ADMISSIBLE SUBMONOIDS OF ARTIN-TITS MONOIDS

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ABSTRACT. We show the analogue of M"uhlherr’s [Coxeter groups in Coxeter groups, Finite Geom. and Combinatorics, Cambridge Univ. Press (1993), 277–287] for Artin-Tits monoids and for Artin-Tits groups of spherical type. That is, the submonoid (resp. subgroup) of an Artin-Tits monoid (resp. group of spherical type) induced by an admissible partition of the Coxeter graph is an Artin-Tits monoid (resp. group).

This generalizes and unifies the situation of the submonoid (resp. subgroup) of fixed elements of an Artin-Tits monoid (resp. group of spherical type) under the action of graph automorphisms, and the notion of LCM-homomorphisms defined by Crisp in [Injective maps between Artin groups, Geom. Group Theory Down Under, Canberra (1996) 119–137] and generalized by Godelle in [Morphismes injectifs entre groupes d’Artin-Tits, Algebr. Geom. Topol. 2 (2002), 519–536].

We then complete the classification of the admissible partitions for which the Coxeter graphs involved have no infinite label, started by M"uhlherr in [Some contributions to the theory of buildings based on the gate property, Dissertation, T"ubingen (1994)]. This leads us to the classification of Crisp’s LCM-homomorphisms.

INTRODUCTION.

In 1993-1994, M"uhlherr introduced the notion of admissible partitions of a Coxeter graph to define subgroups of the associated Coxeter group that inherit a Coxeter group structure from the ambient one [13, 14]. This construction generalizes the situation of the subgroup of fixed elements of a Coxeter group under the action of a group of graph automorphisms, studied by Hée in [11].

The aim of this paper is to show the analogue for Artin-Tits monoids and for Artin-Tits groups of spherical type. Like in the Coxeter case, our construction generalizes the situation of the submonoid (resp. subgroup) of fixed elements of an Artin-Tits monoid (resp. group of spherical type) under the action of a group of graph automorphisms (studied in the early 2000’s in [12, 4, 5, 7]). When only finite Coxeter graphs without infinite labels are involved, our construction — more precisely the underlying notion of morphisms between Artin-Tits monoids (or groups) — is equivalent to the notion of LCM-homomorphisms defined in 1996 by Crisp [3].

For arbitrary Coxeter graphs, our construction is more general than the notion of LCM-homomorphisms developed in 2002 by Godelle [9], which allowed finite Coxeter graphs with infinite labels, as it works for infinite Coxeter graphs and includes all the morphisms coming from actions of graph automorphisms and all the morphisms induced by the bursts of a Coxeter graph used by Paris in [15]. Moreover, we show that some important combinatorial properties of those earlier defined objects (such as their respect of simple elements and of normal forms) are still valid in our more general context.
1. Preliminaries.

1.1. Generalities on monoids.

Let $M$ be a monoid, i.e., a (non-empty) set endowed with an associative binary operation $M \times M \rightarrow M,$ $(x, y) \mapsto xy,$ with an identity element (denoted by 1). An element $x \in M$ is said to be a left (resp. right) unit if there exists $y \in M$ such that $xy = 1$ (resp. $yx = 1$). For example, 1 is a left and right unit. The monoid $M$ is said to be left (resp. right) cancellative if, for all $x, y, z \in M,$ $xy = xz$ (resp. $yx = zy$) implies $y = z$; and $M$ is said to be cancellative if it is left and right cancellative. Note that, in a left or right cancellative monoid, left units and right units coincide.

Let $S = \{ s_e \mid e \in E \}$ be a generating subset of $M$ such that the map $E \rightarrow S,$ $e \mapsto s_e,$ is one-to-one. A word $e_1 \cdots e_n$ on $E$ is a representation (on $E$) of $x \in M$ if $x = s_{e_1} \cdots s_{e_n};$ it is called reduced if it is of minimal length among all the representations of $x.$ We denote by $\ell_S(x)$ this minimal length, and call the function $\ell_S : M \rightarrow \mathbb{N}$ thus defined the length on $M$ with respect to $S.$

We denote by $\preceq$ (resp. $\succeq$) the left (resp. right) divisibility in $M,$ i.e., for $x, y \in M,$ we write $y \preceq x$ (resp. $x \succeq y$) if there exists $z \in M$ such that $x = yz$ (resp. $x = zy$). There are natural notions of gcd’s and lcm’s in $M:$ an element $d$ in $M$ is a left gcd of a non-empty subset $X \subseteq M$ if $d \preceq x$ for all $x \in X$ and if, for every $z \in M$ with this property, we get $z \preceq d$; an element $m$ in $M$ is a right lcm of a non-empty subset $X \subseteq M$ if $x \succeq m$ for all $x \in M$ and if, for every $z \in M$ with this property, we get $z \succeq z.$ The notions of right gcd and left lcm are defined symmetrically. If two elements $x, y \in M$ have a unique left (resp. right) lcm, we denote it by $x \lor y$ (resp. $x \lor_R y$); and if they have a unique left (resp. right) gcd, we denote it by $x \land y$ (resp. $x \land_R y$). Note that in a cancellative monoid with no non-trivial unit, gcd’s and lcm’s are unique when they exist.

For $x_1, \ldots, x_n \in M,$ we denote by $\prod_{k=1}^{n} x_k$ the product $x_1 x_2 \cdots x_n$ in that order. For $x, y \in M$ and $n \in \mathbb{N},$ we denote by $\prod_{k=1}^{n} (x, y)$ the product $xyx \cdots$ of $n$ terms alternatively equal to $x$ and $y$ (starting with $x$). If $M = \mathbb{N}$ endowed with the usual addition, we prefer the notation $\sum_{n} (x, y)$ for the sum $x + y + x + y + \cdots$ of $n$ terms alternatively equal to $x$ and $y$ (starting with $x$).

1.2. Generalities on Coxeter groups and Artin-Tits groups.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix over an arbitrary (non necessarily finite) set $I,$ i.e., with $m_{i,j} = m_{j,i} \in \mathbb{N}_{\geq 1} \cup \{ \infty \}$ and $m_{i,i} = 1 \Longleftrightarrow i = j.$ The matrix $\Gamma$ is usually represented by its Coxeter graph, i.e., the graph with vertex set $I,$ edge set $\{(i,j) \mid m_{i,j} \geq 3\},$ and a label $m_{i,j}$ over the edge $\{i,j\}$ if $m_{i,j} \geq 4.$ We denote by

$$W_\Gamma = \{ s_i, i \in I \mid s_i^2 = 1, \prod_{m_{i,j} \geq 3} (s_i, s_j) = \prod_{m_{i,j} \geq 3} (s_j, s_i), \text{ if } m_{i,j} \neq \infty \},$$

$$B_\Gamma = \{ s_i, i \in I \mid \prod_{m_{i,j} \geq 3} (s_i, s_j) = \prod_{m_{i,j} \geq 3} (s_j, s_i), \text{ if } m_{i,j} \neq \infty \},$$

$$B^+_\Gamma = \{ s_i, i \in I \mid \prod_{m_{i,j} \geq 3} (s_i, s_j) = \prod_{m_{i,j} \geq 3} (s_j, s_i), \text{ if } m_{i,j} \neq \infty \}^+,$$
the Coxeter group, the Artin-Tits group and the Artin-Tits monoid associated with \( \Gamma \) respectively. Note that we may use the same symbols for the generators of \( B_T \) and \( B_T^+ \) since Paris showed in [13] that \( B_T^+ \) identifies with the submonoid of \( B_T \) generated by the \( s_i, i \in I \) (he actually proved this result when \( I \) is finite, but this implies the general case). Set \( S_T = \{ s_i \mid i \in I \} \) and \( S_T = \{ s_i \mid i \in I \} \); we say that the pair \((W_T, S_T)\) (resp. \((B_T, S_T)\), resp. \((B_T^+, S_T)\)) is the Coxeter (resp. Artin-Tits, resp. positive Artin-Tits) system of type \( \Gamma \). Note that \( W_T \) is generated by \( S_T \) as a monoid. We denote by the same letter \( \ell \) the lengths on \( W_T \) with respect to \( S_T \), and on \( B_T^+ \) with respect to \( S_T \), and call them standard lengths.

Let \( \Gamma = \{ (m_{i,j})_{i,j \in I} \} \) and \( \Gamma' = \{ (m'_{i,j})_{i,j \in I}' \} \) be two Coxeter matrices. An isomorphism from \( \Gamma \) onto \( \Gamma' \) is a bijective map \( f : I \to I' \) such that \( m_{i,j} = m'_{f(i),f(j)} \) for all \( i, j \in I \). In particular, we denote by \( \text{Aut}(\Gamma) \) the automorphism group of \( \Gamma \). We say that two pairs \((G_1, S_1)\) and \((G_2, S_2)\), where \( G_i \) is a group (resp. a monoid) generated by \( S_i \) \((i = 1, 2)\), are isomorphic if there exists an isomorphism \( f : G_1 \to G_2 \) that maps \( S_1 \) onto \( S_2 \) for example, the two systems \((W_T, S_T)\) and \((W_T', S_T')\) (resp. \((B_T, S_T)\) and \((B_T', S_T)\), resp. \((B_T^+, S_T)\) and \((B_T^+, S_T')\)) are isomorphic if and only if so are \( \Gamma \) and \( \Gamma' \).

1.2.1. Simple elements.

Let \( \pi_T : B_T \to W_T \) be the canonical morphism sending \( s_i \) on \( s_i \) for all \( i \in I \).

The order of \( s_i s_j \) in \( W_T \) is exactly \( m_{i,j} \) [11] Ch. V, n° 4.3, Prop. 4. In particular, the map \( I \to S_T, i \mapsto s_i \), and hence the map \( I \to S_T, i \mapsto s_i \), are one-to-one. Tits showed in [10] Thm. 3 that two reduced representations on \( I \) of an element \( w \in W \) only differ from a finite sequence of transformations — called braid relations — of the form \( \prod_{m_{i,j}}(i,j) \sim \prod_{m_{i,j}}(j,i) \) with \( i, j \in I \) such that \( i \neq j \) and \( m_{i,j} \neq \infty \). This property makes the following definition allowable:

**Definition 1** (simple elements). The canonical morphism \( \pi_T : B_T \to W_T \) has a section \( w \mapsto w \in B_T^+ \) where \( w \) is represented on \( I \) by one (and hence any) reduced representation of \( w \) on \( I \). We say that such an element \( w \) in \( B_T^+ \) is simple and set \( W_T = \{ w \mid w \in W_T \} = \{ x \in B_T^+ \mid \ell(x) = \ell(\pi_T(x)) \} \).

1.2.2. Standard parabolicity, sphericity and irreducibility.

Let \( J \subseteq I \). We set \( \Gamma_J = \{ (m_{i,j})_{i,j \in J} \} \) (it is a Coxeter matrix) and we denote by \( W_J \) (resp. \( B_J \), resp. \( B_J^+ \)) the subgroup of \( W_T \) (resp. the subgroup of \( B_T \), resp. the submonoid of \( B_T^+ \)) generated by \( \{ s_j \mid j \in J \} \) (resp. \( \{ s_j \mid j \in J \} \)).

**Definition 2** (standard parabolicity). The subgroups \( W_J \) (resp. subgroups \( B_J \), resp. submonoids \( B_J^+ \)), \( J \subseteq I \), of \( W_T \) (resp. \( B_T \), resp. \( B_T^+ \)) are called standard parabolic (with respect to \( \Gamma \)).

The pair \((W_J, \{ s_j \mid j \in J \})\) (resp. \((B_J, \{ s_j \mid j \in J \})\), resp. \((B_J^+, \{ s_j \mid j \in J \})\)) is (isomorphic to) the Coxeter (resp. Artin-Tits, resp. positive Artin-Tits) system of type \( \Gamma_J \) (see [11] Ch. IV, n° 1.8, Thm. 2 for the Coxeter case, [14] Ch. II, Thm. 4.13) for the Artin-Tits case with \( I \) finite — which implies the general result —, the positive Artin-Tits case being obvious.

