Natural connections on the bundle of Riemannian metrics

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Abstract

Let $F_M, \mathcal{M}_M$ be the bundles of linear frames and Riemannian metrics of a manifold $M$, respectively. The existence of a unique $\text{Diff}_M$-invariant connection form on $J^1\mathcal{M}_M \times_M FM \to J^1\mathcal{M}_M$, which is Riemannian with respect to the universal metric on $J^1\mathcal{M}_M \times_M TM$, is proved. Applications to the construction of universal Pontryagin and Euler forms, are given.

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1 Introduction

Let $q: \mathcal{M}_M \to M$ be the bundle of Riemannian metrics of a smooth manifold $M$ of dimension $n$. The goal of this paper is to prove that the bundle $q^*_1 FM \to J^1\mathcal{M}_M$, obtained by pulling the linear frame bundle $FM$ back to the 1-jet bundle of metrics, is endowed with a unique $\text{Diff}_M$-invariant connection form $\omega$—called the universal Levi-Civita connection—with the property of being Riemannian with respect to the universal metric $g$ on $q^*_1 TM$; or equivalently, $\omega$ is the only $\text{Diff}_M$-invariant connection form on the subbundle $OM$ of pairs $(u_x, j^*_x g) \in q^*_1 FM$ such that $u_x$ is $g_x$-orthonormal (see Theorem 5.1 and the precise definitions below). This result is analogous to that proving the existence of a canonical connection on the principal $G$-bundle $J^1 P \to C(P)$, where $C(P)$ is the bundle of connections of a principal $G$-bundle $P \to M$ (see [2]).

As is well known (e.g., see [1]), the Levi-Civita map, which assigns its Levi-Civita connection to every Riemannian metric, is a natural map, i.e., it is $\text{Diff}_M$-equivariant with respect to the natural actions of the diffeomorphism
group on the space of Riemannian metrics on $M$ and on the space of linear connections on $M$. This map induces a $\text{Diff} M$-invariant connection form $\omega_{\text{hor}}$ on $q^*_1FM \to J^1\mathcal{M}_M$, called the horizontal Levi-Civita connection as it is horizontal with respect to the projection $q^*_1FM = J^1\mathcal{M}_M \times_M FM \to FM$; but $\omega_{\text{hor}}$ is not a Riemannian connection, i.e., it is not reducible to $OM$. Surprisingly, $\omega$ is obtained by adding a contact form to $\omega_{\text{hor}}$, thus showing that the contact structure on $J^1\mathcal{M}_M$ plays a crucial role in our construction.

The connection form $\omega$ allows us to construct the universal Pontryagin and Euler differential forms on $J^1\mathcal{M}_M$, which contain more information than the corresponding cohomology classes on $M$; for example the forms of degree greater than $n$ do not vanish necessarily—unlike their cohomology classes. We also remark on the fact that such forms play the same role, in metric theory, than the universal characteristic forms introduced in \[ \mathfrak{K} \] in gauge theories.

2. The bundle of the bundle of metrics

2.1 The bundle of metrics

The bundle of Riemannian metrics $q: \mathcal{M}_M \to M$ is a convex open subset in $S^2(T^*M)$ and every Riemannian metric $g$ is identified to a global section $g: M \to \mathcal{M}_M$ of this bundle.

Every system of coordinates $(U; x^i)$ on $M$ induces a system of coordinates $(q^{-1}U; x^i, y_{ij})$ on $\mathcal{M}_M$ by setting $g_x = y_{ij}(g_x)(dx^i) \otimes (dx^j)$, $\forall g_x \in \mathcal{M}_M$, $x \in U$. We denote by $(y^{ij})$ the inverse matrix of $(y_{ij})$.

The diffeomorphism group of $M$ acts in a natural way on $\mathcal{M}_M$ by automorphisms of this bundle: The natural lift of a diffeomorphism $\phi \in \text{Diff} M$ to the bundle of metrics $\bar{\phi}: \mathcal{M}_M \to \mathcal{M}_M$ is defined by

\begin{equation}
(1) \quad \bar{\phi}(g_x) = (\phi^*)^{-1}(g_x) \in (\mathcal{M}_M)_{\phi(x)};
\end{equation}

$\phi^*: S^2T^*_{\phi(x)}M \to S^2T^*_xM$ being the induced homomorphism. Hence $q \circ \bar{\phi} = \phi \circ q$. In the same way, the lift of a vector field $X \in \mathfrak{X}(M)$ is denoted by $\bar{X} \in \mathfrak{X}(\mathcal{M}_M)$. If $X = X^i \partial / \partial x^i$, then

\begin{equation}
(2) \quad \bar{X} = X^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial X^r}{\partial x^i} y_{kj} + \frac{\partial X^k}{\partial x^i} y_{ki} \right) \frac{\partial}{\partial y_{ij}}.
\end{equation}

2.2 Jets of metrics

Let $q_r: J^r \mathcal{M}_M \to M$ be the $r$-jet bundle of sections of $\mathcal{M}_M$ and, for every $r \geq s$, let $q_{rs}: J^r \mathcal{M}_M \to J^s \mathcal{M}_M$ be the canonical projections. For every $\phi \in \text{Diff} M$ we denote by $\bar{\phi}^{(r)}$ the natural prolongation to $J^r \mathcal{M}_M$ of the lift $\bar{\phi}$ given in (1); precisely, $\bar{\phi}^{(r)}(j^r_x g) = j^r_x(\bar{\phi} \circ g \circ \phi^{-1})$. Similarly, $X^{(r)}$ denotes the jet prolongation of the lift $X \in \mathfrak{X}(\mathcal{M}_M)$ given in (2).

Let $(q_r^{-1}U; x^i, y_{ij}, y_{ij,l}, 1 \leq |l| \leq r, I \in \mathbb{N}^n)$, be the coordinate system induced by $(q^{-1}U; x^i, y_{ij})$; i.e., $y_{ij,l}(j^l_x g) = (\partial^{[l]}(y_{ij} \circ g) / \partial x^j)(x)$. If $(U; x^i)$
is a normal coordinate system for the metric $g$ centered at $x$, then we have $g_{ij}(j^1_x g) = \delta_{ij}$, $g_{ij,k}(j^1_x g) = 0$.

