Near Instance-Optimality in Differential Privacy

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Abstract

We develop two notions of instance optimality in differential privacy, inspired by classical statistical theory: one by defining a local minimax risk and the other by considering unbiased mechanisms and analogizing the Cramér-Rao bound, and we show that the local modulus of continuity of the estimand of interest completely determines these quantities. We also develop a complementary collection mechanisms, which we term the inverse sensitivity mechanisms, which are instance optimal (or nearly instance optimal) for a large class of estimands. Moreover, these mechanisms uniformly outperform the smooth sensitivity framework—on each instance—for several function classes of interest, including $\mathbb{R}$-valued continuous functions. We carefully present two instantiations of the mechanisms for median and robust regression estimation with corresponding experiments.

1 Introduction

We study instance-specific optimality for differentially private release of a function $f(x)$ of a dataset $x \in \mathcal{X}^n$. In contrast to existing notions of optimality for private procedures, which measure mechanisms’ worst case performance over all instances, we develop instance-specific notions to capture the difficulty of—and potential adaptivity of private mechanisms to—the given data $x$, rather than some potential worst case.

The trajectory of differential privacy research and private mechanisms reflects the desire to be adaptive both to the function $f$ to be computed and dataset $x$ at hand. Dwork et al.’s original perspective [21] targets the former, privatizing $f(x)$ by adding noise commensurate with the global sensitivity $GS_f := \sup_{x, x': d_{ham} (x, x') \leq 1} |f(x) - f(x')|$ of $f$ and adapting to the function $f$ at hand; more recent work expands this function adaptive approach [6]. As the classical approach can be conservative—it does not reflect the sensitivity of the underlying data $x$—a natural idea is to add noise that scales with the local sensitivity (or local modulus of continuity) $LS_f(x) := \sup_{x': d_{ham} (x, x') \leq 1} |f(x) - f(x')|$ of $f$ at the dataset $x$. Unfortunately, this fails to protect privacy, as the sensitivity itself may be compromising, leading Nissim et al. [35] to propose mechanisms that rely on smooth upper bounds to the local sensitivity. Yet these mechanisms are complex and, as our results show, may be conservative.

To understand these phenomena, we take a two-pronged approach, presenting both lower bounds on error and complementary (near) optimal mechanisms. We first consider the desiderata a lower bound should satisfy, following a program Cai and Low [11] develop (see also [16]):

(i) the putative lower bound is instance specific, depending on the instance $x$ at hand;

(ii) there is a super-efficiency result, so that if an estimator outperforms the lower bound at one instance $x$, it must be (substantially) worse at another instance $x'$;

(iii) the lower bound is achievable.

Point (ii) is important, as it eliminates trivial mechanisms: consider the mechanism that for all $x \in \mathcal{X}^n$ releases $M(x) = f(x_0)$ for a fixed sample $x_0$; this is optimal at $x_0$ but poor elsewhere. More broadly, to satisfy (i) and (ii), we develop two sets of lower bounds; one for unbiased mechanisms and the other via the private local minimax framework we develop.
For the complementary point (iii) on uniform achievability, we study and develop privacy-preserving mechanisms that adapt to the instance $x$ at hand. Here, rather than measuring the function $f$’s change over datasets $x$, which may itself be sensitive, we consider the inverse of the local sensitivity: for a target $t$, the number of individuals we must change in $x$ to reach $x’$ with $f(x’) = t$, which by definition has limited sensitivity to $x$. Thus, defining the length

$$\text{len}_f(x; t) := \inf_{x’} \{d_{\text{ham}}(x, x’) \mid f(x’) = t\},$$

it is possible to define (nearly) instance-optimal mechanisms. Indeed, consider the case that we wish to compute a discrete $f : X^n \to T$, $|T| < \infty$. Then for a privacy parameter $\varepsilon > 0$, the inverse sensitivity mechanism $M$ instantiates the exponential mechanism [30] with $\text{len}_f$:

$$\mathbb{P}(M(x) = t) := \frac{e^{-\text{len}_f(x; t)\varepsilon / 2}}{\sum_{s \in T} e^{-\text{len}_f(x; s)\varepsilon / 2}}.$$ (M.1)

This abstract mechanism is perhaps folklore; Johnson and Shmatikov’s distance-score mechanism [28, Sec. 5] is an example, and it appears in various exercise sets and lecture notes on differential privacy [e.g. 36, Ex. 3.1]. Yet its strong optimality and adaptivity properties—which we begin to delineate, and which include always outperforming the Laplace and smooth Laplace mechanisms—appear unexplored, and its extension to the continuous case requires nontrivial care. As a motivation for what follows, a consequence of our results is that for every instance $x \in X^n$, mechanism (M.1) is more likely to output $f(x)$ than any other mechanism $\tilde{M}$ satisfying $\mathbb{P}(\tilde{M}(x) = f(x)) \geq \mathbb{P}(\tilde{M}(x) = t)$.

1.1 Problem setting, definitions, and contributions

To situate our contributions, we begin by recalling a few privacy definitions, starting with the standard definition of differential privacy [21, 20]. Central to each is that two datasets $x, x’ \in X^n$ are neighboring if $d_{\text{ham}}(x, x’) \leq 1$, that is, they differ in at most one example.

**Definition 1.1.** A randomized algorithm $M : X^n \to T$ is $(\varepsilon, \delta)$-differentially private if for all neighboring datasets $x, x’ \in X^n$ and all measurable $S \subset T$,

$$\mathbb{P}(M(x) \in S) \leq e^{\varepsilon} \mathbb{P}(M(x’) \in S) + \delta.$$

If $\delta = 0$, then $M$ is $\varepsilon$-differentially private.

We also recall Mironov’s Rényi-differential privacy [32]. For $\alpha \geq 1$ and distributions $P$ and $Q$, the Rényi-divergence of order $\alpha$ is

$$D_{\alpha}(P \| Q) := \frac{1}{\alpha - 1} \log \int \left(\frac{dP}{dQ}\right)^\alpha dQ,$$

where $D_1(P \| Q) = \lim_{\alpha \downarrow 1} D_{\alpha}(P \| Q) = D_{\text{kl}}(P \| Q)$. Adopting a consistent abuse of notation that for random variables $X \sim P$ and $Y \sim Q$ we set $D_{\alpha}(X \| Y) := D_{\alpha}(P \| Q)$, we have

**Definition 1.2.** A randomized algorithm $M : X^n \to T$ is $(\alpha, \varepsilon)$-Rényi-differentially private if for all neighboring datasets $x, x’ \in X^n$

$$D_{\alpha}(M(x) \| M(x’)) \leq \varepsilon.$$
We measure a mechanism’s performance by its expected loss: for a function $f : \mathcal{X}^n \to \mathcal{T}$ and loss $L : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+$, the expected loss of a mechanism $M$ on instance $x$ is

$$\mathbb{E}[L(M(x), f(x))].$$

The first notion of optimality we adopt is the local minimax risk, which for a family $\mathcal{M}$ of mechanisms is

$$\mathcal{R}(x, L, \mathcal{M}) := \sup_{x' \in \mathcal{X}^n} \inf_{M \in \mathcal{M}} \max_{x \in \{x, x'\}} \mathbb{E}[L(M(\tilde{x}), f(\tilde{x}))].$$

(2)

This definition descends from an insight of Stein’s that a problem should be as hard as its “hardest one-dimensional sub-problem” [39]. Accordingly, the infimum over mechanisms $M$ is inside the outer supremum, so that $M$ may use that the data will be either $x$ or $x'$; the local minimax risk (2) measures the difficulty of privately estimating $f(x)$ against the hardest alternative instance $x' \in \mathcal{X}^n$. Cai and Low use an analogous quantity [11, Eq. (1.4)] in nonparametric function estimation.

The local minimax risk (2) is adaptive to the underlying instance $x$, and we show in Section 2 that it satisfies the super-efficiency requirement (ii) we highlight in the introduction. With this, we may define local minimax optimality.

**Definition 1.3.** Let $L : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+$ and $\mathcal{M}$ be a family of mechanisms. A mechanism $M : \mathcal{X}^n \to \mathcal{T}$ is local minimax optimal for the family $\mathcal{M}$ if there exists a universal (numerical) constant $C < \infty$ such that

$$\mathbb{E}[L(M(x), f(x))] \leq C \cdot \mathcal{R}(x, L, \mathcal{M}).$$

In our development, we often consider families of $(\varepsilon, \delta)$-private mechanisms, and show that a $(c\varepsilon, c\delta)$-differentially private mechanism satisfies Definition 1.3, where $c$ is a constant we specify; we say such a mechanism is local minimax $c$-optimal. An alternative way to understand $c$-optimality is that a $c$-optimal mechanism requires a sample $\tilde{x}$ of size approximately $cn$ (rather than $n$) to achieve the local minimax risk $\mathcal{R}(x)$ for $x \in \mathcal{X}^n$. Indeed, privacy amplification by subsampling techniques [5, 3] show that a mechanism $\tilde{M}$ that randomly subsamples a sample $\tilde{x}$ of $cn$ examples down to $x \in \mathcal{X}^n$, then applies $M(x)$, is $\varepsilon$-differentially private.

The local minimax benchmark (2) eliminates the possibility of trivial mechanisms we mention above. An alternative is to consider restricted families of mechanisms. In this context, we borrow from the statistical tradition of unbiased estimators to allow analogues of the classical Cramér-Rao bounds. We may then define unbiased mechanisms.

**Definition 1.4** ([29], Ch. 1.5, Eq. (1.9)). Let $L : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+$. A randomized algorithm $M : \mathcal{X}^n \to \mathcal{T}$ is $L$-unbiased if for any $x \in \mathcal{X}^n$ and $t \in \mathcal{T}$, $\mathbb{E}[L(M(x), f(x))] \leq \mathbb{E}[L(M(x), t)].$

For the squared error $L(s, t) = (s - t)^2$, we recover the familiar equality $\mathbb{E}[M(x)] = f(x)$, while an $\ell_1$-unbiased mechanism has median $f(x)$. A mechanism is unbiased for the 0-1 loss if $\mathbb{P}(M(x) = f(x)) \geq \mathbb{P}(M(x) = t)$ for all $t \in \mathcal{T}$. Most standard mechanisms, including the Laplace and Gaussian mechanisms, are unbiased. We then can parallel Definition 1.3.

**Definition 1.5.** Let $C \geq 1$ and the loss $L : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+$. A randomized algorithm $M : \mathcal{X}^n \to \mathcal{T}$ is $C$-optimal against $L$-unbiased mechanisms if $M$ is $C\varepsilon$-differentially private, and for any $\varepsilon$-differentially private and $L$-unbiased mechanism $M_{\text{unb}} : \mathcal{X}^n \to \mathcal{T}$

$$\mathbb{E}[L(M(x), f(x))] \leq \mathbb{E}[L(M_{\text{unb}}(x), f(x))] \text{ for all } x \in \mathcal{X}^n.$$
1.1.1 Contributions and outline

We highlight the three main contributions in this paper.

**Instance-specific notions of optimality.** We move beyond worst-case (minimax) loss to study instance-optimality in differential privacy, providing two potential definitions and their consequences and satisfying desiderata (i) and (ii). Making this concrete, let $d_{\text{ham}}$ denote the Hamming distance and $(\mathcal{T}, d_{\mathcal{T}})$ be a metric space; then the local modulus of continuity of a function $f: \mathcal{X}^n \rightarrow \mathcal{T}$ at $x \in \mathcal{X}^n$ is

$$\omega_f(x; k) = \sup_{x' \in \mathcal{X}^n} \{d_{\mathcal{T}}(f(x), f(x')) : d_{\text{ham}}(x, x') \leq k\}.$$  

(3)

Our results show that this local modulus of continuity characterizes the optimal error in estimation at each instance $x \in \mathcal{X}^n$ for differentially private mechanisms, including those satisfying only relaxed (e.g. Rényi) definitions of privacy. We provide these results in Section 2.

**Nearly instance-optimal mechanisms.** We show that the inverse sensitivity mechanism (M.1) and its arbitrary-valued extensions (which we develop in Sec. 3.2) are $C$-optimal for reasonable $C$ for many functions $f$ and losses $L$ of interest. For example, in Section 3.1 we show that mechanism (M.1) is 4-optimal against unbiased mechanisms for the 0-1-loss (Def. 1.5). For general losses and functions $f$, we show the inverse sensitivity mechanisms are $O(\log n)$-optimal in both optimality senses we consider (Definitions 1.3 and 1.5).

**Instance-optimality for sample-monotone functions.** We define a class of what we term *sample-monotone* functions (see Section 4), which are reasonably-behaved with respect to changes in the sample $x \in \mathcal{X}^n$, and which includes all $\mathbb{R}$-valued continuous functions on convex domains. We show that the inverse sensitivity mechanism is $O(1)$-optimal for many such functions $f$, and nearly $O(\log\log n)$-optimal for any sample-monotone function. Finally, we show in Section 4 that the inverse sensitivity mechanism uniformly outperforms the Laplace and smooth Laplace mechanisms over all $x$ for any sample-monotone function.

**Applications.** An important component of this work is methodological, and we aim to develop practicable procedures. We include detailed examples for estimating the median of a sample (Sec. 5.1) and robust regression (Sec. 5.2). As a side benefit of this development, we show that the inverse-sensitivity offers a quadratic improvement in sample complexity over standard smooth Laplace mechanisms. We include representative experiments in Section 6.

**Notation** We let bold symbols $x \in \mathcal{X}^n$ denote samples and non-bold symbol $x \in \mathcal{X}$ denote individual examples. For $x, x' \in \mathcal{X}^n$, $d_{\text{ham}}(x, x')$ is the Hamming distance. Using the local modulus (3), the local sensitivity of $f: \mathcal{X}^n \rightarrow \mathcal{T}$ at instance $x$ is $LS_f(x) = \omega_f(x; 1)$, and the global sensitivity of $f$ is $GS_f = \sup_{x \in \mathcal{X}^n} \omega_f(x; 1)$. For $K \in \mathbb{N}$, we let $[K] = \{1, 2, \ldots, K\}$. For a subset $\mathcal{T}$ of a vector space with norm $\|\cdot\|$, we let $\text{diam}(\mathcal{T}) = \sup_{s,t \in \mathcal{T}} \|s - t\|$. We write $a \lesssim b$ or $a = O(b)$ if there is a universal (numerical) constant $c < \infty$ such that $a < cb$. 

4
1.2 Background and related work

The standard method to achieve $\varepsilon$-DP for releasing $\mathbb{R}$-valued $f$ is the Laplace mechanism, which adds Laplace noise proportional to global sensitivity [21],

$$M_{\text{Lap}}(x) := f(x) + \frac{GS_f}{\varepsilon} \text{Lap}(1).$$  \hspace{1cm} (4)

To address that the Laplace mechanism adds noise that must scale with the worst-case sensitivity of $f$, Nissim et al. [35] introduce the smooth sensitivity framework, showing that for appropriate upper bounds $S(x)$ on the local sensitivity, the smooth Laplace mechanism

$$M_{\text{sm-Lap}}(x) := f(x) + \frac{2S(x)}{\varepsilon} \text{Lap}(1)$$  \hspace{1cm} (5)

is $(\varepsilon, \delta)$-DP. Here, one requires that $LS(x) \leq S(x)$ and that $S(x) \leq e^\beta S(x')$ for neighboring instances $x, x' \in X^n$, where $\beta = \frac{\varepsilon}{2 \log(2/\delta)}$. A main motivation of the smooth Laplace mechanism [35] is to calculate the median of a dataset, and there are instances where the mechanism (5) adds noise scaling as $\frac{2}{\varepsilon^2}$, while mechanism (4) adds noise $\frac{1}{\varepsilon^2}$. Yet as the results in this paper demonstrate, even replacing $S$ with the tighter local sensitivity $LS(x)$ in the smooth Laplace mechanism—which is not differentially private—is instance-suboptimal, and the smooth sensitivity framework does not provide $\varepsilon$-DP with exponentially decaying noise.

Our approach is a natural descendant of work that in some sense inverts sensitivity measurements. The two most salient works here are Dwork and Lei’s propose-test-release framework [18] and Smith and Thakurta’s instantiation for high-dimensional regression (Lasso) problems [37]. Briefly, algorithms in this framework test whether the maximal local sensitivity in a neighborhood of the given dataset $x$ is upper bounded by a prespecified value $\beta$ (testing that the dataset $x$ is “far” from any other $x'$ for which the bound $\beta$ fails to hold), then add noise $\text{Lap}(\beta/\varepsilon)$ if the test passes. The framework, however, cannot provide pure $\varepsilon$-DP, and it does not enjoy instance-optimal performance as it adds large noise—or fails—if there is even one dataset in the local neighborhood of the given instance with large local sensitivity. Our examples demonstrating the instance sub-optimality of the smooth sensitivity framework also show sub-optimality of the propose-test-release framework.

The discrete version of the inverse sensitivity mechanism (M.1) is in a sense folklore—indeed, McSherry and Talwar [30] considered it but did not include it in their development of the exponential mechanism (personal communication). Variants of it are also explicit or implicit in a number of papers and lectures [28, 36, 10]. Yet it appears that no work investigates the instance-optimality guarantees of mechanism (M.1), nor does any work explain the advantages of the framework over other approaches to adding data-dependent noise, such as smooth sensitivity or propose-test-release; indeed, most work developing data-dependent noise addition mechanisms is based on these frameworks [37, 25, 9].

Our results also belong to an intellectual tradition, familiar from statistics [41, 29], of optimality relative to classes of algorithms or procedures. Most salient is the existence of uniformly minimum variance unbiased estimators (UMVUs) [29]. In this vein, it is unsurprising that unbiased mechanisms (Definitions 1.4 and 1.5) allow instance-specific optimality results. In more sophisticated settings, instance-optimality—in the sense of estimators optimal for the given population generating the sample $x \in X^n$—is challenging [cf. 41, Ch. 8], necessitating local minimax constructions or regular estimators.

In the differential privacy literature, Hardt and Talwar [26] develop minimax lower bounds for estimating linear functions, which De [12] and Nikolov et al. [34] extend. A different line
of work develops optimality results in the oblivious setting where private mechanisms add noise independent of the underlying instance \cite{38, 23, 22, 24} (a natural restriction on the class of estimators). Soria-Comas and Domingo-Ferrer \cite{38} and Geng et al. \cite{22, 24} independently develop the optimal mechanism in this setting for one-dimensional real valued functions, showing that the optimal noise has a staircase-shaped probability density for symmetric losses. Despite these results, their obliviousness implies the mechanisms are generally suboptimal.

2 Instance-dependent lower bounds

We begin our theoretical investigation by proving lower bounds on the loss of private mechanisms that capture the hardness of the underlying instance.

2.1 Unbiased mechanisms and worst-case bounds

Our first set of lower bounds applies to unbiased mechanisms, where the lower bounds are relatively straightforward to develop. They rely on the idea that if a mechanism is \(\varepsilon\)-differentially private and unbiased, it must assign “enough probability mass” to each possible output \(t\) of \(f(x)\), an idea present in other lower bounds (cf. \cite{26, 4}). This idea allows an essentially immediate corresponding minimax result. We first consider discrete functions \(f: X^n \rightarrow T, |T| < \infty\), with the 0-1 loss \(\ell_{0-1}(s, t) = 1\{s \neq t\}\), so \(\mathbb{E}[L(M(x), f(x))] = 1 - \mathbb{P}(M(x) = f(x))\). See Appendix A.1 for the proof of the next proposition.

**Proposition 2.1** (Lower bounds for 0-1 loss). Let \(f: X^n \rightarrow T\) and let \(M\) be \(\varepsilon\)-DP. Then

\[
\inf_{x \in X^n} \mathbb{P}(M(x) = f(x)) \leq \inf_{x \in X^n} \frac{1}{\sum_{t \in T} e^{-\text{len}_f(x; t)\varepsilon}}.
\]

If \(M\) is also \(\ell_{0-1}\)-unbiased, then for any instance \(x \in X^n\),

\[
\mathbb{P}(M(x) = f(x)) \leq \frac{1}{\sum_{t \in T} e^{-2\text{len}_f(x; t)\varepsilon}}.
\]

Proposition 2.1 shows that worst-case bounds may be pessimistic for many instances \(x\), depending on \(\text{len}_f(x; t)\). Consider an instance \(x\) with a highly stable neighborhood: \(\text{len}_f(x; t) \gg 1\) for any \(t \neq f(x)\). Instance-dependent bounds show that we may hope to get \(\mathbb{P}(M(x) = f(x)) \approx 1\) for this instance (as the mechanism (M.1) achieves), in contrast to worst-case bounds.

Now we extend our previous lower bound to general loss functions. We consider functions \(f: X^n \rightarrow T\) for a metric space \(T\) with distance \(d_T: T \times T \rightarrow \mathbb{R}_+\) and loss function \(L(s, t) = \ell(d_T(s, t))\) where \(\ell: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) is non-decreasing. We have the following lower bound in this setting, which highlights the intuitive centrality of the local modulus (3): the more sensitive the function \(f\) is to changes in the underlying sample \(x\), the more challenging it should be to estimate. We prove the theorem in Appendix A.2.

**Theorem 1** (Lower bound for general loss). If \(M\) is \(\varepsilon\)-DP, then for any \(k \geq 1\),

\[
\sup_{x \in X^n} \mathbb{E}[L(M(x), f(x))] \geq \frac{\ell(\omega_f(x; k)/2)}{e^{k\varepsilon} + 1}.
\]

If \(M\) is additionally \(L\)-unbiased, then for any \(x \in X^n\),

\[
\mathbb{E}[L(M(x), f(x))] \geq \frac{\ell(\omega_f(x; k)/2)}{e^{2k\varepsilon} + 1}.
\]
2.2 Local minimax bounds and super-efficiency

The lower bounds for unbiased mechanisms in Proposition 2.1 and Theorem 1 are the analogues of the Cramér-Rao bound in classical estimation, with the concomitant failure to apply beyond unbiased estimators. Accordingly, we turn now to the local minimax risk (2), which we characterize in the following theorem. The result applies whenever the loss in estimation is measured by a distance \( d_T \) on the target space \( T \). In the theorem, recall the definition (3) of the local modulus of continuity, \( \omega_f(x; k) = \sup_x \{ d_T(f(x), f(x')) : d_{\text{ham}}(x, x') \leq k \} \).

