CHOW MOTIVES OF ELLIPTIC MODULAR SURFACES AND THREEFOLDS

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Abstract. The main result of this paper is the proof for elliptic modular threefolds of some conjectures formulated by the second-named author and shown by Jannsen to be equivalent to a conjecture of Beilinson on the filtration on the Chow groups of smooth projective varieties. These conjectures are known to be true for surfaces in general, but for elliptic modular surfaces we obtain more precise results which are then used in the proof of the conjectures for elliptic modular threefolds.

Let \(\phi : E \to M\) be the universal elliptic curve with level-\(N\) structure, whose smooth completion is an elliptic modular surface \(\overline{E}\). An elliptic modular threefold is a desingularization \(\tilde{E}\) of the fibre product \(E \times_\overline{E} \overline{E}\). The first main result is that there exists a decomposition of the diagonal \(\Delta(\tilde{E})\) modulo rational equivalence as a sum of mutually orthogonal idempotent correspondences \(\pi_i\) which lift the Künneth components of the diagonal modulo homological equivalence. These correspondences act on the Chow groups of \(\tilde{E}\), and secondly we show that \(\pi_i \cdot \text{CH}^j(\tilde{E}) = 0\) for \(i < j\) or \(i > 2j\); the implication of this is that there is a filtration on \(\text{CH}^j(\tilde{E})\) that has \(j\) steps, as predicted by the general conjectures. The third main result is that the first step of this filtration, the kernel of \(\pi_{2j}\) acting on \(\text{CH}^j(\tilde{E})\), coincides with the kernel of the cycle class map from \(\text{CH}^j(\tilde{E})\) into the cohomology \(H^{2j}(\tilde{E})\); which is to say that there is a natural, geometric description for this step of the filtration. We also identify \(F^2 \text{CH}^3(\tilde{E})\) as the Albanese kernel. As a by-product of our methods we also obtain some information about the Chow groups of the Chow motives for modular forms \(kW\) defined by Scholl, for \(k = 1\) and \(2\), for example that \(\text{CH}^1(1W) = \text{CH}^2_{\text{Alb}}(E)\), and that \(\text{CH}^3(2W) = F^3 \text{CH}^3(\tilde{E})\) lives at the deepest level of the filtration, within the Albanese kernel.

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Introduction

Let $X$ be a smooth projective variety of dimension $d$. The second-named author has conjectured that as an element of the Chow group $\text{CH}^d(X \times X) \otimes \mathbb{Q}$ the diagonal $\Delta(X)$ can be decomposed as a sum $\Delta(X) = \sum_{i=0}^{2d} \pi_i$ of mutually orthogonal idempotent correspondences modulo rational equivalence which lift the Künneth components of the diagonal [Murre, 1993]. These Chow-Künneth components of the diagonal, which are not in general canonical, act on the Chow groups of $X$ to give a decomposition of the form $\text{CH}^j(X) \otimes \mathbb{Q} = \bigoplus_{i=j}^{\infty} \pi_i \cdot (\text{CH}^j(X) \otimes \mathbb{Q})$. Then conjecturally $\pi_i \cdot \text{CH}^j(X) \otimes \mathbb{Q} = 0$ for $i < j$ or $i > 2j$; and when this is the case, the filtration defined by $F^\nu \text{CH}^j(X) \otimes \mathbb{Q} := \bigoplus_{i=j}^{\infty} \pi_i \cdot (\text{CH}^j(X) \otimes \mathbb{Q})$ has precisely $j$ steps. A third conjecture asserts that this filtration is independent of the choice of projectors $\pi_i$; and as the first step in this direction, a fourth conjecture proposes that $F^1 \text{CH}^j(X) \otimes \mathbb{Q}$ is precisely the kernel of the cycle class map into cohomology.

U. Jannsen has shown that these conjectures of the second-named author [op. cit] together are equivalent to conjectures of Beilinson on the existence of a canonical filtration on the Chow groups of smooth projective varieties [Jannsen, 1994]; see also [Bloch, 1980] [Beilinson, 1987]. The class of varieties for which the conjectures are known to be true is still very small: For curves, it is elementary (compare [Manin, 1968], [Kleiman, 1972]); for surfaces, see [Murre, 1990]; for products of surfaces and curves, see [Murre, 1993, II]; for uniruled threefolds, see [del Angel and Müller-Stach, 1996]; and the existence of a Chow-Künneth decomposition is known for abelian varieties [Shermenev, 1974] [Deninger and Murre, 1991] [Künnemann, 1994].

The main result of the present paper is the proof of the conjectures (except for some points concerning the canonicity of the filtration) for elliptic modular threefolds. To describe these, let $N \geq 3$ be an integer (which we suppress from the notation), let $M := M_N$ be the modular curve parameterizing elliptic curves with full level-$N$ structure, and let let $\phi : E \to M$ be the universal elliptic curve (with full level-$N$ structure) over $M$. Then the smooth completion $\overline{\phi : E \to M}$ of $E$ over the compactification $\overline{M}$ of $M$ obtained by adjoining the cusps is an elliptic modular surface [Shioda, 1972]. The fibre product $\mathcal{E} := E \times_{\overline{M}} \overline{E}$ over $\overline{M}$ has only rational double points for singularities, and by blowing these up we get the nonsingular elliptic modular threefold $\mathcal{E}$ that is the main focus of our attention; such threefolds have also been studied in [Schoen, 1986]. For the fibre products $\mathcal{E} \times_{\overline{M}} \cdots \times_{\overline{M}} \overline{E}$ ($k \geq 1$ times) there is a natural desingularization $\mathcal{E}$ due to [Deligne, 1969], but see also [Scholl, 1990]; the first-named author of the present paper has looked at the cohomology and the Hodge structure of these $\mathcal{E}$, and verified the generalized Hodge conjecture for them [Gordon, 1993].

To prove the conjectures for elliptic modular surfaces and threefolds, we begin by constructing projectors for $E$ that extend the canonical relative projectors that are known for $E$ as elliptic curve scheme over $M$ [Deninger and Murre, 1991] [Künnemann, 1994]. Using these projectors, we construct a finer Chow motive decomposition of $E$ than the Chow-Künneth decomposition, and thus obtain more...
precise results about the filtration of the Chow groups of $E$ than can be proved for surfaces in general [Murre, 1990]. Then using the fibre product structure of $\tilde{E}$, we can construct projectors for $2\tilde{E}$ which extend the relative tensor products (over $M$) of the canonical relative projectors for $E/M$. The projectors we get this way, together with a detailed knowledge of the cohomology of $\tilde{E}$, then give us a Chow-Künneth decomposition from which we are able to deduce the other conjectures as well. We expect that these methods can be generalized to give a Chow-Künneth decomposition for $k\tilde{E}$ for any $k \geq 1$; however, this becomes technically more complicated, and we intend to return to it later.

For $k \geq 1$ Scholl has constructed Chow motives $\mathcal{W} := \mathcal{W}_N$ supported on $\tilde{E}$, and he has shown that their cohomology groups are the parabolic cohomology groups attached to cusp forms of weight $(k + 2)$ and level $N$ [Scholl, 1990]. Not surprisingly, we also encounter these motives, for $k = 1$ and $2$, and we recover the same results about their cohomology. But then we also study their Chow groups: For $\mathcal{W}$ we show that (modulo torsion) $\text{CH}^1(\mathcal{W}) = \text{CH}^2(\mathcal{W}) = \text{CH}^2_{\text{Alb}}(\tilde{E})$, the kernel in $\text{CH}^2(\tilde{E})$ of the Albanese map; for $\mathcal{W}$ we find that (modulo torsion) it has only two Chow groups, namely $\text{CH}^2(\mathcal{W})$, which is related to the intermediate Jacobian $J^2(\tilde{E})$, and $\text{CH}^3(\mathcal{W})$, which we find lies in the deepest level of the filtration on $\text{CH}^3(\tilde{E})$.

The paper is organized as follows. In section one we recall the definitions and some facts about Chow motives, and give the precise statements of the conjectures. In section two we collect together some of the facts we need about elliptic modular surfaces and threefolds. In section three we construct projectors which extend the canonical relative relative projectors for $E/M$ to the fibre variety $\tilde{E}$ over $\bar{M}$, and we also construct projectors for $2\tilde{E}$ that extend the tensor products of these canonical relative projectors. We also need some extra projectors to account for the degenerate fibres over the cusps, and we introduce these in section three as well. Section four is the technical center of the paper, for there we identify the motives defined by the projectors of section three with motives supported on varieties of lower dimension and the “Scholl motives” $\mathcal{W}$, for $k = 1, 2$. In section five we study the cohomology of the motives from section four, and obtain Chow-Künneth decompositions for $E$ and $2\tilde{E}$. Finally in section six we use the Chow-Künneth decompositions from section five to study the Chow groups of $E$ and $2\tilde{E}$ and obtain the desired results about the filtrations on those Chow groups. The main results are stated precisely in Theorems 4.2, 5.1 and 6.2, and each section has a small introduction of its own.

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1. Chow motives and the conjectures.

Let $k$ be a field, let $\rho : S \to \text{Spec} \ k$ be a smooth, connected, quasi-projective scheme, and let $\mathcal{V}(S)$ be the category of projective $S$-schemes $\lambda : X \to S$ with $\lambda$ smooth. When $S = \text{Spec} \ k$ we write $\mathcal{V}(k)$ for $\mathcal{V}(\text{Spec} \ k)$. Let $\text{CH}^j(X)$ denote the
Chow group of codimension \( j \) algebraic cycles on \( X \) modulo rational equivalence, and 
\( \text{CH}^j(X, \mathbb{Q}) := \text{CH}^j(X) \otimes \mathbb{Q} \). In those cases where we need to consider the
Chow group of a singular variety \( V \) we write \( \text{CH}_i(V) \) for the Chow group in
the sense of [Fulton, 1984] of dimension \( i \) algebraic cycles on \( X \) modulo rational equivalence, and
\( \text{CH}_i(V, \mathbb{Q}) := \text{CH}_i(V) \otimes \mathbb{Q} \). For a cycle \( Z \) on \( X \) we write \([Z]\)
for its class in \( \text{CH}^j(X, \mathbb{Q}) \) or \( \text{CH}^j(X, \mathbb{Q}) \).

1.1. The category of Chow motives. To establish some general notations and
fix ideas, we briefly recall some basic definitions and properties for the category of
Chow motives over \( S \), specifically allowing the possibility that \( S = \text{Spec} \, k \). For
more details see [Scholl, 1994] and [Deninger and Murre, 1991].

1.1.1. Definition of the category of Chow motives. Let \( X \) and \( Y \) in \( \mathcal{V}(S) \),
and for convenience we assume that \( X \) is connected and of relative dimension
\( d_S(X) \) over \( S \). Then the group of relative correspondences of degree \( r \) from \( X \) to
\( Y \) is
\[
\text{Corr}_S^r(X, Y) := \text{CH}^{d_S(X)+r}(X \times_S Y, \mathbb{Q}).
\]
There is also the usual bilinear composition
\[
(\alpha, \beta) \mapsto \beta \circ \alpha := \text{pr}_{13*}(\text{pr}_{12}^*\alpha \cdot \text{pr}_{23}^*\beta),
\]
where \( \text{pr}_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j \) is the projection and the intersection product
is taken in \( \text{CH}^*(X_1 \times_S X_2 \times_S X_3, \mathbb{Q}) \). Then the category \( \mathcal{M}(S) \) of Chow motives
over \( S \) can be defined by: Objects are triples \((X, p, m)\), where \( X \) is in \( \mathcal{V}(S) \), and
\( m \in \mathbb{Z} \), and \( p \circ p = p \in \text{Corr}_S^0(X, X) \) is an idempotent (projector); and morphisms
are given by
\[
\text{Hom}_{\mathcal{M}(S)}((X/S, p, m), (Y/S, q, n)) := q \circ \text{Corr}_S^{n-m}(X, Y) \circ p
\]
\[
= q \circ \text{CH}^{d_S(X)+n-m}(X \times_S Y, \mathbb{Q}) \circ p.
\]
When \( m = 0 \) we usually write \((X/S, p)\) for \((X/S, p, 0)\).

1.1.2. Examples. (a) There is a unit object in \( \mathcal{M}(S) \), namely \( 1_S := (S, \text{id}_S) \).
More generally, when \( X \) in \( \mathcal{V}(S) \) (connected) has a rational section \( e : S \to X \) (or
still more generally, a relative zero-cycle of degree one), then \( 1_S \cong (X/S, (e(S) \times_S X)) \).

(b) By definition, the Lefschetz motive is \( \mathbb{L}_S := (S, \text{id}_S, -1) \), and more generally
we let \( \mathbb{L}_S^d := \mathbb{L}_S \otimes \mathbb{L}_S \cdots \otimes \mathbb{L}_S = (S, \text{id}_S, -d) \). When \( d_S(X) = d \) and there exists
a rational section \( e : S \to X \), then also \( \mathbb{L}_S^d \cong (X/S, (X \times_S e(S))) \). In particular, when
\( S = \text{Spec} \, k \) and \( X \) is any curve with a rational point \( e \), then \( \mathbb{L} \cong (X, X \times \{e\}) \).

(c) For simplicity, let \( S = \text{Spec} \, k \), and let \( \alpha \in \text{CH}^p(X, \mathbb{Q}) \) and \( \beta \in \text{CH}^q(X, \mathbb{Q}) \),
where \( p + q = \dim X \), and suppose the intersection multiplicity \( (\alpha \cdot \beta) = 1 \). Then
\( \alpha \times \beta \in \text{CH}^{p+q}(X \times X, \mathbb{Q}) \) is a projector, and \((X, \alpha \times \beta) \cong \mathbb{L}^q \) in \( \mathcal{M}(k) \). In fact,
\( \alpha \in \text{Corr}^{-q}(X, \text{Spec} \, k) \) and \( \beta \in \text{Corr}^q(\text{Spec} \, k, X) \), and these induce inverse isomorphisms.
1.1.3. The tensor product in $\mathcal{M}(S)$. There is a tensor product in $\mathcal{M}(S)$, induced by the direct product in $\mathcal{V}(S)$. First, for $\alpha \in \text{CH}^*(X_1, Q)$ and $\beta \in \text{CH}^*(X_2, Q)$ let

$$\alpha \times_S \beta := (\text{pr}_1^*(\alpha) \cdot \text{pr}_2^*(\beta)) \in \text{CH}^*(X \times_S Y, Q).$$

Next, for correspondences $\phi \in \text{CH}^*(X_1 \times_S X_2, Q)$ and $\psi \in \text{CH}^*(X_3 \times_S X_4, Q)$, let

$$\phi \otimes_S \psi := t^*(\phi \times_S \psi) \in \text{CH}^*((X_1 \times_S X_3) \times_S (X_2 \times_S X_4), Q),$$

where

$$t : (X_1 \times_S X_3) \times_S (X_2 \times_S X_4) \longrightarrow (X_1 \times_S X_2) \times_S (X_3 \times_S X_4)$$

permutes the factors. This determines the tensor product on morphisms, and then the tensor product of two objects is given by

$$(X/S, p, n) \otimes (Y/S, q, m) := ((X \times_S Y)/S, p \otimes_S q, m + n).$$

1.1.4. The direct sum in $\mathcal{M}(S)$. There is also a direct sum in $\mathcal{M}(S)$, induced by taking disjoint union in $\mathcal{V}(S)$. When $m = n$ it is defined by

$$(X/S, p, m) \oplus (Y/S, q, m) := (X \sqcup Y, p \oplus q, m).$$

If $m < n$, say, then rewrite

$$(X/S, p, m) \equiv (X/S, p, n) \oplus \mathbb{L}_S^{n-m} = (X \times_S (\mathbb{P}_S^1)_{n-m}, p', n)$$

for some projector $p'$, and then the direct sum is defined by

$$(X/S, p, m) \oplus (Y/S, q, n) := ((X \times_S (\mathbb{P}_S^1)^{n-m} \sqcup Y)/S, p' \oplus q, n).$$

1.1.5. $\mathcal{M}(S)$ is pseudoabelian. With these definitions it can be shown that $\mathcal{M}(S)$ is a $\mathbb{Q}$-linear pseudoabelian tensor category. An additive category is said to be pseudoabelian iff for every object $M$ every idempotent $g \in \text{End}_{\mathcal{M}(S)}(M)$ has an image, or equivalently a kernel, and the canonical map

$$(\text{Im}(g) \oplus \text{Im}(\text{id} - g)) \longrightarrow M$$

is an isomorphism. See [Jannsen, 1992] or [Scholl, 1994, Cor.3.5] to see that $\mathcal{M}(S)$ is not in general an abelian category.

1.1.6. The functor $\mathcal{V}(S) \rightarrow \mathcal{M}(S)$. There is a natural contravariant functor from $\mathcal{V}(S)$ to $\mathcal{M}(S)$, given by associating to a morphism $f : X \rightarrow Y$ of smooth projective $S$-schemes the class of the transpose of its graph, $[\Gamma_f] \in \text{CH}^{\text{ds}(Y)}(Y \times_S X, Q)$, and associating to $X$ in $\mathcal{V}(S)$ the object $(X, [\Delta(X)])$, where $\Delta(X)$ denotes the diagonal in $X \times_S X$. When $S = \text{Spec} k$ we write

$$h(X) := (X, [\Delta(X)]).$$
1.1.7. Some formulas. For later use we note that for \( X, Y, Z \) in \( V(S) \), and 
\( f : X \to Y, \ f' : Y \to X, \ g : Y \to Z, \ g' : Z \to Y, \) and \( \alpha \in CH(X \times_S Y, \mathbb{Q}) \) and 
\( \beta \in CH(Y \times_S Z, \mathbb{Q}) \), 

\[
[\Gamma_g] \circ \alpha = (\text{id}_X \times_S g)_*(\alpha) \quad \quad [\Gamma_{g'}] \circ \alpha = (\text{id}_X \times_S g')^*(\alpha)
\]

\[
\beta \circ [\Gamma_f] = (f \times_S \text{id}_Z)^*(\beta) \quad \quad \beta \circ [\Gamma_{f'}] = (f' \times_S \text{id}_Z)_*(\beta)
\]

\[
[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{g \circ f}] \quad \quad [\Gamma_{g'}] \circ [\Gamma_{f'}] = [\Gamma_{f' \circ g'}]
\]

see [Deninger and Murre, 1991, 1.2.1]. In particular, if \( f_1 : X \to Y, \ f_2 : X \to Y, \) then

\[
[\Gamma_{f_2}] \circ [\Gamma_{f_1}] = (f_1 \times f_2)_*([\Delta(X)])
\]

1.1.8. Remark on the relation between relative and absolute motives. When \( S \) is projective the covariant functor \( V(S) \to V(k) \) taking \( \lambda : X \to S \) to \( \rho \circ \lambda : X \to \text{Spec} \ k \) induces a natural covariant functor \( \Psi : M(S) \to M(k) \) which makes the diagram

\[
\begin{array}{ccc}
V(S) & \xrightarrow{\rho_*} & V(k) \\
\downarrow & & \downarrow h \\
M(S) & \xrightarrow{\Psi} & M(k)
\end{array}
\]

commute. Namely, let \( i : X \times_S Y \hookrightarrow X \times Y \) be the inclusion, and consider the morphism

\[
i_* : \text{CH}^{d_S(X)}(X \times_S Y, \mathbb{Q}) \to \text{CH}^{\text{dim}X}(X \times Y, \mathbb{Q}).
\]

Then it is easy to see that the codimensions work out so that a relative correspondence of degree zero maps to an absolute correspondence of degree zero, and it can also be checked that composition of relative correspondences agrees with composition of absolute correspondences under this “pushing forward.” Thus when \( Y = X \) relative projectors map to absolute projectors, and in this way we get a functor \( \Psi \) as claimed. Although this remark does not precisely apply to the situation of this paper, it has been useful as part of the philosophy behind our methods; see also 3.2.7–3.2.8 and 3.3.7–3.3.9 below.

1.1.9. The Chow groups of a Chow motive. Recall that in general a correspondence \( \gamma \in \text{CH}(X_1 \times_k X_2, \mathbb{Q}) \) acts on a cycle class \([Z] \in \text{CH}(X_1, \mathbb{Q})\) by

\[
\gamma([Z]) := (\text{pr}_2)_*(\text{pr}_1^*(Z) \cdot \gamma).
\]

Then the Chow groups of \((X, p, m)\) in \( M(k) \) are defined by

\[
\text{CH}^j((X, p, m), \mathbb{Q}) := p(\text{CH}^{j+m}(X, \mathbb{Q}))
\]

\[
= \text{Hom}_{M(k)}(\mathbb{L}_{k}^j, (X, p, m))
\]
and we let
\[ \text{CH}((X, p, m), \mathbb{Q}) := \bigoplus_{j \in \mathbb{Z}} \text{CH}^j((X, p, m), \mathbb{Q}). \]

1.1.10. The cohomology groups of a Chow motive. In principle the cohomology groups of a Chow motive can be defined with respect to any Weil cohomology theory, cf. [Kleiman, 1968, 1994], but in this paper we will only consider Betti and étale cohomology. For a smooth, projective scheme \( X \) over \( k \), we write \( H^i_\bullet(X, \mathbb{Q}_\bullet) \) to signify either the Betti cohomology of \( X \) (\( \mathbb{C} \)) with coefficients in \( \mathbb{Q}_B := \mathbb{Q} \), if \( k \) comes with an embedding into \( \mathbb{C} \), or the étale cohomology of \( X \times_{\text{Spec} k} \text{Spec} k_{\text{sep}} \) with coefficients in \( \mathbb{Q}_\ell \); after taking Tate twists into account, cf. [Deligne, 1982, §1], we have
\[ H^i_\bullet(X, \mathbb{Q}_\bullet(r)) := \begin{cases} H^i_B(X(\mathbb{C})^{\text{an}}, \mathbb{Q}_B(r)), \\ H^i_{\text{ét}}(X \times_{\text{Spec} k} \text{Spec} k_{\text{sep}}, \mathbb{Q}_\ell(r)). \end{cases} \]

Then the cohomology groups of \((X, p, m)\) in \( M(k) \) are defined by
\[ H^i_\bullet((X, p, m), \mathbb{Q}_\bullet) := p(H^{i+2m}_\bullet(X, \mathbb{Q}_\bullet(m))). \]

Note that the \( i \)-th cohomology group of \((X, p, m)\) has weight \( i \), and for instance \( H^i_\bullet((X, p, m), \mathbb{Q}_\bullet) \neq H^i_\bullet((X, p), \mathbb{Q}_\bullet(m)) \). Let
\[ H_\bullet((X, p, m), \mathbb{Q}_\bullet) := \bigoplus_{i \in \mathbb{Z}} H^i_\bullet((X, p, m), \mathbb{Q}_\bullet). \]

1.2. The conjectures. Continuing to establish general terminology, as well as some of the underlying motivation, we briefly recall the conjectures from [Murre, 1993] about the Chow groups of smooth projective varieties. For more details and a summary of what is known, see op. cit.; for the relationship with conjectures of Beilinson, see [Jannsen, 1994].

1.2.1. Definition of Chow-Künneth decomposition. Let \( X \) be a smooth projective variety of dimension \( d \). A Chow-Künneth decomposition of \( X \) is a collection of mutually orthogonal projectors \( \pi_0(X), \ldots, \pi_{2d}(X) \) in \( \text{CH}^d(X \times X, \mathbb{Q}) = \text{Corr}^0(X, X) \) such that
\[ \sum_{i=0}^{2d} \pi_i(X) = [\Delta(X)], \]
and
\[ \pi_i(X)(H_\bullet(X, \mathbb{Q}_\bullet)) = H^i_\bullet(X, \mathbb{Q}_\bullet). \]

When a Chow-Künneth decomposition of \( X \) exists, we let
\[ h^i(X) := (X, \pi_i(X)). \]
**Conjecture A.** For any smooth projective variety \( X \) there exists a Chow-Künneth decomposition of \( X \).

**Conjecture B.** Let \( X \) be a smooth projective variety, and assume that there exists a Chow-Künneth decomposition of \( X \). Then

\[
\text{CH}^j(h^i(X), \mathbb{Q}) := \pi_i(X)(\text{CH}^j(X, \mathbb{Q})) = 0 \quad \text{for } i < j \text{ or } i > 2j.
\]

1.2.2. A filtration on the Chow groups of \( X \). Let \( X \) be a smooth projective variety, and assume that there exists a Chow-Künneth decomposition of \( X \) such that \( \text{CH}^j(h^i(X), \mathbb{Q}) = 0 \) for \( i < j \) or \( i > 2j \). Then there is a \( j \)-step filtration on \( \text{CH}^j(X, \mathbb{Q}) \) defined by

\[
F^\nu \text{CH}^j(X, \mathbb{Q}) := \text{Ker}\{\pi_{2j-\nu+1}|F^{\nu-1} \text{CH}^j(X, \mathbb{Q})}\}
\]

\[
= \bigoplus_{i=j}^{2j-\nu} \text{CH}^j(h^i(X), \mathbb{Q}),
\]

for \( 0 \leq \nu \leq j \).

**Conjecture C.** Let \( X \) be a smooth projective variety, and assume that there exists a Chow-Künneth decomposition of \( X \) such that \( \text{CH}^j(h^i(X), \mathbb{Q}) = 0 \) for \( i < j \) or \( i > 2j \). Then the filtration \( F^\nu \text{CH}^j(X, \mathbb{Q}) \) is independent of the choice of Chow-Künneth projectors \( \pi_i(X) \).

