Exclusive Decays and Lifetime of $B_c$ Meson in QCD sum rules

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We summarize theoretical calculations of $B_c$ decays and lifetime in the framework of QCD sum rules and compare the results with the estimates by the methods of Operator Product Expansion in the inverse heavy quark masses as well as of potential quark models. The agreement of estimates in the various approaches is discussed. The features of $B_c$ decay modes are considered.

PACS numbers: 13.20.Gd,13.25.Gv,11.55.Hx

I. INTRODUCTION

Decays of long-lived heavy meson $B_c$, containing two heavy quarks of different flavors, were considered in the pioneering paper written by Bjorken in 1986 [1]. The report was devoted to the common view onto the decays of hadrons with heavy quarks: the mesons and baryons with a single heavy quark, the $B_c$ meson, the baryons with two and three heavy quarks. A lot of efforts was recently directed to study the long-lived doubly heavy hadrons\(^1\) on the basis of modern understanding of QCD dynamics in the weak decays of heavy flavors in the framework of today approaches\(^2\): the Operator Product Expansion, sum rules of QCD and NRQCD \(^3\), and potential models adjusted due to the data on the description of hadrons with a single heavy quark. Surprisingly, the Bjorken’s estimates of total widths and various branching fractions are close to what is evaluated in a more strict manner. At present we are tending to study some subtle effects caused by the influence of strong forces onto the weak decays of heavy quarks, which determines our understanding a probable fine extraction of CP-violation in the heavy quark sector.

Various hadronic matrix elements enter in the description of weak decays. So, measuring the lifetimes and branching ratios implies the investigation of quark confinement by the strong interactions, which is important in the evaluation of pure quark characteristics: masses and mixing angles in the CKM matrix, all of which enter as constraints on the physics beyond the Standard Model. More collection of hadrons with heavy quarks provides more accuracy and confidence in the understanding of QCD dynamics and isolation of bare quark values.

So, a new lab for such investigations is a doubly heavy long-lived quarkonium $B_c$ recently observed for the first time by the CDF Collaboration \(^4\). The measured $B_c$ lifetime is equal to

$$\tau[B_c] = 0.46^{+0.18}_{-0.16} \pm 0.03 \text{ ps},$$

which is close to the value expected by Bjorken.

The $B_c$ meson allows us to use such advantages like the nonrelativistic motion of $b$ and $c$ quarks similar to what is known in the heavy quarkonia $bb$ and $cc$, and suppression of light degrees of freedom: the quark-ghon sea is small in the heavy quarkonia. These two physical conditions imply two small expansion parameters for $B_c$:

- the relative velocity of quarks $v$,
- the ratio of confinement scale to the heavy quark mass $\Lambda_{QCD}/m_Q$.

The double expansion in $v$ and $1/m_Q$ generalizes the HQET approach \(^7\) to what is called NRQCD \(^3\). Moreover, the energy release in the heavy quark decays determines the $1/m_Q$ parameter to be the appropriate quantity for the Operator Product Expansion (OPE) and justifies the use of potential models (PM) in the calculations of hadronic matrix elements, too. The same arguments ensure the applicability of sum rules (SR) of QCD and NRQCD.

The $B_c$ decays were, at first, calculated in the PM \(^8\) \(^9\) \(^10\) \(^11\) \(^12\) \(^13\) \(^14\) \(^15\) \(^16\) \(^17\), wherein the variation of techniques results in close estimates after the adjustment on the semileptonic decays of $B$ mesons. The OPE evaluation of inclusive decays gave the lifetime and widths \(^18\), which agree with PM, if one sums up the dominating exclusive modes. That was quite unexpected, when the SR of QCD resulted in the semileptonic $B_c$ widths \(^19\), which are one order of magnitude less than those of PM and OPE. The reason was the valuable role of Coulomb corrections, that implies the summation of $\alpha_s/v$ corrections significant in the heavy quarkonia, i.e. in the $B_c$ \(^10\) \(^21\) \(^22\). At present, all of mentioned approaches give the close results for the lifetime and decay modes of $B_c$ at similar sets of parameters. Nevertheless, various dynamical questions remain open:

- What is the appropriate normalization point of non-leptonic weak lagrangian in the $B_c$ decays, which basically determines its lifetime?
• What are the values of masses for the charmed and beauty quarks?
• What are the implications of NRQCD symmetries for the form factors of $B_c$ decays and mode widths?
• How consistent is our understanding of hadronic matrix elements, characterizing the $B_c$ decays, with the data on the other heavy hadrons?

In the present work we shortly review the $B_c$ decays by summarizing the theoretical predictions including new calculations on the exclusive decays in the framework of QCD sum rules and discuss how direct experimental measurements can answer the questions above.

In Section II we recollect the inclusive estimates of $B_c$ lifetime in various techniques: the Operator Product Expansion combined with the effective theory of heavy quarks, the Potential Model approach and the QCD sum rules. Section III is devoted to the application of QCD sum rules to the exclusive decays of $B_c$ meson, where the estimates of form factors are given. The spin symmetry in the $B_c$ decays is discussed and tested in Section IV.

Our numerical estimates of decay rates are presented in Sections V and VI for the semileptonic and non-leptonic modes, respectively. We summarize our results in section VII.

II. $B_c$ LIFETIME AND INCLUSIVE DECAY RATES

The $B_c$-meson decay processes can be subdivided into three classes:

1) the $\bar{b}$-quark decay with the spectator $c$-quark,
2) the $c$-quark decay with the spectator $\bar{b}$-quark and
3) the annihilation channel $B_c^+ \to l^+\nu_l(c\bar{s}, us)$, where $l = e, \mu, \tau$.

In the $\bar{b} \to c\bar{s}$ decays one separates also the Pauli interference with the $c$-quark from the initial state. In accordance with the given classification, the total width is the sum over the partial widths

$$\Gamma(B_c \to X) = \Gamma(b \to X) + \Gamma(c \to X) + \Gamma(\text{ann.}) + \Gamma(\text{PI}).$$

For the annihilation channel the $\Gamma(\text{ann.})$ width can be reliably estimated in the framework of inclusive approach, where one takes the sum of the leptonic and quark decay modes with account for the hard gluon corrections to the effective four-quark interaction of weak currents. These corrections result in the factor of $a_1 = 1.22 \pm 0.04$. The width is expressed through the lepton constant of $f_{B_c} \approx 400$ MeV. This estimate of the quark-contribution does not depend on a hadronization model, since a large energy release of the order of the meson mass takes place.

From the following expression, one can see that the contribution by light leptons and quarks can be neglected,

$$\Gamma(\text{ann.}) = \sum_{i=\tau, e} \frac{G_F^2}{8\pi} |V_{bc}|^2 f_{B_c}^2 M_{B_c}^2 (1 - m_i^2 / m_{B_c}^2)^2 \cdot C_i,$$

where $C_\tau = 1$ for the $\tau^+\nu_\tau$-channel and $C_e = 3|V_{cs}|^2 a_1^2$ for the $c\bar{s}$-channel.

As for the non-annihilation decays, in the approach of the Operator Product Expansion for the quark currents of weak decays $^{18}$, one takes into account the $\alpha_s$-corrections to the free quark decays and uses the quark-hadron duality for the final states. Then one considers the matrix element for the transition operator over the bound meson state. The latter allows one also to take into account the effects caused by the motion and virtuality of decaying quark inside the meson because of the interaction with the spectator. In this way the $\bar{b} \to c\bar{s}$ decay mode turns out to be suppressed almost completely due to the Pauli interference with the charm quark from the initial state. Besides, the $c$-quark decays with the spectator $\bar{b}$-quark are essentially suppressed in comparison with the free quark decays because of a large bound energy in the initial state.