**Theorem 2** (Local-minimax lower bounds). Let \( L(s, t) = \ell(d_T(s, t)) \) for a non-decreasing function \( \ell : \mathbb{R}_+ \to \mathbb{R}_+ \) and distance \( d_T \) on \( T \). Let \( f : X^n \to T \) and \( x \in X^n \) be an arbitrary sample. If \( \mathcal{M}_\varepsilon \) is the collection of of \( \varepsilon \)-differentially private mechanisms,

\[
\frac{1}{4} \max_{k \leq n} \left\{ \ell(\omega_f(x; k)/2)e^{-k\varepsilon} \right\} \leq \mathcal{R}(x, L, \mathcal{M}_\varepsilon).  
\]

(6a)

If \( \mathcal{M} \) is either the collection of \( (\varepsilon, \delta) \)-DP or \( (\alpha, \varepsilon/2) \)-Rényi-DP mechanisms with \( \alpha \geq 1 + 2\varepsilon^{-1}\log \frac{1}{\delta} \), then for \( K(\varepsilon, \delta) = \min\{\varepsilon^{-1}\log \frac{1}{\delta}, \delta^{-1/2}\} \),

\[
\frac{1}{8} \max_{k \leq K(\varepsilon, \delta)} \left\{ \ell(\omega_f(x; k)/2)e^{-k\varepsilon} \right\} \leq \mathcal{R}(x, L, \mathcal{M}).  
\]

(6b)

If \( \mathcal{M}_{\alpha,2\varepsilon^2} \) is the collection of \( (\alpha, 2\varepsilon^2) \)-Rényi-DP mechanisms, where \( \alpha \geq 1 \), then

\[
\frac{1}{8} \ell(\omega_f(x; 1/(2\varepsilon))/2) \leq \mathcal{R}(x, L, \mathcal{M}_{\alpha,2\varepsilon^2}).  
\]

(6c)

For each of the preceding families of mechanisms \( \mathcal{M} \), we have

\[
\mathcal{R}(x, L, \mathcal{M}) \leq \max_{k \leq n} \left\{ \frac{1}{1 + e^{k\varepsilon/2}} \ell(\omega_f(x; k)) \right\}.
\]

See Appendix A.3 for a proof.

Theorem 2 characterizes, to within numerical constants, the local minimax risk for each collection of mechanisms in Sec. 1.1, showing that for \( \varepsilon \)-private mechanisms, \( \mathcal{R}(x) \) it should scale as \( \max_{k \leq n} \ell(\omega_f(x; k))e^{-k\varepsilon} \). Thus, we see that in an essential way, the local modulus of continuity (3) determines the local minimax risk \( \mathcal{R} \) at a sample \( x \). We expect that the maxima (6) over \( k \) should be generally achieved for \( k \asymp \frac{1}{\varepsilon} \), so that we expect that for \( \varepsilon \)-DP mechanisms (or \( (\varepsilon, \delta) \)- or Rényi-DP mechanisms), the local minimax risk should scale as

\[
\mathcal{R}(x, L, \mathcal{M}) \asymp \ell(\omega_f(x; 1/\varepsilon)).
\]

(7)

We always have the lower bound \( \ell(\omega_f(x; \varepsilon^{-1})) \) by choosing \( k = 1/\varepsilon \) in the inequalities (6). As an example for attainment (7), if \( L(s, t) = d_T(s, t) \) and the modulus cannot grow too exponentially quickly, e.g., \( \omega_f(x; k)/\omega_f(x; k_0) \leq \exp(k/k_0) \) for \( k \geq k_0 = 1/\varepsilon \), we have \( \max_k \omega_f(x; k)e^{-k\varepsilon} \leq \omega_f(x; 1/\varepsilon) \); similar calculations show equality (7) under other conditions.

Regardless of the growth of \( \omega_f \), we have bounds on the local minimax error that satisfy the desideratum (i), that they depend on the given instance \( x \). Yet as the local minimax risk (2) uses (essentially) a one-dimensional sub-problem of the full estimation problem and still requires an outer supremum, it is not immediately clear that we satisfy the other two desiderata: (ii), that no estimator can achieve better performance than the bounds (6) without...
Then there exists a sample \( x \in \mathcal{X} \) with expected error better than our local minimax risk \( R \). Consequently, not only must the expected loss be at least the modulus of continuity (3) at distance 1, as in our expected scaling (7), necessarily result in worse expected losses elsewhere. We focus on the loss of (roughly) the modulus of continuity (3) at distance 1 in Section 3, turning now to a super-efficiency result.

We begin with a general proposition, which shows that any improvements over an expected loss of (roughly) the modulus of continuity (3) at distance \( 1/\varepsilon \), as in our expected scaling (7), necessarily result in worse expected losses elsewhere. We focus on \( \mathbb{R} \)-valued statistics for simplicity, deferring the proof of the next proposition to Appendix A.4.

**Proposition 2.2.** Let \( L(s, t) = \ell(|s - t|) \) for a non-decreasing function \( \ell : \mathbb{R}_+ \to \mathbb{R}_+ \). Let \( f : \mathcal{X}^n \to \mathbb{R} \) and \( x \in \mathcal{X}^n \). Assume the mechanism \( M \) is \( \varepsilon \)-DP and for some \( \gamma \leq \frac{1}{2\varepsilon} \) achieves

\[
\mathbb{E} \left[ \ell(\| M(x) - f(x) \|) \right] \leq \gamma \ell \left( \omega_f(x; 1/\varepsilon) / 2 \right).
\]

Then there exists a sample \( x' \in \mathcal{X}^n \) with \( d_{\text{ham}}(x, x') \leq \frac{\log(1/2\gamma)}{2\varepsilon} \) such that

\[
\mathbb{E} \left[ \ell(\| M(x') - f(x') \|) \right] \geq \frac{1}{4} \ell \left( \frac{1}{4} \omega_f \left( x'; \frac{\log(1/2\gamma)}{2\varepsilon} \right) \right).
\]

Roughly, this result says that if any mechanism achieves expected loss better than the local minimax rate \( R \), it has a nearby \( x' \) such that the expected loss is quantitatively higher—the local modulus at distance \( 1/\varepsilon \). Thus, any method achieving expected error better than our local minimax rate \( R \) at any point \( x \) must necessarily be much worse at other points. As one example, we have the following corollary, which assumes that the modulus \( \omega_f \) grows no faster than exponentially, but captures the behavior we expect.

**Corollary 2.1.** Let \( \gamma \leq \frac{1}{2\varepsilon} \). Let \( f : \mathcal{X}^n \to \mathbb{R} \) and \( x \in \mathcal{X}^n \) be such that for any \( x' \in \mathcal{X}^n \) satisfying \( d_{\text{ham}}(x', x) \leq \frac{\log(1/2\gamma)}{2\varepsilon} \), we have \( \omega_f(x', k) \leq \omega_f(x', k_0) e^{k/k_0} \) for \( k \geq k_0 = \frac{2}{\varepsilon} \). Let \( M_\varepsilon \) denote the collection of \( \varepsilon \)-differentially private mechanisms and \( \varepsilon(\gamma) := 4\varepsilon / \log \frac{1}{2\gamma} \). Then if \( M \) is \( \varepsilon \)-differentially private, whenever

\[
\mathbb{E} \left[ \| M(x) - f(x) \| \right] \leq \gamma \cdot R(x; | \cdot |, M_\varepsilon),
\]

there exists a sample \( x' \in \mathcal{X}^n \) with \( d_{\text{ham}}(x, x') \leq \frac{\log(1/(2\varepsilon^2\gamma))}{2\varepsilon} \) such that

\[
\mathbb{E} \left[ \| M(x') - f(x') \| \right] \geq \frac{1}{16} \cdot R \left( x', | \cdot |, M_\varepsilon(\gamma) \right).
\]

An improvement over the local minimax rate \( R \) at one \( x \) thus implies that the expected loss at some other sample \( x' \) is at least the local minimax rate over the family of mechanisms satisfying \( \varepsilon(\gamma) = 4\varepsilon / \log \frac{1}{2\gamma} \)-differential privacy, which is a stronger privacy guarantee. We note in passing that similar super-efficiency lower bounds hold for \((\varepsilon, \delta)\)- and Rényi-differential privacy as well, but the calculations are tedious.

**Proof** We upper bound the local minimax risk \( R \). By Theorem 2 we have for \( k_0 = \frac{2}{\varepsilon} \) that

\[
R(x', | \cdot |, M_\varepsilon) \leq \max_k e^{-k\varepsilon/2} \omega_f(x'; k) \leq \max_k e^{-k\varepsilon/2} \omega_f(x'; k_0) e^{k/k_0} = \omega_f \left( x'; \frac{2}{\varepsilon} \right) \leq e^{2\varepsilon} \omega_f \left( x'; \frac{1}{\varepsilon} \right)
\]

by the assumptions of the corollary, whenever \( x' \) is close to \( x \). Proposition 2.2 thus implies that there exists \( x' \) such that

\[
\mathbb{E}[\| M(x') - f(x') \|] \geq \frac{1}{16} \omega_f \left( x'; \frac{-\log(-2e^2\gamma)}{2\varepsilon} \right) \geq \frac{1}{16} R(x', | \cdot |, M_\varepsilon(\gamma)),
\]

as desired. \qed
3 The inverse-sensitivity mechanism

We now turn to develop the inverse-sensitivity mechanism in both discrete (§ 3.1) and general (§ 3.2) cases, showing that the mechanism (nearly) achieves the instance-dependent lower bounds of the previous section. In Section 4 to come, we perform a more careful investigation for a set of natural functions $f$ of interest, keeping the development here general.

3.1 Instance-optimality in the discrete case

We begin by considering functions $f : X^n \to T$ for $|T| < \infty$. The inverse sensitivity mechanism (M.1) instantiates the exponential mechanism [30], which we re-display here:

$$P(M_{\text{disc}}(x) = t) = \frac{e^{-\text{len}_f(x;t)\epsilon/2}}{\sum_{s \in T} e^{-\text{len}_f(x;s)\epsilon/2}}. \quad \text{(M.1)}$$

Privacy guarantees for the inverse sensitivity mechanism (M.1) are nearly immediate from those for the exponential mechanism.

Lemma 3.1 (McSherry and Talwar [30], Theorem 6). Let $h : X^n \times T \to \mathbb{R}_+$ be 1-Lipschitz with respect to the Hamming distance on $X$, $|h(x,t) - h(x',t)| \leq 1$ for any $t \in T$ and neighboring instances $x, x' \in X^n$. Let $\mu$ be any measure on $T$. Then the mechanism $M$ with density

$$\frac{d\pi}{d\mu}(t) = \frac{e^{-h(x,t)\epsilon/2}}{\int_T e^{-h(x,s)\epsilon/2} d\mu(s)}$$

is $\epsilon$-differentially private.

As an immediate consequence of Lemma 3.1, mechanism (M.1) is private (see Appendix B.1).

Lemma 3.2. The mechanism (M.1) is $\epsilon$-DP. Moreover, if $f$ is binary then it is $\epsilon/2$-DP.

An immediate bound on the 0-1 loss follows from definition (M.1):

Proposition 3.1. Let $f : X^n \to T$ where $|T| < \infty$. Then the mechanism (M.1) has

$$P(M_{\text{disc}}(x) = f(x)) = \frac{1}{\sum_{t \in T} e^{-\text{len}_f(x;t)\epsilon/2}}.$$

Combining Proposition 3.1 and the lower bound of Proposition 2.1 implies that the inverse sensitivity mechanism is nearly instance optimal for the 0-1 loss.

Corollary 3.1. For the 0-1 loss $\ell_{0,1}$, the mechanism $M_{\text{disc}}$ is 4-optimal against $\ell_{0,1}$-unbiased mechanisms (Definition 1.5) for any discrete function $f : X^n \to T$.

We can extend this analysis to general losses applied to distances in the target space $T$. Let $(T, d_T)$ be a finite metric space, and let $d_T^* = \max_{s,t \in T} d_T(s, t)$. The following theorem upper bounds the loss of the inverse sensitivity mechanism (we defer proof to Appendix B.2).

Theorem 3 (Discrete functions: upper bound for general loss). Let $L(s,t) = \ell(d_T(s,t))$ for a non-decreasing function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : X^n \to T$. Then for any $x \in X^n$ and $\gamma > 0$,

$$\mathbb{E}[L(M_{\text{disc}}(x), f(x))] \leq \ell\left(\omega_f\left(x; \frac{2}{\gamma} \log \frac{2\ell(d_T^*) \text{card}(T)}{\gamma\epsilon}\right)\right) + \gamma.$$

Let $\tilde{M}_{\text{disc}}$ be the discrete mechanism (M.1) except that we replace the privacy parameter $\epsilon$ with $\tilde{\epsilon} = 2\epsilon \log \frac{2\ell(d_T^*) \text{card}(T)}{\gamma\epsilon}$. Then $\mathbb{E}[L(\tilde{M}_{\text{disc}}(x), f(x))] \leq \ell(\omega_f(x; \frac{1}{\tilde{\epsilon}})) + \gamma$ for all $x \in X^n$. 
Theorem 3 and the lower bounds of Theorems 1 and 2 imply the following corollary.

**Corollary 3.2.** Let the conditions of Theorem 3 hold, and let \( \varepsilon > 0 \) and \( \sup_{s,t \in \mathcal{T}} L(s,t) \leq L^* \). Define \( C = 2 \log \frac{2L^* \text{card}(\mathcal{T})}{\omega_f(x_0, \frac{1}{\varepsilon})} \). Then the discrete mechanism \( M_{\text{disc}} \) (M.1) is both local minimax \( C \)-optimal (Def. 1.3) and \( C \)-optimal against unbiased mechanisms (Def. 1.5).

### 3.2 The inverse sensitivity mechanism for arbitrary-valued functions and near optimality

The mechanisms we have thus far developed focus on the discrete case, so we now show how to extend the ideas to functions taking values in an arbitrary measurable space \( \mathcal{T} \). For a function \( f : X^n \rightarrow \mathcal{T} \), the direct approach is to discretize the range of \( f \), then apply our mechanism for discrete functions. Aside from aesthetic inelegance, this mechanism may result in unsatisfactory running time. A natural alternative is to replace sums with integrals in our discrete mechanism, so that for a base measure \( \mu \) on \( \mathcal{T} \), we define \( M(x) \) to have \( \mu \)-density

\[
\pi_M(x)(t) = \frac{e^{-\text{len}_f(x; t)/2}}{\int_{\mathcal{T}} e^{-\text{len}_f(x; s)/2} d\mu(s)}.
\]

Subtleties arise with this direct generalization that do not in the discrete case. The complication is that \( \text{len}_f(x; t) = 0 \) only for \( t = f(x) \), so that for stable \( f \) the denominator may be small except at the point \( f(x) \), yielding large probabilities for \( t \) distant from \( f(x) \). To circumvent this difficulty, we sometimes work with a smoother version of \( \text{len}_f(x; t) \).

Restricting the generality of \( \mathcal{T} \) as an arbitrary measure space a bit, let \( \mathcal{T} \) be a vector space with norm \( \|\cdot\| \). Then for \( \rho > 0 \), we define the \( \rho \)-smooth version of the inverse sensitivity

\[
\text{len}_f^\rho(x; t) = \inf_{s \in \mathcal{T}: \|s - t\| \leq \rho} \text{len}_f(x; s). \tag{8}
\]

For \( \mathcal{T} \)-valued functions, the inverse sensitivity mechanism \( M_{\text{cont}}(x) \) instantiates the exponential mechanism [30] with \( \text{len}_f^\rho \) via the \( \mu \)-density

\[
\pi_{M_{\text{cont}}(x)}(t) = \frac{e^{-\text{len}_f^\rho(x; t)/2}}{\int_{\mathcal{T}} e^{-\text{len}_f^\rho(x; s)/2} d\mu(s)}. \tag{M.2}
\]

The value of \( \rho \) must be small to achieve satisfactory utility yet large enough to accumulate sufficient weight in the denominator. In most statistical applications, estimators converge at rate \( 1/\sqrt{n} \), so a typical rule-of-thumb is to take \( \rho = 1/\text{poly}(n) \ll 1/\sqrt{n} \). Conveniently, it is easy to see that the general mechanism \( M_{\text{cont}} \) is \( \varepsilon \)-DP.

**Proposition 3.2.** The mechanism \( M_{\text{cont}} \) (M.2) is \( \varepsilon \)-differentially private.

**Proof** Lemma 3.1 shows that it is enough to prove that \( \text{len}_f^\rho(x; t) \) is 1-Lipschitz with respect to Hamming distance. Let \( x, x' \) be neighboring datasets, \( t \in \mathcal{T} \), and assume w.l.o.g. that \( l = \text{len}_f^\rho(x; t) \leq \text{len}_f^\rho(x'; t) \). If \( l = \text{len}_f(x; t) \) then \( \text{len}_f^\rho(x'; t) \leq \text{len}_f(x'; t) \leq \text{len}_f(x; t) + 1 = l + 1 \) since \( \text{len}_f \) is 1-Lipschitz. Otherwise, there exists \( s \) such that \( \|t - s\| \leq \rho \) and \( l = \text{len}_f(x; s) \). It follows that \( \text{len}_f^\rho(x'; t) \leq \text{len}_f(x'; s) \leq \text{len}_f(x; s) + 1 = l + 1 \) as desired. \( \square \)

We turn to demonstrate near instance-optimality for general losses and functions \( f \); in the next section, we provide stronger results when \( f \) is reasonable in a sense we make precise
For instance, if \( f : \mathcal{X}^n \rightarrow \mathbb{R} \) is continuous. The main result of this section shows that using a logarithmically larger \( \varepsilon \), mechanism (M.2) achieves optimal error to constant factors. For simplicity, we state our results for the case that \( \mu \) is the Lebesgue measure and \( \mathcal{T} = \mathbb{R} \), though our proofs generalize this a bit. We assume the loss \( L(s, t) = \ell(|s - t|) \) for a non-decreasing function \( \ell : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) where \( \ell(u + v) \leq \ell(u) + C\ell v \) for all \( u, v \) and some \( C\ell < \infty \).\(^{1}\) Under these assumptions and recalling the local modulus of continuity (3), we have the following upper bound, which we prove in Appendix B.3.

**Theorem 4.** Let \( f : \mathcal{X}^n \rightarrow \mathbb{R} \) and assume that the uniformity condition (24) holds. Then for any \( x \in \mathcal{X}^n \),

\[
\mathbb{E}[L(M_{\text{cont}}(x), f(x))] \leq \ell \left( \omega_f(x; \frac{2}{\varepsilon} \left[ \log \frac{1}{\varepsilon} + 2 \log \frac{n\text{GS}_f}{\rho} \right] ) \right) + O(1)C\ell\rho.
\]

Let \( \widetilde{M}_{\text{cont}} \) be the mechanism (M.2) with privacy parameter \( \overline{\varepsilon} = 2\varepsilon[\log \frac{1}{2\varepsilon} + 2 \log \frac{n\text{GS}_f}{\rho}] \). Then

\[
\mathbb{E}[L(\widetilde{M}_{\text{cont}}(x), f(x))] \leq \omega_f(x; \frac{2}{\varepsilon}).
\]

Theorem 4 and the lower bounds of Theorems 1 and 2, with their evident dependence on the modulus \( \omega_f \), demonstrate the near instance optimality of mechanism (M.2) whenever \( \rho \) is small. We summarize this with the following parallel to Corollary 3.2.

**Corollary 3.3.** Let \( f : \mathcal{X}^n \rightarrow \mathbb{R} \), \( \varepsilon > 0 \), \( p \in \mathbb{N} \) and \( \rho = n^{-p} \). Then the mechanism (M.2) is \( C = 2(\log \frac{1}{\varepsilon}) + 2 \log(n^{1+p}\text{GS}_f) \)-optimal to within an additive error \( n^{-p} \) both in a local minimax sense (Def. 1.3) and against unbiased mechanisms (Def. 1.5).

Of course, \( C = O(\log n) \) optimality may be unsatisfying: the privacy parameter \( \varepsilon = \log n \) generally provides limited privacy protections, and collecting a dataset \( \overline{x} \) of size \( n\log n \) and downsampling, as we discuss following Def. 1.3, may be infeasible. In some realms of theoretical analysis an increase in sample complexity by a factor \( \log n \) is benign, but—as we do in the coming section—we will aim for better.

### 4 Instance-optimality for sample-monotone functions

While the previous sections prove that the inverse sensitivity mechanism with privacy parameter \( \overline{\varepsilon} = O(\log n) \cdot \varepsilon \) is instance-optimal against \( \varepsilon \)-DP (or \( (\varepsilon, \delta) \)-DP or Rényi-\( \delta \)-DP) mechanisms, it is important to understand when \( O(1) \)-optimality is possible. To that end, here we prove stronger results for what we call sample-monotone functions, a natural class including continuous functions over convex domains. In Section 4.1 we show that for “most” instances and well-behaved functions, the inverse sensitivity mechanism is \( O(1) \)-optimal. We also show how allowing a degradation in the local minimax rate \( \mathcal{R}(x) \) allows near-optimality: the inversesensitivity mechanisms (M.1) and (M.2) with privacy parameter \( \varepsilon(\log \frac{1}{\varepsilon} + \log \log n) \) achieve, for any \( \tau > 0 \), expected error \( n^{\tau}\mathcal{R}(x) \) (see Section 4.2). We conclude the section with comparisons in Section 4.3, where we show that the inverse sensitivity mechanism uniformly outperforms the Laplace and smooth Laplace mechanisms.

Our function class consists of functions for which changing the function value \( f(x) \) significantly requires changing more elements of \( x \) than does changing \( f(x) \) only slightly.

**Definition 4.1.** Let \( f : \mathcal{X}^n \rightarrow \mathbb{R} \). Then \( f \) is sample-monotone if for every \( x \in \mathcal{X}^n \), and \( s, t \in \mathbb{R} \) satisfying \( f(x) \leq s \leq t \) or \( t \leq s \leq f(x) \), we have \( \text{len}_f(x; s) \leq \text{len}_f(x; t) \).

