DEFORMATIONS OF WEIGHTED HOMOGENEOUS POLYNOMIALS
WITH LINE SINGULARITIES AND EQUIMULTIPLICITY

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ABSTRACT. Consider a family \{f_t\} of complex polynomial functions with line singularities and assume that \(f_0\) is weighted homogeneous. We investigate conditions on the members \(f_t\) of the family that guarantee equimultiplicity. In particular we positively answer the Zariski multiplicity conjecture for new classes of line singularities.

1. INTRODUCTION

Let \(z := (z_1, \ldots, z_n)\) be linear coordinates for \(\mathbb{C}^n (n \geq 2)\), and let
\[ f_0: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \; z \mapsto f_0(z), \]
be a weighted homogeneous polynomial function. We suppose that \(f_0\) is reduced at 0. A deformation of \(f_0\) is a polynomial function
\[ f: (\mathbb{C} \times \mathbb{C}^n, \mathbb{C} \times \{0\}) \to (\mathbb{C}, 0), \; (t, z) \mapsto f(t, z), \]
such that the following two conditions hold:

1. \(f(0, z) = f_0(z)\) for any \(z \in \mathbb{C}^n\);
2. if we write \(f_t(z) := f(t, z)\), then each \(f_t\) is reduced at 0.

Thus a deformation of \(f_0\) may be viewed as a 1-parameter family of polynomial functions \(f_t\) locally reduced at 0 and depending polynomially on the parameter \(t\).

We are looking for easy-to-check conditions on the members \(f_t\) of the family that guarantee equimultiplicity. In the case where the functions \(f_t\) have an isolated singularity at 0, G.-M. Greuel \[\text{[11]}\] and D. O’Shea \[\text{[27]}\] proved the following result.

**Theorem 1.1** (Greuel and O’Shea). Suppose that \{\(f_t\)\} is a family of isolated hypersurface singularities such that the polynomial function \(f_0\) is weighted homogeneous with respect to a given system of weights \(w := (w_1, \ldots, w_n)\) with \(w_i \in \mathbb{N} \setminus \{0\}\). Under these assumptions, if furthermore the family \{\(f_t\)\} is \(\mu\)-constant (i.e., if for all sufficiently small \(t\), the Milnor number of \(f_t\) at 0 is independent of \(t\)), then it is equimultiple (i.e., the multiplicity of \(f_t\) at 0 is independent of \(t\) for all sufficiently small \(t\)).

Here, by the multiplicity of \(f_t\) at 0 (denoted by \(\text{mult}_0(f_t)\)) we mean the number of points of intersection near 0 of the hypersurface \(V(f_t) := f_t^{-1}(0) \subseteq \mathbb{C}^n\) with a generic line in \(\mathbb{C}^n\) passing arbitrarily close to, but not through, the origin. As \(f_t\) is reduced at 0, this number coincides with the order of \(f_t\) at 0 (denoted by \(\text{ord}_0(f_t)\)).

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In the special case where all the weights $w_i$ ($1 \leq i \leq n$) are equal to 1 (i.e., $f_0$ is a homogeneous polynomial), Theorem [14] was first proved by A. M. Gabriélov and A. G. Kušnirenko in [8].

As the Milnor number is a topological invariant (cf. [14, 24, 29, 31]), Theorem [14] implies that any topologically $\mathcal{Y}$-equisingular deformation of an isolated hypersurface singularity defined by a weighted homogeneous polynomial function is equimultiple. Thus Theorem [14] partially answers the famous Zariski multiplicity conjecture for this special class of singularities.

We recall that a family $\{f_t\}$ is said to be topologically $\mathcal{Y}$-equisingular if there exist open neighbourhoods $D$ and $U$ of the origins in $\mathbb{C}$ and $\mathbb{C}^n$, respectively, together with a continuous map

$$
\varphi: (D \times U, D \times \{0\}) \rightarrow (\mathbb{C}^n, 0)
$$

such that for all sufficiently small $t$, there is an open neighbourhood $U_t \subseteq U$ of $0 \in \mathbb{C}^n$ such that the map $\varphi_t: (U_t, 0) \rightarrow (\varphi(t \times U_t), 0)$ defined by $\varphi_t(z) := \varphi(t, z)$ is a homeomorphism sending $V(f_0) \cap U_t$ onto $V(f_t) \cap \varphi(U_t)$.

The proof of Theorem [14] relies on a very deep theorem of A. N. Varchenko [32] which says that if the assumptions of Theorem [14] are satisfied, then the deformation family $\{f_t\}$ is upper. The word “upper” is defined as follows. Expand $f(t, z)$ with respect to the deformation parameter $t$:

$$
f(t, z) = f_0(z) + \sum_{1 \leq j \leq n} t^j g_j(z),
$$

where $g_j: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is a polynomial function. We say that $\{f_t\}$ is upper if each $g_j(z)$ is a linear combination of monomials of weighted degree (with respect to the weights $w$) greater than or equal to the weighted degree of $f_0$.

In the present paper, we investigate the same question as Greuel and O’Shea for the simplest class of hypersurfaces with non-isolated singularities—namely, the hypersurfaces with line singularities. Certainly, for such singularities, the Milnor number is no longer relevant. However the Lê numbers of D. Massey [19, 22] may be used instead. While these numbers are not topological invariants for arbitrary non-isolated singularities, for line singularities they are constant if the local ambient topological type of $V(f_t)$ at $0$ is constant (cf. [17, 18]). In particular, we obtain new partial positive answers to the Zariski multiplicity conjecture for such singularities.

