Estimates for fractional type Marcinkiewicz integrals with non-doubling measures

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Abstract

Under the assumption that $\mu$ is a non-doubling measure on $\mathbb{R}^d$ satisfying the growth condition, the authors prove that the fractional type Marcinkiewicz integral $M$ is bounded from the Hardy space $H^{1,\infty}_{\mu}$ to the Lebesgue space $L^q(\mu)$ for $\frac{1}{q} = 1 - \frac{n}{\alpha}$ with kernel satisfying a certain Hörmander-type condition. In addition, the authors show that for $p = \frac{n}{\alpha}$, $M$ is bounded from the Morrey space $M^{p,q}_{\mu}$ to the space $\text{RBMO}(\mu)$ and from the Lebesgue space $L^{n/\alpha}(\mu)$ to the space $\text{RBMO}(\mu)$.

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1 Introduction

Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$ which satisfies the following growth condition: for all $x \in \mathbb{R}^d$ and all $r > 0$,

$$\mu(B(x,r)) \leq C_0 r^n,$$

(1.1)

where $C_0$ and $n$ are positive constants and $n \in (0,d]$. $B(x,r)$ is the open ball centered at $x$ and having radius $r$. So $\mu$ is claimed to be non-doubling measure. If there exists a positive constant $C$ such that for any $x \in \text{supp}(\mu)$ and $r > 0$, $\mu(B(x,2r)) \leq C \mu(B(x,r))$, the $\mu$ is said to be doubling measure. It is well known that the doubling condition on underlying measures is a key assumption in the classical theory of harmonic analysis. Especially, in recent years, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved still valid if the underlying measure is a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies (1.1) (see [1–8]). The motivation for developing the analysis with non-doubling measures and some examples of non-doubling measures can be found in [9]. We only point out that the analysis with non-doubling measures played a striking role in solving the long-standing open Painlevé’s problem by Tolsa in [10].

Let $K(x,y)$ be a $\mu$-locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\}$. Assume that there exists a positive constant $C$ such that for any $x, y \in \mathbb{R}^d$ with $x \neq y$,

$$|K(x,y)| \leq C|x - y|^{-\frac{n}{\alpha} - 1},$$

(1.2)
and for any \(x, y, y' \in \mathbb{R}^d\),

\[
\int_{|x-y| \geq 2|y-y'|} \left[ |K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \right] \frac{1}{|x-y|} \, d\mu(x) \leq C. \tag{1.3}
\]

The fractional type Marcinkiewicz integral \(M\) associated to the above kernel \(K(x,y)\) and the measure \(\mu\) as in (1.1) is defined by

\[
M(f)(x) = \left( \int_0^\infty \left( \int_{|x-y| \leq t} \frac{|K(x,y)|}{|x-y|^{n-\alpha}} |f(y)\,d\mu(y)|^2 \, \frac{dt}{t^2} \right)^{\frac{1}{2}} \right), \quad x \in \mathbb{R}^d, \quad 0 < \alpha < n. \tag{1.4}
\]

If \(\mu\) is the \(d\)-dimensional Lebesgue measure in \(\mathbb{R}^d\), and

\[
K(x,y) = \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}, \tag{1.5}
\]

with \(\Omega\) homogeneous of degree zero and \(\Omega \in \text{Lip}_{\gamma}(S^{d-1})\) for some \(\gamma \in (0,1]\), then \(K\) satisfies (1.2) and (1.3). Under these conditions, \(M\) in (1.4) is introduced by Si \textit{et al.} in [11]. As a special case, by letting \(\alpha = 0\), we recapture the classical Marcinkiewicz integral operators that Stein introduced in 1958 (see [12]). Since then, many works have appeared about Marcinkiewicz type integral operators. A nice survey has been given by Lu in [13].

In 2007, the Hörmander-type condition was introduced by Hu \textit{et al.} in [14], which was slightly stronger than (1.3) and was defined as follows:

\[
\sup_{\ell>0, y, y' \in \mathbb{R}^d, \atop |y-y'| \leq \ell} \sum_{k=1}^\infty k \int_{\|x-y\| \leq 2^{k+1}\ell} \left[ |K(x,y) - K(x,y')| ight] \\
+ \left| K(y,x) - K(y',x) \right| \frac{1}{|x-y|} \, d\mu(x) \leq C. \tag{1.6}
\]

However, in this paper, we discover that the kernel should satisfy some other kind of smoothness condition to replace (1.6).