Moreover, the standard length on \( W_J \) (resp. \( B_J^+ \)) is induced by the one on \( W_T \) (resp. \( B_T^+ \)) [11] Ch. IV, n° 1.8, Cor. 4. This implies that \( W_J = W_T \cap B_J^+ \).
Definition 3 \textbf{(sphericity).} The Coxeter matrix $\Gamma_J$ is called \textit{spherical} — and the subset $J$ of $I$ is called \textit{spherical (with respect to $\Gamma$)} — if $W_J$ is finite. In that case, the subgroups $W_J$, $B_J$, and submonoid $B^+_J$ are also called \textit{spherical}.

In a finite Coxeter group, there exists a unique element of maximal standard length, which is of order two if not trivial \cite[Ch. IV, § 1, Ex. 22]{1}. If $J$ is spherical, we denote by $r_J$ the unique element of maximal standard length in $W_J$ and by $w_J$ its image in $W_J$ (\textit{i.e.} the unique element of maximal standard length in $W_J$).

Definition 4 \textbf{(irreducibility).} The matrix $\Gamma$ is said to be \textit{reducible} if there exists a partition of cardinality two $\{J, K\}$ of $I$ such that $m_{j,k} = 2$ for every pair $(j,k) \in J \times K$. In that case, we write $\Gamma = \Gamma_J \times \Gamma_K$, as we have $W_\Gamma = W_J \times W_K$, $B_\Gamma = B_J \times B_K$ and $B^+_\Gamma = B^+_J \times B^+_K$. If this is not the case, then $\Gamma$ is said to be \textit{irreducible}; this is precisely when the Coxeter graph of $\Gamma$ is connected.

We assume that the reader is familiar with the list of the irreducible spherical Coxeter graphs, which can be found for example in \cite[Ch. VI, n° 4.1, Thm. 1]{1}.

1.2.3. \textit{Properties of $B^+_\Gamma$.}

Since the defining relations of $B^+_\Gamma$ are homogeneous, the standard length of $B^+_\Gamma$ is \textit{additive}, \textit{i.e.} $\ell(xy) = \ell(x) + \ell(y)$ for all $x, y \in B^+_\Gamma$. This clearly implies that $B^+_\Gamma$ has no non-trivial unit. Moreover, $B^+_\Gamma$ is \textit{cancellative} \cite[Prop. 2.3]{2} (hence gcd’s and lcm’s are unique when they exist), and two elements of $B^+_\Gamma$ always have left and right gcd’s, and have a right (resp. left) lcm as soon as they have a right (resp. left) common multiple \cite[Props. 4.1 and 4.2]{2}.

Example 5. Let $J \subseteq I$ be non-empty. By \cite[Thm. 5.6]{2}, the elements $s_j, j \in J$, have a (left or right) lcm if and only if $\Gamma_J$ is spherical, and in that case their (left and right) lcm is $r_J$ \cite[Prop. 5.7]{2}. In particular, two elements $s_i$ and $s_j$ have a (left or right) lcm if and only if $m_{i,j} \neq \infty$, in which case $s_i \wedge_R s_j = s_i \wedge_L s_j = r_{\{i,j\}} = \prod_{m_{i,j} < \infty}(s_i, s_j) = \prod_{m_{i,j} \neq \infty}(s_j, s_i)$.

In \cite[Prop. 2.1]{12}, Michel showed that for all $x \in B^+_\Gamma$, there exists a unique maximal (for $\preceq$) element $L(x)$ in the set $\{w \in W_\Gamma \mid w \preceq x\}$ of all simple left divisors of $x$. The maximal simple right divisor $R(x)$ of $x$ is defined symmetrically.

Definition 6 \textbf{(normal forms).} The \textit{left normal form} of a non-trivial element $x \in B^+_\Gamma$ is the unique sequence $(x_1, \ldots, x_n)$ of elements of $W_\Gamma$ such that $x = x_1 \cdots x_n$, $x_n \neq 1$ and $x_k = L(x_k x_{k+1} \cdots x_n)$ for $1 \leq k \leq n-1$. \textit{Right normal forms} are defined symmetrically.

It is clear that $B^+_\Gamma$ generates $B_\Gamma$ (as a group). If $\Gamma$ is spherical, $B_\Gamma$ is more precisely the \textit{group of fractions} of $B^+_\Gamma$, \textit{i.e.} every $b \in B_\Gamma$ can be written $b = x^{-1} y = x' y'^{-1}$ for $x, y, x', y' \in B^+_\Gamma$ \cite[Prop. 5.5]{2}.

Definition 7 \textbf{(irreducible fractions).} Assume that $\Gamma$ is spherical and fix $b \in B_\Gamma$. Then \cite[Cor. 7.5]{7} shows that there exists a unique pair $(x, y)$ (resp. $(x', y')$) in $(B^+_\Gamma)^2$ such that $b = x^{-1} y$ and $x \wedge_L y = 1$ (resp. $b = x' y'^{-1}$ and $x' \wedge_R y' = 1$). We say that this pair $(x, y)$ (resp. $(x', y')$) is an \textit{irreducible left (resp. right) fraction}, and is the \textit{irreducible left (resp. right) form} of $b$. 

2. Admissible partitions — The work of Mühlherr.

In this section, we recall the definition of an admissible partition of a Coxeter graph and the principal results of [13] on the subgroup of the associated Coxeter group defined by such a partition. Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix and let $W = W_\Gamma$.

2.1. Definitions.

Definition 8 ([13]). We say that a partition $\tilde{I}$ of $I$ is spherical (with respect to $\Gamma$) — or by abuse of language is a spherical partition of $\Gamma$ — if, for all $\alpha \in \tilde{I}$, $\Gamma_\alpha$ is spherical (i.e., $W_\alpha$ is finite). In that case, we denote by

- $\tilde{S} = \{r_{\alpha} \mid \alpha \in \tilde{I}\}$ the set of all $r_{\alpha}$, $\alpha \in \tilde{I}$ (recall that $r_{\alpha}$ is the unique element of maximal standard length in $W_\alpha$),
- $\tilde{W} = (\tilde{S})$ the subgroup of $W$ generated by $\tilde{S}$,
- $\tilde{l} = l_\tilde{S}$ the length on $W$ with respect to $\tilde{S}$ ($\tilde{W}$ is generated by $\tilde{S}$ as a monoid),
- $\tilde{\Gamma} = (\{r_{\alpha}r_{\beta}\})_{\alpha,\beta \in \tilde{I}}$ the Coxeter matrix of orders of the products $r_{\alpha}r_{\beta}$ in $W$.

We call $\tilde{\Gamma}$ the type of $\tilde{I}$.

Moreover for $\alpha_1, \ldots, \alpha_n \in \tilde{I}$ and $w = \prod_{k=1}^n r_{\alpha_k} \in \tilde{W}$, we say that the word $\prod_{k=1}^n \alpha_k$ on $\tilde{I}$ is compatible — or is a compatible representation of $w$ — (with respect to $\Gamma$), if $\ell(w) = \sum_{k=1}^n \ell(r_{\alpha_k})$.

Note that we always have $\ell(w) \leq \sum_{k=1}^n \ell(r_{\alpha_k})$, and the equality holds precisely when the representation $R_{\alpha_1} \cdots R_{\alpha_n}$ of $w$ on $I$, where for $1 \leq k \leq n$ the word $R_{\alpha_k}$ is a reduced representation of $r_{\alpha_k}$ on $I$, is reduced.

Notation 9. Let $w \in W$. We set

\[
\begin{align*}
I^+(w) &= \{i \in I \mid \ell(ws_i) = \ell(w) + 1\}, \\
I^-(w) &= \{i \in I \mid \ell(ws_i) = \ell(w) - 1\}.
\end{align*}
\]

Note that $I^-(w)$ is a spherical subset of $I$ ([13] Lem. 2.8).

Definition 10 ([13]). Let $\tilde{I}$ be a partition of $I$. We say that $\tilde{I}$ is admissible (with respect to $\Gamma$) — or by abuse of language is an admissible partition of $\Gamma$ — if it is a spherical partition of $\Gamma$ such that, for all $(w, \alpha) \in \tilde{W} \times \tilde{I}$, either $\alpha \subseteq I^+(w)$ or $\alpha \subseteq I^-(w)$.

Remark 11. Let $\alpha$ be a spherical subset of $I$ and $w \in W$. Then $\alpha \subseteq I^-(w)$ (resp. $\alpha \subseteq I^+(w)$) if and only if $\ell(wr_{\alpha}) = \ell(w) - \ell(r_{\alpha})$ (resp. $\ell(wr_{\alpha}) = \ell(w) + \ell(r_{\alpha})$) ([13] Lems. 2.4 and 2.8).

2.2. Admissible partitions and Coxeter groups.

The two main results of [13] are the following theorems:

Theorem 12 ([13] Thm. 1.1]). Let $\tilde{I}$ be an admissible partition of $\Gamma$, of type $\tilde{\Gamma}$. Then the pair $(\tilde{W}, \tilde{S})$ is (isomorphic to) the Coxeter system of type $\tilde{\Gamma}$.

Theorem 13 ([13] Thm. 1.2]). Let $\tilde{I}$ be a partition of $\Gamma$. The following conditions are equivalent :

1. $\tilde{I}$ is an admissible partition of $\Gamma$,
2. for all $\alpha, \beta \in \tilde{I}$ with $\alpha \neq \beta$, $\{\alpha, \beta\}$ is an admissible partition of $\Gamma_{\alpha \cup \beta}$.
So proving the admissibility of a partition reduces to proving the admissibility of partitions of cardinality two. The following lemma gives a criterion for that. It is left as an exercise in [13], but for convenience and because it will be of great importance for our purpose, we prove it below, following [8]. Note that our condition (1b) is slightly weaker than the one of [13, Lem. 3.3]; this formulation simplifies the proof of the second part of the lemma and will be useful later in section 5. From now on, we call 2-partition a partition of cardinality two.

**Lemma 14 ([13, Lem. 3.3]).** Let \( \tilde{I} = \{\alpha, \beta\} \) be a spherical 2-partition of \( \Gamma \).

1. The following conditions are equivalent:
   a. \( \tilde{I} \) is an admissible partition of \( \Gamma \),
   b. for every integer \( 0 \leq n < |r_\alpha r_\beta| + 1 \), the words \( \prod_n (\alpha, \beta) \) and \( \prod_n (\beta, \alpha) \) are compatible.

2. If \( \Gamma \) is spherical, then \( |r_\alpha r_\beta| \neq \infty \) and conditions (1a) and (1b) above are equivalent to the following condition:
   a. the words \( \prod_{|r_\alpha r_\beta|} (\alpha, \beta) \) and \( \prod_{|r_\alpha r_\beta|} (\beta, \alpha) \) are compatible.

Moreover, we get in that case \( \prod_{|r_\alpha r_\beta|} (r_\alpha, r_\beta) = \prod_{|r_\alpha r_\beta|} (r_\beta, r_\alpha) = r_1 \).

**Proof.** The subgroup \( \tilde{W} = \langle r_\alpha, r_\beta \rangle \) of \( W \) is a dihedral group of order \( 2|r_\alpha r_\beta| \), hence the reduced representations on \( \tilde{I} \) of the elements of \( \tilde{W} \) are the words \( \prod_n (\alpha, \beta) \) and \( \prod_n (\beta, \alpha) \) for every integer \( 0 \leq n < |r_\alpha r_\beta| + 1 \).

