ON FRACTIONAL INTEGRALS GENERATED BY
RADON TRANSFORMS OVER PARABOLOIDS

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Abstract. We apply the Fourier transform technique and a modified version of E. Stein’s interpolation theorem communicated by L. Grafakos, to obtain sharp $L^p$-$L^q$ estimates for the Radon transform and more general convolution-type fractional integrals with the kernels having singularity on the paraboloids.

1. Introduction

In the present paper we develop the Fourier transform approach to the Radon-type transform

$$(Pf)(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) \, dy'$$ (1.1)

and more general fractional integrals

$$(P_\alpha^\pm f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (y_n - |y'|^2)^{\alpha-1} f(x - y) \, dy, \quad \text{Re} \alpha > 0, \quad (1.2)$$

which extend analytically to $\text{Re} \alpha \leq 0$ and have a singularity on the paraboloid $y_n = |y'|^2$. Here $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$ (similarly for $y$), and $f$ is a sufficiently good function on $\mathbb{R}^n$; see Section 2 for notation. We call (1.1) the parabolic Radon transform. This operator was also considered in [5, 21, 22]. The limiting case $\alpha = 0$ in (1.2) yields (1.1).

Sharp $L^p$-$L^q$ estimates for $Pf$ can be obtained by making use of the Oberlin-Stein theorem for the usual Radon transform over affine hyperplanes in $\mathbb{R}^n$ [16]. They can also be derived from the similar estimates for the transversal Radon transform

$$(Tf)(x) = \int_{\mathbb{R}^{n-1}} f(y', x_n + x' \cdot y') \, dy', \quad (1.3)$$

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which was introduced by Strichartz [31] in his study of Radon transforms on the Heisenberg group; see [19], [20, Section 4.13], [21] for details.

The purpose of the present paper is to present a straightforward Fourier transform approach to the $L^p$-$L^q$ estimates for the entire analytic family $\{P_\pm^\alpha\}_{\alpha \in \mathbb{C}}$, taking into account that these operators have convolution structure.

This approach becomes possible thanks to a version of Stein’s interpolation theorem [27] communicated by L. Grafakos [10]. An advantage of this version is that the hypotheses of Stein’s theorem, which make this theorem applicable, are presented in [10] in the more convenient terms of smooth compactly supported functions, rather than in terms of simple functions in [27]. We recall that simple functions are finite linear combinations of the characteristic functions of disjoint compact sets; see Section 5.3 for details. A close interpolation theorem in the multi-linear setting was recently proved by Grafakos and Ouhabaz [11].

Some $L^p$-$L^q$ estimates for localized modifications of (1.2) with a smooth cut-off function under the sign of integration were announced by Littman [13] and Tao [32], who referred to Stein’s interpolation theorem [27]. A distinctive feature of $P^\alpha_\pm f$ in comparison with [13, 32] is that our operators are not localized and their Fourier transforms can be explicitly computed. A roundabout approach to the study of operators (1.2) via non-convolution-type fractional integrals generated by the operator $T$ was developed in [22].

It is interesting to note that implementation of the Fourier transform shows that the analytic continuations of $P^\alpha_\pm$ at $\alpha = (1 - n)/2$ extend as unitary operators in $L^2(\mathbb{R}^n)$. This fact plays an important role in the inverse problems for elastic wave equations [4, 15].

To formulate our main result, let $S(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions on $\mathbb{R}^n$, which are rapidly decreasing together with their derivatives of all orders.

**Theorem 1.1.** Let $1 \leq p, q \leq \infty$; $\alpha_0 = \text{Re} \alpha$. The operators $P^\alpha_\pm$, initially defined on functions $\varphi \in S(\mathbb{R}^n)$ by analytic continuation, extend as linear bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if

\[
\frac{1 - n}{2} \leq \alpha_0 \leq 1, \quad p = \frac{n + 1}{n + \alpha_0}, \quad q = \frac{n + 1}{1 - \alpha_0}.
\]

In particular, the Radon transform $P$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $p = (n + 1)/n$ and $q = n + 1$.

Although this statement is not new (cf. [22, Theorem 1.1]), our method of the proof, which relies on explicit computation of the Fourier
transform of $P_{\pm}^\alpha f$ combined with Grafakos’ version of the interpolation theorem, might be of interest and instructive.

**Plan of the Paper.** Sections 2 and 3 contain notation and elementary properties of the convolution operators $P_{\pm}^\alpha$. In Section 4 we compute their Fourier multipliers in the framework of the corresponding distribution theory. Section 5 contains auxiliary material and detailed proof of Theorem 1.1. Section 6 provides some comments and indicates possible generalizations. Some technical calculations are moved to Appendix.

## 2. Notation

In the following, $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n$. The notation $C(\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n)$, and $L^p(\mathbb{R}^n)$ for function spaces is standard; $\| \cdot \|_p = \| \cdot \|_{L^p(\mathbb{R}^n)}$; $C_c^\infty(\mathbb{R}^n)$ is the space of compactly supported infinitely differentiable functions on $\mathbb{R}^n$. The notation $\langle f, g \rangle$ for functions $f$ and $g$ is used for the integral of the product of these functions. We keep the same notation when $f$ is a distribution and $g$ is a test function.

If $m = (m_1, \ldots, m_n)$ is a multi-index, then $\partial^m = \partial_1^{m_1} \cdots \partial_n^{m_n}$, where $\partial_i = \partial/\partial x_i$. The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^n, \quad (2.1)$$

where $x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n$. We denote by $S(\mathbb{R}^n)$ the Schwartz space of $C^\infty$-functions which are rapidly decreasing together with their derivatives of all orders. The space $S(\mathbb{R}^n)$ is equipped with the topology generated by the sequence of norms

$$\| \varphi \|_k = \max_x (1 + |x|)^k \sum_{|j| \leq k} \left| (\partial^j \varphi)(x) \right|, \quad k = 0, 1, 2, \ldots. \quad (2.2)$$

The Fourier transform is an automorphism of $S(\mathbb{R}^n)$. The space of tempered distributions, which is dual to $S(\mathbb{R}^n)$, is denoted by $S'(\mathbb{R}^n)$. The Fourier transform of a distribution $f \in S'(\mathbb{R}^n)$ is a distribution $\hat{f} \in S'(\mathbb{R}^n)$ defined by

$$\langle \hat{f}, \psi \rangle = \langle f, \hat{\psi} \rangle, \quad \psi \in S(\mathbb{R}^n). \quad (2.3)$$

The equality (2.3) is equivalent to

$$\langle \hat{f}, \varphi \rangle = (2\pi)^n \langle f, \varphi_1 \rangle, \quad \varphi \in S(\mathbb{R}^n), \quad \varphi_1(x) = \varphi(-x). \quad (2.4)$$

The inverse Fourier transform of a function (or distribution) $f$ is denoted by $\hat{f}$. 

Given a real-valued quantity $X$ and a complex number $\lambda$, we set $(X)^\pm_\lambda = |X|^\lambda$ if $\pm X > 0$ and $(X)^\pm_\lambda = 0$, otherwise. All integrals are understood in the Lebesgue sense. The letter $c$, sometimes with subscripts, stands for a nonessential constant that may be different at each occurrence.

3. Elementary Properties of Parabolic Convolutions $P^\alpha_\pm f$

For $y = (y_1, \ldots, y_n-1, y_n) = (y', y_n) \in \mathbb{R}^n$, we denote

$$p^\alpha_+(y) = (y_n - |y'|^2)^{\alpha-1}_+ = \begin{cases} (y_n - |y'|^2)^{\alpha-1} & \text{if } y_n > |y'|^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$p^\alpha_-(y) = (y_n - |y'|^2)^{\alpha-1}_- = \begin{cases} (|y'|^2 - y_n)^{\alpha-1} & \text{if } y_n < |y'|^2, \\ 0 & \text{otherwise.} \end{cases}$$

These functions have a singularity on the paraboloid $y_n = |y'|^2$. The corresponding convolution operators

$$(P^\alpha_\pm f)(x) = \int_{\mathbb{R}^n} p^\alpha_\pm(y) f(x - y) \, dy, \quad \text{Re} \alpha > 0, \quad (3.1)$$

are generalizations of the Riemann-Liouville fractional integrals [25]

$$(I^\alpha_\pm f)(t) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} s^{\alpha-1}_\pm f(t - s) \, ds, \quad t \in \mathbb{R}, \quad (3.2)$$

and coincide with them if $n = 1$. We call (3.1) the parabolic fractional integrals.

Lemma 3.1. Let $f \in S(\mathbb{R}^n)$. The following statements hold.

(i) For each $x \in \mathbb{R}^n$, $(P^\alpha_\pm f)(x)$ extend as entire functions of $\alpha$. Moreover,

$$\lim_{\alpha \to 0} (P^\alpha_\pm f)(x) = (Pf)(x), \quad (3.3)$$

where $(Pf)(x)$ is the parabolic Radon transform (1.1).

