LOCALIZATION OF EIGENVALUES OF DOUBLY CYCLIC MATRICES

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ABSTRACT. For a family of doubly cyclic matrices of the form (1.1), a maximum for the number of eigenvalues in the left half-plane is attained by $X \in \{1\}$, with $\alpha, \beta \in \{1\}$. This confirms a conjecture of C. Johnson, Z. Price, and I. Spitkovsky.

Moreover, the complete range of possibilities for the number of eigenvalues in the left half-plane is demonstrated: if $\alpha < \beta$, then any odd number between 1 and the maximum, inclusive, is attainable.

1. Introduction. For $n \in \mathbb{N}$, $n \geq 2$, we consider matrices $X \in M_n(\mathbb{R})$ of a particular form. Defining $\mathbb{R}_{>0} = (0, \infty)$, and fixing vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}_{>0}^n$, we study the matrix

$$
X = \begin{pmatrix}
  a_1 & -b_1 & 0 & \cdots & 0 \\
  0 & a_2 & -b_2 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & a_{n-1} & -b_{n-1} & 0 \\
  -b_n & 0 & \cdots & 0 & a_n
\end{pmatrix}.
$$

Since

$$
\det X = \alpha^n - \beta^n, \quad \alpha \equiv \left( \prod_{k=1}^{n} a_k \right)^{1/n}, \quad \beta \equiv \left( \prod_{k=1}^{n} b_k \right)^{1/n}
$$

the geometric means of the $a_k$’s and $b_k$’s play a key role. We let $DC(\alpha, \beta)$ denote the set of matrices of the form (1.1) with given geometric mean $\alpha$ for the $a_k$’s and $\beta$ for the $b_k$’s.

Inspired by the occurrence of such matrices in the previous paper [JJZ12], C. Johnson, Z. Price, and I. Spitkovsky, in [JPS13], consider the number of eigenvalues of such a matrix in the left half-plane. In particular, they note that for several cases (when $n \leq 4$, or $\cos \left( \frac{2\pi n}{n} \right) < \frac{\alpha}{\beta} < 1$), the number of eigenvalues in the left-half-plane is the same as that for $\alpha I - \beta \Sigma \ast$. Here, $I$ is the identity $n \times n$ matrix and

$$
\Sigma \ast = \begin{pmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & 1 \\
  1 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

is the relevant permutation $n \times n$ matrix.
Numerical evidence presented in [JPS13] suggests that in general, the number of eigenvalues in the left half-plane for any matrix in $DC(\alpha, \beta)$ is bounded above by the corresponding value for $\alpha I - \beta \Sigma_*$. In this paper, we prove this conjecture.

**Theorem 1.1.** Fix $n \in \mathbb{N}$, $n \geq 2$. Fix $\alpha, \beta \in \mathbb{R}_{>0}$. Let $X \in DC(\alpha, \beta)$. Then the number of eigenvalues of $X$ with negative real part does not exceed the number of eigenvalues of $\alpha I - \beta \Sigma_*$ with negative real part, and setting $X = \alpha I - \beta \Sigma_* \in DC(\alpha, \beta)$ allows us to attain this upper bound as a maximum among all elements of $DC(\alpha, \beta)$.

See also Remark 2.2 for an extension of the claim of this theorem.

Conjugating with nonsingular matrices preserves the spectrum. We conjugate $X$ with the diagonal matrix

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{\beta}{b_1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$  

(1.4)

Notice that $1 = \prod_{k=1}^{n} \frac{\beta}{b_k}$. Then with $c_k = \frac{a_k}{\beta}, 1 \leq k \leq n$,

$$Q^{-1}XQ = \begin{pmatrix} a_1 & -\beta & 0 & \cdots & 0 \\ 0 & a_2 & -\beta & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & a_{n-1} & -\beta \\ -\beta & 0 & \cdots & 0 & a_n \end{pmatrix} \beta \begin{pmatrix} c_1 & -1 & 0 & \cdots & 0 \\ 0 & c_2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & c_{n-1} & -1 \\ -1 & 0 & \cdots & 0 & c_n \end{pmatrix} = \beta \tilde{X}.$$  

(1.5)

The factor $\beta > 0$ rescales the spectrum, but does not change the signs of the real parts of the points of the spectrum; therefore, we study $\tilde{X}$. We have simplified the parameter scheme to an $n$-parameter system $c = (c_1, c_2, \ldots, c_n) \in (\mathbb{R}_{>0})^n$, with geometric mean

$$\gamma = \left( \prod_{k=1}^{n} c_k \right)^{1/n} = \left( \prod_{k=1}^{n} \frac{a_k}{\beta} \right)^{1/n} = \frac{\alpha}{\beta}.$$  

(1.6)
By cofactor expansion down the first row,

\[
\det(\tilde{X} - \lambda I) = (c_1 - \lambda) \det \begin{pmatrix}
  c_2 - \lambda & -1 & \cdots & 0 \\
  (c_3 - \lambda) & -1 & & \\
  0 & \cdots & (c_{n-1} - \lambda) & -1 \\
  0 & \cdots & 0 & (c_n - \lambda)
\end{pmatrix} \\
+ (-1)^{n-1} \det \begin{pmatrix}
  -1 & 0 & \cdots & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & \ddots & 0 \\
  0 & \cdots & 0 & -1
\end{pmatrix}
\]

(1.7)

\[
= \prod_{k=1}^{n} (c_k - \lambda) + (-1)^n(-1)^{n-1}
= \prod_{k=1}^{n} (c_k - \lambda) - 1.
\]

As a matter of technical convenience for the later proof, we rewrite \(-\lambda\) as \(z\). Thus, we analyze the roots of an algebraic equation

(1.8) \[ P(z) = 1, \quad \text{where} \quad P(z) \equiv P(z; c) = \prod_{k=1}^{n} (c_k + z). \]

We define

(1.9a) \[ E^- \equiv \{ \xi \in \mathbb{C} : \Re \xi < 0 \}, \]
(1.9b) \[ E^0 \equiv \{ \xi \in \mathbb{C} : \Re \xi = 0 \}, \]
(1.9c) \[ E^+ \equiv \{ \xi \in \mathbb{C} : \Re \xi > 0 \}, \]
and

(1.10) \[ E \equiv E^+ = \{ \xi \in \mathbb{C} : \Re \xi \geq 0 \}. \]

Let \(\nu_-(c)\) (respectively \(\nu_0(c), \nu_+(c), \tilde{\nu}(c)\)) denote the number of solutions to (1.8) in \(E^-\) (respectively \(E^0, E^+, E\)), counted with multiplicity.

If in the above construction, \(X = X_+ \equiv \alpha I - \beta \Sigma_+\), i.e.,

(1.11) \[ X_+ = \begin{pmatrix}
  \alpha & -\beta & 0 & \cdots & 0 \\
  0 & \alpha & -\beta & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & \cdots & \alpha & -\beta \\
  -\beta & 0 & \cdots & 0 & \alpha
\end{pmatrix}, \]

then \(\tilde{X} = \gamma I - \Sigma_+\), and the characteristic polynomial of \(\tilde{X}\) is

(1.12) \[ \det(z + \tilde{X}) = (z + \gamma)^n - 1, \quad \gamma = \frac{\alpha}{\beta}. \]

Letting \(c^* \equiv (\gamma, \gamma, \ldots, \gamma)\), we have the algebraic equation

(1.13) \[ P^*(z) = 1, \quad \text{where} \quad P^*(z) \equiv P(z; c^*) = (\gamma + z)^n. \]

The set of solutions to (1.13) is

(1.14) \[ \{-\gamma + \omega^k : 0 \leq k < n\}, \quad \text{where} \quad \omega = \exp \left( \frac{2\pi i}{n} \right), \]
and therefore we have (for $\gamma > 0$)

$$\nu_+ (c^*) = \# \left\{ k : 0 \leq k < n : \cos \left( \frac{2\pi k}{n} \right) > \gamma \right\},$$

(1.15a)

$$\nu (c) = \# \left\{ k : 0 \leq k < n : \cos \left( \frac{2\pi k}{n} \right) \geq \gamma \right\}.$$ 

(1.15b)

Thus, we have reduced the main theorem to the following proposition, as we are interested in counting the number of roots of (1.8) and (1.13) with positive real part.

**Theorem 1.2.** Fix $n \in \mathbb{N}$. If $c = (c_1, c_2, \ldots, c_n) \in (\mathbb{R}_{>0})^n$, then

$$\nu_+ (c) \leq \nu_+ (c^*), \quad \nu (c) \leq \nu (c^*).$$

(1.16)

In addition, we may describe the range of roots of (1.8) in the open or closed right-half-plane. To state the results succinctly, note by (1.15) that the number of solutions to (1.13) is either 0 or odd, by the evenness of the cosine function, and is nonzero if $\gamma < 1 = \cos(0)$; therefore, if $\gamma < 1$, we may write for some $\kappa_+, \kappa \in \mathbb{N}$ that

$$\nu_+ (c^*) = 2\kappa_+ + 1$$

(1.17a)

$$\nu (c) = 2\kappa + 1.$$ 

(1.17b)

**Theorem 1.3.** Fix $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}^+$. Then the range of $\nu_+ (c)$ among the set of $c \in \mathbb{R}^n_{>0}$ with geometric mean $\gamma$ is:

$$\begin{cases} 
\{0\}, & \text{if } \gamma \geq 1 \\
2\{0, 1, \ldots, \kappa_+\} + 1, & \text{if } \gamma < 1.
\end{cases}$$

(1.18)

Similarly, the range of $\nu (c)$ among these vectors is

$$\begin{cases} 
\{0\}, & \text{if } \gamma > 1 \\
\{1\}, & \text{if } \gamma = 1 \\
2\{0, 1, \ldots, \kappa\} + 1, & \text{if } \gamma < 1.
\end{cases}$$

(1.19)

**Theorem 1.4.** Fix $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}^+$. Then:

(a) If $\alpha > \beta$, then no $X \in DC(\alpha, \beta)$ has an eigenvalue in the closed left half-plane.

(b) If $\alpha = \beta$, then for every $X \in DC(\alpha, \beta)$, 0 is the only eigenvalue in the closed left half-plane.

(c) If $\alpha < \beta$, then $X \in DC(\alpha, \beta)$ has an odd number of eigenvalues in the open left half-plane, but no more than that of $\alpha I - \beta \Sigma_\alpha$. Moreover, for every such odd number $k$, some $X \in DC(\alpha, \beta)$ has exactly $k$ eigenvalues in the open left half-plane. Similarly for the closed left half-plane.

The core of the paper (Sections 2 to 5) is devoted to proving Theorem 1.2. First, we observe that the roots of $P(z; c) = 1$ in the right-half-plane are simple, and are bounded away from $\infty$ and 0 with bounds only depending on $\max_j c_j, \min_j c_j$, and $\gamma$; these statements are recorded in Section 2. Moreover, their number is odd. This allows us to show that the zeroes in the region of interest vary smoothly as $c$ varies, indeed to use the Implicit Function Theorem (our variation is described in Appendix B). We therefore wish to find a path $c(t)$ starting from any $c_0$ to $c^*$, along which $\nu_+(c(t))$ is increasing. We are still wondering if a “direct” path would work, but we choose to build it step-by-step, steadily bringing the most extreme elements to meet with the next most extreme. Our rephrasing
in terms of the number of distinct elements of \( c \), and creating a path from any given \( c_0 \) to one with less extreme \( \max_j c_j \) and \( \min_j c_j \) (and fewer distinct elements) is related in Section 3. In Section 4, we study the effects on the roots with positive real part, showing that they remain in the right half-plane. In the beginning of Section 5, we put the partial paths together to build the desired path from \( c_0 \) to \( c^* \) along which the number of roots of (1.8) with positive (or 0) real part is increasing. Appendix C clarifies a positivity condition used in this work.

The end of Section 5, and Sections 6, 7, present more details about the precise behavior of the zero-counting functions, and complete the proofs of Theorems 1.3 and 1.4.

The remaining sections tighten the bounds on the range of permissible zeroes in the right-half plane, giving dimension-invariant bounds. Section 8 gives the details, and Appendix A clarifies a bound used in this work.

2. Technical Preliminaries. For \( z = x + iy \), (1.8) implies

\[
\prod_{k=1}^{n} (z + c_k) = 1, \tag{2.1a}
\]

\[
\left| \prod_{k=1}^{n} (z + c_k) \right|^2 = \prod_{k=1}^{n} ((x + c_k)^2 + y^2) = 1. \tag{2.1b}
\]

If \( z \in E \) in (1.10), \( z \neq 0 \), it follows from (2.1b) that

\[
1 > \prod_{k=1}^{n} c_k^2 = \gamma^{2n}; \tag{2.2}
\]

if \( z = 0 \),

\[
1 = \prod_{k=1}^{n} c_k^2 = \gamma^{2n}. \tag{2.3}
\]

Therefore, if \( \gamma > 1 \), then \( \nu(c) = 0 \), and with \( \gamma = 1 \), the point \( z^* = 0 \) is the only solution for (1.8) and (1.13) in \( E \). In both cases,

\[
\nu_+(c) \leq \nu_+(c^*), \quad \nu(c) \leq \nu(c^*), \tag{2.4}
\]

i.e., (1.16) holds.

In the sequel, we therefore analyze only the case

\[
0 < \gamma < 1. \tag{2.5}
\]

In this case, \( \nu_+(c) \geq 1 \), since all coefficients are real, and

\[
P(0, c) = \gamma^n < 1 < \prod_{k=1}^{n} (1 + c_k) = P(1, c).
\]

We further note that the function is strictly increasing on \([0, \infty)\), so this root is simple, and unique on \([0, \infty)\).

If \( P(w; c) = 1 \), then by conjugation,

\[
P(\overline{w}; c) = 1,
\]

and so \( \overline{w} \) is also a root of (1.8) (of the same multiplicity). Similarly, if we consider

\[
h(y) = |P(iy, c)|^2 = \prod_{k=1}^{n} (y^2 + c_k^2),
\]
we have that
\[ h(0) = \gamma^{2n} \leq 1 < \prod_{k=1}^{n} (1 + c_k^2) = h(1), \]
and that \( h \) is even, and increasing on \([0, \infty)\), so there exists a unique solution on the positive imaginary axis to (2.1b) (i.e., \( x = 0 \)); call it \( z = iY(c) \). Therefore, we have the following.

