Numerical method for deriving sharp inclusion of the Sobolev embedding constant on bounded convex domain

Kazuaki Tanaka\textsuperscript{1,*}, Kouta Sekine\textsuperscript{2}, Makoto Mizuguchi\textsuperscript{1}, Shin’ichi Oishi\textsuperscript{2,3}

\textsuperscript{1}Graduate School of Fundamental Science and Engineering, Waseda University,
\textsuperscript{2}Faculty of Science and Engineering, Waseda University,
\textsuperscript{3}CREST, JST

Abstract. In this paper we proposed a verified numerical method for deriving a sharp inclusion of the Sobolev embedding constant from $H^1_0(\Omega)$ to $L^p(\Omega)$ on bounded convex domain in $\mathbb{R}^2$. We estimated the embedding constant by computing the corresponding extremal function using verified numerical computation. Some concrete numerical inclusions of the constant on a square domain were presented.

Key words: embedding constant; Sobolev inequality; verifying positiveness; verified numerical computation

1 Introduction

We are concerned with the best constant $C_p(\Omega)$ in the Sobolev type inequality satisfying

$$
\|u\|_{L^p(\Omega)} \leq C_p(\Omega) \|u\|_{H^1_0(\Omega)}, \quad \forall u \in H^1_0(\Omega),
$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2, 3, \cdots$) and $1 < p < \infty$ if $n = 2$, $1 < p < (n + 2)/(n - 2)$ if $n \geq 3$. Since Sobolev type inequalities are important in studies on partial differential equations, there have been a lot of works on such inequalities and their applications, e.g., [1, 2, 3, 4, 5, 10, 13, 18, 21, 20, 23]. The classical Sobolev embedding theorem has been well known. Moreover, a formula giving the best constant in the classical Sobolev inequality on $\mathbb{R}^n$ was independently shown by Aubin [1] and Talenti [23] in 1976 (see Theorem A.1). Since all elements in $H^1_0(\Omega)$ can be regarded as those in $H^1(\Omega)$ by zero extension outside $\Omega$, we can obtain a rough upper bound of $C_p(\Omega)$ for a general domain $\Omega \subset \mathbb{R}^n$ using the formula (see Corollary A.1). One can find another estimation formula in [18] (see Theorem A.2).

In this paper we will propose a numerical method for deriving a verified sharp inclusion of the best constant $C_p(\Omega)$ satisfying (1) for bounded convex domain $\Omega \subset \mathbb{R}^2$, e.g., we proved the following proposition by our method:

Proposition 1.1. The smallest values of $C_p(\Omega)$ ($p = 3, 4, 5$) satisfying (1) are enclosed as follows:

$$
C_3(\Omega) \in [0.25712475017617, 0.25712766496560],
C_4(\Omega) \in [0.28524446071925, 0.28524446071939],
C_5(\Omega) \in [0.31058015094169, 0.31067136032829],
$$

where $\Omega = (0, 1)^2$.

Hereafter, we replace the notation $C_p(\Omega)$ with $C_{p+1}(\Omega)$ for notational convenience. The smallest value of $C_{p+1}(\Omega)$ can be written by

$$
C_{p+1}(\Omega) = \sup_{u \in H^1_0(\Omega)} \Phi(u),
$$

E-mail address: *imahazimari@fuji.waseda.jp
where $\Phi (u) = \|u\|_{L^{p+1}(\Omega)} / \|u\|_{H^1_0(\Omega)}$. It is well known that $C_{p+1}(\Omega)$ in (5) is finite and realized by an extremal function $u^* \in H^1_0(\Omega)$ (see, e.g., [6]). The critical point problem for $\Phi$ is reduced to finding weak solutions to the following problem:

$$
\begin{cases}
-\Delta u = u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(6)

This problem has only one solution if $\Omega \subset \mathbb{R}^2$ is bounded and convex [11]; therefore, in this case, we can derive an inclusion of $C_{p+1}(\Omega) = \|u^*\|_{L^{p+1}(\Omega)} / \|u^*\|_{H^1_0(\Omega)}$ by computing the solution to (6) with verification.

There is a number of numerical methods for verifying solution to semilinear elliptic boundary value problems (e.g., in [14, 17, 18, 22]) and related works, e.g., [15, 24]. These methods enable us to obtain a concrete ball in the senses of $\|\nabla \cdot \|_{L^2(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ containing exact solution and can be applied to the problem:

$$
\begin{cases}
-\Delta u = |u|^{p-1}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(7)

However, positiveness of the solution to (7) should be proven to derive a verified solution to (6). One of the main contributions of this paper is to propose how to verify the positiveness.

