A REMARK ON CARLESON MEASURES OF DOMAINS IN $\mathbb{C}^n$

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Abstract. We provide characterizations of Carleson measures on a certain class of bounded pseudoconvex domains. An example of a vanishing Carleson measure whose Berezin transform does not vanish on the boundary is given in the class of the Hartogs triangles

$$\mathbb{H}_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1 \right\}, \ k \in \mathbb{Z}^+.$$ 

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $A^2(\Omega)$ be the space of holomorphic, square-integrable functions on $\Omega$ (with respect to the Lebesgue measure $dV$ in $\mathbb{C}^n$). A non-negative finite Borel measure $\mu$ on $\Omega$ is called a Carleson measure of $A^2(\Omega)$ if there exists a positive constant $C$ such that

$$\int_{\Omega} |h|^2 \, d\mu \leq C \int_{\Omega} |h|^2 \, dV, \ \forall h \in A^2(\Omega).$$

Equivalently, the inclusion $A^2(\Omega) \hookrightarrow L^2(\Omega, \mu)$ is bounded. A Carleson measure $\mu$ on $\Omega$ is called vanishing if the inclusion $A^2(\Omega) \hookrightarrow L^2(\Omega, \mu)$ is compact.

We are interested in finding a characterization of Carleson and of vanishing Carleson measures on a bounded pseudoconvex domain $\Omega$ in $\mathbb{C}^n$. Let $K$ denote the Bergman kernel of $\Omega$. Given a Borel measure $\mu$ on $\Omega$, the Berezin transform of $\mu$ is defined by

$$B_\mu : \Omega \to \mathbb{R}^+ \cup \{\infty\}$$

$$w \to \int_{\Omega} \frac{|K(z, w)|^2}{K(w, w)} \, d\mu(z).$$

Note that since $\Omega$ is bounded, this function is well-defined. Let $k_w$ be the normalized Bergman kernel, defined by

$$k_w(z) := \frac{K(z, w)}{\sqrt{K(w, w)}}.$$ 

Since $k_w \in A^2(\Omega)$ and

$$\int_{\Omega} |k_w(z)|^2 \, dV(z) = 1, \ \forall w \in \Omega,$$
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a necessary condition for (1) to be satisfied is that $B_\mu \in L^\infty(\Omega)$. This leads us to the question whether this necessary condition is also sufficient.

Carleson measures have been studied for certain classes of pseudoconvex domains, particularly in connection with many important operators in complex analysis such as: Hankel and Toeplitz operators, see e.g. [5, 19, 20, 1]; and its generalisation to other spaces in harmonic analysis [9, 4, 23, 3]. For bounded symmetric domains (e.g. unit polydiscs), Békollé-Berger-Coburn-Zhu [5, Theorem 8] proved that $B_\mu \in L^\infty$ is also a sufficient condition for a Carleson measure $\mu$. For strongly pseudoconvex domains with smooth boundary, the same property was confirmed by H. Li [19, Theorem C]. In this case of domains, Abate and Saracco [2, Theorem 1.1] proved that the boundedness of $B_\mu$ on $\Omega$ is also a necessary and sufficient condition for Carleson measures of $A^p(\Omega)$, for any $p \geq 1$.

Turning to the question of vanishing Carleson measures, it has been shown (see [5, 19, 1]), for any strongly pseudoconvex domain or bounded symmetric domain $\Omega$, $\mu$ is a vanishing Carleson measure if and only if $\lim_{z \to \partial \Omega} B_\mu(z) = 0$. Here $\partial \Omega$ denotes the topological boundary of $\Omega$. The “only if” statement comes easily from the fact that $k_z \to 0$ weakly in $A^2(\Omega)$ as $z \to \partial \Omega$.

To our knowledge, however, it is not yet much studied to what extent these characterizations are still true for a general pseudoconvex domain in $\mathbb{C}^n$. For the sake of illustration, let us first consider the class of bounded convex domains in $\mathbb{C}^n$ (with no boundary regularity assumptions). In general, a domain in this class is neither a strongly pseudoconvex domain nor a symmetric domain (for example, consider the Thullen domains $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^p + |z_2|^q < 1\}$, $p, q \geq 1$). Thus in this case, it is reasonable to ask whether the same behaviours of the Berezin transform would capture the characterisations of Carleson measures. We shall show that this is the case. The method we give here is elementary. The argument combines known results for intrinsic geometry on sublevel sets of the pluricomplex Green function and an estimate due to Békollé [7]. Our approach also extends previous work on strongly pseudoconvex or bounded symmetric domains to a wider class of domains.

Let $G_\Omega(\cdot, w)$ be the pluricomplex Green function with pole $w \in \Omega$, defined by

$$
G_\Omega(\cdot, w) := \sup\left\{ u(\cdot) : u \in PSH^-(\Omega), \limsup_{z \to w} (u(z) - \log |z - w|) < \infty \right\}.
$$

Here $PSH^-(\Omega)$ denotes the set of all negative plurisubharmonic functions on $\Omega$. The following useful estimate was proved by Blocki [7], which is a sharper version of a previous estimate of Herbort [14]: for any bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and $M > 0$,

$$
(3) \quad \int_{\{G_\Omega(\cdot, z) < -M\}} |h(w)|^2 dV(w) \geq e^{-2nM} \frac{|h(z)|^2}{K(z, z)}, \forall z \in \Omega, \forall h \in \mathcal{O}(\Omega).
$$

Using this estimate, it follows that

$$
(4) \quad \int_\Omega |h(z)|^2 d\mu(z) \leq e^{2nM} \int_\Omega \int_\Omega \mathbf{1}_{A_z, M}(w) K(z, z) |h(w)|^2 dV(w) d\mu(z),
$$

where $A_z, M$ denotes the sublevel set of the pluricomplex Green function $G_\Omega(\cdot, z)$ with pole $w = z$ and upper bound $M$. This estimate provides a powerful tool for studying Carleson measures in strongly pseudoconvex and bounded symmetric domains, and it opens up new avenues for further research in this area.
where $A_{z,M} := \{ G_\Omega (\cdot, z) < -M \}$. Note that it is safe to write the iterated integral on the RHS of (4), because $G_\Omega$ is upper semicontinuous on $\Omega \times \Omega$ (see [17, Corollary 6.2.6]), so the integrand is measurable.

