Acceleration for infrared radiation

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The acceleration responsible for soft braking radiation is unknown. We interpret Faddeev-Kulish electrons in QED and construct a black hole remnant analog of the moving mirror model (dynamical Casimir effect) to determine the acceleration analytically. The equation of motion corresponds to deep infrared emission, experimentally observed via classical inner bremsstrahlung during beta decay. The radiation originates from a cloud of soft photons near the dressed electron. A novel prediction is the time evolution and temperature of the emission.

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I. INTRODUCTION

The discovery that black holes radiate light \cite{1}, unified aspects of gravitation, quantum mechanics, and thermodynamics. Subsequently, a single perfectly reflecting accelerating point in a flat-spacetime background, dubbed the ‘moving mirror model’ \cite{2–5}, was developed within the curved spacetime quantum fields program \cite{6, 7}. The primary motivation was to help in understanding features of Hawking radiation by providing a simple model of otherwise subtle behaviour in curved higher dimensions \cite{8}. The moving mirror analog side-steps much of the technical machinery associated with gravitation and provides an intuition on the origin of the radiation with respect to acceleration via the Equivalence Principle \cite{9–11}. Additional significant advantages have been seen: empirical tractability on an Earth-bound lab \cite{12–14}, and a system worthy of study in-and-of-itself \cite{15}, notwithstanding the faint emission \cite{16}.

Beta decay is also accompanied by a weak continuous braking radiation called internal bremsstrahlung (IB). The probabilistic effect is fundamentally quantum and all experimental findings are in agreement with the results of quantum theory whereas even the salient high-frequency cut-off of IB cannot be described classically. Nevertheless, classical and semiclassical calculations are presented in many textbooks (see e.g., \cite{17, 18}) and the limits of applicability are discussed in detail.

Interestingly, the origin of the acceleration radiation is a direct consequence of classical electrodynamics. But even though the dominant contribution, by far, for the energy of the continuous gamma radiation from beta decay is in the classical regime (see e.g. \cite{19}), one still needs to implement the full perturbative mathematical machinery of QED to obtain the very small radiative quantum corrections (see e.g. \cite{20} and references therein). Classically, of course, one cannot account for the nuclear creation of the electron or the source of its energy from radioactive decay. However, once given a continuous time-dependent description of the electron’s motion, one may ask: What is the time-dependent spectral distribution and power radiated by the electron? This is a straightforward problem for relativistic classical electrodynamics, and we sketch its solution.

![FIG. 1. Classical electrodynamics describes the origin story of acceleration radiation as emitted by the electron, known as inner bremsstrahlung (IB). In the original classical computation, the soft emission occurs after the electron is created and has escaped the nucleus. Soft IB is the dominant contribution to the total energy emitted by gamma radiation during beta decay.](image-url)

However, we are interested in the nature of the radiation as well as the dynamics responsible. Part of the unexpected success of the classical contribution, sits in the quantum field explanation which reveals a subtlety associated with time resolution: the electron exists in a virtual state before escape, then emits a photon and afterwards, is itself emitted from the nucleus \cite{21}, but as far as the dominant contribution to energy is concerned, the overall process of beta decay can actually be divided into two steps; (1) $n \rightarrow p + \bar{\nu} + e'$ and (2) $e' \rightarrow e + \gamma$, with the joint probability as the product of both \cite{22}. See Figure 1 for an illustration of the second step. This division is only valid if the two steps are independent of each

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other, and while there is no good reason for assuming this, the identical agreement between the classical and the quantum result justifies 73 years FAPP use of the simpler method [23] (the results are even in agreement for forbidden transitions). A main advantage of the simple approach, utilized well by experimental physicists, has been to avoid the difficulty of dealing with the multitude of beta decay interactions. For observed differences of various kinds between theoretical and experimental quantities of IB accompanying beta decay, it has been important to confirm this fundamental concept [24].

How exactly does a classical trajectory find utility for the quantum excitation of a field mode? An Unruh-DeWitt detector (see e.g. [25]) with center of mass moving along a classical trajectory perceives a thermalized quantum state, radiating particles from the accelerated composite body with quantum degrees of freedom; but the electron is not generally considered composite in this context. Refreshingly, in addendum to the classically understood origins of the radiation, the lowest order IB emission process releases a soft-photon which can be described completely semi-classically [26]. If the frequency of the soft photon is small enough, then it is fine to neglect the emission influence of the quantum on the scattering source. This traces back to the fact that the differential cross section for IB of soft photons is proportional to the differential cross section of the accelerating electrons at the same energy and angle (see also the connection between the Ward identity and the soft photon theorem [27]). This means, the amount of energy and momentum carried off by the photon is so small that the trajectory of the electron remains effectively undisturbed [28]. Thus, quantum field theory justifies the semi-classical regime of the moving mirror (quantum fields + classical trajectory) as suitable for describing soft IB.

