A Lyapunov functional and blow-up results for a class of perturbed semilinear wave equations

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Abstract

We consider in this paper some class of perturbation for the semilinear wave equation with subcritical (in the conformal transform sense) power nonlinearity. We first derive a Lyapunov functional in similarity variables and then use it to derive the blow-up rate. Although the result is similar to the unperturbed case in its statements, this is not the case of our method, which is new to our knowledge.

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1. Introduction

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation:

\[
\begin{align*}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} &= \Delta u + |u|^{p-1} u + f(u) + g(u) , \\
(u(x,0), u_t(x,0)) &= (u_0(x), u_1(x)) \in H_{\text{loc,}u}^1(\mathbb{R}^N) \times L_{\text{loc,}u}^2(\mathbb{R}^N),
\end{cases}
\end{align*}
\]

where \( p > 1, f \) and \( g \) are locally Lipschitz-continuous satisfying the following condition:

\[
\begin{align*}
\text{(H}_f) & \quad |f(x)| \leq M(1 + |x|^q) \quad \text{with} \quad (q < p, \ M > 0), \\
\text{(H}_g) & \quad |g(x)| \leq M(1 + |x|),
\end{align*}
\]

and \( L_{\text{loc,}u}^2(\mathbb{R}^N) \) and \( H_{\text{loc,}u}^1(\mathbb{R}^N) \) are the spaces defined by

\[
L_{\text{loc,}u}^2(\mathbb{R}^N) = \{ u : \mathbb{R}^N \to \mathbb{R} / \sup_{a \in \mathbb{R}^N} \left( \int_{|x-a| \leq 1} |u(x)|^2 \, dx \right) < +\infty \},
\]

and

\[
H_{\text{loc,}u}^1(\mathbb{R}^N) = \{ u \in L_{\text{loc,}u}^2(\mathbb{R}^N), \ |\nabla u| \in L_{\text{loc,}u}^2(\mathbb{R}^N) \}.\]
We assume in addition that

\[ 1 < p < p_c \equiv 1 + \frac{4}{N - 1}. \]

The Cauchy problem of equation (1.1) is wellposed in \( H_{\text{loc}, u}^1 \times L_{\text{loc}, u}^2 \). This follows from the finite speed of propagation and the wellposedness in \( H^1 \times L^2 \), valid whenever \( 1 < p < 1 + \frac{4}{N - 1} \).

The existence of blow-up solutions for the associated ordinary differential equation of (1.1) is a classical result. Using the finite speed of propagation, we conclude that there exists a blow-up solution \( u(t) \) of (1.1). In this paper, we consider a blow-up solution \( u(t) \) of (1.1), and the initial data in \( H^1 \). The set \( u(t) \) is called the maximal influence domain of \( u \). Moreover, from the finite speed of propagation, \( T \) is a 1-Lipschitz function. Let \( T \) be the minimum of \( T(x) \) for all \( x \in \mathbb{R}^N \). The time \( T \) and the graph \( \Gamma \) are called (respectively) the blow-up time and the blow-up graph of \( u \).

We first introduce the following non-degeneracy condition for \( \Gamma \). If we introduce for all \( x \in \mathbb{R}^N, t \leq T(x) \) and \( \delta > 0 \), the cone

\[ C_{x,t,\delta} = \{ (\xi, \tau) : 0 \leq \tau \leq t - \delta |\xi - x| \}, \]

then our non-degeneracy condition is the following: \( x_0 \) is a non-characteristic point if

\[ \exists \delta_0 = \delta_0(x_0) \in (0, 1) \quad \text{such that} \quad u \text{ is defined on } C_{x_0, T(x_0), \delta_0}. \]

We aim at studying the growth estimate of \( u(t) \) near the space–time blow-up graph.

In the case \((f, g) \equiv (0, 0)\), equation (1.1) reduces to the semilinear wave equation:

\[ u_{tt} = \Delta u + |u|^{p-1} u, \quad (x, t) \in \mathbb{R}^N \times [0, T). \]

Merle and Zaag in [8] (see also [6, 7]) have proved that if \( u \) is a solution of (1.4) with blow-up graph \( \Gamma : \{ x \to T(x) \} \) then for all \( x_0 \in \mathbb{R}^N \) and \( t \in [\frac{2}{3}T(x_0), T(x_0)] \), the growth estimate near the space–time blow-up graph satisfies

\[
\begin{align*}
(T(x_0) - t)^{-\frac{2}{p-2}} &\left( \frac{\|u(t)\|_{L^1(B(x_0, T(x_0) - t))}^2}{(T(x_0) - t)^{\frac{p}{2}}} \right) \\
+ &\left( T(x_0) - t \right)^{\frac{2}{p-1}} \left( \frac{\|u(t)\|_{L^1(B(x_0, T(x_0) - t))}^2}{(T(x_0) - t)^{\frac{p}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0) - t))}^2}{(T(x_0) - t)^{\frac{p}{2}}} \right) \leq K,
\end{align*}
\]

where the constant \( K \) depends only on \( N, p \), on an upper bound on \( T(x_0) \), \( \frac{1}{T(x_0)} \) and the initial data in \( H_{\text{loc}, u}^1(\mathbb{R}^N) \times L_{\text{loc}, u}^2(\mathbb{R}^N) \). If in addition \( x_0 \) is non-characteristic (in the sense of (1.3)), then for all \( t \in \left[ \frac{2}{3}T(x_0), T(x_0) \right] \),

\[
\begin{align*}
0 < \epsilon_0(N, p) \leq &\left( T(x_0) - t \right)^{-\frac{2}{p-2}} \frac{\|u(t)\|_{L^1(B(x_0, T(x_0) - t))}^2}{(T(x_0) - t)^{\frac{p}{2}}} \\
+ &\left( T(x_0) - t \right)^{\frac{2}{p-1}} \left( \frac{\|u(t)\|_{L^1(B(x_0, T(x_0) - t))}^2}{(T(x_0) - t)^{\frac{p}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0) - t))}^2}{(T(x_0) - t)^{\frac{p}{2}}} \right) \leq K,
\end{align*}
\]

where the constant \( K \) depends only on \( N, p \), on an upper bound on \( T(x_0) \), \( \frac{1}{T(x_0)} \), \( \delta_0(x_0) \) and the initial data in \( H_{\text{loc}, u}^1(\mathbb{R}^N) \times L_{\text{loc}, u}^2(\mathbb{R}^N) \).

Following this blow-up rate estimate, Merle and Zaag addressed the question of the asymptotic behaviour of \( u(x, t) \) near \( \Gamma \) in one space dimension.
More precisely, they proved in [9, 10] that the set of non-characteristic points $\mathcal{R} \subset \mathbb{R}$ is open and that $x \mapsto T(x)$ is of class $C^1$ on $\mathcal{R}$. They also described the blow-up profile of $u$ near $(x_0, T(x_0))$ when $x_0 \in \mathcal{R}$.