**Definition 1.1** Let \(1 \leq s < \infty, 0 < \varepsilon < 1\). The kernel \(K\) is said to satisfy a Hörmander-type condition if there exist \(c_s > 1\) and \(C_s > 0\) such that for any \(x \in \mathbb{R}^d\) and \(\ell > c_s|x|\),

\[
\sup_{\ell>0, y, y' \in \mathbb{R}^d, \atop |y-y'| \leq \ell} \sum_{k=1}^\infty 2^{ks} \left( 2^k \ell \right)^\varepsilon \left( \int_{\|x-y\| \leq 2^{k+1}\ell} \left[ (|K(x,y) - K(x,y')| \right] \\
+ \left| K(y,x) - K(y',x) \right| \frac{1}{|x-y|} \right)^s \, d\mu(x) \right)^{\frac{1}{s}} \leq C_s. \tag{1.7}
\]

We denote by \(\mathcal{H}^s\) the class of kernels satisfying this condition. It is clear that these classes are nested,

\[
\mathcal{H}^{s_2} \subset \mathcal{H}^{s_1} \subset \mathcal{H}^1, \quad 1 < s_1 < s_2 < \infty.
\]

We should point out that \(\mathcal{H}^1\) is not condition (1.6).
The purpose of this paper is to get some estimates for the fractional type Marcinkiewicz integral \( M \) with kernel \( K \) satisfying (1.2) and (1.7) on the Hardy-type space and the RBMO(\( \mu \)) space. To be precise, we establish the boundedness of \( M \) in \( H^{1,\infty,0}_{\text{loc}}(\mu) \) for \( \frac{1}{q} = 1 - \frac{\alpha}{n} \) in Section 2. In Section 3, we prove that \( M \) is bounded from the space RBMO(\( \mu \)) to the Morrey space \( M^p_q(\mu) \), from the space RBMO(\( \mu \)) to the Lebesgue space \( L^{n/\alpha}(\mu) \) for \( p = \frac{n}{\alpha} \).

Before stating our results, we need to recall some necessary notation and definitions.

For a cube \( Q \subset \mathbb{R}^d \), we mean a closed cube whose sides are parallel to the coordinate axes. We denote its center and its side length by \( x_Q \) and \( \ell(Q) \), respectively. Let \( \eta > 1 \), \( \eta Q \) denote the cube with the same center as \( Q \) and \( \ell(\eta Q) = \eta \ell(Q) \). Given two cubes \( Q \subset R \in \mathbb{R}^d \), set

\[
S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{[\ell(2^k Q)]^n},
\]

where \( N_{Q,R} \) is the smallest positive integer \( k \) such that \( \ell(2^k Q) \geq \ell(R) \). The concept \( S_{Q,R} \) was introduced in [15], where some useful properties of \( S_{Q,R} \) can be found.

**Lemma 1.2** For a function \( b \in L^1_{\text{loc}}(\mu) \), \( 0 < \beta \leq 1 \), conditions (i) and (ii) below are equivalent.

(i) There exist some constant \( C_2 \) and a collection of numbers \( b_Q \) such that these two properties hold: for any cube \( Q \),

\[
\frac{1}{\mu(2Q)} \int_Q |b(x) - b(y)| \, d\mu(x) \leq C_2 \ell(Q)^\beta,
\]

and for any cube \( R \) such that \( Q \subset R \) and \( \ell(R) \leq 2\ell(Q) \),

\[
|b_Q - b_R| \leq C_2 \ell(Q)^\beta.
\]

(ii) For any given \( p, 1 \leq p \leq \infty \), there is a constant \( C(p) \geq 0 \) such that for every cube \( Q \), then

\[
\left[ \frac{1}{\mu(Q)} \int_Q |b(x) - m_Q(b)|^p \, d\mu(x) \right]^{\frac{1}{p}} \leq C(p) \ell(Q)^\beta,
\]

where

\[
m_Q(b) = \frac{1}{\mu(Q)} \int_Q b(y) \, d\mu(y),
\]

and also for any cube \( R \) such that \( Q \subset R \) and \( \ell(R) \leq 2\ell(Q) \),

\[
|m_Q(b) - m_R(b)| \leq C(p) \ell(Q)^\beta.
\]

**Remark 1.3** Lemma 1.2 is a slight variant of Theorem 2.3 in [16]. To be precise, if we replace all balls in Theorem 2.3 of [16] by cubes, we then obtain Lemma 1.2.
Remark 1.4 For $0 < \beta \leq 1$, (1.9) is equivalent to

$$|b_Q - b_R| \leq CS_Q, R \ell(R)^\beta \quad (1.11)$$

for any two cubes $Q \subset R$ with $\ell(R) \leq 2\ell(Q)$ (see Remark 2.7 in [16]).

Lemma 1.5 Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$ and $q \geq \frac{\alpha}{n - \alpha}$. Then the fractional integral operator $I_\alpha$ defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy$$

is bounded from $L^p(\mu)$ to $L^r(\mu)$ (see [17]).