TABLE I: The branching ratios of the $B_c$ decay modes calculated in the framework of inclusive OPE approach, by summing up the exclusive modes in the potential model $^{11,13}$ and according to the semi-inclusive estimates in the sum rules of QCD and NRQCD $^{21,22}$.

| $B_c$ decay mode | OPE, % | PM, % | SR, % |
|------------------|-------|------|------|
| $\bar{b} \to c\bar{d}^+\nu_l$ | 3.9 ± 1.0 | 3.7 ± 0.9 | 2.9 ± 0.3 |
| $\bar{b} \to c\bar{u}d$ | 16.2 ± 1.1 | 16.7 ± 4.2 | 13.1 ± 1.3 |
| $\sum \bar{b} \to \bar{c}c$ | 25.0 ± 6.2 | 25.0 ± 6.2 | 19.6 ± 1.9 |
| $c \to s\bar{l}^+\nu_l$ | 8.5 ± 2.1 | 10.1 ± 2.5 | 9.0 ± 0.9 |
| $c \to su\bar{d}$ | 47.3 ± 11.8 | 45.4 ± 11.4 | 54.0 ± 5.4 |
| $\sum c \to s$ | 64.3 ± 16.1 | 65.6 ± 16.4 | 72.0 ± 7.2 |
| $B_c^+ \to \tau^+\nu_\tau$ | 2.9 ± 0.7 | 2.0 ± 0.5 | 1.8 ± 0.2 |
| $B_c^+ \to c\bar{s}$ | 7.2 ± 1.8 | 7.2 ± 1.8 | 6.6 ± 0.7 |

In the framework of exclusive approach, it is necessary to sum up widths of different decay modes calculated in the potential models. While considering the semileptonic decays due to the $\bar{b} \to c\bar{l}^+\nu_l$ and $c \to s\bar{l}^+\nu_l$ transitions, one finds that the hadronic final states are practically saturated by the lightest bound 1S-state in the $(\bar{c}c)$-system, i.e. by the $\eta_c$ and $J/\psi$ particles, and the 1S-states in the $(\bar{b}s)$-system, i.e. $B_s$ and $B_s^*$, which can only enter the accessible energetic gap.

Further, the $\bar{b} \to cu\bar{d}$ channel, for example, can be calculated through the given decay width of $b \to c\bar{l}^+\nu_l$ with account for the color factor and hard gluon corrections to the four-quark interaction. It can also be obtained as a sum over the widths of decays with the $(ud)$-system bound states.

The results of calculation for the total $B_c$ width in the inclusive OPE and exclusive PM approaches give the values consistent with each other, if one takes into account the most significant uncertainty related to the choice of quark masses (especially for the charm quark), so that
finally, we have
\[ \tau[B_c^+]_{\text{OPE, PM}} = 0.55 \pm 0.15 \text{ ps}, \]  
(1)
which agrees with the measured value of \( B_c \) lifetime.

The OPE estimates of inclusive decay rates agree with recent semi-inclusive calculations in the sum rules of QCD and NRQCD \[21, 22\], where one assumed the saturation of hadronic final states by the ground levels in the \( c\bar{c} \) and \( b\bar{s} \) systems as well as the factorization allowing one to relate the semileptonic and hadronic decay modes. The coulomb-like corrections in the heavy quarkonia states play an essential role in the \( B_c \) decays and allow one to remove the disagreement between the estimates in sum rules and OPE. In contrast to OPE, where the basic uncertainty is given by the variation of heavy quark masses, these parameters are fixed by the two-point sum rules for bottomonia and charmonia, so that the accuracy of SR calculations for the total width of \( B_c \) is determined by the choice of scale \( \mu \) for the hadronic weak lagrangian in decays of charmed quark. We show this dependence in Fig. 1, where \( \frac{\mu}{m_c} < \mu < m_c \) and the dark shaded region corresponds to the scales preferred by data on the charmed meson lifetimes.

![Fig. 1](image)

FIG. 1: The \( B_c \) lifetime calculated in QCD sum rules versus the scale of hadronic weak lagrangian in the decays of charmed quark. The wide shaded region shows the uncertainty of semi-inclusive estimates, the dark shaded region is the preferable choice as given by the lifetimes of charmed mesons. The dots represent the values in OPE approach taken from ref. [18]. The narrow shaded region represents the result obtained by summing up the exclusive channels with the variation of hadronic scale in the decays of beauty anti-quark in the range of \( 1 < \mu_0 < 5 \text{ GeV} \). The arrow points to the preferable prescription of \( \mu = 0.85 \text{ GeV} \) as discussed in [22].

Supposing the preferable choice of scale in the \( c \rightarrow s \) decays of \( B_c \) to be equal to \( \mu^2 \approx (0.85 \text{ GeV})^2 \), putting \( a_1(\mu) = 1.20 \) and neglecting the contributions caused by nonzero \( a_2 \) in the charmed quark decays [22], in the framework of semi-inclusive sum-rule calculations we predict
\[ \tau[B_c]_{\text{SR}} = 0.48 \pm 0.05 \text{ ps}, \]
(2)
which agrees with the direct summation of exclusive channels calculated in the next sections. In Fig. 1 we show the exclusive estimate of lifetime, too.

### III. MACHINERY OF QCD SUM RULES

In this section we present basic points for the procedure of calculating the form factors for the various exclusive semileptonic decays of \( B_c \). We omit some technical details, which can be found in a number of references appropriately cited.

#### A. Form factors

We accept the following convention on the normalization of wave functions for the hadron states under study, i.e. for the pseudoscalar \((P)\) and vector \((V)\) states:
\[ (0)|\mathcal{J}_\mu|P\rangle = -i f_P p_\mu, \]
(3)
\[ (0)|\mathcal{J}_\mu|V\rangle = \epsilon_\mu f_V M_V, \]
(4)
where \( f_{P,V} \) denote the leptonic constants, so that they are positive,
\[ f_{P,V} > 0, \]

\( p_\mu \) is a four-momentum of the hadron, \( \epsilon_\mu \) is a polarization vector of \( V \), \( M_V \) is its mass, and the current is composed of the valence quark fields constituting the hadron
\[ \mathcal{J}_\mu = \bar{q}_1 \gamma_\mu (1 - \gamma_5) q_2. \]

In this respect we can easily apply the ordinary Feynman rules for the calculations of diagrams, so that the quark-meson vertices in the decay channel are chosen with the following spin structures:
\[ \Gamma_P = \frac{i}{\sqrt{2}} \gamma_5, \quad \Gamma_V = -\frac{1}{\sqrt{2}} \epsilon_\mu. \]

Then, we get general expressions for the hadronic matrix elements of weak currents in the exclusive decays of \( P \rightarrow P' \) and \( P \rightarrow V \) with the definitions of form factors given by the formulae
\[ \langle P'(p_2)|\mathcal{J}_\mu|P(p_1)\rangle = f_+ p_\mu + f_- q_\mu, \]
(5)
\[ \frac{1}{V(p_2)}|\mathcal{J}_\mu|P(p_1)\rangle = i F_V \epsilon_\mu \rho_{\alpha \beta} e^{* \mu} p^\alpha q^\beta + F_0^A \epsilon_+ + F_+^A (\epsilon^* p_1) p_\mu + F_-^A (\epsilon^* p_1) q_\mu, \]
(6)
where \( q_\mu = (p_1 - p_2)_\mu \) and \( p_\mu = (p_1 + p_2)_\mu \). The form factors \( f_{\pm} \) are dimensionless, while \( F_V \) and \( F_\pm^A \) have a dimension of inverse energy, \( F_0^A \) is of the energy dimension. In the case of nonrelativistic description for both initial and final meson states we expect that
\[ f_+ > 0, \quad f_- < 0, \quad F_V > 0, \quad F_0^A > 0, \quad F_+^A < 0, \quad F_-^A > 0. \]
It is important to note that for the pseudoscalar state the hermitian conjugation in (3) does not lead to the change of sign in the right hand side of equation because of the prescription accepted, since the conjugation of imaginary unit takes place with the change of sign for the momentum of meson (the transition from the out-state to in-one). The same speculations show that the spin structure of matrix element in the quark loop order does not involve a functional dependence of form factors on the transfer momentum squared except of $F^A_0$, so that we expect that the simplest modelling in the form of the pole dependence can be essentially broken for $F^A_0$, while the other form factors are fitted by the pole model in a reasonable way.

Following the standard procedure for the evaluation of form factors in the framework of QCD sum rules (22), in the $B_c$ decays we consider the three-point functions

\[
\Pi_\mu(p_1, p_2, q^2) = i^2 \int dx \, dy \, e^{i(p_2 \cdot x - p_1 \cdot y)} \cdot \langle 0|T\{\bar{q}_2(x)\gamma_5 q_1(x) J_\mu(0) \bar{b}(y)\gamma_5 c(y)\}|0\rangle, \quad (7)
\]

\[
\Pi_{\mu\nu}(p_1, p_2, q^2) = i^2 \int dx \, dy \, e^{i(p_2 \cdot x - p_1 \cdot y)} \cdot \langle 0|T\{\bar{q}_2(x)\gamma_\mu q_1(x) J_\nu(0) \bar{b}(y)\gamma_5 c(y)\}|0\rangle, \quad (8)
\]

where $\bar{q}_2(x)\gamma_5 q_1(x)$ and $\bar{q}_2(x)\gamma_\mu q_1(x)$ denote the interpolating currents for the final states mesons.

