SYNCHRONIZATION FROM A CATEGORICAL PERSPECTIVE

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ABSTRACT. We introduce a notion of synchronization for higher-dimensional automata, based on coskeletons of cubical sets. Categorification transports this notion to the setting of categorical transition systems. We apply the results to study the semantics of an imperative programming language with message-passing.

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1. Introduction

Traditional labeled transition systems synchronize on labels [19], essentially by a limit construction. Any type of synchronization of transition systems can be represented that way. The situation for higher-dimensional automata or HDA’s [7] is not so straightforward. Indeed, in the one-dimensional case a transition is either in the limiting object or is not. On the other hand, in the case of general HDA’s a transition has a dimension and synchronization may induce a change of dimension in addition to filter the transition out. Since HDA’s live in slices of the category \texttt{cSet} of cubical sets, there is a well-established theory available. In this paper, the relevant ingredients are the \textit{n-coskeleton} (right adjoint to the \textit{n-truncation}) and the \textit{categorification} (left adjoint to the \textit{cubical nerve}). We introduce a notion of $(\Sigma,n)$-coskeleton, a version of the usual $n$-coskeleton which interacts well with the (higher-dimensional) labeling. It turns out that this yields a notion of synchronization of HDA’s suitable for the study of message passing.

Section 2 contains some categorical background. In section 3 we recall the salient facts about cubical sets and prove some relevant technical lemmas. In section 4 we recall what HDA’s are and introduce their 1-coskeletal synchronization. In a nutshell, all potential

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The author wishes to thank the anonymous referee for the constructive part of his criticism. He gratefully acknowledges Ronnie Brown, Eric Goubault and Vincent Schmitt for the stimulating discussions. His debt to Claudio Hermida deserves a separate sentence.

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boundaries made of unsynchronized transitions are filled. This applies in particular to transition systems, since the latter are nothing but 1-dimensional HDA’s. A transition system is acubic if it does not contain boundaries, i.e. paths “around” cubes. Our main result about 1-coskeletal synchronization is theorem 4.14. Without technical noise, this theorem can be stated as follows:

**Theorem** The categorification of 1-coskeletal synchronized acubic transition systems is a limit of free categories.

After having introduced the programming language CIP in section 5, we study its semantics in section 6. The categorification of HDA’s obtained by applying CIP’s operational rules yields categories of evolutions and control categories, technically domains and codomains of ulf functors. The Giraud-Conduché correspondence of section 2 yields a finite presentation in form of categorical transition systems, technically pseudofunctors from free categories to \textbf{Span}. Section 7 takes up the topic of (bi)simulation and process categories. Concluding remarks are to be found in section 8.

2. The Giraud-Conduché Correspondence

In this section, we expose some known and less known facts about functors with the unique lifting of factorizations property and their correspondence with certain pseudofunctors. For general background about bicategories, we refer to the original text [1] and to [2].

2.1. Ulf Functors.

2.2. Definition. A functor \( F : \mathcal{B} \to \mathcal{C} \) has the unique lifting of factorizations property if, given \( u \in \mathcal{B} \) and \( \mathcal{C} \ni f = h \circ g \) such that \( F(u) = f \), there are unique \( v, w \in \mathcal{B} \) such that \( u = w \circ v \) with \( F(v) = g \) and \( F(w) = h \).

We use the acronym \textit{ulf} for such functors.

2.3. Proposition. (Street [18]) Let \( F : \mathcal{B} \to \mathcal{C} \) be a functor, \( J = J_2 \) be the interval category

\[
\begin{array}{ccc}
- & \to & + \\
\end{array}
\]

\( J_3 \) be the category generated by

\[
\begin{array}{ccc}
- & \to & 0 & \to & + \\
\end{array}
\]

and \( d_1 : J_2 \to J_3 \) be the functor such that \(- \mapsto -\) and \(+ \mapsto +\). The following are equivalent

(i) \( F \) is an ulf functor;

(ii) \( F \) has the strict right lifting property with respect to \( d_1 \), i.e. any commuting square
admits a unique filler.

**Proof.** Obvious. \( \square \)

Item (ii) of Proposition 2.3 is actually taken as definition in [18, pp. 532-533].

**2.4. Proposition.** (Street [18]) Let \((\mathbb{N}, +)\) be the additive monoid on the natural numbers. A category \(\mathcal{A}\) is free iff there is an ulf functor \(\ell : \mathcal{A} \rightarrow (\mathbb{N}, +)\).

**Proof.** Obvious. \( \square \)

**2.5. Proposition.** The full subcategory \(\text{Ulf}^- \subseteq \text{Cat}^-\) with ulf functors as objects is complete.

**Proof.** Products are straightforward. Suppose

\[
\begin{array}{ccc}
\mathcal{E} & \xleftarrow{C} & \mathcal{A} \\
\downarrow{K} & & \downarrow{U} \\
\mathcal{E}' & \xleftarrow{C'} & \mathcal{A}' \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{F} & \mathcal{B} \\
\downarrow{\ell} & & \downarrow{\ell} \\
\mathcal{E}' & \xrightarrow{F'} & \mathcal{B}' \\
\end{array}
\]

is an equalizer diagram in \(\text{Cat}^-\), so both rows are equalizer diagrams in \(\text{Cat}\) and \(K = U|_A\). Suppose \(U\) and \(V\) are ulf functors. Let \(f \in \mathcal{E}\) and suppose \(K(f) = f'\) and \(f' = q' \circ p'\) is a factorization in \(\mathcal{E}'\). Since \(U\) is an ulf functor, there is a unique factorization \(f = q \circ p\) in \(A\) such that \(U(p) = p'\) and \(U(q) = q'\). Let \(f_1 \overset{\text{def.}}{=} F(f)\). Since \(f \in \mathcal{E}\), we have \(f_1 = F(f) = G(f)\). Let \(f'_1 \overset{\text{def.}}{=} V(f_1)\), \(p'_1 \overset{\text{def.}}{=} F'(p')\) and \(q'_1 \overset{\text{def.}}{=} F'(q')\). Since \(p' \in \mathcal{E}'\) and \(q' \in \mathcal{E}'\), the factorization

\[
f_1' = q'_1 \circ p'_1
\]

has liftings \(f_1 = F(q) \circ F(p)\) and \(f_1 = G(q) \circ G(p)\). But \(V\) is an ulf functor, hence \(F(p) = G(p)\) and \(F(q) = G(q)\), i.e. the unique lifting of the factorization \(f = q \circ p\) is in \(\mathcal{E}\). \( \square \)

**2.6. Lax Comma Categories.**

**2.7. Definition.** Let \(\mathcal{K}\) be a bicategory. A morphism in \(\mathcal{K}\) is a map if it admits a right adjoint.
2.8. Proposition. Let $\mathcal{B}$ be a category with pullbacks. A morphism in $\text{Span}(\mathcal{B})$ is a map precisely when its left leg is iso.

It follows that the iso class of a map in $\text{Span}(\mathcal{B})$ has a span with identity left leg as a canonical representative.

Let $F, G : \mathcal{K} \rightarrow \mathcal{L}$ be lax functors. Recall that a lax transformation

$$\alpha : F \Rightarrow G$$

is given by the data

(i) for each $x \in \mathcal{K}$ a morphism $\mathcal{L} \ni \alpha_x : F(x) \rightarrow G(x)$;

(ii) for each morphism $\mathcal{K} \ni f : x \rightarrow y$ a 2-cell

$$\begin{array}{c}
\begin{array}{c}
F(x) \\
F(f)
\end{array}
\xrightarrow{\alpha_x} \\
\xleftarrow{\alpha_y}
\begin{array}{c}
G(x) \\
G(f)
\end{array}
\end{array} \xrightarrow{\alpha_f} \\
\begin{array}{c}
F(y) \\
F(f)
\end{array}
\xrightarrow{\alpha_y} \\
\xleftarrow{\alpha_y}
\begin{array}{c}
G(y) \\
G(f)
\end{array}
$$

subject to the coherence conditions

(i) $\alpha_{f'} \circ (G(\theta) \circ \alpha_x) = \alpha_f$ for each 2-cell $\theta : f \Rightarrow f' : x \rightarrow y$;

(ii) $(\alpha_g \circ F(f)) \circ \alpha_f = \alpha_{g \circ f}$ for each $f : x \rightarrow y$ and $g : y \rightarrow z$.

2.9. Definition. Let $F, G : \mathcal{K} \rightarrow \mathcal{L}$ be lax functors. A lax transformation is representable provided its components $\alpha_x$ are maps for all $x \in \mathcal{K}$.

Special cases of interest are lax representable transformations among lax functors from a category to $\text{Span} \overset{\text{def}}{=} \text{Span}(\text{Set})$. Let $p, q : \mathcal{B} \rightarrow \text{Span}$ be such lax functors and let $\alpha : p \Rightarrow q$ be a lax representable transformation. Its data with respect to to $\mathcal{B} \ni f : x \rightarrow y$ is given by the lax square

$$\begin{array}{c}
\begin{array}{c}
p(x) \\
p(f_1)
\end{array}
\xrightarrow{id} \\
\xrightarrow{\alpha_f}
\begin{array}{c}
p(x) \\
p(f_2)
\end{array}
\xrightarrow{\alpha_y}
\begin{array}{c}
q(x) \\
q(f_1)
\end{array}
\xrightarrow{q(f_2)}
\begin{array}{c}
q(y) \\
q(f)
\end{array}
\end{array}$$

i.e. $\alpha_f$ is a morphism of spans

$$\begin{array}{c}
\begin{array}{c}
p(x) \\
p(f_1)
\end{array}
\xleftarrow{p_1} \\
\xleftarrow{\alpha_f}
\begin{array}{c}
p(f) \\
q(x)
\end{array}
\xleftarrow{q(f_2) \circ p_2}
\begin{array}{c}
p(x) \times_{q(x)} q(f)
\end{array}
\end{array}$$
Recall that a lax functor $F : \mathcal{K} \to \mathcal{L}$ is normalized or normal provided it is strict on identities, i.e. the distinguished 2-cell $id_{F(x)} \Rightarrow F(id_x)$ is the identity 2-cell for all $x \in \mathcal{K}$.

2.10. Definition. Let $\mathbf{rGraph}$ be the category of reflexive graphs, i.e. graphs with a distinguished loop at every vertex. Let $\mathcal{K}$ be a bicategory and $F : \mathbf{rGraph} \to \mathbf{Cat}$ the "free" functor. The lax comma-category $F//\mathcal{K}$ is given by

(i) Objects: normal pseudofunctors from a path category to $\mathcal{K}$;

(ii) Morphisms: given $s : F(G) \to \mathcal{K}$ and $t : F(H) \to \mathcal{K}$ normal pseudofunctors, a morphism $\alpha : s \to t$ is a lax representable transformation $\alpha : s \Rightarrow t \circ F(k)$ for a morphism of reflexive graphs $k : G \to H$;

(iii) Composition: $\beta \circ \alpha = (l \circ k, l \beta \circ \alpha)$ where $\alpha : s \Rightarrow t \circ F(k)$ and $\beta : t \Rightarrow u \circ F(l)$ while the vertical composition $l \beta \circ \alpha$ is given by componentwise bicategorical pasting.

2.11. The Correspondence. Let $\mathbf{Ulf}/\mathbb{B}$ be the full subcategory of the slice category $\mathbf{Cat}/\mathbb{B}$ where the objects are ulf functors. Let $\mathbf{Psd}[\mathbb{B}, \mathbf{Span}]$ be the category of normal pseudo-functors from $\mathbb{B}$ to $\mathbf{Span}$ and lax representable transformations. There is the equivalence of categories

$$\mathbf{Ulf}/\mathbb{B} \simeq \mathbf{Psd}[\mathbb{B}, \mathbf{Span}]$$

which is the discrete particular case of the equivalence between functors with liftings of factorizations unique up to connected component and distributors, discovered independently by Giraud and by Conduché in the early 70’s [11].

2.12. Proposition. Let $\mathbf{Ulf}_{\mathcal{F}}$ be the full subcategory of the comma-category $\mathbf{id}_{\mathbf{Cat}} \downarrow \mathcal{F}$ where the objects are ulf functors and

$$A$$

$$u \downarrow \in \mathbf{Ulf}_{\mathcal{F}}$$

$$\mathcal{F}H$$

Then there is a reflexive graph $G$ and a morphism $m : G \to H$ such that $u = F(m)$.

Proof. It is obvious that the composition of two ulf functors is an ulf functor. By proposition 2.4 there is an ulf functor $\ell : \mathcal{F}(H) \to (\mathbb{N}, +)$. Hence $\ell \circ u$ is ulf so $A$ is free again by proposition 2.4. But $u$ is ulf so it preserves length. In particular, generators are mapped on generators. \qed

A variation of the classic equivalence above is

2.13. Theorem. Let $\mathbf{Ulf}_{\mathcal{F}}$ be as in proposition 2.12. There is an equivalence of categories

$$\mathbf{Ulf}_{\mathcal{F}} \simeq \mathcal{F}//\mathbf{Span}$$

Proof. The ulf counterpart $\pi_s : s \to \mathcal{F}(G)$ of a pseudofunctor $s : \mathcal{F}(G) \to \mathbf{Span}$ is obtained by an appropriate Grothendieck construction: a morphism in $s$ is of the form $(k, f) : (a, x) \to (b, y)$ with $a \in s(x)$ and $b \in t(x)$ while $k \in s(f)$ such that $k$ it is mapped on $a$ respectively $b$ by $s(f)$’s left respectively right leg. We further have
where

\[
(a, x) \quad \mapsto \quad (\alpha_x(a), \mathcal{F}(h)(x))
\]

\[
\tilde{\alpha} : \quad (k, f) \quad \mapsto \quad ((p_2 \circ \alpha_f)(k), \mathcal{F}(h)(f))
\]

\[
(b, y) \quad \mapsto \quad (\alpha_y(b), \mathcal{F}(h)(y))
\]

The other way round it is enough to take the fibers. Let \( s : \mathbb{D} \to \mathcal{F}(G) \) be an ulf functor and let \( s_x \) respectively \( s_f \) be the set of objects over \( x \in \mathcal{F}(G)_0 \) respectively the set of arrows over \( f \in \mathcal{F}(G)_1 \). The functor \( s \) determines a pseudofunctor \( \overline{s} : \mathcal{F}(G) \to \text{Span} \) given by

\[
\overline{s} : \quad x \quad \mapsto \quad s_x
\]

\[
f \quad \mapsto \quad s_f
\]

\[
y \quad \mapsto \quad s_y
\]

Let \( (m, n) : s \to t \) be a morphism in \( \text{Ulf}_\mathcal{F}^- \). It determines a lax representable transformation \( \overline{m} : \overline{s} \Rightarrow \overline{n} \circ \overline{t} \) given at \( f \) by \( \overline{m}_x \overset{\text{def.}}{=} m \mid_{s_x} \) and \( \overline{m}_f(p) \overset{\text{def.}}{=} (\text{dom}(p), m(p)) \).

\[\square\]

2.14. \textbf{Remark.} The square (\( \ast \)) is a pullback square provided \( \alpha = \text{id} \).

