A Generalized Bohr–Rogosinski Phenomenon

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Abstract
In this paper, we study a generalized Bohr–Rogosinski sum for the Ma-Minda classes of starlike and convex functions. The phenomenon is also explored for the classes of starlike functions with respect to symmetric and conjugate points along with their convex cases. Further, the connection between the derived results and the known ones are established with the suitable examples.

Keywords Subordination · Radius problem · Starlike and convex functions · Bohr radius · Bohr–Rogosinski radius

Mathematics Subject Classification 30C45 · 30C50 · 30C80

1 Introduction

Let $A$ denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disk $D := \{z : |z| < 1\}$. We denote by $S$, the subclass of $A$ of univalent functions. Using subordination (Miller and Mocanu 2000), Ma and Minda (1992) (also see Kumar and Gangania 2021) introduced the unified class of univalent starlike and convex functions defined as follows:

$$S^*(\psi) := \left\{ f \in A : \frac{zf''(z)}{f'(z)} < \psi(z) \right\}$$

and $C(\psi) := \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} < \psi(z) \right\}$, where $\psi$ is univalent, analytic and starlike with respect to 1 with $\Re\psi(z) > 0$, $\psi'(0) > 0$, $\psi(0) = 1$ and $\psi(D)$ is symmetric about real axis. Note that $\psi \in P$, the class of Carathéodory functions. Also when $\psi(z) = (1 + z)/(1 - z)$, $S^*(\psi)$ and $C(\psi)$ reduce to the standard classes $S^*$ and $C$ of univalent starlike and convex functions, respectively.

In GFT, radius problems have a long and glorious history that continues to this day. For the recent developments involving the Ma-Minda classes, see the articles Gangania and Kumar (2022a, 2022c, 2022d), Janowski (1970/71), Kumar and Kamaljeet (2021), Kumar and Gangania (2021) and Wang et al. (2022, 2021). In 1914, Bohr (1914) proved a significant radius problem for the absolute power series:

**Theorem 1 (Bohr’s Theorem, Bohr 1914)** Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in $D$ and $|g(z)| < 1$ for all $z \in D$, then $\sum_{k=0}^{\infty} |a_k||z|^k \leq 1$, for $|z| \leq 1/3$.

In analogy with Bohr’s Theorem, there is also the notion of Rogosinski radius; however, a little is known about Rogosinski radius (also see Landau and Gaier 1986; Rogosinski 1923; Schur and Szegö 1925) as compared to Bohr radius, which is defined as follows:

**Theorem 2 (Rogosinski Theorem)** If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ with $|g(z)| < 1$, then for every $N \geq 1$, we have $|\sum_{k=0}^{N-1} b_k z^k| \leq 1$, for $|z| \leq \frac{1}{2}$. The radius $1/2$ is called the Rogosinski radius.

Recently, Kumar and Sahoo (2022) obtained the generalized classical Bohr’s Theorem for functions satisfying $\Re f(z) < 1$. Also see, Kayumov et al. (2022).

**Theorem 3 (Kumar and Sahoo 2022, Theorem 2.2)** Let $\left\{ \nu_k(r) \right\}_{k=0}^{\infty}$ be a sequence of non-negative continuous...
functions in \([0, 1]\) such that the series \(v_0(r) + \sum_{k=1}^{\infty} v_k(r)\) converges locally uniformly with respect to \(r \in [0, 1]\). Let \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) with \(\Re f(z) < 1\) and \(p \in (0, 1]\). If
\[
v_0(r) = \frac{2(1 + r^m)}{p(1 - r^m)} \sum_{k=1}^{\infty} v_k(r).
\]
Then the following sharp inequality holds:
\[
|f(z^m)|v_0(r) + \sum_{k=1}^{\infty} |a_k|v_k(r) \leq v_0(r) \quad \text{for all} \quad |z| = r \leq R_1,
\]
where \(R_1\) is the minimal positive root of the equation:
\[
v_0(r) = \frac{2(1 + r^m)}{p(1 - r^m)} \sum_{k=1}^{\infty} v_k(r).
\]
In case when
\[
v_0(r) < \frac{2(1 + r^m)}{p(1 - r^m)} \sum_{k=1}^{\infty} v_k(r)
\]
in some interval \((R_1, R_1 + \epsilon)\), then the number \(R_1\) cannot be improved.

If we choose \(v_1(r) = r^k\) in Theorem 3, we get Theorem 1 with \(\Re g(z) < 1\). At this juncture, it is natural to pose the following problem:

**Problem 1** Can we establish the analog of Theorem 3 for the Ma-Minda classes \(S^*(\psi)\) and \(C(\psi)\)?

In context of the above problem and Muhanna (2010), we now describe the notion of generalized Bohr–Rogosinski phenomenon here below, in terms of subordination, following the recent development as seen in Kayumov et al. (2022), Kumar and Sahoo (2022) and Lin and Liu (2021).

**Definition 1** Let \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) and \(g(z) = \sum_{k=0}^{\infty} b_k z^k\) are analytic in \(D\). Let \(d(f(0), \partial \Omega)\) denotes the Euclidean distance between \(f(0)\) and the boundary of \(\Omega = f(D)\). For a fixed \(f\), consider a class of analytic functions
\[
S(f) := \{g : g \prec f\}
\]
or equivalently,
\[
S(\Omega) := \{g : g(z) \in \Omega\}.
\]
Then we say \(S(f)\) satisfies the **Generalized Bohr–Rogosinski phenomenon**, if there exists a constant \(r_0 \in (0, 1]\) such that
\[
\mathcal{P}(r, g, f) + \sum_{k=1}^{\infty} |b_k|\phi_k(r) \leq d(f(0), \partial \Omega),
\]
holds for all \(|z| = r \leq r_0\), where

(i) \(\mathcal{P}(r, g, f)\) represent some function of \(r\) or certain proper combination of moduli of \(g, f\) and their derivatives.

(ii) \(|\phi_k(r)\|\) be a sequence of non-negative continuous functions in \([0, 1]\) such that the series of the form
\[
p_0\phi_0(r) + \sum_{k=1}^{\infty} p_k\phi_k(r)
\]
converges locally uniformly with respect to \(r \in [0, 1]\), where \(p_k\) depends on the function \(f\) and provide bounds for \(b_k\).

