Certain classes of bi-univalent functions related to Shell-like curves connected with Fibonacci numbers

N. Magesh · C. Abirami · V. K. Balaji

Abstract
In 2010, Srivastava et al. [38] revived the study of coefficient problems for bi-univalent functions. Due to the pioneering work of Srivastava et al. [38], there has been elicit curiosity to study the coefficient problems for various subclasses of bi-univalent functions. Motivated predominantly by Srivastava et al. [38], in this work, we consider certain classes of bi-univalent functions related with shell-like curves connected with Fibonacci numbers to obtain the estimates of second and third Taylor-Maclaurin coefficients and Fekete - Szegö inequalities. Further, special cases are also indicated. Some observations of the results presented here are also discussed.

Keywords Univalent functions · Bi-univalent functions · Shell-like function · Convex shell-like function · Pseudo starlike function · Bazilević function

Mathematics Subject Classification Primary 30C45 · Secondary 30C50

1 Introduction and definitions

Let \( \mathbb{R} = (-\infty, \infty) \) be the set of real numbers, and

\( \mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\} \)
be the set of positive integers. Let $A$ denote the class of functions of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]  
which are analytic in the open unit disk $D = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$, where $\mathbb{C}$ be the set of complex numbers. Further, by $S$, we shall denote the class of all functions in $A$ which are univalent in $D$.

Let $P$ denote the class of functions of the form
\[ p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad z \in D \]
which are analytic with $\Re \{ p(z) \} > 0$. Here $p$ is called as Carathéodory functions [11]. It is well known that the following correspondence between the class $P$ and the class of Schwarz functions $w$ exists: $p \in P$ if and only if $p(z) = 1 + w(z)/1 - w(z)$. Let $P(\beta)$, $0 \leq \beta < 1$, denote the class of analytic functions $p$ in $D$ with $p(0) = 1$ and $\Re \{ p(z) \} > \beta$.

For analytic functions $f$ and $g$ in $D$, $f$ is said to be subordinate to $g$ if there exists an analytic function $w$ such that
\[ w(0) = 0, \quad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)), \quad z \in D, \]
denoted by
\[ f \prec g, \quad z \in D \]
or, conventionally, by
\[ f(z) \prec g(z), \quad z \in D. \]

In particular, when $g$ is univalent in $D$,
\[ f \prec g \quad (z \in D) \iff f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D). \]

Some of the important and well-investigated subclasses of $S$ include (for example) the class $S^*(\alpha)$ of starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) in $D$ and the class $K(\alpha)$ of convex functions of order $\alpha$ ($0 \leq \alpha < 1$) in $D$, the class $S^*(\varphi)$ of Ma-Minda starlike functions and the class $K(\varphi)$ of Ma-Minda convex functions, where $\varphi$ is an analytic function with positive real part in $D$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi$ maps $D$ onto a region starlike with respect to $1$ and symmetric with respect to the real axis (see [11]).

The inverse functions of the functions in the class $S$ may not be defined on the entire unit disc $D$ although the functions in the class $S$ are invertible. However using Koebe one quarter theorem [11] it is obvious that the image of $D$ under every function $f \in S$ contains a disc of radius $1/4$. Hence every function $f \in S$ has an inverse $f^{-1}$, defined by
\[ f^{-1}(f(z)) = z, \quad z \in D \]
and
\[ f(f^{-1}(w)) = w, \quad |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}, \]
where
\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \]

A function $f \in A$ is said to be bi-univalent in $D$ if both $f$ and $f^{-1}$ are univalent in $D$. Let $\Sigma$ denote the class of bi-univalent functions in $D$ given by (1.1). For a brief history and
interesting examples of functions which are in (or which are not in) the class $\Sigma$, together with various other properties of the bi-univalent function class $\Sigma$ one can refer the work of Srivastava et al. [38] and the references therein. We want to strongly, represent the fact that, the pioneering work by Srivastava et al. [38] which actually invigorated the study of analytic and bi-univalent functions in recent years and also that a rather huge flood of sequels to their pioneering work [38] have resulted in the literature on the study of various subclasses of bi-univalent functions were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found (see, for example, [1–8,12–19,21,22,24,26,27,30–37,39–45] and references therein). However, the problem is to find the coefficient bounds on $|a_n|$ ($n = 3, 4, \ldots$) for functions $f \in \Sigma$ is still an open problem. Also, one can refer a very recent work of Srivastava [29] for operators of basic (or $q$-) calculus and fractional $q$-calculus and their applications in Geometric Function Theory of Complex Analysis.

