Narayana polynomials and Hall-Littlewood symmetric functions

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Abstract

We show that Narayana polynomials are a specialization of row Hall-Littlewood symmetric functions. Using \( \lambda \)-ring calculus, we generalize to Narayana polynomials the formulas of Koshy and Jonah for Catalan numbers.

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1 Introduction

There are many possible \( q \)-generalizations of Catalan numbers [20, exercise 34]. In this paper we shall deal with the following one, perhaps less popular than others.

Let \( q \) be an indeterminate. For any nonnegative integer \( n \), we denote by \( C_n(q) \) the polynomial in \( q \) defined by \( C_0(q) = 1 \) and the recurrence formulas

\[
C_n(q) = (1 - q)C_{n-1}(q) + q \sum_{i=0}^{n-1} C_i(q)C_{n-i-1}(q)
\]

\[
= (1 + q)C_{n-1}(q) + q \sum_{i=1}^{n-2} C_i(q)C_{n-i-1}(q),
\]

the second relation valid for \( n \geq 3 \). Clearly we have \( C_n(0) = 1, C_n(1) = C_n \), the ordinary Catalan number, and \( C_n(2) = s_n \), the small Schröder number [7]. The first values of \( C_n(q) \) are given by

\[
C_1(q) = 1, \quad C_2(q) = q + 1, \quad C_3(q) = q^2 + 3q + 1,
\]

\[
C_4(q) = q^3 + 6q^2 + 6q + 1, \quad C_5(q) = q^4 + 10q^3 + 20q^2 + 10q + 1.
\]
Experts will at once recognize the symmetric distribution of the Narayana numbers \( N(n, k) \) defined by

\[
N(n, k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}.
\]

Using known results about generating functions \([20, 23]\), we have

\[
C_n(q) = \sum_{k=1}^{n} N(n, k)q^{k-1},
\]

the Narayana polynomial. It is sometimes more simple to consider the “large” Narayana polynomial, defined by \( C_0(q) = 1 \) and \( C_n(q) = q C_{n+1}(q) \) for \( n \geq 1 \). Then we have \( C_n(2) = R_n \), the large Schröder number \([7]\).

There is a rich combinatorial literature on this subject. Here we shall only refer to \([19, 22]\) and references therein. The purpose of this paper is to show that \( C_n(q) \) has a deep connection with the theory of symmetric functions.

Let \( P_k(x; q) \) denote the one-row Hall-Littlewood symmetric function associated with \( q \) \([13\text{ Chapter 3}]\). It is known that \( P_k(q) \) interpolates between the power-sum \( p_k \) and the complete symmetric function \( h_k \), namely

\[
P_k(0) = h_k, \quad P_k(1) = p_k.
\]

We denote by \( P_k(1^m; q) \) the value of \( P_k(x; q) \) taken at the \( m \)-vector \( x = (1, \ldots, 1) \). In Section 3, we prove

\[
C_n(1 - q) = \frac{1}{n+1} P_n(1^{n+1}; q).
\]

This result might be obtained by using classical properties of symmetric functions \([18]\). However it is more interesting to prove it in the framework of \( \lambda \)-rings.

The powerful formalism of \( \lambda \)-rings not only allows a compact presentation. It gives a quick access to deep properties, which would be much more difficult to get otherwise. We shall illustrate this efficiency through several examples.

A recursive formula for Catalan numbers is given by Koshy as follows \([9\text{ p.322}]\)

\[
C_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k}.
\]

In Section 4, we generalize this relation to Narayana polynomials as

\[
C_n(q) = (1 - q)^{n-1} + q \sum_{k=1}^{n-1} C_{n-k}(q) \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} (1 - q)^{k-m-1}.
\]

Similarly in Section 5, we extend Jonah’s formula for Catalan numbers \([9\text{ p.325}]\)

\[
\sum_{k=0}^{r} \binom{n-2k}{r-k} C_k = \binom{n+1}{r}.
\]
in two ways. The first one is given by
\[
C_r(q) + q \sum_{k=1}^{r-1} C_{r-k}(q) \sum_{m=0}^{k-1} (q-1)^m \binom{k-1}{m} \binom{n-2r+2k-m}{k} = \sum_{m=0}^{r-1} (q-1)^m \binom{r-1}{m} \binom{n-m}{r-1}.
\]

The second generalization is given for large Narayana polynomials and may be written as
\[
\sum_{k=0}^{r} C_k(q) \sum_{m=0}^{r-k} (1-q)^m \binom{n-2k-m}{r-k-m} \binom{k+m}{m} = \binom{n+1}{r}.
\]

Finally in Section 6, we determine the transition matrix between the large Narayana polynomials corresponding to two variables \(q\) and \(q'\), namely
\[
(n+1)C_n(q') = \sum_{k=0}^{n} C_{n-k}(q) \sum_{i+j=0}^{k} (1-q)^i(q' - 1)^j \binom{n-k+i}{i} \binom{n+1}{j} \binom{2k-i-j-1}{k-i-j}.
\]

We emphasize that these four examples are easy consequences of the same elementary \(\lambda\)-ring identity. It would be very interesting to obtain a combinatorial interpretation of these algebraic results.

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## 2 Some identities

We consider the generating function
\[
C_q(u) = 1 + \sum_{r \geq 1} C_r(q) u^r.
\]

By definition it satisfies the relation
\[
(C_q(u) - 1)/u = (1-q)C_q(u) + q C_q(u)^2,
\]

namely
\[
qu C_q(u)^2 + (u(1-q) - 1)C_q(u) + 1 = 0.
\]

The only regular solution at \(u = 0\) is
\[
C_q(u) = \frac{1 - (1-q)u - \sqrt{(1-q)^2u^2 - 2(1+q)u + 1}}{2qu}.
\]
It is proved in [23] that this solution is given by
\[ C_q(u) = \sum_{n \geq 0, k \geq 0} N(n, k + 1) u^n q^k. \]

Another proof may be found in [20, exercise 36], which gives
\[ \sum_{n \geq 1, k \geq 1} N(n, k) u^n q^k = q (C_q(u) - 1). \]

Therefore \( C_r(q) \) is the Narayana polynomial
\[ C_r(q) = \sum_{k=1}^{r} \frac{1}{r} \binom{r}{k-1} \binom{r}{k} q^{k-1}. \]

In this section, we shall derive some alternative expressions of \( C_r(q) \) in a unified way. The latter are very well known, but our proof is quite elementary. It uses the binomial expansion
\[ (1 + a)^{1/2} = 1 - 2 \sum_{i \geq 0} \binom{2i}{i} \left(\frac{-1}{4}\right)^{i+1} \frac{a^{i+1}}{i+1}, \]
together with four possible ways of writing \((1 - q)^2 u^2 - 2(1 + q) u + 1\).