(ii) If $\text{Re} \alpha > 0$, then for any multi-index $m$,

$$\partial^m P^\alpha_\pm f = P^\alpha_\pm \partial^m f. \quad (3.4)$$

(iii) For any positive integer $k$,

$$P^\alpha_\pm f = (\pm 1)^k P^{\alpha + k}_\pm \partial^k_n f = (\pm 1)^k \partial^k_n P^{\alpha + k}_\pm f. \quad (3.5)$$
Proof. To prove (i), we have

\[
(P_\alpha^\pm f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-1}} dy' \int_{-\infty}^{\infty} (y_n - |y'|^2)^{\alpha - 1}_\pm f(x' - y', x_n - y_n) dy_n
= \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} s^{\alpha - 1} A_{x,\pm}(s) ds;
\]

where

\[
A_{x,\pm}(s) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2 \pm s) dy'.
\]

Because the functions \(A_{x,\pm}(s)\) are smooth, rapidly decreasing, and satisfy \(A_{x,\pm}(0) = (Pf)(x)\), the result follows; cf. [7, Chapter I, Section 3.2], [20, Section 2.5]. In (ii) we simply differentiate under the sign of integration. The first equality in (3.5) can be obtained using integration by parts. The second equality is the result of differentiation:

\[
(\pm 1)^k \partial_n^k P_\alpha^{\alpha + k} f = (\pm 1)^k \partial_n^k I_\alpha^k P_{\pm} f = P_\alpha f.
\]

\(\square\)

Remark 3.2. The formula (3.5) gives an explicit expression of the analytic continuation of \(P_\alpha f\) from the domain \(\text{Re}\,\alpha > 0\) to \(\text{Re}\,\alpha > -k\). For example, for any positive integers \(\ell\) and \(k > \ell\), setting \(\alpha = -\ell\), we have

\[
P_{-\ell}^\pm f = (\pm 1)^k \partial_n^k P_{\pm}^{-\ell} f.
\]

Note also that by Fubini’s theorem,

\[
P_\alpha^\beta I_\pm f = I_\pm^\beta P_\alpha f = P_\alpha^{\alpha + \beta} f, \quad \text{Re}\,\alpha > 0, \quad \text{Re}\,\beta > 0,
\]

(3.8)

where \(I_\pm^\beta\) are applied in the last variable. Similarly, for \(\text{Re}\,\alpha > 0\),

\[
P_\alpha f = I_\pm^\alpha Pf,
\]

(3.9)

where \(P\) is the Radon transform (1.1). Here it is assumed that \(f\) is good enough, so that the change of the order of integration is well justified.

3.1. The Dual Fractional Integrals. For \(y = (y', y_n) \in \mathbb{R}^n\) and \(\text{Re}\,\alpha > 0\), denote

\[
p_{\alpha,\pm}^*(y) = \frac{1}{\Gamma(\alpha)} (y_n + |y'|^2)^{\alpha - 1}_\pm = p_{\alpha,\pm}(-y).
\]

(3.10)

The set of singularities of this function is the paraboloid \(y_n = -|y'|^2\) in the negative half-space \(y_n \leq 0\). The corresponding convolutions

\[
\hat{P}_\alpha f = p_{\alpha,\pm}^* f
\]

(3.11)
are called the dual parabolic fractional integrals. The name is motivated by the following lemma.

**Lemma 3.3.** For $\Re \alpha > 0$,

\[
\langle P_\pm^\alpha f, \varphi \rangle = \langle f, \overset{\ast}{P}_\pm^\alpha \varphi \rangle,
\]

provided that either side of this equality exists in the Lebesgue sense.

**Proof.**

\[
l.h.s. = \int_{\mathbb{R}^n} \varphi(x)dx \int_{\mathbb{R}^{n-1}} dy' \int_{-\infty}^{\infty} (y_n - |y'|^2)^{\alpha-1}_\pm f(x' - y', x_n - y_n) dy_n
\]

\[
= \int_{\mathbb{R}^{n-1}} dz' \int_{-\infty}^{\infty} f(z', z_n)dz_n \int_{\mathbb{R}^n} dx' \int_{-\infty}^{\infty} \varphi(x', x_n)(x_n - z_n - |x' - z'|^2)^{\alpha-1}_\pm dx_n
\]

\[
= \int_{\mathbb{R}^n} f(z)dz \int_{\mathbb{R}^{n-1}} dy' \int_{-\infty}^{\infty} \varphi(z' - y', z_n - y_n)(-y_n - |y'|^2)^{\alpha-1}_\pm dy_n = r.h.s.
\]

Clearly,

\[
\overset{\ast}{P}_\pm^\alpha = J P_\pm^\alpha J, \quad (Jf)(x) = f(-x).
\]

An analogue of (3.3) for $f \in S(\mathbb{R}^n)$ is

\[
\lim_{\alpha \to 0} (\overset{\ast}{P}_\pm^\alpha f)(x) = (P^* f)(x),
\]

where $P^* f$ is the dual parabolic Radon transform defined by

\[
(P^* f)(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n + |y'|^2) dy';
\]

cf. (1.1). By Fubini’s theorem,

\[
P_\pm^\alpha I_\pm^\beta f = I_\pm^\beta P_\pm^\alpha f = \overset{\ast}{P}_\pm^{\alpha+\beta} f, \quad \Re \alpha > 0, \quad \Re \beta > 0,
\]

if $f$ is good enough; cf. (3.8). Similarly, for $\Re \alpha > 0$,

\[
\overset{\ast}{P}_\pm^\alpha f = I_\pm^\alpha P^* f.
\]

**4. PARABOLIC CONVOLUTIONS AND DISTRIBUTIONS**

In the following, we invoke the theory of distributions and the Fourier transform. Formally,

\[
(p_{\alpha \pm} * f)^\wedge = \overset{\wedge}{p}_{\alpha \pm} \overset{\wedge}{f}.
\]

Our aim is to give this equality precise meaning and compute $\overset{\wedge}{p}_{\alpha \pm}$. 
4.1. Some Preparations.

**Lemma 4.1.** If $\Re \alpha > 0$, then $p_{\alpha \pm}$ can be viewed as regular tempered distributions, which extend as entire distribution-valued functions of $\alpha \in \mathbb{C}$. In particular, for any integer $k > 0$ and $\varphi \in S(\mathbb{R}^n)$,

$$
\langle p_{\alpha \pm}, \varphi \rangle = (\mp 1)^k \int_{\mathbb{R}^n} p_{(\alpha + k) \pm}(y) (\partial_n^k \varphi)(y) \, dy, \quad \Re \alpha > -k. \quad (4.2)
$$

In the case $\alpha = 0$ we have

$$
p_{0 +}(y) = p_{0 -}(y) \overset{\text{def}}{=} \delta_P(y),
$$

where

$$
\langle \delta_P, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi(y', |y'|^2) \, dy'. \quad (4.3)
$$

**Proof.** If $\Re \alpha > 0$, then $(p_{\alpha \pm}, \varphi)$ are absolutely convergent integrals that can be estimated by norms (2.2); see Appendix. Furthermore, as in (3.6),

$$
\langle p_{\alpha \pm}, \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha - 1} A_{\pm}(s) \, ds, \quad A_{\pm}(s) = \int_{\mathbb{R}^{n-1}} \varphi(y', |y'|^2 \pm s) \, dy',
$$

and therefore $p_{\alpha \pm}$ extend analytically as $S'$-distributions to all $\alpha \in \mathbb{C}$. The equality (4.2) is a consequence of the integration by parts. In the case $\alpha = 0$, as in (3.3), we have

$$
\langle p_{0 \pm}, \varphi \rangle = \lim_{\alpha \to 0} \langle p_{\alpha \pm}, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi(y', |y'|^2) \, dy' = \langle \delta_P, \varphi \rangle.
$$

Note that $\delta_P(y)$ differs from the delta distribution $\delta(y_n - |y'|^2)$, which is defined by a surface integral

$$
\langle \delta(y_n - |y'|^2), \varphi \rangle = \int_{y_n = |y'|^2} \varphi(y) \, d\sigma(y) \quad (4.4)
$$

and expresses through the integral over $\mathbb{R}^{n-1}$ with the corresponding Jacobian factor.

**Corollary 4.2.** For any $\alpha \in \mathbb{C}$ and any positive integer $\ell$,

$$
(\pm 1)^\ell \partial_n^\ell p_{\alpha \pm} = p_{(\alpha - \ell) \pm}. \quad (4.5)
$$
Proof. Suppose that \( \text{Re} \alpha > -k, \ k \in \mathbb{N} \), and let \( \varphi \in S(\mathbb{R}^n) \). By (4.2),

\[
\langle (\pm 1)^{\ell} \partial_n^\ell p_{\alpha \pm}, \varphi \rangle = (\mp 1)^{\ell} \langle p_{\alpha \pm}, \partial_n^\ell \varphi \rangle
\]

\[
= (\mp 1)^{\ell + k} \int_{\mathbb{R}^n} p_{(\alpha + k) \pm}(y) (\partial_n^{\ell + k} \varphi)(y) \, dy.
\]

The same expression can be obtained for \( \langle p_{(\alpha - \ell) \pm}, \varphi \rangle \) if we replace \( \alpha \) by \( \alpha - \ell \) and \( k \) by \( \ell + k \) in (4.2). \( \square \)

**Definition 4.3.** Following Lemma 4.1, we define \( P_{\pm}^\alpha \varphi \) for any \( \alpha \in \mathbb{C} \) as convolutions of the \( S' \)-distributions \( p_{\alpha \pm} \) with the test function \( \varphi \in S(\mathbb{R}^n) \) by the formula

\[
(P_{\pm}^\alpha \varphi)(x) = \langle p_{\alpha \pm}(y), \varphi(x - y) \rangle. \tag{4.6}
\]

