Solving Vertical Body Motion with Quadratic Resistance using Closed-form Solution for Ricatti Differential Equation of Constant Coefficients based on Discriminant Criteria

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Abstract. One of problem in mechanics is vertical body motion in upward and downward direction due to gravitational and resistance forces. In this paper, we are interested to analyse the effect of quadratic resistance force respect to the instantaneouly velocity and position of the body. The equation of the vertical body motion affected by gravitational and quadratic resistance forces derived using the second Newton law generally in the form of Riccati differential equation of constant coefficients. Therefore we need a method for solving the mathematics model. For the purpose we introduce a closed form solution for Ricatti differential equation of constant coefficients based on discriminant criteria that can be used to solve the analytic model of vertical body motion by simple manner.

1. Introduction
The analytic solution of the vertical body motion such as the motion of a projectile in upward and downward direction with zero air drag force in the form of parabolic equation is well known. The parabolic equation can be obtained easily by using basic integral. However, for practical situation, such as a ball thrown vertically experiencing quadratic resistance force besides the gravitational force. Hence, the equation of the vertical body motion derived from the second Newton’s law probably need more mathematical treatment. The equation of the vertical body motion for finding the instantaneouly velocity in a quadratic resistance force generally are in the form of Riccati differential equation of constant coefficients [1]. There are so many methods of solving the Riccati differential equation [2-13] which yeilding the exact and approximate solutions. The method introduced in this paper corresponds to the exact method which solving the Riccati differential equation of constant coefficients by specifying its discriminant. Further, we introduce the exact solution as the closed-form solution of the Riccati differential equation of constant coefficients. For the special application we use the closed-form solution of the Riccati differential equation as analytic tool for solving the equation of vertical body motion in resistive medium, especially for the medium that causes quadratice resistance force for the movement of body.

In order to ensure the validity of the closed-form solution of Riccati differential equation of constant coefficients, firstly we will show the existence of the identity of trigonometric function \( \sec^2(\theta) = \tan^2(\theta) + 1 \) and the identity of hyperbolic function \( \text{sech}^2(\theta) + \tanh^2(\theta) = 1 \) for the case of negative and positive discriminant repectively. Secondly, we verify the validity of closed-form solution of the Riccati differential equation of constant coefficients by proving that the round trip time of vertical body motion from ground-to-ground in the quadratic resistive medium is different to the round trip time in the
medium with zero air drag as obtained in [1]. Finally we will show that the definition of terminal velocity obtained using the closed-form solution of the Riccati differential equation of constant coefficients equals to the definition of terminal velocity obtained using the method of partial fractions in [1,3,14-15].

2. Basic Theory

The Riccati differential equation is an important first order nonlinear ordinary differential equation for applications in science and engineering. Recently the interested applications of the Riccati differential equation broads from random processes and optimal control until on development some of fundamental theoretical. Approximate analytical methods have been widely used to solve the Riccati differential equation, such as homotopy method [2], fractional order method [4], differential transform method [8], and adomian decomposition method [13]. Meanwhile the exact method of solving the Riccati differential equation are covered in [3,7,9,12]. In this paper we create the closed-form solution for Riccati differential equation of constant coefficients by using the change variable method which specifying the solution based on discriminat creteria.

2.1 Riccati differential equation of constant coefficients

The Riccati differential equation of constant coefficients which we call as RDE is of form,

\[ \frac{dy}{dx} = ay^2 + by + c. \]  

(1)

Besides the RDE consists of independent variable \( x \) and the dependent variable \( y(x) \), it also contains three real constant coefficients \( a, b, \) and \( c \). To obtain the exact solution of the RDE we perform the following change variable,

\[ y(x) = u(x) - \frac{b}{2a}, \]  

(2)

and involving the discriminant definition of the RDE,

\[ D = b^2 - 4ac. \]  

(3)

2.2 Solving the RDE for negative discriminant

We start with negative discriminant \( (D < 0) \) that is,

\[ D = -|D|, \]  