1.2.3. The cycle class map. Let

\[
\text{CH}^j_{\text{hom}}(X, \mathbb{Q}) := \text{Ker}(\gamma : \text{CH}^j(X, \mathbb{Q}) \rightarrow H^{2j}_*(X, \mathbb{Q}_*(j))),
\]

where \( \gamma \) is the cycle class map. Then it follows from the commutative diagram

\[
\begin{array}{ccc}
\text{CH}^j(X, \mathbb{Q}) & \xrightarrow{\pi_{2j}(X)} & \text{CH}^j(X, \mathbb{Q}) \\
\gamma \downarrow & & \downarrow \gamma \\
H^{2j}_*(X, \mathbb{Q}_*(j)) & \xrightarrow{\sim} & H^{2j}_*(X, \mathbb{Q}_*(j))
\end{array}
\]

that

\[
F^1 \text{CH}^j(X, \mathbb{Q}) := \text{Ker}(\pi_{2j}(X) | \text{CH}^j(X, \mathbb{Q})) \subseteq \text{CH}^j_{\text{hom}}(X, \mathbb{Q}).
\]

**Conjecture D.** Let \( X \) be a smooth projective variety, and assume that there exists a Chow-Künneth decomposition of \( X \) such that \( \text{CH}^j(h^i(X), \mathbb{Q}) = 0 \) for \( i < j \) or \( i > 2j \). Then

\[
F^1 \text{CH}^j(X, \mathbb{Q}) = \text{CH}^j_{\text{hom}}(X, \mathbb{Q}) \quad \text{for } 1 \leq j \leq \dim(X).
\]
1.2.4. A generalization. Suppose $M = (X,p,m) \in \mathcal{M}(k)$ is a Chow motive, with $X$ equidimensional of dimension $d$. Then one can define a Chow-Künneth decomposition of $M$ as a collection of mutually orthogonal projectors $\pi_i(M) \in \text{End}_{\mathcal{M}(k)}(M) := \text{Corr}^0(M,M)$, with $-2m \leq i \leq 2d - 2m$, such that $\sum_i \pi_i(M) = \text{id}_M = p$ and $\pi_i(M)(H^i_*(M,Q_\bullet)) = H^i_*(M,Q_\bullet)$. It might sometimes be useful, as it is for us below in sections four through six, to decompose a variety as a sum of submotives in some other way than a Chow-Künneth decomposition, and then verify the conjectures on the various submotives, in the sense of the following lemma.

**Lemma 1.2.5.** Suppose $M \simeq M_1 \oplus M_2$ in $\mathcal{M}(k)$.

1. If a Chow-Künneth decomposition of $M_1$ exists and a Chow-Künneth decomposition of $M_2$ exists, then a Chow-Künneth decomposition of $M$ exists.

2. If in addition $\pi_i(M_t)(\text{CH}^j(M_t,Q)) = 0$ whenever $i < j$ or $i > 2j$ for both $t = 1$ and $t = 2$, then with the induced Chow-Künneth decomposition $\pi_i(M)(\text{CH}^j(M,Q)) = 0$ whenever $i < j$ or $i > 2j$.

3. If in addition

$$\text{Ker} (\pi_{2j}(M_t)|\text{CH}^j(M_t,Q)) = \text{Ker} \{\gamma : \text{CH}^j(M_t,Q) \to H^{2j}_*(M_t,Q_\bullet(j))\}$$

for both $t = 1$ and $t = 2$, then with the induced Chow-Künneth decomposition

$$\text{Ker} (\pi_{2j}(M)|\text{CH}^j(M,Q)) = \text{Ker} \{\gamma : \text{CH}^j(M,Q) \to H^{2j}_*(M,Q_\bullet(j))\}.$$

**Proof.** (1) Let $M_1 = (X_1,p_1,m_1)$ and $M_2 = (X_2,p_1,m_2)$, an suppose first for simplicity that $m_1 = m_2 =: m$, say, so that by definition 1.4

$$M \simeq (X_1 \sqcup X_2,p_1 \oplus p_2,m).$$

Then the inclusions $j_1$ and $j_2$ of $X_1$ and $X_2$ respectively into $X_1 \sqcup X_2$ induce orthogonal central idempotents, say $e_1$ and $e_2$, whose sum is the identity in $\text{End}_{\mathcal{M}(k)}(M_1 \oplus M_2)$. Therefore $M_t \cong (X_1 \sqcup X_2,e^*_t(p_1 \oplus p_2),m)$, $t = 1,2$. So if $\text{id}_{M_t} = \sum_i \pi_i(M_t)$ is a Chow-Künneth decomposition for $M_t$, then (up to isomorphism)

$$\text{id}_{(M_1 \oplus M_2)} = \sum_i \left(e_{1*}\pi_i(M_1) + e_{2*}\pi_i(M_2)\right)$$

is a Chow-Künneth decomposition for $M$. In case $m_1 < m_2$, say, then as in 1.4 we have

$$M_1 \simeq M'_1 := (X_1 \times (\mathbb{P}^1)^q,p'_1,m_2),$$

for a suitable choice of $p_1'$ and $q := m_2 - m_1$. Then $M \simeq M'_1 \oplus M_2$, so it suffices to know that the existence of a Chow-Künneth decomposition for $M_1$ implies the existence of a Chow-Künneth decomposition for $M'_1$. However, the isomorphism $M_1 \cong M'_1$ can be used to transform a Chow-Künneth decomposition of $M_1$ into a Chow-Künneth decomposition of $M'_1$ with $\pi_i(M'_1) \simeq \pi_{i-2q}(M_1)$. 
(2) As in (1), first suppose \( m_1 = m_2 =: m \). Then from 1.1.9 we see that
\[
\text{CH}^j(M, \mathbb{Q}) \cong \text{Hom}_{M(k)}(L^j, M_1 \oplus M_2)
\]
\[
\cong \text{Hom}_{M(k)}(L^j, M_1) \oplus \text{Hom}_{M(k)}(L^j, M_1)
\]
\[
= \text{CH}^j(M_1, \mathbb{Q}) \oplus \text{CH}^j(M_2, \mathbb{Q}).
\]
Thus if \( \pi_t(M_t)(\text{CH}^j(M_t, \mathbb{Q})) = 0 \) whenever \( i < j \) or \( i > 2j \) for both \( t = 1 \) and \( t = 2 \), then the same must be true for \( M \) as well. Now if \( m_1 < m_2 \), say, then we need to know that \( \pi_t(M'_t)(\text{CH}^j(M'_t, \mathbb{Q})) = 0 \) whenever \( i < j \) or \( i > 2j \), with \( M'_t \) as above. So consider the diagram
\[
\begin{array}{ccc}
\text{CH}^j(M'_t, \mathbb{Q}) & \longrightarrow & \text{CH}^{j-q}(M_1, \mathbb{Q}) \\
\pi_t(M'_t) & \downarrow & \pi_{i-2q}(M_1) \\
\text{CH}^j(M'_t, \mathbb{Q}) & \longrightarrow & \text{CH}^{j-q}(M_1, \mathbb{Q})
\end{array}
\]
where the equalities follow from 1.1.9. Since \( q > 0 \), if \( i < j \) then \( i - 2q < j - q \) and if \( i > 2j \) then \( i - 2q > 2j - q \), so \( M'_t \) satisfies the hypothesis of (2) whenever \( M_t \) does, as required.

(3) When \( m_1 = m_2 \), then similarly as above we see that the cycle class map \( \gamma : \text{CH}^j(M, \mathbb{Q}) \to H^2_{\text{et}}(M, \mathbb{Q}(j)) \) is the direct sum of the two cycle class maps \( \gamma : \text{CH}^j(M_t, \mathbb{Q}) \to H^2_{\text{et}}(M_t, \mathbb{Q}(j)) \), for \( t = 1, 2 \), and the claim follows directly. And if \( m_1 < m_2 \), then the diagram in (2) above with \( i = 2j \) can be combined with the diagram in 1.2.3 to show that \( M'_t \) satisfies the hypothesis of (3) whenever \( M_t \) does, as required. This completes the proof of the lemma. \( \square \)

2. Elliptic modular surfaces and threefolds

We review the geometric structure of elliptic modular surfaces and threefolds with level-\( N \) structure. To begin, we fix an integer \( N \geq 3 \) once and for all, and a ground field \( K \) in which \( 2N \) is invertible and which contains \( N \)-th roots of unity. When there is no danger of confusion we will drop \( N \) or \( K \) from the notation.

2.1. The elliptic modular curve. Let \( M := M_N \) be the elliptic modular curve over \( K \) that represents the functor which to a \( K \)-scheme \( S \) associates the set of isomorphism classes of elliptic curves \( E/S \) with level-\( N \) structure, where a level-\( N \) structure consists of an isomorphism
\[
\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \times S \xrightarrow{\sim} E[N]/S
\]
of group schemes over \( S \), compare [Deligne and Rapoport, 1973, Ch.IV] or [Katz and Mazur, 1985, Ch.III]. If \( K \) is a subfield of \( \mathbb{C} \), the analytic space \( M^{\text{an}}(\mathbb{C}) \) associated to \( M \) is isomorphic to \( \Gamma(N) \backslash \tilde{H} \), where \( \Gamma(N) \subset \text{SL}_2(\mathbb{Z}) \) is the subgroup of matrices congruent to the identity modulo \( N \). A smooth completion of \( M \)
\[ j : M \leftarrow \overline{M} \]
is obtained by adjoining a finite set of cusps
\[ M^\infty := \overline{M} - M \]
which parameterize generalized elliptic curves.
2.2. The elliptic modular surface. Since $N \geq 3$ there exists a universal elliptic curve with level $N$ structure $\phi : E \to M$. Then the universal generalized elliptic curve with level-$N$ structure $\overline{\phi} : \overline{E} \to \overline{M}$ is the canonical minimal smooth completion of $\phi : E \to M$ [Shioda, 1972], [Deligne and Rapoport, 1973]. Let

$$\overline{\alpha} : \left(\mathbb{Z}/N\mathbb{Z}\right)^2 \times \overline{M} \sim \overline{E}.$$ 

denote the extension of the level-$N$ structure to $\overline{E}$. The Néron model $E^* \to M$ is the canonical minimal smooth completion of $\phi : E \to M$. The following diagram summarizes the notation.

$$\begin{array}{ccc}
E & \longrightarrow & \overline{E} \\
\phi \downarrow & & \downarrow \overline{\phi} \\
M & \longrightarrow & \overline{M}
\end{array} \quad \begin{array}{ccc}
E^* & \longrightarrow & \overline{E} \\
\phi^* \downarrow & & \downarrow \overline{\phi} \\
M^* & \longrightarrow & \overline{M}
\end{array}$$

2.2.1. Description of $E^\infty$. For $c \in M^\infty$, the fibres $E_c := \phi^{-1}(c) \simeq \mathbb{Z}/N\mathbb{Z} \times \mathbb{P}^1$ are standard Néron $N$-gons, where we can number the components by letting $\theta_c(m) \simeq \mathbb{P}^1$ be the component containing $\overline{\alpha}((m,n),c)$, for $(m,n) \in \left(\mathbb{Z}/N\mathbb{Z}\right)^2$. Note that for fixed $m$, as $n$ varies the $\overline{\alpha}((m,n),c)$ all lie in the same component, and may be identified with $N$th roots of unity when $\theta_c(m)$ minus its intersections with $\theta_c(m-1)$ and $\theta_c(m+1)$ is identified with $\mathbb{G}_m$. Sometimes we refer to $\theta_c(0)$ as the identity component. In this notation the intersection relations among the components of $E^\infty$ are

$$\left(\theta_c(m) \cdot \theta_{c'}(m')\right) = \begin{cases} 
-2 & \text{if } c = c' \text{ and } m = m' \\
1 & \text{if } c = c' \text{ and } m - m' = \pm 1 \\
0 & \text{otherwise}
\end{cases}$$

[Kodaira, 1963, III], [Shioda, 1972], [Ash et al., 1975, I.4]. In particular, the rank of intersection matrix for the components of the fibre over a cusp is $(N - 1)$.

Remark 2.2.2. It follows from [Shioda, 1972, Thm.1.1] that a basis for $\text{NS}(\overline{E}) \otimes \mathbb{Q}$ is given by the zero-section $\overline{e} := \overline{\alpha}((0,0),\overline{M})$, a regular fibre, and the components of the cusp fibres other than the identity component.

2.3. The elliptic modular threefold. Consider the fibre products

$$\begin{align*}
^2\phi : \quad &^2E := E \times_M E \longrightarrow M \\
^2\phi^* : \quad &^2E^* := E^* \times_{\overline{M}} E^* \longrightarrow \overline{M} \\
\overline{^2\phi} : \quad &\overline{^2E} := \overline{E} \times_{\overline{M}} \overline{E} \longrightarrow \overline{M} \\
\overline{^2E^\infty} := \quad &E^\infty \times_M E^\infty \longrightarrow M^\infty.
\end{align*}$$
Then $\mathcal{E}$ is not smooth: Using the local coordinates of [Deligne, 1969, Lemme 5.5] or [Scholl, 1990, §2], compare also [Schoen, 1986], one can check that the points over $c \in M^\infty$ that are a product of two double points of $\mathcal{E}_c$ are rational double points in $\mathcal{E}$. If we let $\mathcal{E}_0^\infty = \mathcal{E}_c^\text{sing} \subset \mathcal{E}_0^\infty$ denote the reduced subscheme of $\mathcal{E}$ consisting of all these points, for all $c \in M^\infty$, then applying [Deligne, 1969, Lemmes 5.4, 5.5] or [Scholl, 1990, Prop.2.1.1, Thm.3.1.0(i)] gives us the following description of the desingularization $\tilde{\mathcal{E}}$ of $\mathcal{E}$.

**Proposition 2.3.1.** Let

$$\beta : \tilde{\mathcal{E}} \rightarrow \mathcal{E}$$

be the blowing-up of $\mathcal{E}$ along $\mathcal{E}_0^\infty$. Then $\tilde{\mathcal{E}}$ is nonsingular. Further, let

$$\tilde{\mathcal{E}}^\infty := (\mathcal{E}_0^\infty \circ \beta)^{-1}(M^\infty)$$

be the union of the resulting fibres over $M^\infty$. Then $\tilde{\mathcal{E}}^\infty$ consists of $2N^2 \cdot \#(M^\infty)$ components, half of which are quadric surfaces (isomorphic to $V(xy - zw) \subset \mathbb{P}^3$) that are the components of the exceptional divisor, and half of which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with four (smooth) points blown up, these being the proper transforms with respect to $\beta$ of the components of $\mathcal{E}_0^\infty$. In particular, all the components of $\tilde{\mathcal{E}}^\infty$ are rational surfaces.

**Remark 2.3.2.** In fact $\#(M^\infty) = \frac{1}{2}N^2 \prod_{p | N}(1 - p^{-2})$ [Miyake, 1989], though this will play no explicit role for us.

**2.3.3. Notation.** As a matter of notation, let

$$2\tilde{\phi} := \mathcal{E}_0^\infty \circ \beta : \tilde{\mathcal{E}} \rightarrow \mathcal{M}$$

be the fibre structure map. The following diagram then summarizes the rest of the notation.

\[
\begin{array}{cccccc}
2E & \hookrightarrow & 2E^* & \hookrightarrow & \tilde{\mathcal{E}} & \hookrightarrow & \tilde{\mathcal{E}}^\infty \\
\| & & \| & & \| & & \| \\
2E & \hookrightarrow & 2E^* & \hookrightarrow & \mathcal{E} & \hookrightarrow & \mathcal{E}_0^\infty \\
2E & \downarrow 2\phi & & \downarrow \mathcal{E}_0^\infty & \downarrow \mathcal{E}_0^\infty & & \\
M & \downarrow j & \mathcal{M} & \downarrow \mathcal{M} & \mathcal{M} & \downarrow M^\infty
\end{array}
\]

**2.3.4. Indexing the components of $\tilde{\mathcal{E}}^\infty$.** For use later (in 3.3.11 and 4.4.1) we also index the components $\Theta_c$ of the cusp fibres $\tilde{\mathcal{E}}_c$, for $c \in M^\infty$. According to Proposition 2.3.1, half the components are the proper transforms of the components $\theta_c(m) \times_{\{c\}} \theta_c(n)$ of $\mathcal{E}_c$, so these $\Theta_c(m, n)$ are naturally indexed by pairs $(m, n) \in (\mathbb{Z}/N\mathbb{Z})^2$. The remaining components come from blowing up points which can be described as the (fibre) product (over $c$ in $M^\infty$) of the point where $\theta_c(m)$ intersects $\theta_c(m+1)$ with the point where $\theta_c(n)$ intersects $\theta_c(n+1)$, as $m$ and $n$ run over $\mathbb{Z}/N\mathbb{Z}$. Then the correct incidence relations and symmetries are best described if we call the blowing-up of this point $\Theta_c(m + \frac{1}{2}, n + \frac{1}{2})$, indexed by a pair of half-integers mod $N\mathbb{Z}$; compare [Deligne and Rapoport, 1973, §VII.1].
3. Construction of projectors for $\overline{E}$ and $\tilde{2E}$

3.1. Introduction to the construction. When $S$ is a smooth, connected, quasi-projective base scheme and $A \to S$ is an abelian scheme, then there exist canonical, mutually orthogonal relative projectors $\pi^\text{can}_i(A/S)$ in $\text{CH}^{d_2(A)}(A \times_S A, \mathbb{Q})$ whose sum is the diagonal [Shermenev, 1974], [Deninger and Murre, 1991], [K"unneman, 1994]. These are characterized by the property that

$$[\Gamma_{\mu(n)}] \circ \pi^\text{can}_i(A/S) = \pi^\text{can}_i(A/S) \circ [\Gamma_{\mu(n)}] = n^i \pi^\text{can}_i(A/S),$$

where $\mu(n) : A \to A$ is the multiplication by $n$ endomorphism of $A/S$. In particular, when $S$ is a point, these $\pi^\text{can}_i(A)$ define a Chow-K"unneth decomposition of $A$.

In our situation, even though $\overline{E} \to M$ and $\tilde{2E} \to \tilde{M}$ are not abelian schemes, one of the underlying ideas for the projectors we define in this section is to extend in a suitable sense the canonical relative projectors for $E/M$ and $2E/M$ to projectors for $\overline{E}$ and $\tilde{2E}$, respectively. The idea is that $E \times_M E$ naturally embeds in $\overline{E} \times \overline{E}$, and this embedding factors through the natural embedding of $\overline{E} \times_M \overline{E}$ into $\overline{E} \times \overline{E}$; and likewise the natural embedding of $2E \times_M 2E$ into $\tilde{2E} \times \tilde{2E}$ factors through $\tilde{2E} \times_M \tilde{2E}$. Then what we would like to do is “push forward” $\pi^\text{can}_i(E/M)$ from $\text{CH}^1(E \times_M E, \mathbb{Q})$ to $\text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})$, and similarly “push forward” $\pi^\text{can}_i(2E/M)$ from $\text{CH}^2(2E \times_M 2E, \mathbb{Q})$ to $\text{CH}^3(\tilde{2E} \times \tilde{2E}, \mathbb{Q})$. The trouble is, there is no natural push forward for this situation, and the only alternative seems to be to choose an explicit cycle to represent the rational equivalence class $\pi^\text{can}_i(E/M)$, then take its closure in $\overline{E} \times_M \overline{E}$, and then push that forward to a cycle on $\overline{E} \times \overline{E}$; and likewise for $\tilde{2E} \times \tilde{2E}$. Conceptually this is what we do, but as a matter of logical presentation it seems preferable to begin by describing explicit cycles supported on $\overline{E} \times_M \overline{E}$, and then show that they have nice properties as mutually orthogonal projectors in $\text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})$. The point is that for technical reasons these cycles on $\overline{E} \times_M \overline{E}$ are not simply the closures of the obvious “natural” representatives for $\pi^\text{can}_i(E/M)$, as described in [K"unnemann, 1994] for example. So it requires some work to show that their restrictions to $E \times_M E$ do indeed represent the canonical relative projectors for $E/M$, see Proposition 3.2.8.

Then for $2E$ we use the previously-defined cycles on $\overline{E} \times_M \overline{E}$ in the definition of explicit cycles supported on $\tilde{2E} \times_M \tilde{2E}$ that behave nicely as mutually orthogonal projectors in $\text{CH}^2(\tilde{2E} \times \tilde{2E}, \mathbb{Q})$. One of the ideas underlying this construction is to take advantage of the relative product structure of $2E/M$, for in this way we get nine projectors corresponding (in the sense of 3.3.7–3.3.9 below) to the $\pi^\text{can}_i(E/M) \otimes_M \pi^\text{can}_j(E/M)$, for $0 \leq i, j \leq 2$, in $\text{CH}^2(2E \times_M 2E, \mathbb{Q})$, rather than just the five that correspond to the $\pi^\text{can}_i(2E/M)$, for $0 \leq i \leq 4$.

3.2. Extending canonical relative projectors to $\overline{E}$.

3.2.1. The zero-section and its transpose. Let $e := \alpha((0, 0), M)$ be the zero-
section as curve in $E$. Then
\[
[E \times_M e] = \pi_2^{can}(E/M), \quad \text{in } CH^1(E \times_M E, Q),
\]
\[
[e \times_M E] = \pi_0^{can}(E/M) = \pi_2^{can}(E/M)
\]
[in CH$^1(E \times_M E, Q)$]

Künemann, 1994, 4.1.2(iv). Now let $\bar{e} := \bar{\alpha}((0,0), \mathcal{M})$ be the zero-section as curve in $\mathcal{E}$. Then
\[
\bar{p}_2 := [\bar{e} \times_M \mathcal{E}], \quad \bar{p}_0 := [e \times_M \mathcal{E}] = t \bar{p}_2
\]
in CH$^2(\mathcal{E}, Q)$ are projectors, but unexpectedly, they are not orthogonal, as the next lemma explains. In order to formulate this lemma precisely, and also for later purposes (see 3.2.7), we consider the inclusions
\[
\psi : E \times_M E \xrightarrow{\psi_1} \mathcal{E} \times_M \mathcal{E} \xrightarrow{\psi_2} \mathcal{E} \times \mathcal{E}.
\]

**Lemma 3.2.2.** In CH$^2(\mathcal{E} \times \mathcal{E}, Q)$

1. $\bar{p}_0 \circ \bar{p}_0 = \bar{p}_0$ and $\bar{p}_2 \circ \bar{p}_2 = \bar{p}_2$;
2. $\bar{p}_2 \circ \bar{p}_0 = 0$;
3. $\bar{p}_0 \circ \bar{p}_2 = (\psi_2)_* (\bar{\phi} \times_M \bar{\phi}) \bar{\phi}_* [\bar{e} \cdot \bar{e}] \neq 0$, where $[\bar{e} \cdot \bar{e}]$ denotes the self-intersection cycle in CH$^2(\mathcal{E}, Q)$.

**Proof.** Let $\bar{\mu}(0) := \bar{\alpha}((0,0), \bar{\phi}(\bullet)) : \mathcal{E} \to \mathcal{E}$ be the morphism given by projection onto the zero-section. (The notation is meant to suggest “multiplication by zero,” extending to $\mathcal{E}$ of the fibre-wise group homomorphism that maps everything to the identity element.) Then $\bar{p}_2$ and $\bar{p}_0$ correspond to the graph and transposed graph of $\bar{\mu}(0)$, respectively,
\[
\bar{p}_2 = [\Gamma_{\bar{\mu}(0)}], \quad \bar{p}_0 = [\Gamma_{\bar{\mu}(0)}].
\]

Then (1) follows because $\bar{\mu}(0) \circ \bar{\mu}(0) = \bar{\mu}(0)$, and (2) because by 1.1.7
\[
[\Gamma_{\bar{\mu}(0)}] \circ [\Gamma_{\bar{\mu}(0)}] = (\bar{\mu}(0) \times \bar{\mu}(0))_* ([\Delta(\mathcal{E})]),
\]
which vanishes in CH$^2(\mathcal{E} \times \mathcal{E}, Q)$ for dimension reasons. As for (3), we verify this by direct computation. In order to have proper intersection for this computation, we move the graph $[\Gamma_{\bar{\mu}(0)}]$ on $\mathcal{E} \times \mathcal{E}$ by first moving the divisor $\bar{e}$ in its linear equivalence class on $\mathcal{E}$ to a divisor $\bar{e}'$ intersecting $\bar{e}$ properly on $\mathcal{E}$ (and moreover, for simplicity, also such that over a cusp $\bar{e}'$ passes through neither the crossing points of the components of that fibre nor through the intersection of $\bar{e}$ with the fibre). Also note that the cycle class we finally get is the class of a cycle supported on the singular variety $\mathcal{E} \times_M \mathcal{E}$ and therefore we have to go via CH$^1(\mathcal{E} \times_M \mathcal{E}, Q)$ (in the sense of [Fulton, 1988]). The nonvanishing is a consequence of the fact that the self-intersection number $(\bar{e} \cdot \bar{e}) = -(p_a + 1) < 0$ [Kodaira, 1963, p.15], [Shioda, 1972, p.25].
3.2.3. Definition of $\pi_0(E/\overline{M})$ and $\pi_2(E/\overline{M})$. If we now let, in $\text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})$,

$$
\pi_0(E/\overline{M}) := \overline{p}_0 - \frac{1}{\overline{\phi}} \circ \overline{p}_2 = \tau_{\mu(0)} - \frac{1}{\overline{\phi}} \circ \tau_{\mu(0)}
$$

$$
\pi_2(E/\overline{M}) := \overline{p}_2 - \frac{1}{\overline{\phi}} \circ \overline{p}_0 = \tau_{\mu(0)} - \frac{1}{\overline{\phi}} \circ \tau_{\mu(0)} = \pi_0(E/\overline{M})
$$

then it follows from Lemma 3.2.2 that these are orthogonal projectors. For use below we also choose a zero cycle $a$ on $\overline{M}$ representing $\overline{\phi} \cdot [\bar{e} \cdot \bar{e}]$, i.e.,

$$
[a] = \overline{\phi}_* [\bar{e} \cdot \bar{e}] \in \text{CH}^1(\overline{M}, \mathbb{Q}),
$$

and observe that by doing so we get representative cycles for $\pi_0(E/\overline{M})$ and $\pi_2(E/\overline{M})$ supported on $\overline{E} \times \overline{\pi} \overline{E}$. Also for later reference note that the “correction term”

$$
\frac{1}{\overline{\phi}} \circ \overline{p}_2 = \frac{1}{\overline{\phi}} \circ \tau_{\mu(0)} = \frac{1}{\overline{\phi}} \circ \tau_{\mu(0)}
$$

is nilpotent of order 2 in $\text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})$.