**Lemma 4.4.** Let \( \alpha \in \mathbb{C} \) and \( \varphi \in S(\mathbb{R}^n) \). The following statements hold.

(i) \( (P_{\pm}^\alpha \varphi)(x) \) are infinitely differentiable tempered functions on \( \mathbb{R}^n \). In particular, for any multi-index \( m \),

\[
(\partial^m P_{\pm}^\alpha \varphi)(x) = \langle p_{\alpha \pm}(y), \partial^m_x \varphi(x - y) \rangle. \tag{4.7}
\]

(ii) For any positive integer \( \ell \),

\[
(\pm \partial_n)\ell P_{\pm}^\alpha \varphi = P_{\pm}^{\alpha - \ell} \varphi. \tag{4.8}
\]

Proof. The statement (i) mimics known facts for \( S' \)-convolutions; see, e.g., [6, p. 26, Lemma 2.5], [33, p. 84]. To prove (ii), by making use of (4.6) and (4.5), we have

\[
(\pm \partial_n)\ell (P_{\pm}^\alpha \varphi)(x) = (\pm 1)^{\ell} \langle p_{\alpha \pm}(y), (\partial/\partial x_n)^\ell \varphi(x - y) \rangle
\]

\[
= (\mp 1)^{\ell} \langle p_{\alpha \pm}(y), (\partial/\partial y_n)^\ell \varphi(x - y) \rangle
\]

\[
= (\pm 1)^{\ell} \langle \partial_n^\ell p_{\alpha \pm}, \varphi(x - y) \rangle = \langle p_{(\alpha - \ell) \pm}, \varphi(x - y) \rangle
\]

\[
= (P_{\pm}^{\alpha - \ell} \varphi)(x).
\]

\( \square \)

All results of this subsection can be easily reformulated for the dual fractional integrals introduced in Subsection 3.1. Because the functions \( p_{\alpha \pm}^*(y) \) differ from \( p_{\alpha \pm}(y) \) only by a sign of \( y_n \), they can be viewed as \( S' \)-distributions, which extend analytically to all \( \alpha \in \mathbb{C} \). In particular, for \( \alpha = 0 \),

\[
p_{0+}^*(y) = p_{0-}^*(y) \overset{\text{def}}{=} \delta_p^*(y), \tag{4.9}
\]

where

\[
\langle \delta_p^*, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi(y', -|y'|^2) \, dy', \quad \varphi \in S(\mathbb{R}^n). \tag{4.10}
\]
4.2. The Fourier Transforms $\hat{p}_{\alpha\pm}$ and Spaces of the Semyanistyi Type. Following [25, pp. 98, 137], we fix the branches of the analytic functions $(\pm ix_n)^{\lambda}$, $\lambda \in \mathbb{C}$, by setting
\[
(\pm ix_n)^{\lambda} = \exp \left( \lambda \log |x_n| \pm \frac{\lambda \pi i}{2} \text{sgn} \ x_n \right),
\]
and denote
\[
\omega_+(x) = \pi^{(n-1)/2} \exp \left( -\frac{i|x'|^2}{4x_n} \right), \quad \omega_-(x) = \omega_+(x) \exp \left( \frac{(n-1)\pi i}{2} \text{sgn} \ x_n \right),
\]
\[
q_{\alpha\pm}(x) = (\mp ix_n)^{-\alpha-(n-1)/2} \omega_\pm(x).
\]
Our aim is to show that
\[
\hat{p}_{\alpha\pm} = q_{\alpha\pm}
\]
in a suitable sense.

By the definition of the Fourier transform in $S'(\mathbb{R}^n)$, $\langle \hat{p}_{\alpha\pm}, \psi \rangle = \langle p_{\alpha\pm}, \hat{\psi} \rangle$, where $\psi \in S(\mathbb{R}^n)$ and $\langle p_{\alpha\pm}, \hat{\psi} \rangle$ is understood as analytic continuation of the integral $\int_{\mathbb{R}^n} p_{\alpha\pm}(y) \hat{\psi}(y) \, dy$ from the domain $\text{Re} \ \alpha > 0$. Thus, to prove (4.13), one may be tempted to show that
\[
\int_{\mathbb{R}^n} p_{\alpha\pm}(y) \hat{\psi}(y) \, dy = \int_{\mathbb{R}^n} q_{\alpha\pm}(x) \psi(x) \, dx
\]
for some $\text{Re} \ \alpha > 0$, for which both integrals are absolutely convergent. However, these integrals may not exist simultaneously, rather than $\psi(x)$ vanishes at $x_n = 0$. Note also that multiplication by $q_{\alpha\pm}$ does not preserve the space $S(\mathbb{R}^n)$.

To circumvent this obstacle, one can reduce the space of the test functions in a suitable way and expand the corresponding space of distributions. An idea of this approach originated in the work of Semyanistyi [26] who treated Fourier multipliers having singularity at a single point $x = 0$. This idea was independently used by Helgason [12, p. 162]. It was extended by Lizorkin [14] and later by Samko [23] to more general sets of singularities; see also [20, 24, 25] and references therein. In our case, described below, the set of singularities is the hyperplane $x_n = 0$.

**Definition 4.5.** We denote by $\Psi(\mathbb{R}^n)$ the subspace of functions $\psi \in S(\mathbb{R}^n)$ vanishing with all derivatives on the hyperplane $x_n = 0$. Let also $\Phi(\mathbb{R}^n)$ be the Fourier image of $\Psi(\mathbb{R}^n)$.

\[\text{In the cited literature, except [12], the notation } \Phi, \Psi \text{ is used in the case of singularity at a single point } x = 0. \text{ For the sake of simplicity, we keep the same notation, which does not cause any ambiguity.}\]
Proposition 4.6. (see, e.g., [24, Theorem 2.33]) The space $\Phi(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Every function $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, can be regarded as a regular $\Phi'$-distribution.

Proposition 4.7. (cf. [18, Lemma 3.11], [34]) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $(1 \leq p, q < \infty)$, coincide as $\Phi'$-distributions, then they coincide almost everywhere on $\mathbb{R}^n$.

Recall that a function $\mu$ is called a multiplier on a linear topological space $X$ if the map $X \ni f \rightarrow \mu f \in X$ is continuous in the topology of $X$. One can readily see that multiplication by $q_\alpha\pm$ is an automorphism of $\Psi(\mathbb{R}^n)$. Note also that the space $\Phi(\mathbb{R}^n)$ is invariant under translations, that is, for every $h \in \mathbb{R}^n$, the map $\Phi(\mathbb{R}^n) \ni \varphi(x) \rightarrow \varphi(x - h) \in \Phi(\mathbb{R}^n)$ is continuous in the induced topology of $\Phi(\mathbb{R}^n)$. Indeed, since the Fourier transform maps $\Phi(\mathbb{R}^n)$ onto $\Psi(\mathbb{R}^n)$ isomorphically, this statement follows from the observation that the function $\xi \rightarrow e^{ih\cdot\xi}$ is a multiplier on $\Psi(\mathbb{R}^n)$.

By Theorem 1 from [8, Chapter III, Section 3.7, p. 148], the inverse Fourier transform $\hat{q}_\alpha\pm(x) \in \Phi'(\mathbb{R}^n)$ is a convolutor on $\Phi(\mathbb{R}^n)$, that is the map $\varphi \rightarrow \hat{q}_\alpha\pm * \varphi$ is continuous in $\Phi(\mathbb{R}^n)$, and

$$\langle \hat{q}_\alpha\pm * \varphi \rangle = q_\alpha\pm \hat{\varphi}. \quad (4.14)$$

Note also that the Riemann-Liouville operators (3.2), acting in the last variable and corresponding to the Fourier multipliers $(\mp ix_n)^{-\alpha}$, are automorphism of $\Phi(\mathbb{R}^n)$.

Lemma 4.8. Let $\psi \in \Psi(\mathbb{R}^n)$, $\alpha \in \mathbb{C}$. Then

$$\langle \hat{p}_\alpha\pm, \psi \rangle = \langle q_\alpha\pm, \psi \rangle, \quad (4.15)$$

which means that $\hat{p}_\alpha\pm = q_\alpha\pm$ in the $\Psi'$-sense. In particular, if $\alpha = 0$, for the delta distribution (4.3) we have

$$\langle \hat{\delta}_P, \psi \rangle = \langle q_0, \psi \rangle, \quad (4.16)$$