Let us fix a coordinate system $(U; x^i)$ centered at $x \in M$, and let $\phi \in \text{Diff} M$ be a diffeomorphism such that $\phi(x) \in U$. The equations of the transformation $\tilde{\phi}^{(1)}$ are as follows:

\[
\begin{aligned}
\left( \phi^{-1} \right)^a &= x^a \circ \phi^{-1}, \\
y_{ij} \circ \tilde{\phi}^{(1)} &= y_{ab} \left( \frac{\partial (\phi^{-1})^a}{\partial x^i} \circ \phi \right) \left( \frac{\partial (\phi^{-1})^b}{\partial x^j} \circ \phi \right), \\
y_{ij,k} \circ \tilde{\phi}^{(1)} &= y_{ab,c} \left( \frac{\partial (\phi^{-1})^c}{\partial x^k} \circ \phi \right) \left( \frac{\partial (\phi^{-1})^a}{\partial x^i} \circ \phi \right) \left( \frac{\partial (\phi^{-1})^b}{\partial x^j} \circ \phi \right) \\
&\quad + y_{ab} \left( \frac{\partial (\phi^{-1})^b}{\partial x^j} \circ \phi \right) \left( \frac{\partial^2 (\phi^{-1})^a}{\partial x^i \partial x^k} \circ \phi \right) \\
&\quad + y_{ab} \left( \frac{\partial (\phi^{-1})^a}{\partial x^i} \circ \phi \right) \left( \frac{\partial^2 (\phi^{-1})^b}{\partial x^j \partial x^k} \circ \phi \right).
\end{aligned}
\]

If $M$ is an orientable and connected manifold, we denote by $\text{Diff}^+ M$ the subgroup of orientation preserving diffeomorphisms. The manifold is said to be irreversible if $\text{Diff} M = \text{Diff}^+ M$; otherwise, the manifold is said to be reversible. Every compact, orientable and connected manifold of dimension $\leq 3$ is reversible; e.g., see [7, Chapter 9, §1]. We recall the following result about extending diffeomorphisms (see [7, 10]):

**Lemma 2.1.** Let $\phi: U \to U$ be a diffeomorphism defined on an open neighbourhood of $x$ in an orientable differentiable manifold $M$ such that $\phi(x) = x$. If $M$ is irreversible we further assume $\phi_{*,x} \in \text{Gl}^+ (T_x M)$. Then a global diffeomorphism $\tilde{\phi} \in \text{Diff} M$ exists coinciding with $\phi$ on a neighbourhood of $x$.

$\text{Diff}^+ M$ (and hence $\text{Diff} M$) acts transitively on every orientable connected manifold $M$. Even more, as a simple consequence of the existence of normal coordinates and Lemma 2.1 we have

**Proposition 2.2.** If $M$ is an orientable and connected manifold, then the group $\text{Diff}^+ M$ acts transitively on $J^1 M$.

Let $\theta \in \Omega^1 (J^1 M, S^2 (T^* M))$ be the structure form of $J^1 M$ (see [3]), where we use the canonical identification $V(M_M) \simeq M_M \times_M S^2 (T^* M)$; i.e., $\theta_{j_{1g}} (X) = (q_{10})_*(X) - g_*(q_1_*(X))$, $\forall X \in T_{j_{1g}} J^1 M$. In local coordinates,

\[
\theta = (dy_{ij} - y_{ij,k} dx^k) \otimes dx^i \otimes dx^j.
\]

The bundle $q^*_1 TM = J^1 M \times_M TM \to J^1 M$, obtained by pulling $TM$ back via $q_1 : J^1 M \to M$, is endowed with a universal metric given by $g_\ast \left( (j^1_{2g} X), (j^1_{2g} Y) \right) = g_*(X, Y)$, $\forall X, Y \in T_x M$, which satisfies the following universal property: $(j^1 g)^* g = g$, for every Riemannian metric $g$ on $M$. By means of this metric, we can identify $q^*_1 TM$ to $q^*_1 T^* M$.  

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If \( \alpha \in \Omega^*(J^1\mathcal{M}_M, T^*M) \), the element of \( \Omega^*(J^1\mathcal{M}_M, TM) \) corresponding to \( \alpha \) under this identification, is denoted by \( g^{-1}\alpha \). We have

\[
q_1^*(\text{EndTM}) \simeq q_1^*(S^2T^*M) = q_1^*(S^2T^*M) \oplus q_1^*(\wedge^2T^*M).
\]

Let \( \text{End}_S TM \) (resp. \( \text{End}_A TM \)) be the image of \( q_1^*(S^2T^*M) \) (resp. \( q_1^*(\wedge^2T^*M) \)) in \( q_1^*(\text{EndTM}) \) under the previous isomorphism. We have

\[
(5) \quad \vartheta = g^{-1}\theta \in \Omega^1(J^1\mathcal{M}_M, \text{End}_S TM).
\]

If \( \alpha \in \Omega^r(J^1\mathcal{M}_M, \text{EndTM}) \), then a decomposition exists such that \( \alpha = \alpha_S + \alpha_A \), where the forms \( \alpha_S \in \Omega^r(J^1\mathcal{M}_M, \text{End}_S TM) \), \( \alpha_A \in \Omega^r(J^1\mathcal{M}_M, \text{End}_A TM) \) are called the symmetric and anti-symmetric parts of \( \alpha \), respectively.

