A class of second order
dilation invariant inequalities

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Abstract

We compute the best constants in some dilation invariant inequalities
for the weighted $L^2$-norms of $-\Delta u$ and $\nabla u$, with weights being powers of
the distance from the origin.

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mensions, weighted biharmonic operator

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1 Introduction

In recent years, there has been a growing interest in dilation invariant in-
equalities that are somehow related with the famous Rellich inequality [15],
[16]. We shall not attempt to provide a complete list of references on this
subject. However, among the more recent contributions we cite [1]–[13], [17]
and references therein.

In the present paper we study a class of inequalities for the weighted
$L^2$-norms of $-\Delta u$ and $\nabla u$. More precisely, let $n \geq 2$ be a given integer, let
$\alpha \in \mathbb{R}$ be a varying parameter, and let $\Sigma$ be a regular domain in $\mathbb{S}^{n-1}$. We
are interested in inequalities of the form

$$\int_{\mathcal{C}_\Sigma} |x|^{\alpha} |\Delta u|^2 \, dx \geq c \int_{\mathcal{C}_\Sigma} |x|^{\alpha-2} |\nabla u|^2 \, dx \quad \text{for any } u \in C^2_c(\overline{\mathcal{C}_\Sigma} \setminus \{0\}) \quad (1.1)$$

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where \( C_\Sigma \) denotes the cone in \( \mathbb{R}^n \) spanned by \( \Sigma \), namely
\[
C_\Sigma = \left\{ x \in \mathbb{R}^n \setminus \{0\} \left| \frac{x}{|x|} \in \Sigma \right. \right\}.
\]

Notice that \( C_\Sigma = \mathbb{R}^n \setminus \{0\} \) when \( \Sigma = S^{n-1} \). Our aim is to compute the best constant
\[
\delta_{n,\alpha}(C_\Sigma) := \inf_{u \in C_2^0(C_\Sigma \setminus \{0\}) \atop u \neq 0} \frac{\int_{C_\Sigma} |x|^{\alpha} |\Delta u|^2 dx}{\int_{C_\Sigma} |x|^{\alpha - 2} |\nabla u|^2 dx}.
\]

In fact this goal was already accomplished by Ghoussoub and Moradifam in [11] in the case of the whole space. However we provide alternative proofs which naturally adapt to handle with cone like domains. Even if this generalization to cones seems to have a somehow artificial flavour, in fact in our opinion it contains some deeper features. Firstly it allows us to consider the case of domains, even very regular, like the half-space, such that the singularity stays on the boundary. Moreover our results are stated in a fashion which makes clearer the expression of the best constant even in the case of the whole space. This fact is strongly related to the peculiar approach followed here.

We also mention the papers [2] and [14] dealing with a class of inequalities for radially symmetric functions on \( \mathbb{R}^n \) in the non Hilbertian case, that is, involving the weighted \( L^p \)-norms of \( -\Delta u \) and \( \nabla u \), with \( p > 1 \).

In order to state our main results we put
\[
\gamma_{n,\alpha} = \frac{(n - 4 + \alpha)(n - \alpha)}{4}, \quad h_{n,\alpha} = \left( \frac{n - 4 + \alpha}{2} \right)^2.
\]

Given a domain \( \Sigma \) in \( S^{n-1} \) with \( \partial \Sigma \in C^2 \), we denote by \( \Lambda_\Sigma \) the spectrum of the Laplace-Beltrami operator on \( \Sigma \) with null boundary conditions and by \( \lambda_\Sigma \) the first eigenvalue. Notice that \( \lambda_\Sigma > 0 \) apart from the case \( \Sigma = S^{n-1} \).

**Theorem 1.1** Let \( n \geq 2 \) and let \( \Sigma \) be a domain in \( S^{n-1} \) with \( \partial \Sigma \in C^2 \). Assume \( \alpha \neq 4 - n \). Then the following facts hold.

(i) \( \delta_{n,\alpha}(C_\Sigma) > 0 \) if and only if \( -\gamma_{n,\alpha} \notin \Lambda_\Sigma \). Moreover
\[
\delta_{n,\alpha}(C_\Sigma) \leq M_{n,\alpha}(\Sigma) := \min_{\lambda \in \Lambda_\Sigma} \frac{(\gamma_{n,\alpha} + \lambda)^2}{h_{n,\alpha} + \lambda}. \tag{1.2}
\]

(ii) If \( \gamma_{n,\alpha} - 2h_{n,\alpha} \leq \lambda_\Sigma \) then \( \delta_{n,\alpha}(C_\Sigma) = M_{n,\alpha}(\Sigma) \).
When $\Sigma = \mathbb{S}^{n-1}$ we can be more precise. First of all, as well as in [11], we have the following sharp result for the best constant in the class of radially symmetric functions.

