A VISCOSITY SOLUTION APPROACH TO REGULARITY PROPERTIES OF THE OPTIMAL VALUE FUNCTION

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Abstract. In this paper we analyze the optimal value function $v$ associated to a general parametric optimization problems via the theory of viscosity solutions. The novelty is that we obtain regularity properties of $v$ by showing that it is a viscosity solution to a set of first-order equations. As a consequence, in Banach spaces, we provide sufficient conditions for local and global Lipschitz properties of $v$. We also derive, in finite dimensions, conditions for optimality through a comparison principle. Finally, we study the relationship between viscosity and Clarke generalized solutions to get further differentiability properties of $v$ in Euclidean spaces.

1. Introduction

We consider parametric optimization problems of the form

$$\inf_{x \in \Phi(u)} f(x, u)$$

(1.1)

depending on a parameter vector $u \in U$. Here $x$ belongs to a non empty subset $X$ of a real Banach space $X$, $U$ is a non empty subset of a real Banach space $Y$, $f : X \times U \to \mathbb{R}$ is a continuous function, and $\Phi : U \to 2^X$ is the feasible set mapping. The corresponding optimal value function $v : U \to [-\infty, +\infty]$ is defined as

$$v(u) := \inf_{x \in \Phi(u)} f(x, u), \quad u \in U,$$

and the optimal set function $S : U \to 2^X$ is given by

$$S(u) := \{x \in \Phi(u) : v(u) = f(x, u)\}.$$

Our main regularity result concerns Lipschitz and differentiability properties of $v$ that are obtained by appealing to the theory of viscosity solutions. We study the Lipschitz continuity of $v$ for Banach spaces in two steps. First, we provide conditions related to certain first-order partial differential equation that imply this Lipschitz property, via the viscosity solution concept. Second, we analyze conditions based on the data that give the validity of these sufficient conditions for $v$.

The Lipschitzian properties of the optimal value function and of the optimal set function have been studied by a large number of authors since the 80’s. Indeed, Aubin in 1984, [1], addressed the Lipschitz property of the optimal set function $S$, in Banach spaces. Since then, we can cite, among others, [35], [25], [33], [9].

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and [18] in $\mathbb{R}^n$, and [6], [19], and [20] for Banach spaces, that have also analyzed this regularity property of $S$ in different situations, e.g. linear, convex, non-linear cases, optimal control problems, semi-infinite programming. With respect to the Lipschitz properties of $v$, we can refer to [5], [10], [11], [39], and [26] in $\mathbb{R}^n$; [19], and [27] for Banach spaces. See the surveys [32] and [24], and references therein, for Lipschitz properties of $v$ and $S$ in the case of semi-infinite optimization. The directional lipschitzian optimal solutions and directional derivative for the optimal value function have also been studied, see, e.g., [35], [25], and [38]. We refer to [3] and references therein for a thorough presentation of perturbation analysis of optimization problems for Banach spaces.

In this paper we propose a new approach via the viscosity solution concept. Viscosity solutions were introduced by Crandall and Lions in [15] for first-order differential equations of Hamilton-Jacobi type. In general, nonlinear first-order problems do not have classical solutions, even if the Hamiltonian and boundary conditions are smooth. Indeed, these kind of equations are usually satisfied by value functions, which fail to be $C^1$ or even differentiable. Value functions arise in deterministic control problems, deterministic differential games and the calculus of variations. The book [2] contains a nice presentation of the relationship between value functions and viscosity solutions, and further aspects of the theory of Hamilton-Jacobi-Bellman equations. We point out that the advantage of the weak type of solution introduced in [15] over other notions is that it provides existence ([15], [28], [36]), stability ([12], [36]) and uniqueness ([15], [13]) of solutions. Regularity issues, like Lipschitz and Hölder continuity, $C^1$ regularity and second-order differentiability may also be obtained for viscosity solutions of first-order equations ([2], [28]). The theory was also extended to second-order degenerate elliptic equations in [30] and [31], and to infinite dimensional spaces in [16]. We highlight the reference [29] where a proof of uniqueness by purely PDE methods is provided. A review of the theory of viscosity solutions, with a vast source of references, can be found in [14].

Our assay resembles the analysis in [8] referred to the regularity properties of the value function of infinite horizon problem in optimal control, except that instead of using generalized gradients, we mostly consider super and subgradients in viscosity sense. In particular, the paper [23] that discusses the relationship of locally Lipschitz continuous viscosity solutions and generalized derivative in Clarke sense for Hamilton-Jacobi equations is very close to the final section of the present work. One main difference with the application to optimal control theory is that we postulate a new first-order equation to tackle the problem (1.1) and find conditions on the data so that a viscosity solution is locally Lipschitz.

Furthermore, in the context of Euclidean spaces, we obtain a Comparison Principle that gives the uniqueness property and allows a verification technique to determine sufficient conditions for both, optimality and non optimality. In addition, we provide a necessary condition for optimality. In [40], for optimal control theory, it is developed a verification technique in the framework of viscosity solutions, here we provide somewhat analogous conditions for general parametric optimization problems. Finally, in this finite dimensional setting, we derive relationships between the viscosity solutions and the Clarke generalized solutions. These relations give the possibility of obtaining pointwise differentiability properties of $v$ and deduce, for
convex problems, that $v$ is $C^1$ under some conditions on the data $f$ and $\Phi$. This analysis of going from pointwise differentiability to local differentiability is developed for Hamilton-Jacobi equations in [7].

This paper is organized as follows. Section 2 introduces the necessary notation and preliminaries. In Section 3 we discuss the Lipschitz continuity of viscosity solutions of a certain first-order equation. Section 4 provides conditions on the data so that the optimal value function is a viscosity solution of this equation, and hence Lipschitz continuous. Finally, in Section 5 we restrict our analysis to finite dimensional spaces to find a comparison principle, a uniqueness property, and optimality conditions. Furthermore, we study the relationship between locally Lipschitz continuous viscosity solutions and solutions in the sense of Clark and generalized derivative, obtaining differentiability properties of the optimal value function.

2. Notation and preliminaries

In this section, we introduce the basic notation, definitions, and results that we employ throughout the paper.

2.1. Basic notation and definitions. For given real Banach space $\mathcal{B}$, its dual is $\mathcal{B}^*$ and $\langle \cdot, \cdot \rangle$ is the usual pairing. The underlying norm in $\mathcal{B}$ will be denoted by $\| \cdot \|$ (or $\| \cdot \|_B$ when is needed for clarity) and the closed ball centered at $z \in \mathcal{B}$ with radius $r > 0$ will be $B(z, r)$. Whenever we consider a Cartesian product of topological spaces we will always endow it with the product topology.

For $E \subset \mathcal{B}$, $USC(E)$, $LSC(E)$ and $C(E)$ will stand for the sets of upper semicontinuous, lower semicontinuous, and continuous functions $w : E \to \mathbb{R}$, respectively. We say that $g : \mathcal{B} \to \mathbb{R}$ is Fréchet differentiable at $z \in \mathcal{B}$ if there is $L_z \in \mathcal{B}^*$ so that

$$
\lim_{h \to 0} \frac{g(z + h) - g(z) - \langle L_z, h \rangle}{\|h\|} = 0,
$$

and we write

$$
\nabla g(z) := L_z \quad \text{and} \quad \nabla g(z) \cdot h = \langle \nabla g(z), h \rangle = \langle L_z, h \rangle.
$$

We will mostly use the notation $\nabla g(z) \cdot h$ when the space is of finite dimension. For open $E \subset \mathcal{B}, C^1(E) := \{ w : E \to \mathbb{R}, \nabla w \text{ is continuous in } E \}$. A norm $\| \cdot \|$ in $\mathcal{B}$ is called Fréchet differentiable if $\| \cdot \|$ is Fréchet differentiable at every point of $\mathcal{B} \setminus \{0\}$. It is known, [21 Theorem 8.19], that a Banach space $\mathcal{B}$ admits an equivalent Fréchet differentiable norm whenever $\mathcal{B}^*$ is separable.

