AWLCO: All-Window Length Co-Occurrence

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Abstract
Analyzing patterns in a sequence of events has applications in text analysis, computer programming, and genomics research. In this paper, we consider the all-window-length analysis model which analyzes a sequence of events with respect to windows of all lengths. We study the exact co-occurrence counting problem for the all-window-length analysis model. Our first algorithm is an offline algorithm that counts all-window-length co-occurrences by performing multiple passes over a sequence and computing single-window-length co-occurrences. This algorithm has the time complexity $O(n)$ for each window length and thus a total complexity of $O(n^2)$ and the space complexity $O(|I|)$ for a sequence of size $n$ and an itemset of size $|I|$. We propose AWLCO, an online algorithm that computes all-window-length co-occurrences in a single pass with the expected time complexity of $O(n)$ and space complexity of $O(\sqrt{n|I|})$. Following this, we generalize our use case to patterns in which we propose an algorithm that computes all-window-length co-occurrence with expected time complexity $O(n|I|)$ and space complexity $O(\sqrt{n|I|} + \varepsilon_{max}|I|)$, where $\varepsilon_{max}$ is the length of the largest pattern.

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1 Introduction
Analyzing regularities in streams and event sequences has applications in data analytics as well as programming languages, natural language processing, and genomics. Examples of a event sequence include a sequence of system logs, memory requests by a program, tweets by a user, a series of symptoms, a sequence of words in a document, or an RNA sequence. One metric of regularity is co-occurrence [12, 20] — the number of times that an entire set of items or more broadly of patterns is contained within a sliding window of an arbitrary size. For example, consider the sequence “abcacb” and window size three. This sequence of events contains four such windows: “abc”, “bcb”, “ccb”, and “cba”. We see that both “a” and “b” appear together in two windows. Thus, itemset \{a, b\} co-occurs twice for window size of three. In the sequence “cat dog cat” with window size seven, we see that the words, referred to as patterns, “cat” and “dog” both appear as substrings in two windows, and thus the pattern set \{cat, dog\} have co-occurrence of two with window size seven.
Most applications assume that the window is given by a user or defined in an adhoc manner. Existing counting algorithms for streams often assume the sliding-window model of computation, that is answering queries or mining is done over the last $w$ most recent data elements [6, 5]. Successful pattern-searching tools, such as ShapeSearch, enable the search for desired patterns within a fixed window size in trendlines [19]. However, in certain applications of co-occurrence analysis, the query is about identifying the time windows that satisfy certain conditions on the co-occurrence. For instance, in text analysis, what is the time window in which a set of events are very likely to appear? Or, at which time window does the co-occurrence of a set of words in a document become random? Or, how often do two or multiple gene expression patterns co-occur in an RNA sequence? These applications require the analysis of all possible window lengths, possibly as large as the size of the sequence.

The All-Window-Length Analysis Model In this paper, we consider a new analysis model of computation for streams and sequences, the all-window-length analysis model, where the analysis of a sequence of data elements is done in one pass for all window lengths, starting from the size of a pattern up to the size of a sequence. Unlike single-window-length analysis, in this model, window length becomes a variable. We consider the co-occurrence counting of items and patterns in this analysis model. A pattern is a string with characters drawn from alphabet $A$. Given a sequence $T$ of size $n$, and an itemset $I$ consisting of patterns, find the the number of windows in which every pattern in $I$ occurs for all window lengths $x \in \{1, \ldots, n\}$ in $T$. This model enables us to perform analysis without apriori knowledge of window-size, i.e. a window size can be chosen and analyzed on demand at query time. For a sequence $T$ of size $n$ and an itemset $I$ consisting of $|I|$ unique tokens, the co-occurrence analysis considers $\sum_{x=1}^{n} (n-x+1)$ windows. We propose efficient exact algorithms and theoretical analysis for the co-occurrence counting of sets of items and patterns under this analysis model. Note that this analysis model is different than the setting of counting frequent itemset in a stream, in which data elements arrive in baskets of arbitrary lengths and the goal is to find the itemset that appears in $s$ fraction of the baskets, where $s$ is a support threshold [16, 13, 2, 1].

Applications We expect the all-window-length analysis model to open research opportunities that lead to solving problems in natural language processing, the optimization of the memory layout of programs, and accelerating the search for RNA sequences in genomes. In natural language processing, the co-occurrence of words within a sliding window is the basis for training word embeddings, which are vector representations of a word’s meaning and usage [14] [15]. Different window sizes are useful for different purposes; embeddings derived from smaller windows tend to represent syntactic information while larger windows represent semantic information [17]. Identifying an effective window length for training word embeddings requires the efficient exploration of the relationship between window size and co-occurrence frequency of words [11].

The application of all-window-length co-occurrence analysis in programming languages is in the optimization of the memory layout of programs. Modern processor performance is dependent on cache performance and cache block utilization. A set of data elements belong to the same affinity group if they are always accessed close to each other. This closeness is defined by $k$-linkedness. A reference affinity forms a unique partition of data for every $k$, and the relation between different $k$s is hierarchical, meaning the affinity groups at link length $k$ are a finer partition of the groups at $k+1$. Reference affinity has been used to optimize the memory layout in data structure splitting [23], whole-program code layout [9], and both [21]. Finding affinity groups requires the analysis of the access co-occurrence of data elements in memory access traces for all $k$s.

Research has shown that analyzing nucleotide co-occurrence over the entire human genome
provides a powerful insight into the evolution of viruses \cite{18, 7}. Co-occurrence is a method for tracking cooperative genomic interactions as a major force underlying virus evolution. Existing co-occurrence network construction tools such as cooccurNet \cite{24} consider pairs of nucleotides or amino acids for analysis and apply filters on the significance of the co-occurrence of genes. The distance in a co-occurrence network counts for the relatedness of genes. An all-window-length analysis of the co-occurrence gene sequences provides further insight into pattern analysis in genomics.

**Results** In this paper, we propose an efficient algorithm that computes all-window-length exact co-occurrence of patterns in a single pass. For co-occurrence of itemsets of size one or two, our past work proposed a linear time algorithm (in sequence length) to compute co-occurrence for all window lengths \cite{12}. To analyze co-occurrence, first, we introduce an algorithm to calculate co-occurrence that runs in $O(n)$ time, is easily understood, and uses $O(|I|)$ space for single-window-length co-occurrence, where $n$ is the length of the sequence, and $I$ is the set of co-occurring items. However, to find the co-occurrence across all window lengths the algorithm would require to compute the co-occurrence for each window length separately and use $O(n^2)$ time which is impractical for large datasets. We propose AWLCO, a time- and space-efficient algorithm that computes the exact co-occurrence of itemsets for all window lengths, in a single pass. The algorithm computes co-occurrence by finding gaps in the sequence, or substrings of the sequence that do not contain subsets of the queried pattern. This is a novel approach to compute co-occurrence and provides an improved algorithm since the stored gaps are not bound to any window lengths, thus, the collection of gaps allows the co-occurrence to be determined for all window lengths in a single pass through the gaps. Furthermore, we propose a simple approach for computing all of the gaps for an itemset in a single pass through the sequence. The relevant gaps can be found by iterating through the sequence and keeping track of the items and the orders they last appeared. We theoretically prove that gaps are only relevant and counted if the current item encountered in the sequence is the item that was seen furthest in the past, thus, drastically reducing the amount of space and updates needed. AWLCO enables all-window-length queries in expected $O(n)$ time by using $O(\sqrt{n} |I|)$ additional space, assuming a perfect hashing function.

Finally, we generalize our problem to finding the co-occurrence of a set of patterns. We argue that finding an algorithm that handles multiple elements at the same index of a sequence would solve all window length pattern co-occurrence. We present an algorithm for pattern co-occurrence counting with the expected time complexity $O(n |I|)$ and space complexity $O(\sqrt{n} |I| + e_{\text{max}} |I|)$, where $e_{\text{max}}$ is the length of the largest pattern.

**2 Problem Definition**

We begin by fixing a vocabulary $A$ that we will be working in. Let $T$ be a sequence with elements in $A$. Sequence $T$ can be considered as a stream. Let $n$ be the length of the sequence $T$ and for any natural number $l$, let $[l] = \{1, \ldots, l\}$. A sequence will have its indices zero indexed, i.e. $T[0]$ is the first element that appeared in the sequence and $T[i]$ is the element that appeared at position $i$. We use $T[i \ldots j]$ to denote a sub-string of $T$. For example, $T[0 \ldots j]$ indicates the first $j + 1$ elements of sequence $T$. An itemset $I$ is a finite non-empty subset of $A$. For a sequence $T$, a window is a sub-string of $T$, or a contiguous selection of elements of $T$. For sequence $T$ we define the window at index $i$ of length $x$ where $x \leq i + 1$, $\omega(T, i, x)$, to be the window containing the $i$-th element of $T$ and the $x - 1$ previous elements of $T$. When it is clear what sequence is being referenced we will refer simply to $\omega(i, x)$. For
example, for the sequence $T = \text{“abcdef”}$, $\omega(3, 3)$ is “bcd”. We define the co-occurrence count as the number of windows of length $x$ in sequence $T$ that contain all elements of the itemset $I$.

