I. INTRODUCTION

In condensed matter physics, spectral analysis based on density of state is one of the key ways to understanding the mechanism of material properties\(^1\). For example, the widely recognized mechanism of metal and insulator depends on the spectral information in band theory\(^2\). Of course, it also includes a series of new condensed matter materials represented by high temperature superconductors, which are also inseparable from the mechanism analysis system based on spectrum\(^3\). Whenever a new material is discovered or prepared, we are always eager to know the superconducting gap of the material, hoping to test whether the theoretical model is reliable. However, it is not easy to measure it accurately.

In fact, there is no experimental method to measure the spectrum directly. At present, a feasible experimental strategy is to obtain the tunneling current at low temperature by using scanning tunneling microscope (STM)\(^4\), and then deconvolute the zero temperature density of states according to the scanning tunneling spectrum. The difficulty of this strategy is that the process of solving zero temperature by finite temperature is a typical ill-posed problem in mathematics\(^5\). Because temperature would erase the high-frequency details of the spectrum information, the obtained zero temperature information is unreliable. Infinitely reducing the experimental temperature is a mathematically feasible method to alleviate the ill-posed problem, but it is obviously difficult and unrealistic in the experiment. No matter how high the numerical precision of input data and calculation process is, the reliability of output results is difficult to guarantee, especially when our goal is to find the superconducting gap where the spectrum diverges.

To analyze the causes of this process, we carry out quantitative discussion. Fortunately, the theory of tunneling current \(I(eV)\) at limited low temperature \((T)\) based on the zero temperature density of states \((\text{DOS}, \rho(\omega))\) of materials is clear, so this process can be expressed as the following formula as Eq.(1)\(^6\),

\[
I(eV) = \left(\frac{4\pi e^2}{\hbar}\right) \int_{-\infty}^{\infty} d\omega |M|^2 \rho(\omega) \rho_p(\omega + eV) \\
\times \left[ f(\omega) - f(\omega + eV) \right], \quad (1)
\]

where \(\rho_p(\omega + eV)\) is the DOS of the probe. \(M\) is the tunneling matrix element, \(f(\omega) = 1/(e^{\beta\omega} + 1)\) is Fermi distribution function dependent on \(\beta = 1/k_BT\), \(k_B\), \(e\) and \(\hbar\) represent Boltzmann constant, charge quantity and reduced Planck constant respectively.

According to the theoretical formula of temperature broadening effect from Bardeen approximate formula as Eq.(1), in order to obtain the zero temperature spectrum, it need to deconvolve the differential conductance at finite temperature as Eq.(2) with treated \(M\) and \(\rho_p\) as constants, and eliminate \(f(\omega)\), we have

\[
\sigma(eV) = \left(\frac{\pi e^2 |M|^2 \rho_p \beta}{\hbar}\right) \\
\times \int d\omega \rho(\omega) \cosh^{-2} \frac{\beta(\omega + eV)}{2}. \quad (2)
\]

This deconvolution problem of \(\rho(\omega)\) from \(\sigma(eV)\) is essentially a ill-posed integral equation, can be transformed into the form of solving matrix equation by discretization. The condition number of this problem is very large, thus the result is instability by sawtooth noise of input data \(\sigma(eV)\), cannot be solved directly by the least-squares fit.

If the observed material is superconductor, the information of the DOS cannot only accurately determine whether they enter the superconducting state, but also estimate the superconducting gap. So a series of physical situations taking the above problems as an example, have great significance for the scientific analysis and research. In practical research, we often use the data available \((G(m))\) to deduce the dynamical information \((A(\omega))\) as...
in general form in Eq. (3)\textsuperscript{6,7},

\[ G(m) = \int_{-\infty}^{\infty} d\omega K(m, \omega) A(\omega), m = 1, \ldots, N, \quad (3) \]

where \( K(m, \omega) \) is a known kernel and data \( G(m) \) including \( N \) discrete values in total. This process can usually be mathematically divided into the Fredholm integral equation of the first kind, which is a typical ill-posed problem in Mathematics\textsuperscript{8,9}.

