Remarks on the Sequential Products

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Abstract. In this paper, we show that those sequential products which were proposed by Liu and Shen and Wu in [J. Phys. A: Math. Theor. 42, 185206 (2009), J. Phys. A: Math. Theor. 42, 345203 (2009)] are just unitary equivalent to the sequential product $A \circ B = A^\dagger BA^\dagger$.

Key words. Hilbert space, Lüders operation, Sequential product.

Pacs. 03.65.Aa, 03.65.Db

Quantum measurement theory is one of the key problems in quantum theory, it contains a great many of mathematical problems and philosophical problems. Also it has applications in quantum information theory and quantum correction theory. The essential difference between quantum measurement and classical measurement is that the quantum measurement would make the system collapsed. It has the follows four characteristics:

(1). Randomness. It is unpredictable and uncontrollable.
(2). Irreversibility. In general, measurement is entropy-increasing procedure.
(3). Decoherence. Eliminate all the coherence of the original state.
(4). Nonlocality. The collapse of the wave function is nonlocal.

In history, Heinsenberg, von Neumann, Birkhoff published some important far-reaching fundamental works. In order to state our main results, now, we need to recall some elementary notations.

Let $\mathcal{L}$ be a quantum-mechanical system and it be represented by a complex Hilbert space $H$. Each self-adjoint operator $A$ on $H$ satisfies that $0 \leq A \leq I$ is said to be a quantum effect ([1-2]). Quantum effects represent yes-no measurements that may be unsharp. The set of quantum effects on $H$ is denoted by $\mathcal{E}(H)$. The subset $\mathcal{P}(H)$ of $\mathcal{E}(H)$ consisting of orthogonal projection operators represents sharp yes-no measurements. Let $\mathcal{T}(H)$ be the set of trace class operators on $H$ and $\mathcal{S}(H)$ the set of density operators on $H$, i.e., the state set of quantum system $\mathcal{L}$.

As we knew, a quantum measurement can be described as a quantum operation which is a completely positive linear mapping $\Phi : \mathcal{T}(H) \to \mathcal{T}(H)$ such that for each $T \in \mathcal{S}(H)$,

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*This project is supported by Zhejiang Innovation Program for Graduates (YK2009002) and Natural Science Foundation of China (10771191 and 10471124) and Natural Science Foundation of Zhejiang Province of China (Y6090105).

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0 \leq tr[\Phi(T)] \leq 1 (\text{[3-5]}). For each \( P \in \mathcal{P}(H) \), the so-called Lüders operation \( \Phi^P_L \) is defined by \( T \rightarrow PTP \), in physics, it implied that if the quantum-mechanical system \( L \) is in state \( W \in S(H) \), then the probability that the measurement \( P \) is observed is given by \( p_W(P) = tr(PWP) \), moreover, the resulting state after the measurement \( P \) is observed is \( \frac{PWP}{tr(PWP)} \) whenever \( tr(PWP) \neq 0 \) ([4]). Each quantum effect \( B \in \mathcal{E}(H) \) gives to a general Lüders operation \( \Phi^B_L : T \rightarrow B^\sharp TB^\sharp \). If \( A, B \in \mathcal{E}(H) \), then the composition operation \( \Phi^B_L \circ \Phi^A_L \) defines a new operation and is called a sequential operation as it is obtained by performing first \( \Phi^A_L \) and then \( \Phi^B_L \). It is easily to prove that \( \Phi^B_L \circ \Phi^A_L = \Phi^A_L \circ \Phi^B_L \) ([5, P26-27]). Let us denote \( \Phi^B_L \circ \Phi^A_L \) by \( A \circ B \), then \( A \circ B \in \mathcal{E}(H) \) and \( \circ \) has the following important properties ([6-7]):

(S1). The map \( B \rightarrow A \circ B \) is additive for each \( A \in \mathcal{E}(H) \), that is, if \( B + C \leq I \),
then \( (A \circ B) + (A \circ C) \leq I \) and \( (A \circ B) + (A \circ C) = A \circ (B + C) \).

(S2). \( I \circ A = A \) for all \( A \in \mathcal{E}(H) \).

(S3). If \( A \circ B = 0 \), then \( A \circ B = B \circ A \).

(S4). If \( A \circ B = B \circ A \), then \( A \circ (I - B) = (I - B) \circ A \) and \( A \circ (B \circ C) = (A \circ B) \circ C \)
for all \( C \in \mathcal{E}(H) \).

(S5). If \( C \circ A = A \circ C \), \( C \circ B = B \circ C \), then \( C \circ (A \circ B) = (A \circ B) \circ C \)
and \( C \circ (A + B) = (A + B) \circ C \) whenever \( A + B \leq I \).

