Finite $N$-dependent Multiplicative-Noise Contributions in Finite $N$-unit Langevin Models: Augmented Moment Approach

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Hideo Hasegawa

Department of Physics, Tokyo Gakugei University,
Koganei, Tokyo 184-8501, Japan

Abstract

Finite $N$-unit Langevin models with additive and multiplicative noises have been studied with the use of the augmented moment method (AMM) previously proposed by the author [H. Hasegawa, Phys. Rev E 67, 041903 (2003)]. Original $N$-dimensional stochastic equations are transformed to the three-dimensional deterministic equations for means and fluctuations of local and global variables. Calculated results of our AMM are in good agreement with those of direct simulations (DS). We have shown that although the effective strength of the additive noise of the $N$-unit system is scaled as $\beta(N) = \beta(1)/\sqrt{N}$, it is not the case for multiplicative noise $[\alpha(N) \neq \alpha(1)/\sqrt{N}]$, where $\alpha(N)$ and $\beta(N)$ denote the strength of multiplicative and additive noises, respectively, for the size-$N$ system. It has been pointed out that the naive assumption of $\alpha(N) = \alpha(1)/\sqrt{N}$ leads to result which violates the central-limit theorem and which does not agree with those of DS and AMM.
It has been recognized that stochastic (Langevin) equations subjected to additive and/or multiplicative noises may be good models for discussing the property of many systems not only in physics but also in biology, chemistry, economy and networks. In recent years, much attention has been paid to multiplicative noise whose interesting aspects have been intensively investigated (for a recent review, see ref. 1: related references therein). It has been shown that multiplicative noise induces the phase transition where multiplicative noise creates an ordered state while additive noise generally works to destroy the ordered state. The effect of multiplicative noise in Langevin equation has been discussed in connection with the non-extensive thermodynamics [2, 3]: non-Gaussian distribution is realized for Gaussian multiplicative noise [4, 5, 6].

The property of multiplicative noise is different from that of additive noise in some respects. In order to make our discussion concrete, let’s consider the simple, linear Langevin model given by

\[
\frac{d x_i(t)}{dt} = -\lambda x_i(t) + \alpha x_i(t) \eta_i(t) + \beta \xi_i(t), \quad (i = 1 - N)
\]  

where \(\lambda\) denotes the relaxation rate, \(\alpha\) and \(\beta\) the strengths of multiplicative and additive Gaussian white noises, respectively, given by zero-mean random variables \(\eta_i(t)\) and \(\xi_i(t)\) with \(\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t - t')\). The system given by eq. (1) has been studied by using the Fokker-Plank equation (FPE) mainly for \(N = 1\) or \(N = \infty\): for the latter case, the mean-field and diffusion approximations are employed [1, 7]. In order to discuss the dynamics of the finite-\(N\) Langevin model, we define the global variable given by

\[
X(t) = (\frac{1}{N}) \sum_i x_i(t),
\]

and sum eq. (1) over \(i\) to get

\[
\frac{dX(t)}{dt} = -\lambda X(t) + \alpha X(t) \left( \frac{\sum_i x_i(t) \eta_i(t)}{\sum_i x_i(t)} \right) + \frac{\beta}{\sqrt{N}} \sum_i \xi_i(t).
\]

(2)

It is reasonable to replace the additive noise (third) term in eq. (2) by \(\beta(N)\xi\) where \(\xi\) denotes the white noise with

\[
\beta(N)^2 \equiv \left\langle \left( \frac{\beta}{N} \sum_i \xi_i \right)^2 \right\rangle = \frac{\beta^2}{N}.
\]

(3)

As for the multiplicative noise (second) term in eq. (2), Muñoz, Colaiori and Castellano claimed in a recent paper [8] that it may be replaced by \(\alpha(N)X \eta\) with the white noise \(\eta\) in the weak-noise case with

\[
\alpha(N)^2 \equiv \left\langle \left( \alpha \frac{\sum_i x_i \eta_i}{\sum_i x_i} \right)^2 \right\rangle = \frac{\alpha^2}{N},
\]

