SOME REMARKS ON BI-\(f\)-HARMONIC MAPS AND \(f\)-BIHARMONIC MAPS

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Abstract

In this paper, we prove that the class of bi-\(f\)-harmonic maps and that of \(f\)-biharmonic maps from a conformal manifold of dimension \(\neq 2\) are the same (Theorem 1.1). We also give several results on nonexistence of proper bi-\(f\)-harmonic maps and \(f\)-biharmonic maps from complete Riemannian manifolds into non-positively curved Riemannian manifolds. These include: any bi-\(f\)-harmonic map from a compact manifold into a non-positively curved manifold is \(f\)-harmonic (Theorem 1.6), and any \(f\)-biharmonic (respectively, bi-\(f\)-harmonic) map with bounded \(f\) and bounded \(f\)-bienrgy (respectively, bi-\(f\)-energy) from a complete Riemannian manifold into a manifold of strictly negative curvature has rank \(\leq 1\) everywhere (Theorems 2.2 and 3.3).

1. A RELATIONSHIP BETWEEN \(f\)-BIHARMONIC MAPS AND BI-\(f\)-HARMONIC MAPS

We assume all objects studied in this paper, including manifolds, tension fields, and maps, are smooth unless they are stated otherwise.

Harmonic maps are critical points of the energy functional

\[
E(\phi) = \frac{1}{2} \int_\Omega |d\phi|^2 v_g
\]

for maps \(\phi : (M, g) \longrightarrow (N, h)\) between Riemannian manifolds and for all compact domain \(\Omega \subseteq M\). Harmonic map equation is simply the Euler Lagrange equation of the energy functional which is given (see [ES]) by

\[
\tau(\phi) \equiv \text{Tr}_g \nabla d\phi = 0,
\]
where $\tau(\phi) = \text{Tr}_g \nabla d\phi$ is the tension field of the map $\phi$.

**Biharmonic maps** are critical points of the bienergy functional defined by

$$E_2(\phi) = \frac{1}{2} \int_{\Omega} |\tau(\phi)|^2 v_g,$$

where $\Omega$ is a compact domain of $M$ and $\tau(\phi)$ is the tension field of the map $\phi$. **Biharmonic map equation** (see [Ji]) is the Euler-Lagrange equation of the bienergy functional, which can be written as

$$\tau_2(\phi) := \text{Tr}_g[(\nabla^\phi \nabla^\phi - \nabla_{\nabla^M})\tau(\phi) - R^N(d\phi, \tau(\phi))d\phi] = 0,$$

where $R^N$ denotes the curvature operator of $(N, h)$ defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X, Y]}Z.$$

Clearly, any harmonic map is always a biharmonic map.

**$f$-Harmonic maps** (see [Li]) are critical points of the $f$-energy functional defined by

$$E_f(\phi) = \frac{1}{2} \int_{\Omega} f |d\phi|^2 v_g,$$

where $\Omega$ is a compact domain of $M$ and $f$ is a positive function on $M$. **$f$-Harmonic map equation** can be written (see, e.g., [Co, OND]) as

$$\tau_f(\phi) \equiv f\tau(\phi) + d\phi(\text{grad } f) = 0,$$

where $\tau(\phi) = \text{Tr}_g \nabla d\phi$ is the tension field of $\phi$. It is easily seen that an $f$-harmonic map with $f = C$ is nothing but a harmonic map.

**$f$-Biharmonic maps** are critical points of the $f$-bienergy functional

$$E_{f, 2}(\phi) = \frac{1}{2} \int_{\Omega} f |\tau(\phi)|^2 v_g,$$

for maps $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds and all compact domain $\Omega \subseteq M$. **$f$-Biharmonic map equation** can be written as (see [Lu])

$$\tau_{f, 2}(\phi) \equiv f\tau_2(\phi) + (\Delta f)\tau(\phi) + 2\nabla_{\text{grad } f} \tau(\phi) = 0,$$

where $\tau(\phi)$ and $\tau_2(\phi)$ are the tension and the bitension fields of $\phi$ respectively.

**Bi-$f$-harmonic maps** $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds which are critical points of the bi-$f$-energy functional

$$E_{2, f}(\phi) = \frac{1}{2} \int_{\Omega} |\tau_f(\phi)|^2 v_g,$$
over all compact domain $\Omega$ of $M$. The Euler-Lagrange equation gives the bi-$f$-harmonic map equation (OND)
\begin{equation}
(6) \quad \tau_{2,f}(\phi) \equiv -fJ^\phi(\tau_f(\phi)) + \nabla_{\text{grad } f}^\phi \tau_f(\phi) = 0,
\end{equation}
where $\tau_f(\phi)$ is the $f$-tension field of the map $\phi$ and $J^\phi$ is the Jacobi operator of the map defined by $J^\phi(X) = -\text{Tr}_g[\nabla^\phi \nabla^\phi X - \nabla^\phi_{\nabla^\phi X} X - R^N(d\phi, X)d\phi]$.

We would like to point out that in some literature, e.g., [OND], [Ch], the name “$f$-biharmonic maps” was also used for the critical points of the bi-$f$-energy functional (5).

From the above definitions, we can easily see the following inclusion relationships among the different types of biharmonic maps which give two different paths of generalizations of harmonic maps.

\begin{align*}
\{\text{Harmonic maps}\} & \subset \{\text{Biharmonic maps}\} \subset \{f -\text{Biharmonic maps}\}, \\
\{\text{Harmonic maps}\} & \subset \{f -\text{harmonic maps}\} \subset \{\text{Bi} - f -\text{harmonic maps}\}.
\end{align*}

For the obvious reason, we will call a bi-$f$-harmonic map which is not $f$-harmonic a proper bi-$f$-harmonic map, and an $f$-biharmonic map which is not harmonic a proper $f$-biharmonic map.

Concerning harmonicity and $f$-harmonicity, we have a classical result of Lichnerowicz [Li] which states that a map $\phi : (M^m, g) \to (N^n, h)$ with $m \neq 2$ is an $f$-harmonic map if and only if it is a harmonic map with respect to a metric conformal to $g$. Concerning the relationship between biharmonic maps and $f$-biharmonic maps, it was proved in [Ou2] that a map $\phi : (M^2, g) \to (N^n, h)$ from a 2-dimensional manifold is an $f$-biharmonic map if and only if it is a biharmonic map with respect to the metric $\bar{g} = f^{-1}g$ conformal to $g$.