Next we recall some well-known concepts for set-valued mappings. Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, and a set $U$ in $\mathcal{Y}$, a set-valued mapping $\mathcal{A} : U \to 2^\mathcal{X}$ is said to be closed at $u_0 \in U$ if for any sequences $\{x_k\} \subset \mathcal{X}$, $\{u_k\} \subset U$, such that $x_k \in \mathcal{A}(u_k), x_k \to x_0$, and $u_k \to u_0$, it holds that $x_0 \in \mathcal{A}(u_0)$. Moreover, $\mathcal{A}$ is called closed if its graph

$$
gph \mathcal{A} := \{ (u, x) : u \in U, x \in \mathcal{A}(u) \}
$$

is closed in $U \times \mathcal{X}$. Also, $\mathcal{A}$ is said to be lower semicontinuous at $u_0$ (lsc, in brief), in the sense of Berge, if for each open set $W \subset \mathcal{X}$ such that $W \cap \mathcal{A}(u_0) \neq \emptyset$, there exists an open neighborhood $V$ of $u_0$ in $U$, such that $\mathcal{A}(u) \cap W \neq \emptyset$ for all $u \in V$. $\mathcal{A}$ is said to be upper semicontinuous at $u_0$ (usc, in brief), in the sense of Berge, if
for each open set $W \subset X$ such that $A(u_0) \subset W$, there is an open neighborhood $V$ of $u_0$ in $U$ so that $A(u) \subset W$ for all $u \in V$.

**Definition 2.1** (Inf-compactness property). Given $A : U \to 2^X$, a function $f : X \times U \to \mathbb{R}$, satisfies the inf-compactness property on $\text{gph } A$ if for every $u \in U$, there exist $\alpha \in \mathbb{R}$ and a compact set $C \subset X$ so that

$$\emptyset \neq \{ x \in A(u') : f(x, u') \leq \alpha \} \subset C$$

for all $u'$ in a neighborhood of $u$.

For these functions $f$ and for a fixed $x \in X$, we will write $f_x(u) := f(x, u)$, and use indistinctly the notation $D_d f_x(u)$ or $D_d f(x, u)$ for the directional derivative of the function $f_x(\cdot)$ at a point $u \in U$ in the direction of $d \in Y$. We may also use $D_d f$, for short.

Finally, the following well-known result accounts for continuity of the optimal value function (see, e.g., [22, Theorem 1.2]). We state it in the current framework of Banach spaces, but the result holds in any compactly generated topological space.

**Theorem 2.2** (Continuity of $v$ and usc of $S$). Let the problem (1.1) and suppose that $\Phi$ is lower semicontinuous in $U$ and that $f$ is continuous in $X \times U$ and satisfies the inf-compactness property on $gph \Phi$. Then, the optimal value function $v$ is continuous in $U$ and the optimal set function $S$ is upper semicontinuous in $U$.

2.2. **Viscosity solutions in Banach spaces.** We now recall some useful concepts and results about viscosity solutions. In what follows, $F : U \times \mathbb{R} \times Y^* \to \mathbb{R}$ is a first-order operator.

**Definition 2.3.** Let $U$ be an open subset of a Banach space $Y$. We say that a function $w \in \text{USC}(U)$ is a viscosity subsolution of (2.1)

$$F(u, w, \nabla w) = 0$$

at $u_0 \in U$ if for any $\eta$ in the set

$$J^+ w(u_0) := \left\{ p \in Y^* : \limsup_{u \to u_0} \frac{w(u) - w(u_0) - \langle p, u - u_0 \rangle}{\| u - u_0 \|} \leq 0 \right\},$$

the following holds

$$F(u_0, w(u_0), \eta) \leq 0.$$  

Similarly, we say that a function $w \in \text{LSC}(U)$ is a viscosity supersolution of (2.1)

at $u_0 \in U$ if for any $\eta$ in the set

$$J^- w(u_0) := \left\{ p \in Y^* : \liminf_{u \to u_0} \frac{w(u) - w(u_0) - \langle p, u - u_0 \rangle}{\| u - u_0 \|} \geq 0 \right\},$$

the following holds

$$F(u_0, w(u_0), \eta) \geq 0.$$  

A viscosity solution of (2.1) at $u_0 \in U$ is a viscosity sub- and supersolution at $u_0 \in U$. Finally, a function is a viscosity subsolution (supersolution, solution) of (2.1) in $U$ if it is so for all $u_0 \in U$.

We have the following known equivalence, adapted from Proposition 1 in [16, Part 1]:
Proposition 2.4. Let \( w \in USC(U) \). Then, \( w \) is a viscosity subsolution of (2.1) at \( u_0 \in U \) if and only if for every \( g \in C(U) \) Fréchet differentiable at \( u_0 \) and such that \( w - g \) attains a local maximum at \( u_0 \), the following holds
\[
F(u_0, w(u_0), \nabla g(u_0)) \leq 0
\]
An analogous statement holds true for a supersolution.

The main obstacle in the application of this last proposition is that closed bounded sets are not compact with respect to the strong topology (unless the underlying space is finite dimensional). Hence continuous functions over these sets may not attain their maximum. However, when the domain under consideration satisfies the Radon-Nikodym property, any continuous function attains its maximum under a small linear perturbation. Recall that a closed, convex and bounded set \( D \) of a Banach space \( X \) is said to be a RNP set (a set with the Radon-Nikodym property) if for every finite measure space \((\Omega, \Sigma, \mu)\), every vector measure \( m : \Sigma \to X \) that is of bounded variation, absolutely continuous with respect to \( \mu \), and has average range
\[
\{\mu(E)^{-1}m(E) : E \in \Sigma, \mu(E) > 0\}
\]
contained in \( D \) is representable by a Bochner integrable function (see, e.g., [17] for further details on RNP properties).

According to [37], examples of RNP sets are convex weakly compact sets in Banach spaces. Hence, in a reflexive Banach space, closed balls are RNP sets. The next known theorem is the key in producing maximum of bounded semicontinuous functions in bounded sets (see, e.g., [37, pag. 174-176] for the proof).

Theorem 2.5. Let \( \mathcal{Y} \) be a Banach space and let \( D \subset \mathcal{Y} \) be a RNP set. Assume that \( g : D \to \mathbb{R} \) is upper semicontinuous and bounded above in \( D \). Then, for any \( \delta > 0 \), there exists \( h \in \mathcal{Y}^*, \|h\| \leq \delta \), such that \( g + h \) attains its maximum at a point \( x_0 \in D \).

3. Lipschitz continuity of a viscosity subsolution

In this section, we consider the following first order equation defined in \( U \),
\[
(3.1) \quad - \langle \nabla w(u), d \rangle + \inf_{x \in S(u)} D_d f_x(u) = 0,
\]
where we assume that the second term has sense. We will provide conditions that assure that a viscosity subsolution is locally Lipschitz in \( U \).

In order to state our regularity result, we will need the following assumptions.

Assumptions on the Banach space \( \mathcal{Y} \):

(H1) \( \mathcal{Y} \) is reflexive;

(H2) There exists a function \( M : \mathcal{Y} \times \mathcal{Y} \to [0, \infty) \) so that: \( y_1 \to M(y_1, y_2) \) and \( y_2 \to M(y_1, y_2) \) are Fréchet differentiable at every point except at \( y_2 \) and \( y_1 \) respectively. Also, there are constants \( \lambda \in (0, 1] \) and \( \Lambda \in [1, \infty) \) so that
\[
(3.2) \quad \lambda \|y_1 - y_2\|_{\mathcal{Y}} \leq M(y_1, y_2) \leq \Lambda \|y_1 - y_2\|_{\mathcal{Y}},
\]
for all \( y_1, y_2 \in \mathcal{Y} \), and
\[
(3.3) \quad \lambda \leq \|\nabla M(\cdot, y_2)\|_{\mathcal{Y}^*}, \quad \|\nabla M(y_1, \cdot)\|_{\mathcal{Y}^*} \leq \Lambda,
\]
whenever the quantities on the center are defined.
Next, we state and prove our main tool:

**Theorem 3.1.** Let the problem \([1.7]\), with \(U\) open, and let \(w : U \to \mathbb{R}\) be a continuous, locally bounded function. In addition to (H1)-(H2), suppose that the following holds:

(i) the directional derivative \(D_d f_x(\cdot)\) exists in \(U\) for all unit \(d \in \mathcal{Y}\) and all \(x \in X\). Moreover, there is a constant \(C_0 > 0\) such that for all \(u \in U\) and all unit \(d\)
\[
\inf_{x \in S(u)} D_d f_x(u) > -C_0;
\]

(ii) for all unit \(d \in \mathcal{Y}\), \(w\) is a viscosity subsolution of \([3.7]\) in \(U\). Then \(w\) is locally Lipschitz in \(U\).

