Conjunctive Queries, Existentially Quantified Systems of Equations and Finite Substitutions*

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Abstract

This report presents an elementary theory of unification for positive conjunctive queries. A positive conjunctive query is a formula constructed from propositional constants, equations and atoms using the conjunction $\land$ and the existential quantifier $\exists$. In particular, empty queries correspond to existentially quantified systems of equations — called $\mathcal{E}$-formulas. We provide an algorithm which transforms any conjunctive query into a solved form. We prove some lattice-theoretic properties of queries. In particular, the quotient set of $\mathcal{E}$-formulas under an equivalence relation forms a complete lattice. Then we present another lattice — a lattice of finite substitutions. We prove that the both lattices are isomorphic. Finally, we introduce the notion of application of substitutions to formulas and clarify its relationship to $\mathcal{E}$-formulas. This theory can be regarded as a basis for alternative presentation of logic programming.

1 Introduction

In this paper we present an elementary theory of unification for positive conjunctive queries. A positive conjunctive query (or just a query for short) is a natural generalization of a system of equations. It is a formula constructed from propositional constants, equations and atoms using the conjunction $\land$ and the existential quantifier $\exists$. In particular, empty query corresponds to existentially quantified system of equations — called $\mathcal{E}$-formulas. The aim of this paper is to generalize results presented in [3] concerning the theory of unification for systems of equations.

The paper is organized as follows. Section 2 contains basic definitions and notations. First we give a brief overview of syntax and semantics of first order languages. Then we define positive conjunctive queries and introduce the important concept of

*Reprint of the technical report TR mff-ii-10-1992, September 1992.
an equivalence relation on queries. Finally, we introduce finite substitutions and give a short description of some lattice-theoretic properties of terms.

In Section 3 we give an algorithm, called Solved Form Algorithm, which transforms any query into a solved form. Theorem 3.1 establishes correctness and termination of the algorithm.

In Section 4 we study lattice-theoretic properties of queries. In particular, we show that the relationship between two equivalent queries in solved form is very strong (see Theorem 4.8).

In the last section we present two lattices: the lattice of $E$-formulas and the lattice of finite substitutions. We prove that the both lattices are isomorphic (see Theorem 5.13). Finally, we introduce the notion of application of substitutions to formulas and clarify its relationship to $E$-formulas (see Theorem 5.20).

This theory can be regarded as a basis for alternative presentation of logic programming (see [2]).

2 Notation and definitions

In this section we recall some basic definitions. We refer to [1, 4, 5] for a more detailed presentation of our topics.

**Syntax** The alphabet $L$ for a first order language consists of logical symbols (a denumerable set of variables, punctuation symbols, connectives and quantifiers) and two disjoint classes of nonlogical symbols: (i) a set $Func_L$ of function symbols (including constants) and (ii) a set $Pred_L$ of predicate symbols. Throughout this paper we assume that the set of function symbols contains at least one constant. Moreover we always suppose that the equality symbol $=$ and propositional constants $True$ and $False$ are contained in all alphabets we shall use. We shall write $e_1 \equiv e_2$ to denote the syntactical identity of two strings $e_1$ and $e_2$ of symbols. We denote by $Var$ the set of all variables.

We use $u$, $v$, $x$, $y$ and $z$, as syntactical variables which vary through variables; $f$, $g$ and $h$ as syntactical variables which vary through function symbols; $p$ and $q$ as syntactical variables which vary through predicate symbols excluding $=$; and $a$, $b$, $c$ and $d$ as syntactical variables which vary through constants.

The first order language consists of two classes of strings of symbols over a given alphabet $L$: (i) a set of terms, denoted $Term_L$, and (ii) a set of all well-formed formulas. We use $r$, $s$ and $t$, as syntactical variables which vary through terms; $F$, $G$ and $H$ as syntactical variables which vary through formulas. An equation is a formula of the form $s = t$; and an atom is a formula of the form $p(s_1, \ldots, s_n)$. We use $A$, $B$ and $C$, as syntactical variables which vary through atoms. A formula is called positive if it is constructed from propositional constants $True$ and $False$, and from equations and
Consider a term $s$. Then $\text{vars}(s)$ denotes the set of variables appearing in $s$. If $\text{vars}(s)$ is empty, then the term $s$ is called ground. Similarly, $\text{vars}(F)$ denotes the set of free variables of a formula $F$. $F$ is said to be closed if $\text{vars}(F) = \emptyset$. Let $x_1, \ldots, x_n$ be all distinct variables occurring freely in a formula $F$ in this order. We write $(\forall) F$ or $(\exists) F$ for $(\forall x_1) \ldots (\forall x_n) F$ or $(\exists x_1) \ldots (\exists x_n) F$, respectively. We call $(\forall) F$ or $(\exists) F$ the universal closure or the existential closure of $F$, respectively.

In order to avoid the awkward expression the first order language over an alphabet $L$ we will simply say the first order language $L$ (or the language $L$ for short).

In Section 5.2 we introduce the notion of application of finite substitutions to formulas. The definition is based on a weaker form of this application. We follow here [5]:

(a) $t_{x_1, \ldots, x_n}[s_1, \ldots, s_n]$ denotes a term obtained from $t$ by simultaneously replacing of each occurrence $x_1, \ldots, s_n$ in $t$ by $s_1, \ldots, s_n$, respectively.

(b) $F_{x_1, \ldots, x_n}[s_1, \ldots, s_n]$ denotes a formula obtained from $F$ by simultaneously replacing of each free occurrence $x_1, \ldots, x_n$ in $F$ by $s_1, \ldots, s_n$, respectively.

Whenever $t_{x_1, \ldots, x_n}[s_1, \ldots, s_n]$ or $F_{x_1, \ldots, x_n}[s_1, \ldots, s_n]$ appears, $x_1, \ldots, x_n$ are restricted to represent distinct variables. Moreover, in (b) we always suppose that each term $s_i$ is substitutable for $x_i$ in $F$ i.e. for each variable $y$ occurring in $s_i$, no part of $F$ of the form $(\exists y)G$ (or $(\forall y)G$) contains an occurrence of $x_i$ which is free in $F$. We shall omit the subscripts $x_1, \ldots, x_n$ when they occur freely in $F$ in this order and $\text{vars}(F) = \{x_1, \ldots, x_n\}$.

We say that $F'$ is a variant of $F$, if $F'$ can be obtained from $F$ by a sequence of replacement of the following type: replace a part $(\exists x)G$ or $(\forall x)G$ by $(\exists y)G_x[y]$ or by $(\forall x)G_x[y]$, respectively, where $y$ is a variable not free in $G$.

By free equality axioms for a language $L$ (see [1, 4]), we mean the theory $\text{EQ}_L$ consisting of the following formulas:

(a) $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \iff x_1 = y_1 \land \ldots \land x_n = y_n$ for each $n$-ary function symbol $f$.

(b) $f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m) \iff \text{False}$ for each $n$-ary function symbol $f$ and $m$-ary function symbol $g$ such that $f \neq g$.

(c) $x = t \iff \text{False}$ for each variable $x$ and term $t$ such that $x \neq t$ and $x$ occurs in $t$.

Since we identify constants with $0$-ary function symbols, then (b) includes $a \neq b$ for pairs of distinct constants as a special case.
Semantics A pre-interpretation $J$ for a language $L$ consists of (i) a non-empty universe $U^J_L$, called a domain of $J$, and (ii) a fixed interpretation of all function symbols. The equality $=$ is interpreted as the identity on $U^J_L$. An interpretation $I$ for $L$ is based on $J$ (or just $J$-interpretation for short) if it is obtained from $J$ by selecting some interpretation of predicate symbols.

Consider a pre-interpretation $J$. A variable assignment $h : Var \rightarrow U^J_L$ will be called a valuation over $J$ (or a $J$-valuation for short). The set of all $J$-valuations is denoted by $V^J_L$. Obviously each $J$-valuation $h$ has a unique homomorphic extension $h'$ from $Term$ into $U^J_L$. We shall write $h(s)$, where $s$ is a term, instead of $h'(s)$. We call $h(s)$ a $J$-instance of $s$. Consider an atom $A \equiv p(s_1, \ldots, s_n)$ and a $J$-valuation $h$. The generalized atom $h(A) \equiv p(h(s_1), \ldots, h(s_n))$. will be called a $J$-instance of $A$. The set of all $J$-instances of atoms, called $J$-base, is denoted by $B^J_L$. We shall identify interpretations based on $J$ with subsets of $B^J_L$. We shall use the notation $h[x \leftarrow d]$, where $x$ is a variable and $d \in U^J_L$, to denote a valuation defined as follows:

$$h[x \leftarrow d](y) = \begin{cases} d & \text{if } y \equiv x \\ h(y) & \text{otherwise} \end{cases}$$

Example 2.1 The Herbrand pre-interpretation $H$ for $L$ is defined as follows:

(a) Its domain is the set $U^H_L$ of all ground terms of $L$; called the Herbrand universe.

(b) Each constant in $L$ is assigned to itself.

(c) If $f$ is an n-ary function symbol in $L$ then it is assigned to the mapping from $(U^H_L)^n$ to $U^H_L$ defined by assigning the ground term $f(s_1, \ldots, s_n)$ to the sequence $s_1, \ldots, s_n$ of ground terms.

By a Herbrand interpretation for $L$ we mean any interpretation based on $H$. As remarked above we shall identify Herbrand interpretations with subsets of the set $B^H_L$ called the Herbrand base.

Given a formula $F$ we define its truth in a $J$-valuation $h$ and an interpretation $I$ based on $J$, written as $I \models_h F$, in the obvious way. In particular, $I \models_h s = t$ iff $h(s) = h(t)$. We call $h \in V^J_L$ a solution of $F$ in $I$, if $I \models_h F$. The set of solutions of $F$ in $I$ is denoted by $\langle F \rangle^I_I$. So:

$$\langle F \rangle^I_I = \{h \in V^J_L \mid I \models_h F\}$$

We say that a formula $F$ is true in $I$, written as $I \models F$, when for all valuations $h \in V^J_L$, $I \models_h F$. We say that a formula is valid when it is true in any interpretation.

Consider a theory $T$ over $L$. An interpretation $I$ is a model for $T$ if $I \models F$ for any $F$ from $T$. A formula $F$ (over $L$) is a logical consequence of $T$ if any model $I$ for $T$ is a model for $F$, as well. We write $T \models F$ in $L$ (or $T \models F$ for short).
Consider an equational formula $F$. Note that the semantics of $F$ depends only on an interpretation of function symbols. Consequently, given a pre-interpretation $J$, the truth of $F$ in a $J$-valuation $h$ is well-defined. We shall write $J \models_h F$. We call $h \in V^J_L$ a solution of $F$ in $J$ (or $J$-solution of $F$ for short), if $J \models_h F$. The set of $J$-solutions of $F$ is denoted by $\text{soln}_J(F)$. So:

$$\text{soln}_J(F) = \{ h \in V^J_L \mid J \models_h F \}$$

$F$ is said to be true in $J$, written as $J \models F$, when for all valuations $h \in V^J_L$, is $J \models_h F$. We say that $J$ is a model for $\text{EQ}_L$, when any axiom of $\text{EQ}_L$ is true in $J$.

The following theorems will be used in the sequel (see [5]).

**Theorem 2.1 (Variant Theorem)** If $F'$ is a variant of $F$, then $F' \leftrightarrow F$ is valid.

**Theorem 2.2 (Theorem on Constants)** Let $T$ be a theory and $F$ a formula. If $a_1, \ldots, a_n$ are distinct constants not occurring in $T$ and $F$, then

$$T \models F \iff T \models F[a_1, \ldots, a_n]$$

**Theorem 2.3** Consider a theory $T$ and formulas $F$ and $G$. Then:

1. $F \rightarrow (\exists x)F$ is valid.
2. If $T \models F \rightarrow G$, then $T \models (\exists x)F \rightarrow G$ provided $x$ is not free in $G$.
3. If $T \models F \rightarrow G$, then $T \models (\exists x)F \rightarrow (\exists x)G$

**Queries** A conjunctive query is a formula constructed from propositional constants, equations and atoms using the conjunction $\land$, the negation $\neg$ and the existential quantifier $\exists$. In this paper we deal only with conjunctive queries which are positive formulas. From now on by a query we mean a positive conjunctive query. We denote by $|Q|$ the number of atoms occurring in $Q$. If $|Q| = 0$, then $Q$ is called an empty query.

