New result in NG-Groups

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Abstract: We investigate and present a new result in NG groups that consisting of non-bijective transformations cannot be subset of symmetric groups. In this paper, we revive the concepts of NG groups. Moreover, we introduce the definition of regular NG and inverse NG the new concept as new results in these groups. Hover, some properties are studied.

Key words: Permutation group, Equivalence class, quotient set, NG groups.

1- Introduction:

The transformation group is important and good part of the group theory [1]. We recall a permutation group on a non-empty set A is a group consisting of bijections transformations from A to itself with respect to compositions mapping on anon empty set A which the one most important transformation groups for more details see [2],[3],[4]. It is well known that any permutation group on a set A with cardinality n has order not greater that n!. In [5], Y. Wu, X.Wei present the conditions of the groups generated by nonobjective transformations on a set. Author in [6], introduce find the maximal order of these groups. In this paper, we fined a new result in these groups, which is the inverse of these groups.

2- Preliminary :

We review some basics definitions and properties of the finite group theory that we will used in our paper. For more detailed in lots of abstract algebra and finite group theory can see [7],[8] would be good supplementary sources for the theory needed here.

Definition 2.1. Suppose that A is non-empty set. A binary relation R in A is called an equivalence relation on A if it satisfy the following:

(i) xRx for any x ∈ A;
(ii) for any x, y ∈ A, if xRy then yRx;
(iii) for any x, y, z ∈ A, if xRy and yRz then xRz.

We denoted P(A) to the set of all its transforms. For any f ∈ P(A), we use Im(f) to denoted the image of f. Also, Z and Z+ will respective dented the set of integers and positive integers.
**Definition 2.2.** Suppose that $R$ is an equivalence relation on $A$. For an element $x \in A$, we call $\{x \in A \mid x R\}$ the equivalence class of $x$ determined by $R$, which is denoted by $R[x]$. And $A/R = \{R[x] \mid x \in A\}$ is called the quotient set of $A$ relative to the equivalence relation $R$.

**Lemma 2.3.** ([5], theorem 1) For any $f \in NG$ and the $e$ the identity element of $NG$, $R_e = R_f$.

**Proof.**

For any $x \in A$, our goal is to show that $[a] = [a]_e$. On one hand, if $a \in [x]$, i.e. $f(a) = f(x)$. Since $NG$ is a group with identity element $e$, there is a transformation $f' \in NG$ such that $ff = e = f'f$. Therefore, $e(a) = f'(f(a)) = f'(f(x)) = e(x)$, Which yields that $a \in [x]$. On the other hand, if $y \in [a]_e$ i.e. $e(a)e(y)$. Hence, $f(a) = (fe)(a) = (fe)(y) = (fe)(y) = f(y)$, which implies $y \in [x]$. It follows that $[x]_e = [x]_f$ for any $x \in A$, as wanted.

**Remark 2.4.** For Lemma 2.3, we see that $R_e = R_g$ for any element $f, g \in NG$. The following Theorem is revised version of Theorem 2.[5].

**Theorem 2.5.** Let $f$ be an element in $P(A)$ and $\hat{f}$ be the induced transformation of $f$ on $A/R_f$, i.e. $\hat{f} : A/R_f \rightarrow A/R_f$, $[x]_f \mapsto [f(x)]_f$. Then the following hold:

(i) There exists a groups $NG \subseteq P(A)$ containing $f$ as the identity element iff $f \hat{f} = f$.

(ii) There is a groups $NG \subseteq P(A)$ containing $f$ as the identity element iff $\hat{f}$ is abjective on $A/R_f$.

We make some corrections to the original proofs of corollaries from [5]. Actually, we adopt the restriction of finiteness on $A$ in the first corollary from the original one. And we used the finiteness on $A$ in the second corollary; the original one did not use it.

**Corollary 2.6.** Let $f$ be an element in $P(A)$. Then $f \hat{f} = f$ iff the induced mapping $\hat{f}$ on $A/R_f$ is the identity element.

**Proof.**

$(\Rightarrow)$ we suppose that $f \hat{f} = f$. Then for any $[x]_f \in A/R_f$, as $f(x) = f(f(x))$, we see that $[x]_f = [f(x)]_f$. It follows that $\hat{f}([x]_f) = [f(x)]_f = [x]_f$; which implies that $\hat{f}$ is the identity mapping on $A/R_f$.

$(\Leftarrow)$ we assume that $\hat{f}$ is the identity mapping on $A/R_f$. Then for any $[x]_f \in A/R_f$, the condition that $\hat{f}([x]_f) = [x]_f$ will imply that $[f(x)]_f = [x]_f$ and hence $f(f(x)) = f(x)$. It follows that $f \hat{f} = f$ as required.

