Van’t Hoff’s Law for Active Suspensions: the Role of the Solvent Chemical Potential

Jeroen Rodenburg,1 Marjolein Dijkstra,2 and René van Roij1

1 Institute for Theoretical Physics, Center for Extreme Matter and Emergent Phenomena, Utrecht University, Princetonplein 5, 3584 CC Utrecht, The Netherlands
2 Soft Condensed Matter Group, Debye Institute for Nanomaterials Science, Utrecht University, Princetonplein 5, 3584 CC Utrecht, The Netherlands

In equilibrium, the osmotic pressure of a dilute suspension, in osmotic contact with a solvent reservoir via a semipermeable membrane, is given by Van’t Hoff’s law. We extend Van’t Hoff’s law to a suspension of active Brownian particles and predict the corresponding rise in the meniscus of the suspension to be on the order of micrometers or millimeters for experimentally realized self-propelled colloids. We show that activity increases the osmotic pressure purely by increasing the solvent chemical potential of the suspension with respect to the reservoir. Our Fokker-Planck calculations demonstrate this increase in chemical potential to depend on the colloid-membrane interaction potential. We conclude that the osmotic pressure is a state function of a state that itself is influenced by the colloid-membrane interaction potential.

In 1887, Van’t Hoff formulated his famous law stating that the osmotic pressure $\Pi$ of a dilute suspension of concentration $\rho$ at temperature $T$ equals the pressure $\rho k_B T$ of a dilute gas of the same concentration and temperature, where $k_B$ is the Boltzmann constant \[1\]. Van’t Hoff’s law is the ideal-gas version of the much wider notion of “colloids as big atoms”, which holds in equilibrium also at higher densities where effective (medium-induced) interactions are important \[2\]. In Van’t Hoff’s version, the total pressure of the suspension $P_{\text{col}}(\rho, \mu_s) = \rho k_B T + P_s(\mu_s)$ decomposes as the sum of the effective colloid-only pressure $\rho k_B T$, and a ‘background’ pressure $P_s(\mu_s)$ of the solvent at chemical potential $\mu_s$. In the typical experimental setup to measure osmotic pressure (Fig. \[1\]), the chemical potential $\mu_s$ is set by a solvent reservoir that connects to the suspension via a membrane permeable to solvent only. The net force per unit area exerted on the membrane defines the osmotic pressure, and results from the difference in suspension pressure $P_{\text{col}}(\rho, \mu_s)$ and reservoir pressure $P_s(\mu_s)$ at either side. As this pressure difference induces a height difference $H$ between the two menisci, the osmotic pressure $\Pi \sim H$ can be directly inferred.

Van’t Hoff’s law does not apply to non-equilibrium suspensions of active particles that constantly convert energy into directed motion, such as swimming bacteria \[7\] or artificial microswimmers \[8\]. Not only are these systems promising for applications in e.g. self-assembly \[9\] \[10\] \[11\] and targeted cargo transport \[12\] \[13\], they also display remarkable phase behaviour \[14\] \[15\] that calls for an underpinning thermodynamic framework \[16\] \[17\]. An essential prerequisite for such a framework is that thermodynamic properties can be expressed as functions of variables that characterize the system state, a seemingly trivial condition that was nonetheless questioned for the osmotic pressure \[29\].

In both theoretical and simulation-based investigations of the pressure, the active suspension is typically modelled as an effective one-component system of particles subject to a propulsion force, under the implicit assumption that the underlying solvent state is unaffected. However, as the propulsion force is internal \[12\], the solvent pressure is affected, by the opposite reaction force \[31\].

In this Letter, we apply this insight to extend Van’t Hoff’s law to active suspensions. We predict the corresponding rise $\Delta H$ in the fluid meniscus to be experimentally measurable. We show that the osmotic pressure increases with activity purely due to a difference in solvent chemical potential that develops between the suspension and the reservoir. Remarkably, we demonstrate this difference to generally depend on the details of the colloid-membrane interactions. We conclude that the osmotic pressure is a state function of a state that itself is affected by the colloid-membrane interaction potential.
We model the effective one-component system of suspended particles as overdamped active Brownian particles (ABPs) [43]. Every particle is characterized by a three-dimensional position \( \mathbf{r} \) and by an orientation \( \mathbf{\hat{e}} \) that represents a polar angle \( \theta \) and an azimuthal angle. The density \( \psi(\mathbf{r}, \mathbf{\hat{e}}, t) \) of the probability to find a particle at position \( \mathbf{r} \) with orientation \( \mathbf{\hat{e}} \) at time \( t \) satisfies the Fokker-Planck equation [44]

