Sharp $L^1$ Inequalities for Sup-Convolution

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Abstract: Given a compact convex domain $C \subset \mathbb{R}^k$ and bounded measurable functions $f_1, \ldots, f_n : C \to \mathbb{R}$, define the sup-convolution $(f_1 * \ldots * f_n)(z)$ to be the supremum average value of $f_1(x_1), \ldots, f_n(x_n)$ over all $x_1, \ldots, x_n \in C$ which average to $z$. Continuing the study by Figalli and Jerison and the present authors of linear stability for the Brunn-Minkowski inequality with equal sets, for $k \leq 3$ we find the optimal constants $c_{k,n}$ such that

$$\int_C f^* f(x) - f(x) dx \geq c_{k,n} \int_C \text{co}(f)(x) - f(x) dx$$

where $\text{co}(f)$ is the upper convex hull of $f$. Also, we show $c_{k,n} = 1 - O\left(\frac{1}{n}\right)$ for fixed $k$ and prove an analogous optimal inequality for two distinct functions. The key geometric insight is a decomposition of polytopal approximations of $C$ into hypersimplices according to the geometry of the set of points where $\text{co}(f)$ is close to $f$.

1 Introduction

Let $C \subset \mathbb{R}^k$ be a compact convex domain. For a bounded function $f : C \to \mathbb{R}$, $\text{co}(f)$ is defined to be the upper convex hull of $f$ (the infimum of all concave functions larger than $f$), and for bounded measurable functions $f_1, \ldots, f_n : C \to \mathbb{R}$, the sup-convolution is defined to be

$$(f_1 * \ldots * f_n)(z) := \sup \left\{ \frac{f_1(x_1) + \ldots + f_n(x_n)}{n} : \frac{x_1 + \ldots + x_n}{n} = z \right\}.$$
The operation of sup-convolution, or in its equivalent form inf-convolution $-(( - f_1 ) * ... * ( - f_n ))$, naturally appears in problems of optimization, with $f_1, \ldots, f_n$ utility functions and $C$ representing a cost domain [174]. For a general survey, see [Str96]. Clearly $f_1 * \ldots * f_n \geq \frac{f_1 + \ldots + f_n}{n}$, and equality is attained when for example $f_1, \ldots, f_n$ are scalings of the same concave function $f = \text{co}(f)$.

We can view the sup-convolution operation geometrically in terms of the Minkowski sum of regions in $\mathbb{R}^{k+1}$. Indeed, consider the hypograph

$$A_{f, \lambda} = \{(x,y) \in \mathbb{C} \times \mathbb{R} : \lambda \leq y \leq f(x)\}.$$ 

Then we have the closed convex hull $\text{co}(A_{f,\lambda}) = \text{co}(f,\lambda)$, and for $\lambda$ sufficiently negative we have

$$A_{f_1*\ldots*f_n,\lambda} = \frac{1}{n}(A_{f_1,\lambda} + \ldots + A_{f_n,\lambda}).$$

The study of how close a Minkowski sum is to its convex hull was started by Starr-Shapley-Folkman [Sta69] and Emerson-Greenleaf [EG69], who showed that if $A_1, \ldots, A_n$ are subsets of the unit ball in $\mathbb{R}^k$, then the Hausdorff distance between the Minkowski averages $\frac{1}{n}(A_1 + \ldots + A_n)$ and $\frac{1}{n}(\text{co}(A_1) + \ldots + \text{co}(A_n))$ is bounded above by $\sqrt{k/n}$. Of particular interest for us will be when $A_1 = \ldots = A_n = A$, where we are concerned with how close $\frac{1}{n}(A + \ldots + A)$ is to $\text{co}(A)$; for this equal sets case we refer the reader to the extensive survey [FMMZ18].

Ruzsa [Ruz97, Theorem 5] showed that there is a constant $D_k$ such that for $A \subset \mathbb{R}^k$ of positive measure (taking the outer Lebesgue measure everywhere) and $n > D_k |\text{co}(A)|/|A|$, we have $|\frac{1}{n}(A + \ldots + A)| \geq \left(1 - \frac{D_n}{n} \cdot \frac{|\text{co}(A)|}{|A|}\right)^k |\text{co}(A)|$. In another direction, resolving a conjecture of Figalli and Jerison [FJ19, FJ15] on the stability of the Brunn-Minkowski inequality for homothetic sets, the present authors [vHST20b] showed that for $t \in (0, 1)$ there are constants $c_k(t)$ and $d_k(t)$ such that for subsets $A \subset \mathbb{R}^k$ of positive measure, $|tA + (1-t)A| \geq c_k(t)|\text{co}(A) \setminus A|$ provided $|(tA + (1-t)A) \setminus A| \leq d_k(t)|A|$. A nice feature of this last result is that for $A = A_{f,\lambda}$ the hypograph of a function, the $d_{k+1}(t)$ condition is always satisfied provided we take $\lambda$ to be sufficiently negative. Taking $t = \frac{1}{2}$ allows us to conclude, writing $f^{*n}$ for $f * \ldots * f$, that there exist positive constants $c_{k,n}$ such that

$$\int_C f^{*n}(x) - f(x)dx \geq c_{k,n}\int_C \text{co}(f)(x) - f(x)dx$$

(see Appendix A, where we also give an alternate self-contained proof of this particular inequality).

The constants $c_{k,n}$ one obtains in this way however are not optimal. Our first theorem establishes the optimal constants for $k \leq 3$, making progress towards Question 1.8 from [vHST20a] which asked an analogous question in the discrete setting with $n = 2$.

