On Quantum Operations of Photon Subtraction and Photon Addition

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Abstract—The conventional photon subtraction and photon addition transformations, $\rho \rightarrow t a \rho a^\dagger$ and $\rho \rightarrow t a^\dagger \rho a$, are not valid quantum operations for any constant $t > 0$ since these transformations are not trace nonincreasing. For a fixed density operator $\rho$ there exist fair quantum operations, $N_-$ and $N_+$, whose conditional output states approximate the normalized outputs of former transformations with an arbitrary accuracy. However, the uniform convergence for some classes of density operators $\rho$ has remained essentially unknown. Here we show that, in the case of photon addition operation, the uniform convergence takes place for the energy-second-moment-constrained states such that $\text{tr}[\rho H^2] \leq E_2 < \infty$, $H = a^\dagger a$. In the case of photon subtraction, the uniform convergence takes place for the energy-second-moment-constrained states with nonvanishing energy, i.e., the states $\rho$ such that $\text{tr}[\rho H] \geq E_1 > 0$ and $\text{tr}[\rho H^2] \leq E_2 < \infty$. We prove that these conditions cannot be relaxed and generalize the results to the cases of multiple photon subtraction and addition.

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1. INTRODUCTION

In quantum theory, a system state is described by a density operator\textsuperscript{[1, 2]}, i.e., a positive-semidefinite operator $\rho$ on Hilbert space $\mathcal{H}$ such that its trace $\text{tr}[\rho] = 1$. Denote $\mathcal{S}(\mathcal{H})$ the set of density operators on $\mathcal{H}$. Hereafter in this paper, we consider a separable Hilbert space $\mathcal{H}$ with a countable orthonormal basis $\{|n\rangle\}_{n=0}^{\infty}$ such that $|n\rangle\langle n|$ is a Fock state with the fixed number $n$ of photons is a fixed mode of electromagnetic radiation. The photon annihilation operator $a$ and the photon creation operator $a^\dagger$ are defined through

$$a = \sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n|, \quad a^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1}|n+1\rangle\langle n|$$

and satisfy the commutation relation $[a, a^\dagger] = I$, the identity operator on $\mathcal{H}$. Hereafter, $\dagger$ denotes the Hermite conjugation. The photon creation and annihilation operators are extensively used in quantum optics\textsuperscript{[3]} because many physical operators and characteristics are be expressed through them, for instance in terms of the moments $\text{tr}[\rho(a^\dagger)^m a^n]$, Refs.\textsuperscript{[4–6]}.

Conditional transformations of quantum states in a measurement are conventionally described by a mapping $\mathcal{I} : (\Omega, \mathcal{F}) \rightarrow \mathcal{O}$ that is also referred to as instrument\textsuperscript{[2, 7–9]}. Here, $\Omega$ is a nonempty set of classical measurement outcomes, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$, $\mathcal{O}$ is a set of operations on $\mathcal{T}(\mathcal{H})$, and $\mathcal{T}(\mathcal{H})$ is the set of trace class operators. The definition of quantum operation naturally follows from physical requirements, namely, a mapping $\mathcal{N}$ on $\mathcal{T}(\mathcal{H})$ is an operation if it is linear, completely positive, and trace nonincreasing. The complete positivity of $\mathcal{N}$ means that the mapping $\mathcal{N} \otimes \text{Id}_R$ on $\mathcal{T}(\mathcal{H} \otimes \mathcal{H}_R)$ is

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positive for all finite dimensional extensions $\mathcal{H}_R$. The physical meaning of complete positivity is related with the fact that the system in interest can be potentially entangled with an ancillary system $R$, and the transformation of the total density operator must be positive. Since the ancillary system is not affected by $\mathcal{N}$, the total transformation is $\mathcal{N} \otimes \text{Id}_R$, where $\text{Id}_R$ is the identity map on $\mathcal{T}(\mathcal{H}_R)$.

Let $\varrho$ be the input state and $\mathcal{N} = \mathcal{I}(x)$ be a quantum operation associated with the classical outcome $x$ of instrument $\mathcal{I}$. The quantity $\text{tr}[\mathcal{N}[\varrho]] \leq 1$ is the probability to observe the outcome $x$. Suppose $\text{tr}[\mathcal{N}[\varrho]] > 0$, then

$$
\tilde{\varrho} = \frac{\mathcal{N}[\varrho]}{\text{tr}[\mathcal{N}[\varrho]]}
$$

is a conditional output density operator associated with the outcome $x$ [2, 10].