\[
\partial_t \psi(\mathbf{r}, \mathbf{\hat{e}}, t) = -\nabla \cdot \mathbf{j}(\mathbf{r}, \mathbf{\hat{e}}, t) - \nabla_\mathbf{\hat{e}} \cdot \mathbf{j}_\mathbf{\hat{e}}(\mathbf{r}, \mathbf{\hat{e}}, t).
\]

The translational and rotational fluxes,

\[
\mathbf{j} = [-\beta D_t \nabla \psi(\mathbf{r}, \mathbf{\hat{e}}) + \psi_0 \mathbf{\hat{e}} - D_t \nabla \mathbf{\hat{e}}] \psi(\mathbf{r}, \mathbf{\hat{e}}, t) \quad \text{and} \quad \mathbf{j}_\mathbf{\hat{e}} = [-\beta D_t \nabla_\mathbf{\hat{e}} \psi(\mathbf{r}, \mathbf{\hat{e}}) - D_t \nabla_\mathbf{\hat{e}}] \psi(\mathbf{r}, \mathbf{\hat{e}}, t),
\]

are driven by (i) external forces and torques generated by the external potential \( V(\mathbf{r}, \mathbf{\hat{e}}) \) acting on every particle, (ii) a constant self-propulsion force corresponding to propulsion speed \( \psi_0 \) acting along every particle’s orientation \( \mathbf{\hat{e}} \), and (iii) Brownian forces and torques giving rise to translational and rotational Brownian motion with diffusion coefficients \( D_t \) and \( D_r \), respectively. Furthermore, \( \beta \equiv (k_B T)^{-1} \). In order to focus on the essential physics, we follow Van’t Hoff and ignore effective colloid-colloid interactions. For simplicity, we focus on a steady state with an external potential only due to a semipermeable membrane parallel to the \( xy \)-plane, i.e. \( V(\mathbf{r}, \mathbf{\hat{e}}) = V(z, \theta) \), such that \( \psi(\mathbf{r}, \mathbf{\hat{e}}, t) = \psi(z, \theta) \). The potential \( V(z, \theta) \) corresponds to the setting of Fig. 1 and is assumed to decay from \( \infty \) in the infinitely large reservoir, located at \( z < 0 \) and containing \( z \)-coordinate \( z_{\text{res}} \), to 0 in a range of \( z \)-values \( z_{\text{res}} \ll z \ll 0 \). The suspension, located at \( z > 0 \), contains \( z \)-coordinate \( z_b > 0 \) characterizing its bulk. The zeroth and first moment of the probability density \( \psi(z, \theta) \) define the density \( \rho(z) \equiv \int \psi(z, \theta) \, d\theta \) and the polarization \( m(z) \equiv \int \psi(z, \theta) \cos \theta \, d\theta \) of the ABPs. The zeroth moment of the Fokker-Planck equation [44] yields the continuity equation \( \partial_z J(z) = 0 \), which implies for a no-flux boundary condition that the particle flux \( J(z) = 0 \) for all \( z \). Here the colloid flux \( J(z) \) is such that the associated frictional force,

\[
J(z) = \frac{\psi_0}{\beta D_t} m(z),
\]

is balanced by the external body force \( f^s(z) \), the propulsion body force \( f^p(z) \), and the pressure gradient force \( -\partial_z P(z) \). The form of the propulsion body force,

\[
f^p(z) = \frac{\psi_0}{\beta D_t} m(z),
\]

is easily understood as the \( z \)-component of the sum of propulsion forces \( (\beta D_t)^{-1} \psi_0 \mathbf{\hat{e}} \) acting on individual colloids. The local pressure \( P(z) \equiv \rho(z) k_B T \) is the normal component of the local stress tensor, that represents momentum flux of the active colloids [44].