In [11], they proved that $S = \mathbb{R} \setminus \mathcal{R}$ has an empty interior and that $\Gamma$ is a corner of angle $\frac{\pi}{4}$ near any $x_0 \in S$. They also showed that $u(x, t)$ decomposes into a sum of decoupled solitons near $(x_0, T(x_0))$.

Our aim in this work is to generalize the blow-up rate estimate obtained for equation (1.4) in [6] and [8] in the subcritical case ($p < p_c$) to equation (1.1).

One may think that such a generalization is straightforward and only technical. In fact, that opinion may be valid for all the steps, except for the very first one, that is, the existence of a Lyapunov functional in similarity variables which is far from being trivial. That functional is our main contribution. The existence of the Lyapunov functional is a crucial step towards the derivation of blow-up results for equation (1.1).

As in [6, 7], we want to write the solution $v$ of the associate ordinary differential equation of (1.1). It is clear that $v$ is given by

$$v'' = v^p + f(v) + g(v), \quad v(T) = +\infty,$$

and satisfies

$$v(t) \sim \frac{\kappa}{(T-t)^{-\frac{1}{p-1}}} \quad \text{as } t \to T, \quad \text{where } \kappa = \left(\frac{2p+2}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$  

For this reason, we define for all $x_0 \in \mathbb{R}^N$, $0 < T_0 \leq T_0(x_0)$, the following similarity transformation introduced by Antonini and Merle [3] and used in [6–8]:

$$y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t), \quad u(x, t) = \frac{1}{(T_0 - t)^{\frac{1}{p-1}}} w_{ss, T_0}(y, s).$$

The function $w_{ss, T_0}$ (we write $w$ for simplicity) satisfies the following equation for all $y \in B$ the unit ball of $\mathbb{R}^N$ and $s \geq -\log T_0$:

$$w_{ss} = \frac{1}{\rho} \text{div} (\rho \nabla w - \rho(y \cdot \nabla w)y) - \frac{2p+2}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} w_y - 2y \cdot \nabla w_y + e^{-\frac{2m}{p-1}} f(e^{\frac{2m}{p-1}} w) + e^{-\frac{2m}{p-1}} g(e^{\frac{2m}{p-1}} (w_y + y \cdot \nabla w + \frac{2}{p-1} w)), \quad (1.8)$$

where $\rho = (1 - |y|^2)^{\alpha}$ and $\alpha = \frac{2}{p-1} - \frac{N-1}{2} > 0$.

In the new set of variables $(y, s)$, the behaviour of $u$ as $t \to T_0$ is equivalent to the behaviour of $w$ as $s \to +\infty$.

**Remark.** We note that the corresponding terms of the functions $f(u)$ and $g(u)$ in the problem (1.8) satisfy the following inequalities, for all $s \geq 0$:

$$e^{-\frac{2m}{p-1}} \left| f\left(e^{\frac{2m}{p-1}} w\right) \right| \leq C Me^{-\frac{2m}{p-1} + C Me^{-\frac{2m}{p-1}} |w|^p} \leq C Me^{-\frac{2m}{p-1}} + C Me^{-\frac{2m}{p-1}} |w|^p$$

and

$$e^{-\frac{2m}{p-1}} \left| g\left(e^{\frac{2m}{p-1}} (w_y + y \cdot \nabla w + \frac{2}{p-1} w)\right) \right| \leq C Me^{-\frac{2m}{p-1}} + C Me^{-s} \left| w_y + y \cdot \nabla w + \frac{2}{p-1} w \right|.$$  

For this reason, we can see that in the variables $(y, s)$ problem (1.8) is a perturbation of the particular case where $(f, g) \equiv (0, 0)$, when $s \to +\infty$.  


Equation (1.8) will be studied in the space
\[ \mathcal{H} = \left\{ (w_1, w_2) \mid \int_B \left( w_2^2 + |\nabla w_1|^2(1 - |y|^2) + w_1^2 \right) \rho \, dy < +\infty \right\}. \]

In the whole paper, we denote \( F(u) = \int_0^u f(v) \, dv \).

In the case \((f, g) \equiv (0, 0),\) Antonini and Merle [3] proved that
\[
E_0(w) = \int_B \left( \frac{1}{2} w_s^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2} (y \cdot \nabla w)^2 + \frac{p+1}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho \, dy, \tag{1.9}
\]
is a Lyapunov functional for equation (1.8). When \((f, g) \neq (0, 0),\) we introduce
\[
H(w) = E(w)e^{\frac{N-2}{4} \gamma r} + \theta e^{-2\gamma r}, \tag{1.10}
\]
where \( \theta \) is a sufficiently large constant that will be determined later,
\[
E(w) = E_0(w) + I(w) + J(w), \quad I(w) = -e^{\frac{2N+4}{p+1}} \int_B F(e^{\frac{2}{N+4}} w) \rho \, dy
\]
and
\[
J(w) = -e^{-\gamma r} \int_B w_{s2} \rho \, dy \quad \text{with} \quad \gamma = \min \left( \frac{1}{2} \frac{p-q}{p-1} \right) > 0. \tag{1.11}
\]
We now claim that the functional \( H(w) \) is a decreasing function of time for equation (1.8), provided that \( s \) is large enough.

Here we announce our main result.

**Theorem 1.1 (Existence of a Lyapunov functional for equation (1.8)).** Let \( N, p, q \) and \( M \) be fixed. There exists \( S_0 = S_0(N, p, q, M) \in \mathbb{R} \) such that, for all \( s_0 \in \mathbb{R} \) and \( w \) solution of equation (1.8) satisfying \((w, w_s) \in \mathcal{C}([s_0, +\infty), \mathcal{H}),\) it holds that \( H \) satisfies the following inequality, for all \( s_2 > s_1 \geq \max(s_0, S_0):\)
\[
H(w(s_2)) - H(w(s_1)) \leq -\alpha \int_{s_1}^{s_2} \int_B w_s^2(y, s) \frac{\rho}{1 - |y|^2} \, dy \, ds, \tag{1.12}
\]
where \( \alpha = \frac{2}{p-1} - \frac{N-4}{2} > 0. \)

**Remark.**

1. One may wonder why we take only sublinear perturbations in \( u_t, \) (see hypothesis \( (H_g)) \). It happens that any superlinear terms in \( u_t \) generates in similarity variables \( L^r \) norms of \( w_s \) and \( \nabla w, \) where \( r > 2, \) hence, non-controllable by the terms in the Lyapunov functional \( E_0(w) \) (1.9) of the non-perturbed equation (1.4).
2. Our method breaks down in the critical case \( p = p_c, \) since in the energy estimates in similarity variables, the perturbation terms are integrated on the whole unit ball, hence, difficult to control with the dissipation of the non-perturbed equation (1.4), which degenerates to the boundary of the unit ball.

