ON REALIZABILITY OF GAUSS DIAGRAMS

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ABSTRACT. The problem of which Gauss diagram can be realized by knots is an old one and has been solved in several ways. In this paper, we present a direct approach to this problem. We show that the needed conditions for realizability of a Gauss diagram can be interpreted as follows “the number of exits = the number of entrances” and the sufficient condition is based on Jordan curve Theorem.

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INTRODUCTION

In the earliest time of the Knot Theory C.F. Gauss defined the chord diagram (= Gauss diagram). Gauss observed that if a chord diagram can be realized by a plane curve, then every chord is crossed only by an even number of chords, but that this condition is not sufficient.

The aim of this paper is to present a direct approach to the problem of which Gauss diagram can be realized by knots. This problem is an old one, and has been solved in several ways.

In 1936, M. Dehn found a sufficient algorithmic solution based on the existence of a touch Jordan curve which is the image of a transformation of the knot diagram by successive splits replacing all the crossings. A long time after in 1976, L. Lovasz and M.L. Marx found a second necessary condition and finally during the same year, R.C. Read and P. Rosenstiehl found the third condition which allowed the set of these three conditions to be sufficient. The last characterization is based on the tripartition of graphs into cycles, cocycles and bicycles.

In the notation of oriented chord diagram was introduced and it was showed that these diagrams classify cellular generic curves on oriented surfaces. As a corollary a simple combinatorial classification of plane generic curves was derived, and the problem of realizability of these diagrams was also solved.

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However all these ways are indirect; they rest upon deep and nontrivial auxiliary construction. There is a natural question: whether one can arrive at these conditions in a more direct and natural fashion?

We believe that the conditions for realizability of a Gauss diagram (by some plane curve) should be obtained in a natural manner; they should be deduced from an intrinsic structure of the curve.

In this paper, we suggest an approach, which satisfies the above principle. We use the fact that every Gauss diagram $G$ defines a (virtual) plane curve $C(G)$ (see [4, Theorem 1.A]), and the following simple ideas:

1. For every chord of a Gauss diagram $G$, we can associate a closed path along the curve $C(G)$.

2. For every two non-intersecting chords of a Gauss diagram $G$, we can associate two closed paths along the curve $C(G)$ such that every chord crosses both of those chords correspondences to the point of intersection of the paths.

3. If a Gauss diagram $G$ is realizable (say by a plane curve $C(G)$), then for every closed path (say) $P$ along $C(G)$ we can associate a coloring another part of $C(G)$ into two colors (roughly speaking we get “inner” and “outer” sides of $P$ cf. Jordan curve Theorem). If a Gauss diagram is not realizable then ([4, Theorem 1.A]) it defines a virtual plane curve $C(G)$. We shall show that there exists a closed path along $C(G)$ for which we cannot associate a well-defined coloring of $C(G)$, i.e., $C(G)$ contains a path is colored into two colors.

Using these ideas we solve the problem of which Gauss diagram can be realized by knots.

1. Preliminaries

Recall that classically, a knot is defined as an embedding of the circle $S^1$ into $\mathbb{R}^3$, or equivalently into the 3-sphere $S^3$, i.e., a knot is a closed curve embedded on $\mathbb{R}^3$ (or $S^3$) without intersecting itself, up to ambient isotopy.

The projection of a knot onto a 2-manifold is considered with all multiple points are transversal double with will be call crossing points (or shortly crossings). Such a projection is called the shadow by the knots theorists [1, 9], following [8] we shall also call these projections as plane curves. A knot diagram is a generic immersion of a circle $S^1$ to a plane $\mathbb{R}^2$ enhanced by information on overpasses and underpasses at double points.
1.1. **Gauss Diagrams.** A generic immersion of a circle to a plane is characterized by its Gauss diagram [6].

**Definition 1.1.** *The Gauss diagram* is the immersing circle with the preimages of each double point connected with a chord.

