Erratum

Erratum to
“Integral closure of ideals in excellent local rings”
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We are grateful to Ray Heitmann for pointing out that Theorem 2.7 in the published version is wrong. The proof of the main theorem of the published paper used Theorem 2.7. Here we give new proofs of the main theorem as well as of some intermediate results. We point out that the main results still prove special cases of the Linear Artin Approximation theorem.

The main theorem. Let \((R, m)\) be an excellent local ring. Let \(I\) be an ideal of \(R\). Then there exists a positive integer \(c\) such that

\[ I + m^n \subseteq \overline{I} + m^{\lfloor n/c \rfloor} \]

for all \(n\).

As in [1, Section 2], the proof of this theorem reduces to the case where \((R, m)\) is a complete local normal integral domain and \(I\) is principal. However, contrary to the claims in [1], we may not assume that \(I\) is a radical ideal. In fact, Theorems 2.7 and 2.8 should be cut out of [1].

The following is a slight (and needed) generalization of [1, Theorem 3.9]. The proof here is essentially the same as the one in [1], only more direct.

Theorem 3.9. Let \((R, m)\) be a complete normal local domain and \(f R\) a non-zero principal ideal. In the case when \(R\) does not contain a field, we let \(p\) be a generator of the maximal
ideal in a coefficient ring for $R$, and we assume that $f$ satisfies one of the following properties:

(i) $f$, $p$ is a part of a system of parameters, or 
(ii) $f = ap^c$ for some positive integer $c$ and some element $a$ of $R$ not contained in any minimal prime ideal over $pR$.

Then there exist integers $d$ and $l$ such that for each $n$, every element in $fR + m^n$ satisfies an integral equation of degree $d$ over $fR + m^{[n/l]}$.

Proof. It is sufficient to prove that if $JR$ is $m$-primary, then there exists an integer $d$ such that for each $n$, every element in $fR + J^n$ satisfies an integral equation of degree $d$ over $fR + J^{[n/d]}$. (Note, however, that $d$ depends on $J$)

We use the Cohen structure theorem. Let $f_1, \ldots, f_l$ be a system of parameters in $R$. When $R$ contains a coefficient field $k$, we may assume that $f_1 = f$, and we define $A = k[f_1, \ldots, f_l]$. When $R$ contains a coefficient ring $(V, (p))$ of dimension 1, we may assume that $p$ is $f_1$. In case (i) we may also assume that $f = f_2$ and in case (ii) we may assume that $f_2$ is $a$ if $a$ is not a unit. In case (ii) if $a$ is a unit, as $fR = p^nR$, without loss of generality $a = 1$. We then define $A = V[f_2, \ldots, f_l]$. In either case, set $J = (f_1, \ldots, f_l)A$.

By the Cohen structure theorem, $A$ is a regular local ring contained in $R$, $R$ is module-finite over $A$, and $JR$ is $m$-primary. We will prove the theorem for this $JR$. Furthermore, we will prove that the integral equation of degree $d$ will have coefficients in $A$.

Let $K$ be the fraction field of $A$ and $L$ the fraction field of $R$. By elementary field theory, there exist fields $L'$ and $F$ such that all the inclusions $K \subseteq F \subseteq L'$ and $L \subseteq L'$ are finite, such that $L'$ is Galois over $F$ and such that $F$ is purely inseparable over $K$. To simplify notation, as the coefficients of the integral equation will actually lie in $A$, we may replace $R$ by the integral closure of $R$ in $L'$ and so we may assume that $L = L'$. Let $c = [L : F]$ and $e = [F : K]$. Let $S$ be the integral closure of $A$ in $F$. Then $S$ is a complete normal local domain between $A$ and $R$ and the extension from $S$ to $R$ is Galois.

Let $u \in fR + (JR)^c$. Consider the (at most) $c$ conjugates of $u$ over $S$, say $u = u_1, u_2, \ldots, u_c$. Write an integral equation for $u$ over $fR + (JR)^c$:

$$u^k + \alpha_1u^{k-1} + \alpha_2u^{k-2} + \cdots + \alpha_k = 0$$

with $\alpha_i \in (fR + (JR)^c)^i$. By applying field automorphisms to this equation and by using the fact that $(f)$ and $J$ are ideals of $A$ (and thus of $S$), we obtain that each $u_i$ is integral over $fR + J^cR$. Let $s_i$ be the sum of the products of the $u_i$, taken $h$ at a time ($h$th symmetric function in the $u_i$). Then

$$u^c - s_1u^{c-1} + \cdots + (-1)^cs_c = 0,$$

and $s_h \in (fR + (JR)^c)^h \cap S$. We raise all this to the $e$th power. As $e$ is either 1 or a power of the characteristic $p$ of the given fields, we obtain

$$u^{ce} - s_1u^{e(c-1)} + \cdots + (-1)^cs_c^e = 0$$

and
Let \( R, m \) be a Noetherian local integrally closed integral domain, and \( f \in R \) satisfying the following:

