Classification of Static Plane Symmetric Spacetimes according to their Matter Collineations

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Abstract

In this paper we classify static plane symmetric spacetimes according to their matter collineations. These have been studied for both cases when the energy-momentum tensor is non-degenerate and also when it is degenerate. It turns out that the non-degenerate case yields either four, five, six, seven or ten independent matter collineations in which four are isometries and the rest are proper. There exists three interesting cases where the energy-momentum tensor is degenerate but the group of matter collineations is finite-dimensional. The matter collineations in these cases are either four, six or ten.

Keywords : Matter symmetries, Static Plane Symmetric spacetimes

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1 Introduction

There exists a large body of literature on classification of spacetimes according to their isometries or Killing vectors (KVs) and the groups admitted by them [1]-[4]. These investigations of symmetries played an important role in the classification of spacetimes, giving rise to many interesting results with useful applications. As curvature and Ricci tensors play a significant role in understanding the geometric structure of metrics, the energy-momentum tensor enables us to understand the physical structure of spacetimes. Symmetries of the energy-momentum tensor (also called matter collineations) provide conservation laws on matter fields. These enable us to know how the physical fields, occupying in certain region of spacetimes, reflect the symmetries of the metric [5].

Some recent literature [6]-[12] shows keen interest in the study of matter collineations (MCs). In one of the recent papers [12], the study of MCs has been taken for static spherically symmetric spacetimes and some interesting results have been obtained. In this paper, we address the same problem for static plane symmetric spacetimes. It turns out that static plane symmetric spacetimes admit an MC Lie algebra of 10, 7, 6, 5 and 4 dimensions apart from the infinite dimensional algebras.

Let \((M, g)\) be a spacetime, where \(M\) is a smooth, connected, Hausdorff four-dimensional manifold and \(g\) is smooth Lorentzian metric of signature \((+ - - -)\) defined on \(M\). The manifold \(M\) and the metric \(g\) are assumed smooth \((C^\infty)\). We shall use the usual component notation in local charts, and a covariant derivative with respect to the symmetric connection \(\Gamma\) associated with the metric \(g\) will be denoted by a semicolon and a partial derivative by a comma. A smooth vector field \(\xi\) is said to preserve a matter symmetry [13] on \(M\) if, for each smooth local diffeomorphism \(\phi_t\) associated with \(\xi\), the tensors \(T\) and \(\phi_t^*T\) are equal on the domain \(U\) of \(\phi_t\), i.e., \(T = \phi_t^*T\). Equivalently, a vector field \(\xi^a\) is said to generate a matter collineation if it satisfies the following equation

\[
\mathcal{L}_\xi T_{ab} = 0, \tag{1}
\]

where \(\mathcal{L}\) is the Lie derivative operator, \(\xi^a\) is the symmetry or collineation vector. Every KV is an MC but the converse is not true, in general. Collineations can be proper (non-trivial) or improper (trivial). We define a proper MC to be an MC which is not a KV, or a homothetic vector (HV). The MC Eq.(1) can be written in component form as

\[
T_{ab,c} \xi^c + T_{ac} \xi^c_b + T_{cb} \xi^c_a = 0, \quad (a, b, c = 0, 1, 2, 3). \tag{2}
\]

A plane symmetric spacetime is a Lorentzian manifold possessing a physical stress-energy tensor. This admits \(SO(2) \times \mathbb{R}^2\) as the minimal isometry group in such a way that the group orbits are spacelike surfaces of constant curvature. The metric for static plane symmetric spacetimes is given in the form [3]

\[
ds^2 = e^{\nu(x)}dt^2 - dx^2 - e^{\mu(x)}(dy^2 + dz^2), \tag{3}
\]
where $\nu$ and $\mu$ are arbitrary functions of $x$. The surviving components of the energy-momentum tensor, given in Appendix A, are $T_0$, $T_1$, $T_2$, $T_3$, where $T_3 = T_2$ and we have used the notation $T_{aa} = T_a$ for the sake of simplicity.

The MC equations can be written as follows

\begin{align*}
T_{0,1}\xi^1 + 2T_0\xi^0_0 &= 0, \quad (4) \\
T_{0,1}\xi^0_1 + T_1\xi^1_0 &= 0, \quad (5) \\
T_{0,2}\xi^0_2 + T_2\xi^2_0 &= 0, \quad (6) \\
T_{0,3}\xi^0_3 + T_2\xi^3_0 &= 0, \quad (7) \\
T_{1,1}\xi^1_1 + 2T_1\xi^1_1 &= 0, \quad (8) \\
T_{1,2}\xi^2_2 + T_2\xi^2_1 &= 0, \quad (9) \\
T_{1,3}\xi^3_3 + T_3\xi^3_1 &= 0, \quad (10) \\
T_{2,1}\xi^1_2 + 2T_2\xi^2_2 &= 0, \quad (11) \\
T_{2,3}\xi^3_3 + \xi^3_2 &= 0, \quad (12) \\
T_{2,1}\xi^1_3 + 2T_2\xi^3_3 &= 0. \quad (13)
\end{align*}

