E₀-SEMIGROUPS AND PRODUCT SYSTEMS OF W*-BIMODULES

YUSUKE SAWADA

ABSTRACT. Product systems have been originally introduced to classify E₀-semigroups on type I factors by Arveson. We develop the classification theory of E₀-semigroups on a general von Neumann algebra and the dilation theory of CP₀-semigroups in terms of W*-bimodules. For this, we provide a notion of product system of W*-bimodules. This is a W*-bimodule version of Arveson’s and Bhat-Skeide’s product systems. There exists a one-to-one correspondence between CP₀-semigroups and units of product systems of W*-bimodules. The correspondence implies a construction of a dilation of a given CP₀-semigroup, a classification of E₀-semigroups on a von Neumann algebra up to cocycle equivalence and a relationship between Bhat-Skeide’s and Muhly-Solel’s constructions of minimal dilations of CP₀-semigroups.

1. INTRODUCTION

E₀-semigroups naturally arise in the quantum field theory. An E₀-semigroup is a semigroup of normal ∗-endomorphisms on a von Neumann algebra with σ-weak continuity, and the study of E₀-semigroups have been initiated by Powers in [15]. In [2], Arveson has provided the notion of product system and associated a product system with an E₀-semigroup on a type I factor. A product system \( \{ \mathcal{H}_t \}_{t>0} \) is a measurable family of Hilbert spaces \( \mathcal{H}_t \) parameterized by positive real numbers equipped with isomorphisms \( \mathcal{H}_s \otimes \mathcal{H}_t \cong \mathcal{H}_{s+t} \) with the associativity. Note that this is not his original definition, however they are the essentially same (see [11]). He also classified E₀-semigroups on type I factors by product systems up to cocycle conjugacy. E₀-semigroups on type I factors are roughly divided into type I, II and III by units of associated product systems. Every product systems have a numerical index and type I E₀-semigroups and product systems are completely classified by their indexes. We refer the reader to his monograph [4] for the physical background of E₀-semigroups and the theory of product systems. The theory of E₀-semigroups on von Neumann algebras which are not type I factors, has often been developed in terms of Hilbert modules. A Bhat-Skeide’s product system introduced in [8] is a family \( \{ E_t \}_{t\geq 0} \) of Hilbert bimodules over a C*-algebra satisfying a similar property with Arveson’s one with respect to tensor products of Hilbert bimodules. They classified E₀-semigroups on a C*-algebra by their product systems up to cocycle equivalence. In [1], Alevras has associated a product system of Hilbert bimodules with each E₀-semigroup on a \( \Pi_1 \) factor by a different way from Bhat-Skeide’s one and they form a complete invariant. In Skeide’s monograph [23], we have the classification theory of E₀-semigroups on the algebra \( \mathcal{B}^a(E) \) of all adjointable right \( A \)-linear maps on a Hilbert (von Neumann) \( A \)-module \( E \). On the other hand, Margetts-Srinivasan have introduced other invariants of E₀-semigroups on \( \Pi_1 \)

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factors in [12], and they have investigated non-cocycle conjugate $\mathcal{E}_0$-semigroups on factors in [13] by using the modular conjugation of Tomita-Takesaki theory.

A $\mathcal{CP}_0$-semigroups is a $\sigma$-weakly continuous semigroup of normal completely positive maps on a von Neumann algebra. The theory of Arveson’s product systems influenced the constructions of minimal dilations of $\mathcal{CP}_0$-semigroups. Roughly speaking, a dilation of a $\mathcal{CP}_0$-semigroup is an extension of it to an $\mathcal{E}_0$-semigroup in a suitable sense. Stinespring’s dilation theorem cannot be applied to $\mathcal{CP}_0$-semigroups, and some researchers have shown an existence of the minimal dilation of a given $\mathcal{CP}_0$-semigroup gradually. In [6] and [7], Bhat has shown it in the cases when $M$ is $\mathcal{B}(\mathcal{H})$ and a $\mathcal{C}^*$-algebra, respectively, in which we do not assume the $\sigma$-weakly continuity for semigroups. In [8], Bhat-Skeide constructed minimal dilations by a method which is valid for both of the von Neumann algebra case and the $\mathcal{C}^*$-algebra case. Also, we know Muhly-Solel’s ([14]) and Arveson’s ([4]) constructions, which differ from each other, of the minimal dilation of a $\mathcal{CP}_0$-semigroup on a von Neumann algebra. It is more difficult to construct an example of $\mathcal{E}_0$-semigroups than $\mathcal{CP}_0$-semigroups in general, however the existence of (minimal) dilations gives rise to $\mathcal{E}_0$-semigroups from $\mathcal{CP}_0$-semigroups. This is one of benefits of the dilation theory. Also, in [17], we have clarified a direct relationships between Bhat-Skeide’s and Muhly-Solel’s constructions of the minimal dilation of a discrete $\mathcal{CP}_0$-semigroup, which is different from one described by Skeide’s commutant duality in [20] and [21].

There have been no approaches to the classification theory of $\mathcal{E}_0$-semigroups on a von Neumann algebra and the dilation theory of $\mathcal{CP}_0$-semigroups by the $\mathcal{W}^*$-bimodule (which is not von Neumann bimodule) theory. In this paper, we attempt to give a $\mathcal{W}^*$-bimodule approach to their field by a way reflected by Bhat-Skeide’s works in [8].

We give an outline of this paper. We will recall the notions of $\mathcal{W}^*$-bimodule, relative tensor product, $\mathcal{CP}_0$-semigroup and $\mathcal{E}_0$-semigroup in Section 2.

In Section 3, we will provide a concept of product system of $\mathcal{W}^*$-bimodules, where adopted tensor products are relative tensor product introduced by Connes[9]. This is a direct extension of Arveson’s product system. A unit $\Xi$ of a product system $H$ of $\mathcal{W}^*$-$M$-bimodules induces an $\mathcal{E}_0$-semigroup on End($\mathcal{H}$), where $\mathcal{H}$ is the inductive limit of $H$ with respect to parameters. The $\mathcal{E}_0$-semigroup is called the dilation of the pair $(H, \Xi)$. We prove that cocycles of the dilation of the pair $(H, \Xi)$ and units of $H$ are the essentially same.

In Section 4, we will find a one-to-one correspondence between $\mathcal{CP}_0$-semigroups on a von Neumann algebra $M$ and pairs of product systems of $\mathcal{W}^*$-$M$-bimodules and units up to unit preserving isomorphism. The correspondence enables us to translate the $\sigma$-weak continuity of $\mathcal{CP}_0$-semigroups into a continuity of units. Also, the dilation of the pair associated with a given $\mathcal{CP}_0$-semigroup $T$, gives a dilation of $T$. The product system of $\mathcal{W}^*$-bimodules associated with a $\mathcal{CP}_0$-semigroup $T$ describes a relation between Bhat-Skeide’s and Muhly-Solel’s constructions of the minimal dilation of $T$. This is an extension to the continuous case of the relation in the discrete case in [17]. Some relationships among the two constructions and Arveson’s construction have not been clarified yet.

In Section 5, we consider the product system associated the heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ given by the Laplacian $\Delta$ on a compact Riemannian manifold, and its dilation by the method in Section 4, as an example. We will show that the $\mathcal{W}^*$-bimodules appearing in the construction of the product system associated with $\{e^{t\Delta}\}_{t \geq 0}$ are realized as $L^2$-spaces.
with respect to measures given by the heat kernel. We will reconstruct the dilation in more detail under this identification.

We can get the product system \( H^\alpha \) of \( W^\ast \)-bimodules (and the unit) from an \( E_0 \)-semigroup \( \alpha \) on a von Neumann algebra \( M \) as CP\_0-semigroups by the above correspondence. We will classify \( E_0 \)-semigroups on \( M \) by product systems of \( W^\ast \)-bimodules: two \( E_0 \)-semigroups \( \alpha \) and \( \beta \) on \( M \) are cocycle equivalent if and only if \( H^\alpha \cong H^\beta \) in Section 6. Hence, this enables us to classify \( E_0 \)-semigroups up to cocycle conjugacy by product systems of \( W^\ast \)-bimodules. Also, we will get a unit of a given \( E_0 \)-semigroup \( \theta \) on \( \Pi_1 \) factor from a unit of the product system \( H^\theta \) associated with \( \theta \).

2. Preliminaries

In this section, we recall the notions of \( W^\ast \)-bimodule, relative tensor product, CP\_0-semigroup, \( E_0 \)-semigroup, tensor product related to CP\_0-semigroup and partition, which will be used in the later sections.

\( W^\ast \)-bimodules are Hilbert spaces on which von Neumann algebras act from the left and the right. More precisely, for von Neumann algebras \( N \) and \( M \), a Hilbert space \( \mathcal{H} \) with normal *-representations of \( N \) and the opposite von Neumann algebra \( M^\ast \) of \( M \) is a \( W^\ast \)-\( N \)-\( M \)-bimodule if their representations commute. When \( N = \mathbb{C} \) or \( M = \mathbb{C} \), we call \( \mathcal{H} \) a right \( W^\ast \)-\( M \)-module or a left \( W^\ast \)-\( N \)-module, respectively. We write a \( W^\ast -N \)-\( M \)-bimodule, a right \( W^\ast \)-\( M \)-module and a left \( W^\ast \)-\( N \)-module by \( _N \mathcal{H}_M, \mathcal{H}_M \) and \( _N \mathcal{H} \), respectively.

Let \( N \) be a von Neumann algebra, \( \mathcal{H}_N \) and \( \mathcal{K}_N \) be right \( W^\ast \)-\( N \)-modules, and \( _N \mathcal{H}' \) and \( _N \mathcal{K}' \) be left \( W^\ast \)-\( N \)-modules. Hom\( (\mathcal{H}_N, \mathcal{K}_N) \) and Hom\( (\mathcal{H}_N', \mathcal{K}_N') \) are the sets of all right and left \( N \)-linear bounded maps, respectively. If \( \mathcal{H} = \mathcal{K} \) and \( \mathcal{H}' = \mathcal{K}' \), they are denoted by \( \text{End}(\mathcal{H}_N) \) and \( \text{End}(\mathcal{H}_N') \), respectively.

We denote the standard space of a von Neumann algebra \( M \) by \( L^2(M) \). The standard space \( L^2(M) \) contains all left and right GNS-spaces and we have \( [\phi] M \psi^\ast = \phi^\ast M [\psi] \) in \( L^2(M) \) and \( \phi^\ast M = [\phi] L^2(M) \) for all \( \phi, \psi \in M_\ast^\ast \). In particular, we have \( \phi^\ast M = L^2(M) = _M \phi^\ast \) for each faithful \( \phi \in M_\ast \). The observation will be helpful under the assumption which a von Neumann algebra has a faithful normal state in the later sections. We refer the reader to [25, Chapter IX], [28], [26] and [27] for details of the definition and properties of standard spaces included in the modular theory.

Now, we shall recall (left) relative tensor products. For more details, see [9, Chapter 5, Appendix B], [10] or [25, Chapter IX, Section 3]. Suppose \( \mathcal{H} \) is a \( W^\ast \)-\( M \)-\( N \)-bimodule and \( \mathcal{K} \) is a \( W^\ast \)-\( N \)-\( P \)-bimodule. Let \( \phi \) be a faithful normal state on \( N \). A vector \( \xi \in \mathcal{H} \) is called a (left) \( \phi \)-bounded vector if there is \( c > 0 \) such that \( \| \xi x \| \leq c \| \phi^\ast x \| \) for all \( x \in M \). We denote the set of all \( \phi \)-bounded vectors in \( \mathcal{H} \) by \( \mathcal{D}(\mathcal{H}; \phi) \). The (left) relative tensor product \( \mathcal{H} \otimes^\mathcal{N}_\phi \mathcal{K} \) is the completion \( \overline{\mathcal{D}(\mathcal{H}; \phi) \otimes_{\text{alg}} \mathcal{K}} \) with respect to an inner product defined by

\[
\langle \xi_1 \phi^{-\frac{1}{2}} \eta_1, \xi_2 \phi^{-\frac{1}{2}} \eta_2 \rangle = \langle \eta_1, \pi_\phi(\xi_1)^\ast \pi_\phi(\xi_2) \eta_2 \rangle,
\]

for each \( \xi_1, \xi_2 \in \mathcal{D}(\mathcal{H}; \phi) \) and \( \eta_1, \eta_2 \in \mathcal{K} \), where \( \pi_\phi(\xi) : L^2(N) \ni \phi^\ast x \mapsto \xi x \in \mathcal{H} \) and we usually use a notation \( \xi \phi^{-\frac{1}{2}} \eta \) rather than \( \xi \otimes \eta \). Also, we can define the right relative tensor product by right \( \phi \)-bounded vectors.
Remark 2.1. Left and right relative tensor products can be defined by the way which is independent on a choice of $\phi$ (see [2]). If we denote the left and the right relative tensor product by $\mathcal{H} \otimes^L \mathcal{K}$ and $\mathcal{H} \otimes^R \mathcal{K}$, respectively for $W^*$-bimodules $\mathcal{H}_N$ and $\mathcal{K}_P$, we already know the $W^*$-bimodule isomorphism $\mathcal{H} \otimes^R \mathcal{K} \cong \mathcal{H} \otimes^L \mathcal{K}$. In [13], we have constructed the isomorphism $\mathcal{H} \otimes^L \mathcal{K} \cong \mathcal{H} \otimes^R \mathcal{K}$ by the canonical way, and shown that the two $W^*$-bicategories of $W^*$-bimodules with left and right tensor products as tensor functors are monoidally equivalent.