\(^{1}\)If \( f \) is bounded, this need hold only over the range of \( f \).
Many estimands, including the sample mean and median, are sample-monotone. More, any continuous function over a convex domain is sample monotone; see Appendix C.1 for a proof.

**Observation 4.1.** Let \( f : \mathcal{X}^n \to \mathbb{R} \) be continuous and \( \mathcal{X} \) be convex. Then \( f \) is sample-monotone.

As one consequence of Observation 4.1, partial minimization of regularized convex losses is sample monotone. For example, if we wish to find the multiplier \( \theta \in \mathbb{R} \) on a feature \( j \) in a linear model, then we generically wish to estimate

\[
\theta(x) = \arg\min_{\theta \in \mathbb{R}} \inf_{\beta \in \mathbb{R}^d} \left\{ L(\theta, \beta; x) := \sum_{i=1}^n \ell(\theta, \beta; x_i) + \frac{\lambda}{2} \|\theta, \beta\|_2^2 \right\}.
\]

for some loss \( \ell(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^d \times \mathcal{X} \to \mathbb{R} \), which is continuous in each of its arguments. The continuity of the minimizer \( f(x) \) is immediate from standard stability guarantees [8].

### 4.1 Typical instance optimality

We focus here on optimality results that hold for well-behaved instances \( x \). Key to our upper bounds is the expected modulus of continuity of \( f \) for instance \( x \) at a random distance,

\[
W_f(x; \varepsilon) := \mathbb{E}_{K \sim \text{Geo}(1-\varepsilon)} \left[ \omega_f(x; K)1\{K \leq n\} \right],
\]

where \( K \sim \text{Geo}(\lambda) \) denotes a geometric random variable, \( \mathbb{P}(K = i) = (1 - \lambda)^i \lambda, i \in \mathbb{N} \). Our optimality results depend on the sensitivity of the modulus of continuity, motivating the ratio

\[
R_f(x) := \frac{W_f(x; \varepsilon/4)}{W_f(x; \varepsilon/2)}.
\]

The main result of this section shows that the inverse sensitivity mechanism is \( O(1) \)-optimal for \( x \) whenever

\[
R_f(x) \lesssim 1,
\]

and we have the following theorem, whose proof we defer to Appendix C.2.

**Theorem 5.** Let \( f : \mathcal{X}^n \to \mathbb{R} \) be sample-monotone, \( p \in \mathbb{N} \), and \( \rho > 0 \). The mechanism \( M_{\text{cont}} \) (M.2) satisfies

\[
\mathbb{E} \left| M_{\text{cont}}(x) - f(x) \right|^p \leq 2^{p+1} R_f(x) \max_{1 \leq k \leq n} \omega_f(x; k) P e^{-k\varepsilon/4} + \gamma,
\]

where \( \gamma = 2p \rho + \frac{2\omega_f(x; n) P^{p+1} e^{-(n+1)\varepsilon/2}}{\rho + \sum_{k=1}^n \omega_f(x; k) e^{-k\varepsilon/2}}. \)

Recalling Theorems 1 and 2, Theorem 5 implies that the inverse sensitivity mechanism is \( O(1) \)-optimal to within small additive factors—in both local-minimax (Def. 1.3) and unbiased (Def. 1.5) senses—whenever the modulus is smooth enough that \( R_f(x) \lesssim 1 \). One consequence of Theorem 5 is that as long as \( f : \mathcal{X}^n \to \mathbb{R} \) is sample monotone and grows at most exponentially in the sample distance \( k \), then the inverse sensitivity mechanism (M.2) is \( O(1) \)-optimal. We prove the following corollary in Appendix C.3.

**Corollary 4.1.** Let \( f : \mathcal{X}^n \to \mathbb{R} \) and \( \varepsilon \lesssim 1 \). If \( \frac{\omega_f(x; k)}{\omega_f(x; t)} \lesssim e^{Ck/t} \) for \( \frac{1}{2} \leq t \leq k \) and \( C < \infty \) then \( R_f(x) \lesssim 1 \). In particular, \( \mathbb{E} \left| M_{\text{cont}}(x) - f(x) \right| \lesssim \omega_f(x; \frac{4C}{\varepsilon}) + e^{-n\varepsilon/4}. \)
Additionally, combined with bounds that $R_f(x) \lesssim 1$ for well-behaved functions $f$ and instances $x$, Theorem 5 demonstrates good behavior of the inverse sensitivity mechanism (M.2). For example, when $f$ is the median and $x_i \overset{\text{iid}}{\sim} P$ for a distribution $P$ with density near its median, $R_f(x) \lesssim 1$ with high probability (the proof of this is tedious but similar to our derivations for expected loss of the median in Sec. 5.1). A more sophisticated example is partial minimization (9) of smooth convex losses. We consider a setting with losses $\sum_{i=1}^{n} \ell(\tau; x_i)$, and we wish to estimate the parameter $\theta(x)$ as in Eq. (9) or a similar problem. For shorthand, let $\tau = (\theta, \beta) \in \mathcal{T} \subset \mathbb{R}^{1+d}$, and we consider the following conditions on the loss $\ell$. For notational simplicity, we let $\ell$ and $\ell'$ denote the gradient and Hessian of $\ell$.

**Assumption A.** The losses $\ell : \mathbb{R} \times \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}$ have $G_1$-Lipschitz continuous gradient and $G_2$-Lipschitz continuous Hessian. Additionally, for each $\tau \in \mathcal{T} \subset \mathbb{R} \times \mathbb{R}^d$, the image of their derivatives satisfies

$$G_{0,\text{in}} \mathbb{B} \subset \{\ell(\tau; x)\}_{x \in \mathcal{X}} \subset G_{0,\text{out}} \mathbb{B}.$$  

Assumption A imposes conditions on the losses $\ell$ and data $x$ that are often natural. The final condition on $\ell(\tau; x)$ simply means that $\ell$ is $G_{0,\text{out}}$-Lipschitz with respect to the $\ell_2$-norm and that there exist examples that can move the loss in many directions. For example, for robust regression problems, satisfying Assumption A is nearly immediate.

**Example 1** (Robust regression): Assume the data are of the form $(x, y) \in \mathbb{R}^d \times \mathbb{R}$, where $\|x\|_2 \leq Bx$. We let $h : \mathbb{R} \to \mathbb{R}_+$ be convex, symmetric, and 1-Lipschitz, satisfying $\sup_{t \in \mathbb{R}} |h'(t)| = 1$, and consider the robust regression losses

$$\ell(\tau; x, y) = h(\langle \tau, x \rangle - y);$$

this includes the “$\alpha$-insensitive” losses [13] with $h_\alpha(t) = \alpha \log(1 + e^{t/\alpha}) + \alpha \log(1 + e^{-t/\alpha})$ or the Huber loss [27]. Consider the $\alpha$-insensitive loss for concreteness. In this case, the $\alpha$-insensitive loss for concreteness. In this case, when the variables $y \in \mathbb{R}$ may be arbitrary (natural for robust regression) and $\|x\|_2 \leq r$, we have the equality

$$\{\ell(\tau; x, y)\}_{\|x\|_2 \leq r, y \in \mathbb{R}} = r \mathbb{B}_{2}^{d+1},$$

so that $G_{0,\text{in}} = G_{0,\text{out}}$ in Assumption A. Defining $p(\tau; x, y) = 1/(1 + \exp((y - \langle \tau, x \rangle)/\alpha))$, the losses satisfy $\ell(\tau; x, y) = \alpha^{-1} p(\tau; x, y)(1 - p(\tau; x, y)) xx^T \preceq (4 \alpha)^{-1} xx^T$, so that $\ell$ has Lipschitz gradient and Hessian with $G_1 \leq (4 \alpha)^{-1} r^2$ and $G_2 \leq r^3/\alpha^2$. 

We then have the following example, which shows that under appropriate conditions on the empirical loss $L_n(\tau) = n^{-1} \sum_{i=1}^{n} \ell(\tau; x_i)$ we can almost exactly compute the modulus of continuity of the partial minimizer $\theta(x)$. This is not surprising, as standard statistical regularity conditions show the minimizer $\theta(x)$ should be (asymptotically) linear in the sample [41].

**Example 2** (Smoothness of partial minimization): Let $\mathcal{T} \subset \mathbb{R}^{1+d}$ be a closed convex set, and for losses $\ell$ satisfying Assumption A, consider the partial minimization problem

$$\theta(x) := \arg\min_{\theta} \inf_{\beta : (\theta, \beta) \in \mathcal{T}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \ell(\theta, \beta; x_i) \right\}. $$

Fix a number $k \in \mathbb{N}$ of examples to change. For shorthand, let $\tau = \tau(x) = \arg\min_{\tau \in \mathcal{T}} L_n(\tau)$, and let us assume both the local strong convexity condition that $\tilde{L}_n(\tau) \succeq \lambda I$ and that $\tau$ is interior to $\mathcal{T}$, satisfying $\text{dist}(\tau, \text{bd} \mathcal{T}) \geq \frac{6G_{0,\text{out}}}{\lambda} \frac{k}{n}$. Then we claim both that

$$\omega_{\theta}(x; k) = \left(1 + \frac{24G_{0,\text{out}} G_1 G_2 k}{G_{0,\text{in}} \lambda^3 n} \right) \sup_{d_{\text{ham}}(x, x') \leq k} \left[ \frac{1}{n} \sum_{i=1}^{n} \tilde{L}_n(\tau)^{-1}(\ell(\tau; x_i) - \ell(\tau; x'_i)) \right]_1 \quad (11)$$

13
and for the matrix $A = \tilde{L}_n(\tau)^{-1}$ having first column $a \in \mathbb{R}^{1+d}$, we also have

$$
\frac{k}{n} \|a\|_2 \cdot G_{0,\text{in}} \leq \sup_{d_{\text{ham}}(x',x) \leq k} \left[ \frac{1}{n} \sum_{i=1}^n \tilde{L}_n(\tau)^{-1}(\ell(\tau;x_i) - \ell(\tau;x'_i)) \right] \leq \frac{2k}{n} \|a\|_2 \cdot G_{0,\text{out}}. \tag{12}
$$

We provide proofs of both inequalities in Appendix D.1.

The essential observation here is that so long as the privacy parameter $\varepsilon$ and empirical minimizer $\tau = \tau(x)$ satisfy $\text{dist}(\tau, \text{bd } T) \gg \frac{G_{0,\text{out}}}{\lambda_m}$, then it is immediate the ratio (10) satisfies $R_\theta(x) \lesssim \frac{G_{0,\text{out}}}{G_{0,\text{in}}}$. Theorem 5 then implies that whenever the losses are regular enough that $G_{0,\text{out}}/G_{0,\text{in}} \lesssim 1$, for example, in the case of robust regression (Ex. 1), the continuous inverse sensitivity mechanism (M.2) with smoothing parameter $\rho = n^{-2}$ attains expected loss

$$
\mathbb{E}[\|M_{\text{cont}}(x) - \theta(x)\|^p] \leq O(1) \cdot R(x, |\cdot|^p, \varepsilon) + O(n^{-2p}).
$$

That is, it is $O(1)$-optimal. □

We discuss two points in Example 2. The assumption that the empirical loss $L_n$ satisfies the local strong convexity condition $\tilde{L}_n(\tau(x)) \succeq \lambda I$ is natural assuming the data $x_i \overset{\text{iid}}{\sim} P$ from some distribution $P$, and that the population minimizer $\tau^* = \text{argmin}_\tau \{L_\infty(\tau) := \mathbb{E}[L(\tau; x_i)]\}$ satisfies both $\tau^* \in \text{int } T$ and $\tilde{L}_\infty(\tau^*) \succeq 2\lambda I$. In this case, it is standard that the conditions of the example occur with high probability [41, Ch. 5.8]. The combination of Examples 1 and 2 show that in robust regression, with high probability over the drawn sample, the inverse sensitivity mechanism (M.2) is $O(1)$-optimal.

### 4.2 Near and conditional instance optimality

While the optimality results in the previous section hold for well-behaved monotone functions, in this section we provide conditional—when a good unbiased private estimator exists, the inverse sensitivity mechanism (M.2) is good—and near-optimality results for sample monotone functions. In the first result, we provide an asymptotic guarantee that the inverse sensitivity mechanism (M.2) is nearly $O(\log \frac{1}{\varepsilon} + \log \log n)$-optimal, to within a sub-polynomial multiplicative factor on the expected loss.

**Proposition 4.1.** Let $f: \mathcal{X}^n \to \mathbb{R}$ be sample-monotone and assume $\text{GS}_f \leq n$, $n^{-1} \leq \varepsilon \lesssim 1$, $p > 0$. Let $K = \frac{8(p+2)\log n}{\varepsilon}$, $\lambda = \frac{1}{\log K}$. The mechanism (M.2) with $\rho = n^{-p}$ satisfies

$$
\mathbb{E}[\|M_{\text{cont}}(x) - f(x)\|] \leq 2e \exp \left( \frac{4\log n}{\log 2 + \log \log n} \right) \max_{1 \leq k \leq K} e^{-\lambda k \varepsilon/2} \omega_f(x;k) + O(n^{-p}).
$$

See Appendix C.5 for a proof.

Comparing Proposition 4.1 to our upper bound for general functions in Theorem 4, it is clear that if the privacy parameter $\varepsilon$ is small enough that $\varepsilon = n^{-\beta}$ for some $\beta > 0$, both upper bounds achieve a similar result: the inverse sensitivity mechanism is $O(\log \frac{1}{\varepsilon})$-optimal (in either definition 1.3 or 1.5 of optimality). Proposition 4.1 provides a different bound in the important regime where $\varepsilon = \Theta(1)$. Indeed, letting the mechanism $\tilde{M}_{\text{cont}}$ be (M.2) with privacy parameter $\tilde{\varepsilon} = O(\log \log n) \cdot \varepsilon$, Proposition 4.1 implies that for any $\tau > 0$,

$$
\mathbb{E}[\|\tilde{M}_{\text{cont}}(x) - f(x)\|] \leq O(n^\tau) \max_{1 \leq k \leq n} e^{-k \varepsilon} \omega_f(x;k) + O(n^{-p}).
$$
In typical applications with scaling $\omega_f(x;k) = O(k/n)$ (e.g., of the sample mean) or another polynomial, we have local minimax rate $\mathcal{R}(x) \lesssim \omega_f(x;1/\varepsilon) = O((n\varepsilon)^{-1})$, so that $\tilde{M}_{cont}$ achieves nearly optimal scaling.

We conclude this section with a conditional optimality result for the inverse sensitivity mechanism. As Theorem 2 makes clear, the essential quantity in lower bounds on the local minimax risk is the local modulus $\omega_f(x;k_*)$ at sample radius $k_* = \Theta(\varepsilon^{-1})$. The next result shows if any $\ell_1$-unbiased and $\varepsilon/8$-differentially private mechanism achieves this convergence guarantee for the absolute error, then the inverse sensitivity mechanism (M.2) is also $O(1)$-optimal. We provide the proof in Appendix C.4.

**Proposition 4.2.** Let $f : \mathcal{X}^n \to \mathbb{R}$ be sample monotone and $\varepsilon = O(1)$. If there exists an $\ell_1$-unbiased $\varepsilon/16$-differentially private mechanism $\tilde{M}$ such that $\mathbb{E}[|\tilde{M}(x) - f(x)|] \lesssim \omega_f(x;\tilde{k})e^{-k\varepsilon/16}$ for some $\tilde{k} \in \mathbb{N}$, then $R_f(x) \lesssim e^{\tilde{k}\varepsilon/16}$.

Thus, whenever the postulated mechanism achieves $\mathbb{E}[|\tilde{M}(x) - f(x)|] \lesssim \omega_f(x;O(1))$, we may take $\tilde{k} = O(1/\varepsilon)$, and the ratio (10) satisfies $R_f(x) \lesssim 1$. Then Theorem 5 implies that $\mathbb{E}[|M_{cont}(x) - f(x)|] \lesssim \omega_f(x;O(1)) + e^{-3n\varepsilon/8}$ as $\gamma \lesssim e^{-7n\varepsilon/16}$. In particular, $M_{cont}$ is $O(1)$-optimal whenever an $O(1)$-optimal unbiased mechanism exists. Similar results to Proposition 4.2 hold for alternative losses and unbiased mechanisms.

### 4.3 Comparisons and expected loss

While the previous sections show strong adaptivity guarantees of the inverse sensitivity mechanism to the underlying data $x \in \mathcal{X}^n$ and function $f$, it is not immediately clear how these guarantees differ from more standard and classical mechanisms. It is thus instructive to compare the bounds to those available for the Laplace (4) and smoothed Laplace mechanisms (5). Focusing on sample-monotone functions (Def. 4.1), we show that the inverse sensitivity mechanism uniformly outperforms the Laplace and smooth Laplace mechanisms.

We begin with a stylized example and consider quantizing the average of a binary query $q : \mathcal{X} \to \{0,1\}$ into steps of width $T > 0$, where $Te \gg 1$, defining $f_{step}(x) = \lfloor \sum_{i=1}^n q(x_i)/T \rfloor$. For instance $x$ near discontinuities in $f_{step}$, such as $x$ with $\sum_{i=1}^n q(x_i) = T$, the smooth Laplace mechanism (5) adds Laplace noise with variance at least $1/\varepsilon^2$, as $LS(x) \geq 1$. In contrast, as $Te$ is large, the inverse sensitivity mechanism (M.1) returns the value $f_{step}(x)$ or $f_{step}(x) - 1$ with probability near 1. Fig. 1 illustrates this by plotting 90% confidence sets for each mechanism against the value $\sum_{i=1}^n q(x_i)$. The Laplace and smooth-Laplace mechanisms have much larger confidence sets (regions of plausible release) than mechanism (M.1).

We proceed to a theoretical comparison of the three mechanisms. Focusing for simplicity on the absolute loss for a function $f : \mathcal{X}^n \to \mathbb{R}$, we immediately obtain the lower bounds

$$
\mathbb{E}[|M_{Lap}(x) - f(x)|] = \frac{GS_f}{\varepsilon},
$$

$$
\mathbb{E}[|M_{sm-Lap}(x) - f(x)|] \geq \frac{2}{\varepsilon} \max_{x'} \left\{ LS(x') : d_{ham}(x,x') \leq \frac{2}{\varepsilon} \log \frac{2}{\delta} \right\}.
$$

(13)

Theorem 1 states that any $\ell_1$-unbiased mechanism $M$ satisfies $\mathbb{E}[|M(x) - f(x)|] \geq \Omega(1) \cdot \omega_f(x;1/\varepsilon)$, which may be smaller, and the definition (3) of the modulus of continuity gives $\omega_f(x;k) \leq k \max_{x' : d_{ham}(x,x') \leq k} LS(x')$. Thus, applying either of Theorems 3 or 4, we obtain
Corollary 4.2. Assume that \( f : \mathcal{X}^n \to \mathbb{R} \) and \( \text{diam}(f(\mathcal{X}^n)) \leq \text{poly}(n) \). Then for any numerical constant \( b < \infty \) there is a numerical constant \( C = C(b) < \infty \) such that the following holds. Let \( M \) be either of \( M_{\text{disc}} \) or \( M_{\text{cont}} \) with \( \mu \) the Lebesgue measure. Then

\[
\mathbb{E}[|M(x) - f(x)|] \leq \omega_f \left( x, \frac{C \log n}{\varepsilon} \right) + n^{-b} = \mathcal{O} \left( \frac{\log n}{\varepsilon} \max_{x'} \left\{ \text{LS}_f(x') \mid d_{\text{ham}}(x, x') \leq \frac{C \log n}{\varepsilon} \right\} \right) + \mathcal{O}(n^{-b}).
\]

Of course, we always have \( \text{LS}_f \leq \text{GS}_f \), and so Corollary 4.2 guarantees that the inverse sensitivity mechanisms (M.1) and (M.2) can never have expected loss more than a factor of \( \log n \) larger than either the Laplace or smooth-Laplace mechanisms.

We now show that variants of the continuous mechanism (M.2) (as we change the base measure \( \mu \)) have strong accuracy guarantees for sample-monotone functions (Definition 4.1); taking \( \mu \) to be discrete gives the discrete mechanism (M.1) as a special case. We elaborate the setting somewhat, and assume that the range \( \mathcal{T} \subset \mathbb{R} \) is uniformly spaced, meaning that \( \mathcal{T} \) is either \( \mathbb{R} \) or of the form \( \mathcal{T} = \{ k\beta \mid k \in \mathbb{Z} \} \), where \( \beta \in \mathbb{R} \) is a fixed value. With this condition, we have the following proposition, whose proof we defer to Appendix C.6.

**Proposition 4.3.** Let \( f : \mathcal{X}^n \to \mathcal{T} \) be sample-monotone, \( \mathcal{T} \) be uniformly spaced and the base measure \( \mu \) in mechanism (M.2) be uniform on \( \mathcal{T} \), where the smoothing \( \rho \geq 0 \) is arbitrary. Then for any \( x \in \mathcal{X}^n \),

\[
\mathbb{E}[|M(x) - f(x)|] \lesssim \text{GS}_f/\varepsilon.
\]

Let \( 0 \leq \gamma \leq \frac{\rho \varepsilon}{\text{GS}_f} \) and define

\[
L := \max_{x'} \left\{ \text{LS}(x') : d_{\text{ham}}(x, x') \leq \max \left\{ \frac{4}{\varepsilon}, \frac{2}{\varepsilon}, \frac{\log \frac{1}{\gamma} + \log \frac{\text{GS}_f^2}{\rho \varepsilon}}{} \right\} \right\}.
\]

Then

\[
\mathbb{E}[|M(x) - f(x)|] \lesssim \frac{1}{\varepsilon} \cdot \left[ L + \gamma \log \frac{1}{\gamma} \right].
\]
We know from Eq. (13) that $E[M_{\text{Lap}}(x) - f(x)] = \text{GS}_f/\varepsilon$ and $E[M_{\text{sm-Lap}}(x) - f(x)] \geq \Omega(1) \cdot L/\varepsilon$ if $\delta \leq n^{-1}$. So whenever the global sensitivity $\text{GS}_f = \text{poly}(n)$ and $\varepsilon \gtrsim \text{poly}(n)^{-1}$, for sample-monotone functions, the inverse sensitivity mechanisms must outperform the Laplace and smooth Laplace mechanisms for all instances $x \in \mathcal{X}^n$.