2. Statement of the results

Suppose that $\{f_t\}$ is a family of line singularities. As in [18, §4], by this we mean that for each $t$ near 0 in $\mathbb{C}$ the singular locus $\Sigma f_t$ of $f_t$ near the origin $0 \in \mathbb{C}^n$ is given by the $z_1$-axis and the restriction of $f_t$ to the hyperplane $V(z_1)$ defined by $z_1 = 0$ has an isolated singularity at the origin. Then, by [21, Remark 1.29], the partition of $V(f_t)$ given by

$$
\mathcal{S}_t := \{V(f_t) \setminus \Sigma f_t, \Sigma f_t \setminus \{0\}, \{0\}\}
$$

is a good stratification for $f_t$ in a neighbourhood of 0, and the hyperplane $V(z_1)$ is a prepolar slice for $f_t$ at 0 with respect to $\mathcal{S}_t$ for all $t$ small enough. In particular, combined with [21, Proposition 1.23], this implies that the Lê numbers $\lambda^t_{0, z}(0)$ and

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1The Zariski multiplicity conjecture says that any topologically $\mathcal{Y}$-equisingular family of (possibly non-isolated) hypersurface singularities is equimultiple (cf. [14]). For a survey on this conjecture and related topics, we refer the reader to [2][3].
Deformations of weighted homogeneous line singularities and equimultiplicity

\( \lambda_{f_\ell, z}(0) \) and the polar number \( \lambda_{f_\ell, z}^l(0) \) of \( f_\ell \) at \( 0 \) with respect to the coordinates \( z \) do exist. Note that for line singularities, the only possible non-zero Lê numbers are precisely \( \lambda_{f_0, z}^0(0) \) and \( \lambda_{f_0, z}^1(0) \). All the other Lê numbers \( \lambda_{f_0, z}^k(0) \) for \( 2 \leq k \leq n - 1 \) are defined and equal to zero (cf. [21]).

Here is our first result.

**Theorem 2.1.** Assume that \( \{ f_\ell \} \) is a family of line singularities such that the polynomial function \( f_0 \) is weighted homogeneous with respect to a given system of weights \( w := (w_1, \ldots, w_n) \) with \( w_\ell \in \mathbb{N} \setminus \{0\} \). Also, suppose that the smallest weight \( w_{b_0} := \min\{w_1, \ldots, w_n\} \) divides the weighted degree \( d \) of \( f_0 \). Under these assumptions, if furthermore for all sufficiently small \( t \) the Lê numbers

\[
\lambda_{f_\ell, z}^0(0) \quad \text{and} \quad \lambda_{f_\ell, z}^1(0)
\]

are independent of \( t \) and if

\[
d/w_{b_0} \geq 2 + \lambda_{f_0, z}^0(0),
\]

then the family \( \{ f_\ell \} \) is equimultiplicity.

Theorem 2.1 is proved in Section 4.

By the Iomdine-Lê-Massey formula (cf. [21, Theorem 4.5]), the assumptions of the theorem imply that the function

\[
f_0 + z_{b_0}^{d/w_{b_0}}
\]

has an isolated singularity at \( 0 \). Weighted homogeneous polynomials \( f_0 \) for which \( d/w_{b_0} \in \mathbb{N} \) and \( f_0 + z_{b_0}^{d/w_{b_0}} \) defines an isolated singularity also appear, for instance, in Example 4.2.5 of [23] where D. Massey and D. Siersma investigate the Betti numbers of the Milnor fibre of such polynomials.

Theorem 2.1 may be viewed as a partial generalization of Theorem 1.1 to line singularities in the sense that if \( g_0(z_2, \ldots, z_n) \) is a weighted homogeneous polynomial in \( \mathbb{C}^{n-1} \) with respect to a given system of weights \( (w_2, \ldots, w_n) \) such that:

- the smallest weight \( w_{b_0} \) divides the weighted degree \( g_0 \);
- \( g_0 \) has an isolated singularity at the origin;

and if \( g(t, z_2, \ldots, z_n) \) is a \( \mu \)-constant deformation of \( g_0 \), then the corresponding family of line singularities in \( \mathbb{C}^n \), defined by

\[
f(t, z_1, z_2, \ldots, z_n) := g(t, z_2, \ldots, z_n),
\]

is such that:

1. \( \lambda_{f_\ell, z}^0(0) = 0 \) for all small \( t \);
2. \( \lambda_{f_\ell, z}^1(0) \) is independent of \( t \) for all small \( t \);
3. \( d/w_{b_0} \geq 2 + \lambda_{f_0, z}^0(0) = 2 \);

and therefore, by Theorem 2.1 the family \( \{ f_\ell \} \) (and hence the family \( \{ g_\ell \} \)) is equimultiple. (Note that \( f_0 \) is weighted homogeneous with respect to \( (w_1, w_2, \ldots, w_n) \), where \( w_1 \) is an integer that we have chosen greater than \( w_{b_0} \).) That (1) and (2) hold true is explained by D. Massey in [18, §5]. His argument is as follows. For (1), it suffices to observe that the relative 1st polar variety \( \Gamma_{f_\ell, z}^1 \) of \( f_\ell \) with respect to \( z \)

