ON TWO LETTER IDENTITIES IN LIE RINGS

BORIS BARANOV, SERGEI O. IVANOV, AND SAVELII NOVIKOV

Abstract. Let $L = L(a, b)$ be a free Lie ring on two letters $a, b$. We investigate the kernel $I$ of the map $L \oplus L \to L$ given by $(A, B) \mapsto [A, a] + [B, b]$. Any homogeneous element of $L$ of degree $\geq 2$ can be presented as $[A, a] + [B, b]$. Then $I$ measures how far such a presentation from being unique. Elements of $I$ can be interpreted as identities $[A(a, b), a] = [B(a, b), b]$ in Lie rings. The kernel $I$ can be decomposed into a direct sum $I = \bigoplus_{n, m} I_{n, m}$, where elements of $I_{n, m}$ correspond to identities on commutators of weight $n + m$, where the letter $a$ occurs $n$ times and the letter $b$ occurs $m$ times. We give a full description of $I_{2, m}$; describe the rank of $I_{3, m}$; and present a concrete non-trivial element in $I_{3, 3n}$ for $n \geq 1$.

Introduction

It is easy to check that the following identity is satisfied in any Lie ring (=Lie algebra over $\mathbb{Z}$)

$$[a, b, b, a] = [a, b, a, b],$$

where $[x_1, \ldots, x_n]$ is the left-normed bracket of elements $x_1, \ldots, x_n$ defined by recursion $[x_1, \ldots, x_n] := [[x_1, \ldots, x_{n-1}], x_n]$. We denote by $[a, i, b]$ the Engel brackets of $a, b$:

$$[a, 0, b] = a, \quad [a, i+1, b] = [[a, i, b], b].$$

For example, $[a, 3, b] = [a, b, b, b]$. In [1] the second author together with Roman Mikhailov generalized the identity [1] as follows

$$[[a, i, b], a] = \left[ \sum_{i=0}^{n-1} (-1)^i [[a, i, b], a, i, b], b \right],$$

where $n \geq 1$. This identity is crucial in their proof that the wedge of two circles $S^1 \vee S^1$ is a $\mathbb{Q}$-bad space in sense of Bousfield-Kan. Note that the letter $a$ occurs twice in each commutator of this identity and the letter $b$ occurs $2n$ times. Moreover, the identity has the form $[A, a] = [B, b]$.

We are interested in identities of the form:

$$[A(a, b), a] = [B(a, b), b],$$

where $A$ and $B$ are some expressions on letters $a$ and $b$. These identities can be interpreted as equalities in the free Lie ring $L = L(a, b)$. Note that a description of all identities of such kind would give a full description of the intersection $[L, a] \cap [L, b]$. Consider a $\mathbb{Z}$-linear map

$$\Theta : L \oplus L \to L,$n

$$\Theta(A, B) = [A, a] + [B, b].$$

Then the problem of describing identities of type (3) can be formalised as the problem of describing

$$I := \text{Ker}(\Theta).$$

Any homogeneous element of $L$ of degree $\geq 2$ can be presented as $[A, a] + [B, b]$. So $I$ measures how far this presentation from being unique. The problem of describing of $I$ is different from the problem formulated on the formal language of identities, because $A, B$ here are not just formal expressions but they are elements of the free Lie ring. For example, the identity $[[b, b], a] = [[a, a], b]$ is not interesting for us because $([b, b], [a, a]) = (0, 0)$ in $I$. This work is devoted to the study of $I$.

The Lie ring $L$ has a natural grading by the weight of a commutator: $L = \bigoplus_{n \geq 1} L_n$. Moreover, $L_n = \bigoplus_{k+m=n} L_{k, m}$, where $L_{k, m} \subseteq L_{k+m}$ is an abelian group generated by multiple commutators with $k$ letters $a$ and $m$ letters $b$. We can consider the following restrictions of the map $\Theta$

$$\Theta_n : L_{n-1} \oplus L_{n-1} \to L_n, \quad \Theta_{k, l} : L_{k-1, l} \oplus L_{k, l-1} \to L_{k, l},$$

and set $I_n = \text{Ker}(\Theta_n)$ and $I_{k, l} = \text{Ker}(\Theta_{k, l})$. It is easy to check that

$$I = \bigoplus_{n \geq 1} I_n, \quad I_n = \bigoplus_{k+l=n} I_{k, l}.$$
The main results of the paper are the full description of $I_{2,n}$; the description of the rank of the free abelian group $I_{3,n}$; and the description of a concrete series of elements from $I_{3,3n}$ for any $n \geq 1$.