\textbf{Definition 1.} Single-window length co-occurrence problem: Given a sequence $T$ and an itemset $I$, find the co-occurrence count of itemset $I$ in windows of length $x$ in sequence $T$.

$$\text{co-occurrence}(T, I, x) = |\{\omega(i, x) : i \in \{x-1, \ldots, n-1\}, \forall e \in I, e \in \omega(i, x)\}| \quad (1)$$

\textbf{Example 2.} Consider the sequence $T = \text{“abcabe”}$. The co-occurrence count of itemset $\{a, b\}$ in all windows with size four, $\text{co-occurrence}(abcabe, \{a, b\}, 4)$, is three.

In this paper, we consider the new problem of finding co-occurrence counts of $I$ in $T$ for all window lengths.

\textbf{Definition 3.} All-window length co-occurrence problem: Given a sequence $T$ of size $n$, and an itemset $I$, find the co-occurrence counts of itemset $I$ in all windows of lengths $x \in \{|I|, \ldots, n\}$ in sequence $T$.

In Section \[3\] we define a baseline algorithm for finding all window length co-occurrence counts based on finding the single window length co-occurrence count. In Section \[4\] we describe our algorithm for simultaneously finding co-occurrence counts of all window lengths in expected linear time in the length of the sequence and the space complexity of $O(\sqrt{n|I|})$.

A pattern is a string with characters drawn from alphabet $\mathbb{A}$. A pattern $e$’s $i$th component is denoted $e[j]$ and the length of the pattern is $|e|$. A pattern occurs in a sequence $T$ if there exists $j \in \{0, \ldots, |e| - 1\}$ such that for all $i \in \{0, \ldots, |e| - 1\}$, $T[j + i] = e[i]$.

\textbf{Definition 4.} All-window length pattern co-occurrence problem: Given a sequence $T$ of length $n$, and an itemset $I$ consisting of patterns, find the the number of windows in which every pattern in $I$ occurs for all window lengths $x \in \{1, \ldots, n\}$ in sequence $T$.

\section{Single-Window-Length Co-occurrence}

Consider an item $e \in \mathbb{A}$ and a sequence $T$. The time elapsed since last access of $e$ at index $i$, $\text{tesla}(T, e, i)$, is the difference between $i$ and the greatest index where $e$ occurs in $T$ up to and possibly including $i$, and in the case that there is no occurrence of $e$ in the interval up to $i$ we define it to be $\infty$. When the choice of $T$ is clear we use the shorthand $\text{tesla}(e, i)$ instead. There is a direct connection between the tesla values for items in the itemset and the number of times the items of the itemset co-occur.

\textbf{Lemma 5.} Itemset $I$ co-occurs in a window $\omega(i, x)$ if and only if $\max \{\text{tesla}(e, i) | e \in I\} < x$.

\textbf{Proof.} The statement implies that for each $e \in I$, $\text{tesla}(e, i) < x$, which implies that $e \in \omega(i, x)$. Conversely, if each $e \in \omega(i, x)$, then we have $\text{tesla}(e, i) < x$; therefore, we have $\max \{\text{tesla}(e, i) | e \in I\} < x$. \hfill \qed

\textbf{Example 6.} Consider the sequence $T = \text{“abcabe”}$ and itemset $\{a, b\}$. Suppose we have processed $T[0 \ldots 3]$ and we know $\text{tesla}(a, 3) = 0$ and $\text{tesla}(b, 3) = 2$. Since the max tesla value is two, the itemset does not co-occur in the size two window $\omega(3, 2)$. 
We therefore define a function $r_i(j)$ which refers to the $j$-th item in the sequence $I[i]$. We have $r_i(j) = I[i][j]$. The co-occurrences of an itemset can be calculated for multiple window lengths by updating Algorithm 1 step by step. By an abuse of notation, in our algorithms we refer to $S$ and $S_i$, which are labeled in such a way that at index $i$ in our sequence,

$$\text{tesla}(e^1, i - 1) \leq \text{tesla}(e^2, i - 1) \leq \cdots \leq \text{tesla}(e^{|A|}, i - 1).$$

Now let $r_i^j : Q \rightarrow A$ be given by $r_i^j(A) = e^j$, for $j \in \{1, \ldots, |A|\}$. That is to say that, $r$ arranges the members of $A$ in a finite sequence according to $\text{tesla}(\cdot, i - 1)$. This notation is robust as it allows for weak ordering and will be used to consider a generalized case later on. We call the realization of $r_i^j$ a book-stack, i.e. $S_i = \{(r_i^1(I), \text{tesla}(r_i^1(I), i)), \ldots, (r_i^{|A|}(I), \text{tesla}(r_i^{|A|}(I), i))\}$ based on the above ordering, given a set $A$. We define $S_i, \text{find} : A \rightarrow \{1, \ldots, |A|\}$, such that $\text{find}(a) = j$, where $r_i^j(A) = a$. We define $S_i, \text{update} : \{1, \ldots, |A|\} \rightarrow \times_{|A|} A$, in which $S_i, \text{update}(j) = (r_{i+1}^1(A), \ldots, r_{i+1}^{|A|}(A))$, where we have

$$r_{i+1}^j(A) = \begin{cases} r_i^j(A), & l = 1 \\ r_{i+1}^l(A), & 1 \leq l < j \\ r_i^j(A), & j < l \leq |A|. \end{cases}$$

We therefore define $S_{i+1} = S_i, \text{update}(S_i, \text{find}(T[i]))$. It is straightforward to see that the $\text{update}$ guarantees the correct ordering for $r_{i+1}^j$ based on $\text{tesla}(\cdot, i)$. Figure 1 illustrates $\text{update}$ to a book-stack data structure step by step. By an abuse of notation, in our algorithms we refer to $S_i$ with $S$.

Algorithm 1 SINGLECOUNTING demonstrates co-occurrence count for a specific window length. The co-occurrences of an itemset can be calculated for multiple window lengths by repeating Algorithm 1 and varying the argument $x$.
Algorithm 1 SINGLECOUNTING

Input: Sequence $T$ of length $n$, Itemset $I$, Window Length $x$

Result: $co$-occurrence$(T, I, x)$

1. $count \leftarrow 0$
2. $S \leftarrow$ empty book-stack
3. for each item $e \in I$ do
4.   $S += (e, -\infty)$
5. end
6. for $i = 0$ to $n - 1$ do
7.   if $T[i] \in I$ then
8.     $j \leftarrow S$.find($T[i]$)
9.     $S$.update($j$)
10. end
11. if $i \geq x - 1$ and $i - S$.retrieve($|I|$) < $x$ then
12.   $count \leftarrow count + 1$
13. end
14. end
15. return $count$

Table 1 Book Stack changes for single-window co-occurrence counting.

| max tesla | initial | a (i=0) | b (i=1) | c (i=2) | a (i=3) | b (i=4) | e (i=5) |
|-----------|---------|---------|---------|---------|---------|---------|---------|
| $0 - (-\infty)$ | $a(-\infty)$ | $a(0)$ | $b(1)$ | $a(2)$ | $a(3)$ | $a(4)$ | $b(5)$ |
| $1 - 0$ | $1 - 0 = 1$ | $2 - 0 = 2$ | $3 - 1 = 2$ | $4 - 3 = 1$ | $5 - 3 = 2$ |

Example 7. Consider the sequence $T = abcabe$ and the itemset $I = \{a, b\}$. The algorithm initializes the $S$ by adding $(e, -\infty)$ for each item $e$ in $I$, representing that element $e$ has never been seen. Table 1 shows the state of $S$ and the resultant max tesla value every time an element of $T$ is processed. At any step the max tesla value can be found by taking the current index in the sequence and subtracting the last access time of the item in the bottom of the book-stack.

3.1 Complexity Analysis

The book-stack can be implemented as a doubly linked list of items. Finding elements on the bottom of the book-stack can then be done in constant time. We can maintain a hash table from each element to the corresponding node in the book-stack. Each node can be accessed in constant time. The book-stack will only take $|I|$ space and no additional space is needed, thus the total space is $O(|I|)$. In addition, each element of the sequence is accessed once, and only constant time operations are performed, giving a time complexity of $O(n)$. For co-occurrence of a single window length, this algorithm performs optimally with respect to time complexity. This is because there is an intrinsic linear cost in computing co-occurrence, as each element in the sequence must be examined in the worst case. In the next section we present a solution that in linear time can calculate the co-occurrence for all window lengths.
Figure 2 Gaps for certain elements in a sequence. The uppermost pattern illustrates the three gaps 'a'-gaps, the middle pattern shows the 'b'-gaps, and the bottom pattern shows the three gaps that contain neither 'a' nor 'b'.