A query $Q$ is said to be in solved form if

- $Q$ is a propositional constant True or False or
- $Q$ is of the form

$$(\exists z_1) \ldots (\exists z_k)(x_1 = s_1 \land \ldots \land x_n = s_n \land A_1 \land \ldots \land A_m),$$

where (i) $x_i$'s and $z_j$'s are distinct variables, (ii) $x_i$'s occur nor in the right hand side of any equation nor in any atom, (iii) each $z_j$ has at least one occurrence in the conjunction and (iv) $z_j \not\equiv s_i$ for any $i$ and $j$. 

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The variables \(x_1, \ldots, x_n\) are said to be eliminable and the set \(\{x_1, \ldots, x_n\}\) is denoted by \(\text{elim}(Q)\). The remaining free variables in \(Q\) are called parameters and the set of parameters is denoted by \(\text{param}(Q)\). So \(\text{vars}(Q) = \text{elim}(Q) \cup \text{param}(Q)\). The set \(\{z_1, \ldots, z_k\}\) of (existentially) bound variables in \(Q\) is denoted by \(\text{bound}(Q)\). We put \(\text{elim}(\text{True}) = \text{param}(\text{True}) = \text{bound}(\text{True}) = \emptyset\).

In Section 3 we shall describe an algorithm, called Solved Form Algorithm, which transforms every query into a solved form. The algorithm is sound in the following sense: if \(Q'\) is a computed solved form of a query \(Q\), then \(Q'\) is semantically equivalent to \(Q\) w.r.t. \(\text{EQ}_L\) i.e.

\[
\text{EQ}_L \models Q' \leftrightarrow Q
\] (1)

Actually, we prove that Solved Form Algorithm has a stronger property than (1). In particular, as we see later, the algorithm preserves the number of atoms. Consequently, \(|Q'| = |Q|\). For this purpose we introduce a modified concept of solution sets for queries. Instead of the mapping \(\{Q\}_T\) : \(2^B_J \rightarrow V_J\), which has a single interpretation as an argument, we shall consider a mapping \([Q]_J : 2^B_J \times \cdots \times 2^B_J \rightarrow V_J\) defined on sequences \(T\) of interpretations. Each component in \(T\) will serve as an “input” for the corresponding atom in \(Q\). We follow here ideas developed in [6].

By a multiinterpretation based on a pre-interpretation \(J\) we mean any finite sequences of interpretations based on \(J\). The empty sequence is denoted by \(\Lambda\). We shall use overlined letters to denote multiinterpretations. We write \(\overline{T}_1 \overline{T}_2\) for the concatenation of multiinterpretations \(\overline{T}_1\) and \(\overline{T}_2\). The length of \(\overline{T}\) is denoted by \(|\overline{T}|\). We say that \(\overline{T}\) is for a query \(Q\) if \(|\overline{T}| = |Q|\).

Consider a pre-interpretation \(J\), a query \(Q\) and a multiinterpretation \(\overline{T}\) for \(Q\) based on \(J\). The set of solutions of the query \(Q\) in \(\overline{T}\) is a set of \(J\)-valuations, denoted by \([Q]_{\overline{T}}\), and it is defined inductively as follows:

1. \([\text{True}]_{\Lambda} = V_J\) and \([\text{False}]_{\Lambda} = \emptyset\)
2. \([s = t]_{\Lambda} = \text{soln}_J(s = t)\)
3. \([A]_{\Lambda} = \langle A \rangle_{\Lambda}\), if \(A\) is an atom.
4. \([Q_1 \land Q_2]_{\overline{T}_1 \overline{T}_2} = [Q_1]_{\overline{T}_1} \cap [Q_2]_{\overline{T}_2}\), where \(|\overline{T}_1| = |Q_1|\) and \(|\overline{T}_2| = |Q_2|\)
5. \([\exists x]Q|_{\overline{T}} = \{h \in V_J \mid h[x \leftarrow d] \in \{Q\}_{\overline{T}} \text{ for some } d \in U_J\}\)

We call a \(J\)-valuation \(h\) from \([Q]_{\overline{T}}\) a solution of \(Q\) in \(\overline{T}\). Note that \(\langle Q \rangle_{\overline{T}} = [Q]_{\overline{T}}\), where \(\overline{T} = I, \ldots, I\) and \(|\overline{T}| = |Q|\).

Consider a pre-interpretation \(J\). Then a preorder relation \(\preceq_J\) and an equivalence relation \(\approx_J\) on queries is defined as follows:

(a) \(Q \preceq_J Q'\) iff \(|Q| = |Q'|\) and \([Q]_{\overline{T}} \subseteq [Q']_{\overline{T}}\) for any multiinterpretations \(\overline{T}\) based on \(J\)
(b) \( Q \approx_J Q' \) iff \(|Q| = |Q'|\) and \([Q]^J_T = [Q']^J_T\) for any multiinterpretations \( T \) based on \( J \).

Finally, we say

(a) \( Q' \) is more general than \( Q \), written as \( Q \preceq Q' \), if for any pre-interpretation \( J \), which is a model for \( EQ_L \), we have \( Q \preceq_J Q' \).

(b) \( Q' \) is equivalent to \( Q \), written as \( Q \equiv Q' \), if \( Q \preceq Q' \) and \( Q' \preceq Q \).

Then the precise form of soundness of Solved Form Algorithm is following (see Theorem 3.1): the solved form algorithm applied to a query \( Q \) will return an equivalent one in solved form.

We say that a query \( Q \) is consistent if there is an interpretation \( I \) (based on a pre-interpretation \( J \)), which is a model for \( EQ_L \), such that \( I \models h Q \) for some \( h \in V_L^J \), or equivalently if \( EQ_L \not\models \neg Q \). Note that the consistency of queries in solved form can be checked directly: if \( Q \) is a query in solved form, then \( Q \) is consistent iff \( Q \not\equiv False \).

As we marked above Solved Form Algorithm preserves equivalence and consequently it preserves consistency, as well. Hence the consistency of a query \( Q \) can be reached directly by the the form of its (computed) solved form.

### Finite Substitutions

By a finite substitution we mean any mapping of terms to variables from a finite set \( X \) of variables. Let \( X = \{x_1, \ldots, x_n\} \). We shall use the standard set-theoretic notation \( \sigma = \{x_1 \gets s_1, \ldots, x_n \gets s_n\} \) to denote \( \sigma \), where \( s_i \equiv \sigma(x_i) \). We say that \( \sigma \) is over \( X \). We denote by \( \text{dom}(\sigma) \) or \( \text{range}(\sigma) \) the set \( X \) or the set of variables occurring in terms \( s_1, \ldots, s_n \), respectively. The pair \( x \gets s \) is called a binding. In the sequel by a substitution we always mean a finite substitution.

Let \( \sigma \) and \( \theta \) are substitutions with disjoint domains. By \( \sigma \cup \theta \) we mean a substitution over \( \text{dom}(\sigma) \cup \text{dom}(\theta) \) assigning \( \sigma(x) \) or \( \theta(x) \) to \( x \in \text{dom}(\sigma) \) or \( x \in \text{dom}(\theta) \), respectively. We call \( \sigma \) a permutation if it is one-to-one mapping from \( \text{dom}(\sigma) \) onto \( \text{range}(\sigma) \).

There is unique substitutions \( \varepsilon \) with the empty domains; it will be called the empty substitution.

A substitution \( \sigma = \{x_1 \gets s_1, \ldots, x_n \gets s_n\} \) is applicable to a term \( t \) if \( \text{dom}(\sigma) \) contains all variables occurring in \( t \). Then an application of \( \sigma \) to \( t \), denoted \( t\sigma \), is defined as the term \( t_{x_1,\ldots,x_n}[s_1,\ldots,s_n] \). The term \( t\sigma \) is called an instance of \( t \). If \( \sigma \) is a permutation, then \( t\sigma \) and \( t \) are said to be variants.

A substitution \( \sigma = \{x_1 \gets s_n, \ldots, x_n \gets s_n\} \) is applicable to a formula \( F \) if each \( s_i \) is substitutible for \( x_i \) in \( F \). Then an application of \( \sigma \) to \( F \), denoted \( F\sigma \), is defined as a formula \( F_{x_1,\ldots,x_n}[s_1,\ldots,s_n] \). In Section 5.2 we generalize this operation for arbitrary substitutions.

A substitution \( \theta \) is applicable to \( \sigma \) if \( \text{range}(\sigma) \subseteq \text{dom}(\theta) \). Then a composition of \( \sigma \) and \( \theta \) is a substitution over \( \text{dom}(\sigma) \), denoted \( \sigma\theta \), and it is defined in the obvious way. If \( \theta \) is a permutation, then we say that \( \sigma \) and \( \sigma\theta \) are variants.
We denote by \( \sigma|X \) a restriction of \( \sigma \) onto \( X \). So \( \sigma|X \) is a substitution over \( \text{dom}(\sigma) \cap X \). We call \( \sigma \) an extension of \( \sigma|X \) onto \( \text{dom}(\sigma) \).

**Terms** In Section 4 we shall study lattice-theoretic properties of queries. To prove the claims of that section we need the following simple results concerning terms.

First we introduce the preorder relation \( \preceq \) and the equivalence relation \( \approx \) of terms:

- A term \( t \) is more general than a term \( s \), written as \( s \preceq t \), if there is a substitution \( \theta \) such that \( s \equiv t\theta \).
- A term \( t \) is equivalent to a term \( s \), written as \( s \approx t \), if \( s \preceq t \) and \( t \preceq s \).

On the other hand each pre-interpretation \( J \) defines a preorder \( \preceq_J \) and an equivalence \( \approx_J \) as follows. Let \( \text{inst}_J(s) \) denote the set of all \( J \)-instances of the term \( s \). Then:

- A term \( t \) is more general w.r.t. \( J \) than a term \( s \), written as \( s \preceq_J t \), if \( \text{inst}_J(s) \subseteq \text{inst}_J(t) \).
- A term \( t \) is equivalent w.r.t. \( J \) (or \( J \)-equivalent for short) to a term \( s \), written as \( s \approx_J t \), if \( \text{inst}_J(s) = \text{inst}_J(t) \).

Clearly if \( \sigma \preceq \theta \) or \( \sigma \approx \theta \) then \( \sigma \preceq_J \theta \) or \( \sigma \approx_J \theta \), respectively. The following example shows that \( \preceq_J \) does not coincide with \( \preceq \) in general.

**Example 2.2** Suppose that the Herbrand universe for \( L \) consists of the constant \( a \) only. Let \( x \) be a variable. Then \( x \preceq_H a \) and \( x \not\preceq a \).

Proposition 2.5 states that both preorders \( \preceq_J \) and \( \preceq \) are identical (and hence \( \approx_J \equiv \approx \) as well), provided that \( J \) satisfies the following two conditions: (i) \( J \) is a model for \( EQ_L \) and (ii) its domain has at least two elements (such pre-interpretations will be called non-trivial). The next simple lemma will be used in the sequel.

**Lemma 2.4** Suppose that \( J \) is a non-trivial model of \( EQ_L \). Let \( s \) and \( t \) be both non-variable terms.

1. \( \text{inst}_J(s) \neq V_L^J \).

2. If \( s \) and \( t \) have different principal functors, then \( \text{inst}_J(s) \cap \text{inst}_J(t) = \emptyset \).