These are the first order non-linear partial differential equations in four variables $\xi^a(x^b)$. We solve these equations for the non-degenerate case, when

$$\det(T_{ab}) = T_0 T_1 T_2 \neq 0 \quad (14)$$

and for the degenerate case, where $\det(T_{ab}) = 0$. The rest of the paper is organized as follows. The next section contains a solution of MC equations when the energy-momentum tensor is non-degenerate. In section 3, MC equations are solved for the degenerate energy-momentum tensor and section 4 provides some examples admitting proper MCs for the non-degenerate case. Finally, section 5 contains a summary and discussion of the results obtained.

## 2 Matter Collineations in the Non-Degenerate Case

In this section, we shall evaluate MCs only for those cases which have non-degenerate energy-momentum tensor, i.e., $\det(T_{ab}) \neq 0$. To this end, we set up the general conditions for the solution of MC equations for the non-degenerate case.
When we solve Eqs. (4)-(13) simultaneously, after some algebraic computations, we arrive at the following solution

\[
\xi^0 = -\frac{T_2}{T_0} \left[\frac{1}{2}(y^2 + z^2)\dot{A}_1 + z\dot{A}_2 + y\dot{A}_3\right] + A_4, \quad (15)
\]

\[
\xi^1 = -\frac{T_2}{T_1} \left[\frac{1}{2}(y^2 + z^2)\dot{A}'_1 + z\dot{A}'_2 + y\dot{A}'_3\right] + A_5, \quad (16)
\]

\[
\xi^2 = \frac{1}{2} z^2(c_1 y + c_3) + z(c_2 y + c_4) - \frac{1}{6} c_1 y^3 - \frac{1}{2} c_3 y^2 + yA_1 + A_3, \quad (17)
\]

\[
\xi^3 = -\frac{1}{2} y^2(c_1 z + c_2) - y(c_3 z + c_4) + \frac{1}{6} c_1 z^3 + \frac{1}{2} c_2 z^2 + zA_1 + A_3. \quad (18)
\]

where \( c_1, c_2, c_3, c_4 \) are arbitrary constants and \( A_\mu = A_\mu(t, x), \mu = 1, 2, 3, 4, 5 \) are integration constants. Here dot and prime indicate the differentiation with respect to time and \( x \) coordinate respectively. When we replace these values of \( \xi^a \) in MC Eqs. (4)-(13), we obtain the following constraints on \( A_\mu \)

\[
\frac{T_0}{T_1} A'_i + 2 \dddot{A}_i = 0, \quad (i = 1, 2, 3), \quad (19)
\]

\[
\dot{A}_i = \frac{\sqrt{T_0}}{T_2} f_i(t), \quad A'_i = \frac{\sqrt{T_1}}{T_2} g_i(t), \quad (20)
\]

\[
\dot{A}_4 = -\frac{T_0}{2 T_0 \sqrt{T_1}} g_5(t), \quad A'_4 = -\frac{\sqrt{T_1}}{T_0} g_5(t), \quad (21)
\]

\[
c_1 = 0, \quad T_2' A'_1 = 0, \quad (22)
\]

\[
\frac{T_2'}{T_2} A_5 + 2 A_1 = 0, \quad (23)
\]

\[
\frac{T_2'}{T_2 \sqrt{T_1}} g_2(t) - 2 c_2 = 0, \quad (24)
\]

\[
\frac{T_2'}{T_2 \sqrt{T_1}} g_3(t) + 2 c_3 = 0, \quad (25)
\]

\[
\frac{T_2'}{T_2 \sqrt{T_1}} g_5(t) + 2 A_1 = 0, \quad (26)
\]

\[
\frac{T_0'}{T_2 \sqrt{T_1}} g_i(t) + 2 \dddot{A}_i = 0, \quad (27)
\]

where \( f_i(t), g_i(t), g_5(t) \) are integration functions. Thus the problem of working out MCs for all possibilities of \( A_i, A_4, A_5 \) is reduced to solving the set of Eqs. (15)–(18) subject to the above constraints. We would solve these to classify MCs of the plane symmetry manifolds.