The classes $SL(\tilde{p})$ and $KSL(\tilde{p})$ of shell-like functions and convex shell-like functions are respectively, characterized by $zf'/f(z) < \tilde{p}(z)$ or $1 + z^2f''/f'(z) < \tilde{p}(z)$, where $\tilde{p}(z) = (1 + \tau^2z^2)/(1 - \tau z - \tau^2z^2)$, $\tau = (1 - \sqrt{5})/2 \approx -0.618$. The classes $SL(\tilde{p})$ and $KSL(\tilde{p})$ were introduced and studied by Sokół [28] and Dziok et al. [9] respectively (see also [10,25]). The function $\tilde{p}$ is not univalent in $\mathbb{D}$, but it is univalent in the disc $|z| < (3 - \sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp \arccos(1/4)}) = \sqrt{5}/5$ and it may also be noticed that $1/|\tau| = |\tau|/1 - |\tau|$ which shows that the number $|\tau|$ divides $[0, 1]$ such that it fulfills the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by $\left(10x - \sqrt{5}\right)^2 = \left(5 - 2x\right)^2 \left(\sqrt{5}x - 1\right)^2$, which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop and for $r = 1$, it has a vertical asymptote. Since $\tau$ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers $\tau^n$ as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of $\tau$ and 1. The resulting recurrence relationships yield Fibonacci numbers $u_n$

$$u_n = u_n \tau + u_{n-1}.$$  

Recently Raina and Sokół [25], taking $\tau z = t$, showed that

$$\tilde{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau z - \tau^2z^2} = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1}) \tau^n z^n, \quad (1.2)$$

where

$$u_n = \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618 \quad \text{and} \quad n = 1, 2, \cdots. \quad (1.3)$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers $u_n$, such that

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+2} = u_n + u_{n+1}$$

for $n = 0, 1, 2, 3, \cdots$. Hence

$$\tilde{p}(z) = 1 + \tau z + 3\tau^2z^2 + 4\tau^3z^3 + 7\tau^4z^4 + 11\tau^5z^5 + \cdots. \quad (1.4)$$

We note that the function $\tilde{p}$ belongs to the class $P(\beta)$ with $\beta = \frac{\sqrt{5}}{10} \approx 0.2236$ (see [25]).
Recently, in literature, the initial coefficient estimates are found for functions in the class of bi-univalent functions associated with certain polynomials like the Faber polynomial, the Lucas polynomial, the Chebyshev polynomial and the Gegenbauer polynomial. Motivated in these lines and the works of Ali et al. [1], Güney et al. [15] and Orhan et al. [20,22], we introduce the following new subclasses of bi-univalent functions.

In the following definitions, in its special cases and throughout the paper, we take

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau^2 z^2}, \quad p(w) = \frac{1 + \tau^2 w^2}{1 - \tau^2 w^2} \quad \text{and} \quad \tau = \frac{1 - \sqrt{5}}{2} \approx -0.618$$

one or otherwise mentioned specifically.

**Definition 1.1** A function $f \in \Sigma$ of the form (1.1) belongs to the class $\mathcal{WSL}_\Sigma(\gamma, \lambda, \alpha, \tilde{p})$, $\gamma \in \mathbb{C}\setminus \{0\}$, $\alpha \geq 0$ and $\lambda \geq 0$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \frac{f(z)}{z} + (\alpha - 2\lambda) f'(z) + \lambdazf''(z) - 1 \right) < \tilde{p}(z), \quad z \in \mathbb{D} \quad (1.5)$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \frac{g(w)}{w} + (\alpha - 2\lambda) g'(w) + \lambdawg''(w) - 1 \right) < \tilde{p}(w), \quad w \in \mathbb{D}. \quad (1.6)$$

It is interesting to note that the special values of $\alpha$, $\gamma$ and $\lambda$ lead the class $\mathcal{WSL}_\Sigma(\gamma, \lambda, \alpha, \tilde{p})$ to various subclasses, we illustrate the following subclasses:

1. For $\alpha = 1 + 2\lambda$, we get the class $\mathcal{WSL}_\Sigma(\gamma, \lambda, 1 + 2\lambda, \tilde{p}) \equiv \mathcal{FSL}_\Sigma(\gamma, \lambda, \tilde{p})$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{FSL}_\Sigma(\gamma, \lambda, \tilde{p})$, if the following conditions

$$1 + \frac{1}{\gamma} \left( f'(z) + \lambda zf''(z) - 1 \right) < \tilde{p}(z), \quad z \in \mathbb{D}$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma} \left( g'(w) + \lambda wg''(w) - 1 \right) < \tilde{p}(w), \quad w \in \mathbb{D}$$

hold.

