TESTING THE GENERAL RELATIVISTIC “NO-HAIR” THEOREMS USING THE GALACTIC CENTER BLACK HOLE SAGITTARIUS A*  
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ABSTRACT

If a class of stars orbits the central black hole in our galaxy in short-period (~0.1 yr), high-eccentricity (~0.9) orbits, they will experience precessions of their orbital planes induced by both relativistic frame dragging and the quadrupolar gravity of the hole, at levels that could be as large as 10 μas per year, if the black hole is rotating faster than half of its maximum rotation rate. Astrometric observations of the orbits of at least two such stars can in principle lead to a determination of the angular momentum vector \( \mathbf{J} \) of the black hole and its quadrupole moment \( Q_2 \). This could lead to a test of the general relativistic no-hair theorems, which demand that \( Q_2 = -J^2/M \). Future high-precision adaptive infrared optics instruments may make such a fundamental test of the black hole paradigm possible.

Subject headings: black hole physics — Galaxy: center — relativity  
On-line material: color figures

1. INTRODUCTION

The discovery, using infrared telescopes, of stars orbiting within an arcsecond of the central object Sgr A* in our galaxy, together with accurate determinations of their orbits, has provided strong evidence for the existence there of a massive black hole (MBH) of around \( 3.6 \times 10^6 M_\odot \) (see Alexander 2005 for a review). In addition to opening a window on the innermost region of the Galactic center, the discovery of these stars has made it possible to contemplate using orbital dynamics to probe the curved spacetime of a rotating black hole.

The orbital periods of these stars are on the scale of tens of years, and thus most relativistic effects, such as the advance of the pericenter, are too small to be observed at present (see, however, Zucker et al. 2006). Nevertheless, there seems to be every expectation that, with improved observing capabilities, a population of stars significantly closer to the hole will eventually be discovered, making orbital relativistic effects detectable (Jaroszyński 1998; Fragile & Mathews 2000; Rubilar & Eckart 2001; Weinberg, Milosavljević, & Ghez 2005; Kraníotis 2007). Furthermore, plans are being developed to achieve infrared astrometry on such objects at the level of 10 μas (Eisenhauer et al. 2008). High-precision Doppler measurements may also be possible (Zucker et al. 2006).

This makes it possible to consider doing more than merely detect relativistic effects, but rather to provide the first test of the black hole “no-hair” or uniqueness theorems of general relativity. According to those theorems, an electrically neutral black hole is completely characterized by its mass \( M \) and angular momentum \( J \). As a consequence, all the multipole moments of its external spacetime are functions of \( M \) and \( J \), specifically, the quadrupole moment \( Q_2 = -J^2/M \) (in units where \( G = c = 1 \)).

If the black hole were nonrotating \( (J = 0) \), then its exterior metric would be that of Schwarzschild, and the most important relativistic effect would be the advance of the pericenter. If it is rotating, then two new phenomena occur, the dragging of inertial frames and the effects of the hole’s quadrupole moment, leading not only to an additional pericenter precession but also to a precession of the orbital plane of the star. These precessions are smaller than the Schwarzschild effect in magnitude because they depend on the dimensionless angular momentum parameter \( \chi \equiv J/M^2 \), which is always less than 1, and because they fall off faster with distance from the black hole. However, accumulating evidence suggests that MBHs should be rather rapidly rotating, with \( \chi \) larger than 0.5 and possibly as large as 0.9, so these effects could be significant.

The purpose of this Letter is to point out that if a class of stars were to be found with orbital periods of fractions of a year and with sufficiently large orbital eccentricities, then the quadrupole-induced precessions could be as large as 10 μas yr\(^{-1} \). Figure 1 illustrates this: assuming a black hole with \( \chi = 0.7 \), it shows the orbital period required as a function of eccentricity for the rates of precessions due to Schwarzschild \((S)\), frame-dragging \((J)\), and quadrupole \((Q_2)\) terms to be as large as 10, 5, and 1 μas yr\(^{-1} \).

Figure 2 shows the effect of black hole spin on the amplitudes of the relativistic effects. For orbits with eccentricity 0.9 and periods of 1 yr and 0.1 yr, the amplitudes of the three effects are plotted in μas yr\(^{-1} \).