4 All Window-Length Co-occurrence

4.1 Counting Co-occurring Windows

To find the co-occurrence of an itemset \( I = \{e_1, e_2, \ldots, e_{|I|}\} \) in sequence \( T \) with window length \( x \) we must count how many \( x \)-length windows in \( T \) contain \( I \). We will make use of

\[
\text{co-occurrence} (T, I, x) = |(e_1)_x \cup \ldots \cup (e_{|I|})_x|.
\]

Using the inclusion-exclusion principle we can rewrite the co-occurrence as follows.

\[
\text{co-occurrence} (T, I, x) = (n - x + 1) - \sum_{A \subseteq I: A \neq \emptyset} (-1)^{|A|+1} |A_x|
\]

We know that an \( A \)-gap of size \( k \) contains \( k - x + 1 \) windows of length \( x \) in which none of \( A \) occurs. Thus, \( |A_x| = \sum_{k=x}^n (k - x + 1)N_A(k) \), where \( N_A(k) \) is the number of \( A \)-gaps of length \( k \). Working with the right term of equation (4),

\[
\sum_{A \subseteq I: A \neq \emptyset} (-1)^{|A|+1} |A_x| = \sum_{A \subseteq I: A \neq \emptyset} (-1)^{|A|+1} \sum_{k=x}^n (k - x + 1)N_A(k)
\]

\[
= \sum_{k=x}^n (k - x + 1) \sum_{A \subseteq I: A \neq \emptyset} (-1)^{|A|+1} N_A(k).
\]

Now, let us define:

\[
H[k] = \sum_{A \subseteq I: A \neq \emptyset} (-1)^{|A|+1} N_A(k)
\]

We call the collection of \( H[k] \)'s for all values of \( k \) a gap histogram, \( H \). The co-occurrence of \( I \) in \( x \)-length windows of sequence \( T \) is then calculated as follows.

\[
\text{co-occurrence} (T, I, x) = (n - x + 1) - \sum_{k=x}^n (k - x + 1)H[k]
\]
An elegant property of this equation is that by storing the cumulative counts in a gap histogram we can simultaneously calculate the co-occurrence for all window lengths. Using gap histograms to store cumulative counts has a space complexity of $\sqrt{n|I|}$. In Theorem 10, we will formally discuss the space complexity in more details. Calculating the co-occurrence from the gap histogram instead of directly counting co-occurrent windows is beneficial since calculating gaps does not require a window length as input and yet the gap information is still sufficient to easily calculate the co-occurrence for all window lengths. Thus, all that is needed to calculate all window length co-occurrence is an algorithm to generate the gap histogram.

The simplest way to generate the gap histogram is to iterate through the sequence, keeping track of where gaps begin and end. Whenever an item in the given itemset is found at some index $i$ it marks the end of a gap for any subset of $I$ containing that item and also marks the beginning of a new gap spanning from $T[i+1...k-1]$, where $k$ is either the index of the next occurrence of an element in the subset of $I$ in the sequence or $n$ if another element does not occur before the end of the sequence. Note that if an element in $I$ occurs in two adjacent indices in the sequence ($i$ and $i+1$), we obtain the gap $[i+1,i]$ which we treat as a length 0 gap and discard. The length $l$ of each newly ended gap can be updated in the histogram by either incrementing or decrementing $H[l]$ depending on whether the subset size was odd or even respectively. The pseudocode of this approach can be found in the appendix under, Algorithm 3, GAPCOUNTING. This algorithm has run time $O(n2^{|I|})$ and performs poorly for large itemsets. Algorithm 3 is inefficient since whenever an item from $I$ is encountered in the sequence, we need to consider $2^{|I|}-1$ subsets of $I$ and update the histogram (subtract or add counts) accordingly (Line 9). A better algorithm is presented next.

4.2 Efficient Gap Counting

Since updates to the histogram have negating effects on each other (Equation 7), many of the histogram entries do not change when an item of the itemset is observed in the sequence. It turns out when an item of the itemset is observed in a sequence, we only need to update the histogram for the gaps related to the first and second least recently seen items of $I$. To keep track of the tesla’s, as we iterate through the sequence we can maintain a book-stack data structure that contains each item in $I$ along with the time that it last appeared in the sequence, so that the most recently seen item appears at the top of book-stack.

Observe that, when an item $e$ from $I$ is seen in the sequence at index $i$, a maximal gap representing each subset of $I$ containing $e$ is added to the histogram. Furthermore, for any one of those sets $G$, the length of the added gap is the minimum tesla value attained by an item in $G$ at index $i-1$. Note that in this context we take $\text{tesla}(e, i) = i$ if the element has not yet been encountered in the sequence. These gaps account for all of the gaps in the sequence except for gaps that include the final element of the sequence, these gaps are handled specially.

For the following theorem, we first provide some notation. Let

$$H_i = (H_i[1], H_i[2], \ldots, H_i[n])$$

be the histogram up to index $i$ in the sequence.
Theorem 8. For any $0 < i < n$, suppose $T[i] = r_i^{|I|}(I)$ then

$$H_i[k] = \begin{cases} H_{i-1}[k] + 1 & \text{if } k = \text{tesla}(r_i^{|I|}(I), i - 1) \\ H_{i-1}[k] - 1 & \text{if } k = \text{tesla}(r_{i-1}^{|I|-1}(I), i - 1) \\ H_{i-1}[k] & \text{otherwise.} \end{cases}$$

If $T[i] \neq r_i^{|I|}(I)$ then $H_i[k] = H_{i-1}[k]$ for all $k$. In other words, the histogram is only updated when the next element in the sequence is the item that was just at the bottom of the book-stack.

Proof Sketch. We have a maximal gap for every subset $A$ of $I$ containing $T[i]$. The length of this $A$-gap is $\min_{e \in A} \text{tesla}(e, i - 1)$, hence the addition to the histogram from $A$ is $(-1)^{|A|+1}$ to the $k$th spot where $k = \min_{e \in A} \text{tesla}(e, i - 1)$. Suppose $T[i] \neq \arg\max_{e \in I} \text{tesla}(e, i - 1)$ i.e., $T[i]$ is not the item seen furthest in the past most recently. There are the same number of even and odd subsets of $I$ in which $T[i] = \arg\min_{e \in A} \text{tesla}(e, i - 1)$ hence these subsets contribute no net updates to $H_i$. For the remaining subsets, the same argument follows, hence there are no net updates.

Now suppose that $T[i] = \arg\max_{e \in I} \text{tesla}(e, i - 1)$. Similar to the above, for each item in $I$ not equal to $r_{i-1}^{|I|-1}(I)$ and $T[i]$, there are the same number of even and odd subsets of $I$ in which $T[i] = \arg\min_{e \in A} \text{tesla}(e, i - 1)$. But for $r_i^{|I|-1}(I)$ there is but one subset in which this is satisfied, namely, $\{r_{i-1}^{|I|-1}(I), T[i]\}$, and there is also one subset in which $T[i]$ satisfies this, $\{T[i]\}$. Therefore we have

$$H_i[\text{tesla}(T[i], i - 1)] = H_{i-1}[\text{tesla}(T[i], i - 1)] + 1,$$

$$H_i[\text{tesla}(r_{i-1}^{|I|-1}(I), i - 1)] = H_{i-1}[\text{tesla}(r_i^{|I|}(I), i - 1)] - 1.$$ 

The theorem does not handle the case for $H_n$, which we now address. The argument is similar to the proof above for $H_i$ with $i < n$, except that $T[i]$ is undefined. All gaps necessarily close at the end of the sequence. This means that $|C_k| = \sum_{j=0}^{|I|} \binom{|I|}{j}$, for all but $j = |I|$. For $j = |I|$ there is but one set for which $r_i^{|I|}(I) = \arg\min_{e \in A} \text{tesla}(e, n)$, namely, $\{r_i^{|I|}(I)\}$. Thus $H_n[k] = H_{n-1}[k]$ for all $k$ except when $k = \text{tesla}(n, i - 1)$ in which $H_n[k] = H_{n-1}[k] - 1$.

The incremental updates that we have derived above result in algorithm AWLCO, shown in Algorithm 9.

4.3 Complexity Analysis

The next two theorems assume that the histogram can be implemented as a hashtable with perfect hashing. Without perfect hashing the histogram must contain space for all entries from $1 - n$ and thus will be linear in space to maintain a constant run time or constant histogram updates must be sacrificed to obtain a worst case $n^2$ runtime.

Theorem 9 (Time Complexity). The time complexity of all-window length co-occurrence algorithm is linear in the length of the sequence.

Proof. The algorithm iterates over the sequence once and possibly updates the book-stack and the histogram for each element in the sequence. Since updating the book-stack and updating the histogram are both done in constant time, the generation of the histogram is done in linear time in the length of the sequence. Once a histogram is computed, the co-occurrence for every window length is computed in a linear time by summing the histogram as shown in Equation 8. Thus, the algorithm provides an $O(n)$ method to calculate all window length co-occurrence.
Theorem 10 (Space Complexity). The space complexity of the algorithm is $O(\sqrt{n|I|})$ where $n$ is the length of the sequence and $|I|$ is the size of the itemset.

