ON BÜCHI’S K3 SURFACE

MICHELA ARTEBANI, ANTONIO LAFACE, AND DAMIANO TESTA

Abstract. We study the Büchi K3 surface proving that it belongs to the one dimensional family of Kummer surfaces associated to genus two curves with automorphism group $D_4$. We compute its Picard lattice and show that the rational points of the surface are Zariski-dense. Moreover, we provide analogous results for the Kummer surface associated to any genus two curve whose automorphism group contains a non-hyperelliptic involution.

Introduction

Let $n$ be an integer with $n \geq 3$ and let $\{x_i\}_{i=1}^n$ be a sequence of $n$ integers satisfying the system of second order difference equations

\[
(0.1) \quad (x_{i+2}^2 - x_{i+1}^2) - (x_{i+1}^2 - x_i^2) = 2 \quad \text{for } i \in \{1, \ldots, n-2\}.
\]

Any sequence obtained from a solution of (0.1) by arbitrary sign changes or reversal is still a solution. Also, for any integer $x$, the sequence of consecutive integers $\{x_i = x + i\}_{i=1}^n$ is a solution of (0.1). An integral solution of (0.1) is trivial if it is obtained from a sequence of consecutive integers by arbitrary sign changes or reversal, and it is non-trivial otherwise.

The Büchi problem is the well known question asking whether there exists a positive integer $n$ such that all integral solutions of (0.1) are trivial. It is known that if $n \leq 4$ there are non-trivial integral solutions of (0.1).

A positive answer to Büchi’s problem would imply, using the negative answer to Hilbert’s Tenth Problem by Yu. Matiyasevich, that there is no algorithm to decide whether a system of diagonal quadratic forms with integer coefficients admits an integer solution. Several authors studied Büchi’s problem and related questions. In [Voj00] P. Vojta determined all the one-parameter solutions to (0.1) for $n \geq 8$; combining this result with the Bombieri-Lang conjecture gives a positive answer to Büchi’s problem. An explicit description for length three and four Büchi sequences is provided in [Vid11] and [SV11]. The problem has been studied also on finite fields, see for example the paper by H. Pasten [Pas11]. In [BB06] the authors study the related problem of finding sequences of integer squares with constant second differences. They find infinitely many non-trivial such sequences lying on elliptic curves of positive rank. For a recent survey on the problem, we refer the reader to [PPV10].

2000 Mathematics Subject Classification. 14J28, 11D41.

Key words and phrases. Büchi problem, Kummer surfaces, rational points.

The first author has been partially supported by Proyecto FONDECYT Regular N. 1110249.
The second author has been partially supported by Proyecto FONDECYT Regular N. 1110096.
The third author has been partially supported by EPSRC grant EP/F060661/1.
For any \( n \geq 3 \) the system (0.1) defines a projective surface \( X_n \) of degree \( 2^{n-2} \) in \( \mathbb{P}^n \). This is a rational surface for \( n = 3 \) and \( 4 \), a K3 surface for \( n = 5 \) and a surface of general type otherwise. In this paper we show that the B"uchi K3 surface \( X_5 \) is the minimal resolution of the Kummer surface of a genus two curve whose automorphism group is isomorphic to \( D_4 \). Such curves form a one-dimensional family in \( \mathcal{M}_2 \), so that \( X_5 \) naturally lives in a family of lattice polarized K3 surfaces. We determine the Picard lattice of a general member of this family and show that it coincides with \( \text{Pic}(X_5) \). Moreover, we perform a similar analysis for any curve in \( \mathcal{M}_2 \) whose automorphism group contains a non-hyperelliptic involution. There are five irreducible families of such curves, determined by their automorphism group \([CGLR99]\).

**Theorem 1.** Let \( G \) be one of the groups in Table 1, let \( \Lambda_G \) be the corresponding lattice and let \( \mathcal{F}_G \) be the family of genus two curves whose automorphism group is isomorphic to \( G \). If \( C \in \mathcal{F}_G \) and \( X \) is the minimal resolution of the Kummer surface of \( C \), then \( \text{Pic}(X) \) contains a primitive sublattice isometric to \( \Lambda_G \). Moreover, \( \text{Pic}(X) \) is isomorphic to \( \Lambda_G \) whenever the two lattices have the same rank; this happens for very general curves \( C \) in the family \( \mathcal{F}_G \).

| \( G \) | \( \Lambda_G \) |
|---|---|
| \( V_4 \) | \( U(2) \oplus E_8 \oplus D_7 \oplus (-4) \) |
| \( D_4 \) | \( U(4) \oplus E_8^2 \oplus (-4) \) |
| \( D_6 \) | \( U(2) \oplus E_8^2 \oplus (-12) \) |
| \( 2D_6 \) | \( U \oplus E_8^2 \oplus (-4) \oplus (-12) \) |
| \( S_4 \) | \( U \oplus E_8^2 \oplus (-4) \oplus (-8) \) |

**Table 1.** Automorphism group and Picard lattices of Kummer surfaces

As a consequence of this result we determine the Picard lattice of \( X_5 \) and of other K3 surfaces studied from an arithmetical point of view (see Examples 3.3 and 3.4). We prove that the set of non-trivial rational points of the B"uchi K3 surface \( X_5 \) is infinite and Zariski-dense. The strategy is to find an elliptic fibration defined over \( \mathbb{Q} \) with infinitely many sections defined over \( \mathbb{Q} \). In fact, we prove that the same statement holds for all surfaces in any of the above families (see Theorem 3.7).

The paper is organized as follows. In Section 1 we introduce the B"uchi surfaces \( X_n \), characterise the fields over which they are smooth and show that, for \( n \geq 4 \), the linear automorphism group of \( X_n \) consists of scalar permutations. Section 2 deals with genus two curves admitting a non-hyperelliptic involution. Here we determine the Néron-Severi lattice of their Jacobians. In Section 3 we prove Theorem 1 and Theorem 3.7. In Section 4 we provide a detailed study of the Mordell-Weil groups of some elliptic fibrations \( X_5 \rightarrow \mathbb{P}^1 \) defined over the rational numbers. We also analyse the Galois representation of the elliptic curve \( E \) and its set of supersingular primes: we use these results to compute the local zeta function of \( X \) at such primes.

**Acknowledgements.** We thank the anonymous referee for his careful reading of our paper and for suggesting to us to look at families of lattice polarized K3 surfaces. We would also like to thank Kieran O’Grady and Daniel Huybrechts for
On Büchi’s K3 surface

1. Büchi surfaces

Let \( n \) be an integer with \( n \geq 3 \). The Büchi surface \( X_n \) is the zero locus in \( \mathbb{P}^n \) of the \( n - 2 \) quadrics of equations

\[
x_{i+2}^2 - 2x_i^2 + x_{i+1}^2 - 2x_0^2 = 0 \quad \text{for } i \in \{1, \ldots, n - 2\}.
\]

Observe that \( X_3 \) is a quadric, \( X_4 \) is a del Pezzo surface of degree four, \( X_5 \) is a K3 surface and \( X_n, n \geq 6 \), is a surface of general type (if they are smooth). A first attempt to find rational points on \( X_n \) is to parametrize the surface when \( n \leq 4 \).

The del Pezzo surface \( X_4 \) can be parametrized by the system of plane cubics passing through the five points

\[
[1, 0, 0] \quad [0, 1, 0] \quad [0, 0, 1] \quad [3, -3, 2] \quad [3, -1, 1].
\]

Explicitly we have the following parametrization of \( X_4 \):

\[
\begin{align*}
x_0 &= -2a^2b - 6ab^2 - 3a^2c + 9b^2c + 9ac^2 + 9bc^2 \\
x_1 &= -12ab^2 + 3a^2c - 18abc + 9b^2c - 18ac^2 \\
x_2 &= 2a^2b - 6ab^2 - 24abc - 9ac^2 + 9bc^2 \\
x_3 &= -4a^2b - 3a^2c + 6abc - 9b^2c - 18bc^2 \\
x_4 &= -6a^2b - 6ab^2 - 6a^2c - 18b^2c + 9ac^2 + 27bc^2
\end{align*}
\]

which immediately proves the existence of infinitely many non-trivial rational solutions of the Büchi’s problem for four variables. Indeed, not only the set of rational points of \( X_4 \) is infinite, but even the set of points that are integral with respect to the divisor \( x_0 = 0 \) is infinite (see [PPV10]). The surface \( X_4 \) contains exactly 16 lines whose parametric equations in the affine chart \( x_0 \neq 0 \) are given by

\[ x_i = \pm(t + i), \quad i \in \{1, 2, 3, 4\}. \]

The surface \( X_5 \) is the double cover of \( X_4 \) branched along the smooth genus 5 curve \( B \in \mid -2K_{X_4} \mid \) defined in \( X_4 \) by \( x_1^2 - 2x_2^2 - 2x_3^2 = 2x_0^2 \) (the covering map is the projection on the first four coordinates). Since the lines of \( X_4 \) are tangent to the branch locus \( B \), the inverse image of each line in the double cover \( X_5 \) decomposes as the union of two lines. Thus \( X_5 \) contains 32 lines, with parametric equations

\[
x_i = \pm(t + i), \quad i \in \{1, 2, 3, 4, 5\}.
\]

These are the only lines in \( X_5 \), since the covering map is linear. Observe that the lines on \( X_5 \) contain all the trivial solutions of Büchi’s problem.

1.1. Smoothness of Büchi surfaces. We show that all Büchi surfaces are smooth in characteristic zero and provide a necessary and sufficient condition for their smoothness in positive characteristic. For any positive integer \( n \) we define a pair of
matrices

\[
B_n = \begin{pmatrix}
-2 & 1 & -2 & 1 \\
-2 & 1 & -2 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
-2 & & 1 & -2 & 1
\end{pmatrix}, \quad A_n = \begin{pmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -2 & 1 \\
& & & & 1 & -2 & 1
\end{pmatrix}
\]

where \(B_n\) is a \((n-2) \times (n+1)\) matrix while \(-A_n\) is the Cartan matrix of size \(n\) of a root system of type \(A\). We recall that the determinant of the matrix \(A_n\) equals \((-1)^n(n+1)\): this is well-known and it is also immediate to prove by induction developing the determinant with respect to the first row. In particular, if \(k\) is a field, then the rank of the matrix \(A_n\) satisfies

\[
\text{rk}(A_n) = \begin{cases} 
n & \text{if char}(k) \text{ does not divide } n+1, 
n-1 & \text{otherwise}.
\end{cases}
\]

It is easy to check that the following identity holds

(1.3) \((1 \ 2 \ 3 \ \cdots \ n)A_n = (0 \ \cdots \ 0 \ - (n+1))\).

