PERTURBATIVE GAUGE ANOMALIES 
IN THE HAMILTONIAN FORMALISM: 
A COHOMOLOGICAL ANALYSIS.

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ABSTRACT

The quantum action principle of renormalisation theory is applied to the antibracket-antifield formalism for Hamiltonian systems. General results on the local BRST cohomology allow one to prove that the anomalies appear in the time development of the BRST charge and violate the nilpotency of this charge. Furthermore they are equivalent to those of the Lagrangian formalism. The analysis provides a completely gauge and regularisation independent proof of Faddeev’s conjecture on the relationship between gauge anomalies and Schwinger terms in the context of descent equations.

1. Introduction

The existence of a relationship between gauge anomalies and the appearance of Schwinger terms in the equal time commutation relations between the corresponding currents was first established by perturbative calculations. The discovery that anomalies are constrained by consistency conditions and the use of the quantum action principle allowed to determine the existence of anomalies and to calculate their form by cohomological techniques. It is natural that there should exist a related algebraic principle constraining the existence of possible Schwinger terms. For non-abelian anomalies, the first investigation in this direction in the Hamiltonian formalism showed that the Schwinger terms should be related to the gauge anomaly through descent equations. It was followed by a series of perturbative calculations in order to verify this conjecture.

Recently Hamiltonian BRST methods have been used and an algebraic principle based on locality assumptions and the validity of the Jacobi identity for the equal time commutator involving the BRST charge and the Hamiltonian has been conjectured. The authors of then infer that the anomaly in \([\Omega, H]\) corresponds to

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the Lagrangian anomaly and that the link with the the anomaly in \([\hat{\Omega}, \hat{\Omega}]\) should be understood by some some sort of descent. As they are careful to point out, they are interested above all in the calculational aspect of anomalies and their results are not meant to be rigorous because no discussion of renormalisation in the Hamiltonian formalism is made.

The purpose of this letter is the rigorous treatement of anomalies in the Hamiltonian framework. We will be able to prove the basic equations about Hamiltonian anomalies conjectured in \(^2\) (see also \(^1\)). The main idea is to quantize perturbatively our first order Hamiltonian system like a Lagrangian one. It has already been shown \(^1\) that this gives formally equivalent results to those obtained from a perturbative quantization of the underlying Lagrangian system. There is no need for a new algebraic consistency condition because the quantum action principle applies and, like in the Lagrangian case, it can be used to reduce the whole analysis of the first order renormalisation effects to a local cohomological problem, which amounts to finding the general solution to a set of descent equations.

We show that, by choosing appropriate representatives in the relevant cohomological classes, the content of these descent equations can be expressed in terms of quantities of the Hamiltonian formalism. In terms of these representatives, the relationship through descent equations of the anomaly in the time development of the BRST charge and the violation of its nilpotency naturally follows (Faddeev’s conjecture). Furthermore, the equivalence to the Lagrangian anomalies is also direct, because of a theorem \(^4\) proving the invariance of local BRST cohomology classes with respect to generalized auxiliary fields content, which is precisely what the Langrangian and the Hamiltonian antibracket-antifield formalisms differ in.

Regularisation independence of the analysis is guaranteed by using general results from renormalisation theory, like the quantum action principle, while gauge independence holds for all gauges of the antibracket-antifield formalism.

### 2. Quantum action principle and antibracket-antifield formalism.

Let us briefly recall the application of the quantum action principle (see \(^2\) for a review) in the context of the antibracket-antifield formalism. From a classical gauge theory with a local action \(S_0[\phi]\), one builds a possibly non-minimal proper and local solution of the classical master equation \(^2\) \((S, S) = 0\). A gauge fixed action is obtained by making a canonical (in the antibracket sense) change of variables which consists either of shifting the antifields with the help of some local gauge fixing fermion \(\Psi\) or of exchanging for some field-antifield pair the role of the field and the antifield (including a minus sign) \((\phi^A, \phi^*_A) \longrightarrow (-\phi^*_A, \phi^A)\) in such a way that after putting to zero all the (new) antifields, the action has no gauge freedom left. One then considers the extended action \(S_{ext}[\phi^A; \phi^*_A] = S[\phi^A; \phi^*_A + \delta\Psi/\delta\phi^A]\). The Leg-

\(^2\) For a different point of view on anomalies in this formalism see also \(^3\).