**Proof.** Let \(u_0 \in U\) and take \(\eta > 0\) so that \(w\) is bounded on \(B(u_0, 2\eta) \subset U\). For \(\gamma := \frac{\lambda}{4\Lambda} < 1\), choose any \(u_1, u_2 \in B(u_0, \gamma \eta) \subset B(u_0, \eta)\). Observe that
\[
M(u_1, u_2) \leq \Lambda \|u_1 - u_2\| \leq 2\Lambda \gamma \eta = \frac{\lambda}{2}\eta.
\]

Let \(C_1\) be a positive constant so that
\[
C_1 \lambda - 1 - C_0 > 0,
\]
where \(C_0\) is from \([3.4]\). Let \(h : [0, \infty) \to [0, \infty)\) be a \(C^1\) function so that
\[
h'(r) \geq C_1\) for all \(r > 0\), \(h(r) = C_1 r \in \left[0, \frac{\lambda \eta}{2}\right]\), and \(h(\lambda \eta) \geq 2\|w\|_{C^1(B(u_2, \eta))} + 1\).

To get an appropriate test function, define:
\[
g(u) := w(u_2) + h(M(u, u_2)).
\]

Then \(g\) is Fréchet differentiable in \(U \setminus \{u_2\}\) and \((w - g)(u_2) = 0\). We will prove that
\[
(w - g)(u) \leq 0 \quad \text{for all} \quad u \in B(u_2, \eta).
\]

Arguing by contradiction, suppose that there is \(\tilde{u} \in B(u_2, \eta)\) so that
\[
w(\tilde{u}) > g(\tilde{u}).
\]

Observe that \(w - g\) is bounded above and continuous in \(B(u_2, \eta)\), then by Theorem 2.5 for every \(0 < \delta < 1\), there is \(h_\delta \in \mathcal{Y}\) so that \(\|h_\delta\|_{\mathcal{Y}^*} \leq \delta\) and \(w - g - h_\delta\) attains a maximum in the closed ball \(B(u_2, \eta)\). Let \(u_\delta^* \in B(u_2, \eta)\) be where \(w - g - h_\delta\) attains its maximum over \(B(u_2, \eta)\) and observe that for \(\delta\) small enough
\[
(w - g - h_\delta)(u_\delta^*) \geq (w - g - h_\delta)(\tilde{u}) > 0.
\]

Thus, if \(u_\delta^* = u_2\) for a sequence \(\delta_i \to 0^+\) as \(i \to \infty\), we obtain from \([3.9]\) that
\[
0 = (w - g)(u_2) \geq (w - g)(\tilde{u}) > 0,
\]

which is a contradiction. Hence, there is some \(0 < \delta_0 < 1\) so that \(u_\delta^* \neq u_2\) for any \(\delta \in (0, \delta_0)\). From now on, we assume \(\delta < \delta_0\) and prove that \(u_\delta^*\) belongs to the
interior of the ball $B(u_2, \eta)$. Indeed, if $u_\delta^* \in \partial B(u_2, \eta)$, then by (3.2), (3.7), and the choice of $h$ (3.6), we derive

$$g(u_\delta^*) \geq w(u_2) + h(\lambda \eta) \geq \|w\|_{C(B(u_2, \eta))} + 1 > w(u_\delta^*).$$

This contradicts (3.10) for all small enough $\delta$. Thus, $u_\delta^*$ is in the interior of the ball $B(u_2, \eta)$. Since $u_2 \neq u_\delta^*$ and $w$ is a viscosity subsolution of equation (3.1), the following holds for any unit $d$ and for the test function $g + h_\delta$

$$\langle -h'(M(u_\delta^*, u_2)) \nabla M(\cdot, u_2)(u_\delta^*) - h_\delta, d \rangle + \inf_{x \in \partial B(u_\delta^*)} D_d f(x, u_\delta^*) \leq 0.$$  

By Hahn-Banach theorem ([4, Corollary 1.3]), there is $d_0 \in \mathcal{Y}^{**} = \mathcal{Y}$ such that

$$\langle d_0, \nabla M(\cdot, u_\delta^*) \rangle = \|\nabla M(\cdot, u_2)(u_\delta^*)\|_{\mathcal{Y}}^2.$$

Plugging the unit vector

$$d = \frac{1}{\|\nabla M(\cdot, u_\delta^*)\|_{\mathcal{Y}}^2} d_0$$

into (3.11), and recalling (3.3), the properties of $h$, and (3.12), we obtain

$$C_1 \lambda - \langle h_\delta, d \rangle + \inf_{x \in \partial B(u_\delta^*)} D_d f(x, u_\delta^*) \leq 0.$$  

Since $|\langle h_\delta, d \rangle| \leq 1$ we have

$$C_1 \lambda - 1 - C_0 \leq 0,$$

which is a contradiction with the choice of $C_1$ in (3.5). In this way, we have proved (3.8).

To end the proof of this theorem, observe that by the choice of $\gamma$, it follows that

$$\|u_1 - u_2\| \leq 2\gamma \eta = \frac{\lambda}{2\Lambda} \eta \leq \frac{\lambda}{2} \eta$$

and consequently $u_1 \in B(u_2, \eta)$ and $h(\Lambda \|u_1 - u_2\|) = C_1 \Lambda \|u_1 - u_2\|$. Hence by (3.7) and (3.8), we get

$$w(u_1) - w(u_2) \leq h(M(u_1, u_2)) \leq h(\Lambda \|u_1 - u_2\|) = C_1 \Lambda \|u_1 - u_2\|.$$

Since $u_1$ and $u_2$ are arbitrary in $B(u_0, \gamma \eta)$ and $u_0$ is any point of $U$, we derive the locally Lipschitz regularity of $w$ in $U$. \qed

Notice that the Lipschitz constant, $C_1 \Lambda$, that we have found in the previous proof is the same for all $u_0 \in U$. Hence, it is easy to modify this proof to get the following global property:

**Theorem 3.2.** Under the same assumptions as in Theorem 3.1 if $w$ is bounded and $U = \mathcal{Y}$, then $w$ is Lipschitz in $\mathcal{Y}$.

**Remark 3.3.** We now discuss the feasibility of assumption (H2). This assumption clearly holds when the underlying norm is Fréchet differentiable. Moreover, observe that by [21, Theorem 8.19], if $\mathcal{Y}^{*}$ is separable, then $\mathcal{Y}$ admits an equivalent norm $\| \cdot \|_e$ which is Fréchet differentiable. Hence, we may define

