BIRATIONAL EQUIVALENCES AND GENERALIZED WEYL ALGEBRAS

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ABSTRACT. We calculate suitably localized Hochschild homologies of various quantum groups and Podleś spheres after realizing them as generalized Weyl algebras (GWAs). We use the fact that every GWA is birationally equivalent to a smash product with a 1-torus. We also address and solve the birational equivalence problem, and the birational smoothness problem for GWAs.

INTRODUCTION

A birational equivalence is an algebra morphism that becomes an isomorphism after a suitable localization. In this paper, we show that every generalized Weyl algebra (GWA) is birationally equivalent to a smash product with a rank-1 torus. This fact significantly simplifies their representation theory, and structure problems such as the isomorphism problem [24, 5, 38, 42, 43] and the smoothness problem [4, 24, 41, 31, 32], provided one replaces isomorphisms with suitable noncommutative birational equivalences. We address and solve a relative version of the birational equivalence problem in Section 2.7, and the birational smoothness problem in Section 3.3. We then calculate the Hochschild homology of suitably localized examples of GWAs in Section 4.

Generalized Weyl algebras (See Section 2.1) are defined by Bavula [3, 4], Hodges [24] and Rosenberg [39] independently under different disguises. Their representation theory resembles that of Lie algebras [12, 36] (see Section 2.5), their homologies are extensively studied [13, 11, 31, 32], and they found diverse uses in areas such as noncommutative resolutions of Kleinian singularities [11, 6, 30] and noncommutative geometry of various quantum spheres and lens spaces [9]. Apart from noncommutative resolutions of Kleinian singularities, the class is known to contain the ordinary rank-1 Weyl algebra $A_1$, the enveloping algebra $U(s\mathfrak{l}_2)$ and its primitive quotients, the quantum enveloping algebra $U_q(s\mathfrak{l}_2)$, the quantum monoid $O_q(M_2)$, the quantum groups $O_q(GL_2)$, $O_q(SL_2)$ and $O_q(SU_2)$. We verify that the standard Podleś spheres $O_q(S^2)$ [37] and parametric Podleś spheres $O_{q,c}(S^2)$ of Hadfield [18] are also examples of GWAs. We finish the paper by calculating localized Hochschild homology of all of these examples.

The Hochschild homology of quantum groups $O_q(GL_n)$ and $O_q(SL_n)$ with coefficients in a 1-dimensional character coming from a modular pair in involution is calculated for every $n \geq 1$ in [27], and with coefficients in themselves in specific cases in [33, 40, 19, 20]. The Hochschild cohomology of the Podleś sphere was studied by Hadfield [18], and then in the context of van den Bergh duality [45, 44] by Krähmer [28]. Both Hadfield and Krähmer use twisted Hochschild (co)homology by the Nakayama automorphism with coefficients in themselves. In this paper we only calculate the ordinary Hochschild homology of these algebras with coefficients in themselves since one can always move to and from the ordinary Hochschild homology and the twisted homology via suitable cup and cap products [28, 16].
In this paper we focus on rank-1 GWAs, i.e. algebras that are birationally equivalent to smash products with rank-1 tori. For the higher rank tori, one must consider twisted generalized Weyl algebras (TGWAs) \([35,34,22,23]\) that conjecturally recover enveloping algebras of higher rank Lie algebras and their quantizations. We conjecture that TGWAs are birationally equivalent to smash products with higher rank tori, but we leave this investigation for a future paper.

The celebrated Gelfand-Kirillov Conjecture, on the other hand, states that the universal enveloping algebra \(U(g)\) of a finite dimensional Lie algebra is birationally equivalent to a sufficiently high rank Weyl algebra \([15]\). One of the equivalent forms of the conjecture is that \(U(g)\) is birationally equivalent to the smash product of a polynomial algebra with a torus. The conjecture is known to be false in general \([24,10]\), but is true for a large class of Lie algebras \([15,25,21]\). The quantum analogue of the conjecture (see \([7, pp.19–21 and Sect.II.10.4]\) and references therein) is also known to be true in many instances \([11,14]\). In the light of our conjecture above, we believe that the universal enveloping algebra \(U(g)\) of a rank-\(n\) semi-simple Lie algebra is birationally equivalent to the smash product of a smooth algebra with an \(n\)-torus. We also believe that the same is true for the quantum enveloping algebras \(U_q(g)\) and the quantum groups \(O_q(G)\) where one replaces the \(n\)-torus with a quantum \(n\)-torus.

**Plan of the article.** In Section 1 we recall some basic facts on localizations, relative homology of algebra extensions, smash products and biproducts. In Section 2 we prove two fundamental structure theorems for GWAs in Sections 2.2 and 2.3. Then we state and solve the birational equivalence problem for GWAs in Section 2.7. In Section 3 we investigate the interactions between homology, smash biproducts and noncommutative localizations, and in Sections 3.3 and 3.4 we state and solve the birational smoothness problem for GWAs. Finally, we use our machinery to calculate suitably localized Hochschild homologies of various GWAs in Section 4.

**Notations and conventions.** We assume \(0 \in \mathbb{N}\). We fix an algebraically closed ground field \(k\) of characteristic 0, and we set the binomial coefficients \(\binom{n}{m}\) = 0 whenever \(m > n\) or \(m < 0\). All unadorned tensor products \(\otimes\) are taken over \(k\). All \(k\)-algebras are assumed to be unital and associative, but not necessarily commutative or finite dimensional. For such an algebra \(A\), we always use \(1_A\) for the unit. We work exclusively with domains, i.e. we assume that our algebras are devoid of zero divisors. All modules are assumed to be right modules unless otherwise stated. We use \(\text{Mod}\)-\(A\) to denote a category of modules over \(A\). For an Ore subset \(S\) of an algebra \(A\), we use \(S^{-1}A\) for the localization of \(A\) at \(S\). We extend the notation for \(A\)-modules by using \(S^{-1}X\) for \(X \otimes_A S^{-1}A\). For an algebra \(A\), we use \(A^e\) for the enveloping algebra \(A \otimes A^{\text{op}}\). With this definition at hand, a module in \(\text{Mod}\)-\(A^e\) is a bimodule over \(A\). For (bi)modules, left actions are denoted by \(\triangleright\) and right actions are by \(\triangleleft\). For a \(A\)-bimodule \(X\), we use \(X_A\) for the quotient by the vector subspace generated by elements of the form \([a,x] := a \triangleright x - x \triangleleft a\) with \(a \in A\) and \(x \in X\). We use \(X^A\) for the vector subspace \(X\) generated by elements \(x \in X\) with \([a,x] = 0\) for \(a \in A\). In the case of \(A\) is the group algebra of a group \(G\), and \(X\) is a \(G\)-bimodule, these spaces agree with \(G\)-coinvariants and \(G\)-invariants of the corresponding adjoint representation, respectively. We use \(k[X]\) for the free unital commutative algebra generated by a set \(X\), while we use \(k\{X\}\) for the free unital algebra generated by the same set \(X\). Throughout the paper we use \(\mathbb{T}\) to denote the algebra of Laurent polynomials \(k[x,x^{-1}]\) which we implicitly consider as the group algebra of the additive group of the ring of integers \(\mathbb{Z}\). We refer to this algebra as the rank 1-torus.
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1. Preliminaries

