REPRESENTATIONS OF THE EXCEPTIONAL LIE SUPERALGEBRA

E(3,6): I. DEGENERACY CONDITIONS.

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Abstract. Recently one of the authors obtained a classification of simple infinite-dimensional Lie superalgebras of vector fields which extends the well-known classification of E. Cartan in the Lie algebra case. The list consists of many series defined by simple equations, and of several exceptional superalgebras, among them E(3,6).

In the article we study irreducible representations of the exceptional Lie superalgebra E(3,6). This superalgebra has \( sl(3) \times sl(2) \times gl(1) \) as the zero degree component of its consistent \( \mathbb{Z} \)-grading which leads us to believe that its representation theory has potential for physical applications.

0. Introduction.

Recently V. Kac obtained the classification of infinite-dimensional simple linearly compact Lie superalgebras [K1]. Two of the exceptional algebras, E(3,6) and E(3,8) have the Lie algebra \( sl(3) \times sl(2) \times gl(1) \) as the zero degree component \( g_0 \) in their consistent \( \mathbb{Z} \)-grading. This points to the potential physical applications of representations of these algebras.

We deal with representations of E(3,6) in this article having the main objective to classify and describe irreducible representations.

We follow the approach developed for representations of infinite-dimensional simple linearly compact Lie algebras by A. Rudakov in [R]. The problem reduces quite quickly (Proposition 1.3) to the description of the so-called degenerate modules, and for the latter we have to study singular vectors and secondary singular vectors (see definitions in Section 1).

In this first article on the topic we get an important restriction on the list of degenerate irreducible representations of E(3,6) (Theorem 1.1).

In order to obtain the complete list of these representations and to get a hold on their construction and structure more work is to be done. We describe it in the subsequent articles. In particular, it turns out that the degenerate irreducible representations of E(3,6) fall into four series, and we construct four complexes of E(3,6)-modules which lead to an explicit description of these series.

Let us mention that the existence of exceptional superalgebras was announced by I. Shchepochkina [S1] in 1983, but her construction is implicit and quite difficult to use (see [S2]). We rely on the explicit construction of E(3,6) found by S.J. Cheng and V. Kac ([K1, CK1]).

For the related mathematical development see [K2, K3, CK2]. Basic properties of superalgebras can be found for example in [K4].

All vector spaces, linear maps and tensor products are considered over \( \mathbb{C} \).

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1. General remarks on representations of linearly compact Lie superalgebras.

Representation theory of infinite-dimensional Lie superalgebras was initiated by A. Rudakov some 25 years ago. We will follow the same approach in the Lie superalgebra case. It is a worthwhile undertaking because the list of linearly compact infinite-dimensional simple Lie superalgebras and their irreducible modules turns out to be much richer than in the Lie algebra case and because of their potential applications to quantum physics.

It is most natural to consider continuous representations of linearly compact Lie algebras in linearly compact topological spaces. However, technically it is more convenient to work with the contragredient to these, which are continuous representations in spaces with discrete topology. The continuity of a representation of a linearly compact Lie superalgebra \( L \) in a vector space \( V \) with discrete topology means that the stabilizer \( L_v = \{ g \in L \mid gv = 0 \} \) of any \( v \in V \) is an open (hence of finite codimension) subalgebra of \( L \).

Let \( L_0 \) be an open subalgebra of \( L \). In order to avoid pathological examples we shall always assume that the \( L \)-module \( V \) is \( L_0 \)-locally finite, meaning that any \( v \in V \) is contained in a finite-dimensional \( L_0 \)-invariant subspace. We shall denote by \( \mathcal{P}(L, L_0) \) the category of all continuous \( L \)-modules \( V \), where \( V \) is a vector space with discrete topology, that are \( L_0 \)-locally finite. When talking about representations of \( L \), we shall always mean modules from \( \mathcal{P}(L, L_0) \) (after a suitable choice of \( L_0 \)), unless otherwise is stated.

Suppose that \( L \) is a simple infinite-dimensional linearly compact Lie superalgebra. It is known [K1] that \( L \) has a maximal open subalgebra \( L_0 \) called a maximally even subalgebra which contains all even exponentiable elements of \( L \) (in most of the cases \( L \) has a unique such subalgebra). If \( L_0' \) is another open subalgebra of \( L \), then, since all even elements of \( L_0' \) are exponentiable [K1], we conclude that the even part of \( L_0' \) lies in \( L_0 \). It follows that

\[ \mathcal{P}(L, L_0') \supset \mathcal{P}(L, L_0), \]

and therefore any open subalgebra of \( L \) acts locally finitely on modules from \( \mathcal{P}(L, L_0) \).

Let \( L_- \) be a complementary to \( L_0 \) (finite-dimensional) subspace in \( L \). In most (but not all cases) of simple \( L \) and maximally even \( L_0 \) one can choose \( L_- \) to be a subalgebra. Choosing an ordered basis of \( L_- \) we denote by \( U(L_-) \) the span of all PBW monomials in this basis. We have: \( U(L) = U(L_-) \otimes U(L_0) \), a vector space tensor product. (Here and further \( U(L) \) stands for the universal enveloping algebra of the Lie algebra \( L \).) It follows that any irreducible \( L \)-module \( V \) from the category \( \mathcal{P}(L, L_0) \) is finitely generated over \( U(L_-) \):

\[ V = U(L_-)E \]

for some finite-dimensional subspace \( E \). This last property is very important in the theory of conformal modules [CK2], [K2].

Let \( V \) be an \( L \)-module from the category \( \mathcal{P}(L, L_0) \). Denote by Sing \( V \) the sum of all irreducible \( L_0 \)-submodules of \( V \). This subspace is clearly different from zero. Its vectors are called singular vectors of the \( L \)-module \( V \).

Given an \( L_0 \)-module \( F \), we may consider the associated induced \( L \)-module

\[ M(F) = \text{Ind}_{L_0}^L F = U(L) \otimes_{U(L_0)} F, \]

called also the universal \( L \)-module (associated to \( F \)). Other names used for these kinds of modules are: generalized Verma modules, Weyl modules, etc.

The \( L_0 \)-module \( F \) is canonically an \( L_0 \)-submodule of \( M(F) \), and the sum of its irreducible submodules, that is Sing \( F \), is a subspace of Sing \( M(F) \), called the subspace of trivial singular vectors.
Let us mention that if $F$ is finite-dimensional then being a continuous $L_0$-module it is annihilated by an open ideal $I$ of $L_0$, so, in fact, in this case $F$ is a module over a finite-dimensional Lie superalgebra $L_0/I$.

The following proposition is standard.

**Proposition 1.1.** (a) A finite-dimensional $L_0$-module $F$ is continuous if and only if $\text{Ann}F = \{g \in L_0| gF = 0\}$ is an open ideal of $L_0$.

(b) If $L$ has a filtration by open subalgebras: $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ and $F$ is a continuous finite-dimensional $L_0$-module, then the $L$-module $M(F)$ lies in $\mathcal{P}(L, L_0)$.

**Proof.** (a) is trivial. Hence, if $F$ is a continuous finite-dimensional $L_0$-module, we have: $L_j F = 0$ for $j \gg 0$. Note that $M(F) = U(L_-)F$, hence we can make an increasing filtration of $M(F)$ by finite-dimensional subspaces:

$$F \subset L_- F + F \subset L_-^2 F + L_- F + F \subset \cdots.$$ 

But, clearly, each member of this filtration is annihilated by $L_j$ for $j \gg 0$, which proves the continuity of the $L$-module $M(F)$. Since $\dim L_- < \infty$, we conclude also that $M(F)$ is $L_0$-locally finite, proving (b). □

**Definition 1.2.** An irreducible $L$-module $V$ is called non-degenerate if $V = M(F)$ for an irreducible $L_0$-module $F$. We often call $F$ and $M(F)$ with this property non-degenerate as well.

In many interesting cases $L$ has an element $Y$ with the following properties:

(i) ad $Y$ is diagonalizable,

(ii) the spectrum of ad $Y$ is real and discrete and eigenspaces are finite-dimensional,

(iii) the number of negative eigenvalues is finite.

Such an element is called a hypercharge operator. It defines the triangular decomposition:

$$L = L_- + g_0 + L_+,$$

where $L_-$ is the sum of eigenspaces of ad $Y$ with negative eigenvalues, $g_0$ is the 0-th eigenspace and $L_+$ is the product of eigenspaces with positive eigenvalues.

We let $L_0 = g_0 + L_+$. Both $L_+$ and $L_0$ are open subalgebras of $L$.

Suppose that $Y|_F$ is a scalar operator (which is true if $F$ is finite-dimensional irreducible $L_0$-module). An eigenvector of $Y$ in Sing $M(F) \setminus \text{Sing}$ $F$ is called a non-trivial singular vector of $M(F)$. Denote by $V(F)$ the quotient of the $L$-module $M(F)$ by the submodule generated by all non-trivial singular vectors, that are eigenvectors of $Y$. Their eigenvalues are necessarily different from those of trivial singular eigenvectors so the map of $F$ to $V(F)$ is injective. We will often identify $F$ with its image in $M(F)$ or $V(F)$ depending on the module under consideration.

Clearly $V(F)$ could be irreducible even if $M(F)$ is not, which often happens when $M(F)$ is degenerate, but not always. To study this we are to look at singular vectors in $V(F)$.

Elements of Sing $V(F)$ are called secondary singular vectors. The image of Sing $F \subset F$ in $V(F)$ lies in Sing $V(F)$ and is called the subspace of trivial secondary singular vectors.

**Proposition 1.3.** Let $L$ be a linearly compact Lie superalgebra with a hypercharge operator $Y$. Then

(a) Any finite-dimensional $L_0$-module $F$ is continuous.

(b) If $F$ is a finite-dimensional $L_0$-module, then $M(F)$ is in $\mathcal{P}(L, L_0)$.

(c) In any irreducible finite-dimensional $L_0$-module $F$ the subalgebra $L_+$ acts trivially.
(d) If \( F \) is an irreducible finite-dimensional \( L_0 \)-module, then \( M(F) \) has a unique maximal submodule.

(e) Denote by \( I(F) \) the quotient by the unique maximal submodule of \( M(F) \). The map \( F \mapsto I(F) \) defines a bijective correspondence between irreducible finite-dimensional \( g_0 \)-modules and irreducible \( L \)-modules from \( \mathcal{P}(L, L_0) \), the inverse map being \( V \mapsto \text{Sing } V \).

