On the oscillation spectra of ultracompact stars: An extensive survey of gravitational-wave modes

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ABSTRACT

An extensive survey of gravitational-wave modes for uniform density stars is presented. The study covers stars ranging in compactness from $R/M = 100$ to the limit of stability in general relativity: $R/M = 9/4$. We establish that polar and axial gravitational-wave modes exist for all these stellar models. Moreover, there are two distinct families of both axial and polar modes. We discuss the physics of these modes and argue that one family is primarily associated with the interior of the star, while the second family is mainly associated with the stellar surface. We also show that the problem of axial perturbations has all the essential features of the polar problem as far as gravitational waves are concerned. This means that the axial problem is much more important than has previously been assumed. We also find some surprising features, such as avoided crossings between the polar gravitational-wave modes and the Kelvin f-mode as the star becomes very compact. This seems to suggest that the f-mode should be considered on equal footing with the polar w-modes for ultracompact stars. All modes may have the main character of trapped modes inside the curvature potential barrier for $R/M < 3$. 
1. Introduction

Neutron stars have remarkably rich oscillation spectra (McDermott, Van Horn and Hansen 1988). Essentially every characteristic of the star – such as its density, temperature, rotation, magnetic field et cetera – can be, more or less directly, associated with a unique set of oscillation modes. This makes a study of the full problem of pulsating neutron stars hopelessly difficult. The only reasonable approach is to strip off all the features that are not expected to interfere with the principal physical mechanism behind a certain pulsation mode. Once the “simple” situation is explored one can add, one by one, other features of the star back on and (possibly) verify that the approximation made was a valid one. This approach has been successful in the past, and our understanding of pulsations that are mainly associated with the stellar fluid has been much improved.

The oscillation modes of a star can be divided into two general classes; polar (spheroidal, or even parity) and axial (toroidal, or odd parity). The polar ones correspond to spheroidal deformations of the star, whereas the axial ones are associated with differential rotation. As far as the stellar fluid is concerned, polar pulsation modes exist for all conceivable stellar models, whereas the existence of axial modes rely upon nonzero rotation, magnetic field or shear modulus. To set the scene for this paper, let us discuss a hierarchy of increasingly complex stellar models and the pulsation modes associated with each of them;

1. The simplest possible model for a neutron star is a non-rotating ball of fluid at zero temperature. If the fluid has constant density this model can only support a single pulsation mode for each multipole $\ell$. (The oscillations are typically described in terms of the standard spherical harmonics $Y_{\ell m}$.) This mode, which in Newtonian theory has oscillation frequency

$$\omega(R^3/M)^{1/2} = \sqrt{2(l(l-1))/(2l+1)} \approx 0.894 \quad \text{for } \ell = 2,$$

where $R$ and $M$ are the radius and mass of the star in units where $c = G = 1$, was first studied by Kelvin (Cox 1980) and is usually referred to as the f-mode (fundamental mode) (Cowling 1941). It is distinguishable by the fact that there are no nodes in the corresponding eigenfunctions inside the star. In a way, the f-mode is due to the interface between the star and its surroundings. The eigenfunctions of such modes would typically have maxima at the interface and fall off away from it (McDermott, van Horn and Hansen 1988). This is exactly the character of the f-mode: It reaches maximum amplitude at the surface of the star (see for example Figure 7–8 in Kokkotas and Schutz 1992).

2. A somewhat more realistic star consists of a perfect fluid. Then one must specify the equation of state, and most studies to date have been for simple polytropes. For this stellar model a second set of modes – the p-modes (Cowling 1941), the restoring force of which is pressure – exists. The oscillation frequencies of the p-modes depend directly on the travel time for an acoustic wave across the star. This would typically lead to an oscillation period shorter than a few milliseconds.

3. When the temperature of the star is non-zero a further set of modes come into play. These modes are mainly restored by gravity and are consequently referred to as g-modes (Cowling 1941). For a star in convection, i.e., when the entropy is constant, the g-modes
are all degenerate at zero frequency. In general, however, their oscillation periods depend directly on the central temperature of the star and are typically longer than a few hundred milliseconds.

4. The three families of “fluid” modes discussed so far, the f-, p- and g-modes, all belong to the class of polar modes. For these models there are no “fluid” axial modes. But then, the stellar models discussed are all somewhat unrealistic. In reality, it is expected that the crust of the star will crystallize, and a typical neutron star would have a kilometer-thick crust. When the shear modulus in the crust is nonzero axial modes exist (Schumaker and Thorne 1983). There will also be families of modes directly associated with the various interfaces in the star (McDermott, van Horn and Hansen 1988).

Although a real neutron star will most likely be rapidly rotating, have a strong magnetic field, contain a region of superfluidity et cetera we will not discuss these complications here. What we want to emphasize with the above examples is that each feature of the star is associated with at least one unique set of pulsation modes. The more complicated the stellar model, the richer is its oscillation spectrum.

The picture painted so far is nevertheless not satisfactory since we have not included mechanisms for dissipation. Pulsations of a real neutron star will be damped. For example, displacements close to the surface of the star will affect the external magnetic field, and “shaking” magnetic field lines will generate electromagnetic waves. These can carry pulsation energy away from the star. For pulsars this might result in observable perturbations of the pulsar beam (McDermott, van Horn and Hansen 1988). A second dissipation mechanism is due to general relativity. Gravitational waves will be generated when the stellar fluid sloshes about. Emission of gravitational waves is the dominant dissipation mechanism for many stellar modes.

The (relativistic) Cowling approximation (Cowling 1941; McDermott, Van Horn and Scholl 1983) has been an invaluable tool for achieving the present understanding of neutron star pulsations. In this approximation all perturbations of the gravitational field are neglected, and the pulsation equations simplify considerably. The damping due to gravitational waves can be estimated by means of the celebrated quadrupole formula (Balbingski and Schutz 1982; Balbingski et al. 1985). This leads to a typical damping time of a few seconds for the f- and the p-modes. Compared to this, the gravitational-wave damping of the g-modes is incredibly slow. Very little mass is involved in the g-mode pulsations – the modes are localized close to the surface (or the centre) of the star, and the damping time is estimated to be of the order of the lifetime of the star. For the g-modes the damping due to “shaking” of the magnetic field is more important.

