Exact Hypermultiplet Dynamics in Four Dimensions

Sergei V. Ketov

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
ketov@itp.uni-hannover.de

Abstract

We use N=2 harmonic and projective superspaces to formulate the most general ‘Ansatz’ for the SU(2)R-invariant hypermultiplet low-energy effective action (LEEA) in four dimensions, which describes the two-parametric family of the hyper-Kähler metrics generalizing the Atiyah-Hitchin metric. We then demonstrate in the very explicit and manifestly N=2 supersymmetric way that the (magnetically charged, massive) single hypermultiplet LEEA in the underlying non-abelian N=2 supersymmetric quantum field theory can receive both perturbative (e.g., in the Coulomb branch) and non-perturbative (e.g., in the Higgs branch) quantum corrections. The manifestly N=2 supersymmetric Feynman rules in harmonic superspace can be used to calculate the perturbative corrections described by the Taub-NUT metric. The non-perturbative corrections (due to instantons and anti-instantons) can be encoded in terms of an elliptic curve, which is very reminiscent to the Seiberg-Witten theory. Our four-dimensional results agree with the three-dimensional Seiberg-Witten theory and instanton calculations.
1 Introduction

The exact gauge low-energy effective action (LEEA) in the Coulomb branch of N=2 supersymmetric Yang-Mills theory in four spacetime dimensions (4d), which includes both perturbative (one-loop) and nonperturbative (instanton) quantum corrections, was determined by Seiberg and Witten [1]. The main tools of their construction were the general constraints implied by N=2 extended supersymmetry, the known anomaly structure, and electric-magnetic duality. The N=2 off-shell supersymmetry implies the unique ‘Ansatz’ for the N=2 (abelian) vector multiplet LEEA, in terms of a holomorphic function \( \mathcal{F}(W) \) of the N=2 (restricted chiral) superfield strength \( W \). The chiral anomaly determines the perturbative (logarithmic) contribution to the function \( \mathcal{F}''(W) \). Nonperturbative consistency and duality unambiguously fix the instanton corrections to \( \mathcal{F}(W) \), which are related to the BPS monopoles representing nonperturbative degrees of freedom and belonging to hypermultiplets. The exact Seiberg-Witten (\( SU(2) \)-based) solution can be encoded in terms of an elliptic curve \( \Sigma_{SW} \), by integrating certain abelian differential \( \lambda_{SW} \) over the torus periods.

Since a generic 4d, N=2 gauge field theory has both N=2 vector multiplets and hypermultiplets, the latter may also have their own N=2 supersymmetric LEEA [2]. The hypermultiplet LEEA is highly constrained by N=2 extended supersymmetry and its automorphisms too, so that its exact form can also be determined. For instance, in three spacetime dimensions, Seiberg and Witten [3] used the so-called c-map [4], relating the special Kähler geometry of the N=2 vector multiplet moduli space to the hyper-Kähler geometry of the hypermultiplet moduli space. They further argued that the Atiyah-Hitchin (AH) metric [5] is the only regular exact solution. This proposal was later confirmed by 3d instanton calculations [6], which also discovered a one-parameter family of possible hyper-Kähler metrics generalizing the AH metric. We propose the most general ‘Ansatz’ for the 4d hypermultiplet LEEA, which is compatible with all unbroken symmetries and has two parameters. It allows us to reformulate the solution in the very transparent geometrical way. The earlier approaches [3, 4] are formulated only in 3d on the gauge field theory side, and they are not manifestly supersymmetric, which may cast some doubt on their ultimate consistency.

Our main purpose in this Letter is a derivation of the exact hypermultiplet LEEA directly in four spacetime dimensions, in the manifestly N=2 supersymmetric way. We confine ourselves to a single hypermultiplet for simplicity. We fully exploit the

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3 An abelian gauge vector is dual to a scalar in three dimensions.
restrictions implied by off-shell N=2 supersymmetry and its $SU(2)_R$ automorphisms, by making both of them manifest in HSS. Converting the HSS action into N=2 projective superspace (PSS) allows us to calculate the effective hyper-Kähler metric. The solution can be put into the Seiberg-Witten form after introducing the auxiliary elliptic curve associated with an $O(4)$ projective multiplet in N=2 PSS.

