Seiberg-Witten prepotential from WZNW conformal block:
Langlands duality and Selberg trace formula

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Abstract

We show how $SU(2) N_f = 4$ Seiberg-Witten prepotentials are derived from $\hat{sl}_2, k (k \to 2)$ four-point conformal blocks via considering Langlands duality.

1 Introduction

Last June, Alday, Gaiotto and Tachikawa (AGT) \[1\] claimed that correlation functions of primary states in Liouville field theory (LFT) can get re-expressed in terms of Nekrasov’s partition function $Z_{N_{ck}}$ of 4d $\mathcal{N} = 2$ quiver $SU(2)$ SCFT (at low-energy Coulomb phase). In particular, every Riemann surface $C \equiv C_{g,n}$ on which LFT dwells is responsible for certain SCFT $\mathcal{T}_{g,n}(A_1)$ such that the following equality holds.

$$\text{Conformal block w.r.t. } C_{g,n} = \text{Instanton part of } Z_{N_{ck}} \left( \mathcal{T}_{g,n}(A_1) \right)$$

Their discovery has a profound impact on the unification of many known mathematical corners, say, Hitchin integrable system (including isomonodromic deformation), Selberg-Dotsenko-Fateev $\beta$-ensemble and $Z_{N_{ck}}^2$.

For any $C$, talking about the quantization of its moduli space $\mathcal{M}(C)$ usually relies on a CFT living on it. For instance, a finite-dimensional Hilbert space $\mathcal{H}$ thus obtained is spanned by chiral conformal blocks of, say, WZNW model on $C$ and $\text{dim} \mathcal{H}$ gets computed by Verlinde’s formula. Naively, one wants to ask why only CFTs characterized by $\mathcal{W}$-algebra are more preferred than others governed by, say, affine Kac-Moody algebra? In this letter, we are going to consider the role which WZNW models (or conformal blocks of them) play in connection with 4d $\mathcal{N} = 2$ SCFT. It will be shown that at least from $\hat{sl}_2$ cases, thanks to their intimate relationship with $\mathcal{W}_2$/Virasaro-algebra, the $SU(2) N_f = 4$ Seiberg-Witten (SW) prepotential $\mathcal{F}^{SW}$ is reproduced. It should be emphasized that our approach needs neither knowledge about AGT conjecture nor its ramified (surface operator inserted) version like \[3\]\[4\]\[5\]\[6\]. What we are truly after is our previous work \[7\] together with a piece of independently developed mathematical concept, Langlands duality, relating classical LFT and WZNW model at critical level. As an intermediate step, let us first describe the result of $\hat{sl}_2$.

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2See \[2\] for explanations between the last two topics.
In [7] we noticed an analogy between two seemingly unrelated arenas, say, classical LFT and large-$N$ Hermitian matrix model, by appealing to Polyakov’s conjecture devised for $C_{0,4} \equiv \mathbb{C}\{0,1,q\}$ ($q$: cross-ratio)

$$c(q) = \frac{\partial}{\partial q} \left( f_{\delta} \left[ \begin{array}{cc} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{array} \right](q) \right)_{\delta=\delta_s(q)}$$

$f$: $s$-channel classical conformal block, $\delta_i$: classical conformal dimension

$$\sqrt{\delta_s - \frac{1}{4}} = p_s: s$-channel saddle-point intermediate momentum  \hspace{1cm} (1.1)$$

By translating this formula belonging to LFT into the language of Hermitian matrix model, it had helped us envision

$$f \simeq F^{SW}$$

The reason is listed as below.

(I) Basically, (1.1) stems from Ward’s identity involving the $(2,0)$ stress-tensor $T_L(z) = Q\partial^2_z \phi - (\partial_z \phi)^2$ inserted inside some four-point Liouville primary field $V_\alpha(z)$ correlator $\langle X \rangle$ under the classical limit $b \to 0$. Besides, $\langle X \rangle$ is replaced by $\langle 1 \rangle$ when boundary terms $S_{bdy}$ fixed by $\Delta_\alpha = \alpha(Q-\alpha)$ ($Q = b + b^{-1}$) and positions of $V_\alpha(z)$ are added to the original bulk Liouville action $S_{bul}$. Through $b \to 0$, $\exp(-S_{tot})$ ($S_{tot} = S_{bul} + S_{bdy}$) permits one unique saddle-point $\phi_{cl} = \varphi_{cl}/b$ whose singular behavior is reflected by

