 4000.6  | 0           | 0       | 241.8 | 0         | 0    | 0               |

TABLE 4: Comparison of time (in seconds) required for feature selection (F), classification (C), joint feature selection and classification (F+C), and model training (T). Values corresponding to highest and second highest accuracy are bolded and gray–shaded, and gray–shaded accordingly. Cells are marked with ‘−’ if the corresponding method was unable to generate results within a cutoff time of 12 days.

| Dataset   | ETANA | F–ETANA | Time |
|-----------|-------|---------|------|
| Madelon   | 0.062 | 3.678   | 117  |
| Lung C.   | 0.141 | 4.34    | 037  |
| MLL       | 0.003 | 0.144   | 0.70 |
| Dexter    | 3.680 | 2.392   | 023  |
| Car       | 15.376| 2.28    | 014  |
| Lung2     | 0.001 | 0.001   | 004  |
| News20    | 0.047 | 0.046   | 001  |

5.4 Comparison with Online Feature Selection Methods

In this subsection, we compare ETANA and F–ETANA with the following state–of–the–art feature selection methods: OPS–Density [7], OPS–A3M [9], SAOLA [6], OSFS [5], Fast–OSFS [5], and Alpha–Investing [4]. We use KNN classifier with three neighbours to evaluate a selected feature subset, which has been shown to outperform SVM, CART, and J48 classifiers on the datasets used in [6, 7]. For SAOLA, OSFS, and Fast–OSFS, the parameter α is set to 0.01 [5, 6]. For Alpha–Investing, parameters are set to the values used in [4].

We summarize our observations from Tables 2, 3, 4 by dataset as follows.

Madelon: ETANA achieves the highest accuracy using only ~4 features on average. In fact, ETANA achieves an improvement of 3% in accuracy over Alpha–Investing, which has the highest accuracy among all the baselines using the same number of features. At the same time, ETANA is 7.5%, and 67.8% faster in joint feature selection and classification, and model training respectively, compared to Alpha–Investing.

Lung Cancer: SAOLA and Fast–OSFS achieve the highest accuracy using 52, and 6.8 features, respectively, but require 18.7, and 2.4 times more features respectively, compared to ETANA for a difference of 2% in accuracy. Further, ETANA is much faster in joint feature selection and classification compared to SAOLA and Fast–OSFS.

MLL: ETANA and F–ETANA achieve 100% accuracy using 5.07, and 10.8 features on average, respectively. This corre-
OFSD–Density achieves the highest accuracy with Lung2: compared to SAOLA, the best performing baseline. This corresponds to an improvement of 3% on average. ETANA is a close second while using only 13.85 features compared to ETANA. The rest of the online feature selection methods were unable to generate results within a cutoff time of 12 days. Although SAOLA achieves the highest accuracy, it requires ∼200% more features and is much slower runtime. Dexter: OFSD–Density achieves the highest accuracy, but requires 58.2% more features. This results in a significant slowdown for joint feature selection and classification compared to ETANA. Car: F–ETANA achieves the highest accuracy, however, ETANA is a close second while using only 13.85 features on average. This corresponds to an improvement of 3% and 66.5% in accuracy and average number of features used respectively, while at the same time, leads to a faster runtime compared to SAOLA, the best performing baseline. Lung2: OFSD–Density achieves the highest accuracy with 3.1% more features as compared to ETANA, and much slower runtime.

5.5 Performance Assessment on a High Dimensional Dataset

In this subsection, we discuss the performance of our algorithms, ETANA and F–ETANA, and the state–of–the–art online feature selection methods on the News20 dataset. Experiments on this dataset are conducted using the high performance computing cluster provided by the Information Technology Services at the University of Albany, SUNY. We used one node with 20 Intel(R) Xeon(R) E5-2680 v4 @2.40GHz CPUs with 256 GB memory. Except for SAOLA, the rest of the online feature selection methods were unable to generate results within a cutoff time of 12 days. Although SAOLA achieves the highest accuracy, it requires -200% more features and is -20 times slower in joint feature selection and classification compared to ETANA for c = 0.001 (see the last row in Tables 2, 3, 4).