The rank of a free abelian group $X$ is called “dimension of $X$” in this paper and it is denoted by $\dim X$. It well known that the dimension of $L_n$ can be computed by the Necklace polynomial

$$\dim L_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right)2^d,$$

where $\mu$ is the Mobius function.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $\dim I_n$ | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

Since the map $\Theta_n$ is an epimorphism, we obtain

$$\dim I_n = 2 \cdot \dim L_{n-1} - \dim L_n.$$ 

We prove the following

$$\dim I_{2,m} = \begin{cases} 0, & \text{if } m \text{ is odd} \\
1, & \text{if } m \text{ is even} \end{cases}, \quad \dim I_{3,m} = \left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{m-1}{3} \right\rfloor - 1.$$ 

(Proposition 1.4 Proposition 2.5). For $n \leq 13$ we obtain the following table for dimensions.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|---|---|---|---|---|---|---|----|----|----|----|
| $\dim I_{2,n-2}$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\dim I_{3,n-3}$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 1 | |
| $\dim I_n$ | 3 | 0 | 1 | 0 | 3 | 0 | 6 | 4 | 13 | 12 | 37 | 40 |

The computation of $\dim I_{2,m}$ shows that there are no any other elements in $I_{2,m}$ except those that come from the identities (4). In particular, all non-trivial identities corresponding to elements of $I_{2,n}$ have even weight.

Note that $I_n = 0$ for odd $n < 9$. This can be interpreted as the fact that there is no a non-trivial identity of the type $[A,a] = [B,b]$ on two letters of odd weight lesser than 9. However, we have found a non-trivial identity of this type of weight 9 (Theorem 2.8). If we set

$$C_n = [a_n,b],$$

then the following identity of weight 9 holds in any Lie ring

$$[2[C_5,C_1] + 5[C_4,C_2], a] = [2[C_4,C_1,C_0] + 3[C_3,C_2,C_0] - 2[C_3,C_1,C_1] + [C_2,C_1,C_2], b].$$

The main result of this paper is a concrete series of identities that correspond to non-trivial elements in $I_{3,3n}$ that generalise the identity (4) (Theorem 2.9). Namely, for any $n \geq 1$, the following identity is satisfied in $L_{3,3n}$.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+1-k,k}[C_{2n+k-1-k},C_{n+k-1}] , a = \sum_{i=0}^{n} \sum_{j=0}^{\lfloor \frac{i+1}{2} \rfloor} (-1)^{i+1} \alpha_{i,j} [C_{n+i-j},C_{n+j-1},C_{n-i}] , b,$$

where $\alpha_{0,0} = 1$ and $\alpha_{i,j} = 2(i+j-1) + (i+j-2) - (i+j-2) - 2(i+j-1)$ for $i,j \geq 0$ and $(i,j) \neq (0,0)$. For $n = 2$ we obtain the identity (4). For $n = 1$ we obtain the identity

$$[3[a,b,b,[a,b]] + 2[a,b,a], a] = [-[a,b,a,[a,b]] + 2[a,b,a], b]$$

that holds in any Lie ring.

1. Identities corresponding to elements in $I_{2,m}$

**Definition 1.1.** If $w$ is a Lyndon word, we denote by $[w]$ the corresponding element of the Lyndon-Shirshov basis of the free Lie algebra $L$ (see [2]). If $w$ is a letter, then $[w] = w$. If $w$ is not a letter then $w$ has a standard factorisation $w = uv$ and $[w]$ is defined by recursion $[w] = [[u],[v]]$. For example, $[a] = a$ and $[ab^n] = [ab^{n-1},b] = C_n$.

**Lemma 1.2.** The following set is a basis of $L_{2,n}$ with $n \in \mathbb{N}$.

$$\{ [C_k,C_l] \mid k > l, k + l = n \}$$
Proof. The intersection of the Lyndon-Shirshov basis with \( L_{k,m} \) is a basis of \( L_{k,m} \). The basis of \( L_{2,m} \) consists of commutators of Lyndon words with 2 letters “\( a \)” and \( m \) letters “\( b \)”.

\[
[ab^l ab^k] = [[ab^l], [ab^k]] = -[[ab^l], [ab^k]] = -[C_k, C_l].
\]

Word \( ab^l ab^k \) is a Lyndon word only when \( k > l \). The assertion follows. \( \Box \)

Lemma 1.3. For any \( n \in \mathbb{N} \) the following is satisfied:

\[
\dim L_{2,n} = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n + 1}{2} \right\rceil.
\]