where

$$q_0(x) = q_0\pm(x) = (-ix_n)^{-(n-1)/2}\pi^{(n-1)/2}\exp\left(-\frac{|x'|^2}{4x_n}\right). \quad (4.17)$$
Proof. Let us prove (4.15) for $p_{\alpha^+}$. Denote
\[ e_\varepsilon(y) = e^{-\varepsilon y_n}, \quad \varepsilon > 0, \tag{4.18} \]
and suppose first that $\alpha$ is real-valued. Then, for $\alpha > 0$,
\[ \langle p_{\alpha^+}, \psi \rangle \overset{\text{def}}{=} \langle p_{\alpha^+}, \hat{\psi} \rangle = \lim_{\varepsilon \to 0} \langle p_{\alpha^+ + e_\varepsilon}, \hat{\psi} \rangle = \lim_{\varepsilon \to 0} \langle (p_{\alpha^+ + e_\varepsilon})^\wedge, \psi \rangle. \tag{4.19} \]
Here the integral $\langle p_{\alpha^+}, \hat{\psi} \rangle$ is absolutely convergent (cf. the proof of Lemma 4.1), so that application of the Lebesgue dominated convergence theorem is well justified. Note also that $p_{\alpha^+ + e_\varepsilon} \in L^1(\mathbb{R}^n)$ because
\[
\int_{\mathbb{R}^n} p_{\alpha^+}(y) e^{-\varepsilon y_n} dy_n = \int_{\mathbb{R}^n} dy' \int_{|y'|^2}^\infty (y_n - |y'|^2)^{\alpha - 1} e^{-\varepsilon y_n} dy_n \\
= \int_{\mathbb{R}^n} e^{-\varepsilon |y'|^2} dy' \int_{0}^\infty s^{\alpha - 1} e^{-s} ds < \infty.
\]
Furthermore, as above,
\[
(p_{\alpha^+ + e_\varepsilon})^\wedge(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \frac{e^{ix' \cdot y} dy'}{|y'|^2} \int_{|y'|^2}^\infty e^{ix_n y_n} (y_n - |y'|^2)^{\alpha - 1} e^{-\varepsilon y_n} dy_n \\
= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \frac{e^{ix' \cdot y} e^{-|y'|^2(\varepsilon - ix_n)} dy'}{|y'|^2} \int_{0}^\infty s^{\alpha - 1} e^{-s(\varepsilon - ix_n)} ds.
\]
Both integrals can be explicitly evaluated, and we get
\[ (p_{\alpha^+ + e_\varepsilon})^\wedge(x) = \pi^{(n-1)/2}(\varepsilon - ix_n)^{-\alpha - (n-1)/2} \exp \left(-\frac{|x'|^2}{4(\varepsilon - ix_n)}\right). \tag{4.20} \]
Here the power function is defined by
\[ z^\lambda = e^{\lambda \log |z| + i \arg z}, \quad -\pi/2 < \arg z < \pi/2, \tag{4.21} \]
\[ z = \varepsilon - ix_n, \quad \lambda = -\alpha - (n-1)/2. \]
To justify the passage to the limit in the last expression in (4.19), we take into account that $\psi(x)$ has a strong decay as $x_n \to 0$ and
\[
|(p_{\alpha^+ + e_\varepsilon})^\wedge(x)| = \frac{\pi^{(n-1)/2}}{(\sqrt{\varepsilon^2 + x_n^2})^{\alpha + (n-1)/2}} \exp \left(-\frac{\varepsilon |x'|^2}{4(\varepsilon^2 + x_n^2)}\right) \\
\leq \frac{\pi^{(n-1)/2}}{|x_n|^{\alpha + (n-1)/2}}.
\]
Thus for $\alpha > 0$, owing to (4.11), we obtain
\[
\lim_{\varepsilon \to 0} \langle (p_{\alpha+\varepsilon})^\wedge, \psi \rangle = \lim_{\varepsilon \to 0} \langle (p_{\alpha+\varepsilon})^\wedge, \psi \rangle = \pi^{(n-1)/2} \int_{\mathbb{R}^n} (-ix_n)^{-\alpha-(n-1)/2} \exp\left(-\frac{i|x'|^2}{4x_n}\right) \psi(x) \, dx = \langle q_{\alpha+}, \psi \rangle. 
\]

Reverting to (4.19), for $\alpha > 0$ we obtain $\langle \hat{p}_{\alpha+}, \psi \rangle = \langle q_{\alpha+}, \psi \rangle$. Noting that $\langle \hat{p}_{\alpha+}, \psi \rangle = \langle p_{\alpha+}, \psi \rangle$ is an entire function of $\alpha$ and the same is true for $\langle q_{\alpha+}, \psi \rangle$, we make use of the analytic continuation and conclude that $\langle \hat{p}_{\alpha+}, \psi \rangle = \langle q_{\alpha+}, \psi \rangle$ for all $\alpha \in \mathbb{C}$.

To compute the Fourier transform of $p_{\alpha-}$, unlike (4.18), we set
\[
\eta_{\varepsilon}(y) = e^{\varepsilon y_2 - 2\varepsilon |y|_2^2}, \quad \varepsilon > 0.
\]

Note that $p_{\alpha-} \eta_{\varepsilon} \in L^1(\mathbb{R}^n)$. Indeed,
\[
\int_{\mathbb{R}^n} p_{\alpha-}(y) \eta_{\varepsilon}(y) \, dy_n = \int_{\mathbb{R}^{n-1}} e^{-2\varepsilon |y'|_2^2} \int_{-\infty}^\infty (|y'|_2^2 - y_n)^{\alpha-1} e^{\varepsilon y_n} \, dy_n
\]
\[
= \int_{\mathbb{R}^{n-1}} e^{-\varepsilon |y'|_2^2} \int_0^\infty s^{\alpha-1} e^{-s} \, ds < \infty.
\]

Hence, as in (4.19), for $\alpha > 0$ we have $\langle \hat{p}_{\alpha-}, \psi \rangle = \lim_{\varepsilon \to 0} \langle (p_{\alpha-} \eta_{\varepsilon})^\wedge, \psi \rangle$, where
\[
(p_{\alpha-} \eta_{\varepsilon})^\wedge(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-1}} e^{ix \cdot y - 2\varepsilon |y'|^2} \int_{-\infty}^\infty e^{ix_n y_n + \varepsilon y_n} (|y'|_2^2 - y_n)^{\alpha-1} \, dy_n
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-1}} e^{ix \cdot y'} e^{-|y'|^2(\varepsilon - ix_n)} \int_0^\infty s^{\alpha-1} e^{-s(x + ix_n)} \, ds
\]
\[
= \pi^{(n-1)/2} (\varepsilon + ix_n)^{-\alpha} (\varepsilon - ix_n)^{(1-n)/2} \exp\left(-\frac{|x'|^2}{4(\varepsilon - ix_n)}\right).
\]

Here, as above,
\[
(\varepsilon + ix_n)^{-\alpha} = |\varepsilon + ix_n|^{-\alpha} e^{-i\alpha \arg(\varepsilon + ix_n)}, \quad -\pi/2 < \arg(\varepsilon + ix_n) < \pi/2;
\]
\[
(\varepsilon - ix_n)^{(1-n)/2} = |\varepsilon - ix_n|^{(1-n)/2} e^{i(1-n)/2 \arg(\varepsilon - ix_n)},
\]
\[
-\pi/2 < \arg(\varepsilon - ix_n) < \pi/2.
\]
Assuming \( \varepsilon \to 0 \), owing to (4.11), we obtain
\[
\lim_{\varepsilon \to 0} (p_{\alpha - \eta_\varepsilon})^\wedge(x) = q_{\alpha -}(x),
\]
where
\[
q_{\alpha -}(x) = \pi^{(n-1)/2} (ix_n)^{-\alpha}(-ix_n)^{(1-n)/2} \exp \left( \frac{-ix'|^2}{4x_n} \right)
\]
or, by (4.11),
\[
q_{\alpha -}(x) = (ix_n)^{-\alpha-(n-1)/2} \omega_-(x),
\]
\[
\omega_-(x) = \pi^{(n-1)/2} \exp \left( \frac{(n-1)\pi i}{2} \text{sgn} \, x_n - \frac{i|x'|^2}{4x_n} \right)
\]
\[
= \omega_+(x) \exp \left( \frac{(n-1)\pi i}{2} \text{sgn} \, x_n \right).
\]
Thus, as in the previous case, we obtain \((\hat{p}_{\alpha -}, \psi) = (q_{\alpha -}, \psi)\), as desired. \( \square \)

The following statement follows from (4.14) and Lemma 4.8.

**Lemma 4.9.** The convolution operators \( P^\alpha_{\pm} \) initially defined by (3.1) for \( \text{Re} \alpha > 0 \), extend to all \( \alpha \in \mathbb{C} \) as automorphisms of the space \( \Phi(\mathbb{R}^n) \) by the formula
\[
(P^\alpha_\pm \varphi)(x) = (q^\alpha_\pm \varphi) \wedge(x), \quad \varphi \in \Phi(\mathbb{R}^n).
\]

**Remark 4.10.** The formula (4.17) means that the parabolic Radon transform (1.1) is a composition of the Riemann-Liouville integral operator \( I_{x_n}^{(n-1)/2} \) in the \( x_n \)-variable and a certain convolution operator which corresponds to the Fourier multiplier \( \pi^{(n-1)/2} \exp (-i|x'|^2/4x_n) \) and is bounded in \( L^2(\mathbb{R}^n) \); cf. Lemma 5.1 below.

To compute the Fourier transforms of the distributions \( p^*_{\alpha \pm} \) corresponding to the dual fractional integrals, we set
\[
q^*_{\alpha \pm}(x) \equiv q^*_{\alpha \pm}(x', x_n) = q_{\alpha \pm}(x', -x_n),
\]
or, by (4.12),
\[
q^*_{\alpha \pm}(x) = (\pm ix_n)^{-\alpha-(n-1)/2} \omega^*_{\pm}(x),
\]
\[
\omega^*_{\pm}(x) = \pi^{(n-1)/2} \exp \left( \frac{i|x'|^2}{4x_n} \right), \quad \omega^*_{\pm}(x) = \omega^*_{\pm}(x) \exp \left( -\frac{(n-1)\pi i}{2} \text{sgn} \, x_n \right).
\]

The next statement mimics Lemmas 4.8 and 4.9.