**Theorem 1.2** Let $n \geq 2$ and $\alpha \in \mathbb{R}$. Then

$$\delta_{\text{rad}}^{n,\alpha} := \inf_{\substack{\alpha \in C_c^2(\mathbb{R}^n \setminus \{0\}) \quad \text{u} \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^2 \, dx}{\int_{\mathbb{R}^n} |x|^{\alpha - 2} |\nabla u|^2 \, dx} = \left(\frac{n - \alpha}{2}\right)^2.$$ 

Theorem 1.2 will be proved in Section 2.3. When we allow $u$ to be any function in $C_c^2(\mathbb{R}^n \setminus \{0\})$ non necessarily radial we can estimate the best constant with the aid of Theorem 1.1 and using the explicit knowledge of the spectrum of the Laplace-Beltrami operator on the sphere:

$$\Lambda_{\mathbb{S}^{n-1}} = \{k(n - 2 + k) \mid k \in \mathbb{N} \cup \{0\}\}.$$ 

In particular $\lambda_{\mathbb{S}^{n-1}} = 0$. To simplify the notation, we write $\delta_{n,\alpha}$ instead of $\delta_{n,\alpha}(\mathbb{R}^n \setminus \{0\})$, and $M_{n,\alpha}$ instead of $M_{n,\alpha}(\mathbb{S}^{n-1})$. The results stated in the next theorems are already known (see [11]) but we prove them in a different way.

**Theorem 1.3** Let $n \geq 2$ and assume $\alpha \neq 4 - n$.

(i) If $n = 2$ then $\delta_{2,\alpha} = M_{2,\alpha}$ for any $\alpha \in \mathbb{R}$.

(ii) If $n \geq 3$ then there exists $\alpha^* \in [4 - n, 2)$ such that $\delta_{n,\alpha} = M_{n,\alpha}$ for any $\alpha \notin [4 - n, \alpha^*)$.

(iii) If $n \geq 3$ and $\alpha^* < \alpha < n$ then $\delta_{n,\alpha} = \delta_{\text{rad}}^{n,\alpha}$.

In the “critical case” $\alpha = 4 - n$ a very singular phenomenon can be observed.

**Theorem 1.4** If $\alpha = 4 - n$ then

$$\delta_{n,4-n} = \min \{ (n - 2)^2, \ n - 1 \}.$$ 

In particular,

$\delta_{n,4-n} > 0$ for any $n \geq 3$ and $\delta_{n,4-n} = n - 1 < \delta_{\text{rad}}^{n,4-n}$ for any $n \geq 4.$
It should be emphasized the fact that the function \( \alpha \mapsto \delta_{n,\alpha} \) is not continuous at \( \alpha = 4 - n \), unless \( n = 2 \). Let us make some remarks about the above results in the meaningful case \( \alpha = 0 \). First notice that in two dimensions \( \delta_{2,0} = 0 < \delta_{2,0}^{\text{rad}} = 1 \). In dimension \( n = 3 \) the best constant \( \delta_{3,0} \), already known according to the paper [11] can be computed by means of the formula for \( M_{3,0} \) and yields:

\[
\int_{\mathbb{R}^3} |\Delta u|^2 \, dx \geq \frac{25}{36} \int_{\mathbb{R}^3} |x|^{-2} |\nabla u|^2 \, dx \quad \text{for any } u \in C_c^2(\mathbb{R}^3 \setminus \{0\}).
\]

Notice that \( \delta_{3,0}^{\text{rad}} = 9/4 \) is larger than the best constant on the whole space and breaking symmetry occurs. A similar phenomenon appears in the critical dimension \( n = 4 \). Indeed \( \delta_{4,0}^{\text{rad}} = 4 \), while from Theorem 1.4 it follows that 3 is the best constant in the inequality

\[
\int_{\mathbb{R}^4} |\Delta u|^2 \, dx \geq 3 \int_{\mathbb{R}^4} |x|^{-2} |\nabla u|^2 \, dx \quad \text{for any } u \in C_c^2(\mathbb{R}^4 \setminus \{0\}).
\]

To handle higher dimensions we estimate

\[
\alpha^* < \frac{1}{3} \left( n + 4 - 2\sqrt{n^2 - n + 1} \right). \tag{1.3}
\]

Notice that \( \alpha^* < 0 \) if \( n \geq 5 \). A standard density result can be used to infer the next corollary.