**Lemma 1.1.** Let \(a, b, c\) be integers satisfying \(0 \leq a < b < c \leq n\) and let \(B_n^{a,b,c}\) denote the \((n-2) \times (n-2)\)-matrix obtained from \(B_n\) by removing the three columns with indices \(a+1, b+1, c+1\). If the index \(a\) satisfies \(a \geq 1\), then the determinant of the matrix \(B_n^{a,b,c}\) equals

\[
\det(B_n^{a,b,c}) = (-1)^{a+b+c}(a - b)(a - c)(b - c).
\]

If the index \(a\) satisfies \(a = 0\), then the determinant of the matrix \(B_n^{0,b,c}\) equals

\[
\det(B_n^{b,c}) = (-1)^{b+c}(b - c).
\]

**Proof.** We only treat the case in which \(a\) is different from 0, since the remaining case is similar and simpler. Fix integers \(a, b, c\) satisfying \(1 \leq a < b < c \leq n\) and proceed by induction on \(n\) to prove the assertion. If \(a \geq 2\), then we develop the determinant of \(B_n^{a,b,c}\) with respect to the second column to find \(\det(B_n^{a,b,c}) = -\det(B_n^{a-1,b,c})\). Similarly, if \(c \leq n - 1\), then we develop the determinant of \(B_n^{a,b,c}\) with respect to the last column to find \(\det(B_n^{a,b,c}) = \det(B_n^{a,b-1,c})\). Therefore, by the inductive hypothesis we reduce to the case in which the indices satisfy \(a = 1\), \(c = n\) and the matrix \(B_n^b = B_{n-1,b,n}^n\) is the matrix

\[
B_n^b = \begin{pmatrix}
\vdots & \vdots \\
-2 & 0 & \cdots & 0 & 1 & 0 & \cdots \\
\vdots & \vdots \\
\end{pmatrix}
\]

where the boxes denote square blocks of sizes \(b-2\) and \(n-b-1\) filled by the matrices \(A_{b-2}\) and \(A_{n-b-1}\) respectively. If, for \(i \in \{1, \ldots, b-1\}\), we multiply the \(i\)-th row of the matrix \(B_n^b\) and then sum the resulting row vectors, then, using relation (1.3), we obtain the row vector \(v_1 = -(b-1)((b-2) \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots 0)\). A similar computation shows that row vector \(v_2 = -(n-b)((n-b-1) \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots 0)\) is a linear combination of the last \(n-b-1\) rows. Hence, adding \(\frac{1}{b-1}v_1 + \frac{1}{n-b}v_2\) to
the \((b - 1)\)-st row of the matrix \(B_n^b\), the determinant is not affected and the matrix \(B_n\) becomes the matrix

\[
\begin{pmatrix}
\vdots \\
-(n-1) & 0 & \cdots & 0 & 0 & \cdots \\
\vdots \\
A_{b-2} \\
\vdots \\
A_{n-b-1}
\end{pmatrix},
\]

whose determinant is \((-1)^{1+b+n}(1-n)(1-b)(b-n)\), as required. \(\square\)

Our next result is about the smoothness of the scheme \(X_n\). To simplify the notation, let \(Y_n\) denote the linear subspace of \(\mathbb{P}^n\) defined by the system of equations

\[
Y_n: \begin{cases}
x_i - 2x_{i+1} + x_{i+2} = 2x_n, & \text{for } i \in \{1, \ldots, n-2\}.
\end{cases}
\]

The dimension of the linear subspace \(Y_n\) is two and there is a morphism \(q: X_n \to Y_n\) determined by \(q([x_0, \ldots, x_n]) = [x_0^2, \ldots, x_n^2]\). The morphism \(q\) is finite of degree \(2^n\).

**Corollary 1.2.** The surface \(X_n\) is singular if and only if the characteristic \(p\) of the ground field satisfies \(0 < p < n\). If the surface \(X_n\) is singular, then \(X_n\) has exactly \(2^n - p\) irreducible components each isomorphic to \(X_p\).

**Proof.** An easy induction shows that, for \(i \in \{1, \ldots, n\}\), the equations

\[x_i^2 = (i-1)(i-2)x_{i-1}^2 - (i-2)x_{i}^2 + (i-1)x_{i+1}^2\]

hold on \(X_n\). In particular, if \(i, j \in \{1, \ldots, n\}\) satisfy \(i \equiv j \pmod{p}\), then the equation \(x_i^2 = x_j^2\) holds on \(X_n\). We deduce that to prove the corollary, it suffices to show that \(X_n\) is smooth under the assumption \(p \notin \{2, 3, \ldots, n-1\}\).

Let \(x = [x_0, \ldots, x_n]\) be a point in \(X_n\). To check smoothness, we compute the rank of the Jacobian matrix \(J_n\) of the equations (1.1) at the point \(x\). Observe that the rank of the matrix \(J_n\) at \(x\) depends only on the positions of the vanishing coordinates of the point \(x\). As we observed above, we reduce to the case in which \(p\) satisfies \(p \notin \{2, \ldots, n-1\}\).

If at most two of the coordinates of \(x\) are non-zero, then let \(a, b \in \{0, 1, \ldots, n\}\) be indices such that \(a < b\) and for all \(i \in \{0, \ldots, n\} \setminus \{a, b\}\) we have \(x_i \neq 0\). Let \(c\) be an integer satisfying \(c \in (\{1, \ldots, n\} \cap \{b-1, b+1\}) \setminus \{a\}\). Lemma 1.1 implies that the matrix obtained from \(B_n\) by removing the columns with indices \(a, b, c\) has non-vanishing determinant and the result follows in this case.

To conclude, it therefore suffices to show that, assuming \(p \notin \{2, \ldots, n-1\}\), there are no points on \(X_n\) with three vanishing coordinates. Suppose that the integers \(a, b, c \in \{0, \ldots, n\}\) satisfy \(0 \leq a < b < c \leq n\) and also \(x_a, x_b, x_c = 0\). The determinant of the matrix \(B_n\) with the columns \(a, b, c\) removed is non-zero by Lemma 1.1, since there are no indices in \(\{1, \ldots, n\}\) \(\subset \{1, 2, \ldots, p\}\) with difference divisible by \(p\). It follows that the subscheme of \(Y_n\) defined by \(x_a = x_b = x_c = 0\) is empty and the corollary follows. \(\square\)

1.2. **Linear automorphisms.** A scalar permutation is an automorphism of the projective space \(\mathbb{P}^N\) whose matrix with respect to the natural homogeneous coordinates only has one non-zero entry in each row and in each column. Equivalently, a scalar permutation is an automorphism of projective space fixing the set of coordinate points (namely, those points of projective space with a unique non-zero coordinate).
Theorem 1.3. Let \( X \subset \mathbb{P}^N \) be a non-degenerate smooth projective variety of codimension at least two whose defining ideal is a complete intersection of diagonal hypersurfaces of the same degree \( d \geq 2 \). Then the linear automorphism group of \( X \) consists of scalar permutations.

Proof. If \( Q \subset \mathbb{P}(I(X)_d) \) is a linear space of hypersurfaces of degree \( d \) vanishing on \( X \), denote by \( \Sigma_Q \) the intersection of the vertices of the cones in \( Q \), and note that \( \Sigma_Q \) is a coordinate linear subspace. Therefore, the linear automorphism group of \( X \) permutes the set of linear subspaces \( \{ \Sigma_Q \mid Q \subset \mathbb{P}(I(X)_d) \} \). Hence, to prove the assertion, it suffices to show that for any subset \( S \) of \( \dim X + 2 \) variables of the homogeneous coordinate ring of \( \mathbb{P}^N \) there is a diagonal element \( f \in I(X)_d \) containing exactly the variables in \( S \); given such an \( S \) we eliminate the complementary variables from \( I(X)_d \) obtaining a homogeneous form \( f \in I(X)_d \). The vertex of the cone of \( V(f) \) is a linear subspace \( L \) of dimension \( N - r \). Since \( X \) is a smooth complete intersection it follows that \( X \cap L = \emptyset \) and thus \( r \geq \dim X + 2 \). We conclude that \( r = \dim X + 2 \), as required. \( \square \)

As an application of Theorem 1.3, we deduce the following corollary.

Corollary 1.4. Let \( X_n \) be a smooth Büchi surface with \( n \geq 4 \). Then the linear automorphism group of \( X_n \) consists of scalar permutations.

Remark 1.5. Each surface \( X_n \) is a complete intersection of quadrics. Using the adjunction formula, we see that the canonical divisor \( X_n \) is linearly equivalent to \((n - 5)H\), where \( H \) is the hyperplane section of \( X_n \). Every automorphism of \( X_n \) must preserve the (anti)canonical linear series. Since the Picard group of \( X_n \) is torsion-free, it follows that the automorphism group of \( X_n \) consists just of the linear automorphisms of \( X_n \) when \( n \geq 3 \) and \( n \neq 5 \).

If \( n = 5 \) the previous argument is no longer true, since \( K_{X_5} \) is trivial. In this case the automorphism group of \( X_5 \) has infinite order and we will show in Section 4.3 that this is true even for the subgroup \( \text{Aut}(X_5)_\mathbb{Q} \) of automorphisms defined over the rationals.

Let \( \rho : \mathbb{P}^n \rightarrow \mathbb{P}^n \) be the involution defined by
\[
\rho([x_0, \ldots, x_n]) = [x_0, x_n, x_{n-1}, \ldots, x_2, x_1];
\]
the automorphism \( \rho \) induces an involution of \( X_n \) that we denote by the same symbol.

For \( i \in \{0, \ldots, 5\} \), denote by \( \sigma_i : X_5 \rightarrow X_5 \) the automorphism induced by the sign change of the variable \( x_i \) and by \( \tau \) the involution
\[
\tau : X_5 \rightarrow X_5 \quad [x_0, \ldots, x_4] \mapsto [x_3, 2x_2, x_1, 2x_0, x_5, 2x_4].
\]
As a consequence of Proposition 3.1 (or also by a direct computer analysis), we find that the linear automorphism group of \( X_5 \) is generated by \( \rho, \sigma_0, \ldots, \sigma_5 \) and \( \tau \); these elements are non-symplectic involutions. Thus, any product of an even number of these generators is symplectic. The product of three distinct elements among \( \sigma_0, \ldots, \sigma_5 \) is a non-symplectic involution with no fixed points; the corresponding quotient of \( X_5 \) is an Enriques surface. Finally, the fixed locus of \( \rho \) is a smooth genus one curve and the fixed locus of \( \tau \) is the union of two disjoint conics.
2. Genus two curves

Let \( C \) be a curve of genus two and let \( \sigma \) be the hyperelliptic involution on \( C \). By [Igu60] (see also [CGLR99, Table 1]) the automorphism group of \( C \) is isomorphic to one of the following groups: \( \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, D_4, D_6, 2D_6, \tilde{S}_4, \mathbb{Z}/10\mathbb{Z} \). In all but the first and last case, the group \( \text{Aut}(C) \) contains a non-hyperelliptic involution: we fix one and denote it by \( \tau \). We denote by \( E_1 \) and \( E_2 \) the two quotients

\[
E_1 = C/\langle \tau \rangle \quad \text{and} \quad E_2 = C/\langle \tau \sigma \rangle,
\]

which are both curves of genus one, and by \( \pi_i : C \rightarrow E_i \) the natural quotient maps.