\[\footnote{For the definitions of good stratifications, prepolar slices, Lê numbers and polar numbers, we refer the reader to Chapter 1 of D. Massey’s book [21]. For the reader’s convenience, we also briefly recall these definitions in the appendices below.}
(see Appendix B for the definition) is empty. For (2), Massey observes that for line singularities,
\[ \lambda_{f,t}^1(0) = \bar{\mu}_{f,t}, \]
where \( \bar{\mu}_{f,t} \) is the Milnor number of a generic hyperplane slice of \( f_t \) at a point on \( \Sigma f_t \) sufficiently close to the origin (cf. [15,18,21]). But in our case \( \bar{\mu}_{f,t} \) is nothing but the Milnor number of \( g_t \) at the origin, which is constant. Finally, to show that (3) holds, we argue by contradiction. Suppose that \( d/w_i \) is not constant as \( t \) varies from \( 0 \) to \( t=0 \). Thus, by a theorem of J. Milnor and P. Orlik [25], if \( \mu_{\Sigma f_t}(0) \) denotes the Milnor number of \( g_0 \) at \( 0 \in \mathbb{C}^n \), we have
\[ \mu_{\Sigma f_t}(0) = \prod_{2 \leq i \leq n} \left( \frac{d}{w_i} - 1 \right) = 0 \]
and the origin is not a critical point of \( g_0 \)—a contradiction.

Theorem 2.1 has the following important corollary, which provides a new partial positive answer to the Zariski multiplicity conjecture.

**Corollary 2.2.** Assume that \( \{ f_t \} \) is a family of line singularities such that the polynomial function \( f_0 \) is weighted homogeneous with respect to a given system of weights \( w := (w_1, \ldots, w_n) \) with \( w_i \in \mathbb{N} \setminus \{0\} \). Also, suppose that the smallest weight \( w_0 := \min\{w_1, \ldots, w_n\} \) divides the weighted degree \( d \) of \( f_0 \). Under these assumptions, if furthermore the family \( \{ f_t \} \) is topologically \( \psi \)-equisingular and if
\[ d/w_0 \geq 2 + \lambda_{f_0,t}^0(0), \]
then \( \{ f_t \} \) is equimultiple.

**Proof.** Suppose that \( \{ f_t \} \) is not equimultiple. Then, by Theorem 2.1 either \( \lambda_{f,t}^0(0) \) or \( \lambda_{f,t}^1(0) \) is not constant as \( t \) varies from \( t_0 \neq 0 \) to \( t=0 \). Thus the corollary follows from the following result of D. Massey (cf. [17,18]). \( \square \)

**Theorem 2.3** (Massey). If \( \{ f_t \} \) is a family of line singularities, then the following two conditions are equivalent:

1. The Lê numbers \( \lambda_{f,t}^0(0) \) and \( \lambda_{f,t}^1(0) \) are independent of \( t \) for all small \( t \);
2. The (embedded) topological invariants \( \bar{\mu}_{f,t} \) and \( \tilde{\chi}(F_{f,t},0) \) are independent of \( t \) for all small \( t \).

Here, \( \tilde{\chi}(F_{f,t},0) \) denotes the reduced Euler characteristic of the Milnor fibre \( F_{f,t,0} \) of \( f_t \) at \( 0 \). Note that in our case
\[ \tilde{\chi}(F_{f,t},0) = (-1)^{n-1}\lambda_{f,t}^0(0) + (-1)^{n-2}\lambda_{f,t}^1(0) \]
(see [18, §4] or [21, Theorem 3.3]).

Our second result is as follows.

**Theorem 2.4.** Again assume that \( \{ f_t \} \) is a family of line singularities such that the polynomial function \( f_0 \) is weighted homogeneous with respect to a given system of weights \( w := (w_1, \ldots, w_n) \) with \( w_i \in \mathbb{N} \setminus \{0\} \). Also, suppose that the smallest weight \( w_0 := \min\{w_1, \ldots, w_n\} \) divides the weighted degree \( d \) of \( f_0 \). Under these assumptions, if furthermore for all sufficiently small \( t \) the numbers
\[ \lambda_{f,t}^1(0) \text{ and } \gamma_{f,t}^1(0) + \lambda_{f,t}^0(0) \]
are independent of $t$, and if, at least when $t \neq 0$, the relative 1st polar variety $\Gamma^1_{f_i, z}$ of $f_i$ with respect to $z$ is irreducible, then the family \( \{ f_i \} \) is equimultiple.

Theorem 2.4 is proved in Section 5.

The condition (2.1) is already used by D. Massey in [18, §5] in his first partial generalization of the Lê-Ramanujam theorem to line singularities. Indeed, by [21, Proposition 1.23],

$$\gamma^1_{f_i, z} (0) + \lambda^0_{f_i, z} (0) = \left( \Gamma^1_{f_i, z} \cdot [V(f_i)] \right)_0,$$

where $\Gamma^1_{f_i, z}$ and $[V(f_i)]$ denote the analytic cycles associated to $\Gamma^1_{f_i, z}$ and $V(f_i)$, respectively, and where $\left( \Gamma^1_{f_i, z} \cdot [V(f_i)] \right)_0$ is the intersection number at 0 of these two cycles (cf. Appendix B). In [18, Theorem (5.2)], Massey showed that if $n \geq 5$ and if

$$\lambda^1_{f_i, z} (0) \quad \text{and} \quad \left( \Gamma^1_{f_i, z} \cdot [V(f_i)] \right)_0$$

are constant (as $t$ varies)---equivalently, if (2.1) holds---then the diffeomorphism type of the Milnor fibration of $f_i$ at 0 is constant too. Note that in [21, Theorem 9.4], Massey proved a stronger result, namely he showed that if $n \geq 5$ and if $\lambda^0_{f_i, z} (0)$ and $\lambda^1_{f_i, z} (0)$ are constant, then the diffeomorphism type of the Milnor fibration of $f_i$ at 0 is constant. This is stronger because (2.1) implies that the Lê numbers $\lambda^0_{f_i, z} (0)$ and $\lambda^1_{f_i, z} (0)$ are constant. This latter implication is already explained in [18].

