Knowledge Cores in Large Formal Contexts

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Abstract—Knowledge computation tasks are often infeasible for large data sets. This is in particular true when deriving knowledge bases in formal concept analysis (FCA). Hence, it is essential to come up with techniques to cope with this problem. Many successful methods are based on random processes to reduce the size of the investigated data set. This, however, makes them hardly interpretable with respect to the discovered knowledge. Other approaches restrict themselves to highly supported subsets and omit rare and interesting patterns. An essentially different approach is used in network science, called k-cores. These are able to reflect rare patterns if they are well connected in the data set. In this work, we study k-cores in the realm of FCA by exploiting the natural correspondence to bi-partite graphs.

This structurally motivated approach leads to a comprehensible extraction of knowledge cores from large formal contexts data sets.

Index Terms—k-Cores, Bi-Partite Graphs, Formal Concept Analysis, Lattices, Implications, Knowledge Base

1 INTRODUCTION

Large (binary) relational data sets are a demanding challenge for contemporary knowledge discovery methods using formal concept analysis [10]. This is due to the fact that many considered problems in this realm are computationally intractable, e.g., enumerating formal concepts, i.e., closed sets, or computing the canonical base [6], [18] of the implicational theory. A cause is the potentially exponential large output size of knowledge discovery processes. For example, large knowledge bases may be incomprehensible to human readers.

Different methods were developed to adapt to the growth of data sets. Sophisticated algorithms employ filtering for data reduction. For example, formal concepts can be filtered by their support in the data set. This is done in Apriori like techniques [24], [28]. More recent methods consider the minimum description length [9]. However, all these approaches are unable to cope with large relational data sets for two reasons: first, they cannot discover rare combinations of attributes that are (comparatively) highly supported in the data set; secondly, computations require an infeasible amount of steps. Moreover, random approaches do not succeed either in these cases, since low supported combinations are unlikely to be sampled. Other techniques, such as feature combination or object clustering [3], [4] lack in meaningfulness.

In general, there are two approaches to overcome the requirements of large data sets with respect to knowledge discovery. One line of research is to introduce novel knowledge features apart from closed sets and their related notions. This may lead to results which are not accessible to well studied knowledge procedures, e.g., from formal concept analysis. The other well investigated practice is to develop data reduction procedures that reduce the data sets significantly. For example, latent semantic analysis or unsupervised clustering of attributes [3], [4] is often applied. This, however, does often lead to unexplainable features.

Data set reduction, i.e., k-Cores by Seidman [22], to the realm of formal concept analysis. The inviolable constraint for our investigation is to maintain interpretability as well as explainability of knowledge with respect to the original data set. To this end we study theoretically as well as experimentally the impact of the core reduction process on the conceptual knowledge. Using this we demonstrate a principle method to discover interesting cores of knowledge in large data sets. In detail, we give a formal overview of to be defined pq-cores and their reduction effects on conceptual structures and implicational theories. Furthermore, we provide valuations for choosing interesting cores in large relational data sets.

We complement our findings by introducing knowledge transformation algorithms. For a given data set and an initial pq-core they are able to provide a computationally efficient navigation process in the emerging knowledge structure of all pq-cores. Finally, we argue that our methods are able to cope with arbitrary subsets of binary relational data.

The rest of our work is structured as follows. In Section 2 we first recollect common notations from formal concept analysis and introduce cores in formal contexts thereafter in Section 2.1. The related formal concept lattice and canonical base are investigated in Section 3 and Section 4. This is followed by an extensive experimental study in Section 5 and Section 6 which is concluded by a presentation of efficient algorithms for pq-cores in Section 7. After a discussion of related work in Section 8 we conclude with Section 9.

2 FORMAL CONCEPT ANALYSIS

Formal concept analysis (FCA) deals with binary relational data sets [10], [27]. These are represented in formal context \((G, M, I)\) where the finite sets \(G\) and \(M\) are called objects and attributes, respectively. The binary relation \(I\) between these sets is called incidence, where \((g, m) \in I\) is interpreted as “object \(g\) has attribute \(m\)”. Two derivation operators emerge on the power sets of \(G\) and \(M\): \(\cdot ^{\prime} : \mathcal{P}(G) \rightarrow \mathcal{P}(M)\) where \(A \mapsto A^{\prime} := \{m \in M \mid \forall g \in A : (g, m) \in I\}\) and \(\cdot ^{\prime} : \mathcal{P}(M) \rightarrow \mathcal{P}(G)\) dually. Composing the two operators leads to two closure operators (i.e., idempotent, monotone, and extensive maps) on \(\mathcal{P}(G)\) and \(\mathcal{P}(M)\). We investigate in this
work induced sub-contexts, i.e., \( S = (H, N, J) \) with \( H \subseteq G \), \( N \subseteq M \), and \( J = I \cap (H \times N) \), denoted by \( S \leq K \). When multiple formal contexts are in play we often use the incidence relation for indicating a derivation, e.g., \( \{g\}^J \) for a derivation of \( g \in G \) in \( K \) and \( \{g\}^J \) for a derivation of \( g \in H \) in \( S \). A formal concept is a pair \((A, B) \in \mathcal{P}(G) \times \mathcal{P}(M) \) with \( A' = B \) and \( B = B' \). We call \( A \) the extent and \( B \) the intent of \((A, B)\) and denote with \( \text{Ext}(K) \) and \( \text{Int}(K) \) the sets of all extents and intents respectively. The set of all formal concepts of \( K \) is denoted by \( \mathfrak{B}(K) \). This set can be ordered by \( \leq \) where \((A, B) \leq (C, D) \iff A \subseteq C \) for \((A, B), (C, D) \in \mathfrak{B}(K)\). The ordered set of all formal concepts is denoted by \( \mathfrak{A}(K) \). The fundamental theorem of FCA states that \( \mathfrak{B}(K) \) is a (complete) lattice. Furthermore, we investigate implications in this work, i.e., \( A \rightarrow B \), where \( A, B \subseteq M \). We say \( A \rightarrow B \) is valid iff \( B' \subseteq A' \). The set of all valid implications is denoted by \( \text{Th}(K) \). Usually, one does work with a base of the theory, e.g., Duquenne–Guigues-Base \([13]\) (canonical base), denoted by \( \mathcal{C}_K \). It can be computed using pseudo-intents, i.e., \( P \subseteq M \) with \( P \neq P'' \) and \( Q'' \subseteq P \) holds for every pseudo-intent \( Q \subseteq P \). The recursive nature of this definition is by design. Besides being the minimal base of the implications from \( \text{Th}(K) \), the set of all pseudo-intents can still be exponential in the size of the context \([18]\).

## 2.1 Cores in Formal Contexts

Our theory on \( pq \)-cores is based on bipartite cores by Ahmed, Batagelj, Fu, et al. \([1]\), Section 3.1. We translated their approach to the realm of formal concept analysis, exploiting the natural correspondence between bipartite graphs and formal contexts. This results in the following definition.