**Corollary 1.5** Assume \( n \geq 5 \). Then

\[
\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \geq \frac{n^2}{4} \int_{\mathbb{R}^n} |x|^{-2} |\nabla u|^2 \, dx \quad \text{for any } u \in D^{2,2}(\mathbb{R}^n),
\]

and \( n^2/4 \) is the best constant.

## 2 Proofs

The only tools we use are the Cauchy-Schwarz inequality, integration by parts, the variational characterization of the eigenvalues, and the Emden-Fowler transform \( u \mapsto w = T u \), that is defined via

\[
u(x) = |x|^{2-n-\alpha} w \left( -\log |x|, \frac{x}{|x|} \right).\]
Such a transform $T$ maps functions $u: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ into functions $w = w(s, \sigma)$ on the cylinder $\mathbb{R} \times S^{n-1}$. More generally, given a domain $\Sigma$ in $S^{n-1}$, let us denote

$$Z_\Sigma := \mathbb{R} \times \Sigma$$

the corresponding cylinder. We point out that $w \in C^2_c(Z_\Sigma)$ as $u \in C^2_c(C_\Sigma \setminus \{0\})$. Moreover, by direct computation (see for instance [6]), it can be proved that

$$\delta_{n,\alpha}(C_\Sigma) = \inf_{w \in C^2_c(Z_\Sigma) \setminus \{0\}} \int_{Z_\Sigma} |\Delta_\sigma w + w_{ss} + (\alpha - 2)w - \gamma_{n,\alpha}w|^2 \, dsd\sigma \geq \lambda.$$

Here and in the rest of the paper we denote by $-\Delta_\sigma$, $\nabla_\sigma$ the Laplace-Beltrami operator and the gradient on $S^{n-1}$, respectively, while $w_s$ is the derivative of $w$ with respect to $s \in \mathbb{R}$.

### 2.1 Some notation and technical lemmas

For every eigenvalue $\lambda \in \Lambda_\Sigma$ let

$$Y_\lambda := \{g\varphi \mid g \in C^2_c(\mathbb{R}), \varphi \text{ eigenfunction corresponding to } \lambda\}.$$

Notice that $Y_\lambda \subset C^2_c(Z_\Sigma)$. Moreover set

$$V_\lambda := \{v \in C^2_c(Z_\Sigma) \mid \int_{Z_\Sigma} vw \, dsd\sigma = 0 \forall w \in Y_\lambda, \forall \lambda' \in \Lambda_\Sigma, \lambda' < \lambda\}.$$

Hence $V_\lambda \supset Y_\lambda$ and $V_\lambda = C^2_c(Z_\Sigma)$ when $\lambda = \lambda_\Sigma$.

**Lemma 2.1** For every $\lambda \in \Lambda_\Sigma$ one has that

$$\inf_{w \in V_\lambda \setminus \{0\}} \int_{Z_\Sigma} (|\nabla_\sigma w|^2 + |w_s|^2) \, dsd\sigma \geq \int_{Z_\Sigma} |w|^2 \, dsd\sigma \geq \lambda.$$

**Proof.** Clearly

$$\inf_{w \in V_\lambda \setminus \{0\}} \int_{Z_\Sigma} (|\nabla_\sigma w|^2 + |w_s|^2) \, dsd\sigma = \inf_{w \in V_\lambda \setminus \{0\}} \int_{Z_\Sigma} |\nabla_\sigma w|^2 \, dsd\sigma.$$

Then the conclusion follows from the fact that every mapping $w \in V_\lambda$ is orthogonal to $Y_\lambda'$ for any eigenvalue $\lambda' < \lambda$ and from the variational characterization of the eigenvalues. \qed
For $A, B, C \in \mathbb{R}$ set

$$N_{A,B}(w) = \int_{\Sigma} \left| \Delta_{\sigma}w + w_{ss} + Aw_s - Bw \right|^2 dsd\sigma$$

$$D_C(w) = \int_{\Sigma} \left( \left| \nabla_{\sigma} w \right|^2 + \left| w_s \right|^2 \right) dsd\sigma + C \int_{\Sigma} \left| w \right|^2 dsd\sigma .$$

