Analytic linearization of a generalization of the semi-standard map: radius of convergence and Brjuno sum

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Abstract: One considers a system on \( \mathbb{C}^2 \) close to an invariant curve which can be viewed as a generalization of the semi-standard map to a trigonometric polynomial with many Fourier modes. The radius of convergence of an analytic linearization of the system around the invariant curve is bounded from below by \( \exp\left(-\frac{1}{2} B(d\alpha) - C\right) \), where \( C \geq 0 \) does not depend on \( \alpha \), \( d \in \mathbb{N}^* \) and \( \alpha \) is the frequency of the linear part. For a class of trigonometric polynomials, it is also bounded from above by a similar function. The error function is non decreasing with respect to the smallest coefficient of the trigonometric polynomial.

1 Introduction

Consider the following discrete dynamical system:

\[
\begin{aligned}
x_{n+1} &= x_n + y_n + A(x_n) \\
y_{n+1} &= y_n + A(x_n)
\end{aligned}
\]

where \( A \) is a trigonometric polynomial with positive Fourier modes: \( A(x) = \sum_{K=1}^{N} a_K e^{iKx} \), \( a_K \in \mathbb{C} \). This can be viewed as a generalization of the semi-standard map, which is given by \( A(x) = e^{ix} \). The semi-standard map was introduced as a model which is similar to, but easier to study than, the standard map given by \( A(x) = \sin x \), where the coexistence of both positive and negative Fourier modes makes the linearization problem more difficult to solve.

With the change of variables \( z_n = e^{ix_n}, \lambda_n = e^{iy_n} \), system (1) is conjugated to the system

\[
\begin{aligned}
z_{n+1} &= \lambda_n z_n \prod_{K=1}^{N} e^{ia_K z_n^K} \\
\lambda_{n+1} &= \lambda_n \prod_{K=1}^{N} e^{ia_K z_n^K}
\end{aligned}
\]

for which the set \( \{0\} \times S^1 \) is invariant. Let \( F(\lambda, z) = (\lambda z \prod_{K=1}^{N} e^{ia_K z^K}, \lambda \prod_{K=1}^{N} e^{ia_K z^K}) \) so the system (2) can be written \( (z_{n+1}, \lambda_{n+1}) = F(z_n, \lambda_n) \). One looks for an analytic linearization of the system, that is to say, an analytic map \( H(z, \lambda) = (h(z, \lambda), h_2(z, \lambda)) \) such that \( F \circ H = H \circ R \) where \( R(z, \lambda) = (\lambda z, \lambda) \). If there is such a linearization and if it is analytic, then the invariant curves of the rotation \( R \) are smoothly preserved. Then one has a family of invariant closed curves in \( \mathbb{C}^2 \), corresponding to the numbers \( \lambda \) with...
modulus 1 and \(z\) in a neighbourhood of 0 where the linearization is analytic. Now if \(\lambda\) is rational, there is no way of finding a linearization which is analytic in \(z\). Thus we will have to assign a value to the parameter \(\lambda\), with \(|\lambda| = 1\) and \(\text{arg}(\lambda) \in \mathbb{R} \setminus \mathbb{Q}\), to construct a linearization which is close to the identity and analytic; its radius of convergence will depend on the arithmetical properties of \(\lambda\).

Davie [12] and Marmi [15] proved that concerning the semi-standard map, the radius of convergence \(\rho(\alpha)\) of the linearization is bounded as follows:

\[
\exp(-2B(\alpha) - C) \leq \rho(\alpha) \leq \exp(-2B(\alpha) + C')
\]

where \(C > 0, C' > 0\) do not depend on the complex argument \(\alpha\) of \(\lambda\) and where \(B(\alpha)\) is the Brjuno sum of \(\alpha\). In particular, if \(B(\alpha)\) diverges, then there is no analytic linearization around 0. This can be reformulated as stating that the error function \(\alpha \mapsto 2B(\alpha) + \ln \rho(\alpha)\) is bounded.

After the numerical evidence in [17], a similar result about the standard map, in the perturbative case, was proved in [2] and [3]. However, concerning the semi-standard map and in the present paper, the strong assumption of Fourier modes being only positive makes it possible to remove the perturbative assumption.

The Brjuno sum was first introduced in [5] to give a sufficient condition to the convergence of the linearization for analytic vector fields around a fixed point.

Yoccoz proved in [19] that the Brjuno condition (i.e. the convergence of the Brjuno function, or equivalently of the Brjuno sum) is necessary and sufficient to the analytic linearization of the quadratic polynomial and of germs of diffeomorphisms of \((\mathbb{C}, 0)\). This resulted in the study of the error function \(\Phi + \ln r\), where \(\Phi\) is the Brjuno function and \(r\) is the radius of convergence of the linearization for the quadratic polynomial. It was conjectured in [16] that this function is \(1/2\)-Hölder and Buff and Chéritat showed first that it is bounded in [6], then that it is continuous in [7]. Chéraghi-Chéritat then proved that a restriction of this error function is \(1/2\)-Hölder.

Brjuno’s lower bound on the convergence radius for linearization of analytic vector fields was improved in [13]. In [18], Stolovitch replaced the arithmetical condition on the spectrum of the linear part by a condition of algebraic nature. The Brjuno sum was also proved to play a role in other analytic linearization problems, as for instance linearization of vector fields around an invariant torus (see [1] and [9]), or reducibility of quasiperiodic cocycles ([10]).

The optimality of the Brjuno condition was also studied in other linearization problems. Carletti-Marmi proved in [8] that the Brjuno condition is also necessary to linearize analytically a germ of diffeomorphism around a fixed point, and generalize it to Gevrey classes. However the continuity of the analogue of the error function for linearization problems more general than the quadratic polynomial remains open up to now.

Our main result is stated in the following two theorems:
Theorem 1 Let \( \rho \) be the radius of convergence of the linearization of the system (2). Let \( d \) be the greatest common divisor of the indices of the Fourier modes of \( A \). There exists \( C \geq 0 \), which is a non decreasing function of the greatest coefficient of \( A \) and which does not depend on \( \alpha \), such that

\[
\rho \geq \exp\left(-\frac{2}{d}B(d\alpha) - C\right)
\]

(3)

Theorem 2 Assume that the coefficients of \( A \) satisfy the following assumption: there exists \( \theta \in \mathbb{R} \) such that for all \( k = 1, \ldots, N \) with \( a_k \neq 0 \), the complex argument of the number \( a_k \) is \( k\theta + \frac{\pi}{2} \).

Let \( \rho \) be the radius of convergence of the linearization of the system (2). Let \( d \) be the greatest common divisor of the indices of the Fourier modes of \( A \). There exists \( C' \geq 0 \), which is a non decreasing function of the smallest coefficient of \( A \) and does not depend on \( \alpha \) such that

\[
\rho \leq \exp\left(-\frac{2}{d}B(d\alpha) + C'\right)
\]

(4)

In particular, if \( B(d\alpha) \) diverges, then there is no analytic linearization.

Remark: Let \( \kappa_0 \) be the smallest integer such that \( a_{\kappa_0} \neq 0 \). The assumption in Theorem 2 says that the argument of every non zero \( a_k \) is the following function of \( a_{\kappa_0} \):

\[
\text{Arg}(a_k) = \frac{k}{\kappa_0} \text{Arg}(a_{\kappa_0}) + \frac{\pi}{2} (1 - \frac{k}{\kappa_0})
\]

This assumption is in the spirit of Cremer’s counterexample of non-linearizable germs, except that the arguments are defined from the beginning instead of recursively.

The main result is obtained by a direct analysis of the coefficients of the formal linearization (which always exists, and is unique if one requires it to be formally close to the identity); as in [15] and [12], it appears that there is a link between those coefficients and the Brjuno sum of \( d\alpha \). To get an upper bound of the radius of convergence, that is to say, a lower bound on the coefficients of the linearization, one uses a strong assumption on the complex arguments of the coefficients of \( A \), in order to be able to bound the sum from below by just one of its terms. However the lower bound on the radius of convergence does not use this assumption.

2 Notations

Let \( x \in \mathbb{R} \), one denotes by \( ||x||_\mathbb{Z} \) the distance between \( x \) and the closest integer: \( ||x||_\mathbb{Z} = \min_{p \in \mathbb{Z}} |x - p| \).