Proof. Consider the basis from lemma 1.2 Hence \( L_{2,n} = \{ (\{ C_{n_1}, C_{n_2} \} | n_1, n_2 \in N_0, n_1 > n_2 \text{ and } n_1 + n_2 = n \} \). Total number of words with 2 letters \( a \) and \( n \) letters \( b \) starting with \( a \) is \( n + 1 \). However, in our case \( n \_1 > n_2 \). Hence for odd \( n \) number of such commutators is \( \frac{n+1}{2} \) and for even \( n \) it is \( \frac{n}{2} \). \( \Box \)

Proposition 1.4. For any \( m \in \mathbb{N} \) we have

\[
\dim I_{2,m} = \begin{cases} 
0, & \text{if } m \text{ is odd} \\
1, & \text{if } m \text{ is even}.
\end{cases}
\]

Proof. By definition, \( I_{2,m} = \text{Ker } \Theta_{2,m} \). Then \( \dim I_{2,m} = \text{dim(ker } \Theta_{2,m}) = \dim(L_{1,m} + L_{2,m-1}) - \dim(\text{Im } \Theta_{2,m}) = \dim(L_{1,m} \oplus L_{2,m-1}) - \dim L_{2,m} = \dim L_{1,m} + \dim L_{2,m-1} - \dim L_{2,m} = 1 + \dim L_{2,m-1} - \dim L_{2,m} = 1 + \left\lfloor \frac{m}{2} \right\rfloor - \left\lceil \frac{m+1}{2} \right\rceil \) (see lemma 1.3). Let \( m \) be even, then \( \dim I_{2,m} = 1 + \frac{m}{2} - \left\lfloor \frac{m}{2} \right\rfloor = 1 + \frac{m}{2} - \frac{m}{2} - \frac{1}{2} = 0 \). Consider the case of odd \( m \). Then \( \dim I_{2,m} = 1 + \left\lfloor \frac{m-1}{2} \right\rfloor - \left\lceil \frac{m+1}{2} \right\rceil = 1 + \frac{m}{2} - \frac{m}{2} - \frac{1}{2} = 0. \) \( \Box \)

Theorem 1.5. For any \( m \in \mathbb{N} \) the following is satisfied

\[
I_{2,m} = \begin{cases} 
0, & \text{if } m \text{ is odd} \\
\{(C_m, \sum_{i=1}^{\frac{m}{2}} (-1)^i[C_{m-i}, C_{i-1}] ) \}, & \text{if } m \text{ is even}.
\end{cases}
\]

Proof. Triviality of the kernel for odd \( m \) can be easily proven using lemma 1.3. Consider the case when \( m \) is even. Then basis of \( L_{1,m} \) consists of one element \([ab^m]^m \), i.e. \( L_{1,m} = \{ [ab^m]^m | a \in Z \} \). Basis of \( L_{2,m-1} \) consists of \( \left\lfloor \frac{m}{2} \right\rfloor \) elements (according to lemma 1.3). Because \( m \) is even \( \left\lfloor \frac{m}{2} \right\rfloor = \frac{m}{2} \). Hence the following equality is true

\[
L_{2,m-1} = \{ a_1 [ab^m]^m + a_2 [ab^m]^2 + \cdots + a_n [ab^m - \frac{n-1}{2}] | a_1, a_2, \ldots, a_n \in Z \}.
\]

By definition, \( I_{2,m} = \text{ker } \Theta_{2,m} \). We can apply map \( \Theta_{2,m} \) to arbitrary element of \( L_{1,m} \oplus L_{2,m-1} \) that is expressed as basis elements and equate the obtained to zero. Jacobi identity implies the following

\[
[[ab^m ab^n], b] = [[ab^{n+1}, [ab^m]] + [ab^m ab^{m+1}]].
\]

We can use this equality to transform an image of an element from \( L_{2,m-1} \).

\[
\Theta_{2,m} \left( \left[ ab^n \right], \sum_{i=1}^{\frac{m}{2}} a_i [ab^{i-1}ab^{m-i}] \right) = a_i [ab^m, a] + \sum_{i=1}^{\frac{m}{2}} a_i [ab^{i-1}ab^{m-i}, b] = -a_i [ab^m] + \sum_{i=1}^{\frac{m}{2}} a_i ([ab^i, [ab^{m-i}]] + [ab^{i-1}ab^{m-i+1}]) =
\]

