ON UNCERTAINTY AND INFORMATION PROPERTIES OF RANKED SET SAMPLES

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Abstract

Ranked set sampling is a sampling design which has a wide range of applications in industrial statistics, and environmental and ecological studies, etc.. It is well known that ranked set samples provide more Fisher information than simple random samples of the same size about the unknown parameters of the underlying distribution in parametric inferences. In this paper, we consider the uncertainty and information content of ranked set samples in both perfect and imperfect ranking scenarios in terms of Shannon entropy, Rényi and Kullback-Leibler (KL) information measures. It is proved that under these information measures, ranked set sampling design performs better than its simple random sampling counterpart of the same size. The information content is also a monotone function of the set size in ranked set sampling. Moreover, the effect of ranking error on the information content of the data is investigated.

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1. Introduction and Preliminaries

During the past few years, ranked set sampling has emerged as a powerful tool in statistical inference, and it is now regarded as a serious alternative to the commonly used simple random sampling design. Ranked set sampling and some of its variants have been applied successfully in different areas of applications such as industrial statistics, environmental and ecological studies, biostatistics and statistical genetics. The feature of ranked set sampling is that it combines simple random sampling with other sources of information such as professional knowledge, auxiliary information, judgement, etc., which are assumed to be inexpensive and easily obtained. This extra information helps to increase the chance that the collected sample yields more representative measurements (i.e., measurements that span the range of the value of the variable of interest in the underlying population). In its original form, ranked set sampling involves randomly drawing $k$ units (called

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a set of size \( k \) from the underlying population for which an estimate of the unknown parameter of interest is required. The units of this set are ranked by means of an auxiliary variable or some other ranking process such as judgmental ranking. For this ranked set, the unit ranked lowest is chosen for actual measurement of the variable of interest. A second set of size \( k \) is then drawn and ranking carried out. The unit in the second lowest position is chosen and the variable of interest for this unit is quantified. Sampling is continued until, from the \( k \)th set, the \( k \)th ranked unit is measured. This entire process may be repeated \( m \) times (or cycles) to obtain a ranked set sample of size \( n = mk \) from the underlying population.

Let \( X_{SRS} = \{X_i, i = 1, \ldots, n\} \) be a simple random sample (SRS) of size \( n \geq 1 \) from a continuous distribution with probability distribution function (pdf) \( f(x) \). Let \( F(x) \) denote the cumulative distribution function (cdf) of the random variable \( X \) and define \( F(x) = 1 - F(x) \) as the survival function of \( X \) with support \( S_X \). Also assume that \( X_{RSS} = \{X_{(i)}, i = 1, \ldots, k, j = 1, \ldots, m\} \) denotes a ranked set sample (RSS) of size \( n = mk \) from \( f(x) \) where \( k \) is the set size and \( m \) is the cycle size. Here \( X_{(i)} \) is the \( i \)th order statistic in a set of size \( k \) obtained in cycle \( j \) with pdf

\[
 f_{(i)}(x) = \frac{k!}{(i-1)!(k-i)!} F^{(i-1)}(x) F^{k-i}(x)f(x), \quad x \in S_X.
\]

When ranking is imperfect we use \( X_{RSS}^* = \{X_{[i], i = 1, \ldots, k, j = 1, \ldots, m}\} \) to denote an imperfect RSS of size \( n = mk \) from \( f(x) \). We also use \( f_{[i]}(x) \) to show the pdf of the judgemental order statistic \( X_{[i]} \) which is given by

\[
 f_{[i]}(x) = \sum_{r=1}^{n} p_{i,r} f_{(r)}(x),
\]

(1)

where \( p_{i,r} = P(X_{[i]} = X_{(r)}) \) denotes the probability with which the \( r \)th order statistic is judged as having rank \( i \) with \( \sum_{i=1}^{k} p_{i,r} = \sum_{r=1}^{k} p_{i,r} = 1 \). Readers are referred to Wolfe (2004, 2010), Chen et al. (2004) and references therein for further details.

The Fisher information plays a central role in statistical inference and information theoretic studies. It is well known that RSS provides more Fisher information than SRS of the same size about the unknown parameters of the underlying distribution in parametric inferences (e.g., Chen, 2000, Chapter 3). Park and Lim (2012) studied the effect of imperfect rankings on the amount of Fisher information in ranked set samples. Frey (2013) showed by example that the Fisher information in an imperfect ranked set sample may be higher than the Fisher information in a perfect ranked-set sample. The concept of information is so rich that there is no single definition that will be able to quantify the information content of a sample properly. For example, from an engineering perspective, the Shannon entropy or the Rényi information might be more suitable to be used as measures to quantify the information content of a sample than the Fisher information. In this paper, we study the notions of uncertainty and information content of RSS data in both perfect and imperfect ranking scenarios under the Shannon entropy, Rényi and Kullback-Leibler (KL) information measures and compare them with their counterparts with SRS data. These measures are increasingly being used in various contexts such as order statistics by Wong and Chen (1990) and Park (1995), Ebrahimi et al. (2004), Bratpour et al. (2007a, b), censored data by Abo-Eleneen, (2011), record data and reliability and life testing context by Raqab and Awad (2000, 2001), Zahedi and Shakil (2006), Ahmadi and Fashandi (2008) and in testing hypothesis by Park (2005), Balakrishnan et al. (2007) and Habibi Rad et al. (2011). So, it would be of interest to use these
measures to calculate the information content of RSS data and compare them with their counterparts with SRS data.