For the sake of completeness, let us briefly recall Massey’s argument. By [18, Corollary 2.4], if (2.1) holds, then, for any integer $j$ sufficiently large, the Milnor numbers

$$\mu_{f_i + z_1^i} (0) \quad \text{and} \quad \mu_{f_i V(1)} (0)$$

are both independent of $t$ for all $t$ sufficiently small. Indeed, by the uniform Iomdine-Lê-Massey formulas (see Proposition 2.1 and the relation (2.2) in [18] and Theorem 4.15 in [21]), for all $t$ sufficiently small and all $j$ sufficiently large, $f_i + z_1^i$ has an isolated singularity at the origin and

$$\begin{cases} 
\mu_{f_i + z_1^i} (0) = \lambda^0_{f_i, z} (0) + (j-1) \lambda^1_{f_i, z} (0); \\
\mu_{f_i + V(1)} (0) = \gamma^1_{f_i, z} (0) + \lambda^0_{f_i, z} (0) + j \lambda^1_{f_i, z} (0).
\end{cases}$$

Thus, if (2.1) holds, then the sum

$$\mu_{f_i + z_1^i} (0) + \mu_{f_i V(1)} (0)$$

is independent of $t$, and hence, by the upper-semicontinuity of the Milnor number, $\mu_{f_i + z_1^i} (0)$ and $\mu_{f_i V(1)} (0)$ are both independent of $t$. It follows that $\lambda^0_{f_i, z} (0)$ and $\gamma^1_{f_i, z} (0)$ do not depend on $t$.

It would be nice if (as in Theorem 2.4 above or as in [21, Theorem 9.4]) we could replace the condition (2.1) of Theorem 2.4 by the assumption that $\lambda^0_{f_i, z} (0)$ and $\lambda^1_{f_i, z} (0)$ are independent of $t$ for all small $t$ (see also Remark 2.6 below).

Note that requiring that (2.1) holds does not imply the Whitney conditions along the $t$-axis (cf. [18, §5]). If the family $\{ f_i \}$ were Whitney equisingular (i.e., if there were a Whitney stratification of $V(f) := f^{-1}(0) \subseteq C \times C^n$ with the $t$-axis as a stratum), then the result would follow immediately from a theorem of H. Hironaka which says that any reduced complex analytic space endowed with a Whitney stratification is equimultiple along every stratum (cf. [13 Corollary (6.2)]).
In [10] Corollary 6.6], T. Gaffney and R. Gassler proved that in the special case where \( \{f_t\} \) is a family of surface singularities in \( \mathbb{C}^3 \), then the family \( \{f_t\} \) is Whitney equisingular if, in addition to the condition (2.1), the second polar number \( \gamma_{f_t, z}^2(0) \) is independent of \( t \) as well. Still in the case of surfaces in \( \mathbb{C}^3 \), they even showed that Whitney equisingularity does hold true if and only if the Lê numbers \( \lambda_{f_t, x}^0(0) \), \( \lambda_{f_t, x}^1(0) \) and the polar numbers \( \gamma_{f_t, x}^1(0) \), \( \gamma_{f_t, x}^2(0) \) are all independent of \( t \) for all small \( t \).

Theorem 2.4 has the following important corollary, which provides another new partial positive answer to the Zariski multiplicity conjecture for line singularities.

**Corollary 2.5.** Assume again that \( \{f_t\} \) is a family of line singularities such that the polynomial function \( f_0 \) is weighted homogeneous with respect to a given system of weights \( w := (w_1, \ldots, w_n) \) with \( w_i \in \mathbb{N} \setminus \{0\} \). Also, suppose that the following three conditions hold true:

1. \( w_0 := \min\{w_1, \ldots, w_n\} \) divides the weighted degree of \( f_0 \);
2. \( \gamma_{f_t, x}^1(0) \) is independent of \( t \) for all small \( t \);
3. \( \Gamma_{f_t, x}^1(0) \) is irreducible except perhaps when \( t = 0 \).

Under these assumptions, if furthermore the family \( \{f_t\} \) is topologically \( \mathcal{V} \)-equisingular, then it is equimultiple.

**Proof.** It is similar to the proof of Corollary 2.2. Suppose that \( \{f_t\} \) is not equimultiple. Then, by Theorem 2.4 either \( \lambda_{f_t, x}^1(0) \) or \( \gamma_{f_t, x}^1(0) + \lambda_{f_t, x}^0(0) \) is not constant. As \( \gamma_{f_t, x}^1(0) \) is constant, it follows that either \( \lambda_{f_t, x}^1(0) \) or \( \lambda_{f_t, x}^0(0) \) is not constant. Then again the conclusion follows from Theorem 2.3. \( \square \)

**Remark 2.6.** If we could replace the assumption (2.1) in Theorem 2.4 by the condition that \( \lambda_{f_t, x}^0(0) \) and \( \lambda_{f_t, x}^1(0) \) are constant, then, by Theorem 2.3 we could also avoid the assumption that \( \gamma_{f_t, x}^1(0) \) is constant in Corollary 2.5.

In the special case where the polynomial \( f_0 \) is homogeneous, the first author [4] proved the following (stronger) theorem which generalizes to line singularities the Gabrièlo-Kušnirenko theorem mentioned in the introduction.

