MATRICIAL FORMULAE FOR PARTITIONS

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Abstract. The exponential of the triangular matrix whose entries in the diagonal at distance \( n \) from the principal diagonal are all equal to the sum of the inverse of the divisors of \( n \) is the triangular matrix whose entries in the diagonal at distance \( n \) from the principal diagonal are all equal to the number of partitions of \( n \). A similar result is true for any pair of sequences satisfying a special recurrence.

Let \( \Delta \) be the infinite triangular matrix having zeroes on and below the main diagonal, and whose values on each parallel to the main diagonal are all equal to:

\[
\Delta_{i,i+n} = \sigma(n) \quad (i \geq 0, \ n > 0),
\]

where \( \sigma(n) \) is the sum of the inverses of the divisors of \( n \), for \( n > 0 \).

Proposition 1. The matrix

\[
P = \exp(\Delta)
\]

is triangular, with zero below the diagonal, 1 on the diagonal, and its elements on the parallel to the diagonal at distance \( n \) from the diagonal are equal to the number \( p(n) \) of partitions of the integer \( n \) (being \( p(0) = 1 \)):

\[
P_{i,i+n} = p(n) \quad (i \geq 0, \ n \geq 0).
\]

We prove in the next section a more general result. As byproduct, we find a formula (see eq.(3)) for the number of partitions of \( n \) as sum of the products of the function \( \sigma \) over all elements of each partition, and a formula for \( \sigma \) as combination of the products of the number of partitions over all elements of each partition (see eq.(6)).

Generalisation of Proposition 1.

Let \( \{s_n\}_{n=0}^\infty \) be any infinite sequence, satisfying \( s_0 = 0 \).
Let \( t_n \) be the infinite sequence obtained by the following recurrence:

\[
\begin{align*}
  t_0 &= 1, \\
  t_n &= \frac{1}{n} \sum_{k=0}^{n-1} s_{n-k} t_k
\end{align*}
\]

Consider the infinite triangular matrix \( S \) having zeroes on the diagonal and below the diagonal, and whose value on each parallel to the diagonal and at distance \( n \) from the diagonal are all equal to:

\[
S_{i,i+n} = \frac{s_n}{n} \quad (i \geq 0, \ n > 0).
\]

**Theorem 1.** The matrix

\[
T = \exp(S)
\]

is triangular, with zero below the diagonal, 1 on the diagonal, and the values of its elements on each parallel to the diagonal at distance \( n \) from the diagonal are equal to \( t_n \):

\[
T_{i,i+n} = t_n \quad (i \geq 0, \ n \geq 0).
\]

**Example 1.** Proposition 1 is a particular case of Theorem 1. Indeed, \( \sigma(k) = \sigma(k)/k \), \( \sigma(k) \) being the sum of all divisors of \( k \). The recurrence

\[
np(n) = \sum_{k=0}^{n-1} \sigma(n-k)p(k) \equiv \sum_{h=1}^{n} \sigma(h)p(n-h)
\]

is proved, for instance in [1]. (A proof can be done in a similar way as we do for the recurrence of the next example).

**Example 2.** Let \( p^o(n) \) be the number of partitions of \( n \) into odd integers,

\[
\begin{array}{c|cccccccccccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \ldots \\
p^o(n) & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 6 & 8 & 10 & 12 & 15 & 18 & 22 & 27 & 32 & 38 & 46 & 54 & 64 & \ldots \\
\end{array}
\]

and \( \sigma^o_k \) be the sum of odd divisors of \( k \):

\[
\begin{array}{c|cccccccccccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \ldots \\
\sigma^o(k) & 1 & 1 & 4 & 1 & 6 & 4 & 8 & 13 & 6 & 12 & 4 & 14 & 8 & 24 & 1 & 18 & 13 & 20 & 6 & \ldots \\
\end{array}
\]

**Lemma.**

\[
np^o(n) = \sum_{k=0}^{n-1} \sigma^o(n-k)p^o(k) \equiv \sum_{h=1}^{n} \sigma^o(h)p^o(n-h).
\]

**Proof of Lemma.** Consider all partitions of \( n \) into odd integers

\[
  n = o_1 + o_2 + \cdots + o_r, \quad o_i \text{ odd},
\]

and sum all these equalities: we obtain at left \( np^o(n) \), and at right

\[
\sum_{i=1}^{n} o_i \mu_i, \quad \mu_i \geq 0,
\]
μ_i being the number of times the odd o_i appears in total inside all partitions of n into odd integers. We have

\[ \mu_i = \sum_{m=1}^{n/o_i} p^o(n - m o_i). \]

Indeed, there are possibly several copies of o_i in a partition: p^o(n - o_i) is the number of those containing at least one copy of o_i, and then we count in this summand the first copy of o_i in all partitions; p^o(n - 2o_i) is the number of partitions containing at least 2 copies of o_i, and then we count in this summand the second copy of o_i in all partitions, and so on. We obtain:

\[ np^o(n) = \sum_{i=1}^{n} \sum_{o_i | h} p^o(n - h), \]

the second sum being extended to all odd integers dividing h. Then

\[ p^o(n) = \frac{1}{n} \sum_{h=1}^{n} \sigma^o(h)p^o(n - h) = \frac{1}{n} \sum_{k=0}^{n-1} \sigma^o(n - k)p^o(k). \]