The Lorentz structures in the correlators can be written down as

\[
\Pi_\mu = \Pi_+(p_1 + p_2)_\mu + \Pi - q_\mu, \quad (9)
\]

\[
\Pi^T = i \Pi V \epsilon^{\mu \nu \alpha \beta} p_2^\alpha p_1^\beta + \Pi^0_\mu g_\mu + \Pi^{A_1}_2 p_2, p_1, \nu + \Pi^{A_2}_2 p_1, p_2, \nu + \Pi^{A_4}_2 p_1, p_2, \nu. \quad (10)
\]

The leading QCD term is a triangle quark-loop diagram in Fig. 2, for which we can write down the double dispersion representation at $q^2 \leq 0$

\[
\Pi^T_1(p_1^2, p_2^2, q^2) = \frac{1}{(2\pi)^2} \int \frac{\rho^{\text{pert}}(s_1, s_2, Q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} ds_1 ds_2 + \text{subtractions},
\]

where $Q^2 = -q^2 \geq 0$, and the integration region is determined by the condition

\[
-1 < \frac{2s_1 s_2 + (s_1 + s_2 - q^2)(m_b^2 - m_c^2 - s_1)}{\lambda^{1/2}(s_1, s_2, q^2)\lambda^{1/2}(m_c^2, s_1, m_b^2)} < 1, \quad (11)
\]

where $\lambda(x_1, x_2, x_3) = (x_1 + x_2 - x_3)^2 - 4x_1 x_2$. The expressions for spectral densities $\rho^{\text{pert}}(s_1, s_2, Q^2)$ were calculated in (21) and presented explicitly in Appendix A of ref. (22).

The connection to hadrons in the framework of QCD sum rules is obtained by matching the resulting QCD expressions of current correlators with the spectral representation, derived from a double dispersion relation at $q^2 \leq 0$, so that

\[
\Pi_1(p_1^2, p_2^2, q^2) = \frac{1}{(2\pi)^2} \int \frac{\rho^{\text{phys}}(s_1, s_2, Q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} ds_1 ds_2 + \text{subtractions},
\]

Assuming that the dispersion relation is well convergent, the physical spectral functions are generally saturated by the ground hadronic states and a continuum starting at some effective thresholds $s_1^{th}$ and $s_2^{th}$

\[
\rho^{\text{phys}}_i(s_1, s_2, Q^2) = \rho^{\text{res}}_i(s_1, s_2, Q^2) + \rho^{\text{cont}}_i(s_1, s_2, Q^2),
\]

where the resonance term is expressed through the product of leptonic constant and form factor for the transition under consideration, so that

\[
\rho^{\text{res}}_i(s_1, s_2, Q^2) = \langle 0|\bar{q}_3 \left[\gamma_5 \gamma_\mu\right] q_2|P'(V)\rangle \langle B_c|\bar{b}\gamma_5 c(0)\rangle \cdot F_i(Q^2)(2\pi)^2\delta(s_1 - M_{i,1}^2)\delta(s_2 - M_{i,2}^2)
\]

+ higher state contributions,

where $M_{i,1,2}$ denote the masses of hadrons in the initial and final states. The continuum of hadron states is usually approximated by the perturbative absorptive part of $\Pi_i$ above the continuum threshold, that involves the parametric dependence of sum rule results on the values of threshold energies.
Then, the expressions for the form factors $F_i$ can be derived by equating the representations for the three-point functions $\Pi_i$ in terms of physical densities and those of calculated in QCD, which means the formulation of sum rules. In this work we use the scheme of moments, implying the calculation of $n$-th derivatives of scalar correlators $\Pi(p_1^2, p_2^2, Q^2)$ over the squared momenta of initial and final channels $p_1^2$ and $p_2^2$ at $p_1^2 = p_2^2 = Q^2 = 0$.

B. Coulomb corrections in the heavy quarkonia

For the heavy quarkonium $\bar{b}c$, where the relative velocity of quark movement is small, an essential role is taken by the Coulomb-like $\alpha_s/v$-corrections. They are caused by the ladder diagram, shown in Fig. 3. It is well known that an account for this corrections in two-point sum rules numerically leads to a double-triple multiplication of Born value of spectral density [24]. In our case it leads to the finite renormalization for $\rho_i$ [21], so that

$$\rho_i^\prime = C\rho_i,$$

with

$$C^2 = \left| \frac{\Psi_{bc}^C(0)}{\Psi_{bc}^{\text{free}}(0)} \right|^2 = \frac{4\pi\alpha_C^2}{3v} \frac{1}{1 - \exp \left( -\frac{4\pi\alpha_C^2}{3v} \right)},$$

(13)

where $v$ is the relative velocity of quarks in the $bc$-system,

$$v = \sqrt{1 - \frac{4m_b m_c}{p_1^2 - (m_b - m_c)^2}},$$

(14)

and the coupling constant of effective coulomb interactions $\alpha_C^2$ should be prescribed by the calculations of leptonic constants for the appropriate heavy quarkonia as described in the next section.

A similar coulomb factor appears in the vertex of heavy quarks composing the final heavy quarkonium, in the case of $\bar{c}c$.

C. Leptonic constants of heavy quarkonia

In order to fix such the parameters as the heavy quark masses and effective couplings of coulomb exchange in the nonrelativistic systems of heavy quarkonia with the same accuracy used in the three-point sum rules, we explore the two-point sum rules of QCD for the systems of $\bar{c}c$, $bc$ and $bb$. Thus, we take into account the quark loop contribution with the coulomb factors like that of [16]. We keep this procedure despite of current status of NRQCD sum rules for the heavy quarkonium, wherein the three-loop corrections to the correlators are available to the moment (see [25] for review), since for the sake of consistency, the calculations should be performed in the same order for both the three-point and two-point correlators. This fact follows from the expression for the resonance term in the three-point correlator, so that the form factor of transition involves the normalization by the leptonic constants extracted from the two-point sum rules. This procedure is taken since calculations of two-loop corrections to the three-point correlators are not available to the moment, unfortunately.

Then, the use of experimental values for the leptonic constants of charmonium and bottomonium in addition to the consistent description of spectral function moments in the two-point sum rules allows us to extract the effective couplings of coulomb exchange as well as the heavy quark masses in the heavy quarkonium channels. A characteristic picture for the description of leptonic constant for the $LS$ vector state of bottomonium $\Upsilon$ is shown in Fig. 4 as obtained in the scheme of moments for the spectral function. In this calculations we take

$$f_{\Upsilon}, \text{ MeV}$$

into account the contributions caused by the higher excitations due to the known masses and ratios of leptonic constants to that of ground state. These higher level terms are important at small $n$ in order to get the stable description of leptonic constant for the ground state. We have found that the normalization of leptonic constant is fixed by the appropriate choice of effective constant.
for the coulomb exchange $\alpha_s^C$, while the stability is very sensitive to the prescribed value of heavy quark mass.

The above procedure allows us to extract the following values of effective constants:

$$\alpha_s^C[bc] = 0.45, \quad \alpha_s^C[cc] = 0.60, \quad \alpha_s^C[bb] = 0.37,$$

where we have used the experimental normalization of leptonic constant $26$

$$f_\psi = (409 \pm 8) \, \text{MeV}, \quad f_T = (690 \pm 15) \, \text{MeV},$$

and the scaling relation obtained in the quasilocal sum rules $27$,

$$\frac{f_{nS}^2}{M_{nS}} \left( \frac{M_{nS}(m_1 + m_2)}{4m_1m_2} \right)^2 \approx \frac{c}{n}, \quad (15)$$

where $c$ is a constant value independent of the heavy flavours in the quarkonium as well as its excitation number. Eq. (15) is valid for the leptonic constants of $nS$-levels in the heavy quarkonia composed of heavy quarks with the masses $m_1$ and $m_2$. This relation yields also the leptonic constant for the $B_c$ meson not yet measured to the moment, so that the prediction$^3$ reads off

$$f_{B_c} = (400 \pm 15) \, \text{MeV}.$$

The heavy quark masses are given by

$$m_b = 4.60 \pm 0.02 \, \text{GeV}, \quad m_c = 1.40 \pm 0.03 \, \text{GeV}.$$

The physical meaning of above values for the heavy quark masses is determined by the threshold posing the energy at which the coulomb spectrum starts, so that it is very close to the so-called ‘potential subtracted masses’ of heavy quarks, $m_{PS}$ known in the literature (see $21$) in the context of renormalon in the perturbative pole mass $30$. We have found that the numerical values obtained coincide with the appropriate $m_{PS}$ within the error bars.

### D. Leptonic constants of heavy-light mesons

As was already explained the calculations in the three-point and two-point sum rules should be performed in the same order of $\alpha_s$. So, although the three-loop calculation are available for the leptonic constants of heavy-light mesons to the moment $31$, we will use the quark-loop approximation (see details in $22$). In that case the renormalization constants of quark-meson vertex in both the three-point and two-point sum rules cancel each other in the expression for the transition form factors, while the absolute normalization of leptonic constants of heavy-light mesons is irrelevant in this respect, and it is defined by the loop corrections, which are rather significant $31$.

