A VARIATIONAL APPROACH TO NONLOCAL SINGULAR PROBLEMS

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ABSTRACT. We provide a suitable variational approach for a class of nonlocal problems involving the fractional laplacian and singular nonlinearities for which the standard techniques fail. As a corollary we deduce a characterization of the solutions.

1. INTRODUCTION AND RESULTS

In recent years, considerable attention has been given to equations involving general integrodifferential operators, especially, those with the fractional Laplacian operator. This is related to the fact that the nonlocal structure has connection with many real world phenomena. Indeed, nonlocal operators naturally appear in elasticity problems [35], thin obstacle problem [9], phase transition [11, 8, 34], flames propagation [14], crystal dislocation [25, 39], stratified materials [33], quasi-geostrophic flows [15] and others. Since these operators are also related to Lévy processes and have a lot of applications to mathematical finance, they have been also studied from a probabilistic point of view (see for example [1, 7, 27, 28, 41]). We refer the readers to, for instance, [3, 5, 10, 11, 12, 13, 19, 23, 24, 32] where existence of solutions, qualitative properties of solutions and regularity of solutions are studied for some nonlocal problems. In this paper we aim to provide a variational structure to the following problem

\[(\mathcal{P}_\gamma)\begin{cases}
(-\Delta)\gamma u = \frac{1}{u^\gamma} + \omega & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}\]

where \(\omega \in W^{-s,2}(\Omega)\) i.e. the dual space of \(W_0^{s,2}(\Omega)\) that we will define below, \(\Omega\) is a bounded smooth domain, \(0 < s < 1, N > 2s\), \((-\Delta)^s\) is the fractional Laplacian (see Section 2 for the definition) and the equation is understood as in Definition 2.3.

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In spite of the fact that \((P)\) is formally the Euler equation of the functional

\[
J(u) = \frac{c_{N,s}}{4} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} \Phi(u) \, dx - \langle \omega, u \rangle \quad u \in W^{s,2}_0(\Omega),
\]

where

\[
(1.1) \quad \Phi(s) = \begin{cases} 
- \int_1^s t^{-\gamma} \, dt & \text{if } s \geq 0 \\
+\infty & \text{if } s < 0,
\end{cases}
\]

a standard variational approach is obstructed by the fact that the energy functional might be identically infinity as it is the case when solutions do not belong to \(W^{s,2}_0(\Omega)\). Even in the local case it has been shown in [22, 31] that the solution cannot belong to \(W^{1,2}_0(\Omega)\) if \(\gamma \geq 3\) so that, as remarked, the classical approach cannot be exploited. However many results have been obtained in the literature developing alternative techniques. We only mention here the related results in [6, 16, 17, 18, 29, 30, 31, 26, 40].

The study of nonlocal problems involving singular nonlinearities is quite undertaken in the literature. Existence and uniqueness of the solution to \((P)\) were studied in the recent works [2, 20]. Here, to deal with the nonlocal case, we exploit some ideas introduced in [17] facing the difficulties caused by the nonlocal nature of the problem. In all the paper we shall take into account the fact that the solutions are not in the classical nonlocal Sobolev spaces and the boundary datum has to be understood in a nonstandard way.

Let us now state our main result.

**Theorem 1.1.** Let \(\gamma > 0\), \(\omega \in W^{-s,2}(\Omega)\) and \(u \in W^{s,2}_{\text{loc}}(\Omega)\). If \(u\) satisfies the problem \((P_D)\)

\[
\begin{align*}
& \begin{cases} 
& u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\gamma} \in L^1_{\text{loc}}(\Omega), \\
& \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy - \int_{\Omega} u^{-\gamma} \varphi \, dx = \langle \omega, \varphi \rangle \quad \forall \varphi \in C^\infty_c(\Omega), \\
& u \leq 0 \text{ on } \partial\Omega,
\end{cases}
\end{align*}
\]

then \(u\) is the solution to the problem \((P_V)\)

\[
\begin{align*}
& \begin{cases} 
& u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\gamma} \in L^1_{\text{loc}}(\Omega), \\
& \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - u(x)) - (v(y) - u(y)))}{|x-y|^{N+2s}} \, dx \, dy - \int_{\Omega} u^{-\gamma}(v-u) \, dx \geq \langle \omega, v-u \rangle \quad \forall v \in u + (W^{s,2}_0(\Omega) \cap L^\infty_c(\Omega)) \text{ with } v \geq 0 \text{ a.e. in } \Omega, \\
& u \leq 0 \text{ on } \partial\Omega.
\end{cases}
\end{align*}
\]
Moreover if \( \omega \in W^{-s,2}(\Omega) \cap L^1_{\text{loc}}(\Omega) \), then the problems \( (P_D) \) and \( (P_V) \) are equivalent.

Note that \( u \in W^{s,2}_{\text{loc}}(\Omega) \) is a solution to \( (P_V) \) if and only if \( u \) is the minimum of a suitable functional actually defined in \( (3.11) \). Remarkably, this provides a variational characterization of the solutions that is completely new in this setting and that could be exploited to deduce existence and multiplicity results under suitable assumptions.

Furthermore, as consequence of Theorem 1.1, we also provide a decomposition of the solution \( u \). Namely we deduce that:

\[
  u = u_0 + w
\]

where, \( u_0 \in L^\infty(\Omega) \) is the unique solution to \( (P_\gamma) \) with \( \omega \equiv 0 \) (see Proposition 3.1 below) and \( w \in W^{s,2}_{\text{loc}}(\Omega) \) is a critical point (in the meaning of \( [38] \)) of an associated functional.

