State Agreement in Finite Time

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1 Introduction

The theory of agreement or consensus problems for multi-agent systems has emerged as a challenging new area of research in recent years. It is a basic yet fundamental problem in decentralized control of networks of dynamic agents and has attracted great attention of researchers. This is partly due to its broad applications in cooperative control of unmanned air vehicles, formation control of mobile robots, control of communication networks, design of sensor networks, flocking of social insects, swarm-based computing, etc.

In [1], Vicsek et al. proposed a simple but interesting discrete-time model of $n$ agents all moving in the plane. Each agent’s motion is updated using a local rule based on its own state and the states of its neighbors. The Vicsek model can be viewed as a special case of a computer model mimicking animal aggregation proposed in [2] for the computer animation industry. By using graph theory and nonnegative matrix theory, Jadbabaie et al. provided a theoretical explanation of the consensus behavior of the Vicsek model in [3], where each agent’s set of neighbors changes with time as system evolves. The typical continuous-time model was proposed by Olfati-Saber and Murray in [4], where the concepts of solvability of agreement problems and agreement protocols were first introduced. In [4], Olfati-Saber and Murray used a directed
graph to model the communication topology among agents and studied three agreement problems. They are directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology. And it was assumed that the directed topology is balanced and strongly connected. In [5], Ren and Beard extended the results of [3] and [4] and presented more relaxable conditions for state agreement under dynamically changing directed interaction topology. In the past two years, agreement problems of multi-agent systems have been developing fast and several research topics have been addressed, such as agreement over random networks [6,7], asynchronous information consensus [8], dynamic consensus [9], networks with nonlinear agreement protocols [10], consensus filters [11], and networks with communication time-delays [12,13,14]. For details, see the survey [15] and references therein.

By long-time observation of animal aggregations, such as schools of fish, flocks of birds, groups of bees, and swarms of social bacteria, it is believed that simple, local motion coordination rules at the individual level can result in remarkable and complex intelligent behavior at the group level. We call those local motion coordination rules protocols. In the study of agreement problems, they are called agreement protocols.

In the analysis of agreement problems, convergence rate is an important performance index of the proposed agreement protocol. In [4], a linear agreement protocol was given and it was shown that the second smallest eigenvalue of interaction graph Laplacian, called algebraic connectivity of graph, quantifies the speed of convergence of consensus algorithms. In [16], Kim and Mesbahi considered the problem of finding the best vertex positional configuration so that the second smallest eigenvalue of the corresponding graph Laplacian is maximized, where the weight for an edge between two vertices is a function of the distance between the corresponding two agents. In [17], Xiao and Boyd considered and solved the problem of the weight design by using
semi-definite convex programming, so that algebraic connectiv-
ity is increased. If the communication topology is a small-world
networks, it was shown that large algebraic connectivity can be
obtained \cite{18}. Although by maximizing the second smallest eigen-
value of topology graph Laplacian, we can get better convergence
rate of the linear protocol proposed in \cite{4}, the state agreement
can never occur in a finite time. Therefore, finite-time agreement
is more appealing and there are a number of settings where finite-
time convergence is a desirable property. Our goal in this paper is
to address finite-time agreement problems and present two dis-
tributed protocols that can solve agreement problems in finite
time. The method used in this paper is partly motivated by the
work of \cite{19}, in which continuous finite-time differential equa-
tions were introduced as fast accurate controllers for dynamical
systems.

This paper is organized as follows. In Section II, we state the con-
sidered problem. In Section III, we give our main results. Simula-
tion results are given in Section IV. Finally, concluding remarks
are stated in Section V.

2 Problem Formulation

The distributed dynamic system studied in this paper consists of
$n$ autonomous agents, e.g. particles or robots, labeled 1 through
$n$. All these agents share a common state space $\mathbb{R}$. We use $x_i$ to
denote the state of agent $i$ and suppose that agent $i$ is with the
following dynamics

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n,$$  \hspace{1cm} (1)

where $u_i(t), i \in \mathcal{I}_n$, seen as a whole, is the protocol to be designed,
and $\mathcal{I}_n = \{1, 2, \ldots, n\}$.