5 Methodologies: instantiations and approximations of the inverse sensitivity mechanism

As a second purpose of this paper is to develop practical near-optimal mechanisms, we highlight the methodological possibilities of the inverse sensitivity mechanism here. To that end, we provide two concrete examples: (i) computing the median, where we can efficiently compute the inverse sensitivity and (ii) minimization of robust regression losses. In each example, we show how to calculate the inverse modulus (1) and briefly analyze the procedure. In each case, we recover a mechanism distinct from the traditional exponential mechanism [30].

5.1 Median of a dataset

We begin with the median, where the mechanism is folklore [36, Ex. 3.1], though its consistency properties are not. Given a dataset $x \in \mathbb{R}^n$, we wish to calculate $\text{Median}(x)$. For simplicity we assume $x_i \in [0,R]$ for $R > 0$, though our theory and derivations are completely identical if the data may be unbounded and we redefine $f(x) = \min\{R, \max\{-R, \text{Median}(x)\}\}$. To implement the mechanism (M.2), we calculate $\text{len}_f$:

\textbf{Lemma 5.1.} Let $m = \text{Median}(x)$. Then for $t \in [0,R]$, $\text{len}_f(x; t) = \{x_i : x_i \in (t,m] \cup [m,t)\}$.

\textbf{Proof} If $t = m$, certainly $\text{len}_f(x; t) = 0$. Otherwise, if $t < m$, then to make $t$ a median, we need to replace all the values $x_i$ such that $t < x_i \leq m$. The case $t > m$ follows similarly. \hfill \square

Before stating performance guarantees, we describe a procedure implementing (M.2) in time $O(n \log n)$. Sort the data so that $x_1 \leq \cdots \leq x_n$. For $0 \leq k \leq n$, let $I_k = \{t \in [0,R] : \text{len}_f(x; t) = k\}$, a union of two intervals by Lemma 5.1. As $x$ is sorted, we can find the sets $I_k$ by iterating once over $x$. Then to sample the output of the mechanism, sample an index $k \in [n]$ with probability proportional to $e^{-k \varepsilon/2} |I_k|$ where $|I_k| = \int_{I_k} dt$ is the volume of $I_k$, and return an element uniform in $I_k$.

Let us examine the behavior of mechanism (M.2). Assume $x \in [0,R]^n$ has $x_i \overset{iid}{\sim} P$, and we make the standard assumption [41, Ch. 21] that $P$ has a continuous density $\pi_P$ near $m := \text{Median}(P)$. Then the mechanism $M_{\text{cont}}$ returns an accurate estimate with exponentially high probability, as the following proposition shows. (See Appendix D.2 for a proof.)

\textbf{Proposition 5.1.} Let $\gamma > 0$, $0 \leq u \leq \gamma/4$, and $p_{\text{min}} = \inf_{|t-m| \leq 2\gamma} \pi_P(t)$. Let $\hat{m}_n := \text{Median}(x)$. Under the above conditions, the mechanism $M_{\text{cont}}$ with smoothing parameter $\rho$ satisfies

$$
\mathbb{P}(|M_{\text{cont}}(x) - \hat{m}_n| > 2u + \rho) \leq \frac{R}{\rho} \exp\left(-\frac{np_{\text{min}}u\varepsilon}{4}\right) + 4\exp\left(-\frac{n\gamma^2p_{\text{min}}^2}{4}\right) + 2\gamma \exp\left(-\frac{np_{\text{min}}u}{8}\right),
$$

where the probability is jointly over $x_i \overset{iid}{\sim} P$ and the randomness in $M_{\text{cont}}$. 17
As an alternative to understanding the (conditional on $x$) instance optimality of a mechanism, as Defs. 1.3 and 1.5 identify, we may instead consider the rate at which we can take the privacy parameter $\varepsilon = \varepsilon_n \to 0$ as the sample size $n$ grows while maintaining optimal statistical convergence. To understand this, recall that the empirical median $\hat{m}_n := \text{Median}(x)$ satisfies

$$\sqrt{n}(\hat{m}_n - m) \overset{d}{\to} N\left(0, \frac{1}{4\pi P(m)^2}\right),$$

where the rate $\sqrt{n}$ and variance $1/4\pi P(m)^2$ are optimal [41, Chs. 21, 25.3]. In this case, the inverse sensitivity mechanism (M.2) can achieve the asymptotics (15) whenever $\varepsilon \gg \log n/\sqrt{n}$. Indeed, in Proposition 5.1, take the triple $u, \varepsilon, \gamma$ such that $\gamma = n^{-1/4}$, $\log n/n \ll u \ll 1/\sqrt{n}$, $\rho = 1/n$, and $u \varepsilon \gg \log n/n$. Then the proposition guarantees $\sqrt{n}(M_{\text{cont}}(x) - \hat{m}_n) \overset{p}{\to} 0$, and Slutsky’s lemmas [41, Ch. 2.8] yield

$$\sqrt{n}(M_{\text{cont}}(x) - m) \overset{d}{\to} N\left(0, \frac{1}{4\pi P(m)^2}\right) \quad \text{whenever} \quad \varepsilon \gg \log n/\sqrt{n}.$$ 

In contrast, the following lower bound shows that the smooth Laplace mechanism may achieve this rate only if $\varepsilon \gg n^{-1/4}$, ignoring logarithmic factors, assuming the mechanism is $(\varepsilon, \delta)$ private with $\delta \leq 1/n$. As typically $\delta$ is assumed negligible in $n$ [19], this is no restriction, and we make the problem easier by assuming $P$ has density $\pi_P$ satisfying $p_{\text{min}} \leq \pi_P \leq p_{\text{max}}$.

**Lemma 5.2.** Let the above conditions hold. The smooth Laplace mechanism adds noise $\frac{\alpha}{2} \text{Lap}(1)$ where $\alpha \geq \frac{\log(n)}{2 \pi p_{\text{max}} n \varepsilon}$ with probability at least $q := 1 - 2R n \varepsilon p_{\text{max}} \exp(-\frac{p_{\text{min}} \log n}{10 p_{\text{max}} \varepsilon^2})$, so

$$\mathbb{E}||M_{\text{sm-Lap}}(x) - \text{Median}(x)|| \geq \frac{q}{2 \varepsilon p_{\text{max}}} \frac{\log n}{n \varepsilon^2}.$$ 

See Appendix D.2.2 for a proof. To achieve the asymptotically optimal convergence (15) for the smooth Laplace mechanism we may provide $(\varepsilon, \delta)$-privacy only for $\varepsilon \gg n^{-1/4}$, quadratically worse than the inverse sensitivity mechanism (M.2).

### 5.2 Robust regression problems

We revisit the robust regression problems we identify in Example 1, sketching application of the inverse sensitivity mechanism. As in the example, we have data $(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}$ with $\|x_i\|_2 \leq r$ and loss $\ell(\theta; x_i, y_i) = h((\theta, x_i) - y_i)$ where $h : \mathbb{R} \to \mathbb{R}_+$ is convex, symmetric and 1-Lipschitz. Rather than partial minimization as in Ex. 2, we consider the minimizer

$$\hat{\theta}_n(x, y) = \arg\min_{\theta \in \Theta} L_n(\theta; x, y) := \frac{1}{n} \sum_{i=1}^{n} \ell(\theta; x_i, y_i).$$

For simplicity in the calculations, we consider a slight tweak to our privacy definitions to instead consider addition of a user rather than substitution (a standard alternative [19]). Here, a mechanism is $\varepsilon$-differentially private with respect to user additions if for any two instances $(x, y)$ and $(x', y') = (x, y) \cup \{(x'_{n+1}, y'_{n+1})\}$,

$$\frac{\mathbb{P}(M(x, y) \in S)}{\mathbb{P}(M(x', y') \in S)} \leq e^\varepsilon \quad \text{and} \quad \frac{\mathbb{P}(M(x', y') \in S)}{\mathbb{P}(M(x, y) \in S)} \leq e^\varepsilon.$$
We then define the user-addition length

\[ \text{len}_{\text{add}}(x, y; \theta) = \inf_{x' \in \mathbb{R}^{n \times d}, y' \in \mathbb{R}^n} \{ n' - n : \hat{\theta}_n(x', y') = \theta, x'_i = x_i, y'_i = y_i, 1 \leq i \leq n \}. \quad (16) \]

The inverse sensitivity mechanism for this problem instantiates (M.2) with the user-addition inverse sensitivity (16). The following lemma shows that it is \( \varepsilon \)-private (for addition of users), as \( \text{len}_{\text{add}} \) is 1-Lipschitz with respect to dataset additions.

**Lemma 5.3.** The mechanism (M.2) with \( \text{len}_{\text{add}} \) (16) is \( \varepsilon \)-DP (with respect to user-additions).

The following lemma, proved in Appendix D.3.2, characterizes the inverse sensitivity.

**Lemma 5.4.** Assume \( \|x\|_2 \leq r \) for all \( x \). If \( \hat{\theta}_n(x, y) \in \text{int} \Theta \), then for every \( \theta \in \Theta \),

\[ \text{len}_{\text{add}}(x, y; \theta) = \left\lceil \frac{n \| \nabla L_n(\theta; x, y) \|_2}{r} \right\rceil. \]

Whenever \( \hat{\theta}_n(x, y) \in \text{int} \Theta \), the inverse sensitivity mechanism (M.2) is thus

\[ \pi_{\text{inv}}(\theta) \propto \exp \left( -\frac{n \varepsilon}{2} \left\lceil \frac{n \| \nabla L_n(\theta; x, y) \|_2}{r} \right\rceil \right). \quad (17) \]

No matter the value of \( \hat{\theta}_n \) (even if it is on the boundary \( \text{bd} \Theta \), so that \( \hat{\theta}_n \notin \text{int} \Theta \)), the mechanism (17) is still \( \varepsilon \)-private, as \( \text{bd} \Theta \) has Lebesgue measure 0 (its utility may suffer). The mechanism (17) is distinct from the standard exponential mechanism [30] for empirical risk minimization problems; the exponential mechanism has density

\[ \pi_{\text{exp}}(\theta) \propto \exp \left( -\frac{\varepsilon}{2r^2} L_n(\theta; x, y) \right). \]

For \( \theta = \theta_n + \Delta \) near \( \theta_n = \hat{\theta}_n(x, y) \), if we ignore large \( \Delta \), this is (heuristically) a density

\[ \pi_{\text{exp,heur}}(\theta_n + \Delta) \propto \exp \left( -\frac{\varepsilon}{2r} \langle \nabla L_n(\theta_n; x, y), \Delta \rangle \right), \]

which is similar to but distinct from the inverse mechanism (17).

We proceed with a heuristic analysis of the inverse sensitivity mechanism here, assuming throughout that \( \theta_n := \hat{\theta}_n(x, y) \in \text{int} \Theta \). (As in Example 2, utility crucially relies on this interior assumption.) We assume the losses \( h(\cdot) \) are \( C^2 \), and that the Hessian \( \nabla^2 L_n(\theta_n; x, y) > 0 \). Using the Taylor approximation \( \nabla L_n(\theta_n; x, y) = \nabla^2 L_n(\theta_n; x, y)(\theta - \theta_n) + O(\|\theta - \theta_n\|^2) \), we get that for small \( \Delta \), the density (17) is approximately

\[ \pi(\theta_n + \Delta) \propto \exp \left( -\frac{\varepsilon}{2r} \| \nabla^2 L_n(\theta_n; x, y) \Delta \| \right). \quad (18) \]

We can compute the error of the heuristic mechanism (18). To draw such a distribution, we draw a radius \( R \sim \text{Gamma}(d, 1) \) and \( U \sim \text{Uni}(S^{d-1}) \), then set \( \Delta = \frac{2r}{n \varepsilon^2} \nabla^2 L_n(\theta_n; x, y)^{-1} \cdot R \cdot U \) (see Appendix D.3.3). Our heuristic (18) for mechanism (17) thus achieves

\[ \mathbb{E} \left[ \| \theta - \theta_n \|^2 \right] \leq \left( \frac{r^2 R^2}{n^2 \varepsilon^2} \right) \cdot \mathbb{E} \left[ \| \nabla^2 L_n(\theta_n; x, y)^{-1} U \|^2 \right] = \frac{r^2 (d + 1)}{n^2 \varepsilon^2} \cdot \text{tr} \left( \nabla^2 L_n(\theta_n; x, y)^{-2} \right). \]

While it may be challenging to efficiently sample from the inverse sensitivity mechanism (17), a Metropolis-Hastings (MH) scheme yields an \((\varepsilon, \delta)\)-differentially private mechanism. The MH algorithm samples points \( \theta, t \) as follows, beginning from \( \theta \). Given a transition
kernel \( q(\theta, \cdot) \) satisfying \( q(\theta, A) \geq 0 \) and \( q(\theta, \mathbb{R}^d) = 1 \), we iteratively draw \( T \sim q(\theta, \cdot) \) and accept the move \( T = t \) with probability \( \min \left\{ \frac{\pi(t)}{\pi(\theta)} \frac{q(\theta)}{q(t, \cdot)}, 1 \right\} \), otherwise remaining at \( \theta \). If the proposal \( q \) is independent of the initial point \( \theta \), then we have geometric mixing if \( \frac{q(t)}{\pi(t)} \geq \beta > 0 \) for all \( t \), and \( \|P^n(\theta_0, \cdot) - \pi\|_{TV} \leq (1 - \beta)^n \) under this condition [31, Thm. 2.1].

We now sketch a fast mixing algorithm for sampling from the inverse sensitivity mechanism under a few additional conditions on the losses \( h \). In addition to our previous assumptions, we assume that the losses \( h \) have \( G_1(x) \)-Lipschitz gradient over \( \Theta \). We assume that \( \nabla^2 L_n(\theta_n) \succeq \lambda I \). (It is possible to use the propose-test-release framework [cf. 19, Ch. 7.2] to check that this holds, failing with only a prescribed probability \( \delta \).) Let \( r_n \) be a be a fixed rate satisfying \( n^{-1} \ll r_n \ll n^{-2/3} \). Under these conditions, we have the following lemma, whose proof we sketch in Appendix D.3.4, as the mixing time analysis is not our main focus.

**Lemma 5.5.** Let the conditions above hold, and consider the proposal density

\[
q(\theta) \propto \exp \left( -\frac{n \varepsilon}{2} \left( \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| \wedge r_n \right) \right) 1\{\theta \in \Theta\}.
\]

There exists a numerical constant \( \beta > 0 \) such that \( \frac{q(t)}{\pi(t)} \geq \beta \) for all \( t \in \Theta \).

Assuming that we can compute \( \text{Vol}(\Theta) \), the Lebesgue volume of \( \Theta \), it is possible to efficiently sample from \( q(\theta) \). Indeed, as in Appendix D.3.3, a change of variables allows easy sampling from the density \( f(z) \propto \exp(-\|Az\|) \). Thus, by rejection sampling from the density proportional to \( \exp(-\frac{n \varepsilon}{2} \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\|) \) we can draw exactly from \( q(\theta) \). Running the Metropolis-Hastings algorithm using proposal \( q \) for (say) \( n \) steps then gives an \( (\varepsilon, e^{-\Omega(n)}) \)-differentially private algorithm using [31, Thm. 2.1] and Lemma 5.5.

### 6 Experiments

We conclude with an experimental evaluation of the inverse sensitivity mechanisms for our examples in Section 5. Our theory suggests that the inverse sensitivity mechanism should provide a competitive utility against any private mechanism, especially for the high-privacy regime where \( \varepsilon \ll 1 \), and indeed, the inverse sensitivity mechanism demonstrates strong utility in our two experiments.

![Figure 2. The accuracy of each mechanism \(|M(x) - \text{Median}(x)|\) as a function of the privacy parameter \( \varepsilon \) with 0.9 confidence intervals on the UC salary dataset.](https://example.com/figure2.png)

#### 6.1 Median of a dataset

We begin our experiments with the median example of Section 5.1, where we aim to evaluate the performance of the inverse sensitivity mechanism (M.2) (setting \( K = 10^7 \) and \( \rho = 1/n \))
against the smooth Laplace mechanism (5) (setting the standard value δ = n−1.1) for estimating the median of a dataset. Our experiments use a publicly available dataset consisting of the salaries of all employees in the University of California system. We run each method 50 times and report the median of the absolute loss |M(x) − Median(x)|, with 90% confidence intervals across all experiments. Fig. 2 shows the results of each mechanism as a function of the privacy parameter ε. The plot—as expected—shows a 2–3 order of magnitude improvement in error of the inverse sensitivity mechanism (M.2) over the smooth Laplace mechanism (which additionally only guarantees (ε, δ)-differential privacy).

6.2 Robust regression and empirical risk minimization

In our final experiment, we investigate the robust regression problem of Section 5.2. For a fixed θ∗, the data follows the distribution y = xθ + ε, where x ∼ Unif{−5, 5}, and in each repetition of the experiment we draw θ ∼ Unif{−5, 5}. Following our notation from Example 1, we consider losses ℓθ(x; y) = hα(θx − y) for hα(t) := α log(1 + e^{t/α}) + α log(1 + e^{−t/α}), varying the α parameter as well to induce more smoothness (α large) or less (α small), so that h(t) ≈ |t|.

We compare two algorithms in this experiment. The first is specialization (17) of the inverse sensitivity mechanism (M.2), which we implement by running 500 steps of Section 5.2’s Metropolis-Hastings procedure. We also consider private Stochastic Gradient Descent with a moments accountant [1], which achieves state-of-the-art performance. Briefly, at each iteration, private SGD subsamples a set S ⊂ [n] of users with probability q, then sets

\[ \theta_{t+1} = \theta_t - \frac{\eta_t}{|S|} \left( \sum_{i \in S} \nabla \ell_{\theta}(\theta_t; x_i, y_i) + N(0, \sigma^2 G_{0, \text{out}}) \right), \]  

where \( \eta_t \) is a stepsize rate, \( \sigma \) is a noise parameter, and \( G_{0, \text{out}} \) a bound on the \( \ell_2 \)-norm of the gradient, which in this case is exactly \( \max_i |x_i| \).

The private SGD procedure requires several parameters, and we attempt to optimize them. We vary \( q \in \{0.004, 0.016, 0.064\} \) and set \( \sigma = 2 \); for a desired privacy level ε, we then calculate the maximum number of iterations T that the (computational) moment-accounting technique [1] allows for the given \( q, \sigma \). We used a stepsize \( \eta_t = \eta_0 / \sqrt{t} \) for each step; as SGD is extremely sensitive to the choice of \( \eta_0 \) even in the non-private setting [2], we vary \( \eta_0 \in \{0.05, 1, 3, 1, 3, 10\} \). We run each method 30 times, where to most advantage the private
stochastic gradient algorithm (19), we choose the value of \(\eta_0\) for each distinct privacy level \(\varepsilon\) that yields the best convergence. (The mechanism (19) is sensitive to these values.)

In Figure 3, we report our results. Each plot displays the median \(\ell_1\)-loss \(|M(x, y) - \theta^*|\) with 95\% coverage over all experiments versus the attained privacy level \(\varepsilon\), as \(\varepsilon\) varies from \(10^{-3}\) to 1, and plots (a), (b), and (c) correspond (respectively) to the choice \(\alpha = .5, 1, 4\) in the loss \(\ell_\alpha\). Each plot makes clear that the inverse sensitivity mechanism achieves much better convergence than private SGD (19), which (because of the guarantees the moments-accountant gives) cannot provide privacy for \(\varepsilon \leq .1\) or so. We see, roughly, that there are several orders of magnitude difference in the losses of the inverse sensitivity mechanism (M.2) and the SGD procedure, except when \(\varepsilon\) is (perhaps unacceptably) large. We hope this lends credence to the desiderata we have tried to highlight in this paper, that one should attempt to be optimal for the problem at hand.

A Proofs of lower bounds (Section 2)

A.1 Proof of Proposition 2.1

We begin with a useful lemma formalizing the intuition that if a mechanism \(M\) is unbiased and returns the correct answer \(f(x)\) with high probability, it must as well return \(t\) with high probability—depending on \(\text{len}_f(x; t)\)—to preserve differential privacy.

**Lemma A.1.** Let \(M\) be \(\ell_{0.1}\)-unbiased and \(\varepsilon\)-DP. Then for any \(t \in T\),
\[
\Pr(M(x) = t) \geq e^{-2\text{len}_f(x; t)\varepsilon} \Pr(M(x) = f(x)).
\]

**Proof.** For any two instances \(x, x' \in \mathcal{X}^n\),
\[
\frac{\Pr(M(x) = f(x))}{\Pr(M(x') = f(x'))} = \frac{\Pr(M(x) = f(x)) \Pr(M(x') = f(x))}{\Pr(M(x') = f(x))} \leq e^{d_{\text{ham}}(x, x')\varepsilon} \frac{\Pr(M(x') = f(x))}{\Pr(M(x') = f(x'))} \leq e^{d_{\text{ham}}(x, x')\varepsilon},
\]
where \((i)\) follows from the definition of \(\varepsilon\)-DP and \((ii)\) follows since \(M\) is \(\ell_{0.1}\)-unbiased. Denote \(\ell = \text{len}_f(x; t)\), so there exists an instance \(x' \in \mathcal{X}^n\) such that \(d_{\text{ham}}(x, x') = \ell\) and \(f(x') = t\). To finish the proof, we use the above inequality and the definition of differential privacy to get
\[
\Pr(M(x) = t) \geq e^{-\ell\varepsilon} \Pr(M(x') = t) = e^{-\ell\varepsilon} \Pr(M(x') = f(x')) \geq e^{-2\ell\varepsilon} \Pr(M(x) = f(x)),
\]
as desired. \(\square\)

We now return to prove Proposition 2.1, beginning with the instance-dependent bounds. For any \(x \in \mathcal{X}^n\), Lemma A.1 implies that
\[
1 = \sum_{t \in T} \Pr(M(x) = t) \geq \sum_{t \in T} e^{-2\text{len}_f(x; t)\varepsilon} \Pr(M(x) = f(x)).
\]
The second part of the proposition follows. Now we prove the first part of the proposition. For a mechanism \(M\), let \(p = \min_{x \in \mathcal{X}^n} \Pr(M(x) = f(x))\). For any \(t \in T\), there exists a dataset \(x'\) such that \(f(x') = t\) and \(d_{\text{ham}}(x, x') = \text{len}_f(x; t)\). Therefore, the definition of differential privacy implies
\[
\Pr(M(x) = t) \geq e^{-\text{len}_f(x; t)\varepsilon} \Pr(M(x') = t) = e^{-\text{len}_f(x; t)\varepsilon} \Pr(M(x') = f(x')) \geq e^{-\text{len}_f(x; t)\varepsilon} p.
\]
Using the last inequality, the claim follows as
\[
1 = \sum_{t \in T} \Pr(M(x) = t) \geq \sum_{t \in T} e^{-\text{len}_f(x; t)\varepsilon} p.
\]
A.2 Proof of Theorem 1

As in the proof of Proposition 2.1, we begin with a lemma that nearby datasets must incur similar losses under differential privacy.

