An Integro-Differential Structure for Dirac Distributions

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Abstract

We develop a new algebraic setting for treating piecewise functions and distributions together with suitable differential and Rota-Baxter structures. Our treatment aims to provide the algebraic underpinning for symbolic computation systems handling such objects. In particular, we show that the Green’s function of regular boundary problems (for linear ordinary differential equations) can be expressed naturally in the new setting and that it is characterized by the corresponding distributional differential equation known from analysis.

Keywords: Differential algebra, Rota-Baxter algebra; distribution theory; boundary problems.

1. Introduction

It is indisputable that differential algebra Ritt (1966); Kolchin (1973), differential Galois theory Put, Singer (2003) as well as various other approaches of Symbolic Analysis have made outstanding contributions to the theory of differential equations Seiler (1997). From their particular algebraic-algorithmic vantage points, they provide powerful tools for describing and analyzing the structure of solutions. Interestingly, the theory of distributions—in modern analysis the hard bedrock supporting the theory of linear (ordinary and partial) differential equations—has received comparably little attention in Symbolic Analysis.

One reason for this is perhaps that the standard approach to distributions seems to be inherently topological in nature; even the very definition of distributions involves the continuous dual of certain carefully chosen function spaces Duistermaat, Kolk (2010). Of course, such an objection begs the question: Namely, how much algebraic structure can one extract from the Algebra-Analysis mixture at first encountered? Even differential algebra was in the same situation before basic notions such as differential rings were introduced.

Based on the results of the present paper, we think the main obstacle to an algebraic treatment of distributions is the widespread limitation of differential algebra to structures having only derivations (differential rings / fields / algebras / modules). In a setting thus limited, one can say
A distribution like \( \delta_a \) has arbitrary formal derivatives \( \delta'_a, \delta''_a, \ldots \). In effect, one treats \( \delta_a \) as a differential indeterminate. But the characteristic feature of the Dirac distribution \( \delta_a \) is of course that it effects an evaluation at \( a \) when it “appears under the integral”. For making this central idea precise (in algebraic terms), one has to take recourse to the theory of Rota-Baxter algebras Guo (2012). By the same token, one must also provide an algebraic treatment of evaluation; both of these are linked in the structure of integro-differential algebras Rosenkranz Regensburger (2008) by a crucial relation (3).

Once integration enters the stage, it is clear that we should also consider piecewise functions since the latter may be built up from the Heaviside function \( H_a(x) = H(x - a) \), which is in turn the integral of the Dirac delta distribution \( \delta_a \). In fact, it is natural to start with the theory of piecewise (smooth or continuous) functions since these may usefully be endowed with a Rota-Baxter operator without involving distributions. One might even be tempted to build up distributions via this route, simply adding a derivation that maps \( H_a \) to the Dirac distribution \( \delta_a \). As we shall see subsequently (Remark 13), our actual development must follow a slightly different route, though we shall indeed treat piecewise functions before introducing Dirac distributions.

Assuming one can extract some “algebraic substance” from the theory of distributions, what does it achieve? In particular, does it allow any symbolic computation for practically important applications? We think the answer is yes, as we would like to demonstrate here: While the primary purpose of this paper is to lay out the foundations, we do include an application section to sketch one domain where our algebraic approach to distributions can be employed—boundary problems for linear ordinary differential equations (LODE). Here distributions come into their own: As we shall see in detail, one may actually distinguish three different (though related) roles for the algebraic Heaviside functions and Dirac distributions: (1) The Green’s function of a regular boundary problem is naturally a piecewise function (or even a proper distribution in the case of ill-posed boundary problems). (2) It can be shown to satisfy a differential equation with the Dirac distribution on its right-hand side. (3) The Green’s operator may act on functions which are only piecewise smooth.

Of course this does not exhaust the possible scope of applications. Eventually, computer algebra systems like Maple^TM and Mathematica® should be able to treat distributions and piecewise functions much like any other “functional” terms. They should provide support for all crucial operations on these objects, including many that we cannot address here (e.g. convolution, Fourier/Laplace transforms, composition). For practical applications, one often needs to be able to use piecewise functions and distributions at suitable places in algebraic and differential equations to be solved or simplified. We hope our approach will provide a convenient starting point for further development in this direction.

Structure of the paper. In detail, we will develop the subject matter as follows. After completing this Introduction by explaining some crucial notation, we briefly review the theory of differential Rota-Baxter algebras and modules, which form the basic algebraic framework for the rest of the paper (Section 2). In the next section we build up the algebra of piecewise functions and show that is a Rota-Baxter extension of the ground algebra (Proposition 4), generalizing the familiar setting of piecewise smooth or piecewise analytic functions (Examples 7 and 8). Then distributions are introduced as a differential Rota-Baxter module (Section 4), following an independent route but such that the piecewise functions reappear as a Rota-Baxter subalgebra (Theorem 12). In fact, we shall see that the distributions even form an integro-differential module (Proposition 15), that they can be characterized by a natural universal property (Proposition 17), and that they inherit the shift structure from the ground algebra (Theorem 18). We end the section
by exhibiting the filtration structure of the integro-differential module of distributions. The topic of the next section is a—rather modest—species of bivariate distributions (Section 5), though large enough to cater for the applications of the next section. Just as the univariate distribution module contains the Rota-Baxter subalgebra of piecewise functions, the bivariate version contains the bivariate piecewise functions as a subalgebra relative to both Rota-Baxter structures (Proposition 23). However, the main result of this section is that the bivariate distribution module is a differential Rota-Baxter module with respect to both differential Rota-Baxter structures, containing isomorphic copies of both univariate distribution modules plus “diagonal” distributions (Theorem 24). Equipped with these tools, we turn to the aforementioned applications in the theory of LODE boundary problems (Section 6). Our first goal is to generalize the algorithm extracting Green’s functions from Green’s operators given in Rosenkranz, Serwa (2015) to bivariate distributions over ordinary shifted integro-differential algebras (Theorem 26). Next we show that such a Green’s function also satisfies an algebraic version of the well-known differential equation with \( \delta(x-\xi) \) on the right-hand side (Theorem 29). Finally, we confirm that the corresponding Green’s operator of an arbitrary well-posed boundary problem may actually be applied to piecewise functions (Proposition 30). We conclude with some thoughts about future developments.

**Notation.** With the exception of the ring of integro-differential operators (to be introduced in Section 2), all rings and algebras in this paper are assumed to be commutative and—unless stated otherwise—also unitary. Algebras are over a ground ring \( K \) that will usually be a field (in fact an ordered field for most of the time). The set of nonzero elements of \( K \) is denoted by \( K^\times \).

We write \( \text{Aut}_K(\mathcal{F}) \) for the group of \( K \)-algebra automorphisms of an algebra \( \mathcal{F} \). By a character of \( \mathcal{F} \) we mean an algebra homomorphism \( \mathcal{F} \to K \). If \( P \) and \( Q \) are any linear operators on \( \mathcal{F} \), their commutator is denoted by \( [P,Q] := PQ - QP \). If \( S \) is a semigroup, we write \( \mathcal{F}[S] \) for the semigroup algebra of \( S \) over \( \mathcal{F} \), by which we mean the monoid algebra (Lang, 2002, p. 104) of the unitarization \( S \uplus \{1\} \).

If \( (\mathcal{F}, \partial) \) is a differential algebra, we write as usual \( f' := \partial f \) and \( f^{(k)} = \partial^k f \) for the derivatives of an element \( f \in \mathcal{F} \). For a set of differential indeterminates \( X \), the algebra of differential polynomials \( \mathcal{F}[X] \) is the free object in the category of differential \( \mathcal{F} \)-algebras. Similarly, the \( \mathcal{F} \)-submodule \( \mathcal{F}[X]_1 \) consisting of affine differential polynomials, i.e. those having total degree at most 1, is the free object in the category of differential \( \mathcal{F} \)-modules.

Since in Section 3 we will be dealing with \( K \)-algebras where \( (K,+) \) is an ordered field (hence of characteristic zero), it is useful to introduce some notation for ordered fields. We denote the minimum and maximum of two elements \( a, b \in K \) by \( a \wedge b \) and \( a \vee b \), respectively. We agree that \( \wedge, \vee \) have precedence over +, −. Furthermore, we shall write \( a^+ := a \vee 0 \) and \( a^- := a \wedge 0 \) for the positive and negative part of \( a \in K \); then we have \( a = a^+ + a^- \) and \( |a| = a^+ - a^- \). We observe that both \( (K, \wedge) \) and \( (K, \vee) \) are semigroups, which we denote by \( K_\wedge \) and \( K_\vee \), respectively. We define the **Heaviside operator** \( H : K \to K \) by

\[
H(a) = \begin{cases} 
0 & \text{if } a < 0, \\
\eta & \text{if } a = 0, \\
1 & \text{if } a > 0 
\end{cases}
\]

for \( a \in K \). The appropriate choice of \( \eta \in K \) is somewhat subtle; we will repeatedly come back to this point. From the analytic point of view, we might think of the mapping \( \mathbb{R} \to \mathbb{R}, a \mapsto H(a) \) as
a (representative of an) $L^2$ function\(^1\), and then the choice of $\eta \in \mathbb{R}$ is of course immaterial. For the algebraic treatment, however, we will distinguish three more or less natural possibilities (the terminology is again motivated by the case $K = \mathbb{R}$):

- The left continuous convention uses $\eta = 0$.
- In contrast, the right continuous choice is to put $\eta = 1$.
- Finally, the symmetric setting $\eta = 1/2$ is essentially the sign function in the sense that one has $\text{sgn}(a) = 2H(a) - 1$. It is neither left nor right continuous.

In this paper, we use the left-continuous convention $\eta = 0$, but we will discuss the other possibilities as we develop the corresponding algebraic structures. For convenience we shall use also $H(x) := 1 - H(x)$ for the dual Heaviside operator.

2. Differential Rota-Baxter Algebras and Modules

Just as a differential algebra $(\mathcal{F}, \partial)$ encodes the essence of derivatives, the basic algebraic structure for encoding integration is a Rota-Baxter algebra $(\mathcal{F}, \int)$, meaning a $K$-algebra with a $K$-linear operator $\int : \mathcal{F} \to \mathcal{F}$ satisfying the Rota-Baxter axiom

$$\int f \cdot g = \int f \int g + \int g \int f,$$

which we also call the weak Rota-Baxter axiom in view of an important generalization that we shall explain soon. At this juncture we should point out our parenthesis convention for nested Rota-Baxter operators:\(^2\) The scope of $\int$ extends across all implicit products (denoted by juxtaposition), terminated by $\cdot$ as on the left-hand side of (1). While this saves a host of parentheses, one must be careful to distinguish $\int f \int g$ and $\int f \cdot \int g$.

It is often necessary to combine differential and Rota-Baxter structures, especially for application areas like boundary problems, but also for the algebraic theory of distributions that we are about to build up. There are two important ways of coupling the two structures. The weaker one is called a differential Rota-Baxter algebra $(\mathcal{F}, \partial, \int)$ in Guo, Keigher (2008); by definition this is a differential algebra $(\mathcal{F}, \partial)$ and a Rota-Baxter algebra $(\mathcal{F}, \int)$ such that the Rota-Baxter operator $\int$ is a section of the derivation $\partial$. Thus the differential and Rota-Baxter structures are only coupled by the so-called section axiom $\partial \circ \int = 1_{\mathcal{F}}$.

In many cases the coupling is stronger: We call $(\mathcal{F}, \int, \partial)$ an integro-differential algebra if $(\mathcal{F}, \partial)$ is a differential algebra and $\int : \mathcal{F} \to \mathcal{F}$ is a $K$-linear operator satisfying the strong\(^3\) Rota-Baxter axiom (Rosenkranz Regensburger, 2008, Eqn. (6)), namely

$$f \int g = \int f \int g + \int f' \int g.$$

\(^1\)These remarks are purely motivational. It is important to distinguish $H(a)$ from $H_v$ which we shall introduce below, in Definition 2, as the actual algebraic model of the Heaviside function $x \mapsto H(x-a)$.

\(^2\)Note also that here and henceforth we use operator notation for $\partial$ and $\int$, as it is common in analysis. So the Leibniz rule is $\partial (fg) = f\partial g + g\partial f$ rather than $\partial (fg) = f\partial g + g\partial f$ when using functional notation with $d$. In the same vein, (1) would be $P(f)P(g) = P(fP(g)) + P(gP(f))$ in functional notation with $P$.

\(^3\)In Rosenkranz Regensburger (2008) we have called it the “differential Rota-Baxter axiom”. In the present context we prefer to avoid this terminology as it might be misconstrued as characterizing differential Rota-Baxter algebras.
This terminology stems from the fact that the weak axiom (1) is a consequence (just replace $f$ by $\int f$ in the strong axiom and use the section axiom) while there are differential Rota-Baxter algebras that are not integro-differential algebras. The first such example was found by G. Regensburger, using a quotient of a polynomial ring (Rosenkranz Regensburger, 2008, Ex. 3). In fact, we shall soon encounter a natural example from analysis, namely piecewise smooth functions “interpreted in the $L^2$ style” (Proposition 6).

There are various equivalent characterizations of the difference between differential Rota-Baxter and integro-differential algebras (Guo, Regensburger, Rosenkranz, 2012, Thm. 2.5), for example that $\text{Im} \int \subset \mathcal{F}$ is an ideal rather than a subalgebra, or that $\int$ is linear not just over $K$ but over $\text{Ker} \, \partial$. One reformulation that is important here involves the so-called induced evaluation

$$e := 1_{\mathcal{F}} - \int \partial,$$

which is a just projector onto $K$ along $\text{Im} \int$ for a general differential Rota-Baxter algebra but moreover multiplicative for an integro-differential algebra.

We call an (integro-)differential algebra ordinary if $\text{Ker} \, \partial = K$. In that case, $e$ is a linear functional for a general differential Rota-Baxter algebra and a character for an integro-differential algebra. We will usually start from an ordinary integro-differential algebra $(\mathcal{F}, \partial, \int)$.

In fact, ordinary differential Rota-Baxter algebras are automatically integro-differential (since then linearity over $K$ is actually over $\text{Ker} \, \partial$).

The notion of evaluation is crucial for the algebraic theory of integration. For certain purposes (cf. Definition 3), it will thus be useful to extend it to the more general setting of a plain Rota-Baxter algebra $(\mathcal{F}, \int)$, where an evaluation is any character $e : \mathcal{F} \to K$ with $e \int = 0$. This generalizes also the case of so-called ordinary Rota-Baxter algebras Rosenkranz, Gao, Guo (2015), defined as Rota-Baxter algebras $(\mathcal{F}, \int)$ where $\int : \mathcal{F} \to \mathcal{F}$ is injective and $\text{Im}(\int) + K = \mathcal{F}$; the projector $e$ onto $K$ along $\text{Im}(\int)$ is then a distinguished evaluation. As noted in Rosenkranz, Gao, Guo (2015), each ordinary Rota-Baxter algebra corresponds to a unique integro-differential algebra $(\mathcal{F}, \partial, \int)$ such that (3) holds; $(\mathcal{F}, \partial, \int)$ is thus ordinary in the usual sense of $\text{Ker} \, \partial = K$.

We think of the evaluation as evaluating at a certain point $c$, namely the (implicit) initialization point of the Rota-Baxter operator $\int = \int_c$. While this is only suggestive notation, we can consider an arbitrary character $\varphi : \mathcal{F} \to K$ and turn the given Rota-Baxter operator $\int$ into a new one $\int_\varphi := (1 - \varphi)\int$ that we call initialized at $\varphi$. Its initialized function space $\text{Im} \int_\varphi$ is given by $\text{Ker} \, \varphi$, the functions “vanishing at $\varphi$”. This may be viewed as an algebraic description of integrals $\int_x$ from various fixed initialization points $\varphi$ to the variable upper bound $x$. We will have a more rigid connection when we construct piecewise functions via shift maps (Definition 3), labeling the characters $\varphi = e_c$ by points $c \in K$. In this context we will often use $f(c)$ as a suggestive shorthand for $e_c(f)$, likewise $\int_x$ for $\int_{e_c}$.

**Example 1.** The standard example from analysis is $\mathcal{F} = C^\infty(\mathbb{R})$ with derivation $\partial f(x) = df/dx$ and the Rota-Baxter operator $\int f(x) = \int_0^x f(x) \, dx$. Here the initialized functions $f(x)$ are those with $f(0) = 0$, corresponding to the evaluation $e_0(f) = f(0)$. Any other evaluation $e_c(f) := f(c)$ may be used for generating additional Rota-Baxter operators $\int_c f = \int_0^c f(x) \, dx$.

Within this paper we cannot review the theory and algorithms for linear boundary problems over an ordinary integro-differential algebra $(\mathcal{F}, \partial, \int)$; let us refer the reader to Rosenkranz Regensburger (2008). Here we just recall that, given a collection $\Phi$ of characters $\mathcal{F} \to K$, one constructs the ring of integro-differential operators $\mathcal{F}_\Phi[\partial, \int]$ with canonical direct decomposition

$$\mathcal{F}_\Phi[\partial, \int] = \mathcal{F}[\partial] + \mathcal{F}[\int] + (\Phi)$$
as $K$-vector spaces; here $\mathcal{F}[\partial]$ is the usual ring of differential operators and $\mathcal{F}[\partial]]$ the corresponding (nonunitary) ring of integral operators (generated over $\mathcal{F}$ by $\partial$) while $(\Phi)$ is the two-sided ideal generated by the character set $\Phi$. The latter may be characterized as left $\mathcal{F}$-module generated by Stieltjes conditions, defined as the right ideal $\Phi : \mathcal{F}[\partial, \partial]$.

In the standard example above, these are arbitrary linear combinations of local conditions (derivative evaluations of any order) and global conditions (definite integrals with premultiplied weighting functions).

A boundary problem is a pair $(T, \mathcal{B})$ consisting of a monic differential operator $T \in \mathcal{F}[\partial]$ of order $n$ and a boundary space $\mathcal{B} \subset \mathcal{F}'$ spanned by $n$ linearly independent Stieltjes conditions $\beta_1, \ldots, \beta_n$. We call $(T, \mathcal{B})$ regular iff $\text{Ker} T + \mathcal{B} \subset \mathcal{F}$, where the orthogonal is defined as the admissible function space $\mathcal{B}^\perp := \{ f \in \mathcal{F} \mid \forall \beta \in \mathcal{B} \beta(f) = 0 \}$. Regularity of $(T, \mathcal{B})$ is equivalent to the classical stipulation: There is exactly one solution $u \in \mathcal{F}$ of

$$
\begin{align*}
Tu &= f, \\
\beta(u) &= 0 \quad (\beta \in \mathcal{B})
\end{align*}
$$

for every forcing function $f \in \mathcal{F}$. Having a fundamental system $u_1, \ldots, u_n$ of the homogeneous system, meaning a $K$-basis of $\text{Ker} T$, this may be checked algorithmically: The regularity of $(T, \mathcal{B})$ is equivalent to the regularity of the evaluation matrix $\beta(u) \in K^n$ formed by evaluating each $\beta_i \in \mathcal{B}$ on each $u_j \in \text{Ker} T$.

The Green’s operator $G$ of a regular boundary problem $(T, \mathcal{B})$ is characterized by the relations $TG = 1_\mathcal{F}$ and $\text{Im} G = \mathcal{B}^\perp$; it is the map $f \mapsto u$ for (4) and may be computed as an element of the operator ring $G \in \mathcal{F}[\partial, \partial]$. Using the natural action of $\mathcal{F}[\partial, \partial]$ on $\mathcal{F}$, one may check that $u := Gf$ actually satisfies (4).

The above ring-theoretic notions (differential algebra, Rota-Baxter algebra, differential Rota-Baxter algebra, integro-differential algebra) all have natural module-theoretic analogs. For example, a differential Rota-Baxter module $(M, \partial, \partial)$ over a differential Rota-Baxter algebra $(\mathcal{F}, \partial, \partial)$ consists of a derivation $\partial : M \to M$ in the sense that $\partial \ f \varphi = (\partial f) \varphi + f \partial \varphi$ for $f \in \mathcal{F}$ and $\varphi \in M$, and a Rota-Baxter operator $\partial : M \to M$ characterized by the (weak) Rota-Baxter axiom

$$
\int f \cdot \int \varphi = \int f \int \varphi + \int (\int f) \varphi
$$

for $f \in \mathcal{F}$ and $\varphi \in M$; confer also (Gao, Guo, Rosenkranz, 2015, Ex. 3.7(b)). It is now also clear what one means by a Rota-Baxter module. The notion of integro-differential module, however, is slightly more subtle since we must now distinguish the strong Rota-Baxter axiom (2) for coefficients and the one for module elements; we shall postpone this discussion to later when it is needed (Lemma 14). For now let us just agree to call $(M, \partial, \partial)$ ordinary iff $\text{Ker} \partial = K$.