**Theorem 1.1.** If $f : C \to \mathbb{R}$ is a bounded measurable function with $C \subset \mathbb{R}^k$ a compact convex domain with $k \leq 3$, and $n \geq 1$, then

$$\int_C f^{*n}(x) - f(x)dx \geq c_{k,n}\int_C \text{co}(f)(x) - f(x)dx$$

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1Formally we work with the “upper Lebesgue integral” to avoid the issue of the measurability of $f_1 * \ldots * f_n$. 
with
\[ c_{k,n} = \begin{cases} \frac{n-1}{n} & k = 1 \\ \frac{(2n-1)(n-1)}{2n^2} & k = 2 \\ \frac{(n-1)^2}{n^2} & k = 3. \end{cases} \]

This is sharp, taking \( f \) the indicator function on the vertices of \( C = T \) a simplex. In any dimension \( k \), letting \( e_1, \ldots, e_{k+1} \) be the standard basis vectors in \( \mathbb{R}^{k+1} \) and identifying \( C = T \) with the convex hull of \( ne_1, \ldots, ne_{k+1} \) we will see that the level sets of this particular \( f^{*\alpha} \) induce a subdivision of \( C \) into hypersimplices, where translates of the \( m \)‘th \( k \)-dimensional hypersimplex \( P_{k,m} := [0, 1]^{k+1} \cap \{ \sum x_j = m \} \) appear \( \binom{n+k-m}{k} \) times (see Section 3).

For example for \( k = 2 \) (depicting the case \( n = 4 \) below), \( f^{*\alpha} \) takes value \( \frac{n-1}{n} \) in the shaded region, the union of \( \binom{n+1}{2} \) translates of the triangle \( P_{2,1} \), and \( \frac{n-2}{n} \) in the unshaded region, the union of \( \binom{n}{2} \) translates of the triangle \( P_{2,2} \).

The shaded regions are precisely those parts of \( T \) whose points can be expressed as \( \sum x_i \) with all but one of the \( x_i \) a vertex of \( T \), and the remaining unshaded regions can be expressed with all but two of the \( x_i \) a vertex of \( T \).

For \( k = 3 \), we can subdivide \( T \) into \( \binom{n+3}{3} \) translates of \( \frac{1}{3}T = P_{3,1} \), \( \binom{n+1}{3} \) translates of the octahedron \( P_{3,2} \), and \( \binom{3}{3} \) translates of \( -P_{3,1} = P_{3,3} \), on which \( f^{*\alpha} \) takes the values \( \frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n} \) respectively. The partition is according to whether the maximum number of vertices of \( T \) which can be used to express the point as an \( n \)-average is \( n-1, n-2 \), or \( n-3 \) respectively.

To prove Theorem 1.1, we pass to a piecewise-linear approximation and then triangulate according to the domains of linearity of \( \co(f) \). On each simplex \( T \) we prove a sharp inequality relating \( \int_R \co(f)(x) - f^{*\alpha}(x) \, dx \) and \( \int_T \co(f)(x) - f(x) \, dx \) for \( R \) ranging over the hypersimplices in the subdivision alluded to above. This in turn is encompassed in our notion of an “\( m \)-averageable” subset of \( T \) (Section 4), and showing certain hypersimplices are “\( m \)-averageable” allows us to conclude.

We make the following conjecture for arbitrary \( k, n \geq 1 \). In what follows, write \( A(k, \ell) \) for the Eulerian number counting permutations of \( S_k \) with \( \ell \) descents.

**Conjecture 1.2.** If \( k, n \geq 1 \) and \( f : C \to \mathbb{R} \) is a bounded measurable function with \( C \subset \mathbb{R}^k \) a compact convex domain, then we have
\[
\int_C f^{*\alpha}(x) - f(x) \, dx \geq c_{k,n} \int_C \co(f)(x) - f(x) \, dx,
\]
where
\[ c_{k,n} = \frac{1}{n^k} \sum_{m=1}^k \frac{n-m}{n} \binom{n+k-m}{k} A(k, m-1) = \frac{k+1}{n^{k+1}} (1^k + \ldots + (n-1)^k). \]
If true, this would be sharp by taking $f$ the indicator function on the vertices of $C = T$ a simplex (see Section 3). Here $\frac{n-m}{m}$ is the value of $f^{*n}$ on each hypersimplex $P_{k,m} + x$, $(\binom{n+k-m}{k})$ is the number of such hypersimplices, and $\frac{A(k,m-1)}{m}$ is the volume ratio of $P_{k,m}$ to $T$.

Remark 1.3. Omitting the $\frac{n-m}{m}$ factor, we obtain a geometric proof of the Worpitzky identity $n^k = \sum (\binom{n+k-m}{k})A(k,m-1)$. A similar observation was recently exploited by Early [Ear16, Section 3] to categorify the Worpitzky identity via the representation theory of the symmetric group.

We also show the following asymptotic result for fixed $k$ as $n \to \infty$.

**Theorem 1.4.** For any $k \geq 1$ and $n \geq k + 1$, we have $c_{k,n} \geq 1 - (\frac{n}{k})^{k+1} = 1 - O(\frac{1}{n})$.

This is optimal up to the constant on $\frac{1}{n}$, which this theorem shows can be taken to be $\frac{k+1}{k!} = eO(k)$, though our conjectured extremal example gives a constant of $\frac{k+1}{2}$.

Finally, we consider the sup-convolution of distinct functions $f, g$, showing that $f$ is close to $\text{co}(f)$ provided $f \ast g$ is close to $\frac{f+g}{2}$.

**Theorem 1.5.** If $f, g : C \to \mathbb{R}$ are bounded measurable functions with $C \subset \mathbb{R}^k$ a compact convex domain and $k \leq 3$ then

$$\int_C f \ast g(x) - \frac{f(x) + g(x)}{2} \, dx \geq \frac{k+1}{2k+1} \int_C \text{co}(f)(x) - f(x) \, dx.$$ 

The constant $c_{k,2} = \frac{k+1}{2k+1}$ is again sharp, as for example we can take $f = g$ the indicator function on the vertices of $C = T$ a simplex.