As we said earlier, the existence of this Lyapunov functional (and a blow-up criterion for equation (1.8) based on \( H, \) see lemma 2.3) are a crucial step in the derivation of the blow-up rate for equation (1.1). Indeed, with the functional \( H \) and some more work, we are able to adapt the analysis performed in [8] for equation (1.4) and obtain the following result:

**Theorem 1.2 (Blow-up rate for equation (1.1)).** Let \( N, p, q \) and \( M \) be fixed. Then, there exist \( \tilde{S}_0 = \tilde{S}_0(N, p, q, M) \in \mathbb{R} \) and \( \tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(N, p, M), \) such that if \( w \) is a solution of (1.1) with blow-up graph \( \Gamma : \{ x \rightarrow T(x) \} \) and \( x_0 \) is a non-characteristic point, then
(i) For all \( s \geq \tilde{\varepsilon}_0(\tilde{S}_0) = \max(\tilde{S}_0(N, p, q, M), -\log \frac{T(x_0)}{4}), \)
\[
0 < \varepsilon_0 \leq \| w_{\tilde{\varepsilon}_0T(\tilde{\varepsilon}_0)}(s) \|_{H^1(B)} + \| \tilde{\varepsilon}_0 w_{\tilde{\varepsilon}_0T(\tilde{\varepsilon}_0)}(s) \|_{L^2(B)} \leq K,
\]
where \( w_{\tilde{\varepsilon}_0T(\tilde{\varepsilon}_0)} \) is defined in (1.8) and \( B \) is the unit ball of \( \mathbb{R}^N. \)
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(ii) For all \( t \in [t_0(x_0), T(x_0)) \), where \( t_0(x_0) = \max(T(x_0) - e^{-\frac{\omega_0}{2}}, \frac{3\gamma(x_0)}{4}) \), we have

\[
0 < \varepsilon_0 \leq (T(x_0) - t)^{\frac{s}{2}} \left( \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0) - t)^{\frac{s}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0) - t)^{\frac{s}{2}}} \right) \leq K,
\]

where \( K = K(N, p, q, \delta_0(x_0), \|(u(t_0(x_0)), u_t(t_0(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{3\gamma(x_0)}{4}))} \) and \( \delta_0(x_0) \in (0, 1) \) is defined in (1.3).

RemarK. With this blow-up rate, one can ask whether the results proved by Merle and Zaag for the non-perturbed problem in [9–11], hold for equation (1.1) (blow-up, profile, regularity of the blow-up graph, existence of characteristic points, etc). We believe that it is the case, however, that the proof will be highly technical, with no interesting ideas (in particular, equation (1.1) is not conserved under the Lorentz transform, which is crucial in [9–11], and lots of minor terms will appear in the analysis). Once again, we believe that the key point in the analysis of blow-up for equation (1.1) is the derivation of a Lyapunov functional in similarity variables, which is the object of our paper.

As in the particular case where \((f, g) \equiv (0, 0)\), the proof of theorem 1.2 relies on four ideas (the existence of a Lyapunov functional, interpolation in Sobolev spaces, some Gagliardo–Nirenberg estimates and a covering technique adapted to the geometric shape of the blow-up surface). It happens that adapting the proof of [8] given in the non-perturbed case (1.4) is straightforward, except for a key argument, where we bound the \(L^p+1\) space–time norm of \(w\). Therefore, we only present that argument, and refer to [6] and [8] for the rest of the proof.

This paper is divided into two sections, each of them devoted to the proof of a theorem.

2. A Lyapunov functional for equation (1.8)

This section is divided into two parts:

• We first prove the existence of a Lyapunov functional for equation (1.8).

• Then, we give a blow-up criterion for equation (1.8) based on the Lyapunov functional.

Throughout this section, we consider \((w, w_1) \in C([\delta_0, +\infty), \mathcal{H})\) where \(w\) is a solution of (1.8) and \(\delta_0 \in \mathbb{R}\). We aim at proving that the functional \(H\) defined in (1.10) is a Lyapunov functional for equation (1.8), provided that \(s \geq \delta_0\), for some \(\delta_0 = \delta_0(N, p, q, M)\). We denote by \(C\) a constant which depends only on \((p, q, N, M)\). We denote the unit ball of \(\mathbb{R}^N\) by \(B\).

2.1. Existence of a Lyapunov functional

Lemma 2.1. Let \(N, p, q\) and \(M\) be fixed. There exists \(S_1 = S_1(N, p, q, M) \in \mathbb{R}\) such that, for all \(x_0 \in \mathbb{R}\) and \(w \) solution of equation (1.8) satisfying \((w, w_1) \in C([\delta_0, +\infty), \mathcal{H})\), we have the following inequality, for all \(s \geq \max(x_0, S_1)\):

\[
\frac{d}{ds} (E_0(w) + I(w)) \leq - \frac{3\alpha}{2} \int_B w^2 \frac{\rho}{1 - |y|^2} dy + \Sigma_0(s),
\]

where \(\Sigma_0\) satisfies

\[
\Sigma_0(s) \leq Ce^{-2t} \int_B |\nabla w|^2 (1 - |y|^2) \rho dy + Ce^{-2t} \int_B w^2 \rho dy
+ Ce^{-2t} \int_B \int_B |w|^{p+1} \rho dy + Ce^{-2t} \int_B |w|^{p+1} \rho dy.
\]
Proof. Multiplying (1.8) by \( w_1 \rho \), and integrating over the ball \( B \), we obtain, for all \( s \geq s_0 \), (recall from [3] that in the case where, \( (f, g) \equiv (0, 0) \), we have \( \frac{d}{ds} E_0(w) = -2\alpha \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy \))

\[
\frac{d}{ds}(E_0(w) + I(w)) = -2\alpha \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy + \frac{2(p + 1)}{p - 1} \int_B e^{-\frac{2s+1}{p-1}} \frac{1}{y} \, dy \\
+ \frac{2}{p - 1} \int_B f(e^{\frac{s}{p-1}})w \, dy \\
+ e^{-\frac{2s}{p-1}} \int_B g(e^{\frac{s}{p-1}} (w_2 + y \cdot \nabla w + \frac{2}{p - 1} w))w_1 \rho \, dy.
\]  

(2.2)

By exploiting the fact that \(|F(x)| + |xf(x)| \leq C(1 + |x|^{p+1})\), we obtain