(4)

if \(x_i\) and \(\eta_i\) are uncorrelated. Equations (3) and (4) lead to the Langevin equation for \(X\) given by

\[
\frac{dX(t)}{dt} = -\lambda X(t) + \frac{\alpha}{\sqrt{N}} X(t) \eta(t) + \frac{\beta}{\sqrt{N}} \xi(t),
\]

(5)

from which the FPE for the finite-\(N\) system is easily obtained. The assumption given by eq. (4) naively seems reasonable. It may be, however, not justified because \(x_i\) and \(\eta_i\) are correlated: \(x_i\) may be increased (decreased) for \(x_i > 0\) \((x_i < 0)\) by a positive \(\eta_i\) in eq. (1).
The purpose of this letter is to examine the validity of the approximation given by eq. (4). We will employ the augmented moment method (AMM) (or dynamical mean-field approximation) developed by the author, which has been successfully applied to a study of coupled stochastic systems [9]. In this approach, we have discussed dynamics of coupled stochastic systems in terms of a fairly small number of quantities relevant to local and global variables, by transforming original stochastic equations to deterministic equations. It is essentially the second-order moment method both for local and global variables. In ref. 9, we obtained equations of motions for means, variances and covariances, by expanding variables around their mean values. We have adopted, in this letter, an alternative FPE method to obtain equations of motions of the quantities, in order to avoid the difficulty due to the Ito versus Stratonovich calculus inherent for multiplicative noise. Numerical results calculated by using our AMM approach are in good agreement with direct simulations (DS) for the Langevin equation given by eq. (1). On the contrary, the result obtained from the FPE using the approximation given by eq. (4) disagrees with those of DS and AMM.

We have adopted the finite $N$-unit general Langevin model given by

$$
\frac{dx_i}{dt} = F(x_i) + \alpha G(x_i) \eta_i(t) + \beta \xi_i(t) + I^{(c)}_i(t), \quad (i = 1 - N) \tag{6}
$$

with

$$
I^{(c)}_i(t) = \frac{w}{Z} \sum_{k(\neq i)} [x_k(t) - x_i(t)], \tag{7}
$$

where $F(x)$ and $G(x)$ denote arbitrary functions of $x$, $w$ the coupling strength, $Z = N - 1$, $\alpha$ and $\beta$ denote the strengths of multiplicative and additive noises, respectively, and $\eta_i(t)$ and $\xi_i(t)$ express zero-mean Gaussian white noises with correlations given by

$$
\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t - t'), \quad (8)
$$

$$
\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t - t'), \quad (9)
$$

$$
\langle \eta_i(t) \xi_j(t') \rangle = 0. \quad (10)
$$

The Fokker-Planck equation for the distribution of $p(\{x_i\}, t)$ is given by [10]

$$
\frac{\partial}{\partial t} p(\{x_i\}, t) = -\sum_k \frac{\partial}{\partial x_k} \{ [F(x_k) + \frac{\phi \alpha^2}{2} G'(x_k) G(x_k) + I^{(c)}_k] p(\{x_i\}, t) \}
$$

$$
+ \frac{1}{2} \sum_k \frac{\partial^2}{\partial x_k^2} \{ [\alpha^2 G(x_k)^2 + \beta^2] p(\{x_i\}, t) \}, \tag{11}
$$

where $G'(x) = dG(x)/dx$, and $\phi = 1$ and 0 in the Stratonovich and Ito representations, respectively.