Define the matrix Σ^o whose elements are zero on and below the diagonal and

\[ \Sigma^o_{i,i+n} = \sigma^o(n)/n \quad (i \geq 0, \quad n > 0). \]

By Theorem 1, the matrix P^o = exp(Σ^o) is triangular and satisfies:

\[ P^o_{i,i+n} = p^o(n), \quad (i \geq 0, \quad n \geq 0). \]

**Example 3.** Let p^q(n) be the number of partitions of n into integers non divisible by the prime q, and σ^q_k be the sum of divisors of k non divisible by q:

Like in the above lemma, we prove

\[ np^q(n) = \sum_{h=1}^{n} \sigma^q(h)p^q(n - h). \]

Define the matrix Σ^q whose elements are zero on and below the diagonal and

\[ \Sigma^q_{i,i+n} = \sigma^q(n)/n \quad (i \geq 0, \quad n > 0). \]

By Theorem 1, the matrix P^q = exp(Σ^q) is triangular and satisfies:

\[ P^q_{i,i+n} = p^q(n) \quad (i \geq 0, \quad n \geq 0). \]

**Proof of Theorem 1**

The exponential of S is given by:

\[ \exp(S) = E + S + \frac{1}{2!}S^2 + \frac{1}{3!}S^3 + \frac{1}{4!}S^4 + \ldots \]

where E is the identity matrix.
Consider the sequence $\tau_n$, defined as the first row of the matrix $T = \exp(S)$. We will prove that $T_{i,i+n} = \tau_n$, and $\tau_n$ satisfies the recurrence (i).

We use in the following the notation $s(n)$ instead of $s_n$ in order to avoid many subscript indices. We thus have:

$$S_{i,k} = s(k-i)/(k-i), \quad S_{0,k} = s(k)/k.$$ 

We calculate the powers of $S$.

The square $S^2$ is equal to:

$$S^2_{i,j} = \sum_{k=0}^n S_{i,k}S_{k,j} = \sum_{i<k<j} s(k-i)s(j-k)/(k-i)(j-k),$$

being $S_{i,k} = 0$ for $k \leq i$. Hence $S^2_{i,j} = 0$ for $j < i + 2$.

The matrix $S^3$ is equal to:

$$S^3_{i,j} = \sum_{k=0}^n S_{i,k}S^2_{k,j} = \sum_{k=0}^n S_{i,k} \sum_{h=0}^n S_{k,h}S_{h,j} = \sum_{i<k<h<j} s(k-i)s(h-k)s(j-h)/(k-i)(h-k)(j-h),$$

being $S^2_{i,j} = 0$ for $j \leq i + 1$. Hence $S^3_{i,j} = 0$ for $j < i + 3$.

We obtain, for the $r$-th power of $S$:

$$S^r_{i,j} = \sum_{i=k_0<k_1<k_2<...<k_r=j} \prod_{m=1}^r s(k_m-k_{m-1})/(k_m-k_{m-1}),$$

which is zero if $j < i + r$.

The first elements of the sequence $\tau_0, \ldots, \tau_n$ are the first elements of the first row of the matrix $\sum_{r=0}^n \frac{1}{r!}(S)^r$, since the elements at distance lower than $m$ over the diagonal of $S^m$ are zero, as we have seen.

The element $T_{0,n}$ of the first row of the matrix $E + \sum_{r=1}^n \frac{1}{r!}S^r$ are therefore, for $n > 0$

$$\tau_n = \left\{ E + \sum_{r=1}^n \frac{1}{r!}S^r \right\}_{0,n} = \sum_{r=1}^n \frac{1}{r!} \sum_{0<k_1<k_2<...<k_r=n} \prod_{m=1}^r s(k_m-k_{m-1})/(k_m-k_{m-1}),$$

and $\tau_0 = 1$. For every $r = 1 \ldots n$, the $r$ positive integers $k_m$ satisfy

$$\sum_{m=1}^n (k_m-k_{m-1}) = n;$$

hence the sum is extended to all partitions of $n$ containing exactly $r$ elements $n = h_1, \ldots, h_r$. However, for every partition, the same product $\prod_{m=1}^r s(h_m)/(h_m)$ corresponds to a number of cases with different $k_m$ depending on the ordering of the $h_m$. This number of different cases is equal to

$$r!/\prod_{m=1}^r \rho_m!,$$

where $\rho_m$ is the multiplicity of $h_m$ in the partition.