In the physics literature, the photon subtraction transformation $\mathcal{A}_-$ and the photon addition transformation $\mathcal{A}_+$ are defined through [11–23]

$$
\mathcal{A}_-[\varrho] = ta\varrho a^\dagger, \quad \mathcal{A}_+[\varrho] = ta^\dagger\varrho a,
$$

where $t > 0$ is a real number proportional to the probability of the successful transformation. The conditional output states read

$$
\tilde{\varrho}_- = \frac{a\varrho a^\dagger}{\text{tr}[a\varrho a^\dagger]}, \quad \tilde{\varrho}_+ = \frac{a^\dagger\varrho a}{\text{tr}[a^\dagger\varrho a]}.
$$

The transformations (2) satisfy the conditions of linearity and complete positivity, however, they are not trace nonincreasing. In fact, let $\varrho = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{1}{n^s} |n\rangle \langle n|$, where $\zeta(s)$ is the Riemann zeta function, $s > 2$. Then $\text{tr}[\varrho] = 1$ and $\text{tr}[\mathcal{A}_-[\varrho]] = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{1}{(n+1)^s} = t^{\frac{s-1}{\zeta(s)}} \rightarrow \infty$ if $s \rightarrow 2 + 0$ for all $t > 0$. Similarly, $\text{tr}[\mathcal{A}_+[\varrho]] = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \left( \frac{1}{n^s} + \frac{1}{n^{s-1}} \right) = t^{\frac{s-1}{\zeta(s)}}+\zeta(s) \rightarrow \infty$ if $s \rightarrow 2 + 0$ for all $t > 0$. This means that the transformations (2) are not quantum operations and cannot be exactly implemented in any experiment.

Recently, the quantum operations have been extended to the space of relatively bounded operators [24–26], where the bound is related with the system Hamiltonian. In the case of the one-mode electromagnetic radiation, the Hamiltonian is essentially the photon number operator and reads $H = a^\dagger a$. The goal of this paper is to show that for some classes of states $\varrho$ with specific restrictions on energy moments $\text{tr}[\varrho H^k]$ there exist fair quantum operations $\mathcal{N}_\pm$ such that conditional output states (1) and (3) become indistinguishable in practice. In other words, the transformations (2) can be realized approximately with an arbitrary precision for all states in the class. We also discuss the processes of multiple photon subtraction and photon addition.

2. APPROXIMATE PHOTON SUBTRACTION AND PHOTON ADDITION

The physical model of photon subtraction exploits an ideal beam splitter, with one input being a state $\varrho$ and the other (auxiliary) input being a vacuum. A detection of a single photon in the output auxiliary mode results in the following quantum operation [13]:

$$
\mathcal{N}_-(\gamma)[\varrho] = (e^{2\gamma} - 1)ae^{-\gamma a^\dagger a}\varrho e^{-\gamma a^\dagger a}a^\dagger,
$$

where $\gamma > 0$ and $e^{-2\gamma}$ is the power transmittance. From this viewpoint, this process describes an open quantum dynamics for the system [27, 28]. If $k$ photons are observed in the output auxiliary mode, then one gets the operation

$$
\mathcal{N}_{-k}(\gamma)[\varrho] = \frac{1}{k!}(e^{2\gamma} - 1)^k a^k e^{-\gamma a^\dagger a}\varrho e^{-\gamma a^\dagger a}a^\dagger.
$$

It is not hard to see that $\sum_{k=0}^{\infty} \mathcal{N}_{-k}^\dagger(\gamma) = \mathcal{I}$, i.e., $\sum_{k=0}^{\infty} \mathcal{N}_{-k}(\gamma)$ is trace preserving and each $\mathcal{N}_{-k}(\gamma)$ is trace nonincreasing. Therefore, the transformation $\mathcal{N}_{-k}^\dagger(\gamma)$ is a fair quantum operation.

Similarly, if the auxiliary mode is initially in the single-photon state and no photons are observed at its output, then one obtains the operation of approximate photon addition

$$
\mathcal{N}_+(\gamma)[\varrho] = (e^{2\gamma} - 1)e^{-\gamma a^\dagger a}a^\dagger \varrho e^{-\gamma a^\dagger a}.
$$
The approximate addition of \( k \) photons reads
\[
\mathcal{N}_{+k}(\gamma)[\sigma] = (e^{2\gamma} - 1)e^{-\gamma a^\dagger a}k a^k e^{-\gamma a^\dagger a}.
\]

It is worth mentioning that other realizations of approximate photon addition via the spontaneous parametric down conversion are usually implemented in practice [12, 14].

Let us demonstrate that for a general state \( \rho \) the result of an approximate photon subtraction (4) can significantly differ from the state (3). The distinguishability between two quantum states \( \rho \) and \( \sigma \) reads \( \frac{1}{2}||\rho - \sigma||_1 \) and quantifies the optimal minimum-error discrimination [1, 2]. Here \( ||X||_1 = \text{tr}[\sqrt{X^\dagger X}] \).

**Proposition 1.** For any given \( \gamma > 0 \) there exists a state \( \rho \in \mathcal{S}(\mathcal{H}) \) with finite energy \( \text{tr}[\rho H] < \infty \) such that
\[
\left|\left| \tilde{\rho}_- (s) - \frac{\mathcal{N}_- (\gamma)[\rho(s)]}{\text{tr}[\mathcal{N}_- (\gamma)[\rho(s)]]} \right|\right|_1 \geq \frac{1}{2} \ln (e - 1) \approx 0.27.
\]

**Proof.** We restrict to the case of photon subtraction. The case of photon addition is treated in a similar way. Consider a one-parameter family of states \( \rho(s) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{1}{n^s} |n\rangle \langle n| \) with \( s > 2 \). Then
\[
\left|\left| \tilde{\rho}_- (s) - \frac{\mathcal{N}_- (\gamma)[\rho(s)]}{\text{tr}[\mathcal{N}_- (\gamma)[\rho(s)]]} \right|\right|_1 \geq \frac{e^{-2\gamma}}{\text{Li}_{s-1}(e^{-2\gamma})} - \frac{1}{\zeta(s-1)}.
\] (6)

