Classical chiral kinetic theory and anomalies in even space-time dimensions

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Abstract
We propose a classical action for the motion of massless Weyl fermions in a background gauge field in $(2N + 1) + 1$ space-time dimensions. We use this action to derive the collisionless Boltzmann equation for a gas of such particles, and show how classical versions of the gauge and abelian chiral anomalies arise from the Chern character of the non-abelian Berry connection that parallel transports the spin degree of freedom in momentum space.

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1. Introduction
The axial anomaly—a non-conservation of currents associated with chiral fermions—is usually regarded as a purely quantum effect, so it is rather surprising that it is possible to extract the abelian anomaly from a purely classical Hamiltonian phase-space calculation [1]. The authors of [1] did so by considering the classical dynamics of a finite-density gas of Weyl fermions in background electromagnetic field. They observed that the incompressibility of the phase space allows the anomalous inflow of particles from the negative-energy Dirac sea into the positive-energy Fermi sea [2–4] to be reliably counted by keeping track of the density flux only near the Fermi surface. Near the Fermi surface, and therefore well away from the dangerously quantum Dirac point, a classical Boltzmann equation becomes sufficiently accurate for this purpose. The only quantum input required is the knowledge of how to normalize the phase space measure and a simple computation of the momentum-space gauge field that accounts for the gyroscopic effect of the Weyl particle’s spin. This gauge field is a Berry-phase effect that subtly alters the classical canonical structure so that $x$ and $p$ are no longer conjugate variables, and $d^3p d^3x$ is no longer the element of phase space volume [5, 6].

In a previous paper [7] we extended the analysis of [1] and considered the motion of Weyl particles in a background non-abelian gauge field. In this way we obtained the
non-abelian gauge anomaly in 3+1 dimensions. Our calculation, like that in [1], relied on a number of simplifications peculiar to three spatial dimensions—in particular that the Berry phase is indeed a phase and not a more general unitary matrix. In this paper we derive both the abelian and the gauge anomaly in any even number \((2N+1)+1\) of space-time dimensions. Apart from the intrinsic interest of the role of higher-dimension anomalies in the hydrodynamics of a Weyl gas [8] and other fluids with anomalies [9], it turns out that the structure of the calculation becomes more transparent when we discard the special features of the lower dimensional dynamics.

In section 2 we review the action that describes the motion of a Weyl particle in a background 3+1 abelian gauge field. We then explain how this action can be extended so as to obtain the motion of a Weyl particle in any even-dimensional space-time, and in a background non-abelian field. In section 3 we review the differential form formulation of Hamiltonian dynamics and show how it is modified when the underlying symplectic form becomes time-dependent. We use this language to extend Liouville’s theorem on phase-space volume conservation to the time-dependent case, and identify the vulnerability in the proof where permitting a singularity allows for anomalous conservation laws. In section 4 we temporarily assume that the time-dependent Liouville theorem is \textit{not} compromised and show how it implies the non-anomalous conservation of the abelian number current and the covariant conservation of the gauge current. Section 5 then reveals how the non-trivial Chern character of the momentum-space spin connection provides a source term in Liouville conservation law. We then show that this source term gives rise to both the singlet anomaly and the covariant form of the gauge anomaly. An appendix displays the efficiency of the formalism used in this paper compared to that of our previous method. Two further appendices provide technical details of the results used in the main text.

2. A classical action for Weyl fermions

It was shown in [1] that the classical phase-space action for a 3+1 dimensional Weyl fermion moving in a background electromagnetic field is

\[
S(x, p) = \int dt (p \cdot \dot{x} - |p| - e\phi(x) + eA \cdot \dot{x} - a \cdot \dot{p}).
\]  

(1)

Here \(eA \cdot \dot{x}\) is the standard coupling of the Maxwell vector potential \(A\) to the velocity \(\dot{x}\) of the charge \(e\) particle. The combination \(e\phi(x, t) + |p|\), where \(\phi = -A_0\) is the scalar potential and \(|p|\) is the kinetic energy of the massless fermion, is the classical Hamiltonian \(H(x, p)\). The \(a \cdot \dot{p}\) term accounts for the gyroscopic effect of the spin angular momentum, which for a right handed massless particle is forced to point in the direction of the momentum.

2.1. Berry connection in momentum space

The momentum-space gauge field \(a(p)\) appearing in (1) is the adiabatic Berry connection [10] which has components

\[
a_k = i \left\{ p_k + \left| \frac{\partial}{\partial p_k} \right| p_k, + \right\}.
\]  

(2)

These components are obtained from the \(E = +|p|\) eigenvector \(|p, +\rangle\) of the quantum Hamiltonian

\[
\hat{H}_p = \sigma \cdot p.
\]  

(3)

and the resulting Berry curvature \(b = \nabla \times a\) possesses a monopole singularity

\[
\nabla \cdot b = 2\pi \delta^3(p)
\]  

(4)
at \( p = 0 \). When \( p \gg 0 \) and the \( 2|p| \) energy gap is large we can ignore the negative energy eigenstate \( |p, -\rangle \) and safely make the adiabatic approximation that allows us to forget most quantum effects. We retain only the abelian Berry phase, now interpreted as a classical gauge potential. The adiabatic approximation fails near \( p = 0 \), however, and \ref{eq:berry-connection} should be understood only as shorthand indicating the presence of a non-zero Berry flux through the surfaces surrounding (but distant from) the origin \cite{1}.

We desire to generalize \ref{eq:action-functional} from 3 to \( 2N+1 \) space dimensions. The principal complication is that in dimensions greater than three the Berry phase is replaced by a unitary evolution matrix and \( a(p) \) by a non-abelian Berry connection. Extracting the effects of the Berry connection by taking a suitable limit of the quantum system is complicated, and is ultimately equivalent to the conventional Feynman diagram calculations of anomalies that we wish to avoid. Instead we build on the results of \cite{1} by letting symmetry and gauge invariance dictate the necessary modification to the abelian action functional \ref{eq:action-functional}.

In \( 2N + 1 \) space dimensions the quantum Weyl Hamiltonian \ref{eq:weyl-hamiltonian} becomes
\[
\hat{H}_p = \sum_{i=1}^{2N+1} \Gamma_i p^i,
\]
where the \( \Gamma_i \) are a set of \( 2N \)-by-\( 2N \) Dirac gamma matrices (or more accurately Dirac alpha matrices) obeying
\[
\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}.
\]
The gamma matrices will also obey
\[
\Gamma_1 \Gamma_2 \cdots \Gamma_{2N+1} = \pm i^{2N} \sigma_y,
\]
where \( \pm \) sign depends the Weyl particle’s helicity—i.e. which of the two inequivalent irreducible representations of the Clifford algebra \ref{eq:clifford-algebra} is selected.

The eigenvalues of \( \hat{H}_p \) remain \( \pm |p| \), but each energy level is now \( 2^{2N-1} \)-fold degenerate. The positive energy eigenspaces \( V_+(p) \) form the fibers of a non-trivial spin(\( 2N \)) bundle over the momentum space minus its origin. If \( |p, \alpha, +\rangle, \alpha = 1, \ldots, 2^{2N-1} \) form a basis for \( V_+(p) \), the natural non-abelian Berry connection on the bundle has components \cite{11}:
\[
\alpha_{\alpha \beta, k} = i \left\{ p, \alpha, + \left| \frac{\partial}{\partial p^k} \right| p, \beta, + \right\}.
\]
Its matrix-valued curvature tensor is
\[
\hat{F}_{ij} = \frac{\partial a_i}{\partial p^j} - \frac{\partial a_j}{\partial p^i} - i[a_i, a_j].
\]
We anticipate that \( a(p) \) is to be replaced by the matrix-valued \( a(p) \).

It is often convenient to expand the matrix-valued connection and curvature as
\[
a_i = \sum_{n<m} X_{nm} a_{i,m},
\]
\[
\hat{F}_{ij} = \sum_{n<m} X_{nm} \hat{F}_{ij},
\]
where \( X_{nm} = -X_{mn} \) are the \( 2^{N-1} \)-by-\( 2^{N-1} \) Hermitian matrix generators of the Lie algebra of spin(\( 2N \)). The generator \( X_{nm} \) corresponds to an infinitesimal rotation in the \( n, m \) plane, and the set of all \( 2N^2 - N \) pairs \( (n, m) \), \( n < m \), should be thought of as the range of a single index labeling the generators. We will abbreviate this index as \( (n) \). For example the commutation relation
\[
[X_{ij}, X_{km}] = i(\delta_{ni}X_{jm} - \delta_{nj}X_{im} - X_{mj}\delta_{jn} + X_{mj}\delta_{jn}),
\]
\[3\]
will be abbreviated as

$$[X_{(a)}, X_{(b)}] = i f_{(a)(b)}^{(c)} X_{(c)}. \quad (12)$$

In other words, $f_{(a)(b)}^{(c)}$ (with indices in parentheses) denotes the structure constants of the Lie algebra of spin(2N).