Actually with \( \eta \) and \( \zeta \) equal to \( \pm 1 \), we have
\[(1 - q)^2 u^2 - 2(1 + q) u + 1 = (1 + (\eta + \zeta q) u)^2 \left(1 - 2u \frac{1 + \eta + q(1 + \zeta) + qu(1 + \eta \zeta)}{(1 + (\eta + \zeta q) u)^2}\right),\]
which yields
\[ C_q(u) + \frac{1 + \eta - q(1 - \zeta)}{2q} = q^{-1} \sum_{i \geq 0} \binom{2i}{i} 2^{-i-1} \frac{u^i}{i+1} \left(1 + \eta + q(1 + \zeta) + qu(1 + \eta \zeta)\right)^{i+1} \left(1 + (\eta + \zeta q) u\right)^{-2i-1}. \]

Expanding two times the right-hand side by the binomial formula, it becomes
\[ q^{-1} \sum_{i,j,k \geq 0} \binom{2i}{i} 2^{-i-1} \frac{u^{2i+j-k+1}}{i+1} \binom{i+1}{k} \binom{1+\eta+q(1+\zeta)k}{1+\eta+q(1+\zeta)}(q(1+\eta \zeta))^{i+1-k} (-\eta - \zeta q)^j \binom{2i+j}{j}. \]

This implies
\[ C_r(q) = \sum_{i,j \geq 0} 2^{-i-1} q^{-i-j-1} (1 + \eta + q(1 + \zeta))^{2i+j-r+1} (1 + \eta \zeta)^{r-i-j} (-\eta - \zeta q)^j \binom{i+1}{r-i-j} \binom{2i+j}{j} \frac{1}{i+1} \binom{2i}{i}. \quad (3) \]
In a second step we specialize $\eta$ and $\zeta$ to $\pm 1$. For $\eta = -\zeta = 1$ we obtain

$$C_r(q) = q C_r(q) = \frac{1}{r+1} \sum_{m=0}^{r} (q-1)^m \binom{r+1}{m} \binom{2r-m}{r}. \quad (4)$$

Since $C_r(q)$ is a polynomial in $q$, the right-hand side is divisible by $q$, namely

$$\sum_{m=0}^{r} (-1)^m \binom{r+1}{m} \binom{2r-m}{r} = 0.$$  

Hence (4) can be alternatively written as

$$C_r(q) = \frac{1}{r+1} \sum_{m=0}^{r-1} (-1)^{m-1} \frac{1 - (1-q)^m}{1 - (1-q)} \binom{r+1}{m} \binom{2r-m}{r}. \quad (5)$$

If we apply the identity

$$\sum_{m=k+1}^{r} (-1)^{m-1} \binom{r+1}{m} \binom{2r-m}{r} = (-1)^k \binom{r-1}{k} \binom{2r-k}{r},$$

proved by induction on $k$, we obtain the equivalence of (5) with

$$C_r(q) = \frac{1}{r+1} \sum_{m=0}^{r-1} (q-1)^m \binom{r-1}{m} \binom{2r-m}{r}. \quad (6)$$

For $\eta = -\zeta = -1$ relation (3) may be written as

$$C_r(q) = \sum_{m=0}^{r} q^m (1-q)^{r-m} \binom{r+m}{2m} C_m. \quad (7)$$

For $\eta = \zeta = -1$ it yields

$$C_r(q) = \sum_{m=0}^{r} q^m (q+1)^{r-2m-1} \binom{r-1}{2m} C_m. \quad (8)$$

Finally for $\eta = \zeta = 1$ we obtain

$$C_r(q) = \sum_{i,j \geq 0} (-1)^i q^{-i-j-1} (q+1)^{2i+j-r+1} \binom{i+1}{r-i-j} \binom{2i+j}{j} C_i.$$  

Comparing with (3), by identification of the coefficients of $q^m(q+1)^{r-2m-1}$, and writing $r = m + n + 1$, we get the interesting identity

$$\sum_{i=m}^{n} (-1)^{n-i} \binom{n+i}{2i} \binom{i+1}{m+1} C_i = \binom{m+n}{2m} C_m.$$
A direct proof might be obtained by using
\[ \sum_{i=m}^{n} (-1)^{n-i} \binom{n+i}{i-m} \binom{n}{i} = \binom{n}{m}, \]
a variant of the Chu-Vandermonde identity. The former identity written for \( m = n - 1 \) or \( m = n - 2 \) gives the classical recursive formula
\[ C_{r+1} = 2 \frac{2r + 1}{r + 2} C_r. \]

Relations (4) and (8) are respectively (1.3) and (1.1) of [19], where references to other proofs are given. All identities are trivial for \( q = 1 \), except (8) which gives a quick proof of the celebrated Touchard’s identity [9, p. 319]
\[ C_r = \sum_{m \geq 0} 2^{r-2m-1} \binom{r-1}{2m} C_m. \]

3 A \( \lambda \)-ring exercise

Here we only give a short survey of \( \lambda \)-ring theory. More details and other applications may be found in [11, 12, 14], or (but not explicitly) in some examples of [18] (see pp. 25, 43, 65 and 79).

Let \( S \) denote the ring of symmetric functions. The classical bases of elementary functions \( e_k \), complete functions \( h_k \), and power sums \( p_k \) generate \( S \) algebraically. Schur functions \( s_\mu \), monomial symmetric functions \( m_\mu \), and products \( e_\mu, h_\mu, p_\mu \) are linear bases of \( S \), indexed by partitions.

Let \( X = \{x_1, x_2, x_3, \ldots \} \) be a (finite or infinite) set of independent indeterminates (an “alphabet”). We define an action \( f \to f[\cdot] \) of \( S \) on the ring \( \mathbb{R}[X] \) of polynomials in \( X \) with real coefficients. Since the power sums generate \( S \), it is enough to define the action of \( p_k \) on \( \mathbb{R}[X] \). Writing any polynomial as \( \sum_{c,P} cP \), with \( c \) a “constant” and \( P \) a monomial in “variables”, we define
\[ p_k \left[ \sum_{c,P} cP \right] = \sum_{c,P} cP^k. \]

Of course this action is strongly dependent on the status of any indeterminate, which may be chosen as a “constant” or a “variable”, or a combination of both. Therefore each status must be carefully specified. For instance, if \( Q = 1 - q \) is a variable, we have \( p_k[q] = 1 - Q^k = 1 - (1-q)^k \), but not \( p_k[q] = q^k \). Variables are referred to as “elements of rank 1”.

If all elements of \( X \) are of rank 1, we write \( X^\dagger = \sum_i x_i \). By definition we have
\[ p_k[X^\dagger] = \sum_i x_i^k = p_k(X), \quad \text{hence} \quad f[X^\dagger] = f(X) \]
for any symmetric function $f$. Similarly $f[m] = f(1, \ldots, 1)$, the value of $f$ at the $m$-vector $(1, \ldots, 1)$.

