Non-zero constraints in elliptic PDE with random boundary values and applications to hybrid inverse problems

Giovanni S Alberti

MaLGa Center, Department of Mathematics, University of Genoa, Via Dodecaneso 35, 16146 Genova, Italy
E-mail: giovanni.alberti@unige.it

Received 2 May 2022, revised 20 September 2022
Accepted for publication 11 October 2022
Published 28 October 2022

Abstract

Hybrid inverse problems are based on the interplay of two types of waves, in order to allow for imaging with both high resolution and high contrast. The inversion procedure often consists of two steps: first, internal measurements involving the unknown parameters and some related quantities are obtained, and, second, the unknown parameters have to be reconstructed from the internal data. The reconstruction in the second step requires the solutions of certain PDE to satisfy some non-zero constraints, such as the absence of nodal or critical points, or a non-vanishing Jacobian.

In this work, we consider a second-order elliptic PDE and show that it is possible to satisfy these constraints with overwhelming probability by choosing the boundary values randomly, following a sub-Gaussian distribution. The proof is based on a new quantitative estimate for the Runge approximation, a result of independent interest.

Keywords: hybrid inverse problems, coupled-physics imaging, photoacoustic tomography, non-zero constraints, Runge approximation, elliptic equations

1. Introduction

Many inverse problems for partial differential equations (PDE) are severely ill-posed, meaning that the best possible stability is of logarithmic type [11, 52]. This phenomenon is typically due to the infinite smoothing effect of the PDE involved, and has a big impact in the implementation, because only low-resolution reconstructions are possible, despite the fact that the parameters under investigation exhibit high contrast and are relevant for the applications [64, 65]. Several examples include the Calderón problem for electrical impedance tomography...
Inverse Problems 38 (2022) 124005 G S Alberti

(EIT) [34, 62], inverse scattering problems [39] and optical tomography [20]. On the other hand, other modalities, such as ultrasonography (modelled by the wave equation, which preserves singularities) and magnetic resonance imaging (MRI) exhibit high resolution, but may have low contrast in certain settings.

In order to overcome this limitation, a class of inverse problems has been extensively studied over the last decade. These are called hybrid or coupled-physics problems, because they involve the interplay of two physical modalities, one providing high-contrast, and one providing high resolution [8, 19, 21, 47]. The most known hybrid modality is photoacoustic tomography (PAT) [48, 64], in which light and ultrasounds are combined to image the high-contrast optical absorption by making high-resolution ultrasonic measurements. Many other techniques have been considered, including thermoacoustic tomography (combining microwaves and ultrasounds) [48, 64], acousto-electric tomography (combining ultrasound-induced deformations and electrical measurements) [1, 17, 49, 67] and magnetic resonance electrical impedance tomography (combining MRI and EIT) [58].

Because of the combination of two modalities, the inversion process in hybrid problems usually consists of two steps. In a first step, by solving an inverse problem for the high-resolution/low-contrast modality, some internal data is reconstructed. This typically contains the unknown parameter(s), as well as the solution(s) of the corresponding PDE related to the low-resolution/high-contrast modality (for instance, in PAT, the internal data is the product between the optical absorption $\mu(x)$ and the light intensity $u(x)$). In the second step, the actual unknown of the problem ($\mu(x)$, in PAT) has to be reconstructed from the internal data, by solving an inverse problem for the low-resolution/high-contrast modality. While a lot of research has been done on the first step, from both the theoretical and experimental points of view, the second step has received far less attention.

In this paper, we study a crucial aspect for solving the inverse problem in the second step: the solutions to the PDE under consideration should satisfy certain non-zero constraints, depending on the problem, such as the absence of nodal or critical points, or a non-vanishing Jacobian. It is worth observing that, by using unique continuation estimates, it is possible to have uniqueness and stability of the inverse problem even when these non-zero constraints are not satisfied [12, 13, 31, 37, 38]. However, the non-zero constraints allow for optimal stability estimates (of Lipschitz type) and, very often, explicit reconstruction formulae [8, 21]. In very simplified terms (with the risk of not capturing the complete picture), when these non-zero constraints are satisfied, it is possible to avoid the problem of ‘division by zero’.

There exist several methods to construct suitable solutions that satisfy the relevant non-zero constraints, including methods based on generalisations of the Radó–Kneser–Choquet theorem [10, 14, 15, 30], on complex geometrical optics solutions (CGO) [21, 22, 25, 27, 29], on the use of multiple frequencies [2–5], on dynamical systems [23] and on the Runge approximation [9, 29]. All these techniques have several drawbacks, for example the generalisations of the Radó–Kneser–Choquet theorem are valid only in 2D and for coercive elliptic problems [7, 8, 35], the construction with CGO depends on the unknown coefficients, which have to be very smooth and isotropic, and the use of multiple frequencies works only with frequency-dependent problems.

The approach based on the Runge approximation [50, 51] is very flexible, because it is valid with many PDE, allows for anisotropic coefficients, and the smoothness assumptions are very mild, since only the unique continuation property is needed. However, in the original formulation [29], the suitable solutions are not explicitly constructed (the existence follows from the Hahn–Banach theorem, hence from the axiom of choice) and depend on the unknown coefficients, as with CGO. By combining this approach with the Whitney embedding theorem, it is possible to prove that the set of suitable solutions is open and dense [9], with explicit
estimates on the number of solutions needed (typically very small). However, the ‘open and dense’ condition does not imply anything on the ‘size’ of this set, since an open and dense set may have arbitrarily small measure, which may make it hard to construct, in practice, these solutions. Furthermore, no quantitative lower bound for the constraints is given, potentially causing the problem of ‘division by a very small number’ (again, by oversimplifying the matter).

The focus of this work is on overcoming these limitations. We consider a second-order elliptic PDE and prove that, by choosing random boundary values, sampled with respect to a fixed sub-Gaussian distribution, the corresponding solutions will satisfy the non-zero constraints with overwhelming probability (see theorem 1). In this way, the suitable boundary conditions are explicitly constructed, and independent of the unknown parameters. Furthermore, a quantitative lower bound is derived. The price to pay is a slightly larger number of solutions needed if compared to the one in [9], which is due to the use of concentration inequalities.

A crucial ingredient of the proof of theorem 1 is a new result on quantitative Runge approximation. Classical Runge approximation allows for approximating a local solutions $h$ in $D$ to a PDE by a global solution $u$ in $\Omega$ (with $D \subseteq \Omega$) [32, 50, 51], and is based on the unique continuation property [16]. This construction was made quantitative in [56], where the authors obtain an estimate on $\|u|_{\partial D}\|_{H^1(\partial D)}$. In the second main result of this paper, theorem 2, we obtain an estimate of a stronger norm of $u|_{\partial D}$, which is needed in the proof of theorem 1. This result is also of independent interest, since the Runge approximation finds applications not only to hybrid problems, but also to many other inverse problems for PDE (see [56] and references therein).

This work is structured as follows. In section 2 we describe the setup we consider, and state the main result of this work on satisfying non-zero constraints in elliptic PDE with random boundary measurements, theorem 1. In section 3 we state the result on quantitative Runge approximation, theorem 2. These two theorems are then proved in sections 4 and 5, respectively.

2. Non-zero constraints in PDE with random boundary values

2.1. The elliptic equation

In this paper, we consider a second-order elliptic equation of the form

$$Lu := -\text{div}(a\nabla u) + qu = 0 \quad \text{in} \quad \Omega,$$

where we assume that

- The domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is bounded, Lipschitz and connected;
- The diffusion coefficient $a \in W^{1,\infty}(\Omega; \mathbb{R}^{d \times d})$, is symmetric ($a^T = a$) and satisfies

$$a(x)\xi \cdot \xi \geq \Lambda^{-1}|\xi|^2, \quad x \in \Omega, \, \xi \in \mathbb{R}^d,$$

and

$$\|a\|_{W^{1,\infty}(\Omega)} \leq \Lambda$$

for some $\Lambda \geq 1$;
- The potential $q \in L^{\infty}(\Omega)$ satisfies

$$\|q\|_{\infty} \leq \Lambda.$$

3
0 is not a Dirichlet eigenvalue of \( L \) in \( \Omega \) and
\[
\|L^{-1}\|_{H^{-1}(\Omega)\rightarrow H^1(\Omega)} \leq \Lambda, \tag{5}
\]
where \( H^{-1}(\Omega) \) denotes the dual of \( H^1_0(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \partial \Omega \} \).

Unless specified, all function spaces considered in this paper consist of real-valued functions.

### 2.2. Non-zero constraints

Let \( \Omega' \subset \Omega \) be a Lipschitz domain where we are interested in satisfying the non-zero constraints. The results presented here would work also with \( \Omega' = \Omega \), but with additional technicalities. We are looking for boundary conditions \( \varphi_1, \ldots, \varphi_n \in H^{1/2}(\partial \Omega) \) such that the corresponding solutions \( u_1, \ldots, u_n \in H^1(\Omega) \) to
\[
\begin{cases}
Lu_i = -\text{div}(a \nabla u_i) + qu_i = 0 & \text{in } \Omega, \\
u_i = \varphi_i & \text{on } \partial \Omega,
\end{cases}
\tag{6}
\]
satisfy the constraint
\[
\zeta(u_1, \ldots, u_n)(x) \neq 0, \tag{7}
\]
in \( \Omega' \), locally or globally. Here, \( n \geq 1 \) and the map
\[
\zeta : C^{1,1/2}(\overline{\Omega'})^n \rightarrow C^{0,1/2}(\overline{\Omega'})
\]
is multilinear and bounded. Furthermore, we require \( \zeta \) to be local, in the sense that for every \( x \in \overline{\Omega'} \) and for every \( u, v \in C^{1,1/2}(\overline{\Omega'})^n \) there exists \( r > 0 \) such that, if \( u|_{B(x,r)} = v|_{B(x,r)} \), then \( \zeta(u)(x) = \zeta(v)(x) \).

Note that, thanks to (3), \( u_i \in C^{1,1/2}(\overline{\Omega'}) \) by classical elliptic regularity theory \([41, \text{ theorem 8.32}]\), and so (7) is well-defined. We could have considered \( C^{1,\alpha}(\overline{\Omega'}) \) with any \( \alpha \in (0,1) \), and we chose \( \alpha = 1/2 \) only for simplicity. We decided to consider only Dirichlet boundary values in order to simplify the exposition, but the whole approach would easily extend to other types of boundary conditions.

### 2.3. Examples

As examples of this general framework, we consider here some particular cases of maps \( \zeta \), and we mention the corresponding applications to the reconstruction procedures for several hybrid inverse problems. We present here only simplified settings, we omit the details and provide only local arguments, in order to better illustrate the role of the non-zero constraints with toy models. For more complete models and the full derivations see, e.g., \([3, 8, 19, 21, 28, 29, 47]\) and the references therein.

**Example 1 (Nodal points).** Let \( n = 1 \) and \( \zeta_1(u) = u \). This simply yields the constraint
\[
u(x) \neq 0, \tag{8}
\]
namely, the (local or global) absence of nodal points. This constraint appears, for example, in dynamic elastography \([53]\) and in quantitative photoacoustic tomography (QPAT) \([6, 26, 48]\).
For the sake of illustration, let us consider a simplified model of QPAT in which the absorption coefficient \( \mu \in L^\infty(\Omega) \), appearing as a potential in the diffusion equation
\[
-\Delta u + \mu u = 0 \quad \text{in } \Omega,
\]
has to be reconstructed from the knowledge of the internal energy
\[
H = \mu u \quad \text{in } \Omega.
\]
If the boundary condition of \( u \) is known, by solving the Poisson equation
\[
\Delta u = H \quad \text{in } \Omega,
\]
we can recover \( u \) in \( \Omega \). Finally, \( \mu \) can be recovered in \( x \in \Omega \) by using
\[
\mu(x) = \frac{H(x)}{u(x)},
\]
provided that the constraint (8) is satisfied.