**Lemma 4.11.**

(i) For any \( \psi \in \Psi(\mathbb{R}^n) \) and \( \alpha \in \mathbb{C} \),
\[
\langle (p^*_{\alpha \pm})^\wedge, \psi \rangle = \langle q^*_{\alpha \pm}, \psi \rangle.
\]
which means that $(p^*_\pm) = q^*_\pm$ in the $\Psi'$-sense. In particular, if $\alpha = 0$, then for the delta distribution (4.9) we have

$$\langle (\delta^*)_p, \varphi \rangle = \langle q^*_0, \varphi \rangle,$$

where

$$q^*_0(x) = (ix_n)^{-(n-1)/2} \pi^{(n-1)/2} \exp \left( \frac{i|x|^2}{4x_n} \right).$$

(ii) The convolution operators $P^\alpha_\pm$ initially defined by (3.11) for $\text{Re} \alpha > 0$, extend to all $\alpha \in \mathbb{C}$ as automorphisms of the space $\Phi(\mathbb{R}^n)$ by the formula

$$(P^\alpha_\pm \varphi)(x) = (q^*_\pm \varphi)(x), \quad \varphi \in \Phi(\mathbb{R}^n).$$

Corollary 4.12. (cf. (3.8), (3.16)) If $\varphi \in \Phi(\mathbb{R}^n)$, then the equalities

$$P^\alpha_\pm I^\beta_\pm \varphi = I^\beta_\pm P^\alpha_\pm f = P^{\alpha+\beta}_\pm \varphi, \quad P^\alpha_\pm I_\mp^\beta \varphi = I_\pm^\beta P^\alpha_\pm \varphi = P^{\alpha+\beta}_\pm \varphi$$

hold for all $\alpha, \beta \in \mathbb{C}$. In particular, for all $\alpha \in \mathbb{C}$,

$$P^\alpha \varphi = I^\alpha P \varphi, \quad P^\alpha_\pm \varphi = I^\alpha_\pm P^\varphi,$$

where $P$ and $P^*$ are the Radon transforms (1.1), (3.14).

Corollary 4.13. For all $\alpha, \beta \in \mathbb{C}$,

$$q^*_\pm(x) q^*_{\alpha\pm}(x) = \pi^{n-1}|x_n|^{-\alpha-\beta-n+1} \exp \left( \pm \frac{(\alpha - \beta)\pi i}{2} \text{sgn} x_n \right).$$

In particular, for $\alpha = \beta$,

$$q^*_\pm(x) q^*_{\alpha\pm}(x) = \pi^{n-1}|x_n|^{-2\alpha-n+1}$$

and

$$q^*_\pm(x) q_\alpha(x) = \pi^{n-1}|x_n|^{-n}.$$ 

Remark 4.14. The meaning of (4.34) and (4.35) is that the composition of the corresponding operators is a constant multiple of the Riesz potential in the $x_n$-variable. Specifically,

$$\check{P}^\alpha_\pm P^\alpha_\pm \varphi = P^\alpha_\pm \check{P}^\alpha_\pm \varphi = \pi^{n-1} I^{2\alpha+n-1}_n \varphi, \quad P^* P^\varphi = PP^* = \pi^{n-1} I^{n-1}_n \varphi,$$

where $\varphi \in \Phi(\mathbb{R}^n)$ and, by definition,

$$(I^\lambda_n \varphi)^\wedge(x) = |x_n|^{-\lambda} \hat{\varphi}(x).$$

5. Convolutions with $L^p$-functions

In this section, we obtain necessary and sufficient conditions, under which $P^\alpha_\pm$ extend as linear bounded operators mapping $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$: $1 \leq p, q \leq \infty$. First, we investigate the cases $\text{Re} \alpha = (1-n)/2$ and $\text{Re} \alpha = 1$. Other cases will be treated by interpolation.
5.1. The $L^2-L^2$ Estimate.

**Lemma 5.1.** For the operators $P_{\pm}^{(1-n)/2+i\gamma}$, $\gamma \in \mathbb{R}$, initially defined by (4.6), there exist unique linear bounded extensions

$$P_{\pm}^{(1-n)/2+i\gamma} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

such that for every $f \in L^2(\mathbb{R}^n)$,

$$||P_{\pm}^{(1-n)/2+i\gamma} f||_2 \leq \pi^{(n-1)/2} e^{\pi|\gamma|/2} ||f||_2.$$  

Moreover, in the case $\gamma = 0$, the operators $\pi^{(1-n)/2}P_{\pm}^{(1-n)/2}$ are unitary.

**Proof.** Let us first consider the “+” case and suppress “+” for ease of presentation. By (4.11) and (4.12),

$$q_{(1-n)/2+i\gamma}(x) = \omega(x)(-ix_n)^{-i\gamma} = \omega(x) \exp(-i\gamma \log |x_n| - \frac{\pi\gamma}{2} \operatorname{sgn} x_n),$$

and therefore

$$|q_{(1-n)/2+i\gamma}(x)| = \pi^{(n-1)/2} e^{-\pi|\gamma|/2} \operatorname{sgn} x_n \leq \pi^{(n-1)/2} e^{\pi|\gamma|/2}.$$

Hence there exists a unique linear operator (5.1) defined by

$$(P^{(1-n)/2+i\gamma} f)^\wedge(x) = q_{(1-n)/2+i\gamma}(x) \hat{f}(x), \quad f \in L^2(\mathbb{R}^n),$$

and satisfying

$$||P^{(1-n)/2+i\gamma}|| = ||q_{(1-n)/2+i\gamma}||_{\infty} \leq \pi^{(n-1)/2} e^{\pi|\gamma|/2};$$

see, e.g., Theorem 3.18 in [28, Chapter I, Section 3].

Let us prove that $P^{(1-n)/2+i\gamma}$ is an extension of $P_{\pm}^{(1-n)/2+i\gamma}$, that is,

$$P^{(1-n)/2+i\gamma}\varphi = P_{\pm}^{(1-n)/2+i\gamma}\varphi, \quad \varphi \in S(\mathbb{R}^n).$$

Suppose $n$ is even and make use of (4.5) with $\ell = n/2$, $\alpha = 1/2 + i\gamma$. As in the proof of Lemma 4.8 (cf. (4.19)), we have

$$(P^{(1-n)/2+i\gamma}\varphi)(x) = \langle P_{(1-n)/2+i\gamma}(y), \varphi(y) \rangle = \langle \partial_n^{n/2} p_{1/2+i\gamma}(y), \varphi(y) \rangle$$

$$= \langle p_{1/2+i\gamma}(y), (\partial_n^{n/2} \varphi)(y) \rangle$$

$$= \lim_{\varepsilon \to 0} \langle p_{1/2+i\gamma}(y), (\partial_n^{n/2} \varphi)(x-y) \rangle.$$

By Plancherel’s formula, setting

$$g_x(y) = (\partial_n^{n/2} \varphi)(x-y),$$

we continue:

$$(P^{(1-n)/2+i\gamma}\varphi)(x) = \lim_{\varepsilon \to 0} \langle p_{1/2+i\gamma}(y), g_x(y) \rangle$$

$$= (2\pi)^{-n} \lim_{\varepsilon \to 0} \langle (p_{1/2+i\gamma}e^\varepsilon)^\wedge(\xi), g_x(\xi) \rangle.$$
where
\[
\hat{g}_x(\xi) = \int_{\mathbb{R}^n} e^{iy\cdot\xi} \frac{\nabla_n^{n/2}}{2n} \varphi(x-y) dy = e^{-ix\cdot\xi} (-i\xi)^{n/2} \hat{\varphi}(\xi).
\]

Computing \((p_{1/2+i\gamma}c_\varepsilon)^\wedge(\xi)\), as in (4.20), and passing to the limit under the sign of integration, as in (4.22), we obtain
\[
(P^{(1-n)/2+i\gamma}\varphi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \omega(\xi)(-i\xi)^{-i\gamma} e^{-ix\cdot\xi} d\xi
\]
\[
= (q(1-n)/2+i\gamma \hat{\varphi})(x).
\]
The last expression agrees with (5.3). The equality \(P^{(1-n)/2+i\gamma} - \hat{\varphi} = P^{(1-n)/2+i\gamma} - \hat{\varphi}\) can be proved similarly, following the corresponding reasoning in the proof of Lemma 4.8. If \(n\) is odd, the proof follows the same lines, but with the choice \(\ell = (n+1)/2\) and \(\alpha = 1 + i\gamma\) while using (4.5).

The operators \(\pi^{(1-n)/2}P^{(1-n)/2}\) are obviously isometries. To show that they are unitary, it remains to note that the reciprocals \((\omega_\pm)^{-1}\) are bounded functions, and therefore they generate linear bounded operators from \(L^2(\mathbb{R}^n)\) to \(L^2(\mathbb{R}^n)\).

**Remark 5.2.** The case \(\gamma = 0\) in Lemma 5.1 agrees with Proposition 2 in [15] and Proposition 1 in [4], where the reasoning differs from ours.