We emphasize that the propulsion body force \( f^p(z) \) is an internal force. By Newton’s third law, the force balance of the solvent,

\[
f^s(z) - f^p(z) - \partial_z P_s(z) = 0,
\]

contains the opposite body force \( -f^p(z) \) as a reaction force. In addition, the solvent is subject to an external body force \( f^s(z) \), that we assume to vanish for simplicity (i.e. we assume the solvent does not feel the membrane at all), and to the force associated with the gradient of the solvent pressure \( P_s(z) \). In equilibrium, where \( f^s(z) = 0 \), the solvent force balance [4] guarantees equal solvent pressures in the bulk suspension and solvent reservoir, i.e. \( \Delta P_s \equiv P_s(z_b) - P_s(z_{\text{res}}) = 0 \). In an active system, however, the existence of a nonzero propulsion force \( f^p(z) \) results in a difference in these solvent pressures, derived
in the SM \[44\] to be

\[
\Delta P_s = -\int_{z_{res}}^{z_0} dz f^p(z) = \frac{v_0^2}{6 D_r D_z} \rho k_B T \\
- \frac{v_0}{2 D_r} \int_{z_{res}}^{z_0} dz \int d\theta \psi(z, \theta) \sin(\theta) \partial_\theta V(z, \theta).
\]

The first term on the right-hand side of Eq. (6) corresponds to what is known as the swim pressure \[19\][20][32], \[52\][56] which we thus identify as a difference in solvent pressures. The second term on the right-hand side term of Eq. (6), present for particles experiencing a torque $-\partial_\theta V(z, \theta)$, is of special interest because it generally depends on the potential $V(z, \theta)$. This issue will be discussed later.

The force balance of the total suspension,

\[
f^s(z) + f^p(z) - \partial_z P_{tot}(z) = 0,
\]

simply follows as the sum of the colloid force balance \[3\] and the solvent force balance \[5\], where the pressure of the total suspension $P_{tot}(z) = P(z) + P_s(z)$ \[45\]. From this total force balance (7), the osmotic pressure $\Pi \equiv \int_{z_{res}}^{z_0} dz [f^s(z) + f^p(z)]$, defined as the magnitude of the force per unit area exerted on the membrane, is given by $P_{tot}(z_0) - P_s(z_{res})$. As the total bulk pressure decomposes into colloid and solvent contributions as $P_{tot}(z_m) = \rho k_B T + P_s(z_m)$, the osmotic pressure reads

\[
\Pi = \rho k_B T + \Delta P_s.
\]

In equilibrium, Eq. (8) reduces to Van’t Hoff’s result $\Pi = \rho k_B T$ on account of $\Delta P_s = 0$. Activity increases the osmotic pressure by increasing the solvent pressure with respect to the reservoir by an amount $\Delta P_s$ given by Eq. (8). Together, Eq. (8) and (6) constitute the generalization of Van’t Hoff’s law to active suspensions, which is the key result of this Letter.

To clarify these concepts further, we have solved the Fokker-Planck equation \[1\] numerically for a system of active spheres subject to a propulsion force characterized by Peclet number $\text{Pe} \equiv \frac{v_0}{(D_r D_z)^{1/2}}$, in a planar geometry modelling the setting of Fig. 1. The membrane is modelled by the soft potential $V(z) = \lambda k_B T(z/\ell)^2$ for $z < 0$ and $V(z) = 0$ for $z \geq 0$, acting on the colloids only, i.e. there is no torque. Here $\lambda = 1$ is the potential strength and $\ell \equiv (D_r D_z)^{1/2}$ the microscopic lengthscale. Fig. 2(a) shows the profile of the propulsion body force $f^p(z)$. Whereas $f^p(z) = 0$ for a passive system, an active system displays a nonzero polarization $\rho^a(z)$, and thus by Eq. (4) a propulsion body force $f^p(z)$, in the vicinity of the membrane directed towards the membrane. This well-known effect \[49\][51] is in this case caused by colloids persistently propelling ‘into’ the repulsive membrane. Fig. 2(b) shows the pressure profiles of the colloids $P(z)$, the solvent $P_s(z)$, and the total suspension $P_{tot}(z)$ for a passive system (Pe = 0). Here the reaction body force $-f^p(z) = 0$ for all $z$, and hence the solvent pressure $P_s(z)$ is constant. It is only due to the bulk colloid pressure $P(z_m) = \rho k_B T$ that the total bulk pressure $P_{tot}(z_m) > \rho k_B T$ is higher than the total reservoir pressure $P_{tot}(z_{res})$. The osmotic pressure