Lemma 1.6 Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that $K(x, y)$ satisfies (1.2) and (1.3) and $\mathcal{M}$ is as in (1.4). Then there exists a positive constant $C > 0$ such that for all bounded functions $f$ with compact support,

$$\|\mathcal{M}(f)\|_{L^r(\mu)} \leq C\|f\|_{L^p(\mu)}.$$
where the notation \( \varphi \sim x \) means that \( \varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d) \) and satisfies

1. \( \|\varphi\|_{L^1(\mu)} \leq 1 \),
2. \( 0 \leq \varphi(y) \leq \frac{1}{|x-y|^\alpha} \) for all \( y \in \mathbb{R}^d \),
3. \( |\varphi'(y)| \leq \frac{1}{|x-y|^\alpha} \) for all \( y \in \mathbb{R}^d \).

Based on Theorem 1.2 in [18], we can define the Hardy space \( H^1(\mu) \) as follows (see [15]).

**Definition 2.2** The Hardy space \( H^1(\mu) \) is the set of all functions \( f \in L^1(\mu) \) satisfying that

\[
\int_{\mathbb{R}^d} f \, d\mu = 0 \quad \text{and} \quad M_\varphi f \in L^1(\mu).
\]

Moreover, the norm of \( f \in H^1(\mu) \) is defined by

\[
\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\varphi f\|_{L^1(\mu)}.
\]

We recall the atomic Hardy space \( H^{1,\infty,0}_{\text{atb}}(\mu) \) as follows.

**Definition 2.3** Let \( \rho > 1 \). A function \( h \in L^1_{\text{loc}}(\mu) \) is called an atomic block if

1. there exists some cube \( R \) such that \( \text{supp} \ h \subset R \),
2. \( \int_{\mathbb{R}^d} h(x) \, d\mu(x) = 0 \),
3. for \( i = 1, 2 \), there are functions \( a_i \) supported on cubes \( Q_i \subset R \) and numbers \( \lambda_i \in \mathbb{R} \) such that \( h = \lambda_1 a_1 + \lambda_2 a_2 \), and

\[
\|a_i\|_{L^\infty(\mu)} \leq \left[ \mu(\rho Q_i) S_{Q_i,R} \right]^{-1}.
\]

Then define

\[
|h|_{H^{1,\infty,0}_{\text{atb}}(\mu)} = |\lambda_1| + |\lambda_2|.
\]

Define \( H^{1,\infty,0}_{\text{atb}}(\mu) \) and \( H^{1,\infty,0}_{\text{fin}}(\mu) \) as follows:

\[
\|f\|_{H^{1,\infty,0}_{\text{atb}}(\mu)} = \inf \left\{ \sum_{j=1}^\infty |h_j|_{H^{1,\infty,0}_{\text{atb}}(\mu)} : f = \sum_{j=1}^\infty h_j, \{h_j\}_{j \in \mathbb{N}} \text{ are } (1, \infty, 0)\text{-atoms} \right\}
\]

and

\[
\|f\|_{H^{1,\infty,0}_{\text{fin}}(\mu)} = \inf \left\{ \sum_{j=1}^k |h_j|_{H^{1,\infty,0}_{\text{atb}}(\mu)} : f = \sum_{j=1}^k h_j, \{h_j\}_{j=1}^k \text{ are } (1, \infty, 0)\text{-atoms} \right\},
\]

where the infimum is taken over all possible decompositions of \( f \) in atomic blocks, \( H^{1,\infty,0}_{\text{fin}}(\mu) \) is the set of all finite linear combinations of \( (1, \infty, 0)\text{-atoms} \).

**Remark 2.4** It was proved in [15] that for each \( \rho > 1 \), the atomic Hardy space \( H^{1,\infty,0}_{\text{atb}}(\mu) \) is independent of the choice of \( \rho \).

Applying the theory of Meda et al. in [19], we easily get the result as follows.

**Theorem 2.5** Let \( 0 < \alpha < n \), \( \frac{1}{q} = 1 - \frac{\alpha}{n} \). Suppose that \( K \) satisfies (1.2) and the \( H^q \) condition and \( f \in H^{1,\infty,0}_{\text{fin}}(\mu) \). Then \( M \) is bounded from the Hardy space into the Lebesgue space,
namely there exists a positive constant $C$ such that

$$\|M(f)\|_{L^q(\mu)} \leq C\|f\|_{L^\infty_0(\mu)}^r.$$  

**Proof of Theorem 2.5** Without loss of generality, we may assume that $p = 4$ and $f = \sum h$ as a finite of atomic blocks defined in Definition 2.3. It is easy to see that we only need to prove the theorem for one atomic block $h$. Let $R$ be a cube such that $supp h \subset R$, $\int_{2^d} h(x) \, d\mu(x) = 0$, and

$$h(x) = \lambda_1 a_1(x) + \lambda a_2(x), \quad \lambda_1 a_1(x) + \lambda a_2(x) \leq 1 + I_{1/2}.$$  

where $\lambda_i$ for $i = 1, 2$ is a real number, $|h_i|_{L^\infty(\mu)} = \lambda_1 + \lambda_2$, $a_i$ for $i = 1, 2$ is a bounded function supported on some cubes $Q_i \subset R$ and it satisfies

$$\|a_i\|_{L^\infty(\mu)} \leq \left[\mu(4Q_i)|S_{Q_i,R}|^{-1}\right]. \quad (2.2)$$