**Lemma 2.1.** Fix \( c \in (\mathbb{R}_{>0})^n \) with geometric mean \( \gamma < 1 \). Then \( \nu_+(c) \) and \( \nu_-(c) \) are both odd and positive. \( P(z; c) \) has exactly one root in \((0, 1)\), and the others are not real.

Also, \( |P(z; c)|^2 = 1 \), or \( |P(z; c)| = 1 \), has a unique solution on the positive imaginary axis.

For \( \gamma, D_*, D^* \), satisfying
\[ 0 < D_* \leq \gamma \leq D^* < \infty, \]
we define
\[ C(\gamma; D_*, D^*) = \left\{ c = (c_1, c_2, \ldots, c_n) \in [D_*, D^*]^n : \prod_{k=1}^{n} c_k = \gamma^n \right\}. \]

**Remark 2.2.** In the analysis of polynomials \( P(z, c) \) and related algebraic equations, without loss of generality, we may suppose that the \( c_k \) are in order, i.e.
\[ 0 < D_* \leq c_1 \leq c_2 \leq \ldots \leq c_m \leq D^* < \infty. \]
It will be useful in the technical analysis which follows. But it helps to understand that in Theorem 1.1, we can talk about any \( \Sigma \), not just \( \Sigma_* \), which corresponds to an \( n \)-cycle permutation \( \kappa \).

Indeed, for \( A = \text{diag}(a_1, \ldots, a_n) \),
\[ \det [(A - \lambda I) - \beta \Sigma] = \prod_{k=1}^{n} (a_k - \lambda) + (-1)^{\text{sign } \kappa} (-\beta)^n, \]
and
\[ \text{sign } \kappa = n - 1; \]
see, e.g., [DF04, Section 3.5, p. 110], either Proposition 25 or the line +13.

For \( c \in C(\gamma; D_*, D^*) \), we may uniformly establish a root-free zone for \( P(z; c) \) in a small disk centered at the origin.

**Lemma 2.3.** Fix \( 0 < \gamma < 1 \) and two positive real numbers \( D_* \) and \( D^* \), satisfying (2.6). Then for all \( c \in C(\gamma; D_*, D^*) \), there are no roots to (1.8) in the closed disk \( \{ \xi \in \mathbb{C} : |\xi| \leq d \} \), where
\[ d \equiv d(\gamma, D^*) = (1 - \gamma^n) (1 + D^*)^{-n}. \]

**Proof.** We denote by \( \sigma_j(c) = \sigma_j((c_1, \ldots, c_n)) \) the \( j \)th elementary symmetric polynomial evaluated at \((c_1, c_2, \ldots, c_n)\). If \( z \) is a root of (1.8), then \( z \neq 0 \) because \( \gamma < 1 \), as \( \prod_{k=1}^{n} c_k = \)
\( \gamma^n < 1 \). Then by (2.1a), and Lemma 2.4
\[
1 = \left| \prod_{k=1}^{n} (z + c_k) \right| \leq \prod_{k=1}^{n} (|z| + c_k)
\]
\[
= \sum_{k=0}^{n} \sigma_{n-k}(c)|z|^k
\]
\[
= \sigma_n(c) + \sum_{k=1}^{n} \sigma_{n-k}(c)|z|^k
\]
\[
= \gamma^n + |z| \sum_{k=1}^{n} \sigma_{n-k}(c)|z|^{k-1}
\]
\[
< \gamma^n + |z| \left[ (D^n)^n + \sum_{k=1}^{n} \sigma_{n-k}(D^n, D^n, \ldots, D^n) \right]
\]
\[
= \gamma^n + |z|(1 + D^n)^n,
\]
so
\[
(2.13) \quad \frac{1 - \gamma^n}{(1 + D^n)^n} < |z|.
\]
\[
\square
\]
For \( D > 0 \), define the closed half-plane
\[
(2.14) \quad E_{\text{ext}} = E_{\text{ext}}(D) \equiv \left\{ \xi \in \mathbb{C} : \text{Re} \xi \geq -\frac{D}{3} \right\}
\]
For \( c \in C(\gamma; D_*, D^*) \), we can also bound from above the size of the roots of (1.8) in \( E_{\text{ext}} \).

**Lemma 2.4.** Fix \( n \in \mathbb{N} \) and two positive real numbers \( D_* \) and \( D^* \), satisfying (2.6). Then for all \( c \in C(\gamma; D_*, D^*) \), all roots of (1.8) in \( E_{\text{ext}}(D_*) \) are in the disk \( \{ \xi \in \mathbb{C} : |\xi| < 1 \} \).

**Proof.** If \( z = x + iy \) is a root of (1.8) with \( x \geq 0, y \) real, then by (2.1b),
\[
1 = \left| \prod_{k=1}^{n} (z + c_k) \right|^2 = \prod_{k=1}^{n} ((x + c_k)^2 + y^2)
\]
\[
= \prod_{k=1}^{n} ((x^2 + y^2) + c_k(2x + c_k))
\]
\[
> \prod_{k=1}^{n} (|z|^2 + c_k \left[ -\frac{2D_*}{3} + D_* \right]) > |z|^{2n},
\]
so \( |z| < 1 \). (Indeed, \( x > -\frac{D_*}{2} \) is all that is required here). \( \square \)

(In Section 8 and Appendix A we give better estimates, but Lemma 2.4 is good enough for the proof of our main theorem.)

We define \( \text{Ann}(r, R) = \{ z \in \mathbb{C} : r < |z| < R \} \), the annulus centered at the origin with radii \( r \) and \( R \). We summarize Lemmas 2.4 and 2.3 as follows.

**Corollary 2.5.** Fix \( 0 < \gamma < 1 \), and \( D_*, D^* \) positive reals with \( D_* \leq \gamma \leq D^* \). Then for any \( c \in C(\gamma; D_*, D^*) \), all zeroes of (1.8) in \( E_{\text{ext}}(D_*) \) are also in \( \text{Ann}(d, 1) \), \( d \in (2.11) \).
In the sequel, it is sometimes more convenient to use a bounding box, rather than a bounding semiannulus, for the permissible range of the zeros with positive real part.

**Corollary 2.6.** Fix $0 < \gamma < 1$ and two positive real numbers $D_*$ and $D^*$, satisfying (2.6). Then for all $c \in C(\gamma; D_*, D^*)$, if $z \in E_{\text{ext}}(D_*)$ satisfied $P(z, c) = 1$, then $w$ is inside the box

$$Box = \left\{ \xi \in \mathbb{C} : -\frac{D_*}{3} < \Re \xi < 1, |\Im \xi| < 1 \right\} \setminus \left\{ \xi \in \mathbb{C} : |\Re \xi| \leq \delta, |\Im \xi| \leq \delta \right\},$$

where

$$\delta = \delta(\gamma, D^*) \equiv \frac{2}{3} = \frac{2}{3}(1 - \gamma^n)(1 + D^n)^{-n} \leq \frac{2}{3}.$$

**Proof.** If $z = x + iy$ with $x > -\frac{D_*}{3}$, $y$ real, then by Lemma \ref{lemma2.5} $|z| < 1$, so $x < 1$ and $|y| < 1$. If $|x| \leq \delta$ and $|y| \leq \delta$, $|z|^2 = x^2 + y^2 \leq 2\delta^2 = 2 \cdot \left( \frac{2}{3} d \right)^2 = \frac{8}{9} d^2 < d^2$, so $|z| < d$ and we may apply Lemma \ref{lemma2.3}. \hfill \Box

The containing regions for the roots of $P(z; c) = 1$ are shown in Figure 1.

We will frequently use — even without a reference — the following.

**Remark 2.7.** If $z \in \mathbb{C}$, $z \neq 0$, then $\Re \frac{1}{z} > 0$ (respectively, $\Re \frac{1}{z} < 0$) if and only if $\Re z > 0$ (respectively, $\Re z < 0$).

**Proof.** It immediately follows from the identity

$$\Re \frac{1}{z} = \Re \frac{\overline{z}}{|z|^2} = \frac{1}{|z|^2} \Re \overline{z} = \frac{x}{|z|^2}, \quad z \neq 0, \quad z = x + iy, \quad x, y \text{ real}.$$
Lemma 2.8. Fix $0 < \gamma$, and let $D_*, D^* \in \mathbb{R}_{>0}$ satisfy $D_* \leq \gamma \leq D^*$. Then for all $c \in C(\gamma; D_*, D^*)$, if $w$ is a root of (1.8) and $w \in E_{ext}(D_*)$, then $w$ is a simple root of (1.8).

Proof. Let $w$ be a root of (1.8) in $E_{ext}(D_*)$. Then for all $k$, $1 \leq k \leq n$,

$$\text{Re}(w + c_k) \geq \text{Re}(w + D_*) \geq \frac{2D_*}{3},$$

and by (1.8), (2.18) and Lemma 2.4,

$$\text{Re}\left(\frac{dP}{dz}\bigg|_{z=w}\right) = \frac{n}{\sum_{k=1}^{n} \left|w + c_k\right|^2} \geq \frac{n}{(1 + D^*)^2} > 0$$

so $w$ is a simple root of (1.8). □

3. Restructuring of the sequence $c$. Under (2.8), we will not change the orders of the elements in the vectors, and if $c_k = c_{k+1}$ are identical, we will never do any change to make them nonequal. Therefore, it now behooves us only pay attention to the distinct entries in $(c_1, c_2, \ldots, c_n)$. We choose to write the distinct entries in $(c_1, c_2, \ldots, c_n)$ as $(d_0, \ldots, d_q)$,

$$0 < D_* \leq d_0 < d_1 < \cdots < d_q \leq D^*,$$

so that the number of distinct entries is $1 + q$; i.e., the number of strict inequalities in (2.8), or the number of gaps in (3.1) is $q$. Hereafter, we call $q$ the diversity of the multiset. We therefore reformulate $c \in \mathbb{C}^n$ as a multiset with $q + 1$ distinct entries and total weight $n$.

For a given $c$, let

$$K_j \equiv \{k \in \pi : c_k = d_j\}, \text{ where } \pi = \{1, 2, \ldots, n\}$$

$$\left(\text{so } \bigcup_{j=0}^{q} K_j = \pi, \quad K_j \cap K_{\ell} = \emptyset \text{ if } j \neq \ell\right),$$

and let

$$m_j = |K_j|, \quad 0 \leq j \leq q, \left(\text{so } \sum_{j=0}^{q} m_j = n\right).$$

In short, we present $c$ as $(d, m; q) = \{(d_j, m_j)\}_{j=0}^{q}$. In this language, we have

$$P(z; c) = \prod_{k=1}^{n} (1 + c_k) = \prod_{j=0}^{q} (1 + d_j)^{m_j},$$

$$\gamma^n = \prod_{k=1}^{n} c_j = \prod_{j=0}^{q} d_j^{m_j},$$
and \( c^* \) is presented by \( d^* = \{ \gamma, n; 0 \} \). The family \( C(\gamma; D_*, D^*) \) is presented by the family of multisets

\[
D(\gamma; D_*, D^*) = \bigcup_{q=0}^{n-1} \left\{ (d_j, m_j)_{j=0}^q \in ([D_*, D^*] \times (N \cup \{0\}))^{q+1} : \prod_{j=0}^q d_j^{m_j} = \gamma^n, \sum_{j=0}^q m_j = n \right\}
\]

(3.6)

We now construct the basic elements of our path connecting \( \{d, m; q\} \) to \( d^* \), or \( c \) to \( c^* \). Our goal is to reduce the diversity \( q \), i.e., the number of gaps, and maintain the geometric mean.

If \( q \geq 2 \), we put

\[
\tau^*_n = \frac{1}{m_q} \log \left( \frac{d_1}{d_0} \right)
\]

(3.7a)

\[
\tau^* = \frac{1}{m_0} \log \left( \frac{d_q}{d_{q-1}} \right)
\]

(3.7b)

\[
\tau = \min\{ \tau_n, \tau^* \}
\]

(3.7c)

We have three cases:

(I) \( \tau = \tau_n < \tau^* \),

(II) \( \tau_n > \tau^* = \tau \),

(III) \( \tau_n = \tau^* = \tau \).

We now construct a path for the sequence (2.8), or for the multiset (3.1), parameterized by \( t \) in \([0, \tau]\) and \([0, \tau] \). We define on \([0, \tau]\)

\[
d_j(t) = \begin{cases} 
  d_0 \exp(m_q t), & j = 0, \\
  d_j, & 0 < j < q \\
  d_q \exp(-m_0 t), & j = q 
\end{cases}
\]

(3.8)

We note that the multiplicities are unchanged on \([0, \tau]\): for \( 0 < j < q \), the \( d_j \) do not move, and for \( 0 < t < \tau \leq \tau_n \), by (3.7a),

\[
d_0 < d_0(t) = d_0 \exp(m_q t) < d_0 \exp(m_q \tau_n) = d_1;
\]

similarly, \( d_q(t) > d_q(t) \) for \( t \) in \([0, \tau]\). The geometric mean is preserved:

\[
\prod_{j=0}^q d_j(t)^{m_j} = (d_0 \exp(m_q t))^{m_0} \cdot \left( \prod_{j=1}^{q-1} d_j^{m_j} \right) \cdot (d_q \exp(-m_0 t))^{m_q}
\]

(3.9)

\[
= \left( \prod_{j=0}^q d_j^{m_j} \right) \cdot \exp(m_q m_0 t - m_0 t m_q) = \prod_{j=0}^q d_j^{m_j} = \gamma^n.
\]

Since \( d_0(t) \) is increasing and \( d_q(t) \) is decreasing, we have that if \( \{d(0), m; q\} \) is in \( D(\gamma; D_*, D^*) \), then so is \( \{d(t), m; q\} \) for all \( t \) in \((0, \tau]\). The cases differ in the appropriate extension when \( t = \tau \).
(I) In this case, $\tau = \tau_s$, so $\lim_{t \to -\tau^*} d_{0}(t) = d_1$, but $\tau \neq \tau^*$, so $\lim_{t \to \tau^*} d_{q}(t) = d_q \exp(-m_0 \tau_s) > d_{q-1}$. Therefore, the end multiset \{(d'_{j}, m'_j)\}_{j=0}^{q'} at \(t = \tau\) is defined with\[ q' = q - 1, \tag{3.10} \]
m'_0 = m_0 + m_1, \quad m'_j = m_{j=1}, 1 \leq j \leq q',
\]
\[ d'_0 = d_1 = d_0 \exp(m_q \tau), \quad d'_j = d_{j+1} for 1 \leq j < q', \quad d'_q = d_{q-1} = d_q \exp(-m_0 \tau). \]

In short, the $0$th and $1$st points of the multiset (3.1) have coalesced. Again, the geometric mean is $\gamma$, by continuity, and the end multiset belongs to $D(\gamma; D_*, D^*)$.