Considering the trigonometric polynomial \( A(x) \) defined at the beginning, denote by \( \kappa_0 < \kappa_1 < \cdots < \kappa_N \) all indices of Fourier modes of \( A \): thus for all \( 0 \leq i \leq N \), \( a_{\kappa_i} \neq 0 \), and if \( \forall i = 0, \ldots, N, K \neq \kappa_i \), then \( a_K = 0 \). Also denote by \( \kappa_{\min} \) (resp. \( \kappa_{\max} \)) the number \( \kappa_i \) minimizing (resp. maximizing) \( \{|a_{\kappa_i}|, i = 0, \ldots, N\} \).
Denote by $M \subset \mathbb{N}$ the additive semi-group generated by $\{\kappa_0, \ldots, \kappa_N\}$:

$$M = \{p_0 \kappa_0 + \cdots + p_N \kappa_N \geq 0, p_0, \ldots, p_N \in \mathbb{N}\}$$  \hspace{1cm} (5)

3 Analysis of the linearization

With a reasoning similar to the one in [4] (which is reproduced and adapted to the present model in the appendix), one proves that if $H$ linearizes the system, then, up to a multiplicative constant which will not change the radius of convergence, the first component of the linearization is a function $h(z, \lambda) = iz e^{\Phi(\lambda z)}$ satisfying (mod $2i\pi$):

$$\sum_{K=1}^{N} i a_K (iz)^K e^{K \Phi(z)} = \Phi(\lambda^{-1} z) + \Phi(\lambda z) - 2\Phi(z)$$  \hspace{1cm} (6)

Let $\lambda \in \mathbb{C}$ with modulus 1 and $\alpha = \frac{\arg(\lambda)}{2\pi}$. Denoting $\Phi(\lambda z) = \sum_{l \geq 1} \phi_l z^l$ and

$$d_{l,\lambda} = \lambda^l - \lambda^{-l} = 2i \sin(\pi l \alpha)$$  \hspace{1cm} (7)

one gets the following:

$$\sum_{K \in \{\kappa_0, \ldots, \kappa_N\}} i^{K+1} a_K z^K \left(1 + \sum_{p \geq 1} \frac{1}{p!} \left(\sum_{j \geq 1} \phi_j z^j\right)^p\right) = \sum_{l \geq 1} \phi_l z^l d_{l,\lambda}^2$$  \hspace{1cm} (8)

In what follows, $\lambda$ being fixed with modulus 1 and argument $\alpha$ such that $\alpha^2 \pi \notin \mathbb{Q}$, we will denote for all $l \in \mathbb{N} \setminus \{0\}$,

$$D_l := -4 \sin^2(\pi l \alpha) < 0$$

Then $|D_l|^{-1} \geq \frac{1}{4}$ for all $l \geq 1$. Let $l \geq 1$, then $\phi_l = 0$ if $l < \kappa_0$ and if $l \geq \kappa_0$,

$$\phi_l = \frac{1}{D_l} \sum_{K \in \{\kappa_0, \ldots, \kappa_N\}} a_K [\delta_{K,l} + \sum_{m=1, \ldots, K \atop m \leq l-K} C_m^K \sum_{p_1=1}^{l-K} \cdots \sum_{p_m=1}^{l-K} \frac{1}{p_1! \cdots p_m!} \sum_{j_1, \ldots, j_m} \phi_{j_1} \cdots \phi_{j_{p_1}} \cdots \phi_{j_{p_m}}]$$

Another recurrence relation is required: let $\Psi(\lambda z) = \sum_{K \in \{\kappa_0, \ldots, \kappa_N\}} \Psi_K(z)$, $\Psi_K(z) = i^{K+1} a_K z^K e^{K \Phi(z)}$ and denote by $\sum_{k \geq 1} \psi_k z^k$ the Taylor expansion of $\Psi$ and by $\sum_{l} \psi_{K,l} z^l$ the Taylor expansion of $\Psi_K$. Then

$$\Psi(\lambda z) = \Phi(\lambda^{-1} z) + \Phi(\lambda z) - 2\Phi(z)$$  \hspace{1cm} (10)

which implies that for all $l \geq \kappa_0$, 

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\[ \psi_l = D_l \phi_l \quad (11) \]

Moreover, derivating \( \Psi_K \), one sees that \( \psi_{K,l} = 0 \) for all \( l < K \), \( \psi_{K,K} = i^{K+1}a_K \), and for all \( n \geq K + 1 \),

\[ (n - K)\psi_{K,n} = K \sum_{k=1}^{n-1} k \phi_k \psi_{K,n-k} \quad (12) \]

**Remark:** One sees that \( \phi_l \not= 0 \Rightarrow l \in \mathcal{M} \). This can be shown by recurrence: for \( l = \kappa_0 \), the property holds. Assume it holds for all \( \kappa_0 \leq l' \leq l - 1 \), for a fixed \( l > \kappa_0 \). Assume that \( \phi_l \not= 0 \). If \( l \in \{ \kappa_0, \ldots, \kappa_N \} \), then \( l \in \mathcal{M} \); otherwise, there exists \( K \in \{ \kappa_0, \ldots, \kappa_N \}, m \leq l - K, p_1 \leq l - K, \ldots, p_m \leq l - K \) and non vanishing \( \phi_{j_1}, \ldots, \phi_{j_m} \) such that \( j_1 + \cdots + j_m = l - K \). By recurrence assumption, \( j_1, \ldots, j_m \in \mathcal{M} \) therefore \( l \in \mathcal{M} \).

A similar fact holds for every \( \Psi_K \): if \( \psi_{K,l} \not= 0, l \geq K \), then \( l \in \mathcal{M} \). Indeed \( \psi_{K,l} = 0 \) if \( l < K \) so the recurrence property holds for \( l \leq K \). Assume the property holds up to a fixed \( l \geq K \). If \( \psi_{K,l+1} \not= 0 \) then by (12) there exist \( k \in \{ 1, \ldots, n - 1 \} \) such that \( \phi_k \not= 0 \) and \( \psi_{K,l+1-k} \not= 0 \) so \( k \in \mathcal{M} \) and \( l + 1 - k \in \mathcal{M} \), therefore \( l + 1 \in \mathcal{M} \).

**Example:** If \( A \) is a monomial of order \( K \), then \( \Phi_A(z) \) only has coefficients indexed by multiples of \( K \). Then there exists a function \( \Xi_\lambda : \mathbb{C} \to \mathbb{C} \) such that for all \( z \), \( \Phi_\lambda(z) = \Xi_\lambda(z^K) \).

The following lemma will be used to bound the coefficients of \( \Phi \) from below by one of the terms of the sum determining them.

**Lemma 3** If there exists \( \theta \in \mathbb{R} \) such that for all \( K \in \{ \kappa_0, \ldots, \kappa_N \} \), the complex number \( a_K \) has argument \( K\theta + \frac{\pi}{2} \), then for all \( l \in \mathcal{M} \), \( \phi_l \) has argument \( l(\theta + \frac{\pi}{2}) + \pi \) and for all \( K \in \{ \kappa_0, \ldots, \kappa_N \} \), if \( \psi_{K,l} \not= 0 \) then \( \psi_{K,l} \) has argument \( l(\theta + \frac{\pi}{2}) + \pi \).

**Proof:** First one proves the part of the statement concerning \( \phi_l \):

1. If \( l = \kappa_0 \), then (11) implies that \( \phi_l = \frac{1}{D_{\kappa_0}} i^{\kappa_0+1}a_{\kappa_0} \) which is, by assumption, the product of a number of modulus \( \kappa_0(\theta + \frac{\pi}{2}) \) with \( \frac{1}{D_{\kappa_0}} \), the latter being real and positive.

2. Let \( l \geq \kappa_0 + 1 \). Assume that for all \( l' \in \mathcal{M} \) such that \( l' \leq l - 1 \), \( \phi_{l'} \) has argument \( l'(\theta + \frac{\pi}{2}) \). If \( l \) differs from all \( \kappa_i \), then by (11), \( \phi_l \) is the sum of terms which are the product of a number \( \frac{i^{K+1}a_K}{D_l} \), which by assumption has argument \( K(\theta + \frac{\pi}{2}) \), with a number which by recurrence assumption has argument \( (l - K)(\theta + \frac{\pi}{2}) \). If \( l \) is one of the \( \kappa_i \), then one has to add the term \( \frac{1}{D_l} i^{l+1}a_l \), which has argument \( l(\theta + \frac{\pi}{2}) \).