(f) The \( L \)-module \( M(F) \) is irreducible if and only if the \( L_0 \)-module \( F \) is irreducible and \( \text{Sing } M(F) = \emptyset \).

(g) If the finite-dimensional \( L_0 \)-module \( F \) is irreducible, and all its secondary singular vectors are trivial, then the \( L \)-module \( V(F) \) is irreducible (and coincides with \( I(F) \)).

(h) If \( \tilde{S} \) is an irreducible \( L_0 \)-submodule of \( M(F) \) and \( S \) is the \( L \)-submodule of \( M(F) \) generated by \( \tilde{S} \), then \( S \) is irreducible if \( \text{Sing } S = S \cap \text{Sing } M(F) = \tilde{S} \).

Proof. Let \( F \) be a finite-dimensional \( L_0 \)-module and let \( v \) be a generalized eigenvector of \( Y \) with eigenvalue \( \lambda \). If \( a \) is an eigenvector of \( \text{ad } Y \) with eigenvalue \( j \), it follows that \( a(v) \) is a generalized eigenvector with eigenvalue \( \lambda + j \). Hence \( v \) is annihilated by all but finitely many eigenspaces of \( \text{ad } Y \), proving (a).

One has a filtration of \( L \) by open subspaces given by \( L_j = (\text{ product of eigenspaces of } \text{ad } Y \text{ with eigenvalues } \geq j) \). Now (a) and the proof of Proposition [4] prove (b).

Similarly, one shows that all elements from \( L_+ \) act on \( F \) as nilpotent operators and therefore, by the superanalog of Engel’s theorem, they annihilate a non-zero vector. Since the space spanned by these vectors is \( L_0 \)-invariant, it coincides with \( F \), which proves (c).

If \( F \) is an irreducible \( L_0 \)-module, it is actually an irreducible \( g_0 \)-module (with \( L_+ \) acting trivially) on which therefore \( Y \) acts as a scalar, let it be \( y_0 \). Then clearly \( Y \) acts diagonally on \( M(F) \) in such a way that \( F \) coincides with its eigenspace for the eigenvalue \( y_0 \), and \( \text{Re}(y) < \text{Re}(y_0) \) for any other eigenvalue \( y \) of \( Y \) on \( M(F) \). This implies (d). Then (e), (f) and (g) follow.

The statement (h) follows from (f), as soon as we notice that the inclusion of \( L_0 \)-modules \( \tilde{S} \subset M(F) \) induces the morphism of \( L \)-modules \( M(\tilde{S}) \rightarrow M(F) \) and the map is injective by PBW theorem, therefore \( S = M(\tilde{S}) \).

One has the following well-known corollary of Proposition 1.3.

Corollary 1.4. An \( L \)-module \( M(F) \) is irreducible (hence non-degenerate) if and only if the \( g_0 \)-module \( F \) is irreducible and \( M(F) \) has no non-trivial singular vectors.

Remark 1.5. The correspondence defined by Proposition 1.3e provides the classification of irreducible modules of the category \( \mathcal{P}(L, L_0) \). For the non-degenerate of those modules the definition of \( M(F) \) supplies the construction, and Proposition 1.3g gives a construction of the degenerate modules having only trivial secondary singular vectors, provided that one has a description of singular vectors.

2. Construction and basic properties of \( E(3, 6) \).

One way to construct \( E(3, 6) \) is via its embedding into \( E(5, 10) \). We describe first the geometric construction of \( E(5, 10) \) from [CK1], Section 5.3 or [K1], Section 5.

Let \( x_1, \ldots, x_5 \) be even variables with \( \text{deg } x_i = 2 \), and let \( S_5 \) be the Lie algebra of divergence zero vector fields in these variables. Let \( \Omega^1(5) \) be the space of closed differential 2-forms in these variables. Choosing degrees of the variables and the degree of \( d \) determines a \( \mathbb{Z} \)-grading in vector fields and differential forms. We let the degree of \( d \) be \(-5/2\), so that \( \text{deg } dx_i = -1/2 \).
The Lie superalgebra $E(5, 10)$ is constructed as follows: $E(5, 10)_{\bar{0}} \simeq S_5$ as a Lie algebra, $E(5, 10)_{\bar{1}} \simeq \text{d} \Omega^4(\mathbb{R})$ as an $S_5$-module. The bracket on $E(5, 10)_{\bar{1}}$ is defined as the exterior product of differential forms which is a closed 4-form identified with the vector field whose contraction with the volume form produces this 4-form. This construction gives a $\mathbb{Z}$-grading in $E(5, 10)$ that we will call the consistent $\mathbb{Z}$-grading (since its even and odd numbered pieces are comprised of even and odd elements, respectively).

In order to make explicit calculations we will use the following notations:

$$d_{jk} := dx_j \wedge dx_k, \quad \partial_i := \partial/\partial x_i.$$ 

We assume that the volume form is $dx_1 \wedge \cdots \wedge dx_5$. Now an element $A$ from $E(5, 10)_{\bar{0}} = S_5$ can be written as

$$A = \sum_i a_i \partial_i, \quad \text{where } a_i \in \mathbb{C}[[x_1, \ldots, x_5]], \quad \sum_i \partial_i a_i = 0,$$

and an element $B$ from $E(5, 10)_{\bar{1}}$ is of the form

$$B = \sum_{j,k} b_{jk} d_{jk}, \quad \text{where } b_{jk} \in \mathbb{C}[[x_1, \ldots, x_5]], \ dB = 0.$$ 

In particular the brackets in $E(5, 10)_{\bar{1}}$ can be computed using bilinearity and the rule

$$[\text{ad}_{d_{jk}}, \text{bd}_{lm}] = \varepsilon_{ijklm} ab \partial_i$$

where $\varepsilon_{ijklm}$ is the sign of the permutation $(ijklm)$ when $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$ and zero otherwise.

By definition ([K1], Example 5.4) the algebra $E(3, 6)$ is a consistently $\mathbb{Z}$-graded simple linearly compact Lie superalgebra such that

$$\begin{align*}
\mathfrak{g}_0 &\simeq \mathfrak{sl}(3) + \mathfrak{sl}(2) + \mathfrak{gl}(1), \\
\mathfrak{g}_{-1} &\simeq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}(-1), \\
\mathfrak{g}_{-2} &\simeq \mathbb{C}^3 \otimes 1 \otimes \mathbb{C}(-2), \\
\mathfrak{g}_{-3} &\simeq 0, \\
\mathfrak{g}_1 &\simeq S^2\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}(1) + \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}(1).
\end{align*}$$

We may construct $E(3, 6)$ as a subalgebra of $E(5, 10)$ as follows (we slightly modify the construction from [CK1]). Consider the secondary grading in $E(5, 10)$ defined by the conditions:

$$\begin{align*}
\deg x_1 &= \deg x_2 = \deg x_3 = 0, \\
\deg x_4 &= \deg x_5 = 1, \\
\deg \partial_1 &= \deg \partial_2 = \deg \partial_3 = 0, \\
\deg \partial_4 &= \deg \partial_5 = -1, \ \deg d = -1/2.
\end{align*}$$

**Proposition 2.1.** ([CK1]) For the secondary grading of $E(5, 10)$, the zero-degree subalgebra is the Lie superalgebra $E(3, 6)$. The consistent $\mathbb{Z}$-grading in $E(3, 6)$ is induced by the consistent grading of $E(5, 10)$.

As a result we have for $L = E(3, 6)$ the following description of the first three pieces of its consistent $\mathbb{Z}$-grading $L = \Pi_{j \geq -2} \mathfrak{g}_j$:

$$\begin{align*}
\mathfrak{g}_{-2} &= \langle \partial_i, i = 1, 2, 3 \rangle, & \mathfrak{g}_{-1} &= \langle d_{ij}, i = 1, 2, 3, j = 4, 5 \rangle.
\end{align*}$$
We shall use the following basis of $g_0 = s\ell(3) + s\ell(\mathfrak{s}) + \mathfrak{gl}(1)$:

$$
\begin{align*}
&h_1 = x_1 \partial_1 - x_2 \partial_2, \quad h_2 = x_2 \partial_2 - x_3 \partial_3, \quad e_1 = x_1 \partial_2, \quad e_{12} = x_2 \partial_3, \quad e_3 = x_1 \partial_3, \\
f_1 = x_2 \partial_1, \quad f_2 = x_3 \partial_2, \quad f_{12} = x_3 \partial_1, \quad h_3 = x_4 \partial_4 - x_5 \partial_5, \quad e_3 = x_4 \partial_5, \quad f_3 = x_5 \partial_4, \\
Y = \frac{2}{3}(x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3) - (x_4 \partial_4 + x_5 \partial_5).
\end{align*}
$$

Here $s\ell(3)$ (resp. $s\ell(2)$) is spanned by elements involving indeterminates $x_i$ with $i = 1, 2, 3$ (resp. 4,5) and $\mathfrak{gl}(1) = \mathfrak{CY}$. We use the element $Y$ as the hypercharge operator and we fix the standard Cartan subalgebra $\mathcal{H} = \langle h_1, h_2, h_3, Y \rangle$ and the standard Borel subalgebra $\mathcal{B} = \mathcal{H} \oplus \mathcal{N}$, where $\mathcal{N} = \langle e_i (i = 1, 2, 3), e_{12} \rangle$, of $g_0$. Note that the eigenspace decomposition of $3Y$ coincides with the consistent $Z$-grading of $E(3,6)$. (Incidentally, $E(5, 10)$ has no hypercharge operators.)

The algebra $E(3, 6)$ is generated by $\mathfrak{g}_{-1}, g_0, \mathfrak{g}_1$; moreover it is generated by $g_0$ and the following three elements $e_0, e'_0, f_0$:

$$
\begin{align*}
&f_0 = d_{14}, \\
e'_0 = x_3 d_{35}, \\
e_0 = x_3 d_{25} - x_2 d_{35} + 2x_5 d_{23},
\end{align*}
$$

where the element $f_0$ is the highest weight vector of the $g_0$-module $\mathfrak{g}_{-1}$, while $e'_0, e_0$ are the lowest weight vectors of the $g_0$-module $g_1$, and one has:

$$
\begin{align*}
[e'_0, f_0] &= f_2, \\
[e_0, f_0] &= \frac{2}{3}h_1 + \frac{1}{3}h_2 - h_3 - Y =: h_0.
\end{align*}
$$

So the elements $\{h_i, e_i, f_i (i = 0, 1, 2, 3), e'_0 \}$ generate $E(3, 6)$. We call them the generalized Chevalley generators of $E(3, 6)$ (since apart from $\{0, 2\}$ they satisfy the relations satisfied by the ordinary Chevalley generator of a semisimple Lie algebra).