Operationally, we put the following expression for the leptonic constant of pseudoscalar meson containing the heavy quark $Q$

$$f_{Qq} = K \cdot E_c \sqrt{\frac{E_c}{m_Q} \alpha_s^{1/3}(m_Q) \left( \frac{m_Q}{M_{QQ}} \right)^{3/2}} \left( 1 - \frac{2\alpha_s(m_Q)}{3\pi} + \frac{3}{88} \left( \frac{E_c^2}{m_Q^2} - \frac{\pi^2}{2} \langle \bar{q}q \rangle \right) \right), \quad (16)$$

where we numerically accept

$$\alpha_s(m_c) = 0.35, \quad \alpha_s(m_b) = 0.22,$$

with the dimensional parameters

$$E_c = 1.2 \, \text{GeV}, \quad \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = -(0.27)^3 \, \text{GeV}^3,$$

which was reasonably investigated in $22$. We stress, first, that at the fixed value of energy threshold $E_c$ in the heavy-light channel the total dependence of transition form factors of $B_c$ into the heavy-light meson has a local minimum, which ensures the stability of result with respect to a small variation of $E_c$. Second, the $K$-factor caused by the higher loop-corrections is irrelevant for the extraction of form factors, although it is large $31$.

Next, the fixed value of threshold energy $E_c$ determines the binding energy of heavy quark in the meson, $\Lambda \approx 0.63 \, \text{GeV}$, which yields the same value of mass for the beauty $m_b \approx 4.6 \, \text{GeV}$, as it was determined from the analysis of two-point sum rules in the $bb$ channel. However, taking into account the second order corrections in $1/m_c$, we find that the mass of charmed quark is shifted to the value of $m_c \approx 1.2 \, \text{GeV}$ in the heavy-light channel in comparison with the $cc$ states. Thus, in the transition of $B_c$ to the charmed meson we put the mass of spectator charmed quark equal to $m_c \approx 1.2 \, \text{GeV}$, while for the transition into the charmonium we accept $m_c \approx 1.4 \, \text{GeV}$, as discussed above.

Finally, we take the following ordinary ratios of leptonic constants for the vector and pseudoscalar states and for the heavy-strange mesons:

$$\frac{f_{B^*}}{f_B} \approx \frac{f_{D^*}}{f_D} \approx 1.11, \quad \frac{f_{B_c}}{f_{D_c}} \approx \frac{f_{D_c}}{f_D} \approx 1.16,$$

which agree with the lattice computations $32$. 

### E. Finite energy sum rules

Putting the density of continuum hadronic states equal to that of calculated in the perturbative QCD, we involve the dependence of calculations on the threshold energy $s_{th}$ in the channels of two-quark currents, since this procedure supposes the cancellation of integrals over the spectral functions in both theoretical and physical

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$^3$ The consideration of leptonic constants for the heavy quarkonia in the framework of potential approach was presented in $28$. 
parts of sum rules. At rather high numbers of moments for the spectral density the contributions of excitations like $2S$, are suppressed by the high powers of mass ratio $M_{1S}/M_{2S}$, so that the local stability of form factor value on the variation of moment number allow us extract the form factor for the transition of $1S \rightarrow 1S$, which is sufficient for the $B_c$ decays in the heavy-light mesons. However, the estimates for the decays into the quite narrow $2S$-resonances of charmonium cannot be obtained in this way.

We accept the concept of finite energy sum rules in order to calculate the form factors for the decays of $B_c \rightarrow \bar{c}c[2S]$. In this approach we suggest that with a rather good accuracy the contribution to the correlator evaluated by the theoretical spectral density integrated out over the finite energy interval containing the resonance under consideration gives the approximation for the appropriate physical part excluding the contributions of other resonances. In contrast to the global integration in the standard QCD sum rules, where the result for the $1S$ channel reveals a weak dependence on the continuum threshold, the choice of duality interval for the $2S$ level involves a significant variation of the result on the level of 20%. In order to minimize the variation we explore the principle of stability, implying the presence of local minimum in the ratios of form factors for the decays into the $1S$ and $2S$ states of charmonium: $F_{B_c \rightarrow \bar{c}c[1S]}/F_{B_c \rightarrow \bar{c}c[2S]}$.

while in the finite energy sum rules yielding the form factors for the $F_{B_c \rightarrow \bar{c}c[2S]}$ transition we put

$$k_i[2S] = 1.09 \text{ GeV}, \quad k_{th}[2S] = 1.2 \text{ GeV},$$

which fits the region of $2S$ charmonium. At the above values of thresholds the ratio of form factors

$$\mathcal{R} = \frac{F^A_0(B_c \rightarrow \bar{c}c[1S])}{F^A_0(B_c \rightarrow \bar{c}c[2S])}$$

as well as both form factors themselves reveal the stability versus the variation of moment numbers as shown in Fig. 5. The ratios for the other form factors multiplied by the kinematic factors depending on the total spin $j = 0, 1$ of recoil meson

$$\kappa_j = \frac{f_{1S} M_{1S}^{2-j}}{f_{2S} M_{2S}^{2-j}}$$

as evaluated in the framework of finite energy sum rules are presented in Table II.

![FIG. 5: The ratio of form factors $\mathcal{R}$ for the transitions of $B_c$ meson into the vector charmonium states versus the numbers of moments in the channels of $bc$ and $\bar{c}c$.](image)

In detail, we parameterize the energy $E$ in the bound state channels by the relative momentum $k$, so that

$$E = \frac{k^2}{2m_{\text{red}}},$$

where $m_{\text{red}}$ is the reduced mass of quarks composing the meson. In the QCD sum rules yielding the form factors for the $F_{B_c \rightarrow \bar{c}c[1S]}$ transition we put the initial and final values of momentum determining the resonance range by

$$k_i[1S] = 0, \quad k_{th}[1S] = 1.2 \text{ GeV},$$

Thus, we have found that the amplitudes squared for the decays of $B_c$ into the $2S$ charmonium are suppressed by an order of magnitude in comparison with the appropriate values for the decay into the ground pseudoscalar and vector states.

F. Numerical values

The characteristic dependence of sum rule results in the scheme of moments for the spectral density is shown in Fig. 6 wherein we see the region of stability versus the variation of moment numbers in both channels of initial and final hadron states. Similar results are obtained for the other form factors of decays under consideration.

Our estimates are summarized in Table III where for the sake of comparison we expose the results obtained in the potential model, which parameters are listed in Appendix B of ref. 23. In the potential model the most reliable results are expected at zero recoil of meson in the final state of transition, since the wave functions are rather accurately calculable at small virtualities of quarks composing the meson. We take the predictions of the potential model at zero recoil and evolve the values of form factors to zero transfer squared in the model with the pole dependence

$$F_i(q^2) = \frac{F_i(0)}{1 - q^2/M_{i,\text{pole}}^2},$$

TABLE II: The ratio of form factors $\mathcal{R}_{2S}$ in the decays of $B_c$ into the $1S$ and $2S$ charmonium states.

| Form factor | $f_+^A$ | $f_-^A$ | $F_V^A$ | $F_{1S}^A$ | $F_{2S}^A$ |
|-------------|---------|---------|---------|-----------|-----------|
| $\mathcal{R}_{2S} \cdot \kappa_j$ | 3.9 | 2.2 | 3.1 | 3.5 | 4.9 | 2.3 |

Thus, we have found that the amplitudes squared for the decays of $B_c$ into the $2S$ charmonium are suppressed by an order of magnitude in comparison with the appropriate values for the decay into the ground pseudoscalar and vector states.
TABLE III: The form factors of various transitions calculated in the framework of QCD sum rules at \( q^2 = 0 \) in comparison with the estimates in the potential model (PM) of \( [9] \).

| Transition     | \( f_+ \), [PM] | \( f_- \), [PM] | \( F_{Vr} \), [PM] (GeV\(^{-1}\)) | \( F^{A-}_{0} \), [PM] (GeV) | \( F^{A-}_{0} \), [PM] (GeV\(^{-1}\)) | \( F^{A+}_{0} \), [PM] (GeV) | \( F^{A+}_{0} \), [PM] (GeV\(^{-1}\)) |
|----------------|-----------------|-----------------|-------------------------------|-----------------|-------------------------------|-----------------|-------------------------------|
| \( B_c \to B^{(*)}_s \) | 1.3, [1.1]      | -5.8, [-5.9]    | 1.1, [1.1]                    | 8.1, [8.2]      | 0.2, [0.3]                    | 1.8, [1.4]      |
| \( B_c \to B^{(*)} \)  | 1.27, [1.38]    | -7.3, [-7.3]    | 1.35, [1.37]                  | 9.8, [9.4]      | 0.35, [0.36]                  | 2.5, [1.75]      |
| \( B_c \to D^{(*)} \)  | 0.32, [0.29]    | -0.34, [-0.37]  | 0.20, [0.21]                  | 3.6, [3.6]      | -0.062, [-0.060]             | 0.10, [0.16]    |
| \( B_c \to D^{(*)}_s \) | 0.45, [0.43]    | -0.43, [-0.56]  | 0.24, [0.27]                  | 4.7, [4.7]      | -0.077, [-0.071]             | 0.13, [0.20]    |
| \( B_c \to \bar{c}c[1S] \) | 0.66, [0.7]    | -0.36, [-0.38]  | 0.11, [0.10]                  | 5.9, [6.2]      | -0.074, [-0.070]             | 0.12, [0.14]    |