**Definition 2.1.** Let \( K = (G, M, I), S = (H, N, J) \) be formal contexts with \( S \leq K \). We call \( S \) a \( pq \)-core of \( K \) for \( p, q \in \mathbb{N} \), iff

i) \( S \) is \( pq \)-dense, i.e.,
\[ \forall g \in H, \forall m \in N : \{(g^2)^J \} \geq p \land \{m^J \} \geq q \]

ii) \( S \) is maximal, i.e.,
\[ \exists \mathcal{O} \leq K : \text{O \: pq-dense \: s.t. \: \mathcal{O} \neq \varnothing \land \mathcal{O} \neq \mathcal{S} \leq \varnothing \]

We denote this by \( S \leq_{pq} K \). In particular we call contexts \( S \) with \( S \leq_{pq} K \) an attribute-core and \( S \leq_{p, q} K \) an object-core.

**Proposition 2.2 (Uniqueness).** Let \( K = (G, M, I) \) and \( \mathcal{T = (U, V, L) \) be a formal context and \( p, q \in \mathbb{N} \). Then there exists only one \( S \leq K \) with \( S \leq_{p, q} K \).

**Proof.** Let \( S = (H, N, J) \) and \( \mathcal{T = (U, V, L) \) be two different formal contexts with \( S \leq K \) and \( \mathcal{T} \leq K \). Furthermore, for some \( p, q \in \mathbb{N} \) we have that \( S \leq_{p, q} K \) and \( \mathcal{T} \leq_{p, q} \mathcal{K} \). Construct the context \( \mathcal{D} = (H \cup U, N \cup V, J \cup L) \). Then it follows that
\[ \forall g \in H \cup U, \forall m \in N \cup V : \{(g^2)^J \} \geq p \land \{m^J \} \geq q \]

Hence, \( \mathcal{D} \) is \( pq \)-dense and \( S, \mathcal{T} \) are proper sub-contexts of \( \mathcal{D} \). This contradicts the maximality of \( S \) and \( \mathcal{T} \).

Based on this result we refer to \( S \leq_{pq} K \) as the \( pq \)-core. We depict the formal context of an example \( pq \)-core in Figure 1. On the left is the formal context of the prominent “Living beings and Water” example from \([10]\) and on the right is the 4, 3-core of it. We observe that the objects “Bean” and “Leech” as well as the attributes “suckles its offspring” and “two seed leaves” are removed. Even though \( \{\text{[Bean]}\}^J \geq 4 \) it is removed by a cascading effect triggered by the removal of the attribute “two seed leaves”.

## 3 Concept Lattices of \( pq \)-Cores

In this section we investigate the relation of the concept lattice for a \( pq \)-core to the concept lattice of the originating formal context. We investigate in particular the influence of the parameters \( p \) and \( q \). The computation of the \( pq \)-core for some \( p, q \) can be understood as a sequential removal of objects and attributes in arbitrary order. Based on this observation we analyze the impact of object and attribute removal on concept lattices. To this end, we first take a look at a proposition about structural embeddings. For some \( X \subseteq \mathfrak{B}(K) \) we use the notation \( \bigvee X \) for the supremum of \( X \) in \( \mathfrak{B}(K) \) and \( \bigwedge X \) for the infimum of \( X \) in \( \mathfrak{B}(K) \), cf. \([10]\).

**Proposition 3.1** \((\text{[10], Proposition 31 on page 98}]) \). Let \( K = (G, M, I), \mathcal{T = (U, V, L) \) be formal contexts with \( \mathcal{T} \subseteq K \) and \( S \leq K \). Then the mapping \( \mathfrak{B}(T) \rightarrow \mathfrak{B}(K) \) where \( (A, B) \) is mapped to the formal concept \( (B^1, B) \) is a \( \vee \)-preserving order-embedding of \( \mathfrak{B}(T) \) in \( \mathfrak{B}(K) \).

Dually, the map \( \mathfrak{B}(S) \rightarrow \mathfrak{B}(K) \) with \( (A, B) \rightarrow (A, A') \) is a \( \land \)-preserving order embedding of \( \mathfrak{B}(S) \) in \( \mathfrak{B}(K) \).

For \( K \) we observe that Proposition 3.1 is not applicable since a \( pq \)-core has potentially a modified set of objects and attributes with respect to \( K \). Nonetheless, we can still exploit Proposition 3.1 in the following way. First, there exists an order-embedding of \( \mathfrak{B}(H, M, I \cap H \times M) \) into \( \mathfrak{B}(K) \). Secondly, there is an order-embedding from \( \mathfrak{B}(S) \) into \( \mathfrak{B}(H, M, I \cap H \times M) \). Hence, it is easy to see that the composition of the two maps results in an order-embedding from \( \mathfrak{B}(S) \) into \( \mathfrak{B}(K) \). However, suprema and infima are not necessarily preserved. Nonetheless, the existence of the order-embedding does in particular imply that a significant amount of structural (conceptual) information is preserved by the \( pq \)-core with respect to the lattice \( \mathfrak{B}(K) \) and \( p, q \in \mathbb{N} \).

In the following we want to investigate more thoroughly how concepts change when objects/attributes are deleted or added. We start with recalling a fact from \([10], p. 99\) which is related to \([10], Proposition 30 on p. 98\). It describes how attribute closures alter when attributes are removed.

**Proposition 3.2 (Deleting Attributes).** Let \( K = (G, M, I) \) and \( S = (G, N, J) \) be formal contexts with \( S \leq K \). Then

i) \( \forall D \in \text{Int}(K) : (D \cap N) \subseteq \text{Int}(S) \)
ii) \( \forall D \in \text{Int}(S) : D^I \in \text{Int}(K) \)

**Proof.** i) We refer the reader to \([10], p. 99\) ii) We know that \( D^J \subseteq \text{Ext}(S) \) and \( D^J \subseteq \text{Int}(K) \). Therefore, we know that \( D^J \in \text{Int}(K) \). With \( S \leq K \) we can infer that \( D = D^J \subseteq D^J \) and \( D^J \cap N = D \). Hence, with \( D^J \) there exists a \( B \) as required in ii.)
Corollary 3.3 (Adding Attributes). Let $K = (G, M, I)$ and $S = (G, N, J)$ be formal contexts where $S \leq K$ is true. Then,

i) $\forall D \in \text{Int}(S) \exists B \in \text{Int}(K) : B \cap N = D.$

ii) $\forall D \in \text{Int}(S) \backslash \text{Int}(K) : B \cap N = D.$

Proof. i) Use construction of $B$ from Proposition 3.2, part ii). ii) Assume there is no $B$ in $\text{Int}(K) \backslash \text{Int}(S)$ with $B \cap N = D$. From i) we can then draw that $B \in \text{Int}(S) \cap \text{Int}(K).$ With $B = B \cap N = D$ this yields the contradiction $D \in \text{Int}(K)$. 