The above observations give the following proposition.

**Proposition 2.2.** The elements $e_i (i = 0, 1, 2, 3)$ and $e'_0$ generate $\mathcal{N} + L_+$. Consequently, a $g_0$-highest weight vector $v$ of a $E(3, 6)$-module is singular iff

$$
e_0 \cdot v = 0, \quad e'_0 \cdot v = 0 .
$$

In order to see the action of $g_0$ on the space $\mathfrak{g}_{-1} = \langle d_{ij}, i = 1, 2, 3, j = 4, 5 \rangle$ more clearly we write

$$
d^+_i := d_{i4}, \quad d^-_i := d_{i5}.
$$

and we define $\mathfrak{g}_{-1}^\pm = \langle d^+_i, d^\pm_i, d^+_5 \rangle$. We also use the following shorthand notations for the elements from $\Lambda(d^-_i, d^-_j, d^-_5)$:

$$
d^-_i := d^-_i \cdot d^-_j, \quad d^-_i d^-_j := d^-_i \cdot d^-_j \cdot d^-_5 ,
$$

and similarly for the “$+$” type. We let $\Lambda^\pm := \Lambda(d^+_i, d^+_j, d^+_5)$.

Consider the following abelian subalgebras of $g_1$, normalized by $s\ell(3)$:

$$
g^\pm_1 = \langle x_i d_{j5} + x_j d_{i5} \mid i, j = 1, 2, 3 \rangle, \quad g^\pm_1 = \langle x_i d_{j4} + x_j d_{i4} \mid i, j = 1, 2, 3 \rangle,
$$

and let

$$
S(3)^\pm = g^\pm_{-1} + s\ell(3) + g^\pm_1.
$$
It is easy to see that $S(3)\pm$ are subalgebras of $\mathfrak{g}$ isomorphic to the simple Lie superalgebra $S(3)$ of divergenceless vector fields in three anticommuting indeterminates. Note that

\begin{equation}
[g_{-1}^\pm, g_i^\mp] = 0.
\end{equation}

One can check that $E(3,6)_0 \simeq W_3 \oplus \Omega^0(3) \otimes \mathfrak{sl}(2)$ and $E(3,6)_1 \simeq \Omega^1(3) \otimes \mathbb{C}^2$. Here the first isomorphism maps $D \in W_3$ to $D - \frac{1}{3} \text{div} (D)(x_3 \partial_1 + x_1 \partial_3)$ and is identical on the second summand. The second isomorphism could be chosen according to the following formula (which differs from the one in [CK1])

$$fdx_i \cdot \varepsilon_a \longrightarrow -d(fdx_i \cdot x_{a+3}), i = 1, 2, 3, \quad a = 1, 2,$$

where $\varepsilon_a, a = 1, 2$, is the standard basis in $\mathbb{C}^2$. Of course it is possible to define brackets in $E(3,6)$ in terms of these isomorphisms and this gives the construction of $E(3,6)$ from [CK1].

3. Lemmata about $\mathfrak{sl}(3)$-modules

FROM now on we let $L = E(3,6)$. As before, we use notation $L_\pm = \bigoplus_{j < 0} \mathfrak{g}_j, L_0 = \mathfrak{g}_0 + L_+$. We shall use the realization of this Lie superalgebra as a submodule of $E(5,10)$ described in Section 2. As explained in Section 1, our first main objective is to study irreducibility of the induced $\mathfrak{g}$-modules

\begin{equation}
M(V) = U(L) \otimes_{U(L_0)} V \cong U(L_-) \otimes V,
\end{equation}

where $V$ is a finite-dimensional irreducible $\mathfrak{g}_0$-module extended to $L_0$ by letting $\mathfrak{g}_j$ for $j > 0$ acting trivially. The isomorphism in (3.1) is an isomorphism of $\mathfrak{g}_0$-modules, which can be used to define the action of $L$ on $U(L_-) \otimes V$ (in particular $L_-$ acts by left multiplication).

Recall that $\mathfrak{g}_0 = \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$, where

$$\mathfrak{sl}(3) = \langle x_i \partial_j | 1 \leq i, j \leq 3 \rangle \cap \mathfrak{g}_0, \quad \mathfrak{sl}(2) = \langle x_i \partial_j | i, j = 4, 5 \rangle \cap \mathfrak{g}_0.$$

Hence it is important to have a model for $\mathfrak{sl}(3)$, i.e., an $\mathfrak{sl}(3)$-module in which every finite-dimensional irreducible $\mathfrak{sl}(3)$-module appears exactly once. Note that $\mathfrak{sl}(3)$ acts on the polynomial algebra $\mathbb{C}[\partial_1, \partial_2, \partial_3, x_1, x_2, x_3]$ in a natural way (by derivations $x_i \partial_j (x_k) = [x_i, \partial_j](x_k) = \delta_{jk} x_i$, $x_i \partial_j (\partial_k) = [x_i, \partial_j](\partial_k) = [x_i, \partial_k](\partial_j) = -\delta_{jk} \partial_i$), so that the element $P := \partial_1 x_3 + \partial_2 x_2 + \partial_3 x_1$ is annihilated. Denote by $\mathcal{M}$ the quotient of this polynomial algebra by the ideal generated by $P$, with the induced action of $\mathfrak{sl}(3)$.

**Lemma 3.1.** The $\mathfrak{sl}(3)$-module $\mathcal{M}$ is a model. The irreducible $\mathfrak{sl}(3)$-module with highest weight $(m, n)$ appears in $\mathcal{M}$ by the bigraded component

$$\langle \partial_{1}^{a_1} \partial_{2}^{a_2} \partial_{3}^{a_3} x_1^{b_1} x_2^{b_2} x_3^{b_3} | \sum a_i = m, \sum b_i = n \rangle.$$

The highest weight vector of this submodule is $\partial_{3}^{n} x_{1}^{m}$. 

**Proof.** Let $U$ denote the subgroup of the group $G = SL(3, \mathbb{C})$ consisting of upper triangular matrices with 1’s on the diagonal. It is well-known that in the space $\mathbb{C}[G/U]$ of regular functions on $G/U$ all irreducible finite-dimensional $G$-modules occur exactly once. On the other hand, $G/U$ is isomorphic to the orbit of the sum of highest weight vectors in the $G$-module $C^3 \oplus C^{3+}$ and this orbit is the complement to 0 in the quadric $\sum_i x_i \partial_i = 0$, where $x_i$ (resp. $\partial_i$) are standard coordinates on $C^3$ (resp. $C^{3+}$). Since this quadric is a normal variety, we conclude that the $G$-module $\mathbb{C}[G/U]$ is isomorphic to $\mathcal{M}$. The lemma follows. \qed
Thus, every $L$-module $M(V)$ is contained in $U(L_-) \otimes M \otimes T$, where $T$ is a (finite-dimensional irreducible) $\mathfrak{sl}(2)$-module. We shall use the following shorthand notation:

$$u[m]t = u \otimes m \otimes t \in U(L_-) \otimes M \otimes T.$$  

(This notation also reminds one that elements of $M$ are cosets.) We shall mark the elements $\partial_i \in \mathfrak{g}_{-2}$ by a hat in order to distinguish them from the elements $\partial_i$ used in the construction of $\mathcal{M}$. We let $\mathcal{S} = \mathbb{C}[\hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3]$.

We shall consider the tensor product $U(L_-) \otimes \mathcal{M}$ of associative algebras. It is a $U(\mathfrak{sl}(3))$-module with the usual action on the tensor product. Hence we may consider the smash product

$$U = (U(L_-) \otimes \mathcal{M}) \# U(\mathfrak{sl}(3)).$$

This is an associative algebra which acts on $U(L_-) \otimes \mathcal{M}$ in the obvious way (elements from $U(L_-)\mathcal{M}$ act by left multiplication).

The algebra $U(L_-) \otimes \mathcal{M}$ contains a commutative $U(\mathfrak{sl}(3))$-invariant subalgebra $\mathcal{S} \otimes \mathcal{M}$. In the following proposition and further we shall denote by $\text{wt}_v$ the $\mathfrak{sl}(3)$-weight of a vector $v$.

**Proposition 3.2.** Consider the following elements of $\mathcal{S} \otimes \mathcal{M}$:

$$\begin{align*}
\bar{D}_1 &= \hat{\partial}_1[x_1] + \hat{\partial}_2[x_2] + \hat{\partial}_3[x_3], \\
\bar{D}_2 &= \hat{\partial}_2[\partial_3] - \hat{\partial}_3[\partial_2], \\
\bar{D}_3 &= \hat{\partial}_3[1] = \hat{\partial}_3.
\end{align*}$$

Any $\mathfrak{sl}(3)$-highest weight vector $\bar{w}$ in $\mathcal{S} \otimes \mathcal{M}$ can be uniquely written as

$$\bar{w} = \sum_{\alpha, m, n} \overline{c}_{\alpha, m, n} \bar{D}_{\alpha}^{(3)} \bar{D}_2^{(2)} \bar{D}_1^{(1)} [\partial_3^m x_1^n].$$

If $\text{wt}_3 \bar{w} = (a, b)$ then $(m, n) = (a, b) - (\alpha_2, \alpha_3)$ for non-zero $\overline{c}_{\alpha, m, n}$.

**Proof.** It follows from [Sh] that the algebra of $U$-invariants for the action of $SL(3)$ on the algebra $A := \mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_1, \partial_2, \partial_3, x_1, x_2, x_3]$ is generated by six algebraically independent elements: $D_0 := \sum_i \partial_i x_i, D_1, D_2, D_3, \partial_3$ and $x_1$. Since $\mathcal{S} \otimes \mathcal{M} \cong A/(D_0)$ by Lemma 3.1, the proposition follows. \hfill $\Box$

Suppose $\bar{w} \in \mathcal{S} \otimes \mathcal{M}$, where $M \subset \mathcal{M}$ is an irreducible $\mathfrak{sl}(3)$-module generated by the highest vector $[\partial_3^m x_1^n] \in \mathcal{M}$. Then in the above formula we should have $(\nu_1, \nu_2) = (m, n) + (\alpha_1, \alpha_2)$. So the weights of $\mathfrak{sl}(3)$-highest weight vectors in $\mathcal{S} \otimes \mathcal{M}$ are

$$(a, b) = (\nu_1, \nu_2) - (\alpha_1, \alpha_2) + (\alpha_2, \alpha_3)$$

where $\alpha_i \leq \nu_i$, $i = 1, 2$.