Traditional methods are mainly based on easing the number of conditions\textsuperscript{9,10}, including Pseudo inverse matrix method (Pinv)\textsuperscript{11} by ignoring the minimal eigenvalues, Tikhonov-Phillips regularization method (TPRM)\textsuperscript{12-14} by using regularization idea to increase small eigenvalues, Maximum entropy method (MEM)\textsuperscript{15,16} by regular the problem based on baseline model. Based on the idea of regularization, these methods alleviate the instability of the problem, but still have oscillation and unclear boundaries. We think the common reason is the dense uniform sampling to ensure the approximation. Artificial neural network method (ANN)\textsuperscript{17,18} based on the idea of supervised learning in deep learning, uses a large number of \( \rho(\omega) \) and \( \sigma(eV) \) correlation data for neural network model training. Benefiting from large-scale training data, this method can get accurate and fast results, even better and faster than MEM. However, due to relying too much on training data, once the data sampling changes or the physical model changes, the performance can not be guaranteed. Stochastic optimization method (SOM)\textsuperscript{19-21} introduces dynamic and non-uniform discrete sampling to solve the problems concerned in this paper. But it is difficult to optimize the discrete mode in practice. With a small number of samples, the model is easy to converge, but the result accuracy is limited. Using a large number of samples, the difficulty of convergence increases rapidly and the results are easy to fluctuate.

In order to reduce the above difficulty, on the basis of dynamic optimization of non-uniform sampling, we introduce neural network into this problem and named it as Neural Network Method (NNM). The core idea is to use a neural network to approach the objective function \( \rho(\omega) \), and the training process of the neural network is used to replace the existing deconvolution algorithms. Fully connected neural network with Relu activation function Mathematically equivalent to piecewise linear functions, the training process makes it possible to adjust an appropriate non-uniform sampling distribution. Considering the strong expression ability of neural network\textsuperscript{22} and the priority of approaching low frequency signal\textsuperscript{23}, the experimental results show that, to some extent, our method can eliminate the oscillation error and improve the accuracy of the results. By comparing the results of our NNM with existing methods in theory dataset and experiment data, it shows a strong practical value for deconvolution of spectrum problem.

II. RELATED WORK

In this section, we discuss the technical scheme, advantages, disadvantages, and application of these methods.

A. Pseudo inverse matrix method

The ill-posed problems in this paper can be generalized by Eq. (3), for the STM problems concerned in this paper, the formula is expressed as tunneling current form Eq. (1) or differential conductance form Eq. (2). First of all, the integral of Eq. (2) needs to be discretized, considering that \( \sigma(eV) \) is usually discrete, the formula can be transformed into matrix equation form

\[ \vec{\sigma} = A\vec{\rho}, \quad (4) \]

where \( \vec{\sigma} \) and \( \vec{\rho} \) are vectors of variables \( eV \) and \( \omega \), respectively, matrix \( A(eV, \omega) = \cosh^{-2}(\beta(\omega + eV)/2)\text{dis}(\omega) \), with \text{dis}(\omega) is the distance between adjacent discrete samples, and \( a = (\pi\epsilon^2 |M|^2 \rho_p\beta)/\hbar \). Among them, the calculation from the spectral information \( \rho(\omega) \) to the measured data \( \sigma(eV) \) is good and can be simulated with high precision, but in turn, the process from \( \sigma(eV) \) to \( \rho(\omega) \) is ill-posed. Due to the condition number of matrix \( A \) is large, the result \( \rho(\omega) \) is instability by sawtooth noise of input data \( \sigma(eV) \). In form, it can be summarized as the following loss function minimization with \( \ell_2 \) norm, these kinds of ill-posed problems cannot be solved directly by the least-squares fit methods.

\[ \min_{\rho} ||A\vec{\rho} - \vec{\sigma}||_2^2 \quad \text{or} \quad \vec{\rho} = A^{-1}\vec{\sigma}. \quad (5) \]

Here, Pinv\textsuperscript{11} is used to replace the inverse of the matrix, which can avoid the restriction that the inverse matrix must require a square matrix, so that the spectral information in any case can be obtained. Furthermore, for the very small eigenvalue of the matrix, because it is close to the machine error, the accuracy cannot be guaranteed, so it is not included in the optimization process to avoid the divergence problem.

As a classical linear algebra method, Pinv appears more in textbooks than in practical applications. The Pinv method has been integrated into most linear algebra libraries, such as scipy\textsuperscript{24} in python: scipy.linalg.pinv.

B. Tikhonov-Phillips regularization method

For Eq. (4), considering the error of data, it can solve the pseudo-inverse matrix\textsuperscript{11} by discard eigenvalues less than a threshold. However, only in the case of low temperature, it can be accepted reluctantly, when the temperature rises, there will be violent oscillation by sawtooth noise of input data, especially near the singularity.