To state such a result let us start considering \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfying the growth assumption

\[
  \left\{ \begin{array}{l}
  \text{there exists } a \in L^{\frac{2N}{N+2s}}(\Omega) \text{ and } b \in \mathbb{R} \text{ such that } \\
  |g(x,t)| \leq a(x) + b|t|^{\frac{N+2s}{N-2s}} \text{ for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}.
  \end{array} \right.
\]

Then let \( g_1(x,t) = g(x,u_0(x) + t) \), \( G_1(x,t) = \int_0^t g_1(x,t) \, dt \) and \( \Phi : W^{s,2}_{\text{loc}}(\Omega) \to ]-\infty, +\infty[ \) the \( C^1 \) functional defined by

\[
  \Phi(u) = -\int_\Omega G_1(x,u) \, dx.
\]

Moreover let \( \Psi : W^{s,2}_{\text{loc}}(\Omega) \to ]-\infty, +\infty[ \) be the convex functional defined by

\[
  \Psi(v) = \frac{CN,s}{4} \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} G_0(x,v) \, dx.
\]

Finally define \( F : W^{s,2}_{\text{loc}}(\Omega) \to ]-\infty, +\infty[ \) by

\[
  F(v) = \Psi(v) + \Phi(v).
\]

We have the following

**Theorem 1.2.** Let \( \gamma > 0 \).

The function \( u \in \left. W^{s,2}_{\text{loc}}(\Omega) \cap L^{\frac{2N}{N-2s}}(\Omega) \right. \) is a solution to the problem

\[
  \left\{ \begin{array}{l}
  u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\gamma} \in L^1_{\text{loc}}(\Omega), \\
  \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\Omega} u^{-\gamma} \varphi \, dx + \int_{\Omega} g(x,u) \varphi \, dx \quad \forall \varphi \in C^\infty_c(\Omega), \\
  u \leq 0 \text{ on } \partial \Omega,
  \end{array} \right.
\]

if and only if
\[ u \in u_0 + W_0^{s,2}(\Omega) \text{ and } w := u - u_0 \text{ is a critical point of } F. \]

The paper is organized as follows: in Section 2 we give some preliminaries related to the functional framework associated to problem \((P_\gamma)\), we introduce the proper notion of solution that will be used through this work and some preliminary results. Section 3 deals with the proof of the main result of this work.

2. Notations and Preliminary Results

Let us recall that, given a function \(u\) in the Schwartz’s class \(\mathcal{S}(\mathbb{R}^N)\) we define for \(0 < s < 1\), the fractional Laplacian as

\[ (-\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi), \quad \xi \in \mathbb{R}^N, \]

where \(\hat{u} \equiv \mathcal{F}(u)\) is the Fourier transform of \(u\). It is well known (see for example [37, 41]) that this operator can be also represented, for suitable function \(s\), as a principal value of the form

\[ (-\Delta)^s u(x) := c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy \]

where

\[ c_{N,s} := \left( \int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}} \, d\xi \right)^{-1} = \frac{4^s \Gamma \left( \frac{N}{2} + s \right)}{\pi^{\frac{N}{2}} \Gamma(-s)} > 0, \]

is a normalizing constant chosen to guarantee that (2.1) is satisfied (see [23, 36, 41]).

The symbol \(\| \cdot \|_{L^p(\Omega)}\) stands for the standard norm for the \(L^p(\Omega)\) space. For a measurable function \(u : \mathbb{R}^N \rightarrow \mathbb{R}\), we let

\[ [u]_{D^{s,2}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2} \]

be its Gagliardo seminorm. We consider the space

\[ W^{s,2}(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : [u]_{D^{s,2}(\mathbb{R}^N)} < \infty \right\}, \]

endowed with norm \(\| \cdot \|_{L^2(\mathbb{R}^N)} + [\cdot]_{D^{s,2}(\mathbb{R}^N)}\). For \(\Omega \subset \mathbb{R}^N\) open and bounded, we consider

\[ W_0^{s,2}(\Omega) := \left\{ u \in W^{s,2}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}, \]

endowed with norm \([\cdot]_{D^{s,2}(\mathbb{R}^N)}\). The imbedding \(W_0^{s,2}(\Omega) \hookrightarrow L^r(\Omega)\) is continuous for \(1 \leq r \leq 2_s^*\) and compact for \(1 \leq r < 2_s^*\), where \(2_s^* := 2N/(N - 2s)\) and \(N > 2s\) (as we are assuming throughout the paper). The space \(W_0^{s,2}(\Omega)\) can be equivalently defined as the completion of \(C_0^\infty(\Omega)\) in the norm \(\| \cdot \|_{L^2(\mathbb{R}^N)} + [\cdot]_{D^{s,2}(\mathbb{R}^N)}\), provided \(\partial \Omega\) is smooth enough.

In this context by \(C_0^\infty(\Omega)\) we mean the space

\[ C_0^\infty(\Omega) := \left\{ f : \mathbb{R}^N \rightarrow \mathbb{R} : f \in C^\infty(\mathbb{R}^N), \text{ support } f \text{ is compact and support } f \subseteq \Omega \right\}. \]
Finally define the space
\[ W^{s,2}_c(\Omega) := \{ u : \Omega \to \mathbb{R} : u \in L^2(K), \ |u|_{W^{s,2}(K)} < \infty, \ \text{for all } K \subset \Omega \}. \]

We shall denote the localized Gagliardo seminorm by
\[ [u]_{W^{s,2}(\Omega)} := \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}. \]

We prove that all the three terms on the right-hand side of (2.4) are well defined. In fact
\[ \frac{1}{2} C_{N,s} \int_{K \times K} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy < C, \]
for some positive constant C, since by hypothesis \( u \in W^{s,2}_c(\Omega) \).
We can write the second term as

\[
\frac{1}{2} c_{N,s} \int_{K \times K^c} \frac{(u(x) - u(y))((\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy
\]

\[
= \frac{1}{2} c_{N,s} \int_{K \times K^c} \frac{(u(x) - u(y))((\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]

We observe that, for all points \((x, y) \in K \times K^c\), we have that \(|x - y| \geq \delta > 0\), for some positive constant \(\delta = \delta(K, K^c)\) and therefore

\[
\frac{1}{2} c_{N,s} \int_{K \times K^c} \frac{(u(x) - u(y))((\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \leq C,
\]

with \(C = C(\delta, K, \varphi, \|u\|_{L^1(\mathbb{R}^N)}, \|\varphi\|_{L^\infty(K^c)})\) a positive constant. Here we have used the fact that \(u \in L^1(\mathbb{R}^N)\) (since \(u \in L^1(\Omega)\) and \(u = 0\) a.e. in \(\mathbb{R}^N \setminus \Omega\)) and \(\varphi \in C^\infty(K^c)\). From (2.5) and (2.6) we obtain

\[
\frac{1}{2} c_{N,s} \int_{K \times K^c} \frac{(u(x) - u(y))((\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \leq C.
\]

For the third term we argue in the same way as in (2.5), (2.6) and (2.7). Finally, by (2.4) we obtain the thesis. \(\square\)

Having in mind Proposition 2.2, the basic definition of solution can be formulated in the following

**Definition 2.3.** A positive function \(u \in W^{s,2}_{loc}(\Omega) \cap L^1(\Omega)\) is a weak solution to problem \((P_\gamma)\) if \(u^{-\gamma} \in L^1_{loc}(\Omega)\),

\[
u = 0 \quad \text{for a.e. } x \in \mathbb{R}^N \setminus \Omega
\]

and we have

\[
\frac{c_{N,s}}{2} \int \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))((\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} \varphi u^\gamma \, dx + \langle \omega, \varphi \rangle,
\]

for every \(\varphi \in C^\infty_c(\Omega)\).