The generality of the soft factor is sweeping. In the low-energy photon limit, the amplitude for any scattering process leading to radiation emission is factorized in the same way, regardless of spin or internal structure of the charged particle [26]. The soft-photon theorem tells us that the soft IB expression is universal depending only on charge, mass and magnetic moment of the particle. Our results will corroborate universality, and furthermore demonstrate that accelerating boundaries are subject$^1$ (i.e. the mirror has no mass, charge, magnetic moment, spin or internal structure, yet emits soft scalars). This is also in harmony with the similar generality of low-energy theorems which hold for emission of massive bosons, including pion emission in nuclear collisions, e.g. [29].

Motivation along this line of investigation is found in large part from an unexpected synthesis of systems$^2$ as we shall see; in particular, black hole remnants [32], the dynamical Casimir effect [33] and dressed electrons [34]. When one considers scattering of Faddeev-Kulish electrons in QED, it turns out that there exists a cloud of soft photons in the state of the dressed electron, that also have the same energy emitted in classical IB (see the Appendix for explicit calculation). The fact that rather subtle calculations in different contexts lead to the same energy, suggest the infrared result is even more applicable than currently understood. An acceleration responsible for the infrared radiation could help provide a simple underlying physical as well as formal connection to gravitation via the Equivalence Principle.

The outline sketch is: problem (Sec. II), method (Sec. III), solution (Sec. IV). In Sec. II, we set up the problem by briefly reviewing a previously studied model of the radiation spectrum for classical beta decay. Interestingly, accurate results imply instant creation of an electron at $t = 0$ with constant speed $s$. This classical procedure finds an experimentally verified time-independent angular distribution of energy and total energy by utilizing infinite acceleration in an infinitesimal time. In Sec. III we describe analog aspects of the semi-classical mirror model with the classical electrodynamics of a moving point charge. We motivate its use as an analog method for investigating acceleration radiation of such a violently accelerated electron after its nuclear creation during beta decay. In Sec. IV we provide the solution to the acceleration, obtaining the time-dependent power, time-dependent angular distribution, and particle spectrum consistent with the total energy, including the temperature. In Sec. V we discuss and conclude. $\hbar = c = \mu_0 = k_B = 1$.

II. ACCELERATION PROBLEM

For IB in beta decay, classical electrodynamics bears witness to the origin of the acceleration radiation [23]. The energy of the classical radiation is very small relative to the total energy released but nevertheless has been observed to great accuracy [35], providing important insights [20] [22]. The electron is assumed to be initially at rest at $t = 0$ and imagined to be violently accelerated to

$^1$ Roughly speaking, the long wavelengths of soft particles lack the capability to probe the internal structure of the source so interestingly, and subsequently, the source can be a moving mirror.

$^2$ Direct analog soft electron-hole pair production in graphene [30] with respect to the Sudakov double-log cross section result in photoexcitation cascades and quantum-relativistic jets [31].
a final constant speed, \( v(t) = \begin{cases} s, & t > 0. \\ 0, & t < 0. \end{cases} \) (1)

Using this trajectory, as a mathematical underpinning, the angular differential distribution of radiated energy is found to be (see the Appendix):

\[
\frac{d^2E}{d\omega d\Omega} = \frac{q^2}{16\pi^3} \left( \frac{s \sin \theta}{1 - s \cos \theta} \right)^2,
\]

where \( \theta \) is the angle between the final velocity \( \vec{\beta} \) and the observation point of the radiation. Integration of Eq. (2) over solid angle \( d\Omega = \sin \theta d\theta d\phi \) and over frequencies IR/UV-limited by cutoffs \( \Delta \omega = \omega_{\text{max}} - \omega_{\text{min}} \) gives the energy radiated by the electron. The total energy is rendered finite in this interval,

\[
E = \frac{q^2}{4\pi^2} \left[ \frac{1}{s} \ln \left( \frac{1 + s}{1 - s} \right) - 2 \right] \Delta \omega.
\]

With final rapidity defined by \( \eta \equiv \tanh^{-1} s \), then \( E = q^2 \Delta \omega (\eta/s - 1)/(2\pi^2) \). The detector sets the energy scale sensitivity. Eq. (3) is soft IB energy.

This 1949 computation by Chang and Falkoff [23] is treated in Jackson [17] and Zangwell [18]. Also called the instantaneous collision formalism, Eq. (3) has been used to help compute the gravitational energy released during the quantum creation of pairs of black holes [36] and the electromagnetic radiation released during the high energy collision of a rapidly accelerated charged point particle with a black hole [37].

However useful, both theoretically and experimentally, it is physically desirable to avoid infinite acceleration. The mathematical use of the discrete step velocity of Eq. (1), limits the final results to quantities independent of time. In the above model, the electron is assumed to accelerate during an unsatisfying infinitesimally short time interval. This moment occurs at \( t = 0 \), which is the creation instant of the electron. Since it takes no time at all to form, the model is physically incomplete (being deficient in time), however suitable for the purposes of computing the total observed energy.

In the next section, Sec. III, we will introduce an analog model that reproduces and confirms this energy without appeal to infinite acceleration. This allows for the computation of new important (in principle, testable) time-dependent quantities, like the power or the distribution, as will be shown in Sec. IV.