On the other words, this notation can be defined as follows. Let us walk on a path along the plane curve until returning back to the origin and then generate a word $W$ which is the sequence of the crossings in the order we meet them on the path. $W$ is a double occurrence word. If we put the labels of the crossing on a circle in the order of the word $W$ and if we join by a chord all pairs of identical labels then we obtain a chord diagram (=Gauss diagram) of the plane curve (see **Figure 1**).

A *virtual knot diagram* [4] is a generic immersion of the circle into the plane, with double points divided into real crossing points and virtual crossing points, with the real crossing points enhanced by information on overpasses and underpasses (as for classical knot diagrams). At a virtual crossing the branches are not divided into an overpass and an underpass. The Gauss diagram of a virtual knot is constructed in the same way as for a classical knot, but all virtual crossings are disregarded.

**Theorem 1.2.** [4 Theorem 1.A] *A Gauss diagram defines a virtual knot diagram up to virtual moves.*

Arguing similarly as in the real knot case, one can define a *shadow of the virtual knot* (see **Figure 2**).
1.2. **Conway’s Smoothing.** We frequently use the following notations. Let $K$ be a knot, $C$ its shadow and $\mathcal{G}$ the Gauss diagram of $C$. For every crossing $c$ of $C$ we denote by $c$ the corresponding chord of $\mathcal{G}$.

If a Gauss diagram $\mathcal{G}$ contains a chord $c$ then we write $c \in \mathcal{G}$. We denote by $c_0, c_1$ the endpoints of every chord $c \in \mathcal{G}$. We shall also consider every chord $c \in \mathcal{G}$ together with one of two arcs are between its endpoints, and a chosen arc is denoted by $c_0c_1$.

Further, $c_\times$ denotes the set of all chords cross the chord $c$ and $c_\parallel$ denotes the set of all chords do not cross the chord $c$. We put $c \notin c_\times$ and $c \in c_\parallel$.

Throughout this paper we consider Gauss diagrams such that $c_\times \neq \emptyset$ for every $c \in \mathcal{G}$.

As well known, John Conway introduced a “surgical” operation on knots, called smoothing, consists in eliminating the crossing by interchanging the strands (Figure 3).

![Figure 3](image)

We aim to specialize a Conway smoothing a crossing of a plane curve to an operation on chords of the corresponding Gauss diagram.
Let $K$ be a knot, $C$ its shadow, and $G$ the Gauss diagram of $C$. Take a crossing point $c$ of $C$ and let $D_c$ be a small disk centered at $c$ such that $D_c \cap C$ does not contain another crossings of $C$. Denote by $\partial D_c$ the boundary of $D_c$. Starting from $c$, let us walk on a path along the curve $C$ until returning back to $c$. Denote this path by $L_c$ and let $c_l, c_z$ be the sequence of the points in the order we meet them on $L_c$, where $\{c_l, c_z\} = L_c \cap \partial D_c$. After returning back to $c$ let us keep walking along the curve $C$ in the same direction as before until returning back to $c$. Denote the corresponding path by $R_c$ and let $c_r, c_a$ be the sequence of the points in the order we meet them on $R_c$, where $\{c_r, c_a\} = \partial D_c \cap R_c$.

Next, let us delete the inner side of $D_c \cap C$ and attach $c_l$ to $c_a$, and $c_z$ to $c_r$. We thus get the new plane curve $\hat{C}_c$ (see Figure 4). It is easy to see that this curve is the shadow of the knot, which is obtained from $K$ by

![Figure 4](image-url)
Conway's smoothing the crossing \( c \). Let \( \hat{\mathcal{G}}_c \) be the Gauss diagram of \( \hat{\mathcal{G}}_c \). We shall say that the Gauss diagram \( \hat{\mathcal{G}}_c \) is obtained from the Gauss diagram \( \mathcal{G} \) by Conway's smoothing the chord \( c \).