**Proof:** Straightforward. \( \square \)

**Proposition 2.5** Let \( s \) and \( t \) be terms. If \( J \) is a non-trivial model of \( EQ_L \), then:

1. \( s \preceq_J t \) iff \( s \preceq t \)
2. $s \approx J t$ iff $s \approx t$ iff $s$ and $t$ are variants

To prove the claim we first introduce the notion of difference sets for terms. By a difference set of terms $s$ and $t$ we mean a set of pairs of terms, denoted $Diff(s, t)$, which is defined inductively as follows:

- $Diff(x, x) = \{(x, x)\}$
- $Diff(a, a) = \emptyset$
- $Diff(f(s_1, \ldots, s_n), f(t_1, \ldots, t_n)) = Diff(s_1, t_1) \cup \ldots \cup Diff(s_n, t_n)$
- $Diff(s, t) = \{(s, t)\}$ otherwise

For example

$$Diff(f(x, g(z), z), f(y, a, z)) = \{(x, y), (g(z), a), (z, z)\}$$

Difference sets have the following property:

**Lemma 2.6** Let $s$ and $t$ be terms. Suppose that $J$ is a model of $EQ_L$. If $h$ and $g$ are $J$-valuations such that $h(s) = g(t)$, then $h(s') = g(t')$ for any pair $(s', t') \in Diff(s, t)$.

**Proof:** Straightforward. $\square$

Moreover notice that

$$Diff(s, t\theta) = \bigcup\{Diff(s', t'\theta) \mid (s', t') \in Diff(s, t)\}, \quad (2)$$

for any substitution $\theta$ applicable to $t$. Now we are ready to prove the desired proposition.

**Proof of Proposition 2.5:** First we prove that $s \preceq J t$, implies $s \preceq t$.

Let $(s', t') \in Diff(s, t)$. Then $t'$ must be a variable, since otherwise by Lemma 2.4 there is $h \in V_L^J$ such that $h(s') \notin \text{inst}_J(t')$. Then Lemma 2.6 implies that $h(s) \notin \text{inst}_J(t)$. Contradiction.

Assume that $(r_1, x) \in Diff(s, t)$ and $(r_2, x) \in Diff(s, t)$. Then terms $r_1$ and $r_2$ must be the same, since otherwise there is $h \in V_L^J$ such that $h(r_1) \neq h(r_2)$. Because $\text{inst}_J(s) \subseteq \text{inst}_J(t)$, there is $g \in V_L^J$ such that $h(s) = g(t)$. Lemma 2.6 implies that $h(r_1) = g(x)$ and $h(r_2) = g(x)$ Contradiction.

Now let $\theta$ be a substitution over $vars(t)$ such that for any $x \in vars(t)$ it holds $\theta(x) \equiv r$, where $(r, x) \in Diff(s, t)$. $\theta$ is well defined since of the form $Diff(s, t)$. From (2) we have $Diff(s, t\theta) = \bigcup\{Diff(x, x) \mid x \in vars(s)\}$. Hence $s \equiv t\theta$ i.e. $s \preceq t$.

If $s \approx J t$ or $s \approx t$, then the substitution $\theta$ is a permutation and hence $s$ and $t$ are variants. $\square$
Canonical language  In Section 5 we shall prove the so-called Compactness Theorem. It relies on Proposition 2.8.

We say that the first order language $L_2$ is an extension of a first order language $L_1$ if every nonlogical symbol of $L_1$ is a nonlogical symbol of $L_2$. The following claim will be used in the proof of Proposition 2.8.

**Lemma 2.7 ([5])** Consider a theory $T$ over $L_1$. If $L_2$ is obtained from $L_1$ by adding some constants, then

\[ T \models F \text{ in } L_1 \text{ iff } T \models F \text{ in } L_2 \]

for every formula $F$ over $L_1$.

We restrict our attention only to extensions which are obtained by adding denumerable number of new constants. We say that a first order language $L_c$ is a canonical language for a language $L$ if $L_c$ is obtained by adding denumerable number of new constants to $L$.

**Proposition 2.8** Let $L_c$ be a canonical language for $L$ and $H$ a Herbrand pre-interpretation for $L_c$. Then

\[ H \models F \text{ iff } EQ_L \models F \text{ in } L \]

for every equational formula $F$ over $L$.

**Proof:** Lemma 2.7 implies that it is sufficient to show that

\[ H \models F \text{ iff } EQ_L \models F \text{ in } L_c \]  \hspace{1cm} (3)

holds for every equational formula $F$ over $L$. The proof of IF part in (3) is based on the following claim:

\[ \text{if } H \models F, \text{ then } EQ_L \models F \text{ in } L_c \]  \hspace{1cm} (4)

for every closed equational formula $F$ over $L_c$. We prove (4) by induction on closed formulas.

If $H \models s = t$, where $s$ and $t$ are ground terms, then $s \equiv t$. Thus $EQ_L \models s = t$ in $L_c$. By a straightforward application of the induction hypothesis we prove (4) for a formula $F$ of the form $F \equiv \neg G$, $F \equiv G \land H$, $F \equiv G \lor H$, $F \equiv G \rightarrow H$ or $F \equiv G \leftrightarrow H$, respectively.

Suppose $F \equiv (\exists x)G$. If $H \models F$, then $H \models G_x[s]$ for some ground term $s$. By induction hypothesis we have $EQ_L \models G_x[s]$ in $L_c$. Since $G_x[s] \rightarrow (\exists x)G$ is valid, we have $EQ_L \models F$ in $L_c$.

Assume finally that $F \equiv (\forall x)G$. If $H \models F$, then $H \models G_x[s]$ for any ground term $s$. By induction hypothesis we have for any $s \in U^H$ that $EQ_L \models G_x[s]$ in $L_c$. If $s$ is a constant $a$ occurring nor in $EQ_L$ and nor in $G$, then by Theorem 2.2 we have $EQ_L \models G$ in $L_c$ and therefore $EQ_L \models F$ in $L_c$. 

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Now let $F$ be arbitrary (not necessarily closed) equational formula over $L$. Consider distinct $a_1, \ldots, a_n$ constants not occurring in $EQ_L$ and $F$. If $H \models F[a_1, \ldots, a_n]$ and hence by (4) we have $EQ_L \models F[a_1, \ldots, a_n]$ in $L_c$. Theorem 2.2 implies that $EQ_L \models F$ in $L_c$. This concludes the proof of IF part in (3).

The proof of ONLY-IF part in (3) is straightforward. □

3 Solving queries

In this section we present an algorithm, called Solved Form Algorithm, which transforms any query into an equivalent one in solved form. We follow here the presentation of [3].

**Solved Form Algorithm** For a given query $Q$ non-deterministically apply the following elementary steps (1) - (12). We write $(\exists y)$ instead of $(\exists y_1) \ldots (\exists y_k)$, if $y$ is the sequence $y_1, \ldots, y_k$.

The first group of elementary actions is determined by the form of a selected equation in $Q$.

1. $f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$ replace by $s_1 = t_1 \land \ldots \land s_n = t_n$
2. $f(s_1, \ldots, s_n) = g(t_1, \ldots, t_m)$ replace by False, if $f$ and $g$ are distinct symbols
3. $x = t$ replace by False, if $x$ and $t$ are distinct terms such that $x$ occurs in $t$
4. $x = x$ replace by True

The following two actions eliminate a variable $x$ if the selected equation is of the form $x = t$, where $x \not\equiv t$ and $x$ does not occur in $t$. We suppose that $x = t$ is "surrounded" only by atoms and equations i.e. there is a subquery $(\exists y)(Q' \land x = t \land Q'')$ of the query $Q$, where $Q'$ and $Q''$ are conjunctions only of equations and atoms. The third action redirects $t = x$ according the form of $t$.

5. $(\exists y)(Q' \land x = t \land Q'')$ replace by $(\exists y)(Q_x'[t] \land x = t \land Q_x''[t])$, if $x$ is not in $y$ and it has another (free) occurrence in $Q'$ or in $Q''$
6. $(\exists y)(Q' \land x = t \land Q'')$ replace by $(\exists y)(Q_x'[t] \land True \land Q_x''[t])$, if $x$ is in $y$
7. $(\exists y)(Q' \land t = x \land Q'')$ replace by $(\exists y)(Q' \land x = t \land Q'')$, if (i) $t$ is not a variable or (ii) $t$ is a distinct variable from $x$ not occurring in $y$ and $x$ is in $y$

The operations for eliminating quantifiers are defined as follows:

8. $(\exists y')(\exists y)(\exists y'')(Q$ replace by $(\exists y')(\exists y'')(Q$, if $y$ is not free in $(\exists y'')Q$
(9) \( (\exists \bar{y}_1)Q_1 \land (\exists \bar{y}_2)Q_2 \) replace by \( (\exists \bar{z}_1)(\exists \bar{z}_2)(R_1 \land R_2) \), where \( (\exists \bar{z}_i)R_i \) is a variant of \( (\exists \bar{y}_i)Q_i \), for \( i = 1, 2 \) such that \( \text{vars}(R_1) \cap \bar{z}_2 = \emptyset \) and \( \text{vars}(R_2) \cap \bar{z}_1 = \emptyset \); we suppose that \( \bar{y}_1 \neq \emptyset \) or \( \bar{y}_2 \neq \emptyset \)

Finally, we have:

(10) \( A \land s = t \) replace by \( s = t \land A \), where \( A \) is an atom.

(11) delete any occurrence of the propositional constant True

(12) replace \( Q \) by False, if \( Q \) obtains at least one occurrence of the propositional constant False

The algorithm terminates with \( Q' \) as the output when no step can be applied to \( Q' \) or when False has been returned. We write \( Q \rightarrow Q' \) if \( Q' \) can be obtained from \( Q \) by one step. By \( \rightarrow^* \) we mean the reflexive and transitive closure of \( \rightarrow \).

The following theorem establishes the correctness and the termination of the solved form algorithm.

**Theorem 3.1** The solved form algorithm applied to a query will return an equivalent one in solved form after finite number of steps.

**Proof:** We only outlined the proof (see [3] for details). **Correctness.** Let \( Q' \) is obtained from \( Q \) using the step (1) – (5) or (7) – (12). Clearly \( Q' \approx Q \). The correctness of the step (6) is a trivial consequence of the following claim: if \( x \) does not occur in \( s \), then \( (\exists x)(x = s \land F) \leftrightarrow F_x[s] \) is valid formula (see [5]). **Termination.** Straightforward by using the same arguments as in [3]. □

## 4 The lattice of queries

In this section we investigate lattice–theoretic properties of queries. In particular, we will see in Theorem 4.8 that the relationship between two equivalent queries in solved form is very strong. The theorem states also that \( \preceq_J \equiv \preceq \) and \( \approx_J \equiv \approx \) provided that \( J \) is a non-trivial model for \( EQ_L \). Finally we show that the preorder \( \preceq \) is well-founded (see Theorem 4.9) i.e. there is no infinite increasing sequence \( Q_0 \prec Q_1 \prec Q_2 \prec \ldots \) of queries. We follow here the presentation of [3].

To this purpose we first show how from a given query in solved form it is possible to obtain new queries again in solved form. Consider a query

\[
Q \equiv (\exists z_1)\ldots(\exists z_m)(x_1 = s_1 \land \ldots x_n = s_n \land A_1 \land \ldots A_q)
\]

in solved form. A query \( Q' \) is obtained from \( Q \) (a) by permutation of equations and bound variables, (b) by renaming of bound variables or (c) by redirecting equations if
queries in solved form. We say that $Q'$ is in solved form. Let $Q'$ be consistent queries in solved form. If $Q \preceq_J Q'$, then $|\text{elim}(Q')| \geq |\text{elim}(Q)|$.

