A New $C$-Eigenvalue Localisation Set for Piezoelectric-Type Tensors

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Abstract. A new inclusion set for localisation of the $C$-eigenvalues of piezoelectric tensors is established. Numerical experiments show that it is better or comparable to the methods known in literature.

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1. Introduction

Third order tensors play an important role in physics and engineering, including nonlinear optics [10,12], properties of crystals [6,11,19,20,22,26] and liquid crystals [5,9,24]. In particular, piezoelectric tensors find wide applications in converse piezoelectric and piezoelectric effects [4]. Chen et al. [4] specify the piezoelectric-type tensors as follows.

Definition 1.1 (cf. Chen et al. [4]). A third order $n$-dimensional tensor $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ is called the piezoelectric-type tensor if the last two indices of $A$ are symmetric — i.e. if $a_{ijk} = a_{ikj}$ for all $j,k \in [n]$, where $[n] := \{1,2,\ldots,n\}$.

Qi [21] and Lim [18] introduced the notion of eigenvalues for higher order tensors. It is worth noting that the eigenvalues of the third order symmetric traceless-tensors are widely used in the theory of liquid crystals [5,9,24]. Following these ideas, Chen et al. [4] defined $C$-eigenvalues and $C$-eigenvectors for piezoelectric-type tensors, which turn out to be useful in the study of piezoelectric and converse piezoelectric effects in solid crystals.

Definition 1.2 (cf. Chen et al. [4]). Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a third-order $n$-dimensional tensor. A number $\lambda \in \mathbb{R}$ is called the $C$-eigenvalue of $A$ if there are $x,y \in \mathbb{R}^n$ such that

\[ A y y = \lambda x, \quad x A y = \lambda y, \quad x^T x = 1, \quad y^T y = 1, \quad (1.1) \]
where}

$$(Ayy)_i = \sum_{k,j\in[n]} c_{ikj}y_ky_j, \quad (x.Ay)_i = \sum_{k,j\in[n]} c_{kji}x_ky_j.$$ 

The vectors $x$ and $y$ are referred to as associated left and right $C$-eigenvectors, respectively.

By $\sigma(Ayy)$ we denote the $C$-spectrum of the piezoelectric-type tensor $Ayy$ — i.e. the set of all $C$-eigenvalues of the piezoelectric-type tensor $Ayy$. The $C$-spectral radius of $Ayy$ is defined by

$$\rho(Ayy) := \max\{||\lambda|| : \lambda \in \sigma(Ayy)\}.$$ 

For a piezoelectric tensor $Ayy$, Chen et al. [4] proved the existence of $C$-eigenvalues associated with left and right $C$-eigenvectors. They also showed that the largest $C$-eigenvalue of the piezoelectric tensor represents the highest piezoelectric coupling constant and it can be determined as

$$\lambda^* = \max\{x.Ayy : x^\top x = 1, y^\top y = 1\},$$

where

$$x.Ayy := \sum_{i,k,j\in[n]} c_{ijk}x_iy_j.$$ 

However, the practical calculation of $\lambda^*$ is a challenging problem because of the uncertainty with the $C$-eigenvectors $x$ and $y$ in actual operations. On the other hand, we can capture all eigenvalues of a high order tensor by the eigenvalue localisation. In particular, for real symmetric tensors, Qi [21] considers an eigenvalue localisation set, which is an extension of the Geršgorin matrix eigenvalue inclusion theorem for matrices [23]. For general tensors, Li et al. [16] proposed Brauer-type eigenvalue inclusion sets. Later on, various eigenvalue localisation sets and their applications have been studied in Refs. [1,2,8,13,14,17,25,27].

Recently, C. Li and Y. Li [15] introduced two intervals to estimate all $C$-eigenvalues of a piezoelectric-type tensor.

**Theorem 1.1** (cf. C. Li & Y. Li [15]). If $\lambda$ is a $C$-eigenvalue of the piezoelectric-type tensor $Ayy = (c_{ijk}) \in \mathbb{R}^{n \times n \times n}$, then

$$\lambda \in [-\rho, \rho],$$

where

$$\rho = \max_{i,j\in[n]} \left\{ R_i^{(1)}(Ayy)R_j(Ayy) \right\}^{1/2},$$

$$R_i^{(1)}(Ayy) = \sum_{l,k\in[n]} |c_{ilk}|, R_j(Ayy) = \sum_{l,k\in[n]} |c_{lkj}|, \quad [n] = \{1, 2, \ldots, n\}.$$ 

**Theorem 1.2** (cf. C. Li & Y. Li [15]). If $\lambda$ is a $C$-eigenvalue of the piezoelectric-type tensor $Ayy = (c_{ijk}) \in \mathbb{R}^{n \times n \times n}$ and $S$ is a subset of $[n]$, then

$$\lambda \in [-\rho_S, \rho_S],$$

where

$$\rho_S = \max_{i,j\in[n]} \left\{ R_i^{(1)}(Ayy)R_j(Ayy) \right\}^{1/2},$$

$$R_i^{(1)}(Ayy) = \sum_{l,k\in[n]} |c_{ilk}|, R_j(Ayy) = \sum_{l,k\in[n]} |c_{lkj}|, \quad [n] = \{1, 2, \ldots, n\}.$$