Based on the insights so far we may draw a lemma that will drive our to be proposed pq-core-algorithm. It will employ an identity: For $K = (G, M, I)$ and $S = (G, N, J)$ is $\text{Int}(K) = (\text{Int}(S) \cup \text{Int}(K) \backslash \text{Int}(S)) \backslash \text{Int}(S) \cap \text{Int}(K)$. 

Lemma 3.4. Let $K = (G, M, I)$ and $S = (G, N, J)$ be formal contexts with $S \leq K$. Given $\text{Int}(S)$ we can compute $\text{Int}(K)$ in output polynomial time in size of $\text{Int}(K) \backslash \text{Int}(S) \cap \text{Int}(S) \cap \text{Int}(K)$. 

Proof. We use the well-known next_closure algorithm from [11]. We choose some order $\leq M$ on such that $\forall n \in M \backslash N \exists n \in N : m \leq M n$. We start the algorithm with $N$, which is the largest closure in $\text{Int}(S)$. The set $\text{Int}(K) \backslash \text{Int}(S)$ can be computed output polynomial by next_closure since for every element $X$ of the output we have $X \cap (M \backslash N) \neq \emptyset$. From Corollary 3.3 we know that for every $Y \in \text{Int}(S) \backslash \text{Int}(K)$ there is a $Z \in \text{Int}(K) \backslash \text{Int}(S)$ with $Y \cap N = Z$. We construct the set $\text{Int}(S) \cap \text{Int}(K)$ by \{X $\cap$ N $|$ X $\in$ Int(K) $\cap$ Int(S) $\cap$ (X $\cap$ N) $\supseteq$ Y $\cap$ N\}. From Corollary 3.3 we find that this construction yields at least all closures in $\text{Int}(S)$ \text{Int}(K)$ and from Proposition 3.2 we know that the construction is limited to elements of $\text{Int}(S)$, again limited by the predicate in the construction to only those from $\text{Int}(K) \backslash \text{Int}(S)$. Altogether, we have output polynomial time for $\text{Int}(K) \backslash \text{Int}(S)$ and one additional polynomial time check for every element of this set. 

Another identity that is useful in the experimental section is $(\text{Int}(K) \cup \text{Int}(S) \backslash \text{Int}(K)) \backslash (\text{Int}(S) \cap \text{Int}(K)) = \text{Int}(S)$. Using this the proof from Lemma 3.4 can also be used to show that computing $\text{Int}(S)$ given $\text{Int}(K)$ is possible in output polynomial time in the size of $\text{Int}(K) \backslash \text{Int}(S) \union \text{Int}(S) \backslash \text{Int}(K)$. Since we want to explain the relation of $pq$-cores lattices to the concept lattice of the original lattice we may state how we derive also the extents.

Corollary 3.5. Let $K = (G, M, I)$ and $S = (G, N, J)$ be formal contexts with $S \leq K$. Given $\mathcal{B}(S)$ we can compute $\mathcal{B}(K)$ in output polynomial time in size of $\text{Int}(K) \backslash \text{Int}(S) \cup \text{Int}(S) \backslash \text{Int}(K)$. 

The only task one has to do for this is to additionally compute $X^T$ for $X \in \text{Int}(K) \backslash \text{Int}(S)$, since all the extents from intents $\text{Int}(S) \backslash \text{Int}(K)$ remain unchanged. We also see that all results in this section about attribute operations can be translated to object operations through duality.

After the theoretical consideration on the impact of adding/removing attributes to formal contexts we now want to look into the dependence of $pq$-cores to removing objects.

Proposition 3.6 (Object Cores). For two formal contexts $K$ and $S$ with $S \leq_{p,0} K$ and $F := \{B \in \text{Int}(K) \mid |B| \geq p\}$ the equality $\{\bigcap X \mid X \subseteq F\} = \text{Int}(S)$ holds. 

Proof. $\subseteq$: Since $S$ is $p$-core of $K$ we have that $\forall B \subseteq M : |B| \geq p \Rightarrow B^{II} = B^{JJ}$. Hence, $\forall X \in F : X^{II} = X^{JJ} \subseteq \text{Int}(S)$. Since $\text{Int}(S)$ is closed under intersection [10] we find that for all $X \subseteq F : \forall X \subseteq \text{Int}(S) : \exists B \subseteq \text{Int}(S)$ with $B \neq \bigcap X \subseteq F$. By definition of $F$ we know that $|B| < p$. Without loss of generality $B$ is meet-irreducible, i.e., there is no $Y \subseteq \text{Int}(K) : \exists \forall B \subseteq \text{Int}(S)$ in which every element is a proper super set of $B$. In this set we either find a meet irreducible set or we go to the next representation until we have sets of cardinality $p$. Thus, there exists an object $g$ of the formal context $S = (H, N, J)$ with $g^T = B$. This contradicts $|\{g\}| \geq p$. 

This proof employs the notion of meet irreducible intents. Computing those is computationally challenging, in particular for larger concept lattices. We presume that one does often consider multiple $pq$-cores for investigation. In this case one may resort to the following idea: given a set of $pq$-cores, find a common super core, i.e., some $T$ of their original context, such that all considered cores are $\leq T$. Compute the cover relation of the conceptual order in $\mathcal{B}(T)$. Using this relation one can infer the meet irreducible elements of $\mathcal{B}(T)$, which are also the possible meet irreducible elements in the concept lattices for all sub-core contexts (or induced sub-contexts).

Based on the above we can now draw some conclusions about computing the $pq$-core concept lattice for some formal context $K$ and $p, q \in \mathbb{N}$. But first we may note the following. 

Remark 3.7. For $S = (H, N, J)$ and $K = (G, M, I)$ with $S \leq_{p, q} K$ it holds that $S \subseteq_{p, 0} (G, N, I \cap G \times N)$. 

Taking all the results above together we find a useful correspondence between the concept lattices of a context, its induced sub-contexts and, in particular, its cores. Starting with a $pq$-core $S \leq_{p, q} K$ we are able to indicate stable concepts (with respect to $K$ or a more general core) in the concept lattice of the $pq$-core. Notably, using Lemma 3.4 we are able to compute efficiently the difference of the concepts $S \subseteq_{p, q} T \subseteq_{p, q} \hat{K}$ with $p \leq \hat{p}$ and $q \leq \hat{q}$.

In the last part of this section we may further generalize the findings above. For some formal context $K$ consider an arbitrary set of induced sub-contexts $K$. We may compare their concept lattices efficiently using Lemma 3.4, following their super/sub-context relation, as depicted Figure 2.