When dealing with bivariate distributions, we shall come across algebras with two distinct differential and/or Rota-Baxter structures. In such a case we shall speak of duplex structures. For example, a duplex Rota-Baxter algebra $(\mathcal{F}_2, \partial_2, \partial_2, \partial_2, \int \partial, \int \partial)$ is characterized by requiring both $(\mathcal{F}_2, \partial_2, \partial_2, \int \partial)$ and $(\mathcal{F}_2, \partial_2, \int \partial)$ to be Rota-Baxter algebras. (It should be noted that from our algebraic viewpoint $\partial_2, \partial_2$ and $\int \partial, \int \partial$ are just two pairs of derivations and Rota-Baxter operators that we might as well call $d, e$ and $P, Q$ should we care to.)

### 3. The Piecewise Extension

The passage from smooth functions $C^\infty(\mathbb{R})$ to piecewise smooth functions $PC^\infty(\mathbb{R})$ can be achieved by adding characteristic functions for all intervals $[a, b] \subset \mathbb{R}$, and these can in turn be
generated by the well-known Heaviside function \( H(x) \in PC^\infty(\mathbb{R}) \) as \( 1_{(a,b)}(x) = H(x-a) H(b-x) \). We shall come back to this motivating instance (Example 7).

Our present goal is to describe the passage from a suitable integro-differential algebra \( \mathcal{F} \) to its piecewise extension \( \mathcal{PF} \) in an abstract algebraic manner. As we have just seen, it is sufficient to adjoin algebraic Heaviside functions to \( \mathcal{F} \). These can be defined in a natural way if the ground field \( K \) of the given integro-differential algebra \( \mathcal{F} \) is an ordered field\(^4\); in classical analysis this is of course \( K = \mathbb{R} \). For the algebraic construction it is sufficient to employ the semigroup algebra of \( K_\omega \) over \( \mathcal{F} \).

**Definition 2.** Let \( \mathcal{F} \) be an algebra over an ordered ring \( (K, \prec) \). Then we define its piecewise extension as \( \mathcal{PF} := \mathcal{F}[K_\omega] \).

We denote the identity element of \( \mathcal{PF} \) by 1 and the other generators by \( H_a(a \in K) \). Then \( \mathcal{PF} \) can be viewed as the quotient of the polynomial ring \( \mathcal{F}[H_a \mid a \in K] \) modulo the ideal generated by the relations \( H_a H_b - H_{a \cdot b} \) \((a, b \in K)\). Moreover, linearity of the order on \( K \) gives rise to the exchange law \( H_{a \cdot b} + H_{x \cdot b} = H_{x} + H_{b}(a, b \in K) \), which implies in turn that the piecewise extension \( \mathcal{PF} = \mathcal{F}[K_\omega] \) is isomorphic to its dual \( \mathcal{F}[K_\omega]^\ast \) via \( H_a \mapsto 1 - \hat{H}_a \), where the \( \hat{H}_a \) denote the generators of the dual. We will restrict ourselves to \( \mathcal{PF} = \mathcal{F}[K_\omega] \), using \( \hat{H}_a := 1 - H_a \in \mathcal{F}[K_\omega] \) as shorthand notation. Introducing the alternative notation \( H(x-a) := H_a \) and \( H(a-x) := \hat{H}_a \), the above relations entail

\[
H(x-a) H(x-b) = H((x-a) \sqcup b),
\]
\[
H(a-x) H(b-x) = H((x-a) \cap b),
\]
\[
H(a-x) H(x-b) = 0 \quad \text{if} \ a < b.
\]

In the classical setting (Example 1), this provides a faithful model of the (rising and falling) Heaviside functions based at various points \( a, b \in \mathbb{R} \). We will elaborate on the analysis setting in due course (Example 7).

Nevertheless, one may wonder if there is any intrinsically algebraic characterization. One possibility is this: Call an algebra \( \mathcal{F} \) over an ordered ring \( (K, \prec) \) order-related if it encodes the order of the ground ring within its multiplicative structure, i.e. if there exists a monoid embedding \( H : (K, \sqcup) \hookrightarrow (\mathcal{F}, \cdot) \) so that \( H_a H_b = H_{a \cdot b} \). An order-related morphism between order-related rings is an algebra homomorphism \( \zeta : \mathcal{F} \to \mathcal{T} \) such that \( \zeta(H_a) = \hat{H}_a \). Then the piecewise extension \( \mathcal{PF} \) can be characterized as universal order-related extension algebra of \( \mathcal{F} \), meaning every embedding \( \mathcal{F} \hookrightarrow A \) into an order-related algebra \( A \) factors through the algebra embedding \( \mathcal{F} \hookrightarrow \mathcal{PF} \) via a unique order-related morphism \( \mathcal{PF} \to A \). The verification is straightforward.

In order to introduce a Rota-Baxter operator \([ : \mathcal{PF} \to \mathcal{PF}\) encapsulating the integration of piecewise continuous—in particular: piecewise smooth—functions, we need a notion of algebraic domain with multiple evaluation points (intuitively this is because integrating against a step function based at \( a \in K = \mathbb{R} \) amounts to starting off the integral at \( a \), with integration constant induced by evaluation at \( a \)). One way to make this precise is in terms of a shifted Rota-Baxter algebra: using an action of the additive group of the ground field for shifting evaluation to arbitrary field points.

\(^4\)Distributions are usually defined as generalizations of functions of a real variable, meaning either \( \mathbb{R}^n \to \mathbb{R} \) or \( \mathbb{R}^n \to \mathbb{C} \). The case of a complex variable \( \mathbb{C}^n \to \mathbb{C} \) is effectively treated as \( \mathbb{R}^{2n} \to \mathbb{C} \), ignoring the field structure of \( \mathbb{C} \cong \mathbb{R} \times \mathbb{R} \). Starting from an ordered field thus seems plausible.
Definition 3. By a shift map on an algebra \( \mathcal{F} \) we mean a group homomorphism
\[
S : (K, +) \to (\text{Aut}_K(\mathcal{F}), \circ), \quad \text{written } S_a f = f(x + a) \text{ for } a \in K, f \in \mathcal{F}.
\]

If Rota-Baxter/derivation operators are present, we require compatibility conditions:

1. We call \((\mathcal{F}, \int, S)\) a shifted Rota-Baxter algebra if \(S\) is a shift map on a Rota-Baxter algebra \((\mathcal{F}, \int)\) with evaluation \(\varepsilon\) such that \([S_c, \int] = \varepsilon_c \int\) for all \(c \in K\), where \(\varepsilon_c := \varepsilon \circ S_c\) is called the evaluation at \(c\).

2. We call \((\mathcal{F}, \partial, S)\) a shifted differential algebra if \(S\) is a shift map on a differential algebra \((\mathcal{F}, \partial)\) such that \([S_c, \partial] = 0\) for all \(c \in K\).

3. We call \((\mathcal{F}, \partial, \int, S)\) a shifted differential Rota-Baxter algebra if \((\mathcal{F}, \int, S)\) is a Rota-Baxter algebra such that both \((\mathcal{F}, \int, S)\) and \((\mathcal{F}, \partial, S)\) are shifted.

In the sequel, we suppress the shift map \(S\) when referring to structures such as \((\mathcal{F}, \partial, \int, S)\).

The most important examples are of course the Rota-Baxter algebra \((C(\mathbb{R}), \int_0)\) and the integro-differential algebra \((C^\infty(\mathbb{R}), \int_0^\infty, \partial)\), both with the shift map \(f(x) \mapsto f(x + a)\) and the corresponding evaluations \(\varepsilon_c f(x) = f(c)\).

In a shifted Rota-Baxter algebra \((\mathcal{F}, \int)\), all evaluations \(\varepsilon_c : \mathcal{F} \to K\) are characters but \(\varepsilon_0 = \varepsilon\) is distinguished\(^5\) by annihilating the given Rota-Baxter operator \(\int\). Using the evaluations, we can introduce shifted Rota-Baxter operators \(\int_c : \mathcal{F} \to \mathcal{F}\) and definite integrals \(\int_c^d : \mathcal{F} \to K\) by
\[
\int_c := (1 - \varepsilon_c) \int \quad \text{ and } \quad \int_c^d := \varepsilon_d \int_c.
\]

One checks immediately that \(\int_c = S_{-c} \int S_c\) and \(\int_c^d = \int_c - \int_c\) are equivalent definitions. Obviously, each \((\mathcal{F}, \int_c)\) is a Rota-Baxter algebra with evaluation \(\varepsilon_c\) for \(c \in K\).

Let us now return to the task of defining the Rota-Baxter operator on \(P\mathcal{F}\), assuming a shifted Rota-Baxter algebra \((\mathcal{F}, \int)\). Note first that every element \(\zeta \in \mathcal{P}\mathcal{F}\) can be written uniquely as
\[
\zeta = f + \sum_{a \in K} f_a H_a \quad (f, f_a \in \mathcal{F})
\]
with almost all \(f_a\) zero. Hence it suffices to define \(\int : \mathcal{P}\mathcal{F} \to \mathcal{P}\mathcal{F}\) as the unique extension of \(\int : \mathcal{F} \to \mathcal{F}\) such that
\[
\int f H_a = (\int_a^\infty f) H_a - (\int_a^0 f) \hat{H}_a = (\int_a^\infty f) H_a + \hat{H}(a) \int_0^a f
\]
for all \(f \in \mathcal{F}\) and \(a \in K\). For the sake of symmetry, let us also note that then
\[
\int f \hat{H}_a = (\int_a^\infty f) \hat{H}_a - (\int_a^0 f) H_a = (\int_a^\infty f) \hat{H}_a + H(a) \int_0^a f.
\]

In fact, (6) and (7) are equivalent.\(^6\)

\(^5\)The formulation in terms of a distinguished character \(\varepsilon\) is practical for applications. A more symmetric formulation would be to use the equitable setup described in Rosenkranz, Serwa (2015), where one starts from a whole family of ordinary Rota-Baxter operators \(\int_i : \mathcal{F} \to \mathcal{F}\) whose induced evaluations \(\varepsilon_i\) are required to satisfy the general shift relations \([S_c, \int_i] = \int_i^{\circ c}\) for all \(a, c \in K\), where the right-hand integral is defined as above. In the asymmetric setup used here, these relations can be derived by a straightforward calculation.

\(^6\)Note that the choice of the splitting point 0 \(\in K\) in (6)–(7) is to some extent arbitrary. Any other point of \(K\) would yield the same operator \(\int : \mathcal{P}\mathcal{F} \to \mathcal{P}\mathcal{F}\); in particular one could also choose the initialization point of the given Rota-Baxter operator \(\int\) of \(\mathcal{F}\). Here we have picked out 0 \(\in K\) for convenience.
The motivation for definition (6) comes from the standard example $\mathcal{F} = C^\infty(\mathbb{R})$ where it reproduces the usual Riemann integral $\int = \int_0^a$. This is illustrated in Figure 1, where we have visualized $\int fH_a$ with $f(x) = \cosh x$ in the four different cases corresponding to the signs of $x$ and $a$. While both forms of (6) are obvious from the figure, their identity is a general fact of ordinary shifted integro-differential algebras $\mathcal{F}$ as one can see by a straightforward calculation using the generic relations (9) mentioned below.

In a similar way we can also define the shifts $S_a : \mathcal{P} \mathcal{F} \to \mathcal{P} \mathcal{F}$. Since in the standard examples, $f(x) \mapsto f(x + a)$ shifts the graph of $f$ by $a$ units to the left, we are led to $S_a(H_b) := H_{b-a}$. This fixes $S_a : \mathcal{P} \mathcal{F} \to \mathcal{P} \mathcal{F}$ in view of (5) by requiring it to be an algebra homomorphism extending the given shifts $S_a : \mathcal{F} \to \mathcal{F}$. Obviously, the group law $S_a \circ S_b = S_{a+b}$ is satisfied.

Finally, we define $\varepsilon : \mathcal{P} \mathcal{F} \to K$ as the unique character extending $\varepsilon : \mathcal{F} \to K$ by the equivalent stipulations $\varepsilon(H_a) = \check{H}(a)$ or $\varepsilon(\check{H}_a) = H(a)$. Using again $\varepsilon_c := \varepsilon \circ S_c : \mathcal{P} \mathcal{F} \to K$ as shorthand notation, we have also $\varepsilon_c(H_a) = \check{H}(a-c)$ and $\varepsilon_c(\check{H}_a) = H(a-c)$. At this point it should be noted that we could also use the right continuous convention for the Heaviside operator $H(a)$ but not the symmetric one: Indeed, the relation $H_a \check{H}_b = H_{a \lor b}$ implies $\check{H}(a) \check{H}(b) = \check{H}(a \lor b)$, which is satisfied by all three conventions if $(a, b) \neq (0, 0)$; but the remaining case $\check{H}(0)^2 = \check{H}(0)$ entails $\check{H}(0) \in \{0, 1\}$ and thus rules out the symmetric convention. It is easy to check that $\varepsilon$ is indeed an evaluation on $(\mathcal{P} \mathcal{F}, \int)$.

**Proposition 4.** Let $(\mathcal{F}, \int)$ be an ordinary shifted Rota-Baxter algebra over an ordered field $K$. Then $(\mathcal{P} \mathcal{F}, \int)$ is a shifted Rota-Baxter algebra extending $(\mathcal{F}, \int)$.

---

7We can only see one apparent advantage of the symmetric convention, in trying to build up a derivation with a Leibniz rule for Heavisides—but even this is ultimately doomed to fail: see Remark 13.
Proof. Since \( K \) is of characteristic zero, by the polarization identity, it suffices to prove

\[
(fH_a)^2 = 2 \int fH_a \int fH_a
\]

for \( f \in \mathcal{F} \) and \( a \in K \). Since \( H_a \) is idempotent, the definition of \( \int : \mathcal{P}\mathcal{F} \to \mathcal{P}\mathcal{F} \) and the Rota-Baxter axioms of \( \int_a \) and \( \int_a^{-1} \) give 2 \((\int_a^0 f)H_a + 2(\int_a^{0+} f)H_a \) for the left-hand side of (8). Likewise, we get 2 \((\int_a^0 f)H_a - 2(\int_a^{0-} f)H_a \) for the right-hand side of (8), using twice the definition of \( \int : \mathcal{P}\mathcal{F} \to \mathcal{P}\mathcal{F} \). It remains to check that the second terms are equal on both sides.

For \( a \geq 0 \) both terms vanish while for \( a < 0 \) the problem reduces to checking \( \int_a^0 f = \int_a^{0+} f \) on the left-hand side yields \( (\int_a^{0-} f)^2 - \int_a^{0+} f \) since \( \int_a^0 \) is K-linear and \( \int_a^{0+} f \in K \) by \((\mathcal{F}, \int)\) being ordinary. Then the result follows from the Rota-Baxter axiom of \((\mathcal{F}, \int)\).

As for any monoid algebra (Lang, 2002, p. 106), the map \( \mathcal{F} \to \mathcal{F}[K_a] \) is an embedding, hence \( \mathcal{F} \) is a \( K \)-subalgebra of \( \mathcal{P}\mathcal{F} = \mathcal{F}[K_a] \). Since the Rota-Baxter operator on \( \mathcal{P}\mathcal{F} \) has been defined as an extension, \((\mathcal{P}\mathcal{F}, \int)\) is indeed a Rota-Baxter extension of \((\mathcal{F}, \int)\).

We have already seen that the \( S_a : \mathcal{P}\mathcal{F} \to \mathcal{P}\mathcal{F} \) defined above yield a shift map on \( \mathcal{P}\mathcal{F} \), and that the character \( \varepsilon : \mathcal{P}\mathcal{F} \to K \) defined above is an evaluation on \((\mathcal{P}\mathcal{F}, \int)\). Hence it remains to prove the compatibility relation \([S_c, \int] = \varepsilon_c \int\) for the induced evaluations \( \varepsilon_c = \varepsilon \circ S_c \). By (5), we need only verify the relation on elements of the form \( fH_a \in \mathcal{P}\mathcal{F} \); we know it is satisfied for \( f \in \mathcal{F} \) due to the shift relation on \( \mathcal{F} \). The verification may be done by a four-fold case distinction based on the positivity of \( a \) and \( a - c \). For an alternative direct proof one employs the generic identities (valid for Rota-Baxter algebras over ordered fields)

\[
\int_a f + H(a) \int_a f = \int_a f,
\]

\[
\int_a f - H(a) \int_a f = \int_a f,
\]

\[
\int_a f + H(a - c) \int_a f = \int_a f,
\]

\[
\int_a f - H(a - c) \int_a f = \int_a f,
\]

for both sides of \([S_c, \int]fH_a = \varepsilon_c \int fH_a \).}

We have now an algebraic description of integration on rings of piecewise functions, constructed from Heavisides. If all functions are piecewise smooth (cf. Example 7), we can add a derivation \( \partial \) to obtain a differential Rota-Baxter algebra that is, however, not an integro-differential algebra (Proposition 6). Normally, only in the latter case do we speak of an induced evaluation \( \varepsilon := 1F - \int \partial \), but since the analogous concept is also useful in differential Rota-Baxter algebras we introduce this operation now in the general context.

Definition 5. Let \((\mathcal{F}, \partial, \int)\) be a differential Rota-Baxter algebra. Then \( \varepsilon = 1 - \int \partial \) is called the induced pseudo-evaluation of \((\mathcal{F}, \partial, \int)\).

Assume now that \((\mathcal{F}, \partial, \int)\) is a differential Rota-Baxter algebra such that \((\mathcal{F}, \int)\) satisfies the conditions of Proposition 4. Then we define a derivation on the piecewise extension \( \mathcal{P}\mathcal{F} \) by extending the derivation on \( \mathcal{F} \) by zero. In other words, we set \( \partial H_a = 0 \) for all \( a \in K \); then \( \partial : \mathcal{P}\mathcal{F} \to \mathcal{P}\mathcal{F} \) is uniquely determined by the Leibniz rule. Note that the ring of constants is enlarged to \( \text{Ker}(\partial) = K[H_a | a \in K] \). This reflects the viewpoint of analysis that the derivative of the Heaviside function \( H(x-a) \in L^2(\mathbb{R}) \) vanishes. Of course, this is in stark contrast to the more ambitious treatment via distributions taken up in the next section (where the simple derivation from above is no longer in use).
Proposition 6. Let \((F, \partial, \{\})\) be an ordinary shifted differential Rota-Baxter algebra over the ordered field \((K, <)\). Then \((PF, \partial, \{\})\) is a shifted differential Rota-Baxter extension algebra whose induced pseudo-evaluation

\[
\hat{e}(fH_a) = e_0(f)H_a + e_a(f)\hat{H}_a = \begin{cases} 
   e_0(f) - e_a(f)\hat{H}_a & \text{if } a \leq 0, \\
   e_a(f)H_a & \text{if } a \geq 0,
\end{cases} \tag{10}
\]

is not multiplicative. Hence \((PF, \partial, \{\})\) is not an integro-differential algebra.

Proof. From the definition it is clear that \(\{\} : PF \to PF\) is a section of \(\partial : PF \to PF\), so \((PF, \partial, \{\})\) is a differential Rota-Baxter algebra by Proposition 4. For showing that it is shifted, it remains to prove the compatibility relation \([S, \partial] = 0\). Since it is true by hypothesis on \(F\), we need only check that \(S\partial fH_a = f'(x-c)H_a - c\partial S fH_a\).

One checks immediately that the pseudo-evaluation of \(PF\) is given by \((10)\), using the handy relation \(e_a = e_{a'} - e_0 + e_{a'}\). As for every integro-differential algebra, we have \((K[x], \{\}) \subseteq (F, \{\})\).

Since \(K \supseteq \mathbb{Q}\) is an ordered field, we have \(0 < 1 < 2\) so that

\[
\hat{e}(xH_1 \cdot xH_2) = \hat{e}(x^2H_2) = 4H_2 \neq 2H_2 = H_1 \cdot 2H_2 = \hat{e}(xH_1) \cdot \hat{e}(xH_2),
\]

which shows that \(\hat{e}\) fails to be multiplicative. \(\square\)

As a special case of \((10)\), note that \(\hat{e}(H_a) = H_a\). This is in agreement with the fact that the constants are given by \(\text{Ker}(\partial) = \{H_a : 0 \leq a < 1\}\); we see \(H_a\) as a differential constant that pseudo-evaluates to itself. Of course one must be careful not to confuse the pseudo-evaluation with the distinguished evaluation \(e(H_a) = \hat{H}(a)\), which has image \(K\) rather than \(K[H_a : a \in K]\).

This shows also that Proposition 4 cannot be strengthened to yield an ordinary Rota-Baxter algebra \((PF, \{\})\). Indeed, even when \(\{\}\) is injective as in Proposition 6, the complement of its image is larger than the ground field \(K\).

Example 7. Let us show that for \(K = \mathbb{R}\) and \(F = C(\mathbb{R})\), the piecewise extension \(PC(\mathbb{R})\) yields the usual Rota-Baxter algebra of piecewise continuous functions \(PC(\mathbb{R})\), up to a quotient.\(^8\)

Taking the subalgebra \(F = C^\infty(\mathbb{R})\), we obtain similarly the differential Rota-Baxter algebra \(PC^\infty(\mathbb{R})\) as the algebraic counterpart of the piecewise smooth functions \(PC^\infty(\mathbb{R})\).