\(^4\) It is sufficient for the first order quantum corrections to consider the classical master equation, because the corrections of the quantum master equation are ill-defined and drop out anyway in the course of (BPHZ) renormalisation (see \(^4\)).
endre transform on the sources $J_A$ of the generating functional $Z_c[J_A; \phi^*_A]$ for connected Green’s functions associated to $S_{ext}[\phi^A; \phi^*_A]$ gives the generating functional for one particle irreducible Green’s functions $\Gamma[\phi^A; \phi^*_A]$ for connected Green’s functions associated to $S_{ext}[\phi^A; \phi^*_A]$ which satisfies the master equation $(S_{ext}, S_{ext})$ in the sources $\phi^A$ and $\phi^*_A$. The quantum action principle then states that

$$ (\Gamma, \Gamma) = [\Delta \cdot \Gamma] $$

where $\Delta$ is a local integrated polynomial of ghost number 1, of fixed dimension and of order at least one in $\hbar$. Here $[\Delta \cdot \Gamma]$ denotes the generating functional for one particle irreducible Green’s functions with the insertion $\Delta$. The quantum action principle is a general result from renormalisation theory; it has been proved in various renormalisation schemes and is believed to be scheme independent. At lowest order the identity $((\Gamma, \Gamma), \Gamma) = 0$ yields the consistency condition $(\Delta, S_{ext}) = 0$ in the sources $\phi^A$ and $\phi^*_A$ on the insertion $\Delta$. Solutions of the form $\Delta = (\Lambda, S_{ext})$, with $\Lambda$ a local integrated polynomial of ghost number zero and the dimension of $S_{ext}$, can be absorbed through local counterterms added to $S_{ext}$.

Dropping the subscript in $\phi^A$ and making the canonical change of variables $\phi^*_A \rightarrow \phi^*_A - \delta \Psi / \delta \phi^A$ we have to find all cohomologically non trivial solutions of

$$ (\Delta', S) = 0 $$

with $\Delta'[\phi^A, \phi^*_A] = \Delta[\phi^A, \phi^*_A - \delta \Psi / \delta \phi^A]$. Let $\Delta' = \int b_{[k]}$ where $b_{[k]}$ is a $D$--form valued polynomial in the fields, the antifields and a finite number of their derivatives. The boundary conditions on the fields and antifields (they are sources and as such they are $C^\infty$ fast decreasing functions), imply that (3) is equivalent to (see for instance chapter 12)

$$ sb_{[k]} + db_{[k-1]} = 0. $$(3)

A non trivial solution $b_{[k]}$ is an element of $H(s|d)$ in form degree $D$ whose shortest descent stops after $k$ steps. Indeed, because of the triviality of the cohomology of the algebraic exterior differential $d$, (3) yields a set of descent equations

$$ sb_{[k-1]} + db_{[k-2]} = 0 $$(4)

$$ \vdots $$

$$ sb_{[0]} = 0. $$(5)

The strategy to find the most general non-trivial solution of (3) is the following: first one looks for the most general solution of the bottom equation (5), i.e., one chooses all $s \mod d$ nontrivial solutions of $H(s)$ in all form degrees. Then one tries if the solutions $b_{[0]}$ in form degree $D - k$ can be lifted $k$ steps to yield a non-trivial solution of (3).

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*Footnote:* If the bottom is $s \mod d$ trivial, a redefinition of $b_{[k]}$ allows to get a shorter set of descent equations.
Because the usual non-minimal sectors introduced for gauge fixing purposes do not contain derivatives of the additional fields, the contracting homotopy, which eliminates these fields from \( H(s) \), commutes with derivatives (see chapter 12) and they do not appear in \( H(s|d) \) either, implying that it is enough to consider \( s \) associated to a minimal solution of the master equation.