This paper consists of the following sections: In Section 2 and 3, we propose a method for proving positiveness of a solution to (7) and a method for estimating the embedding constant $C_{p+1}(\Omega)$, respectively. In Section 4, some numerical examples are also presented, which lead Proposition 1.1.

## 2 Verification method for positiveness

In this section, we propose a sufficient condition for positiveness of the solution to (7), which will be summarized in Theorem 2.1. This theorem enables us to numerically check the positiveness.

Let us first introduce the following notation to be used throughout this paper:

- define $\mathbb{N} := \{1, 2, 3, \cdots \}$;
- let $B(x, r ; \| \cdot \|)$ be an open ball whose center is $x$ and whose radius is $r \geq 0$ in the sense of the norm $\| \cdot \|$;
- denote its closure by $\overline{B}(x, r ; \| \cdot \|)$;
- let $L^\infty(\Omega)$ be the functional space of Lebesgue integrable functions over $\Omega$, s.t., $|u(x)| < \infty$ (a.e. $x \in \Omega$) with the norm $\|u\|_{L^\infty(\Omega)} := \text{ess sup}\{|u(x)| \mid x \in \Omega\}$;
- let $H^1(\Omega)$ be the first order $L^2$ Sobolev space on $\Omega$;
- let $H^1_0(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial \Omega \text{ in the trace sense}\}$ which endowed with the norm $\|\cdot\|_{H^1_0(\Omega)} := \|\nabla \cdot\|_{L^2(\Omega)}$.

Let us remark that, for Lebesgue integrable functions, we omit the expression “almost everywhere” for simplicity, e.g., we denote $u > 0$ in the place of $u(x) > 0$ a.e. $x \in \Omega$ for $u \in H^1_0(\Omega)$.

Let us introduce the following lemma for proving Theorem 2.1.

**Lemma 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. All weak solutions $u \in H^1_0(\Omega)$ to (6) satisfies $\text{ess sup}\{u(x)^{p-1} \mid x \in \Omega\} \geq \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the following problem:

$$
(\nabla u, \nabla v)_{L^2(\Omega)} = \lambda (u, v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega).
$$

(8)
Proof. Let $\phi_1 \geq 0$ ($\phi_1 \neq 0$) be the first eigenfunction corresponding to $\lambda_1$, which satisfies

$$\int_{\Omega} u^p (x) \phi_1 (x) \, dx = \lambda_1 \int_{\Omega} u (x) \phi_1 (x) \, dx.$$ 

We have

$$\int_{\Omega} u^p (x) \phi_1 (x) \, dx = \int_{\Omega} \{u (x)\}^{p-1} \{u (x) \phi_1 (x)\} \, dx$$

$$\leq M_u \int_{\Omega} u (x) \phi_1 (x) \, dx$$

$$= \lambda_1^{-1} M_u \int_{\Omega} u^p (x) \phi_1 (x) \, dx,$$

where $M_u := \text{ess sup} \{u (x)^{p-1} \mid x \in \Omega\}$. Positiveness of $\int_{\Omega} u^p (x) \phi_1 (x) \, dx$ implies $M_u \geq \lambda_1$. □

Using lemma 2.1, we are able to prove the following theorem, which gives a sufficient condition for positiveness of the solution to (7).

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain. If a solution $u \in C^2 (\Omega)$ to (7) is positive in a subdomain $\Omega' \subset \Omega$ and if $\sup\{u^- (x)^{p-1} \mid x \in \Omega\} < \lambda_1 (\Omega)$, then $u > 0$ in the original domain $\Omega$, where $\lambda_1 (\Omega) > 0$ is the first eigenvalue of the problem (8) and $u^-$ is defined by

$$u^- (x) := \begin{cases} -u (x), & u (x) < 0, \\ 0, & u (x) \geq 0. \end{cases}$$

Namely, $u$ is also a weak solution to (6).