This action of $S$ on $\mathbb{R}[X]$ has two fundamental properties. Firstly for any polynomials $P, Q$ we have

$$h_n[P + Q] = \sum_{k=0}^{n} h_{n-k}[P] h_k[Q].$$

Denoting the generating series by $H_u = \sum_{k \geq 0} u^k h_k$, this can be also written as

$$H_u[P + Q] = H_u[P] H_u[Q], \quad H_u[P - Q] = H_u[P] H_u[Q]^{-1}.$$  \hfill (10)

Secondly we have three “Cauchy formulas”

$$h_n[\sum_{|\mu| = n} z_{\mu}^{-1} p_{\mu} P p_{\mu} Q] = \sum_{|\mu| = n} h_{|\mu|}[P] m_{\mu}[Q] = \sum_{|\mu| = n} s_{\mu}[P] s_{\mu}[Q],$$

where as usual, we denote by $|\mu|$ the weight of a partition, $l(\mu)$ and $\{m_i(\mu), i \geq 1\}$ its length and multiplicities, and $z_{\mu} = \prod_i i^{m_i(\mu)} m_i(\mu)!$.

**Lemma 1.** (i) For a constant $c$ we have

$$p_k[c] = c, \quad h_k[c] = \binom{c+k-1}{k}, \quad e_k[c] = \binom{c}{k}, \quad m_{\mu}[c] = \binom{c}{l(\mu)} \frac{l(\mu)!}{\prod_i m_i(\mu)!}.$$  

(ii) If $q$ is an element of rank 1, we have

$$p_k[q] = h_k[q] = q^k, \quad e_k[q] = q \delta_{k1}.$$  

(iii) If $1-q$ is an element of rank 1 we have

$$p_k[q] = 1 - (1-q)^k, \quad p_k[-q] = (1-q)^k - 1, \quad h_k[q] = (-1)^k e_k[-q] = q, \quad e_k[q] = (-1)^k h_k[-q] = q(q-1)^{k-1}.$$  

Moreover we have $s_{\mu}[q] = 0$ and $s_{\mu}[-q] = 0$, except if the partition $\mu$ is a hook $(a,1^b)$ in which case $s_{\mu}[q] = q(q-1)^b$ and $s_{\mu}[-q] = (-1)^{b+1} q(1-q)^{b-1}$.

**Proof.** Denoting $E_u = \sum_{k \geq 0} u^k e_k$, we have [IS p.25] $E_u = (H_u)^{-1}$ and

$$h_k = \sum_{|\mu| = k} z_{\mu}^{-1} p_{\mu}, \quad e_k = \sum_{|\mu| = k} (-1)^{k-l(\mu)} z_{\mu}^{-1} p_{\mu}.$$
If $c$ is a constant, one has [18, Example 1.2.1, p.26]

$$h_k[c] = \sum_{|\mu|=k} z_{\mu}^{-1} c^{|\mu|} = \binom{c + k - 1}{k}.$$ 

Hence $H_u[c] = (1 - u)^{-c}$ and $E_u[c] = (1+u)^c$, i.e. $e_k[c] = \binom{c}{k}$. The second Cauchy formula implies

$$h_k[c] = \sum_{|\mu|=k} h_{\mu}[1] m_\mu[c] = \sum_{|\mu|=k} m_\mu[c].$$

The value of $m_\mu[c]$ follows since we have

$$\left( \frac{c + k - 1}{k} \right) = \sum_{p=1}^{k} \sum_{k_1 + \ldots + k_p = k} \binom{c}{p},$$

where the second sum is taken on sequences made of $p$ (strictly) positive integers summing to $k$ (compositions of length $p$ and weight $k$).

If $q$ is of rank 1, we have

$$h_k[q] = \sum_{|\mu|=k} z_{\mu}^{-1} q^{|\mu|} = q^k.$$

Hence $H_u[q] = (1 - uq)^{-1}$ and $E_u[q] = 1 + uq$. If $Q = 1 - q$ is of rank 1, using [10] one has

$$H_u[q] = H_u[1 - Q] = H_u[1] H_u(Q)^{-1} = \frac{1 - uQ}{1 - u} = 1 + \frac{uQ}{1 - u},$$

$$E_u[q] = \frac{1 + u}{1 + uQ} = 1 + \frac{uQ}{1 - u(q - 1)}.$$

For the evaluation of $s_\mu[1 - Q]$ and $s_\mu[Q - 1]$, we refer to [11] p.11. They are respectively equal to $(-Q)^b(1 - Q)$ and $(-1)^bQ^a(1 - Q)$ for $\mu = (a, 1^b)$. \hfill \Box

We consider the one-row Hall-Littlewood symmetric function $P_r(X; q)$. As a straightforward consequence of its definition [18, (2.9), p.209], we have $P_0(X; q) = 1$ and for $r \geq 1$,

$$P_r(X; q) = (1 - q)^{-1} g_r(X; q) \quad \text{with} \quad \sum_{k \geq 0} u^k g_k(X; q) = \prod_{i \geq 1} \frac{1 - qux_i}{1 - ux_i}.$$

If $q$ and all indeterminates $x_i$ are elements of rank 1, using [10] we have

$$\prod_{i \geq 1} \frac{1 - qux_i}{1 - ux_i} = H_u[X^\dagger] H_u[qX^\dagger]^{-1} = H_u[(1 - q)X^\dagger].$$

In other words, for $r \geq 1$ we have

$$P_r(X; q) = (1 - q)^{-1} h_r[(1 - q)X^\dagger].$$
**Theorem 1.** For \( r \geq 1 \) we have

\[
P_r(1^n; q) = \sum_{m=0}^{r-1} (-q)^m \binom{r-1}{m} \binom{n + r - m - 1}{r}.
\]

**Proof.** We must compute

\[
P_r(1^n; q) = (1 - q)^{-1} h_r[(1 - q)n],
\]

with \( q \) of rank 1. Writing \( q = 1 - Q \), by Lemma 1 (iii) together with the last Cauchy formula, we have

\[
h_r[Qn] = \sum_{m=0}^{r-1} s_{(r-m,1^m)}[Q] s_{(r-m,1^m)}[n] = \sum_{m=0}^{r-1} (Q - 1)^m s_{(r-m,1^m)}[n].
\]

The assertion is then a consequence of

\[
s_{(a,1^b)}[c] = \binom{a + b - 1}{b} \binom{a + c - 1}{a + b},
\]

for any real number \( c \). This can be proved in many ways, for instance by induction, using Lemma 1 (i) and the Pieri formula

\[
h_a e_b = s_{(a,1^b)} + s_{(a+1,1^{b-1})},
\]

which imply

\[
s_{(a,1^b)}[c] + s_{(a+1,1^{b-1})}[c] = \binom{a + c - 1}{a} \binom{c}{b}.
\]

\( \square \)

We shall give two proofs of the following theorem. The first one relies on an identity given in Section 2. The second proof merely uses the definition of Narayana polynomials.