3.2.4. Automorphism correspondences on $\overline{E}$. Following [Scholl, 1990] we consider a group of automorphisms acting on $\overline{E}$. Firstly, for $b \in (\mathbb{Z}/\mathbb{N})^2$ translation by $\alpha(b, z)$ in each fibre $\phi^{-1}(z)$ defines an automorphism $\tau(b) : E \to E$ of $E$. Since this depends only on the group structure of $E/\overline{M}$, it extends first to an automorphism $\tau^*(b)$ of $E^*$, and then by Zariski’s Main Theorem [Hartshorne, 1977, V.5.2, p.410], since the invertibility of $\tau^*(b)$ away from the isolated points of $\overline{E} - E^*$ precludes the total transform of any of these points in the closure of the graph of $\tau^*(b)$ having dimension one or more, to an automorphism $\bar{\tau}(b) : \overline{E} \to \overline{E}$. In this way we get a group action of $(\mathbb{Z}/\mathbb{N})^2$ on $\overline{E}$. By the same reasoning, the fibrewise inversion map is an automorphism of $E$ that extends first to an automorphism of $E^*$ and then to an automorphism $\bar{\mu}(1) : \overline{E} \to \overline{E}$ of $\overline{E}$, and together with the identity map this gives a group action of $\mu_2$ on $\overline{E}$. These two group actions together give a group action of the semidirect product

$$
\mathcal{G} := (\mathbb{Z}/\mathbb{N})^2 \rtimes \mu_2
$$

on $\overline{E}$, which can be extended $\mathbb{Q}$-linearly to define an action of the group ring $\mathbb{Q}[\mathcal{G}]$ on $\overline{E}$. In particular, by associating to a group element $g \in \mathcal{G}$ the class of its graph $[\Gamma_g]$ (respectively, transposed graph $[\Gamma^g]$) in $\text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})$, we get a $\mathbb{Q}$-algebra homomorphism

$$
\mathbb{Q}[\mathcal{G}] \to \text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})
$$

(respectively, antihomomorphism $\mathbb{Q}[\mathcal{G}]^{\text{opp}} \to \text{CH}^2(\overline{E} \times \overline{E}, \mathbb{Q})$) from the group ring of $\mathcal{G}$ into the ring of degree-zero correspondences on $\overline{E}$. Further, since the group actions operate fibrewise, these correspondences are supported on $\overline{E} \times \overline{\pi} \overline{E}$. We remark also that for automorphisms of $\overline{E}$ such as those defined by the action of $g \in \mathcal{G}$,

$$
[\Gamma_g] = [\Gamma^{-1}_g].
$$
3.2.5. Definition of $\pi_1(E/M)$. We take for $\pi_1(E/M)$ the projector $\Pi_\epsilon$ defined in [Scholl, 1990, 1.1.2] for $k = 1$, which may be described as follows: Let $\epsilon = \epsilon_1$ be the character of $\mathfrak{G}$ defined by the product of the trivial character on $(\mathbb{Z}/N\mathbb{Z})^2$ and the sign character on $\mu_2$; then one description of $\pi_1(E/M)$ is

$$\pi_1(E/M) := \pi_\epsilon(E/M) = \frac{1}{2N^2} \sum_{g \in \mathfrak{G}} \epsilon(g)^{-1} [\Gamma_g].$$

As the homomorphic image of an idempotent in $\mathbb{Q}[\mathfrak{G}]$, it follows that $\pi_1(E/M)$ is a projector in $\text{CH}^2(E \times E, \mathbb{Q})$, and it is also clear that $\pi_1(E/M) = \pi_1(E/M)$.

Another description of $\pi_1(E/M)$ comes from observing that

$$\lambda := \frac{1}{2} ([\Gamma_{\mu(1)}] - [\Gamma_{\mu(-1)}]), \quad \vartheta := \frac{1}{N^2} \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^2} [\Gamma_{\tau(b)}]$$

are homomorphic images of commuting idempotents in $\mathbb{Q}[\mathfrak{G}]$, and then

$$\pi_1(E/M) = \lambda \circ \vartheta = \vartheta \circ \lambda.$$

Proposition 3.2.6. The $\pi_i(E/M)$, for $i = 0, 1, 2$, are mutually orthogonal projectors in $\text{CH}^2(E \times E, \mathbb{Q})$.

Proof. We have already seen the idempotency of each $\pi_i(E/M)$, and the orthogonality of $\pi_0(E/M)$ and $\pi_2(E/M)$, so it only remains to check that $\pi_1(E/M)$ is orthogonal to the other two. To see this, we can use 3.2.3 that

$$\pi_2(E/M) := [\Gamma_{\mu(0)}] - \frac{1}{2} [\Gamma_{\mu(0)}] \circ [\Gamma_{\mu(0)}] = \pi_0(E/M).$$

Then from the observation that

$$\bar{\mu}(0) \circ \bar{\mu}(\pm 1) = \bar{\mu}(\pm 1) \circ \bar{\mu}(0) = \bar{\mu}(0)$$

and the formulas 1.1.7 it follows immediately that $\lambda$ is orthogonal to both $\pi_2(E/M)$ and $\pi_0(E/M)$, and thus $\pi_1(E/M)$ is as well. □

3.2.7. Notations and definitions related to cycles on $E \times_M \overline{E}$. Suppose $\alpha : \overline{E} \to \overline{E}$ is a morphism such that $\alpha$ respects the fibre structure of $\overline{E} \to \overline{M}$. Then the graph $\Gamma_{\alpha}$ of $\alpha$ is supported on $E \times_M \overline{E}$. In order to emphasize this we may write $\Gamma_{\alpha}^{\text{rel}}$ for the graph of $\alpha$ as a cycle on $E \times_M \overline{E}$, and $[\Gamma_{\alpha}]^{\text{rel}} := [\Gamma_{\alpha}^{\text{rel}}]$ for its class in $\text{CH}_2(E \times_M \overline{E}, \mathbb{Q})$ (the Chow group in the sense of [Fulton, 1984], since $E \times_M \overline{E}$ is singular). Now consider again the inclusions

$$\psi : E \times_M E \xleftarrow{\psi_1} E \times_M \overline{E} \xrightarrow{\psi_2} E \times \overline{E}.\,$$
Then $[\Gamma_\alpha] = (\psi_2)_*([\Gamma_\alpha]^{rel})$. By abuse of notation define

$$\psi^#([\Gamma_\alpha]) := \psi^*_1([\Gamma_\alpha]^{rel}) \quad \text{in } CH^1(E \times_M E, \mathbb{Q}).$$

Then $\psi^#([\Gamma_\alpha])$ is just the class in $CH^1(E \times_M E, \mathbb{Q})$ of the graph of the restriction of $\alpha$ to $E$. Therefore if the morphism $\beta : \overline{E} \to \overline{E}$ also respects the fibre structure of $\overline{E} \to \overline{M}$ then we have

$$\psi^#([\Gamma_\alpha \circ \beta]) = \psi^#([\Gamma_\alpha]) \circ \psi^#([\Gamma_\beta]),$$

and, if we allow the same notations and definitions for the transpose of a graph, also

$$\psi^#([^t\Gamma_\alpha \circ \beta]) = \psi^#([^t\Gamma_\beta]) \circ \psi^#([^t\Gamma_\alpha]).$$

Now if we extend these notations and definitions by linearity and apply them to the cycles and projectors defined in 3.2.5, then we have

$$\bar{\pi}^{rel}_1(E/M) = \frac{1}{2N^2} \sum_{g \in \mathfrak{G}} \varepsilon(g)^{-1} [\Gamma_g]^{rel}$$

in $CH_2(\overline{E} \times_{\overline{M}} \overline{E}, \mathbb{Q})$, and $\bar{\lambda}^{rel}$ and $\bar{\vartheta}^{rel}$ may be defined similarly. We may also apply these notations and definitions to the cycles and projectors in 3.2.3, and let

$$\bar{\pi}^{rel}_0(E/M) := [^t\Gamma_{\mu(0)}]^{rel} - \frac{1}{2}([\bar{\phi} \times_{\overline{M}} \bar{\phi})^*(\alpha)]$$

$$\bar{\pi}^{rel}_2(E/M) := [\Gamma_{\mu(0)}]^{rel} - \frac{1}{2}([\bar{\phi} \times_{\overline{M}} \bar{\phi})^*(\alpha)].$$

Then for $i = 0, 1, 2$,

$$\bar{\pi}_i(E/M) = (\psi_2)_* \bar{\pi}^{rel}_i(E/M),$$

and thus we have elements

$$\psi^#(\bar{\pi}_1(E/M)) := \psi^*_1 \bar{\pi}^{rel}_1(E/M) \in CH^1(E \times_M E, \mathbb{Q}),$$

and we get $\psi^#(\bar{\lambda})$ and $\psi^#(\bar{\vartheta})$ similarly.

**Proposition 3.2.8.** With the notation as above, in $CH^1(E \times_M E, \mathbb{Q})$

$$\psi^#(\bar{\pi}_i(E/M)) = \pi^c_{i*}(E/M)$$

for $0 \leq i \leq 2$.

**Proof.** Consider $i = 1$ first. Then it follows immediately from the considerations in 3.2.7 that

$$\psi^#(\bar{\pi}_1(E/M)) = \psi^#(\bar{\lambda}) \circ \psi^#(\bar{\vartheta}) = \psi^#(\bar{\vartheta}) \circ \psi^#(\bar{\lambda}).$$
Then in $\text{CH}^1(E \times_M E, \mathbb{Q})$ we have
\[
\psi_1^*(\vartheta^{\text{rel}}) \circ [\Gamma_\mu(N)] = \frac{1}{N^2} \sum_{b \in \mathbb{Z}/N\mathbb{Z}^2} [\Gamma_\mu(N)] \circ [\Gamma_\tau(b)]
\]
\[
= \frac{1}{N^2} \sum_{b \in \mathbb{Z}/N\mathbb{Z}^2} [\Gamma_\mu(N) \circ \tau(b)]
\]
\[
= [\Gamma_\mu(N)].
\]

Now apply $\psi^#\pi_1(\overline{E}/M)$ to the relation
\[
[\Delta(E/M)] = \pi_0^{\text{can}}(E/M) + \pi_1^{\text{can}}(E/M) + \pi_2^{\text{can}}(E/M).
\]

Then from the characterizing property 3.1(1) of the $\pi_i^{\text{can}}$ we get that
\[
[\Gamma_\mu(-1)] \circ \pi_i^{\text{can}}(E/M) = \pi_i^{\text{can}}(E/M) \circ [\Gamma_\mu(-1)] = (-1)^i \pi_i^{\text{can}}(E/M).
\]

It follows that $\psi_1^*(\overline{X}^{\text{rel}})$ annihilates $\pi_0^{\text{can}}(E/M)$ and $\pi_2^{\text{can}}(E/M)$, whence the same is true of $\psi^#\pi_1(\overline{E}/M)$, and also that $\psi_1^*(\overline{X}^{\text{rel}}) \circ \pi_1^{\text{can}}(E/M) = \pi_1^{\text{can}}(E/M)$. Hence
\[
\psi^#\pi_1(\overline{E}/M) = \psi_1^*(\vartheta^{\text{rel}}) \circ \pi_1^{\text{can}}(E/M).
\]

Now multiply both sides of this equation by $N$. Then again using 3.1(1) we get
\[
N(\psi^#\pi_1(\overline{E}/M)) = \psi_1^*(\vartheta^{\text{rel}}) \circ [\Gamma_\mu(N)] \circ \pi_1^{\text{can}}(E/M)
\]
\[
= [\Gamma_\mu(N)] \circ \pi_1^{\text{can}}(E/M)
\]
\[
= N\pi_1^{\text{can}}(E/M).
\]

Therefore $\psi^#\pi_1(\overline{E}/M) = \pi_1^{\text{can}}(E/M)$ in $\text{CH}^1(E \times_M E, \mathbb{Q})$ as required.

Now consider $i = 0, 2$. From 3.2.3 and 3.2.7 we have
\[
\psi^#\pi_i(E/M) = \psi^#\pi_i - \frac{1}{2} \psi^#((\overline{\phi} \times \overline{\phi})^*[a]).
\]

As we have already observed (3.2.1) that $\psi^#\pi_i = \pi_i^{\text{can}}(E/M)$ for $i = 0, 2$ [Künemann, 1994, 4.1.2(iv)], what we need to show is that $\psi^#((\overline{\phi} \times \overline{\phi})^*[a]) = 0$.

But from the definition of $\psi^#$ and the commutativity of the diagram
\[
\begin{array}{ccc}
E \times_M E & \xrightarrow{\psi_1} & \overline{E} \times_M \overline{E} \\
\downarrow & & \downarrow \\
M & \xrightarrow{j} & \overline{M}
\end{array}
\]
it follows that
\[ \psi^\#((\phi \times_{\mathcal{M}} \phi)^* [a]) = (\phi \times_{\mathcal{M}} \phi)^* j^* ([a]). \]
Therefore it will suffice to prove that \( j^* ([a]) = 0 \) in \( \text{CH}^1(M, \mathbb{Q}) \), or equivalently, that \([a] \in \text{CH}^1(M, \mathbb{Q})\) can be supported in \( M^\infty \).

To see this, let \( \bar{e}_0 := \bar{e} := \bar{\alpha}(0, 0, \mathcal{M}), \) and let \( \bar{e}_1 := \bar{\alpha}(1, 0, \mathcal{M}), \) and \( \bar{e}_2 := \bar{\alpha}((0, 1, \mathcal{M}). \) Then for distinct \( i, j \in \{0, 1, 2\} \) the intersection cycle \([\bar{e}_i \cdot \bar{e}_j] = 0 \) in \( \text{CH}^2(E, \mathbb{Q}) \), since these sections are distinct in every fibre. Now let \( \eta \) denote the generic point of \( M \). Then \( N\bar{e}_0(\eta) \) and \( N\bar{e}_1(\eta) \) are \( \mathbb{Q}(\eta) \)-rational zero cycles on \( E_\eta \), each summing to \( \bar{e}_0(\eta) \) on \( E_\eta \), whence by Abel’s theorem they are linearly equivalent on \( E_\eta \). More precisely,
\[ N\bar{e}_0(\eta) = N\bar{e}_1(\eta) + \text{div}(f_\eta) \]
for some \( f_\eta \in \mathbb{Q}(\eta) \). But then as cycles
\[ N\bar{e}_0 = N\bar{e}_1 + \phi^* (b) + D + \text{div}(F) \]
for some zero-cycle \( b \) on \( M \) and some divisor \( D \) supported in \( E^\infty \) and some \( F \in \mathbb{Q}(E) \) (corresponding to \( f_\eta \)). If we now intersect both sides of (1) with \( \bar{e}_2 \) and push the resulting cycle down to \( \mathcal{M} \) by \( \phi_* \), then we find that \( b \) is linearly equivalent on \( \mathcal{M} \) to some zero-cycle \( b' \) supported on \( M^\infty \). Therefore we may rewrite (1) as
\[ N\bar{e}_0 \sim_{\text{lin}} N\bar{e}_1 + D' \]
on \( E \), with \( D' \) a divisor supported in \( E^\infty \). Now intersecting (2) with \( \bar{e}_0 = \bar{e} \), it follows that the self-intersection cycle \( N[\bar{e} \cdot \bar{e}] \) can be supported in \( E^\infty \). And since \([a] = \bar{\alpha}([\bar{e} \cdot \bar{e}]) \) in \( \text{CH}^1(M, \mathbb{Q}) \), it follows that \([a] \) can be supported in \( M^\infty \) and \( j^* ([a]) = 0 \) in \( \text{CH}^1(M, \mathbb{Q}) \), which was what we needed to show. \( \square \)

**Remark.** A similar argument can be used to show that \([e \cdot e] = 0 \) in \( \text{CH}^2(E, \mathbb{Q}). \)

### 3.2.9. Definition of \( \pi_\infty(E/M) \).

Let
\[ \pi_f(E/M) := \sum_{i=0}^{2} \pi_i(E/M) \quad \text{in} \quad \text{CH}^2(E \times E, \mathbb{Q}). \]

Then Proposition 3.2.8 implies that \( \psi^\# \pi_f(E/M) = [\Delta(E/M)] \). Let
\[ \pi_\infty(E/M) := [\Delta(E)] - \pi_f(E/M) \quad \text{in} \quad \text{CH}^2(E \times E, \mathbb{Q}). \]

Then it follows from the mutual orthogonality and idempotency of the \( \pi_i(E/M) \), for \( i = 0, 1, 2 \), that \( \pi_f(E/M) \) and \( \pi_\infty(E/M) \) are projectors as well, and \( \pi_\infty(E/M) \) is orthogonal to all the others. In fact we can say more, using Proposition 3.2.8 and the geometry of \( E \).
Lemma 3.2.10 (the structure of \(\pi_\infty(E/\mathcal{M})\)). In \(\text{CH}^2(\overline{E} \times E, \mathbb{Q})\),

\[
\pi_\infty(E/\mathcal{M}) = \sum_{c \in M^\infty} \pi_c(E/\mathcal{M}),
\]

where the \(\pi_c(E/\mathcal{M})\) are mutually orthogonal projectors, orthogonal to the \(\pi_i(E/\mathcal{M})\) for \(0 \leq i \leq 2\), and of the form

\[
\pi_c(E/\mathcal{M}) = \sum_{i,j \in \mathbb{Z}/\mathbb{N}Z} r_c(i,j)[\theta_c(i)] \times_{\{c\}} [\theta_c(j)]
\]

for some rational numbers \(r_c(i,j)\).

Proof. Consider the diagram

\[
\begin{array}{cccc}
\text{CH}_2(E^{\infty} \times_{M^\infty} E^{\infty}, \mathbb{Q}) & \rightarrow & \text{CH}_2(\overline{E} \times \overline{E}, \mathbb{Q}) & \rightarrow \text{CH}_2(E \times_2 E, \mathbb{Q})
\\ & \searrow & \downarrow (\psi_2)_* & \downarrow
ds & \text{CH}_2(\overline{E} \times \overline{E}, \mathbb{Q}) & \rightarrow & \text{CH}_2(E \times E, \mathbb{Q})
\end{array}
\]

whose horizontal rows are exact [Fulton, 1984, 1.8, p.21]. Then in the notation of 3.2.7 \(\pi_f(E/\mathcal{M}) = (\psi_2)_* \pi_{f,2}^\text{rel}(E/\mathcal{M})\), and it follows from Proposition 3.2.8 that the difference \([\Delta(E)]^\text{rel} - \pi_{f,2}^\text{rel}(E/\mathcal{M})\) in \(\text{CH}_2(\overline{E} \times \overline{E}, \mathbb{Q})\) maps to zero in \(\text{CH}_2(E \times_2 E, \mathbb{Q})\). Therefore \(\pi_\infty(E/\mathcal{M}) \in \text{CH}_2(\overline{E} \times \overline{E}, \mathbb{Q})\) must be in the image of \(\text{CH}_2(E^{\infty} \times_{M^\infty} E^{\infty}, \mathbb{Q})\). Thus, since \(E^{\infty} \times_{M^\infty} E^{\infty}\) is 2-dimensional, with components of the form \(\theta_c(i) \times_{\{c\}} \theta_c(j)\), we get that \(\pi_\infty(E/\mathcal{M})\) can be written in the form indicated. On the other hand, the disjointness of the fibres \(\overline{E}_c = E^{\infty}_c\) implies that the distinct \(\pi_c(E/\mathcal{M})\), for \(c \in M^\infty\), are mutually orthogonal, and idempotent; and hence as constituents of \(\pi_\infty(E/\mathcal{M})\), orthogonal also to \(\pi_i(E/\mathcal{M})\) for \(0 \leq i \leq 2\). This proves the lemma.

\[\square\]

3.3. Extending canonical relative projectors to \(\tilde{\overline{E}}\).

3.3.1. Introduction to the method. The basic idea of the projectors we now define for \(\tilde{\overline{E}}\) is that we would like them to be the tensor products over \(\overline{\mathcal{M}}\), in the sense of 1.1.3, of the projectors \(\pi_i(E/\mathcal{M})\) defined above for \(\overline{E}\). But since neither \(\overline{E}\) nor \(\tilde{\overline{E}}\) is smooth over \(\overline{\mathcal{M}}\), and further since after the blowing-up \(\beta : \tilde{\overline{E}} \rightarrow \overline{E}\) \(\tilde{\overline{E}}\) is no longer a product over \(\overline{\mathcal{M}}\), the definition 1.1.3 of the tensor product of correspondences does not directly apply to our situation. For this reason we shall define projectors for \(\tilde{\overline{E}}\) directly as combinations of graphs and transposes of graphs of morphism, as we did for \(\pi_i(E/\mathcal{M})\). In particular this means that again we start with explicit representative cycles.

Firstly we write down correspondences in \(\tilde{\overline{E}} \times \tilde{\overline{E}}\) but supported on \(\tilde{\overline{E}} \times \overline{\mathcal{M}}\tilde{\overline{E}}\) that act on \(\tilde{\overline{E}}\) like \(\pi_i(E/\mathcal{M})\) on one fibre factor and identity on the other (in spite of the fact that the construction of \(\tilde{\overline{E}}\) by desingularizing \(\overline{\mathcal{M}}\) destroyed the fibre
product structure!), for \(0 \leq i \leq 2\). Then in order to get the mutually orthogonal projectors we actually want from these, we have to show that the correspondence that acts as \(\pi_i(E/M)\) on the first factor and identity on the second commutes with the correspondence that acts as \(\pi_i(E/M)\) on the second factor and identity on the first. Once that is done, we check that the restrictions of these projectors to \(2E \times_M 2E\), in a similar sense as 3.2.7 and 3.2.8, see 3.3.7 to 3.3.9 below, are indeed tensor products of the canonical relative projectors. Finally, similarly as for \(E\) we define \(\tilde{\pi}_\infty(2\tilde{E}/\tilde{M})\) for \(2\tilde{E}\).

3.3.2. Definition of \(\tilde{\pi}_0^{(j)}(2\tilde{E}/\tilde{M})\) and \(\tilde{\pi}_2^{(j)}(2\tilde{E}/\tilde{M})\). Recall from 3.2.2(4) that we wrote \(\mathfrak{p}_2 = [\Gamma_{\mu(0)}]\) and \(\mathfrak{p}_0 = [\Gamma_{\bar{\mu}(0)}]\), where \(\mu(0) := \bar{\alpha}(0, 0, \bullet) \circ \bar{\phi} : E \to \overline{E}\) is the projection onto the zero-section morphism. Consider now \(\mu(0) \times_M \text{id}_E : 2E \to 2\overline{E}\).

Since the image of this map is disjoint from the center \(2\mathfrak{T}_0^\infty\) of the blowing up \(\beta : 2\tilde{E} \to 2\overline{E}\), by factoring it through \(\beta\) it lifts to a morphism of \(2\tilde{E}\) that respects the fibre structure of \(2\tilde{E} \to \overline{M}\). More precisely, let

\[
\tilde{\mu}(0, 1) := (\beta')^{-1} \circ (\mu(0) \times_M \text{id}_E) \circ \beta : 2\tilde{E} \longrightarrow 2\overline{E}
\]

where \(\beta'\) is the restriction of \(\beta\) to \(2\tilde{E} - \beta^{-1}(2\mathfrak{T}_0^\infty)\), where it is an isomorphism. If we define \(\tilde{\mu}(0, 0)\) similarly, then the product in either order

\[
\tilde{\mu}(0, 0) := \tilde{\mu}(0, 1) \circ \tilde{\mu}(1, 0) = \tilde{\mu}(1, 0) \circ \tilde{\mu}(0, 1) = \tilde{\alpha}(0, 2\tilde{\phi}(\bullet)) : 2\tilde{E} \longrightarrow 2\overline{E}
\]

is the projection onto the zero-section of \(2\tilde{E}\), where \(\tilde{\alpha} : (\mathbb{Z}/N\mathbb{Z})^4 \times \overline{M} \to 2\tilde{E}\) is the level-\(N\) structure.

Now let

\[
\tilde{\pi}_0^{(1)}(2\tilde{E}/\tilde{M}) := [\Gamma_{\tilde{\mu}(0, 1)}] - \frac{1}{2}[\Gamma_{\tilde{\mu}(0, 1)}] \circ [\Gamma_{\tilde{\mu}(0, 1)}]
\]

\[
\tilde{\pi}_0^{(2)}(2\tilde{E}/\tilde{M}) := [\Gamma_{\tilde{\mu}(1, 0)}] - \frac{1}{2}[\Gamma_{\tilde{\mu}(1, 0)}] \circ [\Gamma_{\tilde{\mu}(1, 0)}]
\]

\[
\tilde{\pi}_2^{(1)}(2\tilde{E}/\tilde{M}) := [\Gamma_{\tilde{\mu}(0, 1)}] - \frac{1}{2}[\Gamma_{\tilde{\mu}(0, 1)}] \circ [\Gamma_{\tilde{\mu}(0, 1)}]
\]

\[
\tilde{\pi}_2^{(2)}(2\tilde{E}/\tilde{M}) := [\Gamma_{\tilde{\mu}(1, 0)}] - \frac{1}{2}[\Gamma_{\tilde{\mu}(1, 0)}] \circ [\Gamma_{\tilde{\mu}(1, 0)}],
\]

where the notation is chosen to suggest that \(\tilde{\pi}_i^{(j)}(2\tilde{E}/\tilde{M})\) acts like \(\pi_i(E/M)\) on the \(j\)th fibre factor and identity on the other. Then the idempotency of each of these, and the orthogonality of \(\tilde{\pi}_0^{(j)}(2\tilde{E}/\tilde{M})\) and \(\tilde{\pi}_2^{(j)}(2\tilde{E}/\tilde{M})\) for fixed \(j\), follows easily from observing that

\[
[\Gamma_{\tilde{\mu}(0, 1)}] \circ [\Gamma_{\tilde{\mu}(0, 1)}] = (\tilde{\mu}(0, 1) \times \tilde{\mu}(0, 1))_* ([\Delta(2\tilde{E})]) = 0
\]

\[
[\Gamma_{\tilde{\mu}(1, 0)}] \circ [\Gamma_{\tilde{\mu}(1, 0)}] = (\tilde{\mu}(1, 0) \times \tilde{\mu}(1, 0))_* ([\Delta(2\tilde{E})]) = 0,
\]

in \(\text{CH}^3(2\tilde{E} \times 2\overline{E}, \mathbb{Q})\).
Similarly as in 3.2.3 now let $a^{(1)}$ and $a^{(2)}$ be two disjoint zero cycles on $\overline{M}$ that both also represent $[a] = \overline{\phi}_a([e \cdot e])$ in $\text{CH}^1(\overline{M}, \mathbb{Q})$. Then the correction term $\frac{1}{2}[\Gamma_{\tilde{\mu}(0,1)}] \otimes [\Gamma_{\tilde{\mu}(1,0)}]$ can be represented by a 3-dimensional cycle $b^{(1)}$ supported on $(2\overline{\phi} \times \overline{\phi})^{-1}([a^{(1)}])$, where $|c|$ denotes the support of a zero cycle $c$ on $\overline{M}$, and similarly there is a cycle $b^{(2)}$ representing $\frac{1}{2}[\Gamma_{\tilde{\mu}(1,0)}] \otimes [\Gamma_{\tilde{\mu}(0,1)}]$ and supported on $(2\overline{\phi} \times \overline{\phi})^{-1}([a^{(2)}])$. This can be seen by a direct computation: In order to have a proper intersection on $\overline{2E} \times 2\overline{E} \times 2\overline{E}$ we can move (similarly as in the proof of 3.2.2) inside the second factor by looking first at the $\overline{E}$ over which it lies and there moving the zero section $e_0$ in the relevant factor $\overline{E}$ to a cycle $e'_0$ such that $e_0$ and $e'_0$ intersect properly and have no common points over the cusps and such that $e'_0$ does not pass through the crossing points of the components over the cusps. This then gives a corresponding moving for the $[\Gamma_{\tilde{\mu}(1,0)}]$ which leads to a proper intersection. Then we get (at least set theoretically)

$$b^{(1)} = \{(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in \overline{2E}, 2\overline{\phi}(\tilde{x}) = 2\overline{\phi}(\tilde{y}) \in a^{(1)}, \beta(\tilde{x}) = (x_1, x_2), \beta(\tilde{y}) = (y_1, y_2), x_2 = y_2\}$$

For later reference note also that the correction terms $[b^{(1)}]$ and $[b^{(2)}]$ are nilpotent of order 2 in $\text{CH}^3(\overline{2E} \times 2\overline{E}, \mathbb{Q})$, and orthogonal to each other.