5.6 F–ETANA versus ETANA

Thus far we have shown that ETANA outperforms all baselines in terms of accuracy, number of features, and time required for joint feature selection and classification. The limitation of ETANA is in its training time (see Table 4), due to the construction of a (K+1) × d matrix which grows exponentially with the number of classes (see Section 3). F–ETANA drastically reduces the time required for model training as compared to ETANA without sacrificing accuracy (see Fig 5). At the same time, F–ETANA requires more features per data instance compared to ETANA (see Table 5).

6 Conclusion

This paper investigated a new research problem, on–the–fly joint feature selection and classification, which aims to minimize the number of feature evaluations per data instance for fast and accurate classification. Specifically, an optimization problem was defined in terms of the cost of evaluating features and the Bayes risk associated with the classification decision. The optimum solution was derived using dynamic programming and it was shown that the corresponding functions are concave, continuous and piecewise linear. Two algorithms, ETANA and F–ETANA were proposed based on the optimum solution and its properties. The proposed algorithms outperformed state–of–the–art feature selection methods in terms of the average number of features used, classification accuracy, and the time required for on–the–fly joint feature selection and classification. Furthermore, F–ETANA resulted in a drastic reduction in model training time compared to ETANA. As a part of our future work, we plan to exploit feature dependencies, which may improve performance even more.

Appendix A

A.1 Proof of Lemma 1

We start from the definition of the a posteriori probability vector, i.e.:

\[
\pi_k = [\pi_k^1, \pi_k^2, \ldots, \pi_k^N]^T,
\]

and consider any element in this vector, i.e., \(\pi_k^i\). Specifically, we use Bayes’ rule and the law of total probability to get the following result:

\[
\pi_k^i = \frac{P(T_i|F_1, \ldots, F_k)}{P(F_1, \ldots, F_k|T_i)} P(T_i) = \frac{P(F_1, \ldots, F_k|T_i) P(T_i)}{\sum_{j=1}^{N} P(F_1, \ldots, F_k|T_j) P(T_j)} = \frac{P(F_1, \ldots, F_k|T_i) P(T_i)}{\sum_{j=1}^{N} P(F_1, \ldots, F_k|T_j) P(T_j)}.
\]

Note that we can further simplify Eq. [21] by exploiting the conditional independence of the features in set \(F\) given the class variable \(T\) as follows:

\[
\pi_k^i = \frac{P(T_i) \prod_{n=1}^{k} P(F_n|T_i)}{\sum_{j=1}^{N} P(T_j) \prod_{n=1}^{k} P(F_n|T_j)} = \frac{p_i \prod_{n=1}^{k} P(F_n|T_i)}{\sum_{j=1}^{N} p_j \prod_{n=1}^{k} P(F_n|T_j)}.
\]

Similarly, \(\pi_{k-1}^i\) will take the following form:

\[
\pi_{k-1}^i = \frac{p_i \prod_{n=1}^{k-1} P(F_n|T_i)}{\sum_{j=1}^{N} p_j \prod_{n=1}^{k-1} P(F_n|T_j)}.
\]
We can now rewrite \( \pi_k^i \) in Eq. (22) in terms of \( \pi_{k-1}^i \) in Eq. (23) as follows:

\[
\begin{align*}
\pi_k^i &= \frac{p_i \prod_{n=1}^k P(F_n | T_i)}{\sum_{j=1}^K p_j \prod_{n=1}^{k-1} P(F_n | T_j)} \\
&= \frac{P(F_k | T_i) \left( p_i \prod_{n=1}^{k-1} P(F_n | T_i) \right)}{\sum_{j=1}^K p_j \prod_{n=1}^{k-1} P(F_n | T_j)} \\
&= \frac{P(F_k | T_i) \left( \pi_{k-1}^i \sum_{j=1}^K p_j \prod_{n=1}^{k-1} P(F_n | T_j) \right)}{\sum_{j=1}^K p_j \prod_{n=1}^{k-1} P(F_n | T_j)} \\
&= \frac{P(F_k | T_i) \pi_{k-1}^i}{\sum_{j=1}^K p_j \prod_{n=1}^{k-1} P(F_n | T_j)} \\
&= \pi_{k-1}^i P(F_k | T_i) \pi_{k-1}^i.
\end{align*}
\]