2.15. \textbf{Remark.} \( \mathcal{F}/\text{Span} \) is complete by theorem 2.13 and proposition 2.5.

We indicate for reference the construction of pullbacks at hand of the case where the involved pseudo-functors have the same domain \( \mathcal{F}(K) \). No generality is lost doing so since it is always possible to reindex. Let thus \( F, G, H : \mathcal{F}(K) \longrightarrow \text{Span} \), \( \alpha : F \Rightarrow G \), \( \beta : H \Rightarrow G \) and \( \mathcal{F}(K) \ni f : a \to b \). We then have

\[
\begin{array}{cccccc}
Fa & \overset{id}{\longrightarrow} & Ga & \overset{\alpha_b}{\longrightarrow} & Ga & \overset{\beta_a}{\longrightarrow} & Ha & \overset{id}{\longrightarrow} & Ha \\
(Ff)_1 & \downarrow & (Gf)_1 & \downarrow & (Hf)_1 & \\
Ff & \overset{\alpha_f}{\longrightarrow} & Gf & \overset{\beta_f}{\longrightarrow} & Hf \\
(Ff)_2 & \downarrow & (Gf)_2 & \downarrow & (Hf)_2 & \\
Fb & \overset{id}{\longrightarrow} & Fb & \overset{\alpha_b}{\longrightarrow} & Gb & \overset{\beta_b}{\longrightarrow} & Hb & \overset{id}{\longrightarrow} & Hb
\end{array}
\]
at $f$. Let $\varphi(s,t)$ be the pullback data of morphisms $s$ and $t$ with common codomain in a category with pullbacks (i.e. the triple consisting of the pullback object and the projections). There is the series

1. $(Q', q'_1, q'_2) \overset{\text{def.}}{=} \varphi(\alpha_a, (Gf)_1)$
2. $(Q'', q''_1, q''_2) \overset{\text{def.}}{=} \varphi(\beta_a, (Gf)_1)$
3. $(Q, q_1, q_2) \overset{\text{def.}}{=} \varphi(q'_2 \circ \alpha_f, q''_2 \circ \beta_f)$
4. $(R, r_1, r_2) \overset{\text{def.}}{=} \varphi(\alpha_b, \beta_a)$
5. $(S, s_1, s_2) \overset{\text{def.}}{=} \varphi(\alpha_b, \beta_b)$

of pullback data in $B$ and the pseudo-functor resulting from pulling back $\alpha$ along $\beta$ evaluates at $f$ as

$$
\begin{array}{ccc}
\langle(Ff)_1 \circ q_1, (Hf)_1 \circ q_2\rangle & \xrightarrow{Q} & \langle(Ff)_2 \circ q_1, (Hf)_2 \circ q_2\rangle \\
R & \xleftarrow{\downarrow} & S
\end{array}
$$

The legs of this span are given by universal property.

3. Cubical Sets

Similarly to simplicial sets which are presheaves over the simplicial category $\Delta$, cubical sets are presheaves over a category of "ideal cubes" $\Box$. They were introduced by Serre in his thesis back in the early 1950’s [16]. They turn out to be more convenient than simplicial sets in some situations, since their product is geometrically simpler. Particularly for topologists of the Bangor school, those devices have been standard tools of the trade since some 25 years [15] 1.

3.1. Cubical Sets.

3.2. Definition. A cubical set $K$ is a sequence

\[
\cdots K_n \xleftarrow{\partial^+_i} K_{n-1} \xrightarrow{\partial^-_i} \cdots
\]

of sets and functions where $1 \leq i \leq n$, subject to the cubical identities

(i) $\partial_i^\omega \circ \partial_j^{\omega'} = \partial_{j-1}^{\omega'} \circ \partial_i^\omega$ for all $i < j$ and $\omega, \omega' \in \{-, +\}$

1This paper only marks the beginning of a long series.
(ii) $\epsilon_i \circ \epsilon_j = \epsilon_{j+1} \circ \epsilon_i$ for all $i \leq j$

(iii) $\partial_i^c \circ \epsilon_j = \begin{cases} 
\epsilon_{j-1} \circ \partial_i^\omega & \text{for all } i < j \text{ and } \omega \in \{-, +\} \\
\epsilon_j \circ \partial_{i-1}^\omega & \text{for all } i > j \text{ and } \omega \in \{-, +\} \\
id & \text{for all } i = j
\end{cases}$

The $\partial$'s are called (positive respectively negative) faces while the $\epsilon$'s are called degeneracies. Again in analogy to simplicial sets, elements of $K_n$ are called $n$-cubes. A cube in the image of a degeneracy is called degenerate. 1-faces of an $n$-cube $y$, obtained from the latter by applying a composite of $n - 1$ face maps, are called $y$'s edges.

3.3. Definition. Let

- $2 = \{0, 1\}$ and $1 = \{\ast\}$;
- $\varepsilon \overset{def}{=} !_{2} : 2 \to 1$;
- $\delta^- : 1 \to 2 \ast \mapsto 0$;
- $\delta^+ : 1 \to 2 \ast \mapsto 1$;
- $2^0 \overset{def}{=} 1$ and $2^p \square 2^q \overset{def}{=} 2^{p+q}$.

The category $\square$ has $\{2^m \mid m \in \mathbb{N}\}$ as its set of objects and is generated by the maps

- $\delta^\omega_i \overset{def}{=} 2^{i-1} \square \delta^\omega \square 2^{n-i} : 2^{n-1} \to 2^n$ for all $1 \leq i \leq n$ and $\omega \in \{-, +\}$;
- $\varepsilon_i \overset{def}{=} 2^{i-1} \square \varepsilon \square 2^{n-i} : 2^n \to 2^{n-1}$ for all $1 \leq i \leq n$

The $\delta^\omega_i$ are called cofaces while the $\varepsilon_i$ are called codegeneracies. $\square$ is strict monoidal (a PRO in fact), yet the tensor $\square$ is not a product although it is induced by the product in $\text{Set}$. We obtain the cocubical identities

(i) $\delta_{j'}^\omega \circ \delta_{i}^\omega = \delta_{i}^\omega \circ \delta_{j'-1}^\omega$ for all $i < j$ and $\omega, \omega' \in \{-, +\}$;

(ii) $\varepsilon_j \circ \varepsilon_i = \varepsilon_i \circ \varepsilon_{j+1}$ for all $i \leq j$;

(iii) $\varepsilon_j \circ \delta_{i}^\omega = \begin{cases} 
\delta_{i}^\omega \circ \varepsilon_{j-1} & i < j \\
\delta_{i-1}^\omega \circ \varepsilon_j & i > j \\
id & i = j
\end{cases}$ for all $\omega \in \{-, +\}$

from the two basic two ones $\varepsilon \circ \delta^\omega = id$ for $\omega \in \{-, +\}$, so a cubical set $K$ is indeed a presheaf in $\text{cSet} \overset{def}{=} \text{Set}^{\square^{op}}$. Morphisms in this category are called cubical maps. As a matter of notation, $K_p$ stands for $K(2^p)$ while $t_p$ stands for the component $t_{2^p}$ of the cubical map $t$ at $2^p$. We write $\text{cSet}_{\ast}$ for the category of pointed cubical sets.
3.4. Lemma. (Grandis and Mauri [9]) Each composite of cofaces and codegeneracies has a canonical form
\[ \delta_{j_1} \circ \cdots \circ \delta_{j_s} \circ \epsilon_{i_1} \cdots \circ \epsilon_{i_t} : 2^m \to 2^{m-t} \to 2^n \]
where \( 1 \leq i_1 < \cdots < i_t \), \( n \geq j_1 > \cdots > j_s \geq n \) and \( m - t = n - s \geq 0 \). This canonical form is unique.

Proof. The cocubical identities act as rewriting rules from left to right. \( \square \)

3.5. Remark. Obviously, each composite of faces and degeneracies of a cubical set has a canonical form too. The factors just reverse.

3.6. Skeletons and Coskeletons. Let \( \square_n \subset \square \) be the full subcategory with objects \( \{2^0 \cdots 2^n\} \), \( J^op : \square^op \leftarrow \square^op \) be the dual of the inclusion and \( tr_n \overset{def.}{=} (-) \circ J^op : \text{Set}^{\square^op} \rightarrow \text{Set}^{\square^op} \) be the \((n-1)\)truncation. Since left and right Kan extensions to \( \text{Set} \) always exist, there are adjunctions
\[ sk_n \overset{def.}{=} \text{Lan}_{J^op}(-) \dashv tr_n \dashv \text{cosk}_n \overset{def.}{=} \text{Ran}_{J^op}(-) \]

In analogy to the simplicial situation, a cubical set in the image of \( sk_n \) is called \( n \)-skeletal. Given an arbitrary cubical set \( K \), its image under \( Sk_n \overset{def.}{=} sk_n \circ tr_n \) is called its \( n \)th skeleton.

On the other hand, a cubical set in the image of \( \text{cosk}_n \) is called \( n \)-coskeletal while applying \( \text{Cosk}_n \overset{def.}{=} \text{cosk}_n \circ tr_n \) yields the \( n \)-coskeleton. Obviously, an \( n \)-coskeletal cubical set is isomorphic to its \( n \)-coskeleton.

The representable presheaf \( \square[k] \overset{def.}{=} \square(-, 2^k) \) is called the standard \( k \)-cube. \( \square[k] \) is \( k \)-skeletal with precisely one non-degenerated cube in dimension \( k \), namely \( s_k \overset{def.}{=} \text{id}_{2^k} \). The faces of this cube are \( \partial^w(s_k) = \delta_i^w \in \square(2^{k-1}, 2^k) \) for \( 1 \leq i \leq k \) and \( \omega \in \{-, +\} \). The \((k-1)\)th skeleton \( \partial \square[k] \) of the standard \( k \)-cube is called boundary.

3.7. Definition. Suppose \( Y \in \text{Set}^{\square^op} \). Its cubical kernel is a set \( Y_{n+1}^* \) equipped with a family of maps
\[ d_i^w : Y_{n+1}^* \rightarrow Y_n \]
for all \( 1 \leq i \leq n+1 \) and \( w \in \{-, +\} \), such that the cubical identities hold and such that \( Y_{n+1}^* \) satisfies the following universal property. Given any set \( Z \) with maps \( z_i^w : Z \rightarrow Y_n \) such that the cubical identities hold, there is a unique map \( z : Z \rightarrow Y_{n+1}^* \) such that
\[ d_i^w \circ z = z_i^\omega \]
for all \( 1 \leq i \leq n+1 \) and \( w \in \{-, +\} \).

\(^2\)Since this piece of terminology is chosen in analogy to the simplicial case, the “co” is on the wrong side due to a “historical accident” [9].
3.8. Lemma. The cubical kernel exists for all $n \in \mathbb{N}$ and all $Y \in \textbf{Set}^\Box_{op}$.

Proof. $Y_n^* \overset{\text{def.}}{=} \left\{ (y_i^\omega) \in \prod_{1 \leq i \leq n+1} \prod_{\omega \in \{-, +\}} Y_n | \forall 1 \leq i < j \leq n+1. \partial_i^\omega(y_j^\omega) = \partial_{j-1}^\omega(y_i^\omega) \right\}$

Elements of $Y_n^*$ are called $n$-shells [15]. Any boundary is a shell.

3.9. Proposition. The following are equivalent:

(i) $X$ is $n$-coskeletal;

(ii) any boundary in $X$ has a unique “lifting”, i.e. any cubical map $\partial \Box[k] \to X$ uniquely extends through the boundary inclusion:

\[
\begin{array}{ccc}
\partial \Box[k] & \to & X \\
\downarrow & & \downarrow \\
\Box[k] & \leftarrow & \\
\end{array}
\]

for all $k > n$;

(iii) $X_k$ is an iterated cubical kernel for all $k > n$.

Proof. (i) $\Rightarrow$ (ii) By hypothesis there is $X' : \Box_{op}^\Box \to \textbf{Set}$ such that $X \cong \cosk_n(X')$. Let $\pi^{(m)} : 2^m \downarrow J \to \textbf{Set}$ be the projection functor. We have $X_m = \text{lim}(X' \circ \pi^{(m)})$. Unraveling yields

$$X_m = \left\{ (x_s) \in \prod_{s : 2^p \to 2^m} X'(\text{dom}(s)) \mid \forall p, q \leq n, h : 2^q \to 2^p, s : 2^p \to 2^m, t : 2^t \to 2^m. t = s \circ h \Rightarrow X'(h)(x_s) = x_t \right\}$$

Let $k > n$. A cubical map $\partial \Box[k] \to X$ amounts to a choice of elements

$$(Y_u^{1^-}), (Y_u^{1^+}), \ldots, (Y_u^{k^-}), (Y_u^{k^+}) \in X_{k-1}$$

such that

$$Y_{\delta_i^\omega \circ f} = Y_{\delta_i^{(i+1)^\omega}}^{j^\omega}$$

for all $f : 2^p \to 2^{k-2}$, $j \leq i$ and $\omega, \omega' \in \{-, +\}$. Let $p \leq n$ and $s : 2^p \to 2^k$. The latter has a canonical form by lemma 3.4. Since $k > n$, this canonical form has at least one factor which is a coface so $s = \delta_i^\omega \circ u$ for some $u : 2^p \to 2^{k-1}$ and $\delta_i^\omega : 2^{k-1} \to 2^k$. The desired unique boundary extension is then given by the element $(x_s) \in X_k$ with coordinate at $s$

$$x_s = y_u^{j^\omega}$$
(ii) ⇒ (iii) Iterating the cubical kernel yields a cubical set. Given $k > n$, faces are given by the projections, i.e. $\partial^ω_i \overset{\text{def.}}{=} \pi_ω \circ π_i$. Degeneracies are given by

$$ ε_j(Y)^ω_i = \begin{cases} (ε_{j-1} \circ \partial^ω_i)(Y) & i < j \\ (ε_j \circ \partial^ω_{i-1})(Y) & i > j \\ Y & i = j \end{cases} $$

(c.f. [15]). But an element of a cubical kernel amounts to a cubical map $\partial^2_k \rightarrow X$.

(iii) ⇒ (i) It is not hard to see that $X_k \simeq \text{Cosk}_n(X)_k$ enjoys the universal property of definition 3.7 for any $k > n$.