![Figure 2](image-url)
Given a formal \( K \), the set \( K := \{ S \subseteq K \} \) constitutes a complete lattice. One can see this using the map \( \mathcal{P}(G) \times \mathcal{P}(M) \rightarrow K, (H, N) \mapsto (H, N, I \cap (H \times N)) \), which is an order isomorphism from the lattice \( \mathcal{P}(G) \times \mathcal{P}(M) \) to \( K \). Hence, for two arbitrary induced sub-context \( S = (H, N, J) \) and \( T = (U, V, L) \) of \( K = (G, M, I) \) on may compute \( \mathcal{B}(\mathcal{V}(S, T)) \) and \( \mathcal{B}(\mathcal{A}(S, T)) \) in order to infer \( \mathcal{B}(T) \) efficiently using \( \mathcal{B}(S) \), or vice versa. The set of all \( pq \)-cores is contained in \( K \), however, it does not constitute a lattice. To see this a counter example is presented in Figure 3.

3.1 A Small Case Study

We apply our notion for \( pq \)-cores on a particularly small example, the Forum Romanum (FR) context ([10, Figure 1.16]), in order to study the applicability to real-world data sets. The data set consists of monuments on the Forum Romanum (objects) and their star ratings by different travel guides (attributes). In Figure 4 we depicted the concept lattice for \( S \) and their star ratings by different travel guides.

There are two essential notions when discussing implications with confidence one are considered valid in \( K \). The sup-

| \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( c_1 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) | \( x \) |
| \( b_1 \) | \( x \) | \( x \) | \( x \) |
| \( b_2 \) | \( x \) | \( x \) | \( x \) |
| \( b_3 \) | \( x \) | \( x \) | \( x \) |
| \( c_1 \) | \( x \) | \( x \) | \( x \) |

Figure 3. An example context (upper) and the order relation of all \( pq \)-cores (lower). Each node in the order diagram represents a \( pq \)-core with its \( p, q \) values written above the node.

Proposition 4.1 (Core Implications). Let \( K, S \) be formal contexts, with \( S \subseteq p, q K \) where \( K = (G, M, I) \) and \( S = (H, N, J) \). For all \( A \rightarrow B \in \text{Th}(S) \) is

i) \( |H|/|G| \cdot \text{sup}_K(A \rightarrow B) \leq \text{sup}_K(A) \rightarrow B) \)

ii) \( \text{sup}_K(A \rightarrow B) \leq |H|/|G| \cdot \text{sup}_K(A) \rightarrow B) \)

iii) \( \text{conf}_K(A \rightarrow B) \geq |A/B|/|A| \cdot |G|/|H| \cdot \text{sup}_K(A) \rightarrow B) \)

iv) \( |A| \geq p \implies \text{conf}_K(A \rightarrow B) = 1 \)

v) \( |A \cup B| \geq p \implies \text{sup}_K(A \rightarrow B) = |H|/|G| \cdot \text{sup}_K(A) \rightarrow B) \).

Proof. i) Since \( J \subseteq I \) we can infer that \( |A| \leq |A| \), we can infer \( |H|/|G| \cdot \text{sup}_K(A) \leq \text{sup}_K(A) \).

ii) With the same argument as in i) we can infer \( |A| \leq |A| + |G| \cdot |H| \), from which one can deduce the statement.

iii) Using i) and ii), which would be the best case / worst case for supports, since all additional objects are counter examples for \( A \rightarrow B \), we find \( \text{conf}_K(A \rightarrow B) = |A/B|/|A| \cdot |G|/|H| \) divided by \( |H|/|G| \cdot |A|/|H| \cdot \text{sup}_K(A) \rightarrow B) \). This can be simplified to \( \text{conf}_K(A \rightarrow B) = |A/B|/|A| \cdot |G|/|H| \cdot |H|/|G| \cdot |A|/|H| \cdot \text{sup}_K(A) \rightarrow B) \). For \( |A| \geq p \) we have \( A \equiv A/J \) by definition of \( pq \)-cores and also \( (A \cup B) ^J = (A \cup B) ^J \). Together with the definition of confidence we obtain the statement.

iv) From \( |A \cup B| \geq p \) we find that \( |A/B|/|A| \cdot |G|/|H| \cdot \text{sup}_K(A) \rightarrow B) \), which results in a equality in i). \( \square \)

Note that i), ii), and iii) are also valid for sub-contexts. We now study minimal representations of implicational theories, i.e., the canonical base of \( \text{Th}(S) \) for some formal context \( K \).

The next logical step would be to partially derive the canonical base for some formal context \( K \) using a \( pq \)-core. However, this endeavor is so far not understood. In the simple case of formal contexts on disjoints attribute sets, i.e., computing the canonical base of \( (G, N_i \cup N_j, J_i \cup J_j) \) using the bases of \( (G, N_1 \cup N_2, J_1 \cup J_2) \), we refer the reader to [26]. Nonetheless, we may yield some results for the canonical direct bases [5], [10] (CDB), i.e., a complete, sound and iteration-free basis. Such a basis for a formal context \( K = (G, M, I) \) is constituted by set of proper premises, i.e., sets \( A \subseteq M \) where \( |A| \) does hold, cf. [12].
Assume \( k \) be a proper premise of \( G \). From this we can conclude that \( B^J \subseteq B^I \). Following, there is an \( n \in N \) with \( n \notin A \) and \( n \notin B^J \) for all \( B \subseteq A \). With the following Lemma 4.3, we find that for all \( B \subseteq A \) we have \( B^J = B^J \cup (B^I \setminus N) \). Therefore, we find that \( n \notin B^J \). From this we can conclude that \( n \in A^I \setminus (A \cup \bigcup_{B \subseteq A} B^I) \) which is therefore not empty.

**Lemma 4.3.** Let \( \mathbb{K} = (G, M, I), \mathbb{S} = (G, N, J) \) be two formal contexts with \( \mathbb{S} \subseteq \mathbb{K} \) and \( B \subseteq N \), then \( B^J = B^J \cup (B^I \setminus N) \).

**Proposition 4.2** (Induced Contexts CDB). Let \( \mathbb{K} = (G, M, I), \mathbb{S} = (G, N, J) \) be two formal contexts with \( \mathbb{S} \subseteq \mathbb{K} \) and let \( \mathcal{L}_p(\mathbb{S}), \mathcal{L}_p(\mathbb{K}) \) be their canonical direct bases, then

\[
\mathcal{L}_p(\mathbb{S}) \subseteq \mathcal{L}_p(\mathbb{K}).
\]

**Proof.** Let \( A \subseteq N \) be a proper premise of \( \mathbb{S} \). Hence, we know that \( A^J \setminus (A \cup \bigcup_{B \subseteq A} B^J) \neq \emptyset \). Following, there is an \( n \in N \) with \( n \notin A \) and \( n \notin B^J \) for all \( B \subseteq A \). With the following Lemma 4.3, we find that for all \( B \subseteq A \) we have \( B^J = B^J \cup (B^I \setminus N) \). Therefore, we find that \( n \notin B^J \). From this we can conclude that \( n \in A^I \setminus (A \cup \bigcup_{B \subseteq A} B^I) \) which is therefore not empty.