### 3.3.3. Definition of $\tilde{\pi}_1^{(2)}(\overline{2E}/M)$.

Now to define a correspondence that acts as $\tilde{\pi}_1(\overline{E}/M)$ on one fibre factor and identity on the other, we first observe that, for $g \in \mathfrak{G}$ acting on $\overline{E}$, the fibre product morphism $g \times \overline{\phi} \text{id}_{\overline{E}} : \overline{E} \to \overline{E}$ scheme-theoretically preserves the center $(2\overline{E})_0$ of the blowing-up $\beta : 2\overline{E} \to \overline{E}$. Therefore it lifts uniquely to a morphism, say $\tilde{\chi}(g, \text{id}) : 2\overline{E} \to 2\overline{E}$, of $2\overline{E}$ [Hartshorne, 1977, II.7.15, p.165]. Similarly $\text{id}_{\overline{E}} \times \overline{\phi} g$ lifts to a morphism, say $\tilde{\chi}(\text{id}, g)$, and moreover, for $g_1, g_2 \in \mathfrak{G}$ we have

$$\tilde{\chi}(g_1, \text{id}) \circ \tilde{\chi}(\text{id}, g_2) = \tilde{\chi}(\text{id}, g_2) \circ \tilde{\chi}(g_1, \text{id}) =: \tilde{\chi}(g_1, g_2).$$

Thus $\mathfrak{G}^2 := \mathfrak{G} \times \mathfrak{G}$ acts as a group of fibrewise automorphisms on $\overline{2E}$, and this action extends $\mathbb{Q}$-linearly to give a homomorphism

$$\mathbb{Q}[\mathfrak{G}^2] \longrightarrow \text{CH}^3(2\overline{E} \times 2\overline{E}, \mathbb{Q}).$$

As special cases, for $a = (a_1, a_2) \in (\mu_2 \times \mu_2)$ we write $\tilde{\mu}(a) : 2\overline{E} \to 2\overline{E}$ for the corresponding morphism, and for $b \in (\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/N\mathbb{Z})^2$ we let $\tilde{\tau}(b) : 2\overline{E} \to 2\overline{E}$ denote the corresponding morphism. Then analogously as in the definition 3.2.5 of $\tilde{\pi}_1(\overline{E}/M)$, let

$$\tilde{\lambda}^{(1)} := \frac{1}{2}(\tilde{\Gamma}_{\tilde{\mu}(1,1)} - \tilde{\Gamma}_{\tilde{\mu}(-1,1)}) \quad \tilde{\lambda}^{(2)} := \frac{1}{2}(\tilde{\Gamma}_{\tilde{\mu}(1,1)} - \tilde{\Gamma}_{\tilde{\mu}(1,-1)})$$

$$\tilde{\gamma}^{(1)} := \frac{1}{N^2} \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^2} [\Gamma_{\tilde{\tau}(b,0)}] \quad \tilde{\gamma}^{(2)} := \frac{1}{N^2} \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^2} [\Gamma_{\tilde{\tau}(0,b)}].$$
and then
\[ \pi_1^{(1)}(2\tilde{E}/\mathcal{M}) := \tilde{\vartheta}^{(1)}(\lambda^{(1)}) = \tilde{\lambda}^{(1)} \circ \tilde{\vartheta}^{(1)} \]
\[ \pi_1^{(2)}(2\tilde{E}/\mathcal{M}) := \tilde{\vartheta}^{(2)}(\lambda^{(2)}) = \tilde{\lambda}^{(2)} \circ \tilde{\vartheta}^{(2)}. \]

As in the definition of \( \pi_1(\mathcal{E}/\mathcal{M}) \), it follows easily from identities in the group ring \( \mathbb{Q}[\mathcal{G} \times \mathcal{G}] \) that \( \tilde{\lambda}^{(j)} \) commutes with \( \tilde{\vartheta}^{(j)} \), and that all the \( \tilde{\lambda}^{(j)} \) and \( \tilde{\vartheta}^{(j)} \) and thus the \( \pi_1^{(j)}(2\tilde{E}/\mathcal{M}) \) are idempotent. Here we also have
\[ \pi_1^{(1)}(2\tilde{E}/\mathcal{M}) \circ \pi_1^{(2)}(2\tilde{E}/\mathcal{M}) = \pi_1^{(2)}(2\tilde{E}/\mathcal{M}) \circ \pi_1^{(1)}(2\tilde{E}/\mathcal{M}), \]
because the two factors of \( \mathcal{G} \times \mathcal{G} \) commute.

The following lemma should be compared with Proposition 3.2.6.

**Lemma 3.3.4.** For fixed \( j = 1 \) or \( 2 \), the \( \pi_1^{(j)}(2\tilde{E}/\mathcal{M}) \), for \( i = 0, 1, 2 \), are mutually orthogonal idempotents in \( \text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q}) \).

**Proof.** All that remains to be checked is that \( \pi_1^{(j)}(2\tilde{E}/\mathcal{M}) \) is orthogonal to both \( \pi_0^{(j)}(2\tilde{E}/\mathcal{M}) \) and \( \pi_2^{(j)}(2\tilde{E}/\mathcal{M}) \). But for this one can argue similarly as for Proposition 3.2.6, that \( \tilde{\lambda}^{(1)} \) is orthogonal to both \( [\Gamma_{\tilde{\mu}(0,1)}] \) and \( [\Gamma_{\tilde{\mu}(0,1)}] \), and \( \tilde{\lambda}^{(2)} \) is orthogonal to both \( [\Gamma_{\tilde{\mu}(1,0)}] \) and \( [\Gamma_{\tilde{\mu}(1,0)}] \).

3.3.5. **Definition of** \( \pi_{i_1,i_2}(2\tilde{E}/\mathcal{M}) \). For \( 0 \leq i_1, i_2 \leq 2 \) define
\[ \pi_{i_1,i_2}(2\tilde{E}/\mathcal{M}) := \pi_{i_1}^{(1)}(2\tilde{E}/\mathcal{M}) \circ \pi_{i_2}^{(2)}(2\tilde{E}/\mathcal{M}). \]

**Proposition 3.3.6.** The \( \pi_{i_1,i_2}(2\tilde{E}/\mathcal{M}) \), for \( 0 \leq i_1, i_2 \leq 2 \), are mutually orthogonal projectors in \( \text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q}) \).

**Proof.** This proposition will follow immediately from the Lemma 3.3.4 as soon as we verify the commutativity relation, that for all \( i_1, i_2 = 0, 1, 2 \),
\[ \pi_{i_1}^{(1)}(2\tilde{E}/\mathcal{M}) \circ \pi_{i_2}^{(2)}(2\tilde{E}/\mathcal{M}) = \pi_{i_2}^{(2)}(2\tilde{E}/\mathcal{M}) \circ \pi_{i_1}^{(1)}(2\tilde{E}/\mathcal{M}). \]

We shall verify this case by case.

**Case** \( i_1 = i_2 = 1 \). We have already seen in 3.3.3 that (1) holds because the \( \pi_1^{(j)}(2\tilde{E}/\mathcal{M}) \), for \( j = 1, 2 \), are homomorphic images of commuting projectors in the group ring \( \mathbb{Q}[\mathcal{G}^2] \).

**Case** \( i_1 = 1 \neq i_2 \) or \( i_1 = 1 = i_2 \). In this case the commutativity relation (1) will follow if we can show that the graph of \( \tilde{\chi}(g_1, \text{id}) \) commutes with both the graph and the transposed graph of \( \tilde{\mu}(0,1) \), and similarly that the graph of \( \tilde{\chi}((\text{id}, g_2) \) commutes with both the graph and the transposed graph of \( \tilde{\mu}(1,0) \). But recalling
that $[\Gamma_\chi] = [\Gamma_{\chi^{-1}}]$ whenever $\chi$ is an automorphism of $2\bar{E}$, and then using 1.1.7, the problem reduces to proving that for any $g \in G$

$$\tilde{\mu}(0, 1) \circ \chi(\text{id}, g) = \chi(\text{id}, g) \circ \tilde{\mu}(0, 1),$$

$$\tilde{\mu}(1, 0) \circ \chi(\text{id}, g) = \chi(\text{id}, g) \circ \tilde{\mu}(1, 0).$$

as endomorphisms of $2\bar{E}$.

To prove the first of these, say, since the argument is the same for both, first recall that by definition $\tilde{\mu}(0, 1) := (\beta')^{-1} \circ (\tilde{\mu}(0, 1) \circ \beta)$, where $\beta'$ is the restriction of $\beta$ to $2\bar{E} - \beta^{-1}(2\bar{E}_0^\infty)$, on which it is an isomorphism, and $\tilde{\mu}(0, 1) := \tilde{\mu}(0) \times_\pi \text{id}_{\bar{E}} : 2\bar{E} \to 2\bar{E}$. On the other hand, the automorphism $\chi(\text{id}, g)$ preserves the exceptional divisor of $2\bar{E}$, as it was lifted to a morphism on $2\bar{E}$ from $\chi(\text{id}, g) := \text{id} \times_\pi g : 2\bar{E} \to 2\bar{E}$, which preserves the center $(2\bar{E}_0^\infty)_0$ of the blowing-up. Therefore by [Hartshorne, 1977, II.7.15, p.165]

$$\beta \circ \chi(\text{id}, g) = \chi(\text{id}, g) \circ \beta.$$ 

Combining this with the definition of $\tilde{\mu}(0, 1)$, we get

$$\tilde{\mu}(0, 1) \circ \chi(\text{id}, g) = (\beta')^{-1} \circ \tilde{\mu}(0, 1) \circ \beta \circ \chi(\text{id}, g)$$

$$= (\beta')^{-1} \circ \tilde{\mu}(0, 1) \circ \chi(\text{id}, g) \circ \beta$$

$$= (\beta')^{-1} \circ \chi(\text{id}, g) \circ \tilde{\mu}(0, 1) \circ \beta$$

$$= \chi(\text{id}, g) \circ \tilde{\mu}(0, 1).$$

Case $i_1 = i_2 \neq 1$. First consider the cases $\bar{\pi}_{0,0}(2\bar{E}/\bar{M})$ and $\bar{\pi}_{2,2}(2\bar{E}/\bar{M})$ which are similar. Take for instance $\bar{\pi}_{2,2}(2\bar{E}/\bar{M})$. Using, as remarked in 3.3.2, that the correction terms are orthogonal, we have

$$\bar{\pi}_{2,2}(2\bar{E}/\bar{M}) = \bar{\pi}_2^{(2)}(2\bar{E}/\bar{M}) \circ \bar{\pi}_2^{(2)}(2\bar{E}/\bar{M})$$

$$= [\Gamma_{\tilde{\mu}(0,0)}] - \frac{1}{2}[\Gamma_{\tilde{\mu}(0,1)}] \circ [\Gamma_{\tilde{\mu}(1,0)}] \circ [\Gamma_{\tilde{\mu}(1,0)}]$$

$$- \frac{1}{2}[\Gamma_{\tilde{\mu}(0,1)}] \circ [\Gamma_{\tilde{\mu}(1,1)}] \circ [\Gamma_{\tilde{\mu}(0,1)}].$$

Thus proving the commutativity relation (1) for $\bar{\pi}_{0,0}(2\bar{E}/\bar{M})$ and $\bar{\pi}_{2,2}(2\bar{E}/\bar{M})$ reduces to proving the relations

$$[\Gamma_{\tilde{\mu}(0,1)}] \circ [\Gamma_{\tilde{\mu}(1,0)}] = [\Gamma_{\tilde{\mu}(1,0)}] \circ [\Gamma_{\tilde{\mu}(0,1)}],$$

$$[\Gamma_{\tilde{\mu}(1,0)}] \circ [\Gamma_{\tilde{\mu}(0,1)}] = [\Gamma_{\tilde{\mu}(0,1)}] \circ [\Gamma_{\tilde{\mu}(1,0)}],$$

which are straightforward to verify by direct computation.
Case 1 \( \neq i_1 \neq i_2 \neq 1 \). It remains to consider \( \tilde{\pi}_{0,2}(\overline{2E}/\overline{M}) \) and \( \tilde{\pi}_{2,0}(\overline{2E}/\overline{M}) \). Take for instance \( \tilde{\pi}_{0,2}(\overline{2E}/\overline{M}) \). Again using the orthogonality of the correction terms, now we get

\[
\tilde{\pi}_{0,2}(\overline{2E}/\overline{M}) := \tilde{\pi}_0^{(1)}(\overline{2E}/\overline{M}) \circ \tilde{\pi}_2^{(2)}(\overline{2E}/\overline{M})
\]

\[
= [\Gamma_{\bar{\mu}}(0,1)] \circ [\Gamma_{\bar{\mu}}(1,0)] - \frac{1}{2}[\Gamma_{\bar{\mu}}(0,1)] \circ [\Gamma_{\bar{\mu}}(1,0)] - \frac{1}{2}[\Gamma_{\bar{\mu}}(0,1)] \circ [\Gamma_{\bar{\mu}}(1,0)].
\]

Then after writing out \( \tilde{\pi}_2^{(2)}(\overline{2E}/\overline{M}) \circ \tilde{\pi}_0^{(1)}(\overline{2E}/\overline{M}) \) we find that the commutativity relation (1) also follows in this case from the relations (3). This completes the proof. \( \square \)

### 3.3.7. Notations and definitions related to cycles on \( \overline{2E} \times \overline{M} \).

As in 3.2.7, when \( \alpha : \overline{2E} \to \overline{2E} \) is a morphism that respects the fibre structure of \( \overline{2E} \to \overline{M} \) then the graph \( \Gamma_\alpha \) of \( \alpha \) is supported on \( \overline{2E} \times \overline{M} \overline{2E} \) and we write \( [\Gamma_\alpha]^{\text{rel}} \) and \( [\Gamma_\alpha]^{\text{rel}} \) for its class and the class of its transpose in \( \text{CH}_3(\overline{2E} \times \overline{2E}, \mathbb{Q}) \). Now consider the inclusions

\[
\overline{2E} \times \overline{2E} \overset{2\psi_1}{\longrightarrow} \overline{2E} \times \overline{2E} \overset{2\psi_2}{\longrightarrow} \overline{2E} \times \overline{2E}.
\]

Then we have \( [\Gamma_\alpha] = (2\psi_2)_* [\Gamma_\alpha]^{\text{rel}} \), and, by abuse of notation we define

\[
2\psi#([\Gamma_\alpha]) := 2\psi^*([\Gamma_\alpha]^{\text{rel}}) \quad \text{ in } \text{CH}^2(\overline{2E} \times \overline{2E}, \mathbb{Q}),
\]

and similarly for the transpose of the graph. If \( \beta : \overline{2E} \to \overline{2E} \) is another morphism respecting the fibre structure of \( \overline{2E} \to \overline{M} \), then

\[
2\psi#([\Gamma_{\alpha\circ\beta}]) = 2\psi#([\Gamma_\alpha]) \circ 2\psi#([\Gamma_\beta])
\]

\[
2\psi#([\Gamma_{\alpha\circ\beta}]) = 2\psi#([\Gamma_\beta]) \circ 2\psi#([\Gamma_\alpha]).
\]

Also as before we extend these definitions by linearity.

Next we apply these definitions to the explicit cycles in 3.3.3. There we defined \( \tilde{\lambda}^{(j)} \) and \( \tilde{\vartheta}^{(j)} \), for \( j = 1, 2 \), as linear combinations of graphs of automorphisms that respect the fibre structure of \( \overline{2E} \to \overline{M} \), so \( \tilde{\lambda}^{(j)} \) and \( \tilde{\vartheta}^{(j)} \) in \( \text{CH}_3(\overline{2E} \times \overline{M} \overline{2E}, \mathbb{Q}) \) and \( 2\psi#(\tilde{\lambda}^{(j)}) \) and \( 2\psi#(\tilde{\vartheta}^{(j)}) \) in \( \text{CH}^2(\overline{2E} \times \overline{2E}, \mathbb{Q}) \) are defined, for \( j = 1, 2 \). If we write, as we may,

\[
\tilde{\pi}_1^{(1)}(\overline{2E}/\overline{M}) = \frac{1}{2N^2} \sum_{g \in \mathfrak{G}} \varepsilon(g)^{-1}[\tilde{\lambda}(g, \text{id})]
\]

\[
\tilde{\pi}_1^{(2)}(\overline{2E}/\overline{M}) = \frac{1}{2N^2} \sum_{g \in \mathfrak{G}} \varepsilon(g)^{-1}[\tilde{\lambda}(\text{id}, g)]
\]
with \( \varepsilon \) as in 3.2.5, then \( \tilde{\pi}_i^{(j)\text{rel}}(2\overline{E}/\mathcal{M}) \) and \( 2\psi^\#(\tilde{\pi}_i^{(j)}(2\overline{E}/\mathcal{M})) \) are defined in the obvious way, for \( j = 1, 2 \), and moreover in \( \text{CH}^2(2E \times_M 2E, \mathbb{Q}) \)

\[
2\psi^\#(\tilde{\pi}_i^{(j)}(2\overline{E}/\mathcal{M})) = 2\psi^\#(\tilde{\lambda}(j)) \circ 2\psi^\#(\tilde{\gamma}(j)) = 2\psi^\#(\tilde{\gamma}(j)) \circ 2\psi^\#(\tilde{\lambda}(j)).
\]

Next we want to apply the definitions above to \( \tilde{\pi}_i^{(j)}(2\overline{E}/\mathcal{M}) \), for \( i = 0, 2 \) and \( j = 1, 2 \), as defined in 3.3.2. Recall that there we chose explicit cycles \( b^{(j)} \) supported on \( 2\overline{E} \times_M 2\overline{E} \) and such that

\[
[b^{(1)}] = \frac{1}{2}[\Gamma_{\tilde{\mu}(0,1)}] \circ [\Gamma_{\tilde{\mu}(0,1)}] \quad \text{and} \quad [b^{(2)}] = \frac{1}{2}[\Gamma_{\tilde{\mu}(1,0)}] \circ [\Gamma_{\tilde{\mu}(1,0)}].
\]

Thus we have elements \( [b^{(j)}]_{\text{rel}} \in \text{CH}_3(2\overline{E} \times_M 2\overline{E}, \mathbb{Q}) \), and therefore also elements \( \tilde{\pi}_i^{(j)\text{rel}}(2\overline{E}/\mathcal{M}) \in \text{CH}_3(2\overline{E} \times_M 2\overline{E}, \mathbb{Q}) \) such that

\[
\tilde{\pi}_i^{(j)}(2\overline{E}/\mathcal{M}) = (2\psi_2)_{\ast}(\tilde{\pi}_i^{(j)\text{rel}}(2\overline{E}/\mathcal{M}))
\]

for \( i = 0, 2 \) and \( j = 1, 2 \). Hence we may also define

\[
2\psi^\#(\tilde{\pi}_i^{(j)}(2\overline{E}/\mathcal{M})) := 2\psi_1^\ast(\tilde{\pi}_i^{(j)\text{rel}}(2\overline{E}/\mathcal{M})) \in \text{CH}^2(2E \times_M 2E, \mathbb{Q}),
\]

for \( i = 0, 2 \) and \( j = 1, 2 \).

Finally, we would like to apply the definitions at the beginning of this section to \( \tilde{\pi}_{i_1,i_2}(2\overline{E}/\mathcal{M}) \), for \( 0 \leq i_1, i_2 \leq 2 \). The following lemma shows how we can do this, even though these projectors were defined in 3.3.5 as a composition of cycle classes, i.e.,

\[
\tilde{\pi}_{i_1,i_2}(2\overline{E}/\mathcal{M}) := \tilde{\pi}_{i_1}^{(1)}(2\overline{E}/\mathcal{M}) \circ \tilde{\pi}_{i_2}^{(2)}(2\overline{E}/\mathcal{M}).
\]

**Lemma 3.3.8.**

1. There exist \( \tilde{\pi}_{i_1,i_2}^{\text{rel}}(2\overline{E}/\mathcal{M}) \in \text{CH}_3(2\overline{E} \times_M 2\overline{E}, \mathbb{Q}) \) such that

\[
(2\psi_2)_{\ast}\tilde{\pi}_{i_1,i_2}^{\text{rel}}(2\overline{E}/\mathcal{M}) = \tilde{\pi}_{i_1,i_2}(2\overline{E}/\mathcal{M}),
\]

for \( 0 \leq i_1, i_2 \leq 2 \).

2. Let

\[
2\psi^\#(\tilde{\pi}_{i_1,i_2}(2\overline{E}/\mathcal{M})) := 2\psi_1^\ast(\tilde{\pi}_{i_1,i_2}^{\text{rel}}(2\overline{E}/\mathcal{M})).
\]

Then in \( \text{CH}^2(2E \times_M 2E, \mathbb{Q}) \) we have

\[
2\psi^\#(\tilde{\pi}_{i_1,i_2}(2\overline{E}/\mathcal{M})) = 2\psi^\#(\tilde{\pi}_{i_1}^{(1)}(2\overline{E}/\mathcal{M}) \circ 2\psi^\#(\tilde{\pi}_{i_2}^{(2)}(2\overline{E}/\mathcal{M}),
\]

for \( 0 \leq i_1, i_2 \leq 2 \).

**Proof.** We will prove this lemma case by case, as we did for Proposition 3.3.6.
Case \( i_1 = i_2 = 1 \). Consider the character \( \varepsilon_2 : \mathfrak{G}^2 \to \{ \pm 1 \} \) defined by \( \varepsilon_2(g_1, g_2) := \varepsilon(g_1)\varepsilon(g_2) \), where \( \varepsilon : \mathfrak{G} \to \{ \pm 1 \} \) is the character defined in 3.2.5.

Then
\[
\pi_{1,1}(\hat{2E}/\mathcal{M}) := \pi_1^{(1)}(\hat{2E}/\mathcal{M}) \circ \pi_1^{(2)}(\hat{2E}/\mathcal{M}) \\
= \frac{1}{4N^4} \sum_{(g_1, g_2) \in \mathfrak{G}^2} \varepsilon_2(g_1, g_2)^{-1}[\Gamma_{\chi(g_1, g_2)}].
\]

We may use this expression to define \( \pi_{1,1}^{rel}(\hat{2E}/\mathcal{M}) \), proving (1), and then (2) follows from observing that
\[
2\psi^\#([\Gamma_{\chi(g_1, g_2)}]) = 2\psi^\#([\Gamma_{\chi(g_1, id)}] \circ 2\psi^\#([\Gamma_{\chi(id, g_2)}]),
\]
see 3.3.7(1).

Case \( i_1 = 1 \neq i_2 \) or \( i_1 \neq 1 = i_2 \). Consider for instance \( i_1 = 1 \) and \( i_2 = 2 \).

Then
\[
\pi_{1,2}(\hat{2E}/\mathcal{M}) := \pi_1^{(1)}(\hat{2E}/\mathcal{M}) \circ \pi_2^{(2)}(\hat{2E}/\mathcal{M}) \\
= \frac{1}{2N^2} \sum_{g \in \mathfrak{G}} \left([\Gamma_{\chi(g, id)} \circ \rho(1, 0)] - [(\text{id}_{2E} \times \mathcal{M} \chi(g, id))_*(\mathfrak{b}^{(2)})]\right),
\]
where the second term in each summand comes from 1.1.7 applied to \( [\Gamma_{\chi(g, id)}] \circ [\mathfrak{b}^{(2)}] \). Now \( [\Gamma_{\chi(g, id)} \circ \rho(1, 0)]^{rel} \in \text{CH}_3(2E \times \mathfrak{M}^{2E}, \mathbb{Q}) \) is defined, because it comes from the graph of a morphism, and the second term is supported on \( 2\hat{E} \times \mathcal{M}^{2\hat{E}} \) (indeed even on \( (\hat{2E} \times \mathcal{M}^{2\hat{E}})^{-1}(\mathfrak{a}^{(2)}) \)), as well. Therefore we may define \( \pi_{1,2}^{rel}(\hat{2E}/\mathcal{M}) \) by the explicit expression (4), and this proves part (1) in this case.