Now, we provide the basic notions related with CP$_0$-semigroups and E$_0$-semigroups. A family $T = \{T_t\}_{t \geq 0}$ of normal UCP-maps $T_t$ on a von Neumann algebra $M$ is called a CP$_0$-semigroup if $T_0 = \text{id}_M$, $T_sT_t = T_{s+t}$ for all $s,t \geq 0$, and for every $x \in M$ and $\phi \in M_*$, the function $\phi(T_t(x))$ on $[0, \infty)$ is continuous. If each $T_t$ is a $*$-homomorphism, $T$ is called an E$_0$-semigroup. A CP$_0$-semigroup (E$_0$-semigroup) without the continuity is called an algebraic CP$_0$-semigroup (algebraic E$_0$-semigroup, respectively).

Example 2.2. Let $\{v_t\}_{t \geq 0}$ be a family of isometries $v_t$ in a von Neumann algebra $M$ such that $v_{s+t} = v_sv_t$ for all $s, t \geq 0$ and $v_0 = 1_M$. Suppose $\{v_t\}_{t \geq 0}$ is strongly continuous with respect to the parameter. If we define $T = \{T_t\}_{t \geq 0}$ by $T_t(x) = v_t^*vxv_t$ for each $x \in M$ and $t \geq 0$, then $T$ is a CP$_0$-semigroup. If each $v_t$ is unitary, $T$ is an E$_0$-semigroup.

Example 2.3. The CCR heat flow is a CP$_0$-semigroup $T$ which has the noncommutative Laplacian $\Delta$ as generators. This will be immediately and concretely defined by the Weyl system. For more details, see [10, Section 7].

Let $\mathcal{H} = L^2(\mathbb{R})$ and $M = \mathcal{B}(\mathcal{H})$. For $x = (x,y) \in \mathbb{R}^2$, the concrete Weyl operator is $W_x = \exp(\frac{ix}{2})U_xV_y$, where $\{U_x\}_{x \in \mathbb{R}}$ and $\{V_y\}_{y \in \mathbb{R}}$ are the unitary groups which have the position operator $Q$ and the momentum operator $P$ as generators, respectively, i.e.

$$(U_tf)(x) = e^{ix}f(x), \quad (V_tf)(x) = f(x+t)$$

for $f \in L^2(\mathbb{R})$ and $t, x \in \mathbb{R}$. Then, the family $\{W_x\}_{x \in \mathbb{R}^2}$ of the unitaries satisfies the Weyl relations

$$W_{x_1}W_{x_2} = \exp\left(\frac{i}{2}(x_2y_2 - x_1y_1)\right)W_{x_1 + x_2}$$

for $x_1 = (x_1,y_1), x_2 = (x_2,y_2) \in \mathbb{R}^2$. The CCR heat flow is defined as the unique CP$_0$-semigroup $T = \{T_t\}_{t \geq 0}$ on $M$ satisfying $T_t(W_x) = \exp(-t\|x\|^2)W_x$ for all $x \in \mathbb{R}^2$ and $t \geq 0$. More precisely, we define $T_t$ for $t \geq 0$ by a weak integral $T_t(x) = \int_{\mathbb{R}^2} W_{\mathbf{x}}W_{\mathbf{x}}d\mu_t(x)$ for each $x \in M$, where $\mu_t$ is the probability measure whose Fourier transformation is $u_t(x) = \exp(-t\|x\|^2)$.

According to Stinespring’s dilation theorem, for a UCP-map $T$ from a C$^*$-algebra $A$ into $\mathcal{B}(\mathcal{H})$, there exist a Hilbert space $\mathcal{K}$, a unital representation of $A$ on $\mathcal{K}$ and an isometry $v : \mathcal{H} \to \mathcal{K}$ such that $T(a) = v^*\pi(a)v$ for all $a \in A$. However, Stinespring’s theorem does not apply to CP$_0$-semigroup. The notion of dilation of CP$_0$-semigroups are introduced as follows:

Definition 2.4. Let $T = \{T_t\}_{t \geq 0}$ be a CP$_0$-semigroup on a von Neumann algebra $M$. A dilation of $T$ consists of a von Neumann algebra $N$, a projection $p \in N$ and an E$_0$-semigroup $\{\theta_t\}_{t \geq 0}$ on $N$ such that $M = pNp$ and $T_t(x) = p\theta_t(x)p$ for all $x \in M$ and
t ≥ 0. In addition, if N is generated by θ_{[0,∞)}(M) and the central support of p in N is 1_N, the dilation is said to be minimal.

Note that a minimal dilation of a CP₀-semigroup is unique (if it exists). The existence of minimal dilations is proved by Bhat–Skeide and Muhly-solo. In Section 4, a relation between the two constructions will be clarified. Arveson also constructed the minimal dilation by other approach in [4] (or [3]).

The notion of cocycle for E₀-semigroups which is useful for the classification of E₀-semigroups, is introduced as the following definition.

**Definition 2.5.** Let θ be an E₀-semigroup on a von Neumann algebra M. A family \( w = \{w_t\}_{t≥0} \subseteq M \) is called a right cocycle for θ if \( w_{s+t} = \theta_t(w_s)w_t \) for all \( s, t ≥ 0 \). If each \( w_t \) is unitary (contractive), then \( w \) is called a right unitary (contractive, respectively) cocycle.

**Definition 2.6.** Two E₀-semigroups \( α \) and \( β \) on a von Neumann algebra M are said to be cocycle equivalent if there exists a strongly continuous right unitary cocycle \( w \) such that \( β_t(x) = w^*_tα_t(x)w_t \) for all \( t ≥ 0 \) and \( x ∈ M \). Then, the E₀-semigroup \( β \) is called the cocycle perturbation of \( α \) with respect to \( w \).

Let \( α \) and \( β \) be E₀-semigroups on von Neumann algebras M and N, respectively, and \( Φ : M → N \) is a *-isomorphism. The conjugation \( β^Φ \) of \( β \) with respect to \( Φ \) is an E₀-semigroup on M defined by \( β^Φ_t = Φ^{-1} ◦ β_t ◦ Φ \) for each \( t ≥ 0 \). If \( β^Φ \) is a cocycle perturbation of \( α \), we say \( α \) and \( β \) are cocycle conjugate.

We will establish a product system of W∗-bimodule from a given CP₀-semigroup in Section 4. For this, we prepare a W∗-bimodule equipped with a information of a given normal UCP-map as follows. Let T be a normal UCP-map on a von Neumann algebra M and \( H \) a W∗-M-N-bimodule. We define \( M ⊗_N H \) as the completion of the algebraic tensor product \( M ⊗_{alg} H \) with respect to an inner product defined by

\[ \langle x ⊗ ξ, y ⊗ η \rangle = \langle ξ, T(x^*y)η \rangle \]

for each \( x, y ∈ M \) and \( ξ, η ∈ H \). The Hilbert space \( M ⊗_N H \) has the canonical W∗-M-N-bimodule structure: \( a(x ⊗ ξ) = (ax) ⊗ (ξb) \) for each \( a, x ∈ M, b ∈ N \) and \( ξ ∈ H \). When \( H = L^2(M) \), we can provide the following formula related to the relative tensor products and normal UCP-maps, which will be used for computing inner products in later arguments.

**Proposition 2.7.** Let T be a normal UCP-map on a von Neumann algebra M. For \( x, y ∈ M \), we have \( x ⊗ yφ^2z = D(M ⊗_T L^2(M); φ) \). Moreover, we have

\[ π_φ(x_1 ⊗ y_1φ^\frac{1}{2}z_1)π_φ(x_2 ⊗ y_2φ^\frac{1}{2}z_2) = y_1^T(x_1^*x_2)y_2 \]

for \( x_1, x_2, y_1, y_2 ∈ M \).

**Proof.** For \( x', y', z ∈ M \), we can compute as

\[ \langle π_φ(x_1 ⊗ y_1φ^\frac{1}{2}z_1)π_φ(x_2 ⊗ y_2φ^\frac{1}{2}z_2), x' ⊗ y'φ^\frac{1}{2}z' \rangle = \langle x_1 ⊗ y_1φ^\frac{1}{2}z_1, x' ⊗ y'φ^\frac{1}{2}z' \rangle = \langle φ^\frac{1}{2}z_1, y_1^T(x_1^*x')y'φ^\frac{1}{2}z' \rangle, \]

and hence we have \( π_φ(x_1 ⊗ y_1φ^\frac{1}{2}z_1)π_φ(x_2 ⊗ y_2φ^\frac{1}{2}z_2) = y_1^T(x_1^*x')y'φ^\frac{1}{2}z' \). Thus, we conclude that \( π_φ(x_1 ⊗ y_1φ^\frac{1}{2}z_1)π_φ(x_2 ⊗ y_2φ^\frac{1}{2}z_2)φ^\frac{1}{2}z = y_1^T(x_1^*x_2)y_2\phi^\frac{1}{2}z \).
Finally, we prepare notations related with partitions. We fix $t > 0$. Let $\mathfrak{P}_t$ be the set of all finite tuples $p = (t_1, \ldots, t_n)$ with $t_i > 0$ such that $\sum_{i=1}^n t_i = t$. For $p = (t_1, \ldots, t_n) \in \mathfrak{P}_t$, we define $\# p = n$. Let $p = (t_1, \ldots, t_n), q = (s_1, \ldots, s_m) \in \mathfrak{P}_t$. We define the joint tuple by $p \vee q = (t_1, \ldots, t_n, s_1, \ldots, s_m)$ for $p = (t_1, \ldots, t_n), q = (s_1, \ldots, s_m) \in \mathfrak{P}_t$. Also, we write $p \triangleright q$ if there exist partitions $q_i \in \mathfrak{P}_i$ for $i = 1, \ldots, m$ such that $p = q_1 \vee \cdots \vee q_m$. Let $\mathfrak{P}_0$ be the singleton of the empty tuple () satisfying $p \triangleright () = () \triangleright p = p$. Note that when we consider partitions of an interval $[0, t]$, treating $\mathfrak{P}_t$ or the set $\mathfrak{P}_i'$ of all finite tuples $(t_1, \ldots, t_n)$ such that $t = t_n > t_{n-1} > \cdots > t_1 > 0$ is equivalent because $\mathfrak{P}_i$ and $\mathfrak{P}_i'$ are order isomorphic via a map $\circ : \mathfrak{P}_i \to \mathfrak{P}_i'$ defined by $\circ(t_1, t_2, \ldots, t_n) = (\sum_{i=1}^t t_i, \sum_{i=1}^{t_2} t_i, \ldots, \sum_{i=1}^{t_n} t_i)$ for each $p = (t_1, \ldots, t_n) \in \mathfrak{P}_i$.

3. PRODUCT SYSTEMS OF $W^*$-BIMODULES AND MAXIMAL DILATIONS

In this section, we will introduce a new concepts of product system of $W^*$-bimodules and their units which are inspired by the definitions of Arveson’s and Bhat-Skeide’s product systems and units. We will construct an algebraic $E_0$-semigroup from a given unit of a product system of $W^*$-bimodules by taking the inductive limit.

Definition 3.1. Let $M$ be a von Neumann algebra and $H = \{\mathcal{H}_t\}_{t \geq 0}$ a family of $W^*$-$M$-bimodules with $\mathcal{H}_0 = L^2(M)$. If there exist bimodule unitaries $U_{s,t} : \mathcal{H}_s \otimes^M \mathcal{H}_t \to \mathcal{H}_{s+t}$ for each $s, t \geq 0$ such that

\[ U_{r,s+t}(id_{\mathcal{H}_s} \otimes^M U_{s,t}) = U_{r+s,t}(U_{r,s} \otimes^M id_{\mathcal{H}_t}) \]

for each $r, s, t \geq 0$, and $U_{0,t}$ and $U_{s,0}$ are the canonical identifications, then the pair $(H, \{U_{s,t}\}_{s \geq 0})$ is called a product system of $W^*$-$M$-bimodules, where the notation $\otimes^M$ denotes the relative tensor product over $M$.

Remark 3.2. Precisely speaking, associativity means that the following diagram commutes.

Here, the morphism $a$ is the associativity isomorphism which is discussed in detail in [15]. By [15, Theorem 3.2], we can choose either the left or the right relative tensor product. We will construct a product system of $W^*$-bimodules from a given CP$_0$-semigroup by left relative tensor products.

Remark 3.3. The notion of product system of $W^*$-bimodules is a direct extension of Arveson’s product system. Indeed, if $M = \mathbb{C}$, then a product system of $W^*$-$\mathbb{C}$-bimodules is just an Arveson’s product system without a measurable structure. For the purpose of this paper, measurable structures for product systems of $W^*$-bimodules are not necessary.

Definition 3.4. Let $M$ be a von Neumann algebra with a faithful normal state $\phi$ and $(H, \{U_{s,t}\}_{s \geq 0})$ a product system of $W^*$-$M$-bimodules. A family $\Xi = \{\xi(t)\}_{t \geq 0}$ of $\xi(t) \in \mathcal{D}(\mathcal{H}_t; \phi)$ is called a unit of $H$ with respect to $\phi$ if $\xi(0) = \phi^\frac{1}{2}$ and

\[ U_{s,t}(\xi(s)\phi^{-\frac{1}{2}}\xi(t)) = \xi(s + t) \]
for all \( s, t \geq 0 \).