In Section 2 we show that Theorem 1.1, Conjecture 1.2, Theorem 1.4, and Theorem 1.5 reduce to the case that $C = T$ is a simplex, $f \leq 0$ on the vertices and $f \equiv 0$ otherwise. In Section 3 we construct our hypersimplex subdivision of $T$. In Section 4 we introduce a new geometric notion of “$m$-averageable subsets of $T$”, and reduce to showing certain hypersimplices in $T$ are “$m$-averageable”. In Section 5 we show that the relevant hypersimplices up to dimension 3 are “$m$-averageable” and conclude Theorem 1.1 and Theorem 1.5. In Section 6 we prove Theorem 1.4. Finally, in Appendix A we show how the existence of a non-sharp constant in Conjecture 1.2 can be derived from [vHST20b], and we also give a quick self-contained proof.

## 2 Reduction to Simplices

Here we reduce Theorem 1.1, Conjecture 1.2, Theorem 1.4, and Theorem 1.5 to the case $C = T$ is a simplex, $f \leq 0$, and $f = 0$ at the vertices.

**Proposition 2.1.** The statements of Theorem 1.1, Conjecture 1.2, Theorem 1.4, and Theorem 1.5, respectively, are equivalent to the corresponding statements with the additional assumption that $C = T$ is a simplex, $f \leq 0$, and $f = 0$ at the vertices.
Proof. The reduction is divided in the following three steps. The first two steps will reduce to the situation that $f$ is nonnegative with piecewise-linear $\text{co}(f)$ with finitely many domains of linearity. Considering a particular domain of linearity $T$, by subtracting the linear function $\text{co}(f)|_T$, we deduce the result. We shall always focus on the reduction of Conjecture 1.2, as the others follow in a similar way.

**Claim 2.2.** Suppose that Theorem 1.1 (resp. Conjecture 1.2, Theorem 1.4, Theorem 1.5) is true when the domain $C$ is a polytope $P$, $f \geq 0$, and $f = 0$ on a neighborhood of $\partial C$. Then Theorem 1.1 (resp. Conjecture 1.2, Theorem 1.4, Theorem 1.5) is true.

**Proof.** We prove this claim for Conjecture 1.2, the other cases are similar. The inequality doesn’t change if we scale $f$ or add a constant so assume that $f(x) \in [0, n + 1]$ for all $x \in C$. Let $P_1, P_2, \ldots$ be a sequence of polytopes with $C \subset P_i^n$ (the interior of $P_i$) and $|P_i| \to |C|$. We extend $f$ to a function $f_i$ on $P_i$ by setting $f_i = 0$ on $P_i \setminus C$. Then we note that for any $x \in C$, $f_i^n(x) \geq f_i(x) \geq n$, but

$$f_i(x_1) + \ldots + f_i(x_n) \leq \frac{(n+1)(n-1)}{n} < n$$

provided any $x_j \in \partial C$, so we conclude that $f_i^n|C = f^n$.

Thus as $\text{co}(f_i) \geq \text{co}(f)$,

$$\int_C f^n - f(x) dx \geq \int_{P_i} f^n(x) - f_i(x) dx - |P_i \setminus C| \cdot ||f^n||_\infty$$

$$\geq c_{k,n} \int_{P_i} \text{co}(f_i)(x) - f_i(x) dx - |P_i \setminus C| \cdot ||f^n||_\infty$$

$$\geq c_{k,n} \int_C \text{co}(f)(x) - f(x) dx - |P_i \setminus C| \cdot ||f^n||_\infty \rightarrow c_{k,n} \int_C \text{co}(f)(x) - f(x) dx,$$

where in the last step we used that $||f^n||_\infty = ||f||_\infty \leq n + 1$. \hfill \Box

**Claim 2.3.** Suppose that Theorem 1.1 (resp. Conjecture 1.2, Theorem 1.5) is true when the domain $C$ is a polytope $P$ and $\text{co}(f)$ is the upper convex hull of finitely many points $(r_i, f(r_i))$ (so is in particular piecewise linear). Then it is true when the domain $C$ is a polytope $P$, $f \geq 0$, and $f = 0$ on a neighborhood of $\partial C$.

**Proof.** We prove this claim for Conjecture 1.2, the other cases are similar. Suppose $C = P$, $f \geq 0$, and $f = 0$ on a neighborhood of $\partial C$ (but we do not necessarily know that $\text{co}(f)$ is piecewise linear).

We’ll show that $\text{co}(f)$ is continuous at all points $x \in C$. First, suppose that $x \in C^0$. Then for $y \in C^0$, let $z_1, z_2$ be the points on $\partial C$ such that $z_1, x, y, z_2$ are collinear in that order. We have

$$\text{co}(f)(x) \geq \frac{|x - z_1|}{|y - z_1|} \text{co}(f)(y) + \frac{|x - y|}{|y - z_1|} \text{co}(f)(z_1) \geq \frac{|x - z_1|}{|y - z_1|} \text{co}(f)(y)$$

$$\text{co}(f)(y) \geq \frac{|y - z_2|}{|x - z_2|} \text{co}(f)(x) + \frac{|y - z_2|}{|x - z_2|} \text{co}(f)(z_2) \geq \frac{|y - z_2|}{|x - z_2|} \text{co}(f)(x).$$
We remark that $|| \cdot ||$ denotes the Euclidean norm, so $co(f)(y) \rightarrow co(f)(x)$ as $y \rightarrow x$. Next, instead suppose that $x \in \partial C$. Then take any linear function $L$ which is 0 at $x$ and positive with $\inf_{\text{supp}(f)} L > 0$, which exists as $f(x)$ is supported on a compact subset of the interior of $C$. We may further assume that $L_{|\text{supp}(f)} > ||f||_\infty$ by replacing $L$ with $(1 + ||f||_\infty)(\inf_{\text{supp}(f)} L)^{-1}L$. Then $co(f)$ is sandwiched between the constant function 0 and the continuous function $L$ which agree at $x$, which implies that $co(f)(x) = 0$ and $co(f)$ is continuous at $x$.