We state a weak comparison principle for sub-super solutions to \((P_\gamma)\). To do this we first give the following
**Definition 2.4.** Given \( z \in W^{s,2}_{\text{loc}}(\Omega) \cap L^1(\Omega) \) with \( s \geq 0 \), we say that \( z \) is a weak supersolution (respectively subsolution) to \([\mathcal{P}_\gamma]\), if
\[
\int \int_{\mathbb{R}^N} \frac{(z(x) - z(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \geq \int_{\Omega} \frac{\varphi}{z^\gamma} \, dx + \langle \omega, \varphi \rangle,
\forall \varphi \in C^\infty_c(\Omega), \varphi \geq 0
\]
(and respectively)
\[
\int \int_{\mathbb{R}^N} \frac{(z(x) - z(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \leq \int_{\Omega} \frac{\varphi}{z^\gamma} \, dx + \langle \omega, \varphi \rangle,
\forall \varphi \in C^\infty_c(\Omega), 0 \leq \varphi \leq z.
\]

**Theorem 2.5.** Let \( \gamma > 0 \) and \( \omega \in W^{-s,2}(\Omega) \). Let \( u \) be a subsolution to \([\mathcal{P}_\gamma]\) such that \( u \leq 0 \) on \( \partial \Omega \) and let \( v \) be a supersolution to \([\mathcal{P}_\gamma]\). Then, \( u \leq v \) a.e. in \( \Omega \).

**Proof.** Theorem 2.5 can be proved as [20, Theorem 4.2]. \( \square \)

3. **Proof of Theorem 1.1**

**Proposition 3.1.** Let us consider the problem
\[
(\Delta)^s u = \frac{1}{u^\gamma} \quad \text{in} \ \Omega,
\]
\[
u \quad \text{in} \ \Omega,
\]
\[u = 0 \quad \text{in} \ \mathbb{R}^N \setminus \Omega.
\]

Then (3.1) has a unique solution \( u_0 \in C^\infty(\Omega) \) (in the sense of Definition 2.3) such that
\begin{enumerate}
\item \( u_0 \in W^{s,2}_0(\Omega) \) if \( 0 < \gamma \leq 1 \), with \( \text{essinf}_K u > 0 \) for any compact \( K \subset \Omega \);
\item \( u_0 \in W^{s,2}_{\text{loc}}(\Omega) \cap L^1(\Omega) \) such that \( u_0^{\gamma/2} \in W^{s,2}_0(\Omega) \) if \( \gamma > 1 \), with \( \text{essinf}_K u > 0 \) for any compact \( K \subset \Omega \).
\end{enumerate}

Moreover
\[
\|u_1\|_{L^\infty(\Omega)} \leq u_0 \leq (\gamma + 1)u_1, \quad \gamma > 0,
\]
where \( u_1 \) is the solution to \((\Delta)^s u = 1\) in \( \Omega \) and \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \). In particular \( u_0 \in C(\Omega) \).

**Proof.** The existence, uniqueness and summability properties of the solution \( u_0 \) to (3.1) follow by [20, Theorem 1.2, Theorem 1.6]. We have to prove (3.2). Let us consider the unique solution \( u_1 \in W^{s,2}_0(\Omega) \cap C^\infty(\Omega) \) to \((\Delta)^s u = 1\) in \( \Omega \) and \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \). In particular we have that \( u_1 > 0 \) for any compact \( K \subset \Omega \) and by standard regularity results [32], it follows that \( u_1 \in C^\delta(\mathbb{R}^N) \). Let us define
\[
\hat{w} = ((\gamma + 1)u_1)^{1/\gamma}, \quad \gamma > 0.
\]

We want to show that \( \hat{w} \) is a supersolution to (3.1), namely
\[
\frac{c_{N,s}}{2} \int \int_{\mathbb{R}^N} \frac{(\hat{w}(x) - \hat{w}(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \geq \int_{\Omega} \frac{\varphi}{\hat{w}^{\gamma}} \, dx,
\]
where \( c_{N,s} \) is a constant.
for every $\varphi \in C_c^\infty(\Omega)$ and $\varphi \geq 0$. By (3.3) it follows that $\check{w} \in W^{s,2}_{\text{loc}}(\Omega) \cap L^1(\Omega)$ and $\check{w} = 0$ in $\mathbb{R}^N \setminus \Omega$. Therefore by Proposition 2.2 we have that the l.h.s. of (3.4) is well defined. Hence

\begin{equation}
(3.5) \quad \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{(\check{w}(x) - \check{w}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy
\end{equation}

\begin{align*}
&= \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) \geq \varphi(y) \}} \frac{(\check{w}(x) - \check{w}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&+ \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) < \varphi(y) \}} \frac{(\check{w}(x) - \check{w}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.
\end{align*}

We estimate the first term on the r.h.s of (3.5). Using a convexity argument, we deduce

\begin{equation}
(3.6) \quad \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) \geq \varphi(y) \}} \frac{\check{w}^{-\gamma}(x) (u_1(x) - u_1(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy
\end{equation}

\begin{align*}
&\geq \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) \geq \varphi(y) \}} \frac{(u_1(x) - u_1(y)) (\check{w}^{-\gamma}(x) \varphi(x) - \check{w}^{-\gamma}(y) \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&+ \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) \geq \varphi(y) \}} \frac{(u_1(x) - u_1(y)) (\check{w}^{-\gamma}(y) - \check{w}^{-\gamma}(x) \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy
\end{align*}