**Proof.** Assume that $u$ is not a positive solution in $\Omega$, which ensures there exists a domain $\Omega'' \subset \Omega \setminus \Omega'$ such that $u < 0$ in $\Omega''$ and $u = 0$ on $\partial \Omega''$. Hopf’s lemma makes it impossible for a subdomain $\Omega_0 \subset \Omega$ such that $u \equiv 0$ in $\Omega_0$ to exist. Therefore, the restricted function $v := -u|_{\Omega''}$ can be regarded as a solution to

$$\begin{cases} -\Delta v = v^p & \text{in } \Omega'', \\ v > 0 & \text{in } \Omega'', \\ v = 0 & \text{on } \partial \Omega''. \end{cases}$$

From Lemma 2.1, we have

$$\sup_{x \in \Omega} u^- (x)^{p-1} \geq \sup_{x \in \Omega''} v (x)^{p-1} \geq \lambda_1 (\Omega'').$$

Since $\Omega'' \subset \Omega$, we have $\lambda_1 (\Omega'') \geq \lambda_1 (\Omega)$. Hence, $u$ is a solution to (6) if $\sup\{u^- (x)^{p-1} \mid x \in \Omega\} < \lambda_1 (\Omega)$.

**Remark 2.1.**

i) Since Hopf’s lemma requires the regularity $u \in C^2 (\Omega)$, we also need this regularity to prove Theorem 2.1. A weak solution $u \in H^1_0 (\Omega)$ to (7) is in $C^2 (\Omega)$, e.g., when $\Omega$ is a bounded convex domain with piecewise $C^2$ boundary (see, e.g., [7]).

ii) The first eigenvalue of the problem (8) can be numerically estimated by, e.g., the method in [12], which enables us to concretely evaluate the eigenvalues of (8) on polygonal domains in $\mathbb{R}^2$. 

3
3 Estimation method for embedding constant

In this section, we propose a method for estimating the embedding constant \( C_{p+1}(\Omega) \) defined in (5). The following theorem gives concrete estimation of the embedding constant from a verified solution to (6).

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex domain. If there exists the solution to (6) in a closed ball \( B(\hat{u}, r ; \| \cdot \|_{H^1_0(\Omega)} ) \) with \( \hat{u} \in H^1_0(\Omega) \) satisfying \( \| \hat{u} \|_{H^1_0(\Omega)} > 2r \), then the embedding constant \( C_{p+1}(\Omega) \) defined in (5) is estimated as

\[
\frac{\| \hat{u} \|_{L^{p+1}(\Omega)}}{\| \hat{u} \|_{H^0_0(\Omega)}} \leq C_{p+1}(\Omega) \leq \frac{\| \hat{u} \|_{L^{p+1}(\Omega)}}{\| \hat{u} \|_{H^0_0(\Omega)} - 2r}.
\]

**Proof.** It is obvious that \( \| \hat{u} \|_{L^{p+1}(\Omega)} / \| \hat{u} \|_{H^0_0(\Omega)} \) is a lower bound of \( C_{p+1}(\Omega) \). The solution to (6) is unique when \( \Omega \subset \mathbb{R}^2 \) is bounded and convex [11]; therefore, the ratio \( \| u \|_{L^{p+1}(\Omega)} / \| u \|_{H^0_0(\Omega)} \) is maximized by the solution \( u \) to (6). Let us write the solution to (6) as \( \hat{u} + rv \) with \( v \in H^0_0(\Omega) \), \( \| v \|_{H^0_0(\Omega)} = 1 \). Then, we have

\[
C_{p+1}(\Omega) = \frac{\| \hat{u} + rv \|_{L^{p+1}(\Omega)}}{\| \hat{u} + rv \|_{H^0_0(\Omega)}} \leq \frac{\| \hat{u} \|_{L^{p+1}(\Omega)} + rC_{p+1}(\Omega)}{\| \hat{u} \|_{H^0_0(\Omega)} - r},
\]

that is,

\[
\left( \| \hat{u} \|_{H^0_0(\Omega)} - 2r \right) C_{p+1}(\Omega) \leq \| \hat{u} \|_{L^{p+1}(\Omega)}.
\]

Hence, when \( \| \hat{u} \|_{H^0_0(\Omega)} > 2r \), \( \| \hat{u} \|_{L^{p+1}(\Omega)} / (\| \hat{u} \|_{H^0_0(\Omega)} - 2r) \) becomes an upper bound of \( C_{p+1}(\Omega) \).

**Remark 3.1.** Theorem 3.1 can be naturally applied to the case that \( n \geq 3 \). For example, if \( \Omega \) is a convex symmetric domain in \( \mathbb{R}^n \) (\( n \geq 3 \)) and \( 1 < p < (n + 2)/(n - 2) \), (6) has only one solution [8, 16].