Suppose now that an $\mathfrak{sl}(3)$-module $M$ is given as an abstract finite-dimensional module. We would like to extend our description of the highest weight vectors in $\mathcal{S} \otimes \mathcal{M}$ to $\mathcal{S} \otimes \mathcal{M}$. Note that $\mathcal{S} \otimes \mathcal{M}$ is an $\mathcal{S}$-module via the left multiplication, and also a $U(\mathfrak{sl}(3))$-module with the usual action on tensor product, so that we have the action of $\mathcal{S} \# U(\mathfrak{sl}(3))$ on $\mathcal{S} \otimes \mathcal{M}$.

Let $h = h_1 + h_2 + 1$. Instead of $D_i$, we define the operators $\bar{D}_i$ from $\mathcal{S} \# U(\mathfrak{sl}(3)) \subset U$ as follows (as before, we drop the tensor signs):

$$\begin{align*}
\bar{D}_1 &= \hat{\partial}_1 h_1 h + \hat{\partial}_2 f_{12} h + \hat{\partial}_3 (f_3 h_1 + f_2 f_1), \\
\bar{D}_2 &= \hat{\partial}_2 h_2 + \hat{\partial}_3 f_2, \\
\bar{D}_3 &= \hat{\partial}_3.
\end{align*}$$  

(3.2)
Lemma 3.7. 

Proof. This is not difficult to check by a straightforward calculation (see below).

The operators \( D_i \) on \( S \otimes M \) is related to the left multiplication by the \( \hat{D}_i \) by the formulae

\[
\begin{align*}
D_1(s[\partial_s^q x_1^p]) &= p(p + q + 1) \hat{D}_1(s[\partial_s^q x_1^{p-1}]), \\
D_2(s[\partial_s^q x_1^p]) &= q \hat{D}_2(s[\partial_s^q x_1^{p-1}]), \\
D_3(s[\partial_s^q x_1^p]) &= \hat{D}_3(s[\partial_s^q x_1^p]).
\end{align*}
\]

(3.3)

Note that equations (3.2) represent the defining property of the operators \( D_i \), and (3.1) is a solution of these equations.

Proposition 3.3. The operators \( D_i \) commute with each other, and while acting on \( S \otimes M \) the operator \( D_i \) commutes with \( D_j \) for \( j < i \).

Proof. This is not difficult to check by a straightforward calculation (see below).

In the following \( A^{[n]} := A(A - 1) \cdots (A - n + 1) \).

Proposition 3.4. Let \( M \) be an irreducible \( sl(3) \)-module with the highest weight vector \( m_0 \). Any highest weight vector in \( S \otimes M \) can be written uniquely as a linear combination of the form

\[
w = \sum_\alpha c_\alpha D_1^\alpha D_2^{\alpha_2} D_3^{\alpha_3} m_0. \tag{3.5}
\]

Proof. Pick a monomorphism \( \mu : M \to \mathcal{M} \) such that \( \mu(m_0) = [\partial_3^q x_1^p] \in \mathcal{M} \) for a highest weight vector \( m_0 \) of \( M \). Clearly, by Proposition 3.3 and equation (3.3), for the expression for \( \bar{w} \) given by Proposition 3.2 we have:

\[
w = \sum_{\alpha,m,n} c_{\alpha,m,n} D_1^\alpha D_2^{\alpha_2} D_3^{\alpha_3} m_0, \quad \text{where} \quad \mu(w) = \bar{w}
\]

and

\[
(\nu_1)^{[\alpha]}(\nu_1 + \nu_2 + 1)^{[\alpha_2]} c_{\alpha,m,n} = \bar{c}_{\alpha,m,n}
\]

Any element \( v \) of \( S \otimes M \) can be written uniquely in the form

\[
v = \sum_{\alpha \in \mathbb{Z}_+^3} \hat{D}_1^{\alpha_1} \hat{D}_2^{\alpha_2} \hat{D}_3^{\alpha_3} t_\alpha = \sum_\alpha \hat{\partial}^\alpha t_\alpha, \quad \text{where} \ t_\alpha \in \mathcal{M}.
\]

We define \( \ell h t v = \hat{\partial}^\sigma t_\sigma \), where \( \sigma \) is the lexicographically highest element of the set \( \{\alpha \in \mathbb{Z}_+^3 | t_\alpha \neq 0\} \). It is immediate to see

\[
\ell h t \hat{D}^\alpha [\partial_3^q x_1^m] = \hat{\partial}^\alpha [\partial_3^{\alpha + \alpha_2} x_1^{m + \alpha_1}]. \tag{3.6}
\]

Proposition 3.5. \( \ell h t D^\alpha = \hat{\partial}^{\alpha h} h^{[\alpha_1]} h^{[\alpha_2]}. \)

Using (3.6), we get the following corollary.

Corollary 3.6.

\[
\ell h t D^\alpha [\partial_3^q x_1^p] = p^{[\alpha_1]}(p + q + 1)^{[\alpha_2]} q^{[\alpha_2]} \ell h t \hat{D}^\alpha [\partial_3^{\alpha_2} x_1^{p - \alpha_1}].
\]

The proof of Proposition 3.3 is based on several lemmas, which also establish some properties of the operators \( D_i \) used in the sequel.

Lemma 3.7. \( D_i \) commute with each other.
We conclude that 

\[ D = \hat{\partial}_1 h_1 + \hat{\partial}_2 f_1, \quad B = f_{12} h_1 + f_2 f_1 \] 

so that \( D_1 = A h + \hat{\partial}_3 B \). We see that 

\[ [A, \hat{\partial}_2] = 0, \quad [A, h_2] = 0, \quad [A, \hat{\partial}_3] = 0 \] 

and \( [A, f_2] = \hat{\partial}_1 f_2 - \hat{\partial}_2 f_{12} \). Therefore 

\[ [A, D_2] = \hat{\partial}_1 \hat{\partial}_3 f_2 - \hat{\partial}_2 \hat{\partial}_3 f_{12}. \]

Also \( [h, D_2] = 0 \), \( [\hat{\partial}_3, D_2] = 0 \) and 

\[
\begin{align*}
[B, \hat{\partial}_2] &= \hat{\partial}_2 f_{12} - \hat{\partial}_1 f_2, \\
[B, h_2] &= B, \\
[B, \hat{\partial}_3] &= -\hat{\partial}_1 h_1 - \hat{\partial}_2 f_1 = -A, \\
[B, f_2] &= [f_{12}, f_2] = 0,
\end{align*}
\]

thus 

\[ [B, D_3] = (\hat{\partial}_2 f_{12} - \hat{\partial}_1 f_2)h_2 + \hat{\partial}_3 B - Af_2 = (\hat{\partial}_2 f_{12} - \hat{\partial}_1 f_2)h. \]

We conclude that \([D_1, D_2] = 0\). We have computed that \([A, \hat{\partial}_3] = 0\), \([B, \hat{\partial}_3] = -A\), and \([h, \hat{\partial}_3] = \hat{\partial}_3\), so \([D_1, D_3] = 0\). Clearly \([D_2, D_3] = 0\) as well.

The following lemma could be applied either to \(D_1\) or \(D_2\). It provides quite a nice expression for \(D^k\) (here we will need only the lexicographically highest term of the sum but later we will also use the second one).

**Lemma 3.8.** Let \(D = ah + \partial b\) where 

\[
\begin{align*}
[a, b] &= 0, & [\partial, b] &= +a, & [\partial, a] &= 0, \\
[h, b] &= -2b, & [h, a] &= -a, & [h, \partial] &= \partial.
\end{align*}
\]

Then 

\[ D^k = \sum_{m=0}^{k} \binom{k}{m} \partial^m b^m a^{k-m}(h - m)^{[k-m]} . \]

**Proof.** The formula is easily proven by induction on \(k\).

One can easily check that \(D_1\) with \(a = A, b = B\) satisfies the above lemma, as well as \(D_2\) with \(a = \hat{\partial}_2, b = f_2\) and \(A\) with \(a = \hat{\partial}_1, b = f_1\). This makes it easy to compute the following lexicographically highest terms.

**Corollary 3.9.** \( \text{ht} D_1^k = \hat{\partial}_1^k h [k] h_1^{[k]}, \) \( \text{ht} D_2^k = \hat{\partial}_2^k h_2^{[k]} \).

**Lemma 3.10.** Let \(D_2\{+m\} = \hat{\partial}_2 (h_2 + m) + \hat{\partial}_3 f_2\). Then \([D_2, \hat{\partial}_3] = [D_2, \hat{\partial}_1] = 0\) and \(D_2 \hat{\partial}_2 = \hat{\partial}_2 D_2 \{+1\}\).

**Proof.** This is a straightforward calculation.

**Corollary 3.11.** If \(\text{ht} f = \hat{\partial}^\alpha u\) then \(\text{ht}(D_2 f) = \text{ht}(\hat{\partial}^\alpha D_2 \{+\alpha_2\}) u\).
Now we describe the $sl(3)$-highest weight vectors in $\Lambda^\pm \otimes F$. We here omit ± because the results are exactly the same for “+” and for “−”.