The main result of this section is the following theorem.

**Theorem 2.1.** Let \( G \) be one of the groups in Table 2, let \( L_G \) be the corresponding lattice and let \( \mathcal{F}_G \) be the family of genus two curves whose automorphism group is isomorphic to \( G \). If \( C \in \mathcal{F}_G \), then the Néron-Severi lattice of its Jacobian surface contains a primitive sublattice isometric to the lattice \( L_G \). Moreover, the lattices \( \text{NS}(JC) \) and \( L_G \) are isomorphic whenever they have the same rank; this happens for very general curves \( C \) in the family \( \mathcal{F}_G \).

| \( G \) | \( L_G \) |
|-------|-------|
| \( V_4 \) | \((2) \oplus (-2)\) |
| \( D_4 \) | \( U(2) \oplus (-2) \) |
| \( D_6 \) | \( U \oplus (-6) \) |
| \( 2D_6 \) | \( U \oplus (-2) \oplus (-6) \) |
| \( \tilde{S}_4 \) | \( U \oplus (-2) \oplus (-4) \) |

**Table 2. Automorphism groups and Picard lattices of Jacobian surfaces**

**Lemma 2.2.** Let \( C \) be a smooth genus two curve with a non-hyperelliptic involution \( \gamma \in \text{Aut}(C) \). The subset \( E_\gamma = \{ p + \gamma(p) - K_C : p \in C \} \) of \( JC \) is an elliptic curve isomorphic to \( C/\langle \gamma \rangle \) which passes through the origin.

**Proof.** The curve \( E_\gamma \) is the image of the morphism \( C \rightarrow JC \) defined by \( p \mapsto [p + \gamma(p) - K_C] \). Since this morphism is a double cover which factors through \( C/\langle \gamma \rangle \) and \( \gamma \) is not the hyperelliptic involution, the curve \( E_\gamma \cong C/\langle \gamma \rangle \) is elliptic. The involution \( \sigma \gamma \) has two fixed points exchanged by \( \gamma \), let us call them \( p, \gamma(p) \). The identity \( [p + \gamma(p) - K_C] = [p + \sigma(p) - K_C] = 0 \) shows that \( E_\gamma \) passes through the origin. \( \square \)

Observe that by Lemma 2.2 the curves \( E_1 \) and \( E_2 \) of (2.1) are isomorphic to \( E_\tau \) and \( E_{\sigma \tau} \) respectively. We introduce the isogeny:

\[
E_\tau \times E_{\sigma \tau} \xrightarrow{\alpha} JC \quad (x, y) \mapsto x + y.
\]

**Lemma 2.3.** With the above notation, the following hold:

(i) \( E_\tau^2 = E_{\sigma \tau}^2 = 0 \) and \( E_\tau \cdot E_{\sigma \tau} = 4 \);
(ii) the isogeny \( \alpha \) has degree 4 and kernel isomorphic to \( V_4 \);
(iii) the divisor \( E_\tau + E_{\sigma \tau} \) is linearly equivalent to twice the theta divisor on \( JC \).
Proof. Since $E_τ$ and $E_{στ}$ are elliptic curves in $JC$, they both have self-intersection zero. Moreover $E_τ$ and $E_{στ}$ intersect transversally, since two elliptic curves meeting non-transversally in an abelian variety coincide; in particular the intersection number $E_τ · E_{στ}$ equals the cardinality of $E_τ ∩ E_{στ}$. This set consists of the origin plus the set of solutions of the equation $p + τ(p) = q + στ(q)$ for $p, q ∈ C$. The last set contains the elements of the form $p + τ(p)$, where $p ∈ Fix(σ)$. These give exactly three points since $τ$ acts with three orbits of cardinality two on the set of Weierstrass points of $C$. Thus $E_τ · E_{στ} = 4$, proving (i).

To show (ii) observe that if $(x, y)$ is in $ker(α)$, then $[p + τ(p) - K_C] = x = -y = [σ(q) + σ^2τ(q) - K_C] = [q' + στ(q') - K_C]$. Thus the equation has four solutions corresponding to the four points in $E_τ ∩ E_{στ}$. This proves (ii).

Now we show (iii). First of all, we calculate the intersection of a theta divisor $Θ$ with $E_τ$ and $E_{στ}$. Let $q ∈ C$ be a fixed point for the automorphism $τ ∈ Aut(C)$: We denote by $C_q := \{[p - q] : p ∈ C\} = \{[p + σ(q) - K_C] : p ∈ C\}$ and observe that $C_q$ is numerically equivalent to $Θ$. By a similar argument as above the set $E_τ ∩ C_q = \{[p + τ(p) - K_C] : p ∈ Fix(τ)\}$ has cardinality two. On the one hand $τ^*_{|E_τ}$ acts as the identity on the tangent space of $E_τ$ at all of its points. On the other hand $τ^*_{|C_q}$ acts as $-id$ on the tangent space of $C_q$ at each fixed point of $τ$. Thus we conclude that the intersection $E_τ ∩ C_q$ is transverse so that $E_τ ∩ C_q = 2$ and similarly one shows $E_{στ} ∩ C_q = 2$. We deduce that the images of $E_τ$ and $E_{στ}$ under the Kummer map $ϕ_{[2Θ]} : JC → P^3$ are two conics. By (i) the two conics intersect at four points since the four intersection points of $E_τ ∩ E_{στ}$ are fixed by $σ$. Thus the two conics are coplanar and hence $2Θ ∼ E_τ + E_{στ}$.

Proof of Theorem 2.1. For each group $G$ in Table 2, we compute the Néron-Severi group of a very general curve $C$ in $\mathcal{F}_G$ (see Remark 2.4).

Case 1: $G = V_4$. In this case, $Aut(C) = \{id, τ, σ, τσ\}$, where the hyperelliptic involution is $σ$. By Lemma 2.3 the Néron-Severi lattice $NS(JC)$ contains the sublattice $Λ_0$ generated by the classes of $\frac{1}{2}(E_τ - E_{στ})$ and $\frac{1}{2}(E_τ + E_{στ})$ $≅ Θ$. Since $E_τ · E_{στ} = 4$, the lattice $Λ_0$ is isometric to the lattice $A_1 ⊕ A_1(-1)$. The discriminant group of $Λ_0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} ⊕ \mathbb{Z}/2\mathbb{Z}$ with quadratic form $diag(-\frac{1}{2}, \frac{1}{2})$, therefore the only proper overlattice of $Λ_0$ of rank two is obtained adding the class of $\frac{1}{2}E_τ$. This possibility is ruled out by [Kan94, Theorem 1.1], so that $Λ_0$ is a primitive sublattice of $NS(JC)$ of rank two.

Case 2: $G = D_4$. In this case, $Aut(C) = \langle τ, η | η^4, τ^2, (τη)^2 \rangle$, where the hyperelliptic involution is $η$. Observe that the automorphism $η$ induces an isomorphism between $E_τ$ and $E_{ητ}$. According to Lemma 2.2 the involution $ητ ∈ Aut(C)$ defines the elliptic curve $E_{ητ}$ in $JC$. The intersection $E_τ ∩ E_{ητ}$ consists of the origin plus the points $[p + τ(p) - K_C]$, where $p ∈ Fix(η)$. The fixed points of $η$ are contained in the Weierstrass points $Fix(σ)$ of $C$. By applying the Riemann-Hurwitz formula to the degree four cyclic covering $C → C/⟨η⟩$ $≅ P^1$ we deduce that $η$ has exactly two fixed points exchanged by $τ$. Thus we deduce $E_τ · E_{ητ} = 2$ and similarly one shows $E_{στ} · E_{ητ} = 2$. The sublattice $Λ_1 ⊆ NS(JC)$ generated by the classes of $E_τ$, $E_{ητ}$, and $Θ - E_τ - E_{ητ}$ is isometric to $U(2) ⊕ (-2)$. The only isotropic and non-zero vectors in the discriminant group of $Λ_1$ are $\frac{1}{2}E_τ$ and $\frac{1}{2}E_{ητ}$. These do not belong to $NS(JC)$ by [Kan94, Theorem 1.1], thus $Λ_1$ is a primitive sublattice of $NS(JC)$ of rank three.
Case 3: \(G = D_6\). In this case, \(\text{Aut}(C) = \langle \tau, \rho \mid \rho^6, \tau^2, (\tau \rho)^2 \rangle\), where the hyperelliptic involution is \(\sigma = \rho^3\). According to Lemma 2.2, the involution \(\rho \tau \in \text{Aut}(C)\) defines the elliptic curve \(E_{\rho \tau}\) in \(JC\). The intersection \(E_{\tau} \cap E_{\rho \tau}\) consists of the origin plus the points \([p + \tau(p) - K_C]\), where \(p \in \text{Fix}(\rho)\). The fixed points of \(\rho\) are contained in the Weierstrass points of \(C\). By applying the Riemann-Hurwitz formula to the degree six cyclic covering \(C \to C/(\rho) \cong \mathbb{P}^1\) we deduce that \(\rho\) has no fixed points. Thus \(E_{\tau} \cap E_{\rho \tau} = 1\). The intersection \(E_{\sigma \tau} \cap E_{\rho \tau}\) consists of the origin plus the points \([p + \tau(p) - K_C]\), where \(p \in \text{Fix}(\rho^2)\). By applying the Riemann-Hurwitz formula to the degree three cyclic covering \(C \to C/(\rho^2) \cong \mathbb{P}^1\) we deduce that \(\rho^2\) has four fixed points exchanged in pairs by \(\sigma \tau\). Thus \(E_{\sigma \tau} \cap E_{\rho \tau} = 3\). The sublattice \(\Lambda_2 \subseteq \text{NS}(JC)\) generated by the classes of \(E_{\tau}, E_{\rho \tau}\) and \(\Theta - 2E_{\tau} - 2E_{\rho \tau}\) is isometric to \(U \oplus (-6)\). Since \(\det(\Lambda_2)\) is square free, we conclude that \(\Lambda_2\) is a primitive sublattice of \(\text{NS}(JC)\) of rank three.