Suppose (1a) and let us show (1b). Let \( w = \prod_n (r_\alpha, r_\beta) \in \tilde{W} \) for some \( 0 \leq n < |r_\alpha r_\beta| + 1 \). We have to show that either \( \alpha \subseteq I^+(w) \) or \( \alpha \subseteq I^-(w) \), and the same for \( \beta \). We can assume \( w \neq 1 \) (because \( \alpha \cup \beta = I = I^+(1) \)). For \( k \in \mathbb{N} \), set \( \alpha_k = \alpha \) if \( k \) is odd and \( \alpha_k = \beta \) if \( k \) is even. Since \( \prod_n (\alpha, \beta) \) is compatible, we get \( \alpha_n \subseteq I^-(w) \).

If \( |r_\alpha r_\beta| \neq \infty \) and if \( n = |r_\alpha r_\beta| \), we thus get by symmetry \( \alpha \cup \beta = I = I^-(w) \). If \( n < |r_\alpha r_\beta| \), then the word \( \prod_{n+1} (\alpha, \beta) \) is compatible, whence \( \alpha_{n+1} \subseteq I^-(wr_{\alpha_{n+1}}) \) and hence \( \alpha_{n+1} \subseteq I^+(w) \).

Suppose (1a) and let us show (1b). We first prove, by induction on \( \ell(w) \), that every \( w \in \tilde{W} \) admits a compatible representation on \( \tilde{I} \). If \( w = 1 \) this is obvious, else let \( i \in I \) be such that \( \ell(ws_i) = \ell(w) - 1 \). There is no loss of generality in assuming that \( i \in \alpha \). Since \( I \) is admissible, we have \( \alpha \subseteq I^-(w) \), and \( \ell(ws_i) = \ell(w) - \ell(r_\alpha) \). By induction, \( wr_\alpha \) admits a compatible representation \( \alpha_1 \cdots \alpha_n \), and \( \alpha_1 \cdots \alpha \) is then a compatible representation of \( w \).

Now, fix an integer \( 0 \leq n < |r_\alpha r_\beta| + 1 \) and consider the word \( \prod_n (\alpha, \beta) \). If \( n < |r_\alpha r_\beta| \), then this word is the unique reduced representation on \( \tilde{I} \) of the element \( w = \prod_n (r_\alpha, r_\beta) \in \tilde{W} \), so it must be the existing compatible representation of \( w \) (it is clear that a non-reduced word on \( \tilde{I} \) cannot be compatible). It remains to prove that, if \( |r_\alpha r_\beta| \neq \infty \) and if \( \prod_{|r_\alpha r_\beta|} (\alpha, \beta) \) is compatible, then so is \( \prod_{|r_\alpha r_\beta|} (\beta, \alpha) \). This is clear if \( |r_\alpha r_\beta| \) is even, so assume \( |r_\alpha r_\beta| \) odd and set \( w = \prod_{|r_\alpha r_\beta|} (r_\alpha, r_\beta) = \prod_{|r_\alpha r_\beta|} (r_\beta, r_\alpha) \) and \( w' = \prod_{|r_\alpha r_\beta| - 1} (r_\beta, r_\alpha) \). The word \( \prod_{|r_\alpha r_\beta| - 1} (\beta, \alpha) \) is the unique reduced representation of \( w' \), hence it is compatible and we have \( \alpha \subseteq I^-(w') \). Since \( w' \) is not the element of maximal standard length in \( W \), we get \( \beta \not\subseteq I^-(w') \), whence \( \beta \subseteq I^+(w') \) by admissibility, and hence \( \prod_{|r_\alpha r_\beta|} (\beta, \alpha) \) is a compatible representation of \( w \).

If \( \Gamma \) is spherical, then it is clear that \( |r_\alpha r_\beta| \neq \infty \) and (1b) implies (2a). Conversely, if (2a) holds, then for all \( 0 \leq n < |r_\alpha r_\beta| \), the prefix \( \prod_n (\alpha, \beta) \) of \( \prod_{|r_\alpha r_\beta|} (\alpha, \beta) \)
(resp. $\prod_n(\beta, \alpha)$ of $\prod_{[r, r]}(\beta, \alpha)$) is necessarily compatible, whence \[\Box\]. Now consider $w = \prod_{[r, r]}(r_\alpha, r_\beta) = \prod_{[r, r]}(r_\beta, r_\alpha)$ in $\tilde{W}$. Since both words $\prod_{[r, r]}(\alpha, \beta)$ and $\prod_{[r, r]}(\beta, \alpha)$ are compatible, we get $\alpha \cup \beta = \Gamma = \Gamma^{-1}(w)$, whence $w = r_I$. \qed

Let us conclude this subsection with some further properties of admissible partitions:

**Proposition 15** ([14 Prop. 3.5, A1] and [14 Lem. 2.5.5]). Let $\tilde{I}$ be an admissible partition of $\Gamma$, of type $\tilde{\Gamma}$, and let $w \in \tilde{W}$.

1. a representation of $w$ on $\tilde{I}$ is reduced if and only if it is compatible,
2. $\Gamma$ is spherical if and only if so is $\tilde{\Gamma}$, in which case $r_I = r_\tilde{I}$.

### 2.3. Examples.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix and $G$ be a subgroup of $\text{Aut}(\Gamma)$. The action of $G$ on $I$ induces an action of $G$ on $W_\Gamma$ which preserves the standard length. If $\alpha$ is an orbit of $I$ under $G$, then $G$ stabilizes $W_\alpha$ and hence, if $\alpha$ is spherical, $G$ fixes $r_\alpha$ (which is the unique element of maximal standard length in $W_\alpha$). So if we denote by $\tilde{J}$ the set of spherical orbits of $I$ under $G$, by $J = \bigcup_{\alpha \in J} \alpha \subseteq I$ their union and if we set $\tilde{S} = \{r_\alpha \mid \alpha \in \tilde{J}\}$ and $\tilde{W} = \langle \tilde{S} \rangle$, we get that $\tilde{W}$ is included in the subgroup $(W_\Gamma)^G$ of fixed points of $W_\Gamma$ under $G$, and that $\tilde{J}$ is an admissible partition of $\Gamma_J$. Let $\tilde{\Gamma}$ be the type of $\tilde{J}$.

In fact, it can be shown that $\tilde{W} = (W_\Gamma)^G$, hence $((W_\Gamma)^G, \tilde{S})$ is (isomorphic to) the Coxeter system of type $\tilde{\Gamma}$ [13 Thm. 1.3]. See [14 Cor. 3.5] for the original proof of that result.

**Example 16.** Here are symbolized the non-trivial automorphisms of the spherical irreducible Coxeter graphs, and the type of the different sets of orbits we get (see [14 section 2.5] or section [4] below for justifications):

\[
\begin{array}{llll}
A_{2n-1} \rightarrow A_{2n} \rightarrow D_{n+1} \\
B_n \downarrow \quad\quad\quad\quad\quad\downarrow \quad B_n \\
(n \geq 2) \quad (n \geq 2) \quad (n \geq 3)
\end{array}
\]

\[
\begin{array}{llllll}
D_4 \quad E_6 \quad F_4 \quad I_2(m) \\
G_2 \quad F_4 \quad I_2(8) \quad A_1 \\
6 \quad 4 \quad 8 \quad m
\end{array}
\]

**Example 17.** Here are two admissible partitions that are not the set of orbits of an action of graph automorphisms (see [14 section 2.5], subsection [33] or section [4] below for justifications):
3. Admissible partitions and Artin-Tits monoids or groups.

In subsection 3.2 below, we introduce the submonoid of an Artin-Tits monoid (resp. the subgroup of an Artin-Tits group), and the morphism between Artin-Tits monoids or groups, induced by an admissible partition of a Coxeter graph, and we establish the analogue of [13, Thm. 1.1] (cf. theorem 12 above) for Artin-Tits monoids and for Artin-Tits groups of spherical type.

In subsection 3.3 we explain how our constructions generalize the situations of the submonoids (resp. subgroups) of fixed elements of an Artin-Tits monoid (resp. group of spherical type) under the action of graph automorphisms, of the LCM-homomorphisms [3, 9], and of the morphisms between Artin-Tits monoids (or groups) induced by the bursts of a Coxeter graph [15].

In subsection 3.4, we show that some important properties of submonoids of fixed elements of an Artin-Tits monoid under the action of graph automorphisms and of LCM-homomorphisms extend to our settings. In particular, we establish them for the morphisms induced by the bursts of a Coxeter graph [15], for which they were not known when Coxeter graphs with infinite labels are involved.

But let us begin this section by recalling the notion of morphisms that respect lcm’s defined by Crisp in [3]. It is the key-tool in the proofs of the injectivity of the LCM-homomorphisms in [3, 9], and plays a similar role for our main result of subsection 3.2.

3.1. Morphisms that respect lcm’s.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ and $\tilde{\Gamma} = (\tilde{m}_{\alpha,\beta})_{\alpha,\beta \in \tilde{I}}$ be two Coxeter matrices (where $\tilde{I}$ is here an arbitrary set). If $x$ and $y$ are two elements of $B^{+}_{\Gamma}$ (resp. $B^{+}_{\tilde{\Gamma}}$), we say for short that $x \lor_R y$ exists in $B^{+}_{\tilde{\Gamma}}$ if and only if $\varphi(x) \lor_R \varphi(y)$ exists in $B^{+}_{\Gamma}$, in which case $\varphi(x) \lor_R \varphi(y) = \varphi(x \lor_R y)$.

**Definition 18 ([3, Def. 1.1])**. We say that a morphism $\varphi : B^{+}_{\Gamma} \to B^{+}_{\tilde{\Gamma}}$ respects right lcm’s if:

1. for all $\alpha \in \tilde{I}$, $\varphi(s_{\alpha}) \neq 1$,
2. for all $\alpha, \beta \in \tilde{I}$, $s_{\alpha} \lor_R s_{\beta}$ exists in $B^{+}_{\Gamma}$ if and only if $\varphi(s_{\alpha}) \lor_R \varphi(s_{\beta})$ exists in $B^{+}_{\tilde{\Gamma}}$, in which case $\varphi(s_{\alpha}) \lor_R \varphi(s_{\beta}) = \varphi(s_{\alpha} \lor_R s_{\beta})$.

Morphisms that respect left lcm’s are defined symmetrically, and we say that such a morphism respects lcm’s if it respects right and left lcm’s.

**Proposition 19 ([3, Thm. 8])**. Let $\varphi : B^{+}_{\Gamma} \to B^{+}_{\tilde{\Gamma}}$ a morphism that respects right lcm’s. Then :

1. for all $x, y \in B^{+}_{\Gamma}$, $x \lor_R y$ exists in $B^{+}_{\tilde{\Gamma}}$ if and only if $\varphi(x) \lor_R \varphi(y)$ exists in $B^{+}_{\Gamma}$, in which case $\varphi(x) \lor_R \varphi(y) = \varphi(x \lor_R y)$,
2. for all $x, y \in B^{+}$, $\varphi(x) \preceq \varphi(y) \Rightarrow x \preceq y$. In particular, $\varphi$ is injective.
Of course, the symmetrical version of proposition 19 is also true. Here is a fundamental example of morphism that respects lcm’s (cf. 3.9 and theorem 23 below):

**Lemma 20.** Let \((J_α)_{α ∈ I}\) be a family of non-empty spherical subsets of \(I\) and assume that, for all \(α, β ∈ I\), \(m_{α, β} ≠ ∞\) implies that \(Γ_{J_α∪J_β}\) is spherical and \(r_{J_α∪J_β} = \prod m_{α, β}(r_{J_α}, r_{J_β})\). Then the map \(s_α ↦ r_{J_α}\) extends to a morphism from \(B_Γ^+\) to \(B_Γ^+\). Moreover, if for all \(α, β ∈ I\), \(m_{α, β} = ∞\) implies that \(Γ_{J_α∪J_β}\) is non-spherical, then this morphism respects lcm’s.