### III. ANALOG METHOD

Here we motivate an analog method by appeal to Lorentz invariance as a underlying postulate of symmetry. Both with respect to quantum and classical phenomena, the invariance manifests itself in analogies relating mirrors, electrons and black holes [38]. For the moving mirror the radiation obeys the quantum power formula [39],

\[
P_{\text{mirror}} = \frac{\hbar \alpha^2}{6\pi c^2} \rightarrow \frac{\alpha^2}{6\pi}.
\]

where \( \alpha \) is the frame-invariant proper acceleration. This measure of power is a Lorentz invariant, independent of dimension (like the speed of light). Compare this to the classical Larmor counterpart with the same relativistic covariant scaling, well-studied in electrodynamics [17],

\[
P_{\text{electron}} = \frac{2}{3} \frac{q^2 \alpha^2}{4\pi \epsilon_0 \epsilon_0 c^3} \rightarrow \frac{q^2 \alpha^2}{6\pi},
\]

where the left side is in SI units and “\( \rightarrow \)” implies conversion to natural units where \( \epsilon_0 = c = \hbar = 1 \), and \( \alpha \) is the magnitude of the proper acceleration of the moving point charge.

The Lorentz invariance cross-over is present in the quantum Lorentz-Abraham-Dirac (LAD) force discovered by Ford-Vilenkin [9] (for a derivation see [40]) whose magnitude is a Lorentz scalar invariant jerk,

\[
F_{\text{mirror}} = \frac{\hbar \alpha'}{6\pi c^2} \rightarrow \frac{\alpha'}{6\pi},
\]

where the prime denotes a derivative with respect to proper time. Recall the classical LAD force with scalar magnitude,

\[
F_{\text{electron}} = \frac{q^2 \alpha'}{6\pi \epsilon_0 c^3} \rightarrow \frac{q^2 \alpha'}{6\pi}.
\]

These are quantum and classical forces (and power) tying together mirror and electrons in close analogy. The moving mirror system, as a general purpose analog [41], thus serves to provide a new way of looking at the electron acceleration radiation problem, permitting novel application of the ideas in the quantum system to the corresponding classical system (see e.g. [42–44]).

The analogy continues. Asymptotic inertia, like that owned by the electron trajectory in Eq. (1), also plays a central role for the mirror. For instance, to obtain solutions to the information loss problem of the moving mirror model (e.g. [45, 46]), one requires asymptotic inertia. In general, trajectories that stop accelerating emit a total finite energy. A carefully chosen asymptotic inertial trajectory [47] could be extremely useful in focusing attention on the specific time-dependent quantities, and would be an unexpected route to a possible solution for the electron’s violent acceleration.

There are several known asymptotically inertial moving mirror trajectories with energy solutions that are black hole remnant analogs [47–52]. These time-dependent motions match the asymptotic behavior of Eq. (1). For instance, consider the CGHS moving mirror model [40] with an asymptotic inertial modification [53]. This remnant version of the CGHS black hole (‘drifting-arcx’) starts from rest and ends at constant velocity,
emitting a total finite energy $E = \kappa(\gamma v)^2/24\pi$, where $\gamma v$ is the final constant celerity (proper velocity). This does not scale as Eq. (3) and will not be considered further.

There is yet no observational evidence for black hole remnants. An experimental analog might be valuable for understanding the radiation and end-state, as well as create opportunities for practical experiments. The method proposed here leverages the well-understood process of classical IB from beta decay as an analog for potential better understanding of black hole remnants.

Vice-versa, black hole evaporation analogs could model classical radiation from specific elementary radioactive decays. In the next section, we use a particular black hole remnant analog via a moving mirror to model the braking radiation emitted during beta decay. We will find it more than a helpful heuristic and propose solutions for the unknown time-dependent quantities of power and angular distribution associated with soft IB.

### IV. DYNAMIC SOLUTION

The study of soft theorems started in 1937 via the seminal work of Block and Nordsieck [54]. They found the universality and classical limit characterize the average radiated energy. In particular, they pointed out that it is equal to the energy radiated classically from the corresponding trajectory. In this section, we present and motivate this time-dependent equation of motion.

Recall two difficulties mentioned in Sec. II. Infinite acceleration is non-physical and time is absent in the emission of radiation. It is desirable to study an exactly solvable model that remedies these concerns.

We have found such a model. Using the analog method proposed in Sec. III, consider the trajectory [47],

$$\frac{dt}{dr} = \frac{1}{\kappa r} + \frac{1}{s}. \quad (8)$$

The asymptotic speeds are $v = (0, s)$ as $r \rightarrow (0, \infty)$ (the electron moves to the right by convention$^3$), matching Eq. (1). The proper acceleration, $\alpha = d\gamma/dr$, has time-dependence, $\alpha(t) = \kappa\beta\gamma^3(1 - \beta/s)^2$, and possesses asymptotic inertia. See Figure 2 for illustration.