**Example 2 (Critical points).** Let \( n = 1 \) and \( \zeta(u) = \partial_{x_1} u \). This choice for \( \zeta \) corresponds to the constraint
\[
\partial_{x_1} u(x) \neq 0,
\]
which implies the absence of critical points. We consider the problem of reconstructing \( a \) in
\[
-\text{div}(a \nabla u) = 0 \quad \text{in } \Omega,
\]
from the knowledge of the potential \( u \) in \( \Omega \), as in [10, 42]. This problem is motivated, for instance, by the inverse problem of aquifer hydrology [54], but is very related to the inverse problems of current density imaging [24, 66] and of acousto-electric tomography [1, 17, 36]. If \( a \) is scalar, we obtain
\[
\nabla (\log a) \cdot \nabla u = -\Delta u \quad \text{in } \Omega.
\]
Since \( \nabla u \) and \( \Delta u \) are known, this is a first-order linear PDE in \( \log a \), and can be solved by using the method of characteristics if \( u \) does not have critical points.

**Example 3 (Jacobian).** Let \( n = d \) and \( \zeta(u_1, \ldots, u_d) = \det [\nabla u_1 \ldots \nabla u_d] \), which yields the constraint
\[
\det [\nabla u_1 \ldots \nabla u_d](x) \neq 0.
\]
In other words, we look for a non-vanishing Jacobian. A simple application of this constraint is in the same inverse problem considered in the previous example. Suppose that we have \( d \) measurements \( u_1, \ldots, u_d \). By (11) we obtain
\[
(\nabla (\log a))^T [\nabla u_1 \ldots \nabla u_d] = -[\Delta u_1 \ldots \Delta u_d] \quad \text{in } \Omega.
\]
If (12) holds true, we can immediately obtain \( \nabla (\log a) \), hence \( a \) up to a multiplicative constant, by inverting the matrix \( [\nabla u_1 \ldots \nabla u_d] \) in the above identity.

**Example 4 (Augmented Jacobian).** Let \( n = d + 1 \) and
\[
\zeta_d(u_1, \ldots, u_{d+1}) = \det \begin{bmatrix} u_1 & \cdots & u_{d+1} \\ \nabla u_1 & \cdots & \nabla u_{d+1} \end{bmatrix}.
\]
This yields the constraint
\[
\text{det} \begin{bmatrix} \nabla u_1 & \cdots & \nabla u_{d+1} \\
\nabla u_1 & \cdots & \nabla u_{d+1} \end{bmatrix}(x) \neq 0, \tag{13}
\]
the so-called non-vanishing augmented Jacobian. This constraint appears, for instance, in QPAT. As in example 1, consider again the inverse problem of recovering the absorption coefficient \( \mu \) in (9) from the internal energy
\[
H(x) = \Gamma(x)\mu(x)u(x),
\]
where \( \Gamma \) is the Grüneisen parameter, which is now supposed unknown. Suppose we have at our disposal \( d + 1 \) measurements \( H_i = \Gamma \mu u_i \) with \( u_i \neq 0 \) in \( \Omega \) (see the constraint in example 1).

Setting \( v_i = u_i / u_1 \), it is easy to show that
\[
-\text{div}(u_1^2 \nabla v_i) = 0 \quad \text{in} \quad \Omega.
\]
Further, (13) implies that
\[
\text{det}[\nabla v_2 \ldots \nabla v_{d+1}](x) \neq 0,
\]
see [8, 18]. Thus, since \( v_i = H_i / H_1 \) is known, \( u_1 \) may be recovered by arguing as in example 3. Once \( u_1 \) is known, \( \mu \) may be recovered as in example 1, see (10).

2.4. Main result

Before giving our main result, we need to state two hypotheses. In the first one, we assume that, at least in the case of constant coefficients, the constraint (7) can be satisfied.

**Assumption 1.** Let \( E > 0 \). We assume that for every \( x_0 \in \overline{\Omega} \) there exist \( u_0^1, \ldots, u_0^n \in C^{1,1/2}(\Omega) \) such that \( \text{div}(a(x_0)\nabla u_0^i) = 0 \) in \( \Omega \), \( \|u_0^i\|_{C^{1,1/2}(\Omega)} \leq E \) for every \( i = 1, \ldots, n \) and
\[
|\zeta(u_0^1, \ldots, u_0^n)(x_0)| \geq 1. \tag{14}
\]

**Remark.** We explicitly observe that this assumption is trivially satisfied for all \( a \) and for all the constraints introduced in section 2.3. Indeed, the functions \( 1, x_1, \ldots, x_d \) trivially satisfy the PDE \( \text{div}(a(x_0)\nabla u_0^i) = 0 \) (because their gradients are constant), and
\[
\zeta(1) = \zeta_2(x_1) = \zeta_3(x_1, \ldots, x_d) = \zeta_d(1, x_1, \ldots, x_d) = 1,
\]
and so (14) is trivially satisfied.

The key aspect of the approach presented in this paper lies in the random choice for the boundary conditions in (6) in order to satisfy (7), with respect to the following distribution.

**Assumption 2.** Let \( \varphi \) be a square-integrable random variable in \( H^1(\partial\Omega) \), and let \( \nu \) denote the corresponding distribution, so that \( \varphi \sim \nu \). We assume that \( \varphi \) is sub-Gaussian, has mean 0 and that its covariance operator \( \Sigma : H^1(\partial\Omega) \to H^1(\partial\Omega) \) is injective.

**Remark.** Let us make this assumption more precise. Given a probability space \((X, \mathcal{F}, \mu)\), a random variable in \( H^1(\partial\Omega) \) is a measurable map \( \varphi : X \to H^1(\partial\Omega) \). We use \( \varphi \) to push-forward the measure \( \mu \) on \( X \) to a measure \( \nu \) on \( H^1(\partial\Omega) \). The covariance operator is defined by \( \Sigma = \mathbb{E}[\varphi \otimes \varphi] \), and is self-adjoint, positive and trace-class.
We follow [40, 43] for defining a sub-Gaussian random variable in a Hilbert space. We say that \( \varphi \) is sub-Gaussian if there exists \( C > 0 \) such that

\[
\tau \left( \langle \varphi, \psi \rangle_{L^2(\partial \Omega)} \right) \leq C \left( E |\langle \varphi, \psi \rangle_{L^2(\partial \Omega)}|^2 \right)^{1/2}, \quad \psi \in H^2(\partial \Omega),
\]

where, for a real random variable \( \xi \), we define

\[
\tau(\xi) = \inf \left\{ a \geq 0 : E e^{t \xi} \leq e^{t^2 a^2} \text{ for every } t \in \mathbb{R} \right\}.
\]

The simplest choice for \( \varphi \) is any non-degenerate Gaussian random variable in \( H^{1/2}(\partial \Omega) \). More explicitly, \( \varphi \) may be expressed as

\[
\varphi = \sum_{k \in \mathbb{N}} a_k e_k,
\]

where \( \{e_k\}_k \) is a fixed orthonormal basis of \( H^{1/2}(\partial \Omega) \) and \( a_k \sim N(0, \sigma_k^2) \) are independent real Gaussian variables, with \( \sigma_k > 0 \) for every \( k \) (since \( \Sigma \) is injective) and \( \sum_k \sigma_k < +\infty \) (since \( \Sigma \) is a trace-class operator). Decompositions of the form (15), which will appear several times in the sequel, are called Karhunen–Loève expansions [61], and converge in norm.

The main result of this section reads as follows.

**Theorem 1.** There exist \( C_1, C_2, C_3 > 0 \) depending only on \( \Omega, \Omega', \Lambda, E, \zeta \) and \( \nu \) such that the following is true. Take \( N \in \mathbb{N}, N \geq n^{d-1} \), and let \( \varphi_l \sim \nu \) be sampled i.i.d. in \( H^{1/2}(\partial \Omega) \) for \( i = 1, \ldots, n \) and \( l = 1, \ldots, N \). Then, with probability greater than

\[
1 - C_1 N^d \exp \left( -C_2 N^{3/2} \right)
\]

we have that

\[
\max_{i=1,\ldots,N} |\zeta(u_1', \ldots, u_n')(x)| \geq C_3, \quad x \in \overline{\Omega'},
\]

where \( u_i' \) is the solution to (6) with boundary condition \( \varphi_i' \).

Several comments on this result are in order.

- The positive constants \( C_1, C_2 \) and \( C_3 \) depend only on the priori data on the problem and are independent of the unknown parameters \( a \) and \( q \) of the PDE.
- The lower bound for the probability, \( 1 - C_1 N^d \exp(-C_2 N^{3/2}) \), converges exponentially to 1 as \( N \) grows, and so a small number \( N \) of measurements is sufficient to enforce the constraint with overwhelming probability. Further, the parameters \( d \) and \( n \) appear directly in the estimate: the larger they are, the worse, and this is expected, because \( d \) and \( n \) can be seen as underlying dimensions of the problem.
- Even if the number \( N \) of measurements is expected to be small, for \( N > 1 \) condition (16) guarantees that the constraint is verified only locally in \( \overline{\Omega'} \). More precisely, by (16) we have the open cover

\[
\overline{\Omega'} \subseteq \bigcup_{i=1}^N \Omega_i, \quad \Omega_i = \{ x \in \Omega : |\zeta(u_1', \ldots, u_n')(x)| > C_3/2 \}.
\]
In other words, the domain $\Omega'$ can be covered by $N$ subdomains, and in each of them the constraint (7) is satisfied for the measurement $l$. Therefore, the local reconstruction procedures described in section 2.3 may be carried out in each subdomain separately, by using the corresponding measurement $l$.

• It is worth observing that (16), or, equivalently, the cover (17), guarantees that (7) is satisfied with a quantitative lower bound $C_3$ depending only on the $a priori$ data. This lower bound is crucial from the numerical point of view, because it makes all the inversion steps discussed in section 2.3 well conditioned.

• The fact that (16) is true only with overwhelming probability, and not with probability 1, is unavoidable with this approach, where we aim for a quantitative lower bound with randomly-chosen (i.i.d.) boundary values. Indeed, in general, the probability that all the boundary conditions are not suitably chosen will be positive. It would be interesting to investigate whether the random choice of the boundary values could be modified to improve upon the number of solutions, possibly in combination with the approach based on the Whitney embedding theorem introduced in [9]: we leave this for future work.

The proof of this result is presented in section 4, and is based on the two following steps:

(a) Letting $\varphi_i \sim \nu$ i.i.d. for $i = 1, \ldots, n$, and denoting the corresponding solutions to (6) by $u_i$, we prove a lower bound of the form

$$E(\zeta(u_1, \ldots, u_n)(x)^2) \geq C > 0, \quad x \in \Omega'.$$

(b) A concentration inequality and regularity estimates for the $u_i$s allow us to move from an estimate in expectation to an estimate in probability in the whole domain.

For step (a), we shall need a quantitative version of the Runge approximation property, which is the main result of the following section.