\(|I_1| + |I_2| \leq C e^{-\frac{2s+1}{p-1}} \int_B (1 + |e^{\frac{s}{p-1}} w|^{p+1}) \rho \, dy \leq C e^{-\frac{2s+1}{p-1}} + C e^{-\frac{2s+1}{p-1}} \int_B |w|^{p+1} \rho \, dy. \)  

(2.3)

Note that \(|x|^{p+1} \leq C(1 + |x|^{p+1})\), we deduce from (2.3) that for all \( s \geq \max(s_0, 0)\),

\(|I_1| + |I_2| \leq C e^{-\frac{2s+1}{p-1}} + C e^{-\frac{2s+1}{p-1}} \int_B |w|^{p+1} \rho \, dy. \)  

(2.4)

Since \(|g(x)| \leq M(1 + |x|)\), we write

\(|I_3| \leq C e^{-s} \int_B w_1^2 \rho \, dy + C e^{-s} \int_B |y \cdot \nabla w| |w_1| \rho \, dy \\
+ C e^{-s} \int_B |w| |w_1| \rho \, dy + C e^{-\frac{2s}{p-1}} \int_B |w_1| \rho \, dy. \)  

(2.5)

By exploiting the inequality \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \), we obtain

\( C e^{-s} \int_B |y \cdot \nabla w| |w_1| \rho \, dy \leq \frac{a}{8} \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy + C e^{-2s} \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy. \)  

(2.6)

Similarly, we prove that

\( C e^{-s} \int_B |w| |w_1| \rho \, dy \leq \frac{a}{8} \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy + C e^{-2s} \int_B w_2^2 \rho \, dy. \)  

(2.7)

We infer from the inequality \(|a| \leq 1 + a^2\) that

\( C e^{-\frac{2s}{p-1}} \int_B |w_1| \rho \, dy \leq C e^{-\frac{2s}{p-1}} + C e^{-\frac{2s}{p-1}} \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy. \)  

(2.8)

Combining (2.5), (2.6), (2.7) and (2.8), we conclude that, for all \( s \geq \max(s_0, 0)\),

\(|I_3| \leq \left( C e^{-s} + \frac{a}{4}\right) \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy + C e^{-2s} \int_B w_2^2 \rho \, dy \\
+ C e^{-2s} \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy + C e^{-\frac{2s}{p-1}}. \)  

(2.9)

Then, using (2.2), (2.4) and (2.9), we deduce that, for all \( s \geq \max(s_0, 0)\),

\[
\frac{d}{ds}(E(w) + I(w)) \leq \left( -\frac{7\alpha}{4} + C e^{-s} \right) \int_B w_1^2 \frac{\rho}{1 - |y|^2} \, dy + C e^{-2s} \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy \\
+ C e^{-2s} \int_B w_2^2 \rho \, dy + C e^{-\frac{2s+1}{p-1}}. \]  

(2.10)

Taking \( S_1 = S_1(N, p, q, M) \) large enough, we have estimate (2.1). This concludes the proof of lemma 2.1.
We are now going to prove the following estimate for the functional $J$:

**Lemma 2.2.** Let $N$, $p$, $q$ and $M$ be fixed. There exists $S_2 = S_2(N, p, q, M) \in \mathbb{R}$ such that, for all $s_0 \in \mathbb{R}$ and $w$ solution of equation (1.8) satisfying $(w, w_s) \in C([s_0, +\infty), \mathcal{H})$, $J$ satisfies the following inequality, for all $s \geq \max(s_0, S_2)$:

$$
\begin{align*}
\frac{d}{ds} J(w) &\leq \frac{\alpha}{2} \int_B w^2 s \rho \frac{\rho}{1-|y|^2} dy + \frac{p+3}{2} e^{-\gamma s} E(s) - \frac{p-1}{4} e^{-\gamma s} \int_B |\nabla w|^2 (1 - |y|^2) \rho dy \\
&\quad - \frac{p+1}{2(p-1)} e^{-\gamma s} \int_B w^2 \rho dy - \frac{p-1}{2(p+1)} e^{-\gamma s} \int_B |w|^{p+1} \rho dy + \Sigma_1(s),
\end{align*}
$$

(2.11)

where $\gamma = \min \left( \frac{1}{2}, \frac{p-q}{p-1} \right) > 0$ and $\Sigma_1(s)$ satisfies

$$
\Sigma_1(s) \leq Ce^{-2\gamma s} \int_B w^2 \rho dy + Ce^{-2\gamma s} \int_B |\nabla w|^2 \rho (1 - |y|^2) dy \\
+ Ce^{-2\gamma s} \int_B |w|^{p+1} \rho dy + Ce^{-2\gamma s}.
$$

(2.12)

**Proof.** Note that $J$ is a differentiable function for all $s \geq s_0$ and that

$$
\frac{d}{ds} J(w) = \gamma e^{-\gamma s} \int_B w w_s \rho dy - e^{-\gamma s} \int_B w^2 \rho dy - e^{-\gamma s} \int_B w w_s \rho dy.
$$

Using equation (1.8) and integrating by parts, we have

$$
\begin{align*}
\frac{d}{ds} J(w) &= -e^{-\gamma s} \int_B w^2 \rho dy + e^{-\gamma s} \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho dy \\
&\quad + \frac{2p+2}{(p-1)^2} e^{-\gamma s} \int_B w^2 \rho dy - e^{-\gamma s} \int_B |w|^{p+1} \rho dy \\
&\quad + \left( \gamma - \frac{p+3}{p-1} - 2N \right) e^{-\gamma s} \int_B w w_s \rho dy - 2e^{-\gamma s} \int_B w w_s (y \cdot \nabla \rho) dy \\
&\quad - 2e^{-\gamma s} \int_B w_s (y \cdot \nabla w) \rho dy - e^{-\gamma s} \int_B w f \left( \frac{y}{|y|^2} + \frac{2}{p-1} \right) \rho dy.
\end{align*}
$$

(2.13)

By combining (1.9), (1.11) and (2.13), we write

$$
\begin{align*}
\frac{d}{ds} J(w) &\leq \frac{p+3}{2} e^{-\gamma s} E(s) - \frac{p-1}{4} e^{-\gamma s} \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho dy \\
&\quad - \frac{p+1}{2(p-1)} e^{-\gamma s} \int_B w^2 \rho dy - \frac{p-1}{2(p+1)} e^{-\gamma s} \int_B |w|^{p+1} \rho dy \\
&\quad + \left( \gamma - \frac{p+3}{p-1} - 2N \right) e^{-\gamma s} \int_B w w_s \rho dy \\
&\quad - 2e^{-\gamma s} \int_B w w_s (y \cdot \nabla \rho) dy - 2e^{-\gamma s} \int_B w_s (y \cdot \nabla w) \rho dy \\
&\quad - e^{-\gamma s} \int_B w f \left( \frac{y}{|y|^2} \right) \rho dy.
\end{align*}
$$
\[-e^{-\frac{2p+3}{p-1}y^s} \int_B w g \left( e^{\frac{p+3}{p-1}} (w + y \cdot \nabla w + \frac{2}{p-1} w) \right) \rho \, dy \]
\[+ \left( \frac{p+3}{2} \right) e^{-\frac{2p+3}{p-1}y^s} \int_B F(e^{\frac{p+3}{p-1}} w) \rho \, dy. \quad (2.14)\]