(2.1)

Moreover, for $\lambda \in \Lambda_{\Sigma}$, set

$$M_{\lambda}(A, B, C) = \inf_{w \in V_{\lambda} \setminus \{0\}} \frac{N_{A,B}(w)}{D_C(w)} \quad \text{and} \quad \bar{M}_{\lambda}(A, B, C) = \inf_{w \in V_{\lambda} \setminus \{0\}} \frac{N_{A,B}(w)}{D_C(w)} .$$

**Lemma 2.2** For every $\lambda \in \Lambda_{\Sigma}$, if $0 < B + \lambda \leq 2(C + \lambda)$, then

$$M_{\lambda}(A, B, C) \geq \frac{(B + \lambda)^2}{C + \lambda} .$$

**Proof.** For every $w \in V_{\lambda}$, integrating by parts and using Cauchy-Schwarz inequality, we obtain

$$D_B(w) = -\int_{\Sigma} w \left( \Delta_{\sigma}w + w_{ss} + Aw_s - Bw \right) dsd\sigma$$

$$\leq \left( \int_{\Sigma} \left| w \right|^2 dsd\sigma \right)^{\frac{1}{2}} \left( \int_{\Sigma} \left| w_s \right|^2 dsd\sigma \right)^{\frac{1}{2}} .$$

Then for $w \in V_{\lambda} \setminus \{0\}$ we have

$$\frac{N_{A,B}(w)}{D_C(w)} \geq \frac{(R(w) + B)}{R(w) + C} \quad \text{where} \quad R(w) = \frac{\int_{\Sigma} \left( \left| \nabla_{\sigma} w \right|^2 + \left| w_s \right|^2 \right) dsd\sigma}{\int_{\Sigma} \left| w \right|^2 dsd\sigma} .$$

Therefore, using Lemma 2.1 we infer that

$$M_{\lambda}(A, B, C) \geq \inf_{r \geq \lambda} \frac{(B + r)^2}{C + r} = \frac{(B + \lambda)^2}{C + \lambda} .$$

where the last equality can be obtained by elementary calculus using the assumptions on $B$ and $C$. 

\[\square\]
Lemma 2.3 For every $\lambda \in \Lambda_\Sigma$, if $A^2 + 2(B + \lambda) > (B + \lambda)^2/(C + \lambda)$ and $C + \lambda > 0$, then

$$
\tilde{M}_\lambda(A, B, C) \geq \frac{(B + \lambda)^2}{C + \lambda}.
$$

Proof. Using the definition of $Y_\lambda$ we obtain that

$$
\tilde{M}_\lambda(A, B, C) = \inf_{g \in C^2(R)} \frac{\int_{-\infty}^{\infty} |g'' + Ag' - (B + \lambda)g|^2 \, ds}{\int_{-\infty}^{\infty} (|g'|^2 + (C + \lambda)|g|^2) \, ds}.
$$

To simplify notation, we can assume that

$$
\int_{-\infty}^{\infty} |g|^2 \, ds = 1.
$$

Integration by parts yields

$$
\int_{-\infty}^{\infty} |g'' + Ag' - (B + \lambda)g|^2 \, ds = \int_{-\infty}^{\infty} |g''|^2 \, ds
$$

$$
+ (A^2 + 2(B + \lambda)) \int_{-\infty}^{\infty} |g'|^2 \, ds + (B + \lambda)^2.
$$

Moreover by Cauchy-Schwarz and Young inequality we estimate

$$
\int_{-\infty}^{\infty} |g'|^2 \, ds = -\int_{-\infty}^{\infty} g'' g \, ds \leq \left( \int_{-\infty}^{\infty} |g''|^2 \, ds \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2} + \frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |g''|^2 \, ds.
$$

Then

$$
\frac{\int_{-\infty}^{\infty} |g'' + Ag' - (B + \lambda)g|^2 \, ds}{\int_{-\infty}^{\infty} (|g'|^2 + (C + \lambda)|g|^2) \, ds}
$$

$$
\geq \frac{(A^2 + 2(B + \lambda + \varepsilon)) \int_{-\infty}^{\infty} |g'|^2 \, ds + (B + \lambda)^2 - \varepsilon^2}{\int_{-\infty}^{\infty} |g'|^2 \, ds + C + \lambda}
$$

and consequently

$$
\tilde{M}_\lambda(A, B, C) \geq (A^2 + 2(B + \lambda + \varepsilon)) \inf_{t \geq 0} \frac{B_2 + t}{C + \lambda + t}
$$