\section{Quantitative Runge approximation}

We consider the PDE introduced in section 2.1. The following result is known as the Runge approximation property [50, 51], which establishes that local solutions to

$$Lu = -\text{div}(a\nabla u) + qu = 0 \quad \text{in} \quad \Omega,$$

(recall (1)) can always be approximated by global solutions.

\textbf{Proposition 1.} Let $D \subset \Omega$ be a Lipschitz domain such that $\Omega \setminus \overline{D}$ is connected. Take $h \in H^1(D)$ such that $Lh = 0$ in $D$ and $\varepsilon > 0$. Then there exists $u \in H^1(\Omega)$ such that $Lu = 0$ in $\Omega$ and

$$\|h - u\|_{L^2(D)} \leq \varepsilon.$$

In the proof of theorem 1, we will need a quantitative version of this result, where a suitable norm of $u|_{\partial D}$ is estimated. This will be the content of the main result of this section.

We need to introduce the class of Lipschitz subdomains $D$ we are going to consider. Following [60, chapter VI, section 3], we say that an open set $D \subset \mathbb{R}^d$ is $\Lambda$-Lipschitz, for a Lipschitz bound $\Lambda \geq 1$, if there exists a family of open sets $\{U_i\}$, such that:

(a) If $x \in \partial D$, then $B(x, \Lambda^{-1}) \subset U_i$ for some $i$, where $B(x, r)$ denotes the open ball in $\mathbb{R}^d$ of centre $x$ and radius $r$, namely, $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$;
(b) The intersection of more than $|\Lambda|$ of the sets $U_i$ is always empty, where $|\Lambda|$ is the integer part of $\Lambda$;
(c) For every $i$, up to a rotation of the axes, $D \cap U_i = D_i \cap U_i$, where $D_i$ is the set of points lying above a hypersurface with Lipschitz graph, with Lipschitz constant smaller than or equal to $\Lambda$.

We are ready to state the main result on quantitative Runge approximation.

**Theorem 2.** Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded, Lipschitz and connected domain, $\Lambda \geq 1$, $\varepsilon > 0$ and $f : H^{1/2}(\partial \Omega) \to [0, +\infty)$ be positively homogeneous and such that $f^{-1}([0, +\infty))$ is dense in $H^{1/2}(\partial \Omega)$. Then, there exists $C > 0$ depending only on $\Omega$, $\Lambda$, $\varepsilon$ and $f$ such that the following is true.

Let $a \in W_1^{1,\infty}(\Omega; \mathbb{R}^{d \times d})$ and $q \in L^\infty(\Omega)$ satisfy $a^T = a$ and $(2)$–$(5)$. Let $D \Subset \Omega$ be a convex domain such that

$$|D| \geq \Lambda^{-1}, \quad d(D, \partial \Omega) \geq \Lambda^{-1}, \quad D \text{ is } \Lambda\text{-Lipschitz.} \quad (18)$$

For every $h \in H^1(D)$ such that $Lh = 0$ in $D$ there exists $u \in H^1(\Omega)$ such that $Lu = 0$ in $\Omega$ and

$$\|h - u\|_{L^2(D)} \leq \varepsilon \|h\|_{H^1(D)}, \quad f(u|_{\partial \Omega}) \leq C\|h\|_{H^1(D)}. \quad (19)$$

Several comments on this result are in order:

- As it will be clear from the proof, the assumption on the convexity of $D$ may be removed, provided that $\Omega \setminus \overline{D}$ is connected. However, in this case, the constant $C$ will depend also on $D$, and not only on its Lipschitz bound $\Lambda$.
- The function $f$ may be chosen as a stronger (semi-)norm on $\partial \Omega$, such as

$$f(\varphi) = \|\varphi\|_{H^{1/2}(\partial \Omega)} \quad (20)$$

for any $s \geq \frac{1}{2}$, because $f(z\varphi) = |z|f(\varphi)$ for all $z \in \mathbb{R}$. Further, the density of $f^{-1}([0, +\infty)) = H^s(\partial \Omega)$ in $H^{1/2}(\partial \Omega)$ follows from the density of smooth functions in $H^{1/2}(\partial \Omega)$.
- For the proof of theorem 1, we will apply this result with

$$f(\varphi) = \left( \sum_{k \in \mathbb{N}} \frac{|a_k|^2}{\sigma_k} \right)^{\frac{1}{2}} \in [0, +\infty], \quad \varphi = \sum_{k \in \mathbb{N}} a_ke_k \in H^s(\partial \Omega), \quad (21)$$

where we use the notation of (15). It is worth observing that, if $\Omega = B(0, 1) \subseteq \mathbb{R}^2$ and the basis $\{e_k\}$ is the Fourier basis on $\partial B(0, 1)$, this map $f$, for a suitable choice of weights $\sigma_k$, corresponds to the one in (20). Thus, in some sense, $f(\varphi)$ may be seen as a generalised Sobolev norm on $\partial \Omega$.
- If we compare this result with the quantitative version of the Runge approximation property derived in [56], we observe that in [56] the second bound in (19) is replaced by

$$\|u|_{\partial \Omega}\|_{H^{1/2}(\partial \Omega)} \leq C e^{C_\varepsilon^{-\mu} \|h\|_{L^2(D)}},$$

where $C$ and $\mu$ are positive constants that are independent of $\varepsilon$. Thus, the estimate in theorem 2 allows for controlling more general norms of $u|_{\partial \Omega}$ (and this will be needed in the proof of theorem 1), but is weaker because no explicit dependence on $\varepsilon$ is given. This is due to the fact that the proof of theorem 2 is based on an argument by
contradiction and is not constructive, in contrast to the argument of [56], which is based on quantitative unique continuation estimates. Another difference lies in the fact that in the results of [56], the constant $C$ depends on $D$, while here it depends only on its Lipschitz bound $\Lambda$.

4. Proof of theorem 1

Let us now express the random variable $\varphi$ more explicitly. By the Karhunen–Loève theorem [61], there exists an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of $H^1_2(\partial \Omega)$ such that

$$\varphi = \sum_{k \in \mathbb{N}} a^k e_k,$$

(22)

where $\{a^k\}_k$ are real random variables, have zero-mean, are pairwise uncorrelated and have variance $\sigma_k^2$. More precisely, we have

$$\mathbb{E}(a^k) = 0, \quad \mathbb{E}(a^k a^{k'}) = \delta_{kk'} \sigma_k^2, \quad k, k' \in \mathbb{N}.$$

(23)

Note that $\sigma_k > 0$ for every $k$, because the covariance of $\varphi$ is injective (assumption 2). It is worth observing that the example of the Gaussian random variable presented in (15) is a particular case of this construction, with $a^k$ independent Gaussian random variables.

As in the statement of theorem 1, let $\varphi_i \sim \nu$ be sampled i.i.d. in $H^1_2(\partial \Omega)$ for $i = 1, \ldots, n$, and let $u_i$ be the corresponding solution to (6):

$$\begin{cases}
Lu_i := -\text{div}(a \nabla u_i) + qu_i = 0 & \text{in } \Omega, \\
u_i = \varphi_i & \text{on } \partial \Omega.
\end{cases}$$

(6)

Note that $u_i$ is itself a random variable in $H^1(\Omega)$, and consequently $\zeta(u_1, \ldots, u_n)$ is a random variable in $C^{0,1/2}(\overline{\Omega})$. The mean of $\zeta(u_1, \ldots, u_n)$ is zero, because $\varphi_i$ has mean zero, $L$ is linear and $\zeta$ is multilinear. Its variance is computed in the following lemma.

**Lemma 1.** We have

$$\mathbb{E}(\zeta(u_1, \ldots, u_n)^2) = \sum_{k_1, \ldots, k_n \in \mathbb{N}} \sigma_{k_1}^2 \cdots \sigma_{k_n}^2 \zeta(z_{k_1}, \ldots, z_{k_n})^2,$$

where $z_k$ is the unique solution to

$$\begin{cases}
-\text{div}(a \nabla z_k) + qz_k = 0 & \text{in } \Omega, \\
z_k = e_k & \text{on } \partial \Omega.
\end{cases}$$

**Proof.** In view of (22), we can write $\varphi_i = \sum_{k \in \mathbb{N}} a_k^i e_k$. Thus, using the fact that the PDE (6) is linear, we have $u_i = \sum_{k \in \mathbb{N}} a_k^i z_k$. Thus, since $\zeta$ is multilinear and bounded, we have

$$\zeta(u_1, \ldots, u_n) = \zeta\left(\sum_{k_1 \in \mathbb{N}} a_{k_1}^1 z_{k_1}, \ldots, \sum_{k_n \in \mathbb{N}} a_{k_n}^n z_{k_n}\right)$$

$$\quad = \sum_{k_1, \ldots, k_n \in \mathbb{N}} a_{k_1}^1 \cdots a_{k_n}^n \zeta(z_{k_1}, \ldots, z_{k_n}).$$
We readily derive
\[
\zeta(u_1, \ldots, u_n)^2 = \left( \sum_{k_1, \ldots, k_n \in \mathbb{N}} a_1^{k_1} \cdots a_n^{k_n} \zeta(z_{k_1}, \ldots, z_{k_n}) \right) \left( \sum_{k_1, \ldots, k_n \in \mathbb{N}} a_1^{k_1} \cdots a_n^{k_n} \zeta(z_{k_1}, \ldots, z_{k_n}) \right) \\
= \sum_{k_1, k_1', \ldots, k_n, k_n' \in \mathbb{N}} \left( a_1^{k_1} a_1^{k_1'} \right) \cdots \left( a_n^{k_n} a_n^{k_n'} \right) \zeta(z_{k_1}, \ldots, z_{k_n}) \zeta(z_{k_1'}, \ldots, z_{k_n'}).
\]

Note that \( z_k \) is a deterministic quantity. Taking the expectation, by using that the random variables \( a_i^k \) and \( a_i' \) are uncorrelated if \( i \neq j \), we obtain
\[
\mathbb{E}(\zeta(u_1, \ldots, u_n)^2) = \sum_{k_1, k_1', \ldots, k_n, k_n' \in \mathbb{N}} \mathbb{E}\left( a_1^{k_1} a_1^{k_1'} \right) \cdots \mathbb{E}\left( a_n^{k_n} a_n^{k_n'} \right) \zeta(z_{k_1}, \ldots, z_{k_n}) \zeta(z_{k_1'}, \ldots, z_{k_n'}).
\]

By (23), we have
\[
\mathbb{E}(\zeta(u_1, \ldots, u_n)^2) = \sum_{k_1, \ldots, k_n} \sigma_{k_1}^2 \cdots \sigma_{k_n}^2 \zeta(z_{k_1}, \ldots, z_{k_n})^2,
\]
as desired. \( \square \)

The next lemma shows that it is possible to approximate locally a solution to the PDE \( Lu = 0 \) by a solution to the PDE with constant coefficients and without the zeroth order term.

