Matroid Intersection for Two Countable Nearly Finitary Matroids

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Abstract
We prove that if $M$ and $N$ are nearly finitary matroids on a common countable edge set $E$ then they admit a common independent set $I$ such that there is a bipartition $E = E_M \cup E_N$ for which $I \cap E_M$ spans $E_M$ in $M$ and $I \cap E_N$ spans $E_N$ in $N$. It answers positively the original form of the Matroid Intersection Conjecture of Nash-Williams in the countable case improving the partial result obtained by Aharoni and Ziv. However the problem for more general matroids remains open.

1 Introduction
An $M = (E, \mathcal{I})$ is a finitary matroid if $\mathcal{I} \subseteq \mathcal{P}(E)$ with
1 $\emptyset \in \mathcal{I}$;
2 $\mathcal{I}$ is downward closed;
3 For every $I, J \in \mathcal{I}$ where $|I| < |J|$, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
F If every finite subset of an $X \subseteq E$ is in $\mathcal{I}$, then $X \in \mathcal{I}$.

Infinite vector spaces with the linear independence and infinite graphs where the independent sets are the subforests are natural examples for infinite finitary matroids. The axiom system above was not considered an entirely satisfying infinite generalization of the concept of matroids because it does not capture a key concept of the finite theory, the duality. Indeed, the class of structures satisfying the axioms above are not closed under taking duals, i.e., the set of subsets of $E$ avoiding some $\subseteq$-maximal element of $\mathcal{I}$ does not necessarily satisfy 1-3 and F.

Rado asked in 1966 if there is a reasonable notion of infinite matroids admitting duality and minors. Among other attempts Higgs introduced [1] a class of structures he called “B-matroids”. Oxley gave an axiomatization for B-matroids and showed that they are the largest class of structures satisfying 1-3 and being closed under taking duals and minors (see [2] and [3]). The investigation of infinite matroids gained a new momentum after a long break when Bruhn, Diestel, Kriesell, Pendavingh, Wollan found a set of cryptomorphic axioms for infinite matroids, generalising the usual independent set-, basis-, circuit-, closure- and rank-axioms for finite matroids (see [4]). Their motivation came from topological infinite graph theory and they found out only later that their infinite matroid concept and the B-matroids of Higgs are actually the same.

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$M = (E, \mathcal{I})$ is a matroid if $\mathcal{I} \subseteq \mathcal{P}(E)$ with

1. $\emptyset \in \mathcal{I}$;
2. $\mathcal{I}$ is downward closed;
3'. For every $I, J \in \mathcal{I}$ where $J$ is $\subseteq$-maximal in $\mathcal{I}$ but $I$ is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;

$M$ For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

The following terminology was introduced by Bowler and Carmesin in [6] in a slightly different but equivalent form.

**Definition 1.1** (Intersection property). Let $M$ and $N$ be matroids on the same ground set $E$. We say that $\{M, N\}$ has the **Intersection property** if there is a common independent set $I$ of $M$ and $N$ and a bipartition $E = E_M \cup E_N$ such that $I_M := I \cap E_M$ spans $E_M$ in $M$ and $I_N := I \cap E_N$ spans $E_N$ in $N$.

**Conjecture 1.2** (Matroid Intersection Conjecture (general form)). If $M$ and $N$ are matroids on the same ground set then $\{M, N\}$ has the Intersection property.

**Conjecture 1.3** (Matroid Intersection Conjecture by Nash-Williams, [5]). If $M$ and $N$ are finitary matroids on the same ground set then $\{M, N\}$ has the Intersection property.

Aharoni and Ziv proved in [5] that if $M$ and $N$ are countable finitary matroids and one of them satisfies the extra property described below, then $\{M, N\}$ has the Intersection property. The extra condition says: whenever $B$ is a base and $e_i \in B$, $f_i \in E \setminus B$ for $i \in \mathbb{N}$ such that $B - e_0 + f_0 - \cdots - e_n + f_n$ is a base for every $n \in \mathbb{N}$, then $(B \setminus \{e_n : n \in \mathbb{N}\}) \cup \{f_n : n \in \mathbb{N}\}$ is also a base.

Let us recall that the finitarization of a matroid $M$ on $E$ is the matroid $M'$ on $E$ whose circuits are exactly the finite circuits of $M$. Cofinitarization defined similarly but with cocircuits. A matroid $M$ is nearly finitary if every base of $M$ can be extended to a base of its finitarization by adding finitely many edges. A matroid is nearly cofinitary if its dual is nearly finitary.

**Theorem 1.4** (Aigner-Horev, Carmesin and Frölich; Theorem 1.5 in [7]). If $M$ is a nearly finitary and $N$ is a nearly cofinitary matroid on a common ground set of arbitrary size then $\{M, N\}$ has the Intersection property.

Theorem 1.4 is based on the work of Bowler and Carmesin in [6] and incomparable with the theorem of Aharoni and Ziv. Our main result is a strengthening of the latter. We show (slightly more than) that the extra condition about one of the matroids can be completely omitted.

**Theorem 1.5.** If $M$ and $N$ are nearly (co)finitary matroids on a common countable edge set $E$, then $\{M, N\}$ has the Intersection property.