### 3 Natural connections

#### 3.1 Linear frame bundles

Let \( \pi: FM \to M \) be the linear frame bundle of \( M \), and let \( (x^i, x^j) \) be the coordinate system induced on \( \pi^{-1}U \) by a coordinate system \( (U; x^h) \) in \( M \); i.e.,

\[
u u = ((\partial/\partial x^j)_x, \ldots, (\partial/\partial x^h)_x) \cdot (x^j(u)), \quad \forall u \in \pi^{-1}(x), \quad \forall x \in U.
\]

The lift of \( \phi \in \text{Diff}M \) to \( FM \) is denoted by \( \hat{\phi}: FM \to FM \), \( \hat{\phi}(u) = \phi(u) \). Analogously, \( \hat{X} \in \mathfrak{X}(FM) \) stands for the lift of \( X \in \mathfrak{X}(M) \). Let \( q_1^*FM = J^1\mathcal{M}_M \times_M FM \) be the pull-back of \( FM \) to \( J^1\mathcal{M}_M \) via \( q_1: J^1\mathcal{M}_M \to M \). There are two canonical projections

\[
q_1^*FM \xrightarrow{\hat{q}_1} FM \quad \xrightarrow{\pi} \quad J^1\mathcal{M}_M \xrightarrow{\hat{q}_1} M
\]

The first projection \( \hat{\pi}: q_1^*FM \to J^1\mathcal{M}_M \) is a principal \( GL(n, \mathbb{R}) \)-bundle with respect to the induced action, given by \( (j_1^h g, u) \cdot A = (j_2^h g, u \cdot A), \quad \forall j_2^h g \in J^1\mathcal{M}_M, \quad \forall u \in F_{\mathcal{M}}, \quad \forall A \in GL(n, \mathbb{R}) \), and the second projection \( \hat{q}_1: q_1^*FM \to FM \) is \( GL(n, \mathbb{R}) \)-equivariant.

The diffeomorphism group \( \text{Diff}M \) acts on \( \mathcal{M}_M \) and on \( FM \) as explained above; hence it acts on \( q_1^*FM \) by the induced action. If \( \phi \in \text{Diff}M \) its lift to \( q_1^*FM \) is \( \hat{\phi} = (\tilde{\phi}^{(1)}, \tilde{\phi}) \). Similarly, if \( X \in \mathfrak{X}(M) \) we denote by \( \hat{X} \) its lift to \( q_1^*FM \). As \( \hat{q}_1 \) is a \( \text{Diff}M \)-equivariant map we have \( (\hat{q}_1)_*(\hat{X}) = \hat{X} \). For every \( t \in \mathbb{R} \), we define \( \varphi_t \in \text{AutFM} \) (resp. \( \varphi_t \in \text{Diff}(J^1\mathcal{M}_M) \), resp. \( \varphi_t \in \text{Aut}(q_1^*FM) \)) by

\[
\varphi_t(u) = \exp(-\frac{t}{2}) \cdot u,
\]

\[
\hat{\varphi}_t(j_1^h g) = j_1^h (\exp(t) \cdot g),
\]

\[
\hat{\varphi}_t(j_2^h g, u) = (j_2^h (\exp(t) \cdot g), \exp(-\frac{t}{2}) \cdot u).
\]

We denote by \( \xi \in \mathfrak{X}(FM) \) (resp. \( \xi \in \mathfrak{X}(J^1\mathcal{M}_M) \), resp. \( \hat{\xi} \in \mathfrak{X}(q_1^*FM) \)) the infinitesimal generator of the 1-parameter group \( (\varphi_t) \) (resp. \( (\varphi_t) \), resp. \( (\hat{\varphi}_t) \)) defined above. We have \( q_1*(\xi) = 0, \hat{q}_1*(\hat{\xi}) = \xi \), and \( \pi_*(\hat{\xi}) = \xi \). Hence, the group
\( \mathcal{G} = \text{Diff}M \times \mathbb{R} \) acts on the principal \( \text{Gl}(n, \mathbb{R}) \)-bundle \( \pi : q^*_1FM \to J^1\mathcal{M}_M \). by automorphisms. If \((\phi, t) \in \mathcal{G}\), we set \( \hat{\phi}_t = \phi \circ \hat{\varphi}_t = \hat{\varphi}_t \circ \phi \). This action induces a \( \mathcal{G} \)-action on the associated bundles to \( q^*_1FM \), (such as \( q^*_1TM \), \( q^*_1T^*M \), etc.), and on the space of sections and differential forms with values on such bundles.

**Proposition 3.1.** The universal metric \( g \in \Omega^0(J^1\mathcal{M}_M, S^2T^*M) \) on the bundle \( q^*_1TM \), is invariant under the action of the group \( \mathcal{G} = \text{Diff}M \times \mathbb{R} \) defined above.

**Proof.** Let \( \tilde{g} \in \Omega^0(q^*_1FM, S^2(\mathbb{R}^n)^*) \) be the \( \text{Gl}(n, \mathbb{R}) \)-invariant function on \( q^*_1FM \) corresponding to \( g \). Let \((e_i)\) be the standard basis on \( \mathbb{R}^n \) and let \((e^i)\) be its dual basis. For every \( j^1_xg \in J^1\mathcal{M}_M \) and every frame \( u_x = (X_1, \ldots, X_n) \in FM_n \) we have \( \tilde{g}(j^1_xg, u_x) = g_x(X_1, X_j)e^i \otimes e^j \). Hence, for every \((\phi, t) \in \mathcal{G}\), we have

\[
\left( \tilde{\phi}^*_t \tilde{g} \right)(j^1_xg, u_x) = \tilde{g}(\phi_t^{-1}(j^1_xg), \phi_t u_x)
= (\phi_t^{-1})^* g_{\phi(tx)}(\phi_t X_1, \phi_t X_j)e^i \otimes e^j
= g_x(X_1, X_j)e^i \otimes e^j
= \tilde{g}(j^1_xg, u_x).
\]

\( \Box \)

### 3.2 The horizontal Levi-Civita connection

We denote by \( \pi : F_qM \to M \) the orthonormal frame bundle with respect to a Riemannian metric \( g \) on \( M \), which is a reduction of group \( O(n) \) of the bundle \( FM \). We denote by \( \Gamma^g \) the Levi-Civita connection of \( g \); i.e., the only symmetric connection on \( F_qM \). If there is no risk of confusion we also denote by \( \Gamma^g \) its direct image with respect to the canonical injection \( F_qM \hookrightarrow FM \) (see [3, II, Proposition 6.1]). Analogously, \( \omega^g \) denotes the connection form of both connections, and \( \nabla^g \) the covariant derivation law with respect to \( \Gamma^g \) on the associated vector bundles.

