Strong convergence for the Euler-Maruyama approximation of stochastic differential equations with discontinuous coefficients

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Abstract

In this paper we study the strong convergence for the Euler-Maruyama approximation of a class of stochastic differential equations whose both drift and diffusion coefficients are possibly discontinuous.

2010 Mathematics Subject Classification: 60H35; 41A25; 60C30;

Keywords: Euler-Maruyama approximation · Strong rate of convergence · Stochastic differential equation · Discontinuous coefficients

1 Introduction

Let us consider the one-dimensional stochastic differential equation (SDE)

\[ X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad x_0 \in \mathbb{R}, \quad t \in [0,T], \]  

where \( W := (W_t)_{0 \leq t \leq T} \) is a standard Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\) satisfying the usual conditions. Since the solution of (1) is rarely analytically tractable, one often approximates \( X = (X_t)_{0 \leq t \leq T} \) by using the Euler-Maruyama (EM) scheme given by

\[ X_t^{(n)} = x_0 + \int_0^t b\left(X_{\eta_n(s)}^{(n)}\right)ds + \int_0^t \sigma\left(X_{\eta_n(s)}^{(n)}\right)dW_s, \quad t \in [0,T], \]

where \( \eta_n(s) = kT/n =: t_k^{(n)} \) if \( s \in [kT/n, (k + 1)T/n) \).

It is well-known that if \( b \) and \( \sigma \) are Lipschitz continuous, the EM approximation for (1) converges at the strong rate of order \( 1/2 \) (see [12]). On the other hand, when \( b \) and \( \sigma \) are not Lipschitz continuous, the strong rate is less known and it has been a subject of extensive study. In the recent articles [11] and [7], it has been shown that

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for every arbitrarily slow convergence speed there exist SDEs with infinitely often differentiable and globally bounded coefficients such that neither the EM approximation nor any approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. The approximation for SDEs with possibly discontinuous drift coefficients was first studied in [5]. It is shown that if the drift satisfies the monotonicity condition and the diffusion coefficient is Lipschitz continuous, then the EM scheme converges at the rate of $1/4$ in pathwise senses. In [8], the strong convergence of EM scheme is shown for SDEs with discontinuous monotone drift coefficients. If $\sigma$ is uniformly elliptic and $(\alpha + 1/2)$-Höder continuous, and $b$ is of locally bounded variation, it has been shown that the strong rate of the EM in $L^1$-norm is $n^{-\alpha}$ for $\alpha \in (0, 1/2]$ and $(\log n)^{-1}$ for $\alpha = 0$ (see [20][22]). The strong rate of convergence for SDEs whose drift coefficient $b$ is Hölder continuous is studied in [6][18][22]. The above mentioned papers contain just a few selected results and a number of further and partially significantly improved approximation results for SDEs with irregular coefficients are available in the literature; see, e.g., [2][3][9][10][13][16][17][21][25] and the references there in.

In this paper we are interested in strong approximation of SDEs with discontinuous diffusion coefficients. These SDEs appears in many applied domains such as stochastic control and quantitative finance (see [4][1]). For such SDEs, the existence and uniqueness of solution was studied in [19][14][4]; the weak convergence of EM approximation was shown in [25]. To the best of our knowledge, the strong convergence of the EM approximation of SDEs with discontinuous diffusion coefficient has not been considered before in the literature. It is worth noting that the key ingredients to establish the strong rate of convergence of EM approximation for SDEs with discontinuous drift are either the Krylov estimate (see [13][6]) or the Gaussian bound estimate for the density of the numerical solution ([15][20][22]). However, these estimates seem no longer available for SDEs with discontinuous diffusion coefficients. Therefore in this paper we develop another method, which is based on an argument with local time, to overcome this obstacle.

The remainder of the paper is structured as follows. In the next section we introduce some notations and assumptions for our framework together with the main results. All proofs are deferred to Section 3.

## 2 Main results

### 2.1 Notations

Throughout this paper the following notations are used. For any continuous semimartingale $Y$, we denote $L^x_t(Y)$ the symmetric local time of $Y$ up to time $t$ at the level $x \in \mathbb{R}$ (see [14]). For bounded measurable function $f$ on
Here are some remarks on the class \( H^{\beta,\kappa} \).

**Remark 2.1.**

1. \( H^{\beta,\kappa} \) is a vector space on \( \mathbb{R} \), i.e., if \( a, b \in \mathbb{R} \) and \( f, g \in H^{\beta,\kappa} \) then \( af + bg \in H^{\beta,\kappa} \).

2. A bounded function \( f \) is called piecewise \( \beta \)-Hölder if there exist a positive constant \( L \) and a sequence 
   \[-\infty = s_0 < s_1 < s_2 < \ldots < s_m < s_{m+1} = \infty \] such that \( |f(u) - f(v)| \leq L|u - v|^\beta \) for any \( u, v \) satisfying 
   \( s_k < u < v < s_{k+1} \). It is easy to verify that such function \( f \in H^{\beta,1} \), \( S(f) = \{ s_1, \ldots, s_m \} \) and \( C_{\beta,1} \leq 2m \).

