A multipoint vorticity mixed finite element method for incompressible Stokes flow

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Abstract
We propose a mixed finite element method for Stokes flow with one degree of freedom per element and facet of simplicial grids. The method is derived by considering the vorticity-velocity-pressure formulation and eliminating the vorticity locally through the use of a quadrature rule. The discrete solution is pointwise divergence-free and the method is pressure robust. The theoretically derived convergence rates are confirmed by numerical experiments.

Keywords: multipoint vorticity, mixed finite element method, Stokes flow, hybridization

2020 MSC: 65N12, 76M10, 65N30

1. Introduction
The velocity field that weakly solves a Stokes flow problem is typically sought in the Hilbert space $H^1$. Stable, mixed finite element methods (MFEM) that satisfy this regularity constraint can be constructed through the use of discrete Stokes complexes \cite{1}. However, we aim to construct a MFEM of low order using the Raviart-Thomas finite element pair, which lacks the regularity necessary to fit into the Stokes complex framework. Instead, we choose to conform to a three-field formulation of the Stokes equations, known as the vorticity-velocity-pressure formulation \cite{2}. Recently, this formulation was discretized using the framework of finite element exterior calculus \cite{3}.

The three-field formulation thus introduces the vorticity as a third variable and the main idea proposed in this letter is to subsequently eliminate this variable, locally, using a low-order quadrature rule. The technique is inspired by the multipoint flux MFEM (MF-MFEM) \cite{4}, and we inherit the nomenclature by referring to our method as a multipoint vorticity MFEM (MV-MFEM). The method is linearly convergent with the velocity in $H(\nabla \cdot, \Omega)$ and the (locally post-computed) vorticity in $H(\nabla \times, \Omega)$.

We remark that multipoint stress MFEM (MS-MFEM)\cite{5} was recently proposed for Stokes flow, based on the stress-velocity-vorticity formulation. In that case, the stress and vorticity are eliminated, which leads to a method that preserves local momentum balance but lacks local mass conservation, in contrast to MV-MFEM. The hybridization techniques in this family of methods were recently identified and generalized as local approximations of the exterior coderivative \cite{6}.

In this letter, we combine a low-order discretization of the three-field formulation from \cite{3} with the computation of local co-derivatives from \cite{6}. Focusing on the lowest order method, we show that the use of the localized quadrature rule does not impact the linear convergence. Moreover, the pressure variable is unaffected, as is the curl of the vorticity and the divergence of the velocity. In fact, in two dimensions, the only influence of the quadrature rule is a second-order error in the velocity. These results are shown theoretically in Section 4 and experimentally in Section 5.

1.1. Notation
Let $L^2(\Omega)$ denote the space of square-integrable functions on $\Omega$ and let $H(\nabla \cdot, \Omega)$ and $H(\nabla \times, \Omega)$ be the spaces of square-integrable vector fields on $\Omega$ with square-integrable divergence and curl, respectively. The $H^s(\Sigma)$ norm is denoted $\| \cdot \|_{s,\Sigma}$ and we use the short-hand notation $\| \cdot \|_\Sigma := \| \cdot \|_{0,\Sigma}$ and $\| \cdot \| := \| \cdot \|_{0,\Omega}$. We use $(\cdot, \cdot)_{\Omega}$ to represent the

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Preprint submitted to Elsevier August 30, 2022
following calculus identity for solenoidal fields $q$
\begin{equation}
\varepsilon \partial_a q = \text{bounded constant}
\end{equation}
\begin{equation}
g \in \nabla \times \text{dual of a Hilbert space}
\end{equation}

Using the definition of the vorticity problem (2.2) to be of the form $B$.

To highlight the structure of this problem, we define the operators $A$, $B_r$, $B_q$, and functionals $f_r$, $f_q$ such that:

\begin{align*}
\langle Ar, \tilde{r} \rangle := (j^{-1} r, \tilde{r})_{\Omega}, & \quad \langle B_r r, \tilde{q} \rangle := \langle \nabla \times r, \tilde{q} \rangle_{\Omega}, & \langle B_q q, \tilde{p} \rangle := \langle \nabla \cdot q, \tilde{p} \rangle_{\Omega}, \\
\langle f_r, \tilde{r} \rangle := -\langle q_0, \tilde{r} \rangle_{\partial \Omega}, & \quad \langle f_q, \tilde{q} \rangle := \langle g, \tilde{q} \rangle_{\Omega} - \langle p_0, \nu \cdot \tilde{q} \rangle_{\partial \Omega}.
\end{align*}