Using a similar argument we get

\begin{equation}
(3.7) \quad \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) < \varphi(y) \}} \frac{(u_1(x) - u_1(y)) (\check{w}^{-\gamma}(x) \varphi(x) - \check{w}^{-\gamma}(y) \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy
\end{equation}

\begin{align*}
&\geq \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N} \cap \{ \varphi(x) < \varphi(y) \}} \frac{(u_1(x) - u_1(y)) (\check{w}^{-\gamma}(x) \varphi(x) - \check{w}^{-\gamma}(y) \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.
\end{align*}

From (3.5), collecting (3.6) and (3.7), we deduce

\begin{align*}
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} &\frac{(\check{w}(x) - \check{w}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&\geq \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{(u_1(x) - u_1(y)) (\check{w}^{-\gamma}(x) \varphi(x) - \check{w}^{-\gamma}(y) \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&= \int_{\Omega} \frac{\varphi}{\check{w}^{\gamma}} \, dx,
\end{align*}

that is (3.4). Defining

\begin{equation}
(3.8) \quad \check{w} = \|u_1\|_{L^\infty(\Omega)}^{-\gamma} u_1, \quad \gamma > 0,
\end{equation}

using the weak formulation (3.1), we can prove as well that $\check{w}$ is a subsolution to (3.1), namely

\begin{align*}
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} &\frac{(\check{w}(x) - \check{w}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \leq \int_{\Omega} \frac{\varphi}{\check{w}^{\gamma}} \, dx,
\end{align*}
for every \( \varphi \in C_c^\infty(\Omega) \) and \( \varphi \geq 0 \). Then using the definitions (3.3) and (3.8), together with Theorem 2.5, we get (3.2). Now it readily follows that \( u_0 \in C(\Omega) \).

Let \( u_0 \) as in Proposition 3.1. Let \( G_0 : \Omega \times \mathbb{R} \to [0, +\infty] \) be defined by
\[
G_0(x, s) = \Phi(u_0(x) + s) - \Phi(u_0(x)) + su_0(x)^{-\gamma},
\]
where \( \Phi(\cdot) \) is defined in (1.1). Then \( G_0(x, 0) = 0 \) and \( G_0(x, \cdot) \) is convex and lower semicontinuous for any \( x \in \Omega \). Moreover \( G_0(x, \cdot) \) is \( C^1 \) on \( ] - u_0(x), +\infty[ \) with
\[
D_sG_0(x, s) = u_0^{-\gamma}(x) - (u_0(x) + s)^{-\gamma}.
\]

Let us define the functional
\[
J_\omega(u) = \frac{c_{N,s}}{4} \int_{\mathbb{R}^{2N}} \frac{((u(x) - u_0(x)) - (u(y) - u_0(y)))^2}{|x - y|^{N+2s}} \, dx \, dy
+ \int_{\Omega} G_0(x, u - u_0) \, dx - \langle \omega, u - u_0 \rangle \quad \text{if } u \in u_0 + W_0^{s,2}(\Omega)
\]
and \( J_\omega(u) = +\infty \) otherwise. We observe that \( J_\omega \) is strictly convex, lower semicontinuous and coercive and that \( J_\omega(u_0) = 0 \). We remark that the real domain of the functional \( J_\omega \) is given by
\[
\{ u \in u_0 + W_0^{s,2}(\Omega) : G_0(x, u - u_0) \in L^1(\Omega) \}.
\]

**Theorem 3.2.** For every \( \omega \in W^{-s,2}(\Omega) \) and \( u \in W_{\text{loc}}^{s,2}(\Omega) \), it follows that \( u \) is the minimum of \( J_\omega \) if and only if \( u \) verifies
\[
\begin{align*}
\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))((v(x) - u(x)) - (v(y) - u(y)))}{|x - y|^{N+2s}} \, dx \, dy &- \int_{\Omega} u^{-\gamma}(v - u) \, dx \geq \langle \omega, v - u \rangle \\
\forall v &\in u + (W_0^{s,2}(\Omega) \cap L_{c}^{\infty}(\Omega)) \text{ with } v \geq 0 \text{ a.e. in } \Omega, \quad u \leq 0 \text{ on } \partial\Omega.
\end{align*}
\]

In particular for every \( \omega \in W^{-s,2}(\Omega) \), problem (3.12) has one and only one solution \( u \in W_{\text{loc}}^{s,2}(\Omega) \).

**Proof.** We start proving (3.12). Given \( \omega \in W^{-s,2}(\Omega) \), using standard minimization techniques, there exits only one minimum \( u \in u_0 + W_0^{s,2}(\Omega) \) of \( J_\omega \). Therefore \( G_0(x, u - u_0) \in L^1(\Omega) \) and (see (1.1) and (3.9))
\[
(3.13) \quad u \geq 0 \quad \text{a.e. in } \Omega.
\]
Let \( v \in u_0 + W_0^{s,2}(\Omega) \) be such that
\[
G_0(x, v - u_0) \in L^1(\Omega).
\]
Then $v \geq 0$ a.e. in $\Omega$ and moreover $v - u \in W^{s,2}_0(\Omega)$. Then since $D_s G_0(\cdot, s)$ is non decreasing (see (3.10)), for $\xi \in ((v-u_0) \wedge (u-u_0), (v-u_0) \vee (u-u_0))$, we deduce

$$L^1(\Omega) \ni G_0(x, v-u_0) - G_0(x, u-u_0) = (u_0^{-\gamma} - (u_0 + \xi)^{-\gamma})(v-u)$$

$$\geq (u_0^{-\gamma} - u^{-\gamma})(v-u),$$

namely

$$(3.14) \quad \left(\frac{1}{u_0^\gamma} - \frac{1}{u^\gamma}\right)(v-u) \in L^1(\Omega).$$