For all \( i \neq \frac{m}{2} \) commutator \([ab^i], [ab^{m-i}] = [ab^m] \). Then the sum can be rewritten as follows

\[
= -a_i [ab^{m-i}] + a_1 [ab^{m-1}] + a_2 [ab^{m-2}] + a_3 [ab^{m-3}] + \cdots + a_{i-1} [ab^{i-2}ab^{m-i+2}] + a_i [ab^{i-1}ab^{m-i+1}] + a_{i+1} [ab^{i}ab^{m-i+1}] + a_{i+2} [ab^{i+1}ab^{m-i+1}] + \cdots + a_{\frac{m}{2}} [ab^{\frac{m}{2}}, [ab^{m-\frac{m}{2}}]] + a_{\frac{m}{2}} [ab^{\frac{m}{2}} - ab^{m-\frac{m}{2}} + 0].
\]

Last but one element of sum is equals to 0 because \( a_\frac{m}{2} [ab^{\frac{m}{2}}, [ab^{m-\frac{m}{2}}]] = a_\frac{m}{2} [ab^{\frac{m}{2}}, [ab^{\frac{m}{2}}]] = 0 \). It is easy to see that for equality we need such coefficients \( a \) and \( \alpha \) that terms of the sum will be reduced. Let \( a = 1 \), hence \( \alpha_1 = 1 \) because we need \( -a_i [ab^{m-i}] + a_1 [ab^{m-1}] \) to be reduced. Other coefficients can be obtained similarly. Commutators with coefficients \( \alpha_i \) and \( \alpha_{i+1} \) will be reduced. Hence ker \( \Theta_{2,m} \) is generated by element \([ab^m] = C_m \) and sum \([ab^{m-1}] - [ab^{m-2}] - \cdots + [ab^{i-2}ab^{m-i+2}] \pm [ab^{i-1}ab^{m-i+1}] = \sum_{i=1}^{\frac{m}{2}} (-1)^{i+1} [ab^{i-1}, [ab^{m-i}]] = \sum_{i=1}^{\frac{m}{2}} (-1)^i [C_{m-i}, C_{i-1}] \) \( \Box \)
Lemma 2.1. For any \( n \in \mathbb{N} \) the following set is a basis of \( L_{3,n} \):

\[
\{ [C_k, C_l, C_m] \mid k > l, k \geq m, k + l + m = n \mid k, l, m \in \mathbb{N} \}.
\]

Proof. Lyndon words commutators of length \( n + 3 \) with 3 letters “a” and \( n \) letters “b” construct the basis of \( L_{3,n} \). It is easy to prove that \( ab'ab'ab' \) is a Lyndon word if and only if \( i \leq j \) and \( i < t \), where \( i, j, t \in \mathbb{N}_0 \). Consider two cases:

1. \( j < t \) then \([ab'] [ab'] [ab'] = [[ab'], [ab'], [ab']] = [[ab'], [ab'], [ab']] = [C_l, C_j, C_t] \). Take \( t = k, j = l, i = m \) then \( k > l, k > m \) and \( l \geq m \).
2. \( j \geq t \) then \([ab'] [ab'] [ab'] = [[ab'], [ab'], [ab']] = [[ab'], [ab'], [ab']] = [C_i, C_j, C_t] \). Take \( j = k, u = t, t = m \) then \( k \geq m, l \leq k \) and \( m > l \). Hence \( k \geq m > l \), so \( k > l \).

If we unite conditions of both cases, we get \( k > l \) and \( k \geq m \) for arbitrary \( k, l, m \in \mathbb{N}_0 \).

2.1. Generalized identity with three letters “a”.

Lemma 2.2. For expression \( \alpha_{i,j} = 2^{(i+j-1) + (i+j-2) - (i+j-2) - 2^{(i+j-1)}} \), where \( i, j \in \mathbb{N}_0 \) the following conditions are satisfied:

1) \( \alpha_{i-1,j} + \alpha_{i,j-1} = \alpha_{i,j} \), when \( i \neq j \) and \( j \neq 0 \)
2) \( \alpha_{i,i} = \alpha_{i,j} \), when \( i \geq 2 \)
3) \( \alpha_{i,0} = 2 \), when \( i \geq 1 \)

Proof. We can use the recurrence relation for binomial coefficient to prove the first condition:

\[
\alpha_{i-1,j} + \alpha_{i,j-1} = 2^{(i+j-2) + (i+j-2) - (i+j-2) - 2^{(i+j-2)}} + 2^{(i+j-2) - (i+j-2) - 2^{(i+j-2)}} + 2^{(i+j-2) - (i+j-2) - 2^{(i+j-2)}} = 2^{(i+j-2)} + 2^{(i+j-2)} + 2^{(i+j-2)} = \alpha_{i,j}.
\]