For \(\mathcal{P}(r, g, f) = |g(z)|, \phi_k(r) = r^k (k \geq N)\) and 0 otherwise in Definition 1, gives the quantity considered by Kayumov et al. (2021), which is known as the Bohr–Rogosinski sum and given by
\[
|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k, \quad |z| = r.
\]
The link between the Bohr–Rogosinski and Bohr phenomenon can be noticed, if we replace \(|g(z)|\) by \(g(0)\) with \(N = 1\). We also refer the readers to see Aizenberg (2012) and Alkhaleefah et al. (2020). Now we see that the family \(S(f)\) has Bohr–Rogosinski phenomenon provided there exists \(r_N^f \in (0, 1]\) such that the inequality:
\[
|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq |f(0)| + d(f(0), \partial \Omega)
\]
holds for \(|z| = r \leq r_N^f\). The largest such \(r_N^f\) is called the Bohr–Rogosinski radius.

In case when the function \(f\) is normalized, then Gangania and Kumar (2022b) considered the class

**Definition 2** Let \(f \in S^*(\psi)\) or \(C(\psi)\) be fixed. Then the class of subordinants functions \(g\) is defined as:
\[
S_f(\psi) := \left\{ g(z) = \sum_{k=1}^{\infty} b_k z^k : g \prec f \right\},
\]
and studied the Bohr–Rogosinski phenomenon for the class of analytic subordinates \(S_f(\psi)\):

The class \(S_f(\psi)\) has a Bohr–Rogosinski phenomenon, if there exists an \(0 < r_0 \leq 1\) such that
\[
|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(f(0), \partial \Omega)
\]
for \(|z| = r \leq r_0\), where \(m, N \in \mathbb{N}, \Omega = f(D)\) and \(d(f(0), \partial \Omega)\) denotes the Euclidean distance between \(f(0)\) and the boundary of \(\Omega\).

**Theorem 4** (Gangania and Kumar 2022b, Theorem 2.3, Page no. 7) Let \(f_0(z)\) be given by the equation (10) and
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\psi).$$ Assume $$f_0(z) = z + \sum_{n=2}^{\infty} f_n z^n$$ and $$\tilde{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n$$. If $$g \in S_f(\psi)$$. Then

$$|g(z^n)| + \sum_{k=N}^{\infty} |b_k| |z|^k \leq d(0, \tilde{c}\Omega)$$

holds for $$|z| = r_b \leq \min\{\frac{1}{2}, r_0\}$$, where $$m, N \in \mathbb{N}$$, $$\Omega = f(\mathbb{D})$$ and $$r_0$$ is the unique positive root of the equation:

$$f_0(r^n) + \tilde{f}_0(r) - p_{\tilde{f}_0}(r) = -f_0(-1),$$

where

$$p_{\tilde{f}_0}(r) = \begin{cases} 
0, & N = 1; \\
N, & N = 2; \\
\frac{1}{\sum_{n=2}^{N-1} |t_n| r^n}, & N \geq 3. 
\end{cases}$$

The result is sharp when $$r_b = r_0$$ and $$t_n > 0$$.

For the class $$C(\psi)$$, see (Gangania and Kumar 2022b, Theorem 2.13, page no. 12) In context of the above, also see Aizenberg (2012), Alkhaleefah et al. (2020), Bhowmik and Das (2018) and Kayumov et al. (2021).

The Bohr Operator

$$M_r(f) = \sum_{n=0}^{\infty} |a_n| |z|^n = \sum_{n=0}^{\infty} |a_n| r^n$$

for the analytic functions $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$ was used by Paulsen and Singh (2022) to provide a simple proof of the Bohr’s Theorem 1 and extended it to the Bunach algebras (for the basic important discussion, see Muhanna et al. 2021; Paulsen and Singh 2022). Later, Muhanna et al. (2021) used this operator to obtain some new Bohr type of inequalities for the $$k$$-th section $$\sum_{n=k}^{\infty} a_n z^n$$ and also obtain simple proofs of the known results.

Gangania and Kumar (2022b) considered the basic operator for $$f$$, given by:

$$M_r^N(f) = \sum_{n=N}^{\infty} |a_n| |z|^n = \sum_{n=N}^{\infty} |a_n| r^n,$$

and thus the following observations hold for $$|z| = r$$ for each $$z \in \mathbb{D}$$

(i) $$M_r^N(f) \geq 0$$, and $$M_r^N(f) = 0$$ if and only if $$f \equiv 0$$
(ii) $$M_r^N(f + g) \leq M_r^N(f) + M_r^N(g)$$
(iii) $$M_r^N(xf) = |x|M_r^N(f)$$ for $$x \in \mathbb{C}$$
(iv) $$M_r^N(fg) \leq M_r^N(f)M_r^N(g)$$
(v) $$M_r^N(1) = 1$$.

For the above properties in terms of the sequence $$\{v_n(r)\}_{n=0}^{\infty}$$, see (Ponnusamy et al. 2021, Lemma 1). Using this operator, a simple proof of Bhowmik and Das (2018, Lemma 1) was achieved by Gangania and Kumar (2022b) to settle the Bohr–Rogosinski Phenomenon for the classes $$S^*(\psi)$$ and $$C(\psi)$$, which in terms of intermix $$k$$th section $$f_k(z) := \sum_{n=k}^{\infty} a_n z^n$$ is as follows:

**Lemma 1** (Gangania and Kumar 2022b) let $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$ and $$g(z) = \sum_{n=0}^{\infty} b_n z^n$$ be analytic in $$\mathbb{D}$$ and $$g \prec f$$, then

$$M_r^N(g_k) \leq M_r^N(f_k)$$

for all $$|z| = r \leq 1/3$$ and $$k, N \in \mathbb{N}$$, when $$N = 1$$ and $$k \to \infty$$, lemma was obtained by Bhowmik and Das (2018). Various interesting applications of this lemma can be seen in Gangania and Kumar (2022a, 2022b), Bhowmik and Das (2018), Hamada (2021) and Kumar and Gangania (2021).

