On condensation of topological defects and confinement

Patricio Gaete
*Departamento de Física, Universidad Técnica F. Santa María, Valparaíso, Chile

Clovis Wotzasek†
Instituto de Física, Universidade Federal do Rio de Janeiro, 21945, Rio de Janeiro, Brazil

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We study the static quantum potential for a theory of anti-symmetric tensor fields that results from the condensation of topological defects, within the framework of the gauge-invariant but path-dependent variables formalism. Our calculations show that the interaction energy is the sum of a Yukawa and a linear potentials, leading to the confinement of static probe charges.

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I. INTRODUCTION

One of the fundamental and long-standing issues of theoretical physics whose solution has evaded complete comprehension despite an intense effort for already many decades is that of the confinement for the fundamental constituents of matter. On the other hand a great many deal of progress has been made towards making an unequivocal distinction between the apparently related phenomena of screening and confinement. In fact such distinction is of considerable importance in our present understanding of gauge theories. Field theories that yield the linear potential are very important to particle physics, since those theories may be used to describe the confinement of quarks and gluons and be considered as effective theories of quantum chromodynamics. Due to intense interest mainly in string related topics, these studies have been extended out of the four dimensional domain and extended to theories of antisymmetric tensors of arbitrary ranks in arbitrary space-time dimensions that appear as low-energy effective field theories of strings. This has also helped us to gain insights over the mechanisms of confinement in different contexts.

It is the main purpose of this work to study the confinement versus screening properties of some theories of massless antisymmetric tensors, magnetically and electrically coupled to topological defects that eventually condense, as a consequence of the Julia–Toulouse mechanism [1]. This mechanism is the dual to the Higgs mechanism and has been shown by Quevedo and Trugenberger [2] to lead to a concrete massive antisymmetric theory with a jump of rank. We clearly show that in the presence of two tensor fields, the condensation induces, not only a mass term and a jump of rank for one of the tensors but a BF coupling will also become manifest which will be responsible for the change from the screening to the confining phase of the theory.

In another circumstances antisymmetric tensor theories have been studied in the past because they are the natural extension of abelian gauge theories, our basic model for a gauge theory, with which they share very interesting properties, particularly the essential property of duality, leading to the strong–weak coupling mapping, known to exist in the electromagnetism. Such features are known to exist also in antisymmetric tensor theories in any dimension which will be essential for us in the sequel. Antisymmetric tensors also appear very naturally in supersymmetric field theories and in string theories where they play an important role in the realization of the various strong-weak coupling dualities among string theories.

An antisymmetric tensor of rank \( p \) couples naturally to an elementary extended \( p-1 \) dimensional object, a \( (p-1) \)-brane since its world-hypersurface is a \( p \)-dimensional object. However, if the antisymmetric tensors are compact fields, there may also appear defects or solitonic excitations in the underlying theory. Defects are classical solutions of the equations of motion and may be classified as topological or nontopological. Topological defects appear in models that support spontaneous symmetry breaking. They are important also in Cosmology and Condensed Matter Physics. Condensation of topological defects may drive phase transitions, particularly from the screening or Coulomb phases to the confining phase, which is our main interest in this paper. The prototype of this phenomenon is the well known Kosterlitz–Thouless transition in two space dimensions driven by vortex condensation. An important question in the analysis of phase transitions induced by topological defects regards the conditions for a topological defect to condense.
and for which values of parameters like temperature and coupling constants. Such questions have been tackled mostly by lattice techniques.

Another important issue regards the nature of the new phase with a finite condensate of topological defects. It is this last aspect, in $D = d + 1$ space-time dimensions for generic antisymmetric tensor field theories, that is of importance for us in this paper. This issue was discussed long time ago by Julia and Toulouse in the framework of ordered solid-state media and more recently in the relativistic context by Quevedo and Trugenberger. The basic idea in Ref. [1] was to consider models with non-trivial homotopy group able to support stable topological defects characterized by a length scale $r$. The long wavelength fluctuations of the continuous distribution of topological defects are the new hydrodynamical modes for the low-energy effective theory that appear when topological defects condense. In Ref. [1] there is a clear cut algorithm to identify these new modes in the framework of ordered solid-state media. However, due to the presence of non-linear terms in the topological currents, the lack of relativistic invariance and the need to introduce dissipation terms it becomes difficult to write down an action for the phase with a condensate of topological defects in this framework.