$$M(y_1, y_2) := \|y_1 - y_2\|_e,$$
and observe that (3.2) is satisfied for some \( \lambda, \Lambda > 0 \) so that
\[
\lambda \|y\|_\gamma \leq \|y\|_e \leq \Lambda \|y\|_\gamma, \quad y \in \gamma.
\]
Now, we have that \( y_1 \to M(y_1, y_2) \) (resp. \( y_2 \to M(y_1, y_2) \)) is Fréchet differentiable at any \( y_1 \neq y_2 \) \((y_2 \neq y_1)\) with respect to \( \| \cdot \|_e \). Next, we prove that these mappings are also Fréchet differentiable with respect to the original norm \( \| \cdot \|_\gamma \) and that in fact their Fréchet derivatives coincide. Indeed, denoting by \( \nabla M(\cdot, y_2)(y_1) \) the Fréchet derivative of \( y_1 \to M(y_1, y_2) \) with respect to \( \| \cdot \|_e \), we get for all \( y \neq 0 \),
\[
0 \leq \frac{|M(y_1 + y, y_2) - M(y_1, y_2) - \nabla M(\cdot, y_2)(y_1)y|}{\|y\|_\gamma} \\
\leq \frac{\Lambda |M(y_1 + y, y_2) - M(y_1, y_2) - \nabla M(\cdot, y_2)(y_1)y|}{\|y\|_e} = o(1),
\]
where we have used (3.13). Furthermore, by [21, pag. 242], we have
\[
\sup_{y \neq 0} \frac{|\nabla M(\cdot, y_2)(y)|}{\|y\|_e} = 1.
\]
Another application of (3.13) gives, for all \( y \neq 0 \), that
\[
\lambda \frac{|\nabla M(\cdot, y_2)(y)|}{\|y\|_e} \leq \frac{|\nabla M(\cdot, y_2)(y)|}{\|y\|_e} \leq \Lambda \frac{|\nabla M(\cdot, y_2)(y)|}{\|y\|_e}.
\]
The same argument applies to \( y_2 \to M(y_1, y_2) \). Therefore, (3.3) holds.

4. The optimal value function as a viscosity solution

For considering regularity results on \( v \), first we state possible conditions on \( f \) and on \( \Phi \).

Set of assumptions on the function \( f \):

(A1) \( f \) is continuous in \( X \times U \) and satisfies the inf-compactness property on \( gph \Phi \).

(A2) For \( u \in U \), there is a unit \( d(u) \) in \( Y \) so that for all \( x \in \Phi(u) \), the function \( f_x(\cdot) = f(x, \cdot) \) is directionally differentiable at \( u \) in the direction \( d(u) \), and
\[
\inf_{x \in S(u)} D_{d(u)} f_x(u) > -\infty.
\]

(A3) If \( \{x_k\} \subset \Phi(u) \) converges to \( x \in X \), then
\[
D_{d(u)} f_x(u) \leq \limsup_{k \to \infty} D_{d(u)} f_{x_k}(u),
\]
for \( u \in U \), and where \( d(u) \) is as in (A2).

(A4) For \( u \in U \), there is a constant \( C_0 > 0 \) and a neighborhood \( N \) of \( u \) such that, for all \( u_1 \in N \) and all unit \( d \in Y \),
\[
\inf_{x \in S(u_1)} D_{d} f_x(u_1) > -C_0.
\]

Notice that (A1) implies that the optimal set \( S(u) \) is non empty and compact for any \( u \in U \).

For short, we will synthesize the first three conditions on the objective function \( f \) in the following way:
Condition (A):
(A1) holds in general, and (A2)-(A3) for all \( d \in \mathbb{R}^m \) and all \( u \in U \).

Sometimes we will only need that (A2) and (A3) hold true at a particular point \( u_0 \in U \), and in this case we will write "Condition (A) at \( u_0 \)." Similarly, "Condition (A) at \( u_0 \) and \( d(u_0) \)" means that (A2) and (A3) are only required to be valid at \( u_0 \in U \) and a unit \( d(u_0) \in Y \).

On the side of the feasible set mapping we will analyze two cases. One for a fixed feasible set \( \emptyset \neq \Phi(u) = \Phi \subset X \), for all \( u \in U \), in a similar fashion as Proposition 4.12 in [3]. The other case considers perturbed feasible sets defined by abstract constraints under the following general assumptions \( (X \subset X \text{ and } U \subset Y \) are as usual).

Set of assumptions for a non constant feasible set mapping \( \Phi \):
(B1) \( \Phi : U \to 2^X \) is defined by
\[
\Phi(u) = \{ x \in X : G(x, u) \in K \},
\]
where \( G : X \times U \to Z \) is a continuous function, \( Z \) is a given topological vector space, and \( K \subset Z \) is a closed set with non-empty interior.

(B2) (Slater-like condition) For \( u \in U \), it holds that \( G(x, u) \in \text{int } K \) for any \( x \in S(u) \).

Observe that (B1) implies that \( \Phi \) is a closed mapping.

Again, we will synthesize these cases and conditions in the following way:

Condition (B):
In the case of a fixed feasible set, it is always closed and, in the case of perturbed feasible sets, (B1) holds in general and (B2) for all \( u \in U \).

As before, we say "Condition (B) at \( u_0 \)" when (B2) is only required to hold at a particular point \( u_0 \in U \).

Remark 4.1 (Inf-compactness property and lsc continuity of \( v \)). Observe that, under the continuity property of \( f \) and, assuming that \( \Phi \) is closed, the fact that \( f \) satisfies the inf-compactness property at some \( u_0 \) guaranties the lsc continuity of \( v \) at that point. Indeed, if there exist a real number \( \alpha \) and a compact set \( C \) such that
\[
\emptyset \neq \{ x \in \Phi(u) : f(x, u) \leq \alpha \} \subset C
\]
for all \( u \) in some neighborhood of \( u_0 \), then for any sequence \( \{ u_k \} \) converging to \( u_0 \) with
\[
\lim \inf_{u \to u_0} v(u) = \lim_{k \to \infty} v(u_k)
\]
we may find, for \( k \) large enough, a sequence \( x_k \in S(u_k) \subset C \), which we assume without loss of generality (w.l.o.g.) that converges to some \( x_0 \in C \). So the closedness of \( \Phi \) gives that \( x_0 \in \Phi(u_0) \). Hence
\[
v(u_0) \leq f(x_0, u_0) = \lim_{k \to \infty} f(x_k, u_k) = \lim_{k \to \infty} v(u_k) = \lim \inf_{u \to u_0} v(u).
\]
Remark 4.2 (Slater-like condition and usc continuity of $v$). Suppose that for any $x_0 \in S(u_0)$, $G(x_0, u_0) \in \text{int } K$, which is a Slater-like condition. Then it follows immediately the usc continuity of $v$ at $u_0$. In fact, from the continuity of the given functions $f$ and $G$, we may find, for any $\varepsilon > 0$, neighborhoods $V$ and $W$ of $x_0 \in S(u_0)$ and $u_0$, respectively, such that

$$G(x, u) \in K \quad \text{and} \quad f(x, u) < f(x_0, u_0) + \varepsilon = v(u_0) + \varepsilon,$$

for all $x \in V$ and $u \in W$. Notice that, for $u \in W$, $x_0 \in \Phi(u)$ and hence $\Phi(u) \neq \emptyset$. Thus

$$v(u) \leq v(u_0) + \varepsilon,$$

in $W$, giving the usc continuity of $v$ at $u_0$.

The next lemma states that $v$ is a viscosity solution of a first-order partial differential equation. This fact will allow us to apply Theorem 3.1 to address the Lipschitz continuity of $v$.

Lemma 4.3. For the problem (1.1) with $U$ open, let $u_0 \in U$. Assume that $f$ satisfies Conditions (A) at $u_0$ and $\inf d(u_0)$, and (B) at $u_0$. Then, the optimal value function $v$ is a viscosity solution of the equation

$$(4.2) \quad - \langle \nabla w(u), d(u) \rangle + \inf_{x \in S(u)} D_{d(u)}f_x(u) = 0$$

at $u_0$.

Proof. First, consider the case of a fixed closed feasible set. In view of Theorem 2.2 and assumption (A1) of $f$, we obtain that $v$ is a continuous function. Let $\eta \in \mathcal{J}^+v(u_0)$. Take any decreasing sequence, $s_k \downarrow 0$ and set $u_k := u_0 + s_k d(u_0)$. Now, consider optimal solutions $x_k \in S(u_k)$. By the inf-compactness property, the fact that $\Phi$ is closed, and the continuity of $v$, $x_k$ converges to some $x_0 \in S(u_0)$ (by passing to a subsequence if necessary).