Write

$$\|M(h)\|_{L^q(\mu)} \leq \left(\int_{2^d} |M(h)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} + \left(\int_{2^d} |M(h)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} + \left(\int_{2^d} |M(h)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} + \left(\int_{2^d} |M(h)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} + \left(\int_{2^d} |M(h)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}$$

By (2.1), we have

$$I = \left(\int_{2^d} |M(h)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}$$

$$\leq |\lambda_1| \left(\int_{2^d} |M(a_1)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} + |\lambda_2| \left(\int_{2^d} |M(a_2)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}$$

$$= I_1 + I_2.$$  

To estimate $I_1$, we write

$$I_1 \leq |\lambda_1| \left(\int_{2Q_1} |M(a_1)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} + |\lambda_2| \left(\int_{2R\cup 2Q_1} |M(a_2)(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}$$

$$= I_{11} + I_{12}.$$  

Choose $p_i$ and $q_i$ such that $1 < p_i < \frac{n}{d}, 1 < q < q_i$ and $\frac{1}{q_i} = \frac{1}{q_1} - \frac{n}{a}$. By the Hölder inequality, the fact that $S_{Q_i,R} \geq 1$ and the $(L^p(\mu), L^q(\mu))$-boundedness of $M$ (see Lemma 1.6), we
have that
\[
I_{11} \leq |\lambda_1| \left[ \int_{2Q_1} |M(a_1)(x)|^q \, d\mu(x) \right]^\frac{1}{q} \mu(2Q_1)^{\frac{1}{q} - \frac{1}{p}}
\]
\[
\leq C|\lambda_1| \|a_1\|_{L^p(\mu)} \mu(2Q_1)^{\frac{1}{q} - \frac{1}{p}}
\]
\[
\leq C|\lambda_1| \|a_1\|_{L^\infty(\mu)} \mu(2Q_1)^{\frac{1}{q} - \frac{1}{p}}
\]
\[
\leq C|\lambda_1|.
\]

Denote \(N_{2Q_1,2\lambda}\) simply by \(N_1\). Invoking the fact that \(\|a_1\|_{L^\infty(\mu)} \leq [\mu(4Q_i)S_{Q_i,R}]^{-1}\), we thus get
\[
I_{12} \leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \int_{2^k Q_1 \setminus 2^{k+1} Q_1} \int_0^\infty \int_{|x-y| \leq t} \frac{|a_1(y)|}{|x-y|^{n-\alpha-1}} \, d\mu(y) \left[ \int_0^\infty \frac{dt}{t^3} \right]^\frac{q}{2} \, d\mu(x) \right\}^{\frac{1}{q}}
\]
\[
\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{\eta(q-\alpha)} \int_{2^k Q_1 \setminus 2^{k+1} Q_1} \int_{Q_1} |a_1(y)| \, d\mu(y) \right\}^{\frac{1}{q}}
\]
\[
\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{\eta(q-\alpha)} \mu(2^{k+1} Q_1) \|a_1\|_{L^\infty(\mu)}^q \mu(Q_1)^{\frac{q}{2}} \right\}^{\frac{1}{q}}
\]
\[
\leq C|\lambda_1| \left\{ \sum_{k=1}^{N_1+1} \ell(2^k Q_1)^{\eta(q-\alpha)} \mu(4Q_i)^{-q} S_{Q_i,R}^{-q} \mu(2^{k+1} Q_1) \|a_1\|_{L^\infty(\mu)}^q \mu(Q_1)^{\frac{q}{2}} \right\}^{\frac{1}{q}}
\]
\[
\leq C|\lambda_1| \left( S_{Q_1,R}^{-q} \sum_{k=2}^{N_1+1} \frac{\mu(2^k Q_1)}{\ell(2^k Q_1)^{\eta}} \right)^\frac{1}{q}
\]
\[
\leq C|\lambda_1|.
\]

Here we have used the fact that
\[
\sum_{k=2}^{N_1+1} \frac{\mu(2^k Q_1)}{\ell(2^k Q_1)^{\eta}} \leq C S_{Q_1,R},
\]
see [16] for details.