In the relativistic context none of the above problems is present. This allowed Quevedo and Trugenberger to show that the Julia–Toulouse prescription can be made much more precise, leading to an explicit form for the action in the finite condensate phase, for generic compact antisymmetric field theories. In this context the Julia–Toulouse mechanism is the natural generalization of the confinement phase for a vector gauge field.

In this paper we make use of the Julia–Toulouse mechanism, as presented by Quevedo and Trugenberger, to study the low-energy field theory of a pair of massless anti-symmetric tensor fields, say $A_p$ and $B_q$, coupled electrically and magnetically to a large set of $(q - 1)$-branes, characterized by charge $e$ and a Chern-Kernel $\Lambda_{p+1}$, that eventually condense. It is shown that the effective theory that results displays the confinement property by computing explicitly the effective potential for a pair of static, very massive point probes.

Basically, we are interested in studying the Julia–Toulouse mechanism in model field theories involving $B_q$ and $A_p$, electrically and magnetically coupled to a $(q - 1)$-brane, respectively, according to the following action

$$S = \int \frac{1}{2} \frac{(-1)^q}{(q + 1)!} [H_{q+1}(B_q)]^2 + e B_q J^q(\Lambda) + \frac{1}{2} \frac{(-1)^p}{(p + 1)!} [F_{p+1}(A_p) - e \Lambda_{p+1}]^2$$

(1)

and consider the condensation phenomenon when the Chern-Kernel $\Lambda_{p+1}$ becomes the new massive mode of the effective theory. Our compact notation here goes as follows. The field strength reads $F_{p+1}(A_p) = F_{\mu_1\mu_2\cdots\mu_{p+1}} = \partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]}$ and $\Lambda_{p+1} = \Lambda_{\mu_1\cdots\mu_{p+1}}$ is a totally anti-symmetric object of rank $(p + 1)$. The conserved current $J^q(\Lambda)$ is given by a delta-function over the world-volume of the $(q - 1)$-brane $[3]$. This conserved current may be rewritten in terms of the kernel $\Lambda_{p+1}$ as

$$J^q(\Lambda) = \frac{1}{(p + 1)!} \epsilon^{q,\alpha,p+1} \partial_{\alpha} \Lambda_{p+1},$$

(2)

and $\epsilon^{q,\alpha,p+1} = \epsilon^{\mu_1\cdots\mu_q,\alpha,\nu_1\cdots\nu_{p+1}}$. This notation will be used in the discussion of the Julia-Toulouse mechanism in the next section as long as no chance of confusion occurs. We will show in the next section how the Quevedo-Trugenberger prescription is used to constructed the effective interacting action, in the condensed phase, between the anti-symmetric tensor field $B_q$ and the degrees of freedom of the condensate $\Lambda_{p+1}$. In Section III we study the confinement properties of this effective action, after the condensate is integrat out, by computing the effective quantum potential for a pair of static probe charges within the framework of the gauge-invariant but path-dependent variables formalism. In particular, we shall be interested in the dependence of the confinement properties with the condensation parameters coming from the Julia–Toulouse mechanism.

II. THE EFFECTIVE ACTION

A. The Julia–Toulouse Mechanism

We begin with a discussion of the Julia–Toulouse mechanism in the relativistic context as originally developed in Ref. [2] which we follow closely. Consider a generic field theory in a $D$-dimensional space-time whose symmetry group $G$ is spontaneously broken down to $H$. Topological defects may arise in this theory according to the values taken by the homotopy group of the quotient-space $\Pi_h(G/H)$. For $h < d$, a non-trivial $\Pi_h(G/H)$ will lead to $(d - h - 1)$–dimensional solitons while instantons appears in the Euclidean version of the model when $h = d$. The characteristic sizes of these extended classical solutions with finite-energy are $r_i = 1/\tilde{M}_i$, where $\tilde{M}_i$ are mass parameters associated with the spontaneous symmetry breaking process. For ordered solid-state media the low-energy excitations are generically described by field theories for some order parameter as in the Ginzburg–Landau theory. For the case at
hand, the effective low-energy theory that has symmetry group \( H \), is meaningful only on scales much bigger than \( \max \{ r_i \} \) and, most important, experiences topological defects essentially as \( d - h - 1 \)-dimensional singularities in the space \( R^d \) (for solitons) or point singularities in \( R^{d+1} \) (for instantons). To our interest here the important fields are the anti-symmetric tensors of rank \( (d-h) \) and \( (h-1) \) that are able to couple electrically and magnetically, respectively, to the singularities. Besides, the most important point to stress here is that the effective action for this low-energy theory is then well-defined only outside these singularities.