We now study each of the last six terms. To estimate \( J_1 \), we use the fact that for all \( s \geq \max(x_0, 0) \),
\[|y + \frac{p+3}{p-1} - 2N + \frac{p+3}{2} e^{-y^s}| \leq C. \quad (2.15)\]
Using (2.15) and the Cauchy–Schwarz inequality we obtain
\[|J_1| \leq Ce^{-ys} \int_B |w w_j| \rho \, dy \leq \frac{\alpha}{8} \int_B w_s^2 \frac{\rho}{1 - |y|^2} \, dy + Ce^{-2ys} \int_B w^2 \rho \, dy. \quad (2.16)\]
Now we estimate the expression \( J_2 \). Since we have \( y \cdot \nabla \rho = -2\alpha \frac{|y|^2}{(1 - |y|^2)} \rho \), we can use the Cauchy–Schwarz inequality to write
\[|J_2| \leq Ce^{-ys} \int_B |w_j|(1 - |y|^2) \frac{2\alpha}{(1 - |y|^2)} |w| |y|(1 - |y|^2) \frac{2\alpha}{(1 - |y|^2)} \, dy \]
\[\leq \frac{\alpha}{8} \int_B w_s^2 \frac{\rho}{1 - |y|^2} \, dy + Ce^{-2ys} \int_B w^2 \frac{|y|^2 \rho}{1 - |y|^2} \, dy. \quad (2.17)\]
Since we have the following Hardy type inequality for any \( w \in H^{1}_{\text{loc}}(\mathbb{R}^N) \) (see appendix B in [6] for details):
\[\int_B w^2 \frac{|y|^2 \rho}{1 - |y|^2} \, dy \leq C \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy + C \int_B w^2 \rho \, dy, \quad (2.18)\]
we use (2.17) and (2.18) to conclude that
\[|J_2| \leq \frac{\alpha}{8} \int_B w_s^2 \frac{\rho}{1 - |y|^2} \, dy + Ce^{-2ys} \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy \quad (2.19)\]
Using the Cauchy–Schwarz inequality, we have
\[|J_3| \leq \frac{\alpha}{8} \int_B w_s^2 \frac{\rho}{1 - |y|^2} \, dy + Ce^{-2ys} \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy \quad (2.20)\]
By exploiting the fact that \(|F(x)| \leq CM(1 + |x|^q + 1)\) and \(|f(x)| \leq M(1 + |x|^q)\), we write
\[|J_4| + |J_6| \leq Ce^{-\frac{2p+3}{p-1}y^s} \int_B (1 + |w|^{q+1}) \rho \, dy \leq Ce^{-2ys} \int_B (1 + |w|^{q+1}) \rho \, dy \]
\[\leq Ce^{-2ys} + Ce^{-2ys} \int_B |w|^{p+1} \rho \, dy. \quad (2.21)\]
In a similar way, using the fact that \(|g(x)| \leq M(1 + |x|)\), we write
\[|J_5| \leq Ce^{-2ys} \int_B w_s^2 \rho \, dy + Ce^{-2ys} \int_B |y \cdot \nabla w| |w| \rho \, dy + Ce^{-2ys} \int_B w^2 \rho \, dy + Ce^{-2ys}. \]
Then, by (2.18), we have
\[|J_5| \leq Ce^{-2ys} \int_B w_s^2 \frac{\rho}{1 - |y|^2} \, dy + Ce^{-2ys} \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy \]
\[+ Ce^{-2ys} \int_B w^2 \rho \, dy + Ce^{-2ys}. \quad (2.22)\]
Finally, using (2.14), (2.16), (2.19), (2.20), (2.21) and (2.22) we deduce that
\[
\frac{d}{ds} J(w) \leq \frac{p + 3}{2} e^{-\gamma s} E(s) - \frac{p - 1}{4} e^{-\gamma s} \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho \, dy \\
- \frac{p + 1}{2(p - 1)} e^{-\gamma s} \int_B w^2 \rho \, dy - \frac{p - 1}{2(p + 1)} e^{-\gamma s} \int_B |w|^{p+1} \rho \, dy \\
+ Ce^{-2\gamma s} \int_B w^2 \rho \, dy + Ce^{-2\gamma s} \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy \\
+ \left(\frac{3\alpha}{8} + Ce^{-2\gamma s}\right) \int_B w^2 \rho (1 - |y|^2) \, dy + Ce^{-2\gamma s} + Ce^{-2\gamma s} \int_B |w|^{p+1} \rho \, dy.
\]
Since $|y \cdot \nabla w| \leq |y||\nabla w|$, it follows that
\[
\int_B (|\nabla w|^2 \rho (1 - |y|^2) \, dy \leq \int_B ((|\nabla w|^2 - (y \cdot \nabla w)^2) \rho \, dy.
\]
Taking $S_2 = S_2(N, p, q, M)$ large enough, we have easily estimate (2.11) and (2.12). This concludes the proof of lemma 2.2.

With lemmas 2.1 and 2.2, we are in a position to prove theorem 1.1.

**Proof of theorem 1.1.** From lemmas 2.1 and 2.2, we obtain for all $s \geq \max(s_0, S_1, S_2)$,
\[
\frac{d}{ds} E(w) \leq Ce^{-2\gamma s} + \frac{p + 3}{2} e^{-\gamma s} E(w) - \alpha \int_B w^2 \rho \frac{1}{1 - |y|^2} \, dy \\
+ \left(\frac{p - 1}{4} e^{-\gamma s}\right) \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho \, dy \\
+ \left(\frac{p + 1}{2(p - 1)} e^{-\gamma s}\right) \int_B w^2 \rho \, dy \\
+ \left(\frac{p - 1}{2(p + 1)} e^{-\gamma s}\right) \int_B |w|^{p+1} \rho \, dy.
\]

We now choose $S_0 \geq \max(S_1, S_2)$, large enough, so that for all $s \geq S_0$, we have
\[
\frac{p - 1}{4} - Ce^{-\gamma s} \geq 0, \quad \frac{p + 1}{2(p - 1)} - Ce^{-\gamma s} \geq 0, \quad \frac{p - 1}{2(p + 1)} - Ce^{-\gamma s} \geq 0.
\]