Calculations for perturbed stars in general relativity were pioneered by Thorne and Campolattaro in 1967. Since then many authors have considered the problem, but there are not many numerical studies of mode-frequencies. The most extensive one, due to Lindblom and Detweiler (1983), considers many different equations of state but only the quadrupole f-mode for each star. Until a few years ago the consensus was that general relativity would not considerably change the results of Newtonian theory. Each mode-frequency would only adopt a small imaginary part (making it complex) to account for damping due to gravitational waves.

It is now becoming clear, however, that general relativity is playing a much more exciting role. According to Einstein’s theory the gravitational field should be considered a dynamical entity, and neutron stars are very compact objects for which relativity should be of some importance. Hence, it seems likely that a neutron star will have pulsation modes mainly
associated with the gravitational field itself: A kind of “spacetime” modes that are, in some sense, coupled to the “fluid” modes of Newtonian theory. These modes would be reminiscent of the quasinormal modes of black holes (Chandrasekhar 1983).

That such modes should exist was first argued by Kokkotas and Schutz (1986). They suggested a simple model problem of a finite string – representing the stellar fluid – coupled by means of a spring to a semi-infinite string – the gravitational field. This system supports two sets of pulsation modes; one that is slowly damped and analogous to the “fluid” modes of a star, and a second one that is rapidly damped and has no analogue in Newtonian theory. This new set of modes has been termed w-modes because of the close association to gravitational waves. That such modes exist also for polytropes has since been demonstrated by several studies (Kojima 1988; Kokkotas and Schutz 1992; Leins, Nollert and Soffel 1993; Andersson, Kokkotas and Schutz 1995a).

The problem of axial perturbations of stars has not attained much interest in the past. The main reason for this is that axial perturbations do not couple to oscillations of the stellar fluid. As was shown by Thorne and Campolattaro in their seminal paper of 1967, axial perturbations are governed by a single homogeneous wave equation for one of the perturbed metric functions. For three of the simple stellar models discussed above there are no axial pulsation modes associated with the fluid. In 1983 Schumaker and Thorne developed a general relativistic description of axial modes for a neutron star with a crust. This is one of very few discussions of axial modes in the literature.

An exciting idea of Chandrasekhar and Ferrari (1991b) recently brought the axial modes more into focus. If the star is made very compact – roughly when $R/M < 3$ – the problem for axial modes becomes similar to that for quasi-bound scattering resonances in quantum mechanics. That is, one gets an effective potential with a well inside the barrier that is familiar from black-hole problems (Chandrasekhar 1983). Clearly, axial modes associated with this potential well should exist and Chandrasekhar and Ferrari found a few such modes. Since then, results of Kokkotas (1994) have indicated that there may, in fact, exist an infinite number of axial modes for these compact stars. Most of these modes are rapidly damped, which explains why they could not be identified with the numerical technique employed by Chandrasekhar and Ferrari.

It is interesting to note here that, the idea of trapped waves for very compact stars is not at all new. That the black-hole potential barrier may affect the f-mode for sufficiently compact stars was first pointed out by Detweiler (1975). Moreover, Vishveshwara and his co-workers discussed ultracompact stars in a series of interesting papers. The first of these (Kembhavi and Vishveshwara 1980), concerns neutrinos trapped by neutron stars. This problem is remarkably similar to that for axial modes and compact stars; there is a well inside a potential barrier. Two following papers (Iyer and Vishveshwara 1985; Iyer, Vishveshwara and Dhurandhar 1985), investigate whether ultracompact stars may exist for “realistic” neutron star equations of state. It is clear that one can only have $R/M < 3$ for extremely stiff equations of state. Nevertheless, using the so-called core-envelope model, the authors show that stable ultracompact objects with causal cores may exist. On the other hand, no such objects were found using the available high-density equations of state.

The present paper mainly concerns gravitational-wave modes for uniform density stars. We present the most extensive survey of such modes carried out so far. Admittedly, we have chosen an unrealistic stellar model, but we have good reasons for doing so. The main advantage is that we can study a sequence of models with continuously varying surface redshift. That is, it is
straightforward to do calculations covering all degrees of compactness, including white dwarfs: 
\( R/M = 10^4 \); neutron stars: 
\( R/M = 5 \) and continuing towards the ultimate limit of relativistic stability at 
\( R/M = 9/4 \). By following individual modes as the stellar models change in this way one may hope to understand better the physical origin of the modes.

Another motivation behind the present study is completeness. We have previously considered the problem for modes that are slowly damped (Kojima, Andersson and Kokkotas 1995), but only for stars close to the ultimate limit posed by general relativity. We wanted to see whether the axial and the polar modes behaved in a similar way as this limit was approached. That was expected since both sets would then behave as “trapped” modes inside very similar potential barriers. That the two sets do “approach” each other as 
\( R/M \rightarrow 9/4 \) was, indeed, one of the conclusions of our previous work. But that study was limited in several ways by the technique we used to identify polar modes; the so-called resonance method. Inherent in that technique is that it will only work as long as the imaginary part of a mode-frequency is much smaller than the real part (Chandrasekhar, Ferrari and Winston 1991), and hence we could not hope to study rapidly damped w-modes with it. Furthermore, our results indicated that the w-modes would cross the f-mode for very compact models. Again due to the nature of the resonance method, we could not resolve these possible mode-crossings.