In this section we want to study experimentally their applicability on real-world data sets. The most pressing question is to identify particularly interesting cores of a given formal context. A commonly used technique to assess the interestingness of k-cores in networks is to investigate the number of connected components depending on the core parameter \( k \). A well-known observation is that the number of connected components increases the greater \( k \) is. Parameters that are considered interesting are those around the steepest rate of increase in the number of components. Also often considered are changes of some valuation function, such as the size of the largest connected component or some network statistical property. We will adapt the former idea and analyze the component structures.

### Data Sets

We conduct our investigation on five various sized data sets from different domains. Living beings in Water is the well known FCA data set [10, Figure 1.1]. It consists of living beings as objects and their properties as attributes. Forum Romanum as already used in Section 3.1, is also taken from [10]. It is made of places of interest as objects and their ratings in different tour guides as attributes. Spices is created by the authors. The objects are dishes and the attributes are spices to be used for these dishes. The incidence relation is extracted from a spices planer [19]. Mushroom is an often used classification data set provided by UCI [8]. The objects are mushrooms and the non-binary attributes are common mushroom properties. Those were scaled using a nominal scale. The Pocket Knives data set was self-created by the authors through crawling the Victorinox AG website \(^1\) in April 2019. The context contains all pocket knives as objects and their features as attributes. Wiki44k was created in an experimental study [16] on finding implications in Wikidata. It is a scaled context drawn from the most dense part of the Wikidata knowledge graph. All presented data sets are available in the FCA software \texttt{conexp-clj} [14] through GitHub.\(^2\) We collected their numerical properties in Table 1.

### Interesting pq-cores

For all data sets we applied different combinations of parameters \( p \) and \( q \) and evaluated to what extent this leads to interesting pq-cores using the steepest increase method. For this we regarded all non-empty pq-cores as bipartite graph and counted the resulting connected components. We observed that no data set has a pq-core with more than one connected component. This is surprising since constructing a formal context falling apart into multiple connected components for some \( p \) and \( q \) is easy. This might indicate that real-world data sets do not exhibiting this property. However, we acknowledge that the number of considered data sets is comparatively low. Nonetheless, this observation might be attributed to the following fact: in

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1. https://www.victorinox.com
2. https://github.com/tomhanika/conexp-clj

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![Diagram of concept lattice](image-url)
all data sets there is a small number of objects with high support, i.e., many attributes, covering in union all attributes and having at least pairwise one attribute in common. These objects are contained in all \( pq \)-cores. Hence, we need to adapt the idea of components to the realm of formal contexts differently. For this we consider the context size distribution among all \( pq \)-cores. In these distribution we may characterize sub-contexts that are removed while computing a \( pq \)-core as structural components. This is in contrast to the classical component analysis for graph \( k \)-cores. Using those we define interesting \( pq \)-cores as those where a further increase of \( p \) or \( q \) would result in a high increase in the size of the removed structural component. In our experiments we find that there are many such critical \( p \) and \( q \) for the investigated data sets. To narrow this set we propose the following pragmatic selection criteria due to computational limitations: 1) The size of a selected core should be in the range of computational feasibility (with respect to the to be employed analysis procedures). 2) The parameters \( p \) and \( q \) of a selected core should differ in magnitudes, i.e., either \( p \ll q \) or \( p \gg q \). The interpretation of either criterion depends on the particular data analysis application. For example, if one is more interested in keeping a larger attribute domain then one should choose an interesting core with low \( q \) and high \( p \). Analogously one might one want to keep more objects.

This being said we want to propose a different approach for characterizing interesting \( pq \)-cores. In contrast to solely considering a \( pq \)-core \( \subseteq_{p,q} K \) of some context \( K \) one might look into the concept lattice that is created by this \( pq \)-core, i.e., \( \mathfrak{B}(\subseteq) \). With this approach the size of the resulting concept lattice could be a criterion to select a \( pq \)-core. The motivation for this is that we rather select a \( pq \)-cores depending on the entailed conceptual knowledge then purely on contextual size. This approach is computational costly since we need to compute a large number of concept lattices. However, relying on Lemma 3.4, Proposition 3.6 and Remark 3.7 we may ease this cost significantly. Analogously we propose selection criteria: 1) The diagram of a selected core lattice should be human readable, (e.g., the number of concepts should be in a human feasible range) 2) The parameters \( p \) and \( q \) of a selected core lattice should differ in magnitudes, i.e., either \( p \ll q \) or \( p \gg q \). Again, the concrete employment of either criterion depends on the particular data analysis application. For example, we find a lattice with more than thirty concepts too large for human comprehension, even if drawn with sophisticated drawing algorithms. Hence, we will consider this number for the rest of this work as bound. On a final note in this section, we consider the special cases of object- and attribute cores not to be interesting. They remove attributes or objects simply by their object/attribute support and do not represent an interesting sub-structure.

**Experiment: Water**

We analyze the **living beings and water** context Figure 1 and present our core analysis in Figure 5. For this we computed the size of all core concept lattices. A first observation is that interesting cores, with respect to our just introduced notion of interestingness, are the 4, 3, 3, 4- and 2, 4-core. We suspect that they include important knowledge. Increasing the core parameters more would lead to an (almost) empty concept lattice. From this list of interesting \( pq \)-cores we present the lattice diagram of 4, 3 in Figure 5. This lattice contains thirteen formal concepts in contrast to the nineteen in the original concept lattice. The 4, 3-core captures a significant portion of knowledge from the original domain, however, only six out of eight objects and seven out of nine attributes are in the picture. We can still infer two different groups of beings, plants and animals. Nonetheless, the original lattice is much more refined. For example, the original concept lattice is more distinct in the subsets of beings that need **chlorophyll** or those who can move around. We consider the \( pq \)-core to be a more coarse representation of the entailed domain knowledge.

**Experiment: Spices**

In this experiment we analyze a spice recommendation data set. This context is derived from a spice planer published in [19]. It contains 56 meals and 37 spices. Meals in the data set cover nine categories which are not part of the formal context. There are fifteen vegetables, nine meat, three poultry, five fish, five potato, four rice dishes, as well as three sauces, eight baked goods and four diverse dishes. The incidence relation is which meal requires which spices. The resulting concept lattice of the original context contains 531 formal concepts. The results of applying \( pq \)-cores to this data set with different parameters is depicted in Figure 6. There is a great number of candidate cores to be considered, i.e., cores with a steep decrease in the number of formal concepts while increasing parameters \( p \) or \( q \). However, many of those are still very large with respect to the number of formal concepts, e.g., 5, 7-core or the 9, 4-core. Following our pragmatic criterion for human readability those are not interesting. In contrast is the 5, 11-core (cf. light red color in figure). In this core lattice the parameters \( p \) and \( q \) are approximately equally sized.

