Knot Invariants from Classical Field Theories

Lorenzo Leal

Departamento de Física, Facultad de Ciencias,
Universidad Central de Venezuela, AP 47270,
Caracas 1041-A, Venezuela
Email: lleal@tierra.ciens.ucv.ve

Abstract

We consider the Non-Abelian Chern-Simons term coupled to external particles, in a gauge and diffeomorphism invariant form. The classical equations of motion are perturbatively studied, and the on-shell action is shown to produce knot-invariants associated with the sources. The first contributions are explicitely calculated, and the corresponding knot-invariants are recognized. We conclude that the interplay between Knot Theory and Topological Field Theories is manifested not only at the quantum level, but in a classical context as well.

1Talk delivered at the Spanish Relativity Meeting, Bilbao, Spain, 1999
1 Introduction

The study of quantum Chern-Simons theory (C.S.T.) and its relationship with Knot Theory (K.T.) is an active field of research both in Mathematics and Theoretical Physics [1]. The bridge between field and knot theories was established when E.Witten discovered that the vacuum expectation value of the Wilson Loop (W.L.) yields a knot invariant closely related with the Jones polynomial [2, 3]. From a physical point of view, this relationship has interesting applications. For instance, the knot invariants obtained from a perturbative expansion of the W.L. average [4] provide solutions to the constraints of Ashtekar Quantum Gravity [5] in the Loop Representation [6, 7]. The purpose of this work is to show that the interplay between C.S.T. and K.T. is manifested also at the classical level. To this end, we shall develop a perturbative study of the classical equations of motion of the C.S.T. coupled to external point particles. As we shall see, this study leads to the obtention of link invariants associated to the world lines of the particles, inasmuch the calculation of the expectation value of the W.L. does in the quantum theory. This approach was explored by the author for the abelian case several years ago [8].

2 Classical Chern-Simons Theory

2.1 Preliminaries

The SU(N) action is given by

$$ S_{CS} = -k \int d^3 x \epsilon^{\mu \nu \rho} tr(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) $$

(1)

with $A_\mu = A^a_\mu T^a$. The $N^2 - 1$ generators $T^a$ are chosen so that $tr T^a T^b = -\frac{1}{2} \delta^{ab}$ and $[T^a, T^b] = \Gamma^{abc} T^c$, $\Gamma^{abc}$ being the (completely antisymmetric) structure constants. We shall add an interaction term

$$ S_I = \int d^3 x tr(A_\mu J^\mu) $$

(2)

with

$$ J^\mu(x) = \sum_{i=1}^n \int d\tau \delta^3(x - z(\tau)) z^\mu_{(i)}(\tau) I_{(i)}(\tau) $$

(3)

In this expression, $z^\mu_{(i)}(\tau)$ denotes the world line of the $i-th$ particle, which acts as an externally prescribed source for the C.S. field. Indeed, since we are going to take these world lines as closed curves in 2+1 dimensions, the analogy with point particles holds only formally. The algebra valued object $I_{(i)}(\tau) = I^a_{(i)}(\tau) T^a$ could be interpreted as the color charge carried by the $i-th$ particle. The current $J^\mu(x)$ was introduced by Wong [8] to deal with point particles interacting with the Yang-Mills field. Varying $S_{CS} + S_I$ w.r.t. $A_\mu$ produces the field equations.
\[ k \epsilon^{\mu \nu \rho} F_{\nu \rho} = J_\mu \] 

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). Consistence of Eq.(4) leads to

\[ D_\mu J_\mu \equiv \partial_\mu J_\mu + [A_\mu, J_\mu] = 0 \] (5)

Suppose now that one is able to solve Eqs.(4),(5) to get \( A_\mu = A_\mu(J) \), and to calculate the on shell action \( S_{OS} = S[A(J)] \). Since both \( S_{CS} \) and \( S_I \) are topological terms, it is immediate to see that \( S_{OS} \) is metric independent too. Moreover, since \( S_{OS} \) depends only on the curves \( \dot{z}_\mu(i)(\tau) \), we have to conclude that it is a link-invariant associated to the curves. Of course, one cannot solve the non-linear equations (4),(5) exactly. Therefore, we shall adopt a perturbative scheme, with \( \Lambda = k^{-1} \) being the expansion parameter. We shall get

\[ S_{OS}(J) = \sum_{p=0}^{\infty} \Lambda^p S^{(p)}(J) \] (6)

and, since \( S_{OS}(J) \) is a link-invariant for all \( \Lambda \), the same will be true for each contribution \( S^{(p)} \).