The estimates for \(I_{11}\) and \(I_{12}\) give the desired estimate for \(I_1\). With a similar argument, we have
\[
I_2 \leq C|\lambda_2|.
\]

Combining the estimates for \(I_1\) and \(I_2\) yields the estimate for \(I\).
For $i = 1, 2, y \in Q_i \subset R$, $x \in \mathbb{R}^d \setminus (2R)$, we have $|x - y| \sim |x - x_R| \sim |x - x_R| + 2\ell(R)$, by Minkowski's inequality, we get

$$\begin{align*}
II & \leq \left\{ \int_{\mathbb{R}^d \setminus (2R)} \int_R \frac{h(y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right\} \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq C \int_{\mathbb{R}^d \setminus (2R)} \int_R \frac{h(y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right\} \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq C \int_{\mathbb{R}^d \setminus (2R)} \int_R \frac{h(y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right\} \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq C \int_{\mathbb{R}^d \setminus (2R)} \int_R \frac{h(y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right\} \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq C \left( \sum_{j=1}^{2} \lambda_j \|a_j\|_{L^1(\mu)} \right) \left( \int_R \frac{h(y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right) \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq C \left( \sum_{j=1}^{2} \lambda_j \right).
\end{align*}$$

For any $y \in R$, we have $|x - y| \leq |x - x_R| + |y - x_R| \leq |x - x_R| + 2\ell(R) \leq t$. It follows that

$$\begin{align*}
III & \leq \left\{ \int_{\mathbb{R}^d \setminus (2R)} \int_R \frac{K(x,y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right\} \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq \left\{ \int_{\mathbb{R}^d \setminus (2R)} \int_R \frac{K(x,y)}{|x - y|^{n-\alpha}} \left( \int_{|x - x_R| + 2\ell(R)} \frac{dt}{t^\delta} \right)^{\frac{q}{2}} |x|^{\frac{\delta}{2}} \frac{dt}{t^\delta} \right\} \frac{1}{2} \frac{\mu(x)}{\mu(y)} \\
& \leq C \left( \sum_{j=1}^{2} \lambda_j \right)
\end{align*}$$
Here we have used the fact that $\frac{1}{q} = 1 - \frac{a}{n}$.

Combining the estimates for I, II and III yields that

$$\|\mathcal{M}(h)\|_{L^q(\mu)} \leq C|h|_{H^{1,\infty,0}(\mu)},$$

and this is the result of Theorem 2.5. □

3 Boundedness of $\mathcal{M}$ in RBMO($\mu$) spaces

In this section, we discuss the boundedness for $\mathcal{M}$ as in (1.4) in the space RBMO($\mu$) for $f \in M^p_q(\mu)$ and $f \in L^{\frac{n}{2}}(\mu)$, respectively.

Firstly, we need to recall the definition of Morrey space with non-doubling measure denoted by $M^p_q(\mu)$, which was introduced by Sawano and Tanaka in [20].

**Definition 3.1** Let $v > 1$ and $1 \leq q \leq p < \infty$. The Morrey space $M^p_q(\mu)$ is defined by

$$M^p_q(\mu) = \{f \in L^q_{\text{loc}}(\mu) : \|f\|_{M^p_q(\mu)} < \infty\},$$

where the norm $\|f\|_{M^p_q(\mu)}$ is given by

$$\|f\|_{M^p_q(\mu)} = \sup_Q \mu(vQ)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f(x)|^q \, d\mu(x)\right)^{\frac{1}{q}}.$$

We should note that the parameter $v > 1$ appearing in the definition does not affect the definition of the space $M^p_q(\mu)$, and $M^p_q(\mu)$ is a Banach space with its norms (see [20]).

Using the Hölder inequality to (1.4), it is easy to see that for all $1 \leq q_2 \leq q_1 \leq p$, then

$$L^p(\mu) = M^p_{p}(\mu) \subset M^p_{q_1}(\mu) \subset M^p_{q_2}(\mu).$$

**Theorem 3.2** Let $0 < \alpha < n$, $1 \leq q < p = \frac{n}{\alpha}$. Suppose that $K(x, y)$ satisfies (1.2) and the $H'$ condition, $\mathcal{M}$ is defined as in (1.4). Then there exists a positive constant $C$ such that for all $f \in M^p_{q_2}(\mu)$,

$$\|\mathcal{M}(f)\|_{\text{RBMO}(\mu)} \leq C\|f\|_{M^p_{q_2}(\mu)}.$$

**Theorem 3.3** Let $0 < \alpha < n$ and $p = \frac{n}{\alpha}$. Suppose that $K(x, y)$ satisfies (1.2) and the $H^{\frac{n}{2\alpha}}$ condition, $\mathcal{M}$ is defined as in (1.4). Then there exists a positive constant $C$ such that for all bounded functions $f$ with compact support,

$$\|\mathcal{M}(f)\|_{\text{RBMO}(\mu)} \leq C\|f\|_{L^\frac{n}{2\alpha}(\mu)}.$$

**Remark 3.4** As a special condition, we take $p = q = \frac{n}{\alpha}$, Theorem 3.3 can be deduced with a similar method of Theorem 3.2.