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where
\[ B_\varepsilon = \frac{(B + \lambda)^2 - \varepsilon^2}{A^2 + 2(B + \lambda + \varepsilon)}. \]

By the assumptions on \( A, B \) and \( C \) we have that \( C + \lambda > B_\varepsilon > 0 \) for \( \varepsilon > 0 \) small enough. Then, by elementary calculus,
\[ \inf_{t \geq 0} \frac{B_\varepsilon + t}{C + \lambda + t} = \frac{B_\varepsilon}{C + \lambda}. \]

Hence for \( \varepsilon > 0 \) small enough
\[ \widetilde{M}(A, B, C) \geq \frac{(B + \lambda)^2 - \varepsilon^2}{C + \lambda}. \]

and letting \( \varepsilon \to 0 \) we get the conclusion. \( \square \)

2.2 Proof of Theorem 1.1

Fix \( \alpha \in \mathbb{R} \) and \( n \in \mathbb{N} \), \( n \geq 2 \). For every \( w \in C^2_c(\Sigma) \) set
\[ N(w) = \int_{\Sigma} |\Delta w + w_{ss} + (\alpha - 2)w - \gamma_{n,\alpha} w_s|^2 \, dsd\sigma \]
\[ D(w) = \int_{\Sigma} (|\nabla w|^2 + |w_s|^2) \, dsd\sigma + h_{n,\alpha} \int_{\Sigma} |w|^2 \, dsd\sigma. \]

Notice that according to the notation (2.1) we have that
\[ A = \alpha - 2 \]
\[ B = \gamma_{n,\alpha} \]
\[ C = h_{n,\alpha}. \]  

(2.2)

Proof of (i). Since \( \alpha \neq 4 - n \), then \( h_{n,\alpha} > 0 \) and therefore the functional \( D \) is the square of an equivalent Hilbertian norm on \( H^1(\Sigma) \). Assume that \( -\gamma_{n,\alpha} \not\in \Lambda_\Sigma \). In this case, by the results in [6], the functional \( N \) is the square of an equivalent Hilbertian norm on \( H^2(\Sigma) \). Therefore, since with the above notation
\[ \delta_{n,\alpha}(C_\Sigma) = \inf \left\{ \frac{N(w)}{D(w)} \mid w \in C^2_c(\Sigma), \ w \neq 0 \right\}, \]

as \( H^2(\Sigma) \) is continuously embedded into \( H^1(\Sigma) \), we obtain \( \delta_{n,\alpha}(C_\Sigma) > 0 \). The fact that \( \delta_{n,\alpha}(C_\Sigma) = 0 \) if \( -\gamma_{n,\alpha} \in \Lambda_\Sigma \) is a consequence of (1.2). To check
we fix \( \lambda \in \Lambda_{\Sigma} \) and we estimate

\[
\delta_{n,\alpha}(C_{\Sigma}) \leq \inf_{w \in Y_{\lambda} \setminus \{0\}} \frac{N(w)}{D(w)}
\]

\[
= \inf_{g \in C^{2}_{c}(\mathbb{R}) \setminus \{0\}} \int_{\mathbb{R}} |g'' + (\alpha - 2)g' - (\gamma_{n,\alpha} + \lambda)g|^2 ds \geq \frac{(\gamma_{n,\alpha} + \lambda)^2}{h_{n,\alpha} + \lambda}.
\]

The last equality can be easily checked taking \( g(s) = g_{0}(\varepsilon s) \) with \( g_{0} \in C^{2}_{c}(\mathbb{R}) \) fixed, \( g_{0} \neq 0 \), and \( \varepsilon > 0 \), and letting \( \varepsilon \to 0 \). Then (1.2) follows from the arbitrariness of \( \lambda \in \Lambda_{\Sigma} \).

**Proof of (ii).** It suffices to study the case \(-\gamma_{n,\alpha} \not\in \Lambda_{\Sigma}\), since otherwise \( \delta_{n,\alpha}(C_{\Sigma}) = M_{n,\alpha}(\Sigma) = 0 \). Let us distinguish the argument according that \(-\gamma_{n,\alpha}\) stays below the spectrum or not.

