The Adiabatic Transport of Bose-Einstein Condensates in a Double-Well Trap: Case a Small Nonlinearity

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A complete adiabatic transport of Bose-Einstein condensate in a double-well trap is investigated within the Landau-Zener (LZ) and Gaussian Landau-Zener (GLZ) schemes for the case of a small nonlinearity, when the atomic interaction is weaker than the coupling. The schemes use the constant (LZ) and time-dependent Gaussian (GLZ) couplings. The mean field calculations show that LZ and GLZ suggest essentially different transport dynamics. Significant deviations from the case of a strong coupling are discussed.

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I. INTRODUCTION

Nowadays the trapped Bose-Einstein condensate (BEC) is one of the most active topics in modern physics, see monograph 1 and reviews 2, 3, 4, 5, 6. Between many branches of this activity, investigation of weakly bound condensates or multicomponent BEC is of keen interest. It involves various aspects of Bose Josephson junction 7,8 including those in periodically modulated traps 9, 10, 11 and optical lattices 5, 12, 13, transport problems in double-well 10, 14, 15, 16, triple-well 16, 17, 18, 19, 20, 21, 22, 23, and multi-well 24 traps, topological states 24, 25, etc. In all these areas the nonlinearity caused by interaction between BEC atoms plays an essential role. It drastically enriches BEC dynamics by new effects and phenomena. At the same time, the nonlinearity complicates and even hampers some process, e.g. the adiabatic transport 18, 19, 20.

In this paper, we investigate the influence of a week nonlinearity on the adiabatic transport in a double-well trap. The transport is produced by a controllable and irreversible tunneling through the barrier separating the wells. It assumes that BEC atoms, being initially in one potential well, are completely transferred to another well and then kept there. The process can be driven by varying the system parameters in time, such as space separation between the wells and relative position of the well depths. Such transport can be realized in in multi-well traps 8 and arrays of selectively addressable traps 27. In this problem we actually deal with two weakly bound (through the barrier) condensates or multicomponent BEC with the components defined as the populations of the wells. Being produced, the complete irreversible transport could serve as a useful tool for general manipulations of the condensate. Besides, it could open interesting perspectives for generation and investigation of various geometric phases 25, 26, creation of topological states 25, 26, etc.

The transport in multi-well traps can be produced by many ways: Landau-Zener 14, 15, 17 and Rosen-Zener 23 methods, periodic time-dependent potential modulation 10, Rabi switch 24, Stimulated Raman Adiabatic Passage (STIRAP) 16, 18, 20, 21, 22, etc.. All these methods provide a robust population transfer of the ideal (without interaction) condensate but often suffer from the detrimental influence of the nonlinearity 10, 15, 20, 21, 22. This especially concerns the adiabatic population transfer methods, like STIRAP for triple-well/level systems 29, 30, 31, which, being generally robust to small variations of the process parameters, are, nevertheless, fragile to the nonlinearity. Then a natural question arises: is it possible to turn the nonlinearity from the detrimental to favorable factor of the adiabatic transport?

As was recently shown 15, the nonlinearity can indeed favor the adiabatic transport if it is modeled within the Landau-Zener (LZ) 22 and Gaussian Landau-Zener (GLZ) 12 protocols. The latter protocol assumes a Gaussian time-dependent monitoring of the coupling between the wells (barrier penetrability) Ω(t) and a linear evolution in time of the difference ∆(t) between the well depths. The GLZ is thus a generalization of both LZ and RZ methods, periodic time-dependent potential modulation 10, Rabi switch 24, Stimulated Raman Adiabatic Passage (STIRAP) 16, 18, 20, 21, 22, etc., which, being generally robust to small variations of the process parameters, are, nevertheless, fragile to the nonlinearity. Then a natural question arises: is it possible to turn the nonlinearity from the detrimental to favorable factor of the adiabatic transport?
regime where not the center but edges of the Gaussian coupling play a decisive role. The LZ and GLZ protocols were analyzed in terms of nonlinear structures arising in the stationary spectra.