Then, we deduce that, for all $s \geq \max(S_0, S_0)$, we have
\[
\frac{d}{ds} E(w) \leq Ce^{-2\gamma s} + \frac{p + 3}{2} e^{-\gamma s} E(w) - \alpha \int_B w^2 \rho \frac{1}{1 - |y|^2} \, dy.
\]

Finally, we prove easily that the function $H$ satisfies, for all $s \geq \max(S_0, S_0)$,
\[
\frac{d}{ds} H(w) \leq (Ce^\frac{3\alpha}{8} e^{-\gamma s} - 2\theta \gamma) e^{-2\gamma s} - \alpha e^\frac{3\alpha}{8} e^{-\gamma s} \int_B w^2 \rho \frac{1}{1 - |y|^2} \, dy \\
\leq (C - 2\theta \gamma) e^{-2\gamma s} - \alpha \int_B w^2 \rho \frac{1}{1 - |y|^2} \, dy.
\]

We now choose $\theta$ large enough, so we have $C - 2\theta \gamma \leq 0$ and then
\[
\frac{d}{ds} H(w) \leq - \theta \int_B w^2 \rho \, dy.
\]

Now (1.12) is a direct consequence of inequality (2.27).

This concludes the proof of theorem 1.1.
2.2. A blow-up criterion in the \(w(y,s)\) variable

We now make the following proposition:

**Lemma 2.3.** Let \(N, p, q, M\) be fixed. There exists \(S_3 = S_3(N, p, q, M) \geq S_0\) such that, for all \(s_0 \in \mathbb{R}\) and \(w\) solution of equation (1.8) defined to the left of \(s_0\), such that \(\|w(s)\|_{L^{p+1}(B)}\) is locally bounded, if \(H(w(s_3)) < 0\) for some \(s_3 \geq \max(S_3, s_0)\), then \(w\) blows up in some finite time \(S > s_3\).

**Remark.** If \(w = w_{s_0, T_0}\) is defined from a solution of (1.1) by (1.7) and \(x_0\) is a non-characteristic point, then \(\|w(s)\|_{H^1(B)}\) is locally bounded and so is \(\|w(s)\|_{L^{p+1}(B)}\) by Sobolev’s embedding.

**Proof.** The argument is the same as in the corresponding part in [3]. We write the proof for completeness. Arguing by contradiction, we assume that there exists a solution \(w\) on \(B\), defined for all time \(s \in [s_3, +\infty[\), where \(H(w(s_3)) < 0\). Since the energy \(H\) decreases in time, we have \(H(w(1 + s_3)) < 0\). Consider now for \(\delta > 0\) the function \(\tilde{w}^\delta(y,s)\) for \((y,s) \in B \times [1 + s_3, +\infty[\) defined by

\[
\forall s \geq 1 + s_3, \quad \forall y \in B, \quad \tilde{w}^\delta(y,s) = \frac{1}{(1 + \delta e^{-y})^{\frac{p}{p+1}}} w \left( \frac{y}{1 + \delta e^{-y}}, -\log(\delta + e^{-y}) \right).
\]

- (A) Note that \(\tilde{w}^\delta\) is defined in \(B \times [1 + s_3, +\infty[\), whenever \(\delta > 0\) is small enough such that \(-\log(\delta + e^{-\gamma s}) \geq s_3\).
- (B) From its construction, \(\tilde{w}^\delta\) is also a solution of (1.8). Indeed, let \(\bar{u}\) be such that \(w = u_{0,0}\) in definition (1.7). Then \(u\) is a solution of (1.1) and \(\tilde{w}^\delta = w_{0, -\delta}\) is defined as in (1.7); so \(\tilde{w}^\delta\) is a solution of (1.8).
- (C) For \(\delta\) small enough, we have \(H(\tilde{w}^\delta(1 + s_3)) < 0\) by continuity of the function \(\delta \mapsto H(\tilde{w}^\delta(1 + s_3))\). Then, we write that \(H(\tilde{w}^\delta(1 + s_3)) < 0\).

Now, we fix \(\delta = \delta_0 > 0\) such that (A), (B) and (C) hold. We note that we have

\[
- e^{-\gamma s} \int_B w_h^\delta w_h^\delta \rho \, dy \geq - \frac{1}{4} \int_B (w_h^\delta)^2 \rho \, dy - e^{-2\gamma s} \int_B (w_h^\delta)^2 \rho \, dy
\]

and from (2.4)

\[
- e^{-2\gamma s} \int_B \|w_h^\delta\|^2 \rho \, dy \geq - C e^{-2\gamma s} \int_B |w_h^\delta|^{p+1} \rho \, dy.
\]

By (1.9), (1.11), (2.28) and (2.29) we deduce

\[
E(w_h^\delta(s)) \geq \frac{1}{4} \int_B (w_h^\delta)^2 \rho \, dy + \frac{p + 1}{(p - 1)^2} - e^{-2\gamma s} \int_B (w_h^\delta)^2 \rho \, dy
\]

\[
- \left( \frac{1}{p + 1} + C e^{-2\gamma s} \right) \int_B |w_h^\delta|^{p+1} \rho \, dy - C e^{-2\gamma s}.
\]

We now choose \(s_4 \geq s_3\) large enough, so that we have \(\frac{p + 1}{(p - 1)^2} - e^{-2\gamma s_4} \geq 0\). Then, we deduce that we have, for all \(s \geq s_4\),

\[
E(w_h^\delta(s)) \geq - \left( \frac{1}{p + 1} + C e^{-2\gamma s} \right) \int_B |w_h^\delta|^{p+1} \rho \, dy - C e^{-2\gamma s}.
\]

Since \(\rho \leq 1\), after a change of variables, we find that

\[
E(w_h^\delta(s)) \geq - \left( \frac{1}{p + 1} + C e^{-2\gamma s} \right) \int_B |w(z, -\log(\delta_0 + e^{-z}))|^{p+1} \, dz - C e^{-2\gamma s}.
\]
Since we have \(-\log(\delta_0 + e^{-r}) \to -\log(\delta_0)\) as \(s \to +\infty\) and since \(\|w(s)\|_{L^{p+1}(B)}\) is locally bounded by hypothesis, by a continuity argument, it follows that the former integral remains bounded and

\[ E(w^s(h)(s)) \geq \frac{C}{(1 + \delta_0 e^{s})^{1+4/N}} - Ce^{-2ys} \to 0, \]

as \(s \to +\infty\) (use the fact that \(\frac{4}{p+1} + 2 - N > 0\) which follows from the fact that \(p < 1 + \frac{4}{N-2}\)). So, from (1.10), it follows that

\[ \lim_{s \to +\infty} \inf H(w^s(s)) \geq 0. \] (2.31)

Inequality (2.31) contradicts the inequality \(H(w^s(s_3)) < 0\) and the fact that the energy \(H\) decreases in time for \(s \geq s_3\). This concludes the proof of lemma 2.3. \(\blacksquare\)

3. Boundedness of the solution in similarity variables

We prove theorem 1.2 here. Note that the lower bound follows from the finite speed of propagation and wellposedness in \(H^1 \times L^2\). For a detailed argument in the similar case of equation (1.4), see lemma 3.1 in [8, p 1136].