Our first theorem shows that although $f$-biharmonic maps and bi-$f$-harmonic maps are quite different by their definitions, they belong to the same class in the sense that any bi-$f$-harmonic map $\phi : (M^m, g) \to (N^n, h)$ with $m \neq 2$ and $f = \alpha$ is an $f$-biharmonic map for $f = \alpha^{m/(m-2)}$ with respect to some metric conformal to $g$, and conversely, any $f$-biharmonic map $\phi : (M^m, g) \to (N^n, h)$ with $m \neq 2$ and $f = \alpha$ is a bi-$f$-harmonic map for $f = \alpha^{(m-2)/m}$ with respect to some metric conformal to $g$.

**Theorem 1.1.** For $m \neq 2$ and a map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds, we have
\begin{equation}
(7) \quad \tau_{2,f}(\phi, g) = \int_{M^2} \tau_{f_{m-2}}(\phi, \bar{g}),
\end{equation}
where \( \bar{g} = f^{\frac{2}{m-2}}g \). In particular, a map \( \phi : (M^m, g) \rightarrow (N^n, h) \) is a bi-\( f \)-harmonic map if and only if it is an \( f^{\frac{m}{m-2}} \)-biharmonic after the conformal change of the metric \( \bar{g} = f^{\frac{2}{m-2}}g \) in the domain manifold, conversely, a map \( \phi : (M^m, g) \rightarrow (N^n, h) \) is an \( f \)-biharmonic map if and only if it is a bi-\( f^{\frac{m}{m-2}} \)-harmonic after the conformal change of the metric \( \bar{g} = f^{\frac{2}{m-2}}g \).

**Proof.** A straightforward computation gives the transformation of the tension fields under the conformal change of a metric \( \bar{g} = F^{2}g \):

\[
\tau(\phi, \bar{g}) = F^{2}\{\tau(\phi, g) - (m - 2)d\phi(\text{grad ln} F)\}.
\]

When \( m \neq 2 \) and \( F^{-2} = f^{\frac{2}{m-2}} \), we have

\[
\tau(\phi, \bar{g}) = f^{\frac{m}{m-2}}\tau(\phi, g) + f^{\frac{m}{m-2}}d\phi(\text{grad} f) = f^{\frac{m}{m-2}}(f\tau(\phi, g) + d\phi(\text{grad} f)) = f^{\frac{m}{m-2}}\tau_f(\phi, g).
\]

It follows that

\[
\tau_f(\phi, g) = f^{\frac{m}{m-2}}\tau(\phi, \bar{g}).
\]

Now we use formula (3) in [Ou1] to have the following formula of changes of Jacobi operators under the conformal change of metrics \( \bar{g} = f^{\frac{2}{m-2}}g \)

\[
J_g(X) = f^{\frac{m}{m-2}}J_{\bar{g}}(X) + f^{-1}\nabla^{\phi}_{\text{grad} f}(X).
\]

By using this and (8), we have

\[
\tau_{2,f}(\phi, g) = -f J^{\phi}(\tau_f(\phi)) + \nabla^{\phi}_{\text{grad} f}\tau_f(\phi) = -f \left( f^{\frac{m}{m-2}}J_g(\tau_f(\phi)) + f^{-1}\nabla^{\phi}_{\text{grad} f}\tau_f(\phi) \right) + \nabla^{\phi}_{\text{grad} f}\tau_f(\phi) = -f^{\frac{m}{m-2}}J_g(\tau_f(\phi)).
\]

Substituting (8) into (9) yields

\[
\tau_{2,f}(\phi, g) = -f^{\frac{m}{m-2}}J_g \left( f^{\frac{m}{m-2}}\tau(\phi, \bar{g}) \right).
\]

By a straightforward computation using (see (7) in [Ou1])

\[
J(fX) = fJ(X) - (\Delta f)X - 2\nabla^{\phi}_{\text{grad} f}X,
\]
we can rewrite (10) as
\[
\tau_2(f) = -f \frac{m-2}{m-4} J_g \left( \frac{m}{m-4} \tau(\phi, \bar{g}) \right)
\]
\[
= f \frac{m-2}{m-4} \left( -f \frac{m}{m-4} J_g \tau(\phi, \bar{g}) + (\Delta g f \frac{m}{m-4}) \tau(\phi, \bar{g}) + 2 \nabla^\phi \frac{m}{m-4} \tau(\phi, \bar{g}) \right)
\]
\[
= f \frac{m-2}{m-4} \tau_{f \frac{m}{m-4}} (\phi, \bar{g}),
\]
where the third equality was obtained by using the \( f \)-bi-tension field (11) and the fact that \(-J_g(\tau(\phi, \bar{g}) = \tau_2(\phi, \bar{g})\). Thus, we obtain the relationship (7) from which the last statement of the theorem follows.

Many examples of proper \( f \)-biharmonic maps were given in [Ou2], from which and Theorem 1.1 we have

Example 1. It was proved in [Ou2] that for \( m \geq 3 \), the map \( \phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\} \) with \( \phi(x) = \frac{x}{|x|^4} \) is a proper \( f \)-biharmonic map for \( f(x) = |x|^4 \). Using Theorem 1.1 we conclude that the map \( \phi : (\mathbb{R}^m \setminus \{0\}, |x|^{-8/m} \delta_{ij}) \to \mathbb{R}^m \setminus \{0\} \) with \( \phi(x) = \frac{x}{|x|^4} \) is a proper bi-\( f \)-harmonic map for \( f(x) = |x|^{4(m-2)/m} \). In particular, when \( m = 4 \), we have a bi-\( f \)-harmonic map \( \phi : (\mathbb{R}^4 \setminus \{0\}, |x|^{-2} \delta_{ij}) \to \mathbb{R}^4 \setminus \{0\} \) with \( \phi(x) = \frac{x}{|x|^2} \) for \( f(x) = |x|^2 \).

Corollary 1.2. A map \( \phi : (M^m, g) \to (N^n, h) \) with \( m \neq 2 \) is a bi-\( f \)-harmonic map if and only if
\[
f \frac{m}{m-2} \tau_2(\phi, \bar{g}) + (\Delta g f \frac{m}{m-2}) \tau(\phi, \bar{g}) + 2 \nabla^\phi \frac{m}{m-2} \tau(\phi, \bar{g}) = 0,
\]
where \( \bar{g} = f \frac{m}{m-2} g \), and \( \tau(\phi, \bar{g}) \) and \( \tau_2(\phi, \bar{g}) \) are the tension and the bi-tension fields of the map \( \phi : (M^m, \bar{g}) \to (N^n, h) \) respectively.