To this end, in Section 2 we obtain the Shannon entropies of RSS and SRS data of the same size. We show that the difference between the Shannon entropy of \( X_{RSS} \) and \( X_{SRS} \) is distribution free and it is a monotone function of the set size in ranked set sampling. In Section 3 similar results are obtained under the Rényi information. Section 4 is devoted to the Kullback-Leibler information of RSS data and its comparison with its counterpart under SRS data. We show that the Kullback-Leibler information between the distribution of \( X_{SRS} \) and distribution of \( X_{RSS} \) is distribution-free and increases as the set size increases. Finally, in Section 5 we provide some concluding remarks.

2. Shannon Entropy of Ranked Set Samples

The Shannon entropy or simply the entropy of a continuous random variable \( X \) is defined by

\[
H(X) = -\int f(x) \log f(x) \, dx,
\]

provided the integral exists. The Shannon entropy is extensively used in the literature as a quantitative measure of uncertainty associated with a random phenomena. The development of the idea of the entropy by Shannon (1948) initiated a separate branch of learning named the “Theory of Information”. The Shannon entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a sample regarding its parent distribution. Indeed, the amount of information which we get when we observe the result on a random experiment can be taken to be equal to the amount of uncertainty concerning the outcome of the experiment before carrying it out. In practice, smaller values of the Shannon entropy are more desirable.

We refer the reader to Cover and Thomas (1991) an references therein for more details. In this section, we compare the Shannon entropy of SRS data with its counterparts under both perfect and imperfect RSS data of the same size. Without loss of generality, we take \( m = 1 \) throughout the paper. From (2), the Shannon entropy of \( X_{SRS} \) is given by

\[
H(X_{SRS}) = -\sum_{i=1}^{n} \int f(x_i) \log f(x_i) \, dx_i = nH(X_1).
\]

Under the perfect ranking assumption, it is easy to see that

\[
H(X_{RSS}) = -\sum_{i=1}^{n} \int f(i)(x) \log f(x) \, dx = \sum_{i=1}^{n} H(X_{(i)}),
\]

where \( H(X_{(i)}) \) is the entropy of the \( i \)th order statistic in a sample of size \( n \). Ebrahimi et al. (2004) explored some properties of the Shannon entropy of the usual order statistics (see also, Park, 1995; Wong and Chen, 1990). Using (2) and the transformation \( X_{(i)} = F^{-1}(U_{(i)}) \) it is easy to prove the following representations for the Shannon entropy of order statistics (see, Ebrahimi et al. 2004, page 177):

\[
H(X_{(i)}) = H(U_{(i)}) - \mathbb{E} [\log [f(F^{-1}(W_i))]],
\]

where \( W_i \) is the uniform random variable.
where \( W_i \) has the beta distribution with parameters \( i \) and \( n - i + 1 \) and \( U_{(i)} \) stands for the \( i \)th order statistic of a random sample of size \( n \) from the Uniform\((0,1)\) distribution.

In the following result, we show that the Shannon entropy of RSS data is smaller than its SRS counterpart when ranking is perfect.

**Lemma 1.** \( H(X_{RSS}) \leq H(X_{SRS}) \) for all set size \( n \in \mathbb{N} \) and the equality holds when \( n = 1 \).

**Proof.** To show the result we use the fact that \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_{(i)}(x) \) (see Chen et al., 2004). Using the convexity of \( g(t) = t \log t \) as a function of \( t > 0 \), we have

\[
\frac{1}{n} \sum_{i=1}^{n} f_{(i)}(x) \log f_{(i)}(x) \geq \left( \frac{1}{n} \sum_{i=1}^{n} f_{(i)}(x) \right) \left( \log \frac{1}{n} \sum_{i=1}^{n} f_{(i)}(x) \right) = f(x) \log f(x).
\]

(5)

Now, the result follows by the use of (3) and (5).

In the sequel, we quantify the difference between \( H(X_{RSS}) \) and \( H(X_{SRS}) \). To this end, by (4), we first get

\[
H(X_{RSS}) = \sum_{i=1}^{n} H(U_{(i)}) - \int \sum_{i=1}^{n} f_{(i)}(x) \log f(x) dx
\]

\[
= \sum_{i=1}^{n} H(U_{(i)}) + H(X_{SRS}).
\]