In this multi-agent system, each agent can communicate with
some other agents which are defined as its neighbors. We use
a weighted undirected graph $G(A) = (\mathcal{V}, \mathcal{E}, A)$ to represent the communication topology, where $A = [a_{ij}]$ is a $n \times n$ nonnegative symmetric matrix, $\mathcal{V} = \{v_i : i \in \mathcal{I}_n\}$ is the vertex set, and $\mathcal{E}$ is the edge set. Vertex $v_i$ corresponds to agent $i$. An edge of $G(A)$ is denoted by $(v_i, v_j)$, which is an unordered pair of vertices. $(v_i, v_j) \in \mathcal{E}$, if and only if $a_{ij} > 0$, if and only if agents $i$ and $j$ can communicate with each other, i.e., they are adjacent. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathcal{I}_n$. We call $A$ the weight matrix and $a_{ij}$ is the weight of edge $(v_i, v_j)$. In consistence with the definition of agents’ neighbors, the set of neighbors of vertex $v_i$ is denoted by $\mathcal{N}_i = \{v_j : (v_i, v_j) \in \mathcal{E}\}$. A path in a graph from $v_i$ to $v_j$ is a sequence of distinct vertices starting with $v_i$ and ending with $v_j$ such that consecutive vertices are adjacent. A graph is connected if there is a path between any two vertices of the graph. More comprehensive discussions about graph can be found in [20].

Protocol $u_i$ is a state feedback, which is designed based on the state information received by agent $i$ from its neighbors.

Given protocol $u_i$, $i \in \mathcal{I}_n$, we say that $u_i$ or this multi-agent system solves an agreement problem if for any given initial states and any $j, k \in \mathcal{I}_n$, $|x_j(t) - x_k(t)| \to 0$, as $t \to \infty$, and we say that it solves a finite-time agreement problem if for any initial states, there exist a time $t^*$ and a real number $\kappa$ such that $x_j(t) = \kappa$ for $t \geq t^*$ and for all $j \in \mathcal{I}_n$. If the final agreement state is the average of the initial states, i.e., $x_j(t) \to \frac{\sum_{k=1}^{n} x_k(0)}{n}$ for all $j \in \mathcal{I}_n$ as $t \to \infty$, we say that it solves the average-agreement problem.

With the above preparation, we present two agreement protocols that solve agreement problems in finite time:

i) $u_i = \text{sign}(\sum_{v_j \in \mathcal{N}_i} a_{ij}(x_j - x_i))|\sum_{v_j \in \mathcal{N}_i} a_{ij}(x_j - x_i)|^\alpha, i \in \mathcal{I}_n; \quad (2)$
ii)

\[ u_i = \sum_{v_j \in N_i} a_{ij} \text{sign}(x_j - x_i)|x_j - x_i|^\alpha, \ i \in \mathcal{I}_n, \]  

where \( 0 < \alpha < 1 \) and \( \text{sign}(\cdot) \) is the sign function, such that

\[
\text{sign}(r) = \begin{cases} 
1, & r > 0 \\
0, & r = 0 \\
-1, & r < 0 
\end{cases}
\]

Remark: If we set \( \alpha = 1 \) in the above protocols (2) and (3), then they will become the typical linear agreement protocol studied in [4] and [5], and solve the average-agreement problem asymptotically provided that \( \mathcal{G}(A) \) is connected. If we set \( \alpha = 0 \) in (2) and (3), they will become noncontinuous. Research of them is beyond the scope of our study. In the next section, we will show that protocols (2) and (3) all solve the finite-time agreement problems if \( \mathcal{G}(A) \) is connected and \( 0 < \alpha < 1 \).

Remark: The information exchange among agents is assumed to be bidirectional and thus the communication topology can be represented by an undirect graph. The study of the case with unidirectional information exchange will be a future research direction.

3 Main Results

We assume in this note that \( \mathcal{G}(A) \) is connected, since if \( \mathcal{G}(A) \) is not connected, then there exist at least two groups of agents, between which there does not exist information exchange, and therefore it is impossible for the system to solve an agreement problem through distributed protocols.