Proof. This follows from Theorem 1.1 and the \( f \frac{m}{m-2} \)-biharmonic map equation (11).

Proposition 1.3. A function \( u : (M^m, g) \to \mathbb{R} \) is bi-\( f \)-harmonic if and only if it is a solution of the following bi-\( f \)-Laplace equation
\[
\Delta_f^2 u = \Delta_f (\Delta_f u) = 0,
\]
where \( \Delta_f \) is the \( f \)-Laplace operator defined by
\[
\Delta_f u = f \Delta u + g(\nabla f, \nabla u).
\]

Proof. For a real-valued function \( u : (M^m, g) \to \mathbb{R} \), one can easily check that the tension field is \( \tau(u) = (\Delta u) \frac{x}{\partial t} \), and hence \( \tau_f(u) = f(\Delta u) \frac{x}{\partial t} + du(\nabla f) = \)
The solutions of the bi-$f$-tension field of $u$ is given by

$$\tau_{2,f}(u) = J^m(\tau_f(u)) - \nabla_{\text{grad} f} \tau_f(u)$$

for $u \in \mathbb{R}$ on a Riemannian manifold, we make the following definitions.

Definition 1.4. For a function $u : (M^m, g) \to \mathbb{R}$ on a Riemannian manifold, we define the bi-$f$-Laplacian $\Delta_{2,f}$ acting on functions by

$$\Delta_{2,f} u := \Delta_f(\Delta_f u),$$

and the $f$-bi-Laplacian by

$$\Delta_{f,2} u = \Delta(f\Delta u) = f\Delta^2 u + (\Delta f)\Delta u + 2g(\nabla f, \nabla \Delta u).$$

The solutions of the bi-$f$-Laplace equation $\Delta_{2,f} u = 0$ and that of the $f$-bi-Laplace equation $\Delta_{f,2} u = 0$ are called bi-$f$-harmonic functions and $f$-biharmonic functions respectively.

Proposition 1.5. For a real-valued function $u$ on a Riemannian manifold $(M^m, g)$ with $m \neq 2$, we have the following relationship between bi-$f$-Laplacian and $f^{m-2}$-bi-Laplacian operators

$$\Delta_{2,f} = \Delta_f^2 = f^{m-2} \Delta_{f^{m-2},2} u = f^{m-2} \Delta_{f^{m-2}} \left( f^{m-2} \Delta_g u \right),$$

where $\Delta_{f^{m-2},2} u := f^{m-2} \Delta_g^2 u + (\Delta_g f^{m-2}) \Delta_g u + 2\tilde{g}(\text{grad}_g f^{m-2}, \text{grad}_g \Delta_g^2 u)$ is the $f^{m-2}$-bi-Laplacian of the conformal metric $\tilde{g} = f^{m-2} g$. In particular, a function $u : (M^m, g) \to \mathbb{R}$ is bi-$f$-harmonic if and only if it is $f^{m-2}$-biharmonic with respect the conformal metric $\tilde{g} = f^{m-2} g$.}

Proof. Using $\tilde{\Delta}$ and $\tilde{\nabla}$ to denote the Laplacian and the gradient operators on $(M^m, \tilde{g} = f^{m-2} g)$, we can easily check that $\Delta_f u = f^{m-2} \tilde{\Delta} u$. A straightforward
computation yields

\[ \Delta_f^2 u = \Delta_f (\Delta_f u) = f^{m-1} \Delta (f^{m-2} \Delta u) = f^{m-1} \left( f^{m-2} \Delta^2 u + (\Delta f^{m-1}) \Delta u + 2\bar{g}(\nabla f^{m-1}, \nabla \Delta^2 u) \right) = f^{m-1} \Delta_f f^{m-2} u, \]

(18)

from which we obtain the proposition. □

In a recent paper [Ch], Chiang proved that any bi-\(f\)-harmonic map from a compact manifold without boundary into a non-positively curved manifold satisfying

\[ \langle f \nabla^\phi \nabla^\phi \tau_f(\phi) - \nabla^\phi f \nabla^\phi \tau_f(\phi), \tau_f(\phi) \rangle \geq 0 \]

is an \(f\)-harmonic map. Our next theorem gives an improvement of this result by dropping the condition (19), and hence gives a generalization of Jiang’s result (Proposition 7 in [Ji]) on biharmonic maps viewed as bi-\(f\)-harmonic maps with \(f\) being a constant.

**Theorem 1.6.** Any bi-\(f\)-harmonic map \(\phi : (M^m, g) \to (N^n, h)\) from a compact Riemannian manifold without boundary into a non-positively curved manifold is an \(f\)-harmonic map.

**Proof.** A straightforward computation yields

\[ \frac{1}{2} \Delta |\tau_f(\phi)|^2 = |\nabla^\phi \tau_f(\phi)|^2 + \langle \Delta^\phi \tau_f(\phi), \tau_f(\phi) \rangle. \]

Using this and Equation (13) we have

\[ \frac{1}{2} \Delta |\tau_f(\phi)|^2 = |\nabla^\phi \tau_f(\phi)|^2 + \langle R^N(d\phi, \tau_f(\phi)) d\phi, \tau_f(\phi) \rangle - f^{-1} \langle \nabla^\phi \tau_f(\phi), \tau_f(\phi) \rangle \]

\[ = |\nabla^\phi \tau_f(\phi)|^2 - R^N(d\phi, \tau_f(\phi), d\phi, \tau_f(\phi)) - \frac{1}{2} \langle \nabla |\tau_f(\phi)|^2, \nabla \ln f \rangle. \]

It follows that

\[ \frac{1}{2} \Delta |\tau_f(\phi)|^2 + \frac{1}{2} \langle \nabla |\tau_f(\phi)|^2, \nabla \ln f \rangle \]

(20)

\[ = |\nabla^\phi \tau_f(\phi)|^2 - R^N(d\phi, \tau_f(\phi), d\phi, \tau_f(\phi)) \geq 0, \]

which implies that

\[ \Delta_f (|\tau_f(\phi)|^2) \geq 0, \]

where \(\Delta_f\) is the \(f\)-Laplacian defined by (13). Applying the maximum principle we conclude that \(|\tau_f(\phi)|\) is constant and \(\nabla^\phi \tau_f(\phi) = 0\) on \(M\).
To complete the proof, we define a vector field on $M$ by $Y = \langle f\nabla \phi, \tau_f(\phi) \rangle$. Then

$$\text{div} Y = |\tau_f(\phi)|^2 + \langle f\nabla \phi, \nabla^\phi \tau_f(\phi) \rangle = |\tau_f(\phi)|^2,$$

which, together with Stokes’s theorem, implies that $|\tau_f(\phi)| = 0$, i.e. $\phi$ is an $f$-harmonic map. Thus, we obtain the theorem.