Since $G_0(x, \cdot)$ is convex (see (3.9)) we deduce also that, for $t \in [0, 1]$,

$$G_0(x, t(v-u_0) + (1-t)(u-u_0)) = G_0(x, u-u_0 + t(v-u)) \in L^1(\Omega).$$

Since $u$ is the minimum point, for $t \in (0, 1]$ we get

$$(3.15) \quad 0 \leq \frac{J_\omega(u+t(v-u)) - J_\omega(u)}{t}$$

$$= \frac{c_{N,s}}{2} \int_\mathbb{R}^2N \frac{((u(x) - u_0(x)) - (u(y) - u_0(y)))((v(x) - u(x)) - (v(y) - u(y)))}{|x-y|^{N+2s}} dx dy$$

$$+ \frac{c_{N,s}}{4} \int_\mathbb{R}^2N \frac{((v(x) - u(x)) - (v(y) - u(y)))^2}{|x-y|^{N+2s}} dx dy$$

$$+ \frac{1}{t} \left( \int_\Omega G_0(x, u-u_0 + t(v-u)) dx - \int_\Omega G_0(x, u-u_0) dx \right) - \langle \omega, (v-u) \rangle$$

$$= \frac{c_{N,s}}{2} \int_\mathbb{R}^2N \frac{((u(x) - u_0(x)) - (u(y) - u_0(y)))((v(x) - u(x)) - (v(y) - u(y)))}{|x-y|^{N+2s}} dx dy$$

$$+ \frac{c_{N,s}}{4} \int_\mathbb{R}^2N \frac{((v(x) - u(x)) - (v(y) - u(y)))^2}{|x-y|^{N+2s}} dx dy$$

$$+ \int_\Omega \left( \frac{1}{u_0^\gamma} - \frac{1}{(u_0 + \xi_t)^\gamma} \right) (v-u) dx - \langle \omega, (v-u) \rangle,$$

with $\xi_t \in ((u-u_0 + t(v-u)) \wedge (u-u_0), (u-u_0 + t(v-u)) \vee (u-u_0))$. Recalling (3.14) and that $v-u \in W^{s,2}_0(\Omega)$, passing to the limit for $t \to 0^+$ in (3.15) we obtain

$$(3.16) \quad \frac{c_{N,s}}{2} \int_\mathbb{R}^2N \frac{((u(x) - u_0(x)) - (u(y) - u_0(y)))((v(x) - u(x)) - (v(y) - u(y)))}{|x-y|^{N+2s}} dx dy$$

$$\geq \int_\Omega \left( \frac{1}{u_0^\gamma} - \frac{1}{(u_0 + \xi_t)^\gamma} \right) (v-u) dx + \langle \omega, (v-u) \rangle,$$
for every \( v \in u_0 + W^{s,2}_0(\Omega) \) such that \( G_0(x, v - u_0) \in L^1(\Omega) \). In particular (3.14) holds for all \( v \in C^\infty_c(\Omega) \) with \( v \geq 0 \). Therefore (since \( v \) is arbitrary) we obtain that
\[
\left( \frac{1}{u_0} - \frac{1}{w} \right) v \in L^1(\Omega) \quad \forall v \in C^\infty_c(\Omega) \text{ with } v \geq 0,
\]
whence \( u^{-} \in L^1_{\text{loc}}(\Omega) \) and (see also (3.13)) \( u > 0 \text{ a.e. in } \Omega \). For \( \varepsilon, \sigma > 0 \) let us define
\[
(3.17) \quad v = \min\{ u - u_0, \varepsilon - (u_0 - \sigma)^+ \}.
\]
Since \( t \to t^+ \), \( t \in \mathbb{R} \) is a Lipschitz function, we remark that
\[
(3.18) \quad w(x) := \varepsilon - (u_0 - \sigma)^+ \in W^{s,2}_0(\Omega).
\]
Moreover, by Proposition 3.1 we know that \( u_0 \in C^{\infty}_0(\Omega) \). Therefore there exists a compact set \( K \subseteq \Omega \) such that \( u < \sigma \) in \( K^c = \mathbb{R}^N \setminus K \). We want to show
\[
(3.19) \quad \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < +\infty.
\]
We have
\[
(3.20) \quad \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = \int_{\mathbb{R}^N \setminus (K^c \times K^c)} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
since
\[
\int_{K^c \times K^c} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0.
\]
By a symmetry argument
\[
(3.21) \quad \int_{\mathbb{R}^N \setminus (K^c \times K^c)} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = \int_{K^c \times K^c} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
\[+ 2 \int_{K^c \times K^c} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
and readily by (3.18)
\[
\int_{K^c \times K^c} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < +\infty.
\]
Let \( \delta = \text{dist}(K, \partial \Omega)/2 \). We have
\[
(3.22) \quad \int_{K^c \times K^c} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
\[= \int_{K} dx \int_{K^c \cap \{y-x \leq \delta\}} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dy + \int_{K} dx \int_{K^c \cap \{y-x \geq \delta\}} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \, dy
\]
\[= I_1 + I_2.
\]
In particular let us consider a compact \( \tilde{K} \subseteq \Omega \) such that
\[
K \subset \tilde{K} \quad \text{and} \quad \text{for } x \in K \text{ fixed } K^c \cap \{|y-x| < \delta\} \subset \tilde{K}.
\]

NONLOCAL SINGULAR PROBLEMS
Using (3.18) we deduce that

\[
I_1 \leq \int_K dx \int_{K \cap \{|y-x| < \delta\}} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dy 
\leq \int_K dx \int_{K} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dy < +\infty.
\]

On the other hand

\[
I_2 \leq C(\|u_0\|_{L^\infty(\Omega)}) \int_K dx \int_{R^N \cap \{|y-x| \geq \delta\}} \frac{1}{|x-y|^{N+2s}} dy 
\leq \int_K dx \int_{R^N \setminus B_\delta(0)} \frac{1}{|y|^{N+2s}} dy < +\infty.
\]