If \( s \to +\infty \), then the right hand side of (7) vanishes. Since \( \text{lim}_{s \to +\infty} \text{Li}_{s-1}(e^{-2\gamma}) = \text{Li}_1(e^{-2\gamma}) = -\ln(1 - e^{-2\gamma}) \) and \( \text{lim}_{s \to 2+0} \zeta(s-1) = +\infty \), there exists \( s_1 > 2 \) such that
\[
\frac{e^{-2\gamma}}{\text{Li}_{s_1-1}(e^{-2\gamma})} - \frac{1}{\zeta(s_1 - 1)} = \frac{e^{-2\gamma}}{2 \ln \left( \frac{1}{1-e^{-2\gamma}} \right)} \geq \frac{e - 1}{2e} > \frac{1}{2} \ln (e - 1).
\]

If \( 0 < \gamma < \frac{1}{2} \ln \left( \frac{e-1}{e-1} \right) \), then we consider the terms in Eq. (6) with \( n \leq N = \left| \frac{1}{2\gamma} \left( 1 - \frac{\text{Li}_{s_1-1}(e^{-2\gamma})}{\zeta(s_1 - 1)} \right) \right| \) as the the expression inside the absolute value bars in Eq. (6) is positive in this case because \( e^{-2\gamma n} \geq 1 - 2\gamma n \). Consequently,
\[
\left|\left| \tilde{\rho}_- (s) - \frac{\mathcal{N}_- (\gamma)[\rho(s)]}{\text{tr}[\mathcal{N}_- (\gamma)[\rho(s)]]} \right|\right|_1 \geq \sum_{n=1}^{N} \frac{1}{n^{s_1 - 1}} \left( \frac{e^{-2\gamma n}}{\text{Li}_{s_1-1}(e^{-2\gamma})} - \frac{1}{\zeta(s_1 - 1)} \right) \to \frac{1}{\text{Li}_1(e^{-2\gamma})} \sum_{n=1}^{\infty} \frac{e^{-2\gamma n}}{n}
\]
if \( s \to 2 + 0 \). Therefore, there exists \( s_2 > 2 \) such that
\[
\left|\left| \tilde{\rho}_- (s_2) - \frac{\mathcal{N}_- (\gamma)[\rho(s_2)]}{\text{tr}[\mathcal{N}_- (\gamma)[\rho(s_2)]]} \right|\right|_1 \geq \frac{1}{2 \text{Li}_1(e^{-2\gamma})} \sum_{n=1}^{\left| \frac{1}{2\gamma} \right| + 1} \frac{e^{-2\gamma n}}{n} \geq \frac{1}{2} \left( 1 - \frac{1}{\text{Li}_1(e^{-2\gamma})} \sum_{n=1}^{\left| \frac{1}{2\gamma} \right| + 1} \frac{e^{-2\gamma n}}{n} \right)
\]
\[
\geq \frac{1}{2} \left( 1 - \frac{2\gamma}{e(1 - e^{-2\gamma}) \text{Li}_1(e^{-2\gamma})} \right) \geq \frac{1}{2} \ln (e - 1).
\]

The energy of states \( \rho(s_1) \) and \( \rho(s_2) \) is finite because \( s_1, s_2 > 2 \).

Proposition 1 reveals that the physically implementable approximation of photon subtraction or addition cannot reproduce the result of an ideal photon subtraction or addition (3) for any input state. Physically, the problem arises due to a high energy of the input. In the next section, we show that the conditional output state (1) for the approximate operation (4) does not converge uniformly to the result of the ideal photon subtraction (3) even in the case of energy-constrained states.
3. ENERGY-CONSTRAINED STATES

Denote $\mathcal{S}_E(\mathcal{H})$ the set of states $\varrho$ such that $0 < \text{tr}[\varrho H] \leq E$ [25, 29, 30].

**Proposition 2.** For any given $\gamma > 0$ and $E > 0$ there exists a state $\varrho \in \mathcal{S}_{E+1}(\mathcal{H})$ such that

$$\left|\left|\tilde{\varrho}_+ - \frac{\mathcal{N}_+(\gamma)[\varrho]}{\text{tr}[\mathcal{N}_+(\gamma)[\varrho]]}\right|\right|_1 \geq \sqrt{\frac{E}{E+1}}.$$  \hspace{1cm} (8)

**Proof.** In the case of photon subtraction, consider a family of states $\varrho(N) = |\psi(N)\rangle\langle\psi(N)|$ with $|\psi(N)\rangle = \sqrt{1 - \frac{E}{N}}|1\rangle + \sqrt{\frac{E}{N}}|N\rangle$, $N \geq \max(E, 2)$. The states in the family have the energy $\text{tr}[\varrho H] = 1 - \frac{E}{N} + E \leq E + 1$, so $\varrho(N) \in \mathcal{S}_{E+1}(\mathcal{H})$. The conditional output density operator for the ideal photon subtraction, $\tilde{\varrho}_-(N)$, has support spanned by vectors $|0\rangle$ and $|N - 1\rangle$, so it is given by the following matrix in the corresponding 2-dimensional subspace:

$$\tilde{\varrho}_-(N) = \frac{1}{1 - \frac{E}{N} + E} \begin{pmatrix} 1 - \frac{E}{N} & \sqrt{E} \sqrt{1 - \frac{E}{N}} \\ \sqrt{E} \sqrt{1 - \frac{E}{N}} & E + 1 \end{pmatrix} \rightarrow \frac{1}{E + 1} \begin{pmatrix} 1 & \sqrt{E} \\ \sqrt{E} & E \end{pmatrix} \text{ if } N \to \infty.$$