From Theorem 1.6 we have

**Corollary 1.7.** Any bi-$f$-harmonic map $\phi : (M^m, g) \rightarrow \mathbb{R}^n$ from a compact manifold into a Euclidean space is a constant map.

From our Theorem 1.6 and Theorem 1.1 in [HLZ] we see that any bi-$f$-harmonic or $f$-biharmonic map from a closed Riemannian manifold into a Riemannian manifold of non-positive sectional curvature is an $f$-harmonic map or harmonic map respectively. Both these generalize Jiang’s nonexistence result which states that biharmonic maps from closed Riemannian manifolds into Riemannian manifolds of non-positive sectional curvature are harmonic maps (see Proposition 7 in [Ji]). When $M$ is complete noncompact, nonexistence results of proper biharmonic maps into Riemannian manifolds of non-positive sectional curvature were first proved in [BFO], [NUG] and later generalized in [Ma] and [Luo1], [Luo2]. In the rest of this paper, we will give some nonexistence results of proper bi-$f$-harmonic maps and $f$-biharmonic maps which generalize the corresponding results for biharmonic maps obtained in [BFO], [NUG], [Ma], [Luo1], and [Luo2].

2. Some nonexistence theorems for proper bi-$f$-harmonic maps

In this section, we give some nonexistence results of proper bi-$f$-harmonic maps from complete noncompact manifolds into a non-positively curved manifold.

**Theorem 2.1.** Let $\phi : (M, g) \rightarrow (N, h)$ be a bi-$f$-harmonic map from a complete Riemannian manifold into a Riemannian manifold of non-positive sectional curvature. Then, we have

(i) If $\left(\int_M f|d\phi|^q dv_g\right)^{\frac{1}{q}} < +\infty$ and $\int_M |\tau_f(\phi)|^p dv_g < \infty$ for some $q \in [1, \infty]$ and $p \in (1, \infty)$, then $\phi$ is an $f$-harmonic map.

(ii) If $\text{Vol}_f(M, g) := \int_M f dv_g = \infty$ and $\int_M |\tau_f(\phi)|^p dv_g < \infty$ for some $p \in (1, \infty)$, then $\phi$ is an $f$-harmonic map.

When the target manifold has strictly negative sectional curvature, we have

**Theorem 2.2.** Let $\phi : (M, g) \rightarrow (N, h)$ be a bi-$f$-harmonic map from a complete Riemannian manifold into a Riemannian manifold of strictly negative sectional curvature with $\int_M |\tau_f(\phi)|^p dv_g < \infty$ for some $p \in (1, \infty)$. If there is some point $x \in M$ such that $\text{rank} \phi(x) \geq 2$, then $\phi$ is an $f$-harmonic map.
To prove these theorems, we will need the following lemma.

**Lemma 2.3.** Let \( \phi : (M, g) \rightarrow (N, h) \) be a bi-\( f \)-harmonic map from a complete noncompact Riemannian manifold into a Riemannian manifold of non-positive sectional curvature. If \( \int_M |\tau_f(\phi)|^p f^g dv_g < +\infty \) for some \( p > 1 \), then \( \nabla^\phi \tau_f(\phi) = 0 \).

**Proof.** For a real number \( \epsilon > 0 \), a straightforward computation shows that

\[
\Delta(|\tau_f(\phi)|^2 + \epsilon) \frac{1}{2} \]

(21) \( = (|\tau_f(\phi)|^2 + \epsilon) \left( \frac{1}{2}(|\tau_f(\phi)|^2 + \epsilon) \Delta|\tau_f(\phi)|^2 - \frac{1}{4} \nabla|\tau_f(\phi)|^2 \right) \).

Since \( \nabla|\tau_f(\phi)|^2 = 2h(\nabla^\phi \tau_f(\phi), \tau_f(\phi)) \), we have

\[
|\nabla|\tau_f(\phi)|^2 |^2 \leq 4(|\tau_f(\phi)|^2 + \epsilon)|\nabla^\phi \tau_f(\phi)|^2,
\]

from which we obtain

\[
\frac{1}{2}(|\tau_f(\phi)|^2 + \epsilon) \Delta|\tau_f(\phi)|^2 - \frac{1}{4} \nabla|\tau_f(\phi)|^2 \geq \frac{1}{2}(|\tau_f(\phi)|^2 + \epsilon)(\Delta|\tau_f(\phi)|^2 - 2|\nabla^\phi \tau_f(\phi)|^2).
\]

(22)

Since \( \phi \) is bi-\( f \)-harmonic, we use (15) to have

\[
\frac{1}{2} \Delta|\tau_f(\phi)|^2 = |\nabla^\phi \tau_f(\phi)|^2 + \langle \Delta^\phi \tau_f(\phi), \tau_f(\phi) \rangle
\]

\[
= |\nabla^\phi \tau_f(\phi)|^2 - Tr_g R^N(\tau_f(\phi), d\phi, \tau_f(\phi), d\phi) - f^{-1} \langle \nabla^\phi \tau_f(\phi), \tau_f(\phi) \rangle
\]

(23) \( \geq |\nabla^\phi \tau_f(\phi)|^2 - \frac{1}{2} \langle \nabla|\tau_f(\phi)|^2, \nabla \ln f \rangle \),

where the inequality was obtained by using the assumption that \( R^N \leq 0 \). Rewriting (23) as

\[
\Delta|\tau_f(\phi)|^2 - 2|\nabla^\phi \tau_f(\phi)|^2 \geq - \langle \nabla|\tau_f(\phi)|^2, \nabla \ln f \rangle
\]

(24) and substituting (24) into the right hand side of (22) we have

\[
\frac{1}{2}(|\tau_f(\phi)|^2 + \epsilon) \Delta|\tau_f(\phi)|^2 - \frac{1}{4} \nabla|\tau_f(\phi)|^2 \geq \frac{1}{2}(|\tau_f(\phi)|^2 + \epsilon) \langle \nabla|\tau_f(\phi)|^2, \nabla \ln f \rangle.
\]