Proof. Space is used to maintain the book-stack and the histogram. The book-stack will use $O(|I|)$ space. Note that for any item $e$ in the itemset the total length of gaps for $\{e\}$ is at most the length of the sequence. Thus, we have that the sum of all of the lengths of single-item gaps is bounded above by $n|I|$. Furthermore, whenever an item of the itemset is on the bottom of the book-stack a maximum of two new gaps are added to the histogram. The length of the gap associated to the bottom item in the book-stack is equal to the length of a single-item gap. The length of the other gap is bounded above by the length of the first gap. Therefore, the sum of the length of all gaps added to the histogram is bounded above by $2n|I|$. Note the size of the histogram is the number of distinct gap lengths added to it. In the worst case, gaps are greedily added to the histogram such that there is a length 1, 2, ..., $k$ size gap added. In this case, if the total number of gaps added is $k$ the total length of the gaps is $\frac{k(k+1)}{2}$. We know that the sum of the gaps length in a histogram is bounded above by $2n|I|$. Thus, we have that $\frac{k(k+1)}{2} \leq 2n|I|$. Solving for $k$, we have that $k^2 + k \leq 4n|I|$ and $k \leq 2\sqrt{n|I|}$. Thus, the total space used is bounded above by $|I| + 2\sqrt{n|I|}$ which gives a space complexity of $O(\sqrt{n|I|})$.

5 Pattern Co-occurrence

We now wish to generalize our algorithm in two ways. The first is to patterns and the second is to a stream in which multiple events can occur at the same index. Pattern co-occurrence is explained first. A pattern is a string with characters drawn from our alphabet $A$. A pattern $e$’s $i$th component is denoted $e[j]$ and the length of the pattern is $|e|$. A pattern occurs in a sequence $T$ if there exists $j \in [|T|]$ such that $T[j...j+|e|-1] = e$, also let all such $j$ be denoted in the set $b(e)$. Thus, pattern co-occurrence for an itemset $I$ is defined as the number of windows in which every pattern in $I$ occurs. We wish to find an algorithm that can compute the co-occurrence for all window lengths in one pass for patterns. It is clear that $b(e)$ is no longer well-defined. Let $e$ be a pattern. So define $b_{tesla}(e, i) = i - \max(|b(e) \cap \{0, \ldots, i\})$, which is the distance between $i$ and the most recent start of the pattern. If $b(e) \cap \{0, \ldots, i\}$ is empty, then let it be $i$.

We can use our previous definition of an $A$-gap for $A \subseteq I$, but the size of an $A$ gap is now found differently. Previously, the size of an $A$-gap closed at time $i$ would be $\min_{e \in A} b_{tesla}(e, i - 1)$, but now it is $\min_{e \in A} b_{tesla}(e, i)$, since an $A$-gap still occurs if all but the tail ends of members of $A$ are within said gap. Supposing that no two patterns in consideration end at the same time, it is easy to see that Theorem 5 still holds in this case, using $b_{tesla}$ in place of $tesla$. Thus finding an algorithm that handles multiple events at the same index would solve all window length pattern co-occurrence as well.

5.1 Multiple Item Co-occurrence

It is now natural to define co-occurrence for sets of items. We let $T[i] \subseteq A$, rather than just one element of $A$, for all $i$. A co-occurring window for some itemset $I \subseteq A$ is a window in which for all $e \in I$, there exists a set $A \in w(x, i)$ such that $e \in A$. Thus the co-occurrence is the sum of these co-occurring windows. This is the natural extension. We will now present the following theorem relating to the updates of $H$. Let $X_i$ denote the set of items that occur at $T[i]$.
Algorithm 2 AWLCO

Input: Sequence $T$, ItemSet $I$

Result: Co-occurrence of all window lengths

1. $H \leftarrow$ empty histogram
2. $cooc \leftarrow []$
3. $S \leftarrow$ empty book-stack
4. for $e \in I$ do
5. \hspace{1em} $S += (e, -\infty)$
6. end

// Read through entire sequence
7. for $i = 0$ to $n - 1$ do
8. \hspace{1em} current$\leftarrow T[i]$
9. \hspace{2em} // When element is seen, update bottom two gaps
10. \hspace{2em} if current$\in I$ then
11. \hspace{3em} $f \leftarrow i - S.$retrieve$(|I|)$
12. \hspace{3em} $s \leftarrow i - S.$retrieve$(|I| - 1)$
13. \hspace{3em} $H[f] \leftarrow H[f] + 1$
14. \hspace{3em} $H[s] \leftarrow H[s] - 1$
15. \hspace{2em} end
16. \hspace{2em} $j \leftarrow S.$find$(current)$
17. \hspace{2em} $S.$update$(j)$
18. \hspace{1em} end

// Final gap from bottom of book-stack
19. $f \leftarrow i - S.$retrieve$(|I|)$
20. $H[f] \leftarrow H[f] + 1$
21. for $x = |I|$ to $|T| - |I| + 1$ do
22. \hspace{1em} $S_x \leftarrow 0$
23. \hspace{2em} for $k = x$ to $|T|$ do
24. \hspace{3em} $S_x \leftarrow S_x + (k - x + 1)H[k]$
25. \hspace{2em} end
26. \hspace{1em} $cooc[x] \leftarrow (|T| - x + 1) - S_x$
27. end
28. return $cooc$
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\(T[i - k_1]\) & \(T[i - k_2]\) & \(T[i]\) \\
\hline
\(x_1\) & \(x_2\) & \(x_1\) \\
\hline
\(y_1\) & \(y_2\) & \\
\hline
\(x_3\) & \(x_4\) & \\
\hline
\end{tabular}
\end{center}

\textbf{Figure 3} Illustration of Theorem [11]. The set of patterns \(I\) consists of \(x_1, x_2, \ldots\), which were seen at \(T[i]\), and all other patterns \(y_1, y_2, \ldots\). Here \(A\) is the set \(\{x_1, x_2\}\) of patterns seen at \(T[i]\) that were last seen further in the past than any of the other patterns \(y_1, y_2, \ldots\). We add one to \(H[k_1]\), where \(k_1\) is the time elapsed since \(x_1\) was last seen. We subtract one from \(H[k_2]\), where \(k_2\) is the time elapsed since \(y_1\) was last seen.

\textbf{Theorem 11.} Suppose that for all \(a \in I \cap X_i\), \(\text{tesla}(a, i - 1) < \text{tesla}(e, i - 1)\) for any \(e \in I \setminus X_i\). Then for any \(1 < i < n\),

\[H_i[k] = \begin{cases} 
H_{i-1}[k] + 1 & \text{for } k = \text{tesla}(r_i^{\mid I \setminus X_i\mid}(I \setminus X_i), i - 1) \\
H_{i-1}[k] - 1 & \text{for } k = \text{tesla}(r_i^{\mid I\mid}(I), i - 1) \\
H_{i-1}[k] & \text{otherwise.}
\end{cases}\]

Otherwise, \(H_i[k] = H_{i-1}[k]\) for all \(k\).

\textbf{Remark 12.} This is a generalization of Theorem [8]. Observe that in the case when \(|A| = 1\), this reduces to that result. Moreover, when \(i = n\) the same update follows.

\textbf{Proof Sketch.} The incremental updates to \(H\) correspond to all the subsets of \(I\) that contain at least one member of \(X_i\). Weakly order \(X_i\) according to \(\text{tesla}\). Now let \(A_j\) where \(j \in [X_i]\), be the set of subsets of \(I\) that contains \(x_j\). Let \(\langle A_i \rangle\) be the updates corresponding to some set \(A\). Therefore

\[H_i - H_{i-1} = \sum_{J \subseteq \mid I \mid} (-1)^{|J|+1} \left\langle \bigcap_{j \in J} A_j \right\rangle.\]

Consider each \(K_j = \bigcap_{i \in J} A_i\). For each one, the update is the same if one removes all members of \(X_i\) besides the one corresponding to the smallest number in \(J\), call this set \(K_j'\). Using Theorem [8], the update is +1 for \(k = \text{tesla}(x, i - 1)\), \(x\) being the item described before, and also is −1 for \(k = \text{tesla}(x, i - 1)\) for \(x\) being the furthest item seen in the past not in \(K_j \cap I\). This implies that \(J\) that are not of the form \(J_m = \{|X_i| - m, \ldots, |X_i|\}\). For any \(0 \leq m < |X_i|\), the positive update corresponding to \(K_{J_m}\) cancels with the negative update corresponding to \(K_{J_{m+1}}\). This process telescopes leaving only the positive update corresponding to \(K_{J_0}\) and the negative update corresponding to \(K_{J_{|X_i|}}\). This gives the desired result. □

A full proof is given in the appendix. Figure [9] provides an illustration of Theorem [11]. With this result we can now construct a similar algorithm to those before, with a few modifications. Maintain a book-stack as before, but notice that it is no longer a strict ordering. For example, if \(X_i = \{e_1, e_2\} \subseteq I\), then one of \(e_1\) and \(e_2\) will occupy the top of the book-stack and the other will occupy the the second to top spot. To check whether \(\max_{e \in X_i \cap I} \text{tesla}(e, i) < \min_{e \in I \setminus X_i} \text{tesla}(e, i)\), we partition the book-stack using \(p \in \{0, \ldots, |I|\}\),
where \( p \) is defined as follows: for all \( j \leq p \), \( \text{tesla}(r^j(I), i - 1) = \text{tesla}(r^j(I), i - 1) \), and for all \( j > p \), \( \text{tesla}(r^j(I), i - 1) > \text{tesla}(r^{j+1}(I), i - 1) \). Thus checking if the non-trivial conditions given in Theorem 11 hold is easy as we just check that \( r^j(I) \leq p \) for every \( j \) corresponding to a member in \( X \). It is also easy to update the histogram if these conditions hold, as we just update according to \( r^p(I) \) and \( r^{p+1}(I) \). The pseudocode is given in algorithm 4 in the appendix.

