Calabi-Yau algebras viewed as deformations of Poisson algebras

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Abstract

From any algebra $A$ defined by a single non-degenerate homogeneous quadratic relation $f$, we prove that the potential $w = fz$ (where $z$ is an extra generator) is 3-Calabi-Yau. It means that the quadratic algebra $B$ defined by the potential $w$ is 3-Calabi-Yau. The algebras $A$ and $B$ are both Koszul. The classification of the algebras $B$ in three generators, i.e., when $A$ has two generators, leads to three types of algebras. The second type (the most interesting one) is viewed as a deformation of a Poisson algebra $S$ whose Poisson bracket is non-diagonalizable quadratic. Although the potential of $S$ has non-isolated singularities, the homology of $S$ is computed. Next the Hochschild homology of $B$ is obtained.

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1 Introduction

The deformation theory offers a way to study non-commutative algebras $A$ by examining associated Poisson algebras $S$. The idea is to get (more or less explicitly) invariants attached to $S$ by the use of differential calculus on the commutative algebra $S$, and then to deduce from them some invariants attached to $A$. Invariants of interest in various domains are of homological nature. In particular, a natural question is the following (there is the same in cohomology).

(Q) Is the Hochschild homology $\text{HH}_\bullet(A)$ of $A$ isomorphic to the Poisson homology $\text{HP}_\bullet(S)$ of $S$?

A positive answer was given by Kontsevich [16] in cohomology when $S$ is the algebra of the $C^\infty$ functions on $\mathbb{R}^n$ endowed with any Poisson bracket $\pi$ extended to the space of formal series $S[[\hbar]]$, and $A$ is the space $S[[\hbar]]$ whose usual commutative product is replaced by the Kontsevich star product $\ast_\pi$ (see e.g. [9] for further developments).

The question (Q) was initiated by Brylinski [8] when the algebra $A$ is filtered such that $\text{gr}(A)$ is assumed to be commutative and smooth. Then $\text{gr}(A)$ is naturally a Poisson algebra $S$, and there is the Brylinski spectral sequence

$$E^2 = HP_\bullet(S) \implies HH_\bullet(A).$$

(1.1)

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In this context, we can replace the question (Q) by the following one.

(Q') Does the Brylinski spectral sequence degenerate at $E^2$?

As shown by Kassel [14], this is the case if $S$ is any polynomial algebra whose generators have all degree 1, so that the Poisson bracket of $S$ is of degree $\leq 1$ and $A$ is a Sridharan enveloping algebra. Actually, Kassel proved more precisely that the symmetrization defines an isomorphism from the Poisson complex of $S$ to the Koszul complex of $A$. The latter result was generalized (in a weaker form) in [13] to the crossed products of enveloping algebras.

Van den Bergh showed that the question (Q') has a positive answer if $A$ is a generic Sklyanin algebra in three generators [27]. In this case, the Poisson bracket of $S$ (which is quadratic) is derived from a “Poisson potential” $\phi$ such that

(IS) the origin is an isolated singularity of the polynomial $\phi$.

In the paper of Van den Bergh, the property (IS) appears as essential in the computation of $\text{HP}_\bullet(S)$, and it is the same for the other examples (quadratic, cubic, or more generally quasi-homogeneous) considered further [19, 22, 21]. In this paper, we present an example of quadratic algebra, called $B$ in the text, for which the previous programme is performed without the property (IS). In other words, in our example of quadratic algebra $B$, the Poisson bracket of $S$ (which is non-diagonalizable quadratic) is derived from a potential $\phi$ having non-isolated singularities. Nevertheless, $\text{HP}_\bullet(S)$ is explicitly computed (Section 5 below). Next we prove that any Poisson cycle can be lifted in a Koszul cycle (according to the fact that $B$ is Koszul, it is more convenient to use Koszul complex, instead of Hochschild complex, in order to define $\text{HH}_\bullet(B)$). Finally we get the computation of $\text{HH}_\bullet(B)$ as stated in the following.

**Theorem 1.1** Let $B$ be the $\mathbb{C}$-algebra defined by the generators $x$, $y$, $z$, and the following relations

$$zy = yz + 2xz, \quad zx = xz, \quad yx = xy + x^3.$$ 

Let $S$ be the polynomial $\mathbb{C}$-algebra in $x, y, z$, endowed with the Poisson bracket derived from the potential $\phi = -x^2z$. Then the Hochschild homology of $B$ is isomorphic to the Poisson homology of $S$ and is given by

$$\text{HH}_0(B) \simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z];$$

$$\text{HH}_1(B) \simeq \mathbb{C}[\phi] \begin{pmatrix} \frac{xz}{0} & \frac{z^2}{0} \\ -xz & \frac{0}{-xz} \end{pmatrix} \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \begin{pmatrix} ky^{k-1}z^{n-k} \\ (n-k)y^kz^{n-1-k} \end{pmatrix} \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \begin{pmatrix} (2n+3)yz \\ -3kz \\ (-2n+3k-1)y \end{pmatrix} \begin{pmatrix} y^{k-1}z^{n+1-k} \\ y^ky^{n+1} \end{pmatrix};$$

$$\text{HH}_2(B) \simeq \mathbb{C}[\phi] \begin{pmatrix} y \\ z \end{pmatrix} \oplus (x\mathbb{C}[\phi] \oplus z\mathbb{C}[z]) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \begin{pmatrix} (k+1)x \\ 2n-k+1 \\ -2(k+1)z \end{pmatrix} \begin{pmatrix} y^{k+1}z^{n-k} \\ y^ky^{n-k} \end{pmatrix};$$

$$\text{HH}_3(B) \simeq \mathbb{C}[\phi];$$

$$\text{HH}_p(B) \simeq 0 \text{ if } p \geq 4.$$
Since $B$ is 3-Calabi-Yau, the Hochschild cohomology is then immediate: $HH^\bullet(B) = HH_{3-\bullet}(B)$. In particular, the center of $B$ is the polynomial algebra generated by $x^2z$. The above theorem shows that the Hochschild homology is not free as a module over the center (in contrast to Sklyanin algebras). Remark that the duality $HF^\bullet(S) = HP_{3-\bullet}(S)$ holds since the Poisson bracket derives from a potential.

Our example of Koszul algebra $B$ belongs to a large class of Koszul 3-Calabi-Yau algebras. Let us explain how we define this class (see Section 2 below for details). For any non-degenerate quadratic relation $f = \sum_{i,j} f_{ij} x_i x_j$ in $n \geq 2$ non-commutative variables $x_i$, the algebra $A$ defined by the single relation $f$ is Koszul and AS-Gorenstein of global dimension 2 [11], and $A$ is Calabi-Yau if and only if $f$ is symplectic [23]. Add an extra generator $z$ to the $x_i$’s and consider the quadratic algebra $B$ defined by the potential $w = fz$ (see e.g. [12] [6] [9] for algebras defined by potentials). The properties of $B$ that we shall obtain in Section 2 work out over any field and are stated in the following.

**Theorem 1.2** Let $f = \sum_{i,j} f_{ij} x_i x_j$ be any non-degenerate quadratic relation over a field $k$ in non-commutative variables $x_1, \ldots, x_n$ ($n \geq 2$) and let $z$ be an extra generator. Let $A$ be the $k$-algebra defined by the generators $x_1, \ldots, x_n$ and the single relation $f$. Let $B$ be the $k$-algebra defined by the generators $x_1, \ldots, x_n, z$ and the potential $w = fz$. Then
1) $B$ is a skew polynomial ring over $A$ in the generator $z$ and defined by an automorphism of $A$.
2) $z$ is normal in $B$, that is $Bz = zB$.
3) If $f$ is alternate, then $z$ is central in $B$. The converse holds if the characteristic of $k$ is $\neq 2$.
4) $B$ is Koszul and 3-Calabi-Yau,
5) the Hilbert series $h_B(t)$ of the graded algebra $B$ is given by
$$h_B(t) = (1 - (n+1)t + (n+1)t^2 - t^3)^{-1},$$
6) the Gelfand-Kirillov dimension $GK.dim(B)$ of $B$ is finite if and only if $n = 2$, and in this case $GK.dim(B) = 3$.
7) If $k$ is algebraically closed of characteristic zero, $B$ is left (or right) noetherian if and only if $n = 2$.

When $n = 2$, the graded $\mathbb{C}$-algebras $B$ are classified in three types (Section 4 below): the polynomial algebra in $x$, $y$ and $z$ (classical type), the algebra of Theorem 1.1 (Jordan type), a family of quantum spaces in $x$, $y$, $z$ (quantum type). The second type is the one of interest for us, since it is well-known that the question (Q) has a positive answer in the first or third type (if the quantum parameter $q$ is not a root of unity). Hochschild homology in the quantum type can be deduced from Wambst’s result ([23] Théorème 6.1).

When $n = 2$, the $\mathbb{C}$-algebras $B$ are AS-regular of global dimension 3, and it is important to notice that their invariants $j$ (in terminology of [11]) are infinite, unlike Sklyanin algebras. The first type and the third type can be considered as limits of Sklyanin algebras by vanishing the parameter $c$ used in [27], but such a process is not possible for the second type which appears really apart.

Recently, Smith has given a detailed study of a remarkable algebra having seven generators and defined in terms of octonions [24]. To avoid confusion with our notation, let us call $C$ the Smith algebra. Actually, $C$ does not belong to the class of algebras $B$ defined in Theorem 1.2 since in characteristic zero $C$ has no normal element except the elements of $k$ ([24] Proposition 11.2). However $C$ has many properties in common with algebras $B : C$ is Koszul and 3-Calabi-Yau, $C$ is defined by an explicit potential, $C$ is a skew polynomial ring over an algebra $A$ in the last generator (but $C$ is defined by a derivation of $A$, not an
automorphism), and the single relation of $A$ is symplectic in the first six generators. Suárez-Alvarez has obtained similar properties for a class of algebras containing $C$ and defined from any oriented Steiner triple system [20]. It would be satisfactory to enlarge naturally the class of algebras $B$ in order to include the Smith algebra $C$ or more generally the Suárez-Alvarez algebras.

2 A family of 3-Calabi-Yau algebras

A down-to-earth approach of non-commutative projective algebraic geometry consists in studying graded algebras defined by generators (assumed to be of degree 1) satisfying some homogeneous non-commutative polynomial relations. Following this naïve approach, the first class to study is certainly the class of non-commutative quadratics, i.e., the class of non-commutative graded algebras defined by a single quadratic relation. It is a bit surprising that this class can be used as a toy model (see [8]) in order to introduce to several duality theories (Koszul duality, AS-Gorenstein duality and Calabi-Yau duality) playing a basic role in more sophisticated approaches. Throughout the paper, the algebras of this class will be denoted $(Koszul duality, AS-Gorenstein duality and Calabi-Yau duality) playin g a basic role in more

Let us fix the notation. For $n \geq 1$, $k\langle x_1, \ldots, x_n \rangle$ denotes the graded free associative algebra over a field $k$, generated by $x_1, \ldots, x_n$, assumed to be of degree 1. Let us give a non-zero element $f = \sum_{1 \leq i,j \leq n} f_{ij} x_i x_j$ homogeneous of degree 2 in this algebra, or equivalently a non-zero $n \times n$ matrix $M = (f_{ij})_{1 \leq i,j \leq n}$ with entries in $k$. Then $A = A(f) = A(M)$ denotes the quadratic graded algebra defined as the quotient of $k\langle x_1, \ldots, x_n \rangle$ by the two-sided ideal generated by $f$. Let us recall the properties of $A$ (see [3] for the proofs and for the definitions of Koszul, AS-Gorenstein or Calabi-Yau algebras).

**Proposition 2.1** Let $A = A(f) = A(M)$ be as above.

1) The quadratic algebra $A$ is Koszul.

2) The global dimension of $A$ is equal to 2, except if $f$ is symmetric of rank 1 (in this case, the global dimension is infinite).

3) $A$ is AS-Gorenstein if and only if $f$ is non-degenerate.

4) $A$ is 2-Calabi-Yau if and only if $f$ is non-degenerate and skew-symmetric.

5) If the global dimension of $A$ is equal to 2, the Hilbert series of the graded algebra $A$ is given by

$$h_A(t) = (1 - nt + t^2)^{-1}.$$ 

Otherwise, one has

$$h_A(t) = (1 - nt + t^2 - t^3 + t^4 - \cdots)^{-1}.$$ 

6) The Gelfand-Kirillov dimension $GK.dim(A)$ of $A$ is equal to 0 if $n = 1$, to $\infty$ if $n > 2$, and if $n = 2$, it is equal to the global dimension.

7) If $k$ is algebraically closed of characteristic zero and if $f$ is non-degenerate, $A$ is left (or right) noetherian if and only if $n = 1$ or $n = 2$.

Set $F = k\langle x_1, \ldots, x_n, z \rangle$ where $z$ is an extra generator of degree 1. We refer to [12, 4, 5] for more details on the definitions and basic properties concerning on algebras defined by a potential. The elements of $F_{cyc} = F/[F,F]$ are called potentials. The $k$-vector space $F_{cyc}$ is
sometimes identified to the space of the cyclic sums $c(a)$ when $a$ runs over $F$. Let us define

our potential as being the class $\overline{w}$ in $F_{ cyc }$ of $w \in F$ where

$$
w = f z = \sum_{1 \leq i, j \leq n} f_{ij} x_i x_j z,
$$
or as being the cyclic sum

$$
c(w) = \sum_{1 \leq i, j \leq n} f_{ij} (x_i x_j z + z x_i x_j + x_j z x_i).
$$

Let us denote by $B = B(f) = B(M)$ the algebra defined by this potential. This is the quotient of the free algebra $F$ by the cyclic partial derivatives $\partial_{x_i}(w), \ldots, \partial_{x_n}(w), \partial_z(w)$, where

$$
\partial_{x_i}(w) = \sum_{1 \leq j \leq n} (f_{ij} x_j z + f_{ji} z x_j), \quad 1 \leq i \leq n, \tag{2.1}
$$

$$
\partial_z(w) = f. \tag{2.2}
$$

So $B$ is a quadratic graded algebra. Let us prove that $A$ is isomorphic to the quotient of $B$ by $z$. Define the morphisms of graded algebras $\varphi$ and $\psi$ by the commutative diagrams

$$
k(x_1, \ldots, x_n) \xrightarrow{\text{can}} F \xrightarrow{\text{can}} A \xrightarrow{\varphi} B/(z) \xrightarrow{\text{can}} B/(z) \xrightarrow{\psi} A
$$

The morphisms $\psi \circ \varphi$ and $\varphi \circ \psi$ leave the generators $x_1, \ldots, x_n$ invariant, hence they coincide with the respective identity maps. Thus $\varphi$ and $\psi$ are isomorphisms of graded algebras, inverse to each other. In all the following, we abbreviate these isomorphisms by $A \cong B/(z)$. The following lemma allows us to consider $A$ as a quadratic subalgebra of $B$.

**Lemma 2.2** The morphism of graded algebras $A \rightarrow B$ induced by the composite

$$
k(x_1, \ldots, x_n) \xrightarrow{\text{can}} F \xrightarrow{\text{can}} B
$$

is injective.

**Proof.** Any element $a \in k\langle x_1, \ldots, x_n \rangle$ belonging to the two-sided ideal of $F$ generated by $\partial_{x_i}(w), \ldots, \partial_{x_n}(w)$ has all its homogeneous components in $z$ of degree $\geq 1$, hence $a = 0$. ■

Denote by $V_A$ and $R_A$ (respectively $V_B$ and $R_B$) the space of generators and the space of relations of $A$ (resp. $B$). One has $V_A = k x_1 + \ldots + k x_n$, $R_A = k f$, $V_B = V_A \oplus k z$ and $R_B = (\sum_{1 \leq i \leq n} k \partial_{x_i}(w)) \oplus R_A$.

**Lemma 2.3** 1) We have $\dim(R_B) \geq \rk(M) + 1$.

2) In particular, if $f$ is non-degenerate (i.e., $M$ is invertible), then $\dim R_B = n + 1$ and $\partial_{x_i}(w), \ldots, \partial_{x_n}(w)$ are $k$-linearly independent in $F$.

3) If $f$ is non-degenerate, the element $z$ is normal in the algebra $B$. It means that $Bz = zB$.

**Proof.** Write down the relations $\partial_{x_i}(w) = 0, \ 1 \leq i \leq n$, as the linear system (in the space $B$)

$$
M \begin{pmatrix}
x_1 z \\
\vdots \\
x_n z
\end{pmatrix} = -^t M \begin{pmatrix}
z x_1 \\
\vdots \\
z x_n
\end{pmatrix} \tag{2.3}
$$
with unknowns $x_1 z, \ldots, x_n z$ viewed in $B$. Reducing this system to a triangular form provides $p = \text{rk}(M)$ equalities in $B$ beginning by $p$ distincts pivots $x_1 z$, so that these equalities viewed as elements of $R_B$ are linearly independent. Hence the inequality in 1). This shows as well that if $M$ is invertible, the elements $x_1 z, \ldots, x_n z$ belong to $zB$, proving that $Bz \subseteq zB$. The opposite inclusion is obtained similarly by reading the linear system (2.3) from the right to the left. 

It is also easy to deduce the inequality in 1) from the following relation

$$\dim(R_{B(M)}) = \text{rk}(\ M \ M^t\ ) + 1,$$

which is immediate from the following

$$\begin{pmatrix}
\partial_{x_1}(w) \\
\vdots \\
\partial_{x_n}(w)
\end{pmatrix} = (\ M \ M^t\ )
\begin{pmatrix}
x_1 z \\
\vdots \\
x_n z
\end{pmatrix} \tag{2.4}
$$

where the latter column is formed by the $2n$ elements $x_1 z, \ldots, x_n z, zx_1, \ldots zx_n$. Note that we have equality in 1) if $M$ is symmetric or antisymmetric (not necessarily invertible), and for $n = 2$ and $f = x_1^2 + x_1 x_2$, the inequality is strict.

**Remark 2.4** Performing the basis change

$$\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix} = P
\begin{pmatrix}
x_1' \\
\vdots \\
x_n'
\end{pmatrix},$$

we see that (2.3) is equivalent to

$$M'
\begin{pmatrix}
x_1' z \\
\vdots \\
x_n' z
\end{pmatrix} = -M'
\begin{pmatrix}
x_1' \\
\vdots \\
x_n'
\end{pmatrix} \tag{2.5}
$$

where $M' = M^t PMP$.

**Example 2.5** Assume that $f$ is symplectic, i.e. non-degenerate and alternate. So $n = 2p$ is even and we can choose generators $x_1, \ldots, x_p, y_1, \ldots, y_p$ such that $f = \sum_{1 \leq i \leq p}(x_i y_i - y_i x_i)$. Then $\partial_{x_i}(w) = y_i z - z y_i$ and $\partial_{y_i}(w) = zx_i - x_i z$ for $1 \leq i \leq p$. In this case, $z$ is central in $B$ and one has

$$c(w) = \sum_{1 \leq i \leq p} \text{Ant}(x_i, y_i, z) \tag{2.6}$$

where $\text{Ant}(a, b, c)$ denotes the antisymmetrizer of $a, b, c$.

We continue the study of the algebra $B = B(f)$. Throughout the sequel of this section, we assume that $f$ is non-degenerate. The proof of Lemma 2.3 shows that $A z = z A$ is a sub-$A$-bimodule of $B$. Let $T_A(A z)$ denote the tensor algebra of the $A$-bimodule $A z$ (on tensor algebras of bimodules, see e.g. [15, p.485]). The $k$-algebra $T_A(A z)$ is generated by $A \oplus A z$, hence by $x_1, \ldots, x_n, z$. Define

$$u : T_A(A z) \to B$$
the natural morphism of $k$-algebras extending the inclusion $A \oplus Az \to B$. The inclusions $x_1, \ldots, x_n \to A$ and $z \to Az$ define a morphism of $k$-algebras

$$F \to T_A(Az)$$

factoring out the relations of $B$. Hence the morphism of $k$-algebras

$$v : B \to T_A(Az)$$

leaving fixed $x_1, \ldots, x_n, z$. Then $v \circ u$ and $u \circ v$ are morphisms of $k$-algebras leaving fixed $x_1, \ldots, x_n, z$. They coincide with the respective identity maps since the $k$-algebras $T_A(Az)$ and $B$ are both generated by $x_1, \ldots, x_n, z$. Thus $u$ and $v$ are isomorphisms, inverse to each other. The following lemma is crucial.