**Proof:** To prove the claim we will establish an injective mapping from $\text{elim}(Q')$ into $\text{elim}(Q)$. If $x$ belongs to both sets then $x$ is assigned to itself. Let $x \in \text{elim}(Q') \setminus \text{elim}(Q)$. Then there is an equation $x = t$ in $Q'$. As $x \notin \text{elim}(Q)$, the term $t$ must be a variable $y$. Really, suppose $t$ is not a variable. Put $\mathcal{T} = B_L^1, \ldots, B_L^j$, where $|\mathcal{T}| = |Q| = |Q'|$. Let $d$ be an element from $U_L^j$ distinct from all $J$-instances of the term $t$. Then there is a solution $h$ of $Q$ in $\mathcal{T}$ such that $h(x) = d$. Clearly $h \notin [Q']_I^j$. As $Q \preceq_J Q'$, this is impossible. Further $y \in \text{elim}(Q)$, since we could assign to $x$ and $y$ distinct $J$-values to obtain a solution of $Q$ in $\mathcal{T}$, but this is not possible since the equation $x = y$ appears in $Q'$. Moreover $y \notin \text{elim}(Q')$. We assign $y$ to $x$. So elements from $\text{elim}(Q') \setminus \text{elim}(Q)$ are mapped onto elements from the set $\text{elim}(Q) \setminus \text{elim}(Q')$.

The mapping so constructed is one-to-one. Consider different variables $x_1$ and $x_2$ from $\text{elim}(Q') \setminus \text{elim}(Q)$. If the same variable $y$ is assigned to both variables then we would have $x_1 = y$ and $x_2 = y$ in $Q'$. Consequently $x_1$ and $x_2$ would be bound to take
the same $J$-values. As both do not occur in $\text{elim}(Q)$, this is impossible. So we proved $|\text{elim}(Q)| \geq |\text{elim}(Q')|$. □

We can strengthen Lemma 4.2. Let $x \in \text{elim}(Q') \setminus \text{elim}(Q)$ and $T$ as before. From the proof above we know that if $x = y$ is an equation in $Q'$, then there is an equation $y = s$ in $Q$ for some $s$. The term $s$ must be the variable $x$, since otherwise there is a solution $h$ of $Q$ in $T$ such that $h(x) \neq h(y)$. But this is impossible since $x$ and $y$ are bound in $Q'$ to take the same $J$-value. So if $\{x_1, \ldots, x_k\} = \text{elim}(Q') \setminus \text{elim}(Q)$ then for each equation $x_i = y_i$ in $Q'$ we have an equation $y_i = x_i$ in $Q$. If we put

$$Q'' = Q_{x_1, \ldots, x_k, y_1, \ldots, y_k, y_1, \ldots, y_k, x_1, \ldots, x_k},$$

then $Q''$ is isomorphic to $Q$ such that $\text{elim}(Q'') \supseteq \text{elim}(Q')$. So we proved the next lemma.

**Lemma 4.3** Let $Q$ and $Q'$ be consistent queries in solved form. If $Q \preceq_J Q'$, then there is a query $Q''$ in solved form isomorphic to $Q$ such that $\text{elim}(Q'') \supseteq \text{elim}(Q')$.

**Example 4.1** Since $(x = f(y)) \prec (\exists v)(x = f(v))$, it is possible that $Q \prec Q'$ and $\text{elim}(Q) = \text{elim}(Q')$.

**Lemma 4.4** Let $Q$ and $Q'$ be consistent queries in solved form. If $Q \preceq_J Q'$, then $\text{vars}(Q) \supseteq \text{vars}(Q')$.

**Proof:** By Lemma 4.3 we can suppose that $\text{elim}(Q) \supseteq \text{elim}(Q')$. Assume that there is $y \in \text{vars}(Q') \setminus \text{vars}(Q)$. Then $y$ is a parameter of $Q'$ and hence there is an equation $x = t$ in $Q'$, where $y$ appears in $t$. Put $T = B^1, \ldots, B^i$, where $|T| = |Q| = |Q'|$. Then there are solutions $h_1$ and $h_2$ of $Q$ in $T$ which differs only on $y$. Hence $h_1$ and $h_2$ are solutions of $Q'$ in $T$, as well. As $h_1(y) \neq h_2(y)$, we obtain that $h_1(x) \neq h_2(x)$. Contradiction. □

**Example 4.2** Since $(\exists u)(x = f(u, u)) \prec (\exists v_1)(\exists v_2)(x = f(v_1, v_2))$, it is possible that $Q \prec Q'$ and $\text{vars}(Q) = \text{vars}(Q')$.

Consider now two consistent queries $Q$ and $Q'$ in solved form and suppose that $Q \equiv (\exists u_1) \ldots (\exists u_k)Q_0$ and $Q' \equiv (\exists v_1) \ldots (\exists v_l)Q'_0$, where

$$Q_0 \equiv (x_1 = s_1 \land \ldots x_n = s_n \land A_1 \land \ldots A_m)$$

and

$$Q'_0 \equiv (x_1 = t_1 \land \ldots x_n = t_n \land B_1 \land \ldots B_m).$$

Put

$$\text{Diff}(Q, Q') =_{\text{def}} \bigcup_{i=1}^n \text{Diff}(s_i, t_i) \cup \bigcup_{j=1}^m \text{Diff}(A_j, B_j) \quad (5)$$

Then the following lemma holds.
Lemma 4.5 If $Q \preceq_J Q'$, then the set $\text{Diff}(Q, Q')$ contains only pairs of the form

1. $(y, y)$, where $y$ is a parameter of both $Q$ and $Q'$.

2. $(s, v)$, where $v$ is a bound variable in $Q'$.

Moreover the following uniqueness condition holds: for any variable $z$ there is at most one term $s$ such that $(s, z) \in \text{Diff}(Q, Q')$.

Proof: Notice first that $\text{Diff}(Q, Q')$ has the following property. Consider $g, h \in V^J_L$, where $h = g$ on $\text{vars}(Q')$. Let $\overline{T}(g)$ be a multinterpreration $g(A_1), \ldots, g(A_m)$ based on $J$. If $g \in [Q_0]_{\overline{T}(g)}$ and $h \in [Q_0]_{\overline{T}(g)}$, then for any pair $(s, t) \in \text{Diff}(Q, Q')$ we have $g(s) = h(t)$. Really, assume first that $(s, t) \in \text{Diff}(s_i, t_i)$. Then $g(x_i) = h(x_i)$ and hence $g(s_i) = h(t_i)$. Therefore $g(s) = h(t)$ by Lemma 2.6. Suppose now that $(s, t) \in \text{Diff}(A_j, B_j)$. Then by the form of $\overline{T}(g)$ we have $g(A_j) = h(B_j)$. Consequently $g(s) = h(t)$.

Now let $(s, t) \in \text{Diff}(Q, Q')$. Then $t$ must be a variable, since otherwise there is a $J$-solution $g$ of $x_1 = s_1 \land \ldots \land x_n = s_n$ such that $g(s) \notin \text{inst}_J(t)$. Put $\overline{T}(g) = g(A_1), \ldots, g(A_m)$. Then $g \in [Q_0]_{\overline{T}(g)}$. Since $Q \preceq_J Q'$, there is a solution $h$ of $Q_0$ in $\overline{T}(g)$ identical to $g$ on $\text{vars}(Q')$. Then $g(s) = h(t)$. Contradiction. So $\text{Diff}(Q, Q')$ may contain only pairs of the form $(s, z)$, where $z$ is a variable.

Assume now that $(s, y) \in \text{Diff}(Q, Q')$, where $y$ is a parameter of $Q'$. Then by Lemma 4.4 $y$ is a parameter of $Q$, as well. We shall prove that $s$ must be the variable $y$. Suppose that $s \neq y$. By similar arguments as in the previous case we find $g \in V^J_L$ such that $g(s) \neq g(y)$ and $Q$ is a solution of $Q_0$ in $\overline{T}(g)$, where $\overline{T}(g) = g(A_1), \ldots, g(A_m)$. Therefore there is $h \in [Q_0]_{\overline{T}(g)}$ such that $h = g$ on $\text{vars}(Q')$. We have $g(s) = h(y)$. But this is impossible since $h(y) = g(y)$ holds.

Let $v$ be a bound variable in $Q'$. We prove that there is just one term $s$ such that $(s, v) \in \text{Diff}(Q, Q')$. Suppose that $(r_1, v) \in \text{Diff}(Q, Q')$ and $(r_2, v) \in \text{Diff}(Q, Q')$, where $r_1$ and $r_2$ are distinct terms. Then we can find $g \in V^J_L$ such that $g(r_1) \neq g(r_2)$ and $g \in [Q_0]_{\overline{T}(g)}$, where $\overline{T}(g) = g(A_1), \ldots, g(A_m)$. Therefore there is $h \in [Q_0]_{\overline{T}(g)}$ such that $h = g$ on $\text{vars}(Q')$. Thus $g(r_1) = h(v)$ and $g(r_2) = h(v)$. Contradiction. □

As a consequence of previous lemma we have the following two claims.

Lemma 4.6 If $Q \approx_J Q'$ in Lemma 4.5, then $Q$ and $Q'$ are isomorphic.

Proof: Straightforward.

Lemma 4.7 If $Q \prec_J Q'$ in Lemma 4.5, then one of the following cases should appear:

1. $(s, v) \in \text{Diff}(Q, Q')$, where $s$ is a nonvariable term and $v \in \text{bound}(Q')$.

2. $(y, v) \in \text{Diff}(Q, Q')$, where $y$ is a parameter of $Q$ and $v \in \text{bound}(Q')$. 

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3. \((u,v_1) \in \text{Diff}(Q,Q')\) and \((u,v_2) \in \text{Diff}(Q,Q')\), where \(u \in \text{bound}(Q)\), \(v_1 \in \text{bound}(Q')\), \(v_2 \in \text{bound}(Q')\) and \(v_1 \neq v_2\).

**Proof:** Straightforward. □

The following example shows that each case in Lemma 4.7 may appear.

**Example 4.3**

\[
(x = f(a)) \prec (\exists v)(x = f(v))
\]
\[
(x = f(y)) \prec (\exists v)(x = f(v))
\]
\[
(\exists u)(x = f(u,u)) \prec (\exists v_1)(\exists v_2)(x = f(v_1,v_2))
\]

The main aim of this section is to prove the following claim.

**Theorem 4.8** Let \(Q\) and \(Q'\) be queries. If \(J\) is a non-trivial model of \(EQL\), then:

1. \(Q \approx_J Q'\) iff \(Q \approx Q'\)
2. \(Q \preceq_J Q'\) iff \(Q \preceq Q'\)

In particular if \(Q\) and \(Q'\) are equivalent queries in solved form, then they are isomorphic.

**Proof:** Suppose first that \(Q\) and \(Q'\) are queries in solved form and \(Q \approx_J Q'\). By a straightforward application of the previous lemmas we obtain that \(Q\) and \(Q'\) are isomorphic and hence equivalent.

Assume now \(Q \approx_J Q'\). Theorem 3.1 implies that there are queries \(Q_1\) and \(Q'_1\) in solved form equivalent to \(Q\) and \(Q'\), respectively. Then \(Q_1 \approx_J Q'_1\) and hence \(Q_1 \approx Q'_1\). Thus \(Q \approx Q'\).

Let \(Q \preceq_J Q'\). Then \((Q \land Q') \approx_J (Q \land Q)\) and hence by 1) \((Q \land Q') \approx (Q \land Q)\). Consequently \(Q \preceq Q'\). □

Finally we establish the following result: the preorder \(\preceq\) is well-founded i.e. there is no infinite increasing sequence \(Q_0 \prec Q_1 \prec Q_2 \prec \ldots\) of queries.