To prove the second condition we can substitute \( j = i \) into \( \alpha_{i,j} \) and express each term using recurrence relation for binomial coefficient:

\[
\alpha_{i,i} = 2^{(2i-2) + (2i-2) - (2i-2) - 2^{(2i-2)}} = 2^{(2i-2)} + 2^{(2i-2)} + 2^{(2i-2)} = \alpha_{i-1,i} + 2^{(2i-2)} + 2^{(2i-2)} + 2^{(2i-2)} = \alpha_{i+1,i}.
\]

All we need to prove now is that \( 2^{(2i-2)} + 2^{(2i-2)} + 2^{(2i-2)} = 0 \). Using symmetric property of binomial coefficient, i.e. \( \binom{n}{k} = \binom{n}{n-k} \), all terms will be reduced:

\[
2^{(2i-2)} + 2^{(2i-2)} + 2^{(2i-2)} = 2^{(2i-2)} + 2^{(2i-2)} + 2^{(2i-2)} = \alpha_{i+1,i}.
\]

Consider the case, when \( j = 0 \). We need to mention that for \( k < 0 \) binomial coefficient \( \binom{n}{k} = 0 \). Then the expression will be as follows:

\[
\alpha_{i,0} = 2^{(i-1)} + 2^{(i-1)} + 2^{(i-1)} + 2^{(i-1)} = 2^{(i-1)} + 2^{(i-1)} = 2.
\]

Theorem 2.3. For any \( n \in \mathbb{N} \), the following identity is satisfied in \( L_{3,3n} \):

\[
\sum_{k=0}^{n} (-1)^{n+1} \alpha_{n+1-k,k} [C_{2n+1-k}, C_{n+k-1}], a = \sum_{i=0}^{n} \sum_{j=0}^{\frac{3n}{2}} (-1)^{i+1} \alpha_{i-j,j} [C_{n+i-j}, C_{n+j-1}, C_{n-i}] b,
\]

where \( \alpha_{0,0} = 1 \) and \( \alpha_{i,j} = 2^{(i+j-1)} + 2^{(i+j-2)} + 2^{(i+j-2)} + 2^{(i+j-2)} \), where \( i, j \in \mathbb{N} \).
Proof. Consider \( n, k \in \mathbb{N} \). Let's prove the following identity for any \( k \leq n \):

\[
\sum_{i=0}^{k} \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^{i+j} \alpha_{i-j,j} [C_{n+i-j}, C_{n+j-1}, C_{n-i}], b] = \sum_{t=0}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{k+1} \alpha_{k+1-t,t} [C_{n+k+1-t}, C_{n-1+t}, C_{n-k}].
\]

Denote the left part as \( \omega_k \) and the right part as \( \theta_k \). We will prove this equality using mathematical induction with variable \( k \).

1. We can expand the sum and use lemma 2.2.

\[
\omega_k = [-\alpha_{0,0} [C_n, C_{n-1}, C_n] + \alpha_{1,0} [C_{n+1}, C_{n-1}, C_n], b] = -1[C_n, C_{n-1}, C_n, b] + 2[C_{n+1}, C_{n-1}, C_n, b] =
\]

\[
= -1[C_{n+1}, C_{n-1}, C_n] + [C_{n+1}, C_{n-1}, C_n, b] + [C_{n+1}, C_{n-1}, C_{n-1}, b] + 2[C_{n+2}, C_{n-1}, C_n, b] + 2[C_{n+1}, C_{n-1}, C_{n-1}, b] =
\]

\[
= 2[C_{n+2}, C_{n-1}, C_n] + 3[C_{n+1}, C_{n-1}, C_n] = \alpha_{2,0} [C_{n+2}, C_{n-1}, C_n] + \alpha_{1,1} [C_{n-1}, C_n] = \theta_1.
\]

2. We need to prove that \( \omega_k = \theta_k \) implies \( \omega_{k+1} = \theta_{k+1} \). We can expand \( \omega_{k+1} \) as a sum of \( \omega_k \) and the last element of the first sum in \( \omega_{k+1} \):

\[
\omega_{k+1} = \omega_k + \left[ \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{k+2} \alpha_{k+1-j,j} [C_{n+k+2-j}, C_{n-j-1}, C_{n-k}][C_{n+k-1-j}, C_{n-j-1}, C_{n-k}] + \right]
\]

By expanding the commutator as follows:

\[
[C_{n+k+2-j}, C_{n-j-1}, C_{n-k}] = [C_{n+k+2-j}, C_{n-j-1}, C_{n-k}] + [C_{n+k+1-j}, C_{n-j-1}, C_{n-k}].
\]

Consider two cases:

1. 