1. There exists a positive integer \( c \) such that for all \( n > 1 \), \((f) + m^n \subseteq (f) + m^{[n/c]}\).
2. For every \( k = 1, \ldots, N \) there exist positive integers \( d \) and \( l \) such that for all \( n \), every element of \((f^k) + m^n\) satisfies an equation of integral dependence of degree \( d \) over \((f) + m^{[n/l]}\).

Then for every \( k = 1, \ldots, N \), there exists a positive integer \( c \) such that \((f) + m^n \subseteq (f^k) + m^{[n/c]}\) for all \( n \).

**Proof.** We prove this by induction on \( k \). The case \( k = 1 \) is assumed. So assume \( k > 1 \). By induction, \((f^k) + m^n \subseteq (f^{k-1}) + m^{[n/c']}\) for some constant \( c' \) independent of \( n \). We pick an element \( u \) in \((f^k) + m^n\). Write \( u = rf^{k-1} + s \) for some \( r \in R \) and \( s \in m^{[n/c']}\). It suffices to prove that \( rf^{k-1} \) lies in \((f^k) + m^{[n/c]}\) for some \( c \) independent of \( n \) and \( u \). Note that \( rf^{k-1} \) is integral over \((f^k) + m^{[n/c']}\). Hence, it suffices to prove that \((f^{k-1}) \cap (f^k) + m^{[n/c]}\) is contained in \((f^k) + m^{[n/c']}\) for some \( c \) independent of \( n \), or even that \((f^{k-1}) \cap (f^k) + m^n\) is contained in \((f^k) + m^{[n/c]}\) for some \( c \) independent of \( n \). Thus, without loss of generality we may assume that \( u = rf^{k-1} \). Our goal is to prove that \( r \in (f) + m^{[n/c']}\) for some integer \( c' \) independent of \( n \) and \( r \), for then we know that \( r \in (f) + m^{[n/c'']}\) for some \( c'' \) independent of \( n \) and \( r \), which proves that \( u \) lies in the desired ideal.

**Claim.** \( r^d \in (f) + m^{[n/l]}\) for some constant \( e \) independent of \( n \).

**Proof of the claim.** By assumption there exists an integer \( d \) independent of \( n \) and \( r \) such that \( rf^{k-1} \) satisfies an integral equation of degree \( d \) over \((f^k) + m^{[n/l]}\), say: 
\[
(rf^{k-1})^d + \alpha_1(rf^{k-1})^{d-1} + \cdots + \alpha_d = 0,
\]
where \( \alpha_i \in ((f^k) + m^{[n/l]})^l \).
We will recursively define \( \beta_{d-i+1} \in ((f^k) + m^{[n/l]})^{d-1} \) for each \( i \in \{0, \ldots, d-1\} \) such that
\[
rf^d (f^{k-1})^{d-i} + \alpha_1 r^{d-1} (f^{k-1})^{d-i-1} + \cdots + \alpha_{d-i} r^i + \beta_{d-i+1} = 0. \tag{\#}
\]
If \( i = 0 \), set \( \beta_{d+1} = 0 \). Now assume we have defined \( \beta_{d-i+1} \) for some \( i < d-1 \). By the Artin–Rees lemma there exists a positive integer \( e \) such that \( m^n \cap (f^{k-1}) \subseteq f^{k-1} m^{n-e} \) for all \( n \geq e \). In the following we may and do assume that \( n/l \geq e \). With this we construct the next \( \beta \) using the equation displayed above and the following:
\[
\alpha_{d-i} r^i + \beta_{d-i+1} \in ((f^k) + m^{[n/l]})^{d-i} \cap ((f^k) + m^{[n/l]} m^{d-i}) = (f^k) + m^{[n/l]}((f^k) + m^{[n/l]}(d-i)) = f^k ((f^k) + m^{[n/l]}(d-i)) \leq f^{k-1} ((f^k) + m^{[n/l]}(d-i) - e) \leq f^{k-1}((f^k) + m^{[n/l]}(d-i) - e)
\]
as \( n/l \geq e \). Thus, we may write \( \alpha_{d-i} r^i + \beta_{d-i+1} = f^{k-1} \beta_{d-i+1} \) for some \( \beta_{d-i+1} \in ((f^k) + m^{[n/l]}(d-i-1)). \) To finish the induction step we only have to divide the displayed equation \( (\#) \) by the nonzerodivisor \( f^{k-1} \).