**Lemma 3.12.** Let $F \subset M$ be an irreducible finite-dimensional $sl(3)$-module with highest weight $(p, q)$. For the $sl(3)$-highest weight element $u \in \Lambda(d_1, d_2, d_3) \otimes F$ of weight $(m, n) = (p, q) + \delta$, there are the following possibilities (up to a constant factor):

\[ (00)': \quad \delta = (0, 0) \quad \text{and} \quad u = [\partial_3^m x_1^m], \]
\[ (+0): \quad \delta = (+1, 0) \quad \text{and} \quad u = d_1 [\partial_3^m x_1^{m-1}], \]
\[ (-+): \quad \delta = (-1, 1) \quad \text{and} \quad u = (d_1 [x_2] - d_2 [x_1]) [\partial_3^{n-1} x_1^n]. \]

\[ (0-): \quad \delta = (0, -1) \quad \text{and} \quad u = (d_1 [\partial_1] + d_2 [\partial_2] + d_3 [\partial_3]) [\partial_3^m x_1^m], \]
\[ (00): \quad \delta = (0, 0) \quad \text{and} \quad u = d_{12} [\partial_3^{n-1} x_1^n]. \]

Proof. This is standard and we leave the proof to the reader. □

**Lemma 3.13.** Let $F$ be an irreducible $sl(3)$-module with highest weight $(p, q)$ and such that the action of $L_+$ on $F$ is trivial. If $u \in \Lambda^\pm \otimes F$ is an $sl(3)$-highest weight vector of weight $(m, n)$ and $e_0' \cdot u = 0$, then there are the following possibilities for $u$ (up to a constant factor):

\[ (T0): \quad (m, n) = (p, q) \quad \text{and} \quad u = [\partial_3^m x_1^m] \in F, \]
\[ (T1): \quad p \geq 0, q = 0, \quad (m, n) = (p + 1, 0) \quad \text{and} \quad u = d_1^3 [x_1^p], \]
\[ (T2): \quad p = 0, \quad q \geq 1, \quad (m, n) = (0, q - 1) \quad \text{and} \quad u = d_1^3 [\partial_1] + d_2^3 [\partial_2] + d_3^3 [\partial_3] = \Delta^+ [\partial_3^m x_1^m], \]
\[ (T3): \quad (p, q) = (0, 1), \quad (m, n) = (1, 0) \quad \text{and} \quad u = d_{12}^3 [\partial_2] + d_{13}^3 [\partial_3] = d_1^3 \Delta^+, \]
\[ (T4): \quad (p, q) = (0, 0) \quad \text{and} \quad u = d_{12} [1]. \]

In particular in all cases except for (T0), either $p = 0$ or $q = 0$.

Proof. We will write $u$ in the form provided by Lemma 3.12 and calculate $e_0' \cdot u$. We are to remember that $e_0' (F) = 0$ and the following relations for the action of $e_0'$ on $[d_1^+, d_2^+, d_3^+]$ is important to have in mind.

\[ e_0' \cdot d_1^+ = f_2, \quad e_0' \cdot d_2^+ = -f_{12}, \quad e_0' \cdot d_3^+ = 0. \]

Case (00): $(p, q) = (m, n)$ and $u = (c_0 + c_1 d_{123}) [\partial_3^n x_1^m]$, hence

\[ 0 = e_0' u = 0 + c_1 \left( f_2 d_1^+ d_3^+ [\partial_3^n x_1^m] - d_1^+ (-f_{12}) d_2^+ [\partial_3^n x_1^m] \right) =
\]
\[ c_1 \left( 0 - n d_2^+ d_3^+ [\partial_2 \partial_3^{n-1} x_1^m] - n d_1^+ d_2^+ [\partial_1 \partial_3^{n-1} x_1^m] + m d_1^+ d_3^+ [\partial_3^n x_1^{m-1} x_3] \right). \]

We see that either $c_1 = 0$, which gives us (T0), or $m = n = 0$, which gives (T4).

Case (+0): $(p, q) = (m - 1, n)$ and $u = d_1^+ [\partial_3^n x_1^{m-1}]$, so

\[ 0 = e_0' u = f_2 [\partial_3^n x_1^{m-1}] = -n [\partial_2 \partial_3^{n-1} x_1^{m-1}]. \]

The solution exists for $n = 0$, $m \geq 1$. This is (T1).

Case (−+): $(p, q) = (m + 1, n - 1)$ and $u = (d_1^+ [x_2] - d_2^+ [x_1]) [\partial_3^{n-1} x_1^n]$. We have:

\[ 0 = e_0' u = f_2 [\partial_3^{n-1} x_1^n x_2] - (f_{12}) [\partial_3^{n-1} x_1^{m+1}] =
\]
\[ - (n - 1) (\partial_2 \partial_3^{n-2} x_1^{m+1}) + [\partial_3^{n-1} x_1^{m+1} x_2] -
\]
\[ - (n - 1) (\partial_1 \partial_3^{n-2} x_1^{m+1}) + (m + 1) [\partial_3^{n-1} x_1^m x_3]. \]
This implies $-(n - 1) = m + 2$, but both $m, n$ are non-negative hence no solution is possible.

Case $(0-)$: $(p, q) = (m, n)$ and $u = \Delta^+ [\partial_3 x_1^m]$. We have:

$$
0 = e^+_0 u = f_2 [\partial_1 \partial_3 x_1^m] - f_{12} [\partial_2 \partial_3 x_1^m] = - n [\partial_1 \partial_2 \partial_3 x_1^{-1} x_1^m + n [\partial_1 \partial_2 \partial_3 x_1^{-1} x_1^m] - m [\partial_2 \partial_3 x_1^{-1} x_1^m x_3] = - m [\partial_2 \partial_3 x_1^{-1} x_1^m x_3].
$$

We conclude that $m = 0$ and this gives $(T2)$.

Case $(0+)$: $(p, q) = (m, n - 1)$ and $u = d^+_{12} [\partial_3 x_1^m]$, therefore

$$
0 = e^+_0 u = f_2 d^+_2 [\partial_3 x_1^m] - d^+_1 [\partial_3 x_1^m] = - n [\partial_1 \partial_2 \partial_3 x_1^{-1} x_1^m + n [\partial_1 \partial_2 \partial_3 x_1^{-1} x_1^m] - m d^+_1 [\partial_2 \partial_3 x_1^{-1} x_1^m x_3].
$$

The $d^+_1$-term shows that there are no solutions here.

Case $(-0)$: $(p, q) = (m + 1, n)$ and $u = (d^+_{12} x_3 - d^+_{13} [x_2] + d^+_{23} [x_1]) [\partial_3 x_1^m]$. Then

$$
0 = e^+_0 u = f_2 d^+_2 [\partial_3 x_1^m x_3] - d^+_1 [\partial_3 x_1^m x_3] = - f_2 d^+_2 [\partial_3 x_1^m x_3] + (-f_{12}) [\partial_3 x_1^{m+1}] = d^+_2 [\partial_3 x_1^{m+1} x_3] - n d^+_1 [\partial_1 \partial_3 x_1^{-1} x_1^m x_3] + m d^+_1 [\partial_3 x_1^{-1} x_1^m x_3] + n d^+_1 [\partial_3 x_1^{-1} x_1^m x_3] + m d^+_1 [\partial_3 x_1^{-1} x_1^m x_3].
$$

Now there is only one $d^+_1$-term and this gives $n = 0$. Then we are left with only one $d^+_1$-term and it gives $m = 0$. Then we are left with a non-zero $d^+_1$-term, so there are no solutions in this case.

Case $(+-)$: $(p, q) = (m + 1, n + 1)$ and $u = (d^+_{12} [\partial_3] + d^+_{13} [\partial_3]) [\partial_3 x_1^{m-1}]$. Then

$$
0 = e^+_0 u = f_2 d^+_2 [\partial_2 \partial_3 x_1^{m-1}] - d^+_1 [\partial_3 x_1^{m-1}] = f_2 d^+_2 [\partial_2 \partial_3 x_1^{m-1}] + f_2 d^+_3 [\partial_3 x_1^{m-1}] = d^+_2 [\partial_2 \partial_3 x_1^{m-1}] = m d^+_1 [\partial_1 \partial_2 \partial_3 x_1^{-1} x_1^m x_3] + (m + 1) d^+_3 [\partial_2 \partial_3 x_1^{-1} x_1^m x_3].
$$

Again there is only one $d^+_1$-term and it gives $n = 0$. Then $d^+_1$-term gives $m = 1$, and this gives $(T3)$.

\[\tag{3.8} e^+_1 \cdot d^-_1 = -f_2, \quad e^+_1 \cdot d^-_2 = +f_3, \quad e^+_1 \cdot d^-_3 = 0\]

\[\text{Lemma 3.14.}\] Let $e^+_1 = x_3 d_{34} \in g_1$. Let $F$ be an irreducible $sl(3)$-module with highest weight $(p, q)$ and such that the action of $L^+ \otimes F$ is trivial. If $u \in \Lambda^+ \otimes F$ is an $sl(3)$-highest weight vector of weight $(m, n)$ and $e^+_1 \cdot u = 0$, then there are the following possibilities for $u$ (up to a constant factor):

\begin{itemize}
  \item[(T0):] $(m, n) = (p, q)$ and $u = [\partial_3^q x_1^p] \in F$,
  \item[(T1):] $p \geq 0, q = 0, (m, n) = (p + 1, 0)$ and $u = d^+_1 [x_1^p]$,
  \item[(T2):] $p = 0, q \geq 1, (m, n) = (0, q - 1)$ and $u = \Delta^- [\partial_3^n]$, \\
  \item[(T3):] $(p, q) = (0, 1), (m, n) = (1, 0)$ and $u = (d^+_{12} [\partial_2] + d^+_{13} [\partial_3]) = d^+_1 \Delta^-$,
  \item[(T4):] $(p, q) = (m, n) = (0, 0)$ and $u = d^+_{123} [1]$.
\end{itemize}

\textbf{Proof.} As relations for $e^+_1$

\[e^+_1 \cdot d^-_1 = -f_2, \quad e^+_1 \cdot d^-_2 = +f_3, \quad e^+_1 \cdot d^-_3 = 0\]
Lemma 4.2. If similar maximum for \( \tilde{w} \) this follows from the commutation relations (4.1) and where the summands are singular vector, which is a g-module. We keep the standard Cartan and Borel subalgebras \( \mathcal{H} \) and \( \mathcal{B} \) of \( g_0 \).

We shall use the following notations:

\[
\Lambda_i^\pm := \Lambda^i(g_{-1}^\pm), \quad \Lambda^\pm := \sum_{i \geq 0} \Lambda_i^\pm, \quad S^k := \text{Sym}^k(g_{-2}), \quad S = \sum_{k \geq 0} S^k.
\]

We know that \( M(V) = U(L-) \otimes V \) and by the PBW theorem we have the isomorphisms of vector spaces (where, as before, we drop the tensor product signs):

\[
M(V) = S \Lambda^-, \quad M(V) = S \Lambda^+ \Lambda - V.
\]

When we use the first isomorphism, we say that the \((-\cdot)-order\) (for elements of \( g_{-1} \)) is chosen and when the second, we speak of the \((+\cdot)-order\).