**Theorem 2.** For \( r \geq 1 \) we have

\[
C_r(1 - q) = \frac{1}{r + 1} P_r(1^{r+1}; q).
\]

Equivalently in \( \lambda \)-ring notation, with \( q \) an element of rank 1, we have

\[
C_r(1 - q) = (1 - q) C_r(1 - q) = \frac{1}{r + 1} h_r[(1 - q)(r + 1)].
\]

**First proof.** Immediate consequence of (6) and Theorem 1. \( \square \)
Second proof. Up to now, we have only used the third Cauchy formula. But we may also use the second one, which writes as

\[ P_r(1^{r+1}; q) = (1 - q)^{-1}h_r[(1 - q)(r + 1)] = (1 - q)^{-1} \sum_{|\mu|=r} h_\mu[1 - q]m_\mu[r + 1], \]

with \( q \) of rank 1. Applying Lemma 1 (i) and (iii), we get

\[ P_r(1^{r+1}; q) = \sum_{|\mu|=r} (1 - q)^{l(\mu)-1}\binom{r + 1}{l(\mu)} \frac{l(\mu)!}{\prod_i m_i(\mu)!}. \]

Thus Theorem 2 amounts to prove

\[ C_r(1 - q) = \frac{1}{r + 1} \sum_{|\mu|=r} (1 - q)^{l(\mu)-1}\binom{r + 1}{l(\mu)} \frac{l(\mu)!}{\prod_i m_i(\mu)!}, \]

\[ = \sum_{k=1}^{r} (1 - q)^{k-1}\binom{r}{k-1} \sum_{|\mu|=r \atop l(\mu)=k} \frac{(k - 1)!}{\prod_i m_i(\mu)!}. \]

This is exactly the definition of Narayana polynomials, because one has

\[ \binom{r - 1}{k - 1} = \sum_{|\mu|=r \atop l(\mu)=k} k! \prod_i m_i(\mu)! \].

The latter identity counts the number of sequences made of \( k \) (strictly) positive integers summing to \( r \) (compositions of length \( k \) and weight \( r \)). See [21, p.25] for a proof and [13, Theorem 1, p.461] for a generalization.

As for the first Cauchy formula, with \( q \) of rank 1, it implies

\[ P_r(1^{r+1}; q) = (1 - q)^{-1}h_r[(1 - q)(r + 1)] \]

\[ = (1 - q)^{-1} \sum_{|\mu|=r} z_\mu^{-1}p_\mu[1 - q]p_\mu[r + 1] \]

\[ = (1 - q)^{-1} \sum_{|\mu|=r} z_\mu^{-1}(r + 1)^{l(\mu)} \prod_{i \geq 1}(1 - q^i)^{m_i(\mu)}. \]

By Theorem 2 we obtain

\[ C_r(q) = q C_r(q) = \sum_{|\mu|=r} z_\mu^{-1}(r + 1)^{l(\mu)-1} \prod_{i \geq 1}(1 - (1 - q)^i)^{m_i(\mu)}. \]

This formula is new. It is obvious for \( q = 1 \), since by applying Lemma 1 (i) we have

\[ h_r[r + 1] = \sum_{|\mu|=r} z_\mu^{-1}(r + 1)^{l(\mu)} = \binom{2r}{r} = (r + 1)C_r. \]
For $q = 2$ it yields the following interesting expression for the small Schröder numbers

$$s_r = \sum_{|\mu|=r, \text{all parts odd}} z_\mu^{-1}(2r + 2)^{l(\mu) - 1}.$$  

**Remark on Lagrange involution:** An involution $f \to f^*$ can be defined on $S$ as follows ([13, Example 1.2.24, p. 35], [11, Section 2.4]). Let

$$u = tH_t = \sum_{k \geq 0} t^{k+1} h_k.$$  

Then $t$ can be expressed as a power series in $u$, its compositional inverse, namely

$$t = uH_u^* = \sum_{k \geq 0} u^{k+1} h_k^*.$$  

The map $h_k \to h_k^*$ extends to an involution of $S$, called “Lagrange involution”. Many identities are obtained in this context (see [15, Section 4, p.2236] for a detailed account). In particular for any polynomial $A$ we have [15] (4.10)

$$(r + 1)h_r^*[A] = h_r[-(r + 1)A].$$  

Therefore Theorem 2 can be equivalently written as

$$C_r(1 - q) = (1 - q) C_r(1 - q) = h_r^*[q - 1],$$

with $q$ an element of rank 1.

### 4 A generalization of Koshy’s formula

The following sections present some examples of the efficiency of $\lambda$-ring calculus. We start from a remarkable $\lambda$-ring identity, already mentioned in [15] (4.14), p.2239.

**Lemma 2.** For any polynomial $A$, any real number $z$ and any integer $n \geq 1$ we have

$$\sum_{k=0}^{n} \frac{1}{z+k} h_k[-(z+k)A] h_{n-k}[(z+k)A] = 0.$$  

**Proof.** We apply Lemma 1 (i) and the second Cauchy formula to get

$$h_k[-(z+k)A] = \sum_{|\mu|=k} m_\mu[-z-k] h_\mu[A] = \sum_{|\mu|=k} \binom{-z-k}{l(\mu)} \frac{l(\mu)!}{\prod_i m_i(\mu)!} h_\mu[A],$$

and similarly

$$h_{n-k}[(z+k)A] = \sum_{|\nu|=n-k} \binom{z+k}{l(\nu)} \frac{l(\nu)!}{\prod_i m_i(\nu)!} h_\nu[A].$$
Therefore for any partition $\rho$, it is equivalent to prove
\[
\sum_{\mu \cup \nu = \rho} \frac{l(\mu)!}{\prod_i m_i(\mu)!} \frac{l(\nu)!}{\prod_i m_i(\nu)!} \frac{1}{z + |\mu|} \left( -z - |\mu| \right) \left( -z - |\nu| \right) = 0,
\]
the sum taken over all decompositions of $\rho$ into two partitions (possibly empty). This relation may be written as
\[
\sum_{\mu \cup \nu = \rho} \prod_i \frac{1}{m_i(\mu)!m_i(\nu)!} (-1)^{l(\mu) - 1} \prod_{j=-l(\nu)+1}^{l(\mu)-1} (z + |\mu| + j) = 0.
\]
Equivalently
\[
\sum_{\mu \cup \nu = \rho} \prod_i \frac{m_i(\rho)}{m_i(\mu)} \frac{1}{m_i(\nu)} (-1)^{l(\mu) - 1} \prod_{j=1}^{l(\rho) - 1} (z + |\mu| - l(\nu) + j) = 0.
\]

The proof of this identity is done in two steps. Firstly we observe that we may restrict to partitions $\rho$ having only parts with multiplicity 1, i.e.
\[
\sum_{\mu \cup \nu = \rho} (-1)^{l(\rho)} \prod_{j=1}^{l(\rho) - 1} (z + |\mu| - l(\nu) + j) = 0.
\]
Actually given a partition $\rho = \mu \cup \nu$ having only parts with multiplicity 1, there are $\frac{m_i(\rho)}{m_i(\mu)}$ contributions becoming equal when $m_i(\rho)$ parts get equal to $i$.