**Theorem 2.7 (cf. [4] Theorem 1.6 and Corollary 1.9]).** If \( \{f_t\} \) is a topologically \( \mathcal{V} \)-equisingular family of line singularities (or even a family of line singularities with constant Lê numbers) and if the polynomial \( f_0 \) is homogeneous, then \( \{f_t\} \) is equimultiple.

We conclude this section with the following remark about Whitney equisingularity for families of parametrized surface singularities in \( \mathbb{C}^3 \).

**Remark 2.8.** Suppose that \( \{f_t\} \) is a family of parametrized surface singularities in \( \mathbb{C}^3 \), that is, a family for which there is an analytic map \( (\mathbb{C} \times \mathbb{C}^2, \mathbb{C} \times \{0\}) \rightarrow (\mathbb{C} \times \mathbb{C}^3, \mathbb{C} \times \{0\}) \) of the form

\[
(t, (z_1, z_2)) \in \mathbb{C} \times \mathbb{C}^2 \mapsto (t, \psi_t(z_1, z_2)) \in \mathbb{C} \times \mathbb{C}^3
\]

satisfying \( V(f_t) = \text{im}(\psi_t) \). Also, assume there is a neighbourhood \( W \) of the origin in \( \mathbb{C}^2 \) such that the following two conditions hold:

(a) \( W \cap \psi_t^{-1}(0) = \{0\} \);
(b) the only singularities of \( \psi_t(W) \setminus \{0\} \) are transverse double points.
Then, by Mather-Gaffney’s criterion (cf. [33]), \( \varphi_t \) is finitely \( \mathcal{A} \)-determined, and it follows from Theorem 5.3 in [16] (see also [9]) that if \( \tilde{\mu}_f \) and \( \mu(D(\varphi_t), 0) \) are constant, then, in a neighbourhood of the origin, the partition of \( V(f) \) given by

\[
\{ V(f) \setminus \Sigma f, \Sigma f \setminus (C \times \{0\}), C \times \{0\} \}
\]

is a Whitney stratification—in particular, the family \( \{ f_t \} \) is Whitney equisingular. (Here, \( \mu(D(\varphi_t), 0) \) denotes the Milnor number of the double point locus \( D(\varphi_t) = \varphi_t^{-1}(\Sigma f_t) \) of \( \varphi_t \) at the origin. As usual, \( \Sigma f \) is the critical locus of \( f_t \)).

Now if we suppose further that \( \{ f_t \} \) is topologically \( \mathcal{V} \)-equisingular and such that the 1st polar number \( \gamma_{f_t}^1(0) \) is constant, then, by [11] Theorem 6.2] (see also [6]), the Milnor number \( \mu(D(\varphi_t), 0) \) is constant, and it follows from [28] Proposition 3.3] and [16] Lemma 5.2] that \( \tilde{\mu}_f \) is constant too. Therefore, if \( \{ f_t \} \) is a topologically \( \mathcal{V} \)-equisingular family of parametrized surface singularities in \( \mathbb{C}^3 \) with constant first polar number and satisfying the above conditions (a) and (b), then \( \{ f_t \} \) is Whitney equisingular. For examples of such families that have, in addition, line singularities, we refer the reader to [25].

### 3. APPLICATION TO TOPOLOGICAL EQUISINGULARITY

Corollaries 2.2 and 2.3 may be very useful to decide whether certain families of hypersurfaces with line singularities are not topologically \( \mathcal{V} \)-equisingular—a question which is, in general, extremely difficult to answer. For example, Corollary 2.5 says that in order to show that a family \( \{ f_t \} \) of line singularities, with \( f_0 \) weighted homogeneous, is not topologically \( \mathcal{V} \)-equisingular, it suffices to observe that it is not equimultiple and such that the conditions (1)–(3) enumerated in the corollary are satisfied—four very simple checks.

**Example 3.1.** Consider the family defined by

\[
f_t(z_1,z_2,z_3) = z_1^2z_2^2 + z_2^4 + z_3^4 + tz_1z_2^2 + r^2z_1^2z_2^2.
\]

A priori, it is far from being obvious to decide whether this family is topologically \( \mathcal{V} \)-equisingular or not. However, this easily follows from Corollary 2.5. Indeed, the polynomial function \( f_0(z_1,z_2,z_3) = z_1^2z_2^2 + z_2^4 + z_3^4 \) is weighted homogeneous with respect to the weights \( w := (6,4,5) \), the singular locus \( \Sigma f_t \) of \( f_t \) near the origin is given by the \( z_1 \)-axis, and the restriction \( f_t|_{V(z_1)} \) has an isolated singularity at the origin. Clearly the smallest weight \( w_{i_0} = 4 \) divides the weighted degree \( d = 20 \) of \( f_0 \). An easy computation also shows that for any \( t \) sufficiently small, the relative 1st polar variety \( \Gamma_{f_t} \) of \( f_t \) with respect to \( z \) is given by

\[
\Gamma_{f_t} = V \left( \frac{\partial f_t}{\partial z_2}, \frac{\partial f_t}{\partial z_3} \right) \setminus \Sigma f_t
\]

\[
= V \left( z_2(2z_1^2 + 5z_3^2 + 2tz_1 + 2r^2z_1^2), 4z_3^4 \right) - V(z_2,z_3)
\]

\[
= V \left( 2z_1^2 + 5z_3^2 + 2tz_1 + 2r^2z_1^2, z_3^4 \right),
\]

and hence \( \Gamma_{f_t} \) is irreducible and

\[
\gamma^1_{f_t}(0) = ([\Gamma_{f_t} \cdot V(z_1)])_0 = 9.
\]

As the family \( \{ f_t \} \) is not equimultiple, it follows from Corollary 2.5 that it is not topologically \( \mathcal{V} \)-equisingular.
Remark 3.2. By [5] Corollary 3.7, we know that if \( \{ f_t \} \) is a non-equimultiple family of line singularities of the form \( f_t(z) = f_0(z) + \xi(t)g(z) \), where \( \xi : (\mathbb{C},0) \to (\mathbb{C},0) \) is a non-constant polynomial function and \( g : (\mathbb{C}^n,0) \to (\mathbb{C},0) \) is any polynomial function, then \( \{ f_t \} \) is not topologically \( \mathcal{E} \)-equisingular. The above example is not a consequence of this result.