As for the statement concerning \( \psi_{K,l} \), since \( \psi_{K,K} = i^{K+1}a_K \) which has argument \( K(\theta + \frac{\pi}{2}) + \pi \), the property holds for \( l \leq K \). Assume this property holds up to a fixed \( l \geq K \). By equation (12), if \( \psi_{K,l+1} \not= 0 \) then \( \psi_{K,l+1} \) is a sum of terms with argument \( l(\theta + \frac{\pi}{2}) + \pi \).
$k(\theta + \frac{\pi}{2}) + (l + 1 - k)(\theta + \frac{\pi}{2}) + \pi$ for $k \in \{1, \ldots, n - 1\}$, therefore it has argument $(l + 1)(\theta + \frac{\pi}{2}) + \pi$. □

**Remark:** For instance, $A(x)$ satisfies the assumption of Lemma 3 if it only has coefficients $a_K \in i\mathbb{R}^+$ (then $\theta = 0$), or such that $i^{k+1}a_K \in \mathbb{R}^-$ (then $\theta = -\frac{\pi}{2}$), or if $A$ is a monomial (without restriction on $\theta$).

### 3.1 The semigroup $\mathcal{M}$

We shall also need the following lemma on the set $\mathcal{M}$:

**Lemma 4** Let $d$ be the greatest common divisor of $\kappa_0, \ldots, \kappa_n$. There exists $N_\mathcal{M}$ such that for all integer $m \geq N_\mathcal{M}$, if $m$ is a multiple of $d$, then $m \in \mathcal{M}$. If $\kappa_0 = 1$ then $N_\mathcal{M} = 1$.

**Proof:** Let $I_0 = [\kappa_0, \kappa_0 + \cdots + \kappa_N]\cap \mathbb{N}$ and for $p \geq 1$, let

$$I_p = [p(\kappa_0 + \cdots + \kappa_N), (p + 1)(\kappa_0 + \cdots + \kappa_N)]\cap \mathbb{N}$$

Then $(I_p)_{p \geq 0}$ is a partition of $[\kappa_0, +\infty]\cap \mathbb{N}$. Moreover, if $jd \in I_p \cap \mathcal{M}$ for some $p \geq 0$, then $jd + \kappa_0 + \cdots + \kappa_N$, which is also a multiple of $d$, belongs to $I_{p+1} \cap \mathcal{M}$; therefore, in order to prove this lemma, it is sufficient to prove that for all $jd \in I_0 \setminus \mathcal{M}$, there exists $p \geq 1$ such that $jd + p(\kappa_0 + \cdots + \kappa_N) \in \mathcal{M}$. After a finite number of steps, one obtains an integer $P$ such that all multiples of $d$ belonging to $I_P$ are also elements of $\mathcal{M}$. Translating by $\kappa_0 + \cdots + \kappa_N$, one will deduce that all multiples of $d$ greater than $p(\kappa_0 + \cdots + \kappa_N)$ are also elements of $\mathcal{M}$.

Now, let $jd \in I_0 \setminus \mathcal{M}$ if it exists (otherwise the proof is finished). By Bezout’s theorem, there are relative integers $b_0, \ldots, b_N$, at least one of which is positive, such that $b_0\kappa_0 + \cdots + b_N\kappa_N = jd$. Sorting the coefficients $b_i$ by sign, one infers that $jd - \sum_{i: b_i < 0} b_i\kappa_i$ is a linear combination of the $\kappa_i$ with non negative coefficients, at least one of which is non zero, therefore $jd - \sum_{i: b_i < 0} b_i\kappa_i \in \mathcal{M}$. Therefore $jd + \max(-b_0, \ldots, -b_N)(\kappa_0 + \cdots + \kappa_N) \in \mathcal{M}$. □

Lemma 4 has the following corollary:

**Corollary 5** There exists $N_\mathcal{M}$ such that for all $a, b \in \mathcal{M}$, if $a - b \geq N_\mathcal{M}$, then $a - b \in \mathcal{M}$.

**Proof:** Let $d$ be the greatest common divisor of $\kappa_0, \ldots, \kappa_N$. Since every element of $\mathcal{M}$ is a multiple of $d$, if $a, b \in \mathcal{M}$, then $a - b$ is a multiple of $d$. If moreover $a - b \geq N_\mathcal{M}$, where $N_\mathcal{M}$ was defined in Lemma 4, then $a - b \in \mathcal{M}$. □

### 4 The Brjuno sum and the Brjuno function

Let $d$ be the greatest common divisor of $\kappa_0, \ldots, \kappa_N$. Let us consider the continued fraction expansion of $da$.

**Notations:** Let $(q_k)$ be the sequence of the denominators of the approximants of $da$. Recall the well-known recurrence relation: for all $j \geq 0$,
$q_{j+2} = a_j q_{j+1} + q_j$  \hfill (13)

where $(a_j)$ is the sequence of integers given by the continued fraction expansion.

The following lemmas are given in order to relate the Brjuno sum with the small divisors of our linearization problem.

**Lemma 6** For all $k \geq 1$, there is

$$\frac{1}{2q_{k+1}} \leq |D_{dq_k}|^{\frac{1}{2}} \leq \frac{3}{q_{k+1}}$$

**Proof:** For all $l \in \mathbb{Z}$,

$$|D_l| = 4|\sin(\pi l\alpha)|^2 = 4(||l\alpha||_Z + R(||l\alpha||_Z))^2$$

(whence $R$ is the remainder in the Taylor-Lagrange formula) whence

$$||l\alpha|| \leq |D_l|^{\frac{1}{2}} \leq 3||l\alpha||$$ \hfill (14)

Now for all $k \geq 1$,

$$||q_k d\alpha||_Z = \min_{p \in \mathbb{Z}} |q_k d\alpha - p| = q_k \min_{p \in \mathbb{Z}} |d\alpha - \frac{p}{q_k}| = q_k |d\alpha - \frac{p_k}{q_k}|$$

Since

$$\frac{1}{2q_k q_{k+1}} \leq |d\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_k q_{k+1}}$$ \hfill (15)

(see for instance [16] remark 1.6), there is

$$\frac{1}{2q_{k+1}} \leq |D_{dq_k}|^{\frac{1}{2}} \leq \frac{3}{q_{k+1}}. \square$$

**Remark:** This cannot be extended to generalized continued fractions since the inequality (15) does not hold anymore for generalized continued fractions.

The following lemmas come from [12]. They concern the structure of the set of small divisors and will be used in the lower bound on the radius of convergence. We recall them here and apply the second to our setting.

**Lemma 7** ([12], lemma 2.2) Let $k \in \mathbb{N}$ and $n \in \mathbb{N}, n \geq 1$, such that $d$ is a divisor of $n$. If $|D_n|^{\frac{1}{2}} < \frac{1}{q_k}$, then $\frac{n}{d} \geq q_k$ and either $q_k$ divides $\frac{n}{d}$ or $\frac{n}{d} \geq \frac{q_k + 1}{4}$.

**Lemma 8** ([12], lemma 2.3) For all $k \geq 0, n \geq 1$, let $A_k(n) = \{dq_k \leq j \leq n, d|j, \frac{1}{6q_k + 1} \leq |D_j|^{\frac{1}{2}} < \frac{1}{q_k}\}$. Let $E = \max(dq_k, \frac{dq_{k+1}}{4})$. Then there is a function $g_k : \mathbb{N} \rightarrow \mathbb{R}^+$ such that:

- $g_k(n) \leq (1 + \frac{2dn}{E}) \frac{n}{dq_k}$;
- for all $n_1, n_2 \in \mathbb{N}$, $g_k(n_1) + g_k(n_2) \leq g_k(n_1 + n_2)$;
• if \( n \in A_k(n) \), then \( g_k(n) \geq g_k(n - 1) + 1 \).

**Remark:** This application of Davie’s lemma is possible because the set \( A_k \) satisfies: if \( j_1 = d_{j_1}' < j_2 = d_{j_2}' \in A_k \), then either \( dq_k \) divides \( j_2 - j_1 \) or \( j_2 - j_1 \geq \frac{dq_k + 1}{4} \). Indeed, letting \( p_1 \) (resp. \( p_2 \)) be the integer closest to \( j_1'\alpha \) (resp. \( j_2'\alpha \)),

\[
|| (j_2 - j_1)\alpha || \leq |p_2 - p_1 - (j_2' - j_1')\alpha | \leq || j_2'\alpha || + || j_1'\alpha || \leq |D_{j_2}|^{\frac{1}{q_k}} + |D_{j_1}|^{\frac{1}{q_k}} < \frac{1}{3q_k}
\]

(the last inequality comes from \( j_1, j_2 \in A_k \)). Therefore, by (13), \( |D_{j_2 - j_1}|^{\frac{1}{q_k}} < \frac{1}{q_k} \). Lemma 7 then implies that \( q_k \) divides \( j_2' - j_1' \) or \( j_2' - j_1' \geq \frac{q_k + 1}{4} \).