In a second step, we observe that the previous identity may be generalized as
\[
\sum_{A \subset E} (-1)^{\text{card } A} \prod_{i=1}^{\text{card } E - 1} (z + |A| - \text{card } E + \text{card } A + i) = 0,
\]
where $E$ is some set of indeterminates and $|A| = \sum_{a \in A} a$. This is equivalent to
\[
\sum_{A \subset E} (-1)^{\text{card } A} \prod_{i=1}^{\text{card } E - 1} (z + |A| + i) = 0,
\]
because we can change $z$ into $z + \text{card } E$, and any $a \in E$ into $a - 1$ so that $|A| + \text{card } A$ is changed into $|A|$. Finally we apply the following lemma with $y_i = z + i$.

\begin{lemma}
Let $E = \{x_1, \ldots, x_n\}$ be a set of $n$ indeterminates. For any indeterminates $y = (y_1, \ldots, y_n)$ we have
\[
\sum_{A \subset E} (-1)^{n - \text{card } A} \prod_{i=1}^{n} (|A| + y_i) = n! \prod_{i=1}^{n} x_i,
\]
\[
\sum_{A \subset E} (-1)^{n - \text{card } A} \prod_{i=1}^{n-1} (|A| + y_i) = 0.
\]
\end{lemma}
Proof. We consider the monomial symmetric functions in the indeterminates \((x_1, \ldots, x_n)\). For any partition \(\mu\) with length \(\leq n - 1\) and weight \(\leq n\) we have
\[
\sum_{A \subset \{x_1, \ldots, x_n\}} (-1)^{n - \text{card } A} m_\mu(A) = 0.
\]
This is proved by induction on \(n\), the sum being evaluated as
\[
\sum_{A \subset \{x_1, \ldots, x_{n-1}\}} (-1)^{n - \text{card } A} m_\mu(A) - \sum_k x_n^k \sum_{A \subset \{x_1, \ldots, x_{n-1}\}} (-1)^{n - \text{card } A} m_{\mu \backslash k}(A),
\]
where \(\mu \backslash k\) denotes the partition obtained by subtracting the part \(k\) of \(\mu\) (if it exists). Since for \(k \leq n - 1\) the function \(e^n_k\) is formed of monomial symmetric functions of weight \(\leq n - 1\), we get
\[
\sum_{A \subset \{x_1, \ldots, x_n\}} (-1)^{n - \text{card } A} |A|^k = 0 \quad (k = 0, \ldots, n - 1).
\]
For \(\mu = 1^n\) we have directly
\[
\sum_{A \subset \{x_1, \ldots, x_n\}} (-1)^{n - \text{card } A} m_{1^n}(A) = m_{1^n}(E) = \prod_{i=1}^n x_i.
\]
Since \(e^n_1 = n! m_{1^n}\) up to monomial symmetric functions of weight \(n\) and length \(\leq n - 1\), we get
\[
\sum_{A \subset \{x_1, \ldots, x_n\}} (-1)^{n - \text{card } A} |A|^n = n! \prod_{i=1}^n x_i.
\]
We conclude by writing the expansions
\[
\prod_{i=1}^n (|A| + y_i) = \sum_{k=0}^n |A|^k e_{n-k}(y),
\]
\[
\prod_{i=1}^{n-1} (|A| + y_i) = \sum_{k=0}^{n-1} |A|^k e_{n-k-1}(y).
\]
in terms of the symmetric functions of \(y\).

Remark: Equation (12) is a generalization (in the framework of partitions) of the classical Rothe identity [5]
\[
\sum_{k=0}^n \frac{x}{x-k} \binom{x-k}{k} \binom{y+k}{n-k} = \binom{x+y}{n},
\]
taken at \(x = -y\).
In view of Lemma 1 (i) we have
\[
\frac{1}{k+1}h_k[k+1] = C_k, \quad h_{n-k}[-(k+1)] = \binom{n-2k-2}{n-k} = (-1)^{n-k}\binom{k+1}{n-k}.
\]
Thus Koshy’s formula [9, p.322]
\[
C_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n-k+1}{k} C_{n-k}
\]
can be written as
\[
\sum_{k=0}^{n} (-1)^{n-k}\binom{k+1}{n-k} C_k = \sum_{k=0}^{n} \frac{1}{k+1} h_k[k+1] h_{n-k}[-(k+1)] = 0,
\]
which is the case \( z = 1, A = -1 \) of Lemma 2. We only need to change \( A = -1 \) into \( A = -q \) to obtain the following generalization.

**Theorem 3.** Narayana polynomials satisfy the recurrence relation
\[
C_n(q) = (1-q)^{n-1} + q \sum_{k=1}^{n-1} C_{n-k}(q) \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} (1-q)^{k-m-1}.
\]

**Proof.** If we apply Lemma 2 with \( z = 1 \) and \( A = -q \), we obtain
\[
\sum_{k=0}^{n} \frac{1}{k+1} h_k[(k+1)q] h_{n-k}[-(k+1)q] = 0.
\]
If we assume \( 1-q \) to be an element of rank 1, Theorem 2 may be written as
\[
\frac{1}{k+1} h_k[(k+1)q] = q C_k(q) \quad (k \geq 1).
\]
Hence the previous relation becomes
\[
q^{-1} h_n[-q] + \sum_{k=1}^{n-1} C_k(q) h_{n-k}[-(k+1)q] + C_n(q) = 0.
\]
Applying Lemma 1 (iii) and the last Cauchy formula, we have
\[
h_{n-k}[(k+1)(-q)] = \sum_{m=0}^{n-k-1} s_{(n-k-m,1,m)}[-q] s_{(n-k-m,1,m)}[k+1]
\]
\[
= \sum_{m=0}^{n-k-1} (-1)^{m+1} q(1-q)^{n-k-m-1} s_{(n-k-m,1,m)}[k+1]
\]
\[
= -q \sum_{m=0}^{n-k-1} (-1)^m (1-q)^{n-k-m-1} \binom{n-k-1}{m} \binom{n-m}{n-k},
\]
where the last equation is a consequence of (11). In particular for \( k = 0 \), we have
\[
h_n[-q] = -q(1-q)^{n-1}. \text{ Changing } k \text{ to } n-k, \text{ we can conclude.} \]
For \( q = 1 \), we recover Koshy’s formula. For \( q = 2 \) we obtain the following recurrence for the small Schröder numbers (which seems to be new)

\[
s_n = (-1)^{n-1} + 2 \sum_{k=1}^{n-1} (-1)^{k-1} s_{n-k} \sum_{m=0}^{k-1} \binom{k-1}{m} \binom{n-m}{k}.
\]

5 Two generalizations of Jonah’s formula

Lemma 2 is the particular case \( B = 0 \) of the following \( \lambda \)-ring identity.

**Lemma 4.** For any polynomials \( A \) and \( B \), any real number \( z \) and any integer \( n \geq 1 \) we have

\[
\sum_{k=0}^{n} \frac{z}{z+k} h_k[-(z+k)A] h_{n-k}[(z+k)A + B] = h_n[B].
\]

**Proof.** Applying (9) we have

\[
h_{n-k}[(z+k)A + B] = \sum_{j=0}^{n-k} h_{n-k-j}[B] h_j[(z+k)A],
\]

which yields

\[
\sum_{k=0}^{n} \frac{z}{z+k} h_k[-(z+k)A] h_{n-k}[(z+k)A + B] = \sum_{i=0}^{n} h_{n-i}[B] \sum_{k=0}^{i} \frac{z}{z+k} h_k[-(z+k)A] h_{i-k}[(z+k)A].
\]

We conclude by applying Lemma 2 for \( i \neq 0 \).