3. The following \( \zeta \) is a non-trivial example of function of \( H^{\beta,\kappa} \) with \( \kappa < 1 \). For each \( \hat{\beta}, \kappa \in (0, 1) \), we denote
   \[
   \zeta(x) = \begin{cases} 
   \frac{x-1}{2x-1} & \text{if } x \leq 0, \\
   1 + \frac{\log 2}{\log(n+1)} x^{\hat{\beta}} & \text{if } (n+1)^{-1/(1-\kappa)} \leq x < n^{-1/(1-\kappa)} \text{ and } n \in \mathbb{N}, \\
   \frac{3x+1}{x+1} & \text{if } x \geq 1.
   \end{cases}
   \]

   It can be shown that \( \zeta \) is a strictly increasing function with an infinite number of discontinuous points
   which are cumulative at 0, \( \frac{1}{2} < \zeta < 3 \), and \( \zeta \in H^{\beta,\kappa} \) with \( \beta = \frac{1+\hat{\beta}-\kappa}{2-\kappa} \), \( S(\zeta) = \{ n^{-1/(1-\kappa)} \\}, n = 1, 2, \ldots \) and
   \( C_{\beta,\kappa} \leq 3 \).

### 2.2 Main results

We need the following assumptions on the diffusion coefficient \( \sigma \).

**Assumption 2.2.**

(i) There exists a bounded and strictly increasing function \( f_\sigma \) such that for any \( x, y \in \mathbb{R} \),

\[
|\sigma(x) - \sigma(y)|^2 \leq |f_\sigma(x) - f_\sigma(y)|.
\]

(ii) \( \sigma \) is bounded and uniformly positive, i.e. there exist positive constants \( \underline{\sigma} \) and \( \overline{\sigma} \) such that for any \( x \in \mathbb{R} \),

\[
\underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}.
\]

Le Gall [14] has shown that if \( b \) is bounded measurable, and \( \sigma \) satisfies Assumption 2.2 then there exists a
unique strong solution to SDE (1) (see also [19]). We now give some remarks on the Assumption 2.2.
Remark 2.3.  1. The function $\sigma(x) = 1 + 1_{x\geq 0}$ satisfies Assumption 2.2 and belongs to $H^{1,1}$.

2. The function $\zeta$ defined in 2 also satisfies Assumption 2.2.

3. If $a,b > 0$ and $\sigma_1, \sigma_2$ satisfies Assumption 2.2 then $a\sigma_1 + b\sigma_2$ also satisfies Assumption 2.2.

4. Let $f_1, f_2$ be two strictly increasing, piecewise 1-Hölder functions. Let $\rho$ be a 1/2-Hölder continuous function satisfying $0 < \inf_{x\in\mathbb{R}} \rho(x) \leq \sup_{x\in\mathbb{R}} \rho(x) < \infty$. Then $\sigma := \rho \circ (f_1 - f_2)$ is piecewise 1/2-Hölder and it satisfies Assumption 2.2 with $f_{\sigma} = C(f_1 + f_2)$ for some positive constant $C$.

We are now in the position to state the main result of this paper.

Theorem 2.4. Let Assumption 2.2 hold, and $b, \sigma \in H^{\beta,\kappa}$ for some $\beta \in (0,1]$ and $\kappa > 0$.

(i) There exists a constant $C$ such that for all $n \geq 3$,

$$\sup_{0 \leq t \leq T} \mathbb{E}[\|X_t - X_{t}^{(n)}\|^q] \leq \frac{CqC\sqrt{\log \log n}}{\log n}. \tag{3}$$

(ii) Moreover, if $b \in L^1(\mathbb{R})$, then there exists a constant $C$ such that for all $n \geq 3$,

$$\sup_{0 \leq t \leq T} \mathbb{E}[\|X_t - X_{t}^{(n)}\|] \leq \frac{C}{\log n}. \tag{4}$$

The estimates (3) and (4) were obtained in [6, 20, 22] under a stronger assumption that $\sigma$ is 1/2-Hölder continuous on $\mathbb{R}$.

3 Proof of main results

3.1 Some auxiliary estimates

In this section, we derive a key estimation (Lemma 3.5) for proving the main theorem. We first introduce the following standard estimation (see Remark 1.2 in [6]).

Lemma 3.1. Suppose that $b$ and $\sigma$ are bounded, measurable. Then for any $q > 0$, there exists $C_q \equiv C(q, \|b\|_\infty, \|\sigma\|_\infty, T)$ such that for all $n \in \mathbb{N}$,

$$\sup_{t \in [0,T]} \mathbb{E}[\|X_{t}^{(n)} - X_{n,\eta(t)}^{(n)}\|^q] \leq \frac{C_q}{n^{q/2}}. \tag{5}$$

The next estimation is a uniform $L^2$-bounded of the local time of solution of SDE (1) and its EM approximation.
Lemma 3.2. Suppose that $b$ is bounded, measurable and $\sigma$ is measurable and satisfies Assumption 2.2(ii). For each $\theta \in [0, 1]$, define
\[
V_t^{(n)}(\theta) := (1 - \theta)X_t + \theta X_t^{(n)}.
\]
Then it holds that
\[
E\left[\sup_{\theta \in [0, 1]} |L_T^2(V^{(n)}(\theta))|^2\right] \leq 12\|b\|_{\infty}^2 T^2 + 6\sigma^2 T.
\]