While we establish the Bohr-type inequalities for the general class $$S^*(\psi)$$ or $$C(\psi)$$, the main difficulty that we come across is the unavailability of the sharp coefficients bounds. Here, we require use of Lemma 1 or its proper modifications. Interestingly, Lemma 1 also implies that if $$f \prec g$$, then within the disk $$|z| \leq 1/3$$, we have $$|a_n| \leq |b_n|$$ for all $$n \in \mathbb{N} \cup \{0\}$$, where $$b_n$$ are the coefficients of the function $$g$$ in Lemma 1. Further, this readily gives the following:

**Lemma 2** Let $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$ and $$g(z) = \sum_{n=0}^{\infty} b_n z^n$$ be analytic in $$\mathbb{D}$$. Let $$\{v_n(r)\}_{n=0}^{\infty}$$ be a sequence of non-negative functions, continuous in $$(0, 1)$$ such that the series

$$\sum_{n=0}^{\infty} |b_n| v_n(r)$$

converges locally uniformly with respect to $$r \in [0, 1)$$. If $$g \prec f$$, then

$$\sum_{n=0}^{\infty} |a_n| v_n(r) \leq \sum_{n=0}^{\infty} |b_n| v_n(r) \quad \text{for all } |z| = r \leq \frac{1}{3}.$$
Now if we take \( 2h(z) = f(z) + \overline{f(z)} \) in Definition 3, we obtain the classes \( S^*_c(\psi) \) and \( C_c(\psi) \) of starlike and convex functions with respect to conjugate points, respectively. For the choice \( 2h(z) = f(z) - \overline{f(z)} \), we have the classes \( S^*_c(\psi) \) and \( C_c(\psi) \) of starlike and convex functions with respect to the symmetric conjugate points, respectively. See Ravichandran (2004).

Motivated, by the class \( S^*_c((1 + z)/(1 - z)) \) (Sakaguchi 1959), Gao and Zhou (2005) studied the class \( K_c(\psi) \) of close-to-convex functions \( f \), which is characterized as:

\[
\Re\left( \frac{zf'(z)}{g(z)g(-z)} \right) > 0,
\]

where \( g \) is some starlike function of order 1/2. In view of the Definition 3, the generalized class \( K_c(\psi) \) was studied by Cho et al. (2011) and Wang et al. (2006).

In this paper, we positively answer the Problem 1 in Sect. 2 for the classes \( S^*(\psi) \) and \( C(\psi) \). In Sect. 3, we study the Bohr–Rogosinski phenomenon for the classes \( K_c(\psi), S^*_c(\psi), C_c(\psi) \) and \( C(\psi) \). For convenience, we denote \( f(z) = \sum_{n=0}^{\infty} |a_n| z^n \), whenever \( f(z) = \sum_{n=0}^{\infty} a_n z^n \).

2 Generalized Bohr’s Sum for Ma-Minda Starlike Functions

We here solve the Problem 1, but as we do not have sharp coefficient’s bound for each \( a_n \) for the given class in general. Thus for certain valid assumptions to solve it, we need the following:

Lemma 3 The families \( S^*(\psi) \) and \( C(\psi) \) are normal and compact.

Proof From Montel’s Theorem (Goodman 1983), we see that the class \( S^*(\psi) \) is a normal family. Now let us prove that \( S^*(\psi) \) is compact. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of functions from \( S^*(\psi) \). Suppose that \( \{f_n\} \) is convergent. Then it is well-known that

\[
\lim_{n \to \infty} f_n := f \in S
\]

We show that \( f \in S^*(\psi) \). If possible, suppose that there exists a nearest point \( z_0 \in \mathbb{D} \) such that \( g(z_0) \notin \psi(\mathbb{D}) \), where \( g(z) = \psi f(z) / f(z) \). Note that the corresponding sequence \( \{g_n\} \) converges to \( g \), where \( g_n(z) = \psi f_n(z) / f_n(z) \). Now let

\[
\epsilon = \text{dist}(g(z_0), \psi(\mathbb{D})).
\]

Then the open ball \( B(g(z_0), \epsilon) \not\subset \psi(\mathbb{D}) \). Since \( g_n \to g \), in particular, \( g_n(z_0) \to g(z_0) \). There exists \( n(\epsilon) \in \mathbb{N} \) such that

\[
g_n(z_0) \in B(g(z_0), \epsilon), \quad \forall n \geq n(\epsilon),
\]

which implies \( g_n(z_0) \not\in \psi(\mathbb{D}), \quad \forall n \geq n(\epsilon) \). But as

\[
f_n \in S^*(\psi),
\]

\[
g_n(z_0) = \frac{z_0 f_n'(z_0)}{f_n(z_0)} \in \psi(\mathbb{D}), \quad \forall n.
\]

Hence, we must have \( f \in S^*(\psi) \), that means the family \( S^*(\psi) \) is compact. With the similar arguments, it is easy to see that the family \( C(\psi) \) is also compact.

Remark 1 (Existence of sharp coefficients Bounds) Let us consider the real-valued functional \( J \) defined on \( S^*(\psi) \) as

\[
J(f) = \max \{|a_n|\} \quad \text{for every} \quad f \in S^*(\psi),
\]

where \( n \) is fixed and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). From Lemma 3, \( S^*(\psi) \) is normal and compact. Further, since \( S^*(\psi) \subseteq S^*_c \), we have \( |a_n| \leq n \), that means \( J \) is a bounded functional. Hence, following the discussion in the Goodman’s Book (Goodman 1983, page no. 44-45), we conclude that

\[
J(f) = \max \{|a_n|\} \quad \text{exists in the family} \quad S^*(\psi).
\]

Thus, let us say that

\[
\max_{f \in S^*(\psi)} \{|a_n|\} := M(n)
\]

for each \( n \in \mathbb{N} \). For instance, \( M(n) = n \) for the class of univalent starlike functions. For the Janowski starlike functions, it is given by Aouf (1987, Theorem 3).

We can now state our result in a general setting whose complement is Theorem 6:

Theorem 5 Let \( \{\phi_n(r)\}_{n=1}^{\infty} \) be a sequence of non-negative continuous functions in \( (0, 1) \) such that the series

\[
\phi_1(r) + \sum_{n=2}^{\infty} M(n) \phi_n(r)
\]

converges locally uniformly with respect to each \( r \in (0, 1) \). If for \( \beta \in [0, 1] \)

\[
\beta f_0'(r^m) + (1 - \beta) f_0(r^m) + \sum_{n=1}^{\infty} M(n) \phi_n(r) < -f_0(-1),
\]

and the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\psi) \). Then the following inequality

\[
|\beta f'(z^m)| + |(1 - \beta) f(z^m)| + \sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq d(0, \Omega)
\]

holds for \( |z| = r \leq r_0 \), where \( m \in \mathbb{N}, \Omega = f(\mathbb{D}) \) and \( r_0 \) is the smallest positive root of the equation:
Remark 1. We have such that its Taylor series coefficients $a_n$ satisfy $M(1) = 1$ with $M(n)$ as described in Remark 1 and $f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} \, dt$.

In case when $f_0$ or its rotation serves as an extremal for the coefficient's bound $M(n)$, then the radius $r_0$ is sharp.