\[
\Pi = P_{tot}(z_0) - P_{tot}(z_{res})
\]

is therefore equal to $\rho k_B T$. The same profiles for an active system (Pe = 3), displayed in Fig. 2(c), show that the solvent bulk pressure $P_s(z)$ exceeds the solvent reservoir pressure $P_s(z_{res})$ by an amount $\Delta P_s$. This is caused by the reaction body force $-f^p(z)$, that pushes solvent towards the bulk, as pictured in Fig. 2(b). As a result, both the total bulk pressure $P_{tot}(z_0)$ and the osmotic pressure $\Pi$ exceed their passive values by an amount $\Delta P_s$.

Our expression for the osmotic pressure allows the prediction of the menisci’s height difference $H = \Pi/(\rho_m g)$ in the experiment sketched in Fig. 1. The mass density $\rho_m$ of the dilute suspension (colloid packing fraction: 0.01) we approximate by the mass density 1 kg/L of water, and $g$ is Earth’s gravitational acceleration. The predicted height differences $H$ are shown in Fig. 9 for spherical particles of radius $R$ that satisfy the Einstein relations for diffusion at room temperature. Whereas the passive osmotic pressure $\rho k_B T$ induces values $H_0$ too small to measure, activity induces an increase $\Delta H \sim \Delta P_s$ that brings $H$ up to the regime of micrometers \[53\][58][60] or even millimeters \[52\] for the larger values of propulsion speed $v_0$ and particle size $R$ of experimentally realized microswimmers.

The increase in solvent pressure $\Delta P_s$ is accompanied by an increase in solvent chemical potential $\Delta \mu_s$. Even though the solvent molecules are strictly speaking out of equilibrium as a result of the reaction forces exerted by the self-propelling colloids, this effect is very small, for the thermal velocity $(2k_B T/\mu_s)^{1/2} \sim 500$ m/s of a water molecule of mass $m_0$ is much larger than the propulsion speed $v_0 \sim 0.1 - 1000$ µm/s. Therefore, a meaningful intrinsic solvent chemical potential $\mu_s^{int}(z)$ can be defined...
in terms of the equilibrium (Gibbs-Duhem like) relation \[ \rho_s(z) \partial_z \mu_s^{\text{int}}(z) = \partial_z P_s(z). \] (9)

Hence, the difference in solvent pressure \( \Delta P_s \) between the bulk and the reservoir is accompanied by a difference in the intrinsic solvent chemical potential \( \Delta \mu_s = \mu_s^{\text{int}}(z_b) - \mu_s^{\text{int}}(z_{\text{res}}) = \int_{z_{\text{res}}}^{z_b} \frac{\partial_z P_s(z)}{\rho_s(z)}. \) (10)

We can thus rephrase our findings as follows. Activity increases the solvent chemical potential of the bulk suspension from the reservoir value \( \mu_s \) to \( \mu_s + \Delta \mu_s \). The total bulk pressure \( P_{\text{tot}}(\rho, \mu_s + \Delta \mu_s) = \rho k_B T + P_s(\mu_s + \Delta \mu_s) \) increases accordingly, such that the osmotic pressure \( \Pi = P_{\text{tot}}(\rho, \mu_s + \Delta \mu_s) - P_s(\mu_s) \), which is the difference between the total bulk pressure and the reservoir pressure, now equals \( \Pi = \rho k_B T + \Delta P_s \), where \( \Delta P_s = P_s(\mu_s + \Delta \mu_s) - P_s(\mu_s) \) is the difference in solvent pressures accompanying the difference in solvent chemical potentials.