Note that since \( H(X_{RSS}) \leq H(X_{SRS}) \) we must have \( \sum_{i=1}^{n} H(U_{(i)}) \leq 0 \), for all \( n \in \mathbb{N} \). Also, \( H(X_{RSS}) - H(X_{SRS}) = \sum_{i=1}^{n} H(U_{(i)}) \) is distribution-free (doesn’t depend on the parent distribution). Ebrahimi et al. (2004) obtained an expression for \( H(U_{(i)}) \) which is given by

\[
H(U_{(i)}) = \log B(i, n - i + 1) - (i - 1)[\psi(i) - \psi(n + 1)] - (n - i)[\psi(n - i + 1) - \psi(n + 1)],
\]

where \( \psi(z) = \frac{d}{dz} \log \Gamma(z) \) is the digamma function and \( B(a,b) \) stands for the complete beta function. Hence, we have

\[
H(X_{RSS}) - H(X_{SRS}) = 2 \sum_{j=1}^{n-1} (n - 2j) \log j - n \log n - 2 \sum_{i=1}^{n} (i - 1)\psi(i) + n(n - 1)\psi(n + 1)
\]

\[
= k(n), \quad \text{say}.
\]

By noting that \( \psi(n + 1) = \psi(n) + 1/n \), for \( n \geq 2 \), we can easily find the following recursive formula for calculating \( k(n) \):

\[
k(n + 1) = k(n) + n + \log \Gamma(n) - (n + 1)\log(n + 1).
\]

Table I shows the numerical values of \( H(X_{RSS}) - H(X_{SRS}) \) for \( n \in \{2, \ldots, 10\} \). From Table I it is observed that the difference between the Shannon entropy of RSS data and its SRS counterpart increases as the set size increases. However, intuitively, this can be explained by the fact that ranked set sampling provides more structure to the observed data than simple random sampling and the amount of the uncertainty in the more structured RSS data set is less than that of SRS.
Table 1: The numerical values of $k(n)$ for $n = 2$ up to 10.

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $k(n)$ | -0.386 | -0.989 | -1.742 | -2.611 | -3.574 | -4.616 | -5.727 | -6.897 | -8.121 |

Now, assume that $X^*_{RSS} = \{X_{[i]}, i = 1, \ldots, n\}$ is an imperfect RSS of size $n$ from $f(x)$. Similar to the perfect RSS we can easily show that

$$H(X^*_{RSS}) = \sum_{i=1}^{n} H(X_{[i]}),$$

(6)

where we assume that the cycle size is equal to one and $k = n$. Also $H(X_{[i]}) = -\int f_{[i]}(x) \log f_{[i]}(x) \, dx$, or equivalently

$$H(X_{[i]}) = -\int \left( \sum_{r=1}^{n} p_{i,r} f_{(r)}(x) \right) \log \left( \sum_{r=1}^{n} p_{i,r} f_{(r)}(x) \right) \, dx.$$

Again, using the convexity of $g(t) = t \log t$ and the equalities $\sum_{r=1}^{n} p_{i,r} = \sum_{i=1}^{n} p_{i,r} = 1$, we find

$$H(X^*_{RSS}) = \sum_{i=1}^{n} H(X_{[i]})$$

$$\leq -n \int \left( \frac{1}{n} \sum_{r=1}^{n} \left( \sum_{i=1}^{n} p_{i,r} f_{(r)}(x) \right) \right) \log \left( \frac{1}{n} \sum_{r=1}^{n} \left( \sum_{i=1}^{n} p_{i,r} f_{(r)}(x) \right) \right) \, dx$$

$$= -n \int f(x) \log f(x) \, dx$$

$$= H(X_{SRS}).$$

So, we have the following result.

**Lemma 2.** $H(X^*_{RSS}) \leq H(X_{SRS})$ for all set size $n \in \mathbb{N}$ and the equality holds when the ranking is done randomly and $p_{i,r} = \frac{1}{n}$, for all $i, r \in \{1, \ldots, n\}$.

In the following result we compare the Shannon entropies of perfect and imperfect RSS data. We observe that the Shannon entropy of $X^*_{RSS}$ is less than the Shannon entropy of $X^*_{RSS}$.

**Lemma 3.** $H(X_{RSS}) \leq H(X^*_{RSS})$ for all set size $n \in \mathbb{N}$ and the equality happens when the ranking is perfect.

*Proof.* Using the inequality $f_{[i]}(x) \log f_{[i]}(x) \leq \sum_{r=1}^{n} p_{i,r} f_{(r)}(x) \log f_{(r)}(x)$, we have

$$H(X_{[i]}) \geq -\sum_{r=1}^{n} p_{i,r} \int f_{(r)}(x) \log f_{(r)}(x) \, dx = \sum_{r=1}^{n} p_{i,r} H(X_{(r)}).$$

Now, the result follows from (6) upon changing the order of summations and using $\sum_{i=1}^{n} p_{i,r} = 1$. \(\square\)

Summing up, we find the following ordering relationship among the Shannon entropies of $X^*_{RSS}, X_{RSS}$ and $X_{SRS}$:

$$H(X_{RSS}) \leq H(X^*_{RSS}) \leq H(X_{SRS}).$$
Example 1. Suppose $X$ has an exponential distribution with pdf $f(x) = \lambda e^{-\lambda x}$, $x > 0$, where $\lambda > 0$ is the unknown parameter of interest. We consider the case where $n = 2$. For an imperfect RSS of size $n = 2$, we use the ranking error probability matrix

$$
P = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}.
$$

Using (1), we have $f_i(x) = 2\lambda e^{-\lambda x} \left[ (p_{i,1} - p_{i,2})e^{-\lambda x} + p_{i,2} \right]$, $i = 1, 2$. Straightforward calculations show that