(II) In this case, $\tau = \tau^*$ so $\lim_{t \to -\tau^*} d_{q}(t) = d_q \exp(m_q \tau^*) = d_{q-1}$, but $\tau \neq \tau_s$, so $\lim_{t \to \tau^*} d_{0}(t) < d_1$. Therefore, the end multiset \{(d'_{j}, m'_j)\}_{j=0}^{q'} at \(t = \tau\) is defined with\[ q' = q - 1, \tag{3.11} \]
m'_0 = m_0 + m_1, \quad m'_j = m_{q-1} + m_q,
\]
\[ d'_0 = d_0 \exp(m_q \tau), \quad d'_j = d_{j+1} for 1 \leq j < q', \quad d'_q = d_{q-1} = d_q \exp(-m_0 \tau). \]

In short, the $(q-1)$st and $q$th points of the multiset (3.1) have coalesced. Again, the geometric mean is $\gamma$, by continuity, and the end multiset belongs to $D(\gamma; D_*, D^*)$.

(III) In this case, we have both the lowest $2$ and upper $2$ points of the multiset (3.1) coalescing. It behooves us to separate out the case $q > 2$ (so $d_1 \neq d_{q-1}$) and $q = 2$ (where $d_1 = d_{q-1}$).
(a) If $q > 2$, then $q' = q - 2$, and the end multiset \{(d'_{j}, m'_j)\}_{j=0}^{q'} at \(t = \tau\) is defined with\[ q' = q - 1, \tag{3.12} \]
m'_0 = m_0 + m_1, \quad m'_j = m_{q-1} + m_q,
\]
\[ d'_0 = d_1 = d_0 \exp(m_q \tau), \quad d'_j = d_{j+1} for 1 \leq j < q', \quad d'_q = d_{q-1} = d_q \exp(-m_0 \tau). \]

(b) If $q = 2$, then $q' = 0$, and $m'_0 = m_0 + m_1 + m_2$, $d'_0 = d_1$. Since the geometric mean is preserved we must have $d^*$, a multi-singleton, our goal.

Finally, we handle the $q = 1$ case.

(IV) If $q = 1$, we find $\tau > 0$ such that\[ d_0 \exp(m_1 \tau) = d_1 \exp(-m_0 \tau), \quad i.e., \]
\[ \tau = \frac{1}{n} \log \frac{d_1}{d_0} \tag{3.13} \]

and set\[ d_{0}(t) = d_0 \exp(m_1 t), \quad d_{1}(t) = d_1 \exp(-m_0 t), \quad 0 \leq t < \tau. \]

For $t = \tau$, we change to the multi-singleton \{(d'_{0}, n; 0)\}, $d'_{0} = d_0 \exp(m_1 \tau) = d_1 \exp(-m_0 \tau)$. Again, since this process does not change the geometric mean, we end up at $d^*$, which we follow (3.1) so
\[ c_k(t) = d_{j}(t), \quad k \in K_j, \quad 0 \leq j \leq q. \tag{3.15} \]

On each step, the coordinates of $c(t)$ have the structure $B \exp(\beta t)$ with $0 < B \leq D^*$ and $|\beta| \leq n$, so the following condition holds:
\[ |c'(t)| \leq n D^*, \quad |c''(t)| \leq n^2 D^*. \tag{3.16} \]
We summarize our desired reduction of steps as follows.

**Proposition 3.1.** Fix positive real numbers $0 < D_s < \gamma \leq D^*$, with $\gamma < 1$. Fix $c_0 \in C(\gamma; D_s, D^*)$, $c_0 \neq e^*$. Then with $\tau$ defined as in (3.13) if $q = 1$ and (3.7c) if $q \geq 2$, we have defined a $C^\infty$ function $c(t) : [0, \tau] \to C(\gamma; D_s, D^*)$ such that

(a) $c(0) = c_0$;
(b) letting $q'$ denote the number of gaps in the d-notation for $c(\tau)$,
\begin{equation}
q' \leq q - 1.
\end{equation}

Moreover, in Cases [III][b] and [IV] $c(t) = e^*$.

**3.0.1. Extension of Path.** For the technical arguments later in the paper, we will need to extend the paths $d(t), c(t)$ beyond $[0, \tau]$; indeed, for the Implicit Function Theorem, we wish to use complex values for $t$. Of course, the formulas in (3.7) – (3.8), (3.14) are valid for all $t \in \mathbb{C}$, but for any $\rho \in \left(0, \frac{\log 3}{2n}\right)$, we may simply extend it to the $\mathbb{C}$-neighborhood
\begin{equation}
J_\rho = \{ \xi \in \mathbb{C} : -\rho \leq \text{Re} \xi \leq \tau + \rho, \ |\text{Im} \xi| \leq \rho \}.
\end{equation}
Of course, (3.9) still holds, so the geometric mean is preserved, and by the bounds on $\rho$, for any real $r \in [-\rho, \tau + \rho] = J_\rho \cap \mathbb{R}$,
\begin{equation}
d_0(r) = d_0 \exp(m_q r) \geq d_0 \exp(-m_q \rho) > D_s \exp\left(-\frac{n \log 3}{2n}\right) = \frac{D_s}{\sqrt{3}}
\end{equation}
and
\begin{equation}
d_0(r) = d_0 \exp(m_q r) \leq d_0 \exp(m_q \tau) \exp(m_q \rho) < d_1 \exp\left(\frac{n \log 3}{2n}\right) \leq \sqrt{3}D^*.
\end{equation}

Similar bounds hold for $d_q$. Therefore, for any $t \in (3.18)$, we have
\begin{align*}
d_0(t) &= d_0 \exp(m_q t) = d_0 \exp(m_q \text{Re} t + i m_q \text{Im} t) \\
&= d_0 \exp(m_q \text{Re} t) \left[\cos(m_q \text{Im} t) + i \sin(\text{Im} t)\right].
\end{align*}

Of course,
\begin{equation}|d_0(t)| = d_0 \exp(m_q \text{Re} t),
\end{equation}
so by (3.19) and (3.20),
\begin{equation}
\frac{D_s}{\sqrt{3}} \leq |d_0(t)| \leq \sqrt{3}D_s,
\end{equation}
but we also wish to bound the real and imaginary parts separately. By $\rho < \frac{\log 3}{2n} < \frac{\pi}{6n}$,
\begin{equation}|m_q \text{Im} t| \leq n \rho < n \frac{\pi}{6n} = \frac{\pi}{6},
\end{equation}
and so with (3.19), we have
\begin{equation}
\text{Re} d_0(t) = d_0 \exp(m_q \text{Re} t) \cos(m_q \text{Im} t) > \frac{D_s}{\sqrt{3}} \cos\left(\frac{\pi}{6}\right) = \frac{D_s}{2},
\end{equation}
and by (3.20) we have
\begin{equation}
\text{Re} d_0(t) = d_0 \exp(m_q \text{Re} t) \cos(m_q \text{Im} t) \leq \sqrt{3}D_s
\end{equation}
and
\begin{equation}
|\text{Im} d_0(t)| = |d_0 \exp(m_q \text{Re} t) \sin(m_q \text{Im} t)| \leq \sqrt{3}D_s \cdot \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}D_s.
\end{equation}

Similar inequalities hold for $d_q(t)$, and if $q > 1$, then $d_1(t), \ldots, d_{q-1}(t)$ are still positive constants in $[D_s, D^*]$. 
We create \( c(t) \) as in (3.15), but using the initial \( m \) and \( K_j \)'s to define the multiplicities, i.e.,

\[
(3.21) \quad c(t) = \left( d_0(t), \ldots, d_0(t), d_1(t), \ldots, d_1(t), \ldots, d_q(t), \ldots, d_q(t) \right),
\]

so that

\[
(3.22) \quad P(z; c(t)) = \prod_{k=1}^n (z + c_k(t)) = \prod_{j=0}^q (z + d_j(t))^{m_j}.
\]

Of course, for \( t > \tau \), \( d_0(t) > d_1(t) \) or \( d_{q-1}(t) < d_q(t) \), but (3.21) still gives the same polynomials as the previous constructions for \( t \in [0, \tau] \), and shows that the diversity is at most \( q \). We therefore have the following.

**Lemma 3.2.** Let \( c_0 \in C(\gamma; D_\ast, D^\ast) \), fix \( \rho \in \left( 0, \frac{\log 3}{2n} \right) \), and with \( J_\rho \) as in (3.18), define \( c(t) \) for \( t \in J_\rho \) as in (3.21), for \( d(t) \) as in (3.8) for \( q \geq 1 \) and (3.14) for \( q = 1 \). Then \( c(t) \) is a holomorphic function on \( J_\rho \), and the image of \( J_\rho \) is inside \([R(D_\ast, D^\ast)]^n \), where

\[
(3.23) \quad R(D_\ast, D^\ast) = \left\{ \xi \in \mathbb{C} : \frac{D_\ast}{2} < \text{Re} \xi \leq \sqrt{3} D^\ast, \ |\text{Im} \xi| \leq \frac{\sqrt{3}}{2} D^\ast \right\},
\]

and

\[
(3.24) \quad \frac{D_\ast}{\sqrt{3}} \leq |c_k(t)| \leq \sqrt{3} D^\ast, \ 1 \leq k \leq n.
\]

In particular, \( c(t) \in C(\gamma; \frac{D_\ast}{2}, 2D^\ast) \) for \( t \in [-\rho, \tau + \rho] \), and \( c(t) \in C(\gamma; D_\ast, D^\ast) \) for \( t \in [0, \tau] \).

In Case [III] part (a) or Case [IV] we will not consider \( c(t) \) for \( t > \tau \).

**4. Reduction of the diversity** \( q \). In the \( d \)-notation (3.1) – (3.3), our polynomial becomes

\[
(4.1) \quad P(z, c) = \prod_{j=0}^q (z + d_j)^{m_j}.
\]

In Section [III] we have chosen the path \( c(t) \) or \( d(t) \), \( 0 \leq t \leq \tau \), which reduces the diversity \( q \) of the initial multiset

\[
(4.2) \quad r = (r_k)_{k=1}^n = c(0)
\]

to \( q' = q - 1 \) or \( q - 2 \) when we move to \( c(\tau) \), i.e., \( d(\tau) \). The polynomial (4.1) changes accordingly, and we want to understand how its roots are changing, in particular, when \( t \) is close to \( 0 \) or \( \tau \). In what follows, as in Subsection [3.0.1], \( c(t) \) is defined by (3.8) or (3.14), i.e. by (3.21), for \(-\rho \leq t \leq \tau \), for small enough \( \rho \).

**Proposition 4.1.** Fix \( 0 < D_\ast \leq \gamma \leq D^\ast \), with \( \gamma < 1 \). Let \( r = c(0) \in C(\gamma; D_\ast, D^\ast) \), and \( w \in E \) be a root of the equation

\[
(4.3) \quad P(z; r) = 1.
\]

Then for sufficiently small \( \eta > 0 \), there exists an unique analytic function \( w(t) \), \( t \in J_\eta \in (3.18) \), such that

\[
(4.4) \quad w(0) = w,
\]

\[
(4.5) \quad P(w(t), c(t)) = 1, \quad t \in J_\eta.
\]
If \( t \in [-\eta, \tau] \), then \( w(t) \in E \cap \text{Ann} \left( \frac{d}{2}, 1 \right) \). If, in addition, \( \text{Re} \, w(t) \in [-\epsilon, \epsilon] \),

(4.6) \[ \epsilon \equiv \min \left\{ \frac{D_*}{12}, \frac{\delta(\gamma, \sqrt{3}D^*)^2}{4(1+\sqrt{3}D^*)} \right\}, \]

then

(4.7) \[ \text{Re} \, \dot{w}(t) > 0. \]

Proof. To use Appendix B Claim B.1, we put

(4.8a) \[ F(z, t) = P(z; c(t)) - 1, \]

(4.8b) \[ \rho = \frac{1}{2} \min \left\{ \epsilon, \frac{\log 3}{2n} \right\}, \]

(4.8c) \[ V = \{ z \in \mathbb{C} : \text{Re} \, z \geq -\epsilon, |z| \leq 1 \}, \]

(4.8d) \[ J = [0, \tau] \]

so that the lozenge-shaped neighborhood \( J(\rho) \) defined as in (B.11) is a subset of the rectangle \( J_\rho \in (5.18) \).

We first note the following estimate: If \( |z| \leq 2 \), and \( |c_k(t)| \leq 2D^* \) for all \( k \in \mathbb{N}, k \leq n \), then

(4.9) \[ |P(z; c(t))| \leq 2^n(1 + D^*)^n \equiv M_0. \]

The estimate is on an appropriate domain: for \( z \in V, |z| \leq 1 \), so for \( z \in V_\rho \) with \( \rho < 1 \), \( |z| < 2 \). For \( t \in J_\rho \), Lemma 3.2 (3.24), ensures \( |c_k(t)| \leq 2D^* \) for all \( k \).

We divide the next part of the proof into smaller claims.