**Lemma 2 ([8, proposition 7.10]).** Take \( \delta, E > 0 \) and \( u^0 \in C^{1,1/2}(\overline{\Omega}) \) such that \( \|u^0\|_{C^{1,1/2}(\overline{\Omega})} \leq E \). There exists \( r > 0 \) depending only on \( \Omega, \Omega', \Lambda, E \) and \( \delta \) such that for every \( x_0 \in \overline{\Omega} \) such that \( \text{div}(a(x_0) \nabla u^0) = 0 \) in \( B(x_0, r) \) there exists \( u \in H^1(B(x_0, r)) \) such that \( Lu = 0 \) in \( B(x_0, r) \) and
\[
\|u - u^0\|_{C^{1,1/2}(B(x_0, r))} \leq \delta.
\]

As anticipated in (21), we now define the following stronger ‘norm’ on \( H^{1/2}(\partial\Omega) \):
\[
\|\varphi\|_\sigma^2 = \sum_{k \in \mathbb{N}} \left| a_k \right|^2 \sigma_k^2 \in [0, +\infty]. \quad \varphi = \sum_{k \in \mathbb{N}} a_k e_k \in H^{1/2}(\partial\Omega). \quad (24)
\]

Note that this is well defined because \( \sigma_k > 0 \) for every \( k \) and that it is stronger than \( \| \cdot \|_{H^{1/2}(\partial\Omega)} \) because \( \sigma_k \to 0 \) (since the covariance of the random variable \( \varphi \) is trace class). The following lemma is a consequence of assumption 1 and of theorem 2.

**Lemma 3.** Take \( x_0 \in \overline{\Omega}' \). There exist \( C > 0 \) depending only on \( \Omega, \Omega', \Lambda, E, \zeta \) and \( \nu \) and \( \tilde{\varphi}_1, \ldots, \tilde{\varphi}_n \in H^{1/2}(\partial\Omega) \) such that
\[
\|\tilde{\varphi}_i\|_\sigma \leq C, \quad i = 1, \ldots, n
\]
and
\[
|\zeta(u_1, \ldots, u_n)(x_0)| \geq 1/2,
\]
where
\[
\begin{cases}
-\text{div}(a \nabla \tilde{u}_i) + q \tilde{u}_i = 0 & \text{in } \Omega, \\
\tilde{u}_i = \tilde{\varphi}_i & \text{on } \partial\Omega.
\end{cases}
\]
Proof. By an abuse of notation, several positive constants depending only on $\Omega$, $\Omega'$, $\Lambda$, $E$, $\zeta$ and $\nu$ will be denoted by the same letter $C$.

Take $\varepsilon \in (0, 1]$ to be chosen later. Let $u_1^0, \ldots, u_n^0 \in C^{1,1/2}(\overline{\Omega})$ be as in assumption 1. By Lemma 2 there exist $r \in (0, d(\Omega', \partial\Omega)/2]$ depending only on $\Omega$, $\Omega'$, $\Lambda$, $E$ and $\varepsilon$ and $u_1, \ldots, u_n \in H^1(B(x_0, r))$ such that $Lu_i = 0$ in $B(x_0, r)$ and

$$\|u_i - u_i^0\|_{C^{1,1/2}(\overline{B(x_0, r)})} \leq \varepsilon, \quad i = 1, \ldots, n. \quad (25)$$

In particular, we have

$$\|u_i\|_{C^{1,1/2}(\overline{B(x_0, r)})} \leq E + 1, \quad i = 1, \ldots, n. \quad (26)$$

Next, note that $f : H^{1/2}(\partial\Omega) \to [0, +\infty]$ defined by $f(\varphi) = \|\varphi\|_\sigma$ is positively homogeneous and such that $f^{-1}([0, +\infty))$ is dense in $H^{1/2}(\partial\Omega)$, since span $\{e_k\}_k \subseteq f^{-1}([0, +\infty))$.

Therefore, by Theorem 2 with $D = B(x_0, r)$ there exists $\tilde{u}_1, \ldots, \tilde{u}_n \in H^1(\Omega)$ such that $\tilde{L}\tilde{u}_i = 0$ in $\Omega$ and

$$\|u_i - \tilde{u}_i\|_{L^2(B(x_0, r))} \leq \varepsilon \|u_i\|_{H^1(B(x_0, r))},$$

$$\|\tilde{u}_i\|_\sigma \leq C(\Omega, \Omega', \Lambda, \nu, \varepsilon, r)\|u_i\|_{H^1(B(x_0, r))}.$$ 

Thus, by (26) we have

$$\|u_i - \tilde{u}_i\|_{L^2(B(x_0, r))} \leq C(E)\varepsilon, \quad \|\tilde{u}_i\|_\sigma \leq C(\Omega, \Omega', \Lambda, E, \varepsilon).$$

Classical elliptic regularity [41, theorem 8.32] yields

$$\|u_i - \tilde{u}_i\|_{C^{1,1/2}(\overline{B(x_0, r)})} \leq C\|u_i - \tilde{u}_i\|_{L^2(B(x_0, r))} \leq C\varepsilon.$$

As a consequence, by (25) we have

$$\|u_i^0 - \tilde{u}_i\|_{C^{1,1/2}(\overline{B(x_0, r)})} \leq \varepsilon + C\varepsilon \leq C\varepsilon.$$

Therefore, by the fact that $\zeta$ is continuous and local, in view of (14) we can choose $\varepsilon \in (0, 1]$ depending only on $C$ and $\zeta$ such that

$$|\zeta(\tilde{u}_1, \ldots, \tilde{u}_n)(x_0)| \geq 1/2.$$

Choosing $\tilde{\varphi}_1 = \tilde{u}_i|_{\partial\Omega}$ concludes the proof.

In the next result we prove a lower bound for the variance of $\zeta(u_1, \ldots, u_n)$.

**Lemma 4.** We have

$$\mathbb{E}(\zeta(u_1, \ldots, u_n)(x)^2) \geq \eta, \quad x \in \overline{\Omega}$$

for some $\eta \in (0, 1]$ depending only on $\Omega$, $\Omega'$, $\Lambda$, $E$, $\zeta$ and $\nu$.

**Proof.** By an abuse of notation, several positive constants depending only on $\Omega$, $\Omega'$, $\Lambda$, $E$, $\zeta$ and $\nu$ will be denoted by the same letter $C > 0$.

Take $x_0 \in \overline{\Omega'}$. Let $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n \in H^{1/2}(\partial\Omega)$ be as in Lemma 3. Thus, in view of (24) we have

$$\|\tilde{\varphi}_i\|_\sigma^2 = \sum_{k \in \mathbb{N}} |\tilde{a}_k|^2 \leq C, \quad \tilde{\varphi}_i = \sum_{k \in \mathbb{N}} \tilde{a}_k^e e_k.$$ 

(27)
In particular, we have \( \tilde{u}_i = \sum_{k_i \in \mathbb{N}} \tilde{a}^{k_i}_i z_{k_i} \), so that

\[
\frac{1}{2} \leq |\zeta(\tilde{u}_1, \ldots, \tilde{u}_n)(x_0)| \leq \left| \sum_{k_1, \ldots, k_n \in \mathbb{N}} \tilde{a}^{k_1}_1 \cdots \tilde{a}^{k_n}_n \zeta(z_{k_1}, \ldots, z_{k_n})(x_0) \right|
\]

\[
= \left| \sum_{k_1, \ldots, k_n \in \mathbb{N}} \frac{\tilde{a}^{k_1}_1 \cdots \tilde{a}^{k_n}_n}{\sigma_{k_1} \cdots \sigma_{k_n}} \zeta(z_{k_1}, \ldots, z_{k_n})(x_0) \right|
\]

By using the Cauchy–Schwarz inequality we obtain

\[
\frac{1}{2} \leq \left( \sum_{k_1, \ldots, k_n \in \mathbb{N}} \frac{(\tilde{a}^{k_1}_1)^2 \cdots (\tilde{a}^{k_n}_n)^2}{\sigma_{k_1}^2 \cdots \sigma_{k_n}^2} \right)^{\frac{1}{2}} \left( \sum_{k_1, \ldots, k_n \in \mathbb{N}} (\sigma_{k_1}^2 \cdots \sigma_{k_n}^2) \zeta(z_{k_1}, \ldots, z_{k_n})(x_0)^2 \right)^{\frac{1}{2}}.
\]

Observe that, by (27) we have

\[
\sum_{k_1, \ldots, k_n \in \mathbb{N}} \frac{(\tilde{a}^{k_1}_1)^2 \cdots (\tilde{a}^{k_n}_n)^2}{\sigma_{k_1}^2 \cdots \sigma_{k_n}^2} = \prod_{i=1}^n \sum_{k_i \in \mathbb{N}} \frac{(\tilde{a}^{k_i}_i)^2}{\sigma_{k_i}^2} = \prod_{i=1}^n \|\tilde{\varphi}_i\|_{\psi^2}^2 \leq C.
\]

As a consequence, by lemma 1 we obtain

\[
\frac{1}{2} \leq C \left( \sum_{i=1}^n |\zeta(u_1, \ldots, u_n)(x_0)|^2 \right)^{\frac{1}{2}},
\]

and the result follows.

We say that a real-valued random variable \( X \) is \( \gamma \)-subexponential if there exists \( K > 0 \) such that \( \mathbb{E} \exp(|X|/K^\gamma) \leq 2 \) \cite{MR644902}. In this case, we write

\[
\|X\|_{\psi^\gamma} = \inf\{t > 0 : \mathbb{E} \exp(|X|^\gamma/t^\gamma) \leq 2\}.
\]

In the case \( \gamma = 2 \), we say that \( X \) is sub-Gaussian. We now show that \( \|\varphi\|_{H^{1/2}(\partial\Omega)} \) is sub-Gaussian.

**Lemma 5.** If \( \varphi \sim \nu \), then the real random variable \( \|\varphi\|_{H^{1/2}(\partial\Omega)} \) is sub-Gaussian. In particular, there exist \( c_1, c_2 > 0 \) depending only on \( \nu \) such that

\[
\mathbb{P}\left( \|\varphi\|_{H^{1/2}(\partial\Omega)} \geq t \right) \leq 2 \exp(-c_1 t^2), \quad t \geq 0,
\]

and

\[
\mathbb{E} \exp\left( \|\varphi\|_{H^{1/2}(\partial\Omega)}^2 / c_2 \right) \leq 2.
\]

**Proof.** Since \( \varphi \) is a sub-Gaussian random variable in \( H^{1/2}(\partial\Omega) \), we have that also \( \|\varphi\|_{H^{1/2}(\partial\Omega)} \) is sub-Gaussian \cite{MR644902, MR793930}. The tail bound (28) follows by \cite[proposition 2.5.2]{MR644900}, and (29) follows from the fact that \( \|\varphi\|_{H^{1/2}(\partial\Omega)} \) is finite by definition.

\[\square\]
Next, we show that the random variable $\zeta(u_1, \ldots, u_n)(x)^2$ is $\frac{1}{n}$-subexponential.

**Lemma 6.** Take $x \in \overline{\Omega'}$. The random variable $\zeta(u_1, \ldots, u_n)(x)^2$ is $\frac{1}{n}$-subexponential and 

$$\|\zeta(u_1, \ldots, u_n)(x)^2\|_{\mathcal{V}^2_{\frac{1}{n}}} \leq C,$$

for some $C$ depending only on $\Omega$, $\Omega'$, $\Lambda$, $E$, $\zeta$ and $\nu$.

**Proof.** By an abuse of notation, several positive constants depending only on $\Omega$, $\Omega'$, $\Lambda$, $E$, $\zeta$ and $\nu$ will be denoted by the same letter $C > 0$.