5.2 Complexity Analysis

Maintaining the partition is at the worst case a linear scan of \( I \) at each index. A state machine is spawned at each element that starts one of the patterns, and is terminated either by the pattern not being completed, or by completion of the pattern. If completed, that pattern is moved from its current level in the structure to the level corresponding to the now found value of \( \text{tesla} \). This requires \( O(|I|) \) operations. Additionally, there can be at most \( 2|I| \) state machines created or terminated at each step. Thus, the time complexity is \( O(n|I|) \).

Space complexity is also the same but including the space for the state machines, giving \( O(\sqrt{n|I|} + e_{\text{max}}|I|) \), where \( e_{\text{max}} \) is the length of the largest pattern.

Continuous co-occurrence

The previous section opens up new opportunities. Because we can now allow multiple items to occur at the same index, we can analyze occurrences of discrete events, which are items in the original sense, in an interval of time, which is the sequence in the original sense. Given a set of events and times, we would like to approximate the probability of two events occurring within some interval of time with generalized AWLCO.

To be more precise, let \( \tau \geq 0 \) and define

\[
T = \{(e_1, \omega_{1, n_1}), \ldots, (e_1, \omega_{1, n_k_1}), (e_2, \omega_{2, 1}), \ldots, (e_{|I|}, \omega_{|I|, n_{|I|}})\}
\]

in which \( e_i \in I \) and \( \omega_{i,j} \in [0, \tau] \) for all \( j \in \{1, \ldots, n_i\} \), for all \( i \in \{1, \ldots, |I|\} \). We call this \( T \) the set of time stamps of our set of events \( I \). Let, for the sake of convenient notation,

\[
\Pr(I \in [a, b] \setminus T, r) = \frac{\Pr}{y \sim U([a, b])} (\forall e \in I, \exists (e, \omega) \in T : |y - \omega| < r).
\]

We will attempt to find a discrete analog of the above probability. Let \( Q_t(T) = \{(e, \lfloor \frac{t}{\tau} \rfloor t) : (e, \omega) \in T\} \). \( Q_t(T) \) is a means of discretizing the possible values of \( \omega \) that could lie in \( T \). Notice that \( \lim_{t \to \infty} Q_t(T) = T \), which says that as \( T \) gets smaller, \( Q_t(T) \) becomes a better and better approximation of \( T \). Now let \( T_{\tau} = \{a_0, a_1, \ldots, a_n\} \) in which \( a_i = \{e \in T : \exists (e, \omega) \in T, \lfloor \frac{\omega}{\tau} \rfloor = i\} \) for all \( i \in \{0, \ldots, n\} \). This is what we will feed into generalized AWLCO in order to approximate the above probability. It is clear that \( T_{\tau} = T_{\tau}(Q_t(T)) \). Let \( T_{\tau} + x = \{a_{-x/2}, \ldots, a_1, a_0, a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+x/2+1}\} \) in which \( a_{-x/2} = a_{-x/2+1} = \cdots = a_{-1} = a_{n+1} = \cdots = a_{n+x/2} = 0 \), but all other \( a_i \) are given as above. We now have the following result which allows us to translate co-occurrence into probabilities.

\textbf{Theorem 13.}

\[
c_{\text{co-occurrence}}(T_{\tau}(T) + x, I, x) = \frac{1}{T} \int_0^T 1(|I| \in \max\{0, y - x/2\}, \min\{\tau, y + x/2\}, Q_t(T))dy
\]

\textbf{Remark 14.} The right hand side is simply the measure of the set of \( y \in [0, \tau] \) such that all of \( I \) occurs within distance \( x/2 \) of \( y \).
Proof. We clearly have that
\[
\text{co-occurrence}(\text{Tr}_t(T) + x, I, x) = \sum_{i=x/2}^{n+x/2} \mathbb{1}(I \subseteq w(x, i)).
\] (9)

Now if \( I \subseteq w(x, i) \), that means for all \( e \in I \) there exists \( (e, \omega) \in T \) such that \( \lceil \frac{e}{\omega} \rceil \in \{i-x+1, \ldots, i\} \). This means that \( \lceil \frac{e}{\omega} \rceil t \in [(i-x+1)t, (i+1)t] \). If \( I \not\subseteq w(x, i) \), then for some \( e \in I \) there is no such \( (e, \omega) \in T \) in which \( \lceil \frac{e}{\omega} \rceil t \in [(i-x+1)t, (i+1)t] \). We can therefore write
\[
\sum_{i=x/2}^{n+x/2} \mathbb{1}(I \subseteq w(x, i)) = \sum_{i=x/2}^{n+x/2} \mathbb{1}(I \in [(i-x+1)t, (i+1)t], Q_t(T))
\] (10)
\[
= \int_{x/2}^{(n+x/2)t} \mathbb{1}(I \in [([y] - x + 1)t, ([y] + 1)t], Q_t(T))dy
\] (11)
\[
= \frac{1}{t} \int_{x/2}^{(n+x/2)t} \left( I \in \left( \left\lceil \frac{y}{t} \right\rceil - x + 1 \right)t, \left( \left\lceil \frac{y}{t} \right\rceil + 1 \right)t, Q_t(T) \right) dy.
\] (12)

But now recall that for every \( (e, \omega) \in Q_t(T) \), that \( \omega = kt \) for some \( k \in \mathbb{Z}_+ \). So suppose that \( I \in [(\left\lceil \frac{y}{t} \right\rceil - x + 1)t, (\left\lceil \frac{y}{t} \right\rceil + 1)t] \). Then for every \( e \in I \), there exists \( (e, \omega) \in Q_t(T) \) such that \( \omega/t \in \{\frac{y}{t} - x + 1, \ldots, \frac{y}{t}\} \). Since \( y/t - x < \left\lceil \frac{y}{t} \right\rceil - x + 1 \) and that \( y/t \geq \left\lceil \frac{y}{t} \right\rceil \), we have that \( I \in (y - xt, y) \). Now suppose that \( I \not\in [(\left\lceil \frac{y}{t} \right\rceil - x + 1)t, (\left\lceil \frac{y}{t} \right\rceil + 1)t] \) then for some \( e \in I \), there is no such \( (e, \omega) \in Q_t(T) \) in which \( \omega/t \in \{\frac{y}{t} - x + 1, \ldots, \frac{y}{t}\} \). Now since \( y/t - x \geq \left\lceil \frac{y}{t} \right\rceil - x \) and \( y/t < \left\lceil \frac{y}{t} \right\rceil + 1 \) we have that \( I \not\in (y - xt, y) \). Therefore
\[
\frac{1}{t} \int_{x/2}^{(n+x/2)t} \mathbb{1}(I \in \left( \left\lceil \frac{y}{t} \right\rceil - x \right)t, \left( \left\lceil \frac{y}{t} \right\rceil \right)t, Q_t(T) \) dy
\] (13)
\[
= \frac{1}{t} \int_{x/2}^{(n+x/2)t} \mathbb{1}(I \in (y - xt, y), Q_t(T)) dy
\] (14)
\[
= \frac{1}{t} \int_{x/2}^{(n+x/2)t} \mathbb{1}(I \in [y - xt, y], Q_t(T)) dy
\] (15)
\[
= \frac{1}{t} \int_{0}^{nt} \mathbb{1}(I \in [y - xt/2, y + xt/2], Q_t(T)) dy
\] (16)
\[
= \frac{1}{t} \int_{0}^{\tau} \mathbb{1}(I \in [\max\{0, y - xt/2\}, \min\{\tau, y + xt/2\}], Q_t(T)) dy,
\] (17)

where the last inequality follows from \( nt = \tau \) and that no members of \( I \) lie below 0 or above \( \tau \).

The previous theorem shows us that we can represent co-occurrence as a continuous sum, which makes it much easier for us to achieve an error bound for continuous co-occurrence. Let us now find this bound.

**Theorem 15.** Let \( I(A) = \int_{0}^{\tau} \mathbb{1}(I \in [\max\{0, y - xt/2\}, \min\{\tau, y + xt/2\}], A)dy \). Then if \( x > t \),
\[
|I(Q_t(T)) - I(T)| < \frac{\tau t}{x}
\] (18)
Proof Sketch. We first write

$$I(Q_t(T)) - I(T) = \int_0^\tau \mathbb{1}(I \in [\max\{0, y-x/2\}, \min\{\tau, y+x/2\}], Q_t(T)) \, dy$$

$$\quad - \int_0^\tau \mathbb{1}(I \in [\max\{0, y-x/2\}, \min\{\tau, y+x/2\}], T) \, dy,$$

so it suffices to bound the length of region where the integrands differ. For the sake of ease, denote

$$\mathbb{1}(y, Q_t(T)) = \mathbb{1}(I \in [\max\{0, y-x/2\}, \min\{\tau, y+x/2\}], Q_t(T))$$

and

$$\mathbb{1}(y, T) = \mathbb{1}(I \in [\max\{0, y-x/2\}, \min\{\tau, y+x/2\}], T).$$

The trick here is to consider only $y \in [kt, (k+1)t]$ at a time. It can be shown that if $t > x/2$, then the measure of the set in each of these segments in which $\mathbb{1}(y, T)$ and $\mathbb{1}(y, Q_t(T))$ differ, is at most $t - x/2$. There are at most $\tau/t$ such segments. Therefore in this case $|I(Q_t(T)) - I(T)| < \frac{\tau}{2}(t-x/2) = \tau - \tau^{x/2}$ And since $x/t > 1/2$, we have that $\tau - \tau^{x/2} < \tau/2 < \frac{\tau}{2}$.