### 5.2. The \(L^1-L^\infty\) Estimate

In the case \(\alpha = 1 + i\gamma, \gamma \in \mathbb{R}\), we have
\[
(P^{1+i\gamma}_\pm \varphi)(x) = \frac{1}{\Gamma(1+i\gamma)} \int_{\mathbb{R}^n} \frac{(y_n - |y'|^2)^{i\gamma}}{\pm} \varphi(x-y) dy.
\]

Note that
\[
|\Gamma(1+i\gamma)|^2 = \frac{\pi\gamma}{\sinh(\pi\gamma)};
\]
see, e.g., [1]. This gives the following statement.

**Lemma 5.3.** For any \(\gamma \in \mathbb{R}\) and \(\varphi \in L^1(\mathbb{R}^n)\),
\[
||P^{1+i\gamma}_\pm \varphi||_\infty \leq e^{\pi|\gamma|/2}||\varphi||_1. \tag{5.4}
\]

### 5.3. Interpolation

The \(L^2-L^2\) boundedness of \(P^\alpha_\pm\) in the case \(Re\alpha = (1-n)/2\) and the \(L^1-L^\infty\) boundedness of these operators with \(Re\alpha = 1\) on functions \(\varphi \in S\) pave the way to the \(L^p-L^q\) boundedness of \(P^\alpha_\pm\) for intermediate values of \(\alpha\) via interpolation. To this end, one may be tempted to employ Stein’s interpolation theorem for analytic families of operators [27]. This theorem can also be found in [3, 9, 28] and many other sources. However, the hypotheses of this theorem and the proof...
are given in terms of simple functions, i.e. finite linear combinations of the characteristic functions of disjoint compact sets. It is unclear how to check these hypotheses for our operators, generated by distributions and defined on Schwartz functions.

A modification of Stein’s theorem, the hypotheses of which do not contain simple functions, was communicated by Grafakos [10]. In order to formulate his result, we first make some assumptions and establish notation.

Given $0 < p_0, p_1 \leq \infty$ and $0 < q_0, q_1 \leq \infty$, we set

$$
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}; \quad 0 < \theta < 1.
$$

Denote

$$
S = \{ z \in \mathbb{C} : 0 < \text{Re} z < 1 \}, \quad \bar{S} = \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \}.
$$

For $z \in S$, let $T_z$ be a family of linear operators mapping $C_c^\infty(\mathbb{R}^n)$ to $L^1_{\text{loc}}(\mathbb{R}^n)$ and satisfying the following conditions.

(A) For all $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$, the function

$$
A(z) = \int_{\mathbb{R}^n} (T_z \varphi)(x) \psi(x) \, dx
$$

is analytic in $S$ and continuous on $\bar{S}$.

(B) There exist constants $\gamma \in [0, \pi)$ and $s \in (1, \infty]$, such that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ and any compact subset $K \subset \mathbb{R}^n$,

$$
\log \| T_z \varphi \|_{L^s(K)} \leq C e^{\gamma |\text{Im} z|}
$$

for all $z \in \bar{S}$ and some constant $C = C(\varphi, K)$.

**Theorem 5.4.** [10, Theorem 5.5.3] Let $T_z, z \in \bar{S}$, be a family of linear operators mapping $C_c^\infty(\mathbb{R}^n)$ to $L^1_{\text{loc}}(\mathbb{R}^n)$ and satisfying (A) and (B) above. Suppose that there exist constants $B_0, B_1$, and continuous functions $M_0(\gamma), M_1(\gamma)$ satisfying

$$
M_0(\gamma) + M_1(\gamma) \leq \exp(c e^{\tau |\gamma|})
$$

with some constants $c \geq 0$ and $0 \leq \tau < \pi$, such that

$$
\| T_{i \gamma} f \|_{q_0} \leq B_0 M_0(\gamma) \| f \|_{p_0}, \quad \| T_{1 + i \gamma} f \|_{q_1} \leq B_1 M_1(\gamma) \| f \|_{p_1}
$$

for all $\gamma \in \mathbb{R}$. Then for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$
\| T_\theta f \|_q \leq B_\theta M_0 \| f \|_p,
$$
where \( B_\theta = B_0^{1-\theta} B_1^\theta \) and

\[
M_\theta = \exp \left\{ \frac{\sin(\pi \theta)}{2} \int_{-\infty}^{\infty} \left[ \frac{\log M_0(\gamma)}{\cosh(\pi \gamma) - \cos(\pi \theta)} + \frac{\log M_1(\gamma)}{\cosh(\pi \gamma) + \cos(\pi \theta)} \right] d\gamma \right\}.
\]

5.4. Proof of Theorem 1.1. The proof consists of two steps, namely, the “if” part and the “only if” part.

**STEP 1.** We define

\[
T_z = P_\pm^{\alpha(z)}, \quad \alpha(z) = \frac{1+n}{2} z + \frac{1-n}{2}.
\]  

(5.11)

Then \( 0 \leq \text{Re } z \leq 1 \) corresponds to \((1 - n)/2 \leq \text{Re } \alpha \leq 1\). In our case,

\[
p_0 = q_0 = 2, \quad p_1 = 1, \quad q_1 = \infty.
\]

If, for real \( \alpha \in [(1-n)/2, 1] \), we set \( \alpha = ((1+n)/2) \theta + (1-n)/2 \), then \( \theta = (2n + 1)/(n+1) \), and (5.5) yields

\[
1/p = (\alpha + n)/(n+1), \quad 1/q = (1 - \alpha)/(n+1).
\]

The latter agrees with (1.4).

To apply Theorem 5.4, we must show that the operator families \( T_z = P_\pm^{\alpha(z)} \) meet all hypotheses of Theorem 5.4.

First, we note that by Lemma 4.4 (i), \( T_z \) maps \( S(\mathbb{R}^n) \) to \( C^\infty(\mathbb{R}^n) \), and therefore the required mapping \( T_z : C^\infty_c(\mathbb{R}^n) \to L^1_{\text{loc}}(\mathbb{R}^n) \) is valid for all \( z \in \mathbb{C} \), not only for \( z \in \mathbb{S} \). To check (A), for any \( \varphi, \psi \in C^\infty_c(\mathbb{R}^n) \),

\[
A(z) = \int_{\mathbb{R}^n} (T_z \varphi)(x) \psi(x) \, dx = \int F(x,z) \, dx,
\]

where

\[
F(x,z) = (T_z \varphi)(x) \psi(x) = (P_\pm^{\alpha(z)} \varphi)(x) \psi(x)
\]

is an entire function of \( z \), which is \( C^\infty \) in the \( x \)-variable; see Lemma 3.1(i) and Lemma 4.4(i). This gives (A).

To check (B), we make use of the first equality in (3.5) combined with (3.6). We obtain

\[
(P_\pm^{\alpha} \varphi)(x) = (\pm 1)^k (P_\pm^{\alpha+k} \partial_n^k \varphi)(x) = \frac{(\pm 1)^k}{\Gamma(\alpha + k)} \int_0^\infty s^{\alpha+k-1} A_{x,\pm}^{(k)}(s) \, ds,
\]

\[
A_{x,\pm}^{(k)}(s) = \int_{\mathbb{R}^{n-1}} (\partial_n^k \varphi)(x' - y', x_n \mp s - |y'|^2) \, dy'.
\]
Choose \( k > -\text{Re}\alpha \) and estimate the obtained expression in the “+” case. For any positive integer \( m \), there is a constant \( c = c(k, m, \varphi) \) such that

\[
|A_{x,+}^{(k)}(s)| \leq c \int_{\mathbb{R}^{n-1}} \frac{dy'}{(1 + |x' - y'|)^m(1 + |x_n - s - |y'|^2)|^m}.
\]

Note that

\[
\frac{1}{1 + |x' - y'|} \leq \frac{1 + |x' - y'| + |x'|}{(1 + |x' - y'|)(1 + |y'|)} \leq \frac{1 + |x'|}{1 + |y'|}. \tag{5.12}
\]

Similarly,

\[
\frac{1}{1 + |x_n - s - |y'|^2|} = \frac{1 + s + |y'|^2}{1 + |x_n - s - |y'|^2|} \times \frac{1}{1 + s + |y'|^2} \leq \left(1 + \frac{|x_n|}{1 + |x_n - s - |y'|^2|}\right) \frac{1}{1 + s} \leq \frac{1 + |x_n|}{1 + s}.
\]

Hence

\[
|A_{x,+}^{(k)}(s)| \leq c \left(\frac{(1 + |x'|)(1 + |x_n|)}{1 + s}\right)^m \int_{\mathbb{R}^{n-1}} \frac{dy'}{(1 + |y'|)^m} \leq c' \left(\frac{1 + |x|}{1 + s}\right)^{2m}
\]

for some constant \( c' = c'(k, m, \varphi) \) with \( k > -\text{Re}\alpha \) and \( m > n - 1 \). It follows that

\[
|(P_{+}^\alpha \varphi)(x)| \leq \frac{c'(1 + |x|)^{2m}}{|\Gamma(\alpha + k)|} \int_0^\infty s^{\text{Re}\alpha+k-1} (1+s)^m ds < \infty
\]

provided \( -k < \text{Re}\alpha < m - k \). In our case, \((1 - n)/2 \leq \text{Re}\alpha \leq 1\). Choosing \( k > (n - 1)/2 \) and \( m > \max(k + 1, n - 1) \) we make this inequalities compatible, and for any compact set \( K \subset \mathbb{R}^n \) obtain

\[
||P_{+}^\alpha \varphi||_{L^\infty(K)} \leq \frac{C}{|\Gamma(\alpha + k)|}, \quad C = C(K, \varphi). \tag{5.13}
\]