**Lemma 2.6** The left (resp. right) sub-$A$-module $Az$ (resp. $zA$) of $B$ is free generated by $z$.

**Proof.** Let $a \in k(x_1, \ldots, x_n)$ be homogeneous of degree $\ell \geq 0$ such that $az$ belongs to the two-sided ideal of $F$ generated by $\partial_{x_1}(w), \ldots, \partial_{x_n}(w), f$. Our aim is to prove that $a$ belongs to the two-sided ideal of $k(x_1, \ldots, x_n)$ generated by $f$. Write

$$az = \sum_{1 \leq i \leq n, \alpha, \beta \in \mathcal{M}} \lambda_{i, \alpha, \beta} \alpha \partial_{x_i}(w)\beta + \sum_{\alpha, \beta \in \mathcal{M}'} \mu_{\alpha, \beta} \alpha f \beta,$$

(2.7)

where $\mathcal{M}$ denotes the set of (non-commutative) monomials in $x_1, \ldots, x_n$ and $\mathcal{M}'$ is the union of $\mathcal{M}$ with the singleton $\{ z \}$, where $\lambda_{i, \alpha, \beta}$ and $\mu_{\alpha, \beta}$ are in $k$, and where the sums are finitely supported. In the second sum, one has $\deg_{z} \alpha + \deg_{z} \beta = 1$ where $\deg_{z}$ denotes the degree in $z$. Viewing the linear system (2.3) in the free algebra $F$ and reducing it to the triangular form, we see that $z$ in $\alpha$ or $\beta$ can be put on the right modulo the elements $\partial_{x_i}(w)$. Moreover the non-commutative Euler relation

$$\sum_{1 \leq i \leq n} \partial_{x_i}(w)x_i + fz = \sum_{1 \leq i \leq n} x_i \partial_{x_i}(w) + zf = c(w)$$

(2.8)

associated to the potential $w$ shows that the $z$ appearing in each $\alpha f \beta$ can be put completely on the right modulo the elements $\partial_{x_i}(w)$. So we can write

$$az = \sum_{1 \leq i \leq n, \alpha, \beta \in \mathcal{M}} \lambda_{i, \alpha, \beta} \alpha \partial_{x_i}(w)\beta + \left( \sum_{\alpha, \beta \in \mathcal{M}} \mu_{\alpha, \beta} \alpha f \beta \right) z,$$

(2.9)

by keeping the same notation for the coefficients. Denote by $S$ the first sum. In order to conclude, it is sufficient to prove that $S = 0$. We have

$$S = \sum_{1 \leq i, j \leq n, \alpha, \beta \in \mathcal{M}} (\lambda_{i, \alpha, \beta} f_{ij} x_j z \beta + \lambda_{i, \alpha, \beta} f_{ji} \alpha x_j \beta),$$

(2.10)

where $\deg \alpha + \deg \beta = \ell - 1$ and $\deg$ denotes the total degree in $x_1, \ldots, x_n$.

From $S = (a - \sum_{\alpha, \beta \in \mathcal{M}} \mu_{\alpha, \beta} \alpha f \beta) z$, we are going to deduce that the coefficients $\lambda_{i, \alpha, \beta}$ vanish inductively. Choose $\beta$ with the maximal degree $\ell - 1$, so that $\alpha = 1$. Since $\deg(x_j \beta) > \deg(\beta')$ for any $\alpha'$, $\beta'$ appearing in a term $\alpha' x_k z \beta'$ such that $\deg \alpha' + \deg \beta' = \ell - 1$, the coefficient of $z x_j \beta$ vanishes. Hence $\sum_{1 \leq i \leq n} \lambda_{i, 1, \beta} f_{ji} = 0$ for $j = 1, \ldots, n$. As $M$ is invertible, this implies that $\lambda_{i, 1, \beta} = 0$ for $i = 1, \ldots, n$. Thus we can remove all the elements $\lambda_{i, 1, \beta}$ when $\deg \beta = \ell - 1$.

Next, choose $\beta$ of degree $\ell - 2$ and $\alpha$ of degree 1. By the same argument, the coefficient of $\alpha x_j \beta$ vanishes, hence $\sum_{1 \leq i \leq n} \lambda_{i, \alpha, \beta} f_{ji} = 0$ for $j = 1, \ldots, n$, and we conclude again that
all the elements $\lambda_{i,a,b}$ vanish when $\deg \beta = \ell - 2$. Continuing the process, we arrive to $S = 0$. One proves similarly that the right $A$-module $zA$ is free generated by $z$. $\blacksquare$

Lemma 2.6 has the following consequence: for any $a \in A$, there exists a unique element $\sigma(a) \in A$ (resp. $\sigma'(a) \in A$) such that $za = \sigma(a)z$ (resp. $az = \sigma'(a)$). Clearly, $\sigma$ and $\sigma'$ are automorphisms of the $k$-algebra $A$, inverse to each other. Now we are going to describe $B$ as a skew polynomial algebra with coefficients in $A$. It suffices to do it for $T_A(Az) = \bigoplus_{p \geq 0}(Az)^{\otimes A^p}$.

Lemma 2.7 The left (or right) $A$-module $(Az)^{\otimes A^p}$ is free generated by $z^{\otimes A^p} = z \otimes_A \cdots \otimes_A z$ (p times) for any $p \geq 1$.

Proof. Proceed by induction on $p$, the case $p = 1$ being obtained by Lemma 2.6. Assume that the left $A$-module $(Az)^{\otimes A^l}$ is free generated by $z^{\otimes A^l}$. Then any element of $(Az)^{\otimes A^{l+1}} = (Az) \otimes_A (Az)^{\otimes A^l}$ is uniquely written $\alpha \otimes_A (z^{\otimes A^l})$ [We use the basic fact: Let $R$ be a ring, $M$ a left $R$-module which is free of basis $(e_i)$, and $N$ a right $R$-module. Then any element of $N \otimes_R M$ is uniquely written as a sum $\sum_i n_i \otimes_R e_i$ where the family $(n_i)$ is finitely supported]. Writing $a = az, a \in A$, we have

$$\alpha \otimes_A (z^{\otimes A^l}) = (az) \otimes_A (z^{\otimes A^l}) = a(z \otimes_A z^{\otimes A^l}) = az^{\otimes A^{l+1}},$$

Hence the left $A$-module $(Az)^{\otimes A^{l+1}}$ is generated by $z^{\otimes A^{l+1}}$ and, if $az^{\otimes A^{l+1}} = 0$ then $a = 0$ by uniqueness in the recalled result, thus $a = 0$ by Lemma 2.6. We proceed similarly on the right. $\blacksquare$

Consequently, any element of $T_A(Az)$ is uniquely written as a finitely supported sum $\sum_{p \geq 0} a_p z^p$ where $a_p \in A$ and where we set $z^0 = z^{\otimes A^0}$. The product in $T_A(Az)$ is determined by the product in $A$ and by the relations

$$z_p a = \sigma_p(a)z^p,$$

for any $a \in A$ and $p \geq 1$. We have obtained (on skew polynomial rings, see e.g. [7] p. 8-9).

Proposition 2.8 For any non-degenerate $f$, the $k$-algebra $B = B(f)$ is isomorphic to the skew polynomial $k$-algebra $A[z; \sigma]$ defined over the $k$-algebra $A$ by $z$ and the $k$-automorphism $\sigma$ of $A$.

The isomorphism $B \cong A[z; \sigma]$ is an isomorphism of graded algebras, knowing that $A$ is graded and $z$ has degree 1. For any non-zero element $a = \sum_{p \geq 0} a_p z^p$ in $B$, its degree $\deg_z(a)$ in $z$ is the highest $p$ such that $a_p \neq 0$. One has $\deg_z(az) = \deg_z(za) = \deg_z(a) + 1$, so that $z$ is not a zero-divisor in $B$. Moreover it is easy to deduce the Hilbert series of $B$ from the Hilbert series of $A$:

$$h_B(t) = \frac{h_A(t)}{1-t}. \quad (2.11)$$

Using Proposition 2.1 we get

$$h_B(t) = (1 - (n+1)t + (n+1)t^2 - t^3)^{-1} \text{ if } n \geq 2, \quad (2.12)$$

$$h_B(t) = (1 - 2t + 2t^2 - 2t^3 + \cdots)^{-1} \text{ if } n = 1. \quad (2.13)$$

Proposition 2.9 For any non-degenerate $f$, the quadratic algebra $B = B(f)$ is Koszul. The global dimension of $B$ is equal to 3 if $n \geq 2$, to $\infty$ if $n = 1$. 

8
Proof. The global dimension of the graded algebra $B$ is immediately derived from the expression (2.12) or (2.13) of its Hilbert series. According to a standard result on Koszul algebras [2], the Koszulity of $B$ comes from the Koszulity of $A = B/(z)$ because the element $z$ of degree 1 in $B$ is normal and is not a zero-divisor (see also Example 1, p.33 in [23]).

Assume now that $f$ is non-degenerate and $n \geq 2$. The Hilbert series (2.12) of the Koszul algebra $B$ shows that

$$\dim((R_B \otimes V_B) \cap (V_B \otimes R_B)) = 1.$$  

Since $c(w)$ is an element of $(R_B \otimes V_B) \cap (V_B \otimes R_B)$ by the non-commutative Euler relation (2.8), one has

$$(R_B \otimes V_B) \cap (V_B \otimes R_B) = k\,c(w).$$

Therefore the bimodule Koszul complex of $B$ is the following complex

$$K_w 0 \to B \otimes k\,c(w) \otimes B \xrightarrow{d_1} B \otimes R_B \otimes B \xrightarrow{d_2} B \otimes V_B \otimes B \xrightarrow{d_3} B \otimes B$$

and $K_w \xrightarrow{\mu} B$ is the Koszul resolution of $B$ where $\mu : B \otimes B \to B$ is the multiplication. To simplify notation, set $x_{n+1} = z$ and $r_i = \partial_{x_i}(w)$ for $i = 1, \ldots, n+1$. Beside the cyclic partial derivative $\partial_{x_i}$ defined on $F_{cyc}$, there is the “ordinary” partial derivative $\frac{\partial}{\partial x_i} : F \to F \otimes F$ (see [28]) defined on any monomial $a$

$$\frac{\partial a}{\partial x_i} = \sum_{a = ux_i v} u \otimes v,$$

which will be written as

$$\frac{\partial a}{\partial x_i} = \sum_{1,2} \left( \frac{\partial a}{\partial x_i} \right)_1 \otimes \left( \frac{\partial a}{\partial x_i} \right)_2.$$  

Then the differential of the Koszul complex $K_w$ is given by

$$d_1(x_i) = x_i \otimes 1 - 1 \otimes x_i,$$  

$$d_2(r_i) = \sum_{1 \leq j \leq n+1} \sum_{1,2} \left( \frac{\partial r_i}{\partial x_j} \right)_1 \otimes x_j \otimes \left( \frac{\partial r_i}{\partial x_j} \right)_2$$  

$$d_3(c(w)) = \sum_{1 \leq j \leq n+1} x_j \otimes r_j \otimes 1 - 1 \otimes r_j \otimes x_j.$$  

Theorem 2.10 For any non-degenerate $f$ and for any $n \geq 2$, the algebra $B = B(f)$ is 3-Calabi-Yau. In other words, the potential $w = fz$ is 3-Calabi-Yau.

Proof. It suffices to prove that the complex $K_w$ is self-dual with respect to the functor $(-)^\vee = Hom_{B-B}(\cdot, B \otimes B)$ ([5], Lemma 3.7). Denote by $E^*$ the dual space of a $k$-linear space $E$. It is easy to compute the complex $K_w^\vee$:

$$B \otimes B \xrightarrow{d_1^*} B \otimes V_B^* \otimes B \xrightarrow{d_2^*} B \otimes R_B^* \otimes B \xrightarrow{d_3^*} B \otimes kc(w)^* \otimes B \to 0$$

where the differential is given on the dual basis by

$$d_1^*(1) = \sum_{1 \leq j \leq n+1} x_j \otimes x_j^* \otimes 1 - 1 \otimes x_j^* \otimes x_j.$$  


\[
\begin{align*}
    d_2^1(x_i^*) &= \sum_{1 \leq j \leq n+1} \sum_{1,2} \left( \frac{\partial r_j}{\partial x_i} \right)_2 \otimes r_j^* \otimes \left( \frac{\partial r_j}{\partial x_i} \right)_1 \quad (2.20) \\
    d_3^1(r_i^*) &= x_i \otimes c(w)^* \otimes 1 - 1 \otimes c(w)^* \otimes x_i. \quad (2.21)
\end{align*}
\]

Define the diagram (in which we have omitted the symbols \( \otimes \))
\[
B(kc(w))B \xrightarrow{d_3} BRB \xrightarrow{d_2} BVB \xrightarrow{d_1} BkB \\
\downarrow f_3 \quad \downarrow f_2 \quad \downarrow f_1 \quad \downarrow f_0 \\
BkB \xrightarrow{d_i^*} BV^*_B \xrightarrow{d^*_i} B^*_RB \xrightarrow{d^*_i} B(kc(w)^*)B
\]

where the \( B \)-bimodule isomorphisms \( f_i \) are given by
\[
f_0(1) = c(w)^*, \quad f_1(x_i) = r_i^*, \quad f_2(r_i) = x_i^*, \quad f_3(c(w)) = 1. \quad (2.23)
\]

The diagram \( (2.22) \) is commutative. In fact, it is immediate to check that the left square and the right square are commutative. Moreover the commutativity of the central square is a straightforward consequence of the following non-commutative Hessian formula \( [28] \)
\[
\tau \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \right) = \frac{\partial^2 w}{\partial x_j \partial x_i} \quad (2.24)
\]

where we set \( \frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \) and where \( \tau : F \otimes F \to F \otimes F \) is the flip. Thus \( f : K_w \to K_w^\vee \) is a complex isomorphism, so that \( K_w \) is self-dual. \( \blacksquare \)

We complete the properties of \( B = B(f) \) by the following.

**Proposition 2.11** Assume that \( f \) is non-degenerate and \( n \geq 2 \).

1) \( B \) is AS-Gorenstein.

2) If \( f \) is alternate, then \( z \) is central in \( B \). The converse holds if the characteristic of \( k \) is \( \neq 2 \).

3) The Gelfand-Kirillov dimension \( GK.dim(B) \) of \( B \) is finite if and only if \( n = 2 \), and in this case \( GK.dim(B) = 3 \).

4) If \( k \) is algebraically closed of characteristic zero, \( B \) is left (or right) noetherian if and only if \( n = 2 \).

**Proof.** 1) It is an immediate consequence of the Calabi-Yau property ([3], Proposition 4.3).

2) The first claim comes from Example 2.5. For the converse, if \( z \) is central in \( B \), then we have in \( B \) the equalities \( \sum_{1 \leq j \leq n} (f_{ij} + f_{ji}) x_j z = 0 \) for \( i = 1, \ldots, n \). But the \( A \)-module \( Az \) is free, hence \( x_1 z, \ldots x_n z \) are linearly independent in \( B \), and we conclude that \( f_{ij} + f_{ji} = 0 \) for any \( i \) and \( j \).

3) For \( n = 2 \), we have \( h_B(t) = (1-t)^{-3} \), hence \( GK.dim(B) = 3 \). If \( n > 2 \), the polynomial \( 1 - (n+1)t + (n+1)t^2 - t^3 \) has a real root between 0 and 1, hence \( GK.dim(B) = \infty \).

4) If \( n = 2 \) and if \( k \) is algebraically closed of characteristic zero, the algebras \( A \) are classified (Section 4 below) and it is known that they are noetherian. Thus \( B \cong A[z; \sigma] \) is noetherian by a standard result [4]. If \( n > 2 \), then \( GK.dim(B) = \infty \), so we conclude that \( B \) is not (left or right) noetherian by using a theorem due to Stephenson and Zhang [25]. Let us note that the equivalence in 4) can be deduced from the analogous equivalence for \( A \) (see 7) in Proposition 2.1, which is proved by the same arguments as for \( B \) and from a general result ([17], Corollary 2.3). \( \blacksquare \)
3 Classification of the algebras $B$

The aim of this section is to classify the algebras $B = B(M)$ up to isomorphisms of graded algebras. Denote by $A(V, R)$ the quadratic graded algebra defined by a space of generators $V$ and a space of relations $R$ (subspace of $V \otimes V$). We write $A(V, R) \cong A(V', R')$ if there exists an isomorphism of graded algebras from $A(V, R)$ to $A(V', R')$. It is equivalent to say that there exists a linear isomorphism $\varphi : V \to V'$ such that $(\varphi \otimes \varphi)(R) = R'$ (this property is the definition of isomorphisms between two quadratic graded algebras given in [18, 4]).

When $M \in M_n(k)$, we want to classify up to isomorphism the graded algebras $A(M)$ defined at the beginning of Section 2. Recall the result ([3], end of Section 5) and how we get it. Two matrices $M$ and $N$ in $M_n(k)$ are said to be cogredient if there exists $P \in GL_n(k)$ such that $N = ^tPMP$.

**Proposition 3.1.** Let $k$ be a field and $n \geq 1$. Let $M$ and $N$ be in $M_n(k)$. Then $A(M) \cong A(N)$ if and only if the matrices $M$ and $N$ are cogredient.

**Proof.** Following the notation of the beginning of Section 2, let $A(M)$ be the quadratic algebra defined as the quotient of $k\langle x_1, \ldots, x_n \rangle$ by the two-sided ideal generated by $f$, where $f = \sum_{1 \leq i,j \leq n} f_{ij} x_i x_j$ and $M = (f_{ij})_{1 \leq i,j \leq n}$. We have

$$f = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad (3.1)$$

where the matrix product is performed in the free algebra $k\langle x_1, \ldots, x_n \rangle$. Denote by $V$ the vector space of basis $(x_1, \ldots, x_n)$. The isomorphism class of $A(M)$ is identified to the class of the algebras

$$k\langle x'_1, \ldots, x'_n \rangle/(f')$$

where $(x'_1, \ldots, x'_n)$ is a basis of $V$ and $f'$ is deduced from $f$ by replacing $x_1, \ldots, x_n$ by their linear expressions in the basis $(x'_1, \ldots, x'_n)$, so that one has

$$f' = \begin{pmatrix} x'_1 & \ldots & x'_n \end{pmatrix} ^tPMP \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}, \quad (3.2)$$

where $P \in GL_n(k)$. Let $A(N) = k\langle x'_1, \ldots, x'_n \rangle/(F)$ be another algebra defined by

$$F = \begin{pmatrix} x'_1 & \ldots & x'_n \end{pmatrix} N \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}, \quad (3.3)$$

where $x'_1, \ldots, x'_n$ can be considered as forming a basis of the same space $V$. Thus $A(M) \cong A(N)$ if and only if $N = ^tPMP$ where

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}.$$

Now we are interested in the classification of the graded algebras $B$ up to isomorphism. We shall prove that this classification is the same as for the algebras $A$. Keeping the algebra

11
with the space of relations $S$ by (3.3). Then $\Lambda$ is an isomorphism from $B$ to $A$ relative to the action of $GL_n$ on matrices, i.e. the orbits relative to the action of $GL_n$ are cogredient. However the condition $B = \Lambda$ if and only if $B = \Lambda' \tilde{M} \tilde{A}$ holds if and only if $B = \Lambda$ holds.

**Lemma 3.2** If $\Lambda = \left( \begin{array}{cc} \tilde{A} & 0 \\ 0 & \lambda \end{array} \right)$ where $\tilde{A} \in GL_n(k)$ and $\lambda \in k^\times$, we have $\Lambda \cdot B(M) = B(\tilde{A} \tilde{M} \tilde{A})$.