From Eqs. (24)–(26), there arises two main cases:

1. \( \left( \frac{T_2'}{T_2 \sqrt{T_1}} \right)' \neq 0 \),
2. \( \left( \frac{T_2'}{T_2 \sqrt{T_1}} \right)' = 0 \).
Case (1): In this case, we have $T'_2 \neq 0$ and hence Eq.(22) gives $A_1 = A_1(t)$. Using these in Eq.(26), it follows that

$$\frac{T'_2}{T_0 \sqrt{T_1}} g_5(t) + 2 A_1(t) = 0$$

(28)

which implies that $g_5 = 0$ and $A_1 = 0$. Thus we have from Eqs.(21) and (23) $A_5 = 0$, $A_4 = c_0$. Also, Eqs.(24) and (25) yield

$$g_2 = 0 = g_3, \quad c_2 = 0 = c_3.$$

(29)

Now from Eqs.(19) and (20), we have

$$A'_j = 0, \quad \dot{A}_j = 0, \quad \dot{f}_j = 0, \quad (j = 2, 3)$$

(30)

which gives

$$A_j(t, x) = \sqrt{\frac{T_0}{T_2}} c_j t + c_{j+2}.$$ 

(31)

Since $A'_j(t, x) = 0$ which implies that either

(a) $\left(\frac{T_2}{T_0}\right)' = 0$,  \quad or (b) $\left(\frac{T_2}{T_0}\right)' \neq 0$.

In the first case 1(a), we have the following MCs

$$\xi(1) = \partial_t, \quad \xi(2) = \partial_y, \quad \xi(3) = \partial_z, \quad \xi(4) = z \partial_y - y \partial_z, \quad \xi(5) = t \partial_z - \frac{T_2}{T_0} \partial_t, \quad \xi(6) = t \partial_y - \frac{T_2}{T_0} \partial_t.$$ 

(32)

Thus we obtain six independent MCs in which four are the usual isometries of the plane symmetry and the rest are the proper MCs. The MCs for the case 1(b) turns out to be the same as the minimal isometries for the plane symmetry.

Case (2): This case implies that $\frac{T'_2}{T_2 \sqrt{T_1}} = \alpha$, where $\alpha$ is an arbitrary constant and can have the following two subcases according as $\alpha$ is non-zero or zero.

(a) $\alpha \neq 0$,  \quad (b) $\alpha = 0$.

For the case 2(a), we use Eqs.(20),(22),(24) and (25) so that

$$g_2 = \frac{2c_2}{\alpha}, \quad g_3 = -\frac{2c_3}{\alpha}, \quad A'_i = 0,$$

(33)

and

$$\left(\frac{T_2}{T_0}\right)' A_i = 0.$$ 

(34)

The last equation further gives us the following two possibilities:

(i) $\left(\frac{T_2}{T_0}\right)' \neq 0$,  \quad (ii) $\left(\frac{T_2}{T_0}\right)' = 0$.

In the first case 2a(i), Eqs.(20),(21),(23),(25),(36) and (37) imply that

$$A_5 = c_5, \quad T'_0 A'_i = 0, \quad A_5 = -\frac{2c_5}{\alpha \sqrt{T_1}}, \quad A'_4 = 0.$$ 

(35)

and

$$\dot{A}_4 = \frac{T'_0}{\alpha T_0 \sqrt{T_1}} c_3.$$ 

(36)
This last equation implies that for \((\frac{T'_0}{\sqrt{T_1}})') \neq 0\), we have the same MCs as KVs. When \(\frac{T'_0}{\sqrt{T_1}} = \beta\), where \(\beta\) is an arbitrary constant, this further gives the following two subcases

\((*) \quad \beta \neq 0, \quad (**) \quad \beta = 0.\)

The case 2ai\((*)\), in addition to the usual isometries of plan symmetry, gives the following one proper MC

\[\xi(5) = \frac{\beta}{\alpha} t \partial_t - \frac{2}{\alpha \sqrt{T_1}} \partial_x + y \partial_y + z \partial_z. \tag{37}\]

For the case 2ai\((**)\), we have \(T_0 = \text{constant}\) and we obtain the following MCs

\[
\begin{align*}
\xi(5) &= yz + \left(\frac{z^2}{2} - \frac{y^2}{2}\right) \frac{2}{\alpha^2 T_2} \partial_z, \\
\xi(6) &= yz - \left(\frac{z^2}{2} - \frac{y^2}{2}\right) \frac{2}{\alpha^2 T_2} \partial_y, \\
\xi(7) &= y \partial_y + z \partial_z. \tag{38}
\end{align*}
\]

This implies that we have seven independent MCs in which three are the proper MCs.