2 N=2 supersymmetry and hyper-Kähler metrics

The scalar kinetic part of the hypermultiplet LEEA is of the second order in spacetime derivatives, so that it has the form of a non-linear sigma-model (NLSM). By N=2 supersymmetry in 4d, the metric of this 4d NLSM has to be hyper-Kähler [7]. Making N=2 supersymmetry manifest (i.e. off-shell) also makes manifest the hyper-Kähler nature of the hypermultiplet LEEA. In the harmonic superspace (HSS) approach [8], both an N=2 vector multiplet and a hypermultiplet can be introduced off-shell on equal footing. For example, the Fayet-Sohnius hypermultiplet is described by an unconstrained complex analytic superfield $q^+$ of $U(1)$ charge (+1), whereas an N=2 vector multiplet is described by an unconstrained analytic superfield $V^{++}$ of $U(1)$ charge (+2).

The general N=2 NLSM Lagrangian in HSS reads [9]

$$\mathcal{L}^{(+4)} = -\sqrt{q}^+ D^{++} q^+ - \mathcal{K}^{(+4)}(\sqrt{q}^+, q^+; u^\pm),$$

(2.1)

where $\mathcal{K}^{(+4)}$ is called a hyper-Kähler potential, while the harmonic covariant derivative $D^{++}$ includes central charges. The N=2 central charge $Z$ can be treated as the abelian N=2 vector superfield background whose N=2 gauge superfield strength is given by $\langle W \rangle = Z$ [10]. The role of the analytic function $\mathcal{K}^{(+4)}$ in the hypermultiplet LEEA (2.1) is similar to the role of the holomorphic Seiberg-Witten potential $\mathcal{F}$ in the N=2 gauge LEEA. Because of manifest N=2 supersymmetry by construction, the equations of motion for the HSS action (2.1) determine (at least, in principle) the component hyper-Kähler NLSM metric in terms of a single analytic function $\mathcal{K}^{(+4)}$. An explicit form of this relation is, however, not known in general. A crucial simplification arises when the $SU(2)_R$ automorphisms of N=2 supersymmetry are also preserved, which implies that the hyper-Kähler potential is independent upon harmonics. Since $SU(2)_R$ is known to be non-anomalous [11, 12], the most general ‘Ansatz’ for the hypermultiplet LEEA takes the form of a real quartic polynomial,

$$\mathcal{K}^{(+4)} = \frac{1}{2} (\sqrt{q}^+)^2 (q^+)^2 + \left[ \gamma \ (q^+)^4 + \beta \ (q^+)^3 q^+ + h.c. \right],$$

(2.2)
with one real ($\lambda$) and two complex ($\beta, \gamma$) parameters. The $Sp(1)_{PG}$ internal symmetry of the free hypermultipet action leaves the form of the quartic (2.2) invariant but not the coefficients. Hence, $Sp(1)_{PG}$ can be used to reduce the number of coupling constants in the family of hyper-K"ahler metrics associated with the hyper-K"ahler potential (2.2) from five to two. Equations (2.1) and (2.2) also imply the conservation law

$$D^{++} K^{(+4)} = 0 \quad (2.3)$$
on the equations of motion, $D^{++} \bar{q}^+ = \partial K^{(+4)}/\partial q^+$ and $D^{++} q^+ = -\partial K^{(+4)}/\partial \bar{q}^+$. 

### 3 Perturbative hypermultiplet LEEA

The manifestly N=2 supersymmetric HSS description of the hypermultiplet LEEA allows us to exploit the constraints imposed by unbroken N=2 supersymmetry and its automorphism symmetry in the very efficient and transparent way. For example, as regards a perturbation theory in 4d, N=2 supersymmetric QED (or in the Coulomb branch of N=2 supersymmetric $SU(2)$ Yang-Mills theory [1]), the unbroken symmetry is given by

$$SU(2)_{R, \text{ global}} \times U(1)_{\text{local}}. \quad (3.1)$$

The unique hypermultiplet self-interaction consistent with N=2 supersymmetry and the internal symmetry (3.1) in HSS is described by the hyper-K"ahler potential

$$K_{TN}^{(+4)} = \frac{\lambda}{2} \left( \frac{\bar{q}^+ + q^+}{2} \right)^2, \quad (3.2)$$

just because it is the only function of $U(1)$ charge (+4) that is independent upon harmonics and invariant under $U(1)_{\text{local}}$.