In view of Lemma 1 (i) we have

\[
\frac{1}{k+1} h_k[k+1] = C_k, \quad h_{r-k}[n-r-k+1] = \binom{n-2k}{r-k}, \quad h_r[n-r+2] = \binom{n+1}{r}.
\]

Thus Jonah’s formula [9, p.325]

\[
\sum_{k=0}^{r} \binom{n-2k}{r-k} C_k = \binom{n+1}{r}
\]

can be written as

\[
\sum_{k=0}^{r} \frac{1}{k+1} h_k[k+1] h_{r-k}[n-r-k+1] = h_r[n-r+2].
\]

This is the case \( z = 1, A = -1, B = n-r+2 \) of Lemma 4. Our first generalization only needs to change \( A = -1, B = n-r+2 \) into \( A = -q, B = q(n-r+2) \).
**Theorem 4.** For any positive integers \( n, r \) we have
\[
C_r(q) + q \sum_{k=1}^{r-1} C_{r-k}(q) \sum_{m=0}^{k-1} (q-1)^m \binom{k-1}{m} \binom{n-2r+2k-m}{k} = \sum_{m=0}^{r-1} (q-1)^m \binom{r-1}{m} \binom{n-m}{r-1}.
\]

**Proof.** If we apply Lemma 4 with \( z = 1, B = q(n-r+2) \) and \( A = -q \) we obtain
\[
\sum_{k=0}^{r-1} \frac{1}{k+1} h_k[(k+1)q] h_{r-k}[(n-r-k+1)q] = h_r[(n-r+2)q].
\]
Assuming \( 1-q \) to be an element of rank 1 and applying Theorem 2, this relation becomes
\[
q \sum_{k=1}^{r-1} C_k(q) h_{r-k}[(n-r-k+1)q] + q C_r(q) = h_r[(n-r+2)q] - h_r[(n-r+1)q].
\]
Applying Lemma 1 (iii) and the last Cauchy formula, we have
\[
\begin{align*}
\sum_{m=0}^{r-1} (q-1)^m s_{r-k-m,1} [n-r-k+1] \sum_{m=0}^{r-k-1} s_{r-k-m,1} [n-r-k+1] &= \sum_{m=0}^{r-k-1} q(q-1)^m s_{r-k-m,1} [n-r-k+1] \\
&= q \sum_{m=0}^{r-k-1} (q-1)^m \binom{r-k-1}{m} \binom{n-2k-m}{r-k},
\end{align*}
\]
where the last equation is a consequence of (11). Exactly in the same way we have
\[
\begin{align*}
\sum_{m=0}^{r-1} (q-1)^m r \binom{r-1}{m} \binom{n-m}{r-1} = q \sum_{m=0}^{r-1} (q-1)^m \binom{r-1}{m} \binom{n-m}{r-1}.
\end{align*}
\]
Summing the contributions and changing \( k \) to \( r-k \), we can conclude.

For \( q = 1 \) we recover Jonah’s formula under the form
\[
\sum_{k=1}^{r} \binom{n-2k}{r-k} C_k = \binom{n}{r-1}.
\]
For \( q = 2 \) we obtain the following identity for the small Schröder numbers (which is probably new)
\[
s_r + 2 \sum_{k=1}^{r-1} s_{r-k} \sum_{m=0}^{k-1} \binom{k-1}{m} \binom{n-2r+2k-m}{k} = \sum_{m=0}^{r-1} \binom{r-1}{m} \binom{n-m}{r-1}.
\]
Our second generalization of Jonah’s formula only needs to change \( A = -1, B = n-r+2 \) into \( A = -q, B = n-r+2 \).
Theorem 5. For any positive integers \( n, r \) we have

\[
\sum_{k=0}^{r} C_k(q) \sum_{m=0}^{r-k} (1-q)^m \binom{n-2k-m}{r-k-m} \binom{k+m}{m} = \binom{n+1}{r}.
\]

Proof. If we apply Lemma 4 with \( z = 1, B = n - r + 2 \) and \( A = -q \) we obtain

\[
\sum_{k=0}^{r} \frac{1}{k+1} h_k[(k+1)q] h_{r-k}[(n-r-k+1) + (k+1)(1-q)] = \binom{n+1}{r}.
\]

Assuming \( 1 - q \) to be an element of rank 1 and applying Theorem 2, the left-hand side becomes

\[
h_r[(n-r+1)+1-q] + q \sum_{k=1}^{r} C_k(q) h_{r-k}[(n-r-k+1) + (k+1)(1-q)].
\]

By (9) we have

\[
h_{r-k}[(n-r-k+1) + (k+1)(1-q)] = \sum_{m=0}^{r-k} h_{r-k-m}[n-r-k+1] h_m[(k+1)(1-q)].
\]

By Lemma 1 (i) we get

\[
h_{r-k-m}[n-r-k+1] = \binom{n-2k-m}{r-k-m}.
\]

On the other hand, since \( 1 - q \) is an element of rank 1, we have

\[
H_u[(k+1)(1-q)] = \left(H_u[1-q]\right)^{k+1}
\]

\[
= (1 - u(1-q))^{-k-1}
\]

\[
= \sum_{m \geq 0} u^m (1-q)^m \binom{k+m}{m},
\]

where the second relation is a consequence of Lemma 1 (ii). Finally we obtain

\[
h_{r-k}[(n-r-k+1) + (k+1)(1-q)] = \sum_{m=0}^{r-k} \binom{n-2k-m}{r-k-m} (1-q)^m \binom{k+m}{m}.
\]

For \( q = 2 \) we obtain the following identity for the large Schröder numbers (which seems to be new)

\[
\sum_{k=0}^{r} R_k \sum_{m=0}^{r-k} (-1)^m \binom{n-2k-m}{r-k-m} \binom{k+m}{m} = \binom{n+1}{r}.
\]

For a very different extension of Koshy’s and Jonah’s identities, see [1].
6 Transition matrix

Theorem 6. For any positive integer $n$ we have

$$(n + 1)C_n(q') = \sum_{k=0}^{n} C_{n-k}(q) \sum_{i+j=0}^{k} (1 - q)^i(q' - 1)^j \binom{n-k+i}{i} \binom{n+1}{j} \binom{2k-i-j-1}{k-i-j}. $$

Proof. If we apply Lemma 4 with $z = 1$, $A = -q$ and $B = (n+1)q'$, we obtain

$$\sum_{k=0}^{n} \frac{1}{k+1} h_k[(k+1)q] h_{n-k}[(n-k) + (k+1)(1-q) - (n+1)(1-q')] = h_n[(n+1)q']. $$

Assuming $1 - q$ and $1 - q'$ of rank 1 and applying Theorem 2, the right-hand side is $(n+1)q'C_n(q')$ and the left-hand side becomes

$$h_n[n+(1-q) - (n+1)(1-q')] + q \sum_{k=1}^{n} C_k(q) h_{n-k}[(n-k) + (k+1)(1-q) - (n+1)(1-q')]. $$

By (9) we have

$$h_{n-k}[(n-k) + (k+1)(1-q) - (n+1)(1-q')]$$

$$= \sum_{i+j=0}^{n-k} h_{n-k-i-j}[(n-k) h_i[(k+1)(1-q)] h_j[(-(n+1)(1-q')]]$$

$$= \sum_{i+j=0}^{n-k} \left(2n-2k-i-j-1\right) \left(1-q\right)^i \binom{k+i}{i} \binom{n+1}{j}. $$

Here the last relation is a consequence of Lemma 1 (i), together with (13) and

$$H_u[(-(n+1)(1-q')]] = \left(H_u[1-q']\right)^{-n-1}$$

$$= (1-u(1-q'))^{n+1}$$

$$= \sum_{j \geq 0} u^j(q'-1)^j \binom{n+1}{j}, $$

since $1 - q'$ is an element of rank 1. Changing $k$ to $n-k$, we can conclude.