**Proof.** From

$$
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n \\
  z
\end{pmatrix}
\Lambda
\begin{pmatrix}
  x'_1 \\
  \vdots \\
  x'_n \\
  z'
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n \\
  z
\end{pmatrix}
\begin{pmatrix}
  x'_1 \\
  \vdots \\
  x'_n \\
  z'
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n \\
  z
\end{pmatrix}
\begin{pmatrix}
  x'_1 \\
  \vdots \\
  x'_n \\
  z'
\end{pmatrix}
$$

and $z = \lambda z'$ and using (3.4), we obtain

$$
\begin{pmatrix}
  \partial_{x_1}(w)' \\
  \vdots \\
  \partial_{x_n}(w)'
\end{pmatrix}
\Lambda
\tilde{A}
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n \\
  z
\end{pmatrix}

\begin{pmatrix}
  x'_1 \\
  \vdots \\
  x'_n \\
  z'
\end{pmatrix}
$$

and the equality (3.2) holds with $P = \tilde{A}$. Thus the spaces of relations of $\Lambda \cdot B(M)$ and $B(\tilde{A} \tilde{M} \tilde{A})$ (considered in the same generators $x_1', \ldots, x_n', z'$) are equal.

**Theorem 3.3** Let $k$ be a field and $n \geq 1$. Let $M$ and $N$ be in $M_n(k)$. Then $B(M) \cong B(N)$ if and only if $M$ and $N$ are cogredient. In other words, the isomorphism classes of the graded algebras $B(M)$ when $M \in M_n(k)$ are in bijection with the orbits in $M_n(k)$ of cogredient matrices, i.e. the orbits relative to the action of $GL_n(k)$ given by $(P, M) \mapsto \tilde{P}M\tilde{P}^{-1}$.

**Proof.** If $N = \tilde{P}M\tilde{P}$ with $P \in GL_n(k)$, the conclusion $B(M) \cong B(N)$ comes from Lemma 3.2 applied to $\tilde{A} = P$ and $\lambda = 1$. Conversely, assume now that $B(M) \cong B(N)$. If $M = 0$, then $N = 0$, so we assume also that $M \neq 0$ and $N \neq 0$. If the isomorphism $\varphi : B(M) \to B(N)$ of graded algebras is such that $\varphi(kz) = kz'$, then $\varphi$ factors out to produce an isomorphism $A(M) \to A(N)$ of graded algebras, and Proposition 3.1 shows that $M$ and $N$ are cogredient. However the condition $\varphi(kz) = kz'$ does not have to be verified, so that we shall proceed differently.

Define $B(M)$ over the generators $x_1, \ldots, x_n, z$ by the potential $w = fz$ as above. For $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq n+1} \in GL_{n+1}(k)$, define $\Lambda \cdot B(M)$ over the generators $x_1', \ldots, x_n', z'$ as above. Define $B(N)$ over the generators $x_1', \ldots, x_n', z'$ by the potential $W = Fz'$ where $F$ is given by (3.3). Then $\Lambda \cdot B(M) = B(N)$ means that the space of relations $R'$ of $\Lambda \cdot B(M)$ coincides with the space of relations $S$ of $B(N)$. One has

$$R' = \left( \sum_{1 \leq i \leq n} (k\partial_{x_i}(w))' \right) \oplus kf'$$
\[ S = ( \sum_{1 \leq i \leq n} k \partial_{x_i}(W)) \oplus kF. \]

Clearly, \( R' \) has the same dimension as \( R_{B(M)} \). Recall that the latter dimension is equal to \( \text{rk} \left( M \quad ^tM \right) + 1 \). Proceeding in three steps, we prove the following claim.

**Claim.** If \( R' = S \), then \( M \) and \( N \) are cogredient.

**First step.** If the claim holds for \( B(N) \) defined by the potential \( F \), it holds for any potential \( \mu F \) where \( \mu \in k^\times \): it suffices to replace \( z' \) by \( \mu z' \).

**Second step.** The claim holds for any invertible \((n + 1) \times (n + 1)\) matrix \( \Lambda \) such that \( \Lambda = \left( \begin{array}{cc} \tilde{A} & c \\ \ell & \lambda \end{array} \right) \) where \( \tilde{A} \in \text{GL}_n(k) \) (this assumption is essential for the proof). In fact, from

\[ X = \left( \begin{array}{c} \tilde{A} \\ \ell \end{array} \right) Y', \]

where \( ^tX = ( x_1 \ldots x_n ) \) and \( ^tY' = ( x'_1 \ldots x'_n \quad z' ) \), and from the expression (3.1) of \( f \), we obtain

\[ f' = ^tY' \left( \begin{array}{cc} ^t\tilde{A}M\tilde{A} & ^t\tilde{A}Mc \\ t\ell M\tilde{A} & t\ell Mc \end{array} \right) Y'. \tag{3.5} \]

Therefore, \( f' \) modulo \( z' \) is equal to \( ^tX'(^t\tilde{A}M\tilde{A})X' \), where \( ^tX' = ( x'_1 \ldots x'_n ) \). One has \( R'/(z') = S/(z') = kF \) and since \( f' \) modulo \( z' \) does not vanish because \( \tilde{A} \) is invertible and \( M \neq 0 \), \( F \) is proportional to \( f' \) modulo \( z' \). One can even assume that \( F \) is equal to \( f' \) modulo \( z' \) by the first step, thus \( N = ^t\tilde{A}M\tilde{A} \).

**Third step.** If the claim holds for the matrices \( \Lambda_1 \) and \( \Lambda_2 \), it holds for the product \( \Lambda_2\Lambda_1 \). Therefore, since any invertible matrix \( \Lambda \) can be written as a product \( T_1PT_2 \) (Bruhat decomposition) where \( T_1 \) and \( T_2 \) are upper triangular (for which the claim holds by the second step) and where \( P \) is a permutation matrix, it remains to solve the case where \( \Lambda \) is a permutation matrix, not leaving \( z' \) fixed (by the second step). Actually, it suffices to examine the case of a transposition exchanging \( z' \) with another generator which can be assumed to be \( x'_1 \) (by the second step). Performing the basis change

\[ x_1 = z', \quad x_2 = x'_2, \ldots, x_n = x'_n, \quad z = x'_1 \]

in \( f \), we obtain

\[ f' = f_{11}z'^2 + \sum_{2 \leq j \leq n} (f_{1j}z'x'_j + f_{j1}x'_1z') + \sum_{2 \leq i, j \leq n} f_{ij}x'_i x'_j. \]

Assume that \( f' \) does not vanish modulo \( z' \) and show a contradiction. In fact the equality \( R'/(z') = kF \) implies that \( F = \sum_{2 \leq i, j \leq n} f_{ij}x'_i x'_j \) (using the first step). Since the space \( R' \) has the following generators

\[ (\partial_{x_i}(w))' = f_{11}z'x'_i + f_{i1}x'_1z' + \sum_{2 \leq j \leq n} (f_{ij}x'_j x'_i + f_{ji}x'_i x'_j), \quad 1 \leq i \leq n, \]

\[ f' = f_{11}z'^2 + \sum_{2 \leq j \leq n} (f_{1j}z' x'_j + f_{j1} x'_j z') + F, \]

\[ x_1 = z', \quad x_2 = x'_2, \ldots, x_n = x'_n, \quad z = x'_1 \]
the equality $R'/(z') = kF$ implies that any element $\sum_{2 \leq j \leq n} (f_{ij}x_j x'_i + f_{ji}x'_j x'_i)$ is proportional to $F$, hence vanishes ($x'_i$ does not occur in $F$). Therefore $f_{ij} = 0$ whenever $2 \leq i, j \leq n$, which contradicts $F \neq 0$.

Thus $f'$ vanishes modulo $z'$ and we have $f_{ij} = 0$ whenever $i \neq 1$ and $j \neq 1$. Consequently, the space $R'$ has the following generators

$$(\partial_{x_i}(w))' = f_{i1}(z'x'_1 + x'_i z') + \sum_{2 \leq j \leq n} (f_{ij}x_j x'_i + f_{ji}x'_j x'_i),$$

$$(\partial_{x_i}(w))' = f_{i1}z'x'_1 + f_{i1}x'_i z', \quad 2 \leq i \leq n,$$

$$f' = f_{11}z'^2 + \sum_{2 \leq j \leq n} (f_{1j}z'x'_j + f_{j1}x'_j z').$$

Viewing $R'$ modulo $z'$ and using the first step, we get that $F = \sum_{2 \leq j \leq n} (f_{1j}x'_j x'_1 + f_{j1}x'_1 x'_j)$, and $S$ has the following generators

$$\partial_{x'_1}(W) = \sum_{2 \leq j \leq n} (f_{1j}z'x'_j + f_{j1}x'_j z'),$$

$$\partial_{x'_i}(W) = f_{i1}z'x'_1 + f_{i1}x'_i z', \quad 2 \leq i \leq n,$$

$$F = \sum_{2 \leq j \leq n} (f_{1j}x'_j x'_1 + f_{j1}x'_1 x'_j).$$

Now the assumption $R' = S$ shows that $f_{11}z'^2$ belongs to $S$, hence $f_{11} = 0$ and necessarily $n$ is $\geq 2$. Therefore $N = tM$ with

$$M = \begin{pmatrix} 0 & f_{12} & \cdots & f_{1n} \\ f_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & 0 & \cdots & 0 \end{pmatrix},$$

and in this case we have $R' = S$. Clearly $r = rk(M \ tM)$ is equal to 2 or 3. We are going to find an invertible matrix $P$ such that $N = tPMP$ in both cases. To simplify the notation, we shall write $f_{i-1}$ for $f_{1i}$, and $g_{i-1}$ for $f_{ii}$, for all $2 \leq i \leq n$, so that

$$M = \begin{pmatrix} 0 & v_1 \\ t v_2 & 0 \end{pmatrix}$$

where $v_1 := (f_1, \ldots, f_{n-1}) \in k^{n-1}$ and $v_2 := (g_1, \ldots, g_{n-1}) \in k^{n-1}$.

**First case: $r=3$.**

In this case, the vectors $v_1 \in k^{n-1}$ and $v_2 \in k^{n-1}$ are linearly independent, so that there exists a invertible matrix $Q \in GL_{n-1}(k)$ interchanging these two elements, i.e., satisfying :

$$Q^t v_1 = t v_2, \quad \text{and} \quad Q^t v_2 = t v_1.$$

We then consider the matrix $P \in GL_n(k)$ given by:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t Q \end{pmatrix}$$

and it is easy to see that it satisfies $tPMP = tM$.

**Second case: $r=2$.**
In this case, the vectors $v_1$ and $v_2$ are linearly dependent, and for all $1 \leq i, j \leq n - 1$, one has $f_i g_j = f_j g_i$. We shall proceed by induction on $n \geq 2$.

If $n = 2$, then $M = \begin{pmatrix} 0 & f_1 \\ g_1 & 0 \end{pmatrix}$ and the matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies the desired property: $^t PMP = ^t M$.

Now, suppose that $n \geq 3$ and that for every matrix of the form

$$M' = \begin{pmatrix} 0 & v'_1 \\ i v_2 & 0 \end{pmatrix} \in M_{n-1}(k)$$

where $v'_1 := e_k^{n-2}$ and $v'_2 \in k^{n-2}$, there exists a matrix $P' \in GL_{n-1}(k)$ satisfying $^t P'M'P' = ^t M'$. Let now

$$M = \begin{pmatrix} 0 & v_1 \\ i v_2 & 0 \end{pmatrix} \in M_n(k)$$

where $v_1 := (f_1, \ldots, f_{n-1}) \in k^{n-1}$ and $v_2 := (g_1, \ldots, g_{n-1}) \in k^{n-1}$. Now, three cases have to be considered: $f_{n-1} = g_{n-1} = 0$, or $g_{n-1} = 0$ and $f_{n-1} \neq 0$, or $g_{n-1} \neq 0$.

1. First, let us suppose that $f_{n-1} = g_{n-1} = 0$, then, by induction hypothesis, there exists a matrix $P' \in GL_{n-1}(k)$ satisfying $^t P'M'P' = ^t M'$, where $M'$ is the truncated $(n-1) \times (n-1)$ matrix:

$$M' = \begin{pmatrix} 0 & f_1 & \cdots & f_{n-2} \\ g_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-2} & 0 & \cdots & 0 \end{pmatrix},$$

and it is clear that the matrix $P = \begin{pmatrix} P' & 0 \\ 0 & 1 \end{pmatrix} \in GL_n(k)$ satisfies $^t PMP = ^t M$.

2. Now, let us assume that $g_{n-1} = 0$ and $f_{n-1} \neq 0$. Note that in this case, one has necessarily $g_k = 0$ for all $1 \leq k \leq n - 1$. It is then straightforward to verify that $P$ given by:

$$P = \begin{pmatrix} 0 & (f_1/f_{n-1}) & \cdots & (f_{n-2}/f_{n-1}) & 1 \\ 0 & Id_{n-2} & 0 \\ 1 & (-f_1/f_{n-1}) & \cdots & (-f_{n-2}/f_{n-1}) & 0 \end{pmatrix} \in M_n(k)$$

is invertible and satisfies $^t PMP = ^t M$.

3. If however $g_{n-1} \neq 0$, then an analogous computation as before shows that the matrix $P$ given by:

$$P = \begin{pmatrix} 0 & (g_1/g_{n-1}) & \cdots & (g_{n-2}/g_{n-1}) & 1 \\ 0 & Id_{n-2} & 0 \\ 1 & (-g_1/g_{n-1}) & \cdots & (-g_{n-2}/g_{n-1}) & 0 \end{pmatrix} \in M_n(k)$$

is invertible and satisfies $^t PMP = ^t M$. □
4 Special properties of $B(M)$ when $n = 2$

Throughout the sequel of this paper, we assume that $k = \mathbb{C}$ (actually, it would be sufficient to assume that $k$ is algebraically closed of characteristic zero). An explicit parameter space for the $GL(2, \mathbb{C})$-orbits of cogredient matrices in $GL(2, \mathbb{C})$ is given by Dubois-Violette in [11]. According to the previous section, this parameter space forms a “moduli space” for the algebras $B(M)$ when $M$ runs over $GL(2, \mathbb{C})$. Recall the description in three types of this parameter space as stated in ([11], end of Section 2). In each type, we give the potential $w$ and the relations of $B(M)$. The generators are denoted by $x$, $y$ and $z$. The types depend on the rank $rk$ of the symmetric part $s(M) = \frac{1}{2}(M + M^T)$ of $M$. It turns out that the corresponding algebras $A(M)$ are exactly the AS-regular algebras of global dimension 2 [1]. Therefore the three types classify the AS-regular algebras of global dimension 2 as well, and we have kept the same standard terminology for the classification of our algebras $B(M)$.

**First type (classical type):** $rk = 0$. There is only one orbit, which is the orbit of $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One has $w = (yx - xy)z$. The relations of $B(M)$ are the following: $zy = yz$, $xz = zx$, $yx = xy$. Hence $B(M) = \mathbb{C}[x, y, z]$ the commutative polynomial algebra. Note that it is the unique orbit such that $z$ is central in $B(M)$ (Proposition 2.11), or such that $A(M)$ is Calabi-Yau (Proposition 2.1).

**Second type (Jordan type):** $rk = 1$. There is only one orbit, which is the orbit of $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. One has $w = (yx - xy - x^2)z$. The relations of $B(M)$ are the following: $zy = yz + 2xz$, $xz = zx$, $yx = xy + x^2$.

**Third type (quantum type):** $rk = 2$. The orbits are parametrized by the set $\{q \in \mathbb{C} \setminus \{0, 1\}\}/(q \sim q^{-1})$. These are the orbits of $M = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$. One has $w = (yz - q^{-1}xy)z$.

The relations of $B(M)$ are the following: $xy = qyx$, $yz = qzy$, $zx = qxz$. So $B(M)$ is a quantum space.

**Proposition 4.1** For any $M \in GL(2, \mathbb{C})$, the algebra $B(M)$ is a quadratic AS-regular algebra of global dimension 3, of type $A$ non-generic (following the classification of Artin and Schelter [2]). For any $M$ in the list of the above classification, $(x^iy^jz^k)_{i\geq 0, j\geq 0, k\geq 0}$ is a basis of the vector space $B(M)$.

**Proof.** According to Proposition 2.11 $B(M)$ is AS-Gorenstein with a finite Gelfand-Kirillov dimension, hence is AS-regular by definition. Since $B(M)$ is Calabi-Yau (Theorem 2.10), $B(M)$ is of type $A$ ([3], Proposition 5.4). Moreover, $B(M)$ is a skew polynomial ring over $A(M)$ in $z$ (Proposition 2.8) and it is clear from each relation $f$ listed above that $A(M)$ is a skew polynomial ring in $x, y$. Therefore $B(M)$ is a skew polynomial ring in $x, y, z$ (hence the basis of monomials $x^iy^jz^k$), thus the invariant $j$ of $B(M)$ is infinite ([1], Theorem 6.11).

Among the AS-regular algebras of type $A$, the generic ones correspond to a finite invariant $j$ and are called Sklyanin algebras [27]. Recall that Sklyanin algebras are defined by three generators $x, y, z$, and three relations $\alpha x + \beta y + \gamma z^2 = 0$, $\alpha x z + \beta z x + \gamma y^2 = 0$, $\alpha xy + \beta y x + \gamma z^2 = 0$, where $(\alpha, \beta, \gamma) \in \mathbb{F}^2 \setminus S$ and

$$S = \{(\alpha, \beta, \gamma) \in \mathbb{F}^2; \alpha^2 = \beta^3 = \gamma^3\} \cup \{(1 : 0 : 0), (0 : 1 : 0), (1 : 0 : 0)\}.$$
So the algebras $B(M)$ of the first or third type can be considered as limits of Sklyanin algebras by vanishing the parameter $\gamma$, but such a process is not possible for the second type.

In the classification of their quadratic regular algebras of global dimension 3, Artin and Schelter define the invariant $j$ as the invariant of a certain cubic curve $C$ in $\mathbb{P}^2$. As we shall see, the equation $\phi = 0$ of $C$ is easily deduced from the potential $w$ of $B(M)$. More importantly, we shall interpret $\phi$ as a Poisson potential whose associated Poisson bracket (defined as usual by $\nabla \phi$) is exactly the semi-classical limit of $B(M)$ viewed as a deformation of $\mathbb{C}[x, y, z]$. Let us begin by the description of the curve $C$ as in [1].

To avoid confusion with our notation, let us replace the notation $M, w, Q$ used in [1] by $\mathcal{M}, W, Q$. The $3 \times 3$ matrix $\mathcal{M}$ is defined by

$$\begin{pmatrix} \frac{\partial_{x}(w)}{f} \\ \frac{\partial_{y}(w)}{f} \\ \frac{\partial_{z}(w)}{f} \end{pmatrix} = \mathcal{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (4.1)$$

so that we have

$$\mathcal{M} = \begin{pmatrix} \overset{t}{M}z & M \begin{pmatrix} x \\ y \end{pmatrix} \\ x & y \end{pmatrix}.$$ 

The element $W$ is defined by $W = XM^tX$, where $X = \begin{pmatrix} x & y & z \end{pmatrix}$. From (4.1) and the non-commutative Euler relation (2.8), we deduce that

$$W = c(w), \quad (4.2)$$

that is, $W$ is precisely the potential $w$ once identified to its cyclic sum. The matrix $Q$ is defined by $XQ = \overset{t}{M} \begin{pmatrix} x \\ y \end{pmatrix} tQ$. Then (2.8) implies that $Q$ is equal to the identity matrix and we recover the fact that $B(M)$ is of type A (by definition).