When we consider the averaged, global variable $X$ given by

$$
X(t) = \frac{1}{N} \sum_i x_i(t), \tag{12}
$$

the Fokker-Planck equation $P(X, t)$ for $X$ is given by

$$
P(X, t) = \int \cdots \int \Pi_i dx_i p(\{x_i\}, t) \delta \left( X - \frac{1}{N} \sum_i x_i \right). \tag{13}
$$
The \( k \)-th moments of local and global variables are defined by

\[
\langle x_i^k \rangle = \int \Pi_i \, dx_i \, p(x_i, t) \, x_i^k, \quad (k = 1, 2, \cdots)
\]

By using eqs. (11) and (14), we get equations of motions for mean, variance and covariance of local variable \( x_i \) given by

\[
\begin{align*}
\frac{d\langle x_i \rangle}{dt} &= \langle F(x_i) \rangle + \frac{w}{Z} \sum_k \langle (x_k) - \langle x_i \rangle \rangle + \frac{\phi \alpha^2}{2} \langle G'(x_i) G(x_i) \rangle, \\
\frac{d\langle x_i x_j \rangle}{dt} &= \langle x_i F(x_j) \rangle + \langle x_j F(x_i) \rangle + \frac{w}{Z} \sum_k \left[ (\langle x_i x_k \rangle + \langle x_j x_k \rangle - \langle x_i^2 \rangle - \langle x_j^2 \rangle) \right] \\
&\quad + \frac{\phi \alpha^2}{2} \left[ (\langle x_i G'(x_j) G(x_j) \rangle + \langle x_j G'(x_i) G(x_i) \rangle) \right] \\
&\quad + \left[ \alpha^2 \langle G(x_i)^2 \rangle + \beta^2 \right] \delta_{ij}.
\end{align*}
\]

Equations of motions of mean and variance of global variable \( X \) are obtainable by using eqs. (12), (16)-(17):

\[
\begin{align*}
\frac{d\langle X \rangle}{dt} &= \frac{1}{N} \sum_i \frac{d\langle x_i \rangle}{dt}, \\
\frac{d\langle X^2 \rangle}{dt} &= \frac{1}{N^2} \sum_i \sum_j \frac{d\langle x_i x_j \rangle}{dt}.
\end{align*}
\]

The mean-field approximation employs only eq. (16) [11]. Equations (16) and (17) are adopted for a discussion on the fluctuation-induced phase transition in infinite-\( N \) stochastic systems [12]. Equations (18) and (19) play a crucial role in discussing finite-\( N \) systems, as will be shown below.

In the AMM [9], we define the three quantities, \( \mu \), \( \gamma \) and \( \rho \), given by

\[
\begin{align*}
\mu &= \langle X \rangle = \frac{1}{N} \sum_i < x_i >, \\
\gamma &= \frac{1}{N} \sum_i < (x_i - \mu)^2 >, \\
\rho &= < (X - \mu)^2 >.
\end{align*}
\]

It is noted that \( \gamma \) expresses the averaged fluctuations in local variables \( x_i \) while \( \rho \) denotes fluctuations in global variable \( X \). Expanding \( x_i \) in eqs. (16)-(19) around the average value of \( \mu \) as

\[
x_i = \mu + \delta x_i,
\]

and retaining up to the order of \( < (\delta x_i)^2 > \), we get equations of motions for \( \mu \), \( \gamma \) and \( \rho \) given by

\[
\begin{align*}
\frac{d\mu}{dt} &= f_0 + f_2 \gamma + \left( \frac{\phi \alpha^2}{2} \right) [g_0 g_1 + 3(g_1 g_2 + g_0 g_3) \gamma],
\end{align*}
\]
\[
\begin{align*}
\frac{d\gamma}{dt} &= 2f_1\gamma + \left(\frac{2wN}{Z}\right)(\rho - \gamma) + (\phi + 1)(g_1^2 + 2g_0g_2)\alpha^2\gamma + \alpha^2g_0^2 + \beta^2, \\
\frac{d\rho}{dt} &= 2f_1\rho + (\phi + 1)(g_1^2 + 2g_0g_2)\alpha^2\rho + \frac{\alpha^2g_0^2}{N} + \frac{\beta^2}{N},
\end{align*}
\]

where \(f_\ell = (1/\ell!)[\partial^\ell F(\mu)/\partial x^\ell\partial x^\ell\partial x^\ell\partial x^\ell]\) and \(g_\ell = (1/\ell!)[\partial^\ell G(\mu)/\partial x^\ell\partial x^\ell\partial x^\ell\partial x^\ell]\). Original \(N\)-dimensional stochastic equations given by eq. (6) is transformed to three-dimensional deterministic equations given by eqs. (24)-(26). In the limit of \(\alpha = 0\), they reduce to those obtained previously for the Langevin model subjected only to additive noises [eqs. (7)-(9) in ref. 9c].