**Proposition 3.2.** The \( \text{gl}(n, \mathbb{R}) \)-valued 1-form on \( q^*_1FM \) defined by

\[
\omega_{\text{hor}}(X) = \omega^g((\bar{q}_1)_*X), \forall X \in T(j^1_xFM),
\]

is a connection form on the principal \( \text{Gl}(n, \mathbb{R}) \)-bundle \( \pi : q^*_1FM \to J^1\mathcal{M}_M \).

**Proof.** The definition makes sense as \( \omega^g|_{\pi^{-1}(x)} \) depends only on \( j^1_xg \). We check the two characteristic properties of a connection form.

1. For every \( A \in \text{gl}(n, \mathbb{R}) \) we have

\[
\omega_{\text{hor}}(A_{j^1_xFM}) = \omega^g((\bar{q}_1)_*A_{j^1_xFM}) \quad \text{by the definition of } \omega_{\text{hor}}
= \omega^g(A_{\bar{q}_1}) \quad \text{as } \bar{q}_1 \text{ is equivariant}
= A \quad \text{as } \omega^g \text{ is a connection form}
\]

2. For every \( A \in \text{Gl}(n, \mathbb{R}) \) and \( X \in TFM_{j^1\mathcal{M}_M} \), we have
\[(R^*_A \omega_{\text{hor}})(X) = \omega_{\text{hor}}((R_A)_*X) = \omega^g((\bar{q}_1)_*(R_A)_*X) = \omega^g((R_A)_*(\bar{q}_1)_*X) \quad \text{as } \bar{q}_1 \text{ is equivariant}
\]
\[= (\text{Ad}_{A^{-1}} \circ \omega^g)((\bar{q}_1)_*X) \quad \text{as } \omega^g \text{ is a connection form}
\]
\[= (\text{Ad}_{A^{-1}} \circ \omega_{\text{hor}})(X).
\]

Remark 3.3. The bundle of orthonormal frames cannot be used in the preceding definition as it depends on the metric chosen. Below, we show that, in fact, \(\omega_{\text{hor}}\) is not reducible to the bundle of orthonormal frames, although a new connection form \(\omega\) can be defined, which will be reducible to this bundle.

The connection \(\omega_{\text{hor}}\) induces a derivation law \(\nabla_{\omega_{\text{hor}}}\) in the associated bundles to \(q_1^*FM\); in particular, on \(q_1^*TM\), \(q_1^*T^*M\), etc. In local coordinates we have
\[
\nabla_{\omega_{\text{hor}}} X = (dX^i + \Gamma^i_{jk} X^j dx^k) \otimes \frac{\partial}{\partial x^i},
\]
\[
\nabla_{\omega_{\text{hor}}} \alpha = (d\alpha_i - \Gamma^i_{jk} \alpha_k dx^k) \otimes dx^i,
\]
(6)
\[
\Omega_{\text{hor}} = (d\Gamma^i_{jk} \wedge dx^k + \Gamma^i_{as} \Gamma^s_{jk} dx^s \wedge dx^r) dx^i \otimes \frac{\partial}{\partial x^i},
\]
where \(X = X^i \partial/\partial x^i \in \Omega^0(J^1M, TM), \alpha = \alpha_i dx^i \in \Omega^0(J^1M, T^*M), \)
(7)
\[
\Gamma^i_{jk} = \frac{1}{2} y^a_{ij,k} + y^a_{ak,j} - y^a_{jk,i},
\]
and \(\Omega_{\text{hor}}\) is the curvature form of \(\omega_{\text{hor}}\).

Proposition 3.4. The connection form \(\omega_{\text{hor}}\) satisfies the following properties:

1. If \(\sigma_g: FM \to q_1^*FM\) is the equivariant section induced by a Riemannian metric \(g\) (i.e., \(\sigma_g(u_x) = (j_x^*g, u_x))\), then \(\sigma_g^*\omega_{\text{hor}}\) is the Levi-Civita connection form of \(g\).
2. The form \(\omega_{\text{hor}}\) is invariant under the action of the group \(G = \text{Diff}M \times \mathbb{R}\) on \(q_1^*FM\).
3. \(\nabla_{\omega_{\text{hor}}} g = \theta\).

Proof. (1) Let \(X \in T_{u_x}FM\). As \(\bar{q}_1 \circ \sigma_g = \text{id}_{FM}\), we have
\[
(\sigma_g^*\omega_{\text{hor}})(X) = \omega_{\text{hor}}(\sigma_g(X)) = \omega^g(\bar{q}_1 \circ \sigma_g(X)) = \omega^g((\bar{q}_1 \circ \sigma_g)_*(X)) = \omega^g(X).
\]
(2) First recall that if \( \phi \in \text{Diff}M \) and \( \omega^g \) is the Levi-Civita connection of the metric \( g \), then \((\tilde{\phi}^{-1})^*\omega^g = \omega^{\phi^*g}\) is the Levi-Civita connection of the metric \( \phi \cdot g = \phi \circ g \circ \phi^{-1} \). In the same way, if \( t \in \mathbb{R} \), then \((\tilde{\phi}^{-1})^*\omega^g\) is the Levi-Civita connection of the metric \( \exp(t) \cdot g \). From the definition of \( \omega_{\text{hor}} \), for every \( X \in T_{(j,g,u)}q^*_1FM \), and \((\phi, t) \in G\) we have

\[
(\tilde{\phi}_t^* \omega_{\text{hor}})(X) = (\omega_{\text{hor}})(\tilde{\phi}_t^* X) \\
= \omega^{\phi^* g}(\tilde{q}_1 \circ \tilde{\phi}_t \circ X) \\
= \left( (\tilde{\phi}_t^{-1})^* \omega^g \right) (\tilde{\phi}_t \circ \tilde{q}_1 \circ X) \\
= \omega^g(\tilde{q}_1 \circ X) \\
= \omega_{\text{hor}}(X).
\]