Next, define the subvariety $C$ of $\mathbb{P}^2$ by its equation $S(W) = 0$, where $S(W)$ denotes the symmetrization of the element $W$. The symmetrization consists in replacing products of variables in the free algebra by the same products in the polynomial algebra. In our situation, $S(W) = 3S(w) = 3S(f)z$. Set $\phi = S(f)z$. The above classification in three types allows us to limit ourselves to the following matrices

$$M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, \quad b \neq 0, \quad (4.3)$$

so that

$$w = (ax^2 + bxy + yx)z, \quad \phi = (ax^2 + (b + 1)xy)z. \quad (4.4)$$

In the classical type, one has $C = \mathbb{P}^2$, and otherwise $C$ is the union of three straight lines whose two ones coincide in the Jordan type.

**Definition 4.2** The polynomial $\phi = \phi(M)$ is called the Poisson potential associated to the algebra $B(M)$.

This definition will be more natural when $B(M)$ will be viewed as a Gerstenhaber deformation whose Poisson bracket $\{\cdot, \cdot\}$ will be defined on $\mathbb{C}[x, y, z]$ by the formulas

$$\{x, y\} = \frac{\partial \phi}{\partial z}, \quad \{y, z\} = \frac{\partial \phi}{\partial x}, \quad \{z, x\} = \frac{\partial \phi}{\partial y},$$

i.e., $\{\cdot, \cdot\} = \frac{\partial \phi}{\partial z} \wedge \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} \wedge \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial y} \wedge \frac{\partial \phi}{\partial x}$.
Definition 4.3 The so-defined Poisson bracket \{·,·\} is called the Poisson bracket derived from the Poisson potential \(\phi\).

It is clear that \(\phi\) belongs to the Poisson center of \{·,·\} (because \(\{x,\phi\} = \{y,\phi\} = \{z,\phi\} = 0\), i.e., \(\phi\) is a Casimir element of the Poisson bracket. In the classical type, \{·,·\} = 0. For the other types, the next proposition shows that the Casimir element \(\phi\) lifts to a non-zero element \(\Phi\) of the center of \(B(M)\). Note that the potential \(w\) lifts \(\phi\) in the free algebra, but \(w\) vanishes in \(B\) (it is a difference with Sklyanin algebras for which the potential \(w\) does not vanish in the algebra). Note that \(c(w)\) always vanishes in the algebra according to the non-commutative Euler relation (2.8).

Proposition 4.4 The element \(\Phi\) of \(B(M)\) defined by

\[\Phi = (ax^2 + (b+1)xy)z\]

belongs to the center of \(B(M)\).

Proof. The relations of \(B\) can be written as follows \(zy = (-a + \frac{a}{b})xz - byz, zx = -\frac{1}{b}xz, yx = -ax^2 - bxy\). These relations allow us to decompose the elements \(x\Phi\) and \(\Phi x\) in the basis \((x^iy^jz^k)_{i,j,k\geq 0}\) of \(B\). The computations are straightforward and they show that \(x\Phi = ax^3z + (b+1)x^2yz = \Phi x\). In the same manner, we get \(y\Phi = a^2/(b-1)x^3z + a(b^2-b-1)x^2yz - b(b+1)xy^2z = \Phi y\) and \(z\Phi = ax^2z^2 + (b+1)xy^2z = \Phi z\). □

We are now interested in the comparison between the Hochschild homology of \(B(M)\) and the Poisson homology of \((\mathbb{C}[x,y,z],\{·,·\})\). In the classical type, the Hochschild homology of \(\mathbb{C}[x,y,z]\) is well-known and coincides with the Poisson homology for the corresponding \{·,·\} (vanishing everywhere in this case). As proved in the next proposition, it is the same for the quantum type if \(q\) is not a root of unity. We are grateful to Wambst for the explicit formulas of Hochschild homology.

Proposition 4.5 Let \(q\) be a non-zero complex number which is not a root of unity. Let \(B\) be the \(\mathbb{C}\)-algebra defined by the potential \(w = (yx - q^{-1}xy)z\), i.e. defined by the generators \(x, y, z\), and the following relations

\[xy = qyx, yz = qzy, zx = qxz.\]

Let \(S = \mathbb{C}[x,y,z]\) be the polynomial \(\mathbb{C}\)-algebra in \(x, y, z\), endowed with the Poisson bracket derived from the Poisson potential \(\phi = (1 - q^{-1})xyz\). Denote by \(Z = \mathbb{C}[xyz]\) the subalgebra of \(S\) generated by \(xyz\). Then the Hochschild homology of \(B\) is isomorphic to the Poisson homology of \(S\) and is given by

\[\text{HH}_0(B) \cong Z \oplus x\mathbb{C}[x] \oplus y\mathbb{C}[y] \oplus z\mathbb{C}[z],\]
\[\text{HH}_1(B) \cong ((\mathbb{C}[x] \oplus yz\mathbb{C}) \oplus y) \oplus ((\mathbb{C}[y] \oplus xz\mathbb{C}) \oplus x) \oplus ((\mathbb{C}[z] \oplus xy\mathbb{C}) \oplus z),\]
\[\text{HH}_2(B) \cong (xz \otimes (y \wedge z)) \oplus (yz \otimes (z \wedge x)) \oplus (zZ \otimes (x \wedge y)),\]
\[\text{HH}_3(B) \cong Z \otimes (x \wedge y \wedge z),\]
\[\text{HH}_p(B) = 0\] for any \(p \geq 4\).

Proof. On one hand, setting \(x_1 = x, x_2 = y, x_3 = z\) and applying Théorème 6.1 in [29], we obtain

\[\text{HH}_p(B) = \bigoplus_{\beta \in \{0,1\}^3, |\beta| = p} \bigoplus_{\alpha \in \mathbb{N}_0^2, \alpha + \beta \in C} x_\alpha \otimes x_\beta,\]

18
where }C = \{ \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3; \alpha_i = 0 \text{ or } \prod_{1 \leq j \leq 3} q_{ij}^\alpha = 1 \}. \text{ As usual, one has } q_{ii} = 1 \text{ and } q_{ij} = q_{ji}^{-1}. \text{ For our quantum space } B, \text{ we have } q_{12} = q_{23} = q_{31} = q, \text{ so that we get } C = \mathbb{N}(1,0,0) \cup \mathbb{N}(0,1,0) \cup \mathbb{N}(0,0,1) \cup \mathbb{N}(1,1,1). \text{ The result for } HH_p(B) \text{ follows easily.}

On the other hand, the Poisson bracket derived from the potential } xyz \text{ is diagonalizable in sense of Monnier, and the Poisson cohomology is computed in [20]. Moreover, the duality } HP^\bullet(S) \cong HP_{3-\bullet}(S) \text{ holds } a \text{ priori, because a Poisson bracket deriving from a potential, is always unimodular, i.e., its modular class vanishes (see the definition of the modular vector field (and its class) in [31], which is called the curl vector field in [10]). Recall the definition of the Poisson homology complex of } S, \text{ associated to a Poisson bracket } \{ \cdot, \cdot \}. \text{ First, the space of Poisson 1-chains is the } S\text{-module of Kähler differentials of } S, \text{ denoted by } \Omega^1(S) \text{ and generated, as an } S\text{-module by the three elements } dx, dy, dz. \text{ Then, for } p \in \mathbb{N}, \text{ the } S\text{-module of Kähler } p\text{-differentials is the space of Poisson } p\text{-chains and is given by } \Omega^p(S) = \bigwedge^p \Omega^1(S). \text{ Of course, one has } \Omega^p(S) = \{ 0 \}, \text{ as soon as } p \geq 4. \text{ Next, the Poisson boundary operator, } \delta_p : \Omega^p(S) \to \Omega^{p-1}(S), \text{ called the Brylinski or Koszul differential, is given, for } F_0, F_1, \ldots, F_p \in S, \text{ by (see [3])}:

\[\delta_p(F_0 \, df_1 \wedge \cdots \wedge df_p) = \sum_{i=1}^p (-1)^{i+1} \{ F_0, F_i \} \, df_1 \wedge \cdots \wedge \hat{df}_i \wedge \cdots \wedge df_p + \sum_{1 \leq i < j \leq p} (-1)^{i+j} F_0 \, df_i \wedge df_j \wedge \cdots \wedge df_p,\]

where the symbol } \hat{df}_i \text{ means that we omit the term } df_i. \text{ The example 2.4 (i) of [20], then gives:}

}\begin{align*}
HP_0(S) & \cong Z \oplus xC[x] \oplus yC[y] \oplus zC[z] \\
HP_1(S) & \cong (\langle C[x] \oplus yzZ \rangle dx) \oplus ((\langle C[y] \oplus xzZ \rangle dy) \oplus ((\langle C[z] \oplus xyZ \rangle dz) \\
HP_2(S) & \cong (xZ \oplus (dy \wedge dz) \oplus (yZ \oplus (dz \wedge dx) \oplus (zZ \oplus (dx \wedge dy)) \\
HP_3(S) & \cong Z \otimes (dx \wedge dy \wedge dz) \\
HP_p(S) & = 0 \text{ for any } p \geq 4.
\end{align*}

It is now clear that the Poisson homology vector spaces of } S \text{ are isomorphic to the Hochschild homology vector spaces of } B. \blacksquare

We have seen that the duality } HP^\bullet(S) \cong HP_{3-\bullet}(S) \text{ holds since the Poisson bracket derives from a potential. Since } B \text{ is 3-Calabi-Yau (Theorem 2.10), one also has the duality } HH^\bullet(B) \cong HH_{3-\bullet}(B). \text{ In particular the center of the algebra } B \text{ is the polynomial algebra generated by the element } \Phi \text{ of Proposition 4.4, i.e. generated by } xyz.

## 5 Poisson homology and Hochschild homology for the second type

In this section, the field } k = \mathbb{C} \text{ and } n = 2. \text{ The aim of this section is to prove Theorem 1.1 of the Introduction, i.e., to determine the Hochschild homology of the algebra } B \text{ of the second type (Jordan type):}

\[B = \mathbb{C}(x, y, z)/(zy = yz + 2xz, zx = xz, yx = x^2 + xy).\]

Let us recall the notation: } f = x^2 + xy - yx, w = f z, r_1 = \partial_x(w), r_2 = \partial_y(w) \text{ and } r_3 = \partial_z(w). \text{ We have }

\[r_1 = xz + zx + yz - yx, \quad r_2 = zx - xz, \quad r_3 = x^2 + xy - yx.\]
In order to obtain the Hochschild homology of \( B \), we will first see \( B \) as the deformation of a Poisson algebra for which we will determine the Poisson homology. To do this, let us consider the filtration \( F \) of \( B \) given by the degree of \( y \). In other words, for this filtration, the degree of \( y \) is 1 while the degrees of \( x \) and \( z \) are 0. It is clear from the relations of \( B \) that \( gr_F(B) \simeq \mathbb{C}[x, y, z] \). So the filtered algebra \( B \) is almost commutative. Moreover, \( gr_F(B) \) is equipped with the Poisson bracket defined by:

\[
\{z, y\} = 2xz, \quad \{z, x\} = 0, \quad \{y, x\} = x^2,
\]

which is the Poisson structure derived from the Poisson potential \( \phi = -x^2z \). In the sequel, we will denote this Poisson algebra by \( T = (\mathbb{C}[x, y, z], \{, \}) \).

From the Koszul resolution \( K_w(2.14) \), we easily get the following Koszul complex \( B \otimes B^e \)

\[
K_w \text{ associated to } B \text{ (where we have omitted the symbols } \otimes \text{)}:
\]

\[
0 \to B(\mathbb{C}(w)) \xrightarrow{\tilde{d}_3} BR_B \xrightarrow{\tilde{d}_2} BV_B \xrightarrow{\tilde{d}_1} B \to 0
\]

where \( c(w) = x^2z + xzx + z^2x^2 + xyz + yzx + zxy - yxz - xzy - yx \) and where the differentials are given, for \( a \in B \), by:

\[
\tilde{d}_1(a \otimes x) = [a, x] = ax - xa, \quad \tilde{d}_1(a \otimes y) = [a, y], \quad \tilde{d}_1(a \otimes z) = [a, z],
\]

while

\[
\tilde{d}_2(a \otimes r_1) = (za + az) \otimes x + (za - az) \otimes y + (ax + xa + ay - ya) \otimes z,
\]

\[
\tilde{d}_2(a \otimes r_2) = (az - za) \otimes x + (xa - ax) \otimes z,
\]

\[
\tilde{d}_2(a \otimes r_3) = (xa + ax + ya - ay) \otimes x + (ax - xa) \otimes y,
\]

and

\[
\tilde{d}_3(a \otimes c(w)) = [a, x] \otimes r_1 + [a, y] \otimes r_2 + [a, z] \otimes r_3.
\]

This complex computes the Hochschild homology \( HH_*(B) \) of \( B \). Notice that one can naturally identify the spaces of chains of this complex with \( B \) or \( B^3 \). Indeed, one can write

\[
B(\mathbb{C}(w)) \simeq B, \quad \text{while } BR_B \simeq B^3 \text{ and } BV_B \simeq B^3,
\]

by identifying, for any \( a_1, a_2, a_3 \in B \), the element \( a_1 \otimes r_1 + a_2 \otimes r_2 + a_3 \otimes r_3 \in BR_B \) or the element \( a_1 \otimes x + a_2 \otimes y + a_3 \otimes z \in BV_B \) with \( (a_1, a_2, a_3) \in B^3 \).

In the previous section, we have recalled the definition of the differential \( \delta \) of the Poisson homology complex. The Poisson chains are given by the Kähler differentials. The \( T \)-module of Kähler differentials of \( T \) is generated by the three elements \( dx, dy, dz \), so that we can naturally identify an element \( F_1dx + F_2dy + F_3dz \in \Omega^1(T) \) with the element \( (F_1, F_2, F_3) \in T^3 \). Similarly, an element \( F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy \in \Omega^2(T) \) can be identified with the element \( (F_1, F_2, F_3) \in T^3 \), and an element \( Fdx \wedge dy \wedge dz \in \Omega^3(T) \) with the element \( F \in T \). (Notice that in the following we will either write an element in \( T^3 \) as a row vector or a column vector.)

Using the previous identifications, the Poisson homology complex of the Poisson algebra \( T \) can be written as:

\[
\begin{array}{ccccccc}
0 & \quad \delta_3 & \quad T^3 & \quad \delta_2 & \quad T^3 & \quad \delta_1 & \quad T & \quad 0 \\
\end{array}
\]

with the differentials given, for \( F \in T \), and \( \vec{F} := (F_1, F_2, F_3) \in T^3 \), by:

\[
\begin{align*}
\delta_1(\vec{F}) &= \nabla \phi \cdot (\nabla \times \vec{F}) = \text{Div}(\vec{F} \times \nabla \phi), \\
\delta_2(\vec{F}) &= -\nabla(\vec{F} \cdot \nabla \phi) + \text{Div}(\vec{F}) \nabla \phi, \\
\delta_3(F) &= -\nabla F \times \nabla \phi,
\end{align*}
\]
where \( \text{Div}(\tilde{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \in T \), while, for all element \( G \in T \), we write \( \nabla G := \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right) \in T^3 \). We also have respectively denoted by ‘•’ and \( \nabla \cdot \times \), the usual inner and cross products in \( T^3 \) and the curl operator. Note that \( \nabla \phi = (-2xz, 0, -x^2) \).

As in \( B \), we assume that in \( V_B \), \( R_B \) and \( c(w) \), \( y \) has degree 1 while \( x \) and \( z \) have degree 0, so that we can consider the total filtration degree in \( B \otimes_{B^c} K_w \). In Proposition 5.2 below, we show that \( \tilde{d}_1, \tilde{d}_2 \) and \( \tilde{d}_3 \) have degree –1 for the total filtration degree. So \( B \otimes_{B^c} K_w \) will be a filtered complex (with differentials preserving the degree) for the following shifts on the total filtration degree:

\[
0 \rightarrow B(\mathbb{C}(w))(3) \xrightarrow{\tilde{d}_3} BR_B(2) \xrightarrow{\tilde{d}_2} BV_B(1) \xrightarrow{\tilde{d}_1} B \rightarrow 0 \tag{5.4}
\]

whose filtration is still denoted by \( F \). In the same proposition, we show that \( gr_F(B \otimes_{B^c} K_w) \) is identified to the Poisson homology complex of the Poisson algebra \( T \). The next lemma will be useful.

Lemma 5.1 For all \( k \in \mathbb{N} \), the following identities hold in \( B \):

\[
y x^k = x^k y + k x^{k+1}, \quad z^k = y z^k + 2 k x z^k, \tag{5.5}
\]

while

\[
y^k x = \sum_{j=0}^{k} \frac{k!}{j!} x^{k-j+1} y^j, \tag{5.6}
\]

and

\[
zy^k = \sum_{\ell=0}^{k} (k - \ell + 1) \frac{k!}{\ell!} x^{k-\ell} y^\ell z. \tag{5.7}
\]

Proof. Straightforward by induction on \( k \in \mathbb{N} \).

Proposition 5.2 Let us consider the filtration \( F \) on the algebra \( B \), given by the degree in \( y \). Then the differential of the Koszul complex \( B \otimes_{B^c} K_w \) has degree –1 for the total filtration degree. Moreover, the graded complex associated to the filtered complex \( [5.4] \) is isomorphic to the Poisson homology complex of the Poisson algebra \( T \).

Proof. As we have already seen, \( gr_F(B) \simeq T \). As the monomials \( x^i y^j z^k \), with \( i, j, k \in \mathbb{N} \), form a \( \mathbb{C} \)-basis of \( B \) (Proposition 5.1), the identifications of the spaces of Hochschild chains with \( B \) or \( B^3 \) and their analogs for the spaces of Poisson chains with \( T \) and \( T^3 \) show that the spaces of Hochschild and Poisson chains are isomorphic. Let us begin to compute the images of \( \tilde{d}_e \) in the basis \( (x^i y^j z^k) \).

