ON SYMMETRIC VERSIONS OF SYLVESTER’S PROBLEM

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Abstract. We consider moments of the normalized volume of a symmetric or nonsymmet-
ric random polytope in a fixed symmetric convex body. We investigate for which bodies
these moments are extremized, and calculate exact values in some of the extreme cases. We
show that these moments are maximized among planar convex bodies by parallelograms.

1. Introduction

Sylvester’s four point problem asks for the probability that the convex hull of four random
points, chosen independently and uniformly from a convex body \( K \subset \mathbb{R}^2 \), is a quadrilateral,
and in particular, for which convex bodies \( K \) this probability is extremal. This is equivalent
to asking what the expected area of the convex hull of three random points in \( K \) is, and
for which bodies this expectation is extremal. This problem was solved by Blaschke [2, 3],
who showed that the expected area achieves its maximum exactly when \( K \) is a triangle and
achieves its minimum exactly when \( K \) is an ellipse.

Since then, various authors have considered several extensions of this problem. Many
of these are special cases of the following general problem. We write \( K^n \) for the set of all
convex bodies in \( \mathbb{R}^n \), that is, all compact convex sets with interior points. Let \( K \in K^n \) and
\( N \geq n + 1 \). Let \( x_1, x_2, \ldots, x_N \) be independent random points distributed uniformly in \( K \).
We define

\[
U_{K,N} = \frac{\text{vol}_n(\text{conv}\{x_1, x_2, \ldots, x_N\})}{\text{vol}_n(K)};
\]

thus the random variable \( U_{K,N} \) is the normalized volume of a random polytope in \( K \). Note
that the distribution of \( U_{K,N} \) is an affine invariant of \( K \). The generalized Sylvester’s problem
asks, for each \( n \geq 2 \), \( N \geq n + 1 \), and \( p \geq 1 \), for which \( K \in K^n \) does the \( p \)th moment \( \mathbb{E}U_{K,N}^p \)
achieve its extremal values? It should be noted at this point that a compactness argument
 guarantees that such extremal bodies do exist; see [11, 7, 9, 6].

Groemer [11, 12] showed that, for each such \( n \), \( N \), and \( p \), \( \mathbb{E}U_{K,N}^p \) is minimized exactly when
\( K \) is an ellipsoid. Dalla and Larman [7] showed for \( n = 2 \) that \( \mathbb{E}U_{K,N} \) is maximized, for each
\( N \geq 3 \), when \( K \) is a triangle; and Campi, Colesanti, and Gronchi [6] extended this to \( \mathbb{E}U_{K,N}^p \)
for all \( p \geq 1 \). Giannopoulos [9] showed that for \( n = 2 \), \( \mathbb{E}U_{K,N} \) is maximized only if \( K \) is a triangle.
Very little is known about maximizing bodies when \( n \geq 3 \). It is widely conjectured
that \( \mathbb{E}U_{K,N}^p \) should achieve its maximum exactly when \( K \) is a simplex, but there are only
partial results in this direction [7, 6]. As noted explicitly in [11] (but see also [16, Proposition
5.6]), this would in particular imply the well-known hyperplane conjecture [10, 10].

In this paper, we consider two “symmetric” variants of this generalized Sylvester’s problem.
We write \( K^n_s \) for the set of symmetric convex bodies in \( \mathbb{R}^n \); that is, all \( K \in K^n \) such that
\( K = -K \). The first variant asks, for which \( K \in K^n_s \) does \( \mathbb{E}U_{K,N}^p \) achieve its extremal values?

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The second variant asks the same question when the random polytope, as well as the fixed body $K$, is symmetric. More precisely, for $K \in \mathcal{K}_s^n$ and $N \geq n$, we again let $x_1, x_2, \ldots, x_N$ be independent random points distributed uniformly in $K$. We define

$$V_{K,N} = \frac{\text{vol}_n(\text{conv}\{\pm x_1, \pm x_2, \ldots, \pm x_N\})}{\text{vol}_n(K)}.$$ 

The distribution of $V_{K,N}$ is a linear invariant of $K$. We now ask, for each $N \geq n$ and $p \geq 1$, for which $K \in \mathcal{K}_s^n$ does $\mathbb{E}V_{K,N}^p$ achieve its extremal values?

The first goal of this paper is to bring the level of knowledge about these symmetric versions of Sylvester’s problem to a level close to that for the nonsymmetric case. Since $\mathbb{E}U_{K,N}^p$ is already known to be minimized over all $K \in \mathcal{K}_s^n$ exactly when $K$ is an ellipsoid, it is in particular minimized over all $K \in \mathcal{K}_s^n$ exactly when $K$ is an ellipsoid. Furthermore, it was noted in [14] that Groemer’s proof also shows that $\mathbb{E}V_{K,N}^p$ is minimized over all $K \in \mathcal{K}_s^n$ exactly when $K$ is an ellipsoid. Thus in this paper we will deal with the question of which $K \in \mathcal{K}_s^n$ maximize $\mathbb{E}U_{K,N}^p$ and $\mathbb{E}V_{K,N}^p$. We show in Theorem 2.9 that when $n = 2$, each maximum is achieved when $K$ is a parallelogram. Our main tools, which we introduce in Section 2, are symmetric adaptations of tools developed by Campi, Colesanti, and Gronchi [6] to study the nonsymmetric generalized Sylvester’s problem. Following [6], we derive some partial results for general $n$, which in particular support the conjecture that the maximizing symmetric convex bodies should be either parallelotopes or crosspolytopes, or bodies built from these.