Now suppose that $t \leq x/2$. It can be shown now that measure of the set of $y$ in each segment that differ are at most $t$. But if this were to happen, it would imply that they do not differ for the following $x/2t$ segments on either side. This means there is error at most $t$ for $x/t$ segments each of length $t$ which gives a total error bound of $\tau \frac{t}{x} \frac{t}{x} = \tau \frac{t}{x}$. 

Now putting these results together, we obtain that

$$|\text{co-occurrence}(Tr(T) + x, I, x)t/\tau - Pr(I \in [a,b], T, x/2)| < \frac{t}{x}. \quad (21)$$

6 Related Work

Counting in Streams - In count-distinct problem, the goal is to know the number of unique elements in a stream [8, 13]. In bit-counting problem, the goal is to maintain the frequency count of 1’s in the last $k$ bits of a bit stream of size $N$. Datar et al. propose an approximate algorithm with for the bit-counting problem with $O(\log^2 k)$ space complexity [5]. Existing counting algorithms for streams assume the sliding-window model of computation, that is answering queries or mining is done over the last $w$ elements seen so far [6]. However,awlco introduces a new analysis model – all-window-length analysis model – which is compelled to analyze and query all windows of all lengths starting from the beginning of a stream or anytime in the the past. To that end, awlco presents an efficient and exact itemset counting algorithm for the all-window-length analysis model.

The frequent itemset mining in stream is a well-studied problem that adheres to the counting problem [4]. The seminal work by Manku and Motwani presents an algorithm for estimating the frequency count of itemsets in a stream and identify those itemsets that occur in at least a fraction $\theta$ of the stream seen so far with some error parameter $\epsilon$ [13]. For example, when the input is a stream of transactions where each transaction is a set of items, the goal is to find the most frequent itemsets within transactions. The challenge is to consider variable-length itemsets and avoid the combinatorial enumeration of all possible itemsets. Many existing frequent itemset mining algorithms (with exception of [10, 3]) obtain
approximate results with error bounds. A variation of frequent itemset mining is the problem of mining frequent co-occurrence patterns across multiple data streams \[20\]. The definition of co-occurrence patterns is slightly different than co-occurrence itemsets considered by AWLCO. A co-occurrence pattern is a group of items that appear consecutively showing tight correlations between these items. A frequent co-occurrence pattern is the pattern that appears in at least \(\theta\) streams within a time period of length \(\tau\) and the appearance of the pattern in each stream happens within a time window of \(\delta\) or smaller. In this paper, AWLCO presents an all-window length frequency counting for a query itemset. A natural extension of the itemset frequency counting of presented by AWLCO is mining frequent itemsets in all window-lengths.

**Affinity Analysis** - Zhong et al. defined reference affinity for data elements on an access trace. A set of data elements belong to the same affinity group if they are always accessed close to each other \[23\]. The closeness is defined by \(k\)-linked-ness. They proved that reference affinity forms a unique partition of data for every \(k\), and the relation between different \(k\)s is hierarchical, i.e. the affinity groups at link length \(k\) are a finer partition of the groups at \(k+1\). This definition requires strict co-occurrence in that every occurrence of a group element must be accompanied by all other elements of the group. Weak reference affinity \[22\] introduces a second parameter, affinity threshold. It adheres to the unique and hierarchical partition properties with respect to both parameters. Zhang et al. showed that neither strict reference affinity, nor weak reference affinity can efficiently be computed \[21\]. Thus they gave a heuristic solution and adapted it to use sampling. The average time complexity of their algorithm is \(O(N\delta\omega^2 + N\delta\pi)\), where \(N\) is the length of the trace, \(\delta\) is the sampling rate, \(\omega\) is the size of the affinity group, and \(\pi\) is the average time length of windows containing accesses to all members of the group \(\omega\). Lavaee et al. gave an \(O(L\delta\omega^2)\) algorithm to compute the affinity for all sub-groups of sizes up to \(\omega\) \[9\]. Reference affinity has been used to optimize the memory layout in data structure splitting \[23\], whole-program code layout \[9\], and both \[21\].

### 7 Discussion and Future Work

**Applications** The all-window-length co-occurrence has applications in text analysis, the optimization of the memory layout of programs, and accelerating the search for RNA sequences in genomes. In terms of practical applications, our plan is to develop interactive tools that enable the exploration of sequences of events and genomics data. Projects such as cooccurNet \[24\] provide a basis that can be extended with all-window-length co-occurrence analysis functionalities.

**Mining Problems** In this paper, we expounded co-occurrence counting of itemsets and patterns in the all-window-length analysis model. Going forward, we study mining algorithms in this analysis model, including mining frequent closed itemsets, i.e. given a sequence \(T\) find the top-\(k\) itemsets that have highest co-occurrences in an arbitrary window size and for a frequent itemset \(X\), there exists no super-pattern \(X \subset Y\), with the same co-occurrence as \(X\). The algorithm requires to mine frequent itemsets for all window lengths in one pass.

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References

1. Rakesh Agrawal, Tomasz Imielinski, and Arun N. Swami. Mining association rules between sets of items in large databases. In SIGMOD, pages 207–216, 1993.

2. Rakesh Agrawal and Ramakrishnan Srikant. Fast algorithms for mining association rules in large databases. In VLDB, pages 487–499, 1994.

3. Joong Hyuk Chang and Won Suk Lee. Finding recent frequent itemsets adaptively over online data streams. In SIGKDD, pages 487–492.

4. Graham Cormode and Marius Hadjieleftheriou. Finding frequent items in data streams. PVLDB, 1(2):1530–1541, 2008.

5. Mayur Datar, Aristides Gionis, Piotr Indyk, and Rajeev Motwani. Maintaining stream statistics over sliding windows. SIAM J. Comput., 31(6):1794–1813, 2002.

6. Mayur Datar and Rajeev Motwani. The sliding-window computation model and results. In Data Stream Management - Processing High-Speed Data Streams, pages 149–165. 2016.

7. Xiangjun Du, Zhuo Wang, Aiping Wu, Lin Song, Yang Cao, Haiying Hang, and Taijiao Jiang. Networks of genomic co-occurrence capture characteristics of human influenza a (h3n2) evolution. 18(1), January 2008.

8. Philippe Flajolet and G. Nigel Martin. Probabilistic counting. In FOCS, pages 76–82, 1983.

9. Rahman Lavacce, John Criswell, and Chen Ding. Codesitcher: inter-procedural basic block layout optimization. In Proceedings of the International Conference on Compiler Construction, pages 65–75, 2019.

10. Carson Kai-Sang Leung and Quamrul I. Khan. Dstree: A tree structure for the mining of frequent sets from data streams. In ICDM, pages 928–932, 2006.

11. Omer Levy and Yoav Goldberg. Dependency-based word embeddings. In ACL, pages 302–308, 2014.

12. Yumeng (Lucinda) Liu, Daniel Busaba, Chen Ding, and Daniel Gildea. All timescale window co-occurrence: Efficient analysis and a possible use. In Proceedings of the 28th Annual International Conference on Computer Science and Software Engineering, CASCON ’18, pages 289–292, Riverton, NJ, USA, 2018. IBM Corp.

13. Gurmeet Singh Manku and Rajeev Motwani. Approximate frequency counts over data streams. In VLDB, pages 346–357, 2002.

14. Tomas Mikolov, Ilya Sutskever, Kai Chen, Greg S Corrado, and Jeff Dean. Distributed representations of words and phrases and their compositionality. In Advances in Neural Information Processing Systems, pages 3111–3119, 2013.

15. Jeffrey Pennington, Richard Socher, and Christopher Manning. Glove: Global vectors for word representation. In Proceedings of the 2014 Conference on Empirical Methods in Natural Language Processing (EMNLP), pages 1532–1543, Doha, Qatar, 2014.

16. Anand Rajaraman, Jure Leskovec, and Jeffrey D. Ullman. Mining Massive Datasets. 2014.

17. Hinrich Schütze. Ambiguity Resolution in Language Learning – Computational and Cognitive Models. Number 10 in CSLI Lecture Notes Series. Center for the Study of Language and Information, Stanford, California, 1997.

18. Jason W. Shapiro and Catherine Putonti. Gene co-occurrence networks reflect bacteriophage ecology and evolution. mBio, 9(2), 2018.

19. Tarique Siddiqui, Paul Luh, Zesheng Wang, Karrie Karahalios, and Aditya G. Parameswaran. ShapeSearcH: A flexible and efficient system for shape-based exploration of trendlines. In SIGMOD, pages 51–65, 2020.

20. Ziqiang Yu, Xiaohui Yu, Yang Liu, Wenzhu Li, and Jian Pei. Mining frequent co-occurrence patterns across multiple data streams. In EDBT, pages 73–84, 2015.