The precession of the orbital plane is the most important effect here because it depends only on \( J \) and \( Q_2 \); the Schwarzschild part of the metric affects only the pericenter advance. The orbital plane is determined by its inclination angle \( i \) relative to the plane of the sky and by the angle of nodes \( \Omega \) between a reference direction and the intersection of the two planes. Standard astrometric and Doppler observations can determine \( \Omega, i, \) the pericenter angle \( \omega \), the semimajor axis \( a \), and the orbital eccentricity \( e \) and, given sufficient observation time, the secular rates of change \( \partial \omega/\partial t, \partial i/\partial t, \) and \( \partial a/\partial t \).

However, in order to test the no-hair theorems, one must determine five parameters: the mass of the black hole, the magnitude and two angles of its spin, and the value of the quadrupole moment. The “Kepler-measured” mass is determined from the orbital periods of stars but may require data from a number of stars to fix it separately from any extended distribution of mass. Then, to measure \( J \) and \( Q_2 \), it is necessary and sufficient to measure \( \partial \omega/\partial t \) and \( \partial i/\partial t \) for two stars in nondegenerate orbits. A test of the no-hairness of the central object in our galaxy would be compelling evidence that it is truly a black hole of general relativity.
2. ORBIT PERTURBATIONS IN THE FIELD OF A ROTATING BLACK HOLE

For the purpose of this rough analysis, it suffices to work in the post-Newtonian limit. The equation of motion of a body of negligible mass in the field of a body with mass \( M \), angular momentum \( \mathbf{J} \), and quadrupole moment \( Q_2 \) is given by

\[
\mathbf{a} = -\frac{Mx}{r^3} + \frac{Mx}{r^3} \left( 4 \frac{M}{r} - \frac{v^2}{r^2} \right) + 4 \frac{M}{r^3} \mathbf{v} \\
- \frac{2J}{r^3} [2v \times \mathbf{J} - 3\mathbf{n} \times \mathbf{J} - 3(\mathbf{h} \cdot \mathbf{J}) \mathbf{r}]
+ \frac{3Q_2}{2r^4} [5(n \cdot \mathbf{J})^2 - 2(n \cdot \mathbf{J}) \mathbf{n} - n],
\]

where \( x \) and \( v \) are the position and velocity of the body, \( n = x/r, \ r = n \cdot v, h = x \times v, \) and \( \mathbf{J} = J/|J| \) (see, e.g., Will 1993). The first line of equation (1) corresponds to the Schwarzschild part of the metric (at post-Newtonian order), the second line is the frame-dragging effect, and the third line is the effect of the quadrupole moment (formally a Newtonian-order effect). For an axisymmetric black hole, the symmetry axis of the hole’s quadrupole moment coincides with its rotation axis, given by the unit vector \( \mathbf{J} \). The magnitude of the quadrupole moment will be left free.

Using standard orbital perturbation theory, we find that the precessions per orbit of the orientation variables are given by

\[
\Delta \omega = A_s - 2A_s \cos \alpha
\]

\[
\sin i \Delta \Omega = \sin \alpha \sin \beta (A_y - A_{Q_2} \cos \alpha),
\]

\[
\Delta i = \sin \alpha \cos \beta (A_y - A_{Q_2} \cos \alpha),
\]

where \( A_s = 6\pi M/p, A_y = 4\pi J/(mp^3)^{1/2}, A_{Q_2} = 3\pi Q_2/mp^2 \), and \( \Delta \omega = \Delta \omega + \cos i \Delta \Omega \) is the precession of pericenter relative to the fixed reference direction and \( p = a(1 - e^2) \) is the semilatus rectum. The quantities \( \alpha \) and \( \beta \) are the polar angles of the black hole’s angular momentum vector with respect to the star’s orbital plane defined by the line of nodes \( e_p \) and the vector in the orbital plane \( e_o \), orthogonal to \( e_p \) and \( h \).