Hence we obtain

**Proposition 2.** The elements $\tau_n$, for $n > 0$, are equal to:
\[
\tau_n = \sum_{r=1}^{n} \sum_{X_r(n)} \prod_{m=1}^{r} \frac{s(h_m)}{h_m \rho_m!}
\]

where \(X_r(n)\) is the set of partitions of \(n\) into exactly \(r\) summands \(h_m\), and \(\rho_m\) is the multiplicity of \(h_m\).

The formula expressing the elements of \(S^r\) shows that \(S_{i,j}^r\) depends only on the difference \((j - i)\), and therefore the elements \(T_{i,j}\) are equal on each parallel at distance \(r\) from the diagonal, where \(j = i + r\).

The proof of Theorem 2 is now completed proving that formula (3) satisfies the recurrence:

\[
\tau_N = \frac{1}{N} \sum_{n=0}^{N-1} s(N - n) \tau_n.
\]

We write the left member of (4) by substituting the terms \(\tau_n\) given by formula (3):

\[
\frac{1}{N} \left( s(N) + \sum_{n=1}^{N-1} s(N - n) \sum_{r=1}^{n} \sum_{X_r(n)} \prod_{m=1}^{r} \frac{s(h_m)}{h_m \rho_m!} \right).
\]

Each summand in the expression from any \(\tau_n, 0 < n < N\), is the product of \(r\) factors \(s(h_m)\), such that \(\sum_{m=1}^{r} h_m \rho(m) = n\). In expression (5) it is multiplied by \(s(N - n)\) and becomes therefore the product of \(r + 1\) factors \(s(h_m)\) such that their sum is \(N\), as in the expression of \(\tau_N\) by (3).

Consider now any one of such terms, containing \(\prod_{m=1}^{r+1} s(h_m)\), in which one factor, say \(s(h_{m^*})\), is \(s(N - n)\).

If \(h_{m^*}\) is different from the other \(r\ \ h_m\), the contribution of the corresponding term is equal to

\[
\frac{1}{N} \frac{1}{\prod_{m=1}^{r+1} h_m \rho_m!} = \frac{1}{N} \frac{1}{\prod_{m=1}^{r+1} h_m \rho_m! / h_{m^*}}.
\]

If \(h_{m^*}\) is equal to one of the other \(h_m\), i.e., in the product has multiplicity \(\rho_m > 1\), the coefficient of the corresponding term is equal to

\[
\frac{1}{N} \frac{1}{\prod_{m=1}^{r+1} h_m \rho_m! / (h_{m^*} \rho_{m^*})},
\]

since \(s(h_{m^*} = N - n)\) had multiplicity \(\rho_{m^*} - 1\) in the original term in \(\tau_n\).

If among the \(r + 1\) integers \(h_m, d \leq r\ \ \)integers \(h_{m^*}\) are different, in correspondence of the same product we are considering, there will be in \(t_N\) the contribution of \(d\) terms, namely for every value of \(n\) such that \(N - n = h_{m^*}\).

The coefficient of the terms \(\prod_{m=1}^{r+1} s(h_m)\) is thus

\[
\frac{1}{N} \sum_{k=1}^{d} \frac{1}{\prod_{m=1}^{r+1} h_m \rho_m! / (h_{m^*} \rho_{m^*})},
\]
which is equal to

\[ \sum_{k=1}^{d} h_{m_k} \rho_{m_k} = \frac{N}{N \prod_{m=2}^{r+1} h_m \rho_m} = \frac{1}{\prod_{m=1}^{r+1} h_m \rho_m}. \]

Hence we obtain from eq. (5), denoting \( q = r + 1 \):

\[ \tau_N = \frac{s(N)}{N} + \sum_{r=1}^{N-1} \sum_{X_{r+1}(N)} \prod_{m=1}^{r+1} \frac{s(h_m)}{h_m \rho_m} = \frac{s(N)}{N} + \sum_{q=2}^{N} \sum_{X_q(N)} \prod_{m=1}^{q} \frac{s(h_m)}{h_m \rho_m}, \]

i.e., the expression of \( \tau_N \) by eq. (3).

**Inverse formula**

We see now how the sequence \( s_n \) can be written in terms of the sequence \( t_n \) satisfying the recurrence (2).

**Proposition 3.** The following identity holds:

\[ \frac{s_n}{n} = \sum_{r=1}^{n} (-1)^{r-1}(r-1)! \sum_{X_r(n)} \prod_{m=1}^{r} \frac{t_{h_m}}{\rho_m}. \]

**Proof.**

From the identity \( T = \exp(S) \) we obtain

\[ S = \ln T = \ln(E + (T - E)) = \sum_{r \geq 1} (-1)^{r-1} \frac{1}{r}(T - E)^r. \]

The first row of matrix \( T - E \) contains, but the initial elements which is zero, the elements of the sequence \( t_n \). Hence we calculate the elements of \( (T - E)^r \), as in the proof of Theorem 1, and we add them with coefficients \( (-1)^{r+1}/r \), to obtain the sequence \( s_n \).

\[ \square \]

**References**

[1] E. Grosswald, *Topics from the theory of numbers*, Macmillan, New York, 1966, p. 226.

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