In [7] we also replaced the scalar-valued abelian $(A_0, A)$ fields in (1) by a non-abelian Hermitian matrix gauge field

$$A_{\mu} = \hat{\lambda}_a A^a_{\mu}, \quad F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} - i [A_\mu, A_\nu] \quad (13)$$

for a compact simple gauge group $G$. Here the $\hat{\lambda}_a$ are the matrices corresponding to the generator $\lambda_a \in \text{Lie}(G)$ in the representation that determines the ‘charge’ of our Weyl particle. The generators obey

$$[\lambda_a, \lambda_b] = i f_{ab}^c \lambda_c. \quad (14)$$

Thus, $f_{ab}^c$ (indices without parentheses) are the structure constants of Lie($G$). To obtain a classical version of the gauge and spin quantum degrees of freedom, we must somehow replace the matrix-valued gauge and spin connection fields with scalar-quantities. This is because any action functional must be the integral of scalar. However the simplest scheme of taking traces over the matrix indices in $a_i$ and $A_\mu$ will not preserve gauge invariance.

### 2.2. Dequantising spin

In [7] we resolved the problem of gauge invariance by de-quantizing the gauge group representation matrices by introducing an internal group-valued degree of freedom $g(t) \in G$ whose classical mechanics, when re-quantized by using the geometric quantization procedure of Kostant, Kirillov and Souriau [12] would give us back the matrices. How to do this is explained in physicist’s language in [13, 14] and also in [7]. Briefly stated, a representation of $G$ with highest weight $\Lambda$ determines a Lie algebra element $\alpha/\Lambda$ that lies in the Cartan subalgebra. The quantum representation of Hilbert space is replaced by a classical phase space $\mathcal{O}/\Lambda$ that is the adjoint orbit of $\alpha/\Lambda$. This orbit can be identified with the coset $G/H$, where the isotropy group $H$ is the subgroup of $G$ whose adjoint action $\alpha/\Lambda \mapsto g \alpha/\Lambda g^{-1}$ leaves $\alpha/\Lambda$ fixed. The matrix-valued gauge field components $A_\mu$ and $F_{\mu\nu}$ are replaced by functions $\mathcal{O}/\Lambda \rightarrow \mathbb{R}$ as

$$A_\mu \mapsto \widetilde{A}_\mu \equiv \text{tr}(QA_\mu), \quad F_{\mu\nu} \mapsto \widetilde{F}_{\mu\nu} \equiv \text{tr}(QF_{\mu\nu}), \quad (15)$$

where

$$Q = g \alpha/\Lambda g^{-1} = Q^a \lambda_a. \quad (16)$$

Here the trace is taken in some fixed faithful representation (most conveniently the defining representation of $G$). We can use this fixed trace to define a metric

$$\gamma_{ab} = \text{tr}(\lambda_a \lambda_b), \quad (17)$$

that we will use to raise or lower indices on Lie-algebra tensors such as the structure constants $f_{ab}^c$. In particular $Q_a = \gamma_{ab} Q^b$ is the classical analogue of the generator $\lambda_a$, and the commutation relations (14) are replaced by classical Poisson-bracket relations

$$\{Q_a, Q_b\} = i f_{ab}^c Q_c. \quad (18)$$

The obvious way to treat the non-abelian Berry connection is to repeat this scheme. We therefore introduce $\sigma(t) \in \text{spin}(2N)$ to accommodate the spin dynamics. We then have an
element $\beta_s \in \text{Cartan}(\text{spin}(2N))$ that is determined by the particle's spin $s$ (see appendix C for details), an orbit $O_s$, and the substitution

$$a_i \mapsto \tilde{a}_i \overset{\text{def}}{=} \text{tr}(\tilde{\mathcal{G}} a_i),$$

$$\tilde{g}_{ij} \mapsto \tilde{\tilde{g}}_{ij} \overset{\text{def}}{=} \text{tr}(\tilde{\mathcal{G}} \tilde{g}_{ij}),$$

(19)

where

$$\mathcal{G} \overset{\text{def}}{=} \sigma \beta_s \sigma^{-1} = \mathcal{G}^{(m)} \chi_{(m)},$$

(20)

### 2.3. The action and the equations of motion

We now assemble these ingredients to write down the natural candidate for the classical action that describes the motion of a Weyl fermion in a background non-abelian gauge field in $2N+2$ space-time dimensions:

$$S[x, p, g, \sigma] = \int dt \left( p^i \dot{x}^i - |p| + i \text{tr} \left( \alpha_A g^{-1} \left( \frac{d}{dt} - i(A_0 + x^i A_i) \right) g \right) \right.$$ 

$$- i \text{tr} \left( \beta_s \sigma^{-1} \left( \frac{d}{dt} - i\dot{p}^j a_j \right) \sigma \right) \right).$$

(21)

The proposed action is clearly invariant under the gauge transformation

$$g(t) \rightarrow h^{-1}(x(t), t) g(t),$$

$$A_\mu \rightarrow h^{-1} A_\mu h + i h^{-1} \frac{\partial}{\partial x^\mu} h.$$

(22)

It is also invariant under a $p$-dependent change of basis $|p, \beta, +\rangle \rightarrow |p, \alpha, +\rangle U_{\alpha\beta}(p)$ that takes

$$\sigma(t) \rightarrow U^{-1}(p) \sigma(t),$$

$$a_j \rightarrow U^{-1} a_j U + i U^{-1} \frac{\partial}{\partial p^j} U.$$

(23)

The equations of motion that arise from varying $g \rightarrow g(1+g^{-1} \delta g)$ and $\sigma \rightarrow \sigma (1+\sigma^{-1} \delta \sigma)$ are

$$[\alpha_A, g^{-1} (\partial_t - i(A_0 + x^i A_i)) g] = 0,$$

$$[\beta_s, \sigma^{-1} (\partial_t - i\dot{p}^j a_j) \sigma] = 0.$$  

(24)

These equations do not uniquely determine $g$ and $\sigma$. Indeed any solution $g(t)$ and $\sigma(t)$ can be multiplied on the right by an arbitrary time-dependent element of the corresponding isotropy subgroup and still satisfy the equation. As a result $g$ and $\sigma$ must be thought of as living in the cosets $O_A$ and $O_s$. The time evolution of the Lie-algebra-valued quantities $Q$ and $\mathcal{G}$ is insensitive to the ambiguity, and from (24) we find that

$$\dot{Q} = -i [Q, A_0 + x^i A_i],$$

$$\dot{\mathcal{G}} = -i [\mathcal{G}, a_j \dot{p}^j].$$

(25)

In components these equations read

$$\dot{Q}^a = f_{ab}^{\ \ c} Q^b (A_0^c + x^i A_i^c),$$

$$\dot{\mathcal{G}}^{(c)} = f_{a(b)}^{\ \ (c)} \mathcal{G}^{(a)} a_j^{(b)} \dot{p}^j.$$  

(26)

The remaining equations of motion

$$\dot{p}^j = \text{tr}(Q (F_{0j} + F_{ij} x^i)), $$

$$\dot{x}^i = \dot{p}^j - \text{tr}(\mathcal{G} \tilde{g}_{ij}) \dot{p}^j$$

(27)
are more straightforward, although in principle we still have to solve them for $\dot{x}^i$ and $\dot{p}_i$ in terms of the other degrees of freedom as we did in [7]. This task is easy in three space dimensions, but complicated in dimensions higher than three. For this reason, in subsequent sections we develop tools that allow us to deduce the consequences of (27) without finding explicit expressions for $\dot{x}^i$ and $\dot{p}_i$.

3. A generalized Liouville theorem

Both the gauge and abelian chiral anomalies can be understood physically as a spectral flow of states from the infinitely deep negative-energy Dirac sea, through the diabolical [15] Dirac point, and into the finite-depth positive-energy Fermi sea. If we monitor the phase-space density only in the positive energy region, the influx of states will appear as a source term located at $p = 0$ in the Liouville phase-space volume conservation law. We therefore begin by recalling the proof of Liouville’s theorem, and how the theorem is modified when the symplectic structure is allowed to depend on time.