The 1-truncation of a cubical set is a reflexive graph, i.e. a graph with a distinguished loop at every vertex. The other way round, a reflexive graph is a 1-skeletal cubical set. We use both terms interchangeably. As customary, we do not make a notational difference between a set $Σ$ and a (here reflexive) graph $G$ with one vertex and such that $G_1 = Σ$.

3.10. Cubical Nerve and Categorification.

3.11. Lemma. (Kan [12]) Let $F : C \rightarrow A$ be a functor, $y : C \rightarrow \hat{C}$ be the Yoneda embedding and

$$ F_* : A \rightarrow \hat{C} $$

$$ A \mapsto A(F_-, A) $$

If it exists, $\text{Lan}_y F$ is left adjoint to $F_*$.

3.12. Proposition. Let $J = \tau \rightarrow \mapsto \nu$ be the interval category, $d^ω_{i,p} : J^{p-1} \rightarrow J^p$ the functor inserting $ω \in \{−, +\}$ respectively $\text{id}_ω$ at $i$th coordinate and $e^ω_{i,p} : J^p \rightarrow J^{p-1}$ the functor deleting the $i$th coordinate. These data organize to a functor

$$ c : \square \rightarrow \text{Cat} $$

$$ 2^p \mapsto J^p $$

$$ δ^ω_{i,p} \mapsto d^ω_{i,p} $$

$$ ε^ω_{i,p} \mapsto e^ω_{i,p} $$

Let $y : \square \rightarrow \text{Set}^{\text{op}}$ be the Yoneda embedding. There is the adjunction

$$ \text{Lan}_y c \dashv c_* $$

Proof. By lemma 3.11

3.13. Definition. With the notation of proposition 3.12, $C \overset{\text{def.}}{=} \text{Lan}_y c$ is called categorification while $N \overset{\text{def.}}{=} c_*$ is called cubical nerve.

3.14. Lemma. Let $K$ be a cubical set and $y \in K_n$. Let $f, g \in K_1$ be $y$’s edges with canonical forms $f = (\partial^ω_{n-1} \circ \cdots \circ \partial^ω_{1})(y)$ and $g = (\partial^ω_{n-1} \circ \cdots \circ \partial^ω_{j_1})(y)$. The following are equivalent:

(i) $\partial^-(g) = \partial^+(f)$;

(ii) there is precisely one $1 \leq k \leq n - 1$ such that $\forall 1 \leq l \leq n - 1. i_k \neq j_l$ and $ω^k = −$. 
3.15. Corollary. There are exactly $n!$ paths from $\text{dom}_n(y)$ to $\text{cod}_n(y)$.

3.16. Proposition. Under the notation of lemma 3.14, let

$$\text{dom}_n(y) = \text{def.} (\partial_0 \circ \cdots \circ \partial_n)(y)$$

and

$$\text{cod}_n(y) = \text{def.} (\partial_0^+ \circ \cdots \circ \partial_{n-1}^+)(y)$$

Suppose $P_y$ be the set of paths from $\text{dom}_n(y)$ to $\text{cod}_n(y)$. The categorification $\mathcal{C}(K)$ of a cubical set $K$ has the set of objects $\mathcal{C}(K)_0 = K_0$. It is generated by the reflexive graph $tr_1(K)$ and is subject to the cubical relations

$$\forall r, s \in P_y, r = s$$

for all non-degenerate $y \in K_n$ in dimensions $n > 1$. The action of $\mathcal{C}$ on a cubical map $t$ is determined by $t_1$.

Proposition 3.16 is essentially a consequence of lemma 3.14. The details of the proofs consist of tedious, yet entirely standard combinatorics.

3.17. Remark. The $n!$ paths “around a cube” are all different, yet their categorifications do not in general have the same length. Consider for instance the non-degenerate cube

$$\begin{array}{ccc}
\partial_0 & \partial_1 & \partial_2 \\
\partial_3 & \partial_4 & \partial_5 \\
\partial_6 & \partial_7 & \partial_8
\end{array}$$

In particular, given a cubical set $K$ such that $K_0 = 1_{\text{Set}}$, we have

$$(\mathcal{C} \circ \text{Cosk}_1)(K) \cong 1_{\text{Cat}}$$

since in this case all paths are equated.

3.18. Definition. A cocubical object in category $\mathcal{C}$ is a functor $\square \rightarrow \mathcal{C}$. A cubical object in a category $\mathcal{C}$ is a functor $\square^{\text{op}} \rightarrow \mathcal{C}$.

3.19. Proposition. Let $\mathcal{C}$ be locally small. $C \in \mathcal{C}$ and a cocubical object $K : \square \rightarrow \mathcal{C}$ determine a cubical set

$$\text{Cubes}_K(C) \text{ def.} \mathcal{C}(K(\_), C)$$

Proof. A cubical set is just a presheaf on $\square$. $\square$
3.20. **Proposition.** Let $C = - \xrightarrow{c} +$ be the interval reflexive graph and

$$K : (\square, \square) \longrightarrow (\mathbf{rGraph}, \times)$$

be the monoidal functor such that

\[ - K(2) = C ; \]
\[ - K(\delta^-)(C) = - ; \]
\[ - K(\delta^+)(C) = + . \]

Then

(i) $\text{Cubes}_K(R)_m = \mathbf{rGraph}(C^m, R) ;$

(ii) there is the isomorphism of cubical sets

$$\cosk_1(R) \cong \text{Cubes}_K(R)$$

for all $R \in \mathbf{rGraph}.$

**Proof.**

(i) $\text{Cubes}_K(R)_m = \mathbf{rGraph}(C^m, R)$ since $K(2^m) = C^m ;$

(ii) A morphism $h : C^2 \longrightarrow R$ determines the 2-shell

$$((h(-c), h(c-)), (h(+c), h(c+))) \in Y^*_2(R)$$

Conversely, the 2-shell

$$((y_1^-, y_2^-), (y_1^+, y_2^+)) \in Y^*_2(R)$$

determines the morphism

$$l : C^2 \longrightarrow R$$

\[ -c \longmapsto y_1^- \]
\[ c- \longmapsto y_2^- \]
\[ +c \longmapsto y_1^+ \]
\[ c+ \longmapsto y_2^+ \]

The assertion follows by induction on $m.$

\[ \square \]
3.21. Corollary. Let

$$NCubes_K(R) \overset{\text{def.}}{=} \bigcup_{m \in \mathbb{N}} \{ h \in Cubes_K(R) | h \text{ non-degenerate} \}$$

and $\sim^K$ be the congruence on $\mathcal{F}(R)$ generated by

$$s \sim t \overset{\text{def.}}{\iff} \exists h \in NCubes_K(R). s \in \text{im}(\mathcal{F}(h)) \land t \in \text{im}(\mathcal{F}(h)).$$

Then

$$(C \circ \text{cosk}_1(R)) \cong \mathcal{F}(R)/\sim^K$$

4. Synchronization of Higher-Dimensional Automata

For the computer scientist, cubical sets are a convenient tool to organize independence relations among transitions \[8\]. Indeed, traditional labeled transition systems over an alphabet $\Sigma$ only capture the interleavings of potentially parallel computational paths. If one is interested in “true parallelism”, additional information needs to be provided. A possible way to realize this program is to label transitions with sequences over $\Sigma$.

4.1. Automata with concurrency.

4.2. Proposition. (Goubault \[7\]) Given a set $\Sigma$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Sigma^*$, let $|\alpha| \overset{\text{def.}}{=} n$ be $\alpha$’s length. Suppose from now on $\Sigma$ totally ordered and let

$$\alpha \models (\Sigma, \leq) \overset{\text{def.}}{\iff} \alpha_1 \leq \cdots \leq \alpha_n$$

Let further $* \notin \Sigma$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\Sigma \cup \{*\})^*$, $\|\alpha\| \leq \alpha$ be the word obtained from $\alpha$ by removing all the occurrences of $*$ and

$$!\Sigma_n \overset{\text{def.}}{=} \{ \alpha \in (\Sigma \cup \{*\})^* \mid |\alpha| = n \land \|\alpha\| \models (\Sigma, \leq) \}$$

($(\Sigma)_n \in \mathbb{N}$ along with faces

$$\partial^-_i (\alpha_1, \ldots, \alpha_n) = \partial^+_i (\alpha_1, \ldots, \alpha_n) \overset{\text{def.}}{=} (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$$

and degeneracies

$$\epsilon_i (\alpha_1, \ldots, \alpha_n) \overset{\text{def.}}{=} (\alpha_1, \ldots, \alpha_{i-1}, *, \alpha_i, \ldots, \alpha_n)$$

determine a cubical set.
4.3. Definition. A higher-dimensional automaton or HDA (over the alphabet $\Sigma$) is an object in the slice category $c\text{Set}/!\Sigma$. As a convention, let $!\Sigma_0 \defeq \{\ast\}$. An HDA $t : T \rightarrow !\Sigma$ is $n$-skeletal provided $T$ is.

Cubes in higher dimensions encode independence relations for each arity while face maps encode the coherence of these relations. Indeed, computational intuition says that if a set of transitions is pairwise independent so each subset thereof has to be pairwise independent as well. Put differently, an $n$-cube $k$ can be seen as an $n$-dimensional transition from state $\text{dom}_n(k)$ to state $\text{cod}_n(k)$ (c.f. lemma 3.14), the dimension being the number of processes operating without interaction. A traditional labeled transition system is thus a pointed 1-skeletal HDA $t : T \rightarrow !\Sigma$ such that $t_1$ is locally injective.

4.4. Coskeletal HDA’s. Is there a notion of $n$-coskeleton suitable for HDA’s, i.e. compatible with labeling? Given an HDA $t : T \rightarrow !\Sigma$, the first reaction would of course be to consider $\text{Cosk}_n(t)$. However, $!\Sigma$ is obviously not $n$-coskeletal. Consider on the other hand the transition system

$$
\begin{array}{ccc}
b & \overset{\alpha}{\rightarrow} & d \\
\beta \downarrow & & \downarrow \theta \\
a & \overset{\psi}{\rightarrow} & c
\end{array}
$$

It’s underlying reflexive graph is a 1-shell, yet there is no canonical way to label this 1-shell.

4.5. Definition. Let $t : T \rightarrow !\Sigma$ be an HDA. A $\Sigma_n$-shell of $t$ is an element

$$
y = ((y^-_1, \ldots, y^-_{n+1}), (y^+_1, \ldots, y^+_{n+1})) \in T^*_{n+1}
$$

such that $t_n(y^-_i) = t_n(y^+_i)$ for all $1 \leq i \leq n + 1$. $T^*_{n+1}$ is the set of all $\Sigma_n$-shells and $\text{Cosk}_{\Sigma,n}(T) \rightarrow \text{Cosk}_n(T)$ is the subobject determined by the $T^*_{n+j+1}$’s for all $j \in \mathbb{N}$. This subobject is called the $n$-th $\Sigma$-coskeleton of $T$.

In short, a $\Sigma$-coskeleton verifies the Kan condition of proposition 3.9 for the “good” shells.

4.6. Lemma. Let $y$ be a $\Sigma_n$-shell of the HDA $t : T \rightarrow !\Sigma$. The sequence of labels

$$
(t_n(y^-_i)) \in \prod_{1 \leq i \leq n+1} !\Sigma_n
$$

is determined by $t_n(y^+_i)$ and $(\partial^-_1 \circ \cdots \circ \partial^-_n \circ t_n)(y^-_2)$.

Proof. Let $t_n(y^-_i) = \alpha_i \defeq (\alpha^{(1)}_i, \ldots, \alpha^{(n)}_i)$ for $1 \leq i \leq n + 1$. Since $t$ is a cubical map, we have

$$
\partial^{-}_k(\alpha_1) = \partial^+_1(\alpha_{k+1})
$$
for all $1 \leq k \leq n$, i.e.

$$(\alpha_1^{(2)}, \ldots, \alpha_i^{(n)}) = (\alpha_1^{(1)}, \ldots, \alpha_1^{(k-1)}, \alpha_1^{(k+1)}, \ldots, \alpha_1^{(n)})$$

for all $2 \leq i \leq n+1$. Similarly,

$$\partial^-_i (\alpha_i) = (\alpha_1^{(1)}, \ldots, \alpha_1^{(k-1)}, \alpha_1^{(k+1)}, \ldots, \alpha_1^{(n)}) = (\alpha_1^{(1)}, \ldots, \alpha_1^{(k-1)}, \alpha_1^{(k+1)}, \ldots, \alpha_1^{(n)}) = \partial^-_i (\alpha_{i+1})$$

so in particular

$$\alpha_i^{(1)} = \alpha_{i+1}^{(1)}$$

for all $2 \leq i \leq n$. □

4.7. Proposition. Let $t : T \longrightarrow !\Sigma$ be an HDA. There is the HDA

$$\text{Cosk}_{\Sigma,n}(t) : \text{Cosk}_{\Sigma,n}(T) \longrightarrow !\Sigma$$

given in dimension $k > n$ by

$$(\text{Cosk}_{\Sigma,n}(t))_k(y) \overset{\text{def.}}{=} (t_{k-1}(y_1^-), (\partial^-_n \circ \cdots \circ \partial^-_2 \circ t_{k-1})(y_2^-))$$

Proof. By lemma 4.6. □

When $n = 1$, we write $\text{Cosk}_\Sigma(t) : \text{Cosk}_\Sigma(T) \rightarrow !\Sigma$ respectively $\text{cosk}_\Sigma(r) : \text{cosk}_\Sigma(R) \rightarrow !\Sigma$. Clearly, both lemma 4.6 and proposition 4.7 could also be formulated with the “positive” parts of the $\Sigma$-shells.

4.8. Coskeletal Synchronization. Synchronization of transition systems can be presented by a table also (glamorously) called synchronization algebra [19]. The idea is to filter out and to relabel transitions out of a product. This amounts to the construction of a pullback. Similarly, synchronization of HDA’s can be presented along those lines. We introduce here the simplest case called 1-coskeletal synchronization, the general treatment will appear elsewhere.