Theorem 4.1. If an \( E(3,6) \)-module is degenerate then the \( sl(3) \)-highest weight of \( V \) is either \((p,0)\) or \((0,q)\).

Proof. Suppose that the \( sl(3) \)-highest of \( V \) is \((p,q)\) and \( pq \neq 0 \) and that the module \( M(V) \) is degenerate. We have to show that this is impossible. Let \( w \) be a non-trivial singular vector, which is a \( g_0 \)-highest weight vector.

Using the \((-\cdot)\) order we write

\[
M(V) = \sum_{m,i,j} S^m \Lambda_i^- \Lambda_j^+ V,
\]

where the summands are \( sl(3) \)-modules, let \( w = \sum w_{m,i,j} \) be the corresponding decomposition of \( w \). Similarly

\[
M(V) = \sum_{m,i,j} S^m \Lambda_j^+ \Lambda_i^- V,
\]

and \( w = \sum \tilde{w}_{m,j,i} \) is the decomposition for \((+\cdot)\)-order.

Let \( n \) be the maximum value of \( m \) such that there exists \( w_{m,i,j} \neq 0 \), and let \( n' \) be the similar maximum for \( \tilde{w}_{m,j,i} \).

Lemma 4.2. If \( j \neq 0 \) then \( w_{n;i,j} = 0 \), and if \( i \neq 0 \) then \( \tilde{w}_{n';j,i} = 0 \).

Proof. Notice that

\[
eq 0 \quad \sum_{i \geq 0} \Lambda_i^+ \Lambda_{i+1}^- \Lambda_i^+ V + \sum_{i \geq 0} \Lambda_i^- \Lambda_{i+1}^+ V.
\]

This follows from the commutation relations

\[
\begin{align*}
[e_0', \hat{\partial}_1] &= 0, & [e_0', \hat{\partial}_1] &= 0, & [e_0', \hat{d}_1^+] &= +f_2, \\
[e_0', \hat{\partial}_2] &= 0, & [e_0', \hat{d}_2^+] &= 0, & [e_0', \hat{d}_2^+] &= -f_2, \\
[e_0', \hat{\partial}_3] &= -\hat{d}_3^- & [e_0', \hat{d}_3^-] &= 0, & [e_0', \hat{d}_3^+] &= 0.
\end{align*}
\]
We denote by $P_{(m;i,j)}$ the projection onto $S^n \Lambda_i^- \Lambda_j^+ V$ (in the $(-\pm)$ decomposition), and we see from (4.1) that for any $i \geq 0$ and $j \geq 1$ we have:

$$0 = P_{(n;i,j-1)} e'_0 w = P_{(n;i,j-1)} e'_0 w_{n;i,j}.$$ 

Let us write $w_{n;i,j} = \sum \delta^a t^J_+ w^J_0$ where $a, I, J$ are multi-indices and $|a| = n$, $|I| = i$, $|J| = j$. We get

$$P_{(n;i,j-1)} e'_0 w_{n;i,j} = \sum \delta^a t^J_+ (e'_0 (t^J_+ w^J_0)) = 0.$$ 

So we conclude that for any given $a, I$,

$$\sum_{|J|=j} e'_0 t^J_+ w^J_0 = 0.$$ 

Since each $w_{n;i,j}$ is a highest weight vector for $sl(3)$, the coefficient $\sum_{|J|=j} e'_0 t^J_+ w^J_0$ of $\hat{\delta}^a t^J_+$ of lowest weight in the expression for $w_{n;i,j}$ is an $sl(3)$-highest weight vector. Hence, by Lemma 3.13, $w_{n;i,j} = 0$ for $j > 0$.

In a similar way the commutation relations for $e'_a$ and Lemma 3.14 imply the second statement of the lemma. $\square$

**Lemma 4.3.**

(a) $w_{n-k;i,j} = 0$ for $j > k$ and $w_{n';-k;j,i} = 0$ for $i > k$.

(b) $n = n'$.

(c) If $w_{n-k;i,j} \neq 0$ or $w_{n-k;j,i} \neq 0$ then $i + j = 2k$.

(d) If $w_{n-k;i,j} \neq 0$ then $j \leq k \leq i$, and if $w_{n-k;j,i} \neq 0$ then $i \leq k \leq j$.

(e) If $w_{n-k;i,j} \neq 0$ then $i = j = k$.

(f) $sl(2)$ acts trivially on $V$.

**Corollary 4.4.** $w = w_{n;0,0} + w_{n-1;1,1} + w_{n-2;2,2} + \ldots$ and $w_{n;0,0} \neq 0$.

**Proof.** (a) Let us use induction on $k$. The case $k = 0$ follows from Lemma 4.2. Equation (4.1) shows that

$$0 = P_{(n-k;i,j-1)} e'_0 w = P_{(n-k;i,j-1)} e'_0 w_{n-k;i,j} + P_{(n-k;i,j-1)} e'_0 w_{n-k+1;i-1,j-1},$$

but for $j > k$ the last summand is zero by induction. Now we can apply Lemma 3.13 as we did above and conclude that $w_{n-k;i,j} = 0$. The other statement follows in the same way from Lemma 3.14.

To prove (b) let us notice first that

$$\Lambda^+_j \Lambda^-_i V \subset \Lambda^+_j \Lambda^-_i V + S^1 \Lambda^-_{i-1} \Lambda^+_j V + S^2 \Lambda^-_{i-2} \Lambda^+_j V + \ldots.$$ 

We know that if $\bar{w}_{(n';-k);j,i} \neq 0$ then $i \leq k$, therefore from (4.2) it follows that

$$\bar{w}_{(n';-k);j,i} \in \sum_{s \leq k} S^s \Lambda^-_{i-s} \Lambda^+_j V.$$ 

As $s \leq k$, this implies $n \leq n'$, but the arguments can be reversed so $n = n'$.

For (c) let us notice that the $Y$-eigenvalue of $w_{n-k;i,j}$ and of $\bar{w}_{n-k;i,j}$ is equal to

$$y_V - \frac{1}{3} (i + j) - \frac{2}{3} (n - k)$$

where $y_V$ is defined by $Y |_V = y_V Id_V$. But the eigenvalues are all the same whatever $i, j, k$ so (c) follows. Now (d) follows immediately from (a-c).

To get (e) let us consider $P(w)$ where

$$P = \sum_{m,i \leq j} P_{(m;i,j)}.$$
If \( \tilde{w}_{n-k;j,i} \neq 0 \) then \( i \leq k \leq j \) by (d) and from (4.2) it follows that
\[
\mathbb{P} \tilde{w}_{n-k;j,i} = \tilde{w}_{n-k;j,i}.
\]
We conclude that \( \mathbb{P} w = w \). But at the same time we have:
\[
\mathbb{P} w_{n-k;i,j} = 0, \text{ if } i > j
\]
and because \( i > j \) implies \( \mathbb{P} w_{n-k;i,j} = 0 \) for \( i > j \). This proves (e).

Corollary 4.4 follows from (e). To establish (f) we need the following lemma.

**Lemma 4.5.** Let \( h \in \mathbb{C}[x_1, x_2, x_3] \) of degree \( n \) and \( g = hx_5 \partial_4 \). Then \( g(\Lambda^{n-k} \Lambda_1^+ \Lambda_k^+ V) = 0 \) for \( k > 0 \).

**Proof.** One has to check it for \( n = k = 1, 2, 3 \), then the relation \( [hx_5 \partial_4, \partial_i] = - (\partial_i h)(x_5 \partial_4) \) makes it easy to organize induction on \( n - k \). □

Now from the lemma it follows that \( f_3 w_{n,0,0} = (x_5 \partial_4) w_{n,0,0} = 0 \). On the other hand \( e_3 w = 0 \), and using the expression for \( w \) from Corollary 4.4 we conclude that \( e_3 w_{n,0,0} = 0 \), but \( w_{n,0,0} \neq 0 \).

As a result, because \( e_3 \) and \( f_3 \) act trivially on \( g_{-2} \), we conclude that they act trivially on all coefficients in \( w_{n,0,0} \), which are elements from \( V \). But we know that \( V \) is isomorphic to the tensor product of irreducible representations of \( \mathfrak{sl}(2) \) and \( \mathfrak{sl}(3) \). Therefore the existence of a trivial \( \mathfrak{sl}(2) \) submodule in \( V \) means that \( \mathfrak{sl}(2) \) acts trivially on \( V \), which gives (f). □

Unless otherwise stated, we use the \((+-)\)-order. In the following we can suppose that \( V \) is realized as a submodule in \( \mathcal{M} \), i.e., that elements of \( V \) are linear combinations of monomials
\[
\prod_{i,j} \partial_j \alpha_i x_i^{m_i},
\]
because the action of \( \mathfrak{sl}(2) \) is trivial due to Lemma 4.3f. For \( \alpha \in \mathbb{Z}_+^3 \) we let, as before, \( D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \).

According to Proposition 3.4 one has
\[
w = \sum_\alpha D^\alpha T_\alpha,
\]
where \( T_\alpha \) are highest weight vectors in \( \Lambda^{-} \Lambda^+ V \), and for their weights we have the relation
\[
\text{wt \ }_3 w = (-\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) + \text{wt \ }_3 T_\alpha.
\]

If \( |\sigma| = n \) and \( T_\sigma \neq 0 \), then, because of Corollary 4.4 \( T_\sigma \in \mathcal{M} \), so \( \text{wt \ }_3 T_\sigma = (p, q) \), thus
\[
\text{wt \ }_3 w = (-\sigma_1 + \sigma_2, -\sigma_2 + \sigma_3) + (p, q).
\]

This means that given \( n, \text{wt} \ )\ w \) and \( (p, q) \) we have a unique choice for \( \sigma \), and we can write \( T_\sigma = [\partial_3^q x_1^p] s \), where \( s \) is a non-zero scalar. Therefore
\[
w_{n,0,0} = D^\sigma[\partial_3^q x_1^p] s, \quad s \in \mathbb{C}.
\]

At the same time, due to (3.3)–(3.5) we have:
\[
D^\sigma[\partial_3^q x_1^p] s = \tilde{D}^\sigma[\partial_3^q x_1^{p-\sigma_1}] \tilde{s},
\]
where \( \tilde{s} = p^{[\sigma_1]}(p + q + 1)^{[\sigma_2]} q^{[\sigma_2]} s \).