If a unit \( \Xi = \{\xi(t)\}_{t \geq 0} \) satisfies \( \pi_\phi(\xi(t))^*\pi_\phi(\xi(t)) = 1_M \) \( (\|\pi_\phi(\xi(t))^*\pi_\phi(\xi(t))\| \leq 1) \), it is said to be unital (contractive, respectively).

We introduce a natural notion of isomorphism between product systems of \( W^*-\text{M-bimodules} \) and generating property for units as follows:

**Definition 3.5.** Let \( (H, \{U_{s,t}\}_{s,t \geq 0}) \) and \( (K, \{V_{s,t}\}_{s,t \geq 0}) \) be product systems of \( W^*-\text{M-bimodules} \). An isomorphism is a family \( u^\otimes = \{u_t\}_{t \geq 0} \) of \( M^*\)-bilinear unitaries \( u_t : \mathcal{H}_t \rightarrow \mathcal{K}_t \) satisfying

\[
V_{s,t}(u_s \otimes^M u_t) = u_{s+t} U_{s,t}
\]

for all \( s, t \geq 0 \). Then, the product system \( H \) is said to be isomorphic to \( K \) and we denote as \( H \cong K \).

**Definition 3.6.** Let \( H = \{\mathcal{H}_t\}_{t \geq 0} \) be a product system of \( W^*-\text{M-bimodules} \) with \( M\)-bilinear unitaries \( \{U_{s,t}\}_{s,t \geq 0} \). For \( t \geq 0 \) and \( p = (t_1, \cdots, t_n) \in \mathcal{P}_t \), we denote the \( M\)-bilinear unitary

\[
U(p) = U_{t_1,t_1'}(\text{id}_{t_1} \otimes U_{t_2,t_2'})(\text{id}_{t_1} \otimes U_{t_3,t_3'}) \cdots (\text{id}_{t_{n-2}} \otimes U_{t_{n-1},t_{n-1}'})(\text{id}_{t_{n-1}'} \otimes \cdots \otimes \text{id}_t)
\]

from \( \mathcal{H}_{t_1} \otimes^M \cdots \otimes^M \mathcal{H}_{t_{n-1}} \) onto \( \mathcal{H}_t \), where \( t'_i = t_{i+1} + \cdots + t_n \) and \( \text{id}_{t_1,\cdots,t_i} = \text{id}_{t_1} \otimes \cdots \otimes \text{id}_{t_i} \).

A unital unit \( \Xi = \{\xi(t)\}_{t \geq 0} \) with respect to \( \phi \) is said to be generating when the set

\[
\{U(p)(x_1 \xi(t_1)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} x_{n-1} \xi(t_{n-1})\phi^{-\frac{1}{2}} x_n \xi(t_n)y) \mid p \in \mathcal{P}_t, x_1, \cdots, x_n, y \in M\}
\]

is dense in \( \mathcal{H}_t \) for all \( t \geq 0 \).

**Remark 3.7.** Like Arveson’s and Bhat-Skeide’s product system, every product system of \( W^*-\text{M-bimodules} \) does not always have a unit with respect to a faithful normal state \( \phi \) on \( M \). We say that a product system of \( W^*-\text{M-bimodules} \) equipped with a unit with respect to \( \phi \) is \( \phi \)-spatial. From now, we fix a faithful normal state \( \phi \) on a von Neumann algebra \( M \) through this paper. When we say a unit merely, suppose that it is a unit with respect to \( \phi \) of a \( \phi \)-spatial product system.

Let \( \Xi \) be a unital unit of a \( \phi \)-spatial product system \( H \) of \( W^*-\text{M-bimodules} \). We shall define the inductive limit \( \mathcal{H} \) of \( H \) which depends on \( \Xi \) and an algebraic \( E_0 \)-semigroup on \( \text{End}(\mathcal{H}_M) \) as follows: for \( 0 \leq s \leq t \), we define a right \( M \)-linear isometry \( b_{t,s} : \mathcal{H}_s \rightarrow \mathcal{H}_t \) by

\[
b_{t,s}(\xi) = U_{t-s,s}(\xi(t-s)\phi^{-\frac{1}{2}} \xi)
\]

for each \( \xi \in \mathcal{H}_s \). Note that \( b_{t,s} \circ b_{r,s} = b_{t,r} \) for \( 0 \leq r \leq s \leq t \). Let \( \mathcal{H} \) be the inductive limit of the inductive system \( (H, \{b_{t,s}\}_{s \leq t}) \) and \( \kappa_t : \mathcal{H}_t \rightarrow \mathcal{H} \) the canonical embedding for each \( t \geq 0 \). The right \( W^*-\text{M-module} \) \( \mathcal{H} \) is called the inductive limit of the pair \( (H, \Xi) \).

**Theorem 3.8.** Fix \( t \geq 0 \). There exists a right \( M \)-linear unitary \( U_t : \mathcal{H} \otimes^M \mathcal{H}_t \rightarrow \mathcal{H} \)

**Proof.** For \( s \geq 0 \), \( \xi_s \in \mathcal{D}(\mathcal{H}_t; \phi) \) and \( \eta_t \in \mathcal{H}_t \), we define

\[
U_t((\kappa_s \xi_s)\phi^{-\frac{1}{2}} \eta_t) = \kappa_{s+t} U_{s,t}(\xi_s \phi^{-\frac{1}{2}} \eta_t).
\]
We shall show that

\[ \langle U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t), U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t') \rangle = \langle \xi_s\phi^{-\frac{1}{2}}\eta_t, \xi_s'\phi^{-\frac{1}{2}}\eta_t' \rangle \]

\[ = \langle \eta_t, \pi_\phi(\xi_s)\pi_\phi(\xi_s')\eta_t' \rangle = \langle \eta_t, \pi_\phi(\kappa_s\xi_s)\pi_\phi(\kappa_s\xi_s)'\eta_t' \rangle = \langle ((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t, (\kappa_s\xi_s)'\phi^{-\frac{1}{2}}\eta_t' \rangle. \]

This implies that for \( s \geq 0, \xi_s, \xi'_s \in \mathcal{H}_s \) and \( \eta_t, \eta'_t \in \mathcal{H}_t \), in the general case, we have

\[ \langle U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t), U_t((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t') \rangle \]

\[ = \langle U_t((\kappa_s+b_{s+r,s}\xi_s)\phi^{-\frac{1}{2}}\eta_t), U_t((\kappa_s+b_{s+r,s}\xi_s')\phi^{-\frac{1}{2}}\eta_t') \rangle \]

\[ = \langle ((\kappa_s+b_{s+r,s}\xi_s)\phi^{-\frac{1}{2}}\eta_t, (\kappa_s+b_{s+r,s}\xi_s')\phi^{-\frac{1}{2}}\eta_t') = ((\kappa_s\xi_s)\phi^{-\frac{1}{2}}\eta_t, (\kappa_s\xi_s')\phi^{-\frac{1}{2}}\eta_t'). \]

We shall check that \( U_t \) is surjective. In the case when \( s \leq t \), for \( \eta = \kappa_s\eta_s \in \mathcal{H} \), we can conclude that the image of \( (\kappa_0\phi^{\frac{1}{2}})\phi^{-\frac{1}{2}}\kappa_s\eta_s \) by \( U_t \) is \( \eta \). In the case when \( s > t \), let \( \mathcal{D} \) be a subspace of \( \mathcal{H}_{s-t} \otimes M \mathcal{H}_t \) spanned by vectors \( \eta_{s-t}\phi^{-\frac{1}{2}}\eta_t \) for all \( \phi \)-bounded vectors \( \eta_{s-t} \in \mathcal{H}_{s-t} \) and \( \eta_t \in \mathcal{H}_t \). For \( \eta = \kappa_s\eta_s \in \mathcal{H} \), \( \eta_s \) can be approximated by vectors \( U_{s-t}\eta_s \) for some \( \zeta \in \mathcal{D} \) and we have \( U_t((\kappa_s-\chi)\otimes \chi) = \kappa_s U_{s-t}\eta_t \) on \( \mathcal{D} \).

The von Neumann algebra \( M \) can be represented faithfully on \( \mathcal{H} \) by

\[ \pi(x)\xi = \kappa_0(x(x_0^*\xi)) \]

for each \( x \in M \) and \( \xi \in \mathcal{H} \). Note that \( \pi(M) \subset \text{End}(\mathcal{H}_M) \). Then, we can define an algebraic \( E_0 \)-semigroup \( \theta \) on the von Neumann algebra \( \text{End}(\mathcal{H}_M) \) by

\[ \theta_t(a) = U_t(a \otimes M \text{id}_{\mathcal{H}_t})U_t^* \]

for each \( a \in \text{End}(\mathcal{H}_M) \). The algebraic \( E_0 \)-semigroup \( \theta \) is called the dilation of the pair \( (H, \Xi) \). If we assume a continuity to the unit \( \Xi \), then it becomes an \( E_0 \)-semigroup as follows:

**Proposition 3.9.** If the unital unit \( \Xi = \{ \xi(t) \}_{t \geq 0} \) of the \( \phi \)-spatial product system \( H \) of \( W^* \)-\( M \)-bimodules satisfies that

\[ U_t(\xi\phi^{-\frac{1}{2}}\xi(t)) \to \xi \quad (t \to +0) \]

for all \( \xi \in \mathcal{D}(\mathcal{H}; \phi) \), then the dilation \( \theta \) of the pair \( (H, \Xi) \) is an \( E_0 \)-semigroup.

**Proof.** For each \( a \in \text{End}(\mathcal{H}_M) \) and each \( \phi \)-bounded vector \( \xi \in \mathcal{H} \), we have

\[ \theta_t(a)\xi - a\xi = \theta_t(a)\xi - U_t(a\xi\phi^{-\frac{1}{2}}\xi(t)) + U_t(a\xi\phi^{-\frac{1}{2}}\xi(t)) - a\xi \]

\[ = \theta_t(a)(\xi - U_t(\xi\phi^{-\frac{1}{2}}\xi(t))) + U_t(a\xi\phi^{-\frac{1}{2}}\xi(t)) - a\xi \to 0 \]

when \( t \to 0 \). Thus, the map \( t \mapsto \theta_t(a) \) is \( \sigma \)-weakly continuous for each \( a \in \text{End}(\mathcal{H}_M) \). \( \square \)

**Definition 3.10.** We say that a unit \( \Xi \) with \( (3.6) \) is continuous.

A right cocycle \( w = \{ w_t \}_{t \geq 0} \) for \( \theta \) is said to be adapted if \( \kappa_t\kappa_t^* w_t \kappa_t \kappa_t^* = w_t \) for all \( t \geq 0 \), where \( \kappa_t \) is the canonical embedding from \( \mathcal{H}_t \) into \( \mathcal{H} \). The following theorem describing a correspondence between cocycles and units, will enable as to classify \( E_0 \)-semigroups by product systems of \( W^* \)-bimodules in Section 6.
Theorem 3.11. Let $\theta = \{\theta_t\}_{t\geq 0}$ be the dilation of a pair $(H, \Xi)$ of a $\phi$-spatial product system $H = \{\mathcal{H}_t\}_{t\geq 0}$ of $W^*$-M-bimodules and a continuous unital unit $\Xi = \{\xi(t)\}_{t\geq 0}$. There exists a one-to-one correspondence between contractive adapted right cocycles $w = \{w_t\}_{t\geq 0}$ on $\operatorname{End}(\mathcal{H}_t)$ and contractive units $\Lambda = \{\lambda(t)\}_{t\geq 0}$ of $H$ by relations $\lambda(t) = \kappa_t^* w_t \kappa_0^* \phi^\frac{1}{2}$ and $w_t = \pi_\phi (\kappa_t \lambda(t)) \pi_\phi (\kappa_0 \phi^\frac{1}{2})^*$ for all $t \geq 0$.

Proof. Let $\{U_{s,t}\}_{s,t\geq 0}$ be a family giving the relative product system structure of $H$ and $\mathcal{H}$ the inductive limit of $(H, \Xi)$.