It is easy to see that the stationary solutions of \(\gamma\) and \(\rho\) for \(w = 0\) satisfy the central-limit theorem:

\[
\begin{align*}
\rho &= \frac{\gamma}{N}, \\
\gamma &= \frac{\alpha^2g_0^2 + \beta^2}{[-2f_1 - (1 + \phi)(g_1^2 + 2g_0g_2)\alpha^2]}.
\end{align*}
\]

It is noted that even for \(N = \infty\), local fluctuations exist \((\gamma \neq 0)\), although global fluctuations vanish \((\rho = 0)\).

Now we consider the result when the approximation given by eq. (4) is employed (such result is referred to as the APP hereafter). The APP leads to equations of motion for \(\mu\) and \(\rho\) given by

\[
\begin{align*}
\frac{d\mu}{dt} &= f_0 + f_2\gamma + \left(\frac{\phi}{2N}\right)[g_0g_1 + 3(g_2 + g_3)\gamma], \\
\frac{d\rho}{dt} &= 2f_1\rho + (1 + \phi)(g_1^2 + 2g_0g_2)\alpha^2\rho + \frac{\alpha^2g_0^2}{N} + \frac{\beta^2}{N}.
\end{align*}
\]

We note that the third term of eq. (29) and the second term of eq. (30) are different from their counterparts in eqs. (24) and (26). The ratio between \(\rho\) and \(\gamma\) in the stationary state becomes

\[
\begin{align*}
\frac{\rho^{\text{APP}}}{\gamma^{\text{APP}}} &= \frac{1}{N}\left(\frac{-2f_1 - (1 + \phi)(g_1^2 + 2g_0g_2)\alpha^2}{-2f_1 - (1 + \phi)(g_1^2 + 2g_0g_2)\alpha^2/N}\right), \\
&\to \frac{1}{N}\left(\frac{-2f_1 - (1 + \phi)(g_1^2 + 2g_0g_2)\alpha^2}{-2f_1}\right), \quad \text{as } N \to \infty
\end{align*}
\]

which is in contradiction to the central-limit theorem.

In order to make numerical calculations, we have adopted the linear Langevin model given by eq. (1):

\[
F(x) = -\lambda x, \quad G(x) = x.
\]

In the Stratonovich representation, equations of motion given by eqs. (24)-(26) become

\[
\begin{align*}
\frac{d\mu}{dt} &= -\lambda\mu + \frac{\alpha^2\mu}{2}, \\
\frac{d\gamma}{dt} &= -2\lambda\gamma + \frac{2wN}{Z}N(\rho - \gamma) + 2\alpha^2\gamma + \alpha^2\mu^2 + \beta^2, \\
\frac{d\rho}{dt} &= -2\lambda\rho + 2\alpha^2\rho + \frac{\alpha^2\mu^2}{N} + \frac{\beta^2}{N}.
\end{align*}
\]
The stationary values for \( w = 0 \) are given by

\[
\begin{align*}
\mu &= 0, \\
\gamma &= \frac{\beta^2}{2(\lambda - \alpha^2)}, \\
\rho &= \frac{\gamma}{N}.
\end{align*}
\]  

(37) \hspace{1cm} (38) \hspace{1cm} (39)

Equation (38) shows that \( \alpha^2 < \lambda \) because \( \gamma > 0 \).