$$\lim_{b \to 0} b^2 T_L(z) \to T(z) \equiv \frac{1}{2} \partial^2_z \varphi_{cl} - \frac{1}{4} (\partial_z \varphi_{cl})^2 = \sum_{i=1}^{3} \frac{\delta_i}{(z - z_i)^2} + \sum_{i=1}^{3} \frac{c(z_i)}{z - z_i}$$

where the only unknown accessory parameter turns out to be $c(q)$. A conjectured determination of $c(q)$ is therefore (1.1). This procedure is rigid because of a unique $\varphi_{cl}$ which leads to a factorized form of $\exp(-S_{tot}[b^{-1}\varphi_{cl}]) = \exp(-b^{-2}f) \cdot \exp(-b^{-2}\bar{f}) \cdots$. Only the holomorphic part, $f$, survives $\partial/\partial q$ thereof.

(II) Call the LHS of a second-order Fuchsian equation (Baxter equation)

$$\partial^2_z + T(z) = 0$$

$G$-oper. In fact, (1.3) is also obtainable by imposing $b \to 0$ on the null-vector decoupling equation:

$$\left( L_{-1}^2 + b^2 L_{-2} \right) V_{-\frac{3}{2}} = 0$$

realized at the conformal block level. It seems natural to regard (1.3), doubly-sheeted cover of $C_{0,4}$, as the genus-zero spectral curve of some Hermitian matrix model $Z_M$ through $\partial_z \to iy$ due to the same governing Virasoro algebra in both situations.

\[^{3}\text{For the sake of brevity, we will simply use } f \text{ which abbreviates the classical conformal block.}\]
By doing so, (1.1) necessarily acquires certain interpretation within the matrix model context. To conclude, when (1.3) gets identified with the large-$N$ spectral curve \( N \) (matrix rank)

\[ y^2 = \lim_{N \to \infty} \langle T_M(z) \rangle = \langle \partial \phi_{KS} \rangle^2 \]

\( T_M(z) \): stress-tensor constructed via Kodaira-Spencer free boson \( \phi_{KS}(z) \) associated with \( Z_M \) inevitably \( f \simeq F_0 \), which represents genus-zero free energy of \( Z_M = \exp(\hbar^{-2}F_0 + F_1 + \cdots) \). This sounds plausible because as \( \hbar = 1/N \to 0 \), by replacing \( f \) with \( F_0 \), (1.1) arises from Ward’s identity of \( T_M(z) \) and thus accounts for the accessory parameter of the meromorphic function \( \lim_{N \to \infty} \langle T_M(z) \rangle \). Moreover,

\[ f \simeq F_0 = F^{SW} \quad (1.5) \]

when (1.3) \( \simeq G \) (Gaiotto’s curve \( \mathbb{S} \)) is assumed such that \( Z_M \) containing \( G \) (rewritten SW curve) as its own spectral curve \( \gamma \) has to be recognized as the instanton part of \( Z_{N,\epsilon} \) at \( \epsilon_1 = -\epsilon_2 = \hbar \). In \( [7] \) by taking the large-\( p_s \) limit a perfect agreement \( f = F^{SW} \) was observed.

Equipped with these arguments, in the next section we start to see \( F^{SW} \) indeed also hides behind \( \widehat{sl}_2 \) conformal blocks via Langlands duality.

2 Derivation of KZ equation at critical level

To begin with, what we will mainly rely on is the statement encountered in geometric Langlands correspondence according to Feigin, Frenkel and Reshetikhin \([14] \):

\( \blacklozenge \) “The space of opers, associated with Langlands-dual Lie algebra \( ^Lg \) on \( \mathbb{P}^1 \) with regular singularities at marked points \( \{ z_i \} \), can be identified with the spectrum of a corresponding Gaudin algebra denoted by \( Z_{(z_i)}(g) \).”

Remarkably, the problem of fixing accessory parameters of a \( ^L G \)-oper (\( ^L G \): adjoint group of \( ^L g \)) is switched to solving the spectrum of \( Z_{(z_i)}(g) \) (or equivalently Gaudin Hamiltonian \( \Xi \)). Let us first see how a Knizhnik-Zamolodchikov (KZ) equation can be reached by means of the proposal \( \blacklozenge \). Ultimately, we are capable of claiming that \( \mathcal{N} = 2 \; SU(2) \) \( N_f = 4 \) SW prepotentials are encoded in \( \widehat{sl}_2 \) four-point conformal blocks at critical level \( k \to 2 \). We stress again that any recently developed argument from either AGT conjecture or its ramified version will not be borrowed. Some mathematical aspects will be postponed until Sec. 2.2.