As discussed in the introduction we wish to focus on anti-symmetric tensor field theories which are simple, yet highly non-trivial generalization of the usual Maxwell theory. Furthermore, we consider compact antisymmetric field theories, which are the generalizations of the compact QED, as put forward in \[ [3, 4] \], to higher-rank tensor theories. In compact antisymmetric field theories \( p \)-branes appear as topological defects of the original theory and their world-volume can be viewed as closed \( (d-h) \)-dimensional singularities excluded from the (space-time) domain of the model. They constitute the charges of the effective theory with strength \( e_i \) and Chern-Kernels \( \Lambda_h^{(i)} \) such that \( d_i^{d-h} = e^{d-h \cdot 1} \partial \Lambda_h^{(i)} \). These charges together with the anti-symmetric tensors are the basic building blocks in the construction of the low-energy effective field theory outside these singularities.

To understand the mechanism proposed by Julia and Toulouse one has to observe that the presence of these singularities induces non-trivial homology cycles for the antisymmetric tensor fields. There are then topological quantum number given in terms of the fields and the singularities, that are essentially \( (d - h - 1) \)-dimensional holes defined on \( D \)-dimensional Minkowski space-time while instantons would correspond to a model on \( (d+1) \)-dimensional Euclidean space with holes at the location of the instantons. To realize the significance of this we consider the intersection of these singularities with an \( (h+1) \)-dimensional hyperplane \( \Sigma_{h+1} \) perpendicular to it. Consider a sphere \( S_h \) on \( \Sigma_{h+1} \) such that it isolates one of the two intersection points with the singularities world-volume (this is a \( (d-h) \)-dimensional object). To see the topological quantum numbers, given in terms of \( S_h \), generated by stable topological defects, for which the relevant homotopy groups are \( \Pi_h (G/H) = Z \), we proceed as follows. Let us define an \( h \)-form \( \Omega_h \) which is exact outside \( S_h \) whose components read (except for normalization) \( \Omega_{\mu_1 \cdots \mu_h} = \partial_{[\mu_1} \phi_{\mu_2 \cdots \mu_h]} \) where \( \phi_{h-1} \) is an \( (h-1) \)-anti-symmetric tensor field. Then there exists topological invariants that can be written as \( \Phi = \int_{S_h} \Omega_h da^h \) where the compact notation of the preceding section has been invoked.

Let us consider next the effect of many topological defects on the low-energy effective theory since topological defects can condense leading to drastic changes in the infra-red structure of the underlying theory \[ [5, 6] \]. Therefore, the relevant question to address regards the change of the physical scenario when the number of topological defects grows making the manifold on which the low-energy effective action is defined to become very complicated. It may happens then that, for a certain range of parameters, dynamics favors the formation of a macroscopic number of topological excitations (the condensate) with a finite density. The condensate is then described by new long-lived modes. To identify the additional hydrodynamical modes of a solid state medium due to the continuous distribution of topological defects one uses the Julia–Toulouse theory, while in the framework of relativistic field theories, Quevedo and Trugenberger have shown that it leads to simple demands of the closed form \( \Omega_h \) into \( \Omega_h = \partial H_{h+1} (\phi_{h-1}) \) where \( \partial = \frac{\partial}{\partial x}\mu \) and Chern-Kernels \( \Lambda_h^{(i)} \). Upon fixing this gauge invariance one can drop all considerations over \( \phi_{h-1} \) after absorbing \( H_{h+1} (\phi_{h-1}) \) into \( \Omega_h \), so that the action describes the exact number of
degrees of freedom of a massive field whose mass parameter reads \( m = \Delta/e \). This process, named as Julia-Toulouse mechanism by Quevedo and Trugenberger, is the process dual to the well known Higgs mechanism of electric charges. Here on the other hand, the new modes generated by the condensation of magnetically coupled topological defects absorbs the original variables of the effective field theory, thereby acquiring a mass while in the Higgs mechanism it is the original field that incorporates the degrees of freedom of the electric condensate to acquire mass. This difference explains the change of rank in the JT mechanism that is not present in the Higgs process.