Proof. By using the symmetric Itô-Tanaka formula, we have
\[
L_T^2(V^{(n)}(\theta)) = |V_T^{(n)}(\theta) - x| - |x_0 - x| - \int_0^T \left(1(V_s^{(n)}(\theta) > x) - 1(V_s^{(n)}(\theta) < x)\right) dV_s^{(n)}(\theta)
\]
\[
\leq \left|V_T^{(n)}(\theta) - x_0\right| + \int_0^T \left(1(V_s^{(n)}(\theta) > x) - 1(V_s^{(n)}(\theta) < x)\right) dV_s^{(n)}(\theta)
\]
\[
\leq 2\int_0^T \left|(1 - \theta)b(X_s) + \theta b(X_{\eta_n(s)})\right| ds + \int_0^T \left|(1 - \theta)\sigma(X_s) + \theta \sigma(X_{\eta_n(s)})\right| dW_s
\]
\[
+ \int_0^T \left|(1 - \theta)\sigma(X_s) + \theta \sigma(X_{\eta_n(s)})\right| dW_s.
\]
Since $b$ and $\sigma$ are bounded, it follows from inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and the $L^2$-isometry that,
\[
E\left[\sup_{\theta \in [0, 1]} |L_T^2(V^{(n)}(\theta))|^2\right] \leq 12\|b\|_{\infty}^2 T^2 + 6\sup_{\theta \in [0, 1]} \int_0^T \left|(1 - \theta)\sigma(X_s) + \theta \sigma(X_{\eta_n(s)})\right|^2 ds
\]
\[
\leq 12\|b\|_{\infty}^2 T^2 + 6\sigma^2 T.
\]
This concludes the statement.

The following lemma, which is similar to Lemma 2.2 in [20], plays a crucial role in our argument.

Lemma 3.3. Assume that $b$ and $\sigma$ are bounded measurable. For any $\varepsilon, \chi > 0$ such that $\delta := \frac{\chi}{8(T\|b\|_{\infty}^4 + 2^7\sigma^4)} \leq T$, it holds that for any $t \geq 0$ and $n \in \mathbb{N}$, $\mathbb{P}(\sup_{t \leq \tau \leq t + \delta} |X_{t + \tau}^{(n)} - X_t^{(n)}| \geq \varepsilon) \leq \delta \chi$.

Proof. Let $t \in [0, T]$ be fixed. We define $Z_{t+s}^{(n)} := X_{t+s}^{(n)} - X_t^{(n)}$. Then using Burkholder-Davis-Gundy's inequality, it holds that for any $\delta \in [0, T]$,
\[
E\left[\sup_{0 \leq s \leq \delta} |Z_{t+s}^{(n)}|^4\right] \leq 8E\left[\sup_{0 \leq s \leq \delta} \int_t^{t+s} b(X_{\eta_n(r)}^{(n)}) dr\right]^4 + 8E\left[\sup_{0 \leq r \leq \delta} \int_t^{t+s} \sigma(X_{\eta_n(r)}^{(n)}) dW_r\right]^4
\]
\[
\leq 8\delta^3 E\left[\int_t^{t+s} b(X_{\eta_n(r)}^{(n)})^4 dr\right] + 2^{10}\delta^2 E\left[\int_t^{t+s} \sigma(X_{\eta_n(r)}^{(n)})^4 dr\right]
\]
\[
\leq 8\|b\|_{\infty}^4 \delta^4 + 2^{10}\sigma^4 \delta^2 \leq 8(\|b\|_{\infty}^4 T^2 + 2\sigma^4) \delta^2.
\]
Hence, for any \( \varepsilon, \chi > 0 \) such that \( \delta := \frac{\chi^2}{8(T^2 \| b \|_\infty^4 + \sigma^2)} \leq T \), from Markov’s inequality, we have

\[
\mathbb{P} \left( \sup_{t \leq s \leq t + \delta} |X_s^{(n)} - X_t^{(n)}| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^4} \mathbb{E} \left[ \sup_{t \leq s \leq t + \delta} |X_s^{(n)} - X_t^{(n)}|^4 \right] \leq \frac{1}{\varepsilon^4} \mathbb{E} \left[ \sup_{0 \leq s \leq \delta} |Z_s^{(n)}|^4 \right] \leq \frac{8}{\varepsilon^4} \left( \|b\|_\infty^4 T^2 + 2\sigma^4 \right) \delta^2 = \delta \chi,
\]

which concludes the statement. \( \square \)

Lemma 3.3 directly implies the following result.