We now turn to the second term of Eq. (6) for colloids experiencing a torque \(-\partial_\theta V(z, \theta)\). To investigate its implications, we have solved the Fokker-Planck equation (4) for active dumbbells, consisting of two point particles with separation \( \ell \). Both point particles are subject to the same potential \( V(z) \) as before (but for several strength parameters \( \lambda \)), such that a nonzero torque \(-\partial_\theta V(z, \theta)\) is exerted on the dumbbell. More details can be found in the SM (4). Fig. 4 shows the resulting increase in solvent pressure \( \Delta P_s \) as a function of the potential strength \( \lambda \) for different activities \( \text{Pe} \). For \( \text{Pe} > 0 \), the increase \( \Delta P_s \) depends on \( \lambda \). The reason is that the torque, which is proportional to \( \lambda \), acts on the particles that propel ‘into’ the membrane. Thereby, the torque influences the shape of the polarization profile \( m(z) \), and thus the reaction body force \(-f^p(z)\) that pushes solvent towards the suspension. Consequently, the increase in solvent pressure \( \Delta P_s \) is influenced by the torque, and thus by the potential strength \( \lambda \). The same potential dependence was found in Ref. [29] for ellipsoidal particles under the assumptions that the distribution \( \psi(z, \theta) \) attains its bulk value already at \( z = 0 \), and that the effect of ellipses that feel the potential only partially is negligible. We thus confirm the conclusion of Ref. [29] that the second term of Eq. (6) depends on the precise form of the colloid-membrane interaction potential \( V(z, \theta) \), by a numerical solution \( \psi(z, \theta) \) that does not require any further assumptions.

Whereas in Ref. [29] this finding was reason to question whether the osmotic pressure is a state function, we emphasize it is the bulk state of the suspension itself that depends on the colloid-membrane potential. To appreciate its consequences, we note that in equilibrium the ensemble of reservoir and suspension is specified by the state variables \((\mu_s, \rho, T)\), since the solvent chemical potential of the reservoir \( \mu_s \) sets the same chemical potential in the suspension. The fact that for an active system the solvent pressure difference \( \Delta P_s \) and thereby also the chemical potential difference \( \Delta \mu_s \) generally depends on the colloidal-membrane interaction potential, implies that a complete specification of the ensemble requires an additional state variable, e.g. the bulk solvent chemical potential \( \mu^b_s \equiv \mu_s + \Delta \mu_s \). In fact, this whole framework can be extended to include effective colloid-colloid interactions (not shown here), in which case the activity, in e.g. the form of \( v_0 \), is furthermore required as a state variable. A complete set of (intensive) state variables therefore reads \((\mu_s, \mu^b_s, \rho, T, v_0)\). All the mentioned pressures, including the osmotic pressure, are state functions of these variables.

Crucial in our approach is that activity enters the colloid force balance (3) as the body force \( f^p(z) \), conform Ref. [31], whereas the local pressure \( P(z) = \rho(z) k_B T \) is of the same form as in equilibrium. This contrasts the approach of some authors who account for activity by modifying the local pressure \([30, 37, 39, 41]\). Our approach follows Speck and Jack [37], who showed the bulk colloid pressure \( \rho k_B T \) to represent momentum flux of colloids (in the non-interacting case) and thereby the normal component of the bulk stress tensor [62]. We show in the SM that \( P(z) = \rho(z) k_B T \) represents momentum flux outside the bulk as well [44].

Our results hold for a steady state in a planar geometry, with boundary conditions dictating both the particle flux \( J(z) \) and the solvent flow to vanish for all \( z \). For a suspension confined on one side by a semipermeable membrane in e.g. a pipe with open boundaries, we expect an associated fluid flow that is being pumped by the colloidal activity at the membrane.

Our work highlights active matter in a solvent as a
special class of out-of-equilibrium systems particularly amenable to a thermodynamic description. In contrast to non-equilibrium systems that are driven externally, the driving agent - the solvent - is part of the system itself. The consequence, that overall momentum is conserved, makes a thermodynamic framework based on conserved quantities more likely. This is exemplified by our result that the osmotic pressure is a state function, in sharp contrast to the mechanical pressure exerted on a wall by active matter on a substrate \[29\]. Applying a similar reasoning to the energy suggests, as a direction for future research, to take the fuel explicitly into account.

In conclusion, we generalize Van’t Hoff’s law and predict the osmotic pressure of an active suspension, along with an experimentally measurable rise of its meniscus. We show activity to increase the osmotic pressure of the suspension by increasing its intrinsic solvent chemical potential with respect to the reservoir. This increase, and thereby the bulk solvent state itself, generally depends on the colloid-membrane interaction potential.