$$
H(X_{RSS}) = 2 - 2 \log(2\lambda), \quad H(X_{SRS}) = 3 - 2 \log(2\lambda), \quad \text{and}
$$

$$
H(X^*_{RSS}) = 2 - 2 \log(2\lambda) + (p_{2,2} - p_{1,1}) + \eta(p_{1,1}) + \eta(p_{2,2}),
$$

where

$$
\eta(a) = \frac{2}{1 - 2a} \int_a^{1-a} u \log u \, du = \frac{1}{2} + \frac{1}{1 - 2a} \left[ (1 - a)^2 \log(1 - a) - a^2 \log a \right],
$$

with $0 < a < 1$. It is easy to show that

$$
H(X_{RSS}) - H(X_{SRS}) = 1 - 2 \log 2 \approx -0.3863 < 0,
$$

$$
H(X^*_{RSS}) - H(X_{RSS}) = \eta(p_{1,2}) + \eta(1 - p_{1,2}) = 2\eta(p_{1,2}) - 2 \log 2 < 0,
$$

$$
H(X_{RSS}) - H(X^*_{RSS}) = 1 - \eta(p_{1,2}) - \eta(1 - p_{1,2}) = 1 - 2\eta(p_{1,2}) < 0.
$$

Figure 1 shows the differences between $H(X^*_{RSS})$ and $H(X_{RSS})$ with $H(X_{SRS})$. It also presents the effect of ranking error on the amount of the Shannon entropy of the resulting RSS data by comparing $H(X^*_{RSS})$ with $H(X_{RSS})$. It is observed that, the maximum difference occurs for $p_{1,2} = 0.5$.

![Figure 1: Computed values of the difference between the Shannon entropies of $X_{RSS}$ and $X^*_{RSS}$ compared with that of $X_{SRS}$ of the same size as a function of the ranking error probability $p_{1,2}$.](image-url)
3. Rényi Information of Ranked Set Samples

A more general measure of entropy with the same meaning and similar properties as that of Shannon entropy has been defined by Rényi (1961) as follows

\[ H_\alpha(X) = \frac{1}{1 - \alpha} \log \int f^\alpha(x) d\nu(x) = \frac{1}{1 - \alpha} \log \mathbb{E} \left[ f^{\alpha - 1}(X) \right], \]

(7)

where \( \alpha > 0, \alpha \neq 1 \) and \( d\nu(x) = dx \) for the continuous and \( d\nu(x) = 1 \) for discrete cases. It is well known that

\[ \lim_{\alpha \to 1} H_\alpha(X) = -\int f(x) \log f(x) dx = H(X). \]

Rényi information is much more flexible than the Shannon entropy due to the parameter \( \alpha \). It is an important measure in various applied sciences such as statistics, ecology, engineering, economics, etc. In this section, we obtain the Rényi information of \( X_{RSS} \) and \( X^*_{RSS} \) and compare them with the Rényi information of \( X_{SRS} \). To this end, from (7), it is easy to show that the Rényi information of a SRS of size \( n \) from \( f \) is given by

\[ H_\alpha(X_{SRS}) = \sum_{i=1}^{n} H_\alpha(X_i) = nH_\alpha(X_1). \]

(8)

Also, for a RSS of size \( n \), we have

\[ H_\alpha(X_{RSS}) = \sum_{i=1}^{n} H_\alpha(X_{(i)}). \]

(9)

To compare \( H_\alpha(X_{SRS}) \) with \( H_\alpha(X_{RSS}) \) and \( H_\alpha(X^*_{RSS}) \), we consider two cases, i.e. \( 0 < \alpha < 1 \) and \( \alpha > 1 \). First, we find the results for \( 0 < \alpha < 1 \) which are stated in the next lemma.

**Lemma 4.** For any \( 0 < \alpha < 1 \) and all \( n \in \mathbb{N} \), we have

\[ H_\alpha(X_{RSS}) \leq H_\alpha(X^*_{RSS}) \leq H_\alpha(X_{SRS}). \]

**Proof.** We first show that for any \( 0 < \alpha < 1 \), \( H_\alpha(X_{RSS}) \leq H_\alpha(X^*_{RSS}) \). To this end, using

\[ H_\alpha(X^*_{RSS}) = \frac{1}{1 - \alpha} \sum_{i=1}^{n} \log \int \left( \sum_{j=1}^{n} p_{i,j} f_{(j)}(x) \right)^\alpha dx, \]