Finally, using the above result, the a posteriori probability vector takes the following form:

\[
\pi_k = [\pi_k^1, \pi_k^2, \ldots, \pi_k^N]^T = \text{diag}(\{P(F_k | T_1), \ldots, P(F_k | T_N)\}) [\pi_{k-1}^1, \ldots, \pi_{k-1}^N]^T
\]

where \( \text{diag}(A) \) denotes a diagonal matrix with diagonal elements being the elements in vector \( A \), and \( \pi_0 = [p_1, p_2, \ldots, p_N]^T \).

**A.2 Proof of Lemma 2**

Using the law of total probability, we can write the probability \( P(D_R = j, T_i) \) as follows:

\[
P(D_R = j, T_i) = \sum_{k=1}^K P(R = k, D_k = j, T_i)
\]

Using the fact that the event \( \{ R = k, D_k = j \} \) depends only on the set \( \{F_1, F_2, \ldots, F_k\} \), and by the definition \( P(R = k, D_k = j) = E\{1_{\{R=k\}} \mathbb{1}_{\{D_k=j\}}\} \), Eq. (26) can be written as follows:

\[
P(D_R = j, T_i) = \sum_{k=1}^K \sum_{F_1, F_2, \ldots, F_k} 1_{\{R=k\}} \mathbb{1}_{\{D_k=j\}} P(T_i) P(F_1, \ldots, F_k | T_i).
\]

Then, using the result in Eq. (20), we can incorporate \( \pi_k^i \) in Eq. (27) as follows:

\[
P(D_R = j, T_i) = \sum_{k=1}^K \sum_{F_1, F_2, \ldots, F_k} 1_{\{R=k\}} \mathbb{1}_{\{D_k=j\}} \pi_k^i \times P(F_1, \ldots, F_k | T_i).
\]

Further, from the definition of the expectation operator (i.e., if a random variable \( Y \) has set \( D \) of possible values and probability mass function \( P(Y) \), then the expected value \( \mathbb{E}\{H(Y)\} \) of any function \( H(Y) \) equals \( \sum_{Y \in D} P(Y)H(Y) \)), Eq. (28) can be rewritten as follows:

\[
P(D_R = j, T_i) = \sum_{k=1}^K \mathbb{1}_{\{R=k\}} \mathbb{1}_{\{D_k=j\}} \pi_k^i.
\]

By the linearity of expectation in Eq. (29), we get:

\[
P(D_R = j, T_i) = \sum_{k=1}^K \mathbb{1}_{\{R=k\}} \mathbb{1}_{\{D_k=j\}} \pi_k^i.
\]

Finally, using the fact that \( x_R = \sum_{k=0}^K x_k \mathbb{1}_{\{R=k\}} \), Eq. (30) will end up in the desired form as shown below:

\[
P(D_R = j, T_i) = \mathbb{1}_{\{\pi_R \mathbb{1}_{\{D_R=j\}}\}}.
\]

**A.3 Proof of Theorem 1**

At any stopping time \( R \), the optimum decision \( D_R \) takes only one out of \( N \) possibilities such that the misclassification cost is minimum. In the process of finding this optimum decision \( D_R \), it is important to note that \( \sum_{j=0}^N \mathbb{1}_{\{D_R=j\}} = 1 \), where \( \mathbb{1}_A \) is the indicator function for event \( A \) (i.e., \( \mathbb{1}_A = 1 \) when \( A \) occurs, and \( \mathbb{1}_A = 0 \) otherwise), which implies that only one of the terms in the sum becomes 1, while the remaining terms are equal to zero. We note that:

\[
M_j^T \pi_R \geq g(\pi_R), \quad \forall j \in \{1, 2, \ldots, N\},
\]

where \( g(\pi_R) = \min_{1 \leq j \leq N} [M_j^T \pi_R] \). Then, using the fact that \( \mathbb{1}_{\{D_R=j\}} \) is non-negative, we get the following result:

\[
N \sum_{j=1}^N (M_j^T \pi_R) \mathbb{1}_{\{D_R=j\}} = g(\pi_R).
\]