Denote commutators in sums as \( a_j \) and \( b_j \) correspondingly. We can show that for any \( j \geq 0 \) it is satisfied that \( a_{j+1} = b_j \). Because of that, the expression can be written as a sum of \( a_0, b_{\lfloor \frac{k+1}{2} \rfloor} \) with corresponding coefficients and one sum on index \( j \) with summed coefficients.

\[
\omega_{k+1} = \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor-1} (-1)^{k+2} (a_{k-j-1} + a_{k-1-j}) [C_{n+k+1-j}, C_{n-j-1}, C_{n-k}] + (-1)^{k+2} a_{k+1,0} [C_{n+k+2}, C_{n-1}, C_{n-k}] -
\]

\[
+ (-1)^{k+2} a_{k+1-\lfloor \frac{k+1}{2} \rfloor} [C_{n+k+1-\lfloor \frac{k+1}{2} \rfloor}, C_{n-1-\lfloor \frac{k+1}{2} \rfloor}, C_{n-k}].
\]

Coefficients of commutators, in obtained sum on index \( j \), can be transformed using the first case of lemma 2.2. Also, we can change index of sum by subtracting 1 from it. Coefficient of \( a_0 \) can be rewritten using the third case of lemma 2.2. Then \( b_{\lfloor \frac{k+1}{2} \rfloor} \) will become a zero element of sum on index \( j \).

\[
\omega_{k+1} = \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{k+2} a_{k-2-j} [C_{n+k+2-j}, C_{n-1-j}, C_{n-k-1}] + (-1)^{k+2} a_{k+2,0} [C_{n+k+2}, C_{n-1}, C_{n-k}] +
\]

\[
+ (-1)^{k+2} a_{k+1-\lfloor \frac{k+1}{2} \rfloor} [C_{n+k+1-\lfloor \frac{k+1}{2} \rfloor}, C_{n-1-\lfloor \frac{k+1}{2} \rfloor}, C_{n-k}] =
\]

\[
= \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{k+2} a_{k+2-j} [C_{n+k+2-j}, C_{n-1-j}, C_{n-k-1}] + (-1)^{k+2} a_{k+1-\lfloor \frac{k+1}{2} \rfloor} [C_{n+k+1-\lfloor \frac{k+1}{2} \rfloor}, C_{n-1-\lfloor \frac{k+1}{2} \rfloor}, C_{n-k}] b_{\lfloor \frac{k+1}{2} \rfloor}.
\]

Consider two cases:
1) $k$ is odd. Then $\left\lfloor \frac{k+2}{2} \right\rfloor = \left\lfloor \frac{k+1}{2} + \frac{1}{2} \right\rfloor = \frac{k+1}{2} + \left\lfloor \frac{1}{2} \right\rfloor = \frac{k+1}{2} + 1$, hence $b\left\lfloor \frac{k+1}{2} \right\rfloor = 0$. It is true because $b\left\lfloor \frac{k+1}{2} \right\rfloor = [C_{n+k+1-k\frac{k+1}{2}}, C_{n+k\frac{k+1}{2}}, C_{n-k-1}] = [0, C_{n-k-1}] = 0$. Consequently, the identity above is satisfied for any $n, k \in \mathbb{N}$, such that $k \leq n$. If we substitute $k = n$, we will get the original identity.

$$\omega_{k+1} = \sum_{j=0}^{\left\lfloor \frac{k+2}{2} \right\rfloor} (-1)^{k+2} \alpha_{k+2-j,j} [C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] = \theta_{k+1}.$$ 

2) $k$ is even. Then $\left\lfloor \frac{k+2}{2} \right\rfloor = \frac{k+2}{2} = \frac{k+1}{2} + 1$, hence $a_{k+2-\left\lfloor \frac{k+2}{2} \right\rfloor} = a_{\left\lfloor \frac{k+1}{2} \right\rfloor + 1, \left\lfloor \frac{k+1}{2} \right\rfloor}$. Consequently, the identity above is satisfied for any $n, k \in \mathbb{N}$, such that $k \leq n$. If we substitute $k = n$, we will get the original identity.