Since $\eta \in \mathcal{J}^+v(u_0)$, by considering that $f(x_k, u_0) \geq f(x_0, u_0) = v(u_0)$, it follows that

$$s_k D_{d(u_0)}f_{x_k}(u_0) = f(x_k, u_k) - f(x_k, u_0) + o(s_k)$$

$$\leq v(u_k) - v(u_0) + o(s_k)$$

$$\leq s_k \langle \eta, d(u_0) \rangle + o(s_k).$$

Dividing by $s_k$, taking $k \to \infty$, and making use of the assumption (A3) of $f$, we obtain

$$- \langle \eta, d(u_0) \rangle + \inf_{x \in S(u_0)} D_{d(u_0)}f_x(u_0) \leq 0.$$

Therefore, $v$ is a viscosity subsolution to (4.2) at $u_0$.

Next, in order to show that $v$ is also a viscosity supersolution to (4.2) at $u_0$, assume that $\eta \in \mathcal{J}^-v(u_0)$. Take any $x \in S(u_0)$, then

$$s_k D_{d(u_0)}f_x(u_0) = f(x, u_k) - f(x, u_0) + o(s_k)$$

$$\geq v(u_k) - v(u_0) + o(s_k)$$

$$\geq s_k \langle \eta, d(u_0) \rangle + o(s_k).$$

Once again, divide by $s_k$ and let $k \to \infty$, to get $D_{d(u_0)}f_x(u_0) \geq \langle \eta, d(u_0) \rangle$ for all $x \in S(u_0)$. Hence

$$-\langle \eta, d(u_0) \rangle + \inf_{x \in S(u_0)} D_{d(u_0)}f_x(u_0) \geq 0,$$

as desired.

Now, consider the case of perturbed feasible sets satisfying Condition (B) at $u_0$. In view of the two remarks above, the optimal value function $v$ is continuous at $u_0$. As before, let $\{s_k\}$ be a decreasing sequence, $s_k \downarrow 0$, and put $u_k = u_0 + s_kd(u_0)$. Taking into account the continuity of $f$ and $G$, the inf-compactness property, and that $\Phi$ is a closed mapping, we can obtain, w.l.o.g, optimal solutions $x_k \in S(u_k)$ converging to some $x_0 \in S(u_0)$. From (B2), there are open neighborhoods $V$ and $W$ of $x_0$ and $u_0$, respectively, so that $G(x, u) \in \text{int } K$, i.e. $x \in \Phi(u)$, for all $x \in V, u \in W$. In particular, for $k$ large enough, $x_k \in \Phi(u_k)$. Therefore, we can follow similar steps as above in the case of fixed feasible sets to get that $v$ is a viscosity subsolution to (4.2) at $u_0$.

To prove that $v$ is a viscosity supersolution to (4.2) at $u_0$, take any $x \in S(u_0)$. Once again, (B2) provides open neighborhoods $V$ and $W$ of $x$ and $u_0$, respectively, so that $G(x', u) \in \text{int } K$, i.e. $x' \in \Phi(u)$, for all $x' \in V, u \in W$. In particular, for $k$ large enough, $x_k \in \Phi(u_k)$. Once more, we obtain that $v$ is a viscosity supersolution to (4.2) at $u_0$ following the same steps as in the first case of fixed feasible sets. \(\square\)

**Corollary 4.4.** Assume all the above conditions and that $\dim \mathcal{Y} < \infty$. If $v$ is differentiable at $u_0$ and $f_x$ is differentiable with respect to $u$ at $u_0$ for all $x \in S(u_0)$, then $D_{d}f_x(u_0)$ is constant on $S(u_0)$ and

$$\nabla v(u_0) = \nabla f(x, u_0)$$

for any $x \in S(u_0)$.

**Proof.** From Lemma [4.3] it follows that

$$-\nabla v(u_0) \cdot d + \inf_{x \in S(u_0)} D_{d}f_x(u_0) = 0$$

for all unit $d \in \mathcal{Y}$, because $\nabla v(u_0) \in \mathcal{J}^+v(u_0) \cap \mathcal{J}^-v(u_0)$. Furthermore, taking into account that $D_{d}f_x(u_0) = \nabla f(x, u_0) \cdot d$, we have

$$\inf_{x \in S(u_0)} \nabla f(x, u_0) \cdot d = \nabla v(u_0) \cdot d = -\nabla v(u_0) \cdot (-d) = \inf_{x \in S(u_0)} \nabla f(x, u_0) \cdot (-d) = \sup_{x \in S(u_0)} \nabla f(x, u_0) \cdot d.$$

Hence, $D_{d}f_x(u_0)$ is constant on $S(u_0)$ and $\nabla v(u_0) = \nabla f(x, u_0)$ for any $x \in S(u_0)$ as desired. \(\square\)

**Theorem 4.5.** Let the optimization problem (4.7) where $U$ is open in $\mathcal{Y}$. Assume hypotheses (H1)-(H2) on $\mathcal{Y}$, Conditions (A) and (B), and (A4) for all $u \in U$. Then $v$ is locally Lipschitz in $U$. 


Proof. It is a straightforward consequence of Theorem 3.1 and Lemma 4.3. □

Corollary 4.6. Assume all the above conditions. If dim \( Y < \infty \), then \( v \) is differentiable a.e. in \( U \) and, for all unit \( d \in Y \),

\[ \nabla v(u) \cdot d = \inf_{x \in S(u)} D_d f_x(u), \text{ a.e. in } U. \]

Proof. It follows immediately from Lemma 4.3 whenever \( v \) is differentiable at \( u \), because \( \nabla v(u) \in J^+ v(u) \cap J^- v(u) \). □

Remark 4.7. In the case of \( U = Y \), and a bounded function \( f \), if the constant \( C_0 \) works globally in (4.1), we also obtain from Theorem 3.2 and Lemma 4.3 that \( v \) is Lipschitz in \( Y \).

5. The Case of Finite Dimensional Spaces

In this section we restrict our analysis to the euclidean space \( Y = \mathbb{R}^m \). As usual, we identify \( Y^* \) with \( \mathbb{R}^m \), and use the notations \( |\cdot| \) and \( p \cdot d \) for the euclidean norm and the inner product, respectively. Also, \( V \subset \subset U \) stands for \( V \subset V \subset U \).

5.1. Comparison Principle and Uniqueness of the Viscosity Solution.

The following lemma gives a comparison principle which will allow us to characterize \( v \) as the unique viscosity solution of a set of equations.

Lemma 5.1 (Comparison Principle). Let the problem (1.1) where \( U \) is an open subset of \( \mathbb{R}^m \), and let \( U_1 \) be any bounded and open subset such that \( U_1 \subset \subset U \). Assume:

(i) \( f : X \times U \to \mathbb{R} \) has directional derivatives \( D_d f_x(u) \) for all \( x \in X, u \in U \), and all unit \( d \in \mathbb{R}^m \);

(ii) For all \( u \in U_1 \), all sequences \( \{d_n\} \subset \mathbb{R}^m, \{u_n\} \subset U_1 \) with

\[ |d_n| = 1 \quad \text{and} \quad \lim_{n \to \infty} u_n = \pi, \]

contain subsequences, which are not relabel, such that

\[ \inf_{x \in S(u_n)} D_{d_n} f_x(u_n) - \inf_{x \in S(\pi)} D_{d_n} f_x(\pi) \to 0 \quad \text{as } n \to \infty, \]

and

(iii) \( w_1 \in C(\overline{U_1}) \) and \( w_2 \in C(\overline{U_1}) \) are sub- and supersolutions of

\[ -\nabla w(u) \cdot d + \inf_{x \in S(u)} D_d f_x(u) = 0 \quad \text{in } U_1, \]

respectively, for all unit \( d \in \mathbb{R}^m \), and that

\[ w_1(u) \leq w_2(u) \quad \text{for } u \in \partial U_1. \]

Then

\[ w_1 \leq w_2 \quad \text{in } U_1. \]
Proof: Let \( u^* \notin \overline{U}_1 \). For \( \varepsilon_n, \delta > 0 \), with \( \varepsilon_n \downarrow 0 \) as \( n \to \infty \), define \( \Psi_n : \overline{U}_1 \times \overline{U}_1 \to \mathbb{R} \) by

\[ \Psi_n(u_1, u_2) = w_1(u_1) - w_2(u_2) - \frac{|u_1 - u_2|^2}{2\varepsilon_n} - \delta|u_1 - u^*|. \]

There are points \((u_{1,n}, u_{2,n}) \in \overline{U}_1 \times \overline{U}_1\) (depending on \( \varepsilon_n \) and \( \delta \)) so that

\[ \Psi_n(u_1, u_2) \leq \Psi_n(u_{1,n}, u_{2,n}) \]

for all \((u_1, u_2) \in \overline{U}_1 \times \overline{U}_1\). In particular

(5.4)

\[ w_1(u_1) - w_2(u_2) = \Psi_n(u_1, u_1) + \delta|u_1 - u^*| \leq \Psi(u_{1,n}, u_{2,n}) + \delta|u_1 - u^*| \]

for all \( u_1 \in U_1 \).