4. Proof of Theorem 2.1

As \( d/w_0 \geq 2 + \lambda^0_{f_0,z}(0) \) and since the Lê numbers \( \lambda^0_{f_1,z}(0) \) and \( \lambda^1_{f_1,z}(0) \) are constant, we have \( d/w_0 \geq 2 + \lambda^0_{f_1,z}(0) \) for all small \( t \). Thus, by the Iomdine-Lê-Massey formula (cf. [21] Theorem 4.5]), the function

\[
\Gamma_{f_0 + z_0^{d/w_0}}
\]

has an isolated singularity at 0 and the family \( \{ f_t + z_0^{d/w_0} \} \) is \( \mu \)-constant. Now, as for \( t = 0 \) the function \( f_0 + z_0^{d/w_0} \) is weighted homogeneous with respect to \( w = (w_1, \ldots, w_n) \), the Greuel-O’Shea theorem (cf. Theorem 1.1) says that

\[
\text{ord}_0(f_t + z_0^{d/w_0}) := \text{ord}_0(f_0 + z_0^{d/w_0} + \sum_{1 \leq j \leq f_0} t^j g_j(z)) = \text{ord}_0(f_0 + z_0^{d/w_0}),
\]

where, as above, the polynomial functions \( g_j \) are defined by

\[
f_t(z) = f_0(z) + \sum_{1 \leq j \leq f_0} t^j g_j(z).
\]

If \( \text{ord}_0(f_0) \leq d/w_0 \), then the relation (4.1) implies \( \text{ord}_0(f_t) = \text{ord}_0(f_0) \), and the theorem is proved. Now we claim that we always have \( \text{ord}_0(f_0) \leq d/w_0 \). Indeed, take any monomial \( \alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) of the initial polynomial \( \text{in}(f_0) \) of \( f_0 \). Then we have

\[
\sum_{1 \leq i \leq n} \alpha_i = \text{deg in}(f_0) = \text{ord}_0(f_0),
\]

and since \( f_0 \) (and hence \( \text{in}(f_0) \)) is weighted homogeneous with respect to the weights \( w = (w_1, \ldots, w_n) \), we also have

\[
\sum_{1 \leq i \leq n} \alpha_i w_i = d.
\]

As \( w_0 = \min\{w_1, \ldots, w_n\} \), it follows that

\[
d = \sum_{1 \leq i \leq n} \alpha_i w_i \geq \sum_{1 \leq i \leq n} \alpha_i w_0 = w_0 \sum_{1 \leq i \leq n} \alpha_i = w_0 \text{ord}_0(f_0).
\]

5. Proof of Theorem 2.4

As already observed in Section 2 the constancy of \( \lambda^1_{f_1,z}(0) \) and \( \gamma^1_{f_1,z}(0) + \lambda^0_{f_1,z}(0) \) implies that of \( \lambda^0_{f_1,z}(0) \) and \( \gamma^1_{f_1,z}(0) \). Moreover, as \( \gamma^1_{f_1,z}(0) \) is defined, the relative 1st polar variety \( \Gamma_{f_1,z} \) is purely 1-dimensional or empty at 0 and the intersection \( \Gamma_{f_1,z} \cap V(z_1) \) is 0-dimensional or empty at 0. Now the proof divides into two cases depending on whether \( \gamma^1_{f_0,z}(0) \) is zero (i.e., \( \Gamma_{f_0} = \emptyset \)) or not.
5.1. **The case** \( \gamma_{f_0,z}(0) \neq 0 \). If the 1st polar number \( \gamma_{f_0,z}(0) \) is not zero, then, as \( \gamma_{f_0,z}(0) \) is constant, \( \gamma_{f_0,z}(0) \neq 0 \) for all small \( t \), and by [21] Proposition 1.23,\footnote{The reference to [21] and [22] is implied here.}

\[
\rho_0 := \frac{\langle [\Gamma_{f_0,z}] \cdot [V(f_t)] \rangle_0}{\langle [\Gamma_{f_0,z}] \cdot [V(z_1)] \rangle_0} = \frac{\gamma_{f_0,z}(0) + \lambda_{f_0,z}(0)}{\gamma_{f_0,z}(0)}.
\]

**Lemma 5.1.** For each irreducible component \( \eta \) of \( \Gamma_{f_0,z}(0) \), endowed with its reduced structure, we have the following equalities:

\[
\langle [\eta] \cdot [V(f_0)] \rangle_0 = d \quad \text{and} \quad \langle [\eta] \cdot [V(z_1)] \rangle_0 = w_1.
\]

In particular, \( \rho_0 = d/w_1 \).