Using that $\zeta$ is bounded we obtain 

$$\|\zeta(u_1, \ldots, u_n)(x)^2\|_{C^{0,1/2}(\overline{\Omega'})} \leq C \prod_{i=1}^n \|u_i\|_{C^{1,1/2}(\overline{\Omega'})}.$$ 

Thus, classical elliptic regularity [41, theorem 8.32] yields 

$$|\zeta(u_1, \ldots, u_n)(x)| \leq \|\zeta(u_1, \ldots, u_n)\|_{C^{0,1/2}(\overline{\Omega'})} \leq C \prod_{i=1}^n \|\varphi_i\|_{H^2(\partial\Omega)}; \quad (30)$$

Take now $K > 0$ to be chosen later. We now argue as in [63, lemma 2.7.7]. By using (30) and the inequality of arithmetic and geometric means we readily derive 

$$\exp\left(\frac{|\zeta(u_1, \ldots, u_n)(x)|^2}{K}\right) \leq \exp\left(\frac{C}{K} \prod_{i=1}^n \|\varphi_i\|_{H^2(\partial\Omega)}\right)^{2/n} \leq \exp\left(\frac{C}{K} \left(\sum_{i=1}^n \|\varphi_i\|_{H^2(\partial\Omega)}\right)^2\right) \leq \exp\left(\frac{C}{Kn} \sum_{i=1}^n \|\varphi_i\|_{H^2(\partial\Omega)}^2\right) = \prod_{i=1}^n \exp\left(\frac{C}{Kn} \|\varphi_i\|_{H^2(\partial\Omega)}^2\right).$$

Using again the inequality of arithmetic and geometric means and Hölder inequality in $\mathbb{R}^n$ we obtain 

$$\exp\left(\frac{|\zeta(u_1, \ldots, u_n)(x)|^2}{K}\right) \leq \left(\frac{1}{n} \sum_{i=1}^n \exp\left(\frac{C}{Kn} \|\varphi_i\|_{H^2(\partial\Omega)}^2\right)\right)^n \leq \frac{n^{n-1}}{n^n} \sum_{i=1}^n \exp\left(\frac{C}{K} \|\varphi_i\|_{H^2(\partial\Omega)}^2\right) \leq \frac{1}{n} \sum_{i=1}^n \exp\left(\frac{C}{K} \|\varphi_i\|_{H^2(\partial\Omega)}^2\right).$$
Therefore, in view of (29), we have
\[ \mathbb{E} \exp \left( \frac{\zeta(u_1, \ldots, u_n)(x)^{2/n}}{c_2 C} \right) \leq 2, \]
and the result follows.

We recall the following concentration inequality for \( \gamma \)-subexponential random variables, which immediately follows from [44, theorem 1.3] (see also [45]).

**Lemma 7.** Let \( X_1, \ldots, X_N \) be i.i.d. \( \gamma \)-subexponential real random variables such that \( \mathbb{E} X_l = \mu \) and \( \| X_l \|_{\psi\gamma} \leq M \) for some \( \gamma \in (0, 1) \). Then
\[ \mathbb{P} \left( \left| \frac{1}{N} \sum_{l=1}^{N} X_l - \mu \right| \geq t \right) \leq 2 \exp \left( -C \min \left( \frac{N t^2}{M^2}, \frac{t^\gamma N^{1/\gamma}}{M^{1/\gamma}} \right) \right). \]

We now apply this result to the random variable \( \zeta(u_1, \ldots, u_n)(x)^2 \): as in the statement of theorem 1, let \( \varphi'_l \sim \nu \) be sampled i.i.d. and denote the corresponding solutions to (6) by \( u'_l \).

**Lemma 8.** There exist \( C', C'' > 0 \) depending only on \( \Omega, \Omega', \Lambda, E, \zeta \) and \( \nu \) such that
\[ \mathbb{P} \left( \frac{1}{N} \sum_{l=1}^{N} \zeta(u'_1, \ldots, u'_n)(x)^2 \leq C'' \right) \leq 2 \exp \left( -C N^{1/n} \right), \quad x \in \Omega'. \]

**Proof.** By an abuse of notation, several positive constants depending only on \( \Omega, \Omega', \Lambda, E, \zeta \) and \( \nu \) will be denoted by the same letter \( C > 0 \).

Take \( x \in \Omega' \) and set \( X_l = \zeta(u'_1, \ldots, u'_n)(x)^2 \). By lemma 4 we have \( \mu := \mathbb{E} X_l \geq \eta \). By lemma 6, \( X_l \) is \( \frac{1}{n} \)-subexponential and \( \| X_l \|_{\psi\gamma} \leq C \). We readily derive
\[ \mathbb{P} \left( \frac{1}{N} \sum_{l=1}^{N} X_l \leq \frac{\eta}{2} \right) \leq \mathbb{P} \left( \frac{1}{N} \sum_{l=1}^{N} X_l \leq \mu - \frac{\eta}{2} \right) \]
\[ = \mathbb{P} \left( \mu - \frac{1}{N} \sum_{l=1}^{N} X_l \geq \frac{\eta}{2} \right) \]
\[ \leq \mathbb{P} \left( \left| \frac{1}{N} \sum_{l=1}^{N} X_l - \mu \right| \geq \frac{\eta}{2} \right). \]

By lemma 7 we have
\[ \mathbb{P} \left( \left| \frac{1}{N} \sum_{l=1}^{N} X_l - \mu \right| \geq \frac{\eta}{2} \right) \leq 2 \exp \left( -C \min \left( \frac{N \eta^2}{C^2}, \frac{\eta^{1/n} N^{1/n}}{C^{1/n}} \right) \right) \]
\[ \leq 2 \exp \left( -C \min \left( \frac{N \eta^2}{\eta^{1/n} N^{1/n}} \right) \right) \]
\[ \leq 2 \exp \left( -CN^{1/n} \right). \]

Setting \( C'' = \frac{\eta}{2} \), by these two estimates we obtain
\[ \mathbb{P} \left( \frac{1}{N} \sum_{l=1}^{N} X_l \leq C'' \right) \leq 2 \exp \left( -CN^{1/n} \right), \]
as desired. \( \square \)
We are now ready to prove theorem 1.

**Proof of theorem 1.** By an abuse of notation, several positive constants depending only on \( \Omega, \Omega', \Lambda, E, \zeta \) and \( \nu \) will be denoted by the same letter \( C \).

Take \( t \geq 0 \) to be chosen later. By (28) we have that

\[
\mathbb{P}\left( \|\phi_i^l\|_{H^\frac{1}{2}(\Omega)} \geq t \right) \leq 2 \exp(-Ct^2), \quad i = 1, \ldots, n, \ l = 1, \ldots, N.
\]

Thus, the union bound yields

\[
\mathbb{P}\left( \max_i \|\phi_i^l\|_{H^\frac{1}{2}(\Omega)} \geq t \right) \leq 2nN \exp(-Ct^2).
\]

In other words, with probability greater than or equal to \( 1 - 2nN \exp(-Ct^2) \), we have \( \max_i \|\phi_i^l\|_{H^\frac{1}{2}(\Omega)} \leq t \). By (30) we have

\[
\|\zeta(u'_1, \ldots, u'_n)\|_{L^{0.5}(\Omega')} \leq C \prod_{i=1}^n \|\phi_i\|_{H^\frac{1}{2}(\Omega)} \leq Cr^n.
\]

In other words, using for simplicity the equivalent sup norm in \( \mathbb{R}^d \), we have

\[
|\zeta(u'_1, \ldots, u'_n)(x) - \zeta(u'_1, \ldots, u'_n)(y)| \leq C r^\nu \|x - y\|_\infty^{1/2}, \quad x, y \in \Omega'.
\]

Let \( C' > 0 \) be given by lemma 8 and set \( r = \frac{C'}{(2C')^d} \). It is possible to cover \( \Omega' \) with \( M \leq Cr^{-d} \) balls (with respect to the norm \( \| \cdot \|_\infty \)):

\[
\overline{\Omega'} \subseteq \bigcup_{j=1}^M B_{\infty}(x_j, r),
\]

where \( x_1, \ldots, x_M \in \overline{\Omega'} \). In view of (31), for \( j = 1, \ldots, M \) and \( x \in B_{\infty}(x_j, r) \cap \overline{\Omega'} \) we have

\[
|\zeta(u'_1, \ldots, u'_n)(x) - \zeta(u'_1, \ldots, u'_n)(x_j)| \leq C r^\nu \|x - x_j\|_\infty^{1/2} \leq C r^{\nu - 1/2} = \sqrt{C'/2},
\]

so that

\[
|\zeta(u'_1, \ldots, u'_n)(x)| \geq |\zeta(u'_1, \ldots, u'_n)(x_j)| - |\zeta(u'_1, \ldots, u'_n)(x) - \zeta(u'_1, \ldots, u'_n)(x_j)|
\]

\[
\geq |\zeta(u'_1, \ldots, u'_n)(x_j)| - \sqrt{C'/2}.
\]

Next, by lemma 8,

\[
\mathbb{P}\left( \frac{1}{N} \sum_{j=1}^N \zeta(u'_1, \ldots, u'_n)(x_j)^2 \leq C' \right) \leq 2 \exp\left(-C'N^{1/2}\right), \quad j = 1, \ldots, M.
\]

Thanks to the union bound, we have that

\[
\frac{1}{N} \sum_{j=1}^N \zeta(u'_1, \ldots, u'_n)(x_j)^2 \geq C', \quad j = 1, \ldots, M,
\]

with probability greater than or equal to \( 1 - 2M \exp(-C'N^{1/2}) \). Note that the previous condition implies that for every \( j = 1, \ldots, M \) there exists \( l = 1, \ldots, N \) such that \( |\zeta(u'_1, \ldots, u'_n)(x_j)| \geq \)
\( \sqrt{C^n} \). Thus, in view of (32) and (33) we have that for every \( x \in \Omega \) there exists \( l = 1, \ldots, N \) such that \( |\zeta(u'_1, \ldots, u'_N)(x)| \geq \sqrt{C^n}/2 \). In other words, we proved (16) with \( C_3 = \sqrt{C^n}/2 \). This happens with probability greater than or equal to

\[
1 - 2M \exp\left(-C'N^{1/n}\right) - 2nN \exp(-Ct^2).
\]

Now, choose \( t = N^{1/2} \), so that \( r = \frac{C}{N} \) and \( M \leq C r^{-d} \leq CN^d \), so that

\[
1 - 2M \exp\left(-C'N^{1/n}\right) - 2nN \exp(-Ct^2) \\
\geq 1 - CN^d \exp\left(-C'N^{1/n}\right) - 2nN \exp(-CN^{1/n}) \\
\geq 1 - C_1 N^d \exp\left(-C_2 N^{1/n}\right),
\]

where we have used that \( N \geq n^{1/4} \). This concludes the proof. \( \square \)

5. Proof of theorem 2

We need three technical lemmata.

**Lemma 9.** Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded Lipschitz domain, \( p \in (1, +\infty) \) and consider the canonical embedding \( I : L^\infty(\Omega) \to W^{-1,p}(\Omega) := \left(W^{1,p}_0(\Omega)\right)' \) defined by

\[
\langle Iq, v \rangle_{W^{-1,p}(\Omega) \times W^{1,p}_0(\Omega)} = \int_\Omega qv \, dx, \quad q \in L^\infty(\Omega), \, v \in W^{1,p}_0(\Omega).
\]

If \( q_n, q \in L^\infty(\Omega) \) and \( q_n \rightharpoonup q \) weak* in \( L^\infty(\Omega) \), then \( Iq_n \rightharpoonup Iq \) in \( W^{-1,p}(\Omega) \).