The first real way to deal with ill-posed problems is TPRM, named from the first applications of the ideas to
ill-posed integral equations by Tikhonov and Phillips\textsuperscript{12–14}, formalized as
\[
\min_{\bar{\rho}} \| \bar{\rho} - \bar{\sigma} \|^2 + \lambda \| \bar{\Gamma} \rho \|^2.
\]

TPRM method introduce the regularization constraint \( \bar{\Gamma} \) on \( \bar{\rho} \). The introduction of this trick is equivalent to synchronously increasing the values of all eigenvalues of matrix \( \bar{A} \), e.g. \( \bar{A} \to \bar{A} + \bar{\Lambda} \). Thus the condition number of the problem is improved but the solution will deviate.

The main difficulty of this method is the selection of regularization constraints and correlation coefficients \( \lambda \), which requires a lot of human effort to optimize. Even so, the results suppress with large derivatives by the regularization especially when the spectral function has sharp edges or narrow peaks.

At last, TPRM is widely used in engineering, it is relatively simple to implement TPRM in the algorithm. It only needs to change the loss function from \( \| \bar{A} \bar{\rho} - \bar{\sigma} \|^2 \) to \( \| \bar{A} \bar{\rho} - \bar{\sigma} \|^2 + \lambda \| \bar{\Gamma} \rho \|^2 \) at the optimization objective level.

C. Maximum entropy method

To solve the shortage of TPRM, MEM\textsuperscript{15,16} searches for the most probable solution \( \hat{\rho} \) among variational space with assuming the prior knowledge that \( \hat{\rho} \) is close to a predefined function \( D(\omega) \), called default model. The regularization constraints are replaced by entropy,
\[
S[\bar{\rho}] = \int d\omega \bar{\sigma}(\omega) \ln \left( \frac{\bar{\rho}(\omega)}{D(\omega)} \right).
\]
The entropy characterizes the deviation of \( \bar{\rho}(\omega) \) from the default model \( D(\omega) \). If a lot of information is known about \( \bar{\rho}(\omega) \), a good default model \( D(\omega) \) can be defined, then MEM is better than TPRM. However, the method highly relies on the default model, which is a serious drawback if the interesting features of the spectra are very sensitive to the chosen default model\textsuperscript{9}.

So far, MEM can be said to be one of the most widely used methods. A large number of similar problems have achieved good results by using MEM, such as using for a real-valued single-orbital problem\textsuperscript{25}. At the same time, many literatures have studied the properties and improvement optimization of MEM\textsuperscript{36–28}.

Up to now, a large number of libraries have realized the algorithm function of MEM, such as Maxent\textsuperscript{29} with code available in https://github.com/TRIQS/maxent.

D. Artificial neural network

With the development of deep learning, especially supervised learning technology, using the neural network to solve this problem has become a potential direction. Reference\textsuperscript{37,18} introduce ANN based on supervised learning, and each the same level of accuracy with MEM and reduce the computational cost by almost three orders of magnitude. The advantage of this method is that the process from \( \bar{\rho}(\omega) \) to \( \bar{\sigma}(eV) \) corresponds to matrix operation without error, so a large enough dataset can be generated for training,
\[
\bar{\sigma}(eV) \to \text{ANN} \to \bar{\rho}(\omega).
\]

And this kind of algorithm has fast computing speed, that is, once the neural network training is completed, the subsequent application only needs a single network inference process without iteration, the speed of solving new data is very fast. In fact, for the determined physical model, increasing the training set is a completely feasible and effective method to improve the accuracy, because the equivalent parameter degrees of freedom in the actual physical model is usually low, and the neural network is completely possible to learn the physical laws.

However, just because it requires training steps, the training time is relatively long. And in order to ensure sufficient accuracy, training requires more than \( 10^5 \) training samples. Such a large number of training samples may not easy to obtain. Even if the model is well trained, the generalization ability of the model is also insufficient\textsuperscript{30}, which means the model is dependent on the training data. Once the case is not included in the training set, the trained model is likely to fail, including the calculation of new sampling data points and the calculation of new physical models.

Although this kind of method has great application potential, it needs the academic community to jointly improve the training dataset and training model base. Only when the model base has sufficient scale, this kind of method can play an ideal effect. Considering that the training cost of a large number of data is too high, the numerical experiment part of this paper does not include the comparison of ANN methods.