for all $(r, q, p, (\tilde{r}, \tilde{q}, \tilde{p}) \in X$. Recall that unsubscripted, angled brackets $(\cdot, \cdot)$ denote duality pairings. We then identify problem (2.2) to be of the form $\mathcal{B}(r, q, p) = f$ with $\mathcal{B} : X \to X^*$ and $f \in X^*$ given by

\begin{equation}
\mathcal{B} := \begin{bmatrix} A & -B_r' \\ B_r & -B_q' \end{bmatrix}, \quad f := \begin{bmatrix} f_r \\ f_q \end{bmatrix}.
\end{equation}

For the analysis of this problem, we define the energy norm as

\begin{equation}
\| (r, q, p) \|^2 := \| r \|^2 + \| \nabla \times r \|^2 + \| q \|^2 + \| \nabla \cdot q \|^2 + \| p \|^2.
\end{equation}

It was shown in [3, Lem. 3.4] that problem (2.2) admits a unique solution that is bounded in the energy norm (2.3).
3. Mixed finite element discretization
3.1. A three-field mixed finite element method

Let \( \Omega \) be a shape-regular, simplicial tessellation of \( \Omega \). For \( n = 3 \), the finite element spaces are chosen as the linear Nédélec elements of the second kind, the Raviart-Thomas space of lowest order, and the piecewise constants:

\[
R_h := \mathbb{N}_1 \cap R, \quad Q_h := \mathbb{R}T_0 \cap Q, \quad P_h := \mathbb{P}_0 \cap P.
\]

We clarify that this choice of spaces leads to one degree of freedom per face for the velocity, one degree of freedom per element for the pressure, but two degrees of freedom per edge for the vorticity. This choice facilitates the hybridization introduced in the next section. For \( n = 2 \), we only adapt the vorticity space to the linear Lagrange elements, \( R_h := \mathbb{L}_1 \cap R \). It is important to note that \( \nabla \times R_h \subseteq Q_h \) and \( \nabla \cdot Q_h = P_h \). Finally, we define the composite space as \( X_h := R_h \times Q_h \times P_h \).

The three-field mixed finite element method (3F-MFEM) of problem (2.2) is: Find \( \tilde{x} \in X_h \) such that

\[
(\mathcal{B} \tilde{x}, \tilde{x}) = (f, \tilde{x}), \quad \forall \tilde{x} \in X_h. \tag{3.1}
\]

Lemma 3.1 ([3, Thm. 3.7]). The 3F-MFEM is stable and convergent with the error bounded by

\[
\| \tilde{x} - x \| \lesssim \inf_{\tilde{x} \in X_h} \| \tilde{x} - x \|. \tag{3.2}
\]

3.2. A quadrature rule for the vorticity

Following [6, Eq. (4.6)], we introduce a quadrature rule for the vorticity variable and define the associated norm:

\[
(r, \tilde{r})_h := \sum_{\omega \in \Omega_h} \frac{|\omega|}{n+1} \sum_{x \in V(\omega)} (r_x \cdot \tilde{r}_x)(x), \quad \|r\|_h^2 := (r, r)_h, \quad \forall r, \tilde{r} \in \mathbb{P}_j^0(\Omega_h) \supseteq R_h, \tag{3.3}
\]

with \( r_x \) the restriction of \( r \) on the simplex \( \omega \) and \( V(\omega) \) the set of vertices of \( \omega \). We use \( \mathbb{P}_j(\Omega_h) \) to denote the space of discontinuous, elementwise polynomials of order \( j \) and the coefficient is given by \( k_3 = 3 \) and \( k_2 = 1 \). Finally, let us define the auxiliary space \( W_h := \mathbb{P}_0^1(\Omega_h) \) containing elementwise constant (vector) fields.

Two key properties of the quadrature rule are highlighted in the following lemma.