In the limit \( \Delta \to 0 \) the only relevant field configurations are those that satisfy the constraint \( F_{h+1}(\Omega_h) = 0 \) whose solution reads \( \Omega_{h+1} = \partial_{[\mu_1} \psi_{\mu_2 \cdots \mu_h]} \) where \( \psi_{h-1} \) is an \( (h-1) \)-anti-symmetric tensor field. The field \( \psi_{h-1} \) can then be absorbed into \( \phi_{h-1} \) this way recovering the original low-energy effective action before condensation.

**B. The Action in the Condensed Phase**

We are now ready to discuss the consequences of the JT mechanism over the action (1). The distinctive feature is that after condensation the Chern-Kernel \( \Lambda_{p+1} \) is elevated to the condition of propagating field. The new degree of freedom absorbs the degrees of freedom of the tensor \( A_p \) this way completing its longitudinal sector. The new mode is therefore explicitly massive. Since \( A_p \to \Lambda_{p+1} \) there is a change of rank with dramatic consequences. The last term in (1), displaying the magnetic coupling between the field-tensor \( F_{p+1}(A_p) \) and the \((q-1)\)-brane, becomes the mass term for the new effective theory in terms of the tensor field \( \Lambda_{p+1} \) and a new dynamical term is induced by the condensation. Another important feature is that the minimal coupling of the \( B_q \) tensor becomes responsible for another contribution for the mass, this time of topological nature. Indeed the second term in (1) becomes an interacting \( BF \)-term in the form \( B \wedge F(\Lambda) \) term between the remaining propagating modes, inducing the appearance of topological mass, in addition to the induced condensate mass. The final result reads

\[
S_{\text{cond}} = \int \frac{(-1)^q}{2(q + 1)!} [H_{q+1}(B_q)]^2 + e B_q \epsilon^{q,\alpha,p+1} \partial_\alpha \Lambda_{p+1} + \frac{(-1)^{p+1}}{2(p+2)!} [F_{p+2}(\Lambda_{p+1})]^2 - \frac{(-1)^{p+1}(p+1)!}{2} m^2 \Lambda_{p+1}^2 \tag{4}
\]

where \( m = \Delta/e \).

Recall that the initial theory, before condensation, displayed two independent fields coupled to a \((q-1)\)-brane. The nature of the two couplings were however different with important consequences. The \( A_p \) tensor, that was magnetically coupled to the brane, was then absorbed by the condensate after phase transition. On the other hand, the electric coupling, displayed by the \( B_q \) tensor, became a “\( B \wedge F(\Lambda) \)” topological term after condensation.

There has been a drastic change in the physical scenario. To show that the new systems displays a confining phase is the goal of this work. In this section we want to obtain an effective action for the \( B_q \) tensor. To this end we shall next integrate out the field describing the condensate. The implications of the resulting effective action will be studied in the next section.

To integrate out the condensate field \( \Lambda \) we rewrite this sector of the action as,

\[
S_{\Lambda} = \int \frac{(-1)^{p+1}}{2(p+2)!} [F_{p+2}(\Lambda_{p+1})]^2 - \frac{(-1)^{p+1}(p+1)!}{2} m^2 \Lambda_{p+1}^2 + e B_q \epsilon^{q,\alpha,p+1} \partial_\alpha \Lambda_{p+1} \]

\[
= \int \frac{(-1)^{p+2}(p+1)!}{2} \Lambda_{p+1} (\Delta^2 + m^2) \Lambda_{p+1}^2 + e B_q \epsilon^{q,\alpha,p+1} \partial_\alpha \Lambda_{p+1} \tag{5}
\]

where we have made use of (i) the identity

\[
\epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} \epsilon^{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} = (-1)^{D+1} p! \delta_{[\nu_1 \cdots \nu_q]}^{\nu_1 \cdots \nu_q} \tag{6}
\]

which in our compact notation reads

\[
\epsilon_{p,q} \epsilon^{p,q} = (-1)^{D+1} p! \delta_{[q]}^{q} \tag{7}
\]

and (ii) of an integration by parts, such that

\[
(e^{q,\alpha,p+1} \partial_\alpha \Lambda_{p+1})^2 = (-1)^{D} q! p! \Lambda_{p+1} (\Delta^2) \Lambda_{p+1} \tag{8}
\]

Next we solve the equations of motion

\[
\frac{\delta S_{\Lambda}}{\delta \Lambda_{p+1}} = 0 \tag{9}
\]
to obtain

$$\Lambda_{p+1} = \frac{e (-1)^p \varphi(\epsilon \partial)}{(p+1)!} \frac{1}{\Delta^2 + m^2} \epsilon_{p+1,\alpha,\rho} \partial^\rho B^\alpha$$