On the other hand, the conditional output state for the approximate photon subtraction has support in the same subspace and reads

$$\frac{\mathcal{N}_-(\gamma)[\varrho(N)]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho(N)]]} = \frac{1}{(1 - \frac{E}{N}) e^{-2\gamma} + E e^{-2\gamma N}} \begin{pmatrix} (1 - \frac{E}{N}) e^{-2\gamma} & \sqrt{E} \sqrt{1 - \frac{E}{N} e^{-\gamma(N+1)}} \\ \sqrt{E} \sqrt{1 - \frac{E}{N} e^{-\gamma(N+1)}} & E e^{-2\gamma N} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } N \to \infty.$$

Therefore, $\lim_{N \to \infty} \left|\left|\tilde{\varrho}_-(N) - \frac{\mathcal{N}_-(\gamma)[\varrho(N)]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho(N)]]}\right|\right|_1 = 2\sqrt{\frac{E}{E+1}}$ and there exists a finite $N < \infty$ such that (8) is fulfilled.

In the case of photon addition, similarly consider a family of states $\varrho(N) = |\psi(N)\rangle\langle\psi(N)|$ with $|\psi(N)\rangle = \sqrt{1 - \frac{E}{N}}|0\rangle + \sqrt{\frac{E}{N}}|N\rangle$, $N \geq \max\{E, 1\}$. \hspace{1cm} $\square$

The physical meaning of Proposition 2 is that in a fixed experimental scheme it is impossible to obtain the uniform convergence of the approximate photon subtraction (addition) to the ideal one within the set of energy-constrained states with fixed $E$. In other words, there exist states with the same energy such that for one of them the approximate photon subtraction is very close to the ideal photon subtraction, whereas for another one it is quite far from ideal.

Note that the mapping (3) transforms the energy-constrained states in the proof of Proposition 2 to the states $\tilde{\varrho}_\pm$ with unbounded energy, i.e., for any $E > 0$ and $E' > 0$ there exists a state $\varrho \in \mathcal{S}_{E+1}(\mathcal{H})$ such that $\tilde{\varrho}_\pm \notin \mathcal{S}_{E'}(\mathcal{H})$.

Analyzing the states in the proof of Proposition 2, we observe that $\lim_{N \to \infty} \text{tr}[\varrho H] = E + 1$ whereas $\lim_{N \to \infty} \text{tr}[\varrho H^2] = \infty$. This allows one to make a conjecture that if the second moment of Hamiltonian, $\text{tr}[\varrho H^2]$, would be bounded from above, there could be a uniform convergence within the set of such states. This is indeed the case for the photon addition; however, this is not the case for the photon subtraction as we show in the next section.
4. ENERGY-SECOND-MOMENT-CONSTRAINED STATES

Denote $S_{E}^{(2)}(\mathcal{H})$ the set of states $\rho$ such that $0 < \text{tr}[\rho H^2] \leq E$. Note that $\rho \in S_{E}^{(2)}(\mathcal{H})$ implies $\rho \in S_{E}(\mathcal{H})$ because $\sum_{n=0}^{\infty} p_n n^2 \leq \sum_{n=0}^{\infty} p_n n^2 \leq E$ for any probability distribution $\{p_n\}$. The mapping (3) transforms the energy-second-moment-constrained states to the energy-constrained states.

**Proposition 3.** For any $\varepsilon > 0$ and $E < \infty$ there exists $\gamma > 0$ such that

$$
\left\| \tilde{\rho}_+ - \frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} \right\|_1 < \varepsilon
$$

for all $\rho \in S_{E}^{(2)}(\mathcal{H})$.

**Proof.** Consider a pure state $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, $\sum_{n}|c_n|^2 = 1$. Note that $\tilde{\rho}_+ = |\varphi\rangle\langle\varphi|$ with $|\varphi\rangle = \left[\sum_{k=1}^{\infty} |c_k|^2 (k+1)\right]^{-1/2} \sum_{n=1}^{\infty} c_n \sqrt{n+1} |n+1\rangle$ and $\frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} = |\chi\rangle\langle\chi|$ with $|\chi\rangle = \left[\sum_{k=1}^{\infty} |c_k|^2 (k+1)e^{-\gamma(k+1)}\right]^{-1/2} \sum_{n=1}^{\infty} c_n \sqrt{n+1} |n+1\rangle |n+1\rangle$. Since $|||\varphi\rangle\langle\varphi| - |\chi\rangle\langle\chi||||_1 = 2 \sqrt{1 - |\langle\varphi|\chi\rangle|^2}$, we have

$$
\left\| \tilde{\rho}_+ - \frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} \right\|_1 = 2 \sqrt{1 - \left(\frac{F_1 + 1 - \gamma(1 + 2F_1 + F_2)}{F_1 + 1}\right)^2} < \sqrt{\frac{8\gamma(1 + 2F_1 + F_2)}{F_1 + 1}}. \quad (9)
$$