Conjecture 1.3 contains as a special case Menger’s theorem for infinite graphs (see Theorem 1.1 of [7]). The latter was shown by Aharoni and Berger in [8] and filled more than 60 pages (while the countable case needed only 10). It suggests that in the contrast of the finitary-cofinitary “mixed” case the generalization of Theorem 1.5 for arbitrary large matroids might increase the complexity of the proof significantly.

We also prove the following side results which are closely related to the main Theorem 1.5. Let $\text{cond}(M, N)$ stand for the condition “for every $W \subseteq E$ for which there is a base of $M \upharpoonright W$ independent in $N \downharpoonright W$, there exists a base of $N \upharpoonright W$ which independent in $M$". The next theorem
gives a necessary and sufficient condition for the existence of a set which is independent in $M$ and spanning in $N$. Such a set was called a matroid matching by Aharoni and Ziv in the late ’80s but this phrase is used nowadays for a different phenomenon. This reformulation is a standard way to attack the Matroid Intersection Conjecture.

**Theorem 1.6.** Let $M$ and $N$ be matroids on a common countable edge set $E$ such that each of them is either nearly finitary or nearly cofinitary. Then there is a base of $N$ which is independent in $M$ if and only if $\text{cond}(M,N)$.

The nearly finitary-nearly cofinitary mixed case of Theorem 1.6 can be written as a covering problem for two nearly finitary matroids by taking the dual of the nearly cofinitary one. This case was proved in [7] even for arbitrary large matroids.

Looking for an $M$-independent $N$-base can be rephrased as searching for an $N$-base contained in an $M$-base. It seems natural to ask about a characterisation for having a common base.

**Theorem 1.7.** Let $M$ and $N$ be matroids on a common countable edge set $E$ such that each of them is either finitary or cofinitary. Then $M$ and $N$ have a common base if and only if $\text{cond}(M,N) \land \text{cond}(N,M)$.

Maybe surprisingly, the generalization of Theorem 1.7 for arbitrary countable matroids is consistently false (take $U$ and $U^*$ from Theorem 5.1 of [10]). In the contrast of our other results, we do not even know if “finitary or cofinitary” can be relaxed to “nearly finitary or nearly cofinitary” in Theorem 1.7.

The the paper is structured as follows. After introducing a few notation we recall the augmenting path method in Edmonds’ proof of the Matroid Intersection Theorem and remind the so called wave technique developed by Aharoni. In section 4 we show that the restriction of Theorem 1.6 to finitary matroids implies all the theorems we are intended to prove and from that point we focus only on this theorem. In section 5 we investigate feasible sets, i.e., common independent sets $I$ with $\text{cond}(M/I,N/I)$. The intended meaning of “feasible” is being extendable to an $M$-independent base of $N$. The main result is proved in section 6 and its core is Lemma 6.1 which enables as to find a feasible extension of a given feasible set which spans in $N$ a prescribed edge.

### 2 Notation

The minor of $M$ obtained by the contraction of $X$ and deletion of $Y$ is denoted by $M/X - Y$, furthermore, $M \restriction X$ and $M.Y$ are abbreviations for $M - (E \setminus X)$ and $M/(E \setminus Y)$ respectively. We write $\text{span}_M(X)$ for the set of edges spanned by $X \subseteq E$ in matroid $M$. If $e \in \text{span}_M(I) \setminus I$ where $I$ is independent in $M$ then $C_M(e, I)$ is the unique circuit of $M$ through $e$ contained by $I + e$. Let us define $\|X\|$ to be $|X|$ if it is finite and $\infty$ otherwise. Let $B$ and $B_X$ be a base of $M$ and $M \restriction X$ respectively with $B_X \subseteq B$. Then $\|B \setminus B_X\|$ does not depend on the choice of $B$ and $B_X$ and called the corank $c_M(X)$ of $X$ in $M$. An $e \in E$ is an $M$-loop if $\{e\}$ is dependent in $M$.

### 3 Preliminaries

#### 3.1 Augmenting paths

The Matroid Intersection Theorem states (using our terminology) that every pair of finite matroids on the same ground set has the Intersection property. It is a fundamental tool in combinatorial optimization and has a great importance since it has been discovered by Edmonds
Proof. We show by induction on \( n \) that \( C_N(f, I) \) exists and contains \( e \) by definition. For \( k \leq n \), let us denote \( I + x_0 + x_1 + x_2 + \ldots + x_{2k-1} + x_{2k} \) by \( I_k \). Observe that \( I_n = I \triangle P \). We show by induction on \( k \) that \( I_k \) is \( N \)-independent and \( e \in C_N(f, I_k) \). Since \( I + x_0 \) is \( N \)-independent by definition we obtain \( C_N(f, I) = C_N(f, I_0) \). Suppose that we already know the statement for some \( k < n \). We have \( C_N(x_{2k+2}, I_k) = C_N(x_{2k+2}, I) \supseteq x_{2k+1} \) because there is no jumping arc in the augmenting path. It follows that \( I_{k+1} \) is \( N \)-independent. If \( x_{2k+1} \notin C_N(f, I_k) \) then \( C_N(f, I_{k+1}) = C_N(f, I_k + x_{2k+1}) \) and the induction step is done. Suppose that \( x_{2k+1} \in C_N(f, I_k) \). We apply strong circuit elimination with \( C_N(f, I_k) \) and \( C_N(x_{2k+2}, I_k) \) keeping \( e \) and removing \( x_{2k+1} \). The resulting circuit \( C \) must contain at most one element out of \( I_{k+1} \), namely \( f \). Since