**Proof.** The first point is clear since the hypothesis implies \(\prod m_{α, β}(r_{J_α}, r_{J_β}) = \prod m_{α, β}(r_{J_α}, r_{J_β})\) if \(m_{α, β} ≠ ∞\). Let us show the second point. We get \(φ(s_α) = r_{J_α} ≠ 1\) since \(J_α\) is non-empty. Moreover, we have the following sequence of equivalences (where the symbol \(∈\) stands for \(∀\) or \(∀_R\)):

- \(s_α ∧ s_β = φ(\prod m_{α, β}(s_α, s_β)) = \prod m_{α, β}(r_{J_α}, r_{J_β}) = r_{J_α} ∨ r_{J_β} = φ(s_α) ∨ φ(s_β)\).

### 3.2. Admissible morphisms, submonoids and subgroups.

Let \(Γ = (s_{i,j})_{i,j ∈ I}\) be a Coxeter matrix.

The admissibility of a spherical partition \(I\) of \(Γ\) can naturally be expressed in terms of simple elements in \(B_Γ^+\). Indeed, if we denote by \(W\) the image of the subgroup \(W = \{r_α \ni α ∈ I\}\) of \(W_Γ\) in \(W_Γ \subseteq B_Γ^+\), then we get that \(I\) is admissible if and only if, for all \((w, α) ∈ W × I\), either the products \(w · s_i\) are simple for all \(i ∈ α\), or \(w ≻ s_i\) for all \(i ∈ α\). In the same way, the compatibility of words on \(I\) is easy to characterize:

**Lemma 21.** Let \(I\) be a spherical partition of \(Γ\) and fix \(α_1, . . . , α_n ∈ I\). Then the word \(π_{k=1}^n α_k\) is compatible \(⇔\) the element \(π_{k=1}^n r_α_k\) is simple.

In that case, if \(w = π_{k=1}^n r_α_k\) in \(W\), then \(w = π_{k=1}^n r_α_k\) in \(W_Γ\).

**Proof.** Set \(w = \prod_{k=1}^n r_α_k = π(\prod_{k=1}^n r_α_k)\). Assume that \(π_{k=1}^n α_k\) is compatible, i.e. \(ℓ(w) = π_{k=1}^n ℓ(r_α_k)\), and fix a reduced representation \(R_α_k\) of each \(r_α_k\) on \(I\). Then the representation \(π_{k=1}^n R_α_k\) of \(w\) on \(I\) is reduced and hence, by definition of \(w\), we get \(w = \prod_{k=1}^n r_α_k\) in \(W_Γ\). Conversely, if the product \(π_{k=1}^n r_α_k\) is simple, then \(ℓ(w) = ℓ(π_{k=1}^n r_α_k) = π_{k=1}^n ℓ(r_α_k) = π_{k=1}^n ℓ(r_α_k)\) (the first and third equalities by definition of \(W_Γ\), and the second by additivity of the standard length on \(B_Γ^+\)), whence the compatibility of \(π_{k=1}^n α_k\).

This lemma allows us to reformulate the characterizations of the admissibility of a 2-partition of \(Γ\) (cf. lemma 14 above) in terms of simple elements of \(B_Γ^+\):

**Lemma 22.** Let \(I = \{α, β\}\) be a spherical 2-partition of \(Γ\).

1. The following conditions are equivalent:
   a. \(I\) is an admissible partition of \(Γ\).
   b. for every integer \(0 ≤ n < |r_α r_β| + 1\), the two elements \(π_{n}(r_α, r_β)\) and \(π_{n}(r_β, r_α)\) of \(B_Γ^+\) are simple.
2. If \(Γ\) is spherical, then \(|r_α r_β| ≠ ∞\) and conditions (1a) and (1b) above are equivalent to the following:
Remark 26. \( \text{(see theorems 28 and 33 below).} \)

under the action of graph automorphisms and of LCM-homomorphisms of \([3, 9]\) of standard parabolic submonoids or subgroups, of submonoids of fixed elements.

We are now able to prove the analogue of theorem \[\text{[22]}\] for Artin-Tits monoids and for Artin-Tits groups of spherical type:

**Theorem 23.** Let \( \tilde{I} \) be an admissible partition of \( \Gamma \), of type \( \tilde{\Gamma} \). Then:

1. the map \( S_\tilde{\Gamma} \to B_\tilde{\Gamma}^+, s_\alpha \mapsto r_\alpha \), extends to a morphism \( \varphi : B_\tilde{\Gamma}^+ \to B_\Gamma^+ \),

2. this morphism respects lcm’s, hence is injective.

In particular, if we set \( \tilde{S} = \{ r_\alpha \mid \alpha \in \tilde{I} \} \) and denote by \( \tilde{B}^+ = \langle \tilde{S} \rangle^+ \) the submonoid of \( B_\Gamma^+ \) generated by the \( r_\alpha, \alpha \in \tilde{I} \), then the pair \( (\tilde{B}^+, \tilde{S}) \) is (isomorphic to) the positive Artin-Tits system of type \( \tilde{\Gamma} \).

**Proof.** We can apply lemma \[\text{[20]}\] to the set \( \tilde{I} \), since it consists of non-empty spherical subsets of \( I \), and since we have \( |r_\alpha r_\beta| \neq \infty \) if and only if \( \Gamma_{\alpha \cup \beta} \) is spherical (by proposition \[\text{[13]}\]), in which case we get \( \prod_{|r_\alpha r_\beta|}(r_\alpha, r_\beta) = r_{\alpha \cup \beta} \) by lemma \[\text{[22]}\]

The morphism \( \varphi : B_\tilde{\Gamma}^+ \to B_\Gamma^+ \) clearly extends to a group homomorphism \( \varphi_{\text{gr}} : B_\tilde{\Gamma} \to B_\Gamma \) whose image is the subgroup \( \tilde{B} = \langle r_\alpha, \alpha \in \tilde{I} \rangle \) of \( B_\Gamma \).

When \( \tilde{\Gamma} \) is spherical, the injectivity of \( \varphi \) implies the following:

**Theorem 24.** Let \( \tilde{I} \) be an admissible partition of \( \Gamma \), of spherical type \( \tilde{\Gamma} \). Then the homomorphism \( \varphi_{\text{gr}} : B_\tilde{\Gamma} \to B_\Gamma \) is injective. In other words, the pair \( (\tilde{B}, \tilde{S}) \) is (isomorphic to) the Artin-Tits system of type \( \tilde{\Gamma} \).

**Proof.** Since \( \tilde{\Gamma} \) is spherical, every \( b \in B_\tilde{\Gamma} \) can be written \( b = x^{-1}y \) for \( x, y \in B_\Gamma^+ \) (cf. subsection \[\text{[12]}\]), and the equality \( \varphi_{\text{gr}}(b) = 1 \) hence implies \( \varphi(x) = \varphi(y) \), whence the result thanks to the injectivity of \( \varphi \).

Let us name the objects we have just defined:

**Definition 25.** Let \( J \subseteq I \) be a subset of \( I \) and let \( \tilde{J} \) be an admissible partition of \( \Gamma_J \), of type \( \tilde{\Gamma} \). Let \( \tilde{S} = \{ s_\alpha \mid \alpha \in \tilde{J} \} \). Then we say that:

- the submonoid \( \tilde{B}^+ = \langle \tilde{S} \rangle^+ \) of \( B_\tilde{\Gamma}^+ \) (resp. the subgroup \( \tilde{B} = \langle \tilde{S} \rangle \) of \( B_\Gamma \)) is induced by \( \tilde{J} \), or, by abuse of language, is an admissible submonoid (resp. subgroup) of \( B_\Gamma^+ \) (resp. \( B_\Gamma \)),

- the morphism \( \varphi : B_\tilde{\Gamma}^+ \to B_\Gamma^+ \) (resp. \( \varphi_{\text{gr}} : B_\tilde{\Gamma} \to B_\Gamma \)), which sends each \( s_\alpha \in S_\tilde{\Gamma} \) on \( r_\alpha \in \tilde{S} \), is induced by \( \tilde{J} \), or, by abuse of language, is an admissible morphism.

**Remark 26.** In our definitions, we allow partitions of subsets of \( I \). This generalization does not change the conclusions of theorems \[\text{[23]}\] and \[\text{[24]}\] and allows the notion of standard parabolic submonoids or subgroups, of submonoids of fixed elements under the action of graph automorphisms and of LCM-homomorphisms of \[\text{[3]} \] \[\text{[9]}\] (see theorems \[\text{[28]}\] and \[\text{[43]}\] below).

**Remark 27.** If the partition \( \tilde{J} \) of \( \Gamma_J \) is only supposed to be spherical, then the map \( S_\tilde{\Gamma} \to B_\tilde{\Gamma}^+, s_\alpha \mapsto r_\alpha \), does not necessarily extend to a morphism from \( B_\Gamma^+ \) to \( B_\tilde{\Gamma}^+ \) : for example, if \( \Gamma = \{ 1, 2, 3 \} \) with \( \alpha = \{ 1 \} \) and \( \beta = \{ 2, 3 \} \), then \( |r_\alpha r_\beta| = 3 \) but \( r_\alpha r_\beta r_\alpha \neq r_\beta r_\alpha r_\beta \) in \( B_\Gamma^+ \) (look at the standard length).
3.3. Admissibility and Artin-Tits monoids or groups in the literature.

In this subsection, we show how our notions of admissible submonoids, subgroups or morphisms generalize and unify three situations that have been studied earlier.

3.3.1. Submonoids of fixed points under the action of graph automorphisms.

Here is the analogue of [11, Cor. 3.5] and [13, Thm. 1.3] (cf. subsection 2.3 above) for Artin-Tits monoids and for Artin-Tits groups of spherical type. Hence we recover the results [7, Thm. 9.3], [12, Cor. 4.4] and [4, Lem. 10 and Thm. 11].

Theorem 28. Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix and $G$ be a subgroup of $\text{Aut}(\Gamma)$. Let $J$ be the set of all spherical orbits of $I$ under $G$ and let $J \subseteq I$ be their union. Let $\tilde{\Gamma}$ be the type of the admissible partition $\tilde{J}$ of $\Gamma$, and let $\tilde{S} = \{r_\alpha | \alpha \in \tilde{J}\}$, $B^+ = (\tilde{S})^+$ and $\tilde{B} = (\tilde{S})$. Then:

1. $(B^+_1)^G = \tilde{B}^+$ and hence the pair $((B^+_1)^G, \tilde{S})$ is (isomorphic to) the positive Artin-Tits system of type $\tilde{\Gamma}$,
2. if $\Gamma$ is spherical, then $(B_1)^G = \tilde{B}$ and hence the pair $((B_1)^G, \tilde{S})$ is (isomorphic to) the Artin-Tits system of type $\tilde{\Gamma}$.

Proof. We already know that $\tilde{J}$ is an admissible partition of $\Gamma_J$ (cf. subsection 2.3). Thanks to theorems 23 and 24 above, the only things to prove are $(B^+_1)^G = B^+$ and, when $\Gamma$ is spherical, $(B_1)^G = \tilde{B}$.

For $\alpha \in \tilde{J}$, the group $G$ stabilizes $B^+_1$ and the induced action respects the standard length, so $G$ fixes $r_\alpha$ (which is the unique element of maximal standard length in $W_\alpha$). Hence we get $B^+ \subseteq (B^+_1)^G$ and $\tilde{B} \subseteq (B_1)^G$.

Let $x$ be an element of $(B^+_1)^G$ and let us show by induction on $\ell(x)$ that $x \in \tilde{B}^+$. There is nothing to prove if $x = 1$, so assume $x \neq 1$ and consider an element $i \in I$ such that $s_i \leq x$. Then, for all $g \in G$, $s_g(i) \leq x$. This implies that the orbit $\alpha$ of $i$ under $G$ is spherical and that $r_\alpha \leq x$. So there exists $x' \in B^+_1$ such that $x = r_\alpha x'$, and $\ell(x') < \ell(x)$. By cancellativity in $B^+_1$, we get $x' \in (B^+_1)^G$, hence $x' \in \tilde{B}^+$ by induction, and finally $x \in \tilde{B}^+$.

