Rescattering Effects in the Hadronic-Light-by-Light Contribution to the Anomalous Magnetic Moment of the Muon

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We present a first model-independent calculation of \( \pi \pi \) intermediate states in the hadronic-light-by-light (HLBL) contribution to the anomalous magnetic moment of the muon \((g - 2)_{\mu}\) that goes beyond the scalar QED pion loop. To this end, we combine a recently developed dispersive description of the HLBL tensor with a partial-wave expansion and demonstrate that the known scalar-QED result is recovered after partial-wave resummation. Using dispersive fits to high-statistics data for the pion vector form factor, we provide an evaluation of the full pion box \( a_{\mu}^{\text{box}} = -15.9(2) \times 10^{-11} \). We then construct a suitable input for the \( \gamma^* \gamma^* \to \pi \pi \) helicity partial waves, based on a pion-pole left-hand cut and show that for the dominant charged-pion contribution, this representation is consistent with the two-loop chiral prediction and the COMPASS measurement for the pion polarizability. This allows us to reliably estimate S-wave rescattering effects to the full pion box and leads to our final estimate for the sum of these two contributions \( a_{\mu}^{\text{box}} + a_{\mu, J=0}^{\pi \pi, S\text{-pole LHC}} = -24(1) \times 10^{-11} \).

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Introduction.—The final report of the BNL E821 experiment [1] dominates the world average for the experimental value of the anomalous magnetic moment of the muon \((g - 2)_{\mu}\), establishing a departure from its standard-model (SM) expectation by about 3σ and thus providing an intriguing hint for new physics that makes the improved measurement at FNAL E989 [2], as well as a potential independent determination at J-PARC E34 [3], highly anticipated (see [4] for a detailed comparison of the two methods). However, the significance of the deviation crucially depends on the details of the SM evaluation. Even more so, a sound interpretation of the future experiments demands that also the theory uncertainties be carefully reassessed and ideally reduced in parallel with the experimental improvement.

The by-far dominant uncertainties in the SM prediction arise from hadronic contributions: hadronic vacuum polarization (HVP) at second order in the fine structure constant \( \alpha \) and hadronic-light-by-light scattering (HLBL) at \( \mathcal{O}(\alpha^2) \) [5]. With higher-order iterations of the same topologies already under good control [6–9], most theoretical efforts are concentrated on reducing the uncertainties in the calculations of the HVP and HLBL contributions. But while analyticity and unitarity allow one to express the former in terms of \( \sigma(e^+e^- \to \text{hadrons}) \) [10,11], which is well measured, an expression of the HLBL contribution in terms of measurable quantities was not known until recently. So, traditionally, HLBL scattering has been estimated using hadronic models relying on different limits of QCD—large \( N_c \), chiral symmetry, perturbative expansion—as guiding principles [12–26], which, however, complicates the assessment of the theoretical uncertainty as well as the identification of strategies for systematic improvements, making it emerge as a potential roadblock [27,28].

In a series of recent papers [29–33], we have shown that also the HLBL contribution can be expressed in terms of measurable quantities, albeit not in a form as compact as for HVP. In our model-independent approach based on dispersion relations, we have organized the calculation of the HLBL tensor in terms of its singularities, i.e., single-particle poles and unitarity cuts, by expanding in the mass of intermediate states [34]. Individual terms in this expansion can be uniquely defined in terms of form factors and scattering amplitudes, which, at least in principle, are accessible to experiment. In this way, the notion of pion-pole and pion-box contributions becomes unambiguous.
and the first terms in the expansion—pseudoscalar poles from $\pi^0$, $\eta$, $\eta'$ intermediate states—are fully determined by the corresponding doubly virtual transition form factors. Progress on the pseudoscalar-pole contributions hinges on improved input for these form factors, in combination with constraints on the asymptotic behavior [23], and only concerns a few of the scalar functions that are necessary for a full description of the HLBL tensor. A program to reconstruct the transition form factors based on a combination of unitarity, analyticity, and perturbative QCD with experimental data is currently under way [41–49].