3.1. Extended phase space

Consider a general even-dimensional phase space $M$ with co-ordinates $\xi = (\xi^1, \ldots, \xi^{2n})$ and equipped with a Hamiltonian action functional

$$ S[\xi] = \int dt \left\{ \sum_{i=1}^{2n} \eta_i(\xi, t) \dot{\xi}^i - H(\xi, t) \right\}. \tag{28} $$

Demanding that $\delta S = 0$ under a variation $\xi^i \rightarrow \xi^i + \delta \xi^i$ results in the equations of motion

$$ \left( \frac{\partial \eta_j}{\partial \xi^i} - \frac{\partial \eta_i}{\partial \xi^j} \right) \dot{\xi}^j = \left( \frac{\partial H}{\partial \xi^i} + \frac{\partial \eta_i}{\partial t} \right). \tag{29} $$

We will use the notation

$$ \omega_{ij} = \frac{\partial \eta_j}{\partial \xi^i} - \frac{\partial \eta_i}{\partial \xi^j}. \tag{30} $$

For the slightly generalized set of Hamilton’s equations (29) to be solvable for $\dot{\xi}^i$ without constraints, the symplectic matrix $\omega_{ij}$ must be invertible at every point in $M$. We assume that this condition is satisfied. Now, from (29) and the antisymmetry of $\omega_{ij}$, we see that the $\dot{\xi}^i$ automatically satisfy the condition

$$ \dot{\xi}^i \left( \frac{\partial H}{\partial \xi^i} + \frac{\partial \eta_i}{\partial t} \right) = 0. \tag{31} $$

As with the usual Hamiltonian formalism, many results are most compactly obtained with the tools of vector fields and differential forms. To use these in the time-dependent setting it is convenient to extend the even-dimensional phase space $M$ to the odd-dimensional space $M' = M \times \mathbb{R}$, where $\mathbb{R}$ is the time co-ordinate. We may then combine the $\eta_i$ and the Hamiltonian $H$ into a one-form

$$ \eta_H = \sum_{i=1}^{2n} \eta_i d\xi^i - H dt, \tag{32} $$

on $T^*M'$. Let $\omega_H = d\eta_H$ be its exterior derivative

$$ \omega_H = \frac{1}{2} \omega_{ij} d\xi^i d\xi^j - \left( \frac{\partial \eta_i}{\partial t} + \frac{\partial H}{\partial \xi^i} \right) d\xi^i dt. \tag{33} $$
Define a vector field
\[ \mathbf{v} = \frac{\partial}{\partial t} + \dot{\xi}^j \frac{\partial}{\partial \xi^j}, \] (34)
and take the interior product of \( \mathbf{v} \) with the two-form \( \omega_H \) to get
\[ \iota_\mathbf{v} \omega_H = \left( -\omega_{ij} \dot{\xi}^j + \frac{\partial H}{\partial \xi^i} + \frac{\partial \eta}{\partial t} \right) \text{d}\xi^i - \dot{\xi}^i \left( \frac{\partial H}{\partial \xi^i} + \frac{\partial \eta}{\partial t} \right) \text{d}t. \] (35)
From (29) and (31) we see that the compact equation
\[ \iota_\mathbf{v} \omega_H = 0 \] (36)
is completely equivalent to the equations of motion.

Any odd-dimensional two-form such as \( \omega_H \) must possess at least one null vector. Here, the matrix \( \omega_{ij} \) being invertible ensures that the null space of \( \omega_H \) is precisely one-dimensional. Collectively the null spaces compose the characteristic bundle over \( M' \) of the contact structure \( \omega_M \) [16], and the \( \mathbf{v}(\xi, t) \) determined by the equations of motion is the unique vector in the fiber over \( (\xi, t) \) that has unity as the coefficient of \( \frac{\partial}{\partial t} \).

### 3.2. The Liouville measure

We can define a volume \((2n + 1)\)-form on the extended phase space by
\[ \Omega \overset{\text{def}}{=} \frac{1}{n!} \omega^n_H \text{d}t, \]
\[ = \frac{1}{n!} \omega^n \text{d}t. \] (37)
Here
\[ \omega = \frac{1}{2} \omega_{ij} \text{d}\xi^i \text{d}\xi^j, \] (38)
is the usual phase space symplectic form, now allowed to be time-dependent. The second line of (37) follows from the first because the explicit factor of \( \text{d}t \) excludes all \( \text{d}\xi^i \text{d}t \) terms in \( \omega_H^n \).

We can use the first line of (37) to compute Lie derivative of \( \Omega \) with respect to \( \mathbf{v} \). We find
\[ \mathcal{L}_\mathbf{v} \Omega = (\iota_\mathbf{v} \text{d} + \text{d}\iota_\mathbf{v}) \Omega, \]
\[ = \text{d}\iota_\mathbf{v} \Omega, \quad (\text{d}\Omega = 0 \text{ because } \Omega \text{ is a top form}) \]
\[ = \frac{1}{n!} \text{d}((\iota_\mathbf{v} \omega_H^n) \text{d}t + \omega_H^n), \quad (\iota_\mathbf{v} \text{ is an antiderivation, and } \omega_H^n \text{ is even}) \]
\[ = \frac{1}{n!} \text{d}(\omega_H^n), \quad (\iota_\mathbf{v} \omega_H^n \equiv \omega_H^{n-1} \wedge \iota_\mathbf{v} \omega_H = 0 \text{ by the equations of motion}) \]
\[ = 0. \quad (\text{d}^2 \eta_H = 0 \text{ provided } \eta_H \text{ is nowhere-singular}) \] (39)

We can alternatively compute the Lie derivative of \( \Omega \) from the second line of (37). We observe that
\[ \frac{1}{n!} \omega^n \text{d}t = \sqrt{\omega} \text{d}\xi^1 \cdots \text{d}\xi^{2n} \text{d}t, \] (40)
where \( \sqrt{\omega} \equiv \sqrt{\det(\omega)} \) is the Pfaffian \( \text{Pf}(\omega) \) of the skew-symmetric matrix \( \omega_{ij} \). From the derivation property of the Lie derivative we now find
\[ \mathcal{L}_\mathbf{v} \Omega = (\mathcal{L}_\mathbf{v} \sqrt{\omega}) \text{d}\xi^1 \cdots \text{d}\xi^{2n} \text{d}t + \sqrt{\omega} \left( \mathcal{L}_\mathbf{v} \text{d}\xi^1 \cdots \text{d}\xi^{2n} \text{d}t \right) \]
\[ = \left( \frac{\partial \sqrt{\omega}}{\partial t} + \dot{\xi}^i \frac{\partial \sqrt{\omega}}{\partial \xi^i} \right) \text{d}\xi^1 \cdots \text{d}\xi^{2n} \text{d}t + \sqrt{\omega} \left( \frac{\partial \dot{\xi}^i}{\partial t} \text{d}\xi^1 \cdots \text{d}\xi^{2n} \text{d}t \right) \]
\[ = \left( \frac{\partial \sqrt{\omega}}{\partial t} + \frac{\partial \sqrt{\omega} \dot{\xi}^i}{\partial \xi^i} \right) \text{d}\xi^1 \cdots \text{d}\xi^{2n} \text{d}t. \] (41)
Comparing the two computations shows that our generalized Hamilton equations lead to

$$\frac{\partial \sqrt{\omega}}{\partial t} + \frac{\partial \sqrt{\omega} \xi^i}{\partial \xi^i} = 0. \tag{42}$$

This last equation is the time-dependent version of Liouville’s theorem. In our application, the manipulations in (39) will fail at the last step because a generalization of the Berry-phase monopole leads \(\omega_H^0\) to be singular. Consequently (42) will be violated by a source term at the Dirac point. The remaining sections of this paper will be devoted to finding the strength of this source term in terms of the external fields acting on our particles.

4. The Boltzmann equation

Now we apply the formalism of section 3 to the action functional (21). The extended phase space we need is \(M' = \mathbb{R}^{2N+2} \times \mathcal{O}_\Lambda \times \mathcal{O}_s \times \mathbb{R}\) with \(\mathbb{R}^{2N+2}\) being the particle’s (x, p) coordinates, \(\mathcal{O}_\Lambda, \mathcal{O}_s\) being the internal gauge and spin spaces, and \(\mathbb{R}\) time. We will avoid as much as possible the use explicit co-ordinates \(\xi^i\) on the internal spaces, and instead use intrinsic geometric quantities.