(3) In local coordinates, we have

\[
\nabla^{\omega_{\text{hor}}} g = \left( dy_{ij} - \left( y^a_{ij} \Gamma^a_{ki} + y^a_{ai} \Gamma^a_{kj} \right) dx^k \right) \otimes dx^j \otimes dx^i \\
= \left( dy_{ij} - y_{ij,k} dx^k \right) \otimes dx^j \otimes dx^i,
\]

where we have used the equation

\[
y_{ij,k} = y_{a,j} \Gamma^a_{ki} + y_{a,i} \Gamma^a_{kj},
\]

and we conclude by virtue of the formula (4).

\[
(8)
\]

3.3 The universal Levi-Civita connection

The connections on \( q^*_1FM \) are an affine space modelled over \( \Omega^1(J^1M, \text{End} T M) \). Furthermore, as \( \vartheta \in \Omega^1(J^1M, \text{End} T M) \), we can define a connection form on \( q^*_1FM \) as follows:

\[
\omega = \omega_{\text{hor}} + \frac{1}{2} \vartheta.
\]

The connection form \( \omega \) is called the universal Levi-Civita connection.

The following lemma can be proved by computing in local coordinates:

**Lemma 3.5.** If \( \alpha \in \Omega^1(J^1M, \text{End} T M) \), then \( \nabla^{\omega_{\text{hor}} + \alpha} g = \nabla^{\omega_{\text{hor}}} g - 2 \alpha_S \).

The next theorem states the basic properties of the universal Levi-Civita connection and it is analogous to Proposition 3.4.

**Theorem 3.6.** The connection form \( \omega \) satisfies the following properties:

1. With the notations of Proposition 3.4 for any Riemannian metric \( g \), the form \( \sigma^*_g \omega \) is the Levi-Civita connection form of \( g \).
2. The form \( \omega \) is invariant under the action of \( G = \text{Diff}M \times \mathbb{R} \) on \( q^*_1FM \).
3. If \( \nabla^\omega \) is the derivation law induced by \( \omega \), then

\[
(9) \quad \nabla^\omega g = 0.
\]
Proof. (1) Follows from Proposition $\S 3.4$ and from the fact that $(j^1 g)^* \vartheta = 0$.

(2) From Proposition $\S 3.1$ and Proposition $\S 3.1$ (2), we know that $\omega_{\text{hor}}$ and $g$ are $G$-invariant. Moreover, from Proposition $\S 3.1$ (3) we have $\theta = \nabla^{\omega_{\text{hor}}} g$, and hence $\theta$, as well as $\vartheta = g^{-1} \theta$, are also $G$-invariant. Hence $\omega = \omega_{\text{hor}} + \frac{1}{2} \vartheta$ is $G$-invariant.

(3) It is a consequence of Lemma $\S 3.5$ and Proposition $\S 3.1$ (3). □

Let $OM \to J^1 M_M$ be the reduction of $q_1^* FM$ to the subgroup $O(n)$ given by $OM = \{(u_x, j_x^1 g) \in q_1^* FM: u_x \text{ is } g_x\text{-orthonormal}\}$.

The following result shows the advantage of $\omega$ over $\omega_{\text{hor}}$.

**Proposition 3.7.** The connection $\omega$ is reducible to a connection on $OM$.

Proof. It follows from $\S$ III, Proposition 1.5 and the formula (9). □

**Proposition 3.8.** If $\Omega$ is the curvature form of $\omega$, then we have

\begin{equation}
\Omega = (\Omega_{\text{hor}})_A - \frac{1}{2} \vartheta \wedge \vartheta.
\end{equation}

Proof. As $\omega = \omega_{\text{hor}} + \frac{1}{2} \vartheta$, we have $\Omega = \Omega_{\text{hor}} + \frac{1}{2} d^\omega_{\text{hor}} \vartheta + \frac{1}{2} \vartheta \wedge \vartheta$. Hence, it suffices to prove $d^\omega_{\text{hor}} \vartheta = -\vartheta \wedge \vartheta - 2 (\Omega_{\text{hor}})_S$. Let $(x^i)$ be a system of normal coordinates for $g$ centered at $x$. By taking the covariant exterior differential with respect to $\omega_{\text{hor}}$ in the local expression $\vartheta = y^a (dy_{a_j} - y_{a_j,k} \wedge dx^k) \otimes dx^j \otimes \partial / \partial x^i$, and evaluating it at $j_x^1 g$, we have

\begin{equation}
(d^\omega_{\text{hor}} \vartheta)_{j_x^1 g} = (dy^a \wedge dy_{a_j} - dy_{a_j,k} \wedge dx^k)_{j_x^1 g} \otimes \left( dx^j \otimes \frac{\partial}{\partial x^i} \right)_{j_x^1 g}.
\end{equation}

Taking the exterior differential in $y^a y_{a_j} = \delta^a_j$ we obtain $dy^a = -y^b y_{b_j} dy^b$, and taking the exterior differential in the formula (8) and evaluating at $j_x^1 g$ we have $(dy_{a_j,k})_{j_x^1 g} = (d\Gamma^a_{jk})_{j_x^1 g} + (d\Gamma^a_{jk})_{j_x^1 g}$. Substituting these expressions in (11) and taking the formula (6) into account we obtain

\begin{align*}
(d^\omega_{\text{hor}} \vartheta)_{j_x^1 g} = & - \left( dy_{a_j} \wedge dy_{a_j} + (d\Gamma^a_{jk} + d\Gamma^a_{jk}) \wedge dx^k \right)_{j_x^1 g} \otimes \left( dx^j \otimes \frac{\partial}{\partial x^i} \right)_{j_x^1 g} \\
& = -\vartheta_{j_x^1 g} \wedge \vartheta_{j_x^1 g} - 2 ((\Omega_{\text{hor}})_S)_{j_x^1 g}.
\end{align*}