TABLE IV: The pole masses used in the model for the form factors in various transitions.

| Transition     | \( M_{pole}[f_+] \), GeV | \( M_{pole}[f_-] \), GeV | \( M_{pole}[F_{VR}] \), GeV | \( M_{pole}[F^{A-}_{0}] \), GeV | \( M_{pole}[F^{A+}_{0}] \), GeV | \( M_{pole}[F^{A+}_{0}] \), GeV |
|----------------|---------------------------|---------------------------|-------------------------------|---------------------------|-------------------------------|---------------------------|
| \( B_c \to \bar{c}c \) | 4.5                       | 4.5                       | 4.5                           | 4.5                       | 4.5                           | 4.5                       |
| \( B_c \to B^{(*)}_s \) | 1.8                       | 1.8                       | 1.8                           | 1.8                       | 1.8                           | 1.8                       |
| \( B_c \to B^{(*)} \)  | 1.7                       | 1.7                       | 2.2                           | 3.2                       | 3.2                           | 3.2                       |
| \( B_c \to D^{(*)} \)  | 5.0                       | 5.0                       | 6.2                           | \( \infty \)              | 6.2                           | 6.2                       |
| \( B_c \to D^{(*)}_s \) | 5.0                       | 5.0                       | 6.2                           | \( \infty \)              | 6.2                           | 6.2                       |

FIG. 6: The form factor \( F^{A}_{0} \) (GeV) at \( q^2 = 0 \) for the transition of \( B_c \) meson into the vector \( B^{*} \) state versus the numbers of moments in the channels of \( bc \) and \( bu \).

The sum rule estimates of form factors are taken at zero transfer squared, while the dependence on \( q^2 \) is beyond the reliable accuracy of the method, particularly, because at positive transfer squared some non-Landau singularities could be important. So, the slopes of form factors are not under control in the sum rules. Nevertheless, due to a suggestion on the pole dependence we can restore the behaviour of form factors in the whole physical region with the parameters shown in Table XIV. Of course, the numerical values of pole masses can vary in reasonable ranges, which involves the uncertainty in the estimates of width about 5–10%, depending on the energy release: at a small energy release the errors of modelling become less.

IV. RELATIONS BETWEEN THE FORM FACTORS

In the limit of infinitely heavy quark mass, the NRQCD and HQET lagrangians possess the spin symmetry, since the heavy quark spin is decoupled in the leading approximation. The most familiar implication of such symmetry is the common Isgur-Wise function determining the form factors in the semileptonic decays of singly heavy hadrons.

In contrast to the weak decays with the light spectator quark, the \( B \) decays to both the charmonia \( \psi \) and \( \eta_c \) and \( B^{(*)}_c \) involve the heavy spectator, so that the spin symmetry works only at the recoil momenta close to zero, where the spectator enters the heavy hadron in the final state with no hard gluon rescattering. Hence, in a strict consideration we expect the relations between the form factors in the vicinity of zero recoil. The normalization of common form factor is not fixed, as was in decays of hadrons with a single heavy quark, since the heavy quarkonia wave-functions are flavour-dependent. Nevertheless, in practice, the ratios of form factors as fixed at a given zero recoil point are broken only by the different dependence on the transfer squared, that is not significant in real numerical estimates in the restricted region of physical phase space.

As for the implications of spin symmetry for the form factors of decay, in the soft limit for the transitions \( B_c^{(*)} \to \psi(\eta_c)e^+\nu \)

\[
v_1^\mu \neq v_2^\mu, \quad w = v_1 \cdot v_2 \to 1, \tag{17}
\]

where \( v_{1,2}^\mu = p_{1,2}^\mu/\sqrt{p_{1,2}^2} \) are the four-velocities of heavy quarkonia in the initial and final states, we derive the relations \( [21] \)
so that \( m_1 \) is the mass of decaying quark, \( m_2 \) is the quark mass of decay product, and \( m_3 \) is the mass of spectator quark, while \( M_1 = m_1 + m_2, M_2 = m_2 + m_3 \).

The SR estimates of form factors show a good agreement with the relations, whereas the deviations can be basically caused by the difference in the \( q^2 \)-evolution of form factors from the zero recoil point, that can be neglected within the accuracy of SR method for the transitions of \( B_c \to \bar{c}c \) as shown in \([21]\).

In the same limit for the semileptonic modes with a single heavy quark in the final state we find that the ambiguity in the ‘light quark propagator’ (strictly, we deal with the uncertainty in the spin structure of amplitude because of light degrees of freedom) restricts the number of relations, and we derive

\[
\begin{align*}
&F_+(c_1^p \cdot M_2 - c_2^p \cdot M_1) - F_-(c_1^p \cdot M_2 + c_2^p \cdot M_1) = 0, \\
&F_0^A \cdot c_2 = 2 c_e \cdot F_V M_1 M_2 = 0,
\end{align*}
\]

(18)

\[
\begin{align*}
&F_0^A (c_1 + c_2) - c_e M_1 (F_+(M_1 + M_2) + F_-(M_1 - M_2)) = 0, \\
&F_0^A c_1^p + c_e \cdot M_1 (f_+ + f_-) = 0,
\end{align*}
\]

(19)

where

\[
\begin{align*}
c_e &= -2, \\
c_1 &= - \frac{m_3(3m_1 + m_3)}{4m_1 m_2}, \\
c_1^p &= 1 + \frac{m_3}{2m_1} - \frac{m_3}{2m_2}, \\
c_2^p &= 1 - \frac{m_3}{2m_1} + \frac{m_3}{2m_2}, \\
c_2 &= \frac{1}{2m_1 m_2} (2m_1 m_2 + m_1 m_3 + m_2 m_3), \\
c_V &= - \frac{1}{2m_1 m_2} (2m_1 m_2 + m_1 m_3 + m_2 m_3),
\end{align*}
\]

(20)

so that \( m_2 \) is the mass of the light quark. The parameter \( \tilde{B} \) has the form

\[
\tilde{B} = - \frac{2m_1 + m_3}{2m_1} + \frac{4m_3(m_1 + m_3)}{F_0^A} F_V.
\]

(23)

The \( 1/m_Q \)-deviations from the symmetry relations in the decays of \( B_+^+ \to B_+^{(*)} \epsilon^+ \nu \) are about 10-15 \%, as found in the QCD sum rules considered in \([22]\).

Next, we investigate the validity of spin-symmetry relations in the \( B_c \) decays to \( B^{(*)} \), \( D^{(*)} \), and \( D_s^{(*)} \). The results of estimates for the \( f_\pm \) evaluated by the symmetry relations with the inputs given by the form factors \( F_V \) and \( F_0^A \) extracted from the sum rules are presented in Table \( V \) in comparison with the values calculated in the framework of sum rules.

We have found that the uncertainty in the estimates is basically determined by the variation of pole masses in the \( q^2 \)-dependencies of form factors, which govern the evolution from the zero recoil point to the zero transfer squared. So, the variation of \( M_{pole} \) in the range of 4.8 - 5 GeV for the transitions of \( B_c \to D^{(*)} \) and \( B_c \to D_s^{(*)} \) results in 30%-uncertainty in the form factors presented in Table \( V \). Analogously, the variation of \( M_{pole} \) in the range of 1.5 - 1.9 GeV for the transitions of \( B_c \to D^{(*)} \) results in the uncertainty about 35%.

Note, that the combinations of relations given above reproduce the only equality \([34]\), which was found for each mode in the strict limit of \( \epsilon_1 = \epsilon_2 \).

V. SEMILEPTONIC AND LEPTONIC MODES

A. Semileptonic decays

The semileptonic decay rates are underestimated in the QCD SR approach of ref. \([19]\), because large coulomb-like corrections were not taken into account. The recent analysis of SR in \([20, 21, 22]\) decreased the uncertainty, so that the estimates agree with the calculations in the potential models.