Now, given a \( PGL_2 \)-oper \( (PGL_2 = ^L G \): adjoint group of \( ^L g \) with \( g = sl_2 \) on \( C = \mathbb{C}\backslash \{ z_1, \cdots, z_N \} \), i.e.

\[ \partial^2_z + \sum_{i=1}^{N} \frac{\delta_i}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{\epsilon_i}{z - z_i} \quad (2.1) \]

4See \([9, 10, 11, 12, 13] \) for recent publications towards this direction.
subject to $\sum c_i = 0$ (no further pole at $z_0 = \infty$), the pair $(\delta_i, c_i)$ can get read off from $(C_i, \Xi_i)$ which stands for, respectively, quadratic Casimir operators

$$C_i = \frac{1}{2} \sum_{a=1}^{d} J_a^{(i)} J_a^{(i)} = j_i(j_i + 1) = -\delta_i = \xi_i(\xi_i - 1)$$  \hspace{1cm} (2.2)$$

where $\{J_a\}$ denotes the basis of $\mathfrak{sl}_2$ whilst $J^a J^b \equiv \kappa_{ab} J^a J^b$ with $\kappa_{ab}$ being the Cartan-Killing form and Gaudin Hamiltonians

$$\Xi_i = \sum_{j \neq i} \sum_{a=1}^{d} J_a^{(i)} J_a^{(j)} \frac{1}{z_i - z_j} = c_i$$  \hspace{1cm} (2.3)$$

subject to $\sum_i \Xi_i = 0$. (2.3) reflects faithfully the above ♣.

Notice that (2.2) implies $\xi = j + 1$ which had appeared in the celebrated $H_3^+$-WZNW/Liouville dictionary [] near $b \to 0$. Generally, the momentum of Liouville primary fields $V_\alpha(z)$ is related to the spin-$j$ WZNW primary field $\Phi^j(y|z)$ by

$$\alpha = \frac{1}{b} (j + 1) + \frac{b}{2}.$$  

While $y$’s are (isospin) variables of the $SL(2,\mathbb{R})$ group manifold, a convenient spin-$j_r$ representation of $\{J_a^{(r)}\}$ $(a = \pm, 3)$ is usually chosen like

$$J_+^{(r)} = y_r^2 \frac{\partial}{\partial y_r} - 2j_r y_r, \hspace{1cm} J_-^{(r)} = \frac{\partial}{\partial y_r}, \hspace{1cm} J_3^{(r)} = y_r \frac{\partial}{\partial y_r} - j_r.$$  

### 2.1 $\widehat{\mathfrak{sl}}_2$ KZ equation

We are ready to show how $\widehat{\mathfrak{sl}}_{2,k}$ KZ equations on a four-punctured $\mathbb{P}^1$ ($q$: cross-ratio)

$$(k - 2) \frac{\partial}{\partial q} \Psi = \Xi_r \Psi, \hspace{1cm} k - 2 = \tilde{b}^{-2}, \hspace{1cm} \tilde{b} = \frac{1}{b}$$  \hspace{1cm} (2.4)$$

are obtained. In general, $m$-point spheric $\widehat{\mathfrak{sl}}_{2,k}$ conformal blocks satisfy the following KZ equation

$$\left( (k - 2) \frac{\partial}{\partial z_r} - \Xi_r \right) \Psi_m = 0,$$

$$\Psi_m(y|z) \equiv \langle \Phi^{j_1}(y_1|z_1) \cdots \Phi^{j_m}(y_m|z_m) \rangle.$$  \hspace{1cm} (2.5)$$

At $k \to 2$, we anticipate the solution to (2.4) factorizes as

$$\Psi_4(y|q) \to \mathfrak{S}(q) \psi(y)$$

where $\psi(y)$ depends only on isospin variables and

$$\mathfrak{S} = \exp \left( b^{-2} f \right), \hspace{1cm} f \equiv f_{\delta} \left( \begin{array}{cc} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{array} \right) (q)$$
is referred to as the *quantum* Belavin-Polyakov-Zamolodchikov (BPZ) conformal block \[^{15}\] \( F_{\Delta} \left( \begin{array}{cc} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{array} \right) \) with \( \Delta \equiv b^{-2}\delta \) at \( b \to 0 \). The self-dual symmetry \( b \leftrightarrow \bar{b} = 1/b \) respected by LFT leads to \( (k \to 2) \leftrightarrow (b \to 0) \). While KZ equations are analogous to decoupling equations of null-vectors at the second level (or BPZ systems), a relation between (1.4) and (2.5) (or equivalently duality between KZ and BPZ \( D \)-modules over \( C \)) had emerged in the framework of quantized geometric Langlands-Drinfeld correspondence. See \[^{16}\] for an excellent lecture note.