As the name suggests, 1-coskeletal synchronization means that everything is determined by dimension 1. Suppose for instance $\Sigma = \{\alpha, \beta, \bar{\alpha}, \bar{\beta}, \tau\}$ and consider the synchronization table

|   | $\star$ | $\alpha$ | $\beta$ | $\bar{\alpha}$ | $\bar{\beta}$ | $\tau$ |
|---|-------|---------|---------|------------|------------|-------|
| $\star$ | $\star$ | $\alpha$ | $\beta$ | $\bar{\alpha}$ | $\bar{\beta}$ | $\tau$ |
| $\alpha$ | $\alpha$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $\bar{\beta}$ | $\beta$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $\bar{\alpha}$ | $\bar{\alpha}$ | $\tau$ | $T$ | $T$ | $T$ | $T$ |
| $\bar{\beta}$ | $\beta$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $\tau$ | $\tau$ | $T$ | $T$ | $T$ | $T$ | $T$ |

indexed by $\Sigma \cup \{\star\}$ and with entries in $\Sigma \cup \{\star, T\}$. This table prescribes the following “synchronized product” of HDA’s:
an idle transition (with label ⋆) is synchronized with any transition and the result is relabeled as the latter;

− α-transitions are synchronized with ¯α-transitions and the result is a τ-transition; similarly for labels β and ¯β;

− all other pairs of transitions are not synchronized (i.e. filtered out); this is indicated by ⊤ and suggests that this information shall propagate to higher dimensions.

Consider the (2-skeletal) HDA

\[ u : \square[2] \rightarrow !\Sigma \]

\[ s_2 \mapsto (\alpha, \beta) \]

depicted as

\[
\begin{array}{ccc}
  b & \xrightarrow{\alpha} & d \\
  \beta & (\alpha, \beta) & \beta \\
  a & \xrightarrow{\alpha} & c
\end{array}
\]

The picture does not include the degenerate cubes. It illustrates the fact that labeling the standard 2-cube with (α, β) implies the labels of its faces, this since \( u \) is technically a cubical map. Consider further the (1-skeletal) HDA

\[ v : \square[1] \rightarrow !\Sigma \]

\[ s_1 \mapsto \bar{\beta} \]

depicted as

\[
x \xrightarrow{\bar{\beta}} y
\]

The synchronized product \( u \otimes v \) of \( u \) and \( v \) with respect to the above table is

\[
\begin{array}{ccc}
  b & \xrightarrow{\alpha} & d \\
  \tau & (\alpha, \tau) & \tau \\
  a & \xrightarrow{\alpha} & c
\end{array}
\]

4.9. Lemma. Let \( i : S \twoheadrightarrow A \times B \) be a relation in a topos. The pushout square determined by the projections \( \pi_1 \circ i \) and \( \pi_2 \circ i \) is a pullback square.

Proof. Let \( s : A \times B \rightarrow \Omega \) be the classifying predicate of \( S \) and \( P \overset{\text{def.}}{=} A +_S B \) with coprojections \( j_1 \) and \( j_2 \). Then

\[
A \times_P B \cong \{(a,b) \in A \times B | j_1(a) = j_2(b)\} = \{(a,b) \in A \times B | s(a,b)\} \cong S.
\]

□
4.10. Proposition. Let \( s : S \to \Sigma \) be an HDA and \( \tilde{s} : tr_1(S) \to \Sigma_s \) be the morphism of reflexive graphs given by the image factorization

\[
\begin{array}{ccc}
tr_1(S) & \xrightarrow{\tilde{s}} & \Sigma_s \\
\downarrow & & \downarrow \\
tr_1(s) & \xrightarrow{\tilde{s}} & \Sigma_s \\
\downarrow & & \downarrow \\
tr_1(!\Sigma) & & \\
\end{array}
\]

in the topos \( \text{rGraph} \). Let \( s_{(-,-)} : (\Sigma \cup \{\ast\})^2 \to \Sigma \cup \{\ast, \top\} \) be a synchronization table. Given a further HDA \( t : T \to \!\Sigma \), let

\[
\Upsilon'_{s,t} \overset{\text{def.}}{=} \{ (\theta, \psi) \in \Sigma_s \times \Sigma_t \mid s(\theta, \psi) \neq \top \land \theta \neq * \land \psi \neq * \}
\]

and

\[
\Upsilon_{s,t} \overset{\text{def.}}{=} \Upsilon'_{s,t} \cup ((\{\ast\} \times \Sigma_t) \setminus \{\ast\} \times \pi_2(\Upsilon'_{s,t})) \cup ((\Sigma_s \times \{\ast\}) \setminus (\pi_1(\Upsilon'_{s,t}) \times \{\ast\}))
\]

Under these assumptions, the pushout square

\[
\begin{array}{ccc}
\Sigma_s & \xrightarrow{} & \Sigma_{s,t} \\
\downarrow & & \downarrow \\
\Sigma_t & \xrightarrow{} & \Sigma_{s,t} \\
\end{array}
\]

is also a pullback square.

Proof. By lemma 4.9.

If \( s \) is 1-skeletal, we slightly abuse notation and write \( s : S \to \Sigma_s \) for \( \tilde{s} \) of proposition 4.10.

4.11. Definition. Under the notation of proposition of proposition 4.10, let

\[
S \boxtimes_s T \overset{\text{def.}}{=} tr_1(S) \times_{\Sigma_s} tr_1(T)
\]

and \( \langle s, t \rangle \) be the comparison morphism in

\[
\begin{array}{ccc}
S \boxtimes_s T & \xrightarrow{} & tr_1(T) \\
\downarrow_{\langle s, t \rangle} & & \downarrow_{t} \\
\Upsilon_{s,t} & \xrightarrow{} & \Sigma_t \\
\downarrow_{\pi_1} & & \downarrow_{\pi_1} \\
\Sigma_s & \xrightarrow{j_{s,t}} & \Sigma_{s,t} \\
\end{array}
\]

The formula for the synchronized product is

\[
s \otimes_s t \overset{\text{def.}}{=} \cosk\Sigma (s_{(-,-)}|_{\Upsilon_{s,t} \circ \langle s, t \rangle})
\]
4.12. Categorification of Synchronized HDA’s.

4.13. Definition. Let $C = -\to \to +$ be the interval reflexive graph and $m \in \mathbb{N}$. An $m$-cube in a reflexive graph $K$ is a morphism $c : C^m \to K$ (c.f. proposition 3.20). An $m$-cube is rigid if $e \neq \text{id} \Rightarrow c(e) \neq \text{id}$ for all edges $e \in (C^m)_1$ and contractible if $c(e) = \text{id}$ for all edges $e \in (C^m)_1$. $K$ is acubic if all the $m$-cubes in $K$ are contractible for all $m \in \mathbb{N}$.

In definition 4.13, $\text{id}$ stands of course for a distinguished self-loop in a reflexive graph. We stick to this abuse of terminology and notation. We write $\partial^- u$ for $\text{dom}(u)$ respectively $\partial^+ u$ for $\text{cod}(u)$ when convenient and use the “turtle graphics” notation for vertices and edges of $C^m$, e.g.

4.14. Theorem. Let $\Sigma$ be an alphabet, $F : \text{rGraph} \to \text{Cat}$ be the “free” functor and $t_i : T_i \to \Sigma$ be morphisms of reflexive graphs for $1 \leq i \leq m$. Suppose that $T_i$ is acubic for all $1 \leq i \leq m$. Let $\Sigma_{i,i+1} \defeq \Sigma_{t_i,t_{i+1}}$ for all $1 \leq i \leq m - 1$. Then

$$(C \circ \text{cosk}_\Sigma)(T_1 \boxtimes \cdots \boxtimes T_m) \cong F(T_1) \times_{F(\Sigma_{1,2})} F(T_2) \times_{F(\Sigma_{2,3})} \cdots \times_{F(\Sigma_{m-1,m})} F(T_m)$$

Proof. Let

$$t^-_i \defeq \begin{cases} j_{t_{i-1},t_i} \circ t_i & 1 < i \leq m \\ j_{t_1,t_2} \circ t_1 & i = 1 \end{cases}$$

and

$$t^+_i \defeq \begin{cases} j_{t_{i},t_{i+1}} \circ t_i & 1 \leq i < m \\ j_{t_{m-1},t_m} \circ t_m & i = m \end{cases}$$

$T_1 \boxtimes \cdots \boxtimes T_m$ is the limit of the diagram

```
...  T_{i-1}  T_i  T_{i+1}  ...
|     |     |     |
| v  t^-_i   v   t^-_i  v   t^-_i  v |
|     |     |     |
...  Sigma_{i-1,i}  Sigma_i  Sigma_{i+1}  ...
```
functor

Suppose $p \leq m$ and let $\{i_1, \ldots, i_p\} \subseteq \{1, \ldots, m\}$ be a set of indices. Let $(u_{i_1}) \in \prod_{1 \leq q \leq p} (T_{i_1})_1$ be a family of edges such that $u_{i_1} \neq id$ and

$$t^{-1}_{i_1}(u_{i_1}) = t^+_{i_1}(u_{i_1}) = id$$

for all $1 \leq q \leq p$. Let $(x_j) \in \prod_{j \in \{i_1, \ldots, i_p\}} T_j$ be a family of vertices. Analyzing the above wide pullback, one sees that the data $(u_{i_1})$ and $(x_j)$ determine a rigid $p$-cube

$$c : C^p \rightarrow T_1 \boxtimes \cdots \boxtimes T_m$$

such that $c(\omega_1 \cdots \omega_{r-1} \kappa \omega_{r+1} \cdots \omega_p) = (id(y_1), \ldots, id(y_{i_1-1}), u_{i_1}, id(y_{i_1+1}), \ldots, id(y_m))$ with

$$y_j \overset{\text{def.}}{=} \begin{cases} \partial_{\omega_j}^{\pm}(u_{i_1}) \quad \exists q \in \{1, \ldots, r-1, r+1, \ldots, p\}, \; j = i_q \\ x_j \end{cases}$$

Conversely, any non-contractible cube in $T_1 \boxtimes \cdots \boxtimes T_m$ arises as above since the $T_i$’s are acubic by hypothesis. In particular, any $p$-cube in $T_1 \boxtimes \cdots \boxtimes T_m$ is contractible for $p > m$ and

$$cosk(\Sigma_1) \boxtimes \cdots \boxtimes T_m) \cong cosk(T_1 \boxtimes \cdots \boxtimes T_m) \quad (**)$$

Suppose now $(e_1, \ldots, e_m) \in T_1 \boxtimes \cdots \boxtimes T_m$ and let $(e_i) \in F(T_i)$ be the path of length 1 determined by $e_i$. We have

$$((e_1), \ldots, (e_m)) \in (F(T_1) \times_{F(\Sigma_{1,2})} F(T_2) \times_{F(\Sigma_{2,3})} \cdots \times_{F(\Sigma_{m-1,m})} F(T_m))_1$$

as a consequence of $(*)$. Let $\sim$ be the congruence on $F(T_1 \boxtimes \cdots \boxtimes T_m)$ generated by all non-contractible cubes in $T_1 \boxtimes \cdots \boxtimes T_m$. Then

$$(C \circ cosk_\Sigma)(T_1 \boxtimes \cdots \boxtimes T_m) \cong (C \circ cosk)(T_1 \boxtimes \cdots \boxtimes T_m) \cong F(T_1 \boxtimes \cdots \boxtimes T_m)/ \sim$$

by $(**)$ and corollary 3.21. But the non-contractible cubes in $T_1 \boxtimes \cdots \boxtimes T_m$ are the rigid ones, hence the map $t$ given on generators by

$$t : ((C \circ cosk_\Sigma)_1 \rightarrow (F(T_1) \times_{F(\Sigma_{1,2})} F(T_2) \times_{F(\Sigma_{2,3})} \cdots \times_{F(\Sigma_{m-1,m})} F(T_m))_1$$

$$(e_1, \ldots, e_m) \rightarrow ((e_1), \ldots, (e_m))$$

is well defined. On the other hand, both categories have the same set of objects $\prod_{1 \leq i \leq m} (T_i)_0$ since the reflexive graphs $\Sigma_{i,i+1}$ have one vertex for all $1 \leq i \leq m - 1$. Hence there is the functor

$$I : (C \circ cosk_\Sigma)(T_1 \boxtimes \cdots \boxtimes T_m) \rightarrow F(T_1) \times_{F(\Sigma_{1,2})} F(T_2) \times_{F(\Sigma_{2,3})} \cdots \times_{F(\Sigma_{m-1,m})} F(T_m)$$
with $I_0$ the identity map on $\prod_{1 \leq i \leq m} (T_i)_0$ and $I_1 = \iota$. This functor has the obvious inverse. \hfill \Box

Theorem 4.14 is not the most general statement of the kind, the proof actually works for any wide pullback over reflexive graphs with one vertex. There could also be a “labeled” version, yet in our applications we categorify precisely in order to get rid of the labels (it is in the nature of a categorical semantics to be “syntax free”). Philosophy aside:

4.15. **Remark.** Theorem 4.14 does not claim that

$$ (C \circ \cosk_\Sigma)(\Sigma_{t_i, t_{i+1}}) \cong \mathcal{F}(\Sigma_{t_i, t_{i+1}}) $$

in which case $C \circ \cosk_\Sigma$ would preserve wide pullbacks of acyclic reflexive graphs. Consider for instance the case $m = 2$. What we do have is merely the diagram

by functoriality and remark 3.17

4.16. **Change of Alphabet.** Suppose

$$ S \xrightarrow{u} \hat{S} $$
$$ s \downarrow \quad \downarrow \hat{s} $$
$$ \Sigma \xrightarrow{w} \hat{\Sigma} $$

and

$$ T \xrightarrow{v} \hat{T} $$
$$ t \downarrow \quad \downarrow \hat{t} $$
$$ \Sigma \xrightarrow{w} \hat{\Sigma} $$

commute. Let

$$ w_T : \Upsilon_{s,t} \longrightarrow \hat{\Upsilon}_{\hat{s},\hat{t}} $$
$$ w_{s,\hat{s}} : \Sigma_s \longrightarrow \hat{\Sigma}_{\hat{s}} $$
$$ w_{t,\hat{t}} : \Sigma_t \longrightarrow \hat{\Sigma}_{\hat{t}} $$
be the maps determined by $w$ and by universal property. "Morally", this says that synchronization under this change of alphabet is still governed by $s(-,-)$. Let

$$
\begin{align*}
\pi_1 &: \Upsilon_{s,t} \to \Sigma_s \\
\pi_2 &: \Upsilon_{s,t} \to \Sigma_t \\
\hat{\pi}_1 &: \Upsilon_{\hat{s},\hat{t}} \to \hat{\Sigma}_{\hat{s}} \\
\hat{\pi}_2 &: \Upsilon_{\hat{s},\hat{t}} \to \hat{\Sigma}_{\hat{t}}
\end{align*}
$$

be the obvious projections. There is a commuting diagram

with $\bar{w}$ given by universal property and thus

$$
\begin{align*}
\bar{w} &\overset{\text{def.}}{=} \bar{w}_{t_i,t_{i+1}} : \Sigma_{t_i,t_{i+1}} \to \hat{\Sigma}_{t_i,t_{i+1}} \\
\bar{w} &\overset{\text{def.}}{=} \bar{w}_{t_i,t_{i+1}} : \Sigma_{t_i,t_{i+1}} \to \hat{\Sigma}_{t_i,t_{i+1}}
\end{align*}
$$

with $u \boxtimes v$ given by universal property. It is straightforward that this fact generalizes to wide pullbacks, so theorem 4.14 admits a “relative” version:

4.17. THEOREM. Let $w : \Sigma \to \hat{\Sigma}$ be a map and $(t_i : T_i \to \Sigma)_{1 \leq i \leq m}$ respectively $(\hat{t}_i : \hat{T}_i \to \hat{\Sigma})_{1 \leq i \leq m}$ be morphisms of reflexive graphs with $T_i$ and $\hat{T}_i$ acubic for all $1 \leq i \leq m$. Suppose there are morphisms $(u_i : T_i \to \hat{T}_i)_{1 \leq i \leq m}$ such that $t_i \circ u_i = w \circ t_i$ for all $1 \leq i \leq m$. Let

$$
\bar{w}_{t_i,t_{i+1}} \overset{\text{def.}}{=} \bar{w}_{t_i,t_{i+1}} : \Sigma_{t_i,t_{i+1}} \to \hat{\Sigma}_{t_i,t_{i+1}}
$$

for all $1 \leq i \leq m - 1$. Then

$$(C \circ \cosk_{\Sigma})(u_1 \boxtimes \cdots \boxtimes u_m) \cong F(u_1) \times_{\mathcal{F}(\bar{w}_{1,2})} \mathcal{F}(u_2) \times_{\mathcal{F}(\bar{w}_{2,3})} \cdots \times_{\mathcal{F}(\bar{w}_{m-1,m})} \mathcal{F}(u_m)$$
Proof. By chasing a series of diagrams as in remark 4.15. The wide pullback
\[ F(u_1) \times_{\mathcal{F}(\bar{w}_{1,2})} F(u_2) \times_{\mathcal{F}(\bar{w}_{2,3})} \cdots \times_{\mathcal{F}(\bar{w}_{m-1,m})} F(u_m) \]
is in \( \text{Cat}^- \). \qed

4.18. Remark. The wide pullback
\[ F(u_1) \times_{\mathcal{F}(\bar{w}_{1,2})} F(u_2) \times_{\mathcal{F}(\bar{w}_{2,3})} \cdots \times_{\mathcal{F}(\bar{w}_{m-1,m})} F(u_m) \]
is actually in \( \text{Ulf}_\mathcal{F}^- \) (c.f. section 2.11) since everything in sight is in the image of \( \mathcal{F} \).

5. The Programming Language CIP

In this section we introduce the syntax and an operational semantics of the programming language CIP. The acronym stands for “communicating imperative programs”. CIP is a variant of CONCURRENT PASCAL with CSP-style message-passing primitives. It is a nice little language, as handy as an assembly language with a decent macro expansion mechanism. Its syntax is presented by typing rules formulated with respect to an underlying type theory \( \mathcal{A} \). We assume \( \mathcal{A} \) is algebraic in order to fix the ideas and keep the setup simple, yet more elaborated theories do also work \cite{10}. It’s operational semantics is a set of rewrite rules over a data structure consisting of instruction stacks, instruction registers and stores.

5.1. CIP’s Syntax. Let \( V \) be a countable set of variables and \( P \) a countable set of port names. Let \( \mathcal{A} \) be an algebraic theory with \( V \) as set of variables.

5.2. Grammar. CIP consists of expressions, statements and terms. CIP’s expressions are \( \mathcal{A} \)’s terms while statements and terms are given by the productions

\[
\begin{align*}
\text{stat} & ::= \text{nop} \mid x := e \mid p! e \mid p? x \mid \\
& \quad \text{stat}_1; \text{stat}_2 \mid \\
& \quad \text{if } e \text{ then } \text{stat} \text{ else } \text{stat} \text{ end} \mid \\
& \quad \text{while } e \text{ do } \text{stat} \text{ end} \\
\text{term} & ::= \text{stat} \mid \text{term} \ll p_1 \simeq q_1, \ldots, p_n \simeq q_n \gg \text{term}
\end{align*}
\]

where \( e \) is an expression, \( x \in V \), \( n \in \mathbb{N} \) and \( p_i, q_i \in P \) for all \( 0 \leq i \leq n \).

5.3. Typing Sequents. Let \( T_{\mathcal{A}} \) be \( \mathcal{A} \)’s set of types. The general format of a typing sequent for CIP is

\[
(x_1 : \tau_1, \ldots, x_n : \tau_n) \vdash \text{term} : (p_1 : \theta_1^{s_1}, \ldots, p_m : \theta_m^{s_m})
\]

where

- \((x_i, \tau_i) \in V \times T_{\mathcal{A}} \) for all \( 1 \leq i \leq n \)
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- $(p_j, \theta_j) \in \mathcal{P} \times \mathbb{T}_A$ for all $1 \leq j \leq m$
- $s_j \in \{-, +\}$ for all $1 \leq j \leq m$

The $x_i$'s are names of variables while the entries $p_k : \theta_k^{s_k}$ are pairs consisting of a port name $p_k$ designating an interface port of type $\theta_k$. The latter is decorated with a polarity $s_k$ which discriminates if $p_k$ is an input or an output port. Their list represents the signature of the program as seen by its environment.

A well-formed context is a context without a repetition of a variable name. A well-formed signature is a signature without a repetition of a port name. In other words, well-formed contexts and signatures encode partial functions of types $\mathcal{V} \to \mathbb{T}_A$ respectively $\mathcal{P} \to \mathbb{T}_A \times \{-, +\}$. Let $\mathcal{K}$ be the set of all well-formed contexts and $\mathcal{S}$ be the set of all well-formed signatures.

5.4. Typing Rules. We assume well-formed all the contexts and signatures occuring in the premises of the following typing rules.

Structural Rules. Let $S_n$ be the $n$th symmetric group. CIP’s structural rules are

\[
\text{Weak} \quad \frac{\Gamma \vdash \text{term} : \Delta}{\Gamma, x : \tau \vdash \text{term} : \Delta} \quad \text{Perm} \quad \frac{\Gamma \vdash \text{term} : \Delta \pi_v \in S|\pi_s \in S_{|\Delta|}}{\pi_v \Gamma \vdash \text{term} : \pi_s \Delta}
\]

Sequential Fragment. CIP’s sequential fragment is given by the rules

\[
\text{Asg} \quad \frac{\Gamma \vdash e : \tau}{\Gamma, x : \tau \vdash x := e : \langle \rangle} \quad \text{Nop} \quad \frac{}{\text{nop} : \langle \rangle}
\]

\[
\text{If} \quad \frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash \text{stat}_1 : \Delta \quad \Gamma \vdash \text{stat}_2 : \Theta}{\Gamma \vdash \text{if} e \text{ then stat}_1 \text{ else stat}_2 \text{ end} : \Delta \otimes \Theta}
\]

\[
\text{While} \quad \frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash \text{stat} : \Delta}{\Gamma \vdash \text{while} e \text{ do stat} \text{ end} : \Delta}
\]

\[
\text{Seq} \quad \frac{\Gamma \vdash \text{stat}_1 : \Delta \quad \Gamma \vdash \text{stat}_2 : \Theta}{\Gamma \vdash \text{stat}_1; \text{stat}_2 : \Delta \otimes \Theta}
\]

where $\tau_i \in \mathbb{T}_A$ for all $i \in \mathbb{N}$ while $\Delta \otimes \Theta \in \mathcal{S}$ is the concatenation of $\Delta$ and $\Theta$, possibly after $\alpha$-conversion.

Concurrent Fragment. CIP’s concurrent fragment is given by the input/output rules

\[
\text{Out} \quad \frac{\Gamma \vdash e : \tau}{\Gamma \vdash p! e : \langle p : \tau^+ \rangle}
\]

\[
\text{In} \quad \frac{\Gamma, x : \tau \vdash p? x : \langle p : \tau^- \rangle}{\Gamma, x : \tau \vdash p? x : \langle p : \tau^- \rangle}
\]

Composition rule. CIP’s composition is given by the rule

\[
\frac{\gamma \vdash \text{term}_1 : \Delta, \langle p_1 : \tau_1^+, \ldots, p_n : \tau_n^+ \rangle \quad \Xi \vdash \text{term}_2 : \Theta, \langle q_1 : \tau_1^-, \ldots, q_n : \tau_n^- \rangle}{\gamma \oplus \Xi \vdash \text{term}_1 \ll p_1 \simeq q_1, \ldots, p_n \simeq q_n \gg \text{term}_2 : \Delta \otimes \Theta}
\]
where $\Gamma \oplus \Xi \in \mathcal{K}$ is the concatenation (of both components) of $\Gamma$ and $\Xi$, possibly after $\alpha$-conversion.

A pair $p_i \backsimeq q_i$ is a restricted channel of type $\tau_i$, connecting ports $p_i$ and $q_i$. The notation resembles the one used in linear logic since it is where the inspiration comes from, just think of the interface ports as resources. In more practical terms, $\text{MCut}$ allows to define interactions of processes connected by typed channels.

5.5. Equality on Terms. We are in fact more interested in the “end product” of a series of applications of the typing rules than in order the latter were performed. Order-sensitive rules are $\text{Seq}$ and $\text{MCut}$, applications of the latter possibly requiring $\alpha$-conversion. Given $\Gamma \in \mathcal{K}$ and $\Delta \in \mathcal{S}$, let $\mathcal{T}_{\Gamma,\Delta}$ be the set of well-formed terms $\Gamma \vdash t : \Delta$. The equivalence relation $\approx \in \mathcal{T}_{\Gamma,\Delta} \times \mathcal{T}_{\Gamma,\Delta}$ is generated by

(i) $(t; t'); t'' \approx t; (t'; t'')$.

(ii) $(t \ll L \gg t') \ll K \gg t'' \approx t \ll L \gg (t' \ll K \gg t'')$.

In particular, we can write $t; t', t''$ and $t \ll L \gg t' \ll K \gg t''$ without ambiguity. Let $\mathcal{L}_{\Gamma,\Delta} \subseteq \mathcal{T}_{\Gamma,\Delta}$ be $\mathcal{T}_{\Gamma,\Delta}$’s subset of well-formed statements. The equivalence class $[t]$ of $t \in \mathcal{T}_{\Gamma,\Delta}$ can be characterized by a pair $([s_1, \ldots, s_n], [p_1 \backsimeq q_1, \ldots, p_m \backsimeq q_m])$ for some $n, m \in \mathbb{N}$, such that $s_i \in \mathcal{L}_{\Gamma,\Delta}$ for all $i \in \mathbb{N}$. All the (sub)statements are flattened with respect to the sequencing operator “;”. Such a pair, called $t$’s normal form, is unique up to $\alpha$-conversion.

5.6. Proposition. Let $\Gamma \vdash t : \Delta$ be a term. The following are equivalent

(i) $t$’s normal form is $([s_1, \ldots, s_n], [p_1 \backsimeq q_1, \ldots, p_m \backsimeq q_m])$;

(ii) there is a family of statements $(\Gamma \vdash s_i : \Delta_i)_{1 \leq i \leq n}$ such that $$(\Gamma \vdash t : \Delta) = (\Gamma_1 \oplus \cdots \oplus \Gamma_n \vdash s_1 \ll \Phi_{1,2} \gg s_2 \cdots s_{n-1} \ll \Phi_{n-1,n} \gg s_n : \Delta_1' \otimes \cdots \otimes \Delta_n')$$

with $\Delta_1', \cdots, \Delta_n' = (\Delta_1, \cdots, \Delta_n) \setminus [p_1 : \tau_1^+, \cdots, p_n : \tau_n^+, q_1 : \tau_1^-, \cdots, q_n : \tau_n^-]$ and $[\Phi_{1,2}, \cdots, \Phi_{n-1,n}] = [p_1 \backsimeq q_1, \ldots, p_n \backsimeq q_n]$ up to $\alpha$-conversion.

Proof. Obvious. \hfill $\square$

Let

$$\mathcal{T}_\Gamma \overset{\text{def.}}{=} \bigcup_{\Delta \in \mathcal{S}} \mathcal{T}_{\Gamma,\Delta}$$

$$\mathcal{T} \overset{\text{def.}}{=} \bigcup_{\Gamma \in \mathcal{K}} \mathcal{T}_\Gamma$$

and $\mathcal{L}_\Gamma \subseteq \mathcal{T}_\Gamma$ and $\mathcal{L} \subseteq \mathcal{T}$ be the corresponding subsets of well-formed statements.
5.7. CIP’s Operational Semantics. CIP’s operational semantics is given in continuation style, i.e. by an abstract machine.

5.8. The Abstract Machine.
Stores. Let $[\_]$ be an interpretation of $\mathcal{A}$ in $\mathbf{Set}$ and

$$\Theta_\mathcal{A} \overset{\text{def.}}{=} \bigcup_{\tau \in \mathcal{T}_\mathcal{A}} [\tau]$$

A store $\sigma$ is a partial function $\sigma \in \mathcal{V} \rightarrow \Theta_\mathcal{A}$ with finite domain of definition. A $\Gamma$-store is a store defined on $\text{Var}(\Gamma)$ such that

$$\forall (x : \tau) \in \Gamma. \sigma(x) \in [\tau]$$

We write $S_-$ for their set.

Locations. A $\Gamma$-location is the address of a node in the syntax tree of some $\text{stat} \in L_\Gamma$. Let $L_\Gamma$ be the set of the $\Gamma$-locations under some encoding, e.g. a path in the tree. Let $\rho(\text{stat})$ be the root of the corresponding tree. Let further

$$\eta, \xi_1, \xi_2, \phi : L_\Gamma \times L_\Gamma \rightarrow L_\Gamma$$

be the partial operations such that

- $\xi_1(\text{if } e \text{ then } \text{stat}_1 \text{ else } \text{stat}_2 \text{ end}, l)$ returns the location of $\text{stat}_1$ given the location $l$ of the if-statement

- $\xi_1(\text{if } e \text{ then } \text{stat}_1 \text{ else } \text{stat}_2 \text{ end}, l)$ returns the location of $\text{stat}_2$ given the location $l$ of the if-statement

- $\phi(\text{while } e \text{ do } \text{stat} \text{ end}, l)$ returns the location of $\text{stat}$ given the location $l$ of the while-statement

We write $\xi_1(l), \xi_2(l)$ respectively $\phi(l)$ when it is clear from the context what the statements are.