In particular, because $co(f)$ is continuous and concave, it is approximated in the supremum norm by concave piecewise-linear functions from above. Let $c$ be a concave piecewise-linear approximation to $co(f)$ with $c \geq co(f)$, and $||c - co(f)||_\infty \leq \varepsilon$ for some fixed $\varepsilon$. Let $x_1, \ldots, x_N \in C$ be a finite collection of points for which the graph of $c$ is the upper convex hull of the points $(x_i, c(x_i))$ (note that here we use the fact that the domain is a polytope).

We note that

$$co(f)(x) = \sup \{ \lambda_1 f(x_1) + \ldots + \lambda_\ell f(x_\ell) : \ell \in \mathbb{N}, \lambda_1, \ldots, \lambda_\ell \in [0, 1], \sum \lambda_i = 1, \sum \lambda_i x_i = x \}. $$

Hence, there exists $M$, points $\{x_i\}_{1 \leq i \leq M}$ and parameters $\lambda_{i,j} \in [0, 1]$ with $\sum_{j=1}^{M} \lambda_{i,j} = 1$, $\sum_{j=1}^{M} \lambda_{i,j} x_{i,j} = x_i$, and

$$co(f)(x_i) \leq \sum_{j=1}^{M} \lambda_{i,j} f(x_{i,j}) + \varepsilon. $$

Let

$$f_\varepsilon(x) = \begin{cases} f(x) + 2\varepsilon & \text{if } x = x_{i,j} \text{ for some } i, j, \\ f(x) & \text{otherwise.} \end{cases}$$

We remark that

$$\sum_{j=1}^{M} \lambda_{i,j} f_\varepsilon(x_{i,j}) = 2\varepsilon + \sum_{j=1}^{M} \lambda_{i,j} f(x_{i,j}) \geq co(f)(x_i) + \varepsilon \geq c(x_i). $$

Hence letting $g$ be the upper convex hull of the points $(x_{i,j}, f_\varepsilon(x_{i,j}))$, we have $g \geq c \geq f$.

We claim that $co(f_\varepsilon)(x) = g$. Indeed, we trivially have $g \leq co(f_\varepsilon)$, so it suffices to show $g \geq co(f_\varepsilon)$. For $x = x_{i,j}$ we clearly have $g(x) \geq f_\varepsilon(x)$ and for $x \neq x_{i,j}$, we have $g(x) \geq f(x) = f_\varepsilon(x)$. Hence $g \geq f_\varepsilon$, so as $g$ is concave, $g \geq co(f_\varepsilon)$.

Hence, $co(f_\varepsilon)$ is the upper convex hull of finitely many points $(r_i, f_\varepsilon(r_i))$. As $||f_\varepsilon - f||_\infty \leq 2\varepsilon$ and $f_\varepsilon \geq f$, we have by our hypothesis,

$$\int_C f^{u,n}(x) - f(x)dx + 2\varepsilon |C| \geq \int_C (f_\varepsilon)^{u,n}(x) - f_\varepsilon(x)dx \geq c_{k,n} \int_C co(f_\varepsilon)(x) - f_\varepsilon(x)dx \geq c_{k,n} \int_C co(f)(x) - f(x)dx - 2\varepsilon c_{k,n} |C|,$n

where the first inequality follows from the fact that $f^{u,n} + 2\varepsilon = (f + 2\varepsilon)^{u,n} \geq (f_\varepsilon)^{u,n}$. Letting $\varepsilon \rightarrow 0$ we conclude. 

\[ \square \]
Claim 2.4. Suppose that Theorem 1.1 (resp. Conjecture 1.2, Theorem 1.4, Theorem 1.5) is true when the domain $C$ is a simplex $T$, $f = 0$ at the vertices of $T$ and $f \leq 0$. Then it is true when the domain $C$ is a polytope $P$ and $\text{co}(f)$ is the upper convex hull of finitely many points $(r_i, f(r_i))$.

Proof. We prove this claim for Conjecture 1.2, the other cases are similar. Let $f$ be defined on a polytopal domain $C = P$ with $\text{co}(f)$ the upper convex hull of finitely many points $(r_i, f(r_i))$. The domains of linearity of $\text{co}(f)$ decompose $C$ into convex polytopes with vertices a subset of the $r_i$. Further subdivide this decomposition into triangulation $\mathcal{T}$. Then $\text{co}(f|_{\mathcal{T}}) = \text{co}(f)|_{\mathcal{T}}$ for all $T \in \mathcal{T}$, so $\int_C \text{co}(f)(x) - f(x)dx = \sum_{T \in \mathcal{T}} \int_T \text{co}(f|_{T})(x) - f|_{T}(x)dx$ and

$$\int_{C} f^{**}(x) - f(x)dx \geq \sum_{T} \int_{T} (f|_{T})^{**}(x) - f|_{T}(x)dx.$$  

Hence it suffices to prove for every $T \in \mathcal{T}$ that

$$\int_{T} (f|_{T})^{**}(x) - f|_{T}(x)dx \geq c_{k,n} \int_{T} \text{co}(f|_{T})(x) - f|_{T}(x)dx.$$  

As $\text{co}(f|_{T})$ is linear, and the inequality is preserved by subtracting linear functions from $f$, we may subtract $\text{co}(f|_{T})$ from $f$, after which $f = 0$ at the vertices of $T$ and $f \leq 0$ on $T$. Thus by hypothesis we are done.