(4)

therefore the RDE of (1) reduces to the form,

\[ \frac{du}{dx} = au^2 + \frac{|D|}{4a}. \]  

(5)

In the equation (4) and (5), the symbol of \( |D| \) denotes absolute of the discriminant. Next, we solve (5) by integration method viz the change variable,

\[ v = \frac{u}{\sqrt{|D|}}, \]  

(6)

therefore the equation (5) transforms into the following integral form :

\[ \frac{dv}{1+v^2} = \frac{\sqrt{|D|}}{2} \, dx. \]  

(7)

Integrating both sides of (7) gives,

\[ \tan^{-1} v = \frac{\sqrt{|D|}}{2} x + C, \]  

(8)
where \( C \) is the integrating constant. Next, from (8) we find the solution of \( u(x) \) of (5) is in tangent function,

\[
 u(x) = \frac{\sqrt{|D|}}{2a} \tan \left( \frac{\sqrt{|D|}}{2} x + C \right),
\]

(9)

Substituting the \( u(x) \) of (9) into equation (2) yielding,

\[
 y(x) = -\frac{b}{2a} + \frac{\sqrt{|D|}}{2a} \tan \left( \frac{\sqrt{|D|}}{2} x + C \right),
\]

(10)

here the integrating constant \( C \) is determined from applying both initial conditions at \( x = x_0 \) in which \( y(x_0) = y_0 \) i.e.,

\[
 C = -\frac{\sqrt{|D|}}{2} x_0 + \tan^{-1} \left( \frac{2a y_0 + b}{\sqrt{|D|}} \right).
\]

(11)

Hence, the exact solution of the RDE for negative discriminant \( (D < 0) \) as of form,

\[
 y(x) = -\frac{b}{2a} + \frac{\sqrt{|D|}}{2a} \tan \left( \frac{\sqrt{|D|}}{2} (x - x_0) + \tan^{-1} \left( \frac{2a y_0 + b}{\sqrt{|D|}} \right) \right), \quad D < 0.
\]

(12)

One can verify the validity of the RDE’s solution by inserting (12) into (1) to prove the following trigonometric identity

\[
 \sec^2 \left( \frac{\sqrt{|D|}}{2} (x - x_0) + \tan^{-1} \left( \frac{2a y_0 + b}{\sqrt{|D|}} \right) \right) = \tan^2 \left( \frac{\sqrt{|D|}}{2} (x - x_0) + \tan^{-1} \left( \frac{2a y_0 + b}{\sqrt{|D|}} \right) \right) + 1.
\]

(13)

2.3 Solving the RDE for positive discriminant

In order to obtain the solution of RDE for positive discriminant \( (D > 0) \) we use RDE’s solution of negative discriminant in (12). Refer to the discriminant definition of (4) we take the square root of the absolute of discriminant as,

\[
 \sqrt{|D|} = -i \sqrt{D},
\]

(14)

where \( i = \sqrt{-1} \) is the imaginary number. Inserting (14) into (12), i.e.,

\[
 y(x) = -\frac{b}{2a} + \frac{-i \sqrt{|D|}}{2a} \tan \left[ -\frac{i \sqrt{|D|}}{2} (x - x_0) + \tan^{-1} \left( \frac{2a y_0 + b}{-i \sqrt{|D|}} \right) \right], \quad D > 0.
\]

(15)

However, when substituting (14) into (5) we find that for positive discriminant the RDE of (1) reduces to the form,

\[
 \frac{du}{dx} = au^2 - \frac{D}{4a}, \quad D > 0,
\]

(16)

which having solution in the form of hyperbolic tangent function. It means that we must convert the second term on R.H.S of (15) into hyperbolic tangent function, For this purpose we use the well known relations of hyperbolic functions for \( \theta \) argument [12], i.e.,

\[
 \tanh(\theta) = -i \tan(i \theta),
\]

(17)

and

\[
 \tanh^{-1}(\theta) = -i \tan^{-1}(i \theta).
\]