**Figure 5.** Figure on the left shows the concept lattice sizes for all \( pq \)-cores of living beings and water data set, the abscissa indicates \( p \) and the ordinate \( q \). On the right we present the 4, 3-core.

**Figure 6.** Concept lattice sizes for all \( pq \)-cores of spices data set, the abscissa indicates \( p \) and the ordinate \( q \).
Hence, it only covers a dense object and attribute selection. In particular there are twelve dishes using six spices.

As another selection we present two different cores exhibiting a large attribute coverage and large object coverage respectively. A real-world motivation for this is: one wants to cook lots of different dishes with possibly fewer spices; one is focused on a diverse usage of spices with possibly fewer meals. We exemplify this with the 2, 18-core and the 14, 1-core, as depicted in Figure 7. The 2, 18-core includes 28 concepts with 33 out of the 56 dishes. The 14, 1-core has 32 concepts with 29 out of the 37 spices. While having less than 10% of the size of the original concept lattice, both concept lattices cover a vast amount of human recognizable knowledge. A thorough investigation with respect to implications is done later in this work.

6 The Problem of Large Contexts

Large formal contexts constitute an infeasible problem for classical formal concept analysis. This is in particular true when computing implicational theories of them. Applying FCA notions only to \(pq\)-cores may be a possible resort. However, this results in a large number of \(pq\)-cores to be considered, which constitutes a problem in its own, see Table 1. Since our ultimate goal in this work is to present a novel method for coping with large formal contexts, we demonstrate and evaluate an approach for reducing the search space for \(p\) and \(q\) in this section. For \(S \leq_{p,q} K\) we know from Proposition 3.1 that \(|B(S)|\) decreases monotonously in \(p\) and \(q\). Let \(\hat{p} \in \mathbb{N}\) be the maximal number such that for all \(S \leq_{p,q} K\) with \(p \geq q\) and \(|B(S)| \leq 30\) we have that \(S \leq_{p,q} T \leq_{\hat{p},1} K\). Furthermore, let \(\hat{q} \in \mathbb{N}\) be the maximal number such that for all \(S \leq_{p,q} K\) with \(p < q\) and \(|B(S)| \leq 30\) we have that \(S \leq_{p,q} T \leq_{1,\hat{q}} K\). This implies that cores with human readable sized concept lattices are sub-contexts of particular object- and attribute cores. Our computational approach now is based on finding those particular cores. Equipped with these contexts we do only need to consider \(pq\)-cores \(S \leq_{p,q} K\) that are sub-contexts of \(T \leq_{\hat{p},1} K\) or \(T \leq_{1,\hat{q}} K\). Since a direct computation of \(\hat{p}\) and \(\hat{q}\) is infeasible we suggest an estimation. A naïve solution for this would be to examine the derivation size distribution of all objects or attributes. For the data sets investigated in this work this approach was unsuccessful. More fruitful is a binary search among the parameters. We set for this the bound for the concept lattice size to 60 as threshold (which is twice as large as what we consider as readable). Therefore, even if the \(\hat{p}\), 1-core is not human readable, we may encounter \(\hat{q}\), \(q\)-core with \(q > 1\) that is readable. A general observation for large formal contexts in the following experiments is that cores with readable concept lattice tend to having extreme values for parameters \(p, q\), i.e., either \(p \ll q\) or \(q \ll p\).

**Binary Search For Cores In Mushroom**

Due to its size (in context as well as in concept lattice terms) the Mushroom data set is an ideal candidate for the just proposed binary search. Computing the sizes of all core concept lattices is infeasible. We search as an initial core for our search paradigm \(\hat{q}\) with \(p = 1\). We start with \(\hat{q} = |G|\), which results almost surely in an empty context for real-world data sets. The binary search in \([1, |G|]\) gives a \(pq\)-core with \(p = 1\) and \(q = 4937\). With 38 formal concepts the concept of this sub-context has less than two times 30 concepts, which we considered human readable. Using this core we reduce the search space to 12832 different \(p, q\), which are all bound by 38 in the number of formal concepts. We may note that searching for some \(\hat{q}\) is impractical for this data set. This is due to the fact that it was created by scaling twenty-three non-binary attributes into 119 nominal-scaled attributes. Hence, there are only two sub-contexts of the mushroom context which are in core relation for \(q = 1\). More accurately, these are the mushroom context and the empty context. We depicted a heat-map of the core concept lattices in Figure 8 for...
Binary Search for Cores in Wiki44k

To provide another example, we perform the same search in the Wiki44k data set. The corresponding concept lattice contains 21,923 formal concepts and we were able to compute that there are approximately 98,000 non-empty \( pq \)-core contexts. Hence, computing all interesting (Section 5) cores is computationally costly. Therefore, we resort again to the binary search approach. As the largest attribute core with a readable concept lattice we identified 1,5202-core, having 54 formal concepts. We display a heat-map for the concept lattice size distribution of all sub-cores starting from this bound in Figure 9. As for the object core we discovered that the 15,1-core has 139 concepts. However, the 16,1-core is empty, thus we are constrained to employ the 15,2-core. Starting from this we can report that the 15,3-core and the 15,4-core have twenty-five concepts and beyond that the cores are empty. Hence, those two are interesting candidates. Despite having more concepts than we considered readable we looked more thorough into the 15,2-core. Using background knowledge about the Wikidata properties we are able to present a well-drawn diagram of its lattice, as depicted in Figure 9. We realized that in this core we do only cover eighteen out of 101 attributes. This is, for example, in contrast to our observations for the Spice data set, where more than 50% were covered using a similar sized \( pq \)-core. Nonetheless, the 15,2-core provides a rough overview about the most important properties in the Wiki44k data set, in terms of usage for items, and how they are connected.

Coming back to the object core investigation, we start with the 1,5202-core. From there we find two candidates for interesting \( pq \)-cores, namely the 1,8290-core on 41735 objects, seven attributes with 34 concepts and the 4,7115-core on 20748 objects, eight attributes with 38 concepts. Although the latter covers more attributes we decided to look into the former. The reason for this is the increased readability (due to a lower number of concepts) and the higher object coverage. Cores with a higher object coverage entail implications with a higher confidence in the original concept lattice, see Proposition 4.1. For the visualizations of Figure 9 we decided to indicate the objects using their Wikidata item numbers instead of their labels. This core describes a majority of the WikiData entities contained in the dataset. The Wiki44k data set employs properties used for countries or people for the majority of statements. Using our proposed core analysis we are able to provide a human readable diagram representing how these properties are related. This, in turn, enables us to identify logical errors. For example, we found that there entities which are countries with an occupation and a gender, see the concept in Figure 9 indicated in red. The Wikidata description of these properties, however, states that the country property should not be used on human. By a closer look into the data set we found that one of these entities is “Alfred A. Knopf”, which is both a person (Q61108) and the name of an American book publisher (Q1431868). Hence, someone added claim to Wikidata on a wrong item. Besides the study of property usage we can also employ our analysis method for the identification of missing information, i.e., missing statements in Wikidata. We see in Figure 9 that all properties that are depicted on the right part of the diagram describe human features, e.g.,

$q \in [4937, 8123]$ and $p \in [1, 5]$. We are interested in cores with as much readable conceptual information as possible, which are cores with $4937 < p < 5100$, that are also interesting. Out of those we find the 5,5176-core is interesting. This core contains seven distinct attributes and 7930 mushrooms. In the depiction of the corresponding concept Figure 8 we refrained from annotating all objects and indicated the number of mushrooms instead (using short-hand notation from FCA). Hence, to get the total number of objects associated to some concept one has to add to the object count all numbers from concepts in the order ideal of that concept. When comparing the core lattice with the original lattice we notice that the object number for all concepts with at least five attributes is similar, which is expected from our theoretical considerations.