In this paper we continue the development of the LZ and GLZ transport schemes but now for a case of a weak nonlinearity \( UN < \Omega \) where \( N \) is the total number of BEC atoms and \( U \) is the atomic interaction. This case is not covered in [15] and hence devotes an additional analysis. We will show that, at small nonlinearity, a complete population transfer is possible for both LZ and GLZ but, unlike the strong nonlinearity case, it takes place at sufficiently low process rates (which was actually expected following the previous studies [12, 13, 14, 15, 17]). At even lower rates, i.e., close to the adiabatic limit, the LZ transfer is heavily spoiled by strong and slowly damped Rabi oscillations while the GLZ protocol is reduced to the Rabi switch. The nonlinear structures (loops, etc) can appear in the stationary spectra but, unlike the strong coupling case [15], they cannot be already used as reliable tools for the analysis of transport features. The transport asymmetry weakens for both LZ and GLZ.

This paper is outlined as follows. In Sec. II the population transfer methods are sketched. In Sec. III the relevant mean field formalism is done. The numerical results are discussed in Sec. IV. The conclusions are given in the Sec. V.

II. TRANSPORT PROTOCOLS

The LZ tunneling between energy levels \[ \text{S2} \] is a general physical process which can be straightforwardly recast for the nonlinear transport of BEC atoms in a double-well trap [12, 13, 14, 15, 17]. The resulting scheme is illustrated in Figure 1 a)-b).

As seen from Fig. 1a)-b), the LZ transfer is controlled by the constant coupling \( \Omega \) and time-dependent difference between the well depths \( \Delta(t) = \alpha t \) where \( \alpha \) is the detuning rate. Following Fig.1c), in the GLZ protocol, the coupling is defined as a time-dependent Gaussian pulse

\[
\Omega(t) = K \tilde{\Omega}(t), \quad \tilde{\Omega}(t) = \exp\left(-\frac{(t - \bar{t})^2}{2\Gamma^2}\right), \tag{1}
\]

where \( K \) is the amplitude, \( \Gamma \) is the pulse width, and \( \bar{t} \) is the centroid time. In the experimental setup, the detuning can be monitored by varying the well depths while the Gaussian coupling by the proper change of the separation distance between the wells [27].

In the linear (without nonlinearity) case, the final probability of the LZ transfer reads \[ \text{S2} \]

\[
P = 1 - e^{-\frac{-\bar{t}^2}{2m}}, \tag{2}
\]

It allows a complete transition \( P = 1 \) only in the adiabatic limit \( \alpha \to 0 \). However, as is seen from the rough estimation [13]

\[
P \approx 1 - e^{-2\pi \frac{\bar{t}^2}{2m}(1 + \frac{\Delta t}{\Lambda})}, \tag{3}
\]

inclusion of the nonlinearity factor \( \Lambda \sim U \) makes possible the complete transport in much wider range of \( \alpha \). Moreover, the effect is obviously asymmetric with respect to the \( \Lambda \) (interaction) sign. This peculiarity of the nonlinear LZ scheme was used in [15] to turn the nonlinearity from the detrimental to helpful factor for the transport.

The introduction of the GLZ protocol is motivated by the well-known fact that the LZ transfer actually takes place only within a finite time interval near the symmetry point \( \Delta(t) \approx 0 \) when \( \Delta(t) < \Omega \). Then it is natural to use a time-dependent coupling of a certain duration, say of the Gaussian form. Note that in fact the GLZ protocol is a generalization of LZ (constant coupling and time-dependent detuning) [32] and RZ (constant detuning and time-dependent coupling) [34] schemes.