**Proof.** For \( \varepsilon > 0 \) small enough, \( \eta \cap V(z_1 - \varepsilon) \) contains at least one point \( a = (a_1, \ldots, a_n) \). Thus, as \( f_0 \) is weighted homogeneous with respect to the weights \( w = (w_1, \ldots, w_n) \), we may pick a parametrization of \( \eta \) (with its reduced structure) of the form

\[
s \mapsto \phi(s) := (a_1 s^{w_1}, \ldots, a_n s^{w_n}).
\]

Observe that \( \eta \not\subset V(f_0) \). Indeed, the \( \mathbb{C}^* \)-action on \( \Gamma_{f_0,z} \) (or on \( \eta \)) gives

\[
\frac{\partial f_0}{\partial z_i}(\phi(s)) = \frac{\partial f_0}{\partial z_i}(a_1 s^{w_1}, \ldots, a_n s^{w_n}) = s^{d-w_i} \cdot \frac{\partial f_0}{\partial z_i}(a_1, \ldots, a_n) = 0
\]

for all \( i \geq 2 \). Thus, if \( f_0 \circ \phi \) identically vanishes, then, for all \( s \),

\[
0 = (f_0 \circ \phi)'(s) = a_1 w_1 s^{w_1 - 1} \frac{\partial f_0}{\partial z_1}(\phi(s)),
\]

and hence,

\[
\frac{\partial f_0}{\partial z_1}(\phi(s)) = 0.
\]

It follows that \( \eta \) is contained in \( \Sigma f_0 \)—a contradiction. Now, since \( \eta \not\subset V(f_0) \), a classical result in intersection theory shows that

\[
\langle [\eta] \cdot [V(f_0)] \rangle_0 = \text{ord}_0(f_0 \circ \phi(s))
\]

(cf. [7] or [22] Appendix A.9), and clearly \( \text{ord}_0(f_0 \circ \phi(s)) = d \).

Similarly, we show \( \langle [\eta] \cdot [V(z_1)] \rangle_0 = w_1 \).

Now if we write the cycle \( [\Gamma_{f_0,z}(0)] \) as \( \sum \kappa \eta \), where the sum is taken over all the irreducible components \( \eta \) of \( \Gamma_{f_0,z}(0) \), then

\[
\rho_0 = \frac{\sum k \eta \langle [\eta] \cdot [V(f_0)] \rangle_0}{\sum k \eta \langle [\eta] \cdot [V(z_1)] \rangle_0} = \frac{d}{w_1}
\]

as desired. \( \square \)

**Remark 5.2.** Note that the above lemma implies that for each irreducible component \( \eta \) of \( \Gamma_{f_0,z}(0) \) (with its reduced structure), the **polar ratio** of \( \eta \), which is defined by

\[
\frac{\langle [\eta] \cdot [V(f_0)] \rangle_0}{\langle [\eta] \cdot [V(z_1)] \rangle_0},
\]

is equal to \( d/w_1 \).
Since \( \lambda^0_{f_0,z}(0) \) and \( \gamma^1_{f_0,z}(0) \) are constant, it follows from Lemma 5.1 that \( \rho_t \) is constant too equal to \( \rho_0 = d/w_1 \). As \( \Gamma^1_{f_0,z} \) is irreducible when \( t \neq 0 \), the function \( f_t \) has only one polar ratio which is nothing but \( \rho_t \). Thus, since \( w_{i0} \) is the smallest weight (and hence \( d/w_{i0} \geq d/w_1 = \rho_0 \)), it follows from the Iomdine-Lê-Massey formula (cf. [21] Theorem 4.5) that for all \( t \) sufficiently small and for all but a finite number of non-zero numbers \( a \), the function

\[
    f_t + az^{d/w_{i0}}
\]

has an isolated singularity at \( 0 \) and the family \( \{ f_t + az^{d/w_{i0}} \} \) is \( \mu \)-constant. Then we conclude exactly as in Theorem 2.1.

5.2. The case \( \gamma^1_{f_0,z}(0) = 0 \). If the 1st polar number \( \gamma^1_{f_0,z}(0) \) vanishes, that is, if the relative 1st polar variety \( \Gamma^1_{f_0,z} \) is empty, then

\[
    0 = (\Gamma^1_{f_0,z} : [V(f_0)])_0 = \gamma^1_{f_0,z}(0) + \lambda^0_{f_0,z}(0) = \lambda^0_{f_0,z}(0).
\]

Again the proof divides into two cases according to either \( d/w_{i0} \geq 2 \) or \( d/w_{i0} = 1 \). If \( d/w_{i0} \geq 2 = 2 + \lambda^0_{f_0,z}(0) \), then, by the Iomdine-Lê-Massey formula, the family \( \{ f_t + az^{d/w_{i0}} \} \) is \( \mu \)-constant and we conclude as above. (We can also apply Theorem 2.1.) If \( d/w_{i0} = 1 \), then the polynomial \( f_0 \) is necessarily of the form \( f_0(z) = cz^i \) for some \( i \) (\( c \) is a constant), that is, \( f_0 \) is homogeneous. In this case the result follows from Theorem 2.7.

**Appendix A. Good stratifications and prepolar slices**

Let \( h : (\mathbb{C}^n,0) \to (\mathbb{C},0) \) be a polynomial function, let \( \Sigma h \) be its the critical locus, and let \( V(h) := h^{-1}(0) \) be the hypersurface in \( \mathbb{C}^n \) defined by \( h \).

A good stratification for \( h \) at a point \( p \in V(h) \) is an analytic stratification \( \mathscr{S} \) of \( V(h) \) in a neighbourhood \( U \) of \( p \) such that the following two conditions hold:

1. the (trace of the) smooth part of \( V(h) \) is a stratum;
2. \( \mathscr{S} \) satisfies Thom’s \( a_k \) condition with respect to \( U \setminus \Sigma h \), that is, if \( \{ q_k \} \) is a sequence of points in \( U \setminus \Sigma h \) such that

\[
    q_k \to q \in S \in \mathscr{S} \quad \text{and} \quad T_{q_k}V(h-h(q_k)) \to T,
\]

then \( T_{q}S \subseteq T \).

