Abstract. — Let $X$ be a complex projective Fano 4-fold. Let $D \subset X$ be a prime divisor. Let us consider the image $N_1(D,X)$ of $N_1(D)$ in $N_1(X)$ through the natural push-forward of one-cycles. Let us consider the following invariant of $X$ given by $\delta_X := \max\{\text{codim}N_1(D,X) \mid D \subset X \text{ prime divisor}\}$, called Lefschetz defect. We find a characterization for Fano 4-folds with $\delta_X = 3$: besides the product of two del Pezzo surfaces, they correspond to varieties admitting a conic bundle structure $f: X \to Y$ with $\rho_X - \rho_Y = 3$. Moreover, we observe that all of these varieties are rational. We give the list of all possible targets of such contractions. Combining our results with the classification of toric Fano 4-folds due to Batyrev and Sato we provide explicit examples of Fano conic bundles from toric 4-folds with $\delta_X = 3$.

Contents

1. Introduction ..................................................... 2
2. Preliminaries .................................................... 4
3. Main results ..................................................... 6
4. Case of $\rho_X = 5$ ................................................ 11
5. Toric Fano conic bundles with relative Picard dimension 3 ...... 15
6. Open Questions .................................................. 19
Acknowledgements ................................................. 20
References .......................................................... 20

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1. Introduction

Let $X$ be a complex projective smooth and Fano variety of dimension $n$. Let us denote by $\mathcal{N}_1(X)$ the $\mathbb{R}$-vector space of one-cycles with real coefficients, modulo numerical equivalence, whose dimension is the Picard number $\rho_X$.

Let $D \subset X$ be a prime divisor. The inclusion $i: D \hookrightarrow X$ induces a pushforward of one-cycles $i_*: \mathcal{N}_1(D) \to \mathcal{N}_1(X)$. We set $\mathcal{N}_1(D, X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X)$, which is the linear subspace of $\mathcal{N}_1(X)$ spanned by numerical classes of curves contained in $D$. In [8] Casagrande introduced the following invariant, called Lefschetz defect:

$$\delta_X := \max\{\text{codim}\mathcal{N}_1(D, X)|D \subset X \text{ prime divisor}\}.$$ 

This invariant allows to deduce many features about the variety. Indeed, in [8, Theorem 1.1] the author proved that if $X$ is a Fano manifold of arbitrary dimension $n$, with $\delta_X \geq 4$ then $X \cong S \times T$, where $S$ is a del Pezzo surface and $T$ is a $(n-2)$-dimensional Fano manifold. We refer the reader to [8] and [11] for the properties of $\delta_X$.

In [27] the author studied non-elementary Fano conic bundles of $X$, namely fiber type contractions $f: X \to Y$ with one-dimensional fibers and such that $\rho_X - \rho_Y > 1$.

An important point for our purposes is that given a Fano manifold $X$ admitting a conic bundle structure $f: X \to Y$ with relative Picard dimension $r := \rho_X - \rho_Y \geq 3$, there is a relation between $r$ and $\delta_X$. More precisely, it is possible to find some lower-bounds for $\delta_X$ in terms of the relative Picard dimension of $f$. For instance, under this assumption we get $\delta_X \geq r - 1$ by [27, Lemma 3.10].

In this paper we focus on the case in which $X$ is a Fano 4-fold. Our main goal is to find a characterization in terms of the Lefschetz defect of some Fano 4-folds admitting a conic fibration.

The first case of main interest is when $\delta_X = 3$. Indeed if $\delta_X \geq 4$, we have already observed that $X \cong S_1 \times S_2$ where each $S_i$ is a del Pezzo surface. In this situation all conic bundle structures are well known, and it is easy to find bounds for $\delta_X$. We refer the reader to Proposition 3.1 and Remark 3.2 for details.

When $X$ is a Fano manifold with $\delta_X = 3$, in the proof of [8, Theorem 1.1 (ii)] Casagrande showed that there exists a conic bundle structure $f: X \to Y$ where $\rho_X - \rho_Y = 3$. In this paper we prove that the converse holds for Fano 4-folds such that $X \not\cong S_1 \times S_2$, where each $S_i$ is a del Pezzo surface. Combining what is already known with our results, we formulate the main Theorem of this paper as follows.

**Theorem 1.1.** — Let $X$ be a projective Fano 4-fold such that $X \not\cong S_1 \times S_2$, where each $S_i$ is a del Pezzo surface. Then $\delta_X = 3$ if and only if there exists a conic bundle $f: X \to Y$ such that $\rho_X - \rho_Y = 3$.

As we have already stressed, in order to prove Theorem 1.1 we are left to investigate in more details conic bundles of Fano 4-folds, with relative Picard dimension 3. By [33, Corollary of Proposition 4.3] the target of such contraction is a Fano 3-fold. One of the main ingredient is the following result where we give the list of all possible targets of such contractions.

**Proposition 1.2.** — Let $f: X \to Y$ be a Fano conic bundle, where $\dim(X) = 4$, and $\rho_X - \rho_Y = 3$. Then $5 \leq \rho_X \leq 13$. Moreover:

(a) If $\rho_X = 5$, then $Y$ is one of the following Fano 3-folds: $Y \cong \mathbb{P}^1 \times \mathbb{P}^2$; $Y \cong \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(1))$; $Y \cong \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(2))$. 

(b) If $\rho_X = 6$, then $Y$ is one of the following Fano 3-folds: $Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; $Y \cong \mathbb{F}_1 \times \mathbb{P}^1$; $Y \cong \mathbb{P}_{1	imes \mathbb{P}_1}(\mathcal{O}(-1,-1) \oplus \mathcal{O})$; $Y \cong \mathbb{P}_{1	imes \mathbb{P}_1}(\mathcal{O}(0,-1) \oplus \mathcal{O}(-1,0))$.

(c) If $\rho_X \geq 7$, then $X \cong S_1 \times S$, where $S_1$ is a del Pezzo surface with $\rho_{S_1} = 4$, $Y \cong \mathbb{P}^1 \times S$, and $f$ is induced by a conic bundle $S_1 \to \mathbb{P}^1$.

From the previous results, we are able to deduce the following:

**Corollary 1.3.** — Let $X$ be a Fano 4-fold with $\delta_X = 3$ or such that there exists a conic bundle $X \to Y$ with $\rho_X - \rho_Y = 3$. Then $X$ is rational.

Using the definition of $\delta_X$, and Theorem 1.1 we give a characterization of Fano 4-folds admitting a conic bundle with relative Picard number 3, in terms of the Picard number of prime divisors on $X$.

**Theorem 1.4.** — Let $X$ be a projective Fano 4-fold, such that $X \not\cong S_1 \times S_2$, where each $S_i$ is a del Pezzo surface. Then $X$ admits a conic bundle structure $f: X \to Y$ with $\rho_X - \rho_Y = 3$ if and only if there exists a prime divisor $D$ on $X$ such that $\rho_D = \rho_X - 3$. If one of the above equivalent conditions holds, then $X$ is rational.

Finally we study in more detail the case in which $X$ is a Fano 4-fold with $\rho_X = 5$ admitting a conic bundle structure of relative Picard dimension 3.

**Theorem 1.5.** — Let $f: X \to Y$ be a Fano conic bundle with $\rho_X = 5$, and $\rho_X - \rho_Y = 3$. Then $X$ is obtained as the blow-up along two smooth disjoint surfaces contained in one of the following 4-folds:

1. $\mathbb{P}^1 \times M$, where $M$ is a Fano $\mathbb{P}^1$-bundle over $\mathbb{P}^2$;
2. $\mathbb{F}_1 \times \mathbb{P}^2$;
3. $\mathbb{P}_Y(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle on $Y$, where either $Y \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ or $Y \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$.

**Methods and outline of the article.** — Let us describe in more detail the content of this paper. In Section 2 we set up notation and terminology. In Subsection 2.2 and 2.3 we recall some crucial facts about the Lefschetz defect and Fano conic bundles.

Section 3 contains the central part of the paper: we prove Theorem 1.1, Proposition 1.2, and we deduce Corollary 1.3. We conclude this section by proving Theorem 1.4.

Let us summarize the strategy to show Theorem 1.1. If $\delta_X = 3$, the thesis follows by the proof of [8, Theorem 1.1 (ii)], as we observe in Proposition 2.6.

Hence the main point to reach the statement consists in the study of non elementary Fano conic bundles $f: X \to Y$ with $\rho_X - \rho_Y = 3$. To this end, we use Proposition 1.2. The key to prove Proposition 1.2 is to look at the discriminant divisor $\Delta_f$ of $f$, which is a divisor of $Y$ that was introduced in [3] and studied further in [30]:

$$\Delta_f = \{ y \in Y \mid f^{-1}(y) \text{ is singular} \}.$$  

More precisely, we look at the components of $\Delta_f$ that in our case are two smooth disjoint divisors of $Y$, as we recall in Subsection 2.3. Let us denote them by $A_i$ for $i = 1, 2$. Since $Y$ is Fano, by [4, Corollary 1.3.2] we know that it is a Mori Dream Space (see [18] for more details about Mori Dream Spaces).

Applying some techniques of Mori Dream Spaces to these divisors $A_i$, we deduce what kind of contractions $Y$ could have.
Then by considering some results on conic bundles we are able to deduce that $Y$ is a Fano bundle of rank 2 over $S$, that means that $Y \cong \mathbb{P}_S(\mathcal{F})$, where $\mathcal{F}$ is a rank 2 vector bundle over $S$. At this point, using the main Theorem of [32] which gives the list of Fano bundles of rank 2 on surfaces we get the proof of Proposition 1.2. Moreover, we are able to deduce an explicit description of the discriminant divisor of $f$ (see Corollary 3.4). Then in each case we compute $\rho_{A_i}$ and these information, together with the conic bundle structure of $X$ allow to deduce that $\delta_X = 3$, and hence Theorem 1.1.