From this we immediately obtain:

**Theorem 1.1.** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that there are $M > 0$ and $c < 1$ such that $d_S (z, w) < c$, for any $w \in A_{z,M}$ and any $z \in \Omega$; where $d_S$ is the Skwarczyński distance defined by

$$d_S (z, w) := \left( 1 - \frac{|K(z, w)|}{\sqrt{K(z, z)} \sqrt{K(w, w)}} \right)^{\frac{1}{2}} \ ; z, w \in \Omega.$$ 

Then

$$\int_{\Omega} |h(z)|^2 d\mu (z) \leq C \int_{\Omega} B_\mu (z) |h(z)|^2 dV (z) , \forall h \in A^2 (\Omega).$$ 

Here $C := \frac{2^{2nM}}{(1-c^2)^2}$. As a consequence, $\mu$ is a Carleson measure iff $B_\mu \in L^\infty (\Omega)$.

**Proof.** For $w \in A_{z,M}$, we have

$$\frac{|K(z, w)|}{\sqrt{K(z, z)} \sqrt{K(w, w)}} > 1 - c^2.$$ 

Using (4), we see that

$$\int_{\Omega} |h(z)|^2 d\mu (z) \leq C \int_{\Omega} \int_{\Omega} \frac{|K(z, w)|^2}{K(w, w)} |h(w)|^2 d\mu (z) dV (w)$$

$$= C \int_{\Omega} B_\mu (w) |h(w)|^2 dV (w) ,$$ 

as desired. \qed

Let us apply this to the case of bounded convex domains $\Omega$ in $\mathbb{C}^n$. It is known that (see [17, Corollary 6.5.3])

$$A_{z,M} \subset \{ w \in \Omega : d_C (w, z) < \text{arctanh} \left( e^{-M} \right) \} ,$$

where $d_C$ is the Carathéodory distance. Since $\Omega$ is convex, by a well-known result of Lempert [18], we have $d_C \equiv d_K$, where $d_K$ is the Kobayashi distance. It follows that

$$A_{z,M} \subset \{ w \in \Omega : d_K (w, z) < \text{arctanh} \left( e^{-M} \right) \} .$$

On the other hand, $d_S \leq d_B / \sqrt{2}$, see [15, Corollary 6.4.7], where $d_B$ denotes the Bergman distance. Note that this fact is true for a general bounded pseudoconvex domain. Finally, it is also known that the Bergman and the Kobayashi metrics are equivalent on bounded convex domains, e.g. [22]. Thus $d_B \leq C d_K$, for some positive constant $C$. Therefore, for $M$ large enough, $d_S (w, z) < 1/2$, for any $w \in A_{z,M}$, so the condition of Theorem 1.1 is satisfied. Thus we obtain the following corollary:
Corollary 1.1. A measure $\mu$ on a bounded convex domain $\Omega$ is a Carleson measure iff $B_\mu \in L^\infty(\Omega)$.

Using the estimate (3), we now claim the characterization of vanishing Carleson measures on convex domains:

Corollary 1.2. A Carleson measure $\mu$ on a bounded convex domain $\Omega$ is vanishing iff $B_\mu(z) \to 0$ as $z \to \partial \Omega$.

Proof. Since $k_w \to 0$ weakly in $A^2(\Omega)$ as $w \to \partial \Omega$ (see [24, Lemma 4.9]), the “only if” part follows.

In the converse direction, notice that $B_\mu$ is continuous on $\Omega$. This follows from the dominated convergence theorem. The hypothesis thus gives $B_\mu \in C(\overline{\Omega})$. Choose a sequence of domains $\{\Omega_j\}$ such that $\Omega_j \supset \Omega_{j+1} \supset \Omega$ and $\cup_{j=1}^\infty \Omega_j = \Omega$. For each $j$, the function $R_j(w) := \int_\Omega \frac{|K(z,w)|^2}{K(w,w)} (1 - 1_{\Omega_j}(z)) \, d\mu(z), w \in \Omega,$
is also continuous on $\overline{\Omega}$ for the same reason. Again, by the dominated convergence theorem, $R_j \to 0$ pointwise. From Dini’s theorem, we conclude that $R_j \to 0$ uniformly on $\Omega$ as $j \to \infty$. On the other hand, for $h \in A^2(\Omega)$, we have

$$\int_{\Omega \setminus \Omega_j} |h|^2 \, d\mu \leq C 2^{nM} \int_{\Omega \setminus \Omega_j} (1 - 1_{\Omega_j}(z)) 1_{A_{z,M}}(w) K(z,z) |h(w)|^2 \, d\mu(z) \, dV(w)$$

$$\leq C \int_{\Omega \setminus \Omega_j} \frac{|K(z,w)|^2}{K(w,w)} (1 - 1_{\Omega_j}(z)) |h(w)|^2 \, d\mu(z) \, dV(w)$$

$$= C \int_{\Omega} R_j(w) |h(w)|^2 \, dV(w).$$

Therefore the operators

$$i_{\Omega_j} : A^2(\Omega) \to L^2(\Omega, \mu)$$

$h \to 1_{\Omega_j}h$

converge in norm to $i_\Omega$ as $j \to \infty$. The desired claim now follows since (by Montel’s theorem) $i_{\Omega_j}$ are compact operators.