**Lemma A.2.** Let $M$ be $L$-unbiased and $\varepsilon$-DP. Then for instances $x, x' \in \mathcal{X}^n$,

$$
\mathbb{E} [L(M(x), f(x))] \leq e^{d_{\text{ham}}(x, x')}\varepsilon \mathbb{E} [L(M(x'), f(x'))].
$$

**Proof** The definition of $\varepsilon$-DP implies that for any $t \in T$

$$
\mathbb{E} [L(M(x), t)] = \sum_{s \in T} \mathbb{P}(M(x) = s)L(s, t)
$$

$$
\leq \sum_{s \in T} e^{d_{\text{ham}}(x, x')}\varepsilon \mathbb{P}(M(x') = s)L(s, t) = e^{d_{\text{ham}}(x, x')}\varepsilon \mathbb{E} [L(M(x'), t)].
$$

Now, we use that $M$ is $L$-unbiased and the previous inequality to get

$$
\frac{\mathbb{E} [L(M(x), f(x))]}{\mathbb{E} [L(M(x'), f(x'))]} = \frac{\mathbb{E} [L(M(x), f(x))]}{\mathbb{E} [L(M(x'), f(x'))]} \leq e^{d_{\text{ham}}(x, x')}\varepsilon
$$
as claimed. □

We use the previous lemma to prove the lower bound on unbiased mechanisms in Theorem 1.

**Lemma A.3.** Let $M$ be $\varepsilon$-DP and $L$-unbiased. Then for $k \geq 1$ and any $x \in \mathcal{X}^n$,

$$
\mathbb{E} [L(M(x), f(x))] \geq \frac{L(\omega_f(x; k)/2)}{e^{2k\varepsilon} + 1}.
$$

**Proof** Let $\alpha$ be such that $\mathbb{E}[L(M(x), f(x))] \leq \alpha$. We shall prove a lower bound on $\alpha$. Using Markov’s inequality, we have

$$
\mathbb{P} \left( d_T(M(x), f(x)) \geq \frac{\omega_f(x; k)}{2} \right) \leq \mathbb{P}(L(M(x), f(x)) \geq \ell(\omega_f(x; k)/2) \leq \frac{\alpha}{\ell(\omega_f(x; k)/2)},
$$
where the first inequality holds since $\ell$ is non-decreasing. The definition of $\omega_f(x; k)$ implies the existence of $x' \in \mathcal{X}^n$ such that $d_{\text{ham}}(x, x') = k$ and $d_T(f(x), f(x')) = \omega_f(x; k)$. Now, we prove that $\mathbb{P}(d_T(M(x), f(x')) \geq \frac{\omega_f(x; k)}{2})$ is also small. Lemma A.2 implies that the loss of the mechanism $M$ for $x'$ is also small, i.e., $\mathbb{E}[L(M(x'), f(x'))] \leq e^{k\varepsilon}\alpha$. Using Markov’s inequality for $x'$ and the definition of $\varepsilon$-DP, we have

$$
\mathbb{P} \left( d_T(M(x), f(x')) \geq \frac{\omega_f(x; k)}{2} \right) \leq e^{k\varepsilon} \mathbb{P} \left( d_T(M(x'), f(x')) \geq \frac{\omega_f(x; k)}{2} \right)
$$

$$
\leq e^{k\varepsilon} \frac{\mathbb{E}[L(M(x'), f(x'))]}{\ell(\omega_f(x; k)/2)} \leq \frac{e^{k\varepsilon}\alpha}{\ell(\omega_f(x; k)/2)}.
$$

As $d_T(f(x), f(x')) = \omega_f(x; k)$, we have

$$
1 \geq \mathbb{P} \left( d_T(M(x), f(x)) < \frac{\omega_f(x; k)}{2} \right) + \mathbb{P} \left( d_T(M(x), f(x')) < \frac{\omega_f(x; k)}{2} \right)
$$

$$
\geq 2 - \frac{\alpha}{\ell(\omega_f(x; k)/2)} - \frac{e^{k\varepsilon}\alpha}{\ell(\omega_f(x; k)/2)}.
$$
The lemma follows by rearranging terms in the last inequality. \hfill \Box

Finally, we have the minimax bound we claim in Theorem 1:

**Lemma A.4.** Let $M$ be $\varepsilon$-DP. Then for $k \geq 1$,
\[
\sup_{x \in \mathcal{X}^n} \mathbb{E}[L(M(x), f(x))] \geq \sup_{x \in \mathcal{X}^n} \frac{\ell(\omega_f(x; k)/2)}{e^{k\varepsilon} + 1}.
\]

**Proof** The proof of the worst-case bound is nearly identical to that of Lemma A.3. Noting that instead of using $\mathbb{E}[L(M(x'), f(x'))] \leq e^{k\varepsilon}$ we may use $\mathbb{E}[L(M(x'), f(x'))] \leq \alpha$ as we seek a uniform bound, we repeat the argument *mutatis-mutandis* to obtain that if $\alpha \geq \sup_{x \in \mathcal{X}^n} \mathbb{E}[L(M(x), f(x))]$, then
\[
1 \geq 2 - \frac{\alpha}{\ell(\omega_f(x; k)/2)} - \frac{e^{k\varepsilon}\alpha}{\ell(\omega_f(x; k)/2)}.
\]
Rearranging gives the result. \hfill \Box

### A.3 Proof of Theorem 2

We divide the proof into two parts; the first on lower bounds and the second proving the (nearly) matching upper bounds on $\mathcal{R}$.

#### A.3.1 Lower bounds

We begin with a simple lemma, which upper bounds the variation distance between private mechanisms.

**Lemma A.5.** Let $M$ be $(\varepsilon, \delta)$-differentially private. Then for $x, x' \in \mathcal{X}^n$ with $d_{\text{ham}}(x, x') \leq k$,
\[
\|M(x) - M(x')\|_{\text{TV}} \leq 1 - e^{-k\varepsilon} + ke^{-\varepsilon}\delta \leq k(\varepsilon + e^{-\varepsilon}\delta).
\]

**Proof** We have from group privacy [19, Theorem 2.2] that for any measurable $S \subset \mathcal{T}$,
\[
\mathbb{P}(M(x) \in S) \leq e^{k\varepsilon}\mathbb{P}(M(x') \in S) + ke^{(k-1)\varepsilon}\delta
\]
or, rearranging,
\[
\mathbb{P}(M(x') \in S) \geq e^{-k\varepsilon}\mathbb{P}(M(x) \in S) - ke^{-\varepsilon}\delta.
\]
Consequently we obtain
\[
\|M(x) - M(x')\|_{\text{TV}} = \sup_S \mathbb{P}(M(x) \in S) - \mathbb{P}(M(x') \in S)
\]
\[
\leq \sup_S \mathbb{P}(M(x) \in S) - e^{-k\varepsilon}\mathbb{P}(M(x) \in S) + ke^{-\varepsilon}\delta
\]
\[
\leq 1 - e^{-k\varepsilon} + ke^{-\varepsilon}\delta,
\]
as desired. The final inequality is simply that $e^t \geq 1 + t$ for all $t \in \mathbb{R}$. \hfill \Box

Similar upper bounds hold for Rényi-differential privacy.

**Lemma A.6.** Let $\alpha \geq 1$. Let $M$ be $(\alpha, 2\varepsilon^2)$-Rényi-DP. Then for any $x, x' \in \mathcal{X}^n$ with $d_{\text{ham}}(x, x') \leq k$,
\[
\|M(x) - M(x')\|_{\text{TV}} \leq k\varepsilon.
\]
Alternatively, if $M$ is $(\alpha, \varepsilon/2)$-Rényi-DP with $\alpha \geq 1 + 2\varepsilon^{-1}\log \frac{1}{\delta}$, then
\[
\|M(x) - M(x')\|_{\text{TV}} \leq 1 - e^{-k\varepsilon} + ke^{-\varepsilon}\delta \leq k(\varepsilon + e^{-\varepsilon}\delta).
\]
Proof The definition of \((\alpha, \varepsilon^2)\)-Rényi-DP implies that \(D_\alpha(M(x^0)|M(x^1)) \leq \varepsilon^2\) for any neighboring datasets. As \(d_{\text{ham}}(x, x') \leq k\), there exist datasets \(x = x^0, x^1, \ldots, x^k = x' \in \mathcal{X}^n\) such that \(d_{\text{ham}}(x^i, x^{i+1}) \leq 1\) for each \(i\), and thus

\[
\|M(x) - M(x')\|_{\text{TV}} \leq \sum_{i=0}^{k-1} \|M(x^i) - M(x^{i+1})\|_{\text{TV}}
\]

\[
\leq \sum_{i=1}^{(i) k-1} \sqrt{D_{\text{kl}}(M(x^i)|M(x^{i+1}))}/2 \leq k\sqrt{\varepsilon^2},
\]

where inequality (i) is Pinsker’s inequality and (ii) follows because \(D_\alpha(\cdot | \cdot)\) is decreasing in \(\alpha\).

For the second result, we recall Mironov [32, Prop. 3], which shows that an \((\alpha, \varepsilon)\)-Rényi-DP mechanism is also \((\varepsilon + \log \frac{\delta-1}{\alpha-1}, \delta)\)-DP for any \(0 < \delta < 1\). Thus, in the case that \(M\) is \((\alpha, \varepsilon/2)\)-Rényi-DP with \(\alpha \geq 1 + 2\varepsilon^{-1}\log \frac{1}{\delta}\), Lemma A.5 implies the second result. \(\square\)

We can now lower bound the loss of private mechanisms for two instances. We adapt Le Cam’s two-point method for lower bounds [cf. 15, Ch. 5.2]. For \(x, x' \in \mathcal{X}^n\), we denote

\[
d_L(x, x') = \inf_{t \in \mathcal{T}} \{L(f(x), t) + L(f(x'), t)\}.
\]

Then we have the following bound.

Lemma A.7. For any mechanism \(M\), and instances \(x, x' \in \mathcal{X}^n\),

\[
\max_{\bar{x} \in \{x, x'\}} \mathbb{E}[L(M(\bar{x}), f(\bar{x}))] \geq \frac{1}{4} d_L(x, x') (1 - \|M(x) - M(x')\|_{\text{TV}}).
\]

Proof Let \(S_x = \{t : L(f(x), t) < \frac{d_L(x, x')}{2}\}\), so we have that \(L(f(x), t) \geq \frac{d_L(x, x')}{2}\) for \(t \not\in S_x\). Further, the definition of \(d_L\) implies that \(L(f(x'), t) \geq \frac{d_L(x, x')}{2}\) for \(t \in S_x\). Hence, we have

\[
\max_{\bar{x} \in \{x, x'\}} \mathbb{E}[L(M(\bar{x}), f(\bar{x}))] \geq \frac{1}{2} \mathbb{E}[L(M(x), f(x))] + \frac{1}{2} \mathbb{E}[L(M(x'), f(x'))]
\]

\[
\geq \frac{d_L(x, x')}{4} (\mathbb{P}(M(x) \not\in S_x) + \mathbb{P}(M(x') \in S_x))
\]

\[
= \frac{d_L(x, x')}{4} (1 - \mathbb{P}(M(x) \in S_x) + \mathbb{P}(M(x') \in S_x))
\]

\[
\geq \frac{d_L(x, x')}{4} (1 - \|M(x) - M(x')\|_{\text{TV}})
\]

by definition of the variation distance. \(\square\)

In our setting, \(L(s, t) = \ell(d_T(s, t))\) for a non-decreasing function \(\ell : \mathbb{R}_+ \to \mathbb{R}_+\), so that recalling the definition (3) of the modulus \(\omega_f(x; k) = \sup_{x', t}(d_T(f(x), f(x')) : d_{\text{ham}}(x, x') \leq k\), for the \(x' \in \mathcal{X}^n\) attaining \(d_T(f(x), f(x')) = \omega_f(x; k)\) (or one arbitrarily close to obtaining it), we have

\[
d_L(x, x') = \inf_t \{\ell(d_T(f(x), t)) + \ell(d_T(f(x'), s))\} \geq \ell\left(\frac{\omega_f(x; k)}{2}\right)
\]

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because for any \( t \), at least one of \( d_T(f(x), t) \geq \omega_f(x; k)/2 \) or \( d_T(f(x'), t) \geq \omega_f(x; k)/2 \). Lemma A.7 then implies that for any \( k \in \mathbb{N} \), there exists \( x' \in X^n \) with \( d_{\text{ham}}(x, x') \leq k \) such that for any mechanism \( M \), we have

\[
\max_{\tilde{x} \in \{x, x'\}} \mathbb{E}[L(M(\tilde{x}), f(\tilde{x}))] \geq \frac{1}{4} \left( \frac{1}{2} \omega_f(x; k) \right) \left( 1 - \|M(x) - M(x')\|_{TV} \right). \tag{20}
\]

The lower bounds in Theorem 2 now each follow from inequality (20) and our bounds on the total variation distance of each family of mechanisms via Lemmas A.5 and A.6. The lower bound (6a) follows from inequality (20) with Lemma A.5. The lower bound (6b) for the family of \((\epsilon, \delta)\)-differentially private mechanisms and \((\alpha, \epsilon/2)\)-Rényi-DP mechanisms with \( \alpha \geq 1 + 2\varepsilon^{-1}\log \frac{1}{\delta} \) follows from Lemma A.5 and A.6, as as \( k \leq \min\{\log \frac{1}{\varepsilon}, \frac{1}{\sqrt{\delta}}\} \) implies that \( 1 - \|M(x) - M(x')\|_{TV} \geq e^{-k\varepsilon} - ke^{-\varepsilon} \geq \frac{1}{2} e^{-k\varepsilon} \). The lower bound (6c) for Rényi-DP follows by setting \( k = \frac{1}{2\varepsilon} \) in Lemma A.6.

### A.3.2 Upper bounds

For the upper bounds, given samples \( x^0, x^1 \in X^n \) with \( d_{\text{ham}}(x^0, x^1) \leq k \), we may construct a mechanism guaranteed to be differentially private. We set

\[
M_\ast(x) = \begin{cases} 
  f(x^0) & \text{with probability } e^{-\varepsilon d_{\text{ham}}(x, x^0)/2} + e^{-\varepsilon d_{\text{ham}}(x, x^1)/2} \\
  f(x^1) & \text{with probability } e^{-\varepsilon d_{\text{ham}}(x, x^1)/2} 
\end{cases}
\]

The mechanism \( M_\ast \) is evidently \( \varepsilon \)-differentially private, and we have

\[
\mathbb{E}[L(M_\ast(x^0), f(x^0))] = \mathbb{E}[L(M_\ast(x^1), f(x^1))] = \frac{1}{1 + e^{\varepsilon d_{\text{ham}}(x^0, x^1)/2}} \ell(d_T(f(x^0), f(x^1))).
\]

Fixing \( x^0 \) and taking a supremum over \( x^1 \) gives upper bounds nearly matching each of the lower bounds (6). The first (6a) is immediate. For the second (6b), any \( \varepsilon \)-DP mechanism is also \((\epsilon, \delta)\)-DP, and moreover, satisfies \((\alpha, \varepsilon)\)-differential privacy for all \( \alpha \in [1, \infty] \). For the final bound (6c), if \( Q(\cdot \mid x) \) denotes the distribution of the mechanism \( M_\ast \), we have for any neighboring samples \( x, x' \) that

\[
\int \left( \frac{dQ(z \mid x)}{dQ(z \mid x')} \right)^2 dQ(z \mid x') = 1 + \int \left( \frac{dQ(z \mid x)}{dQ(z \mid x')} - 1 \right)^2 dQ(z \mid x') \leq 1 + (e^\varepsilon - 1)^2.
\]

In particular, for \( \alpha = 2 \), we have \( D_\alpha(M_\ast(x) \mid M_\ast(x')) \leq \log(1 + (e^\varepsilon - 1)^2) \leq 2 \min\{e^\varepsilon, \varepsilon\} \).

### A.4 Proof of Proposition 2.2

Our proof is similar to the proof of Proposition 2 in [17]. We start with some notation. For distributions \( P_0, P_1 \) we define the \( \chi^2 \)-affinity \( \rho(P_1 \mid P_0) := \mathbb{E}_{P_1}[d_{\chi^2}^2] = D_{\chi^2}(P_1 \mid P_0) + 1 \). We have the following constrained risk inequality.

**Lemma A.8** (Duchi and Ruan [16], Corollary 1). Assume \( \ell : \mathbb{R} \rightarrow \mathbb{R} \) is a non-decreasing function and let \( x, x' \in X^n \). Define \( \Delta = \ell(\frac{1}{2} | f(x) - f(x') |) \). If \( \mathbb{E}[|M(x) - f(x)|] \leq \gamma \) then

\[
\mathbb{E}[\ell(|M(x') - f(x')|)] \geq \left[ \Delta^{1/2} - (\rho(M(x') \mid M(x)) \gamma)^{1/2} \right]^2.
\]
We can now use Lemma A.8 to prove Proposition 2.2. Denote $\Delta_k^x = \ell(\frac{1}{2}\omega_f(x; k))$ and let $x'$ be such that $\omega_f(x; k) = |f(x) - f(x')|$. Since $M$ is $\epsilon$-DP, we have that $\rho(M(x)||M(x')) \leq e^{\epsilon x}$. As $\mathbb{E}[\ell(|M(x) - f(x)|)] \leq \gamma \Delta_k^x$, Lemma A.8 implies that for any $k \in \mathbb{N}$, if we choose $x'$ such that $d_{\text{ham}}(x, x') \leq k$ and $|f(x') - f(x)| = \omega_f(x; k)$, then
\[
\mathbb{E}[\ell(|M(x') - f(x')|)] \geq \left[ \sqrt{\Delta_k^x - (e^{\epsilon x} \Delta_k^x \gamma)^{1/2}} \right]^2.
\]
Taking $k = \epsilon^{-1} \log \frac{1}{2\gamma}$, we obtain
\[
\mathbb{E}[\ell(|M(x) - f(x')|)] \geq \left[ \sqrt{\Delta_k^x \log(1/2\gamma) - \sqrt{\Delta_k^x / 2}} \right]^2 \geq \frac{2}{7} \Delta_k^x \gamma \log \frac{1}{\gamma}, \quad (21)
\]
where the second inequality follows because $1 - 1/\sqrt{2} > \frac{2}{7}$ and $\Delta_k^x$ is increasing in $k$ (we have assumed $\gamma \leq e^{-1/2}$). Now we prove that if $d_{\text{ham}}(x, x') \leq k$ then
\[
2\omega_f(x; 2k) \geq \omega_f(x'; k). \quad (22)
\]
Assume w.l.o.g. that $f(x) \leq f(x')$, and let $\tilde{x}$ be such that $d_{\text{ham}}(x', \tilde{x}) \leq k$ and $\omega_f(x'; k) = |f(x') - f(\tilde{x})|$. Note that $d_{\text{ham}}(x, \tilde{x}) \leq 2k$. We consider two cases: (i) that $f(\tilde{x}) \geq f(x)$ and (ii) that $f(\tilde{x}) \leq f(x)$. In the former case (i), if $f(\tilde{x}) \geq f(x)$ then $\omega_f(x; 2k) \geq |f(\tilde{x}) - f(x)| = \omega_f(x'; k)$, while if $f(\tilde{x}) \leq f(x')$ then $f(\tilde{x}) \in \{f(x), f(x')\}$, so that we must have $f(\tilde{x}) = f(x)$ as $d_{\text{ham}}(x, x') \leq k$, and again inequality (22) holds. In case (ii) that $f(\tilde{x}) < f(x)$, we have
\[
2\omega(x; 2k) \geq f(x') - f(x) + f(x) - f(\tilde{x}) = f(x') - f(\tilde{x}) = \omega_f(x'; k),
\]
as desired. As $\ell$ is non-decreasing, inequalities (21) and (22) imply
\[
\mathbb{E}[\ell(|M(x') - f(x')|)] \geq \frac{1}{4} \ell \left( \frac{1}{4} \omega_f \left( x'; \frac{\log(1/2\gamma)}{2\epsilon} \right) \right).
\]

\section*{B Proofs of general upper bounds (Section 3)}

\subsection*{B.1 Proof of Lemma 3.2}

The first claim is immediate, as $x \mapsto \text{len}_f(x; t)$ is 1-Lipschitz with respect to the Hamming metric on $X^n$. The binary case is more subtle; we assume w.l.o.g. that $T = \{0, 1\}$. Let $\ell(x) = \text{len}_f(x; 1 - f(x))$ be the distance to the closest instance $x'$ with $f(x') \neq f(x)$. Then
\[
\mathbb{P}(M_{\text{disc}}(x) = t) = \begin{cases} 2^\ell(x) & \text{if } t = f(x) \\ 2^{-\ell(x)} & \text{otherwise.} \end{cases}
\]
For any neighboring instances $x, x' \in X^n$, if $f(x) \neq f(x')$, then we have $\mathbb{P}(M_{\text{disc}}(x) = f(x)) = e^{\epsilon/2}$. Conversely, when $f(x) = f(x')$, we have $|\ell(x) - \ell(x')| \leq 1$, and therefore
\[
\mathbb{P}(M_{\text{disc}}(x) = f(x)) = \frac{e^{\ell(x)\epsilon/2}}{e^{\ell(x')\epsilon/2} + 1} = \frac{e^{-\ell(x')\epsilon/2} + 1}{e^{-\ell(x)\epsilon/2} + 1} \leq e^{\epsilon/2}.
\]
We also have
\[
\mathbb{P}(M_{\text{disc}}(x) = 1 - f(x)) = \frac{e^{\ell(x)\epsilon/2} + 1}{e^{\ell(x)\epsilon/2} + 1} \leq e^{\epsilon/2}.
\]
B.2 Proof of Theorem 3

We begin with the following lemma, which we use in the proofs of later results as well.