The above sequence of reductions gives the desired conclusion.

3 Hypersimplex Covering

We take $T$ to be the convex hull of the standard basis vectors $e_1, \ldots, e_{k+1} \in \mathbb{R}^{k+1}$. Recall that the $m$th $k$-dimensional hypersimplex for $1 \leq m \leq k$ is defined to be the region in $\mathbb{R}^{k+1}$ given by

$$P_{k,m} := \left\{(x_1, \ldots, x_{k+1}) \in [0,1]^{k+1} : \sum x_i = m \right\}.$$  

Definition 3.1. Let

$$\mathcal{B}_{k,\ell} = \left\{(x_1, \ldots, x_{k+1}) \in \mathbb{Z}_{\geq 0}^{k+1} : \sum x_i = \ell \right\}.$$  

Proposition 3.2. For $k, n \geq 1$ we have a polytopal subdivision

$$T = \bigcup_{m=1}^{\min(k,n)} \bigcup_{v \in \mathcal{B}_{k,n-m}} \frac{1}{n} P_{k,m} + \frac{1}{n} v.$$  

Proof. Note that because $\bigcup_{v \in \mathbb{Z}^{k+1}} [0,1]^{k+1} + v$ subdivide $\mathbb{R}^{k+1}$, the intersections $\bigcup_{v \in \mathbb{Z}^{k+1}} ([0,1]^{k+1} + v) \cap nT$ form a polytopal subvision of $nT$. Let $\mathcal{B}$ be the set of such $v$ for which $([0,1]^{k+1} + v) \cap nT$ is $k$-dimensional, such that $\bigcup_{v \in \mathcal{B}} ([0,1]^{k+1} + v) \cap nT$ also forms a polytopal subvision of $nT$.  

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Let $v \in \mathcal{B}$, and set $m = n - \sum v_i$. We first claim that $1 \leq m \leq k$ and

$$([0, 1]^{k+1} + v) \cap \{ \sum x_i = n \} = P_{k,m} + v.$$ 

Indeed, as $nT$ lies in the $\sum x_i = n$ hyperplane and $([0, 1]^{k+1} + v) \cap nT$ is $k$-dimensional, we must have $n - k \leq \sum v_i \leq n - 1$, i.e. $1 \leq m \leq k$. Then it is easy to see that $([0, 1]^{k+1} + v) \cap \{ \sum x_i = n \} = P_{k,m} + v$ by definition.

Next, we claim that $v \in \mathcal{B}_{k,n-m}$, i.e. that $v$ has no negative coordinates. Indeed, suppose that $v_1 \leq -1$. Then $([0, 1]^{k+1} + v) \cap nT \subset \{ x_1 = 0 \} \cap nT$ which is at most $k - 1$-dimensional.

Finally, we claim that

$$([0, 1]^{k+1} + v) \cap \{ \sum x_i = n \} \subset nT.$$ 

Indeed, as $v \in \mathcal{B}_{k,n-m}$, all coordinates are non-negative.

Conversely, suppose that we can write $y \in \mathcal{B}_{k,n-m}$ with $n - m$ of the $x_i$ being vertices of $T$, but not with at least $n - m + 1$ of the $x_i$ being vertices of $T$.

The key property of an $m$-averageable subset of a simplex $T$.

**Definition 4.1.** Given a simplex $T$, say that a subset $S \subset T$ is “$m$-averageable” if there are mappings $H_1, \ldots, H_m : T \to T$ which are generically bijective of Jacobian 1 such that $\frac{H_1 + \ldots + H_m}{m}$ is a generically bijective map $T \to S$ with constant Jacobian $|S|/|T|$.

The key property of an $m$-averageable set $S \subset T$ is the observation that

$$\int_S f_1 \ast \ldots \ast f_m dx \geq \frac{|S|}{|T|} \sum_{i=1}^m \int_T f_i(x) dx.$$ 

This observation will be used directly, and in a slightly modified form, in the propositions below.

**Proposition 3.3.** For $1 \leq m \leq \min(k,n)$ and $v \in \mathcal{B}_{k,n-m}$, the points in $\frac{1}{n} P_{k,m} + \frac{1}{n} v$ (the interior of $\frac{1}{n} P_{k,m} + \frac{1}{n} v$) can be written as $\frac{x_1 + \ldots + x_n}{n}$ with $n - m$ of the $x_i$ being vertices of $T$, but not with at least $n - m + 1$ of the $x_i$ being vertices of $T$.

**Proof.** For $y = \frac{1}{n} w + \frac{1}{n} v \in \frac{1}{n} P_{k,m} + \frac{1}{n} v$, we can write $y = \frac{m}{n} \left( \frac{1}{n} w + \sum v_i e_i \right)$, and $\frac{1}{n} w \in \frac{1}{m} P_{k,m} \subset T$.

Conversely, suppose that we can write $y = \frac{x_1 + \ldots + x_n}{n}$ with $x_1, \ldots, x_{n-m+1}$ vertices of $T$ and $x_{n-m+2}, \ldots, x_n \in T$. Then $\lfloor ny \rfloor = v$, so we obtain the contradiction

$$n - m = \sum_{i=1}^{k+1} \lfloor ny_i \rfloor \geq \sum_{i=1}^{k+1} \sum_{j=1}^n |(x_j)_i| \geq \sum_{i=1}^{k+1} \sum_{j=1}^{n-m+1} |(x_j)_i| = n - m + 1.$$

**4 m-averageable sets**

We now define a new notion of “$m$-averageable” subset of a simplex $T$.