Next in the expansion are two-pion intermediate states. As demonstrated in [33], the one-loop diagrams evaluated in scalar QED (sQED), including pion vector form factors at each vertex to account for the photon virtuality, provide an exact representation of the contribution of two-pion intermediate states, where only the pion-pole contribution to the left-hand cut (LHC) of the $\gamma^*\gamma^* \to \pi\pi$ amplitudes is retained. Thus, the dispersive approach unambiguously defines the gauge-invariant pion-box topology in terms of the pion vector form factor, a very well-measured quantity. Here, we present a numerical evaluation of the pion box, using a form factor fit to high-statistics data, in turn using a dispersive representation to analytically continue the timelike data into the spacelike region required for the $(g-2)_\mu$ integral and show that this contribution can be calculated with negligible uncertainties.

Extending our formalism beyond the pion box to account for two-pion rescattering effects is not easy. Here, we briefly review the technical challenges, along with their solutions, to be faced when doing this extension and present a first numerical evaluation of $S$-wave $\pi\pi$-rescattering effects, which unitarize the pion-pole contribution to $\gamma^*\gamma^* \to \pi\pi$. This constitutes the first step towards a full treatment of the $\gamma^*\gamma^* \to \pi\pi$ partial waves [50–52]. Our calculation settles the role of the pion polarizability, which enters at next-to-leading order in the chiral expansion of the HLBL amplitude [53–55] and has been suspected to produce sizable corrections in [54]. In this Letter, we illustrate the general strategy and present first numerical results. Details of the formalism are relegated to [56].

Dispersion relation for HLBL.—The central object in the calculation of the HLBL contribution to $(g-2)_\mu$ is the hadronic four-point function

$$\Pi^{\mu\lambda\alpha}(q_1, q_2, q_3) = -i \int d^4xd^4yd^4z e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \times \langle 0| T \{j_{\text{em}}^\mu(x)j_{\text{em}}^\lambda(y)j_{\text{em}}^\alpha(z)j_{\text{em}}^\sigma(0)\}|0 \rangle,$$

(1)

of four electromagnetic currents

$$j_{\text{em}}^\mu = \bar{q}Q^\mu q,$$

$$Q = \text{diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right),$$

(2)

with momenta $q_i$ as indicated, $q_4 = q_1 + q_2 + q_3$, and quark fields $q = (u, d, s)^T$.

To be able to reconstruct the HLBL tensor $\Pi^{\mu\lambda\alpha}$ with dispersion relations, it is imperative to use a decomposition into scalar functions that are free of kinematic singularities and zeros. Such a representation can be obtained following the general recipe put forward by Bardeen, Tung [57], and Tarrach [58] (BTT), resulting in

$$\Pi^{\mu\lambda\alpha} = \sum_{i=1}^{54} T_i^{\mu\lambda\alpha} \Pi_i,$$

(3)

with scalar functions $\Pi_i$ depending on the Mandelstam variables $s = (q_1 + q_2)^2$, $t = (q_1 + q_3)^2$, $u = (q_2 + q_3)^2$ as well as the virtualities $q_i^2$ and Lorentz structures $T_i^{\mu\lambda\alpha}$ [32,33]. This decomposition fulfills gauge invariance manifestly

$$\{q_1^\mu, q_2^\nu, q_3^\sigma, q_4^\rho\} T_i^{\mu\lambda\alpha} = 0,$$

(4)

is highly crossing symmetric (with only 7 distinct structures, all remaining 47 being related to these by crossing transformations), and ensures that the coefficient functions $\Pi_i$ do not contain kinematic singularities and zeros. In addition, the BTT decomposition typically allows for a very economical representation of HLBL amplitudes; e.g., one of the structures coincides with the amplitude for a pseudoscalar pole, while even the sQED amplitude becomes very compact once expressed in terms of BTT functions [56]. For the contribution to $(g-2)_\mu$, a three-dimensional integral representation is available [56]

$$d_{\mu}^{\text{HLBL}} = \frac{\alpha^3}{432\pi^2} \int_0^\infty d\Sigma \int_0^1 dr \frac{1}{\sqrt{-r^2}} \int_0^{2\pi} d\phi \times \sum_{i=1}^{12} T_i(Q_1, Q_2, Q_3) \bar{\Pi}_i(Q_1, Q_2, Q_3),$$