In the final step \( i = d - 1 \) we thus obtain \( rf^{k-1} + \alpha_1 r^{d-1} + \beta_2 = 0 \). Therefore,
\[
rf^{k-1} = -\alpha_1 r^{d-1} - \beta_2 \in (f^k) \cap ((f^k) + m^{[n/l]}) = (f^k) + f^{k-1} m^{[n/l]-e}.
\]
It follows that \( rf^{k-1} \subseteq (f) + m^{[n/l]-e} \). This completes the proof of the claim. \( \square \)

Now we are ready to prove that \( r \) is integral over \((f) + m^{[n/dlk(e+1)]}\). Recall that \( rf^{k-1} \subseteq (f^k) + m^n \). It suffices to prove that for any valuation \( v \) on the field of fractions of \( R \), \( v(r) \geq \min\{v(f), [n/dlk(e+1)]v(m)\}\).

Since \( r f^{k-1} \subseteq (f^k) + m^n \), \( v(r) + (k-1)v(f) = v(r f^{k-1}) \geq \min\{kv(f), nv(m)\} \), therefore \( v(r) \geq \min\{v(f), nv(m) - (k-1)v(f)\} \). If \( v(r) \geq v(f) \), there is nothing to show, so we may assume that
\[
[n/(e+1)]v(m) - (k-1)v(f) \leq nv(m) - (k-1)v(f) \leq v(r) < v(f).
\]
This implies that \( [n/(e+1)]v(m) \leq kv(f) \). Now we use our detour: as \( r^d \) lies in \((f) + m^{[n/l]-e} \subseteq (f) + m^{[n/l(e+1)]}\), then
\[
dv(r) \geq \min\{v(f), [n/l(e+1)]v(m)\} \geq \min\{v(f), [n/lk(e+1)]v(m)\}.
\]
If $dv(r) \geq [n/\ell k(e + 1)]v(m)$, we are done, so we may assume instead that

$$[n/\ell k(e + 1)]v(m) > dv(r) \geq v(f).$$

Thus,

$$[n/\ell k(e + 1)]v(m) > dv(r) \geq v(f) > \frac{1}{k}[n/(e + 1)]v(m),$$

which is a contradiction. This finishes the proposition. \(\square\)

The following is Theorem 3.10 of [1], presented here with a new proof.

**Theorem 3.10.** Let $(R, m)$ be a complete local normal domain and let $(f)$ be a principal radical ideal. In case $R$ does not contain a field, let $(V, (p))$ be a general coefficient ring of $R$ and we also assume that either $fR = pR$ or that $f, p$ is part of a system of parameters in $R$. Then for all $k$, $(f^k) + mn \subseteq (f^k) + m\lfloor n/k \rfloor$ for some constant $c$ independent of $n$.

**Proof.** The case $k = 1$ holds by [2, Theorem 1.4]. Thus, condition (1) of the previous proposition is satisfied. Condition (2) of the previous proposition is satisfied by Theorem 3.9, so that the corollary follows by Proposition A. \(\square\)

Before we prove the main theorem, we need one new lemma.

**Lemma B.** Let $R$ be an integral domain, $x$ and $y$ non-zero elements of $R$ and $d, l$ positive integers such that for every positive integer $n$, every element of $(xy)^n + m^{[n/l]}$ satisfies an integral equation of degree $d$ over $(xy)^n + m^{[n/l]}$. Then there exists a positive integer $k$ such that for every positive integer $n$, every element of $(x)^n + m^{[n/k]}$ satisfies an integral equation of degree $d$ over $(x)^n + m^{[n/k]}$.