Note that $0 \leq F_1 \leq F_2 \leq E$ so

$$
\left\| \tilde{\rho}_+ - \frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} \right\|_1 \leq \sqrt{\frac{8\gamma(3E + 2)}{F_1 + 1}} < \varepsilon \quad \text{if} \quad \gamma = \frac{\varepsilon^2}{8(3E + 2)}.
$$

For a mixed state $\rho$ with the spectral decomposition $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ we use the purification $|\Psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle \otimes |\psi_i\rangle \in \mathcal{H} \otimes \mathcal{H}$ such that $\rho = \text{tr}_2[|\Psi\rangle\langle\Psi|]$, where $\text{tr}_2$ is a channel describing the partial trace over the second subsystem, $\text{tr}_2[\cdot] = \sum_{n=0}^{\infty} I \otimes |n\rangle \langle n|$. Denote $|\Phi\rangle = \sum_i \sqrt{p_i} a |\psi_i\rangle \otimes |\psi_i\rangle$, $|X\rangle = \sum_i \sqrt{p_i} e^{-\gamma a^\dagger a} |\psi_i\rangle \otimes |\psi_i\rangle$, then $\tilde{\rho}_+ = \text{tr}_2[|\Phi\rangle\langle\Phi|]$ and $\frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} = \text{tr}_2[|X\rangle\langle X|]$. One can readily see that $|\langle X|\langle X||_1 < 8 \frac{\gamma^2}{3E + 2}$ when $F_1 = \text{tr}[\rho H]$ and $F_2 = \text{tr}[\rho H^2]$, so $|||\Phi\rangle\langle\Phi| - |X\rangle\langle X||||_1$ is bounded from above by the same quantity as in Eq. (9). By the contractivity property ([31], Theorem 9.2),

$$
\left\| \tilde{\rho}_+ - \frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} \right\|_1 \leq \sqrt{\frac{8\gamma(3E + 2)}{F_1 + 1}} < \varepsilon
$$

if $\gamma = \frac{\varepsilon^2}{8(3E + 2)}$.

The proof of Proposition 3 also provides the accuracy of the physical implementation of the photon addition. For a state $\rho$ with a finite energy $F$ and the energy variance $\sigma_F^2$, the trace distance

$$
\left\| \tilde{\rho}_+ - \frac{N_+(\gamma)[\rho]}{\text{tr}[N_+(\gamma)[\rho]]} \right\|_1 \leq \sqrt{\frac{2\gamma}{F + 1}} \left[(F + 1)^2 + \sigma_F^2\right].
$$

The claim of Proposition 3 cannot be extended to the case of photon subtraction as we demonstrate below.
Proposition 4. For any given \( \gamma > 0 \) and \( E > 0 \) there exists a state \( \varrho \in S_E^{(2)}(\mathcal{H}) \) such that
\[
\left\| \varrho - \frac{\mathcal{N}_-(\gamma)[\varrho]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho]]} \right\|_1 \geq 1.
\] (10)

Proof. Consider a family of states \( \varrho(N) = |\psi(N)\rangle\langle \psi(N)| \) with \( |\psi(N)\rangle = \sqrt{1 - \frac{E}{N^2}}|0\rangle + \sqrt{\frac{E}{2N^2}}|1\rangle + \sqrt{\frac{E}{2N^2}}|N\rangle \), \( N \geq \max(\sqrt{E}, 1) \). The states in this family have the energy second moment \( \text{tr}[\varrho H^2] = \frac{E}{2N^2}(1 + N^2) \leq E \), so \( \varrho(N) \in S_E^{(2)}(\mathcal{H}) \). The conditional output density operator for the ideal photon subtraction, \( \varrho_-(N) \), has support spanned by vectors \( |0\rangle \) and \( |N - 1\rangle \), so it is given by the following matrix in the corresponding 2-dimensional subspace:
\[
\varrho_-(N) = \frac{1}{N + 1} \begin{pmatrix} \sqrt{N} & 0 \\ \sqrt{N} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if} \quad N \rightarrow \infty.
\]

On the other hand, the conditional output state for the approximate photon subtraction has support in the same subspace and reads
\[
\frac{\mathcal{N}_-(\gamma)[\varrho(N)]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho(N)]]} = \frac{1}{e^{-2\gamma} + Ne^{-2\gamma N}} \begin{pmatrix} e^{-2\gamma} & \sqrt{Ne^{-\gamma(N+1)}} \\ \sqrt{Ne^{-\gamma(N+1)}} & Ne^{-2\gamma N} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if} \quad N \rightarrow \infty.
\]

Therefore, \( \lim_{N \rightarrow \infty} \left\| \varrho_-(N) - \frac{\mathcal{N}_-(\gamma)[\varrho(N)]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho(N)]]} \right\|_1 = 2 \) and there exists a finite \( N < \infty \) such that (10) is fulfilled. \( \square \)

The feature of states used in the proof of Proposition 4 is that their energy \( \text{tr}[\varrho(N)H] \rightarrow 0 \) if \( N \rightarrow \infty \). Finally, we can formulate the necessary conditions for the uniform convergence of \( \varrho_- \) to \( \frac{\mathcal{N}_-(\gamma)[\varrho]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho]]} \) within a given set of states \( S'(\mathcal{H}) \): the set \( S'(\mathcal{H}) \) should be isolated from the states with infinite energy second moment and isolated from the states with infinitesimal energy. We show in the next section, that these conditions are also sufficient.