Claim 4.2. Fix \( 0 < D_* \leq \gamma \leq D^* \), with \( \gamma < 1 \). Fix \( r = c(0) \in C(\gamma; D_*, D^*) \), define \( c(t) \) as in Section 3. Fix \( t_0 \in [0, \tau] \), let \( s = c(t_0) \), and let \( w \in V \in (4.8c) \) be a root of

(4.10) \[ P(z; s) = 1. \]

Then there exists a unique continuous function \( w(t) : \mathbb{D}_r \to \mathbb{C} \), analytic in the interior of \( \mathbb{D}_r \), with range in \( \mathbb{D}_\kappa \), where \( \kappa, \rho \) depend only on \( \gamma, D^*, D_* \), and \( \rho \), such that

(4.11a) \[ w(t_0) = w, \]

(4.11b) \[ P(w(t); c(t)) = 1 \text{ for all } t \in \mathbb{D}_r. \]

Proof. To use Appendix B Claim B.1 on \( F(z; t) \in (4.8a) \), we find appropriate estimates for the inequalities (B.5a), (B.5b), (B.4). Note that by \( \epsilon + \rho \leq \frac{3}{2} \left( \frac{D_*}{12} \right) \leq \frac{D_*}{6} \), \( V(\rho) \subseteq E_{\text{ext}}(D_*/2) \), and as mentioned above, \( t \in J_\rho \) implies by Lemma 3.2 that \( \text{Re} \, c_k(t) \geq \frac{D_*}{2} \) and \( |c_k(t)| \leq \sqrt{3}D^* \) for all \( k \).

For \( M_1 \), by \( P(z; c(t)) \in (3.22) \), for all \( t \in J_\rho \) and \( z \in E_{\text{ext}} \left( \frac{D_*}{2} \right), |z| \leq 1 + \rho < 2, \)

(4.12) \[ \frac{\partial F}{\partial z} = \frac{\partial P}{\partial z} = \sum_{j=0}^q m_j(z + d_j(t))^{m_j-1} \prod_{k=1 \atop k \neq j}^q (z + d_k(t))^{m_k} = \left[ \sum_{j=0}^q \frac{m_j}{z + d_j(t)} \right] P(z; c(t)) \]
and thus by (3.24)

\[
\left| \frac{dP}{dz} \right| \leq (q + 1) \left( \sum_{j=0}^{n} m_j \right) \times (1 + \rho + \sqrt{3}D^*)^{n-1}
\]

\[
\leq n^2 2^n (1 + D^*)^n = n^2 M_0.
\]

For \( \frac{\partial F}{\partial t} \), and for all cases (I) – (IV), we need only two terms:

\[
\frac{\partial F}{\partial t} = \frac{\partial}{\partial t} P(z; c(t)) = \left[ m_0 \dot{d}_0(t) + m_q \dot{d}_q(t) \right] P(z; c(t))
\]

\[
\quad = m_0m_q \left[ \frac{d_0(t)}{z + d_0(t)} - \frac{d_q(t)}{z + d_q(t)} \right] P(z; c(t))
\]

\[
\quad = -m_0m_q z(d_q(t) - d_0(t)) \cdot \frac{P(z; c(t))}{(z + d_0(t))(z + d_q(t))}.
\]

For \( t \in J, 0 < \rho < \frac{\log 3}{2n} \), we have by (3.23) that \( \text{Re} c_j(t) > D^* \), or \( \text{Re} d_j(t) > \gamma \), so for all \( j, 0 \leq j \leq q \), and \( z \in E_{\text{ext}}(D^*/2) \),

\[
\text{Re}(z + d_j) \geq \text{Re}(z + d_*) > \frac{D_*}{6} + \frac{D_*}{2} = \frac{D_*}{3},
\]

Using (4.15), (4.9), and \( |z| < 2 \) in the final line of (4.14),

\[
\left| \frac{\partial F}{\partial t} \right| \leq \frac{36n^2 \cdot M_0}{D_*^2}.
\]

Therefore, we can choose

\[
M_1 = n^2 M_0 + \frac{36n^2 M_0}{D_*^2} = n^2 M_0 \left( 1 + \frac{36}{D_*^2} \right).
\]

As above, we can bound the second derivatives of \( F(z; t) \in (4.8a) \) and it suffices to choose

\[
M_2 = 216n^3(1 + D^*) \left[ 1 + \frac{M_1}{D_*} + \frac{M_0}{D_*^2} \right]
\]

By (4.12), we have that for any particular root \( \bar{w} \in E_{\text{ext}}(D_*/2) \) of

\[
P(z; c(t)) = 1
\]

that

\[
\left. \frac{\partial P}{\partial z} \right|_{z=\bar{w}} = \sum_{j=0}^{q} \frac{m_j}{\bar{w} + d_j(t)},
\]

so defining

\[
\Omega = \frac{nD_*}{6(1 + D^*)^2},
\]

we have that

\[
\text{Re} \left( \left. \frac{dP}{dz} \right|_{z=\bar{w}} \right) = \sum_{j=0}^{q} \text{Re} \left( \frac{m_j}{\bar{w} + d_j} \right) = \sum_{j=0}^{q} \frac{m_j \text{Re}(\bar{w} + d_k)}{|\bar{w} + d_k|^2} > n \frac{D_*}{(1 + D^*)^2} = \Omega.
\]
With \( c(0) = r \) and \( c(t) \) defined in Section 3 and \( t_0 \in [0, t] = J \), we choose \( \tilde{w} \in V \subseteq E_{\text{ext}}(D_*/2) \) among the roots of (4.10). We choose

\[
2\kappa = \min \left\{ \rho, \frac{\Omega}{8M_2} \right\},
\]

(4.23a)

\[
2r = \min \left\{ \kappa \cdot \frac{\Omega}{8(M_1 + M_2)}, \rho \right\}.
\]

(4.23b)

Then by the Implicit Function Theorem, i.e., by Claim B.1, there exists a continuous function \( w(t) : D_r(t) \to \mathbb{C} \), analytic in the interior, with image contained on \( D_k(\tilde{w}) \), such that

\[
P(w(t), c(t)) = 1, \quad w(0) = \tilde{w},
\]

(4.24)

and with \( F(z, t) \in \{4.8a\} \),

\[
\dot{w}(t) = -\frac{\partial F}{\partial t} \left. \frac{\partial F}{\partial z} \right|_{z=w(t)}.
\]

(4.25)

\[\square\]

**Claim 4.3.** In the setting of Claim 4.2 whenever \( t \in D_r(t_0) \), \( t < \tau \), and \( w(t) \) is in the set

\[
\text{Wall} \equiv \{ \xi \in \mathbb{C} : |\text{Re} \xi| \leq \epsilon, \delta \leq |\text{Im} \xi| \leq 1 \}, \quad \delta \in (2.12), \epsilon \in (4.6)
\]

we have that

\[
\text{Re} \dot{w}(t) > 0
\]

(4.27)

**Proof.** We now wish to demonstrate that if \( t \in (t_0 - r, t_0 + r) \), \( t < \tau \), and \( \text{Re} w(t) \leq \epsilon, \epsilon \in (4.6) \), then \( \dot{w} > 0 \). For real \( t \) in this domain, by Lemma 3.3, \( c(t) \in C(\gamma; \frac{D_3}{\sqrt{3}}, \sqrt{3}D^*) \), so when invoking Corollaries 2.5 and Corollary 2.6, we will use \( d(\gamma, \sqrt{3}D^*) \) and \( \delta(\gamma, \sqrt{3}D^*) \).

Consider first the easier case (1V) i.e., \( q = 1 \). The sum (4.20) has only two terms so

\[
\frac{\partial F}{\partial z} \left|_{z=w(t)} \right. = \frac{m_0}{w(t) + d_0(t)} + \frac{m_q}{w(t) + d_1(t)} = \left( m_0 + m_1 \right) \frac{w(t) + \tilde{d}(t)}{(w(t) + d_0(t))(w(t) + d_1(t))},
\]

(4.28)

where

\[
d_0(t) < \tilde{d}(t) < d_1(t), \quad \tilde{d}(t) \equiv \frac{1}{m_0 + m_1} \left( m_1 d_0(t) + m_0 d_1(t) \right)
\]

(4.29)

and since \( P(w(t), t) = 1 \), we have by (4.14) that

\[
\frac{\partial F}{\partial t} \left|_{z=w(t)} \right. = -m_0 m_1 \frac{z(d_1(t) - d_0(t))}{(z + d_0(t))(z + d_1(t))}
\]

(4.30)

Therefore, with \( m_0 + m_1 = n \),

\[
\dot{w}(t) = \frac{m_0 m_1}{n} \left[ d_1(t) - d_0(t) \right] \left( 1 + \frac{\tilde{d}(t)}{w(t)} \right)^{-1}.
\]

(4.31)

If \( w(t) = u(t) + iv(t) \), then

\[
\text{Re} \frac{1}{w(t)} = \frac{u(t)}{u(t)^2 + v(t)^2}
\]

(4.32)
and so
\begin{equation}
(4.33) \quad \text{Re} \left( \frac{\bar{\tilde{d}}(t)}{w(t)} \right) = \bar{\tilde{d}}(t) \cdot \frac{u(t)}{u(t)^2 + v(t)^2}.
\end{equation}

Since $|u(t)| \leq \frac{1}{2} \cdot \frac{d(\gamma, \sqrt{3D^*})^2}{\sqrt{3D^*}}$, by Lemma 2.3
\begin{equation}
(4.34) \quad \left| \text{Re} \left( \frac{\bar{\tilde{d}}(t)}{w(t)} \right) \right| \leq \bar{\tilde{d}}(t) \cdot |u(t)| \leq \frac{\sqrt{3D^*}}{d(\gamma, \sqrt{3D^*})^2} |u(t)| \leq \frac{1}{2}.
\end{equation}

Then $\text{Re} \left( 1 + \frac{\bar{\tilde{d}}(t)}{w(t)} \right) \geq \frac{1}{2} > 0$ and by Remark 2.7
\begin{equation}
(4.35) \quad \text{Re} \tilde{w}(t) > 0 \text{ if } t < \tau.
\end{equation}

In the cases \([1] - [III] 4.14\), with the simplification $P(w(t), t) = 1$, gives
\begin{equation}
(4.36) \quad \frac{\partial F}{\partial t} \bigg|_{z=w(t)} = -m_0m_q \frac{z(d_q(t) - d_0(t))}{(z + d_0(t))(z + d_q(t))}.
\end{equation}

By \(4.20\)
\begin{equation}
(4.37) \quad \frac{\partial F}{\partial z} \bigg|_{z=w(t)} = m_0 \frac{w(t) + d_0(t)}{w(t) + d_q(t)} + \frac{m_q}{w(t) + d_0(t)} + \sum_{j=1}^{q-1} \frac{m_j}{w(t) + d_j},
\end{equation}
and the third term needs special attention, even with constant $d_j$ for $0 < j < q$. By \(4.37\) and \(4.14\),
\begin{equation}
(4.38) \quad \tilde{w}(t) = -\frac{\partial F}{\partial t} / \frac{\partial F}{\partial z} \bigg|_{z=w(t)} = \frac{m_0m_q(d_q(t) - d_0(t))}{H(w(t))}
\end{equation}
where
\begin{equation}
(4.39) \quad H(z) \equiv (m_0 + m_q) \left( 1 + \frac{\bar{\tilde{d}}(t)}{z} + \sum_{j=1}^{q-1} \frac{(z + d_0(t))(z + d_q(t))}{z(z + d_j)} \right) + \sqrt{3D^*} d(t) = \frac{m_qd_0(t) + m_0d_q(t)}{m_0 + m_q}.
\end{equation}

Notice that for $0 < a < c < b,
\begin{equation}
(4.40) \quad \rho \equiv \frac{(z + a)(z + b)}{z(z + c)} = 1 + \frac{a + b - c}{z} + \frac{(c - a)(b - c)}{(-z)(z + c)},
\end{equation}
so
\begin{equation}
(4.41) \quad H(z) = n + \frac{1}{z} \left( (m_0 + m_q)\bar{\tilde{d}}(t) + \sum_{j=1}^{q-1} (d_0(t) + d_q(t) - d_j) \right) + \sum_{j=1}^{q-1} \frac{(d_j - d_0(t))(d_q(t) - d_j)}{-z(z + d_j)}
\end{equation}
\begin{equation}
= n + T_2 + T_3.
\end{equation}
The second term $T_2$ in \(4.41\) — compare \(4.34\) —
\begin{equation}
|\text{Re} T_2| \leq 2n \cdot \sqrt{3D^*} \cdot \frac{|u(t)|}{u(t)^2 + v(t)^2} \leq \frac{2\sqrt{3nD^*}}{d(\gamma, \sqrt{3D^*})^2} \cdot |u(t)| \leq \frac{1}{4} n
\end{equation}
Figure 2. The Wall of (4.26) and the Box of (2.16)

by

\[ |u(t)| \leq \frac{\delta(\gamma, \sqrt{3}D^*)^2}{4(1 + \sqrt{3}D^*)} < \frac{\delta(\gamma, \sqrt{3}D^*)^2}{4\sqrt{3}D^*} \]

For the estimates of the sum \( T_3 \) notice that, with \( z = x + iy, \)

\[ z(z + c) = (x + iy)(x + c + iy) = x(x + c) - y^2 + iy(2x + c) \]

and

\[ \text{Re} \left[ -\frac{1}{z(z + c)} \right] = \frac{y^2 - x(x + c)}{(y^2 - x(x + c))^2 + y^2(2x + c)^2}. \]

With \( c = d_j, 0 < j < q, \)

\[ y^2 - x(x + c) \geq 0 \]

if \( |y| \geq \delta(\gamma, \sqrt{3}D^*) \) by

\[ \delta(\gamma, \sqrt{3}D^*)^2 \geq \epsilon(\epsilon + 2D^*), \quad |x| \leq \epsilon. \]

Therefore, for \( z \in \{4.26\}, \text{Re} T_3 > 0; \) moreover, since \( t \) real and in \([-\rho, \tau]\) implies \( c(t) \in C(\gamma; \frac{D^*}{\sqrt{3}}, \sqrt{3}D^*), \) so by Corollary 2.6 \( |w(t)| \geq \delta(\gamma, \sqrt{3}D^*). \) Together with \( 4.44 \) and \( 4.41, \) this implies that

\[ \text{Re} H(z) \geq \frac{3}{4} \epsilon > 0, \quad z \in \text{Wall}. \]

and by \( 4.43, 4.40, \) and \( 4.38, \text{Re} \hat{w}(t) > 0 \) if the trajectory \( w(t) \) is in the \( \text{Wall}, \) so the root \( w(t) \) cannot leave the \( \text{Box} \) by crossing the \( \text{Wall} \) to the left (see Figure 2). \( \square \)

Claim 4.4. In the setting of Claim 4.3 suppose that \( t_0 < \tau \) and \( \text{Re} \hat{w} \geq 0. \) Then \( w(t), \)

restricted to \([t_0, t_0 + r], \) extends uniquely to a function on \([t_0, \tau]\) such that

\[ \text{Re} w(t) > 0 \text{ for all } t \in (t_0, \tau). \]
Proof. By Corollary 2.5, any root of $w(t)$, $t$ real, with $|\text{Re} \, w(t)| < \epsilon$ is in the Wall. By Claim 4.3 we have that $\text{Re} \, \dot{w}(t) > 0$ if $w(t)$ is in the Wall and $t < \tau$. Thus, whenever $|\text{Re} \, w(t)| < \epsilon$ and $t < \tau$, $\text{Re} \, \dot{w}(t) > 0$.