The same inequality can be obtained for \( P_{-}^\alpha \varphi \) if we slightly change calculations. Specifically, for any positive integers \( m_1 \) and \( m_2 \),

\[
|A_{x,-}^{(k)}(s)| \leq c \int_{\mathbb{R}^{n-1}} \frac{dy'}{(1 + |x' - y'|)^{m_1}(1 + |x_n + s - |y'|^2|)^{m_2}},
\]
where the first factor is estimated as in (5.12) and for the second one we have
\[
\frac{1}{1 + |x_n + s - |y'|^2|} = \frac{1 + |s - |y'|^2|}{1 + |x_n + s - |y'|^2|} \times \frac{1}{1 + |s - |y'|^2|} \leq \frac{1 + |x_n|}{1 + |s - |y'|^2|} \times \frac{1}{1 + s} \leq \frac{(1 + |x_n|)(1 + |y'|^2)}{1 + s}.
\]
This gives
\[
|A^{(k)}_{x,-}(s)| \leq c \frac{(1 + |x'|)^m (1 + |x_n|)^{m_2}}{(1 + s)^{m_2}} \int_{R^{n-1}} (1 + |y'|^{m_2}) \frac{dy'}{(1 + |y'|^{m_1})}.
\]
Choosing \(k, m_1,\) and \(m_2\) so that
\[-k < \Re \alpha < m_2 - k, \quad m_1 > 2m_2 + n - 1,\]
we obtain
\[
|\big(P_{\alpha} - \psi\big)(x)| \leq C \frac{c'(1 + |x|)^{m_1 + m_2}}{|\Gamma(\alpha + k)|} \int_0^\infty s^{Re \alpha + k - 1} (1 + s)^{m_2} ds < \infty.
\]
This gives an analogue of (5.13) for \(P_{\alpha} - \psi.\) Assuming additionally \(k > (n - 1)/2\) and \(m_2 > k + 1,\) we make our reasoning compatible with \((1 - n)/2 \leq \Re \alpha \leq 1.\)

Now, let us revert to the notation of the interpolation theorem. If \(\alpha = \alpha(z)\) and \(T_{\zeta} \varphi = P_{\pm}^{\alpha(z)} \varphi,\) then for all \(0 \leq \Re z \leq 1,\) the inequality (5.13) and its analogue for \(P_{\alpha} - \psi\) give
\[
\|T_{\zeta} \varphi\|_{L^\infty(K)} \leq C \frac{c'(1 + |x|)^{m_1 + m_2}}{|\Gamma(\zeta)|} \int_0^{\infty} s^{Re \alpha + k - 1} (1 + s)^{m_2} ds < \infty.
\]
If \(z = x + iy,\) then \(\zeta = a + ib,\) where
\[
a = \frac{1 + n}{2} x + \frac{1 - n}{2} + k, \quad b = \frac{1 + n}{2} y; \quad 0 \leq x \leq 1.
\]
It is known that if \(a_1 \leq a \leq a_2\) and \(|b| \rightarrow \infty,\) then
\[
|\Gamma(\zeta)| = \sqrt{2\pi} |b|^{a-1/2} e^{-\pi|b|^2/2} [1 + O(1/|b|)],
\]
where the constant implied by \(O\) depends only on \(a_1\) and \(a_2;\) see, e.g., [2, Corollary 1.4.4]. Taking into account that \(1/\Gamma(\zeta)\) is an entire function and using (5.14), after simple calculations we obtain an estimate of the form
\[
\log \|T_{\zeta} \varphi\|_{L^\infty(K)} \leq \begin{cases} c_1 & \text{if } |y| \leq 10, \\ c_2 + c_3 \log |y| + c_4 |y| & \text{if } |y| \geq 10, \end{cases}
\]
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for some positive constants $c_1, c_2, c_3, c_4$. This estimate yields (5.7) for any $\gamma > 0$. Thus verification of the assumption (B) for Theorem 5.4 is complete.

Let us check (5.8). By (5.11),

$$ T_i^\gamma = P_{\pm}^{-1/2 + i\gamma(1+n)/2}, \quad T_{1+i}^\gamma = P_{\pm}^{1+i\gamma(1+n)/2}. $$

Hence the results of Lemmas 5.1 and 5.3 can be stated as

$$ ||T_i^\gamma \varphi||_2 \leq B_0 M_0(\gamma)||\varphi||_2, \quad ||T_{1+i}^\gamma \varphi||_\infty \leq B_1 M_1(\gamma)||\varphi||_1, $$

with some constants $B_0, B_1$. These estimates yield (5.8).

Thus, by Theorem 5.4, the “if” part of Theorem 1.1 is proved for $P_{\pm}^\alpha$ when $\alpha$ is real. If $\alpha$ has a nonzero imaginary part, say, $t$, the above reasoning can be repeated almost verbatim if we re-define $T_z$ in (5.11) by setting $T_z = P_{\pm}^{\alpha + it}$.

**STEP 2.** Let us prove the “only if” part of Theorem 1.1. First we show that the left bound $\Re \alpha = (1-n)/2$ in (1.4) is sharp. Suppose the contrary, assuming for simplicity that $\alpha$ is real. Then there is a triple $(p_0, q_0, \alpha_0)$ with $1 \leq p_0 < q_0 < \infty$ and $\alpha_0 < (1-n)/2$, such that at least one of the operators $P_{\pm}^{\alpha_0}$, say $P_{+}^{\alpha_0}$, is bounded from from $L^{p_0}(\mathbb{R}^n)$ to $L^{q_0}(\mathbb{R}^n)$. Then, interpolating the triples $(p_0, q_0, \alpha_0)$ and $(1, \infty, 1)$, as we did above, we conclude that for any $\alpha \in [\alpha_0, 1]$, the operator $P_{+}^{\alpha}$ is bounded from from $L^{p_\alpha}(\mathbb{R}^n)$ to $L^{q_\alpha}(\mathbb{R}^n)$, where

$$ \frac{1}{p_\alpha} = \frac{\alpha - \alpha_0}{1 - \alpha_0} + \frac{1 - \alpha}{p_0(1 - \alpha_0)}, \quad \frac{1}{q_\alpha} = \frac{1 - \alpha}{q_0(1 - \alpha_0)}. $$

In particular, for $\alpha = (1-n)/2$ and $\alpha_0 = (1-n)/2 - \varepsilon$, $0 < \varepsilon < (1+n)/2$, we obtain

$$ p(1-n)/2 = \frac{2(1 + n)}{1 + n - 2\varepsilon}, \quad q(1-n)/2 = \frac{2(1 + n)}{1 + n + 2\varepsilon}. $$

The latter agrees with the known $L^2$-$L^2$ boundedness of $P_{+}^{(1-n)/2}$ only if $\varepsilon = 0$, that is, $\alpha_0 = (1-n)/2$.

The necessity of $p$ and $q$ in (1.4) can be proved using the scaling argument, which is applied to the equivalent operator families

$$ (T_{\pm}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)^{\alpha - 1}_\pm f(y', y_n + x' \cdot y') dy, \quad \Re \alpha > 0. \quad (5.15) $$

The limiting case $\alpha = 0$ in (5.15) yields the transversal Radon transform (1.3). There is a remarkable connection between $P_{\pm}^{\alpha} f$ and $T_{\pm}^{\alpha} f$. 


which is realized by the maps

\[(B_1 f)(x) = f(x', x_n - |x'|^2), \quad (B_2 F)(x) = F(2x', x_n - |x'|^2). \quad (5.16)\]

Their inverses have the form

\[(B_1^{-1} u)(x) = u(x', x_n + |x'|^2), \quad (B_2^{-1} v)(x) = v \left( \frac{x'}{2}, x_n + \frac{|x'|^2}{4} \right). \quad (5.17)\]

One can easily show that

\[\|B_1 f\|_p = \|f\|_p, \quad \|B_2 F\|_q = 2^{(1-n)/q} \|F\|_q. \quad (5.18)\]

**Lemma 5.5.** [22, Lemma 7.5] The equality

\[P_\pm^\alpha f = B_\pm^\alpha B_1 f, \quad \text{Re} \, \alpha > 0, \quad (5.19)\]

holds, provided that either side of it exists in the Lebesgue sense. If \(f \in S(\mathbb{R}^n)\), then (5.19) extends to all \(\alpha \in \mathbb{C}\) by analytic continuation.

**Proof.** We recall the proof for the sake of completeness. In the “+” case we have

\[
(B_2^{-1} P_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (y_n - |y'|^2)_{+}^{\alpha - 1} f \left( \frac{x'}{2} - y', x_n + \frac{|x'|^2}{4} - y_n \right) dy,
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty s^{\alpha - 1} f \left( \frac{x'}{2} - y', x_n + \frac{|x'|^2}{4} - s - |y'|^2 \right) ds
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha - 1} ds \int_{\mathbb{R}^{n-1}} f(z', x_n - s + x' \cdot z' - |z'|^2) dz'. \quad (5.20)
\]

On the other hand,

\[(T_+^\alpha B_1 f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)_{+}^{\alpha - 1} (B_1 f)(y', y_n + x' \cdot y') dy\]

\[
= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha - 1} ds \int_{\mathbb{R}^{n-1}} f(y', x_n - s + x' \cdot y' - |y'|^2) dy',
\]

which coincides with (5.20). In the “−” case, the proof is similar. \(\Box\)

An analogue of (5.19) for the Radon transforms (1.1) and (1.3) can be found in [5, Lemma 2.3] and [21, Lemma 3.2].