(25)

Using (21) and (25) we deduce that

\[
\Delta(|\tau_f(\phi)|^2 + \epsilon) \frac{1}{2} \geq - \frac{1}{2}(|\tau_f(\phi)|^2 + \epsilon) \langle \nabla|\tau_f(\phi)|^2, \nabla \ln f \rangle.
\]
Now taking the limit on both sides of the above inequality as $\epsilon \to 0$, we have that
\begin{equation}
\Delta |\tau f(\phi)| + \langle \nabla |\tau f(\phi)|, \nabla \ln f \rangle \geq 0.
\end{equation}
This, by using the notation $\Delta_\alpha := \Delta - \langle \nabla \alpha, \nabla \cdot \rangle$, where $\alpha(x)$ is a function on $M$ as in [WX], can be written as
\begin{equation}
\Delta \ln f |\tau f(\phi)| \geq 0,
\end{equation}
which means that $|\tau f(\phi)|$ is a positive $(-\ln f)$-subharmonic function on $M$. By Theorem 4.3 in [WX], we see that if $\int_M |\tau f(\phi)|^p dv_g < \infty$ for some $p > 1$, then $|\tau f(\phi)|$ is a constant on $M$. Moreover from (23) we have $\nabla \phi \tau f(\phi) = 0$. This completes the proof of the lemma.

**Proof of Theorem 2.1.** From the above lemma we conclude that $|\tau f(\phi)| = c$ is a constant. It follows that if $Vol_f(M) = \infty$, then we must have $c = 0$, this proves (ii) of Theorem 2.1. To prove (i) of Theorem 2.1 we consider two cases. If $c = 0$, then we are done. If $c \neq 0$, then the hypothesis implies that $Vol_f(M) < \infty$ and we will derive a contradiction as follows. Define a 1-form on $M$ by
\begin{equation}
\omega(X) := \langle fd\phi(X), \tau f(\phi) \rangle, \quad (X \in TM).
\end{equation}
Then we have
\begin{align*}
\int_M |\omega| dv_g &= \int_M \left( \sum_{i=1}^m |\omega(e_i)|^2 \right)^{\frac{1}{2}} dv_g \\
&\leq \int_M f|\tau f(\phi)||d\phi| dv_g \\
&\leq cVol_f(M)^{1-\frac{1}{q}} \left( \int_M |d\phi|^q dv_g \right)^{\frac{1}{q}} \\
&< \infty,
\end{align*}
where if $q = \infty$, we denote $\|d\phi\|_{L^\infty(M)} = (\int_M f|d\phi|^q dv_g)^{\frac{1}{q}}$. Now, we compute $-\delta \omega = \sum_{i=1}^m (\nabla_{e_i} \omega)(e_i)$ to have
\begin{align*}
-\delta \omega &= \sum_{i=1}^m \nabla_{e_i} (\omega(e_i)) - \omega(\nabla_{e_i} e_i) \\
&= \sum_{i=1}^m \{ \langle \nabla_{e_i}^\phi(fd\phi(e_i)), \tau f(\phi) \rangle - \langle fd\phi(\nabla_{e_i} e_i), \tau f(\phi) \rangle \} \\
&= \sum_{i=1}^m \{ \langle f\nabla_{e_i}^\phi d\phi(e_i) - fd\phi(\nabla_{e_i} e_i) + \nabla_{e_i} f, \nabla_{e_i} \phi, \tau f(\phi) \rangle \} \\
&= |\tau f(\phi)|^2,
\end{align*}
(28)
where in obtaining the second equality we have used $\nabla^\phi \tau_f(\phi) = 0$. Now by Yau’s generalized Gaffney’s theorem (see Appendix) and (28), we have

$$0 = -\int_M \delta \omega dv_g = \int_M |\tau_f(\phi)|^2 dv_g = c^2 \text{Vol}(M),$$

which implies that $c = 0$, a contradiction. Therefore we must have $c = 0$, i.e. $\phi$ is an $f$-harmonic map. This completes the proof of theorem.

**Proof of Theorem 2.2.** By Lemma 2.3, we know that $|\tau_f(\phi)| = c$, a constant. We only need to prove that $c = 0$. Assume that $c \neq 0$, we will derive a contradiction. It follows from (6) that at $x \in M$, we have

$$0 = -\frac{1}{2} \Delta |\tau_f(\phi)|^2 = -\langle \Delta^\phi \tau_f(\phi), \tau_f(\phi) \rangle - |\nabla^\phi \tau_f(\phi)|^2 = \sum_{i=1}^m \langle R^N(\tau_f(\phi), d\phi(e_i))d\phi(e_i), \tau_f(\phi) \rangle + f^{-1}\langle \nabla^\phi \tau_f(\phi), \tau_f(\phi) \rangle - |\nabla^\phi \tau_f(\phi)|^2 = \sum_{i=1}^m \langle R^N(\tau_f(\phi), d\phi(e_i))d\phi(e_i), \tau_f(\phi) \rangle,$$

where in obtaining the first and fourth equalities we have used Lemma 2.3. Since the sectional curvature of $N$ is strictly negative, we conclude that $d\phi(e_i)$ with $i = 1, 2, \cdots, m$ are parallel to $\tau_f(\phi)$ at any $x \in M$ and hence $\text{rank}\phi(x) \leq 1$. This contradicts the assumption that $\text{rank}\phi(x) \geq 2$ for some $x$. The contradiction shows that we must have $c = 0$. Thus, we obtain the theorem.

From Theorem 2.1, we obtain the following corollary.

**Corollary 2.4.** Let $\phi : (M, g) \to (N, h)$ be a bi-$f$-harmonic map with a bounded $f$ from a complete Riemannian manifold into a Riemannian manifold of nonpositive sectional curvature. If

(i) $\int_M f|d\phi|^2 dv_g < +\infty$ and $\int_M |\tau_f(\phi)|^2 dv_g < \infty$,

or

(ii) $Vol_f(M, g) := \int_M f dv_g = \infty$ and $\int_M |\tau_f(\phi)|^2 dv_g < \infty$,

then $\phi$ is $f$-harmonic.

Similarly, from Theorem 2.2 we have the following corollary which shows that a proper bi-$f$-harmonic map with bounded $f$ and bounded bi-$f$-energy from a complete noncompact manifold into a negatively curved manifold must have $\text{rank} \leq 1$ at any point.