Let \( i, j, k \in \mathbb{N} \). By definition of \( \tilde{d}_1 \), we have \( \tilde{d}_1(x^i y^j z^k \otimes x) = x^i y^j x z^k - x^{i+1} y^j z^k \). Using formula (5.6), this gives:

\[
\tilde{d}_1(x^i y^j z^k \otimes x) = \sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j-\ell+1} y^\ell z^k. \tag{5.8}
\]

From \( \tilde{d}_1(x^i y^j z^k \otimes y) = x^i y^j z^k y - y x^i y^j z^k \) and (5.5), we get

\[
\tilde{d}_1(x^i y^j z^k \otimes y) = 2 k x^i y^j x z^k - i x^{i+1} y^j z^k,
\]

which, with (5.6), gives:

\[
\tilde{d}_1(x^i y^j z^k \otimes y) = (2k - i)x^{i+1} y^j z^k + 2k \sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j-\ell+1} y^\ell z^k. \tag{5.9}
\]
From $\tilde{d}_1(x^i y^j z^k \otimes z) = x^i y^j z^{k+1} - x^i z y^j z^k$ and (5.7), we get
\[
\tilde{d}_1(x^i y^j z^k \otimes z) = -\sum_{\ell=0}^{j-1} (j - \ell + 1) \frac{j!}{\ell!} x^{i+j-\ell} y^\ell z^{k+1}.
\] (5.10)

We now compute the elements $\tilde{d}_2(x^i y^j z^k \otimes r_1)$, where $1 \leq \ell \leq 3$. By definition, $\tilde{d}_2(x^i y^j z^k \otimes r_1) = (z x^i y^j z^k + x^i y^j z^{k+1}) \otimes x + (z x^i y^j z^k - x^i y^j z^{k+1}) \otimes y + (x^i y^j z^k x + x^{i+1} y^j z^k + x^i y^{j+1} z^k y - y x^i y^j z^k) \otimes z$. Using the relations of $B$ together with (5.5), (5.6) and (5.7), we get:
\[
\begin{align*}
\tilde{d}_2(x^i y^j z^k \otimes r_1) &= \left( 2x^i y^j z^{k+1} + \sum_{\ell=0}^{j-1} (j - \ell + 1) \frac{j!}{\ell!} x^{i+j-\ell} y^\ell z^{k+1} \right) \otimes x \\
&\quad + \left( -\sum_{\ell=0}^{j-1} (j - \ell + 1) \frac{j!}{\ell!} x^{i+j-\ell} y^\ell z^{k+1} \right) \otimes y \\
&\quad + \left( (2k - i + 2)x^{i+1} y^j z^k + (2k + 1) \sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j-\ell+1} y^\ell z^k \right) \otimes z. 
\end{align*}
\] (5.11)

Next $\tilde{d}_2(x^i y^j z^k \otimes r_2) = (x^i y^j z^{k+1} - x i y^j z^k) \otimes x + (z x^i y^j z^k - x^i y^j z^{k+1}) \otimes y$, which, according to (5.7) and (5.6), can be written as:
\[
\begin{align*}
\tilde{d}_2(x^i y^j z^k \otimes r_2) &= \left( -\sum_{\ell=0}^{j-1} (j - \ell + 1) \frac{j!}{\ell!} x^{i+j-\ell} y^\ell z^{k+1} \right) \otimes x \\
&\quad + \left( -\sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j-\ell+1} y^\ell z^k \right) \otimes y. 
\end{align*}
\] (5.12)

Similarly, from $\tilde{d}_2(x^i y^j z^k \otimes r_3) = (x^{i+1} y^j z^k + x^i y^j z^{k+1} x + y x^i y^j z^k - x^i y^j z^k) \otimes x + (x^i y^j z^k x - x^{i+1} y^j z^k) \otimes y$, we have
\[
\begin{align*}
\tilde{d}_2(x^i y^j z^k \otimes r_3) &= \left( (i + 2 - 2k)x^{i+1} y^j z^k + (1 - 2k) \sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j+1-\ell} y^\ell z^k \right) \otimes x \\
&\quad + \left( \sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j+1-\ell} y^\ell z^k \right) \otimes y. 
\end{align*}
\] (5.13)

Finally, let us look at $\tilde{d}_3$. By definition of $\tilde{d}_3$ and according to previous computations, we get
\[
\begin{align*}
\tilde{d}_3(x^i y^j z^k \otimes c(w)) &= (x^i y^j z^k x - x^{i+1} y^j z^k) \otimes r_1 + (x^i y^j z^k y - y x^i y^j z^k) \otimes r_2 \\
&\quad + (x^i y^j z^{k+1} - x x^i y^j z^k) \otimes r_1 + (x^i y^j z^k y - y x^i y^j z^k) \otimes r_2 \\
&= \left( \sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j+1-\ell} y^\ell z^k \right) \otimes r_1 + (x^i y^j z^k y - y x^i y^j z^k) \otimes r_2 \\
&\quad + \left( -\sum_{\ell=0}^{j-1} (j - \ell + 1) \frac{j!}{\ell!} x^{i+j-\ell} y^\ell z^{k+1} \right) \otimes r_3.
\end{align*}
\]
It remains to use (5.5) to obtain:

\[
\tilde{d}_3(x^iy^jz^k \otimes c(w)) = \left(\sum_{\ell=0}^{j-1} \frac{j!}{\ell!} x^{i+j+1-\ell} y^\ell z^k\right) \otimes r_1
\]

\[
+ \left(2k - i\right) x^{i+1} y^j z^k + 2k \sum_{\ell=0}^{i-1} \frac{j!}{\ell!} x^{i+j+1-\ell} y^\ell z^k\right) \otimes r_2
\]

\[
+ \left(-\sum_{\ell=0}^{i-1} (j - \ell + 1) \frac{j!}{\ell!} x^{i+j-i} y^\ell z^{k+1}\right) \otimes r_3.
\]

(5.14)

Thus the so-obtained formulas show that the differential \(\tilde{d}_3\) has degree \(-1\) for the total filtration degree. It remains to show that the images of \(gr_F(\tilde{d}_3)\) are equal to the images of \(\delta_3\) in the basis \((x^iy^jz^k)\). Let us compute the images of \(\delta_3\), \(1 \leq \ell \leq 3\). The computations are now considered in \(T = \mathbb{C}[x, y, z]\). According to (5.3), we obtain easily:

\[
\delta_1(x^iy^jz^k dx) = \nabla \phi \cdot \begin{pmatrix} x^iy^jz^{k-1} \\ -x^iy^jz^k \end{pmatrix} = jx^{i+2}y^{j-1}z^k,
\]

\[
\delta_1(x^iy^jz^k dy) = \nabla \phi \cdot \begin{pmatrix} -kx^iy^jz^{k-1} \\ 0 \end{pmatrix} = (2k - i)x^{i+1}y^jz^k,
\]

\[
\delta_1(x^iy^jz^k dz) = \nabla \phi \cdot \begin{pmatrix} jx^iy^{j-1}z^k \\ -ix^iy^jz^k \end{pmatrix} = -2jx^{i+1}y^{j-1}z^{k+1}.
\]

Comparing these formulas to (5.8), (5.9) and (5.10), we see that

\[
gr_F(\tilde{d}_1(x^iy^jz^k \otimes x)) = \delta_1(x^iy^jz^k dx),
\]

\[
gr_F(\tilde{d}_1(x^iy^jz^k \otimes y)) = \delta_1(x^iy^jz^k dy),
\]

\[
gr_F(\tilde{d}_1(x^iy^jz^k \otimes z)) = \delta_1(x^iy^jz^k dz).
\]

Also according to (5.3), we compute (with the identifications \(\Omega^1(T) \simeq T^3\), \(\Omega^2(T) \simeq T^3\) and \(\Omega^3(T) \simeq T\), explained above)

\[
\delta_2(x^iy^jz^k dy \wedge dz) = -\nabla \left( \begin{pmatrix} x^iy^jz^k \\ 0 \\ 0 \end{pmatrix} \cdot \nabla \phi \right) + \text{Div} \left( \begin{pmatrix} x^iy^jz^k \\ 0 \\ 0 \end{pmatrix} \right) \nabla \phi
\]

\[
= \begin{pmatrix} 2x^iy^jz^{k+1} \\ 2jx^{i+1}y^jz^{k+1} \\ (2k - i + 2)x^{i+1}y^jz^k \end{pmatrix},
\]

which can also be written as:

\[
\delta_2(x^iy^jz^k dy \wedge dz) = (2x^iy^jz^{k+1})dx + (2jx^{i+1}y^jz^{k+1})dy + ((2k - i + 2)x^{i+1}y^jz^k)dz.
\]
Similarly, we have
\[
\delta_2(x^i y^j z^k dz \wedge dx) = -\nabla \left( \begin{pmatrix} 0 & x^i y^j z^k \\ x^i y^j z^k & 0 \end{pmatrix} \cdot \nabla \phi \right) + \text{Div} \left( \begin{pmatrix} 0 & x^i y^j z^k \\ x^i y^j z^k & 0 \end{pmatrix} \right) \nabla \phi
\]
\[
= \begin{pmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial y^j} \\
\dfrac{\partial}{\partial z^k} & 0 \end{pmatrix}\end{pmatrix}
\]
\[
= (-2j x^{i+1} y^j z^{k+1}) \, dx + (-j x^{i+2} y^j z^k) \, dz,
\]
while
\[
\delta_2(x^i y^j z^k dx \wedge dy) = \begin{pmatrix} (i + 2 - 2k)x^{i+1} y^j z^k \\
(j x^{i+2} y^j z^k) \\
0 \end{pmatrix}
\]
\[
= ((i + 2 - 2k)x^{i+1} y^j z^k) \, dx + (j x^{i+2} y^j z^k) \, dy.
\]
The above formulas, together with the formulas (5.11), (5.12) and (5.13) and under the identifications explained in (5.2), show that:
\[
gr_F(\delta_2(x^i y^j z^k \otimes r_1)) \simeq \delta_2(x^i y^j z^k dy \wedge dz),
\]
\[
gr_F(\delta_2(x^i y^j z^k \otimes r_2)) \simeq \delta_2(x^i y^j z^k dz \wedge dx),
\]
\[
gr_F(\delta_2(x^i y^j z^k \otimes r_3)) \simeq \delta_2(x^i y^j z^k dx \wedge dy).
\]
Finally,
\[
\delta_3(x^i y^j z^k dx \wedge dy \wedge dz) = -\nabla(x^i y^j z^k) \times \nabla \phi = \begin{pmatrix}
\dfrac{\partial}{\partial z^k} & \dfrac{\partial}{\partial y^j} \\
\dfrac{\partial}{\partial x^i} & 0 \end{pmatrix}
\]
\[
= (j x^{i+2} y^j z^k) dy \wedge dz + ((2k - i)x^{i+1} y^j z^k) dz \wedge dx
\]
\[
+ (-2j x^{i+1} y^j z^{k+1}) dx \wedge dy,
\]

hence
\[
gr_F(\delta_3(x^i y^j z^k \otimes c(w)) \simeq \delta_3(x^i y^j z^k dx \wedge dy \wedge dz).
\]

In order to obtain the Poisson homology of the Poisson algebra \( T \), we need to compute the homology of a certain complex, called the Koszul complex associated to the polynomial \( \phi \). By definition, this complex is given by
\[
0 \to \Omega^0(T) \to \Omega^1(T) \to \Omega^2(T) \to \Omega^3(T) \to 0,
\]
where the differential is the map \( \wedge d\phi : \Omega^k(T) \to \Omega^{k+1}(T) \). This complex is identified to the following
\[
0 \to T \to T^3 \to T^3 \to T \to 0
\]

\[
F \to F \nabla \phi \quad \hat{F} \to \hat{F} \cdot \nabla \phi \quad (5.15)
\]

For \( 0 \leq p \leq 3 \), let us denote by \( H^\phi_p(T) \) the \( p \)-th homology space of this complex.
Remark 5.2 Let us recall from [22] that if $\varphi \in T$ is a weight-homogeneous polynomial with an isolated singularity at the origin, then the homology of the Koszul complex associated to $\varphi$ is given by: $H_p^\varphi(T) = \{0\}$, for $p = 0, 1, 2$, while $H_3^\varphi(T) = \{0\}^T$ is the so-called Milnor algebra associated to $\varphi$ and is, in this case, a finite dimensional vector space. In the following lemma, the homology of the Koszul complex associated to $\phi = -x^2z$ (admitting a non-isolated singularity at the origin) does not satisfy the same properties.

Lemma 5.3 Let $\phi = -x^2z$. The homology of the Koszul complex (5.13) associated to $\phi$ is given by:

\[
H_0^\phi(T) = \{F \in T \mid F\nabla_\phi = 0\} = \{0\}; \\
H_1^\phi(T) = \{\bar{F} \in T^3 \mid \bar{F} \times \nabla_\phi = 0\} \supseteq \mathbb{C}[y,z] \left(\begin{array}{c} 2z \\
-2x \end{array}\right); \\
H_2^\phi(T) = \{\bar{F} \in T^3 \mid \bar{F} \cdot \nabla_\phi = 0\} \supseteq \mathbb{C}[y,z] \left(\begin{array}{c} x \\
0 \\ -2z \end{array}\right) \oplus (x\mathbb{C}[y] \oplus \mathbb{C}[y,z]) \left(\begin{array}{c} 0 \\
0 \\
0 \end{array}\right); \\
H_3^\phi(T) = T \supseteq x\mathbb{C}[y] \oplus \mathbb{C}[y,z].
\]

Proof. The fact that $H_0^\phi(T) \cong \{0\}$ is clear. In order to compute the space $H_1^\phi(T)$, let us consider an element $\bar{F} = (F_1, F_2, F_3) \in T^3$ satisfying $\bar{F} \times \nabla_\phi = 0$. This is equivalent to $F_2 = 0$ and $xF_1 = 2xF_3$. So there exists $G \in T$ such that $F_1 = -2zG$ and $F_3 = -xG$. Let us now write $G = xH + K$, with $H \in T$ and $K \in \mathbb{C}[y,z]$, so that $\bar{F} = H\nabla_\phi + \begin{pmatrix} -2z \\ 0 \\ -x \end{pmatrix}$.

Moreover, if $K \in \mathbb{C}[y,z]$ satisfies $K \left(\begin{array}{c} 2z \\
-2x \end{array}\right) = H\nabla_\phi = -H \left(\begin{array}{c} 2xz \\
x^2 \end{array}\right)$, with $H \in T$, then necessarily $K \in \mathbb{C}[y,z] \cap xT = \{0\}$. This permits us to conclude that $H_1^\phi(T) \cong \mathbb{C}[y,z] \left(\begin{array}{c} 2z \\
0 \\
0 \end{array}\right)$.

Let us now compute the space $H_2^\phi(T)$. Every polynomial $G \in T$ can be written as $G = xG_1 + H$, where $G_1 \in T$ and $H \in \mathbb{C}[y,z]$. Then, $G_1$ can also be written as $G_1 = xA + zB + C$, with $A, B \in T$ while $C \in \mathbb{C}[y]$, so that

\[
G = x^2A + xzB + xC + H \\
\in \bar{F} \cdot \nabla_\phi + x\mathbb{C}[y] + \mathbb{C}[y,z],
\]

where $\bar{F} = -(1/2B, 0, A) \in T^3$. This shows that $H_2^\phi(T) \cong x\mathbb{C}[y] + \mathbb{C}[y,z]$. It is also clear that this sum is a direct one, because an equality of the form $xC + H = \bar{G} \cdot \nabla_\phi$, with $C \in \mathbb{C}[y]$, $H \in \mathbb{C}[y,z]$ and $\bar{G} \in T^3$ implies that $x$ divides $H$, so $H = 0$ and then it remains that $C \in (xT + zT) \cap \mathbb{C}[y]$ which means that $C = 0$. We finally have obtained $H_2^\phi(T) \cong x\mathbb{C}[y] \oplus \mathbb{C}[y,z]$.

We determine the space $H_3^\phi(T)$. To do this, let us consider an element $\bar{F} = (F_1, F_2, F_3) \in T^3$ such that $0 = \bar{F} \cdot \nabla_\phi = -x(2zF_1 + xF_3)$. Then, there exists $G \in T$ satisfying $F_1 = xG$ and $F_3 = -2zG$. We write $G = xH + K$, with $H \in T$ and $K \in \mathbb{C}[y,z]$. Moreover, according to the previous computation of $H_3^\phi(T)$, there exist $L = (L_1, L_2, L_3) \in T^3$, $A \in \mathbb{C}[y]$ and $B \in \mathbb{C}[y,z]$ such that we can write $F_2 = \bar{L} \cdot \nabla_\phi + xA + B = -(2xzL_1 + x^2L_3) + xA + B$. 

25
This gives
\[
\vec{F} = \left( \frac{x^2}{0} \right) H + \left( \frac{x}{0} \right) K + \left( -(2xzL_0^0 + x^2L_3) \right) + \left( xA+B \right) \\
= \left( \frac{-L_3}{L_1} \right) \times \nabla \phi + \left( \frac{0}{0} \right) K + \left( xA+B \right) \\
\in \left( \frac{-3}{H} \right) \times \nabla \phi + C[y, z] \left( \frac{x}{0} \right) + \left( xC[y] + C[y, z] \right) \left( \frac{1}{0} \right).
\]

We have shown that \( HP^2(T) \subseteq C[y, z] \left( \frac{x}{0} \right) + (xC[y] + C[y, z]) \left( \frac{0}{0} \right) \) and the other inclusion is clear. It remains to show that this sum is direct. Let \( A \in C[y] \) and \( B, K \in C[y, z] \), and assume that there exists \( \vec{H} = (H_1, H_2, H_3) \in T^3 \) such that

\[
\left( \frac{0}{0} \right) K + (xA + B) \left( \frac{0}{1} \right) = \vec{H} \times \nabla \phi = \left( -2xzH_1 + x^2H_1 \right).
\]

In particular, \( xA + B = -2xzH_3 + x^2H_1 \), so that \( B \in xT \cap C[y, z] = \{0\} \) and then \( A \in (zT + xT) \cap C[y] = \{0\} \). It remains \( xK = -x^2H_2 \), so that \( x \) divides \( K \in C[y, z] \) and \( K = 0 \). This permits us to conclude that \( HP^2(T) \simeq C[y, z] \left( \frac{x}{0} \right) \oplus (xC[y] \oplus C[y, z]) \left( \frac{0}{1} \right) \).

**Remark 5.3** In [22], for every weight-homogeneous polynomial \( \varphi \in C[x, y, z] \), admitting an isolated singularity, it is shown, using the fact that the Koszul complex associated to \( \varphi \) is exact, that \( \{ \vec{F} \in T^3 \mid \nabla \varphi \cdot (\nabla \times \vec{F}) = 0 \} = \{ \nabla G + H \nabla \varphi \mid G, H \in T \} \). In the following lemma, this is not true anymore if \( \varphi \) is replaced by \( \phi = -x^2z \).

In the following, we will say that an element \( \vec{F} = (F_1, F_2, F_3) \in T^3 \) is homogeneous of degree \( d \in \mathbb{Z} \), if \( F_1, F_2, \) and \( F_3 \) are three homogeneous polynomials of the same degree \( d \). (Notice that we use the convention that a polynomial of degree \( d < 0 \) is zero).

**Lemma 5.4** Let \( \phi = -x^2z \in T = C[x, y, z] \). Let \( \vec{F} \in T^3 \) be a homogeneous element of degree \( n \in \mathbb{N} \). If \( \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \), then there exist homogeneous polynomials \( G, H \in T \) (of respective degrees \( n + 1 \) and \( n - 2 \)), \( C, P \in C[y, z] \) (of respective degrees \( n - 1 \) and \( n - 2 \)) satisfying

\[
\frac{\partial C}{\partial y} = P + 2x \frac{\partial P}{\partial z} \tag{5.17}
\]

and \( \alpha \in C \), such that

1. If \( n = 0 \), then \( \vec{F} = \nabla G \);
2. If \( n \notin 2 + 3\mathbb{N} \), then

\[
\vec{F} = \nabla G + H \nabla \phi + \left( \frac{z}{0} \right) C + \left( \frac{2yz}{-3xz} \right) P; \tag{5.18}
\]

3. If \( n = 2 + 3k \), with \( k \in \mathbb{N} \), then

\[
\vec{F} = \nabla G + H \nabla \phi + \left( \frac{z}{0} \right) C + \left( \frac{2yz}{-3xz} \right) P + \alpha(x^2z)^k \left( \frac{xz}{0} \right). \tag{5.19}
\]
Proof. We will prove this result by recursion on \( n \in \mathbb{N} \). As we have to distinguish whether \( n \in 2 + 3\mathbb{N} \) or not, we first have to show the desired result, for \( n = 0, 1, 2 \).

\( \textcircled{1} \) If \( n = 0 \), then there exist \( a, b, c \in \mathbb{C} \), such that \( \vec{F} = (a, b, c) \), and it is clear that \( \vec{F} = \nabla G \), with \( G = ax + by + cz \in T \).

\( \textcircled{2} \) If \( n = 1 \), there exist \( a_i, b_i, c_i \in \mathbb{C} \), for \( i = 1, 2, 3 \), such that \( \vec{F} = \left( \frac{a_1 x + b_1 y + c_1 z}{a_2 x + b_2 y + c_2 z}, \frac{a_3 x + b_3 y + c_3 z}{a_3 x + b_3 y + c_3 z} \right) \) and the condition \( \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \) is equivalent to: \( b_3 = c_2 \) and \( a_2 = b_1 \). Using this, it is easy to verify that we can write
\[
\vec{F} = \nabla G + \left( \binom{zC}{\frac{z}{-x^2}} \right),
\]
with \( G = \frac{1}{2} \left( a_1 x^2 + 2b_1 xy + (c_1 + a_3)xz + b_2y^2 + 2c_2yz + c_3z^2 \right) \in T \) and \( C = \frac{1}{2}(c_1 - a_3) \). Note also that \( \frac{\partial C}{\partial y} = 0 \), so that the equation (5.17) is satisfied (here \( P = 0 \)).