(10)

where $\varphi(\epsilon \partial) = (-1)^{p(q+1)}$ is the parity of the generalized curl operator, in the sense that

$$\int \psi_p \epsilon_{p,\alpha,\rho} \partial_\alpha \phi_q = \varphi(\epsilon \partial) \int \phi_q \epsilon_{q,\alpha,\rho} \partial_\alpha \psi_p.$$  

(11)

Substituting (10) back into the action (5) and using that,

$$\epsilon_{p+1,\alpha,\rho} \partial^\rho B^\alpha = (-1)^{D+1} \frac{(p+1)!}{(q+1)} H_{q+1}(B_q) H^{q+1}(B_q)$$

(12)

gives, after the inclusion of the free term for the $B_q$ tensor, our final effective theory as

$$S_{\text{eff}} = \int (-1)^{q+1} \frac{2}{2(q+1)!} H_{q+1}(B_q) \left(1 + \frac{e^2}{\Delta^2 + m^2}\right) H^{q+1}(B_q)$$

(13)

In the next section we shall examine the screening versus confinement issue. To this end we shall consider a specific example involving two Maxwell tensors coupled electrically and magnetically to a point-charge (a zero-brane) such that after the condensation we end up with a Maxwell and a Kalb-Ramond field (the condensate) coupled topologically to each other, besides the presence of an explicit mass term for the Kalb-Ramond.

### III. INTERACTION ENERGY

Our aim in this Section is to calculate the interaction energy for the effective theory computed above between external probe sources in an specific model. To do this, we will compute the expectation value of the energy operator $H$ in the physical state $|\Phi\rangle$ describing the sources, which we will denote by $\langle H | \Phi \rangle$. It is worth mentioning that our starting point, Eq. (11), with $p = q = 1$ is the Lagrangian obtained in [5]:

$$L = \frac{1}{12} H_{\mu\nu} H^{\mu\nu} - \frac{1}{4} m^2 \Lambda_{\mu\nu} \Lambda^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu} F^{\mu\nu} - A_0 J^0,$$

(14)

where the Kalb-Ramond field $A_{\mu\nu}$ carries the degrees of freedom of the condensate, as discussed at the end of the last Section. Here $H_{\mu\nu} = \partial_{\mu} \Lambda_{\nu\rho} + \partial_{\nu} \Lambda_{\mu\rho} + \partial_\rho \Lambda_{\mu\nu}$, $F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_{\nu} A_\mu$ and $J^0$ is an external current. As stated, our objective will be to calculate the potential energy for this theory. As in the previous subsection, the first step in this direction is to carry out the integration over $\Lambda_{\mu\nu}$ in Eq. (13). This allows us to write the following effective Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} \left(1 + \frac{e^2}{\Delta^2 + m^2}\right) F^{\mu\nu} - A_0 J^0.$$  

(15)

which is a particular case of [13]. We observe, either from (14) or from (15), that the limits $e \to 0$ or $m \to 0$ are well defined and lead (from the point of view of the probe charges) to a pure Maxwell theory or to a (topologically) massive model. Since the probe charges only couple to the Maxwell fields, the Kalb-Ramond condensate will not contribute to their interaction energy in the first case because in the limit where the parameter $e \to 0$ the Maxwell field and the condensate decouple. The second limit means that we are back to the non-condensed phase. As so the confinement of the probe charges are expected to disappear being taken over by an screening phase controlled by the parameter $e$ playing the role of topological mass.

Once this is done, the canonical quantization of this theory from the Hamiltonian point of view follows straightforwardly. The canonical momenta read $\Pi^\mu = -\left(1 + \frac{e^2}{\Delta^2 + m^2}\right) F^{0\mu}$ with the only nonvanishing canonical Poisson brackets being

$$\{A_\mu(t,x), \Pi^\nu(t,y)\} = \delta_\mu^\nu \delta(x-y).$$

(16)

Since $\Pi_0$ vanishes we have the usual primary constraint $\Pi_0 = 0$, and $\Pi^i = \left(1 + \frac{e^2}{\Delta^2 + m^2}\right) F^{i0}$. The canonical Hamiltonian is thus

$$H_C = \int d^3x \left\{-\frac{1}{2} \Pi^i \left(1 + \frac{e^2}{\Delta^2 + m^2}\right)^{-1} \Pi_i + \Pi^i \partial_j A_0 + \frac{1}{4} F_{ij} \left(1 + \frac{e^2}{\Delta^2 + m^2}\right) F^{ij} + A_0 J^0 \right\}.$$  