**Proposition 4.9** Any consistent query has a finite number of generalizations, modulo \(\approx\).

**Proof:** Consider two consistent query \(Q\) and \(Q'\) in solved form and suppose that \(Q \preceq Q'\), \(Q \equiv (\exists x)Q_0\) and \(Q' \equiv (\exists x)Q'_0\), where

\[
Q_0 \equiv (x_1 = s_1 \land \ldots x_n = s_n \land A_1 \land \ldots \land A_m)
\] (6)
and

\[ Q'_0 \equiv (x_1 = t_1 \land \ldots x_k = s_k \land B_1 \land \ldots B_m) \]  

(7)

By Lemma 4.2 we have \( n \geq k \). Put as in (5)

\[ \text{Diff}(Q, Q') = \bigcup_{i=1}^k \text{Diff}(s_i, t_i) \cup \bigcup_{j=1}^m \text{Diff}(A_j, B_j) \]

We can prove as in Lemma 4.5 that \( \text{Diff}(Q_0, Q'_0) \) contains only pairs of the form \((r, z)\), where \( z \) is a variable, and the uniqueness condition holds i.e. for any \( z \) there is at most one \( r \) such that \((r, z) \in \text{Diff}(Q_0, Q'_0)\). Really, choose arbitrary but fixed non-trivial model \( J \) of \( \text{EQ}_L \). By similar arguments as in the proof of Lemma 4.5 it is possible to show that (i) if \((s, t) \in \text{Diff}(Q_0, Q'_0)\), then \( t \) must be a variable and (ii) if \((r_1, z) \in \text{Diff}(Q_0, Q'_0)\) and \((r_2, z) \in \text{Diff}(Q_0, Q'_0)\), then \( r_1 \) and \( r_2 \) are the same terms.

Now let \( Q \) be a given consistent query and \( Q \preceq Q' \). To prove the claim we can suppose that \( Q' \) is not equivalent to \( \text{True} \). Then there are queries \( Q_1 \) and \( Q'_1 \) in solved form such that (i) \( Q_1 \) is equivalent to \( Q \), (ii) \( Q'_1 \) is equivalent to \( Q' \) (iii) \( Q_1 \equiv (\exists \pi)Q_0 \), where \( Q_0 \) is of the form (6) and (iv) \( Q'_1 \equiv (\exists \pi)Q'_0 \), where \( Q'_0 \) is of the form (7). Then the size of terms in \( Q'_1 \) is bounded by the size of corresponding terms in \( Q_1 \) and only the function symbols of \( Q_1 \) can appear in \( Q'_1 \). It follows that there are only a finite, modulo \( \approx \), number of generalizations of \( Q \) in solved form and hence \( Q \) has a finite, modulo \( \approx \), number of generalizations. \( \Box \)

5 Finite substitutions and existentially quantified systems of equations

In this section we shall deal with finite substitutions and empty queries. Recall that empty queries are formulas constructed from propositional constants and equations using the conjunction \( \land \) and the existential quantifier \( \exists \). Actually, empty queries correspond to existentially quantified systems of equations. From now such formulas will be called \( E \)-formulas.

We shall study two lattices: the lattice of \( E \)-formulas and the lattice of finite substitutions. We prove that the both lattices are isomorphic (see Theorem 5.13). Finally, we introduce the notion of application of substitutions to formulas and clarify its relationship to \( E \)-formulas (see Theorem 5.20).

5.1 Lattice of \( E \)-formulas

Now we restrict our attention only to \( E \)-formulas. We shall use \( E \) as a syntactical variable which varies through \( E \)-formulas. The set of \( E \)-formulas (over \( L \)) will be denoted by \( \text{Eqn}_L \).
Consider a pre-interpretation \( J \). By the definition of the preorder \( \preceq_J (\preceq) \) and the equivalence \( \approx_J (\approx) \) we have:

(a) \( E \preceq_J E' \) iff \( \text{soln}_J(E) \subseteq \text{soln}_J(E') \) and \( E \approx_J E' \) iff \( \text{soln}_J(E) = \text{soln}_J(E') \)

(b) \( E \preceq E' \) iff \( EQ_L \models E \rightarrow E' \) and \( E \approx E' \) iff \( EQ_L \models E \leftrightarrow E' \)

As a straightforward corollary of Theorem 4.8 and Theorem 3.1 we obtain the following claims.

**Theorem 5.1** Let \( E \) and \( E' \) be \( \mathcal{E} \)-formulas. If \( J \) is a non-trivial model of \( EQ_L \), then:

1. \( E \approx_J E' \) iff \( E \approx E' \)
2. \( E \preceq_J E' \) iff \( E \preceq E' \)

In particular if \( E \) and \( E' \) are equivalent \( \mathcal{E} \)-formulas in solved form, then they are isomorphic.

**Theorem 5.2** The solved form algorithm applied to an \( \mathcal{E} \)-formula will return an equivalent one in solved form after finite number of steps.

Remember that consistent \( \mathcal{E} \)-formulas in solved form have the same set of free variables. Thus for a given consistent \( \mathcal{E} \)-formula \( E \) there is a unique set of variables, which is the set of free variables of its arbitrary solved form. We shall denote this set as \( \text{kernel}(E) \).

Let \( Eqn_L/\approx \) be the quotient set of \( \mathcal{E} \)-formulas under the equivalence \( \approx \). By \( E_{\approx} \) we mean the class of \( \mathcal{E} \)-formulas equivalent with \( E \). Let \( \preceq \) be the partial order obtained from the preorder \( \preceq \). Then:

**Theorem 5.3** \( Eqn_L/\approx \) is a complete lattice, where the smallest element is \( \text{False}_{\approx} \) and the greatest element is \( \text{True}_{\approx} \).

**Proof:** First notice that \( E \land E' \) is a greatest lower bound of the set \( \{ E, E' \} \). Therefore each nonempty finite set of \( \mathcal{E} \)-formulas has a greatest lower bound. Now let \( S \) be a nonempty (possible infinite) set of \( \mathcal{E} \)-formulas. Then from Proposition 4.9 the set \( Z \) of upper bounds of \( S \) is nonempty and finite. If \( Z = \{ E_1, \ldots, E_n \} \), then \( E_1 \land \ldots \land E_n \) is a lower upper bound of \( S \). A greatest lower bound of \( S \) we obtain as a lower upper bound of the set of lower bounds of \( S \). \( \Box \)

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Now we introduce the notion of projection for $E$-formulas. Let $E \neq \text{False}$ be an $E$-formula and $X$ a finite set of variables. Let $\{z_1, \ldots, z_k\} = \text{vars}(E) \setminus X$, where $z_1, \ldots, z_k$ occur in $\text{Var}$ in this order. The projection of $E$ onto $X$, denoted by $E|X$, is defined as the $E$-formula $(\exists z_1) \ldots (\exists z_k)E$. We put $\text{False}|X = \text{False}$. By Lemma 2.3 $E \rightarrow E|X$ is valid and hence $E \models E|X$.

**Lemma 5.4** Let $E$ and $E'$ be $E$-formulas and $X$ a finite set of variables. Then:

1. If $E \models E'$, then $E|X \models E'|X$.

2. If $E \models E'$, then $E|X \models E'|X$.

**Proof:** Without loss of generality we can suppose that $E$ and $E'$ are consistent $E$-formulas. We first prove 2). Let $E \approx E'$. We can suppose that $E'$ is in solved form. Then $\text{vars}(E') \subseteq \text{vars}(E)$. Let $\{z_1, \ldots, z_k\} = \text{vars}(E') \setminus X$ and $\{z_1, \ldots, z_k, \ldots, z_n\} = \text{vars}(E) \setminus X$. Then

$$\{z_{k+1}, \ldots, z_n\} \cap \text{vars}(E') = \emptyset$$

(8)

By Lemma 2.3 $E \rightarrow (\exists z_{k+1}) \ldots (\exists z_n)E$ is valid and since $E|_L \models E' \rightarrow E$ we obtain $E|_L \models E' \rightarrow (\exists z_{k+1}) \ldots (\exists z_n)E$. Because $E|_L \models E \rightarrow E'$, then by Lemma 2.3 $E|_L \models (\exists z_{k+1}) \ldots (\exists z_n)E \rightarrow E$ since (8) holds. Therefore $E' \approx (\exists z_{k+1}) \ldots (\exists z_n)E$. By applying Lemma 2.3 again we have

$$E|_L \models (\exists z_1) \ldots (\exists z_k)E' \leftrightarrow (\exists z_1) \ldots (\exists z_k) \ldots (\exists z_n)E$$

i.e. $E'|X \models E|X$.

Now assume that $E \leftrightarrow E'$. By 2) we can suppose that $E$ and $E'$ are in solved form. Then $\text{vars}(E') \subseteq \text{vars}(E)$. By similar arguments as in the proof above we can show that $E|X \models E'|X$ holds. □

**Lemma 5.5** Consider arbitrary formula $F$ and a consistent $E$-formula $E$. Let $E'$ be a projection of $E$ onto $\text{vars}(F)$. Then $(\forall)(E \rightarrow F) \leftrightarrow (\forall)(E' \rightarrow F)$ is a valid formula.

**Proof:** Let $I$ be an interpretation based on a pre-interpretation $J$. Suppose that $I \models (\forall)(E \rightarrow F)$ or equivalently $\text{soln}_J(E) \subseteq \langle F \rangle^I_J$. Let $h \in \text{soln}_J(E')$. Since $E' \equiv (\exists z_1) \ldots (\exists z_k)E$, where $\{z_1, \ldots, z_k\} = \text{vars}(E) \setminus \text{vars}(F)$, then there is $g \in \text{soln}_J(E)$ identical to $h$ on $\text{Var} \setminus \{z_1, \ldots, z_k\}$. By assumptions $g \in \langle F \rangle^I_J$. Since $g = h$ on $\text{vars}(F)$, we have $h \in \langle F \rangle^I_J$. We proved that $(\forall)(E \rightarrow F) \rightarrow (\forall)(E' \rightarrow F)$ is a valid formula.

Since $E \rightarrow E'$ is a valid formula, then $(\forall)(E' \rightarrow F) \rightarrow (\forall)(E \rightarrow F)$ is a valid formula, as well. □
As in [3] we establish the so-called compactness theorem. We need to extend the definition of \( \preceq \) and \( \approx \) to equational formulas. Let \( F \) and \( F' \) be equational formulas (over \( L \)). Then:

(a) \( F \preceq F' \) iff \( EQ_L \models F \rightarrow F' \)

(b) \( F \approx F' \) iff \( EQ_L \models F \leftrightarrow F' \)

We shall write \( F \prec F' \) if \( F \preceq F' \) and \( F \not\approx F' \).

**Theorem 5.6 (Strong Compactness)** Let \( E \) and \( E_1, \ldots, E_n \) be \( \mathcal{E} \)-formulas.

1. If \( E_1 \prec E, \ldots, E_n \prec E \), then \( E_1 \lor \ldots \lor E_n \preceq E \).

2. If \( E \approx E_1 \lor \ldots \lor E_n \), then \( E \approx E_j \) for some \( E_j \).

3. If \( E \preceq E_1 \lor \ldots \lor E_n \), then \( E \preceq E_j \) for some \( E_j \).

First we prove the following proposition. Here

\[
F \preceq_f F' \text{ iff } J \models F \rightarrow F' \text{ and } F \approx F' \text{ iff } J \models F \leftrightarrow F',
\]

where \( J \) is a pre-interpretation and \( F, F' \) are equational formulas.

**Proposition 5.7** Suppose that \( L \) contains infinitely many constants and \( H \) is the Herbrand pre-interpretation for \( L \). Let \( E \) and \( E_1, \ldots, E_n \) be \( \mathcal{E} \)-formulas over \( L \).

1. If \( E_1 \prec_H E, \ldots, E_n \prec_H E \), then \( E_1 \lor \ldots \lor E_n \prec_H E \).

2. If \( E \approx_H E_1 \lor \ldots \lor E_n \), then \( E \approx_H E_j \) for some \( E_j \).

3. If \( E \preceq_H E_1 \lor \ldots \lor E_n \), then \( E \preceq_H E_j \) for some \( E_j \).

The assumption that \( L \) contains infinitely many constants is essential. Consider an alphabet \( L \) consisting only from a constant \( a \) and an unary function symbol \( f \). Then we have \( (x = a) \prec_H \text{True} \) and \((\exists z)(x = f(z)) \prec_H \text{True} \). On the other side \(((x = a) \lor (\exists z)(x = f(z))) \approx_H \text{True} \).