**Definition 4.1.** Given a simplex $T$, say that a subset $S \subset T$ is “$m$-averageable” if there are mappings $H_1, \ldots, H_m : T \to T$ which are generically bijective of Jacobian 1 such that $\frac{H_1 + \ldots + H_m}{m}$ is a generically bijective map $T \to S$ with constant Jacobian $|S|/|T|$.
Example 4.2. In three dimensions, the subsimplex $S \subset T$ defined by a vertex of $T$ and the opposite medial triangle is 2-averageable.

Indeed, we can take $H_1$ to be the identity map and $H_2$ to be the linear map fixing the common vertex $v$ of $S$ and $T$ and cycling the remaining vertices $v_1 \mapsto v_2 \mapsto v_3 \mapsto v_1$. Then $\frac{H_1 + H_2}{2}$ is a linear map sending the vertices of $T$ to the vertices of $S$, and is thus a constant Jacobian $\frac{1}{4}$ map $T \to S$.

Recall that for $m \in \{1, \ldots, k\}$, we denote by $P_{k,m} = [0,1]^{k+1} \cap \{\sum x_i = m\}$ for the $m$'th $k$-dimensional hypersimplex. The following two propositions reduce the theorems from the introduction to showing that certain hypersimplices embedded in $T$ are $m$-averageable.

Proposition 4.3. Let $k \geq 1$ and $T$ the convex hull of the standard basis vectors in $\mathbb{R}^{k+1}$, and suppose that $\frac{1}{2}P_{k,2}$ is 2-averageable. Then for bounded functions $f, g : T \to \mathbb{R}$ with $f \leq 0$ and $f(x_i) = 0$ for the vertices $x_i$ of $T$, we have

$$\int_T f \ast g(x) - \frac{f(x) + g(x)}{2} dx \geq \frac{k+1}{2^{k+1}} \int_T \co(f)(x) - f(x) dx.$$ 

Proposition 4.4. Let $T$ be the convex hull of the standard basis vectors in $\mathbb{R}^{k+1}$, and suppose that $\frac{1}{m}P_{k,m}$ is $m$-averageable for $m \leq \min(k, n)$. Then for a bounded function $f : T \to \mathbb{R}_{\leq 0}$ with $f(x_i) = 0$ for the vertices $x_i$ of $T$, we have

$$\int_T f \ast^n (x) - f(x) dx \geq c_{k,n} \int_T \co(f)(x) - f(x) dx$$

where $c_{k,n}$ is as in Conjecture 1.2.

Proof of Proposition 4.3. Consider the $n = 2$ polytope decomposition from Proposition 3.2

$$T = \frac{1}{2}P_{k,2} \cup \bigcup_{i=1}^{k+1} e_i + \frac{T}{2}.$$ 

Because by hypothesis $\frac{1}{2}P_{k,2}$ is 2-averageable, there are functions $H_1, H_2$ such that $H_1, H_2 : T \to T$ are generically bijective with Jacobian 1, and $\frac{H_1 + H_2}{2} : T \to S$ is generically bijective with Jacobian $\frac{1}{4}$.
\[ \frac{1}{|T|} = 1 - \frac{k+1}{2^k}. \] Then the result follows by adding the inequality
\[
\int_{\frac{1}{2} P_{k,2}} (f \ast g)(x)dx = \left( 1 - \frac{k+1}{2^k} \right) \int_T f \ast g \left( \frac{H_1(x) + H_2(x)}{2} \right) dx \\
\geq \left( 1 - \frac{k+1}{2^k} \right) \int_T f(H_1(x))dx + \int_T g(H_2(x))dx \\
= \left( 1 - \frac{k+1}{2^k} \right) \int_T f(x)dx + \int_T g(x)dx
\]
to the inequality
\[
\int_{\frac{1}{2} T} (f \ast g)(x)dx = \frac{1}{2^k} \int_T f \ast g \left( \frac{e_i + x}{2} \right) dx \geq \frac{1}{2^{k+1}} \int_T f(e_i) + g(x)dx = \frac{1}{2^{k+1}} \int_T g(x)dx
\]
for \( i = 1, \ldots, k + 1. \)

**Proof of Proposition 4.4.** By Proposition 3.2, there is a polytope subdivision
\[
T = \bigcup_{m=1}^{\min(k,n)} \bigcup_{x \in \mathbb{Z}_{\geq 0}} \frac{1}{n} P_{k,m} + \frac{1}{n} x
\]
where
\[
\mathcal{B}_{k,\ell} = \left\{ x = (x_1, \ldots, x_{k+1}) \in \mathbb{Z}_{\geq 0}^{k+1} : \sum x_i = \ell \right\}.
\]
Let \( H_{k,1}^1, \ldots, H_{k,m}^m \) be the functions associated to the \( k \)-averageable set \( \frac{1}{m} P_{k,m} \). Then
\[
\int_{\frac{1}{2} P_{k,m} + \frac{1}{n} x} f^m(x)dx = \frac{1}{n} \frac{|P_{k,m}|}{|T|} \int_T f^m \left( \frac{H_{k,m}^1(x) + \ldots + H_{k,m}^m(x) + x_1 e_1 + \ldots + x_{k+1} e_{k+1}}{n} \right) dx \\
\geq \frac{1}{n} \frac{|P_{k,m}|}{|T|} \int_T f(H_{k,m}^1(x)) + \ldots + f(H_{k,m}^m(x)) + x_1 f(e_1) + \ldots + x_{k+1} f(e_{k+1}) dx \\
= A(k,m-1) \cdot \frac{m}{n^{k+1}} \int_T f(x)dx.
\]
Recalling that \( \text{co}(f) = 0 \), summing these inequalities and using the Worpitzky identity that \( \sum_{m} |\mathcal{B}_{k,n-m}| A(k,m-1) = \sum_{m} \binom{n+k-m}{k} A(k,m-1) = n^k \) yields the desired result. \( \square \)
5 Proofs of Theorem 1.1 and Theorem 1.5

By the propositions in the previous section, it will suffice to show that

\[ P_{1,1} \subset \mathbb{R}^2 \]
\[ P_{2,1} + \frac{1}{2} P_{2,2} \subset \mathbb{R}^3 \]
\[ P_{3,1} + \frac{1}{2} P_{3,2} + \frac{1}{3} P_{3,3} \subset \mathbb{R}^4 \]

are all \( m \)-averageable in the corresponding convex hull of standard basis vectors \( T \) for \( m = 1 \), \( m = 1, 2 \), and \( m = 1, 2, 3 \) respectively.