The coupling constant $\lambda$ in eq. (3.2) (in the one-loop approximation) is determined by the HSS graph shown in Fig. 1. The analytic propagator (wave lines in Fig. 1) of the N=2 vector superfield (in N=2 Feynman gauge) is (see [10] for details)

$$i \langle V^{++}(1)V^{++}(2) \rangle = \frac{1}{\Box_1} (D_1^+)^4 \delta^{12}(Z_1 - Z_2) \delta^{-2,2}(u_1, u_2). \quad (3.3)$$

The hypermultiplet analytic propagator (solid lines in Fig. 1) with non-vanishing central charges (in the pseudo-real notation) reads [10]

$$i \langle q^+(1)q^+(2) \rangle = \frac{-1}{\Box_1 + m^2} \frac{(D_1^+)^4(D_2^+)^4}{(u_1^+ u_2^+)^3} e^{\tau_3 [\nu(2) - \nu(1)]} \delta^{12}(Z_1 - Z_2), \quad (3.4)$$
Fig. 1. The one-loop harmonic supergraph contributing to the induced hypermultiplet self-interaction.

where $m^2 = |Z|^2$ is the hypermultiplet (bare) BPS mass and

$$ iv = -Z(\tilde{\theta}^+ \tilde{\theta}^-) - \bar{Z}(\theta^+ \theta^-). $$

In the one-loop approximation one finds [10] the predicted form (3.2) with

$$ \lambda = \frac{g^4}{\pi^2} \left[ \frac{1}{m^2} \ln \left(1 + \frac{m^2}{\Lambda^2}\right) - \frac{1}{\Lambda^2 + m^2} \right] $$

in terms of the gauge coupling constant $g$, the BPS mass $m^2$ and the IR-cutoff $\Lambda$. Note that $\lambda \neq 0$ only when $Z \neq 0$. The dependence of $\lambda$ upon the IR-cutoff is expected to disappear after summing up all contributions from higher loops.

To understand the hyper-Kähler geometry associated with the hyper-Kähler potential (3.2), it is convenient to rewrite the HSS action into PSS, by going partially on-shell, in terms of an N=2 tensor multiplet. Unlike the Fayet-Sohnius hypermultiplet, the N=2 tensor superfield $L^{(ij)}$ has a finite number of auxiliary fields. In the standard N=2 superspace ($Z$) the $L^{(ij)}$ is defined by the off-shell constraints

$$ D_\alpha^{(ik)} L^{jk} = \bar{D}_\alpha^{(ij)} L^{jk} = 0, $$

and the reality condition

$$ L^{ij} = \varepsilon_{ik} \varepsilon_{jl} L^{kl}. $$
It is not difficult to verify that eq. (3.7) implies

\[ \nabla_\alpha G \equiv (D^1_\alpha + \xi D^2_\alpha)G = 0 , \quad \Delta^{\bullet}_\alpha G \equiv (\bar{D}^1_\alpha + \xi \bar{D}^2_\alpha)G = 0 , \quad (3.9) \]

for any function \( G(Q^2_2(\xi), \xi) \) depending upon \( Q^2_2(\xi) \equiv \xi \xi_j L^{ij}(Z) \), \( \xi_j \equiv (1, \xi) \),

\[ Q^2_2(\xi) \equiv \xi_i \xi_j L^{ij}(Z) , \quad \xi_i \equiv (1, \xi) , \quad (3.10) \]

where the \( CP(1) \) inhomogeneous coordinate \( \xi \) has been introduced. It follows from eq. (3.9) that one can construct \( N=2 \) invariant actions by integrating the potential \( G(Q^2_2(\xi), \xi) \) over the rest of the \( N=2 \) superspace coordinates [7],

\[ S = \int d^4x \frac{1}{2\pi i} \oint_C \frac{d\xi}{(1 + \xi^2)^2} \bar{\nabla}^2 \bar{\Delta}^2 G(Q^2_2(\xi), \xi) \text{ h.c.} , \quad (3.11) \]

where the new Grassmann superspace derivatives,

\[ \bar{\nabla}_\alpha = \xi D^1_\alpha - D^2_\alpha , \quad \bar{\Delta}^{\bullet}_\alpha = \xi \bar{D}^1_\alpha - \bar{D}^2_\alpha , \quad (3.12) \]

‘orthogonal’ to those of eq. (3.9), have been introduced. After being reduced to 4d, \( N=1 \) superspace, eqs. (3.10) and (3.11) take the form

\[ Q^2_2(\xi) \big| = \Phi + \xi H - \xi^2 \Phi , \quad (3.13) \]

and

\[ S = \int d^4x d^4\theta \frac{1}{2\pi i} \oint_C \frac{d\xi}{\xi^2} G(Q^2_2(\xi) \big| , \xi) \text{ h.c.} , \quad (3.14) \]

in terms of the \( N=1 \) chiral superfield \( \Phi \) and the \( N=1 \) real linear superfield \( H \).