Lemma 3.2 ([6, Thm. 4.1]). The quadrature rule from (3.3) satisfies the following properties

\[
\|r\|_h \approx \|r\|, \quad \forall r \in R_h, \quad (\tilde{r}, w)_h = (\tilde{r}, w), \quad \forall \tilde{r} \in \mathbb{P}_j^0, w \in W_h. \tag{3.4}
\]

Let \( A_h : R_h \to R_h^* \) be the operator associated with the quadrature rule such that \( (A_hr, \tilde{r}) := (\mu ^{-1}r, \tilde{r})_h \). Next, we replace \( A \) by \( A_h \) in the system which leads us to define a new operator \( \mathcal{B}_h : X_h \to X_h^* : \)

\[
(\mathcal{B}_h(r, q, p), (\tilde{r}, \tilde{q}, \tilde{p})) := (\mathcal{B}(r, q, p), (\tilde{r}, \tilde{q}, \tilde{p})) + (A_hr, \tilde{r}) - (Ar, \tilde{r}), \quad \forall (r, q, p), (\tilde{r}, \tilde{q}, \tilde{p}) \in X_h. \tag{3.5}
\]

With this substitution in place, we consider the augmented system: Find \( x_h \in X_h \) such that

\[
(\mathcal{B}_h x_h, \tilde{x}) = (f, \tilde{x}), \quad \forall \tilde{x} \in X_h. \tag{3.6}
\]

Similar to (2.3), the natural energy norm for this problem is given by

\[
\|(r, q, p)\|_h^2 := \|r\|_h^2 + \|\nabla \times r\|^2 + \|q\|^2 + \|\nabla \cdot q\|^2 + \|p\|^2, \tag{3.7}
\]

and Lemma 3.2 directly gives us the equivalence relation \( \|x\| \approx \|x\|_h \) for all \( x \in X_h \).

Lemma 3.3 (Well-posedness). Problem (3.6) admits a unique and bounded solution. In particular, the operator \( \mathcal{B}_h \) from (3.5) satisfies

\[
\sup_{\tilde{x} \in X_h} \frac{(\mathcal{B}_h x_h, \tilde{x})}{\|\tilde{x}\|_h} \approx \|x_h\|_h, \quad \forall x_h \in X_h. \tag{3.8}
\]

Proof. For the continuity bound \("\lesssim\",\) we note that \( \mathcal{B} \) is continuous in the energy norm \( \|\cdot\| \), which is equivalent to \( \|\cdot\|_h \). Moreover, \( A_h \) is continuous in the norm \( \|\cdot\|_h \) and so \( \mathcal{B}_h \), given by (3.5), is continuous in the norm \( \|\cdot\|_h \). For the lower bound \("\gtrsim\",\) we follow the same steps as in the proof of [3, Lem. 3.4] and use the equivalence relation from Lemma 3.2 where necessary. Finally, we follow [3, Lem. 3.5] to obtain the final requirements to invoke the Babuška-Lax-Milgram theorem for existence of the unique and bounded solution. □
3.3. A multipoint vorticity mixed finite element method

The quadrature rule (3.3) is local in the sense that \((r_i, \tilde{r}_i)_h\) is only non-zero for basis functions \(r_i, \tilde{r}_i \in R_h\) that have a degree of freedom at the same vertex. As can thus be inverted by solving local systems around the vertices which allows us to eliminate the vorticity \(r_h\). After elimination, we obtain the multipoint vorticity mixed finite element method (MV-MFEM): Find \((q_h, p_h) \in Q_h \times P_h\) such that

\[
\begin{bmatrix}
    B_h A_h^{-1} B_h' & -B_h' \\
    B_q & 0
\end{bmatrix}
\begin{bmatrix}
    q_h \\
    p_h
\end{bmatrix} =
\begin{bmatrix}
    f_q - R_h A_h^{-1} f_r \\
    0
\end{bmatrix}.
\]

(3.9)

4. A priori analysis of MV-MFEM

For the analysis of the MV-MFEM (3.9), we consider its equivalent formulation (3.6). We start by introducing suitable interpolants. Let \(\Pi_R\) and \(\Pi_Q\) be the canonical interpolants onto the finite element spaces \(R_h, Q_h\), respectively, defined for sufficiently regular \(r \in \mathbb{R}^d\) and \(q \in Q\). Similarly, let \(\Pi_P\) and \(\Pi_{\mathbb{R}}\) be the \(L^2\) projections onto \(P_h\) and \(W_h\), respectively. These interpolants satisfy

\[
\langle \nabla \times (I - \Pi_R) r, q \rangle_{\Omega} = 0 \quad \text{and} \quad \langle \nabla \cdot (I - \Pi_Q) q, p \rangle_{\Omega} = 0 \quad \text{for all} \quad (r, q, p) \in \mathbb{X}_h,
\]

and have the approximation properties

\[
\| (I - \Pi_R) r \| + \| \nabla \times (I - \Pi_R)^{1/2} r \| \lesssim h \| r \|_{1, \Omega}, \quad \| (I - \Pi_Q) q \| + \| \nabla \cdot (I - \Pi_Q)^{1/2} q \| \lesssim h \| q \|_{1, \Omega}.
\]

(4.1a)

In turn, the composite interpolant \(\Pi_X\) is given by \(\Pi_X(r, q, p) := (\Pi_R r, \Pi_Q q, \Pi_P p)\) for sufficiently regular \((r, q, p) \in \mathbb{X}\).