As an immediate consequence of the preceding discussion, we get the following proposition.

**Proposition 1.1.** Let \( \mathcal{G} \) be a Gauss diagram and \( c \) be its arbitrary chord. Then \( \hat{\mathcal{G}}_c \) is obtained from \( \mathcal{G} \) as follows: (1) delete the chord \( c \), (2) if two chords \( a, b \in c_x \) intersect (resp. do not intersected) in \( \mathcal{G} \) then they do not intersect in \( \hat{\mathcal{G}}_c \) (resp. intersected), (3) another chords keep their positions.

**Proof.** Indeed, let \( W \) be the word which is the sequence of the crossings in the order we meet them on the curve \( \mathcal{C} \). Since \( c_x \neq \emptyset \), \( W \) can be written as follows \( W = W_1cW_2cW_3 \), where \( W_1, W_2, W_3 \) are subwords of \( W \) and at least one of \( W_1, W_3 \) is not empty. Define \( W_2^R \) as the reversal of the word \( W_2 \). Then, from the preceding discussion, the word \( \hat{W}_c := W_1W_2^RW_3 \) gives \( \hat{\mathcal{G}}_c \) (see Figure 4) and the statement follows.

\[ \square \]

2. **Partitions of Gauss Diagrams**

In this section we introduce notations, whose importance will become clear as we proceed.

**Definition 2.1.** Let \( \mathcal{G} \) be a Gauss diagram and \( a \) a chord of \( \mathcal{G} \). A \( C \)-contour, denoted \( C(a) \), consists of the chord \( a \), a chosen arc \( a_0a_1 \), and all chords of \( \mathcal{G} \) such that all their endpoints lie on the arc \( a_0a_1 \). We call a chord from the set \( a_x \) the door chord of the \( C \)-contour \( C(a) \).

Let us consider a plane curve \( \mathcal{C} : S^1 \to \mathbb{R}^2 \) and let \( \mathcal{G} \) be its Gauss diagram. Every chord \( c \in \mathcal{G} \) corresponds to the crossing \( c \) of \( \mathcal{C} \). Thus for every \( C \)-contour \( C(c) \), we can associate a closed path \( \mathcal{C}(c) \) along the curve \( \mathcal{C} \). We call \( \mathcal{C}(c) \) the loop of the curve \( \mathcal{C} \). It is obviously that there is the one-to-one correspondence between self-intersection points of \( \mathcal{C}(c) \) and all chords from \( C(c) \).

**Example 2.2.** In Figure 5 the plane curve \( \mathcal{C} \) and its Gauss diagram are shown. Consider the (cyan) \( C \)-contour \( C(5) \). We see that \( \mathcal{C}(5) \) is the closed path along the curve. It is the self-intersecting path and we see that the crossing 6 corresponds to the chord from the set 5\( || \). Further, the red closed path \( \mathcal{C}(0) \) corresponds to the red \( C \)-contour \( C(0) \).

**Definition 2.3.** Let \( \mathcal{G} \) be a Gauss diagram, \( a, b \) its intersecting chords. An \( X \)-contour, denoted \( X(a, b) \), consists of two non-intersecting arcs \( a_0b_0, a_1b_1 \) and all chords of \( \mathcal{G} \) such that all their endpoints lie on \( a_0b_0 \) or on \( a_1b_1 \). A chord is called the door chord of the \( X \)-contour \( X(a, b) \) if only one
Figure 5. Every $C$-contour of the Gauss diagram corresponds to the closed path along the plane curve and vice versa. We see that the chord 6 corresponds to the self-intersection point 6 of the cyan loop.

of its endpoints belongs to $X(a, b)$. We say that the $X$-contour $X(a, b)$ is non-degenerate if it has at least one door chord, and it does not contain all chords of $\mathcal{G}$.