**Proof of Theorem 3.2** For any cubes $Q$ and $R$ in $\mathbb{R}^d$ such that $Q \subset R$ satisfies $\ell(R) \leq 2\ell(Q)$, let

$$a_Q = m_Q[M(f \chi_{\mathbb{R}^d \setminus \frac{3}{2}Q})]$$
and
\[ a_R = m_R \left[ \mathcal{M}(f \chi_{R^{n\setminus \frac{3}{2}R}}) \right]. \]

It is easy to see that \( a_Q \) and \( a_R \) are real numbers. By Lemma 1.2, we need to show that for some fixed \( r > q \), there exists a constant \( C > 0 \) such that
\[
\left( \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f(x) - a_Q)|^r \, d\mu(x) \right)^\frac{1}{r} \leq C \|f\|_{\mathcal{M}_{q}^p(\mu)} (3.1)
\]
and
\[
|a_Q - a_R| \leq C \|f\|_{\mathcal{M}_{q}^p(\mu)} (3.2)
\]

Let us first prove estimate (3.1). For a fixed cube \( Q \) and \( x \in Q \), decompose \( f = f_1 + f_2 \), where \( f_1 = f_{\frac{3}{2}Q} \) and \( f_2 = f - f_1 \). Write that
\[
\frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f(x) - a_Q)|^r \, d\mu(x)
= \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f_1 + f_2)(x) - a_Q|^r \, d\mu(x)
\leq \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f_1)(x)|^r \, d\mu(x) + \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f_2)(x) - a_Q|^r \, d\mu(x)
= I_1 + I_2.
\]

For \( \frac{q}{r} = \frac{1}{q} - \frac{q}{n} \) and \( p = \frac{q}{n} \), it follows that
\[
I_1 = \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f_1)(x)|^r \, d\mu(x)
\leq C \frac{1}{\mu(2Q)} \left( \int_{\frac{3}{2}Q} |f(x)|^q \, d\mu(x) \right)^{\frac{r}{q}}
\leq C \frac{1}{\mu(2Q)} \left( \mu(2Q)^{1 - \frac{1}{q}} \int_{\frac{3}{2}Q} |f(x)|^q \, d\mu(x) \right)^{\frac{r}{q}} \mu(2Q)^{\frac{1}{q} - \frac{1}{p}}
\leq C \|f\|_{\mathcal{M}_{q}^p(\mu)}^r \mu(2Q)^{\frac{1}{q} - \frac{1}{p}}
\leq C \|f\|_{\mathcal{M}_{q}^p(\mu)}^r.
\]

Now let us estimate the term \( I_2 \),
\[
I_2 = \frac{1}{\mu(2Q)} \int_Q |\mathcal{M}(f_2)(x) - a_Q|^r \, d\mu(x)
= \frac{1}{\mu(2Q)} \int_Q \left| \mathcal{M}(f_2)(x) - \frac{1}{\mu(Q)} \int_Q \mathcal{M}(f \chi_{R^{n\setminus \frac{3}{2}Q}})(y) \, d\mu(y) \right|^r \, d\mu(x)
= \frac{1}{\mu(2Q)} \int_Q \frac{1}{\mu(Q)} \int_Q \mathcal{M}(f_2)(x) \, d\mu(y) - \frac{1}{\mu(Q)} \int_Q \mathcal{M}(f \chi_{R^{n\setminus \frac{3}{2}Q}})(y) \, d\mu(y) \right|^r \, d\mu(x)
\leq \frac{1}{\mu(2Q)} \frac{1}{\mu(Q)} \int_Q \int_Q |\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)|^r \, d\mu(x) \, d\mu(y).
\]
In order to estimate $|\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)|$, we write

$$D_1(x, y) = \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{|K(x, z)|}{|x-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

$$D_2(x, y) = \left( \int_0^\infty \left[ \int_{|y-z|\leq t < |x-z|} \frac{|K(y, z)|}{|y-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

and

$$D_3(x, y) = \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{|K(x, z)|}{|x-z|^{\alpha}} - \frac{K(y, z)}{|y-z|^{\alpha}} \right] f_2(z) d\mu(z) \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