Making $q = 1$ or $q' = 1$, we obtain

$$(n + 1)C_n(q) = \sum_{k=0}^{n} C_{n-k}(q) \sum_{j=0}^{k} (q-1)^j \binom{n+1}{j} \binom{2k-j-1}{k-j}, $$

$$(n + 1)C_n = \sum_{k=0}^{n} C_{n-k}(q) \sum_{i=0}^{k} (1-q)^i \binom{n-k+i}{i} \binom{2k-i-1}{k-i}. $$

which can be also transformed to known results (respectively (1.3) and (1.4) of [19]). Making \( q = 2 \) or \( q' = 2 \), analogous identities are

\[
(n + 1)C_n(q) = \sum_{k=0}^{n} R_{n-k} \sum_{i+j=0}^{k} (-1)^i (q - 1)^j \binom{n-k+i}{i} \binom{n+1}{j} \binom{2k-i-j-1}{k-i-j},
\]

\[
(n + 1)R_n = \sum_{k=0}^{n} C_{n-k}(q) \sum_{i+j=0}^{k} (1-q)^i \binom{n-k+i}{i} \binom{n+1}{j} \binom{2k-i-j-1}{k-i-j},
\]

which connect Narayana polynomials with the large Schröder numbers.

7 Narayana alphabet

Since the complete symmetric functions are algebraically independent, they may be specialized in any way. Given a family of functions \( \{f_n, n \geq 0\} \), it is possible to write \( f_n = h_n(A) \) for some (at least formal) alphabet \( A \), provided \( f_0 = 1 \). Equivalently the generating function for the \( f_n \)'s is then \( H_u(A) = \sum_{n \geq 0} u^n h_n(A) \).

Let us perform such a specialization for the Narayana polynomials and denote \( A \) the “Narayana alphabet” defined by \( h_n(A) = C_n(q) \). Equivalently we have \( H_u(A) = C_q(u) \). Similarly we denote \( A_1 \) the “Catalan alphabet” defined by \( h_n(A_1) = C_n \) and \( H_u(A_1) = H_u(A)|_{q=1} \).

We may compute some classical bases of symmetric functions for \( A \), for instance the power sums \( \{p_r(A), r \geq 1\} \) or the Schur functions \( s_\mu(A) \).

**Theorem 7.** We have

\[
s_{k\k}(A) = s_{(k-1)k}(A) = (-q)\binom{k}{2}.
\]

**Proof.** Equation (11) implies

\[
1 - \frac{1}{C_q(u)} = u(1 - q + qC_q(u)) = \frac{u}{1 - quC_q(u)}.
\]

Therefore the generating function \( C_q(u) \) may be written as

\[
C_q(u) = \frac{1}{1 - \frac{u}{1 - \frac{qu}{1 - \frac{u}{1 - \ldots}}}},
\]

which yields a very simple expression as the continued fraction [2, Section 3.5]
whose coefficients are alternatively 1 and \( q \). We then apply [11, (5.3.5)] according which, in the expression of \( H_u(A) \) as a continued fraction, the coefficients are given in terms of \( s_k(A) \) and \( s_{(k-1)k}(A) \).

By computer calculations, Lascoux noticed that the power sums \( \{ p_r(A), r \geq 1 \} \) are polynomials in \( q \) with positive integer coefficients. For instance

\[
p_1(A) = 1, \quad p_2(A) = 2q + 1, \quad p_3(A) = 3q^2 + 6q + 1, \\
p_4(A) = 4q^3 + 18q^2 + 12q + 1, \quad p_5(A) = 5q^4 + 40q^3 + 60q^2 + 20q + 1.
\]

**Theorem 8.** The power sum \( p_r(A) \) is given by

\[
p_r(A) = \sum_{k=0}^{r-1} \binom{r-1}{k} \binom{r}{k} q^k.
\]

**Proof.** By a classical formula [18, p. 23], we have

\[
\sum_{r \geq 1} p_r(A) u^{r-1} = \frac{d/du \ C_q(u)}{C_q(u)}.
\]

Using (2), by explicit computation the right-hand side is

\[
\frac{1}{\sqrt{(1-q)^2 u^2 - 2(1+q)u + 1}} \left( \frac{1}{u} - \frac{2q}{(1 - (1-q)u - \sqrt{(1-q)^2 u^2 - 2(1+q)u + 1})} \right).
\]

In view of (14), this can be transformed into

\[
\frac{1}{u \sqrt{(1-q)^2 u^2 - 2(1+q)u + 1}} \left( 1 - \frac{1}{C_q(u)} \right) = \frac{1 + q(C_q(u) - 1)}{\sqrt{(1-q)^2 u^2 - 2(1+q)u + 1}}
\]

\[
= -\frac{1}{2u} \left( 1 - \frac{1 + (1-q)u}{\sqrt{(1-q)^2 u^2 - 2(1+q)u + 1}} \right)
\]

\[
= -\frac{1}{2u} \left( 1 - \left( 1 - \frac{4u}{(1 + (1-q)u)^{1/2}} \right)^{-1/2} \right).
\]

Using

\[
(1 - a)^{-1/2} = 1 + 2 \sum_{i \geq 1} \binom{2i - 1}{i} \left( \frac{a}{4} \right)^i,
\]

followed by the binomial expansion of \( (1 + (1-q)u)^{-2i} \), we get

\[
p_r(A) = \sum_{k=1}^{r} \binom{2k-1}{k} \binom{r+k-1}{r-k} (q-1)^{r-k}
\]

\[
= \sum_{k=1}^{r} \binom{2k-1}{k} \binom{r+k-1}{r-k} \sum_{m=0}^{r-k} (-1)^{r-k-m} \binom{r-k}{m} q^m.
\]
Therefore it remains to prove
\[
\binom{r-1}{m} \binom{r}{m} = \sum_{k=1}^{r} \binom{2k-1}{k} \binom{r+k-1}{r-k} (-1)^{r-k-m} \binom{r-k}{m}.
\]
Since we have
\[
\binom{2k-1}{k} \binom{r+k-1}{r-k} \binom{r-k}{m} = \binom{r}{m} \binom{r+k-1}{r} \binom{r-m}{k},
\]
this is equivalent to
\[
\binom{r-1}{m} = \sum_{k=1}^{r} \binom{r+k-1}{r} \binom{r-m}{k} (-1)^{r-k-m}
\]
\[
= \sum_{k=1}^{r} \binom{r+k-1}{r} \binom{-k-1}{r-m-k}
\]
\[
= \sum_{k=1}^{r} \binom{r+k-1}{r} \binom{-k-1}{-r+m-1},
\]
a variant of the Chu-Vandermonde identity. □