Case 1. If $r \geq \tau - t_0$, for each $\eta > 0$, we may apply Claim C.1 with $h(t) = \text{Re} \, w(t)$, $[a, b] = [t_0, t_0 + \tau - \eta]$, $\Delta = \epsilon$, so that $\text{Re} \, w(t) > 0$ for all $t \in (t_0, t_0 + \tau - \eta]$. Thus, $\text{Re} \, w(t) > 0$ for all $t \in (t_0, \tau)$. In addition, $\text{Re} \, w(\tau) > 0$: if for some interval $(\tau - \eta, \tau)$, $\text{Re} \, w(t) < \epsilon$ for $t \in (\tau - \eta, \tau)$, then $\text{Re} \, \dot{w}(t) > 0$ for $t \in (\tau - \eta, \tau)$, so $\text{Re} \, w(t) > 0$. Otherwise, for all $\eta > 0$, there exists $t \in (\tau - \eta, \tau)$ with $\text{Re} \, w(t) \geq \epsilon$, so there exists an increasing sequence $\{t_j\}_{j=1}^\infty$ in $(t_0, \tau)$ with $\text{Re} \, w(t_j) \geq \epsilon$ for all $j \geq 1$, and

$$\text{Re} \, w(\tau) = \lim_{t \to \tau^-} \text{Re} \, w(t) = \lim_{j \to \infty} \text{Re} \, w(t_j) \geq \epsilon.$$ 

In all cases, $\text{Re} \, w(t) > 0$ on $(t_0, \tau)$.

Case 2. If $r < \tau - t_0$, define

$$K = \inf \{k \in \mathbb{N} : \tau - t_0 \leq k \cdot \frac{r}{2} \};$$

$$K \geq 3 \text{ by } r \geq \frac{2r}{\tau - t_0}. \text{ Then let}$$

$$(4.49) \quad t_k = t_0 + k \cdot \frac{r}{2}, \quad 0 \leq k \leq K - 1,$$

and we inductively define $w(t)$ on $\bigcup_{k=0}^{K-2} \mathbb{D}_r(t_k)$ as follows. Put $w(t) = w_0(t)$ on $\mathbb{D}_r(t_0)$ as in Claim 4.2. We may apply Claim C.1 with $h(t) = \text{Re} \, w_0(t)$, $[a, b] = [t_0, t_1]$, $\Delta = \epsilon$, to ensure $\text{Re} \, w_0(t) > 0$ on $(t_0, t_1)$.

Suppose that we have defined $w(t)$ on $\bigcup_{k=0}^{j} \mathbb{D}_r(t_k)$, $j \leq K - 3$, and ensured that $\text{Re} \, w(t) > 0$ on $(t_0, t_{j+1}]$: we now show how to extend the definition to $\bigcup_{k=0}^{j+1} \mathbb{D}_r(t_k)$ and ensure positive real part on $(t_0, t_{j+2}]$. Since $\text{Re} \, w(t_{j+1}) > 0$ by hypothesis, we may define $w_{j+1}(t)$ on $\mathbb{D}_r(t_{j+1})$ by Claim 4.2 the unique function such that $w_{j+1}(t_{j+1}) = w(t_{j+1})$ and $P(w_{j+1}(t), c(t)) = 1$ for all $t \in \mathbb{D}_r(t_{j+1})$. We have $w(t_{j+1}) = w_{j+1}(t_{j+1}) \in \mathbb{D}_r(t_j) \cap \mathbb{D}_r(t_{j+1})$, so by the uniqueness statement for $w_{j+1}$, $w_{j+1}(t) = w_j(t)$ for all $t$ in $\mathbb{D}_r(t_j) \cap \mathbb{D}_r(t_{j+1})$. We extend the definition of $w(t)$ by

$$w(t) = \begin{cases} w_{\text{old}}(t), & t \in \bigcup_{k=0}^{j} \mathbb{D}_r(t_k), \\ w_{j+1}(t), & t \in \mathbb{D}_r(t_{j+1}) \end{cases},$$

which is a valid definition by the equality on the overlap. Moreover, $t_{j+2} \in \mathbb{D}_r(t_{j+1})$, and $j \leq K - 3$, so $j + 2 \leq K - 1$, so $t_{j+2} \leq t_{K-1} < \tau$ by definition of $K$, and so we may apply Claim C.1 to $h(t) = \text{Re} \, w(t)$, $[a, b] = [t_0, t_{j+2}]$, $\Delta = \epsilon$ to demonstrate that $\text{Re} \, w(t) > 0$ on $(t_0, t_{j+2}]$. 

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By induction, we have defined \( w(t) \) on \( \bigcup_{k=0}^{K-2} \mathbb{D}_r (t_k) \), such that \( \Re w(t) > 0 \) on \( (t_0, t_{K-1}] \). As in our induction argument, we may expand the definition of \( w(t) \) to include \( \mathbb{D}_r (t_{K-1}) \), but now \( \tau - t_{K-1} \leq t_0 + \tilde{K}(r/2) - [t_0 + (K-1)(r/2)] = r/2 < r \), so as in Case 1, we may argue that \( \Re w(t) > 0 \) on \( [t_{k-1}, \tau] \), hence on \( (t_0, \tau] \), in this last step.

\[ \]

Remark 4.5. In Claim 4.4, we could replace “\( \Re \tilde{w} \geq 0 \)” by “\( \Re \tilde{w} \geq -\epsilon \)” for the starting point and “\( \Re w(t) > 0 \)” by “\( \Re w(t) > -\epsilon \)” for \( t > t_0 \) − for in the invocations of Claim C.1, we could have taken \( h(t) = \Re w(t) + \epsilon \) and \( \Delta = 2\epsilon \), since Claim 4.3 ensures that \( \tilde{w}(t) > 0 \), hence \( h'(t) > 0 \), if \( |\Re w(t)| \leq \epsilon \), i.e. \( 0 \leq h(t) \leq 2\epsilon \). Thus, we can start a little to the left of the imaginary axis and have a path on \( [t_0, \tau] \).

\[%\]

Completion of the proof of Proposition 4.7 The primary step remaining is to show an appropriate choice of \( \eta \) such that I can extend to all \( t \in J_\eta \). By Claim 4.4 if \( \Re \tilde{w} \geq 0 \), we have a nice function \( w(t) \) on \([0, \tau] \) with \( w(0) = \tilde{w} \) and \( \Re w(t) > 0 \) for \( t \in (0, \tau] \). At each point \( t \in [0, \tau] \), we have an \( r \)-radius ball where the function \( w(t) \) can be extended, and the uniqueness from the Implicit Function Theorem ensures that these extensions are consistent. Therefore, \( w(t) \) exists on \( J(r) \) according to the model of \([B.1] \). Setting \( \eta = \frac{2r}{3} \), \( J_\eta \in \{B.1\} \) is a subset of \( J(r) \in \{3.18\} \), by the same reasoning as in the proof of Corollary 2.6.

Since \( \eta = \frac{2r}{3} \leq \frac{\rho}{3} < \frac{\log 3}{2n} \), Lemma 3.2 ensures that for \( t \in [-\eta, \tau + \eta] \), \( c(t) \in C(\gamma; \frac{D_2}{2}, 2D^*) \), and hence by Corollary 2.5 all nonnegative roots are in \( \text{Ann} (d(\gamma, 2D^*), 1) \subset \text{Ann} (d(\gamma, D^*/2, 1) \). The result on the sign of the derivative follows from Claim 4.3.

\[\]

5. Movement of the Roots.

Claim 5.1. \( \nu_+ (c(t)) \) and \( \overline{\nu}(c(t)) \) are nondecreasing on \([0, \tau] \)

Proof. First, we prove the inequality on \([0, \tau] \).

\[
(5.1) \quad \nu_+ (c(0)) \leq \nu_+ (c(t)), \quad \nu_+ (c(0)) \leq \nu_+ (c(t)) \]

So far, we have talked about the trajectory \( w(t) \) of one root \( w(0) = \tilde{w} \). All roots in the Box are simple by Lemma 2.8 so at no instant \( t \) do two of the \( \nu_+ (c(0)) \) trajectories with the initial \( \nu_+ (c(0)) \) root-points could coalesce; yet they remain in the Box and \( E^+ \) (or \( E \)). New roots could come from the left, i.e. from \( E^- = \{ \xi \in \mathbb{C} : \Re \xi < 0 \} \), but this only pushes up the number \( \nu_+ (c(t)) \) in the right half-plane so \( \nu_+ (c(0)) \leq \nu_+ (c(t)) \), \( 0 \leq t \leq \tau \). The same can be said about roots in \( \overline{E} \) so \( \overline{\nu}(c(0)) \leq \overline{\nu}(c(t)) \).

We will get the full claim if we show

\[
(5.2) \quad \nu_+ (c(t')) \leq \nu_+ (c(t)) \quad \text{and} \quad \overline{\nu}(c(t')) \leq \overline{\nu}(c(t)), \quad 0 \leq t' < t \leq \tau. \]

Without changing the structure or diversity \( q \) of the multiset \([3.1] - [3.3] \) let us only change \( d_0 \) to \( \tilde{d}_0 = d_0 \exp(m_q t') \) and \( d_q \) to \( \tilde{d}_q = d_q \exp(-m_q t') \). With \( t' < \tau \) the inequalities

\[
(5.3) \quad \tilde{d}_0 < d_1 \quad \text{and} \quad d_{q-1} < \tilde{d}_q \quad \text{in Cases} \ {[I]} - {[III]} \quad \text{or} \quad \tilde{d}_0 < \tilde{d}_1 \quad \text{in Case} \ {[IV]} \]

will be preserved. If we proceed by the scheme of Section 3 with the initial multiset
\[ d_0 < d_1 < \cdots < d_q < \widetilde{d}_q \quad (\text{or } \tilde{d}_0 < \tilde{d}_1), \]
and the old multiplicities \( \{m_j\}_{j=0}^q \), recalculation of \( \tau \) leads to
\[
\tilde{\tau}_* = \frac{1}{m_q} \log \left( \frac{d_1}{d_0 \exp(m_q t')} \right) = \tau_* - t',
\]
\[
\tilde{\tau}^* = \frac{1}{m_q} \log \left( \frac{d_q \exp(-m_0 t')} {d_{q-1}} \right) = \tau_* - t',
\]
and \( \bar{\tau} = \tau - t' \), in Cases [I]–[III] or
\[
\bar{\tau} = \frac{1}{m_0 + m_1} \log \left( \frac{d_1 \exp(-m_0 t')} {d_0 \exp(m_1 t')} \right) = \tau - t'
\]
in Case [IV].

Now we can apply the work of Section 4 with the understanding that \([t', \tau]\) is shifted by \( t' \) to \([0, \tilde{\tau}]\), and (5.1) becomes the inequalities (5.2).

In Section 3 we made one step \( \{c(t) \in \mathbb{R}^n, \quad 0 \leq t \leq \tau\} \), or \( \{d(t) \in \mathbb{R}^{q+1}; 0 \leq t \leq \tau\}\), to bring the diversity \( q \) down by 1 or 2, with numbers of zeroes \( \nu_+(c), \nu(c) \) of \( P(z; c) - 1 \) in \( E^+ \) and \( E \) not decreasing.

We can repeat the same construction (many times, but at most \( q \) times) if \( q' > 0 \) still, treating the end–multiset of the previous set as the initial sequence (2.8), or multiset (3.1) for the next step. In this way, we get the intervals \([\tau_i, \tau_{i+1}]\), \( \tau_0 = 0, \Delta_i = \tau_{i+1} - \tau_i > 0, \)
\( i = 0, 1, \ldots, p - 1; p \leq q \), and the following holds.

**Proposition 5.2.** Fix \( c_0 \in (\mathbb{R}_{>0})^n \), with geometric mean \( \gamma \). There exists a continuous, piecewise–\( C^\infty \) function
\[ c(t) : [0, T] \rightarrow (\mathbb{R}_{>0})^n, \quad T = \sum_{i=0}^{p-1} \Delta_i, \quad p \leq q, \quad \Delta_i > 0, \]
such that
(a) \( c(0) = c_0 \in (2.8) \)
(b) \( c(T) = c^* = (\gamma, \gamma, \ldots, \gamma). \)
(c) \( \nu_+(c(t)) \) and \( \nu(c(t)) \) are non-decreasing functions on \([0, T]\), i.e., for all \( t, t' \),
\[ 0 \leq t' < t \leq T, \]
\[ \nu_+(c(t')) \leq \nu_+(c(t)) \text{ and } \nu(c(t')) \leq \nu(c(t)). \]

**Proof.** We explained these claims in Section 4.

The \( t = 0, t' = T \) case of [c] is precisely Theorem 1.2.

We now describe more precisely the movement of the roots. Define for \( t \in [0, T] \)
\[ \omega_-(t) = \nu_-(c(t)) \]
\[ \omega_0(t) = \nu_0(c(t)) \]
\[ \omega_+(t) = \nu_+(c(t)) \]
\[ \bar{\omega}(t) = \bar{\nu}(c(t)) \]
(5.4)
Claim 5.3. The counting function $\nu_+(t)$ has a point of discontinuity at $t = t^*$ if and only if
\begin{equation}
P(z; c(t^*)) - 1 = 0
\end{equation}
has roots on $i\mathbb{R}$.