5. ENERGY-SECOND-MOMENT-CONstrained STATES WITH NONVANISHING ENERGY

Denote \( S_{E_1;E_2}^{(1;2)}(\mathcal{H}) \) the set of states \( \varrho \) such that \( \text{tr}[\varrho H] \geq E_1 \) and \( \text{tr}[\varrho H^2] \leq E_2 \).

Proposition 5. For any \( \varepsilon > 0 \), \( E_1 > 0 \), and \( E_2 < \infty \) there exists \( \gamma > 0 \) such that
\[
\left\| \varrho_0 - \frac{\mathcal{N}_-(\gamma)[\varrho]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho]]} \right\|_1 < \varepsilon
\]
for all \( \varrho \in S_{E_1;E_2}^{(1;2)}(\mathcal{H}) \).

Proof. Consider a pure state \( \varrho = |\psi\rangle\langle \psi| \), where \( |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \), \( \sum_n |c_n|^2 = 1 \). Note that \( \varrho_- = |\varphi\rangle\langle \varphi| \) with \( |\varphi\rangle = \left( \sum_{k=1}^{\infty} |ck|^{-1/2} \right)^{-1/2} \sum_{n=1} c_n \sqrt{n} |n - 1\rangle \) and \( \frac{\mathcal{N}_-(\gamma)[\varrho]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho]]} = |\chi\rangle\langle \chi| \) with \( |\chi\rangle = \left( \sum_{k=1}^{\infty} |ck|^2 k^{-2\gamma k} \right)^{-1/2} \sum_{n=1} c_n \sqrt{n} e^{-\gamma n} |n - 1\rangle \). Since \( |||\varphi\rangle\langle \varphi| - |\chi\rangle\langle \chi|||| = 2 \sqrt{1 - |\langle \varphi|\chi\rangle|^2} \), we have
\[
\left\| \varrho_0 - \frac{\mathcal{N}_-(\gamma)[\varrho]}{\text{tr}[\mathcal{N}_-(\gamma)[\varrho]]} \right\|_1 = 2 \sqrt{1 - \frac{\left( \sum_{n=1}^{\infty} |c_n|^2 n e^{-\gamma n} \right)^2}{\left( \sum_{k=1}^{\infty} |ck|^2 k^{-2\gamma k} \right) \left( \sum_{k=1}^{\infty} |ck|^2 k^2 \right)}}.
\]
Denote $F_1 = \sum_{k=0}^{\infty} |c_k|^2k \geq E_1 > 0$ the energy of the input state and $F_2 = \sum_{k=0}^{\infty} |c_k|^2k^2 \leq E_2 < \infty$ the energy second moment. Then

$$F_1 \geq \sum_{n=0}^{\infty} |c_n|^2 n\epsilon^{-n} \geq \sum_{n=0}^{\infty} |c_n|^2 n(1-\gamma n) = F_1 - \gamma F_2.$$ 

Therefore

$$\left\| \tilde{\rho} - \frac{\mathcal{N}_- (\gamma)[\rho]}{\text{tr}[\mathcal{N}_- (\gamma)[\rho]]} \right\|_1 \leq 2\sqrt{1 - \frac{(F_1 - \gamma F_2)^2}{F_1^2}} < \sqrt{\frac{8\gamma F_2}{F_1}} \leq \sqrt{\frac{8\gamma E_2}{E_1}} = \epsilon \quad \text{if} \quad \gamma = \frac{E_1 \epsilon^2}{8E_2}. \quad (11)$$

For a mixed state $\rho$ with the spectral decomposition $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \in \mathcal{H} \otimes \mathcal{H}$. Denote

$$|\Phi\rangle = \frac{\sum_i \sqrt{p_i}|\psi_i\rangle \otimes |\psi_i\rangle}{\sum_i \sqrt{p_i}|\psi_i\rangle \otimes |\psi_i\rangle}, \quad |X\rangle = \frac{\sum_i p_i |\psi_i\rangle a^\dagger e^{-\gamma a^\dagger a}|\psi_i\rangle}{\sum_i \sqrt{p_i}|\psi_i\rangle a^\dagger e^{-2\gamma a^\dagger a}|\psi_i\rangle},$$

then $\tilde{\rho}_+ = \text{tr}_2|\Phi\rangle \langle \Phi| \text{ and } \frac{\mathcal{N}_+ (\gamma)[\rho]}{\text{tr}[\mathcal{N}_+ (\gamma)[\rho]]} = \text{tr}_2|X\rangle \langle X|$. One can readily see that $\langle \Phi|X\rangle \geq \frac{F_1 - \gamma F_2}{F_1}$, where $F_1 = \text{tr}[a^\dagger a H^\dagger]$ and $F_2 = \text{tr}[a^\dagger a H^\dagger]^2$, so $|||\Phi\rangle \langle \Phi| - |X\rangle \langle X||||_1$ is bounded from above by the same quantity as in Eq. (11). By the contractivity property ([31], Theorem 9.2),

$$\left\| \tilde{\rho}_+ - \frac{\mathcal{N}_+ (\gamma)[\rho]}{\text{tr}[\mathcal{N}_+ (\gamma)[\rho]]} \right\|_1 \leq |||\Phi\rangle \langle \Phi| - |X\rangle \langle X||||_1 < \epsilon \quad \text{if} \quad \gamma = \frac{E_1 \epsilon^2}{8E_2}. \quad \square$$