(17)
Time conservation of the primary constraint $\Pi_0$ leads to the secondary Gauss-law constraint

$$\Gamma_1 (x) \equiv \partial_t \Pi^i - J^0 = 0. \quad (18)$$

The preservation of $\Gamma_1$ for all times does not give rise to any further constraints. The theory is thus seen to possess only two constraints, which are first class, therefore the theory described by (14) is a gauge-invariant one. The extended Hamiltonian that generates translations in time then reads $H = H_C + \int d^3x (c_0 (x) \Pi_0 (x) + c_1 (x) \Gamma_1 (x))$, where $c_0 (x)$ and $c_1 (x)$ are the Lagrange multiplier fields. Moreover, it is straightforward to see that $A_0 (x) = [A_0 (x), H] = c_0 (x)$, which is an arbitrary function. Since $\Pi^0 = 0$ always, neither $A^0$ nor $\Pi^0$ are of interest in describing the system and may be discarded from the theory. Then, the Hamiltonian takes the form

$$H = \int d^3x \left\{ -\frac{1}{2} \Pi_i \left(1 + \frac{e^2}{\Delta^2 + m^2}\right)^{-1} \Pi^i + \frac{1}{4} F_{ij} \left(1 + \frac{e^2}{\Delta^2 + m^2}\right) F^{ij} + c(x) \left(\partial_i \Pi^i - J^0\right)\right\}, \quad (19)$$

where $c(x) = c_1 (x) - A_0 (x)$.

The quantization of the theory requires the removal of nonphysical variables, which is done by imposing a gauge condition such that the full set of constraints becomes second class. A convenient choice is found to be $\mathcal{S}$

$$\Gamma_2 (x) \equiv \int_{C_{\xi}} dz^\nu A_\nu (z) \equiv \int_0^1 d\lambda x^i A_i (\lambda x) = 0, \quad (20)$$

where $\lambda (0 \leq \lambda \leq 1)$ is the parameter describing the spacelike straight path $x^i = \xi^i + \lambda (x - \xi)^i$, and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. In this case, the only nonvanishing equal-time Dirac bracket is

$$\{A_i (x), \Pi^j (y)\} = \delta^j_i \delta^{(3)} (x - y) - \partial^j_i \int_0^1 d\lambda x^j \delta^{(3)} (\lambda x - y). \quad (21)$$

In passing we recall that the transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{A, B\} \rightarrow (-i) [A, B]. \quad (22)$$

We now turn to the problem of obtaining the interaction energy between pointlike sources in the model under consideration. The state $|\Phi\rangle$ representing the sources is obtained by operating over the vacuum with creation/annihilation operators. We want to stress that, by construction, such states are gauge invariant. In the case at hand we consider the gauge-invariant stringy $|\overline{\Psi} (y) \Psi (y')\rangle$, where a fermion is localized at $y'$ and an antifermion at $y$ as follows $\mathcal{F}$,

$$|\Phi\rangle \equiv |\overline{\Psi} (y) \Psi (y')\rangle = \overline{\psi} (y) \exp \left(\frac{iq}{\hbar} \int_{y'}^y dz^i A_i (z)\right) \psi (y') |0\rangle, \quad (23)$$

where $|0\rangle$ is the physical vacuum state and the line integral appearing in the above expression is along a spacelike path starting at $y'$ and ending $y$, on a fixed time slice. It is worth noting here that the strings between fermions have been introduced in order to have a gauge-invariant function $|\Phi\rangle$. In other terms, each of these states represents a fermion-antifermion pair surrounded by a cloud of gauge fields sufficient to maintain gauge invariance. As we have already indicated, the fermions are taken to be infinitely massive (static).

From our above discussion, we see that $\langle H \rangle_\Phi$ reads

$$\langle H \rangle_\Phi = \langle \Phi | \int d^3x \left\{ -\frac{1}{2} \Pi_i \left(1 + \frac{e^2}{\Delta^2 + m^2}\right)^{-1} \Pi^i + \frac{1}{4} F_{ij} \left(1 + \frac{e^2}{\Delta^2 + m^2}\right) F^{ij}\right\} |\Phi\rangle. \quad (24)$$

Consequently, we can write Eq. (24) as

$$\langle H \rangle_\Phi = \langle \Phi | \int d^3x \left\{ -\frac{1}{2} \Pi_i \left(1 - \frac{e^2}{\Delta^2 - m^2}\right)^{-1} \Pi^i\right\} |\Phi\rangle, \quad (25)$$
where, in this static case, $\Delta^2 = -\nabla^2$. Observe that when $e = 0$ we obtain the pure Maxwell theory, as mentioned after (15). From now on we will suppose $e \neq 0$.