W.l.o.g, we may assume that \((u_{1,n}, u_{2,n}) \to (\overline{u}_1, \overline{u}_2) \in \overline{U}_1 \times \overline{U}_1 \). Observe that

\[ \Psi_n(u_{1,n}, u_{1,n}) \leq \Psi_n(u_{1,n}, u_{2,n}), \]

which implies that

\[ w_2(u_{2,n}) - w_2(u_{1,n}) \leq -\frac{|u_{1,n} - u_{2,n}|^2}{2\varepsilon_n}. \]

Thus, the boundedness of the function \( w_2 \) and the fact that \( \varepsilon_n \downarrow 0 \) give

(5.5)

\[ |u_{1,n} - u_{2,n}| \to 0 \quad \text{as} \quad n \to \infty, \]

and hence \( \overline{u}_1 = \overline{u}_2 \).

Now, we shall prove that

(5.6)

\[ \liminf_{n \to \infty} \Psi_n(u_{1,n}, u_{2,n}) \leq 0. \]

Indeed, for each \( \varepsilon_n \), we have two possibilities:

(i) \((u_{1,n}, u_{2,n}) \in \partial(U_1 \times U_1)\)

(ii) \((u_{1,n}, u_{2,n}) \in U_1 \times U_1\).

Suppose that (i) holds for a subsequence \( \varepsilon_{n_k} \downarrow 0 \). In this case, we may have that \( u_{1,n_k} \in \partial U_1 \), and (5.2) gives

\[ \Psi_n(u_{1,n_k}, u_{2,n_k}) \leq w_2(u_{1,n_k}) - w_2(u_{2,n_k}). \]

Also, it may hold that \( u_{2,n_k} \in \partial U_1 \) and in this case we obtain

\[ \Psi_n(u_{1,n_k}, u_{2,n_k}) \leq w_1(u_{1,n_k}) - w_1(u_{2,n_k}). \]

In either case, (5.6) follows from (5.5) and the continuity of \( w_1 \) and \( w_2 \).

Next, suppose that \((u_{1,n}, u_{2,n}) \in U_1 \times U_1\) for all \( \varepsilon_n \) small enough. Let define the test functions

\[ g_{1,\varepsilon_n} : \quad u_1 \to w_2(u_{2,n}) + \frac{|u_1 - u_{2,n}|^2}{2\varepsilon_n} + \delta|u_1 - u^*|, \]

\[ g_{2,\varepsilon_n} : \quad u_2 \to w_1(u_{1,n}) - \frac{|u_2 - u_{1,n}|^2}{2\varepsilon_n} - \delta|u_{1,n} - u^*|. \]

Then \( w_1 - g_{1,\varepsilon_n} = \Psi_n(\cdot, u_{2,n}) \) attains a maximum at \( u_{1,n} \), and \( w_2 - g_{2,\varepsilon_n} = -\Psi_n(u_{1,n}, \cdot) \) attains a minimum at \( u_{2,n} \). Moreover,

\[ \nabla g_{1,\varepsilon_n}(u_{1,n}) = \frac{u_{1,n} - u_{2,n}}{\varepsilon_n} + \delta \frac{u_{1,n} - u^*}{|u_{1,n} - u^*|}. \]
and
\[ \nabla g_{2,\varepsilon_n}(u_{2,n}) = \frac{u_{1,n} - u_{2,n}}{\varepsilon_n}. \]

Since \( w_1 \) is a subsolution and \( w_2 \) is a supersolution, we obtain for all unit \( d \) that
\[ (5.7) \quad -\left( \frac{u_{1,n} - u_{2,n}}{\varepsilon_n} + \delta \frac{u_{1,n} - u^*}{|u_{1,n} - u^*|} \right) \cdot d + \inf_{x \in S(u_{1,n})} D_d f_x(u_{1,n}) \leq 0, \]
and
\[ (5.8) \quad -\frac{u_{1,n} - u_{2,n}}{\varepsilon_n} \cdot d + \inf_{x \in S(u_{2,n})} D_d f_x(u_{2,n}) \geq 0. \]

From (5.7) and (5.8) we derive
\[ -\delta (u_{1,n} - u^*) \cdot d + \inf_{x \in S(u_{1,n})} D_d f_x(u_{1,n}) - \inf_{x \in S(u_{2,n})} D_d f_x(u_{2,n}) \leq 0. \]

Choosing
\[ d_n = -\frac{u_{1,n} - u^*}{|u_{1,n} - u^*|}, \]
we obtain
\[ \delta + \inf_{x \in S(u_{1,n})} D_d f_x(u_{1,n}) - \inf_{x \in S(u_{2,n})} D_d f_x(u_{2,n}) \leq 0. \]

Letting \( n \to \infty \), recalling that \( \lim_{n \to \infty} u_{1,n} = \lim_{n \to \infty} u_{2,n} \), and appealing to (5.1), we obtain the contradiction \( \delta \leq 0 \). Therefore, just case (i) above may hold and so (5.6) follows.

Finally, taking \( \lim \inf \) in (5.4) gives
\[ w_1(u_1) - w_2(u_1) \leq \delta |u_1 - u^*| \]
for all \( u_1 \in U_1 \), and (5.3) follows by letting \( \delta \to 0 \).

Theorem 5.2. Let the problem (1.1) where \( U \subset \mathbb{R}^m \) is open, and assume conditions (A) and (B). Let \( U_1 \subset \subset U \) be any bounded and open subset. If (ii) in Lemma 5.1 holds, then the optimal value function \( v \) is the only continuous function in \( U_1 \) which satisfies
\[ w(u) = \min_{x \in S(u)} f(x, u) \quad \text{on} \quad \partial U_1 \]
and
\[ -\nabla w(u) \cdot d + \inf_{x \in S(u)} D_d f_x(u) = 0 \quad \text{in} \quad U_1 \]
in the viscosity sense for all unit \( d \in \mathbb{R}^m \).

Proof. It follows immediately from Lemmas 4.3 and 5.1.
5.2. Conditions for optimality: verification technique. Now, we obtain necessary and sufficient conditions for optimality by following the line of the verification method for optimal control problems summarized in [2, Chapter III].

We consider the problem (1.1) and assume all the hypotheses developed in Section 4 for making \( v \) a locally Lipschitz viscosity solution of the equation
\[
- \nabla w(u) \cdot d + \inf_{x \in S(u)} Df_x(u) = 0
\]
in \( U \), i.e. \( U \) is open and conditions (A), (A4) for all \( u \in U \), and (B) are valid.

Fix \( u_0 \in U \). We start with the following necessary condition for optimality.

**Proposition 5.3.** If \( x_0 \in S(u_0) \), then, the function \( w : U \to \mathbb{R} \) defined by
\[
w(u) = f(x_0, u)
\]
is a viscosity subsolution of equation (5.9) at \( u_0 \).