III. MEAN FIELD MODEL

BEC transport is studied in mean-field approximation within the Gross-Pitaevskii equation (GPE) \[ \text{S3} \]

\[
\dot{\Psi}(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\vec{r}, t) + g_0 |\Psi(\vec{r}, t)|^2 \right] \Psi(\vec{r}, t) \tag{4}
\]

where the dot means time derivative, \( \Psi(\vec{r}, t) \) is the order parameter of the system, \( V_{\text{ext}}(\vec{r}, t) \) is the external trap potential involving both (generally time-dependent) confinement and coupling, \( g_0 = 4\pi a / n \) is the parameter of the interaction between BEC atoms, \( a \) is the scattering length, and \( n \) is the atomic mass.

In the two-mode approximation [33], the order parameter in a double-well trap can be written as

\[
\Psi(\vec{r}, t) = \sqrt{N} (\psi_1(t) \Phi_1(\vec{r}) + \psi_2(t) \Phi_2(\vec{r})) \tag{5}
\]
where \( \Phi_k(\vec{r}) \) is the static ground state solution of (1) for the isolated k-th well and
\[
\psi_k(t) = \sqrt{N_k(t)} e^{i\phi_k(t)}
\]
is the amplitude, related to the corresponding population \( N_k(t) \) and phase \( \phi_k(t) \). The total particle number \( N \) is conserved, i.e. \( \int d\vec{r}\Psi(\vec{r}, t)^2/N = \sum_{k=1}^M N_k(t) = 1 \).

Note that the approximation (5) is generally valid for a weak interaction and small number of atoms, say \( N < 1000 \), see discussion [3, 14]. In this connection, using the GPE within the two-mode approximation is somewhat contradictory, since the former assumes a large number of atoms \( N \) while the latter is valid for small condensates. Nevertheless, a reasonable balance between these conditions is possible and their combination is widely used in studies of BEC dynamics in double-well traps, see e.g. [1, 14, 20].

By using the linear canonical transformation [20]
\[
z = N_1 - N_2, \quad Z = N_1 + N_2 = 1, \quad \theta = \frac{1}{2}(\phi_2 - \phi_1), \quad \Theta = -\frac{1}{2}(\phi_1 + \phi_2)
\]
it is convenient to turn the unknowns \( N_k \) and \( \phi_k \) to new variables, population imbalance \( z \) and phase difference \( \theta \). This allow to extract from the equations the integral of motion \( Z \) and corresponding total phase \( \Theta \).

Then, by substituting (3)-(10) into (11), performing the spatial integration, and using (11)-[13], we obtain equations of motion [2, 20]
\[
\dot{z} = -\tilde{\Omega}(t)\sqrt{1 - z^2}\sin 2\theta, \\
\dot{\theta} = \frac{1}{2}([\Delta(t) + \Lambda z + \tilde{\Omega}(t)]\frac{z}{\sqrt{1 - z^2}}\cos 2\theta),
\]
where \( \tilde{\Omega}(t) = \Omega(t)/K \) is the normalized coupling [1],
\[
\Delta(t) = \frac{1}{2K}(E_1(t) - E_2(t)) = \alpha t
\]
is the scaled detuning,
\[
\Lambda = \frac{UN}{2K}
\]
is the key nonlinearity parameter determining the ratio between the coupling amplitude \( K \) and interaction \( U \). In [9]-[10], the time \( t \) is rescaled as \( 2Kt \rightarrow t \). Eqs. (9)-(10) are solved to get the numerical results presented in the next section.

In principle, the values \( \Omega(t), E_{1,2}(t) \) and \( U_{1,2} = U \) are determined from the GPE as
\[
\Omega(t) = -\frac{1}{\hbar} \int d\vec{r} \left[ \frac{\hbar^2}{2m} \nabla \Phi_1 \cdot \nabla \Phi_2 + \Phi_2^* V(t) \Phi_1 \right], \\
E_k(t) = \frac{1}{\hbar} \int d\vec{r} \left[ \frac{\hbar^2}{2m} \nabla \Phi_k^* \cdot \nabla \Phi_k + \Phi_k^* V(t) \Phi_k \right],
\]
and
\[
U_k = \frac{g_0}{\hbar} \int d\vec{r} |\Phi_k|^4.
\]

However, in the present study they come as input parameters. In GLZ, the coupling is approximated by the Gaussian function [1]. The energies \( E_{1,2} \) enter the detuning (11) and the interaction \( U \) between the atoms inside every well is included into the nonlinearity parameter (12).