\( \textcircled{3} \) If \( n = 2 \), there exist \( a_{ij}, b_{ij}, c_{ij} \in \mathbb{C} \), for \( 1 \leq i, j \leq 3 \), such that
\[
\vec{F} = \left( \frac{a_{11} x^2 + a_{12} xy + a_{13} xz + a_{22} y^2 + a_{33} z^2}{a_{11} x^2 + a_{22} y^2 + a_{33} z^2}, \frac{b_{11} x^2 + b_{12} xy + b_{13} xz + b_{22} y^2 + b_{33} z^2}{b_{11} x^2 + b_{22} y^2 + b_{33} z^2} \right).
\]
The condition \( \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \) is equivalent to the following identities: \( 2c_{12} - b_{13} - a_{23} = 0, 2c_{22} - b_{23} = 0, c_{23} - 2b_{33} = 0, 2b_{11} - a_{12} = 0 \) and \( b_{12} - 2a_{22} = 0 \). Using this, it is straightforward to verify that
\[
\vec{F} = \nabla G + \left( \binom{2 y z P}{-3 z P} \right) + \alpha \left( \binom{x}{0} \right),
\]
with
\[
G = \frac{1}{3} a_{11} x^3 + \frac{1}{2} a_{12} x^2 y + \frac{1}{3} (a_{13} + c_{11}) x^2 z + \frac{1}{2} (a_{22} + b_{13}) x y z + a_{22} y^2 + \frac{1}{3} (a_{33} + c_{11}) x z^2 + \frac{1}{2} b_{22} y^2 z + b_{33} y z^2 + \frac{1}{3} b_{22} y^3 + \frac{1}{3} c_{33} z^3,
\]
\( \alpha = \frac{1}{3} (a_{13} - 2c_{11}), C = \frac{1}{6} (2c_{23} - b_{13}) y + \frac{1}{6} (2c_{33} - c_{13}) z \) and \( P = \frac{1}{6} (a_{23} - b_{13}) \). Notice that the equation (5.17) is clearly satisfied in this case.

\( \textcircled{4} \) Let now \( m \in \mathbb{N} \), such that \( m \geq 3 \), and suppose the lemma is proved for all \( n \in \mathbb{N} \) such that \( n < m \). Let now \( \vec{F} \in T^3 \) be a homogeneous element of degree \( m \), satisfying \( \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \).

According to Lemma 5.3, this hypothesis implies that there exist homogeneous elements: \( \vec{K} \in T^3, P, C \in \mathbb{C}[y, z] \) and \( E \in \mathbb{C}[y] \) such that
\[
\nabla \times \vec{F} = \vec{K} \times \nabla \phi + \left( \binom{0}{0} \right) P + \left( \binom{0}{1} \right) (xE + C),
\]
with the degrees of \( \vec{K}, P, E \) and \( C \) respectively equal to \( m - 3, m - 2, m - 2 \) and \( m - 1 \). Computing the divergence of this, we obtain
\[
0 = \text{Div}(\nabla \times \vec{F}) = (\nabla \times \vec{K}) \cdot \nabla \phi - 2z \frac{\partial P}{\partial z} + xE' + \frac{\partial C}{\partial y},
\]

27
where \( E' = \frac{dE}{dy} \) and where we have used that \( \text{Div}(\vec{K} \times \nabla \phi) = (\nabla \times \vec{K}) \cdot \nabla \phi \). This implies that \( x \) divides the polynomial \(-P - 2z \frac{\partial P}{\partial z} + \frac{\partial C}{\partial y} \) which lies in \( \mathbb{C}[y,z] \), so that it is zero:

\[
\frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z},
\]

It remains that \((\nabla \times \vec{K}) \cdot \nabla \phi + xE' = 0\), which gives \( E' \in (xT + zT) \cap \mathbb{C}[y] = \{0\} \), and \( E' = 0 \). This fact, together with the hypothesis that \( E \) is supposed to be homogeneous of degree \( m - 2 \), and \( m \geq 3 \), imply \( E = 0 \). We have obtained:

\[
\nabla \times \vec{F} = \vec{K} \times \nabla \phi + \left( \begin{array}{c} xP \\ -2zP \end{array} \right).
\]

Now, \( \vec{K} \) is of degree \( m - 3 < m \) and satisfies \((\nabla \times \vec{K}) \cdot \nabla \phi = 0\), so that we can apply the recursion hypothesis to obtain the existence of homogeneous elements \( G,H \in T, D,Q \in \mathbb{C}[y,z], \alpha \in \mathbb{C} \), and \( k \in \mathbb{N} \), such that

\[
\frac{\partial D}{\partial y} = Q + 2z \frac{\partial Q}{\partial z}, \quad \text{(5.20)}
\]

and

\[
\vec{K} = \nabla G + H \nabla \phi + \left( \begin{array}{c} zD+2yzQ \\ -3xzQ \\ -xD+xyQ \end{array} \right) + \alpha(x^2z)^k \left( \begin{array}{c} xz \\ 0 \\ 0 \end{array} \right).
\]

The polynomials \( G,H,D \) and \( Q \) are respectively of degree \( m - 2 \), \( m - 5 \), \( m - 4 \) and \( m - 5 \). Notice that, by hypothesis, \( \alpha \) is supposed to be zero, except if \( m - 3 = 2 + 3k \). We now compute

\[
\vec{K} \times \nabla \phi = \nabla G \times \nabla \phi + \left( \begin{array}{c} 3x^2zQ \\ 3z^2D \\ -6xz^2Q \end{array} \right) + \alpha(x^2z)^k \left( \begin{array}{c} 0 \\ 3z^2 \\ 0 \end{array} \right).
\]

Denoting by \( \vec{e} \) the so-called Euler vector \( \vec{e} := (x,y,z) \in T^3 \), we use the following general result (Proposition 3.5 in [22]), which is due to the exactness of the De Rham complex of \( \mathbb{C}[x,y,z] \): if \( \vec{A} = (A_1, A_2, A_3) \in T^3 \) is a homogeneous element of degree \( \ell \in \mathbb{N} \), such that \( \text{Div}(\vec{A}) = 0 \), then the Euler formula \( \nabla A_i \cdot \vec{e} = d A_i \) (1 \( \leq i \leq 3 \)), implies that \( (d + 2)\vec{A} = \nabla \times (\vec{A} \times \vec{e}) \).

As

\[
\text{Div} \left( \begin{array}{c} 3x^2zQ \\ 3z^2D \\ -6xz^2Q \end{array} \right) + \alpha(x^2z)^k \left( \begin{array}{c} 0 \\ 3z^2 \\ 0 \end{array} \right) = \text{Div} \left( \vec{K} \times \nabla \phi - \nabla G \times \nabla \phi \right) = \left( \nabla \times (\vec{K} - \nabla G) \right) \cdot \nabla \phi = 0,
\]

we can write \( \vec{K} \times \nabla \phi = \nabla \times \vec{L} \), where \( \vec{L} \) is given by:

\[
\vec{L} = G \nabla \phi + \frac{1}{m + 1} \left( \begin{array}{c} 3x^2zQ \\ 3z^2D \\ -6xz^2Q \end{array} \right) \times \vec{e} + \frac{1}{m + 1} \alpha(x^2z)^k \left( \begin{array}{c} 0 \\ 3z^2 \\ 0 \end{array} \right) \times \vec{e}
\]

\[
= G \nabla \phi + \frac{1}{m + 1} \left( \begin{array}{c} 3x^2z^2D+6xy^2zQ \\ -9x^2z^2Q \\ 3x^2yQ-3x^2zD \end{array} \right) + \frac{3}{m + 1} \alpha(x^2z)^k+1 \left( \begin{array}{c} xz \\ 0 \\ -xz \end{array} \right).
\]

Moreover, we have also

\[
\left( \begin{array}{c} xP \\ -2zP \end{array} \right) = \frac{1}{m + 1} \nabla \times \left( \left( \begin{array}{c} xP \\ -2zP \end{array} \right) \times \vec{e} \right) = \frac{1}{m + 1} \nabla \times \left( \left( \begin{array}{c} zC+2yzP \\ -3xzP \\ -xC+xyP \end{array} \right) \right).
\]
We finally obtain
\[ \nabla \times \vec{F} = \nabla \times \left( \vec{L} + \frac{1}{m+1} \left( \frac{zC+2yzP}{-3xzP-xC+xyP} \right) \right). \]

This permits us to apply another general result (Proposition 3.5 in \cite{22}): if \( \vec{A} = (A_1, A_2, A_3) \in T^3 \) is a homogeneous element of degree \( d \in \mathbb{N} \), such that \( \nabla \times \vec{A} = 0 \), then the Euler formula \( \nabla A_i \cdot \vec{e}_i = d A_i \) (1 \( \leq i \leq 3 \)), implies that \( (d+1)\vec{A} = \nabla (\vec{A} \cdot \vec{e}) \).

This implies that there exists a homogeneous element \( S \in T \) (of degree \( m+1 \), such that
\[ \vec{F} = \nabla S + \vec{L} + \frac{1}{m+1} \left( \frac{zC+2yzP}{-3xzP-xC+xyP} \right) \]
\[ = \nabla S + \nabla \phi + \frac{1}{(m+1)} \left( \frac{3x^2z^2D+6x^2yz^2Q}{3x^2yzQ-3x^2zD} \right) \]
\[ + \frac{3}{m+1} \alpha (x^2z)^{k+1} \left( \frac{xz}{-x^2z} \right) + \frac{1}{m+1} \left( \frac{zC+2yzP}{-3xzP-xC+xyP} \right). \]

Let now \( V := \frac{3}{(2m-7)} xz(D + 2yQ) \) (as \( 2m - 7 \neq 0 \)). The polynomial \( V \) is homogeneous of degree \( m - 2 \). Now, using (5.20) and the Euler formulas \( y \frac{\partial D}{\partial y} + z \frac{\partial D}{\partial z} = (m - 4)D \) and \( y \frac{\partial Q}{\partial y} + z \frac{\partial Q}{\partial z} = (m - 5)Q \), it is straightforward to verify that we have:
\[ \nabla \left( -\frac{3}{m+1} x^2zV \right) - V \nabla \phi = \frac{1}{(m+1)} \left( \frac{3x^2z^2D+6x^2yz^2Q}{3x^2yzQ-3x^2zD} \right). \]

Denoting by \( \tilde{\alpha} := \frac{3}{m+1} \alpha \), \( \tilde{G} := G - V \) and by \( \tilde{S} := S - \frac{3}{(m+1)} x^2zV \), \( \tilde{C} = \frac{1}{m+1} C \) and \( \tilde{P} = \frac{1}{m+1} P \), we can write:
\[ \vec{F} = \nabla \tilde{S} + \nabla \phi + \tilde{\alpha} (x^2z)^{k+1} \left( \frac{\alpha}{-x^2z} \right) + \left( \frac{\tilde{C}z+2yz\tilde{P}}{-3xz\tilde{P}} \right). \]

Moreover, \( \tilde{S} \) and \( \tilde{G} \) are homogeneous polynomials and we have already seen that \( C \) and \( P \) satisfy the identity (5.17) so that this identity is also satisfied by \( \tilde{C} \) and \( \tilde{P} \).

In the sequel, we will several times need the following technical result.

**Lemma 5.5** If \( r \in \mathbb{N} \), \( c \in \mathbb{C} \) and if \( K \in \mathbb{C}[x,y,z] \) is a homogeneous polynomial such that \( cx^r \phi \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) = \nabla K \times \nabla \phi \), then \( K \in \mathbb{C}[x,y,z] \). Moreover, \( K \) is a constant, or \( \phi \) divides \( K \).

**Proof.** Denote by \( m \) the degree of the polynomial \( K \). Suppose that \( K \notin \mathbb{C} \), so that \( m \geq 1 \). The hypothesis on \( K \) is equivalent to \( \frac{\partial K}{\partial y} = 0 \) and \( c \phi = -2z \frac{\partial K}{\partial z} + x \frac{\partial K}{\partial x} \), hence \( K \in \mathbb{C}[x,z] \). If \( r = 0 \), then for degree reasons, \( c = 0 \) and \( 0 = -2z \frac{\partial K}{\partial z} + x \frac{\partial K}{\partial x} \). If \( r \geq 1 \), then the degree of \( K \) is \( m = 3r \). The hypothesis implies that \( x \) divides the polynomial \( z \frac{\partial K}{\partial x} \) and using the Euler formula \( mK = x \frac{\partial K}{\partial x} + z \frac{\partial K}{\partial z} \), we conclude that \( x \) divides \( K \). Similarly, we obtain that \( z \) divides \( K \), so that we can write \( K = xzK_1 \), with \( K_1 \) a homogeneous polynomial in \( \mathbb{C}[x,z] \), (of degree \( m - 2 \)). We write \( K_1 = xK_2 + c' z^{m-2} \), where \( c' \in \mathbb{C} \) and \( K_2 \in \mathbb{C}[x,z] \) is a homogeneous polynomial of degree \( m - 3 \).

Hence
\[ -c \phi^{-1} x = -c' (2m-3)z^{m-2} - 2xz \frac{\partial K_2}{\partial z} + xz^2 \frac{\partial K_2}{\partial x}, \]
which implies that \( c' = 0 \), so that \( K = xzK_2 \) and \( \phi \) divides \( K \).
Corollary 5.6 Let $\phi = -x^2z \in T = \mathbb{C}[x, y, z]$. We have
\[
\frac{\{ \vec{F} \in T^3 \mid \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \}}{\{ \nabla G + H \nabla \phi \mid G, H \in T \}} \simeq \mathbb{C}[\phi] \left( \frac{xz}{x^2} \right) \oplus \mathbb{C}[z] \left( \frac{z}{-xz} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \left( \frac{(2n+3)yz}{-3kxz} \right) \left( \frac{(-2n+3(k-1))xy}{(-2n+3(k-1))xy} \right) y^{k-1}z^{n+1-k}.
\]

Proof. First of all, using (5.17), it is straightforward to verify that an element $\vec{F} \in T^3$ of the form given in (5.18) or in (5.19) satisfies the equation $\nabla \phi \cdot (\nabla \times \vec{F}) = 0$. This, together with Lemma 5.4 gives:
\[
\frac{\{ \vec{F} \in T^3 \mid \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \}}{\{ \nabla G + H \nabla \phi \mid G, H \in T \}} = \left\{ \left( \begin{array}{c} zC + 2yzP \\ -3xzP \\ -xC + xyP \end{array} \right) + \alpha (x^2z)^k \left( \begin{array}{c} xz \\ 0 \\ -x^2 \end{array} \right) \mid \alpha \in \mathbb{C}, k \in \mathbb{N}, C, P \in \mathbb{C}[y, z] \text{ satisfying } \frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z} \right\}
\]

Fix now $n \in \mathbb{N}$. Let $C, P \in \mathbb{C}[y, z]$ be homogeneous polynomials satisfying $\frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z}$. We suppose that $P$ is of degree $n$, so that $C$ is zero or of degree equal to $n+1$. We write $C$ and $P$ as
\[
P = \sum_{k=0}^{n} a_k y^k z^{n-k}, \quad C = \sum_{k=0}^{n+1} b_k y^k z^{n+1-k},
\]
where $a_k, b_k \in \mathbb{C}$. Then, compute
\[
0 = \frac{\partial C}{\partial y} - P - 2z \frac{\partial P}{\partial z} = \sum_{k=0}^{n} ((k+1)b_{k+1} - (2(n-k)+1)a_k) y^k z^{n-k},
\]
so that, necessarily, for all $k = 1, \ldots, n+1$, we have $b_k = \frac{(2(n-k)+3)}{k} a_{k-1}$. We then can write
\[
\left( \begin{array}{c} zC + 2yzP \\ -3xzP \\ -xC + xyP \end{array} \right) = b_0 \left( \begin{array}{c} z^2 \\ 0 \\ -xz \end{array} \right) z^n + \sum_{k=1}^{n+1} \frac{1}{k} a_{k-1} \left( \begin{array}{c} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{array} \right) y^{k-1}z^{n-k+1}
\]
\[
\in \mathbb{C}[z] \left( \begin{array}{c} z^2 \\ 0 \\ -xz \end{array} \right) + \sum_{k=1}^{n+1} \mathbb{C} \left( \begin{array}{c} (2n+3)yz \\ -3kxz \\ (-2n+3(k-1))xy \end{array} \right) y^{k-1}z^{n-k+1-k}
\]

We then have shown that
\[
\frac{\{ \vec{F} \in T^3 \mid \nabla \phi \cdot (\nabla \times \vec{F}) = 0 \}}{\{ \nabla G + H \nabla \phi \mid G, H \in T \}} \simeq \mathbb{C}[\phi] \left( \frac{xz}{x^2} \right) \oplus \mathbb{C}[z] \left( \frac{z}{-xz} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \left( \frac{(2n+3)yz}{-3kxz} \right) \left( \frac{(-2n+3(k-1))xy}{(-2n+3(k-1))xy} \right) y^{k-1}z^{n+1-k},
\]
and it remains to show that this sum is a direct one. To do this, it suffices to show that each homogeneous component of this sum is a direct sum. Notice that an element of the space of the right hand side of the previous equation is at least of degree 2, so that, we fix
Proposition 5.7. Set \( G, H \in T \), of respective degrees equal to \( n + 3 \) and \( n \), satisfying:

\[
\alpha \phi^r \left( \frac{xz}{-x^2} \right) + cz^n \left( \frac{xz}{-x^2} \right) + \sum_{k=1}^{n+1} a_k \left( \frac{(2n+3)ry^{n+2-k} - 3kx^{k-1}y^{n+2-k}}{(-2n+3(k-1))x^{k}z^{n+1-k}} \right) = \nabla G + H \nabla \phi. \tag{5.21}
\]

Notice that \( \alpha = 0 \) if \( n \neq 3r \). Applying the curl operator to this identity permits us to obtain

\[
- \sum_{k=1}^{n+1} k(2n+6) a_k y^{k-1}z^{n+2-k} = 2xz \frac{\partial H}{\partial y},
\]

which implies that \( a_k = 0 \), for all \( 1 \leq k \leq n + 1 \). This, together with \( \check{5.21} \) imply that \( \frac{\partial G}{\partial y} = 0 \), so that \( G \in \mathbb{C}[x, z] \), while, this together with the result obtained by applying the curl operator to the previous identity \( \check{5.21} \), give:

\[
3(r+1)\alpha x \phi^r + (n+3)cz^{n+1} = -2xz \frac{\partial H}{\partial z} + x^2 \frac{\partial H}{\partial x}.
\]

This shows that \( x \) divides \( (n+3)cz^{n+1} \), which means that \( c = 0 \) and the equation \( \check{5.21} \) becomes:

\[
\alpha \phi^r \left( \frac{z}{-x^2} \right) = \nabla G + H \nabla \phi. \tag{5.22}
\]

It remains to show that \( \alpha = 0 \). Of course, it is true if \( n \neq 3r \). Suppose now that \( n = 3r \). Applying to the equation \( \check{5.22} \) the inner product with \( \check{e} \) and using the Euler formula, gives \( \check{0} = (n+3)G + 3H \phi \), so that \( \phi \) divides the polynomial \( G \). Now, either \( G \) is zero or we can write \( G = \phi^r G_1 \) and \( H = - (r+1) \phi^{r-1} G_1 \), where \( r \in \mathbb{N}^* \) and \( G_1 \in \mathbb{C}[x, z] \) is a homogeneous polynomial of degree \( n + 3 - 3r = 3(r - r) + 3 \), not divisible by \( \phi \). In the case \( G = \phi^r G_1 \), we can write \( \check{5.22} \) as

\[
\alpha \phi^r \left( \frac{z}{-x^2} \right) = \phi^r \nabla G_1 + (r - r - 1) \phi^{r-1} G_1 \nabla \phi, \tag{5.23}
\]

and also

\[
\alpha \phi^r \left( \frac{0}{3xz} \right) = \phi^r \nabla G_1 \times \nabla \phi = \phi^r \left( -2xz \frac{\partial G_1}{\partial y} + x^2 \frac{\partial G_1}{\partial x} \right). \tag{5.24}
\]

As the degree of \( G_1 \) is equal to \( 3(r - r + 1) \), if \( r \geq r + 1 \), then \( G_1 \) is a constant and equation \( \check{5.24} \) implies \( \alpha = 0 \). Suppose that \( r \leq r \), then \( \alpha \phi^{r-1} \left( \frac{0}{3xz} \right) = -3\alpha x \phi^{r-1} \left( \frac{0}{1} \right) = \nabla G_1 \times \nabla \phi \). Then, Lemma 5.5 shows that \( G_1 \) is a constant and \( \check{5.24} \) implies that \( \alpha = 0 \). \( \blacksquare \)

We now determine the Poisson homology of the Poisson algebra \( (T, \{\cdot, \cdot\}) \).