Section 4 is entirely devoted to the case in which $X$ is a Fano 4-fold with $\rho_X = 5$ admitting a conic bundle structure of relative Picard dimension 3. Here we prove some lemmas (see Lemma 4.1, 4.2) which we need to show Theorem 1.5. The aim of Section 5 is to provide examples of Fano conic bundles $f: X \to Y$ where $X$ is a toric Fano 4-fold with $\rho_X \in \{5, 6\}$, and $\rho_X - \rho_Y = 3$. In particular we deduce that every $Y$ in Proposition 1.2 appears as target of some conic bundle of relative Picard dimension 3.

In Section 6 we conclude with some open questions for further investigations.

2. Preliminaries

2.1. Notations and Conventions. — We work over the field of complex numbers. Let $X$ be a smooth projective variety of arbitrary dimension $n$. We call $X$ a Fano variety if $-K_X$ is an ample divisor.

$\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) is the $\mathbb{R}$-vector space of one-cycles (respectively, divisors) with real coefficients, modulo numerical equivalence.

$\dim \mathcal{N}_1(X) = \dim \mathcal{N}^1(X) = \rho_X$ is the Picard number of $X$.

Let $C$ be a one-cycle of $X$, and $D$ a divisor of $X$. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class in $\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$). Moreover we denote by $\mathbb{R}[C]$ the linear span of $[C]$ in $\mathcal{N}_1(X)$, and by $\mathbb{R}_{\geq 0}[C]$ the corresponding ray. The symbol $\equiv$ stands for numerical equivalence (for both one-cycles and divisors).

$\text{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves.

We denote by $\text{Eff}(X)$ the effective cone of $X$, that is the convex cone inside $\mathcal{N}^1(X)$ spanned by classes of effective divisors.

Denote by $\text{Bs}(|\cdot|)$ the base locus of a linear system. Let $D \subset X$ be a Cartier divisor. The divisor $D$ is movable if there exists $m > 0$, $m \in \mathbb{Z}$ such that $\text{codim} \text{Bs}|mD| \geq 2$. We denote by $\text{Mov}(X)$ the movable cone of $X$, that is the closure of the convex cone of $\mathcal{N}^1(X)$ spanned by classes of movable divisors. We denote by $\text{Nef}(X)$ the nef cone of $X$, that is the convex cone of $\mathcal{N}^1(X)$ spanned by classes of nef divisors.

A contraction of $X$ is a surjective morphism $\varphi: X \to Y$ with connected fibers, where $Y$ is normal and projective. We denote by $\text{Exc}(\varphi)$ the exceptional locus of $\varphi$, i.e. the locus where $\varphi$ is not an isomorphism. We say that $\varphi$ is of type $(a, b)$ if $\dim \text{Exc}(\varphi) = a$ and $\varphi(\text{Exc}(\varphi)) = b$.

We denote by $\text{NE}(\varphi)$ the relative cone of $f$, namely the convex subcone of $\text{NE}(X)$ generated by classes of curves contracted by $\varphi$.

A contraction of $X$ is called $K_X$-negative (or simply $K$-negative) if $-K_X \cdot C > 0$ for every curve $C$ contracted by $\varphi$.

A $\mathbb{P}^1$-bundle over a projective variety $Z$ is the projectivization of a rank 2 vector bundle on $Z$. We say that a vector bundle $\mathcal{E}$ of rank $r$ on a projective variety $Z$ is a Fano bundle of rank $r$ if $\mathbb{P}_Z(\mathcal{E})$ is a Fano manifold and. By abuse of notation, we will often say in this case that the variety $\mathbb{P}_Z(\mathcal{E})$ is a Fano bundle as well.
2.2. Lefschetz defect. — Let $X$ be a smooth Fano variety and take $D \subset X$ a prime divisor. The inclusion $i: D \hookrightarrow X$ induces a pushforward of one-cycles $i_*: \mathcal{N}_1(D) \to \mathcal{N}_1(X)$, that does not need to be injective nor surjective.

We set $\mathcal{N}_1(D, X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X)$. Equivalently, $\mathcal{N}_1(D, X)$ is the linear subspace of $\mathcal{N}_1(X)$ spanned by classes of curves contained in $D$.

Working with $\mathcal{N}_1(D, X)$ instead $\mathcal{N}_1(D)$ means that we consider curves in $D$ modulo numerical equivalence in $X$, instead of numerical equivalence in $D$. Note that $\dim \mathcal{N}_1(D, X) \leq \rho_D$.

In [8] Casagrande introduced the following invariant of $X$, called Lefschetz defect:

$$\delta_X := \max \{ \text{codim} \mathcal{N}_1(D, X) \mid D \text{ is a prime divisor of } X \}$$

Note that if $n \geq 3$ and $D$ is ample, then $i_*: \mathcal{N}_1(D) \to \mathcal{N}_1(X)$ is surjective by the Lefschetz Theorem on hyperplane sections. Hence $\delta_X$ measures the failure of this Theorem for non ample divisors. The following theorem is related to this invariant and will be crucial for our purposes.

**Theorem 2.1** ([8], Theorem 1.1). — For any Fano manifold $X$ we have $\delta_X \leq 8$.

Moreover:

(a) If $\delta_X \geq 4$ then $X \cong S \times T$, where $S$ is a del Pezzo surface, $\rho_S = \delta_X + 1$, and $\delta_Y \leq \delta_X$;

(b) If $\delta_X \geq 3$ then there exists a flat fiber type contraction $\psi: X \to Z$ where $Z$ is an $(n-2)$-dimensional Fano manifold, $\rho_X - \rho_Z = 4$, and $\delta_Z \leq 3$.

2.3. Fano conic bundles. — In this subsection we recall some definitions and the main properties on Fano conic bundles which we need throughout the paper. We refer the reader to [27, 28, 29] for a more complete exposition about Fano conic bundles. For the convenience of the reader we recall some results from [27] without proofs, thus making our exposition self-contained.

**Definition 2.2.** — Let $X$ be smooth, projective variety and let $Y$ be a normal, projective variety. A conic bundle $f: X \to Y$ is a fiber type $K$-negative contraction where every fiber is one-dimensional. A Fano conic bundle is a conic bundle $f: X \to Y$ where $X$ is a Fano variety.

**Definition 2.3.** — Let $f: X \to Y$ be a conic bundle. If $\rho_X - \rho_Y > 1$, $f$ is called non-elementary. Otherwise $f$ is called elementary (or standard).

**Proposition 2.4** ([27], Proposition 3.5). — Let $f: X \to Y$ be a Fano conic bundle with $\rho_X - \rho_Y := r$. Then:

(a) There exists a factorization: $X \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_{r-1}} X_r \xrightarrow{g} Y$, where each $f_i$ is a blow-up along a smooth subvariety of $X_{i+1}$ of codimension 2, and $g$ is an elementary conic bundle;

(b) There exist prime divisors $A_1, \ldots, A_{r-1}$ of $Y$ such that $f^*(A_i) = E_i + \hat{E}_i$, where $E_i, \hat{E}_i$ are smooth prime divisors on $X$ such that $E_i \to A_i$ and $\hat{E}_i \to A_i$ are $\mathbb{P}^1$-bundle for every $i = 1, \ldots, r$;

(c) Set $\Delta_f := \{ y \in Y \mid f^{-1}(y) \text{ is singular} \}$ the discriminant divisor of $f$. Then $\Delta_f = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_{r-1} \sqcup \Delta_g$, where $A_i$ are smooth components of $\Delta_f$ for $i = 1, \ldots, r - 1$, and $\Delta_g$ is the discriminant divisor of $g$.

The geometric situation is represented in Figure 1.
Finally, we recall a weaker version of [27, Theorem 4.2], which we need in the next sections.

**Theorem 2.5.** — Let $f: X \to Y$ be a Fano conic bundle. Then $\rho_X - \rho_Y \leq 8$. Moreover:

(a) If $\rho_X - \rho_Y \geq 4$, then $X \cong S \times T$, where $S$ is a del Pezzo surface, $T$ is a $(n-2)$-dimensional Fano manifold, $Y \cong S \times \mathbb{P}^1$, and $f$ is induced by a conic bundle $S \to \mathbb{P}^1$;

(b) Assume that $\rho_X - \rho_Y = 3$. Let us take a factorization for $f$ as in Proposition 2.4(a), and let us denote by $g: X_2 \to Y$ the elementary conic bundle in this factorization. Then $g$ is smooth, $Y$ is also Fano and there exists a smooth $\mathbb{P}^1$-fibration\(^{(1)}\) $\xi: Y \to Y'$, where $Y'$ is a $(n-2)$-dimensional Fano manifold.

The following Proposition is a consequence of the proof of [8, Theorem 1.1 (ii)].

**Proposition 2.6.** — Let $X$ be a Fano manifold of dimension $n \geq 3$. If $\delta_X = 3$, then it admits a conic bundle structure $f: X \to Y$ where $\rho_X - \rho_Y = 3$.

**Proof.** — By Theorem 2.1 (b) there exists a flat fiber type contraction $\psi: X \to Z$ where $Z$ is a $(n-2)$-dimensional Fano manifold and $\rho_X - \rho_Z = 4$. By the proof of [8, Theorem 1.1 (ii)] it follows that this fiber type contraction $\psi$ factors in the following way $\psi = h \circ f$ where $f: X \to Y$ and $h: Y \to Z$ are contractions. In particular, $f$ is a Fano conic bundle with $\rho_X - \rho_Y = 3$, and our claim follows.