The second goal of this paper is to obtain information about the extremal values of $\mathbb{E}V_{K,N}^p$, which we do in Section 3. When $n = 2$, we derive the exact distributions of the random variables $V_{P,2}$ and $V_{E^2,2}$, where $P$ denotes a parallelogram and $E^2$ denotes an ellipse; and we calculate $\mathbb{E}V_{P,2}^p$ and $\mathbb{E}V_{E^2,2}^p$ for all $N \geq 2$. We also calculate $\mathbb{E}V_{E^3,2}^p$ for all $N \geq 3$, where $E^3$ denotes an ellipsoid in $\mathbb{R}^3$. The corresponding extremal values of $\mathbb{E}U_{K,N}^p$ are already available in the literature.

2. RS- and SRS-decomposability

In this section, we recall the notions of RS-movements and RS-decomposability of a convex body, which were introduced by Campi, Colesanti, and Gronchi in [6], and introduce complementary notions for symmetric convex bodies. These tools will be used to address the problem of identification of maximizers of $\mathbb{E}V_{K,N}^p$ and $\mathbb{E}U_{K,N}^p$ for $K \in \mathcal{K}_s^n$.

We first recall the notion of a linear parameter system, due to Rogers and Shephard [20]. For $n \geq 2$, let $K \in \mathcal{K}_s^n$, $\alpha : K \to \mathbb{R}$, and $v \in \mathbb{R}^n \setminus \{0\}$. Then for each $t$ in some interval in $\mathbb{R}$, we set

$$K_t = \text{conv}\{x + t\alpha(x)v : x \in K\}.$$ 

The family of sets $K_t$ is called a linear parameter system with speed function $\alpha$. The most important property of linear parameter systems is the following, proved in [20].

**Theorem 2.1** (Rogers - Shephard). Let $K_t$ be a linear parameter system for $K \in \mathcal{K}_s^n$. Then $\text{vol}_n(K_t)$ is convex as a function of $t$.

As in [6], the interest here is in the case in which the speed function is constant on each chord of $K$ which is parallel to $v$. Let $\pi_v : \mathbb{R}^n \to v^\perp$ denote orthogonal projection, and let $\beta : \pi_v(K) \to \mathbb{R}$. In the terminology of [6], a family of sets

$$K_t = \{x + t\beta(\pi_v(x))v : x \in K\},$$

(1)
for all \( t \) in some interval containing 0, is called an RS-movement of \( K \) if \( K_t \) is convex for each allowed \( t \). A more convenient way to describe \( K_t \) is the following. Given \( v \in \mathbb{R}^n \), there exist functions \( f_v, g_v : \pi_v(K) \to \mathbb{R} \) with \( f_v \) convex and \( g_v \) concave such that

\[
K = \{ x + rv : x \in \pi_v(K), f_v(x) \leq r \leq g_v(x) \}.
\]

Then if \( \beta : \pi_v(K) \to \mathbb{R} \) is given, (1) is equivalent to

\[
K_t = \{ x + rv : x \in \pi_v(K), f_v(x) + t\beta(x) \leq r \leq g_v(x) + t\beta(x) \}.
\]

Note that a necessary and sufficient condition for \( \beta : \pi_v(K) \to \mathbb{R} \) to define an RS-movement is that \( f_v + t\beta \) is convex and \( g_v + t\beta \) is concave for each allowed \( t \).

It is noted in [6] that if \( \beta \) is any affine function on \( \pi_v(K) \), then \( K_t \) as defined by (1) is an RS-movement of \( K \) such that each \( K_t \) is an affine image of \( K \). Moreover, Steiner symmetrization is related to a particular RS-movement as follows. If \( \beta = -(f_v + g_v) \), we obtain an RS-movement of \( K \) such that \( K_1 \) is the reflection of \( K \) with respect to \( v^\perp \), and \( K_{1/2} \) is the Steiner symmetrization of \( K \) with respect to \( v^\perp \).

Now let \( K \in \mathcal{K}_s^n \). We say that an RS-movement \( K_t \) of \( K \) is an SRS-movement if the speed function \( \beta : \pi_v(K) \to \mathbb{R} \) is odd, that is, if \( \beta(-x) = -\beta(x) \). Note that this is precisely the condition which ensures that \( K_t \in \mathcal{K}_s^n \) for each \( t \in [a, b] \). Note that if \( \beta \) is any linear function on \( v^\perp \), then \( K_t \) as defined by (1) is an SRS-movement of \( K \) such that each \( K_t \) is a linear image of \( K \). Furthermore, if \( K \) is symmetric, then for any \( v \in \mathbb{R}^n \), the functions \( f_v, g_v \) in (2) satisfy \( g_v(-x) = -f_v(x) \). For example, the RS-movement with speed function \( \beta = -(f_v + g_v) \), which gives rise to reflection and Steiner symmetrization with respect to \( v^\perp \), is an SRS-movement of \( K \).