(10)

and concavity of \( h_1(t) = t^\alpha \), for \( 0 < \alpha < 1, t > 0 \), we have

\[ H_\alpha(X^*_{RSS}) \geq \frac{1}{1 - \alpha} \sum_{i=1}^{n} \log \int \sum_{j=1}^{n} p_{i,j} f_{(j)}^\alpha(x) dx \]

\[ \geq \frac{1}{1 - \alpha} \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j} \log \int f_{(j)}^\alpha(x) dx \]

\[ = \frac{1}{1 - \alpha} \sum_{j=1}^{n} \log \int f_{(j)}^\alpha(x) dx = H_\alpha(X_{RSS}), \]

where the second inequality is obtained by using the concavity of \( h_2(t) = \log t \), for \( t > 0 \). This, with (8), shows the result. To complete the proof we show that \( H_\alpha(X^*_{RSS}) \leq H_\alpha(X_{SRS}) \) for any \( 0 < \alpha < 1 \) and all \( n \in \mathbb{N} \). To
this end, from (10), and using \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \), we have

\[
H_\alpha(\mathbf{X}_{RSS}) = \frac{1}{1 - \alpha} \sum_{i=1}^{n} \log \int f_i^\alpha(x) \, dx \\
\leq \frac{n}{1 - \alpha} \log \sum_{i=1}^{n} \frac{1}{n} \int f_i^\alpha(x) \, dx \\
\leq \frac{n}{1 - \alpha} \log \left( \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right)^\alpha \, dx \\
= \frac{n}{1 - \alpha} \log \int f^\alpha(x) \, dx = H(\mathbf{X}_{SRS}). 
\]

\[\blacksquare\]

In Lemma 4, we were able to show analytically an ordering relationship among the Rényi information of \( \mathbf{X}^*_{RSS} \), \( \mathbf{X}_{RSS} \) and \( \mathbf{X}_{SRS} \) when \( 0 < \alpha < 1 \). It would naturally be of interest to extend such a relationship to the case where \( \alpha > 1 \). It appears that similar relationship as in Lemma 4 holds when \( \alpha > 1 \). However, we have not analytical proof here.

\textbf{Conjecture 1.} For any \( \alpha > 1 \) and all \( n \in \mathbb{N} \), we have

\[
H_\alpha(\mathbf{X}_{RSS}) \leq H_\alpha(\mathbf{X}^*_{RSS}) \leq H_\alpha(\mathbf{X}_{SRS}).
\]

In Example 2, we compare the Rényi information of \( \mathbf{X}^*_{RSS}, \mathbf{X}_{RSS} \) and \( \mathbf{X}_{SRS} \) as a function of \( \alpha \) in the case of an exponential distribution. The results are presented in Figure 2, which do support Conjecture 1.

**Example 2.** Suppose the assumptions of Example 1 hold, then the Rényi information of a SRS of size \( n = 2 \) is given by

\[
H_\alpha(\mathbf{X}_{SRS}) = -2 \log \lambda - \frac{2}{1 - \alpha} \log \alpha, \quad \alpha \neq 1.
\]

Straightforward calculations show that

\[
H_\alpha(X_{(1)}) = - \log \lambda - \log 2 - \frac{1}{1 - \alpha} \log \alpha, \\
H_\alpha(X_{(2)}) = - \log \lambda + \frac{\alpha}{1 - \alpha} \log 2 + \frac{1}{1 - \alpha} \log \left\{ \frac{\Gamma(\alpha + 1)\Gamma(\alpha)}{\Gamma(2\alpha + 1)} \right\},
\]

and so the Rényi information of \( \mathbf{X}_{RSS} \) is given by \( H_\alpha(\mathbf{X}_{RSS}) = H_\alpha(X_{(1)}) + H_\alpha(X_{(2)}) \). Now,

\[
H_\alpha(\mathbf{X}_{RSS}) - H_\alpha(\mathbf{X}_{SRS}) = \frac{\alpha}{1 - \alpha} (1 - \log 2) + \frac{1}{1 - \alpha} \log \left\{ \frac{\Gamma(\alpha + 1)\Gamma(\alpha)}{\Gamma(2\alpha + 1)} \right\}.
\]

To obtain \( H_\alpha(\mathbf{X}^*_{RSS}) \), let

\[
U_{i,\lambda}(x,t) = a_i(t)e^{-\lambda^x} + b_i(t), \quad i = 1, 2,
\]

where

\[
a_i(t) = (-1)^i(1 - 2t) \quad \text{and} \quad b_i(t) = t^{(1-i)(1-t)^{(2-i)}}
\]

and \( p_{1,1} = P(X_{(1)} = X_{[1]}) \) is defined in Example 1. Now, the Rényi information of \( \mathbf{X}^*_{RSS} \) is obtained as follows

\[
H_\alpha(\mathbf{X}^*_{RSS}) = \frac{\alpha}{1 - \alpha} \log 2\lambda + \frac{1}{1 - \alpha} \sum_{i=1}^{2} \log \int_{0}^{\infty} e^{-\alpha \lambda x}U_{i,\lambda}^\alpha(x,p_{1,1}) \, dx, \quad \alpha \neq 1,
\]
which can be calculated numerically. Figure 2(a) shows the values of \( H_\alpha(X^*_{RSS}) - H_\alpha(X_{SRS}) \) as a function of \( \alpha \) for \( p_{1,1} \in \{0.8, 0.9, 0.95, 1\} \). When \( p_{1,1} = 1 \), \( H_\alpha(X^*_{RSS}) - H_\alpha(X_{SRS}) = H_\alpha(X_{RSS}) - H_\alpha(X_{SRS}) \). In Figure 2(b) we show the effect of the ranking error on the Rényi information of \( X_{RSS} \) by comparing \( H_\alpha(X^*_{RSS}) \) and \( H_\alpha(X_{RSS}) \) as functions of \( \alpha \) for different values of \( p_{1,1} \).