3. Main results

In this section we discuss the main results of this paper which give a characterization of some Fano 4-folds, in terms of the Lefschetz defect of these varieties. We first analyze the easier case in which the Fano variety $X$ is a product between two del Pezzo surfaces. The first proposition is an immediate consequence of the structure

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\(^{(1)}\) A smooth $\mathbb{P}^1$-fibration is a smooth morphism such that every fiber is isomorphic to $\mathbb{P}^1$. 

of conic bundles induced on del Pezzo surfaces, and of the very definition of $\delta_X$. We point out that in this case we know all possible non-elementary conic bundles of $X$.

**Proposition 3.1.** — Let $X \cong S_1 \times S_2$ where $S_i$ are del Pezzo surfaces $i = 1, 2$, and $X \ncong \mathbb{P}^2 \times \mathbb{P}^2$. Then $X$ has a conic bundle structure $f: X \rightarrow Y$ with $\delta_X \geq r := \rho_X - \rho_Y$.

**Proof.** — Let $X \cong S_1 \times S_2$ where $S_i$ are del Pezzo surfaces $i = 1, 2$. Take one between $S_i$ such that $S_i \ncong \mathbb{P}^2$. Assume for simplicity that it is $S_1$. Then we can consider the following contraction $f = (\tilde{f}, \text{id}_{S_2}): X \rightarrow \mathbb{P}^1 \times S_2$, where $\tilde{f}: S_1 \rightarrow \mathbb{P}^1$ is a conic bundle with $r = \rho_{S_1} - 1$, and $\text{id}_{S_2}: S_2 \rightarrow S_2$ is the identity. The statement follows because $\delta_X = \max\{\rho_{S_1} - 1, \rho_{S_2} - 1\}$ (see [8, Example 3.1]).

**Remark 3.2.** — Let $X$ be as in the above Proposition. We know all possible conic bundles of $X$. Indeed, if $f: X \rightarrow Y$ is a Fano conic bundle then by [27, Lemma 2.10] $Y$ is also a product of two smooth varieties. In particular, the same proof of [27, Theorem 4.2 (a)] allows to deduce that $Y \cong \mathbb{P}^1 \times S_2$, and $f$ is induced by a conic bundle $S_1 \rightarrow \mathbb{P}^1$.

Our next goal will be to analyze the case in which $X$ is not isomorphic to a product of two del Pezzo surfaces, by proving Theorem 1.1. To this end, we need two preliminary results given by the following Lemma and Proposition 1.2.

**Lemma 3.3.** — Let $Y = \mathbb{P}_S(E)$ be a Fano bundle of rank 2 on a del Pezzo surface $S$. Assume that $Y$ has an elementary divisorial contraction $\psi: Y \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ of type (2, 1), and that there exists a prime effective divisor $D$ of $Y$ such that $D \cap \text{Exc}(\psi) = \emptyset$. Let us denote by $\ell$ a fiber of the natural projection $\xi: Y \rightarrow S$. If $\text{Exc}(\psi) \cdot \ell > 0$ and $D \cdot \ell > 0$, then $S \cong \mathbb{F}_1$, and $Y \cong \mathbb{P}^1 \times \mathbb{F}_1$.

**Proof.** — Let us denote by $A := \text{Exc}(\psi)$. It is a prime divisor of $Y$, and by Mori’s classification of divisorial contractions of smooth Fano 3-folds [21, Theorem 3.3], $A$ is a $\mathbb{P}^1$-bundle over smooth curve $\tilde{\gamma}$. We set $C := \psi(A)$ and $B := \psi(D)$ which by our assumption are respectively a curve and a divisor in $\mathbb{P}^1 \times \mathbb{P}^2$. Note that $C \cap B = \emptyset$ because $A \cap D = \emptyset$.

Let us consider the projections $\pi_1: \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1$ and $\pi_2: \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$, and set $\Psi := \pi_1 \circ \psi: Y \rightarrow \mathbb{P}^1$. We have the following morphisms:

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\pi_1} \mathbb{P}^1 \\
\downarrow{\xi} & & \downarrow{\pi_2} \\
S & & \mathbb{P}^2 \\
\end{array}
$$

If $\Psi$ is finite on the fibers of $\xi: Y \rightarrow S$, then the claim follows by [6, Lemma 4.9]. Moreover, in this case $S \cong \mathbb{F}_1$.

Assume that there exists a fiber $\ell$ of $\xi$ such that $\Psi(\ell) = \{pt\}$. Since $\psi$ is an elementary contraction different by $\xi$, it cannot contract the same fibers of $\xi$, then $\tilde{\ell} := \psi(\ell)$ is a curve on $\mathbb{P}^1 \times \mathbb{P}^2$ and $\pi_1(\tilde{\ell})$ is a point. We observe that $C$ is not numerically proportional to $\tilde{\ell}$. Indeed if we consider the intersection with the divisor
B, by the projection formula one has that for some $m \in \mathbb{Z}_{>0}$, $B \cdot \ell = mD \cdot \ell > 0$, instead $B \cdot C = 0$ because $C \cap B = \emptyset$. It means that $\pi_1(C) = \mathbb{P}^1$.

Let us prove that in this case $B$ is of the form $\mathbb{P}^1 \times \gamma$, where $\gamma \subset \mathbb{P}^2$ is a curve, and that $C$ is of the form $\mathbb{P}^1 \times \{q\}$, where $q \notin \gamma$. Since $B$ is a non-zero effective divisor, it is linearly equivalent to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(a, b)$ with $a, b \geq 0$ and $(a, b) \neq (0, 0)$.

We note that $a = 0$ or $b = 0$, otherwise $B$ is ample and by Kleiman ampleness criterion we get $B \cdot C > 0$, against our assumption.

On one hand, If $b = 0$ then $B$ is the pullback of a divisor on $\mathbb{P}^1$ and hence it is given by a plane of the form $\{pt\} \times \mathbb{P}^2$. On the other hand, $\pi_1(C) = \mathbb{P}^1$ and hence $B$ and $C$ cannot be disjoint in this case, a contradiction. We have therefore that $a = 0$ and hence $B$ is the pullback of a divisor on $\mathbb{P}^2$, i.e., $B \cong \mathbb{P}^1 \times \gamma$, where $\gamma \subset \mathbb{P}^2$ is a curve.

Moreover, the projection formula gives $0 = C \cdot \pi_2^*L = (\pi_2)_*C \cdot L$, where $L$ is a line on $\mathbb{P}^2$ and thus $C$ is of the form $\mathbb{P}^1 \times \{q\}$, where $q \notin \gamma$. In particular, since $\psi: Y \to \mathbb{P}^1 \times \mathbb{P}^2$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along $C = \mathbb{P}^1 \times \{q\}$, we deduce that $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the conclusion follows. \hfill \Box

Now we prove Proposition 1.2. The strategy is to look at the two components of $\Delta_f$, which are smooth divisors of $Y$, and that we denote by $A_i$ as in Proposition 2.4 (c). Since by [33, Corollary of Proposition 4.3] $Y$ is Fano, using [4, Corollary 1.3.2] we know that it is a Mori Dream Space (see [18] for details about Mori Dream Spaces). Applying some techniques of Mori Dream Spaces to these divisors $A_i$, we deduce that $\rho_X = 5$. Since $\xi$ is smooth with fibers isomorphic to $\mathbb{P}^1$ and $S$ is rational, then there exists a rank 2 vector bundle $E$ on $S$ such that $Y \cong \mathbb{P}(E)$ (see for instance [14, Proposition 4.3]).

Proof of Proposition 1.2. — By Theorem 2.5 (b), there exists a smooth $\mathbb{P}^1$-fibration $\xi: Y \to S$, with $S$ a del Pezzo surface. By our assumption $\rho_X - p_S = 4$ so that $\rho_X \geq 5$. Since $\xi$ is smooth with fibers isomorphic to $\mathbb{P}^1$ and $S$ is rational, then there exists a rank 2 vector bundle $E$ on $S$ such that $Y \cong \mathbb{P}(E)$ (see for instance [14, Proposition 4.3]).

Assume that $\rho_X = 5$. This means that in the smooth $\mathbb{P}^1$-fibration $\xi: Y \to S$, $S \cong \mathbb{P}^2$. As we have already observed, $Y \cong \mathbb{P}(E)$. We find all possible $E$.

Let us denote by $A_i$ for $i = 1, 2$ the two smooth components of the discriminant divisor of $f$, as in Proposition 2.4 (c). Suppose that both divisors $A_i$ are nef. We prove that in this case $Y \cong \mathbb{P}^1 \times \mathbb{P}^2$.

Since by [33, Corollary of Proposition 4.3] $Y$ is Fano, applying [4, Corollary 1.3.2] it is a MDS, hence $A_i$ are semiample divisors. Using that $A_1 \cap A_2 = \emptyset$, it is easy to check that they are numerically proportional divisors of $Y$ which give a contraction $\Psi: Y \to \mathbb{P}^1$ such that $\Psi(A_i)$ is a point for $i = 1, 2$. By the proof of [27, Theorem 4.2 (b)] we know that $\xi(A_i) = S$ for $i = 1, 2$. Then $\Psi$ is finite on the fiber of $\xi$. The claim follows by [6, Lemma 4.9].

Assume that only one divisor between $A_i$‘s is nef. For instance, suppose that it is $A_1$. Since Nef($Y$) $\neq$ Mov($Y$), then $A_2$ is not a movable divisor and by [7, Remark 4.7] we have that $A_2$ is an exceptional divisor of a birational contraction of $Y$. 

In particular, $A_1$ gives this birational contraction which contracts the other divisor $A_2$ at a point. By [9, Lemma 3.9] and the main Theorem of [32] it follows that either $Y \cong P_{22}(O \oplus O(1))$ or $Y \cong P_{22}(O \oplus O(2))$.

We observe that there are no other possibilities: if neither $A_1$ or $A_2$ are nef divisors of $Y$, then they should be two exceptional divisors of two different extremal divisorial contractions of $Y$, but $Y$ is a Fano bundle of rank 2 over $P^2$ and we get a contradiction. Then claim (a) follows.