Following [6], we say that \( K \in \mathcal{K}_s^n \) is RS-decomposable if there exists an RS-movement \( K_t, t \in (-\varepsilon, \varepsilon) \) for some \( \varepsilon > 0 \), such that \( K_0 = K \) and such that the speed function is not affine. \( K \in \mathcal{K}_s^n \) is called RS-indecomposable if it is not RS-decomposable. In analogy, we say that \( K \in \mathcal{K}_s^n \) is SRS-decomposable if there exists an SRS-movement \( K_t, t \in (-\varepsilon, \varepsilon) \) for some \( \varepsilon > 0 \), such that \( K_0 = K \) and such that the speed function is not linear. \( K \in \mathcal{K}_s^n \) is called SRS-indecomposable if it in not SRS-decomposable.

We remark at this point that to avoid ambiguity, we maintain a strict distinction between affine and linear functions, even in one dimension, so that a linear function \( f : \mathbb{R} \to \mathbb{R} \) required to satisfy \( f(0) = 0 \).

**Example 2.2.** A symmetric parallelogram \( \mathcal{P} \in \mathcal{K}_s^2 \) is SRS-indecomposable.

**Proof.** We may identify \( v^\perp \) with \( \mathbb{R} \). Then there exist \( 0 \leq a \leq b \) such that \( \pi_v(\mathcal{P}) = [-b, b] \) and such that the functions \( f_v, g_v \) as in (2) are affine on each of the intervals \([-b, -a], [-a, a] \), and \([a, b] \). Moreover, one of \( f_v, g_v \) is affine on \([-a, b] \) and the other is affine on \([-b, a] \). Assume without loss of generality that \( f_v \) is affine on \([-a, b] \). In order for \( f_v + t\beta \) to be convex for both positive and negative values of \( t \), \( \beta \) must be linear on \([-a, b] \). Since \( \beta \) is odd, this implies that \( \beta \) is linear on \( \pi_v(\mathcal{P}) \). \( \square \)

If \( K = \text{conv}(K' \cup \{x, -x\}) \), where \( K' \) is a symmetric convex body in a hyperplane \( H \) and \( x \notin H \), then we call \( K \) a double cone with base \( K' \).

**Proposition 2.3.** Let \( K \in \mathcal{K}_s^n \) be either a symmetric cylinder or a double cone. Then \( K \) is SRS-decomposable if and only if its base is.
The proof of this is an almost verbatim repetition of [6, Example 2.6], which shows that a
cylinder or cone is RS-decomposable if and only if its base is. Combined with Example 2.2
Proposition 2.3 implies the following.

Corollary 2.4. Any symmetric parallelootope or crosspolytope in \( \mathbb{R}^n \), \( n \geq 2 \), is SRS-indecomposable.

We note that the results of [6] imply that every simplex is RS-indecomposable, whereas
every parallelootope is RS-decomposable.

The proof of [6, Theorem 3.3] also yields the following.

Proposition 2.5. Let \( K \in \mathcal{K}^n_s \) be such that \( \partial K \) has a nonempty open subset of class \( C^2 \) on
which all the principal curvatures are positive. Then \( K \) is SRS-decomposable.

The main technical result of [6] is the following.

Proposition 2.6 (Campi-Colesanti-Gronchi). Let \( K_t \) be an RS-movement of \( K \in \mathcal{K}^n \). Then
\( E^\mathcal{U}_{K_t,N}^p \) is a convex function of \( t \), for every \( p \geq 1 \) and \( N \geq n + 1 \). Furthermore, \( E^\mathcal{U}_{K_t,N}^p \) is
strictly convex if and only if the speed function is not affine.

Theorem 2.1 is the main tool used to prove this. With minor modifications, the same
proof yields the following.

Proposition 2.7. Let \( K_t \) be an SRS-movement of \( K \in \mathcal{K}^n_s \). Then
\( E^\mathcal{V}_{K_t,N}^p \) is a convex function of \( t \), for every \( p \geq 1 \) and \( N \geq n \). Furthermore, \( E^\mathcal{V}_{K_t,N}^p \) is strictly convex if and only if the speed function is not linear.

As an immediate consequence of Proposition 2.7 and the definition of SRS-decomposability,
we have the following.

Corollary 2.8. Let \( p \geq 1 \) and \( N \geq n \). If \( K \) maximizes \( E^\mathcal{U}_{K,N+1}^p \) or \( E^\mathcal{V}_{K,N}^p \) for all \( K \in \mathcal{K}^n_s \),
then \( K \) is SRS-indecomposable.

Corollary 2.8 and Proposition 2.5 suggest (but do not imply) that the maximizers of
\( E^\mathcal{U}_{K,N+1}^p \) and \( E^\mathcal{V}_{K,N}^p \) are polytopes. Furthermore, Corollary 2.4 shows that the present
method will not rule out the obvious candidates.