E. Stochastic optimization method

Comparing with the previous methods, based on a large amount of prior knowledge of the model and uniformly fixed discrete sampling, SOM\textsuperscript{19–21} does not use any default model nor impose any smoothening, only restrict prior knowledge to normalization and positivity of the solution. SOM use a likelihood functional
\[
\bar{\rho} = \int d\bar{\rho} \bar{P}(\bar{\rho} | \bar{\sigma}),
\]
where \( \bar{\rho} \) is obtained as an average of particular solutions \( \hat{\rho} \) with weight of likelihood function \( \bar{\rho}(\hat{\rho} | \bar{\sigma}) \). \( \bar{\rho}(\hat{\rho} | \bar{\sigma}) \) describes the corresponding probability by residual \( \| \bar{A} \bar{\rho} - \bar{\sigma} \| \), although \( \hat{\rho} \) with too small residual over-fit the data \( \bar{\sigma} \) with sawtooth noise, in practice the sawtooth noise can be self-averaging in a sum over a large enough number of particular solutions if keep residual not too restrictive, which sets up an implicit regularization procedure. In some cases, if the residual can be ensured to be less than the given error threshold, then the influence of the weight
can be ignored, and the above formula becomes a simple algebraic average of $N$ simples,

$$\tilde{\rho} = \frac{1}{N} \sum \rho_i.$$  \hspace{1cm} (10)

Although SOM has less prior knowledge dependence, the calculation process of a large number of particular solutions has brought many difficulties. How to obtain independent solutions efficiently and how to ensure the ergodicity of solutions are difficult to strictly guarantee. Therefore, this method is still difficult to achieve satisfactory results.\(^{10}\)

SOM method may be a widely used solution second only to the MEM method. You can easily find TRIQS/SOM library in GitHub to achieve the computing goal.\(^{31}\) The software library contains a large number of commonly used physical models, which are easy to use. At the same time, many other scholars have also developed software packages with different implementation paths on GitHub.

III. METHODOLOGY

In this section, we first discuss the possible advantages of non-uniform discretization from the perspective of discretization strategy, and then propose our NNM method based on the expression ability of the neural network. We need to note that ANN is based on the idea of supervised learning, while our method adopts the idea of function approximation without a large number of training data. The two methods are essentially different.

A. Condition number versus discrete sample rate

As in Fig. 1(Upper), the condition number of matrix $\tilde{A}$ in Eq.(4) is directly related to the temperature and the sampling rate. At the same sampling rate, lower temperature (larger $\beta$) will effectively reduce the condition number, which can be understood by the broadening properties of $\tilde{A}(eV = 0, \omega)$ in Fig. 1(Lower). And fixed temperature, the condition number of the model will diverge with the increase of sampling rate at the speed of transcendental exponential function. This means that a more dense sampling rate will bring disastrous unstable results. However, the sampling rate is a direct means to improve the resolution of results, which undoubtedly leads to a contradiction between resolution and stability.

Therefore, it is very important to select the appropriate sampling rate according to the needs. MEM and others only consider the case of uniform mesh, to reduce the loss in Eq.(5), it needs a high sampling rate with a large condition number. This requires the introduction of regularization to avoid oscillation, the choice of regularization and sampling rate is important and difficult. We consider choosing the appropriate high sampling rate for key areas and low sampling rate for other areas of the non-uniform sampling method. Although it does not significantly improve the condition number, which depends more on the region with dense sampling, it has a significant effect on improving the accuracy of the solution. SOM can use non-uniform mesh and can solve a good mesh adaptively. However, it is still necessary to achieve a given strategy of meshes in the calculation process, which is usually difficult.

B. Neural network method

Therefore, how to build a proper mesh is a problem that needs further exploration. In this regard, traditional methods may be difficult to achieve ideal results, but the
We introduce neural network into the ill-posed problem the physical principle of DOS. It should note that we activation function, which is mathematically equivalent multi-layer fully connected neural network with the ReLU activation function. Considering the characteristics of the \( \rho \) corresponding to \( \omega \), we can use its inference procedure to give \( \rho \) corresponding to the experimental data \( \sigma(eV) \) with broadening. The black line with temperature \( \beta = \text{Inf} \) corresponding to objective function \( \rho(\omega) \) with diverge at \( \omega = \pm \Delta \), the red and blue lines with \( \beta = 3, 10 \) corresponding to experimental data \( \sigma(eV) \) with broadening, with the experimental dataset \( \{eV, \sigma(eV)\} \), the first step is to sampling \( \omega \). We formally use the uniform discretization of \([\omega_{\text{min}}, \omega_{\text{max}}]\) interval to sample \( N_o \) \( \omega \), where \( N_o = N_{eV} \). We set \( \omega_{\text{max/min}} = \pm 6, 20\% \) expand about \( eV_{\text{max/min}} \), in order to ensure that the process of \( eV \) broadening is not affected by \( \omega \) at the boundary. Note that simpling \( \omega \) is not equal to the non-uniform piecewise function. NNM can adopt more flexible sampling methods, including non-uniform sampling and even different sampling in each iteration step. The sampling method has no significant impact on the results but increases the complexity of the algorithm, so for the sake of intuitiveness, uniform sampling is adopted in this paper.