4.1. Boltzmann and Liouville

We begin by writing (21) as an integral along the phase-space trajectory. We need to define the differential forms

\[ A = A_i dx^i + A_0 dt, \quad a = a_i dp^i, \tag{43} \]

as well as

\[ \omega_R^0 = dgg^{-1}, \quad \omega_R^0 = d\sigma \sigma^{-1}, \tag{44} \]

which are pullbacks to the trajectory in (x, p, g, \(\sigma\)) space of the right-invariant Maurer–Cartan forms on \(G\) and on Spin(2N) respectively. The action (21) then becomes

\[ S[x, p, g, \sigma] = \int (p^i dx^i - |p| dt + tr[Q(i dgg^{-1} + A)] - tr[\mathcal{G}(id\sigma \sigma^{-1} + a)]) \]

\[ = \int (p^i dx^i - |p| dt + tr[Q(i\omega^0_R + A)] - tr[\mathcal{G}(i\omega^0_R + a)]). \tag{45} \]

Now (45) is of the general form (28) with

\[ \eta_H = p^i dx^i - |p| dt + tr[Q(i\omega^0_R + A)] - tr[\mathcal{G}(i\omega^0_R + a)]. \tag{46} \]

Using \(F = dA - iA^2, \tilde{g} = da - ia^2\) and \(d\omega_R = (\omega_R)^2\) we find that

\[ \omega_H = dp^i dx^i - |p| dt + F - \tilde{g} - i tr[Q(\omega^0_R - iA)^2] + i tr[\mathcal{G}(\omega^0_R - iA)^2], \tag{47} \]

and the volume form (37) becomes

\[ \Omega = \frac{1}{M!} \omega_H^M dt, \tag{48} \]

where \(M = (2N + 1) + m_\Lambda + m_s\) with \(m_\Lambda = \text{dim}(\mathcal{O}_\Lambda)/2, m_s = \text{dim}(\mathcal{O}_s)/2\).

There are potentially many terms in the high power of \(\omega_H\) appearing in (48). A considerable simplification arises, however, because all terms involving explicit A’s and a’s must cancel. This cancellation can be tediously verified in 3+1 dimensions, but can be shown to occur in general by observing that at any chosen point in x, p space we can make gauge transformations so that both A and a (but not their derivatives) vanish. The transformations (22) and (23) that return us to the original gauge will leave this \(\Omega(a = A = 0)\) volume form unchanged—either because
traces are invariant under adjoint actions, or because the inhomogeneous $g^{-1}dg$ and $\sigma^{-1}d\sigma$ terms will seek to introduce extra $dx^\mu$'s, $dt$'s, or $dp'$'s that are forbidden by antisymmetry.

Thus we find that

$$\frac{1}{M!} \Omega_X^M = \frac{1}{(2N+1)!} (dp' dx' + \tilde{F} - \tilde{g})^{2N+1} d\mu_\Lambda d\mu_\Sigma,$$

(49)

where

$$d\mu_\Lambda = \frac{1}{m_\Lambda!} [i \text{tr} \{ \alpha_\Lambda (\omega_\Lambda^0)^2 \}]^{m_\Lambda}, \quad d\mu_\Sigma = \frac{1}{m_\Sigma!} [-i \text{tr} \{ \beta_\Sigma (\omega_\Sigma^0)^2 \}]^{m_\Sigma},$$

(50)

are, up to factors, the Kirillov–Kostant measures on the co-adjoint orbits $\mathcal{O}_\Lambda$ and $\mathcal{O}_\Sigma$ respectively. We have taken advantage of the absence of the gauge fields to introduce the left-invariant Maurer–Cartan forms $\omega_\Lambda^0 = g^{-1}dg$ and $\omega_\Sigma^0 = \sigma^{-1}d\sigma$.

We also find that

$$\nu = \frac{\partial}{\partial t} + \dot{x}' \frac{\partial}{\partial x'} + \dot{p}' \frac{\partial}{\partial p'} + (g^{-1}\dot{g})^a L^\text{gauge}_a + (\sigma^{-1}\dot{\sigma})^a L^\text{spin}_a$$

(51)

is the appropriate form for the Hamiltonian vector field $\nu$ of equation (34). Here $L^\text{gauge}_a$ is the left-invariant vector field dual to the left-invariant Maurer–Cartan form. We have

$$\omega_\Lambda^0 (L^\text{gauge}_a) = \lambda_a,$$

(52)

so $\omega_\Lambda^0 (\nu) = (g^{-1}\dot{g})^a \lambda_a = g^{-1}\dot{g}$ is the analogue of $dx'(\nu) = \dot{x}'$. Similarly $L^\text{spin}_a$ is dual to the Spin(2N) Maurer–Cartan form:

$$\omega_\Sigma^0 (L^\text{min}_a) = X_{(a)}$$

(53)

In order to write down the Liouville theorem we will need to know how to compute the Lie derivative of the adjoint orbit measures. After a little effort and the use of (26) we find

$$L_\nu d\mu_\Lambda = f_{abc} \dot{c}^b A^c \frac{\partial}{\partial Q^c} d\mu_\Lambda + \cdots,$$

(54)

where the dots indicate terms involving $dx'$ or $dt$ that have no effect in $d\mu_\Lambda dx^1 \cdots dx^{2N+1} dt$. There is an analogous expression for the Lie derivative of $\mu_\Sigma$.

We now define $\sqrt{\omega}$ by

$$\Omega = \sqrt{\omega} \, d\mu_\Lambda \, d\mu_\Sigma \, dp^1 \cdots dp^{2N+1} \, dx^1 \cdots dx^{2N+1} \, dt.$$

(55)

If—for the duration of this section only—we ignore the singularity at $p = 0$ and follow the procedure in section 3 we end up with $L_\nu \Omega = 0$ being equivalent to

$$\nabla_t \sqrt{\omega} + \nabla_{x'} \sqrt{\omega} x' + \nabla_{p'} \sqrt{\omega} p' = 0,$$

(56)

where

$$\nabla_t = \frac{\partial}{\partial t} + f_{abc} A^b \frac{\partial}{\partial Q^c},$$

$$\nabla_{x'} = \frac{\partial}{\partial x'} + f_{abc} Q^b \frac{\partial}{\partial Q^c},$$

$$\nabla_{p'} = \frac{\partial}{\partial p'} + f_{(a)(b)} c^a \frac{\partial}{\partial Q^b},$$

(57)

are ‘covariant derivatives’ that ensure invariance under gauge transformations.

We introduce a phase-space density $f(x, p, Q, \Sigma, t)$ and let it be advected with the flow

$$\left( \frac{\partial}{\partial t} + \dot{x}' \frac{\partial}{\partial x'} + \dot{p}' \frac{\partial}{\partial p'} + \dot{Q}^a \frac{\partial}{\partial Q^a} + \dot{\Sigma}^{(a)} \frac{\partial}{\partial \Sigma^{(a)}} \right) f = 0.$$
This advection condition is the collisionless single-particle Boltzmann equation for our system. It generalizes the 3+1-dimensional Boltzmann equation in [17].

We can use (26) and (57) to group these terms as

\[(\nabla_t + \dot{x} \nabla_x + \dot{p} \nabla_p) f = 0,\]  

and so find that

\[L_v (f \Omega) = 0,\]

which expresses the conservation of probability.

4.2. Conservation laws

The continuity equation (60) for the phase-space density is the origin of various conservation laws. From (60) we can derive the conservation

\[\frac{\partial F_0}{\partial \tau} + \frac{\partial F_i}{\partial x^i} = 0 \quad (61)\]

of the particle-number current

\[F_0^0(t, x) = \int f(x, p, Q, S, t) \sqrt{\omega} d\tilde{\mu}_\Lambda \, d\tilde{\mu}_\sigma \frac{d^{2N+1}p}{(2\pi)^{2N+1}}, \quad F_0^i(t, x) = \int \dot{x}^i f(x, p, Q, S, t) \sqrt{\omega} d\tilde{\mu}_\Lambda \, d\tilde{\mu}_\sigma \frac{d^{2N+1}p}{(2\pi)^{2N+1}}, \]

and the covariant conservation

\[\frac{\partial F_0^0}{\partial \tau} - f_{ab} e A^a c F_0^c + \frac{\partial F_i^0}{\partial x^i} - f_{ab} e A^a c F_i^c = 0 \quad (63)\]

of the gauge current

\[F_0^0(t, x) = \int Q_{\alpha} f(x, p, Q, S, t) \sqrt{\omega} d\tilde{\mu}_\Lambda \, d\tilde{\mu}_\sigma \frac{d^{2N+1}p}{(2\pi)^{2N+1}}, \quad F_0^i(t, x) = \int Q_{\alpha} \dot{x}^i f(x, p, Q, S, t) \sqrt{\omega} d\tilde{\mu}_\Lambda \, d\tilde{\mu}_\sigma \frac{d^{2N+1}p}{(2\pi)^{2N+1}}.\]

In each of these definitions the integral is over all of momentum space and over both adjoint orbits \(O_\Lambda, O_\sigma\). While writing down the expressions for the currents we have taken the opportunity to normalize the measure factors. We have put a 1/2 \(\pi\) with each \(dp\) and rescaled the adjoint orbit measure so that

\[d\tilde{\mu}_\Lambda = \frac{1}{(2\pi)^{m_\Lambda} m_\Lambda!} \text{tr} \{\alpha A^\alpha (\omega^0)\}, \quad d\tilde{\mu}_\sigma = \frac{1}{(2\pi)^{m_\sigma} m_\sigma!} \text{tr} \{\beta A^\beta (\omega^\mu)\}. \]

This rescaling is one place where we need knowledge of quantum mechanics: the normalized Liouville measure counts (approximately) one quantum state per unit volume of the classical phase space. Consequently the exclusion principle says that the maximum allowed value of \(f(x, p, Q, S, t)\) is unity.