1.1. Noncommutative localizations. Our main reference for noncommutative localizations is [29] §10.

Let $A$ be an algebra. A multiplicative submonoid $S \subseteq A$ is called a right Ore set if for every $s \in S$ and $u \in A$

(i) there are $s' \in S$ and $u' \in A$ such that $su' = us'$, and
(ii) if $su = 0$ then there is $u' \in A$ such that $u's = 0$.

If $S \subseteq A$ is a right Ore set then there is an algebra $S^{-1}A$ and a morphism of algebras $\iota_S: A \to S^{-1}A$ such that $\phi(S) \subseteq (S^{-1}A)^\times$. The algebra $S^{-1}A$ is called the localization of $A$ with respect to $S$. The morphism $\iota_S$ is universal among such $S$ inverting morphisms where if $\phi: A \to B$ is an algebra morphism that satisfies $\phi(S) \subseteq B^\times$ then there is a unique algebra morphism $\phi': S^{-1}A \to B$ with $\phi = \phi' \circ \iota_S$.

1.2. Birational equivalences of algebras. We call an algebra morphism $\phi: A \to B$ as a birational equivalence if there are two Ore sets $S \subseteq A$ and $T \subseteq B$ such that $\phi(S) \subseteq T$ and the extension of $\phi$ to the localization $\varphi_S: S^{-1}A \to T^{-1}B$ is an isomorphism of algebras.

Remark 1.1. The motivation for the definition comes from algebraic geometry. Given two irreducible algebraic varieties $X$ and $Y$, an algebraic map $f: X \to Y$ is called a birational equivalence if $f$ restricted to an open subvariety is an isomorphism [17] §4.2]. This equivalent to $f$ being an isomorphism on function fields (the maximal localization) on the coordinate functions of the open subvarieties.

1.3. Hochschild homology. Let $A$ be an algebra, and let $M$ be an $A$-bimodule. Consider the graded $k$-vector space

\[ \text{CH}_*(A, M) = \bigoplus_{n \geq 0} M \otimes A^\otimes n \]

together with linear maps $b_n: \text{CH}_n(A, M) \to \text{CH}_{n-1}(A, M)$ defined for $n \geq 1$ via

\[ b_n(m \otimes a_1 \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n \]
\[ + \sum_{i=1}^{n-1} (-1)^i m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \]
\[ + (-1)^n a_nm \otimes a_1 \otimes \cdots \otimes a_{n-1}. \]

These maps satisfy $b_nb_{n+1} = 0$ for every $n \geq 1$, and we define $H_n(A, M) = \ker(b_n)/\text{im}(b_{n+1})$. We use the notation $HH_*(A)$ for $H_*(A, A)$.

1.4. Smash products with Hopf algebras. Assume $H$ is a Hopf algebra. We use the Sweedler notation for its coproduct $\Delta: H \to H \otimes H$ where we write $\Delta(h) = h^{(1)} \otimes h^{(2)}$ for any $h \in H$. We also assume $A$ is an $H$-module algebra, i.e. $A$ is an algebra and a left $H$-module $\triangleright: H \otimes A \to A$ such that we have

\[ h \triangleright (ab) = (h^{(1)} \triangleright a)(h^{(2)} \triangleright b) \]
for every $h \in H$ and $a, b \in A$. The smash product $A \# H$ is defined as the tensor product space $A \otimes H$ together with the product

$$(a \otimes h)(a' \otimes h') = a(h_{(1)} \triangleright a') \otimes h_{(2)}h'$$

for every $h, h' \in H$ and $a, a' \in A$. With this product, the unit element is $1_A \otimes 1_H$. For convenience, we are going to write $ah$ for a homogeneous element $a \otimes h$ in $A \# H$.

**Example 1.2.** Let $T$ be $\mathbb{k}[x, x^{-1}]$ the Hopf algebra of the group ring of the additive group of the ring of integers $\mathbb{Z}$. This Hopf algebra is also known as the algebra of Laurent polynomials with the standard Hopf algebra structure on a group ring where the coproduct is defined as $\Delta(x^i) = (x^i \otimes x^i)$ for every $i \in \mathbb{Z}$. Now, assume $A$ is an algebra with a fixed automorphism $\sigma$. Thus we obtain a $T$-module algebra structure on $A$, and $A \# T$ is the vector space $A \otimes T$ together with the multiplication $(ax^i)(a'x^j) = a\sigma^i(a')x^{i+j}$ for any $a, a' \in A$ and $i, j \in \mathbb{Z}$. The unit for the algebra is $1_Ax^0$ for which we simply write $1_A$.

1.5. **Smooth algebras.** An algebra $B$ is said to be smooth if it has finite Hochschild homological dimension, i.e. when

$$hh.dim(B) := \sup \{n \in \mathbb{N} \mid H_n(B, M) \neq 0, \ M \in \text{Mod-}B^e\}$$

is finite. We call an algebra $B$ $m$-smooth if $hh.dim(B) = m$, for $m \in \mathbb{N}$.

The canonical examples for 0-smooth algebras are the group ring $\mathbb{k}[G]$ over finite groups where $|G|$ does not divide the characteristic of $\mathbb{k}$, and quotients of polynomial algebras $\mathbb{k}[x]/\langle f(x) \rangle$ where $f(x)$ is a separable polynomial. For $m \geq 1$, the polynomial algebras $\mathbb{k}[t_1, \ldots, t_m]$ and the Laurent polynomial algebras $\mathbb{k}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ with $m \geq 0$ can be given as examples.