**Proposition 5.7** Set \( T = \mathbb{C}[x, y, z] \) and consider \( \phi = -x^2 \in T \). The algebra \( T \) becomes a Poisson algebra when equipped with the Poisson bracket \( \{\cdot, \cdot\} \), defined by:

\[
\{y, z\} = \frac{\partial \phi}{\partial x} = -2xz, \quad \{z, x\} = \frac{\partial \phi}{\partial y} = 0, \quad \{x, y\} = \frac{\partial \phi}{\partial z} = -x^2.
\]

Using the identifications \( \Omega^1(T) \simeq T^3 \), \( \Omega^2(T) \simeq T^3 \), \( \Omega^3(T) \simeq T^3 \) explained above, the
Poisson homology spaces of the Poisson algebra \( T \) are given by:

\[
HP_0(T) \simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z];
\]

\[
HP_1(T) \simeq \mathbb{C}[\phi] \left( \frac{x}{0} - \frac{z}{-xz} \right) \oplus \mathbb{C}[z] \left( \frac{z^2}{0} - \frac{x}{-xz} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
+ \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
= \mathbb{C}[\phi] \left( \frac{y^n}{n \in \mathbb{N}} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
+ \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
\subset \mathbb{C}[\phi] \left( \frac{y^n}{n \in \mathbb{N}} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
\subset \mathbb{C}[\phi].
\]

\[
HP_2(T) \simeq \mathbb{C}[\phi] \left( \frac{\tilde{y}}{\tilde{z}} \right) \oplus (x\mathbb{C}[\phi] \oplus 2\mathbb{C}[z]) \left( \frac{0}{1} \right)
\]

\[
+ \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
\subset \mathbb{C}[\phi] \left( \frac{\mathbb{C}}{\mathbb{C}} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C}
\]

\[
\subset \mathbb{C}[\phi].
\]

\[
HP_3(T) \simeq \mathbb{C}[\phi].
\]

\[
Proof.\text{ Remark that it is not possible to apply the results of Monnier for the computation of Poisson homology as we have seen for the third type (proof of Proposition 14). Actually, it is easy to check that the Poisson bracket derived from the potential \( x^2z \) is not diagonalizable.}
\[
\text{To determine the Poisson homology spaces, notice that the polynomial} \ \phi \ \text{is homogeneous of degree 3, so that, considering} \ T \ \text{graded by the total degree of the polynomials, the operator} \ \delta_p \ (1 \leq p \leq 3) \ \text{is homogeneous of degree 1. This permits one to determine the Poisson homology spaces, degree by degree.}
\]

The 0-th Poisson homology space \( HP_0(T) \).

According to (5.3),

\[
HP_0(T) \simeq \frac{T}{\delta_1(\tilde{G}) | \tilde{G} = (G_1, G_2, G_3) \in T^3},
\]

where \( \delta_1(\tilde{G}) = 2xz \left( \frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y} \right) + x^2 \left( \frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x} \right) \). We have already seen that every polynomial \( G \in T \) can be written as in (5.10), with \( A, B \in T \) while \( C \in \mathbb{C}[y] \) and \( H \in \mathbb{C}[y, z] \).

Now, there exist \( G_1 \in T \) and \( G_3 \in T \), such that \( A = \frac{\partial G_3}{\partial y} \) and \( B = -2\frac{\partial G_2}{\partial y} \), so that

\[
G = x^2A + xzB + xC + H = -2xz\frac{\partial G_3}{\partial y} + x^2\frac{\partial G_1}{\partial y} + xC + H = \delta_1(\tilde{G}) + xC + H,
\]

with \( \tilde{G} = (G_1, 0, G_3) \in T^3 \). This shows that \( HP_0(T) \simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z] \) and as for the determination of the space \( HP_1(T) \), we show that this sum is a direct one. We finally have obtained \( HP_0(T) \simeq x\mathbb{C}[y] \oplus \mathbb{C}[y, z] \).

The third Poisson homology space \( HP_3(T) \).

First, it is easy to see that \( \mathbb{C}[\phi] \simeq \mathbb{C}[\phi]dx \wedge dy \wedge dz \subseteq HP_3(T) \). Conversely, we consider a homogeneous element \( F \in HP_3(T) \), i.e., \( F \in T \simeq \Omega^3(T) \) satisfying \( \nabla F \times \nabla(x^2z) = 0 \).
We write $F = \phi^r F_1$, with $r \in \mathbb{N}$ and $F_1 \in T$, a homogeneous polynomial, not divisible by $\phi = -x^2 z$. Then

$$0 = \nabla F \times \nabla \phi = \nabla (\phi^r F_1) \times \phi = r \phi^{r-1} F_1 \nabla \phi \times \nabla \phi + \phi^r \nabla F_1 \times \nabla \phi = \phi^r \nabla F_1 \times \nabla \phi,$$

so that $\nabla F_1 \times \nabla \phi = 0$. Lemma 5.3 (with $c = 0$) then implies that $F_1 \in \mathbb{C}$. We finally have obtained that $HP_3(T) \simeq \mathbb{C}[x^2 z] = \mathbb{C}[\phi]$.

**The first Poisson homology space $HP_1(T)$.**

By (5.3),

$$HP_1(T) = \left\{ \tilde{F} \in T^3 \mid \nabla \phi \cdot (\nabla \times \tilde{F}) = 0 \right\} \delta \left\{ -\nabla \left( \tilde{G} \cdot \nabla \phi \right) + \text{Div}(\tilde{G}) \nabla \phi \mid \tilde{G} \in T^3 \right\}.$$

Let $\tilde{F} \in T^3$ be an element satisfying $\nabla \phi \cdot (\nabla \times \tilde{F}) = 0$. According to corollary 5.16 there exist $G, H \in T$ such that:

$$\tilde{F} \in \nabla G + H \nabla \phi + \mathbb{C}[\phi] \left( \begin{array}{c} \frac{x^2}{0} \\ \frac{0}{-x^2} \end{array} \right) + \mathbb{C}[z] \left( \begin{array}{c} \frac{2}{0} \\ \frac{0}{-2} \end{array} \right) + \sum_{n \in \mathbb{N} \ k = 1}^{n+1} \mathbb{C} \tilde{u}_{n,k},$$

where $\tilde{u}_{n,k} = \left( \begin{array}{c} (2n+3) yz \\ -3k xz \\ (-2n+3k-1) xz \end{array} \right) y^{k-1} z^{n+1-k}$, for $n \in \mathbb{N}$ and $1 \leq k \leq n + 1$.

Now, according to the determination of $HP_0(T)$, there exist $\bar{L}, \bar{K} \in T^3$, $A, \bar{A} \in \mathbb{C}[y]$ and $B, \bar{B} \in \mathbb{C}[y, z]$ such that:

$$G = \nabla \phi \cdot (\nabla \times \bar{L}) + xA + B, \quad H = \nabla \phi \cdot (\nabla \times \bar{K}) + x\bar{A} + \bar{B}.$$ 

As $\delta_2 (-\nabla \times \bar{L}) = \nabla \left( \nabla \phi \cdot (\nabla \times \bar{L}) \right)$, and $\delta_2 (\bar{K} \times \nabla \phi) = \left( \nabla \phi \cdot (\nabla \times \bar{K}) \right) \nabla \phi$, we obtain

$$\tilde{F} \in \delta_2 (-\nabla \times \bar{L} + \bar{K} \times \nabla \phi) + \nabla (xA + B) + (x\bar{A} + \bar{B}) \nabla \phi$$

$$+ \mathbb{C}[\phi] \left( \begin{array}{c} \frac{x^2}{0} \\ \frac{0}{-x^2} \end{array} \right) + \mathbb{C}[z] \left( \begin{array}{c} \frac{2}{0} \\ \frac{0}{-2} \end{array} \right) + \sum_{n \in \mathbb{N} \ k = 1}^{n+1} \mathbb{C} \tilde{u}_{n,k}.$$

Now, let us consider $A_1 \in \mathbb{C}[y]$ satisfying $\frac{\partial A_1}{\partial y} = \bar{A}$ and $B_1 \in \mathbb{C}[y, z]$ satisfying $\frac{\partial B_1}{\partial y} = \bar{B}$. It is then straightforward to verify that

$$\delta_2 \left( \left( \begin{array}{c} -xB \\ xA_1 - 2z \frac{\partial B_1}{\partial y} \\ 2zB \end{array} \right) \right) = (x\bar{A} + \bar{B}) \nabla \phi,$$

so that

$$\tilde{F} \in \text{Im}(\delta_2) + \nabla (xA + B) + \mathbb{C}[\phi] \left( \begin{array}{c} \frac{x^2}{0} \\ \frac{0}{-x^2} \end{array} \right) + \mathbb{C}[z] \left( \begin{array}{c} \frac{2}{0} \\ \frac{0}{-2} \end{array} \right) + \sum_{n \in \mathbb{N} \ k = 1}^{n+1} \mathbb{C} \tilde{u}_{n,k}.$$

This permits us to write

$$HP_1(T) \simeq \mathbb{C}[\phi] \left( \begin{array}{c} \frac{x^2}{0} \\ \frac{0}{-x^2} \end{array} \right) + \mathbb{C}[z] \left( \begin{array}{c} \frac{2}{0} \\ \frac{0}{-2} \end{array} \right) + \sum_{n \in \mathbb{N} \ k = 1}^{n+1} \mathbb{C} \tilde{u}_{n,k}$$

$$+ \left\{ \nabla (xA + B) \mid A \in \mathbb{C}[y], B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z] \right\}. \quad (5.25)$$

33
Let us show that this sum is a direct one. Suppose that there exist \( P \in \mathbb{C}[X] \) a polynomial in one variable, \( Q \in \mathbb{C}[z] \), \( A \in \mathbb{C}[y] \), and a polynomial \( B \in \mathbb{C}[y, z] + z\mathbb{C}[y, z] \) (i.e., \( B \in \mathbb{C}[y, z] \) with no constant term), \( \vec{K} \in T^3 \), and for every \( n \in \mathbb{N} \) and every \( 1 \leq k \leq n + 1 \), suppose that \( \alpha_k^n \in \mathbb{C} \) are constants, such that:

\[
\nabla(xA + B) + P(\phi) \left( \frac{x^2}{-x^2} \right) + Q(z) \left( \frac{z^2}{-xz} \right) + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \alpha_k^n \vec{u}_{n,k} \\
= -\nabla(\vec{K} \cdot \nabla \phi) + \text{Div}(\vec{K}) \nabla \phi.
\]

This implies that

\[
P(\phi) \left( \frac{x^2}{-x^2} \right) + Q(z) \left( \frac{z^2}{-xz} \right) + \sum_{n \in \mathbb{N}} \sum_{k=1}^{n+1} \alpha_k^n \vec{u}_{n,k} \in \{ K\nabla \phi + \nabla L \mid K, L \in T \}.
\]

According to Corollary 5.6, necessarily \( P = 0 \), \( Q = 0 \) and \( \alpha_k^n = 0 \), for all \( n \in \mathbb{N} \) and all \( 1 \leq k \leq n + 1 \). It then remains \( \nabla(xA + B + \vec{K} \cdot \nabla \phi) = \text{Div}(\vec{K}) \nabla \phi \), which implies that \( \nabla(xA + B + \vec{K} \cdot \nabla \phi) \times \nabla \phi = 0 \) and according to the determination of the space \( HP_3(T) \), this gives the existence of polynomial in one variable \( R \in \mathbb{C}[X] \), such that \( xA + B + \vec{K} \cdot \nabla \phi = R(\phi) \). As there is no constant term in \( B \) and for degree reason, necessarily \( \phi = -x^2z \) divides \( R(\phi) \) and \( x \) divides \( B \in \mathbb{C}[y, z] \), which implies that \( B = 0 \). Moreover, last equation permits us to obtain \( A \in \mathbb{C}[y] \cap (xT + zT) = \{ 0 \} \). This permits us to conclude that the sum in (5.25) is a direct one, i.e., we can write

\[
HP_1(T) \simeq \mathbb{C}[\phi] \left( \frac{x^2}{-x^2} \right) \oplus \mathbb{C}[z] \left( \frac{z^2}{-xz} \right) \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{k=1}^{n+1} \mathbb{C} \vec{u}_{n,k} \\
\oplus \{ \nabla(xA + B) \mid A \in \mathbb{C}[y], B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z] \}.
\]

Finally, it is clear that

\[
\{ \nabla(xA + B) \mid A \in \mathbb{C}[y], B \in y\mathbb{C}[y, z] + z\mathbb{C}[y, z] \} = \\
\bigoplus_{n \in \mathbb{N}} \mathbb{C} \left( nxy^{n-1} \right) \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{0 \leq k \leq n} \mathbb{C} \left( ky^{k-1}z^{n-k} \right),
\]

and this finishes the determination of \( HP_1(T) \).

The second Poisson homology space \( HP_2(T) \).

According to [5.3], we have

\[
HP_2(T) = \left\{ \vec{F} \in T^3 \mid -\nabla(\vec{F} \cdot \nabla \phi) + \text{Div}(\vec{F}) \nabla \phi = 0 \right\} / \{ \nabla G \times \nabla \phi \mid G \in T \}.
\]

Let \( \vec{F} \in T^3 \) be a homogeneous element of degree \( n \in \mathbb{N} \), satisfying

\[
-\nabla(\vec{F} \cdot \nabla \phi) + \text{Div}(\vec{F}) \nabla \phi = 0.
\]

This implies that \( \nabla(\vec{F} \cdot \nabla \phi) \times \nabla \phi = 0 \). This, together with the determination of \( HP_1(T) \) gives the existence of \( \alpha \in \mathbb{C} \) and \( r \in \mathbb{N} \) such that \( \vec{F} \cdot \nabla \phi = \alpha \phi^r \). Notice that if \( r = 0 \), then for degree reasons, \( \alpha = 0 \). According to the Euler formula \( \nabla \phi \cdot \vec{e} = 3\phi \) (where we recall that
\( \vec{e} = (x, y, z) \in T^3 \), we get \( \vec{F} \cdot \nabla \phi = \alpha \phi^{-1} \vec{e} \cdot \nabla \phi \). With the help of the determination of \( H^\phi_2(T) \) in Lemma \( 5.3 \), this gives the existence of a homogeneous element \( \vec{G} \in T^3 \) of degree \( n - 2 \), and homogeneous polynomials \( B, D \in \mathbb{C}[y, z] \) and \( A \in \mathbb{C}[y] \) such that

\[
\vec{F} = \frac{\alpha}{3} \phi^{-1} \vec{e} + \vec{G} \times \nabla \phi + \left( \frac{xD}{-2zD} \right).
\]

We now compute the divergence of \( \vec{F} \):

\[
\text{Div}(\vec{F}) = \alpha \phi^{-1} + (\nabla \times \vec{G}) \cdot \nabla \phi - D + xA' + \frac{\partial B}{\partial y} - 2z \frac{\partial D}{\partial z}.
\]

The equation (5.26) then becomes

\[
0 = (\nabla \times \vec{G}) \cdot \nabla \phi - D + xA' + \frac{\partial B}{\partial y} - 2z \frac{\partial D}{\partial z},
\]

which shows that \( x \) divides the polynomial \( \frac{\partial B}{\partial y} - 2z \frac{\partial D}{\partial z} - D \in \mathbb{C}[y, z] \). This implies that \( \frac{\partial B}{\partial y} = D + 2z \frac{\partial D}{\partial z} \) and \( 0 = (\nabla \times \vec{G}) \cdot \nabla \phi + xA' \), so that we also have \( A' \in (xT + zT) \cap \mathbb{C}[y] = \{0\} \).

We have obtained that \( A = \beta \in \mathbb{C} \) is a constant and

\[
(\nabla \times \vec{G}) \cdot \nabla \phi = 0.
\]

Lemma [5.2] leads to the existence of homogeneous polynomials \( G, H \in T, \) and \( C, P \in \mathbb{C}[y, z], \)

of respective degrees equal to \( n - 3 \) and \( n - 4 \), satisfying

\[
\frac{\partial C}{\partial y} = P + 2z \frac{\partial P}{\partial z}
\]

and \( \gamma \in \mathbb{C}, k \in \mathbb{N} \), such that

\[
\vec{G} = \nabla G + H \nabla \phi + \left( \frac{z}{x} \right) C + \left( \frac{2yz}{-3xz} \right) P + \gamma \phi^k \left( \frac{xz}{0} - x \right).
\]

Now, this permits us to write:

\[
\vec{F} = \frac{\alpha}{3} \phi^{-1} \vec{e} + \nabla G \times \nabla \phi + \left( \frac{xD}{-2zD} \right) + \left( \frac{3x^3zP}{3x^2zC} - \frac{6x^2z^2P}{-6xz^2P} \right) + (-3\gamma x \phi^{k+1} + \beta x) \left( \frac{0}{0} \right).
\]

Let us now fix a homogeneous polynomial \( P_3 \in \mathbb{C}[y, z] \) verifying \( \frac{\partial P_3}{\partial y} = P \). Then the equation (5.27) gives \( \frac{\partial}{\partial y} \left( C - P_1 - 2z \frac{\partial P_3}{\partial z} \right) = 0 \), i.e., \( C - P_1 - 2z \frac{\partial P_3}{\partial z} \) is a homogeneous polynomial in \( \mathbb{C}[z] \), of degree \( n - 3 \), which means that there exists \( \eta \in \mathbb{C} \) such that \( C = P_1 + 2z \frac{\partial P_3}{\partial z} + \eta z^{n-3} \).

Now compute

\[
\delta_3 \left( -3xzP_1 + \frac{-3\eta}{2n-5} xz^{n-2} \right) = -3 \nabla \left( xzP_1 + \frac{\eta}{2n-5} xz^{n-2} \right) \times \nabla \phi
\]

\[
= 3 \left( \left( \frac{z}{x} \right) P_1 + \eta z^{n-2} \frac{z}{x} \frac{z}{x} \frac{\partial P_3}{\partial y} \right) \times \left( \frac{2xz}{6} \right).
\]

\[
= 3 \left( \frac{z^2x^2P_3}{-2x^2z^2} + \eta z^{n-2} \frac{z}{x} \frac{z}{x} \frac{\partial P_3}{\partial y} \right) - 6x^2z^2P.
\]

\[
= \left( \frac{3x^3zP}{3x^2zC} - \frac{6x^2z^2P}{-6xz^2P} \right).
\]

35
This implies: \( \vec{F} \in \text{Im}(\delta_3) + \mathbb{C}[\phi]\vec{e} + \left( \frac{xD}{-2zD} \right) + x\mathbb{C}[\phi] \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \). Conversely, it is straightforward to see that an element of this space lies in the kernel of \( \delta_2 \). In other words,

\[
HP_2(T) = \mathbb{C}[\phi]\vec{e} + x\mathbb{C}[\phi] \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + \left\{ \left( \frac{xD}{B - 2zD} \right) \mid D, B \in \mathbb{C}[y, z] \text{ satisfying } \frac{\partial B}{\partial y} = D + 2z\frac{\partial D}{\partial z} \right\}.
\]

Let us show that the previous sum is a direct one. To do this, let us consider a homogeneous element of this sum. Let \( n \in \mathbb{N}, \alpha, \beta \in \mathbb{C}, \) and homogeneous polynomials \( D, B \in \mathbb{C}[y, z] \) of respective degrees equal to \( 3n \) and \( 3n + 1 \) satisfying \( \frac{\partial B}{\partial y} = D + 2z\frac{\partial D}{\partial z} \) and a homogeneous \( G \in T \) of degree \( 3n \), such that

\[
\alpha \phi^n \vec{e} + \beta x\phi^n \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + \left( \frac{xD}{B - 2zD} \right) = \nabla G \times \nabla \phi = \left( \begin{array}{c} -2z \frac{\partial G}{\partial y} + 2 \frac{\partial G}{\partial z} \\ 2z \frac{\partial G}{\partial y} - \frac{\partial G}{\partial z} \end{array} \right).
\]

Computing the inner product of this identity with \( \nabla \phi \) leads to \( 3\alpha \phi^{n+1} = 0 \), so that \( \alpha = 0 \). Moreover, this gives \( xD = -x^2 \frac{\partial G}{\partial y} \), so that \( D \in xT \cap \mathbb{C}[y, z] = \{0\} \) and \( \frac{\partial B}{\partial y} = 0 \), i.e., \( G \in \mathbb{C}[x, z] \). Moreover, the second row of the previous equation implies that \( B \in xT \cap \mathbb{C}[y, z] = \{0\} \), and \( B = 0 \). It remains to show that \( \beta = 0 \), while we have \( \beta x\phi^n \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = \nabla G \times \nabla \phi \).