![Figure 2](image_url)

Figure 2: Comparison of the Rényi information of \( X_{RSS} \) and \( X^*_{RSS} \) with that of \( X_{SRS} \) as a function of \( \alpha \). The value of \( H_\alpha(X^*_{RSS}) - H_\alpha(X_{SRS}) \) are presented in (a) while \( H_\alpha(X^*_{RSS}) - H_\alpha(X_{RSS}) \) are given in (b).

Note that for \( \alpha > 1 \) the difference between the Rényi information of \( X_{RSS} \) with its counterpart under SRS can be written as follows

\[
H_\alpha(X_{RSS}) - H_\alpha(X_{SRS}) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \int f_{(i)}^\alpha(x)dx - \frac{n}{1-\alpha} \log \int f^\alpha(x)dx
\]

\[
= \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \left( \frac{\int f_{(i)}^\alpha(x)dx}{\int f^\alpha(x)dx} \right)
\]

\[
= \frac{\alpha}{1-\alpha} n \log n + \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \mathbb{E} \left[ \{P_{F(W)}(T = i - 1)\}^\alpha \right], \tag{11}
\]

where \( T|W = w \sim Bin(n-1,F(w)) \) and \( W \) has a density proportional to \( f^\alpha(w) \), i.e. \( g(w) = \frac{f^\alpha(w)}{\int f^\alpha(w)dw} \). Since \( P_{F(w)}(T = i - 1) \leq 1 \) for all \( i = 1, \ldots, n-1 \) and fixed \( w \), we have

\[
\log \mathbb{E} \left[ \{P_{F(W)}(T = i - 1)\}^\alpha \right] \leq 0,
\]

for all \( \alpha > 1 \). This results in a lower bound for the difference between the Rényi information of \( X_{RSS} \) and \( X_{SRS} \) as \( H_\alpha(X_{RSS}) - H_\alpha(X_{SRS}) \geq \frac{\alpha}{1-\alpha} n \log n \). In the following result, we find a sharper lower bound for \( H_\alpha(X_{RSS}) - H_\alpha(X_{SRS}) \) when \( \alpha > 1 \).

**Lemma 5.** For any \( \alpha > 1 \) and all \( n \geq 2 \), we have \( H_\alpha(X_{RSS}) - H_\alpha(X_{SRS}) \geq \Psi(\alpha, n) \), with

\[
\Psi(\alpha, n) = \frac{\alpha}{1-\alpha} \sum_{i=1}^{n} \log \left\{ n \binom{n-1}{i-1} \left( \frac{i-1}{n-1} \right)^{i-1} \left( \frac{n-i}{n-1} \right)^{n-i} \right\},
\]

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where $\Psi(\alpha, n) \in \left[ \frac{\alpha}{1-\alpha} \log n, 0 \right]$.

**Proof.** Using (9), the pdf of $X_{(i)}$ and the transformation $F(X) = U$, we have

$$H_\alpha(X_{RSS}) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \int_0^1 \left\{ f_{n-i+1}^*(u) \right\}^\alpha f^{\alpha-1}(F^{-1}(u)) du,$$

where $f_{n-i+1}^*(u)$ is the pdf of a Beta($i, n - i + 1$) random variable with its mode at $u^* = \frac{i-1}{n-1}$. Now, since $f_{n-i+1}^*(u) \leq f_{i,n-1+1}^*(\frac{i-1}{n-1})$, we have

$$H_\alpha(X_{RSS}) \geq \frac{\alpha}{1-\alpha} \sum_{i=1}^{n} \log \left\{ n \binom{n-1}{i-1} (\frac{i-1}{n-1})^{i-1} (\frac{n-i}{n-1})^{n-i} \right\} + \frac{n}{1-\alpha} \log \int_0^1 f^{\alpha-1}(F^{-1}(u)) du$$

$$= \Psi(\alpha, n) + H(X_{SRS}),$$

where

$$\Psi(\alpha, n) = \frac{\alpha}{1-\alpha} \sum_{i=1}^{n} \log \left\{ n \binom{n-1}{i-1} (\frac{i-1}{n-1})^{i-1} (\frac{n-i}{n-1})^{n-i} \right\}. \quad (12)$$

It is easy to show that for all $n \in \mathbb{N}$ and $\alpha > 1$, $\Psi(\alpha, n) < 0$. To do this, one can easily check that $\Psi(\alpha, n+1) \leq \Psi(\alpha, n)$ for all $n \geq 2$ with $\Psi(\alpha, 2) = \frac{2\alpha}{1-\alpha} < 0$. Also, since $(\frac{n-1}{i-1}) (\frac{i-1}{n-1})^{i-1} (\frac{n-i}{n-1})^{n-i} \leq 1$ for all $i = 1, \ldots, n$, we have $\Psi(\alpha, n) \geq \frac{\alpha}{1-\alpha} \sum_{i=1}^{n} \log n = \frac{\alpha}{1-\alpha} \log n$. \qed