(5)

where the $T_i$ are known kernel functions, the $\bar{\Pi}_i$ suitable linear combinations of the BTT $\Pi_i$, and the Euclidean momenta squared are given by [59]

$$Q_{1,2}^2 = \frac{\Sigma}{3} \left( 1 - \frac{r}{2} \cos \phi \mp \frac{r}{2\sqrt{3}} \sin \phi \right),$$

(6)

$$Q_3^2 = \frac{\Sigma}{3} (1 + r \cos \phi).$$

There are only 6 distinct functions $\bar{\Pi}_i$; the remaining ones are again related to these by crossing symmetry. It suffices to calculate the $\bar{\Pi}_i$ in the kinematic limit, where $q_4 \to 0$; the transition to $(g-2)_\mu$ then proceeds by means of (5).

Two-pion intermediate states.—In a dispersive approach, two-pion intermediate states comprise all contributions that involve a two-pion cut, generically represented by the left (unitarity) diagram in Fig. 1. The dominant term is obtained if in the $\gamma^*\gamma^* \to \pi\pi$ subamplitudes, in turn, the pion is put on shell, i.e., if the pion-pole contribution to the LHC is
isolated. In this case, the remaining hadronic amplitudes are
given by pion vector form factors, and as demonstrated in [33],
this class of two-pion intermediate states, the pure pion box
diagram in diagram (a) in Fig. 1, reproduces the sQED
pion loop with vertices augmented by the appropriate pion
form factors. The reason for this behavior can be traced
back to the fact that only the singularities of the box
diagrams in sQED matter, while the triangle and bulb
diagrams are simply required to restore gauge invariance.
Because of the high degree of crossing symmetry, this pion-
box contribution can be expressed in terms of either fixed-
s, -t, or -u dispersion relations or in a symmetrized form

$$\Pi^\Box_{i} (s,t,u) = \frac{1}{3} \left( \int_{4M_s^2}^{\infty} dt \text{Im} \Pi^\Box_{\text{box}}(s,t',u') \right. $$

$$\left. + \int_{4M_s^2}^{\infty} du \text{Im} \Pi^\Box_{\text{box}}(s,t',u') \right) u' - u $$

$$+ \text{fixed } t + \text{fixed } u. \quad (7)$$

In this case, the representation is exact.

Once heavier intermediate states are considered, generically
denoted by the double lines in diagrams (b) and (c) in
Fig. 1, a more detailed investigation of the double spectral
functions is required. In practice, such contributions can be
included using a partial-wave expansion, in which case the
subprocess becomes a polynomial in the crossed variable,
and the crossed-channel cuts are neglected. Writing down
all crossed versions of the unitarity diagrams shown in
Fig. 1, one sees that each double spectral region appears
exactly twice in a symmetrized form as in (7) so that the
prefactor has to be changed from 1/3 → 1/2 [56], with
corrections suppressed by the mass scale of the neglected
LHC. In particular, this representation becomes exact for
\(\pi\pi\)-rescattering effects, which, by definition, are
polynomial in the crossed Mandelstam variable.

**Partial-wave expansion.**—Constraints from unitarity are
most conveniently formulated in a partial-wave expansion
for HLBL helicity amplitudes \(h^J_{\lambda_1\lambda_2\lambda_3\lambda_4}\),
with angular momentum \(J\) and helicity labels \(\lambda_i\). In this case, the
unitarity relation becomes diagonal

$$\text{Im} h^J_{\lambda_1\lambda_2\lambda_3\lambda_4} (s) = \frac{\sigma_\mu}{16\pi S} h^J_{\lambda_1\lambda_2} (s) h^J_{\lambda_3\lambda_4} (s), \quad (8)$$

where \(\sigma_\mu (s) = \sqrt{1 - 4M^2_s/s}\) gives the phase space, \(S = 2\)
a symmetry factor in case of indistinguishable particles, and
\(h^J_{\lambda_1\lambda_2} (s)\) the helicity partial waves for \(\gamma^\gamma \rightarrow \pi \pi\). Once
formulated in isospin basis, Watson’s theorem [60] [see
also (10) below] guarantees that the phases on the right-
hand side cancel to produce a real imaginary part. The
partial-wave expansion of the pion box is obtained if both
\(h^J_{\lambda_1\lambda_2\lambda_3\lambda_4} (s)\) and \(h^J_{\lambda_1\lambda_2} (s)\) are identified with the partial-wave-
projected Born terms, while the rescattering effects corre-
spond to the unitarity corrections to either subamplitude
derived from (10).