Here $\kappa$ is the dimensionful acceleration parameter of the model which we find corresponds to the sensitivity in frequency range and charge of the electron. With large $\kappa$, the speed of Eq. (8) approaches the unit step function, Eq. (1). However, as we shall see, no such approximation is needed to obtain Eq. (2) or Eq. (3).

The time-dependent power distribution is computed using Eq. (8) with straightforward vector algebra (see the procedure in [55, 56]),

$$\frac{dP}{d\Omega} = \frac{\kappa^2\beta^2(\beta - s)^4\sin^2\theta}{16\pi^2 s^4(1 - \beta\cos\theta)^5}. \quad (9)$$

where $s$ again, is the final constant speed, and $\beta = \beta(t)$ is the time-dependent velocity of the mirror (or electron). Integration over time, gives the scaled time-independent angular differential distribution of energy, Eq. (2). This confirms that the proposed trajectory, Eq. (8), yields the identical distribution in space of radiated energy, supporting the notion that Eq. (8) is a good candidate for the motion of the electron.

Moreover, using the Lorentz-invariant proper acceleration in $P = \alpha^2/6\pi$, we can obtain the quantum power,

$$P = \frac{\kappa^2\gamma^6\beta^2}{6\pi} \left(1 - \frac{\beta}{s}\right)^4. \quad (10)$$

Integration of Eq. (10), over space with the appropriate Jacobian, $dt/dr$ of Eq. (8), and correct bounds, $(r; 0, \infty)$, gives the total energy, Eq. (3), scaled by $\kappa \leftrightarrow 12\gamma^2\Delta_{\omega}/\pi$. This is additional confirmation that the violent acceleration of the electron is appropriately modeled by trajectory Eq. (8).

Interestingly, a period of constant emission is present in the power measured by a far away observer. Best represented as the change of energy with respect to retarded time $u = t - r$, and written as $\bar{P} = \frac{d\bar{E}}{du}$, such that

$$E = \int_{-\infty}^{\infty} \bar{P}(u)\,du, \quad (11)$$

we write $\bar{P} = P\frac{dt}{du} = P/(1 - v)$. Formulating $\bar{P}(u)$ in terms of retarded time, gives a lengthy result, but we plot the measure $\bar{P}(u)$ at high final asymptotic speeds.

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$^3$ In the moving mirror model of [47] and others, the usual convention is to move to the left (see Figure 5). The difference is a sign change in the angular distribution. The energy remains invariant.
s \sim 1 \text{ and reveal a constant energy flux plateau indicative of thermal emission. Additionally, beta Bogolubov coefficients will corroborate this power plateau via an explicit Planck distribution in Eq. (14). See a plot of the power plateau in Figure 3.}

![Plot of the power, $P(u)$, with a notable plateau demonstrating constant emission when the final speed of the electron is ultra-relativistic, $s = 1 - 10^{-11} = 0.9999999999$. The acceleration parameter is $\kappa = 1$. This plateau corroborates the conclusion that at high speeds the photons find themselves in a Planck distribution, Eq. (14) with temperature $T = \kappa/2\pi$, Eq. (16). The constant power plateau here has been normalized by $\kappa^2/48\pi$ so that the plateau is at height $P(u) = 1$. The integral under the curve here, Eq. (11) is the experimentally observed soft IB energy, Eq. (3).}

Having computed the power, $P = \alpha^2/6\pi$, from Eq. (4), we now turn to the self-force, $F = \alpha'(\tau)/6\pi$, from Eq. (7). The jerk is analytically tractable, and a concise expression is given in terms of speed $\beta$,

$$F = \frac{\kappa^2 s^6 \beta^2}{6\pi} \left( 1 - \frac{\beta}{s} \right)^3 \left( 2\beta + 1 - \frac{3}{s} \right). \quad (12)$$

The self-force is zero at maximum power. Integrating over distance gives the work done,

$$W = \int_0^\infty F(\tau) \, d\tau = -\int_{-\infty}^\infty P(t) \, dt = -E. \quad (13)$$

That is, taking Eq. (12) over $d\beta$ using, $d\beta/d\tau = \kappa(1 - \beta/s)^2$, where $\beta$ ranges from $(0, s)$, one obtains the energy associated with the self-force. The resulting work is $W = -E$, the equal and opposite of Eq. (3). This demonstrates consistency between the radiation reaction and conservation of energy.