**Proof of Proposition 5.7:** Let \( E_1 \prec_H E, \ldots, E_n \prec_H E \). We can suppose without loss of generality (see Section 4) that \( E_1, \ldots, E_n, E \) are consistent \( \mathcal{E} \)-formulas in solved form such that \( \text{elim}(E_1) \supseteq \text{elim}(E), \ldots, \text{elim}(E_n) \supseteq \text{elim}(E) \) and that any bound variable in \( E \) does not occur freely in \( E_1, \ldots, E_n \). Put

\[
X = \text{bound}(E) \cup \bigcup_{i=1}^{n} (\text{vars}(E_i) \setminus \text{elim}(E))
\]
Note that \( \text{param}(E) \subseteq X \), \( \text{param}(E_1) \subseteq X \), \ldots, \( \text{param}(E_n) \subseteq X \). Moreover \( \text{elim}(E) \cap X = \emptyset \). Let \( X = \{x_1, \ldots, x_k\} \) and \( a_1, \ldots, a_k \) be distinct constants not occurring in \( E_1, \ldots, E_n \). Let \( E \) be of the form \((\exists \forall)D\). Then there is a \( H \)-solution \( h \) of \( D \) (and hence of \( E \) as well) such that \( h(x_1) = a_1, \ldots, h(x_k) = a_k \). To prove 1) it suffices to show that \( h \) is not a \( H \)-solution of any \( E_i \). Really, let \( E_i \) be of the form \((\exists \forall)D_i\) and suppose that there is \( g \in \text{soln}_H(D_i) \) such that \( g = h \) on \( \text{vars}(E_i) \). We show that this is impossible. There are two cases to consider.

Suppose that \( |\text{elim}(E_i)| > |\text{elim}(E)| \). Then there is an equation \( x = s \) in \( E_i \) such that \( x \notin \text{elim}(E) \). We have \( x \in X \) and \( g(x) = h(x) \). If \( s \) is not a variable, then \( g(x) \notin \text{inst}_H(s) \) and therefore \( g \notin \text{soln}_H(D_i) \). If \( s \) is a variable \( z \) then \( z \in X \). Hence \( g \notin \text{soln}_H(D_i) \), since \( x \neq z \).

Assume now that \( |\text{elim}(E_i)| = |\text{elim}(E)| \). We can suppose that \( D \) and \( D_i \) are of the form

\[
D \equiv (x_1 = t_1 \land \ldots \land x_m = t_m) \quad \text{and} \quad D_i \equiv (x_1 = s_1 \land \ldots \land x_m = s_m)
\]

Consider \( \text{Diff}(E_i, E) \) introducing in (5). If \( (s, t) \in \text{Diff}(E_i, E) \), then \( g(s) = h(t) \) (see the proof of Lemma 4.5). We show that this is impossible. According to Lemma 4.7 there are three cases to consider:

(i) \( (r, v) \in \text{Diff}(E_i, E) \), where \( r \) is a nonvariable term and \( v \in \text{bound}(E) \). Then \( v \in X \) and hence \( g(r) \neq h(v) \). Contradiction.

(ii) \( (y, v) \in \text{Diff}(E_i, E) \), where \( y \in \text{param}(E_i) \) and \( v \in \text{bound}(E) \). Then \( v \in X \), \( y \in X \) and hence \( g(y) \neq h(v) \). Contradiction.

(iii) \( (u, v_1) \in \text{Diff}(E_i, E) \) and \( (u, v_2) \in \text{Diff}(E_i, E) \), where \( v_1 \) and \( v_2 \) are distinct bound variables in \( E \). From \( g(u) = h(v_1) \) and \( g(u) = h(v_2) \) we have \( h(v_1) = h(v_2) \). But this is impossible since \( v_1 \in X \) and \( v_2 \in X \).

So we proved that if \( E_1 \prec_H E, \ldots, E_n \prec_H E \), then \( E_1 \lor \ldots \lor E_n \prec_H E \).

Clearly the second part of the claim immediately follows from the first. The third part is a trivial consequence from the second by noting that the hypothesis implies that \( E \approx_H (E \land E_1) \lor \ldots \lor (E \land E_n) \). \( \square \)

Now we are ready to prove Theorem 5.6.

**Proof of Theorem 5.6:** Let \( F \) and \( F' \) be equational formulas over \( L \). Let \( L_c \) be a canonical language for \( L \) and \( H \) a Herbrand pre-interpretation for \( L_c \). Then by Proposition 2.8 we have:

\[
F \prec F' \iff F \prec_H F'
\]  

(9)

Assume now that \( E_1 \prec E, \ldots, E_n \prec E \). Then by (9) we have \( E_1 \prec_H E, \ldots, E_n \prec_H E \).

Proposition 5.7 yields that \( E_1 \lor \ldots \lor E_n \prec_H E \). By applying (9) again we obtain \( E_1 \lor \ldots \lor E_n \prec E \). The proof of the second and the third part of the claim is similar to the proof of Proposition 5.7. \( \square \)
5.2 Lattice of Substitutions

In this section we introduce the lattice of (finite) substitutions. Theorem 5.13 states that the quotient set of substitutions under an equivalence forms a complete lattice. Moreover, this theorem shows us that there is a strong relationship between \( E \)-formulas and substitutions. The crux of this relationship is the mapping between an \( E \)-formula

\[
E \equiv (\exists z_1) \ldots (\exists z_m)(x_1 = s_1 \land \ldots x_n = s_n),
\]

in solved form and a substitution

\[
\sigma = \{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, y_1 \leftarrow y_1, \ldots, y_k \leftarrow y_k\},
\]

where \( y_1, \ldots, y_k \) are parameters of \( E \). Actually this mapping gives us an isomorphism (up to an equivalence relation) between \( E \)-formulas and substitutions.

First we introduce the notion of \( J \)-instances of a substitution. Let \( \sigma \) be a substitution over \( X \) and \( J \) a pre-interpretation. A \( J \)-valuation \( h \) is said to be a \( J \)-instance of \( \sigma \) if there is \( g \in V_J^L \) such that \( h(x) = g(\sigma(x)) \) for all \( x \in X \). The set of all \( J \)-instances of \( \sigma \) is denoted by \( \text{inst}_J(\sigma) \).

\( J \)-instances of substitutions induce a preorder \( \preceq_J \) and an equivalence \( \approx_J \) as follows.

Consider two substitutions \( \sigma \) and \( \theta \) and let \( J \) be a pre-interpretation. Then:

- \( \theta \) is said to be more general w.r.t. \( J \) than \( \sigma \), written as \( \sigma \preceq_J \theta \), if \( \text{inst}_J(\sigma) \subseteq \text{inst}_J(\theta) \)
- \( \theta \) is said to be equivalent w.r.t. \( J \) (or \( J \)-equivalent for short) to \( \sigma \), written as \( \sigma \approx_J \theta \), if \( \text{inst}_J(\sigma) = \text{inst}_J(\theta) \)

Note that if \( \sigma \) and \( \theta \) are variants, then they are \( J \)-equivalent. Further if \( \theta \) is an extension of \( \sigma \), then \( \text{inst}_J(\theta) \subseteq \text{inst}_J(\sigma) \). Proposition 5.8 gives a syntactic characteristic of extensions of a given substitution \( \sigma \) which have the same set of \( J \)-instances as \( \sigma \).

Definition 5.1 A substitution \( \theta \) over \( Y \) is a regular extension of a substitution \( \sigma \) over \( X \) if (i) \( \theta \) is an extension of \( \sigma \) (ii) \( \theta \) maps the set \( X \setminus \text{dom}(\sigma) \) injectively into the set \( \text{Var} \setminus \text{range}(\sigma) \) of variables.

Proposition 5.8 Let \( \theta \) be an extension of \( \sigma \). If \( J \) is a non-trivial model of \( EQ_L \), then:

\( \theta \approx_J \sigma \) iff \( \theta \) is a regular extension of \( \sigma \)

Proof: Suppose first that \( \theta \) is a regular extension of \( \sigma \). Let \( h \in \text{inst}_J(\sigma) \). Then by the definition there is \( g \in V_J^L \) such that \( h(x) = g(\sigma(x)) \) for each \( x \in \text{dom}(\sigma) \). Put

\[
g'(y) = \begin{cases} 
g(\theta^{-1}(y)) & \text{if } y \in \text{range}(\theta) \setminus \text{range}(\sigma) \\
g(y) & \text{otherwise}
\end{cases}
\]

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The definition of \( g' \) is correct since \( \theta \) is a regular extension of \( \sigma \). We have \( h(x) = g'(\theta(x)) \) for each \( x \in \text{dom}(\theta) \). Hence \( \text{inst}_J(\sigma) \subseteq \text{inst}_J(\theta) \) and therefore \( \sigma \approx_J \theta \).

Assume now that \( \theta \) is an extension of \( \sigma \) and \( \text{inst}_J(\sigma) \subseteq \text{inst}_J(\theta) \). We shall prove that \( \theta \) must be the regular extension of \( \sigma \). Note first that if for some \( g, g' \in V_L^J \) is \( h(x) = g(\sigma(x)) \) for all \( x \in \text{dom}(\sigma) \) and \( h(x) = g'(\theta(x)) \) for all \( x \in \text{dom}(\theta) \), then \( g'(y) = g(y) \) for all \( y \in \text{range}(\sigma) \). Now suppose that \( x \in X \), where \( X = \text{dom}(\theta) \setminus \text{dom}(\sigma) \). Then \( \theta(x) \) must be a variable, since otherwise there is a \( J \)-instance \( h \) of \( \sigma \) such that \( h(x) \not\in \text{inst}_J(\theta(x)) \). Clearly \( h \not\in \text{inst}_J(\theta) \). Contradiction. Further, since there is \( h \in \text{inst}_J(\sigma) \) such that \( h(x_1) \neq h(x_2) \) for any distinct variables \( x_1 \) and \( x_2 \) from \( X \), \( \theta \) is one-to-one on the set \( X \). Now suppose that there is \( x \in X \) and \( x' \in \text{dom}(\sigma) \) such that \( \sigma(x') \) and \( \theta(x) \) are the same variable \( y \). Consider arbitrary \( J \)-valuation \( g \). Then there is \( h \) such that \( h(z) = g(\sigma(z)) \) for all \( z \in \text{dom}(\sigma) \) and \( h(x) \neq g(y) \). By assumptions for some \( g' \), where \( h(z) = g'(\theta(z)) \) for all \( z \in \text{dom}(\theta) \), we have \( h(x) = g'(\theta(x)) = g'(y) = g(y) \). Contradiction. Finally suppose that there are variables \( x \in X \) and \( x' \in \text{dom}(\sigma) \) such that \( \theta(x) \) is a variable \( y \) which does not occur in a nonvariable term \( t = \sigma(x') \). Consider arbitrary \( g \in V_L^J \). Then there is \( h \) such that \( h(z) = g(\sigma(z)) \) for all \( z \in \text{dom}(\sigma) \) and \( h(x) = g(t) \). By assumptions for some \( g' \), where \( h(z) = g'(\theta(z)) \) for all \( z \in \text{dom}(\theta) \), we have \( g(y) = g'(y) = g'(\theta(x)) = h(x) = g(t) \). Therefore the equation \( y = t \) is \( J \)-solvable. Contradiction. So we proved that \( \theta \) is a regular extension of \( \sigma \). \( \square \)

Now we introduce standard preorder \( \preceq \) and equivalence \( \approx \) using a composition of substitutions:

- \( \theta \) is more general than \( \sigma \), written as \( \sigma \preceq \theta \), if there is \( \tau \) such that \( \sigma' = \theta' \tau \), where \( \sigma' \) and \( \theta' \) are regular extensions of \( \sigma \) and \( \theta \) respectively over the same domain
- \( \theta \) is equivalent to \( \sigma \), written as \( \sigma \approx \theta \), if \( \theta \preceq \sigma \) and \( \sigma \preceq \theta \).