Theorem 2.9. For any \( K \in \mathcal{K}^n_s \), \( p \geq 1 \), and \( N \geq 2 \),
\[
E^\mathcal{U}_{K,N+1}^p \leq E^\mathcal{U}_{P,N+1}^p,
\]
\[
E^\mathcal{V}_{K,N}^p \leq E^\mathcal{V}_{P,N}^p,
\]
with strict inequality in both of the above if \( K \) is a symmetric polygon with more than 4
vertices.

Proof. Suppose that \( K \) is a symmetric polygon with vertices \( \pm P_1, \pm P_2, \ldots, \pm P_m \), \( m \geq 3 \),
ordered so that \( P_i, P_{i+1} \) are adjacent for each \( i = 1, 2, \ldots, m-1 \). Then \( P_2, -P_m \) are the
vertices adjacent to \( P_1 \). We set
\[
K_t = \text{conv}\{ \pm (P_1 + t(P_2 + P_m)), \pm P_2, \pm P_3, \ldots, \pm P_m \}.
\]
There exist an \( \varepsilon_1 > 0 \) such that \( P_1 + \varepsilon_1(P_2 + P_m) \) lies on the line through \( P_2 \) and \( P_3 \), and
an \( \varepsilon_2 > 0 \) such that \( P_1 - \varepsilon_2(P_2 + P_m) \) lies on the line through \( -P_m \) and \( -P_{m-1} \). Then
$K_t, t \in [-\varepsilon_2, \varepsilon_1]$ is an SRS-movement such that $K_{\varepsilon_1}$ and $K_{-\varepsilon_2}$ have $2(m-1)$ vertices. Furthermore, this SRS-movement fixes

$$\text{conv}\{\pm P_2, \pm P_3, \ldots, \pm P_m\},$$

and therefore the corresponding speed function is not linear. Thus Proposition 2.7 implies

$$\mathbb{E}U_{K,N+1}^p < \max\{\mathbb{E}U_{K_{\varepsilon_1},N+1}^p, \mathbb{E}U_{K_{-\varepsilon_2},N+1}^p\},$$

$$\mathbb{E}V_{K,N}^p < \max\{\mathbb{E}V_{K_{\varepsilon_1},N}^p, \mathbb{E}V_{K_{-\varepsilon_2},N}^p\}.$$ 

Iterating this argument, we obtain

$$\mathbb{E}U_{K,N+1}^p < \mathbb{E}U_{P,N+1}^p$$

and

$$\mathbb{E}V_{K,N}^p < \mathbb{E}V_{P,N}^p.$$ 

The statement for a general $K \in \mathcal{K}_s$ now follows by the continuity of $\mathbb{E}U_{K,N+1}^p$ and $\mathbb{E}V_{K,N}^p$ as functions of $K$. □

**Corollary 2.10.** For $K \in \mathcal{K}^n$, let $L_K$ denote the isotropic constant of $K$.

1. For any $K \in \mathcal{K}_s^n$, $L_K \leq L_P = (12)^{-1/2}$.
2. For any $K \in \mathcal{K}_s$ with centroid at the origin, $L_K \leq L_\Delta = (108)^{-1/4}$, where $\Delta$ denotes a triangle with centroid at the origin.

**Proof.** The first claim follows directly from Theorem 2.9 and the formula

$$L_K^2 = \frac{1}{4} \left(n! \mathbb{E}V_{K,n}^2\right)^{1/n}$$

for any $K \in \mathcal{K}_s^n$ (see [14]). The second claim follows from the fact that $\mathbb{E}U_{K,N}^p \leq \mathbb{E}U_{\Delta,N}^p$ for $N \geq 3$ and $p \geq 1$ [6] and the formula

$$L_K^2 = \left(\frac{n! \mathbb{E}U_{K,n+1}^2}{n+1}\right)^{1/n}$$

for any $K \in \mathcal{K}_s^n$ with centroid at the origin, due (essentially) to Kingman [13]. □

Schmuckenschläger proved [21] that for all $n$, $L_{B_p^n} \leq L_{B_\infty^n} = (12)^{-1/2}$ for all $1 \leq p \leq \infty$, where $B_p^n$ is the unit ball of $\ell_p^n = (\mathbb{R}^n, \| \cdot \|_p)$. This fact and Corollary 2.10 support the conjecture $L_K \leq (12)^{-1/2}$ for all $K \in \mathcal{K}_s^n, n \in \mathbb{N}$. This may be considered an isometric form of the hyperplane conjecture.

### 3. Calculations for parallelograms and ellipsoids

In this section we calculate some extremal values of $\mathbb{E}V_{K,N}^p$. In Section 3.1, we derive the exact distributions of $V_{K,2}$ when $K$ is either a parallelogram or an ellipse, making essential use of the symmetries of those bodies. In Sections 3.2 and 3.3, we derive general formulas for $\mathbb{E}V_{K,N}$ for $N \geq n$ and $n = 2, 3$ respectively. When $n = 2$ we use these to derive simple expressions in the cases of parallelograms and ellipses; when $n = 3$ we derive an expression for ellipsoids. We also indicate where the corresponding values of $\mathbb{E}U_{K,N}^p$ may be found in the literature.