**Corollary 2.5.** Let $\phi : (M, g) \to (N, h)$ be a bi-$f$-harmonic map with bounded $f$ from a complete Riemannian manifold into a Riemannian manifold of strictly
negative sectional curvature. If \( \int_M |\tau_f(\phi)|^2 dv_g < \infty \) and there is a point \( x \in M \) such that \( \text{rank}\phi(x) \geq 2 \), then \( \phi \) is an \( f \)-harmonic map.

3. Some nonexistence theorems for proper \( f \)-biharmonic maps

In this section we give some nonexistence theorems for proper \( f \)-biharmonic maps from a complete manifold into a non-positively curved manifold. Such a study was started in [Ou2] where it was proved that there exists no proper \( f \)-biharmonic map with constant \( f \)-bienergy density from a compact Riemannian manifold into a nonpositively curved manifold. Later in [HLZ], it was shown that the condition of having constant \( f \)-bienergy density can be dropped. When the domain manifold is complete noncompact, the authors in [HLZ] also proved the following result.

Theorem 3.1 (HLZ). Let \( \phi : (M, g) \rightarrow (N, h) \) be an \( f \)-biharmonic map from a complete Riemannian manifold \( (M, g) \) into a Riemannian manifold \( (N, h) \) with non-positive curvature. If

(1) \( \int_M |d\phi|^2 dv_g < \infty \), \( \int_M |\tau(\phi)|^2 dv_g < \infty \), and \( \int_M f^p|\tau(\phi)|^p dv_g < \infty \)

for some \( p \geq 2 \), or

(2) \( \text{Vol}(M, g) = \infty \) and \( \int_M f^p|\tau(\phi)|^p dv_g < \infty \),

then \( \phi \) is harmonic.

Our next theorem gives a generalization of Theorem 3.1.

Theorem 3.2. An \( f \)-biharmonic map \( \phi : (M, g) \rightarrow (N, h) \) from a complete Riemannian manifold into a Riemannian manifold of non-positive curvature is a harmonic map if either

(i) \( (\int_M |d\phi|^q dv_g)^{\frac{1}{q}} < +\infty \) and \( \int_M f^p|\tau(\phi)|^p dv_g < \infty \) for some \( q \in [1, \infty) \) and \( p \in (1, \infty) \), or

(ii) \( \text{Vol}(M, g) = \infty \) and \( \int_M f^p|\tau(\phi)|^p dv_g < \infty \) for some \( q \in [1, \infty) \) and \( p \in (1, \infty) \).

Remark 1. Clearly, our Theorem 3.2 improves Theorem 3.1 since (I) and (II) in Theorem 3.1 (in which \( p \geq 2 \) and finite bienergy are required) implies (i) and (ii) in Theorem 3.2 respectively, but not conversely.

When the target manifold has strictly negative sectional curvature, we have

Theorem 3.3. Let \( \phi : (M, g) \rightarrow (N, h) \) be an \( f \)-biharmonic map from a complete Riemannian manifold into a Riemannian manifold of strictly negative curvature with \( \int_M f^p|\tau(\phi)|^p dv_g < \infty \) for some \( p \in (1, \infty) \). Assume that there is a point \( x \in M \) such that \( \text{rank}\phi(x) \geq 2 \), then \( \phi \) is a harmonic map.
The proofs of Theorems 3.2 and 3.3 are very similar to those of Theorems 2.1 and 2.2. The main ideas and outlines are given as follows.

**Proof of Theorem 3.2** First, we rewrite the $f$-biharmonic map equation (4) as

$$\Delta \phi(f\tau(\phi)) - \text{Trace}_g R^N(d\phi, f\tau(\phi))d\phi = 0,$$

and use it to compute $\Delta(f^2|\tau(u)|^2 + \epsilon)^{\frac{1}{2}}$ for a constant $\epsilon > 0$. Then, we use the result and an argument similar to that used in the proof of Lemma 2.3 to have

$$\Delta(f^2|\tau(u)|^2 + \epsilon)^{\frac{1}{2}} \geq 0.$$

By taking limit on both sides of the above inequality as $\epsilon \to 0$, we have $\Delta(f|\tau(\phi)|) \geq 0$. Now, by Yau’s classical $L^p$ Liouville type theorem (see [Yau]) and the assumption that $\int_M f^p|\tau(\phi)|^p dv_g < \infty$, we conclude that $f|\tau(\phi)| = c$, a constant, which is used to deduce that $\nabla^\phi(f\tau(\phi)) = 0$.

It is easy to see that $f|\tau(\phi)| = c$ together with hypothesis (ii) implies that $c = 0$, which mean $\tau(\phi) = 0$ and hence $\phi$ is a harmonic map. To prove the theorem under hypothesis (i), we assume $c \neq 0$ and define a field of 1-form $\omega$ on $M$ by

$$\omega(X) := \langle d\phi(X), f\tau(\phi) \rangle, \ (X \in TM).$$

Then a straightforward computation shows that $-\delta \omega = f|\tau(\phi)|^2$ which, together with the assumption (i) of Theorem 3.2 implies that

$$\int_M |\omega| dv_g < \infty.$$

Using Yau’s generalized Gaffney’s theorem, we have

$$0 = \int_M \delta \omega dv_g = -\int_M f|\tau(\phi)|^2 dv_g = -c^2 \int_M f^{-1} dv_g,$$

which implies that $c = 0$, a contradiction. The contradiction shows that we must have $c = 0$, and hence $\phi$ is a harmonic map. Thus, we obtain the theorem.

**Proof of Theorem 3.3** As in the first part of the proof of Theorem 3.2, we have $f|\tau(\phi)| = c$, a constant. It is enough to prove that $c = 0$. Assume that
$c \neq 0$, we will derive a contradiction. Using (29) we have at $x \in M$:

$$0 = -\frac{1}{2} \Delta (|f \tau (\phi)|^2)$$

$$= -\langle \Delta^\phi (f \tau (\phi)), f \tau (\phi) \rangle - |\nabla^\phi (f \tau (\phi))|^2$$

$$= \sum_{i=1}^{m} \langle R^N (f \tau (\phi), d\phi(e_i))d\phi(e_i), f \tau (\phi) \rangle - |\nabla^\phi f \tau (\phi)|^2$$

$$= \sum_{i=1}^{m} \langle R^N (f \tau (\phi), d\phi(e_i))d\phi(e_i), f \tau (\phi) \rangle.$$

Since the sectional curvature of $N$ is strictly negative, we must have that $d\phi(e_i)$ with $i = 1, 2, \ldots, m$ are parallel to $f \tau (\phi)$ at any $x \in M$, so $\text{rank}\phi(x) \leq 1$. This contradicts the assumption that $\text{rank}\phi(x) \geq 2$ for some $x \in M$. The contradiction shows that $c = 0$, and hence $\phi$ is a harmonic map. This completes the proof of Theorem 3.3.