The \( N=2 \) tensor multiplet constraints (3.7) and (3.8) read in HSS as

\[ D^{++} L^{++} = 0 \quad \text{and} \quad \mathbf{L}^{++} = L^{++} , \quad (3.15) \]

respectively, where \( L^{++} = u_i^+ u_j^+ L^{ij}(Z) \). Let’s substitute (we temporarily set \( \lambda = 1 \))

\[ K^{(++)}_{TN} = \frac{1}{2} \left( \bar{q}^+ + q^+ \right)^2 = -2(L^{++})^2 , \quad \text{or, equivalently,} \quad \bar{q}^+ q^+ = 2iL^{++} , \quad (3.16) \]

which is certainly allowed because of eq. (2.3). The constraints (3.7) can be incorporated off-shell by using extra real analytic superfield \( \omega \) as the Lagrange multiplier. Changing the variables from \( (\bar{q}^+, q^+) \) to \( (L^{++}, \omega) \) amounts to an \( N=2 \) duality transformation in HSS. The explicit solution to eq. (3.16) reads

\[ q^+ = -i \left( 2u^+_i + if^{++} u^-_1 \right) e^{-i\omega/2} , \quad \bar{q}^+ = i \left( 2u^+_2 - if^{++} u^-_2 \right) e^{i\omega/2} , \quad (3.17) \]
where the function $f^{++}$ is given by

$$f^{++}(L, u) = \frac{2(L^{++} - 2i u_1^+ u_2^+)}{1 + \sqrt{1 - 4u_1^+ u_2^+ u_1^- u_2^- - 2iL^{++} u_1^- u_2^-}} . \quad (3.18)$$

It is straightforward to rewrite the free (massless) HSS action (2.7) in terms of the new variables. It results in the improved (i.e. N=2 superconformally invariant) N=2 tensor multiplet action

$$S_{\text{impr.}} = \frac{1}{2} \int d\zeta (-4) du (f^{++})^2 . \quad (3.19)$$

The action dual to the NLSM action defined by eqs. (2.1) and (3.2) is thus given by a sum of the non-improved (quadratic) and improved (non-polynomial) HSS actions for the N=2 tensor multiplet [11, 12],

$$S_{TN}[L; \omega] = S_{\text{impr.}} + \frac{1}{2} \int d\zeta (-4) du \left[(L^{++})^2 + \omega D^{++} L^{++}\right] . \quad (3.20)$$

The equivalent PSS action is given by eq. (3.11) with

$$\oint G = M \oint_{C_0} \frac{(Q_2^2)^2}{2\xi} + \oint_{C_r} Q_2^2 (\ln Q_2^2 - 1) , \quad (3.21)$$

where we have restored the dependence upon $\lambda$ by setting $M = \frac{1}{2} \lambda^{-1/2}$. The contour $C_0$ in complex $\xi$-plane goes around the origin, whereas the contour $C_r$ encircles the roots of the quadratic equation

$$Q_2^2(\xi) = 0 . \quad (3.22)$$

The hyper-Kähler metric of the N=2 NLSM defined by eqs. (3.21) and (3.22) is equivalent to the Taub-NUT metric with the mass parameter $M = \frac{1}{2} \lambda^{-1/2}$ [4, 9].

The $SU(2)_R$ transformations act in PSS in the form of projective (fractional) transformations [12]

$$\xi' = \frac{\bar{a}\xi - \bar{b}}{a + b\xi} , \quad |a|^2 + |b|^2 = 1 , \quad (3.23)$$

while a generic PSS action (3.11) is not invariant under these transformations. Nevertheless, eq. (3.11) with some non-trivial contour $C_r$ is going to be invariant under the transformations (3.23) provided that

$$G(Q_2^2(\xi'), \xi') = \frac{1}{(a+b\xi)^2} G(Q_2^2(\xi), \xi) , \quad \text{where} \quad Q_2^2(\xi') = \frac{1}{(a+b\xi)^2} Q_2^2(\xi) . \quad (3.24)$$

Eq. (3.24) implies that the invariant PSS potential $G(Q_2^2)$ should be ‘almost’ linear in $Q_2^2$, like in the second term of the action (3.21). The transition $u_i \to \xi_i = (1, \xi)$ describes the holomorphic projection of HSS on PSS, where the analytic superfield $L^{++}(\zeta, u)$ is replaced by the holomorphic (with respect to $\xi$) section $Q_2^2(L, \xi)$ of the line bundle $O(2)$ whose fiber is parametrized by constrained superfields. The equation (3.13) defines the Riemann sphere in $\mathbb{C}^2$ parametrized by $(Q_2, \xi)$. 