Now we will compute the mirror analog spectrum and demonstrate consistency with the total energy emitted. We combine the results for each side of the mirror [47] using a valid technique based on asymptotic inertia [39]. The overall spectrum, using the Bogolubov coefficients, is found to be,

$$|\beta_{\omega'}|^2 = \frac{2\omega\omega' \left( \omega^2 + \omega'^2 \right)}{\pi \kappa \omega_T \omega_s^2 \left( e^{\frac{\pi}{\kappa} \omega} - 1 \right)}. \quad (14)$$

Here $\omega_s = (\frac{1}{2} - 1) \omega + (\frac{1}{2} + 1) \omega'$, and $\omega_s = (-\frac{1}{2} - 1) \omega' + (1 - \frac{1}{2}) \omega$. The total frequency is $\omega_T = \omega + \omega'$. A numerical integration of

$$E = \int_0^\infty \int_0^\infty \omega |\beta_{\omega'}|^2 \, d\omega \, d\omega', \quad (15)$$

confirms the total energy radiated, Eq. (3). Thus, the new expression, Eq. (14), plotted in Figure 4, for the particle spectrum of the radiation, may help characterize the color of the light from the accelerated electron. The explicit Planck factor demonstrates the particles, $N(\omega) = \int d\omega' |\beta_{\omega'}|^2$, are distributed with a temperature,

$$T = \frac{\kappa}{2\pi} = \frac{6q^2}{\pi^2} \Delta \omega, \quad (16)$$

in the high frequency approximation $\omega' \gg \omega$ [1]. See further explanation about the high frequency approximation regime in [57]. Again, $\Delta \omega \equiv \omega_{\max} - \omega_{\min}$, which is the scale set by the sensitivity of detection. In light of these results, soft IB from beta decay is an observational analog example of the dynamical Casimir effect (DCE).

![Plot of $|\beta_{\omega'}|^2$ spectrum of Eq. (14). Here $\omega' = \kappa = 1$ and $s = 1/2$. The vertical axis has been scaled by $10^5$ for visual clarity. The spectrum does not suffer from an infrared divergence as is typical for black hole remnants models. The qualitative black-body shape is indicative of the explicit Planck factor in Eq. (14).]

V. CONCLUSION

We have used the semi-classical moving mirror model (quantum fields under the influence of a classical trajectory) to find the trajectory responsible for soft (deep infrared) braking radiation, which corresponds to lowest order inner bremsstrahlung emitted during beta decay. We have derived novel time-dependent power and time-dependent angular distribution formula for the acceleration radiation. The soft self-force was computed...
and the Bogolubov coefficients and spectrum were obtained, demonstrating consistency with the observed total energy. The temperature of the gamma emission is found from the Planck distribution. The key result is an analytic and continuous equation of motion for infrared acceleration radiation.

A few remarks are in order regarding the impact of this work with respect to past and future experiments. There is no experimental evidence\(^4\) for the classical radiation reaction of an individual electric charge \(^{[58]}\). Our results indicate that soft IB is experimental evidence of the self-force. Moreover, soft IB is an observational analogue example of the DCE. These results highlight soft IB as potentially more central to acceleration radiation studies and physically more interesting than previously recognized. If future experiments confirm the proposed time-dependent angular distribution, power or spectrum, then a few claims are feasible:

- Quantum radiation from moving mirrors have predictive power for classical radiation from electrons\(^5\).
- Classical radiation from electrons reveal insights into certain models of decaying black holes\(^6\).
- Discontinuous (‘wild’) velocity parallels a continuous (‘mild’) analog\(^7\).

If the time-dependence presented here is experimentally dis-confirmed, then this particular system with identical energy distribution might be regarded as ‘degenerate’. That is, some other trajectory\(^8\)(or none at all\(^9\)) which also gives the same energy distribution, could be capable of providing the correct time dependent quantities. This is a general problem closely related to how classical radiation from an accelerated charge emerges from the thermal vacuum in quantum field theory \(^{[60]}\).

A future extension might exploit crossing symmetry: treating the \(S\)-matrix as a function of energy variables relating the different analytic regions non-trivially (rotating the Feynman diagrams around). Under this symmetry, the results found for IB are relevant to pair production, \(\gamma + \gamma \to e^- + e^+\).

Relentless interest in new and established gravitational analogue models like the moving mirror continue to deliver insights into the nature of acceleration radiation. We hope that there are further central but surprising physical issues whose resolution will be significantly aided by the analysis of the connection between dynamic acceleration and the emission of light.

**APPENDIX**

A. Derivation 1: Zero Frequency Limit

To compute the angular distribution of Eq. (2), where \(d^2E/d\omega d\Omega = dI(\omega)/d\Omega\), the fastest route is to assume infrared light only; see Jackson \(^{[17]}\). We start with the general angular distribution formula (see, e.g. Zangwill \(^{[18]}\)):

\[
\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \int_{-\infty}^{\infty} dt \left| \frac{\hat{n} \times (\hat{n} - \hat{\beta}) \times \hat{\beta}}{(1 - \beta \cos \theta)^2} e^{i \phi} \right|^2 ,
\]  

and use the differential identity:

\[
\frac{\hat{n} \times (\hat{n} - \hat{\beta}) \times \hat{\beta}}{(1 - \beta \cos \theta)^2} = \frac{d}{dt} \left[ \hat{n} \times (\hat{n} \times \hat{\beta}) \right],
\]

\(\hat{n}\) is the unit vector parallel to the velocity of the electron.

\(^4\) However, radiation reaction for oscillating currents is manifest in the radiation resistance of antennas.

\(^5\) The shape, color, and time-dependence of the light from electrons can be deduced from mirror radiation.

