Global uniqueness of the compact support source identification problem

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Abstract. In this paper, we formulate a global theorem of uniqueness of the solution of the inverse problem for the reconstruction of a special-kind source in heat equation. As an overdetermination we take compact-support trace of the solution on an internal segment inside the domain at the final moment of time. We assume the symmetry of the domain along the coordinate axis and the validity of some restrictions on the signs of a given term in the source and its derivatives. No restrictions on the size of the domain or on the absolute value of the given functions assumed. We prove corresponding facts for the solution of the direct problem. These are maximum principle and belonging of the solution to the peculiar class of functions.

1. Direct problem statement

Let $T > 0$, $0 < \alpha < 1$ – fixed numbers, on the plane with Cartesian coordinates of points $x = (x_1, x_2)$ a bounded domain $\Omega$ with a smooth boundary of class $C^{2,\alpha}$. $\Omega$ is given. In the point space $(x, t)$ we define a cylinder $\Omega_T = \Omega \times (0, T]$ with a side boundary $\Gamma_T = \partial \Omega \times [0, T]$.

In a closed cylinder $\overline{\Omega}_T = \overline{\Omega} \times [0, T]$ consider the first initial-boundary value problem for the heat equation with a source of a special form, namely, the problem of determining the function $u : \Omega_T \mapsto \mathbb{R}$ from the conditions:

\[ (Lu)(x, t) = u_t(x, t) - \Delta u(x, t) = f(x_1)h(x, t) + g(x, t), \quad (x, t) \in \Omega_T, \quad (1) \]
\[ u(x, t) = \mu(x, t), \quad (x, t) \in \Gamma_T, \quad u(x, 0) = \phi(x), \quad x \in \Omega. \quad (2) \]

In the equation (1) and conditions (2) functions $f : \mathbb{R} \mapsto \mathbb{R}$, $h, g : \overline{\Omega}_T \mapsto \mathbb{R}$, $\mu : \Gamma_T \mapsto \mathbb{R}$, $\phi : \Omega \mapsto \mathbb{R}$ are considered to be known.

On the function $f$ we assume that it has a compact support. We assume that there are such numbers $a, b$ that this function differs from 0 only on the segment $[a, b]$. In addition, we assume that there is such a number $c$ that the interval $\omega = (a, b) \times c$ lies entirely in the region $\Omega$ and moreover $\overline{\omega} = [a, b] \times c \subset \Omega$.

2. Direct problem solution properties

If the functions defined in the conditions (1) – (2) satisfy the constraint $h, g \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$, $\mu \in C(\Gamma_T)$, $\phi \in C(\overline{\Omega})$, $f \in C^\alpha(\mathbb{R})$, the conditions of matching $\mu(x, 0) = \phi(x), x \in \partial \Omega$, it is known (see, e.g., [1], p. 95), task (1) – (2) has a unique solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T) \cap C(\overline{\Omega}_T)$.

In the future, when studying the inverse problem for the problem (1) – (2), we assume that $f$ satisfies only the condition $f \in C^\alpha[a, b]$, that is, generally speaking $f \notin C^\alpha(\mathbb{R})$. In this case, as
it is known ([2]), the function \( u(x, t) \) — solution of the problem (1) – (2) may not have second derivatives of \( x_1 \). In this case, the solution of problem 1 should be understood in a generalized sense. For the convenience of further reasoning, we define the sets:

\[
P = \left\{ (x_1, x_2) : a < x_1 < b, \ x_2 \in \mathbb{R} \right\},
\]

\[
\overline{P} = \left\{ (x_1, x_2) : a \leq x_1 \leq b, \ x_2 \in \mathbb{R} \right\},
\]

\[
P_t = P \times (0, T], \quad \overline{P}_T = \overline{P} \times [0, T].
\]

We define the space of functions \( U(\Omega_T) \) by the rule:

\[
U(\Omega_T) = C(\overline{\Omega}_T) \cap C^{2+\alpha, 1+\alpha}(\overline{P}_T \cap \Omega_T) \cap C^{2+\alpha}(\Omega_T \setminus \overline{P}_T) \cap C^{1,0}_x(\Omega_T).
\]

We consider the solution of the problem (1) – (2) in the class of functions \( U(\Omega_T) \).

**Remark 1.** Everywhere further when writing the equation 1 as it is usually accepted we will write \( (x, t) \in \Omega_T \) though as it follows from definition of class \( U(\Omega_T) \) at \( x_1 = a, \ x_1 = b \) the equation (1) can be not executed.

**Remark 2.** the solution of equation (1), generally speaking, is not a generalized solution of the problem (1) – (2) in the space \( W^{1,0}_2(\Omega_T) \), since the functions \( \phi, \mu \) can only be continuous. But at \( \phi = 0, \mu = 0 \) this solution will be, of course.

**Remark 3.** an Important property of the class of functions \( U(\Omega_T) \), which will then be used principally in the formulation of the inverse problem for equation(1) is that \( u(x_1, c, T) \in C^{2,\alpha}[a, b] \) at \( u \in U(\Omega_T) \).

To consider the inverse problem for equation (1), we impose additional restrictions on the domain \( \Omega \). We say that the domain \( \Omega \) satisfies the condition (A) when

(i) there is a function \( x_2 = \nu(x_1), \ 0 \leq x_1 \leq l \), such that \( \Omega = \{(x_1, x_2) : x_2 = \nu(x_1), \ 0 < x_1 < l\} \),

(ii) \( \partial \Omega \in C^{2,\alpha} \).