The absolute values of semileptonic widths are presented in Table \( V \) in comparison with the estimates obtained in potential models.

In practice, the most constructive information is given by the \( \psi \) mode, since this charmonium is clearly detected in experiments due to the pure lepton decays \([3]\).
addition to the investigation of various form factors and their dependence on the transfer squared, we would like to stress that the measurement of decay to the excited state of charmonium, i.e. $\psi'$, could answer the question on the reliability of QCD predictions for the decays to the excited states. We see that to the moment the finite energy sum rules predict the width of $B_s \to \psi / l^+ l^- \nu$ decays in a reasonable agreement with the potential models if one takes into account an uncertainty about 50%.

TABLE VI: Exclusive widths of semileptonic $B_s^+$ decays, $\Gamma$ in \(10^{-15}\) GeV, the symbol $\star$ marks the result of this work.

| Mode                | $\Gamma$ [a] | $\Gamma$ [b] | $\Gamma$ [c] | $\Gamma$ [d] | $\Gamma$ [e] | $\Gamma$ [f] |
|---------------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $B_s^+ \to \eta e^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to \eta \tau^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to \eta' e^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to \eta' \tau^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to J/\psi e^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to J/\psi \tau^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to \psi e^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to \psi \tau^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to D^0 e^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to D^0 \tau^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to D^0 e^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |
| $B_s^+ \to D^0 \tau^+ \nu$ | 0.15         | 0.11         | 0.12         | 0.12         | 0.12         | 0.12         |

B. Leptonic decays

The dominant leptonic decay of $B_s$ is given by the $\tau \nu_\tau$ mode (see Table IV). However, it has a low experimental efficiency of detection because of hadronic background in the $\tau$ decays or a missing energy. Recently, in refs. [33], the enhancement of muon and electron channels in the radiative modes was studied. The additional photon allows one to remove the helicity suppression for the leptonic decay of pseudoscalar particle, which leads, say, to the double increase of muonic mode.

VI. NON-LEPTONIC MODES

In comparison with the inclusive non-leptonic widths, which can be estimated in the framework of quark-hadron duality (see Table I), the calculations of exclusive modes usually involves the approximation of factorization [35], which, as expected, can be quite accurate for the $B_c$, since the quark-gluon sea is suppressed in the heavy charmonium. Thus, the important parameters are the factors $a_1$ and $a_2$ in the non-leptonic weak lagrangian, which depend on the normalization point suitable for the $B_c$ decays.

TABLE VII: Exclusive non-leptonic decay widths of the $B_c$ meson, $\Gamma$ in \(10^{-15}\) GeV, the symbol $\star$ marks the result of this work. The $\bar{b}$-quark decays with $c$-quark spectator.

| Class | Mode            | $\Gamma$ [a] | $\Gamma$ [b] | $\Gamma$ [c] | $\Gamma$ [d] | $\Gamma$ [e] | $\Gamma$ [f] |
|-------|-----------------|--------------|--------------|--------------|--------------|--------------|--------------|
| I     | $B_c^+ \to \eta c \pi^0$ | 0.15         | 0.11         | 0.12         | 0.15         | 0.15         | 0.15         |
| I     | $B_c^+ \to \eta c \rho^+$ | 0.15         | 0.11         | 0.12         | 0.15         | 0.15         | 0.15         |
| I     | $B_c^+ \to J/\psi \pi^+$ | 0.15         | 0.11         | 0.12         | 0.15         | 0.15         | 0.15         |
| I     | $B_c^+ \to J/\psi \rho^+$ | 0.15         | 0.11         | 0.12         | 0.15         | 0.15         | 0.15         |
| II    | $B_c^+ \to J/\psi \pi^+$ | 0.15         | 0.11         | 0.12         | 0.15         | 0.15         | 0.15         |
| II    | $B_c^+ \to J/\psi \rho^+$ | 0.15         | 0.11         | 0.12         | 0.15         | 0.15         | 0.15         |

The QCD SR estimates for the non-leptonic decays of charmed quark in $B_c$ give the widths represented in Tables VII and VIII. The agreement of results with the
values predicted by the potential models is rather good for the direct transitions with no permutation of colour lines, i.e. the class I processes with the factor of $a_1$ in the non-leptonic amplitude determined by the effective lagrangian. In contrast, the sum rule predictions are significantly enhanced in comparison with the values calculated in the potential models for the transitions with the colour permutation, i.e. for the class II processes with the factor of $a_2$.

Further, for the transitions, wherein the interference is significantly involved, the class III processes, we find that the absolute values of different terms given by the squares of $a_1$ and $a_2$ calculated in the sum rules are in agreement with the estimates of potential models. We stress that under fixing the definitions of hadron state phases as described in section III A, we have found that the Pauli interference has determined the negative sign of two amplitudes with $a_1$ and $a_2$, however, the relevant Fierz transformation has led to the complete cancellation of the Pauli interference effect, and the relative sign of two amplitude in the modes under consideration is positive in agreement with the results of potential models listed in Table VIII. Taking into account the negative value of $a_2$ with respect to $a_1$, we see that all of decays shown in Table VIII should be suppressed in comparison with the case of the interference switched off. The characteristic values of effects caused by the interference is presented in Table IX where we put the widths in the form

$$\Gamma = \Gamma_0 + \Delta \Gamma, \quad \Gamma_0 = x_1 a_1^2 + x_2 a_2^2, \quad \Delta \Gamma = za_1 a_2.$$ 

Then, we conclude that the interference can be straightforwardly tested in the listed decays, wherein its significance reaches about 50%.

At large recoils as in $B^+_c \to \psi \pi^+(\rho^+)$, the spectator picture of transition can be broken by the hard gluon exchanges. However, we emphasize that the significant rates of $B_c$ decays to the P- and D-wave charmonium states point out that the corrections in the second order of the heavy-quark velocity in the heavy quarkonia under study could be quite essential and they can suppress the corresponding decay rates, since the relative momentum of heavy quarks inside the quarkonium if different from zero should enhance the virtuality of gluon exchange, which suppresses the decay amplitudes.

The widths of non-leptonic $c$-quark decays are listed in Table IX in comparison with the predictions of potential models. We see that the sum rule estimates are greater than those of potential models. In this respect we check that our calculations are consistent with the inclusive ones. So, we sum up the calculated exclusive widths and estimate the total width of $B_c$ meson as shown in Fig. II which points to a good agreement of our calculations with those of OPE and semi-inclusive estimates.

Another interesting point is the possibility to extract the factorization parameters $a_1$ and $a_2$ in the $c$-quark decays by measuring the branching ratios

$$\frac{\Gamma[B^+_c \to B^+ \bar{K}^0]}{\Gamma[B^+_c \to B^0 K^+]} = \frac{\Gamma[B^+_c \to B^+ \bar{K}^{*0}]}{\Gamma[B^+_c \to B^0 K^{*+}]} = \frac{\Gamma[B^+_c \to B^{+*} \bar{K}^0]}{\Gamma[B^+_c \to B^{0*} K^+]}, \quad (24)$$

\[ \begin{array}{|c|c|c|c|c|c|} \hline \text{Class} & \text{Mode} & \Gamma [\text{GeV}] & \Gamma [\text{OPE}] & \Gamma [\text{Semi}] & \Gamma [\text{SUM}] \\ \hline \text{I} & B^+_c \to \psi \pi^+ & (2.39 a_1 + 3.50 a_2) & (2.16 a_1 + 2.57 a_2) & (1.13 a_1 + 1.98 a_2) & (2.58 a_1 + 3.40 a_2) \\ & B^+_c \to \eta \pi^+ & (2.39 a_1 + 2.51 a_2) & (2.03 a_1 + 2.16 a_2) & (1.04 a_1 + 1.90 a_2) & (2.36 a_1 + 1.99 a_2) \\ & J/\psi \pi^+ & (1.92 a_1 + 3.11 a_2) & (1.62 a_1 + 1.72 a_2) & (1.02 a_1 + 1.95 a_2) & (1.65 a_1 + 2.92 a_2) \\ \hline \text{II} & \eta \pi^+ & (13.3 a_1 + 42.2 a_2 + 48.3 a_2^2) & (3.13 a_1 + 3.67 a_2^2) & (3.31 a_1 + 3.89 a_2^2) & \\ & J/\psi \pi^+ & (0.50 a_1 + 0.56 a_2^2) & (0.485 a_1 + 0.528 a_2^2) & (0.193 a_1 + 0.440 a_2^2) & (0.47 a_1 + 0.73 a_2^2) \\ & J/\psi K^+ & (0.44 a_1 + 0.59 a_2^2) & (0.466 a_1 + 0.452 a_2^2) & (0.181 a_1 + 0.430 a_2^2) & (0.37 a_1 + 0.66 a_2^2) \\ & J/\psi K^{*+} & (0.41 a_1 + 0.53 a_2^2) & (0.372 a_1 + 0.338 a_2^2) & (0.177 a_1 + 0.442 a_2^2) & (0.30 a_1 + 0.44 a_2^2) \\ \hline \end{array} \]

\[ \begin{array}{|c|c|} \hline \text{Class} & \Delta \Gamma/\Gamma_0, \% \\ \hline \text{I} & \| \text{II} & \| \text{III} \\ \hline B^+_c \to \psi \pi^+ & -48 & -39 & -53 & -83 & -45 & -43 & -46 \\ B^+_c \to \eta \pi^+ & -48 & -39 & -53 & -83 & -45 & -43 & -46 \\ B^+_c \to J/\psi \pi^+ & -50 & -45 & -43 & -46 \\ B^+_c \to J/\psi K^+ & -50 & -45 & -43 & -46 \\ B^+_c \to J/\psi K^{*+} & -50 & -45 & -43 & -46 \\ \end{array} \]
TABLE X: Exclusive non-leptonic decay widths of the \( B_c \) meson, \( \Gamma \) in \( 10^{-15} \) GeV, the symbol \( \ast \) marks the result of this work. The c-quark decays with b-quark spectator.