Despite the restrictions, the results of our previous work were very interesting. It seemed clear that the origin of the axial and the polar modes must be the same for these very compact models. If so, it seems reasonable that both sets are due to the same physical mechanism in general; that they are “spacetime” modes that do not depend on the fluid at all for their existence. This does not mean that the fluid would not affect the polar modes. It certainly will, but the modes exist even if we freeze the fluid motion (Andersson, Kokkotas and Schutz 1995b). If this idea is correct, the existence of rapidly damped axial gravitational-wave modes for all stellar models is unavoidable. In this context it is relevant to point out that the established idea that axial modes will not exist for less compact stars relies heavily on Newtonian theory where the fluid plays the main dynamic role. It is quite easy to argue that axial perturbations should be as relevant as the polar ones as far as the gravitational field is concerned. Evidence for this is certainly provided by the black-hole problem, where the axial and polar spectra are identical (Chandrasekhar 1983). The present study aims to establish the existence of axial gravitational-wave modes for all stellar models.

2. On the problem

The equations governing perturbed stars in general relativity have been considered in great detail by many authors. Thus, we do not find it necessary to go into much detail here. We will simply refer the interested reader to the original sources for the exact form of most equations. Nevertheless, an understanding of the general form of some of these equations will be useful for a discussion of the present results and we introduce the necessary material in this section. We will also comment on our present numerical work and assess its reliability.

2.1. Perturbed stars in general relativity
We use the unperturbed metric
\[ ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{2} \]
where \( \nu \) and \( \lambda \) are functions of \( r \) only. In fact,
\[ e^{-\lambda} = 1 - \frac{2m(r)}{r}, \tag{3} \]
where \( m(r) \) represents the mass inside radius \( r \). For a fluid ball of uniform density it is easy to show that
\[ m(r) = M \left( \frac{r}{R} \right)^3, \tag{4} \]
inside the star. The total mass of the star is \( M = m(R) \). The metric variable \( \nu \) is determined by the equations of hydrostatic equilibrium, and for uniform density we get (Schutz 1985)
\[ e^{\nu} = \frac{1}{4} \left[ 3 \left( 1 - \frac{2M}{R} \right)^{1/2} - \left( 1 - \frac{2m(r)}{r} \right)^{1/2} \right]^2. \tag{5} \]
In the vacuum outside the star this should be replaced by
\[ e^{\nu} = \left( 1 - \frac{2M}{r} \right), \tag{6} \]
and the metric becomes the standard Schwarzschild metric.

We now wish to consider linear perturbations of (2). That is, we introduce a perturbed metric
\[ g_{\mu\nu} = g_{\mu\nu}^{\text{background}} + h_{\mu\nu}^{\text{polar}} + h_{\mu\nu}^{\text{axial}}, \tag{7} \]
where \( h_{\mu\nu} \) are (in some sense) small. As we have already stated, the polar perturbations correspond to spheroidal deformations of the star, whereas the axial ones correspond to differential rotation. The equations governing these two classes will decouple. We also need variables describing the perturbations of the fluid. We could, for example, use displacement components \( \xi^r \) and \( \xi^\theta \) together with the Eulerian variation in the density \( \delta \rho \). We essentially have thirteen undetermined functions. The equations that govern the evolution of the perturbed metric and the stellar fluid are (i) ten perturbed Einstein equations, and (ii) three equations of motion for the fluid. Clearly, this is a well posed problem. However, because of the Bianchi identities only ten of our variables will be independent. For example, if we decide to use the perturbed Einstein equations to determine all ten metric perturbations the three equations of motion for the fluid will automatically be satisfied. Hence, several different sets of perturbation equations can be used, depending on one’s favourite variables. The possibilities also involve various gauge choices.

### 2.2. The perturbation equations

Although it is by no means clear what the “best” choice of variables may be, it is straightforward to prescribe the expected form of the final perturbation equations. In the case of axial perturbations the expectations are, in fact, easily realized since these perturbations do not
couple to oscillations of the fluid. Three of the perturbed metric components are axial, and as was shown by Thorne and Campolattaro (1967) one of these is governed by an equation of form

$$\frac{\partial^2 X}{\partial t^2} + \frac{\partial^2 X}{\partial r_*^2} - V X = 0,$$

where the tortoise coordinate $r_*$ is defined by

$$\frac{\partial}{\partial r_*} = e^{(\nu - \lambda)/2} \frac{\partial}{\partial r}.$$

The exact form of the function $X = X(h_{r\theta})$ and the potential $V$ are not of great concern to us here. They are given explicitly by, for example, Chandrasekhar and Ferrari (1991b) who also graphed the potential for a few very compact uniform density stars. Some general properties of the potential are useful to keep in mind, however; $V$ is positive for all $r$, and in the exterior vacuum it reduces to the Regge-Wheeler potential (Chandrasekhar 1983). This potential reaches a maximum at $R/M \approx 3$ and vanishes as $1/r^2$ as $r \to \infty$. Inside the star, the potential diverges as $\ell(\ell + 1)/r^2$ as $r \to 0$.

In the exterior vacuum, the polar perturbation problem also simplifies to a single equation of form (8). This is the so-called Zerilli equation (Chandrasekhar 1983). The polar effective potential is similar, albeit not identical, to that for the axial case. The description of the polar perturbation problem inside the star is not as straightforward. Several different sets of equations have been derived and used. It has long been accepted that the problem is one of fourth order. For example, Lindblom and Detweiler (1983) used Regge-Wheeler gauge and derived four coupled first-order equations for $[h_{tt}, h_{rr}, \xi_r, \xi_\theta]$. This system was used in their numerical study of f-modes for many different equations of state. They later realized (Detweiler and Lindblom 1985) that this system of equations could become singular inside the star and that it was preferable to use $h_{tr}$ instead of $h_{tt}$. Recently, Chandrasekhar and Ferrari (1991a) approached the problem in diagonal gauge and derived a fifth-order system of equations. Interestingly, their new set of equations does not explicitly contain the fluid variables and hence the problem of gravitational waves scattered off a star can be considered in much the same way as that for black holes. Price and Ipser (1991) have shown that the Chandrasekhar-Ferrari equations can also be reduced to fourth order. The fifth equation corresponds to a solution that is ruled out by physical requirements and is thus superfluous. A fourth-order system of equations without explicit dependence on the fluid variables in Regge-Wheeler gauge was first derived Ipser and Price (1991). It has been extended to the case of slow rotation by Kojima (1992,1993). In our opinion, these latter equations are the most appealing ones derived so far. They consist of two coupled wave equations, corresponding to the two dynamical degrees of freedom; the fluid oscillations and the gravitational waves. The equations for a barotropic case are given by

