Maximum principle and one-sign solutions for the elliptic $p$-Laplacian $^*$

Guowei Dai$^\dagger$

Department of Mathematics, Northwest Normal University, Lanzhou, 730070, PR China

Abstract

In this paper, we prove a maximum principle for the $p$-Laplacian with a sign-changing weight. As an application of this maximum principle, we study the existence of one-sign solutions for a class of quasilinear elliptic problems.

Keywords: Maximum principle; Bifurcation; One-sign solution

MSC(2000): 35B50; 35B32; 35D05; 35J70

1 Introduction

Recently, Dai and Ma $^\ddagger$ studied the existence of one-sign solutions for the following $p$-Laplacian problem

$$\begin{cases} -\Delta_p u = \tilde{f}(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < N$ is the $p$-Laplacian of $u$, $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Under the assumption that $\tilde{f}(x, s)$ satisfies signum condition $\tilde{f}(x, s)s > 0$ for $s \neq 0$ and crosses the first eigenvalue of $-\Delta_p$, they showed that the above problem possesses at least a positive solution and a negative one.

Naturally, one may ask what will happen if $\tilde{f}$ does not satisfy the signum condition. The main purpose of this paper is to establish a result similar to that of $^\ddagger$ for the following $p$-Laplacian problem

$$\begin{cases} -\Delta_p u = m(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

(1.1)

where $m : \Omega \to \mathbb{R}$ is a continuous negative function with $m \neq 0$ in $\Omega$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies:

(f1) $f(s)/\varphi_p(s)$ is bounded for $s \in \mathbb{R} \setminus \{0\}$.

$^*$Research supported by the NSFC (No.11061030).
$^\dagger$Corresponding author. Tel: +86 931 7971297.
E-mail address: daiguowei@nwnu.edu.cn (G. Dai).
It is well-known that
\[
\{-\Delta_p u = \lambda m(x)\varphi_p(u) \text{ in } \Omega,
\quad u = 0 \quad \text{on } \partial \Omega \}
\tag{1.2}
\]
possesses one principal eigenvalues \(\lambda_1\) (see [2]). Moreover, if \(m\) changes sign in \(\Omega\), problem (1.2) possesses two principal eigenvalues \(\lambda^+_1\) and \(\lambda^-_1\) (see [3]) with
\[
\lambda^+_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p \, dx \bigg| u \in W^{1,p}_0(\Omega), \int_{\Omega} m|u|^p \, dx = 1 \right\}
\]
and
\[
\lambda^-_1 = \max \left\{ -\int_{\Omega} |\nabla u|^p \, dx \bigg| u \in W^{1,p}_0(\Omega), \int_{\Omega} -m|u|^p \, dx = 1 \right\}.
\]
Define
\[
f_0 = \lim_{s \to 0} \frac{f(s)}{\varphi_p(s)} \quad \text{and} \quad f_\infty = \frac{f(s)}{\varphi_p(s)}.
\]
Furthermore, we also suppose that
\[
(f2) \quad f_0 < \lambda_1 < f_\infty \quad \text{or} \quad f_0 > \lambda_1 > f_\infty.
\]

Obviously, the methods used in [4] cannot be used to deal with problem (1.1) because we do not require that \(f\) satisfies the signum condition \(f(s)s > 0\) for \(s \neq 0\), which raises the essential difficulty. In order to overcome this difficulty, we use a maximum principle which will be proved in Section 2. More precisely, we consider the following problem
\[
\{-\Delta_p u - \lambda m(x)\varphi_p(u) = h(x) \text{ in } \Omega,
\quad u = 0 \quad \text{on } \partial \Omega, \}
\tag{1.3}
\]
where \(m\) is a changing-sign function.

Let \(p' = p/(p - 1)\). The main result of this work is the following maximum principle.

**Theorem 1.1.** Let \(h \in L^{p'}(\Omega)\) with \(h \geq 0 \quad (\leq 0)\), \(\not\equiv 0 \) in \(\Omega\) and \(u\) be a solution of problem (1.3). Then \(u > 0 \quad (< 0)\) in \(\Omega\) if and only if \(\lambda \in (\lambda^-_1, \lambda^+_1)\).

For the case of \(p = 2\), Hess and Kato [8, Corollary 2] proved that the condition is sufficient. For \(m(x) \equiv 1\) in \(\Omega\), Fleckinger et al. [7, Theorem 5] showed that the condition is sufficient and necessary. Thus, the result of Theorem 1.1 has extended and improved the corresponding ones to [7, 8].

In particular, we have the following corollary.