**Lemma 3.4.** Assume that \( b \) and \( \sigma \) are bounded measurable. Let \( (\gamma_n)_{n \in \mathbb{N}} \) be a decreasing sequence such that \( \gamma_n \in (0, 1] \) and \( \gamma_n \downarrow 0 \) and \( \gamma_n n^2 \to \infty \) as \( n \to \infty \). Denote \( \varepsilon_n := \frac{\delta^2}{\gamma_n^{1/4}} \), \( \bar{c} := 2^{3/4} T^{1/2} \{T^2 \| b \|_\infty^4 + 2\sigma^4\}^{1/4} \), \( \chi_n := \frac{\gamma_n}{\bar{c}} \), \( \delta_n := \frac{\chi_n \delta^2}{8(T^2 \| b \|_\infty^4 + \sigma^2)} = \frac{T}{n} \). For each \( k = 0, \ldots, n - 1 \), we define

\[
\Omega_{k,n,\varepsilon_n} := \left\{ \omega \in \Omega \left| \sup_{t_{k-1}^{(n)} \leq s \leq t_k^{(n)}} |X_s^{(n)}(\omega) - X_{t_k^{(n)}}(\omega)| \geq \varepsilon_n \right. \right\}.
\]

Then it holds that \( \mathbb{P}(\Omega_{k,n,\varepsilon_n}) \leq \delta_n \chi_n = \gamma_n \).

Now we state the a key lemma of our demonstration.

**Lemma 3.5.** Let Assumption 2.2 (ii) hold and the drift coefficient \( b \) be bounded and measurable. Let \( f \in H^{\beta, \kappa} \) for some \( \beta \in (0, 1) \). Then for any \( p \geq 1 \) and \( 0 < \alpha < \frac{\beta^2}{2 + \beta} \wedge \frac{2\alpha}{\kappa + 1} \), there exists a positive constant \( C_p(f) = C^*(p, \alpha, \beta, \kappa, T, x_0, \|f\|_\beta, C_{\beta, \kappa}; \|b\|_\infty, \sigma, \bar{c}) \) which does not depend on \( n \) such that for each \( n \geq 3 \),

\[
\int_0^T \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)}) \right|^p \right] ds \leq \frac{C_p(f)}{n^\alpha \log n}.
\]

**Proof.** From Lemma 3.4 and the boundedness of \( f \), it holds that

\[
\begin{align*}
\int_0^T \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{\eta_n(s)}^{(n)}) \right|^p \right] ds &= \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p \left( 1_{\Omega_{k,n,\varepsilon_n}} + 1_{\Omega_{k,n,\varepsilon_n}^c} \right) \right] ds \\
&\leq 2^p \|f\|_\infty^p \|X\|_\infty T \gamma_n + \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p 1_{\Omega_{k,n,\varepsilon_n}} \right] ds \\
&\quad + \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p 1_{\Omega_{k,n,\varepsilon_n}^c} 1_{X_s^{(n)} \notin S^*(f)} \right] ds.
\end{align*}
\]

We estimate the second term of (7) as follows

\[
\begin{align*}
&\sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p 1_{\Omega_{k,n,\varepsilon_n}} \right] ds \\
&= \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p 1_{\Omega_{k,n,\varepsilon_n}} 1_{X_s^{(n)} \notin S^*(f)} \right] ds \\
&\quad + \sum_{k=0}^{n-1} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \mathbb{E} \left[ \left| f(X_s^{(n)}) - f(X_{t_k^{(n)}}^{(n)}) \right|^p 1_{\Omega_{k,n,\varepsilon_n}^c} \right] ds.
\end{align*}
\]
On the set $\Omega_{k,n} \cap \{ X_s^{(n)} \not\in S^e(f) \}$, it holds that $S(f) \cap [ X_s^{(n)} \wedge X_{t_k}^{(n)}, X_s^{(n)} \vee X_{t_k}^{(n)} ] = \emptyset$, thus,

$$
| f(X_s^{(n)}) - f(X_{t_k}^{(n)}) |^p \mathbb{1}_{\Omega_{k,n}} \leq \| f \|^p_{\beta} \left( X_s^{(n)} - X_{t_k}^{(n)} \right)^{p\beta}.
$$

This implies the second term of (8) is bounded by

$$
\| f \|^p_{\beta} \sum_{k=0}^{n-1} \int_{t_{k+1}}^{t_k} \mathbb{E} \left[ \left( X_s^{(n)} - X_{t_k}^{(n)} \right)^{p\beta} \right] ds \leq \| f \|^p_{\beta} T C_{p\beta} n^{-p\beta/2},
$$

where the last inequality follows from Lemma 3.1. For each constant $K_n \geq 1 \vee (|x_0| + T\|b\|_\infty)$, the first term of (8) is bounded by

$$
2^p \| f \|^p_{\beta} \sum_{k=0}^{n-1} \int_{t_{k+1}}^{t_k} \left( \mathbb{E} \left[ 1_{X_s^{(n)} \in S^e(f) \cap [-K_n,K_n]} \right] + \mathbb{E} \left[ 1_{X_s^{(n)} \in S^e(f) \cap [-K_n,K_n]} \right] \right) ds
\leq 2^p \| f \|^p_{\beta} \int_0^T \mathbb{E} \left[ 1_{X_s^{(n)} \in S^e(f) \cap [-K_n,K_n]} \right] ds + 2^p \| f \|^p_{\beta} \int_0^T \mathbb{E} \left[ 1_{|X_s^{(n)}| \geq K_n} \right] ds.
$$