We remark that if $\mathcal{E}^n$ denotes an ellipsoid in $\mathbb{R}^n$, $\mathbb{E}V_{\mathcal{E}^n,n}^p$ was computed for $n \in \mathbb{N}$ and $p > 0$ by the author in [14], and $\mathbb{E}U_{\mathcal{E}^n,n+1}^p$ was computed for $n, p \in \mathbb{N}$ by Miles in [15].
3.1. Densities when \( n = N = 2 \).

**Proposition 3.1.** \( V_{P,2} \) has density

\[
I_{[0,1]}(t) \int_{2t-1}^{1} (\log |s|)(\log |2t-s|)ds.
\]

**Proof.** We may assume that \( P \) is the square \([-1,1]^2\). Since the symmetric convex hull of two points \( x, y \in \mathbb{R}^2 \) has area \( 2|x_1y_2 - x_2y_1| \), \( V_{P,2} \) has the same distribution as \( \frac{1}{2} |X_1X_2 - X_3X_4| \), where \( X_i, 1 \leq i \leq 4 \), are independent random variables uniformly distributed in \([-1,1]\). By symmetry, \( V_{P,2} \) also has the same distribution as \( \frac{1}{2} |X_1X_2 + X_3X_4| \). We begin by calculating the distribution of \( X_1X_2 \). First note that \( X_1X_2 \) is symmetric. Now, for \( t > 0 \),

\[
\mathbb{P}[X_1X_2 \leq t] = \frac{1}{2} (1 + a_t),
\]

where \( a_t \) is the area of \( \{(x_1, x_2) \in [0,1]^2 : x_1x_2 \leq t\} \). By elementary integration, we obtain \( a_t = t(1 - \log t) \) for \( 0 < t \leq 1 \), and \( a_t = 1 \) for \( t > 1 \). From this we obtain that \( X_1X_2 \) has density

\[
d \frac{d}{dt} \mathbb{P}[X_1X_2 \leq t] = -\frac{1}{2} \log |t|,
\]

supported on \([-1,1]\). The distribution of \( X_1X_2 + X_3X_4 \) is then the convolution of this distribution with itself, so its density is

\[
f(t) = \frac{1}{4} \int_{\max\{-1,-t\}}^{\min\{1,t+1\}} (\log |s|)(\log |t-s|)ds,
\]

supported on \([-2,2]\). Finally, \( \frac{1}{2} |X_1X_2 - X_3X_4| \) has density

\[
4f(2t) = \int_{2t-1}^{1} (\log |s|)(\log |2t-s|)ds,
\]

supported on \([0,1]\). \( \square \)

**Proposition 3.2.** \( V_{E^2,2} \) has density

\[
\pi t I_{[0,\frac{2\pi}{t}]}(t) \int_{\frac{\pi}{2t}}^{1} s^{-2}\sqrt{1-s^2}ds.
\]

**Proof.** We may assume that \( E^2 \) is the unit disc. By the rotational invariance of the uniform measure on \( E^2 \), \( V_{E^2,2} \) has the same distribution as the \( \frac{1}{\pi} \) times the area of the symmetric convex hull of two independent random points, one uniformly distributed in \( E^2 \), the other distributed in the interval \([0,1]\) on the \( x \)-axis with density \( 2t \). Note that since one of the random points lies on the \( x \)-axis, the area of their symmetric convex hull depends only on the absolute value of the \( y \)-coordinate of the other point, which is distributed in \([0,1]\) with density \( \frac{2}{\pi} \sqrt{1-t^2} \). Therefore \( V_{E^2,2} \) has the same distribution as \( \frac{2}{\pi}XY \), where \( X \) and \( Y \) are independent random variables in \([0,1]\) such that \( X \) has density \( 2t \) and \( Y \) has density \( \frac{2}{\pi} \sqrt{1-t^2} \). \( V_{E^2,2} \) then has density supported on \([0,\frac{2\pi}{t}]\) given by

\[
\frac{d}{dt} \mathbb{P}[XY \leq \frac{\pi}{2}t] = \frac{d}{dt} \int_{0}^{1} \mathbb{P}\left[ X \leq \frac{\pi t}{2s} \right] \frac{2}{\pi} \sqrt{1-s^2}ds
\]

\[
= \int_{0}^{1} I_{[0,1]} \left( \frac{\pi t}{2s} \right) \pi ts^{-2}\sqrt{1-s^2}ds.
\]
Exact densities of $U_{K,N}$ have not been derived; however, $\mathbb{E}U_{\Delta,3}^p$ and $\mathbb{E}U_{P,3}^p$ were calculated for all $p \in \mathbb{N}$ by Reed \cite{17}. The values of $\mathbb{E}U_{E^2,3}^p$ for $p \in \mathbb{N}$ are a special case of the above mentioned result of Miles \cite{15}.

3.2. Expected area in an ellipse or parallelogram. In this and the next section we derive general formulas for $\mathbb{E}V_{K,n}$ when $n = 2, 3$. The derivations make use of standard arguments for geometric probability, adapted for the symmetric case; see for example the papers of Rényi and Sulanke \cite{18,19} and Buchta and Reitzner \cite{5} for related formulas derived using similar ideas. Our derivations follow the outline of Buchta and Reitzner’s proof of a nonsymmetric analogue of Proposition 3.6 below.