**Proof.** From Lemma 3 and Remark 1, we see that: (a) $\mathcal{J}$ is a bounded real valued continuous functional, (b) $\mathcal{S}^r(\psi)$ is a normal family, and (c) $\mathcal{S}^n(\psi)$ is a compact family in $\mathbb{D}$. Thus, the sharp bounds for each $a_n$ exists. In view of Remark 1, we have $|a_n| \leq M(n)$, which yields that

$$\sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq \sum_{n=1}^{\infty} M(n) \phi_n(r).$$

The Koebe-radius for the functions in $\mathcal{S}^r(\psi)$ satisfies

$$d(0, \partial f(\mathbb{D})) \geq -f_0(-1).$$

Now combining it with the growth and distortion theorems (Ma and Minda 1992), and using the condition 2, the inequalities (4) and (5) gives

$$\beta |\phi'|(z) + (1 - \beta) |f'|(z) + \sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq \beta \phi_0(r) + (1 - \beta) f_0(r) + \sum_{n=1}^{\infty} M(n) \phi_n(r) \leq d(0, \partial \Omega),$$

which holds in $|z| = r \leq r_0$, where $r_0$ is the minimal positive root of the equation (3). Existence of the root $r_0$ follows from the Intermediate value theorem for continuous function in $(0, 1)$. To see the sharpness case, let us consider the function

$$f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} \, dt,$$

such that it’s Taylor series coefficients $a_n(f_0)$ satisfies $|a_n(f_0)| = M(n)$. For this function we have

$$d(0, \partial f(\mathbb{D})) = -f_0(-1),$$

and the following equality holds for $|z| = r_0$:

$$\beta \phi_0(r) + (1 - \beta) f_0(r) + \sum_{n=1}^{\infty} M(n) \phi_n(r) = d(0, \partial f(\mathbb{D})), \quad \text{and therefore, if } f_0 \text{ is extremal for each coefficient's bound, then the radius } r_0 \text{ cannot be improved.}$$

**Question 1.** What if we do not have $M(n)$?

For such cases, the following result completes Theorem 5:

**Theorem 6.** Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a non-negative sequence of continuous functions in $[0, 1]$ such that the series

$$\phi_1(r) + \sum_{n=2}^{\infty} \frac{f_0^{(n)}(0)}{n!} \phi_n(r)$$

converges locally uniformly with respect to each $r \in [0, 1]$. If

$$\beta |\phi'|(z) + (1 - \beta) |f'|(z) + \sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq d(0, \partial \Omega)$$

holds for $|z| = r \leq r_0 = \min\{1/3, r_0\}$, where $m \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the smallest positive root of the equation:

$$\beta \phi_0(r) + (1 - \beta) f_0(r) + \sum_{n=1}^{\infty} M(n) \phi_n(r) = d(0, \partial \Omega),$$

where

$$f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} \, dt.$$

Moreover, the inequality (6) also holds for the class $S_f(\psi)$ in $|z| = r \leq r_0$. When $r_0 = r_0$, then the radius is best possible.

**Proof.** Since $f \in \mathcal{S}^r(\psi)$, it is known that $f(z)/z \prec f_0(z)/z$. Applying Lemma 1, we see that

$$\sum_{n=1}^{m} \frac{|a_n|^n}{n!} \leq \sum_{k=1}^{m} \frac{|f_0^{(k)}(0)|}{k!} |z|^k \quad \text{for } |z| = r \leq 1/3.$$

Now choosing $N = m$, we conclude that

$$|a_n| \leq \left| \frac{|f_0^{(n)}(0)|}{n!} \right|$$

holds for each $n$ with in the disk $|z| = r \leq 1/3$. Hence, it suffices to see
\[
\beta f'(z^n) + (1 - \beta) f(z^n) \geq \sum_{n=1}^{\infty} |a_n| \phi_n(r) \\
\leq \beta f_0'(r^n) + (1 - \beta) f_0(r^n) + \phi_1(r) \\
+ \sum_{n=2}^{\infty} \frac{|f_0(n)|}{n!} \phi_n(r) \\
\leq -f_0(-1) \\
\leq d(0, \partial \Omega),
\]
holds in \(|z| = r \leq \min\{r_0, 1/3\}\). If \(r_0 \leq 1/3\), then equality case can be seen for the function \(f = f_0\), whenever Taylor coefficients of \(\psi\) are positive. \(\square\)

The result for the class \(C(\psi)\) also follows on the similar lines, so we omit the details of proof.

**Theorem 7** Let \(\{\phi_n(r)\}_{n=1}^{\infty}\) be a non-negative sequence of continuous functions in \([0, 1]\) such that the series

\[
\phi_1(r) + \sum_{n=2}^{\infty} \frac{|f_0(n)|}{n!} \phi_n(r)
\]
converges locally uniformly with respect to each \(r \in [0, 1]\).

If

\[
\beta f'(z^n) + (1 - \beta) f(z^n) + \phi_1(r) + \sum_{n=2}^{\infty} \frac{|f_0(n)|}{n!} \phi_n(r) < -f_0(-1)
\]
and \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\psi)\). Then

\[
\beta f'(z^n) + (1 - \beta) f(z^n) + \sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq d(0, \partial \Omega)
\]
holds for \(|z| = r \leq r_0\), where \(m \in \mathbb{N}, \Omega = f(D)\) and \(r_0\) is the minimal positive root of the equation:

\[
\beta f_0'(r^n) + (1 - \beta) f_0(r^n) + \sum_{n=2}^{\infty} \frac{|f_0(n)|}{n!} \phi_n(r) = -f_0(-1) - \phi_1(r),
\]
where \(f_0\) satisfy

\[
1 + \frac{z f_0''(z)}{f_0(z)} = \psi(z).
\]
Moreover, the inequality (7) also holds for the class \(S_f(\psi)\) in \(|z| = r \leq r_b\). When \(r_b = r_0\), then the radius is best possible.

**Remark 2**

1. By taking \(\phi_n(r) = r^n\) in Theorem 6 give (Kumar and Gangania 2020, Theorem 5.1), and (Hamada 2021, Theorem 3.1) for the choice \(g = f\) with Taylor coefficients of \(\psi\) being positive.

2. Taking \(\phi_n = r^n\) for \(n \geq N\) and 0 elsewhere in Theorem 6 yields (Gangania and Kumar 2022b, Theorem 5, Corollary 3).

Let us discuss the generalized Bohr–Rogosinski phenomenon for the celebrated Janowski class of univalent starlike functions. For simplicity, write \(S'(1 + Dz) / (1 + Ez) \equiv S[D, E]\), where \(-1 \leq E < D \leq 1\).