$$- \frac{1}{C^2} \frac{\partial^2 Y}{\partial t^2} + \frac{\partial^2 Y}{\partial r_*^2} + F \left( \frac{\partial Y}{\partial r_*}, \frac{\partial Z}{\partial r_*}, Y, Z \right) = 0,$$

$$- \frac{\partial^2 Z}{\partial t^2} + \frac{\partial^2 Z}{\partial r_*^2} + G \left( \frac{\partial Y}{\partial r_*}, \frac{\partial Z}{\partial r_*}, Y, Z \right) = 0,$$

where $C^2 = \delta p/\delta \rho$ and the functions $Y$ and $Z$ are linear combinations of the metric perturbations $h_{tt}$ and $h_{\theta\theta}$. These equations clearly represent the hydrodynamical and gravitational wave propagation. In this paper, we adopt the approximation $\delta \rho = 0$, or equivalently $C^2 \to \infty$. 
Then the basic equations become one constraint equation and one wave equation. There are no singularities in the equations introduced in this approximation since they include only $1/C^2$ and not $C^2$.

### 2.3. Comments on the present work

The quasinormal modes of a stellar system are solutions to the perturbation equations that satisfy the physical condition of regularity at the centre of the star, appropriate matching conditions at the surface and also correspond to purely outgoing gravitational waves at infinity. Let us assume that standard Fourier-decomposition leads to solutions to (8) that depend on time as $\exp(5\omega t)$. In the following we will refer to the corresponding time-independent equation as (9). It follows, since both the axial and the polar potential vanishes as $r \to \infty$, that a quasinormal mode characterised by a frequency $\omega_n$ is a solution to (8) that behaves as

$$X \sim e^{-i\omega_n r_*} \quad \text{as } r_* \to \infty.$$  \hspace{0.5cm} (12)

Identifying such a solution is a far from a trivial problem. We expect the modes of a star to be damped as gravitational waves are emitted. This means that $\omega_n$ must be complex with a positive imaginary part. An observer sitting at a constant value of $r_*$ will then see a gravitational wave that is exponentially damped with time. The boundary condition (12) thus requires that we identify a solution that diverges as $r_* \to \infty$. This is clearly not easy to do since we must ensure that no (exponentially small) ingoing waves remains at infinity.

In our previous study, of slowly damped modes for ultracompact stars with uniform density, we used the so-called resonance method to identify polar modes. In this method one constructs the ratio of the asymptotic amplitudes for out- and ingoing waves for real values of $\omega$. The quasinormal modes with small imaginary part can then be identified as singularities [see Figure 1 in Kojima, Andersson and Kokkotas (1995)]. This method clearly fails for rapidly damped modes and is therefore not useful for a survey of the present kind. In a study of rapidly damped modes the frequency must be considered complex.

The divergence-problem is well-known from black-hole studies and several ways to overcome it have been developed. Some of these have also been extended to the problem of rapidly damped stellar modes. Kokkotas and Schutz (1992) used a WKB approximation at the surface of the star, whereas Leins, Nollert and Soffel (1993) employed Leaver’s continued fraction method (Leaver 1985) as well as the Wronskian scheme developed by Nollert and Schmidt (1992). Here we employ a complex-coordinate approach that was developed for black holes by Andersson (1992) and recently applied to the stellar problem by Andersson, Kokkotas and Schutz (1995a). The general idea behind this scheme is very simple. By allowing the coordinate $r_*$ to assume complex values we can suppress the divergence at $|r_*| \to \infty$. Specifically, for each complex frequency $\omega_n$ one can always find a contour in the complex $r_*$-plane such that the out- and ingoing waves, i.e., the two asymptotic solutions $\exp(-i\omega_n r_*)$ and $\exp(i\omega_n r_*)$ to (9), are of the same order of magnitude. Along such a contour it is straightforward to ensure that the outgoing-wave boundary condition is satisfied. In fact, since the potentials vanishes as $|r_*| \to \infty$ it is clear that such contours will asymptotically be straight lines with a slope equal to $-\text{Im } \omega_n/\text{Re } \omega_n$. In order to use this method one must, of course, prove that a boundary condition introduced for complex $r_*$ actually corresponds to the desired behaviour on the real
r-axis – where the physical condition is imposed. The simplest argument for this appeals to analyticity. This hand-waving argument has been supported by a semiclassical demonstration by Araújo, Nicholson and Schutz (1993).

The method developed by Andersson, Kokkotas and Schutz (1995a) must be used with some care for ultracompact stars, however. This is an important point and we will try to explain it without going into too much detail here. Solutions to (8) are constructed as linear combinations of two numerically determined functions which asymptotically become equal to the two WKB solutions, and thus represent out- and ingoing waves at infinity. These functions can be split into an amplitude and a phase, which are related because the Wronskian of any two linearly independent solutions to (8) must be constant. Hence, it is sufficient to consider the phase-function. In the Andersson-Kokkotas-Schutz method this phase-function is numerically determined along a straight line in the complex r-plane. This contour ends at the surface of the star, and is chosen such that the out- and ingoing wave solutions are of the same order of magnitude far away from the star, i.e., with a slope $-\text{Im} \omega_n/\text{Re} \omega_n$.

With this method the validity of the geometrical optics argument – that waves are not backscattered by the curvature in the exterior of the star – that is the essence of the WKB approximation that Kokkotas and Schutz (1992) used in their study of w-modes, can be tested. It was shown (Andersson, Kokkotas and Schutz 1995a), that the geometrical optics argument is perfectly valid for most w-modes of polytropes. However, if wave-reflection by the exterior curvature becomes considerable, for example when the star becomes very compact and the black-hole potential barrier is unveiled (as $R/M \approx 3$), geometrical optics will clearly no longer be reliable.