3. Application to Hamiltonian systems

3.1. Descent equations in the Hamiltonian framework

Consider a local extended Hamiltonian formalism satisfying suitable regularity conditions. The BRST charge \( \Omega \) and the Hamiltonian \( H \) can be constructed as local functionals in space, \( \Omega = \int d^{D-1}x \omega, \ H = \int d^{D-1}x h \), where \( \omega \) and \( h \) depend on the fields \( \{ q^i(t, x^m), \ p_i(t, x^m), \ \eta^a(t, x^m), \ P_a(t, x^m) \} \equiv \phi^A(t, x^m), \ m = 1, ..., D - 1 \) and a finite number of their spatial derivatives \( \partial_m \). A local proper solution to the master equation is given by

\[
S_H[\phi^A, \phi^*_A] = \int d^Dx L_H = \int dt \int d^{D-1}x - \frac{1}{2} \dot{\phi}^A \sigma_{AB} \dot{\phi}^B - h + \frac{\partial \omega}{\partial (\phi^A)^0(k)} \sigma^{AB}(\phi^*_B)^0(k), \tag{6}
\]

where we have made the identification \( P_a = -\lambda^*_a \), with \( \lambda^a \) the Lagrange multipliers for the constraints. For notational simplicity we consider only irreducible constraints, the reducible case can be analyzed along the same lines. If one puts the antifields to zero, the gauge is completely fixed (multiplier gauge \( \lambda^a = 0 \)). The local nilpotent BRST symmetry

\[
s_H = \frac{\partial}{\partial (\phi^A)^l(k)} (\sigma^{AB} \frac{\delta}{\delta \phi^B} (\phi^*_A)^l(k)) - \frac{\partial}{\partial (\phi^*_A)^l(k)} (\sigma^{AB} \frac{\delta}{\delta \phi^B} (\phi^A)^l(k)) + \frac{\partial}{\partial \omega} (\sigma^{AB} (\phi^*_B)^0(m)) l(k). \tag{7}
\]

splits according to the antifield number into the sum of \( \delta \), the Koszul-Tate differential of the Hamiltonian stationary surface, and \( \sigma = s_\omega + \gamma \), where

\[
s_\omega = \frac{\partial}{\partial (\phi^A)^0(k)} (\sigma^{AB} \frac{\delta}{\delta \phi^B} (\phi^*_A)^0(k)) \tag{8}
\]

is the local version of the usual Hamiltonian BRST symmetry. Take

\[
\rho = -\frac{\partial}{\partial (\phi^A)^{l+1(k)}} \sigma^{AB} (\phi^*_B)^l(k). \tag{9}
\]

\(^4\)In the index \( l(k) \), \( l \) refers to the time while \( (k) \) is a space multiindex.

\(^*\)\( \delta/\delta \) is the Euler-Lagrange derivative in space.
Consider a representative of an element of $H(s_H)$, $b \in \mathcal{FA}$, where $\mathcal{FA}$ is the space of form valued polynomials in the fields, the antifields and a finite number of their derivatives, and apply the anticommutator of $\rho$ with $s_H$ to homogeneous terms in $b$ containing $n$ antifields and $m$ fields with at least one time derivative, $n + m \neq 0$, to get

$$(n + m)b - s_H \rho b = -\frac{\bar{\delta} b}{\partial (\phi^A)^{l+1}(k)} \sigma^{AB}(-\frac{\delta h}{\delta \phi^A} + \frac{\bar{\delta}}{\delta \phi^A} \partial (\phi^A)^{0(m)}(k)),$$

This means that, by repeated redefinitions, one can first absorb all time derivatives of the fields and then the antifields. A representative of an element of $H(s_H)$ can be chosen from $\tilde{\mathcal{F}}$, the space of form valued polynomials in the fields and a finite number of their spatial derivatives. Because on such representatives, the cocycle condition $s_H b = 0$ reduces to $s_\omega b = 0$, and the freedom of adding $s_H$-exact terms is reduced to $s_\omega$-exact terms we get that $H(s_H)$ in $\mathcal{FA}$ is isomorphic to $H(s_\omega)$ in $\tilde{\mathcal{F}}$.