Next, go to the loop iteration. Firstly, \( \tilde{\rho}_{NN}(\omega) \) is obtained by forward inference of neural network. Then combine with matrix \( \tilde{A}(eV, \omega) \) to calculate \( \bar{\sigma}_{NN}(eV) = \tilde{A}(eV, \omega)\tilde{\rho}_{NN}(\omega) \) as Eq.(4). At the end of each loop, the derivative of the model parameters \( \theta \) is calculated according to the loss function \( \mathcal{L} = ||\tilde{\sigma}(eV) - \bar{\sigma}_{NN}(eV)||_2^2 \) and then the model parameters are updated. Here we use the widely used minimum square error with \( \ell_2 \) norm, and generally we can choose more appropriate error formula according to the characteristics of the problem. We use Adam optimizer with a learning rate 0.001 during training. And finally evaluation the residual error between \( \tilde{\rho}_{NN}(\omega) \) and groundtruth \( \tilde{\rho}(\omega) \).

### IV. EXPERIMENTS

#### A. Theoretical datasets

We use the theoretical formula of isotropic s-wave superconductivity to build the datasets. We can evaluate
the model by comparing the solution error of DOS with
that of strict solution. The exact solution of DOS is

$$\rho(\omega) = \frac{N_F \omega \theta(|\omega| - \Delta)}{\sqrt{\omega^2 - \Delta^2}}, \quad (12)$$

where $N_F$ is the density of states on Fermi surface, $\Delta$ is superconducting gap and $\theta(|\omega| - \Delta) = 1$ if $|\omega| > \Delta$, $\theta(|\omega| - \Delta) = 0$ otherwise. In the process of building a dataset, for the convenience of calculation, we take the following parameter as $N_F = 1$, $\Delta = 1$ and $a = 1$ (in Eq.(4)).

Figure 2 shows the curves of DOS $\rho(\omega)$ and differential conductance $\sigma(eV)$ with $\beta = 3$ and 10. It is clear that the zero temperature DOS divergent at $\omega = \pm \Delta$, and the broadening of $\sigma(eV)$ increases with the temperature increase. The gap in the middle become smooth, the gradient become slow and the divergent at $\omega = \pm \Delta$ disappear. So the dataset contains $\{eV, \sigma(eV)\}$ for training, and $\{\omega, \rho(\omega)\}$ for calculation the approximation error of evaluation the algorithms. $eV_{\text{min}}, eV_{\text{max}}, N_{eV}, \omega_{\text{min}}, \omega_{\text{max}}, N_\omega$ represents the upper and lower limits of variables $eV, \omega$ and the number of samples, respectively. We set $eV_{\text{max}}/eV_{\text{min}} = \pm 5$, $\omega_{\text{max}}/\omega_{\text{min}} = \pm 6$ and $N_{eV}/\omega = 1201$ if not specified.

**B. Numerical results**

First, we use theoretical datasets to compare the residuals of different approximation functions as in Fig.3. For low temperature ($\beta = 10$), it’s easier to get more accurate results, all the methods show that the result is closer to the exact solution from low temperature. Pinv method shows the highest peak around $\pm \Delta$ but with the most severe oscillation. Due to the introduction of regularization constraints, the peaks of TPRM, MaxEnt, and SOM...
methods are not that high than Pinv and appear a weaker oscillation. Will for NNM, it has a good performance in the height and position of the peak and amplitudes of oscillation.

The core of NNM is to use a neural network to approximate the actual function $\rho(\omega)$. Compared with the other methods, it has many advantages. (1) First of all, because the neural network is a piecewise linear function mathematically, it can find the appropriate piecewise method independently based on the ability to approach the objective function. Therefore, the neural network can encrypt the piecewise density in the changing interval according to the details of the objective function, and reduce the piecewise density in other intervals, so as to achieve better approximation mesh by using the finite number of segments effect. (2) Secondly, the obtained neural network does not depend on the sample points used in training, and has considerable generalization ability, that is to say, it can give all the output results corresponding to the input in the specified interval. (3) Thirdly, the Relu activation function can ensure that the model is always greater than 0, strict positive.