Next, from our above Hamiltonian analysis, we note that

$$\Pi_i (x) \langle \Psi (y) \Psi (y') \rangle = \langle \Psi (y) \Psi (y') \Pi_i (x) | \Phi \rangle + q \int d \tau \delta^{(3)} (z - x) \langle \Phi \rangle, \quad \text{(26)}$$

As a consequence, Eq. (26) becomes

$$\langle H \rangle_{\Phi} = \langle H \rangle_0 + V^{(1)} + V^{(2)}, \quad \text{(27)}$$

where $\langle H \rangle_0 = \langle 0 | H | 0 \rangle$. The $V^{(1)}$ and $V^{(2)}$ terms are given by:

$$V^{(1)} = -\frac{q^2}{2} \int d^3 x \int \int d z \delta^{(3)} (x - z') \frac{1}{\sqrt{z^2 - M^2}} \int \int d z' \delta^{(3)} (x - z), \quad \text{(28)}$$

and

$$V^{(2)} = \frac{q^2 m^2}{2} \int d^3 x \int \int d z \delta^{(3)} (x - z') \frac{1}{\sqrt{z^2 - M^2}} \int \int d z' \delta^{(3)} (x - z), \quad \text{(29)}$$

where $M^2 \equiv m^2 + e^2$ and the integrals over $z^i$ and $z'_i$ are zero except on the contour of integration.

The $V^{(1)}$ term may look peculiar, but it is nothing but the familiar Yukawa interaction plus self-energy terms. In effect, as was explained in Ref. [10], the expression (28) can also be written as

$$V^{(1)} = \frac{e^2}{2} \int \int d z \delta^{(3)} \int \int d z' \delta^{(3)} G (z', z), \quad \text{(30)}$$

where $G$ is the Green function

$$G(z', z) = \frac{1}{4\pi} \frac{e^{-M|z' - z|}}{|z' - z|}. \quad \text{(31)}$$

Employing Eq. (31) and remembering that the integrals over $z^i$ and $z'_i$ are zero except on the contour of integration, the expression (30) reduces to the Yukawa-type potential after subtracting the self-energy terms, that is,

$$V^{(1)} = -\frac{q^2}{4\pi} \frac{e^{-M|y - y'|}}{|y - y'|}. \quad \text{(32)}$$

We now turn our attention to the calculation of the $V^{(2)}$ term, which is given by

$$V^{(2)} = \frac{q^2 m^2}{2} \int \int d z \int \int d z' G (z', z). \quad \text{(33)}$$

It is appropriate to observe here that the above term is similar to the one found for the system consisting of a gauge field interacting with a massive axion field. Notwithstanding, in order to put our discussion into context it is useful to summarize the relevant aspects of the calculation described previously. In effect, as was explained in Ref. [10], by using the Green function (31) in momentum space

$$\frac{1}{4\pi} \frac{e^{-M|z' - z|}}{|z' - z|} = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i k (z' - z)}}{k^2 + M^2}, \quad \text{(34)}$$

the expression (33) can also be written as

$$V^{(2)} = q^2 m^2 \int \frac{d^3 k}{(2\pi)^3} \left[ 1 - \cos (k \cdot r) \right] \frac{1}{(k^2 + M^2)} \frac{1}{(\hat{n} \cdot k)^2}, \quad \text{(35)}$$
where \( \hat{n} \equiv \frac{y - y'}{|y - y'|} \) is a unit vector and \( r = y - y' \) is the relative vector between the quark and antiquark. Since \( \hat{n} \) and \( r \) are parallel, we get accordingly

\[
V^{(2)} = \frac{q^2 m^2}{8\pi^3} \int_{-\infty}^{\infty} \frac{dk_r}{k_r^2} \left[ 1 - \cos(k_r r) \right] \int_0^{\infty} d^2k_T \frac{1}{(k_r^2 + k_T^2 + M^2)},
\]  

(36)

where \( k_T \) denotes the momentum component perpendicular to \( r \). We may further simplify Eq. (36) by doing the \( k_T \) integral, which leads immediately to the result

\[
V^{(2)} = \frac{q^2 m^2}{8\pi^2} \int_{-\infty}^{\infty} \frac{dk_r}{k_r^2} \left[ 1 - \cos(k_r r) \right] \ln \left( 1 + \frac{\Lambda^2}{k_r^2 + M^2} \right),
\]  

(37)

where \( \Lambda \) is an ultraviolet cutoff. We also observe at this stage that similar integral was obtained independently in Ref. [11] in the context of the dual Ginzburg-Landau theory by an entirely different approach.