To see that (60) leads to the conservation laws (61) and (63) recall that if we have a q-dimensional manifold \(M\) and integrate a p-form \(\theta\) over a smooth p-dimensional region \(N(\tau)\), each point of which is being advected by a flow \(v = dx/d\tau\), then Liebniz’ integral formula reads

\[\frac{d}{d\tau} \int_{N(\tau)} \theta = \int_{N(\tau)} L_v \theta. \]
An immediate corollary is that when we integrate a q-form $\Theta$ over the entirety of $M$ we have
\[ \int_M \mathcal{L}_v \Theta = 0. \] (67)

Now (61) is equivalent to the vanishing of
\[ \int_{\mathbb{R}^{2N+1} \times \mathbb{R}} \psi(x, t) \left( \frac{\partial J^0}{\partial t} + \frac{\partial J^i}{\partial x^i} \right) d^{2N+1}x \, dt \] (68)
for any test function $\psi(x, t)$. But $\mathcal{L}_v (f \Omega) = 0$ and (67) gives us
\[ 0 = \int_M \mathcal{L}_v (\psi f \Omega) \]
\[ = \int_M \left( \frac{\partial \psi}{\partial t} + \dot{x} \frac{\partial \psi}{\partial x} \right) f \Omega \]
\[ = - \int_{\mathbb{R}^{2N+1} \times \mathbb{R}} \psi(x, t) \left( \frac{\partial J^0}{\partial t} + \frac{\partial J^i}{\partial x^i} \right) d^{2N+1}x \, dt. \] (69)

A similar calculation gives (63).

### 5. The Liouville anomaly

The extended phase-space differential-form language introduced in the previous section may seem rather abstruse, but a glance at appendix A will reveal that it provides a more compact and transparent derivation of the abelian anomaly than that in [1, 7]. In this section we will generalize the derivation in the appendix by identifying the Liouville theorem source term in any even space-time dimension and use it to derive the anomalous conservation laws.

We begin with an observation about the Liouville measure. In
\[ \frac{1}{(2N+1)!} \left( dp^0 dx^i + \tilde{F} - \tilde{\Xi} \right)^{2N+1} \, dt = \sqrt{\omega} dp^1 \cdots dp^{2N+1} \, dx^1 \cdots dx^{2N+1} \, dt \] (70)
the measure factor $\sqrt{\omega}$ is the Pfaffian of the $(4N+2)$-by-$(4N+2)$ matrix
\[ \omega_{ij} = \left( \begin{array}{cc} -\tilde{\Xi} & \mathbb{I} \\ -\mathbb{I} & \tilde{F} \end{array} \right)_{ij}. \] (71)

The factor $dt$ excludes $\tilde{F}_{0j}$ from being an entry in the block submatrix $\tilde{F}$ appearing here, so it has the same number of rows and columns as does $\tilde{\Xi}$. The Schur determinant formula now shows that
\[ \det \left( \begin{array}{cc} -\tilde{\Xi} & \mathbb{I} \\ -\mathbb{I} & \tilde{F} \end{array} \right) = \det(\mathbb{I} - \tilde{F} \tilde{\Xi}), \] (72)
and the Liouville measure is a square root of this determinant. This root can be expressed in terms of a ‘double Pfaffian’
\[ \text{Pf}(\tilde{\Xi}, \tilde{F}) \overset{\text{def}}{=} \sum_{k=0}^N \sum_{I_{2k}} \text{Pf}(\tilde{\Xi}_{I_{2k}}) \text{Pf}(\tilde{F}_{I_{2k}}), \] (73)
where each $I_{2k}$ is a cardinality-$2k$ subset of the indices on the matrices, and the term with $k = 0$ is understood to be unity. The square root is ambiguous, and we actually have
\[ \sqrt{\omega} = \pm \text{Pf}(\tilde{\Xi}, \tilde{F}), \] (74)
where the $\pm$ sign depends on how many $dx^i$'s and $dp^j$'s have to be interchanged to turn $(dx^i dp^j)^{2N+1}$ into $d^{2N+1}p \, d^{2N+1}x$. This sign is unimportant provided we use a consistent definition in the current and anomaly measures.
While it is nice to have an explicit expression for $\sqrt{\omega}$ that generalizes the 3+1 dimensional $\sqrt{\omega} = 1 + \mathbf{b} \cdot \mathbf{B}$ from [1], we do not really need it because the source term in the Liouville theorem comes from $\omega^M_M / M!$. When we expand out $\omega^M_M / M!$ there is no longer an explicit $dr$ and so $\tilde{F}_{00}$ is allowed. In particular one of the many terms that appear in $\omega^M_M / M!$ is
\[
\frac{(-1)^N}{N!} \left( \frac{\pi}{2\pi} \right)^N \frac{1}{(N+1)!} \left( \tilde{F} \right)^{N+1} d\tilde{\mu}_\Lambda d\tilde{\mu}_s
\]  
(75)
where the $2\pi$s come from the factors of $1/2\pi$ that we put with each $dp$.

It is this term that causes $d\omega^M_M / M! \neq 0$ in the last line of (39) and hence gives rise to the anomaly. We show in the appendix that the integral of the Chern character
\[
\frac{1}{N!} \left( \frac{2\pi}{N} \right)^N \int_S \text{tr}(s^N)
\]  
(76)
of the spin-connection curvature over a closed 2N-surface $S$ in momentum space is $(-1)^N$ (for positive-helicity) when $S$ encloses $p = 0$, and is zero otherwise. As with the abelian monopole, we can conveniently abbreviate this statement as
\[
d \left( \frac{1}{N!} \left( \frac{2\pi}{N} \right)^N \text{tr}(s^N) \right) = (-1)^N \delta^{2N+1}(p) dp \cdots dp^{2N+1},
\]  
(77)
exhibiting a higher-dimensional version of (4). For the gauge field, however, we have
\[
d \left( \frac{1}{(N+1)!} \text{tr} \left( \frac{F}{2\pi} \right)^{N+1} \right) = 0
\]  
(78)
by well known properties of characteristic classes.

In (77) of course we have the matrix-valued curvature $\tilde{\xi}$ and a trace over its quantum indices. Similarly in (78). In (75) we have the function-valued $\tilde{\xi}$ and $\tilde{F}$ and are to be integrated over the adjoint orbits $O_\lambda$ and $O_s$. However we also show in the appendix that provided we take integral not over the naive orbit associated with the greatest weight of the spin representation but instead over the orbit associated with the Weyl shifted weight, then the quantum and classical traces coincide. Similarly the classical trace $O_\lambda$ is always proportional to the quantum trace, and so (78) remains true when $F$ is replaced by $\tilde{F}$. We also show that (75) is the only contribution to $d\omega^M_M / M!$ that survives the trace operation.

One additional ingredient is required. It is not the integral of $\mathcal{L}_s \Omega$ that is needed in the conservation law, but the integral of $\mathcal{L}_s (f \Omega)$. If we are to get the standard expression for the anomaly we must have $f$ be identically unity over and within a surface sufficiently far from $p = 0$ that the adiabatic approximation and the resultant classical machinery be applicable. Thus we need a Fermi sea that is deep enough that finite temperature effects do not depopulate the sea too close to the Dirac point.