There are 41 independent helicity amplitudes for the full
HLBL tensor, which reduce to 27 if one photon is taken on
shell. Rewriting the representation of the contribution to
\((g - 2)_\mu\), Eq. (5), in such a way that only dispersive
integrals over imaginary parts of these 27 helicity ampli-
tudes appear is highly nontrivial. By explicitly requiring
that unphysical amplitudes drop out in the final result and
that the two redundancies which appear in four space-time
dimensions \(d = 4\) [59] do not affect the result, one can
derive a set of sum rules for the scalar functions. (In [61],
sum rules for the special case of forward HLBL scattering
have been derived.) These sum rules apply to the full
amplitudes but not necessarily at the level of the partial-
wave-expanded ones, producing an apparent dependence
on unphysical amplitudes that would only disappear after a
resummation of all partial waves.

To avoid such pathologies, we were able to construct a
set of 27 amplitudes \(\tilde{\Pi}_i\), related to the 27 singly on-shell
helicity amplitudes by a basis change that we have derived
in explicit analytic form. In the limit \(q_\perp \rightarrow 0\), a subset of the
\(\tilde{\Pi}_i\) includes all the scalar functions needed as input in (5)
[56]. Moreover, this set of 27 amplitudes is manifestly free
of Tarrach [58] or \(d = 4\) ambiguities [59]. For singly on-
shell kinematics, there still exist 15 sum rules among the 27
helicity amplitudes, which we have exploited to optimize to
certain degree the representation with respect to the
convergence of the partial-wave expansion. This formalism
is now ready to be applied to the evaluation of rescattering
effects, but before doing that, we test it with the help of the
pion box and study how well we are able to reproduce its
numerical value by resumming the partial-wave expansion.

**Pion box.**—The formalism for dealing with the pion box
has been developed in [33]. Here, we provide a first
numerical evaluation thereof, with a realistic pion form
factor. The latter has been obtained by fitting a dispersive
representation as suggested in [62,63] to both spacelike [64]
and timelike [65–70] form factor data (similar representa-
tions have been used before in [71–76]), with the result

$$\sigma_\mu^\Box = -15.9(2) \times 10^{-11}, \quad (9)$$

and an uncertainty determined from the differences between
the timelike data sets as well as the details of the fit
representation. The main reduction in uncertainty compared
to earlier evaluations of a “pion loop” [13,16] is due to the
insight that the pion box, defined as two-pion intermediate
states with a pion-pole left-hand cut, is the unambiguous first
term in the expansion and can be expressed in terms of a
hadronic observable, the pion vector form factor, which is
very well known phenomenologically.

The pion box also provides an ideal test case for the
framework presented in the previous section since the full
result is known, and explicit expressions for all BTT scalar
functions are available. As a first step, we verified that the
sum rules encountered in the context of the partial-wave
expansion are fulfilled. Second, in the special case of the
pion box, a fixed-s, -t, -u representation should each hold,
combining to the symmetrized version in (7), so that the
convergence can be studied in each channel separately. The
results, for simplicity, obtained by using a vector-meson-
dominate pion form factor \( F^\pi_\mu(q^2) = M_\rho^2/(M_\rho^2 - q^2) \),
with \( \mu_{\text{box},\text{VMD}}^\rho = -16.4 \times 10^{-11} \), are shown in Table I,
demonstrating that each representation approaches the full
result (going up to \( J_{\text{max}} = 20 \), we checked that also the
remaining differences disappear after partial-wave resum-
mation). The vanishing S-wave contribution for fixed s is
well understood and partly a matter of convention in the
choice of the six functions \( \Pi_i \), see [56]. In concrete
applications, the prescription of changing the prefactor in (7),
as explained above, combines the three representa-
tions in a way that best captures the physics (such as a
resonance) in all channels at once, which means that the
convergence patterns for fixed \( t \) or \(-u \) are more represen-
tative of realistic cases, and the average of the three should
be viewed as a worst-case scenario. But even that displays a
very reasonable convergence behavior.

