Generalized Second Law of Thermodynamics on the Event Horizon for Interacting Dark Energy

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Abstract Here we are trying to find the conditions for the validity of the generalized second law of thermodynamics (GSLT) assuming the first law of thermodynamics on the event horizon in both cases when the FRW universe is filled with interacting two fluid system— one in the form of cold dark matter and the other is either holographic dark energy or new age graphic dark energy.

Keywords Holographic dark energy · New age graphic dark energy

1 Introduction

The equivalence between a black body and a black hole (BH) both emitting thermal radiation (using semi-classical description) put a new era for black hole physics. The black hole behaves as a thermodynamical system with the temperature (known as Hawking temperature) and the entropy proportional to the surface gravity at the horizon and the area of the horizon \([1, 2]\) respectively. Further, this temperature, entropy and mass of a black hole are related by the first law of thermodynamics \([3]\). On the other hand, the thermodynamic parameters namely, the temperature and entropy are characterized by the space time geometry. So a natural speculation about some relationship between black hole thermodynamics and Einstein equations is legitimate. In fact, Jacobson \([4]\) showed that Einstein equations can be derived from the first law of thermodynamics: \(\delta Q = TdS\), for all local Rindler Causal horizons with \(\delta Q\) and \(T\) as the energy flux and Unruh temperature measured by an accelerated observer just inside the horizon, while on the other way Padmanabhan \([5, 6]\) derived the first law of thermodynamics on the horizon, starting from Einstein equations for general static spherically symmetric space-time.
This equivalence between the thermodynamical laws and the Einstein gravity subsequently leads to generalize this idea in cosmology, treating universe as a thermodynamical system. More precisely, if we assume that the universe is bounded by the apparent horizon $R_A$ with temperature $T_A = \frac{1}{2\pi R_A}$ and entropy $S_A = \frac{\pi R_A^2}{G}$ then the Friedmann equations and the first law of thermodynamics (on the apparent horizon) are equivalent [7]. Usually, the universe bounded by the apparent horizon is termed as a Bekenstein system because Bekenstein’s entropy-mass bound ($S \leq 2\pi \pi R_A$) and entropy-area bound ($S \leq \frac{\pi}{4}$) are obeyed in this region. On the other hand, the cosmological event horizon does not exist in the usual standard big bang model while it exists in an accelerating universe dominated by dark energy $\omega_D \neq -1$. However, both the first and second law of thermodynamics break down on the event horizon [8]. Using the usual definition of temperature (as on the apparent horizon) Wang et al. [8] have argued that the applicability of the first law of thermodynamics is restricted to nearby states of local thermodynamic equilibrium while event horizon characterizes the global features of space time.

The present observational evidences obtained from Wilkinson-Microwave-Anisotropy-Probe (WMAP) strongly suggest that the current expansion of the universe is accelerating [9–16]. There are two possible ways [17–25] of explaining this accelerated expansion of the universe. In the frame work of general relativity it can be explained by introducing the dark energy having negative pressure. The other possibility is to consider modified gravity theory such as $f(R)$ gravity [23–28], where the action is an arbitrary function ($f(R)$) of the scalar curvature $R$. As a result, the Friedmann equations [29, 30] become complicated by including powers of Ricci scalar $R$ and its time derivatives. In the present work, we examine the validity of the generalized second law of thermodynamics of the universe bounded by the event horizon (which exists due to present accelerating phase of the universe). The matter in the universe is taken in the form of interacting two fluid system- one component is dust and the other is in the form of dark energy.

The model of dark energy obeying holographic principle is termed as holographic dark energy. From the effective quantum field theory, the energy density for the holographic dark energy is given by [31]

$$\rho_D = 3c^2 M_p^2 L^{-2}$$

where $L$ is an IR cut-off in units $M_p^2 = 1$, $c$ is any free dimension less parameter determined from observational data [32]. Li [33] has argued that (also from the present context) the cut off length $L$ can be chosen as the radius of the event horizon to get correct equation of state and desired accelerating universe. Very recently, Wei and Cai [34] proposed another model of dark energy known as new age graphic dark energy (NADE) which may have interesting cosmological consequences (Originally Cai [35] proposed an age graphic dark energy model (ADE) which is unable to describe the matter dominated era). These new dark energy models are based on the uncertainty relation in quantum mechanics and gravitational effect due to Einstein gravity. Also the NADE models are constrained by various astronomical observations [36]. Although the evolution behavior of the NADE has similarity [37–40] with the holographic dark energy but the causality problem in the holographic dark energy model can be overcome in ADE by choosing the age of the universe as the measure of the length (instead of the horizon distance) while the conformal time $'\eta'$ is chosen as the time scale in NADE. Thus the energy density of the NADE can be written as

$$\rho_{ND} = \frac{3m^2 M_p^2}{\eta^2}$$

(1)
where the conformal time $\eta$ has the expression

$$ \eta = \int \frac{dt}{a} = \int_0^a \frac{da}{Ha^2} $$

and the numerical factor $3n^2$ is taken care of uncertainties in quantum theory and the effect of curved space time.