It is easy to get that for any $x, y \in Q$,

$$|\mathcal{M}(f_2)(x) - \mathcal{M}(f_2)(y)|$$

$$= \left| \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{K(x, z)}{|x-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} - \left( \int_0^\infty \left[ \int_{|y-z|\leq t < |x-z|} \frac{K(y, z)}{|y-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \right|$$

$$\leq \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{K(x, z)}{|x-z|^{\alpha}} f_2(z) d\mu(z) - \int_{|y-z|\leq t < |x-z|} \frac{K(y, z)}{|y-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{K(x, z)}{|x-z|^{\alpha}} f_2(z) d\mu(z) + \int_{|y-z|\leq t < |x-z|} \frac{K(x, z)}{|x-z|^{\alpha}} f_2(z) d\mu(z) \right. \right.$$ 

$$\left. - \int_{|y-z|\leq t < |x-z|} \frac{K(y, z)}{|y-z|^{\alpha}} f_2(z) d\mu(z) - \int_{|x-z|\leq t < |y-z|} \frac{K(y, z)}{|y-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{K(x, z)}{|x-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$+ \left( \int_0^\infty \left[ \int_{|y-z|\leq t < |x-z|} \frac{K(y, z)}{|y-z|^{\alpha}} f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$+ \left\{ \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \left( \frac{K(x, z)}{|x-z|^{\alpha}} - \frac{K(y, z)}{|y-z|^{\alpha}} \right) f_2(z) d\mu(z) \right]^2 \frac{dt}{t^3} \right\}^{\frac{1}{2}}$$

$$\leq \sum_{j=1}^3 D_j(x, y).$$

For $D_1(x, y)$, since $x, y \in Q$, $z \in \frac{1}{2}Q$, thus we get

$$D_1(x, y) \leq C \left( \int_0^\infty \left[ \int_{|x-z|\leq t < |y-z|} \frac{|f_2(z)|}{|x-z|^{\alpha-1}} d\mu(z) \right]^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq C \int_{|x-z|\leq |y-z|} \frac{|f_2(z)|}{|x-z|^{\alpha-1}} \left( \int_{|x-z|}^{|y-z|} \frac{dt}{t^3} \right)^{\frac{1}{2}} d\mu(z)$$

$$\leq Ct \left( Q \right)^{\frac{1}{2}} \int_{|x-z|\leq |y-z|} \frac{|f_2(z)|}{|x-z|^{\alpha-\frac{1}{2}}} d\mu(z).$$
\[
\begin{align*}
&\leq C\ell(Q)^{\frac{1}{2}} \int_{|x-z|^{\alpha-\frac{a}{2}}} |f(x) - f(z)| d\mu(z) \\
&\leq C\ell(Q)^{\frac{1}{2}} \sum_{k=1}^{\infty} \int_{x^{k-1}Q \cap |x-z|^{\alpha-\frac{a}{2}}} |f(x) - f(z)| d\mu(z) \\
&\leq C\ell(Q)^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{\ell(\frac{3}{2}2^k Q)^{\alpha-a}} \int_{x^{k-1}Q} |f(z)| d\mu(z) \\
&\leq C \sum_{k=1}^{\infty} 2^{-\frac{a}{2}} \frac{1}{\ell(\frac{3}{2}2^k Q)^{\alpha-a}} \left( \int_{x^{k-1}Q} |f(z)|^q d\mu(z) \right)^{\frac{1}{q}} \mu \left( \frac{3}{2}2^k Q \right)^{\frac{1}{q}} \\
&\leq C \|f\|_{M^p_{\alpha}(\mu)} \sum_{k=1}^{\infty} 2^{-\frac{a}{2}} \\
&\leq C \|f\|_{M^p_{\alpha}(\mu)}.
\end{align*}
\]

By a similar argument, it follows that

\[D_2(x,y) \leq C \|f\|_{M^p_{\alpha}(\mu)}.\]

Finally, by the condition \(H^p\), which the kernel \(K(x,y)\) conditions, applying Minkowski's inequality, and the fact that \(\alpha = \frac{n}{p}\), we have

\[D_3(x,y) = \int_0^{\infty} \left( \int_{|x-z|^{\alpha-\frac{a}{2}}} \left| \frac{K(x,z)}{|x-z|^{\alpha}} - \frac{K(y,z)}{|y-z|^{\alpha}} \right|^2 d\mu(z) \right)^{\frac{1}{2}} dt \]

\[\leq C \int_{|x-z|^{\alpha-\frac{a}{2}}} \left| \frac{K(x,z)}{|x-z|^{\alpha}} - \frac{K(y,z)}{|y-z|^{\alpha}} \right| d\mu(z) \]

\[\leq C \sum_{k=1}^{\infty} \int_{\frac{1}{2}2^{k+1}Q \cap |x-z|^{\alpha-\frac{a}{2}}} \left| \frac{K(x,z)}{|x-z|^{\alpha}} - \frac{K(y,z)}{|y-z|^{\alpha}} \right| d\mu(z) \]

\[\leq C \|f\|_{M^p_{\alpha}(\mu)} \sum_{k=1}^{\infty} \mu \left( 2^k Q \right)^{\frac{1}{2} - \frac{1}{p}} \]