Since $\sigma$ is uniformly elliptic, $(X^{(n)})_t \geq \sigma^2 t$, we obtain

$$
\int_0^T \mathbb{E} \left[ 1_{X_s^{(n)} \in S^e(f) \cap [-K_n,K_n]} \right] ds \leq \sigma^{-2} \mathbb{E} \left[ \int_0^T 1_{X_s^{(n)} \in S^e(f) \cap [-K_n,K_n]} d(X^{(n)})_s \right]
= \sigma^{-2} \mathbb{E} \left[ \int_\mathbb{R} 1_{S^e(f) \cap [-K_n,K_n]}(x) L_T^2(X^{(n)}) dx \right],
$$

where the last equation follows from the occupation time formula. Moreover, it follows from Lemma 3.2 that

$$
\mathbb{E} \left[ \int_\mathbb{R} 1_{S^e(f) \cap [-K_n,K_n]}(x) L_T^2(X^{(n)}) dx \right] \leq \int_\mathbb{R} 1_{S^e(f) \cap [-K_n,K_n]}(x) \mathbb{E}[L_T^2(X^{(n)})] dx
\leq \sup_{x \in \mathbb{R}} \mathbb{E}[L_T^2(X^{(n)})] \lambda \left( S^e(f) \cap [-K_n,K_n] \right)
\leq \{12\|b\|_\infty^2 T^2 + 6\sigma^2 T\}^{1/2} C_{\beta,\alpha} K_n^{\epsilon_n}.\]

Now we consider the second term of (10). For each $s \in [0,T]$,

$$
\mathbb{E} \left[ 1_{|X_s^{(n)}| \geq K_n} \right] \leq \mathbb{P} \left( \left| \int_0^s \sigma(X^{(n)}_{\eta_s(u)}) dW_u \right| \geq K_n - |x_0| + \int_0^s b(X^{(n)}_{\eta_s(u)}) du \right),
\leq \mathbb{P} \left( \left| \int_0^s \sigma(X^{(n)}_{\eta_s(u)}) dW_u \right| \geq K_n - \|b\|_\infty T - |x_0| \right).
$$

Since $(\int_0^s \sigma(X^{(n)}_{\eta_s(u)}) dW_u)_t \leq \sigma^2 T$ almost surely, from Proposition 6.8 of [23] and the inequality $(a - b)^2 \geq a^2 / 2 - b^2$ for any $a, b \in \mathbb{R}$, we have

$$
\mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(X^{(n)}_{\eta_s(u)}) dW_s \right| \geq K_n - \|b\|_\infty T - |x_0| \right)
\leq 2 \exp \left( - \frac{(K_n - |x_0| - \|b\|_\infty T)^2}{2\sigma^2 T} \right) \leq 2 \exp \left( \frac{(|x_0| + \|b\|_\infty T)^2}{2\sigma^2 T} \right) \exp \left( - \frac{K_n^2}{4\sigma^2 T} \right).
$$

This implies

$$
\int_0^T \mathbb{E} \left[ 1_{|X_s^{(n)}| \geq K_n} \right] ds \leq 2T \exp \left( \frac{(|x_0| + \|b\|_\infty T)^2}{2\sigma^2 T} \right) \exp \left( - \frac{K_n^2}{4\sigma^2 T} \right).
$$

(11)
Gathering together the estimates (7) –(11), we get
\[
\int_0^T \mathbb{E} \left[ |f(X_s^{(n)}) - f(X_{\eta_n(s)})|^p \right] ds \leq 2^p \|f\|_{\infty}^p T \gamma_n + \|f\|_{\beta}^p T C_\beta n^{-p}\beta/2
+ 2^p \|f\|_{\beta}^p \sigma^{-2} \{12 |b|_{\infty}^2 T^2 + 6\sigma^2 T \}^{1/2} C_{\beta,\kappa} K_n \varepsilon_n^n
+ 2^{p+1} \|f\|_{\beta}^p T \exp \left( \frac{(|x_0| + \|b\|_{\infty} T)^2}{2\sigma^2 T} \right) \exp \left( -\frac{K_n^2}{4\sigma^2 T} \right). \tag{12}
\]
For each \(0 < \alpha < \frac{p\beta}{2} \wedge \frac{2\kappa}{\kappa + 4}\), by choosing \(K_n = (1 + |x_0| + T \|b\|_{\infty} + 2\sqrt{T \alpha})\sqrt{\log n}\) and \(\gamma_n = \frac{1}{n^{\alpha \log n}}\), we obtain (6) from (12).