**2.2. Additional results.**

**Lemma 2.4.** For any $n \in \mathbb{N}$ the following is satisfied:

$$\dim L_{3,n} - \dim L_{3,n-1} = \left\lfloor \frac{n-1}{3} \right\rfloor + 1.$$ 

**Proof.** To calculate this expression, we need to count all Lyndon words of form $ab^{n_1}ab^{n_2}ab^{n_3}$, where $n_1, n_2, n_3 \in \mathbb{N}$ and $n_1 + n_2 + n_3 = n$. Let $n_1 = i$ and $n_2 = j$, hence $n_3 = n - i - j$. As it was mentioned before, $ab^{n_1}ab^{n_2}ab^{n_3}$ is a Lyndon word if and only if $n_1 \leq n_2$ and $n_1 < n_3$, where $n_1, n_2, n_3 \in \mathbb{N}$. We can portray integer points that satisfy these conditions on coordinate plane by drawing plots of functions $y = x$ and $y = n - 2x$. 

```
1) k is odd. Then [k+2/2] = [k+1/2 + 1/2] = k+1/2 + [1/2] = k+1/2, hence b[k+1/2] = 0. It is true because b[k+1/2] = [C_{n+k+1-k[k+1/2]}, C_{n+k[k+1/2]}, C_{n-k-1}] = [0, C_{n-k-1}] = 0. Consequently, the identity above is satisfied for any n, k âˆ N, such that k â‰¤ n. If we substitute k = n, we will get the original identity.

$$\omega_{k+1} = \sum_{j=0}^{[k+2/2]} (-1)^{k+2} \alpha_{k+2-j,j} [C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] = \theta_{k+1}.$$ 

2) k is even. Then [k+2/2] = k+2/2 = k+1/2 + 1, hence a_{k+2-[k+2/2]} = a_{[k+1/2] + 1, [k+1/2]}. Consequently, the identity above is satisfied for any n, k âˆ N, such that k â‰¤ n. If we substitute k = n, we will get the original identity.

**2.2. Additional results.**

**Lemma 2.4.** For any n âˆ N the following is satisfied:

$$\dim L_{3,n} - \dim L_{3,n-1} = \left\lfloor \frac{n-1}{3} \right\rfloor + 1.$$ 

**Proof.** To calculate this expression, we need to count all Lyndon words of form ab^{n_1}ab^{n_2}ab^{n_3}, where n_1, n_2, n_3 âˆ N_0 and n_1 + n_2 + n_3 = n. Let n_1 = i and n_2 = j, hence n_3 = n - i - j. As it was mentioned before, ab^{n_1}ab^{n_2}ab^{n_3} is a Lyndon word if and only if n_1 â‰¤ n_2 and n_1 < n_3, where n_1, n_2, n_3 âˆ N. We can portray integer points that satisfy these conditions on coordinate plane by drawing plots of functions y = x and y = n - 2x.
Abscissa of functions intersection point is $\frac{n}{2}$: $ab^iab^{n-i-j}$ is a Lyndon word if point $(i,j)$ belongs to $\Delta ABC$ (without point on the line $y = n - 2x$). Then $\dim L_{3,n}$ equals to number of integer points in $\triangle DEC$. Hence $\dim L_{3,n} - \dim L_{3,n-1}$ equals to number of integer points on segment $DE$, i.e. $\left\lfloor \frac{n-1}{3} \right\rfloor + 1$.

**Proposition 2.5.** For any $m \in \mathbb{N}$ the following is satisfied:

$$\dim I_{3,m} = \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m-1}{3} \right\rfloor - 1.$$ 

**Proof.** By definition, $I_{3,m} = \ker \Theta_{3,m}$. Hence, according to lemmas [4.4] and [2.4] $\dim I_{3,m} = \dim (L_{2,m} \oplus L_{3,m-1}) - \dim L_{3,m} = \dim L_{2,m} - (\dim L_{3,m} - \dim L_{3,m-1}) = \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m-1}{3} \right\rfloor - 1$. 

**Lemma 2.6.** For $k > l$, $k \geq m$ the following is satisfied:

$$[C_k, C_l, C_m, b] = \begin{cases} 
[C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m], & \text{if } k > l + 1, k \geq m + 1 \\
[C_{k+1}, C_l, C_m] + [C_k, C_l, C_{m+1}], & \text{if } k > l + 1, k < m + 1 \\
2[C_{k+1}, C_l, C_m] - [C_{k+1}, C_{l+1}, C_m], & \text{if } k = l + 1, k < m \\
2[C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] - [C_{k+1}, C_k, C_l], & \text{if } k > l + 1, k = m.
\end{cases}$$