**Proof.** Let \( q_n, q \in L^\infty(\Omega) \) be such that \( q_n \rightharpoonup q \) weak* in \( L^\infty(\Omega) \), namely,

\[
\int_\Omega q_n v \, dx \to \int_\Omega q v \, dx, \quad v \in L^1(\Omega).
\]

In particular, since \( \Omega \) is bounded, we have \( L^p(\Omega) \subseteq L^1(\Omega) \), so that

\[
\int_\Omega q_n v \, dx \to \int_\Omega q v \, dx, \quad v \in L^p(\Omega).
\]

In other words, \( q_n \rightharpoonup q \) weakly in \( L^p(\Omega) \). Let \( \iota : W^{1,p}_0(\Omega) \to L^p(\Omega) \) be the canonical embedding, which is compact thanks to the Rellich–Kondrachov theorem, and

\[
T : L^p(\Omega) \to \left(L^p(\Omega)\right)' = \left(W^{1,p}_0(\Omega)\right)' \subseteq L^p(\Omega),
\]

be the canonical isomorphism. Since \( \iota^* : \left(L^p(\Omega)\right)' \to W^{-1,p}(\Omega) \) is compact, we have that \( \iota^* T : L^p(\Omega) \to W^{-1,p}(\Omega) \) is compact too. As a consequence, using that \( q_n \rightharpoonup q \) weakly in \( L^p(\Omega) \), we
have $t^* Tq_\rho \to t^* Tq$ in $W^{-1,q}(\Omega)$. It remains to show that $t^* T|_{L^\infty(\Omega)} = I$, namely, that it is the canonical embedding. For every $u \in L^\infty(\Omega)$ and $v \in W_0^{1,p}(\Omega)$ we have

$$\langle t^* Tu, v \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p}(\Omega)} = \langle Tu, v \rangle_{L^p(\Omega) \times L^p(\Omega)}$$

$$= \int_{\Omega} u(t) \, dx$$

$$= \int_{\Omega} uv \, dx$$

$$= \langle Ju, v \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p}(\Omega)}$$

as desired. \(\square\)

Lemma 10. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. The map

$$(u, v) \in H^1(\Omega) \times H^1(\Omega) \mapsto uv \in W_0^{1,d+1}(\Omega)$$

is well-defined and bounded.

Proof. The proof is split into three steps.

Step 1: the map is well defined in the case $d \geq 3$. Take $(u, v) \in H^1(\Omega) \times H^1(\Omega)$. By Sobolev embedding, $u, v \in L^{\frac{2p}{p-2}}(\Omega)$. By Cauchy–Schwartz inequality, this implies that $uv \in L^{\frac{2p}{p-2}}(\Omega)$. Thus, $uv \in L^{d+1}(\Omega)$ since $(d+1)\frac{d}{d+1} = \frac{d}{2} \leq \frac{d}{p-2}$.

Next, we show that $\nabla (uv) \in L^{d+1}(\Omega; \mathbb{R}^d)$. We have $\nabla (uv) = u \nabla v + v \nabla u$, with $u, v \in L^{\frac{2p}{p-2}}(\Omega)$ and $\nabla u, \nabla v \in L^2(\Omega; \mathbb{R}^d)$. Hölder inequality implies that $u \nabla v, v \nabla u \in L^p(\Omega; \mathbb{R}^d)$ where

$$\frac{1}{p} = \frac{d - 2}{2d} + \frac{1}{2} = \frac{d - 1}{d} \frac{1}{d}$$

Thus $u \nabla v, v \nabla u \in L^p(\Omega; \mathbb{R}^d) \subseteq L^{d+1}(\Omega; \mathbb{R}^d)$.

Finally, since $v|_{\partial \Omega} = 0$, we have that $(uv)|_{\partial \Omega} = 0$, so that $uv \in W_0^{1,d+1}(\Omega)$.

Step 2: the map is well defined in the case $d = 2$. Take $(u, v) \in H^1(\Omega) \times H^1(\Omega)$. By Sobolev embedding, $uv \in L^p(\Omega)$ for every $p \in (1, +\infty)$, and in particular $uv \in L^{d+1}(\Omega)$. Next, we show that $\nabla (uv) \in L^{d+1}(\Omega; \mathbb{R}^d)$. We have $\nabla (uv) = u \nabla v + v \nabla u$, with $u, v \in L^2(\Omega)$ for every $p \in (1, +\infty)$ and $\nabla u, \nabla v \in L^2(\Omega; \mathbb{R}^d)$. Thus $u \nabla v, v \nabla u \in L^2(\Omega; \mathbb{R}^d)$, as desired. As above, $(uv)|_{\partial \Omega} = 0$.

Step 3: the map is bounded. The boundedness of the bilinear map $(u, v) \mapsto uv$ follows immediately from the argument above, by using the continuity of Sobolev embeddings. \(\square\)

Lemma 11. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. The map $T: W^{-1,d+1}(\Omega) \times H^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle T(q, u), v \rangle_{H^{-1}(\Omega) \times H^1(\Omega)} = \langle q, uv \rangle_{W^{-1,d+1}(\Omega) \times W_0^{1,d+1}(\Omega)}$$

for $q \in W^{-1,d}(\Omega), u \in H^1(\Omega)$ and $v \in H^1(\Omega)$, is well-defined and bounded.
Proof. The map $T$ is well defined, because $uv \in W^{1,d+1}_0(\Omega)$, thanks to lemma 10. We now show that $T$ is bounded. By using again lemma 10 we readily obtain

$$
\|T(q,u)\|_{H^{-1}(\Omega)} = \sup_{\|v\|_{H^1(\Omega)} = 1} \|q,v\|_{W^{-1,d+1}(\Omega),H^{1,d+1}_0(\Omega)} \\
\leq \sup_{\|v\|_{H^1(\Omega)} = 1} \|q\|_{W^{-1,d+1}(\Omega)} \|uv\|_{H^{1,d+1}_0(\Omega)} \\
\leq C\|q\|_{W^{-1,d+1}(\Omega)} \sup_{\|v\|_{H^1(\Omega)} = 1} \|u\|_{H^1(\Omega)} \|v\|_{H^1_{loc}(\Omega)} \\
= C\|q\|_{W^{-1,d+1}(\Omega)} \|u\|_{H^1(\Omega)} 
$$

as desired. □

We are now ready to prove theorem 2.

Proof of theorem 2. The proof is split into several steps. For $\Omega' \subset \Omega$, we use the notation $H^1_2(\Omega') = \{u \in H^1(\Omega') : Lu = 0 \text{ in } \Omega'\}$. Note that $H^1_2(\Omega')$ is a vector subspace of $H^1(\Omega')$ because $L$ is linear, namely, (1) is a linear PDE.

Step 1: without loss of generality we can assume $\|h\|_{H^1_2(D)} = 1$. Suppose that the result is true if $\|h\|_{H^1_2(D)} = 1$. Observe that, since $f$ is positively homogeneous, (19) is equivalent to

$$
\left| \frac{h}{\|h\|_{H^1_2(D)}} - \frac{u}{\|u\|_{H^1_2(D)}} \right| \leq \varepsilon, \quad f\left( \frac{u}{\|u\|_{H^1_2(D)}} \right) \leq C. \quad (34)
$$

Since the PDE under consideration is linear, $\frac{h}{\|h\|_{H^1_2(D)}}$ is still a local solution in $D$, and so there exists $\tilde{u} \in H^1_2(\Omega)$ such that

$$
\frac{\tilde{u}}{\|\tilde{u}\|_{H^1_2(D)}} - \frac{u}{\|u\|_{H^1_2(D)}} \leq \varepsilon, \quad f(\tilde{u}) \leq C.
$$

Then $u = \frac{\|h\|_{H^1_2(D)}}{\|h\|_{H^1_2(D)}} \tilde{u} \in H^1_2(\Omega)$ and satisfies (34), as desired.

Step 2: a proof by contradiction. By contradiction, assume that such constant $C > 0$ does not exist. Thus, for every $n \in \mathbb{N}$ there exist $a_n \in W^{1,\infty}(\Omega;\mathbb{R}^{d \times d})$ and $q_n \in L^\infty(\Omega;\mathbb{R})$ satisfying $a_n^\top = a_n$ and (2)–(5), $D_n$ as in the statement and $h_n \in H^1_2(D_n)$, where $L_nu = -\text{div}(a_n \nabla u) + q_n u$, satisfying $\|h_n\|_{H^1_2(D_n)} = 1$ such that

$$
\forall u \in H^1_2(D_n) \quad \left( f(u|_{\partial D_n}) \leq n \implies \|u|_{D_n} - h_n\|_{L^2(D_n)} > \varepsilon \right). \quad (35)
$$

In the rest of the proof, we will consider several subsequences: in order to simplify the exposition, with an abuse of notation, we will never denote them.

Step 3: $q_n \to q$ weak$^*$ in $L^\infty(\Omega)$. Viewing $L^\infty(\Omega)$ as the dual of $L^1(\Omega)$, by the Banach–Alaoglu theorem the closed ball $\overline{B}_{L^\infty(\Omega)}(0,\Lambda)$ is compact with respect to the weak$^*$ topology. Since $\overline{B}_{L^\infty(\Omega)}(0,\Lambda)$ is metrisable and $q_n \in \overline{B}_{L^\infty(\Omega)}(0,\Lambda)$ by (4), there exist $q \in \overline{B}_{L^\infty(\Omega)}(0,\Lambda)$ and a subsequence of $(q_n)$ such that $q_n \to q$ weak$^*$ in $L^\infty(\Omega)$, namely,

$$
\int_\Omega q_n v \, dx \to \int_\Omega q v \, dx, \quad v \in L^1(\Omega). \quad (36)
$$

(This is the so-called sequential Banach–Alaoglu theorem.)
Step 4: $a_n \to a$ in $L^\infty(\Omega; \mathbb{R}^{d \times d})$. Recall that $a_n \in W^{1,\infty}(\Omega; \mathbb{R}^{d \times d})$ and, by (3), $\|a_n\|_{W^{1,\infty}(\Omega)} \leq \Lambda$ for every $n$. By the Ascoli–Arzelà theorem, there exists $a \in W^{1,\infty}(\Omega; \mathbb{R}^{d \times d})$ such that $\|a_n - a\|_{W^{1,\infty}(\Omega)} \leq \Lambda$ and $a_n \to a$ in $L^\infty(\Omega; \mathbb{R}^{d \times d})$. In particular, since $a_n$ satisfies $a_n = a$ and (2) for every $n \in \mathbb{N}$, then $a^1 = a$ and $a$ satisfies (2) too.

Now that we have found limit points for $(q_n)$ and $(a_n)$, let us denote $Lu := -\div(a\nabla u) + qu$.