The following lemma handles all cases except for \( \frac{1}{2} P_{3,2} \).

**Lemma 5.1.** \( P_{k,1} \) is 1-averageable and \( \frac{1}{k} P_{k,k} \) is \( k \)-averageable for all \( k \geq 1 \).

**Proof.** For \( n = 1 \), \( P_{k,1} = T \) so we may take \( H \) to be the identity map and there is nothing to prove.

For \( n = k \), let \( \sigma \) be the linear map taking \( e_1 \mapsto e_2 \mapsto \ldots \mapsto e_{k+1} \mapsto e_1 \). Then \( \sigma \) is an isometry, and so \( H_i = \sigma^i \) is also an isometry. The average

\[ \frac{H_1 + \ldots + H_k}{k} \]

is the linear map taking \( e_i \mapsto \frac{1}{k} \sum_{j \neq i} e_j \), which is a linear bijection from the simplex \( T \) to the simplex \( \frac{1}{k} P_{k,k} \). \( \square \)

The following lemma therefore completes the proofs of Theorem 1.1 and Theorem 1.5.

**Lemma 5.2.** \( \frac{1}{2} P_{3,2} \) is 2-averageable.

**Proof.** Decompose \( T = R_{12} \cup R_{23} \cup R_{34} \cup R_{41} \) where \( R_{i(i+1)} \) is the simplex

\[ R_{i(i+1)} = \text{co} \left( e_i, e_{i+1}, \frac{e_1 + e_3}{2}, \frac{e_2 + e_4}{2} \right) . \]

Indeed, viewing \( T \) as the 1-dimensional cycle connecting \( e_1 \to e_2 \to e_3 \to e_4 \to e_1 \) coned off at the points \( \frac{e_1 + e_3}{2} \) and \( \frac{e_2 + e_4}{2} \), \( R_{i(i+1)} \) corresponds to the line segment connecting \( e_i \to e_{i+1} \) coned off at the points \( \frac{e_1 + e_3}{2} \) and \( \frac{e_2 + e_4}{2} \).
Let $H_1$ be the identity map and $H_2 : T \to T$ be the piecewise linear local isometry defined by taking $R_{i(i+1)} \mapsto R_{(i+1)(i+2)}$, sending the vertices $e_i, e_{i+1}, \frac{e_i + e_{i+1}}{2}, \frac{e_i + e_{i+2}}{2}$ to $e_{i+1}, e_{i+2}, \frac{e_{i+1} + e_{i+2}}{2}, \frac{e_{i+1} + e_{i+3}}{2}$, respectively. Then $H_1 + H_2$ takes $R_{i(i+1)}$ to

$$S_{i(i+1)(i+2)} = \operatorname{co}\left(\frac{e_i + e_{i+1}}{2}, \frac{e_{i+1} + e_{i+2}}{2}, \frac{e_{i+2} + e_{i+3}}{2}, \frac{e_{i+3} + e_{i+4}}{2}\right),$$

and the simplices $S_{123}, S_{234}, S_{341}, S_{412}$ subdivide $\frac{1}{2} P_{3,2}$. Indeed, the octahedron $\frac{1}{2} P_{3,2}$ can be described as the one-dimensional cycle around the boundary of the square $\frac{e_1 + e_2}{2} \to \frac{e_2 + e_3}{2} \to \frac{e_3 + e_4}{2} \to \frac{e_4 + e_1}{2} \to \frac{e_1 + e_2}{2}$ coned off at the points $\frac{e_1 + e_2}{2}$ and $\frac{e_2 + e_3}{2}$, and $S_{i(i+1)(i+2)}$ is the segment connecting $\frac{e_{i+1} + e_{i+2}}{2}$ and $\frac{e_{i+2} + e_{i+3}}{2}$ coned off at the points $\frac{e_{i+1} + e_{i+2}}{2}$ and $\frac{e_{i+2} + e_{i+3}}{2}$.

Hence $H_1 + H_2$ is a bijection, and by symmetry has almost everywhere constant Jacobian. This shows $\frac{1}{2} P_{3,2}$ is $2$-averageable as desired.

\[\square\]

6 Asymptotics for $c_{n,k}$ for $k$ fixed and $n$ large

In this section we prove Theorem 1.4 that for $n \geq k + 1$ we have

$$c_{k,n} \geq 1 - \binom{n}{k} \frac{k^{k+1}}{n^{k+1}}.$$

Proof of Theorem 1.4. Indeed, it suffices to show this $c_{k,n}$ works for functions $f$ on a simplex $C = T$ with $f = 0$ at the vertices and $f \leq 0$ everywhere by Section 2. Set $T$ to be the convex hull of the standard basis vectors $e_1, \ldots, e_{k+1}$ in $\mathbb{R}^{k+1}$.