\[\times \left\{ \int_{\frac{1}{2}2^{k+1}Q \cap |x-z|^{\alpha-\frac{a}{2}}} \left| \frac{1}{|y-z|^{\alpha}} \right| d\mu(z) \right\}^{\frac{1}{q}} \]

\[\leq C \|f\|_{M^p_{\alpha}(\mu)} \sum_{k=1}^{\infty} \ell \left( \frac{3}{2}2^k Q \right)^{\frac{n}{q}} \]

\[\times \left\{ \int_{\frac{1}{2}2^{k+1}Q \cap |x-z|^{\alpha-\frac{a}{2}}} \left| \frac{1}{|y-z|^{\alpha}} \right| d\mu(z) \right\}^{\frac{1}{q}} \]

\[\leq C \|f\|_{M^p_{\alpha}(\mu)} \sum_{k=1}^{\infty} \ell \left( \frac{3}{2}2^k Q \right)^{\frac{n}{q}} \ell \left( \frac{3}{2}2^k Q \right)^n \]

\[\times \left\{ \frac{1}{\ell(\frac{3}{2}2^k Q)^{n}} \int_{\frac{1}{2}2^{k+1}Q \cap |x-z|^{\alpha-\frac{a}{2}}} \left| K(x,z) - K(y,z) \right| \frac{1}{|y-z|^{\alpha}} d\mu(z) \right\}^{\frac{1}{q}} \]
\[ C \| f \|_{M_{\mu}^p} + C \| f \|_{M_{\mu}^p} \sum_{k=1}^{\infty} (2^k \ell(\frac{3}{2}2^k Q))^\frac{n}{n-\alpha} \ell(Q)^\alpha \left( \int_{2^{k+1}Q \cap \frac{3}{2}2^k Q} \frac{1}{|y-z|^{\alpha q}} \, d\mu(z) \right)^\frac{1}{q} \]

\[ \leq C \| f \|_{M_{\mu}^p}. \]

Combining these estimates, we conclude that

\[ I_2 \leq C \| f \|_{M_{\mu}^p}, \]

and so estimate (3.1) is proved.

We proceed to show (3.2). For any cubes \( Q \subset R \) with \( x \in Q \), denote \( N_{Q,R+1} \) simply by \( N \).

Write

\[ |a_Q - a_R| \leq \left| m_R[M(f \chi_{R \cap 2N}Q)] - m_Q[M(f \chi_{R \cap 2N}Q)] \right| + \left| m_Q[M(f \chi_{2N \cap Q} \frac{3}{2}Q)] \right| \]

\[ = E_1 + E_2 + E_3. \]

As in the estimate for the term \( I_2 \), then

\[ E_2 \leq C \| f \|_{M_{\mu}^p}. \]

We conclude from \( y \in R, z \in 2NQ \setminus \frac{3}{2}Q \) that

\[ M(f \chi_{2NQ \setminus \frac{3}{2}R})(y) \leq C \int_{2NQ \setminus \frac{3}{2}R} \left| \frac{K(y,z)}{|y-z|^{\alpha q}} \left( \int_{\frac{1}{|y-z|}}^\infty \frac{dt}{t^3} \right)^\frac{1}{2} \right| \, d\mu(z) \]

\[ \leq C \int_{2NQ \setminus \frac{3}{2}R} \frac{|f(z)|}{|y-z|^{\alpha q-a}} \, d\mu(z) \]

\[ \leq C \ell(R)^{\alpha-\alpha} \int_{2NQ \setminus \frac{3}{2}R} |f(z)|^a \, d\mu(z) \]

\[ \leq C \ell(R)^{\alpha-\alpha} \left( \int_{2NQ \setminus \frac{3}{2}R} |f(z)|^q \, d\mu(z) \right)^\frac{1}{q} \mu(2NQ)^{\frac{1}{q}} \mu(2NQ)^{1-\frac{1}{q}} \]

\[ \leq C \ell(R)^{\alpha-\alpha} \mu(2NQ)^{\frac{1}{q}} \left( \int_{2NQ} |f(z)|^q \, d\mu(z) \right)^\frac{1}{q} \mu(2NQ)^{1-\frac{1}{q}} \]

\[ \leq C \| f \|_{M_{\mu}^p} \ell(2NQ)^{\alpha-\frac{\alpha}{2}} \]

\[ \leq C \| f \|_{M_{\mu}^p}. \]

Taking mean over \( y \in R \), we obtain

\[ E_3 \leq C \| f \|_{M_{\mu}^p}. \]

Analysis similar to that in the estimates for \( E_3 \) shows that

\[ E_2 \leq C \| f \|_{M_{\mu}^p}. \]

Finally, we get (3.2) and this is precisely the assertion of Theorem 3.2. \( \square \)
Competing interests
The authors declare that they do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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