\( \pi\pi \)-rescattering effects.—We now turn to the evalua-
tion of rescattering effects as a first important step to go beyond
the pion-box contribution. The helicity amplitudes
\( h_{J_1 J_2}(s) \) entering (8) satisfy themselves a unitarity
relation

\[
\text{Im} h_{J_1 J_2}^I(s) = \sin \delta_J^I(s) e^{-i \delta_J^I(s)} h_{J_1 J_2}(s),
\]

with isospin labels \( I \) and \( \pi\pi \) phase shifts \( \delta_J^I \). This relation is
clearly violated for the (real) Born terms alone, but this
deficiency can be easily repaired by solving the dispersion
relation for the subprocess \( \gamma^+ \gamma^+ \rightarrow \pi\pi \).

In contrast to the on-shell and singly virtual case [50–
52], the calculation of the \( \gamma^+ \gamma^+ \rightarrow \pi\pi \) partial waves for two
off-shell photons is complicated by the fact that even for
S waves two different helicity partial waves, \( h_{0, e} \) and \( h_{0, 0} \),
become coupled, including off-diagonal kernel functions
required to eliminate kinematic singularities [30,33]. Here,
we apply this framework to construct the \( \gamma^+ \gamma^+ \rightarrow \pi\pi \)
amplitudes that correspond to the rescattering corrections to
the Born terms, whose solution can still be derived based
on Muskhelishvili–Omnès methods [77,78]. We use \( \pi\pi \)
phase shifts, based on the modified inverse-amplitude
method [79], for the main reason that it has a simple
analytic expression, which is convenient to use in combi-
nation with Muskheleshvili–Omnès methods, while at the
same time, it reproduces accurately the low-energy prop-
ties of the phase shifts as well as pole position and
couplings of the \( f_0(500) \) resonance. This phase shift
departs from the correct one just below the \( K\bar{K} \) threshold
because it does not feature the sharp rise due to the
(980) resonance but continues flat with a smooth high-energy
behavior. A full-fledged evaluation of the \( f_0(980) \) resonance
would require a proper treatment of the \( K\bar{K} \) channel, which is
beyond the scope of this first estimate. We can, on the other
hand, test the sensitivity to the asymptotic part of the
dispersive integrals by studying solutions with different
cutoff values \( \Lambda = [1 \text{ GeV}, \infty) \), constructed with finite-
point techniques [51,80–83]. Moreover, we
checked that for low values of \( \Lambda \), phase shifts obtained by
solving Roy equations [84–86] lead to equivalent results.

The results for the rescattering contribution, summarized
in Table II, are indeed stable over a wide range of cutoffs,
indicating that our input for the \( \gamma^+ \gamma^+ \rightarrow \pi\pi \) partial waves
reliably unitarizes the Born-term LHC, which should
indeed dominate at low energies. In addition, we checked
that the only sum rule that receives S-wave contributions is
already saturated at better than 90%, completely in line
with the expectation that the sum rules will be fulfilled
only after partial-wave resummation. The isospin-0 part
of the result can be interpreted as a model-independent
implementation of the contribution from the \( f_0(500) \) of
about \(-9 \times 10^{-11} \) to HLBL scattering in \((g-2)_\mu \). In total,
we obtain for the \( \pi\pi \)-rescattering effects related to the
pion-pole LHC,

\[
a_{\mu,J=0}^{\pi\pi,\text{pole-LHC}} = -8(1) \times 10^{-11}.
\]

| \( J_{\text{max}} \) | Fixed s | Fixed t | Fixed u | Average |
|----------------|--------|--------|--------|---------|
| 0              | 0.0%   | 102.6% | 102.6% | 70.8%   |
| 2              | 73.9%  | 102.3% | 92.7%  | 89.6%   |
| 4              | 89.2%  | 101.5% | 96.4%  | 95.7%   |
| 6              | 94.3%  | 100.7% | 97.9%  | 97.6%   |
| 8              | 96.5%  | 100.4% | 98.7%  | 98.5%   |

TABLE II. \( S \)-wave rescattering corrections to \( a_{\mu}^{\text{box}} \) in units of
\( 10^{-11} \) for both isospin components and in total.
where the error is dominated by the uncertainties related to the asymptotic parts of the integral, see Table II. Improving the energy region \( \gtrsim 1 \text{ GeV} \) requires the inclusion of the \( K\bar{K} \) channel as well as higher contributions to the LHC, neither of which can be expressed in terms of the pion vector form factor. Very likely, such effects beyond pion states will be less precisely estimated.