**Theorem 4.1** (Convergence). If the continuous solution \(x \in \mathbb{X}\) to (2.2) is sufficiently regular, then the discrete solution \(x_h \in X_h\) to (3.6) converges linearly in the energy norm (2.3):

\[
\| x_h - x \| \lesssim h.
\]

**Proof.** We closely follow [6] and start, as in [6, Eq. (3.11)], by evaluating the difference between \(x_h := (r_h, q_h, p_h) \in X_h\), the solution to (3.6), and \(\hat{x} := (\hat{r}, \hat{q}, \hat{p}) \in \mathbb{X}_h\), the solution to (3.1). Using the equivalences from Lemma 3.2 and Lemma 3.3, we derive

\[
\| \hat{x} - x_h \| \approx \| \hat{x} - x_h \|_{h} \approx \sup_{\bar{x} \in X_h} \frac{\langle B_h (\hat{x} - x_h), \bar{x} \rangle}{\| \bar{x} \|_h} = \sup_{\bar{x} \in X_h} \frac{\langle B_h \hat{x}, \bar{x} \rangle - \langle f, \bar{x} \rangle}{\| \bar{x} \|_h} = \sup_{\bar{r} \in R_h} \frac{\langle A_h \bar{r}, \bar{r} \rangle - \langle A_h \hat{r}, \hat{r} \rangle}{\| (\bar{r}, 0, 0) \|_h}.
\]

(4.2a)

Next, we follow the steps from [6, Thm. 3.2] to further bound the final term. Scaling the numerator by \(\mu\) for ease of presentation, we use Lemma 3.2 and a Cauchy-Schwarz inequality to obtain

\[
\mu \langle A_h \hat{r}, \hat{r} \rangle - \langle A_h \hat{r}, \hat{r} \rangle_{\Omega} \approx \langle \hat{r} - \Pi_{\mathbb{R}} \hat{r}, \hat{r} \rangle_{\Omega} \lesssim \| \hat{r} - \Pi_{\mathbb{R}} \hat{r} \| \| \hat{r} \|.
\]

(4.2b)

Combining (4.2), the triangle inequality leads us to

\[
\| \hat{x} - x_h \| \leq \mu^{-1} \sup_{\bar{r} \in R_h} \frac{\| \hat{r} - \Pi_{\mathbb{R}} \hat{r} \| \| \hat{r} \|_{h}}{\| (\bar{r}, 0, 0) \|_{h}} \lesssim \| \hat{r} - \Pi_{\mathbb{R}} \hat{r} \| \| \hat{r} \| + \| r - \Pi_{\mathbb{R}} r \|.
\]

Next, we use another triangle inequality and Lemma 3.1 to derive

\[
\| x_h - x \| \leq \| x_h - \hat{x} \| + \| \hat{x} - x \| \lesssim \| r - \Pi_{\mathbb{R}} r \| + \inf_{\bar{x} \in X_h} \| x - \bar{x} \| \leq \| r - \Pi_{\mathbb{R}} r \| + \| x - \Pi_{\mathbb{X}} x \|.
\]

Finally, the properties of the interpolants (4.1) and the assumed regularity of \(x\) provides the result.

**Lemma 4.2** (Solenoidal). The discrete solution \(q_h \in R_h\) is point-wise solenoidal.
Finally, we combine (4.4) and use the stability of the 3F-MFE $M$ [3, Lem. 3.4] to bound the equality follows from Lemma 4.2, Theorem 4.4 and Corollary 4.4.1. For the final estimate, we note that Lemma 4.5.