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4 Exact hypermultiplet LEEA and $O(4)$ bundle

In an abelian quantum field theory there are no instantons, so that the one-loop results of sect. 3 are, in fact, exact in that case. If, however, the underlying N=2 gauge field theory has a non-abelian gauge group whose rank is more than one, one expects nonperturbative contributions to the LEEA of a single (magnetically charged) hypermultiplet from instantons and anti-instantons [13]. It may happen, e.g., in the Higgs branch where the gauge symmetry is completely broken.

Given the most general $SU(2)_R$-invariant hyper-Kähler potential (2.2), let’s make a substitution

$$K^{(+4)}(q, \bar{q}) \equiv \frac{1}{2}(\bar{q}^+ q^+)^2 (q^+)^2 + \left[ \gamma \left( q^+ \right)^4 + \beta \left( q^+ \right)^3 q^+ + \text{h.c.} \right] = L^{++++}(\zeta, u) \ , \quad (4.1)$$

where the real analytic superfield $L^{++++}$ satisfies the conservation law (2.3),

$$D^{++}L^{++++} = 0 \ . \quad (4.2)$$

Eq. (4.2) can be recognized as the off-shell N=2 (standard) superspace constraints

$$D_\alpha (i L^{jklm}) = \bar{D}_\alpha (i L^{jklm}) = 0 \ , \quad (4.3)$$

where $L^{++++} = u^+_i u^+_j u^+_k u^+_l L^{ijkl}(Z)$, while eq. (4.1) implies the reality condition

$$\bar{L}^{ijkl} = \varepsilon_{imn} \varepsilon_{jkl} \varepsilon_{pq} L^{mnop} \ , \quad (4.4)$$

defining together the $O(4)$ projective multiplet [14]. Unlike the $O(2)$ tensor supermultiplet (sect. 3), the $O(4)$ supermultiplet does not have a conserved vector (or a gauge antisymmetric tensor) amongst its field components.

The N=2 invariant PSS action construction (3.11) in terms of a PSS potential $G(Q_4^2(\xi), \xi)$ equally applies to the projective $O(4)$ supermultiplets, while $L^{ijkl}$ should enter the action via the argument [12]

$$Q_4^2(\xi) = \xi_i \xi_j \xi_k \xi_l L^{ijkl}(Z) \ , \quad (4.5)$$

The N=1 superspace projections of the N=2 superfield (4.5) and the N=2 invariant PSS action are given by

$$Q_4^2(\xi) = \Phi + \xi H + \xi^2 V - \xi^3 \bar{H} + \xi^4 \bar{\Phi} \ , \quad (4.6)$$

4In fact, it also applies to any projective $O(2k)$ multiplets satisfying the off-shell N=2 superspace constraints generalizing those of eqs. (4.3) and (4.4) with $k > 2$ [12].
and
\[ S = \int d^4x d^4\theta \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{d\xi}{\xi^2} G(Q^2_4(\xi)) \cdot \xi + \text{h.c.} \tag{4.7} \]
respectively, in terms of the N=1 chiral superfield \( \Phi \), the N=1 complex linear superfield \( H \), and the N=1 general (unconstrained) real superfield \( V \) \[7, 12\].

The N=1 superfield \( V \) enters the action (4.7) as the Lagrange multiplier, whose elimination via its ‘equation of motion’ implies the algebraic constraint \[12\]
\[ \text{Re} \oint \frac{\partial G}{\partial Q^2_4} = 0 . \tag{4.8} \]
Eq. (4.8) reduces the number of independent N=2 NLSM physical real scalars from five to four, which is consistent with the well-known fact that the real dimension of any hyper-Kähler manifold is a multiple of four \[7\]. After solving the constraint (4.8), the complex linear N=1 superfield \( H \) can be traded for yet another N=1 chiral superfield \( \Psi \), by the use of the N=1 superfield Legendre transform that results in the N=1 superspace Kähler potential \( K(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) \) associated with the N=2 supersymmetric NLSM of eq. (4.7) \[7\].

The most straightforward procedure of calculating the dependence \( q(L) \), as well as performing an explicit N=2 transformation of the unconstrained HSS action into the PSS action in terms of the constrained N=2 superfield defined by eq. (4.1), use roots of the quartic polynomial. Remarkably, the N=2 PSS action in question can be fixed without calculating the roots in the manifestly N=2 supersymmetric approach.