Processes. Given $\Gamma \in \mathcal{K}$ let $\mathcal{M}_\Gamma$ be the set of $\Gamma$-stores. A $\Gamma$-process is a triple

$$(S, \langle \text{stat}, l \rangle, \sigma) \in \text{List}(L_\Gamma \times L_\Gamma) \times (L_\Gamma \times L_\Gamma) \times \mathcal{M}_\Gamma$$

$S$ is the instruction stack while $\text{stat}$ is the instruction register. The reason to tie the instructions to their locations will become apparent in section 6.

Configurations and Rewrites. A $\Gamma$-configuration is a vector of $\Gamma$-processes. CIP’s operational semantics consists of a set of conditional rewrite rules

$$\forall [P_1, \ldots, P_n] \rightarrow [P'_1, \ldots, P'_n]$$
on $\Gamma$-configurations. We consider fixed communication topologies, i.e. $n$ remains constant under the rewrites. The vector of channels, $\Gamma$ itself as well as $[,]$ will be needed to evaluate the rewrite conditions $\varpi$.

Observe that, although we need $\Gamma$ for rewrite conditions, we can keep it as a constant datum much like the list of channels. It is a design choice since we do not introduce nested scopes or (remote) procedure calls, a side-issue here. In presence such features though, rewrites may not leave $\Gamma$ constant.

5.9. Rewrite Rules for CIP’s Sequential Fragment. Let $f$ and $f’$ be partial functions and

\[
(f \uplus f')(x) \stackrel{\text{def.}}{=} \begin{cases} f(x) & x \in \text{dom}(f) \setminus \text{dom}(f’) \\ f’(x) & x \in \text{dom}(f’) \end{cases}
\]

Given a well-formed expression $\Gamma \vdash e : \tau$ such that $[x_1 : \tau_1, \ldots, x_n : \tau_n] \subseteq \Gamma_0$ and $\sigma$ a $\Gamma$-store, let further

\[
\sigma(e) \stackrel{\text{def.}}{=} [\Gamma \vdash e] (\sigma(x_1)\ldots\sigma(x_n)) \in [\tau]
\]

CIP’s sequential fragment is given by the rather self-explaining rules

\[
\begin{align*}
\text{Nop} & \quad [P_1, \ldots, P_{i-1}, (S \ast [(\text{stat}, l)], (\text{nop}, k), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S, (\text{stat}, l), \sigma), \ldots, P_n] \\
\text{Asg} & \quad [P_1, \ldots, P_{i-1}, (S \ast [(\text{stat}, l)], (x := e, k), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S, (\text{stat}, l), \sigma \uplus [x \mapsto \sigma(e)]), \ldots, P_n] \\
\text{If1} & \quad [P_1, \ldots, P_{i-1}, (S, (\text{if} \ e \ \text{then} \ stat_1 \ \text{else} \ stat_2 \ \text{end}, l), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S, (\text{stat}_1(\xi_1(l))), \sigma), \ldots, P_n] \\
\text{If2} & \quad [P_1, \ldots, P_{i-1}, (S, (\text{if} \ e \ \text{then} \ stat_1 \ \text{else} \ stat_2 \ \text{end}, l), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S, (\text{stat}_2(\xi_2(l))), \sigma), \ldots, P_n] \\
\text{While1} & \quad [P_1, \ldots, P_{i-1}, (S, (\text{while} \ e \ \text{do} \ stat \ \text{end}, l), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S \ast [(\text{while} \ e \ \text{do} \ stat \ \text{end}, l)], (\text{stat}, \phi(l)), \sigma), \ldots, P_n] \\
\text{While2} & \quad [P_1, \ldots, P_{i-1}, (S \ast [(\text{stat}_1(\eta(l))], (\text{while} \ e \ \text{do} \ stat \ \text{end}, l), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S, (\text{stat}_1(l)), \sigma), \ldots, P_n]
\end{align*}
\]

There is no explicit rewriting rule corresponding to the sequence operator “;”. Such a rule, e.g

\[
[P_1, \ldots, P_{i-1}, (S, (\text{stat}_1; \text{stat}_2, l), \sigma), \ldots, P_n] \rightarrow [P_1, \ldots, P_{i-1}, (S \ast [(\text{stat}_2, \eta(l))], (\text{stat}_1, l), \sigma), \ldots, P_n]
\]
(where \(\eta(stat_1; stat_2, l)\) returns the location of \(stat_2\) given the location \(l\) of \(stat_1\) involves only the management of the instruction stack. We therefore assume the rule implicit to the abstract machine under consideration (“hardwired” to use a real-worldish jargon).

It is intuitively clear that transitions obtained by the sequential rules shall not be observable.

5.10. Rewrite Rules for CIP’s Concurrent fragment. CIP’s message-passing mechanism is given by the rules detailed as follows.

**Rendez-vous.** The “rendez-vous” rule

\[
\begin{align*}
\text{R}\text{V} & \quad p \preceq q \\
\begin{bmatrix}
P_1 \\
\vdots \\
P_i \\
(S_1 * [(stat_1, l_1)], \langle p!e, l \rangle, \sigma_1) \\
\vdots \\
P_{i+m} \\
(S_2 * [(stat_2, k_2)], \langle q?x, k \rangle, \sigma_2) \\
\vdots \\
P_n
\end{bmatrix} & \rightarrow \\
\begin{bmatrix}
P_1 \\
\vdots \\
P_i \\
(S_1, \langle stat_1, l_1 \rangle, \sigma_1) \\
\vdots \\
P_{i+m} \\
(S_2, \langle stat_2, k_2 \rangle, \sigma_{i+m} \cup \{x \mapsto \sigma_i (e)\}) \\
\vdots \\
P_n
\end{bmatrix}
\end{align*}
\]

specifies the transmission of a datum. Process \(P_{i+1}\) sends the expression \(e\) through the port \(p\) and proceeds by popping the next statement from the stack. On the other side, process \(P_{i+m+1}\) is ready to receive a datum from the port \(q\). Both ports are connected, i.e. they form a restricted channel \(p \preceq q\), which allows \(P_{i+m+1}\) to receive the expression \(e\) evaluated in its original context. \(P_{i+m+1}\) stores it in the variable \(x\) and proceeds by popping the next statement from the stack. Transitions obtained by \(\text{R}\text{V}\) shall not be observable since the communication channel is restricted.

**Send.** The “send” rule

\[
\begin{align*}
\text{S} & \quad \not\exists q \quad p \preceq q \\
\begin{bmatrix}
P_1, \ldots, P_{i-1}, (S * [(stat , l)], \langle p!e, k \rangle, \sigma), \ldots, P_n \\
P_1, \ldots, P_{i-1}, (S, \langle stat , l \rangle, \sigma), \ldots, P_n
\end{bmatrix} & \rightarrow \\
\begin{bmatrix}
P_1, \ldots, P_{i-1}, (S, \langle stat , l \rangle, \sigma \cup \{x \mapsto \sigma_i (e)\}), \ldots, P_n
\end{bmatrix}
\end{align*}
\]

specifies the behavior of a program where process \(P_i\) performs a “send” without a rendezvous partner. Basically nothing happens, yet a transition obtained by an application of \(\text{S}\) shall be observable.

**Receive.** The “receive” rules

\[
\begin{align*}
\text{R}_w & \quad \not\exists p \quad p \preceq q \\
\begin{bmatrix}
P_1, \ldots, P_{i-1}, (S * [(stat , l)], \langle q?x, k \rangle, \sigma), \ldots, P_n \\
P_1, \ldots, P_{i-1}, (S, \langle stat , l \rangle, \sigma \cup \{x \mapsto w\}), \ldots, P_n
\end{bmatrix} & \rightarrow \\
\end{align*}
\]

\(w \in \Gamma(x)\)
specify the behavior of a program where process $P_i$ performs a “receive” without a rendezvous partner. They collectively say that the variable $x$ can then be assigned to any value $w$ of the right type. Transitions obtained applications of the $R_w$’s shall be observable.

5.11. STATE SPACE.

5.12. DEFINITION. Let $t \in T_i$ with normal form

$$([s_1, \ldots, s_n], \left[p_1 \simeq q_1, \ldots, p_m \simeq q_m\right])$$

(c.f. 5.5) and $\mu \overset{\text{def.}}{=} [\mu_1, \ldots, \mu_n]$ be a vector of $\Gamma$-stores (c.f. 5.8). Let

$$t_\mu \overset{\text{def.}}{=} \left(([\left], \langle s_1, \rho(s_1) \rangle, \mu_1), \ldots, ([\left], \langle s_n, \rho(s_n) \rangle, \mu_n)\right)$$

be a $\Gamma$-configuration. A $(\Gamma, t, \mu)$-configuration is $t_\iota$ or is obtained from $t_\iota$ by applications of therewrite rules of section 5.9 and section 5.10. Let $R_{t, \mu}$ be the set of all $(\Gamma, t, \mu)$-configurations. The state space of $t$ is defined as

$$\Psi_t \overset{\text{def.}}{=} \bigcup_{\mu \in S_\g} R_{t, \mu}$$

5.13. PROPOSITION. Under the notation of definition 5.12, there is a unique

$$\left\lfloor \mu \right\rfloor : \Psi_t \rightarrow \prod_{1 \leq i \leq n} [\tau_i]$$

such that $\sigma_j \in S_\g$ is minimal for $1 \leq j \leq n$.

PROOF. Let $\Gamma = x_1 : \tau_1, \ldots, x_k : \tau_k$. By proposition 5.6 we can assume that there is a unique decomposition

$$x_1 : \tau_1, \ldots, x_k : \tau_k = \left(x_i^{(1)} : \tau_i^{(1)}, \ldots, x_i^{(m_i)} : \tau_i^{(m_i)}\right)_{1 \leq i \leq k}$$

such that each $\sigma_i$ is a $(x_i^{(1)} : \tau_i^{(1)}, \ldots, x_i^{(m_i)} : \tau_i^{(m_i)})$-store defined precisely on $\{x_i^{(1)}, \ldots, x_i^{(m_i)}\}$. Each $\sigma_i$ is thus minimal with this property with respect to the standard order on partial functions. Hence

$$\left\lfloor P_1, \ldots, P_k \right\rfloor \overset{\text{def.}}{=} \left(\sigma_i(x_i^{p_i})\right)_{1 \leq i \leq k, 1 \leq p_i \leq n_i} \in \prod_{1 \leq i \leq k, 1 \leq p_i \leq n_i} [\tau_i^{p_i}] \cong \prod_{1 \leq j \leq n} [\tau_j]$$
5.14. **Definition.** The map $| - |$ of proposition 5.13 is called the store-map. The map
\[
\| - \| : \Psi_t \longrightarrow \prod_{1 \leq i \leq n} L_\Gamma
\]

\[
\begin{bmatrix}
P_1 \\
\vdots \\
P_n \\
\end{bmatrix} \longmapsto \begin{bmatrix}
l_1 \\
\vdots \\
l_n \\
\end{bmatrix}
\]

is called the location-map.

6. **CIP Evolutions**

In this section, we construct HDA’s from CIP-programs applying the operational rules. We then explain how categorification of morphisms of such HDA’s gives rise to ulf-functors. This leads to a finite presentation in terms of pseudofunctors into $\text{Span}$.

6.1. **From CIP Programs to HDA’s.**

6.2. **Alphabets.** Let
\[
S \overset{\text{def.}}{=} \{ \alpha, \gamma_1, \gamma_2 \} \\
O \overset{\text{def.}}{=} \{ !_{w,p} | \Gamma \in \mathcal{K}, x \in \text{Var}(\Gamma), w \in [\Gamma(x)], p \in \mathcal{P} \} \\
I \overset{\text{def.}}{=} \{ ?_{w,p} | \Gamma \in \mathcal{K}, x \in \text{Var}(\Gamma), w \in [\Gamma(x)], p \in \mathcal{P} \}
\]

and
\[
\mathcal{E} \overset{\text{def.}}{=} S \cup O \cup I
\]

Given $\mathcal{E} = \{ \alpha, \gamma_1, \gamma_2, !, ? \}$, there is the obvious “erasure” map
\[
\| - \| : \mathcal{E} \longrightarrow \widehat{\mathcal{E}}
\]

\[
\alpha \longmapsto \alpha \\
\gamma_1 \longmapsto \gamma_1 \\
\gamma_2 \longmapsto \gamma_2 \\
!_{w,p} \longmapsto ! \quad \text{for all } w \in [\Gamma(x)] \text{ and } p \in \mathcal{P} \\
?_{w,p} \longmapsto ? \quad \text{for all } w \in [\Gamma(x)] \text{ and } p \in \mathcal{P}
\]

6.3. **HDA’s.** Let $\Gamma \in \mathcal{K}$ be a context, $t \in \mathcal{T}_\Gamma$ be a CIP term and

\[
([s_1, \ldots, s_n], [p_1 \succ q_1, \ldots, q_m \prec q_m])
\]

be its normal form (c.f. 5.5). The HDA’s
\[
[t]^{\Gamma} : [t]^{\Gamma,c} \longrightarrow !\mathcal{E}
\]

respectively
\[
\widehat{[t]}^{\Gamma} : \widehat{[t]}^{\Gamma,c} \longrightarrow !\widehat{\mathcal{E}}
\]

are constructed as follows.
1. The sets of states are

\[
[t]^{\Gamma, c}_0 \overset{\text{def.}}{=} \{ |P_1, \ldots, P_n| \mid [P_1 \ldots, P_n] \in \Psi_t \}
\]

respectively

\[
\widehat{[t]}^{\Gamma, c}_0 \overset{\text{def.}}{=} \{ ||P_1, \ldots, P_n|| \mid [P_1 \ldots, P_n] \in \Psi_t \}
\]

2. The 1-skeletons are given by the applications of the rewriting rules. Each state

\[
|P_1, \ldots, P_n| \in [t]^{\Gamma, c}_0
\]

is a vertex of a graph \([t]^{\Gamma, c}_0 \overset{\text{def.}}{=} ([t]^{\Gamma, c}_0, [t]^{\Gamma, c}_1)\). Each edge of this graph comes from an application of some rewrite rule

\[
\varpi
\]

\[
[P_1, \ldots, P_n] \longrightarrow [P'_1, \ldots, P'_n]
\]

so it can be labeled according to the latter. This labeling, written

\[
[t]^{\Gamma}_1 : [t]^{\Gamma, c}_0 \longrightarrow \mathcal{E}
\]

is given by

| Asg | If1/While1 | If2/While2 | RV | S | R \_w |
|-----|-------------|-------------|----|---|------|
| \(\alpha\) | \(\gamma_1\) | \(\gamma_2\) | \(\alpha\) | ! \(_{\sigma(e), p}\) | ? \(_{w, q}\) |

Similarly, the labeling of the graph \(\widehat{[t]}^{\Gamma, c}_0\), written

\[
\widehat{[t]}^{\Gamma}_1 : \widehat{[t]}^{\Gamma, c}_0 \longrightarrow \widehat{\mathcal{E}}
\]

is given by

| Asg | If1/While1 | If2/While2 | RV | S | R \_w |
|-----|-------------|-------------|----|---|------|
| \(\alpha\) | \(\gamma_1\) | \(\gamma_2\) | \(\alpha\) | ! | ? |

Degeneracies are added freely and labeled with \(*\)

3. \([t]^{\Gamma}\) is the 1-coskeletal HDA

\[
[t]^{\Gamma} \overset{\text{def.}}{=} \text{cosk}_\mathcal{E} ([t]^{\Gamma}_0, 1)
\]

while \(\widehat{[t]}^{\Gamma}\) is the 1-coskeletal HDA

\[
\widehat{[t]}^{\Gamma} \overset{\text{def.}}{=} \text{cosk}_\widehat{\mathcal{E}} (\widehat{[t]}^{\Gamma}_0, 1)
\]
6.4. Proposition. \([s]_{0,1}^{\Gamma, c}\) and \(\hat{[s]}_{0,1}^{\Gamma, c}\) are acubic for all \(s \in \mathcal{L}_\Gamma\).