□

4 Universal Pontryagin and Euler forms

Let $T^d_g$ denote the Weil invariant polynomials of degree $d$ for the Lie group $G$, see $\S$ XIII. As $OM \to J^1 M_M$ is a principal $O(n)$-bundle and $\omega$ is a connection form on this bundle, the Chern-Weil construction of the characteristic classes provides us a closed differential $(2d)$-form $f(\Omega)$ on $J^1 M_M$, by applying a Weil
polynomial \( f \in I_d^{O(n)} \) to the curvature \( \Omega \) of \( \omega \). As is well known (e.g., see [6, 8]), \( I_d^{O(n)} \) is spanned by the polynomials \( p_k \in I_{2k}^{O(n)} \), \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \), characterized by
\[
\det \left( \lambda I - \frac{1}{2\pi} X \right) = \sum p_k(X)\lambda^{n-2k}, \quad \forall X \in \mathfrak{so}(n).
\]

We define the universal \( k \)-Pontryagin form of \( M \) as \( p_k(\Omega) \in \Omega^{4k}(J^1M_M) \). Moreover, assuming \( M \) is connected and oriented, we have a principal \( SO(n) \)-bundle over \( J^1M_M \), \( O^+M = \{(u_x,j_x^2g) \in OM: u_x \text{ is positively oriented} \} \), and \( \omega \) is reducible to \( O^+M \). A well-known result (e.g., see [6] Chapter 8, [8] XII, Theorem 2.7) states that \( \mathcal{T}^{SO(n)} \) is generated by the polynomials \( \{p_k\} \) for odd \( n \), and by \( \{p_k, Pf\} \) for even \( n \), where \( Pf \in \mathcal{T}_{n/2}^{SO(n)} \) denotes the Pfaffian. For every even dimension \( n = \dim M \), we define the universal Euler form of \( M \) by setting
\[
E = (2\pi)^{-\frac{n}{2}} Pf(\Omega) \in \Omega^n(J^1M_M).
\]

From the identity \( (2\pi)^{-n} Pf^2 = p_{n/2} \) we deduce \( E \wedge E = p_{n/2}(\Omega) \). The properties of \( \omega \) lead us readily to the following

**Proposition 4.1.** We have

1. The universal Pontryagin forms and the universal Euler form are closed.
2. The universal Pontryagin forms \( p_k(\Omega) \) (resp. the universal Euler form \( E \)) are invariant under the action of \( Diff^\ast M \times \mathbb{R} \) (resp. \( Diff^\ast M + \mathbb{R} \)) on \( J^1M_M \).
3. For any Riemannian metric \( g \) on \( M \), we have
   \[
   \begin{align*}
   (j^1g)^*(p_k(\Omega)) &= p_k(\Omega^g), \\
   (j^1g)^*(E) &= (2\pi)^{-\frac{n}{2}} Pf(\Omega^g),
   \end{align*}
   \]
   where \( \Omega^g \) denotes the curvature of the Levi-Civita connection \( \omega^g \) of \( g \).

**Remark 4.2.** The Euler form is not invariant under the elements of \( Diff^\ast M \) (recall that \( O^+(M) \) is invariant under \( Diff^\ast M \), but not under \( Diff^\ast (M) \)). In fact, for any \( \phi \in Diff^\ast M \) we have \( \phi^{(1)} \ast (E) = -E \).

The relation between the universal Pontryagin and Euler forms on \( J^1M_M \) and the usual Pontryagin and Euler classes on \( M \) is the same as the relation between characteristic forms and classes on the bundle of connections of a principal bundle (e.g., see [3]): The map \( q^1_\ast: H^\ast(M) \rightarrow H^\ast(J^1M_M) \) is an isomorphism with inverse map \( (j^1g)^\ast: H^\ast(J^1M_M) \rightarrow H^\ast(M) \), for any Riemannian metric \( g \) on \( M \), and by [1] (3) the \( k \)-Pontryagin (resp. Euler) class of \( M \) is the image under this isomorphism of the cohomology class of the universal \( k \)-Pontryagin (resp. Euler) form.

As in the case of the bundle of connections of a principal bundle, the Pontryagin forms contain more information than the Pontryagin classes. For example, if \( 4k > n \), the \( k \)-Pontryagin class vanishes, but the corresponding form does not
necessarily, as \( \dim J^1M_M > n \). For example, if \( n = 2 \), then the first Pontryagin form \( p_1(\Omega) \in \Omega^2(J^1M_M) \) does not vanish, whereas the first Pontryagin class vanishes by dimensional reasons. According to [5], these higher-order Pontryagin forms can be interpreted as closed Diff\(^+M \)-invariant differential forms on the space of Riemannian metrics on \( M \). In a forthcoming paper, we shall study these forms and their extension to equivariant cohomology in a similar way as done in [5] for the characteristic forms on the bundle of connections.

5 The universal Levi-Civita connection characterized

**Theorem 5.1.** The universal Levi-Civita connection \( \omega \) is the only Diff\( M \)-invariant connection form on \( q^*_1FM \to J^1M_M \) satisfying the condition \( (9) \). In other words, the form \( \omega \) is the only Diff\( M \)-invariant connection form on the bundle \( OM \to J^1M_M \).

The proof of this theorem is based in the following

**Lemma 5.2.** The Diff\( M \)-invariant 1-forms on \( J^1M_M \), with values on \( \otimes^2T^*M \), are \( \lambda \theta + \mu \mathrm{tr} \theta \otimes g \), \( \lambda, \mu \in \mathbb{R} \). Equivalently, the only Diff\( M \)-invariant connection forms on the bundle \( q^*_1FM \to J^1M_M \) are \( \omega + \lambda \theta + \mu \mathrm{tr} \theta \otimes \text{id}_{TM} \), \( \lambda, \mu \in \mathbb{R} \).