4 Numerical example

In this section, we present some numerical examples of proving positiveness of a solution to (7) and estimating the corresponding embedding constant, which will lead Proposition 1.1.

All computations were carried out on a computer with Intel Xeon E5-2687W 3.10 GHz, 512 GB RAM, CentOS 6.3, and MATLAB 2012a. All rounding errors were strictly estimated using toolboxes for verified numerical computations: INTLAB version 6 [19] and KV library version 0.4.16 [9]. Therefore, the accuracy of all results is mathematically guaranteed.

We first treated the case that \( p = 3 \) and \( \Omega = (0, 1)^2 \), which corresponds to the critical point problem for the embedding constant \( C_4(\Omega) \). In this case, (7) has an infinite number of solution, while it has only one positive solution [11]. We computed an approximate solution to (7) with the Fourier basis \( \phi_{ij} := a_{ij} \sin(\pi ix) \sin(\pi jy) \), \( 1 \leq i, j \leq N \), \( a_{ij} \in \mathbb{R} \). Moreover, we proved existence of the solution to (7) in a \( H^0_0(\Omega) \)-ball and \( L^\infty \)-ball whose center is the approximation using the method in [17] with the method in [24]. Figure 1 shows an approximate solution \( \hat{u} \) to (7) such that existence of the positive solution to (7) in a neighbourhood of \( \hat{u} \) was proven. Table 1 shows verification results in this case: The first eigenvalue \( \lambda_1(\Omega) \) of (8) is \( 2\pi \); therefore,
\[ \sup \{ u_\lambda (x)^{p-1} \mid x \in \Omega \} < \lambda_1 (\Omega) \] holds, this means positiveness of the solution to (7) is proved, in all cases in the table. Moreover, the last row shows intervals containing \( C_4 (\Omega) \), e.g., \( 1.23783 \) represents the interval \([1.23456, 1.23789]\); the case that \( N = 34 \) corresponds to (3).

We also treated the cases that \( p = 2, 4 \) on the same domain \( \Omega \). Remark that the maximum principal guarantees positiveness of all solutions to (7) with even \( p > 1 \) except for the trivial solution. We derived the estimation results (2) and (4) by computation with bases of \( N = 140 \) and \( N = 20 \), respectively.

In Table 2, one can find a comparison between lower and upper bounds by our method, upper bounds by Plum’s formula [18] (Theorem A.2), and the classical bounds by Corollary A.1.

\[
\begin{align*}
\text{Table 1: Verification result of the case that } p = 3. \\
N & H_0^1\text{-error} & L^\infty\text{-error} & \sup \{ u_{\lambda} (x)^2 \mid x \in \Omega \} & C_4 (\Omega) \\
10 & 2.449623e-02 & 3.223795e-02 & 3.458360e-03 & 0.2871011044062756445768010 \\
20 & 1.531402e-06 & 1.973787e-06 & 7.071145e-04 & 0.28524445117578346071925 \\
30 & 6.227678e-11 & 8.015445e-11 & 7.070096e-04 & 0.2852444607122331925 \\
34 & 2.284208e-12 & 2.939286e-12 & 7.070096e-04 & 0.28524446071925 \\
\end{align*}
\]

![Figure 1: An approximate solution to (7) in the case that \( p = 3 \).](image)

\[
\begin{align*}
\text{Table 2: Upper bounds of the embedding constant by our method, by Corollary A.1, and by Theorem A.2.} \\
C_p (\Omega) & \text{Our method} & \text{Corollary A.1} & \text{Theorem A.2} \\
C_3 (\Omega) & 0.25712775664986560475107617 & 0.27991104681667 & 0.32964899322075 \\
C_4 (\Omega) & 0.285244460719253992 & 0.31830988618379 & 0.3989228040144 \\
C_5 (\Omega) & 0.32186713603282958015094169 & 0.35780388458050 & 0.4890903972535 \\
\end{align*}
\]
5 Conclusion

In this paper we proposed a numerical method for deriving a sharp inclusion of the best constant in the Sobolev inequality (1). We derived inclusions of the constants by computing the solution to problem (6) with verification. The positiveness of a verified solution to (7) was proved using the method proposed in Section 2. The accuracy of all results, e.g., those in Proposition 1.1, is mathematically guaranteed using toolboxes for verified numerical computations [19, 9].

A Simple bounds for the embedding constant

The following theorem gives the best constant in the classical Sobolev inequality.