Clearly if \( \sigma \) and \( \theta \) are variants, then they are equivalent. The following theorem shows that as in the case of terms and \( E \)-formulas precedors \( \preceq_J \) and \( \preceq \) (\( \approx_J \) and \( \approx \) as well) are identical, provided that \( J \) is a non-trivial model of \( \text{EQ}_L \).

**Theorem 5.9** Let \( \sigma \) and \( \theta \) be substitutions. If \( J \) is a non-trivial model of \( \text{EQ}_L \), then:

1. \( \sigma \preceq_J \theta \) iff \( \sigma \preceq \theta \)
2. \( \sigma \approx_J \theta \) iff \( \sigma \approx \theta \)

**Proof:** Let \( \sigma \preceq_J \theta \). Due to Proposition 5.8 we can suppose that \( \sigma \) and \( \theta \) are substitutions over the same domain \( X = \{ x_1, \ldots, x_n \} \). Put

\[
\text{Diff}(\sigma, \theta) = \text{def} \bigcup_{i=1}^{n} \text{Diff}(s_i, t_i),
\]

where \( s_i \equiv \sigma(x_i) \) and \( t_i \equiv \theta(x_i) \).
Let \((s, t) \in \text{Diff}(\sigma, \theta)\). Then \(t\) must be a variable since otherwise there is \(g \in V^L_I\) such that \(g(s) \not\in \text{inst}_J(t)\). If \(h(x) = g(\sigma(x))\) for \(x \in X\) then \(h \in \text{inst}_J(\sigma)\) and \(h \not\in \text{inst}_J(\theta)\). Contradiction.

Let \((s, y) \in \text{Diff}(\sigma, \theta)\) and \((s', y) \in \text{Diff}(\sigma, \theta)\). Then \(s\) and \(s'\) must be the same terms since otherwise there is \(g \in V^L_I\) such that \(g(s') \neq g(s)\). If \(h(x) = g(\sigma(x))\) for \(x \in X\) then \(h \in \text{inst}_J(\sigma)\) and \(h \not\in \text{inst}_J(\theta)\). Contradiction.

Now let \(\tau\) be a substitution over range(\(\theta\)) such that for any \(y \in \text{range}(\theta)\) it holds \(\tau(y) \equiv s\), where \((r, y) \in \text{Diff}(\sigma, \theta)\). Clearly \(\tau\) is well defined. Then we have \(\sigma = \theta \circ \tau\) and hence \(\sigma \preceq \theta\). □

As a straightforward corollary of the proof above we have the following claim.

**Corollary 5.10** If in the previous theorem \(\sigma\) and \(\theta\) are over same domain, then:

\[ \sigma \approx_J \theta \iff \sigma \approx \theta \iff \sigma\text{ and }\theta\text{ are variants} \]

The following claim states that the projection preserves the preorder \(\preceq\) and the equivalence \(\approx\).

**Theorem 5.11** Consider substitutions \(\sigma\) and \(\theta\). Then:

1. If \(\sigma \preceq \theta\), then \(\sigma|X \preceq \theta|X\).
2. If \(\sigma \approx \theta\), then \(\sigma|X \approx \theta|X\).

**Proof:** Obviously if \(\theta\) is a regular extension of \(\sigma\) then they are equivalent. Now let \(\sigma \preceq \theta\). Then there is \(\tau\) such that \(\sigma' = \theta' \tau\), where \(\sigma'\) and \(\theta'\) are regular extensions of \(\sigma\) and \(\theta\) respectively over the same domain. We have \(\sigma'|X = \theta'|X \circ \tau|Y\), where \(Y = \text{range}(\theta'|X)\), and hence \(\sigma'|X \preceq \theta'|X\). For \(\sigma'|X\) and \(\theta'|X\) are regular extensions of \(\sigma|X\) and \(\theta|X\) respectively, we obtain that \(\sigma|X \preceq \theta|X\) holds. □

Clearly equivalent substitutions may have different domains. Nevertheless we can find for any substitutions \(\sigma\) a minimal set \(X\) of variables (under set inclusion), denoted \(\text{kernel}(\sigma)\), having the following property: the restriction of \(\sigma\) onto \(X\) is equivalent to \(\sigma\). Really, consider a a set

\[ \mathcal{K}_\sigma = \{X \subseteq \text{Var} \mid \sigma|X \approx \sigma\} \]

Then the existence of such set is a straightforward consequence of the following properties of \(\mathcal{K}_\sigma\):

- \(\mathcal{K}_\sigma\) is nonempty, since \(\text{dom}(\sigma) \in \mathcal{K}_\sigma\)
- if \(X \in \mathcal{K}_\sigma\), then \(\sigma\) is a regular extension of \(\sigma|X\)
• $\mathcal{K}_\sigma$ has a finite intersection property i.e. if $X_i \in \mathcal{K}_\sigma$, then $X_1 \cap X_2 \in \mathcal{K}_\sigma$.

The last follows by noting that $\sigma$ is a regular extension of $\sigma | (X_1 \cap X_2)$. Now we put

$$\text{kernel}(\sigma) = \bigcap \{X | X \in \mathcal{K}_\sigma\}$$

The following theorem establishes properties of kernels.

**Theorem 5.12** Consider substitutions $\sigma$ and $\theta$. Then:

1. $\sigma$ is a regular extension of $\sigma | \text{kernel}(\sigma)$.
2. If $\sigma \preceq \theta$, then $\text{kernel}(\sigma) \supseteq \text{kernel}(\theta)$.
3. If $\sigma \approx \theta$, then $\text{kernel}(\sigma) = \text{kernel}(\theta)$.

**Proof:** Let $\sigma \preceq \theta$. To prove 2) it sufficient to show that $\mathcal{K}_\sigma \subseteq \mathcal{K}_\theta$ or equivalently for every $X$ if $\sigma$ is a regular extension of $\sigma | X$, then $\theta$ is a regular extension of $\theta | X$ as well. Now let $\sigma$ is a regular extension of $\sigma | X$. By assumptions there is $\tau$ such that $\sigma' = \theta' \tau$, where $\sigma'$ and $\theta'$ are regular extensions of $\sigma$ and $\theta$ respectively over the same domain. Then $\sigma'$ is a regular extension of $\sigma | X$ and hence $\theta'$ is a regular extension of $\theta' | (X \cap \text{dom}(\theta))$. But $\theta' | (X \cap \text{dom}(\theta)) \equiv \theta | X$ and therefore $\theta$ is a regular extension of $\theta | X$. $\Box$

Let $\bot$ be an arbitrary object that is not element of $\text{Sub}_L$. Let $\text{Sub}_L^\bot$ be the set $\text{Sub}_L \cup \{\bot\}$. We extend the partial ordering $\preceq$ and the equivalence $\approx$ to $\text{Sub}_L^\bot$ by requiring $\bot$ to be the smallest element of $\text{Sub}_L^\bot$. We denote by $\text{Sub}_L^\bot / \approx$ the new quotient set and by $\sigma_\approx$ the equivalence class in which $\sigma$ lies. We put $\text{inst}_J(\bot) = \emptyset$.

**Theorem 5.13** The lattice $\text{Sub}_L^\bot / \approx$ is isomorphic to the lattice $\text{Eqn}_L / \approx$. In particular, $\text{Sub}_L^\bot / \approx$ is a complete lattice with $\varepsilon_\approx$ as the greatest element and with $\bot$ as the smallest element.

**Proof:** The isomorphic mapping $\Phi$ between $\text{Eqn}_L / \approx$ and $\text{Sub}_L^\bot / \approx$ is defined as follows:

1. $\Phi(\text{True}_\approx) = \varepsilon_\approx$ and $\Phi(\text{False}_\approx) = \bot$

2. $\Phi(E_\approx) = \sigma_\approx$, if

$$E \equiv (\exists z_1) \ldots (\exists z_m)(x_1 = s_1 \land \ldots x_n = s_n)$$

is in solved form and

$$\sigma = \{x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n, y_1 \leftarrow y_1, \ldots, y_k \leftarrow y_k\},$$

where $y_1, \ldots, y_k$ are parameters of $E$. 25
To prove that \( \Phi \) is an isomorphism between \( Eqn_{L/\approx} \) and \( Sub_{L/\approx} \) it is sufficient to show that (i) if \( J \) is a model of \( EQ_{L} \), then \( soln_{J}(E) = inst_{J}(\sigma) \), where \( E \) and \( \sigma \) are from 2) and (ii) for any \( \sigma \) there is \( E \) such that \( \Phi(E_{\approx}) = \sigma_{\approx} \).

(i) Suppose \( h \in soln_{J}(E) \). Then there is \( g \) identical to \( h \) on \( vars(E) \) such that \( g(x_{i}) = g(s_{i}) \) for \( i = 1, \ldots, n \). Then \( h(x) = g(\sigma(x)) \) for \( x \in dom(\sigma) \). Thus we have \( h \in inst_{J}(\sigma) \). Assume now that \( h \in inst_{J}(\sigma) \). Then there is \( g \) such that \( h(x_{i}) = g(s_{i}) \) and \( h(y_{j}) = g(y_{j}) \) for any \( x_{i} \) and \( y_{j} \). Put \( g'(u) = h(u) \) on \( vars(E) \) and \( g'(u) = g(u) \) otherwise. Then \( g'(y_{j}) = g(y_{j}) \) and hence \( g'(s_{i}) = g(s_{i}) \) for any \( s_{i} \). We have \( g'(x_{i}) = h(x_{i}) = g(s_{i}) = g'(s_{i}) \) and therefore \( h \in soln_{J}(E) \).

(ii) Consider arbitrary substitution \( \sigma = \{ x_{1} \leftarrow s_{1}, \ldots, x_{n} \leftarrow s_{n} \} \).

We can suppose due to Corollary 5.10 that \( dom(\sigma) \) and \( range(\sigma) \) are disjoint sets. Put

\[
E \equiv (\exists z_{1}) \ldots (\exists z_{m})(x_{1} = s_{1} \land \ldots x_{n} = s_{n}),
\]

where \( \{ z_{1}, \ldots, z_{m} \} = range(\sigma) \). By similar arguments as in (i) we obtain that \( soln_{J}(E) = inst_{J}(\sigma) \), where \( J \) is arbitrary model \( EQ_{L} \), and hence \( \Phi(E_{\approx}) = \sigma_{\approx} \). □

As a straightforward corollary from the proof above we obtain:

**Corollary 5.14** Suppose that \( \Phi(E_{\approx}) = \sigma_{\approx} \). Then:

1. \( soln_{J}(E) = inst_{J}(\sigma) \) for any model \( J \) of \( EQ_{L} \).
2. \( kernel(E) = kernel(\sigma) \), provided that \( E \) is consistent.
3. \( \Phi((E|X)_{\approx}) = (\sigma|X)_{\approx} \).

### 5.3 Application of Substitutions to Formulas

In Section 2 we introduced the notion of application of substitutions to formulas. Recall that if \( \sigma = \{ x_{1} \leftarrow s_{n}, \ldots, x_{n} \leftarrow s_{n} \} \) then \( F\sigma \) is defined as \( F_{x_{1},\ldots,x_{n}}[s_{1}, \ldots, s_{n}] \), provided that \( \sigma \) is applicable to \( F \). Now we generalize this notion for arbitrary substitutions.

Consider a formula \( F \) and a substitution \( \sigma \). Suppose first that \( dom(\sigma) = vars(F) \). The application is defined by structural induction as follows:

1. Let \( F \) is an atom \( p(s_{1}, \ldots, s_{n}) \). Then \( F\sigma =_{def} p(s_{1}\sigma, \ldots, s_{n}\sigma) \).
2. Let \( F \) is a formula of the form \( \neg G \). Then \( F\sigma =_{def} \neg(G\sigma) \).
3. Let \( F \) is a formula of the form \( G \ b \ H \), where \( b \) is \( \land, \lor \) or \( \rightarrow \). Then \( F\sigma =_{def} G\sigma_{1} \ b \ H\sigma_{2} \), where \( \sigma_{1} \) or \( \sigma_{2} \) is a restriction of \( \sigma \) onto \( vars(G) \) or \( vars(H) \), respectively.
4. Let $F$ is a formula of the form $(\exists x)G$.