Proof. Induction on syntax. \qed

6.5. Proposition. The 1-skeletal HDA’s \([s]^{\Gamma}\) and \(\hat{[s]}^{\Gamma}\) are transition systems, i.e.

\[\forall e, f \in [s]_1^{\Gamma, c}. [s]_1^{\Gamma}(e) = [s]_1^{\Gamma}(f) \Rightarrow e = f\]

respectively

\[\forall e, f \in \hat{[s]}_1^{\Gamma, c}. \hat{[s]}_1^{\Gamma}(e) = \hat{[s]}_1^{\Gamma}(f) \Rightarrow e = f\]

for all \(s \in \mathcal{L}_\Gamma\).

Proof. Induction on syntax. \qed

6.6. Definition. Let \(t \in \mathcal{T}_\Gamma\) be a term with normal form

\([(s_1, \ldots, s_n), [p_1 \asymp q_1, \ldots, p_1 \asymp q_1])\]

The term \(t\) determines the synchronization table \(s_t((-,-))\) such that

(i) \(\text{Tab}_t(\ast, \theta) = \text{Tab}_t(\theta, \ast) \overset{\text{def.}}{=} \theta\) for all \(\theta \in \mathcal{E}\);

(ii) \(\text{Tab}_t(\!, v, p), (?_v, q) = \text{Tab}_t(?_v, !, v, p) \overset{\text{def.}}{=} \alpha\) if \(p \asymp q\) and \(v = w\);

(iii) all other entries are \(\top\).

We write \(\otimes_t\) for the synchronized product of HDA’s with respect to \(s_t\) and \(\boxdot_t\) for its counterpart on the underlying reflexive graphs.

6.7. Proposition. Suppose \(t \in \mathcal{T}_\Gamma\) with normal form

\([(s_1, \ldots, s_n), [p_1 \asymp q_1, \ldots, p_1 \asymp q_1])\]

and let \(\Gamma_i \in \mathcal{K}\) be the contexts such that \(s_i \in \mathcal{T}_{\Gamma_i}\) for all \(1 \leq i \leq n\). Then

\([t]^\Gamma \cong [s_1]_{0,1}^{\Gamma_i} \otimes_t \cdots \otimes_t [s_n]_{0,1}^{\Gamma_n}\)

respectively

\(\hat{[t]}^\Gamma \cong [s_1]_{0,1}^{\Gamma_i} \boxdot_t \cdots \boxdot_t [s_n]_{0,1}^{\Gamma_n}\)

Proof. Analyzing the rewrite rules, it is easy to see that

\([t]_{0,1}^{\Gamma, c} \cong [s_1]_{0,1}^{\Gamma_i, c} \boxdot_t \cdots \boxdot_t [s_n]_{0,1}^{\Gamma_n, c}\)

respectively

\(\hat{[t]}_{0,1}^{\Gamma, c} \cong [s_1]_{0,1}^{\Gamma_i, c} \boxdot_t \cdots \boxdot_t [s_n]_{0,1}^{\Gamma_n, c}\)

\qed
6.8. Proposition. There is a morphism of reflexive graphs

$$\pi_s : \left[ s \right]_{\Gamma_0,1}^{\Gamma,c} \rightarrow \left[ \hat{s} \right]_{\Gamma_0,1}^{\Gamma,c}$$

for all $$s \in L\Gamma$$.

Proof. The morphism $$\pi_s$$ is well-defined on edges by proposition 6.5.

6.9. Remark. The diagram

$$\begin{array}{ccc}
\left[ s \right]_{\Gamma_0,1}^{\Gamma,c} & \xrightarrow{\pi_s} & \left[ \hat{s} \right]_{\Gamma_0,1}^{\Gamma,c} \\
\downarrow & & \downarrow \\
\left[ s \right]_{\Gamma_0,1}^{\Gamma} & \xrightarrow{\|} & \hat{\left[ s \right]_{\Gamma_0,1}}
\end{array}$$

commutes by construction.

6.10. Categorification as a generalized Relational Semantics.

6.11. Theorem. Suppose $$t \in T\Gamma$$ with normal form $$([s_1, \ldots, s_n], [p_1 \sim q_1, \ldots, q_m \sim q_m])$$. Let $$\Gamma_i \in K$$ be the contexts such that $$s_i \in T\Gamma_i$$ for all $$1 \leq i \leq n$$ and $$\pi_{s_i}$$ be the morphism of proposition 6.8 for all $$1 \leq i \leq n$$. Let

$$\bar{w}_{i,i+1} : \mathcal{E}_{[s_i]_{\Gamma_0,1}[s_{i+1}]_{\Gamma_0,1}} \rightarrow \hat{\mathcal{E}}_{[s_i]_{\Gamma_0,1}[s_{i+1}]_{\Gamma_0,1}}$$

be as in theorem 4.17. Then

$$(\mathcal{C} \circ \cosk_{\Sigma})(\pi_{s_1} \boxtimes \cdots \boxtimes \pi_{s_n}) \cong \mathcal{F}(\pi_{s_1}) \times_{\mathcal{F}(\bar{w}_{1,2})} \mathcal{F}(\pi_{s_2}) \times_{\mathcal{F}(\bar{w}_{2,3})} \cdots \times_{\mathcal{F}(\bar{w}_{n-1,n})} \mathcal{F}(\pi_{s_n})$$

Proof. By proposition 6.4, remark 6.9 and theorem 4.17.

Theorem 6.11 produces a workable abstraction of the more precise HDA-semantics, the degree of precision being measured by the unit of the adjunction $$\mathcal{C} \dashv \mathcal{N}$$ (c.f. proposition 3.12). This abstraction is the limit $$\pi_t : E_t \rightarrow C_t$$ of the diagram.
in $\textbf{Cat}^-$ where

\[ E_t \cong \mathcal{F} \left( \left[ s_1 \right]_{0,1}^{\Gamma_i} \right) \times_{\mathcal{F}(\varepsilon_{1,1})} \mathcal{F} \left( \left[ s_2 \right]_{0,1}^{\Gamma_2} \right) \times_{\mathcal{F}(\varepsilon_{1,2})} \cdots \times_{\mathcal{F}(\varepsilon_{n-1,n})} \mathcal{F} \left( \left[ s_n \right]_{0,1}^{\Gamma_n} \right) \]

$t$'s category of evolutions and

\[ C_t \cong \mathcal{F} \left( \left[ s_1 \right]_{0,1}^{\Gamma_i} \right) \times_{\mathcal{F}(\varepsilon_{1,2})} \mathcal{F} \left( \left[ s_2 \right]_{0,1}^{\Gamma_2} \right) \times_{\mathcal{F}(\varepsilon_{1,2})} \cdots \times_{\mathcal{F}(\varepsilon_{n-1,n})} \mathcal{F} \left( \left[ s_n \right]_{0,1}^{\Gamma_n} \right) \]

$t$'s control category. Accordingly, we call $E_i \overset{\text{def}}{=} \mathcal{F} \left( \left[ s_i \right]_{0,1}^{\Gamma_i} \right)$ and $C_i \overset{\text{def}}{=} \mathcal{F} \left( \left[ s_i \right]_{0,1}^{\Gamma_i} \right)$ $s_i$'s category of evolutions respectively $s_i$'s control category. $I_{i,i+1} \overset{\text{def}}{=} \mathcal{F} \left( \hat{\varepsilon}_{i,i+1} \right)$ is the control category of the corresponding interface $\phi_{i,i+1} \overset{\text{def}}{=} \mathcal{F} \left( \hat{w}_{i,i+1} \right)$.
6.12. Proposition. The control category $C_i$ is finite for all $1 \leq i \leq n$. In particular, the control category $C_t$ is finite.

Proof. Induction on syntax. □

Let us abuse notation by writing $\pi_{s_i}$ for $F(\pi_{s_i})$. The Giraud-Conduché correspondence (c.f. section 2.11) produces a finite presentation of $\pi_t$ as a pseudofunctor

$$\pi_t : C_t \longrightarrow \text{Span}$$

This pseudofunctor is a wide pullback object

$$\pi_t \cong \pi_{s_1} \times_{\phi_{1,2}} \pi_{s_2} \times_{\phi_{2,3}} \cdots \times_{\phi_{n-1,n}} \pi_{s_n}$$

in $\mathcal{F}/\text{Span}$. On the other hand, the hypothesis on $t$ and proposition 5.6 entail that

$$t \approx s_1 \ll \Phi_1 \gg s_2 \ll \Phi_2 \gg \cdots \ll \Phi_{n-1} \gg s_n$$

for lists of channels $\Phi_{i,i+1}$. Suppose $\Gamma_i = x_1 : \tau_1, \ldots, x_{m_i} : \tau_{m_i}$ and let

$$[\Gamma_i] \overset{\text{def.}}{=} \prod_{1 \leq j \leq m_i} [\tau_j]$$

for all $1 \leq i \leq n$. Unraveling theorem 2.13 and theorem 6.11 gives the following characterization of $\pi_t$.

1. There is

$$\phi_{i,i+1} : I_{i,i+1} \longrightarrow \text{Span}$$

for all $1 \leq i \leq n$. $I_{i,i+1}$ has one vertex and is generated by $\Phi_{i,i+1}$. Each generator $p \bowtie q \in \Phi_{i,i+1}$ has an associated type $\tau$ since it comes from an application of MCut. We have

$$\phi_{i,i+1}(p \bowtie q) = (!, [\tau], !)$$

where (!, [$\tau$], !) is the span

$$1 \overset{1}{\longleftarrow} [\tau] \overset{1}{\longrightarrow} 1$$

in $\text{Set}$. We have further $\phi_{i,i+1}(id) = (!, [\Gamma_i], !)$.

2. There is

$$\pi_{s_i} : C_{s_i} \longrightarrow \text{Span}$$

for all $1 \leq i \leq n$. $C_{s_i}$’s vertices are program locations. Then

$$\left(\pi_{s_i}\right)_0(l) = [\Gamma_i]$$

for all $1 \leq i \leq n$. Let $l \in (C_{s_i})_0$ and $\psi(l)$ be the statement at location $l$. Given the syntax and the operational rules, there are the following possible cases.
(a) \( \text{out}(l) = \emptyset \); 

(b) if \( \text{out}(l) = \{l_1\} \) then there are four cases; 

i. if \( \psi(l) \equiv \text{"nop"} \) then 
\[
\pi_s(l, l_1) = (id, [\Gamma_i], id);
\]

ii. if \( \psi(l) \equiv \text{"x}_j := e" \) then 
\[
\pi_s(l, l_1) = (id, [\Gamma_i], \pi_1 \times \cdots \times \pi_{j-1} \times [e] \times \pi_{j+1} \times \cdots \times \pi_m);
\]

iii. if \( \psi(l) \equiv \text{"p!e"} \) then 
\[
\pi_s(l, l_1) = (id, [\Gamma_i], id);
\]

iv. if \( \psi(l) \equiv \text{"p?x"} \) then 
\[
\pi_s(l, l_1) = (\pi_1, [\Gamma_i] \times [\Gamma_i], \pi_2);
\]

(c) if \( \text{out}(l) = \{l_1, l_2\} \) then there are two cases; 

i. if \( \psi(l) \equiv \text{"if } e \text{ then } \text{stat}_1 \text{ else } \text{stat}_2 \text{ end"} \); 

ii. \( \psi(l) \equiv \text{"while } e \text{ do } \text{stat} \text{ end"} \); 

in both cases, suppose w.l.o.g that \( [\widehat{l}_{0,1}(l, l_1)] = \gamma_1 \) and \( [\widehat{l}_{0,1}(l, l_2)] = \gamma_2 \) (c.f. section 6.3). Let \( \kappa : S_e \mapsto [\Gamma_i] \) be the subobject classified by \( [e] : [\Gamma_i] \mapsto \{tt, ff\} \), that is 
\[
S_e \overset{\text{def.}}{=} \{x \in [\Gamma_i] \mid [e](x)\}
\]

Let \( \bar{\kappa} : \overline{S}_e \mapsto [\Gamma_i] \) be its complement. Then 
\[
\pi_s(l, l_1) = (\kappa, S_e, \kappa)
\]
and 
\[
\pi_s(l, l_2) = (\bar{\kappa}, \overline{S}_e, \bar{\kappa}).
\]

3. There are lax representable transformations 
\[
\iota^\omega_{i,i+1} \overset{\text{def.}}{=} \left( \mathcal{F}(j^-_{i,i+1}), \mathcal{F}(\hat{j}^-_{i,i+1}) \right) : \pi_{s_{i+1}} \Rightarrow \phi_{i,i+1} \circ \mathcal{F}(\hat{j}^-_{i,i+1})
\]
and 
\[
\iota^\omega_{i,i+1} \overset{\text{def.}}{=} \left( \mathcal{F}(j^+_{i,i+1}), \mathcal{F}(\hat{j}^+_{i,i+1}) \right) : \pi_s \Rightarrow \phi_{i,i+1} \circ \mathcal{F}(\hat{j}^+_{i,i+1})
\]
for all \( 1 \leq i < n \). Let 
\[
\Gamma^\omega_{i,i+1} \overset{\text{def.}}{=} \begin{cases} 
\Gamma_{i+1} & \omega = - \\
\Gamma_i & \omega = +
\end{cases}
\]