Let \( \mathbf{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^n \) and let \( x = [x_1, x_2, \ldots, x_n]^T \). We first show that the equilibrium point set of the considered differential equations \( \dot{x}_i = u_i, \ i \in \mathcal{I}_n \), is the set of all agreement states.
Property 1 With protocol (2) or (3), the equilibrium point set of the differential equations $\dot{x}_i = u_i, i \in \mathcal{I}_n$, is $\text{span}(1)$.

Proof: We only consider protocol (3).

Let $y_1, y_2, \ldots, y_n$ satisfy the equations $u_i = 0, i \in \mathcal{I}_n$, and let $y_k = \min_{i \in \mathcal{I}_n} y_i$. From $u_k = 0$, we have that $y_i = y_k, v_i \in N_k$. Let $N^{(1)} = \{k\} \cup N_k$. By the same arguments, $y_i = y_k, v_i \in N^{(1)} \cup \bigcup_{v_j \in N^{(1)}} N_j$. Let $N^{(2)} = N^{(1)} \cup \bigcup_{v_j \in N^{(1)}} N_j$, and if $N^{(i)}$ is defined, let $N^{(i+1)} = N^{(i)} \cup \bigcup_{v_j \in N^{(i)}} N_j$. By induction, we have $y_j = y_k$ for all $v_j \in N^{(i)}, i = 1, 2, \ldots$. Since $\mathcal{G}(A)$ is connected, there exists some $i$, such that $N^{(i)} = \{v_1, v_2, \ldots, v_n\}$. And therefore $y = [y_1, y_2, \ldots, y_n]^T \in \text{span}(1)$.

We hope that all the equilibrium points are stable and state of the system reaches $\text{span}(1)$ in finite time.

Apparently, protocol (2) and (3) are continuous with respect to state variables $x_1, x_2, \ldots, x_n$. Therefore, if we adopt one of these two protocols, then for any initial state $x(0)$, by Peano’s Existence Theorem and Extension Theorem [21], there exists at least one solution of differential equation (1) on $[0, \infty)$. Moreover, one notices that (2) and (3) are not Lipschitz at some points. As all solutions reach subspace $\text{span}(1)$ in finite time, there is nonuniqueness of solutions in backwards time. This, of course, violates the uniqueness condition for solutions of Lipschitz differential equations.

In order to establish our main results, we need the following Lemmas.

Lemma 1 Let $y_1, y_2, \ldots, y_n \geq 0$ and let $p > 0$. Then there exists $m(n, p) > 0$, which is a function of $n$ and $p$, such that

$$\sum_{i=1}^{n} y_i^p \geq m(n, p) \left(\sum_{i=1}^{n} y_i\right)^p.$$
Proof: Obviously
\[ \sum_{i=1}^{n} y_i = 0 \iff \sum_{i=1}^{n} y_i^p = 0. \]

Let \( y = [y_1, y_2, \ldots, y_n]^T \) and let \( U = \{y : \sum_{i=1}^{n} y_i = 1 \text{ and } y \geq 0\} \).

If \( \sum_{i=1}^{n} y_i \neq 0 \),
\[ \frac{\sum_{i=1}^{n} y_i^p}{(\sum_{i=1}^{n} y_i)^p} = \sum_{i=1}^{n} \left( \frac{y_i}{\sum_{i=1}^{n} y_i} \right)^p \geq \inf_{y \in U} \sum_{i=1}^{n} y_i^p. \]

Let \( m(n, p) = \inf_{y \in U} \sum_{i=1}^{n} y_i^p \). Since \( \sum_{i=1}^{n} y_i^p \) is continuous and \( U \) is a bounded closed set, \( m(n, p) \) exists and \( m(n, p) > 0 \). And thus the inequality holds.

\( m(n, p) \) can be calculated directly. Precisely,
\[ m(n, p) = \min\{n^{1-p}, 1\}. \]

**Lemma 2** [4] Let \( L(A) = [l_{ij}] \in \mathbb{R}^{n \times n} \) denote the graph Laplacian of \( G(A) \), which is defined by
\[
l_{ij} = \begin{cases} \sum_{k=1, k \neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases}. \]

\( L(A) \) has the following properties:

i) 0 is an eigenvalue of \( L(A) \) and \( 1 \) is the associated eigenvector;

ii) \( x^T L(A) x = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} (x_j - x_i)^2 \), and the semi-positive definiteness of \( L(A) \) implies that all eigenvalues of \( L(A) \) are real and not less than zero;

iii) If \( G(A) \) is connected, the second smallest eigenvalue of \( L(A) \), which is denoted by \( \lambda_2(L_A) \) and called the algebraic connectivity of \( G(A) \), is larger than zero;
iv) The algebraic connectivity of $G(A)$ is equal to $\min_{x \neq 0, 1^T x = 0} x^T L(A) x$, and therefore, if $1^T x = 0$, we have that
\[ x^T L(A) x \geq \lambda_2(L_A) x^T x. \]

Now, we present our first main result.