Clearly, the class of such areas is not empty. All conditions are satisfied, for example, a circle of radius \( \frac{l}{2} \) centered at \( \left( \frac{l}{2}, 0 \right) \).

We assume that the interval \( \omega \) lies on the axis \( Ox_1 \), that is \( c = 0, \ 0 < a < b < l \).

In the space \( U(\Omega_T) \) for solutions of the equation (1), as well as for classical solutions, the maximum principle is valid.

**Lemma.** Let the domain \( \Omega \) satisfy the condition (A), function \( u \in U(\Omega_T) \). Then if \( (Lu)(x, t) \leq 0 \) \( ((Lu)(x, t) \geq 0) \), \( (x, t) \in \Omega_T \), the function \( u \) takes the value of the positive maximum (negative minimum) (if any of them, of course) on the parabolic boundary, i.e. either at \( t = 0 \), or if \( x \in \partial \Omega \).

The formulated Lemma implies the uniqueness of the solution of the problem (1) – (2) in the class \( U(\Omega_T) \) and the positivity of the solution of the problem (1) – (2) in \( \Omega_T \) at \( (Lu)(x, t) \geq 0 \), \( (Lu)(x, t) \neq 0 \) in \( \Omega_T \).

Using thermal potentials (volume and double layer) (see [2]), we can prove the following theorem of existence and uniqueness of the solution of the problem (1) – (2) in the class \( \Omega_T \).
Theorem 1. Let $\mu \in C(\Gamma_T)$, $\phi \in C(\Omega)$, $f \in C^\alpha[a, b]$, $h, g \in C^{\alpha, \frac{\alpha}{2}}(\Omega_T)$, the matching conditions $\mu(x, 0) = \phi(x)$, $x \in \partial\Omega$. Then there is a unique solution to the problem (1) – (2) in the class of functions $U(\Omega_T)$.

3. Uniqueness for the inverse problem solution

For the formulation of the inverse problem for equation (1) we assume that not only the function $u(x, t)$ is unknown in equations (1), but also the function $f(x_1)$. In this case, we assume that for the solution of equation (1) the information about the function $u(x, t)$ is given not only in the form of condition (2), but also in the form of additional conditions (so-called overrides) in the following form:

$$u(x_1, 0, T) = \chi(x_1), \quad x_1 \in [a, b]. \quad (3)$$

Thus, we obtain the following inverse problem:

Define a pair of functions $(u, f) \in U(\Omega_T) \times C^\alpha[a, b]$ from conditions (1) – (3).

The inverse problem of determining the source with two compact carriers and a slightly different redefinition for the heat equation was studied in [3]. In [4] the inverse problem (1) – (3) for the case of a source with two compact supports was considered, and for it in the class $U(\Omega_T)$ a local (a little $b - a$) theorem of existence and uniqueness was proved. In this paper, we formulate a global theorem of uniqueness of the solution of the inverse problem (1) – (3). A little $b - a$ is not assumed. Instead, we assume the symmetry of the $\Omega$ domain along the $Ox_1$ axis and the validity of some restrictions on the signs of the function $h(x, t)$ and its derivatives.

Other formulations of inverse problems of determining the source of the history of the study of these problems with him in [5] – [10].

For the problem (1) – (3) the following global uniqueness theorem holds.

Theorem 2. Let $h, h_t, h_{x_2} \in C^{\alpha, \frac{\alpha}{2}}(\Omega_T)$, $|h(x_1, 0, T)| \geq h_T > 0$, $x_1 \in [a, b]$, function

$$\tilde{h}(x, t) = \frac{1}{2}(h(x_1, x_2, t) + h(x_1, -x_2, t)), \quad (x_1, x_2) \in \Omega$$

satisfies the following conditions: $\tilde{h}(x, t) \geq 0$, $\tilde{h}_t(x, t) \geq 0$, $(x, t) \in \Omega$, $x_2 \leq 0$. Then problem (1) – (3) cannot have 2 different solutions.

References

[1] Friedman A 1967 Partial differential Equations of parabolic type (in Russian) (Moscow: Nauka)
[2] Ladyzhenskaya O, Solonnikov V and Ural'tseva N 1967 Linear and quasi-linear equations of parabolic type (in Russian) (Moscow: Nauka)
[3] Denisov A 2016 J. of Comp. Math. and Math Phys. 56 No. 10 1754-59
[4] Soloviev V 2018 J. of Comp. Math. and Math Phys. 58 No. 5 778-89
[5] Denisov A 1994 Introduction to the theory of inverse problems (in Russian) (Moscow: MSU publishing)
[6] Lavrent'ev M 1973 Conditionally well-posed problems for differential equations (in Russian) (Novosibirsk: NSU publishing)
[7] Prilepko A., Orlovsky D. and Vasin I 2010 Methods for solving problems in mathematical physics (New-York, Basel: Marcel Dekker, Inc.)
[8] Isakov V 2006 Inverse problems for differential equations (New-York: Springer)
[9] Kostin A 2013 Matematicheskie Sbornik (in Russian) 204 No. 10 3-46
[10] Soloviev V 1989 Differential equations (in Russian) 29 No. 9 1577-83.