Let \(f : D \to \mathbb{R}\) be continuous/smooth on an open set \(D \subseteq \mathbb{R}\). Then we call \(f\) piecewise continuous/smooth if \(D\) has finite complement in \(\mathbb{R}\) and \(f\) has one-sided limits at each \(x \in \mathbb{R}\setminus D\).

We call \(x \in \mathbb{R} \setminus D\) regular if \(f\) extends to a continuous/smooth map \(f_x : D \cup \{x\} \to \mathbb{R}\); in that case \(f_x(x) = \lim_{\xi \to x} f(\xi)\).

For a piecewise function \(f : D \to \mathbb{R}\) we shall write \(\widehat{f} : D \to \mathbb{R}\) for its maximal continuous/smooth extension. We define \(PC(\mathbb{R})\) and \(PC^\infty(\mathbb{R})\) as the set of piecewise functions \(f : D \to \mathbb{R}\) with \(\widehat{f} = f\). They become rings by setting

\[
f_1 + f_2 := \widehat{f_1 \oplus f_2}, \quad f_1 \cdot f_2 := \widehat{f_1 \odot f_2},
\]

where \(f_1 \oplus f_2\) and \(f_1 \odot f_2\) denote the pointwise sum and product of functions \(f_1 : D_1 \to \mathbb{R}\) after restricting each to their common domain \(D_1 \cap D_2\). We endow \(PC(\mathbb{R})\) and its subalgebra \(PC^\infty(\mathbb{R})\), with the usual Rota-Baxter operator \(\widehat{f} = f_0^\chi\); it is clear that this yields Rota-Baxter

---

\(^8\)Note the difference between \(P\) and \(\mathcal{P}\) in this example; the latter stands for the algebraic construction described above while the former denotes the standard notion of piecewise functions in real analysis.
algebras $(PC^{c}(\mathbb{R}), \{\}) \subset (PC(\mathbb{R}), \{\})$. Moreover, we can use the standard derivative $\partial = \frac{d}{dx}$ on the piecewise smooth functions, obtaining a differential Rota-Baxter algebra $(PC^{c}(\mathbb{R}), \partial, \{\})$.

There is an algebra homomorphism $\pi : PC(\mathbb{R}) \to PC(\mathbb{R})$ that fixes $C(\mathbb{R})$ and that sends each $H_a (a \in \mathbb{R})$ to $H(x - a) \in PC(\mathbb{R})$. Clearly, we have also $PC^{c}(\mathbb{R}) \to PC^{c}(\mathbb{R})$ by restriction. We show that both homomorphisms $\pi$ are surjective: Each $f \in PC(\mathbb{R})$ or $f \in PC^{c}(\mathbb{R})$ with regular part $f : D \to \mathbb{R}$ can be written as

$$f(x) = \sum_{i=0}^{n} f_i(x) H(x - x_i) H(x_{i+1} - x)$$

where $\mathbb{R} \setminus D = \{x_1 < \cdots < x_n\}$ and $f_i : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous/smooth extension of the function pieces $f_i(x_{[x_i,x_{i+1})})$. Here we set $x_0 = -\infty$ and $x_{n+1} = +\infty$ with the understanding that $H(x + \infty) = H(\infty - x) = 1$. With this choice of pieces $f_0, \ldots, f_n$ we have $f = \pi(\sum f_i H_i H_{i+1})$, so $\pi$ is indeed surjective. The ideals

$$\mathcal{R} := \text{Ker} (\pi : PC(\mathbb{R}) \to PC(\mathbb{R})) \quad \text{and} \quad \mathcal{R}_\infty := \text{Ker} (\pi : PC^{c}(\mathbb{R}) \to PC^{c}(\mathbb{R}))$$

encode the algebraic relations between continuous/smooth functions and Heavisides, for instance $b(x) H(x - 2) = 0$ where $b(x)$ is any bump function supported in $[-1, 1]$. Hence we obtain the quotient representations $PC(\mathbb{R}) \cong PC(\mathbb{R})/\mathcal{R}$ and $PC^{c}(\mathbb{R}) \cong PC^{c}(\mathbb{R})/\mathcal{R}_\infty$.

Example 8. The case of piecewise real-analytic functions is essentially different since analytic continuation breeds multi-valued functions (or Riemann surfaces) whose proper treatment involves sheaf-theoretic methods combined with integro-differential structures. This would lead us too far afield but may provide interesting substance for future research.

For keeping things simple, let us consider the complex algebra $PC^{c}(\mathbb{R})$ of piecewise real-analytic functions, in the sense that each function piece $f_i : (x_i, x_{i+1}) \to \mathbb{C}$ extends to an entire function.\footnote{This is a very restricted setting since even $\frac{1}{x} \notin PC^{c}(\mathbb{R})$. Indeed, keeping $\frac{1}{x}$ creates analytic and algebraic complications: multi-valued logarithms and quasi-antiderivatives (Guo, Regensburger, Rosenkranz, 2012, Ex. 4.3), respectively.} Apart from this distinction, the construction of $PC^{c}(\mathbb{R})$ is completely analogous to that of $PC^{c}(\mathbb{R})$ in Example 7. Taking now the algebra $\mathcal{F} = C^{c}(\mathbb{R})$ of global real-analytic functions (real restrictions of entire functions) as coefficient algebra, we can apply the construction of Example 7. But now the relation ideal $\mathcal{R}$ is trivial because each real-analytic function piece extends uniquely to a global real analytic function, and we obtain $PC^{c}(\mathbb{R}) \cong PC^{c}(\mathbb{R})$.

Piecewise defined functions are a major motivation for introducing distributions, via generalized derivatives. In particular, we will no longer view $\partial H_a$ as identically zero but as a “Dirac delta” $\delta_a$, sometimes written $\delta_a(x) = \delta(x - a)$. Again we shall pursue a purely algebraic route to introduce these quantities along with an integro-differential structure.

4. Construction of the Distribution Module

The basic property of the Dirac distribution $\delta_a$ concentrated at a source point $a$ is that its only nonzero “value” is assumed for $x = a$, in the sense that $f \delta_a$ vanishes identically when $f(a) = 0$. In other words, $f \delta_a$ only depends on $f(a)$ and not on all of $f$, and one has the sifting property

$$f \delta_a = f(a) \delta_a$$

(11)
for “extracting” the source value. This will be the basis of our algebraic construction.\textsuperscript{10}

**Definition 9.** Let \((\mathcal{F}, \partial)\) be a differential algebra over a ring \(K\). We define the distribution module \((\mathcal{DF}, \partial)\) as the differential \(\mathcal{F}\)-module \(\mathcal{F} \{ H_a \mid a \in K \}_1 / \mathcal{Z}\), where \(\mathcal{Z}\) denotes the differential \(\mathcal{F}\)-submodule generated by \(\{ f \delta_a - \varepsilon_a(f) \delta_a \mid f \in \mathcal{F}, a \in K \}\).

Recall that \(\mathcal{F}[X]_1\) denotes the module of affine differential polynomials in \(X\). We have also employed the abbreviation \(\delta_a := H_a',\) which we shall continue to use throughout this paper (of course derivatives \(\delta_\varphi\) of \(\varphi \in \mathcal{DF}\) are also denoted by \(\varphi'\)). The order on \(K\) induces an elimination ranking \(<\) on \(\mathcal{F} \{ H_a \mid a \in K \}\) and thus a Noetherian term order on the \(\mathcal{F}\)-module \(\mathcal{F} \{ H_a \mid a \in K \}_1\). We have \(H_a^{(n)} < H_b^{(m)}\) iff \(a < b\) or otherwise \(a = b\) and \(m < n\). In the sequel we shall always employ this term order on the free differential module underlying \(\mathcal{DF}\). It is easy to get a kind of Gröbner basis for \(\mathcal{F}\) with respect to this term order. Moreover, the direct decomposition

\[ \mathcal{F} \{ H_a \mid a \in K \}_1 = \bigoplus_{a \in K} \mathcal{F} \{ H_a \}_1 \]

of differential \(\mathcal{F}\)-modules induces the direct decomposition \(\mathcal{Z} = \bigoplus Z_a\), and we write

\[ \zeta = \sum_{a \in K} \zeta_a \quad (\zeta_a \in Z_a) \]

for the corresponding sum representation of an arbitrary \(\zeta \in \mathcal{Z}\). Let us now proceed to the crucial Presentation Lemma for exhibiting the Gröbner basis.

**Lemma 10.** The differential \(\mathcal{F}\)-module \(\mathcal{Z}\) in Definition 9 is generated as an \(\mathcal{F}\)-module by

\[ \left\{ f \delta_a^{(k)} - \sum_{i=0}^{k} \binom{k}{i} \varepsilon_a(f^{(i)}) \delta_a^{(k-i)} \mid a \in K, f \in \mathcal{F}, k \geq 0 \right\}, \quad (12) \]

which forms a Gröbner basis of \(\mathcal{Z}\). For every element \(\zeta \in \mathcal{Z}\), the leading coefficient \(f_a\) of each \(\zeta_a\) has the property \(\varepsilon_a(f_a) = 0\). Relative to this Gröbner basis, the elements \(\varphi + \zeta \in \mathcal{DF}\) of the quotient have the canonical representatives

\[ \varphi = f + \sum_{a \in K} f_a H_a \quad \sum_{a \in K} \sum_{k \geq 0} \lambda_{a,k} \delta_a^{(k)} \quad (f, f_a \in \mathcal{F}; \lambda_{a,k} \in K) \quad (13) \]

with only finitely many \(f_a\) and \(\lambda_{a,k}\) nonzero.

**Proof.** We split the proof in several steps.

1. Let us first show that \(\mathcal{Z}\) contains the \(\mathcal{F}\)-module generated by \((12)\). Since the components \(Z_a\) are independent, we fix an \(a \in K\) and abbreviate the corresponding elements of \((12)\) by \(\zeta_{f,k}\).

We prove by induction on \(k\) that all \(\zeta_{f,k}\) are contained in \(\mathcal{Z}\). For \(k = 0\) this is clear since \(\zeta_{f,0}\) is a (differential) generator of \(\mathcal{Z}\). Assume that all \(\zeta_{f,j}\) with \(j < k\) and arbitrary \(f \in \mathcal{F}\)

\textsuperscript{10}Using \(\mathcal{DF} \{ \delta_a \mid a \in K \}_1\) instead of \(\mathcal{F} \{ H_a \mid a \in K \}_1\) may seem more natural and incremental, but it runs into problems with the Leibniz rule: see Remark 13.
are contained in Z; we show that \( \zeta_{f,k} \in Z \) for a fixed \( f \in \mathcal{F} \). Differentiating an arbitrary generator \( f \delta_a = e_a(f) \delta_a \) of Z, we obtain

\[
\partial^k \zeta_{f,0} = f \delta_a^{(k)} + \sum_{i=0}^{k-1} \binom{k}{i} (\partial^{k-i} f) \delta_a^{(i)} - e_a(f) \delta_a^{(k)} \in Z.
\]

Eliminating the terms \( f^{(i)} \delta_a^{(k-i)} \) yields

\[
\partial^k \zeta_{f,0} - \sum_{i=0}^{k-1} \binom{k}{i} \zeta_{f^{i-1}a} = f \delta_a^{(k)} + \sum_{i=0}^{k-1} \sum_{j=i}^{k-1} \binom{k}{i} \binom{k}{j} (-1)^i e_a(f^{k-j}) \delta_a^{(j)} - e_a(f) \delta_a^{(k)}
\]

after an index transformation. The double sum simplifies to

\[
\sum_{j=i}^{k-1} \sum_{i=0}^{k-1} \binom{k}{i} \binom{k}{j} (-1)^i e_a(f^{k-j}) \delta_a^{(j)} = - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j e_a(f^{(j)}) \delta_a^{(k-j)},
\]

using the fact that the inner sum above evaluates to \((-1)^{k+1}(k)\). Extending the range of the last sum to include \( j = 0 \) incorporates the remaining term so that

\[
\partial^k \zeta_{f,0} - \sum_{i=0}^{k-1} \binom{k}{i} \zeta_{f^{i-1}a} = f \delta_a^{(k)} - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j e_a(f^{(j)}) \delta_a^{(k-j)} = \zeta_{f,k},
\]

which shows that \( \zeta_{f,k} \in Z \) since all \( \zeta_{f^{i-1}a} \in Z \) by the induction hypothesis.

2. For establishing the converse inclusion that \( Z \) is contained in the \( \mathcal{F} \)-module generated by \( (12) \), it suffices to show that all the derivatives \( \partial^k \zeta_{f,0} \) are \( \mathcal{F} \)-linear combination of the \( \zeta_{f,k} \). But this is clear from the last identity of the previous item.

3. We proceed now to the statement about the leading coefficients. To this end, we rewrite the module generators as

\[
\zeta_{f,k} = (f - e_a(f)) \delta_a^{(k)} - \sum_{j=1}^{k} \binom{k}{j} (-1)^j e_a(f^{(j)}) \delta_a^{(k-j)},
\]

from which the claim is evident.

4. Next we must show that \( (12) \) forms a Gröbner bases for the \( \mathcal{F} \)-module \( Z \). This involves a slight variation of the usual setting of Gröbner bases for commutative polynomials Buchberger (2006) since we have infinitely many indeterminates and the coefficient ring \( \mathcal{F} \) may have zero divisors (it is certainly not a field). Since we need only the linear fragment of the polynomial ring, we may use the approach of (Bergman, 1978, §9.5a), which also allows for infinitely many generators. In the notation of (Bergman, 1978, §9.5a), we set \( k = K \) and \( R = \mathcal{F} \) with trivial presentation (every element of \( \mathcal{F} \) is a generator, and there are no relations) and the module \( M = Z \) with generators \( \delta_a^{(k)} \) and
relations (12). The only S-polynomials \( \sigma \) arise from the self-overlaps of (12), namely \( {\ddot{\delta}}^{(k)}_a \), and this yields

\[
\sigma = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left( E_a(f^{(i)}) \bar{f} - E_a(\bar{f}^{(i)}) f \right) {\ddot{\delta}}^{(k-i)}_a
\]

\[
\rightarrow \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{k}{j} \binom{k-j}{i} (-1)^{i+j} E_a(f^{(i)} \bar{f}^{(j)} - \bar{f}^{(i)} f^{(j)}) {\ddot{\delta}}^{(k-i-j)}_a = \sum_{i \neq j} e_{ij} \eta_{ij},
\]

which vanishes since the summation is over a triangle \( i + j \leq k \), symmetric with respect to \( i \leftrightarrow j \), while the evaluation term \( e_{ij} = E_a(\ldots) \) is antisymmetric and the trinomial term \( \eta_{ij} = k! / i! (k - i - j)! (-1)^{i+j} {\ddot{\delta}}^{(k-i-j)}_a \) symmetric.

5. The analog of the Diamond Lemma in (Bergman, 1978, §9.5a) ensures that the normal forms of (12) are canonical representatives of the congruence classes \( \varphi + Z \in DF \). Hence it suffices to characterize the normal forms of an arbitrary (noncanonical) representative \( \varphi \). Clearly, every such \( \varphi \) is reducible as long as it contains any \( {\ddot{\delta}}^{(k)}_a \) with a coefficient in \( \mathcal{F} \setminus K \); hence we can achieve (13), which is clearly irreducible with respect to (12).

This completes the proof of the Presentation Lemma. \( \square \)

We identify the Heavisides \( H_a \in DF \) with the corresponding \( H_a \in PF \). As a consequence, we have \( PF \subset DF \) as plain \( \mathcal{F} \)-modules\(^{11} \) but not as differential \( \mathcal{F} \)-modules: Indeed, the derivation \( \partial : PF \rightarrow PF \) just annihilates the Heavisides, \( \partial H_a = 0 \), whereas \( \partial : DF \rightarrow DF \) sends them to \( \partial H_a = \delta_a \). The situation for the Rota-Baxter structure is very different—in fact, we shall see that \( PF \subset DF \) as Rota-Baxter \( \mathcal{F} \)-modules. To this end we define \( \bar{f} : DF \rightarrow DF \) as an extension of \( \bar{f} : PF \rightarrow PF \) via the recursion

\[
\bar{f} \delta_a^{(k)} = \begin{cases} E_a(f) \bar{f} \delta_a^{(k-1)} - \int f' \delta_a^{(k-1)} & \text{for } k > 0, \\ \int f \delta_a^{(0)} & \text{for } k = 0, \end{cases}
\] (14)

where \( E_a \) denotes the evaluation in \( \mathcal{F} \) and the integral in the base case is given in terms of the (rising or falling) Heaviside function via

\[
\int \delta_a = H_a - \bar{H}(a) = H(a) - \bar{H}_a,
\] (15)

which may also be written symmetrically as \( \int \delta_a = H(a) H_a - \bar{H}(a) \bar{H}_a \). Setting \( f = 1 \) in (14), we obtain the higher Dirac antiderivatives \( \int \delta_a^{(k)} = \delta_a^{(k-1)} \) for \( k > 0 \). Hence the induced evaluation \( \bar{\xi} = 1_{DF} - \int \bar{\delta} \) of the module generators is given by

\[
\bar{\xi}(H_a) = \bar{H}(a) \quad \text{and} \quad \bar{\xi}(\delta_a^{(k)}) = 0 \quad (k \geq 0),
\] (16)

which—unlike in the piecewise extension—do go to the ground field \( K \). This should be contrasted to the pseudo-evaluation \( \xi(H_a) = H_a \) we introduced earlier (after Proposition 6). We shall come back to \( \bar{\xi} : DF \rightarrow DF \) in due course (see Lemmas 14 and 15).

\(^{11}\)The total order presupposed in the definition of \( PF \) is irrelevant for the identification of modules: It is only needed for the ring structure of \( PF \), which is momentarily ignored but incorporated later (Remark 13).
Remark 11. The definition of the Rota-Baxter operator \( \mathcal{f} : \mathcal{D}F \to \mathcal{D}F \) in (6) and (14)–(15) may be rephrased more economically by joining (6) with the single formula

\[
\mathcal{f} \mathcal{H}_u^{(k+1)} = f \mathcal{H}_u^{(k)} - \mathcal{f} \mathcal{H}_u^{(k)} \quad (k \in \mathbb{N}).
\]  

(17)

While this is evident for \( k > 0 \), it requires a small calculation to confirm in the case \( k = 0 \). The main point is to use (6) in conjunction with the relation \( \varepsilon_a = \varepsilon_{e_r} - \varepsilon_0 + \varepsilon_{e_r} \) already used in the proof of Proposition 6 and the simple fact that \( f(a^*) = f(a) H(a) + f(0) \bar{H}(a) \). We have chosen the split definition (14)–(15) above since we find it more intuitive.

Our main result states that the distribution module \( \mathcal{D}F \) is an extension of the ground algebra \( \mathcal{F} \) that contains the piecewise extension \( \mathcal{P}F \) qua Rota-Baxter module; see the figure nearby, where \( \iota \) is the embedding of Rota-Baxter \( \mathcal{F} \)-modules while \( u_\mathcal{D} \) and \( u_\mathcal{D}^\prime \) are the structure maps of the \( \mathcal{F} \)-modules \( \mathcal{P}F \) and \( \mathcal{D}F \), respectively.

Theorem 12. Let \( (\mathcal{F}, \partial, \int) \) be an ordinary shifted integro-differential algebra. Then the distribution module \( (\mathcal{D}F, \partial, \int) \) is a differential Rota-Baxter module over \( \mathcal{F} \) that extends \( (\mathcal{P}F, \partial, \int) \) as a Rota-Baxter module.

Proof. It suffices to prove the following statements:

1. The map \( \mathcal{f} : \mathcal{D}F \to \mathcal{D}F \) is well-defined. For this we have to show that \( \mathcal{f} Z \subseteq Z \), which we do by the aid (and with the notation) of Lemma 10. So for \( a \in \mathcal{K} \) fixed, we prove \( \mathcal{f} \xi_{f,k} \in Z \) for all \( f \in \mathcal{F} \) and \( k \geq 0 \). Using induction on \( k \), the base case \( k = 0 \) follows immediately from (14). For the induction step it suffices to prove that \( \mathcal{f} \xi_{f,k+1} = \xi_{f,k} - \mathcal{f} \xi_{f',k} \) for all \( f \in \mathcal{F} \). Using the generators (12) we have

\[
\mathcal{f} \xi_{f,k+1} = \mathcal{f} \delta_a^{(k+1)} - \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i \varepsilon_a (f^{(i)}) \delta_a^{(k-i)},
\]

which simplifies by (14) and the binomial recursion \( \left( \binom{k+1}{i} = \left( \binom{k}{i} + \binom{k}{i-1} \right) \right) \) to

\[
\mathcal{f} \delta_a^{(k)} = - \sum_{i=0}^{k} \binom{k}{i} (-1)^i \varepsilon_a (f^{(i)}) \delta_a^{(k-i)} - \left( \mathcal{f} f'\delta_a^{(k)} - \sum_{i=0}^{k} \binom{k}{i} (-1)^i \varepsilon_a (f'^{(i)}) \delta_a^{(k-i)} \right) = \xi_{f,k} - \mathcal{f} \xi_{f',k}
\]

and thus completes the induction.