**Theorem 1** If communication topology $G(A)$ is connected, then system (1) solves an agreement problem in finite time when decentralized protocol (2) is applied.

Proof: Take semi-positive definite function
\[ V_1(x(t)) = \frac{1}{4} \sum_{i,j=1}^{n} a_{ij} (x_j(t) - x_i(t))^2, \]
which will be proven to be a valid Lyapunov function for agreement analysis.

Since $G(A)$ is connected, $V_1(x) = 0$ implies that for any $i, j \in \mathcal{I}_n$, $x_i = x_j$. And since $A$ is symmetric, we have that
\[ \frac{\partial V_1(x)}{\partial x_i} = - \sum_{v_j \in N_i} a_{ij} (x_j - x_i), \]
and
\[ \frac{dV_1(t)}{dt} = \sum_{i=1}^{n} \frac{\partial V_1(x)}{\partial x_i} \dot{x}_i \]
\[ = - \sum_{i=1}^{n} \left| \sum_{v_j \in N_i} a_{ij} (x_j - x_i) \right|^{1+\alpha} \]
\[ = - \sum_{i=1}^{n} \left( \sum_{v_j \in N_i} a_{ij} (x_j - x_i) \right)^{1+\alpha} x_i. \]
By Lemma 1, we have

$$\frac{dV_1(t)}{dt} \leq -m(n, \frac{1 + \alpha}{2}) \left( \sum_{i=1}^{n} \left( \sum_{v_j \in N_i} a_{ij}(x_j - x_i) \right)^2 \right)^{\frac{1+\alpha}{2}},$$

where $m(n, \frac{1+\alpha}{2}) = 1$.

If $V_1(x) \neq 0$,

$$\frac{\sum_{i=1}^{n} \left( \sum_{v_j \in N_i} a_{ij}(x_j - x_i) \right)^2}{V_1(x)} = \frac{x^T L(A)^T L(A) x}{\frac{1}{2} x^T L(A) x}.$$

Let the eigenvalues of $L(A)$ be $\lambda_1(L_A), \lambda_2(L_A), \ldots, \lambda_n(L_A)$ in the increasing order. Since $\mathcal{G}(A)$ is connected, by Lemma 2, $\lambda_1(L_A) = 0, \lambda_2(L_A) > 0$.

Let

$$D = \begin{bmatrix} 0 \\ \lambda_2(L_A) \\ \vdots \\ \lambda_n(L_A) \end{bmatrix}.$$ 

Since $L(A)$ is symmetric, there exists an orthogonal matrix $T \in \mathbb{R}^{n \times n}$ such that

$$L(A) = T^T DT.$$

Therefore,

$$\frac{x^T L(A)^T L(A) x}{\frac{1}{2} x^T L(A) x} = \frac{2x^T T^D DTT^D T x}{x^T T^T D T x} = \frac{2x^T T^T D^2 T x}{x^T T^T D T x} \geq 2\lambda_2(L_A).$$
And we have that
\[
\frac{dV_1(t)}{dt} \leq - \left( \frac{\sum_{i=1}^{n} \left( \sum_{v_j \in N_i} a_{ij} (x_j - x_i) \right)^2}{V_1(t)} \right)^{\frac{1+\alpha}{2}} V_1(t) \leq - (2\lambda_2(L_A))^{\frac{1+\alpha}{2}} V_1(t)^{\frac{1+\alpha}{2}}.
\]