The paper is organized as follows. In Sect. 2 the holographic dark energy model has been used while recently formulated new age graphic dark energy model is taken in Sect. 3. The conclusions are presented in Sect. 4.

### 2 Interacting Holographic Dark Energy Model and Generalized Second Law of Thermodynamics

In this section, we take the FRW universe bounded by the event horizon and the matter in the universe is taken as the holographic dark energy (HDE) interacting with dust. So the energy density of the HDE model has the expression

$$ \rho_D = 3c^2 M_p^2 R_E^{-2} $$

The individual continuity equations for the HDE and dust are of the form

$$ \dot{\rho}_D + 3H(1 + \omega_D)\rho_D = -Q $$

and

$$ \dot{\rho}_m + 3H\rho_m = Q $$

where $Q = \Gamma\rho_D$ [41] is the interaction term and the decay rate $\Gamma$ corresponds to conversion of dark energy to dust. The above conservation equations can be written in non-interacting form as [41]

$$ \dot{\rho}_D + 3H(1 + \omega_{D \text{eff}})\rho_D = 0 $$

and

$$ \dot{\rho}_m + 3H(1 + \omega_{m \text{eff}})\rho_m = 0 $$

These equations show that the interacting matter system is equivalent to non-interacting two fluid system with variable equation of state

$$ \omega_{D \text{eff}} = \omega_D + \frac{\Gamma}{3H} \quad \text{and} \quad \omega_{m \text{eff}} = -\frac{\Gamma}{3Hu} $$

where $u = \frac{\rho_m}{\rho_D}$ is the ratio of two energy densities. So combining (5) and (6) we get

$$ \dot{\rho}_i + 3H(\rho_i + p_i) = 0 $$

where

$$ \rho_i = \rho_D + \rho_m, \quad p_D = \rho_D\omega_{D \text{eff}}, \quad p_m = \rho_m\omega_{m \text{eff}} \quad \text{and} \quad p_i = p_D + p_m $$
For FRW Universe with line element

$$ds^2 = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + a^2(t)r^2 d\Omega_2^2$$

the Friedmann equations are

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_i$$

and

$$\dot{H} - \frac{k}{a^2} = -4\pi G (\rho_i + p_D)$$

with $\rho_i = \rho_m + \rho_D$. Now as usual the density parameters are

$$\Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_D = \frac{\rho_D}{3H^2}, \quad \Omega_k = \frac{k}{3H^2}$$

and due to the first Friedmann equation they are related by the relation

$$\Omega_D + \Omega_m = 1 + \Omega_k$$

Now using the Friedmann equations, the conservation relations and the expression for the energy density of the holographic dark energy (2), the equation of state parameter $\omega_D$ for the HDE can be obtained as [41]

$$\omega_D = -\frac{1}{3} - \frac{2\sqrt{\Omega_D - \Omega_k}}{3c} - \frac{b^2(1 + \Omega_k)}{\Omega_D}$$

where the decay rate is chosen as [41]

$$\Gamma = 3b^2(1 + u)H$$

with $b^2$ as the coupling constant.

The deceleration parameter $q = -(1 + \frac{\dot{H}}{H^2})$ can be expressed in terms of density parameters (using Friedmann equations and the conservation relations) as

$$q = -\frac{\Omega_D}{2} - \frac{\Omega_D \sqrt{\Omega_D - \Omega_k}}{c} + \frac{1}{2} (1 - 3b^2)(1 + \Omega_k)$$

Now using the expression for the energy density of holographic dark energy (2) and the modified conservation (5) the change in the radius of the event horizon is given by [42]

$$dR_E = \frac{3}{2} R_E H (1 + \omega_D^{\text{eff}}) dt$$

Assuming the validity of the first law of thermodynamics on the event horizon and using the expression for the amount of energy crossing the event horizon in time $dt$ [7, 43, 44] i.e.

$$-dE = 4\pi R_E^3 H (\rho_i + p_i) dt$$

we obtain

$$dS_E = \frac{4\pi R_E^3 H (\rho_i + p_i) dt}{T_E}$$
where $S_E$ is the entropy of the event horizon and $T_E$ is the temperature on the event horizon.