2. The map \( \mathcal{f} : \mathcal{D}F \to \mathcal{D}F \) is a Rota-Baxter operator. Hence we must prove, for any \( f, g \in \mathcal{F} \) and \( a \in \mathcal{K} \) and \( k \geq 0 \), the Rota-Baxter axiom

\[
\mathcal{f} f \cdot \mathcal{f} g \delta_a^{(k)} = \mathcal{f} \mathcal{f} \mathcal{g} \delta_a^{(k)} + \mathcal{f} (\mathcal{f} g) \delta_a^{(k)}.
\]

(18)

We fix \( a \in \mathcal{K} \) and use induction on \( k \) to prove (18) for all \( f, g \in \mathcal{F} \). In the base case, exploring definition (14) reveals that \( \varepsilon_a (g) \) factors on both sides of (18); hence it suffices to take \( g = 1 \). The left-hand side is then \( \mathcal{f} f \cdot \mathcal{f} \delta_a \) while we obtain

\[
(H(a) \mathcal{f} H_a - \bar{H}(a) \mathcal{f} \bar{H}_a) + \mathcal{f}^a f \cdot \mathcal{f} \delta_a
\]
for the right-hand side. Using the definition (6), (7) of the Rota-Baxter operator on the piecewise extension $\mathcal{P}F \subset \mathcal{D}F$ and properties of the Heaviside operator, the first parenthesized term becomes $\int f \cdot \delta_a$ and then combines with the remaining term to $\int f \cdot \delta_a$; this completes the base case of the induction. Assume now that (18) holds for $k$; we show that it holds for $k + 1$. Using the definition (14) once, the left-hand side is $\int f \cdot (g\delta_a^{(k)} - g'\delta_a^{(k)})$. On the right-hand side we use (14) on each summand to get

$$\int fg\delta_a^{(k)} - \int f g'\delta_a^{(k)} + (\int f) g\delta_a^{(k)} - \int f g\delta_a^{(k)} - (\int f) g'\delta_a^{(k)}$$

$$= (\int f) g\delta_a^{(k)} - \int f g'\delta_a^{(k)} - \int (f) g'\delta_a^{(k)}.$$

Canceling the first terms on both sides, we end up with (18) where $g$ is replaced by $g'$, and this holds by the induction hypothesis.

3. The map $\partial: \mathcal{D}F \to \mathcal{D}F$ is a well-defined derivation. In fact, it suffices to prove well-definedness since the derivation property then follows immediately from the definition of $\mathcal{D}F$ as a quotient of a differential module. Hence we must prove $\partial Z \in Z$, but this follows directly from $\partial Z = \partial Z$, obtained by differentiating the identity of Item (1).

4. The Rota-Baxter operator $\mathcal{R}$ is a section of the derivation $\partial$. We start by showing that $\partial \int f H_a = f H_a$ holds for all $f \in F$. Using definition (6) for the Rota-Baxter operator on $\mathcal{P}F$ and the Leibniz rule together with the basic relation $f\delta_a = \epsilon_\delta(f)\delta_a$ of $Z$ yields

$$\partial \int f H_a = f H_a + (\int f \cdot f)\delta_a + \int f_0(f)\delta_a \quad (19)$$

whose last two terms combine to $0 + 0$ in the case $a \geq 0$ and again to $\int f f_0 = 0$ in the case $a \leq 0$. Hence the right-hand side of (19) is indeed $f H_a$. Now for elements of the form $f\delta_a^{(k)}$ we use induction on $k$. In the base case we have

$$\partial \int f\delta_a = \epsilon_\delta(f)\partial(H_a - H(a)) = \epsilon_\delta(f)\delta_a = f\delta_a,$$

where the last step uses again the basic relation of $Z$. Now assume $\partial \int f\delta_a^{(k)} = f\delta_a^{(k)}$ for a fixed $k$. Then we have

$$\partial \int f\delta_a^{(k+1)} = \partial(f\delta_a^{(k)}) = \partial \int f'\delta_a^{(k)} = f\delta_a^{(k+1)},$$

where the last step uses the Leibniz rule for $\partial$ and the induction hypothesis. This completes the proof of the section axiom for $\mathcal{R}$.

Before analyzing some further properties of $\mathcal{D}F$, let us digress briefly for addressing an important “design question” that has come up repeatedly in the course of building up the algebraic structure of Heaviside functions and Dirac distributions.

**Remark 13.** It sounds tempting to introduce distributions as a differential ring extension of $\mathcal{P}F$. However, the famous negative result Schwartz (1954) serves as a warning signal that we should not be overly optimistic in that respect. In the algebraic setup, we see that things are in a sense worse—we cannot even expect a Leibniz rule that involves Heavisides: Since $H_a^a = H_a$ in $\mathcal{P}F$, differentiation would yield $2H_a\delta_a = \delta_a$ as a new relation. Hence we would need $\mathcal{D}F$ to be a module over $\mathcal{P}F$, though it would not be a differential module since the derivation does not restrict to a map $\partial: \mathcal{P}F \to \mathcal{P}F$. Furthermore, we would now expand the relations $Z$ of Definition 9.
to include \( F \delta_a = \varepsilon_a(F) \delta_a \) for all \( F \in \mathcal{PF} \) and not just for \( F \in \mathcal{F} \); in conjunction with the new relation this forces on us the symmetric convention for the Heaviside operator. We have discarded the latter (see before Proposition 4) solely for ensuring multiplicative evaluations on \( \mathcal{PF} \), so let us momentarily assume the symmetric convention. At any rate, the new relation \( 2H_a \delta_a = \delta_a \) implies that
\[
\delta_a = 2H_a \delta_a = 2H_a (2H_a \delta_a) = (4H_a^2) \delta_a = 4H_a \delta_a = 2\delta_a,
\]
which means \( \delta_a = 0 \) and hence \( \mathcal{DF} = \mathcal{PF} \).

It is now clear why our construction of \( \mathcal{DF} \) was based on a free differential module over \( \mathcal{F} \) rather than some module over \( \mathcal{PF} \). On the other hand, it is clear that \( \mathcal{PF} \subset \mathcal{DF} \), and we may export the product structure of the piecewise extension \( \mathcal{PT} \) to the distribution module \( \mathcal{DF} \). Hence we may say \( H_a^2 = H_a \in \mathcal{DF} \) but we are barred from differentiating this relation since \( \mathcal{DF} \) is a differential module over \( \mathcal{F} \) and not over \( \mathcal{PF} \). \( \square \)

In Theorem 12 we use the rather strong assumption that the ground algebra \( (\mathcal{F}, \partial, \gamma) \) is an ordinary integro-differential algebra since this is what we need in our applications. This has the nice consequence that the distribution module itself has similar properties. However, for a general differential Rota-Baxter module one must distinguish between the strong Rota-Baxter axiom (2) for coefficients and for module elements (whether one may pull out constants of either kind from the integral). In the sequel, we shall write \( \hat{e} := 1_M - \oint \delta \) for the induced (pseudo)evaluation in an arbitrary differential Rota-Baxter module \( (M, \partial, \oint) \).

**Lemma 14.** Let \( (M, \partial, \oint) \) be a differential Rota-Baxter module over the integro-differential algebra \( (\mathcal{F}, \partial, \gamma) \). Then we have the following equivalences (where \( f, c \in \mathcal{F} \) and \( \varphi, \gamma \in M \)):

1. \( \oint c \varphi = c (\oint \varphi) \) (for all \( c \in \text{Ker} \partial \)) \( \iff \oint f \varphi = f \oint \varphi - \oint f' \oint \varphi \)
2. \( \oint f' \gamma = (\oint f) \gamma \) (for all \( \gamma \in \text{Ker} \partial \)) \( \iff \oint f \varphi = (\oint f) \varphi - f (\oint \varphi) \gamma \)
3. \( \oint (f \varphi) = \oint f \oint (\varphi) \) \( \iff (1a) \&(2a) \iff (1b) \&(2b) \)

If \( M \) is ordinary, property (1a) and hence (1b) is automatic; if \( \mathcal{F} \) is ordinary, the same holds for properties (2a) and (2b).

**Proof.** The implications are similar to the corresponding ones given in (Guo, Regensburger, Rosenkranz, 2012, Thm. 2.5) for noncommutative rings, provided one splits the properties of the ring into its left-hand and right-hand versions.

Let us start with (1). The implication from right to left is obvious, so assume the homogeneity condition (1a) for \( c \in \text{Ker} \partial \). Then we have
\[
f \oint \varphi = (f - \oint f') \oint \varphi + (\oint f')(\oint \varphi) = \oint f \varphi - \oint (\oint f') \varphi + (\oint f')(\oint \varphi),
\]
where we have used the homogeneity condition for \( c = f - \oint f' \in \text{Ker} \partial \). By the (plain) Rota-Baxter axiom the last term above is \( (\oint f')(\oint \varphi) = (\oint f')(\varphi) + f' \oint \varphi \), hence one immediately obtains (1b). The proof of the equivalence (2a) \( \iff \) (2b) is completely analogous. Turning to (3), let us first assume the multiplicativity condition \( \oint (f \varphi) = \oint f \oint (\varphi) \). Specializing to \( f = c \in \text{Ker} \partial \) yields \( \oint c \varphi = c \oint \varphi \), which is (1a) since \( \partial \) is surjective; likewise specializing to \( \varphi = \gamma \in \text{Ker} \partial \) gives \( \oint f' \gamma = (\oint f') \gamma \), which is (2a) since \( \partial \) is surjective as well. For the converse statement, we
may assume (1b) and (2b) to prove the multiplicativity condition for the evaluations. From the plain Rota-Baxter axiom we have

\[(\int f')\, (f \varphi') = \int (f' \varphi') + \int f' \int f \varphi' = \left(\int f'\right) \varphi - \int f \varphi' + \left(\int f \varphi - \int f' \varphi\right)\]

where the first and the second parenthesized terms come from applying (2b) and (1b), respectively. Subtracting \(f \varphi\) from both sides of the above identity and rearranging, one obtains exactly \(\mathcal{E}(f \varphi) = \mathcal{E}(f) \mathcal{E}(\varphi)\). □

If \((M, \partial, \{\})\) satisfies the multiplicativity requirement of (3) above, we shall call it an integro-differential module (similar terms could be introduced for the weaker properties (1) and (2) but will not be needed for our purposes). It is now easy to see that the distribution module \(\mathcal{D}\mathcal{F}\) of Theorem 12 is indeed an ordinary integro-differential module in this sense.

**Proposition 15.** If \((\mathcal{F}, \partial, \{\})\) is an ordinary shifted integro-differential algebra, \((\mathcal{D}\mathcal{F}, \partial, \{\})\) is an ordinary integro-differential module over \(\mathcal{F}\).

**Proof.** Let us first prove that \(\mathcal{D}\mathcal{F}\) is ordinary, meaning \(\text{Ker} \partial = K\). Hence assume \(\partial \varphi = 0\) for an arbitrary element \(\varphi \in \mathcal{D}\mathcal{F}\). By Lemma 10 we may assume

\[\varphi = f + \sum_{a \in K} f_a H_a + \sum_{a \in K} \lambda_{a,k} \delta_a^{(k)}\]

for some \(f, f_a \in \mathcal{F}\) and \(\lambda_{a,k} \in K\) so that

\[f' + \sum_{a \in K} (f'_a H_a + f_a \delta_a) + \sum_{a \in K} \lambda_{a,k} \delta_a^{(k+1)} = 0.\]

Since the above representation is canonical by Lemma 10, we obtain \(f' = f'_a = f_a = \lambda_{a,k} = 0\). But then we have \(\varphi = f \in \text{Ker} \partial = K\), so the differential module \((\mathcal{D}\mathcal{F}, \partial, \{\})\) is ordinary. From Lemma 14 it follows immediately that \((\mathcal{D}\mathcal{F}, 0, \{\})\) is also an integro-differential module. □

The distribution module \((\mathcal{D}\mathcal{F}, \partial, \{\})\) over the ordinary shifted integro-differential algebra \((\mathcal{F}, \partial, \{\})\) can also be characterized in terms of a universal mapping property. First we encapsulate the minimal requirements for adjoining a family of distributions \(\delta_a (a \in K)\) to the given integro-differential algebra \((\mathcal{F}, \partial, \{\})\). Algebraically, they are characterized by the sifting property (11), the integro-differential relation

\[\delta^{(k)}_{\partial} \frac{\partial}{f} \delta^{(k+1)}\]

for \(k \geq 0\), and the stipulation that \(\delta_a\) has the Heaviside function \(H_a\) as its antiderivative with integration constant \(-H(a)\). From the latter stipulation, it is clear that the resulting structure must contain the Rota-Baxter submodule \(\mathcal{P}\mathcal{F}\). Finally, we hold fast to the analysis tradition of barring multiplication of distributions (see Remark 13 for the algebraic view of this proscription). The universal property stated below can then be construed as exhibiting the distribution module \((\mathcal{D}\mathcal{F}, \partial, \{\})\) as the most economic solution to the task of adjoining Dirac distributions subject to these minimal requirements.
Definition 16. Let $(\mathcal{F}, \partial, \int)$ be an integro-differential algebra. An integro-differential module $(M, \delta_M, \int_M)$ over $\mathcal{F}$ is called a Dirac module if $\mathcal{P}\mathcal{F} \hookrightarrow M$ as Rota-Baxter modules such that (11) holds and $\delta_a := \delta_M H_a$ satisfies $\int_M \delta_a = H_a - \mathcal{H}(a)$ as well as $\int_M \delta_a^{(k+1)} = \delta_a^{(k)}$, for all $a \in \mathcal{K}$ and $k \geq 0$.

Proposition 17. The differential Rota-Baxter module $(\mathcal{D}\mathcal{F}, \partial, \int)$ is the universal Dirac module over $(\mathcal{F}, \partial, \int)$ that extends $(\mathcal{P}\mathcal{F}, \int)$ as a Rota-Baxter module. In other words, for every Dirac module $M$ there is a unique integro-differential morphism $\Phi: \mathcal{D}\mathcal{F} \to M$ that respects the canonical embedding of $\mathcal{P}\mathcal{F}$.

Proof. Let $\kappa: \mathcal{P}\mathcal{F} \hookrightarrow M$ be the embedding of Rota-Baxter modules from Definition 16, and let $u_F$, $u_{\mathcal{D}F}$, $i$ be as in the diagram before Theorem 12.

Furthermore, we will write $u_M$ for the structure map of the $\mathcal{F}$-module $M$. We construct a morphism of integro-differential modules $\Phi: \mathcal{D}\mathcal{F} \to M$ that makes the right-hand diagram commute. It suffices to show $\Phi_i = \kappa$ since then $\Phi u_M = u_M$ follows from the module structures $u_{\mathcal{F}} = u_{\mathcal{D}F}$ and $u_M = u_M$.

If the required map $\Phi$ exists, it must be $\mathcal{F}$-linear and send $(\kappa H_a)'$ to $(i \kappa H_a)'$. But this defines $\Phi$ uniquely because $\mathcal{D}\mathcal{F}$ is generated by $(i \kappa H_a)'$ as an $\mathcal{F}$-module. Defining first $\Phi_i: \mathcal{F}[H_a \mid K]_1 \to M$ by these requirements, it follows at once that $\Phi$ is in fact a morphism of differential $\mathcal{F}$-modules. For seeing that it lifts to a map $\Phi: \mathcal{D}\mathcal{F} \to M$, we must show $\Phi(\mathcal{Z}) = 0$. Since $\Phi$ respects the derivation, it suffices to prove that $\Phi$ annihilates the differential generators $f \delta_a - \varepsilon_n(f) \delta_a$ or, more precisely, the corresponding elements $u_{\mathcal{F}}(f)i(H_a)' - \varepsilon_n(f)i(H_a)'$. But this follows immediately from the sifting property (11) of the Dirac module $M$.

We have now a differential morphism $\Phi: \mathcal{D}\mathcal{F} \to M$ that clearly satisfies the required commutation property $\Phi_i = \kappa$. Moreover, it is clear from the construction that $\Phi$ is unique. Hence it only remains to prove that $\Phi$ is also a morphism of Rota-Baxter algebras over $\mathcal{F}$. To this end, we show first that

$$\int_M \Phi(f \kappa H_a) = \Phi(\int f \kappa H_a).$$

Note that the left-hand side may be written as $\int_M \kappa(f \kappa H_a)$ since $\Phi_i = \kappa$. Since by hypothesis we have $\mathcal{P}\mathcal{F} \to M$ as Rota-Baxter $\mathcal{F}$-modules, we may now apply $\int_M \kappa = \kappa \int$ and then expand the integral $\int$ of $\mathcal{P}\mathcal{F}$ to obtain

$$\kappa(\int f \kappa H_a - \kappa(\int f \kappa H_a)) = \Phi(\int f \kappa H_a - \kappa(\int f \kappa H_a))$$

for the left-hand side of (20), using again $\Phi_i = \kappa$ for the last step. Recalling that $\int$ on $\mathcal{D}\mathcal{F}$ was defined as an extension of $\int$ on $\mathcal{D}\mathcal{F}$, this yields the right-hand side of (20). It remains to prove

$$\int_M \Phi(f \delta_a^{(k)}) = \Phi(\int f \delta_a^{(k)})$$

for all $k \geq 0$. By the sifting property (11), valid in $\mathcal{D}\mathcal{F}$ as well as $M$, we may replace $f$ by $\varepsilon_n(f)$ on both sides of (21). Hence we may set $f = 1$ for the proof of (21). For $k = 0$, we use the antiderivative relation of the Dirac module $M$ in the precise form $\int(\kappa H_a)' = \kappa H_a - \mathcal{H}(a)$ to obtain

$$\int_M \Phi \delta_a = \int(\kappa H_a)' = \kappa(\kappa H_a - \mathcal{H}(a)) = \Phi(\kappa H_a - \mathcal{H}(a)) = \Phi(\delta_a).$$

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as required. For \(k > 0\), Equation (21) follows immediately from \(\int_M(k\mathcal{H}_a)^{(k)} = (k\mathcal{H}_a)^{(k-1)}\), which holds since \(\mathcal{M}\) is a Dirac module. \(\square\)

As in the piecewise extension \((\mathcal{P}\mathcal{F}, \{\})\), we can also provide shifted evaluations on the distribution module \((\mathcal{D}\mathcal{F}, \{\})\) if we have a shift map \(\hat{S}: K \to \text{Aut}_K(\mathcal{F})\) on the ground algebra \((\mathcal{F}, \{\})\). Then we define \(\hat{S}: K \to \text{Aut}_K(\mathcal{D}\mathcal{F})\) by extending \(\hat{S}_a H_a = H_{a-c}\) and \(\hat{S}_a \delta_a = \delta_{a-c}\) \((a, c \in K)\) through linearity and multiplicativity. It is immediate that \(\hat{S}\) is a shift map on the distribution module \(\mathcal{D}\mathcal{F}\). The latter is an ordinary integro-differential module if \(\mathcal{F}\) is an ordinary integro-differential algebra (Proposition 15), hence we get shifted evaluations on \(\mathcal{D}\) by setting \(\hat{\varepsilon}_c := \varepsilon_c \hat{S}_c\). Clearly, this yields \(\hat{\varepsilon}_c H_a = H(a - c)\) and \(\hat{\varepsilon}_c \delta_a = 0\) on the generators as per (16). As usual we write \(\int_{b} \mathcal{F}\) for the resulting shifted Rota-Baxter operators.

**Theorem 18.** If \((\mathcal{F}, \partial, \{\})\) is a differential algebra, \((\mathcal{D}\mathcal{F}, \partial, \{\})\) is an ordinary shifted integro-differential module over \(\mathcal{F}\). Its shifted Rota-Baxter operators are given by the recursion (14), with \(\int\) replaced by \(\int_{b}\) and by the base case (15), with \(H(a)\) replaced by \(H(a - b)\) or \(H(a)\) replaced by \(H(a - b)\).

**Proof.** The recursive description of the shifted Rota-Baxter operators follows immediately from the definition \(\int_{b} := (1 - \varepsilon_b)\int\). In view of Proposition 15, it then remains to prove the compatibility relations \([\hat{S}_a, \{\}] = \varepsilon_c \{\} + [\hat{S}_c, \partial] = 0\). Let us start with the former.

Since \(\hat{S}_a\) and \(\{\}\) as well as \(\varepsilon_c\) agree on \(\mathcal{P}\mathcal{F} \subset \mathcal{D}\mathcal{F}\) by definition, it suffices to consider elements of the form \(f \delta_a^{(k)}\) \((k \geq 0)\). We apply induction on \(k\). For the base case \(k = 0\), we obtain \(\varepsilon_c(f) (H(a-c) - H(a))\) for both left-hand and right-hand side of the relation \([\hat{S}_a, \{\}] = \varepsilon_c \{\}\) applied to \(f \delta_a\). Now assume the relation for all \(f \delta_a^{(k)}\) with fixed \(k \geq 0\); we must show it for \(f \delta_a^{(k+1)}\). A straightforward computation, using the induction hypothesis on \(f \delta_a^{(k)}\), yields \(\varepsilon_c(f) f \delta_a^{(k+1)}\) for both sides of \([\hat{S}_a, \{\}] = \varepsilon_c \{\}\) as applied to \(f \delta_a^{(k+1)}\).