In the previous study [20], the stationary spectra (chemical potentials)
\[
\mu = \frac{1}{2}[(\Delta(t) + z^2 - \tilde{\Omega}(t))\sqrt{1 - z^2}\cos 2\theta],
\]
were employed for the analysis of the transport. These spectra are obtained by using the equations \( \dot{z} = \dot{\theta} = 0 \) which explicitly read
\[
\theta = \frac{\pi}{2} n, \\
\Delta(t) + z(\Lambda + (-1)^n(\tilde{\Omega}(t) \sqrt{1 - z^2}) = 0
\]
where \( n \) is an integer real number. Substituting numeric solutions of (17)-(18) into (16), we get the chemical potentials \( \mu_- \) and \( \mu_+ \) as the eigenvalues of the stationary states. They will be used below to demonstrate the nonlinear structures arising in the stationary spectra.

IV. RESULTS AND DISCUSSIONS

Results of our calculations are presented in the Figs. 2 and 3. In Fig. 2, the populations of the first well, \( N_1(t) \), and the second well, \( N_2(t) \), are shown for LZ and GLZ protocols. BEC atoms are initially placed in the first well, e. g. \( N_1(t \rightarrow -\infty) = 1 \), \( N_2(t \rightarrow -\infty) = 0 \), and then transferred to the second well. The weak nonlinearity \( \Lambda = \pm 0.5 \) covering both repulsive \( \Lambda > 0 \) and attractive \( \Lambda < 0 \) interaction is used. Following [13, 20], a weak nonlinearity allows a complete adiabatic transport only for rather small detuning rates \( \alpha \). Here we consider even smaller rates \( 0.01 \leq \alpha \leq 0.1 \) to test the vicinity of the adiabatic limit. As discussed below, just this rate region demonstrates the major differences between LZ and GLZ protocols and is thus most interesting for our aims. Following Fig. 2, these differences vanish while approaching the upper value \( \alpha = 0.1 \), where the complete transport is most robust, see panels a)-i). At even higher \( \alpha \), the process is already too rapid to keep the adiabaticity at the given \( \Lambda \) and the transport becomes incomplete [20].

Figure 2 shows that the major LZ-GLZ differences take place at the lowest rate \( \alpha = 0.001 \) (panels a)-d)). The LZ demonstrates here high-amplitude slowly-damped Rabi oscillations of the populations. This is because, at so
small rate, the energies $E_1(t)$ and $E_2(t)$ are almost parallel and the condition $\Delta \leq \Omega$ for Rabi oscillations is fulfilled for very long time. The transport process in this case is vague. However, as seen from panels c) and g), it becomes more distinctive with increasing $\alpha$.

Much better transport results take place for GLZ (panels b),d),h)) where the Rabi oscillations exist only for a short time determined by the duration of the Gaussian coupling $\bar{\Omega}(t)$. In this case, the completeness of the transport depends on the instant when the coupling is over and Rabi oscillations are switched off. In other words, it is determined by the width $\Gamma$ of the Gaussian coupling pulse $\bar{\Omega}$. Panels b), d) and d) show that if the Rabi switch is done at the proper time, then the transport is effective. If not (panel f)), then the transport fails. The process can be controlled by monitoring both $\Gamma$ and $\alpha$. It is similar to other Rabi switch techniques, e.g. [24].