As for showing that, with the definitions as given here,
\[
(\ref{equation:2psi}) \quad 2\psi^\#(\pi_{1,2}(\hat{2E}/\mathcal{M})) = 2\psi^\#(\pi_1^{(1)}(\hat{2E}/\mathcal{M}) \circ 2\psi^\#(\pi_2^{(2)}(\hat{2E}/\mathcal{M})�
\]
first we claim that \( 2\psi^\#([\mathfrak{b}^{(2)}]) = 0 \). From the explicit computation of \( [\mathfrak{b}^{(2)}]^{rel} \) in 3.3.2 we get
\[
2\psi^\#([\mathfrak{b}^{(2)}]) = 2\psi^*([\mathfrak{b}^{(2)}]^{rel}) \\
= \frac{1}{2}((\phi \times \mathcal{M} \phi)^* j^* (\mathfrak{a}^{(2)})) \times \mathcal{M} \Delta(E/\mathcal{M})�
\]
and we have already seen in the proof of 3.2.8 that \( j^*([\mathfrak{a}^{(2)}]) = 0 \). On the other hand,
\[
2\psi^\#([\text{id}_{2E} \times \mathcal{M} \chi(g, id))_*(\mathfrak{b}^{(2)})]^{rel}) = (\text{id}_{2E} \times \mathcal{M} \chi(g, id))_*(2\psi^*([\mathfrak{b}^{(2)}]^{rel}) = 0,\]
where the first equality follows because \( 2\psi_1 \) is an open immersion which is preserved by the action of \( (g, \text{id}) \in \mathfrak{G}^2 \). Now (5) follows for \( (i_1, i_2) = (1, 2) \), and the other
cases are similar, except that when $i_1 = 0$ or $i_2 = 0$ we use transposed graphs throughout.

Case $i_1 = i_2 \neq 1$. Take for instance $(i_1, i_2) = (2, 2)$, the other case $(i_1, i_2) = (0, 0)$ will be similar. Then

$$
\bar{\pi}_{2,2}(\bar{E}/\overline{M}) = \bar{\pi}_2^{(1)}(2\bar{E}/\overline{M}) \circ \bar{\pi}_2^{(2)}(2\bar{E}/\overline{M})
$$

$$
= [\Gamma_{\bar{\mu}(0,0)}] - [\Gamma_{\bar{\mu}(0,1)}] \circ [\mathbf{\hat{b}}(2)] - [\mathbf{\hat{b}}(1)] \circ [\Gamma_{\bar{\mu}(1,0)}]
$$

(6)

$$
= [\Gamma_{\bar{\mu}(0,0)}] - (\operatorname{id}_{2\bar{E}} \times_{\overline{M}} \bar{\mu}(0,1))_* ([\mathbf{\hat{b}}(2)]) - (\bar{\mu}(1,0) \times_{\overline{M}} \operatorname{id}_{2\bar{E}})^* ([\mathbf{\hat{b}}(1)]).
$$

This last expression gives us explicit cycles with which to define $\bar{\pi}_{2,2}^{\text{rel}}(2\bar{E}/\overline{M})$, proving part (1) for this case. To prove part (2) we must verify by straightforward computation that $2\psi_1^*(\operatorname{id}_{2\bar{E}} \times_{\overline{M}} \bar{\mu}(0,1))_* ([\mathbf{\hat{b}}(2)]^{\text{rel}}) = 0$, and similarly mutatis mutandis; the proofs are similar to the previous ones.

Case $1 \neq i_1 \neq i_2 \neq 1$. Take for instance $(i_1, i_2) = (0, 2)$. Then similarly as in the previous case we have

$$
\bar{\pi}_{1,2}(\bar{E}/\overline{M}) = \bar{\pi}_1^{(1)}(2\bar{E}/\overline{M}) \circ \bar{\pi}_2^{(2)}(2\bar{E}/\overline{M})
$$

(7)

$$
= [\Gamma_{\bar{\mu}(0,1)}] \circ [\Gamma_{\bar{\mu}(1,0)}] - (\operatorname{id}_{2\bar{E}} \times_{\overline{M}} \bar{\mu}(0,1))_* ([\mathbf{\hat{b}}(2)])
$$

$$
- (\bar{\mu}(1,0) \times_{\overline{M}} \operatorname{id}_{2\bar{E}})^* ([\mathbf{\hat{b}}(1)]).
$$

Now to see that $[\Gamma_{\bar{\mu}(0,1)}] \circ [\Gamma_{\bar{\mu}(1,0)}]$ is or can be supported on $2\bar{E} \times_{\overline{M}} 2\bar{E}$ we can compute at the level of cycles where

$$
[\Gamma_{\bar{\mu}(0,1)}] \circ [\Gamma_{\bar{\mu}(1,0)}] = [\operatorname{pr}_{13*} (\{(\bar{\mu}(0,1) \times 2\bar{E}) \cdot (2\bar{E} \times \operatorname{pr}_{13})\}])
$$

Then we see that this can be represented by a cycle supported on the set

$$
\{(\bar{x}, \bar{y}) : 2\phi^*(\bar{x}) = 2\phi^*(\bar{y}), \beta(\bar{x}) = (0, x_2), \beta(\bar{y}) = (y_1, 0), \text{ with } x_2, y_1 \in \overline{E}\}
$$

(8)

contained in $2\bar{E} \times_{\overline{M}} 2\bar{E}$. Using this we can define $\bar{\pi}_{0,2}^{\text{rel}}(2\bar{E}/\overline{M})$ via formula (7).

For part (2) we use firstly that the correction terms vanish after applying $2\psi_1^*$, as above, and that if we use a cycle representative for $[\Gamma_{\bar{\mu}(0,1)}] \circ [\Gamma_{\bar{\mu}(1,0)}]$ supported on the set (8) then with the obvious notation we get

$$
2\psi_1^*[\Gamma_{\bar{\mu}(0,1)}] \circ [\Gamma_{\bar{\mu}(1,0)}] = 2\psi_1^*[\Gamma_{\bar{\mu}(0,1)}]^{\text{rel}} \circ 2\psi_1^*[\Gamma_{\bar{\mu}(1,0)}]^{\text{rel}}.
$$

Set-theoretically this is immediate, and in order to see that the intersection multiplicities are correct use [Weil, 1948, VIII.4, Thm.10, p.233]. This completes the proof of the lemma. □
Proposition 3.3.9. In $\text{CH}^2(2E \times_M 2E, \mathbb{Q})$ we have

\begin{equation}
2 \psi^# \widetilde{\pi}_{i_1,i_2}((2\bar{E}/M)) = \pi_{i_1}^\text{can}(E/M) \otimes_M \pi_{i_2}^\text{can}(E/M),
\end{equation}

for $0 \leq i_1, i_2 \leq 2$, and moreover

\begin{equation}
2 \psi^# \left( \sum_{i_1+i_2=i} \widetilde{\pi}_{i_1,i_2}((2\bar{E}/M)) \right) = \pi_i^\text{can}(2E/M),
\end{equation}

for $0 \leq i \leq 2$.

Proof. Firstly we claim that

$$2 \psi^# \left( \pi_{i_1}((2\bar{E}/M)) \right) = \psi^# \left( \pi_{i_1}(E/M) \right) \otimes_M [\Delta(E/M)],$$

with the tensor product defined as in 1.1.3, and similarly for $2 \psi^# \left( \pi_{i_2}((2\bar{E}/M)) \right)$. For $i_1 = 1$ this comes immediately from the expression 3.3.7(2). If $i_1 = 0$, say, then we have seen in the proof of Lemma 3.3.8 that $2 \psi^*_1((\text{b}^{(j)})^\text{rel}) = 0$, from which it follows that \( \pi_0((2\bar{E}/M)) = [\Gamma_{\mu(0,1)}]^\text{rel} \). Therefore, $2 \psi_1^* \left( \pi_0((2\bar{E}/M)) \right) = [\Gamma_{\mu(0)}] \otimes_M [\Delta(E/M)]$ as claimed. The argument is the same if $i_1 = 2$ or if $i_1$ is replaced by $i_2$. Hence (1) now follows from Lemma 3.3.8(2) and Proposition 3.2.8. Then (2) follows from (1), the Künneth formula for relative Chow motives over $M$, and the characterizing property 3.1(1) of the canonical relative projectors for abelian schemes.

3.3.10. Definition of $\pi_\infty((2\bar{E}/M))$. Let

$$\pi_f((2\bar{E}/M)) := \sum_{i_1,i_2=0}^2 \pi_{i_1,i_2}(2\bar{E}/M).$$

Then by Proposition 3.3.8 $2 \psi^# \pi_f((2\bar{E}/M)) = [\Delta(2E/M)]$ in $\text{CH}^2(2E \times_M 2E, \mathbb{Q})$.

Let

$$\pi_\infty((2\bar{E}/M)) := [\Delta(2E)] - \pi_f((2\bar{E}/M)) \quad \text{in} \quad \text{CH}^3(2\bar{E} \times 2\bar{E}, \mathbb{Q}).$$

Then it is immediate from the orthogonality and idempotency of the $\pi_{i_1,i_2}(2\bar{E}/M)$ that $\pi_f((2\bar{E}/M))$ and $\pi_\infty((2\bar{E}/M))$ are mutually orthogonal projectors, and that $\pi_\infty((2\bar{E}/M))$ is orthogonal to all the $\pi_{i_1,i_2}(2\bar{E}/M)$, for $0 \leq i_1, i_2 \leq 2$. Similarly as for $\pi_\infty(E/M)$, we can say more about the structure of $\pi_\infty((2\bar{E}/M))$.

Lemma 3.3.11 (structure of $\pi_\infty((2\bar{E}/M))$). For $c \in M^\infty$ let $\Theta_c(m)$ denote the components of the fibre $2\bar{E}_c$ over $c$ (as $m$ runs through pairs of integers and pairs of half-integers mod $NZ$, as in 2.3.4). Then in $\text{CH}^3(2\bar{E} \times 2\bar{E}, \mathbb{Q})$,

$$\pi_\infty((2\bar{E}/M)) = \pi^{(2)}(2\bar{E}/M) + \pi^{(4)}(2\bar{E}/M),$$

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with
\[ \tilde{\pi}_\infty^{(2)}(\tilde{2}E/M) := \sum_{c \in M} \tilde{\pi}_\infty^{(2)}(\tilde{2}E/\tilde{M}) \]
\[ \tilde{\pi}_\infty^{(4)}(\tilde{2}E/\tilde{M}) := \sum_{c \in M} \tilde{\pi}_\infty^{(4)}(\tilde{2}E/\tilde{M}) = \tilde{t}\tilde{\pi}_\infty^{(2)}(\tilde{2}E/\tilde{M}) \]
where
\[ \tilde{\pi}_c^{(2)}(\tilde{2}E/\tilde{M}) := \sum_{m \in I} [Z_c \times \{c\} \Theta_c(m)], \]
\[ \tilde{\pi}_c^{(4)}(\tilde{2}E/\tilde{M}) := \sum_{m \in I} [\Theta_c(m) \times \{c\} Z_c(m)], \]
for some \([Z_c(m)] \in \text{CH}^2(\tilde{2}E, \mathbb{Q})\) supported in \(\tilde{2}E_c\). Moreover, all the \(\tilde{\pi}_c^{(2)}(\tilde{2}E/\tilde{M})\) and \(\tilde{\pi}_c^{(4)}(\tilde{2}E/\tilde{M})\) are projectors, mutually orthogonal, and also orthogonal to all \(\tilde{\pi}_{i_1,i_2}(\tilde{2}E/\tilde{M})\), for \(0 \leq i_1,i_2 \leq 2\).

\textbf{Proof.} Consider the diagram
\[
\begin{array}{ccc}
\text{CH}_3(\tilde{2}E^\infty \times_M 2\tilde{E}^\infty, \mathbb{Q}) & \rightarrow & \text{CH}_3(2\tilde{E} \times \tilde{M} 2\tilde{E}, \mathbb{Q}) \xrightarrow{\partial(\tilde{\psi}_2)_*} \text{CH}_3(2\tilde{E} \times_M 2\tilde{E}, \mathbb{Q}) \rightarrow 0 \\
\downarrow & & \downarrow \\
\text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q}) & \rightarrow & \text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q}) \rightarrow 0
\end{array}
\]
whose horizontal rows are exact [Fulton, 1984, 1.8, p.21]. Then \(\tilde{\pi}_c^{(2)}(\tilde{2}E/\tilde{M}) = (\tilde{\pi}_c^{(2)})(\tilde{2}E/\tilde{M})\), in the notation of 3.3.7, and it follows from Proposition 3.3.9 that the difference \([\Delta(2\tilde{E})]_\text{rel} \rightarrow \tilde{\pi}_c^{(2)}(\tilde{2}E/\tilde{M})\) in \(\text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q})\) maps to zero in \(\text{CH}_3(2\tilde{E} \times M 2\tilde{E}, \mathbb{Q})\). Hence \(\tilde{\pi}_\infty^{(2)}(\tilde{2}E/\tilde{M}) \in \text{CH}_3(2\tilde{E} \times \tilde{M} 2\tilde{E}, \mathbb{Q})\) must be in the image of \(\text{CH}_3(2\tilde{E}^\infty \times_M 2\tilde{E}^\infty, \mathbb{Q})\). On the other hand, the components of \(2\tilde{E}^\infty \times_M 2\tilde{E}^\infty\) are of the form \([\Theta_c(m)] \times \{c\} \Theta_c(m')\), which by Proposition 2.3.1 are products of rational surfaces. Therefore, for each of these components, linear equivalence coincides with homological equivalence, and thus the Künneth formula for homology allows us to conclude that \(\text{CH}_3(2\tilde{E}^\infty \times_M 2\tilde{E}^\infty, \mathbb{Q})\) is generated by elements of the form \([\Theta_c(m)] \times \{c\} [Z_c]\) and \([Z'_c] \times \{c\} [\Theta_c(m)]\), for \(c \in M^\infty\) and \(m \in I\) and \([Z_c]\), \([Z'_c]\) \(\in \text{CH}_1(2\tilde{E}_c, \mathbb{Q})\). Hence, \(\tilde{\pi}_\infty^{(2)}(\tilde{2}E/\tilde{M})\) can be written in the form claimed. But in addition, every class of the form \([\Theta_c(m)] \times \{c\} [Z_c]\) is orthogonal to every class of the form \([Z'_c] \times \{c\} [\Theta_c(m)]\) for reasons of dimension, and cycles which can be supported over distinct \(c \in M^\infty\) are orthogonal, as they are disjoint. Therefore all the \(\tilde{\pi}_c^{(i)}(\tilde{2}E/\tilde{M})\) are mutually orthogonal. However, they must also be idempotent and orthogonal to all the \(\tilde{\pi}_{i_1,i_2}(\tilde{2}E/\tilde{M})\), for \(i_1,i_2 = 0,1,2\), since this is true for \(\tilde{\pi}_\infty^{(2)}(\tilde{2}E/\tilde{M})\). \(\square\)
3.3.12. Splitting $\tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$ into symmetric and antisymmetric parts. Before leaving this section there is one further refinement we need. First, observe that fibrewise permutation of the fibre factors of $\mathcal{E} \to \mathcal{M}$ preserves the center of the blowing-up $\beta$ scheme-theoretically, whence it lifts uniquely to a morphism, say $\sigma : 2\tilde{E} \to 2\tilde{E}$ of $2\tilde{E}$. Thus we get an action of the permutation group $\mathfrak{S}_2$ on $2\tilde{E}$, which together with the action of $\Gamma$ gives a group action of the semidirect product $\mathfrak{S}_2 \ltimes \mathfrak{S}_2$ on $2\tilde{E}$. (This is the group $\Gamma_2$ of [Scholl, 1990, 1.1.1].)

Next, let

$$A_2 := \frac{1}{2}([\Delta(2\tilde{E})] + [\Gamma_2]) \quad \text{in } \text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q}).$$

$$S_2 := \frac{1}{2}([\Delta(2\tilde{E})] - [\Gamma_2])$$

Then $A_2$ and $S_2$ are mutually orthogonal projectors whose sum is the identity in $\text{CH}^3(2\tilde{E} \times 2\tilde{E}, \mathbb{Q})$. Moreover, the restrictions (in the sense of 3.2.7 and 3.2.8) $2\psi^#A_2$ and $2\psi^#S_2$ of $A_2$ and $S_2$ respectively to $\text{CH}^2(2E \times_M 2E, \mathbb{Q})$, in the notation of Proposition 3.3.8, project the tensor square of a correspondence in $\text{CH}^1(E \times_M E, \mathbb{Q})$ to its exterior and symmetric square parts, respectively, cf. [Künnemann, 1994, [del Baño Rolla, 1995].

Now we compose these projectors with $\tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$, and write $A_2 \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$ for $A_2 \circ \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$ and $S_2 \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$ for $S_2 \circ \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$. Then it is easy to check (by looking in $\mathbb{Q}[\mathfrak{S}_2 \ltimes \mathfrak{S}_2]$) that $A_2$ and $S_2$ commute with $\tilde{\lambda}^{(1)} \circ \tilde{\lambda}^{(2)}$ as well as with $\tilde{\vartheta}^{(1)} \circ \tilde{\vartheta}^{(2)}$, and therefore with $\tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$. Thus, in addition to

$$A_2 \tilde{\pi}_{1,1} + S_2 \tilde{\pi}_{1,1} = \tilde{\pi}_{1,1},$$

we also have that $A_2 \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$ and $S_2 \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})$ are orthogonal to each other as well as to all the $\tilde{\pi}_{i_1,i_2}(2\tilde{E}/\mathcal{M})$, for $(i_1, i_2) \neq (1, 1)$. Furthermore, from the definitions and Proposition 3.3.8,

$$2\psi^#(S_2 \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})) = \text{Sym}^2_M \pi^\text{can}_1(E/M),$$

whereas

$$2\psi^#(A_2 \tilde{\pi}_{1,1}(2\tilde{E}/\mathcal{M})) = \wedge^2_M \pi^\text{can}_1(E/M) \simeq \pi^\text{can}_2(E/M),$$

as follows from the definitions, Proposition 3.3.8, and the result of [Shermenev, 1974] and [Künnemann, 1994, Thm.3.3.1].

4. Analysis of the Chow motives $h(E)$ and $h(2\tilde{E})$

This section is the technical center of the paper, for here we analyze the Chow motives determined by the projectors defined in section three in order to identify them up to isomorphism, when we can, with Chow motives that can be defined in terms of lower-dimensional varieties. For example, we view $\mathbb{L}^d \simeq (\text{Spec } K, \text{id}_K, -d)$
as being supported on a point, and \( h(\overline{M}) \simeq 1 \oplus \mathbb{L} \oplus h^1(\overline{M}) \) as consisting of a constituent submotive belonging essentially to the curve together with two constituent submotives supported on points; the precise isomorphisms that we prove in this section are stated in Theorem 4.2, below. We reserve exploring the implications of this theorem for Chow-Künneth decompositions and filtrations on the Chow groups of \( \overline{E} \) and \( \overline{2E} \) until the next two sections.

4.1. Notation. Let \( ^1W := (\overline{E}, \pi_1(\overline{E}/\overline{M})) \) and \( ^2W := (\overline{2E}, S_2 \overline{\pi}_{1,1}(\overline{2E}/\overline{M})) \). Then these are the Chow motives for modular forms constructed in [Scholl, 1990], for \( k = 1, 2 \); and modulo homological equivalence, they are the motives for modular forms constructed in [Deligne, 1969].

In the statement of the next theorem, a positive integer coefficient on a motive indicates the multiplicity with which that motive, up to isomorphism, occurs.

Theorem 4.2. As Chow motives in \( \text{M}(K) \),

\[
(1) \quad h(\overline{E}) \simeq 1 \oplus m \mathbb{L} \oplus \mathbb{L}^2 \\
\quad \quad \quad \oplus h^1(\overline{M}) \oplus (h^1(\overline{M}) \otimes \mathbb{L}) \\
\quad \quad \quad \oplus ^1W
\]

for some positive integer \( m \), and

\[
(2) \quad h(\overline{2E}) \simeq 1 \oplus n \mathbb{L} \oplus n \mathbb{L}^2 \oplus \mathbb{L}^3 \\
\quad \quad \quad \oplus h^1(\overline{M}) \oplus 3 (h^1(\overline{M}) \otimes \mathbb{L}) \oplus (h^1(\overline{M}) \otimes \mathbb{L}^2) \\
\quad \quad \quad \oplus 2 (^1W) \oplus 2 (^1W \otimes \mathbb{L}) \\
\quad \quad \quad \oplus ^2W
\]

for some positive integer \( n \).

Remark 4.2.1. It will follow from the proof together with 2.3.2 that \( m = \frac{1}{2} N^2 (N - 1) \prod_{p|N} (1 - p^{-2}) \). Unfortunately, we don’t have equally precise information about \( n \).

4.2.2. Organization of the proof. The rest of this section is devoted to the proof of Theorem 4.2, and is divided into five parts. In the first part we analyze the motives defined by \( \pi_0(\overline{E}/\overline{M}) \) and \( \pi_0(\overline{2E}/\overline{M}) \) for \( \overline{E} \), and by \( \overline{\pi}_{0,0}(\overline{2E}/\overline{M}) \) and \( \overline{\pi}_{2,2}(\overline{2E}/\overline{M}) \) for \( \overline{2E} \); these are the constituents of lowest and highest weights. Then we prove a proposition that describes the action of the extended relative projectors on the components of the cusp fibres; we need this in the analysis of all the remaining projectors. Next we look at the remaining \( \overline{\pi}_{1,1}(\overline{2E}/\overline{M}) \), for it turns out that they can be treated together. After that we describe the motives defined by \( \overline{\pi}_{\infty}(\overline{E}/\overline{M}) \) and \( \overline{\pi}_{\infty}(\overline{2E}/\overline{M}) \), and then finally we put everything together to complete the proof of the theorem.
4.3. All zeroes or all twos. We begin with a little lemma to help deal with the nuisance of the correction terms occurring in the projectors with zeroes or twos. This lemma may be compared with the lemma of Beilinson on the lifting of idempotents by a nilpotent ideal [Jannsen, 1994, p.289].

**Lemma 4.3.1.** When $X$ is a smooth (connected) projective variety, and $p,p' \in \text{CH}^{\dim X}(X \times X, \mathbb{Q})$ are projectors such that $(p-p') \circ (p-p') = 0$, then as Chow motives $(X,p) \simeq (X,p')$.

*Proof.* The identity map, i.e., $[\Delta(X)]$, induces the isomorphism. Write $p' = p + n$, with $n \circ n = 0$. Then from $p \circ p = p$ and $(p+n) \circ (p+n) = (p+n)$ it is elementary to deduce that $p \circ (p+n) \circ p = p$ and $(p+n) \circ p \circ (p+n) = (p+n)$, as required. \hfill \square

**Proposition 4.3.2.** As Chow motives in $\mathcal{M}(K)$,

1. $(\bar{E},\pi_0(\bar{E}/\bar{M})) \simeq h(\bar{M})$.
2. $(\bar{E},\pi_2(\bar{E}/\bar{M})) \simeq h(\bar{M}) \otimes L$.
3. $(^2\bar{E},\pi_{0,0}(^2\bar{E}/\bar{M})) \simeq h(\bar{M})$.
4. $(^2\bar{E},\pi_{2,2}(^2\bar{E}/\bar{M})) \simeq h(\bar{M}) \otimes L^2$.

*Proof.* All the isomorphisms are induced by the graphs or transposed graphs of the structure maps onto $\bar{M}$ and the zero-sections. We give first the argument for (3), as (1) is similar but simpler. From the lemma it follows that $(^2\bar{E},\pi_{0,0}(^2\bar{E}/\bar{M})) \simeq (^2\bar{E},[\Gamma_{\bar{M}(0,0)}])$ since, as we have observed (3.2.3 and 3.3.2), all the correction terms are nilpotent of order 2. Then to obtain that $(^2\bar{E},[\Gamma_{\bar{M}(0,0)}]) \simeq (\bar{M},[\Delta(\bar{M})])$, it suffices to show

\[
[\Gamma_{\bar{M}(0,0)}] \circ [\Gamma_{\bar{2}}] \circ [\Delta(\bar{M})] = [\Gamma_{\bar{M}(0,0)}],
\]

\[
[\Delta(\bar{M})] \circ [\Gamma_{\bar{A}(0)}] \circ [\Gamma_{\bar{M}(0,0)}] \circ [\Delta(\bar{M})] = [\Delta(\bar{M})].
\]

But these follow from the identities

\[
\tilde{\mu}(0,0) \circ \tilde{\alpha}(0) \circ 2_{\bar{\phi}} \circ \tilde{\mu}(0,0) = \tilde{\mu}(0,0),
\]

\[
2_{\bar{\phi}} \circ \tilde{\mu}(0,0) \circ \tilde{\alpha}(0) = \text{id}_{\bar{M}},
\]

where $\tilde{\alpha}$ is the extension of the level-$N$ structure of $^2\bar{E}$ to $^2\bar{E}$, as in 3.3.2. Now transposing everything proves (4), and likewise the correspondences that prove (2) are the transposes of those that prove (1). \hfill \square

4.4. Action of projectors on fibres and components at infinity. Next we consider the action of the our projectors on fibres and the components of the fibres at infinity. Roughly speaking, $\pi_f(\bar{E}/\bar{M})$ and $\pi_f(\bar{E}/\bar{M})$ annihilate the components of the cusp fibres—indeed, it was so that this would be the case that $\pi_f(\bar{E}/\bar{M})$,
and consequently the \( \pi_{i_1, i_2}(\overline{2E/M}) \) with \( i_1 \) or \( i_2 = 1 \), were chosen as they were—and \( \pi_\infty(E/M) \) and \( \pi_\infty(\overline{2E/M}) \) act as the identity on those components, but there are some nuances involving the identity components; the next proposition gives a precise statement. As a matter of notation, for any \( t \in M \) we let \( E_t := \phi^{-1}(t) \) and \( \overline{2E_t} := \overline{2}\phi^{-1}(t) \). Further, for any cusp \( c \in M^\infty \) we let \( \theta_c(0) \) be the identity component of \( \overline{2E_c} \), i.e., the component containing \( \bar{\alpha}(0, 0, c) \), and similarly let \( \Theta_c(0) \) be the identity component of \( \overline{2E_c} \), the component containing \( \bar{\alpha}(0, c) \).