Let us discuss an extension of this approach to other classes of bounded pseudoconvex domains. We will see in later that the condition in Theorem [14] is true for the classical Hartogs triangle $H = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$, which is an example of a non-hyperconvex domain. Note also that this condition is also satisfied for strongly pseudoconvex domains. This comes from the fact that for this class of domains, the Carathéodory metric, the Kobayashi metric and the Bergman metric are equivalent, see [10, 13]. Also, in view of the above example, the condition in Theorem [14] is satisfied if the Bergman distance $d_B(z,w)$ can be made small for any $w$ in sublevel sets $A_{z,M}$, for
some fixed (large) constant $M$. Such an estimate has been recently obtained by Zimmer \cite[Theorem 1.10]{Zimmer2016} for domains having bounded intrinsic geometry. This class includes in particular homogeneous domains (so, in particular, bounded symmetric domains), finite type domains in $\mathbb{C}^2$, strongly pseudoconvex domains, convex domains, $\mathbb{C}$-convex domains which are Kobayashi hyperbolic, simply connected domains which have a complete Kähler metric with pinched negative sectional curvature, and any domain that is biholomorphic to one of the previously mentioned domains. Let $\Omega$ be a bounded domain having bounded intrinsic geometry, then $\Omega$ must be pseudoconvex \cite[Corollary 1.3]{Zimmer2016}. From \cite[Theorem 1.10]{Zimmer2016}, there are constants $C, \tau > 0$ such that for any $M > 0$, we have $d_B(w, z) < e^{C-M}$, for all $w \in A_{z,M}$ with $d_B(w, z) < \tau$. On the other hand, it can be seen from the proof of Theorem 6.4 on page 19 that the condition $d_B(w, z) < \tau$ is automatically satisfied for any $w \in A_{z,M}$, provided that $M$ is large enough. We conclude that any bounded domain with bounded intrinsic geometry enjoys the property stated in Theorem 1.1. Therefore $B_\mu \in L^\infty(\Omega)$ is a necessary and sufficient condition for a Carleson measure $\mu$ on any domain $\Omega$ in this class.

Now let us generalize Corollary 1.2 to other classes of domains. The following statement is clear from the proof of Corollary 1.2.

**Theorem 1.2.** Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain satisfying the condition in Theorem 1.1. Then the following statements hold:

(i) If a Carleson measure $\mu$ on $\Omega$ satisfies the condition: $B_\mu(z) \to 0$ as $z \to \partial\Omega$ then $\mu$ is vanishing.

(ii) If $\mu$ is a vanishing Carleson measure on $\Omega$, and assume further that

\[
|z| \to 0 \text{ weakly in } A^2(\Omega) \text{ as } z \to \partial\Omega,
\]

then $B_\mu(z) \to 0$ as $z \to \partial\Omega$.

The condition (5) is true for pseudoconvex domains with smooth boundary, and convex domains \cite{Zimmer2016}. It is also satisfied for domains having a certain upper bound estimate on the Bergman kernel, such as bounded symmetric domains \cite{Zimmer2016}. Consequently, the characterization $B_\mu(z) \to 0$ as $z \to \partial\Omega$ applies to these classes of domains.

However, unlike (2), which is true for any pseudoconvex domain and used to obtain the necessary condition for a Carleson measure, the condition (5) is not the case for any pseudoconvex domain. An example was already given in \cite{Zimmer2016}. This motivates the need to examine characterizations of Carleson measures on pseudoconvex domains with an irregular boundary.

In the rest of this note, we study the analogous questions for the fat Hartogs triangles:

$$\mathbb{H}_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1 \right\}, \ k \in \mathbb{Z}^+.$$  

It can be seen that the condition (5) is not satisfied for $\mathbb{H}_k$. To check this, for simplicity of exposition, let us restrict to the case $k = 1$. Since $g(z) := 1/|z_2| \in A^2(\mathbb{H})$, we have

$$\left| \int_{\mathbb{H}} \frac{K(z, w)}{\sqrt{K(w, w)}} g(z) dV(z) \right| = \frac{1}{|w_2| \sqrt{K(w, w)}}, \ \forall w \in \mathbb{H}.$$
By using the explicit formula (6), the desired claim now follows since
\[
\frac{1}{|w_2|} \sqrt{K(w, w)} \not\to 0 \text{ as } w \to \partial\mathbb{H},
\]
for example, by considering \( w_j = \left( \frac{1}{j}, \frac{1}{j^2} \right) \to (0, 0) \in \partial\mathbb{H} \) as \( j \to \infty \).

We will show, in particular, that a measure \( \mu \) is a vanishing Carleson measure on \( \mathbb{H}_k \) if and only if there exists \( \delta > 0 \) such that
\[
|z_2|^\delta B_{\mu}(z) \not\to 0 \text{ as } z \to \partial\mathbb{H}_k.
\]
And we can construct a vanishing measure \( \mu \) on \( \mathbb{H}_k \) such that \( B_{\mu}(z) \not\to 0 \text{ as } z \to \partial\mathbb{H}_k \) (for example, on \( \mathbb{H} \), consider \( d\mu(z) = \left(1 - |z_2|^2\right)^3 \left(1 - \frac{|z_1|^2}{|z_2|^2}\right)^3 dV(z) \), see Remark 3.1). The appearance of \( |z_2|^\delta \) can be explained as a weighted distance to the singular point \((0,0)\). It therefore illustrates a different property compared to the previously known examples, and indicates that in general, characterizations of vanishing Carleson measures rely heavily on boundary regularity data of the domain.