| Class | Mode | \( \Gamma \) [\( \ast \)] | \( \Gamma \) [12] | \( \Gamma \) [13] | \( \Gamma \) [14] |
|-------|------|------------------|-------------|-------------|-------------|
| I     | \( B_c^+ \rightarrow B_s^0 \pi^+ \) | 167 a17 | 15.8 a17 | 58.4 a17 | 34.8 a17 |
|       | \( B_c^+ \rightarrow B_s^0 \rho^+ \) | 72.5 a17 | 39.2 a17 | 44.8 a17 | 23.6 a17 |
|       | \( B_c^+ \rightarrow B_s^\ast^0 \pi^+ \) | 66.3 a17 | 12.5 a17 | 51.6 a17 | 19.8 a17 |
|       | \( B_c^+ \rightarrow B_s^\ast^0 \rho^+ \) | 204 a7 | 171. a7 | 150. a7 | 123 a7 |
|       | \( B_c^+ \rightarrow B_s^\ast^0 K^+ \) | 10.7 a7 | 1.70 a7 | 4.20 a7 |
|       | \( B_c^+ \rightarrow B_s^\ast^0 K^+ \) | 3.8 a7 | 1.34 a7 | 2.96 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^+ \) | 1.06 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^+ \) | 11.6 a7 |
|       | \( B_c^+ \rightarrow B_s^0 \pi^+ \) | 10.6 a7 | 1.03 a7 | 3.30 a7 | 1.50 a7 |
|       | \( B_c^+ \rightarrow B_s^0 \rho^+ \) | 9.7 a7 | 2.81 a7 | 5.97 a7 | 1.93 a7 |
|       | \( B_c^+ \rightarrow B_s^\ast^0 \pi^+ \) | 9.5 a7 | 0.77 a7 | 2.90 a7 | 0.78 a7 |
|       | \( B_c^+ \rightarrow B_s^\ast^0 \rho^+ \) | 26.1 a7 | 9.01 a7 | 11.9 a7 | 6.78 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^+ \) | 0.70 a7 | 0.105 a7 | 0.255 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^+ \) | 0.15 a7 | 0.125 a7 | 0.180 a7 |
|       | \( B_c^+ \rightarrow B_s K^+ \) | 0.56 a7 | 0.064 a7 | 0.195 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^+ \) | 0.59 a7 | 0.065 a7 | 0.374 a7 |
|       | \( B_c^+ \rightarrow B_s K^+ \) | 13.1 a7 | 4.50 a7 | 5.96 a7 | 4.56 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 286 a7 | 39.1 a7 | 96.5 a7 | 24.0 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 64 a7 | 46.8 a7 | 68.2 a7 | 13.8 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 231 a7 | 24.0 a7 | 73.3 a7 | 8.9 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 242 a7 | 247 a7 | 141 a7 | 82.3 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 5.3 a7 | 0.51 a7 | 1.65 a7 | 1.03 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 4.4 a7 | 1.40 a7 | 2.98 a7 | 1.28 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 4.8 a7 | 0.38 a7 | 1.45 a7 | 0.53 a7 |
|       | \( B_c^+ \rightarrow B_s^0 K^0 \) | 13.1 a7 | 4.50 a7 | 5.96 a7 | 4.56 a7 |

This procedure can give the test for the factorization approach itself.

The suppressed decays caused by the flavor changing neutral currents were studied in 39.

The CP-violation in the \( B_c \) decays can be investigated in the same manner as made in the \( B \) decays. The expected CP-asymmetry of \( \mathcal{A}(B_c^+ \rightarrow J/\psi D^\pm) \) is about \( 4 \cdot 10^{-3} \), when the corresponding branching ratio is suppressed as \( 10^{-4} \) 40. Thus, the direct study of CP-violation in the \( B_c \) decays is practically difficult because of low relative yield of \( B_c \) with respect to ordinary \( B \) mesons: \( \sigma(B_c)/\sigma(B) \sim 10^{-3} \). A model-independent way to extract the CKM angle \( \gamma \) based on the measurement of two reference triangles was independently offered by Masetti, Fleischer and Wyler in 44 by investigating the modes with the neutral charmed meson in the final state.

Another possibility is the lepton tagging of \( B_s \) in the \( B_c^\pm \rightarrow B_s^\pm \nu \) decays for the study of mixing and CP-violation in the \( B_s \) sector 41.

VII. DISCUSSION AND CONCLUSIONS

We have calculated the exclusive decay widths of the \( B_c \) meson in the framework of QCD sum rules and reviewed the current status of theoretical predictions for the decays of \( B_c \) meson. We have found that the various approaches: OPE, Potential models and QCD sum rules, result in the close estimates, while the SR as explored for the various heavy quark systems, lead to a smaller uncertainty due to quite an accurate knowledge of the heavy quark masses. So, summarizing we expect that the dominant contribution to the \( B_c \) lifetime is given by the charm quark decays (\( \sim 70\% \)), while the b-quark decays and the weak annihilation add about 20% and 10%, respectively. The coulomb-like \( \alpha_s/\nu \) corrections play an essential role in the determination of exclusive form factors in the QCD SR. The form factors obey the relations dictated by the spin symmetry of NRQCD and HQET with quite a good accuracy expected.

The predictions of QCD sum rules for the exclusive decays of \( B_c \) are summarized in Table XI at the fixed values of factors \( a_{1,2} \) and lifetime. In addition to the decay channels with the heavy charmonium \( J/\psi \) well detectable through its leptonic mode, one could expect a significant information on the dynamics of \( B_c \) decays from the channels with a single heavy mesons, if an experimental efficiency allows one to extract a signal from the cascade decays. An interesting opportunity is presented by the relations for the ratios in 21, which can shed light to characteristics of the non-leptonic decays in the explicit form.

We have found that the \( \bar{b} \) decay to the doubly charmed states gives

\[
\text{Br}[B_c^+ \rightarrow c\bar{c}c\bar{s}] \approx 1.39\%,
\]

so that in the absolute value of width it can be compared with the estimate of spectator decay 18,

\[
\Gamma[B_c^+ \rightarrow c\bar{c}c\bar{s}]_{\text{SR}} \approx 20 \cdot 10^{-15} \text{GeV},
\]

\[
\Gamma[B_c^+ \rightarrow c\bar{c}c\bar{s}]_{\text{spect}} \approx 90 \cdot 10^{-15} \text{GeV},
\]

and we find the suppression factor of about 1/4.5. This result is in agreement with the estimate in OPE 18, where a strong dependence of negative term caused by the Pauli interference on the normalization scale of non-leptonic weak lagrangian was emphasized, so that at moderate scales one gets approximately the same suppression factor, too.

To the moment we certainly state that the accurate direct measurement of \( B_c \) lifetime can provide us with the information on both the masses of charmed and beauty quarks and the normalization point of non-leptonic weak lagrangian in the \( B_c \) decays (the \( a_1 \) and \( a_2 \) factors). The experimental study of semi leptonic decays and the extraction of ratios for the form factors can test the spin symmetry derived in the NRQCD and HQET approaches.
and decrease the theoretical uncertainties in the corresponding theoretical evaluation of quark parameters as well as the hadronic matrix elements, determined by the nonperturbative effects caused by the quark confinement. The measurement of branching fractions for the semileptonic and non-leptonic modes and their ratios can inform on the values of factorization parameters, which depend again on the normalization of non-leptonic weak lagrangian. The charmed quark counting in the $B_c$ decays is related to the overall contribution of $b$ quark decays as well as with the suppression of $b \to c\bar{s}s$ transition because of the destructive interference, which value depends on the nonperturbative parameters (roughly estimated, the leptonic constant) and non-leptonic weak lagrangian.