**ACKNOWLEDGMENTS**

We acknowledge Bob Evans for helpful discussions. This work is part of the D-ITP consortium, a program of the Netherlands Organisation for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW).

[1] J. van’t Hoff, Z. Phys. Chem. 1, 481 (1887); Phil. Mag. 26, 81 (1888); in *The Modern Theory of Solution, Memoirs by Pfeffer, van't Hoff, Arrhenius, and Raoult*, edited by H. C. Jones (Harper & Brothers Publishers, New York and London, 1899).

[2] W. Poon, Science 304, 830 (2004).

[3] M. Dijkstra, R. van Roij, and R. Evans, J. Chem. Phys. 113, 4799 (2000).

[4] M. Dijkstra, R. van Roij, and R. Evans, Phys. Rev. E 59, 5744 (1999).

[5] N. Likos, Phys. Rep. 348, 267 (2001).

[6] A. A. Louis, J. Phys. Condens. Matt. 14, 1987 (2002).

[7] H. C. Berg, *E. coli in Motion* (Springer, 2003).

[8] J. Schwarz-Linek, J. Arlt, A. Jepson, A. Dawson, T. Vissers, D. Miroli, T. Pilizota, V. A. Martinez, and W. C. K. Poon, Coll. Surf. B 137, 1 (2016).

[9] S. J. Ebbens and J. R. Howe, Soft Matter 6, 726 (2010).

[10] H. H. Wensink, V. Kantsler, R. E. Goldstein, and J. Dunkel, Phys. Rev. E 89, 010302 (2014).

[11] V. Prymidis, H. Sielckcn, and L. Filion, Soft Matter 21, 4158 (2015).

[12] A. Snezhko and I. S. Aranson, Nat. Mater. 10, 698 (2011).

[13] L. Baraban, M. Tasinkevych, M. N. Popescu, S. Sanchez, S. Dietrich, and O. G. Schmidt, Soft Matter 8, 48 (2012).

[14] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Phys. Rev. Lett. 75, 1226 (1995).

[15] Y. Fily and M. C. Marchetti, Phys. Rev. Lett. 108, 235702 (2012).

[16] J. Palacci, S. Sacanna, A. P. Steinberg, D. J. Pine, and P. M. Chaikin, Science 339, 936 (2013).

[17] G. S. Redner, A. Baskaran, and M. F. Hagan, Phys. Rev. E 88, 012305 (2013).

[18] M. E. Cates and J. Tailleur, Annu. Rev. Condens. Matter Phys. 6, 219 (2015).

[19] X. Yang, M. L. Manning, and M. C. Marchetti, Soft Matter 10, 6477 (2014).

[20] S. C. Takatori, W. Yan, and J. F. Brady, Phys. Rev. Lett. 113, 028103 (2014).

[21] S. C. Takatori and J. F. Brady, Soft Matter 10, 9433 (2014).

[22] F. Ginot, I. Theurkauff, D. Levis, C. Ybert, L. Bocquet, L. Berthier, and C. Cottin-Bizonne, Phys. Rev. X.

[23] S. C. Takatori and J. F. Brady, Phys. Rev. E 91, 032117 (2015).

[24] A. Tiribocchi, R. Wittkowski, D. Marenduzzo, and M. E. Cates, Phys. Rev. Lett. 115, 188302 (2015).

[25] T. F. F. Farage, P. Krinninger, and J. M. Brader, Phys. Rev. E 91, 042310 (2015).

[26] C. Maggi, U. M. B. Marconi, N. Gnan, and R. D. Leonardo, Sci. Rep. 5, 10742 (2015).

[27] S. C. Takatori and J. F. Brady, Curr. Op. Colloid Interface Sci. 21, 24 (2016).

[28] A. P. Solon, J. Stenhammar, R. Wittkowski, M. Kardar, Y. Kafri, M. E. Cates, and J. Tailleur, Phys. Rev. Lett. 114, 198301 (2015).

[29] A. P. Solon, Y. Fily, A. Baskaran, M. E. Cates, Y. Kafri, M. Kardar, and J. Tailleur, Nature Phys. 11, 673 (2015).

[30] U. M. B. Marconi and C. Maggi, Soft Matter 11, 8768 (2015).

[31] W. Yan and J. F. Brady, Soft Matter 11, 6235 (2015).