In the infinite case these augmenting paths are working in the same way and will play an important role in our proof. However they are not sufficient alone to prove our main result. Indeed, applying augmenting paths recursively yields a sequence of common independent sets where a reasonable limit object cannot be guaranteed in general. In this subsection we introduce our terminology about augmenting paths and point out some properties which were irrelevant for Edmonds’ proof but they are necessary for our arguments.

Let \( N \) and \( M \) be fixed arbitrary matroids on the same ground set \( E \). For a common independent set \( I \), let \( D(I, N, M) \) be a digraph on \( E \) with the following arcs. For \( e \in I \) and \( f \in E \setminus I \), \( ef \) is an arc if \( f \in \text{span}_N(I) \) with \( e \in C_N(f, I) \) and \( fe \) is an arc if \( f \in \text{span}_M(I) \) with \( e \in C_M(f, I) \). Note that \( D(I, M, N) \) is obtained from \( D(I, M, N) \) by reversing all the arcs. An augmenting path with respect to the triple \(( I, N, M ) \) is a \( \subseteq \)-minimal \( P \subseteq E \) of odd size admitting a linear ordering \( P = \{ x_0, \ldots, x_{2n} \} \), for which

1. \( x_0 \in E \setminus I \) and \( I + x_0 \) is independent in \( N \),
2. \( x_{2n} \in E \setminus I \) and \( I + x_{2n} \) is independent in \( M \),
3. \( x_{k+1} \) is an arc of \( D(I, N, M) \) for \( k < 2n \).

Observe that by the minimality of \( P \) each \( x_k \) with \( 0 < k < 2n \) is spanned by \( I \) in both matroids and there cannot be \( k + 1 < \ell \) for which \( x_kx_{\ell} \) is an arc in \( D(I, N, M) \) (i.e., there are no jumping arcs). Therefore the linear order witnessing that \( P \) is an augmenting path for \(( I, N, M ) \) is unique. Clearly augmenting paths for \(( I, N, M ) \) and \(( I, M, N ) \) are the same (the witnessing orderings are the reverse of each other) thus being augmenting path for \( I \) and \( \{ M, N \} \) is well-defined. If there is no augmenting path then the set \( E_M \) of elements reachable from \( \{ e \in E \setminus I : I + e \) is independent in \( N \} \) in \( D(I, N, M) \) together with \( E_N := E \setminus E_M \) and \( I \) witnessing the intersection property of \( \{ N, M \} \). This duality allows us to give the following simple characterization of the common independent sets in Definition 1.1. An element \( I \) of a set family \( \mathcal{F} \) is called strongly maximal in \( \mathcal{F} \) if \( \| J \setminus I \| \leq \| J \setminus I \| \) for every \( J \in \mathcal{F} \). On the one hand, if \( I \) is as in Definition 1.1, then its strong maximality among the common independent sets is ensured by the properties of the bipartition \( E = E_N \sqcup E_M \). On the other hand, an augmenting path \( P \) has always one more element in \( E \setminus I \) than in \( I \) and \( I \triangle P \) is a common independent set. Hence if \( I \) is a strongly maximal common independent set, then there cannot exist any augmenting path which yields to a desired bipartition.

Let an augmenting path \( P = \{ x_0, \ldots, x_{2n} \} \) for \(( I, N, M ) \) be fixed.

Lemma 3.1. If \( ef \) is an arc of \( D(I, N, M) \) for some \( e, f \in E \setminus P \), then \( ef \) remains an arc in \( D(I \triangle P, N, M) \).

Proof. Assume first that \( e \in I \) and \( f \in E \setminus I \). Then \( C_N(f, I) \) exists and contains \( e \) by definition. For \( k \leq n \), let us denote \( I + x_0 + x_1 + x_2 + \ldots + x_{2k-1} + x_{2k} \) by \( I_k \). Observe that \( I_n = I \triangle P \). We show by induction on \( k \) that \( I_k \) is \( N \)-independent and \( e \in C_N(f, I_k) \). Since \( I + x_0 \) is \( N \)-independent by definition we obtain \( C_N(f, I) = C_N(f, I_0) \). Suppose that we already know the statement for some \( k < n \). We have \( C_N(x_{2k+2}, I_k) = C_N(x_{2k+2}, I) \supseteq x_{2k+1} \) because there is no jumping arc in the augmenting path. It follows that \( I_{k+1} \) is \( N \)-independent. If \( x_{2k+1} \notin C_N(f, I_k) \) then \( C_N(f, I_{k+1}) = C_N(f, I_k + x_{2k+1}) \) and the induction step is done. Suppose that \( x_{2k+1} \in C_N(f, I_k) \). We apply strong circuit elimination with \( C_N(f, I_k) \) and \( C_N(x_{2k+2}, I_k) \) keeping \( e \) and removing \( x_{2k+1} \). The resulting circuit \( C \supseteq e \) can have at most one element out of \( I_{k+1} \), namely \( f \). Since
$I_{k+1}$ is $N$-independent it must be actually exactly one element and therefore $C = C_N(f, I_{k+1})$. In the case $e \in E \setminus I$ and $f \in I$ one can justify the statement by a similar induction going the other direction on the augmenting path.