Assuming that $f$ is indeed unity in the region where it is needed, we immediately find that contribution of the delta function (77) to $\mathcal{L}_s \Omega$ modifies the conservations laws to read
\[
\frac{\partial \tilde{J}^0}{\partial t} + \frac{\partial \tilde{J}^i}{\partial x^i} = \frac{\epsilon^{ab} \tilde{F}_{ab} \tilde{F}^0_0}{(2\pi)^{N+2} \left( N + 1 \right)!} \int_{O_\lambda} (\tilde{F}_{1i_1} \cdots \tilde{F}_{2N_i + 12N_s + 2}) d\tilde{\mu}_\Lambda
\]  
(79)
and
\[
\frac{\partial \tilde{J}^0}{\partial t} - f_{ab} \tilde{A}^0_a \tilde{P}^0_b + \frac{\partial \tilde{J}^i}{\partial x^i} - f_{ab} \tilde{A}^i_a \tilde{P}^i_b = \frac{\epsilon^{12} \tilde{F}_{12} \tilde{F}^0_0}{(2\pi)^{N+2} \left( N + 1 \right)!} \int_{O_\lambda} \tilde{Q}_a (\tilde{F}_{1i_1} \cdots \tilde{F}_{2N_i + 12N_s + 2}) d\tilde{\mu}_\Lambda.
\]  
(80)
The phase-space integrals are classical approximations to the symmetrized traces
\[
\int_{O_\lambda} (\tilde{F}_{1i_1} \cdots \tilde{F}_{2N_i + 12N_s + 2}) d\tilde{\mu}_\Lambda \sim \text{str}_\Lambda \{ \tilde{F}_{1i_1} \cdots \tilde{F}_{2N_i + 12N_s + 2} \}
\]  
(81)
\[ \int_{\Omega_\Lambda} \mathcal{Q}_a (\tilde{F}_{i_1 i_2} \cdots \tilde{F}_{i_{2N+1} i_{2N+2}}) \, d\bar{\mu}/\Lambda_1 \sim \text{str} /\Lambda_1 \{ \lambda^a F_{i_1 i_2} \cdots F_{i_{2N+1} i_{2N+2}} \} \]  

(82)

and

\[ \partial_\mu J^\mu = \frac{1}{(4\pi)^{n!}} \epsilon^{\nu_1 \cdots \nu_{2n}} \text{tr}_\Lambda \{ \hat{\lambda}^a_1 \cdots \hat{\lambda}^a_n \} F^{\nu_1}_{\nu_2} \cdots F^{\nu_{2n}}_{\nu_{2n+1}}, \]  

(83)

and equation (80) is the covariant form

\[ D_\mu J^\mu_a = \frac{1}{(4\pi)^{n!}} \epsilon^{\nu_1 \cdots \nu_{2n}} \text{tr}_\Lambda \{ \hat{\lambda}^a_1 \cdots \hat{\lambda}^a_n \} F^{\nu_1}_{\nu_2} \cdots F^{\nu_{2n}}_{\nu_{2n+1}}, \]  

(84)

of the non-abelian gauge anomaly.

It is exhibited explicitly in [7] for the case \( G = \text{SU}(3) \), that the accuracy of the classical approximation to the symmetrized trace is again greatly increased if the classical trace integral is actually taken over the orbit corresponding to the Weyl-shifted weight \( \Lambda \rightarrow \Lambda + \rho \), where the Weyl vector \( \rho \) is half the sum of the positive roots.

6. Discussion

We have shown that the classical equations of motion derived from the single-particle action functional (21) give rise to the same anomalous conservation laws as a Weyl fermion in a background gauge field. That we get the correct form and coefficients in the anomalies gives strong evidence that we correctly identified (21) as the appropriate generalization of the 3+1 abelian action (1).

The anomaly we have obtained is the covariant anomaly as opposed to the consistent anomaly that satisfies the Wess–Zumino consistency condition [18]. This is perhaps inevitable because while the Hamiltonian formalism only makes manifest the canonical structure, gauge covariance is being tacitly maintained at all points of the calculation. The lack of ‘consistency’ is not a deficiency of the calculation, however. In general, when an anomalous chiral gauge theory makes physical sense, the Weyl particles will be domain-wall fermions residing on the boundary of some higher-dimensional space. The anomaly is then accounted for by the inflow of gauge current from the bulk, via the Callan–Harvey mechanism [20]. This inflowing current can obtained by functionally differentiating a bulk Chern–Simons action and the integrated out boundary variation of the Chern–Simons term is then precisely the Bardeen–Zumino polynomial [19] that converts the consistent gauge current to the covariant current (see for example [21]). A similar argument shows that in an anomalous theory the current that appears in the Lorentz-force contribution to the energy-momentum conservation law is the covariant current [22, 23]. Thus the physical anomaly is usually the covariant one.

The principal quantum input into the equations of motion is the non-abelian Berry connection that performs parallel transport of the degenerate helicity states. On its own the Berry transport provides the form of the chiral anomaly. Additional quantum knowledge is required to obtain the correct coefficient. We at least need to normalize the measure on the phase-space so that there is one quantum state per unit volume. Further, we have seen that to get the exact coefficients for ‘small’ spin and gauge representations we have to Weyl shift the weight vector that defines the co-adjoint phase space. This shift is the generalization of the \( 2j \rightarrow 2j + 1 \) replacement in the dimension of the spin-\( j \) angular momentum representation. That we need some sort of correction is not surprising because we only expect a classical calculation to be exact in the classical regime of large spins and large gauge groups. It is,
However, remarkable that the quantum correction is as simple as it is. This is one of the miracles and themes of the Kirillov orbit method in the group representation theory [12]. It is interesting, but perhaps not surprising, that quantum Berry-phase effects survive in the classical limit. The original example of the abelian Berry phase is a spin that is being forced to change its direction by a strong magnetic field. Here Berry’s gauge field $\mathbf{a}(\mathbf{p})$ provides an analogue Lorentz force that causes Larmor precession. The gauge field is simply the way in which the spin’s gyroscopic stiffness manifests itself, and so is best understood as a classical effect. What our calculation shows is that the quantum effects of the Fermi-surface Berry flux [2–4] might also be best thought of as being the result of gyroscopic forces. In this way the classical derivation of the anomaly gives us insight into why the quantum system behaves as it does.

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Appendix A. The abelian anomaly in 3+1 dimensions

In this appendix we use the extended phase-space formalism to obtain the abelian anomaly in 3 + 1 dimensions. The reason for our doing so is to illustrate how much more powerful is this formalism compared to the method used in [1] and [7].

The action is

$$S[\mathbf{x}, \mathbf{p}] = \int \eta_H = \int (p_i \, dx^i - |\mathbf{p}| \, dt + A - a).$$  \hspace{1cm} (A.1)

Then

$$\omega_H = d\eta_H = d(p_i \, dx^i - |\mathbf{p}| \, dt + A - a) = dp_i \wedge dx^i - \hat{p}_i \, dp^i \wedge dt + F - \tilde{\mathbf{g}},$$  \hspace{1cm} (A.2)

where $F = dA$ is the Maxwell tensor and $\tilde{\mathbf{g}} = da$ is the corresponding curvature for the Berry connection. Because $dF = 0$ from Maxwell’s equations, we have

$$d \omega_H = dF - d\tilde{\mathbf{g}} = - d\tilde{\mathbf{g}}.$$  \hspace{1cm} (A.3)

Here, $d\tilde{\mathbf{g}} \neq 0$ because of the Berry monopole at the origin $\mathbf{p} = 0$.

The Lie derivative of the Liouville volume form, using equation (39), is simply

$$L_v \Omega = \frac{1}{3!} \, d\omega_H = \frac{1}{2!} \, d\omega_H \wedge \omega_H
= - \frac{1}{2} \, d\tilde{\mathbf{g}} \wedge (dp_i \wedge dx^i - \hat{p}_i \, dp^i \wedge dt + F - \tilde{\mathbf{g}})^2
= - \frac{1}{2} \, d\tilde{\mathbf{g}} \wedge F^2
= - \frac{1}{2} \, \frac{\partial \mathcal{B}_i}{\partial p_i} \left[ \frac{1}{2} \left( \epsilon_{ijk} \frac{\partial}{\partial p_i} \left( \frac{1}{2} \tilde{\mathbf{g}}_{jk} \right) \right) \right] \frac{1}{2} \, e^{\lambda^{i\mu\nu} F_{\mu\nu}} \, d^3 p \, d^3 x \, dt
= - \left( \frac{\partial \mathcal{B}_i}{\partial p_i} \right) \frac{1}{2} \left[ \epsilon_{\mu\nu\lambda} \, F_{\mu\nu} F_{i\nu} \right] \, d^3 p \, d^3 x \, dt
= - (\nabla_p \cdot \mathcal{B})(-E_{in}) B_{ni} \, d^3 x \, dt
= (\nabla_p \cdot \mathcal{B})(\mathbf{E} \cdot \mathbf{B}) \, d^3 p \, d^3 x \, dt.$$  \hspace{1cm} (A.4)
Comparison with equation (42) gives
\[ \frac{\partial \sqrt{\omega}}{\partial t} + \frac{\partial \sqrt{\omega} x_i}{\partial x_i} + \frac{\partial \sqrt{\omega} p_i}{\partial p_i} = (\nabla_p \cdot \mathbf{B})(E, B) = 2\pi \delta^3(\mathbf{p})(E, B) \] (A.5)
which is the result obtained in [1]. Defining the currents as in [1], leads immediately to
\[ \partial_\mu J^\mu = \frac{1}{(2\pi)^2} E \cdot B, \] (A.6)
which is the expression for chiral anomaly in 3+1 dimensions. Comparison with the labour in [1, 7] shows the compactness of the present derivation.