Now assume that $\Gamma$ is spherical and fix $b \in (B_1)^G$. Let $(x, y) \in (B^+_1)^2$ be the irreducible left form of $b$ (i.e. the unique pair such that $b = x^{-1}y$ and $x \wedge_L y = 1$, cf. definition 7 above). Since the action of $G$ on $B^+_1$ respect divisibility (hence gcd’s), we get by unicity that $x, y \in (B^+_1)^G$. The first point then gives $x, y \in \tilde{B}^+$, whence $b \in \tilde{B}$.

□

Remark 29. On the work of Crisp [4].

1. Our proof of theorem 25 is very similar to those of [4, Lem. 10 and Thm. 11], and indeed, the results [4, Lem. 6], [5] and lemma 32 below show that the Coxeter matrix $(m_{BC})_{B,C \in S}$ constructed by Crisp in [4, 5] is precisely our matrix $\tilde{\Gamma}$.
2. Crisp actually established the second point of theorem 28 for a wider class of Coxeter graphs than the spherical ones, namely the type FC ones, i.e. the finite Coxeter graphs for which every complete subgraph with no infinite label is spherical [4, Thm. 4].
3.3.2. LCM-homomorphisms.

We recall in definition 31 below the notion of LCM-homomorphisms as defined in [9, Def. 2.1], which generalizes the one of [3, Def. 2.1] by allowing finite Coxeter graphs with infinite labels. We adapt these definitions to our settings by defining the notion of LCM-partitions of a Coxeter graph, which will turn out to be nothing else but special cases of admissible partitions (cf. proposition 33 below). We do not suppose that the Coxeter graphs involved are finite.

**Definition 30.** Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix and let $\tilde{I}$ be a spherical partition of $\Gamma$. Let $\Omega = (n_{\alpha,\beta})_{\alpha,\beta \in \tilde{I}}$ be a Coxeter matrix over $\tilde{I}$. We say that $\tilde{I}$ is an LCM-partition of $\Gamma$, of type $\Omega$, if, for each pair $(\alpha, \beta) \in \tilde{I}^2$, we have the following alternative:

(Fi) $n_{\alpha,\beta} \neq \infty$, $\Gamma_{\alpha \cup \beta}$ is spherical and $r_{\alpha \cup \beta} = \prod_{\alpha,\beta}(r_{\alpha}, r_{\beta})$.

(In) $n_{\alpha,\beta} = \infty$ and for all $i \in \alpha$, $\Gamma_{\{i\} \cup \beta}$ is non-spherical.

**Definition 31 ([3, 9] Defs. 2.1]).** Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix. Let $J \subseteq I$ be a subset of $I$ and let $\tilde{J}$ be an LCM-partition of $\Gamma_J$, of type $\Omega_J = (n_{\alpha,\beta})_{\alpha,\beta \in \tilde{J}}$. Lemma 20 above shows that the map $S_{\Omega} \to B_{\tilde{I}}^+$, $s_{\alpha} \mapsto r_{\alpha}$, extends to a morphism that respects lcm’s from $B_{\tilde{I}}^+$ to $B_I^+$, which we call, after [3, 9], an LCM-homomorphism.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix, and let $\tilde{I}$ be an LCM-partition of $\Gamma$, of type $\Omega = (n_{\alpha,\beta})_{\alpha,\beta \in \tilde{I}}$. We show in proposition 33 below that $\tilde{I}$ is an admissible partition of $\Gamma$, and that its type (as an LCM-partition) $\Omega$ is necessarily its type (as a spherical partition) $\tilde{\Gamma} = (\{r_{\alpha}r_{\beta}\})_{\alpha,\beta \in \tilde{I}}$.

**Lemma 32.** Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix and let $\alpha$ and $\beta$ be two spherical subsets of $I$.

1. If $\Gamma_{\alpha \cup \beta}$ is spherical and if there exists an integer $n \in \mathbb{N}$ such that $r_{\alpha \cup \beta} = \prod_n(r_{\alpha}, r_{\beta}) = \prod_n(r_{\beta}, r_{\alpha})$, then $n = |r_{\alpha}r_{\beta}|$.
2. If, for all $n \in \mathbb{N}$, the product $\prod_n(r_{\alpha}, r_{\beta})$ is simple, then $|r_{\alpha}r_{\beta}| = \infty$ and $\Gamma_{\alpha \cup \beta}$ is non-spherical.

**Proof.** Under the hypothesis of assertion (1), we get $(r_{\alpha}r_{\beta})^n = \prod_{2n}(r_{\alpha}, r_{\beta}) = (r_{\alpha \cup \beta})^2 = 1$ in $W_\Gamma$, hence $|r_{\alpha}r_{\beta}|$ divides $n$. If $|r_{\alpha}r_{\beta}| < n$, then we can replace a factor $\prod_{r_{\alpha}r_{\beta}}(r_{\alpha}, r_{\beta})$ of $\prod_n(r_{\alpha}, r_{\beta})$ by $\prod_{r_{\alpha}r_{\beta}}(r_{\beta}, r_{\alpha})$ and then simplify $2|r_{\alpha}r_{\beta}|$ terms, whence $\ell(\prod_n(r_{\alpha}, r_{\beta})) < \sum_n(\ell(r_{\alpha}), \ell(r_{\beta})) = \sum_n(\ell(r_{\alpha}), \ell(r_{\beta})) = \ell(\prod_n(r_{\alpha}, r_{\beta}))$, and a contradiction since $\prod_n(r_{\alpha}, r_{\beta})$ is simple. Under the hypothesis of assertion (2), the dihedral group $(r_{\alpha}, r_{\beta})$, which is included in $W_{\alpha \cup \beta}$, is infinite, hence $|r_{\alpha}r_{\beta}| = \infty$ and $\Gamma_{\alpha \cup \beta}$ is non-spherical. □

**Theorem 33.** Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix, and let $\tilde{I}$ be an LCM-partition of $\Gamma$, of type $\Omega = (n_{\alpha,\beta})_{\alpha,\beta \in \tilde{I}}$. Then $\tilde{I}$ is an admissible partition of $\Gamma$, and $\Omega = \tilde{\Gamma} = (\{r_{\alpha}r_{\beta}\})_{\alpha,\beta \in \tilde{I}}$.

**Proof.** A consequence of [9, Lem. 2.5] is that, if $n_{\alpha,\beta} = \infty$, then for all $n \in \mathbb{N}$, the product $\prod_n(r_{\alpha}, r_{\beta})$ is simple. Lemma 32 then shows that $\Omega = \tilde{\Gamma}$ and the characterizations of lemma 22 show that for all $\alpha, \beta \in \tilde{I}$, $\{\alpha, \beta\}$ is an admissible partition of $\Gamma_{\alpha \cup \beta}$. We conclude that $\tilde{I}$ is an admissible partition of $\Gamma$ thanks to theorem 13. □
So, as announced, an LCM-partition is an admissible partition (and hence an LCM-homomorphism is an admissible morphism): the converse is false in general (cf. example 34, remark 39 and example 45 below), but is true for example if:

1. the matrix $\tilde{\Gamma}$ has no infinite coefficient,
2. the matrix $\Gamma$ is right angled, i.e. $m_{i,j} \in \{1, 2, \infty\}$ for all $i, j \in I$ (to see this, use [14, Lem. 2.5.15], recalled in proposition 48 below),
3. the matrix $\Gamma$ is of type FC (this notion is defined in remark 29) and $\tilde{I}$ is the set of orbits of $I$ under the action of a subgroup of $\text{Aut}(\Gamma)$.

**Example 34.** Consider the Coxeter graph $\Gamma$ of affine type $\tilde{A}_3$, and its 2-partition formed by pairs of opposite vertices:

![Diagram of a Coxeter graph]

This spherical 2-partition is admissible since it is the set of orbits of $\Gamma$ under the action of the "central symmetry", and its type is $\tilde{\Gamma} = I_2(\infty)$ since $\Gamma$ is non-spherical. It is not an LCM-partition (condition (In) of definition 30 is not satisfied): indeed, if $i$ is one of the vertices of $\Gamma$ and if $\beta$ is the orbit that does not contain $i$, then $\Gamma \{i\} \cup \beta$ is of spherical type $A_3$.

**Remark 35.** The results [3, Prop. 2.3] and [9, Cor. 2.7] on the injectivity of the morphism between Coxeter groups induced by an LCM-homomorphism now appear as special cases of [13, Thm. 1.1] (recalled in theorem 12 above). In fact, one can check that the proof of [9, Cor. 2.7] works for general admissible partitions and hence gives a new proof of [13, Thm. 1.1].

### 3.3.3. The bursts of a Coxeter graph.

We recall here a construction of Mühlherr [14, section 2.6], a quasi-identical version of which has independently been obtained by Crisp and Paris for Coxeter graphs with no infinite label [6, section 6], and by Paris in general [15, section 5]. The differences between the two approaches rely essentially in the choice of the integer $N$ in definition 36 below.

Let $\delta : \mathbb{N}_{\geq 2} \cup \{\infty\} \to \mathbb{N}_{\geq 1}, m \mapsto \begin{cases} m - 1 & \text{if } m \text{ is even}, \\ \frac{m - 1}{2} & \text{if } m \text{ is odd}, \\ 2 & \text{if } m = \infty. \end{cases}$

**Definition 36** ([14, section 2.6]). Suppose that $\Gamma = (m_{i,j})_{i,j \in I}$ is a Coxeter matrix such that the subset $\{m_{i,j} \mid i, j \in I\}$ of $\mathbb{N} \cup \{\infty\}$ is finite. Set $N_0 = \text{lcm}\{\delta(m_{i,j}) \mid i, j \in I, i \neq j\}$ and let $N$ be a multiple of $N_0$. A $N$-burst, or simply a burst, of $\Gamma$ is a Coxeter graph $\tilde{\Gamma}$ with vertex set the disjoint union $\tilde{I} = \bigcup_{i \in I} T(i)$ of sets $T(i) = \{i^{(1)}, \ldots, i^{(N)}\}$ of cardinality $N$, and with edges displayed as follows:

1. there is no edge between two elements of a same $T(i)$,
2. if $m_{i,j} \in \mathbb{N}_{\geq 2}$ is even, the graph $\tilde{\Gamma}_{T(i) \cup T(j)}$ is the disjoint union of $\frac{N}{\delta(m_{i,j})}$ copies of the following graph:

```
1 ———— 2 ———— 3 ———— ... ———— \delta(m_{i,j})
```

```
where the vertices • constitute \( T(i) \) and the vertices \( o \) constitute \( T(j) \).

(3) if \( m_{i,j} \in \mathbb{N}_{\geq 3} \) is odd, the graph \( \tilde{\Gamma}_{T(i) \cup T(j)} \) is the disjoint union of \( \frac{N}{\delta(m_{i,j})} \) copies of the following graph:

\[
\begin{array}{cccc}
1 & 2 & 3 & \cdots & \delta(m_{i,j}) \\
\circ & \circ & \circ & \cdots & \circ
\end{array}
\]

where the vertices • constitute \( T(i) \) and the vertices \( o \) constitute \( T(j) \).

(4) if \( m_{i,j} = \infty \), the graph \( \tilde{\Gamma}_{T(i) \cup T(j)} \) is the disjoint union of \( \frac{N}{\delta(m_{i,j})} \) copies of the following graph:

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \delta & \circ
\end{array}
\]

where the vertices • constitute \( T(i) \) and the vertices \( o \) constitute \( T(j) \).

**Theorem 37** ([14 Thm. 2.6.1 and its proof]). Let \( \Gamma = (m_{i,j})_{i,j \in I} \) be a Coxeter matrix with \( \{ m_{i,j} \mid i, j \in I \} \) finite, and let \( \tilde{\Gamma} \) be a \( N \)-burst of \( \Gamma \). Then the partition \( \{ T(i) \mid i \in I \} \) of \( \tilde{\Gamma} \) is an admissible partition of \( \tilde{\Gamma} \), of type (isomorphic to) \( \Gamma \).

**Proof.** It is enough to check that, for all \( i, j \in I, i \neq j \), \( \{ T(i), T(j) \} \) is an admissible partition of \( \tilde{\Gamma}_{T(i) \cup T(j)} \), of type \( I_2(m_{i,j}) \) (with \( I_2(2) = A_1 \times A_1 \)).

If \( m_{i,j} = 2 \), then there is no edge between a vertex of \( T(i) \) and a vertex of \( T(j) \). If \( m_{i,j} \in \mathbb{N}_{\geq 3} \), then the graph \( \tilde{\Gamma}_{T(i) \cup T(j)} \) is the disjoint union of \( \frac{2N}{m_{i,j}-1} \) copies of the spherical Coxeter graph of type \( A_{m_{i,j}-1} \), and the partition \( \{ T(i), T(j) \} \) induces on each of these connected components the bipartite partition of \( A_{m_{i,j}-1} \). If \( m_{i,j} = \infty \), then the graph \( \tilde{\Gamma}_{T(i) \cup T(j)} \) is the disjoint union of \( \frac{N}{2} \) copies of the affine Coxeter graph of type \( \widetilde{A}_3 \), and the partition \( \{ T(i), T(j) \} \) induces on each of these connected components the partition of \( \widetilde{A}_3 \) described in example 34 above. We conclude by applying results of [14] section 2.5 recalled in propositions 47, 49 and 50 below (note that we really need our stronger version, prop. 49 of [14] Lem. 2.5.4 when \( m_{i,j} = \infty \)).

**Example 38.** If \( \Gamma \) is of type \( H_3 \) (resp. \( H_4 \)), then \( N_0 = 2 \) and every 2-burst \( \tilde{\Gamma} \) of \( \Gamma \) is of type \( D_6 \) (resp. \( E_8 \)). We thus recover the figures of example 17.

**Remark 39.** When \( \Gamma \) has an infinite coefficient, then \( \{ T(i) \mid i \in I \} \) is not an LCM-partition of \( \tilde{\Gamma} \) (condition (In) of definition 30 is not satisfied) : indeed, if \( m_{i,j} = \infty \), then for \( i^{(k)} \in T(i) \), we get that the graph \( \tilde{\Gamma}_{T(i^{(k)})} \) is the disjoint union of \( N - 2 \) connected components of type \( A_1 \) and one connected component of type \( A_3 \), hence is spherical.

### 3.4. Some properties of admissible morphisms.

In this subsection, we show that some properties established in \([3, 4, 9]\) for their special cases of admissible morphisms are in fact satisfied by all admissible morphisms.
3.4.1. Respect of the combinatorics.

Let \( \Gamma = (m_{i,j})_{i,j \in I} \) be a Coxeter matrix, \( J \subseteq I \) be a subset of \( I \), and \( \tilde{J} \) be an admissible partition of \( \Gamma_J \) of type \( \tilde{\Gamma} \). We consider the admissible morphism \( \varphi : B^+_{\tilde{\Gamma}} \rightarrow B^+_{\Gamma} \) induced by \( \tilde{J} \), and we denote by \( \tilde{W} \) the image of the subgroup \( W = \langle r_\alpha, \alpha \in \tilde{J} \rangle \) of \( W_\Gamma \) in \( W_\Gamma \subseteq B^+_{\Gamma} \).

We know that \( \varphi \) respects lcm’s and divisibility, in the sense of theorem [19] above. The following lemma establishes that \( \varphi \) respects the notions of simple elements in \( B^+_{\tilde{\Gamma}} \) and in \( B^+_{\Gamma} \); it is a generalization of the well-known analogous result for the standard parabolic subgroups, and of [3, Lem. 2.2] and [9, Prop. 2.6].

**Lemma 40.** With the above notations, we get \( \varphi(W_\Gamma) = \tilde{B}^+ \cap W_\Gamma = \tilde{W} \). Moreover, if \( \tilde{\Gamma} \) (or equivalently \( \Gamma_J \)) is spherical, then \( \varphi(r_J) = r_J \).

**Proof.** This is a direct consequence of proposition [15] and lemma [21].

Let us mention two consequences of that result, given by [9, Thm. 2.10 and Cor. 2.11], which apply to our settings; note however that for the proofs of [9, Lem. 2.9 and Thm. 2.10] to be correct, we have to add to their hypothesis the following condition, which is satisfied by any admissible morphism : \( \text{Im}(\varphi) \subseteq B^+_{\bigcup_{a \in J} p(a)} \), where \( p(a) = \{ i \in I \mid s_i \preceq \varphi(s_a) \} = \{ i \in I \mid \varphi(s_a) \succ s_i \} \).

**Proposition 41** ([9, Thm. 2.10]). Let \( \varphi \) be as above. Then :

1. the morphism \( \varphi \) respects (left and right) normal forms, i.e. if \( (x_1, \ldots, x_n) \) is the left (resp. right) normal form of a non trivial element \( x \in B^+_{\tilde{\Gamma}} \), then \( (\varphi(x_1), \ldots, \varphi(x_n)) \) is the left (resp. right) normal form of \( \varphi(x) \in B^+_{\Gamma} \).
2. the morphism \( \varphi \) respects (left and right) gcd’s, i.e. for all \( (x, y) \in (B^+_{\tilde{\Gamma}})^2 \), we get \( \varphi(x \land_L y) = \varphi(x) \land_L \varphi(y) \) and \( \varphi(x \land_R y) = \varphi(x) \land_R \varphi(y) \).

**Corollary 42** ([9, Cor. 2.11]). Assume that \( \Gamma \) and \( \tilde{\Gamma} \) are spherical. Then the morphism \( \varphi_{gr} : B_{\tilde{\Gamma}} \rightarrow B_{\Gamma} \) respects (left and right) irreducible fractions, i.e. if \( (x, y) \in (B^+_{\tilde{\Gamma}})^2 \) is the left (resp. right) irreducible form of an element \( g \in B_{\tilde{\Gamma}} \), then \( (\varphi(x), \varphi(y)) \) is the left (resp. right) irreducible form of \( \varphi_{gr}(g) \in B_{\Gamma} \).

3.4.2. Composition of admissible morphisms.

In proposition [13] below, we recall the result [14, Lem. 2.5.6] on admissible partitions of an admissible partition. This result implies that the class of admissible morphisms is closed by composition (see corollary [14] below) and offers a criterion to test the admissibility of some spherical partitions, which we use in example [15] below and further in section 4.

**Proposition 43** ([14, Lem. 2.5.6]). Let \( \Gamma = (m_{i,j})_{i,j \in I} \) be a Coxeter matrix and let \( I' \) be an admissible partition of \( \Gamma \), of type \( \Gamma' \). Let \( I'' \) be a spherical partition of \( \Gamma' \), of type \( \Gamma'' \). Set \( \Phi = \bigcup_{\alpha \in \Phi} \alpha \) for \( \Phi \in I'' \) and \( \overline{\Phi} = \{ \Phi \mid \Phi \in I'' \} \). Then \( \overline{\Phi} \) is a spherical partition of \( \Gamma \), of type (isomorphic to) \( \Gamma'' \), and \( \overline{\Phi} \) is admissible if and only if \( I'' \) is admissible.

The following result has been established for the LCM-homomorphisms of [3] (cf. [3, page 134]). It can be shown that it is not true for the LCM-homomorphisms of [9].

**Corollary 44.** The composition of two admissible morphisms is an admissible morphism.
Proof. Let $\Gamma$, $\Gamma'$ and $\Gamma''$ be three Coxeter matrices and let $\varphi : B^+_{\Gamma'} \to B^+_{\Gamma}$ and $\varphi' : B^+_{\Gamma''} \to B^+_{\Gamma'}$ be two admissible morphisms. In other words, $\Gamma'$ is the type of an admissible partition $J'$ of $J \subseteq I$, and $\Gamma''$ is the type of an admissible partition $K''$ of $K' \subseteq J'$. But $K'$ is then an admissible partition of $K = \bigcup_{\alpha \in K} \alpha \subseteq J$ (cf. theorem 13), and proposition 43 tells us that $\overline{K} = \{ \Phi \mid \Phi \in K'' \}$ is an admissible partition of $K$. Moreover we get $\varphi \circ \varphi'(s_\Phi) = \varphi(r_\Phi) = \varphi(\text{lcm}\{ s_\alpha \mid \alpha \in \Phi \}) = \text{lcm}\{ \varphi(s_\alpha) \mid \alpha \in \Phi \} = \text{lcm}\{ r_\alpha \mid \alpha \in \Phi \} = r_{\overline{\Phi}}$ for every $\Phi \in K''$ (we use proposition 19 for the third equality). Hence $\varphi \circ \varphi'$ is the admissible morphism induced by the admissible partition $\overline{K}$ of $K$. \hfill $\Box$

Example 45. Consider the two following Coxeter graphs, where $m \in \mathbb{N}_{\geq 3}$ :

$$\Gamma = \begin{array}{c}
\bullet \\
m \\
m \\
m \\
i
\end{array} \quad \quad \hat{\Gamma} = \begin{array}{c}
\bullet \\
\infty \\
m \\
m \\
m \\
1 \\
2 \\
3
\end{array}$$

The graph $\hat{\Gamma}$ (which is of type FC) is the type of the admissible partition of $\Gamma$ composed of orbits of $\Gamma$ under the action of the automorphisms of $\Gamma$ that fix the vertex $i$. Proposition 43 then implies that the spherical partition $\{ \{1, 3\}, \{2\} \}$ of $\hat{\Gamma}$ is admissible since it "lifts" to the admissible partition of $\Gamma$ composed of orbits of $\Gamma$ under the action of the whole group $\text{Aut}(\Gamma)$. This admissible 2-partition of $\hat{\Gamma}$ is of type $I_2(\infty)$ (since $\hat{\Gamma}$ is not spherical) and is not an LCM-partition (condition (In) of definition 30 is not satisfied) since $\hat{\Gamma}_{\{2, 3\}}$ is spherical.

3.4.3. Geometrical point of view.

In [3, section 3] (resp. in [4, section 5] and in [9, section 3.2]), the authors gave a geometrical interpretation of their special case of admissible morphism between Artin-Tits groups in terms of a map between the associated Salvetti complexes (resp. modified Deligne complexes). One can check that these constructions are still valid for general admissible morphisms.