In the case 2a(ii), we obtain \(T_2 = \gamma T_0\), where \(\gamma\) is an arbitrary constant and this yields the following MCs

\[
\begin{align*}
\xi(5) &= \frac{1}{2} \left( t^2 - \frac{4}{\alpha^2 T_0} - \gamma y^2 - \gamma z^2 \right) \partial_t + \frac{2}{\alpha \sqrt{T_1}} \partial_x + ty \partial_y + tz \partial_z, \\
\xi(6) &= \frac{1}{\gamma} t \partial_t + \frac{2}{\alpha^2 \sqrt{T_1}} z \partial_x + yz \partial_y - \frac{1}{2} \frac{t^2}{\gamma} + \frac{4}{\alpha^2 T_2} + y^2 - z^2 \partial_z, \\
\xi(7) &= \gamma \partial_t - t \partial_z, \\
\xi(8) &= \frac{1}{\gamma} ty \partial_t + \frac{2}{\alpha^2 \sqrt{T_1}} y \partial_x - \frac{1}{2} \frac{t^2}{\gamma} + \frac{4}{\alpha^2 T_2} - y^2 + z^2 \partial_y - yz \partial_z, \\
\xi(9) &= \partial_y - t \partial_y, \\
\xi(10) &= t \partial_t + \frac{2}{\alpha \sqrt{T_1}} \partial_x + y \partial_y + z \partial_z. \tag{39}
\end{align*}
\]

This shows that we have ten independent MCs including six proper MCs.

The case 2b implies that \(T_2 = \text{constant}\) which yields that either

\((i) \quad \left(\frac{\sqrt{T_0}}{\sqrt{T_1}}\right)' = 0 \quad \text{or} \quad (ii) \quad \left(\frac{\sqrt{T_0}}{\sqrt{T_1}}\right)' \neq 0.\)

For the first possibility 2b(i), we have \(\frac{\sqrt{T_0}}{\sqrt{T_1}} = \delta\), where \(\delta\) is an arbitrary constant and gives two possibilities according as it is non-zero or zero

\((*) \quad \delta \neq 0, \quad (**) \quad \delta = 0.\)

For the case 2bi\((*)\), we obtain the following MCs

\[\xi(5) = \frac{T_2}{\sqrt{T_0}} z \sin \delta \partial_t - \frac{T_2}{\sqrt{T_1}} z \cos \delta \partial_x + \frac{T_0}{\delta} \cos \delta t \partial_z, \quad 6\]
\[ \xi_{(6)} = \frac{T_2}{\sqrt{T_0}} z \cos \delta t \partial_z + \frac{T_2}{\sqrt{T_1}} z \sin \delta t \partial_x - \frac{T_0}{\delta} \sin \delta t \partial_z, \]
\[ \xi_{(7)} = \frac{T_2}{\sqrt{T_0}} y \sin \delta t \partial_z - \frac{T_2}{\sqrt{T_1}} y \cos \delta t \partial_x + \frac{T_0}{\delta} \cos \delta t \partial_y, \]
\[ \xi_{(8)} = \frac{T_2}{\sqrt{T_0}} y \cos \delta t \partial_z + \frac{T_2}{\sqrt{T_1}} y \sin \delta t \partial_x - \frac{T_0}{\delta} \sin \delta t \partial_y, \]
\[ \xi_{(9)} = \frac{1}{\sqrt{T_0}} \sin \delta t \partial_t - \frac{1}{\sqrt{T_1}} \cos \delta t \partial_x, \]
\[ \xi_{(10)} = \frac{1}{\sqrt{T_0}} \cos \delta t \partial_t + \frac{1}{\sqrt{T_1}} \sin \delta t \partial_x, \]
which yields ten independent MCs having six proper MCs.

In the case of 2bi(**), we have the following MCs
\[ \xi_{(5)} = \frac{T_2}{\sqrt{T_1}} z \partial_x - \int \sqrt{T_1} dx \partial_z, \]
\[ \xi_{(6)} = \frac{T_2}{\sqrt{T_1}} y \partial_x - t \partial_z, \]
\[ \xi_{(7)} = \frac{T_2}{\sqrt{T_1}} z \partial_t - \int \sqrt{T_1} dx \partial_y, \]
\[ \xi_{(8)} = \frac{T_2}{\sqrt{T_1}} y \partial_t - t \partial_y, \]
\[ \xi_{(9)} = \frac{1}{T_0} \int \sqrt{T_1} dx \partial_t - \frac{1}{\sqrt{T_1}} t \partial_x, \]
\[ \xi_{(10)} = \frac{1}{\sqrt{T_1}} \partial_x, \]
giving ten independent MCs with six proper MCs.