\(^6\) That is, the total energy emitted by a black hole remnant system is given, in scale, by total energy emitted by an electron produced during beta decay. At least in this one case, the electron acts as an analog for the remnant.

\(^7\) No approximation on the continuous Eq. (8) is needed to obtain Eq. (2), originally obtained from discontinuous Eq. (1).

\(^8\) A Möbius transform of Eq. (8), while preserving the total energy emitted \(^{[59]}\), does not give a novel trajectory with the same asymptotic inertia requirement of Eq. (1).

\(^9\) Perhaps a fully realized quantum field method, utilizing no classical trajectory, could give consistent time-dependent results.
where the derivatives are evaluated at retarded time. The distribution is then expressed as

$$\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \left[ \int_{-\infty}^{\infty} dt_r \frac{d}{dt_r} \left( \hat{n} \times (\hat{n} \times \beta) \right) e^{i\phi} \right]^2, \quad (19)$$

For low frequencies, $e^{i\phi} \rightarrow 1$, where $\phi = \omega t_r - k \cdot r_0(t_r) \rightarrow 0$ for $\omega \rightarrow 0$. The integral is now a perfect differential, and one has (assuming the charge is initially at rest):

$$\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \left[ \hat{n} \times (\hat{n} \times \beta) \right]^2. \quad (20)$$

The numerator simplifies because $|\hat{r} \times (\hat{r} \times \beta)|^2 = |\hat{r} \times \beta|^2$ for any vector $\beta$. At late times, when $\beta(t) = \beta_t$ and the final magnitude is speed, $s = |\beta_t|$, the energy distribution per frequency per solid angle is exactly Eq. (2):

$$\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \left( \frac{s \sin \theta}{1 - s \cos \theta} \right)^2. \quad (21)$$

This low-frequency result holds both quantum mechanically and classically in accordance with the soft-photon theorem for infrared scattering.

### B. Derivation 2: Zero Time Limit

The computation of the angular distribution in Eq. (2), can be done without direct appeal to low-frequency limit. It involves strict reliance on the step function trajectory Eq. (1). That is, with infinite acceleration in zero time, the problem is tractable, albeit, with a nearly unwieldy integral.

To compute the angular distribution, $d^2E/d\omega d\Omega = dI(\omega)/d\Omega$, without the low-frequency approximation, we instead perform an integration by parts inside the absolute signs of Eq. (17) after using Eq. (18), which gives

$$\frac{\hat{n} \times (\hat{n} \times \beta)}{1 - \beta \cos \theta} e^{i\phi} + \int_{-\infty}^{\infty} dt_r \hat{n} \times (\hat{n} \times \beta) e^{i\phi}, \quad (22)$$

where the boundary terms vanish. Here $\phi = \omega t_r - k \cdot r_0(t_r)$, where $k = \omega$. Using $\hat{n} \approx \hat{r}$ as a constant vector means it comes outside the integral. Since $|\hat{r} \times (\hat{r} \times \beta)|^2 = |\hat{r} \times \beta|^2$ for any vector $\beta$, we obtain the angular spectrum of radiated energy as:

$$\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \left| \hat{r} \times \beta(t) \exp[-i(k \cdot r_0(t) - \omega t)] \right|^2. \quad (23)$$

Here is where the approximation of infinite acceleration is made. As can be seen, the integral was already considered zero inside $(-\infty, 0)$. Now we focus on the non-zero contribution inside $(0, +\infty)$. The electron is moving after $t > 0$, with $\beta(t) = \beta_t$. The trajectory function is $r_0(t) = \beta tt$. Using $k = \omega \hat{r}$,

$$\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \left| \hat{r} \times \beta_t \right|^2 \left[ \int_{0}^{\infty} dt \exp[-i\omega(\hat{r} \cdot \beta_t - 1)t] \right]^2. \quad (24)$$

The pre-factor $\omega^2$ frequency dependence will ultimately cancel out after integration. The integral diverges at late times (upper limit), so a convergence regulator $e^{-\epsilon t}$ is applied and $\epsilon \rightarrow 0$ after integration. Using $\hat{r} \cdot \beta_t = s \cos \theta$, where $\theta$ is the angle between $\beta_t$ and the observation point, the integral is:

$$\int_{0}^{\infty} dt \exp[i\omega(1 - s \cos \theta)t - \epsilon t] = \frac{i}{i\epsilon + \omega(1 - s \cos \theta)^2}, \quad (25)$$

where now $\epsilon \rightarrow 0$. The square of the integral is:

$$\left| \int_{0}^{\infty} dt \exp[-i\omega(\hat{r} \cdot \beta_t - 1)t] \right|^2 = \frac{1}{\omega^2(1 - s \cos \theta)^2}, \quad (26)$$

demonstrating that frequency dependence cancels out exactly from Eq. (24). Using $\hat{r} \times \beta_t = s \sin \theta$, the result is Eq. (2):

$$\frac{dI(\omega)}{d\Omega} = \frac{q^2}{16\pi^3} \left( \frac{s \sin \theta}{1 - s \cos \theta} \right)^2. \quad (27)$$

The angular distribution of energy radiated per unit frequency (and the total energy radiated per unit frequency) are independent of frequency.