If $n$ is solenoidal, which allows us to take $\tilde{\mathbf{v}}_q$ such that $\nabla \times \tilde{\mathbf{v}}_q = \tilde{\mathbf{q}} - q_h$. Setting $\tilde{x} := (\tilde{\mathbf{v}}_q, 0)$, we derive

$$0 = \langle B\tilde{x} - B_h\tilde{x}, \tilde{x} \rangle = \langle (A\mathbf{v} - A_h\mathbf{v}, \tilde{r} - r_h) - (\tilde{\mathbf{q}} - q_h, \nabla \times (\tilde{\mathbf{r}} - r_h)) \rangle_{\Omega} = (A(\tilde{\mathbf{r}} - r_h), \tilde{\mathbf{r}} - r_h) = \mu^{-1}\|\tilde{\mathbf{r}} - r_h\|^2,$$

where we used that $\tilde{\mathbf{r}} - r_h \in \mathbb{R} \subset W_h$ and Lemma 3.2.

In 2D, the definition of the curl and the use of the divergence of the two systems and setting now the test function as $\tilde{x} := (\tilde{\mathbf{r}}, r_h, 0, 0)$, we derive

$$0 = \langle B\tilde{x} - B_h\tilde{x}, \tilde{x} \rangle = \langle (A\mathbf{v} - A_h\mathbf{v}, \tilde{r} - r_h) - (\tilde{\mathbf{q}} - q_h, \nabla \times (\tilde{\mathbf{r}} - r_h)) \rangle_{\Omega} = (A(\tilde{\mathbf{r}} - r_h), \tilde{\mathbf{r}} - r_h) = \mu^{-1}\|\tilde{\mathbf{r}} - r_h\|^2.$$

Hence, we continue by using (4.3) to obtain the following bound

$$\langle (A - A_h)\mathbf{v}, \tilde{\mathbf{r}} \rangle = \langle (A - A_h)(I - \Pi_W)\mathbf{v}, (I - \Pi_W)\tilde{\mathbf{r}} \rangle \lesssim \| (I - \Pi_W)\mathbf{v} \| \| (I - \Pi_W)\tilde{\mathbf{r}} \| \lesssim \| \nabla \times \mathbf{v} \| \| q_h \|.$$

Finally, we combine (4.4) and use the stability of the 3F-MFEM [3, Lem. 3.4] to bound $\| \nabla \times \mathbf{r} \|$.  

**Theorem 4.6** (Improved estimate). If $n = 2$ and $r$ is sufficiently regular, then $r_h$ converges as $\| r_h - r \| \lesssim h^2$.

**Proof.** We first consider the error equations for the 3F-MFEM with test function $\tilde{x} = \tilde{x} - \Pi \tilde{x}$. Due to the properties of the canonical interpolants, the off-diagonal components cancel and we obtain:

$$0 = \langle B(\tilde{x} - \tilde{x}), \tilde{x} \rangle = \mu^{-1}(\tilde{r} - r - \Pi \tilde{r} - r) = \mu^{-1}\left(\| \tilde{r} - r \|^2 + (\tilde{r} - r - \Pi \tilde{r} - r) \right).$$

Next, Corollary 4.4.1 gives us that $r_h = \tilde{r}$ and so we conclude, using (4.3), that

$$\| r_h - r \| = \| \tilde{r} - r \| = \frac{\| (I - \Pi_W)\mathbf{v} \|}{\| \tilde{r} - r \|} \lesssim \| (I - \Pi_W)\mathbf{v} \| \lesssim h^2\| r \|_2.$$
5. Numerical results

In this section, we perform a convergence study for \(n = 2\) and \(3\) to verify the theoretical results of Section 4. The results are obtained with the library PorePy [7] and PyGeoN [8]. Let \(\Omega := (0, 1)^2\), with the known solutions

\[
\begin{align*}
q(x, y) &= \nabla \times ((x^2)^2 (x - 1)^2 (y - 1)^2), \\
p(x, y) &= xy(1-x)(1-y), \\
q(x, y, z) &= \nabla \times [(1-x)\lambda(1-y)^2 y^2(1-z)^2 z^2 0 0]^T, \\
p(x, y, z) &= xyz(1-x)(1-y)(1-z),
\end{align*}
\]

We set boundary conditions \(q_0 = 0\) and \(p_0 = 0\) on \(\partial \Omega = \partial_p \partial \Omega\), let \(\mu := 1\) and compute the force term as \(g = \nabla \cdot (2e q - p I)\).

Table 1 compares MV-MFEM with 3F-MFEM in terms of numbers of degrees of freedom and relative \(L^2\)-errors.

For fairness of comparison in the case \(n = 3\), we choose \(R_h\) in 3F-MFEM as the lowest-order Nédélec elements of the first kind, that has one degree of freedom per mesh edge. The numbers of degrees of freedom in MV-MFEM are nevertheless smaller due to the complete elimination of the vorticity. The error rates are in agreement with Theorem 4.1, exhibiting at least linear convergence of all variables with respect to \(h\). As proven in Theorem 4.4, we observe numerically that the pressure, and therewith Error\(_p\), is unaffected by the quadrature rule.