It is the \( SU(2)_R \) invariance that is powerful enough to fix the PSS action equivalent to the HSS action of eqs. (2.1) and (2.2) (cf. ref. \[14\]). The one real and two complex constants, \( (\lambda, \beta, \gamma) \), respectively, parametrizing the hyper-Kähler potential (2.2), are naturally united into the \( SU(2) \) 5-plet \( c^{ijkl} \) subject to the reality condition (4.4). After extracting a constant piece out of \( q^+ \), say, \( q_+^a = u_+^a + \tilde{q}_+^a \) and \( u_a = (1, \xi) \), and collecting all constant pieces on the left-hand-side of eq. (4.1), we can identify their sum with a constant piece \( c^{++++} = c^{ijkl} u_i^+ u_j^+ u_k^+ u_l^+ \) of \( L^{++++} \) on the right-hand-side of eq. (4.1), representing the constant vacuum expectation values of the N=1 superfield components of \( L^{++++} \) defined by eq. (4.6), i.e.
\[ \lambda = \langle V \rangle , \quad \beta = \langle H \rangle , \quad \gamma = \langle \Phi \rangle . \tag{4.9} \]

The \( SU(2)_R \) transformations in PSS are the projective transformations (3.24), so that the PSS potential \( G \) of the ‘improved’ \( O(4) \) multiplet action having the form (3.11) must be proportional to \( Q_4 \equiv \sqrt{Q^{2}_4} \) because of the relations
\[ G(Q_4^{2}(\xi'), \xi') = \frac{1}{(a + b\xi)^2} G(Q^{2}_4(\xi), \xi) \] and \( Q_4^{2}(\xi') = \frac{1}{(a + b\xi)^4} Q^{2}_4(\xi) \). \tag{4.10}
The most general non-trivial contour $C_r$ in complex $\xi$-plane, whose definition is compatible with the projective $SU(2)$ symmetry, is the one encircling the roots of the quartic (cf. sect. 3),

$$Q_4^2(\xi) = p + \xi q + \xi^2 r - \xi^3 \bar{q} + \xi^4 \bar{p}, \quad (4.11)$$

with one real ($r$) and two complex ($p, q$) additional parameters belonging to yet another 5-plet of $SU(2)$. The projective $SU(2)$ invariance of the PSS action defined by eqs. (4.7) and (4.11) can be used to reduce the number of independent parameters in the corresponding family of hyper-Kähler metrics from five to two, which is consistent with the HSS predictions of sect. 2. We didn’t attempt to establish an explicit relation between the HSS coefficients ($\lambda, \gamma, \beta$) and the PSS coefficients ($r, q, p$). The most natural (non-trivial) contour $C_r$ surrounds the roots of the equation

$$Q_4^2(\xi) = 0, \quad (4.12)$$

and it leads to the only non-singular hyper-Kähler NLSM metric (sect. 5).

The $SU(2)$-invariant PSS action, equivalent to the one defined by eqs. (2.1) and (4.1), is therefore given by

$$\frac{1}{2\pi i} \oint G = -\frac{1}{2\pi i} \oint_{C_0} \frac{Q_4^2}{\xi} + \oint_{C_r} Q_4. \quad (4.13)$$

The constraint (4.8) in the case (4.12) takes the form

$$\oint_{C_r} \frac{d\xi}{\sqrt{Q_4^2}} = 1. \quad (4.14)$$

The component form of the metric associated with eqs. (4.13) and (4.14) was found in ref. [14]. Because of the reality condition (4.4), the quartic (4.12) has two pairs of roots ($\rho, -1/\bar{\rho}$) related by an $SL(2, \mathbb{Z})$ transformation and satisfying the defining relation

$$Q_4^2(\xi) = c(\xi - \rho_1)(\bar{\rho}_1\xi + 1)(\xi - \rho_2)(\bar{\rho}_2\xi + 1). \quad (4.15)$$

The branch cuts of the root in eq. (4.14) can be chosen to run from $\rho_1$ to $-1/\bar{\rho}_2$ and from $\rho_2$ to $-1/\bar{\rho}_1$. The contour integration in eq. (4.14) can thus be reduced to the complete elliptic integral (in the Legendre normal form) over the branch cut [14],

$$\frac{4}{\sqrt{c(1 + |\rho_1|^2)(1 + |\rho_2|^2)}} \int_0^1 \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)}} = 1, \quad (4.16)$$