**Proof.** Let $r \in (x)^n + m^n$. Then $ry \in (xy)^n + m^n$. Thus, there exist elements $r_i \in (xy)^n + m^{[n/l]}$ such that

$$(ry)^d + r_1(ry)^{d-1} + \cdots + r_{d-1}ry + r_d = 0.$$ 

Write $r_i = s_i(xy)^{d-1}$ for some $s_i \in R$ and some $t_i \in m^{[n/l]}$. Then

$$(ry)^d + s_1(xy)(ry)^{d-1} + \cdots + s_{d-1}(xy)^{d-1}ry + s_d(xy)^d$$

$$+ t_1(ry)^{d-1} + \cdots + t_{d-1}ry + t_d = 0.$$ 

Thus, $t_1(ry)^{d-1} + \cdots + t_{d-1}ry + t_d \in (y^d) \cap m^{[n/l]}$. By the Artin–Rees lemma there exists an integer $c$ such that $t_1(ry)^{d-1} + \cdots + t_{d-1}ry + t_d \in y^d m^{[n/c]}$. But then dividing the integral equation above by $y^d$ shows that $r$ satisfies an integral equation of degree $d$ over $(x)^n + m^{[n/cd]}$. \(\square\)
Finally, we can prove the general result for principal ideals in complete normal local domains. The theorem below is Theorem 3.12 of [1], presented here with a new proof.

**Theorem 3.12.** Let \((R, m)\) be a complete normal local domain. Let \(f\) be an element in \(R\). Then there exists a positive integer \(c\) such that

\[
\overline{(f)} + m^n \subseteq (f) + m^{[n/c]} \quad \text{for all } n.
\]

**Proof.** If \(f = 0\), the theorem is known by [2, Theorem 1.4]. So we may assume that \(f\) is not zero.

As \(R\) is normal, all the associated prime ideals of the ideal \((f)\) are minimal over \((f)\).

By [1, Corollary 3.4] it suffices to prove the theorem for the primary components of \((f)\) in place of \((f)\). Let \(P\) be an associated prime ideal of \((f)\). As \(R\) is normal, the localization \(R_P\) is a one-dimensional regular local ring, so \(f R_P = P^{(k)} R_P\) for some integer \(k\). Thus, the \(P\)-primary component of \(f R\) equals the \(k\)th symbolic power \(P^{(k)}\) of \(P\) and it suffices to prove the theorem for all \(P^{(k)}\) in place of \((f)\).

Let \(P = (a_1, \ldots, a_l)\). Let \(X_1, \ldots, X_l\) be indeterminates over \(R\) and let \(S\) be the faithfully flat extension \(R[X_1, \ldots, X_l]/mR[X_1, \ldots, X_l]\) of \(R\). Note that all the associated primes of \(x\) have height one and as \(S\) localized at height one prime ideals is a principal ideal domain, the ideal generated by \(x = a_1X_1 + \cdots + a_lX_l\) is radical.

Suppose that this \(x\) satisfies the conditions of Theorem 3.10. Namely, either \(R\) contains a field, or instead if \((V, (p))\) is a coefficient ring of \(R\), then either \(x = p\) or \(x = p\) is a part of a system of parameters. Then by Theorem 3.10, for every positive integer \(k\) there exists an integer \(c\) such that \(x^kS + m^n S \subseteq x^kS + m^{[n/c]} S\) for all \(n\). Note also that \(PS\) is associated to \(xS\) and that the \(PS\)-primary component of \(x^kS\) is \(P^{(k)}S\) (as \(SP_S\) is a principal ideal domain). Thus there exists an element \(y\) in \(S\) such that \(x^kS : y = P^{(k)}S\). As \(R\) is normal, then so is \(S\), so that \(x^kS = x^kS\). An application of [1, Lemma 3.11] shows that there exists an integer \(c'\) such that \(P^{(k)}S + m^n S \subseteq P^{(k)}S + m^{[n/c']} S\) for all \(n\). Finally,

\[
\frac{P^{(k)}S + m^n}{P^{(k)}S + m^{n/c'}} \subseteq \frac{P^{(k)}S + m^{[n/c']} S}{P^{(k)}S + m^{[n/c']} S} \cap R \subseteq \left( P^{(k)}S + m^{[n/c']} S \right) \cap R = P^{(k)} + m^{[n/c']}
\]

as \(S\) is faithfully flat over \(R\).

This finishes the theorem for rings containing fields.

Now assume that \(R\) contains a coefficient field \((V, (p))\). The above proves the theorem for all \(f\) which are not contained in any minimal prime ideal over \(pV\). Thus, by [1, Lemma 3.11], for all height one prime ideals \(P\) of \(R\) not containing \(p\) and all positive integers \(k\) there exists an integer \(c\) such that \(P^{(k)} + m^n \subseteq P^{(k)} + m^{[n/c]}\).