2. For $\lambda = 0$, we obtain the class $\mathcal{WSL}_\Sigma(\gamma, 0, \alpha, \tilde{p}) \equiv \mathcal{BSL}_\Sigma(\gamma, \alpha, \tilde{p})$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{BSL}_\Sigma(\gamma, \alpha, \tilde{p})$, if the following conditions

$$1 + \frac{1}{\gamma} \left( (1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) - 1 \right) < \tilde{p}(z), \quad z \in \mathbb{D}$$

and for $g(w) = f^{-1}(w)$

$$1 + \frac{1}{\gamma} \left( (1 - \alpha) \frac{g(w)}{w} + \alpha g'(w) - 1 \right) < \tilde{p}(w), \quad w \in \mathbb{D}$$

hold.

3. For $\lambda = 0$ and $\alpha = 1$, we have the class $\mathcal{WSL}_\Sigma(\gamma, 0, 1, \tilde{p}) \equiv \mathcal{HSL}_\Sigma(\gamma, \tilde{p})$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{HSL}_\Sigma(\gamma, \tilde{p})$, if the following conditions

$$1 + \frac{1}{\gamma} \left( f'(z) - 1 \right) < \tilde{p}(z), \quad z \in \mathbb{D}$$
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and for \( g(w) = f^{-1}(w) \)

\[
1 + \frac{1}{\gamma} \left( g'(w) - 1 \right) \prec \widetilde{p}(w), \quad w \in \mathbb{D}
\]

hold.

**Definition 1.2** A function \( f \in \Sigma \) of the form (1.1) belongs to the class \( \mathcal{RSL}_\Sigma(\gamma, \lambda, \tilde{p}) \), \( \gamma \in \mathbb{C} \setminus \{0\} \) and \( \lambda \geq 0 \), if the following conditions are satisfied:

\[
1 + \frac{1}{\gamma} \left( z^{1-\lambda} f'(z) - 1 \right) \prec \widetilde{p}(z), \quad z \in \mathbb{D} \tag{1.7}
\]

and for \( g(w) = f^{-1}(w) \)

\[
1 + \frac{1}{\gamma} \left( w^{1-\lambda} g'(w) - 1 \right) \prec \widetilde{p}(w), \quad w \in \mathbb{D}. \tag{1.8}
\]

(1) For \( \lambda = 0 \), we let the class \( \mathcal{RSL}_\Sigma(\gamma, 0, \tilde{p}) \equiv \mathcal{SL}_\Sigma(\gamma, \tilde{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( \mathcal{SL}_\Sigma(\gamma, \tilde{p}) \), if the following conditions

\[
1 + \frac{1}{\gamma} \left( z f'(z) - 1 \right) \prec \widetilde{p}(z), \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
1 + \frac{1}{\gamma} \left( w g'(w) - 1 \right) \prec \widetilde{p}(w), \quad w \in \mathbb{D}
\]

hold.

**Remark 1.1** For \( \gamma = 1 \) the class \( \mathcal{SL}_\Sigma(1, \tilde{p}) \equiv \mathcal{SL}_\Sigma(\tilde{p}) \) was introduced and studied Güney et al. [15].

(2) For \( \lambda = 1 \), we have \( \mathcal{RSL}_\Sigma(\gamma, 1, \tilde{p}) \equiv \mathcal{SL}_\Sigma(\gamma, \tilde{p}) \).

**Definition 1.3** A function \( f \in \Sigma \) of the form (1.1) belongs to the class \( \mathcal{SLB}_\Sigma(\lambda, \tilde{p}) \), \( \lambda \geq 1 \), if the following conditions are satisfied:

\[
z \left[ f'(z) \right]^\lambda \prec \widetilde{p}(z), \quad z \in \mathbb{D} \tag{1.9}
\]

and for \( g(w) = f^{-1}(w) \)

\[
w \left[ g'(w) \right]^\lambda \prec \widetilde{p}(w), \quad w \in \mathbb{D}. \tag{1.10}
\]