There was no zero frequency limit taken in this computation. The basis of this computation is the step-function velocity, Eq. (1), given to the electron at the moment of its creation.

### C. Derivation 3: Dressing Formulism

The action, $S = \epsilon R^f$, for Faddeev-Kulish dressing is given in Tomaras-Toumbas [34], where the soft dressing function depends on the photon momentum $\bar{q}$ and the electron momentum $\bar{p}$,

$$f^\mu(\bar{q}, \bar{p}) = e \left( \frac{p^\mu - \epsilon^\mu}{pq} \right). \quad (28)$$

The phase has been set to unity to ensure only soft photons are radiated. Inclusion of a UV and IR cut-offs on the integration, allows us to write the coherent electron particle state as

$$|\bar{p}\rangle = N_{\bar{p}} \exp \left[ \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} f^\mu(\bar{k}, \bar{p}) a^\dagger(\bar{k}) \right] |0\rangle. \quad (29)$$
Here the normalization is given by, \(N_p \equiv e^{-\frac{\theta}{\kappa}}\), where
\[
N \equiv \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega q} f^\mu(q, \hat{p}) f^*_\mu(q, \hat{p}).
\] (30)

This integrand is detailed as follows:
\[
dN = \frac{d^3q}{(2\pi)^3} \frac{e^2}{2\omega q} \frac{p^\mu}{pq} (p\frac{\mu}{pq} - c\nu) (p\frac{\mu}{pq} - c\nu).
\] (31)

Since \(c^2 = 0\) as a null vector, \(2q^0c^\mu = (-1, \hat{q})\), one has, dropping the subscript on \(\omega\),
\[
\frac{d^2N}{d\omega d\Omega} = \frac{\omega^2}{(2\pi)^3} \frac{e^2}{2\omega} \left( \frac{p^2}{(pq)^2} - 2 \frac{cp}{pq} \right)
\] (32)

Using \(2q^0cp = 0 + \hat{p} \cdot \hat{q}\), and \(p^2 = -(p^0)^2 + (\hat{p})^2\), and \(pq = q^0(p^0 - \hat{p} \cdot \hat{q})\), for the distribution, we then have
\[
\frac{\omega^2 e^2}{(2\pi)^3} \frac{1}{2\omega} \left( \frac{-(p^0)^2 + (\hat{p})^2}{(p^0 - \hat{p} \cdot \hat{q})^2} + \frac{p^0 + \hat{p} \cdot \hat{q}}{2\omega (p^0 - \hat{p} \cdot \hat{q})} \right)
\] (33)

Cleaning up the \(\omega\)'s and factors of 2,
\[
\frac{d^2N}{d\omega d\Omega} = \frac{1}{16\pi^3} \left( \frac{1}{(\omega^0 - \hat{p} \cdot \hat{q})^2} \right),
\] (34)

allows us to grab the common denominator,
\[
\frac{d^2N}{d\omega d\Omega} = \frac{1}{16\pi^3} \left( \frac{1}{(\omega^0 - \hat{p} \cdot \hat{q})^2} \right),
\] (35)

and with the fact that the speed of the electron is \(s = |\hat{p}|/p^0\), we have
\[
\frac{d^2N}{d\omega d\Omega} = \frac{1}{16\pi^3} \left( \frac{1}{(1 - s \cos \theta)^2} \right).
\] (36)

With full spherical coordinates,
\[
\frac{d^2N}{d\omega d\Omega} = \frac{1}{\omega 16\pi^3} \left( \frac{1}{(1 - s \cos \theta)^2} \right),
\] (37)

The integral can readily be done with the aforementioned cut-offs,
\[
N = \frac{e^2}{2\pi^2} \left[ \frac{\eta}{s} - 1 \right] \ln \left( \frac{\omega_{\max}}{\omega_{\min}} \right).
\] (38)

It is straightforward to see that the number of photons in this state,
\[
\langle f^\mu | N_{ph} | f \rangle = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega q} f^\mu(q, \hat{p}) f^*_\mu(q, \hat{p}) = N.
\] (39)

The energy of the state is also straightforward,
\[
\langle f^\mu | H_{ph} | f \rangle = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega q} f^\mu(q, \hat{p}) f^*_\mu(q, \hat{p}) = \frac{e^2}{2\pi^2} \left[ \frac{\eta}{s} - 1 \right] \Delta_{\omega},
\] (40)

where \(\Delta_{\omega} = \omega_{\max} - \omega_{\min}\). This confirms the classically obtained energy of soft IB, Eq. (3), and the mirror-electron trajectory Eq. (8) leading to the same, Eq. (54).