**Lemma B.1.** Let $f : \mathcal{X}^n \to \mathcal{T}$ and $T \geq 1$ an integer. For any $x \in \mathcal{X}^n$, the mechanism $M_{\text{disc}}$ (M.1) has

$$
\mathbb{E}[L(M_{\text{disc}}(x), f(x))] \leq \ell(\omega_f(x; T)) + \frac{2\ell(d_T^*) \card(T)}{\varepsilon} e^{-T\varepsilon/2}.
$$

**Proof** First, we define the slice of $\mathcal{T}$ containing those $t \in \mathcal{T}$ satisfying $\len_f(x; t) = k$ by

$$
S_{x}^{k} := \{ t \in \mathcal{T} : \len_f(x; t) = k \}. \quad (23)
$$

The slices $S_{x}^{k}$ are disjoint for varying $k$, and we have $\card(S_{x}^{k}) \leq \card(\mathcal{T})$ for all $k \geq 1$. Therefore

$$
\mathbb{P}\left(M_{\text{disc}}(x) \in S_{x}^{k}\right) \leq e^{-k\varepsilon/2} \mathbb{P}\left(M_{\text{disc}}(x) = f(x)\right) \card(S_{x}^{k}) \leq e^{-k\varepsilon/2} \card(\mathcal{T}).
$$

By construction, for $t \in S_{x}^{k}$ we have $d_{\mathcal{T}}(t, f(x)) \leq \omega_f(x; k)$, so that $L(t, f(x)) \leq \ell(\omega_f(x; k))$. We use the previous inequality to get

$$
\mathbb{E}[L(M_{\text{disc}}(x), f(x))] = \sum_{t \neq f(x)} \mathbb{P}(M_{\text{disc}}(x) = t) L(t, f(x))
$$

$$
\leq \sum_{k=1}^{n} \mathbb{P}\left(M_{\text{disc}}(x) \in S_{x}^{k}\right) \ell(\omega_f(x; k))
$$

$$
\leq \ell(\omega_f(x; T)) + \card(\mathcal{T}) \sum_{k=T+1}^{n} e^{-k\varepsilon/2} \ell(\omega_f(x; k))
$$

$$
\leq \ell(\omega_f(x; T)) + \frac{2\ell(d_T^*) \card(T)}{\varepsilon} e^{-T\varepsilon/2},
$$

where the last inequality follows as $\omega_f(x; k) \leq d_T^*$ and $\sum_{i=0}^{\infty} e^{-i\varepsilon} = \frac{e^{\varepsilon}}{e^{\varepsilon}-1} \leq \frac{e^\varepsilon}{\varepsilon}$. \hfill \Box

Setting $T = \frac{\varepsilon}{2}(\log \gamma + \log \frac{2\ell(d_T^*) \card(T)}{\varepsilon})$ in Lemma B.1, we get the first part of Theorem 3. We prove the second claim. If $\bar{\varepsilon} = 2\varepsilon \log \frac{2\ell(d_T^*) \card(T)}{\gamma \varepsilon}$, then

$$
\frac{2}{\bar{\varepsilon}} \log \frac{2\ell(d_T^*) \card(T)}{\gamma \bar{\varepsilon}} = \frac{1}{\varepsilon} \log \frac{2\ell(d_T^*) \card(T)}{\gamma \varepsilon} \left[ \log \frac{2\ell(d_T^*) \card(T)}{\gamma \varepsilon} - \log \frac{2 \log \frac{2\ell(d_T^*) \card(T)}{\gamma \varepsilon}}{\gamma \varepsilon} \right]_{\geq 0},
$$

so that $\mathbb{E}[L(\widetilde{M}_{\text{disc}}(x), f(x))] \leq \gamma + \ell(\omega_f(x; \varepsilon^{-1}))$ as claimed.

B.3 Proof of Theorem 4

We prove the result in a somewhat more general setting than claimed in the theorem. We allow $\mathcal{T}$ to be a subset of a vector space, and instead of Lebesgue measure we assume that the measure $\mu$ approximates a 1-dimensional uniform measure on $\mathcal{T}$, meaning that for the unit ball $B := \{ t \in \mathcal{T} : \| t \| \leq 1 \}$,

$$
\frac{\mu(S)}{\mu(t + \rho B)} \leq \frac{\text{diam}(S)}{\rho} \text{ for all } S \subset \mathcal{T} \text{ and } t \in \mathcal{T}. \quad (24)
$$
Certainly the Lebesgue measure on $\mathbb{R}$ satisfies this, but so too does any discrete measure on equi-spaced points in $\mathbb{R}$. We also assume the loss function $L$ satisfies $L(s,t) = \ell(||s-t||)$ for a non-decreasing function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$, and $||s-t|| \leq K$ for all $s,t \in \mathcal{T}$.

We start with the following lemma, which proves the first part of Theorem 4.

**Lemma B.2.** Let the conditions of Theorem 4 hold. There exists a numerical constant $c < \infty$ such that for any $T \in \mathbb{N}$ and $x \in \mathcal{X}^n$,

$$
\mathbb{E}[L(M_{\text{cont}}(x), f(x))] \leq \ell(\omega_f(x; T)) + C_T \rho + cC_T \rho \exp\left(-T\varepsilon/2 + \log \frac{n^2GS_f^2}{\rho^2\varepsilon}\right).
$$

Before proving the lemma, we show how it gives the theorem. Setting $T = \frac{2}{\varepsilon}(\log \frac{1}{\varepsilon} + 2 \log \frac{nGS_f}{\rho})$ in Lemma B.2 gives the first part of Theorem 4. For the second claim, note that if $\varepsilon = 2\varepsilon[\log \frac{1}{\varepsilon} + 2 \log \frac{nGS_f}{\rho}]$, then

$$
\frac{2}{\varepsilon} \left[ \log \frac{1}{\varepsilon} + 2 \log \frac{nGS_f}{\rho} \right] = \frac{1}{\varepsilon(\log \frac{1}{\varepsilon} + 2 \log \frac{nGS_f}{\rho})} \left[ \log \frac{1}{2\varepsilon} + 2 \log \frac{nGS_f}{\rho} \right] - \log \left( \log \frac{1}{2\varepsilon} + 2 \log \frac{nGS_f}{\rho} \right) \leq \frac{1}{\varepsilon},
$$

so that the first claim of the theorem gives the result.

**Proof.** As in the proof of Lemma B.1, we define the slices

$$
S^k_x = \{ t \in \mathcal{T} : \text{len}_f(x; t) = k \},
$$

so that the $S^k_x$ partition $\mathcal{T}$. Since $f$ has global sensitivity $GS_f$, we have $\text{diam}(S^k_x) \leq 2(kGS_f + \rho)$ for all $k \geq 1$, by coupling the discretization $\rho$ with the fact that changing one element in $x$ changes $f(x)$ by at most $GS_f$. The definition of $M_{\text{cont}}$ and the uniformity (24) then imply

$$
\mathbb{P}\left(M_{\text{cont}}(x) \in S^k_x\right) = \frac{\int_{S^k_x} e^{-\text{len}_f(x; t)/2} d\mu(t)}{\int_{\mathcal{T}} e^{-\text{len}_f(x; s)/2} d\mu(s)} \leq e^{-k\varepsilon/2} \frac{\text{diam}(S^k_x)}{\rho} \leq 2 \left( k \frac{GS_f}{\rho} + 1 \right) e^{-k\varepsilon/2}.
$$

(25)

The previous inequality implies

$$
\mathbb{E}[L(M_{\text{cont}}(x), f(x))] = \int_{t \in \mathcal{T}} \pi_{M_{\text{cont}}(x)}(t)L(t, f(x))d\mu(t)
$$

$$
\leq \sum_{k=1}^n \mathbb{P}\left(M_{\text{cont}}(x) \in S^k_x\right) \ell(\omega_f(x; k) + \rho)
$$

$$
\leq \ell(\omega_f(x; T)) \sum_{k=1}^T \mathbb{P}\left(M(x) \in S^k_x\right) + \sum_{k=T+1}^n \mathbb{P}\left(M_{\text{cont}}(x) \in S^k_x\right) \ell(\omega_f(x; k)) + C_T \rho
$$

$$
\leq \ell(\omega_f(x; T)) + 2C_T \sum_{k=T+1}^n (kGS_f/\rho + 1)kGS_f e^{-k\varepsilon/2} + C_T \rho.
$$
Now, we recall the incomplete gamma function \( \Gamma(a, b) := \int_a^\infty z^{b-1}e^{-z}dz \), which satisfies \( \Gamma(a, b) = O(1)a^{b-1}e^{-a} \) as \( a \) grows [7, Eq. (1.5)], so that

\[
\int_T^\infty \left( \frac{\text{GS}_f^2}{\rho} \right) e^{-z/2} dz = \frac{2}{\varepsilon} \left( \frac{4\text{GS}_f^2}{\rho\varepsilon^2} \right) \int_T^{T/2} u^2 e^{-u} du = O(1) \frac{2}{\varepsilon} \left( \frac{4\text{GS}_f^2}{\rho\varepsilon^2} \right) \left( \frac{T^2}{2} \right)^2 e^{-T/2}
\]

Returning to our string of inequalities, we obtain

\[
\mathbb{E}[L(M_{\text{cont}}(\mathbf{x}), f(\mathbf{x}))] \leq \mathcal{C} \ell + 2 + \mathcal{C} \ell \rho + O(1) \mathcal{C} \ell \rho \exp \left( -T\varepsilon/2 + \log \frac{T^2\text{GS}_f^2}{\rho^2\varepsilon} \right),
\]

and noting that we necessarily have \( T \leq n \) gives the result. \( \square \)

### C Proofs for sample-monotone estimands (Section 4)

#### C.1 Proof of Observation 4.1

Let \( \mathbf{x} \in \mathcal{X}^n \) and assume without loss of generality that \( f(\mathbf{x}) \leq s \leq t \). We need to prove that \( \text{len}_{\mathbf{x}}(\mathbf{x}; t) \geq \text{len}_{\mathbf{x}}(\mathbf{x}; t_1) \). If \( \text{len}_{\mathbf{x}}(\mathbf{x}; t) = \infty \) then we are done. Otherwise there exists \( \mathbf{x}' \) such that \( f(\mathbf{x}') = t \) and \( d_{\text{ham}}(\mathbf{x}, \mathbf{x}') = \text{len}_{\mathbf{x}}(\mathbf{x}; t) \). We define the function \( g(\lambda) = f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{x}') \) for \( \lambda \in [0, 1] \). The function \( g(\cdot) \) is continuous as \( f(\cdot) \) is continuous. We also know that \( g(0) = f(\mathbf{x}) \) and \( g(1) = f(\mathbf{x}') = t \). The intermediate value theorem implies that there exists \( 0 \leq \lambda \leq 1 \) such that \( g(\lambda s) = s \). Since \( \mathcal{X} \) is convex, we get that \( \mathbf{x}_s = \lambda \mathbf{x} + (1 - \lambda) \mathbf{x}' \in \mathcal{X}^n \) and \( f(\mathbf{x}_s) = s \). To finish the proof, we note that \( d_{\text{ham}}(\mathbf{x}, \mathbf{x}_s) \leq d_{\text{ham}}(\mathbf{x}, \mathbf{x}') \) immediately by the definition of \( \mathbf{x}_s \) as a convex combination of \( \mathbf{x} \) and \( \mathbf{x}' \).

#### C.2 Proof of Theorem 5

We first define the right and left moduli of continuity \( \omega_f(\mathbf{x}; k^+) = \sup_{\mathbf{x}'} \{ f(\mathbf{x}') - f(\mathbf{x}) : d_{\text{ham}}(\mathbf{x}, \mathbf{x}') \leq k \} \) and \( \omega_f(\mathbf{x}; k^-) = \sup_{\mathbf{x}'} \{ f(\mathbf{x}) - f(\mathbf{x}') : d_{\text{ham}}(\mathbf{x}, \mathbf{x}') \leq k \} \). Denote \( \omega_f(\mathbf{x}; k) = \omega_k; \omega_f(\mathbf{x}; k^+) = \omega_{k^+} \) and \( \omega_f(\mathbf{x}; k^-) = \omega_{k^-} \) for shorthand. We have the following lemma.

**Lemma C.1.** Let \( f : \mathcal{X}^n \to \mathbb{R} \) be monotone. Then

\[
\mathbb{E} [ |M_{\text{cont}}(\mathbf{x}) - f(\mathbf{x})|^p ] \leq 2^{p-1} \rho + 2^{p-1} \frac{\sum_{k=1}^{n} \omega_k^p (\omega_k - \omega_{k-1}) e^{-k\varepsilon/2}}{\rho + \sum_{k=1}^{n} (\omega_k - \omega_{k-1}) e^{-k\varepsilon/2}}
\]

\[
+ 2^{p-1} \frac{\sum_{k=1}^{n} \omega_k^p (\omega_k - \omega_{k-1}) e^{-k\varepsilon/2}}{\rho + \sum_{k=1}^{n} (\omega_k - \omega_{k-1}) e^{-k\varepsilon/2}}.
\]

**Proof** Let \( S_{\mathbf{x}}^{k^+} = \{ t > f(\mathbf{x}) : \text{len}_{\mathbf{x}}(\mathbf{x}; t) = k \} \) and \( S_{\mathbf{x}}^{k^-} = \{ t < f(\mathbf{x}) : \text{len}_{\mathbf{x}}(\mathbf{x}; t) = k \} \).

Clearly we have \( \rho \leq \text{diam}(S_{\mathbf{x}}^{0}) \) and \( \text{diam}(S_{\mathbf{x}}^{k^+}) = \omega_{k^+} - \omega_{k-1}^+ \) and \( \text{diam}(S_{\mathbf{x}}^{k^-}) = \omega_{k^-} - \omega_{k-1}^- \) for \( k \geq 1 \). As \( f \) is monotone, we have that for \( k \geq 1 \),

\[
\mathbb{P} ( M(\mathbf{x}) \in S_{\mathbf{x}}^{k^+} ) \leq \frac{\omega_{k^+}^p (\omega_{k^+} - \omega_{k-1}^+) e^{-k\varepsilon/2}}{\rho + \sum_{k=1}^{n} (\omega_{k^+} - \omega_{k-1}^+) e^{-k\varepsilon/2}}.
\]
As identical bound holds for $S^{k_-}_x$ except that we swap $\omega_{k^+}$ with $\omega_{k^-}$. Thus
\[
\mathbb{E} [ |M_{\text{cont}}(x) - f(x)|^p ]
\leq \mathbb{P} \left( M(x) \in S^{k_+}_x \right) \rho^p + \sum_{k=1}^{n} \mathbb{P} \left( M(x) \in S^{k_+}_x \right) (\omega_{k^+} + \rho)^p + \sum_{k=1}^{n} \mathbb{P} \left( M(x) \in S^{k_-}_x \right) (\omega_{k^-} + \rho)^p
= 2^{p-1} \rho^p + 2^{p-1} \sum_{k=1}^{n} \mathbb{P} \left( M(x) \in S^{k_+}_x \right) \omega_{k^+}^p + 2^{p-1} \sum_{k=1}^{n} \mathbb{P} \left( M(x) \in S^{k_-}_x \right) \omega_{k^-}^p.
\]
Substituting the bound (26) for $\mathbb{P}(M(x) \in S^{k_+}_x)$ and the symmetric bound for $\mathbb{P}(M(x) \in S^{k_-}_x)$ gives the lemma.

We can now use Lemma C.1 to prove the theorem. Indeed, we have
\[
\sum_{k=1}^{n} \left( (\omega_{k^+} - \omega_{k^-}) + (\omega_{k^-} - \omega_{k^+}) \right) e^{-ke^2/2} \geq (1 - e^{-\epsilon/2}) \sum_{k=1}^{n} (\omega_{k^+} + \omega_{k^-}) e^{-ke^2/2}
\geq (1 - e^{-\epsilon/2}) \sum_{k=1}^{n} \omega_{k} e^{-ke^2/2}.
\]
We also have
\[
\sum_{k=1}^{n} \omega_{k^+}^p (\omega_{k^+} - \omega_{k^-}) e^{-ke^2/2} \leq \sum_{k=1}^{n} (\omega_{k^+}^p - \omega_{k^-}^p) e^{-ke^2/2}
= (1 - e^{-\epsilon/2}) \sum_{k=1}^{n} \omega_{k^+}^p e^{-ke^2/2} + \omega_{n^+}^p e^{-(n+1)\epsilon/2}
\leq (1 - e^{-\epsilon/2}) \sum_{k=1}^{n} \omega_{k}^p e^{-ke^2/2} + \omega_{n}^p e^{-(n+1)\epsilon/2}.
\]

Using the same argument for $\omega_{k^-}$, we have
\[
\mathbb{E} [ |M_{\text{cont}}(x) - f(x)|^p ] \leq 2^{p-1} \rho + 2^{p} \sum_{k=1}^{n} \omega_{k}^p e^{-ke^2/4} + \frac{(2\omega_{n})^p}{\rho} e^{-(n+1)\epsilon/2}
\leq 2^{p-1} \rho + 2^{p} \max_{1 \leq k \leq n} \omega_{k}^p e^{-ke^2/4} + \frac{(2\omega_{n})^p}{\rho} e^{-(n+1)\epsilon/2}.
\]

Theorem 5 now follows from the definition of $W_f(x; \epsilon)$ and that $\frac{1-e^{-\epsilon/2}}{1-e^{-\epsilon/4}} \leq 2$.

C.3 Proof of Corollary 4.1

We need to prove that $R_f(x) = \frac{W_f(x; \epsilon/4)}{W_f(x; \epsilon/2)} \lesssim 1$. The assumptions of the corollary imply that $\omega_f(x; k) \lesssim \omega_f(x; \frac{8C}{\epsilon}) e^{ke^2/8}$ for $k \geq k' = \frac{8C}{\epsilon}$. Therefore we have
\[
W_f(x; \epsilon/4) = (1 - e^{-\epsilon/4}) \sum_{i=1}^{n} \omega_f(x; i) e^{-i\epsilon/4}
\leq (1 - e^{-\epsilon/4}) \sum_{i=1}^{n} \omega_f(x; k') e^{-i\epsilon/8} \leq \omega_f(x; k') \frac{1-e^{-\epsilon/4}}{1-e^{-\epsilon/8}} \leq 2 \omega_f(x; k')
\]
Here inequality (⋆) follows by Theorem 1, which states that any $\varepsilon$-DP $\ell_1$-unbiased mechanism must satisfy $\frac{1}{4} e^{-2\kappa e} \omega_k \leq \mathbb{E}[|M(x) - f(x)|]$, so that the postulated $\varepsilon/16$-DP unbiased mechanism $\tilde{M}$ must therefore satisfy

$$\frac{1}{2} \max_k \omega_k e^{-\kappa e/4} \leq \mathbb{E}[|\tilde{M}(x) - f(x)|] \leq \omega_k e^{-\kappa e/16},$$

or $\omega_i e^{-\kappa e/4} \leq \omega_i e^{-\kappa e/8} \omega_k e^{-\kappa e/16}$ for each $i$. We also have the lower bound

$$W_f(x; \varepsilon/2) = (1 - e^{-\varepsilon/2}) \sum_{i=1}^{n} \omega_i e^{-\kappa e/2} \geq (1 - e^{-\varepsilon/2}) \sum_{i=k}^{n} \omega_i e^{-\kappa e/2}$$

$$= (1 - e^{-\varepsilon/2}) \omega_k e^{-\kappa e/2} \sum_{i=0}^{n-k} e^{-\kappa e/2}$$

$$= (1 - e^{-\varepsilon/2}) \omega_k e^{-\kappa e/2} \left( \frac{e^{\kappa e/2} - e^{-(n-k)\kappa e/2}}{2\varepsilon} \right) \geq \omega_k e^{-\kappa e/2}.$$

Overall we have that the ratio (10) satisfies $R_f(x) \leq e^{7\kappa e/16}$.

### C.5 Proof of Proposition 4.1

We start with the following useful lemma.