**Proof.** It is easy to rewrite the expression in the first case using Jacobi identity:

$$[C_k, C_l, C_m, b] = [C_k, C_l, b, C_m] + [C_k, C_l, [C_m, b]] = [C_k, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_l, C_{m+1}].$$

Second case:

$$[C_k, C_l, C_m, b] = (C_{l+2}, C_l, C_m) + (C_{l+1}, C_l, C_m) + (C_{l+1}, C_l, C_{m+1}) = [C_{k+1}, C_l, C_m] + [C_k, C_l, C_{m+1}].$$

Third case:

$$[C_k, C_l, C_m, b] = [C_{k+1}, C_l, C_m] + [C_k, C_l, C_{m+1}] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_m, C_l, C_k] = 2[C_{k+1}, C_l, C_m] - [C_{k+1}, C_{l+1}, C_m].$$

Fourth case:

$$[C_k, C_l, C_m, b] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_l, C_{m+1}] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_m, C_l] + [C_l, C_m, k] = 2[C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] - [C_{k+1}, C_k, C_l].$$

**Theorem 2.7.** The kernel of $\Theta_{3,3}$ is generated by the following element:

$$(3[C_2, C_1] + 2[C_3, C_0], [C_1, C_0, C_1] - 2[C_2, C_0, C_0]).$$

**Proof.** According to lemma [2.5] $\dim I_{3,3} = \left\lfloor \frac{3}{2} \right\rfloor - \left\lfloor \frac{3}{3} \right\rfloor - 1 = 1$. Consequently, we have to provide only one identity to describe the whole $I_{3,3}$. Substitute $n = 1$ into identity from theorem [2.3]

$$[\alpha_{2,0}[C_3, C_0] + \alpha_{1,1}[C_2, C_1], a] = [-\alpha_{0,0}[C_1, C_0, C_1] + \alpha_{1,0}[C_2, C_0, C_0], b].$$

By definition of $\alpha_{i,j}$, $\alpha_{2,0} = 2$, $\alpha_{1,1} = 3$, $\alpha_{0,0} = 1$ and $\alpha_{1,0} = 2$. We can move right part of the equality to the left side and it will become an image of the element from $L_{2,3} \oplus L_{3,2}$:

$$[2[C_3, C_0] + 3[C_2, C_1], a] + [[C_1, C_0, C_1] - 2[C_2, C_0, C_0], b] = \Theta_{3,3}(2[C_3, C_0] + 3[C_2, C_1], [C_1, C_0, C_1] - 2[C_2, C_0, C_0]) = 0$$

As a result, we obtained the element that generates all identities in $I_{3,3}$ that is equivalent to description of $I_{3,3}$.

**Theorem 2.8.** The abelian group $I_{3,6}$ is generated by the following element

$$(-2[C_5, C_1] - 5[C_4, C_2], 2[C_4, C_1, C_0] + 3[C_3, C_2, C_0] - 2[C_3, C_1, C_1] + [C_2, C_1, C_2]).$$

**Proof.** Similarly to proof of the theorem [2.7] $\dim I_{3,6} = \left\lfloor \frac{9}{2} \right\rfloor - \left\lfloor \frac{5}{3} \right\rfloor - 1 = 1$. Substitute $n = 2$ into identity from theorem [2.3]

$$[-\alpha_{3,0}[C_5, C_1] - \alpha_{2,1}[C_4, C_2], a] = [-\alpha_{0,0}[C_2, C_1, C_2] + \alpha_{1,0}[C_3, C_1, C_1] - \alpha_{2,0}[C_4, C_1, C_0] - \alpha_{1,1}[C_3, C_2, C_0], b].$$

Coefficients will be $\alpha_{3,0} = 2$, $\alpha_{2,1} = 5$, $\alpha_{0,0} = 1$, $\alpha_{1,0} = 2$, $\alpha_{2,0} = 2$ and $\alpha_{1,1} = 3$. Again, we’ve found an element of $L_{2,6} \oplus L_{3,5}$ that generates all possible identities. Coefficients in the right part will be multiplied by $-1$ because of moving to the left side.
REFERENCES

[1] Sergei O. Ivanov, Roman Mikhailov: A finite \( \mathbb{Q} \)-bad space. arXiv:1708.00282

[2] C. Reutenauer: Free Lie algebras, Oxford University Press, 1993

Laboratory of Continuous Mathematical Education (School 564 of St. Petersburg), nab. Obvodnogo kanala 143, Saint Petersburg, Russia.
E-mail address: BBBOOORRIISSS@mail.ru

Laboratory of Modern algebra and Applications, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia.
E-mail address: ivanov.s.o.1986@gmail.com

Laboratory of Continuous Mathematical Education (School 564 of St. Petersburg), nab. Obvodnogo kanala 143, Saint Petersburg, Russia.
E-mail address: novikov.savellii00@gmail.com