Therefore, recalling (3.20), (3.21), (3.23) and (3.24), we obtain (3.19). By the definition (3.17), we deduce that

\[
(v = 0 \text{ a.e. in } R^N \setminus \Omega \text{ and } v \in L^2(\Omega). \text{ Finally by (3.19) and recalling also that } u - u_0 \in W^{s,2}_0(\Omega), \text{ we get that } v \in W^{s,2}_0(\Omega). \text{ Using (3.17) we infer that either } v = u - u_0 \text{ or } \varepsilon = v \leq u - u_0 \text{ or } v = \varepsilon + \sigma - u_0 \text{ and } u_0 \geq \sigma. \text{ In all three cases (see (3.9)) we have that } G_0(x, v) \in L^1(\Omega) \text{ and that }
\]

\[
((u_0 - \sigma)^+ + u - u_0 - \varepsilon)^+ = u - u_0 - v \in W^{s,2}_0(\Omega)
\]

and

\[
(\frac{1}{u_0^\gamma} - \frac{1}{u^\gamma})(v + u_0 - u) \in L^1(\Omega),
\]

where we used a similar argument already used to get (3.14). Then we use (3.16), (replacing \(v\) with \(u_0 + v\))

\[
\frac{c_{N,s}}{2} \int_{R^{2N}} \left( (u(x) - u_0(x)) - (u(y) - u_0(y)) \cdot \frac{(v(x) + u_0(x) - u(x)) - (v(y) + u_0(y) - u(y))}{|x-y|^{N+2s}} \right) dx dy 
\geq \int_{\Omega} \left( \frac{1}{u_0^\gamma} - \frac{1}{u^\gamma} \right) (v + u_0 - u) dx + \langle \omega, (v + u_0 - u) \rangle.
\]

In particular by (3.17), since \(u \neq u_0 + v\) implies \(u > \varepsilon\), from (3.26), we have that both

\[
\frac{1}{u^\gamma}(v + u_0 - u) \in L^1(\Omega) \quad \text{and} \quad \frac{1}{u_0^\gamma}(v + u_0 - u) \in L^1(\Omega).
\]

We know that \(u_0\) (see Proposition 1.1) satisfies

\[
\frac{c_{N,s}}{2} \int_{R^{2N}} \left( \frac{u_0(x) - u_0(y)}{|x-y|^{N+2s}} \right) \varphi(x) - \varphi(y) \right) dx dy = \int_{\Omega} \frac{\varphi}{u_0^\gamma} dx,
\]
for every $\varphi \in C^\infty_c(\Omega)$. Using the nonlocal Kato inequality [21] we get
\begin{equation}
\frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{((u_0(x) - \sigma)^+ - (u_0(y) - \sigma)^+)(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \leq \int_{\Omega} \frac{\varphi}{u_0^\gamma} \, dx,
\end{equation}
for every $\varphi \in C^\infty_c(\Omega)$, $\varphi \geq 0$. Using standard arguments, we point out that the inequality (3.30) holds true for non negative $\varphi \in W^{s,2}_0(\Omega)$ with compact support contained in $\Omega$. By density, let $\varphi_n \in C^\infty_c(\Omega)$ such that $\varphi_n^+ \to u - u_0 - v$ in $W^{s,2}_0(\Omega)$. Let us define
\begin{equation}
\tilde{\varphi}_n := \min\{u - u_0 - v, \varphi_n^+\}.
\end{equation}
As we did above (see (3.19)) we can deduce that $(u_0 - \sigma)^+ \in W^{s,2}_0(\Omega)$. Therefore, using (3.30) with $\tilde{\varphi}_n$ defined in (3.31), we pass to the limit using (3.28) and dominate convergence theorem, getting
\begin{equation}
\frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{((u_0(x) - \sigma)^+ - (u_0(y) - \sigma)^+)(u(x) - u_0(x) - v(x) - (u(y) - u_0(y) - v(y)))}{|x - y|^{N+2s}} \, dx \, dy
\leq \int_{\Omega} \frac{u - u_0 - v}{u_0^\gamma} \, dx.
\end{equation}
Combining (3.32) with (3.27) we deduce
\begin{equation}
\frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{((u_0(x) - \sigma)^+ + u(x) - u_0(x) - \varepsilon) - ((u_0(y) - \sigma)^+ + u(y) - u_0(y) - \varepsilon))}{|x - y|^{N+2s}} \, dx \, dy
\leq \int_{\Omega} \frac{1}{u_0^\gamma}(u - u_0 - v) \, dx + \langle \omega, (u - u_0 - v) \rangle.
\end{equation}
\begin{equation}
\leq \varepsilon^{-\gamma} \int_{\Omega} (u - u_0 - v) \, dx + \langle \omega, (u - u_0 - v) \rangle.
\end{equation}
Let us set $f := (u_0 - \sigma)^+ + u - u_0 - \varepsilon$ and observe that by (3.25), one has that $f^+ = u - u_0 - v$. We have that
\begin{equation}
\frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(f(x) - f(y))(f^+(x) - f^+(y))}{|x - y|^{N+2s}} \, dx \, dy
\end{equation}
\begin{equation}
= \frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{((f(x) - f(x))^+ - (f(y) - f(y))^+)(f^+(x) - f^+(y))}{|x - y|^{N+2s}} \, dx \, dy
\end{equation}
\begin{equation}
+ \frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|f^+(x) - f^+(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\end{equation}
\begin{equation}
\geq \frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|f^+(x) - f^+(y)|^2}{|x - y|^{N+2s}} \, dx \, dy,
\end{equation}
where we used the fact that
\[
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{((f(x) - f(x)^{+}) - (f(y) - f(y)^{+})) (f^{+}(x) - f^{+}(y))}{|x - y|^{N+2s}} \, dx \, dy \geq 0.
\]

In fact let \(K_{f}^{+} = \text{support} \, (f^{+})\) and \(K_{f}^{-} = \text{support} \, (f^{-})\). Therefore (using also a symmetry argument)
\[
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{((f(x) - f(x)^{+}) - (f(y) - f(y)^{+})) (f^{+}(x) - f^{+}(y))}{|x - y|^{N+2s}} \, dx \, dy
= c_{N,s} \int_{K_{f}^{+} \times K_{f}^{-}} \frac{-f(y)f(x)}{|x - y|^{N+2s}} \, dx \, dy \geq 0.
\]

Collecting (3.33) and (3.34) we finally deduce
\[
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u_{0}(x) - v(x)) - (u(y) - u_{0}(y) - v(y))|^{2}}{|x - y|^{N+2s}} \, dx \, dy
\leq \varepsilon^{-\gamma} \int_{\Omega} (u - u_{0} - v) \, dx + \langle \omega, (u - u_{0} - v) \rangle.
\]

Hence for any \(\varepsilon > 0\) (see also (3.25)),
\[
((u_{0} - \sigma)^{+} + u - u_{0} - \varepsilon)^{+} = u(x) - u_{0}(x) - v(x)
\]
is uniformly bounded w.r.t. \(\sigma\) in \(W_{0}^{s,2}(\Omega)\). By Fatou’s Lemma, for \(\sigma \to 0^{+}\) we have that \((u - \varepsilon)^{+} \in W_{0}^{s,2}(\Omega)\), that is \(u \leq 0\) on \(\partial \Omega\) according to Definition 2.1.