Assume that $\rho_X = 6$. Applying again [33, Corollary of Proposition 4.3], $Y$ is Fano and by Theorem 1.1 (b) there exists a smooth $P^1$-fibration $\xi: Y \to S$, where either $S \cong P^1 \times P^1$ or $S \cong F_1$. As above, we denote by $A_1$ the two smooth components of the discriminant divisor of $f$, as in Proposition 2.4 (c).

Suppose that both divisors $A_i$ are nef. The same proof of point (a) allows us to deduce that $Y \cong S \times P^1$ where $S$ is isomorphic to $P^1 \times P^1$ or to $F_1$.

Assume that only one divisor between the $A_i$’s is nef. For simplicity, suppose that it is $A_1$. As above, we deduce that $A_1$ gives this birational contraction which contracts $A_2$ at a point. By [9, Lemma 3.9] and the main Theorem of [32] it follows that $Y \cong P_{p1 \times P^1}(O(-1, -1) \oplus O)$.

Assume that neither $A_1$ nor $A_2$ are nef divisors of $Y$. As we have already observed, this means that both divisors are exceptional divisors of two elementary divisorial contractions of $Y$ which we denote by $\Psi_i: Y \to Z_i$, for $i = 1, 2$. Since $A_1 \cap A_2 = \emptyset$ this means that $\text{Exc}(\Psi_1) \cap \text{Exc}(\Psi_2) = \emptyset$.

Using the main Theorem of [32] and [22, Table 3] we show that in this case the only possible candidate is $Y \cong P_{p1 \times P^1}(O(0, -1) \oplus O(-1, 0))$. To this end, we prove that the other varieties with two divisorial contractions in the list given by the main Theorem of [32] cannot be target of $f$. These varieties correspond to No. 17, 24, 30 in [22, Table 3]. Let us recall that No. 17 is a smooth divisor of tridegree $(1, 1, 1)$ in $P^1 \times P^1 \times P^2$, No. 24 is $P_{F_1}(\beta^*(T_{P^2}(-2)))$ and No. 30 is $P_{F_2}(\beta^*(O_{P^2} + O_{P^2}(-1)))$, where $\beta: F_1 \to P^2$ is the blow-up map.

Assume by contradiction that $Y$ is isomorphic to No. 30. Using the Mori and Mukai’s description of this Fano 3-fold as the blow-up of the strict transform of a line passing through the center of the blow-up map $P(O_{P^2} + O_{P^2}(-1)) \to P^3$ (see [22, Table 3]), it is easy to check that $\text{Exc}(\Psi_1) \cap \text{Exc}(\Psi_2) = \emptyset$, hence a contradiction.

Mori and Mukai’s classification shows that No. 17 has two elementary divisorial contractions, $\Psi_i: Y \to P^1 \times P^2$ of type $(2, 1)$ for $i = 1, 2$. Assume by contradiction that the target $Y$ is isomorphic to No. 17. This means that $\text{Exc}(\Psi_1) \cap \text{Exc}(\Psi_2) = \emptyset$.

In particular, taking as $\Psi$ the contraction $\Psi_1$ and $D := \text{Exc}(\Psi_2)$ the hypothesis of Lemma 3.3 are satisfied, because by the proof of [27, Theorem 4.2 (b), Step 2 and Step 3] it follows that $A_i$ have positive intersection with the fibers of $\xi$. Then, by Lemma 3.3 we reach a contradiction because $Y$ is not a product.

We use the same procedure to exclude the Fano 3-fold No. 24. Indeed, by Mori and Mukai’s classification we know that this variety has two divisorial contractions $\Psi_i$, where $\Psi_i: Y \to P^1 \times P^2$ of type $(2, 1)$. If we assume that the target $Y$ is isomorphic to No. 24 we reach a contradiction by applying Lemma 3.3 as above, and we get (b).

Now we show (c). Let us consider the morphism $\xi \circ f: X \to S$. Assume that $\rho_X \geq 7$, so that $\rho_S \geq 3$. By [5, Theorem 1.1 (i)] one has $X \cong S_1 \times S$, where $S_1, S$
are del Pezzo surfaces. Then by Remark 3.2 one has $Y \cong \mathbb{P}^1 \times S$, and $f$ is induced by a conic bundle $S_1 \to \mathbb{P}^1$. Then we get (c).

Finally, since $\rho_X - \rho_Y = 3$, then $\rho_{S_1} = 4$, so that $\rho_X \leq 13$. We have already proved that $\rho_X \geq 5$, and this complete the proof. \hfill \square

As an immediate consequence of the proof of the above proposition we get an explicit description of the discriminant divisor of our conic bundle. Then we have all ingredients that we need to show Theorem 1.1.

**Corollary 3.4.** Setting as in Proposition 1.2. As usual, denote by $\triangle_f = A_1 \sqcup A_2$ the discriminant divisor of $f$, with $A_i$ smooth components of $\triangle_f$. If $\rho_X = 5$ then $A_i \cong \mathbb{P}^2$ for $i = 1, 2$. If $\rho_X = 6$ and $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ then $A_i \cong \mathbb{P}^1$, otherwise $A_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2$.

**Proof of Theorem 1.1.** The first implication follows by Proposition 2.6.

Assume now that X is a projective Fano 4-fold such that $X \not\cong S_1 \times S_2$, where each $S_i$ is a del Pezzo surface, and that there exists a conic bundle $f : X \to Y$ such that $\rho_X - \rho_Y = 3$. By Proposition 1.2 (c) we are left to study the two possible cases: $\rho_X \in \{5, 6\}$. Let us denote by $A_i$ the two smooth components of the discriminant divisor of $f$, as in Proposition 2.4 (c).

Assume that $\rho_X = 5$. By Corollary 3.4 it follows that $A_i \cong \mathbb{P}^2$, so that $\rho_{A_i} = 2$ for each $i = 1, 2$. To get our claim it is enough to consider one between the $A_i$’s. For simplicity, take $A_1$. Let us consider the divisor $E_1$ of $X$ as in Proposition 2.4 (b) such that $E_1 \to A_1$ is a $\mathbb{P}^1$-bundle, so that $\rho_{E_1} = 2$. Since $\dim N_1(E_1, X) \leq \rho_{E_1}$ it follows that $\delta_X \geq 3$. Then $\delta_X = 3$ by Theorem 2.1 (a).

Assume that $\rho_X = 6$. We apply the same argument of the previous case. The only difference here is that by Corollary 3.4 it follows that either $A_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $A_i \cong \mathbb{P}^1$. In any case $\rho_{A_i} = 2$ for each $i = 1, 2$. If we take $A_1$, and the divisor $E_1 \subset X$ as in Proposition 2.4 (b) we get $\rho_{E_1} = 3$, and hence the statement follows as above. \hfill \square

**Proof of Corollary 1.3.** It is enough to prove the statement when $X \not\cong S_1 \times S_2$ where $S_i$ are del Pezzo surfaces for $i = 1, 2$. By Theorem 1.1 the two conditions required in the assumption are equivalent. Take a conic bundle $f : X \to Y$ where $\rho_X - \rho_Y = 3$. By Proposition 1.2 we know all possible targets $Y$. In particular, all of these $Y$ are rational varieties. Let us take a factorization for $f$ as in Proposition 2.4 (a), and let us denote by $g : X_2 \to Y$ the standard conic bundle of this factorization. By Theorem 2.5 (b) $g$ is a smooth conic bundle. By [14, Proposition 4.3] it follows that $X_2 \cong \mathbb{P}^1(F)$, where $F$ is a rank 2 vector bundle on $Y$. Hence $X_2$ is rational and so is $X$. \hfill \square

We conclude this subsection by giving the proof of Theorem 1.4, in which due to the definition of $\delta_X$, and Theorem 1.1 we find a characterization of Fano 4-folds admitting a conic bundle with relative Picard number 3, in terms of the Picard number of prime divisors on $X$.

**Proof of Theorem 1.4.** Assume that such an X admits a conic bundle structure $f : X \to Y$ with $\rho_X - \rho_Y = 3$. By Proposition 1.2, one has $\rho_X \in \{5, 6\}$. Take a divisor $E_1 \subset X$ as in Proposition 2.4 (b), and set $D := E_1$. In the proof of Theorem 1.1 we have already observed that $\rho_D = 2$ if $\rho_X = 5$, and $\rho_D = 3$ if $\rho_X = 6$, then we get the statement.
Conversely, if there exists a prime divisor \( D \) on \( X \) such that \( \rho_D = \rho_X - 3 \). Since \( \dim \mathcal{N}_1(D,X) \leq \rho_D \), and \( X \) is not a product of del Pezzo surfaces, by Theorem \( 2.1 \) (a) we get \( \delta_X = 3 \). Hence the statement follows by Theorem \( 1.1 \).

\[ \square \]

4. Case of \( \rho_X = 5 \)

The goal of this section is to prove Theorem 1.5. To this end, we start focusing on case in which \( X \) is a Fano 4-fold that admits a conic bundle \( f : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 \) with \( \rho_X - \rho_Y = 3 \).

**Lemma 4.1.** — Let \( X \) be a Fano 4-fold which admits a smooth \( \mathbb{P}^1 \)-fibration \( g : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 \). Let us consider the contraction \( \Psi : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1 \) and let \( F_i \) be two fibers of \( \Psi \) with \( i = 1, 2 \). Assume that both \( F_i \rightarrow \mathbb{P}^2 \) have a section \( B_i \), such that the blow-up of \( X \) along \( B_i \) is Fano. Then either \( X \cong \mathbb{F}_1 \times \mathbb{P}^2 \) or \( X \cong \mathbb{P}^1 \times Z \), with \( Z \) a Fano \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \).