21. Chengliang Zhang, Chen Ding, Mitsunori Ogihara, Yutao Zhong, and Youfeng Wu. A hierarchical model of data locality. In Proceedings of the ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, pages 16–29, 2006.
22 Chengliang Zhang, Yutao Zhong, Chen Ding, and Mitsunori Ogihara. Finding reference affinity groups in trace using sampling method. Technical report, Department of Computer Science, University of Rochester, 2004.

23 Yutao Zhong, Maksim Orlovich, Xipeng Shen, and Chen Ding. Array regrouping and structure splitting using whole-program reference affinity. In Proceedings of the ACM SIGPLAN Conference on Programming Language Design and Implementation, pages 255–266, 2004.

24 Yuanqiang Zou, Zhiqiang Wu, Lizong Deng, Aiping Wu, Fan Wu, Kenli Li, Taijiao Jiang, and Yousong Peng. cooccurnet: an R package for co-occurrence network construction and analysis. Bioinformatics, 33(12):1881–1882, 2017.
Appendix

Proof of Theorem 8

**Proof.** We have a maximal gap for every subset of $I$ containing $T[i]$. This collection of subsets can be written as

$$C = \{ A \subseteq I | A = \{T[i]\} \cup B, B \subseteq I \setminus \{T[i]\} \}.$$

For each $A \in C$, the update to $H$ is $\{-1\}^{\vert A \vert + 1}$ to $H[j_{\text{tesla}}(r_{1}^{i}(A), i - 1)]$ as we have found an $A$-gap of size $\text{tesla}(r_{1}^{i}(A), i - 1)$ at index $i$. Let $C_{k} = \{ A \in C | \text{tesla}(r_{1}^{i}(A), i - 1) = k \}$. The incremental updates can be expressed by

$$H_{i}[k] = H_{i-1}[k] + \sum_{A \in C_{k}} (-1)^{\vert A \vert + 1}, \quad (22)$$

for each $k$. Suppose $T[i] = r_{j_{0}}^{i}(I)$ where $j_{0} < \vert I \vert$, i.e., $T[i]$ is not the item seen furthest in the past most recently. Then there are $\binom{\vert I \vert - j}{\ell}$ sets $A \in C$ of length $\ell + 1$ in which $T[i] = r_{1}^{i}(A)$. Thus for $k = \text{tesla}(T[i], i - 1)$, we have

$$H_{i}[k] - H_{i-1}[k] = \sum_{A \in C_{k}} (-1)^{\vert A \vert + 1} = \sum_{\ell=0}^{\vert I \vert - j} \binom{\vert I \vert - j}{\ell} (1 - 1)^{\ell - j - 1} = 0. \quad (23)$$

Now for every $j < j_{0}$ (which means that $j$ never equals $\vert I \vert - 1$ in this case), we have that there are $\binom{\vert I \vert - j - 1}{\ell}$ members $A \in C$ of length $\ell + 2$ in which, $r_{1}^{i}(I) = r_{1}^{i}(A)$. Therefore for $k = \text{tesla}(r_{1}^{i}(I), i - 1)$,

$$H_{i}[k] - H_{i-1}[k] = \sum_{A \in C_{k}} (-1)^{\vert A \vert + 1} = \sum_{\ell=0}^{\vert I \vert - j - 1} \binom{\vert I \vert - j - 1}{\ell}(1 - 1)^{(\ell + 1)} = -(-1)^{\vert I \vert - j - 1}. \quad (24)$$

But for $\vert I \vert \geq j > j_{0}$, there are no such sets $A \in C$ in which $r_{1}^{i}(I) = r_{1}^{i}(A)$, as $T[i] = r_{j_{0}}^{i}(I)$ is contained in all $A \in C$.

But if $j_{0} = \vert I \vert$, i.e., $T[i] = r_{\vert I \vert}^{i}(I)$, then for each $j < \vert I \vert - 1$, there are again $\binom{\vert I \vert - j - 1}{\ell}$ members $A \in C$ of length $\ell + 2$ in which, $r_{1}^{i}(I) = r_{1}^{i}(A)$, so again equation $\text{(23)}$ holds for $k = \text{tesla}(r_{1}^{i}(I), i - 1)$, giving no net updates for such $k$. But there is exactly one $A \in C$ in which $r_{1}^{i}(I) = r_{1}^{i}(A)$, namely, $r_{1}^{i}(I), T[i]$, and there is also one $A \in C$ in which $T[i] = r_{1}^{i}(I) = r_{1}^{i}(A)$, which is $T[i]$. Therefore we have

$$H_{i}[\text{tesla}(r_{1}^{i}(I), i - 1)] = H_{i-1}[\text{tesla}(r_{1}^{i}(I), i - 1)] + 1,$$

$$H_{i}[\text{tesla}(r_{1}^{i}(I), i - 1)] = H_{i-1}[\text{tesla}(r_{1}^{i}(I), i - 1)] - 1.$$ 

\[\square\]

Proof of Theorem 11

**Proof.** Suppose without loss of generality that $X_{i} \subseteq I$. We wish to find $H_{i} - H_{i-1}$. Denote

$$X_{i} = \{r_{1}^{i}(X_{1}), r_{2}^{i}(X_{1}), \ldots, r_{\vert X_{i} \vert}^{i}(X_{i})\} = \{x_{1}, x_{2}, \ldots, x_{\vert X_{i} \vert}\},$$

as $r$ defined before. Now let

$$U_{i} = \{ A \subseteq I : A = B \cup \{ x_{j} \}, B \subseteq I \setminus \{ x_{j} \}, j \in \{ \vert X_{i} \vert \} \}.$$
which in words, is all subsets of $I$ that contain at least one member of $X_i$. Observe that

$$U_i = \bigcup_{j=1}^{|X_i|} \{ A \subseteq I : A = B \cup \{ x_j \}, B \subseteq I \setminus \{ x_j \} \}.$$ 

Now let $K_j = \{ A \subseteq I : A = B \cup \{ x_j \}, B \subseteq I \setminus \{ x_j \} \}$ for all $j$. Therefore $U_i = \bigcup_{j=1}^{|X_i|} K_j$.

The update rule is known for each $K_j$ based on our previous result. The remains the of the proof is as follows. We can leverage the update rule currently known to compute the total update. But the intersection of $K_j$’s is non-empty, meaning if we update according to each $K_j$, we would be overcounting some members of $U$. Once this is determined, we will find the update rule according for each arbitrary intersection of these $K_j$’s, which completes the proof.

Define $\langle \cdot \rangle_j$ to be a mapping from subsets of $I$ to an integer valued $n$ dimensional vector. $\langle A \rangle_j^k$ is the sum of the number of maximal gaps of length $k$ ending at index $i$ given by even subsets of $A$, minus the sum of the number of maximal gaps of length $k$ ending at index $i$ given by the odd subsets of $A$. Using this new definition, $\langle U_i \rangle_j^k = H_i[k] - H_{i-1}[k]$. We can now appeal to the inclusion exclusion principle to write that

$$H_i - H_{i-1} = \langle U_i \rangle = \left( \bigcup_{j=1}^{|X_i|} K_j \right) = \sum_{J \subseteq [|X_i|]} (-1)^{|J|+1} \left( \bigcap_{j \in J} K_j \right). \quad (24)$$

The right hand side of the above equality will now be used.

Denote for any $J \subseteq [|X_i|]$,

$$K_J = \bigcap_{j \in J} K_j,$$

Let $X_J$ be the set of members of $X_i$ that lie in every member of $K_J$. Observe that

$$X_J = \bigcap_{G \in K_J} G.$$

It also follows that $X_J = \bigcup_{j \in J} \{ x_j \}$. Moreover, we can write

$$K_J = \{ A \subseteq I : A = X_J \cup B, B \subseteq I \setminus X_J \}.$$ 

For each set $A \in K_J$, there is a corresponding set $A'$ in $K_J' = \{ A \subseteq I : A = \{ r_1^* (X_J) \} \cup B, B \subseteq I \setminus X_J \}$, in which $\langle A \rangle = (-1)^{|J|+1} \langle A' \rangle$. This correspondence is easy to find. Let $A \in K_J$. Thus $A = X_J \cup B$, for some $B \in I \setminus X_J$. Then the corresponding set $A' \in K_J'$ is $\{ r_1^* (X_J) \} \cup B$. This is clear, because items that lie in every $A \in K_J$ that never satisfy $\arg \min_{e \in A} \text{tesla}(e, i-1)$ for all $A$ never contribute towards any updates and hence can be ignored, except they may change the parity of the set and hence change the sign of the update. From here, we can apply the first theorem taking $I$ in that theorem to be $I \setminus X_J$, which gives

$$\langle K_J \rangle_i^k = (-1)^{|J|+1} \langle K'_J \rangle_i^k = \begin{cases} (-1)^{|J|+1} & \text{1 for } k = \text{tesla}(r_i^1|X_J|(I \setminus X_J) \cup \{ x^* \}), i - 1 \\ -1 & \text{for } k = \text{tesla}(r_i^1(X_J), i - 1) \\ 0 & \text{otherwise,} \end{cases} \quad (25)$$
when \( r^i_1(X_J) = r^i_1[X_J] + 1((I \setminus X_J) \cup \{r^1_1(X_J)\}) \). Every update is 0 otherwise.