Corollary 3.2. $\text{span}_N(I \triangle P) = \text{span}_N(I + x_0)$ and $\text{span}_M(I \triangle P) = \text{span}_M(I + x_{2n})$.

Proof. By symmetry it is enough to prove the first equality. In the proof of Lemma 3.1, $I_{k+1}$ is obtained from $I_k$ by replacing $x_{2k+1} \in I_k$ by $x_{2k+2}$ for which $x_{2k+1} \in C_N(x_{2k+2}, I_k)$ thus $\text{span}_N(I_k) = \text{span}_N(I_{k+1})$. Since $I_0 = I + x_0$ and $I_n = I \triangle P$ we are done by induction.

Fact 3.3. If $ef$ is an arc of $D(I, N, M)$ and $J \supseteq I$ is a common independent set of $N$ and $M$ with $\{e, f\} \cap J = \{e, f\} \cap I$, then $ef$ remains an arc in $D(J, N, M)$.

3.2 Waves

Waves were introduced by Aharoni to solve problems in infinite matching theory. These techniques turned out to be useful in the proof of the Erdős-Menger conjecture by Aharoni and Berger [8] and in the already mentioned result [5] about the Matroid Intersection Conjecture. Let $M$ and $N$ be arbitrary matroids on the same ground set $E$. An $(M, N)$-wave is a $W \subseteq E$ such that there is a base of $M \setminus W$ which is independent in $N \setminus W$. If $(M, N)$ is clear from the context we write simply wave. A set $W$ of $M$-loops is a wave witnessed by $\emptyset$. We call such a wave trivial.

Proposition 3.4. The union of arbitrary many waves is a wave.

Proof. Suppose that $W_\beta$ is a wave for $\beta < \kappa$ and let $W_{<\alpha} := \bigcup_{\beta < \alpha} W_\beta$ for $\alpha \leq \kappa$. We fix a base $B_\beta \subseteq W_\beta$ of $M \setminus W_\beta$ which is independent in $N \setminus W_\beta$. Let us define $B_{<\alpha}$ by transfinite recursion for $\alpha \leq \kappa$ as follows.

\[
B_{<\alpha} := \begin{cases} 
\emptyset & \text{if } \alpha = 0 \\
B_{<\beta} \cup (B_\beta \setminus W_{<\beta}) & \text{if } \alpha = \beta + 1 \\
\bigcup_{\beta < \alpha} B_{<\beta} & \text{if } \alpha \text{ is limit ordinal.}
\end{cases}
\]

First we show by transfinite induction that $B_{<\alpha}$ is spanning in $M \setminus W_{<\alpha}$. For $\alpha = 0$ it is true since $\emptyset$ is spanned by $\emptyset$. For a limit $\alpha$ it follows directly from the induction hypothesis. If $\alpha = \beta + 1$, then by the choice of $B_\beta$, the set $B_\beta \setminus W_{<\beta}$ spans $W_{<\beta + 1} \setminus W_{<\beta}$ in $M / W_{<\beta}$. Since $W_{<\beta}$ is spanned by $B_{<\beta}$ in $M$ by induction, it follows that $W_{<\beta + 1}$ is spanned by $B_{<\beta + 1}$ in $M$.

The independence of $B_{<\alpha}$ in $N \setminus W_{<\alpha}$ for $\alpha \leq \kappa$ can be reformulated as “$W_{<\alpha} \setminus B_{<\alpha}$ is spanning in $N^+ \setminus W_{<\alpha}$”, which can be proved similarly as above.

By Proposition 3.4 there exists a $\subseteq$-largest $(M, N)$-wave that we denote by $W(M, N)$.

Observation 3.5. If $W_0$ is an $(M, N)$-wave and $W_1$ is an $(M/W_0, N - W_1)$-wave, then $W_0 \cup W_1$ is an $(M, N)$-wave.

Corollary 3.6. For $W = W(M, N)$, the largest $(M/W, N - W)$-wave is $\emptyset$.

Proposition 3.7. If $W$ is a wave and $L \subseteq W$ is a set of $N$-loops, then $W \setminus L$ is a wave. Moreover, if $W$ witnesses the violation of $\text{cond}(M, N)$ then so does $W \setminus L$.