   (a) Let $x \notin \text{range}(\sigma)$. We put $F\sigma = (\exists x)(G\sigma')$, where

   $$\sigma' = \begin{cases} 
   \sigma \cup \{x \leftarrow x\} & \text{if } x \in \text{vars}(G) \\
   \sigma & \text{otherwise}
   \end{cases}$$

   (b) Let $x \in \text{range}(\sigma)$. Let $(\exists y)G_x[y]$ be a variant of $F$, where $y$ is the first variable in $\text{Var}$ such that $y$ does not occur in $\text{range}(\sigma)$. We put $F\sigma = (\exists y)(G_x[y]\sigma')$, where

   $$\sigma' = \begin{cases} 
   \sigma \cup \{y \leftarrow y\} & \text{if } x \in \text{vars}(G) \\
   \sigma & \text{otherwise}
   \end{cases}$$

5. The case when $F$ is a formula of the form $(\forall x)G$ is similar to previous one.

Example 5.1 Let $\text{Var} = \{x, y, z, u, \ldots\}$. Consider a formula $F = (\exists z)p(x, y, z)$ and a substitution $\sigma = \{x \leftarrow z, y \leftarrow x\}$. Then $F\sigma = (\exists u)p(z, x, u)$.

Now consider an arbitrary substitution $\sigma$. First we restrict $\sigma$ to the set $\text{vars}(F)$ to obtain a substitution $\sigma'$ over $\text{dom}(\sigma) \cap \text{vars}(F)$. Now let $\sigma''$ be a regular extension of $\sigma'$ onto $\text{vars}(F)$. We put $F\sigma = (\exists u)p(x, y, z)$, where the right hand side is correctly defined since $\text{dom}(\sigma'') = \text{vars}(F)$. Clearly the application of $\sigma$ to $F$ depends on the choice of $\sigma''$. Nevertheless we can select $\sigma''$ uniquely by applying the following procedure:

$$\sigma'' := \sigma'$$

while $\text{vars}(F) \setminus \text{dom}(\sigma'')$ is nonempty do

let $x$ be the first variable occurring in $\text{vars}(F) \setminus \text{dom}(\sigma'')$

let $y$ be the first variable not occurring in $\text{range}(\sigma'')$

$$\sigma'' := \sigma'' \cup \{x \leftarrow y\}$$
enddo

Obviously $\sigma''$ is a regular extension of $\sigma'$.

Example 5.2 Let $\text{Var} = \{x, y, z, u, \ldots\}$.

1. Suppose that $F = (\exists z)p(x, y, z)$ and $\sigma = \{x \leftarrow f(z, x), z \leftarrow g(x, y, z)\}$. Then $\sigma' = \{x \leftarrow f(z, x)\}$ and $\sigma'' = \{x \leftarrow f(z, x), y \leftarrow y\}$. We have $F\sigma = (\exists u)p(f(z, x), y, u)$.

2. Suppose that $F = (\exists z)p(x, y, z)$ and $\sigma = \{x \leftarrow f(z, y), z \leftarrow g(x, y, z)\}$. Then $\sigma' = \{x \leftarrow f(z, y)\}$ and $\sigma'' = \{x \leftarrow f(z, y), y \leftarrow x\}$. We have $F\sigma = (\exists u)p(f(z, y), x, u)$.

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The main aim of this section is to prove Theorem 5.19. To prove it we first need the following lemmas.

**Lemma 5.15** Consider formulas $F$ and $F'$ and a substitution $\sigma$.

1. Let $\sigma = \{ x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n \}$ be applicable to $F$. Then
   \[ F\sigma = F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]. \]

2. If $F$ and $F'$ are variants, then $F\sigma$ and $F'\sigma$ are variants, as well.

**Proof:** (1) Straightforward. (2) If $F$ and $F'$ are variants, then $\text{vars}(F) = \text{vars}(F')$ and thus we can suppose that $\text{dom}(\sigma) = \text{vars}(F)$. The claim is obtained by structural induction on formulas. The proof is long and straightforward. We shall omit them. $\square$

**Lemma 5.16** Let $I$ be an interpretation over $J$. Consider a formula $F$ and a substitution $\sigma$. Suppose that $\sigma = \{ x_1 \leftarrow s_1, \ldots, x_n \leftarrow s_n \}$ is applicable to $F$ and $\text{dom}(\sigma) = \text{vars}(F)$. Let $g, h \in V_L$. If $h(x) = g(\sigma(x))$ for any $x \in \text{dom}(\sigma)$, then:

\[ I \models h \iff I \models g F_{x_1,\ldots,x_n}[s_1,\ldots,s_n] \]

**Proof:** We prove this claim by the structural induction on formulas. (i) Let $F$ is an equation $r = t$. Then we obtain that $I \models r = t$ iff $h(r) = h(t)$ iff $g(r\sigma) = g(t\sigma)$ iff $I \models g(r = t)$. (ii) Let $F$ is an atom $p(s_1, \ldots, s_m)$. Then we have $I \models p(s_1, \ldots, s_m) \iff p(h(s_1), \ldots, h(s_m)) \in I \iff p(g(s_1), \ldots, g(s_m)) \in I$ iff $I \models p(s_1, \ldots, s_m)$. (iii) Let $F \equiv \neg G$. Then $I \models \neg G$ iff $I \not\models G$ iff $I \not\models G_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$ iff $I \models g F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$. (iv) Let $F$ is a conjunction of $G$ and $H$. Denote by $\sigma'$ or $\sigma''$ the restriction of $\sigma$ to $\text{vars}(G)$ or $\text{vars}(H)$, respectively. Let

\[ \sigma' = \{ y_1 \leftarrow s'_1, \ldots, y_k \leftarrow s'_k \} \quad \text{and} \quad \sigma'' = \{ z_1 \leftarrow s''_1, \ldots, z_l \leftarrow s''_l \}. \]

Then using induction hypothesis we have $I \models \neg G \wedge H$ iff $I \models G$ and $I \models H$ iff $I \models g_{y_1,\ldots,y_k} s'_1,\ldots,s'_k$ and $I \models h_{z_1,\ldots,z_l} s''_1,\ldots,s''_l$ iff $I \models g_{x_1,\ldots,x_n} s_1,\ldots,s_n$. (v) By similar arguments we can prove the claim if $F$ is of the form $G \vee H$, $G \rightarrow H$ or $G \leftrightarrow H$. (vi) Let $F \equiv (\exists y)G$. Then $I \models (\exists y)G$ iff $I \models h_{[y \leftarrow d]} G$ for some $d \in V_L$. Let

\[ \tau = \begin{cases} \sigma \cup \{ y \leftarrow y \} & \text{if } y \in \text{vars}(G) \\ \sigma & \text{otherwise} \end{cases} \]

Then $h[y \leftarrow d](x) = g[y \leftarrow d](\tau(x))$ for any $x \in \text{dom}(\tau)$. We have by induction hypothesis

\[ I \models h_{[y \leftarrow d]} G \iff \begin{cases} I \models g_{[y \leftarrow d]} G_{x_1,\ldots,x_n,y}[s_1,\ldots,s_n] & \text{if } y \in \text{vars}(G) \\ I \models g_{[y \leftarrow d]} G_{x_1,\ldots,x_n}[s_1,\ldots,s_n] & \text{otherwise} \end{cases} \]

Hence $I \models (\exists y)G$ iff $I \models g F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$. (vii) Using the same arguments we can prove that the claim holds when $F \equiv (\forall y)G$. $\square$
Lemma 5.17 Let $I$ be an interpretation over $J$. Consider a formula $F$ and a substitution $\sigma$. Let $\sigma'$ be a restriction of $\sigma$ onto $\text{vars}(F)$. Then:

$$\text{inst}_J(\sigma) \subseteq \langle F \rangle^J_I \iff \text{inst}_J(\sigma') \subseteq \langle F \rangle^J_I$$

**Proof:** Since $\sigma \preceq \sigma'$, $\text{inst}_J(\sigma') \subseteq \langle F \rangle^J_I$ implies $\text{inst}_J(\sigma) \subseteq \langle F \rangle^J_I$. Let $\text{inst}_J(\sigma) \subseteq \langle F \rangle^J_I$ and $h \in \text{inst}_J(\sigma')$. Then there is $g \in \text{inst}_J(\sigma)$ such that $g = h$ on $\text{vars}(F)$. By assumption $g \in \langle F \rangle^J_I$ and hence $h \in \langle F \rangle^J_I$. □

Lemma 5.18 Let $I$ be an interpretation over $J$. Consider a formula $F$ and a substitution $\sigma$. Then:

$$I \models F\sigma \iff \text{inst}_J(\sigma) \subseteq \langle F \rangle^J_I$$

**Proof:** By the definition of application and Lemma 5.17 we can suppose without lost of generality that $\text{dom}(\sigma) = \text{vars}(F)$. Let $\sigma = \{x_1 \leftarrow s_n, \ldots, x_n \leftarrow s_n\}$. Using Lemma 5.15(2) we can assume $\sigma$ is applicable to $F$. Consequently by Lemma 5.15(1) $F\sigma = F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$. So we must prove that

$$I \models F_{x_1,\ldots,x_n}[s_1,\ldots,s_n] \iff \text{inst}_J(\sigma) \subseteq \langle I \rangle^F_J$$

Suppose that $I \models F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$. Let $h \in \text{inst}_J(\sigma)$. Then there is $g$ such that $h(x) = g(\sigma(x))$ for $x \in \text{dom}(\sigma)$. By Lemma 5.16 $I \models_g F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$ and hence $I \models_h F$. Assume now that $\text{inst}_J(\sigma) \subseteq \langle I \rangle^F_J$. Consider arbitrary $g \in V^J_f$. Then there is $h \in \text{inst}_J(\sigma)$ such that $h(x) = g(\sigma(x))$ for $x \in \text{dom}(\sigma)$. By Lemma 5.16 again we have $I \models_h F$ and hence $I \models_g F_{x_1,\ldots,x_n}[s_1,\ldots,s_n]$. □

We are now in position to prove the desired theorem.

**Theorem 5.19** The application of substitutions to formulas has the following properties:

(a) If $\sigma \preceq \theta$, then $(\forall) F\sigma \rightarrow (\forall) F\theta$ is valid.

(b) If $\sigma \approx \theta$, then $(\forall) F\sigma \leftrightarrow (\forall) F\theta$ is valid.

(c) If $\sigma'$ is a restriction of $\sigma$ onto $\text{vars}(F)$, then $(\forall) F\sigma \leftrightarrow (\forall) F\sigma'$ is valid.

**Proof:** Straightforward by applying the previous lemmas. □
We conclude this section by showing that there is an interesting relationship between applications of substitutions and $E$-formulas.

**Theorem 5.20** Let $F$ be a formula. Suppose that a substitution $\sigma$ corresponds to an $E$-formula $E$ in the isomorphic mapping between lattices $\text{Eqn}_L/\approx$ and $\text{Sub}_L/\approx$. Then

\[ EQ_L \models F\sigma \text{ iff } EQ_L \models E \rightarrow F, \]

or equivalently

\[ EQ_L \models (\forall)F\sigma \leftrightarrow (\forall)(E \rightarrow F). \]

**Proof:** Let $I$ over $J$ be a model of $EQ_L$. Then by Lemma 5.17 and Corollary 5.14 we have $I \models F\sigma$ iff $\text{inst}_J(\sigma) \subseteq \langle I \rangle_J^F$ iff $\text{soln}_J(E) \subseteq \langle I \rangle_J^F$ iff $I \models E \rightarrow F$. \qed

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