Without loss of generality we can assume that \( \tilde{s} = 1 \). Let \( t_\sigma = [\partial_3^q x_1^{p-\sigma_1}] \). Using relations (4.2), we compute:
\[
eg_0 \cdot w_n = \neg_0 \tilde{D}^\sigma t_\sigma = -\sigma_1 d_3 \tilde{D}_1^{\sigma_1-1} \tilde{D}_2^{\sigma_2} \tilde{D}_3^{\sigma_3} [x_3] t_\sigma + \sigma_2 d_3 \tilde{D}_1^{\sigma_1} \tilde{D}_2^{\sigma_2-1} \tilde{D}_3^{\sigma_3} [\partial_2] t_\sigma - \sigma_3 d_3 \tilde{D}_1^{\sigma_1} \tilde{D}_2^{\sigma_2} \tilde{D}_3^{\sigma_3-1} t_\sigma.
\]
Let $P_m = \sum_{i,j} P_{(m,i,j)}$. It follows from (4.5) that
\[
e_0 \cdot w_k = P_{n-1} e_0 \cdot w_n.
\]
We will use this formula in the following way. As (4.1) shows,
\[
P_{n-1} e_0 w = P_{n-1} e_0 w_n + P_{n-1} e_0 w_{n-1}
\]
and we have $e_0 w_n = 0$, hence
\[
(*)
P_{n-1} e_0 w_n = -P_{n-1} e_0 w_{n-1}.
\]
We already have quite an explicit expression for the left-hand side. We will write a similar expression for the right-hand side and study the restrictions imposed by the equality ($*$). We will see that there are very few solutions for these equations in our context and in the end that no one of them makes it to the singular highest weight vector.
We know that
\[
w_n-1 = w_{n-1;11} = \sum_{|\beta|=n-1} D^\beta T_\beta \in SL_1^+ \Lambda_1^+ V,
\]
where $T_\beta$ are the $\text{sl}(3)$ highest weight vectors in $\Lambda_1^+ \Lambda_1^+ V$.

**Lemma 4.6.** Let $|\beta| = n-1$ and $T_\beta \neq 0$. There are at most six choices for $\sigma - \beta$: $(-1,1,1)$, $(0,0,1)$, $(0,1,0)$, $(1,-1,1)$, $(1,0,0)$, $(1,1,-1)$.

**Proof.** It is clear that
\[
\text{wt} 3 w = \text{wt} 3 w_{n-1} = (-\beta_1 + \beta_2, -\beta_2 + \beta_3) + (\lambda_1, \lambda_2) + (p, q),
\]
where $\lambda = (\lambda_1, \lambda_2)$ is a weight of $\Lambda_1^+ \Lambda_1^+$. There are six of these weights: $(2,0)$, $(0,1)$, $(1,-1)$, $(-2,2)$, $(-1,0)$, $(0,-2)$. But, by (4.4), $\text{wt} 3 w = (-\sigma_1 + \sigma_2, -\sigma_2 + \sigma_3) + (p, q)$ as well, so given $\lambda$ we have two linear equations on $\beta$. The fact that $|\beta| = n-1$ provides the third equation and thus the difference $\sigma - \beta$ is determined. We get the six values for $\sigma - \beta$ that correspond to the above six choices for $\lambda$.

**Lemma 4.7.** There are the following possibilities for $T_\beta$ (where $t_i \in \mathbb{C}$):

1. $\beta^{(1)} = \sigma - (-1,1,1)$, $\text{wt} \beta^{(1)} = (p,q) + (2,0)$, and $T_\beta^{(1)} = d_1^+ \Delta^+ [\partial_3^{-1} x_1^p] t_1^p$.
2. $\beta^{(2)} = \sigma - (0,0,1)$, $\text{wt} \beta^{(2)} = (p,q) + (0,1)$, and $T_\beta^{(2)} = d_1^+ (d_1^+ [x_2] - d_2^+ [x_1]) [\partial_3^+ x_1^p] t_1^p + (d_2^+ d_2^+ - d_2^+ d_1^+ [\partial_3^+ x_1^p]) t_2^p$.
3. $\beta^{(3)} = \sigma - (0,1,0)$, $\text{wt} \beta^{(3)} = (p,q) + (1,-1)$, and $T_\beta^{(3)} = d_1^+ \Delta^+ [\partial_3^{-1} x_1^p] t_3^p + \Delta^- d_1^+ [\partial_3^{-1} x_1^p] t_3^p$.
4. $\beta^{(4)} = \sigma - (1,-1,1)$, $\text{wt} \beta^{(4)} = (p,q) + (-2,2)$, and $T_\beta^{(4)} = (d_1^+ [x_2] - d_2^+ [x_1]) (d_1^+ [x_2] - d_2^+ [x_1]) [\partial_3^+ x_1^{p-2}] t_4$.
5. $\beta^{(5)} = \sigma - (1,0,0)$, $\text{wt} \beta^{(5)} = (p,q) + (-1,0)$, and $T_\beta^{(5)} = (d_1^+ (d_1^+ [x_3] - d_2^+ [x_2]) + d_2^+ (d_1^+ [x_1] - d_2^+ [x_3])$ $+ d_2^+ (d_1^+ [x_2] - d_2^+ [x_1]) [\partial_3^+ x_1^{p-1}] t_5^p$ $+ (d_1^+ [x_3] - d_2^+ [x_1]) \Delta^+ [\partial_3^{-1} x_1^{p-1}] t_5^p$.
6. $\beta^{(6)} = \sigma - (1,1,1)$, $\text{wt} \beta^{(6)} = (p,q) + (0,-2)$, and $T_\beta^{(6)} = \Delta^- \Delta^+ [\partial_3^{-2} x_1^p] t_6$.

**Proof.** The fact that $T_\beta$ is the highest weight vector in $\Lambda_1^+ \Lambda_1^+ V$ and Lemma 3.12 permit us to write $T_\beta$ explicitly as soon as its weight is known. This directly leads us to the above expressions.
Our next step is to look at the lexicographically highest terms on the left and right hand sides of (\(*\)).

**Lemma 4.8.** \(\ell h t P_{n-1} e'_0(D^3 T_\beta) = \delta^\beta e'_0 h^{|\beta|} h^{-1}_1 h^{|\beta|}_2 T_\beta\).

**Proof.** This follows from \(|\beta| = n - 1\) and Proposition 3.3.

We can rewrite the lemma as

\[ \ell h t P_{n-1} e'_0(D^3 T_\beta) \sim \delta^\beta e'_0 T_\beta, \]

where \(\sim\) means equality up to a constant multiple, because as we know from (3.3)-(3.5), \(h^{|\beta|}_1, h^{|\beta|}_2 \) multiplies \(T_\beta\) by a non-zero constant as long as \(T_\beta \neq 0\).

Thus we see that if \(t_1 \neq 0\), then the \(\ell h t\) of the right-hand side of \((*)\) comes from \(T_{\beta(1)}\) and is proportional to

\[ \delta^\beta(1) d_1 (-q [\partial_2 \partial_3^{-1} x_1'] t_1) \]

(cf. proof of Lemma 3.13). But the \(\ell h t\) of the left-hand side of \((*)\) is smaller, as one concludes immediately from (4.3) and (3.6). This implies that \(t_1 = 0\), and then the \(\ell h t\) on the right side of \((*)\) comes from \(T_{\beta(3)}\) (if it is non-zero), and the \(\ell h t\) in (4.5) are clearly terms with \(\delta^\beta(1) \delta^\beta(2) \delta^\beta(3)\). Comparing the coefficients of this monomial in (4.5) we get:

\[ \sigma_3 d_3 [\partial_3^p x_1'] \sim c_0 (d_1^p \partial_3^p x_1') t_2' + (d_1^p - d_1^p d_2^p [\partial_3^p x_1'] t_2') \]

or

\[ \sigma_3 d_3 [\partial_3^p x_1'] \sim -d_1^p (f_2 [\partial_3^p x_1'] x_1') t_2' + (d_1^p f_2 + d_2^p f_2 [\partial_3^p x_1']) t_2'. \]

This clearly implies that \(\sigma_3 = 0\), \(t_2' = t_2'' = 0\).

Taking this into account, we can rewrite \((*)\):

\[
\sigma_2 d_3 [\partial_2 \partial_3^{-1} x_1'] = -P_{n-1} e'_0 \left( D_{1}^{2} D_{2}^{2-1} T_{\beta(3)} \right)
\]

where \(T_{\beta(4)}\) is absent because there are no such terms when \(\sigma_3 = 0\), as the components of \(\beta(4)\) are non-negative.

Let us be more careful with the constant factors here. In computing the \(\ell h t\) on the right side we apply Lemma 4.8 to \(\beta(3)\) and we get:

\[ \ell h t P_{n-1} e'_0(D^3 T_{\beta(3)}) = \delta^\beta(1) \delta^\beta(2) \delta^\beta(3) e'_0 h^{|\beta|}_1 h^{|\beta|}_2 D_{\beta(3)} \]

because \(w_3 T_{\beta(3)} = (p+1, q-1)\) (and of course \((q-1)^{\sigma_2-1} \neq 0\) as \(\sigma_2 = q\)). Letting \(b = ((p+1)^{\sigma_2}, (p+q+1)^{\sigma_1})(q-1)^{\sigma_2-1}\), we arrive at the following equation:

\[ \sigma_2 q d_3 [\partial_2 \partial_3^{-1} x_1'] b = d_1^p c_0 \left( d_1^p [\partial_1 \partial_3^{-1} x_1'] t_3' + d_2^p [\partial_2 \partial_3^{-1} x_1'] t_2' \right) + d_3^p c_0 \left( [\partial_3^p x_1'] t_3' \right) + d_2^p f_2 [\partial_2 \partial_3^{-1} x_1'] t_2' + f_2 [\partial_2 \partial_3^{-1} x_1'] t_2' \]

Looking at the coefficients of \(d_1^p\), we conclude that \(pt_3' = 0\) and since \(p \neq 0\), \(t_3' = 0\). From the coefficients of \(d_2^p\) we see that \((q-1)t_2'' = 0\), and from the coefficients of \(d_3^p\) we conclude that \(q\sigma_2 b = -qt_3''\).
Thus, either \( \sigma_2 = t_3' = t_3'' = 0 \) or \( \sigma_2 > 0, q = 1, t_3' = 0, t_3'' = -\sigma_2 b \). Since \( \sigma_2 \leq q \), in the latter case we have \( \sigma_2 = 1, t_3'' = -b \).