**Proof.** — Let us consider the contraction \( \xi : \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \). Take another extremal contraction \( h \) of \( X \), different by \( g \), and such that \( \text{NE}(h) \subset \text{NE}(\xi \circ g) \). We get another factorization for \( \psi := \xi \circ g : X \rightarrow \mathbb{P}^2 \), given by \( X \xrightarrow{F_i} Z \xrightarrow{\xi} \mathbb{P}^2 \). We have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow{\psi} & & \downarrow{\xi}' \\
\mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{g} & \mathbb{P}^2 \\
\downarrow{\Psi} & & \downarrow{\pi} \\
\mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^2
\end{array}
\]

Let \( R \) be the extremal ray corresponding to \( h \). We prove that it is not contracted by \( \Psi \). To this end, we first observe that every fiber of \( \Psi \) is a nef divisor of \( X \). Indeed, take a fiber \( F_0 \) of \( \Psi \), and \( C_0 \subset F_0 \) an irreducible curve. Since all fibers of \( \Psi \) are numerically proportional, it follows that \( F_0 \cdot C_0 = m\bar{F} \cdot C_0 \), where \( m \in \mathbb{R} \), and \( \bar{F} \) is another fiber of \( \Psi \). Using that all fibers are disjoint we get \( F_0 \cdot C_0 = 0 \).

It follows that \( F_0 \cdot R > 0 \). Indeed, if \( F_0 \cdot R = 0 \) then there exists a curve \( \tilde{C} \subset F_0 \) such that \( \tilde{C} \in R \) and hence \( g \) and \( \xi \) should contract the same fibers, a contradiction. Applying [33, Corollary 1.4] there are only two possibilities for \( h \): either it is a conic bundle or it is a blow-up of \( Z \) along a smooth surface.

Assume that \( h \) is a conic bundle. By [33, Corollary of Proposition 4.3] it follows that \( Z \) is Fano. Moreover, since all fibers of \( \psi \) are isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), and \( h \) is an equidimensional morphism it follows that \( Z \) is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \), and the fibers of \( h \) are isomorphic to \( \mathbb{P}^1 \cdot \mathbb{P}^1 \). Since \( g \) cannot contracts the fibers of \( h \), by \( [5, \text{ Lemma 4.9}] \) we get \( X \cong \mathbb{P}^1 \times Z \), with \( Z \) a Fano \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \).

From now on we assume that we are in the case in which \( h \) is birational. For the reader’s convenience, the proof is subdivided into several steps:

**Step 1.** We show that all fibers of \( \Psi \) are Fano manifolds.
Proof of Step 1. — Let $F_0$ be a fiber of $\Psi$. By the adjunction formula, for every irreducible curve $\Gamma \subset X$ we get $-K_{F_0} \cdot \Gamma = -K_X \cdot \Gamma - F_0 \cdot \Gamma > 0$ because $X$ is Fano, and $F_0$ is a nef divisor of $X$. 

Step 2. Let us denote by $E = \text{Exc} (h)$. We prove that $E$ cannot be a fiber of $\Psi$. 

Proof of Step 2. — Assume by contradiction that it is. Then $E$ is a nef divisor of $X$. But $E$ is covered by curves $\tilde{C}$ such that $E \cdot \tilde{C} < 0$ and we get a contradiction. 

Step 3. Let us denote by $B_i$ the sections inside $F_i$ as in the assumption. Then $E \cap (B_1 \cup B_2) = \emptyset$. 

Proof of Step 3. — We know that $h$ is a contraction $(3,2)$, and $E$ is covered by rational curves of anticanonical degree 1 (cf. [19, Lemma 1]). Since the blow-up along $B_1$ is Fano, we observe that if $B_1$ intersects a fiber of $h$, this fiber must be contained in $B_1$. Indeed, let us denote by $\ell$ a non-trivial fiber of $h$, by $X_1$ the blow-up of $X$ along $B_1$ and by $\tilde{\ell} \subset X_1$ the strict transform of $\ell$ on $X_1$. Using [6, Lemma 3.8] it follows that if $\ell$ intersects $B_1$ but $\ell \not\subset B_1$ we obtain that 

$$1 = -K_{X_1} \cdot \tilde{\ell} > -K_{X_1} \cdot \ell,$$

which is impossible if $X_1$ is Fano.

Then either $B_1$ is union of fibers of $h$ or $E \cap B_1 = \emptyset$. But if $B_1$ is union of fibers, then $\xi(g(B_1)) = \{p\}$ with $p \in \mathbb{P}^2$ and we get a contradiction. Applying the same method with $B_2$ we deduce that $E \cap (B_1 \cup B_2) = \emptyset$. 

Step 4. Let us denote by $F$ any fiber of $\Psi$. Then $h_F: F \to h(F)$ is an isomorphism. Moreover if we denote by $B$ the center of the blow-up $X \to Z$, then $E \cap F \cong B$. In particular, $E \cong \mathbb{P}^1 \times B$. 

Proof of Step 4. — We observe that $h_F$ is a birational morphism. Indeed, if $h_F$ is a fiber type contraction, then $F \subseteq E$ and hence $F = E$, which contradicts Step 2. Hence we have two cases: 

(a) $h_F$ contracts a curve; 

(b) $h_F$ is a finite morphism.

We observe that case (a) is not possible: indeed, in this case $h(F)$ is a Fano 3-fold with $p_h(F) = 1$, and it cannot dominate $\mathbb{P}^2$. Then (b) holds, $h_F$ is a finite and birational morphism with connected fibers onto a normal variety, then $h$ is an isomorphism by Zariski’s Main Theorem. In particular, $h(F)$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

We prove that $E \cap F \cong B$. We have already observed that $F \cdot R > 0$, with $R$ extremal ray whose contraction is $h$, then $E \cap F \neq \emptyset$, and $S := E \cap F$ is a surface.

Since $h_F: F \to h(F)$ is an isomorphism, it sends $S$ isomorphically to a surface $\tilde{S} \subseteq B$ then $\tilde{S} = B$. Moreover, being all fibers of $\Psi$ disjoint, we obtain three disjoint sections $E \cap F_i$, with $i = 1,2,3$, of the bundle $E \cong \mathbb{P}_B (\mathbb{N}^\vee_{B/Z}) \to B$ and hence $E \cong \mathbb{P}^1 \times B$. 

Step 5. We prove that for every fiber $F$ of $\Psi$, $B$ is a section of $\xi^e_{h(F)}: h(F) \to \mathbb{P}^2$. In particular, $B \cong \mathbb{P}^2$, $F_1 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}_1)$, $F_2 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}_2)$, where $\mathcal{E}_1$, $\mathcal{E}_2$ are decomposable rank 2 vector bundles on $\mathbb{P}^2$.

Proof of Step 5. — We first prove that $B \cong \mathbb{P}^2$. Since $g: X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle over a rational 3-fold it admits a section $D \cong \mathbb{P}^1 \times \mathbb{P}^2 \subseteq X$ (see [14, Proposition 4.3]).
The divisor $D$ intersects transversally every fiber $F$ of $\Psi$ and $D \cap F$ is a section of the $\mathbb{P}^1$-bundle $F \rightarrow \mathbb{P}^2$. Then $D \cap F \cong \mathbb{P}^2$, and $h(D \cap F) \cong \mathbb{P}^2$ by Step 4.

We analyze two different cases. Assume that $D = E$. Using Step 4 we know that $E \cong \mathbb{P}^1 \times B$, thus in this case $B \cong \mathbb{P}^2$ by [15, Theorem 6].

Suppose that $D \neq E$ so that $S = D \cap E$ is a surface. We prove that there exists a fiber $\tilde{F}$ of $\Psi$ such that $\tilde{F} \cap D = S$. Take $\ell \subset S$ a curve and set $\tilde{F} := \Psi^{-1}(\Psi(\ell))$.

In particular, $\tilde{F} \cap D \neq \emptyset$. Take $\gamma \subset \tilde{F} \cap D$ a curve. Then $h(\gamma) \subset B$. But $h(\tilde{F} \cap D) \cong \mathbb{P}^2$, hence by Bézout’s theorem every curve $C \subset \mathbb{P}^2$ intersects $h(\gamma)$, in particular it intersects $B$. Since the blow-up of $Z$ along $B$ is Fano we use the same argument of Step 3, and we conclude that every curve of $h(D \cap \tilde{F})$ must be contained in $B$, hence $B \cong \mathbb{P}^2$.

By Step 4 it follows that $E$ intersect every fiber $F$ of $\Psi$ along a section which is isomorphic to $\mathbb{P}^2$. Let us consider the fibers $F_i$ for $i = 1, 2$ as in the assumption. Since $h(B_i)$ is another section of $h(F_i)$ disjoint by $B$, then $F_i \cong h(F_i) \cong \mathbb{P}^2(\mathcal{E}_i)$ where $\mathcal{E}_i$ is a decomposable rank 2 vector bundle on $\mathbb{P}^2$.

Step 6. We show that $Z \cong \mathbb{P}^2 \times \mathbb{P}^2$, and $X \cong F_1 \times \mathbb{P}^2$.

**Proof of Step 6.** — The morphism $\xi'$ is a $K$-negative and equidimensional contraction with general fiber isomorphic to $\mathbb{P}^2$. We observe that all the fibers of $\xi'$ are isomorphic to $\mathbb{P}^2$, so that $Z \cong \mathbb{P}^2(\mathcal{F})$ with $\mathcal{F}$ a rank 3 vector bundle on $\mathbb{P}^2$. This follows because the fibers of $\psi := \xi \circ g$ are isomorphic to $F_1$, and if $\tilde{F}$ is any fiber of $\psi$, then $h|_{\tilde{F}} \cong \mathbb{F}_1$.