[32] R. G. Winkler, A. Wysocki, and G. Gomper, Soft Matter 11, 6680 (2014).

[33] S. C. Takatori and J. F. Brady, Soft Matter 11, 7920 (2015).

[34] J. Białké, J. T. Siebert, H. Löwen, and T. Speck, Phys. Rev. Lett. 115, 098301 (2015).

[35] G. Falasco, F. Baldovin, K. Kroy, and M. Bärisi, New J. Phys. 18, 093043 (2016).

[36] M. Joyceux and E. Bertin, Phys. Rev. E 93, 032605 (2016).

[37] T. Speck and R. L. Jack, Phys. Rev. E 93, 062605 (2016).

[38] M. Dijkstra, S. Palival, J. Rodenburg, and R. van Roij, (2016), arXiv:1609.02773

[39] J. Blaschke, M. Maurer, K. Menon, A. Zöttl, and H. Stark, (2016), arXiv:1609.02478

[40] A. P. Solon, J. Stenhammar, M. E. Cates, Y. Kafri, and J. Tailleur, (2016), arXiv:1609.03483

[41] B. van der Meer, V. Prymidis, M. Dijkstra, and L. Filion, (2016), arXiv:1609.03867

[42] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, Rev. Mod. Phys. 85, 1143 (2013).

[43] P. Romanczuk, M. Bär, W. Ebeling, and B. Lindner, Eur. Phys. J. ST 202, 1 (2012).

[44] See Supplemental Material at [...] for the derivation of
Eq. (1) and Eq. (6): a proof that $\rho(z)k_B T$ represents momentum flux; and details of numerical solutions to Eq. (1).

[45] This decomposition of the total pressure is possible only upon neglecting colloid-solvent interactions, which is justified in the dilute regime.

[46] L. Rothschild, Phys. Rev. Lett. **198**, 1221 (1963).

[47] A. P. Berke, L. Turner, H. C. Berg, and E. Lauga, Phys. Rev. Lett. **101**, 038102 (2008).

[48] G. Li and J. X. Tang, Phys. Rev. Lett. **103**, 078101 (2009).

[49] J. Elgeti and G. Gompper, Europhys. Lett. **101**, 48003 (2013).

[50] C. Lee, New J. Phys. **15**, 055007 (2013).

[51] Y. Fily, A. Baskaran, and M. F. Hagan, Phys. Rev. E **91**, 012125 (2015).

[52] P. Moerman, H. Moyses, E. van der Wee, D. Grier, A. van Blaaderen, W. Kegel, J. Groenewold, and J. Brujic, “Interactions between self-propelled droplets mediated by concentration gradients,” (2016), in preparation.

[53] J. Vicario, R. Eelkema, W. R. Browne, A. Meetsma, R. M. L. Crois, and B. L. Feringa, Chem. Commun. **31**, 3936 (2005).

[54] D. A. Wilson, R. J. M. Nolte, and J. C. M. van Hest, Nat. Chem. **4**, 212 (2012).

[55] J. R. Howse, R. A. L. Jones, A. J. Ryan, T. Gough, R. Vafabakhsh, and R. Golestanian, Phys. Rev. Lett. **99**, 048102 (2007).

[56] J. Palacci, C. Cottin-Bizonne, C. Ybert, and L. Bocquet, Phys. Rev. Lett. **105**, 088304 (2010).

[57] G. Volpe, I. Buttinoni, D. Vogt, H. Kümmerer, and C. Bechinger, Soft Matter **7**, 8810 (2011).

[58] M. Mussler, S. Rafaï, P. Peyla, and C. Wagner, Euro. Phys. Lett. **101**, 54004 (2013).

[59] K. Drescher, J. Dunkel, L. H. Cisneros, S. Ganguly, and R. E. Goldstein, Proc. Natl. Acad. Sci. U.S.A. **108**, 10940 (2011).

[60] A. Thutupalli, R. Seemann, and S. Herminghaus, New J. Phys. **13**, 073021 (2011).

[61] R. Evans, Adv. Phys. **28**, 143 (1978).

[62] J. S. Rowlinson and B. Widom, Molecular Theory of Capillarity (Dover Publications, 1982).

[63] In addition, the volumes of the suspension and the reservoir are required, to specify the extensivity.