The probability \( p(x) \) for the linear Langevin model \( (N = 1) \) in the stationary state is given by \([\text{1, 5}]\)

\[
p(x) = \frac{1}{Z} \left[ 1 - (1 - q)rx^2 \right]^{\frac{1}{1-q}},
\]

(40)

with

\[
\begin{align*}
Z &= \frac{1}{\sqrt{(q-1)r}} B \left( \frac{1}{2}, \frac{1}{q-1} - \frac{1}{2} \right), \\
r &= \frac{2\lambda + \alpha^2}{2\beta^2}, \\
q &= \frac{2\lambda + 3\alpha^2}{2\lambda + \alpha^2}.
\end{align*}
\]

(41) \hspace{1cm} (42) \hspace{1cm} (43)

where \( B(a, b) \) is the beta function. By using \( p(x) \) given by eqs. (40)-(43) with eq. (14), the first and second moments are given by

\[
\begin{align*}
< x > &= 0, \\
< x^2 > &= \left[ \frac{1}{(q-1)r} \right] B \left( \frac{3}{2}, \frac{1}{q-1} - \frac{3}{2} \right) B \left( \frac{1}{2}, \frac{1}{q-1} - \frac{1}{2} \right), \\
&= \frac{1}{r(5-3q)} = \frac{\beta^2}{2(\lambda - \alpha^2)},
\end{align*}
\]

(44) \hspace{1cm} (45) \hspace{1cm} (46)

which agree with the result given by eqs. (37) and (38).

If we employ the APP, eqs. (30) yields

\[
\begin{align*}
\rho^{\text{APP}} &= \frac{\beta^2/2N}{\lambda - \alpha^2/N}, \\
&\to \frac{\beta^2}{2N\lambda}, \quad \text{(as } N \to \infty) \\
\frac{\rho^{\text{APP}}}{\gamma^{\text{APP}}} &= \frac{1}{N} \left( \frac{\lambda - \alpha^2}{\lambda} \right), \\
&\to \frac{1}{N} \left( \frac{\lambda - \alpha^2}{\lambda} \right), \quad \text{(as } N \to \infty)
\end{align*}
\]

(47) \hspace{1cm} (48) \hspace{1cm} (49) \hspace{1cm} (50)

The value of \( \rho^{\text{APP}} \) in eq. (48) is independent of \( \alpha \) for \( N \to \infty \). Equation (50) is in contradiction to the central-limit theorem.
In order to examine the validity of our AMM approach, we have made direct simulations for the $N$-unit Langevin model given by eq. (1). The Heun method is employed with a time step of 0.0001. Figure 1 shows the $N$ dependence of the stationary value of $\rho$ for the three sets of parameters: (1) $\alpha = 0.8$, $\beta = 1.0$, (2) $\alpha = 0.5$, $\beta = 1.0$, and (3) $\alpha = 0.5$, $\beta = 0.5$ with $\lambda = 1$ and $w = 0$. Squares, circles and triangles show results of direct simulations (DS) for the sets (1), (2) and (3), respectively. We note that $\rho$ follows the relation: $\rho \propto N^{-1}$. In the case of $w = 0$ under consideration, we get $\mu = 0$ and $\gamma(N) = \rho(N = 1)$ independent of $N$. Solid curves show the result of the AMM, which are in good agreement with those of DS. In contrast, the result with the APP shown by dotted curves disagrees with the result of DS and AMM. For $N > 10$, $\rho_{APP}$ for the sets (1) and (2) becomes almost independent of $\alpha$ as eq. (48) shows.