Thus, the progress in measuring the $B_c$ lifetime and decays could enforce the theoretical understanding of what really happens in the heavy quark decays at all.

We point also to the papers, wherein some aspects of $B_c$ decays and spectroscopy were studied: non-leptonic decays in $^{[42]}$, polarization effects in the radiative leptonic decays $^{[43]}$, relativistic effects $^{[44]}$, spectroscopy in the systematic approach of potential nonrelativistic QCD in $^{[45]}$, nonperturbative effects in the semileptonic decays $^{[46]}$, exclusive and inclusive decays of $B_c$ states into the lepton pair and hadrons $^{[47]}$, rare decays in $^{[48]}$, the spectroscopy and radiative decays in $^{[49]}$.

This work is supported in part by the Russian Foundation for Basic Research, grants 01-02-99315, 01-02-16585, and 00-15-96645.

\[ \text{TABLE XI: Branching ratios of exclusive } B_c^+ \text{ decays at the fixed choice of factors: } a_1^+ = 1.20 \text{ and } a_2^+ = -0.317 \text{ in the non-leptonic decays of } c \text{ quark, and } a_1^+ = 1.14 \text{ and } a_2^+ = -0.20 \text{ in the non-leptonic decays of } b \text{ quark. The lifetime of } B_c \text{ is appropriately normalized by } \tau[B_c] \approx 0.45 \text{ ps.} \]

| Mode               | BR, % | Mode               | BR, % | Mode               | BR, % |
|--------------------|-------|--------------------|-------|--------------------|-------|
| $B_c^+ \to \eta c e^+ \nu$ | 0.75  | $B_c^+ \to J/\psi K^+$ | 0.011 | $B_c^+ \to B_d^0 \nu$ | 1.06  |
| $B_c^+ \to \eta c \tau^+ \nu$ | 0.23  | $B_c \to J/\psi K^{*+}$ | 0.022 | $B_c^+ \to B_s^0 \nu$ | 0.37  |
| $B_c^+ \to \eta c e^+ \nu$ | 0.020 | $B_c^+ \to D^+ \overline{\tau}^0$ | 0.0053 | $B_c^+ \to B_s^0 K^+$ | –     |
| $B_c^+ \to \eta c \tau^+ \nu$ | 0.0016 | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.0075 | $B_c^+ \to B_s^0 K^{*+}$ | –     |
| $B_c^+ \to J/\psi e^+ \nu$ | 1.9   | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.0049 | $B_c^+ \to B_s^0 \pi^+$ | 1.06  |
| $B_c^+ \to J/\psi \tau^+ \nu$ | 0.48  | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.033  | $B_c^+ \to B_s^0 \rho^+$ | 0.96  |
| $B_c^+ \to \psi^+ e^+ \nu$ | 0.094 | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.00048 | $B_c^+ \to B_s^0 \pi^+$ | 0.95  |
| $B_c^+ \to \psi \tau^+ \nu$ | 0.008 | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.00071 | $B_c^+ \to B_s^0 \rho^+$ | 2.57  |
| $B_c^+ \to D^0 e^+ \nu$ | 0.004 | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.00045 | $B_c^+ \to B_s^0 K^+$ | 0.07  |
| $B_c^+ \to D^0 \tau^+ \nu$ | 0.002 | $B_c^+ \to D^+ \overline{\tau}^{*0}$ | 0.00026 | $B_c^+ \to B_s^0 K^{*+}$ | 0.015 |
| $B_c^+ \to D^0 c e^+ \nu$ | 0.18  | $B_c^+ \to \eta_c D^+_c$ | 0.28  | $B_c^+ \to B_s^0 K^+$ | 0.055 |
| $B_c^+ \to D^0 \tau^+ \nu$ | 0.008 | $B_c^+ \to \eta_c D^+_c$ | 0.27  | $B_c^+ \to B_s^0 K^{*+}$ | 0.058 |
| $B_c^+ \to B_s^0 e^+ \nu$ | 4.03  | $B_c^+ \to J/\psi D^+_c$ | 0.17  | $B_c^+ \to B_s^0 K^{*0}$ | 1.98  |
| $B_c^+ \to B_s^0 e^+ \nu$ | 5.06  | $B_c^+ \to J/\psi D^+_c$ | 0.67  | $B_c^+ \to B_s^0 K^{*0}$ | 0.43  |
| $B_c^+ \to B_s^0 c e^+ \nu$ | 0.34  | $B_c^+ \to \eta_c D^+_c$ | 0.015 | $B_c^+ \to B_s^0 K^{*0}$ | 1.60  |
| $B_c^+ \to B_s^0 \tau^+ \nu$ | 0.58  | $B_c^+ \to \eta_c D^+_c$ | 0.010 | $B_c^+ \to B_s^0 K^{*0}$ | 1.67  |
| $B_c^+ \to B_s^0 \pi^+$ | 0.20  | $B_c^+ \to J/\psi D^+$ | 0.009 | $B_c^+ \to B_s^0 \pi^+$ | 0.037 |
| $B_c^+ \to B_s^0 \rho^+$ | 0.42  | $B_c^+ \to J/\psi D^+$ | 0.028 | $B_c^+ \to B_s^0 \rho^+$ | 0.034 |
| $B_c^+ \to J/\psi \pi^+$ | 0.13  | $B_c^+ \to B_s^0 \pi^+$ | 16.4 | $B_c^+ \to B_s^0 \pi^+$ | 0.033 |
| $B_c^+ \to J/\psi \rho^+$ | 0.40  | $B_c^+ \to B_s^0 \rho^+$ | 7.2  | $B_c^+ \to B_s^0 \rho^+$ | 0.09  |
| $B_c^+ \to \eta K^+$ | 0.013 | $B_c^+ \to B_s^0 \pi^+$ | 6.5  | $B_c^+ \to \tau^+ \nu$ | 1.6  |
| $B_c^+ \to \eta K^{*+}$ | 0.020 | $B_c^+ \to B_s^0 \rho^+$ | 20.2 | $B_c^+ \to c\bar{s}$ | 4.9  |

[1] J.D.Bjorken, draft report 07/22/86 (1986) [unpublished].
[2] K. Anikeev et al., arXiv:hep-ph/0201071
[3] S. S. Gershtein, V. V. Kiselev, A. K. Likhoded and A. V. Tkabladze, Phys. Usp. 38, 1 (1995) [Usp. Fiz. Nauk 165, 3 (1995)] arXiv:hep-ph/9504139
[4] V. V. Kiselev and A. K. Likhoded, Usp. Fiz. Nauk 172, 497 (2002) [arXiv:hep-ph/0103109];
V. V. Kiselev, A. K. Likhoded, O. N. Pakhomova and V. A. Saleev, Phys. Rev. D 66, 034030 (2002) arXiv:hep-ph/0206140.
[5] C.T.Bodwin, E.Braaten, G.P.Lepage, Phys. Rev. D51, 1125 (1995) [Erratum-ibid. D55, 5853 (1995)];
V. A. Saleev, Yad. Fiz. 64 (2001) (in press) [hep-ph/0007352];
V. V. Kiselev, O. N. Pakhomova and V. A. Saleev, J. Phys. G 28, 595 (2002) [arXiv:hep-ph/0110180];
G. Lopez Castro, H. B. Mayorga and J. H. Munoz, J. Phys. G 28, 2241 (2002) [arXiv:hep-ph/0205273].

[39] D. s. Du, X. l. Li and Y. d. Yang, Phys. Lett. B 380, 193 (1996) [arXiv:hep-ph/9603291];
S. Fajfer, S. Prelovsek and P. Singer, Phys. Rev. D 59, 114003 (1999) [Erratum-ibid. D 64, 099903 (2001)]
[arXiv:hep-ph/9901252];
T. M. Aliev and M. Savci, Phys. Lett. B 480, 97 (2000) [arXiv:hep-ph/9908203].

[40] M. Masetti, Phys. Lett. B 286, 160 (1992);
Y. S. Dai and D. S. Du, Eur. Phys. J. C 9, 557 (1999) [arXiv:hep-ph/9809386];
J. F. Liu and K. T. Chao, Phys. Rev. D 56, 4133 (1997),
R. Fleischer and D. Wyler, Phys. Rev. D 62, 057503 (2000) [arXiv:hep-ph/0004010].

[41] C. Quigg, FERMILAB-CONF-93-265-T, in Proc. of the Workshop on B Physics at Hadron Accelerators, Snowmass, Colorado, 1993, eds. P. McBride and C. S. Mishra [Fermilab, Batavia (1994)].