**Lemma 9** There exists \( C_0, C_1 > 0 \) such that

\[
\sum_{l \geq 0} \frac{1}{q_l} \leq C_0
\]

and

\[
\sum_{l \geq 0} \frac{1}{q_l} \ln q_l \leq C_1
\]

**Proof:** Let us give a bound on \( \sum_{l \geq 0} \frac{1}{q_l} \ln q_l \). Because of the recurrence relation (13), the sequence \( (q_l) \) increases at least as fast as a Fibonacci sequence the first two terms of which are in \( \{\kappa_0, \ldots, \kappa_N\} \): denoting by \( (f_k) \) the Fibonacci sequence with \( f_0 = f_1 = 1 \), one recursively proves that

\[
q_k \geq f_k
\]

(16)

Indeed, \( q_0 \geq 1 \) and \( q_1 \geq 1 \). Assume that \( q_{k-1} \geq f_{k-1} \) and \( q_k \geq f_k \), then

\[
q_{k+1} = a_{k+1}q_{k+1-1} + q_{k+1-2} \geq q_k + q_{k-1} \geq (f_k + f_{k-1}) = f_{k+1}.
\]

Therefore, since the function \( t \mapsto \frac{\ln t}{t} \) decreases on \([e, +\infty[\),

\[
\sum_{l \geq 0} \frac{1}{q_l} \ln q_l \leq \ln 2 + \sum_{l \geq 0} \frac{\ln f_l}{f_l} \leq C_1
\]

where \( C_1 \) is a numerical constant. From the inequality (16), one also infers that

\[
\sum_{l \geq 0} \frac{1}{q_l} \leq \sum_{l \geq 0} \frac{1}{f_l} \leq C_0
\]

where \( C_0 \) is a numerical constant. \( \square \)
4.1 Subsequence of fast increasing denominators

Given the number \( N_M \) which was defined in Lemma 4, let \((n_k)\) the subsequence containing all indices such that

\[
\begin{align*}
q_{n_0} &\geq \max(N_M + 2, 1 + \kappa_{\text{max}}(2 + \frac{1}{d})) > q_{n_0-1}, \\
q_{n_k+1} &\geq q_{n_k}^2 + \zeta(\kappa_{\text{max}}, d)q_{n_k} + \eta(\kappa_{\text{max}}, d)
\end{align*}
\]

where \( \zeta, \eta \) are given by

\[
\zeta(\kappa_{\text{max}}, d) = \frac{\kappa_{\text{max}}}{d} + 3\kappa_{\text{max}} + 2, \quad \eta(\kappa_{\text{max}}, d) = (\frac{\kappa_{\text{max}}}{d} + 2\kappa_{\text{max}})(\kappa_{\text{max}} + 1)
\]

**Lemma 10** For all \( k \geq 0 \), it holds that \( q_{n_k} \geq \max(N_M + 2, \kappa_{\text{max}})^{2^k} \).

**Proof:** By definition, \( q_{n_0} \geq \max(N_M + 2, \kappa_{\text{max}}) \).
Assume that \( q_{n_k} \geq \max(N_M + 2, \kappa_{\text{max}})^{2^k} \), then \( q_{n_k+1} \geq q_{n_k+1} \geq q_{n_k}^2 \geq \max(N_M + 2, \kappa_{\text{max}})^{2^{k+1}} \). □

**Corollary 11** For all \( k \geq 0 \),

\[
\sum_{k \geq 0} \frac{1}{q_{n_k}} \leq \frac{1}{\max(N_M + 2, \kappa_{\text{max}}) - 1}.
\]

**Proof:** Indeed

\[
\sum_{k \geq 0} \frac{1}{q_{n_k}} \leq \sum_{k \geq 0} \frac{1}{\max(N_M + 2, \kappa_{\text{max}})^{2^k}} \leq \sum_{k \geq 1} \frac{1}{\max(N_M + 2, \kappa_{\text{max}})^k}
\]

\[
\leq \frac{1}{\max(N_M + 2, \kappa_{\text{max}}) - 1}. \quad (17)
\]

**Lemma 12** There exists \( C_2 > 0 \) such that

\[
B(d\alpha) - C_2 \leq \sum_{l \geq 0} \frac{\ln |D_{q_n}|}{2q_n} \leq B(d\alpha) + C_2'
\]

**Proof:** Lemma 6 implies that

\[
\sum_{l \geq 0} \frac{\ln q_{n_{l+1}} - \ln 3}{q_{n_l}} \leq \sum_{l \geq 0} \frac{\ln |D_{q_{n_l}}|}{2q_{n_l}} \leq \sum_{l \geq 0} \frac{\ln q_{n_{l+1}}}{q_{n_l}} + \sum_{l \geq 0} \frac{\ln 2}{q_{n_l}} \leq B(d\alpha) + C_0 \ln 2
\]

(where we have also used Lemma 9). Now

\[
\sum_{l \geq 0} \frac{\ln q_{n_{l+1}}}{q_{n_l}} = B(d\alpha) - \sum_{l \geq 0, q_l^2 + \zeta q_l + \eta > q_{l+1}} \frac{\ln q_{l+1}}{q_l}
\]

(where \( \zeta, \eta \) were defined at the beginning of the section). Thus
\[ \sum_{l \geq 0} \ln q_{n+1} \geq B(d\alpha) - \sum_{l \geq 0} \frac{\ln(q_l^2 + \zeta q_l + \eta)}{q_l} \geq B(d\alpha) - \sum_{l \geq 0} \frac{\ln q_l + \ln(q_l + \zeta + \eta)}{q_l} \]

By Lemma 9,
\[ \sum_{l \geq 0} \frac{\ln(\zeta + \eta + 1)}{q_l} \leq C_0 \ln(\zeta + \eta + 1) \]

Therefore one can define
\[ C_2 = 2C_1 + C_0 (\ln 3 + \ln(\zeta + \eta + 1)) \] and \[ C_2' = C_0 \ln 2. \]

5 Recursively defined lower bound

In this section we introduce a function \( F \) which will be used in giving a lower bound on the coefficients of the linearization. More precisely, we shall prove that
\[ \limsup_{k \to +\infty} \frac{F(q_{n_k})}{q_{n_k}} - C \leq \limsup_{k \to +\infty} \frac{\ln|\phi_{q_{n_k}}|}{q_{n_k}} \]

where \( C \) is a constant not depending on \( \alpha \).

Then we shall bound \( \limsup_{k \to +\infty} \left( \frac{F(q_{n_k})}{q_{n_k}} \right) \) by means of the Brjuno sum in the Lemma 13 below.

The function \( F \) is recursively defined on the set \( \{ q_{nk}, k \geq 0 \} \) as follows:

\[ F(q_{n_0}) = 0 \]
\[ \forall k \geq 1, \quad F(q_{n_k}) = p_k^1|\ln|D_{q_{n_0}}|| + p_k^1 F(q_{n_0}) + \cdots + p_k^{k-1} |\ln|D_{q_{n_{k-1}}}||| + p_k^k F(q_{n_{k-1}}) \] \hspace{1cm} \text{(19)}

where the integers \( p_k^i \) are given by successive euclidean divisions on \( dq_{nk} - \kappa_{\text{max}} \):

\[ dq_{nk} - \kappa_{\text{max}} = p_{k-1} dq_{nk-1} + r_{k-1}, \quad r_{k-1} < dq_{nk-1}, \]
\[ r_{k-1} = p_{k-2} dq_{nk-2} + r_{k-2}, \quad r_{k-2} < dq_{nk-2}, \]
\[ \vdots, \]
\[ r_1 = p_0 dq_{n_0} + r_0, \quad r_0 < dq_{n_0}. \]

Notice that the \( p_i^k \) and \( r_0^k \) do depend on \( k \).

**Remark 1** Let \( i \geq 0 \). For all \( k \geq i + 1 \),

\[ p_i^k \leq \frac{q_{n_{i+1}}}{q_{n_i}} \] \hspace{1cm} \text{(21)}
Indeed, the integer $p_i^k$ are given by

$$p_i^{k-1} = E\left(\frac{dq_{n_k} - \kappa_{\text{max}}}{dq_{n_{k-1}}}\right)$$

and for $i = 0, \ldots, k-2$, $p_i^k = E\left(\frac{r_{i+1}}{dq_{n_i}}\right)$, which satisfies \[21\] by definition of $r_{i+1}$.