**Case** \(-\gamma_{n,\alpha} < \lambda_{\Sigma}\).

Since \( C^{2}_{c}(\overline{Z_{\Sigma}}) = V_{\lambda_{\Sigma}} \), we have that \( \delta_{n,\alpha}(C_{\Sigma}) = M_{\lambda_{\Sigma}}(A, B, C) \) with \( A \), \( B \), and \( C \) given as in (2.2). We apply Lemma 2.2 with \( \lambda = \lambda_{\Sigma} \). The condition \( B + \lambda > 0 \) is fulfilled since we are dealing with the case \(-\gamma_{n,\alpha} < \lambda_{\Sigma}\). The condition \( B + \lambda \leq 2(C + \lambda) \) is equivalent to say \( \gamma_{n,\alpha} \leq 2h_{n,\alpha} + \lambda_{\Sigma} \). Hence if \(-\lambda_{\Sigma} < \gamma_{n,\alpha} \leq 2h_{n,\alpha} + \lambda_{\Sigma}\) then

\[
\delta_{n,\alpha}(C_{\Sigma}) \geq \frac{(\gamma_{n,\alpha} + \lambda_{\Sigma})^2}{h_{n,\alpha} + \lambda_{\Sigma}} \geq M_{n,\alpha}(\Sigma).
\]

Hence, in this case, by (1), \( \delta_{n,\alpha}(C_{\Sigma}) = M_{n,\alpha}(\Sigma) \).

**Case** \(-\gamma_{n,\alpha} > \lambda_{\Sigma}\).

We can find two consecutive eigenvalues \( \lambda_{k-1} \) and \( \lambda_{k} \) such that

\[
\lambda_{k-1} < -\gamma_{n,\alpha} < \lambda_{k}.
\]

Any \( w \in C^{2}_{c}(\overline{Z_{\Sigma}}) \) can be written according to the following decomposition

\[
w = v_{1} + ... + v_{k}
\]

with \( v_{j} \in Y_{\lambda_{j}} \) for \( j = 1, ..., k-1 \), and \( v_{k} \in V_{\lambda_{k}} \). One easily checks that

\[
\frac{N(w)}{D(w)} = \sum_{j=1}^{k} \theta_{j} \frac{N(v_{j})}{D(v_{j})} \quad \text{where} \quad \theta_{j} = \frac{D(v_{j})}{D(w)}.
\]
Since \( \theta_j \geq 0 \) for all \( j = 1, \ldots, k \) and \( \theta_1 + \ldots + \theta_k = 1 \), we have that

\[
\frac{N(w)}{D(w)} \geq \min_{j=1,\ldots,k} \frac{N(v_j)}{D(v_j)}. \tag{2.3}
\]

We estimate \( N(v_j)/D(v_j) \) for \( j = 1, \ldots, k - 1 \) by means of Lemma 2.3 with \( \lambda = \lambda_j \) and \( A, B, \) and \( C \) as in (2.2). The condition \( C + \lambda > 0 \) is fulfilled as \( \lambda_j \geq 0 \) and \( h_{n,\alpha} > 0 \) since, by hypothesis, \( \alpha \neq 4 - n \). The condition \( A^2 + 2(B + \lambda) > (B + \lambda)^2/(C + \lambda) \) can be checked by considering the function

\[
\Phi(t) = \left(2t + \frac{(n-2)^2}{2} + \frac{(\alpha-2)^2}{2}\right)(t + h_{n,\alpha}) - (t + \gamma_{n,\alpha})^2.
\]

One has that \( \Phi(0) = h_{n,\alpha}^2 > 0 \) and \( \Phi'(0) = 2h_{n,\alpha} + (\alpha-2)^2 > 0 \). Then \( \Phi(t) > 0 \) for all \( t \geq 0 \). In particular \( \Phi(\lambda_j) > 0 \) and

\[
A^2 + 2(B + \lambda_j) - \frac{(B + \lambda_j)^2}{C + \lambda_j} = \frac{\Phi(\lambda_j)}{h_{n,\alpha} + \lambda_j} > 0.
\]

Hence Lemma 2.3 applies and yields

\[
\frac{N(v_j)}{D(v_j)} \geq \frac{(\gamma_{n,\alpha} + \lambda_j)^2}{h_{n,\alpha} + \lambda_j} \geq M_{n,\alpha}(\Sigma) \quad \forall j = 1, \ldots, k - 1. \tag{2.4}
\]