Case 4: \(G = 2D_6\). In this case, \(\text{Aut}(C) = \langle \tau, \rho, \eta \mid \tau^2, \eta^2 \rho^3, \eta^4, (\rho \eta)^2, (\tau \eta)^2, \eta^{-1} \rho \eta \rho \rangle\), where \(\sigma = \rho^3 = \eta^2\) is the hyperelliptic involution. Observe that \((\tau, \eta) \cong D_4\) and \((\tau, \rho) \cong D_4\). Both \(E_{\tau}\) and \(E_{\rho \tau}\) admit complex multiplication of order three induced by the automorphism \(\rho^3\), thus we obtain \(\text{Hom}(E_{\tau}, E_{\rho \tau}) \cong \mathbb{Z}^2\), so that \(\text{NS}(E_{\tau} \times E_{\rho \tau})\) has rank four. Let \(\Lambda_3 \subset \text{NS}(JC)\) be generated by the classes of \(E_{\tau}, E_{\rho \tau}, \Theta - E_{\eta}\) and \(\Theta - 2E_{\tau} - 2E_{\rho \tau}\). The only intersection number we need to compute is the one between \(E_{\eta} \cap E_{\rho \tau}\) since the other ones appear in Case 2 and Case 3. The intersection \(E_{\eta} \cap E_{\rho \tau}\) consists of the origin plus the points \([p + \eta \tau(p) - K_C]\), where \(p \in \text{Fix}(\rho \eta^3)\). Observe that \((\rho \eta^3)^3 = \sigma\), so that its fixed points are contained in the Weierstrass points of \(C\). As in Case 2, we deduce that it has exactly two fixed points which are exchanged in pairs by \(\eta \tau\). Thus \(E_{\eta \tau} \cap E_{\rho \tau} = 2\). The lattice \(\Lambda_3\) is isometric to \(U \oplus (-2) \oplus (-6)\). The only isotropic and non-zero vector in the discriminant group of \(\Lambda_3\) is \(\frac{1}{2} E_{\eta \tau}\). This does not belong to \(\text{NS}(JC)\) by [Kan94, Theorem 1.1], thus \(\Lambda_3 = \text{NS}(JC)\) since \(\Lambda_3\) is a primitive sublattice of \(\text{NS}(JC)\) of the same rank. Observe that an equation for \(C\) in this case is \(y^2 = x^5 - x\).

Case 5: \(G = \tilde{S}_4\). In this case, \(\text{Aut}(C) = \langle \tau, \rho, \eta \mid \tau^2, \eta^2 \rho^3, \eta^4, (\rho \tau)^2, (\tau \eta)^2, (\rho \eta)^3 \rangle\), where \(\sigma = \rho^3 = \eta^2\) is the hyperelliptic involution. In the previous case we only need to compute the intersection number between \(E_{\eta \tau} \cap E_{\rho \tau}\). The intersection \(E_{\eta \tau} \cap E_{\rho \tau}\) consists of the origin plus the points \([p + \eta \tau(p) - K_C]\), where \(p \in \text{Fix}(\rho \eta^3)\). In this case \((\rho \eta^3)^3 = \sigma\), thus we deduce as in Case 3 that it has no fixed points. Thus \(E_{\eta \tau} \cap E_{\rho \tau} = 1\). Let \(\Lambda_4 \subset \text{NS}(JC)\) be the lattice generated by the classes of \(E_{\tau}, E_{\rho \tau}, \Theta - E_{\tau} - E_{\rho \tau}\) and \(E_{\tau} - E_{\eta \tau} + 2E_{\rho \tau}\), which is isometric to \(U \oplus (-2) \oplus (-4)\). Since the discriminant group of \(\Lambda_4\) has no non-zero isotropic vectors, we conclude that \(\Lambda_4 = \text{NS}(JC)\) since \(\Lambda_4\) is a primitive sublattice of \(\text{NS}(JC)\) of the same rank. Observe that an equation for \(C\) in this case is \(y^2 = x^5 - x\).

To conclude our proof observe that since \(\Lambda_3\) and \(\Lambda_4\) are non-isometric lattices of rank four and both of them are Néron-Severi lattices of Jacobian of curves of the irreducible family given in Case 3, the very general curve \(C\) of this irreducible family must have \(\text{NS}(JC)\) of rank three. Hence we deduce the equality \(\Lambda_2 = \text{NS}(JC)\) for such a curve \(C\) since \(\Lambda_2\) is a primitive sublattice of \(\text{NS}(JC)\) of the same rank. Similarly, we deduce the equality \(\Lambda_1 = \text{NS}(JC)\) for the very general curve \(C\) in Case 2. Since \(\Lambda_1\) and \(\Lambda_2\) are non-isometric lattices and the family of curves in
Case 1 is irreducible, we deduce by a similar argument the equality $\Lambda_0 = \text{NS}(JC)$ for the very general curve $C$ in Case 1.

**Remark 2.4.** Observe that the ranks of the Néron-Severi groups of $JC$ and $E_\tau \times E_{\sigma \tau}$ coincide, since the two abelian surfaces are isogenous. The generality conditions on $C$ in Theorem 2.1 can be made explicit using the formula

$$\text{rk}(\text{NS}(JC)) = \text{rk}(\text{NS}(E_\tau \times E_{\sigma \tau})) = \text{rk}(\text{Hom}(E_\tau, E_{\sigma \tau})) + 2.$$  

For a proof of this formula see [Kan08, Proposition 22]. Thus the rank of $\text{NS}(JC)$ is two if and only if the two elliptic curves $E_\tau$ and $E_{\sigma \tau}$ are non-isogenous and the rank of $\text{NS}(JC)$ is three if and only if the two elliptic curves $E_\tau$ and $E_{\sigma \tau}$ are isogenous and do not have complex multiplication.

3. **Kummer surfaces**

Let $K$ be a field of characteristic different from two and denote by $\mathbb{A}^2_{t,s}$ the affine plane over $K$ with affine coordinates $t, s$. Let $\mathcal{C} \to \mathbb{A}^2_{t,s}$ be the projective family of genus two curves given by the affine equation

$$\mathcal{C}: \quad y^2 = x(x-2t)(x-2s)(x-1)^2(x-\frac{1}{t}).$$

Setting $\gamma_1 = 2s^2-1$, $\gamma_2 = t-s$, $\gamma_3 = 2ts-1$ and $\gamma_4 = 2t^2-1$, we find that the family $\mathcal{C}$ is smooth over the open subset of $\mathbb{A}^2_{t,s}$ where $s\gamma_1\gamma_2\gamma_3\gamma_4 \neq 0$. Observe that the automorphism group of the family contains the hyperelliptic involution $\sigma$ and an involution $\tau$ defined by $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x^2})$. Conversely, let $C$ be a smooth genus two curve whose automorphism group $\text{Aut}(C)$ contains the hyperelliptic involution $\sigma$ and an involution $\tau \neq \sigma$. The action of $\tau$ descends to an action of $\mathbb{Z}/2\mathbb{Z} = \langle \tau \rangle$ on the quotient curve $C/\langle \sigma \rangle \cong \mathbb{P}^1$. The automorphism $\tau$ does not fix any Weierstrass point since the curve $C$ is smooth. After a linear change of coordinates we can assume that $\tau$ is the map $x \mapsto \frac{1}{x}$ and that the images of the six Weierstrass points of $C$ in $\mathbb{P}^1$ are $\{0, \infty, 2t, 1/t, 2s, 1/s\}$ for some $t, s \in \mathbb{C}$. Hence $C$ appears in the family $\mathcal{C}$.

3.1. **Klein form and automorphisms.** Let $\mathcal{X} \to \mathbb{A}^2_{t,s}$ be the relative minimal resolution of the quartic Kummer surface associated to the relative Jacobian $J\mathcal{C}$. The equations of $\mathcal{X}$ are

$$\mathcal{X}: \quad \begin{cases} x_1^2 - 2t^2x_2^2 + \gamma_4x_3^2 = 2\gamma_4x_0^2 \\ t^2sx_2^2-ts\gamma_2x_3^2-ts^2x_4^2 = -\gamma_2x_0^2 \\ \gamma_1x_3^2 - 2s^2x_4^3 + x_5^2 = 2\gamma_1x_0^2. \end{cases}$$

These equations are obtained from the Klein equations for the resolution of the Kummer surface of the Jacobian of a genus two curve of the form $y^2 = f_5(x)$ (see [PV00, §5.3, (36)]). The surfaces in the family $\mathcal{X}$ contain exactly 32 lines: they are cut out by the four hyperplane sections with equations $x_1 \pm x_5 \pm 2\gamma_2x_0 = 0$. Let $\ell \subset \mathcal{X}$ denote the family of lines with equations

$$\ell: \quad \begin{cases} x_2 = x_1 - \frac{1}{2}\gamma_2x_0 \\ x_3 = x_1 - 2tx_0 \\ x_4 = x_1 - \frac{1}{2}\gamma_3x_0 \\ x_5 = x_1 - 2\gamma_2x_0. \end{cases}$$
For \((t,s) = (-1,1)\) the line \(\ell\) contains the arithmetic progressions mentioned in the introduction. Denote by \([6]\) the set \(\{0, \ldots, 5\}\). For each subset \(\mathcal{P} \subset [6]\), denote by \(\sigma_{\mathcal{P}}\) the automorphism of the family \(\mathcal{X}\) that changes the sign of all the variables with indices in \(\mathcal{P}\); clearly we have \(\sigma_{\mathcal{P}} = \sigma_{[6]\setminus \mathcal{P}}\). To simplify the notation, we sometimes denote the involution \(\sigma_{\{a_1, \ldots , a_r\}}\) by \(\sigma_{a_1 \cdots a_r}\). It is straightforward to check that the lines corresponding to two distinct partitions \(\mathcal{P} \cup \mathcal{Q} \) and \(\mathcal{P}' \cup \mathcal{Q}'\) of \([6]\) intersect if and only if the automorphism \(\sigma_{\mathcal{P}\sigma_{\mathcal{Q}}}\) is the sign change of a single variable. 