**Lemma 13** There exist $C_4 > 0$ such that the function $F$ satisfies for all $k \geq 0$,

$$2B(d\alpha) - C_4 \leq \limsup_{k \to +\infty} \frac{F(q_{n_k})}{q_{n_k}}$$

**Proof:** This can be recursively shown. Assume that for a fixed $k \geq 1$ and for all $k' \leq k - 1$, one has

$$\frac{F(q_{n_{k'}})}{q_{n_{k'}} + \frac{\kappa_{\text{max}}}{d}} \geq \sum_{0 \leq i \leq k'-1} |\ln |D_{dq_{n_i}}||\left(\frac{1}{q_{n_i}} - \frac{2}{q_{n_{i+1}}}\right)$$

(23)

(which holds for $k = 1$). Then

$$\frac{F(q_{n_k})}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}} \geq \frac{p_{k-1}}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}} \left(F(q_{n_{k-1}}) + |\ln |D_{dq_{n_{k-1}}}|\right)$$

$$\geq \frac{p_{k-1}(q_{n_{k-1}} + \frac{\kappa_{\text{max}}}{d})}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}} \sum_{i \leq k-2} |\ln |D_{dq_{n_i}}||\left(\frac{1}{q_{n_i}} - \frac{2}{q_{n_{i+1}}}\right) + \frac{p_{k-1}}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}} |\ln |D_{dq_{n_{k-1}}}|$$

(24)

Now by definition of $p_{k-1}$,

$$\frac{p_{k-1}(q_{n_{k-1}} + \frac{\kappa_{\text{max}}}{d})}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}} \geq \frac{(dq_{n_k} - \frac{\kappa_{\text{max}}}{d})}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}}$$

$$= \frac{(q_{n_k} - q_{n_{k-1}} - \frac{\kappa_{\text{max}}}{d})(q_{n_{k-1}} + \frac{\kappa_{\text{max}}}{d})}{q_{n_{k-1}}(q_{n_k} + \frac{\kappa_{\text{max}}}{d})}$$

(25)

and this quantity is greater than 1 since, by assumption on the subsequence $n_k$,

$$q_{n_k} \geq q_{n_{k-1}}^2 + \zeta q_{n_{k-1}} + \eta$$

where $\zeta, \eta$ were defined at the beginning of Section 4.1. Therefore

$$\frac{F(q_{n_k})}{q_{n_k} + \frac{\kappa_{\text{max}}}{d}} \geq \sum_{i \leq k-2} |\ln |D_{dq_{n_i}}||\left(\frac{1}{q_{n_i}} - \frac{2}{q_{n_{i+1}}}\right) + |\ln |D_{dq_{n_{k-1}}}|\left(\frac{1}{q_{n_{k-1}}} - \frac{2}{q_{n_k}}\right)$$

\[22\]
Assumption 1. In this section, one shall assume the following:

- An upper bound on the radius of convergence exists.

Lemma 9 then implies

\[ \min(1, \kappa) \leq 4 \ln 2 \]

Lemma 9 then implies

\[ \frac{F(q_{nk})}{q_{nk} + \kappa_{max} C} \geq \sum_{0 \leq l \leq k} \frac{\ln |D_{dq_{n_l}}|}{q_{n_l}} - 4 \sum_{i \geq 1} \frac{\ln |D_{q_{n_i}}|}{q_{n_i}} - 4 \ln 2 \]

Finally, by Lemma 12,

\[ \limsup_{k \to +\infty} \frac{F(q_{nk})}{q_{nk} + \kappa_{max} C} = \limsup_{k \to +\infty} \frac{F(q_{nk})}{q_{nk} + \kappa_{max} C} \geq 2B(d\alpha) - 2C_2 - 4C_1 - 4 \ln 2C_0 \]

where \( C_2 \) was defined in Lemma 12. Thus one can define \( C_4 = 2C_2 + 4C_1 + 4 \ln 2C_0 \). \( \square \)

6. An upper bound on the radius of convergence

In this section, one shall assume the following:

**Assumption 1**. Assume that there exists \( \theta \in \mathbb{R} \) such that for all \( k = 1, \ldots, N \) with \( a_k \neq 0 \), the complex number \( a_k \) has argument \( k\theta + \frac{\pi}{2} \).

In this case, one can prove a lower bound on the coefficients \( \phi_i \) of the linearization in order to bound the radius of convergence from above.

One needs the following simple lower bound on all coefficients, including those not corresponding to a small divisor.

**Lemma 14**. For all \( r \in \mathbb{N} \), if \( |\phi_r| \neq 0 \), then \( |\phi_r| \geq \min(1, \frac{|a_{\infty}}{4})^\frac{r}{r_0} \).

**Proof**: For all \( K \) such that \( a_K \neq 0 \), it holds that \( |\phi_K| \geq K|a_K| \geq \frac{|a_{\infty}|}{4} \). If \( \frac{|a_{\infty}|}{4} < 1 \), then \( \phi_K \geq \left( \frac{|a_{\infty}|}{4} \right)^K \). If \( \frac{|a_{\infty}|}{4} \geq 1 \) then \( |\phi_K| \geq 1 \).

Let \( r \in \mathbb{M}, r \geq 2 \). Assume that the property holds for all \( r' \leq r - 1 \). Then, either there exists \( i \in \{0, \ldots, N\} \) such that \( r = \kappa_i \), and in this case, \( |\phi_r| \geq \frac{a_{r_i}}{4} \geq \min(1, \frac{|a_{\infty}|}{4}) \), or there exists \( i \in \{0, \ldots, N\} \) such that

\[ |\phi_r| \geq \frac{|a_{\infty}|}{4} |\phi_{r-\kappa_i}| \geq \frac{|a_{\infty}|}{4} |\phi_{r-\kappa_i}| \]

If \( \frac{|a_{\infty}|}{4} < 1 \), then the recurrence assumption implies that \( |\phi_r| \geq \left( \frac{|a_{\infty}|}{4} \right)^{\frac{r}{r_0}} \), or \( \frac{|a_{\infty}|}{4} \geq 1 \), then \( |\phi_r| \geq 1 \). \( \square \)
Lemma 15  For all $j > \kappa_0$ and $p \in \mathbb{N}^*$, there is

$$|\phi_{pj}| \geq \frac{1}{p} \left(\frac{1}{4}\right)^p |D_j \phi_j|^p.$$  

Proof:  Notice that

$$|\phi_{pj}| = \frac{1}{|D_{pj}|} |\psi_{pj}| = \frac{1}{|D_{pj}|} \sum_{K \in \{\kappa_0, \ldots, \kappa_N\}} |\psi_{K,pj}|$$  \hspace{2cm} (27)

(one uses the fact that the $\psi_{K,pj}$ have the same argument for every $K$). Thus, using (12) with $n = pj$, $k = (p - 1)j$,

$$|\phi_{pj}| \geq \frac{1}{|D_{pj}|} \sum_{K} \frac{(p - 1)j}{pj - K} |\phi_{(p-1)j}| |\psi_{K,j}| \geq \frac{(p - 1)j}{|D_{pj}| |pj|} |\phi_{(p-1)j}| |\psi_{j}|$$  \hspace{2cm} (28)

Iterating this, one obtains

$$|\phi_{pj}| \geq \frac{1}{p|D_{pj}| \cdots |D_{2j}|} |\phi_{j}| |\psi_{j}|^{p-1} = \frac{1}{p|D_{pj}| \cdots |D_{2j}|} |\phi_{j}|^p |D_j|^{p-1} \geq \frac{1}{p} \left(\frac{1}{4}\right)^p |D_j \phi_j|^p. \quad \Box$$  \hspace{2cm} (29)

The following lemma states a better lower bound for the coefficients of the linearization corresponding to a small divisor.

Lemma 16  For all $k \geq 0$, there is

$$|\phi_{d_{q_{k+1}}}| \geq \frac{1}{|D_{d_{q_{k+1}}}|} \left(\frac{1}{4}\right)^{k+1} \prod_{l=0}^{k+1} |D_{d_{q_{l-1}}}| \frac{|a_{\kappa_{\max}}|}{|a_{\kappa_{\min}}|} \min(1, \frac{1}{k!}) \sum_{i=0}^{k} |\phi_{d_{q_{i+1}}} q_{i+1}|.$$  \hspace{2cm} (30)

where the integers $p_{k+1}^{k+1}, \ldots, p_{k}^{k+1}, q_{k+1}^{k+1}$ were defined in equation (19).