We underscore that the lower bound \( g(\pi_R) \) derived above is independent of the decision \( D_R \). Thus, it is obvious that this lower bound can be achieved only by the rule defined in Eq. (9), which is therefore the optimum decision for a given stopping time \( R \).

**A.4 Proof of Theorem 2**

At the end of the \( K \)th stage, assuming that all the features have been examined, the only remaining expected cost is the optimum misclassification cost of selecting among \( N \) decision choices at stage \( k = K \), which is \( J_K(\pi_K) = g(\pi_K) \) (see Theorem 1).

Then, consider any intermediate stage \( k = 0, 1, \ldots, K-1 \). Being at stage \( k \), with the available information \( \pi_k \), the optimum strategy has to choose between, either to terminate the feature evaluation process and incur cost \( g(\pi_k) \), which is the optimum misclassification cost of selecting among \( N \) decision choices (see Theorem 1), or continue and incur cost of \( c_{k+1} \) to evaluate feature \( F_{k+1} \) and an additional cost \( \hat{J}_{k+1}(\pi_{k+1}) \) to continue optimally. Thus, the total cost of continuing optimally (referred to as optimum cost-to-go \( [40] \)) is \( c_{k+1} + \hat{J}_{k+1}(\pi_{k+1}) \). It is important to note that at stage \( k \), we do not know the outcome of examining feature \( F_{k+1} \). Thus, we need to consider the expected optimum cost-to-go, which is equal to \( c_{k+1} + \mathbb{E}\{\hat{J}_{k+1}(\pi_{k+1})|\pi_{k}\} \). Using Lemma 1 to express \( \pi_{k+1} \) in terms of \( \pi_k \), and by the definition of the expectation operator (i.e., if a random variable \( Y \) has set
of this function. Finally, minimization over finite minimum of linear functions is a concave, piecewise linear function.

Finally, substituting Eq. (37) in Eq. (34), we get the desired

\[
\tilde{c}_{k+1}(\pi_k) = c_{k+1} + \sum_{F_{k+1}} P(F_{k+1}|F_1, F_2, \ldots, F_k) \\
\times \tilde{J}_{k+1}\left(\frac{\text{diag}(\Delta_{k+1}(F_{k+1})) \pi_k}{\Delta_{k+1}(F_{k+1}) \pi_k}\right)
\]  

(34)

Let us simplify the term \(P(F_{k+1}|F_1, F_2, \ldots, F_k)\) separately. Specifically, using Bayes’ rule and the law of total probability, we observe that:

\[
P(F_{k+1}|F_1, F_2, \ldots, F_k) = \frac{P(F_{k+1})}{P(F_1, F_2, \ldots, F_k)} = \frac{\sum_{j=1}^{N} P(F_{k+1}) \prod_{n=1}^{k+1} P(F_n|T_j)}{\sum_{j=1}^{N} \prod_{n=1}^{k} P(F_n|T_j) P(T_j)}
\]

(35)

Note that we can further simplify Eq. (35) by exploiting the fact that the random variables \(F_k\) are independent under each class \(T_j\) as follows:

\[
P(F_{k+1}|F_1, F_2, \ldots, F_k) = \frac{\sum_{j=1}^{N} P(T_j) \prod_{n=1}^{k+1} P(F_n|T_j)}{\sum_{j=1}^{N} \prod_{n=1}^{k} P(F_n|T_j) P(T_j)}
\]

(36)