Example 2.4. Let us consider the Gauss diagram in Figure 6. The green chords are the door chords of the orange $X$-contour $X(1, 3)$. We see that the door chords correspondence to “entrances” and “exits” of the orange closed path along the curve. We also see that this $X$-contour is non-degenerate.

The previous Example implies a partition of a Gauss diagram (resp. a plane curve) into two parts.

Definition 2.5 (An $X$-contour coloring). Given a Gauss diagram $\mathcal{G}$ and its an $X$-contour. Let us walk along the circle of $\mathcal{G}$ in a chosen direction and color all arcs of $\mathcal{G}$ until returning back to the origin as follows: (1) we don’t colors the arcs of the $X$-contour, (2) we use only two different colors, (3) we change a color whenever we meet an endpoint of a door chord.

Similarly, one can define a $C$-contour coloring of a Gauss diagram $\mathcal{G}$.

Remark 2.6. Let $\mathcal{G}$ be a Gauss diagram and $\mathcal{C}$ the corresponding (may be virtual) plane curve, i.e., $\mathcal{G}$ determines the curve $\mathcal{C}$. For every $X$-contour $X(a, b)$ in $\mathcal{G}$, we can associate the closed path along the curve $\mathcal{C}$. We call this path the $X$-contour and denote by $\mathcal{X}(a, b)$. Similarly one can define door crossing for $\mathcal{X}(a, b)$. 

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Further, for the $X(a, b)$-contour coloring of $G$, we can associate $X'(a, b)$-contour coloring of the curve $C$.

Next, let $G$ be a realizable Gauss diagram determines the plane curve $C$ and let $X(a, b)$ be an $X$-contour of $G$ such that $X'(a, b)$ is the non-self-intersecting path (= the Jordan curve). Then the $X'(a, b)$-contour coloring of $C$ divides the curve $C$ into two colored parts, cf. Jordan curve Theorem.

![Figure 6](image.png)

Figure 6. For the $X(1, 3)$-contour coloring of the Gauss diagram, we associate the plane curve coloring. We see that the $X'$-contour $X'(1, 3)$ (= orange loop) divides the plane curve into two colored parts.

3. The Even and The Sufficient Conditions

If a Gauss diagram can be realized by a plane curve we then say that this Gauss diagram is realizable, and non-realizable otherwise. So, in this section, we give a criterion allowing verification and comprehension of whether a given Gauss diagram is realizable or not. Moreover, we give an explanation allowing comprehension of why the needed condition is not sufficient for realizability of Gauss diagrams.

3.1. The Even Condition.

**Proposition 3.1.** Let $C : S^1 \to \mathbb{R}^2$ be a plane curve and $G$ its Gauss diagram. Then

1. $|a \times b| \equiv 0 \text{ mod } 2$ for every two non-interesting chords $a, b \in G$,

2. $|c_x| \equiv 0 \text{ mod } 2$ for every chord $c \in G$. 
Proof. Let \( a, b \in \mathcal{G} \) be two non-intersecting chords of \( \mathcal{G} \). Take two \( C \)-contours \( C(a) \), \( C(b) \) such that their arcs \( a_0a_1 \), \( b_0b_1 \) do not intersect. It is obvious that for the loops \( C(a) \), \( C(b) \), we can associate the one-to-one correspondence between the set \( C(a) \cap C(b) \) and the set \( a \times b \).

Because, by Proposition 1.1, all chord from the set \( a \times b \) keep their positions in Gauss diagram \( \hat{\mathcal{G}} \) (= Conway’s smoothing the chord \( c \)) for every \( c \in a \parallel b \ \{ a, b \} \), it is sufficient to prove the statement in the case \( a \parallel b \ \{ a, b \} = \{ a, b \} \), i.e., the loops \( C(a) \), \( C(b) \) are non-self-intersecting loops (= the Jordan curves).