On the other hand, by induction, the power sum \( p_r(A_1) \) may be independently shown to be
\[
p_r(A_1) = \binom{2r-1}{r-1}.
\]
Actually the classical formula [18, p. 23]
\[
 nh_n = \sum_{r=0}^{n-1} h_r p_{n-r}
\]
written for the alphabet \( A_1 \) gives
\[
\frac{n}{n+1} \binom{2n}{n} = \sum_{r=0}^{n-1} \frac{1}{r+1} h_r[r+1] h_{n-r-1}[n-r+1] = h_{n-1}[n+2] = \binom{2n}{n-1}.
\]
This is obtained by using Lemma 4 written with \( z = 1, A = -1, B = n + 2 \) together with three applications of Lemme 1 (i), namely
\[
h_r(A_1) = \frac{1}{r+1} h_r[r+1], \quad p_{n-r}(A_1) = h_{n-r-1}[n-r+1], \quad h_{n-1}[n+2] = \binom{2n}{n-1}.
\]

In other words, in the same way than Narayana polynomials \( C_r(q) \) are a \( q \)-refinement of Catalan numbers \( C_r = C_r(1) \) because
\[
\sum_{k=1}^{r} N(r, k) = C_r,
\]
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the polynomials $p_r(A)$ are a refinement of $p_r(A_1) = p_r(A)|_{q=1}$, because

$$
\sum_{m=0}^{r-1} \binom{r-1}{m} \binom{r}{m} = \binom{2r-1}{r-1}.
$$

Moreover Lascoux also noticed, by computer calculations, that up to a sign, the Schur functions $s_\mu(A)$ are polynomials in $q$ with positive integer coefficients. For instance

$$
s_6 = q^5 + 15q^4 + 50q^3 + 50q^2 + 15q + 1, \quad -s_{51} = q^5 + 14q^4 + 40q^3 + 30q^2 + 5q,
$$

$$
-s_{42} = 3q^3 + 8q^2 + 3q, \quad s_{411} = q^5 + 13q^4 + 34q^3 + 24q^2 + 4q,
$$

$$
-s_{33} = q^3 + q^2 + q, \quad s_{321} = q^5 + 7q^2 + 2q,
$$

$$
-s_{213} = q^5 + 12q^4 + 30q^3 + 20q^2 + 3q, \quad -s_{23} = q^3, \quad -s_{222} = 3q^3 + 5q^2 + q,
$$

$$
s_{212} = q^5 + 11q^4 + 26q^3 + 16q^2 + 2q, \quad -s_{16} = q^5 + 10q^4 + 20q^3 + 10q^2 + q.
$$

It would be interesting to investigate this property, which might perhaps be interpreted in terms of some statistics.

## 8 Final remarks

Let $P_n^{(1,1)}$ denote the classical Jacobi (actually Gegenbauer) polynomial defined by

$$
P_n^{(1,1)}(x) = \sum_{m=0}^{n} \binom{n+1}{m} \binom{n+1}{n-m} \left( \frac{x-1}{2} \right)^{n-m} \left( \frac{x+1}{2} \right)^m.
$$

In a recent paper [10] it is proved that one has

$$
C_n(x) = \frac{(x-1)^{n-1}}{n} P_{n-1}^{(1,1)} \left( \frac{x+1}{x-1} \right).
$$

Taking into account Theorem 2, this yields

$$
P_n^{(1^\nu+1);q} = \frac{n+1}{n} (-q)^{n-1} P_{n-1}^{(1,1)} (1 - 2/q).
$$

This specialization of Hall-Littlewood polynomials seems to be new. In view of Theorem 1, it amounts to the identity

$$
\sum_{m=0}^{n-1} \binom{n-1}{m} \binom{2n-m}{n} (-q)^m = \sum_{m=0}^{n-1} \binom{n+1}{m+1} \binom{n-1}{m} (1-q)^m,
$$

which is a consequence of the Chu-Vandermonde formula.

Our second remark is devoted to generalized Narayana numbers, which have been introduced in [6, Section 5.2] in the context of the non-crossing partition lattice for the reflection group associated with a root system. Ordinary Narayana polynomials correspond to a root system of type $A$. 

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For a root system of type $B$, generalized Narayana polynomials are defined \cite[Example 5.8]{ref} by $W_0(z) = 1$ and
\[
W_r(z) = \sum_{k=0}^{r} \binom{r}{k}^2 z^k.
\]
For their combinatorial study we refer to \cite{ref1, ref2} and references therein. We have $W_r(1) = W_r$, the central binomial coefficient, since
\[
W_r = \binom{2r}{r} = \sum_{k=0}^{r} \binom{r}{k}^2.
\]
Moreover \cite[equation (2.1)]{ref3} the Narayana polynomial $W_r(z)$ can be expressed in terms of central binomial coefficients as
\[
W_r(z) = \sum_{m \geq 0} z^m (z + 1)^{r-2m} \binom{r}{2m} W_m.
\]
It is an open problem whether the polynomials $W_r(z)$ can be obtained by specialization of some classical symmetric function. The Hall-Littlewood polynomial of type $B$ \cite{ref4} might be a good candidate.

Finally Christian Stump (private communication) pointed out the existence of a combinatorial proof of Theorem 2, which is sketched below.

Using the description of Hall-Littlewood polynomials given in \cite{ref5} (see Definitions 2.1 – 2.2 and Theorem 2.3 of \cite{ref6}), it is known that
\[
P_n(1^{n+1}; q) = \sum_{w} (1 - q)^{\text{strinc}(w)}.
\]
Here the sum is taken over all weakly increasing sequences $w$ having length $n$ and entries bounded by $n + 1$, and $\text{strinc}(w)$ is the number of strictly increasing positions of $w$.

For instance for $n = 3$ such sequences are

\begin{verbatim}
111 112 113 114 122 123 124 133 134 144
222 223 224 233 234 244
333 334 344
444
\end{verbatim}

and the corresponding strinc statistics are

\begin{verbatim}
0 1 1 1 1 2 2 1 2 1
0 1 1 1 2 1
0 1 1
0
\end{verbatim}

hence we have $P_3(1^4; q) = 4 + 12(1 - q) + 4(1 - q)^2$.

Theorem 2 is a consequence of a bijection between such sequences and Grand-Dyck paths counted by double rises. Actually it is well known that $\binom{2n}{n}$ is the number of Grand-Dyck paths of semi-length $n$ and $(n + 1)N(n, k)$ is the number of Grand-Dyck paths of semi-length $n$ that have $k - 1$ double rises.
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