1.6. **Homology of smash biproducts.** Assume $A$ and $B$ are two algebras. A $\mathbb{k}$-linear map $R: B \otimes A \to A \otimes B$ is called a distributive law if the tensor product space $A \otimes B$ is an algebra with unit $1_A \otimes 1_B$ with the following product structure

$$(a \otimes b)(a' \otimes b') = aR_{(1)}(b \otimes a') \otimes R_{(2)}(b \otimes a')b'$$

where we write $R(b \otimes a') = R_{(1)}(b \otimes a') \otimes R_{(2)}(b \otimes a')$ for every $a' \in A$ and $b \in B$. We denote this new algebra by $A \#_R B$, or by $A \# B$ if the distributive law is clear in the context.

**Example 1.3.** The smash product $A \# H$ of a Hopf algebra $H$ with a $H$-module algebra $A$ is a special example of a smash biproduct where the distributive law is defined as

$$R(h \otimes a) = (h_{(1)} \triangleright a) \otimes h_{(2)}$$

for every $h \in H$ and $a \in A$. Notice that $R$ has an inverse of the form

$$R^{-1}(a \otimes h) = h_{(2)} \otimes S^{-1}(h_{(1)}) \triangleright a$$

for every $h \in H$ and $a \in A$.

We recall the following fact from [29].

**Proposition 1.4.** Let $A$ and $B$ be two algebras, and let $R: B \otimes A \to A \otimes B$ be an invertible distributive law. For any $A \# B$-bimodule $M$ and for all $n \geq 0$ we have isomorphisms of vector spaces of the form

$$H_n(A \# B, M) \cong H_n(CH_* (A, M)_B)$$

when $B$ is 0-smooth, and isomorphisms of vector spaces of the form

$$H_n(A \# B, M) \cong H_n(CH_* (A, M)_B) \oplus H_{n-1}(CH_* (A, M)_B)$$
when $B$ is 1-smooth where
\[
\text{CH}_n(A, M)_B := \{(a \otimes m \triangleright b) - (R(1)(b \otimes a) \otimes R(2)(b \otimes a) \triangleright m) : b \in B, \ a \otimes m \in \text{CH}_n(A, M)\},
\]
and
\[
\text{CH}_n(A, M)^B := \{(a \otimes m) \in \text{CH}_n(A, M) : (a \otimes m \triangleright b) = R(1)(b \otimes a) \otimes R(2)(b \otimes a) \triangleright m, b \in B\}.
\]
In the definition above $R(b \otimes a)$ indicates the $n$-fold composition of the distributive law
\[
B \otimes A^{\otimes n} \to A \otimes B \otimes A^{\otimes n-1} \to \cdots \to A^{\otimes n-1} \otimes B \otimes A \to A^{\otimes n} \otimes B
\]
where the first leg $R(1)$ takes values in $A^{\otimes n}$ while the second leg $R(2)$ takes values in $B$ for $h \in H$ and for $a = \sum_i a_{1,i} \otimes \cdots \otimes a_{n,i} \in A^{\otimes n}$.
Next, let us recall the following result from \[27\], Prop.1.5:

**Proposition 1.5.** Assume $P$ and $Q$ are two algebras together with a left flat algebra morphism $\varphi : Q \to P$. Let $M$ be a $P$-bimodule. Then there is a spectral sequence whose first page is given by
\[
E^1_{i,j} = H_j(Q, M \otimes_P P \otimes \cdots \otimes \otimes_P P)
\]
that converges to the Hochschild homology $H_4(P, M)$.

**Corollary 1.6.** Let $A$ and $B$ be two algebras, and let $R : B \otimes A \to A \otimes B$ be any distributive law. Then for any $A\#B$-bimodule $M$ and for all $n \geq 0$ we still have Equation (1.2) when $B$ is 0-smooth and Equation (1.3) when $B$ is 1-smooth.

**Proof.** We set $P = A\#B$ and $Q = B$ together with the embedding $\varphi : B \to A\#B$ where $\varphi(b) = 1 \otimes b$ for any $b \in B$. Then we use Proposition 1.3. \[\square\]

# 2. Generalized Weyl algebras

## 2.1. The definition.
Assume $A$ is an algebra, let $a \in Z(A)$ and $\sigma \in \text{Aut}(A)$ be fixed. Define a new algebra $W_{a,\sigma}$ as a quotient of the algebra generated by $A$ and two non-commuting indeterminates $x$ and $y$ subject to the following relations:
\[
yx - a, \ xy - \sigma(a), \ xu - \sigma(u)x, \ y\sigma(u) - uy
\]
for every $u \in A$. The algebra $W_{a,\sigma}$ is called *generalized Weyl algebra* \[3, 5\].

## 2.2. A structure theorem for GWAs.
When $a$ is non-zero one can also identify $W_{a,\sigma}$ as a subalgebra of the smash product $A\#\mathbb{T}$. For this we consider the morphism of $\mathbb{k}$-algebras $\varphi : W_{a,\sigma} \to A\#\mathbb{T}$ given by
\[
\varphi(u) = u, \ \varphi(x) = x, \ \varphi(y) = ax^{-1}
\]
for every $u \in A$.

**Theorem 2.1.** For every non-zero $a \in Z(A)$, the algebra $W_{a,\sigma}$ is isomorphic to the subalgebra of the smash biproduct $A\#\mathbb{T}$ generated by $A, x$ and $ax^{-1}$. Hence $W_{a,\sigma}$ is isomorphic to $A\#\mathbb{T}$ for every $a \in Z(A^\times)$.

**Proof.** The result follows from the fact that the image of $\varphi$ (as $\mathbb{k}$-vector spaces) is the direct sum
\[
A \otimes \mathbb{k}[x] \oplus \bigoplus_{n=0}^{\infty} \langle a\sigma^{-1}(a) \cdots \sigma^{-n}(a) \rangle \otimes \text{Span}_\mathbb{k}(x^{-n-1})
\]
where \( \langle u \rangle \) denotes the two sided ideal in \( A \) generated by an element \( u \in A \).

In specific cases, the fact that GWAs can be realized as subalgebras of smash products was already known [6, Lem.2.3]. However, to the best of our knowledge, the fact that a GWA is isomorphic to the full smash product when the distinguished element \( a \in A \) is a unit is not fully taken advantage of in the literature.

From now on, we assume \( a \in Z(A) \) is non-zero and we identify \( W_{a,\sigma} \) with \( im(\varphi) \) in \( A\#T \).

2.3. Localizations of smash products with tori. Let \( A \) be an algebra with a fixed automorphism \( \sigma \in Aut(A) \). Let \( S \subseteq Z(A) \) be any multiplicative submonoid which is stable under the action of \( \sigma \).

**Lemma 2.2.** Any multiplicative monoid \( S \) in \( Z(A) \) which is \( \sigma \)-stable is a right Ore subset in \( A\#T \). Moreover, the algebras \( S^{-1}(A\#T) \) and \( (S^{-1}A)\#T \) are isomorphic.