The lines corresponding to the 16 two-set partitions with even size are pairwise non-adjacent, and similarly for the 16 partitions into sets of odd size. Hence, the intersection graph of the lines on \(\mathcal{X}\) is connected bipartite of type \((16,16)\), and it is regular of valence 6. Observe that one set of pairwise non-adjacent lines is the set of the exceptional divisors of the resolution \(X \to \operatorname{Kum}(JC)\), while the other set contains the images of the 16 translates of a theta divisors in \(JC\).

Let \(\Sigma\) be the group of sign changes of the coordinates: the group \(\Sigma\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^5\) and is a subgroup of the linear automorphism group of any surface in the family \(\mathcal{X}\).

**Proposition 3.1.** Let \(X \subset \mathbb{P}^5\) be a fiber in the family \(\mathcal{X}\) and let \(C\) be the corresponding genus two curve. The linear automorphism group of \(X\) is isomorphic to a semidirect product of \(\operatorname{Aut}(C)/\langle \sigma \rangle\) with \(\Sigma\).

**Proof.** Let \(G\) be the linear automorphism group of \(X\). Denote by \(G_C\) the subgroup of \(G\) induced by \(\operatorname{Aut}(C)\): the group \(G_C\) is isomorphic to \(\operatorname{Aut}(C)/\langle \sigma \rangle\). We first show that the group \(G\) is generated by \(G_C\) and \(\Sigma\). By what we have seen, the group \(\Sigma\) acts simply transitively on the set of lines of \(X\). Let \(\gamma\) be a linear automorphism of \(X\). Composing \(\gamma\) with an element of \(\Sigma\) if necessary, we reduce to the case when \(\gamma\) fixes the line \(\ell\), in particular it preserves each of the two sets of pairwise disjoint lines of \(X\). We deduce that \(\gamma\) comes from an automorphism of \(JC\). Moreover, since it preserves the theta divisor corresponding to \(\ell\), then it is induced by an automorphism of \(C\) by the Torelli Theorem [Lau01, Théoréme 3].

Each element of \(G_C\) is a scalar permutation by Theorem 1.3 and fixes at least one of the lines. We deduce that the intersection \(G_C \cap \Sigma\) consists of only the identity, and we are left to show that \(\Sigma\) is a normal subgroup of the group \(G\). This follows from the fact that if \(D \in \Sigma\) and \(M \in G_C\), then \(M^{-1}DM\) is a diagonal involution and hence in \(\Sigma\). \(\square\)

**Remark 3.2.** The linear automorphism group of the general fiber \(X\) of \(\mathcal{X}\) is generated by \(\Sigma\) and by the involution

\[\tau: X \to X \quad [x_0, \ldots, x_4] \mapsto [tx_3, 2tx_2, sx_1, 2tx_0, tx_5, 2tx^2_4].\]

In fact, the group \(\Sigma\) is contained in the linear automorphism group of the resolution of any Jacobian Kummer surface defined by the Klein equations in \(\mathbb{P}^5\). The involutions changing an even number of signs preserve each set of pairwise disjoint lines and do not preserve any line, this implies that they are induced by the translations by 2-torsion points on \(JC\).

If \(X\) corresponds to a genus two curve \(C\) with \(\operatorname{Aut}(C) \cong D_4\), then the linear automorphism group of \(X\) is generated by \(\Sigma\), \(\tau\) and the involution \(\rho\) defined in Subsection 1.2

### 3.2. Special families

In this subsection we study the special loci in the affine plane \(\mathbb{A}^2_{t,s}\) where the corresponding genus two curve admits extra automorphisms.
We compute the Picard group of the minimal resolution of the Kummer surface over each component.

Proof of Theorem 1. In each case we determine the intersection matrix of $\text{Pic}(X)$ as follows. By means of Theorem 2.1 we first compute the transcendental lattice of $JC$ which is the orthogonal complement of $\text{NS}(JC)$ in $H^2(JC, \mathbb{Z}) \cong U^{\oplus 3}$. This easily gives the transcendental lattice $T(X)$, which is isometric to $T(JC)(2)$ by [Mor84, Proposition 4.3]. Finally, we obtain $\text{Pic}(X)$ as the orthogonal complement of $T(X)$ in the K3 lattice. Let $(A_L, q_L)$ denote the discriminant group of a sublattice $L$ of a unimodular lattice. When computing the orthogonal complement of either $\text{NS}(JC)$ or $T(X)$ we use the fact that the discriminant group of $A_L$ is isomorphic to $(A_L, -q_L)$ and that, under suitable conditions, a lattice with a given signature and discriminant quadratic form is unique up to isometries. This uniqueness property follows from [Nik79, Theorem 1.13.2] and from the classification of positive definite binary quadratic forms [CS99, Table 15.1]. Observe that in case $G = 2D_6$ the quadratic form $\text{diag}(2, 6)$ is the only positive definite binary form with discriminant group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ and having a non-zero isotropic vector. In case $G = S_4$ the quadratic form $\text{diag}(2, 4)$ is the only positive definite binary form with discriminant group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

We will explain the case $G = D_4$, the other cases being similar. The discriminant group of $\text{NS}(JC)$ is isometric to $(\mathbb{Z}/2\mathbb{Z})^3$ with quadratic form

$$q_{\text{NS}(JC)} = \left( \begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right) \oplus \left( -\frac{1}{2} \right).$$

Thus $T(JC)$ has signature $2, (1, 1)$ and discriminant group $((\mathbb{Z}/2\mathbb{Z})^3, -q_{\text{NS}(JC)})$. The only lattice with such property, up to isometries, is $U(2) \oplus (2)$. Thus $T(X)$ is isometric to $U(4) \oplus (4)$ and its discriminant group is $(\mathbb{Z}/4\mathbb{Z})^3$ with quadratic form

$$q_{T(X)} := \left( \begin{array}{cc} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{array} \right) \oplus \left( \frac{1}{4} \right).$$

The Picard lattice of $X$ has signature $(1, 18)$ and discriminant quadratic form $-q_{T(X)}$; again by the uniqueness property, it is isometric to $U(4) \oplus (-4) \oplus E_8^{\oplus 2}$. 

Recall that the hyperelliptic involution $\sigma$ in coordinates is $(x, y) \mapsto (x, -y)$, while the involution $\tau$ is $(x, y) \mapsto \left( \frac{x}{2}, \frac{y + \sqrt{2}x}{2} \right)$. The corresponding families of quotient curves are smooth of genus one and admit the following affine equations:

$$E_1 := \mathcal{C}_{t,s}/\langle \tau \rangle \quad y^2 = (x + 2\sqrt{2}) \left( x - 2t - \frac{1}{t} \right) \left( x - 2s - \frac{1}{s} \right),$$

$$E_2 := \mathcal{C}_{t,s}/\langle \sigma \tau \rangle \quad y^2 = (x - 2\sqrt{2}) \left( x - 2t - \frac{1}{t} \right) \left( x - 2s - \frac{1}{s} \right).$$

The quotient maps are $(x, y) \mapsto \left( \frac{x}{2}, \frac{y + \sqrt{2}x}{2} \right)$, where the positive sign is for $E_1$ and the negative sign is for $E_2$. For special values of $t$ and $s$ the hyperelliptic curve $\mathcal{C}_{t,s}$ acquires more automorphisms: see Table 3 and also [CGLR99]. The curves in the affine $(t, s)$-plane corresponding to $D_4$ and $D_6$ intersect in 17 points. One of them is $(0, 0)$ which corresponds to a singular curve. The remaining sixteen points appear in Table 3. The eight points for $2D_6$ correspond to isomorphic curves of genus two and the same holds for the eight points in the case $S_4$. 
Example 3.3. In [Bre81] A. Bremner investigated the geometry of the K3 surface defined by

\[
X_B : \begin{cases}
x^2 + xu - u^2 = w^2 + wz - z^2 \\
y^2 + yv - v^2 = u^2 + ux - x^2 \\
z^2 + zw - w^2 = v^2 + vy - y^2.
\end{cases}
\]

His main motivation was the search for rational points of \(X_B\), since these give (cubing and summing the equations) rational solutions to the classical Diophantine equation \(x^6 + y^6 + z^6 = u^6 + v^6 + w^6\). Recently M. Kuwata [Kuw07, Theorem 5.1, Corollary 5.2] proved that \(X_B\) is the Kummer surface of \(JC_B\), where \(C_B : y^2 = (x^2 + 1)(x^2 + 2x + 5)(x^2 - 2x + 5)\), and that \(\text{Pic}(X_B)\) has rank 19. An easy computation shows that \(C_B = \mathcal{C}_{t,s}\), with \((t, s) = (\sqrt{3}, \sqrt{2})\), so that its automorphism group is isomorphic to \(D_6\). Since the \(j\)-invariants of the quotient curves \(E_1, E_2\) are not rational integers, the curves do not admit complex multiplication so that \(\text{rk Hom}(E_1, E_2) = 1\). By Theorem 1 the Picard lattice of \(X_B\) is isometric to \(U(2) \oplus E_8^\perp\). We observe that a basis of \(\text{Pic}(X_B)\) had been given in [Bre81, Theorem 3.3].

Example 3.4. In [BB06] J. Browkin and J. Brzeziński study the solutions of the system

\[
X_{BB} : \begin{cases}
x_1^2 - 2x_2^2 + x_3^2 = x_4^2 - 2x_5^2 + x_6^2 \\
x_2^2 - 2x_3^2 + x_4^2 = x_5^2 - 2x_6^2 + x_7^2 \\
x_3^2 - 2x_4^2 + x_5^2 = x_6^2 - 2x_7^2 + x_8^2
\end{cases}
\]

consisting of the sequences of six integer squares with constant second differences. The surface \(X_{BB}\) is a smooth K3 surface. An easy computation shows that it is the Kummer surface of the genus two curve \(\mathcal{C}_{t,s}\) with \((t, s) = (2\sqrt{2}, \sqrt{2})\). The automorphism group of the curve is \(V_4\). Moreover the \(j\)-invariants of the two curves \(E_1\) and \(E_2\) are \(2\sqrt{3}, \sqrt{3}\) and \(2\sqrt{3}, \sqrt{3}\) respectively. Since the prime 5 is of bad reduction just for \(E_1\), the curves \(E_1\) and \(E_2\) are not isogenous and thus \(\text{rk Hom}(E_1, E_2) = 0\). By Theorem 1 the Picard lattice of \(X_{BB}\) is isometric to \(U \oplus E_8 \oplus D_7 \oplus (-4)\).

Remark 3.5. Let \(X\) be any smooth surface in the family \(\mathcal{X}\). The surface \(X\) is a moduli space of rank two vector bundles on itself. This gives a natural modular
interpretation to the rational points on $X$. We do not know if there is a similar interpretation for the integral points on $X$.