### D. Light-cone Trajectories

It is tractable to write the spacetime and timespace trajectories of Eq. (8),
\[
r(t) = \frac{s}{\kappa} W(e^{\kappa t}), \quad t(r) = \frac{1}{\kappa} \ln \left( \frac{\kappa r}{s} \right) + \frac{r}{s},
\] (42)

using null coordinates, \(v = t + r\) and \(u = t - r\). The advanced time trajectory is,
\[
p(u) = u + \frac{2s}{\kappa} W \left[ (1-s)e^{\kappa u} \right],
\] (43)

and the retarded time trajectory is,
\[
f(v) = v - \frac{2s}{\kappa} W \left[ (1+s)e^{\kappa v} \right].
\] (44)

These expressions find utility in application to Möbius transforms and mapping to curved spacetime.

### E. Non-relativistic derivation of cloud energy

The worldline of the electron is detector-dependent. This range-dependent quantum trajectory class of equations,
\[
r(t) = \frac{\hbar c s^2 \pi s}{12q^2 \Delta_{\omega}} W \left( e^{\frac{12q^2 \Delta_{\omega}}{s^2}} \right),
\] (45)

In natural units, \(\hbar = \epsilon_0 = c = 1\),
\[
r(t) = \frac{\pi s}{12q^2 \Delta_{\omega}} W \left( e^{\frac{12q^2 \Delta_{\omega}}{s^2}} \right),
\] (46)

with non-relativistic acceleration, \(a = \ddot{r}(t)\), for Larmor power, \(P = a^2/6\pi\), can be integrated over time, to give the total energy radiated by the slow moving electron (with final speed \(s\)), carried off by the photon radiation,
\[
E = \int_{-\infty}^{+\infty} P(t) \, dt = \frac{q^2 s^2}{6\pi^2} \Delta_{\omega}.
\] (47)

Here \(\Delta_{\omega} \equiv \omega_{\max} - \omega_{\min}\) is the frequency range of the gamma radiation as dictated by the sensitivity of the detector.

### F. Accelerating Distance

There is no birth moment in Eq. (8) like there is in Eq. (1). The eternal electron escapes the nucleus by slowly asymptotically accelerating from rest, eventually reaching its peak emission. Despite the infinite time, the electron travels a finite distance starting at \(r = 0\) and emitting maximum power at
\[
r_0 = \frac{2s}{\kappa \left( \sqrt{9 - 8s^2} + 1 \right)}.\] (48)
At small speeds the distance is \( r_0 = s/2\kappa \). At large speeds it is \( r_0 = 1/\kappa \). The speed at the max acceleration, or max power, is
\[
v_0(r_0) = \frac{2s}{\sqrt{9 - 8s^2 + 3}}, \quad (49)
\]
which is half the speed of light at ultra-relativistic final speeds, and \( v_0(r_0) = s/3 \) at small speeds, to lowest order.

For perspective, the distance traveled before max power emission, at ultra-relativistic final speeds, \( r_0 = h_\gamma \pi c \epsilon / (12q^2 \Delta_\omega) \), using \( \Delta E_\gamma \sim 65 \text{ keV} \) [24], is about 9 picometers, or about a third of the atomic radius of hydrogen (3.6 the electron Compton wavelength).

G. Large \( \kappa \) spectrum \( N(\omega) \)

The \( |\beta_{\omega'}|^2 \) spectrum of Eq. (14) at large \( \kappa \) has an asymptotic value of
\[
\lim_{\kappa \to \infty} |\beta_{\omega'}|^2 = \frac{\omega' (\omega_0^2 + \omega'_{\text{max}}^2)}{\pi \omega_0^2 \omega'_{\text{max}}^2}. \quad (50)
\]
This is integrated over \( \omega' \) to get the particle spectrum \( N(\omega) \) in the limit of large \( \kappa \) to get the same scaling as in Eq. (3),
\[
N(\omega) = \frac{1}{\pi^2} \left[ \frac{\eta}{s} - 1 \right] \frac{1}{\omega}. \quad (51)
\]
The total particle count in this large \( \kappa \) approximation is
\[
N = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} N(\omega) d\omega = \frac{1}{\pi^2} \left[ \frac{\eta}{s} - 1 \right] \ln \left( \frac{\omega_{\text{max}}}{\omega_{\text{min}}} \right). \quad (52)
\]

For energy integrand \( \omega N(\omega) \), and IR/UV-cutoffs in frequency \( \Delta_\omega = \omega_{\text{max}} - \omega_{\text{min}} \), one retrieves via
\[
E = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \omega N(\omega) d\omega = \frac{1}{\pi^2} \left[ \frac{\eta}{s} - 1 \right] \Delta_\omega. \quad (53)
\]
Here if \( \Delta_\omega \leftrightarrow q^2 \Delta_\omega / \pi \) one gets the same pre-factor as Eq. (3). Regardless of the large \( \kappa \) spectrum, a direct numerical integration of \( |\beta_{\omega'}|^2 \) Eq. (14) via Eq. (15), without approximation or a IV/UV-cutoffs, confirms
\[
E = \frac{\kappa}{24\pi} \left[ \frac{\eta}{s} - 1 \right], \quad (54)
\]
which is the energy, Eq. (3), using the identification \( \kappa \leftrightarrow 12q^2 \Delta_\omega / \pi \) and cloud energy, Eq. (41).

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