(1) For \( \lambda = 1 \), we have the class \( \mathcal{SLB}_\Sigma(1, \tilde{p}) \equiv \mathcal{SL}_\Sigma(\tilde{p}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( \mathcal{SL}_\Sigma(\tilde{p}) \), if the following conditions

\[
z f'(z) \prec \widetilde{p}(z), \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
w g'(w) \prec \widetilde{p}(w), \quad w \in \mathbb{D},
\]

hold.
Definition 1.4 A function \( f \in \Sigma \) of the form (1.1) belongs to the class \( \mathcal{PSL}_\Sigma(\lambda; \tilde{\rho}) \), 0 \( \leq \lambda \leq 1 \), if the following conditions are satisfied:

\[
\frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} < \tilde{p}(z), \quad z \in \mathbb{D}
\]  

(1.11)

and for \( g(w) = f^{-1}(w) \)

\[
\frac{wf'(w) + \lambda w^2 g''(w)}{(1 - \lambda)g(w) + \lambda wg'(w)} < \tilde{p}(w), \quad w \in \mathbb{D}.
\]  

(1.12)

(1) For \( \lambda = 0 \), we have the class \( \mathcal{PSL}_\Sigma(0; \tilde{\rho}) \equiv \mathcal{SL}_\Sigma(\tilde{\rho}) \).

(2) For \( \lambda = 1 \), we have the class \( \mathcal{PSL}_\Sigma(1; \tilde{\rho}) \equiv K\mathcal{SL}_\Sigma(\tilde{\rho}) \). A function \( f \in \Sigma \) of the form (1.1) is said to be in \( K\mathcal{SL}_\Sigma(\tilde{\rho}) \), if the following conditions

\[
1 + \frac{z^2 f''(z)}{f'(z)} < \tilde{p}(z), \quad z \in \mathbb{D}
\]

and for \( g(w) = f^{-1}(w) \)

\[
1 + \frac{w^2 g''(w)}{g'(w)} < \tilde{p}(w), \quad w \in \mathbb{D},
\]

hold.

Remark 1.2 For \( \gamma = 0 \), \( \mathcal{PSL}_\Sigma(0, \tilde{\rho}) \equiv \mathcal{SL}_\Sigma(\tilde{\rho}) \) and \( \gamma = 1 \), \( \mathcal{PSL}_\Sigma(1, \tilde{\rho}) \equiv K\mathcal{SL}_\Sigma(\tilde{\rho}) \) the classes were introduced and studied Güney et al. [15].

In order to prove our results for the functions in the classes \( W\mathcal{SL}_\Sigma(\gamma, \lambda, \alpha, \tilde{\rho}) \), \( S\mathcal{LB}_\Sigma(\lambda; \tilde{\rho}) \) and \( \mathcal{PSL}_\Sigma(\lambda; \tilde{\rho}) \), we need the following lemma.

Lemma 1.1 [23] If \( p \in \mathcal{P} \), then \( |p_i| \leq 2 \) for each \( i \).

2 Initial Coefficient Estimates and Fekete-Szegö Inequalities

In this section, we obtain initial coefficient estimates and Fekete-Szegö inequalities for functions in the aforementioned classes.

Theorem 2.1 Let \( f \in f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in the class \( W\mathcal{SL}_\Sigma(\gamma, \lambda, \alpha, \tilde{\rho}) \). Then

\[
|a_2| \leq \frac{\gamma |\tau|}{\sqrt{\gamma \tau (1 + 2\alpha + 2\lambda) + (1 - 3\tau)(1 + \alpha)^2}},
\]

\[
|a_3| \leq \frac{\gamma |\tau| [(1 - 3\tau)(1 + \alpha)^2]}{(1 + 2\alpha + 2\lambda) [\gamma \tau (1 + 2\alpha + 2\lambda) + (1 - 3\tau)(1 + \alpha)^2]}
\]
and

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\gamma||\tau|}{(1 + 2\alpha + 2\lambda)}; \\
0 \leq |\mu - 1| \leq \frac{(1 + 2\alpha + 2\lambda) + (1 + \alpha)^2 (1 - 3\tau)}{|\gamma||\tau|(1 + 2\alpha + 2\lambda)} \\
|\mu - 1| \leq \frac{\gamma\tau (1 + 2\alpha + 2\lambda) + (1 + \alpha)^2 (1 - 3\tau)}{|\gamma||\tau|(1 + 2\alpha + 2\lambda)}. 
\end{cases} \]