Let $w = \{w_t\}_{t\geq 0}$ be a contractive adapted right cocycle for $\theta$. Note that each $\lambda(t) = \kappa_t^* w_t \kappa_0^* \phi^\frac{1}{2}$ is $\phi$-bounded. Moreover, for each $t \geq 0$, we have

$$\begin{align*}
\pi_\phi (\lambda(t))^* \pi_\phi (\lambda(t)) \phi^\frac{1}{2} x &= \kappa_0^* w_t^* w_t \kappa_0 \phi^\frac{1}{2} x \\
\text{for each } x \in M, \text{ and hence the contractivity of } w_t \text{ implies that } \|\pi_\phi (\lambda(t))^* \pi_\phi (\lambda(t))\| \leq 1.
\end{align*}$$

We shall show that $\Lambda$ is a unit. For each $s, t \geq 0$, $\kappa_{s+t} = U_t (\kappa_s \otimes \text{id}_t) U_{s,t}^*$ implies the following calculations.

$$\begin{align*}
\lambda(s + t) &= \kappa_{s+t}^* w_{s+t} \kappa_0 \phi^\frac{1}{2} = \kappa_{s+t}^* \theta_t (w_s) w_t \kappa_0 \phi^\frac{1}{2} = \kappa_{s+t}^* U_t (w_s \otimes \text{id}_t) U_t^* w_t \kappa_0 \phi^\frac{1}{2} \\
&= \kappa_{s+t}^* U_t (w_s \otimes \text{id}_t) ((\kappa_0 \phi^\frac{1}{2}) \phi^\frac{1}{2} w_t \kappa_0 \phi^\frac{1}{2} ) = \kappa_{s+t}^* U_t ((w_s \kappa_0 \phi^\frac{1}{2}) \phi^\frac{1}{2} (\kappa_t^* w_t \kappa_0 \phi^\frac{1}{2} )) \\
&= U_{s,t} (\kappa_s^* \otimes \text{id}_t) U_t^* U_t (w_s \kappa_0 \phi^\frac{1}{2}) \phi^\frac{1}{2} (\kappa_t^* w_t \kappa_0 \phi^\frac{1}{2} )) = U_{s,t} (\lambda(s) \phi^\frac{1}{2} \lambda(t)).
\end{align*}$$

Conversely, let $\Lambda = \{\lambda(t)\}_{t\geq 0}$ be a contractive unit of $H$, and for each $t \geq 0$, $w_t = \pi_\phi (\kappa_t \lambda(t)) \pi_\phi (\kappa_0 \phi^\frac{1}{2})^* \in \operatorname{End}(\mathcal{H}_t)$. For all $\xi \in \mathcal{H}$, the equation

$$w_t \xi = \kappa_t U_{t,0} (\lambda(t) \phi^\frac{1}{2} \kappa_0^* \xi)$$

is implied from the approximation of $\kappa_0^* \xi$ by vectors as the form of $\phi^\frac{1}{2} x$. In particular,

$$w_t (\kappa_0 \phi^\frac{1}{2} x) = \kappa_t \lambda(t) x$$

for all $t \geq 0$, $x \in M$, and hence $w_t = 0$ on the orthogonal complement of the closed subspace $\kappa_0 \mathcal{H}$. Thus, computations

$$\begin{align*}
\theta_t (w_s) w_t &= U_t (w_s \otimes \text{id}_t) U_t^* \pi_\phi (\kappa_t \lambda(t)) \pi_\phi (\kappa_0 \phi^\frac{1}{2})^* \kappa_0 \phi^\frac{1}{2} x \\
&= U_t (w_s \otimes \text{id}_t) U_t^* \kappa_0 \lambda(t) x = U_t (w_s \otimes \text{id}_t) ((\kappa_0 \phi^\frac{1}{2}) \phi^\frac{1}{2} \lambda(t)) \\
&= U_t ((w_s \kappa_0 \phi^\frac{1}{2}) \phi^\frac{1}{2} (\eta(t)) = U_t ((\kappa_s \lambda(s)) \phi^\frac{1}{2} (\lambda(t)) x) \\
&= \kappa_{s+t} U_{s,t} (\lambda(s) \phi^\frac{1}{2} (\lambda(t)) x) = w_t (\kappa_0 \phi^\frac{1}{2} x)
\end{align*}$$

for every $x \in M$, implies that $w$ is a right cocycle. We shall show that $w$ is adapted. For all $t \geq 0$ and all $\xi \in \mathcal{H}$, by (3.8), we have

$$w_t \kappa_t^* w_0 \kappa_0^* = w_t \kappa_t^* U_{t,0} (\lambda(t) \phi^\frac{1}{2} \kappa_0^* \xi) = w_t \kappa_0^* \xi = w_t \xi.$$
4. CP₀-semigroups and units of product systems of W*-bimodules

In this section, we will obtain a one-to-one correspondence between algebraic CP₀-semigroups on M and pairs of (φ-spatial) product systems of W*-M-bimodules and generating unital units up to unit preserving isomorphism. It will be shown that the dilation of the pair associated with a given CP₀-semigroup T is a dilation of T. Also, we will discuss a relation between the continuity of CP₀-semigroups and one of units as follows: the unit associated with a CP₀-semigroup is continuous, and conversely, a continuous unit gives rise to a CP₀-semigroup.

First, we shall establish an algebraic CP₀-semigroup from a unit. Let Ξ = {ξ(t)}_{t≥0} be a unital unit of a φ-spatial product system H = {H_t}_{t≥0} of W*-M-bimodules. We define a unital linear map \( T^Ξ_t \) on M by

\[
T^Ξ_t(x) = \pi_φ(ξ(t))^*π_φ(xξ(t)) ∈ M
\]

for each \( t ≥ 0 \) and \( x ∈ M \).

**Lemma 4.1.** The family \( T^Ξ = \{T^Ξ_t\}_{t≥0} \) is an algebraic CP₀-semigroup.

**Proof.** By the definition, it is clear that each \( T^Ξ_t \) is normal completely positive map.

For \( s, t ≥ 0 \) and \( x, y, z ∈ M \), we can compute as

\[
\langle T^Ξ_s(T^Ξ_t(x))φ^±y, φ^±z \rangle = \langle \pi_φ(ξ(s))^*π_φ(π_φ(ξ(t))^*π(ξ(t))ξ(s))φ^±y, φ^±z \rangle
\]

\[
= \langle \pi_φ(ξ(s))^*π(ξ(t))ξ(s)y, ξ(s)z \rangle = \langle xξ(t)φ^±ξ(s)y, ξ(t)φ^±ξ(s)z \rangle
\]

\[
= \langle xU_s(ξ(t))φ^±ξ(s)y, U_s(ξ(t))φ^±ξ(s)z \rangle = \langle xξ(t+s)y, ξ(t+s)z \rangle
\]

\[
= \langle π_φ(ξ(s + t))^*π_φ(xξ(s + t))φ^±y, φ^±z \rangle = \langle T^Ξ_{s+t}(x)φ^±y, φ^±z \rangle,
\]

and hence \( T^Ξ_s ◦ T^Ξ_t = T^Ξ_{s+t} \).

We can describe the continuity for the algebraic CP₀-semigroup \( T^Ξ \) as the one for the unit \( Ξ \) as the following theorem.

**Theorem 4.2.** Let \( H \) be the inductive limit of the pair \((H, Ξ)\) and \( U_t : \mathcal{H}_t \otimes^H \mathcal{H}_t \to \mathcal{H}_t \) the unitary in Theorem 3.8. The semigroup \( T^Ξ \) associated with \( Ξ \) is a CP₀-semigroup if and only if

\[
U_t(κ_0(xφ^±t)φ^±ξ(t)) = κ_t(xξ(t)) → κ_0(xφ^±t) \quad (t → +0)
\]

holds for each \( x ∈ M \).

**Proof.** Suppose \( U_t(κ_0(xφ^±t))φ^±ξ(t) → κ_t(xξ(t)) \) for all \( x ∈ M \). For \( t ≥ 0 \) and \( x, y, z ∈ M \), we have

\[
\langle T^Ξ_t(x)φ^±y, φ^±z \rangle = \langle \pi_φ(ξ(t))^*π_φ(xξ(t))φ^±y, φ^±z \rangle = \langle xξ(t)y, ξ(t)z \rangle
\]

\[
= \langle U_t(κ_0(xφ^±t))φ^±ξ(t)y, U_t(κ_0(φ^±t))φ^±ξ(t)z) \rangle.
\]

Thus, when \( t → +0 \), the inner product \( \langle T_t(x)φ^±y, φ^±z \rangle \) tends to \( \langle κ_0(xφ^±t)y, κ_0(φ^±t)z) \rangle = \langle xφ^±y, φ^±z \rangle \). We conclude that for every \( x ∈ M \), \( T^Ξ_t(x) → x \) weakly when \( t → +0 \), and hence \( T^Ξ \) is a CP₀-semigroup by the boundedness of \( \{||T^Ξ_t(x)||\}_{t≥0} \).
Conversely, we assume that $T^\Xi$ is a $C_{0}$-semigroup. We can compute as
\[
\langle x\xi(t), x\xi(t) \rangle = \langle xU_{t,0}(\xi(t)\phi^{-\frac{1}{2}}\phi_{s}^{1}), xU_{t,0}(\xi(t)\phi^{-\frac{1}{2}}\phi_{s}^{2}) \rangle = \langle \xi(t)\phi^{-\frac{1}{2}}\phi_{s}^{1}, x^*x\xi(t)\phi^{-\frac{1}{2}}\phi_{s}^{2} \rangle = \langle \phi_{s}^{1}, \pi_{\phi}(\xi(t)) \rangle = \phi_{s}^{1}, T_{t}(x^*x)\phi_{s}^{2},
\]
\[
\langle \kappa_{t}(x\xi(t)), \kappa_{0}(x\phi_{s}^{2}) \rangle = \langle \kappa_{t}(x\xi(t)), \kappa_{0}(x\phi_{s}^{2}) \rangle = \langle \kappa_{t}(x\xi(t)), \kappa_{0}(x\phi_{s}^{2}) \rangle = \langle T_{t}(x^*x)\phi_{s}^{2}, \kappa_{0}(x\phi_{s}^{2}) \rangle = \phi_{s}^{1}, T_{t}(x^*x)\phi_{s}^{2}.
\]
Thus, when $t \to +0$, we have $\|\kappa_{t}(x\xi(t)) - \kappa_{0}(x\phi_{s}^{2})\|^2 \to 0$. □

**Definition 4.3.** We say that a unit $\Xi$ with $[1,2]$ is weakly continuous.

Next, we shall construct a product system and a unit from a given algebraic $C_{0}$-semigroup $T$ on $M$. For $t > 0$ and $p = (t_{1}, \cdots, t_{n}) \in \mathcal{P}_{t}$, we define a $W^{*}$-$M$-bimodule
\[
(4.3) \quad \mathcal{H}^{T}(p, t) = (M \otimes_{t_{1}} L^{2}(M)) \otimes M \cdots \otimes M (M \otimes_{t_{n}} L^{2}(M)),
\]
where $M \otimes_{s} L^{2}(M) = M \otimes_{T_{s}} L^{2}(M)$ for each $s \geq 0$. Suppose $p = (t_{1}, \cdots, t_{n}) > q = (s_{1}, \cdots, s_{m})$ in $\mathcal{P}_{t}$, $p = q(s_{1}) \vee \cdots \vee q(s_{m})$ and $q(s_{i}) = (s_{i,1}, \cdots, s_{i,k_{i}})$ in $\mathcal{P}_{s_{i}}$. We define a map $a_{q(s_{i})} : M \otimes_{s_{i}} L^{2}(M) \to \mathcal{H}^{T}(q(s_{i}), s_{i})$ by
\[
a_{q(s_{i})}(x \otimes s_{i}, \phi_{s_{i}}^{2}) = (x \otimes s_{i}, \phi_{s_{i}}^{2}) = (1 \otimes s_{i}, \phi_{s_{i}}^{2}) = (1 \otimes s_{i}, s_{i,k_{i}}(y) \phi_{s_{i}}^{2})
\]
for each $x, y \in M$. We can check that $a_{q(s_{i})}$ is an $M$-bilinear isometry by Proposition 2.7.

We define an isometry
\[
(4.4) \quad a_{p,q} = a_{q(s_{1})} \otimes M \cdots \otimes M a_{q(s_{m})} : \mathcal{H}^{T}(q, t) \to \mathcal{H}^{T}(p, t).
\]
Then, the pair $\{\mathcal{H}^{T}(p, t)\}_{p \in \mathcal{P}_{t}}\{a_{p,q}\}_{p \geq q}$ is an inductive system of $W^{*}$-$M$-bimodules. Let $\mathcal{H}^{T}_{t}$ be the inductive limit and $\kappa_{p,t} : \mathcal{H}^{T}(p, t) \to \mathcal{H}^{T}_{t}$ the canonical embedding. Put $\mathcal{H}^{T}_{0} = L^{2}(M)$. Also, we define a family $\Xi^{T} = \{\xi^{T}(t)\}_{t \geq 0}$ by $\xi^{T}(t) = \kappa_{t}(1 \otimes \phi_{2}^{2})$ for each $t > 0$ and $\xi^{T}(0) = \phi_{2}^{2}$.

**Theorem 4.4.** The family $\mathcal{H}^{T} = \{\mathcal{H}^{T}_{t}\}_{t \geq 0}$ is a $\phi$-isometric product system of $W^{*}$-$M$-bimodules and $\Xi^{T}$ is a generating unital unit of $\mathcal{H}^{T}$.