(18)

The relation of (17) and (18) are performed after writing (15) in the following form,
\[ y(x) = -\frac{b}{2a} + \frac{\sqrt{D}}{2a} \left\{ -i \tan \left[ i \left( \frac{1}{2} \sqrt{D} (x - x_0) - i \tan^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right) \right] \right\} \quad D > 0. \]  

Here we obtain respectively,

\[ -i \tan^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) = \tanh \left( \frac{2ay_0 + b}{\sqrt{D}} \right), \quad (20) \]

and

\[ -i \tan \left[ i \left( \frac{1}{2} \sqrt{D} (x - x_0) - i \tan^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right) \right] = \tanh \left[ -\frac{\sqrt{D}}{2} (x - x_0) - i \tan^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right]. \quad (21) \]

Next, by using the relation of (20) and (21) therefore form (19) we get the solution of RDE for positive discriminant in the form,

\[ y(x) = -\frac{b}{2a} + \frac{\sqrt{D}}{2a} \tanh \left[ -\frac{\sqrt{D}}{2} (x - x_0) + \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right], \quad D > 0. \quad (22) \]

Because both hyperbolic and archyperbolic tangent functions are odd function, therefore the RDE’s solution of (22) can also be written in the following form:

\[ y(x) = -\frac{b}{2a} - \frac{\sqrt{D}}{2a} \tanh \left[ \frac{\sqrt{D}}{2} (x - x_0) - \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right], \quad D > 0. \quad (23) \]

Again, one can verify the validity of the RDE’s solution of (22) and (23) by inserting each solution into (1) to get the following trigonometric identities,

\[ 1 = \sec h^2 \left[ -\frac{\sqrt{D}}{2} (x - x_0) + \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right] + \tanh^2 \left[ -\frac{\sqrt{D}}{2} (x - x_0) + \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right], \quad (24) \]

for the validity of (22), and

\[ 1 = \sec h^2 \left[ \frac{\sqrt{D}}{2} (x - x_0) - \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right] + \tanh^2 \left[ \frac{\sqrt{D}}{2} (x - x_0) - \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right], \quad (25) \]

for the validity of (23) respectively.

### 2.4 Examples of utilising the RDE’s solutions for certain form of RDE

\[ \frac{dy}{dx} = ay^2 + c. \quad (26) \]

Firstly we solve (26) for the case positive discriminant \((D > 0)\), here we use the RDE’s solution of (23) that gives solution as the form,

\[ y(x) = -\frac{\sqrt{D}}{2a} \tanh \left[ \frac{\sqrt{D}}{2} (x - x_0) - \tanh^{-1} \left( \frac{2ay_0 + b}{\sqrt{D}} \right) \right], \quad D > 0. \quad (27) \]

The following two examples correspond to the RDE of positive discriminant, i.e.,

**Example 1:**

\[ \frac{dy}{dx} = -3y^2 + 2; \quad x_0 = 0, y_0 = 0. \quad (28) \]