Now assume for all \( x \in X_i \) and \( e \in I \setminus X_i \), \( \text{tesla}(x, i - 1) \geq \text{tesla}(e, i - 1) \). For if this does not hold for some \( x' \in X_i \), then by the above, no updates occur due to \( x' \), so analysis is the same.

We now wish to compute the right hand side of equation (24). We can employ equation (25) for each \( K_j \). If \( r^1_1(X_J) \neq r^1_1[X_J] + 1((I \setminus X_J) \cup \{r^1_1(X_J)\}) \), that is, the first ranked item of \( X_J \) is not ranked below all of \( I \setminus X_J \), then \( \langle K_j \rangle = 0 \). We claim that the \( J \) in which \( \langle K_j \rangle \neq 0 \) are of the following form:

\[
J_m = \{X_i | b : b \in [m]\},
\]

for \( 0 \leq m < |X_i| \). We first show that if \( J \neq J_m \) for some \( m \), then \( \langle K_j \rangle = 0 \). If \( J \neq J_m \) for some \( m \), then there exists \( b_0 \) such that \( r^{i[X_i] - b_0}_i(X_i) \notin X_J \), and there is some \( b_1 \) such that \( b_1 > b_0 \) and \( r^{i[X_i] - b_1}_i(X_i) \notin X_J \). Since \( b_1 \leq |X_i| - 1 \), \( b_0 < |X_i| - 1 \) which gives that \( |X_i| - b_0 > 1 \). Let \( c_0 \) and \( c_1 \) be such that \( r^{i[X_i] - b_0}_i(X_i) = r^{i[X_i] - b_1}_i(X_i) \) and \( r^{i[X_i] - b_1}_i(X_i) \). We have that \( c_0 = c_1 \). Now since \( c_0 \leq |I \setminus X_J| + 1 \), \( c_1 \neq |I \setminus X_J| + 1 \). Therefore \( r^i_1(X_J) \neq r^i_1[X_J] + 1((I \setminus X_J) \cup \{r^1_1(X_J)\}) \), hence \( \langle K_j \rangle = 0 \).

Now suppose that \( J = J_m \) for some \( m \). Let \( c_1 \) be such that \( r^{i_1}(I) = r^{i_1}(X_J) \). We then have that for any \( c < c_1 \), \( c \in J \), moreover, \( r^i(I) \neq r^i_1(I) \cup \{r^1_1(X_J)\} \). Thus \( r^i_1(X_J) = r^i_1[X_J] + 1((I \setminus X_J) \cup \{r^1_1(X_J)\}) \), for not, then there would be \( c_0 > c_1 \) in which \( r^{i_1}(I) \in (I \setminus X_J) \cup \{r^1_1(X_J)\} \), a contradiction.

From this, the right hand side of equation (24) becomes

\[
\sum_{J \subseteq |X_i|} (-1)^{|J|+1} \langle K_j \rangle = \sum_{m=0}^{|X_i|-1} (-1)^m \langle K_{J_m} \rangle.
\]  

(27)

We now have for \( J = J_m \), \( r^i_1(X_J) = r^i_1[X_i] - m(X_i) \). Also when \( m < |X_i| - 1 \), we have that

\[
r^i_1[X_i]((I \setminus X_J) \cup \{r^1_1(X_J)\}) = r^i_1[X_i]((I \setminus X_J) \cup \{r^1_1(X_J)\}) = r^i_1[X_i]((I \setminus X_J) \cup \{r^1_1(X_J)\}).
\]

(28)

But when \( m = |X_i| - 1 \), \( J = |X_i| \), therefore

\[
r^i_1[X_i]((I \setminus X_J) \cup \{r^1_1(X_J)\}) = r^i_1[X_i]((I \setminus X_J) \cup \{r^1_1(X_J)\}) = r^i_1[X_i]((I \setminus X_J) \cup \{r^1_1(X_J)\}).
\]

(29)

Now for \( m < |X_i| - 1 \), we can rewrite equation (24) to get

\[
\langle K_{J_m} \rangle = \begin{cases} 
1 & \text{for } k = \text{tesla}(r^i_1[X_i] - (m+1)(X_i), i - 1) \\
-1 & \text{for } k = \text{tesla}(r^i_1[X_i] - m(X_i), i - 1) \\
0 & \text{otherwise}.
\end{cases}
\]

(30)

Now define

\[
u(m)_i^k = \begin{cases} 
1 & \text{for } k = \text{tesla}(r^i_1[X_i] - (m+1)(X_i), i - 1) \\
-1 & \text{for } k = \text{tesla}(r^i_1[X_i] - m(X_i), i - 1) \\
0 & \text{otherwise}.
\end{cases}
\]

(31)

for \( m < |X_i| - 1 \) and

\[
u(|X_i| - 1)_i^k = \begin{cases} 
1 & \text{for } k = \text{tesla}(r^i_1[X_i] - (m+1)(X_i), i - 1) \\
-1 & \text{for } k = \text{tesla}(r^i_1[X_i] - m(X_i), i - 1) \\
0 & \text{otherwise}.
\end{cases}
\]

(32)
Taking, \( u(m)_i = (u_1^1(m), u_2^1(m), \ldots, u_n^1(m)) \), we can write

\[
\sum_{m=0}^{|X_i|-1} (-1)^m \langle K_j m \rangle = \sum_{m=0}^{|X_i|-1} (-1)^m (-1)^m u(m)_i = \sum_{m=0}^{|X_i|-1} u(m)_i^k.
\] (33)

Observe that

\[
u(0)^k_i + u(1)^k_i = \begin{cases} 
1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
-1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
0 & \text{otherwise}
\end{cases}
\] (34)

Applying this for all \( m < |X_i| - 1 \) gives

\[
\sum_{m=0}^{|X_i|-2} u(m)_i^k = \begin{cases} 
1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
-1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
0 & \text{otherwise}
\end{cases}
\] (35)

So combining this with \( u(|X_i| - 1)_i^k \), we get

\[
\sum_{m=0}^{|X_i|-1} u(m)_i^k = \begin{cases} 
1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
-1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
0 & \text{otherwise}
\end{cases}
\] (36)

since \( r_{i}^{X_i}|_{X_i}(X_i) = r_{i}^{X_i}|_{I}(I) \). Combining equation (36) with equations (33), (27), and (24) (and considering the components of each of those equations), we finally get,

\[
H_i[k] - H_{i-1}[k] = \begin{cases} 
1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{X_i}, i - 1) \\
-1 & \text{for } k = \text{tesla}(r_{i}^{X_i}|_{I}(I), i - 1) \\
0 & \text{otherwise}
\end{cases}
\] (37)

proving the result (\( X_i = A \) in the statement of the theorem).

\[\] ▶

**Algorithms**

Here we place pseudocode for the preliminary algorithms mentioned in the main text.
Here we present the pseudocode for the pattern co-occurrence analysis.
Algorithm 4 PAWLCO

Input: Trace $T$, ItemSet $I$

Result: Co-occurrence of all window lengths

1. $H \leftarrow$ empty histogram, $cooc \leftarrow \emptyset$, $S \leftarrow$ empty book-stack
2. for $e \in I$ do
   3. $S += (e, -\infty)$
   4. end
5. $p \leftarrow |I|$, $m \leftarrow 1$
6. while $S$.retrieve($|I|$) = $S$.retrieve($|I| - m$) do
   7. $m \leftarrow m + 1$
   8. end
9. $p \leftarrow |I| - m$
10. for $i = 0$ to $n - 1$ do
11.   $C \leftarrow \{e \in I | e[0] = T[i - |e| + 1]...e[|e| - 1] = T[i]\}$
12.   $\text{min} \leftarrow i$ for $e \in C$ do
13.     if $S$.find($e$) < $\text{min}$ then
14.        $\text{min} \leftarrow S$.find($e$)
15.     end
16. end
17. if $\text{min} > S$.retrieve($p + 1$) then
18.   $f \leftarrow i - S$.retrieve($|I|$), $s \leftarrow i - S$.retrieve($p + 1$)
19.   $H[f] \leftarrow H[f] + 1$, $H[s] \leftarrow H[s] - 1$
20. end
21. for $\text{current} \in C$ do
22.   $j \leftarrow S$.find($\text{current}$)
23.   $S$.update($j$)
24.   // Maintain partition
25.   if $j = p = |I|$ then
26.     $m \leftarrow 2$
27.     while $S$.retrieve($|I| - 1$) = $S$.retrieve($|I| - m$) do
28.        $m \leftarrow m + 1$
29.     end
30.   end
31.  if $p \leq j < |I|$ then
32.   $p \leftarrow p + 1$
33. end
34. end
35. // Final gap from bottom of book-stack
36. $f \leftarrow i - S$.retrieve($|I|$)
37. $H[f] \leftarrow H[f] + 1$
38. for $x = 0$ to $|T| - |I| + 1$ do
39.   $S_x \leftarrow 0$
40.   for $k = x$ to $|T|$ do
41.     $S_x \leftarrow S_x + (k - x + 1)H[k]$
42.   end
43.   $cooc[x] \leftarrow (|T| - x + 1) - S_x$
44. end
45. return $cooc$