Let us now turn to the commutation identity \(\hat{S}_a \partial = \partial \hat{S}_a\). Since \(\mathcal{F}\) is a shifted integro-differential algebra by hypothesis, we need only consider elements of the form \(f H_a^{(k)}\) \((k \geq 0)\). For those one obtains indeed \(\hat{S}_a \partial f H_a^{(k)} = \partial \hat{S}_a f H_a^{(k)} = S_c(f) \delta_a^{(k)} + S_c(f) H^{(k+1)}\), making use of the commutation identity on \(\mathcal{F}\). \(\square\)

It is gratifying that all the required properties of the ground algebra \(\mathcal{F}\) are inherited by the module \(\mathcal{D}\mathcal{F}\): the integro-differential structure, orderliness, and the shift structure. We end this section by endowing the distribution module \(\mathcal{D}\mathcal{F}\) with an ascending filtration. Indeed, let us start by writing \(\mathcal{D}\mathcal{F}_a\) for the differential \(\mathcal{F}\)-submodule generated by \(H_a\). By (13), its elements have the canonical form \(f + f_a H_a + \sum \lambda_{a,b} \delta_a^{(k)}\) with \(f, f_a \in \mathcal{F}\) and \(\lambda_{a,b} \in K\). A glance at (14) confirms at once that such elements are also closed under the Rota-Baxter operator, so we have a differential Rota-Baxter submodule \((\mathcal{D}\mathcal{F}_a, \partial, \{\})\) and indeed a direct sum \(\mathcal{D}\mathcal{F} = \bigoplus_a \mathcal{D}\mathcal{F}_a\) of differential Rota-Baxter submodules. The \(\mathcal{D}\mathcal{F}_a\) are of course not shifted submodules, but the shift map restricts to isomorphisms \(S_c: \mathcal{D}\mathcal{F}_a \to \mathcal{D}\mathcal{F}_b\), for any \(c \in K\) and \(b := a - c\). Next we define \(\mathcal{D}\mathcal{F}_a^{(k)} \subseteq \mathcal{D}\mathcal{F}_a\) as the \(\mathcal{F}\)-submodule generated by all \(H_a^{(j)}\) with \(j \leq k\); note also that the piecewise extension is given by \(\mathcal{P}\mathcal{F} = \bigoplus_a \mathcal{D}\mathcal{F}_a^{(0)}\).
It is obvious that \( \partial \) maps \( DF_a^{k} \) to \( DF_a^{k+1} \), and one see from (14) that \( \int \) restricts to a map from \( DF_a^{k+1} \) to \( DF_a^{k} \). It is also clear that \( (DF_a^{k})_{k \geq 0} \) forms an ascending \( \mathcal{F} \)-module filtration of \( DF_a \). We conclude that each \( DF_a \) as well as the entire distribution module \( DF \) is a filtered differential Rota-Baxter module (see the figure above). Moreover, the restricted shift maps \( S_c : DF_a \rightarrow DF_b \) restrict further to \( S_c : DF_a^{k} \rightarrow DF_b^{k} \).

For some purposes one needs only a few Heavisides (and Diracs), rather than the whole gamut \( H_a (a \in K) \); in the extreme case one gets the *slim* distribution module \( \hat{DF} \), which is differentially generated by a single Heaviside that we shall denote by \( \hat{H} \), its derivative being written \( \hat{\partial} := \hat{H}' \). The whole construction given in this section may obviously be repeated verbatim to obtain the differential Rota-Baxter module \( \hat{DF} \). Alternatively, one may achieve the same result by slimming the hierarchy of the above figure, namely by setting \( \hat{DF} = DF / N_\mathcal{D} \) with \( \hat{H} := H_0 + N_\mathcal{D} \), where \( N_\mathcal{D} \subset DF \) is the differential Rota-Baxter submodule generated by the set \( \mathcal{N} := \{ H_a \mid a \in K^\ast \} \). Similarly, one gets the *slim piecewise extension* \( \hat{PF} = PF / N_\mathcal{P} \) where \( N_\mathcal{P} \subset PF \) is the ideal generated by \( \mathcal{N} \). Obviously, we may view \( \hat{DF} \) as a module over \( \hat{PF} \).

We shall need the slim distribution module \( \hat{DF} \) and the slim piecewise extension \( \hat{PF} \) in the next section for obtaining the bivariate “diagonal” distribution \( \delta (x - \xi) \). In fact, we shall only need the \( K \)-subspace generated by \( \hat{H} \) and its derivatives; let us denote this space by \( \hat{DK} \subset \hat{DF} \). Likewise, we shall write \( \hat{PK} \subset \hat{PF} \) for the \( K \)-subalgebra generated by \( \hat{H} \) alone.

\section{5. Bivariate Distributions}

Since one of our main applications in Section 6 will be to provide an algebraic model of the bivariate Green’s function corresponding to a given boundary problem, it is now necessary to expand the distribution module \( DF \). While the latter contains only univariate Heavisides \( H(x-a) \) and their derivatives (with \( a \in K \) fixed), we shall also need their counterparts \( H(\xi - a) \) in another variable \( \xi \), and moreover the “diagonal” Heaviside function \( H(x - \xi) \) with its derivatives.\textsuperscript{12} Let us first concentrate on the former.

We start with the tensor product \( \mathcal{F}_2 := \mathcal{F} \otimes_k \mathcal{F} \), writing its elements \( f_1 \otimes f_2 \) as \( f_1(x) f_2(\xi) \). Note that \( \mathcal{F}_2 \) is an \( \mathcal{F} \)-bimodule with two derivations and two Rota-Baxter operators

\[
\partial_1 (f_1 \otimes f_2) = (\partial f_1) \otimes f_2, \quad \partial_2 (f_1 \otimes f_2) = f_1 \otimes (\partial f_2),
\]

\[
\int_1 (f_1 \otimes f_2) = (f_1) \otimes f_2, \quad \int_2 (f_1 \otimes f_2) = f_1 \otimes (f_2).
\]

We have two embeddings \( \iota_1, \iota_2 : \mathcal{F} \rightarrow \mathcal{F}_2 \) with \( \iota_1(f) = f \otimes 1 \) and \( \iota_2(f) = 1 \otimes f \); we denote their images by \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. For a ground element \( f \in \mathcal{F} \), their embeddings are also written as \( f(x) := \iota_1(f) \in \mathcal{F}_1 \) and \( f(\xi) := \iota_2(f) \in \mathcal{F}_2 \).

Note that both \((\mathcal{F}_2, \partial_1, \int_1)\) and \((\mathcal{F}_2, \partial_2, \int_2)\) are integro-differential algebras over \( K \), though not ordinary ones since \( \operatorname{Ker} \partial_1 = \mathcal{F}_1 \) and \( \operatorname{Ker} \partial_2 = \mathcal{F}_1 \). In addition to the duplex differential Rota-Baxter structure, \( \mathcal{F}_2 \) has two shift operators \( S_1(f_1 \otimes f_2) := (S_1 f_1) \otimes f_2 \) and \( S_2(f_1 \otimes f_2) := f_1 \otimes (S_2 f_2) \). If \( \tau : \mathcal{F}_2 \rightarrow \mathcal{F}_2 \) is the usual exchange automorphism \( \tau(f_1 \otimes f_2) = f_2 \otimes f_1 \), the derivations, Rota-Baxter and shift operators are conjugate under \( \tau \), meaning \( \partial_2 = \tau \partial_1 \tau, \quad \int_1 = \tau \int_2 \tau, \quad S_1 = \tau S_2 \tau \).

\textsuperscript{12}Our present treatment of bivariate distributions is very limited. A more comprehensive algebraic theory will allow more general distributions, containing at least \( \delta(x_1 x_2 + \cdots + a_n x_n) \). While such a development should properly be given in an LPDE context, we are here only interested in LODE boundary problems where the three distribution families \( \delta_a(x), \delta_a(\xi), \delta(x - \xi) \) turn out to be sufficient (Section 6).
Definition 19. The pure distribution modules are introduced by \( D_x \mathcal{F} := D(\mathcal{F}_2, \partial_x, \mathcal{F}) \) and \( D_\xi \mathcal{F} := D(\mathcal{F}_2, \partial_\xi, \mathcal{F}) \). We write \( H(x - a) \in D_x \mathcal{F} \) and \( H(\xi - a) \in D_\xi \mathcal{F} \) for the corresponding differential generators \( a \in K \).

In this context, we will revive our abbreviations \( H(a - x) \in D_x \mathcal{F} \) and \( H(a - \xi) \in D_\xi \mathcal{F} \). By virtue of \( \delta = H' \), we have likewise \( \delta(x - a) \in D_x \mathcal{F} \) and \( \delta(\xi - a) \in D_\xi \mathcal{F} \). Finally, we define the action of \( \delta \), \( \mathcal{F} \), and \( \mathcal{F} \xi \) on \( D_x \mathcal{F} \) by regarding the \( H(\xi - a) \) as constants, meaning we set \( \delta, fH^{(\delta)} := (\delta, f)H^{(\delta)} \), \( \mathcal{F} \xi fH^{(\delta)} := (\mathcal{F} \xi f)H^{(\delta)} \) and \( \mathcal{F} \xi fH^{(\delta)} := (\mathcal{F} \xi f)H^{(\delta)} \). The action of \( \delta, \mathcal{F} \), and \( \mathcal{F} \xi \) on \( D_\xi \mathcal{F} \) is defined analogously. Altogether we obtain the two duplex shifted differential Rota-Baxter modules \((D_x \mathcal{F}, \partial_x, \partial_x, \mathcal{F}, \mathcal{F}^\xi, \mathcal{F} \xi)\) and \((D_\xi \mathcal{F}, \partial_\xi, \partial_\xi, \mathcal{F}, \mathcal{F} \xi, \mathcal{F}^\xi)\). Their induced evaluations are written as \( e_x := 1 - f^\xi \partial_x \) and \( e_\xi := 1 - f^\xi \partial_\xi \), along with the shifted versions \( \mathcal{F}e_x := e_x \mathcal{F} \) and \( \mathcal{F}e_\xi := e_\xi \mathcal{F} \).

Note that both pure distribution modules contain the corresponding piecewise extension algebras \( \mathcal{P} \mathcal{F} \subset D_x \mathcal{F} \) and \( \mathcal{P} \mathcal{F} \subset D_\xi \mathcal{F} \). These rings can be combined into the bivariante piecewise extension \( \mathcal{P} \mathcal{F} := \mathcal{P}_x \mathcal{F} \otimes_{\mathcal{F} \mathcal{F}} \mathcal{P}_\xi \mathcal{F} \), which is useful for representing the characteristic functions of a rectangle \((x, \xi) \in [a, b] \times [c, d]\) by the tensor product \([a \leq x \leq b] \otimes [c \leq \xi \leq d]\) with Heaviside factors \([a \leq x \leq b] := H(x - a)H(b - x) \in \mathcal{P} \mathcal{F} \) and \([c \leq \xi \leq d] := H(\xi - c)H(d - \xi) \in \mathcal{P} \mathcal{F} \); this is needed in Section 6. By analogy to the situation in \( \mathcal{F} \), we shall drop the \( \otimes \) symbol, thus writing \( H(x - a)H(\xi - b) \) for what is strictly speaking \( H_a \otimes H_b \in \mathcal{P} \mathcal{F} \). Note that \( \mathcal{P} \mathcal{F} \) is a duplex shifted differential Rota-Baxter algebra over \( \mathcal{F} \), analogous to the univariate case.

We will now combine the univariate distribution modules \( D_x \mathcal{F} \) and \( D_\xi \mathcal{F} \) along with the bivariate piecewise extension \( \mathcal{P} \mathcal{F} \) into a single module. To this end, note that both \( D_x \mathcal{F} \otimes_{\mathcal{F} \mathcal{F}} \mathcal{P}_x \mathcal{F} \) and \( \mathcal{P}_\xi \mathcal{F} \otimes_{\mathcal{F} \mathcal{F}} D_\xi \mathcal{F} \) contain isomorphic copies of the \( \mathcal{F} \)-submodule \( \mathcal{P} \mathcal{F} \) with which they are identified. With this identification, we form the direct sum of \( D_x \mathcal{F} \otimes_{\mathcal{F} \mathcal{F}} \mathcal{P}_x \mathcal{F} \) and \( \mathcal{P}_\xi \mathcal{F} \otimes_{\mathcal{F} \mathcal{F}} D_\xi \mathcal{F} \) which we call the tensorial distribution module and denote by \( D_{\mathcal{F}} \mathcal{F} \). Regarding the “foreign” tensor factors as constants (see the comments after Definition 19), all structures combine into a duplex shifted differential Rota-Baxter module \((D_{\mathcal{F}} \mathcal{F}, \partial_x, \partial_x, \mathcal{F}, \mathcal{F}^\xi, \mathcal{F} \xi)\) over \( \mathcal{F} \), which is simultaneously a module over \( \mathcal{P} \mathcal{F} \). Thus far, the situation is parallel to that of Theorem 12.

The algebraic description of the diagonal Heavisides \( H(x - \xi) \) and diagonal Diracs \( \delta(x - \xi) \) is somewhat more complicated. At the level of elements, we insert them essentially by tacking a slim distribution module on top of \( D_{\mathcal{F}} \mathcal{F} \). However, the crucial question is how to combine the diagonal Heavisides with the univariate ones to form a uniform Rota-Baxter structure on the resulting module. The required relation is easy to find if we want to keep touch with analysis. Indeed, for a moment let us think of \( K = \mathbb{R} \) with a fixed \( a \in \mathbb{R} \) and variables \( x, \xi \) ranging over \( \mathbb{R} \). We have \( x \geq a \land x \geq \xi \) iff \((x \geq a \land a \geq \xi) \lor (x \geq \xi \land a \leq \xi)\) since we may split the cases \( a > \xi \) and \( a < \xi \), the remaining possibility \( a = \xi \) holding in both cases above. Translating into Heavisides, this yields

\[
H(x - a)H(x - \xi) = H(x - a)H(a - \xi) + H(x - \xi)H(\xi - a)
\]

or \( H_a(x) \hat{H} = H_a(x) \hat{H}_a(\xi) + H_a(\xi) \hat{H} \) in our algebraic language. We can formulate this into a proper definition of the module providing diagonal Heavisides and Diracs. Here we must take recourse to our earlier interpretations \( H_a(x) := H_a \otimes 1 \in \mathcal{P} \mathcal{F} \) and \( H_a(\xi) := 1 \otimes H_a \in \mathcal{P} \mathcal{F} \).

---

13As in Rosenkranz, Serwa (2015) we use the Iverson bracket notation (Graham, Knuth, Patashnik, 1994, §2.2) for characteristic functions of intervals.
Definition 20. Let \( \hat{\mathcal{Z}} \) be the \( \mathcal{P}_\xi \mathcal{F} \)-submodule of \( \mathcal{P}_\xi \mathcal{F} \otimes_K \hat{\mathcal{D}} \mathcal{K} \) that is generated by the set \( \{(H_a(x) - H_a(\xi))\hat{H} - H_a(x) H_a(\xi) \mid a \in K\} \). Then the \( \mathcal{P}_\xi \mathcal{F} \)-module

\[
\mathcal{D}_{\xi-\xi} \mathcal{F} := \frac{\mathcal{P}_\xi \mathcal{F} \otimes_K \hat{\mathcal{D}} \mathcal{K}}{\hat{\mathcal{Z}}}
\]

is called the diagonal distribution module. We shall denote the (congruence class of) its slim generator \( \hat{H} \in \hat{\mathcal{D}} \mathcal{K} \) by \( H(x - \xi) \), and its derivative \( \hat{\delta} \in \hat{\mathcal{D}} \mathcal{K} \) by \( \delta(x - \xi) \). Analogously to the univariate case, we set also \( H(\xi - x) := 1 - \hat{H} \).

It should also be emphasized that the submodule \( \hat{\mathcal{Z}} \) is not differentially generated. In other words, one is not supposed to differentiate the relation (22) as this would once again lead to inconsistencies. (The situation is completely analogous to the univariate case where one is not supposed to differentiate the relation \( H_a^2 = H_a \); confer Remark 13.)

At this point we have two \( \mathcal{P}_\xi \mathcal{F} \)-modules \( \mathcal{D}_{\xi} \mathcal{F} \) and \( \mathcal{D}_{\xi-\xi} \mathcal{F} \). Since \( \mathcal{F}_2 \subset \mathcal{P}_\xi \mathcal{F} \), we may also view them as \( \mathcal{F}_2 \)-modules. It is easy to see that as such they are free modules just as \( \mathcal{P}_\xi \mathcal{F} \) itself is free as an \( \mathcal{F}_2 \)-module. Indeed, the bivariate piecewise extension \( \mathcal{P}_\xi \mathcal{F} \) has the \( \mathcal{F}_2 \)-basis \( \mathcal{B} := \{1, H_a(x), H_a(\xi), H_a(x) H_a(\xi) \mid a, b \in K\} \), while the tensorial distribution module \( \mathcal{D}_{\xi} \mathcal{F} \) has \( \mathcal{B}_{\xi} := \mathcal{B} \cup \{H_a(x) \delta^0(b - \xi), H_a(\xi) \delta^0(b - x) \mid a, b \in K; n \in \mathbb{N}\} \) as an \( \mathcal{F}_2 \)-basis. Finally, using the relation (22), the diagonal distribution module \( \mathcal{D}_{\xi-\xi} \mathcal{F} \) can be equipped with the "left-focused" \( \mathcal{F}_2 \)-basis \( \mathcal{B}_{\xi} := \mathcal{B} \cup \{H^0(x - \xi), H_a(x) H^0(x - \xi) \mid a \in K, n \geq 0\} \) or with its "right-focused" companion \( \mathcal{B}_{\xi} := \mathcal{B} \cup \{H^0(x - \xi), H_a(\xi) H^0(x - \xi) \mid a \in K, n \geq 0\} \).

We can now put together the tensorial and the diagonal distribution module to obtain the full bivariate distribution module. The latter is already equipped with a duplex differential Rota-Baxter structure, which we shall soon extend to the whole bivariate distribution module in such a way that \( \mathcal{D}_{\xi} \mathcal{F} \) but not \( \mathcal{D}_{\xi-\xi} \mathcal{F} \) will occur as a duplex differential Rota-Baxter submodule.

Definition 21. The bivariate distribution module is given by \( \mathcal{D}_2 \mathcal{F} := \mathcal{D}_{\xi} \mathcal{F} \oplus \mathcal{D}_{\xi-\xi} \mathcal{F} \), as a direct sum of \( \mathcal{P}_\xi \mathcal{F} \)-modules.

Let us first extend the two derivations \( \partial_\xi, \partial_{\xi-\xi} : \mathcal{D}_{\xi} \rightarrow \mathcal{D}_{\xi} \) to the diagonal distribution module \( \mathcal{D}_{\xi-\xi} \mathcal{F} \). For defining \( \partial_{\xi-\xi} \) we use the \( \mathcal{F}_2 \)-basis \( \mathcal{B}_\xi \) to set \( \partial_{\xi} H^0(x - \xi) := H^{0+1}(x - \xi) \).

Regarding the \( H_a(\xi) \) as constants, the map \( \partial_{\xi} : \mathcal{D}_{\xi} \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \) is uniquely determined as an extension of \( \partial_{\xi} : \mathcal{P}_\xi \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \). Analogously, the map \( \partial_{\xi-\xi} : \mathcal{D}_{\xi-\xi} \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \) is introduced as an extension of \( \partial_{\xi} : \mathcal{P}_\xi \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \) with \( \partial_{\xi} H^0(x - \xi) := -H^{0+1}(x - \xi) \), via the \( \mathcal{F}_2 \)-basis \( \mathcal{B}_\xi \). The resulting maps \( \partial_{\xi}, \partial_{\xi-\xi} : \mathcal{D}_{\xi-\xi} \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \) are now combined with the existing derivations \( \partial_{\xi}, \partial_{\xi-\xi} : \mathcal{D}_{\xi} \rightarrow \mathcal{D}_2 \mathcal{F} \) on the tensorial distribution modules into the canonical derivations on the direct sum \( \partial_{\xi}, \partial_{\xi-\xi} : \mathcal{D}_2 \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \). It is clear that \( (\mathcal{D}_2 \mathcal{F}, \partial_\xi, \partial_{\xi-\xi}) \) is then a duplex differential module just over \( \mathcal{F}_2 \), although \( \mathcal{D}_2 \mathcal{F} \) is a module over the duplex differential ring \( (\mathcal{P}_\xi \mathcal{F}, \partial_{\xi}, \partial_{\xi-\xi}) \); this is completely analogous to the univariate structures \( (\mathcal{D} \mathcal{F}, \partial, \partial^\prime) \) and \( (\mathcal{P} \mathcal{F}, \partial) \).