The case 2b(ii) further implies the following two possibilities:
\[ (\text{(*)}) \quad \left( \frac{T_2}{\sqrt{T_1}} \left( \frac{T_0}{T_0 \sqrt{T_1}} \right) \right)' = 0, \quad \left( \text{(**)} \right) \quad \left( \frac{T_2}{\sqrt{T_1}} \left( \frac{T_0}{T_0 \sqrt{T_1}} \right) \right)' \neq 0. \]

For 2bii(*), we have \( \frac{T_2}{\sqrt{T_1}} \left( \frac{T_0}{T_0 \sqrt{T_1}} \right)' = \epsilon \), where \( \epsilon \) is an integration constant
and gives further two cases when
\[ (\text{+}) \quad \epsilon = 0 \quad \text{and} \quad (\text{++}) \quad \epsilon \neq 0. \]

In the case 2bii*(+), we have \( \frac{T_2}{\sqrt{T_1}} \left( \frac{T_0}{T_0 \sqrt{T_1}} \right)' = \chi \neq 0 \) and this gives the following MCs
\[ \xi_{(5)} = \left( \frac{1}{\sqrt{T_0}} - \frac{\chi}{4} t^2 \right) \partial_t + \frac{1}{\sqrt{T_1}} t \partial_x, \]
\[ \xi_{(6)} = \frac{\chi}{2} t \partial_t + \frac{1}{\sqrt{T_1}} \partial_x, \]
yielding six independent MCs.

For the case 2bii*(++), we obtain
\[ \xi_{(5)} = \left( \frac{1}{T_0} \partial_t - \frac{1}{\sqrt{T_1}} \partial_x \right) e^{\sqrt{T_1}}, \]
\[ \xi_{(6)} = \left( \frac{1}{T_0} \partial_t + \frac{1}{\sqrt{T_1}} \partial_x \right) e^{-\sqrt{-m}} \]  

(43)

giving six independent MCs.

In the case 2bii(**), we get MCs equal to the KVs.

3 Matter Collineations in the Degenerate Case

In this section only those cases will be considered for which the energy-momentum tensor is degenerate, i.e., \( \det(T_{ab}) = 0 \). Thus we would discuss the spacetimes when at least one of the \( T_a \) or their combination is zero. When \( T_a = 0 \), we have trivially every direction is an MC. The remaining cases can be classified as follows:

1. When only one of \( T_a \) is non-zero;
2. When two of \( T_a \) are non-zero;
3. When three of \( T_a \) are non-zero.

Case (1): This can further be grouped as follows:

(a) \( T_0 \neq 0, \ T_i = 0 \), (b) \( T_1 \neq 0, \ T_j = 0, \ (i = 1, 2, 3), (j = 0, 2, 3) \).

The case 1(a) yields two possibilities according as \( T'_0 = 0 \) or \( T'_0 \neq 0 \). For the first possibility, we get

\[ \xi^0 = c_0, \ \xi^i = \xi^i(x^a). \]  

(44)

The second possibility implies that

\[ \xi^0 = \xi^0(t), \ \xi^1 = -2T_0 \xi^0(t), \ \xi^k = \xi^k(x^a), \ (k = 2, 3). \]  

(45)

Thus we have infinite dimensional MCs.

The case 1(b) can be solved trivially and gives

\[ \xi^1 = \frac{c_1}{\sqrt{T_1}}, \ \xi^j = \xi^j(x^a) \]  

(46)

which implies infinite dimensional MCs.

Case (2): This case can be divided into the following cases:

(a) \( T_l = 0, \ T_k \neq 0 \) (\( l = 0, 1 \) and \( k = 2, 3 \)),
(b) \( T_l \neq 0, \ T_k = 0 \).

In the first case, if we take \( T_2 = \text{constant} \), then we have the following MCs

\[ \xi^l = \xi^l(x^a), \ \xi^2 = c_0 z + c_1, \ \xi^3 = -c_0 y + c_2 \]  

(47)

which gives infinite dimensional MCs. For \( T'_2 \neq 0 \), we again have infinite dimensional MCs given by

\[ \xi^0 = \xi^0(x^a), \]  

(48)

\[ \xi^1 = -\frac{T'_2}{T_2} (f'(u) + g'(u)), \]  

(49)

\[ \xi^2 = f(u) + g(v), \]  

(50)

\[ \xi^3 = \xi(-f(u) + g(v)) + c_0. \]  

(51)
where \( u = y + \iota z \) and \( v = y - \iota z \).

For the second case 2(b), it follows from Eqs.(4)-(7) and (9)-(10) that \( \xi^l = \xi^l(t, x), \xi^k = \xi^k(x^a) \). Also, Eq.(8) yields \( \xi^1 = \frac{f(t)}{T_1} \). If we use this value in Eqs.(4)-(5) and eliminate \( \xi^0 \), we have

\[
\ddot{f}(t) = \frac{T_0}{\sqrt{T_1}} \left( \frac{T_0'}{2T_0\sqrt{T_1}} \right)'f(t).
\]

From this equation, we see that for \( f = 0 \), we have infinite dimensional MCs given by

\[
\xi^0 = c_0, \quad \xi^l = 0, \quad \xi^k = \xi^k(x^a).
\]