**Proof.** For $s, t > 0$, we define a map $U^{T}_{s,t} : \mathcal{H}^{T}_{s} \otimes_{M} \mathcal{H}^{T}_{t} \to \mathcal{H}^{T}_{s+t}$ by
\[
U^{T}_{s,t}((\xi_{q}, \xi_{q}^{s}), \eta_{q}, \eta_{q}^{s}) = \kappa_{q \lor p, s+t}(\xi_{q}, \eta_{q}^{s}),
\]
for each $q = (s_{1}, \cdots, s_{m}) \in \mathcal{P}_{q}$, $p = (t_{1}, \cdots, t_{n}) \in \mathcal{P}_{t}$, $\xi_{q} \in \mathcal{D}(\mathcal{H}(q, s); \phi)$ and $\eta_{p} \in \mathcal{H}^{T}(p, t)$ Here, note that $\kappa_{q,s} \xi_{q}$ is $\phi$-bounded. We shall show that $U^{T}_{s,t}$ is an isometry, i.e. the equation
\[
\langle (\kappa_{q,s} \xi_{q}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p}), (\kappa_{q',s} \xi_{q'}) \phi^{-\frac{1}{2}}(\kappa_{p',t'} \eta_{p'}) \rangle = \langle (\kappa_{q \lor p', s+t} \xi_{q} \phi^{-\frac{1}{2}} \eta_{p}, (\kappa_{q',s} \xi_{q'}) \phi^{-\frac{1}{2}} \eta_{p'}) \rangle
\]
holds for all $q, q' \in \mathcal{P}_{s}$, $p, p' \in \mathcal{P}_{t}$, $\xi_{q} \in \mathcal{D}(\mathcal{H}(q, s); \phi)$, $\xi_{q'} \in \mathcal{D}(\mathcal{H}(q', s); \phi)$, $\eta_{p} \in \mathcal{H}^{T}(p, t)$ and $\eta_{p'} \in \mathcal{H}^{T}(p', t)$. If $q = q'$ and $p = p'$, we have
\[
\langle (\kappa_{q,s} \xi_{q}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p}), (\kappa_{q,s} \xi_{q'}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p'}) \rangle = \langle (\kappa_{q,s} \xi_{q}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p}), (\kappa_{q,s} \xi_{q'}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p'}) \rangle = \langle (\kappa_{q,s} \xi_{q}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p}), (\kappa_{q,s} \xi_{q'}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p'}) \rangle
\]
\[
= \langle (\kappa_{q,s} \xi_{q}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p}), (\kappa_{q,s} \xi_{q'}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p'}) \rangle = \langle (\kappa_{q,s} \xi_{q}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p}), (\kappa_{q,s} \xi_{q'}) \phi^{-\frac{1}{2}}(\kappa_{p,t} \eta_{p'}) \rangle.
\]
In general case, since $\kappa_{a,s} = \kappa_{a',s'}$ for all $a, s, s' \in \mathcal{P}_s$ with $s' > s$, if we take $q \in \mathcal{P}_s, p \in \mathcal{P}_t$, such that $q > q', p > p', p'$, then

$$\langle (\kappa_{a,s}q)\phi^-\frac{1}{T}(\kappa_{p,t}U_p)q, (\kappa_{a',s'}q'\phi^-\frac{1}{T}(\kappa_{p',t})'p'\rangle$$

$$= \langle (\kappa_{a,s}a_q\eta, (\kappa_{a',s'}q'\phi^-\frac{1}{T}(\kappa_{p',t})'p')p, \rangle$$

$$= \langle (\kappa_{a,s}a_{q'\phi^-\frac{1}{T}(\kappa_{p',t})'p')}q, (\kappa_{a',s'}q'\phi^-\frac{1}{T}(\kappa_{p',t})'p')\rangle$$

$$= \langle \kappa_{q'\phi^-\frac{1}{T}(\kappa_{p',t})'p'}q, \kappa_{a',s'}q'\phi^-\frac{1}{T}(\kappa_{p',t})'p') \rangle.$$  

In particular, we conclude that $U^T$ is well-defined and can be extended to an isometry from $H^T \otimes M \rightarrow H^T_{s,t}$, and also denote the isometry by $U^T$ again. The surjectivity and the two-sides linearity of $U^T$ are obvious.

To show (3.1), it is enough to check it for $\phi$-bounded vectors with the form $\kappa_{p,t}(x_1 \otimes t_1, y_t \phi^{\frac{1}{2}})\phi^-\frac{1}{T}(x_m \otimes t_m, y_m \phi^{\frac{1}{2}})$ for some $x_1, y_t \in M$.

We conclude that $H^T$ is a product system of $W^\alpha$-bimodules, and it is clear that $\Xi^T$ is a generating unital unit of $H^T$ by the definition of $\{U^T\}_{s,t} \geq 0$.

Example 4.5. Let $T = \{T_t\}_{t \geq 0}$ be an algebraic $CP_0$-semigroup obtained by a semigroup $\nu$ of isometries in a von Neumann algebra $M$ in Example 2.3. For each $t \geq 0$, we identify $M \otimes L^2(M)$ with $L^2(M)$ by a bilinear unitary $U^t(x \otimes y \phi^{\frac{1}{2}}) = xv, y \phi^{\frac{1}{2}}$ for $x, y \in M$. For $t \geq 0$ and $p \in \mathcal{P}_t$, the unitaries induce a bilinear unitary $U^p : H^T(p, t) \rightarrow L^2(M)$ such that $U^p a_{p,q} = U^q$ for all $p \geq q$.

Example 4.6. We consider the $CP_0$-semigroup generated by a family of stochastic matrices. Let $M = \mathbb{C} \oplus \mathbb{C}$ be a von Neumann algebra regarded as a von Neumann subalgebra of $M_2(\mathbb{C})$. Then, we have $L^2(M) = \mathbb{C} \oplus \mathbb{C}$, $L^2(M)$ is isomorphic to $\mathcal{C}_2(\mathbb{C})$. For $t \geq 0$, $x, x' \in M$ and $\xi, \eta \in \xi \otimes \eta^*$, the inner product on $M \otimes L^2(M)$ is given by $\langle x \otimes (\xi \otimes \eta^*), x' \otimes (\xi' \otimes \eta'^*) \rangle = \langle \eta', \eta \rangle \int_{\mathbb{R}} \int_{\mathbb{R}} x^* W_{x} \xi, x' \overline{W}_{x} \xi' \rangle d\mu(x)$. Let $p = (t_1, \ldots, t_n) \in \mathcal{P}_t$. Fix a faithful normal state $\phi$ on $M$ and suppose $\rho \in \mathcal{C}_1(\mathbb{H})$ is associated with $\phi$ by $\phi(x) = tr(p \rho x)$ for all $x \in M$. In terms of $C_2(\mathbb{H})$, the inner product
that of the pair \( x \) and \( y \) is the minimal dilation of \( H_{\xi} \) and the unitary:

\[
\kappa_{\xi} = (x_1 \otimes_{t_1} y_1 \phi_{\frac{1}{2}}) \phi_{-\frac{1}{2}} + \cdots + \phi_{-\frac{1}{2}}(x_m \otimes_{t_m} y_m \phi_{\frac{1}{2}})
\]

for some \( s \geq 0 \), \( q = (x_1, \ldots, x_m) \in \mathcal{P}_s \). If we put \( p'(t) = \min\{s_1, \ldots, s_m\} \) and \( \xi = \kappa_{\xi}(x_1 \otimes_{t_1} y_1 \phi_{\frac{1}{2}}) \phi_{-\frac{1}{2}} + \cdots + \phi_{-\frac{1}{2}}(x_m \otimes_{t_m} y_m \phi_{\frac{1}{2}}) \), then we have \( p'(t) = \min\{s_1, \ldots, s_m\} \). Now, we have

\[
\kappa_{\xi} = \kappa_{\xi}(x_1 \otimes_{t_1} y_1 \phi_{\frac{1}{2}}) \phi_{-\frac{1}{2}} + \cdots + \phi_{-\frac{1}{2}}(x_m \otimes_{t_m} y_m \phi_{\frac{1}{2}}) = \kappa_{\xi}(x_1 \otimes_{t_1} y_1 \phi_{\frac{1}{2}}) \phi_{-\frac{1}{2}} + \cdots + \phi_{-\frac{1}{2}}(x_m \otimes_{t_m} y_m \phi_{\frac{1}{2}})
\]

On the other hand, we have

\[
U_t(\xi \phi_{-\frac{1}{2}} \xi T(t)) = \kappa_{\xi}(x_1 \otimes_{t_1} y_1 \phi_{\frac{1}{2}}) \phi_{-\frac{1}{2}} + \cdots + \phi_{-\frac{1}{2}}(x_m \otimes_{t_m} y_m \phi_{\frac{1}{2}}) \phi_{-\frac{1}{2}}(1_M \otimes \phi_{\frac{1}{2}})
\]

By calculations of inner products and Lemma A.2, when \( t \) tends to 0, we conclude that \( U_t(\xi \phi_{-\frac{1}{2}} \xi T(t)) \) converges to \( \xi \).

The dilation of the pair \( (H^T, \Xi^T) \) associated with a \( C_0 \)-semigroup gives a dilation of \( T \) as follows:

**Theorem 4.9.** Let \( T \) be a \( C_0 \)-semigroup on a von Neumann algebra \( M \) and \( \theta \) the dilation of the pair \( (H^T, \Xi^T) \). The triple \((\text{End}(\mathcal{S}^T_M), \pi^T(1_M), \theta)\) is a dilation of \( T \). Moreover, if we denote by \( N \) the von Neumann algebra generated by \( \bigcup_{t \geq 0} \theta_t(\pi^T(M)) \), the triple \((N, \pi^T(1_M), \theta|_N)\) is the minimal dilation of \( T \).
Proof. We shall show that $\pi^T(T_t(x)) = p\theta_t(\pi^T(x))p$ for all $t \geq 0$ and $x \in M$. For all $y, z \in M$, we have
\[
\langle \phi^T y, \kappa_0^t \theta_t(\pi^T(x))\kappa_0^t z \rangle = \langle (\kappa_0^t \phi^T)\phi^{-\frac{1}{2}} t^* \kappa_0^t \kappa_0^t y, (\pi^T(x) \otimes M) \text{id}_{\mathcal{H}_t}(\kappa_0^t \phi^T)\phi^{-\frac{1}{2}} t^* \kappa_0^t \kappa_0^t z \rangle
\]
\[
= \langle (\kappa_0^t \phi^T)\phi^{-\frac{1}{2}} t^* \kappa_0^t \phi^T y, \kappa_0^t (\phi^T \phi^{-\frac{1}{2}} t^* \kappa_0^t \kappa_0^t z) \rangle = \langle (\kappa_0^t \phi^T)\phi^{-\frac{1}{2}} b_t^* \kappa_0^t \phi^T y, \kappa_0^t (\phi^T \phi^{-\frac{1}{2}} b_t^* \kappa_0^t \kappa_0^t z) \rangle
\]
\[
= \langle (\kappa_0^t \phi^T)\phi^{-\frac{1}{2}} \kappa_0^t \phi^T y, \phi^T \phi^{-\frac{1}{2}} \kappa_0^t \phi^T z \rangle = \langle \phi^T x, T_t(x) \phi^T z \rangle.
\]
Thus we have $T_t(x) = \kappa_0^t \theta_t(\pi(x)) \kappa_0^t$.

For $t \geq 0$, let $p = (t_1, \ldots, t_n) \in \mathcal{P}_t$, $x_1, \ldots, x_n, y \in M$, we can check that
\[
\theta_t(\pi^T(x_1))\theta_{t_1}(\pi^T(x_2)) \cdots \theta_{t_n}(\pi^T(x_n))\theta_{t}(\pi^T(x_{n-1}))\theta_{t_n}(\pi^T(x_n)) \kappa_0^t y
\]
\[
= \kappa_0^t \kappa_t(x_1 \otimes t_1, \phi^T)\phi^{-\frac{1}{2}} (x_2 \otimes t_2, \phi^T) \cdots \phi^{-\frac{1}{2}} (x_n \otimes t_n, \phi^T) y
\]
Hence, we have $\overline{\text{span}(N\pi(1_M)\mathcal{S}^T)} \supset \overline{\text{span}(N\kappa_0 L^2(1_M))} = \mathcal{S}^T$. Since the central support $c(\pi^T(1_M))$ of $\pi^T(1_M)$ in $N$ is the projection onto $\overline{\text{span}(N\pi(1_M)\mathcal{S}^T)}$, we have $c(\pi^T(1_M)) = 1_N$. We conclude that the triplet $(N, \pi(1_M), \theta)$ is the minimal dilation of $T$. \qed

Remark 4.10. Let $\theta$ be the dilation of a CP$^0$-semigroup $T$ defined by (3.3). When $s \leq t$ and $\xi_s \in \mathcal{H}_t^\pi$, we have
\[
\theta_t(a)\kappa_s \xi_s = U_t(a \otimes \text{id}_{\mathcal{H}_t^\pi})U_t^* \kappa_s \xi_s = U_t(a(\kappa_0^t \phi^T)\phi^{-\frac{1}{2}} b_t \kappa_s \xi_s)
\]
for all $a \in \text{End}(\mathcal{S}_M^T)$, and hence the operator $\theta_t(a)\kappa_s \xi_s$ is depend on only the image $a(\kappa_0^t \phi^T)$.

The following theorem asserts that the correspondence between algebraic CP$^0$-semigroups and generating unitals is one-to-one up to unit preserving isomorphism.

Theorem 4.11. Let $T$ be an algebraic CP$^0$-semigroup on $M$ and $\Xi$ a generating unital of a $\phi$-spatial product system $H$ of $W^*-M$-bimodule. Then, we have $T^\Xi = T$ and there exists an isomorphism from $H^\Xi_T$ onto $H$, which preserves the units $\Xi^\Xi_T$ and $\Xi$.

Proof. It is clear that $T^\Xi_T = T$. For $t \geq 0$ and a partition $p = (t_1, \ldots, t_n) \in \mathcal{P}_t$, we define a map $u_t : \mathcal{H}_t^\Xi_T \rightarrow \mathcal{H}_t$ by
\[
u_t(\kappa_{p_1}(x_1 \otimes t_1, \phi^T)\phi^{-\frac{1}{2}} (x_2 \otimes t_2, \phi^T) \cdots \phi^{-\frac{1}{2}} (x_n \otimes t_n, \phi^T) y) = U(p)(x_1 \xi(t_1) \phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} x_n \xi(t_n) y)
\]
for each $x_1, \ldots, x_n, y \in M$, where $\kappa_{p_1} : \mathcal{H}_t^{\Xi_T}(p, t) \rightarrow \mathcal{H}_t^{\Xi_T}$ is the canonical embedding. We can check that $u_t$ is an isometry. Since $\Xi$ is generating, $u_t$ can be extended as unitary from $\mathcal{H}_t^{\Xi_T}$ onto $\mathcal{H}_t$.