Proof. Take a $J \subseteq W$ witnessing that $W$ is a wave. Since $J$ is a common independent set, we have $J \cap L = \emptyset$ and therefore $J$ is spanning in $M \setminus (W \setminus L)$. On the other hand the edges in $L$ are loops in $N \setminus W$ as well thus $N(W \setminus L) = N \setminus W - L$. In particular $J$ remains independent in $N(W \setminus L)$. It follows that $W \setminus L$ is a wave.
Assume that $B$ is an $M$-independent base of $N,W$. Then $B \cap L = \emptyset$ and hence $B$ is a base of $N,(W \setminus L)$. Therefore $W \setminus L$ cannot be a witness for the violation of \textit{cond}(M,N) unless $W$ is also a witness.

\begin{observation}
If \textit{cond}(M,N) holds and $L$ consists of $M$-loops, then $E \setminus L$ spans $L$ in $N$ (otherwise wave $L$ would violate \textit{cond}(M,N)).
\end{observation}

\begin{corollary}
Assume that \textit{cond}(M,N) holds, $X \subseteq E$ and $L \subseteq X$ consists of $M$-loops. Then any base $B$ of $(N,X) - L$ is a base of $N,X$.
\end{corollary}

\begin{proof}
For a base $B'$ of $N - X$, the set $B \cup B'$ spans $E \setminus L$ and hence by Observation 3.8 spans the whole $E$ as well.
\end{proof}

Let us write $\text{cond}^+(M,N)$ for the condition that \textit{cond}(M,N) holds and $W(M,N)$ is trivial (i.e., consists of $M$-loops).

\begin{lemma}
If $\text{cond}^+(M,N)$ holds and $e \in E$ then $\text{cond}(M/e,N/e)$ holds.
\end{lemma}

\begin{proof}
Let $W$ be an $(M/e,N/e)$-wave. Note that $(N/e).W = N.W$ by definition. Pick an $I \subseteq W$ which is an $N,W$-independent base of $(M/e) \upharpoonright W$. We may assume that $e \in \text{span}_{M}(W)$ and $e$ is not an $M$-loop otherwise $M \upharpoonright W = (M/e) \upharpoonright W$ holds and $\text{cond}(M/e,N/e)$ follows directly from $\text{cond}(M,N)$. Then $I$ is not a base in $M \upharpoonright W$ but “almost”, namely $e_{M,W}(I) = 1$. It is enough to show that $I$ is a base of $N.W$. We apply the augmenting path method with $I, M \upharpoonright W, N.W$. The augmentation cannot be successful because otherwise we would obtain a non-trivial wave with respect to $(M,N)$. Thus we get a bipartition $W = W_0 \cup W_1$ where $I \cap W_0$ spans $W_0$ in $M$ and $I \cap W_1$ spans $W_1$ in $N.W$. Observe that $W_0$ is an $(M,N)$-wave and hence it must be trivial. By applying Corollary 3.9 with $X = W$, $L = W_0$ and $B = I$, we may conclude that $I$ is a base of $N,W$.
\end{proof}

The next theorem allows us to simplify some arguments significantly.

\begin{theorem}[Corollary 4.5 in [10]]
Let $M_i$ be a finitary or cofinitary matroid on the ground set $E$ for $i \in \{0,1\}$. If there are bases $B_i, B'_i$ of $M_i$ such that $B_0 \subseteq B_1$ and $B'_1 \subseteq B'_0$, then $M_0$ and $M_1$ share some base.
\end{theorem}

\begin{corollary}
If $M_i$ is a finitary or cofinitary matroid on the ground set $E$ for $i \in \{0,1\}$, then $\text{cond}(M_0,M_1)$ implies that for every wave $W$ there is a $B \subseteq W$ which is a common base of $M_0 \upharpoonright W$ and $M_1 \upharpoonright W$.
\end{corollary}

\section{Reductions}

The first reduction is standard, it connects Theorems 1.5 and 1.6 even in a more general form.

\begin{proposition}
Let $M$ and $N$ be matroids on the common ground set $E$ such that $\{M,N\}$ has the Intersection property. Then $\text{cond}(M,N)$ is equivalent with the existence of an $M$-independent base of $N$.
\end{proposition}

\begin{proof}
The condition $\text{cond}(M,N)$ is clearly necessary even without any further assumption. To show its sufficiency, let $I = I_M \cup I_N$ and $E = E_M \cup E_N$ as in Definition 1.1. Then $I_M$ is an $N,E_M$-independent base of $M \upharpoonright E_M$. Therefore $E_M$ is a wave and by $\text{cond}(M,N)$ we can pick a $J$ which is a base of $N,E_M$ and independent in $M$. Then $B := I_N \cup J$ is a base of $N$ and it is also independent in $M$ because $I_N$ is independent in $M,E_N$.
\end{proof}
Proposition 4.2. Assume that $\mathcal{C}$ is a class of matroids closed under taking minors such that for every $(M, N) \in \mathcal{C} \times \mathcal{C}$, $\cond(M, N)$ implies the existence of a base of $N$ which is independent in $M$. Then every pair $(M, N)$ from $\mathcal{C}$ has the Intersection property.

Proof. Let $E_M := W(M, N)$ and let $I_M$ be a base of $M \setminus E_M$ which is independent in $N, E_M$, i.e., $I_M$ is a witness that $E_M$ is a wave. Then $W(M/E_M, N - E_M) = \emptyset$ by Corollary 3.6, in particular $\cond(M/E_M, N - E_M)$. Since $\mathcal{C}$ is closed under taking minors we have $M/E_M, N - E_M \in \mathcal{C}$ and therefore by assumption we can find a base $I_N$ of $N - E_M$ which is independent in $M/E_M$. \square

Corollary 4.3. Let $\mathcal{C}$ be a class of matroids which is closed under taking minors. The following are equivalent:

1. For every $M, N \in \mathcal{C}$ with $E(M) = E(N)$, $(M, N)$ has the Intersection property.
2. For every $M, N \in \mathcal{C}$ with $E(M) = E(N)$, there is a base of $N$ which is independent in $M$ if and only if $\cond(M, N)$.