3.2 Method of removal of drift

The following removal of drift transformation plays a crucial role in our argument. Suppose that \(b \in L^1(\mathbb{R})\). The function \(\varphi(x) := \int_0^x \exp \left( -2 \int_0^y \frac{b(z)}{\sigma^2(z)} dz \right) dy\) is well-defined since \(\sigma^2\) is uniformly elliptic. Define \(Y_t := \varphi(X_t)\) and \(Y_t^{(n)} := \varphi(X_t^{(n)})\). Then by Itô’s formula we have
\[
Y_t = \varphi(x_0) + \int_0^t \varphi'(X_s) \sigma(X_s) dW_s,
\]
and
\[
Y_t^{(n)} = \varphi(x_0) + \int_0^t \left( \varphi'(X_s^{(n)}) b(X_{\eta_n(s)}^{(n)}) + \frac{1}{2} \varphi''(X_s^{(n)}) \sigma^2(X_{\eta_n(s)}^{(n)}) \right) ds + \int_0^t \varphi'(X_s^{(n)}) \sigma(X_{\eta_n(s)}^{(n)}) dW_s.
\]
To simplify the notation, we denote \(K_\sigma = \sigma \vee \frac{2}{\kappa} - 1\) and \(C_0 = e^{2K_\sigma^2 \|b\|_{L^1(\mathbb{R})}}\). We will make repeated use of the following elementary lemma.

**Lemma 3.6.** \((\text{[22]})\) Suppose that \(b \in L^1(\mathbb{R})\) and Assumption2.2(ii) holds.

(i) For any \(x \in \mathbb{R}\), \(C_0^{-1} \leq \varphi'(x) = \exp \left( -2 \int_0^x \frac{b(z)}{\sigma^2(z)} dz \right) \leq C_0\).

(ii) For any \(x \in \mathbb{R}\), \(|\varphi''(x)| \leq 2K_\sigma^2 \|b\|_{\infty} \|\varphi'\|_{\infty} \leq 2 \|b\|_{\infty} K_\sigma^2 C_0\).

(iii) For any \(z, w \in \text{Dom}(\varphi^{-1})\),
\[
|\varphi^{-1}(z) - \varphi^{-1}(w)| \leq C_0 |z - w|.
\tag{13}
\]

3.3 Yamada and Watanabe approximation technique

Under the Assumption 2.2, by using the Yamada-Watanabe approximation technique, Le Gall \([14]\) show that the pathwise uniqueness holds for SDE (1). We also use this technique to prove the main result (see \([24]\) or \([8]\)). For each \(\delta \in (1, \infty)\) and \(\varepsilon \in (0, 1)\), we define a continuous function \(\psi_{\delta,\varepsilon} : \mathbb{R} \to \mathbb{R}^+\) with \(\text{supp} \psi_{\delta,\varepsilon} \subset [\varepsilon/\delta, \varepsilon]\) such that
From (13) and (14), for any useful properties:

\[ \int_{\varepsilon/\delta} \psi_{\delta,\varepsilon}(z) dz = 1 \text{ and } 0 \leq \psi_{\delta,\varepsilon}(z) \leq \frac{2}{z \log \delta}, \quad z > 0. \]

Since \( \int_{\varepsilon/\delta} \frac{2}{z \log \delta} dz = 2 \), there exists such a function \( \psi_{\delta,\varepsilon} \). We define a function \( \phi_{\delta,\varepsilon} \in C^2(\mathbb{R}; \mathbb{R}) \) by \( \phi_{\delta,\varepsilon}(x) := \int_0^x \int_0^y \psi_{\delta,\varepsilon}(z) dz dy \). It is easy to verify that \( \phi_{\delta,\varepsilon} \) has the following useful properties:

\[ |x| \leq \varepsilon + \phi_{\delta,\varepsilon}(x), \text{ for any } x \in \mathbb{R}, \quad (14) \]

\[ 0 \leq |\phi'_{\delta,\varepsilon}(x)| \leq 1, \text{ for any } x \in \mathbb{R}, \quad (15) \]

\[ \phi''_{\delta,\varepsilon}(\pm|x|) = \psi_{\delta,\varepsilon}(|x|) \leq \frac{2}{|x| \log \delta} 1_{[\varepsilon/\delta,\varepsilon]}(|x|), \text{ for any } x \in \mathbb{R} \setminus \{0\}. \quad (16) \]

From (13) and (14), for any \( t \in [0, T] \), we have

\[ |X_t - X_t^{(n)}| \leq C_0 |Y_t - Y_t^{(n)}| \leq C_0 \left( \varepsilon + \phi_{\delta,\varepsilon}(Y_t - Y_t^{(n)}) \right). \quad (17) \]

Using Itô's formula, we have

\[ \phi_{\delta,\varepsilon}(Y_t - Y_t^{(n)}) = M_{t,\delta,\varepsilon}^{n} + I_t^{(n)} + J_t^{(n)}, \quad (18) \]

where

\[ M_{t,\delta,\varepsilon}^{n} := \int_0^t \phi'_{\delta,\varepsilon}(Y_s - Y_s^{(n)}) \left\{ \varphi'(X_s) \sigma(X_s) - \varphi'(X_s^{(n)}) \sigma(X_s^{(n)}) \right\} dW_s, \]

\[ I_t^{(n)} := -\int_0^t \phi'_{\delta,\varepsilon}(Y_s - Y_s^{(n)}) \left\{ \varphi'(X_s^{(n)}) b(X_s^{(n)}) + \frac{1}{2} \varphi''(X_s^{(n)}) \sigma^2(X_s^{(n)}) \right\} ds, \]

\[ J_t^{(n)} := \frac{1}{2} \int_0^t \phi''_{\delta,\varepsilon}(Y_s - Y_s^{(n)}) \left| \varphi'(X_s) \sigma(X_s) - \varphi'(X_s^{(n)}) \sigma(X_s^{(n)}) \right|^2 ds. \]

### 3.4 Proof of Theorem [2.4]

We will only present the detail proof for the case that \( b \in L^1(\mathbb{R}) \). The proof for the case \( b \not\in L^1(\mathbb{R}) \) is based on the localisation technique given in [22] and it will be omitted.