**Proof of Theorem 5.1.** The universal Levi-Civita connection \( \omega \) satisfies the conditions of the statement by virtue of Theorem 3.6. Conversely, let us suppose that \( \omega \) is another Diff\( M \)-invariant connection on \( q^*_1FM \). Then, we have \( \omega = \omega + \alpha \), with \( \alpha \in \Omega^1(J^1M_M, \text{End} TM) \). Clearly, \( \alpha \) is Diff\( M \)-invariant, and from Lemma 5.2 we have \( \alpha = \lambda \theta + \mu \mathrm{tr} \theta \otimes \text{id}_{TM} \) for some \( \lambda, \mu \in \mathbb{R} \). Hence \( \nabla^\omega g = -2(\lambda \theta + \mu \mathrm{tr} \theta \otimes g) \), and consequently, \( \omega \) satisfies the condition \( (9) \) if and only if \( \lambda = \mu = 0 \); i.e., if and only if \( \omega = \omega \).

Next, we state some necessary results to prove Lemma 5.2. First of all, we recall the following:

**Theorem 5.3 ([11]).** (Fundamental theorem of the invariant theory for the orthogonal group) Let \( (v, w) \mapsto \langle v, w \rangle \) be the standard scalar product on \( V = \mathbb{R}^n \), allowing us to identify \( V \) with its dual space. We consider the tensorial representation of \( O(n) \) on \( \otimes^kV \). Then, we have

1. For \( k \) odd, the unique \( O(n) \)-invariant element of \( \otimes^kV \) is the zero element.
2. For \( k = 2l \) even, the subspace of \( O(n) \)-invariant elements on \( \otimes^kV \) is generated by the following invariant linear forms

\[
\varphi_{i_1, i_2, \ldots, i_{2l-1}, i_{2l}}(v_1, \ldots, v_{2l}) = \langle v_{i_1}, v_{i_2} \rangle \cdots \langle v_{i_{2l-1}}, v_{i_{2l}} \rangle,
\]

where \( i_1, i_2, \ldots, i_{2l-1}, i_{2l} \) stands for an arbitrary permutation of the set of indices \( 1, 2, \ldots, 2l - 1, 2l \).
Theorem 5.4 ([11]). (Fundamental theorem of the invariant theory for the special orthogonal group) With the previous notations, for $k < n$, the $SO(n)$-invariants on $\otimes^k V$ coincide with $O(n)$-invariants. For $k = n$, the space of $SO(n)$-invariants is generated by the space of $O(n)$-invariants and $\wedge^n V$.

Remark 5.5. For $k = 4$, from Theorem 5.3 we conclude that the $O(n)$-invariants are generated by the following tensors:

\[
\begin{align*}
\xi_1 &= \sum_{i,j} e_i \otimes e_i \otimes e_j \otimes e_j, \\
\xi_2 &= \sum_{i,j} e_i \otimes e_j \otimes e_i \otimes e_j, \\
\xi_3 &= \sum_{i,j} e_i \otimes e_j \otimes e_j \otimes e_i.
\end{align*}
\]

Proposition 5.6. Let $(e_1, \ldots, e_n)$ be the standard orthonormal base in $V = \mathbb{R}^n$. Consider the $O(n)$-module $E = \otimes^3 V \oplus (S^2 V \otimes (\otimes^2 V)) \oplus (S^2 V \otimes (\otimes^3 V))$.

1. The invariant elements under the action of $O(n)$ on $E$ are

\[
(12) \quad \eta = \lambda \sum_{i,j} e_i \otimes e_i \otimes e_j \otimes e_j + \mu \sum_{i,j} e_i \otimes e_j \otimes e_i \otimes e_j, \quad \lambda, \mu \in \mathbb{R},
\]

where $\otimes$ denotes the symmetric product.

2. For $n \geq 4$ the $SO(n)$-invariants on $E$ coincide with the $O(n)$-invariants.

Proof. (1) Every direct summand in $E$ is $O(n)$-invariant; hence we need only to analyze the invariants on each summand.

From Theorem 5.3 it follows that there are no invariants on $\otimes^3 V$ and on $S^2 V \otimes (\otimes^3 V)$, and the invariants on $S^2 V \otimes (\otimes^2 V)$ are obtained by linear combination of the elements cited in Remark 5.5. Hence, they are of the form $\xi = \lambda \xi_1 + \mu \xi_2 + \nu \xi_3$. Then, $\xi \in S^2 V \otimes (\otimes^2 V)$ if and only if $\mu = v$, and we obtain (12).

(2) For $n > 5$ the result follows from Theorem 5.3. For $n = 5$, from Theorem 5.3 it follows that $O(n)$-invariants and $SO(n)$-invariants coincide on $\otimes^3 V$ and on $S^2 V \otimes (\otimes^3 V)$. Moreover, since we have $\wedge^3 V \cap (S^2 V \otimes (\otimes^3 V)) = 0$, the same conclusion holds for the remaining summand $S^2 V \otimes (\otimes^2 V)$. Finally, for $n = 4$, again from Theorem 5.3 and the fact that $\wedge^4 V \cap (S^2 V \otimes (\otimes^3 V)) = 0$, it follows that there are no new invariant on $\otimes^3 V$ or on $S^2 V \otimes (\otimes^3 V)$. Also, as $-id_V \in SO(4)$ and $-(id_V) \cdot \eta = -\eta$ for all $\eta \in S^2 V \otimes (\otimes^3 V)$, we conclude that no new invariant appears on $S^2 V \otimes (\otimes^3 V)$, and the result follows.