**Proof.** Assume \( s \in S \) and \( x^i \in T \) where \( i \in \mathbb{Z} \). Then \( sx^i = x^i \sigma^i(s) \) and \( \sigma^i(s) \in S \) since we assumed \( S \) is \( \sigma \)-stable. Moreover, if \( sx^i = x^i \sigma^i(s) = 0 \) then so is \( s \). Thus \( S \) is a right Ore set and the localization \( S^{-1}(A\#rT) \) is well-defined. Since elements of the algebra \( S^{-1}A \) are (equivalence classes) of elements of the form \( us^{-1} \) where \( s \in S \) and \( u \in A \), we see that \( \sigma \) extends to an automorphism of \( S^{-1}A \) since \( S \) is stable under the action of \( \sigma \) by construction. Thus the smash biproduct \( (S^{-1}A)\#T \) is the localization \( S^{-1}(A\#T) \) by the universality of the localization.

2.4. Localizations of GWAs. As before, assume \( A \) is an algebra, \( a \) is a non-zero element in \( Z(A) \) and \( \sigma \in Aut(A) \). Recall that by Theorem 2.1 we identified the GWA \( W_{a,\sigma} \) with the subalgebra of the smash product \( A\#T \) generated by the algebra \( A \), and the elements \( x \) and \( ax^{-1} \).

**Theorem 2.3.** Consider the set \( S \subset Z(A) \) of the elements of the form \( \sigma^m(a^n) \) where \( n \in \mathbb{N} \) and \( m \in \mathbb{Z} \). Then the embedding of algebras \( W_{a,\sigma} \subset A\#T \) is a birational equivalence with respect to the Ore set generated by \( S \).

**Proof.** Now, by Lemma 2.2 we have that \( S^{-1}(A\#T) \) is naturally identified with \( (S^{-1}A)\#T \), and by Theorem 2.1 we see that the algebra \( (S^{-1}A)\#T \) is itself generated by \( S^{-1}A \), \( x \) and \( ax^{-1} \) since \( a \in S^{-1}A \) is now a unit.

2.5. Highest weight modules of GWAs. Assume \( A \) is an algebra with a distinguished non-zero element \( a \in Z(A) \) and an automorphism \( \sigma \in Aut(A) \). Let \( V \) be a representation over the GWA \( W_{a,\sigma} \). We have an (not necessarily exhaustive) increasing filtration of submodules of the form

\[
V^{[\ell]} = \{ v \in V \mid v \prec a\sigma^{-1}(a) \cdots \sigma^{-\ell}(a) = 0 \}
\]

defined for \( \ell \in \mathbb{N} \). Let us also define

\[
V^{[\infty]} = \bigcup_{\ell \geq 0} V^{[\ell]}.
\]

We define \( ht_{a,\sigma}(V) \) the height of \( V \) as the smallest integer \( \ell \) such that \( V^{[\ell]} = V^{[\infty]} \), and if no such integer exists we set \( ht_{a,\sigma}(V) = \infty \).

Assume \( V \) is a finite dimensional representation. Then \( h = ht_{a,\sigma}(V) \) is necessarily finite. Furthermore, if the height filtration satisfies \( V^{[h]} = V \), then we get the analogue of a highest weight module for the GWA \( W_{a,\sigma} \). Approaches for such cases can be seen in [12, 36].
Proposition 2.4. Let $S \subseteq Z(A)$ be the subset of elements of the form $\sigma^n(a^m)$ with $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, and let $S^{-1}W_{a,\sigma}$ be the localization of $W_{a,\sigma}$ at $S$. Assume $V$ is an arbitrary $W_{a,\sigma}$-module, and let $h = h_{t_a,\sigma}(V)$ be finite. Then $S^{-1}V := V \otimes_{W_{a,\sigma}} S^{-1}W_{a,\sigma}$ is isomorphic to $S^{-1}(V/V[h])$.

Proof. We consider the following short exact sequence of $W_{a,\sigma}$-modules

$$0 \to V[h] \to V \to V/V[h] \to 0$$

and use the fact that the functor $S^{-1}(\cdot)$ is exact. \hfill \Box

2.6. Morphisms of algebra extensions. An algebra $C$ together with a algebra monomorphism $A \to C$ is called an algebra extension. Given two extensions $i: A \to C$ and $i': A \to C'$ of a fixed algebra $A$, a morphism $f: (C, i) \to (C', i')$ of extensions is a commutative triangle of algebra morphisms of the form:

$$
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow{i} & & \downarrow{i'} \\
A & & A
\end{array}
$$

2.7. Isomorphisms of smash products with tori. In this section we consider the isomorphism problem for smash products with $\mathbb{T} = \mathbb{k}[x, x^{-1}]$ since all GWAs $W_{a,\sigma}$ are birationally equivalent to such products when $a$ is non-zero.

Theorem 2.5. Assume $\sigma$ and $\eta$ are two algebra automorphisms of $A$. Then the algebra extensions $A \subseteq A#_\sigma \mathbb{T}$ and $A \subseteq A#_\eta \mathbb{T}$ are isomorphic if and only if $\eta = u\sigma^\pm u^{-1}$ for some $u \in A^\times$.

Proof. Assume for now that $\sigma = u\eta u^{-1}$ or $\sigma = u\eta^{-1}u^{-1}$. Consider an arbitrary $v \in A$. In the first case define $\delta: A#_\sigma \mathbb{T} \to A#_\eta \mathbb{T}$ by letting $\delta(x) = ux$ and we get

$$\delta(xv) = u xv = u\eta(v)x = \sigma(v)ux = \delta(\sigma(v)x)$$

which implies $\delta$ is an algebra isomorphism. In the second case let $\delta(x) = ux^{-1}$ and we get

$$\delta(xa) = u x^{-1} a = u\eta^{-1}(a)x^{-1} = \delta(\sigma(a)x) = \sigma(a)ux^{-1}$$

which again implies $\delta$ is an isomorphism of algebras. On the opposite direction, assume $\delta: A#_\sigma \mathbb{T} \to A#_\eta \mathbb{T}$ is an isomorphism of algebra extensions. The one easily see that $\delta$ restricted to $A$ is identity, and $\mathbb{T}$ yields an algebra monomorphism where $\delta(x) = ux^{\pm 1}$ for some $u \in A^\times$. Note that the exponent of $x$ is necessarily $\pm 1$ since $\delta^{-1}$ is also such a monomorphism. Thus $\sigma = u\eta^{\pm 1}u^{-1}$ as expected. \hfill \Box

Notice that the automorphism $\sigma$ and its inverse $\sigma^{-1}$ extended to $A#_\sigma \mathbb{T}$ are now inner automorphisms. Now, we have the following result:

Corollary 2.6. If $\sigma \in Aut(A)$ is an inner automorphism given by a unit $u \in A^\times$ then the smash product $A#_\sigma \mathbb{T}$ is isomorphic to the direct product $A \times \mathbb{T}$. 