From the Gibb’s equation [45]

$$T_E dS_I = dE_I + p_t dV,$$

the variation of the entropy ($S_I$) of the fluid inside the event horizon is given by

$$\frac{dS_I}{dt} = \frac{4\pi R_E^3}{3} H (\rho_t + p_D) \left( \frac{3}{2} (\omega_{D}^\text{eff} + 1) - 1 \right)$$

(19)

In deriving (19) we have used (9) and the following expressions

$$V = \frac{4}{3} \pi R_E^3, \quad E_I = V \rho_t$$

Hence combining (10) and (19) the resulting change of total entropy is given by

$$\frac{d}{dt} (S_I + S_E) = \frac{6\pi R_E^3 H}{T_E} (\rho_t + p_D) (\omega_{D}^\text{eff} + 1)$$

or more explicitly

$$= \frac{6\pi R_E^3 H}{T_E} \left[ \rho_D (\omega_D^\text{eff} + 1)^2 + \rho_m (\omega_D^\text{eff} + 1) (\omega_m^\text{eff} + 1) \right]$$

(20)

3 Interacting New Age Graphic Dark Energy Model and Generalized Second Law of Thermodynamics

Similar to the previous section, the matter in the universe bounded by the event horizon is taken as interacting two fluid system- one component is in the form of recently formulated new age graphic dark energy and the other is the dark matter in the form of dust. So as before the time variation of the entropy of the horizon can be obtained from the first law of thermodynamics with expression

$$dS_E = \frac{4\pi R_E^3 H (\rho_t + p_t) dt}{T_E}$$

(21)

where $\rho_t = \rho_{ND} + \rho_m$, $p_{ND} = \rho_{ND} \alpha_{ND}^\text{eff}$, $p_m = \rho_m \alpha_{m}^\text{eff}$ and $p_t = p_{ND} + p_m$. Here the energy density of the dust component ($\rho_m$) satisfies the conservation equation (4)(or (6)) while the NADE has energy density and pressure $\rho_{ND}$ and $P_{ND}$ with equation of state $P_{ND} = \rho_{ND} \alpha_{ND}$. This matter component satisfies the continuity (3) or (5). Also the effective state parameters have the same form as in (7). Further, an explicit form of $\rho_{ND}$ is given in (1) in terms of conformal time. In contrast to HDE, the NADE energy density (given in equation(1)) is not related to the radius of the event horizon. So from the definition of the event horizon, the time variation of $R_E$ is given by [46]

$$\frac{dR_E}{dt} = \left( R_E - \frac{1}{H} \right) H$$

(22)
Then using Gibbs equation, the time variation of the entropy of the matter inside event horizon is given by

\[ \frac{dS_I}{dt} = -\frac{4\pi R_E^2}{T_E} (\rho_t + p_D) \]  

(23)

Thus combining (21) and (23) the change of total entropy is given by

\[ \frac{d}{dt} (S_I + S_E) = 4\pi (\rho_t + p_t) \frac{R_E^2 H}{T_E} \left( R_E - \frac{1}{H} \right) \]  

(24)

Further for the new age graphic dark energy the expressions of the equation of state parameters \( \omega_{ND} \) and the deceleration parameter \( q \) in terms of the density parameters are the following:

\[ \omega_{ND} = -1 + \frac{2\sqrt{\Omega_{ND}}}{3na} - \frac{b^2(1 + \Omega_k)}{\Omega_{ND}} \]  

(25)

and

\[ q = -\frac{3\Omega_{ND}}{2} - \frac{\Omega_{ND}^\frac{3}{2}}{na} + \frac{1}{2} (1 - 3b^2)(1 + \Omega_k) \]  

(26)

with \( \Omega_{ND} = \frac{\rho_{ND}}{3H^2} \) as the density parameter.

4 Discussion and Concluding Remarks

In this paper, validity of the second law of thermodynamics on the event horizon has been analyzed for interacting two fluid system. Here dust is one component of the matter while the other component of the matter is holographic dark energy in Sect. 2 and in Sect. 3 the new age graphic dark energy is chosen as the other component.