For \( f(t) \neq 0 \), we have

\[
\frac{\dot{f}(t)}{f(t)} = \frac{T_0}{\sqrt{T_1}} \left( \frac{T_0'}{2T_0\sqrt{T_1}} \right)' = \alpha,
\]

where \( \alpha \) is an arbitrary constant. This gives two possibilities either \( \alpha = 0 \) or \( \alpha \neq 0 \). For the first possibility, we obtain \( \frac{T_0'}{2T_0\sqrt{T_1}} = \beta \), an arbitrary constant. This again yields the infinite dimensional MCs given by

\[
\xi^0 = c_1(-\frac{\beta}{2}t^2 - \int \sqrt{T_1} \frac{T_0'}{T_0} dx) - c_2 \beta t + c_0,
\]

\[
\xi^l = \frac{1}{T_1}(c_1t + c_2),
\]

\[
\xi^k = \xi^k(x^a).
\]

When \( \alpha \neq 0 \), we have infinite dimensional MCs as follows

\[
\xi^0 = -\frac{T_0'}{2T_0\sqrt{T_1}}(c_1e^{\sqrt{\alpha}t} - c_2e^{-\sqrt{\alpha}t}) + c_3
\]

\[
\xi^l = \frac{1}{T_1}(c_1e^{\sqrt{\alpha}t} + c_2e^{-\sqrt{\alpha}t}),
\]

\[
\xi^k = \xi^k(x^a).
\]

Case (3): This case can be divided as follows:

(a) \( T_0 = 0, \quad T_i \neq 0 \); (b) \( T_1 = 0, \quad T_j \neq 0 \).

In the case 3(a), it is easy to see that Eqs.(4)-(7) imply that \( \xi^0 \) is an arbitrary function of four variables while \( \xi^i = \xi^i(x, y, z) \). Further, it follows from Eqs.(8)-(11) and (13) that

\[
A_1(y, z),_{kk} - \left( \frac{T_2'}{2T_2\sqrt{T_1}} \right)^2 \frac{T_2}{\sqrt{T_1}} A_1(y, z) = 0.
\]

From here we have two possibilities either \( A_1 = 0 \) or \( A_1 \neq 0 \). For the first possibility, we have the following MCs

\[
\xi^0 = \xi^0(x^a), \quad \xi^1 = 0, \quad \xi^2 = c_1z + c_2, \quad \xi^3 = -c_1y + c_3.
\]
When $A_1 \neq 0$, we obtain
\[
\frac{A_1(y, z),_{kk}}{A_1(y, z)} = \left( \frac{T_2'}{2T_2\sqrt{T_1}} \right)' \frac{T_2}{\sqrt{T_1}} = \alpha,
\] (63)
where $\alpha$ is an arbitrary constant which may be zero or non-zero. The possibility $\alpha = 0$ implies that $\frac{T_2'}{2T_2\sqrt{T_1}} = \beta$, an arbitrary constant and we have the following MCs
\[
\xi^0 = \xi^0(x^a),
\]
(64)
\[
\xi^1 = \frac{1}{\sqrt{T_1}}(c_1y + c_2)z + c_3y + c_4),
\]
(65)
\[
\xi^2 = -\left[ \left( \frac{\beta}{4}y^2 + \int \sqrt{T_1T_2}dx \right)c_1z + \frac{\beta c_2}{2}yz \\
+ \left( \frac{\beta}{2}(y^2 - z^2) + \int \sqrt{T_1T_2}dx \right)c_3 + \frac{\beta c_4}{2}y - c_5z + c_7 \right],
\]
(66)
\[
\xi^3 = -\left[ \left( \frac{\beta}{4}z^2 + \int \sqrt{T_1T_2}dx \right)c_1y + \frac{\beta c_3}{2}yz \\
+ \left( \frac{\beta}{2}(z^2 - y^2) + \int \sqrt{T_1T_2}dx \right)c_2 + \frac{\beta c_1}{2}z + c_3y + c_5 \right].
\]
(67)
For $\alpha \neq 0$, the MCs are given by
\[
\xi^0 = \xi^0(x^a), \quad \xi^1 = 0, \quad \xi^2 = \xi^2(x^a), \quad \xi^3 = -c_0y + c_2.
\]
(68)
In the case 3(b), when $T_0 = \gamma$ and $T_2 = \delta$, where $\gamma$ and $\delta$ are arbitrary constants, we have the following MCs
\[
\xi^0 = c_4y + c_5z + c_0,
\]
(69)
\[
\xi^1 = \xi^1(x^a),
\]
(70)
\[
\xi^2 = c_1z - \frac{\delta c_4}{\gamma} t + c_3,
\]
(71)
\[
\xi^3 = -c_1y - \frac{\delta c_5}{\gamma} t + c_2.
\]
(72)
If $T_0' \neq 0$ and $T_2' = 0$, the MCs are given by
\[
\xi^0 = f(t), \quad \xi^1 = -\frac{2T_0'}{T_0}f(t), \quad \xi^2 = c_1z + c_2, \quad \xi^3 = -c_1y + c_2.
\]
(73)
When $T_2' \neq 0$ and $\frac{T_0'T_2}{T_0T_2} = \epsilon = 0$, we obtain the following MCs
\[
\xi^0 = \epsilon,
\]
(74)
\[
\xi^1 = -\frac{2T_2'}{T_2}(f'(u) + g'(v)),
\]
(75)
\[
\xi^2 = f(u) + g(v),
\]
(76)
\[
\xi^3 = -\epsilon(f(u) - g(v)) + c_1.
\]
(77)
where $u = y + \nu z$ and $v = y - \nu z$. For $T'_2 \neq 0$, $\epsilon \neq 0$ and $(\frac{T_0 T'_2}{T_1})' \neq 0$, the proper MCs are given by