**Corollary 1.1.** Assume that \(m \geq 0\) and \(m \not\equiv 0\) in \(\Omega\). Let \(h \in L^{p'}(\Omega)\) with \(h \geq 0 \quad (\leq 0)\), \(\not\equiv 0 \) in \(\Omega\) and \(u\) be a solution of problem (1.3). Then \(u > 0 \quad (< 0)\) in \(\Omega\) if and only if \(\lambda \in (0, \lambda_1)\).

**Remark 1.1.** Note that the result of Corollary 1.1 is enough for this work. We prove more general result like as Theorem 1.1 because we believe that it will be useful in dealing with nonlinear problems with indefinite weight. We shall discuss this kind of problems in our future work.
On the basis of Corollary 1.1, we obtain the following result.

**Theorem 1.2.** Let (f1) and (f2) hold. Then problem (1.1) possesses at least a positive and one negative solution.

** Remark 1.2.** Obviously, the result of Theorem 1.2 extends and improves the corresponding one of [5, Theorem 3.1]. Meanwhile, the result of Theorem 1.2 also extends the results of Theorem 5.1 and 5.2 of [4] in some sense.

From the results of Theorem 1.2, we can easily deduce the following corollary.

**Corollary 1.2.** Besides the assumptions of Theorem 1.2, we also suppose that $f_0$, $f_\infty \in (0, +\infty)$. Then for each

$$\lambda \in \left( \frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty} \right) \cup \left( \frac{\lambda_1}{f_\infty}, \frac{\lambda_1}{f_0} \right),$$

the problem

$$\begin{cases} -\Delta_p u = \lambda m(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

possesses at least a positive and one negative solution.

## 2 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. As usually, we use $\| \cdot \|$ to denotes the norm of $W_0^{1,p}(\Omega)$.

**Proof of Theorem 1.1.** We first prove that the condition is necessary. We divide the proof into two cases.

** Case 1.** $\lambda \geq 0$.

If $u$ is a positive solution of problem (1.3) for $h \geq 0$, multiplying (1.3) by $u$ and integration over $\Omega$, we obtain that

$$\int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} m|u|^p dx = \int_{\Omega} h(x)u dx \geq 0.$$ 

It follows that

$$\int_{\Omega} |\nabla u|^p dx \geq \lambda \int_{\Omega} m|u|^p dx.$$

This relationship together with the variational characterization of $\lambda_1^+$ implies that $\lambda \leq \lambda_1^+$. We claim that $\lambda < \lambda_1^+$. Suppose on the contrary that $\lambda = \lambda_1^+$. Let $u_1$ be the corresponding eigenfunction to $\lambda_1^+$ with $\|u_1\| = 1$. Obviously, one has

$$\begin{cases} -\Delta_p u_1 = \lambda_1^+ m\varphi_p (u_1) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial \Omega. \end{cases}$$

For any $\varepsilon > 0$, we apply Picone’s identity to the pair $u_1$, $u + \varepsilon$. We obtain that

$$0 \leq \lambda_1^+ \int_{\Omega} m u_1^p dx - \int_{\Omega} (\lambda_1^+ m\varphi_p(u + \varepsilon) + h(x)) \frac{u_1^p}{(u + \varepsilon)^{p-1}}.$$
It follows that
\[ \lambda^+_1 \int_{\Omega} m u_1^p \, dx < \int_{\Omega} \left( \lambda^+_1 m \varphi_p (u + \varepsilon) + h(x) \right) \frac{u_1^p}{(u + \varepsilon)^{p-1}} \leq \lambda^+_1 \int_{\Omega} m u_1^p \, dx. \]
We get a contradiction.

**Case 2.** \( \lambda < 0 \).

We restate problem (1.3) as the following form
\[
\begin{align*}
-\Delta_p u - \hat{\lambda} \hat{m} \varphi_p (u) &= h \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( \hat{\lambda} = -\lambda \), \( \hat{m} = -m \). Let \((\hat{\lambda}^+_1, v_1)\) with \( v_1 > 0 \) in \( \Omega \) and \( \|v_1\| = 1 \) be the corresponding principal eigen-pairs of the problem
\[
\begin{align*}
-\Delta_p u &= \hat{\lambda} \hat{m} \varphi_p (u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
By reasoning as above, we obtain \( \hat{\lambda} < \hat{\lambda}^+_1 \). It is well-known that \( \hat{\lambda}^+_1 - 1 = -\lambda - 1 \). So we get \( \lambda > \lambda^+_1 - 1 \).

Conversely, assume that \( \lambda \in (\lambda^+_1, \lambda^-_1) \) and \( u \) is a solution of problem (1.3) for \( h \geq 0 \). We also divide the proof into two cases.