For defining the Rota-Baxter operators \( \int^\xi \) and \( \int^\xi \) on \( \mathcal{D}_2 \mathcal{F} \), it suffices to define them on the diagonal summand \( \mathcal{D}_{\xi-\xi} \mathcal{F} \), using the existing Rota-Baxter operators on the tensorial summand \( \mathcal{D}_{\xi} \mathcal{F} \). Thus we define first \( \int^\xi : \mathcal{D}_{\xi-\xi} \mathcal{F} \rightarrow \mathcal{D}_2 \mathcal{F} \) using the \( \mathcal{F} \)-basis \( \mathcal{B}_\xi \) of \( \mathcal{D}_{\xi-\xi} \mathcal{F} \), considered as a \( \mathcal{D}_2 \mathcal{F} \)-module over the \( \mathcal{F}_2 \)-summand \( \mathcal{B}_\xi \).
Thus we obtain the following partial bivariate analog to Proposition 1.

By our usual convention, we have then \( \int f(x) = (1 - \tau) \int f(x) \) and \( \int g(x) = (1 - \tau) \int g(x) \). Here it is important to distinguish carefully \( H(x - \xi) = \hat{H} \in D_{\mathcal{E}}^\mathcal{F} \) and \( H(\xi - x) = 1 - \hat{H} \in D_{\mathcal{E}}^\mathcal{F} \) from \( H_0(\xi) = 1 \otimes H_0 \neq P_{\mathcal{E}} \mathcal{F} \subset D_{\mathcal{E}}^\mathcal{F} \). Furthermore, it should be noted that while the \( x \)-integral (23) corresponds to (6), the \( \xi \)-integral (24) corresponds to (7) since \( H(x - \xi) \) behaves like \( \hat{H}(\xi) \) from the \( \xi \) perspective; this is the reason for having \( H_0(\xi) \) in (24) as opposed to \( H(\xi) \) in (23).

Before returning to the definition of \( \int f, \int g : D_{\mathcal{E}}^\mathcal{F} \to D_{\mathcal{E}}^\mathcal{F} \), we present a bivariate analog of Proposition 4, introducing the diagonal piecewise extension as the \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \)-submodule

\[
\mathcal{P}_{x,\xi}^\mathcal{F} := \frac{\mathcal{P}_{\mathcal{E}}^\mathcal{F} \otimes_k K \hat{H}}{\hat{Z}} \subset D_{x,\xi}^\mathcal{F},
\]

where \( \hat{K} \hat{H} \subset \hat{D}K \) is the \( K \)-subspace generated by \( \hat{H} = H(x - \xi) \in \hat{D}K \). It is clear that \( \mathcal{P}_{x,\xi}^\mathcal{F} \) is free over \( \mathcal{F}_2 \) with basis \( B^0_\xi := \{ H(x - \xi), H_0(x) H(x - \xi) \mid a \in K \} \subset \mathcal{B}_\xi \) or again alternatively \( B^0_\xi := \{ H(x - \xi), H_a(\xi) H(x - \xi) \mid a \in K \} \subset \mathcal{B}_\xi \).

We note that both \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \) and \( \hat{K} \hat{H} \) are endowed with a multiplication but unlike the former, \( \hat{K} \hat{H} \) is a nonunitary \( K \)-algebra. In fact, its unitarization is just the slim piecewise extension \( \hat{P}K = K \oplus \hat{K} \hat{H} \). At any rate, the numerator of (25) is naturally a nonunitary \( K \)-algebra, and it turns out that the whole quotient module is as well.

**Lemma 22.** The diagonal piecewise extension \( \mathcal{P}_{x,\xi}^\mathcal{F} \) is a nonunitary \( K \)-algebra.

**Proof.** It suffices to prove that \( \hat{Z} \) is an ideal in the nonunitary ring \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \otimes_k K \hat{H} \). Hence let us take an arbitrary \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \)-generator \( \gamma_a := H_a(x) \hat{H} - H_a(\xi) \hat{H} \in D_{\mathcal{E}}^\mathcal{F} \) and show \( \gamma_a \hat{Z} \subset \hat{Z} \).

Since \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \otimes_k K \hat{H} \) is generated over \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \) by \( \hat{H} \), we need only verify \( \gamma_a \hat{H} \in \hat{Z} \). One checks immediately that \( \gamma_a \hat{H} = -H_a(x) H_a(\xi) \hat{H} + \gamma_a H_a(x) \in \hat{Z} \).

In analogy to Definition 21, the bivariate piecewise extension \( \mathcal{P}_2^\mathcal{F} := \mathcal{P}_{\mathcal{E}}^\mathcal{F} \otimes \mathcal{P}_{x,\xi}^\mathcal{F} \) is a \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \)-module consisting of tensorial and diagonal components. But we may also view \( \mathcal{P}_{x,\xi}^\mathcal{F} \) as a nonunitary algebra over \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \), and as such its unitarization is \( \mathcal{P}_{2}^\mathcal{F} \). Therefore the latter is naturally a (unitary) \( \mathcal{P}_{\mathcal{E}}^\mathcal{F} \)-algebra. It is free over \( \mathcal{F}_2 \) with basis \( \mathcal{B} \cup B^0_\xi \subset \mathcal{B}_\xi \), or equivalently with basis \( \mathcal{B} \cup B^0_\xi \subset \mathcal{B}_\xi \). Moreover, it is clear that (23)–(24) restrict to yield operators \( \int f, \int g : \mathcal{P}_2^\mathcal{F} \to \mathcal{P}_2^\mathcal{F} \), which turn out to be Rota-Baxter operators. Thus we obtain the following partial bivariate analog to Proposition 4.
Proposition 23. Let $(\mathcal{F}, \int)$ be an ordinary shifted Rota-Baxter algebra over an ordered field $K$. Then $(\mathcal{P}_x \mathcal{F}, \int^x, \int^\xi)$ is a duplex Rota-Baxter algebra that extends $(\mathcal{F}_2, \int^x, \int^\xi)$.

Proof. The extension property is immediate from the definition of $\int^x$ and $\int^\xi$. Since (23)–(24) are symmetric under exchange of $x$ and $\xi$, it suffices to verify the Rota-Baxter axiom for $\int^x$, say. For this purpose it will be proficient to view $\mathcal{P}_x \mathcal{F}$ as a free module over $\mathcal{P}_x \mathcal{F}$ having the basis $\{b(x), b(x) H_0(x), b(x) H(x - \xi) \mid a \in K, b(x) \in \mathcal{B}_0\}$, where $\mathcal{B}_0$ is an arbitrary but fixed $K$-basis of $\mathcal{F}_x$. Since the Rota-Baxter (1) axiom is bilinear and symmetric in the arguments $f$ and $g$, it suffices to let both arguments range over the basis given above. This leads to $3 + 2 + 1 = 6$ cases. The three cases without $H(x - \xi)$ are covered since $(\mathcal{P}_x \mathcal{F}, \int^x)$ is a Rota-Baxter ring.

Hence it suffices to consider $f = f(x) H(x - \xi)$ for arbitrary $f(x) \in \mathcal{F}_x$ in conjunction with the cases $g = g(x), g(x) H_0(x), g(x) H(x - \xi)$ for arbitrary $g(x) \in \mathcal{F}_x$. We can subsume the first two cases for $g$ by allowing an arbitrary $g(x) \in \mathcal{P}_x \mathcal{F}$.

Let us start with the diagonal-univariate case $f = f(x) H(x - \xi)$ and $g \in \mathcal{P}_x \mathcal{F}$ of the Rota-Baxter axiom (1). Using the defining equation (23), the left-hand side $\int^x f \int^x g$ is given by

$$\left(\int^\xi f(x) \int^\xi g(x)\right) H(x - \xi) + \left(\int^\xi f(x) \int^\xi g(x)\right) H_0(\xi) \tag{26}$$

where $\int^x$ denotes the Rota-Baxter operator of $\mathcal{P}_x \mathcal{F}$ and $\int^\xi$ is as defined after (24). Likewise, a single application of (23) determines the term $\int^x f \int^x g$ on the right-hand side of (1) as

$$\int^x f \int^x g \int^x f \int^x g$$

Thus it remains to compute the other term $\int^x g \int^x f$ on the right-hand side of (1). Here we invoke (23) three times to get

$$\left(\int^\xi g(x) \int^\xi f(x) - \int^\xi f(x) \int^\xi g(x)\right) H(x - \xi) + \left(\int^\xi f(x) \int^\xi g(x) + \int^\xi f(x) \int^\xi g(x)\right) H_0(\xi)$$

Adding the last two equations yields (26) as one sees by a straightforward calculation using the Rota-Baxter axiom of $\mathcal{P}_x \mathcal{F}$, suitably combined with the action of $\tau$. Hence the Rota-Baxter axiom (1) is verified in this case.

We are left with the diagonal-diagonal case $f = f(x) H(x - \xi)$ and $g = g(x) H(x - \xi)$, which is of a more symmetric nature. Calculating according to (23), we obtain for $\int^x f \int^x g$ the expression

$$\left(\int^\xi g(x) \int^\xi f(x) - \int^\xi f(x) \int^\xi g(x)\right) H(x - \xi) + \left(\int^\xi f(x) \int^\xi g(x)\right) H_0(\xi)$$

which combined with its symmetric counterpart $\int^x g \int^x f$ yields

$$\left(\int^\xi f(x) \int^\xi g(x) - \int^\xi f(x) \int^\xi g(x)\right) H(x - \xi) + \left(\int^\xi f(x) \int^\xi g(x)\right) H_0(\xi)$$

Expanding the product $\int^x f \int^x g$ one confirms that this indeed coincides with the expression above, so the Rota-Baxter axiom (1) is again verified. □

We return now to the definition of the Rota-Baxter operators $\int^x$ and $\int^\xi$ on the bivariate distribution module $\mathcal{D}_x \mathcal{F}$, which is in fact dictated by the Rota-Baxter axiom for modules.
Having settled the base case in (23)–(24), we apply the reasoning of Remark 11 to continue the definition by setting

$$\int^x f(x)H^{(n+1)}(x - \xi) := f(x)H^{(n)}(x - \xi) - \int^x f'(x)H^{(n)}(x - \xi),$$

(27)

$$- \int^x g(\xi)H^{(n+1)}(x - \xi) := g(\xi)H^{(n)}(x - \xi) - \int^x g'(\xi)H^{(n)}(x - \xi)$$

(28)

for all $f(x) \in \mathcal{F}_x$, $g(\xi) \in \mathcal{F}_\xi$ and $n \in \mathbb{N}$. Note the distinct sign in (28), due to the fact that $\partial_\xi = -\partial_x$ on the diagonal distribution module $\mathcal{D}_{\cdot, \cdot}\mathcal{F}$. We obtain now the following kind of analog to Theorem 18.

**Theorem 24.** Let $(\mathcal{F}, \partial, \int)$ be an ordinary shifted integro-differential algebra. Then the bivariate distribution module $(\mathcal{D}_2\mathcal{F}, \partial_x, \partial_\xi, \int^x, \int^\xi)$ is a duplex differential Rota-Baxter module containing two isomorphic copies $(\mathcal{D}_2\mathcal{F}, \partial_x, \int^x)$ and $(\mathcal{D}_2\mathcal{F}, \partial_\xi, \int^\xi)$ of the given $(\mathcal{F}, \partial, \int)$. As a duplex Rota-Baxter module, $\mathcal{D}_2\mathcal{F}$ extends $\mathcal{P}_2\mathcal{F}$.

**Proof.** It is obvious from the construction of $\mathcal{D}_2\mathcal{F}$ that it contains the two isomorphic copies $(\mathcal{D}_2\mathcal{F}, \partial_x, \int^x)$ and $(\mathcal{D}_2\mathcal{F}, \partial_\xi, \int^\xi)$, so clearly $\mathcal{D}_2\mathcal{F} \supset \mathcal{D}_n\mathcal{F}$ contains them as well. Furthermore, the Rota-Baxter module extension $\mathcal{D}_2\mathcal{F} \supset \mathcal{P}_2\mathcal{F}$ is immediate from the definition of $\int^x$ and $\int^\xi$.

By symmetry, it suffices to consider the other claims for $\int^x, \partial_\xi$, say. As mentioned after Definition 21, $(\mathcal{D}_2, \partial_x, \int^x)$ is a differential module over $\mathcal{F}_2$ and hence over $K$. We check now that it is also a Rota-Baxter module over $\mathcal{F}_2$, meaning it satisfies the module Rota-Baxter axiom

$$\int^x f \cdot \int^\xi \varphi = \int^x f \int^x \varphi + \int^x (\int^\xi f) \varphi$$

(29)

for all $f \in \mathcal{F}_2$ and $\varphi \in \mathcal{D}_2\mathcal{F}$. Since both $\int^x$ and $\int^\xi$ treat $\mathcal{F}_x < \mathcal{P}_x\mathcal{F}$ as constants, it suffices to consider $f \in \mathcal{F}_x$. Moreover, we may also restrict ourselves to $\varphi \in \mathcal{D}_2\mathcal{F} \cap \mathcal{D}_{\cdot, \cdot}\mathcal{F} < \mathcal{D}_x\mathcal{F}$ since $\int^x$ treats $\mathcal{D}_x\mathcal{F}$ as constants (here we view $\mathcal{D}_x\mathcal{F} = \mathcal{D}_x\mathcal{F} \otimes K < \mathcal{D}_n\mathcal{F}$). The first case $\varphi \in \mathcal{D}_n\mathcal{F}$ is already settled since we know from Theorem 18 that $(\mathcal{D}_n\mathcal{F}, \int^x) \cong (\mathcal{D}\mathcal{F}, \int^x)$ is a Rota-Baxter module. Thus remains to consider the diagonal case $\varphi \in \mathcal{D}_{\cdot, \cdot}\mathcal{F}$, and we can use the $\mathcal{F}_2$-basis $\mathcal{B}_n$. By definition, $\int^x$ coincides for basis elements in $\mathcal{B} \subset \mathcal{B}_n$ with the Rota-Baxter operator $\int^x$ on $\mathcal{P}_n\mathcal{F}$, hence we may restrict ourselves to $\varphi = H^{(n)}(x - \xi)$ and $\varphi = H_n(\xi)H^{(n)}(x - \xi)$ with arbitrary $n \geq 0$ and $a \in K$. But the latter case follows immediately from the former since the $H_n(\xi)$ are constants for $\int^x$. We are now left to prove (29) for $f \in \mathcal{F}_x$ and $\varphi = H^{(n)}(x - \xi)$, which we do by induction on $n$. The base case $n = 0$ is covered by Proposition 23, so we consider (29) with $\varphi = H^{(n+1)}(x - \xi)$ for the induction step. From (27) we get $\int^x \varphi = H^{(n+1)}(x - \xi)$, so the first summand on the right of (29) cancels the second term of expanding (27) with $\int^x f(x)$ in place of $f(x)$; the remaining term $\int^x f(x)H^{(n)}(x - \xi)$ equals the left-hand side of (29).

Note that we have not set up shift operators on the diagonal distributions of $\mathcal{D}_{\cdot, \cdot}\mathcal{F}$ since $\mathcal{D}_{\cdot, \cdot}\mathcal{F}$ would take us outside of $\mathcal{D}_{\cdot, \cdot}\mathcal{F}$. While it is certainly possible to set up a larger domain allowing this (cf. Footnote 12), we do not need it for our present purposes. However, we will need evaluation operators $\mathcal{E}_n^\xi$ and $\mathcal{E}_n^x$ on $\mathcal{D}_{\cdot, \cdot}\mathcal{F}$; since such operators are already defined on $\mathcal{D}_n\mathcal{F}$, this determines $\mathcal{E}_n^\xi$ and $\mathcal{E}_n^x$ on the bivariate distribution module $\mathcal{D}_2\mathcal{F}$ by linearity. The intuitive idea is to define $\mathcal{E}_n^\xi : \mathcal{D}_{\cdot, \cdot}\mathcal{F} \to \mathcal{D}_2\mathcal{F}$ by analogy to the univariate definition $\mathcal{E}_n^\xi := H(\xi - a) \in K$ for $H_\xi \in \mathcal{P}\mathcal{F}$ given earlier (cf. Proposition 4 and the paragraph above it): Since for evaluation it should not play a role whether one views $\xi$ as a parameter or as a variable, we can interpret $H_\xi$ heuristically as $H(x - \xi) \in \mathcal{P}_{\cdot, \cdot}\mathcal{F}$ and the right-hand side
as \( \hat{H}(\xi - a) = 1 \otimes \hat{H}_a \in \mathcal{P}_\xi \mathcal{F} \). For the evaluation with respect to \( \xi \), the reasoning is analogous. Hence we give the definitions

\[
\hat{e}_a^\xi H(x - \xi) := \hat{H}(\xi - a), \quad \hat{e}_a^\xi H(x - \xi) := H(x - a).
\]

(30)

For evaluating diagonal Diracs, we use again the analogy to our earlier definition \( \hat{e}_a \delta(\xi) := 0 \) set up earlier (see the paragraph before Theorem 18). Thus we set

\[
\hat{e}_a^\xi \delta(\xi) = 0, \quad \hat{e}_a^\xi \delta(x - \xi) = 0,
\]

(31)

completing the definition of \( \hat{e}_a^\xi \) and \( \hat{e}_a^\xi \) on the diagonal distribution module \( \mathcal{D}_{\chi, \xi} \mathcal{F} \subset \mathcal{D}_2 \mathcal{F} \) and hence yielding evaluation operators \( \hat{e}_a^\xi, \mathcal{D}_2 \mathcal{F} \to \mathcal{D}_\xi \mathcal{F} \) and \( \hat{e}_a^\xi, \mathcal{D}_2 \mathcal{F} \to \mathcal{D}_\xi \mathcal{F} \).

6. Application to Boundary Problems

As mentioned earlier, the treatment of boundary problems (for linear ordinary differential equations) is a major application area for our algebraic approach to piecewise smooth functions and Dirac distributions. We refer to Rosenkranz Regensburger (2008); Regensburger, Rosenkranz (2009) for basic notions and algorithms in the algebraic theory of boundary problems. Consider a regular boundary problem \((T, B)\) over an ordinary shifted integro-differential algebra \((\mathcal{F}, \partial, \int)\) and let \( G := (T, B)^{-1} \) be its Green’s operator. Assuming a well-posed two-point boundary value problem, classical analysis (Stakgold, Holst, 2011, §3) will inform us that \( G \) is an integral operator \( Gf(x) = \int_{a-\varepsilon}^{a+\varepsilon} g(x, \xi) f(\xi) d\xi \) with the so-called Green’s function \( g(x, \xi) \) as its integral kernel. We shall denote the initialization point of \( \int \) by \( o \in K \) so that \( \int = \int_o \) and \( \varepsilon = \varepsilon_o \).

We distinguish now three essentially independent applications of the algebraic theory developed in Sections 2–5 to such boundary problems, which we elaborate in this section:

1. The Green’s function \( g(x, \xi) \) is a (bivariate) piecewise smooth function, usually described by a case distinction; we would like to express it in the algebraic language of Heaviside functions. For ill-posed boundary problems, \( g(x, \xi) \) may be a Dirac distribution that we wish to express in terms of the distribution module.

2. The very definition of the Green’s function \( g_\xi(x) := g(x, \xi) \) is typically cast in the language of distributions (Stakgold, Holst, 2011, (3.3.4)). Subject to suitable smoothness constraints, it is described uniquely by requiring it, as a function of \( x \), to satisfy the differential equation \( T g_\xi = \delta_\xi \) and the boundary conditions \( \beta(g_\xi) = 0 (\beta \in B) \).

3. A specific instance of the boundary problem \((T, B)\) arises by choosing a forcing function \( f \). Thus one wants to find \( u \in \mathcal{F} \) such that \( Tu = f \) and \( \beta(u) = 0 (\beta \in B) \). In terms of the Green’s operator \( G \), the solution is expressed by the action \( u = Gf \), which has been defined when \( f \in \mathcal{F} \). For a piecewise smooth forcing function \( f \), no choice of integro-differential algebra \( \mathcal{F} \) will enable \( f \in \mathcal{F} \) since piecewise smooth functions do not form an integro-differential algebra (Proposition 6).

\[\text{This is a sensible hypothesis for applications. The usual requirement is piecewise continuity (Stakgold, Holst, 2011, §3.1.1), but continuous functions failing } C^0 \text{ except on isolated singularities are bizarre (Weierstrass function).}\]
For a still more ambitious generalization, see our remarks in the Conclusion.