**Proof.** From condition (B) there is a neighborhood \( N \subset U \) of \( u_0 \) such that if \( u \in N \), then \( x_0 \in \Phi(u) \). Next, consider
\[
w(u) = f(x_0, u), \quad u \in N,
\]
and let \( g \) be any \( C(N) \) function, differentiable at \( u_0 \), such that
\[
\max_N (w - g) = (w - g)(u_0).
\]
Then, for \( u \in N \),
\[
v(u) - g(u) \leq w(u) - g(u) \leq w(u_0) - g(u_0) = v(u_0) - g(u_0),
\]
which yields that
\[
\max_N (v - g) = (v - g)(u_0).
\]
Since \( v \) is a viscosity subsolution, we have
\[
- \nabla g(u_0) \cdot d + \inf_{x \in S(u_0)} Df_x(u_0) \leq 0,
\]
and the proof is completed. \( \square \)

A sufficient condition for optimality is the following:

**Proposition 5.4.** Let \( x_0 \in \Phi(u_0) \). If there exist a bounded neighborhood \( N \subset U \) of \( u_0 \) and a viscosity subsolution of (5.9) in \( N \), \( w \in C(N) \), that satisfies
\[
w(u) \leq \min_{x \in S(u)} f(x, u) \text{ on } \partial N \quad \text{and} \quad w(u) \geq f(x_0, u) \text{ for all } u \in N,
\]
then \( x_0 \in S(u_0) \).

**Proof.** By the Comparison Principle (Lemma 5.1), \( w(u) \leq v(u) \) for all \( u \in U \). In particular, for \( u_0 \),
\[
v(u_0) \leq f(x_0, u_0) \leq w(u_0) \leq v(u_0),
\]
Hence, \( v(u_0) = f(x_0, u_0) \) and so \( x_0 \in S(u_0) \). \( \square \)

We also provide a sufficient condition for non-optimality.
Proposition 5.5. Let \( x_0 \in \Phi(u_0) \). If there exist a bounded neighborhood \( \mathcal{N} \subset U \) of \( u_0 \) and a viscosity supersolution of (5.9) in \( \mathcal{N} \), \( w \in C(\overline{\mathcal{N}}) \), that satisfies
\[
w(u) \geq \min_{x \in S(u)} f(x, u) \quad \text{on} \quad \partial \mathcal{N} \quad \text{and} \quad w(u_0) < f(x_0, u_0),
\]
then \( x_0 \notin S(u_0) \).

Proof. By the comparison principle, \( v \leq w \) in \( \mathcal{N} \). Hence,
\[
v(u_0) \leq w(u_0) < f(x_0, u_0),
\]
which implies that \( x_0 \notin S(u_0) \).

Remark 5.6. Observe that the existence of a function \( w \in C(\overline{\mathcal{N}}) \) such that \( w \geq v \) on \( \partial \mathcal{N} \) and
\[
- \nabla w(u) \cdot d + \inf_{x \in \Phi(u)} D_d f_x(u) \geq 0 \quad \text{in} \quad \mathcal{N},
\]
in the viscosity sense, implies, by Proposition 5.5, that \( x_0 \notin S(u_0) \). The inequality (5.10) does not require to find \( S(u) \).

5.3. Generalized derivative and differentiability properties of \( v \). Here we utilize the viscosity solution approach to consider generalized solutions, in Clarke sense, of the optimal value function \( v \), for \( Y = \mathbb{R}^m \). All the definitions and properties stated with no proof can be found in [12].

Definition 5.7. Let \( u_0 \in U \) and let \( w : U \to \mathbb{R} \) be Lipschitz in a neighborhood of \( u_0 \). The generalized derivative \( D^0 w(u_0) : \mathbb{R}^m \to \mathbb{R} \) is defined by
\[
D^0 w(u_0)y := \limsup_{u \to u_0, h \downarrow 0} \frac{w(u + hy) - w(u)}{h},
\]
for \( y \in \mathbb{R}^m \).

The generalized gradient \( \partial w(u_0) \) of \( w \) at \( u_0 \) is given by
\[
\partial w(u_0) = \{ p \in \mathbb{R}^m : p \cdot y \leq D^0 w(u_0)y \quad \text{for all} \quad y \in \mathbb{R}^m \}.
\]

The generalized gradient coincides with the derivative when the function \( w \) is strictly differentiable, e.g. when \( w \) is \( C^1 \). If \( w \) is convex on \( U \), then \( \partial w(u) \) is the subdifferential at \( u \) in the sense of convex analysis, and \( D^0 w(u)y \) is the usual directional derivative \( D_y w(u) \), see [12] Proposition 2.2.7).

By [12, Proposition 2.1.5] the set valued mapping \( \partial w : U \rightrightarrows \mathbb{R}^m \) is a closed and usc mapping. Moreover, we have the next result related to super- and subgradients (see [23] Theorem 1.4).

Proposition 5.8. If \( w \) is Lipschitz in a neighborhood of \( u_0 \), then
\[
J^+ w(u_0) \cup J^- w(u_0) \subset \partial w(u_0).
\]

An alternative definition of the generalized gradient is given by the following proposition (see [12] Theorem 2.5.1)].
Proposition 5.9. If $w$ is Lipschitz in a neighborhood of $u_0$, then
\[ \partial w(u_0) = \co \left\{ \lim_{l \to \infty} \nabla w(u_l) : u_l \to u_0 \right\}, \]
where the limit is taken over all sequences $\{u_l\}$ converging to $u_0$ so that $\nabla w(u_l)$ does exist and $\{\nabla w(u_l)\}$ is a converging subsequence.

Next, given an operator $F : U \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, we introduce the definition of generalized solutions.

Definition 5.10. A locally Lipschitz function $w : U \to \mathbb{R}$ is a generalized solution of
\[ F(\cdot, w, \nabla w) = 0 \]
at $u_0 \in U$, if
\[ \max_{p \in \partial w(u_0)} F(u_0, w(u_0), p) = 0. \]

We now consider the optimal value function $v$ and provide conditions for $v$ to be a generalized solution of the equation (5.9). For that recall that Lemma 4.3 and Theorem 4.5 give conditions under which $v$ is a locally Lipschitz viscosity solution of this equation.

Theorem 5.11. Let the problem (1.1), where $f$ satisfies (A1). Let $u_0 \in U$ and $d \in \mathbb{R}^m$. Suppose that $D_a f$ is continuous in $X \times N$ for a neighborhood $N$ of $u_0$, and assume that $w$ is a locally Lipschitz viscosity solution of
\[ -\nabla w(u) \cdot d + \inf_{x \in S(u)} D_d f_x(u) = 0 \]
in $N$. Moreover, suppose that $J^- w(u_0) \neq \emptyset$. Then, $w$ is also a generalized solution of (5.11) at $u_0$.

Proof. Since $w$ is a viscosity supersolution, the fact that $J^- w(u_0) \neq \emptyset$ and Proposition 5.8 yield
\[ \max_{p \in \partial w(u_0)} \left( -p \cdot d + \inf_{x \in S(u)} D_d f_x(u_0) \right) \geq 0. \]
To prove the opposite inequality, take any $p_0 \in \partial w(u_0)$. By Proposition 5.9 we may write
\[ p_0 = \sum_{i=1}^{k} \lambda_i p_i, \]
where $\lambda_i \geq 0$ for all $i$, $\sum_{i=1}^{k} \lambda_i = 1$, and
\[ p_i = \lim_{l \to \infty} \nabla w(u^i_l), \]
with $u^i_l \to u_0$ as $l \to \infty$, for all $i$. W.l.o.g., assume that $u^i_l \in N$ for all $i$ and $l$. Since $w$ is a viscosity solution in $N$ of (5.11) we have
\[ -\nabla w(u^i_l) \cdot d + \inf_{x \in S(u^i_l)} D_d f_x(u^i_l) = 0. \]
because \( \nabla w(u_i^j) \in \mathcal{J}^+ w(u_i^j) \cup \mathcal{J}^- w(u_i^j) \). Now, \( S(u_i^j) \) is compact and \( D_d f(\cdot)(u_i^j) \) is continuous, hence there is some \( x_i^j \in S(u_i^j) \) so that

\[
\inf_{x \in S(u_i^j)} D_d f_x(u_i^j) = D_d f_{x_i^j}(u_i^j),
\]

and then

\[
(5.13) \quad - \nabla w(u_i^j) \cdot d + D_d f_{x_i^j}(u_i^j) = 0.
\]