From Jordan curve Theorem, it follows that the loop \( C(a) \) divides the curve \( C \) into two regions, say, \( I \) and \( O \). Assume that \( b \in O \) and let us walk along the loop \( C(b) \). We say that an intersection point \( c \in C(a) \cap C(b) \) is the entrance (resp. the exit) if we shall be in the region \( I \) (resp. \( O \)) after meeting \( c \) with respect to our walk. Since a number of entrances has to be equal to the number of exits, then \( |a \times b| \equiv 0 \bmod 2 \). Arguing similarly, we prove \( |c| \equiv 0 \bmod 2 \) for every chord \( c \). □

As an immediate consequence of Proposition 3.1 we get the following.

**Corollary 3.1 (The Even Condition).** If a Gauss diagram is realizable then the number of all chords that cross a both of non-intersecting chords and every chord is even (including zero).

We conclude this subsection with an explanation why the even condition is not sufficient for realizability of Gauss diagrams.

Roughly speaking, from the proof of Proposition 3.1 it follows that every plane curve can be obtained by attaching its loops to each other by given points. Conversely, if a Gauss diagram satisfies the even condition then it may be non-releasable. Indeed, when we attach a loop, say, \( C(b) \) to a loop \( C(a) \), where \( b \in a \parallel b \), by given points (=elements of the set \( a \times b \)) then the loop \( C(b) \) can be self-intersected curve, which means that we get new crossings (= virtual crossings), see Figure 7.

To be more precisely, we have the following proposition.

**Proposition 3.2.** Let \( \mathcal{G} \) be a non-realisable Gauss diagram which satisfies the even condition. Let \( \mathcal{G} \) defines a virtual plane curve \( C \) (up to virtual moves). There exist two non-intersecting chords \( a, b \in \mathcal{G} \) such that there are paths \( c \to x \to d, e \to x \to f \) on a loop \( C(b) \), where \( c, d, e, f \in a \times b \) are different chords and \( x \) is a virtual crossing of \( C \).

**Proof.** Let \( a, b \in \mathcal{G} \) be two non-intersecting chords. Take non-intersecting \( C \)-contours \( C(a) \), \( C(b) \). Hence we may say that the loop \( C(b) \) attaches to the loop \( C(a) \) by the given points \( p_1, \ldots, p_n \), where \( \{ p_1, \ldots, p_n \} = a \times b \).
Since $\mathcal{G}$ is not realizable and satisfies the even condition then a virtual crossing may arise only as a self-intersecting point of, say, the loop $\mathcal{C}(b)$. Indeed, when we attach $\mathcal{C}(b)$ to $\mathcal{C}(a)$ by $p_1, \ldots, p_n$ we may get self-interesting points, say, $q_1, \ldots, q_m$ of the loop $\mathcal{C}(b)$. If $\mathcal{G}$ contains all chords $q_1, \ldots, q_m$ for every such chords $a, b$, then $\mathcal{G}$ is realizable. Thus, a virtual crossing $x$ does not belong to $\{p_1, \ldots, p_n\} = \mathcal{C}(a) \cap \mathcal{C}(b)$ for some non-intersecting chords $a, b \in \mathcal{G}$. Then we get two paths $c \to x \to d, e \to x \to f$, where $c, d, e, f \in \mathcal{C}(a) \cap \mathcal{C}(b)$ are different chords, as claimed.

\[ \square \]

**Figure 7.** It shows that the even condition is not sufficient for realizability of Gauss diagrams. We see that the plane curve can be obtained by attaching cyan loop to olive loop by the points 3, 4, 5, 6, and thus the olive loop has to have “new” crossings (= self-intersections) $x_1, x_2, x_3$. 

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3.2. The Sufficient Condition.

**Definition 3.1.** Let $G$ be a Gauss diagram (not necessarily realizable) and $X(a, b)$ its $X$-contour. Take the $X$-contour coloring of $G$. A chord of $G$ is called *colorful for $X(a, b)$* if its endpoints are in arcs which have different colors.