3. Homology of GW As

3.1. Homology of smash products with tori. We have the following result since \( T \) is a 1-smooth algebra.

Proposition 3.1. Let \( \sigma \in \text{Aut}(A) \) and assume \( \sigma \) acts on \( CH_*(A) \) diagonally extending the action on \( A \). Let \( CH_*(A)_T \) and \( CH_*(A)^T \) respectively be the complex of coinvariants and invariants of \( \sigma \). Then

\[
HH_n(A\# T) \cong H_n(CH_*(A)_T) \otimes T \oplus H_{n-1}(CH_*(A)^T) \otimes T.
\]

Proof. By Corollary 1.4 we get

\[
HH_n(A\# T) = H_n(CH_*(A, A\# T)_T) \oplus H_{n-1}(CH_*(A, A\# T)^T)
\]

since \( T \) is 1-smooth. We start by splitting \( CH_*(A, A\# T) \) as

\[
CH_*(A, A\# T) = CH_*(A) \otimes T.
\]

Then the difference between the left and right actions is given by

\[
(a \otimes a' x^m) \triangleleft x - x \triangleright (a \otimes a' x^m) = (a \otimes a' x^{m+1}) - (\sigma(a) \otimes \sigma(a') x^{m+1}) =
\]

for every \( a \otimes a' x^m \) in \( CH_*(A, A\# T) \). This means

\[
CH_*(A, A\# T)_T = CH_*(A)_T \otimes T \quad \text{and} \quad CH_*(A, A\# T)^T = CH_*(A)^T \otimes T.
\]

The result follows. \( \square \)

3.2. Algebraic and separable endomorphisms. We call an algebra endomorphism \( \sigma \in \text{End}(A) \) algebraic if there is a polynomial \( f(t) \in \mathbb{k}[t] \) such that \( f(\sigma) = 0 \) in \( \text{End}(A) \). For an algebraic endomorphism \( \sigma \) of \( A \), the monic polynomial \( f(t) \) with the minimal degree that satisfies \( f(\sigma) = 0 \) is called the minimal polynomial of \( \sigma \). We call an algebraic endomorphism \( \sigma \in \text{End}(A) \) as separable if the minimal polynomial of \( \sigma \) is separable.

Notice that all endomorphisms of a finite dimensional \( \mathbb{k} \)-algebra are algebraic. Regardless of the dimension, all automorphisms of finite order and all nilpotent non-unital endomorphisms are also algebraic. If \( \mathbb{k} \) has characteristic 0, automorphisms of finite order are separable, but nilpotent non-unital endomorphisms are not.

3.3. Algebras with separable automorphisms. For a fixed algebraic automorphism \( \sigma \in \text{Aut}(A) \), let \( \text{Spec}(\sigma) \) be the set of unique eigen-values of \( \sigma \), and let \( A^{(\lambda)} \) be the \( \lambda \)-eigenspace of \( \sigma \) corresponding to \( \lambda \in \text{Spec}(\sigma) \).

Theorem 3.2. Assume \( \sigma \in \text{Aut}(A) \) is separable with minimal polynomial \( f(x) \), and let \( B \) be the quotient \( \mathbb{k}[x]/(f(x)) \). Then

\[
H_n(A\# T) = H_n(CH_*(A)) \otimes T \oplus H_{n-1}(CH_*(A)) \otimes T
\]

and

\[
H_n(A\# B) = H_n(CH_*(A)) \otimes B
\]

where \( CH_*(A) \) is generated by homogeneous tensors of the form

\[
a_0 \otimes \cdots \otimes a_n \quad \text{with} \quad a_i \in A^{(\lambda_i)} \quad \text{and} \quad \lambda_1 \cdots \lambda_n = 1
\]

for every \( n \geq 0 \).
Proof. Notice that the distributive law $T \otimes A \rightarrow A \otimes T$ extends to a distributive law $B \otimes A \rightarrow A \otimes B$. Thus $A \# B$ is well-defined. Moreover, since $f(x)$ is separable, $B$ is a product of a finite number of copies of $k$, and therefore, is 0-smooth. Then the result for $A \# B$ immediately follows from Corollary [1.6]. On the other hand, $\text{CH}_*(A)_T = \text{CH}_*(A)_B = \text{CH}_*(A)^B = \text{CH}_*(A)^T$. Then the result for $A \# T$ follows from Proposition [3.1]. □

Note that Theorem 3.2 solves the smoothness problem (namely Hochschild homological dimension being finite) for smash products with $T$, and therefore the birational smoothness (namely Hochschild homological dimension being finite up to birational equivalence) problem for all GWAs, provided that the action is implemented via a separable automorphism. Namely, a smash product with $Z$ via a separable automorphism is smooth if and only if the complex subcomplex of invariants $\text{CH}_*(A)$ has bounded homology.

3.4. Localization of GWAs in homology. In the this subsection we solve the birational smoothness problem for all GWAs without requiring automorphism to be separable.

Consider the set $S$ of elements of the form $\sigma^m(a^n)$ in $Z(A)$ where $n \in N$ and $m \in Z$. Let $k(S)$ be the (commutative) subalgebra of $A$ generated by $S$, and let $k(S)_S$ be its localization at $S$. Then we have that $A_S = A \otimes_{k(S)} k(S)_S$. Now let $k(S)_T$ be the algebra of coinvariants of $k(S)$ which is given by the following quotient

$$k(S)_T := \frac{k(S)}{(\sigma(s) - s \mid s \in S)}$$

Corollary 3.3. We have

$$HH_n(S^{-1}W_{a,\sigma}) \cong HH_n((S^{-1}A)\#T)$$

$$\cong H_n(\text{CH}_*(A)_T \otimes_{k(S)_T} S^{-1}(k(S)_T) \otimes T)$$

$$\quad \oplus H_{n-1}(\text{CH}_*(A) \otimes_{k(S)} S^{-1}k(S))^T \otimes T$$

where we view $\text{CH}_*(A)$ as an $k(S)$-module and $k(S)_T$-module on the coefficient.