The proof of Proposition 5 also provides the accuracy of the physical implementation of the photon subtraction. For a state $\rho$ with a finite energy $F$ and the energy variance $\sigma_F^2$, the trace distance

$$\frac{1}{2} \left\| \tilde{\rho}_+ - \frac{\mathcal{N}_+ (\gamma)[\rho]}{\text{tr}[\mathcal{N}_+ (\gamma)[\rho]]} \right\|_1 < 2\sqrt{\frac{\epsilon^2}{F^2}} \sqrt{F^2 + \sigma_F^2}.$$ 

6. DISCUSSION AND CONCLUSIONS

We have clarified that the ideal transformations (2) cannot be realized in any experiment because the corresponding maps are not trace nonincreasing. However, it is experimentally feasible to implement the operations (4) and (5) of approximate photon subtraction and addition, respectively. However, in an experiment the transmittance parameter $e^{-2\gamma}$ is usually fixed and the natural question arises: What are the input states $\rho$ such the conditional output states of approximate operations are $\epsilon$-close to the ideal states (3)? This formulation of the problem assumes the uniform convergence of conditional output quantum states to the ideal states (3). In this paper, we sequentially imposed restrictions on input quantum states $\rho$. Firstly, we showed that states $\rho$ should have finite energy. Secondly, we demonstrated that the finite energy second moment is also necessary. This turned out to be sufficient for the photon addition operation, however, not sufficient for the photon subtraction operation, for which one more restriction is to be imposed: the input states must not have vanishing energy. The proofs of Propositions 3 and 5 provide the upper bound on the error of approximate photon addition and subtraction, respectively.

Finally, the multiple photon addition and subtraction operations can be treated in the same way because

$$\mathcal{N}_-^k (\gamma)[\rho] \propto \mathcal{N}_- (k\gamma)[\rho], \quad \mathcal{N}_+^k (\gamma)[\rho] \propto \mathcal{N}_+ (k\gamma)[\rho], \quad (12)$$

where the notation $X \propto Y$ for operators $X$ and $Y$ means $X = cY$ for some constant $c$. Eq. (12) implies that $\frac{\mathcal{A}_k^b [\rho]}{\text{tr}[\mathcal{A}_k^b [\rho]]}$ converges uniformly to $\frac{\mathcal{N}_k^b [\rho]}{\text{tr}[\mathcal{N}_k^b [\rho]]}$ for energy-$(k+1)$th-moment-constrained states $\rho$ such that $\text{tr}[\rho H^{k+1}] \leq E < \infty$. Similarly, Eq. (12) implies that $\frac{\mathcal{A}_k^b [\rho]}{\text{tr}[\mathcal{A}_k^b [\rho]]}$ converges uniformly to $\frac{\mathcal{N}_k^b [\rho]}{\text{tr}[\mathcal{N}_k^b [\rho]]}$ for energy-$(k+1)$th-moment-constrained states $\rho$ with nonvanishing energy such that $\text{tr}[\rho H] \geq E_1 > 0$ and $\text{tr}[\rho H^{k+1}] \leq E_2 < \infty$. 

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Interestingly, in contrast to the quantum channels [32–35], the quantum informational properties of quantum operations such as capacities and entanglement degradation remain essentially unstudied. From this viewpoint, the fair quantum operations (4) and (5) can be analyzed as paradigmatic examples of operations on continuous-variable quantum states. In turn, the quantum operations (4) and (5) can be replaced by simpler transformations (2) in the domain of second-moment-energy-constrained states with non-vanishing energy.