First, using the notation from Definition 3.1, we claim that we have a covering

$$T = \bigcup_{v \in \mathcal{B}_{k,n-k}} \frac{vT + v}{n}.$$

Indeed, take $y \in T$, and consider $ny$. We can write $ny = w_1 + \lfloor ny \rfloor$, and $\sum(\lfloor ny \rfloor)_i \geq n - k$. Write $\lfloor ny \rfloor = v + w_2$ with $v, w_2$ non-negative integral vectors such that $\sum v_i = n - k$. Then

$$y = \frac{(w_1 + w_2) + v}{n},$$

with $w_1 + w_2 \in kT$ and $v \in \mathcal{B}_{k,n-k}$.
We can then write
\[
\int_T f^n \geq \sum_{v \in \mathcal{B}_{k,n-k}} \frac{k}{n} \int_T f^n \left( \frac{kx + v_1 e_1 + \ldots + v_n e_n}{n} \right) dx
\]
\[
= \sum_{v \in \mathcal{B}_{k,n-k}} \left( \frac{k}{n} \right)^k \int_T f^n \left( \frac{kv_1 f(e_1) + \ldots + v_{k+1} f(e_{k+1})}{n} \right) dx
\]
\[
= \left( \frac{n}{k} \right)^{k+1} \int_T f(x) dx.
\]
As co(f) = 0 we can rearrange this to
\[
\int_T f^n(x) - f(x) dx \geq \left( 1 - \left( \frac{n}{k} \right)^{k+1} \right) \int_T co(f)(x) - f(x) dx.
\]

\[\square\]

Appendix

A Non-sharp \( c_{k,n} \) for \( f^n \) for all \( k, n \)

We now discuss the existence of a non-sharp constant \( c_{k,n} > 0 \) in all dimensions, i.e. that for all compact convex \( C \subset \mathbb{R}^k \) and bounded measurable \( f : C \to \mathbb{R} \), we have
\[
\int_T f^n(x) - f(x) dx \geq c_{k,n} \int_C co(f)(x) - f(x) dx.
\]

We can immediately deduce the existence of such constants from following result on the stability of Brunn-Minkowski for homothetic regions.

\textbf{Theorem A.1} ([vHST20b]). For any \( k \in \mathbb{N} \) and \( t \in (0, 1) \), there are constants \( c(k,t), d(k,t) > 0 \) such that for any \( A \subset \mathbb{R}^{k+1} \) of positive measure if \( |tA + (1-t)A| - |A| \leq d(k,t)|A| \), then
\[
|tA + (1-t)A| - |A| \geq c(k,t)|co(A) \setminus A|,
\]
where we write \( co(A) \) for the convex hull of \( A \).

Indeed, the existence of the constant \( c_{k,n} \) for sup-convolution then follows by applying this theorem to the set \( A = A_{f,-N} \) where
\[
A_{f,\lambda} = \{(x,y) \in C \times \mathbb{R} : \lambda \leq y \leq f(x)\},
\]
We make the following observations.

1. \( T \) is good.

2. If \( T' \) is good and \( v \) is a vertex of \( T \), then \( \frac{(n-1)v+T'}{n} \) is good.

3. If \( T', T'' \) are good and of the same size, then \( \frac{T'+(n-1)T''}{n} \) is good.

The first observation is trivial. For the second, we note that

\[
\int_{\frac{(n-1)v+T'}{n}} f(x)dx \geq \int_{\frac{(n-1)v+T'}{n}} f^{*n}(x)dx - \int_{\frac{(n-1)v+T'}{n}} f^n(x) - f(x)dx
\]

\[
\geq \frac{1}{n^k} \int_{T'} (n-1)f(v) + f(x)dx - \int_{\frac{(n-1)v+T'}{n}} f^n(x) - f(x)dx
\]

\[
= \frac{1}{n^{k+1}} \int_T f(x)dx - \int_{\frac{(n-1)v+T'}{n}} f^n(x) - f(x)dx
\]

\[
\geq \frac{1}{n^\ell(n+1)(k+1)} \int_T f(x)dx - \left( 1 + \frac{C'}{n^{k+1}} \right) \int_T f^n(x) - f(x)dx
\]

For the third observation, we note that

\[
\int_{\frac{(n-1)v+T'}{n}} f(x)dx \geq \int_{\frac{(n-1)v+T'}{n}} f^{*n}(x)dx - \int_{\frac{(n-1)v+T'}{n}} f^n(x) - f(x)dx
\]

\[
\geq \frac{(n-1)}{n} \int_T f(x)dx + \int_{T'} f^n(x)dx - \int_{\frac{(n-1)v+T'}{n}} f^n(x) - f(x)dx
\]

\[
\geq \frac{1}{n^\ell(k+1)} \int_T f(x)dx - \left( 1 + \frac{(n-1)C_T + C_T'}{n} \right) \int_T f^n(x) - f(x)dx.
\]

If for some \( \ell \) we have a family \( A \) of good translates of \( \frac{1}{n}T \) which cover \( T \), then adding the inequalities together, we obtain (recalling \( \text{co}(f) = 0 \))

\[
\left( \sum_{T' \in A} C_T \right) \int_T f^{*n}(x) - f(x)dx \geq \left( 1 - \frac{|A|}{n^\ell(k+1)} \right) \int_T \text{co}(f) - f(x)dx.
\]
Hence if the total number of the simplices $|A|$ is strictly less than $n^{\ell(k+1)}$, we are done.

From the second and third observations, for every face $F$ of $T$ (including $T$), the set of good translates of $\frac{1}{n^\ell}T$ is dense among the set of all translates of $\frac{1}{n^\ell}T$ incident to $F$. Together with the fact that simplices have a bounded inefficiency of covering space, we will be able to accomplish this task for a sufficiently large $\ell$. Indeed, as each simplex $\frac{1}{n^\ell}T$ covers a $\frac{1}{n^\ell}$ volume of $T$, standard results from covering theory imply that we can find a family $A$ with $|A| = O(n^{\ell}) < n^\ell \cdot n^{\ell(k)} = n^{\ell(k+1)}$ for $\ell$ sufficiently large.

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