However, Godelle’s proof of the injectivity of LCM-homomorphisms between type FC Artin-Tits groups — more precisely the proof of [9, Prop. 3.7] — does not work for an admissible morphism between type FC Artin-Tits groups that is not an LCM-homomorphism (and such a morphism exists, cf. example 45). I do not know whether such a morphism is injective or not.

4. Classification.

The aim of this section is to complete the classification of admissible partitions whose type has no infinite label, began in [14, section 2.5]. Thanks to our results of subsection 3.3.2 above, this will in particular give us the classification of LCM-homomorphisms of $\Gamma$.

The results [13, Thm. 1.2] and [14, Lem. 2.5.5] (cf. theorem 13 and proposition 15 above) reduce this classification to the classification of admissible 2-partitions of spherical Coxeter graphs. In subsection 4.1 we deal with the case $|r_\alpha r_\beta| = 2$ and then recall some results of [14, section 2.5] which allow to reduce again the problem into the classification of admissible 2-partitions of irreducible spherical Coxeter graphs.
In subsection 4.2 we recall the classification of admissible 2-partitions of Coxeter graphs of types $A_n$, $B_n$ and $D_n$, obtained by M"uhlherr in [14, section 2.5], and complete it by examining the exceptional cases.

Finally, in subsection 4.3 we compare this classification with the notion of foldings of a Coxeter graph, defined by Crisp in [3, Def. 4.1] in order to provide examples of LCM-homomorphisms and to begin their classification. This leads us to a generalization (and simplification) of the notion of foldings, which becomes equivalent to the notion of admissible partitions, and allows us to complete the list of cases of the original definition [3, Def. 4.1].

4.1. Admissibility and reducibility.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a Coxeter matrix.

Using Tits’ solution of the word problem [16, Thm. 3], one obtains the following result, where the support of $w \in W_\Gamma$ — denoted by Supp$(w)$ — is the set of letters of any reduced representation of $w$ on $I$ (this set does not depend of the choice of the reduced representation of $w$ since two such words only differ from a finite sequence of braid relations $\prod_{i,j} (i, j) \sim \prod_{i,j} (j, i)$ with $i, j \in I$ such that $i \neq j$ and $m_{i,j} \neq \infty$, which do not change the set of letters involved).

**Lemma 46.** Let $v, w \in W_\Gamma$ such that Supp$(v) \cap$ Supp$(w) = \emptyset$. Then:

1. $\ell(vw) = \ell(v) + \ell(w)$.
2. $vw = wv \iff \forall (i, j) \in$ Supp$(v) \times$ Supp$(w)$, $m_{i,j} = 2$.

We can now deal with the case of the admissible 2-partitions $\{\alpha, \beta\}$ of $\Gamma$ with $|r_\alpha r_\beta| = 2$:

**Proposition 47.** Let $\bar{I} = \{\alpha, \beta\}$ be a spherical 2-partition of $\Gamma$. Then we have $|r_\alpha r_\beta| = 2 \iff \Gamma = \Gamma_\alpha \times \Gamma_\beta$. In that case, $\bar{I}$ is an admissible partition of $\Gamma$.

**Proof.** If $\Gamma = \Gamma_\alpha \times \Gamma_\beta$, then we obviously have $|r_\alpha r_\beta| = 2$. If $|r_\alpha r_\beta| = 2$, then $r_\alpha r_\beta = r_\beta r_\alpha$ and, by the previous lemma, we get that $\Gamma = \Gamma_\alpha \times \Gamma_\beta$ and $\ell(r_\alpha r_\beta) = \ell(r_\alpha) + \ell(r_\beta)$. The result [13, Lem. 3.3] (cf. lemmas [14] or [22] above) then implies that $\bar{I}$ is an admissible partition of $\Gamma$. \qed

We will need the following proposition to limit the "forms" that an admissible 2-partition $\{\alpha, \beta\}$ of $\Gamma$ can have when $|r_\alpha r_\beta| \geq 3$. For convenience, we sketch the proof of M"uhlherr below.

**Proposition 48** ([14, Lem. 2.5.15]). Let $\bar{I} = \{\alpha, \beta\}$ be an admissible 2-partition of $\Gamma$. Assume that there exists $i_0 \in \alpha$ such that $m_{i_0, j} = 2$ for all $j \in \beta$. Then $\Gamma = \Gamma_\alpha \times \Gamma_\beta$ (and hence $|r_\alpha r_\beta| = 2$).

**Proof.** Since $r_\beta s_{i_0} = s_{i_0} r_\beta$, we get by lemma 46 (first assertion) that $\ell(r_\alpha r_\beta s_{i_0}) = \ell(r_\alpha s_{i_0} r_\beta) = \ell(r_\alpha i_0) + \ell(r_\beta) - \ell(r_\alpha) - 1$. Hence $I = \alpha \cup \beta \subseteq \Gamma^{-1}(r_\alpha r_\beta)$ and $r_\alpha r_\beta = r_I = r_\beta r_\alpha$. We conclude by lemma 46 (second assertion). \qed

The following proposition allows us to reduce our classification problem to the irreducible case. It is given in [14, Lem. 2.5.4] for spherical Coxeter graphs $\Gamma_1, \ldots, \Gamma_\alpha$, but in order to complete the proof of theorem 37 above, we need it for general Coxeter graphs. So we prove it below in this more general context, using our characterizations of the admissibility of a 2-partition of $\Gamma$ in terms of simple elements in $B^+_\Gamma$ (cf. lemma [22]).
Proposition 49 ([14], Lem. 2.5.4]). Assume that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. For $1 \leq k \leq n$, let $\{\alpha_k, \beta_k\}$ be a spherical 2-partition of $\Gamma_k$ and set $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ and $\beta = \beta_1 \cup \cdots \cup \beta_n$. Then $\{\alpha, \beta\}$ is a spherical 2-partition of $\Gamma$ with $r_\alpha = r_\alpha_1 \cdots r_\alpha_n$, $r_\beta = r_\beta_1 \cdots r_\beta_n$ and $|r_\alpha r_\beta| = \text{lcm}\{r_\alpha_k r_\beta_k \mid 1 \leq k \leq n\}$. Moreover, the following conditions are equivalent:

1. $\{\alpha, \beta\}$ is an admissible partition of $\Gamma$,
2. $\{\alpha_k, \beta_k\}$ is an admissible partition of $\Gamma_k$ for $1 \leq k \leq n$, and $|r_\alpha_k r_\beta_1| = |r_\alpha_2 r_\beta_2| = \cdots = |r_\alpha_n r_\beta_n|$.

In that case, we get $|r_\alpha r_\beta| = |r_\alpha_1 r_\beta_1| = |r_\alpha_2 r_\beta_2| = \cdots = |r_\alpha_n r_\beta_n|$.

Proof. It is enough to prove the result for $n = 2$. The firsts observations are clear (if needed with the help of lemma [40]). Note that, thanks to lemma [40], we get that $W_\Gamma \approx W_{\Gamma_1} \times W_{\Gamma_2}$. For example, we have $r_\alpha = r_\alpha_1 r_\alpha_2$ and $r_\beta = r_\beta_1 r_\beta_2$. Hence, for all $m \in \mathbb{N}$, we get $\prod_m (r_\alpha, r_\beta) = \prod_m (r_\alpha_1, r_\beta_1) \prod_m (r_\alpha_2, r_\beta_2)$ in $B^+_1 \approx B^+_1 \times B^+_2$.

Suppose (2) and let us show (1). We get $|r_\alpha r_\beta| = |r_\alpha_1 r_\beta_1| = |r_\alpha_2 r_\beta_2|$. For $k = 1, 2$, lemma [22] gives us that $\prod_m (r_\alpha_k, r_\beta_k)$ are simple for all $0 \leq m < |r_\alpha_k r_\beta_k| + 1$. Then $\prod_m (r_\alpha, r_\beta)$ and $\prod_m (r_\beta, r_\alpha)$ are simple for all $0 \leq m < |r_\alpha_1 r_\beta_1| + 1$ and we are done by applying lemma [22] again.

Suppose (1) and let us show (2). If $|r_\alpha r_\beta| \neq \infty$, then necessarily $|r_\alpha_k r_\beta_k| \neq \infty$ for $k = 1, 2$. Moreover, $\Gamma$ is then spherical (by proposition [15]), hence so is $\Gamma_k$ for $k = 1, 2$. Lemma [22] gives us that $r_\iota_1 = \prod_{r_\alpha r_\beta} (r_\alpha, r_\beta) = \prod_{r_\alpha r_\beta} (r_\beta, r_\alpha)$. Let us denote by $I_k$ the vertex set of $\Gamma_k$ for $k = 1, 2$. Since we have $r_\iota_1 = r_\iota_2$, $\prod_{r_\alpha r_\beta} (r_\alpha, r_\beta) = \prod_{r_\alpha r_\beta} (r_\alpha_1, r_\beta_1) \prod_{r_\alpha r_\beta} (r_\alpha_2, r_\beta_2)$, and similarly if we exchange the roles of $\alpha$ and $\beta$, and the roles of $\alpha_k$ and $\beta_k$, we conclude, by identifying the terms in $B^+_1$ and $B^+_2$, that $r_\iota_k = \prod_{r_\alpha r_\beta} (r_\alpha_k, r_\beta_k) = \prod_{r_\alpha r_\beta} (r_\beta_k, r_\alpha_k)$, for $k = 1, 2$. If $|r_\alpha r_\beta| = \infty$, then lemma [22] shows us that the element $\prod_m (r_\alpha, r_\beta) = \prod_m (r_\alpha_1, r_\beta_1) \prod_m (r_\alpha_2, r_\beta_2)$ is simple for all $m \in \mathbb{N}$, and similarly if we exchange the roles of $\alpha$ and $\beta$, and the roles of $\alpha_k$ and $\beta_k$. We then have that $\prod_m (r_\alpha_1, r_\beta_1)$ and $\prod_m (r_\beta_1, r_\alpha_1)$ are simple for all $m \in \mathbb{N}$ (and $k = 1, 2$). In both cases, lemma [32] shows that $|r_\alpha r_\beta| = |r_\alpha_k r_\beta_k|$, for $k = 1, 2$, and we conclude thanks to lemma [32].

4.2. Admissible 2-partition of irreducible spherical Coxeter graphs.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a spherical Coxeter matrix and let $I = \{\alpha, \beta\}$ be a 2-partition of $\Gamma$. Let us denote by $\Gamma_1, \ldots, \Gamma_n$ the connected components of $\Gamma$, and by $I_k$ the vertex set of $\Gamma_k$ for $1 \leq k \leq n$.

- If there exists $1 \leq k \leq n$ such that $I_k$ is included in $\alpha$ or in $\beta$, then $I$ is admissible if and only if $\Gamma = \Gamma_\alpha \times \Gamma_\beta$, in which case $|r_\alpha r_\beta| = 2$ (by proposition [18]).

- If not, then $|r_\alpha r_\beta| \neq 2$ and $\alpha$ and $\beta$ meet every connected component of $\Gamma$, and we are in the situation of proposition [40] with $\alpha_k = \alpha \cap I_k$ and $\beta_k = \beta \cap I_k$ for $1 \leq k \leq n$. So we get that $I = \{\alpha, \beta\}$ is admissible if and only if $\{\alpha_k, \beta_k\}$ is an admissible 2-partition of $\Gamma_k$ for $1 \leq k \leq n$, and $|r_\alpha r_\beta| = |r_\alpha_1 r_\beta_1| = |r_\alpha_2 r_\beta_2| = \cdots = |r_\alpha_n r_\beta_n|$.

Hence we are left with the classification of admissible 2-partitions of irreducible spherical Coxeter graphs and their corresponding coefficient $|r_\alpha r_\beta|$. The first result in this direction is the following proposition. Since irreducible spherical Coxeter
graphs are finite trees (hence bipartite), each of them has a unique bipartite partition, which is a 2-partition except for the type $A_1$.