Proof:  Taking in the recurrence relation (9) the term with $l = q_{k+1}$, $K = \kappa_{\max}$, $m = 1, p_{k+1} = k + 2, j_{k+1}^{1} = dp_{k+1}^{k+1} q_{0}, \ldots, j_{k+1}^{1} = dp_{k+1}^{k+1} q_{k}$ and $j_{k+2}^{1} = r_{0}^{k+1}$, in order to have $j_{1}^{1} + \cdots + j_{k+2}^{1} = dq_{k+1} - \kappa_{\max}$, one obtains

$$|\phi_{d_{q_{k+1}}}| \geq \frac{1}{|D_{d_{q_{k+1}}}|} \left(\frac{1}{4}\right)^{k+1} \prod_{l=0}^{k+1} |D_{d_{q_{l-1}}}| \frac{|a_{\kappa_{\max}}|}{|a_{\kappa_{\max}}|} |\phi_{d_{q_{i+1}}} q_{i+1}| \cdots |\phi_{d_{q_{k+1}}} q_{k}|$$  \hspace{2cm} (31)

Now apply Lemma 15 with $j = dq_{n}$ and $p = p_{i}^{k+1}$ to every factor in the right hand side, except the last one, and apply Lemma 14 to the last factor. One obtains

$$|\phi_{d_{q_{k+1}}}| \geq \frac{1}{|D_{d_{q_{k+1}}}|} \left(\frac{1}{4}\right)^{k+1} \prod_{l=0}^{k+1} |D_{d_{q_{l-1}}}| \frac{|a_{\kappa_{\max}}|}{|a_{\kappa_{\max}}|} \min(1, \frac{1}{k!}) \sum_{i=0}^{k} |\phi_{d_{q_{i+1}}} q_{i+1}|.$$  \hspace{2cm} \Box
The following proposition links the radius of convergence of the linearization to the function \( F \).

**Proposition 1** There exists \( C \geq 0 \) such that for all \( k \),

\[
\frac{1}{d q_{n_k}} \ln |D_{q_{n_k}} \phi_{d q_{n_k}}| \geq \frac{F(q_{n_k})}{d q_{n_k}} - C
\]

Moreover, it is possible to define

\[
C = |\ln(\min(1, |a_{\kappa_{\min}}|/4))| + \frac{1}{1 + N_M} (2 + 9 \ln 4) + C_1 + \left[ |\ln(\min(1, |a_{\kappa_{\min}}|/4))| + \frac{1}{d} |\ln |a_{\kappa_{\max}}|| \right] C_0
\]

**Proof:** Denote \( \tilde{C} = -\ln(\frac{|a_{\kappa_{\min}}|}{4}) \) if \( \frac{|a_{\kappa_{\min}}|}{4} < 1 \) and \( \tilde{C} = 0 \) otherwise. Let

\[
S_0 = 0;
\forall k \geq 1, \quad S_k = \sum_{1 \leq j \leq k} \frac{(j + 1) \ln(j + 1) + \ln(q_{n_j})}{q_{n_j}} + \sum_{q_{n_j} \leq j \leq k} \frac{\ln 4}{q_{n_j}} (1 + p^j_0 + \cdots + p^{j-1}_0)
\]

\[
+ \left[ \frac{r^j_0}{\kappa_0} \ln(\frac{|a_{\kappa_{\min}}|}{4}) \right] + \frac{1}{d} |\ln |a_{\kappa_{\max}}|| \sum_{1 \leq j \leq k} \frac{1}{q_{n_j}}
\]

Thus, for all \( k \geq 0 \), \( S_k \geq 0 \).

Let \( k \geq 0 \). Let us formulate the following recurrence property:

\[
\frac{1}{d q_{n_k}} \ln |\phi_{d q_{n_k}}| \geq \frac{F(q_{n_k})}{d q_{n_k}} + |\ln |D_{d q_{n_k}}| - S_k - \tilde{C}
\]

First note that this property holds for \( k = 0 \). Indeed, since by Corollary \( q_{n_0} \geq N_M \), which implies that \( d q_{n_0} \in M \), then there exists \( b_0, b_1, \ldots, b_N \geq 0 \) such that \( d q_{n_0} = \sum_{i=0}^{N} b_i \kappa_i \). Let \( I \) be such that \( b_I > 0 \). Then

\[
|\phi_{d q_{n_0}}| \geq |D_{d q_{n_0}}|^{\kappa_I} |a_{\kappa_I}| |\phi_{\kappa_I}|^{b_I - 1} \prod_{j \neq I} |\phi_{\kappa_J}|^{-b_J}
\]

Now on the other side, for all \( 0 \leq i \leq N \),

\[
|\phi_{\kappa_i}| \geq \frac{|a_{\kappa_i}|}{|D_{\kappa_i}|} \geq \frac{|a_{\kappa_{\min}}|}{4}
\]

hence, if \( \frac{|a_{\kappa_{\min}}|}{4} < 1 \),

\[
|\phi_{d q_{n_0}}| \geq \frac{1}{|D_{d q_{n_0}}|} |a_{\kappa_I}| \left( \frac{|a_{\kappa_{\min}}|}{4} \right)^{d q_{n_0} - 1} \geq \frac{1}{|D_{d q_{n_0}}|} \left( \frac{|a_{\kappa_{\min}}|}{4} \right)^{d q_{n_0}}
\]

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and if \( \frac{|a_{\text{min}}|}{4} \geq 1 \),
\[
\left| \phi_{d_{n_0}} \right| \geq \frac{1}{|D_{d_{n_0}}|} |a_{\kappa}| \geq \frac{1}{|D_{d_{n_0}}|}
\]
Thus if \( \frac{|a_{\text{min}}|}{4} < 1 \),
\[
\frac{1}{dq_{n_0}} \ln \left| \phi_{d_{n_0}} \right| \geq \left| \ln \frac{|D_{d_{n_2}}|}{dq_{n_0}} \right| + \ln \left( \frac{|a_{\kappa_{\text{max}}}|}{4} \right) \geq \left| \ln \frac{|D_{d_{n_0}}|}{dq_{n_0}} \right| - S_0 - \tilde{C}
\]
and if \( \frac{|a_{\text{min}}|}{4} \geq 1 \),
\[
\frac{1}{dq_{n_0}} \ln \left| \phi_{d_{n_0}} \right| \geq \left| \ln \frac{|D_{d_{n_2}}|}{dq_{n_0}} \right|
\]
therefore the property (33) holds for \( k = 0 \).

Now assume that the recurrence property holds for all \( 0 \leq k' \leq k \), for a fixed \( k \geq 0 \). Lemma 16 implies
\[
\ln \left| D_{d_{n_{k+1}}} \phi_{d_{n_{k+1}}} \right| \geq \ln \left( \frac{|a_{\kappa_{\text{max}}}|}{(k+2)!} \right) - (p_0^{k+1} + \cdots + p_k^{k+1}) \ln 4 - \ln(p_0^{k+1} \cdots p_k^{k+1})
\]

\[
+ p_0^{k+1} \ln \left| D_{d_{n_0}} \phi_{d_{n_0}} \right| + \cdots + p_k^{k+1} \ln \left| D_{d_{n_k}} \phi_{d_{n_k}} \right| + \frac{r_0^{k+1}}{\kappa_0} \ln \min(1, \frac{|a_{\text{min}}|}{4})
\]

By recurrence assumption, one infers
\[
\frac{1}{dq_{n_{k+1}}} \ln \left| D_{d_{n_{k+1}}} \phi_{d_{n_{k+1}}} \right| \geq -\frac{(k+2) \ln(k+2)}{dq_{n_{k+1}}} + \frac{\ln |a_{\kappa_{\text{max}}}|}{dq_{n_{k+1}}} - \frac{p_0^{k+1} + \cdots + p_k^{k+1}}{dq_{n_{k+1}}} \ln 4
\]

\[
+ \frac{p_0^{k+1}}{dq_{n_{k+1}}} (F(q_{n_0}) + \left| \ln \frac{|D_{d_{n_0}}|}{dq_{n_0}} \right| - S_0 - \tilde{C}) + \cdots
\]

\[
+ \frac{p_k^{k+1}}{dq_{n_{k+1}}} (F(q_{n_k}) + \left| \ln \frac{|D_{d_{n_k}}|}{dq_{n_k}} \right| - S_k - \tilde{C})
\]

\[
+ \frac{r_0^{k+1}}{d\kappa_0 q_{n_{k+1}}} \ln \min(1, \frac{|a_{\text{min}}|}{4}) - \ln(p_0^{k+1} \cdots p_k^{k+1})
\]

(34)
thus by definition of \( F \),
\[
\frac{1}{dq_{n_{k+1}}} \ln \left| D_{d_{n_{k+1}}} \phi_{d_{n_{k+1}}} \right| \geq \frac{F(q_{n_{k+1}})}{dq_{n_{k+1}}} - (k+2) \ln(k+2) + \frac{\ln |a_{\kappa_{\text{max}}}|}{dq_{n_{k+1}}} - \frac{p_0^{k+1} + \cdots + p_k^{k+1}}{dq_{n_{k+1}}} \ln 4
\]

\[
+ \frac{r_0^{k+1}}{d\kappa_0 q_{n_{k+1}}} \ln \min(1, \frac{|a_{\text{min}}|}{4}) - \tilde{C} - \frac{p_0^{k+1}}{dq_{n_{k+1}}} S_0 - \cdots - \frac{p_k^{k+1}}{dq_{n_{k+1}}} S_k
\]

(36)
Now for all $i = 0, \ldots, k$, $p_i \leq \frac{q_{n+1}}{q_{n_j}}$ therefore
\[
\frac{\ln(p_0^{k+1} \ldots p_k^{k+1})}{dq_{n+1}} \leq \frac{\ln(q_{n+1}/q_{n_j})}{dq_{n+1}}
\]
and
\[
\frac{1}{dq_{n+1}} \ln |D dq_{n+1} \phi dq_{n+1}| \geq \frac{F(q_{n+1})}{dq_{n+1}} - \tilde{C} - S_{k+1}
\]
Therefore, the property (33) holds for all $k \geq 0$.