According to Lemma 5.5, \( G \) is a constant, which implies that \( \beta = 0 \), or there exist \( \ell \in \mathbb{N}^* \) and a homogeneous polynomial \( H \in \mathbb{C}[x, z] \) (of degree \( 3n - 3\ell \)) and not divisible by \( \phi \), such that \( G = \phi^\ell H \). Necessarily, we have \( \ell \leq n \) and

\[
\beta x\phi^{n-\ell} \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = \nabla H \times \nabla \phi.
\]

Using once more Lemma 5.5 and because \( \phi \) does not divide \( H \), we get \( H \in \mathbb{C} \) and \( \beta x\phi^{n-\ell} \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = 0 \), so that \( \beta = 0 \).

We then have obtained

\[
HP_2(T) = \mathbb{C}[\phi]\vec{e} \oplus x\mathbb{C}[\phi] \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \oplus \left\{ \left( \frac{xD}{B - 2zD} \right) \mid D, B \in \mathbb{C}[y, z] : \frac{\partial B}{\partial y} = D + 2z\frac{\partial D}{\partial z} \right\},
\]

so that it remains to show that

\[
\left\{ \left( \frac{xD}{B - 2zD} \right) \mid D, B \in \mathbb{C}[y, z] : \frac{\partial B}{\partial y} = D + 2z\frac{\partial D}{\partial z} \right\} = \mathbb{C}[z] \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C} \left( \begin{array}{c} (k+1)x \\ (2(n-k)+1) y \\ -2(k+1) z \end{array} \right) y^k z^{n-k}.
\]

To do this, let us consider \( n \in \mathbb{N} \) and homogeneous polynomials \( D, B \in \mathbb{C}[y, z] \) of respective degrees \( n \) and \( n + 1 \), satisfying \( \frac{\partial B}{\partial y} = D + 2z\frac{\partial D}{\partial z} \). We have already seen in the proof of Corollary 5.6 that this implies the existence of complex numbers \( a_k \in \mathbb{C}, 0 \leq k \leq n \) and \( b_0 \in \mathbb{C} \), such that

\[
D = \sum_{k=0}^{n} a_k y^k z^{n-k}, \quad \text{and} \quad B = b_0 z^{n+1} + \sum_{k=1}^{n+1} \frac{2(n-k)+2}{x} a_{k-1} y^k z^{n+1-k}.
\]

This gives

\[
\left( \frac{xD}{B - 2zD} \right) = b_0 z^{n+1} \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + \sum_{k=0}^{n} \frac{a_k}{k+1} \left( \begin{array}{c} (k+1)x y^k z^{n-k} \\ (2(n-k)+1) y^{k+1} z^{n-k} \\ -2(k+1) y^k z^{n+1-k} \end{array} \right) \in \mathbb{C}[z] \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \oplus \bigoplus_{k=0}^{n} \mathbb{C} \left( \begin{array}{c} (k+1)x \\ (2(n-k)+1) y \\ -2(k+1) z \end{array} \right) y^k z^{n-k}.
\]

36
Notice that the previous sum is a direct one, because of degree reasons. This permits us to conclude that (5.25) holds, which finishes the determination of $HP_2(T)$. 

**Remark 5.7** Recall that the Poisson structure which equips the algebra $T$ is unimodular, i.e., its modular class (see 31) vanishes (here, even the curl vector field (see 10) is zero), which implies that there is a duality between the Poisson cohomology and the Poisson homology of the Poisson algebra $T$: $HP^\bullet(T) \simeq HP_{-\bullet}(T)$.

Now, we show that each Poisson cycle can be lifted in a Koszul cycle. By definition, a **Koszul cycle** is a cycle of the filtered complex (5.4).

**Proposition 5.8** We denote by $T$ the polynomial algebra $T = \mathbb{C}[x, y, z]$ and we consider $\phi = -x^2z \in T$. The algebra $T$ is equipped with the Poisson bracket $\{\cdot, \cdot\}$, defined by:

$$\{y, z\} = \frac{\partial \phi}{\partial x} = -2xz, \quad \{z, x\} = \frac{\partial \phi}{\partial y} = 0, \quad \{x, y\} = \frac{\partial \phi}{\partial z} = -x^2.$$

Let also $B$ be the non-commutative algebra

$$B = \mathbb{C}(x, y, z)/(zy = yz + 2xz, zx = xz, yx = x^2 + xy).$$

Let us consider the filtration $F$ on the algebra $B$, given by the degree in $y$. For every Poisson cycle $X$, there exists a Koszul cycle (i.e. a cycle of the filtered complex (5.4)) $\tilde{X}$ such that $gr_F(\tilde{X}) \simeq X$.

**Proof.** Proposition 5.7 gives a basis for each Poisson homology vector space of $T$. For every Poisson boundary $\delta_k(X)$, we have seen in Proposition 5.2 that $gr_F(\tilde{d}_k(Y)) = \delta_k(X)$, where $Y$ is the element $X$, viewed in the algebra $B$ and written in the basis $(x^iy^jz^k)$, $i, j, k \in \mathbb{N}$. So that $\tilde{X} := \tilde{d}_k(Y)$ is a Koszul cycle satisfying $gr_F(\tilde{X}) \simeq \delta_k(X)$. This implies that it suffices to show that each element $X$ of the bases of the Poisson homology spaces given in Proposition 5.7 can be lifted to a Koszul cycle, i.e., for each element $X$ of the bases, we will give a Koszul cycle $\tilde{X}$, satisfying $gr_F(\tilde{X}) = X$. Notice that we will use here the identifications explained before: $B(k(c(w))) \simeq B$, $BR_B \simeq B^3$, $BV_B \simeq B^3$, and $\Omega^3(T) \simeq T$, $\Omega^2(T) \simeq T^3$, $\Omega^1(T) \simeq T$ and $\Omega^0(T) \simeq T$.

**Lifting of the Poisson 0-cycles.**

As every element of $B$ is a Koszul 0-cycle (and similarly for the Poisson 0-cycles), every Poisson 0-cycle can be lifted in a Koszul 0-cycle.

**Lifting of the Poisson 3-cycles.**

According to Proposition 5.7, $HP_3(T) \simeq \mathbb{C}[\phi] = \mathbb{C}[x^2 z]$. For each $c \in \mathbb{C}$ and $k \in \mathbb{N}$, (5.14) clearly leads to $\tilde{d}_3(cx^{2k}z^k \otimes c(w)) = 0$, so that $cx^{2k}z^k \otimes c(w)$, which is identified to $cx^{2k}z^k$, is a Koszul 3-cycle satisfying $gr_F(cx^{2k}z^k) = cx^{2k}z^k$.

**Lifting of the Poisson 1-cycles.**

According to Proposition 5.7, the space $HP_1(T)$ is generated as a $\mathbb{C}$-vector space by the following elements: $A_k := (x^2 z)^k \left(\begin{smallmatrix} x^2 \\ -z \end{smallmatrix}\right)$, $B_r := z^r \left(\begin{smallmatrix} x^2 \\ 0 \end{smallmatrix}\right)$, $\bar{u}_{n,k} = \left(\begin{array}{c} (2n+3) y^2 z^{n+2-k} \\ -3k x y^{k-1} z^{n+2-k} \\ (-3k x y^{k-1} z^{n+2-k}) \\ (2k+3)(3k+1) x y^{k-1} z^{n+2-k} \end{array}\right)$, $\bar{v}_{m,s} := \left(\begin{smallmatrix} 0 \\ y^{m-1} \\ \cdots \\ (m-s) y^{m-1-s} \end{smallmatrix}\right)$ and $\tilde{u}_p := \left(\begin{smallmatrix} y^p \\ p x y^{p-1} \end{smallmatrix}\right)$, where $n, p \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $1 \leq k \leq n+1$, $0 \leq s \leq m$. 

37
Now, by definition of $\tilde{d}_1$ and because $xz = zx$ in $B$, it is clear that:

$$\tilde{d}_1(x^{2k+1}z^{k+1} \otimes x - x^{2k+2}z^k \otimes z) = 0,$$

and

$$\tilde{d}_1(z^{r+2} \otimes x - xz^{r+1} \otimes z) = 0,$$

so that $\tilde{A}_k := x^{2k+1}z^{k+1} \otimes x - x^{2k+2}z^k \otimes z$ and $\tilde{B}_r := z^{r+2} \otimes x - xz^{r+1} \otimes z$ are Koszul 1-cycles satisfying $gr_F(\tilde{A}_k) = A_k$ and $gr_F(\tilde{B}_r) = B_r$.

Next, according to (5.8) and (5.9), it is easy to see that, for all $a, b, c \in \mathbb{N}$,

$$\tilde{d}_1(\mathcal{X}_{a,b,c}) = (2c - a) x^{a+1}y^by^c.$$  \hspace{1cm} (5.29)

where we have denoted by $\mathcal{X}_{a,b,c} := x^ay^bz^c \otimes y - 2c x^ay^bz^c \otimes x$.

Let $n \in \mathbb{N}$ and $1 \leq k \leq n+1$. Using formulas (5.8), (5.9) and (5.10), we get

$$\tilde{d}_1((2n+3)y^kz^{n+2-k} \otimes x - 3ky^kz^{n+2-k} \otimes y$$

$$\hspace{2cm} + (−2n + 3(k − 1))xy^kz^{n+1-k} \otimes z) = \sum_{\ell=0}^{k-2} a_{n,k,\ell} \frac{k!}{\ell!(k-\ell)!} x^{k-\ell+1}y^\ell z^{n+2-k},$$

where for all $0 \leq \ell \leq k - 2$, $a_{n,k,\ell} := -2n - 6 + 6k - 3\ell + 2n(k - \ell) - 3k(k - \ell)$. Now, according to (5.29), for all $0 \leq \ell \leq k - 2$,

$$\tilde{d}_1(\mathcal{X}_{k-\ell,n+2-k}) = (2n+2-k) - (k-\ell)) x^{k+1-\ell}y^\ell z^{n+2-k}$$

$$\hspace{2cm} = (2n+4-3k+\ell) x^{k+1-\ell}y^\ell z^{n+2-k}.$$  \hspace{1cm} (5.29)

Moreover, because $k \leq n+1$, we have $3k-4-2n \leq k - 2$ and if $3k-4-2n \geq 0$, then we denote by $\ell_{(n,k)} := 3k-4-2n$, and it is straightforward, using once more (5.8), (5.9) and (5.10), to verify that

$$\tilde{d}_1 \left( \frac{x^{\ell_{(n,k)}-1}y^{\ell_{(n,k)}+1}z^{n+2-k}}{(\ell_{(n,k)}+1)!} \otimes x - \sum_{\ell=0}^{\ell_{(n,k)}-1} \frac{1}{\ell!} (2n+2-k-\ell) \mathcal{X}_{k-\ell,n+2-k} \right) = \frac{1}{\ell_{(n,k)!}} x^{2n+2k+5} y^{3k-4-2n} z^{n+2-k}.$$  \hspace{1cm} (5.29)

Finally, we let

$$\tilde{U}_{n,k} :=$$

$$(2n+3)y^kz^{n+2-k} \otimes x - 3ky^kz^{n+2-k} \otimes y + (−2n + 3(k − 1))xy^kz^{n+1-k} \otimes z$$

$$\hspace{2cm} - \sum_{\ell=0}^{k-2} \frac{k!}{\ell!(k-\ell)!} a_{n,k,\ell} \mathcal{X}_{k-\ell,n+2-k}, \text{ if } 3k-2(n+2) < 0;$$

$$\hspace{2cm} + \sum_{\ell=0}^{\ell_{(n,k)}-1} \frac{k!}{\ell!(k-\ell)!} a_{n,k,\ell} \mathcal{X}_{k-\ell,n+2-k} - \sum_{\ell=\ell_{(n,k)}+1}^{k-2} \frac{k!}{\ell!(k-\ell)!} a_{n,k,\ell} \mathcal{X}_{k-\ell,n+2-k}$$

$$\hspace{2cm} - \frac{k!}{(\ell_{(n,k)}+1)!} a_{n,k,\ell_{(n,k)}} x^{2n+3-2k} y^{3k-3-2n} z^{n+2-k} \otimes x, \text{ if } 3k-2(n+2) \geq 0.$$  \hspace{1cm} (5.29)

Then, we have $gr_F(\tilde{U}_{n,k}) = \tilde{u}_{n,k}$ and $\tilde{U}_{n,k}$ is a Koszul 1-cycle.
Now, let \( m \in \mathbb{N}^* \), \( 0 \leq s \leq m \) and consider
\[
\hat{V}_{m,s} := s y^{s-1}z^{m-s} \otimes y + (m-s) y^s z^{m-1-s} \otimes z
+ \sum_{\ell=0}^{s-2} \frac{(m-s)}{2(m-s)-\ell} \frac{\ell!}{\ell!} x^{s-\ell-1} y^{\ell} z^{m-s} \otimes y.
\]
Then we have \( \text{gr}_F(\hat{V}_{m,s}) = \tilde{v}_{m,s} \) and it is straightforward, using \((5.9)\) and \((5.10)\) to verify that \( \hat{V}_{m,s} \) is a Koszul 1-cycle.

Finally, let \( p \in \mathbb{N} \) and consider
\[
\hat{W}_p := \sum_{k=0}^{p} \frac{p!}{k!} x^{p-k} y^k \otimes x + \sum_{k=0}^{p-1} \frac{p!}{k!} x^{p-k} y^k \otimes y.
\]
It is clear that \( \text{gr}_F(\hat{W}_p) = \tilde{w}_p \) and moreover, using \((5.8)\) and \((5.9)\), we obtain that \( d_1(\hat{W}_p) = 0 \), i.e., \( \hat{W}_p \) is a Koszul 1-cycle.

Lifting of the Poisson 2-cycles.
According to Proposition 5.7, the space \( HP_2(T) \) is generated as a \( \mathbb{C} \)-vector space by the following elements: \( C_r := (a^2 z)^r \left( \begin{smallmatrix} x \\ y \end{smallmatrix} \right) \), \( D_s := x(a^2 z)^s \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right) \), \( E_t := z^{t+1} \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right) \), \( \tilde{o}_{n,k} := \left( \frac{(k+1)xy^kz^{n-k}}{(2(n-k)+1)y^{k+1}z^{n-k}} \frac{x^kz^{n-k+1}}{2(k+1)y^kz^{n-k+1}} \right) \), where \( r, s, t, n \in \mathbb{N} \) and \( 0 \leq k \leq n \).

First, if \( r \in \mathbb{N} \), then, using \((5.11)\), \((5.12)\) and \((5.13)\), we obtain easily:
\[
d_2(x^{2r+1}z^r \otimes r_1 + x^{2r}yz^{r} \otimes r_2 + x^{2r}z^{r+1} \otimes r_3) = 0,
\]
so that \( \hat{C}_r := x^{2r+1}z^r \otimes r_1 + x^{2r}yz^{r} \otimes r_2 + x^{2r}z^{r+1} \otimes r_3 \) is a Koszul 2-cycle satisfying
\[\text{gr}_F(\hat{C}_r) = C_r.\]
Moreover, if \( s, t \in \mathbb{N} \), then, because \( xz = zx \) in \( B \) and by definition of \( d_2 \), we have
\[
d_2(x^{2s+1}z^s \otimes r_2) = 0, \quad \text{and} \quad d_2(z^{t+1} \otimes r_2) = 0,
\]
so that \( \hat{D}_s := x^{2s+1}z^s \otimes r_2 \) and \( \hat{E}_t := z^{t+1} \otimes r_2 \) are Koszul 2-cycles and satisfy \( \text{gr}_F(\hat{D}_s) = D_s \) and \( \text{gr}_F(\hat{E}_t) = E_t \).

Now, let \( n \in \mathbb{N} \) and \( 0 \leq k \leq n \) and
\[
\hat{O}_{n,k} := (k+1) xy^kz^{n-k} \otimes r_1 + (2(n-k)+1)y^{k+1}z^{n-k} \otimes r_2
+ \left( -2(k+1)y^{k}z^{n-k+1} - \sum_{j=0}^{k-1} \frac{(k+1)!}{j!} x^{k-j}y^jz^{n+1-k} \right) \otimes r_3.
\]
Then, using once more formulas \((5.11)\), \((5.12)\) and \((5.13)\), it is straightforward to verify that \( d_2(\hat{O}_{n,k}) = 0 \). Moreover, we have of course, \( \text{gr}_F(\hat{O}_{n,k}) = \tilde{o}_{n,k} \), which finishes the proof. ■

Proof of Theorem 5.9. Actually, it remains to prove that the Hochschild homology of the algebra \( B \) is isomorphic to the Poisson homology of \( T \) obtained in Proposition 5.7. Following the same method as in [27], we use the Brylinski spectral sequence of the almost commutative algebra \( B \) \( [8, 14] \). Denote by \( C \) the filtered complex \((5.30)\) and denote by \((F_pC)_{p \in \mathbb{Z}}\) its filtration. The complex \( F_pC \) is the following
\[
0 \longrightarrow F_{p+3}(B(\mathbb{C}(w))) \xrightarrow{d_3} F_{p+2}(BR_B) \xrightarrow{d_2} F_{p+1}(BV_B) \xrightarrow{d_1} F_p(B) \longrightarrow 0 \quad (5.30)
\]
where $F_p$ denotes the filtration of the total degree. Let us consider the spectral sequence associated to the filtered complex $C$ (Section 5.4 in [30]). The term $E^0$ of this spectral sequence is the graded complex naturally associated to the filtered complex $C$. By Proposition 5.2, $E^0$ is isomorphic to the Poisson complex of the Poisson algebra $T$. Since the filtration $F$ of the complex $C$ is increasing, exhaustive and bounded below ($F_{-1}B = 0$), the spectral sequence converges to $H_* (C)$ (Theorem 5.5.1.2 in [30]):

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C).$$

(5.31)

Thus, in order to conclude that the Hochschild homology of the algebra $B$ is isomorphic to the Poisson homology of $T$, it is sufficient to prove the following.

**Proposition 5.9** The spectral sequence associated to the filtered complex $C$ degenerates at $E^1$.

**Proof.** We apply a standard criterion for degeneration of spectral complex sequences (Lemma 5.2 in [27]) with $r = 1$ (we use the notation of [27]). This criterion consists in proving that the natural map

$$\phi^1_p : H_* (F_pC) \rightarrow E^1_p$$

is surjective for any $p$. Since the term $E^1$ is isomorphic to the Poisson homology of $T$, surjectivity is given by Proposition 5.8.

Since $B$ is 3-Calabi-Yau (Theorem 2.10), we deduce Hochschild cohomology of $B$ from Theorem 1.1. $HH^* (B) \cong HH_{3-*}(B)$. In particular the center of the algebra $B$ is the polynomial algebra generated by the element $\Phi$ of Proposition 4.4, i.e. generated by $x^2z$.

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