Let $d = \tilde{d} + d^0$ where $\tilde{d} = dx^m \partial_m$ and $d^0 = dt \partial_t$. The bottom $b_0 \in \tilde{\mathcal{F}}$ of a set descent equations has to be a $s_\omega \mod \tilde{d}$ non-trivial element of $H(s_\omega)$. Trying to lift $b_0$, we have to solve the equation

$$s_H b_{[1]} + db_{[0]} = 0. \tag{11}$$

Applying $\rho s_H + s_H \rho$ to $b_{[1]}$, we get (11) with the additional term $-\rho db_{[0]}$ on the right hand side. Because this term contains no time derivatives of the fields, $b_{[1]}$ can be chosen to be independent of the time derivatives of the fields as well. The acyclicity of $\delta$ in $\mathcal{FA}$, then implies that all terms with antifield number higher than 2 can be absorbed. If $b_{[1]} = dt b^0_{[1]} + \tilde{b}_{[1]}$ where $b^0_{[1]}, \tilde{b}_{[1]}$ are independent of $dt$, (11) splits according to antifield number and $dt$ into

$$\delta b^0_{[1]} + s_\omega b^0_{[1]} + \partial_t b_{[0]} - \tilde{d} b_{[0]} = 0 \tag{12}$$
$$\delta \tilde{b}_{[1]} + s_\omega \tilde{b}_{[1]} + \tilde{d} b_{[0]} = 0 \tag{13}$$
$$\sigma \tilde{b}_{[1]} = 0 \sigma b^0_{[1]} = 0. \tag{14}$$

A necessary condition is that the terms multiplying the time derivatives of the fields agree on both sides of these equations. This implies that

$$b^0_{[1]} = \frac{\partial}{\partial (\phi^A)^{0(k)}} \sigma^{AB} (\phi^A)^{0(k)}$$

and $\delta \tilde{b}_{[1]} = 0$ meaning that $\tilde{b}_{[1]}$ can be absorbed through redefinitions of $b_{[1]}$. We then get equations involving elements of $\mathcal{F}$ alone:

$$s_\omega b^0_{[1]} + \frac{\partial}{\partial (\phi^A)^{0(k)}} \sigma^{AB} (\phi^A)^{0(k)} - \tilde{d} b^0_{[0]} = 0 \tag{16}$$
$$s_\omega \tilde{b}_{[1]} + \tilde{d} b_{[0]} = 0 \tag{17}$$
Furthermore, \( \sigma^{0}_{b[1]} = 0 \) can be shown to be satisfied because \( s_{\omega}\tilde{b}[0] = 0 \).

Trying to lift \( b[1] \) in the same way, we find that each \( b[l] \), \( 0 \leq l \leq k \), can be chosen independent of the time derivatives of the fields and at most linear in the antifields (acting with \( d \), no time derivative acts on the antifield dependent part of \( b[l-1] \)), with

\[
b[l] = dt \left( \frac{\delta}{\partial(\phi^{A})^{0(k)}} \sigma^{AB}(\phi^{A})^{0(k)} \right) + dtb^{0}_{l} + \tilde{b}[l] \quad (18)
\]

where \( \tilde{b}[k] = 0 \) and verifying

\[
s_{\omega}b_{l}^{0} + \frac{\delta}{\partial(\phi^{A})^{0(k)}} \sigma^{AB}(\phi^{A})^{0(k)} = 0 \quad (19)
\]

The equation

\[
s_{\omega}b_{l}^{0} + \frac{\delta}{\partial(\phi^{A})^{0(k)}} \sigma^{AB}(\phi^{A})^{0(k)} = 0 \quad (20)
\]

is again satisfied because \( s_{\omega}\tilde{b}[l-1] + \tilde{b}[l-2] = 0 \).

Hence solving the descent equation for \( s_{H} \) is equivalent to finding first the most general solution of the (spatial) descent equations associated to \( s_{\omega} \), in maximal spatial form degree \( D - 1 \), and then selecting those for which there exist solutions \( b^{0}_{k} \) and \( b^{0}_{k-1} \) satisfying (19).