Proof. If such roots do not exist, then define $h > 0$ by 
\begin{equation}
2h = \min \left\{ \frac{D_+}{3}, \min_{z \in \mathbb{C}} \{ |\Re z| \} \right\}.
\end{equation}
Define the region
\begin{equation}
G_h = \{ z \in \mathbb{C} : \Re z \geq h, |z| \leq 1 \},
\end{equation}
and note that by the Cauchy Integral Formula, e.g., [Con00 Section 4.7, pp. 97 – 99],
\begin{equation}
\omega_+(t^*) = \frac{1}{2\pi i} \int_{\partial G_h} \frac{dP/dz}{P(z; c(t^*)) - 1} dz.
\end{equation}

Let
\begin{equation}
\min_{z \in \partial G(h)} |P(z; c(t)) - 1| = \mu(t),
\end{equation}
then $\mu(t^*) > 0$, and $\mu(t)$ is continuous at $t^*$, so there exists $\rho > 0$, $\rho < 1$ such that
\begin{equation}
|\mu(t) - \mu(t^*)| \leq \frac{1}{2} \mu(t^*) \quad \text{if} \quad |t - t^*| \leq \rho.
\end{equation}
Thus, $\mu(t) > 0$ for $t \in [t^* - \rho, t^* + \rho]$, and the Cauchy integral
\begin{equation}
\eta_+(t) = \frac{1}{2\pi i} \int_{\partial G_h} \frac{dP/dz}{P(z; c(t)) - 1} dz.
\end{equation}
is continuous on $[t^* - \rho, t^* + \rho]$, but it is integer-valued, being the counting-function for the roots of
\begin{equation}
P(z; c(t)) - 1 = 0
\end{equation}
in the interior of $G_h$, so $\eta_+(t)$ is constant. Therefore, the number of roots of (5.11) in $G_h$
is $\omega(t^*)$ for all $t$ in $[t^* - \rho, t^* + \rho]$.

To show that $\eta_+(t) = \omega_+(t)$, we must show that no roots enter from the left. We know
that there are no roots to (5.5) in the strip $\{ \xi \in \mathbb{C} : |\Re \xi| < 2h \}$, in particular on the imaginary axis, so we consider
\begin{equation}
G_0 = \{ z \in \mathbb{C} : \Re z \geq 0, |z| \leq 1 \}.
\end{equation}
In the same way, shrinking $\rho$ if necessary, for $t \in [t^* - \rho, t^* + \rho]$ there are no roots of (5.5)
on $\partial G_0$ (in particular, on the imaginary axis), and on $[t^* - \rho, t^* + \rho]$, the function
\begin{equation}
\eta(t) = \frac{1}{2\pi i} \int_{\partial G_0} \frac{dP/dz}{P(z; c(t)) - 1} dz
\end{equation}
is a continuous counting-function, hence constant. $\eta(t^*) = \eta_+(t^*) = \omega_+(t^*)$, as there
are no roots at time $t^*$ with small real part, so as $\eta$ and $\eta_+$ are constant on this interval,
and there are no roots on the imaginary axis, the number of roots in $G_0 \setminus G_h$ is 0 for all $t \in [t^* - \rho, t^* + \rho]$. All roots in the closed right-half-plane must be in $G_0$ by Lemma [2.3]
so we must have $\omega_+(t) = \eta_+(t)$ for $t \in [t^* - \rho, t^* + \rho]$. Thus, $\omega_+(t)$ is constant for $t \in [t^* - \rho, t^* + \rho]$ for some small $\rho$. (Since we showed there were no roots on the
imaginary axis, \( \omega_+(t) = \overline{\omega}(t) \) for \( t \in [t^* - \rho, t^* + \rho] \), so the counting-function on the closed half-plane is also constant.

Suppose that (5.5) has a pure imaginary root; since solutions to (1.3) imply solutions to (2.11), we have by Lemmas 2.1 and 2.3 that the root can only be \( \pm iY \), \( 1 > Y > d > 0 \), \( d \in (2.11) \), only two roots, and they are simple by Lemma 2.8. Now \( t^* \in [0, T] \) can be one of three types of points:

(i) \( t^* \in (t_k, t_{k+1}) \) for some \( k, 0 \leq k < p \leq q - 1 \);
(ii) \( t^* = t_k, 0 \leq k < p \);
(iii) \( t^* = T \).

We need to know well the behaviour of the root \( \bar{\omega}(t) \) with \( \bar{\omega}(t^*) = iY \), for \( t \) around \( t^* \), a well-determined function for small \( \rho \) by the Implicit Function Theorem.

The case \([i]\) is easier: with \( \rho, 0 < \rho < \frac{1}{2} \min\{t_{k+1} - t^*, t^* - t_k\} \), the path \( c(t) \), or \( d(t) \), is defined by formulas (3.8), (3.15); this is analytic on \( I_\rho = [t^* - \rho, t^* + \rho] \), or even if we talk about complex \( t \) in a neighborhood of \( I_\rho \subseteq \mathbb{C} \). By (4.35), \( K = \text{Re} \bar{\omega}(t) \bigg|_{t=t^*} > 0 \), so for \( \rho \) small enough,

\[
\text{Re} \bar{\omega}(t) \geq \frac{1}{2} K > 0, \quad t \in I_\rho,
\]

so

\[
\text{(5.12)}
\]

\[
\text{Re} \bar{\omega}(t) = \text{Re}(\bar{\omega}(t) - \bar{\omega}(t^*)) = \text{Re} \int_{t^*}^{t} \bar{\omega}(\sigma) d\sigma \geq \frac{1}{2} K(t - t^*), \quad \text{if } t > t^*,
\]

\[
\text{(5.13b)}
\]

\[
\text{Re} \bar{\omega}(t) - \text{Re}(\bar{\omega}(t^*)) = - \text{Re} \int_{t^*}^{t} \bar{\omega}(\sigma) d\sigma \leq - \frac{1}{2} K(t^* - t), \quad \text{if } t < t^*.
\]

Now we know the past and the future of the roots \( \pm iY(c(t^*)) \): they are in \( E^+ \) if \( t^* \leq t \leq t^* + \rho \), and they are in \( E^- \) if \( t^* - \rho \leq t < t^* \). All other roots remain in their half-planes; it can be explained as in (5.6) - (5.9) (with \( G_{-h} \) in the place of \( G_0 \)). Therefore,

\[
\omega_+(t) = \begin{cases} 
\omega_+(t^*) + 2, & t^* < t \leq t^* + \rho \\
\omega_+(t), & t^* - \rho \leq t \leq t^*
\end{cases}
\]

and

\[
\overline{\omega}(t) = \begin{cases} 
\overline{\omega}(t^*) = \omega_+(t^*) + 2, & t^* \leq t \leq t^* + \rho \\
\overline{\omega}(t^*) - 2 = \omega_+(t^*), & t^* - \rho \leq t < t^*
\end{cases}
\]

The case \([ii]\), i.e. \( t^* = t_k, 0 \leq k \leq p - 1 \), is more delicate because the function \( c(t) \), or \( d(t) \), at \( t^* \) is only continuous. It is defined by different \( C^\infty \) (or analytic) functions on \( [t_k, t_{k+1}] \) and \( (t_k, t_{k+1}) \) separately. Claim 4.3 gives us all the information; the later case \( (t_k, t_{k+1}) \) is an analogue of \( (0, \tau] \), so

\[
\text{Re} \bar{\omega}(t_k) > 0
\]

and we can repeat (5.12) and (5.13a) to justify the claim

\[
\text{(5.17)}
\]
by (4.34), (4.47), and (5.18) to (5.19),
and by (4.31) or (4.38) has two factors
eigenvalues in the right-half-plane to be equal to
and we have
d

\begin{equation}
\frac{\partial \Delta}{\partial t}(t) \leq -L_* (t^* - t), \quad \text{for some } L_*>0 \text{ and } \rho \ll 1.
\end{equation}

Therefore, for \( t = t^* - h, 0 \leq h \leq \rho \), by (5.19) and (5.20),

\begin{equation}
\Re \bar{w}(t) = -\Re[\bar{w}(t_\tau) - \bar{w}(t)] = - \int_{t_\tau - h}^{t_\tau} \Re \bar{w}(\xi) \, d\xi \geq K_* L_* \int_0^h |\eta| \, d\eta = \frac{h^2}{2} K_* L^* > 0.
\end{equation}

Therefore, as in Case (iv) the inequalities (5.16) and (5.20) justify (5.14) and (5.15) if \( t^* = \tau_k, 0 \leq k < p \).

The case (iii) is special; it happens only if \( c(t) = c^* \), i.e., \([\tau_{p-1}, T]\) is an analogue of \([0, \tau]\) in Cases (iii) b or (iv). As in (5.18), a pure imaginary root comes from the left, so

\begin{equation}
\omega_+(t) = \omega_+(T) \text{ if } T - \rho \leq t \leq T
\end{equation}

\begin{equation}
\bar{w}(t) = \bar{w}(T) - 2 \text{ if } T - \rho \leq t < T.
\end{equation}

6. Construction of multisets such that \( \nu_+(c) = 1 \) and \( \overline{\nu}(c) = 1 \). We have demonstrated that for all \( c \in (\mathbb{R}_{>0})^n \) with geometric mean \( \gamma < 1 \), the maximum values for \( \nu_+(c) \) and \( \overline{\nu}(c) \) are achieved by \( c^* \). The question is what the minimum value is, or could be.

As per Lemma 2.1 if the geometric mean \( \gamma < 1 \), then \( \nu_+(c) \geq 1 \) and \( \overline{\nu}(c) \geq 1 \) by the positive real root. We now show that this lower bound is the minimum.

**Proposition 6.1.** Fix \( n \in \mathbb{N} \) and fix \( \gamma \in (0,1) \). There exists \( c \in (\mathbb{R}_{>0})^n \) with geometric mean \( \gamma \) such that \( \nu_+(c) = \overline{\nu}(c) = 1 \).

We note that for \( 1 \leq n \leq 4, 0 < \gamma < 1, \)

\( (z + \gamma)^n = 1 \)

has only one root with positive real part, as follows from (1.14), so \( \nu_+(c^*) = \overline{\nu}(c^*) = 1 \).

In the sequel, we assume that \( n \geq 5 \).

It turns out that control of the 2 extreme coordinates in \( c \) suffices to force the number of eigenvalues in the right-half-plane to be equal to 1. For convenience, let \( n' = n - 2 \).

**Proposition 6.2.** Let \( n \in \mathbb{N}, n \geq 5, \) and fix \( c' \in (\mathbb{R}_{>0})^{n'} \). Then for any \( \gamma \in (0,1), \) there exists a vector \( c_{ext} = (d_0, c', d_q) \in (\mathbb{R}_{>0})^n \) with geometric mean \( \gamma \) and \( \nu_+(c_{ext}) = \overline{\nu}(c_{ext}) = 1 \).
To begin the proof, we write \( c' \) in \( d \)-notation as \( \{(d'_j, m'_j)\}_{j=0}^{q'} \). We extend \( c' \) to \( c_{ext} \) by setting \( q = q' + 2 \) and choosing \( d_0 < d'_0 \) and \( d_q > d'_q \) such that \( d_0 \left[ \prod_{j=0}^{q'} (d'_j)^{m'_j} \right] d_q = \gamma^n \).

Set

\[
d_j = \begin{cases}
  d_0, & j = 0 \\
  d'_{j-1}, & 1 \leq j \leq q - 1 = q' + 1, \\
  d_q, & j = q
\end{cases}
\]

\[
m_j = \begin{cases}
  1, & j = 0 \\
  m'_{j-1}, & 1 \leq j \leq q - 1 = q' + 1, \\
  1, & j = q
\end{cases}
\]

and let \( c_{ext} \) be the resulting vector in \((\mathbb{R}_{>0})^n\) as created by (3.15). Altogether, setting \( P(z; c_{ext}) = 1 \), we have

\[
(z + d_0) \left[ \prod_{j=1}^{q-1} (z + d_j)^{m_j} \right] (z + d_q) = 1;
\]

(6.1)

\[
d_0 \left[ \prod_{j=1}^{q-1} d_j^{m_j} \right] d_q = \gamma^n, \quad n' = \sum_{j=1}^{q-1} m_j,
\]

We rescale the coefficients to make \( \gamma \) evident:

(6.2)

\[
d_0 = \frac{\gamma}{A}, \quad d_q = M \gamma; \quad d_j = b_j \gamma, \quad 1 \leq j \leq q - 1,
\]

and (6.1) becomes

(6.3)

\[
\left( z + \frac{\gamma}{A} \right) \left[ \prod_{j=1}^{q-1} (z + b_j \gamma)^{m_j} \right] (z + M \gamma) = 1.
\]

We rescale \( z \) as \( z = \gamma w \) to move all the \( \gamma \) terms to the other side, which does not change the signs of the real parts of any zeroes; letting \( G = \frac{1}{\gamma} \) we have

(6.4a)

\[
\left( w + \frac{1}{A} \right) \left[ \prod_{j=1}^{q-1} (w + b_j \gamma)^{m_j} \right] (w + M) = G^n,
\]

(6.4b)

\[
\frac{1}{A} B^{n'} M = 1, \quad B^{n'} = \prod_{j=1}^{q-1} b_j^{m_j}.
\]

We will fix \( d_j, 1 \leq j \leq q - 1 \), i.e., the \( (b_j)_{j=1}^{q-1} \) of (6.4b), and in (6.4a)

(6.5)

\[
b_0 = \frac{1}{A} < b_1 < \cdots < b_{q-1} < b_q = M.
\]

and (thinking of \( M \) as our parameter, and \( A \) varying as in (6.4b) to balance the geometric mean) study the polynomial

(6.6)

\[
P_M \equiv P(w; b, M) \text{ of the left-hand-side of (6.4a).}
and the roots of (6.4a). Similarly to the previous, we define $\mu_+(M)$ and $\overline{\tau}(M)$ to be the number of roots of $P_M(z) = G^n$ in the open right-half-plane $E^+$ and the closed half-plane $E$, respectively.