**Corollary 2.7.** Suppose that $A$ is a finite set and $f$ is an element in $P(A)$. Then there is a group $NG \subseteq P(A)$ containing $f$ as an element iff $\text{Im}(f) = \text{Im}(f \hat{f})$.

**Proof.**

$(\Rightarrow)$ we suppose that there is a group $NG \subseteq P(A)$ containing $f$ as an element. Let $e$ be the identity element of $NG$. Then by Theorem 2.5, the induced mapping $\hat{f}$ is a bijection on $A/R_f$. In particular, $\hat{f}$ is surjective and thus for any $x \in A$, there is a $[y]_f \in A/R_f$ such that $\hat{f}([y]_f) = [x]_f = [f(y)]_f$; which yields that $f(x) = f(f(y)) = (f \hat{f})(y)$. As a result, $\text{Im}(f) \subseteq \text{Im}(f \hat{f})$ and thus $\text{Im}(f) = \text{Im}(f \hat{f})$. 

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The groups of order 2 are: \( G_1 = \{(1,1,3), (3,3,1)\}, \ G_2 = \{(1,2,1), (2,1,2)\}, \ G_3 = \{(1,2,2), (2,1,1)\}, \ G_4 = \{(1,2,3), (1,3,2)\}, \ G_5 = \{(1,2,3), (2,1,3)\}, \ G_6 = \{(1,2,3), (3,2,1)\}, \ G_7 = \{(1,3,3), (3,1,1)\}, \ G_8 = \{(2,2,3), (3,3,2)\}, \) and \( G_9 = \{(2,3,2), (3,2,3)\}. \) But, \( G_4 = \{(1,2,3), (1,3,2)\}, \ G_5 = \{(1,2,3), (2,1,3)\}, \) and \( G_6 = \{(1,2,3), (3,2,1)\} \) are subsets of \( \text{Sym}(X) \).

**Proposition 2-11:** Let \( NG_1 \) and \( NG_2 \) be two \( NG \)-groups, then \( NG_1 \cup NG_2 \) and \( NG_1 \cap NG_2 \) are not necessary to be \( NG \)-groups.
Proof:
Let $X = \{1, 2, 3\}$, we know, $NG_1$ and $NG_2$ are $NG$-groups. We assume $NG_1 \cup NG_2$ is $NG$-groups. But, it contradiction with example 2-10. Since order of $NG_1 \cup NG_2$ is four. In addition, $NG_1 \cap NG_2 = \phi$, and $\phi \subset S_3$. Therefore, $NG_1 \cap NG_2$ is not $NG$-group.

3- New results

In this section, we introduce the new concepts in $NG$ groups.

Definition 3-1: An element $f$ of $NG$ is regular if there exists $y \in NG$ such that $fyf = y$. The $NG$ is regular if all its elements are regular.

Note that $NG$-groups are regular, since we may choose $y = f^{-1}$.

Example 3-2: we consider the example 2-10. If we take $NG = G_1 = \{(1,1,3),(3,3,1)\}$; If we take $f = (1,1,3)$ and $y = (3,3,1)$, then $fyf = (1,1,3) (3,3,1) (1,1,3) = (3,3,1) = y$.

And if we take $f = (3,3,1)$ and $y = (1,1,3)$, then $fyf = (3,3,1) (1,1,3) (3,3,1) = (1,1,3) = y$.

From definition 3-1, $G_1$ is regular; we can see all $NG$ are regular.

Regularity is equivalent to a condition which appears formally to be stronger: We can make this a strong condition.

Proposition 3.3 If $f \in NG$ is regular, then there exists $y \in NG$ such that $fyf = y$ and $yfy = f$.

Proof
Choose $f$ such that $fyf = y$, and set $y = yfy$.
Then $fyf = f yfyf = fyf = y$. $yfy = yfyfyf = fyf = yfy = y = f$.

Definition 3-4: Suppose that $NG$ group, then it is an inverse $NG$ if for each $a$ there is a unique $y$ such that $fyf = y$ and $yfy = f$.

The element $y$ is called the inverse of $f$.

Proposition 3.4: Let $NG$ be an inverse group, each element of $NG$ has a unique inverse.

Conclusion: We investigate and present a new result in $NG$ groups such as the definition of regular $NG$ and inverse $NG$.

ACKNOWLEDGMENTS I would like to thank staff of Higher institute of Suluq for help on this paper.

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