Figure 8. Heat-map for the core concept lattice sizes (above) and the concept lattice of the 5,5176-core of the mushroom context (below).
occurrence (P106), country of citizenship (P27), and gender (P21). Honoring the constraint that occupation is only to be used for instances of (P31) human (Q5), we find 66 items having P106 but missing the property P27. For example, one is "James Blunt" (Q130799), an English singer-songwriter.

The approach described above can be conducted for arbitrary combinations of Wikidata properties. Hence, pq-cores enable the user to validate or contradict reasonable constraints in incomprehensible sized data sets, at least to some confidence. Furthermore, the pq-core approach enables an automated procedure for checking implicational bases, cf. Proposition 4.1. In particular, one could employ methods from [16] to investigate implicational bases in Wikidata through pre-computing feasible sized pq-core contexts.

6.1 Comparison with the TITANIC approach

TITANIC [25] is an Apriori based approach that computes all formal concepts having a minimum support in the data set. Like Apriori, TITANIC computes these concepts in a bottom-up fashion, with respect to the attributes. This results in an ordered set of concepts which constitutes a join-semilattice. An example of such a result, here based on the Mushroom data set, is depicted in Figure 10. In the following we compare concept lattices arising from pq-cores to the join-semilattices computed through TITANIC. We reuse for our analysis the pre-identified interesting 5,176-core \( S \) of Mushroom (see Figure 10, above) and indicated support-values (in \( S \)) for all object concepts, i.e., for all concepts that fulfill \( (g^{\ell}, g^f) \) for \( g \in H \). These numbers are to be read as follows: the true support value for some concept \( c \) is the sum of all support values of concepts in the order ideal \( \downarrow c \) from \( c \). How support values of \( S \) relate to support values in \( K \) was discussed in Proposition 4.1. We observe that \( S \) comprises seven attributes compared to the TITANIC semilattice which has twelve. Both conceptual structures are built-on thirty-two formal concepts. In particular, twenty-one intents of \( S \) are present in the TITANIC semilattice. Hence, the pq-core data reduction approach exhibits a different notion for selecting important subsets of data. Nonetheless, a more thorough investigation of the differences in applicability to real-world problems is deemed future work.

For the rest of this section we investigate the implications one can draw from the TITANIC semilattice \( T \) and compare them to the ones valid in the pq-core \( S \). We know from Proposition 4.1 that all implications \( A \rightarrow B \) with premise length at least five are also valid in the mushroom context. In the pq-core we have 70 such implications. However, there are no non-trivial implications with premise length greater or equal five entailed in \( T \). We consider this a major advantage of the novel pq-core approach in contrast to TITANIC. As for valid implications in \( S \) with premise length less than five we know from Proposition 4.1 that those are implications with high confidence in the Mushroom context. The support value in the mushroom context of such an (valid) implication can also be computed according to Proposition 4.1. For example, since \(|H| = 7930\) and \(|G| = 8123\) we know that the valid implication \( A \rightarrow B \) with \( s := \sup_s (A \rightarrow B) \) has in the Mushroom data set at least 7930 / 8123 \( \cdot s \) support. Due to \( S \) being a 5,176-core we know that support of \( A \rightarrow B \) is \( s \) in the Mushroom context if \( |A \cup B| \geq 5 \). From our analysis we
conclude: while the $pq$-core of the Mushroom context does not have as much attributes as the TITANIC semilattice, it may contain more information in terms of implications.

7 ALGORITHMS

For a novel data reduction approach it is essential to have efficient algorithms available. In this section we present two computational problems concerned with $pq$-cores and their algorithmic solution. We start with the fundamental problem of computing the $pq$-core $S$ for a given formal context $K$. Our solution to this problem is an adaption of an algorithm by Matula and Beck 1983 [20] for computing $k$-cores of graphs. Given some graph $G = (V, E)$ with $E \subseteq \binom{V}{2}$ it uses bucket queues to repeatedly find and remove vertices of small degree. The bucket queue $Q$ is generated with $Q[k] := \{v \in V \mid deg_G(v) = k\}$. After that, the algorithm removes iteratively all vertices in buckets with index smaller than $k$ and reassigns the remaining vertices to buckets of corresponding degree. Our adaption to $pq$-cores employs this algorithm. However, due to the bipartite nature of our data we provision two bucket queues, for objects and attributes, respectively. The computational cost for initializing these buckets queues for a context $(G, M, I)$ is $O(|G| \cdot |M|)$. The worst case cost for one removal iteration on both queues is bound by $O(|G|p + |M|q)$. In this particular case the algorithm has to update the remaining derivation size of at most $p$ attributes for each removed object and $q$ objects for each removed attribute respectively. Hence, the total computation complexity for our algorithm, as presented in Algorithm 1, is $O(|G| \cdot |M|)$. A worst case context is one of ordinal scale as seen in Figure 11.

Navigating Between $pq$-core Lattices

In Section 5 we characterized the interestingness of cores. This required knowledge about the corresponding concept lattice sizes of $pq$-cores. However, every computation of such an concept lattice is (possibly) costly and the number of these computations is large. For example, we have seen that the Wiki44k data set has 97,773 non-empty $pq$-cores. To overcome this issue (to some extent), we developed an algorithm based on the theory presented in Figure 2 (right).

Problem 7.1 (Core Lattice). Given $K$ and the set of all its concepts $B(K)$ compute for $S \subseteq p, q \subseteq K$ the set of concepts $B(S)$.