**Theorem A.1** (T. Aubin, 1976 [1] and G. Talenti, 1976 [23]). Let \( u \) be a function in \( H^1(\mathbb{R}^n) \) (\( n = 2, 3, \cdots \)). Moreover, let \( q \) be a real number such that \( 1 < q < n \), and set \( p = nq/(n-q) \). Then,

\[
\|u\|_{L^p(\mathbb{R}^n)} \leq T_p \|\nabla u\|_{L^q(\mathbb{R}^n)}
\]

holds for

\[
T_p = \pi^{-\frac{1}{2}} n^{-\frac{q}{2}} \left( \frac{q-1}{n-q} \right)^{1-\frac{q}{n}} \left\{ \frac{\Gamma \left(1 + \frac{n}{2}\right) \Gamma (n)}{\Gamma \left(\frac{n}{2}\right) \Gamma \left(1 + n - \frac{n}{q}\right)} \right\}^{\frac{1}{p}}
\]  

with the Gamma function \( \Gamma \).

The following corollary, which comes from Theorem A.1, gives simple bounds for the embedding constant from \( H^1_0(\Omega) \) to \( L^p(\Omega) \) for a bounded domain \( \Omega \).

**Corollary A.1.** Let \( \Omega \subset \mathbb{R}^n(n = 2, 3, \cdots) \) be a bounded domain. Let \( p \) be a real number such that \( p \in (n/(n-1), 2n/(n-2)) \) if \( n \geq 3 \) and \( p \in (n/(n-1), \infty) \) if \( n = 2 \). Moreover, set \( q = np/(n+p) \). Then, (1) holds for

\[
C_p(\Omega) = |\Omega|^\frac{2-n}{2q} T_p,
\]

where \( T_p \) is the constant in (9).

**Proof.** By zero extension outside \( \Omega \), we may regard \( u \in H^1_0(\Omega) \) as a element \( u \in H^1_0(\mathbb{R}^n) \). Therefore, from Theorem A.1,

\[
\|u\|_{L^p(\Omega)} \leq T_p \|\nabla u\|_{L^q(\mathbb{R}^n)}.
\]

Hölder’s inequality gives

\[
\|\nabla u\|_{L^q(\Omega)}^q \leq \left( \int_{\Omega} |\nabla u(x)|^q u^2 dx \right)^{\frac{q}{2}} \left( \int_{\Omega} |1|^\frac{2-q}{2} dx \right)^{\frac{2-q}{2}}
\]

\[
= |\Omega|^{\frac{2-n}{2q}} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{q}{2}},
\]

that is,

\[
\|\nabla u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{2-n}{2q}} \|\nabla u\|_{L^2(\Omega)}.
\]

(11)

where \( |\Omega| \) is the measure of \( \Omega \). From (10) and (11), it follows that

\[
\|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{2-n}{2q}} T_p \|\nabla u\|_{L^2(\Omega)}.
\]

\[\square\]
Using the following theorem, one can also obtain an upper bound of the embedding constant when the minimal point of the spectrum of $-\Delta$ on $H^1_0(\Omega)$ is concretely computed or estimated.

**Theorem A.2** (M. Plum, 2008 [18]). Let $\rho \in [0, \infty)$ denote the minimal point of the spectrum of $-\Delta$ on $H^1_0(\Omega)$ for a domain $\Omega \subset \mathbb{R}^n$.

a) Let $n = 2$ and $p \in [2, \infty)$. With the largest integer $\nu$ satisfying $\nu \leq p/2$, (1) holds for

$$C_p(\Omega) = \left(\frac{1}{2}\right)^{\frac{1}{2} + \frac{2\nu - 3}{p}} \left[\frac{p}{2} \left(\frac{p}{2} - 1\right) \cdots \left(\frac{p}{2} - \nu + 2\right)\right]^{\frac{2}{p}} \rho^{-\frac{n}{2}},$$

where $\frac{p}{2} \left(\frac{p}{2} - 1\right) \cdots \left(\frac{p}{2} - \nu + 2\right) = 1$ if $\nu = 1$.

b) Let $n \geq 3$ and $p \in [2, 2n/(n-2)]$. With $s := n(p^{-1} - 2^{-1} + n^{-1}) \in [0, 1]$, (1) holds for

$$C_p(\Omega) = \left(\frac{n-1}{\sqrt{n(n-2)}}\right)^{1-s} \rho^{-\frac{n}{2}}.$$

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