Finally note that the partial sums $S_k$ converge, since from one side, by choice of the subsequence $q_{n_j}$ and corollary (11)
\[
\sum_{1 \leq j \leq k} (j + 1) \ln(j + 1) + \ln(q_{n_j}) \leq \sum_{1 \leq j \leq k} \frac{1}{\sqrt{q_{n_j}}} + C_1 \leq \sum_{1 \leq j \leq k} \frac{1}{q_{n_j-1}} + C_1 \leq \frac{1}{1 + N_M} + C_1
\]
and from the other side, by Remark (1)
\[
\sum_{j=1}^{k} \frac{\ln 4}{q_{n_j}} (1 + p_0^j + \cdots + p_{j-1}^j)
\]
\[
\leq \sum_{j=1}^{k} \frac{\ln 4}{q_{n_j}} (1 + \frac{q_n}{q_0} + \cdots + \frac{q_{n_j}}{q_{n_j-2}} + \frac{q_{n_j}}{q_{n_j-1}})
\]
\[
\leq \ln 4(\sum_{j=1}^{k} \frac{(j + 1)q_{n_j-1}}{q_{n_j}} + \frac{1}{q_{n_j-1}})
\]
\[
\leq \ln 4(\sum_{j=1}^{k} \frac{j + 2}{q_{n_j-1}})
\]
\[
\leq \ln 4(\frac{8}{q_{n_0}} + \sum_{j=2}^{k-1} \frac{j + 3}{q_{n_j}}) \leq \ln 4(\frac{8}{q_{n_0}} + \sum_{j=2}^{k-1} \frac{1}{q_{n_j}})
\]
\[
\leq \frac{9 \ln 4}{1 + N_M}
\]
(this last inequality used Corollary (11). Also,
\[
\frac{r_0^2}{\kappa_0} \ln(|a_{\kappa_{\min}}|/4) \sum_{1 \leq j \leq k} \frac{1}{q_{n_j}} \leq \ln(|a_{\kappa_{\min}}|/4) \sum_{1 \leq j \leq k} \frac{q_{n_0}}{q_{n_j}} \leq \ln(|a_{\kappa_{\min}}|/4) \sum_{j=0}^{1} \frac{1}{q_{n_j}} \leq C_0 |\ln(|a_{\kappa_{\min}}|/4)|
\]
Thus, let
\[
C = \tilde{C} + \frac{1}{1 + N_M} (2 + 9 \ln 4) + C_1 + |\ln(|a_{\kappa_{\min}}|/4)| + \frac{1}{d} |\ln |a_{\kappa_{\max}}|| C_0
\]
then $C$ satisfies the statement of this proposition. \( \Box \)
Theorem 17 The radius of convergence of \( \Phi \) is bounded from above by \( \exp\left(-\limsup_{k \to +\infty} \frac{F(q_{nk})}{dq_{nk}} + C\right) \) where \( C \geq 0 \) was defined in Proposition 1. It is also bounded from above by \( \exp\left(-\frac{2}{d}B(\alpha) + C_M\right) \) where \( C_M = C_4 + C \), with \( C_4 \) defined in Lemma 13.

Proof: Let \( \rho \) be the radius of convergence of \( \Phi \), then

\[
- \ln \rho \geq \limsup_{k \to +\infty} \frac{1}{dq_k} \ln |\phi_{dq_k}| \geq \limsup_{k \to +\infty} \frac{1}{dq_k} \ln |\phi_{dq_{nk}}| \geq \limsup_{k \to +\infty} \frac{\ln |D_{q_{nk}}|}{dq_{nk}} + \frac{F(q_{nk})}{dq_{nk}} - C
\]

where \( C \) was defined in Proposition 1. By Lemma 13,

\[
- \ln \rho \geq \frac{2}{d}B(\alpha) - \frac{C_4}{d} - C.
\]

One can then define \( C_M = \frac{C_4}{d} + C \). \( \square \)

7 A lower bound on the radius of convergence

In this section, the second part of the main result is proved. The assumption 1 on the coefficients \( a_K \) is relaxed.

Theorem 18 The radius of convergence is at least \( \exp\left(-C' - \sum_{l \geq 0} \frac{2 \ln q_{l+1}}{d_{l+1}}\right) \), where \( C' \geq 0 \) is defined by

\[
C' = \ln(|a_{\kappa_{\text{max}}}|) + r + C_0
\]

if \( |a_{\kappa_{\text{max}}}| > 1 \) and

\[
C' = r + C_0
\]

otherwise, with \( r > 0 \) only depending on the Fourier modes of the trigonometric polynomial \( A \) and \( C_0 \) defined in Lemma 9.

Proof: Let \( w \) be an analytic solution of the functional equation

\[
w(z) = \sum_{k \in \{\kappa_0, \ldots, \kappa_N\}} (ze^{w(z)})^k
\]

and let \( R \) its radius of convergence. Expanding \( w \) in its Taylor series, \( w(z) = \sum_{n \geq 0} \sigma_n z^n \), one obtains the following relation between the coefficients \( \sigma_n \) (it is the same relation as between the coefficients \( \phi_j \), only replacing \( a_k \) by \( i^{k+1} \) and without small divisors):

\[
\sigma_l = \sum_{K \in \kappa_0, \ldots, \kappa_N} \sum_{m=1}^{K} C_K^{m} \prod_{p=1}^{l-K} \left( \sum_{p_m=1}^{1} \frac{1}{p_1! \cdots p_m!} \right) \sum_{j_1+\cdots+j_m=l-K} \sigma_{j_1^{p_1}} \cdots \sigma_{j_m^{p_m}}
\]

\[ + j_1^{p_1} + \cdots + j_m^{p_m} = l-K \]

(38)
One can recursively show that the $\sigma_n$ are non-negative real numbers. Moreover the function $w$ is analytic, therefore $\limsup_{n \to +\infty} \frac{1}{n} \ln \sigma_n$ is equal to $-\ln R$.

The function $g$ giving the upper bound is defined as follows: for all $k \geq 0$, let $g_k$ be the function defined by Davie’s lemma. Then, for all integer $\kappa_0 \leq j < q_0$, let

$$g(j) = j \ln(|a_{\kappa_{max}}|) + 36j$$

if $|a_{\kappa_{max}}| > 1$, and

$$g(j) = 36j$$

otherwise. Now let $j \geq q_0$ and assume that $k$ is the greatest index such that $q_k \leq j$; let

$$g(j) = \sum_{l=0}^{k} 2g_l(j) \ln q_{l+1} + j \ln(|a_{\kappa_{max}}|) + 36j$$

if $|a_{\kappa_{max}}| > 1$, and

$$g(j) = \sum_{l=0}^{k} 2g_l(j) \ln q_{l+1} + 36j$$

otherwise. The function $g$ is increasing. Moreover for all $j_1, j_2 \geq \kappa_0$, as a consequence of Davie’s lemma,

$$g(j_1) + g(j_2) \leq g(j_1 + j_2) \quad (39)$$

Now let us prove that $|\phi_j| \leq \sigma_j e^{g(j)}$ for all $j \geq \kappa_0$. First, if $|D_{\kappa_0}^{-1}a_{\kappa_{max}}| > 1$, then

$$|\phi_{\kappa_0}| = |D_{\kappa_0}^{-1}a_{\kappa_0}| \leq |D_{\kappa_0}^{-1}a_{\kappa_{max}}| \leq e^{g(\kappa_0)}$$