We must show that $U_{s,t}((u_s \xi_s) \phi^{-\frac{1}{2}} (u_t \eta_t)) = U_{s+t}U_{s,t}^\Xi (\xi_s \phi^{-\frac{1}{2}} \eta_t)$ for all $\xi_s \in \mathcal{D}(\mathcal{H}_s^{\Xi_T}; \phi)$ and all $\eta_t \in \mathcal{H}_t^{\Xi_T}$. It enough to show it for
\[
\xi_s = \kappa_{p,s}(x_1 \otimes t_1, \phi^T)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} (x_m \otimes t_m, \phi^T), \eta_t = \kappa_{p,t}(z_1 \otimes t_1, \phi^T)\phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} (z_n \otimes t_n, \phi^T),
\]
where $q = (s_1, \ldots, s_m) \in \mathcal{P}_s$, $p = (t_1, \ldots, t_n) \in \mathcal{P}_t$ and $x_1, \ldots, x_m, z_1, \ldots, z_n, w \in M$. We put $\zeta_1 = x_1 \xi(s_1) \phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} x_m \xi(s_m)$, $\zeta_2 = z_1 \xi(t_1) \phi^{-\frac{1}{2}} \cdots \phi^{-\frac{1}{2}} z_n \xi(t_n) w$. 


By the associativity of \(\{U_{s,t}\}_{s,t \geq 0}\), we have
\[
u_{s,t}U_{s,t}^\Xi(\xi_s \phi^{-\frac{1}{2}} \eta_t) = U_{s_1,s_1'}(id_{s_1} \otimes U_{s_2,s_2'})(id_{s_1,s_2} \otimes U_{s_3,s_3'}) \cdots (id_{s_1,\cdots,s_{m-1}} \otimes U_{s_m,t})(\xi_t \phi^{-\frac{1}{2}} U(p) \eta_t)
\]
= \[
U_{s,t}(id_{s_1} \otimes id_{s_2} \otimes id_{s_3} \otimes id_{s_m}) (\xi_t \phi^{-\frac{1}{2}} U(p) \eta_t) 
\]
\[
\cdots (id_{s_1,\cdots,s_{m-2}} \otimes U_{s_{m-1},s_{m-1}})(id_{s_1,\cdots,s_m} \otimes U_{s_1,t'})(id_{s_1,\cdots,s_m} \otimes id_{t_1,\cdots,t_{n-2}} \otimes U_{t_{n-1},t_{n-1}})(\xi_t \phi^{-\frac{1}{2}} \eta_t).
\]
We conclude that \(\{u_t\}_{t \geq 0}\) gives an isomorphism.

As a corollary of Theorem 4.11, it will turn out that every weakly continuous generating unit is continuous. For this, we shall show that an isomorphism between product systems of \(W^*-\)bimodules induces a unitary between their inductive limits as follows: let \(H = \{\mathcal{H}_t\}_{t \geq 0}\) and \(K = \{\mathcal{K}_t\}_{t \geq 0}\) be product systems of \(W^*-\)bimodules with unital units \(\Xi\) and \(\Lambda\), respectively. Suppose a family \(\{u_t\}_{t \geq 0}\) is an isomorphism from \(H\) onto \(K\). Then, we can define the canonical right \(M\)-linear unitary \(u\) from the inductive limit \(\mathcal{H}\) of \((H, \Xi)\) onto the one \(\mathcal{K}\) of \((K, \Lambda)\) by \(u(\kappa)^H_t(\xi_t) = \kappa^K_t u_t(\xi_t)\) for each \(t \geq 0\) and \(\xi_t \in \mathcal{H}_t\), where \(\kappa^H_t : \mathcal{H}_t \rightarrow \mathcal{H}\) and \(\kappa^K_t : \mathcal{K}_t \rightarrow \mathcal{K}\) are the canonical embeddings.

**Corollary 4.12.** If \(\Xi\) is a weakly continuous generating unital unit of a product system \(H\) of \(W^*-\)bimodules, then \(\Xi\) is continuous.

**Proof.** Let \(U_t^\Xi\) be the unitary giving the right \(W^*-M\)-module isomorphism \(\mathcal{H}_t^\Xi \otimes^M \mathcal{H}_t^\Xi \rightarrow \mathcal{H}_t^\Xi\) for each \(t \geq 0\). By Proposition 4.8, the unit \(\Xi^\Xi(t)\) satisfies
\[
U_t^\Xi(\xi \phi^{-\frac{1}{2}} \xi^\Xi(t)(t) \rightarrow \xi (t \rightarrow +0)
\]
for all \(\xi \in \mathcal{D}(\mathcal{H}_t^\Xi; \phi)\). Suppose \(\mathcal{H}\) and \(\mathcal{K}\) are the inductive limit of \((H, \Xi)\) and \((H^\Xi, \Xi^\Xi)\), respectively, and \(\kappa_t : \mathcal{H}_t \rightarrow \mathcal{H}\) and \(\kappa_t : \mathcal{H}_t^\Xi \rightarrow \mathcal{H}_t^\Xi\) are the canonical embedding for each \(t \geq 0\). Let \(u\) be the unitary from \(\mathcal{H}_t^\Xi\) onto \(\mathcal{H}\) induced from the isomorphism \(\{u_t\}_{t \geq 0}\) which is obtained by Theorem 4.11.

\[
u_{t}^\Xi ((u_t^\Xi(\kappa_t^\Xi \xi_t) \phi^{-\frac{1}{2}} \xi_t^\Xi(t)) = U_t((u_{t_1}^\Xi(\kappa_t^\Xi \xi_t) \phi^{-\frac{1}{2}} \xi_t^\Xi(t)))
\]
by (3.2) and (3.2). Thus, by (4.5) and the fact that \(\{u_t\}_{t \geq 0}\) is unit preserving, we have \(\|U_t(\xi \phi^{-\frac{1}{2}} \eta_t) - \xi\| = \|U_t((u_{t_1}^\Xi(\kappa_t^\Xi \eta_t) \phi^{-\frac{1}{2}} \xi_t^\Xi(t)) - \xi\| = \|U_t^\Xi((u_{t_1}^\Xi(\kappa_t^\Xi \eta_t) \phi^{-\frac{1}{2}} \xi_t^\Xi(t)) - \xi\|\), and hence \(\Xi\) is continuous.

**Remark 4.13.** Let \(T\) be a \(CP_0\)-semigroup on a von Neumann algebra acting on a separable Hilbert space \(\mathcal{H}\). The product system \(H^T = \{H_t^T\}_{t \geq 0}\) of \(W^*-\)bimodules associated with \(T\) gives a relation between Bhat-Skeide’s \((8)\) and Muhly-Solel’s \((14)\) constructions of the minimal dilation of \(T\) as follows: a common point of the two methods is to establish a product system of von Neumann bimodules which has an information by \(T\) by taking the inductive limits with respect to refinements of partitions. Let \(\{E_t^T\}_{t \geq 0}\) be the product systems of von Neumann bimodules associated with \(T\) appearing in Bhat-Skeide’s and Muhly-Solel’s constructions, respectively. Note that each \(E_t^T\) is a von Neumann \(M\)-bimodule and each \(E_t^{M'}\) is a von Neumann \(M'\)-bimodule. By the
correspondence between $W^*$-bimodules and von Neumann bimodules (see \[20\] Section 2), we have a correspondence.

\begin{equation}
E_t \leftrightarrow \mathcal{H}_t^T, \quad E(t) \leftrightarrow \mathcal{H}^* \otimes^M \mathcal{H}_t^T \otimes^M \mathcal{H}.
\end{equation}

for each $t \geq 0$. This is an extension to the continuous case of the relation in the discrete case given by \[17\] and the proof of the correspondence \[4.6\] is essentially the same.

5. Heat semigroups on manifolds and product systems

In this section, we will consider the product system associated a heat semigroup $T$ and a dilation of $T$.

We shall recall the concept of heat semigroup on a compact Riemannian manifold. We refer the reader to \[10\] for their general theory. Let $\mathcal{M}$ be a compact Riemannian manifold with the normalized Riemannian measure $\mu$ associated with $\mathcal{M}$. We can define the self-adjoint positive (unbounded) operator $\Delta$ on $L^2(\mathcal{M})$ like the Laplacian on the Euclid space. The operator $\Delta$ is called the Laplacian (or Dirichlet Laplacian) on $\mathcal{M}$. For convenience, we define as a dilation of $T$ the self-adjoint positive (unbounded) operator $\Delta$ on $\mathcal{M}$.

Heat semigroups on manifolds and product systems

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For $t > 0$, it is known that there exists a measurable function $p_t$ on $\mathcal{M} \times \mathcal{M}$ such that

\begin{equation}
(T_t f)(x) = \int_{\mathcal{M}} p_t(x, y) f(y) d\mu(y)
\end{equation}

for each $f \in L^2(\mathcal{M})$. The family $\{p_t\}_{t > 0}$ called the heat kernel on $\mathcal{M}$ has the following properties:

1. $p_t(x, y) = p_t(y, z) \geq 0$ ($t > 0$, $x, y \in \mathcal{M}$).
2. $\int_{\mathcal{M}} p_t(x, y) d\mu(y) = 1$ ($t > 0$, $x \in \mathcal{M}$).
3. $p_{s+t}(x, z) = \int_{\mathcal{M}} p_t(x, y) p_s(y, z) d\mu(z)$ ($s, t > 0$, $x, z \in \mathcal{M}$).

The equation \[5.1\] enables as to extend the heat semigroup $T$ to a semigroup on $L^p(\mathcal{M})$ for each $1 \leq p \leq \infty$. We have the following continuity with respect to parameters:

\begin{equation}
T_t f \xrightarrow{L^p} f \quad (t \to +0)
\end{equation}

for each $f \in L^p(\mathcal{M})$ and $p = 1, 2$.

The heat semigroup $T$ on $\mathcal{M}$ is a CP$_0$-semigroup on the commutative von Neumann algebra $M = L^\infty(\mathcal{M})$. A Noncommutative Laplacian and the associated CP$_0$-semigroup on a type I factor are discussed in \[4\], Chapter 7. They are noncommutative analogies of the Laplacian on and heat semigroup on a manifold.

Now, we shall compute the product system $H^T$ of $W^*$-bimodules associated with the heat semigroup $T$ on $M = L^\infty(\mathcal{M})$. For this, we introduce the following notations.

**Definition 5.1.** For $t > 0$ and $\mathbf{p} = (t_1, \cdots, t_n) \in \mathbb{P}_t$, we define a probability measure $\mu_\mathbf{p}$ on $\mathcal{M}_\mathbf{p} = \mathcal{M}^{n+1}$ by

$$
\mu_\mathbf{p} = p_{t_1}(x_1, x_2) p_{t_2}(x_2, x_3) \cdots p_{t_{n-1}}(x_{n-1}, x_n) p_{t_n}(x_n, x_{n+1}) \mu_{n+1}.
$$

For convenience, we define as $\mathcal{M}_() = \mathcal{M}$ and $\mu() = \mu$ for the empty partition ( ).
Definition 5.2. Let $s, t > 0$, $p \in \mathfrak{P}_s$, $q \in \mathfrak{P}_t$ with $\#p = m$, $\#q = n$, and $f_p, g_q$ and $h$ be functions on $\mathcal{M}_p, \mathcal{M}_q$ and $\mathcal{M}$, respectively. We define functions $f_p \square g_q, f_p \square h$ and $h \square g_q$ on $\mathcal{M}_{p \square q}, \mathcal{M}_p$ and $\mathcal{M}_q$, respectively, by

$$(f_p \square g_q)(x_1, \ldots, x_m, y_1, \ldots, y_n, z) = f_p(x_1, \ldots, x_m, y_1, \ldots, y_n, z),$$

$$(f_p \square h)(x_1, \ldots, x_m, x_{m+1}) = f_p(x_1, \ldots, x_m, x_{m+1})h(x_{m+1}),$$

$$(h \square g_q)(y_1, \ldots, y_n, y_{n+1}) = h(y_1)g_q(y_1, \ldots, y_n, y_{n+1}).$$

for $x_i, y_j, z \in \mathcal{M}$.

For $t > 0$ and $p \in \mathfrak{P}_t$, the Hilbert space $L^2(\mathcal{M}_p, \mu_p)$ has a canonical $W^*-M$-bimodule structure given by $gf = g \square f$ and $f \square g$ for each $f \in L^2(\mathcal{M}_p, \mu_p)$ and $g \in M$. Then, we can obtain the following identification as $W^*-M$-bimodules.

Proposition 5.3. For $t > 0$ and $p \in \mathfrak{P}_t$, we have an isomorphism $H^T(p, t) \cong L^2(\mathcal{M}_p, \mu_p)$ as $W^*-M$-bimodules.

Proof. Let $\tau$ be the canonical faithful normal trace on $M = L^\infty(\mathcal{M})$ given by integrals on $\mathcal{M}$. Suppose $p = (t_1, \ldots, t_n)$. We define a $M$-bilinear map $u_{p,t} : H^T(p, t) \to L^2(\mathcal{M}_p, \mu_p)$ by

$$u_{p,t}((f_1 \otimes_{t_1} g_1 \tau^\frac{1}{2}) \tau^{-\frac{1}{2}} \cdots \tau^{-\frac{1}{2}}(f_n \otimes_{t_n} g_n)) = f_1(x_1)g_1(x_2)f_2(x_2) \cdots g_{n-1}(x_n)f_n(x_n)g_n(x_{n+1})$$

for each $f_1, g_1, \ldots, f_n, g_n \in M$, where $f_i(x_i)$ denotes the function $f_i$ on $\mathcal{M}$ with variables $x_i$ and $g_i(x_{i+1})$ is similar. By the formula in Proposition 2.7, we have

$$\langle(f_1 \otimes_{t_1} g_1 \tau^\frac{1}{2}) \tau^{-\frac{1}{2}} \cdots \tau^{-\frac{1}{2}}(f_n \otimes_{t_n} g_n), (f'_1 \otimes_{t_1} g'_1 \tau^\frac{1}{2}) \tau^{-\frac{1}{2}} \cdots \tau^{-\frac{1}{2}}(f'_n \otimes_{t_n} g'_n)\rangle = \int_{\mathcal{M}_p} f_1(x_1)g_1(x_2)f_2(x_2) \cdots g_{n-1}(x_n)f_n(x_n)g_n(x_{n+1})d\mu_p,$$

and hence $u_{p,t}$ is an isometry.