Let us show that “nearly finitary” and “nearly cofinitary” cases of the Matroid Intersection Conjecture 1.2 can be reduced to the “finitary” and “cofinitary” cases even if the matroids are not countable, more precisely:

Proposition 4.4. For $i \in \{0, 1\}$, let $M_i$ be a nearly finitary or nearly cofinitary matroid on $E$ and let $M_i'$ be its finitarization. If $\{M_0, M_1\}$ has the Intersection property then so does $\{M_0', M_1'\}$.

Proof. Let $I'$ be a common independent set of $M_0'$ and $M_1'$ and let $E = E_0 \cup E_1$ be a bipartition such that they are witnessing the Intersection property for $\{M_0, M_1\}$ (see Definition 1.1). By definition, for $I_i := I' \cap E_i$ we have $c_{M_i'|E_i}(I_i') = 0$. Observe that $c_{M_i'|E_i}(X) < \infty$ implies $c_{M_i'|E_i}(X) < \infty$ for arbitrary $X \subseteq E$. Indeed, $c_{M_i'|E_i} \leq c_{M_i'|E_i}$ point-wise if $M_i$ is nearly finitary and $c_{M_i'|E_i}(X)$ is finitely larger than $c_{M_i'|E_i}(X)$ if $M_i$ is nearly cofinitary. By deleting finitely many elements of $I'$, we obtain a common independent set $I$ of $M_0$ and $M_1$. Then for $I_i := I \cap E_i$ we have $c_{M_i'|E_i}(I_i) < \infty$ and hence by the observation above $c_{M_i'|E_i}(I_i) < \infty$ as well.

We use the augmenting path method with $M_0, M_1$ and $I$. If there is no augmenting path then $I$ is as desired and we are done. Otherwise we take an augmenting path $P$. Since $P$ has one more element in $E \setminus I$ than in $I$, for $J := I \cup P$ we have $|J \setminus I| = |I \setminus J| + 1 < \infty$. Then $\sum_{i=0,1} |I_i \setminus J_i| = 1 + \sum_{i=0,1} |I_i \setminus J_i|$ where $J_i := J \cap E_i$. Therefore $\sum_{i=0,1} c_{M_i'|E_i}(J_i) < \sum_{i=0,1} c_{M_i'|E_i}(I_i)$. It follows that after finitely many iterative application of augmenting paths we must obtain the desired strongly maximal common independent set. \square

What we actually prove is the restriction of Theorem 1.6 to finitary matroids. Let us show how the other results follow from it. Note that the existence of an $M$-independent $N$-base can be reformulated as the existence of an $N$-base contained in some $M$-base or alternatively the existence of an $M$-independent set which is spanning in $N$ which is equivalent with the existence of an $N^*$-independent set which is spanning in $M^*$. Furthermore, $\cond(M, N)$ is equivalent with $\cond(N^*, M^*)$ by definition. By combining these, we obtain through a simple dualization argument that the restriction of Theorem 1.6 to cofinitary matroids is equivalent with the restriction to finitary ones. The mixed nearly finitary-nearly cofinitary case of Theorem 1.6 is already known even without size restriction as we already mentioned. The relaxation “nearly” can be added by Proposition 4.4. Then Theorem 1.5 is obtained by applying Proposition 4.2 where class $\mathcal{C}$ consists of the nearly finitary and nearly cofinitary matroids. Finally Theorem 1.7 follows from Theorem 1.6 via Theorem 3.11.
5 Feasible sets

Through this section $M$ and $N$ are matroids on the same ground set $E$ satisfying $\text{cond}(M,N)$. An $I \subseteq E$ is a feasible set (with respect to $(M,N)$) if $I$ is a common independent set of $M$ and $N$ such that $\text{cond}(M/I,N/I)$. Note that $\text{cond}(M,N)$ says that $\emptyset$ is feasible, moreover, if Theorem 1.6 is true, then exactly the feasible sets can be extended to a base of $N$ which is independent in $M$. A feasible set $I$ is called nice if $\text{cond}^+(M/I,N/I)$.

Observation 5.1. If $I_0$ is feasible with respect to $(M,N)$ and $I_1$ is feasible with respect to $(M/I_0,N/I_0)$, then $I_0 \cup I_1$ is feasible with respect to $(M,N)$. If in addition $I_1$ is a nice feasible set with respect to $(M/I_0,N/I_0)$, then so is $I_0 \cup I_1$ for $(M,N)$.

Lemma 5.2. If each of $M$ and $N$ is either finitary or cofinitary, then every feasible set $I$ can be extended to a nice feasible set.