For solving this problem we present Algorithm 2, which is based on Propositions 3.2 and 3.6. This algorithm employs a so for not recollected notion in FCA, duality. We say the dual of a formal context $S = (H, N, J)$ is $S^d := (N, H, J^{-1})$. Furthermore, by abuse of notation, we denote by $B(S)^d$ the

Algorithm 1: Compute $p, q$-core

Input: A context $K = (G, M, I)$ and $p, k \in N$
Output: $S$, with $S \subseteq \mu K$

// initialize core context
1 init output $(H, N, J)$ as $(G, M, I)$
2 init $A$, with $A[i] = \{g \in U \mid |g^d| = i\}$
3 init $B$, with $B[i] = \{m \in V \mid |m^d| = i\}$
4 while $\exists g \in A[i < p]$ or $\exists m \in B[i < k]$ do
5 $U = U \setminus \{g \in A[i < p]\}$
6 $V = V \setminus \{m \in B[i < k]\}$
7 $J = J \cap U \times V$
8 update $A$ and $B$
9 return $S = (H, N, J)$
We omit the simple case of $D$ in $\text{Int}(T)$ and have therefore $D \not\in \text{Int}(T)$, $\not\supseteq$: Since $D \subseteq B$ it follows that $B^L \subseteq D^L$. We also know that $D^L = D^J$ because of $D \subseteq N$ and the fact that $S$ is a induced sub-context of $K$ on the same object set. Thus, $B^L \subseteq D^J$, $\subseteq$: For each $D \in \text{Int}(S)$, $D^J \in \text{Ext}(T)$, according to $[10$, Proposition 30]. Therefore, we know that $D^J \subseteq \text{Int}(S)$. With $S \subseteq K$ we know that $D \subseteq D^J$ and following $D_{\text{Int}}^J \cap N = D_{\text{Int}}^K \cap D$. Therefore, for each $D \in \text{Int}(S)$ there exists a $B \in \text{Int}(T)$ with $B \cap N = D$ and $B^L = D^J$. Hence, $D^J \subseteq \bigcup_{B \in \text{Int}(T), B \cap N = D} B^L$. \hfill \Box

Secondly, we remove all objects that are not contained in $S$ from the extents of $B$ and apply the same remove_attributes method to the duals (see Line 4).

The overall run-time complexity of this algorithm is linear in the number of concepts, since the computation of duals is linear and the overall iteration consumes the set of concepts. This is an improvement compared to the output polynomial time complexity of the common computation of $\text{B}(S)$.

In case we only require to compute the set of all concept intents of a pq-core, we can apply Proposition 3.6 in combination with the cover relation of the concept lattice. This relation of $(\text{B}(K), \leq)$ is given by $\prec \leq$ such that for all $c, d \in \text{B}(K)$ we have $c \prec d$ if $c < d$ and there is no $e \in \text{B}(K)$ with $c < e < d$. Using both Proposition 3.6 and $\prec$ we can remove all attributes through intersecting with $N$ (cf. Algorithm 2). Afterwards it is sufficient to remove meet-irreducible intents with cardinality $< p$. These can be identified easily using the cover relation, i.e., the elements with exactly one upper neighbor.

In Figure 2 we illustrate a generalization of Algorithm 2 to arbitrary sub-contexts as stated by the following problem:

**Problem 7.3 (Lattices of Sub-contexts).** Let $S = (H, N, J)$ be a formal context and $\text{B}(S)$ its concepts. Compute the set of concepts $\text{B}(T)$ of $T = (U, V, L)$, with $L \cap H \times N = J \cap U \times V$.
or unsupervised clustering algorithms on the object set/attribute set [3], [4]. However, we find the resulting concept lattices do lack on meaningfulness. Since all mentioned approaches introduces new attributes, e.g., as linear combination of the original attributes, they often loose their human explainability. Contrary there are also procedures to automatically/manually select attributes and objects of relevance to the user [2], [15]. However, these approaches may require a fair amount of domain knowledge, which is not always available. Furthermore, such processes are very often time consuming for large data sets, e.g., with hundreds of attributes, when done manually. A major shortfall of these techniques is that they do not provide proper estimations for their impact on the concept lattice of the original data set.

Another course of action to cope with large formal contexts are techniques such as TITANIC [24]. They address the computational and knowledge size issue by omitting rare attribute combinations, i.e., less supported ones. We consider this a problem as discussed in the first paragraph. Nonetheless, an advantage of TITANIC is that the resulting iceberg ‘lattice’ is sized comprehensively and does not introduce any error with respect to the original concept lattice. Nonetheless, when dealing with implicational knowledge of the investigated domain we can draw less knowledge from iceberg concept lattices, as observed in Section 6.

A well-established method for data set reduction originates from the research field of network analysis, called cores [7], [17]. The original idea for this goes back to Seidman [22]. In there, a network is reduced to a densely connected part. A variation for bipartite networks are pq-cores [1]. Cores are also applied in the realm of pattern structures [23]. Our presented work on pq-cores is based on the research results mentioned in this paragraph and extends them to knowledge cores in formal contexts. Notions, like the impact of pq-cores on concept lattices and the canonical bases are so far not investigated, to the best of our knowledge.

9 Conclusion

In this work we presented an approach to define and investigate the knowledge core of a formal context. For this we employed a notion from two-mode networks, called pq-cores. We transferred the idea from graph theory to formal concept analysis and introduced the notion of pq-core formal contexts in a formal manner. Based on that, we identified essential differences of pq-core lattices and their originating concept lattice. In particular we investigated conceptual differences for general sub-contexts and demonstrated their application to cores. Secondly, we demonstrated different approaches to data analysis using pq-cores. Crucial here was the characterization of interestingness among core lattices.

As for practical demonstration we analyzed different data sets. We could show that our method is able to compute two meaningful core lattices for the spices data set that are also human comprehensible in size. For the wiki44k data set, we were able to pinpoint wrongly used properties as well as missing information using a core lattice diagram.

Furthermore, we found theoretical results enabling us to depict different algorithms for computing and transforming core structures from formal context data sets. As for knowledge bases we were able to provide different estimations for the validity of implicational knowledge in a concept lattice based on core concept lattice computations. We notably showed that some transformations can be done in time linear in the size of the original concept lattice. An exceptionally interesting result is the now achieved ability to navigate efficiently between arbitrary core lattices of a data set without recalculating partially shared concepts. The more these contexts have in common, with respect to their closure systems, the faster a transformation will perform. All algorithms presented in this work are implemented and provided via the FCA software conexp-clj [14], a free and open-source research tool written in Clojure.

For future work we identify different meaningful lines of research. First of all a large experimental study on real-world data sets is required. In such a study domain experts from different fields should evaluate the meaningfulness of core knowledge to their research investigations. Second, we envision a combination of pq-cores with other data reduction approaches. For example, one could couple the TITANIC approach with pq-cores. In such a setup one could compute an initial interesting core with our method and employ in a second step TITANIC to compute an highly supported fraction. In a third research thread we propose a more thorough investigation of the set of all pq-cores. Although we could show that this set does not constitute a lattice structure one may draw meaningful knowledge from investigating the shown order relation with tools from directed graph analysis. Finally, we anticipate an application of pq-cores in temporal knowledge settings. Due to the shown efficient adaptability to small changes in objects or attributes pq-cores are an ideal candidate to maintain the dynamic knowledge of a domain.

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