**Proof** Since \( f \in \mathcal{WSEP}(\gamma', \lambda, \alpha, \tilde{p}) \), from the Definition 1.1, we have

\[ 1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \frac{f(z)}{z} + (\alpha - 2\lambda) f'(z) + \lambda z f''(z) - 1 \right) = \tilde{p}(u(z)) \tag{2.1} \]

and

\[ 1 + \frac{1}{\gamma} \left( (1 - \alpha + 2\lambda) \frac{g(w)}{w} + (\alpha - 2\lambda) g'(w) + \lambda w g''(w) - 1 \right) = \tilde{p}(v(w)), \tag{2.2} \]

where \( z, w \in \mathbb{D} \) and \( g = f^{-1} \). Using the fact that the function \( p \) of the form

\[ p(z) = 1 + p_1 z + p_2 z^2 + \cdots \]

and \( p < \tilde{p} \). Then there exists an analytic function \( u \) such that \( |u(z)| < 1 \) in \( \mathbb{D} \) and \( p(z) = \tilde{p}(u(z)) \). Therefore, the function

\[ h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \]

is in the class \( \mathcal{P} \). It follows that

\[ u(z) = \frac{h(z) - 1}{h(z) + 1} = \frac{c_1}{2} z + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \]

and

\[ \tilde{p}(u(z)) = 1 + \tilde{p} \left( \frac{c_1}{2} z + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right) \]

\[ + \tilde{p}_2 \left( \frac{c_1}{2} z + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right)^2 \]

\[ + \tilde{p}_3 \left( \frac{c_1}{2} z + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \right)^3 \]

\[ + \cdots \]

\[ = 1 + \frac{\tilde{p}_1 c_1}{2} z + \left( \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^3}{4} \tilde{p}_2 \right) z^2 \]

\[ + \left( \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} c_1 \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_2 + \frac{c_1^3}{8} \tilde{p}_3 \right) z^3 \]

\[ + \cdots. \tag{2.3} \]
Similarly, there exists an analytic function \( v \) such that \( |v(w)| < 1 \) in \( \mathbb{D} \) and \( p(w) = \tilde{v}(v(w)) \). Therefore, the function
\[
k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \cdots
\]
is in the class \( \mathcal{P} \). It follows that
\[
v(w) = \frac{k(w) - 1}{k(w) + 1} = \frac{d_1}{2} w + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots
\]
and
\[
\tilde{p}(v(w)) = 1 + \tilde{p} \left( \frac{d_1}{2} w + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots \right)
\]
\[
+ \tilde{p}_2 \left( \frac{d_1}{2} w + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots \right)^2
\]
\[
+ \tilde{p}_3 \left( \frac{d_1}{2} w + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots \right)^3
\]
\[+ \cdots
\]
\[
= 1 + \frac{\tilde{p}_1 d_1}{2} w + \left( \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_1 + \frac{d_2}{8} \tilde{p}_3 \right) w^2
\]
\[
+ \left( \frac{1}{2} \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \tilde{p}_1 + \frac{1}{2} d_1 \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_2 + \frac{d_3}{8} \tilde{p}_3 \right) w^3
\]
\[+ \cdots .
\] (2.4)

By virtue of (2.1), (2.2), (2.3) and (2.4), we have
\[
\frac{1}{\gamma} (1 + \alpha)a_2 = \frac{c_1 \tau}{2},
\] (2.5)
\[
\frac{\alpha}{\gamma} (1 + 2 \alpha + 2 \lambda) = \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tau + \frac{3 c_1^2}{4} \tau^2,
\] (2.6)
\[
-\frac{1}{\gamma} (1 + \alpha)a_2 = \frac{d_1 \tau}{2},
\] (2.7)
and
\[
(1 + 2 \alpha + 2 \lambda) \gamma \frac{(2 a_2^2 - a_3)}{2} = \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tau + \frac{3 d_1^2}{4} \tau^2.
\] (2.8)

From (2.5) and (2.7), we obtain
\[
c_1 = -d_1,
\]
and
\[
\frac{2}{\gamma^2} (1 + \alpha)^2 a_2^2 = \frac{(c_1^2 + d_1^2) \tau^2}{4}
\]
\[
a_2^2 = \frac{\gamma^2 (c_1^2 + d_1^2) \tau^2}{8(1 + \alpha)^2}.
\] (2.9)
By adding (2.6) and (2.8), we have
\[
\frac{2}{\gamma} (1 + 2\alpha + 2\lambda) a_2^2 = \frac{1}{2} (c_2 + d_2) \tau - \frac{1}{4} (c_1^2 + d_1^2) \tau + \frac{3}{4} (c_1^2 + d_1^2) \tau^2. 
\tag{2.10}
\]