**Proposition 4.4.1.**

1. For all \( t \in M \), in \( CH^1(E, \mathbb{Q}) \)

\[
\pi_0(E/M)([E_t]) = [E_t] \\
\pi_i(E/M)([E_t]) = 0 \quad \text{for} \quad i \neq 0.
\]

2. For all \( t \in M \), in \( CH^1(\overline{2E}, \mathbb{Q}) \)

\[
\tilde{\pi}_{0,0}(\overline{2E/M})([\overline{2E_t}]) = [\overline{2E_t}] \\
\tilde{\pi}_{i_1, i_2}(\overline{2E/M})([\overline{2E_t}]) = 0 \quad \text{for} \quad (i_1, i_2) \neq (0, 0).
\]

3. For \( c \in M^\infty \), in \( CH^1(E, \mathbb{Q}) \)

\[
\pi_i(E/M)([\theta_c(m)]) = 0 \quad \text{unless} \quad m = 0 \quad \text{and} \quad i = 0, \\
\pi_c(E/M)([\theta_c(m)]) = [\theta_c(m)] \quad \text{for} \quad m \neq 0, \\
\pi_0(E/M)([\theta_c(0)]) = [E_c].
\]

4. For \( c \in M^\infty \), in \( CH^1(\overline{2E}, \mathbb{Q}) \)

\[
\tilde{\pi}_{i_1, i_2}(\overline{2E/M})([\Theta_c(m)]) = 0 \quad \text{unless} \quad m = 0 \quad \text{and} \quad (i_1, i_2) = (0, 0), \\
\tilde{\pi}_c(\overline{2E/M})([\Theta_c(m)]) = [\Theta_c(m)] \quad \text{for} \quad m \neq 0, \\
\tilde{\pi}_{0,0}(\overline{2E/M})([\Theta_c(0)]) = [\overline{2E_c}].
\]

**Proof.** To begin, we can write \( \pi_0(E/M) = [\Gamma_{\mu(0)}] - \frac{1}{2}(\psi_2)_*(\phi \times_{\overline{M}} \phi)^*(a) \), for a certain class \( a \in CH^1(M, \mathbb{Q}) \), see Lemma 3.2.2. Thus, on any fibre, or any component of a fibre, \( \pi_0(E/M) \) acts as \( \mu(0)^* \), which acts by mapping (the class of) the identity component of a fibre to (the class of) that entire fibre. By orthogonality, we also get that the other projectors defined in section three annihilate the class of an entire fibre. Similarly \( \pi_{0,0}(\overline{2E/M}) \) acts on fibres or components of fibres as \( \overline{\mu}(0, 0)^* \), likewise mapping (the class of) the identity component of any fibre to (the class of) that entire fibre. And again, by orthogonality, we also get that the other projectors annihilate the class of an entire fibre. This proves parts (1) and (2),
and also the statements about the action of $\pi_0(\mathcal{E}/\mathcal{M})$ or $\tilde{\pi}_{0,0}(2\mathcal{E}/\mathcal{M})$ in parts (3) and (4).

Next consider $\pi_2(\mathcal{E}/\mathcal{M}) = [\Gamma_{\hat{\mu}(0)}]$ plus a vertical correction term. This acts on (the class of) any component of any fibre as $\hat{\mu}(0)_*$, thereby annihilating (the class of) that component. Similarly $\pi_1(2\mathcal{E}/\mathcal{M})$ and $\tilde{\pi}_2(2\mathcal{E}/\mathcal{M})$ act on vertical two-dimensional cycles, in particular (classes of) components of fibres. Thus $\pi_{i1, i2}(2\mathcal{E}/\mathcal{M})([\Theta_c(m)]) = 0$ whenever $i_1 = 2$ or $i_2 = 2$, for as we saw in the proof of Proposition 3.3.6 we may write $\tilde{\pi}_{i, 2}(2\mathcal{E}/\mathcal{M}) = \pi_i(2\mathcal{E}/\mathcal{M}) \circ \tilde{\pi}_2(2\mathcal{E}/\mathcal{M})$ and $\tilde{\pi}_2, (2\mathcal{E}/\mathcal{M}) = \tilde{\pi}_1(2\mathcal{E}/\mathcal{M}) \circ \tilde{\pi}_2(2\mathcal{E}/\mathcal{M})$.

Now consider $\pi_1(\mathcal{E}/\mathcal{M}) = \bar{\lambda} \circ \tilde{\vartheta}$, as in 3.2.5. Then $\tilde{\vartheta}$ acts on a component $\theta_c(m)$ of $\mathcal{E}^\infty$ by $\tilde{\vartheta}([\theta_c(m)]) = \frac{1}{2}[\bar{\lambda}[\theta_c(m)]]$, while $\bar{\lambda}([\theta_c(m)]) = \frac{1}{2}([\theta_c(m)] - [\theta_c(-m)])$, as follows from 2.2.1 and 3.2.5. So it is easy to see that their combined effect is to annihilate any $[\theta_c(m)]$.

Finally we consider $\tilde{\pi}_{i1, i2}(2\mathcal{E}/\mathcal{M})$ with $i_1 = 1$ or $i_2 = 1$; as above, we will be finished if we can show that $\pi_{i1}(2\mathcal{E}/\mathcal{M})([\Theta_c(m)]) = 0$ for any component $\Theta_c(m)$ of $2\mathcal{E}^\infty$, for $j = 1$ or 2. For definiteness, suppose for the moment that $j = 2$, and write $\pi_{i1}(2\mathcal{E}/\mathcal{M}) = \lambda^{(2)} \circ \tilde{\vartheta}^{(2)}$, as in 3.3.3. Then letting $\Theta_c(m, n)$ represent the components of $2\mathcal{E}_c$, with the indexing described in 2.3.4, we find that $\lambda^{(2)}([\Theta_c(m, n)]) = \frac{1}{2}([\theta_c(m, n)] - [\theta_c(m, -n)])$, and $\tilde{\vartheta}^{(2)}([\Theta_c(m, n)]) = \frac{1}{2N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} [\Theta_c(m, n)]$. Thus the combined effect of the two is to annihilate $[\Theta_c(m, n)]$, as required. Since the argument is the same $j = 1$, this completes the proof.

4.5. Isomorphisms between submotives of $h(2\mathcal{E})$ and submotives of $h(\mathcal{E})$. The next proposition identifies several of the motivic constituents of $h(2\mathcal{E})$ defined by the projectors defined in section three with motives supported on lower dimensional varieties. Although some of these can be supported on $\mathcal{M}$ or Spec $K$, what we actually verify is that some of the submotives of $2\mathcal{E}$ are isomorphic to submotives of $\mathcal{E}$, so we state the proposition this way and defer further reduction until the last part of the section.

Proposition 4.5.1. As Chow motives in $\mathcal{M}(K)$,

$$
(1) \quad (2\mathcal{E}, \pi_{0,1}(2\mathcal{E}/\mathcal{M})) \simeq (2\mathcal{E}, \tilde{\pi}_{0,1,0}(2\mathcal{E}/\mathcal{M})) \simeq (\mathcal{E}, \pi_1(\mathcal{E}/\mathcal{M}));
$$

$$
(2) \quad (2\mathcal{E}, \pi_{0,2}(2\mathcal{E}/\mathcal{M})) \simeq (2\mathcal{E}, \tilde{\pi}_{0,2,0}(2\mathcal{E}/\mathcal{M})) \simeq (\mathcal{E}, \pi_2(\mathcal{E}/\mathcal{M}));
$$

$$
(3) \quad (2\mathcal{E}, A_2 \pi_{1,1}(2\mathcal{E}/\mathcal{M})) \simeq (\mathcal{E}, \pi_2(\mathcal{E}/\mathcal{M}));
$$

$$
(4) \quad (2\mathcal{E}, \pi_{1,2}(2\mathcal{E}/\mathcal{M})) \simeq (2\mathcal{E}, \tilde{\pi}_{1,2,0}(2\mathcal{E}/\mathcal{M})) \simeq (\mathcal{E}, \pi_1(\mathcal{E}/\mathcal{M}), -1).
$$

Proof. Since the proofs of these isomorphisms between a submotive of $2\mathcal{E}$ and a submotive of $\mathcal{E}$ all follow a similar pattern, when an argument applies generally we will use $\tilde{\pi}$ to represent any of the seven projectors for $2\mathcal{E}$ above, and $\pi$ for the corresponding projector on $\mathcal{E}$, and $m$ for the corresponding Tate twist (when
present); but when the differences in detail require it, we will refer to the specific cases (1)–(4).

With this notation and that in Propositions 3.2.8 and 3.3.9, the first observation is that in each case there is an isomorphism \( (2E, 2\psi\# \pi) \simeq (E, \psi\# \pi, m) \) of relative Chow motives over \( M \). For parts (1), (2) and (4) this follows more or less formally from the tensor structure of the category \( M(M) \), as in 1.1.2 and 1.1.3; whereas for part (3) it follows from the theorem of [Shermenev, 1974] and [K"unnemann, 1994, Thm.3.3.1], as mentioned in 3.3.12. So let \( \alpha \) on \( 2E \times M \) and \( \beta \) on \( E \times M \) be cycles inducing this isomorphism in each direction, and let \( \tilde{\alpha} \) and \( \tilde{\beta} \) denote their closures in \( 2\tilde{E} \times \tilde{E} \) and \( \tilde{E} \times 2\tilde{E} \), respectively. Then we claim that 

\[
\tilde{\alpha} \in \text{Corr}^{-m}(2\tilde{E}, E) \quad \text{and} \quad \tilde{\beta} \in \text{Corr}^{-m}(E, 2\tilde{E})
\]

induce inverse isomorphisms between \( (2\tilde{E}, \tilde{\pi}) \) and \( (E, \pi, m) \). To verify this we must show that 

\[
\pi \circ [\tilde{\alpha}] \circ \tilde{\pi} \circ \tilde{\beta} = \pi \circ \tilde{\beta} \circ \pi \circ [\tilde{\alpha}] = \pi
\]

Consider (5) first. What we already know is that the correspondences on both sides of the equation can be supported on \( E \times _{\pi} \tilde{E} \), and that their restrictions (in the sense of 3.2.7 and 3.2.8) to \( E \times _{\pi} M \) coincide. Thus the exactness of the sequence 

\[ \text{CH}_2(\tilde{E}^\infty \times _{M} E^\infty, \mathbb{Q}) \to \text{CH}_2(\tilde{E} \times _{\pi} \tilde{E}, \mathbb{Q}) \to \text{CH}_2(E \times _{M} E, \mathbb{Q}) \to 0 \]

implies that the difference 

\[
(\pi \circ [\tilde{\alpha}] \circ \tilde{\pi} \circ [\tilde{\beta}] \circ \pi) - \pi = \sum_{c \in M, m, n \in (\mathbb{Z}/N\mathbb{Z})^2} a_c(m, n)[\theta_c(m)] \times \{c\} [\theta_c(n)]
\]

since it lies in the image of \( \text{CH}_2(\tilde{E}^\infty \times _{M} E^\infty, \mathbb{Q}) \) in \( \text{CH}_2(\tilde{E} \times _{\pi} \tilde{E}, \mathbb{Q}) \). Then composing with \( \pi \) on the left and right gives 

\[
(\pi \circ [\tilde{\alpha}] \circ \tilde{\pi} \circ [\tilde{\beta}] \circ \pi) - \pi = \sum_{c \in M, m, n \in (\mathbb{Z}/N\mathbb{Z})^2} a_c(m, n)^{t\pi}([\theta_c(m)]) \times \{c\} \pi([\theta_c(n)]) = 0
\]

by applying Proposition 4.4.1. This proves (5).

The argument for (6), using the right-exact sequence 

\[ \text{CH}_3(2\tilde{E}^\infty \times _{M} 2\tilde{E}^\infty, \mathbb{Q}) \to \text{CH}_3(2\tilde{E} \times _{\pi} 2\tilde{E}, \mathbb{Q}) \to \text{CH}_3(2E \times _{M} 2E, \mathbb{Q}) \to 0, \]
runs in a completely parallel manner up to the point where
\[
(\overline{\pi} \circ [\beta] \circ \pi \circ [\alpha] \circ \overline{\pi}) - \overline{\pi}
\]
\[= \sum_{c \in \mathcal{M} \infty, m \in I} \left( a_c(m)([Z'_c(m)] \times \{c\} [\Theta_c(m)]) + b_c(m)([\Theta_c(m)] \times \{c\} [Z''_c(m)]) \right), \]

for some one-cycles \(Z'_c(m), Z''_c(m)\) on \(\tilde{E}_c\) and rational numbers \(a_c(m), b_c(m)\), with \(\Theta_c(m)\) running over the components of \(\tilde{E}_c\) and \(I\) the indexing described in 2.3.4. Now composing with \(\pi\) on both left and right leaves \((\pi \circ [\beta] \circ \pi \circ [\alpha] \circ \pi) - \pi\) fixed, but on the other terms,
\[
\overline{\pi} \circ ([Z'_c(m)] \times \{c\} [\Theta_c(m)]) \circ \overline{\pi} = \overline{\pi}([Z'_c(m)]) \times \{c\} \pi([\Theta_c(m)]) = 0
\]
\[
\overline{\pi} \circ ([\Theta_c(m)] \times \{c\} [Z''_c(m)]) \circ \overline{\pi} = \overline{\pi}([\Theta_c(m)]) \times \{c\} \pi([Z''_c(m)]) = 0,
\]

since \(\overline{\pi} = \pi\) and \(\overline{\pi}_{0,0}(\tilde{E}/\mathcal{M}) \neq \pi \neq \overline{\pi}_{2,2}(\tilde{E}/\mathcal{M})\), so that Proposition 4.4.1 applies. This proves (6), and concludes the proof of the proposition.

\[\Box\]

**4.6. The motives defined by \(\pi_\infty(E/M)\) and \(\pi_\infty(2\tilde{E}/\mathcal{M})\).** Finally we must analyze the motives defined by \(\pi_\infty(E/M)\) and \(\pi_\infty(2\tilde{E}/\mathcal{M})\). Since these were each defined as the difference between the diagonal and the sum of the \(\pi_i(E/M)\) or \(\tilde{\pi}_{i_1,i_2}(2\tilde{E}/\mathcal{M})\) respectively, it requires some care to get a good grip on them. However, in the end the motives themselves have a rather simple form, as a sum of powers of Lefschetz motives, essentially because all the components of the cusp fibres supporting these projectors are rational varieties.

**Proposition 4.6.1.** As Chow motives in \(\mathcal{M}(K)\),

1. \((E, \pi_\infty(E/M)) \simeq (N - 1)L; \)
2. \((2\tilde{E}, \tilde{\pi}_c^{(2)}(2\tilde{E}/\mathcal{M})) \simeq s\mathbb{L}; \)
3. \((2\tilde{E}, \tilde{\pi}_c^{(4)}(2\tilde{E}/\mathcal{M})) \simeq s\mathbb{L}^2\)

for \(c \in \mathcal{M} \infty\) and some \(0 < s \in \mathbb{Z}\).

**Proof.** We give first the proof for (2) and (3), which come together, and comment at the end on (1), since it can be proved similarly, and even more easily. For convenient reference, recall that \(\tilde{\pi}_c^{(2)}(2\tilde{E}/\mathcal{M})\) and \(\tilde{\pi}_c^{(4)}(2\tilde{E}/\mathcal{M})\) respectively have the form
\[
\tilde{\pi}_c^{(2)}(2\tilde{E}/\mathcal{M}) = \sum_{m \in I} [Z_c(m) \times \{c\} \Theta_c(m)] = \tilde{\pi}_c^{(4)}(2\tilde{E}/\mathcal{M})
\]
where the \(Z_c(m)\) are some one-cycles supported on \(\tilde{E}_c\), about which à priori we know nothing else, and \(I\) is the indexing described in 2.3.4. The proof will proceed in several steps.
Step one. Firstly, we claim that \( m = 0 \), if it occurs, can be eliminated from the expression for \( \tilde{\pi}_c(2\tilde{E}/\tilde{M}) \) and \( \tilde{\pi}_c(4\tilde{E}/\tilde{M}) \), where (as in the proof of Proposition 4.4.1) \( \Theta_c(0) \) denotes the identity component of \( 2\tilde{E}_c \). For observe that in \( \text{CH}^2(2\tilde{E}, \mathbb{Q}) \) the class of the fibre over \( c \in M^\infty \) can be written as

\[
[2\tilde{E}_c] = [\Theta_c(0)] + \sum_{m \neq 0} [\Theta_c(m)],
\]

where the sum runs over all components of \( 2\tilde{E}_c \) other than the identity component. Then using this to give an alternate expression for \( [\Theta_c(0)] \), we rewrite

\[
\tilde{\pi}_c(2\tilde{E}/\tilde{M}) = [Z_c(0)] \times \{ c \} \{ 2\tilde{E}_c \} + \sum_{m \neq 0} [Z'_c(m)] \times \{ c \} [\Theta_c(m)],
\]

for suitable one-cycles \( Z'_c(m) \) supported on \( 2\tilde{E}_c \). Next, there exists \( \Theta_c(c) \in \text{CH}^1(M, \mathbb{Q}) \) rationally equivalent to \( [c] \) but with support disjoint from \( M^\infty \). Since from Proposition 4.4.1 we know that \( \tilde{\pi}_c(2\tilde{E}/\tilde{M}) \) annihilates \( 2\tilde{E}_x \) for any \( x \in M \), it follows that

\[
\tilde{\pi}_c(2\tilde{E}/\tilde{M})([Z'_c(m)]) = \tilde{\pi}_c(2\tilde{E}/\tilde{M})(2\tilde{E}_x)((\Theta_c(c))) = 0.
\]

We also know from Proposition 4.4.1(4) that

\[
\tilde{\pi}_c(2\tilde{E}/\tilde{M})([\Theta_c(m)]) = [\Theta_c(m)] \text{ for } m \neq 0.
\]

Now we compose both sides of (4) with \( \tilde{\pi}_c(2\tilde{E}/\tilde{M}) \) on the left. Since \( \tilde{\pi}_c(2\tilde{E}/\tilde{M}) \) is idempotent, the left-hand side is unchanged. As for the right-hand side, from (5), (6) and the general observation that the composition of a correspondence \( \pi \) with a correspondence of the form \( [Z] \times [T] \) is \( \pi \circ ([Z] \times [T]) = [Z] \times \pi([T]) \), we conclude that

\[
\tilde{\pi}_c(2\tilde{E}/\tilde{M}) = \sum_{m \neq 0} [Z'_c(m)] \times \{ c \} \Theta_c(m)] = 1\tilde{\pi}_c(4\tilde{E}/\tilde{M}),
\]

with \( Z'_c(m) \) as in (4). Thus we have an expression for \( \tilde{\pi}_c(2\tilde{E}/\tilde{M}) \) with no \( m = 0 \) term, as claimed. Indeed, by comparing this with (4), it follows that \([Z_c(0)] = 0\).

Step two. Next, we claim that without loss of generality, we can replace the one-cycles \([Z'_c(m)]\) by one-cycles \([Z''_c(m)]\) with the property that

\[
\tilde{\pi}_c(4\tilde{E}/\tilde{M})([Z''_c(m)]) = [Z''_c(m)],
\]

where, by virtue of step one, \( m \neq 0 \). In fact, if we replace \([Z'_c(m)]\) by

\[
[Z''_c(m)] := [Z'_c(m)] - \tilde{\pi}_f(2\tilde{E}/\tilde{M})([Z'_c(m)])
\]
For any \( \Theta \), the left-hand sides are contained in the right-hand sides because by Proposition 4.4.1, the inclusion of \( \pi_c^{(2)}(\mathcal{E}/\mathcal{M}) \) remains unchanged. For using the orthogonality of \( ^1\pi_f(\mathcal{E}/\mathcal{M}) = \pi_f(\mathcal{E}/\mathcal{M}) \) with \( \pi_c^{(2)}(\mathcal{E}/\mathcal{M}) \), we can write
\[
\pi_c^{(2)}(\mathcal{E}/\mathcal{M}) = \pi_c^{(2)}(\mathcal{E}/\mathcal{M}) - \pi_c^{(2)}(\mathcal{E}/\mathcal{M}) \circ ^1\pi_f(\mathcal{E}/\mathcal{M})
\]
\[
= \sum_{m \neq 0} [Z'_c(m)] \times_{\{c\}} [\Theta_c(m)] - \sum_{m \neq 0} \pi_f(\mathcal{E}/\mathcal{M})([Z'_c(m)]) \times_{\{c\}} [\Theta_c(m)],
\]
from which it follows that
\[
\pi_c^{(2)}(\mathcal{E}/\mathcal{M}) = \sum_{m \neq 0} [Z''_c(m)] \times_{\{c\}} [\Theta_c(m)] = \pi_c^{(2)}(\mathcal{E}/\mathcal{M}).
\]
Furthermore, it’s clear that \( \pi_f(\mathcal{E}/\mathcal{M})([Z''_c(m)]) \) remains unchanged. For any cycle \( \xi \) on \( \mathcal{E}_c \),
\[
\pi_c^{(2)}(\mathcal{E}/\mathcal{M})(\xi) = pr_{2*} \left[ (\xi \times \mathcal{E}) \cdot \left( \sum_{m \neq 0} Z''_c(m) \times \Theta_c(m) \right) \right]
\]
\[
= \sum_{m \neq 0} (\xi \cdot Z''_c(m)) [\Theta_c(m)].
\]
The inclusion of \( \pi_f(\mathcal{E}/\mathcal{M})([Z''_c(m)]) \) in the span of the \( [Z''_c(m)] \) for \( m \neq 0 \) follows similarly.

**Step three.** Next we claim that the Chow groups of the motives defined by \( \pi_c^{(2)}(\mathcal{E}/\mathcal{M}) \) and \( \pi_c^{(4)}(\mathcal{E}/\mathcal{M}) \) respectively are
\[
\text{CH}(\mathcal{E}/\pi_c^{(2)}(\mathcal{E}/\mathcal{M})), \mathcal{Q}) = \text{Span}_\mathcal{Q} \{[\Theta_c(m)] \mid m \neq 0\},
\]
\[
\text{CH}(\mathcal{E}/\pi_c^{(4)}(\mathcal{E}/\mathcal{M})), \mathcal{Q}) = \text{Span}_\mathcal{Q} \{[Z''_c(m)] \mid m \neq 0\},
\]
and thus, in particular, these are finite-dimensional vector spaces. For the right-hand sides are contained in the left-hand sides because by Proposition 4.4.1 \( \pi_c^{(2)}(\mathcal{E}/\mathcal{M}) \) acts on \( [\Theta_c(m)] \) as the identity for \( m \neq 0 \), and similarly, by step two above \( \pi_c^{(4)}(\mathcal{E}/\mathcal{M}) \) acts on \( [Z''_c(m)] \) as the identity, \( m \neq 0 \). On the other hand, \( \text{CH}(\mathcal{E}/\pi_c^{(2)}(\mathcal{E}/\mathcal{M})), \mathcal{Q}) \) is contained in the span of the \( [\Theta_c(m)] \) other than the identity component because for any cycle \( \xi \) on \( \mathcal{E}_c \),
\[
\pi_c^{(2)}(\mathcal{E}/\mathcal{M})(\xi) = pr_{2*} \left[ (\xi \times \mathcal{E}) \cdot \left( \sum_{m \neq 0} Z''_c(m) \times \Theta_c(m) \right) \right]
\]
\[
= \sum_{m \neq 0} (\xi \cdot Z''_c(m)) [\Theta_c(m)].
\]

**Step four.** We claim that the intersection pairing on \( ^2\mathcal{E} \) restricts nondegenerately to a pairing
\[
\text{CH}^1(\mathcal{E}/\pi_c^{(2)}(\mathcal{E}/\mathcal{M})), \mathcal{Q}) \otimes \text{CH}^2(\mathcal{E}/\pi_c^{(4)}(\mathcal{E}/\mathcal{M})), \mathcal{Q}) \to \text{CH}^3(\mathcal{E}, \mathcal{Q}) \simeq \mathcal{Q}.
\]
For any \( \Theta \in \text{CH}^1(\mathcal{E}/\pi_c^{(2)}(\mathcal{E}/\mathcal{M})), \mathcal{Q}) \), consider
\[
[\Theta] = \pi_c^{(2)}(\mathcal{E}/\mathcal{M})([\Theta])
\]
\[
= pr_{2*} \left[ (\Theta \times ^2\mathcal{E}) \cdot \left( \sum_{m \neq 0} [Z''_c(m)] \times \Theta_c(m) \right) \right]
\]
\[
= \sum_{m \neq 0} (\Theta \cdot Z''_c(m)) [\Theta_c(m)].
\]
Thus, unless it is already zero, $[\Theta]$ cannot be orthogonal to all $[Z''(m)]$ for $m \neq 0$. Similarly, no $[Z] \in \text{CH}^2((\tilde{2\tilde{E}}, \pi_c(4)(\tilde{2\tilde{E}}/M)), Q)$ can be orthogonal to all $[\Theta_c(m)]$ for $m \neq 0$.

It also follows that $\text{CH}^1((\tilde{2\tilde{E}}, \pi_c(2)(\tilde{2\tilde{E}}/M)), Q)$ and $\text{CH}^2((\tilde{2\tilde{E}}, \pi_c(4)(\tilde{2\tilde{E}}/M)), Q)$ must have the same dimension.

**Conclusion of the proof for parts (2) and (3).** Now choose any convenient basis for $\text{CH}^1((\tilde{2\tilde{E}}, \pi_c(2)(\tilde{2\tilde{E}}/M)), Q)$, say $\{\omega_l \mid l = 1, \ldots, s\}$, for some $s$, and replace each $[\Theta_c(m)]$ in the last expression for $\pi_c(2)(\tilde{2\tilde{E}}/M)$ by a linear combination of these $\omega_l$. The outcome is then

$$\pi_c(2)(\tilde{2\tilde{E}}/M) = \sum_{l=1}^{s} \zeta_l \times \omega_l = \pi_c(4)(\tilde{2\tilde{E}}/M)$$

for some $\zeta_l \in \text{CH}^2((\tilde{2\tilde{E}}, \pi_c(4)(\tilde{2\tilde{E}}/M)), Q)$. Then for $1 \leq l_0 \leq m$ we have, similarly as above,

$$\omega_{l_0} = \pi_c(2)(\tilde{2\tilde{E}}/M)(\omega_{l_0}) = \text{pr}_{2s}((\omega_{l_0} \times \tilde{2\tilde{E}}) \cdot (\sum_{l=1}^{s} \zeta_l \times \omega_l)) = \sum_{l=1}^{s} (\omega_{l_0} \cdot \zeta_l) \omega_l.$$

But since $\{\omega_l, 1 \leq l \leq s\}$ is a basis of $\text{CH}^1((\tilde{2\tilde{E}}, \pi_c(2)(\tilde{2\tilde{E}}/M)), Q)$, the intersection multiplicity

$$\langle \omega_{l_0} \cdot \zeta_l \rangle = \begin{cases} 1 & \text{when } l = l_0, \\ 0 & \text{when } l \neq l_0. \end{cases}$$

This means that $\{\zeta_l, 1 \leq l \leq s\}$ is the dual basis of $\text{CH}^2((\tilde{2\tilde{E}}, \pi_c(4)(\tilde{2\tilde{E}}/M)), Q)$, and that the individual terms in the expression above for $\pi_c(2)(\tilde{2\tilde{E}}/M)$ and $\pi_c(4)(\tilde{2\tilde{E}}/M)$ are mutually orthogonal idempotents. And as we saw in 1.1.2(c), projectors of this form define powers of Lefschetz motives. Thus the motives defined by $\pi_c(2)(\tilde{2\tilde{E}}/M)$ and $\pi_c(4)(\tilde{2\tilde{E}}/M)$ have the form asserted.