5The generalization of eq. (3.22) similarly to eq. (4.11) is ‘empty’ since the quadratic polynomial $c_2^2(\xi) = p + \xi r - \xi^2 \bar{p}$ can always be removed by an $SU(2)$ transformation.
with the modulus
\[ k^2 = \frac{(1 + \rho_1 \rho_2)(1 + \rho_2 \rho_1)}{(1 + |\rho_1|^2)(1 + |\rho_2|^2)} \quad (4.17) \]
The constraint (4.16) can be explicitly solved in terms of the complete elliptic integrals,
\[ K(k) = \int_0^{\pi/2} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}} \quad \text{and} \quad E(k) = \int_0^{\pi/2} d\gamma \sqrt{1 - k^2 \sin^2 \gamma} \quad (4.18) \]
of the first and second kind, respectively, by using the following parametrization \[14\]:
\[ \Phi = 2e^{2i\varphi} \left[ \cos(2\psi)(1 + \cos^2 \vartheta) + 2i \sin(2\psi) \cos \vartheta + (2k^2 - 1) \sin^2 \vartheta \right] K^2(k) , \]
\[ H = 8e^{i\varphi} \sin \vartheta \left[ \sin(2\psi) - i \cos(2\psi) \cos \vartheta + i(2k^2 - 1) \cos^2 \vartheta \right] K^2(k) , \]
\[ V = 4 \left[ -3 \cos(2\psi) \sin^2 \vartheta + (2k^2 - 1)(1 - 3 \cos^2 \vartheta) \right] K^2(k) , \]
in terms of the Euler ‘angles’ \((\vartheta, \psi, \varphi)\) and the modulus \(k\) representing the independent (superfield) coordinates of the \(N=2\) NLSM under consideration. Applying the generalized Legendre transform \[14\] to the function (4.13) with respect to \(H\) gives rise to the Atiyah-Hitchin (AH) metric \[5\]
\[ ds^2_{\text{AH}} = \frac{1}{4} A^2 B^2 C^2 \left( \frac{dk}{kk'K^2} \right)^2 + A^2(k)\sigma_1^2 + B^2(k)\sigma_2^2 + C^2(k)\sigma_3^2 , \quad (4.20) \]
whose coefficient functions satisfy the relations \[5\]
\[ AB = -K(k) \left[ E(k) - K(k) \right] , \]
\[ BC = -K(k) \left[ E(k) - k'^2 K(k) \right] , \quad (4.21) \]
\[ AC = -K(k) E(k) . \]
while \(\sigma_i\) stand for the \(SO(3)\)-invariant one-forms
\[ \sigma_1 = +\frac{1}{2} (\sin \psi d\vartheta - \sin \vartheta \cos \psi d\varphi) , \]
\[ \sigma_2 = -\frac{1}{2} (\cos \psi d\vartheta + \sin \vartheta \sin \psi d\varphi) , \quad (4.22) \]
\[ \sigma_3 = +\frac{1}{2} (d\psi + \cos \vartheta d\varphi) , \]
and \(k'\) is known as the complementary modulus, \(k'^2 = 1 - k^2\).

In the limit \(k \to 1\) (or, equivalently, \(k' \to 0\), one has an asymptotic expansion
\[ K(k) \approx -\log k' \left[ 1 + \frac{(k')^2}{4} \right] + \ldots \quad (4.23) \]
Eq. (4.23) suggests us to make a redefinition
\[
k' = \sqrt{1 - k^2} \approx 4 \exp \left( \frac{1}{\gamma} \right),
\] (4.24)
and describe the same limit at $\gamma \to 0^-$. After substituting eq. (4.23) into eq. (4.21) one finds that the AH metric becomes exponentially close to the Taub-NUT metric in the form (4.20) subject to the additional relations:
\[
A^2 \approx B^2 \approx \frac{1 + \gamma}{\gamma^2}, \quad C^2 \approx \frac{1}{1 + \gamma}.
\] (4.25)
The extra $U(1)$ symmetry of the Taub-NUT metric is the direct consequence of the relation $A^2 = B^2$ arising from the AH metric in the asymptotic limit described by eq. (4.25). The vicinity of $k' \approx 0^+$ describes the region of the hypermultiplet moduli space where quantum perturbation theory applies, with the exponentially small AH corrections to the Taub-NUT metric being interpreted as the one-instanton and anti-instanton contributions to the hypermultiplet LEEA [3, 4].

From the N=2 PSS viewpoint, the transition from the perturbative hypermultiplet LEEA to the nonperturbative one corresponds to the transition from the $O(2)$ holomorphic line bundle associated with the standard N=2 tensor supermultiplet to the $O(4)$ holomorphic line bundle associated with the $O(4)$ N=2 supermultiplet. The two holomorphic bundles are topologically different: with respect to the standard covering of $CP(1)$ by two open affine sets, the $O(2)$ bundle has transition functions $\xi^{-1}$, whereas the $O(4)$ bundle has transition functions $\xi^{-2}$. The variable $Q$ is the coordinate of the corresponding fiber over $CP(1)$.