Step 5: 0 is not a Dirichlet eigenvalue of $L$ in $\Omega$ and $L$ satisfies (5). Let $F \in H^{-1}(\Omega)$, and let $u \in H_0^1(\Omega)$ be a solution to

$$-\div(a\nabla u) + qu = F.$$ 

This problem may be rewritten as

$$-\div(a_n\nabla u) + q_n u = \div((a - a_n)\nabla u) + (q_n - q)u + F.$$ 

Since $L_n$ satisfies (5), by lemma 11 we have

$$\|u\|_{H^1(\Omega)} \leq \Lambda \|\div((a - a_n)\nabla u) + (q_n - q)u + F\|_{H^{-1}(\Omega)}$$

$$\leq \Lambda \left(\|(a - a_n)\nabla u\|_{L^2(\Omega)} + \|(q_n - q)u\|_{H^{-1}(\Omega)} + \|F\|_{H^{-1}(\Omega)}\right)$$

$$\leq \Lambda \left(\|a - a_n\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|\nabla u\|_{L^2(\Omega)} + \|I(q_n - q)\|_{W^{-1,d+1}(\Omega)} \|\mu_1(\Omega)\| + \|F\|_{H^{-1}(\Omega)}\right),$$

where $\|a\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} = \sup_{x \in \Omega} \|a(x)\|_2$, $\|A\|_2$ denotes the operator norm of the matrix $A \in \mathbb{R}^{d \times d}$ and $I : L^\infty(\Omega) \to W^{-1,d+1}(\Omega)$ is the canonical embedding (see lemma 9). Note that $\|a - a_n\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \to 0$ by step 4. Furthermore, $q_n \to q$ weak* in $L^\infty(\Omega)$ by step 3, so that $\|I(q_n - q)\|_{W^{-1,d+1}(\Omega)} \to 0$ by lemma 9 with $p = d + 1$. Altogether, we obtain

$$\|u\|_{H^1(\Omega)} \leq \Lambda \|F\|_{H^{-1}(\Omega)}.$$ 

This shows that, if $F = 0$, then $u = 0$, so that 0 is not a Dirichlet eigenvalue of $L$. Furthermore, $L$ satisfies (5).

Step 6: $D_n \to D$ with respect to the Hausdorff distance and the volume distance. By Blaschke’s selection theorem [55], there exists a convex domain $D \subseteq \Omega$ such that

$$d_H(D_n, D) \to 0,$$

where $d_H$ denotes the Hausdorff distance and is defined by

$$d(X, Y) = \max \left(\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\right).$$

For later use, we also point out that in this setting the convergence with respect to the Hausdorff distance is equivalent to the convergence in the volume distance, namely,

$$|D_n \setminus D| + |D \setminus D_n| \to 0,$$

see [59], where $|\cdot|$ denotes the Lebesgue measure. In particular, by (18) we have that $|D| \geq \Lambda^{-1}$ and that $D \subseteq \Omega$. 


Step 7: \(\tilde{h}_n \to h\) weakly in \(H^1(\Omega)\) with \(h|_D \in H^1_1(D)\). In view of the classical extension theorem for Sobolev spaces (see, e.g., [60, chapter VI, section 3] and [33, lemma 6.4]), we can extend \(h_n \in H^1(D_n)\) to \(h_n \in H^1(\Omega)\). Since \(\|h_n\|_{H^1(D)} \leq 1\) for every \(n \in \mathbb{N}\), by (18) we have \(\|h_n\|_{H^1(\Omega)} \leq E\), for some \(E > 0\) independent of \(n\). Therefore, using again the Banach–Alaoglu theorem, we obtain that there exists \(h \in H^1(\Omega)\) such that

\[
\tilde{h}_n \to h \quad \text{weakly in } H^1(\Omega). \tag{39}
\]

By the Rellich–Kondrachov theorem,

\[
\tilde{h}_n \to h \quad \text{in } L^2(\Omega). \tag{40}
\]

We now claim that \(h|_D \in H^1_1(D)\). Recall that \(h_n \in H^1_1(D_n)\) for every \(n \in \mathbb{N}\), namely,

\[
\int_{D_n} a_n \nabla h_n \cdot \nabla v \, dx + \int_{D_n} q_n h_n v \, dx = 0, \quad v \in C^\infty_c(D_n). \tag{41}
\]

We need to show that

\[
\int_D a \nabla h \cdot \nabla v \, dx + \int_D q h v \, dx = 0, \quad v \in C^\infty_c(D). \tag{42}
\]

Take \(v \in C^\infty_c(D)\). Thus, by (37), using that \(D_n\) and \(D\) are Lipschitz domains, in view of [57, equation (1.5) and lemma 7.4] there exists \(n_0 \in \mathbb{N}\) such that \(\text{supp } v \subseteq D_n\) for all \(n \geq n_0\). Hence, for \(n \geq n_0\) we have

\[
\left| \int_{D_n} q_n h_n v \, dx - \int_D q h v \, dx \right| = \left| \int_D q_n \tilde{h}_n v \, dx - \int_D q h v \, dx \right| \\
\leq \left| \int_D q_n \tilde{h}_n v \, dx - \int_D q_n h v \, dx \right| + \left| \int_D q_n h v \, dx - \int_D q h v \, dx \right| \\
\leq \int_D |\tilde{h}_n - h| q_n v \, dx + \left| \int_D (q_n - q) h v \, dx \right| \\
\leq \|q_n\|_{L^\infty(D)} \|\tilde{h}_n - h\|_{L^2(D)} \|v\|_{L^2(D)} + \left| \int_\Omega (q_n - q) h v \, dx \right|,
\]

where \(\tilde{v} \in L^2(\Omega)\) is the extension by zero of \(v\). The first of these two factors goes to zero thanks to (40) and (4), while the second factor goes to zero by (36), since \(h \tilde{v} \in L^1(\Omega)\). Therefore

\[
\left| \int_{D_n} q_n h_n v \, dx - \int_D q h v \, dx \right| \to 0 \quad \text{as } n \to +\infty. \tag{43}
\]
Next, we consider the leading order term. Since the matrices $a$ and $a_n$ are symmetric, for $n \geq n_0$ we have

\[
\left| \int_{D_n} a_n \nabla h_n \cdot \nabla v \, dx - \int_D a \nabla h \cdot \nabla v \, dx \right|
\]

\[
= \left| \int_D a_n \nabla h_n \cdot \nabla v \, dx - \int_D a \nabla h \cdot \nabla v \, dx \right|
\]

\[
\leq \left| \int_D a_n \nabla h_n \cdot \nabla v \, dx - \int_D a_n \nabla h_n \cdot \nabla v \, dx \right|
\]

\[
+ \left| \int_D a \nabla h_n \cdot \nabla v \, dx - \int_D a \nabla h \cdot \nabla v \, dx \right|
\]

\[
\leq \int_D |a_n - a| \nabla h_n \cdot \nabla v \, dx + \left| \int \nabla h_n - \nabla h \cdot a \nabla v \right|_{L^2(\Omega)}
\]

\[
\leq ||a_n - a||_{L^\infty(D \times [0,1])} \|h_n\|_{H^1(\Omega)} \|v\|_{H^1(D)} + \left| \int \nabla (h_n - h) \cdot a \nabla v \right|_{L^2(\Omega)}.
\]

Using that $||h_n||_{H^1(\Omega)} \leq E$ for every $n$ and that $a_n \to a$ in $L^\infty(\Omega; \mathbb{R}^{d \times d})$, we obtain that the first factor goes to 0 as $n \to +\infty$. The second term goes to zero thanks to (39). Therefore

\[
\left| \int_{D_n} a_n \nabla h_n \cdot \nabla v \, dx - \int_D a \nabla h \cdot \nabla v \, dx \right| \to 0 \quad \text{as} \quad n \to +\infty.
\]

Finally, (42) follows directly by (41), (43) and (44).

**Step 8:** There exists $u \in H^1_0(\Omega)$ such that $f(u|_{\partial \Omega}) < +\infty$ and $||u|_\Omega - h|_\Omega||_{L^2(\partial \Omega)} \leq \frac{\varepsilon}{4}$. Note that $L$ and $D$ satisfy all the assumptions of proposition 1, because $\Omega \setminus \overline{\Omega}$ is connected (since $D$ is convex and $D \in \Omega$). Thus, there exists $u' \in H^1_0(\Omega)$ such that $||u'|_\Omega - h|_\Omega||_{L^2(\partial \Omega)} \leq \varepsilon/8$. By density of $f^{-1}(0, +\infty)$ in $H^{1/2}(\partial \Omega)$, there exist $g_k \in H^{1/2}(\partial \Omega)$ such that $f(g_k) < +\infty$ and $g_k \to u'|_{\partial \Omega}$ in $H^{1/2}(\partial \Omega)$. Let $u_k \in H^1(\Omega)$ be the unique solution to

\[
\begin{align*}
Lu_k &= 0 & \text{in} & \Omega, \\
u_k &= g_k & \text{on} & \partial \Omega.
\end{align*}
\]

Then $u_k \to u'$ in $H^1(\Omega)$ because $L^{-1}$ is continuous. Take $k_0 \in \mathbb{N}$ such that $||u_{k_0} - u'||_{H^1(\Omega)} \leq \varepsilon/8$, so that

\[
||u_{k_0}|_\Omega - h|_\Omega||_{L^2(\partial \Omega)} \leq ||u_{k_0} - u'||_{L^2(\partial \Omega)} + ||u'|_\Omega - h|_\Omega||_{L^2(\partial \Omega)} \leq \varepsilon/4.
\]

It is enough to set $u = u_{k_0}$.

**Step 9:** $||h - u||_{L^2(D_{\delta_0},D)} \to 0$. Set $w = h - u \in H^1(\Omega)$. By Sobolev embedding, we have $w \in L^q(\Omega)$ for some $q > 2$ (depending on $d$). By Hölder inequality, setting $p = q/2 > 1$, we obtain

\[
||w||_{L^2(D_{\delta_0},D)}^2 \leq ||\chi_{D_{\delta_0}} w||_{L^2(D_{\delta_0})} ||w||_{L^p(\Omega)}^2 = |D_{\delta_0}|^{\frac{1}{p'}} ||w||_{L^p(\Omega)}^2,
\]

where $p' = \frac{p}{p-1} \in (1, \infty)$. The right-hand side goes to zero by (38), whence $||h - u||_{L^2(D_{\delta_0},D)} \to 0$. 

22
Step 10: $u_n \to u$ in $H^1(\Omega)$, where $u_n \in H^1_0(\Omega)$. For $n \in \mathbb{N}$, let $u_n \in H^1(\Omega)$ be the unique solution to the Dirichlet boundary value problem

\[
\begin{align*}
L_{n} u_n &= 0 \quad \text{in } \Omega, \\
u_n &= u \quad \text{on } \partial \Omega.
\end{align*}
\]

Set $v_n = u_n - u \in H^1_0(\Omega)$. We have $L_n v_n = - L_n u = (L - L_n) u$. Arguing as in step 5, we obtain that

\[
\|v_n\|_{H^1(\Omega)} \leq \Lambda \left( \|a - a_n\|_{L^\infty(\Omega; \mathbb{R}^d)} \|\nabla u\|_{L^2(\Omega)} + \|I(q_n - q)\|_{W^{-1,1}(\Omega)} \|u\|_{H^1(\Omega)} \right),
\]

where the right-hand side goes to 0 as $n \to +\infty$, as desired.

Step 11: the contradiction. Recall that, by (40), step 9 and step 10 we have $\|\tilde{h}_n - h\|_{L^2(\Omega)} \to 0$, $\|h - u\|_{L^2(D, \partial) \setminus \Omega} \to 0$ and $\|u - u_n\|_{H^1(\Omega)} \to 0$. Choose $\bar{n} \in \mathbb{N}$ such that $f(u|\partial \Omega) \leq \bar{n}$ and $\|\tilde{h}_n - h\|_{L^2(\Omega)} \leq \varepsilon/4$, $\|h - u\|_{L^2(D, \partial) \setminus \Omega} \leq \varepsilon/4$, $\|u - u_n\|_{H^1(\Omega)} \leq \varepsilon/4$.