We now proceed to compute the integral (37). For this purpose we introduce a new auxiliary parameter \( \varepsilon \) by making

\[
V^{(2)} \equiv \lim_{\varepsilon \to 0} \widetilde{V}^{(2)} = \lim_{\varepsilon \to 0} \frac{q^2 m^2}{8\pi^2} \int_{-\infty}^{\infty} \frac{dk_r}{k_r^2} \left[ 1 - \cos(k_r r) \right] \ln \left( 1 + \frac{\Lambda^2}{k_r^2 + M^2} \right).
\]  

(38)

We further note that the integration on the \( k_r \)-complex plane yields

\[
\widetilde{V}^{(2)} = \frac{q^2 m^2}{8\pi} \left( 1 - e^{-\varepsilon |y - y'|} \right) \ln \left( 1 + \frac{\Lambda^2}{M^2 - \varepsilon^2} \right).
\]  

(39)

Taking the limit \( \varepsilon \to 0 \), expression (39) then becomes

\[
V^{(2)} = \frac{q^2 m^2}{8\pi} |y - y'| \ln \left( 1 + \frac{\Lambda^2}{M^2} \right).
\]  

(40)

This, together with Eq. (32), immediately shows that the potential for two opposite charges located at \( y \) and \( y' \) is given by

\[
V(L) = -\frac{q^2}{4\pi} \frac{e^{-ML}}{L} + \frac{q^2 m^2}{8\pi} L \ln \left( 1 + \frac{\Lambda^2}{M^2} \right),
\]  

(41)

where \( L \equiv |y - y'| \). In this context it may be recalled the calculation reported in Ref. [12] by taking into account topological nontrivial sectors in \( U(1) \) gauge theory is given by

\[
V(L) = -\frac{q^2}{4\pi} \frac{1}{L} + \sigma L.
\]  

(42)

Notice that the result (41) agrees with (42) in the limit of large \( M \). Thus one is led to the conclusion that, although both calculations lead to confinement, the physical mechanism of obtaining a linear potential is quite different. In other terms, our result it may be considered as a physical realization of the topological nontrivial sectors studied in [12]. Let us also mention here that the result (41) is exactly the one obtained in Ref. [11] in the context of the dual Landau-Ginzburg theory. But we do not think that the agreement is an accidental coincidence. As we mentioned before, a gauge theory in the presence of external fields and axions displays the same behavior [10]. It seems a challenging work to extend to a class of models the above analysis, which can predict the same interaction energy. We expect to report on progress along these lines soon.

IV. FINAL REMARKS

We have studied the confinement versus screening issue for a pair of antisymmetric tensors coupled to topological defects that eventually condense, giving a specific realization of the Julia–Toulouse phenomenon. We have seen that
the Julia–Toulouse mechanism for a couple of massless antisymmetric tensors is responsible for the appearance of mass and the jump of rank in the magnetic sector while the electric sector becomes a BF–type coupling. The condensate absorbs and replaces one of the tensors and becomes the new massive propagating mode but does not couple directly to the probe charges. The effects of the condensation are however felt through the BF coupling with the remaining massless tensor. It is therefore not surprising that they become manifest in the interaction energy for the effective theory. We have obtained the effective theory for the condensed phase in general and computed the interaction energy between two static probe charges, in a specific example, in order to test the confinement versus screening properties of the effective model. Our results show that the interaction energy in fact contains a linear confining term and an Yukawa type potential. It can be observed that confinement completely disappears in the limit $m \to 0$ while the screening takes over controlled by the topological mass parameter instead. Although we have considered the case where the effective model consists of the BF–coupling between a Kalb-Ramond field (that represents the condensate) and a Maxwell field, our results seem to be quite general. A direct calculation for tensors of arbitrary rank in the present approach is however a quite challenging problem that we hope to be able to report in the future.

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