We shall check that $u_{p,t}$ is surjective. For an arbitrary $\varepsilon > 0$ and $f \in L^2(\mathcal{M}_p, \mu_p)$, there exists $g \in C(\mathcal{M}^{n+1})$ such that $\|f - g\|_{L^2(\mathcal{M}_p, \mu_p)} < \varepsilon$. Since the space

$$\text{span}\{f_1(x_1) \cdots f_{n+1}(x_{n+1}) \in C(\mathcal{M}^{n+1}) : f_i \in C(\mathcal{M})\}$$

is dense in $C(\mathcal{M}^{n+1})$ with respect to the uniform convergence topology, there exist $N \in \mathbb{N}$ and functions $f_{i,j} \in C(\mathcal{M})$ for each $i = 1, \ldots, n + 1$ and $j = 1, \ldots, N$ such that $\|g - \sum_{j=1}^{N} f_{1,j}(x_1) \cdots f_{n+1,j}(x_{n+1})\|_{\infty} < \varepsilon$. Now, equations

$$f_{1,j}(x_1) \cdots f_{n+1,j}(x_{n+1}) = u_{p,t}((f_1 \otimes_{t_1} 1_M \tau^\frac{1}{2}) \tau^{-\frac{1}{2}} \cdots \tau^{-\frac{1}{2}}(f_{n+1\otimes_{t_n} 1_M \tau^\frac{1}{2}} \tau^{-\frac{1}{2}}(f_{n+1} \otimes_{t_n} f_{n+1,j} \tau^\frac{1}{2})))$$

imply that the image $u_{p,t}(D_{p,t})$ of

$$D_{p,t} = \text{span}\{(f_1 \otimes_{t_1} g_1 \tau^\frac{1}{2}) \tau^{-\frac{1}{2}} \cdots \tau^{-\frac{1}{2}}(f_n \otimes_{t_n} g_n) : f_1, g_1, \ldots, f_n, g_n \in M\}$$

by $u_{p,t}$ is dense in $L^2(\mathcal{M}_p, \mu_p)$, and hence $u_{p,t}$ is unitary.

Remark 5.4. There is a connection between heat kernels and Brownian motions on Riemannian manifolds. Let $T$ be a heat semigroup on a compact Riemannian manifold $\mathcal{M}$.
and \( \{p_t\}_{t>0} \) the heat kernel associated with \( T \). For \( x \in \mathcal{M} \), there exist a probability space \((\Omega, \mathbb{P}, \mathbb{X})\) and an \( \mathcal{M} \)-valued stochastic process \( \{X_t\}_{t \geq 0} \) such that

\[
\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y) d\mu(y) = (T_t 1_A)(x)
\]

for all Borel set \( A \subset \mathcal{M} \), where \( 1_A \) is the characteristic function on \( A \). Also, the family \( \{\mu_p | p \in \mathcal{P}_1\} \) of probability measures describes joint distributions for \( \{X_t\}_{t \geq 0} \) as follows: for a partition \( p = (t_1, \ldots, t_n) \in \mathcal{P}_1 \) and Borel sets \( A_1, \ldots, A_n \), if we denote \( \tilde{p} = (t_2, t_3, \ldots, t_n) \), then we have

\[
\mathbb{P}_x(X_{t_1} \in A_1, X_{t_1+t_2} \in A_2, \ldots, X_{t_1+\ldots+t_{n-1}} \in A_{n-1}, X_t \in A_n) = \int_{\mathcal{M}_{\tilde{p}}} p_t(x, x_2) 1_{A_1}(x_2) \cdots 1_{A_{n-1}}(x_{n-1}) 1_{A_n}(x_{n+1}) d\mu_{\tilde{p}}.
\]

Now, we shall reconstruct the product system associated with the heat semigroup \( T \) and a dilation of \( T \) under the identification in Proposition 5.3.

Let \( q \geq p \) with \( p = (t_1, \ldots, t_n) \in \mathcal{P}_1 \) and \( q = p(t_1) \vee \cdots \vee p(t_n) \) with \( p(t_i) = (t_{i,1}, \ldots, t_{i,k(i)}) \in \mathcal{P}_{t_i} \). The isometry \( a_{q,p} : L^2(\mathcal{M}_p, \mu_p) \rightarrow L^2(\mathcal{M}_q, \mu_q) \) is given by

\[
a_{q,p}(f_p)(x_{1,1}, \ldots, x_{1,k(1)}, x_{2,1}, \ldots, x_{2,k(2)}, \ldots, x_{n,1}, \ldots, x_{n,k(n)}, y) = f_p(x_{1,1}, x_{2,1}, \ldots, x_{n,1}, y)
\]

for each \( f_p \in L^2(\mathcal{M}_p) \) and \( x_{i,j}, y \in \mathcal{M} \). By \( \mathcal{H}_t^T \), we denote the inductive system \((\{L^2(\mathcal{M}_p, \mu_p)\}_{p \in \mathcal{P}_1}, \{a_{q,p} \}_{p \geq q})\). Let \( \kappa_{p,t} : L^2(\mathcal{M}_p, \mu_p) \rightarrow \mathcal{H}_t^T \) be the canonical embedding.

The family \( \{U_{s,t} : \mathcal{H}_s^T \otimes^\mathcal{M} \mathcal{H}_t^T \rightarrow \mathcal{H}_{s+t}^T \}_{s,t \geq 0} \) of \( M \)-bilinear unitaries giving the structure of the product system \( \mathcal{H}_t^T = \{\mathcal{H}_t^T\}_{t \geq 0} \) associated with the heat semigroup \( T \) satisfies the follows: for \( s, t \geq 0 \), \( p \in \mathcal{P}_s \), \( q \in \mathcal{P}_t \) with \#\( p = m \), \#\( q = n \) and \( f_p \in L^\infty(\mathcal{M}_p, \mu_p) \), \( g_q \in L^2(\mathcal{M}_q, \mu_q) \), \( h \in L^2(\mathcal{M}) = \mathcal{H}_0^T \), \( h' \in M = L^\infty(\mathcal{M}) \), we have

\[
U_{s,t}(\langle \kappa_{p,s} f_p, \tau^{-1}(\kappa_{q,t} g_q) \rangle) = \kappa_{p \vee q, s+t} (f_p \square g_q), \quad U_{s,0} (\langle \kappa_{p,s} f_p, \tau^{-1} h \rangle) = \kappa_{p,s} (f_p \square h), \quad U_{0,t} (\langle \kappa_{p,s} f_p, \tau^{-1} h' \rangle) = \kappa_{p,s} (h' \square g_q).
\]

Also, the unit \( \Xi^T = \{\xi^T(t)\}_{t \geq 0} \) associated with \( T \) is given by \( \xi^T(t) = \kappa_{t,1} f_1 \) for each \( t > 0 \) and \( \xi^T(0) = 1_{\mathcal{M}} \).

For \( 0 < s \leq t \), \( p = (s_1, \ldots, s_m) \in \mathcal{P}_s \) and \( f_p \in L^2(\mathcal{M}_p, \mu_p) \), the image of \( \kappa_{p,s} f_p \) by the isometry \( b_{t,s} : \mathcal{H}_s^T \rightarrow \mathcal{H}_t^T \) is given by \( b_{t,s}(\kappa_{p,s} f_p) = \kappa_{(t-s)\vee p, t} f_p \), where \( f_p \in L^2(\mathcal{M}_{(t-s)\vee p}, \mu_{(t-s)\vee p}) \) in the right side is a function defined by \( \tilde{f}_p(x, x_1, \ldots, x_m, y) = f_p(x_1, \ldots, x_m, y) \) for each \( x, x_i, y \in \mathcal{M} \). If we also define \( \tilde{f} \in L^2(\mathcal{M}_{(t-s)}, \mu_{(t-s)}) \) for \( f \in L^2(\mathcal{M}) \) by \( \tilde{f}(x, y) = f(y) \) for each \( x, y \in \mathcal{M} \), then \( b_{t,s} f = \kappa_{(t-s), t} \tilde{f} \). We denote the inductive limit of \((\{\mathcal{H}_t^T\}_{t \geq 0}, \{b_{t,s}\}_{s \leq t})\) by \( \mathcal{S}_t^T \) and let \( \kappa_t : \mathcal{H}_t^T \rightarrow \mathcal{S}_t^T \) be the canonical embedding. We can describe the isometry \( b_{t,0}^* \) by heat kernels as follows:

**Proposition 5.5.** For \( t > 0 \), \( p = (t_1, \ldots, t_n) \in \mathcal{P}_t \) and \( f_p \in L^2(\mathcal{M}_p, \mu_p) \), we have a formula

\[
(b_{t,0}^* \kappa_p, f_p)(y) = \int_{\mathcal{M}_{p'}} f_p(x_1, \ldots, x_n, y) p_{p'}(x_n, y) d\mu_{p'}(x_1, \ldots, x_n)
\]

for each \( y \in \mathcal{M} \), where \( p' = (t_1, \ldots, t_{n-1}) \).
Proof. Note that the function defined by the right hand belongs to \( L^2(\mathcal{M}) \) by Jensen’s inequality with respect to the convex function \( h \) defined by \( h(z) = z^2 \) for each \( z \in \mathbb{R} \).

For each \( g \in L^2(\mathcal{M}) \), we can compute as

\[
\langle b_t, g \rangle = \langle f, g \rangle = \int_{\mathcal{M}_p} f_p(x_1, \ldots, x_n, y)g(y)d\mu_p(x_1, \ldots, x_n, y)
\]

Thus, we have shown the desired equation.

\[ \square \]

Note that the function defined by the right hand belongs to \( L^2(\mathcal{M}) \) by Jensen’s inequality with respect to the convex function \( h \) defined by \( h(z) = z^2 \) for each \( z \in \mathbb{R} \).

The right action of \( \mathcal{M} \) on the right \( W^* \)-module \( \mathcal{S}_T \) is given by \( (\kappa_t \kappa_p, f)g = \kappa_t \kappa_p(f \square g) \) for each \( t \geq 0 \), \( p \in \mathcal{P}_t \), \( f \in L^2(\mathcal{M}_p, \mu_p) \) and \( g \in \mathcal{M} = L^\infty(\mathcal{M}) \). Clearly, for \( t > 0 \), the identification \( \mathcal{S}_T \otimes^\mathcal{M} \mathcal{H}_t \cong \mathcal{S}_T \) as right \( W^* \)-modules is obtained by the right \( M \)-linear unitary

\[ U_t : \mathcal{S}_T \otimes^\mathcal{M} \mathcal{H}_t \ni \kappa_s \kappa_p, f \mapsto \kappa_{s+t} \kappa_p \circ \kappa_{s+t}(f \square g) \in \mathcal{S}_T, \]

where \( f \in L^\infty(\mathcal{M}_p, \mu_p) \) and \( g \in L^2(\mathcal{M}_q, \mu_q) \). Also, the unitary \( U_0 : \mathcal{S}_T \otimes^\mathcal{M} L^2(\mathcal{M}) \to \mathcal{S}_T \) is the canonical isomorphism.

By the embedding \([3.4]\), we regard as \( \mathcal{M} \subset \text{End}(\mathcal{S}_M^T) \). Then, the following direct computations imply that the triple of \( \text{End}(\mathcal{S}_M^T) \), the \( E_0 \)-semigroup \( \theta \) on \( \text{End}(\mathcal{S}_M^T) \) defined by \([3.5]\) and the projection \( p = \kappa_0 \kappa_0^* \) becomes a dilation of \( T \). For \( t > 0 \), \( x \in \mathcal{M} \) and \( f \in \mathcal{M} = L^\infty(\mathcal{M}), \ g \in L^2(\mathcal{M}) \), we have

\[
(\kappa_0^* \theta_t(f)\kappa_0 g)(x) = (\kappa_0^* U_t(f \otimes M \text{id}_{\mathcal{H}_t}))(\kappa_0 1_M)(\kappa_0 \kappa(t))(x)
\]

\[
= (\kappa_0^* U_t f \circ (\kappa(t)))(x) = (\kappa_0^* \kappa(t) \kappa_0 \circ (f \square g))(x) = \int_{\mathcal{M}} f(y)(y, x)p_t(y, x)d\mu(y)
\]

\[
= \int_{\mathcal{M}} f(y)g(y)p_t(y, x)d\mu(y) = (T_t f)(x).
\]

Here, the forth equality is implied from the formula in Proposition \([5.5] \) in the case when \( p = (t) \).

6. Classification of \( E_0 \)-semigroups

In this section, we shall classify \( E_0 \)-semigroups on a von Neumann algebra up to cocycle equivalence by the product systems of \( W^* \)-bimodules associated with their \( E_0 \)-semigroups as \( C_P_0 \)-semigroups.