In order to have $Z \cong \mathbb{P}^2 \times \mathbb{P}^2$ we show that $Z$ has four disjoint sections. By the previous steps, there are many fibers of $\Psi$ such that their images throughout $h$ in $Z$ are decomposable $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ which intersect themselves along $B$. Indeed by Horrocks' cohomological criterion of splitness [24, Theorem 2.3.1] and by the semicontinuity of the dimension of cohomology groups in flat families [17, Theorem III.12.8], there exists an open subset $U \subset \mathbb{P}^1$ such that if $F = \Psi^{-1}(y)$ for $y \in U$, then $F = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a decomposable rank 2 vector bundle on $\mathbb{P}^2$.

In particular, it is possible to find four disjoint sections of $Z$, so that $Z \cong \mathbb{P}^2 \times \mathbb{P}^2$. We are assuming that $h$ is the blow-up of $Z$ along $B$ which by Step 5 is isomorphic to $\mathbb{P}^2$, then $X \cong F_1 \times \mathbb{P}^2$.

Hence the statement follows.

**Lemma 4.2.** — Let $f: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ be a Fano conic bundle with $\rho_X = 5$. Then there exists a factorization as in Proposition 2.4 (a), where $X_1, X_2$ are Fano 4-folds.

**Proof.** — Let us take a factorization for $f$ as in Proposition 2.4 (a), given by:

![Diagram](image)

We denote by $A_i$ the two components of the discriminant divisor of $f$ as in Proposition 2.4 (c). By Proposition 2.4 (b) we know that there exist two prime divisors $E_i, \tilde{E}_i$ of $X$ such that $f^*(A_i) = E_i + \tilde{E}_i$, and $E_i \rightarrow A_i, \tilde{E}_i \rightarrow A_i$ are $\mathbb{P}^1$-bundles for $i = 1, 2$. Let us denote by $e_i \subset E_i, \tilde{e}_i \subset \tilde{E}_i$ the fibers for $i = 1, 2$.

By the conic bundle structure, $D_i := \tilde{g}^*(A_i)$ are $\mathbb{P}^1$-bundles over $A_i$, and each $D_i$ contains a section $A'_i \cong A_i$, such that $f_i$ is the blow-up of $X_i$ along $A'_i$ for $i = 1, 2$. 

By [27, Remark 3.7] the factorization of \( f \) is not unique and depends on the choice of the extremal rays spanned by \([e_1], [e_i]\). To fix the notation, assume that \( \tilde{f}_1 \) corresponds to the contraction of the extremal ray \( R_1 = \mathbb{R}_{\geq 0}[e_1] \), and \( \tilde{f}_2 \) is the contraction given by the extremal ray \( R_2 = \mathbb{R}_{\geq 0}[e_2] \). Using these facts, we first prove that up to changing the factorization of \( f \) we obtain that the target of the first blow-up of the factorization of \( f \) is a Fano variety.

Assume that \( \tilde{X}_1 \) is not Fano. Then there exists an irreducible curve \( C \subset \tilde{X}_1 \) such that \( -K_{\tilde{X}_1} \cdot C < 0 \), and \( C \subset A'_i \). We show that \( D_1 \) is not Fano.

It is easy to check that \( D_1 \) is a nef divisor which has intersection zero with every curve contained in it. Then by the adjunction formula we get \( -K_D \cdot C = -K_{\tilde{X}_1} \cdot C - D_1 \cdot C < 0 \). Since \( \tilde{E}_1 \) is the strict transform in \( X \) of the divisor \( D_1 \), it follows that \( \tilde{E}_1 \) is not a Fano variety.

By the proof of Proposition 1.2 (a), we deduce that \( A'_i \cong \mathbb{P}^2 \), so that \( \rho_{\tilde{E}_2} = 2 \). Now we replace \( f_1 \) with the contraction of the extremal ray spanned by \([e_1]\). In this way we get another blow-up \( f_1 : X \to X_1 \) where \( X_1 \) is Fano by [33, Proposition 3.4].

Let us consider the factorization of \( f \) given by \( X \xrightarrow{\tilde{f}_1} X_1 \xrightarrow{\tilde{f}_2} \tilde{X}_2 \to Y \).

If \( \tilde{X}_2 \) is Fano we are done. Assume that it is not. We can proceed in the same way, replacing \( A'_1 \) with \( A'_2 \), which is the center of the blow-up \( \tilde{f}_2 \). With a similar method, it is easy to check that \( D_2 \) is not Fano, and since its strict transform in \( X_1 \) is isomorphic to \( \tilde{E}_2 \), we deduce that \( \tilde{E}_2 \) is not Fano.

Replacing \( \tilde{f}_2 \) with \( f_2 : X_1 \to X_2 \) which corresponds to the contraction of the extremal ray spanned by \([e_2]\), we complete the proof applying again [33, Proposition 3.4].

\[ \square \]

Remark 4.3. — The proof of Lemma 4.2 depends neither on the dimension of \( X \) nor on \( \rho_X \). Indeed, the same argument shows the following more general statement. Assume that \( X \) is a Fano manifold of arbitrary dimension, and let \( f : X \to Y \) be a non-elementary Fano conic bundle with \( r := \rho_X - \rho_Y \). Let us take a factorization for \( f \) as in Proposition 2.4 (a), and let us denote by \( g : X_{r-1} \to Y \) the standard conic bundle in this factorization. If \( \rho_{A_i} = 1 \), and \( D_i := g^*(A_i) \) is a nef divisor for \( i = 1, \ldots, r-1 \), then there exists a factorization of \( f \) as in Proposition 2.4 (a), where each of the \( X_i \) is a Fano variety.

\[ \square \]

Proof of Theorem 1.5. — Let us take a factorization for \( f \) as in Lemma 4.2. As usual we denote by \( g : X_2 \to Y \) the standard conic bundle in this factorization. Since \( X \) is obtained as the blow-up along two smooth disjoint surfaces contained in \( X_2 \), it is enough to show that \( X_2 \) is isomorphic to one of the varieties at points (1), (2), (3).

By Proposition 1.2 (a), we know all possible \( Y \). Assume that \( Y \cong \mathbb{P}^1 \times \mathbb{P}^2 \). Applying Lemma 4.1 to \( X_2 \) we obtain that \( X_2 \) is isomorphic either to one of the variety of point (1) or of point (2).

Finally, assume that \( Y \cong \mathbb{P}^2(\mathcal{O}(1)+\mathcal{O}(1)) \) or \( Y \cong \mathbb{P}^2(\mathcal{O}(2)+\mathcal{O}(2)) \). By Theorem 2.5 (b) we know that \( g : X_2 \to Y \) is a smooth \( \mathbb{P}^1 \)-fibration. Moreover \( Y \) is rational, then applying [14, Proposition 4.3] we deduce that \( X_2 \) is as in point (3) and hence the statement.

\[ \square \]
5. Toric Fano conic bundles with relative Picard dimension 3

The purpose of this section is to provide examples by proving that all Fano 3-folds $Y$ listed in Proposition 1.2 appear as target of some Fano conic bundle $f : X \to Y$ with $\rho_X - \rho_Y = 3$. In particular, we give explicit examples of toric Fano conic bundle, i.e. Fano conic bundle $f : X \to Y$ where $X$ is toric.

We refer the reader to [10] for the general theory of toric varieties and to [12, 13, 20, 26] for details on the toric MMP.

Let $N \cong \mathbb{Z}^n$ be a lattice and let $N_\mathbb{R} \cong \mathbb{R}^n$ be its real scalar extension. Let $\Delta_X \subseteq N_\mathbb{R}$ be a fan and let us denote by $\Delta_X(k)$ the set of $k$ dimensional cones in $\Delta_X$ and by $X = X(\Delta_X)$ the associated toric variety. If $\sigma \in \Delta_X(k)$ is a cone generated by the primitive vectors $\{u_1, \ldots, u_k\}$ we will denote by $V(\sigma)$ or $V(u_1, \ldots, u_k)$ the closed invariant subvariety of codimension $k$ in $X$. In particular, each $\rho \in \Delta_X(1)$ corresponds to an invariant prime divisor $V(\rho)$ on $X$; such a cone is called a ray. Similarly, each cone of codimension one $\omega \in \Delta_X(n - 1)$ corresponds to an invariant rational curve on $X$; such a cone is called a wall.

If $X$ is a smooth projective toric variety of dimension $n$ then every extremal ray $R \subseteq \text{NE}(X)$ corresponds to an invariant curve $C_\omega$ such that $R = \mathbb{R}_{\geq 0}[C_\omega]$ or, equivalently, to a wall $\omega \in \Delta_X(n - 1)$.

Let us suppose that such a wall $\omega$ is generated by the primitive vectors $\{u_1, \ldots, u_{n-1}\}$. Since the fan $\Delta_X$ is simplicial, $\omega$ separates two maximal cones $\sigma = \text{cone}(u_1, \ldots, u_{n-1}, u_n)$ and $\sigma' = \text{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$, where $u_n$ and $u_{n+1}$ are primitive on rays on opposite sides of $\omega$. The $n + 1$ vectors $u_1, \ldots, u_{n+1}$ are linearly dependent. Hence, they satisfy a so called wall relation (see [10, page 303]):

$$b_n u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n = b_{n+1} = 1$ and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, n - 1$. By reordering if necessary, we can assume that

$$b_i < 0 \quad \text{for} \quad 1 \leq i \leq \alpha$$
$$b_i = 0 \quad \text{for} \quad \alpha + 1 \leq i \leq \beta$$
$$b_i > 0 \quad \text{for} \quad \beta + 1 \leq i \leq n + 1.$$

The wall relation and the signs of the coefficients involved allow us to describe the nature of the associated contraction. More precisely,

(1) Fiber type contractions correspond to $\alpha = 0$
(2) Divisorial contractions correspond to $\alpha = 1$
(3) Small contractions correspond to $\alpha > 1$.