### 2.2 Perturbative Solution

Next we sketch how the perturbative procedure goes on, and calculate \( S_{OS} \) explicitly up to first order in \( \Lambda \). It can be seen that the consistence equation (5) leads to

\[ \frac{dI_\mu(i)(\tau)}{d\tau} + R^{ac}_{(i)}(\tau) I_c(i)(\tau) = 0 \] (7)

with

\[ R^{ac}_{(i)}(\tau) \equiv \Gamma^{abc} \dot{z}_\mu(i)(\tau) B^b_\mu(z(i)(\tau)) \] (8)

and \( B_\mu \equiv \Lambda^{-1} A_\mu \). Eq. (7)is solved by

\[ I_\mu(i)(\tau) = Texp \left[ -\Lambda \int_0^\tau d\tau' R^{ac}_{(i)}(\tau') \right] I_\mu(i)(0) \] (9)

where \( T \) denotes the path ordered exponential along the curve \( z_{(i)}^\mu(\tau) \). Putting Eq.(7) into Eq.(4) one obtains, after developing the ordered exponential

\[ e^{\mu \nu \rho}(\partial_\nu B_\rho^a - \partial_\rho B_\nu^a) = \]

\[ \sum_{i=1}^n \oint dz_\mu(i) \delta^3(x - z_i) I_\mu^a(i)(0) + \Lambda \epsilon^{\mu \nu \rho} \Gamma^{abc} B_\nu^b B_\rho^c \]

\[ -\Lambda \sum_{i=1}^n \oint dz_\mu(i) \delta^3(x - z_i) \int_0^z dz_\mu(i1) R^{a1}_{\mu1}(z_{(i1)}^a) I_{(i1)}^a(0) \]
To solve this equation perturbatively, we introduce the power expansion

$$B^a_\mu = \sum_{p=0}^\infty \Lambda^p B^a_\mu^{(p)}$$

into Eq. (10). This yields, for the 0th order

$$\epsilon^{\mu\nu\rho} (\partial_\nu B^a_\rho^{(0)} - \partial_\rho B^a_\nu^{(0)}) = \sum_{i=1}^n \oint dz^{(i)}(x - z^{(i)}) I^a_i (0)$$

The first order equation is given by

$$\epsilon^{\mu\nu\rho} (\partial_\nu B^a_\rho^{(1)} - \partial_\rho B^a_\nu^{(1)}) = \epsilon^{\mu\nu\rho} \Gamma^{abc} B^b_\nu^{(0)} B^c_\rho^{(0)} - \sum_{i=1}^n \oint dz^{(i)}(x - z^{(i)}) \int_0^z dz^{(i)}_1 I^a_i^{(0)} (z^{(i)}_1) I^a_i (0)$$

For the sake of brevity we omit the $p$th equation. Its general structure is given by

$$\epsilon^{\mu\nu\rho} \partial_\nu B^a_\rho^{(p)} = J^{\mu a}$$

where $J^\mu$ depends on $B^a_\mu$, with $q < p$. This feature allows one to solve Eq. (10) order by order in a straightforward manner. In fact, $B^a_\mu$ is given by

$$B^a_\alpha (x) = -\frac{1}{4\pi} \int d^3 x' \epsilon_{\alpha\beta\gamma} J^{\beta\gamma} (x') \frac{(x - x')^\gamma}{|x - x'|^3}$$

plus the gradient of an arbitrary function, which can be set equal to zero. This amounts to choosing the gauge $\partial^\nu B^a_\nu = 0$. Observe that for the first time, a metric (the Euclidean one) appears into the discussion. This will not spoil the diffeomorphism invariance of the on-shell action, because the latter is not sensible to metric choices. It could be said that choosing a metric to solve the equations amounts to fixing a gauge, or a ”geometry”, that will not break the topological invariance of the ”geometry-independent” on-shell action. As before, there are consistency requirements which must be studied: from Eq. (14) it is immediate that $J^{\mu a}$ must be conserved. This is clearly fulfilled in the 0th order case, but as we shall discuss later, it is by no means trivial for higher orders. We now turn to the on-shell action. We set
\[ S \equiv -\frac{2}{\Lambda} S_{OS} \]
\[ = \int d^3x \varepsilon^{\mu \nu \rho} (B^a_\mu \partial_\nu B^a_\rho + \frac{2}{3} \Lambda B^a_\mu B^b_\nu B^c_\rho \Gamma^{abc} I_{B=0}) \]

where we have used the equation of motion Eq. (16) to eliminate \( J^\mu \) in terms of \( B^\mu \).

Using equations (12) and (15), we have, up to 0th order

\[ S^{(0)} = \int d^3x \varepsilon^{\mu \nu \rho} (0) B^a_\mu (0) B^a_\rho B_{B=0} \]
\[ = \frac{1}{4} \sum_{i,j} I^a_i (0) I^a_j (0) \mathcal{L}(i,j) \]

where

\[ \mathcal{L}(i,j) = \frac{1}{4\pi} \int dz^\mu \int \varepsilon^{\mu \nu \rho} \left( \frac{(z-z')^\nu}{|z-z'|^3} \right) \]

is the Gauss Linking Number of the curves \( i, j \), which is a link invariant. (There are some subtleties about the case \( i = j \), whose discussion is out the scope of this paper [4]).