and $|\phi_{\kappa_0}| \leq 1 = e^{g(\kappa_0)}$ otherwise. Assume that this holds for all $j' \leq j - 1$ and consider $|\phi_j|$. The relation (39) implies

$$|\phi_j| \leq |D_j^{-1}| \sum_{K \in \kappa_0, \ldots, \kappa_N, K \leq j} |a_K| |\delta_{K,j}| + \sum_{m=1, \ldots, K \leq j-K} \sum_{p_1=1}^{j-K} \sum_{p_m=1}^{j-K} \frac{1}{p_1! \cdots p_m!} \sum_{j_1, \ldots, j_p_1, \ldots, j_p_m} |\phi_{j_1}| \cdots |\phi_{j_{p_1}}| \cdots |\phi_{j_{p_m}}|$$

hence, by recurrence assumption,
\[ |\phi_j| \leq |D_j^{-1}| \sum_{K \in \kappa_0, \ldots, \kappa_N, K \leq j} a_K |[\delta_{K,j} + \sum_{m=1, \ldots, K \leq j-K} C_K^m \sum_{p_1=1}^{j-K} \cdots \sum_{p_m=1}^{j-K} \frac{1}{p_1! \cdots p_m!} \sum_{j_1, \ldots, j_m \geq 1, j_1 + \cdots + j_m = j-K} \sigma_{j_1} \cdots \sigma_{j_m} e^{g(j_1)+\cdots+g(j_m)}] | \]

therefore

\[ |\phi_j| \leq |D_j^{-1}| |a_{\kappa_{\max}}| e^{g(j-\kappa_0)} \sigma_j \]

We shall distinguish two cases:

- if \( j \geq \kappa_0 \) and \( |D_j|^{1/2} \geq \frac{1}{6} \), then
  \[ |D_j^{-1}| |a_{\kappa_{\max}}| e^{g(j-\kappa_0)} \sigma_j \leq 36 |a_{\kappa_{\max}}| e^{g(j-\kappa_0)} \sigma_j \leq e^{g(j)} \sigma_j \]

- otherwise there exists \( k \geq 0 \) such that \( \frac{1}{q_{k+1}} \leq |D_j|^{1/2} < \frac{1}{q_k} \), and in this case, \( |D_j^{-1}| \leq 36 q_k^2 \). Moreover, \( j \in M \) implies that \( d \) divides \( j \); then \( j \in A_k(j) \) therefore, by construction of \( g_k \),
  \[ g_k(j) = g_k(j - \kappa_0) + 1 \]

therefore

\[ \ln |D_j^{-1}| + \ln |a_{\kappa_{\max}}| + g(j - \kappa_0) \leq 2 \ln q_{k+1} + \ln 36 + \ln |a_{\kappa_{\max}}| + g(j - \kappa_0) \leq g(j) \]

thus

\[ |\phi_j| \leq |D_j^{-1}| |a_{\kappa_{\max}}| e^{g(j-\kappa_0)} \sigma_j \leq e^{g(j)} \sigma_j \]

The recurrence is finished. Thus, for all \( j \geq \kappa_0 \),

\[ \frac{1}{j} \ln |\phi_j| \leq \frac{1}{j} g(j) + \frac{1}{j} \ln \sigma_j \]  

(42)

Now

\[ \frac{1}{j} g(j) \leq \sum_{0 \leq l \leq k} \frac{2}{j} q_l g_l(j) \ln q_{l+1} + \delta \ln(|a_{\kappa_{\max}}|) + 36 \]

where \( \delta = 1 \) if \( |a_{\kappa_{\max}}| > 1 \) and 0 otherwise. Therefore by Lemma \( \Box \)
\[ \frac{1}{j} g(j) \leq \sum_{0 \leq l \leq k} \left( \frac{2}{dq_l} + \frac{16}{d_{q_{l+1}}} \right) \ln q_{l+1} + \delta \ln(|a_{\kappa_{\text{max}}}|) + 36 \]

Thus, \( \limsup_{j \to +\infty} \frac{1}{j} \ln |\phi_j| \leq \limsup_{j \to +\infty} \frac{1}{j} g(j) + \limsup_{j \to +\infty} \frac{1}{j} \ln \sigma_j \)

\[ \leq \sum_{l \geq 0} \frac{2 \ln(q_{l+1})}{dq_l} + \sum_{l \geq 0} \frac{16 \ln(q_{l+1})}{d_{q_{l+1}}} + \delta \ln(|a_{\kappa_{\text{max}}}|) + 36 + \limsup_{j \to +\infty} \frac{1}{j} \ln \sigma_j \]

Thus, \( \limsup_{j \to +\infty} \frac{1}{j} \ln |\phi_j| \leq 2 \sum_{l \geq 0} \frac{\ln q_{l+1}}{dq_l} + C' \)

where \( C' = \ln(|a_{\kappa_{\text{max}}}|) + \ln R + \frac{16C_1}{d} + 36 \)

if \( |a_{\kappa_{\text{max}}}| > 1 \) and \( R > 1 \), \( C' = \ln R + \frac{16C_1}{d} + 36 \)

if \( R > 1 \) and \( |a_{\kappa_{\text{max}}}| \leq 1 \), and \( C' = \frac{16C_1}{d} + 36 \)

otherwise. It only remains to use the characterization of the radius as a function of \( \limsup_{j \to +\infty} \frac{1}{j} \ln |\phi_j| \). \( \Box \)

8 Appendix

Here we give a proof of equation (44) in two lemmas.

Lemma 19 Let \( H(z, \lambda) = (h(z, \lambda), h_2(z, \lambda)) \) be the linearization. Then \( h_2(z, \lambda) = \frac{h(z, \lambda)}{h(\lambda^{-1} z, \lambda)} \).

**Proof:** Let \( F(z, \lambda) = (\lambda \prod_{k=1}^{N} e^{ia_k z^k}, \lambda \prod_{k=1}^{N} e^{ia_k z^k}) \) be the function generating the system, and \( R(z, \lambda) = (z, \lambda) \). Note that if \( f(z, \lambda) = \prod_{k=1}^{N} e^{ia_k z^k} \) then \( F(z, \lambda) = (zf(z, \lambda), f(z, \lambda)) \). We shall expand the identity \( H = F \circ H \circ R^{-1} \) to infer the desired identity. On one side,

\[
F \circ H \circ R^{-1}(z, \lambda) = F \circ H(\lambda^{-1} z, \lambda) = F(h(\lambda^{-1} z, \lambda), h_2(\lambda^{-1} z, \lambda))
\]

\[
= (h(\lambda^{-1} z, \lambda)f(h(\lambda^{-1} z, \lambda), h_2(\lambda^{-1} z, \lambda)), f(h(\lambda^{-1} z, \lambda), h_2(\lambda^{-1} z, \lambda)))
\]

(44)
By matching the components of $F \circ H \circ R^{-1}$ with those of $H$, one infers that
\[ h(z, \lambda) = h(\lambda^{-1} z, \lambda) f(h(\lambda^{-1} z, \lambda), h_2(\lambda^{-1} z, \lambda)) \]
and
\[ h_2(z, \lambda) = f(h(\lambda^{-1} z, \lambda), h_2(\lambda^{-1} z, \lambda)) \]
Those two identities imply that for all $(z, \lambda)$,
\[ h_2(z, \lambda) = \frac{h(z, \lambda)}{h(\lambda^{-1} z, \lambda)}. \]
\[ \square \]

**Lemma 20** Let $h(z, \lambda) = i z e^{\Phi_\lambda(z)}$, then equation (6) holds.

**Proof:** Let $F(z, \lambda) = (\lambda z \prod_{k=1}^N e^{ia_k z^k}, \lambda \prod_{k=1}^N e^{ia_k z^k})$ be the function generating the system, and let $R(z, \lambda) = (\lambda z, \lambda)$. By definition, $F \circ H = H \circ R$. Now using Lemma 19,
\[ F \circ H(z, \lambda) = \left( \frac{h_2(z, \lambda)}{h(\lambda^{-1} z, \lambda)} \prod_{k=1}^N e^{ia_k h(z, \lambda) k}, \frac{h(z, \lambda)}{h(\lambda^{-1} z, \lambda)} \prod_{k=1}^N e^{ia_k h(z, \lambda) k} \right) \]
and $H \circ R(z, \lambda) = (h(\lambda z, \lambda), h_2(\lambda z, \lambda))$. By matching the components and taking the logarithm modulo $2i\pi$, one obtains equation (6). \[ \square \]

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