Let us first recall some known facts about the Bergman kernel of \( \mathbb{H}_k \). Using Bell’s transformation rule under proper holomorphic maps, L. Edholm [12] established the following formula for the Bergman kernel of \( \mathbb{H}_k \):
\[
(6) \quad K(z, w) = \frac{p_k(s) t^2 + q_k(s) t + s^k p_k(s)}{k \pi^2 (1-t)^2 (t-s^k)^2},
\]
for \( z = (z_1, z_2), w = (w_1, w_2) \) in \( \mathbb{H}_k \), where \( s := z_1 \omega_1, t := z_2 \omega_2, \)
\[
p_k(s) := \sum_{j=1}^{k-1} j (k-j) s^{j-1}, \quad \text{and} \quad q_k(s) := \sum_{j=1}^{k} (j^2 + (k-j)^2 s^k) s^{j-1}.
\]
If \( k = 1 \) then we set \( p_1 \equiv 0 \) in the formula (6).

Let
\[
P(z, w) := \frac{t}{(1-t)^2 (t-s^k)^2},
\]
and consider the function \( T_{\mu} : \mathbb{H}_k \to \mathbb{R}^+ \cup \{\infty\} \) defined by
\[
T_{\mu}(w) := \int_{\mathbb{H}_k} \frac{|P(z, w)|^2}{K(w, w)} d\mu(z).
\]

Our results can be stated as follows:

**Theorem 1.3.** Let \( \mu \) be a Borel measure on \( \mathbb{H}_k \). Then the following statements are equivalent:

(i) \( \mu \) is a Carleson measure on \( \mathbb{H}_k \).
(ii) \( B_{\mu} \in L^\infty(\mathbb{H}_k) \).
(iii) \( T_{\mu} \in L^\infty(\mathbb{H}_k) \).

**Theorem 1.4.** Let \( \mu \) be a Carleson measure on \( \mathbb{H}_k \). Then the following statements are equivalent:

(i) \( \mu \) is vanishing on \( \mathbb{H}_k \).
(ii) There exists \( \delta > 0 \) such that \( |w_2|^\delta B_{\mu}(w) \to 0 \text{ as } w \to \partial\mathbb{H}_k \).
(iii) There exists \( \delta > 0 \) such that \( |w_2|^\delta T_{\mu}(w) \to 0 \text{ as } w \to \partial\mathbb{H}_k \).
Remark 1.1. The advantage of introducing the function $T_\mu$ is that it is easier to work with than $B_\mu$ because the kernel function $\mathcal{P}$ is simpler. For example, when $k = 2$, we have
\[
\mathcal{P}(z, w) = \frac{z_2 \bar{w}_2}{(1 - z_2 \bar{w}_2)^2 (z_2 \bar{w}_2 - z_1 \bar{w}_1^2)^2},
\]
whereas
\[
K(z, w) = \frac{1 + 4z_1 \bar{w}_1 + z_1^2 \bar{w}_1^2}{2\pi^2 (1 - z_2 \bar{w}_2)^2 (z_2 \bar{w}_2 - z_1 \bar{w}_1^2)^2}.
\]
Note also that $|K(z, w)| \lesssim |\mathcal{P}(z, w)|$ for any $k \in \mathbb{Z}^+$. When $k = 1$, it is clear that $|K(z, w)| \approx |\mathcal{P}(z, w)|$. However, for $k \geq 2$, $|K(z, w)| \neq |\mathcal{P}(z, w)|$ since $|\mathcal{P}(z, w)| \geq 1$, while $|K(z, w)|$ may vanish inside $\mathbb{H}_k \times \mathbb{H}_k$, see [11]. We will later use $T_\mu$ to construct some examples.

We shall prove Theorem 1.3 and Theorem 1.4 in much the same above arguments. We use an idea from [16] to obtain estimates on sublevel sets $A$. The main new ingredient now is the analysis at the singular point $(0, 0)$ in the proof of Theorem 1.4. Throughout the proofs we use the notation $F \lesssim G$ to indicate that $F \leq cG$, for some positive constant $c$, and the notation $F \approx G$ for the fact $c_1G \leq F \leq c_2G$, for some positive constants $c_1, c_2$.

2. Proof of Theorem 1.3

(i) $\Rightarrow$ (iii). Since $\mathcal{P}(\cdot, w)$ is holomorphic, it suffices to verify that
\[
\int_{\mathbb{H}_k} \frac{\left|\mathcal{P}(z, w)\right|^2}{K(z, w)} dV(z) \leq C, \quad \forall w \in \mathbb{H}_k,
\]
for some positive constant $C$. We have that
\[
\int_{\mathbb{H}_k} |\mathcal{P}(z, w)|^2 dV(z) = \int_{\mathbb{H}_k} \left|\frac{z_2 \bar{w}_2}{1 - z_2 \bar{w}_2^4 (z_2 \bar{w}_2 - (z_1 \bar{w}_1)^k)^4}\right| dV(z)
= \int_{\mathbb{D} \setminus \{0\}} \frac{dV(z_2)}{|z_2 \bar{w}_2|^2 |1 - z_2 \bar{w}_2|^4} \int_{|z_1| < |z_2|} \frac{dV(z_1)}{|1 - \frac{z_1 \bar{w}_1}{z_2 \bar{w}_2^k}|^4}.
\]
Here $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. Consider the change $\xi := z_2^k / z_2$, we get that
\[
\int_{\mathbb{H}_k} |\mathcal{P}(z, w)|^2 dV(z) = \frac{|w_2|^{-2}}{k} \left(\int_{\mathbb{D} \setminus \{0\}} \frac{|z_2|^2 - 2}{|1 - z_2 \bar{w}_2|^2} dV(z_2)\right) \left(\int_{\mathbb{D}} \frac{dV(\xi)}{|1 - \xi a|^4}\right)
\leq \frac{|w_2|^{-2}}{k} J (w_2) J (\frac{w_1}{w_2}),
\]
where
\[
J(a) := \int_{\mathbb{D}} \frac{|\xi|^2 - 2}{|1 - \xi a|^4} dV(\xi).
\]
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$J(a)$ can be estimated as