Professor Gudder called \( A \circ B \) the sequential product of \( A \) and \( B \), it represents the quantum effect produced by first measuring \( A \) then measuring \( B \) ([6-7]). In [8], Gudder asked: is \( A \circ B = A^\sharp BA^\sharp \) the only operation on \( \mathcal{E}(H) \) which satisfies the properties (S1)-(S5)? In [9], Liu and Wu showed that if \( H \) is a finite dimensional complex Hilbert space, \( f_z(u) \) is the complex-valued function defined on \([0, 1]\), where \( f_z(u) = \exp(z \ln u) \) if \( u \in (0, 1) \) and \( f_z(0) = 0 \), and denote \( A^i = f_z(A) \), \( A^{-i} = f_{-i}(A) \), then \( A \circ_1 B = A^{1/2}A^i B A^{-i} A^{1/2} \) defined a new sequential product which satisfies the properties (S1)-(S5), thus, Gudder’s problem was answered negatively.

Note that the sequential product \( A \circ B = A^\sharp BA^\sharp = A^\sharp B(A^\sharp)^* \) of \( A \) and \( B \) can only describe the instantaneous measurement, that is, the measurement \( B \) is completed at once after the measurement \( A \) is performed. In order to describe a more complicated process where we allow a duration between the measurement \( A \) with the measurement \( B \), then we need to replace \( A^\sharp \) with \( f(A) \), \( (A^\sharp)^* \) with \( (f(A))^* \), where \( f(A) \) is a function of \( A \) which describe the change of \( A \) was made by the duration between \( B \) with \( A \). Thus, we need to consider the following general sequential product \( f(A) B(f(A))^* \).

By the above motivation, in [10], Shen and Wu proved the following result:

**Theorem 1.** Let \( H \) be a finite dimensional complex Hilbert space, for each \( A \in \mathcal{E}(H) \), \( sp(A) \) the spectra of \( A \) and \( B(sp(A)) \) the set of all bounded complex Borel functions on \( sp(A) \). Take a \( f_A \in B(sp(A)) \). Define \( A \circ B = f_A(A) B(f(A))^* \) for \( B \in \mathcal{E}(H) \). Then \( \circ \) has the properties (S1)-(S5) iff the set \( \{f_A\}_{A \in \mathcal{E}(H)} \) satisfies the following conditions:

(i) For every \( A \in \mathcal{E}(H) \) and \( t \in sp(A)_c \), \( |f_A(t)| = \sqrt{7} \).
(ii) For any \(A, B \in \mathcal{E}(H)\), if \(AB = BA\), then there exists a complex constant \(\xi\) such that \(|\xi| = 1\) and \(f_A(A) f_B(B) = \xi f_{AB}(AB)\).

Note that for each \(A \in \mathcal{E}(H)\), we can take many \(f_A \in \mathcal{B}(\text{sp}(A))\) satisfies the conditions (i) and (ii), so, Theorem 1 told us that for each given finite dimensional complex Hilbert space \(H\), there are many sequential products on \((\mathcal{E}(H), 0, I, \oplus)\).

In this note, we show that these sequential products are unitary equivalent to the sequential product \(A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}\).

Firstly, we need the following:

**Lemma 1.1 ([10]).** If \(\{f_A\}_{A \in \mathcal{E}(H)}\) satisfies the conditions (i) and (ii) of Theorem 1, then we have

1. \(f_A(A) f_A(A) = A\), \((f_A(A))^* = f_A(A)\).
2. If \(0 \in \text{sp}(A)\), then \(f_A(0) = 0\).
3. If \(A = \sum_{k=1}^{n} \lambda_k E_k\), where \(\{E_k\}_{k=1}^{n}\) are pairwise orthogonal projections and \(\lambda_k \neq 0\), then \(f_A(A) = \sum_{k=1}^{n} f_A(\lambda_k) E_k\).

Our main result is:

**Theorem 2.** Let \(H\) be a finite dimensional complex Hilbert space. Then the sequential product \(f_A(A) B(f_A(A))^*\) on \((\mathcal{E}(H), 0, I, \oplus)\) is unitary equivalent to the sequential product \(A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}\).

**Proof.** Let \(A = \sum_{k=1}^{n} \lambda_k E_k\) be the spectra decomposition of \(A\), where \(\{E_k\}_{k=1}^{n}\) be pairwise orthogonal projection operators and \(\lambda_k > 0, k = 1, 2, \cdots, n\). By condition (i) of Theorem 1, we have \(|f_A(\lambda_k)| = \sqrt{\lambda_k}\), so \(f_A(\lambda_k) = \sqrt{\lambda_k} e^{i\theta_k}\) for some real number \(\theta\). Let \(E_0 = I - \sum_{k=1}^{n} E_k\) and \(U = \sum_{k=1}^{n} e^{i\theta_k} E_k + E_0\). Then \(U\) is an unitary operator and it is easy to see that \(AU = UA\), so by Lemma 1.1, we have \(f_A(A) = A^{1/2} U\). Thus, \(A \circ B = f_A(A) B f_A(A)^* = A^{1/2} UB(A^{1/2} U)^* = U(A^{1/2} BA^{1/2}) U^* = U(A \circ B) U^*\) and the conclusion is proved.

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