Assuming that there exists an eigenstate \( \psi_r(y) \) of \( \Xi_r \) and using the emphasized relation \( c_r = \Xi_r \) in (2.3), combined with (1.1) one can straightforwardly write down

\[
\left( (k - 2) \frac{\partial}{\partial q} - \Xi_r \right) \hat{\Phi}(q)\psi_r(y) \bigg|_{\delta \to \delta, (q)} = 0, \quad b \to 0. \tag{2.6}
\]

Immediately, this says that \( \lim_{k \to 2} \Psi_4(y|q) = F_{\text{SW}}(q)\psi_r(y) \) encodes \( F_{\text{SW}} \).

As a matter of fact, at the level of complete \((2m-2)\)-point spheric correlators \( \Omega^L \) in LFT as \( b \to 0 \)

\[
\Omega^L(z_1, \cdots, z_m|y_1, \cdots, y_{m-2}) \to \exp \left(-S_{\text{tot}}[b^{-1}\varphi_{ct}]\right) \prod_{i=1}^{m-2} \exp \left(-\frac{1}{2}\varphi_{ct}(y_i, \bar{y}_i)\right) \tag{2.7}
\]

where again \( S_{\text{tot}} \) receives the boundary contribution from \( V_{\alpha_i}(z_i) \) insertions in addition to \( S_{\text{bdy}} \). Besides, \((m-2)\) indicates the total number of degenerate \( V_{2,1}(z, \bar{z}) \) evaluated at \( \varphi_{ct} \). Furthermore, quoting the map of Ribault-Teschner \[^{17}\] we know

\[
\Omega(z_1, \cdots, z_m) = \Omega^L(z_1, \cdots, z_m|y_1, \cdots, y_{m-2}),
\]

\[
\Omega(z_1, \cdots, z_m): m\text{-point spheric } \hat{\mathfrak{sl}}_2\text{-WZNW correlator}
\]

which might, together with (2.7), play a role in fixing the eigenstate \( \psi_r(y) \).

### 2.2 Langlands duality

We have seen that \( \hat{\mathfrak{sl}}_{2,k} \) conformal blocks at \( k \to 2 \) necessarily carries a *classical* piece of LFT through \( \hat{\Phi} \) which in turn encodes \( F_{\text{SW}} \). We want to consequently link two concepts, \( G \)-oper and KZ connection, in order to better understand (2.3) by resorting to a series of arguments below. Namely, this junction joining classical LFT and WZNW model at critical level is truly a piece of Langlands duality in disguise.

\(^{5}\)Note that the saddle-point intermediate momentum \( p_s \) depending on \( q \) should be substituted after \( \partial / \partial q \) is performed.
where $F_A$ denotes the curvature of the connection $d_A = d + A$. The second line implies $\mathfrak{g} \to \mathfrak{g}_C = sl(2, \mathbb{C})$ such that $\phi$ takes values in Lie algebra of $\mathfrak{g}_C$. Alternatively, $\mathcal{M}_H(C)$ is also encoded in its spectral curve ($PGL_2$-oper) written as

$$\Sigma : \det(y - \phi(z)) = 0 \to y^2 + t(z) = 0$$

with $t(z)$ being a quadratic differential. The characteristic polynomial manifest itself as a doubly-sheeted cover of $C$; namely, each pair $(y, z)$ of $T^*C$ is constrained in $\Sigma$.