It can be shown that the phase-function that is the key quantity in the Andersson-Kokkotas-Schutz method will also be affected as waves are backscattered by the curvature. It is quite obvious that it must, since all information about the exact solution to (8) will be encoded in the phase-function. The function is generally smooth and well behaved – which makes it ideally suited for numerical studies – as long as the wave reflection does not play a dominant role. But if it does, the nice properties of the phase-function will not be guaranteed. Strong backscattering can have three effects on the phase-function; (i) it may oscillate (often more rapidly than the solution to (8) itself). (ii) its amplitude may drop by several orders of magnitude (iii) it may have poles (see Appendix A of Andersson 1993). It is clear that this may severely affect numerical calculations.

Consequently, the Andersson-Kokkotas-Schutz method must either be adapted to this situation or used with some care. Here we have chosen the second route. Thus the fact that we lose numerical precision because of effects due to wave reflection in the exterior limits our calculations somewhat. For example, for $M/R = 0.44$ we are restricted to $\text{Im} \omega M < 2.5$ if we want six digit precision in our final iterations. This restriction is much less severe for less compact stars since the surface of the star is then well outside the peak of the black-hole curvature potential. In some cases we also find poles in the phase-function. These can masquerade as “purely ingoing-wave modes” for certain complex frequencies. They are, of course, spurious and should not be taken seriously.

However, these effects are not the main ones that limit the present study. Due to numerical difficulties in the interior calculation we have been restricted to $\text{Im} \omega M < 1.25$ or so. The reason for these difficulties is not well understood at the moment. This issue must be studied in greater detail if more complete mode-surveys are to be at all possible in the future. We have performed calculations using both the equations of Detweiler and Lindblom (1985) and those of Ipser, Price
and Kojima (Ipser and Price 1991; Kojima 1992) for polar perturbations. It seems that both sets suffer from these numerical difficulties, although the latter formulation performs slightly better. In general, we have obtained identical numerical results from the two formulations of the interior problem.

Finally, it is worth mentioning the possibility of anti-damped modes. A linear instability of the star would correspond to an outgoing-wave mode with a negative imaginary part of $\omega_n$. Our numerical technique has allowed us to search for such modes and verify that they do not exist. At least not in the part of the complex frequency plane covered here; $-2 < \text{Im} \omega M < 0$.

3. Results

We present here the most extensive survey of gravitational-wave modes for compact stars carried out to date. Our calculations cover stars ranging in compactness from $R/M = 100$, i.e., something like a hundred times as compact as a typical white dwarf, to ones very close to the ultimate limit of stability in general relativity: $R/M = 9/4$. The numerical results basically consist of a table describing how various pulsation-mode frequencies change as the star is made more compact. These results agree perfectly with our previous ones (Kojima, Andersson and Kokkotas 1995) for the relevant cases.

In order to extract as much of the underlying physics as possible, it is useful to consider these results in various units. For example, it is natural to discuss the f-mode in terms of the density of the star $\sim M/R^3$. It is well-known that the oscillation period of the f-mode depends on the average density of the star. But at present we have no such understanding of the gravitational-wave modes. One could, for example, imagine that the periods of these modes will be associated with the travel time for a gravitational wave that crosses the star, and thus be related to the radius ($R$). Or maybe these modes are similar to the quasinormal modes of black holes, and are best studied in units of mass ($M$)? In fact, we will find all these units useful in the following discussion.

3.1. In units of density

It seems natural to begin a discussion of the pulsation modes of uniform density stars with the single mode that exists in Newtonian theory. This is the f-mode and, as mentioned above, it is best studied in units of density. This mode has already been studied in some detail by Detweiler (1975), and hence some of our results may not be too surprising. In Figure 1 we show how the complex frequency of the f-mode changes as the star is made increasingly relativistic. For large values of $R/M$ the oscillation frequency is very close to the Newtonian value $\omega$. In fact, it differs from this value by less than 1% for stars less compact than $R/M \approx 5$. This means that the prediction of Newtonian theory is remarkably accurate also for neutron stars, even though these are very relativistic objects. As the star is made more compact the damping of the f-mode reaches a maximum. This feature was also noticed by Detweiler. We find that this maximum occurs for $R/M \approx 3.7$ and corresponds to $\text{Im} \omega (R^3/M)^{1/2} \approx 4.6 \times 10^{-4}$. These values are in good agreement with Detweiler’s results. After this maximum the damping of
the f-mode decreases, as does the pulsation frequency. Detweiler suggests that the complex frequency becomes exactly zero at $R/M = 9/4$. This would indicate that the f-mode becomes secularly unstable in the extreme limit of relativity.

How are we to understand the maximum in the f-mode damping rate? A reasonable answer is provided by Detweiler (1975). At first, as the star is made more relativistic, emission of gravitational radiation becomes a more efficient mechanism for energy release and thus the imaginary part of $\omega_n$ increases. However, as $R/M$ decreases more and more of the black-hole potential barrier is unveiled. Finally, at $R/M \approx 3$ the peak of the barrier emerges and the problem becomes analogous to one of barrier scattering in quantum mechanics. A gravitational wave trying to escape to infinity will be partially reflected by the barrier, and thus the damping time of the modes will be longer. This effect is amplified by the fact that the oscillation frequency decreases with $R/M$.

The idea that the potential barrier might be able to trap gravitational waves led Chandrasekhar and Ferrari (1991b) to the discovery of axial modes for very compact stars. That similar trapped polar modes exist was recently shown by us (Kojima, Andersson and Kokkotas 1995). In fact, we argued that the axial and polar modes behave in a similar way as the star becomes very compact. If so, it would seem likely that the two sets of modes rely on the same physical mechanism for their existence. In our view, both sets are gravitational-wave modes that depend mainly on the character of the curved spacetime in the vicinity of the star. In Figure 1 we show the first of these axial and polar modes. As the star is made more compact the oscillation frequencies and the damping rates for these modes generally decrease.