Proof. By Theorem [2.3] we have $S^{-1}W_{a,\sigma} \cong (S^{-1}A)\#T$. Now, we consider the algebra extension $S^{-1}A \subseteq (S^{-1}A)\#T$ for which by [27] there is a spectral sequence whose first page is

$$E_{p,q}^1 = H_q(S^{-1}A, \text{CH}_p(S^{-1}A\#T|S^{-1}A)) = H_q(S^{-1}A, \text{CH}_p(T, S^{-1}A\#T))$$

that converges to $HH_*(S^{-1}A\#T)$. Since $S \subseteq Z(A)$, by [8] we know that

$$E_{p,q}^1 \cong H_q(A, \text{CH}_p(T, S^{-1}A\#T)) \cong H_q(A, \text{CH}_p(T, S^{-1}A\#T)).$$

Thus we have an isomorphism of graded vector spaces of the form $HH_*(S^{-1}A\#T) \cong H_*(A\#T, S^{-1}A\#T)$. Then by Proposition [3.1] we get

$$HH_n(S^{-1}W_{a,\sigma}) \cong H_n(\text{CH}_*(A, S^{-1}A)_T) \otimes T \oplus H_{n-1}(\text{CH}_*(A, S^{-1}A)^T) \otimes T.$$
4. Homology Calculations

4.1. The rank-1 Weyl algebra. The ordinary rank-1 Weyl algebra $A_1$ is the $k$-algebra defined on two non-commuting indeterminates $x$ and $y$ subject to the relations

$$xy - yx = 1.$$ 

One can define $A_1$ as a GWA if we let $A = \mathbb{k}[t]$ where we set the distinguished element $a = t$. We define $\sigma$ to be the algebra automorphism of $A$ given by $(\sigma f)(t) = f(t-1)$ for every $f(t) \in A$. Then the GWA $W_{a,\sigma}$ is the ordinary Weyl algebra $A_1$. See [5, Ex.2.3].

Since $a = t$ is not a unit in $A$ we see that $W_{t,\sigma}$ is the proper subalgebra generated by $x$ and $tx^{-1}$ of the smash product $\mathbb{k}[t]\#T$.

Now, let $S$ be the multiplicative system generated by elements of the form $(t-m)$ where $m \in \mathbb{Z}$. Since there is no non-constant rational function invariant under the action $\sigma(f(t)) = f(t-1)$, we get that $\text{CH}_*(S^{-1}\mathbb{k}[t]^T) = \text{CH}_*(\mathbb{k})$. Next, we see that the subalgebra generated by $S$ is $A = \mathbb{k}[t]$ itself. Moreover, since $\sigma(t) - t = 1$ we get that $\mathbb{k}\langle S \rangle^T$ is zero, and therefore, we get

$$HH_n(S^{-1}A_1) = \begin{cases} \mathbb{T} & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

for every $n \geq 0$.

4.2. The enveloping algebra $U(\mathfrak{sl}_2)$. The universal enveloping algebra of $\mathfrak{sl}_2$ is given by the presentation

$$\mathbb{k}\{E, F, H\} / \langle EH - (H - 2)E, FH - (H + 2)F, EF - FE - H \rangle.$$ 

The center of this algebra is generated by the Casimir element

$$\Omega = 4FE + H(H + 2) = 4EF + H(H - 2).$$

In this Subsection, we would like to write a generalized Weyl algebra isomorphic to $U(\mathfrak{sl}_2)$.

Let $A = \mathbb{k}[c, t]$ and $a = c - t(t + 1)$. Define $\sigma$ to be the algebra automorphism defined by $(\sigma f)(c, t) = f(c, t-1)$ for every $f(c, t) \in A$. In this case $W_{a,\sigma}$ is generated by $c, t, x$ and $(c - t(t + 1))x^{-1}$ in the smash product algebra $A^\#T$. The GWA $W_{a,\sigma}$ is isomorphic to $U(\mathfrak{sl}_2)$ via an isomorphism defined as

$$H \mapsto 2t, \quad E \mapsto x, \quad F \mapsto (c - t(t + 1))x^{-1},$$

see [13, Ex. 2.2].

Let us define $S$ to be the multiplicative system generated by elements of the form

$$c - (t - n)(t - n - 1), \text{ for } n \in \mathbb{Z}.$$ 

Then $S^{-1}W_{t,\sigma}$ is isomorphic to $(S^{-1}\mathbb{k}[c, t])^T$ and $\text{CH}_*(S^{-1}A^T) = \text{CH}_*(\mathbb{k}[c])$. Moreover, the subalgebra of $A = \mathbb{k}[c, t]$ generated by $S$ is $A$ itself and since $\sigma(t) - t = 1$, we again get that $\mathbb{k}(S)^T = 0$. Therefore

$$HH_n(S^{-1}U(\mathfrak{sl}_2)) = \begin{cases} \mathbb{k}[c] \otimes \mathbb{T} & \text{if } n = 1, 2 \\ 0 & \text{otherwise} \end{cases}.$$
4.3. **Primitive quotients of** $U(\mathfrak{sl}_2)$. One can also consider $B_\lambda := W_{a,\sigma}/\langle c - \lambda \rangle$ where $W_{a,\sigma}$ is $U(\mathfrak{sl}_2)$ as we defined above. These algebras are also GWAs since we can realize them using $A = k[t], a = \lambda - t(t + 1)$ with the $\sigma$ given by $t \mapsto t - 1$. See [5, Sect. 3].

In this case, using a similar automorphism we used for $U(\mathfrak{sl}_2)$, we can replace $S$ with the multiplicative system generated by elements of the form $\mu - (t - n)$ and $\mu + (t - n)$ where $\mu \in k$ is fixed and $n$ ranges over $\mathbb{Z}$. Then $k\langle S \rangle = k[t]$ and $S^{-1}B_\lambda \cong (S^{-1}k[t])\# \mathbb{T}$. In this case, $CH_*(S^{-1}k[t])$ is $CH_*(k)$ and $k\langle S \rangle_T = 0$ since $\sigma(t) = t - 1$ as before. Then we get

$$HH_n(S^{-1}B_\lambda) \cong \begin{cases} \mathbb{T} & \text{if } n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

for every $n \geq 0$.

4.4. **Quantum 2-torus.** Fix an element $q \in k^\times$ which is not a root of unity. Let $A = k[t, t^{-1}]$ and let $a = t$ as in the case of the ordinary Weyl algebra. But this time, let us define $\sigma \in Aut(A)$ to be the algebra automorphism given by $(\sigma f)(t) = f(qt)$ for every $f(t) \in A$. The smash product algebra $A\# \mathbb{T}$ is the algebraic quantum 2-torus $\mathbb{T}_q^2$ and the GWA $W_{a,\sigma}$ is the quantum torus itself since $a = t$ is a unit.