The structure of equations (2b) and (2c) can be understood as follows: equation (1) implies that the orbital angular momentum \( \mathbf{h} \) precesses according to \( d\mathbf{h}/dt = \omega \times h \), where the orbit-averaged \( \omega \) is given by \( \omega = J (A_y - A_{Q_2} \cos \alpha) / (mp^2) \); the orbit element variations are given by \( d\mathbf{d}/dt = \omega \times e_p \) and \( \sin i d\Omega/dt = \omega \cdot e_o \). As a consequence, we have the purely geometric relationship

\[
\sin i d\Omega/dt = \tan \beta.
\]

To get an idea of the astrometric size of these precessions, we define an angular precession rate amplitude \( \Theta_i = (aD)A_i/P, \) where \( D \) is the distance to the Galactic center and \( P = 2\pi(a/|M|)^{1/2} \) is the orbital period. Using \( M = 3.6 \times 10^6 \) \( M_\odot \) and \( D = 8 \) kpc, we obtain the rates, in microarcseconds per year,

\[
\Theta_s \approx 13.3 P^{-1} (1 - e^2)^{-1},
\]

\[
\Theta_y \approx 0.847 P^{-3/2} (1 - e^2)^{3/2},
\]

\[
\Theta_{Q_2} \approx 9.68 \times 10^{-3} \chi^2 P^{-5/3} (1 - e^2)^{-2},
\]

where we have assumed \( |Q_2| = M^2 \chi^2 \). The observable precessions will be reduced somewhat from these raw rates because the orbit must be projected onto the plane of the sky. For example, the contributions to \( \Delta i \) and \( \sin i \Delta \Omega \) are reduced by a factor \( \sin i \); for an orbit in the plane of the sky, the plane precessions are unmeasurable.
For the quadrupole precessions to be observable, it is clear that the black hole must have a decent angular momentum ($\chi > 0.5$) and that the star must be in a short-period, high-eccentricity orbit. Figures 1 and 2 show the quantitative requirements, based on these rate amplitudes.

3. TESTING THE NO-HAIR THEOREMS

Although the pericenter advance is the largest relativistic orbital effect, it is not the most suitable effect for testing the no-hair theorems. The frame-dragging and quadrupole effects are small corrections of the leading Schwarzschild pericenter precession, and thus one would need to know $M$, $a$, and $e$ to sufficient accuracy to be able to subtract that dominant term to reveal the smaller effects of interest. Furthermore, the pericenter advance is affected by a number of complicating phenomena: (1) For such relativistic orbits, Schwarzschild contributions to the pericenter precession at the second post-Newtonian order may be needed. (2) Any distribution of mass (such as dark matter or gas) within the orbit, even if it is spherically symmetric, will generally contribute to the pericenter advance. (3) Tidal distortions of the stars are likely to occur near the pericenters of the highly eccentric orbits, leading to additional contributions to the pericenter advance of the form $30\pi(MmR/a)^3(k_2(1 + 3e^2/2 + e^3/8)(1 - e^2)^3)$, where $M$, $R$, and $k_2$ are the mass, radius, and “apsidal constant,” or Love number, of the star, respectively. Tidal contributions could be significant for sufficiently close and eccentric orbits.

Of course, if a star gets too close to the black hole, it could be tidally disrupted. This possibility sets a lower bound on the orbital period $P_{\text{max}} \sim 2\sqrt[3]{3\pi(R/m)^{1/2}}(1 - e)^{-3/2}$, set by requiring that the pericenter distance exceed the Roche radius of the star. This is illustrated by the dotted curves in Figure 1.

By contrast, the precessions of the node and inclination are relatively immune from such effects. Any spherically symmetric distribution of mass has no effect on these orbit elements. As long as any tidal distortion of the star is quasi-equilibrium with negligible tidal lag, the resulting perturbing forces are purely radial and thus have no effect on the node or inclination.

From the measured orbit elements and their drifts for a given star, equation (4) gives the angle $\beta$, independently of any assumption about the no-hair theorems. This measurement then fixes the spin axis of the black hole to lie on a plane perpendicular to the star’s orbital plane that makes an angle $\beta$ relative to the line of nodes. The equivalent determination for another stellar orbit fixes another plane; as long as the two planes are not degenerate, their intersection determines the direction of the spin axis, modulo a reflection through the origin.

This information is then sufficient to determine the angles $\alpha$ and $\beta$ for each star. Then, from the magnitude

$$\left[\left(\sin i \frac{d\theta}{dt}\right)^2 + \left(\frac{d\alpha}{dt}\right)^2\right]^{1/2} = \sin \alpha (A_\gamma - A_{Q_2} \cos \alpha)$$

(8)
determined for each star, together with the orbit elements, one can solve for $J$ and $Q_2$.