Let us continue the proof of the main theorem. In view of (5.16) and Lemma 5.5, it suffices to work with \(T_\pm^\alpha\) in place of \(P_\pm^\alpha\), where the case
\( \alpha = 0 \) corresponds to the Radon transform (1.3). Let, for example \( \alpha \) be real. For \( \lambda = (\lambda_1, \lambda_2), \lambda_1 > 0, \lambda_2 > 0, \) denote
\[
(A_\lambda f)(x) = f(\lambda_1 x', \lambda_2 x_n), \quad (B_\lambda F)(x) = \frac{\lambda_1^{1-n}}{\lambda_2^{\alpha}} F \left( \frac{\lambda_2}{\lambda_1} x', \lambda_2 x_n \right).
\]
Then \( T_\alpha^\pm A_\lambda f = B_\lambda T_\alpha^\pm f \) and we have
\[
||A_\lambda f||_p = \lambda_1^{1-n+p(1-1/p)} ||f||_p, \quad ||B_\lambda F||_q = \lambda_1^{1-n+(n-1)/q} \lambda_2^{-\alpha-n/q} ||F||_q.
\]
If \( ||T_\pm^\alpha f||_q \leq c ||f||_p \) is true for all \( f \in L^p \), then it is true for \( A_\lambda f \), that is, \( ||T_\pm^\alpha A_\lambda f||_q \leq c ||A_\lambda f||_p \) or \( ||B_\lambda T_\pm^\alpha f||_q \leq c ||A_\lambda f||_p \). The latter is equivalent to
\[
\lambda_1^{1-n+(n-1)/q} \lambda_2^{-\alpha-n/q} ||T_\pm^\alpha f||_q \leq c \lambda_1^{1(n-1)/p} \lambda_2^{-1/p} ||f||_p.
\]
Assuming that \( \lambda_1 \) and \( \lambda_2 \) tend to zero and to infinity, we conclude that the last inequality is possible only if
\[
p = \frac{n + 1}{n + \alpha}, \quad q = \frac{n + 1}{1 - \alpha}.
\]
The above reasoning shows that the right bound \( \Re \alpha = 1 \) is sharp, too.

Now, the proof of Theorem 1.1 is complete.

6. Conclusion

Some comments are in order.

1. In the present article we explored parabolic convolutions associated with the Riemann-Liouville operators (3.2). The same method can be applied to convolutions of the Riesz type
\[
(P_\alpha f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} |y_n - |y'|^2|^{\alpha-1} f(x - y) dy,
\]
\[
(P_*^\alpha f)(x) = \frac{1}{\gamma'(\alpha)} \int_{\mathbb{R}^n} |y_n - |y'|^2|^{\alpha-1} \text{sgn}(y_n - |y'|^2) f(x - y) dy,
\]
where
\[
\gamma(\alpha) = 2\Gamma(\alpha) \cos(\alpha \pi/2), \quad \gamma'(\alpha) = 2i\Gamma(\alpha) \sin(\alpha \pi/2);
\]
cf. [7, Chapter I, Section 3], [25].

2. Theorem 1.1 guarantees the existence of the \( L^p-L^q \) bounded extensions of the operators \( P_\pm^\alpha \) provided by interpolation. We denote these extensions by \( \mathcal{P}_\pm^\alpha \). It is natural to ask:

What are the explicit analytic formulas for the \( L^q \)-functions \( \mathcal{P}_\pm^\alpha f \)?
The case $\text{Re}\alpha \leq 0$ is especially intriguing because our operators are not represented by absolutely convergent integrals and need a suitable $L^q$-regularization. A similar question for solutions of the wave equation was studied in [17].

Using the same reasoning as in [22], one can show that in the cases $\alpha = 0$ (for the Radon transform $P$) and $0 < \text{Re}\alpha < 1$, we have $(P^\pm f)(x) = (P^\alpha f)(x)$ for almost $x$. If $(1 - n)/2 \leq \text{Re}\alpha \leq 0$ with $\text{Im}\alpha \neq 0$, the expression $(P^\pm f)(x)$ can be represented by hypersingular integrals, converging in the $L^q$-norm and in the a.e. sense. We leave the details to the interested reader.

3. Analytic families of fractional integral of convolution type arise in the context of non-Euclidean harmonic analysis, when the concept of the convolution is determined by the corresponding Lie group of motions. Examples of such convolutions on the unit sphere and the hyperbolic space can be found, e.g., in [20, 29, 30]. It might be of interest to adjust the hypotheses of Stein’s interpolation theorem from [27] for these cases and obtain the corresponding $L^p-L^q$ estimates (or complete the existing proofs).

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7. **Appendix**

Let us show that the functions $p_{\alpha \pm}$, $\text{Re}\alpha > 0$, can be viewed as regular tempered distributions. Given a test function $\varphi \in S(\mathbb{R}^n)$, for any positive integers $\ell$ and $m$ there is a constant $c_{\ell,m}$ such that

$$\sup_y (1 + |y'|)^\ell (1 + |y_n|)^m |\varphi(y',y_n)| \leq c_{\ell,m}.$$  

Hence, setting $\alpha_0 = \text{Re}\alpha > 0$ and passing to polar coordinates, for sufficiently large $\ell$ and $m$ we have

$$|\langle p_{\alpha+}, \varphi \rangle| \leq \frac{1}{|\Gamma(\alpha)|} \int_0^{\infty} \int_{S^{n-2}} (y_n - r^2)^{\alpha_0 - 1} dy_n \int_{S^{n-2}} |\varphi(r\theta,y_n)| d\theta \leq \sigma_{n-2} c_{\ell,m} \frac{\int_0^{\infty} \int_{S^{n-2}} (y_n - r^2)^{\alpha_0 - 1} dy_n}{(1 + y_n)^m} \int_{S^{n-2}} |\varphi(r\theta,y_n)| d\theta \leq \frac{\sigma_{n-2} c_{\ell,m}}{|\Gamma(\alpha)|} \int_{0}^{\infty} \int_{0}^{1} \frac{s^{\alpha_0 - 1}}{(1 + s)^m} ds < \infty.$$
For $p_{\alpha -}$, the calculations are a little bit more sophisticated. We have

$$|\langle p_{\alpha -}, \varphi \rangle| \leq \frac{1}{|\Gamma(\alpha)|} \int_0^\infty r^{n-2} dr \int_{-\infty}^{\infty} (r^2 - y_n)^{\alpha_0 - 1} dy_n \int_{S^{n-2}} |\varphi(r, y_n)| d\theta$$

$$\leq \text{const} \int_0^\infty r^{n-2} dr \int_0^\infty \frac{s^{\alpha_0 - 1}}{(1 + |r^2 - s|)^m} ds \leq \text{const}(I_1 + I_2).$$

Here

$$I_1 = \int_0^\infty \frac{r^{n-2} dr}{(1 + r)\ell} \int_0^\infty \frac{s^{\alpha_0 - 1}}{(1 + r^2 - s)^m} ds \leq \int_0^\infty \frac{r^{n-2} dr}{(1 + r)\ell} \int_0^\infty s^{\alpha_0 - 1} ds < \infty$$

if $\ell$ is large enough. Further,

$$I_2 = \int_0^\infty \frac{r^{n-2} dr}{(1 + r)\ell} \int_{r^2}^\infty \frac{s^{\alpha_0 - 1}}{(1 + s - r^2)^m} ds = \int_0^\infty \frac{r^{n-2} dr}{(1 + r)\ell} \int_0^\infty \frac{(r^2 + t)^{\alpha_0 - 1}}{(1 + t)^m} dt.$$  

If $\alpha_0 \leq 1$, then

$$I_2 \leq \int_0^\infty \frac{r^{n-2} dr}{(1 + r)\ell} \int_0^\infty \frac{dt}{1 - \alpha_0 (1 + t)^m} < \infty.$$  

If $\alpha_0 > 1$, then $I_2 = I_{2,1} + I_{2,2}$, where $I_{2,1} = \int_0^1 (\ldots) dr$, $I_{2,2} = \int_1^\infty (\ldots) dr$,

$$I_{2,1} \leq \int_0^1 \frac{r^{n-2} dr}{(1 + r)\ell} \int_0^\infty (1 + t)^{\alpha_0 - 1 - m} dt < \infty,$$

$$I_{2,2} \leq \int_1^\infty \frac{r^{n-2} dr}{(1 + r)\ell} \left( \int_0^r (r^2 + t)^{\alpha_0 - 1} dt + \int_r^\infty \frac{(r^2 + t)^{\alpha_0 - 1}}{t^m} dt \right)$$

$$= \int_1^\infty \frac{r^{n-2}}{(1 + r)\ell} (c_1 r^{2\alpha_0} + c_2 r^{2\alpha_0 - m}) dr < \infty.$$  

Thus $p_{\alpha -} \in S'(\mathbb{R}^n)$ for all $\Re \alpha > 0$.  
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