Let \( K_1 = (2\lambda_2(L_A))^{\frac{1+\alpha}{2}} \) and let \( t_1 = \frac{(2V_1(0))^{\frac{1-\alpha}{2}}}{(1-\alpha)\lambda_2(L_A)^{\frac{1-\alpha}{2}}} \). Given initial state \( x(0) \), if \( V_1(0) \neq 0 \),
\[
V_1(t) \leq \left( -K_1 \frac{1-\alpha}{2} t + V_1(0)^{\frac{1-\alpha}{2}} \right)^{\frac{2}{1-\alpha}}, t < t_1,
\]
and
\[
\lim_{t \to t_1} V_1(t) = 0.
\]

Therefore, \( V_1(t) \) will reach 0 in finite time \( t_1 \), i.e., all states of agents will reach a consensus in finite time, and thus system (1) solves a finite-time agreement problem.

The finite time \( t_1 \) is an upper bound of time which guarantee that all agents’ states reach a consensus. Nevertheless, we can use it to establish a connect between convergence time and parameters: \( \alpha \) and initial state. Since \( 0 < \alpha < 1 \), \( 0 < \frac{1-\alpha}{2} < \frac{1}{2} \) and \( \frac{1}{2} < \frac{1+\alpha}{2} < 1 \). If the initial value \( V_1(0) \) increases, then \( t_1 \) increases accordingly. This is obvious, since \( V_1(0) \) can be seen as the total potential energy among agents [22] and thus it will take much time to reduce it to zero.

In [4], it was shown that if \( \alpha = 1 \), protocol (2) solves an agreement problem asymptotically and the convergence in such system is exponential with infinite settling time. If \( \alpha = 0 \), the whole system becomes a non-smooth system. In this case, we can prove that the states of agents reach an agreement in finite time \( \frac{\max_i x_i(0) - \min_i x_i(0)}{2} \), which is independent of graph weights. All those properties are reflected in our model, that is,
\[
\lim_{\alpha \to 1} t_1 = \infty,
\]
and
\[ \lim_{a \to 0} t_1 = \frac{\sqrt{2V_1(0)}}{\lambda_2(L_A)}. \]

Let \( \beta = \frac{1}{n} \mathbf{1}^T x(0) \) and let \( x(0) = y + \beta \mathbf{1} \). Since \( \mathbf{1}^T x(0) = \mathbf{1}^T y + n\beta \), we have that \( \mathbf{1}^T y = 0 \).

\[
\frac{\sqrt{2V_1(0)}}{\lambda_2(L_A)} = \sqrt{\frac{x^T(0)L(A)x(0)}{\lambda_2(L_A)}} = \sqrt{\frac{y^T L(A)y}{\lambda_2(L_A)}} \\
\geq \sqrt{\frac{y^T y}{\lambda_2(L_A)}} \geq \frac{\max_i y_i - \min_i y_i}{\sqrt{2}} \\
\geq \frac{\max_i x_i(0) - \min_i x_i(0)}{2}.
\]

Next, we consider protocol (3).

**Theorem 2** Consider system (1) with communication topology \( G(A) \) that is connected. Given protocol (3), system (1) solves the average-agreement problem in finite time.

Proof: Since \( a_{ij} = a_{ji} \) for all \( i, j \in \mathcal{I}_n \), we get that
\[
\sum_{i=1}^{n} \dot{x}_i(t) = 0.
\]

Let
\[ \kappa = \frac{1}{n} \sum_{i=1}^{n} x_i(t), \]
and we have that \( \kappa \) is time-invariant. Let \( x_i(t) = \kappa + \delta_i(t) \). Consequently, \( \sum_{i=1}^{n} \delta_i(t) = 0 \) and \( \dot{\delta}_i(t) = u_i(t) \). We take Lyapunov function
\[ V_2(\delta(t)) = \frac{1}{2} \sum_{i=1}^{n} \delta_i^2(t), \]
where \( \delta = [\delta_1, \delta_2, \ldots, \delta_n]^T \). In [4], \( \delta \) is referred to as the group
disagreement vector. Differentiate $V_2(t)$ with respect to $t$

$$\frac{dV_2(t)}{dt} = \sum_{i=1}^{n} \delta_i(t) \cdot \dot{\delta}_i(t)$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} (a_{ij} \delta_i \text{sign}(\delta_j - \delta_i) |\delta_j - \delta_i|^\alpha + a_{ji} \delta_j \text{sign}(\delta_i - \delta_j) |\delta_i - \delta_j|^\alpha)$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \delta_i \text{sign}(\delta_j - \delta_i) |\delta_j - \delta_i|^\alpha$$

$$= -\frac{1}{2} \sum_{i,j=1}^{n} a_{ij} |\delta_j - \delta_i|^{1+\alpha}$$

$$= -\frac{1}{2} \sum_{i,j=1}^{n} (a_{ij} \frac{2}{1+\alpha} (\delta_j - \delta_i)^2)^{\frac{1+\alpha}{2}}.$$