In all cases we have that the extremal contraction associated to $R$ is of type $(n - \alpha, \beta)$. We refer the reader to [26, §2] or [20, §14.2] for details. Moreover, Fano 4-folds were classified by Batyrev and Sato into 124 families by using the language of primitive collections [1, 2, 31].

In our analysis, we are mainly interested in extremal $K$-negative contractions between smooth projective varieties $\varphi_R : X \to X_R$ of type $(n - 1, n - 2)$ and type $(n, n - 1)$ (i.e., a standard conic bundle). In the toric case, the associated wall relations are the following:

(a) Type $(n-1, n-2)$: $u_n - u_1 + u_{n+1} = 0$. It corresponds geometrically to the blow-up of the codimension two subvariety $A = V(u_n, u_{n+1}) \subset X_R$, with exceptional divisor $V(u_1) \subset X$. The toric morphism $\varphi_R : X \to X_R$ corresponds to the star
subdivision of $\Delta_{X_R}$ relative to the ray generated by the vector $u_1 = u_n + u_{n+1}$ (see [10, Definition 3.3.7]).

(b) Type $(n, n-1)$: $u_n + u_{n+1} = 0$. It corresponds geometrically to a projective vector bundle structure $X \cong \mathbb{P}_{X_R}(E)$ where $E$ is a decomposable rank 2 vector bundle on $X_R$, for which $V(u_n) \cong V(u_{n+1}) \cong X_R$ are disjoint sections of $X$. The toric morphism $\varphi_R : X \to X_R$ is induced by the quotient map $\Phi_R : N_R \to N_R/\text{Span}(u_n)$.

Let us recall that in the Fano conic bundle case $X \to Y$ with $\rho_X - \rho_Y = 3$, by Proposition 2.4 (a) and Theorem 2.5 (b) it follows that there exists a factorization of $f$

$$
\xymatrix{ X \ar[r]^{f_1} & X_1 \ar[r]^{f_2} & X_2 \ar[r]^g & Y }
$$

where each $f_i$ is a blow-up along a smooth surface of $X_i$ for $i = 1, 2$, and $g$ is a smooth standard conic bundle. By [27, Proposition 3.5 (1)] the centers of the blow-ups are isomorphic to the smooth components of the discriminant divisor of $f$. When $X$ is a Fano 4-fold the possible blow-up centers are described in Corollary 3.4.

We denote by $\text{Bl}_A(X_i)$ the blow-up of $X_i$ along $A_i$. From now on we keep the above notation for a factorization of $f$ and we use Batyrev-Sato notation as in [2, §4] and [31, Table 1].

Case of $\rho_X = 5$. — For a toric Fano conic bundle $f : X \to Y$ with $\rho_X = 5$ and $\rho_X - \rho_Y = 3$, we know by Proposition 1.2 (a) that $Y \cong \mathbb{P}_{g^2}(\mathcal{O} \oplus \mathcal{O}(a))$ for some $a \in \{0, 1, 2\}$. In particular $X$ admits a locally trivial toric bundle over $\mathbb{P}^2$ whose fiber is a Del Pezzo surface with Picard rank 4. Toric Fano 4-folds admitting such a fibration onto $\mathbb{P}^2$ were considered by Batyrev in [2, §3.2.9] and their combinatorial type is denoted by $K$ in [2, §4]. More precisely, we have the following:

**Proposition 5.1.** — Let $X$ be a toric Fano 4-fold with $\rho_X = 5$. Assume that there exists a conic bundle $f : X \to Y$ such that $Y \cong \mathbb{P}_{g^2}(\mathcal{O} \oplus \mathcal{O}(a))$ for some $a \in \{0, 1, 2\}$. Then $X \cong K_i$ for some $i \in \{1, 2, 3, 4\}$.

**Examples 5.2.** — The following examples show that all the varieties $K_1, K_2, K_3, K_4$ admit a conic bundle $f : X \to Y$ onto $Y \cong \mathbb{P}_{g^2}(\mathcal{O} \oplus \mathcal{O}(a))$ for some $a \in \{0, 1, 2\}$.

Let us denote by $\{e_1, e_2, e_3, e_4\}$ the canonical basis of $N_R \cong \mathbb{R}^4$. Let us explain in more detail the method used to provide the examples.

We start by considering suitable toric Fano 4-folds $X_2$ with $\rho_X - \rho_{X_2} = 2$ such that there is a standard smooth conic bundle $g : X_2 \to Y$, where $Y$ is one of the above listed varieties (2). Afterwards, we have to blow-ups two suitable smooth surfaces from $X_2$ which by Corollary 3.4 are isomorphic to $\mathbb{P}^2$.

Let us denote by $A_1, A_2$ these centers. By the conic bundle structure, $A_i$’s must be transversal with respect to the fibers of the standard conic bundle $g : X_2 \to Y$. To this end, it can be checked by using Macaulay2 [16] whether the blow-up $X$ along $A_1, A_2$ is Fano or not. Then by computing the self-intersection number $(-K_X)^4$ and comparing with [2, §4] and [31, Table 1] we can determine the corresponding varieties $X_1$ and $X$ in Batyrev-Sato classification.

---

2. This can be easily checked by looking at the corresponding fans. We refer the interested reader to the auxiliary files accompanying the Macaulay2 package NormalToricVarieties for the complete list of fans of the 124 toric Fano 4-folds in Batyrev-Sato classification.
1. Let $X_2 := D_1 \cong \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^2}(O \oplus O(1, 2))$. The primitive generators of the rays of the fan of $X_2$ are given by

\[ \Delta_{X_2}(1) = \{ u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2 + 2e_4, u_4 = e_3, u_5 = -e_3 + e_4, u_6 = e_4, u_7 = -e_4 \}. \]

The standard conic bundle $g: X_2 \to \mathbb{P}^1 \times \mathbb{P}^2$ is induced by the quotient map $G: \mathbb{R}^4 \to \mathbb{R}^4/\langle e_4 \rangle$. Let $A_1 = V(u_4, u_7)$ and $A_2 = V(u_5, u_7)$, where $A_i \cong \mathbb{P}^2$ for $i = 1, 2$. We verify with Macaulay2 that $X_1 := Bl_{A_1}(X_2) \cong Bl_{A_2}(X_2) \cong H_3$ and $X := Bl_{A_1, A_2}(X_2) \cong K_2$ are Fano. We get the following factorization of $f : X \to Y$.

\[ X \cong K_2 \xrightarrow{f_1} X_1 \cong H_3 \xrightarrow{f_2} X_2 \cong D_1 \xrightarrow{g} Y \cong \mathbb{P}^1 \times \mathbb{P}^2. \]

2. Let $X_2 := D_3 \cong \mathbb{P}_Y(\mathcal{E})$, where $\mathcal{E}$ is a decomposable rank 2 vector bundle on $Y \cong \mathbb{P}_{\mathbb{P}^2}(O \oplus O(1))$. The primitive generators of the rays of the fan of $X_2$ are given by

\[ \Delta_{X_2}(1) = \{ u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2 + e_3 + e_4, u_4 = e_3, u_5 = -e_3 + e_4, u_6 = e_4, u_7 = -e_4 \}. \]

The standard conic bundle $g: D_3 \to \mathbb{P}_{\mathbb{P}^2}(O \oplus O(1))$ is induced by the quotient map $G: \mathbb{R}^4 \to \mathbb{R}^4/\langle e_4 \rangle$. Let $A_1 = V(u_4, u_7)$ and $A_2 = V(u_5, u_7)$, where $A_i \cong \mathbb{P}^2$ for $i = 1, 2$. We verify with Macaulay2 that $X_1 := Bl_{A_1}(X_2) \cong H_2$ and $X := Bl_{A_1, A_2}(X_2) \cong K_2$ are Fano. We get the following factorization of $f : X \to Y$.

\[ X \cong K_2 \xrightarrow{f_1} X_1 \cong H_2 \xrightarrow{f_2} X_2 \cong D_3 \xrightarrow{g} Y \cong \mathbb{P}_{\mathbb{P}^2}(O \oplus O(1)). \]

3. Let $X_2 := D_{16} \cong \mathbb{P}_Y(\mathcal{E})$, where $\mathcal{E}$ is a decomposable rank 2 vector bundle on $Y \cong \mathbb{P}_{\mathbb{P}^2}(O \oplus O(2))$. The primitive generators of the rays of the fan of $X_2$ are given by

\[ \Delta_{X_2}(1) = \{ u_1 = e_1, u_2 = e_2, u_3 = -e_1 - e_2 + e_3 - e_4, u_4 = e_3, u_5 = -e_3 + e_4, u_6 = e_4, u_7 = -e_4 \}. \]

The standard conic bundle $g: D_{16} \to \mathbb{P}_{\mathbb{P}^2}(O \oplus O(2))$ is induced by the quotient map $G: \mathbb{R}^4 \to \mathbb{R}^4/\langle e_4 \rangle$. Let $A_1 = V(u_4, u_7)$ and $A_2 = V(u_5, u_7)$, where $A_i \cong \mathbb{P}^2$ for $i = 1, 2$. We verify with Macaulay2 that $X_1 := Bl_{A_1}(X_2) \cong H_3$ and $X := Bl_{A_1, A_2}(X_2) \cong K_2$ are Fano. We get the following factorization of $f : X \to Y$.

\[ X \cong K_2 \xrightarrow{f_1} X_1 \cong H_3 \xrightarrow{f_2} X_2 \cong D_{16} \xrightarrow{g} Y \cong \mathbb{P}_{\mathbb{P}^2}(O \oplus O(2)). \]

4. The conic bundle structure on $X := K_4 \cong Bl_{p_1, p_2, p_3}(\mathbb{P}^2) \times \mathbb{P}^2$ is induced by the first factor as follows (see Remark 3.2).

\[ K_4 \cong Bl_{p_1, p_2, p_3}(\mathbb{P}^2) \times \mathbb{P}^2 \xrightarrow{f_1} X_1 \cong Bl_{p_1, p_2}(\mathbb{P}^2) \times \mathbb{P}^2 \xrightarrow{f_2} X_2 \cong \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{g} Y \cong \mathbb{P}^1 \times \mathbb{P}^2. \]

\textbf{Case of} $\rho_X = 6$. — Given a Fano conic bundle $f : X \to Y$ with $\rho_X = 6$ and $\rho_Y - \rho_Y = 3$, we know all possible targets $Y$ using Proposition 1.2 (b). The following examples show that all of these Fano 3-folds $Y$ appears as targets of some toric Fano conic bundle $f : X \to Y$ with $\rho_X - \rho_Y = 3$.