If \( \sigma_2 = 0 \), then (4.7) reduces to
\[
\sigma_1 d_3^\prime D_1^{\prime\prime} \left[ \partial_3^\beta x_1^\sigma \right] x_3 = P_{n-1} e_0' D_1^{\prime\prime} - T_{\beta(5)}.
\]

Now we look at the coefficients of \( \sigma_1^{\prime\prime} \) in the equation. In the left-hand side we get:
\[
\sigma_1 d_3^\prime \left[ \partial_3^\beta x_1^\sigma \right] x_3.
\]

Furthermore, since \( \sigma_1 = n \), we can use Lemma 4.8 on the right of (4.8), hence the coefficient of \( \partial_1^{\prime\prime} \) on the right of (4.8) is equal to \( e_1' T_{\beta(5)} \), which we now compute:
\[
e_1' T_{\beta(5)} = -d_1^{-1} e_0' [\partial_1^\beta (\partial_3^\beta x_1^\prime x_3 - \partial_3^\beta (\partial_3^\beta x_1^\prime x_2)] T_1' - d_2^{-1} e_0' [\partial_1^\beta (\partial_3^\beta x_1^\prime x_3)]' T_1' \]
\[
- d_3^{-1} e_0' [\partial_1^\beta (\partial_3^\beta x_1^\prime x_2)] T_1' \]
\[
- d_1^{-1} e_0' \Delta^\prime [\partial_1^\beta (\partial_3^\beta x_1^\prime x_2)] T_1' \]
\[
= d_1^{-1} f_1 [\partial_1^\beta x_1^\prime x_3] T_1' + d_2 [\partial_1^\beta (\partial_3^\beta x_1^\prime x_2)] T_1' \]
\[
- d_3^{-1} f_1 [\partial_1^\beta (\partial_3^\beta x_1^\prime x_2)] T_1' \]
\[
- d_1^{-1} f_1 [\partial_1^\beta (\partial_3^\beta x_1^\prime x_2)] T_1' \]
\[
- d_2^{-1} f_1 [\partial_1^\beta (\partial_3^\beta x_1^\prime x_2)] T_1'.
\]

At the end we get for \( e_1' T_{\beta(5)} \):
\[
e_1' T_{\beta(5)} = d_1^{-1} \left( (-q_1 T_1' - p_1 T_1' \right) [\partial_1 x_1] + (p-1) T_1' [\partial_2 x_2] + (p-1) T_1' [\partial_3 x_3] \left[ \partial_3^\beta - x_1^\prime - 2 x_3 \right] \]
\[
+ d_2^{-1} (-q_1 T_1' - p_1 T_1' \right) [\partial_2^\beta x_1^\prime - x_1^\prime x_3] + d_3^{-1} (p-q+1) T_1' [\partial_3^\beta x_1^\prime - x_3].
\]

We conclude that the terms with \( d_1^{-1} \) and \( d_2^{-1} \) disappear iff either \( p = 1, q_1 T_1' + p_1 T_1' = 0 \) or when \( T_{\beta(5)} = 0 \). Now for the case when \( \sigma_3 \neq \sigma_2 = 0 \) and \( T_{\beta(5)} = 0 \) we get \( \sigma_1 = 0 \) and this is a contradiction.

If \( \sigma_2 = \sigma_3 = 0 \) and \( p = 1 \), then \( \sigma_1 = 1 \) because \( \sigma_1 \leq p \) and it could not be 0 as this gives \( |\sigma| = 0 \). So it becomes \( |\sigma| = n = 1 \) and \( \sigma = w = w_n + w_{n-1} = w_1 + w_0 = D_1[\partial_1^\beta] + T_{\beta(5)} \left( T_{\beta(6)} \right. \) disappears for \( \sigma_2 = 0 \). We can check \( e_3 \cdot w \) now:
\[
e_3 \cdot w = 0 + e_3 T_{\beta(5)} \]
\[
= 2 (d_2^{-1} [x_3] + d_2^{-1} [x_1]) [\partial_1^\beta] T_1' + (d_3^{-1} [x_2] - d_3^{-1} [x_1]) \Delta^\prime [\partial_3^\beta - x_3] T_1' \]

therefore \( e_3 \cdot w = 0 \) implies \( T_1' = T_1'' = 0 \) and we arrive at a contradiction.

If \( \sigma_2 = 1 \), we are left with the situation when \( q = 1, \sigma_2 = 1, t_3' = 0, t_3'' = -b \). Then (4.7) reduces to:
\[
d_3^{-1} D_1^{\prime\prime} \left[ \partial_2^\beta x_1^\prime \right] - \sigma_1 d_3^{-1} D_1^{\prime\prime} D_2 [x_1^\prime x_3] = \]
\[
= -P_{n-1} e_0' D_1^{\prime\prime} T_{\beta(5)} + D_1^{\prime\prime} T_{\beta(5)}.\]

Note that \( \sigma_1 \neq 0 \) because \( \sigma_1 + 1 = n \), and, if \( \sigma_1 = 0 \), then \( n = 1 \), the term with \( T_{\beta(5)} \) disappears and \( w = w_n + w_{n-1} \), hence we have:
\[
w = D_1 T_1 - D_1 \Delta^\prime d_1' [x_1^\beta] b.
\]

Since \( e_3 w = 0 \) and \( e_3 \) annihilates the first summands but does not annihilate the second one, we arrive at a contradiction. Therefore \( \sigma_1 \neq 0 \).
We already know that the \( \ell \)th in both sides of (1.10) are equal, so let us look at the next ones, i.e., the coefficients of \( \partial_1^{\sigma_1-1} \partial_2 \). In the left-hand side we get:

\[
(4.11) \quad \sigma_1 d_3^-[x_1^{p-1}]([\partial_2 x_2] - [\partial_3 x_3]).
\]

In order to do the same for the right-hand side, we need the second lexicographically ordered term of \( D_1^{\sigma_1} \). Using Lemma 3.8 twice, we have

\[
D_1^{\sigma_1} = A^{\sigma_1} h^{[\sigma_1]} + \ldots + \partial_1^{\sigma_1} h^{[\sigma_1]} + \sigma_1 \partial_2 \partial_1^{\sigma_1-1} f_1(h_1 - 1)^{[\sigma_1]} h^{[\sigma_1]} + \ldots
\]

Hence the coefficient of \( \partial_1^{\sigma_1-1} \partial_2 \) on the right-hand side of (4.10) is:

\[
-e_0'(h_1 + 1)^{[\sigma_1-1]}(h + 1)^{[\sigma_1]} f_1 T_{\beta(3)} + (\sigma_1 - 1)^{[\sigma_1-1]}(p + 1)^{[\sigma_1-1]}.
\]

Since the weights of \( f_1 T_{\beta(3)} \) and \( T_{\beta(3)} \) are both equal to \( (p - 1, 1) \), this becomes:

\[
(4.12) \quad -(p(p + 2)e_0' f_1 T_{\beta(3)} + (p - \sigma_1 + 1)e_0' T_{\beta(3)})(p - 1)^{[\sigma_1-1]}(p + 1)^{[\sigma_1-1]}.
\]

Comparing (1.11) and (4.12) gives:

\[
(4.13) \quad \sigma_1 d_3^-[x_1^{p-1}]([\partial_2 x_2] - [\partial_3 x_3]) = -(p - 1)^{[\sigma_1-1]}(p + 1)^{[\sigma_1-1]} (p(p + 2)e_0' f_1 T_{\beta(3)} + (p - \sigma_1 + 1)e_0' T_{\beta(3)}).
\]

Since \( e_0' \) commutes with \( f_1 \), we can use our previous calculation of \( e_0' T_{\beta(3)} \):

\[
e_0' T_{\beta(3)} = -d_3^- \partial_2 x_1^p b.
\]

Hence

\[
(4.14) \quad e_0' f_1 T_{\beta(3)} = f_1 e_0' T_{\beta(3)} = d_3^- \partial_2 x_2^p b - p d_3^- \partial_2 x_2 x_1^p b.
\]

Also, our previous calculation of \( e_0' T_{\beta(3)} \) shows that in general terms with \( d_1^- \), \( d_2^- \) are present in \( e_0' T_{\beta(3)} \). But (4.13) shows that these terms have to be zero because there are no such terms in the other entries in (1.13). We conclude that \( p = 1, \sigma_3 = 0, \sigma_2 = 1 \) and \( \sigma_1 \neq 0 \). The latter condition implies \( \sigma_1 = 1 \) because \( \sigma_1 \leq p \). Let us write again the coefficients of \( \partial_1^{\sigma_1-1} \partial_2 = \partial_2 \) in (4.10), which are given by (4.12), for our specific data:

\[
d_3^- ([\partial_2 x_2] - [\partial_3 x_3]) = 2e_0' f_1 T_{\beta(3)} - e_0' T_{\beta(3)}.
\]

Together with (4.14) and (4.13), we arrive at

\[
d_3^- ([\partial_2 x_2] - [\partial_3 x_3]) = d_3^- (3([\partial_1 x_1] - [\partial_2 x_2])(-1/6) - 3t_5[\partial_3 x_3])
\]

which gives \( t_5 = 1/2 \).

Let us calculate also the terms with \( \partial_3 \) at both sides of (4.10). We get

\[
\hat{\partial}_3 d_3^- [\partial_2 x_2] = -P_3(e_0' \partial_3 (f_1 h_1 + f_2 f_1) T_{\beta(3)} + \partial_3 f_2 T_{\beta(3)})
\]

Clearly

\[
(f_1 h_1 + f_2 f_1) d_3^- [\partial_2 x_1] = d_3^- (f_1 h_1 + f_2 f_1) [\partial_2 x_1] = d_3^- [\partial_2 x_2] 3
\]

and

\[
f_2 d_3^- [\partial_3 x_3] = d_3^- f_2 [\partial_3 x_3] = -d_3^- [\partial_2 x_3].
\]
Combining these equations, we get:
\[ d^{-3} [\partial_3 x_3] 2 = d^{-3} [\partial_2 x_3] \frac{1}{2} + d^{-3} [\partial_2 x_3] \frac{1}{2}, \]
a contradiction. This closes the last case and ends the proof of Theorem 4.1.

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