$$J(a) \leq \int_{|\xi| > \frac{1}{2}} \frac{|\xi|^2}{1 - |a|^2} dV(\xi) + \int_{|\xi| \leq \frac{1}{2}} \frac{|\xi|^2}{1 - |a|^2} dV(\xi)$$

$$\lesssim \int_{D} \frac{1}{|1 - \xi a|^2} dV(\xi) + \int_{|\xi| \leq \frac{1}{2}} |\xi|^2 dV(\xi)$$

$$\lesssim (1 - |a|^2)^{-2} + 1$$

$$\lesssim (1 - |a|^2)^{-2}.$$ 

Here the third inequality follows from Forelli-Rudin estimates, see e.g. [21, Theorem 1.3]. Applying this to (8), we obtain

$$\int_{H_k} |P(z, w)|^2 dV(z) \approx |w_2|^2 \left(1 - |w_2|^2\right)^{-2} \approx K(w, w).$$

Thus the estimate (7) holds.

(iii) $\Rightarrow$ (i). We first observe the following elementary fact:

**Fact.** For any $a, b \in D$ such that $|a - b| < 1/e$ then

$$\left|\frac{a - b}{1 - ab}\right| < 1$$

then

(9) $$|1 - ab| \approx 1 - |b|^2,$$

and

(10) $$1 - |a|^2 \approx 1 - |b|^2.$$

Note that (10) has been verified in [16]. To see (9), set $z = \frac{a - b}{1 - ab}$ then

$$|1 - ab| = \frac{1 - |b|^2}{1 + |z||b|} \approx 1 - |b|^2,$$

since $|z| < 1/e$. From [17, Proposition 6.1.1], we see that

$$G_{D \times D} (F(w_1, w_2), F(z_1, z_2)) \leq G_{H_k} ((w_1, w_2), (z_1, z_2)),$$

for any $z = (z_1, z_2), w = (w_1, w_2) \in H_k$, where $F : H_k \to D \times D$ is the holomorphic map defined by

$$F(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{z_1}{z_2}\right).$$

Recall that

$$G_{D \times D} (w, z) = \max \left\{ \log \left|\frac{z_1 - w_1}{1 - w_1 \overline{z_1}}\right|, \log \left|\frac{z_2 - w_2}{1 - w_2 \overline{z_2}}\right| \right\}.$$
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It follows that on the set $A_z := \{w \in \mathbb{H}_k : G_{\mathbb{H}_k} (w, z) < -1 \}$, one has

$$|\mathcal{P}(z, w)|^2 = \frac{1}{|z_2 w_2|^2 |1 - z_2 w_2|^4 \left| 1 - \frac{z_1 w_1}{z_2 w_2} \right|^4}$$

Therefore for $w \in A_z$,

$$|\mathcal{P}(z, w)|^2 = \frac{1}{|z_2 w_2|^2 |1 - z_2 w_2|^4 \left| 1 - \frac{z_1 w_1}{z_2 w_2} \right|^4}$$

$$\approx \frac{1}{|z_2 w_2|^2 \left( 1 - |w_2|^2 \right)^4 \left( 1 - \frac{|w_1|^{2k}}{|w_2|^2} \right)^4}$$

$$= \frac{|w_2|^2 \left( 1 - |w_2|^2 \right)^{-2} \left( 1 - \frac{|w_1|^{2k}}{|w_2|^2} \right)^{-2}}{|z_2|^2 \left( 1 - |z_2|^2 \right)^2 \left( 1 - \frac{|z_1|^{2k}}{|z_2|^2} \right)^2}$$

$$\approx \frac{1}{|z_2|^2 \left( 1 - |z_2|^2 \right)^2 \left( 1 - \frac{|z_1|^{2k}}{|z_2|^2} \right)^2}$$

$$\approx K(z, z) K(w, w),$$

here we have used the elementary fact (9)-(10).

For any $h \in A^2(\mathbb{H}_k)$, from the estimate (8), we have

$$|h(z)|^2 \leq e^4 K(z, z) \int_{A_z} |h(w)|^2 dV(w), \quad \forall z \in \mathbb{H}_k.$$ 

Thus

$$\int_{\mathbb{H}_k} |h(z)|^2 d\mu(z) \lesssim \int_{\mathbb{H}_k} \mathbb{1}_{A_z}(w) K(z, z) |h(w)|^2 dV(w) d\mu(z)$$

$$\lesssim \int_{\mathbb{H}_k} \int_{\mathbb{H}_k} |\mathcal{P}(z, w)|^2 |h(w)|^2 d\mu(z) dV(w)$$

$$= \int_{\mathbb{H}_k} T_\mu(w) |h(w)|^2 dV(w)$$

$$\leq \|T_\mu\|_{L^\infty} \int_{\mathbb{H}_k} |h(w)|^2 dV(w).$$

We conclude that $\mu$ is a Carleson measure on $\mathbb{H}_k$.