**Claim 6.3.** If $M \in (6.3)$ is sufficiently large, and $A = M : B''$ as in (6.4b), then

\begin{equation}
\mu_+(M) = \overline{\tau}(M) = 1.
\end{equation}

**Proof.** As in Lemma 2.1 there is a guaranteed root in $(0, G)$ by the Intermediate Value Theorem, as by (6.4b),

\begin{equation}
P_M(0) = \frac{1}{A} B'' M = 1 < G^n < \left( G + \frac{1}{A} \right) \left[ \prod_{j=1}^{q-1} (G + b_j)^{m_j} \right] (G + M) = P_M(G).
\end{equation}

Thus, $\overline{\tau}(M) \geq \mu_+(M) \geq 1.$

Let

\begin{equation}
2\beta = \min \left\{ \frac{b_1}{2}, \min \{ b_{j+1} - b_j : 1 \leq j \leq q - 2 \} \right\};
\end{equation}

then the interiors of the disks

\begin{equation}
\mathbb{D}_j = \{ w \in \mathbb{C} : |w + b_j| \leq \beta \}, \quad 1 \leq j \leq q - 1,
\end{equation}

do not intersect, and their closures are in $E^- \in (1.9).$ If, in addition, $M \geq \frac{2}{B'' b_1},$ then

\begin{equation}
b_1 - b_0 = b_1 - \frac{1}{A} \geq b_1 - \frac{b_1}{2} = \frac{b_1}{2} \geq 2\beta
\end{equation}

and the disks $\mathbb{D}_0$ and $\mathbb{D}_1$ have disjoint interior. Similarly, if $M \geq b_{q-1} + 2\beta,$ then the interiors of $\mathbb{D}_{q-1}$ and $\mathbb{D}_q$ do not intersect, and $\mathbb{D}_q$ is contained in the open left-half-plane.

We wish to show that for $1 \leq j \leq q,$

\begin{equation}
\min \{|P_M(w)| : |w + b_j| = \beta\} = \min_{w \in \partial \mathbb{D}_j} \{|P_m(w)| \} \geq 2G^n
\end{equation}

if $M \geq \max \left\{ \frac{2}{\beta B''}, 2(b_{q-1} + \beta), \frac{8}{\beta^{n-1}} G^{n+2} \right\}.$ Indeed, if $k \neq j, 0 \leq j, k \leq q,$

\begin{equation}
|w + b_k| = |(w + b_j) + (b_k - b_j)| \geq |b_k - b_j| - |w + b_j| \geq 2\beta - \beta = \beta,
\end{equation}

so for $w \in \partial \mathbb{D}_j, 1 \leq j \leq q,$

\begin{equation}
|P_M(w)| \geq \left| w + \frac{1}{A} b'' |w + M| \right| \\
\quad \geq \left( \beta - \frac{1}{A} \right) b'' (M - b_{q-1} - \beta) \\
\quad \geq \frac{1}{2} \beta \cdot b'' \cdot \frac{1}{2} M \geq \frac{\beta^{n-1}}{4} M
\end{equation}

if $\frac{1}{A} \leq \frac{1}{2} \beta,$ or $M \geq \frac{2}{\beta B''},$ and $M \geq 2(\beta + b_{q-1}).$ We choose

\begin{equation}
M \geq \max \left\{ \frac{2}{\beta B''}, 2(b_{q-1} + \beta), \frac{8}{\beta^{n-1}} G^{n+2} \right\} \quad \text{Then (6.10) holds.}
\end{equation}

The first term does not exceed the third (because $b_j \geq 2j\beta, j \geq 1,$ by (6.8)), so we choose

\begin{equation}
M = 8 \max \left\{ b_{q-1}, \frac{G^n}{\beta^{n-1}} \right\}.
\end{equation}
Then the number of roots of $P_M(w) = \xi$ does not depend on $\xi$ if $\xi \leq \frac{1}{2} \min_{w \in \partial D_j} \{|P_M(w)|\}$, as this is the integral

$$N(D_j) = \frac{1}{2\pi i} \int_{\partial D_j} \frac{dP_M}{P_m(w) - \xi} \, dw \quad \text{(with the counterclockwise orientation)}$$

(see, e.g., [Con00 Section 4.7, pp. 97–99]). In particular, when $\xi = 0$, this is $m_j, 1 \leq j \leq q$ (this includes the case $j = q$, i.e., $b_q = M$). Therefore, counted with multiplicity, the number of roots of $P_m(w) = G^n$ is also $m_j$ for $M$ as above. This holds for $j = 1, 2, \ldots, q$.

For such $j$, all such $D_j$ are in the left half-plane, as $\beta \leq b_1 - b_0 < b_1$. Hence, there are

$$\sum_{j=1}^q m_j = n' + 1 = n - 1$$

roots in the left half-plane. Since we have already located the zero in the right-half plane, we have accounted for all roots of the $n$th-degree polynomial $P_M(w) = G^n$: $n - 1$ in the open left-half plane, none on the imaginary axis, and 1 in the right half-plane. Thus, $\mu_+(M) = \pi(M) = 1$ for $M$ as in (6.11). □

### 7. The Range of $\nu_+(c)$ and $\overline{\nu}(c)$

For each $(n, \gamma)$ pair, we have shown the maximum and minimum values for $\nu_+(c)$ and $\overline{\nu}(c)$ for all $c \in (\mathbb{R}_{>0})^n$ with geometric mean $\gamma$. By Lemma \ref{lemma2.1}, $\nu_+(c)$ and $\overline{\nu}(c)$, are odd integers, but we wish to show that every odd number between the minimum values and maximum values of $\nu_+(c)$ and $\overline{\nu}(c)$ is achieved.

For convenience, combining Lemma \ref{lemma2.1} and (1.15), write

$$\nu_+(c^\ast) = 2\kappa_+ + 1$$

$$\overline{\nu}(c^\ast) = 2\pi + 1.$$ \hfill (7.1a, 7.1b)

**Corollary 7.1.** Fix $n \in \mathbb{N}$, and $\gamma \in (0, 1)$, and fix $c_0 \in (\mathbb{R}_{>0})^n$ with geometric mean $\gamma$. Let $D_* = \min c_j$ and $D^* = \max c_j$. Construct $c(t) : [0, T] \rightarrow C(\gamma; D_*, D^*)$ as defined in Sections 3 and 5. Then:

(i) for any odd $k$, $\nu_+(c) \leq k \leq 2\kappa_+ + 1$, there exists $t = t(k) \in [0, T]$ with $\omega_+(t(k)) = k$.

(ii) for any odd $\ell$, $\overline{\nu}(c) \leq \ell \leq 2\pi + 1$, there exists $\overline{t}(\ell) \in [0, T]$ with $\overline{\omega}(\overline{t}(\ell)) = k$.

**Proof.** By Claim \ref{claim5.3} or its proof, the jumps of $\omega_+(t)$ and $\overline{\omega}(t)$ are of size 2, and the points of discontinuity $t^\ast$ are where the equation \ref{5.5} has pure imaginary roots. There are $\mu_+ = \frac{1}{2} [(2\gamma + 1) - \nu_+(c)]$ point of discontinuity, by Lemma \ref{lemma2.1} named $\{\eta_j\}_{j=1}^{\mu_+}$, and

$$\omega_+(t) = \omega_j = 2p_j + 1, \quad \eta_j < t \leq \eta_{j+1}, \quad p_j + 1 = p_j + 1, \quad \text{or } 0 < t \leq \eta_1.$$ \hfill (7.2)

$$\overline{\omega}(t) = \omega_j, \quad \eta_j \leq t < \eta_{j+1}.$$ \hfill (7.3)

These facts on the structure of the functions $\omega_+(t), \overline{\omega}(t)$ imply (i), (ii). □

Since by Proposition \ref{6.1} we know for all $(n, \gamma)$ pairs with $n \in \mathbb{N}$, $\gamma \in (0, 1)$, there exists $c_0 \in (\mathbb{R}_{>0})^n$ with geometric mean $\gamma$ and $\nu_+(c_0) = \overline{\nu}(c_0) = 1$, we may apply Corollary \ref{7.1} to such a $c_0$ and achieve all positive odd values less than the maximum. This proves the following.

**Proposition 7.2.** Fix $n \in \mathbb{N}$, and $\gamma \in (0, 1)$. Then:

(i) for all odd $k$, $1 \leq k \leq 2\kappa_+ + 1$, there exists $c \in (\mathbb{R}_{>0})^n$ with $\nu_+(c) = k$.

(ii) for all odd $\ell$, $1 \leq \ell \leq 2\pi + 1$, there exists $c \in (\mathbb{R}_{>0})^n$ with $\overline{\nu}(c) = \ell$. 
In the context of doubly cyclic matrices, we have the following.

**Proposition 7.3.** Fix \( n \in \mathbb{N}, \) and \( 0 < \alpha < \beta < 1. \) Then:

(i) for all odd \( k, 1 \leq k \leq 2\kappa + 1, \) there exists \( X \in DC(\alpha, \beta) \) with \( k \) roots in the open left half-plane with \( \nu_+(c) = k. \)

(ii) for all odd \( \ell, 1 \leq \ell \leq 2\kappa + 1, \) there exists \( X \in DC(\alpha, \beta) \) with \( \ell \) roots in the closed left half-plane.

Since by Lemma 2.1 and Theorem 1.2, \( \nu_+ \) (respectively, \( \nu_+ \)) is odd and less than \( \nu_+ \) (respectively, \( \nu_+ \)), the range is no larger than that demonstrated in Proposition 7.2, so this completes the proof of the \( \gamma < 1 \) case of Theorem 1.3; the \( \gamma \geq 1 \) case was handled at the beginning of Section 2. Similarly, we have proven Theorem 1.4.

### 8. Further Comments

In the proof of the main theorem we used the localization

\[
(8.1) \quad w \in Ann(d, 1), \quad d = \frac{1 - \gamma^n}{(1 + D^*)^n},
\]

of roots of (1.8) in the right half-plane \( E, \) or in Box \( \in (2.16). \) We want now to improve the upper bound 1 in (8.1) and make explicit the dependence on \( \gamma = \left( \prod_{k=1}^{n} c_j \right)^{1/n}. \) As in (2.15),

\[
1 = \prod_{j=1}^{n} \left[ (x^2 + y^2) + 2xc_j + c_j^2 \right]
\]

\[
> \prod_{j=1}^{n} (|z|^2 + c_j^2) = |z|^{2n} \prod_{j=1}^{n} \left( 1 + \left( \frac{c_j}{|z|} \right)^2 \right)
\]

\[
\geq |z|^{2n} \left( 1 + \frac{\gamma^2}{|z|^2} \right)^n \geq \left( |z|^2 + \gamma^2 \right)^n,
\]

so

\[
(8.3) \quad 1 - \gamma^2 > |z|^2,
\]

and by \( \gamma < 1, \)

\[
(8.4) \quad |z|^2 \leq 2(1 - \gamma), \quad |z| \leq \frac{3}{2}(1 - \gamma)^{1/2}.
\]

The last inequality in (8.2) follows from

\[
(8.5) \quad \prod_{k=1}^{n} (1 + t_k) \geq (1 + T)^n, \quad T^n = \prod_{k=1}^{n} t_k, \quad \text{if } t_k \geq 0, \quad 1 \leq k \leq n,
\]

with strict inequality unless all \( t_k \) are equal. See [HLP52, #64, p. 61].

The lower bound \( d \in (8.1) \) can be improved also. Notice that

\[
(8.6) \quad 2ab \leq \omega^2 a^2 + \Omega^2 b^2, \quad \omega \Omega = 1, \quad a, b \geq 0, \quad 1 \geq \omega > 0,
\]

so for all \( j \)

\[
(8.7) \quad 2xc_j \leq \Omega^2 x^2 + \omega^2 c_j^2,
\]
and

\[(8.8)\]

\[
1 = \prod_{j=1}^{n} \left[ (x^2 + y^2) + 2xc_j + c_j^2 \right]
\]

\[
\leq \prod_{j=1}^{n} \left( y^2 + x^2(1 + \Omega^2) + c_j^2(1 + \omega^2) \right) = (1 + \omega^2)^2 \prod_{j=1}^{n} \left[ 1 + \frac{y^2 + (1 + \Omega^2)x^2 1}{1 + \omega^2 c_j^2} \right].
\]

Notice that

\[(8.9)\]

\[
\log(1 + u) < u, \quad \text{for all } u > 0, \quad \text{and } \log(1 + u) \geq \frac{3}{4}u, \quad \text{if } 0 < u \leq \frac{1}{3}.
\]

Then taking logarithms of both sides in (8.8),

\[(8.10)\]

\[
0 \leq \gamma^2 \left[ \log((1 + \omega^2)^2) + \frac{1}{n} \sum_{j=1}^{n} \log \left( 1 + \frac{y^2 + (1 + \Omega^2)x^2 1}{1 + \omega^2 c_j^2} \right) \right].
\]

and

\[(8.11)\]

\[
\log \frac{1}{\gamma^2(1 + \omega^2)} \leq \frac{y^2 + (1 + \Omega^2)x^2}{(1 + \omega^2)} L^2, \quad L^2 = \frac{1}{n} \left( \sum_{j=1}^{n} \frac{1}{c_j^2} \right).
\]

With \(\gamma^2 < 1\) choose \(\omega > 0\) by

\[(8.12)\]

\[
(1 + \omega^2)\gamma^2 = \frac{1}{2} (1 + \gamma^2);
\]

then

\[(8.13)\]

\[
\frac{1 + \Omega^2}{1 + \omega^2} = \frac{1}{\omega^2} = \frac{2\gamma^2}{1 - \gamma^2} = \frac{2\gamma^2}{1 + \gamma}, \quad \frac{1}{1 - \gamma} < \frac{1}{1 - \gamma}.
\]

and (8.11) becomes

\[(8.14)\]

\[
\log \frac{2}{(1 + \gamma^2)} \leq \left( \frac{y^2 + \frac{x^2}{1 - \gamma}}{1 - \gamma} \right) L^2, \quad \text{where } L^2 \leq \frac{1}{D_*^2}.
\]

But \(\frac{2}{1 + \gamma^2} = 1 + \frac{1 - \gamma^2}{1 + \gamma^2}\), and if \(\gamma \geq \frac{1}{2}\), by (8.9),

\[(8.15)\]

\[
\log \frac{2}{1 + \gamma^2} \geq \frac{3}{4} \frac{1 - \gamma^2}{1 + \gamma^2} \geq \frac{3}{4}(1 - \gamma)^2,
\]

and by (8.14),

\[(8.16)\]

\[
\frac{3}{4}(1 - \gamma) \leq \left[ \frac{y^2 + \frac{x^2}{1 - \gamma}}{1 - \gamma} \right] L^2,
\]

and

\[(8.17)\]

\[
\frac{3}{4} D_*^2 \leq \frac{x^2}{(1 - \gamma)^2} + \frac{y^2}{1 - \gamma}.
\]

It implies with \(0 < \gamma < 1\) that

\[(8.18)\]

\[
\frac{3}{4} D_*^2 \leq \frac{x^2 + y^2}{(1 - \gamma)^2}.
\]

Therefore (in conjunction with (8.14)), as an analogue or an improvement of Lemmas 2.3, 2.3, and Corollary 2.5, we can state the following.
Claim 8.1. If \( c \in C(\gamma; D_*, D^*) \), \( 1 > \gamma \geq \frac{1}{2} \), and \( z \) is a root of (1.8) in the right half-plane, then

(i) \( z \) lies outside of the ellipsoid

\[
\frac{x^2}{(1-\gamma)^2} + \frac{y^2}{1-\gamma} < \frac{3}{4} D_*,
\]

or

(ii) in a weaker claim,

\[
\frac{4}{5} D_*(1-\gamma) \leq |z|.
\]

Therefore \( z \in \text{Ann} \left( \frac{4}{5} D_*(1-\gamma), \frac{3}{2} (1-\gamma)^{1/2} \right) \).