Note that for every $u \in A$ and $m \in \mathbb{Z}$ we have $\sigma^m(u) \neq u$ unless $m = 0$ since $q$ is not a root of unity. Thus $CH_*(A)_T = CH_*(A) = CH_*^{(0)}(A)$ where

$$CH_*^{(0)}(A) = Span_k \left( t^{n_0} \otimes \cdots \otimes t^{n_m} \mid n_1, \ldots, n_m \in \mathbb{Z} \text{ with } 0 = \sum_i n_i \right)$$

which gives us just the group homology of $\mathbb{Z}$. Then by Proposition 3.1 we get

$$HH_n(\mathbb{T}_q^2) \cong k^{(2)} \otimes \mathbb{T}$$

for every $n \geq 0$ as expected.

4.5. **The quantum enveloping algebra** $U_q(\mathfrak{sl}_2)$. For a fixed $q \in k^\times$, the quantum enveloping algebra of the lie algebra $\mathfrak{sl}_2$ is given by the presentation

$$k\{K, K^{-1}, E, F\}$$

$$\langle KE - q^2EK, KF - q^{-2}FK, EF - FE = K - K^{-1} \rangle$$

As before, we assume $q$ is not a root of unity. There is an element $\Omega$ in the center of $U_q(\mathfrak{sl}_2)$ called the quantum Casimir element defined as

$$\Omega = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}.$$  

See [7, Sect.I.3]. Our first objective is to give a GWA that is isomorphic to $U_q(\mathfrak{sl}_2)$.

We start by setting $A = k[c, t, t^{-1}]$ together with

$$a = c - (q^{-1}t + qt^{-1})$$

and $\sigma \in Aut(A)$ given by $(\sigma f)(c, t) = f(c, q^2t)$ for every $f(c, t) \in k[c, t, t^{-1}]$. Define an algebra map $\gamma: W_{a,\sigma} \to U_q(\mathfrak{sl}_2)$ given on the generators by

$$t \mapsto K, \quad c \mapsto (q - q^{-1})\Omega, \quad x \mapsto (q - q^{-1})F, \quad ax^{-1} \mapsto (q - q^{-1})E.$$

Notice that the inverse of $\gamma$ is defined easily as

$$K \mapsto t, \quad E \mapsto \frac{ax^{-1}}{q - q^{-1}}, \quad F \mapsto \frac{x}{q - q^{-1}}.$$
One can show that both $\gamma$ and its inverse are well-defined by showing the relations are preserved.

Now, let $S$ be the multiplicative system in $A$ generated by the elements of the form
\[ c - (q^{-2n+1}t + q^{2n-1}t^{-1}), \quad n \in \mathbb{Z}. \]
In this case too, the subalgebra of $A$ generated by $S$ is $A$ itself. Then we have
\[ \text{CH}_*(S^{-1}A)^T \cong \text{CH}_*(k[c, t, t^{-1}]) \cong \text{CH}_*(A)^T. \]
On the other hand, since $\sigma(t) - t = (q^2 - 1)t$ and $t$ is a unit, we get that $k\langle S \rangle_T = 0$. Thus, as in the case of $U(sl_2)$ we get
\[ HH_n(S^{-1}W_{a,\sigma}) \cong HH_n(S^{-1}U_q(sl_2)) \cong \begin{cases} k[c] \otimes \mathbb{T} & \text{if } n = 1, 2, \\ 0 & \text{otherwise} \end{cases} \]
for every $n \geq 0$.

4.6. **The quantum matrix algebra** $\mathcal{O}_q(M_2)$. For a fixed $q \in k^\times$ the algebra $\mathcal{O}_q(M_2)$ of quantum $2 \times 2$ matrices is given by the presentation
\[ bc = cb, \quad ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad db = qbd, \quad dc = qcd, \quad ad - da = (q^{-1} - q)bc. \]
The quantum determinant
\[ \Omega = ad - q^{-1}bc = da - qbc \]
generates the center of this algebra. See [7] pp.4–8.

Now, let $A = k[u, v, w]$ with the distinguished element $u + qvw \in A$ where we set
\[ (\sigma f)(u, v, w) = f(u, q^{-1}v, q^{-1}w) \]
for every $f(u, v, w) \in A$. Then the GWA $W_{a,\sigma}$ is isomorphic to the subalgebra of $A\# \mathbb{T}$ generated by $A$, $x$ and $(u + qvw)x^{-1}$, and it is isomorphic to $\mathcal{O}_q(M_2)$ via
\[ u \mapsto \Omega, \quad v \mapsto b, \quad w \mapsto c, \quad x \mapsto a, \quad (u + qvw)x^{-1} \mapsto d, \]
and its inverse is
\[ a \mapsto x, \quad b \mapsto v, \quad c \mapsto w, \quad d \mapsto (u + qvw)x^{-1}. \]
Since $\mathcal{O}_q(GL_2)$ is obtained by localizing $\mathcal{O}_q(M_2)$ at the quantum determinant $u$, we see that $\mathcal{O}_q(GL_2)$ is isomorphic to the localization of $W_{a,\sigma}$ at the monoid generated by $u$, which itself is a GWA with $A$ replaced by $k[u, u^{-1}, v, w]$ with the remaining datum unchanged.

On the other hand, $\mathcal{O}_q(SL_2)$ is the quotient of $\mathcal{O}_q(M_2)$ by the two sided ideal generated by $u - 1$, and therefore, is again a GWA with the same datum where this time we replace $A$ by $k[u, v, w]/(u - 1)$. We also know that $\mathcal{O}_q(GL_2)$ is isomorphic (as algebras only) to $\mathcal{O}_q(SL_2) \times k[\Omega]$.

For the remaining of the section we are going to concentrate on $\mathcal{O}_q(SL_2)$ only given as the subalgebra of $k[v, w]\# \mathbb{T}$ generated by $v, w, x$ and $(1 + qvw)x^{-1}$.