Using the result in Eq. (22), we can simplify Eq. (36) as follows:

\[
P(F_{k+1}|F_1, F_2, \ldots, F_k) = \sum_{j=1}^{N} \pi_j P(F_{k+1}|T_j) = \Delta_{k+1}(F_{k+1}) \pi_k.
\]

(37)

Finally, substituting Eq. (37) in Eq. (34), we get the desired result:

\[
\tilde{c}_{k+1}(\pi_k) = c_{k+1} + \sum_{F_{k+1}} \Delta_{k+1}(F_{k+1}) \pi_k \\
\times \tilde{J}_{k+1}\left(\frac{\text{diag}(\Delta_{k+1}(F_{k+1})) \pi_k}{\Delta_{k+1}(F_{k+1}) \pi_k}\right).
\]

(38)

**APPENDIX B**

**B.1 Proof of Lemma 3**

Let us consider the definition of \(g(\varpi)\):

\[
g(\varpi) \triangleq \min_{1 \leq j \leq N} \left[ M_j^T \varpi \right], \varpi \in [0, 1]^N.
\]

(41)

The term \(M_j^T \varpi\) is linear with respect to \(\varpi\), and since the minimum of linear functions is a concave, piecewise linear function, we conclude that \(g(\varpi)\) is a concave, piecewise linear function as well. Concavity also assures the continuity of this function. Finally, minimization over finite \(N\) hyperplanes guarantees that the function \(g(\varpi)\) is made up of at most \(N\) hyperplanes.

**B.2 Proof of Lemma 4**

First, let us consider the function \(\tilde{J}_{K-1}(\varpi)\) given by:

\[
\tilde{J}_{K-1}(\varpi) = \min_{\varpi} \left[ g(\varpi), c_K + \sum_{F_{K-1}} \Delta_{K-1}(F_{K-1}) \varpi \right. \\
\times \tilde{J}_{K-1}\left(\frac{\text{diag}(\Delta_{K-1}(F_{K-1})) \varpi}{\Delta_{K-1}(F_{K-1}) \varpi}\right).
\]

(39)

Using the fact that \(\tilde{J}_K(\pi_K) = g(\pi_K)\), we can rewrite Eq. (39) as follows:

\[
\tilde{J}_{K-1}(\varpi) = \min_{\varpi} \left[ g(\varpi), c_K + \sum_{F_{K-1}} \Delta_{K-1}(F_{K-1}) \varpi \right. \\
\times g\left(\frac{\text{diag}(\Delta_{K-1}(F_{K-1})) \varpi}{\Delta_{K-1}(F_{K-1}) \varpi}\right).
\]

(40)

We focus our attention on the following function inside the summation of Eq. (40):

\[
Q(\varpi) \triangleq \Delta_{K-1}(F_{K-1}) \varpi g\left(\frac{\text{diag}(\Delta_{K-1}(F_{K-1})) \varpi}{\Delta_{K-1}(F_{K-1}) \varpi}\right).
\]

(41)

Using the definition of \(g(\varpi)\) in Eq. (41), we can rewrite Eq. (41) as follows:

\[
Q(\varpi) = \Delta_{K-1}(F_{K-1}) \varpi \min_{1 \leq j \leq N} \left[ M_j^T \text{diag}(\Delta_{K-1}(F_{K-1})) \varpi \right. \\
\times \Delta_{K-1}(F_{K-1}) \varpi g\left(\frac{\text{diag}(\Delta_{K-1}(F_{K-1})) \varpi}{\Delta_{K-1}(F_{K-1}) \varpi}\right).
\]

(42)

Note that the term \(M_j^T \text{diag}(\Delta_{K-1}(F_{K-1})) \varpi\) is linear with respect to \(\varpi\). Using the fact that the minimum of linear functions is a concave, piecewise linear function, implies that \(Q(\varpi)\) is a concave, piecewise linear function. Furthermore, we recall: i) the non–negative sum of concave/piecewise linear functions is also a concave/piecewise linear function, and ii) the minimum of two concave/piecewise linear functions is also a concave/piecewise linear function. Based on these two facts, and the fact that \(\Delta_{K-1}(F_{K-1})\) is a probability vector which is non–negative, we conclude that the function \(\tilde{J}_{K-1}(\varpi)\) in Eq. (39) is concave and piecewise linear. Concavity also assures the continuity of this function.