$$
\xi_{(5)} = t \partial_t - \frac{2T_0}{T'_0} \partial_x,
$$
$$
\xi_{(6)} = \nu (z \partial_y - y \partial_z).
$$

(78)

This gives two proper MCs. If $T'_2 \neq 0$, $T_0 = \lambda T_2$ and $T'_0 \neq 0$, we get

$$
\xi_{(5)} = \frac{1}{\lambda} [ty \partial_t - \frac{2T_2}{T'_2} y \partial_x + \frac{1}{2} (y^2 - z^2 - \lambda t^2) \partial y + yz \partial z],
$$
$$
\xi_{(6)} = tz \partial_t - \frac{2T_2}{T'_2} z \partial_x + yz \partial y - \frac{1}{2} (y^2 - z^2 - \lambda t^2) \partial z,
$$
$$
\xi_{(7)} = t \partial t - \frac{2T_2}{T'_2} \partial_x + y \partial y + z \partial z,
$$
$$
\xi_{(8)} = \frac{1}{2\lambda} (y^2 + z^2 - \lambda t^2) \partial t + \frac{2T_2}{T'_2} t \partial_x - ty \partial y - tz \partial z,
$$
$$
\xi_{(9)} = \frac{1}{\lambda} y \partial t - t \partial y,
$$
$$
\xi_{(10)} = \frac{1}{\lambda} z \partial t - t \partial t.
$$

(79)

which yields six proper MCs. Finally, when $T'_2 \neq 0$ and $(\frac{T'_0 T_2}{T_1})' \neq 0$, we have four independent MCs which are exactly the isometries of the plane symmetry. It is interesting to note that the last three subcases of this case give finite dimensional MCs even for the degenerate case.

4 Examples Admitting Proper MCs

In this section we construct examples which admit proper MCs for the non-degenerate energy-momentum tensor. It can be seen from Eq.(A3) that energy-momentum tensor will be non-zero when neither of the metric functions $\nu$ and $\mu$ are constants. If we choose these $\nu$ and $\mu$ such that $\nu = a \ln x + b$, $\mu = c \ln x + d$, where $a,b,c,d$ are constants such that $a \neq c$ then

$$
\frac{T'_2}{T_2 \sqrt{T_1}} = \frac{2(c-2)}{\sqrt{c(c+2a)}} = constant = \alpha \neq 0,
$$

(80)

$$
\frac{T'_0}{T_0 \sqrt{T_1}} = \frac{2(a-2)}{\sqrt{c(c+2a)}} = constant = \beta \neq 0.
$$

(81)

This shows that $\alpha \neq \beta$ as $a \neq c$ and hence the metric

$$
ds^2 = x^2 dt^2 - dx^2 - x^4 (dy^2 + dz^2).
$$

(82)

admits five MCs.
If we choose \( c = 2 \) and \( a \geq 0 \) but \( a \neq 2 \) in Eq.(82), it admits six MCs. If we choose \( a = 2 \) and \( c \geq 0 \) but \( c \neq 2 \) in the metric given by Eq.(82), it yields seven MCs. Finally, if we choose \( \nu = ax = \mu \), then the constraint equations corresponding to ten MCs are satisfied and we obtain the following metric

\[
ds^2 = e^{ar}dt^2 - dx^2 - e^{ar}(dy^2 + dz^2).
\]

This is the well known anti-de Sitter metric.

## 5 Discussion and Conclusion

In a recent paper [12], some interesting results have been obtained when we classify static spherically symmetric spacetimes according to their energy-momentum tensor. In this paper, we have extended the same procedure to classify static plane symmetric spacetimes according to their MCs.