Similarly, one can define *a colorful chord for a $C$-contour $C(a)$* of $G$.

**Example 3.2.** Let us consider the Gauss diagram, which is shown in Figure 8. One can easy check that this Gauss diagram is not realizable. Let us consider the orange $X$-contour $X(1, 3)$ and the $X(1, 3)$-coloring of $G$. The chord with the endpoints 5 is colorful for the $X$-contour $X(1, 3)$. It is interesting to consider the corresponding coloring of the virtual plane curve: one can think that we forget to change color when we cross the orange loop, *i.e.*, the orange loop “does not divide” the curve into two parts. We shall show that this observation is typical for every non-realizable Gauss diagram.

![Figure 8](image_url)

**Figure 8.** This Gauss diagram satisfies even condition but is non-realizable. There are colorful chords (e.g. the chord with endpoints 5).

We have seen that if a Gauss diagram is realizable then there is no colorful chord, with respect to every $X$-contour. We shall show that it is sufficient condition for realizability of a Gauss diagram.

**Proposition 3.3.** Let $G$ be a non-realizable Gauss diagram but satisfy the even condition. Then there exists an $X$-contour and a colorful chord for this $X$-contour.
Proof. By Theorem 1.2, $\mathcal{G}$ defines a virtual curve $\mathcal{C}$ (= the shadow of a virtual knot diagram) up to virtual moves. Starting from a crossing, say, $o$, let us walk along $\mathcal{C}$ till we meet the first virtual crossing, say, $x$. Denote this path by $\mathcal{P}$. Just for convinces, let us put the labels, say, $x_0, x_1$ of the virtual crossing $x$ on the circle of $\mathcal{G}$ in the order we meet them on $\mathcal{P}$.

From Proposition 3.2 it follows that for a chord $o \in \mathcal{G}$ we can find $n \geq 1$ chords $o_1, \ldots, o^n$ such that, for every $1 \leq i \leq n$, we have: (1) the chords $o, o^i$ do not intersect, (2) the loop $\mathcal{C}(o)$ contains the following paths $a^i \to x \to b^i$, $c^i \to x \to d^i$ where $a^i, b^i, c^i, d^i \in o_x \cap o_\times$ are different chords. Thus, for some $1 \leq i, j \leq n$ we have an arc, say, $a_i b_j$ contains only one of $x_0, x_1$, no endpoints of another chords, and $a^i, b^j \in o_x$. Denote this arc by $a_i b_j$, and assume that $x_0$ lies on $a_i b_j$. We have to consider the following two cases.

(1) The chords $a, b$ do not intersect. Let us walk along the circle of $\mathcal{G}$ in the direction $b_1 \to x_0 \to a_1$ (Figure 9) till we meet the first chord, say, $\tilde{o}$ such that $a, b, o \in \tilde{o}_x$. If we cannot find such chord we then choose the chord $o$, i.e., $\tilde{o} := o$. Take the $X$-contour $X(\tilde{o}, b)$ contains the arc $a_i b_j$. By the direct verification one can easy check that at least one of $c, d$ has to cross the both of $a, b$, i.e., this $X$-contour is non-degenerate.

(2) The chords $a, b$ intersect. Take the $X$-contour $X(a, b)$ which contains the arc $a_i b_j$. It is easy to see that this $X$-contour has at least one real door chord, because its another arc $a_i b_0$ has no virtual crossing thus it has to have at least one real crossing (see Figure 10).

So, we have a non-degenerate $\mathcal{X}$-contour of the curve $\mathcal{C}$ such that $x$ is its virtual door. By the construction, this $\mathcal{X}$-contour is the closed curve without self-intersections. Hence, by Jordan curve Theorem, it divides $\mathcal{P}$ into two parts. Let us color these parts into two different colors. If the
crossing \( x \) would be a real door of this \( \mathcal{X} \)-contour, then by Definition 2.5 we can take such \( \mathcal{X} \)-contour coloring of \( \mathcal{P} \) as before. But since \( x \) is not real door then after meeting it at the second time we do not change the color and thus the next crossing is the intersection point of two lines which have different colors, \( \text{i.e.,} \) the corresponding chord is colorful. This completes the proof.