Mukai studied the moduli spaces of stable vector bundles on abelian and K3 surfaces in its influential paper [Muk84]. In particular, he showed in [Muk84, Example 0.9] the existence of a beautiful correspondence between the K3 surfaces that are intersections of three quadrics in $\mathbb{P}^5$ and certain moduli spaces of vector bundles on the K3 surfaces themselves. The correspondence is obtained as follows. Let $Y$ be a smooth complete intersection of three quadrics in $\mathbb{P}^5$; denote by $\mathbb{P}^2$ the net of quadrics containing $X$. In the net of quadrics $\mathbb{P}^2$, the discriminant locus $\Delta_Y$ is the locus of quadrics of rank at most 5: the discriminant $\Delta_Y$ is a plane curve of degree six. Denote by $M_Y$ minimal resolution of the double cover of $\mathbb{P}^2$ branched over the sextic $\Delta_Y$. The general point of $M_Y$ corresponds to a quadric $Q$ containing $Y$, together with the choice of a ruling by 2-planes of the quadric $Q$. Identifying smooth quadrics in $\mathbb{P}^5$ with the Grassmannian $Gr(2, 4)$, we see that the two rulings by 2-planes of a quadric $Q$ correspond to the second Chern classes of the rank 2 vector bundles on $Q$ determined by the dual of the tautological vector bundle and by the universal quotient bundle. Thus, to each point of $M_Y$ corresponding to a smooth quadric $Q$ containing $X$ together with a choice of ruling on $Q$, we associate a rank two vector bundle on $X$ by restricting to $X$ itself the rank two vector bundle on $Q$ determined by the chosen ruling. This correspondence induces an isomorphism between $M_Y$ and the irreducible component of the moduli space $\mathcal{M}(2, \mathcal{O}_X(1), 2)$ of stable rank two vector bundles $E$ on $X$ with Chern classes $c_1(E) = [\mathcal{O}_X(1)]$ and $\deg(c_2(E)) = 4$.

It is classically known that the K3 surfaces that are complete intersections of a net of quadrics and the double cover of the net of quadrics branched over the discriminant locus are isomorphic as soon as the surface contains a line; this applies to our case as $X$ contains 32 lines. A modern reference for this result is [MN04].

### 3.3. Elliptic fibrations.

**Theorem 3.6.** Let $X$ be a smooth surface in the family $\mathcal{X}$; the following statements hold.

(i) There are no quadrics of rank three containing $X$.

(ii) The quadrics of rank four containing $X$ appear in Table 4.

(iii) Every quadric of rank four containing $X$ determines two elliptic fibrations on $X$ defined by pencils of hyperplane sections.

| $\gamma_1(x_1^2 - 2t^2x_2^3) + \gamma_4(2s^2x_1^2 - x_2^2)$ | $\gamma_1x_1^2 + \gamma_2x_1^2 - tx_1 - 4s\gamma_2x_2^2$ | $t(2s^2x_1^2 - s\gamma_2x_2^2 - s^2x_2^4) + \gamma_2x_2^2$ |
|---------------------------------|---------------------------------|---------------------------------|
| $x_1^2 - 2t^2x_2^3 + \gamma_4(x_1^2 - 2x_2^3)$ | $ts(\gamma_2x_2^2 + \gamma_3x_2^3 - \gamma_4x_2^4) - \gamma_2\gamma_3x_2^5$ | $s\gamma_3x_1^2 + 4t^3s\gamma_2x_2^2 - \gamma_2\gamma_3x_2^3 - t\gamma_4x_2^4$ |
| $\gamma_1x_1^2 - 2s^2x_1^2 + x_2^2 - 2\gamma_1x_2^3$ | $ts(\gamma_1x_1^2 - s\gamma_2x_2^2 + \gamma_2x_2^3) - \gamma_1\gamma_2\gamma_3x_2^5$ | $s\gamma_1x_1^2 - \gamma_1\gamma_2\gamma_3x_2^3 + 4t^3s\gamma_2x_2^2 - t\gamma_3x_2^4$ |
| $s(x_1^2 + \gamma_3x_1^2 - 2tx_2^2) - 2t\gamma_3x_2^3$ | $t(\gamma_3x_1^2 + 2t\gamma_2x_2^2 - \gamma_4x_2^4) - \gamma_2\gamma_3x_2^5$ | $\gamma_2x_1^2 + 2t^3\gamma_3x_2^2 - \gamma_2\gamma_3x_2^3 - 2ts^2\gamma_4x_2^4$ |
| $t(2s^2x_1^2 - \gamma_3x_1^2 - x_2^2) + 2s\gamma_1x_2^2$ | $s(\gamma_1x_1^2 + 2s\gamma_2x_2^2 - \gamma_3x_2^3) - 2\gamma_1\gamma_2\gamma_3x_2^5$ | $2t^2\gamma_2x_1^2 - \gamma_1\gamma_2\gamma_3x_2^3 - 2s^3\gamma_1x_2^4 + 2t\gamma_3x_2^3$ |

Table 4: Quadrics of rank four containing $X$

**Proof.** (i) Suppose that $Q$ is a quadric of rank at most three containing $X$, and hence the singular locus of $Q$ contains a two-dimensional plane $\Pi$. Let $Q_1, Q_2$ be two quadrics such that $X = Q \cap Q_1 \cap Q_2$; the intersection $X \cap \Pi$ contains $\Pi \cap Q_1 \cap Q_2$. 


and is therefore not empty and consists of singular points of \( X \), contradicting the assumption that \( X \) is smooth. We deduce that every quadric vanishing on \( X \) has rank at least four.

(ii) Since the surface \( X \) is defined by diagonal quadrics, it follows that the quadrics of rank at most four containing \( X \) are obtained from the equations of \( X \) by eliminating two or more of the variables. Eliminating a variable corresponds to a linear equation on the net of quadrics containing \( X \), so that for each pair \( \mathcal{I} \) of variables there is a quadric containing \( X \) and not involving the variables in \( \mathcal{I} \). An easy computation shows that these are exactly the quadrics defined by the polynomials in Table 4.

(iii) Let \( Q \subset \mathbb{P}^5 \) be a quadric of rank four containing the surface \( X \) (see Table 4); since \( X \) is smooth and it is a complete intersection of quadrics, it follows that \( X \) does not meet the vertex of \( Q \). In particular, the rational map \( \mathbb{P}^5 \dashrightarrow \mathbb{P}^3 \) obtained by projecting away from the singular subscheme of \( Q \) induces a morphism \( \pi_Q: X \rightarrow \overline{Q} \), where \( \overline{Q} \subset \mathbb{P}^3 \) is the image of the projection of \( Q \) from its vertex; composing the morphism \( \pi_Q \) with one of the two projections \( \overline{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) we obtain a fibration \( X \rightarrow \mathbb{P}^1 \). It is immediate to check that these fibrations have connected fibers and therefore they are elliptic fibrations: we deduce from items (i) and (ii) that there are 30 elliptic fibrations on \( X \) arising in this way.

\[ \top \]

**Theorem 3.7.** The family of Kummer surfaces \( \mathcal{X} \rightarrow \mathbb{A}_t^2 \) admits the structure of a family of elliptic fibrations over \( \mathbb{P}^1 \) with Mordell-Weil group of positive rank. In particular, if \( X \) is a smooth surface in the family \( \mathcal{X} \) defined over a field \( K \), then the set of \( K \)-rational points of \( X \) is Zariski-dense.

**Proof.** The family \( \mathcal{X} \) is contained in the rank four quadric \( \mathcal{Q} \) of equation

\[ \mathcal{Q}: \quad \gamma_1(x_1^3 - 2t^2x_2^3) + \gamma_4(2s^2x_1^3 - x_2^3) = 0. \]

This quadric is a cone over \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}_t^2 \) with vertex a line. The two rulings of \( \mathbb{P}^1 \times \mathbb{P}^1 \) induce two elliptic fibrations \( \pi_{\pm}: \mathcal{X} \rightarrow \mathbb{P}^1 \times \mathbb{A}_t^2 \). We partition the set \( \mathcal{L} \) of lines of \( \mathcal{X} \) into the sets

\[ \mathcal{L}_+ = \begin{pmatrix} \ell & \sigma_0 \ell & \sigma_3 \ell & \sigma_03 \ell \\ \sigma_14 \ell & \sigma_014 \ell & \sigma_134 \ell & \sigma_25 \ell \\ \sigma_24 \ell & \sigma_024 \ell & \sigma_234 \ell & \sigma_15 \ell \\ \sigma_45 \ell & \sigma_045 \ell & \sigma_345 \ell & \sigma_12 \ell \end{pmatrix}, \quad \mathcal{L}_- = \begin{pmatrix} \sigma_1 \ell & \sigma_01 \ell & \sigma_13 \ell & \sigma_013 \ell \\ \sigma_2 \ell & \sigma_01 \ell & \sigma_13 \ell & \sigma_013 \ell \\ \sigma_3 \ell & \sigma_02 \ell & \sigma_23 \ell & \sigma_15 \ell \\ \sigma_4 \ell & \sigma_02 \ell & \sigma_23 \ell & \sigma_12 \ell \end{pmatrix}; \]

each row of \( \mathcal{L}_+ \) forms a square in \( \mathcal{X} \) and similarly for the rows of \( \mathcal{L}_- \). The lines of \( \mathcal{L}_+ \) form four fibers of \( \pi_+ \) of type \( I_4 \) and are sections of \( \pi_- \); similarly, the lines of \( \mathcal{L}_- \) form four fibers of \( \pi_- \) of type \( I_4 \) and are sections of \( \pi_+ \). Thus both elliptic fibrations have Mordell-Weil group of positive rank, since they admit two intersecting sections [Shi90, Lem. 8.2 and Thm. 8.4].

The last statement follows at once since all the lines of \( X \) are defined over the field \( K \).

\[ \top \]

**4. The Büchi K3-surface**

From now on we denote by \( X \) the Büchi K3 surface \( X_t \) or equivalently \( \mathcal{X}_{-1,1} \). In particular \( X \) is the Kummer surface of the genus two curve \( C_{t,s} \) with \( (t, s) = (-1, 1) \) which has affine equation \( y^2 = x(x^2 - 4)(x^2 - 1) \). Thus \( \text{Aut}(C) \) is isomorphic to \( D_4 \). Since the \( j \)-invariants of the quotient curves \( E_1, E_2 \) are not rational integers,
then the curves do not admit complex multiplication so that \( \text{rk} \text{Hom}(E_1, E_2) = 1 \).