A slight advantage over Corollary 2.5 and Lemma 2.3 is that there is no \( n \) in Claim 8.1, at least explicitly. Speaking loosely, we can say that the area of localization changes continuously when \( \gamma \) goes from \( \gamma > 1 \) to \( \gamma < 1 \).

Appendix A. Proof of Inequalities (8.5). To make our paper self-contained, we will explain the inequality (8.5).

Let us consider the elementary symmetric polynomials

\[
\sigma_k(x) = \sum_{|K|=k} X(K); \quad X(K) = \prod_{j \in K} x_j, \quad x \in \mathbb{R}^n, \quad \pi \in \mathcal{P}_n
\]

and the averages

\[
S_k(x) = \sigma_k(x) \left( \frac{n}{k} \right) = \# \{ K \subset \pi : |K| = k \}.
\]

C. Maclaurin, in 1729, in [Mac29], proved a chain of inequalities in the case \( x \in (\mathbb{R}_{>0})^n \):

\[
S_1(x) \geq S_2^{1/2}(x) \geq \ldots \geq S_k^{1/k}(x) \geq \ldots \geq S_n^{1/n}(x) = g(x) \equiv \left( \prod_{k=1}^{n} x_k \right)^{1/n},
\]

with strict inequality (at least once) if and only if \( x_j \neq x_k \) for some \( j, k \in \pi, j \neq k \). To explain (8.5) we need just the individual inequalities

\[
S_k^{1/k}(x) \geq g(x), \quad 1 \leq k < n.
\]

For \( x \in (\mathbb{R}_{>0})^n \),

\[
\prod_{j=1}^{n} (1 + x_j) = \sum_{k=0}^{n} \sigma_k(x) = \sum_{k=0}^{n} \left( \frac{n}{k} \right) S_k(x)
\]

\[
\geq \sum_{k=0}^{n} \left( \frac{n}{k} \right) [g(x)]^k = (1 + g(x))^n.
\]

To prove (A.4), let us notice that by (A.2), \( S_k \) is an arithmetic mean of \( \left( \frac{n}{k} \right) \) positive numbers \( X(K) \), but their product is a homogeneous polynomial

\[
\prod_{K} X(K) = \left( \prod_{j=1}^{n} x_j \right)^G,
\]
of degree \( nG = k \binom{n}{k} \), so the AM-GM inequality implies

\[
S_k(x) \geq \left(\frac{[g(x)^n]^G}{n^G}\right)^{n-1} = g(x)^k.
\]

We have proven (A.4).

See more on the Newton and Maclaurin Inequalities in [Nic00], [Nic04], and references therein.

**Appendix B. Implicit Function Theorem.** Of course, the Implicit Function Theorem is well-known (see, e.g., [Rud76, Thm. 9.28, pp. 224] or [FG02, Thm. 7.6, p. 34]), but we use a version with explicit lower bounds on the neighborhoods of validity, so we give the full details below.

For a convex, closed, bounded set \( V \subseteq \mathbb{C} \), put for \( 0 < \rho < 1 \)

\[
\rho\text{-neighborhood of } V,
\]

\[
V(\rho) = \{ z \in \mathbb{C} : |z - v| \leq \rho \text{ for some } v \in V \}.
\]

Let \( F(z, t) \) be an analytic function of two variables in

\[
G = V(\rho) \times J(\rho), \quad J = [a, b] \in \mathbb{R},
\]

where \( J(\rho) \) is the \( \mathbb{C} \)-neighborhood of \( J \) as in (B.1). Assume that \( Z = \{ (z, t) \in V \times J : F(z, t) = 0 \} \) is not empty, and

\[
0 < \Omega = \min \left\{ \left| \frac{\partial F}{\partial t}(z, t) \right| : (z, t) \in Z \right\}.
\]

Put

\[
M_1 = \max \left\{ \left| \frac{\partial F}{\partial z}(z, t) \right| + \left| \frac{\partial F}{\partial t}(z, t) \right| : (z, t) \in G \right\}
\]

and

\[
M_2 = \max \left\{ \left| \frac{\partial^2 F}{\partial z^2}(z, t) \right| + \left| \frac{\partial^2 F}{\partial z \partial t}(z, t) \right| + \left| \frac{\partial^2 F}{\partial t^2}(z, t) \right| : (z, t) \in G \right\}
\]

Now, choose and fix \( \kappa, r \) such that

\[
0 < \kappa \leq \min \left\{ \rho, \frac{\Omega}{8M_2} \right\}
\]

and

\[
0 < r \leq \min \left\{ \kappa \cdot \frac{\Omega}{8(M_1 + M_2)}, \rho \right\}
\]

**Claim B.1.** Under the assumptions and notation (B.1) – (B.7), if

\[
* = (z_0, t_0) \in V \times J \quad \text{and} \quad F(z_0, t_0) = 0
\]

then there exists a unique continuous function \( z(t) \) in the closed disc

\[
D_r(t_0) = \{ t \in \mathbb{C} : |t - t_0| \leq r \},
\]

analytic in the open disk \( \overset{\circ}{D}_r(t_0) \), such that for \( t \in D_r \),

\[
|z(t) - z_0| \leq \kappa,
\]

\[
F(z(t), t) = 0.
\]
Moreover, for \( t \in \mathbb{D}_r(t_0) \)

\[(B.11)\]

\[
\dot{z}(t) = - \frac{\partial F}{\partial t} \frac{\partial F}{\partial z} 
\]

**Proof.** With

\[(B.12)\]

\[
A = \left. \frac{\partial F}{\partial z} \right|_s, \quad B = \left. \frac{\partial F}{\partial t} \right|_s
\]

by the Taylor formula and \((B.8)\)

\[(B.13)\]

\[
F(z, t) = A(z - z_0) + B(t - t_0) + g(z, t), \quad g \text{ analytic in } V(\rho) \times J(\rho).
\]

Put

\[(B.14)\]

\[
z = z_0 + \zeta, \quad t = t_0 + s;
\]

we want (see \((B.10a)\)) to find \( \zeta(s) \in X \), where \( X \) is the Banach space \( (\mathcal{A}(\mathbb{D}_r(0)), \| \cdot \|_{\infty}) \)

of functions analytic in the open disk \( \mathbb{D}_r(0) \) and continuous on the closed disk \( \mathbb{D}_r(0) \), such that

\[(B.15)\]

\[
0 = A\zeta(s) + Bs + G(\zeta(s), s), \quad s \in \mathbb{D}_r,
\]

where

\[(B.16)\]

\[
G(\zeta, s) = F(z_0 + \zeta, t_0 + s) - A\zeta - Bs, \zeta \in \mathbb{D}_\rho, \ s \in \mathbb{D}_\rho.
\]

Define in \( X \) the mapping

\[(B.17)\]

\[
\Phi : \zeta(\cdot) \mapsto - \frac{B}{A} \zeta - \frac{1}{A} G(\zeta(s), s),
\]

at least for functions with \( |\xi| \leq \rho \) for all \( s \in \mathbb{D}_r(0) \). The ball

\[(B.18)\]

\[
K(\kappa) = \{ \zeta \in \mathcal{A}(\mathbb{D}_r(0)) : |\xi(s)| \leq \kappa \text{ if } |s| \leq r \}
\]

is invariant under \( \Phi \): indeed, by \((B.6)\), \((B.7)\), if \( |s| \leq r \leq 1 \),

\[(B.19)\]

\[
\Phi[\xi](s) \leq \frac{|B|}{A} |r + 1/A | M_2(\kappa + r)^2 \leq \frac{1}{\Omega} \left[ M_1 r + 2 M_2 r^2 + 2 M_2 \kappa^2 \right]
\]

\[
\leq \frac{1}{\Omega} \left[ r(M_1 + 2 M_2) + (2 M_2 \kappa) \cdot \kappa \right] \leq \left( \frac{1}{4} + \frac{1}{3} \right) \kappa = \frac{1}{2} \kappa.
\]

Moreover, on \( K(\kappa) \) this mapping is contractive in the uniform norm: If \( \xi(\cdot), \zeta(\cdot) \in K(\kappa) \), then

\[(B.20)\]

\[
\Phi[\xi](s) - \Phi[\zeta](s) = \frac{1}{A} \left( G(\zeta(s), s) - G(\xi(s), s) \right)
\]

\[
= \frac{1}{A} \left( F(z_0 + \zeta, t_0 + s) - A\zeta - F(z_0 + \xi, t_0 + s) - A\xi \right)
\]

\[
= \frac{1}{A} \int_0^1 \frac{d}{du} \left[ F(z_0 + \zeta + u(\zeta - \xi), t_0 + s) - A(\xi + u(\zeta - \xi)) \right] du
\]

\[
= \frac{L}{A}(\zeta - \xi),
\]

where

\[(B.21)\]

\[
L = \int_0^1 \left\| \frac{\partial F}{\partial z}(z_0 + \zeta + u(\zeta - \xi), t_0 + s) - \left. \frac{\partial F}{\partial z} \right|_s \right\|.
\]
Appendix C. Condition for Positivity of Function.

Fix \( h \in C^2([a,b]) \), so
\[
|\Phi(\xi)(s) - \Phi(\zeta)(s)| \leq \frac{M_2 \kappa}{\Omega} \|\zeta - \xi\|_\infty \leq \frac{1}{2} \|\zeta - \xi\|_\infty;
\]
i.e., \( \Phi \) is contractive on \((K(\kappa), \|\cdot\|_\infty)\).

By the Contractive Mapping Principle, we have a solution \( \zeta(s) \) of the equation \((B.15)\), or the solution \( z(t) \) of \((B.10b)\), \( z(t_0) = t_0 \), and \((B.10a)\). The solution of \((B.15)\) is unique

The form of the derivative \((B.11)\) follows from implicit differentiation. \(\square\)

**Appendix C. Condition for Positivity of Function.** Fix \( a < b \) and let \( h \in C^2[a,b] \) be a real-valued function. Suppose that there exists a positive constant \( \Delta \) such that

(C.1a) \quad 0 \leq h(a) \\
(C.1b) \quad \text{If } 0 \leq h(c) \leq \Delta, \text{ then } h'(c) > 0.

**Claim C.1.** If \( h \in C^2[a,b] \) and \( \Delta > 0 \) satisfies \((C.1)\), then \( h(x) > 0 \) for all \( x \in (a,b) \).

**Proof.** If \( h(a) < \Delta \), we have by \((C.1b)\) that \( h'(a) > 0 \), so we have \( \Delta \geq h(x) > h(a) \) if \( a \leq x \leq a + \rho, 0 < \rho \ll 1 \). Define \( \omega^* = \sup\{\omega \in (a,b) : \text{ if } x \in (a,\omega), \text{ then } 0 < h(x) < \Delta\} \). Since \( h'(x) > 0 \) on \((a,\omega)\), \( h(x) > h(a) \geq 0 \) for all \( x \) in \((a,\omega)\), and we are done; also, either \( \omega^* = b \), or \( \omega^* < b \) and \( h(\omega^*) = \Delta \). In the former case, \( h(x) > h(a) \) for all \( x \) in \((a,b)\). In the latter case, i.e., \( \omega^* < b \), we claim that

(C.2) \quad \frac{\Delta}{2} \leq h(x) \quad \text{if } x \in (\omega^*, b].

Otherwise, for some \( y \in (\omega^*, b] \),
\[
\frac{\Delta}{2} \geq h(y) \geq \frac{\Delta}{2}.
\]

Then the set
\[
T = \left\{ t \in [\omega^*, y] : h(t) = \frac{3}{4} \Delta \right\}
\]
is not empty, and closed; therefore, it contains \( t^* = \sup T \), and \( \omega^* < t^* < y \). For \( t, t^* < t < y \), \( h(t) \leq \frac{3}{4} \Delta = h(t^*) \), so \( h'(t) > 0 \) if \( t^* < t < y \). In particular, letting \( k = h'(t^*) > 0 \), by \( h \in C^1 \) there exists a small interval \([t^*, t^1]\) with \( h' \geq \frac{k}{2} \) on \([t^*, t^1]\).

Therefore,
\[
0 > -\frac{1}{4} \Delta \geq h(y) - h(t^*) = \int_{t^*}^{y} h'(u) du > \int_{t^*}^{t^1} k dt + \int_{t^1}^{y} 0 dt > k(t^1 - t^*) > 0.
\]

This contradiction shows that no such \( y \) can exist, so \( h(x) \geq \frac{\Delta}{2} \) for all \( x \in [\omega^*, b] \).

If \( h(a) \geq \Delta \), we set \( \omega^* = a \), and we see that \( h(x) \geq \frac{\Delta}{2} \) for all \( x \) in \((a,b] = (\omega^*, b] \) as in the above proof. \(\square\)

**Remark C.2.** With a slight adjustment of the proof, we may also replace \((C.2)\) with the stronger inequality \( h(x) \geq \Delta \).
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