By Lemma [II]

$$\frac{dV_2(t)}{dt} \leq -\frac{1}{2} m(n^2, \frac{1+\alpha}{2}) \left( \sum_{i,j=1}^{n} a_{ij}^{\frac{2}{1+\alpha}} (\delta_i - \delta_j)^2 \right)^{\frac{1+\alpha}{2}},$$

where $m(n^2, \frac{1+\alpha}{2}) = 1$.

Let $G(\delta) = \sum_{i,j=1}^{n} a_{ij}^{\frac{2}{1+\alpha}} (\delta_i - \delta_j)^2$. Since communication topology $\mathcal{G}(A)$ is connected, we have that $G(\delta) = 0$ if and only if $\delta \in \text{span}(1)$. Since $\sum_{i=1}^{n} \delta_i(t) = 0$, $G(\delta) = 0$ if and only if $V_2(\delta) = 0$.

Let $U = \{ \delta : V_2(\delta) = 1 \text{ and } 1^T \delta = 0 \}$. Then $U$ is a bounded closed set. Let

$$K' = \inf_{\delta \in U} G(\delta).$$

Since $G(\delta)$ is continuous, $K'$ exists and $K' > 0$.

We can calculate $K'$ by matrix theory.

Let $B = [b_{ij}]_{n \times n}$, where $b_{ij} = a_{ij}^{\frac{2}{1+\alpha}}$. Then

$$G(\delta) = \sum_{i,j=1}^{n} b_{ij} (\delta_i - \delta_j)^2 = 2 \delta^T L(B) \delta,$$

where $L(B)$ is the graph Laplacian of $\mathcal{G}(B)$. 

12
Since $1^T\delta = 0$, we have $G(\delta) \geq 2\lambda_2(L_B)\delta^T\delta$, where $\lambda_2(L_B) > 0$ is the second smallest eigenvalue of $L(B)$. Since $V_2(\delta) = \frac{1}{2}\delta^T\delta = 1$,

$$G(\delta)|_{\delta \in U} \geq 4\lambda_2(L_B).$$

And thus $K' = 4\lambda_2(L_B)$. Therefore, if $V_2(t) \neq 0$,

$$\frac{dV_2(t)}{dt} \leq -\frac{1}{2} \left( \frac{G(\delta)}{V_2(t)} \right)^{\frac{1+\alpha}{2}}$$

$$\leq -\frac{1}{2} \left( 4\lambda_2(L_B)V_2(t) \right)^{\frac{1+\alpha}{2}}.$$

Let $K_2 = 2^\alpha \lambda_2(L_B)^{\frac{1+\alpha}{2}}$ and $t_2 = \frac{2^{1-\alpha}V_2(0)^{\frac{1-\alpha}{2}}}{(1-\alpha)\lambda_2(L_B)^{\frac{1-\alpha}{2}}}$. Then

$$\frac{dV_2(t)}{dt} \leq -K_2V_2(t)^{\frac{1+\alpha}{2}}.$$

And with the same arguments as in the proof of Theorem 1, we have that $V_2(t)$ reaches zero in finite time $t_2$, and system (1) solves the finite-time average-agreement problem.

We can see that the convergence time is closely related to the second smallest eigenvalue of $L(B)$, that is, the algebraic connectivity of $G(B)$. Maximizing it, we can get smaller convergence time.

Besides that protocol (3) solves the average-agreement problem, another main difference between the two protocols is that the Lyapunov function $V_2(\delta(t))$ used in the proof of Theorem 2 does not depend on the network topology. This property of $V_2(\delta(t))$ makes it a possible candidate as a common Lyapunov function for convergence analysis of the system with switching topology.