We consider \(u\) a solution of (1.1) which is defined under the graph of \(x \mapsto T(x)\), and \(x_0\) a non-characteristic point. Given some \(T_0 \in (0, T(x_0))\), we introduce \(w_{x_0, T_0}\) defined in (1.7), and write \(w\) for simplicity, when there is no ambiguity. We aim at bounding \(\|(w, \partial_s w)(s)\|_{H^1 \times L^2(B)}\) for \(s\) large.

As in [6], by combining theorem 1.1 and lemma 2.3 (use, in particular, the remark after that lemma) we obtain the following bounds:

**Corollary 3.1 (Bounds on \(E\)).** For all \(s \geq \tilde{s}_3 = \tilde{s}_3(T_0) = \max(S_3, -\log(T_0)), s_2 \geq s_1 \geq \tilde{s}_3\), it holds that

\[ -C_0 \leq E(w(s)) \leq M_0, \]

\[ \int_{s_1}^{s_2} \int_B w^2(y, s) \frac{\rho}{1 + |y|^2} \, dy \, ds \leq M_0, \] (3.1)

where \(M_0 = M_0(N, p, q, M, \tilde{s}_3(T_0)), \|(u(t_3), u_s(t_3))\|_{H^1 \times L^2(B(x_{0}, e^{\tilde{s}_3(T_0)}))}, t_3 = t_3(T_0) = T_0 - e^{-\tilde{s}_3(T_0)}, C_0 = C_0(N, p, q, M)\) and \(\delta_0(x_0) \in (0, 1)\) is defined in (1.3).

Starting from these bounds, the proof of theorem 1.2 is similar to the proof in [8] except for the treatment of the perturbation terms. In our opinion, handling these terms is straightforward in all the steps of the proof, except for the first step, where we bound the time averages of the \(L^{p+1}(B)\) norm of \(w\). For that reason, we only give that step and refer to [6, 8] for the remaining steps in the proof of theorem 1.2. This is the step we prove here (In the following \(K_1\) denotes a constant that depends only on \(p, q, N, M, C_0, M_0\) and \(\varepsilon\) is an arbitrary positive number in \((0, 1)\)).

**Proposition 3.2 (Control of the space–time \(L^{p+1}\) norm of \(w\)).** For all \(s \geq 1 + \tilde{s}_3\),

\[ \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho \, dy \, ds \leq K_1(M_0, C, N, p, q, M). \] (3.2)

**Proof.** For \(s \geq 1 + \tilde{s}_3\), we work with time integrals between \(s_1\) and \(s_2\) where \(s_1 \in [s-1, s]\) and \(s_2 \in [s+1, s+2]\). By integrating expression (1.9) of \(E\) in time between \(s_1\) and \(s_2\), where...
s_2 > s_1 > \hat{s}_1$, we obtain

$$
\int_{s_1}^{s_2} E(s) \, ds = \int_{s_1}^{s_2} \left( \frac{1}{2} w^2 + \frac{p + 1}{(p - 1)^2} w^2 - \frac{1}{p + 1} |w|^{p+1} \right) \rho \, dy \, ds 
$$

(3.3)

By combining the identities (3.3) and (3.4), we obtain

$$
\text{Lemma 3.3.}
$$

By multiplying equation (1.8) by $\int_B \rho \, dy$ and integrating both in time and in space over $B \times [s_1, s_2]$, we obtain the following identity, after some integration by parts:

$$
\left[ \int_B \left( \frac{p + 3}{2(p - 1)} - N \right) w^2 \right]_{s_1}^{s_2} = \int_{s_1}^{s_2} \int_B w^2 \rho \, dy \, ds 
$$

(3.3)

$$
- \int_{s_1}^{s_2} \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho \, dy \, ds - \frac{2p + 2}{(p - 1)^2} \int_{s_1}^{s_2} \int_B \nabla w \cdot \nabla \rho \, dy \, ds 
$$

$$
+ \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho \, dy \, ds + 2 \int_{s_1}^{s_2} \int_B w_\rho (y \cdot \nabla w) \rho \, dy \, ds 
$$

$$
+ 2 \int_{s_1}^{s_2} \int_B w_\rho (y \cdot \nabla w) \rho \, dy \, ds 
$$

(3.4)

By combining the identities (3.3) and (3.4), we obtain

$$
\frac{(p - 1)}{2(p + 1)} \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho \, dy \, ds 
$$

$$
= \frac{1}{2} \left[ \int_B \left( w w_\rho + \left( \frac{p + 3}{2(p - 1)} - N \right) w^2 \right) \rho \, dy \right]_{s_1}^{s_2} - \int_{s_1}^{s_2} \int_B \nabla w \cdot \nabla \rho \, dy \, ds 
$$

(3.5)

We claim that proposition 3.2 follows from the following lemma where we control all the terms on the right-hand side of relation (3.5) in terms of the space–time $L^{p+1}$ norm of $w$:

**Lemma 3.3.** For all $s \geq 1 + \hat{s}_2$, for some $\hat{s}_2 \geq \hat{s}_1$, for all $\epsilon > 0$, for all $s \in [s_1, s_2]$,

$$
\int_{s_1}^{s_2} \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy \, ds \leq K_1 + C \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho \, dy \, ds, 
$$

(3.6)

$$
\sup_{s \in [s_1, s_2]} \int_B w^2 (y, s) \rho \, dy \leq \frac{K_1}{\epsilon} + K_1 \epsilon \int_{s_1}^{s_2} \int_B |w|^{p+1} \rho \, dy \, ds. 
$$

(3.7)
Using (3.1), we write

\[ \int_{\tau_1}^{\tau_2} \int_B |w| |\nabla w| \rho \, dy \, ds \leq K_1 \varepsilon + K_1 \varepsilon \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds, \]  

(3.8)

\[ \int_{\tau_1}^{\tau_2} \int_B |w| |w| \rho \, dy \, ds \leq K_1 \varepsilon + K_1 \varepsilon \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds, \]  