**Corollary 1** Let \(\{\phi_n(r)\}_{n=1}^{\infty}\) be a sequence of non-negative continuous functions in \((0, 1)\) such that the series

\[
\phi_1(r) + \sum_{n=2}^{\infty} \frac{|E - D + Ek|}{k + 1} \phi_n(r)
\]
converges locally uniformly with respect to each \(r \in [0, 1]\).

If \(E \in [0, 1]\)

\[
\beta f_0'(r^n) + (1 - \beta) f_0(r^n) + \sum_{n=1}^{\infty} |a_n| \phi_n(r) < -f_0(-1)
\]
and the function \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S[D, E]\). Then the following sharp inequality

\[
\beta f'(z^n) + (1 - \beta) f(z^n) + \sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq d(0, \partial \Omega)
\]
holds for \(|z| = r \leq r_0\), where \(m \in \mathbb{N}, \Omega = f(D)\) and \(r_0\) is the minimal positive root of the equations:

- If \(E \neq 0\)

\[
\beta (1 + Dr^n) (1 + Ez)^{\frac{n-1}{n}} + (1 - \beta) r^n (1 + Ez)^{\frac{n-1}{n}}
\]

\[
+ \sum_{n=2}^{\infty} \frac{|E - D + Ek|}{k + 1} \phi_n(r) = (1 - E)^{\frac{n-1}{n}} - \phi_1(r),
\]

and if \(E = 0\)

\[
\phi_1(r) + \sum_{n=2}^{\infty} \frac{D}{k + 1} \phi_n(r) = e^D - \phi_1(r),
\]

where

\[
f_0(z) = \begin{cases} 
z(1 + Ez)^{\frac{n-1}{n}}, & E \neq 0; \\
z e^{Dz}, & E = 0.
\end{cases}
\]

The radius \(r_0\) cannot be improved.

**Proof** Let \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S[D, E]\). Then for \(n \geq 2\), Aouf (1987, Theorem 3) states that:

\[
|a_n| \leq \prod_{k=0}^{n-2} \frac{|E - D + Ek|}{k + 1} = M(n),
\]

where the function \(f_0\) given by (9) gives equality. The result now follows by Theorem 5. \(\square\)
Remark 3  Taking \( m \to \infty \) and \( \beta = 0 \) in Corollary 1 yields: If
\[
\sum_{n=1}^{\infty} |a_n| \phi_n(r) < -f_0(-1).
\]

Then the sharp inequality \( \sum_{n=1}^{\infty} |a_n| \phi_n(r) \leq d(0, \tilde{\Omega}) \) holds for \( |z| = r \leq r_0 \), where \( m \in \mathbb{N}, \ \tilde{\Omega} = f(\mathcal{D}) \) and \( r_0 \) is the minimal positive root of the equations:
\[
\beta(1 + (1 - 2\alpha)|z|^m) + \left( 1 - \beta \right)|z|^m
\]
\[
\frac{(1 - r^m)^{2(1-z)+1}}{(1 - r^m)^{2(1-z)}} + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} k \frac{2(1-z)}{k+1} \phi_n(r)
\]
\[
< \frac{1}{4^{1-z}} - \phi_1(r).
\]

and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\alpha) \). Then the sharp inequality (8) holds for \( |z| = r \leq r_0 \), where \( m \in \mathbb{N}, \ \tilde{\Omega} = f(\mathcal{D}) \) and \( r_0 \) is the minimal positive root of the equations:
\[
\beta(1 + (1 - 2\alpha)|z|^m) + \left( 1 - \beta \right)|z|^m
\]
\[
\frac{(1 - r^m)^{2(1-z)+1}}{(1 - r^m)^{2(1-z)}} + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} k \frac{2(1-z)}{k+1} \phi_n(r)
\]
\[
= \frac{1}{4^{1-z}} - \phi_1(r).
\]

The radius \( r_0 \) is sharp.

Putting \( \alpha = 0 \) in Corollary 2, we get the following:

Corollary 3  Let the sequence \( \{ \phi_n(r) \}_{n=1}^{\infty} \) satisfy the hypothesis of Corollary 2 with \( \alpha = 0 \). If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^* \). Then the inequality (8) holds for \( |z| = r \leq r_0 \), where \( m, n \in \mathbb{N}, \ \Omega = f(\mathcal{D}) \) and \( r_0 \) is the smallest positive root of the equations:
\[
\beta(1 + (1 - 2\alpha)|z|^m) + \left( 1 - \beta \right)|z|^m
\]
\[
\frac{(1 - r^m)^{2(1-z)+1}}{(1 - r^m)^{2(1-z)}} + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} k \frac{2(1-z)}{k+1} r^n
\]
\[
= \frac{1}{4^{1-z}}.
\]

The radius \( r_0 \) is sharp.

Example 1  Let us consider \( \phi_n(r) = 0 \) for \( 1 \leq n \leq N \), and \( \phi_n(r) = r^n \) for \( n \geq N \) in Corollary 1. Then the following sharp inequality
\[
\beta |f(z)| + (1 - \beta) |f(z^m)| + \sum_{n=1}^{\infty} |a_n| r^n \leq d(0, \tilde{\Omega})
\]
holds for \( |z| = r \leq R_{m, n} \), where \( m, n \in \mathbb{N}, \ \tilde{\Omega} = f(\mathcal{D}) \) and \( R_{m, n} \) is the unique positive root of the equations:
\[
\beta(1 + D r^m)(1 + E r^m) + (1 - \beta) r^m(1 + E r^m)
\]
\[
+ \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} E + k \frac{2(1-z)}{k+1} r^n
\]
\[
= 1 - D r^m.
\]

Example 2  Taking \( \phi_{2n-1}(r) = 0 \) and \( \phi_{2n}(r) = r^{2n} \) in Corollary 2 yields
\[
\beta |f(z)| + (1 - \beta) |f(z^m)| + \sum_{n=1}^{\infty} |a_{2n}| r^{2n} \leq d(0, \tilde{\Omega})
\]
which holds for \( |z| = r \leq R_{m, \beta, 2} \), where \( m \in \mathbb{N}, \ \beta \in [0, 1], \ \Omega = f(\mathcal{D}) \) and \( R_{m, \beta, 2} \) is the unique positive root of the equations:
\[
\beta(1 + (1 - 2\alpha)|z|^m) + \left( 1 - \beta \right)|z|^m
\]
\[
\frac{(1 - r^m)^{2(1-z)+1}}{(1 - r^m)^{2(1-z)}} + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} k \frac{2(1-z)}{k+1} r^{2n}
\]
\[
= \frac{1}{4^{1-z}}.
\]

The radius is sharp.