Suppose that $a_{ij}$ is a piece-wise constant right continuous function of time, denoted by $a_{ij}(t)$, and takes value in a finite set, such that $a_{ij}(t) = a_{ji}(t) \geq 0$ and $a_{ii}(t) = 0$ for all $i, j \in I_n$. And therefore the topology is changing as time evolves, denoted by $G(A(t))$. 

13
Theorem 3 If $G(A(t))$ is connected all the time, then protocol (3) solves the finite-time average-agreement problem.

Proof: Since all $a_{ij}(t), i, j \in \mathcal{I}_n$, take value in a finite set,

$$\lambda = \min_{t \geq 0} \lambda_2(t)$$

exists and $\lambda > 0$, where $\lambda_2(t)$ is the second smallest eigenvalue of the graph Laplacian of $G([a_{ij}^{\frac{2}{1+\alpha}}(t)])$.

Also let $\delta(t) = x(t) - \kappa \mathbf{1}$ and $V_2(\delta(t)) = \frac{1}{2} \delta^T(t) \delta(t)$. Then we have $1^T \delta(t) = 0$ and

$$\frac{dV_2(t)}{dt} \leq -2^\alpha \lambda \frac{1+\alpha}{2} V_2(t)^{\frac{1+\alpha}{2}}.$$

Therefore, $V_2(t)$ will reach zero in finite time $t_3 = \frac{2^{1-\alpha} V_2(0)^{\frac{1-\alpha}{1-\alpha}}}{(1-\alpha) \lambda \frac{1+\alpha}{2}}$ and the switching system solves the finite-time average-agreement problem.

Remark: The conditions in Theorem 3 can be relaxed in several ways. For example, we can assume that the sum of time intervals, in which $G(A(t))$ is connected, is larger than $t_3$.

\begin{center}
\begin{tabular}{cccc}
\text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} \\
\begin{tabular}{c}
 1 \\
 6 \\
\end{tabular} & \begin{tabular}{c}
 2 \\
 3 \\
 4 \\
 5 \\
\end{tabular} & \begin{tabular}{c}
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\end{tabular} & \begin{tabular}{c}
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 4 \\
 5 \\
\end{tabular} \\
\end{tabular}
\end{center}

Fig. 1. Four graphs: (a) $G_1$, (b) $G_2$, (c) $G_3$, (d) $G_4$

4 Simulations

In this section, we present some numerical simulations to illustrate our theoretical results.
Fig. 2. State trajectories of agents with communication topology $G_1$ and protocol (2)

Fig. 3. State trajectories of agents with communication topology $G_1$ and protocol (3)

Fig. 4. State trajectories of agents with switching communication topology
These simulations are performed with six agents. Fig. 1 shows four graphs and the weight of each edge is 2. For the multi-agent system with fixed topology, we assume that $\alpha = 0.5$, the communication topology is $G_1$, and initial state $x(0) = [-5, -3, 7, 9, 4, 5]^T$. By calculation, $V_1(0) = 338, t_1 = 11.7681$. The trajectories of agents’ states with protocol (2) are shown in Fig. 2. If we apply protocol (3), we obtain that the average of initial states is $2.8333, V_2(0) = 78.4167$, algebraic connectivity of the corresponding graph $G(B)$ is 1.0409, and thus $t_2 = 8.1673$. The trajectories of agents’ states are shown in Fig. 3. If the communication topology is switching from $G_1$, to $G_2$, to $G_3$, to $G_4$, and back to $G_1$, periodically, and each of them lasts for 0.25 seconds, then we apply protocol (3) and the states of agents achieve the average-agreement in finite-time $t_3 = 11.3000$. The trajectories of them are shown in Fig. 4.

5 Conclusion

We have considered the finite-time agreement problems of multi-agent systems with bidirectional information exchange. Two continuous finite-time agreement protocols have been presented. Furthermore, the relations between upper bound of convergence time of each protocol and some parameters are also analyzed.

The work of this paper is the first step toward finite-time consensus analysis of multi-agent systems, and there are still some other interesting and important topics need to be addressed. For example, do our protocols still work when the information exchange among agents is unidirectional? If the system is with switching topology and communication time-delays, do there exist similar results? And do there exist other effective finite-time protocols?
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