By substituting (2.9) in (2.10), we reduce that
\[
a_2^2 = \frac{\gamma^2 (c_2 + d_2) \tau^2}{4 \left[ \gamma \tau (1 + 2\alpha + 2\lambda) + (1 - 3\tau)(1 + \alpha)^2 \right]}.
\tag{2.11}
\]

Now, applying Lemma 1.1, we obtain
\[
|a_2| \leq \frac{|\gamma| |\tau|}{\sqrt{\gamma \tau (1 + 2\alpha + 2\lambda) + (1 - 3\tau)(1 + \alpha)^2}}.
\tag{2.12}
\]

By subtracting (2.8) from (2.6), we obtain
\[
a_3 = \frac{\gamma (c_2 - d_2) \tau}{4 (1 + 2\alpha + 2\lambda)} + a_2^2.
\tag{2.13}
\]

Hence by Lemma 1.1, we have
\[
|a_3| \leq \frac{|\gamma| (|c_2| + |d_2|) |\tau|}{4 (1 + 2\alpha + 2\lambda)} + |a_2|^2 \leq \frac{|\gamma| |\tau|}{(1 + 2\alpha + 2\lambda)} + |a_2|^2.
\tag{2.14}
\]

Then in view of (2.12), we obtain
\[
|a_3| \leq \frac{|\gamma| |\tau|}{(1 + 2\alpha + 2\lambda) \left[ \gamma \tau (1 + 2\alpha + 2\lambda) + (1 - 3\tau)(1 + \alpha)^2 \right]}.
\]

From (2.13), we have
\[
a_3 - \mu a_2^2 = \frac{\gamma (c_2 - d_2) \tau}{4 (1 + 2\alpha + 2\lambda)} + (1 - \mu) a_2^2.
\tag{2.15}
\]

By substituting (2.11) in (2.15), we have
\[
a_3 - \mu a_2^2 = \frac{\gamma (c_2 - d_2) \tau}{4 (1 + 2\alpha + 2\lambda)} + (1 - \mu) \left( \frac{\gamma^2 (c_2 + d_2) \tau^2}{4 \left[ \gamma \tau (1 + 2\alpha + 2\lambda) + (1 - 3\tau)(1 + \alpha)^2 \right]} \right)
= \left( h(\mu) + \frac{\gamma |\tau|}{4 (1 + 2\alpha + 2\lambda)} \right) c_2 + \left( h(\mu) - \frac{\gamma |\tau|}{4 (1 + 2\alpha + 2\lambda)} \right) d_2,
\tag{2.16}
\]

where
\[
h(\mu) = \frac{(1 - \mu) \gamma^2 \tau^2}{4 \left[ \gamma \tau (1 + 2\alpha + 2\lambda) + (1 + \alpha)^2 (1 - 3\tau) \right]}.
\]

Thus by taking modulus of (2.16), we conclude that
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\gamma| |\tau|}{1 + 2\alpha + 2\lambda} : 0 \leq |h(\mu)| \leq \frac{|\gamma| |\tau|}{4 (1 + 2\alpha + 2\lambda)} \\
4|h(\mu)| : |h(\mu)| \geq \frac{|\gamma| |\tau|}{4 (1 + 2\alpha + 2\lambda)}.
\end{cases}
\]

\[\square\]

\[\text{Springer}\]
Theorem 2.2 Let \( f \in RSL_{\Sigma}(\gamma, \lambda, \tilde{p}) \). Then

\[
|a_2| \leq \frac{\sqrt{2} |\gamma| |\tau|}{\sqrt{\gamma \tau (2 + \lambda) (1 + \lambda) + 2(1 - 3\tau)(1 + \lambda)^2}},
\]

\[
|a_3| \leq \frac{|\gamma| |\tau| \{\gamma \tau (2 + \lambda) (1 + \lambda) + 2(1 - 3\tau)(1 + \lambda)^2 + 2 (2 + \lambda) \gamma \tau \}}{(2 + \lambda) \{\gamma \tau (2 + \lambda) (1 + \lambda) + 2(1 - 3\tau)(1 + \lambda)^2\}}
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\gamma| |\tau|}{2 + \lambda} : 0 \leq |\mu - 1| \leq \frac{\gamma \tau (2 + \lambda) (1 + \lambda) + 2(1 + \lambda)^2 (1 - 3\tau)}{2 |\gamma| |\tau| (2 + \lambda)} \\
\frac{|\gamma| |\tau|}{(2 + \lambda) (1 + \lambda) + 2(1 + \lambda)^2 (1 - 3\tau)} : |\mu - 1| \geq \frac{\gamma \tau (2 + \lambda) (1 + \lambda) + 2(1 + \lambda)^2 (1 - 3\tau)}{2 |\gamma| |\tau| (2 + \lambda)}
\end{cases}
\]

Proof of the Theorem 2.2 is similar to Theorem 2.1, so details are omitted here.