**Proof of part (1).** The proof of part (1) can be carried out in the same way, with a few small differences and simplifications. Starting with the expression

$$\pi_c(\tilde{E}/M) = \sum_{m,n \in \mathbb{Z}/\mathbb{N}\mathbb{Z}} r_c(m,n)[\theta_c(m)] \times_{\{c\}} [\theta_c(n)],$$

for some $r_c(m,n) \in \mathbb{Q}$, the same argument as step one applied twice leads to

$$\pi_c(\tilde{E}/M) = \sum_{m \neq 0, n \neq 0} s_c(m,n)[\theta_c(m)] \times_{\{c\}} [\theta_c(n)],$$
for some $s_c(m,n) \in \mathbb{Q}$. Then steps three and four are replaced and made more precise by [Shioda, 1972, Thm.1.1 and Lemma 1.3], which imply that $\{\theta_c(m)\mid 0 \neq m \in \mathbb{Z}/N\mathbb{Z}\}$ is already algebraically independent and has a nondegenerate intersection matrix $(\langle \theta_c(m) \cdot \theta_c(n) \rangle)$, i.e., of rank $(N-1)$, see remark 2.2.2. From this it follows that $(s_c(m,n))$ is the inverse of the intersection matrix. Then if we rewrite

$$
\pi_c(E/M) = \sum_{0 \neq m \in \mathbb{Z}/N\mathbb{Z}} [\theta_c(m)] \times_{\langle c \rangle} (\sum_{n \neq 0} s_c(m,n)[\theta_c(n)]),
$$
we see $\pi_c(E/M)$ as the sum of $(N-1)$ mutually orthogonal projectors of the form $[A] \times [B]$ with $(A \cdot B) = 1$. This proves part (1), and concludes the proof of the proposition.

4.7. Proof of Theorem 4.2. Now we prove Theorem 4.2. Consider first $\overline{E}$: From Propositions 4.3.2 and 4.6.1 we get

$$
h(E) \simeq (\overline{E}, \pi_0(E/M)) \oplus (\overline{E}, \pi_1(E/M)) \oplus (\overline{E}, \pi_2(E/M)) \oplus (\overline{E}, \pi_\infty(E/M)) \simeq h(M) \oplus 1W \oplus (h(M) \otimes L) \oplus rL,
$$
where it follows from 4.6.1 that $r = (N-1) \cdot \#(M^\infty)$. Then by using that

$$
h(M) \simeq 1 \oplus L \oplus h^1(M),
$$
the decomposition asserted in the statement of the theorem follows. The argument for $h(2\overline{E})$, using in addition Proposition 4.5.1, is entirely similar. □

5. Chow-Künneth decompositions and the cohomology of $\overline{E}$ and $2\overline{E}$

We can now give two proofs of the existence of Chow-Künneth decompositions for $\overline{E}$ and $2\overline{E}$. The first proof very quickly deduces the existence and a description of the Chow-Künneth decompositions for $\overline{E}$ and $2\overline{E}$ from Theorem 4.2 using [Scholl, 1990, Thm.1.2.1] to tell us the cohomology of $1W$ and $2W$. The second proof also starts with Theorem 4.2, but then uses a description of the total cohomology spaces $H_*(\overline{E}, \mathbb{Q}_*)$ and $H_*(2\overline{E}, \mathbb{Q}_*)$ to obtain the Chow-Künneth decompositions for $\overline{E}$ and $2\overline{E}$, and at the same time compute the cohomology of $1W$ and $2W$, i.e., the cases $k = 1$ and $k = 2$ of [Scholl, 1990, Thm.1.2.1].

Recall that a positive integer coefficient on a motive indicates the multiplicity with which that motive, up to isomorphism, occurs.

**Theorem 5.1.** With $m$ and $n$ as in Theorem 4.2,

1. $\overline{E}$ has a Chow-Künneth decomposition, with

$$
\begin{align*}
    h^0(\overline{E}) &\simeq 1, \\
    h^4(\overline{E}) &\simeq L^2, \\
    h^1(\overline{E}) &\simeq h^1(M), \\
    h^2(\overline{E}) &\simeq mL \oplus 1W, \\
    h^3(\overline{E}) &\simeq h^1(M) \otimes L.
\end{align*}
$$
(2) $2\tilde{E}$ has a Chow-Künneth decomposition, with

\[
\begin{align*}
    h^0(2\tilde{E}) &\simeq 1 & h^6(2\tilde{E}) &\simeq \mathbb{L}^3 \\
    h^1(2\tilde{E}) &\simeq h^1(\mathcal{M}) & h^5(2\tilde{E}) &\simeq h^1(\mathcal{M}) \otimes \mathbb{L}^2 \\
    h^2(2\tilde{E}) &\simeq n\mathbb{L} \oplus 2(1W) & h^4(2\tilde{E}) &\simeq n\mathbb{L}^2 \oplus 2(3W \otimes \mathbb{L}) \\
    h^3(2\tilde{E}) &\simeq 3(h^1(\mathcal{M}) \otimes \mathbb{L}) \oplus 2W
\end{align*}
\]

Remark 5.1.1. The existence of Chow-Künneth decompositions for surfaces in general is proved in [Murre, 1990]. Proposition 5.1 describes what it looks like specifically for $\tilde{E}$, and also gives a more refined decomposition for this surface.

5.1.2. The first proof. After Lemma 1.2.5, it is only necessary to verify that all of the submotives given by theorem 4.2 have Chow-Künneth decompositions. It is clear that $L^d$ has a Chow-Künneth decomposition, and easy to see that $h^*(\mathcal{M}) \otimes L^d$ does, as well. But $1W$ and $2W$ also have Chow-Künneth decompositions, for by [Scholl, 1990, Thm.1.2.1],

\[
\begin{align*}
    H_*(1W, \mathbb{Q}_*) &\simeq H^1_*(\mathcal{M}, j_* R^1 \phi_* \mathbb{Q}_*) \subset H^2_*(\mathcal{E}, \mathbb{Q}_*) \\
    H_*(2W, \mathbb{Q}_*) &\simeq H^1_*(\mathcal{M}, j_* \text{Sym}^2 R^1 \phi_* \mathbb{Q}_*) \subset H^3_*(2\tilde{E}, \mathbb{Q}_*)
\end{align*}
\]

which means in particular that the cohomology of $1W$ is purely of weight 2, so $\text{id}_{(1W)} = \pi_2(1W)$ is a Chow-Künneth decomposition for $1W$, and similarly the cohomology of $2W$ is purely of weight 3, so $\text{id}_{(2W)} = \pi_3(2W)$ is a Chow-Künneth decomposition for $2W$. Thus Theorem 4.2 gives $h(\mathcal{E})$ and $h(2\tilde{E})$ respectively as direct sums of motives with Chow-Künneth decompositions, therefore by Lemma 1.2.5, both $\mathcal{E}$ and $2\tilde{E}$ have Chow-Künneth decompositions. By collecting together the components of each given weight, we get the Chow-Künneth decompositions for $\mathcal{E}$ and $2\tilde{E}$ as claimed. \hfill \Box

5.2. The cohomology of $\mathcal{E}$ and $2\tilde{E}$. In the proof just given, the nontrivial cohomology computations were already taken care of by [Scholl, 1990, Thm.1.2.1]. But we can also prove the existence of Chow-Künneth decompositions for $\mathcal{E}$ and $2\tilde{E}$ independently of that theorem, while at the same time computing the cohomology of $1W$ and $2W$, which are the cases $k = 1$ and $k = 2$ of [Scholl, 1990, Thm.1.2.1]. Toward this end, we recall some facts about the cohomology of $\mathcal{E}$ and $2\tilde{E}$.

Proposition 5.2.1.

(1) \[
H_*(\mathcal{E}, \mathbb{Q}_*) \simeq \bigoplus_{p=0}^2 \left( H^p_*(\mathcal{M}, \mathbb{Q}_*) \oplus H^p_*(\mathcal{M}, \mathbb{Q}_*(-1)) \right) \oplus H^1_*(\mathcal{M}, j_* R^1 \phi_* \mathbb{Q}_*) \oplus H^2_*(\mathcal{M}, \mathbb{Q}_*(-1))
\]
where $\mathcal{U}_\infty$ is a skyscraper sheaf supported over $M^\infty$ that contributes to cohomology only in degree 2. Moreover, the intersection form on $\mathcal{E}$ induces perfect pairings

$$H^p_\bullet(M, \mathbb{Q}_s(j)) \otimes H^{2-p}_\bullet(M, \mathbb{Q}_s(-(j+1))) \quad \text{for } 0 \leq p \leq 2$$

$$H^1_\bullet(M, j_* R^1 \phi_* \mathbb{Q}_s) \otimes H^1_\bullet(M, j_* R^1 \phi_* \mathbb{Q}_s)$$

$$H^2_{M^\infty}(M, \mathcal{U}_\infty) \otimes H^2_{M^\infty}(M, \mathcal{U}_\infty)$$

into $H^4_\bullet(\mathcal{E}, \mathbb{Q}_s) \simeq \mathbb{Q}_s(-2)$.

(2) $H_\bullet(2\mathcal{E}, \mathbb{Q}_s) \simeq \bigoplus_{p=0}^2 \left( H^p_\bullet(M, \mathbb{Q}_s) \oplus 3 H^p_\bullet(M, \mathbb{Q}_s(-1)) \oplus H^p_\bullet(M, \mathbb{Q}_s(-2)) \right)$

$$\oplus \left( 2 H^1_\bullet(M, j_* R^1 \phi_* \mathbb{Q}_s) \oplus H^2_\bullet(M, j_* R^1 \phi_* \mathbb{Q}_s(-1)) \right)$$

$$\oplus H^2_{M^\infty}(M, \mathcal{U}_\infty^{(2)}) \oplus H^4_{M^\infty}(M, \mathcal{U}_\infty^{(4)})$$

where $\mathcal{U}_\infty^{(j)}$ is a skyscraper sheaf supported over $M^\infty$ that contributes to cohomology only in degree $j$, for $j = 2,4$. Moreover, the intersection form on $2\mathcal{E}$ induces perfect pairings into $H^6_\bullet(2\mathcal{E}, \mathbb{Q}_s) \simeq \mathbb{Q}_s(-3)$ on the isotypic components corresponding to

$$H^p_\bullet(M, \mathbb{Q}_s(j)) \otimes H^{2-p}_\bullet(M, \mathbb{Q}_s(-(j+2)))$$

$$H^1_\bullet(M, j_* R^1 \phi_* \mathbb{Q}_s(j)) \otimes H^1_\bullet(M, j_* R^1 \phi_* \mathbb{Q}_s(-(j+1)))$$

$$H^2_{M^\infty}(M, \mathcal{U}_\infty^{(2)}) \otimes H^4_{M^\infty}(M, \mathcal{U}_\infty^{(4)})$$

Proof. All of this is well-known, but as we do not know of a convenient reference, we sketch the argument for $2\mathcal{E}$, the argument for $\mathcal{E}$ being similar. Firstly, the decomposition theorem of [Beilinson et al., 1983] implies that

$$H_\bullet(2\mathcal{E}, \mathbb{Q}_s) \simeq \bigoplus_{p=0}^2 \bigoplus_{q=0}^4 H^p_\bullet(M, j_* R^q(2\phi)_s \mathbb{Q}_s) \oplus \bigoplus_{s=1}^2 H^{2s}_{M^\infty}(M, \mathcal{U}_\infty^{(2s)})$$

where $\mathcal{U}_\infty^{(2s)}$ is a skyscraper sheaf supported on $M^\infty$ contributing in degree $2s$, as well as the Poincaré duality pairings

$$H^p_\bullet(M, j_* R^q(2\phi)_s \mathbb{Q}_s) \otimes H^{2-p}_\bullet(M, j_* R^{4-q}(2\phi)_s \mathbb{Q}_s)$$

$$H^2_{M^\infty}(M, \mathcal{U}_\infty^{(2)}) \otimes H^4_{M^\infty}(M, \mathcal{U}_\infty^{(4)})$$

The next observation is that as a sheaf on $M$,

$$R^q(2\phi)_s \mathbb{Q}_s \simeq \bigoplus_{r=0}^2 m(2, q, r) \text{Sym}^r R^1 \phi_* \mathbb{Q}_s(\frac{r-q}{2}),$$
where
\[ m(2, q, r) := \binom{2}{q+r} \binom{2}{q+r} - \binom{2}{q+r} - 1 \left( \binom{2}{q+r} + 1 \right), \]
with the convention that any of these binomial coefficients vanish if its argument is negative or non-integral. This is easily computed by observing that \( R^q(\overline{\phi}_* Q_*) \) is the locally constant sheaf associated to the action of the fundamental group of \( M \) on \( H^q_* (\overline{E_t}, Q_*) \), for general \( t \in M \), and that the fundamental group of \( M \) is a form of \( SL(2) \). Via this last identification, sym\( ^r R^1 \phi_* Q_* \) is the locally constant sheaf associated to the symmetric tensor representation of \( SL(2) \) of degree \( r \). When \( r > 0 \) this is an irreducible representation of dimension greater than 1, so in particular there are no invariants or coinvariants. Therefore \( H^q_* (\overline{M}, j_* \text{Sym}^r R^1 \phi_* Q_*) \) vanishes when \( r > 0 \) and \( p = 0 \) or 2. Furthermore, Schur’s lemma implies that \( j_* \text{Sym}^r R^1 \phi_* Q_*) \) can only be Poincaré dual to a Tate twist of itself, and this completes the proof.

5.3. The second derivation of the Chow-Künneth decompositions of \( \overline{E} \) and \( \overline{2E} \), and computation of the cohomology of \( ^1W \) and \( ^3W \). Using Proposition 5.2.1 we derive the Chow-Künnet decompositions of \( \overline{E} \) and \( \overline{2E} \) without using the result of [Scholl, 1990, Thm.1.2.1], and determine the cohomology of \( ^1W \) and \( ^3W \).

5.3.1. The proof for \( \overline{E} \) and \( ^1W \). We consider first the Chow motive decomposition of \( \overline{E} \) given by Theorem 4.2, and begin by matching the cohomology groups of the constituent motives whose cohomology we know with the constituents of \( H_* (\overline{E}, Q_*) \) as given in Proposition 5.2.1. By matching weights also, we obtain

\[ H_* (1, Q_*) \simeq H^0_* (\overline{M}, Q_*) \simeq H^0_* (\overline{E}, Q_*) \]
\[ H_* (h^1(\overline{M}), Q_*) \simeq H^1_* (\overline{M}, Q_*) \simeq H^1_* (\overline{E}, Q_*) \]
\[ H_* (h^1(\overline{M}) \otimes \mathbb{L}, Q_*) \simeq H^1_* (\overline{M}, Q_*(-1)) \simeq H^2_* (\overline{E}, Q_*) \]
\[ H_* (\mathbb{L}^2, Q_*) \simeq H^2_* (\overline{M}, Q_*(-1)) \simeq H^1_* (\overline{E}, Q_*) \]

It therefore follows that the motives \( ^1W \) and \( (\overline{E}, \pi_\infty (\overline{E}/M)) \) have cohomology purely of weight 2, even if we had not already computed that \( (\overline{E}, \pi_\infty (\overline{E}/M)) \) is isomorphic to a sum of Lefschetz motives. This already proves the existence of a Chow-Kühnnet decomposition for \( \overline{E} \), and that the cohomology of the sum \( ^1W \oplus (\overline{E}, \pi_\infty (\overline{E}/M)) \) must be isomorphic to the sum \( H^1_* (\overline{M}, j_* R^1 \phi_* Q_* ) \oplus H^2_* (\overline{M}, \mathcal{U}_\infty) \). Then to compute the cohomology of \( ^1W = (\overline{E}, \pi_1 (\overline{E}/M)) \), and of \( (\overline{E}, \pi_\infty (\overline{E}/M)) \) as a constituent of \( H_* (\overline{E}, Q_*) \), we observe first that \( \pi_1 (\overline{E}/M) (H^2_* (\overline{M}, \mathcal{U}_\infty)) = 0 \), since \( \mathcal{U}_\infty \) is supported over \( M_\infty \) and \( \pi_1 (\overline{E}/M) \) acts as zero on all components of \( E_\infty \), by Proposition 4.4.1. Therefore

\[ H_* (^1W, Q_*) \subseteq H^1_* (\overline{M}, j_* R^1 \phi_* Q_*) . \]
Conversely, it follows from Lemma 3.2.10 that $H_*((\overline{E}, \pi_\infty(\overline{E} / M)), Q_*)$ is generated by the classes of some $\theta_e(m)$ (modulo homological equivalence), and thus consists entirely of algebraic cohomology classes in $H^2_*(\overline{E}, Q_*(1))$. On the other hand, by virtue of the Galois($K^{sep}/K$)-module structure of $H^1_{et}(M \otimes K^{alg}, j_* R^1 \phi_\ast Q_\ell)$ [Deligne, 1969], or the Hodge structure of $H^1_B(\overline{M}(\mathbb{C})^{an}, j_* R^1 \phi_\ast Q)$ [Shioda, 1972] [Sokurov, 1976 and 1981] [Zucker, 1979], $H^1_*(\overline{M}, j_* R^1 \phi_\ast Q_\ast)$ cannot contain any algebraic cohomology classes. Therefore the only possibility is that $H_*((\overline{W}, Q_\ast) \simeq H^1_*(\overline{M}, j_* R^1 \phi_\ast Q_\ast)$ and $H_*((\overline{E}, \pi_\infty(\overline{E} / M)), Q_*) \simeq H^2_{M \otimes (\mathbb{M}, \mathcal{U}_\infty)}$.

5.3.2. The proof for $2 \overline{E}$ and $2 \mathcal{W}$. The argument computing the Chow-Künneth decomposition of $2 \overline{E}$ and the cohomology of $2 \mathcal{W}$ follows similar lines. From the Chow motive computations in section four, using known cohomology groups and matching weights we get

$$H_* (1, Q_\ast) \simeq H^0_*(\overline{M}, Q_\ast) \simeq H^0_*(2 \overline{E}, Q_\ast)$$

$$H_* (h^1(\overline{M}), Q_\ast) \simeq H^1_*(\overline{M}, Q_\ast) \simeq H^1_*(2 \overline{E}, Q_\ast)$$

$$H_* (h^2(\overline{M}), Q_\ast) \simeq H^2_*(\overline{M}, Q_\ast) \subset H^2_*(2 \overline{E}, Q_\ast)$$

$$H_* (2(\mathcal{W}), Q_\ast) \simeq H^1_*(\overline{M}, j_* R^1 \phi_\ast Q_\ast) \otimes \mathbb{Q} \subset H^2_*(2 \overline{E}, Q_\ast)$$

$$H_* (3(h^0(\overline{M}) \otimes \mathbb{L}), Q_\ast) \simeq H^1_*(\overline{M}, Q_\ast(-1)) \otimes \mathbb{Q} \subset H^2_*(2 \overline{E}, Q_\ast)$$

$$H_* (3(h^1(\overline{M}) \otimes \mathbb{L}), Q_\ast) \simeq H^2_*(\overline{M}, Q_\ast(-1)) \otimes \mathbb{Q} \subset H^3_*(2 \overline{E}, Q_\ast)$$

$$H_* (3(h^2(\overline{M}) \otimes \mathbb{L}), Q_\ast) \simeq H^2_*(\overline{M}, Q_\ast(-1)) \otimes \mathbb{Q} \subset H^4_*(2 \overline{E}, Q_\ast)$$

$$H_* (2(\mathcal{W} \otimes \mathbb{L}), Q_\ast) \simeq H^1_*(\overline{M}, j_* R^1 \phi_\ast Q_\ast(-1)) \otimes \mathbb{Q} \subset H^4_*(2 \overline{E}, Q_\ast)$$

$$H_* (h^0(\overline{M}) \otimes \mathbb{L}^2, Q_\ast) \simeq H^1_*(\overline{M}, Q_\ast(-2)) \subset H^4_*(2 \overline{E}, Q_\ast)$$

$$H_* (h^1(\overline{M}) \otimes \mathbb{L}^2, Q_\ast) \simeq H^1_*(\overline{M}, Q_\ast(-2)) \simeq H^5_*(2 \overline{E}, Q_\ast)$$

$$H_* (h^2(\overline{M}) \otimes \mathbb{L}^2, Q_\ast) \simeq H^1_*(\overline{M}, Q_\ast(-2)) \simeq H^6_*(2 \overline{E}, Q_\ast).$$

Therefore the cohomology of the sum of motives $2 \mathcal{W} \oplus (2 \overline{E}, \pi_\infty(2 \overline{E} / M)) \oplus (2 \overline{E}, \pi_\infty(2 \overline{E} / M))$ is the sum of the cohomology groups $H^1_*(\overline{M}, j_* Sym^2 \phi_\ast Q_\ast) \oplus H^2_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(2))} \oplus H^4_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(4))}$. Then by Proposition 4.4.1 $S_2 \bar{\pi}_{1,1}(2 \overline{E} / M)$ annihilates (the classes of) the components of $2 \overline{E} \otimes \mathbb{Q}$, which means that $H^4_*(2 \mathcal{W}, Q_\ast)$ is disjoint from $H^2_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(2))}$. Therefore we have $H_* ((2 \overline{E}, \pi_\infty(2 \overline{E} / M)), Q_\ast) \simeq H^2_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(2))}$. But we also know not only that $H^4_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(2))}$ pairs nondegenerately with $H^1_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(4))}$, but also that the Chow groups of $(2 \overline{E}, \pi_\infty(2 \overline{E} / M))$ and $(2 \overline{E}, \pi_\infty(2 \overline{E} / M))$ pair nondegenerately, by step four in the proof of Proposition 4.6.1, so we must also have that $H_* ((2 \overline{E}, \pi_\infty(2 \overline{E} / M)), Q_\ast) \simeq H^4_{M \otimes (\mathbb{M}, \mathcal{U}_\infty(4))}$. 

Therefore the only remaining possibility is that
\[ H_\ast(\mathcal{E}, \mathbb{Q}) \simeq H_\ast(M, j_\ast \text{Sym}^2 \phi_\ast \mathbb{Q}), \]
as claimed, from which it also follows that \( \mathcal{E} \) has a Chow-Künneth decomposition. \( \square \)

6. The filtration on the Chow groups of \( \mathcal{E} \) and \( \mathcal{E} \)

Recall that Conjecture A predicts the existence of a Chow-Künneth decomposition; for \( \mathcal{E} \) and \( \mathcal{E} \) this is proved in Theorem 5.1 (see also Theorem 4.2). In this section we start with those Chow-Künneth decompositions, and then for \( \mathcal{E} \) and \( \mathcal{E} \) we prove Conjectures B, that \( \text{CH}_j(X,\mathbb{Q}) = 0 \) for \( i < j \) or \( i > 2j \), and D, that \( F^1 \text{CH}^j(X,\mathbb{Q}) = \text{CH}^j_{\text{hom}}(X,\mathbb{Q}) \), and a large part of Conjecture C, that the filtration on the Chow groups induced by these Chow-Künneth decompositions is the natural one. Although the conjectures have been proved for surfaces in general [Murre, 1990], here we give a different proof for \( \mathcal{E} \), using the extra structure that Theorems 5.1 and 4.2 reveal. In particular, we find the Chow groups of \( \mathcal{E} \), which we then use in the proof of Conjectures B and D for the threefold \( \mathcal{E} \). As for proving Conjecture C for \( \mathcal{E} \), precise statements are given in Theorem 6.2 below, but our results may be summarized by observing first that it is trivially true for \( \text{CH}^0(\mathcal{E},\mathbb{Q}) \), and it is equivalent to Conjecture D, which we prove, for \( \text{CH}^1(\mathcal{E},\mathbb{Q}) \); but then we also prove that \( F^1 \text{CH}^j(\mathcal{E},\mathbb{Q}) = \text{CH}^j_{\text{hom}}(\mathcal{E},\mathbb{Q}) \) for \( j = 2, 3 \), and that \( F^2 \text{CH}^3(\mathcal{E},\mathbb{Q}) = \text{CH}^3_{\text{Alb}}(\mathcal{E},\mathbb{Q}) \). So what’s missing is \( F^2 \text{CH}^2(\mathcal{E},\mathbb{Q}) \), which is contained in the kernel of an Abel-Jacobi map defined on \( \text{CH}^2_{\text{hom}}(\mathcal{W},\mathbb{Q}) \) (Proposition 6.5.6), and \( F^3 \text{CH}^3(\mathcal{E},\mathbb{Q}) \), which we show equals \( \text{CH}^3(\mathcal{W},\mathbb{Q}) \) (Proposition 6.6.1).