Let now \(v \in u + (W_{0}^{s,2}(\Omega) \cap L^{\infty}(\Omega))\) with \(v \geq 0\) a.e. in \(\Omega\) and \(v_{0} \in C_{c}^{\infty}(\Omega), v_{0} \geq 0\) in \(\Omega\) such that \(v_{0} = 1\) where \(v \neq u\). Then, for any \(\varepsilon > 0\), \(G_{0}(x, v + \varepsilon v_{0} - u_{0}) \in L^{1}(\Omega)\) and therefore by (3.16)
\[
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{((u(x) - u_{0}(x)) - (u(y) - u_{0}(y))) \cdot ((v(x) + \varepsilon v_{0}(x) - u(x)) - (v(y) + \varepsilon v_{0}(y) - u(y)))}{|x - y|^{N+2s}} \, dx \, dy
\geq \int_{\Omega} \left( \frac{1}{u^{\gamma}} - \frac{1}{u_{0}^{\gamma}} \right) (v + \varepsilon v_{0} - u) \, dx + \langle \omega, (v + \varepsilon v_{0} - u) \rangle,
\]

namely for \(\varepsilon \to 0\)
\[
\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{((u(x) - u_{0}(x)) - (u(y) - u_{0}(y))) ((v(x) - u(x)) - (v(y) - u(y)))}{|x - y|^{N+2s}} \, dx \, dy
\geq \int_{\Omega} \left( \frac{1}{u^{\gamma}} - \frac{1}{u_{0}^{\gamma}} \right) (v - u) \, dx + \langle \omega, (v - u) \rangle.
\]
By (3.29) we also have that
\[
\frac{c_{N,s}}{2} \int \int_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(v(x) - u(x)) - (v(y) - u(y)))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} \frac{1}{u_0}(v - u) \, dx
\]
and together with (3.35) this gives the second line in problem (3.12).

Conversely, let \( u \) be a solution to (3.12) and let \( \hat{u} \in W^{s,2}_\text{loc}(\Omega) \) be the minimum of the functional \( J_\omega \). Therefore, as we just proved above, \( \hat{u} \) verifies (3.12). Both \( u \) and \( \hat{u} \) are sub-supersolution to the problem (3.12), according to the Definition (2.4). Hence by Theorem 2.5, it follows that \( u \equiv \hat{u} \), namely \( u \) is the minimum of \( J_\omega \). \( \square \)

**Proof of Theorem 1.1** If \( u \) satisfies (P_D), we can use a density argument to show that
\[
\frac{c_{N,s}}{2} \int \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega} u^{-\gamma} \varphi \, dx = \langle \omega, \varphi \rangle,
\]
for all \( \varphi \in W^{s,2}_0(\Omega) \cap L^\infty(\Omega) \). In fact we can select a sequence \( \{ \varphi_\varepsilon \} \) of approximating functions, such that for \( \varepsilon \) that goes to zero, we have \( \| \varphi_\varepsilon - \varphi \|_{W^{s,2}(\Omega)} \to 0 \) and for any \( \rho > 0 \)
\[
\text{support } (\varphi_\varepsilon) \subseteq \mathcal{N}_\rho(\text{support } (\varphi)),
\]
where \( \mathcal{N}_\rho(\text{support } \varphi) \) denotes a \( \rho \)-neighborhood of the support \( \varphi \). Then we use \( \varphi_\varepsilon \) as test function in (P_D) and we pass to the limit.

Assume now \( \omega \in W^{-s,2}(\Omega) \cap L^1_\text{loc}(\Omega) \) and that (P_Y) holds. Obviously for every \( v \in C^\infty_c(\Omega) \) such that \( v \geq 0 \) we deduce
\[
(3.36) \quad \frac{1}{2}c_{N,s} \int \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\Omega} u^{-\gamma} v \, dx \geq \int_{\Omega} \omega v \, dx.
\]
If \( v \in C^\infty_c(\Omega) \) with \( v \leq 0 \), for \( t > 0 \), let us define \( v_t = (u + tv)^+ \). Let us denote \( K_{v_t} = \text{supp}(v_t) \), \( K^c_{v_t} := \mathbb{R}^N \setminus K_{v_t} \) and use the decomposition
\[
\mathbb{R}^N \times \mathbb{R}^N = (K_{v_t} \cup K^c_{v_t}) \times (K_{v_t} \cup K^c_{v_t})
\]
Thus setting
\[
(3.37) \quad \varphi_t = (v_t - u)/t,
\]
we have

\[
\begin{align*}
(3.38) \quad c_{N,s} & \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy \\
& = \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}\backslash(K_{v_t} \times K_{v_t})} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy \\
& \quad - \frac{c_{N,s}}{2t} \int_{K_{v_t} \times K_{v_t}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \\
& \quad \leq \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}\backslash(K_{v_t} \times K_{v_t})} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy \\
& \quad = \frac{c_{N,s}}{2} \int_{K_{v_t} \times K_{v_t}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
& \quad + c_{N,s} \int_{(K_{v_t} \times K_{v_t}) \cap \{u(x) \geq u(y)\}} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy \\
& \quad + c_{N,s} \int_{(K_{v_t} \times K_{v_t}) \cap \{u(x) < u(y)\}} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy := A_1 + A_2 + A_3.
\end{align*}
\]

We examine the last three terms in (3.38). Using a similar argument as in equations (3.22), (3.23) and (3.24), since \( u \in W^{s,2}_{\text{loc}}(\Omega) \) and \( v \in C^\infty_c(\Omega) \), we obtain that

\[
(3.39) \quad A_1 = \frac{c_{N,s}}{2} \int_{K_{v_t} \times K_{v_t}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
\leq \frac{c_{N,s}}{2} \int_{\Omega \times \Omega} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \, dx \, dy < +\infty.
\]