In order to estimate \( N(v_k)/D(v_k) \) we apply Lemma 2.2 with \( \lambda = \lambda_k \) and \( A, B, \) and \( C \) as in (2.2). The condition \( B + \lambda > 0 \) is satisfied since \( -\gamma_{n,\alpha} < \lambda_k \). The other condition \( B + \lambda \leq 2(C + \lambda) \) is also fulfilled since

\[
2(C + \lambda) - (B + \lambda) = 2h_{n,\alpha} - \gamma_{n,\alpha} + \lambda_k > 2h_{n,\alpha} - \gamma_{n,\alpha} + \lambda_{\Sigma} > 0
\]

by the assumption made in (ii). Therefore Lemma 2.2 applies and thus

\[
\frac{N(v_k)}{D(v_k)} \geq \frac{(\gamma_{n,\alpha} + \lambda_k)^2}{h_{n,\alpha} + \lambda_k} \geq M_{n,\alpha}. \tag{2.5}
\]

In conclusion by (2.3)–(2.5) and by the arbitrariness of \( w \in C_c^2(\overline{Z_{\Sigma}}) \) one concludes as in the first case. □
2.3 Proof of Theorem 1.2

For a fixed radial function $u \in C^2_c(\mathbb{R}^n \setminus \{0\})$ we introduce the radially symmetric function

$$v(x) = |x|^{2-n-\alpha/2}u_r(x),$$

where $u_r$ is the radial derivative of $u$. Then

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^2 \, dx = \int_{\mathbb{R}^n} |x|^\alpha \left| \frac{n - \alpha}{2} |x|^{-1} u_r + |x|^{\frac{2-n-\alpha}{2}} v_r \right| \, dx$$

$$= \left( \frac{n - \alpha}{2} \right)^2 \int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^2 \, dx + \int_{\mathbb{R}^n} |x|^{2-n}|\nabla v|^2 \, dx,$$

since the double product vanishes:

$$\int_{\mathbb{R}^n} |x|^{\frac{2-n-\alpha}{2}} v_r u_r \, dx = \int_{\mathbb{R}^n} |x|^{1-n} v_r \, dx = c \int_0^\infty (v^2) \, dr = 0.$$

The conclusion is immediate. □

2.4 Proof of Theorem 1.3

We apply Theorem 1.1 considering that we deal with the case $\lambda_\Sigma = 0$.

Proof of (i). If $n = 2$ and $\alpha \neq 2$ then $\gamma_{n,\alpha} = -(\alpha - 2)^2/4 < 0$. The condition

$$\gamma_{n,\alpha} - 2h_{n,\alpha} \leq \lambda_\Sigma$$

holds true for every $\alpha \neq 2$ and thus one can conclude.

Proof of (ii). Consider now the case $n \geq 3$. Suppose $\gamma_{n,\alpha} > 0$ i.e. $\alpha \in (4-n, n)$. In this case the condition (2.6) holds true if and only if $\alpha \geq (n-8)/3$. When $\gamma_{n,\alpha} < 0$, the condition (2.6) always holds true. Hence (ii) is proved with $\alpha^* < (n-8)/3$.

Proof of (iii). If $n \geq 3$ and $\alpha \in (\alpha^*, n)$ then $\gamma_{n,\alpha} > 0$ and the mapping $t \mapsto (\gamma_{n,\alpha} + t)^2/(h_{n,\alpha} + t)$ is increasing in $[0, \infty)$. Hence $M_{n,\alpha} = \gamma_{n,\alpha}^2/h_{n,\alpha} = 2\delta_{n,\alpha}^{rad}$, by Theorem 1.2. □

2.5 Proof of Theorem 1.4

First notice that $\delta_{n,4-n} \leq \delta_{n,4-n}^{rad} = (n-2)^2$ by Theorem 1.2. Now we prove that $\delta_{n,4-n} \leq n-1$. Notice that $\gamma_{n,4-n} = h_{n,4-n} = 0$. We estimate $\delta_{n,4-n}$ with a family of mappings $w(s, \sigma) = g(\varepsilon s)\varphi(\sigma)$ where $g \in C^2_c(\mathbb{R})$ is any
nontrivial fixed function, \( \varepsilon > 0 \) and \( \varphi \) is an eigenfunction for \(-\Delta_\sigma \) on \( S^{n-1} \) relative to the first positive eigenvalue \((n-1)\). In this way we obtain

\[
\delta_{n,4-n} \leq \frac{\int_{\mathbb{R}} \left| \varepsilon^2 g'' + (\alpha - 2)\varepsilon g' - (n-1)g \right|^2 ds}{\varepsilon^2 \int_{\mathbb{R}} |g'|^2 ds + (n-1) \int_{\mathbb{R}} |g|^2 ds}.
\]