By Theorem 1 we conclude

\[ \text{Pic}(X) \cong U(4) \oplus (-4) \oplus E_8^{\oplus 2}. \]

Over the field \( \mathbb{Q}(\sqrt{-6}) \) we can absorb more of the coefficients of the equation of the Kummer surface \( S \) in \( \mathbb{P}^3 \) to obtain the equation

\[ 9(x_0^4 + x_1^4 + x_2^4 + x_3^4) + 6(x_0^2 x_2^2 + x_2^2 x_3^2) + 6(x_0^2 x_2^2 + x_1^2 x_3^2) - 14(x_0^2 x_3^2 + x_1^2 x_2^2) = 0. \]

4.1. Elliptic fibrations. In this section we determine the elliptic fibrations on the surface \( X \) induced by the quadrics of rank four containing \( X \). Such quadrics are given in Table 4 for \( X_t,s \). We describe here the corresponding singular fibers and the Mordell-Weil groups for the Büchi K3 surface \( X \). As a consequence, we find all the conics contained in \( X \). Observe that each of the 15 quadrics in Table 4 does not involve a pair of variables of \( \mathbb{P}^5 \). In the three columns of Table 4 the first quadric is invariant under the linear automorphism group, while the remaining four form two pairs of orbits.

We later analyze in detail the fibrations associated to the first quadric in the first column of Table 4. In general, using the Shioda-Tate formula ([Shi72, Theorem 1.1]) and an explicit set of generators of the Picard group of \( X \), described in Proposition 4.2, we computed with Magma [BCP97] the Mordell-Weil groups of the fibrations associated to the quadrics in Table 4:

(i) the Mordell-Weil groups associated to the first quadric in the first column are isomorphic to \( \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}) \), with singular fibers \( 4I_4 + 4I_2 \);

(ii) the Mordell-Weil groups associated to the remaining quadrics in the first column are isomorphic to \( \mathbb{Z}^3 \oplus (\mathbb{Z}/4\mathbb{Z}) \), with singular fibers \( 4I_4 + 2I_2 + 4I_1 \);

(iii) the Mordell-Weil groups associated to the quadrics in the last two columns are isomorphic to \( \mathbb{Z}^5 \oplus (\mathbb{Z}/2\mathbb{Z}) \), with singular fibers \( 4I_4 + 8I_1 \).

Each fibration contains four fibers of type \( I_4 \), represented by four squares consisting of lines. We also checked that the torsion elements of the Mordell-Weil group are represented by lines on \( X \) and that the quotient of the Mordell-Weil group by the subgroup generated by the lines is free of rank 0 in case (i), rank 1 in case (ii), and rank 2 in case (iii).

**Proposition 4.1.** Every conic on \( X \) is contained in a reducible fiber of an elliptic pencil on \( X \), and viceversa, every fiber of type \( I_2 \) in an elliptic pencil on \( X \) arising from the quadrics in Table 4 is the union of two conics.

**Proof.** Let \( C \subset X \) be a conic on \( X \) and denote by \( \Pi \subset \mathbb{P}^5 \) the two-dimensional plane containing \( C \); it suffices to show that there is a quadric of rank four containing \( X \) and the plane \( \Pi \). Restricting the equations of \( X \) to the plane \( \Pi \) we find that they are all proportional to an equation defining the conic \( C \). We obtain that there is a pencil \( \mathcal{P} \) of quadrics containing \( X \) and the plane \( \Pi \). It follows that the base locus of the pencil of quadrics \( \mathcal{P} \) is singular and hence that there is a quadric \( Q \) in the pencil \( \mathcal{P} \) of rank at most four (see for instance [Dom80, Subsection 1.2]). By Theorem 3.6(i) the rank of the quadric \( Q \) is four and we conclude by Theorem 3.6(iii).

The converse follows from the explicit analysis of the elliptic fibrations arising from the quadrics of rank four of Table 4. \( \square \)
We now analyze the elliptic fibrations on $X$ associated to the first quadric $x_1^2 - 2x_2^2 + 2x_3^2 - x_5^2 = 0$ in Table 4: we concentrate on these fibrations, since the Mordell-Weil group in these cases is generated by the lines on $X$. The two corresponding elliptic fibrations are defined by:

$$\pi_{\pm}: X \to \mathbb{P}^1 \quad [x_0, \ldots, x_5] \mapsto [x_2 \pm x_4, x_1 + x_5].$$

Observe that linear automorphisms of $X$ permutes these two fibrations. For example, $[x_2 - x_4, x_1 + x_5]$ and $[x_2 + x_4, x_1 - x_5]$ give fibrations defining the same elliptic pencil on $X$ due to the following relation in $I(X)$:

$$(x_1 + x_5)(x_1 - x_5) = 2(x_2 + x_4)(x_2 - x_4).$$

The types of the singular fibers of two fibrations $\pi_+$ and $\pi_-$ coincide (since $\pi_+$ and $\pi_-$ are exchanged by an automorphism of the surface) and they are

- 4 fibers of type $I_4$, whose components are the trivial lines of $X$ lying above the points $[1, 1], [-1, 1], [2, 1], [-2, 1]$;
- 2 fibers of type $I_2$, whose components are conics defined over $\mathbb{Q}(\sqrt{2})$ lying above the points $[\sqrt{2}, 1], [-\sqrt{2}, 1]$; and
- 2 fibers of type $I_2$ whose components are conics defined over $\mathbb{Q}(\sqrt{-2})$ lying above the points $[\sqrt{-2}, 1], [-\sqrt{-2}, 1]$.

### 4.2. The Picard group

Let $\zeta_8$ be a primitive complex 8-th root of unity. The Galois group $G$ of the extension $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ acts on $X$ and on its Picard lattice. Let $C_1$ and $C_2$ be the two conics of equations:

- **$C_1$:**
  
  $$\begin{align*}
  2x_1^2 - x_5^2 + 12x_0^2 &= 0 \\
  x_1 + \sqrt{2}x_2 &= 0 \\
  x_4 + \sqrt{2}x_5 &= 0 \\
  2x_3 - \sqrt{2}x_0 &= 0
  \end{align*}$$

- **$C_2$:**
  
  $$\begin{align*}
  2x_1^2 - x_5^2 + 4x_3^2 &= 0 \\
  x_1 - \sqrt{-2}x_4 &= 0 \\
  2x_2 - \sqrt{-2}x_5 &= 0 \\
  x_3 + \sqrt{-2}x_0 &= 0
  \end{align*}$$

which are contracted by $\pi_-$ and are defined over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$ respectively.

**Proposition 4.2.** Let $\text{Pic}(X)_G$ be the subgroup of $\text{Pic}(X)$ fixed by the Galois group $G$. There is a decomposition of $G$-modules

$$\text{Pic}(X) = \text{Pic}(X)_G \oplus \langle C_1, C_2 \rangle,$$

where $\text{Pic}(X)_G$ has rank 17 and is spanned by the classes of the lines of $X$.

**Proof.** Since any line of $X$ is defined over the rationals, the classes of the 32 lines of $X$ span a sublattice $\Lambda$ of $\text{Pic}(X)_G$. By calculating the Smith form of the intersection matrix of these lines plus $C_1$ and $C_2$ one deduces that the discriminant of the lattice $\Lambda \oplus \langle C_1, C_2 \rangle$ is $2^6$, which is also the discriminant of $\text{Pic}(X)$ by Theorem 1. Thus $\text{Pic}(X) = \Lambda \oplus \langle C_1, C_2 \rangle$ and the statement follows. \(\square\)

### 4.3. Rational points

We apply the results of the previous sections to the study of the rational points on $X$. Several authors used elliptic curves of positive rank to prove Zariski density of rational points on elliptic surfaces, see for instance [BT00, HT00]. We can use the results of the previous section to explicitly determine an element of infinite order on the generic fiber $X_\eta$ of $\pi_-$. By putting $\alpha := \frac{x_1 + x_5}{x_2}$ we get that $X_\eta$ is a quartic curve contained in the three dimensional linear subspace of $\mathbb{P}^5$. We chose an origin for the Mordell-Weil group of $X_\eta$ to be the point $O$ whose coordinates are:

$$O := [-2\alpha - 2, 2\alpha + 1, -2\alpha, -2\alpha + 1, 2\alpha - 2, 1].$$
The point $O$ corresponds to a line which is a section of the fibration $\pi$. Now we take the point $Q$ which corresponds to the line $R_2$. Since $R_1$ and $R_2$ intersect at one point, then by [Shi90] the point $Q$ has infinite order with respect to the origin $O$. Moreover we know $Q$ to be the generator of the free part of the Mordell-Weil group of $X_Q$. Its coordinates are the following:

$$Q := [2\alpha + 2, -2\alpha - 1, 2\alpha, 2\alpha - 1, -2\alpha + 2, 1].$$

By calculating $2Q$ we obtain the following result.

**Proposition 4.3.** A one-parameter family of non-trivial rational solutions of the Büchi problem is

$$
\begin{align*}
x_0 &= 12\alpha^4 + 5\alpha^2 - 1 \\
x_1 &= 8\alpha^5 + 8\alpha^4 + 22\alpha^3 - 2\alpha^2 + 2\alpha + 2 \\
x_2 &= -8\alpha^5 - 4\alpha^4 + 2\alpha^3 - 11\alpha^2 - 2\alpha - 1 \\
x_3 &= 8\alpha^5 - 10\alpha^3 - 6\alpha \\
x_4 &= 8\alpha^5 - 4\alpha^4 - 2\alpha^3 - 11\alpha^2 + 2\alpha - 1 \\
x_5 &= -8\alpha^5 + 8\alpha^4 - 22\alpha^3 - 2\alpha^2 - 2\alpha + 2.
\end{align*}
$$

4.4. **Supersingular primes.** In this subsection we will prove a congruence property for the supersingular primes for the elliptic curve $E$ of equation $y^2 = x^3 - 8x^2 - 2x$, isomorphic over the rational numbers to the elliptic curve of equation $y^2 = x^3 - 35x^2 + 35x - 1$.

**Lemma 4.4.** The number field $K_4$ obtained by extending $\mathbb{Q}$ with the abscissas of all the 4-torsion points of the curve $E$ is the 24th cyclotomic field $\mathbb{Q}(\mu_{24})$.