(3.9)

\[ \int_{\tau_1}^{\tau_2} \int_B |w| \rho \, dy \, ds \leq \int_{\tau_1}^{\tau_2} \int_B |w|^2 \rho \, dy \, ds + K_1 \varepsilon \int_{\tau_1}^{\tau_2} \int_B |w|^p \rho \, dy \, ds, \]  

(3.10)

\[ \int_B (w^2(y, s_1) + w^2(y, s_2)) \rho \, dy \leq K_1, \]  

(3.11)

\[ |A_1| \leq K_1 \varepsilon + (K_1 \varepsilon + Ce^{-\gamma s_1}) \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds, \]  

(3.12)

\[ |A_2| \leq K_1 \varepsilon + K_1 \varepsilon \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds, \]  

(3.13)

\[ |A_3| + |A_4| \leq C + Ce^{-\gamma s_1} \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds. \]  

(3.14)

Indeed, from (3.5), corollary 3.1 and this lemma, we deduce that

\[ \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds \leq K_1 \varepsilon + (K_1 \varepsilon + Ce^{-\gamma s_1} + Ce^{-\gamma s_1}) \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds. \]

Taking \( \tilde{s}_3 \) large enough and \( \varepsilon \) small enough so that \( Ce^{-\gamma \tilde{s}_3} + Ce^{-\gamma \tilde{s}_3} \leq \frac{1}{4} \) and \( K_1 \varepsilon \leq \frac{1}{4} \), we obtain (3.2).

It remains to prove lemma 3.3.

**Proof of lemma 3.3.** For estimates (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11), we can adapt with no difficulty the proof given in the case of the wave equation treated in [6].

Now, we control the terms \( A_1, A_2, A_3 \) and \( A_4 \). Since \( |g(x)| \leq M(1+|x|) \), we write

\[ |A_1| \leq C \int_{\tau_1}^{\tau_2} \int_B e^{-s} \int_B w^2 \rho \, dy \, ds + C \int_{\tau_1}^{\tau_2} \int_B |y \cdot \nabla w| |w| \rho \, dy \, ds \]  

\[ + C \int_{\tau_1}^{\tau_2} \int_B e^{-s} \int_B w^2 \rho \, dy \, ds + C \int_{\tau_1}^{\tau_2} \int_B e^{-\frac{3s}{2}} \int_B |w| \rho \, dy \, ds. \]  

(3.15)

Using (3.1), we write

\[ C \int_{\tau_1}^{\tau_2} \int_B e^{-s} \int_B w^2 \rho \, dy \, ds \leq K_4 \]  

(3.16)

Using the fact that \( e^{-\gamma s} \leq 1 \) and inequality (3.7), we obtain

\[ C \int_{\tau_1}^{\tau_2} \int_B e^{-s} \int_B w^2 \rho \, dy \, ds \leq C \sup_{t \in [\tau_1, \tau_2]} \int_B w^2 \rho \, dy \leq K_1 \varepsilon + K_1 \varepsilon \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds. \]  

(3.17)

We infer from (3.17) and the inequality \( |a| \leq 1 + a^2 \) that

\[ C \int_{\tau_1}^{\tau_2} \int_B |w| \rho \, dy \, ds \leq C \int_{\tau_1}^{\tau_2} w^2 \rho \, dy \, ds \leq K_1 \varepsilon + K_1 \varepsilon \int_{\tau_1}^{\tau_2} \int_B |w|^{p+1} \rho \, dy \, ds. \]  

(3.18)
Using the Cauchy–Schwarz inequality, we write
\[ C \int_{s_1}^{s_2} e^{-s} \int_B |y| \cdot \nabla w \rho \, dy \, ds \leq C \int_{s_1}^{s_2} e^{-s} \int_B |y| |\nabla w| \rho \, dy \, ds \]
\[ \leq C \int_{s_1}^{s_2} e^{-s} \int_B w^2 \frac{|y|^2}{1 - |y|^2} \rho \, dy \, ds + Ce^{-s_1} \int_{s_1}^{s_2} |\nabla w|^2 (1 - |y|^2) \rho \, dy \, ds. \]
(3.19)

By combining (3.17), (3.19), (2.18), (3.6) and (3.7), we obtain
\[ C \int_{s_1}^{s_2} e^{-s} \int_B |y| \cdot \nabla w \rho \, dy \, ds \]
\[ \leq C \int_{s_1}^{s_2} e^{-s} \int_B w^2 \rho \, dy \, ds + Ce^{-s_1} \int_{s_1}^{s_2} |\nabla w|^2 (1 - |y|^2) \rho \, dy \, ds \]
\[ \leq Ce^{-s_1} \int_{s_1}^{s_2} \int_B w^2 \rho \, dy \, ds + K_1 e^{-s_1} + Ce^{-s_1} \int_{s_1}^{s_2} \int_B |w|^p \rho \, dy \, ds \]
\[ \leq \frac{K_1}{\varepsilon} + (K_1 \varepsilon + Ce^{-s_1}) \int_{s_1}^{s_2} \int_B |w|^p \rho \, dy \, ds. \]
(3.20)

Using (3.15), (3.16), (3.17), (3.18) and (3.20), we obtain
\[ |A_1| \leq \frac{K_1}{\varepsilon} + (K_1 \varepsilon + Ce^{-s_1}) \int_{s_1}^{s_2} \int_B |w|^p \rho \, dy \, ds. \]
(3.21)

Similarly, we deduce by (3.1) and (3.7) that
\[ |A_2| \leq C \int_{s_1}^{s_2} \int_B w \nabla w_s \rho \, dy \, ds \leq \int_{s_1}^{s_2} \int_B w^2 \rho \, dy \, ds + C \int_{s_1}^{s_2} \int_B w^2 \rho \, dy \, ds \]
\[ \leq K_1 + C \sup_{s \in [s_1, s_2]} \int_B w^2 \rho \, dy \leq \frac{K_1}{\varepsilon} + K_1 \varepsilon \int_{s_1}^{s_2} \int_B |w|^p \rho \, dy \, ds. \]
(3.22)

Finally, by (2.4), we obtain
\[ |A_3| + |A_4| \leq C \int_{s_1}^{s_2} e^{-2p+\frac{2p+4\varepsilon}{p-1}} \, ds + C \int_{s_1}^{s_2} e^{-2p+\frac{2p+4\varepsilon}{p-1}} \int_B |w|^p \rho \, dy \, ds \]
\[ \leq C + Ce^{-\gamma s_1} \int_{s_1}^{s_2} \int_B |w|^p \rho \, dy \, ds. \]
(3.23)

This concludes the proof of lemma 3.3 and proposition 3.2 too.

Since the derivation of theorem 1.2 from proposition 3.2 is the same as in the non-perturbed case treated in [8] (apart from some very minor changes), this concludes the proof of theorem 1.2.

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