Panels i)-l) show that at higher rates, namely at $\alpha = 0.1$, both LZ and GLZ demonstrate a robust and complete transport (with some advantage of the GLZ protocol). The convergence of LZ and GLZ results is explained by the fact that at this rate the condition $\Delta \leq \Omega$ is kept in LZ already for much shorter time comparable with the GLZ Gaussian pulse duration. This time interval is already not enough for development of high-amplitude Rabi oscillations. The oscillations also vanish in GLZ thus signifying the conversion of the Rabi switch mechanism to the robust adiabatic population transfer. As was mentioned above, a considerable further increase of $\alpha$ is not desirable since then the process will be too rapid to support the adiabatic following.

Comparison of the results in Fig. 2 for the repulsive ($\Lambda = +0.5$) and attractive ($\Lambda = -0.5$) interaction shows that the asymmetry in BEC transport, i.e. detrimental or favorable effect of the nonlinearity depending on the interaction sign, is generally small (though it can manifest itself in particular fragile cases of the Rabbi switch, see panels f) and h)). At least the effect is much weaker than for a strong nonlinearity when it causes a drastic support or suppression of the transport [20]. Since the asymmetry effect depends on the nonlinearity magnitude, its insignificance for a weak nonlinearity is natural.

Altogether, Fig. 2 allows to conclude that the cases of weak and strong nonlinearity are quite different. Though
FIG. 3: Chemical potentials $\mu_-$ (upper lines and structures) and $\mu_+$ (lower lines and structures) for LZ (left) and GLZ (right) protocols. The $\mu_-$ is depicted by solid (dash) curves for repulsive (attractive) BEC and vise versa for $\mu_+$. The detuning rate $\alpha$ and nonlinearity $\Lambda$ have the same values as in Fig. 2.

the complete adiabatic transport is possible in both cases, a weak nonlinearity is distinguished by an essential role of the Rabi oscillations (hence specific transport regimes like the Rabi switch) and faint asymmetry effect.

The peculiarities of weak nonlinearity are additionally demonstrated in Fig. 3 where stationary spectra $\mu_+$ and $\mu_-$ are depicted. It is seen that the GLZ spectra exhibit nonlinear structures (panels b), d), f), h)) despite a weak nonlinearity. However, these stationary spectra in general and nonlinear structures in particular do not display the crucial role of Rabi oscillations pertinent to a weak nonlinearity and so can hardly be used for the reliable treatment of the transport regimes. In this sense, the LZ stationary spectra are not instructive as well. They do not exhibit nonlinear structures at all and so might assume a robust transport. However, as seen from Fig. 2 a), c), e), d), g), the LZ transport is heavily damaged by the Rabi oscillations. Here we see again a big difference with the case of a strong nonlinearity where the solid nonlinear structures in the stationary spectra allow to do a reliable analysis of the transport features.

V. CONCLUSIONS

The complete adiabatic transport of Bose-Einstein condensate in a double-well trap is investigated within the conventional Landau-Zener (LZ) protocol and its generalization to time-dependent Gaussian coupling (GLZ) for a case of a small nonlinearity. The relevant range of slow detuning rates when such nonlinearity manifests its main peculiarities is analyzed in detail. The essential role of Rabi oscillations in the vicinity of the adiabatic limit is demonstrated. These oscillations hamper the LZ transport but make possible the Rabi switch transport for GLZ. For higher detuning rates, both LZ and GLZ converge to a robust adiabatic transfer. The asymmetry effect is found generally small.

The nonlinear structures in the stationary spectra arise at very low detuning rates despite a small interaction. However, these structures do not reveal the significant role of Rabi oscillations and so, unlike the case of a strong nonlinearity, cannot be used as a reliable tool for the analysis of transport features.
Altogether, the calculations show a considerable difference between slightly and strongly nonlinear transports. They demonstrate different mechanisms. What is most important, a weak nonlinearity with its negligible asymmetry effect cannot be used as a powerful tool to enforce or suppress the adiabatic transport. In this sense a strong nonlinearity is certainly more promising.

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