**Proof.** The field $K_4$ is the splitting field of the polynomial

$$x(x^2 - 8x - 2)(x^2 + 2)(x^4 - 16x^3 - 12x^2 + 32x + 4).$$

The roots of the degree four factor are

$$4 + 2\sqrt{3} \pm (2\sqrt{6} + 3\sqrt{2}) \quad \text{and} \quad 4 - 2\sqrt{3} \pm (2\sqrt{6} - 3\sqrt{2}),$$

hence an easy calculation shows $\mathbb{Q}(\mu_{24}) = K_4$, as required. $\square$

A calculation using Magma shows that the abelianization of the Galois group of the field $\mathbb{Q}(E[8])$ is the 24th cyclotomic field, but we do not need this more precise information: for us it is enough to know that the field $\mathbb{Q}(\mu_{24})$ is contained in $\mathbb{Q}(E[8])$.

In the next theorem we adapt a strategy that we learned from [Bra09]. In the proof we work with the 8-torsion subgroup, since the information coming from the 4-torsion elements is not enough for our purposes.

**Theorem 4.5.** Let $p$ be a supersingular prime for the elliptic curve $E$. Then $p \equiv 5$ or $23 \mod 24$.

**Proof.** We concentrate on the Galois representation

$$\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/8\mathbb{Z})$$

obtained by acting with the absolute Galois group on the set of 8-torsion points of the curve $E$. By Lemma 4.4 the image $G(8)$ of $\rho$ admits a quotient $A(8)$ isomorphic to the Galois group of the extension $\mathbb{Q}(\mu_{24})/\mathbb{Q}$. Given a prime of good reduction $p$, the Frobenius automorphism acting on the reduction of $X$ modulo $p$ lifts to an element $\Phi_p \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The image of $\Phi_p$ in the quotient $A(8) \cong (\mathbb{Z}/24\mathbb{Z})^*$ of the representation $\rho$ is a congruence class modulo 24. If $p$ is a supersingular prime for $E$, then $\Phi_p$ has null trace. We will show that the image in $A(8)$ of elements with
null trace is congruent to either 5 or 23 modulo 24, concluding the proof. To prove
the last step, we made use of Magma to show that the image \( G(8) \) of the Galois
representation \( \rho \) is generated by the following matrices:

\[
\begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}.
\]

The group \( G(8) \) is a solvable group of order \( 2^6 \) and index \( 2^3 \cdot 3 \) in \( \text{GL}_2(\mathbb{Z}/8\mathbb{Z}) \). It
contains 16 elements of null trace whose images in the quotient \( A(8) \) form a subset
\( S \) of cardinality 2. Since the primes 5 and 167 are supersingular for \( E \) and the
second is congruent to 23 modulo 24, the images of \( \Phi_5 \) and \( \Phi_{167} \) in \( A(8) \) exhaust
the whole \( S \). This concludes the proof. \( \square \)

**Theorem 4.6.** Let \( p \) be a supersingular prime for the elliptic curve \( E \) such that
\( p \equiv 23 \pmod{24} \). Then the number of points of the reduction of \( X \) modulo \( p \) equals
\( p^2 + 18p + 1 \).

**Proof.** Let \( JC \) be the Jacobian of \( C \) and let \( Y := JC/(\sigma) \subset \mathbb{P}^3 \) be its quotient
with respect to the involution \( \sigma : w \mapsto -w \). Observe that \( \#X(\mathbb{F}_p) = \#Y(\mathbb{F}_p) + 16p \)
for any prime \( p \geq 5 \) since all the sixteen singular points of \( Y \) are defined over \( Z \).
Consider two subsets of \( JC \) over \( \mathbb{F}_p^2 \):

\[
S_1 := \{ w \in JC(\mathbb{F}_p^2) : \Phi_p(w) = w \}, \quad S_2 := \{ w \in JC(\mathbb{F}_p^2) : \Phi_p(w) = -w \},
\]

where \( \Phi_p \) is the Frobenius map. We have \( \#Y(\mathbb{F}_p) = \frac{1}{2}(\#S_1 + \#S_2) \). We now want
to show that \( S_1 = JC(\mathbb{F}_p) \) and that

\[
\#S_2 = \#J\tilde{C}(\mathbb{F}_p),
\]

where \( \tilde{C} \) is the quadratic twist of \( C \). The first equality is obvious. About the second
equality, let \( y^2 = f(x) \) be an equation for \( C \) and \( dy^2 = f(x) \) be an equation for \( \tilde{C} \).
Consider the isomorphism

\[
f : C \to \tilde{C} \quad (x, y) \mapsto (x, \frac{1}{\sqrt{d}} y)
\]

defined over \( \mathbb{F}_p^2 \) and observe that \( \Phi_p \circ f = \tilde{f} \circ f \circ \Phi_p \), where \( \tilde{f} \) is the hyperelliptic
involution of \( \tilde{C} \). Let \( f_* : JC \to J\tilde{C} \) be the push-forward map defined by \( f_*([\sum n_i q_i]) = [\sum n_i f(q_i)] \).
Then by the above equality we get \( \Phi_p \circ f_* = -f_* \circ \Phi_p \).
To prove \( (4.1) \) it is enough to show that \( f_*(S_2) = JC(\mathbb{F}_p) \), which immediately
follows from

\[
\Phi_p(f_*(w)) = -f_*(\Phi_p(w)) = f_*(w),
\]

where \( w \in S_2 \).

Assume now that \( p \) is a supersingular prime for \( E \) such that \( p \equiv 23 \pmod{24} \).
Hence the Jacobian variety \( JC \) is isogenous to \( E_{\sqrt{7}} \times E_{-\sqrt{7}} \), by \( (2.2) \) and the fact
that 2 is a square in \( \mathbb{F}_p \). This gives

\[
\#S_1 = \#JC(\mathbb{F}_p) = (p+1)^2.
\]

Moreover the trace of the Frobenius morphism \( \Phi_q \), with \( q = p^n \), on \( H^1_{\text{et}}(JC, \mathbb{Q}_l) \)
equals the trace of the Frobenius morphism on the first étale cohomology group of \( E_{\sqrt{7}} \times E_{-\sqrt{7}} \). Since \( E_{\pm\sqrt{7}} \mod p \) is supersingular then its zeta function is:

\[
Z(E_{\pm\sqrt{7}}, t) = \frac{qt^2 + 1}{(1-t)(1-qt)}.
\]
By means of the isogeny $JC \sim E_{\sqrt{2}} \times E_{-\sqrt{2}}$ we deduce that the zeta function of $C$ is

\begin{equation}
Z(C, t) = \frac{(qt^2 + 1)^2}{(1 - t)(1 - qt)}.
\end{equation}

Observe that $\#C(F_p) = \#\tilde{C}(F_p)$ since $\#C(F_p) + \#\tilde{C}(F_p) = 2p + 2$ and $\#C(F_p) = p + 1$, by the above calculation of the zeta function of $C$. Hence $C$ and $\tilde{C}$ have the same zeta function, since the two curves have the same number of points over $F_p$, and same number of points over $F_{p^2}$, being isomorphic over this field. This gives $\#JC(F_p) = \#J\tilde{C}(F_p) = (p + 1)^2$, so that $\#Y(F_p) = (p + 1)^2$ and $\#X(F_p) = (p + 1)^2 + 16p$ as claimed. \hfill \Box

\textbf{Remark 4.7.} Theorem 4.5 shows that the supersingular primes for the curve $E$ are congruent to 5 or 23 modulo 24. For those congruent to 5 modulo 24 the zeta functions of $C$ and of $\tilde{C}$ coincide and are equal to

\begin{equation}
Z(C, t) = Z(\tilde{C}, t) = \frac{(qt^2 - 1)^2}{(1 - t)(1 - qt)},
\end{equation}

analogous to formula (4.2). We verified this formula for the primes 5, 149, 173, 461, 1229, 2213, 2237 that are the first 7 supersingular primes of $E$ congruent to 5 modulo 24. As a consequence, the number of points of $X$ modulo a prime $p$ congruent to 5 modulo 24 and supersingular for $E$ is $p^2 + 14p + 1$.
[Kan94] _____, Elliptic curves on abelian surfaces, Manuscripta Math. 84 (1994), no. 2, 199–223. ↑8, 9

[Kuw07] M. Kuwata, Equal sums of sixth powers and quadratic line complexes, Rocky Mountain J. Math. 37 (2007), no. 2, 497–517. ↑13

[Igu60] J. Igusa, Arithmetic variety of moduli for genus two, Ann. of Math. (2) 72 (1960), 612–649. ↑7

[Lau01] K. Lauter, Geometric methods for improving the upper bounds on the number of rational points on algebraic curves over finite fields, J. Algebraic Geom. 10 (2001), no. 1, 19–36. With an appendix in French by J.-P. Serre. MR1795548 (2001j:11047) ↑11

[MN04] C. Madonna and Viacheslav V. Nikulin, On a classical correspondence between K3 surfaces. II, Strings and geometry, 2004, pp. 285–300. ↑14

[Mor84] D. R. Morrison, On K3 surfaces with large Picard number, Invent. Math. 75 (1984), no. 1, 105–121. ↑12

[Muk84] S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. Math. 77 (1984), no. 1, 101–116. ↑14

[Nik79] V. V. Nikulin, Integer symmetric bilinear forms and some of their geometric applications, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177, 238. ↑12

[Pas11] H. Pasten, Büchi’s problem in any power for finite fields, Acta Arith. 149 (2011), no. 1, 57–63. ↑1

[PPV10] H. Pasten, T. Pheidas, and X. Vidaux, A survey on Büchi’s problem: new presentations and open problems, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 377 (2010), no. Issledovaniya po Teorii Chisel, 10, 111–140, 243. ↑1, 3

[PV00] L. A. Piovan and P. Vanhaecke, Integrable systems and projective images of Kummer surfaces, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 4 serie 29 (2000), no. 2, 351–392. ↑10

[SV11] P. Sáez and X. Vidaux, A characterization of Büchi’s integer sequences of length 3, Acta Arith. 149 (2011), no. 1, 37–56. ↑1

[Shi72] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan 24 (1972), 20–59. ↑16

[Shi90] _____, On the Mordell-Weil lattices, Comment. Math. Univ. St. Paul. 39 (1990), no. 2, 211–240. ↑15, 18

[Vid11] X. Vidaux, Polynomial parametrizations of length 4 Büchi sequences, Acta Arith. 150 (2011), no. 3, 209–226. ↑1

[Voj00] P. Vojta, Diagonal quadratic forms and Hilbert’s tenth problem, Hilbert’s tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), 2000, pp. 261–274. ↑1

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD DE CONCEPCIÓN, CASILLA 160-C, CONCEPCIÓN, CHILE
E-mail address: michela.artebari@gmail.com, antonio.lafac@gmail.com

MATHEMATICS INSTITUTE, ZEEMAN BUILDING, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UNITED KINGDOM
E-mail address: adomani@gmail.com