Let us return to the given regular boundary problem \((T, B)\). We allow \((T, B)\) to be an arbitrary Stieltjes boundary problem (Rosenkranz, Serwa (2015)), meaning: (1) It may have more than two evaluation points; (2) it may involve definite integrals in the boundary conditions; (3) it may be ill-posed. We assume now that \((\mathcal{F}, \partial, \int)\) is an ordinary shifted integro-differential algebra over the ordered field \(K\); then all the results of Sections 3 and 4 on \(\mathcal{PF} \subset \mathcal{DF}\) are available. The corresponding set of evaluations will be denoted by \(\Phi := \{e_a | a \in K\}\). We may form the standard integro-differential operator ring \(\mathcal{F}_g[\partial, \int]\) and its equitable variant \(\mathcal{F}[\partial, \int_g]\), as described in Rosenkranz, Serwa (2015). Let \(J = \{a_1, \ldots, a_k\} \subseteq K\) be the evaluations actually occurring in the boundary conditions \(B\), in the sense that all \(\beta \in B\) are contained in the right ideal generated by the evaluations \(e_a\) \((a \in J)\). Picking an \(a \in \{a_1, \ldots, a_k\}\) as initialization point \(a\) of the Rota-Baxter operator \(\int\) on \(\mathcal{F}\) will avoid spurious case distinctions in \(g(x, \xi)\), but this is not required for correct extraction (Rosenkranz, Serwa, 2015, Rem. 1).

The setting described in Rosenkranz, Serwa (2015) took the standard integro-differential algebra \(\mathcal{F} = \mathcal{C}^\infty(\mathbb{R})\) over the real field \(K = \mathbb{R}\) as a starting point for an algorithm extracting the Green’s function \(g(x, \xi)\) from the Green’s operator \(G\), which may itself be computed as in Rosenkranz Regensburger (2008). Since \(g(x, \xi)\) is at best piecewise smooth (for well-posed problems) and in general even distributional (for ill-posed problems), a concrete distribution module\(^\text{16}\) from analysis was chosen. For the algebraic framework of boundary problems \((T, B)\) it is more appropriate to provide a purely algebraic construction for accommodating the Green’s function. We shall now show that we may indeed consider \(g(x, \xi) \in \mathcal{DF}\) for regular Stieltjes boundary problems (Theorem 26) and \(g(x, \xi) \in \mathcal{PF}_2\) for well-posed problems (Proposition 27).

The procedure to achieve this goal is rather straightforward: The algorithm of Rosenkranz, Serwa (2015) can be used verbatim, provided we interpret all Heavisides and Diracs in the sense of \(\mathcal{DF}\). We need only prove that the latter have the properties required for the proof of the Structure Theorem for Green’s Functions (Rosenkranz, Serwa, 2015, Thm. 1). We start with the extraction map \(\eta: \mathcal{F}_g[\partial, \int] \to \mathcal{DF}\), which we shall write \(G \mapsto G_{\eta}\) as in the corresponding definition given in (Rosenkranz, Serwa, 2015, §5) before Lemma 1 (but we forgo the modified equitable form, which may sometimes lead to further simplifications). For convenience, we write out the definition of \(\eta\) in Table 1 below, using the natural \(K\)-basis of \(\mathcal{F}_g[\partial, \int]\).

\[
\begin{array}{|c|c|}
\hline
G \in \mathcal{F}_g[\partial, \int] & G_{\eta} \in \mathcal{DF} \\
\hline
u \partial^i & u(x) \delta^{(i)}(x - \xi) \\
u \big| \bigg) v & u(x) v(\xi) [0 \leq \xi \leq x]_+ \\
u e_a \partial^i & (-1)^i u(x) \delta^{(i)}(\xi - a) \\
u e_a \int v & u(x) v(\xi) [0 \leq \xi \leq a]_+ \\
\hline
\end{array}
\]

Table 1: Extraction Map \(\eta: \mathcal{F}_g[\partial, \int] \to \mathcal{DF}\)

Here we have employed the abbreviation \([a \leq \xi \leq b] := H(\xi - a) \tilde{H}(\xi - b)\) for the characteristic function of the interval \([a, b]\) and \([a \leq \xi \leq x] := H(\xi - a) H(x - \xi)\) for that of \([a, x]\). Note that this presupposes \(a < b\) and \(a < x\). While \(a\) and \(b\) are on an equal footing, we must define

\(^{16}\) It is essentially a certain \(\mathcal{C}^\infty(\mathbb{R})\)-submodule of the dual space of the smooth compactly supported test functions, namely the one is generated by the Heavisides \(H_a\) and the Diracs \(\delta_a\) for all \(a \in J\).
the characteristic function \([x \leq \xi \leq a]\) := \(H(\xi - a) H(x - \xi)\) separately for the interval \([x, a]\) with \(x < a\). For the extraction one actually needs signed versions for recording the relative order, namely \([a \leq \xi \leq b]_\pm := [a \leq \xi \leq b] - [b \leq \xi \leq x] \) and \([a \leq \xi \leq x]_\pm := [a \leq \xi \leq x] - [x \leq \xi \leq a]\). One checks immediately that this simplifies to \([a \leq \xi \leq b]_\pm = H(\xi - a) - H(\xi - b)\) and to \([a \leq \xi \leq x]_\pm = H(x - \xi) + H(\xi - a) - 1\), respectively.

**Remark 25.** The first row in Table 1 might make the impression of missing an alternating sign, which was indeed—erroneously—present in our original formulation Rosenkranz, Serwa (2015). Acting on a function \(f(x)\) and setting \(u(x) \equiv 1\) for simplicity, the rule is \(\int_a^b \delta^0(x-\xi) f(\xi) d\xi = f(\xi)\). In analysis this is usually written as

\[
\int_{-\infty}^{\infty} \delta^0(\xi - x) f(\xi) d\xi = (-1)^{i} f^{(i)}(x) \tag{32}
\]

or \(\delta^0[f] = (-1)^{i} f^{(i)}(x)\) in the language of functionals. In the example \(\mathcal{F} = C^\infty(\mathbb{R})\) with standard integro-differential structure, both formulations are equivalent by the well-known Dirac symmetry \(\delta^0(x - \xi) = (-1)^{i} \delta^0(\xi - x)\). In contrast, the alternating sign in the third row of Table 1 cannot be avoided since this corresponds directly to (32). It should also be noted that the sign in the fourth row of Table 1 has been corrected with respect to Rosenkranz, Serwa (2015), where \(\text{sgn}(a)\) was used instead of the signed characteristic function.

For simplicity we assume the initialization point to be \(o = 0\). Otherwise some computations would only become more cumbersome without providing additional insight (producing various intermediate terms that eventually all cancel out). A nonzero initialization point \(o\) is best handled by adapting the splitting point of (6) and (23)–(24); confer Footnote 6.

Let us now prove that the extraction procedure preserves the meaning of the Green’s operator. Since the following result is applicable to the standard example \((C^\infty(\mathbb{R}), \partial, f)\) over \(K = \mathbb{R}\), it includes the setting of Rosenkranz, Serwa (2015) and may be seen as an algebraic abstraction of the distribution setup customarily used in this context. For achieving a smooth formulation, let us introduce an algebraic generalization of functional equality restricted to intervals \([\alpha, \beta]\) of the real line. Given piecewise functions \(f, g \in \mathcal{P}\mathcal{F}\) and \(\alpha < \beta \in \mathbb{K}\), we say that \(f = g\) on \([\alpha, \beta]\) iff \(f \equiv g \mod Z_{\alpha, \beta})\) where \(Z_{\alpha, \beta})\) is the ideal of \(\mathcal{P}\mathcal{F}\) generated by \(H_\alpha\) and \(H_\beta\). In the standard example \(\mathcal{F} = C^\infty(\mathbb{R})\) this corresponds to the familiar notion of analysis. Since \(\mathcal{P}\mathcal{F}\) is isomorphic to the rings \(\mathcal{P}\mathcal{P}\mathcal{F}\) and \(\mathcal{P}\mathcal{F}\mathcal{F}\), we may apply analogous interval restriction to the two latter rings. This allows us to give a precise meaning to the colloquial statement: “The Green’s function provides a faithful realization of the Green’s operator.”

**Theorem 26.** Let \(\mathcal{F}\) be an ordinary shifted integro-differential algebra over any ordered field \(K\), and let \(\eta: \mathcal{F}_0[\partial, f] \to \mathbb{D}\mathcal{F}\) be as in Table 1. Choose \(\alpha, \beta \in \mathbb{K}\) with \(\alpha \leq a_1 < \cdots < a_k \leq \beta\). Writing \(\{f\} := \int f^\delta\) for brevity, we have

\[
Gf(x) = \int_a^b g(x, \xi) f(\xi) \in \mathcal{F}_x \tag{33}
\]

on \([\alpha, \beta]\), for all \(f \in \mathcal{F}\) and \(G \in \mathcal{F}_0[\partial, f]\) with extraction \(g(x, \xi) := G_{x\xi}\). If \(G\) is the Green’s operator of a regular Stieltjes boundary problem, \(g(x, \xi)\) is thus its Green’s function.

**Proof.** Let us start by recalling the exact meaning of (33), for an arbitrary Green’s operator \(G \in \mathcal{F}_0[\partial, f]\) arising from a regular Stieltjes boundary problem and a forcing function \(f \in \mathcal{F}\). On the left-hand side we have the usual action of the operator ring \(\mathcal{F}_0[\partial, f]\) on the underlying integro-differential algebra \(\mathcal{F}\); thus \(Gf \in \mathcal{F}\) and \(Gf(x) = \iota_x(Gf) \in \mathcal{F}_x\) via the embedding \(\iota_x: \mathcal{F} \hookrightarrow \mathcal{F}_x\). On
the right-hand side of (33) we have the associated Green’s function \(g(x, \xi) \in \mathcal{D}_2\mathcal{F}\) and the given function \(f(\xi) = \epsilon(\xi) \in \mathcal{F}_\xi\) via the embedding \(\iota: \mathcal{F} \hookrightarrow \mathcal{F}_\xi\); their product \(g(x, \xi) f(\xi) \in \mathcal{D}_2\mathcal{F}\) is then integrated via \(\mathcal{I}^\alpha_\beta := \int_\alpha^\beta f(\xi)\) where we have as usual set \(\mathcal{I}^\xi_\beta := (1-\epsilon^\xi_\beta)\mathcal{I}^\xi\) and \(\epsilon^\xi_\beta := \xi^\xi_\beta \bar{S}^\xi\) for arbitrary \(c \in \mathbb{K}\).

Let us now go through the rows of Table 1. The first case is \(G = u \partial_i\) so that we obtain immediately \(Gf(x) = u(x) f^{(i)}(x)\) for the left-hand side of (33). Since \(u(x)\) is constant with respect to \(\mathcal{I}^\xi\), the right-hand side is given by \(u(x) \mathcal{I}^\alpha_\beta f(\xi) \delta^{(i)}(x - \xi),\) and it suffices to show

\[
\mathcal{I}^\alpha_\beta f(\xi) \delta^{(i)}(x - \xi) \quad \text{on} \quad [\alpha, \beta],
\]

which one does by induction on \(i\). For the base case \(i = 0\) we compute

\[
\mathcal{I}^\xi f(\xi) \delta(x - \xi) = \int_\xi^\xi f'(\xi) H(x - \xi) - f(\xi) H(x - \xi)
= \left(\int_\xi^\xi f'(\xi)\right) H(x - \xi) + \left(\int_\xi^\xi f'(\xi)\right) H_0(x) - f(\xi) H(x - \xi),
\]

using first (28) and then (24). Since \(\int_\xi^\xi f'(\xi) = f(\xi) - f(x)\) and \(\int_\xi^\xi f'(\xi) = f(x) - f(0),\) this simplifies to

\[
\int_\xi^\xi f(\xi) \delta(x - \xi) = -f(x) H(x - \xi) + r(x) \quad \text{where the term } r(x) \in \mathcal{P}_x\mathcal{F}\text{ is invariant}
\]

under \(\xi^\xi\). Hence the latter term cancels in \(\mathcal{I}^\alpha_\beta = (\xi^\alpha - \xi^\beta)\mathcal{I}^\xi\) so that

\[
\mathcal{I}^\alpha_\beta f(\xi) \delta(x - \xi) = \left(\xi^\alpha - \xi^\beta\right) f(x) H(x - \xi) = f(x) \left(H(x - \alpha) - H(x - \beta)\right),
\]

where in the last step we have applied (30). As a consequence, \(f(x) - \mathcal{I}^\alpha_\beta f(\xi) \delta(x - \xi)\) is given by \(f(x)(H_\alpha(x) + H_\beta(x)) \in Z_{[\alpha, \beta]} \subset \mathcal{P}_x\mathcal{F},\) which means that \(f(x) = \mathcal{I}^\alpha_\beta f(\xi) \delta(x - \xi)\) on \([\alpha, \beta]\) as claimed. For the induction step, we compute

\[
\mathcal{I}^\alpha_\beta f(\xi) \delta^{(i+1)}(x - \xi) = \left(\xi^\alpha - \xi^\beta\right) \left(\int_\xi^\xi f'(\xi) \delta^{(i)}(x - \xi) - f(\xi) \delta^{(i)}(x - \xi)\right)
\]

by (28), which reduces to \(\mathcal{I}^\alpha_\beta f'(\xi) \delta^{(i)}(x - \xi)\) since diagonal Diracs evaluate to zero by (31). Using the induction hypothesis (34) with \(f'\) in place of \(f\), the latter integral equals \(f^{(i+1)}(x)\) on \([\alpha, \beta]\); this is indeed (34) for \(i + 1\).

Next we treat the second row of Table 1. Since \(u(x)\) is constant with respect to \(\mathcal{I}^\xi\), it suffices to show \(\int_\xi^\xi v(x) f(x) = \int_\xi^\xi v(\xi) [0 \leq \xi \leq x]_x f(\xi)\) on \([\alpha, \beta].\) Obviously, we may set \(v = 1\) without loss of generality. Using \([0 \leq \xi \leq x]_x = H(x - \xi) + H_\xi(x) - 1\) we have

\[
\int_\xi^\xi f(\xi) [0 \leq \xi \leq x]_x = \int_\xi^\xi f(\xi) H(x - \xi) + \int_\xi^\xi f(\xi) H_\xi(x) - \int_\xi^\xi f(\xi)
= \left(\int_\xi^\xi f(\xi)\right) H(x - \xi) + \left(\int_\xi^\xi f(\xi)\right) H_\xi(x) - \left(\int_\xi^\xi f(\xi)\right) H_\beta(x)
\]

by applying (24) and (6). The middle summand cancels in \(\mathcal{I}^\alpha_\beta = (\xi^\alpha - \xi^\beta)\mathcal{I}^\xi,\) and one obtains after a few simplifications

\[
\mathcal{I}^\alpha_\beta f(\xi) [0 \leq \xi \leq x]_x = (\mathcal{I}^\alpha_\beta) f(\xi) + (\mathcal{I}^\alpha_\beta) f(\xi) H_\beta(x) + (\mathcal{I}^\alpha_\beta) f(\xi) H_\xi(x) - \mathcal{I}^\alpha_\beta f(\xi) H_\beta(x).
\]

Here we have used the facts \(H(-\alpha) = 1\) and \(H(-\beta) = 0,\) which follow from our assumption that the interval \([\alpha, \beta]\) contains the initialization point \(\alpha = 0\) so that \(\alpha < 0 < \beta.\) Since the first two summands on the right-hand side above are in \(Z_{[\alpha, \beta]} \subset \mathcal{P}_x\mathcal{F},\) we obtain finally

\[
\int_\xi^\xi f - \mathcal{I}^\alpha_\beta f(\xi) [0 \leq \xi \leq x]_x \equiv \mathcal{I}^\xi f(\xi) H_\alpha(x) + H_\beta(x) \equiv 0 \quad \text{(mod } Z_{[\alpha, \beta]}),
\]
Hence we must show that 

$$\int_0^a f(x) \, dx = \int_a^b g(x) \, dx$$

and 

$$\int_0^b f(\xi) \, [0 \leq \xi \leq x]_a$$

is indeed equal to 

$$\int f(x)$$
on \, as was claimed.

Turning to the third row of Table 1, we can again set \( u(x) = 1 \) without loss of generality. Hence we must show that 

$$f^{(i)}(a) = \int_0^a g^{(i)}(\xi-a) \, f(\xi)$$
holds on \([a, \beta]\). In fact, it turns out to hold without constraints. We can now work purely in \( \mathcal{D}_F \) and use (14) to calculate

$$\int f^\xi (\xi-a) \, f(\xi) = \sum_{k=0}^i (-1)^k \int (\xi-a)^k H^{i-k}(\xi-a) - (-1)^i \int f^{(i+1)}(\xi) \, H(\xi-a)$$

by a straightforward induction on \( i \geq 0 \). Applying again \( \hat{f}_\xi^\alpha = (\hat{\xi}_\beta - \hat{\xi}_\alpha) \hat{f}^\xi \), all terms in the sum except for \( i = 0 \) cancel since \( \hat{\xi}_\beta \) and \( \hat{\xi}_\alpha \) annihilate the Diracs, and we get

$$\int f^{(i)}(\xi-a) \, f(\xi) = (-1)^i \left( f^{(i)}(\beta) H(\alpha-\beta) - f^{(i)}(\alpha) H(\alpha-\beta) - \int f^{(i+1)}(\xi) \, H(\xi-a) \right).$$

By our assumption \( \alpha < \beta \) we have \( H(\alpha-a) = 0 \) and \( H(\alpha-\beta) = 1 \). Now we compute the remaining integral according to (6) to obtain

$$(-1)^i \int f^{(i)}(\xi-a) \, f(\xi) = f^{(i)}(\beta) + \left( \int f^{(i)}(\xi-a) \, f(\xi) \right).$$

The last term in the right parenthesis cancels since it is invariant under both \( \hat{\xi}_\beta \) and \( \hat{\xi}_\alpha \). Since we have \( \int f^{(i+1)} = f^{(i)} \), we get for \( (-1)^i \int \hat{f}_\xi^\alpha(\xi-a) \, f(\xi) \) the expected result

$$f^{(i)}(\beta) + \hat{H}(\alpha-a) \left( f^{(i)}(\alpha) - f^{(i)}(\alpha) \right) - \hat{H}(\alpha-\beta) \left( f^{(i)}(\beta) - f^{(i)}(\alpha) \right) = f^{(i)}(\alpha),$$

using again \( \hat{H}(\alpha-a) = 0 \) and \( \hat{H}(\alpha-\beta) = 1 \).

It remains to consider the fourth row of Table 1. As for the second row, we may omit \( u(x) \) and \( v(x) \) without loss of generality, and it suffices to prove 

$$\int_0^a f = \int_0^b f(\xi) \, [0 \leq \xi \leq a]_a.$$ 

Using 

$$\int_0^b f(\xi) = H_0(\xi) - H(\xi) \quad \hat{H}(\alpha-a) \left( f^{(i)}(\alpha) - f^{(i)}(\alpha) \right) - \hat{H}(\alpha-\beta) \left( f^{(i)}(\beta) - f^{(i)}(\alpha) \right) = f^{(i)}(\alpha)$$

according to (6). As before, the last term cancels when computing \( \hat{f}_\xi^\beta = (\hat{\xi}_\beta - \hat{\xi}_\alpha) f^\xi \), and we obtain the desired equality

$$\int_0^b f(\xi) \, [0 \leq \xi \leq a]_a = \left( \int f(\xi) \, H(\beta) - \left( \int_0^a f \right) \hat{H}(\alpha-a) + \left( \int_0^a f - \int_0^a f \right) \hat{H}(\alpha-a) \right)$$

using again \( \hat{H}(\alpha-a) = 0 \) and \( \hat{H}(\alpha-\beta) = 1 \).

**Proposition 27.** Let \( \mathcal{F} \) and \( \eta: \mathcal{F}_a[\partial, \int] \to \mathcal{D}_F \) be as in Theorem 26. If the regular boundary problem \((T, B)\) is well-posed, then we have \( g_{(x, \xi)} \in \mathcal{P}_2 \mathcal{F} \) for the Green’s function \( g(x, \xi) = G_{x, \xi} \) extracted from its Green’s operator \( G = (T, B)^{-1} \).

**Proof.** We use the crucial fact (Rosenkrantz, Serwa, 2015, Thm. 1) that the Green’s function \( g(x, \xi) = \hat{g}(x, \xi) + \hat{g}(x, \xi) \) splits into a functional part \( \hat{g}(x, \xi) \in \mathcal{P}_2 \mathcal{F} \) and a distributional part \( \hat{g}(x, \xi) \in \mathcal{D}_F \mathcal{P}_2 \mathcal{F} \). Hence it suffices to show that \( \hat{g}(x, \xi) = 0 \). In the proof of (Rosenkrantz, Serwa, 2015, Thm. 1), the splitting of the Green’s function is induced by a corresponding splitting of the
Green’s operator $G = \hat{G} + \tilde{G}$ into a functional part $\hat{G}$ with $\hat{G}_{x,\xi} = \hat{g}(x, \xi)$ and a distributional part $\tilde{G}$ with $\tilde{G}_{x,\xi} = \tilde{g}(x, \xi)$, so our goal is to show $\tilde{G} = 0$. From the proof of (Rosenkranz, Serwa, 2015, Lem. 2) we see that the only possible contributions to $\tilde{G}$ come from terms of the form $f E_\alpha \partial^\beta$ in the kernel projector $P$. Moreover, such a term will go to $\hat{G}$ if $k < n$ since in this case the second sum in (Rosenkranz, Serwa, 2015, Lem. 1) is absent, as has been observed after Equation (8) of Rosenkranz, Serwa (2015).