Appendix B. Spin Chern number

Here we compute the Chern number for the bundle of positive-energy eigenstates of
\[ \hat{H}_p = \sum_{i=1}^{2N+1} \Gamma_i p^i \] (B.1)
over the 2N-sphere in momentum space.
\( \hat{H} \) has eigenvalues \( \pm|\mathbf{p}| \) and the projectors on to the positive and negative energy eigenspaces \( V_+ (\mathbf{p}), V_- (\mathbf{p}) \) are
\[ P = \sum_\Lambda |\mathbf{p}, \alpha, + \rangle \langle \mathbf{p}, \alpha, + | = \frac{1}{2} (1 + \hat{p}\Gamma_i), \]
\[ P^\perp = \sum_\Lambda |\mathbf{p}, \alpha, - \rangle \langle \mathbf{p}, \alpha, - | = \frac{1}{2} (1 - \hat{p}\Gamma_i), \] (B.2)
respectively. Here \( \hat{p} \) is the unit vector \( \mathbf{p}/|\mathbf{p}| \).

For any eigenvalue problem with positive and negative energy spaces the matrix-valued two-form \( PdPdP \) has matrix elements only in \( V_+ (\mathbf{p}) \), and coincides there with the matrix elements of the Berry curvature \( \mathbf{F} = \mathbf{d}a - i a^2 \) divided by \( i \). In other words
\[ i P dP dP = \sum_{\alpha, \beta} |\mathbf{p}, \alpha, + \rangle \delta_{\alpha \beta} (\mathbf{p}, \beta, + |. \] (B.3)
In our case
\[ P dP dP = \frac{i}{4} (1 + \hat{p}\Gamma_i) \frac{1}{4i} \{ \Gamma_i, \Gamma_j \} d\hat{p}^i d\hat{p}^j. \] (B.4)
For positive-helicity we have
\[ \Gamma_1 \Gamma_2 \cdots \Gamma_{2N+1} = \gamma^N I_{2N}, \] (B.5)
and can chose a basis in which
\[ \Gamma_{2N+1} = \begin{bmatrix} I_{2N-1} & 0 \\ 0 & -I_{2N-1} \end{bmatrix}. \] (B.6)

To relate our set of \( 2N + 1 \) gamma matrices with the \( 2N \) needed for the representations of spin(2N), consider the fiber over a generic point in \( \mathbf{p} \) space that we may as well take \( \mathbf{p} = (0, \ldots, 0, 1) \). Then, at that point
\[ dP dP = \sum_{i,j=1}^{2N} \frac{i}{2} \begin{bmatrix} \hat{X}_{ij, \sigma \epsilon} & 0 \\ 0 & \hat{X}_{ij, \sigma \epsilon} \end{bmatrix} d\hat{p}^i d\hat{p}^j, \quad 1 \leq i, j \leq 2N; \] (B.7)
where
\[ \hat{X}_{ij, \sigma \epsilon} = \frac{1}{4i} P[\Gamma_i, \Gamma_j] P, \quad \hat{X}_{ij, \sigma \epsilon} = \frac{1}{4i} P^\perp [\Gamma_i, \Gamma_j] P^\perp, \] (B.8)
are the matrices representing the generator $X_{ij}$ in the two inequivalent spin representations of spin(2$N$). The projector $P \equiv P_{\sigma_+} = (\mathbb{1} + \Gamma_{2N+1})/2$ ensures that the trace over the 2$N$ indices in $PdPdP$ coincides with the trace over the 2$N-1$ indices in the $\sigma_+$ representation. and
\[
\text{tr}_{\sigma_+}(X_{i_1j_1,\ldots,X_{i_{2N-1}j_{2N-1}}}e^{iI_{12\ldots2N-1}2N}) = (-i)^N 2^{-N} (2N)! \text{tr}[P_{\sigma_+} \Gamma_1 \ldots \Gamma_{2N}]
\]
\[
= 2^{-N} (2N)! \frac{1}{2} \text{tr}[\mathbb{1}_{2N}]
\]
\[
= (2N)!/2.
\]
In the $\sigma_-$ representation, the RHS becomes $-(2N)!/2$. For any other point this works the same way—except that $P_p = \frac{1}{2}(\mathbb{1} + \hat{p}\Gamma_i)$ projects onto an equivalent set of spin(2$N$) generators.

The Chern number is given by the integral of the Chern-character class over the unit sphere in momentum space
\[
\text{ch}_N(V_+) \equiv \frac{1}{N!} \left( \frac{1}{2\pi} \right)^N \int_{S^2N} \text{tr}_{\sigma_+}(\tilde{g}^N)
\]
\[
= \frac{1}{N!} \left( \frac{i}{2\pi} \right)^N \int_{S^2N} \text{tr}((PdPdP)^N).
\]
To evaluate this set
\[
Z = \sum_{\mu=1}^{2N+1} \hat{p}_i \Gamma_i
\]
so that $Z = P - P_\perp$ and
\[
(2 \times 4^N) \text{tr}((iPdPdP)^N) = i^N \text{tr}[Z(dZ)^{2N}]
\]
\[
= 2^N 2^N \epsilon_{i_1 \ldots i_{2N+1}} \hat{p}^{i_1} \hat{p}^{i_2} \ldots \hat{p}^{i_{2N+1}}
\]
\[
= (-1)^N 2^N (2N)!d[\text{Area on } S^{2N}].
\]
Thus
\[
\frac{i^N}{2^N (2N)!} \int_{S^2N} \text{tr}[Z(dZ)^{2N}] = (-1)^N |S_{2N}|
\]
\[
= (-1)^N 2\pi^{N+1/2}/\Gamma \left( \frac{2N + 1}{2} \right).
\]
Here we have used a standard formula for the surface area $|S^{2N}|$ of the 2$N$-sphere. On further using $x\Gamma(x) = \Gamma(x + 1)$ etc we find
\[
\text{ch}_N(V_+) = \frac{1}{N!} \left( \frac{1}{2\pi} \right)^N \int_{S^2N} \text{tr}((iPdPdP)^N)
\]
\[
= \frac{1}{2(2N)} \left( \frac{i}{8\pi} \right)^N \int_{S^2N} \text{tr}[Z(dZ)^{2N}]
\]
\[
= (-1)^N.
\]

Appendix C. Classical and quantum traces for Spin(2N)

The computation of the Chern character in appendix B required us to evaluate the trace
\[
T^{\text{quantum}}_{\sigma_+} = \epsilon^{i_{12} \ldots i_{2N-1} i_{2N}} \text{tr}_{\sigma_+}[X_{i_1j_1} X_{i_2j_2} \ldots X_{i_{2N-1}j_{2N}}]
\]

\[
(C.1)
\]
where

\[ \hat{X}_{ij} = \frac{1}{4i} P_{\sigma_i} [\Gamma_i, \Gamma_j] P_{\sigma_j} \]  

is the matrix representing the spin(2N) generator \( X_{ij} \) in the positive-helicity representation \( \sigma_+ \). Using

\[ \Gamma_1 \Gamma_2 \cdots \Gamma_{2N+1} = i^{N} \mathbb{1}_{2N} \]  

we found that

\[ T^\text{quantum}_{\sigma_+} = (2N)!/2. \]  

For the classical calculation we need to evaluate the corresponding phase-space integral

\[ T^\text{classical}_{\sigma_+} = \int_{\mathcal{O}_{\sigma_+}} \epsilon^{i_1 i_2 \cdots i_{2N}} \mathcal{S}_{i_1 i_2} \mathcal{S}_{i_3 i_4} \cdots \mathcal{S}_{i_{2N-1} i_{2N}} d\bar{\mu}_{\sigma_+}. \]  

To do this we begin by summarizing some basic facts about the Lie algebra of spin \( (2N) \), which is the double cover of SO \((2N)\) and so shares the same Lie algebra.