We fix \( n \geq 3 \) and a constant \( 0 < \alpha < \frac{\beta}{2} \land \frac{2 \kappa}{\kappa + 4} \). We first consider \( I_t^{(n)} \). Since \( \varphi'' = -\frac{2b \varphi'}{\sigma^2} \),

\[ |I_t^{(n)}| \leq \int_0^T \left| \phi'_{\delta,\varepsilon}(Y_t - Y_t^{(n)}) \varphi'(X_t^{(n)}) \right| b(X_s^{(n)}) - \frac{b(X_s^{(n)}) \sigma^2(X_s^{(n)})}{\sigma^2(X_s^{(n)})} \right| ds. \]

Thanks to Lemma 3.6 and estimate (15), we have

\[ |I_t^{(n)}| \leq K_2^2 C_0 \int_0^T \left| b(X_s^{(n)}) \sigma^2(X_s^{(n)}) - b(X_s^{(n)}) \sigma^2(X_s^{(n)}) \right| ds \]

\[ \leq K_2^2 C_0 \int_0^T \left\{ K_2^2 \left| b(X_s^{(n)}) - b(X_s^{(n)}) \right| + \| b \|_\infty \left| \sigma^2(X_s^{(n)}) - \sigma^2(X_s^{(n)}) \right| \right\} ds. \]

It follows from Lemma 3.5 that

\[ \mathbb{E}[|I_t^{(n)}|] \leq \frac{C_1}{n^\alpha \log n}, \quad (19) \]
where $C_I := K_2^2C_0\{K_2^2C_1^*\|b\|_\infty \sigma C_1^*(\sigma)\}$. Now we estimate $J_t^{(n)}$. From (16), we have

$$J_t^{(n)} \leq \int_0^T \frac{1_{[\varepsilon,\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} \left| \sigma'(X_s) \sigma(X_s) - \varphi'(X_s^{(n)}) \sigma(X_s^{(n)}) \right|^2 ds \leq 3(J_T^{1,n} + J_T^{2,n} + J_T^{3,n}),$$

where

$$J_T^{1,n} := \int_0^T \frac{1_{[\varepsilon,\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} \left| \sigma(X_s) \right|^2 \left| \varphi'(X_s) - \varphi'(X_s^{(n)}) \right|^2 ds,$$

$$J_T^{2,n} := \int_0^T \frac{1_{[\varepsilon,\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} \left| \sigma'(X_s) - \sigma'(X_s^{(n)}) \right|^2 ds,$$

$$J_T^{3,n} := \int_0^T \frac{1_{[\varepsilon,\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}| \log \delta} \left| \sigma'(X_s)(X_s) - \sigma'(X_s^{(n)})(X_s^{(n)}) \right|^2 ds.$$

From Lemma 3.6 (ii), $\varphi'$ is Lipschitz continuous with Lipschitz constant $\|\varphi''\|_\infty$. Hence, we have

$$J_T^{1,n} \leq \frac{K_2^2\|\varphi''\|_\infty^2}{\log \delta} \int_0^T \frac{1_{[\varepsilon,\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|)}{|Y_s - Y_s^{(n)}|} \left| X_s - X_s^{(n)} \right|^2 ds \leq \frac{K_2^2\|\varphi''\|_\infty^2 C_0^2}{\log \delta} \int_0^T 1_{[\varepsilon,\delta,\varepsilon]}(|Y_s - Y_s^{(n)}|) \left| Y_s - Y_s^{(n)} \right| ds \leq \frac{CJ_1 \varepsilon}{\log \delta},$$

(20)

where $C_{J,1} := 4K_2^2C_0^4\|b\|^2 \infty T$. Next we consider $J_T^{2,n}$. We first note that by (13),

$$J_T^{2,n} \leq \frac{C_0^3}{\log \delta} \int_0^T \frac{\sigma(X_s) - \sigma(X_s^{(n)})}{|X_s - X_s^{(n)}|} \left| 1_{|X_s - X_s^{(n)}| \geq \varepsilon/(C_0 \delta)} \right| ds.$$

Recall that by Assumption 2.2 (i), there exists a bounded and strictly increasing function $f_\sigma : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)|^2 \leq |f_\sigma(x) - f_\sigma(y)|.$$