Let \( \theta \) be an \( E_0 \)-semigroup on \( \mathcal{M} \). For each \( t \geq 0 \), let \( \mathcal{H}_t^\theta = L^2(\mathcal{M}) \) as sets with a left and a right actions of \( \mathcal{M} \) defined by \( \delta_x \xi y = \theta_t(x)\xi y \) for each \( x, y \in \mathcal{M} \) and \( \xi \in \mathcal{H}_t^\theta \), and \( \xi^\theta(t) = \phi^\theta \).

Then, the family \( \mathcal{H}_t^\theta = \{ \mathcal{H}_t^\theta \}_{t \geq 0} \) is a product system of \( W^* \)-bimodules and \( \Xi^\theta = \{ \xi^\theta(t) \}_{t \geq 0} \) is a continuous generating unital unit.

**Proposition 6.1.** Let \( (\mathcal{H}_t^\theta, \Xi^\theta) \) be the pair associated with \( \theta \) as \( C_P_0 \)-semigroups. There is an isomorphism \( \varphi^\theta = \{ \varphi(t) \}_{t \geq 0} \) from \( \mathcal{H}_t^\theta \) onto \( \mathcal{H}_t^\theta \) preserving the units \( \Xi^\theta \) and \( \Xi^\theta \).
show that

\[ u_t^\theta : \mathcal{H}_t^\theta \to \tilde{\mathcal{H}}_t^\theta \]

by

\[ u_t^\theta (\kappa_{p,t}(x_1 \otimes_t y_1 \phi_j^\frac{1}{2}) \phi^{\frac{1}{2}} \cdot \ldots \cdot (x_n \otimes_t y_n \phi_j^\frac{1}{2})) \]

\[ = \theta_{t_n}(\theta_{t_{n-1}}(\cdots (\theta_{t_1}(x_1 y_1 x_2 y_2 x_3) \cdots )y_{n-1}x_n)y_n) \]

for each \( p = (t_1, \ldots, t_n) \in \mathcal{P}_t \) and \( x_1, \ldots, x_n, y_1, \ldots, y_n \in M \), where \( \kappa_{p,t} : \mathcal{H}_t^\theta(p,t) \to \mathcal{H}_t^\theta \)

is the canonical embedding. Put \( u_0^\theta = \text{id} \). The family \( u^\theta = \{u_t^\theta\}_{t \geq 0} \) is the desired isomorphism. \( \square \)

**Example 6.2.** For an \( E_0 \)-semigroup \( \theta \) on \( \mathcal{B}(\mathcal{H}) \), the product system \( H^\theta \) associated with \( \theta \) is isomorphic to the product system \( \tilde{H}^\theta \) of \( \mathcal{W}^\theta \)-bimodules and \( \mathcal{H}_t^\theta = \mathcal{H} \otimes \mathcal{H}^\theta \) with left and right actions of \( \mathcal{B}(\mathcal{H}) \) defined by \( x(\xi \otimes \eta^*)y = (\alpha_t(x)\xi) \otimes (\eta^*)^* \) for each \( t \geq 0 \), \( \xi, \eta \in \mathcal{H} \) and \( x, y \in \mathcal{B}(\mathcal{H}) \).

For an \( E_0 \)-semigroup \( \theta \) on \( M \), a family \( \{f_t^\theta\}_{t \geq 0} \) of the trivial right \( M \)-linear unitaries \( f_t^\theta : \mathcal{H}_t^\theta \ni \xi \mapsto \xi \in L^2(M) \) induces the right \( M \)-linear unitary \( f^\theta : \tilde{\mathcal{H}}^\theta \to L^2(M) \), where \( \tilde{\mathcal{H}}^\theta \) is the inductive limit of \( (H^\theta, \Xi^\theta) \). Note that the all canonical embeddings \( \kappa_t^\theta : \mathcal{H}_t^\theta \to \tilde{\mathcal{H}}^\theta \) are unitaries and equal to each other. We have \( \theta = T^\Xi \), and the \( E_0 \)-semigroup \( \{(f_t^\theta)\theta((f_t^\theta)^*f_t^\theta)\}_{t \geq 0} \) coincides with the dilation \( \tilde{\theta} \) of the pair \( (H^\theta, \Xi^\theta) \). We have the following classification of \( E_0 \)-semigroups.

**Theorem 6.3.** Two \( E_0 \)-semigroups \( \alpha \) and \( \beta \) on a von Neumann algebra \( M \) are cocycle equivalent if and only if \( H^\alpha \cong H^\beta \).

**Proof.** We will use the above notations for \( \alpha \) and \( \beta \) in this proof. If \( w = \{w_t\}_{t \geq 0} \subset M \) is a unitary right cocycle for \( \alpha \) and \( \beta_t(\cdot) = w_t^*\alpha_t(\cdot)w_t \) for each \( t \geq 0 \), then \( u_t : \mathcal{H}_t^\beta \ni \phi^{\frac{1}{2}} \mapsto w_t \phi^{\frac{1}{2}} \in \mathcal{H}_t^\alpha \) gives an isomorphism \( H^\beta \cong H^\alpha \). Thus, we have \( H^\beta \cong H^\beta \cong H^\alpha \cong H^\alpha \).

Conversely, suppose a family \( \{u_t\}_{t \geq 0} \) of \( M \)-bilinear unitaries gives an isomorphism from \( H^\beta \) onto \( H^\alpha \). Let \( u^\alpha \) be the isomorphism from \( H^\alpha \) onto \( H^\alpha \) in Proposition 6.1. Put \( \lambda(t) = u^\alpha_t \lambda^\beta(t) \in \mathcal{H}_t^\alpha \) for each \( t \geq 0 \), and then \( \Lambda = \{\lambda(t)\}_{t \geq 0} \) is a unital unit of \( H^\alpha \). We have

\begin{equation}
(6.1) \quad \beta_t(x) = \pi_\alpha(\xi^\beta(t))^{*} \pi_\alpha(x \xi^\beta(t)) = \pi_\alpha(\lambda(t))^{*} \pi_\alpha(x \lambda(t))
\end{equation}

for all \( x \in M \), that is, \( \beta = T^\Lambda \). We can check that \( \Lambda \) is weakly continuous and generating. Hence the unit \( \Lambda \) is continuous by Corollary 11.12. We denote the right cocycle for \( \alpha \) associated with \( \Lambda \) by \( u^0 = \{u_t^0 = \pi_\alpha(\kappa_t^\alpha \lambda(t)) \pi_\alpha(\kappa_0^\alpha \phi^{\frac{1}{2}})^*\}_{t \geq 0} \) as Theorem 6.6.

By (3.8), each \( u_t^0 \) is isometry. Since the map \( \phi^{\frac{1}{2}} x \mapsto \lambda(t)x \) is isometry, we have \( \text{span}\{\lambda(t)x : x \in M\} = L^2(M) \). Thus, by (3.9), each \( u_t^0 \) is surjective. Now, we shall show that \( u^0 \) is strongly continuous. For \( s \geq 0 \), by the continuity of \( \Lambda \), we can check that \( \kappa_s^\alpha \lambda(t) \to \kappa_s^\alpha \lambda(s) \) when \( t \to s \). Let \( \xi \in \tilde{\mathcal{H}}^\alpha \) and \( t \geq s \), by (3.8), we have

\begin{align}
(6.2) & \quad \langle u_t^0 \xi, u_s^0 \xi \rangle = \langle U_{t,t}^0(\lambda(t)\phi^{-\frac{1}{2}}(\kappa_0^\alpha)^*\xi), b_{t,s}^0 U_{s,0}^\alpha(\lambda(s)\phi^{-\frac{1}{2}}(\kappa_0^\alpha)^*\xi) \rangle \\
& = \langle U_{t,t}^0(\lambda(t)\phi^{-\frac{1}{2}}(\kappa_0^\alpha)^*\xi), b_{t,s}^0 U_{s,0}^\alpha(\xi^\alpha(t-s) \phi^{-\frac{1}{2}}(\kappa_0^\alpha)(\kappa_0^\alpha)^*\xi) \rangle \\
& = \langle U_{t,t}^0(\lambda(t)\phi^{-\frac{1}{2}}(\kappa_0^\alpha)^*\xi), U_{t,s}^\alpha U_{s,0}^\alpha(\xi^\alpha(t-s) \phi^{-\frac{1}{2}}(\kappa_0^\alpha)(\kappa_0^\alpha)^*\xi) \rangle \\
& = \langle \lambda(t)\phi^{-\frac{1}{2}}(\kappa_0^\alpha)^*\xi, b_{t,s}^0 \lambda(s)\phi^{-\frac{1}{2}}(\kappa_0^\alpha)^*\xi \rangle = (\kappa_0^\alpha)^*\xi, \pi_\alpha(\kappa_t^\alpha \lambda(t))^{*} \pi_\alpha(\kappa_s^\alpha \lambda(s))\cdots \pi_\alpha(\kappa_0^\alpha \lambda(s))\cdots (\kappa_0^\alpha)^*\xi). \end{align}
Since \( \pi_\phi(\kappa^a_\lambda(t))^{\ast} \pi_\phi(\kappa^a_\lambda(s)) \to 1_M \) weakly when \( t \to s \) or \( s \to t \), (6.2) tends to \( \langle \xi, \xi \rangle \) when \( t \to s + 0 \), and by the symmetry, \( \langle w_t^1 \xi, w_t^0 \xi \rangle \) also tends to \( \langle \xi, \xi \rangle \) when \( t \to s - 0 \). We conclude that \( w_t^0 \xi \to w_0^0 \xi \) when \( t \to s \).

Put \( w_t = f^a u_t(f^a)^{\ast} \in M \). Then, the family \( w = \{w_t\}_{t \geq 0} \) is a strongly continuous right cocycle for \( \alpha \). For all \( t \geq 0 \) and \( x, y, z \in M \), since \( w_t \phi^z_\lambda x = \lambda(t)x \) and \( \beta \) is given as (6.1), we have \( \langle w_t^1 \alpha_t(x) w_t \phi^z_\lambda y, \phi^z_\lambda z \rangle = \langle \pi_\phi(\lambda(t))^{\ast} \pi_\phi(x \cdot \lambda(t)) \phi^z_\lambda y, \phi^z_\lambda z \rangle = \langle \beta_t(x) \phi^z_\lambda y, \phi^z_\lambda z \rangle \), and hence \( \beta_t(x) = w_t^1 \alpha_t(x) w_t \).

\[ \square \]

**Corollary 6.4.** Two \( E_0 \)-semigroups \( \alpha \) and \( \beta \) on von Neumann algebras \( M \) and \( N \), respectively, are cocycle conjugate if and only if there exists a \(*\)-isomorphism \( \Phi : M \to N \) such that \( H^\alpha \) and \( H^{\beta^\Phi} \) are isomorphic.

**Example 6.5.** Let \( u = \{u_t\}_{t \geq 0} \) be a strongly continuous semigroup of unitaries \( u_t \) in a von Neumann algebra \( M \). We define \( \theta_t(x) = u_t^* xu_t \) for each \( t \geq 0 \) and \( x \in M \). The product system of \( W^\ast \)-bimodules associated with \( \theta \) is isomorphic to the trivial product system \( L^2(M) \) for each \( t \geq 0 \).

**Proposition 6.6.** Let \( \theta \) be an \( E_0 \)-semigroup on a von Neumann algebra \( M \). For a unit \( \Xi = \{\xi(t)\}_{t \geq 0} \) of \( H^0 \) and \( t \geq 0 \), there exists a unique \( a_t \in M \) such that \( \xi(t) = a_t \phi^z_\lambda \). The family \( \{a_t\}_{t \geq 0} \) is a right cocycle for \( \theta \).

**Proof.** Fix \( s, t \geq 0 \). Since we have

\[ \phi(a_{s+t}^t \theta_t(a_s)a_t)) = (a_{s+t}, U_{s,t}(a_s \phi^z_\lambda \phi^z_\lambda^{-1}(a_t \phi^z_\lambda))) = \phi(a_{s+t}^t a_{s+t}), \]

\[ \phi((\theta_t(a_s)a_t)^* a_{s+t}))) = (a_{s+t}, (a_s \phi^z_\lambda \phi^z_\lambda^{-1}(a_t \phi^z_\lambda)), U_{s,t}(\xi(s) \phi^z_\lambda \phi^z_\lambda (\xi(t)))) = \phi(\theta_t(a_s)a_t \theta_t(a_s)a_t) \]

and \( \phi \) is faithful, the equation \( \theta_t(a_s)a_t = a_{s+t} \) holds.

\[ \square \]

**Example 6.7.** Let \( \theta \) be an \( E_0 \)-semigroup on a \( \Pi_1 \) factor \( M \) and \( \Xi \) be a unit of the product system \( H^0 \) of \( W^\ast \)-bimodules associated with \( \theta \). For each \( t \geq 0 \), we define an operator \( X^\Xi_t \in B(L^2(M)) \) by \( X^\Xi_t(x \phi^z_\lambda) = \theta_t(x)a_t \phi^z_\lambda \) for each \( x \in M \), where \( \{a_t\}_{t \geq 0} \) is the right cocycle for \( \theta \) associated with \( \Xi \) in Proposition 6.6. Then, the family \( X^\Xi = \{X^\Xi_t\}_{t \geq 0} \) is a unit of \( \theta \), that is, \( X^\Xi_t \) is a semigroup satisfying \( X^\Xi_t x = \theta_t(x) X^\Xi_t \) for all \( t \geq 0 \).

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Graduate school of mathematics, Nagoya University, Chikusaku, Nagoya, 464-8602, Japan

E-mail address: m14017c@math.nagoya-u.ac.jp