For \(n = 2\), Corollary 4.4.1 is reflected in the fact that the columns Error\(_r\) are identical. The observed quadratic convergence in \(r\) was shown in Theorem 4.6. Finally, superconvergence of the pressure in the cell centers was observed (not reported).

| \(h\) | \(N_{dof}\) | Error\(_r\) | Rate\(_r\) | Error\(_q\) | Rate\(_q\) | Error\(_p\) | Rate\(_p\) | \(N_{dof}\) | Error\(_r\) | Rate\(_r\) | Error\(_q\) | Rate\(_q\) | Error\(_p\) | Rate\(_p\) |
|------|-------------|-----------|----------|-----------|----------|-----------|----------|-------------|-----------|----------|-----------|----------|-----------|----------|
| \(\varepsilon_1\) | 6.42e-2 | 1.91e3 | 7.73e-3 | - | 6.62e-2 | - | 1.12e-1 | - | 1.57e3 | 7.73e-3 | - | 6.95e-2 | - | 1.12e-1 | - |
| \(\varepsilon_1\) | 3.17e-2 | 7.33e3 | 1.94e-3 | 1.96 | 3.28e-2 | 0.99 | 5.67e-2 | 0.97 | 6.06e3 | 1.94e-3 | 1.96 | 3.33e-2 | 1.04 | 5.67e-2 | 0.97 |
| \(\varepsilon_1\) | 1.57e-2 | 2.88e4 | 4.86e-4 | 1.96 | 1.65e-2 | 0.98 | 2.84e-2 | 0.98 | 2.39e4 | 4.86e-4 | 1.96 | 1.65e-2 | 0.99 | 2.84e-2 | 0.98 |
| \(\varepsilon_1\) | 7.38e-3 | 1.14e5 | 1.22e-4 | 1.99 | 8.26e-3 | 0.99 | 1.42e-2 | 0.99 | 9.52e4 | 1.22e-4 | 1.99 | 8.26e-3 | 1.00 | 1.42e-2 | 0.99 |
| \(\varepsilon_1\) | 3.94e-3 | 4.56e5 | 3.05e-5 | 2.00 | 4.13e-3 | 1.00 | 7.12e-3 | 1.00 | 3.80e5 | 3.05e-5 | 2.00 | 4.13e-3 | 1.00 | 7.12e-3 | 1.00 |
| \(\varepsilon_2\) | 1.81e-1 | 1.59e4 | 2.30e-1 | - | 1.45e-1 | - | 3.27e-1 | - | 1.10e4 | 1.42e-1 | - | 1.68e-1 | - | 3.27e-1 | - |
| \(\varepsilon_2\) | 1.65e-1 | 2.11e4 | 2.11e-1 | 0.90 | 1.32e-1 | 0.83 | 2.92e-1 | 1.19 | 1.46e4 | 1.25e-1 | 1.34 | 1.48e-1 | 1.31 | 2.92e-1 | 1.19 |
| \(\varepsilon_2\) | 1.48e-1 | 2.86e4 | 1.80e-1 | 1.03 | 1.17e-1 | 1.13 | 2.58e-1 | 1.16 | 1.99e4 | 1.08e-1 | 1.32 | 1.28e-1 | 1.35 | 2.58e-1 | 1.16 |
| \(\varepsilon_2\) | 1.36e-1 | 3.62e4 | 1.73e-1 | 1.03 | 1.07e-1 | 1.05 | 2.38e-1 | 0.90 | 2.52e4 | 1.00e-1 | 0.90 | 1.16e-1 | 1.15 | 2.38e-1 | 0.90 |
| \(\varepsilon_2\) | 1.27e-1 | 4.43e4 | 1.61e-1 | 1.00 | 1.01e-1 | 0.83 | 2.22e-1 | 1.06 | 3.09e4 | 9.28e-2 | 1.14 | 1.09e-1 | 0.97 | 2.22e-1 | 1.06 |

Table 1: Relative \(L^2(\Omega)\)-errors and convergence rates for the vorticity \(r\), velocity \(q\), and pressure \(p\). The presented results are in agreement with the developed theory and show the potential of the proposed approach.

Acknowledgments

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101031434 – MiDiROM.

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