As usual, \( T_{q_k}V(h-h(q_k)) \) denotes the tangent space at \( q_k \) to the level hypersurface defined by \( h(z) = h(q_k) \), and \( T_{q}S \) is the tangent space at \( q \) to the stratum \( S \). Note that good stratifications always exist (cf. [12]).

Now if \( \mathscr{S} \) is a good stratification for \( h \) at a point \( p \in V(h) \), then a hyperplane \( H \) of \( \mathbb{C}^n \) through \( p \) is called a prepolar slice for \( h \) at \( p \) with respect to \( \mathscr{S} \) if it transversely intersects all the strata of \( \mathscr{S} \)—perhaps with the exception of the stratum \( \{ p \} \) itself—in a neighbourhood of \( p \).

For details, see Chapter 1 of Massey’s book [21].

**Appendix B. Lê numbers and polar numbers**

The Lê numbers generalize to non-isolated hypersurface singularities the data given by the Milnor number for an isolated singularity. They were introduced about 25 years ago by D. Massey [19, 22]. For the convenience of the reader, we
briefly recall the definitions in this appendix. We follow the presentation given in Massey’s book [21].

Throughout, we use the following notation.

**Notation B.1.** Let \((X, \mathcal{O}_X)\) be a complex analytic space, and let \(\mathcal{I}\) be a coherent sheaf of ideals in \(\mathcal{O}_X\). We denote by \(V(\mathcal{I})\) the complex analytic space defined by the vanishing of \(\mathcal{I}\). If \(W \subseteq X\) is any analytic subset of \(X\), then we denote by \(\mathcal{I}\cap W\) the gap sheaf associated to \(\mathcal{I}\) and \(W\). As usual, the scheme \(V(\mathcal{I}\cap W)\) defined by the vanishing of the gap sheaf will be also denoted by \(V(\mathcal{I})\cap W\). Finally, we write \([X]\) for the analytic cycle associated to \(X\)—that is, the formal sum \(\sum m_V[V]\), where the \(V\)'s run over all the irreducible components of \(X\) and where \(m_V\) represents the number of times the component \(V\) should be counted.

Consider a polynomial function \(h: (U, 0) \to (\mathbb{C}, 0)\), where \(U\) is an open neighbourhood of \(0\) in \(\mathbb{C}^n\), and fix a system of linear coordinates \(z = (z_1, \ldots, z_n)\) for \(\mathbb{C}^n\). As usual, write \(\Sigma h\) for the critical locus of \(h\). For \(0 \leq k \leq n-1\), the relative \(k\)th polar variety of \(h\) with respect to the coordinates system \(z\) is the scheme

\[
\Gamma^k_{h,z} := V\left(\frac{\partial h}{\partial z_{k+1}}, \ldots, \frac{\partial h}{\partial z_n}\right) \cap \Sigma h.
\]

The \(k\)th Lê cycle of \(h\) with respect to \(z\) is the analytic cycle

\[
[A^k_{h,z}] := \left[\Gamma^k_{h,z} \cap V\left(\frac{\partial h}{\partial z_{k+1}}\right)\right] - \left[\Gamma^k_{h,z}\right].
\]

**Definition B.2.** The \(k\)th Lê number \(\lambda^k_{h,z}(p)\) of \(h\) at \(p = (p_1, \ldots, p_n)\) with respect to the coordinates system \(z\) is defined to be the intersection number

\[
(\lambda^k_{h,z}(p)) := (\lambda^k_{h,z}: [V(z_1-p_1, \ldots, z_k-p_k)])_p
\]

provided that this intersection is 0-dimensional or empty at \(p\); otherwise, \(\lambda^k_{h,z}(p)\) is undefined.

For \(k = 0\), the relation (B.1) means

\[
\lambda^0_{h,z}(p) = ([A^0_{h,z}] : U)_p = \left[\Gamma^1_{h,z} \cap V\left(\frac{\partial h}{\partial z_1}\right)\right]_p.
\]

For any \(k\), with \(\dim_p \Sigma h < k \leq n - 1\), the corresponding Lê number \(\lambda^k_{h,z}(p)\) always exists and is equal to zero. Note that if \(p\) is an isolated singularity of \(h\), then the 0th Lê number \(\lambda^0_{h,z}(p)\) (which is the only possible non-zero Lê number) is equal to the Milnor number \(\mu_h(0)\) of \(h\) at \(p\).

**Definition B.3.** The \(k\)th polar number \(\gamma^k_{h,z}(p)\) of \(h\) at \(p = (p_1, \ldots, p_n)\) with respect to the coordinates system \(z\) is defined to be the intersection number

\[
\gamma^k_{h,z}(p) := ([\Gamma^k_{h,z}] : [V(z_1-p_1, \ldots, z_k-p_k)])_p
\]

provided that this intersection is 0-dimensional or empty at \(p\); otherwise, \(\gamma^k_{h,z}(p)\) is undefined.

Note that \(\gamma^0_{h,z}(p)\) is always defined and equal to zero.

**Remark B.4.** For a generic choice of coordinates \(z\), for any point \(p \in V(h)\) near \(0\) and for any \(0 \leq k \leq \dim_p \Sigma h\), the Lê number \(\lambda^k_{h,z}(p)\) and the polar number \(\gamma^k_{h,z}(p)\) do exist (cf. [21] Proposition 10.2 and Theorem 1.28]).
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