(iii) $\Rightarrow$ (ii). Since $|K(z, w)| \lesssim |\mathcal{P}(z, w)|$, this is immediate.
(ii) $\Rightarrow$ (i). Let 

$$R(z, w) := q_k(s) + p_k(s)t + \frac{s^k}{t}p_k(s),$$

where $s := z_1\overline{w}_1$ and $t := z_2\overline{w}_2$. Then $R(\cdot, w)$ is holomorphic and $R(w, w) \geq 1$, $\forall w \in \mathbb{H}_k$.

For any $h \in A^2(\mathbb{H}_k)$, we thus have

$$|h(z)|^2 \leq |h(z)|^2 |R(z, z)|^2 \leq e^{4K(z, z)} \int_{A_z} |h(w)|^2 |R(w, z)|^2 dV(w).$$

So

$$\int_{\mathbb{H}_k} |h(z)|^2 d\mu(z) \lesssim \int \int_{\mathbb{H}_k |A_z| K(z, z) |R(w, z)|^2 |h(w)|^2 dV(w) d\mu(z)$$

$$\lesssim \int \int_{\mathbb{H}_k |A_z| K(w, w) |P(z, w)|^2 |R(w, z)|^2 |h(w)|^2 dV(w) d\mu(z)$$

$$\lesssim \int \int_{\mathbb{H}_k |K(z, w)|^2 |h(w)|^2 dV(w) d\mu(z)$$

$$\lesssim \|B_{\mu}\|_{L^\infty} \int_{\mathbb{H}_k} |h(w)|^2 dV(w),$$

as desired. \(\square\)

3. Proof of Theorem 1.4

(i) $\Rightarrow$ (iii). It suffices to show that

$$\frac{|P(z, w)| |w_2|}{\sqrt{K(w, w)}} \to 0 \text{ weakly in } A^2(\mathbb{H}_k) \text{ as } w \to \partial \mathbb{H}_k.$$

Take any $g \in A^2(\mathbb{H}_k)$, and choose a sequence of domains $\{\Omega_j\}$ such that $\overline{\Omega}_j \subset \Omega_{j+1} \subset \mathbb{H}_k$ and $\cup_{j=1}^\infty \Omega_j = \mathbb{H}_k$. For each $j$, since

$$|P(z, w)| = \frac{1}{|z_2w_2|\left|1 - z_2w_2\right|^2 \left|1 - \frac{z_1\overline{w}_1}{z_2\overline{w}_2}\right|^2},$$

there exists $c_j > 0$ such that

$$|P(z, w)| < \frac{c_j}{|w_2|}, \forall z \in \Omega_j, \forall w \in \mathbb{H}_k.$$
Thus
\[
\int_{\mathbb{H}_k} \frac{\mathcal{P}(z, w) |w_2|}{\sqrt{K(w, w)}} g(z) \, dV(z) \leq \frac{c_i \sqrt{|\mathbb{H}_k|}}{\sqrt{K(w, w)}} \|g\|_{L^2(\mathbb{H}_k)} + \|\mathcal{P}(\cdot, w)\|_{L^2(\mathbb{H}_k)} \|g\|_{L^2(\mathbb{H}_k \setminus \Omega_j)}
\]
(12)
\[
\leq \frac{c_i \sqrt{|\mathbb{H}_k|}}{\sqrt{K(w, w)}} \|g\|_{L^2(\mathbb{H}_k)} + C \|g\|_{L^2(\mathbb{H}_k \setminus \Omega_j)}.
\]

Note that the right hand side of (12) can be made arbitrarily small as \(w \to \partial \mathbb{H}_k\) and \(j \to \infty\) because \(g \in L^2(\mathbb{H}_k)\) and
\[
\lim_{w \to \partial \mathbb{H}_k} \frac{1}{\sqrt{K(w, w)}} = 0.
\]

It follows that
\[
\int_{\mathbb{H}_k} \frac{\mathcal{P}(z, w) |w_2|}{\sqrt{K(w, w)}} g(z) \, dV(z) \to 0 \text{ as } w \to \partial \mathbb{H}_k,
\]
as desired.

(i) \(\Rightarrow\) (ii). Since \(|K(z, w)| \lesssim |\mathcal{P}(z, w)|\), this is straightforward from the argument in (i) \(\Rightarrow\) (iii).

(ii) \(\Rightarrow\) (i). Following the proof of Theorem 1.3, we first establish the following estimate:

**Lemma 3.1.** For any \(\varepsilon > 0\), there exists \(\delta_\varepsilon > 0\) such that
\[
\int_{W_{\delta_\varepsilon}} |h(z)|^2 \, dV(z) < \varepsilon \int_{\mathbb{H}_k} |h(z)|^2 \, dV(z), \forall h \in A^2(\mathbb{H}_k),
\]
where \(W_{\delta_\varepsilon} := \{z \in \mathbb{H}_k : |z_2| < \delta_\varepsilon\}\).

**Proof of Lemma 3.1.** Repeating the argument used in the proof of Theorem 1.3 we obtain that
\[
\int_{W_{\delta_\varepsilon}} |h(z)|^2 \, dV(z) \lesssim \int_{\mathbb{H}_k} \int_{W_{\delta_\varepsilon}} (z) \mathbb{1}_{A_z} (w) K(z, z) |h(w)|^2 dV(w) dV(z)
\]
\[
\lesssim \int_{\mathbb{H}_k} \left( \int_{W_{\delta_\varepsilon}} |\mathcal{P}(z, w)|^2 \frac{dV(z)}{K(w, w)} \right) |h(w)|^2 dV(w).
\]

It remains to verify the existence of \(\delta_\varepsilon\) such that
\[
\int_{\mathbb{H}_k} \mathbb{1}_{W_{\delta_\varepsilon}} (z) |\mathcal{P}(z, w)|^2 \frac{dV(z)}{K(w, w)} \lesssim \varepsilon, \forall w \in \mathbb{H}_k.
\]
We have
\[
\int_{W_{\delta_k}} |P(z, w)|^2 dV(z) \leq \frac{|w_2|^{-2}}{k} 2^4 \int_{|z_2|<\delta_k} |z_2|^{-2} dV(z_2) \left( \int_{D} |\xi|^{-2} \frac{dV(\xi)}{1 - |\xi|\frac{|w_2|}{|w_1|^2}} \right) \\
\lesssim \delta_k^2 |w_2|^{-2} \left( 1 - |w_2|^2 \right)^{-2} \left( 1 - \frac{|w_1|^{2k}}{|w_2|^2} \right)^{-2} \\
\lesssim \varepsilon K(w, w),
\]
provided that \( \delta_k < \min \{ \varepsilon^{k/2}, 1/2 \} \), as desired.