6.1. Notation. With the present state of knowledge about Chow groups we can at best prove the naturality of a step in the filtration when there is a clear, geometrically described candidate for it. If there are such natural candidates and if the filtration is this natural one, then by abuse of language we will say that Conjecture C is true. For a smooth projective variety \( X \) over a field \( k \), we have
\[ \text{CH}^j_{\text{hom}}(X,\mathbb{Q}) := \text{Ker}\{ \gamma : \text{CH}^j(X,\mathbb{Q}) \to H^{2j}_\ast(X,\mathbb{Q}(j)) \}, \]
where \( \gamma \) is the cycle class map. Further, let
\[ \text{CH}^d_{\text{Alb}}(X,\mathbb{Q}) := \text{Ker}\{ \text{Alb} : \text{CH}^d_{\text{hom}}(X,\mathbb{Q}) \to \text{Alb}(X) \otimes \mathbb{Q} \}, \]
where \( \text{Alb}(X) \) is the Albanese of \( X \) and \( d = \dim X \). Finally, supposing for simplicity that \( \text{char} k = 0 \), let
\[ \text{CH}^j_{\text{AJ}}(X,\mathbb{Q}) := \text{Ker}\{ \text{AJ} : \text{CH}^j_{\text{hom}}(X,\mathbb{Q}) \to J^j(X) \otimes \mathbb{Q} \}, \]
where \( AJ \) is the Abel-Jacobi map to the \( j \)-th intermediate Jacobian \( J^j(X) \).
Theorem 6.2.
(1) For the Chow-Künneth decomposition of $\overline{E}$ described in Theorem 5.1(1) we have
(i) $\text{CH}^j(h^i(\overline{E}),\mathbb{Q}) = 0$ for $i < j$ or $i > 2j$, i.e., Conjecture B is true for $\overline{E}$;
(ii) $F^1 \text{CH}^j(\overline{E},\mathbb{Q}) = \text{CH}^j_{\text{hom}}(\overline{E},\mathbb{Q})$ for $1 \leq j \leq 2$, i.e., Conjecture D is true for $\overline{E}$;
(iii) $F^2 \text{CH}^2(\overline{E},\mathbb{Q}) = \text{CH}^2_{\text{Alb}}(\overline{E},\mathbb{Q})$, and therefore the filtration is independent of the choice of Chow-Künneth projectors $\pi_i(\overline{E})$, i.e., Conjecture C is true for $\overline{E}$.
In particular, for the Chow groups of $W$ we have
$\text{CH}^0(W,\mathbb{Q}) = \text{CH}^1(W,\mathbb{Q}) = 0$
$\text{CH}^2(W,\mathbb{Q}) = F^2 \text{CH}^2(W,\mathbb{Q}) = \text{CH}^2_{\text{Alb}}(\overline{E},\mathbb{Q})$.

(2) For the Chow-Künneth decomposition of $\overline{\tilde{E}}$ described in Theorem 5.1(2) we have
(i) $\text{CH}^j(h^i(\overline{\tilde{E}}),\mathbb{Q}) = 0$ for $i < j$ or $i > 2j$, i.e., Conjecture B is true for $\overline{\tilde{E}}$;
(ii) $F^1 \text{CH}^j(\overline{\tilde{E}},\mathbb{Q}) = \text{CH}^j_{\text{hom}}(\overline{\tilde{E}},\mathbb{Q})$ for $1 \leq j \leq 3$, i.e., Conjecture D is true for $\overline{\tilde{E}}$.
(iii) Towards Conjecture C we also have
(a) $F^2 \text{CH}^2(\overline{\tilde{E}},\mathbb{Q}) \subseteq \text{CH}^2_{\text{Alb}}(\overline{\tilde{E}},\mathbb{Q})$, when $\text{char } K = 0$.
(b) $F^2 \text{CH}^3(\overline{\tilde{E}},\mathbb{Q}) = \text{CH}^3_{\text{Alb}}(\overline{\tilde{E}},\mathbb{Q})$.
In particular, for the Chow groups of $W$ we have
$\text{CH}^0(W,\mathbb{Q}) = \text{CH}^1(W,\mathbb{Q}) = 0$
$\text{CH}^2(W,\mathbb{Q}) = F^1 \text{CH}^2(W,\mathbb{Q}) = \text{CH}^2_{\text{hom}}(W,\mathbb{Q})$
$\text{CH}^3(W,\mathbb{Q}) = F^3 \text{CH}^3(W,\mathbb{Q}) = F^3 \text{CH}^3(\overline{\tilde{E}},\mathbb{Q})$.

6.3. Preliminaries to the proof of Theorem 6.2. Before getting into the proof of Theorem 6.2, we begin with some elementary but useful observations.

6.3.1. The conjectures for $\text{CH}^0(X,\mathbb{Q})$. For any smooth projective $X$ with a Chow-Künneth decomposition, $\text{CH}^0(X,\mathbb{Q})$ trivially satisfies Conjectures B, C and D. For $\text{CH}^0(X,\mathbb{Q}) \otimes \mathbb{Q}_s = H^0_s(X,\mathbb{Q}_s)$, from which it follows that $\pi_0(X)$ is the identity on $\text{CH}^0(X,\mathbb{Q})$. Then by orthogonality, $\pi_i(X)(\text{CH}^0(X,\mathbb{Q})) = 0$ for $i > 0$.

6.3.2. The Chow groups of a motive. Recall from the definitions in section 1 that for any Chow motive $M_0$ we have
$\text{CH}^j(M_0 \otimes L^m,\mathbb{Q}) = \text{CH}^{j-m}(M_0,\mathbb{Q})$.  


6.3.3. The Chow groups of $\text{Spec } K$ and $h^1(M)$. Two special cases of 6.3.2 which we will use in the proof of Theorem 6.2 are

$$\text{CH}^j(L^m, \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}, & \text{if } j = m, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\text{CH}^j(h^1(M) \otimes L^m, \mathbb{Q}) \simeq \begin{cases} \text{Jac}(M) \otimes \mathbb{Q}, & \text{if } j = m + 1, \\ 0, & \text{otherwise} \end{cases}.$$ 

Moreover, these motives satisfy Conjectures B and D, in an obvious sense (cf. Proposition 1.2.5), and they satisfy Conjecture C in the sense that the filtrations on their Chow groups are the natural ones.

6.3.4. The motives of $1W$ and $2W$. Given the Chow-Künneth decompositions in Theorem 5.1, the previous paragraph together with Lemma 1.2.5 imply that to prove Conjectures B and D for $E$ and $2\tilde{E}$ it would suffice to prove them for $1W$ and $2W$ (with the obvious understanding of what that means). However, as in section 5.3, the reality is that it works the other way around: Anything nontrivial that we are able to say about the Chow groups of $1W$ and $2W$ comes indirectly, via analyzing the Chow groups of $E$ and $2\tilde{E}$. As Chow motives, we have

$$1W \simeq h^2(1W), \quad 1W \otimes L \simeq h^4(1W \otimes L), \quad 2W \simeq h^3(2W),$$

since these motives have cohomology only in these degrees, see 5.1 and 5.3. Then by applying 6.3.1 for $E$ and $2\tilde{E}$ we find that

$$\text{CH}^0(1W, \mathbb{Q}) = \text{CH}^0(2W, \mathbb{Q}) = 0.$$ 

6.3.5. Organization of the proof. The rest of this section is devoted to the proof of Theorem 6.2. In the next subsection we consider $\text{CH}^1(E, \mathbb{Q})$ and $\text{CH}^1(2\tilde{E}, \mathbb{Q})$, in the following, $\text{CH}^2(E, \mathbb{Q})$ and $\text{CH}^2(2\tilde{E}, \mathbb{Q})$, and in the last, $\text{CH}^3(2\tilde{E}, \mathbb{Q})$.

6.4. Analysis of $\text{CH}^1(E, \mathbb{Q})$ and $\text{CH}^1(2\tilde{E}, \mathbb{Q})$. As the proof of Conjectures B, C and D is the same for both $\text{CH}^1(E, \mathbb{Q})$ and $\text{CH}^1(2\tilde{E}, \mathbb{Q})$, the details are written out only for $2\tilde{E}$. The proof is based on two lemmas, the first of which describes a general approach to verifying the conjectures for $\text{CH}^1(M_0, \mathbb{Q})$ for any Chow motive $M_0$, while the second identifies the Picard (as well as the Albanese) variety of an elliptic modular variety with the Jacobian of the elliptic modular curve over which it lies.

**Lemma 6.4.1.** Let $M_0$ be a Chow motive in $\mathcal{M}(k)$, and assume $M_0$ has a Chow-Künneth decomposition. Suppose that $\pi_1(M_0)(\xi) = \xi$ for all $\xi \in \text{CH}^1_{\text{hom}}(M_0, \mathbb{Q})$. Then for $\xi \in \text{CH}^1(M_0, \mathbb{Q})$,

1. $\xi = \pi_1(M_0)(\xi) + \pi_2(M_0)(\xi)$.
2. $\pi_i(M_0)(\xi) = 0$ for $i \neq 1, 2$.
3. $\text{Ker } \pi_2(M_0) = \text{CH}^2_{\text{hom}}(M_0, \mathbb{Q})$. 


Proof. To begin with,
\[ \xi - \pi_2(M_0)(\xi) \in \text{Ker} \pi_2(M_0) \subseteq \text{CH}^1_{\text{hom}}(M_0, \mathbb{Q}), \]
see 1.2.3. Then applying the hypothesis,
\[ \xi - \pi_2(M_0)(\xi) = \pi_1(M_0)(\xi - \pi_2(M_0)(\xi)) = \pi_1(M_0)(\xi), \]
where the second equality follows from the orthogonality of \( \pi_1(M_0) \) and \( \pi_i(M_0) \).
This proves part (1), and part (2) follows from the mutual orthogonality of all the Chow-Künneth projectors. To prove part (3), if \( \xi \in \text{CH}^1_{\text{hom}}(M_0, \mathbb{Q}) \), then \( \xi = \pi_1(M_0)(\xi) \) by assumption, and therefore \( \pi_2(M_0)(\xi) = 0 \), once more by orthogonality. \( \square \)

A special case of the following lemma already occurs in [Shioda, 1972, p.24].

Lemma 6.4.2.

(1) \( \text{Pic}^0(\overline{E}) \simeq \text{Pic}^0(2\overline{E}) \simeq \text{Pic}^0(M) = \text{Jac}(M) \)
(2) \( \text{Alb}(\overline{E}) \simeq \text{Alb}(2\overline{E}) \simeq \text{Alb}(M) = \text{Jac}(M) \)

Proof. Consider for instance \( 2\overline{E} \). Letting \( \tilde{\alpha}(0) : \overline{M} \to 2\overline{E} \) denote the extended identity section, then
\[ \tilde{\alpha}(0)^* \circ (\tilde{\phi})^* : \text{Jac}(\overline{M}) \to \text{Pic}^0(2\overline{E}) \to \text{Jac}(\overline{M}) \]
is the identity map. Then (1) follows from the fact that \( \dim H^1_*(2\overline{E}, \mathbb{Q}_*) = \dim H^1_*(\overline{M}, \mathbb{Q}_*) \), see Proposition 5.2.1. By duality, (2) follows as well. The argument is the same for \( \overline{E} \). \( \square \)

Proposition 6.4.3. Conjectures B, C and D are true for \( \text{CH}^1(\overline{E}, \mathbb{Q}) \) and \( \text{CH}^1(2\overline{E}, \mathbb{Q}) \).

Proof. Consider for instance \( 2\overline{E} \). From Lemma 6.4.2 we get the following commutative diagram.

\[
\begin{array}{ccc}
\text{CH}^1_{\text{hom}}(2\overline{E}, \mathbb{Q}) & \xrightarrow{\pi_1(2\overline{E})} & \text{CH}^1_{\text{hom}}(2\overline{E}, \mathbb{Q}) \\
\| & & \| \\
\text{Pic}^0(2\overline{E}) \otimes \mathbb{Q} & \xrightarrow{\pi_1(2\overline{E})} & \text{Pic}^0(2\overline{E}) \otimes \mathbb{Q} \\
(\tilde{\phi})^* \sim & & (\tilde{\phi})^* \sim \\
\text{Jac}(\overline{M}) \otimes \mathbb{Q} & \sim_{\pi_1(\overline{M})} & \text{Jac}(\overline{M}) \otimes \mathbb{Q}
\end{array}
\]

Therefore we can apply Lemma 6.4.1, with \( M_0 = h(2\overline{E}) \), and the conclusions of that lemma give the Conjectures B, C and D for \( \text{CH}^1(2\overline{E}) \). The same argument works for \( \overline{E} \). \( \square \)
Corollary 6.4.4.

1. \( \text{CH}^1(1W, \mathbb{Q}) = 0 \).

2. \( \text{CH}^1(2W, \mathbb{Q}) = 0 \).

Proof. Consider the cycle map \( \gamma : \text{CH}^1(1W, \mathbb{Q}) \to H^2_*(\mathcal{E}, \mathbb{Q}_*) \). Since from 5.3.1 we know that \( 1W \) has no algebraic cohomology, \( \text{CH}^1(1W, \mathbb{Q}) \subset \text{CH}^1_{\text{hom}}(\mathcal{E}, \mathbb{Q}) \). But \( \pi_2(\mathcal{E}) \), which acts as the identity on \( 1W \), also acts as zero on \( \text{CH}^1_{\text{hom}}(\mathcal{E}, \mathbb{Q}) \). Hence \( \text{CH}^1(1W, \mathbb{Q}) = 0 \).

Similarly but even easier, \( \text{CH}^1(2W, \mathbb{Q}) = 0 \) because the identity of \( 2W \) is a part of \( \pi_3(\tilde{\mathcal{E}}) \), which acts as zero on \( \text{CH}^1(2\tilde{\mathcal{E}}, \mathbb{Q}) \). \( \Box \)

6.5. Analysis of \( \text{CH}^2(\mathcal{E}, \mathbb{Q}) \) and \( \text{CH}^2(\tilde{\mathcal{E}}, \mathbb{Q}) \).

6.5.1. The Albanese kernel. For any smooth projective variety \( X \) of dimension \( d \) the Chow-K"unneth projector \( \pi_{2d-1}(X) \) acts as the identity on the Albanese variety \( \text{Alb}(X) \) [Murre, 1990] (this is proved by looking at the torsion points). Hence from the commutative diagram

\[
\begin{array}{ccc}
\text{CH}^d_{\text{hom}}(X, \mathbb{Q}) & \xrightarrow{\pi_{2d-1}} & \text{CH}^d_{\text{hom}}(X, \mathbb{Q}) \\
\text{Alb} \downarrow & & \downarrow \text{Alb} \\
\text{Alb}(X) \otimes \mathbb{Q} & \xrightarrow{\sim} & \text{Alb}(X) \otimes \mathbb{Q}
\end{array}
\]

it follows that \( \text{Ker}(\pi_{2d-1}(X)) \subseteq \text{CH}^d_{\text{Alb}}(X, \mathbb{Q}) \); this may be compared with 1.2.3.

Proposition 6.5.2. Conjectures B, C and D are true for \( \text{CH}^2(\mathcal{E}, \mathbb{Q}) \). Moreover

\( \text{CH}^1(1W, \mathbb{Q}) = \text{CH}^2(1W, \mathbb{Q}) = \text{CH}^2_{\text{Alb}}(\mathcal{E}, \mathbb{Q}). \)

Proof. Consider the Chow-K"unneth decomposition of \( \mathcal{E} \) in Theorem 5.1(1) (see also Theorem 4.2): Other than \( 1W \), all the submotives of \( h(\mathcal{E}) \) are of the form \( L^m \) or \( h^1(M) \otimes \mathbb{L}^m \), and thus satisfy the conjectures, as in 6.3.3. Thus, by Lemma 1.2.5, to prove Conjecture B for \( \text{CH}^2(\mathcal{E}, \mathbb{Q}) \) it suffices to verify that \( \pi_0(\mathcal{E}) \) and \( \pi_1(\mathcal{E}) \) act as zero on \( \text{CH}^2(1W, \mathbb{Q}) \). But this is immediate, since by our construction of the Chow-K"unneth decomposition in Theorem 5.1 we have that \( \text{id}(1W) \) is orthogonal to \( \pi_0(\mathcal{E}) \) and \( \pi_1(\mathcal{E}) \). Thus Conjecture B follows. Moreover, as \( \text{id}(1W) \) is part of \( \pi_2(\mathcal{E}) \) we have

\[
\text{CH}^2(\mathcal{E}, \mathbb{Q}) \simeq \text{CH}^2(h^2(\mathcal{E}), \mathbb{Q}) \oplus \text{CH}^2(h^3(\mathcal{E}), \mathbb{Q}) \oplus \text{CH}^2(h^4(\mathcal{E}), \mathbb{Q})
\]

\[
\simeq \text{CH}^2(1W, \mathbb{Q}) \oplus \text{Jac}(M) \otimes \mathbb{Q} \oplus \mathbb{Q}.
\]

To prove Conjecture D, observe that an element \( \alpha \in \text{CH}^2(\mathcal{E}, \mathbb{Q}) \) is contained in \( \text{CH}^2_{\text{hom}}(\mathcal{E}, \mathbb{Q}) \) if and only if the cycle class map acts on the component of \( \alpha \) in
Conjectures B and D are true for Proposition 6.5.3.

Consider the Chow-Künneth decomposition of \( \text{CH}^{2}(E, \mathbb{Q}) \) described in Theorem 6.2. To prove Conjecture B for \( \text{CH}^{2}(E, \mathbb{Q}) \) it suffices to check that \( \pi_{3}(E) \) acts as zero on both \( \text{CH}^{2}(\mathbb{W} \otimes \mathbb{L}, \mathbb{Q}) \) and \( \text{CH}^{2}(\mathbb{W}, \mathbb{Q}) \), for \( i < 2 \) or \( i > 4 \). But this is true, since by our Chow-Künneth decomposition \( \text{id}_{(1\mathbb{W} \otimes \mathbb{L})} \) and \( \text{id}_{(2\mathbb{W})} \) are both orthogonal to these \( \pi_{i}(E) \). Moreover, \( \text{id}_{(1\mathbb{W} \otimes \mathbb{L})} \) is part of \( \pi_{3}(E) \) and \( \text{id}_{(2\mathbb{W})} \) is part of \( \pi_{3}(E) \).

To prove Conjecture D we must show that equality holds in the inclusion \( \text{Ker}(\pi_{3}(E)) \subseteq \text{CH}^{2}_{\text{hom}}(E, \mathbb{Q}) \): it suffices to see that the cycle class map \( \gamma \) is injective on \( \text{CH}^{2}(h^{4}(E), \mathbb{Q}) \). But from Theorem 5.1 we know that \( h^{4}(E) = 2(\mathbb{W} \otimes \mathbb{L}) \oplus n\mathbb{L}^{2} \). Then from 6.3.2 and Corollary 6.4.4 we find that \( \text{CH}^{2}(\mathbb{W} \otimes \mathbb{L}, \mathbb{Q}) = \text{CH}^{1}(\mathbb{W}, \mathbb{Q}) = 0 \), whereas from 6.3.3 and the definitions we get that \( \text{CH}^{2}(\mathbb{L}, \mathbb{Q}) = \text{CH}^{0}(\text{Spec} \ K, \mathbb{Q}) \), on which \( \gamma \) is injective. Conjecture D follows.

6.5.4. The Abel-Jacobi kernel. Let \( X \) be a smooth projective threefold over a field \( k \), and assume for simplicity \( \text{char} \ k = 0 \). Then when \( X \) has a Chow-Künneth decomposition that satisfies Conjectures B and D, there is a commutative diagram

\[
\begin{array}{ccc}
\text{CH}^{2}_{\text{hom}}(X, \mathbb{Q}) & \xrightarrow{\pi_{3}(X)} & \text{CH}^{2}_{\text{hom}}(X, \mathbb{Q}) \\
\text{AJ}_{X} \downarrow & & \downarrow \text{AJ}_{X} \\
\text{J}^{2}(X) \otimes \mathbb{Q} & \xrightarrow{\sim} & \text{J}^{2}(X) \otimes \mathbb{Q}
\end{array}
\]

where \( \text{J}^{2}(X) \) is the intermediate Jacobian; the lower homomorphism is an isomorphism because algebraic correspondences respect Hodge structure and \( \pi_{3}(X) \) is an
isomorphism on $H^3_B(X, \mathbb{Q}_B)$ (which is the starting point for the construction of $J^2(X)$). From the diagram it follows that

$$\text{Ker}(\pi_3(X)) \subseteq \text{Ker}(AJ_X),$$

or, equivalently,

$$F^2 \text{CH}^2(X, \mathbb{Q}) \subseteq \text{CH}^2_{AJ}(X, \mathbb{Q});$$

this may be compared with 1.2.3 and 6.5.1.

**Conjecture 6.5.5.** When $X$ is a smooth projective threefold over a field $k$ of characteristic zero, and there exists a Chow-Künneth decomposition for $X$ such that $\text{CH}^i(h^j(X, \mathbb{Q})) = 0$ for $i < j$ or $i > 2j$, and $F^1 \text{CH}^2(X, \mathbb{Q}) = \text{CH}^2_{\text{hom}}(X, \mathbb{Q})$, then $F^2 \text{CH}^2(X, \mathbb{Q}) = \text{CH}^2_{AJ}(X, \mathbb{Q})$ (or equivalently, $\text{Ker}(\pi_3(X)) = \text{Ker}(AJ_X)$).

**Proposition 6.5.6.**

1. $\text{CH}^2(2W, \mathbb{Q}) = \text{CH}^2_{\text{hom}}(2W, \mathbb{Q})$.
2. Assume $\text{char} \ K = 0$. Then there is a map $AJ(2E) : CH^2(2W, \mathbb{Q}) \rightarrow J^2(2\tilde{E}) \otimes \mathbb{Q}$ and $F^2 \text{CH}^2(2\tilde{E}, \mathbb{Q}) = \text{CH}^2_{AJ}(2\tilde{E}, \mathbb{Q})$ if and only if $AJ(2W)$ is injective.

**Proof.** The first statement follows directly from the definitions and the fact that $\text{id}(2W) = \pi_3(2W)$, as observed in 6.3.4. Then the existence of $AJ(2W)$ comes by composing $AJ(2E)$ with $S_2 \tilde{\pi}_{1,1} = \text{id}(2W)$. To prove the last statement of part (2), we first note that $h^3(2\tilde{E}) = 2W \oplus 3(h^1(M) \otimes L)$, by Theorem 5.2. and next that $AJ(2E)$ is injective on the summand

$$\text{CH}^2(h^1(M) \otimes L, \mathbb{Q}) \cong \text{CH}^1(h^1(M), \mathbb{Q}) \cong (\text{Jac}(M) \otimes \mathbb{Q}) \otimes \mathbb{Q},$$

since it coincides with (three copies of) the usual map from divisors on a curve to the Jacobian. 

**6.6. Analysis of $\text{CH}^3(2\tilde{E}, \mathbb{Q})$.**

**Proposition 6.6.1.**

1. Conjectures B and D are true for $\text{CH}^3(2\tilde{E}, \mathbb{Q})$.
2. $F^2 \text{CH}^3(2\tilde{E}, \mathbb{Q}) = \text{CH}^3_{\text{Alb}}(2\tilde{E}, \mathbb{Q})$.
3. $\text{CH}^3(2W, \mathbb{Q}) = F^3 \text{CH}^3(2\tilde{E}, \mathbb{Q})$.

**Proof.** For part (1), consider the Chow-Künneth decomposition of $2\tilde{E}$ in Theorem 5.1(2) (see also Theorem 4.2): In view of 6.3.3 and Proposition 1.2.5, to prove Conjecture B for $\text{CH}^3(2\tilde{E}, \mathbb{Q})$, we need only verify it for $1W$, $1W \otimes L$ and $2W$. But $\text{CH}^3(1W, \mathbb{Q}) = 0$. Thus for Conjecture B to be true we must have
\[ \pi_i(1^W \otimes L)(CH^3(1^W \otimes L, Q)) = 0 \text{ for } 0 \leq i \leq 2, \text{ which is the case since } \text{id}_{1^W \otimes L} \text{ is orthogonal to } \tilde{\pi}_i(2\tilde{E}) \text{ for } i < 3 \text{ and moreover is part of } \pi_4(2\tilde{E}). \]

We must also have that \( \pi_i(2W)(CH^3(2W, Q)) = 0 \text{ for } 0 \leq i \leq 2, \text{ which is the case since } \text{id}_{2W} \text{ is orthogonal to } \tilde{\pi}_i(2\tilde{E}) \text{ for } i < 3 \text{ and moreover is part of } \pi_3(2\tilde{E}). \)

Conjecture D follows for \( CH^3(2\tilde{E}, Q) \) similarly as for \( CH^2(E, Q) \): \( \text{Ker}(\pi_6(2\tilde{E})) \subseteq CH^3_{\text{hom}}(2\tilde{E}, Q) \)

\[ \text{Ker}(\pi_6(2\tilde{E})) \subseteq CH^3_{\text{hom}}(2\tilde{E}, Q) \]

\[ \text{Ker}(\pi_6(2\tilde{E})) \subseteq CH^3_{\text{hom}}(2\tilde{E}, Q) = \text{CH}^3(L^3, Q) \]

\[ \sim \rightarrow Q \]

is the degree map.

To prove (2), first observe that by 6.5.1 \( \text{Ker}(\pi_5(2\tilde{E})) \subseteq CH^3_{\text{Alb}}(2\tilde{E}, Q) \).

But here we have equality because of the commutative diagram

\[ \begin{array}{ccc}
\text{CH}^3_{\text{Alb}}(h^5(2\tilde{E}), Q) & \xrightarrow{\sim} & \text{CH}^3(h^1(\overline{M}) \otimes L^2, Q) \simeq \text{CH}^1(h^1(\overline{M}), Q) \\
\downarrow \text{Alb} & & \downarrow \text{Alb} \\
\text{Alb}(2\tilde{E}) \otimes Q & \xrightarrow{\sim} & J(\overline{M}) \otimes Q
\end{array} \]

where the the top row is isomorphism from Theorem 5.1 and the bottom row is an isomorphism by Lemma 6.4.2(2).

Part (3) follows from observing that

\[ F^3 \text{CH}^3(2\tilde{E}, Q) = \text{CH}^3(h^3(2\tilde{E}), Q) = \text{CH}^3(3(h^1(\overline{M}) \oplus 2W, Q), \]

and

\[ \text{CH}^3(h^1(\overline{M}) \otimes L, Q) = \text{CH}^2(h^1(\overline{M}), Q) = 0. \]

\[ \square \]

**Remark 6.6.2.** We remark that

\[ F^2 \text{CH}^3(2\tilde{E}, Q)/F^3 \text{CH}^3(2\tilde{E}, Q) = \text{CH}^3(h^4(2\tilde{E}), Q) \simeq \text{CH}^2_{\text{Alb}}(\overline{E}, Q)^\oplus 2. \]

For by Theorem 5.2 (see also Theorem 4.2)

\[ h^4(2\tilde{E}) \simeq 2h^4(1W \otimes L) \oplus h^4(nL^2). \]

Then by 6.3.2,

\[ \text{CH}^3(h^4(2\tilde{E}), Q) \simeq \text{CH}^3(h^4(1W \otimes L), Q)^\oplus 2 \oplus \text{CH}^3(h^4(L^2), Q)^\oplus n \]

\[ \simeq \text{CH}^2(1W, Q)^\oplus 2 \]

\[ \simeq \text{CH}^2_{\text{Alb}}(\overline{E}, Q)^\oplus 2 \]

by Proposition 6.5.2.
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