**Lemma C.2.** Let $0 \leq a_1 \leq \cdots \leq a_K$ and $\lambda = \frac{1}{\log K}$. Then for $t \geq 1$,

$$\frac{\sum_{k=1}^{K} a_k e^{-\kappa e}}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-\kappa e}} \leq e \exp \left( \frac{4 \log n}{\log \frac{1}{2\varepsilon} + \log \log n} \right) \max_{1 \leq k \leq K} e^{-\kappa e} a_k.$$
Proof We have:

\[
\frac{\sum_{k=1}^{K} a_k^t (a_k - a_{k-1}) e^{-k\varepsilon}}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-k\varepsilon}} = \frac{\sum_{k=1}^{K} e^{-\lambda k\varepsilon} a_k^t (a_k - a_{k-1}) e^{-(1-\lambda)k\varepsilon}}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-k\varepsilon}}
\]

\[
(i) \leq \frac{\left(\sum_{k=1}^{K} e^{-(\lambda k\varepsilon) a_k^t} q \right)^{\frac{1}{p}} \left(\sum_{k=1}^{K} ((a_k - a_{k-1}) e^{-(1-\lambda)k\varepsilon})^p \right)^{\frac{1}{p}}}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-k\varepsilon}}
\]

\[
(ii) \leq e \max_{1 \leq k \leq K} e^{-\lambda k\varepsilon} a_k^t \left(\sum_{k=1}^{K} ((a_k - a_{k-1}) e^{-(1-\lambda)k\varepsilon})^p \right)^{\frac{1}{p}} \frac{1}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-k\varepsilon}}
\]

\[
(iii) = e \max_{1 \leq k \leq K} e^{-\lambda k\varepsilon} a_k^t \left(\sum_{k=1}^{K} ((a_k - a_{k-1}) e^{-(1-\lambda)k\varepsilon})^p \right)^{\frac{1}{p}} \frac{1}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-k\varepsilon}}
\]

where (i) follows from Hölder’s inequality where we set \( q = \log K \) and \( p = \frac{q}{q - 1} \), (ii) follows since \( \|z\|_q \leq d^{1/q} \|z\|_\infty \leq e \|z\|_\infty \) for \( z \in \mathbb{R}^d \) when \( q \geq \log d \), (iii) follows since \( \lambda = \frac{1}{q} \). Let us now separately consider the last term. We have:

\[
\frac{\left(\sum_{k=1}^{K} ((a_k - a_{k-1}) e^{-(1-\lambda)k\varepsilon})^p \right)^{\frac{1}{p}}}{\sum_{k=1}^{K} (a_k - a_{k-1}) e^{-k\varepsilon}} \leq \left( \frac{1}{K} e^{-t\varepsilon} \right)^{\frac{1}{q}} = e^{\frac{te^{-t\varepsilon}}{q}} \leq e^{-\frac{K \log K}{\log K}} = e^{\frac{4 \log \log K}{\log K}}
\]

where (i) follows since \( \frac{a_k}{a_{K}} \leq 1 \) and \( p > 1 \), and (ii) follows since there exists \( 1 \leq t \leq K \) such that \( a_t - a_{t-1} \geq \frac{a_K}{K} \).

\[
E[|M_{\text{cont}}(x) - f(x)|] \leq 2\rho + \frac{\sum_{k=1}^{n} \omega_k + (\omega_k + - \omega_{k-1}) e^{-k\varepsilon/2}}{\rho + \sum_{k=1}^{n} (\omega_k - - \omega_{k-1}) e^{-k\varepsilon/2}} + \frac{\sum_{k=1}^{n} \omega_k + (\omega_k + - \omega_{k-1}) e^{-k\varepsilon/2}}{\rho + \sum_{k=1}^{n} (\omega_k - - \omega_{k-1}) e^{-k\varepsilon/2}}
\]

We only bound the second term as the third follows using similar arguments. For \( K > 0 \) to be chosen presently, we have the following upper bound on the second term

\[
\sum_{k=1}^{K} \omega_k + (\omega_k + - \omega_{k-1}) e^{-k\varepsilon/2} + \frac{1}{\rho} \sum_{k=K+1}^{n} \omega_k + (\omega_k + - \omega_{k-1}) e^{-k\varepsilon/2}
\]

Since \( \omega_k \leq \omega_k \), Lemma C.2 gives an upper bound on the first term in (27) where \( \lambda = \frac{1}{\log K} \).

\[
\frac{\sum_{k=1}^{K} \omega_k + (\omega_k + - \omega_{k-1}) e^{-k\varepsilon/2}}{\sum_{k=1}^{K} (\omega_k + - \omega_{k-1}) e^{-k\varepsilon/2}} \leq \exp \left( \frac{4 \log n}{\log \frac{2}{\varepsilon} + \log \log n} \right) \max_{1 \leq k \leq K} e^{-\lambda k\varepsilon/2} \omega_k.
\]
We can also upper bound the second term in (27),
\[
\frac{1}{\rho} \sum_{k=K+1}^{n} \omega_{k+}(\omega_{k} - \omega_{k-1+})e^{-k\varepsilon/2} \leq \frac{\text{GS}_f^2 e^{-K\varepsilon/2}}{\rho} \sum_{k=1}^{n-K} (k + K)e^{-k\varepsilon/2}
\]
\[
\leq O(1) \frac{\text{GS}_f^2}{\rho \varepsilon^2} e^{-K\varepsilon/2} + O(1) \frac{K \text{GS}_f^2}{\rho \varepsilon^2} e^{-K\varepsilon/2}
\]
\[
\leq O(1) \frac{\text{GS}_f^2}{\rho \varepsilon^2} e^{-K\varepsilon/2} + O(1) \frac{\text{GS}_f^2}{\rho \varepsilon^2} e^{-K\varepsilon/4} \leq O(1) \rho,
\]
where the third inequality follows since \( K\varepsilon \geq 4 \) and \( z e^{-z} \leq e^{-z/2} \) for \( z \geq 4 \), and the last inequality follows by setting \( \rho = n^{-p} \), \( K = \frac{8(p+2) \log n}{\varepsilon} \) and the assumptions that \( \text{GS}_f \leq n \) and \( \varepsilon \geq n^{-1} \). Proposition 4.1 now follows.

### C.6 Proof of Proposition 4.3

We prove inequalities (14a) and (14b) in turn. Our proofs in fact prove a stronger result for higher moments \( \mathbb{E}[|M(x) - f(x)|^p] \) for any \( p \in \mathbb{N} \).

**Proof of inequality (14a)** As in the proof of Theorem 3, Eq. (23), define the slices \( S_{x}^k := \{ t \in \mathcal{T} : \text{len}_f(x; t) = k \} \), and let \( k_0 \) be the minimal integer satisfying \( \omega_f(x; k) \geq \frac{\text{GS}_f}{\varepsilon} \). (If no such \( k_0 \) exists, then inequality (14a) is immediate.) By monotonicity, there exists \( t \in \mathbb{R} \) (w.l.o.g. we take \( t \geq f(x) \)) satisfying \( t - f(x) = \omega_f(x; k_0) \) and \( \text{len}_f(x; t) = k_0 \). Then by monotonicity, we have \( \text{len}_f(x; s) \leq k_0 \) for all \( f(x) \leq s \leq t \), and using the uniformity of \( \mu \) on \( \mathcal{T} \) we obtain
\[
\int_{\mathcal{T}} e^{-\text{len}_f(x; t)} d\mu(t) \geq \mu([0, \omega_f(x; k_0)]) e^{-k_0\varepsilon/2} \geq \mu([0, \text{GS}_f/\varepsilon]) e^{-k_0\varepsilon/2} \geq \frac{1}{\varepsilon} \mu([0, \text{GS}_f]) e^{-k_0\varepsilon/2}.
\]
At the same time, for each slice \( S_{x}^k \) we have by monotonicity (Def. 4.1) that there exist (at most) two points \( t_-, t_+ \) satisfying \( t_- \leq f(x) \leq t_+ \) and
\[
S_{x}^k \subset [t_- - \text{GS}_f, t_+ + \text{GS}_f] \cup [t_- - \text{GS}_f, t_+ + \text{GS}_f],
\]
so that \( \mu(S_{x}^k) \leq 4\mu([0, \text{GS}_f]) \) by uniformity. This gives the probability bound
\[
\mathbb{P}(M(x) \in S_{x}^k) = \frac{\mu(S_{x}^k) e^{-k\varepsilon/2}}{\sum_{l=0}^{n} \mu(S_{x}^l) e^{-l\varepsilon/2}} \leq \frac{\mu(S_{x}^k) e^{-k\varepsilon/2}}{\mu([0, \text{GS}_f/\varepsilon]) e^{-k_0\varepsilon/2}} \leq 4\varepsilon e^{-(k-k_0)\varepsilon/2}.
\]
Now, note that \( \omega_f(x; k) \leq \omega_f(x; k_0) + GS_f(k - k_0) \) for all \( k \geq k_0 \) to obtain

\[
\mathbb{E} [ |M(x) - f(x)|^p ] \leq \sum_{k=1}^{n} \mathbb{P} \left( M(x) \in S_x^k \right) \omega_f(x; k)^p \\
\leq \omega_f(x; k_0)^p + \sum_{k=k_0+1}^{n} \mathbb{P} \left( M(x) \in S_x^k \right) \omega_f(x; k)^p \\
\leq \omega_f(x; k_0)^p + 4 \varepsilon \sum_{k=k_0+1}^{n} e^{-(k-k_0)\varepsilon/2} (\omega_f(x; k_0) + (k - k_0)GS_f)^p \\
\leq \omega_f(x; k_0)^p + 2^{p+1} \varepsilon \sum_{k=k_0+1}^{n} e^{-(k-k_0)\varepsilon/2} \left( \omega_f(x; k_0)^p + (k - k_0)^p GS_f^p \right) \\
\leq C \cdot 2^p \cdot \left[ \omega_f(x; k_0)^p + \Gamma(p + 1) \frac{GS_f^p}{\varepsilon^p} \right],
\]

where inequality (i) used the probability bound (28) and inequality (ii) used that \( (a + b)^p \leq 2^{p-1} a^p + 2^{p-1} b^p \) for \( p \geq 1 \). Using that \( \omega_f(x; k_0) \leq \frac{GS_f}{\varepsilon} + GS_f = \frac{GS_f(1+\varepsilon)}{\varepsilon} \) by definition of \( k_0 \) then gives inequality (14a).

**Proof of inequality (14b)** Let \( T \in \mathbb{N} \) be (for now) arbitrary, and for shorthand define the quantity \( L := \max_{x : d_{ham}(x, x') \leq T} LS_f(x') \). We claim that for \( p \geq 1 \),

\[
\mathbb{E} [ |M(x) - f(x)|^p ]^{1/p} \leq C \left[ \frac{L}{\varepsilon} + GS_f T e^{-\frac{T \varepsilon}{2p}} \left( \frac{GS_f}{\rho} \frac{T}{\varepsilon/(2p) - 1} \right)^{1/p} \right]. \tag{29}
\]

To see how inequality (14b) follows from the bound (29), set

\[
T = \max \left\{ \left\lfloor \frac{4 p}{\varepsilon} \right\rfloor, \left\lfloor \frac{2 p}{\varepsilon} \right\rfloor \left( \log GS_f + \frac{1}{p} \log \frac{GS_f}{\rho \varepsilon} + \log \frac{1}{\gamma_f} \right) \right\}
\]

and note that in this case \( \frac{T \varepsilon}{2p} - 1 \geq \frac{T \varepsilon}{2p} \).

We thus prove inequality (29). Let \( k_0 \leq T \) denote the minimal integer such that \( \omega_f(x; k_0) \geq \frac{L}{T} \), as inequality (29) is immediate if no such \( k_0 \) exists. As in inequality (28), we have

\[
\mathbb{P} (M(x) \in S_x^k) \leq 4 \varepsilon e^{-(k-k_0)\varepsilon/2} \text{ for } k \geq k_0, \text{ and so}
\]

\[
\mathbb{E} [ |M(x) - f(x)|^p ] \leq \sum_{k=1}^{n} \mathbb{P} \left( M(x) \in S_x^k \right) \omega_f(x; k)^p \\
\leq \omega_f(x; k_0)^p + \sum_{k=k_0+1}^{T} \mathbb{P} \left( M(x) \in S_x^k \right) \omega_f(x; k)^p + \sum_{k=T+1}^{n} \mathbb{P} \left( M(x) \in S_x^k \right) \omega_f(x; k)^p. \tag{30}
\]

For the second term in the sum (30), the same derivation as in the proof of inequality (14a) yields

\[
\sum_{k=k_0+1}^{T} \mathbb{P} \left( M(x) \in S_x^k \right) \omega_f(x; k)^p \leq C \cdot 2^p \cdot \left[ \omega_f(x; k_0)^p + \Gamma(p + 1) \frac{\Gamma(p + 1)}{\varepsilon^p} \right].
\]

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For the third term in the sum (30), we use a utility inequality [7, Thm. 2.1] for the incomplete gamma function that
\[
\Gamma(\alpha, \tau) := \int_{\tau}^{\infty} e^{-u} u^{\alpha-1} du \leq \tau^\alpha \exp(-\tau) \cdot \frac{1}{[\tau + 1 - \alpha]_+}.
\]
An analogue of inequalities (25) in the proof of Lemma B.2 with \( d = 1 \) and (28) above implies that
\[
\mathbb{P}(M(x) \in S_0^k) \leq Ce^{-k\varepsilon/2} \frac{\mu([-GS_f, GS_f])}{\mu([0, \rho])} \leq C \frac{GS_f e^{-k\varepsilon/2}}{\rho}.
\]
As we always have \( \omega_f(x; k) \leq kGS_f \), the third term of inequality (30) has bound
\[
\sum_{k=T+1}^{n} \mathbb{P} \left( M(x) \in S_0^k \right) \omega_f(x; k)^p \leq C \frac{GS_f^{p+1}}{\rho} \sum_{k=T+1}^{n} k^p e^{-k\varepsilon/2} \leq C \frac{GS_f^{p+1}}{\rho} \left( \frac{2}{\varepsilon} \right)^{p+1} \int_{T\varepsilon/2}^{\infty} e^{-u} u^{p+1-1} du \leq C \frac{GS_f^{p+1}}{\rho} \frac{T^{p+1}}{[T\varepsilon/2 - p]_+} \exp(-T\varepsilon/2).
\]
Substituting into inequality (30) and using \( (p+1)^{1/p} \leq p \) yields
\[
\mathbb{E} [M(x) - f(x)]^{1/p} \leq \omega_f(x; k_0) + \frac{L}{\varepsilon} + GS_f T e^{-T\varepsilon/2p} \left( \frac{GS_f}{\rho} \frac{T}{[T\varepsilon/2 - p]_+} \right)^{1/p}.
\]
The claim (29) follows as \( \omega_f(x; k_0) \leq \frac{L(1+\varepsilon)}{\varepsilon} \) by construction and because \( p^{-1/p} \leq 1 \) for \( p \geq 1 \).

D Proofs associated with examples (Sections 4.1 and 5)

D.1 Partial minimization

We prove that both inequalities (11) and (12) hold in Example 2. We adopt standard empirical process notation for simplicity in the derivation, so that for a sample \( x = \{x_i\}_{i=1}^n \), \( P_n = n^{-1} \sum_{i=1}^n 1_{x_i} \) denotes the empirical measure on \( x \), and \( P_n f = n^{-1} \sum_{i=1}^n f(x_i) \). Given a modified sample \( x' = \{x'_i\}_{i=1}^n \) with \( d_{\text{ham}}(x', x) \), we use the shorthand
\[
Q_k = \frac{1}{n} \sum_{i=1}^n (1_{x'_i} - 1_{x_i}) = \frac{1}{n} \sum_{i: x'_i \neq x_i} (1_{x'_i} - 1_{x_i})
\]
for the difference measure, so that \( P'_n = n^{-1} \sum_{i=1}^n 1_{x'_i} = P_n + Q_k \). With this, we have that statistic of interest is \( \theta(x) = \arg\min_{\theta} P_n \ell(\theta, \beta; X) \), and given a modified sample \( x' \), we have \( \theta(x') = \arg\min_{\theta} \inf_{\beta} P_n \ell(\theta, \beta; X) \). For shorthand throughout the proof, we recall that \( \tau = (\theta, \beta) \) and \( L_n(\tau) = P_n \ell(\tau; X) \). First, we recapitulate inequalities (11) and (12) in the notation here. Inequality (11) becomes
\[
\omega_\theta(x; k) = \left( 1 + \frac{2AG_{0, \text{out}}^2 G_1 G_2 k}{G_{0, \text{in}}^3} \right) \frac{\sup_{Q_k \in Q_k} \left[ \int_{-1}^{Q_k} L_n(\tau)^{-1} Q_k \ell(\tau; X) \right]}{n}.
\]
while for $A = \bar{L}_n(\tau)^{-1}$ having first column $a$, inequality (12) becomes

$$\frac{k}{n} \|a\|_2 \cdot G_{0,\text{in}} \leq \sup_{Q_k \in Q_k} \left| \left[ \bar{L}_n(\tau)^{-1} Q_k \ell(\tau; X) \right]_1 \right| \leq \frac{2k}{n} \|a\|_2 \cdot G_{0,\text{out}}. \quad (32)$$

We prove each in turn. We begin with a few arguments that provide quantitative bounds on the change in solutions to the empirical risk minimization problem; these are more or less standard arguments in stability and implicit function theorems [14]. The main result we use is the following, which provides an expansion of $\tau$ in terms of the perturbation measure $Q_k$.

**Lemma D.1.** For $x \in \mathcal{X}^n$, define $\tau(x) = \operatorname{argmin}_x P_n \ell(\tau; x)$, and assume that $\bar{L}_n(\tau) \succeq \lambda I$ for some $\lambda > 0$. Let $x' \in \mathcal{X}^n$ satisfy $d_{\text{ham}}(x', x) = k$ and have corresponding difference measure $Q_k$, and assume that $k \leq \frac{\lambda^2}{6G_{0,\text{out}}G_2} \cdot n$. Then

$$\tau(x') - \tau(x) = -\bar{L}_n(\tau)^{-1} Q_k \ell(\tau; X) + e(x', x),$$

where

$$\|\tau(x') - \tau(x)\|_2 \leq \frac{6G_{0,\text{out}} k}{\lambda} \frac{k}{n} \quad \text{and} \quad \|e(x', x)\|_2 \leq \frac{24G_{0,\text{out}}^2 G_2}{\lambda^3} \cdot \frac{k^2}{n^2}.$$

The proof of the lemma is more or less standard convex analysis and perturbation theory and is somewhat tedious; we defer it to Section D.1.1.

With Lemma D.1 in place, inequalities (31) and (32) are straightforward. Under the assumptions of Example 2 we know that $A^{-1} = \bar{L}_n(\tau) \succeq G_1 I$, so that $A \succeq G_1^{-1} I$. In particular, the first diagonal entry $A_{11} \geq G_1^{-1}$. Thus, using the gradient containment guarantees of Assumption A, by considering the sign of the first coordinate $|\bar{L}_n(\tau)^{-1} \sum_{i \neq i'} \ell(\tau; x_i)|_1$ we know that at least one of $\pm \frac{k}{n} G_1^{-1} G_{0,\text{in}}$ must be contained in $|\bar{L}_n(\tau)^{-1} Q_k \ell(\tau; X)|_1$ as $Q_k$ varies over $Q_k$. Thus, we obtain that for each $k \leq \frac{\lambda^2}{6G_{0,\text{out}}G_2} \cdot n$, there exists $x'$ with $d_{\text{ham}}(x, x') = k$ such that

$$|\theta(x) - \theta(x')| \geq \sup_{Q_k \in Q_k} \left| \left[ \bar{L}_n(\tau)^{-1} Q_k \ell(\tau; X) \right]_1 \right| - \frac{24G_{0,\text{out}}^2 G_2}{\lambda^3} \cdot \frac{k^2}{n^2} \geq \left( 1 - \frac{24G_{0,\text{out}}^2 G_1 G_2}{\lambda^3 G_{0,\text{in}}} \cdot \frac{k}{n} \right) \sup_{Q_k \in Q_k} \left| \left[ \bar{L}_n(\tau)^{-1} Q_k \ell(\tau; X) \right]_1 \right|.$$

We obtain the upper bound on $\omega_{\theta}(x; k) = \sup_{d_{\text{ham}}(x', x) \leq k} |\theta(x) - \theta(x')|$ claimed in inequality (31) similarly. The final claim (32) is immediate by Cauchy-Schwarz.

**D.1.1 Proof of Lemma D.1**

We prove Lemma D.1 via series of lemmas on perturbations of solutions and inverses.

**Lemma D.2.** Let $L$ be convex, have $G_2$-Lipschitz Hessian, and satisfy $\bar{L}(\tau) \succeq \lambda I$. If $\|\bar{L}(\tau)\|_2 \leq \epsilon$ and $\epsilon < \frac{\lambda^2}{3G_2}$, then the minimizer $\tau^*$ of $L$ satisfies $\|\tau^* - \tau\|_2 \leq \frac{2\epsilon}{\lambda}$.

**Proof** Whenever $\|\tau' - \tau\|_2 \leq \frac{\lambda}{G_2}$, we have by convexity and smoothness that

$$L(\tau') \geq L(\tau) + \langle \bar{L}(\tau), \tau' - \tau \rangle + \frac{\lambda}{2} \|\tau' - \tau\|_2^2 - \frac{G_2}{6} \|\tau' - \tau\|_2^3$$

$$\geq L(\tau) + \langle \bar{L}(\tau), \tau' - \tau \rangle + \frac{\lambda}{3} \|\tau' - \tau\|_2^2.$$
If $\|\hat{L}(\tau)\|_2 \leq \epsilon$, we have for all $\tau'$ satisfying $\|\tau' - \tau\|_2 > \frac{3\epsilon}{\lambda}$ that

$$\langle \hat{L}(\tau), \tau' - \tau \rangle + \frac{\lambda}{3} \|\tau' - \tau\|_2^2 \geq -\epsilon \|\tau' - \tau\|_2 + \frac{\lambda}{3} \|\tau' - \tau\|_2^2 > 0.$$  

By convexity, $\alpha \mapsto L(\tau + \alpha(\tau' - \tau))$ is increasing in $\alpha \geq 1$ when $\|\tau' - \tau\|_2 \geq \frac{3\epsilon}{\lambda}$, so $L(\tau') > L(\tau)$ whenever $\|\tau' - \tau\|_2 > 3\epsilon/\lambda$. Verifying that $3\epsilon/\lambda < \lambda/G_2$ completes the proof.

**Lemma D.3.** Let the conditions and notation of of Lemma D.1 hold, and let $\tau = \tau(x)$ and $\tau' = \tau(x')$. Then there is an error matrix $E$ satisfying $\|E\|_2 \leq \frac{6G_{0,\text{out}}G_2 k}{\lambda n}$ such that

$$\tau' - \tau = -\left(\hat{L}_n(\tau) + E\right)^{-1} Q_k \hat{\ell}(\tau; X),$$

and $\|\tau' - \tau\|_2 \leq \frac{6G_{0,\text{out}} k}{\lambda n}$.