Proof of Lemma 5.2. Let us fix a point $z_0 = j^{1}_{x_0} g_0 \in J^1 \mathcal{M}_M$, and let us consider a normal coordinate system $(U; x^i)$ centered at $x_0$ for the metric $g_0$. The expression of a covector $\eta \in \Omega^1(J^1 \mathcal{M}_M; \otimes^2 T^* M)$ at $z_0$ on this coordinate system is $\eta_{z_0} = (\lambda_{ah,i} dx^i + \lambda_{ab,j} dy_{ij} + \lambda_{ab,k} dy_{ij,k})_{z_0} \otimes (dx^a)_{z_0} \otimes (dx^b)_{z_0}$.

For reversible $M$ we set $G = SO(n)$ and for irreversible $M$, $G = O(n)$. Given $A \in G$, we define a local diffeomorphism $\varphi_A : U \to M$ around $x_0$ by
\[ \varphi_A(x) = A_j^i x^j, \forall x \in U. \] As \( \varphi_A \) is a linear transformation, from the expression (3) we deduce \( \varphi_A(j_{x_0}^1 g_0) = j_{x_0}^1 g_0 \) and we have
\[
\varphi_A^{-1} (dx^i)_{x_0} = (A^{-1})_a^i (dx^a)_{x_0} = \sum_a A_i^a (dx^a)_{x_0}, \\
\varphi_A^{-1} (dy_{ij})_{x_0} = A_i^a A_j^b (dy_{ab})_{x_0}, \\
\varphi_A^{-1} (dy_{ij,k})_{x_0} = A_i^a A_j^b A_k^c (dy_{abc})_{x_0}.
\]
Hence the map
\[
(dx^i)_{x_0} \mapsto e_i, \\
(dy_{ij})_{x_0} \mapsto e_i \otimes e_j, \\
(dy_{ij,k})_{x_0} \mapsto e_i \otimes e_j \otimes e_k,
\]
determines a \( G \)-module isomorphism between \( T_{x_0} J^1 M \otimes (\otimes^2 T^*_{x_0} M) \) and the space \( E \) in the statement of Proposition 5.6. The local diffeomorphism \( \varphi_A \) satisfies the conditions in Lemma 5.1, and, hence, there exists \( \phi_A \in \text{Diff}_M \) extending \( \varphi_A \) on a neighbourhood of \( x_0 \). As \( \eta \) is \( \text{Diff}_M \)-invariant we have \( \varphi_A^{-1} (\eta) = \eta \), and hence \( \varphi_A^{-1} (\eta_{x_0}) = \eta_{x_0} \) for every \( A \in G \). As \( n = \text{dim} \ M \geq 4 \) for an irreversible \( M \), from Proposition 5.6 for some \( \lambda, \mu \in \mathbb{R} \), we obtain
\[
\eta_{x_0} = \lambda \sum_{i,j} (dy_{ij})_{x_0} \otimes (dx^i)_{x_0} \otimes (dx^j)_{x_0} + \mu \sum_{i,j} (dy_{ij})_{x_0} \otimes (dx^i)_{x_0} \otimes (dx^j)_{x_0}
\]
As the point \( x_0 \in J^1 M \) is arbitrary, for certain smooth functions \( a, b \) on \( J^1 M \) we have \( \eta = a \theta + b \text{tr} \theta \otimes \mathbf{g} \). As \( \eta, \theta \) and \( \text{tr} \theta \otimes \mathbf{g} \) are \( \text{Diff}_M \)-invariant, \( a \) and \( b \) are also \( \text{Diff}_M \)-invariant and, by virtue of Proposition 2.2, they are constant.

\[ \square \]

**Remark 5.7.** The characterization of the connection \( \omega \) given on Theorem 5.1 does not hold for higher-order jet bundles. In fact, below we sketch the proof of the existence of a natural 1-form \( \alpha \in \Omega^1 (J^3 M, \text{End}_A T M) \). Hence, \( \omega + \alpha \) is a \( \text{Diff}_M \)-invariant connection form on \( q_3^* FM \), which also satisfies the condition (9), by virtue of Lemma 3.6. Let
\[
\theta = \langle dy_{ij} - y_{ij,k} dx^k \rangle \otimes \partial/\partial y_{ij} + \langle dy_{ij,k} - y_{ij,kr} dx^r \rangle \otimes \partial/\partial y_{ij,k},
\]
be the \( (q_3)^* V(q_3) \)-valued 1-form on \( J^3 M \) defining its contact structure (cf. [9]). We first notice the natural exact sequence of vector bundles over \( J^2 M \),
\[
0 \to (q_2)^* (S^2 T^* M \otimes S^2 T^* M) \overset{\iota_2}{\longrightarrow} (q_2)^* J^2 (S^2 T^* M) \to (q_2)^* J^1 (S^2 T^* M) \to 0
\]
splits naturally, as a retract \( \rho_2 : (q_2)^* J^2 (S^2 T^* M) \to (q_2)^* J^1 (S^2 T^* M) \) exists of \( \iota_2 \) given by,
\[
\rho_2 (j_2^2 g, j_2^2 h) (X_1, X_2, X_3, X_4) = \frac{1}{2} \langle \nabla \theta \rangle^2 (h) (X_3, X_4, X_1, X_2)
\]
\[ \quad + \frac{1}{2} \langle \nabla \theta \rangle^2 (h) (X_4, X_3, X_1, X_2), \]

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for all \( j^2 g \in J^2_2(\mathcal{M}) \), \( j^2 h \in J^2_2(S^2 T^*M) \), and \( X_1, \ldots, X_4 \in T_x M \). Let 
\( c_{24} : (q_2)^* \otimes^4 T^*M \to (q_2)^* \otimes^2 T^*M \)
be the metric contraction of the second and fourth arguments, i.e.,
\( c_{24}(j^2 g, X_1 \otimes X_2 \otimes X_3 \otimes X_4) = (j^2 g, g(X_2, X_4)X_1 \otimes X_3) \).
By using the canonical vector-bundle isomorphism \( V(q_2) \cong (q_2)^* J^2(S^2 T^*M) \),
the form we are looking for, is defined as follows: 
\( \alpha = (c_{24} \circ \rho_2 \circ \theta^3)_\lambda \).

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