**Case 1.** \( \lambda \geq 0 \).

We obtain by multiplying problem (1.3) by \( u^- \) and integration over \( \Omega \):
\[
\int_{\Omega} -\Delta_p u^- \, dx = \lambda \int_{\Omega} m \varphi_p (u) u^- \, dx + \int_{\Omega} h(x) u^- \, dx. \tag{2.1}
\]
Then, it follows from (2.1) that
\[
\lambda^+_1 \int_{\Omega} m |u^-| \, dx \leq \int_{\Omega} |\nabla u^-|^p \, dx
\]
\[
\leq \lambda \int_{\Omega} m |u^-|^p \, dx - \int_{\Omega} h(x) u^- \, dx
\]
\[
\leq \lambda \int_{\Omega} m |u^-|^p \, dx. \tag{2.2}
\]
So we have
\[
(\lambda^+_1 - \lambda) \int_{\Omega} m |u^-|^p \, dx \leq 0 \quad \text{and} \quad \int_{\Omega} m |u^-|^p \, dx \geq 0.
\]
Hence \( u^- \equiv 0 \), so that \( u \geq 0 \).

We rewrite problem (1.3) as the following form
\[
\begin{align*}
-\Delta_p u + \lambda m^- \varphi_p (u) &= h + \lambda m^- \varphi_p (u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
The strong maximum principle of [9] implies that \( u > 0 \) in \( \Omega \).

**Case 2.** \( \lambda < 0 \).

By an argument similar to (2.2), we obtain
\[
(\lambda^-_1 - \lambda) \int_{\Omega} m |u^-|^p \, dx \leq 0 \quad \text{and} \quad \int_{\Omega} m |u^-|^p \, dx \leq 0.
\]
Thus \( u^- \equiv 0 \) which follows \( u \geq 0 \). We rewrite problem (1.3) as
\[
\begin{align*}
-\Delta_p u - \lambda m^+ \varphi_p (u) &= f - \lambda m^+ \varphi_p (u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
The strong maximum principle of [9] implies that \( u > 0 \) in \( \Omega \).
3 Proof of Theorem 1.2

In this section, based on Theorem 1.1 and the bifurcation result of [6], we study the existence of one-sign solutions for problem (1.1). From now on, for simplicity, we write $X := W^{1,p}_0(\Omega)$.

Let us define $g(s) = f(s) - f_0\varphi_p(s)$, then

$$\lim_{s \to 0} \frac{g(s)}{\varphi_p(s)} = 0 \quad \text{and} \quad \lim_{|s| \to +\infty} \frac{g(s)}{\varphi_p(s)} = f_\infty - f_0.$$  

The existence of one-sign solutions of problem (1.1) is related to the following eigenvalue problem

$$\begin{cases}
-\Delta_p u = \lambda m \varphi_p(u) + mg(u) \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \partial\Omega,
\end{cases}$$

where $\lambda \in \mathbb{R}$ is a parameter. Thus showing that problem (1.1) has a solution is equivalent to show that problem (3.1) has a solution for $\lambda = f_0$.

Applying Theorem 4.5 of [6] to problem (3.1), we obtain that there are two distinct unbounded sub-continua $C^+$ and $C^-$, consisting of the continuum $C$ emanating from $(\lambda_1, 0)$. Moreover, $u > 0$ in $\Omega$ if $(\lambda, u) \in C^+ \setminus \{0\}$ and $u < 0$ if $(\lambda, u) \in C^- \setminus \{0\}$.

Lemma 3.1. Let $\sigma \in \{+, -\}$. There is a $C > 0$ such that if $(\lambda_*, u) \in C^\sigma$ then $|\lambda_*| \leq C$.

Proof. We have that $(\lambda_*, u)$ satisfies

$$\begin{cases}
-\Delta_p u = \left(\lambda_* + \frac{g(u)}{\varphi_p(u)}\right) m \varphi_p(u) \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \partial\Omega.
\end{cases}$$

We do the proof for the case $u > 0$ in $\Omega$, the case $u < 0$ being similar.

Problem (3.2) can be written as

$$\begin{cases}
-\Delta_p u - (\lambda_* - \lambda^*) m \varphi_p(u) = \left(\lambda^* + \frac{g(u)}{\varphi_p(u)}\right) m \varphi_p(u) \quad &\text{in } \Omega, \\
u = 0 \quad &\text{on } \partial\Omega.
\end{cases}$$

Choosing $\lambda^*$ such that

$$\left(\lambda^* + \frac{g(u)}{\varphi_p(u)}\right) m \geq 0 \quad \text{a.e. in } \Omega.$$  

It follows from Corollary 1.1 that $0 < \lambda_* - \lambda^* < \lambda_1$. Thus, we get

$$|\lambda_*| < \max\{|\lambda_1 + \lambda^*|, |\lambda^*|\} =: C.$$

This completes the proof.