Theorem 2.3 Let \( f \in SLB_{\Sigma}(\lambda; \tilde{p}) \). Then

\[
|a_2| \leq \frac{|\tau|}{\sqrt{(2\lambda - 1)[\tau (3 - 5\lambda) + 2\lambda - 1]}}.
\]

\[
|a_3| \leq \frac{|\tau| [(2\lambda - 1)^2 - 2 (5\lambda^2 - 4\lambda + 1) \tau]}{(2\lambda - 1)(3\lambda - 1)[(3 - 5\lambda)\tau + 2\lambda - 1]}
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\gamma| |\tau|}{3\lambda - 1} : 0 \leq |\mu - 1| \leq \frac{(2\lambda - 1) [\tau (3 - 5\lambda) + 2\lambda - 1]}{|\tau| (3\lambda - 1)} \\
\frac{|\gamma| |\tau|}{(2\lambda - 1) [\tau (3 - 5\lambda) + 2\lambda - 1]} : |\mu - 1| \geq \frac{(2\lambda - 1) [\tau (3 - 5\lambda) + 2\lambda - 1]}{|\tau| (3\lambda - 1)}
\end{cases}
\]

Proof of the Theorem 2.3 is similar to Theorem 2.1, so details are omitted here.

Theorem 2.4 Let \( f \in PSL_{\Sigma}(\lambda; \tilde{p}) \). Then

\[
|a_2| \leq \frac{|\tau|}{\sqrt{(1 + \lambda)^2 - 2\tau (2\lambda^2 + 2\lambda + 1)}},
\]

\[
|a_3| \leq \frac{|\tau| (1 - 4\tau) (1 + \lambda)^2}{2(1 + 2\lambda) [(1 + \lambda)^2 - 2\tau (2\lambda^2 + 2\lambda + 1)]}
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\tau|}{2 + 4\lambda} : 0 \leq |\mu - 1| \leq \frac{(1 + \lambda)^2 - 2\tau (2\lambda^2 + 2\lambda + 1)}{2 |\tau| (1 + 2\lambda)} \\
\frac{|\gamma| |\tau|}{(1 + \lambda)^2 - 2\tau (2\lambda^2 + 2\lambda + 1)} : |\mu - 1| \geq \frac{(1 + \lambda)^2 - 2\tau (2\lambda^2 + 2\lambda + 1)}{2 |\tau| (1 + 2\lambda)}
\end{cases}
\]

Proof of the Theorem 2.4 is similar to Theorem 2.1, so details are omitted here.
3 Corollaries and consequences

In this section, we obtain initial coefficient estimates and Fekete-Szegö inequalities for special cases of our defined classes.

**Corollary 3.1** Let \( f \in \mathcal{FSL}_\Sigma(\gamma, \lambda, \tilde{p}) \). Then

\[
|a_2| \leq \frac{\gamma | \gamma | \tau}{\sqrt{3 \gamma \tau (1 + 2 \lambda) + 4(1 - 3 \tau)(1 + \lambda)^2}},
\]

\[
|a_3| \leq \frac{4 | \gamma | | \tau| (1 - 3 \tau)(1 + \lambda)^2}{3(1 + 2 \lambda) \left[ 3 \gamma \tau (1 + 2 \lambda) + 4(1 - 3 \tau)(1 + \lambda)^2 \right]}.
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{| \gamma | | \tau|}{3 + 6 \lambda} & : 0 \leq | \mu - 1 | \leq \frac{3 \gamma \tau (1 + 2 \lambda) + 4(1 - 3 \tau)(1 + \lambda)^2}{(3 + 6 \lambda) | \gamma | | \tau|} \\
\frac{| 1 - \mu | \gamma^2 \tau^2}{3 \gamma \tau (1 + 2 \lambda) + 4(1 - 3 \tau)(1 + \lambda)^2} & : | \mu - 1 | \geq \frac{3 \gamma \tau (1 + 2 \lambda) + 4(1 - 3 \tau)(1 + \lambda)^2}{(3 + 6 \lambda) | \gamma | | \tau|}.
\end{cases}
\]