By the inf-compactness property, there exists \( x^i \) so that \( x_i^j \to x^i \), passing to a subsequence if necessary. Observe that \( x^i \in S(u_0) \). Finally, by letting \( l \to \infty \),

\[-p_i \cdot d + D_d f_{x^i}(u_0) = 0.\]

Hence,

\[-p_0 \cdot d + \inf_{x \in S(u_0)} D_d f_x(u_0) \leq \sum_{i=1}^k \lambda_i (-p_i \cdot d + D_d f_{x^i}(u_0)) = 0.\]

Thus,

\[
(5.14) \quad \max_{p \in \partial w(u_0)} \left( -p \cdot d + \inf_{x \in S(u_0)} D_d f_x(u_0) \right) \leq 0.
\]

(5.12) and (5.14) complete the proof. \( \square \)

**Corollary 5.12.** Let the problem (1.1) that satisfies (A1) and condition (B). Let \( u_0 \in U \) and suppose the existence of a neighborhood \( \mathcal{N} \) of it such that \( D_d f \) is continuous in \( X \times \mathcal{N} \) for all unit \( d \in \mathbb{R}^m \). If \( \mathcal{J}^- v(u_0) \neq \emptyset \), then

(i) \(-p \cdot d + \inf_{x \in S(u_0)} D_d f_x(u_0) = 0,\)

for all \( p \in \mathcal{J}^- v(u_0) \) and all unit \( d \in \mathbb{R}^m \);

(ii) \( \mathcal{J}^- v(u_0) \) is a singleton;

(iii) \( D_d f(\cdot, u_0) \) is constant in \( S(u_0) \) for all unit \( d \in \mathbb{R}^m \); and,

(iv) if \( v \) is differentiable at \( u_0 \), then

\[
(5.15) \quad \nabla v(u_0) = \nabla_u f(x, u_0),
\]

where \( x \) is any point in \( S(u_0) \).

**Proof:** Notice that the hypotheses imply all the conditions in Lemma 4.3 and Theorem 4.5 that make \( v \) a locally Lipschitz viscosity solution of (5.11) in \( \mathcal{N} \). Hence, an application of (5.12) and (5.14) in the proof of Theorem 5.11 gives (i).

(ii) Let \( p, q \in \mathcal{J}^- v(u_0) \), then

\[
p \cdot d = \inf_{x \in S(u_0)} D_d f_x(u_0) = q \cdot d,
\]

for all unit \( d \in \mathbb{R}^m \), which yields \( p = q \).

(iii) Here \( D_d f_x(u_0) = \nabla_u f(x, u_0) \cdot d \). Now, in view of (i), we have for \( p \in \mathcal{J}^- v(u_0) \)
and for all unit $d \in \mathbb{R}^m$ that
\[
\inf_{x \in S(u_0)} \nabla u f(x, u_0) \cdot d = -p \cdot (-d) = -\inf_{x \in S(u_0)} \{ \nabla u f(x, u_0) \cdot (-d) \} = \sup_{x \in S(u_0)} \nabla u f(x, u_0) \cdot d
\]
Thus, $D_d f(\cdot, u_0)$ is constant in $S(u_0)$.

(iv) Observe that when $v$ is differentiable at $u_0$, we have
\[
\nabla v(u_0) \cdot d = \nabla u f(x, u_0) \cdot d
\]
for all unit $d$ and any $x \in S(u_0)$. Hence, (5.15) follows.

Observe that we have already presented in Corollary 4.4 a similar result to (iii) and (iv) under slight different conditions. Moreover, [3, Remark 4.14] discusses conditions so that $D_d f(\cdot, u_0)$ is constant in $S(u_0)$ when $U$ is any Banach space.

Finally, we include some results for the convex setting. Notice that $v$ is convex in $U$ whenever the optimization problem (1.1) is convex, i.e. $X$ and $U$ are convex sets, $f = f(x, u)$ is convex in $(x, u) \in X \times U$, and $\text{gph} \Phi \subset X \times U$ is also convex.

**Proposition 5.13.** Let the problem (1.1) with $U$ open and let $u_0 \in U$. Assume (A1) and condition (B). Suppose the existence of a neighborhood $N$ of $u_0$ such that $D_d f$ is continuous in $X \times N$ for all unit $d \in \mathbb{R}^m$. If $v$ is convex near $u_0$, then $v$ is differentiable at $u_0$.

**Proof:** The convexity of $v$ gives that it is locally Lipschitz and, moreover, that $J^{-v}(u_0) \neq \emptyset$. By virtue of Corollary 5.12, $J^{-v}(u_0)$ is a singleton, which yields the differentiability of $v$ at $u_0$ (see, e.g., [34, Theorem 25.1]).

**Remark 5.14.** Proposition 2.2.7 in [12] yields that for $v$ convex on $U$, the generalized gradient $\partial v(u_0)$ is the usual subdifferential in convex analysis and, moreover, $D^0 \partial v(u_0) y$ is the usual directional derivative $D_d v(u_0)$. Proposition 5.13 goes further and provides conditions so that a convex optimal value function is actually differentiable at the considered point $u_0$.

**Theorem 5.15.** Assume that both $\mathcal{Y}$ and $\mathcal{X}$ are finite dimensional spaces. Let the problem (1.1), with $X$ and $U$ open sets, and $f \in C^1(X \times U)$. Suppose that (A1) and condition (B) hold, and let $u_0 \in U$. If $v$ is convex near $u_0$, then it is $C^1$ in a neighborhood of $u_0$.

**Proof.** Let $O \subset \subset X$ be an open and bounded neighborhood of $S(u_0)$, and let $N \subset \subset U$ be an open and bounded neighborhood of $u_0$ where $v$ is convex. Fix a unit $d \in \mathcal{Y}$. Then $D_d f(x, u)$ is uniformly continuous in $\overline{O} \times N$. Hence, for fixed $\eta > 0$, there is $\delta = \delta(\eta) > 0$ so that
\[
|D_d f(x_1, u_1) - D_d f(x_2, u_2)| < \eta
\]
for any \((x_1, u_1), (x_2, u_2) \in \overline{O} \times \overline{N}\) with max \(|x_1 - x_2|, |u_1 - u_2|\) < \(\delta\). Moreover, we assume that the \(\delta\)-neighborhood of the compact set \(S(u_0)\) given by 
\[
O_\delta := \{x \in \mathbb{R}^n : \text{dist}\ (x, S(u_0)) < \delta\}
\]
is contained in \(O\). Now, \(S\) is usc because it is a closed and compact valued mapping, thus there is a neighborhood \(N_1 \subset N \cap B(u_0, \delta)\) of \(u_0\) such that 
\[
S(u) = \{\eta \in \mathbb{R} : |\eta - \eta_0(u)| < \delta, u \in N_1\} \quad \text{implies} \quad S(u) \subset O_\delta.
\]
In particular, for all \(x_u \in S(u)\), with \(u \in N_1\), there is \(x(u) \in S(u_0)\) such that 
\[
|\eta - \eta_0(u)| = \text{dist}\ (x_u, S(u_0)) < \delta.
\]
Also, observe that, by Proposition 5.13, \(v\) is differentiable in a neighborhood \(N_2 \subset N_1\) of \(u_0\). Thus, for all \(u \in N_2\) and all unit \(d\), it holds that 
\[
|\nabla v(u) \cdot d - \nabla v(u_0) \cdot d| = |D_d f(x_u, u) - D_d f(x(u), u_0)| < \eta,
\]
where we have used Corollary 5.12 (iv), (5.17), and (5.16). Finally, if \(u_1, u_2 \in N_2\), then 
\[
|\nabla v(u_1) \cdot d - \nabla v(u_2) \cdot d| \leq |\nabla v(u_1) \cdot d - \nabla v(u_0) \cdot d| + |\nabla v(u_2) \cdot d - \nabla v(u_0) \cdot d| < 2\eta,
\]
which completes the proof. \(\square\)

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