Proof of Theorem 1.2. We only prove the isolated property of $f_0 < \lambda_1 < f_\infty$ since the case $f_0 > \lambda_1 > f_\infty$ is completely analogous. We shall show that $C^\sigma$ crosses the hyperplane $\{f_0 \times X\}$ in $\mathbb{R} \times X$. Obviously, we have $f_0 < \lambda_1$. Let $(\lambda_n, u_n) \in C^\sigma$ where $u_n \neq 0$ satisfies $|\lambda_n| + ||u_n|| \to +\infty$. Lemma 3.1 implies that there exists a positive constant $M$ such that $|\lambda_n| \leq M$ for each $n \in \mathbb{N}$. It follows that $||u_n|| \to +\infty$ as $n \to +\infty$.

We divide the equation

$$-\Delta_p u_n - \lambda_n m \varphi_p(u_n) = mg(u_n)$$
by \( \|u_n\| \) and set \( \overline{u}_n = u_n/\|u_n\| \). Theorem 4.5 of [6] implies that \( \overline{u}_n > 0 \) (or \( < 0 \)) in \( \Omega \). It follows that \( u_n = \overline{u}_n \|u_n\| \to +\infty \) (or \( -\infty \)) as \( n \to +\infty \). Thus, we have

\[
\lim_{n \to +\infty} \frac{f(u_n)}{\varphi_p(u_n)} = f_\infty - f_0. \tag{3.3}
\]

Since \( \overline{u}_n \) is bounded in \( X \), after taking a subsequence if necessary, we have that \( \overline{u}_n \rightharpoonup \overline{u} \) for some \( \overline{u} \in X \) and \( \overline{u}_n \to \overline{u} \) in \( L^p(\Omega) \). By (3.3) and the compactness of \( R_p : L^p(\Omega) \to X \) (see [5, p.229]) we obtain that

\[
-\Delta_p \overline{u} = (\overline{\lambda} + f_\infty - f_0) m\varphi_p(\overline{u}),
\]

where \( \overline{\lambda} = \lim_{n \to +\infty} \lambda_n \), again choosing a subsequence and relabeling it if necessary.

It is clear that \( \|\overline{u}\| = 1 \) and \( \overline{u} \in \mathcal{C}_\sigma \subseteq \mathcal{C}^\sigma \) since \( \mathcal{C}^\sigma \) is closed in \( \mathbb{R} \times X \). Therefore \( \overline{u} \neq 0 \), i.e., \( \overline{\lambda} + f_\infty - f_0 \) is an eigenvalue of problem (1.2). So we have \( \overline{\lambda} = \lambda_1 + f_0 - f_\infty < f_0 \). Therefore, \( \mathcal{C}^\sigma \) crosses the hyperplane \( \{f_0\} \times X \) in \( \mathbb{R} \times X \).

\section*{References}

[1] W. Allegretto and Y.X. Huang, A Pocone’s identity for the \( p \)-Laplacian and applications, Nonlinear Anal. 32(7) (1998) 819–830.

[2] A. Anane, Simplicité et isolation de la première valeur propre du \( p \)-Laplacien avec poids, Comptes Rendus Acad. Sc. Paris, Série I, 305 (1987) 725–728.

[3] M. Cuesta, Eigenvalue problems for the \( p \)-Laplacian with indefinite weights, Electron. J. Differential Equations, 2001(33) (2001) 1–9.

[4] G. Dai and R. Ma, Unilateral global bifurcation phenomena and nodal solutions for \( p \)-Laplacian, J. Differential Equations 252 (2012) 2448–2468.

[5] M. Del Pino and R. Manásevich, Global bifurcation from the eigenvalues of the \( p \)-Laplacian, J. Differential Equations 92 (1991) 226–251.

[6] P. Drábek and Y.X. Huang, Bifurcation problem for the \( p \)-Laplacian in \( \mathbb{R}^N \), Trans. Amer. Math. Soc. 349(1) (1997) 171–188.

[7] J. Fleckinger, J. Hernández and F.de Thélin, On maximum principles and existence of positive solutions for some cooperative elliptic systems, Differential Integral Equations 7(3–4) (1994) 689–698.

[8] P. Hess and T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, Comm. Partial Differential Equations 5(10) (1980) 999–1030.

[9] M. Montenego, Strong maximum principles for super-solutions of quasilinear elliptic equations, Nonlinear Anal. 37 (1999) 431–448.