Proof. By Observation 5.1, it is enough to prove for $I = \emptyset$. Let $W := W(M,N)$ and let $B \subseteq W$ be a common base of $M \upharpoonright W$ and $N \upharpoonright W$ (exists by Corollary 3.12). Then $B$ is a common independent set, furthermore, $W' := W(M/B,N/B) = W \setminus B$. Indeed, on the one hand $W \setminus B$ consists of $M/B$-loops because $B$ is spanning in $M \upharpoonright W$ which gives $W' \supseteq W \setminus B$. On the other hand, if $J$ witnesses that $W'$ is an $(M/B,N/B)$-wave then $B \cup J$ ensures that $W \cup W'$ is an $(M,N)$-wave therefore $W' \subseteq W$ which yields to $W' \subseteq W \setminus B$.

Lemma 5.3. If $I$ is a nice feasible set and $P$ is an augmenting path for it, then $I \triangle P$ is feasible.

Proof. Let $e \in E$ be the unique element of $P$ for which $I + e$ is independent in $M$. By Lemma 3.10 we conclude $\text{cond}(M/(I + e),N/(I + e))$. Corollary 3.2 ensures that $I + e$ and $I \triangle P$ span each other in $M$. Moreover, $I \triangle P$ spans $I + e$ in $N$ because either $P = \{e\}$ and hence $I \triangle P = I + e$ or $e \in \text{span}_N(I) \subseteq \text{span}_N(I \triangle P)$. We take a wave $W$ with respect to $(M/(I \triangle P),N/(I \triangle P))$ which does not contain any $N/(I \triangle P)$-loop (see Proposition 3.7.), in particular $W \cap (I + e) = \emptyset$. Then $M/(I + e) | W = M/(I \triangle P) | W$. Using again that $W$ is disjoint from $I + e$ (and from $I \triangle P$ ) we also have $N \cdot W = (N/(I + e)) \cdot W = (N/(I \triangle P)) \cdot W$. It follows that $\text{cond}(M/(I \triangle P),N/(I \triangle P))$ is implied by $\text{cond}(M/(I + e),N/(I + e))$.

6 The proof of the main result

Lemma 6.1. If $M$ and $N$ are finitary matroids on the common countable ground set $E$ such that $\text{cond}(M,N)$ holds then for every $e \in E$, there exists a feasible $I$ with $e \in \text{span}_N(I)$.

We fix an enumeration $\{e_n : n \in \mathbb{N}\}$ of $E$ and well-order $E$ according to it. Theorem 1.6 for finitary matroids follows from Lemma 6.1 by a straightforward recursion. Indeed, we build an $\subseteq$-increasing sequence $(I_n)$ of feasible sets starting with $I_0 := \emptyset$ in such a way that $e_n \in \text{span}_N(I_{n+1})$. If $I_n$ is already defined, then we apply Lemma 6.1 with $(M/I_n,N/I_n)$ and $e_n$ and take the union of the resulting $J$ with $I_n$ to obtain $I_{n+1}$ (see Observation 5.1). Using that $M$ and $N$ are finitary, we conclude that $\bigcup_{n=0}^\infty I_n$ is a base of $N$ which is independent in $M$.

proof of Lemma 6.1. Without loss of generality we may assume that $\text{cond}^+(M,N)$ also holds. Indeed, otherwise we can pick first a nice feasible set $J$ (see Proposition 5.2). Thus we have $\text{cond}^+(M/J,N/J)$ and applying the lemma with $(M/J,N/J)$ and $e$ (unless $e$ is already in $J$ in which case $I := J$ is suitable) gives a $K$ for which $I := J \cup K$ is as desired by Observation 5.1.

It is enough to build a sequence $(I_n)$ of nice feasible sets such that $\text{span}_N(I_n)$ is monotone $\subseteq$-increasing in $n$ and $\bigcup_{n=0}^\infty \text{span}_N(I_n) = E$. We start with $I_0 = \emptyset$ and apply augmenting paths at each step. Corollary 3.2 ensures that $\text{span}_N(I_n)$ is monotone $\subseteq$-increasing. Suppose $I_n$ is
already defined. Assume first that there is no augmenting path for $I_n$. Then $I_n$ is a strongly maximal common independent set witnessed by some bipartition $E = E_M \cup E_N$. Then $E_M$ is a wave and it must be trivial by $\text{cond}^\triangledown(M, N)$. Therefore $I_n \subseteq E_N$ and it is spanning in $N$ by Observation 3.8. Hence $B := I_n$ is a base of $N$ which is independent in $M$.

We can assume that there exists some augmenting path for $I_n$. Consider the smallest $e \in E \setminus \text{span}_N(I_n)$ for which there is an augmenting path $P_n$ such that $e \in \text{span}_N(I_n \Delta P_n)$. Lemma 5.3 ensures that $I_n \Delta P_n$ is feasible. We obtain $I_{n+1}$ by extending $I_n \Delta P_n$ to a nice feasible set applying Corollary 5.2. The recursion is done.

Suppose for a contradiction that $X := E \setminus \bigcup_{n=0}^{\infty} \text{span}_N(I_n) \neq \emptyset$. Observation 6.2. Since $N$ is finitary, Observation 3.8 ensures that there is an edge in $X$ which is not an $M$-loop.