As there is no explicit expression for the radius of the event horizon for the new age graphic dark energy model so validity of the GSLT restricts both the matter as well as the geometry. On the other hand, for the holographic dark energy model, there exists an explicit expression for the radius of the event horizon and consequently, validity of GSLT demands restrictions on matter only. The restrictions in compact form can be written as the following:

(a) Holographic DE:

\[ \omega_D > \max \left[ -(1 + u), u - (1 + b^2)(1 + u) \right] \]  

or

\[ \omega_D < \min \left[ -(1 + u), u - (1 + b^2)(1 + u) \right] \]

or equivalently, the parameters \( b^2 \) and \( c \) are constrained as

\[ b^2 \leq \frac{u + \frac{2}{3}(1 - \sqrt{\Omega_D - \Omega_k})}{(1 + u)} \quad \text{and} \quad c < \sqrt{\Omega_D - \Omega_k} \]

or

\[ b^2 \geq \frac{u + \frac{2}{3}(1 - \sqrt{\Omega_D - \Omega_k})}{(1 + u)} \quad \text{and} \quad c > \sqrt{\Omega_D - \Omega_k} \]
(b) **New age graphic DE:**
For open or flat model $R_E \geq R_H = \frac{1}{H}$ so we have

$$\omega_{ND} > -(1 + u) \quad \text{i.e. } b^2 \leq \frac{[u + 2\sqrt{\Omega_{ND}}]}{(1 + u)}.$$ 

However for closed model if $R_E \geq R_H$ then the above inequalities hold, but if $R_E \leq R_H$ then the inequalities will be reversed i.e.

$$\omega_{ND} < -(1 + u) \quad \text{i.e. } b^2 \geq \frac{[u + 2\sqrt{\Omega_{ND}}]}{(1 + u)}.$$ 

One may note that the second alternative for holographic DE and the closed model in NAGDE (with $R_E \leq R_H$) corresponds to a possible phantom area.

We shall now discuss the validity of GSLT from the present observational point of view. From the recent observations we have [36, 47–52]

$$\Omega_D = 0.72, \quad \Omega_k = 0.02, \quad a = 1 \text{ (present time)}, \quad n = 2.7$$

(a) **HDE:** The explicit form of $q$ and $\omega_D$ are given by

$$q = -0.57 - 1.53b^2$$

$$\omega_D = -1.00005 - 1.4167b^2$$

Now the validity of GSLT demands $\rho_t + p_t \geq 0$ and $1 + \omega_D^{\text{eff}} \geq 0$ which gives

$$b^2 \leq 0.294 \quad \text{and} \quad c \geq 0.8366$$

(which agree with [52]) and consequently

$$q \geq -1.01982 \quad \text{and} \quad \omega_D \geq -1.4166.$$ 

Note that the above restriction on ‘c’ agrees with that in Ref. [52]. Thus the holographic dark energy model may be of phantom nature for the validity of the GSLT as shown in Fig 1.

**Fig. 1** The variation of $\omega_D$ for holographic dark energy model against the parameter $b$ which ranges in $[0, 0.542218]$ i.e. $b^2 \leq 0.294$
Fig. 2 The variation of $\omega_{ND}$ for new age graphic dark energy model against the parameter $b$ whose range is $[0, 0.66483]$ i.e. $b^2 \leq 0.442$

(b) **NAGDE**: From the observed data, $q$ and $\omega_{ND}$ are the following

$$q = -0.34 - 1.53b^2$$

$$\omega_{ND} = -0.79 - 1.42b^2$$

Now $\rho_t + p_t \geq 0$ gives an upper bound for $b'$ as $b^2 \leq 0.442$ which gives the explicit restrictions on $q$ and $\omega_{ND}$ in the following manner:

$$\geq -1.01626 \quad \text{and} \quad \omega_{ND} \geq -1.41764.$$  

Hence in this case also [we can find from Fig. 2] the GSLT may be valid in quintessence era as well as in phantom era. Therefore, the phantom divide line has no influence on the validity of the GSLT in both the dark energy models i.e. there may be a smooth transition between the quintessence and phantom era in the new age graphic dark energy model.

Finally, we note that to examine the validity of GSLT we have not used any explicit form of entropy or temperature at the event horizon only we have imposed the condition that the first law of thermodynamics is valid here which may be considered as a conservation equation (since we have assumed that the universe is in thermal equilibrium). However in Ref. [8] it has been shown that the usual definition of the temperature (Hawking temperature) and entropy (Bekenstein entropy) do not hold on the event horizon and consequently the first law of thermodynamics is not satisfied on the event horizon. Therefore, for future work we examine the validity of the first law thermodynamics with appropriate choice of entropy and temperature on the event horizon.

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