**Proposition 50** ([14, Lem. 2.5.13]). The bipartite partition $\{\alpha, \beta\}$ of an irreducible spherical Coxeter graph (distinct from $A_1$) is admissible, and the coefficient $|r_\alpha r_\beta|$ is the Coxeter number of the graph.

**Proof.** The result [2, Lem. 5.8] gives our characterization of lemma 22. □

**Remark 51.** These considerations justify all cases of examples 16 and 17 above except the ones concerning the non-trivial automorphisms of $A_2$, $F_4$ and $I_2(m)$ (this last one being obvious), and reduce the justifications for $A_2$ to the $A_4$ case. These last two cases (non-trivial automorphisms of $A_4$ and $F_4$) can be dealt with by direct computations.

Let us now investigate the different situations case-by-case.

### 4.2.1. Admissible 2-partitions of $A_n$, $B_n$, $D_n$.

The admissible 2-partitions of Coxeter graphs of type $A_n$ ($n \geq 2$), $B_n$ ($n \geq 2$) and $D_n$ ($n \geq 4$) have been classified by Mühlherr in [14, section 2.5]. In those cases, the only admissible 2-partitions are the bipartite ones and, for every $n \geq 2$, the following 2-partition of $A_{2n}$ (where the vertices are numbered in the natural order):

$$ (1) \quad \cdots
\begin{array}{c}
\downarrow \\
n + 1
\end{array}
\cdots \quad \text{with } |r_\alpha r_\beta| = 2n. $$

The admissibility of this 2-partition is a consequence of [14, Lem. 2.5.6] (cf. proposition 43 above) applied to the admissible partition of $A_{2n}$ induced by its non-trivial automorphism and the bipartite partition of $B_n$.

Mühlherr first established the classification for the $A_n$ case by explicit computations in the symmetric group. He inferred from this the classification for the $B_n$ case using [14, Lem. 2.5.6], which shows that every admissible 2-partition of $B_n$ “lifts” to an admissible 2-partition of $A_{2n}$ (or $A_{2n-1}$). In the same vein, since the automorphism of $D_n$ that permutes the vertices $n-1$ and $n$ (for the standard numbering of [1 Planche IV]) gives an admissible partition of type $B_{n-1}$, and since [14, Lem. 2.5.15] (cf. proposition 48 above) shows that for every admissible 2-partition of $D_n$, the vertices $n-1$ and $n$ must be in the same part of the partition, we get by [14, Lem. 2.5.6] that every admissible 2-partition of $D_n$ induces an admissible 2-partition of $B_{n-1}$, whence the classification for the $D_n$ case.

### 4.2.2. Admissible 2-partitions of $E_6$, $E_7$ and $E_8$.

Mühlherr showed in [14, Lem. 2.5.14] that the following 2-partition of $E_6$ is admissible : this is a consequence of [14, Lem. 2.5.6] (cf. proposition 48 above) applied to the admissible partitions of $E_6$ and $F_4$ induced by their non-trivial automorphism.

$$ (2) \quad \begin{array}{c}
\downarrow \\
n + 1
\end{array}
\begin{array}{c}
\downarrow \\
n + 1
\end{array}
\cdots \quad \text{with } |r_\alpha r_\beta| = 8 $$
Proposition 52. The only admissible 2-partitions of the Coxeter graphs $E_n$ ($n = 6, 7, 8$) are the bipartite ones and the 2-partition (2) above.

Proof. Let $\Gamma$ be a Coxeter graph of type $E_6$, $E_7$ or $E_8$ and let $\{\alpha, \beta\}$ be an admissible 2-partition of $\Gamma$. Since $\Gamma$ is connected, $\{\alpha, \beta\}$ does not satisfy the condition of proposition 48 above. Hence, apart from the bipartite partitions and the 2-partition (2) above, there are fifteen other possibilities:

- one for $E_6$:

[Diagram of $E_6$]

- five for $E_7$:

[Diagram of $E_7$]

- and nine for $E_8$:

[Diagram of $E_8$]

By lemma 14, there exist $n \in \mathbb{N}$ such that $\prod_n(r_{\alpha}, r_{\beta}) = \prod_n(r_{\beta}, r_{\alpha}) = r_I$ and $\sum_n(\ell(r_{\alpha}), \ell(r_{\beta})) = \sum_n(\ell(r_{\beta}), \ell(r_{\alpha})) = \ell(r_I)$. Since we have $\ell(r_I) = 36$ (resp. 63, resp. 120) if $\Gamma = E_6$ (resp. $E_7$, resp. $E_8$), cf. Planches V-VII], the consideration on lengths eliminates the last candidate for $E_6$ and leaves only one candidate for $E_7$ (the second one, with $n = 14$) and four for $E_8$ (the first one with $n = 20$, and the third, fourth and sixth ones with $n = 24$). We then verify, if needed with the help of a computation software like GAP or Maple, that the equality $\prod_n(r_{\alpha}, r_{\beta}) = \prod_n(r_{\beta}, r_{\alpha}) = r_I$ occurs in none of the five remaining cases, hence those 2-partitions are not admissible. \hfill \Box

4.2.3. Admissible 2-partitions of $F_4$, $H_3$, $H_4$ (and $I_2(m)$, $m \geq 3$).

The orbits of $F_4$ under the action of its non-trivial automorphism form the following admissible 2-partition:

(3) [Diagram of $F_4$] with $|r_{\alpha}r_{\beta}| = 8$

Proposition 53. The only admissible 2-partitions of the Coxeter graphs $F_4$, $H_3$, $H_4$ and $I_2(m)$, $m \geq 3$, are the bipartite ones and the 2-partition (3) above.

Proof. There is nothing to prove for the dihedral graphs. So assume that $\Gamma$ is a Coxeter graph of type $F_4$, $H_3$ or $H_4$, and let $\{\alpha, \beta\}$ be an admissible 2-partition of $\Gamma$. Since $\Gamma$ is connected, $\{\alpha, \beta\}$ does not satisfy the condition of proposition 48 above and hence is either a bipartite partition, or the 2-partition (3) above, or possibly the following 2-partition of $H_4$:

[Diagram of $H_4$]
To show that this last 2-partition is non-admissible, one can follow the same lines as in the proof of proposition \[ P \] Otherwise, note that \( H_4 \) is the type of an admissible partition of \( E_8 \) (cf. examples \[ 17 \] or \[ 38 \]) so, thanks to proposition \[ 43 \] the admissibility of the above 2-partition of \( H_4 \) is equivalent to the admissibility of a certain 2-partition of \( E_8 \) (not the bipartite one), which has been shown to be non-admissible in proposition \[ P \] \( \square \)

4.3. Foldings.

Let \( \Gamma = (m_{i,j})_{i,j \in I} \) and \( \Gamma' = (m'_{i,j'})_{i',j' \in I'} \) be two Coxeter matrices with no infinite coefficient. Crisp defined in \[ 3 \] Def. 4.1] the notion of a folding of \( \Gamma' \) onto \( \Gamma \), in order to give examples of LCM-homomorphisms and to begin their classification. With our terminology, a folding of \( \Gamma' \) onto \( \Gamma \) is a surjective map \( f : I' \to I \) that satisfy a list of conditions made for the partition \( \{ f^{-1}(\{i\}) \mid i \in I \} \) of \( I' \) to be an LCM-partition of type (isomorphic to) \( \Gamma \) \[ 3 \] Prop. 4.2. Crisp concluded \[ 3 \] section 4] by asking essentially if every LCM-partition is obtained from a folding. The classification we have just established shows that the answer is no, with the definition \[ 3 \] Def. 4.1] for a folding, and indicates how to complete the list of cases of \[ 3 \] Def. 4.1] to turn the answer to yes.

In definition \[ 54 \] below, we propose a generalisation of the notion of foldings that fit to our new point of view, and in proposition \[ 56 \] we rephrase in the manner of \[ 3 \] Def. 4.1] the classification established above.

**Definition 54** (foldings). Let \( \Gamma = (m_{i,j})_{i,j \in I} \) and \( \Gamma' = (m'_{i',j'})_{i',j' \in I'} \) be two Coxeter matrices. A folding of \( \Gamma' \) onto \( \Gamma \) is a map \( f : I' \to I \) such that the set \( \{ f^{-1}(\{i\}) \mid i \in I \} \) is an admissible partition of \( \Gamma' \), of type (isomorphic to) \( \Gamma \).

**Notation 55.** Let \( \Gamma \) be any Coxeter graph. For \( n \in \mathbb{N}_{\geq 1} \), we denote by \( n\Gamma \) the disjoint union of \( n \) copies of \( \Gamma \).

**Proposition 56.** Let \( \Gamma = (m_{i,j})_{i,j \in I} \) and \( \Gamma' = (m'_{i',j'})_{i',j' \in I'} \) be two Coxeter matrices and \( f : I' \to I \) be any map from \( I' \) to \( I \). Assume that \( \Gamma \) has no infinite coefficient. Then \( f \) is a folding from \( \Gamma' \) onto \( \Gamma \) if and only if \( f \) satisfies the following conditions for every \( i, j \in I \):

1. the subset \( f^{-1}(\{i\}) \) of \( I' \) is non-empty and spherical,
2. if \( m_{i,j} = 2 \), then there is no edge between a vertex of \( f^{-1}(\{i\}) \) and a vertex of \( f^{-1}(\{j\}) \), i.e. \( \Gamma'_{f^{-1}(\{i\})} \cap \Gamma'_{f^{-1}(\{j\})} = \Gamma'_{f^{-1}(\{i\})} \times \Gamma'_{f^{-1}(\{j\})} \),
3. if \( m_{i,j} \geq 3 \), then one of the following occurs:
   - (A) \( \Gamma'_{f^{-1}(\{i,j\})} = nI_2(m_{i,j}) \) for some \( n \in \mathbb{N}_{\geq 1} \), and each connected component of \( \Gamma'_{f^{-1}(\{i,j\})} \) (of type \( I_2(m_{i,j}) \)) meets \( f^{-1}(\{i\}) \) and \( f^{-1}(\{j\}) \),
   - (B) \( \Gamma'_{f^{-1}(\{i,j\})} \) is an irreducible and spherical Coxeter graph with Coxeter number \( m_{i,j} \), and the 2-partition \( \{ f^{-1}(\{i\}), f^{-1}(\{j\}) \} \) of \( f^{-1}(\{i,j\}) \) is the bipartite partition of \( \Gamma'_{f^{-1}(\{i,j\})} \);
   - (C1) \( m_{i,j} = 2n \) for some \( n \in \mathbb{N}_{\geq 2} \), \( \Gamma'_{f^{-1}(\{i,j\})} = A_{2n} \), and the 2-partition \( \{ f^{-1}(\{i\}), f^{-1}(\{j\}) \} \) of \( f^{-1}(\{i,j\}) \) is the admissible 2-partition (1) of subsection \[ 4.2.1 \];
   - (C2) \( m_{i,j} = 8 \), \( \Gamma'_{f^{-1}(\{i,j\})} = E_6 \), and the 2-partition \( \{ f^{-1}(\{i\}), f^{-1}(\{j\}) \} \) of \( f^{-1}(\{i,j\}) \) is the admissible 2-partition (2) of subsection \[ 4.2.2 \].
(C3) $m_{i,j} = 8$, $\Gamma'_{f^{-1}\{i,j\}} = F_4$, and the 2-partition \{ $f^{-1}\{i\}, f^{-1}\{j\}$\} of $f^{-1}\{i,j\}$ is the admissible 2-partition (3) of subsection 4.2.3.

(D) the map $f^{-1}\{i,j\} \to \{i,j\}$ induced by $f$ is a composition $h \circ g$, where $g$ is a folding from $\Gamma'_{f^{-1}\{i,j\}}$ onto $nI_2(m_{i,j})$ ($n \in \mathbb{N} \geq 2$) defined only with cases (B) to (C3) and $h$ is a folding from $nI_2(m_{i,j})$ onto $\Gamma_{\{i,j\}} = I_2(m_{i,j})$ of case (A).

Proof. This is a reformulation of the classification obtained above. □

Remark 57. We have added to the list of [3, Def. 4.1] the cases (C1) for $n > 2$, (C2) and (C3). Note that [10, Def. 1.11] already includes case (C3).

Remark 58. The cases (A) to (D) imply, for a non-isolated vertex $i$ of $\Gamma$, that $\Gamma'_{f^{-1}\{i\}}$ is non-empty and spherical, hence our condition (1) can be relaxed to the weaker condition (implicit in [3, Def. 4.1] and [10, Def. 1.11]) :

(1') if $i$ is isolated in $\Gamma$, then $f^{-1}\{i\}$ is non-empty and spherical.

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