\( \square \)

We are ready to verify the implication \((ii) \Rightarrow (i)\). Choose a sequence of domains \( \{\Omega_j\} \) such that \( \overline{\Omega_j} \subset \Omega_{j+1} \subset \mathbb{H}_k \) and \( \cup_{j=1}^\infty \Omega_j = \mathbb{H}_k \). For each \( j \), let
\[
R_j(w) := \int_{\mathbb{H}_k} \frac{|K(z, w)|^2}{K(w, w)} (1 - \mathbbm{1}_{\Omega_j}(z)) d\mu(z), \quad w \in \mathbb{H}_k.
\]
Note that \( R_j \) and \( B_\mu \) are continuous on \( \mathbb{H}_k \). Consider the operators
\[
i_{\Omega_j} : A^2(\mathbb{H}_k) \rightarrow L^2(\mathbb{H}_k, \mu) \quad h \rightarrow i_{\Omega_j}h.
\]
By Montel’s theorem, \( i_{\Omega_j} \) are compact operators. It remains to show that
\[
(13) \quad \|i_{\Omega_j} - i_{\mathbb{H}_k}\| \xrightarrow{j \to \infty} 0.
\]
Fix \( \varepsilon > 0 \), and choose \( \delta_k > 0 \) as in Lemma [3.1]. Then for any \( h \in A^2(\mathbb{H}_k) \) such that \( \|h\|_{L^2(\mathbb{H}_k)} = 1 \), we have
\[
\int_{W_{\delta_k}} |h(w)|^2 dV(w) < \varepsilon.
\]
The key point we need here is that \( \delta_k \) is independent of \( h \). Note that \( R_j \leq B_\mu \) on \( \mathbb{H}_k \). And, on \( \mathbb{H}_k \setminus W_{\delta_k} \), we have \( |z_2| \geq \delta_k \). The hypothesis implies that \( R_j \) can be extended continuously to the compact set \( \overline{\mathbb{H}_k} \setminus W_{\delta_k} \) (with zero value on the part \( \partial \mathbb{H}_k \setminus W_{\delta_k} \)). Moreover, \( R_j \setminus_\circ \) on \( \mathbb{H}_k \setminus W_{\delta_k} \) by the dominated convergence theorem. From Dini’s theorem, there exists \( j_0 > 0 \) such that \( |R_j(w)| < \varepsilon \), for any \( w \in \mathbb{H}_k \setminus W_{\delta_k} \) and \( j > j_0 \). Following the last lines in the proof of Theorem (1.3), we see that
\[
\left\| (1 - \mathbbm{1}_{\Omega_j}) h \right\|_{L^2(\mathbb{H}_k, \mu)}^2 \leq \int_{\mathbb{H}_k} \int_{\mathbb{H}_k} \frac{|K(z, w)|^2}{K(w, w)} (1 - \mathbbm{1}_{\Omega_j}(z)) |h(w)|^2 d\mu(z) dV(w)
\]
\[
\leq \int_{\mathbb{H}_k \setminus W_{\delta_k}} B_\mu(w) |h(w)|^2 dV(w) + \int_{W_{\delta_k}} R_j(w) |h(w)|^2 dV(w)
\]
\[
\leq \|B_\mu\|_{L^\infty(\mathbb{H}_k)} \varepsilon + \varepsilon,
\]
Remark 3.1. With the characterisations in Theorem 13 and Theorem 14, we now discuss some specific examples. Consider a special measure \( \mu \) on \( \mathbb{H}_k \) in the form

\[
d\mu(z) = f \left( \frac{z_1}{z_2} \right) g(z_2) \, dV(z),
\]

where \( f \) and \( g \) are some non-negative, measurable functions on \( \mathbb{D} \). Then we get that

\[
T_\mu(w) \approx \frac{1}{K(w, w)|w_2|^2} \left( \int_\mathbb{D} \frac{g(z_2) |z_2|^{k-2}}{|1 - z_2 \overline{w_2}|^4} \, dV(z_2) \right) \left( \int_\mathbb{D} \frac{f(\xi) |\xi|^{\frac{k}{2}-2}}{|1 - \xi \overline{w_2}|^4} \, dV(\xi) \right).
\]

Note that

\[
\frac{1}{K(w, w)|w_2|^2} \approx \left( 1 - |w_2|^2 \right)^2 \left( 1 - \frac{|w_1|^2}{|w_2|^2} \right)^2.
\]

It follows that \( \mu \) is a (non-zero) Carleson measure on \( \mathbb{H}_k \) if and only if both measures

\[
d\mu_1(\omega) := f(\omega) |\omega|^{\frac{k}{2}-2} \, dV(\omega) \quad \text{and} \quad d\mu_2(\omega) := g(\omega) |\omega|^{\frac{k}{2}-2} \, dV(\omega)
\]

are Carlesons measures on \( \mathbb{D} \).