Again, that makes sense because of the increasing importance of the potential barrier. In Figure 1 the axial and the polar mode seem to behave quite differently for very compact stars, however. The damping of the axial mode decreases monotonically, whereas the polar-mode damping reaches a local minimum, a local maximum, and then falls off rapidly in a way similar to the f-mode damping rate. The obvious question is, does this not contradict our previous conclusions (Kojima, Andersson and Kokkotas 1995) that were based on the observed similarity between the axial and the polar modes?

The answer to this question is provided if one plots the real and the imaginary parts of $\omega_n$ separately as functions of the compactness of the star. An attempt to do this is Figure 2 of Kojima, Andersson and Kokkotas (1995), but because of the method we used to identify modes in that study the finer details could not be resolved. When the data obtained by the recent calculations is used a remarkable picture emerges. In Figure 2 it is clear that the polar pulsation mode frequencies – a set that includes the f-mode – show a series of avoided crossings (cf. for example Figure 17.7 in Cox 1980) as the compactness increases. Meanwhile, the axial spectrum does not have such features. This is a totally unexpected behaviour. Even more surprising is the way in which the “agreement” between axial and polar modes prevails. Notice, for example, that the first axial w-mode is very close to the first polar w-mode for $M/R < 0.439$. This was the kind of evidence on which we based our previous conclusions on the nature of these modes. But as the star is made more compact a “re-ordering” occurs. The first axial mode becomes similar to the f-mode, while the first polar w-mode approaches the second axial mode. At first this seems very peculiar, but it does make sense if these modes are all considered as trapped inside the peak of a potential barrier. The two potentials governing axial and polar perturbations in the exterior vacuum are very similar (Chandrasekhar 1983) and they should support similar sequences of trapped modes. The only surprising feature that remains is that the f-mode must be considered on equal footing with the polar w-modes. It is only the first in a
sequence of trapped polar gravitational-wave modes. This is intriguing evidence for the richness of this problem. Equally interesting is the behaviour of the damping rates as the star gets more compact. At present we have no good explanation for the features in the damping rates that are apparent in Figure 2, however.

Finally, it may be worthwhile commenting on the fact that Detweiler (1975) did not find the trapped w-modes. It seems likely that the reason for this is that his method was based on a variational principle. Thus it requires an initial guess for the f-mode eigenfunction, and modes that do not have similar eigenfunctions will not be identified.

3.2. In units of mass

The present evidence supports the idea that the same physical mechanism gives rise to polar and axial gravitational-wave modes. In fact, the fluid must play a minor role since axial perturbations do not couple to pulsations in the fluid here (Thorne and Campolattaro 1967). We must look to the curvature of spacetime if we want to explain these modes. Then it is natural to wonder whether the w-modes are in some sense similar to the quasinormal modes of black holes. These are pure “spacetime” modes. In order to explore this issue, it is helpful to display our results in units of mass.

Two things follow immediately from the present results for rapidly damped modes: (i) Polar and axial w-modes exist for all stellar models and (ii) There are two distinct families of such modes. Both these results were anticipated from previous evidence (Kojima, Andersson and Kokkotas 1995; Leins, Nollert and Soffel 1993). Nevertheless, the present study brings these results beyond the realm of mere possibilities.

The first result follows immediately from Figure 3, where we show the extension of the data in Figure 1 for less relativistic stars. These modes – that make up the first family of w-modes – approach $\omega M = 0$ as $R/M \to \infty$. As the star becomes more compact $|\omega_n M|$ reaches a maximum. This maximum is greater for the higher overtones (larger $n$). After this maximum $|\omega_n M|$ decreases until, for very compact stars, the behaviour is that shown in Figure 1. The family of polar modes shown in Figure 3 is the one discussed by Kokkotas and Schutz (1992).

The qualitative behaviour of the axial and polar modes in Figure 3 is almost identical. If these were “pure” spacetime modes this similarity would be expected, especially since the axial and the polar spectra are identical for black holes (Chandrasekhar 1983). The present result therefore strongly supports the idea that the stellar fluid is of very little importance for these modes. Since these modes depend mainly on the spacetime curvature (this is especially clear for extremely compact stars) we will from now on refer to them as the “curvature modes”.

We cannot see that the curvature modes in Figure 3 have anything whatsoever in common with the modes of a black hole. The second family of w-modes, shown in Figure 4, is somewhat more reminiscent of the black-hole modes. As $R/M \to 9/4$ these modes approach constant complex frequencies. These values are approximated in Table 1. It is clear that these modes do not behave in the same way as the ones in Figure 3 as we vary the compactness of the star. We can only find one polar mode of the second type for all values of $R/M$ in the range $9/4 < R/M \leq 100$, but further ones (both axial and polar) exist for sufficiently compact stars. These new modes seem to emerge from the imaginary frequency axis. We can find the first axial mode in the family for $M/R > 0.127$, and the second polar one for $M/R > 0.227$. A second
axial mode appears for $M/R > 0.253$. The calculation becomes increasingly difficult as $\text{Im} \, \omega M$ increases and therefore we have only been able to obtain partial results for the third polar mode in this sequence. It seems to exist for $M/R > 0.299$. Anyway, the results imply that only a (small) finite number of such modes exist for each star. For reasons that will be discussed in more detail in section 4.2 we will call this second set of modes the “interface modes”.

As can be seen in Table 1, the interface modes do not have a clear relation to the black-hole modes. This is, of course, a difficult conclusion to prove. We certainly do not expect the stellar modes to approach the black hole ones as $R/M \rightarrow 9/4$. The star does not smoothly transform into a black hole in that limit. However, it is conceivable that these modes can be somehow associated with the peak of the exterior potential barrier for very compact stars (rather than the potential well, as the “trapped” modes of the first family). If so, these modes would be closely related to the black hole ones.

The behaviour of the first polar interface mode as $R/M \rightarrow 9/4$ is very peculiar. We do not profess to understand the “wiggles” that are apparent in Figure 3 at all.