**Corollary 3.2** Let \( f \in \mathcal{BSL}_\Sigma(\gamma, \alpha, \tilde{p}) \). Then

\[
|a_2| \leq \frac{\gamma | \gamma | \tau}{\sqrt{\gamma \tau (1 + 2 \alpha) + (1 - 3 \tau)(1 + \alpha)^2}},
\]

\[
|a_3| \leq \frac{\gamma | \gamma | \tau}{(1 + 2 \alpha) \left[ \gamma \tau (1 + 2 \alpha) + (1 - 3 \tau)(1 + \alpha)^2 \right]}.
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{| \gamma | | \tau|}{1 + 2 \alpha} & : 0 \leq | \mu - 1 | \leq \frac{\gamma \tau (1 + 2 \alpha) + (1 + \alpha)^2(1 - 3 \tau)}{(1 + 2 \alpha) | \gamma | | \tau|} \\
\frac{| 1 - \mu | \gamma^2 \tau^2}{\gamma \tau (1 + 2 \alpha) + (1 + \alpha)^2(1 - 3 \tau)} & : | \mu - 1 | \geq \frac{\gamma \tau (1 + 2 \alpha) + (1 + \alpha)^2(1 - 3 \tau)}{(1 + 2 \alpha) | \gamma | | \tau|}.
\end{cases}
\]

**Corollary 3.3** Let \( f \in \mathcal{HSL}_\Sigma(\gamma, \tilde{p}) \). Then

\[
|a_2| \leq \frac{\gamma | \gamma | \tau}{\sqrt{3 \gamma \tau + 4(1 - 3 \tau)}},
\]

\[
|a_3| \leq \frac{4 | \gamma | | \tau| (1 - 3 \tau)}{3 \left[ 3 \gamma \tau + 4(1 - 3 \tau) \right]}.
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{| \gamma | | \tau|}{3} & : 0 \leq | \mu - 1 | \leq \frac{3 \gamma \tau + 4(1 - 3 \tau)}{3 | \gamma | | \tau|} \\
\frac{| 1 - \mu | \gamma^2 \tau^2}{3 \gamma \tau + 4(1 - 3 \tau)} & : | \mu - 1 | \geq \frac{3 \gamma \tau + 4(1 - 3 \tau)}{3 | \gamma | | \tau|}.
\end{cases}
\]

**Corollary 3.4** Let \( f \in \mathcal{SL}_\Sigma(\gamma, \tilde{p}) \). Then

\[
|a_2| \leq \frac{\gamma | \gamma | \tau}{\sqrt{\gamma \tau + (1 - 3 \tau)}},
\]

\[
|a_3| \leq \frac{| \gamma | | \tau| \left[ 3 \gamma \tau + (1 - 3 \tau) \right]}{2 \gamma \tau + 2(1 - 3 \tau)}.
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{| \gamma | | \tau|}{2} & : 0 \leq | \mu - 1 | \leq \frac{\gamma \tau + (1 - 3 \tau)}{2 | \gamma | | \tau|} \\
\frac{| 1 - \mu | \gamma^2 \tau^2}{\gamma \tau + (1 - 3 \tau)} & : | \mu - 1 | \geq \frac{\gamma \tau + (1 - 3 \tau)}{2 | \gamma | | \tau|}.
\end{cases}
\]
Corollary 3.5 [15] Let \( f \in {\text{SL}}_{\Sigma}(\tilde{\rho}) \). Then

\[
|a_2| \leq \frac{1}{\sqrt{1-2\tau} \tau}, \quad |a_3| \leq \frac{|\tau|(1-4\tau)}{2-4\tau}
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\tau|}{2}; & 0 \leq |\mu - 1| \leq \frac{1-2\tau}{2|\tau|} \\
\frac{|1-\mu| \tau^2}{1-2\tau}; & |\mu - 1| \geq \frac{1-2\tau}{2|\tau|}.
\end{cases}
\]

Corollary 3.6 [15] Let \( f \in K{\text{SL}}_{\Sigma}(\tilde{\rho}) \). Then

\[
|a_2| \leq \frac{1}{\sqrt{4-10\tau} \tau}, \quad |a_3| \leq \frac{|\tau|(1-4\tau)}{6-15\tau}
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\tau|}{6}; & 0 \leq |\mu - 1| \leq \frac{2-5\tau}{3|\tau|} \\
\frac{|1-\mu| \tau^2}{4-10\tau}; & |\mu - 1| \geq \frac{2-5\tau}{6|\tau|}.
\end{cases}
\]

Remark 3.1 Results discussed in Corollaries 3.5 and 3.6 are coincide with bounds obtained in [15, Corollary 1, Corollary 2, Corollary 4 and Corollary 5].

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