Next we shall prove the theorem in the special case \(f = p\). Let \(P_1, \ldots, P_N\) be all the prime ideals in \(R\) minimal over \(pR\). Let \(W = R \setminus (P_1 \cup \cdots \cup P_N)\). As \(R\) is normal, \(W^{-1}R\) is a one-dimensional semi-local regular ring, thus a principal ideal domain. Let \(x_i \in R\) be such that \(x_iW^{-1}R = P_iW^{-1}R\). Therefore, we may write \(p = u'x_1^{n_1} \cdots x_N^{n_N}\) for some unit \(u' \in W^{-1}R\). But then there exist \(u, v \in W\) such that in \(R\), \(up = vx_1^{n_1} \cdots x_N^{n_N}\).

Note that either \(u\) is a unit in \(R\) or else \(p, u\) is a part of a system of parameters. Thus, by Theorem 3.9, for each positive integer \(k\) there exist integers \(d\) and \(l\) such that every
element of $(up)^k + m^n$ satisfies an equation of integral dependence of degree $d$ over $(up)^k + m^{\lfloor n/l \rfloor}$. Thus, by Lemma B, for each positive integer $k$ there exist integers $d$ and $l$ such that for all $i = 1, \ldots, N$, every element of $(x_i)^k + m^n$ satisfies an equation of integral dependence of degree $d$ over $(x_i)^k + m^{\lfloor n/l \rfloor}$. This means that condition (2) of Proposition A is satisfied for each $x_i$. But $x_i R = P_i \cap Q_i$, where $Q_i$ is either the unit ideal or a height one ideal modulo which $p$ is a non-zerodivisor. As $P_i$ is a radical ideal (even prime), by [2, Theorem 1.4], there exists a positive integer $c$ such that for all $n \geq 1$, $P_i + m^n \subseteq P_i + m^{\lfloor n/c \rfloor}$. By what we have proved, there exists a positive integer $c'$ such that for all $n \geq 1$, $Q_i + m^n \subseteq Q_i + m^{\lfloor n/c' \rfloor}$. Thus, by [1, Lemma 3.3], the theorem holds for $x_i$.

In particular, condition (1) of Proposition A is satisfied for $x_i$. Thus, by Proposition A, the theorem holds for all $x_k^i$, as $k$ varies over all positive integers. Then by [1, Lemma 3.11], there exists an integer $c$ such that for all $i = 1, \ldots, N$,

$$P_i^{(k)} + m^n \subseteq P_i^{(k)} + m^{\lfloor n/c \rfloor}.$$ 

Thus, by [1, Lemma 3.3], the theorem holds for $f = p$.

Hence, condition (1) of Proposition A is satisfied for $p$, and condition (2) is satisfied by Theorem 3.9. Thus, by Proposition A, the theorem holds for each $f = p^k$.

It remains to examine the case when $f$ and $p$ do not form a system of parameters. In this case there exist an integer $e$ and elements $u \in W$ and $h \in R$ such that $fh = up^e$. We know the theorem for $u R$ and $p^k R$. Since $u R$ and $p^k R$ are part of a system of parameters, by [1, Lemma 3.3] we also know the theorem for $(u) \cap (p^e) = (up^e) = (fh)$. This means that there exists an integer $c$ such that $fh R + m^n \subseteq fh R + m^{\lfloor n/c \rfloor}$.

Now pick $u \in fh R + m^n$. Then $hu \in fh R + m^n \subseteq fh R + m^{\lfloor n/c \rfloor}$, so $hu \in fh R + m^{\lfloor n/c \rfloor} \cap fh R$. By the Artin–Rees lemma there exists an integer $k$ independent of $u$ and $n$ such that $m^{\lfloor n/c \rfloor} \cap fh R \subseteq hm^{\lfloor n/c \rfloor - k}$. Thus, $hu \in fh R + hm^{\lfloor n/c \rfloor - k}$, so $u \in fh R + m^{\lfloor n/c \rfloor - k}$. Thus, also in this last case, $fh R + m^n \subseteq fh R + m^{\lfloor n/c \rfloor - k}$.

This finishes the proof of the theorem. \(\square\)

References

[1] D. Delfino, I. Swanson, Integral closure of ideals in excellent local rings, J. Algebra 187 (1997) 422–445.
[2] D. Rees, A note on analytically unramified local rings, J. London Math. Soc. 36 (1961) 24–28.