Let $K \in \mathcal{K}_3^2$. For $r \geq 0$ and $0 \leq \theta < 2\pi$, let

$$\ell(r, \theta) = \text{vol}_1(\{(x, y) : (x, y) \cdot (\cos \theta, \sin \theta) = r\} \cap K),$$

$$A(r, \theta) = \text{vol}_2(\{(x, y) : |(x, y) \cdot (\cos \theta, \sin \theta)| \leq r\} \cap K)$$

$$= 2 \int_0^r \ell(s, \theta) ds.$$

**Proposition 3.3.** Let $K \in \mathcal{K}_3^2$ and $N \geq 2$. If $|K| = 1$, then

$$\mathbb{E}V_{K,N} = 1 - \frac{N}{3} \int_0^{2\pi} \int_0^\infty A(r, \theta)^{N-1} \ell(r, \theta)^3 dr d\theta.$$

**Proof.** We consider a random convex polygon $\Pi_{N+1}$ which is the symmetric convex hull of $N + 1$ independent random points distributed uniformly in $K$. Each of these random points is a vertex of $\Pi_{N+1}$ iff it is not contained in the symmetric convex hull of the other $N$ random points, therefore it is a vertex with probability $1 - \mathbb{E}V_{K,N}$. Each of the random points is also a vertex iff its antipode also is. Therefore the expected number $v_{N+1}$ of vertices of $\Pi_{N+1}$ is

$$v_{N+1} = 2(N + 1)(1 - \mathbb{E}V_{K,N}),$$

and thus

$$\mathbb{E}V_{K,N} = 1 - \frac{v_{N+1}}{2(N + 1)}.$$

The expected number of vertices of $\Pi_{N+1}$ is equal to the expected number of edges of $\Pi_{N+1}$. We thus consider the probability that 2 points $P_1, P_2$ chosen from the $N + 1$ random points and their antipodes define an edge of $\Pi_{N+1}$. If $P_1 = -P_2$, then they define an edge with probability 0. Otherwise, the probability that they define an edge is the probability that the other random points and their antipodes lie on the same side of the line $P_1P_2$, which is the case if the $N - 1$ other random points all lie in the strip between this line and its reflection in the origin. There are $\binom{2(N+1)}{2} - (N + 1) = 2N(N + 1)$ pairs of points which are not antipodal. Therefore we have

$$V_{N+1} = 2N(N + 1) \int_K \int_K A(P_1, P_2)^{N-1} dP_1 dP_2,$$

where $A(P_1, P_2)$ is the area of the intersection of $K$ with the strip described above.

$A(P_1, P_2)$ depends only on the line $P_1P_2$. If this is the line $\{(x, y) : (x, y) \cdot (\cos \theta, \sin \theta) = r\}$ for $r \geq 0, 0 \leq \theta < 2\pi$, then $A(P_1, P_2) = A(r, \theta)$. Now

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
so the rotation above takes the line $P_1P_2$ to the vertical line through $(r, 0)$. Now if $P_i = (x_i, y_i)$ for $i = 1, 2$, we denote 

$$
\begin{pmatrix}
  r \\
  s_i
\end{pmatrix}
= 
\begin{pmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  x_i \\
  y_i
\end{pmatrix},
$$

so that

$$
\begin{pmatrix}
  x_i \\
  y_i
\end{pmatrix}
= 
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  r \\
  s_i
\end{pmatrix}
= 
\begin{pmatrix}
  r \cos \theta - s_i \sin \theta \\
  r \sin \theta + s_i \cos \theta
\end{pmatrix}.
$$

From this follows

$$
dx_1dy_1dx_2dy_2 = |s_1 - s_2|drd\theta ds_1ds_2.
$$

Since

$$
\int_a^b \int_a^b |s_1 - s_2|ds_1ds_2 = \frac{1}{3}(b - a)^3,
$$

we have

$$
\int_K \int_K A(P_1, P_2)^{N-1}dP_1dP_2 = \frac{1}{3} \int_0^{2\pi} \int_0^\infty A(r, \theta)^{N-1}\ell(r, \theta)^3drd\theta,
$$

since $\ell(r, \theta)$ is the length of the intersection of the line $\{(x, y) : (x, y) \cdot (\cos \theta, \sin \theta) = r\}$ with $K$. Note that there is no need to restrict the domain of the integrals on the right hand side above, since the integrand is automatically 0 outside the domain of integration. \hfill \Box

**Corollary 3.4.**

$$
\mathbb{E} V_{P, N} = 1 - \frac{4}{3(N + 1)} \sum_{k=1}^{N+1} \frac{1}{k}
$$

for each $N \geq 2$.