Since $f(u|\partial \Omega) = f(u|\partial \Omega) \leq \bar{n}$, by (35) we have $\|\tilde{h}_n - u|D, \partial\|_{L^2(D, \partial) \setminus \Omega} > \varepsilon$. By step 8 we have

\[
\|h|D, \partial - u|D, \partial\|_{L^2(D, \partial) \setminus \Omega} \leq \|h|D, \partial - u|D, \partial\|_{L^2(D, \partial) \setminus \Omega} + \|h|D, \partial - u|D, \partial\|_{L^2(D, \partial) \setminus \Omega} \\
\leq 2(\varepsilon/4)^2 \\
\leq (\varepsilon/2)^2,
\]

whence, by the triangle inequality, we obtain

\[
\|\tilde{h}_n - u|D, \partial\|_{L^2(D, \partial) \setminus \Omega} \leq \|\tilde{h}_n - h\|_{L^2(\Omega)} + \|h|D, \partial - u|D, \partial\|_{L^2(D, \partial) \setminus \Omega} + \|u - u_n\|_{L^2(\Omega)} \leq \varepsilon,
\]
a contradiction. This concludes the proof. \(\Box\)

Acknowledgments

This material is based upon work supported by the Air Force Office of Scientific Research under Award Number FA8655-20-1-7027. The author is member of the ‘Gruppo Nazionale per l’Analisi Matematica, la Probabilit`a e le loro Applicazioni’ (GNAMPA), of the ‘Istituto Nazionale di Alta Matematica’ (INdAM).

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Giovanni S Alberti \(\text{https://orcid.org/0000-0002-8612-3663}\)
References

[1] Adesokan B J, Knudsen K, Krishnan V P and Roy S 2018 A fully non-linear optimization approach to acousto-electric tomography Inverse Problems 34 104004
[2] Alberti G S 2013 On multiple frequency power density measurements Inverse Problems 29 115007
[3] Alberti G S 2015 Enforcing local non-zero constraints in PDEs and applications to hybrid imaging problems Commun. Partial Differ. Equ. 40 1855–83
[4] Alberti G S 2015 On multiple frequency power density measurements: II. The full Maxwell’s equations J. Differ. Equ. 258 2767–93
[5] Alberti G S 2016 Absence of critical points of solutions to the Helmholtz equation in 3D Arch. Ration. Mech. Anal. 222 879–94
[6] Alberti G S and Ammari H 2017 Disjoint sparsity for signal separation and applications to hybrid inverse problems in medical imaging Appl. Comput. Harmon. Anal. 42 319–49
[7] Alberti G S, Bal G and Di Cristo M 2017 Critical points for elliptic equations with prescribed boundary conditions Arch. Ration. Mech. Anal. 226 117–41
[8] Alberti G S and Capdeboscq Y 2018 Lectures on Elliptic Methods for Hybrid Inverse Problems vol 25 (Paris: Société Mathématique de France)
[9] Alberti G S and Capdeboscq Y 2022 Combining the Runge approximation and the Whitney embedding theorem in hybrid imaging Int. Math. Res. Not. 2022 4387–406
[10] Alessandrini G 1986 An identification problem for an elliptic equation in two variables Ann. Mat. Pura Appl. 145 265–95
[11] Alessandrini G 1988 Stable determination of conductivity by boundary measurements Appl. Anal. 27 153–72
[12] Alessandrini G 2014 Global stability for a coupled physics inverse problem Inverse Problems 30 075008
[13] Alessandrini G, Di Cristo M, Francini E and Vessella S 2017 Stability for quantitative photoacoustic tomography with well-chosen illuminations Ann. Mat. Pura Appl. 196 395–406
[14] Alessandrini G and Nesi V 2001 Univalent σ-harmonic mappings Arch. Ration. Mech. Anal. 158 155–71
[15] Alessandrini G and Nesi V 2015 Quantitative estimates on Jacobians for hybrid inverse problems Series. Mathematical Modelling, Programming & Computer Software vol 8 (Bulletin of the South Ural State University) pp 25–41
[16] Alessandrini G, Rondi L, Rosset E and Vessella S 2009 The stability for the Cauchy problem for elliptic equations Inverse Problems 25 123004
[17] Ammari H, Bonnetier E, Capdeboscq Y, Tanter M and Fink M 2008 Electrical impedance tomography by elastic deformation SIAM J. Appl. Math. 68 1557–73
[18] Ammari H, Capdeboscq Y, de Gournay F, Rozanova-Pierrat A and Triki F 2011 Microwave imaging by elastic deformation SIAM J. Appl. Math. 71 2112–30
[19] Ammari H, Garnier J, Kang H, Nguyen L H and Seppecher L 2017 Multi-Wave Medical Imaging (Singapore: World Scientific)
[20] Arridge S R 1999 Optical tomography in medical imaging Inverse Problems 15 R41–93
[21] Bal G 2013 Hybrid inverse problems and internal functionals Inverse Problems and Applications: Inside Out: II (Math. Sci. Res. Inst. Publ. vol 60) (Cambridge: Cambridge University Press) pp 325–68
[22] Bal G, Bonnetier E, Bonnetier F, Monard F and Triki F 2013 Inverse diffusion from knowledge of power densities Inverse Problems Imaging 7 353–75
[23] Bal G and Coudurier M 2013 Boundary control of elliptic solutions to enforce local constraints J. Differ. Equ. 255 1357–81
[24] Bal G, Guo C and Monard F 2014 Inverse anisotropic conductivity from internal current densities Inverse Problems 30 025001
[25] Bal G and Ren K 2011 Multi-source quantitative photoacoustic tomography in a diffusive regime Inverse Problems 27 075003
[26] Bal G and Ren K 2012 On multi-spectral quantitative photoacoustic tomography in diffusive regime Inverse Problems 28 025010
[27] Bal G and Uhlmann G 2010 Inverse diffusion theory of photoacoustics Inverse Problems 26 085010
[28] Bal G and Uhlmann G 2012 Reconstructions for some coupled-physics inverse problems Appl. Math. Lett. 25 1030–3
[29] Bal G and Uhlmann G 2013 Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions Commun. Pure Appl. Math. 66 1629–52
[30] Bauman P, Marini A and Nesi V 2001 Univalent solutions of an elliptic system of partial differential equations arising in homogenization Indiana Univ. Math. J. 50 747–57
[31] Bonnetier E, Choulli M and Triki F 2022 Stability for quantitative photoacoustic tomography revisited Res. Math. Sci. 9 24
[32] Browder F E 1962 Approximation by solutions of partial differential equations Am. J. Math. 84 134–60
[33] Burenkov V I 1999 Extension theory for Sobolev spaces on open sets with Lipschitz boundaries Nonlinear Analysis, Function Spaces and Applications (Acad. Sci. Czech Inst. Math. vol 6) (Prague) pp 1–49
[34] Calderón A-P 1980 On an inverse boundary value problem Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980) (Soc. Brasil. Mat.) pp 65–73
[35] Capdeboscq Y 2015 On a counter-example to quantitative Jacobian bounds J. Éc. Polytech.Math. 2 171–8
[36] Capdeboscq Y, Fehrenbach J, de Gournay F and Kavian O 2009 Imaging by modification: numerical reconstruction of local conductivities from corresponding power density measurements SIAM J. Imaging Sci. 2 1003–30
[37] Choulli M and Triki F 2015 New stability estimates for the inverse medium problem with internal data SIAM J. Math. Anal. 47 1778–99
[38] Choulli M and Triki F 2019 Hölder stability for an inverse medium problem with internal data Res. Math. Sci. 6 15
[39] Colton D and Kress R 1998 Inverse Acoustic and Electromagnetic Scattering Theory (Applied Mathematical Sciences vol 93) 2nd edn (Berlin: Springer)
[40] Fukuda R 1990 Exponential integrability of sub-Gaussian vectors Probab. Theory Relat. Fields 85 505–21
[41] Gilbarg D and Trudinger N S 2001 Elliptic partial differential equations of second order Classics in Mathematics (Berlin: Springer) reprint of the 1998 edn
[42] Giordano M and Nickl R 2020 Consistency of Bayesian inference with Gaussian process priors in an elliptic inverse problem Inverse Problems 36 085001
[43] Gilbarg D and Trudinger N S 2001 Elliptic partial differential equations of second order Classics in Mathematics (Berlin: Springer) reprint of the 1998 edn
[44] Kuchment P and Kunyansky L 2011 Mathematics of photoacoustic and thermoacoustic tomography Handbook of Mathematical Methods in Imaging ed O Scherzer (New York: Springer) pp 817–65
[45] Lavandier B, Jossinet J and Cathignol D 2000 Experimental measurement of the acousto-electric interaction signal in saline solution Ultrasonics 38 929–36
[46] Lax PD 1956 A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations Commun. Pure Appl. Math. 9 747–66
[47] Malgrange B 1955–1956 Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution Ann. Inst. Fourier, Grenoble 6 271–355
[48] Mandache N 2001 Exponential instability in an inverse problem for the Schrödinger equation Inverse Problems 17 1435
[49] McLaughlin J, Oberai A and Yoon J-R 2012 Formulas for detecting a spherical stiff inclusion from interior data: a sensitivity analysis for the Helmholtz equation Inverse Problems 28 084004
[50] Neuman S P and Yakowitz S 1979 A statistical approach to the inverse problem of aquifer hydrology: I. Theory Water Resour. Res. 15 845–60
[51] Price G B 1940 On the completeness of a certain metric space with an application to Blaschke’s selection theorem Bull. Am. Math. Soc. 46 278–80
[56] Rüland A and Salo M 2019 Quantitative Runge approximation and inverse problems \textit{Int. Math. Res. Not.} \textbf{2019} 6216–34

[57] Savaré G and Schimperna G 2002 Domain perturbations and estimates for the solutions of second order elliptic equations \textit{J. Math. Pures Appl.} \textbf{81} 1071–112

[58] Seo J K and Woo E J 2011 Magnetic resonance electrical impedance tomography (MREIT) \textit{SIAM Rev.} \textbf{53} 40–68

[59] Shephard G C and Webster R J 1965 Metrics for sets of convex bodies \textit{Mathematika} \textbf{12} 73–88

[60] Stein E M 1970 \textit{Singular Integrals and Differentiability Properties of Functions} (Princeton Mathematical Series) (Princeton, NJ: Princeton University Press)

[61] Steinwart I 2019 Convergence types and rates in generic Karhunen–Loève expansions with applications to sample path properties \textit{Potential Anal.} \textbf{51} 361–95

[62] Uhlmann G 2009 Electrical impedance tomography and Calderón’s problem \textit{Inverse Problems} \textbf{25} 123011

[63] Vershynin R 2018 \textit{High-Dimensional Probability} (Cambridge Series in Statistical and Probabilistic Mathematics vol 47) (Cambridge: Cambridge University Press) An introduction with applications in data science, With a foreword by Sara van de Geer

[64] Wang K and Anastasio M A 2011 Photoacoustic and thermoacoustic tomography: image formation principles \textit{Handbook of Mathematical Methods in Imaging} ed O Scherzer (New York: Springer) pp 781–815

[65] Widlak T and Scherzer O 2012 Hybrid tomography for conductivity imaging \textit{Inverse Problems} \textbf{28} 084008

[66] Woo E J, Lee S Y and Mun C W 1994 Impedance tomography using internal current density distribution measured by nuclear magnetic resonance \textit{Proc. SPIE} \textbf{2299} 377–85

[67] Zhang H and Wang L V 2004 Acousto-electric tomography \textit{Proc. SPIE. Photons Plus Ultrasound: Imaging and Sensing} vol 5320 pp 145–9