C. Network hyper-parameters

Because the training process of neural network needs a large number of hyper-parameters, the influence of these parameters is briefly discussed here. On the whole, our method has little dependence on parameters.

Firstly, in terms of network structure, this method uses a fully connected neural network, mainly including the number of neural network layers and the number of neurons in each layer. Obviously, when the number of neurons is too small ($< O(10)$), the degree of freedom of the neural network is too low, which is equivalent to that the discrete sampling points are too sparse. At this time, the segmentation characteristics of the training results are obvious. As long as the network complexity is increased, whether the number of layers or the number of neurons is increased, the training results can be effectively improved. Most of the examples in this experiment use a 3-layer fully connected network with 1000 neurons in each layer.

NNM method only depends on 1 set of data, and does not need a large-scale dataset composed of thousands of sets of data in the field of traditional computer vision. Therefore, for the model architecture and iteration times that are not exaggerated, the calculation results are enough to meet the needs and the training time is still short.

In the training process, the learning rate and the number of epochs are usually a pair of very key and interactive parameters. Firstly, in order to converge quickly, starting from the randomly initialized neural network at the beginning of training, an appropriate larger learning rate should be used to speed up the network update and evolution. When the network learns an approximate result, it is necessary to gradually reduce the learning rate to enable the network to learn finely and further converge. In practical experience, the learning rate of 0.01 can be used for the first 100 epochs, and then the learning rate of 0.001 can be used to continue training for 500 epochs. If you want to get more refined results, you can further reduce the learning rate to 0.0001 for another 1000 epochs.

Finally, it should be noted that although the result of this method is robust in most cases, it still depends on the initial value of the neural network. That is, if the random initial value is too bad, the training result is likely to be unsatisfactory. Therefore, the training is usually repeated 2 – 3 times, and the best or most stable result is taken as the final result. Considering that each training time is about minutes, the time cost caused by multiple training is completely acceptable.

D. Real experimental data

The previous results from the theoretical curve can show the convergence of the algorithm, but the actual experimental data usually have a certain degree of noise error. This subsection observe the robustness of these methods to the error data through the actual experimental data.

Figure 4 shows the deconvolution results of several algorithms for real experimental data (the differential conductivity at $T = 2.5K$ without magnetic field at crystal surface measured by STM). The biggest difference between experimental data and exact data is that they contain nonnegligible random noise, so Pinv method cannot get stable results and will not be shown. In comparison, the regularization constraints introduced by TPRM and MEM play a key role in noise suppression, and smooth results are obtained. SOM method mainly combines TPRM results with different parameters, the result is not outstanding than TPRM as a result.

It can be seen from the figure that the curve peak obtained by NNM method is the clearest and can be used to estimate the superconducting energy gap. This fully reflects the advantages of NNM method in solving this problem and mining information. Using the characteristics of the neural network first low-frequency information, we can get more accurate and nonoscillatory results.

V. CONCLUSION

Physical experiments or numerical methods cannot get target physical dynamical properties directly. It must use the available data to deconvolution the dynamical information, such as to get the density of state from differential conductance in scanning tunneling microscope. This problem is essentially ill-posed, is instability by sawtooth noise of input data. The existing methods can get stable results but rely on prior knowledge and are unsatisfactory. In this way, we propose Neural Network Method (NNM) for this problem, by using a fully connected neural network...
FIG. 4. Results of experimental dataset\textsuperscript{36} with temperature $\beta = 2.32$ of several algorithms as shown in the figure (Upper) Full scale and (Lower) amplification. The neural network used in this figure is a fully connected neural network with 3 layers and 1000 neurons in each layer with out activation function on the output layer. NNM performs the best peak position and high, while Pinv diverges and is not drawn on the figure.

Our approach can be extended to many other similar problems in Physics, such as experimental areas like getting Lehmann functions from the spectral response observed in experiments on angle resolved photoelectron spectroscopy (ARPES)\textsuperscript{37} and numerical areas like analytic continuation problems about getting dynamic correlation functions from quantum Monte Carlo simulations with finite lattice size and imaginary time\textsuperscript{38}. Moreover, a similar equation has to be solved for medical X-ray and impedance tomography, image deblurring, and many other practical applications\textsuperscript{39}.

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