\textbf{Examples 5.3.} — We proceed as in Examples 5.2. Let us denote by $\{e_1, e_2, e_3, e_4\}$ the canonical basis of $N_{\mathbb{R}} \cong \mathbb{R}^4$. 
1. Let $X_2 := L_1 \cong \mathbb{P}^{P_1 \times P_1 \times P_1}(O \oplus O(1,1,1))$. The primitive generators of the rays of the fan of $X_2$ are given by
\[
\Delta X_2(1) = \{u_1 = e_1, u_2 = e_2, u_3 = e_1 - e_2, u_4 = e_3, \\
u_5 = e_1 - e_3, u_6 = e_4, u_7 = e_1 - e_4, u_8 = -e_1\}.
\]
The standard conic bundle $g : L_1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is induced by the quotient map $G : \mathbb{R}^4 \to \mathbb{R}^4/(e_1)$. Let $A_1 = V(u_2, u_8), A_2 = V(u_3, u_8)$, where $A_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2$. We verify with Macaulay2 that $X_1 := \text{Bl}_{A_1}(X_1) \cong \text{Bl}_{A_2}(X_2) \cong \mathbb{Q}_3$ and $X := \text{Bl}_{A_1,A_2}(X_2) \cong U_1$ are Fano. We get the following factorization of $f : X \to Y$.
\[
X \cong U_1 \xrightarrow{f_1} X_1 \cong \mathbb{Q}_3 \xrightarrow{f_2} X_2 \cong L_1 \xrightarrow{g} Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.
\]

2. Let $X_2 := L_2 \cong \mathbb{P}^1_Y(E)$, where $E$ is a decomposable rank 2 vector bundle on $Y \cong \mathbb{P}^{P_1 \times P_1}(O(-1,1) \oplus O)$. The primitive generators of the rays of the fan of $X_2$ are given by
\[
\Delta X_2(1) = \{u_1 = e_1, u_2 = e_2, u_3 = e_1 - e_2, u_4 = e_3, \\
u_5 = e_1 - e_2 - e_3, u_6 = e_4, u_7 = e_1 - e_2 - e_4, u_8 = -e_1\}.
\]
The standard conic bundle $g : L_2 \to \mathbb{P}^{P_1 \times P_1}(O(-1,1) \oplus O)$ is induced by the quotient map $G : \mathbb{R}^4 \to \mathbb{R}^4/(e_1)$. Let $A_1 = V(u_2, u_8), A_2 = V(u_3, u_8)$, where $A_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2$. We verify with Macaulay2 that $X_1 := \text{Bl}_{A_1}(X_2) \cong \mathbb{Q}_3$ and $X := \text{Bl}_{A_1,A_2}(X_2) \cong U_1$ are Fano. We get the following factorization of $f : X \to Y$.
\[
X \cong U_1 \xrightarrow{f_1} X_1 \cong \mathbb{Q}_3 \xrightarrow{f_2} X_2 \cong L_2 \xrightarrow{g} Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.
\]

3. Let $X_2 := L_3 \cong \mathbb{P}^1_Y(E)$, where $E$ is a decomposable rank 2 vector bundle on $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. The primitive generators of the rays of the fan of $X_2$ are given by
\[
\Delta X_2(1) = \{u_1 = e_1, u_2 = e_2, u_3 = e_1 - e_2, u_4 = e_3, \\
u_5 = e_1 - e_3, u_6 = e_4, u_7 = e_3 - e_4, u_8 = -e_1\}.
\]
The standard conic bundle $g : L_3 \to \mathbb{P}^1 \times \mathbb{P}^1$ is induced by the quotient map $G : \mathbb{R}^4 \to \mathbb{R}^4/(e_1)$. Let $A_1 = V(u_2, u_8), A_2 = V(u_3, u_8)$, where $A_i \cong \mathbb{F}_1$ for $i = 1, 2$. We verify with Macaulay2 that $X_1 := \text{Bl}_{A_1}(X_2) \cong \mathbb{Q}_5$ and $X := \text{Bl}_{A_1,A_2}(X_2) \cong U_2$ are Fano. We get the following factorization of $f : X \to Y$.
\[
X \cong U_2 \xrightarrow{f_1} X_1 \cong \mathbb{Q}_5 \xrightarrow{f_2} X_2 \cong L_3 \xrightarrow{g} Y \cong \mathbb{F}_1 \times \mathbb{P}^1.
\]

4. Let $X_2 := L_{11} \cong \mathbb{P}^1 \times \mathbb{P}^{P_1 \times P_1}(O(0,-1) \oplus O(-1,0))$. The primitive generators of the rays of the fan of $X_2$ are given by
\[
\Delta X_2(1) = \{u_1 = e_1, u_2 = e_2, u_3 = -e_2, u_4 = e_3, \\
u_5 = -e_2 - e_3, u_6 = e_4, u_7 = e_2 - e_4, u_8 = -e_1\}.
\]
The standard conic bundle $g : L_{11} \to \mathbb{P}^1 \times \mathbb{P}^{P_1 \times P_1}(O(0,-1) \oplus O(-1,0))$ is induced by the quotient map $G : \mathbb{R}^4 \to \mathbb{R}^4/(e_1)$. Let $A_1 = V(u_1, u_3), A_2 = V(u_2, u_8)$, where $A_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2$. We verify with Macaulay2 that

---

3. Let us remark that there is a misprint in [2, §4]: the variety $L_{11}$ is isomorphic to $\mathbb{P}^1 \times V(C_5)$ in Batyrev's notation, i.e. $L_{11} \cong \mathbb{P}^1 \times \mathbb{P}^{P_1 \times P_1}(O(0,-1) \oplus O(-1,0))$. 


$X_1 := \text{Bl}_{A_1}(X_2) \cong Q_{16}$ and $X := \text{Bl}_{A_1,A_2}(X_2) \cong U_8$ are Fano. We get the following factorization of $f: X \to Y$.

$X \cong U_8 \xrightarrow{f_1} X_1 \cong Q_{16} \xrightarrow{f_2} X_2 \cong L_{11} \xrightarrow{g} Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(\mathcal{O}(0,-1) \oplus \mathcal{O}(-1,0))$.

**Remark 5.4.** — As pointed out in Remark 3.2, the case of products of two del Pezzo surfaces is well-understood. In the toric case, there are examples given by $\text{Remark 5.4}$.

**Question 6.1.** — In all known examples of Fano conic bundle $f: X \to Y$ which satisfy the assumption of Theorem 1.1, $X$ is a toric variety (see Section 5 for details). Are there examples where $X$ is not toric?

Note that Theorem 1.1 holds in dimension 3. Assume that $f: X \to S$ is a Fano conic bundle from a Fano 3-fold $X$, and $\rho_X - \rho_S = 3$. By [23, Proposition 4.16], $S$ is a del Pezzo surface. Using Theorem 2.5 (b) there exists a smooth $\mathbb{P}^1$-fibration $S \to \mathbb{P}^1$, then either $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $S \cong F_1$ and hence $\rho_X = 5$. We recall by Proposition 2.4 that the discriminant divisor $\Delta_f$ is given by two smooth disjoint curves $C_i$ on $S$, such that $f^*(C_i) = E_i + \bar{E}_i$ where $E_i \to C_i$ and $\bar{E}_i \to C_i$ are $\mathbb{P}^1$-bundle for $i = 1,2$. Since $\rho_{C_i} = 1$, we get $\rho_{E_i} = 2$, and hence $\delta_{X} \geq 3$.

If $X \not\cong S \times \mathbb{P}^1$ with $S$ a del Pezzo surface, then by Theorem 2.1 (a) we deduce that $\delta_{X} = 3$. In particular, if $\rho_X \geq 6$ then $X \cong S \times \mathbb{P}^1$ with $S$ a del Pezzo surface (see [22, Theorem 2]). In this case, by Remark 3.2 we know all conic bundle structure on $X$, and $\delta_{X} = \rho_{S} - 1$.

Moreover, there exists an example of a Fano conic bundle $X \to F_1$ where $X$ is a toric Fano 3-fold which is not a product, $\rho_X = 5$ and $\delta_{X} = 3$ (see [29, § 4.3.1] for details). The following is a natural question.

**Question 6.2.** — Does Theorem 1.1 hold for higher dimensions?

Using [27, Lemma 3.10], and Theorem 2.1 (a) it follows that if $f: X \to Y$ is a Fano conic bundle with $\rho_X - \rho_Y = 3$, and $X \not\cong S \times T$ where $S$ is a del Pezzo surface and $T$ is a $(n-2)$-dimensional Fano manifold, then $\delta_{X} \not\in \{2,3\}$. Hence Theorem 1.1 holds for higher dimensions if we can exclude the case $\delta_{X} = 2$.

We conclude with a more general question:

**Question 6.3.** — Is it possible to characterize in terms of $\delta_{X}$ all Fano manifolds admitting a conic bundle structure $f: X \to Y$ with $\rho_X - \rho_Y \leq 2$?
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