From Theorems 3.2 and 3.3 we have the following corollaries.

**Corollary 3.4.** Any $f$-biharmonic map $\phi : (M, g) \rightarrow (N, h)$ with a bounded $f$ from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with non-positive curvature is harmonic if either

(i) $\int_{M} |d\phi|^2 dv_g < \infty$ and $\int_{M} f |\tau(\phi)|^2 dv_g < \infty$,

or

(ii) $\text{Vol}(M, g) = \infty$ and $\int_{M} f |\tau(\phi)|^2 dv_g < \infty$.

**Corollary 3.5.** Any $f$-biharmonic map $\phi : (M, g) \rightarrow (N, h)$ with a bounded $f$ and bounded $f$-bienergy from a complete Riemannian manifold into a Riemannian manifold with strictly negative curvature is harmonic if there exists a point $x \in M$ such that $\text{rank}(\phi)(x) \geq 2$.

To close this section, we give the following example which shows that there does exist proper $f$-biharmonic map from a complete manifold into a non-positively curved manifold.

**Example 2.** It was proved in [Ou1] (Page 141, Example 3) that the map $\phi : (\mathbb{R}^2, g = e^{\frac{\pi}{R}}(dx + dy^2)) \rightarrow \mathbb{R}^3, \phi(x, y) = (R \cos \frac{x}{R}, R \sin \frac{x}{R}, y)$ is a proper biharmonic conformal immersion of $\mathbb{R}^2$ into Euclidean space $\mathbb{R}^3$. Then by Theorem 2.3 of [Ou2] we see that $\phi$ is a proper $f$-biharmonic map from a complete manifold $(\mathbb{R}^2, g = dx^2 + dy^2)$ into Euclidean space $\mathbb{R}^3$ (a non-positively curved space) with $f = e^{-\frac{\pi}{R}}$. 

Direct computations show that \(d\phi = (-\sin \frac{x}{R}, \cos \frac{x}{R}, 0)dx + (0, 0, 1)dy\) and \(\tau(\phi) = -\left(\frac{R}{R^2}\cos \frac{x}{R}, \frac{R^2}{R^2}\sin \frac{x}{R}, 0\right)\). Therefore, one can easily check that in this example we have

\[
||d\phi||_{L^\infty(\mathbb{R}^2)} = \sqrt{2} < \infty, \\
||d\phi||_{L^q(\mathbb{R}^2)} = \sqrt{2} \text{Vol}(\mathbb{R}^2)^{\frac{1}{q}} = \infty \text{ for } q \in [1, \infty), \\
\text{Vol}(\mathbb{R}^2) = \infty, \\
\int_{\mathbb{R}^2} f^p|\tau(\phi)|^p dv_g = \infty (p > 1), \text{ and } \int_{\mathbb{R}^2} f|\tau(\phi)|^2 dv_g = \infty.
\]

So, the example shows that the hypothesis “\(\int_M f^p|\tau(\phi)|^p dv_g < \infty\)” in Theorem 3.2 and the hypothesis “\(f\) is bounded, \(\int_M f|\tau(\phi)|^2 dv_g < \infty\)” in Corollary 3.4 cannot be dropped. Note also that \(\text{rank}(\phi)(x) = 2\) for any \(x \in \mathbb{R}^2\) and hence the hypothesis “\(\int_M f^p|\tau(\phi)|^p dv_g < \infty\) and the target manifold has strictly negative sectional curvature” in Theorem 3.3 and the hypothesis “\(f\) is bounded, \(\int_M f|\tau(\phi)|^2 dv_g < \infty\) and the target manifold has strictly negative sectional curvature” in Corollary 3.5 cannot be dropped.

### 4. Further nonexistence results on proper \(f\)-biharmonic and bi-\(f\)-harmonic maps

It would be interesting to know how and to what extent the nature of the function \(f\) would affect the existence of proper \(f\)-biharmonic or bi-\(f\)-harmonic maps from a complete manifold into a non-positively curved manifold. In particular, one would like to know whether the boundedness assumption on the function \(f\) is essential in Corollaries 2.4, 2.5, 3.4, 3.5. In this section, we will show that we can weaken the boundedness assumption on \(f\) by replacing it with

\[
\sup_{B_r} f(x) = o(r^2), \text{ as } r \to \infty, \tag{30}
\]

or

\[
\sup_{B_r} f(x) \leq Cr^2 F(r), \tag{31}
\]

where \(B_r\) is a geodesic ball of radius \(r\) centered at some point on \(M\) and \(F(r)\) is a nondecreasing function such that \(\int_a^\infty \frac{1}{r F(r)} dr = \infty\) for some positive constant \(a > 0\). Here, we illustrate how to generalize Corollary 2.4 by a weaker assumption. Recall that in the proof of Lemma 2.3, we have \(\Delta_{-\infty} f |\tau_f(\phi)| \geq 0\), then by Theorem 4.3 of [WX], we conclude that if

\[
\lim_{r \to \infty} \frac{1}{r^2 F(r)} \int_{B_r} |\tau_f(\phi)|^2 f dv_g < \infty, \tag{32}
\]
then $|\tau_f(\phi)|$ is constant and furthermore $\nabla^\phi \tau_f(\phi) = 0$. It is easy to see that under the assumption of (31) and $\int_M |\tau_f(\phi)|^2 dv_g < \infty$, we have

$$\lim_{r \to \infty} \frac{1}{r^2 F(r)} \int_{B_r} f |\tau_f(\phi)|^2 dv_g \leq \lim_{r \to \infty} \sup_{B_r} f(x) \int_M |\tau_f(\phi)|^2 dv_g < \infty,$$