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REFERENCES

1. A. S. Holevo, Quantum Systems, Channels, Information. A Mathematical Introduction (Walter de Gruyter, Berlin, Boston, 2012).
2. T. Heinosaari and M. Ziman, The Mathematical Language of Quantum Theory (Cambridge Univ. Press, Cambridge, 2012).
3. W. Vogel and D.-G. Welsch, Quantum Optics, 3rd ed. (Wiley-VCH, Weinheim, 2006).
4. S. N. Filippov and V. I. Man’ko, “Measuring microwave quantum states: Tomogram and moments,” Phys. Rev. A 84, 033827 (2011).
5. S. N. Filippov and V. I. Man’ko, “Star product and ordered moments of photon creation and annihilation operators,” J. Phys. A: Math. Theor. 45, 015305 (2012).
6. S. N. Filippov and V. I. Man’ko, “Evolution of microwave quantum states in terms of measurable ordered moments of creation and annihilation operators,” Opt. Spectrosc. 112, 365–372 (2012).
7. E. B. Davies, Quantum Theory of Open Systems (Academic, London, 1976).
8. E. B. Davies and J. T. Lewis, “An operational approach to quantum probability,” Commun. Math. Phys. 17, 239–260 (1970).
9. M. Ozawa, “Quantum measuring processes of continuous observables,” J. Math. Phys. 25, 79–87 (1984).
10. I. A. Luchnikov and S. N. Filippov, “Quantum evolution in the stroboscopic limit of repeated measurements,” Phys. Rev. A 95, 022113 (2017).
11. J. Wenger, R. Tuille-Broui, and P. Grangier, “Non-Gaussian statistics from individual pulses of squeezed light,” Phys. Rev. Lett. 92, 153601 (2004).
12. A. Zavatta, S. Viciani, and M. Bellini, “Quantum-to-classical transition with single-photon-added coherent states of light,” Science (Washington, DC, U. S.) 306, 660–662 (2004).
13. M. S. Kim, “Recent developments in photon-level operations on travelling light fields,” J. Phys. B: At. Mol. Opt. Phys. 41, 133001 (2008).
14. A. Zavatta, V. Parigi, M. S. Kim, H. Jeong, and M. Bellini, “Experimental demonstration of the bosonic commutation relation via superpositions of quantum operations on thermal light fields,” Phys. Rev. Lett. 103, 140406 (2009).
15. A. V. Dodonov and S. S. Mizrahi, “Smooth quantum-classical transition in photon subtraction and addition processes,” Phys. Rev. A 79, 023821 (2009).
16. M. Bellini and A. Zavatta, “Manipulating light states by single-photon addition and subtraction,” Prog. Opt. 55, 41–83 (2010).
17. S. Wang, H.-Y. Fan, and L.-Y. Hu, “Photon-number distributions of non-Gaussian states generated by photon subtraction and addition,” J. Opt. Soc. Am. B 29, 1020–1028 (2012).
18. R. Kumar, E. Barrios, C. Kupchak, and A. I. Lvovsky, “Experimental characterization of bosonic creation and annihilation operators,” Phys. Rev. Lett. 110, 130403 (2013).
19. S. N. Filippov, V. I. Man’ko, A. S. Coelho, A. Zavatta, and M. Bellini, “Single photon-added coherent states: estimation of parameters and fidelity of the optical homodyne detection,” Phys. Scr. T 153, 014025 (2013).
20. E. Agudelo, J. Sperling, L. S. Costanzo, M. Bellini, A. Zavatta, and W. Vogel, “Conditional hybrid nonclassicality,” Phys. Rev. Lett. 119, 120403 (2017).
21. Yu. I. Bogdanov, K. G. Katamadze, G. V. Avosopians, L. V. Belinsky, N. A. Bogdanova, A. A. Kalinkin, and S. P. Kulik, “Multiphoton subtracted thermal states: description, preparation, and reconstruction,” Phys. Rev. A 96, 063803 (2017).
22. G. V. Avospiants, K. G. Katamadze, Yu. I. Bogdanov, B. I. Bantysh, and S. P. Kulik, “Non-Gaussianity of multiple photon-subtracted thermal states in terms of compound-Poisson photon number distribution parameters: theory and experiment,” Laser Phys. Lett. 15, 075205 (2018).
23. S. M. Barnett, G. Ferenczi, C. R. Gilson, and F. C. Speirits, “Statistics of photon-subtracted and photon-added states,” Phys. Rev. A 98, 013809 (2018).
24. M. E. Shirokov, “On extension of quantum channels and operations to the space of relatively bounded operators,” arXiv:1903.06086 [math-ph].
25. M. E. Shirokov, “On the energy-constrained diamond norm and its application in quantum information theory,” Problems Inform. Transmiss. 54, 20–33 (2018).
26. M. E. Shirokov, “On completion of the cone of CP linear maps with respect to the energy-constrained diamond norm,” arXiv:1810.10922 [math.FA].
27. I. A. Luchnikov, S. V. Vintskevich, H. Ouerdane, and S. N. Filippov, “Simulation complexity of open quantum dynamics: connection with tensor networks,” Phys. Rev. Lett. 122, 160401 (2019).
28. S. N. Filippov and D. Chruściński, “Time deformations of master equations,” Phys. Rev. A 98, 022123 (2018).
29. S. Becker and N. Datta, “Convergence rates for quantum evolution and entropic continuity bounds in infinite dimensions,” arXiv:1810.00863 [quant-ph].
30. A. Winter, “Energy-constrained diamond norm with applications to the uniform continuity of continuous variable channel capacities,” arXiv:1712.10267 [quant-ph].
31. M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge Univ. Press, Cambridge, 2000).
32. S. N. Filippov and K. Yu. Magadov, “Spin polarization-scaling quantum maps and channels,” Lobachevskii J. Math. 39 (1), 65–70 (2018).
33. S. N. Filippov, V. V. Frizev, and D. V. Kolobova, “Ultimate entanglement robustness of two-qubit states against general local noises,” Phys. Rev. A 97, 012322 (2018).
34. S. N. Filippov, “Lower and upper bounds on nonunital qubit channel capacities,” Rep. Math. Phys. 82, 149–159 (2018).
35. S. N. Filippov and K. V. Kuzhamuratova, “Quantum informational properties of the Landau–Streater channel,” J. Math. Phys. 60, 042202 (2019).