Finally, it is instructive to consider the separate contributions not in the isospin but in the charge basis. In this case, the unitarity relation (8) is no longer diagonal, and it is not possible to define unambiguously the contribution of each of the charge states. Irrespective of the detailed convention for the separation, charged-pion states are expected to strongly dominate; e.g., in the chiral expansion, neutral-pion intermediate states first appear at three-loop order. The derivative of the Born-term-subtracted amplitude \( h_{0,\pm}(s) \) is related to the pion dipole polarizability \( \alpha_1 - \beta_1 \), to which the unitarized pion-pole LHC contributes

\[
(\alpha_1 - \beta_1)^{\pi^+,\text{pole LHC}} = (5.4 \ldots 5.8) \times 10^{-4} \text{ fm}^3,
\]

\[
(\alpha_1 - \beta_1)^{\pi^{\pm},\text{pole LHC}} = (11.2 \ldots 8.9) \times 10^{-4} \text{ fm}^3 \quad (12)
\]

for \( \Lambda = 1 \text{ GeV} \ldots \infty \). For the charged pion, this result is in perfect agreement with the chiral two-loop prediction 5.7(1.0) [87] (in the same units) as well as the recent COMPASS measurement 4.0(1.2)_{\text{stat}}(1.4)_{\text{sys}} [88]. In contrast, the two-loop prediction for the neutral pion \(-1.9(0.2)\) [89] is substantially smaller in size and has the opposite sign of what we get from our representation. This failure, however, is not reason for much concern because we are not yet including here the main contributions to the LHC of the amplitude for neutral pions, i.e., vector-meson exchange involving \( V = \omega, \rho \). Because of the scaling with \( \Gamma_{V \to \pi\gamma} \), the relative impact on the neutral \( \Lambda \) wave unitarization

\[
\frac{\alpha_0 \times \text{BR}[\omega \to \pi^0\gamma] + \Gamma_\rho \times \text{BR}[\rho^0 \to \pi^0\gamma]}{\Gamma_\rho \times \text{BR}[\rho^0 \to \pi^0\gamma]} \sim 12 \quad (13)
\]

Is an order of magnitude larger so that heavier intermediate states allow one to repair \((\alpha_1 - \beta_1)^{\pi^0}\) without spoiling agreement in the charged channel. In summary, the rescattering effects in (11) are dominated by the charged pion, with input for the \( \gamma^*\gamma^* \to \pi\pi \) partial waves fully consistent with its dipole polarizability. For this reason, (11) can be considered a model-independent implementation of effects related to the low-energy constants \( L_9 \) and \( L_{10} \), which were suspected to produce large effects in [54]. Our calculation proves that this is not the case and that the related rescattering corrections are indeed of very reasonable size (a similar conclusion was reached within a model approach in [55]). In this context, it should be stressed that our analysis does not rely on chiral operators, thus avoiding the pathologies in their high-energy behavior and the need to cure them. The polarizabilities enter here as the limit of our \( \gamma^*\gamma^* \to \pi\pi \) amplitudes at a particular kinematic point that does not contribute to the dispersive integrals directly, providing an important cross-check of the low-energy limit.

In conclusion, we have shown that our framework allows us to estimate, very accurately, the combined effect of two-pion intermediate states generated by a pion-pole LHC and its S-wave unitarization

\[
a_\mu^{\text{pole LHC}} = -24(1) \times 10^{-11}, \quad (14)
\]

which is considered to be among the most important contributions after the dominant pseudoscalar poles but was so far affected by significant uncertainties. This first numerical result based on the dispersive approach lays the foundation for extensions towards higher partial waves, an improved LHC in the \( \gamma^*\gamma^* \to \pi\pi \) subamplitudes, as well as higher-mass intermediate states, all important prerequisites for a model-independent evaluation of the complete HLBL contribution to \((g-2)\).}

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