### 3.2. Identification of the anomalies

Let us analyze in more detail the left hand side of (1) to identify the classical relations which acquire quantum corrections:

\[
\frac{\delta}{\partial\phi^{A}} \mathcal{L}_{H} \frac{\delta}{\partial(\phi^{A})^{0(l)}} \mathcal{L}_{H} = -\dot{\omega} + \partial_{k}((\dot{\phi}^{A})^{0(l)}) \frac{\delta}{\partial(\phi^{A})^{0(l) + c_{k}}} - [h, \omega]_{loc} + \frac{1}{2} \frac{\delta}{\partial(\phi^{A})^{0(l)}} [\omega, \omega]_{loc} \sigma^{AB}(\phi^{A})^{0(l)}. \quad (22)
\]

Here \([ \cdot, \cdot ]_{loc} \) is the extended local Poisson bracket defined in terms of the spatial Euler-Lagrange derivatives. This means that after integration over space and putting the antifields to zero

\[
\frac{d}{dt} \Omega_{op} = \frac{1}{2} \hbar \int d^{d-1}x \ dt(b^{0}_{k})_{op} + O(\hbar^{2}). \quad (23)
\]

On the other hand, writing explicitly the Ward identities associated to (1), we get

\[
\int d^{D}x < TN_{p} \left( \frac{\delta}{\partial\phi^{A}} \mathcal{L}_{H}(x) \right) \frac{\delta}{\partial\phi^{A}(x)} \prod_{j} \phi^{A}(y_{j}) >
\]
\[ i \int d^Dx \left< TN_{p+4-d_A} \frac{\delta}{\delta \phi^A} \mathcal{L}_H \frac{\delta}{\delta \phi^*_A} \mathcal{L}_H(x) + \frac{1}{2} \hbar \delta_{[k]}(x) + O(\hbar^2) \right> \prod_j \phi^{A_j}(y_j) >. \] (24)

Using (18,22), we find, identifying the terms according to the antifields, that the classical relations

\[ [H, \Omega] = 0, \quad [\Omega, \Omega] = 0, \] (25)

where the extended Poisson bracket \([\cdot, \cdot]\) is defined with functional derivatives in space for local integrated functionals, get modified inside Green’s functions by the following quantum corrections

\[ \int d^Dx [H, \Omega]_Q(x) = \frac{1}{2} \hbar \int dt \delta_{[k]} + O(\hbar^2) \] (26)

\[ \int d^Dx [\Omega, \Omega]_Q(x) = \hbar \int dt \delta_{[k-1]} + O(\hbar^2). \] (27)

### 3.3. Equivalence with Lagrangian anomalies

The Lagrangian formalism and the total Hamiltonian formalism are related by auxiliary fields, i.e., the momenta and the Lagrange multipliers can be eliminated from the total Hamiltonian action in an algebraic way by means of their equations of motion to yield the Lagrangian action. For the passage to the extended Hamiltonian formalism, a generalization of the concept of auxiliary fields on the level of the solution of the master equation has to be used. This is possible under general regularity conditions, excluding in particular systems which do not satisfy Dirac’s conjecture[13,15]. If the passage from the Lagrangian to the extended Hamiltonian formalism can be done by means of generalized auxiliary fields preserving the locality of both formalisms, then the following theorem proved in[16] guarantees the equivalence of the first order anomalous corrections in both formalisms:

The local cohomology classes \( H(s|d) \) are isomorphic for theories differing in generalized auxiliary field content.

### 4. Conclusion

The use of the antibracket-antifield formalism and its cohomological properties allows to show the equivalence of the first order anomalies of the Lagrangian and the Hamiltonian formalism and to prove, in a completely gauge and regularisation independent way, that the anomalies in the Hamiltonian framework occur only in the time development of the BRST charge and in the violation of nilpotency of this charge. Within the descent equations adapted to the Hamiltonian formalism, the normal product operator insertions of the BPHZ renormalisation scheme going beyond the scope of this letter.
Anomalies in the Hamiltonian Formalism.

The relationship of both anomalies, the gauge anomaly and the Schwinger terms in the Poisson bracket algebra of the constraints naturally follows.

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