**Lemma 3.1.** Let \( \mathfrak{G} \) be a Gauss diagram. Consider a \( C(a) \)-contour coloring of \( \mathfrak{G} \) for some chord \( a \in \mathfrak{G} \). If there exists a colorful chord for the \( C \)-contour \( C(a) \) then the diagram \( \mathfrak{G} \) does not satisfy the even condition.

**Proof.** Indeed, let \( b \) be a colorful chord for the \( C \)-contour \( C(a) \). First note that, the chord \( b \) cannot cross \( a \) because otherwise \( b \) should be a door chord of the \( C \)-contour \( C(a) \). Next, if the chord \( b \) is colorful then it crosses an odd number of door chords of \( C(a) \). Hence \( |a_x \cap b_x| \equiv 1 \mod 2 \), as claimed.

**Example 3.3.** Let us consider the Gauss diagram which is shown in Figure 11. We have the orange \( \mathcal{X} \)-contour \( \mathcal{X}(1) \) and the \( \mathcal{X}(1) \)-contour coloring of the Gauss diagram and the corresponding (virtual) curve. We see that there are two chords (namely 5 and 6) which are colorful and \( 5_x \cap 1_x = 6_x \cap 1_x = \{2\} \).

**Lemma 3.2.** Let \( \mathfrak{G} \) be a Gauss diagram and \( a, b \in \mathfrak{G} \) be its intersecting chords. Suppose that there exists a colorful chord \( c \) for an \( X \)-contour \( X(a, b) \). Then there exists a \( C \)-contour of the Gauss diagram \( \widehat{\mathfrak{G}}_b \) (= Conway’s smoothing the chord \( b \)) such that the chord \( c \) is colorful for this \( C \)-contour in \( \widehat{\mathfrak{G}}_b \).

**Proof.** Indeed, consider the Gauss diagram \( \widehat{\mathfrak{G}}_b \). From Proposition 1.1 it follows that after Conway’s smoothing the chord \( b \), the chord \( c \) does not intersect \( a \) and intersects the same door chords of the \( X \)-contour \( X(a, b) \) as in \( \mathfrak{G} \). Further, let us consider the \( C \)-contour \( C(a) \) in \( \widehat{\mathfrak{G}}_b \) such that it
does not contain the chord $c$. By Proposition 1.1, the chord $a$ crosses in $\hat{G}_b$ only the chord that are door chords of the $X$-contour $X(a, b)$. Hence, by Definition 2.5, we may take the $C(a)$-contour coloring of $\hat{G}_b$ such that $c$ is the colorful chord for this $C$-contour.

**Proposition 3.4.** Let a Gauss diagram $G$ satisfy the even condition. $G$ is realizable if and only if for every chord $c \in G$, $\hat{G}_c$ satisfies the even condition.

**Proof.** Indeed, let $G$ be a non-realizable Gauss diagram and let $G$ satisfy the even condition. By Proposition 3.3, there exists a colorful chord (say $c$) for a $X$-counter $X(a, b)$ of $G$. By Lemma 3.2, the chord $c$ is the colorful chord in $\hat{G}_b$. Hence from Lemma 3.1 it follows that $\hat{G}_b$ does not satisfy the even condition, and the statement follows.

We can summarize our results in the following theorem.

**Theorem 3.4.** A Gauss diagram $G$ is realizable if and only if the following conditions hold:

1. the number of all chords that cross a both of non-intersecting chords and every chord is even (including zero),

2. for every chord $c \in G$ the Gauss diagram $\hat{G}_c$ (= Conway’s smoothing the chord $c$) also satisfies the above condition.
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