For $x \in X$, let $E(x, n)$ be the set of edges that are reachable from $x$ in $D_n := D(I_n, N, M)$ by a directed path. Let $n_x$ be the smallest natural number such that whenever an $y \in E(x, n) \setminus X$ is smaller than $x$ then $y \in \text{span}_N(I_{n_x})$.

**Claim 6.3.** For every $x \in X$ and $\ell \geq m \geq n_x$,

1. $I_m \cap E(x, m) = I_\ell \cap E(x, m)$,
2. for every $e \in E(x, m) \setminus I_m$ we have $C_M(e, I_\ell) = C_M(e, I_m) \subseteq E(x, m)$,
3. $D_m[E(x, m)]$ is a subdigraph of $D_\ell[E(x, m)]$,
4. $E(x, m) \subseteq E(x, \ell)$.

**Proof.** Suppose that there is an $n \geq n_x$ such that we know already the statement whenever $m, \ell \leq n$. For the induction step it is enough to show that the claim holds for $m = n$ and $\ell = n + 1$.

**Proposition 6.4.** $P_n$ avoids $E(x, n)$.

**Proof.** A meeting of $P_n$ and $E(x, n)$ would show that there is also an augmenting path in $D_n$ starting at $x$ which is impossible since $x \in X$ and $n \geq n_x$.

**Corollary 6.5.** $I_n \cap E(x, n) = (I_n \Delta P_n) \cap E(x, n)$.

**Proposition 6.6.** $(I_n \Delta P_n) \cap E(x, n) = I_{n+1} \cap E(x, n)$.

**Proof.** The edges $I_{n+1} \setminus (I_n \Delta P_n)$ are independent in $M/(I_n \Delta P_n)$ but by the definition of $D_n$ for every $e \in E(x, n) \setminus I_n$ we have $E(x, n) \supseteq C_M(e, I_n) = C_M(e, I_n \Delta P_n)$ witnessing that $e$ is an $M/(I_n \Delta P_n)$-loop.

**Corollary 6.7.** $I_n \cap E(x, n) = I_{n+1} \cap E(x, n)$ and for every $e \in E(x, n) \setminus I_n$ we have $C_M(e, I_n) = C_M(e, I_{n+1}) \subseteq E(x, n)$.

Finally applying Lemma 3.1 and Fact 3.3 consecutively shows that $D_n[E(x, n)]$ is a subdigraph of $D_{n+1}[E(x, n)]$ from which $E(x, n) \subseteq E(x, n + 1)$ follows by definition.

Beyond Claim 6.3 we need the following technical statement.

**Proposition 6.8.** Let $I$ be an independent set in some fixed finitary matroid. Suppose that there is a circuit $C \subseteq \text{span}(I)$ with $e \in I \cap C$. Then there is an $f \in C \setminus I$ with $e \in C(f, I)$. 

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Proof. We apply induction on \(|C \setminus I|\). If \(C \setminus I\) is a singleton, then its only element is suitable for \(f\) since \(C(f,I) = C\). Suppose that \(|C \setminus I| \geq 2\) and pick a \(g \in C \setminus I\). By applying strong circuit elimination with \(C\) and \(C(g,I)\) keeping \(e\) and removing \(g\), for the resulting \(C'\) we have \(C' \setminus I \subseteq C \setminus I\) and we are done by induction. 

By Claim 6.3 we have \(E(x,n) \subseteq E(x,n+1)\) for \(n \geq n_x\). Let \(W := \bigcup_{x \in X} \bigcup_{n=n_x}^\infty E(x,n)\). Note that Claim 6.3 guarantees that for each \(e \in W\) either \(\{n \in \mathbb{N} : e \in I_n\}\) or its complement is finite. Let \(J\) consists of the latter type of edges of \(W\), i.e., that are elements of \(I_n\) for every large enough \(n\). Since \(M\) and \(N\) are finitary, \(J\) is a common independent set. By Claim 6.3, \(W \subseteq \text{span}_M(J)\). We show that \(J\) is independent in \(N.W\). Suppose for a contradiction that there exists an \(N\)-circuit \(C\) that meets \(J\) and avoids \(W \setminus J\). Since \(J\) is \(N\)-independent and \(C\) does not meet \(W \setminus J\), we have \(C \setminus J = C \setminus W \neq \emptyset\). Let us pick some \(e \in C \cap J\). For every large enough \(n\) we have \(C \setminus J = C \setminus I_n\) and \(I_n\) spans \(C\) in \(N\) because \(X \subseteq W \setminus J\). Applying Proposition 6.8 with \(I_n, N, C\) and \(e\) tells that \(e \in C_N(f,I_n)\) for some \(f \in C \setminus W\) whenever \(n\) is large enough. We take an \(x \in X\) and \(n \geq n_x\) such that \(e \in E(x,n)\) and \(e \in C_N(f,I_n)\) for some \(f \in C \setminus W\). Then \(f \in E(x,n) \subseteq W\) contradicts \(f \in C \setminus W\). Thus \(J\) is indeed independent in \(N.W\) and hence \(W\) is a wave. Observation 6.2 guarantees that \(W\) is not a trivial wave which contradicts \(\text{cond}^+(M,N)\).

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