Example 3  Letting \( \phi_{2n-1}(r) = 0 \) and \( \phi_{2n}(r) = r^{2n-1} \) in Corollary 3 gives the following sharp inequality
\[
\beta(1 + (1 - 2\alpha)|z|^m) + \left( 1 - \beta \right)|z|^m
\]
\[
\frac{(1 - r^m)^{2(1-z)+1}}{(1 - r^m)^{2(1-z)}} + \sum_{n=2}^{\infty} \prod_{k=0}^{n-2} k \frac{2(1-z)}{k+1} r^{2n-1}
\]
\[
= \frac{1}{4^{1-z}}.
\]
in \( |z| = r \leq R_{m, \beta} \) where \( m \in \mathbb{N}, \beta \in [0,1], \Omega = f(\mathbb{D}) \) and \( R_{m, \beta} \) is the unique positive root of the equations:
\[
\beta (1 + r^m)/(1 - r^m)^{2+1} + (1 - \beta) r^m + r(1 + r^2)/(1 - r^2)^2 = 1/4.
\]
The radius is sharp.

3 Bohr–Rogosinski Sum for Starlike and Convex Functions with Respect to Conjugate and Symmetric Points

To discuss the Bohr–Rogosinski phenomenon, we first need to recall some related basic definitions:

Definition 4 (Cho et al. 2011) Let us consider the subclass of close-to-convex functions given by
\[
\mathcal{K}_a(\psi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z) + f(-z)} < \psi(z) \right\}
\]
for some \( h \in \mathcal{S}^*(1/2) \).

Definition 5 (Ravichandran 2004) The class of starlike functions with respect to conjugate points is given by
\[
\mathcal{S}^*_c(\psi) = \left\{ f \in \mathcal{A} : \frac{2zf'(z)}{f(z) + f(-z)} < \psi(z) \right\}.
\]

Definition 6 (Ravichandran 2004) The class of convex functions with respect to conjugate points given by
\[
\mathcal{C}_c(\psi) = \left\{ f \in \mathcal{A} : \frac{2zf'(z)'}{f(z) + f(-z)} < \psi(z) \right\}.
\]

Definition 7 (Ravichandran 2004) The class of convex function with respect to symmetric points is given by
\[
\mathcal{C}_s(\psi) = \left\{ f \in \mathcal{A} : \frac{2zf'(z)'}{f(z) + f(-z)} < \psi(z) \right\}.
\]

We also need to recall the following fundamental results for the classes of Ma-Minda starlike and convex functions:

Lemma 4 (Ma and Minda 1992) Let \( f \in \mathcal{S}^*(\psi) \) and \( |z_0| = r < 1 \). Then \( f(z)/z \prec f_0(z)/z \) and \(-f_0(-r) \leq |f(z_0)| \leq f_0(r)\).

Equality holds for some \( z_0 \neq 0 \) if and only if \( f \) is a rotation of \( f_0 \), where
\[
f_0(z) = z \exp \int_0^z \frac{\psi(t)-1}{t} \, dt.
\]

Lemma 5 (Ma and Minda 1992) Let \( f \in \mathcal{C}(\psi) \). Then \( z^m/f'(z) \prec z^m_0/f'(z) \) and \( f'(z) \prec f'(z) \).

Also, for \( |z| = r \) we have \(-l_0(-r) \leq |f(z)| \leq l_0(r)\), where
\[
1 + |z^m_0/f'(z)| = \psi(z).
\]

Remark 4 Throughout this section, we shall assume that \( \min_{|z|=r} |\psi(z)| = \psi(-r) \) and \( \max_{|z|=r} |\psi(z)| = \psi(r) \), as under these conditions growth theorems for the above defined classes in Definitions 4, 5, 6 and 7 are known.

The following result yields the Bohr–Rogosinski radius for the analytic functions subordinated by close-to-convex functions.

Theorem 8 Let \( f \in \mathcal{K}_a(\psi) \) and \( \Omega = f(\mathbb{D}) \). If \( g \in \mathcal{S}_f(\psi) \). Then
\[
|g(z^m)| + \sum_{k=N}^{\infty} |b_k| |z|^k \leq d(0, \partial \Omega)
\]
holds for \( |z| = r_b \leq \min\{1, r_0\} \), where \( m, N \in \mathbb{N} \) and \( r_0 \) is the minimal positive root of the equation:
\[
\int_0^{r_b} \frac{\psi(t)}{1-t^2} \, dt + R_N(r) = \int_0^1 \frac{\psi(-t)}{1-t^2} \, dt,
\]
where
\[
R_N(r) = \int_0^r \frac{M_N(\psi(t)^2}{(1-t^2)} \, dt \quad \text{and} \quad f_0(z) = \int_0^z \frac{\psi(t)}{1-t^2} \, dt.
\]

Proof Let \( g(z) = \sum_{k=1}^{\infty} b_k z^k \prec f(z) \). Then by Lemma 1, for \( r \leq 1/3 \)
\[
\sum_{k=N}^{\infty} |b_k| r^k \leq \sum_{n=N}^{\infty} |a_n| r^n = \sum_{n=N}^{\infty} \frac{|\hat{b}_n|}{n} r^n,
\]
where \( \hat{b}_n \) are the power series coefficient of \( \hat{G}(z) \) defined below. From Definition 4, we have
\[
zf'(z) = G(z)\psi(\omega(z)) =: \tilde{G}(z),
\]
where
\[
G(z) = \frac{-h(z) h(-z)}{z} =: z + \sum_{n=2}^{\infty} 2n h_{2n-1} z^{2n-1},
\]
which is an odd starlike function. Now a simple integration gives that
\[ f(z) = \int_0^z \frac{G(t)\psi(\omega(t))}{t} \, dt. \]

For \( f \in \mathcal{A} \), let us consider the operator
\[ M^N_r(f) = \sum_{n=N}^{\infty} |a_n|z^n = \sum_{n=N}^{\infty} |a_n|r^n. \]

Then
\[ M^N_r(G) \leq M^N_r(G)M^N_r(\psi \circ \omega). \quad (14) \]

Since \( \psi \circ \omega \sim \psi \), we get
\[ M^N_r(\psi \circ \omega) \leq M^N_r(\psi), \quad \text{for } r \leq 1/3. \quad (15) \]