**Proof.** By symmetry, the integral over $0 \leq \theta \leq 2\pi$ in Proposition 3.3 is 8 times the integral over $0 \leq \theta \leq \pi/4$. For $0 < \theta < \pi/4$, we have

$$
\ell(r, \theta) = \sec \theta, \\
A(r, \theta) = 2r \sec \theta
$$

for $0 < r < \frac{1}{2}(\cos \theta - \sin \theta)$;

$$
\ell(r, \theta) = \left(\frac{1}{2} - \frac{r}{\sin \theta} \frac{1}{\cos \theta}\right) \sec \theta, \\
A(r, \theta) = 1 - \left(\frac{1}{2} - \frac{r}{\sin \theta} \frac{1}{\cos \theta}\right)^2 \tan \theta
$$

for $\frac{1}{2}(\cos \theta - \sin \theta) < r < \frac{1}{2}(\cos \theta + \sin \theta)$; and $\ell(r, \theta) = 0$ for $r > \frac{1}{2}(\cos \theta + \sin \theta)$. Using these, the remainder of the proof is elementary integration. \hfill \Box

Similar expressions for $\mathbb{E} U_{P, N}$ and $\mathbb{E} U_{\Delta, N}$ for $N \geq 3$ were derived by Buchta [4].

**Corollary 3.5.**

$$
\mathbb{E} V_{P, N} = 1 - \frac{2N}{3\pi^N} \int_0^{\pi} (t + \sin t)^{N-1}(1 + \cos t)^2 dt
$$

for each $N \geq 2$. 

Proof. We may assume that $\mathcal{E}^2$ is the disc of radius $R = \pi^{-1/2}$. Then $\ell(r, \theta)$ and $A(r, \theta)$ are independent of $\theta$. To apply Proposition 3.3 we need to compute

$$
\int_0^R A(r)^{N-1} \ell(r)^3 dr = R \int_0^{\pi/2} A(R \sin t)^{N-1} \ell(R \sin t)^3 \cos t \, dt.
$$

Now $\ell(R \sin t) = 2R \cos t$, and we have

$$
A(R \sin t) = 2 \int_0^{R \sin t} \ell(s) \, ds = R^2 (2t + \sin 2t).
$$

The claim now follows from Proposition 3.3.

From this we calculate the first few values of $\mathbb{E}V_{\mathcal{E}^2, N}$:

$$
\begin{align*}
\mathbb{E}V_{\mathcal{E}^2, 2} &= \frac{16}{9\pi^2} \approx 0.1801, \\
\mathbb{E}V_{\mathcal{E}^2, 3} &= \frac{35}{12\pi^2} \approx 0.2955, \\
\mathbb{E}V_{\mathcal{E}^2, 4} &= \frac{-5632 + 1575\pi^2}{270\pi^4} \approx 0.3796, \\
\mathbb{E}V_{\mathcal{E}^2, 6} &= \frac{7(-3289 + 600\pi^2)}{432\pi^4} \approx 0.4380.
\end{align*}
$$

A similar expression for $\mathbb{E}V_{\mathcal{E}^2, N}$ for $N \geq 3$ was derived by Efron.

3.3. Expected volume in an ellipsoid. Now let $K \in \mathcal{K}_s^3$. For $r \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi < \pi$, let

$$
\begin{align*}
H(r, \theta, \phi) &= \{(x, y, z) : (x, y, z) \cdot (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) = r\}, \\
A(r, \theta, \phi) &= \text{vol}_2(K \cap H(r, \theta, \phi)), \\
V(r, \theta, \phi) &= \text{vol}_3\{(x, y, z) : |(x, y, z) \cdot (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)| \leq r\} \\
&= 2 \int_0^r A(s, \theta, \phi) \, ds,
\end{align*}
$$

and let $a(r, \theta, \phi) = \mathbb{E}U_{K \cap H(r, \theta, \phi), 3}$.

**Proposition 3.6.** Let $K \in \mathcal{K}_s^3$ and $N \geq 3$. If $|K| = 1$ then

$$
\mathbb{E}V_{K, N} = 1 - \frac{1}{N + 1} - \frac{2N(N - 1)}{3} \int_0^{2\pi} \int_0^\pi \int_0^\infty V(r, \theta, \phi)^{N-2} A(r, \theta, \phi)^3 a(r, \theta, \phi) \sin \phi \, dr \, d\phi \, d\theta.
$$

**Proof.** The basic approach is the same as in the two-dimensional case. We consider a random polyhedron $\Pi_{N+1}$ in $K$ which is the symmetric convex hull of $N+1$ independent random points uniformly distributed in $K$. Let $v_{N+1}$, $e_{N+1}$, $f_{N+1}$ denote the expected number of vertices, edges, and faces, respectively, of $\Pi_{N+1}$. Each of the $N+1$ random points is a vertex of $\Pi_{N+1}$ if it is not contained in the symmetric convex hull of the other $N$ random points, therefore it is a vertex with probability $1 - \mathbb{E}V_{K, N}$. Therefore

$$
V_{N+1} = 2(N + 1)(1 - \mathbb{E}V_{K, N}).
$$

$\Pi_{N+1}$ is simplicial with probability 1, which implies $e_{N+1} = \frac{3}{2} f_{N+1}$. Together with Euler’s formula $v_{N+1} - e_{N+1} + f_{N+1} = 2$, these facts imply