If we set

\[
f(\xi) := \left( 1 - |\xi|^2 \right)^3 |\xi|^{2-\frac{k}{2}} \quad \text{and} \quad g(z_2) := \left( 1 - |z_2|^2 \right)^3 |z_2|^{2-\frac{k}{2}}
\]

then Forelli-Rudin estimates (21) give that

\[
T_\mu(w) \approx \frac{1}{K(w, w)|w_2|^2}.
\]

Thus \( T_\mu \in L^\infty(\mathbb{H}_k) \) and \( |w_2|^2 T_\mu(w) \to 0 \) as \( w \to \partial\mathbb{H}_k \). So the measure

\[
d\mu(z) = |z_1|^{2k-2} \left( 1 - |z_2|^2 \right)^3 \left( 1 - \frac{|z_1|^2}{|z_2|^2} \right)^3 dV(z_1, z_2)
\]

is a vanishing Carleson measure on \( \mathbb{H}_k \).

If we consider

\[
f(\xi) := |\xi|^{2-\frac{k}{2}} \quad \text{and} \quad g(z_2) := |z_2|^{2-\frac{k}{2}}
\]

then

\[
T_\mu(w) \approx \frac{1}{K(w, w)|w_2|^2} \left( 1 - |w_2|^2 \right)^{-2} \left( 1 - \frac{|w_1|^{2k}}{|w_2|^2} \right)^{-2} \approx 1.
\]

On the other hand, consider the point \( M(1/2, 1) \in \partial\mathbb{H}_k \). For any \( \delta > 0 \), since

\[
|w_2|^\delta T_\mu(w) \approx |w_2|^\delta,
\]

it follows that \( |w_2|^\delta T_\mu(w) \not\to 0 \) as \( w \to M \). So

\[
d\mu(z) = |z_1|^{2k-2} dV(z_1, z_2)
\]

for \( j > j_0 \). This demonstrates the claim (13).

(iii) \( \Rightarrow \) (ii). Since \( B_\mu \leq T_\mu \), this is immediate. \( \square \)
is a Carleson measure but not a vanishing Carleson measure on \( \mathbb{H}_k \).

4. Concluding remarks

For the classical Hartogs triangle \( \mathbb{H} \), since \( |K (z, w)| \approx |\mathcal{P} (z, w)| \), it can be seen from the proof of Theorem 1.3 that the Skwarczyński distance \( d_S (z, w) \) is bounded by some constant \( c < 1 \) for any \( w \in A_{z,1} \). However, it is not clear to us whether this is true for \( \mathbb{H}_k \), with \( k \geq 2 \). Note that if we rely only on the weaker estimates (11) then this is not the case. For example, consider \( k = 2 \) and \( z_j = \left( - \left( 1 - \frac{1}{2j} \right), 1 - \frac{1}{2j} \right) \), \( w_j = \left( 1 - \frac{1}{2j}, 1 - \frac{1}{2j} \right) \), for \( j \geq 1 \), then the estimates (11) hold, however \( d_S (z_j, w_j) \to 1 \) as \( j \to \infty \).

We do not know an example of a bounded pseudoconvex domain such that the condition \( \mathcal{B}_\mu \in L^\infty \) is not sufficient for \( \mu \) being a Carleson measure. Here we make the trivial observation that for any bounded smooth domain (not necessarily pseudoconvex) \( \Omega \) in \( \mathbb{C}^n \), the condition \( \mathcal{B}_\mu \in L^\infty (\Omega) \) implies certain \( L^2 \) regularity. Indeed, for any Borel measure \( \mu \), the inclusion \( W^s (\Omega) \cap \mathcal{O} (\Omega) \hookrightarrow L^2 (\Omega, \mu) \) is bounded if \( s \geq (3n + 1)/2 \). Here \( W^s (\Omega) \) is the standard \( L^2 \) Sobolev space of order \( s \) on \( \Omega \). To check this, from [3] Lemma 2 and the remark after it, there exists a bounded linear operator \( \Phi^s : W^s (\Omega) \cap \mathcal{O} (\Omega) \to W^s_0 (\Omega) \) such that \( P \Phi^s = I \), where \( P \) denotes the Bergman projection of \( \Omega \). For \( h \in W^s (\Omega) \cap \mathcal{O} (\Omega) \), it thus follows that

\[
\int_\Omega |h (z)|^2 d\mu (z) = \int_\Omega \left( \int_\Omega \left( \int_\Omega |K (z, w)|^2 |\Phi^s (h) (w)|^2 d\mu (z) \right)^{\frac{1}{2}} dV (w) \right)^2 \leq \|\mathcal{B}_\mu\|_{L^\infty} \left(\int_\Omega K^\frac{1}{2} (w, w) |\Phi^s (h) (w)| dV (w) \right)^2.
\]

Since \( \Phi^s (h) \in W^s_0 (\Omega) \), by Sobolev embedding theorem we have

\[
|\Phi^s (h) (w)| \leq C_\Omega \|\Phi^s (h)\|_{W^{(3n+1)/2} (\Omega)} \delta_\Omega^{-\frac{1}{2}} (w), \forall w \in \Omega.
\]

Here \( \delta_\Omega \) denotes the distance function to the boundary. On the other hand,

\[
K^\frac{1}{2} (w, w) \leq C_\Omega \delta_\Omega^{-\frac{1}{2}} (w), \forall w \in \Omega.
\]

We arrive at

\[
\int_\Omega |h (z)|^2 d\mu (z) \leq \text{const.} \|\Phi^s (h)\|^2_{W^{(3n+1)/2} (\Omega)} \leq \text{const.} \|h\|^2_{W^s (\Omega)}
\]
as desired.

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