The Hermitian matrix generators in the defining vector representation of SO \((2N)\) are

\[ X_{ij} = -i (e_{ij} - e_{ji}). \]

Here \( e_{ij} \) is the matrix with unity at site \( i, j \), so that

\[ e_{ij} e_{ik} e_{kj} = \delta_{jk}. \]  

Thus the commutation relations are

\[ [X_{ij}, X_{mn}] = i (\delta_{mn} X_{jm} - \delta_{mj} X_{jm} - X_{mj} \delta_{jm} + X_{jm} \delta_{jm}). \]  

These relations can be understood as saying that \( X_{mn} \) transforms a skew 2-tensor under the adjoint action of the group.

With traces taken in the defining vector representation, we have

\[ \text{tr} \{X_{ij} X_{kl}\} = 2(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}). \]

If we take as our basis set the \( X_{ij} \) with \( i < j \), the second term in this trace is always zero. The spin(2N) analogue of the metric \((17)\) is therefore

\[ g_{ij,kl} = \text{tr} \{X_{ij} X_{kl}\} = 2\delta_{ik}\delta_{jl}. \]

The Cartan algebra is generated by

\[ h_n = X_{2n-1,2n}, \quad n = 1, \ldots, N, \]

and a general weight will be of the form \( \omega = (m_1, m_2, \ldots, m_N) \) where \( m_n \) is the eigenvalue of \( h_n \). The roots are \( \pm e_i, \pm e_j \) where \( e_i = (1, 0, \ldots, 0) \) etc. In this basis the \( N \) fundamental weights are

\[ \omega_1 = (1, 0, \ldots, 0, 0, 0), \]
\[ \omega_2 = (1, 1, \ldots, 0, 0, 0), \]
\[ \vdots \]
\[ \omega_{N-2} = (1, 1, \ldots 1, 0, 0), \]
\[ \omega_{N-1} \equiv \sigma_+ = \frac{1}{2} (1, 1, \ldots, 1, 1, 1), \]
\[ \omega_N \equiv \sigma_- = \frac{1}{2} (1, 1, \ldots, 1, 1, -1). \]

The highest weight of any unitary representation is a positive-integer linear combination of the fundamental weights. The weight \( \omega_i \) is that of the \( 2N \)-dimensional defining vector representation of SO(2N). The last two fundamental weights are those of the two inequivalent spin representations of spin(2N). They are not representations of SO(2N). The Weyl vector \( \rho \)
is defined to be half the sum of the positive roots, or equivalently the sum of the fundamental weights, and is therefore given by
\[ \rho = (N - 1, N - 2, \ldots, 1, 0). \]  
(C.11)

Consider the representation with highest weight \( s \) and highest weight vector \( |s\rangle \). The Cartan algebra element \( \beta_s \) of equation (20) is defined so that
\[ \text{tr}(\beta_s X) = \langle s | X | s \rangle \]  
(C.12)
for any \( X \) in the Lie algebra. In the representation \( \sigma_+ \) with weight \( s = \sigma_+ \), we have
\[ \langle \sigma_+ | X_{12} | \sigma_+ \rangle = \langle \sigma_+ | X_{13} | \sigma_+ \rangle = \cdots = \langle \sigma_+ | X_{2N-1,2N} | \sigma_+ \rangle = \frac{1}{2} \]  
(C.13)
with all non-Cartan elements giving zero. Thus
\[ \beta_{\sigma_+} = \frac{1}{2} (X_{12} + X_{13} + \cdots + X_{2N-1,2N}). \]  
(C.14)
The orbit co-ordinates are \( \Theta^{ij}, i < j \) given by
\[ \sum_{i<j} \Theta^{ij} X_{ij} = \sigma \beta_{\sigma_+} \sigma^{-1}. \]  
(C.15)
In the Weyl chamber \( \Theta^{12} = \Theta^{34} = \cdots = \Theta^{2N-1,2N} = 1/4 \) with all other components being zero. Using the metric to lower the indices gives
\[ \Theta_{12} = \Theta_{34} = \cdots = \Theta_{2N-1,2N} = 1/2. \]  
(C.16)
We know therefore that
\[ C_{\sigma_+}^{\text{geom}} \defeq \epsilon^{i_1 \ldots i_{2N}} \Theta_{i_1 i_2} \cdots \Theta_{i_{2N-1} i_{2N}} = N! \]  
(C.17)
at the point in which \( \mathcal{O}_{\sigma_+} \) intersects the Weyl chamber. Now \( \epsilon^{i_1 \ldots i_{2N}} \) is an invariant tensor for the adjoint action and so \( C_{\sigma_+}^{\text{geom}} \) is a classical Casimir invariant and takes the same value everywhere on the orbit. Thus we find that
\[ T_{\alpha_{\sigma_+}}^{\text{classical}} \defeq \frac{2}{N!} \int_{\mathcal{O}_{\sigma_+}} \epsilon^{i_1 \ldots i_{2N}} \Theta_{i_1 i_2} \cdots \Theta_{i_{2N-1} i_{2N}} d\mu_{\sigma_+} = \text{Vol}(\mathcal{O}_{\sigma_+}) N!. \]  
(C.18)
In the case of a representation with weight \( s = (m_1, m_2, \ldots, m_n) \) we have \( \Theta_{12} = m_1, \Theta_{34} = m_2, \) etc so this integral generalizes to
\[ T_{\alpha_s}^{\text{classical}} \defeq \frac{2}{N!} \int_{\mathcal{O}_s} \epsilon^{i_1 \ldots i_{2N}} \Theta_{i_1 i_2} \cdots \Theta_{i_{2N-1} i_{2N}} d\mu_s = \text{Vol}(\mathcal{O}_s) 2^N N! m_1 m_2 \cdots m_n. \]  
(C.19)
The reason for the ‘?’ over the equal signs is that \( \text{Vol}(\mathcal{O}_s) \) should correspond to \( \text{tr}_{\mathcal{O}_s}(I_{\dim(s)}) = \dim(s) \), and since \( \dim(\sigma_+) = 2^{N-1} \) the right-hand side of (C.18) would give \( 2^{N-1} N! \). This is not a very good approximation \( (2N)!/2 \).

It turns out that the naïve orbit volumes are often poor approximations to the dimension of the representation—particularly on the edges of the Weyl Chamber where the orbit degenerates and reduces in dimension. It is, however, a corollary of the Kirillov character formula [12] that the volume of the orbit corresponding the Weyl-shifted weight \( s \rightarrow s + \rho \) gives the correct dimension:
\[ \int_{\mathcal{O}_{s+\rho}} d\mu_{s+\rho} = \text{Vol}(\mathcal{O}_{s+\rho}) = \dim(s). \]  
(C.20)
The Weyl shift is a quantum correction that accounts for normal-ordering of generators in the Casimir operators. In the spin-\( j \) representation of \( SU(2) \) for example, the Weyl shift changes the orbit volume from \( 2j \) to \( 2j + 1 \). For large spins this is a negligible effect, but it is important for the fundamental spin representation in high dimensions.
Suppose we make a similar Weyl shift in (C.19) so that it is replaced by
\[ T_{\text{classical}} = \int_{O_{\theta,\varphi}} \epsilon_{i_1 \cdots i_{2N}} \mathcal{S}_{i_1 i_2} \cdots \mathcal{S}_{i_{2N-1} i_{2N}} d\mu_{\sigma + \rho} \]
\[ = \dim(s) 2^N (m_1 + N - 1)(m_2 + N - 2) \cdots m_1. \quad (C.21)\]

The formulae for the spin\((2N)\) Casimir operators in \[24\] now show that the classical trace formula (C.21) coincides with the quantum trace for all \(s\). In particular (C.18) becomes
\[ T_{\text{classical}} = \int_{\mathcal{O}_{\alpha,\beta}} \epsilon_{i_1 \cdots i_{2N}} \mathcal{S}_{i_1 i_2} \cdots \mathcal{S}_{i_{2N-1} i_{2N}} d\mu_{\sigma + \rho} \]
\[ = 2^{N-1} (2^N N!) (N - N) (N - 2) \cdots 1 \]
\[ = (2N)!/2, \]
\[ = T_{\text{quantum}}. \quad (C.22)\]

There are other terms involving \(F_n, n < N\), in the expansion of \(\omega_H\), but the structure of \(F_n\) (skew-symmetric form indices coincide with the Lie algebra indices) requires all pairs of \(i, j\) indices on the \(S_{ij}\) in these products to be distinct. Now \(\epsilon_{i_1 \cdots i_{2N}}\) is the only adjoint-action invariant tensor that allows for distinct indices. All other invariant tensors are sums of products of \(\delta_{ij}\), and so give zero unless pairs of indices coincide. As a consequence no other trace, either quantum or classical, can make a contribution to the anomaly.

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