In the non-degenerate case, we obtain either four, five, six, seven or ten independent MCs. These contain the usual four isometries of the plane symmetry and the rest are the proper MCs. For the degenerate energy-momentum tensor, most of the cases give infinite dimensional MCs. The worth mentioning cases are those where we have got finite number of MCs even when the energy-momentum tensor is zero. We obtain three such different cases having either four, six or ten independent MCs. The results are summarized in the form of tables given below.

### Table 1. MCs for the Non-degenerate Case

| Cases | MCs | Constraints |
|-------|-----|-------------|
| 1a    | 6   | \((\frac{T}{T_0})' \neq 0, \ (\frac{T}{T_0})' = 0\) |
| 1b    | 4   | \((\frac{T}{T_0})' \neq 0, \ (\frac{T}{T_0})' = 0\) |
| 2ai*  | 5   | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} \neq 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} \neq 0\) |
| 2ai** | 7   | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} \neq 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0\) |
| 2aii  | 10  | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} \neq 0, \ (\frac{T}{T_0})' = 0\) |
| 2bi*  | 10  | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0\) |
| 2bi** | 10  | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0\) |
| 2bi**+| 6   | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0\) |
| 2bi**++| 6  | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0\) |
| 2bi**| 4   | \((\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0, \ (\frac{T}{T_0})' = 0, \ \frac{T}{T_0} = 0\) |
Table 2. MCs for the Degenerate Case (only finite cases)

| Cases | MCs | Constraints |
|-------|-----|-------------|
| 3bi   | 6   | $T_1 = 0$, $T_j \neq 0 (j = 0, 2, 3)$, $T_2' \neq 0$, $\frac{T_2}{T_0} \neq 0$, $(\frac{T_2}{T_0})' \neq 0$ |
| 3bii  | 10  | $T_1 = 0$, $T_j \neq 0$, $T_2' \neq 0$, $T_0 = \lambda T_2$, $T_0' \neq 0$ |
| 3biii | 4   | $T_1 = 0$, $T_j \neq 0$, $T_2' \neq 0$, $\frac{T_0 T_2}{T_0 T_2} \neq 0$ |

From these tables, it follows that each case has different constraints on the energy-momentum tensor. Finally, we have constructed some examples satisfying the given constraints.

When the rank of $T_a$ is 3, i.e. $T_1 = 0$, we obtain the following metric

$$ds^2 = e^\nu dt^2 - dx^2 - e^{-2\nu}(dy^2 + dz^2),$$

(84)

where $\nu$ is an arbitrary function of $x$ only. It can be easily verified that this class of metrics represent perfect fluid dust solutions. The energy-density for the above metrics is given as

$$\rho = (2\nu'' - 3\nu'^2)e^{\frac{\nu}{2}},$$

(85)

It would be interesting to solve the constraints involved or more examples should be constructed to check the dimensions of the MCs.
Appendix A

The surviving components of the Ricci tensor are

\[ R_0 \equiv R_{00} = \frac{1}{4} e^{\nu}(2\nu'' + \nu'^2 + 2\nu' \mu'), \]
\[ R_1 \equiv R_{11} = -\frac{1}{4}(2\nu'' + \nu'^2 + 4\mu'' + 2\mu'^2), \]
\[ R_2 \equiv R_{22} = -\frac{1}{4} e^{\mu}(2\mu'' + 2\mu'^2 + \nu' \mu'), \]
\[ R_{33} = R_{22}. \quad (A1) \]

The Ricci scalar is given by

\[ R = \frac{1}{2}(2\nu'' + \nu'^2 + 2\nu' \mu' + 3\mu'^2 + 3\mu''). \quad (A2) \]

Using Einstein field equations, the non-vanishing components of energy-momentum tensor \( T_{ab} \) are

\[ T_0 \equiv T_{00} = -\frac{1}{4} e^{\nu}(4\mu'' + 3\mu'^2), \]
\[ T_1 \equiv T_{11} = \frac{1}{4}(\mu'^2 + 2\nu' \mu'), \]
\[ T_2 \equiv T_{22} = \frac{1}{4} e^{\mu}(2\nu'' + \nu'^2 + \nu' \mu' + \mu'^2 + 2\mu''), \]
\[ T_{33} = T_{22}. \quad (A3) \]

Appendix B

The four independent KVs associated with the plane symmetric spacetimes are given by \[3\]

\[ \xi_{(1)} = \partial_t, \quad \xi_{(2)} = \partial_y, \quad \xi_{(3)} = \partial_z, \quad \xi_{(4)} = z\partial_y - y\partial_z. \quad (B1) \]
Acknowledgment

I would like to thank Ministry of Science and Technology (MOST), Pakistan for providing postdoctoral fellowship at University of Aberdeen, UK.

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