(II) Each Higgs-bundle corresponds to a flat connection $\nabla$ whose moduli space denoted by $\mathcal{M}_{\text{flat}}(\mathfrak{g}_C, C)$ is actually the space of homomorphisms $\pi_1(C) \to \mathfrak{g}_C$ modulo conjugation, i.e.

$$\mathcal{M}_{\text{flat}}(\mathfrak{g}_C, C) \simeq \mathcal{M}_H(C) \simeq \text{Hom}(\pi_1(C), \mathfrak{g}_C).$$

One is capable of choosing a flat connection $\nabla$ ($t^a$: $g = sl_2$ generator)

$$\nabla = \partial_z - \Theta(z), \quad \Theta(z) = -\sum_{i=1}^N \frac{A_i}{z - z_i}, \quad A_i = t^a J^{(i)}_a$$

such that $\det(\nabla)$ gives rise to the RHS of (2.9) conventionally called Drinfeld-Sokolov form.$^6$

(III) As the last step, we can construct Schlesinger (isomonodromic deformation) equations from (2.10), i.e.

$$\partial_z \Upsilon = \sum_{i=1}^N \frac{A_i}{x - z_i} \Upsilon, \quad \partial_{z_i} \Upsilon = -\frac{A_i}{x - z_i} \Upsilon.$$  

(2.11)

In fact, Schlesinger’s system bears another equivalent Poisson form:

$$\partial_{z_i} A_j = \{ \Xi_i, A_j \}, \quad \Xi_i = \sum_{j \neq i} \frac{\text{tr}(A_i A_j)}{z_i - z_j} = \sum_{j \neq i} \sum_a \frac{J^{(i)}_a J^{(j)}_a}{z_i - z_j}$$  

where the covariant derivative $\partial_{z_i} - \{ \Xi_i, \cdot \}$ w.r.t. the matrix-valued $A_i$ is exactly of KZ type. Of course, going from (2.10) to its corresponding KZ equation can be resorted to an inverse procedure of Sklyanin’s separation of variables.$^9$ Now the desired link is completed. See $^{19}$ (and references therein) for very detailed treatment for these subjects.

3 Discussion: Selberg trace formula vs AGT dictionary

In $^7$ through $p_s \equiv a$ ($SU(2)$ Coulomb parameter) we have correctly reproduced $\mathcal{F}^{SW}$ using the $s$-channel classical conformal blocks $f$. This relation actually resembles AGT dictionary$^1$ classically. The key identification $p_s \equiv a$ says equivalently

$$\sqrt{\delta_s - \frac{1}{4}} = p_s \equiv \ell_s$$

$^6$A properly gauge-transformed $\nabla \to C(z) \nabla C(z)^{-1}$ leads to the diagonal $\Theta(z)$ called Miura form.
due to

\[ \ell_s = \oint dz \sqrt{T(z)} = a. \]

Note that \( \ell \) satisfying \( 2 \cosh(\ell/2) = \text{tr}(\rho(\gamma)) \) is the geodesic length of some hyperbolic geometry \( \mathbb{H}/\Gamma \) with \( \Gamma \subset \text{Aut}(\mathbb{H}) \) given a homomorphism \( \rho : \Gamma = \pi_1(C) \to \text{PSL}(2, \mathbb{R}) \) where \( \gamma \)'s are generators of the fundamental group \( \pi_1(C) \).

Taking \( C = \mathbb{C}\setminus\{0, 1, q\} \) for example, one has a geodesic \( \gamma_{12} = \gamma_1 \gamma_2 \) encircling two marked points \( (z_1, z_2) \) such that \( \ell_{12} = \ell(\gamma_1 \gamma_2) = \ell_s \)

\[ 2 \cosh(\ell_s/2) = \text{tr}(\rho(\gamma_1) \rho(\gamma_2)). \]

When \( \delta_s = \xi_s(1 - \xi_s) \) is regarded as the eigenvalue of the hyperbolic Laplacian \( \Delta_C \) defined on \( C \), (3.1) looks like the celebrated Selberg trace formula which conjectures that for an arbitrary Riemann surface \( C = \mathbb{H}/\Gamma \) one has

\[ \sum_{\delta \in \text{Spec}(\Delta_C)} h(\sqrt{\delta} - \frac{1}{4}) = \sum_{\gamma \in \pi_1(C)} \tilde{h}(\gamma) \]  

(3.2)

where \( h \) is certain test function and \( \tilde{h} \) its suitably-transformed counterpart. As (3.2) equates summations over the spectrum of \( \Delta_C \) and closed geodesic on \( C \), it is not actually (3.1) until some good test function \( h \) which reproduces (3.1) is selected. Nevertheless, we believe that (3.1) might shed new light on understanding AGT dictionary. We hope to return to this issue soon in another publication.

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