To obtain the first order contribution \( S^{(1)} \), we have to study the consistence of Eq. (13). Taking the divergence on both sides, and after some calculations, one is lead to the condition

\[ \sum_{i,j} \Gamma I^a_i (0) I^a_j (0) \mathcal{L}(i,j) \delta^3(x-z_i(0)) = 0 \]

where \( z_i(0) \) is the starting point of the curve \( i \). From the last equation we see that a sufficient condition for Eq. (13) being consistent is that \( \mathcal{L}(i,j) = 0 \forall i,j \). A careful look at Eq. (19) reveals that if the curves do not intersect each other, the former is indeed a necessary condition too. Under these conditions, Eq. (13) gives the solution of Eq. (13), which when substituted into Eq. (16) yields the first order contribution to \( S \)

\[ S^{(1)} = \sum_{i,j,k} G_{i,j,k} M(i,j,k) \]

where \( G_{i,j,k} \) is a group theoretical factor and

\[ M(i,j,k) = \int d^3x \varepsilon^{\mu \nu \rho} C_{(i)\mu}(x) C_{(j)\nu}(x) C_{(k)\rho}(x) \]
\[ + \int d^3x \int d^3y \left[ T^{\mu \nu}_{(i)}(x,y) C_{(j)\mu}(x) C_{(k)\nu}(y) + \text{cyclic perm. of } i,j,k \right] \]
\[ C_{(j)\mu}(x) \equiv \frac{1}{8\pi} \oint dz^\gamma \frac{(x - z)^3}{|x - z|^3} \epsilon_{\mu\beta\gamma} \]  

(22)

and

\[ T^{\mu\nu}_{(i)}(x, y) \equiv \oint dz^\mu \int_0^z dz'^\nu \delta^3(z - x) \delta^3(z' - y) \]  

(23)

The quantity defined in Eq. (21) is the Triple Milnor’s Linking Number \([10, 11, 12]\) associated to the curves \(i, j, k\), which is known to be a link-invariant, provided the Gauss Linking Numbers vanish for every \(i, j\). Hence, we obtain the following nice feature: the consistence condition for solving the first order equation of motion, is precisely the condition which allows the first order contribution to the on-shell action to coincide with a non-trivial link-invariant. As an application of this invariant, we recall that the Borromean Rings have non-vanishing \(M(i, j, k)\) while their Gauss Linking Number vanish \([11]\).

3 Discussion

We have presented results supporting the following claim: Classical (not only Quantum) Topological Field Theories are an interesting tool for the study of link invariants. Concretely, we found that the on-shell action of the C.S.T. coupled with string-like sources, admits a perturbative expansion in powers of the inverse coupling constant, whose first and second order contributions give respectively the Gauss Linking Number and the Triple Milnor’s Linking Number. We want to underline that our proposal and results are completely independent of quantum considerations.

The perturbative scheme we have reported could certainly be carried out further. We conjecture that as far as the links allowed do not intersect each other, the corresponding higher order contributions to the on-shell action will yield Milnor’s higher-order linking coefficients. This should be compared with similar results of reference \([12]\), obtained within the quantum framework. It would also be interesting to study the case where the curves intersect. These and other related issues are under work.

4 Acknowledgments

I wish to thank to the Organizers of the ERE-99 and to the City of Bilbao for their warm hospitality. This work was supported by CDCH, Universidad Central de Venezuela.

References

[1] J.M.F. Labastida, Chern-Simons Gauge Theory: Ten Years After, USC-FT-7/99, hep-th/9905057.
[2] E. Witten, Commun. Math. Phys. \textbf{121}, 351 (1989).
[3] V. F. R. Jones, Bull. AMS \textbf{12}, 103 (1985).
[4] E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B330, 575 (1990).
[5] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986).
[6] C. Di Bartolo, R. Gambini, J. Griego, J. Pullin, Consistent Canonical Quantization of General Relativity in the Space of Vassiliev Knots Invariants, gr-qc/9909063.
[7] C. Rovelli, L. Smolin, Nucl. Phys. B331, 80 (1990).
[8] L. Leal, Mod. Phys. Lett. A7, 541 (1992).
[9] S. K. Wong, Nuov. Cim. A65, 689 (1970).
[10] J. Milnor, Ann. of Math. 59, 177 (1954).
[11] M. I. Monastyrsky, V. S. Retakh, Comm. Math. Phys. 103, 445 (1986).
[12] L. Rozansky, J. Math. Phys. 35, 5219 (1994).