We have 
\[
(x_{i,i+1})_l = (id, [\Gamma^\omega_{i,i+1}], !)
\]
for all \( l \in (C_{s_i})_0 \), \( \omega \in \{-, +\} \) and \( 1 \leq i < n \). Given a generator \( (l, l') \in C_{s_i} \), there are five cases:
(a) if $\pi_s(l, l') = (id, \Gamma_{i,i+1}^\omega, id)$ and $\psi(l) \equiv \text{"nop"}$ then

$$\langle \iota^\omega_{i,i+1}(l,l') \rangle = id_{\Gamma_{i,i+1}^\omega} : (id, \Gamma_{i,i+1}^\omega, id) \rightarrow (id, \Gamma_{i,i+1}^\omega, id);$$

this morphism of spans corresponds to the case 2.(b).i above;

(b) if $\pi_s(l, l') = (id, \Gamma_{i,i+1}^\omega, \pi_1 \times \cdots \times \pi_{j-1} \times [e] \times \pi_{j+1} \times \cdots \times \pi_m)$ where

$$m = \begin{cases} m_{i+1} & \omega = - \\ m_i & \omega = + \end{cases}$$

then

$$\langle \iota^\omega_{i,i+1}(l,l') \rangle = id_{\Gamma_{i,i+1}^\omega} : (id, \Gamma_{i,i+1}^\omega, !) \rightarrow (id, \Gamma_{i,i+1}^\omega, !);$$

this morphism of spans corresponds to the case 2.(b).ii above;

(c) if $\pi_s(l, l') = (s, S, s)$ for a subobject $s : S \rightarrow \Gamma_{i,i+1}^\omega$ then

$$\langle \iota^\omega_{i,i+1}(l,l') \rangle = s : (s, S, !) \rightarrow (id, \Gamma_{i,i+1}^\omega, !);$$

this morphism of spans corresponds to cases 2.(c).i and 2.(c).ii above;

(d) if $\pi_s(l, l') = (id, \Gamma_{i,i+1}^\omega, id)$ and $\psi(l) \equiv \text{"ple"}$ then

$$\langle \iota^\omega_{i,i+1}(l,l') \rangle = \langle id, [e] \rangle : (id, \Gamma_{i,i+1}^\omega, !) \rightarrow (\pi_1, \Gamma_{i,i+1}^\omega \times [\tau], !);$$

this morphism of spans corresponds to the case 2.(a).iii above;

(e) if $\pi_s(l, l') = (\pi_1, \Gamma_{i,i+1}^\omega \times \Gamma_{i,i+1}^\omega, \pi_2)$ then

$$\langle \iota^\omega_{i,i+1}(l,l') \rangle = id \times \pi_j : (\pi_1, \Gamma_{i,i+1}^\omega \times \Gamma_{i,i+1}^\omega, !) \rightarrow (\pi_1, \Gamma_{i,i+1}^\omega \times [\tau], !);$$

this morphism of spans corresponds to the case 2.(b).iii above so $\psi(l) \equiv \text{"p?x}_j$" and $\pi_j : [\Gamma_{i,i+1}^\omega] \rightarrow [\tau_j].$

In short, we label the control in the pseudo-monoid $\text{Span}([\Gamma_i, [\Gamma_i]])$. The span labeling a control transition encodes all the evolutions “above” the latter. The lax natural transformations to the interface are functional simulations encoding the observability of the control transitions. For this reason, objects of the lax comma category $\mathcal{F} // \text{Span}$ are also called categorical transition systems [6, 5, 20, 22, 21, 14].
6.13. Remark. $C_{s_i}$ can be obtained directly from the syntax, by a series of pushouts starting from the base cases. The above characterization can thus be taken as the definition of a generalized relational semantics of CIP. The Giraud-Conduché correspondence guarantees “soundness and completeness” with respect to the semantics formulated in terms of categories of evolutions hence ”soundess” with respect to the semantics based on HDA’a.

For example, the purely sequential CIP-program

$$x : \text{nat} \vdash x := 20; \text{while } x > 0 \text{ do } x := x - 1 \text{ end} : \langle \rangle$$

gives rise to the categorical transition system

while the parallel composition

$$x : \text{nat}, z : \text{nat} \vdash x := 5; p! (x + x) \ll p = q \gg q?z; z := z * z : \langle \rangle$$

gives rise to the limit of the diagram

with $\alpha_v = \langle \text{id}, \lambda x : \text{nat}. x + x \rangle$ and $\beta_m = \langle \pi_1, \pi_2 \rangle = \text{id}$. The pullback object is
7. Simulation

In this section we define a basic notion of simulation of HDA’s. We show that it carries over to categorical transition systems via categorification.

7.1. Simulations of HDA’s.

7.2. Definition. Let $s : S \to \Sigma$ and $t : T \to \Sigma$ be pointed HDA’a with points $S_\bullet \in S_0$ respectively $T_\bullet \in T_0$. A simulation $s \not\rightarrow t$ is a relation

$$r \subseteq S_0 \times T_0$$

such that

(i) $S_\bullet(r)T_\bullet$;

(ii) $\forall x \in S_0. \forall x' \in T_0. \forall n \in \mathbb{N}. \forall k \in S_n.

$$x(r)x' \land \text{dom}_n(k) = x \Rightarrow \exists k' \in T_n. \text{dom}_n(k') = x' \land \text{cod}_n(k) = \text{cod}_n(k') \land s_n(k) = t_n(k').$$

The notion of simulation of definition 7.2 extrapolates the notion of strong simulation of transition systems. It admits a characterization in terms of an appropriate notion of open maps.

7.3. Definition. Let $(S, <)$ be a strict total order. Given $x, y \in S$ let

$$x < y \overset{\text{def.}}{\iff} x < y \land \exists z \in S. x < z < y$$
7.4. Definition. Let $\textbf{pcSet}$ be the category of pointed cubical sets. Let $P \subseteq \textbf{pcSet}$ be the full subcategory with objects $P \in \mathbb{P}$ such that

1. there is a subset $S \subseteq P_0$ which is a finite strict total order;
2. the point is given by the smallest element of $S$;
3. given $x, y \in S$
   
   (a) $x \not< y \Rightarrow \forall n \in \mathbb{N}. P_n (x, y) = \emptyset$;
   
   (b) given $x < y$ there is $n_{x,y} \in \mathbb{N}$ such that there is precisely one $T_{x,y} \in P_{n_{x,y}} (x, y)$. This $T_{x,y}$ is not a face and $P_n (x, y) = \emptyset$ for $n < n_{x,y}$;
   
   (c) given $z \in S$ and $x < y < z$, $T_{x,y}$ and $T_{y,z}$ have $y$ as the only common face;
4. any other non-degenerate $n$-cube in $P$ is a face of precisely one $T_{x,y}$.

Objects of $\mathbb{P}$ are called paths.

Definition 7.4 states that paths are series of $n$-transitions glued at the endpoints of their main diagonals as for instance in

\[
\begin{array}{ccccccc}
\quad & y_2 & \rightarrow & x_2 & \rightarrow & x_3 & \rightarrow & z_2 \\
\quad & y_1 & \rightarrow & y_3 & \rightarrow & y_4 & \rightarrow & y_6 \\
\quad & x_1 & \rightarrow & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

where $S = \{ x_1 < x_2 < x_3 < x_4 \}$. It is easily seen from this example that $S$ induces a partition on $K_0 \setminus S$.

7.5. Definition. A morphism $h : M \rightarrow K$ of pointed cubical sets is open provided $h_0$ is surjective and $h$ has the right lifting property with respect to $\mathbb{P}_1$.

7.6. Lemma. Let $h : M \rightarrow K$ be a morphism of pointed cubical sets such that $h_0$ is surjective. The following are equivalent.

1. $h$ is open ;
2. $\forall n \in \mathbb{N}. \forall b \in K_n. \text{dom}_n (b) = h_0 (x) \Rightarrow \exists a \in M_n. h_n (a) = b$.

Proof. Obvious. \qed
7.7. Proposition. Let \( s : S \rightarrow !\Sigma \) and \( t : T \rightarrow !\Sigma \) be pointed HDA’a. The following are equivalent.

1. there is a simulation \( s \not\rightarrow t \);

2. there is a span

\[
\begin{array}{ccc}
R & \xrightarrow{r_2} & T \\
\downarrow{r_1} & & \downarrow{t} \\
S & \xleftarrow{s} & !\Sigma
\end{array}
\]

in \( \text{pcSet} / !\Sigma \) with \( r_1 \) open.

Proof. Lemma 7.6 allows the usual ”lift and project” argument for any dimension. \( \square \)

A simulation of HDA’a can be characterized as a right homotopy with respect to a certain model structure on \( \text{pc2} - \text{Cat} \), the category of pointed cubical 2-categories [13].

7.8. Simulations of Categorical Transition Systems.

7.9. Definition. A functor is \( \mathcal{C}(\mathbb{P}) \)-open if it has the right lifting property with respect to any morphism in \( \mathcal{C}(\mathbb{P}) \).

7.10. Lemma. The following are equivalent:

(i) functor \( \mathbb{B} \xrightarrow{F} \mathcal{C} \) is \( \mathcal{C}(\mathbb{P}) \)-open;

(ii) any \( \mathcal{C} \ni F(B) \rightarrow C \) lifts under \( F \).

Proof. Let \( \text{Ord} \subseteq \text{Cat} \) be the full subcategory with finite ordinals as objects. Then

\[ \mathcal{C}(\mathbb{P}) \cong \text{Ord} \]

and the assertion follows by the usual “lift and project” argument. \( \square \)

7.11. Definition. Suppose \( G \) and \( H \) are pointed reflexive graphs with points \( G_0 \in G \) respectively \( H_0 \in H \). Let \( s : \mathcal{F}(G) \rightarrow \text{Span} \) and \( t : \mathcal{F}(H) \rightarrow \text{Span} \) be pointed categorical transition systems. A path simulation from \( s \) to \( t \) is a relation \( r \subseteq G_0 \times H_0 \) such that

1. \( G_0 \ (r) \ H_0 \);

2. \( x \ (r) \ x' \ \& \ \ x \xrightarrow{f} y \in \mathcal{F}(G) \Rightarrow \exists \ x' \xrightarrow{f'} y' \in \mathcal{F}(H) \ y \ (r) \ y' \ \& \ s (f) = t (f') \).
7.12. **Proposition.** Let \( s : \mathcal{F}(G) \to \text{Span} \) and \( t : \mathcal{F}(H) \to \text{Span} \) be pointed categorical transition systems. The following are equivalent:

(i) there is a path simulation from \( s \) to \( t \);

(ii) there is a graph \( R \), a surjective graph homomorphism \( d : R \to G \) and a functor \( C : \mathcal{F}(R) \to \mathcal{F}(H) \) such that

\[
\begin{array}{ccc}
\mathcal{F}R & \xrightarrow{\mathcal{F}d} & \mathcal{F}G \\
\downarrow C & \downarrow \mathcal{F}t & \downarrow \mathcal{F}s \\
\mathcal{F}H & \xleftarrow{\mathcal{F}g} & \text{Span}
\end{array}
\]

commutes and \( \mathcal{F}d \) is \( \mathcal{C}(P) \)-open.

**Proof.** Lemma 7.10 allows the usual "lift and project" argument for any dimension. \(\square\)

Proposition 7.12 shows that path simulation is formally a strong one. However, categorification brings free categories over the control graphs into the game so we have composition of individual transitions. In a sense, this blurs the difference between the strong and the weak flavor of simulation. Consider for instance the CIP programs

\[
x : \text{nat} \mid \ x := 7
\]

and

\[
x : \text{nat} \mid \ x := 5; x := x + 2
\]

There is the obvious path simulation

\[
\pi_{x:=7} \leftrightarrow \pi_{x:=5; x:=x+2}
\]

while there would be no simulation at all if we had more conservatively rephrased strong simulation in terms of individual transitions only. As illustrated by this example, the notion of path simulation is appropriate for the study of refinement of specification.

However, the above path simulation is not a *path bisimulation*. Indeed, in a path bisimulation, individual transitions need to be matched. The notion of path bisimulation is thus quite crude, as crude as the traditional notion of bisimulation. It is nonetheless a useful notion, yet one ingredient is still missing. Consider for instance the CIP programs

\[
x : \text{nat} \mid p : \text{nat} \mid \nop
\]

and

\[
x : \text{nat} \mid p : \text{nat} \mid p!(2 * x)
\]
There is the nonsense path bisimulation

\[ r : \pi_{\text{nop}} \rightarrow \pi_{p!(2\ast x)} \]

since we have

\[ C_{\text{nop}} \cong C_{p!(2\ast x)} \cong J \]

with \( J \) the interval category \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \right\}

Indeed, it is necessary to classify the transitions in addition to the computations they carry. Since we got rid of the “syntactic” labels, a different classifying device is required. The latter is given by the map to the interface. This leads to considering bisimulation in the bicategory \( \text{Span}(\mathcal{F}/\text{Span(Set)}) \) as in [3]. Roughly, the semantics of \( x : \text{nat} \mid p : \text{nat} \vdash \text{nop} \) is then the span of categorical transition systems

\[ t \leftarrow \pi_{\text{nop}} \rightarrow t \]

with \( t \) the free category with one object \( \ast \) and one generator \( p \). Both legs of the span are the identity lax (representable) transformation. On the other hand, the semantics of \( x : \text{nat} \mid p : \text{nat} \vdash p!(2\ast x) \) is the span of categorical transition systems

\[ t \leftarrow \pi_{p!(2\ast x)} \rightarrow t \]

The left leg is the identity lax transformation while the component of the right one at \( t \) is the morphism of spans

\[ \langle id, \lambda x \in \mathbb{N}. 2 \ast x \rangle : (id, \mathbb{N}, !) \longrightarrow (\pi_1, \mathbb{N} \times \mathbb{N}, !) \]

(c.f. section [3,10]). Hence, the span

\[ \pi_{\text{nop}} \leftarrow \rho \rightarrow \pi_{p!(2\ast x)} \]

of \( C(P) \)-open maps witnessing the nonsense path bisimulation \( r \) does NOT make the diagram commute so there is no bisimulation in this setting. Since \( C(P) \)-open maps form a cover system, i.e. isomorphisms are \( C(P) \)-open maps and \( C(P) \)-open maps are composition- as well as pullback-stable, the construction of process categories [3] applies.
8. Concluding Remarks

We introduced the 1-coskeletal synchronization and showed how it can be applied to the study of message-passing at hand of a realistic application. We have good reasons to believe that other synchronization paradigms e.g. semaphores may turn out to be expressible this way. The present paper provides a thorough harnessing of the semantics of CIP-like message-passing programming languages. What makes the strength of the approach is that the semantics in question is given in a direct way, i.e. without coding an imperative language into some process calculus. These principles have actually been used in the design and implementation of a deductive model-checking tool [17], yet the potential has only been scratched on the surface. What remains to do is to set up a relevant modal logic.

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