### 5 Atiyah-Hitchin metric and elliptic curve

The quadratic dependence of $Q_2$ on $\xi$ in eqs. (3.10) and (3.13) allows us to globally interpret it as a holomorphic (of degree 2) section of PSS, fibered by the superfields $(\Phi, H)$ and topologically equivalent to a complex line (or Riemann sphere of genus 0). Similarly, the quartic dependence of $Q_4$ on $\xi$ in eqs. (4.5) and (4.6) allows us to globally interpret it as a holomorphic (of degree 4) section of PSS, fibered by the superfields $(\Phi, H, V)$ and topologically equivalent to an elliptic curve $\Sigma_{\text{hyper}}$ (or a torus of genus 1). The non-perturbative hypermultiplet LEEA can therefore be encoded in terms of the genus-one Riemann surface $\Sigma_{\text{hyper}}$, in close analogy to the exact N=2 gauge LEEA in terms of the elliptic curve $\Sigma_{\text{SW}}$ of Seiberg and Witten [1].

The classical twistor construction of hyper-Kähler metrics [3] is known to be closely related to the *Hurtubise* elliptic curve $\Sigma_{\text{H}}$ [15]. This curve can be identified with $\Sigma_{\text{hyper}}$. 
that carries the same information and whose defining eq. (4.6) can be put into the Hurtubise form,

\[ \tilde{Q}_4^2(\tilde{\xi}) = K^2(k)\tilde{\xi} \left[ kk'(\tilde{\xi}^2 - 1) + (k^2 - k'^2)\tilde{\xi} \right], \]

by a projective \( SU(2) \) transformation. In its turn, eq. (5.1) is simply related (by a linear transformation) to another standard (Weierstrass) form, \( y^2 = 4x^3 - g_2x - g_3 \). Therefore, in accordance with ref. [5], the real period \( \omega \) of \( \Sigma_H \) is

\[ \omega \equiv 4k_1, \quad \text{where} \quad 4k_1^2 = kk'K^2(k), \]

whereas the complex period matrix of \( \Sigma_H \) is given by

\[ \tau = \frac{iK(k')}{K(k)}. \]

At generic values of the AH modulus \( k, 0 < k < 1 \), the roots of the Weierstrass form are all different from each other, while they all lie on the real axis, say, at \( e_3 < e_2 < e_1 < \infty = (e_4) \). Accordingly, the branch cuts are running from \( e_3 \) to \( e_2 \) and from \( e_1 \) to \( \infty \). The \( C_r \) integration contour in the PSS formulation of the exact hypermultiplet LEEA in eq. (4.13) can now be interpreted as the contour integral over the non-contractible \( \alpha \)-cycle of the elliptic curve \( \Sigma_H \) [13], again in the very similar way as the Seiberg-Witten solution to the \( SU(2) \)-based \( N=2 \) gauge LEEA is written down in terms of the abelian differential \( \lambda_{SW} \) integrated over the periods of \( \Sigma_{SW} \) [1]. The most general (non-trivial) integration contour \( C_r \) in eq. (4.13) is given by a linear combination of the non-contractible \( \alpha \) and \( \beta \) cycles of \( \Sigma_H \), while an integration over \( \beta \) is known to lead to a singularity [5]. This simple observation implies that the AH metric is the only regular solution.

The perturbative (Taub-NUT) limit \( k \to 1 \) corresponds to the situation when \( e_2 \to e_1 \), so that the \( \beta \)-cycle of \( \Sigma_H \) degenerates. The curve (5.1) then asymptotically approaches a complex line, \( \tilde{Q}_4 \sim \pm K\tilde{\xi} \). Another limit, \( k \to 0 \), leads to a (coordinate) bolt-type singularity of the AH metric in the standard parameterization (4.20) [5]. In the context of monopole physics, this corresponds to the coincidence limit of two centered monopoles. In the context of the hypermultiplet LEEA, \( k \to 0 \) implies \( e_2 \to e_3 \), so that the \( \alpha \)-cycle of \( \Sigma_H \) degenerates, as well as the whole hypermultiplet action associated with eq. (4.13). The two limits, \( k \to 1 \) and \( k \to 0 \), are related by the modular transformation exchanging \( k \) with \( k' \), and \( \alpha \)-cycle with \( \beta \)-cycle [14]. The non-perturbative corrections to the hypermultiplet LEEA are therefore dictated by the hidden (in 4d) elliptic curve parametrizing the exact solution.
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