\section*{3.3. In units of radius}

Finally, we want to see whether the gravitational-wave modes discussed here depend on the physical size of the star. This would, for example, be the case if they depend on the time it takes a gravitational wave to cross the star. This can easily be tested by displaying our results in units scaled to the radius of the star. When we do that it is clear that the pulsation frequencies of the curvature modes discussed above do depend on the radius of the star. From Figure 5 it follows that $\text{Re} \, \omega_n R$ approach constant values as $R/M \rightarrow \infty$. Estimated values (for $R/M = 100$) are given in Table 2. In fact, the oscillation frequencies for more compact stars typically differ from those in Table 2 by less than 1% for $R/M > 60$. An interesting observation is that the separation of the various pulsation frequencies is not too different from $\pi$. This fact is important for the discussion of the physics of these modes in the following section.

\section*{4. On the two families of w-modes}

In the previous section we presented extensive results for gravitational-wave modes of uniform density stars. These results should help us understand the origin and the nature of these modes.

The first obvious observation is that, as far as the gravitational-wave modes are concerned, the axial and the polar spectra are very similar. As we understand it, this implies that the spacetime curvature plays the main role in this game. In fact, the qualitative similarity between axial and polar modes provides an enormous advantage for this discussion. The axial modes are described by a single wave-equation, whereas the polar ones are governed by two coupled equations. Hence, it is much easier to interpret the axial modes.

Any conclusions drawn from the present data must, of course, also agree with previous results. We have found that two distinct families of w-modes exist. This idea was first proposed by Leins, Nollert and Soffel (1993), who found a few extra modes that had not been identified
by Kokkotas and Schutz (1992). That these modes do, indeed, exist for polytropes has recently been verified by Andersson, Kokkotas and Schutz (1995a). Leins et al. argue that the new modes are distinct for two reasons. The first is the one that we have relied upon in this investigation: Modes belonging to different families behave differently as the compactness of the star changes. The second argument is based on the eigenfunctions inside the star. Leins et al. found that the eigenfunctions of the w-modes of Kokkotas and Schutz were concentrated at the centre of star whereas the new modes seemed localized at the surface. We feel that an argument based on the eigenfunctions inside the star must be used with some care, however. There is no apparent reason why the eigenfunctions pertaining to the gravitational-wave degrees of freedom could not be localized outside the star. In fact, it seems likely that this is the case for the trapped modes that occur when the star is very compact.

Neither of the above arguments for why the two families of modes are distinct is really satisfactory, however. We must also understand the physical differences between the two families. Based on the evidence provided by the numerical results discussed here we suggest the following: There exist two different families of w-modes for both axial and polar perturbations. One of these families is primarily associated with the spacetime curvature inside the star. The second family of modes is mainly associated with the surface of the star and arises because of the “discontinuity” there. Below we offer arguments that support this and indicate what we believe is the physics behind these modes. These arguments need be supported by more detailed, preferably analytical, work in the future.

4.1. Modes associated with the spacetime curvature

Let us first discuss the modes associated with the stellar interior. These are the ones depicted in Figures 1-3 and 5 above. They were discovered by Kokkotas and Schutz (1992). We suggest that these modes arise, not because of the coupling between the spacetime and the fluid, as has been suggested previously, but rather as gravitational waves that are “trapped” by the spacetime curvature inside the star. It is easy to see how this may happen if one plots the gravitational-wave speed as measured by an observer at infinity: $e^\nu$ as a function of $r$. This has a minimum at the centre of the star, and the interior w-modes would be trapped in this “bowl” of curvature. Moreover, such modes would naturally be concentrated at the centre of the star, which agrees well with the eigenfunctions constructed by Leins, Nollert and Soffel (1993). Hence, it makes sense to refer to these modes as “curvature modes”.

It seems reasonable that the discontinuity provided by the surface of the star will be able to partially reflect gravitational waves, and therefore that the curvature modes will somehow be confined to $r < R$. We intentionally use the word “discontinuity” in a very vague sense here. In principle, some sort of discontinuity should always be present at the stellar surface. In the case of uniform density stars this is obvious, but also for more realistic stellar models, where the density falls off towards the surface, will there be discontinuities (even though these may appear in the higher derivatives of the relevant quantities). More useful than an actual mathematical discontinuity is a rapid change in the variables, for example the gravitational-wave speed, that can act as an effective discontinuity for waves of certain wavelengths.

Anyway, a naive and useful argument leads to standing wave solutions essentially of form $\sin(\omega r)$ inside the star. If this were the true form of the solutions, and the modes only leak out
slowly through the surface (so that we can assume a “zero” boundary condition at the surface) we would get

$$\omega_n R = n\pi, \quad n = 1, 2, ...$$

This argument is far too simplistic, but it is rewarding to find that two of its predictions agree well with our results for the curvature modes: There would exist an infinite sequence of such modes, and their pulsation frequencies $\text{Re} \omega_n R$ would be separated by $\pi$. A similar dependence on the size of the star can be inferred for the modes in the simple toy model of Kokkotas and Schutz (1986).

When the star is made increasingly compact, the curvature modes discussed here clearly change character and should be considered as trapped in the potential well that arises inside the black-hole potential barrier. In that regime the fluid $f$-mode should also be considered as a trapped mode. The evidence for this from Figures 1 and 3 are clear.

### 4.2. Modes associated with the surface of the star

The existence of the second family of w-modes may be more directly due to the discontinuity at the surface of the star. Then the mode-eigenfunctions need not be localized in the star. Rather, these modes would be similar to modes for acoustic waves scattered off a hard sphere. One would typically expect such modes to be short-lived compared to modes trapped inside the star. This agrees well with the evidence of Figures 3 and 4. As Jensen (1989) has shown, the problem of acoustic waves and a hard sphere can be approached analytically. It is intriguing to find that the modes in that model problem are quite similar our second family of w-modes. Only a finite number of modes exist for each multipole $\ell$, and there may be purely imaginary ones. The latter feature suggests that one should perhaps not be surprised to find stellar modes “emerge” from points on the imaginary-frequency axis as in Figure 4 here. In our opinion, this evidence is compelling and makes the association of the second family of w-modes and the interface at the stellar surface likely. Hence, we have decided to call these modes “interface modes”. In principle, our conjecture that only a finite number of these modes exist should be testable. But at present numerical difficulties restrict calculations to $\text{Im} \omega M < 1.25$ or so. Much better numerical schemes, or other formulations of the problem, are required to test this prediction.