It is known that odd starlike functions satisfies \(|h_{2n-1}| \leq 1\). Thus
\[ M^N_r(G) \leq \sum_{n=N}^{\infty} r^{2n-1} = \frac{1}{r} \left( \frac{r^{2N}}{1 - r^2} \right). \quad (16) \]

Now combining the inequalities (13), (14), (15) and (16), the following sequence of inequalities holds for \( r \leq 1/3 \):
\[ \sum_{k=N}^{\infty} |b_k|^j \leq \sum_{n=N}^{\infty} |b_n|^j = \int_0^r M^N_r(G) \, dt \]
\[ \leq \int_0^r M^N_r(G)M^N_r(\psi \circ \omega) \, dt \]
\[ \leq \int_0^r M^N_r(\psi) \frac{r^{2N}}{r^2(1 - r^2)} \, dr := R^N(r). \quad (17) \]

Since, also see growth theorem in Cho et al. (2011, Theorem 2, page no. 4),
\[ |g(z)| = |f(\omega(z))| \leq \max_{|t| = r} |f(|z| \leq r)| \]
\[ \leq \int_0^r \frac{\psi(t)}{1 - t^2} \, dt = f_0(r), \]

it follows that
\[ |g(z^m)| \leq r^m \leq f_0(r^m), \quad \text{where } f_0(z) = \int_0^z \frac{\psi(t)}{1 - t^2} \, dt. \quad (18) \]

Finally, note that
\[ d(0, \Omega) \geq \int_0^1 \frac{\psi(-t)}{1 + t^2} \, dt. \]

Hence, from (17) and (18), we get
\[ |g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \Omega) \]

for \( |z| = r \leq r_b = \min\{1/3, r_0\} \), where \( r_0 \in (0, 1) \) is the minimal root of the equation (12). Existence of the root follows by Intermediate value theorem in the interval \([0, 1]\).

**Remark 5** Taking \( m \to \infty \) and \( N = 1 \), then Theorem 8 reduces to (Allu and Halder 2021, Theorem 2.2).

**Corollary 4** If \( f \in \mathcal{K}_r(\psi) \) and \( \Omega = f(\mathbb{D}) \). Then
\[ |f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(0, \Omega) \]

holds for \( |z| = r_b \leq \min\{1/3, r_0\} \), where \( m, N \in \mathbb{N} \) and \( r_0 \) is the minimal positive root of the equation:
\[ \int_0^{r_0} \frac{\psi(t)}{1 - t^2} \, dt + R_N(r) = \int_0^{1/3} \frac{\psi(-t)}{1 - t^2} \, dt, \]

where
\[ R_N(r) = \int_0^r \frac{M^N_r(\psi(r^2N)}{r^2(1 - r^2)} \, dr \quad \text{and} \quad f_0(z) = \int_0^z \frac{\psi(t)}{1 - t^2}. \]

Our next result provides the Bohr–Rogosinski radius for the analytic functions subordinated by starlike function with respect to conjugate points.

**Theorem 9** Let \( \psi \) be given by (10) and \( f(z) = z + \sum_{n=1}^{\infty} a_nz^n \in S_1(\psi) \). If \( g \in S_1(\psi) \). Then
\[ |g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \Omega) \quad (19) \]

holds for \( |z| = r_b \leq \min\{1/3, r_0\} \), where \( m, N \in \mathbb{N} \), \( \Omega = f(\mathbb{D}) \) and \( r_0 \) is the unique positive root of the equation:
\[ h_\psi(r^m) + R_N(r) + h_\psi(-1) = 0, \quad (20) \]

where
\[ R_N(r) = \int_0^r \frac{M^N_r(h_\psi)M^N_r(\psi)}{r} \, dr \]

The result is sharp when \( r_b = r_0 \) and \( t_r > 0 \).

**Proof** Since the function \( G(z) = (f(z) + f(\bar{z}))/2 \) belongs to \( S_1(\psi) \). Therefore, by Lemma Ma and Minda (1992) we have
\[ G(z) \prec h_\psi(z) \]

which using Lemma 1 yields
\[ M^N_r(G) \leq M^N_r(h_\psi) \quad \text{for } r \leq 1/3. \quad (21) \]

From Definition 5, we get \( zf'(z) = G(z)\psi(\omega(z)) \) which after integration gives
\[ f(z) = \int_{0}^{r} \frac{G(t) \psi(\omega(t))}{t} \, dt. \]  

(22)

Since \( \psi \circ \omega < \psi \),

\[ M_{\psi}^{N}(\psi \circ \omega) \leq M_{\psi}^{N}(\psi) \quad \text{for} \quad r \leq \frac{1}{3}. \]  

(23)

Thus, combining (21), (22) and (23), we see that

\[
\sum_{k=N}^{\infty} |b_{k}| |z|^{k} = M_{\psi}^{N}(g) \leq M_{\psi}^{N}(f)
\]  

\[
= \int_{0}^{r} M_{\psi}^{N}(G) M_{\psi}^{N}(\psi \circ \omega) \, dt
\]  

\[
\leq \int_{0}^{r} M_{\psi}^{N}(h_{\psi}) M_{\psi}^{N}(\psi) \, dt =: R^{N}(r),
\]

holds for \( r \leq 1/3 \). Also, using Maximum-principle of modulus and growth theorem Ravichandran (2004), \( g \prec f \) implies that

\[
|g(\{z \leq r\})| = |f(\{\omega(\{z \leq r\})\})| \leq \max |f(\{z \leq r\})| = h_{\psi}(r),
\]

which yields \( |g(z^{m})| \leq h_{\psi}(r^{m}) \). Finally, note that

\[ d(0, \Omega) \geq -h_{\psi}(-1). \]

Hence, the Bohr–Rogosinski inequality (19) holds for \( |z| \leq \min\{1/3, r_{0}\} \), where \( r_{0} \) is the root of the equation (20). The existence of the root follows by Intermediate value theorem for continuous function in \([0, 1]\). For the sharpness, note that for the function \( h_{\psi} \), \( d(0, \Omega) = -h_{\psi}(-1) \) such that if \( r_{b} = r_{0} \), then for the choice \( g = f = h_{\psi} \):

\[
h_{\psi}(z^{m}) + \sum_{n=N}^{\infty} |a_{n}| |z|^{n} = d(0, \Omega)
\]

holds for \( |z| = r_{b} \) with \( t_{n} > 0 \), where \( h_{\psi}(z) = z + \sum_{n=2}^{\infty} t_{n} z^{n} \) as given in (10).