**Proof** As $\hat{L}_n(\tau) = 0$, we have

$$\|\left(P_n + Q_k\right)\hat{\ell}(\tau; X)\|_2 = \|Q_k \hat{\ell}(\tau; X)\|_2 \leq \frac{1}{n} \sum_{i: x_i \neq x_i'} \|\hat{\ell}(\tau; x_i) - \hat{\ell}(\tau; x_i')\|_2 \leq \frac{2k}{n} G_{0,\text{out}}.$$

Let $\tau' = \text{argmin}_{\tau'} (P_n + Q_k)\hat{\ell}(\tau; X)$. Using the assumption $\hat{L}(\tau) \succeq \lambda I$, Lemma D.2 implies that $\|\tau' - \tau\|_2 \leq \frac{6kG_{0,\text{out}}}{n\lambda}$. A Taylor expansion gives an error matrix $E: \mathbb{R}^d \times \mathbb{R}^d \times X \rightarrow \mathbb{R}^{d \times d}$ with $\|E(\tau, \tau'; x)\|_2 \leq G_2 \|\tau - \tau'\|_2$ such that

$$0 = (P_n + Q_k)\hat{\ell}(\tau'; X) = (P_n + Q_k)\hat{\ell}(\tau; X) + (P_n + Q_k)\hat{\ell}(\tau; X)(\tau' - \tau) + (P_n + Q_k)E(\tau', \tau; X)(\tau' - \tau) = Q_k \hat{\ell}(\tau; X) + \left(\hat{L}_n(\tau) + E_{n,k}\right)(\tau' - \tau),$$

where we use the shorthand $E_{n,k} = (P_n + Q_k)E(\tau', \tau; X)$. Moreover, $\|E_{n,k}\|_2 \leq G_2 \|\tau - \tau'\|_2 \leq \frac{6kG_2 G_{0,\text{out}}}{n\lambda}$ by our bounds on $\|\tau - \tau'\|_2$; inverting the preceding equality gives the lemma.

To control the error in Lemma D.3, we use a standard matrix perturbation result [40].

**Lemma D.4.** Let $A \succeq \lambda I$ and $\|E\|_2 \leq \epsilon$. Then there is $\Delta$ with $\|\Delta\|_2 \leq \frac{\epsilon^2}{2\|A\| - \epsilon}$ satisfying

$$(A + E)^{-1} = A^{-1} - A^{-1}EA^{-1} + \sum_{i=2}^{\infty} (-1)^i (A^{-1}E)^i A^{-1} = A^{-1} - A^{-1}(E + \Delta) A^{-1}.$$

Revisiting Lemma D.3, we have $\tau' - \tau = -\hat{L}_n(\tau)^{-1} Q_k \hat{\ell}(\tau; X) + e_{n,k}$ where for a matrix $\Delta$ satisfying $\|\Delta\|_2 \leq \|E\|_2 / (\lambda - \|E\|_2)$, the error $e_{n,k}$ satisfies

$$\|e_{n,k}\|_2 \leq \left\|\hat{L}_n(\tau)^{-1}(E + \Delta) \hat{L}_n(\tau)^{-1}\right\|_2 \|Q_k \hat{\ell}(\tau; X)\|_2 \leq 3 \|E\|_2 \lambda^2 \|Q_k \hat{\ell}(\tau; X)\|_2 \leq \frac{24G_{0,\text{out}}^2 G_2 k^2}{\lambda^3 n^2}.$$  

In inequality (i) we use that $\|E\|_2 \leq \lambda/2$ and in inequality (ii) that $\|E\|_2 \leq \frac{6G_{0,\text{out}} G_2 k}{\lambda n}$ and $\|Q_k \hat{\ell}(\tau; X)\|_2 \leq 2G_{0,\text{out}} k/n$. This gives Lemma D.1.
D.2 Proofs for the median example (Section 5.1)

We provide proofs for the median example here. We frequently use the following standard Chernoff bound.

**Lemma D.5** ([33], Ch. 4.2.1). Let $X = \sum_{i=1}^{n} X_i$ for $X_i \sim \text{Bernoulli}(p)$. Then for $\delta \in [0, 1]$,\[
\mathbb{P}(X > (1 + \delta)np) \leq e^{-np\delta^2/3} \quad \text{and} \quad \mathbb{P}(X < (1 - \delta)np) \leq e^{-np\delta^2/2}.
\]

Throughout this section, we let $\gamma > 0$, $m = \text{Median}(P)$, $\hat{m} = \text{Median}(x)$ be the empirical median, and $p_{\min} = \inf_{|t-m| \leq 2\gamma} \pi_P(t)$. First, we start with the following lemma, which proves that the empirical median is close to the true median with high probability.

**Lemma D.6.** Under the setting of Proposition 5.1, for any $0 < u \leq \gamma$,
\[
\mathbb{P}(|\hat{m} - m| > u) \leq 2e^{-nu^2p_{\min}^2}.
\]

**Proof** Let $B_i = 1\{x_i > m + u\}$ and $B = \sum_{i=1}^{n} B_i$ denote the number of elements larger than $m + u$. The definition of $p_{\min}$ implies that $\hat{p} = \mathbb{P}(B_i = 1) \leq 1/2 - up_{\min}$. If $\hat{m} > m + u$ then $B \geq n/2$, therefore we get\[
\mathbb{P}(\hat{m} > m + u) \leq \mathbb{P}(B \geq n/2) = \mathbb{P}(B \geq \hat{p}n(1 + 1/2\hat{p} - 1)) \leq e^{-nu^2p_{\min}^2},
\]
where the last inequality follows by using Chernoff bound of Lemma D.5 and $\hat{p} \leq 1/2 - up_{\min}$.

The same steps give that $\mathbb{P}(\hat{m} < m - u) \leq e^{-nu^2p_{\min}^2}$, which proves the claim. \(\square\)

D.2.1 Proof of Proposition 5.1

Let us first divide the interval $[m - \gamma, m + \gamma]$ to blocks of size $u$: $I_1, I_2, \ldots, I_T$. Let $N_i$ denote the number of elements in $I_i$ and $A$ denote the event that $N_i \geq nu_{\min}/2$ for all $i$, and $B$ denote the event that $|m - \hat{m}| \leq \gamma/2$.

**Lemma D.7.** Under the above setting,
\[
\mathbb{P}(A \mid B) \geq 1 - \frac{2\gamma}{u} e^{-nu_{\min}/8} - 2e^{-nu^2_{\min}^2/4}.
\]

**Proof** Let $Z_i = 1\{x_i \in I_j\}$ and $N_j = \sum_{i=1}^{n} Z_i$ be the number of elements inside block $I_j$. As $\mathbb{P}(Z_i = 1) \geq up_{\min}$, we use the Chernoff bound of Lemma D.5 to get\[
\mathbb{P}(N_j < nu_{\min}/2) \leq e^{-nu_{\min}/8}.
\]
Thus using a union bound we have\[
\mathbb{P}(A^c) \leq \frac{2\gamma}{u} e^{-nu_{\min}/8}.
\]
Using that for any events $A, B$ we have $\mathbb{P}(A \mid B) \geq \mathbb{P}(A \mid B)\mathbb{P}(B) = \mathbb{P}(A) - \mathbb{P}(A, B^c) \geq \mathbb{P}(A) - \mathbb{P}(B^c)$, the preceding display and Lemma D.6 give $\mathbb{P}(A \mid B) \geq \mathbb{P}(A) - \mathbb{P}(B^c) \geq 1 - \frac{2\gamma}{u} e^{-nu_{\min}/8} - 2e^{-nu^2_{\min}^2/4}$. \(\square\)
Now, we complete the proof of Proposition 5.1. First, if events $A$ and $B$ occur, then we know that for any $t$ such that $|t - \hat{m}| > 2u$, there are at least $nup_{\text{min}}/2$ elements between $\hat{m}$ and $t$. Therefore $\text{len}_f(x; t) \geq nup_{\text{min}}/2$ for any $t$ such that $|t - \hat{m}| > 2u$, which implies that $\text{len}_f(x; s) \geq nup_{\text{min}}/2$ for $s$ such that $|s - \hat{m}| > 2u + \rho$. The definition (M.2) of the mechanism implies then that for $t$ such that $|t - \hat{m}| > 2u + \rho$,

$$
\pi_{M_{\text{cont}}}(x)(t \mid A, B) = \frac{e^{-\text{len}_f(x; t)\varepsilon/2}}{\int_{s \in T} e^{-\text{len}_f(x; s)\varepsilon/2}} \leq \frac{e^{-nup_{\text{min}}\varepsilon/4}}{\rho}.
$$

Using a union bound to gives

$$
\mathbb{P}(|M_{\text{cont}}(x) - \hat{m}| > 2u + \rho) \leq \mathbb{P}(|M_{\text{cont}}(x) - \hat{m}| > 2u + \rho \mid A, B) + \mathbb{P}(B) + \mathbb{P}(A \mid B)
$$

$$
\leq \frac{R}{\rho} e^{-nup_{\text{min}}\varepsilon/4} + 4e^{-n\gamma^2 p_{\text{min}}^2/4} + \frac{2\gamma}{u} e^{-nup_{\text{min}}/8}.
$$

### D.2.2 Proof of Lemma 5.2

Recall the definition (5) of the smooth Laplace mechanism as $M_{\text{am-Lap}}(x) = f(x) + \frac{2S(x)}{\varepsilon} \text{Lap}(1)$, where $S(x)$ satisfies $\text{LS}(x) \leq S(x)$ and $S(x) \leq \varepsilon \beta S(x')$ for neighboring instances $x, x' \in X^n$ and $\beta = \frac{\varepsilon}{2 \ln(2/\delta)}$. To prove the lemma, we show that $S(x) \geq \frac{\log(n)}{2p_{\text{max}}n\varepsilon}$ with high probability.

The main idea is to show that there exists an instance $x'$ such that $d_{\text{ham}}(x, x') \leq 1/\beta$ and $\text{LS}(x') \geq \frac{\log(n)}{2p_{\text{max}}n\varepsilon}$. To find $x'$, we show that—with high probability—there exist at most $1/\beta$ elements between $\text{Median}(x)$ and $\text{Median}(x) + \frac{\log(n)}{2p_{\text{max}}n\varepsilon}$, so we get our desired $x'$ by increasing the values of elements $x_i$ in this range.

Let $c > 0$, to be chosen, and define $\gamma = \frac{c \log(n)}{n\varepsilon}$, $p_{\text{max}} = \sup_t \pi_P(t)$, $p_{\text{min}} = \inf_t \pi_P(t)$, and

$$
Z_i := 1\{ |x_i - \text{Median}(x)| \leq \gamma \}.
$$

Then $\sum_{i=1}^n Z_i$ upper bounds the number of elements between $\text{Median}(x) - \gamma$ and $\text{Median}(x) + \gamma$. Now, we show that on the event that $\sum_{i=1}^n Z_i \leq \frac{2\log(n)}{\varepsilon}$, we have that $S(x) \geq \gamma$. Indeed, assume $x = (x_1 \leq x_2 \leq \cdots \leq x_n)$ such that $x_i = \text{Median}(x)$. Since $\sum_{i=1}^n Z_i \leq \frac{2\log(n)}{\varepsilon}$, there exists $x_j \geq x_i + \gamma$ such that $|i - j| \leq \sum_{i=1}^n Z_i$. Consider the instance $x'$ with

$$
\begin{cases}
    x'_{i} = x_j & \text{if } i < k < j \\
    x'_{k} & \text{otherwise}
\end{cases}
$$

To prove a lower bound on $S(x)$, we need to show that $x'$ has large local sensitivity and is not too far from $x$. Indeed, we have $\text{LS}(x') \geq |x_i - x_j| \geq \gamma$ as $\text{Median}(x') = x_i$ but we can change it to $x_j$ by changing one user, i.e., $x'_i = x_j$. Moreover, $d_{\text{ham}}(x, x') \leq \sum_{i=1}^n Z_i \leq \frac{2\log(n)}{\varepsilon}$, so that as $\beta = \frac{\varepsilon}{2 \ln(2/\delta)} \leq \frac{\varepsilon}{2 \log(n)}$, we have

$$
S(x) \geq e^{-d_{\text{ham}}(x, x')/\beta} S(x') \geq e^{-\beta \sum_{i=1}^n Z_i \gamma} \geq e^{-1} \gamma = \frac{c \log(n)}{ene\varepsilon}.
$$

The following lemma thus completes the proof of Lemma 5.2.

**Lemma D.8.** Under the above setting, if $c = \frac{1}{2p_{\text{max}}}$,

$$
\mathbb{P}\left( \sum_{i=1}^n Z_i > \frac{2\log(n)}{\varepsilon} \right) \leq \exp\left(-\frac{p_{\text{min}} \log(n)}{16p_{\text{max}} \varepsilon} + \log(2Rn\varepsilon p_{\text{max}})\right).
$$

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Proof Let us divide the interval \([0, R]\) to \(R/\gamma\) intervals, each of length \(\gamma\). Let \(B_i\) denote the number of points inside the \(i\)th such interval. Then, as \(n\gamma p_{\min} \leq E[B_i] \leq n\gamma p_{\max}\), the Chernoff bound of Lemma D.5 yields

\[
P \left( B_i > \frac{\log(n)}{\varepsilon} \right) = P \left( B_i > 2n\gamma p_{\max} \right) \leq e^{-n\gamma p_{\min}/8}.
\]

As evidently \(\sum_{i=1}^{n} Z_i \leq 2\max_i B_i\), we apply union bound over the intervals to see that

\[
P \left( \sum_{i=1}^{n} Z_i > 2\frac{\log(n)}{\varepsilon} \right) \leq P \left( \max_i B_i > \frac{\log(n)}{\varepsilon} \right) \leq \frac{R}{\gamma} e^{-n\gamma p_{\min}/8} = \frac{2Rn\gamma p_{\max}}{\log n} e^{-\frac{p_{\min}n}{16p_{\max}e}}
\]
as desired. \(\square\)

D.3 Proofs for robust regression (Section 5.2)

We collect the proofs of results associated with the robust regression examples in Section D.3 here. In our proofs, we use the shorthand \(L(\theta; x, y) = \sum_{i=1}^{n} h(\langle \theta, x_i \rangle - y_i)\) so that \(L_n(\theta; x, y) = \frac{1}{n} L(\theta; x, y)\)

### D.3.1 Proof of Lemma 5.3

Building on Lemma 3.1, it is enough to prove that \(\text{len}_{\text{add}}\) (16) is 1-Lipschitz. Let \((x, y) = \{(x_i, y_i)\}_{i=1}^{n}\) and \((x', y') = (x, y) \cup \{(x'_{n+1}, y'_{n+1})\}\). Then for every \(\theta \in \Theta\), we prove that \(|\text{len}_{\text{add}}(x, y; \theta) - \text{len}_{\text{add}}(x', y'; \theta)| \leq 1\). It is clear that \(\text{len}_{\text{add}}(x, y; \theta) \leq \text{len}_{\text{add}}(x', y'; \theta) + 1\) as we can add one user to \((x, y)\) to get the dataset \((x', y')\). Hence we only need to prove that \(\text{len}_{\text{add}}(x', y'; \theta) \leq \text{len}_{\text{add}}(x, y; \theta) + 1\). Let \(k = \text{len}_{\text{add}}(x, y; \theta)\). Therefore there exist \(\bar{x}, \bar{y} = \{(\bar{x}_i, \bar{y}_i)\}_{i=1}^{k}\) such that \(\theta\) becomes a minimizer of the loss function when we add them to the dataset \((x, y)\), hence

\[
\nabla L(\theta; x, y) + \nabla L(\theta; \bar{x}, \bar{y}) = 0.
\]

Therefore we have that

\[
\nabla L(\theta; x', y') + \nabla L(\theta; \bar{x}, \bar{y}) = \nabla g h(\langle \theta, x'_{n+1} \rangle - y'_{n+1}).
\]

Therefore we can add \(x_{k+1}, y_{k+1}\) that makes \(\theta\) a minimizer. Indeed, we only need to guarantee that \(\nabla g h(\langle \theta, x'_{n+1} \rangle - y_{n+1}) = -\nabla g h(\langle \theta, x_{k+1} \rangle - y_{k+1}).\) Setting \(x_{k+1} = -x_{n+1}\) and \(y_{k+1} = -2(\theta, x_{n+1}) + y_{n+1}\) proves the lemma.

### D.3.2 Proof of Lemma 5.4

A vector \(\theta \in \text{int} \Theta\) minimizes \(L(\theta; x', y')\) if and only if

\[
\nabla L(\theta; x', y') = \nabla L(\theta; x, y) + \sum_{i=1}^{k} \nabla g h(\langle x'_i, \theta \rangle - y'_i) = 0.
\]

As \(h\) is 1-Lipschitz and \(\|x\|_2 \leq r\), we have that \(\|\nabla g h(\langle x'_i, \theta \rangle - y'_i)\| \leq r\) for every \(1 \leq i \leq k\), so \(\text{len}_{\text{add}}(x, y; \theta) \geq \|\nabla L(\theta; x, y)\|_2 / r\). Now we show that adding \(k = \|\nabla L(\theta; x, y)\|_2 / r\) points is enough. First, denote \(g = \nabla L(\theta; x, y)\), and let \(x'_i = -r \frac{g}{\|g\|_2}\) for every \(1 \leq i \leq k\) and choose \(y'_i \in \mathbb{R}\) such that \(h'(\langle x'_i, \theta \rangle - y'_i) = 1\), for \(i \leq k - 1\), which is possible as \(h\) is 1-Lipschitz and symmetric. We now have \(\nabla L(\theta; x', y') = \nabla L(\theta; x, y) - (k - 1)r \frac{g}{\|g\|_2} = \gamma g / \|g\|_2\) for some \(\gamma \in [0, 1]\). Take \(y'_k\) such that \(h'(\langle x'_k, \theta \rangle - y'_k) = \gamma\).
D.3.3 Sampling from gamma-like distributions

We wish to sample a vector \( T \in \mathbb{R}^d \) with density \( \pi(t) = \exp(-\|At\|) \) for a matrix \( A > 0 \). The change of variables \( u = At \) and then using rotational symmetry gives that

\[
\int \pi(t)dt = \frac{1}{\det(A)} \int \exp(-\|u\|)du = \frac{1}{\det(A)} \int_0^\infty \exp(-r)Vol_{d-1}(rS^{d-1})dr
\]

\[
= \frac{1}{\det(A)} \frac{d\pi^{d/2}}{\Gamma(d/2 + 1)} \int_0^\infty r^{d-1}e^{-r}dr = \frac{d\pi^{d/2}\Gamma(d)}{\det(A)\Gamma(d/2 + 1)}.
\]

In particular, to sample \( T \) with the density \( \pi(t) = \exp(-\|At\|) \), we draw \( R \sim \text{Gamma}(d, 1) \), then \( U = RA^{-1}U \), and set \( T = RA^{-1}U \).

D.3.4 Proof sketch of Lemma 5.5

To simplify notation, we analyze the case when \( \|x_i\|_2 \leq 1 \), and we follow the empirical process notation of Sec. D.1.

By Taylor’s theorem, the Lipschitz continuity of \( \nabla L \) implies there exists an error function \( E_n : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \) satisfying \( \|\nabla E_n(\theta)\| \leq (P_nG_1\|\theta - \theta_n\|) \) for which

\[
\nabla L_n(\theta) = \nabla L_n(\theta_n) + (\nabla^2 L_n(\theta_n) + E_n(\theta))(\theta - \theta_n).
\]

Let \( Z_\pi = \int_\Theta \exp(-n\xi \|\nabla L_n(\theta)\|/2)d\theta \) and \( Z_q = \int_\Theta \exp(-n\xi (\|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| \wedge r_n))d\theta \) be the normalizing constants for the densities \( \pi \) and \( q \). We would like to demonstrate that \( q(t)/\pi(t) \geq \beta > 0 \) for all \( t \). We consider two cases.

(i) Assume that \( \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| \leq r_n \). In this case, the triangle inequality implies that

\[
\frac{q(\theta)}{\pi(\theta)} \geq \frac{Z_\pi}{Z_q} \exp \left( -\frac{\xi n}{2} \|\nabla E_n(\theta)(\theta - \theta_n)\| \right)
\]

\[
\geq \frac{Z_\pi}{Z_q} \exp \left( -\frac{\xi n}{2} (P_nG_1\|\theta - \theta_n\|^2) \right) \geq \frac{Z_\pi}{Z_q} \exp(-\xi n\|n\theta_n\|^2) = \frac{Z_\pi}{Z_q}(1 - o(1)).
\]

(ii) Assume that \( \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| \geq r_n \). This case is a bit more subtle. We consider two regimes. In the first, we have \( \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| \in [r_n, r_n^{2/3}] \). Then

\[
\|\nabla L_n(\theta_n)\| \geq \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| - P_nG_1\|\theta - \theta_n\|^2 \geq r_n - O(r_n^{4/3}).
\]

We consequently obtain

\[
\frac{q(\theta)}{\pi(\theta)} = \frac{Z_\pi}{Z_q} \exp \left( -\frac{\xi n r_n}{2} + \frac{\xi r_n n}{2} \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| - O(nr_n^{4/3}) \right) \geq \frac{Z_\pi}{Z_q} \exp(-O(nr_n^{4/3})).
\]

In the regime that \( \|\nabla^2 L_n(\theta_n)(\theta - \theta_n)\| \geq r_n^{2/3} \), we require a bit more work. First, we define \( f_n(\theta) = (\nabla L_n(\theta), \theta - \theta_n) / \|\theta - \theta_n\| \), noting by convexity that \( r \mapsto f(\theta_n + r\theta - \theta_n) \) is monotone increasing. As \( r_n \leq 1 \), if we define \( u = r_n\frac{\theta - \theta_n}{\|\theta - \theta_n\|} \), this monotonicity implies

\[
\|\nabla L_n(\theta)\| \geq f_n(\theta) = f_n(\theta_n + (\theta - \theta_n)) \geq f_n(\theta_n + u) = (\nabla L_n(\theta_n + u), u / \|u\|) = (\nabla^2 L_n(\theta_n)u, u / \|u\|) \pm (E_n(\theta_n + u)u, u / \|u\|)
\]

\[
\geq (\nabla^2 L_n(\theta_n)u, u / \|u\|) - P_nG_1\|u\|\|u\| \geq \lambda\|u\| - P_nG_1\|u\|^2 = \lambda r_n^{2/3} - O(r_n^{4/3}).
\]

In particular, \( q(\theta)/\pi(\theta) \geq \frac{Z_\pi}{Z_q} \exp\left(-\frac{\xi n r_n}{2} + \frac{\xi r_n n}{2} - O(nr_n^{4/3})\right) \gg 1 \), as \( n^{-1} \ll r_n \ll n^{-3/4} \).
In either case, our condition that $n^{-1} \ll r_n \ll n^{-3/4}$ gives that \( \frac{q(\theta)}{\pi(\theta)} \geq \frac{Z_\pi}{Z_q}(1 - o(1)) \) as $n$ grows. A similar calculation to the two cases above gives that $Z_\pi/Z_q \gtrsim 1$.

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