We have tried to obtain the Fokker-Plank equation $P(X,t)$ for the global variable $X$. We assume that the FPE for $P(X,t)$ is expressed by

$$
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial X} \left\{ \left[ F(X) + \frac{(\phi + A)a^2}{2}G'(X)G(X) \right] P \right\} + \frac{\partial^2}{\partial X^2} \left[ \left( \frac{B\alpha^2}{2}G(X)^2 + C\beta^2 \right) P \right],
$$

where $\phi = 1$ and 0 in the Stratonovich and Ito representations, respectively, and $A$, $B$ and $C$ are parameters whose values will be determined such as to yield the correct first and second moments, as shown below. By using $P(X,t)$ in eq. (51), we get the equation of motion for $\mu$ and $\rho$ given by

$$
\frac{d\mu}{dt} = f_0 + f_2\gamma + \left( \frac{(\phi + A)a^2}{2} \right) [g_0g_1 + 3(g_1g_2 + g_0g_3)\gamma],
$$

$$
\frac{d\rho}{dt} = 2f_1\rho + (\phi + A + B)(g_1^2 + 2g_0g_2)a^2\rho + B\alpha^2g_0^2 + C\beta^2.
$$

A comparison between eqs. (52) and (53) with eqs. (24) and (26) leads to

$$
\phi + A = \phi,
$$

$$
\phi + A + B = \phi + 1,
$$

$$
B = C = \frac{1}{N}.
$$

Unfortunately, conditions (54)-(56) are satisfied only for $N = 1$ with $A = 0$ and $B = C = 1$, but have no solutions for $N \neq 1$. This implies that the FPE for the global variable $X$ in $N$-unit Langevin model is not expressed by the form given by eq. (51) with $A \propto N^{-\delta}$ for $N > 1$ ($\delta$: an index). It is probable that $\sum_i x_i(t) \eta_i(t)$ in the second term of eq. (2) may yield non-Gaussian multiplicative noise because the distribution of $x_i(t)$ does not follow Gaussian.

To summarize, we have studied finite $N$-unit Langevin model including additive and multiplicative noises by using AMM [9], which has been reformulated with the use of FPE. It has been pointed out that the scaling assumption given by eq. (4) adopted in a recent paper (ref. 4) leads to results violating the central-limit theorem [eq. 32] and
disagreeing with those of DS and AMM. The scaling relation of the effective strength against $N$ of multiplicative noise is different from that of additive noise. Our AMM may be applied to the general Langevin model with arbitrary forms of $F(x)$ and $G(x)$, and also to multi-variable stochastic models like FitzHugh-Nagumo neuronal model.

A disadvantage of our AMM is that its applicability is limited to weak-noise cases. On the contrary, an advantage of the AMM is that we can easily discuss dynamical property of the finite $N$-unit Langevin system by solving the three-dimensional ordinary differential equations. In contrast, within direct simulation and the FPE approach we have to solve the $N$-dimensional stochastic Langevin equations and the $(N+1)$-dimensional partial differential equations, respectively. Although the discussion presented in this letter is confined to the stationary solution of equations of motions given by eqs. (24)-(26) [or eqs. (34)-(36)], it is possible to discuss the dynamical property of the coupled Langevin model by solving them. Actually in our previous papers [9], we have studied the $N$-dependent synchronization in networks described by Langevin, FitzHugh-Nagumo and Hodgkin-Huxley models subjected only to additive noises with global, local or small-world couplings (with and without transmission delays). It is interesting to make such calculations by including also multiplicative noises, which is left our future study.

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Figure 1: (Color online) The $N$ dependence of the stationary $\rho$, fluctuations in the global variable of $X$, for the three sets of parameters: (1) $\alpha = 0.8$, $\beta = 1.0$, (2) $\alpha = 0.5$, $\beta = 1.0$, and (3) $\alpha = 0.5$, $\beta = 0.5$ with $\lambda = 1$ and $w = 0$, calculated by direct simulations with 1000 trials (DS: squares, circles and triangles for (1), (2) and (3), respectively), the AMM (solid curves), and the APP (dotted curves), bars denoting the root-mean-square values of DS results.
\[ \log_{10}(N) \]

\[ \log_{10}(\rho) \]

\[ \alpha = 0.5, \beta = 1.0 \]
\[ \alpha = 0.5, \beta = 0.5 \]
\[ \alpha = 0.8, \beta = 1.0 \]

AMM

APP

DS