In practice, of course, the analysis of the astrometric data will be carried out in a more sophisticated, if less transparent, manner. Using data from all detected stars, one carries out a multiparameter least-squares fit, standard in solar system celestial mechanics, to determine their orbit elements. Their motions would be based on equation (1) but with $M$, $J$, and $Q_2$ treated as parameters to be fitted along with the orbit elements of each star. If necessary, the model can be extended to include effects of an additional matter distribution, tidal effects, and so on.

4. CONCLUDING REMARKS

We have shown that a class of stars orbiting a rotating central black hole in our galaxy in short-period, high-eccentricity orbits will experience precessions of their orbital planes induced by both frame dragging and the quadrupolar gravity of the hole, at levels that could be as large as $10 \mu$as yr$^{-1}$. Observations of the orbits of at least two such stars can in principle lead to a determination of the angular momentum vector $J$ and quadrupole moment $Q_2$ of the black hole and could provide a test of the no-hair theorems of general relativity.

Alternative possibilities for no-hair tests involve timing measurements of pulsars orbiting black hole companions (Wex & Kopeikin 1999), gravitational-wave measurements of compact objects spiralling into massive black holes (Ryan 1997; Glampedakis & Babak 2006; Hughes 2006), or detection of quasi-normal “ringdown” gravitational radiation of perturbed black holes (Dreyer et al. 2004; Berti, Cardoso, & Will 2006).

Detecting such stars so close to the black hole and carrying out infrared astrometry to 10 mas accuracy will be a challenge. However, if this challenge can be met with future improved adaptive optics systems currently under study, such as GRAVITY (Eisenhauer et al. 2008), it could lead to a powerful test of the black hole paradigm.

In future work, we plan to study in detail such complicating effects as second post-Newtonian (2PN) corrections to the Schwarzschild part of the pericenter advance, tidal effects, effects of unseen mass distributions within the observed stellar orbits, and light deflection and Shapiro time delay effects (Rubilar & Eckart 2001; Weinberg et al. 2005). For example, a torus of matter of mass $m$ orbiting the black hole at a distance $R$ will induce fractional changes in the apparent angular momentum and quadrupole moment of order $\delta J/J \sim (m/M)(R/m)^{1/2}(1/\chi)$ and $\delta Q/Q \sim (m/M)(R/m)^{3/2}(1/\chi)^2$, so only a very massive and/or very distant torus will be relevant. We also plan to carry out covariance analyses to obtain more realistic estimates of the accuracies that might be obtained for the no-hair test for given raw astrometric accuracies and for a range of observing schedules.

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REFERENCES

Alexander, T. 2005, Phys. Rep., 419, 65
Berti, E., Cardoso, V., & Will, C. M. 2006, Phys. Rev. D, 73, 064030
Dreyer, O., Kelly, B., Krishnan, B., Finn, L. S., Garrison, D., & Lopez-Aleman, R. 2004, Class. Quantum. Grav., 21, 787
Eisenhauer, F., Perrin, G., Rabien, S., Eckart, A., Léna, P., Genzel, R., Abuter, R., Paumard, T., & Brandner, W. 2008, in The Power of Optical/IR Interferometry: Recent Scientific Results and Second Generation Instrumentation, ed. A. Richichi, F. Delplancke, F. Paresce, & A. Chelli (Berlin: Springer)
Fragile, P. C., & Mathews, G. J. 2000, ApJ, 542, 328
Glampedakis, K., & Babak, S. 2006, Class. Quantum Grav., 23, 4167
Hughes, S. A. 2006, in AIP Conf. Proc. 873, Laser Interferometer Space Antenna, 6th International LISA Symposium, ed. S. M. Merkowitz & J. C. Livas (New York: AIP), 233
Jaroszynski, M. 1998, Acta Astron., 48, 653
Kraniotis, G. V. 2007, Class. Quantum Grav., 24, 1775
Rubilar, G. F., & Eckart, A. 2001, A&A, 374, 95
Ryan, F. D. 1997, Phys. Rev. D, 56, 1845
Weinberg, N. N., Milosavljević, M., & Ghez, A. M. 2005, ApJ, 622, 878
Wex, N., & Kopeikin, S. M. 1999, ApJ, 514, 388
Will, C. M. 1993, Theory and Experiment in Gravitational Physics (2d. ed.; Cambridge: Cambridge Univ. Press)
Zucker, S., Alexander, T., Gillessen, S., Eisenhauer, F., & Genzel, R. 2006, ApJ, 639, L21