Thus it suffices to prove that $k < n$ for all terms $f E_\alpha \partial^\beta$ occurring in the kernel projector $P$. But this is clear from the form of $P$ as given e.g. in the proof of Theorem 26 in Rosenkranz Regensburger (2008). Indeed, if $u = (u_1, \ldots, u_n) \in F^n$ is a fundamental system for Ker $T$ and $\beta = (\beta_1, \ldots, \beta_n) \in (\Phi)^n$ a basis of Stieltjes conditions for the boundary space $B$, the kernel projector is given by $P = u^1 \beta(u)^{-1} \beta \in F[d, \int]$, and since $(T, B)$ is well-posed by hypothesis, all local terms $E_\alpha \partial^\beta$ occurring in the boundary conditions $\beta_1, \ldots, \beta_n$ must have $k < n$. □

We turn now to the application labeled (2) in the above introduction. We continue to assume that $(F, \partial, \int)$ is an ordinary shifted integro-differential algebra. Recall that any integro-differential algebra $(F, \partial, \int)$ contains an isomorphic copy of the polynomial ring $(K[x], \partial, \int)$ with its standard integro-differential structure (Buchberger, Rosenkranz, 2012, Prop. 3); we may use the identification $x := \int 1$. Since $(F, \partial, \int)$ is ordinary, it is not difficult to see that Ker $\partial^m = [1, \ldots, x^{m+1}]$ so that dim Ker $\partial^m = n$, as one would expect (Regensburger, Rosenkranz, 2009, Lem. 3.21). However, we need an additional condition to ensure similar behavior for arbitrary differential operators (including nonmonic ones). Hence let us call a differential algebra $(F, \partial)$ strongly ordinary if dim Ker $T < \infty$ for any $T \in F[\partial]$. Note that all the usual examples of ordinary differential algebras in analysis are strongly ordinary, in particular our standard example $C^\infty(\mathbb{R})$. In fact, there is an explicit upper bound in the real-analytic theory (Kato, Struppa, 1999, Thm. 1.3.6), namely dim Ker $T \leq m + d$, where $m$ is the order of the differential operator $T$ and $d$ counts the zeros of its leading coefficient with multiplicities (using hyperfunctions the estimate becomes an identity). We must first ensure that we can uniquely recover Dirac distributions. To achieve this we will need some analytic assumption that plays the role of the Fundamental Lemma of the Variational Calculus, often ascribed to Paul du Bois-Reymond. Hence we say that $\phi \in F$ is degenerate on $[\alpha, \beta]$ if $\int_\alpha^\beta \phi(x) f(x) \, dx = 0$ for all $f \in F$. In the language of (Rosenkranz, Korp, 2013, §3), this says that the Stieltjes condition $\int_\alpha^\beta \phi$ is degenerate. In the same vein, we call $k(x, \xi) \in D_2 F$ nondegenerate if it does not contain any degenerate $\phi(x) \in F_\xi$; we do not need a similar condition on its $F_\tau$ parts. This restriction is clearly no loss of generality since our goal is to integrate over $[\alpha, \beta]$, so degenerate functions may as well be discarded from the outset.

Proposition 28. Let $(F, \partial, \int)$ be a strongly ordinary shifted integro-differential algebra and choose any bivariate distribution $k(x, \xi) \in D_2 F$ that is nondegenerate on $[\alpha, \beta]$. If one has $\int_\alpha^\beta k(x, \xi) f(x) = f(x)$ on $[\alpha, \beta]$ for all $f \in F$, then necessarily $k(x, \xi) = \delta(x - \xi)$. Proof. We have $D_2 F = (D_1 F \otimes_F P_1 F) \oplus (P_1 F \otimes_F D_1 F) \oplus D_{x-\xi} F$ by the definition of bivariate distributions, hence we may assume

$$
 k(x, \xi) = \sum_{i, a} \Xi_i a(\xi) \delta_i^a(x - a) + \sum_{i, a} X_i a(x) \delta_i^a(\xi - a) + \sum_{i=0}^N M_i(x, \xi) \delta_i^a(x - \xi),
$$

(35)

where $N \in \mathbb{N}$, the summations are over $i \in \mathbb{N}$ and $a \in K$, containing only finitely many nonzero coefficients $\Xi_i a(\xi) \in P_\xi F$, $X_i a(x) \in P_x F$ and $M_i(x, \xi) \in P_{x, \xi} F$. Since the latter ring is by
on \([\alpha,\beta]\) where \( \Xi,\alpha,\beta := \int_0^1 \Xi,\alpha,\beta f(\xi) \in K \). Since the distributions \( \delta^{(i)}(x-a) \) are by construction linearly independent from any element of \( \mathcal{P}_x \mathcal{F} \), the assumption \( \hat{f}(x) = 0 \) forces \( \Xi,\alpha,\beta = 0 \) and hence also \( \Xi,\alpha,\beta f(\xi) = 0 \) by the hypothesis on nondegeneracy.

As noted above, we have \( K[x] \subset \mathcal{F} \). It is easy to see that in such circumstances, the well-known algorithm for Hermite interpolation applies to construct polynomials \( p(\xi) \) with arbitrary values prescribed for \( p^{(j)}(\xi_i) \), where \( j \in \{1, \ldots, m\} \) and \( \{\xi_1, \ldots, \xi_n\} \subset K \). Here \( m \) and \( n \) may be arbitrarily large positive integers since we can construct polynomials of indefinitely high degree involving powers of \( x-a \) for any \( a \in K \). In particular, we can construct polynomials \( f_c(\xi) \) with \( f_c^{(i)}(0) = 0 \) for all those \( (i, a) \) that occur in (36) with nonzero coefficients \( X,\alpha,\beta \). We may further assume that they carry \( l \) arbitrary parameters \( c_1, \ldots, c_l \) by adding suitable interpolation data for “unused” higher derivatives; the parameters are collected into \( c = (c_1, \ldots, c_l) \in K^l \). Note that \( l \in \mathbb{N} \) is arbitrary since we may add indefinitely many interpolation values some of which may be frozen to zero if needed.

Substituting the interpolation polynomial thus obtained into (36), also the middle sum now vanishes by our construction, and we are left with the differential equation

\[
\sum_{i=0}^N L_i(x) \partial^i\left(R_i(x) f_c(x)\right) = f_c(x)
\]  

(37)

on \([\alpha,\beta]\). Let us write \( L_i(x) = l_i(x) + \sum_{j<k} l_{j,k}(x) H(x-b) \) with \( l_i(x), l_{j,k}(x) \in \mathcal{F}_x \). Then the set \( B := \{ b \in K \mid \exists_{i=0,\ldots,N} L_i,b(x) \neq 0 \} \) is clearly finite and contained in \([\alpha,\beta]\), so we may rewrite (37) as

\[
\sum_{i=0}^N l_i(x) \partial^i\left(R_i(x) f_c(x)\right) + \sum_{b \in B} H(x-b) \sum_{i=0}^N l_{i,b}(x) \partial^i\left(R_i(x) f_c(x)\right) = f_c(x) + z_{\alpha,\beta}
\]

for some \( z_{\alpha,\beta} \in \mathcal{F}_x \). But the set \( \{H(x-a), H(x-b) \mid b \in B\} \) is linearly independent over \( \mathcal{F} \); hence (37) splits into the \( |B| + 1 \) separate differential equations

\[
\sum_{i=0}^N l_i(x) \partial^i\left(R_i(x) f_c(x)\right) = f_c(x), \quad \sum_{i=0}^N l_{i,b}(x) \partial^i\left(R_i(x) f_c(x)\right) = 0 \quad (b \in B)
\]

Unless the underlying differential operators vanish, each of these differential equations has an infinite-dimensional solution space containing all \( f_c(x) \) with \( c \in K^l \) for \( l \in \mathbb{N} \) indefinitely large. Since \( (\mathcal{F},\partial) \cong (\mathcal{F}_x,\partial_x) \) is assumed to be strongly ordinary, we conclude that we must in fact have \( l_0(x) = R_i(x) = 1 \) and \( l_i(x) = 0 \) for \( i > 0 \) as well as \( l_{i,b}(x) = 0 \).

At this point we have reduced (35) to \( k(x,\xi) = \sum_{i=0}^N X_i,a(x) \delta^{(i)}(\xi-a) + \delta(x-\xi) \), hence the action yields \( \sum_{i=0}^N (-1)^i X_i,a(x) f^{(i)}(a) = 0 \) for all \( f \in \mathcal{F} \). Since for each fixed \( i \), there are only
finitely many nonzero coefficients \( \{X_{ia} \mid a \in A_i\} \) in (35), and these may be assumed to be linearly independent over \( K \) since we may always combine them if needed. But applying again Hermite interpolation, one may choose \( f \in K[x] \subset \mathcal{F}_\xi \) such that \( f^{(i)}(a') = 1 \) for some fixed pair \((i', a')\) and \( f^{(i)}(a) = 0 \) for all other pairs \((i, a)\). This implies immediately that \( X_{i',a'} = 0 \). Since \((i', a')\) is arbitrary, we conclude that indeed \( k(x, \xi) = \delta(x - \xi) \). \( \square \)

For an integro-differential operator \( U \in \mathcal{F}_\partial[\partial, \int] \) we shall write \( U_s : \mathcal{D}_2 \mathcal{F} \to \mathcal{D}_2 \mathcal{F} \) and \( U_\xi : \mathcal{D}_2 \mathcal{F} \to \mathcal{D}_2 \mathcal{F} \) for the two operators induced by the action with respect to \( x \) and with respect to \( \xi \). In other words: If \( U = f \in \mathcal{F} \) then \( U_s \) is the multiplication operator induced by \( f(x) \in \mathcal{F}_s \) while \( U_\xi \) is induced by \( f(\xi) \in \mathcal{F}_\xi \). Likewise, if \( U = \partial \) then \( U_s \) acts as \( \partial_s \) and \( U_\xi \) as \( \partial_\xi \), and if \( U = \int \) then \( U_s \) acts as \( \int_s \) and \( U_\xi \) accordingly as \( \int_\xi \). Finally, for an evaluation \( U = \epsilon_n \), the action \( U_s \) is \( \epsilon_n s \) and the action \( U_\xi \) is correspondingly \( \epsilon_n \xi \).

**Theorem 29.** Let \((\mathcal{F}, \partial, \int)\) be a strongly ordinary shifted integro-differential algebra and \((T, \mathcal{B})\) a regular Stieltjes boundary problem over \((\mathcal{F}, \partial, \int)\). Then there exists a bivariate distribution \( g(x, \xi) \in \mathcal{D}_2 \mathcal{F} \) such that

\[
T_s g(x, \xi) = \delta(x - \xi),
\beta_\gamma g(x, \xi) = 0 \quad (\beta \in \mathcal{B}).
\]

Moreover, this \( g(x, \xi) \) coincides with the Green’s function of Theorem 26.

**Proof.** With \( G \) the Green’s operator of the boundary problem \((T, \mathcal{B})\), set \( g(x, \xi) := G_{s, \xi} \). For existence it suffices to show that \( g(x, \xi) \) satisfies the distributional boundary problem (38), and we may furthermore assume that \( g(x, \xi) \) is nondegenerate (we may discard any degenerate functions occurring in it since the induced action still represents the same \( G \)). Since \( G \) is the Green’s operator of \((T, \mathcal{B})\), the function \( u := Gf \in \mathcal{F} \) satisfies

\[
f(x) = T_s u(x) = T_s \int_s^\beta g(x, \xi) f(\xi) = \int_s^\beta \left( T_s g(x, \xi) \right) f(\xi) \quad \text{on } [\alpha, \beta],
\]

where the second step follows from Theorem 26 and the last step from the fact that \( \partial_s \) and all \( g(x) \in \mathcal{F}_s \) commute with \( \int_s \) and the evaluations \( \epsilon_n s, \epsilon_n \xi \). Note that \( T_s g(x, \xi) \) is still nondegenerate since \( T_s \) does not affect the functions of \( \mathcal{F}_s \). Hence we may apply Proposition 28 to (39) to obtain \( T_s g(x, \xi) = \delta(x - \xi) \), which is the first line of (38). For verifying the second line, take any \( \beta \in \mathcal{B} \). Again we have \( \beta(u) = 0 \) since \( G \) is the Green’s operator of \((T, \mathcal{B})\). But then \( 0 = \beta_\gamma \int_s^\beta g(x, \xi) f(\xi) = \int_s^\beta \left( \beta_\gamma g(x, \xi) \right) f(\xi) \), which implies the second line of (38) since the action of \( \beta_\gamma \) again preserves the nondegeneracy of \( g(x, \xi) \). \( \square \)

Note that the Green’s function \( g(x, \xi) \) of Theorems 26 and 29 is unique on \([\alpha, \beta]\), meaning unique after discarding all degenerate functions \( \delta(\xi) \in \mathcal{F}_\xi \). This is clear since if \( \tilde{g}(x, \xi) \) is another such Green’s function then \( k(x, \xi) \) := \( g(x, \xi) - \tilde{g}(x, \xi) \) would also be nondegenerate but since they induce the same Green’s operator we have \( \int_s^\beta k(x, \xi) f(\xi) = 0 \) for all \( f \in \mathcal{F} \), and this implies \( k(x, \xi) = 0 \) so \( g(x, \xi) = \tilde{g}(x, \xi) \).

Finally, let us now turn to the last goal (3) outlined at the opening of this section. It is relatively easy to achieve using the tools we have now at hand. If we have computed a Green’s operator in the usual setting of the ordinary shifted integro-differential algebra \((\mathcal{F}, \partial, \int)\), we may immediately apply it to a piecewise forcing function by restricting the action defined above to \( G_s : \mathcal{P}_s \mathcal{F} \to \mathcal{P}_s \mathcal{F} \). In this case, however, we should restrict ourselves to well-posed boundary problems so that \( g(x, \xi) \in \mathcal{P}_2 \mathcal{F} \) by Proposition 27. Otherwise the Green’s function \( g(x, \xi) \)
would contain Diracs whose multiplication with the Heavisides of \( f(\xi) \in \mathcal{P}_4 \mathcal{F} \) in Theorem 26 is undefined.\(^{17}\)

With these reservations in mind, we can now make a simple but precise statement about piecewise forcing functions. The basic message is that we can use essentially the same method as for the usual forcing functions taken from the ground algebra \( \mathcal{F} \). In particular, *existence and uniqueness* go through unscathed.

**Proposition 30.** Let \( (\mathcal{F}, \partial, \{ \} ) \) be a strongly ordinary shifted integro-differential algebra and let \( (T, \mathcal{B}) \) be a well-posed Stieltjes boundary problem. Then (4) admits exactly one solution \( u \in \mathcal{P} \mathcal{F} \) for any given forcing function \( f \in \mathcal{P} \mathcal{F} \). If \( G \in \mathcal{F}_0[\partial, \{ \} ] \) is the corresponding Green’s operator with Green’s function \( g(x, \xi) \), we can compute the solution either via \( u = Gf \) or via \( u(x) = \int_0^1 g(x, \xi) f(\xi) \).

**Proof.** From (Rosenkranz, Serwa, 2015, Lem. 2) we know that \( G \in \mathcal{F}[\{g\}] \), so the action of \( G \) involves only integral operators and multiplication by elements of \( \mathcal{F} \). But this means that the correctness proof for Green’s operators (Rosenkranz Regensburger, 2008, Thm. 26) is applicable, and all required reduction rules for the operator ring \( \mathcal{F}_0[\partial, \{ \} ] \) are valid on \( \mathcal{P} \mathcal{F} \). In fact, if we accept the derivation of Proposition 6, the entire action on \( \mathcal{P} \mathcal{F} \) would be well-defined except for the \( \int \partial \) rule of (Rosenkranz Regensburger, 2008, Table 1), which breaks down because the strong Rota-Baxter axiom does not hold in \( \mathcal{P} \mathcal{F} \). While even this could repaired by constructing the differential Rota-Baxter operator ring (Gao, Guo, Rosenkranz, 2015, §4) instead of the usual integro-differential operator ring, we do not need this here since no differential operators are involved in computing \( u = Gf \in \mathcal{P} \mathcal{F} \).

This settles the question of existence. For proving uniqueness, it is sufficient to show that the homogeneous problem (4) with \( f = 0 \) has only the trivial solution \( u \in \mathcal{D} \mathcal{F} \). But we know that \( \mathcal{D} \mathcal{F} = \mathcal{F} \oplus \mathcal{D}^0 \mathcal{F} \) as differential \( K \)-vector spaces (i.e. differential modules over the ground field \( K \)), temporarily setting \( \mathcal{D}^0 \mathcal{F} := (\mathcal{D} \mathcal{F} \setminus \mathcal{F}) \cup \{0\} \). Moreover, the derivation \( \partial : \mathcal{D} \mathcal{F} \rightarrow \mathcal{D}^0 \mathcal{F} \) respects the filtration outlined earlier (at the end of Section 5). Therefore \( Tu = 0 \) implies \( u \in \mathcal{F} \), and this in turn implies \( u = 0 \) since \( (T, \mathcal{B}) \) was assumed to be a regular boundary problem over \( \mathcal{F} \).

Finally, note that Theorem 26 is still valid when restricted to Green’s operators of well-posed boundary problems (hence the unnecessary—and now invalid—cases for the first and third row in Table 1 can be removed). This can be seen by a straightforward generalization of the computations in the proof of Theorem 26 (for the second and fourth case). \( \square \)

Our treatment of forcing functions in \( \mathcal{P} \mathcal{F} \) includes the classical case of piecewise smooth functions by using the standard example \( \mathcal{F} = C^\infty(\mathbb{R}) \). With the reservations made above (cf. Footnote 15), this includes in particular the *calculus for functions with jumps* outlined in Example 11 of (Stakgold, Holst, 2011, §2.1).

7. Conclusion

Our algebraic treatment of *piecewise functions and distributions* is no more than a starting point. Future work might also consider the two constructions in separate developments. Indeed,\(^{17}\) Also the operator interpretation is at best dubious in this case: The Diracs \( \delta^{(k)}(\xi - \alpha) \) engender evaluations \( \delta_\alpha \delta \) whose action on Heavisides is questionable. This reflects the problematic nature of a boundary problem constraining derivative “values” for solutions that will be distributional. Only the borderline case of piecewise solutions—obtaining when the order of the boundary conditions reaches but does not exceed the order of the differential equation—may still make sense when interpreted with caution.

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we have pointed out in Remark 13 that the multiplicative structure exported from the piecewise extension \( \mathcal{F} \) is independent of the other structures on the distribution module \( \mathcal{D} \); we might impose any product whatsoever. While this might be construed as a weakness of the algebraic approach, it clarifies at least the complementary character of the Diracs \( \delta_a \) and the Heavisides \( H_a \): While the multiplication of the latter reflects an order structure in the ground field, the former encode point evaluations without any relation to the order. The only link between the two structures is the defining relation \( H'_a = \delta_a \).

A more ambitious treatment would also allow piecewise continuous coefficients of the differential operator \( T \); this is what is typically encountered in interface problems (Stakgold, Holst, 2011, §1.4). However, it would be difficult to accommodate such a case directly into our present approach since generalizing \( \mathcal{F} \) to be a differential Rota-Baxter (rather than an integro-differential) algebra entails the loss of the strong Rota-Baxter axiom (2). In that case, Green’s operators/functions cannot be computed as usual (at least it needs a different justification).

We have constructed bivariate distributions only in so far as needed for describing Green’s functions (cf. Remark 12). It would be very interesting, and highly important for practical applications in LPDE problems, to generalize the present algebraic approach to the (truly) multivariate distributions. In particular, the LPDE analog of the distributional differential equation in (38), without the boundary conditions, is a crucial tool for the analytic treatment of LPDE, known as the fundamental solution \( \Psi \). For example in the Laplace equation with \( T = -\Delta = -\partial^2_x - \partial^2_y \) one finds \( \Psi(x, y; \xi, \eta) = -\log \sqrt{(x-\xi)^2 + (y-\eta)^2} / 2\pi \).

On another note, one may also contemplate substitution of functions in distributions from an algebraic viewpoint (in the multivariate case this would subsume cases such as the diagonal distribution introduced in Section 5). Analysis tells us the key relation \( \delta(f(x)) = \delta(x-z)/|f'(z)| \) if \( f \) is suitably regular and has one simple root \( z \in \mathbb{R} \) within the domain of consideration. However, it is not clear at this point in how far such a relation can be mapped to an algebraic setting unless one has a suitable algebraic treatment of composing functions with each other. Not much seems to be available in terms of general settings (as far as we are aware), apart from some promising new developments like (Robertz, 2014, §3.3). Future work might bring up some interesting new connections.

Acknowledgment. This work was supported by the Austrian Science Fund under the FWF Grant No. P30052. We thank the anonymous referees for their valuable hints.

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