**Remark 6** Let \( \psi(z) = (1 + z)/(1 - z) \), then Theorem 9 reduces to (Kayumov et al. 2021, Theorem 6).

The following result is explicitly for the class \( S_{\rho}^{\psi}(\psi) \).

**Corollary 5** Let \( f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n} \in S_{\rho}^{\psi}(\psi) \). Then

\[
|f(z^{m})| + \sum_{n=N}^{\infty} |a_{n}| |z|^{n} \leq d(0, \Omega)
\]

holds for \( |z| = r_{b} \leq \min\{1/4, r_{0}\} \), where \( m, N \in \mathbb{N}, \Omega = f(\mathbb{D}) \) and \( r_{0} \) is the unique positive root of the equation:

\[
h_{\psi}(r^{m}) + R^{N}(r) + h_{\psi}(-1) = 0,
\]

where

\[
R^{N}(r) = \int_{0}^{r} \frac{M_{\psi}^{N}(h_{\psi}) M_{\psi}^{N}(\psi)}{t} \, dt
\]

and \( h_{\psi} \) be given by (10). The result is sharp when \( r_{b} = r_{0} \) and \( t_{n} > 0 \).

**Remark 7** Taking \( m \to \infty \) and \( N = 1 \) in Theorem 9 and Corollary 5 establish the Bohr phenomenon for the classes \( S_{\rho}^{\psi}(\psi) \) and \( S_{\rho}^{\psi}(\psi) \), respectively, given in (Allu and Halder 2021, Lemma 2.12) and (Allu and Halder 2021, Theorem 2.9).

In the following, we obtain Bohr–Rogosinski radius for the analytic functions subordinated by convex function with respect to conjugate points.

**Theorem 10** Let \( f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n} \in C_{c}(\psi) \). If \( g \in S_{\rho}^{\psi}(\psi) \). Then

\[
|g(z^{m})| + \sum_{n=N}^{\infty} |b_{n}| |z|^{n} \leq d(0, \Omega)
\]

holds for \( |z| = r_{b} \leq \min\{1/4, r_{0}\} \), where \( m, N \in \mathbb{N}, \Omega = f(\mathbb{D}) \) and \( r_{0} \) is the minimal positive root of the equation:

\[
k_{\psi}(r^{m}) + R^{N}(r) + k_{\psi}(-1) = 0,
\]

(24)

where

\[
R^{N}(r) = \int_{0}^{r} \frac{1}{t} \int_{0}^{s} M_{\psi}^{N}(k_{\psi}) M_{\psi}^{N}(\psi) \, ds \, dt
\]

and \( k_{\psi}(z) = z + \sum_{n=2}^{\infty} t_{n} z^{n} \) is given by (11). The result is sharp when \( r_{b} = r_{0} \) and \( t_{n} > 0 \).

**Proof** Consider the function

\[ G(z) = \frac{f(z) + \overline{f(z)}}{2}. \]

Then \( G \in C(\psi) \). Now from Definition 6, we see that

\[
(zf'(z))' = G'(z) \psi(\omega(z)).
\]

(25)

This gives

\[
f(z) = \int_{0}^{z} \frac{1}{y} \int_{0}^{y} G'(t) \psi(\omega(t)) \, dt \, dy.
\]

(26)

As \( G' < k_{\psi} \), see Lemma 5, it follows using Lemma 1 that

\[ M_{\psi}^{N}(G') \leq M_{\psi}^{N}(k_{\psi}) \quad \text{for} \quad r \leq \frac{1}{3}. \]

(27)

Hence, using (25), (26) and (27)
holds for $|z| = r \leq r_0 = \min\{1/3, r_0\}$, where $r_0$ is minimal root of the equation (24). The existence of $0 < r_0 < 1$ can be seen by Intermediate value theorem for continuous function in $[0, 1]$. The case of equality

Corollary 6 If $f(z) = z + \sum_{k=N}^{\infty} a_n z^n \in C_c(\psi)$. Then

$$|g(z^n)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \omega),$$

follows with the choice $g = f = k_\psi$, whenever $r_b = r_0$ and $l_n > 0$.

Corollary 7 If $f(z) = z + \sum_{k=N}^{\infty} a_n z^n \in C_c(\psi)$. Then

$$|f(z^n)| + \sum_{k=N}^{\infty} |a_k||z|^k \leq d(0, \omega),$$

holds for $|z| = r_b \leq \min\{1/3, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(D)$ and $r_0$ is the minimal positive root of the equation:

$$k_\psi(r_0^m) + R_N(r) + k_\psi(-1) = 0,$$

where

$$R_N(r) = \int_0^r \frac{1}{s} \int_0^s M_N^\psi(k_\psi^N) d\psi ds.$$

The result is sharp when $r_b = r_0$ and $l_n > 0$.

Remark 8 Taking $m \to \infty$ and $N = 1$ in Corollary 6 gives (Allu and Halder 2021, Theorem 2.23).

Now, we omit the details of the proof of our next result as it works on the similar lines discussed in the above theorems.

Theorem 11 Let $k_\psi(z) = z + \sum_{n=2}^{\infty} l_n z^n$ be given by (11) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C_c(\psi)$. If $g \in S_\psi(\psi)$. Then

$$|g(z^n)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \omega),$$

holds for $|z| = r_b \leq \min\{1/3, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(D)$ and $r_0$ is the unique positive root of the equation:

$$\int_0^r \frac{1}{s} \int_0^s \psi(t)(k_\psi^N(t^2))^{1/2} d\psi ds + R_N(r)$$

$$= \int_0^1 \frac{1}{s} \int_0^s \psi(t)(k_\psi^N(t^2))^{1/2} d\psi ds$$

where $K'(z) = (k_\psi(z^2))^{1/2}$ and

$$R_N(r) = \int_0^r \frac{1}{s} \int_0^s M_N^\psi(K'(r)) M_N^\psi(\psi) d\psi ds.$$

The result is sharp when $r_b = r_0$ and $l_n > 0$.

Corollary 7 If $f(z) = z + \sum_{n=N}^{\infty} a_n z^n \in C_c(\psi)$. Then

$$|f(z^n)| + \sum_{k=N}^{\infty} |a_k||z|^k \leq d(0, \omega),$$

holds for $|z| = r_b \leq \min\{1/3, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(D)$ and $r_0$ is as given in Theorem 11. The result is sharp when $r_b = r_0$ and $l_n > 0$.

Remark 9 Taking $m \to \infty$ and $N = 1$ in Corollary 7 gives (Allu and Halder 2021, Theorem 2.25).

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Data availability None.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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