Then, let us consider the function \(\tilde{J}_{K-2}(\varpi)\) given by:

\[
\tilde{J}_{K-2}(\varpi) = \min_{\varpi} \left[ g(\varpi), c_{K-1} + \sum_{F_{K-1}} \Delta_{K-1}(F_{K-1}) \varpi \right. \\
\times \tilde{J}_{K-1}\left(\frac{\text{diag}(\Delta_{K-1}(F_{K-1})) \varpi}{\Delta_{K-1}(F_{K-1}) \varpi}\right).
\]

(43)

We have already proved that the functions \(\tilde{J}_{K-1}(\varpi)\) and \(g(\varpi)\) are concave and piecewise linear. Using the facts that i) the non–negative sum of concave/piecewise linear functions is also a concave/piecewise linear function, and ii) the minimum of two concave/piecewise linear functions is also a concave/piecewise linear function, we conclude that the function \(\tilde{J}_{K-2}(\varpi)\) is also concave and piecewise linear. Concavity also assures the continuity of this function. Using similar arguments, the concavity, the continuity and the piecewise linearity of functions \(\tilde{J}_k(\varpi), n = 0, \ldots, K-3,\)
can also be guaranteed.

B.3 Proof of Theorem 3

Let us start the proof by showing that all \( N \) corners of the \( N - 1 \) dimensional unit simplex always correspond to stopping irrespective of the stage. In other words, when \( \omega = e_i, g(\omega) < c_{k+1}(\omega) \), for all \( k = 0, \ldots, K - 1 \), where \( e_i \) denotes the column vector with a 1 in the \( i \)th coordinate and 0’s elsewhere. At stage \( k = K - 1 \), we have that:

\[
\begin{align*}
\hat{C}_K(\omega) &= c_K + \sum_{F_K} \Delta_T (F_K) e_i e_i, \hat{J}_K \left( \frac{\Delta_T (F_K) e_i}{\Delta_T (F_K) e_i} \right) \\
&= c_K + \sum_{F_K} \Delta_T (F_K) e_i g \left( \frac{\Delta_T (F_K) e_i}{\Delta_K (F_K) e_i} \right) \\
&= c_K + \sum_{F_K} P(F_K | T_i) g \left( \frac{P(F_K | T_i) e_i}{P(F_K | T_i)} \right) \\
&= c_K + g(e_i) \sum_{F_K} P(F_K | T_i) \\
&= c_K + g(e_i),
\end{align*}
\]

where the last inequality holds since \( c_K > 0 \). From Eq. (39), we see that \( J_{K-1}(e) = g(e) \). Then, let us consider the case \( k = K - 2 \) as follows:

\[
\begin{align*}
\hat{C}_{K-1}(\omega) &= c_{K-1} + \sum_{F_{K-1}} \Delta_T (F_{K-1}) e_i \\
&= \hat{c}_{K-1} - \sum_{F_{K-1}} P(F_{K-1} | T_i) \hat{J}_{K-1} \left( \frac{P(F_{K-1} | T_i) e_i}{P(F_{K-1} | T_i)} \right) \\
&= c_{K-1} + \sum_{F_{K-1}} P(F_{K-1} | T_i) \hat{J}_{K-1} \left( \frac{P(F_{K-1} | T_i) e_i}{P(F_{K-1} | T_i)} \right) \\
&= c_{K-1} + g(e_i) \sum_{F_{K-1}} P(F_{K-1} | T_i) \\
&= c_{K-1} + g(e_i) \\
&= c_{K-1} + g(e_i),
\end{align*}
\]

where the last inequality holds since \( c_{K-1} > 0 \). Using similar arguments, the latter result can be proven for all \( k = 0, \ldots, K - 3 \). The rest of the proof is very intuitive.

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