Then, passing to the limit as \( \varepsilon \to 0 \), we conclude that \( \delta_{n,4-n} \leq n-1 \). Thus \( \delta_{n,4-n} \leq \min \{ (n-2)^2, n-1 \} \). To prove the opposite inequality we argue by contradiction. We assume that there exists \( w \in C^2_c(\mathbb{R} \times S^{n-1}) \), \( w \neq 0 \), such that

\[
\min \{ (n-2)^2, n-1 \} > \frac{N(w)}{D(w)},
\]

where \( N(w) \) and \( D(w) \) are as in the proof of Theorem 1.1. We can write \( w \) as \( w(s, \sigma) = g(s) + v(s, \sigma) \), where

\[
\int_{S^{n-1}} v(s, \sigma) d\sigma = 0
\]

for any \( s \in \mathbb{R} \). Notice that \( v \neq 0 \), otherwise (2.7) would contradict Theorem 1.2. Thus

\[
\xi_v := \frac{\int_{\mathcal{Z}} |\nabla_\sigma v|^2 ds d\sigma}{\int_{\mathcal{Z}} |v|^2 ds d\sigma} \geq n-1.
\]

Clearly, \( N(g) \geq (n-2)^2 D(g) \) by Theorem 1.2. Arguing as in the proof of Theorem 1.1 and using (2.8) we can estimate \( N(v) \geq (n-1)D(v) \). Therefore

\[
\min \{ (n-2)^2, n-1 \} > \frac{N(w)}{D(w)} = \frac{N(g) + N(v)}{D(g) + D(v)} \geq \frac{(n-2)^2 D(g) + (n-1)D(v)}{D(g) + D(v)},
\]

that readily leads to a contradiction. Thus equality holds and the theorem is completely proved. \( \square \)

### 2.6 Proof of (1.3)

Let \( \alpha \in (4-n,2) \) such that \( \delta_{n,4-n,\alpha} \geq \delta_{n,\alpha} \). Thus there exist \( g \in C^2_c(\mathbb{R}) \) and \( v \in C^2_c(\mathcal{Z}) \) such that \( v(s, \cdot) \) has zero mean value on the sphere for any \( s \in \mathbb{R} \), and such that

\[
\delta_{n,\alpha}^{rad} > \frac{N(g + v)}{D(g + v)}.
\]
where $N(\cdot)$ and $D(\cdot)$ are defined as in the proof of Theorem 1.1. In addition, it holds that $v \neq 0$ and that $v$ satisfies (2.8). Clearly, $N(g) \geq \delta_{n,\alpha}^\text{rad} D(g)$. Arguing as in the proof of Theorem 1.1 we can estimate

$$\frac{N(v)}{D(v)} \geq \frac{(\xi_v + \gamma_{n,\alpha})^2}{\xi_v + h_{n,\alpha}},$$

where $\xi_v$ is defined in (2.8). Therefore

$$\delta_{n,\alpha}^\text{rad} > \frac{N(g) + N(v)}{D(g) + D(v)} = \frac{\delta_{n,\alpha}^\text{rad} D(g) + \frac{(\xi_v + \gamma_{n,\alpha})^2}{\xi + h_{n,\alpha}} D(v)}{D(g) + D(v)}.$$

Noticing that $h_{n,\alpha} \delta_{n,\alpha}^\text{rad} = \gamma_{n,\alpha}^2$, we infer that $\delta_{n,\alpha}^\text{rad} - 2\gamma_{n,\alpha} > \xi_v \geq n - 1$. Hence $\alpha < 2$ satisfies

$$3\alpha^2 - 2(n + 4)\alpha - n^2 + 4n + 4 > 0,$$

that is,

$$\alpha < \frac{1}{3} \left( n + 4 - 2\sqrt{n^2 - n - 1} \right).$$

Conversely, if

$$\frac{1}{3} \left( n + 4 - 2\sqrt{n^2 - n - 1} \right) \leq \alpha < n,$$

then it necessarily holds that $\delta_{n,\alpha} = \delta_{n,\alpha}^\text{rad}$. □

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