which implies (32). Therefore, $|\tau_f(\phi)|$ is a constant $c$ and furthermore $\nabla^\phi \tau_f(\phi) = 0$. Thus, if $Vol_f(M) = \infty$, we must have $c = 0$, i.e. $\phi$ is an $f$-harmonic map. This shows that under the hypothesis (ii) of Corollary 2.4 and assumption (31), $\phi$ is an $f$-harmonic map. To see that Corollary 2.4 holds under hypothesis (i) of Corollary 2.4 and the assumption (30), we only need to prove that $c = 0$ in this case. If otherwise, we see that $Vol(M) < \infty$ and we will derive a contradiction as follows. Define a 1-form on $M$ by

$$\omega(X) := \langle fd\phi(X), \tau_f(\phi) \rangle, \ (X \in TM).$$

Then we have

$$\lim_{r \to \infty} \frac{1}{r} \int_{B_r} |\omega| dv_g = \lim_{r \to \infty} \frac{1}{r} \int_{B_r} \left( \sum_{i=1}^{m} |\omega(e_i)|^2 \right)^{\frac{1}{2}} dv_g$$

$$\leq \lim_{r \to \infty} \frac{1}{r} \int_{B_r} f |\tau_f(\phi)||d\phi| dv_g$$

$$\leq \lim_{r \to \infty} \frac{1}{r} \left( \int_{B_r} f dv_g \right)^{\frac{1}{2}} \left( \int_M f|d\phi|^2 dv_g \right)^{\frac{1}{2}}$$

$$\leq \lim_{r \to \infty} \sup_{B_r} \sqrt{f(x)} \frac{Vol(M)^{\frac{1}{2}}}{r} \left( \int_M f|d\phi|^2 dv_g \right)^{\frac{1}{2}}$$

$$= 0.$$
where in obtaining the second equality we have used $\nabla^2 \tau_f(\phi) = 0$. Now by Yau’s generalized Gaffney’s theorem (see Appendix) and the above equality we have that
\[
0 = -\int_M \delta \omega dv_g = \int_M |\tau_f(\phi)|^2 dv_g = c^2 \text{Vol}(M),
\]
which implies that $c = 0$, a contradiction. Therefore we must have $c = 0$, i.e. $\phi$ is an $f$-harmonic map.

Summarizing the above discussion, we have the following theorem which gives a generalization of Corollary 2.4.

**Theorem 4.1.** Let $\phi : (M, g) \rightarrow (N, h)$ be a bi-$f$-harmonic map from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ of non-positive sectional curvature. If

(i)
\[
\lim_{r \to \infty} \frac{\sup_{B_r} f(x)}{r^2} = 0,
\]
where $B_r$ is a geodesic ball of radius $r$ around some point on $M$, and
\[
\int_M f|d\phi|^2 dv_g < +\infty \text{ and } \int_M |\tau_f(\phi)|^2 dv_g < \infty;
\]
or

(ii)
\[
\sup_{B_r} f(x) \leq CF(r)r^2
\]
for a nondecreasing function $F(r)$ such that $\int_a^\infty \frac{1}{F(r)} dr = \infty$ for some positive constant $a > 0$, and
\[
\text{Vol}_f(M, g) := \int_M f dv_g = \infty \text{ and } \int_M |\tau_f(\phi)|^2 dv_g < \infty,
\]
then $\phi$ is $f$-harmonic.

Similar arguments apply to obtain the following theorems, which give generalizations of the corresponding results in Corollaries 2.5, 3.4 and 3.5.

**Theorem 4.2.** Let $\phi : (M, g) \rightarrow (N, h)$ be a bi-$f$-harmonic map from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ of strictly negative sectional curvature. Assume that

(i)
\[
\sup_{B_r} f(x) \leq CF(r)r^2
\]
where $B_r$ is a geodesic ball of radius $r$ around some point on $M$ and $F(r)$ is a nondecreasing function such that $\int_a^\infty \frac{1}{rF(r)} \, dr = \infty$ for some positive constant $a > 0$. Then if
\[
\int_M |\tau f(\phi)|^2 \, dv_g < \infty
\]
and there is some point $x \in M$ such that rank$\phi(x) \geq 2$, $\phi$ is an $f$-harmonic map.

**Theorem 4.3.** Let $\phi : (M, g) \longrightarrow (N, h)$ be an $f$-biharmonic map from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ of non-positive curvature. If
\[(i)\]
\[
\lim_{r \to \infty} \frac{\sup_{B_r} f(x)}{r^2} = 0,
\]
where $B_r$ is a geodesic ball of radius $r$ around some point on $M$,
\[
\int_M |d\phi|^2 \, dv_g < \infty, \text{ and } \int_M f|\tau(\phi)|^2 \, dv_g < \infty;
\]
or
\[(ii)\]
\[
\sup_{B_r} f(x) \leq CF(r)r^2
\]
for a nondecreasing function $F(r)$ such that $\int_a^\infty \frac{1}{rF(r)} \, dr = \infty$ for some positive constant $a > 0$,
\[
\text{Vol}(M, g) = \infty, \text{ and } \int_M f|\tau(\phi)|^2 \, dv_g < \infty,
\]
then $\phi$ is a harmonic map.

**Theorem 4.4.** Let $\phi : (M, g) \longrightarrow (N, h)$ be an $f$-biharmonic map from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ of strictly negative curvature. Assume that
\[(38)\]
\[
\sup_{B_r} f(x) \leq CF(r)r^2,
\]
where $B_r$ is a geodesic ball of radius $r$ around some point on $M$ and $F(r)$ is a nondecreasing function such that $\int_a^\infty \frac{1}{rF(r)} \, dr = \infty$ for some positive constant $a > 0$. Then if $\int_M f|\tau(\phi)|^2 \, dv_g < \infty$ and there exists a point $x \in M$ such that rank$\phi(x) \geq 2$, $\phi$ is a harmonic map.
5. Appendix

Theorem 5.1 (Yau’s generalized Gaffney’s theorem). Let $(M, g)$ be a complete Riemannian manifold. If $\omega$ is a $C^1$ 1-form such that $\lim_{r \to \infty} \int_{B_r} |\omega| \, dv_g = 0$, or equivalently, a $C^1$ vector field $X$ defined by $\omega(Y) = \langle X, Y \rangle$, $(\forall Y \in TM)$ satisfying $\lim_{r \to \infty} \int_{B_r} |X| \, dv_g = 0$, where $B_r$ is a geodesic ball of radius $r$ around some point on $M$, then

$$\int_M \delta \omega \, dv_g = \int_M \text{div} X \, dv_g = 0.$$

Proof. See Appendix in [Yau].

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