$$
\mathbb{E}V_{K, N} = 1 - \frac{1}{N + 1} - \frac{1}{4(N + 1)} f_{N+1}.
$$
Now choose three points \( P_1, P_2, P_3 \) from the \( N + 1 \) random points and their antipodes, such that no two of the chosen points are antipodes. There are \( 2^3 \binom{N+1}{3} \) such possible choices. The points \( P_1, P_2, P_3 \) span a face of \( \Pi_{N+1} \) iff all of the other random points and their antipodes lie in the slab between the plane \( H(P_1, P_2, P_3) \) containing \( P_1, P_2, P_3 \) and its opposite. Therefore

\[
f_{N+1} = 8 \left( \frac{N + 1}{3} \right) \int_K \int_K \int_K V(P_1, P_2, P_3)^{N-2} dP_1 dP_2 dP_3,
\]

where \( V(P_1, P_2, P_3) \) is the volume of the intersection of \( K \) with the slab described above.

\( V(P_1, P_2, P_3) \) depends only on the plane \( H(P_1, P_2, P_3) \). If \( H(P_1, P_2, P_3) = H(r, \theta, \phi) \), then we change variables by first rotating by \( \theta, \phi \) (need geometric description here). This will take \( H(P_1, P_2, P_3) \) to the plane parallel to the \( xy \) plane through the point \( (r, 0, 0) \), that is, to the plane \( H(r, 0, \pi/2) \). If \( P_i = (x_i, y_i, z_i) \) is taken to \( (r, s_i, t_i) \) by these rotations for \( i = 1, 2, 3 \), then we have

\[
\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin \phi & 0 & -\cos \phi \\ 0 & 1 & 0 \\ \cos \phi & 0 & \sin \phi \end{pmatrix} \begin{pmatrix} r \\ s_i \\ t_i \end{pmatrix}.
\]

This change of variables has the Jacobian

\[
\begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} \sin \phi.
\]

The claim now follows since

\[
\begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix}
\]

is twice the area of the convex hull of \( P_1, P_2, P_3 \). As in the proof of Proposition 3.3 there is no need to restrict the domain of integration at this point.

\[\square\]

**Corollary 3.7.**

\[
\mathbb{E} V_{\mathcal{E}^3, N} = 1 - \frac{1}{N + 1} - \frac{105N(N - 1)}{2N^5} \int_0^1 (1 - t^2)^4 (3t^2 - t^3)^{N-2} dt.
\]

**Proof.** We may assume that \( \mathcal{E}^3 \) is the ball of radius \( R = (\frac{3}{4\pi})^{1/3} \). Then for any \( \theta, \phi \), and \( r < R \), \( H(r, \theta, \phi) \) is a disc of radius \( \sqrt{R^2 - r^2} \), so

\[
A(r) = \pi (R^2 - r^2),
\]

\[
a(r) = \frac{35}{48\pi^2} A(r) = \frac{35}{48\pi} (R^2 - r^2),
\]

\[
V(r) = 2\pi \int_0^r (R^2 - s^2) ds = 2\pi \left( R^2 r - \frac{1}{3} r^3 \right).
\]

The claim then follows from Proposition 3.6. \( \square \)

From this we calculate the first few values of \( \mathbb{E} V_{\mathcal{E}^3, N} \):

\[
\begin{align*}
\mathbb{E} V_{\mathcal{E}^3, 3} &= \frac{27}{512}, & \mathbb{E} V_{\mathcal{E}^3, 4} &= \frac{72}{715}, \\
\mathbb{E} V_{\mathcal{E}^3, 5} &= \frac{585}{4096}, & \mathbb{E} V_{\mathcal{E}^3, 6} &= \frac{58104}{323323}.
\end{align*}
\]
A similar expression for $E U_{3,N}$ for $N \geq 4$ was derived by Efron \cite{8}.

Buchta and Reitzner \cite{5} use a nonsymmetric analogue of Proposition \ref{proposition5.6} to derive an expression for $E U_{T,N}$ for $N \geq 4$, where $T$ is a tetrahedron. It is natural to ask whether Proposition \ref{proposition5.6} can be used to calculate $E V_{K,N}$ when $K$ is a cube or octahedron. The chief difficulty comes from the appearance of the quantity $a(r, \theta, \phi)$ in the integrand, which depends in general on the shape of the planar sections of $K$. In the case of the tetrahedron, these sections are either triangles or quadrilaterals, for which formulas for the expected area of the convex hull of three random points are known. For polyhedra with more facets, planar sections can be polygons for which the necessary values of $a(r, \theta, \phi)$ are not known.

Unfortunately, it does not seem feasible to extend directly the approach in this and the previous section to $n \geq 4$. The reason is that the proofs of Propositions \ref{proposition3.3} and \ref{proposition5.6} actually calculate the expected number of facets of $\Pi_{N+1}$, whereas $E V_{K,N}$ is directly related to the expected number of vertices of $\Pi_{N+1}$. In the plane, these are equal, and in $\mathbb{R}^3$ they are related via Euler’s formula with the fact that $\Pi_{N+1}$ is almost surely simplicial. If $n \geq 4$ however, the number of facets of a simplicial polytope does not uniquely determine the number of vertices.

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