**4. Kullback-Leibler Information of Ranked Set Samples**

In 1951 Kullback and Leiber introduced a measure of information from the statistical point of view by comparing two probability distributions associated with the same experiment. The Kullback-Leibler (KL) divergence is a measure of how different two probability distributions (over the same sample space) are. The KL divergence for two random variables $X$ and $Y$ with cdfs $F$ and $G$ and pdfs $f$ and $g$, respectively, is given by

$$K(X, Y) = \int f(t) \log \left( \frac{f(t)}{g(t)} \right) dt. \quad (13)$$

Using the same idea, we define the KL discrimination information between $X_{RSS}$ and $X_{SRS}$ as follows:

$$K(X_{SRS}, X_{RSS}) = \int_{X_{SRS}} f(x_{SRS}) \log \left( \frac{f(x_{SRS})}{f(x_{RSS})} \right) dx_{SRS}.$$ 

It is easy to see that

$$K(X_{SRS}, X_{RSS}) = \sum_{i=1}^{n} \int f(x) \log \left( \frac{f(x)}{f_{(i)}(x)} \right) dx = \sum_{i=1}^{n} K(X, X_{(i)}). \quad (14)$$
By substituting the pdf of $X_{(i)}$ in (14), we find
\[
K(X_{SRS}, X_{RSS}) = -\sum_{i=1}^{n} \int f(x) \log \left( \frac{f(x)}{g(i)(x)} \right) dx
\]
\[
= -\sum_{i=1}^{n} \int_{0}^{1} \log \left( i \binom{n}{i} u^{i-1}(1-u)^{n-i} \right) du
\]
\[
= -\sum_{i=1}^{n} \log i \binom{n}{i} + n(n-1)
\]
\[
:= d_n. \tag{15}
\]

Note that $K(X_{SRS}, X_{RSS})$ is distribution-free, and \{\(d_n, n = 1, 2, \ldots\}\) is a nondecreasing sequence of non-negative real values for all $n \in \mathbb{N}$. That is, the KL information between the distribution of SRS and the distribution of RSS of the same size increases as the set size $n$ increases.

**Remark 1.** It is well known that the KL divergence is non-symmetric and cannot be considered as a distance metric. In our problem, note that
\[
K(X_{RSS}, X_{SRS}) = \sum_{i=1}^{n} K(X_{(i)}, X) = \sum_{i=1}^{n} K(U_{(i)}, U) = -\sum_{i=1}^{n} H(U_{(i)}) = -k(n),
\]
where $U$ has uniform distribution. Various measures have been introduced in the literature generalizing this measure. For example, in order to have a distance metric, the following symmetric Kullback-Leibler distance (KLD) is proposed.
\[
KLD(X, Y) = K(X, Y) + K(Y, X).
\]

**Lemma 6.** Suppose $X_{SRS}$ is a SRS of size $n$ from $f(x)$ and let $Y_{RSS}$ and $Y_{SRS}$ be independent RSS and SRS samples of the same size from another distribution with pdf $g(x)$, respectively. Then,
\[
K(X_{SRS}, Y_{SRS}) \leq K(X_{SRS}, Y_{RSS}).
\]

**Proof.** To show the result, by the use of the fact that $g(x) = \frac{1}{n} \sum_{i=1}^{n} g(i)(x)$, we have
\[
K(X_{SRS}, Y_{RSS}) = \sum_{i=1}^{n} \int f(x) \log \left( \frac{f(x)}{g(i)(x)} \right) dx
\]
\[
\geq n \int f(x) \left\{ -\log \left( \sum_{i=1}^{n} \frac{g(i)(x)}{n f(x)} \right) \right\} dx
\]
\[
= n \int f(x) \log \left( \frac{f(x)}{g(x)} \right) dx
\]
\[
= K(X_{SRS}, Y_{SRS}),
\]
where the inequality is due to the convexity of $h(t) = -\log t$. \qed

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Now, let \( X_{RSS}^* = \{X[i], i = 1, \ldots, n\} \) be an imperfect RSS of size \( n \) from \( f(x) \). Then,

\[
K(X_{SRS}, X_{RSS}^*) = \sum_{i=1}^{n} \int f(x) \log \left( \frac{f(x)}{f_{[i]}(x)} \right) dx
\]

\[
= -\sum_{i=1}^{n} \int f(x) \log \left( \sum_{j=1}^{n} p_{i,j} \binom{n}{j} F(j-1)(x) \tilde{F}^{(n-j)}(x) \right) dx
\]

\[
= -n \sum_{i=1}^{n} \int_0^1 \log \left( \sum_{j=1}^{n} p_{i,j} \binom{n}{j} u^{j-1}(1-u)^{n-j} \right) du.
\]

Therefore, the KL discrimination information between the distribution of \( X_{SRS} \) and \( X_{RSS}^* \) is also distribution free and it is only a function of the set size \( n \) and the ranking error probabilities \( p_{i,j} = P(X[i] = X(j)) \).

In the following lemma, we show that the KL information between the distribution of a SRS and a perfect RSS of the same size is greater that the one with imperfect RSS.