### 5. Concluding remarks

We have presented and discussed the results of an extensive survey of gravitational-wave modes for the simplest possible stellar model; that of uniform density. The results are truly exciting and contribute considerably to our understanding of the gravitational-wave modes of compact stars. Nevertheless, it is clear that much work remains in this area.

We have shown that the problem of axial perturbations has all the essential features of the polar problem, as far as gravitational waves are concerned. This establishes the existence of axial modes for all stellar models, contrary to what has long been the accepted view (Thorne and Campolattaro 1967; Chandrasekhar and Ferrari 1991b). The present results show that
there exists two distinct families of gravitational-wave modes. We suggest that one of these families, the curvature modes, corresponds to waves trapped in the interior of the star, whereas the second, the interface modes, is associated with the “discontinuity” at the surface of the star. The first set of modes is similar to those of the toy model proposed by Kokkotas and Schutz (1986). The latter set of modes would be analogous to the acoustic modes associated with a hard sphere (Jensen 1989). While there should exist an infinite number of curvature modes, the number of interface modes may well be finite and possibly small.

Considering that the uniform density star can only support a single fluid mode – the Kelvin f-mode – the unveiled richness of the spectrum of gravitational-wave modes is remarkable. Indeed, it may be anticipated that many features of a more realistic star, such as discontinuities associated with a crust and a superfluid stellar interior, will also affect the w-modes and (probably) give rise to further sets of such modes.

The present work raises many questions that future work must satisfactorily answer. For example, although our suggestions for the physical nature of the two w-mode families are plausible, they must be supported by more detailed studies. Fortunately, the axial problem has all the essential features of the w-modes, and is considerably simpler than the polar problem. Furthermore, this problem can be approached semi-analytically, by (say) the WKB method. Hence, the axial problem may prove invaluable for future exploration of the nature of the w-modes.

A second issue that must be addressed by future work concerns the excitation of stellar pulsation modes in various dynamical situations. Such work is of extreme importance if we are to properly understand a pulsating star as source of gravitational waves. Such waves may be detectable within the near future. By detecting a gravitational wave that carries the signature of the pulsation modes of a neutron star, we can hope to probe not only the interior of the star but also the nature of spacetime itself. This is undoubtedly an intriguing prospect.

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Fig. 1.— Im $\omega (R^3/M)^{1/2}$ vs Re $\omega (R^3/M)^{1/2}$ for the f-mode (solid and denoted by f). Also shown are the first w-mode for polar (solid) and axial (dashed) perturbations. Higher order modes behave in an almost identical way. The box indicates the part of the frequency plane that was covered in the study of Kojima, Andersson and Kokkotas (1995). That study could not resolve the features in the lower left corner of that box. Arrows indicate the direction of increasing compactness.

Table 1: Comparing the frequencies of interface modes in the limit $R/M \rightarrow 9/4$ to the quasinormal modes of a Schwarzschild black hole. For the black hole the axial and the polar spectra are identical. The stellar modes are all for $M/R = 0.44$ but are not expected to deviate much from these values in the true limit. All entries are listed in units of $M^{-1}$.

|       | polar   | axial   | black hole |
|-------|---------|---------|------------|
| 0.67+0.13i | 0.74+0.29i | 0.37+0.09i |
| 0.79+0.57i | 0.87+0.73i | 0.35+0.27i |
| 0.91+1.04i | 0.30+0.48i | 0.30+0.48i |
Fig. 2.— (left) Re $\omega(R^3/M)^{1/2}$ and (right) Im $\omega(R^3/M)^{1/2}$ as functions of the compactness of the star $M/R$ for the $f$-mode (solid and denoted by $f$), and the first few $\omega$-modes for polar (solid) and axial (dashed) perturbations. Notice the beautiful example of avoided crossings in the pulsation frequencies.

Table 2: The constant values approached by Re $\omega_n R$ as $R/M \to \infty$. These results are for the curvature modes, and specifically $R/M = 100$. But the results would typically differ with less than 1% as long as $R/M > 60$. It is interesting to note that the separation between consecutive modes is quite close to $\pi$.

| $n$ | polar modes | axial modes |
|-----|--------------|-------------|
|     | Re $\omega_n R$ | Re $(\omega_n - \omega_{n-1}) R$ | Re $\omega_n R$ | Re $(\omega_n - \omega_{n-1}) R$ |
| 1   | 3.64         |              |              | (1.50)               |
| 2   | 7.02         | 3.38         | 5.35         | (3.85)               |
| 3   | 10.27        | 3.25         | 8.65         | 3.30                 |
| 4   | 13.46        | 3.19         | 11.87        | 3.22                 |
Fig. 3.— Im $\omega_M$ vs Re $\omega_M$ for the polar (solid) and axial (dashed) curvature modes. These modes are the extension of Figure 1 for less compact stars. Although we only show the first four axial and polar modes, it seems likely that an infinite number of such modes exist. Note the remarkable qualitative agreement between the two sets of modes. Arrows indicate the direction of increasing compactness.
Fig. 4.— Im $\omega M$ vs Re $\omega M$ for the polar (solid) and axial (dashed) interface modes. Only the first of these modes exist for all values of stellar compactness. Further modes arise for sufficiently compact stars. The numerical calculation becomes difficult for large Im $\omega M$ and consequently we have only partial data for the third of the polar modes modes. Arrows indicate the direction of increasing compactness.
Fig. 5.— $\Re \omega R$ vs $R/M$ for the polar (solid) and axial (dashed) curvature modes.