We consider approximation $f_{\sigma,\ell} \in C^1(\mathbb{R})$ of $f_\sigma$ which is also strictly increasing function and satisfies $\|f_{\sigma,\ell}\|_\infty \leq \|f_\sigma\|_\infty$ and $f_{\sigma,\ell} \uparrow f_\sigma$ as $\ell \to \infty$ on $\mathbb{R}$. Then by using Fatou’s lemma and the mean value theorem, we have

$$J_T^{2,n} \leq \frac{C_0^3}{\log \delta} \int_0^T \frac{f_{\sigma}(X_s) - f_{\sigma}(X_s^{(n)})}{|X_s - X_s^{(n)}|} \left| 1_{|X_s - X_s^{(n)}| \geq \varepsilon/(C_0 \delta)} \right| ds \leq \liminf_{\ell \to \infty} \frac{C_0^3}{\log \delta} \int_0^T \frac{f_{\sigma,\ell}(X_s) - f_{\sigma,\ell}(X_s^{(n)})}{|X_s - X_s^{(n)}|} \left| 1_{|X_s - X_s^{(n)}| \geq \varepsilon/(C_0 \delta)} \right| ds \leq \liminf_{\ell \to \infty} \frac{C_0^3}{\log \delta} \int_0^T \int_0^1 d\varepsilon f_{\sigma,\ell}(V_s^{(n)}(\theta)),$$

(21)

where $V^{(n)}(\theta) = (V_t^{(n)}(\theta))_{0 \leq t \leq T}$ is defined in Lemma 3.2. Since $\sigma \geq \sigma$, the quadratic variation of $V^{(n)}(\theta)$ satisfies

$$\langle V^{(n)}(\theta) \rangle_t = \int_0^t \left\{ (1 - \theta)\sigma(X_s) + \theta\sigma(X_s^{(n)}(\sigma)) \right\}^2 ds \geq \sigma^2 t,$$
which implies
\[
\int_0^T ds \int_0^1 d\theta f_{\sigma,\ell}^\prime (V_s^{(n)}(\theta)) \leq \sigma^{-2} \int_0^1 d\theta \int_0^T \int_0^1 d\theta f_{\sigma,\ell}^\prime (V_s^{(n)}(\theta))
\]
\[
= \sigma^{-2} \int_\mathbb{R} dx f_{\sigma,\ell}^\prime (x) \int_0^1 d\theta L_T^x (V_s^{(n)}(\theta)),
\]
where the last equality is implied from the occupation time formula. Using Lemma 3.2 and the estimate \( \| f_{\sigma,\ell}^\prime \| \leq 2 \| f_{\sigma,\ell} \| \) we have
\[
\mathbb{E} \left[ \int_0^T ds \int_0^1 d\theta f_{\sigma,\ell}^\prime (V_s^{(n)}(\theta)) \right] \leq \sigma^{-2} \int_\mathbb{R} dx f_{\sigma,\ell}^\prime (x) \int_0^1 d\theta \mathbb{E}[L_T^x (V_s^{(n)}(\theta))]
\]
\[
\leq \sigma^{-2} \| f_{\sigma,\ell}^\prime \| \leq \sup_{\theta \in [0,1], x \in \mathbb{R}} \mathbb{E}[L_T^x (V_s^{(n)}(\theta))]^{1/2}
\]
\[
\leq 2\sigma^{-2} \| f_{\sigma,\ell} \| \{12 \| b \|_\infty^2 T^2 + 6\sigma^2 T \}^{1/2}.
\]
By plugging this estimate to (21) and using Fatou’s lemma, we get the following estimate for the expectation of \( J_{T}^{2,n} \),
\[
\mathbb{E}[J_{T}^{2,n}] \leq \frac{C_{J,2}}{\log \delta}, \quad \text{(22)}
\]
where \( C_{J,2} := 2C_0^3 \sigma^{-2} \| f_{\sigma,\ell} \| \leq \{12 \| b \|_\infty^2 T^2 + 6\sigma^2 T \}^{1/2} \). Finally, we estimate \( J_{T}^{3,n} \) as follows
\[
\mathbb{E}[J_{T}^{3,n}] \leq \frac{C_{J,3}}{\varepsilon \log \delta} \int_0^T \mathbb{E} \left[ \left| \sigma(X_s^n) - \sigma(X_{n,s}^{(n)}) \right|^2 \right] ds.
\]
Applying Lemma 3.5 we get
\[
\mathbb{E}[J_{T}^{3,n}] \leq \frac{\delta}{\varepsilon \log \delta} \frac{C_{J,3}}{n^\alpha \log n}, \quad \text{(23)}
\]
where \( C_{J,3} := C_2^3 C_2^2 (\sigma) \). Since \( \mathbb{E}[M_{T}^{n,\delta,\varepsilon}] = 0 \), it follows from (17) – (23) that there exists a positive constant \( C \) which do not depend on \( n \) such that
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \left| X_t - X_t^{(n)} \right| \right] \leq C \left( \varepsilon + \frac{1}{n^\alpha \log n} + \frac{\varepsilon}{\log \delta} + \frac{1}{\log \delta} + \frac{\delta}{\varepsilon \log \delta} \frac{1}{n^\alpha \log n} \right).
\]
By choosing \( \varepsilon = \frac{1}{\log n} \) and \( \delta = n^\alpha \), we obtain the desired result. \( \square \)

**Acknowledgements**

The authors thank Arturo Kohatsu-Higa, Miguel Martinez and Toshio Yamada for their helpful comments. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED). The second author was supported by JSPS KAKENHI Grant Number 16J00894.
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