**Lemma 7.** Suppose \( X_{SRS} \) is a SRS of size \( n \) from the pdf \( f(x) \) and denote \( X_{RSS}^* \) and \( X_{RSS} \) as independent perfect and imperfect RSS data of the same size from \( f \), respectively. Then,

\[
K(X_{SRS}, X_{RSS}) \geq K(X_{SRS}, X_{RSS}^*).
\]

**Proof.** To show the result note that

\[
K(X_{SRS}, X_{RSS}) = -\sum_{i=1}^{n} \int_0^1 \log \left( \sum_{j=1}^{n} p_{i,j} \binom{n}{j} u^{j-1}(1-u)^{n-j} \right) du
\]

\[
\leq -\sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j} \int_0^1 \log \left( j \binom{n}{j} u^{j-1}(1-u)^{n-j} \right) du
\]

\[
= -\sum_{j=1}^{n} \int_0^1 \log \left( j \binom{n}{j} u^{j-1}(1-u)^{n-j} \right) du
\]

\[
= K(X_{SRS}, X_{RSS}),
\]

which completes the proof.

Another result which is of interest is to compare \( K(X_{RSS}, Y_{RSS}) \) with \( K(X_{SRS}, Y_{SRS}) \). To this end, we have

\[
K(X_{RSS}, Y_{RSS}) = \sum_{i=1}^{n} \int f(i)(x) \left\{ \log \left( \frac{f(x)}{g(x)} \right) + \log \left( \frac{F^{(i-1)}(x) \tilde{F}^{(n-i)}(x)}{G^{(i-1)}(x) \tilde{G}^{(n-i)}(x)} \right) \right\} dx
\]

\[
= n \int f(x) \log \left( \frac{f(x)}{g(x)} \right) dx + \sum_{i=1}^{n} \int f(x) i \binom{n}{i} F^{(i-1)}(x) \tilde{F}^{(n-i)}(x) \log \left( \frac{F^{(i-1)}(x) \tilde{F}^{(n-i)}(x)}{G^{(i-1)}(x) \tilde{G}^{(n-i)}(x)} \right) dx
\]

\[
= K(X_{SRS}, Y_{SRS}) + A_n(F, G),
\]

where

\[
A_n(F, G) = \sum_{i=1}^{n} \int_0^1 i \binom{n}{i} u^{i-1}(1-u)^{n-i} \log \left( \frac{u^{i-1}(1-u)^{n-i}}{\{G(F^{-1}(u))\}^{i-1}\{G(F^{-1}(u))\}^{n-i}} \right) du.
\]
Here again $K(\mathbf{X}_{SRS}, \mathbf{Y}_{SRS}) \leq K(\mathbf{X}_{RSS}, \mathbf{Y}_{RSS})$ if $A_n(F, G) \geq 0$. Furthermore, it is easy to show that

$$A_n(F, G) = -\frac{n(n-1)}{2} - n(n-1) \int_0^1 \{u \log G(F^{-1}(u)) + (1-u) \bar{G}(F^{-1}(u))\} \, du.$$ 

Note that in this case $A_n(F, G)$ depends on the parent distributions of $X$ and $Y$ samples.

**Example 3.** Suppose that $X$ and $Y$ have the exponential distributions with parameters $\lambda_1$ and $\lambda_2$, and pdfs $f(x) = \lambda_1 e^{-\lambda_1 x}$ and $g(y) = \lambda_2 e^{-\lambda_2 y}$, respectively. Then

$$A_n(F, G) = -\frac{n(n-1)}{2} - \frac{n(n-1)}{3} \left(\frac{\lambda_2}{\lambda_1}\right)^2 - n(n-1) \left\{\sum_{i=1}^{\infty} \frac{1}{i(i+1)} - \frac{\lambda_2}{\lambda_1} \sum_{i=1}^{\infty} \frac{1}{i(i+2)}\right\}$$

$$= n(n-1) \left[\frac{\lambda_2}{\lambda_1} \left(\frac{1}{4} - \frac{\lambda_2}{3\lambda_1}\right) - \frac{3}{2}\right] < 0.$$ 

So, for the exponential distributions $K(\mathbf{X}_{SRS}, \mathbf{Y}_{SRS}) > K(\mathbf{X}_{RSS}, \mathbf{Y}_{RSS})$.

## 5. Concluding Remarks

In this paper, we have considered the information content of the perfect and imperfect RSS data using the Shannon entropy, Rényi and Kullback-Leibler information measures. First, we have compared the Shannon entropy of a SRS data ($\mathbf{X}_{RSS}$) to the Shannon entropy of perfect RSS ($\mathbf{X}_{RSS}$) and imperfect RSS ($\mathbf{X}_{RSS}^*$) of the same size. In this case, we have established analytically that the Shannon entropies of $\mathbf{X}_{RSS}$ and $\mathbf{X}_{RSS}^*$ are less that the Shannon entropy of $\mathbf{X}_{SRS}$. We also showed that the Shannon entropy of the RSS data will increase in the presence of the ranking error. Next, we have established similar behaviour under the Rényi information when $0 < \alpha < 1$, while the results for the case were $\alpha > 1$ remain unsolved. We conjectured that similar results hold for the case where $\alpha > 1$ and provided examples to support the conjecture. Similar results are obtained under the Kullback-Leibler information measure. The results of this paper show some desirable properties of ranked set sampling compared with the commonly used simple random sampling in the context of the information theory.

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