To get (3.39), we point out that, since (P) holds, thanks to Theorem 3.2, we have that \( u \in W^{s,2}_{\text{loc}}(\Omega) \) is the minimum of \( J_\omega \) defined in (3.11). Therefore \( u \in u_0 + W^{s,2}_{0}(\Omega) \) and by Proposition 3.1 it follows that \( u \in L^1(\Omega) \). From (3.39) we deduce also that

\[
(3.40) \quad \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \cdot \chi_{K_{v_t} \times K_{v_t}}(x, y) \\
\leq \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \in L^1(\Omega \times \Omega),
\]
where by $\chi_A$ we denote the characteristic function of a set $A$. Using the definition (3.37), we infer that

\begin{align}
A_2 &= c_{N,s} \int_{(K_{v_t} \times K_{v_t}) \cap \{u(x) \geq u(y)\}} \frac{(u(x) - u(y))(v(x) + u(y)/t)}{|x - y|^{N+2s}} \, dx \, dy \\
&\leq c_{N,s} \int_{(K_{v_t} \times K_{v_t}) \cap \{u(x) \geq u(y)\}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&\leq c_{N,s} \int_{\Omega \times \mathbb{R}^N \setminus \Omega} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \, dx \, dy \\
&+ c_{N,s} \int_{\Omega \times \Omega} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \, dx \, dy.
\end{align}

Therefore we have

\begin{align}
&\int_{\Omega \times \mathbb{R}^N \setminus \Omega} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \, dx \, dy \\
&\leq C(s, N, \Omega) \int_{\Omega} |u(x)||v(x)| \, dx \int_{|y| \geq R} \frac{1}{|y|^{N+2s}} \, dy < +\infty,
\end{align}

where we used the fact that $u(x) = v(x) = 0$ for a.e. $x \in \mathbb{R}^N \setminus \Omega$ and dist($\partial K_{v_t}, \partial \Omega) = \bar{R}$, since $v$ has compact support contained in $\Omega$ and $u \in L^1(\Omega)$. Arguing as in (3.39), we have

\[ c_{N,s} \int_{\Omega \times \Omega} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \, dx \, dy < +\infty. \]

Hence, from (3.41) we deduce that

\begin{align}
A_2 &\leq c_{N,s} \int_{\Omega \times \mathbb{R}^N} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \, dx \, dy < +\infty.
\end{align}

Actually we deduce that

\begin{align}
&\frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \cdot \chi_{(K_{v_t} \times K_{v_t}) \cap \{u(x) \geq u(y)\}} \\
&\leq \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} \in L^1(\Omega \times \mathbb{R}^N).
\end{align}

By the definition (3.37) we also get

\begin{align}
A_3 &= c_{N,s} \int_{(K_{v_t} \times K_{v_t}) \cap \{u(x) < u(y)\}} \frac{(u(x) - u(y))(v(x) + u(y)/t)}{|x - y|^{N+2s}} \, dx \, dy \\
&\leq -c_{N,s} \int_{(K_{v_t} \times K_{v_t}) \cap \{u(x) < u(y)\}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \leq 0.
\end{align}
Using (3.38) and (3.44) we deduce
\[
\frac{c_{N,s}}{2} \int_{K_v \times K_v} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
+ c_{N,s} \int_{K_v \times K_v} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy \\
\geq \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]
Observe that $|\varphi_t| \leq |v|$. Since (P1) holds, we infer that
\[
(3.45) \quad \frac{c_{N,s}}{2} \int_{K_v \times K_v} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
+ c_{N,s} \int_{K_v \times K_v} \frac{(u(x) - u(y))(\varphi_t(x) - \varphi_t(y))}{|x - y|^{N+2s}} \, dx \, dy \\
\geq \int_{\Omega} u^{-\gamma} \varphi_t \, dx + \int_{\Omega} \omega \varphi_t \, dx.
\]
Recalling (3.39), (3.40), (3.41), (3.42), (3.43) and that $u > 0$ a.e. in $\Omega$, using the dominate convergence theorem in (3.45), we finally get
\[
(3.46) \quad \frac{c_{N,s}}{2} \int_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
+ c_{N,s} \int_{(\Omega \times \mathbb{R}^N) \setminus \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
\geq \int_{\Omega} u^{-\gamma} v \, dx + \int_{\Omega} \omega v \, dx.
\]
Up to a change of variables in the second integrale in the l.h.s of (3.46), we deduce
\[
(3.47) \quad \frac{c_{N,s}}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \geq \int_{\Omega} u^{-\gamma} v \, dx + \int_{\Omega} \omega v \, dx,
\]
for all $v \in C^\infty_c(\Omega)$ with $v \leq 0$. Thanks to equations (3.36) and (3.47) we deduce that $u$ satisfies (P4), concluding the proof.

**Proof of Theorem 1.2.** Let $u \in W^{s,2}_{loc}(\Omega) \cap L^{\frac{2N}{N+2s}}(\Omega)$ such that (1.4) holds. Let $\omega = g(x, u) = g_1(x, u - u_0)$. Therefore $\omega \in W^{-s,2}(\Omega)$. By Theorem 1.1 and Theorem 3.2 we have that $u \in u_0 + W^{s,2}(\Omega)$ and $u - u_0$ minimizes (3.11), i.e. for all $v \in W^{s,2}(\Omega)$ we have
\[
(3.48) \quad \frac{c_{N,s}}{4} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega} G_0(x, v) \, dx \\
\geq \frac{c_{N,s}}{4} \int_{\mathbb{R}^N} \frac{(u(x) - u_0(x)) - (u(y) - u_0(y))^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega} G_0(x, u - u_0) \, dx \\
- \langle \Phi'(u - u_0), v - (u - u_0) \rangle,
\]
where
that is
\[ \langle \Phi'(u - u_0), v - (u - u_0) \rangle + \Psi(v) - \Psi(u - u_0) \geq 0. \]
Recalling (1.3), \( u - u_0 \) is a critical point of \( F \) in the sense of [38].

Let us assume that (1.5) holds. Then we have (3.48). From (1.5) and Proposition 3.1 we deduce that
\( u \in W^{s,2}_\text{loc}(\Omega) \cap L^{2N/(N-2s)}(\Omega) \) and therefore \( \omega = g(x, u) = g_1(x, u - u_0) \in W^{-s,2}(\Omega) \cap L^{1}_\text{loc}(\Omega) \). By Theorem 1.1 and Theorem 3.2 we deduce that \( u \) is a solution to (1.4). \( \square \)

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