Now, let $S$ be the Ore set generated by elements of the form $1 + q^{2n+1}vw$ for $n \in \mathbb{Z}$. Then $S^{-1}\mathcal{O}_q(SL_2)$ is isomorphic to $(S^{-1}k[v, w])\# \mathbb{T}$. In this case, since $q$ is not a root of unity, we get that
\[ \text{CH}_*(A, S^{-1}A)^T = \text{CH}_*(k) = \text{CH}_*(A)^T. \]
The subalgebra of $k[v, w]$ generated by $S$ is the polynomial algebra $k[vw]$ over the indeterminate $vw$. Since $\sigma(vw) - vw = (q^{-2} - 1)vw$ we get that $k[vw]_T = k$ Hence
\[ HH_n(S^{-1}\mathcal{O}_q(SL_2)) \cong k^{(2)} \otimes \mathbb{T} \]
for every $n \geq 0$. 
4.7. Quantum group \( \mathcal{O}_q(SU_2) \). Let us fix \( q \in \mathbb{k}^\times \). The algebraic quantum group \( \mathcal{O}_q(SU_2) \) is the noncommutative \(*\)-algebra generated by two non-commuting indeterminates \( s \) and \( x \) subject to the following relations

\[
(4.3) \quad x^*x = 1 - s^*s, \quad xx^* = 1 - q^2s^*s, \quad s^*s = ss^*, \quad xs = gsx, \quad xs^* = gs^*x.
\]

See [15] pg.4. One can write \( \mathcal{O}_q(SU_2) \) as a GWA \( W_{a,\sigma} \) by letting \( A = \mathbb{k}[s, s^*] \) with the distinguished element \( a \in A \) is defined as \( 1 - s^*s \) and \( (\sigma f)(s, s^*) = f(qs, qs^*) \) for every \( f(s, s^*) \in \mathbb{k}[s, s^*] \).

Let \( S \) be the multiplicative system in \( A \) generated by elements of the form \( q^{2n}s^*s - 1 \) for \( n \in \mathbb{Z} \). Then \( S^{-1}\mathcal{O}_q(SU_2) \) is isomorphic to \( (S^{-1}A)\#\mathbb{T} \) by Theorem 4.3. If we assume that \( q \in \mathbb{k}^\times \) is not a root of unity we get that

\[
\text{CH}_s(S^{-1}A)^\mathbb{T} = \text{CH}_s(\mathbb{k}) = \text{CH}_s(A)_\mathbb{T}
\]

We also see that the subalgebra of \( \mathbb{k}[s, s^*] \) generated by \( S \) is the polynomial algebra \( \mathbb{k}[ss^*] \), and since \( \sigma(ss^*) - ss^* = (q^2 - 1)ss^* \) we get that \( \mathbb{k}\langle S \rangle_\mathbb{T} = \mathbb{k} \). Then

\[
HH_n(S^{-1}\mathcal{O}_q(SU_2)) \cong \mathbb{k}^2(2) \otimes \mathbb{T}
\]

for every \( n \geq 0 \).

4.8. PodleÅ¡ spheres. For a fixed \( q \in \mathbb{k}^\times \), the algebra of functions \( \mathcal{O}_q(S^2) \) on standard PodleÅ¡ quantum spheres [13, 15] is the subalgebra of \( \mathcal{O}_q(SU_2) \) generated by elements \( s^*s, xs \) and \( s^*x^* \). This means \( \mathcal{O}_q(S^2) \) is the subalgebra of the smash product \( \mathbb{k}[s, s^*]#_\mathbb{R}\mathbb{T} \) generated by the elements \( s^*s, sx \) and \( s^* (1 - s^*s)x^{-1} \). One can give a presentation for the PodleÅ¡ sphere as

\[
(4.4) \quad xt = q^2tx, \quad yt = q^{-2}ty, \quad yx = -t(t-1), \quad xy = -q^2t(q^2t-1)
\]

then we get a GWA structure if we let \( A = \mathbb{k}[t] \) and where we set \( t = s^*s \) with \( a = -t(t-1) \) and \( (\sigma f)(t) = f(q^2t) \) for every \( f(t) \in A \).

Let \( S \) be the multiplicative system in \( A \) generated by the set \( \{ t(t - q^{2n}) \mid n \in \mathbb{Z} \} \) then \( S^{-1}\mathcal{O}_q(S^2) \cong (S^{-1}A)\#\mathbb{T} \). Instead of this generating set one can use \( \{ t \} \cup \{ (t - q^{2n}) \mid n \in \mathbb{Z} \} \) to get the same localization. Then we get that \( \mathbb{k}\langle S \rangle \) is \( A \) itself. If we assume that \( q \in \mathbb{k}^\times \) is not a root of unity we get that

\[
\text{CH}_s(S^{-1}A)^\mathbb{T} = \text{CH}_s^0(\mathbb{k}[t, t^{-1}]), \quad \text{and} \quad \text{CH}_s(A)_\mathbb{T} = \text{CH}_s(\mathbb{k}).
\]

In this case \( \mathbb{k}\langle S \rangle_\mathbb{T} = \mathbb{k} \) since \( \sigma(t) - t = (q^2 - 1)t \). Thus

\[
HH_n(S^{-1}\mathcal{O}_q(S^2)) \cong \mathbb{k}^2(2) \otimes \mathbb{T}
\]

for every \( n \geq 0 \).

4.9. Parametric PodleÅ¡ spheres. In [15] Hadfield defines another family of PodleÅ¡ spheres \( \mathcal{O}_{q,c}(S^2) \) given by a presentation equivalent to the following:

\[
xt = q^2tx, \quad x^*t = q^{-2}tx^*, \quad x^*x = c - t(t-1), \quad xx^* = c - q^2t(q^2t-1).
\]

If we set \( A = \mathbb{k}[c, t] \), and let the distinguish element \( a \in A \) be \( c - t(t-1) \) together with \( (\sigma f)(c, t) = f(c, q^2t) \) for every \( f(c, t) \in A \) we get a GWA structure on \( \mathcal{O}_{q,c}(S^2) \) similar to the GWA structure on \( U(\mathfrak{sl}_2) \) where we changed only the algebra automorphism from \( (\sigma f)(c, t) = f(c, t-1) \) to \( (\sigma f)(c, t) = f(c, q^2t) \).

Let \( S \) be the multiplicative system in \( A \) generated by the elements of the form \( c - q^{2n}t(q^{2n}t-1) \). If we assume that \( q \in \mathbb{k}^\times \) is not a root of unity we conclude that

\[
\text{CH}_s(S^{-1}A)^\mathbb{T} = \text{CH}_s(\mathbb{k}[c]) = \text{CH}_s(A)_\mathbb{T}
\]
which allows us to conclude

\[ HH_n(S^{-1}\mathcal{O}_{q,c}(S^2)) \cong \mathbb{k}^{(2)}_n \otimes \mathbb{k}[c] \otimes \mathbb{T} \]

for every \( n \geq 0 \).

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