Here, \(a = -3, b = 0, c = 2\), and from (3) the discriminant is \(D = 0^2 - 4(-3)(2) = 24 > 0\), then by using (27) we find the solution of (28) in the form:
\[
y(x) = -\frac{\sqrt{24}}{2(-3)} \tanh \left( \frac{\sqrt{24}}{2} \right) = \frac{2}{\sqrt{3}} \tanh(\sqrt{6}x)
\]  
\]  
\[
\text{Example 2:}
\]
\[
\frac{dy}{dx} = 3y^2 - 2; \quad x_0 = 0, y_0 = 0.
\]
\]  
Here, \(a = 3, b = 0, c = -2\), and from (3) the discriminant is \(D = 0^2 - 4(3)(-2) = 24 > 0\), then by using (27) we find the solution of (30) in the form,
\[
y(x) = -\frac{\sqrt{24}}{2(3)} \tanh \left( \frac{\sqrt{24}}{2}x \right) = -\frac{2}{\sqrt{3}} \tanh(\sqrt{6}x)
\]  
\]  
Meanwhile, solving the RDE of (26) for negative discriminant \((D < 0)\) is based on RDE’s solution of (12), i.e.,
\[
y(x) = \frac{\sqrt{|D|}}{2a} \tan \left[ \frac{\sqrt{|D|}}{2}(x-x_0) + \tan^{-1}\left( \frac{2ay_0}{\sqrt{|D|}} \right) \right], \quad D < 0.
\]  
\]  
There are two examples that correspond to the RDE for negative discriminant, i.e.,
\]  
\[
\text{Example 3:}
\]
\[
\frac{dy}{dx} = 3y^2 + 2; \quad x_0 = 0, y_0 = 0.
\]
\]  
Here, \(a = 3, b = 0, c = 2\), and from (3) the discriminant is \(D = 0^2 - 4(3)(2) = -24 < 0\), then by using (32) we get the solution of (33) in the form,
\[
y(x) = \frac{\sqrt{24}}{2(3)} \tan \left( \frac{\sqrt{24}}{2}x \right) = \frac{2}{\sqrt{3}} \tan(\sqrt{6}x)
\]  
\]  
\[
\text{Example 4:}
\]
\[
\frac{dy}{dx} = -3y^2 - 2; \quad x_0 = 0, y_0 = 0.
\]
\]  
Here, \(a = -3, b = 0, c = -2\), and from (3) the discriminant is \(D = 0^2 - 4(-3)(-2) = -24 < 0\), then by using (32) we get the solution of (35) in the form,
\[
y(x) = \frac{\sqrt{24}}{2(-3)} \tan \left( \frac{\sqrt{24}}{2}x \right) = -\frac{2}{\sqrt{3}} \tan(\sqrt{6}x)
\]  
\]
\]  
3. Applying closed-form solution of RDE for solving the equation of vertical body motion and Discussion

In this section we begin our discussion with the governing equation of vertical motion for a body which experiences gravitational and quadratic resistance forces. The equation of vertical body motion for both upward and downward directions are derived based on the second Newton’s law.

3.1 Upward vertical body motion

Important to be known that in this paper the motion is considered occurs in the common cartesian coordinate, and the equation of motion is governed vectorially. Here the direction of motion is taken in positive y-axis. As we know, the gravitational force \(mg\) (the quantity \(m\) is the mass of body, while \(\xi\) is the gravitational acceleration) always leads to negative y-axis. The gravitational force is denoted as \(mg\mathbf{j}\), where \(\mathbf{j}\) is the unit vector of y-axis. Meanwhile the resistance force is in opposite to the direction of motion, thus in this case it leads to negative y-axis. Here the magnitude of the quadratic resistance force is defined as \(f_r = km(\mathbf{v}_0^+ + \mathbf{v}_0^-)\), where \(k\) is constant and \(\mathbf{v}_0\) in the bracket corresponds to
upward direction of velocity. According to second Newton’s law the corresponding equation of upward motion is of form

$$mg\hat{j}+km\left(v_{up}\right)^2\hat{j}=m\frac{dv_{up}}{dt}. \tag{37}$$

In the R.H.S of (37) the velocity is taken in vector symbol. Because the body moving in upward direction it means that the velocity of the body $\vec{v}_{up}$ leads to positive $y$-axis. With this convention we can remove vector symbol in (37), hence it can be written as following form,

$$\frac{dv_{up}}{dt}=-kv_{up}^2-g. \tag{38}$$

Appear that the form of (38) satisfies the RDE’s form of (26), therefore we can solve (38) for obtaining the velocity $v_{up}(t)$ by using (27) and/or (32) depending on the value of its discriminant. Starting with checking the RDE’s discriminant of (38) by using the discriminant definition in (3). The RDE having two coefficients $a=-k$ and $c=-g$, and having negative discriminant, i.e., $D=0^2-4(-k)(-g)=-4kg$. Therefore we solve (38) by using (32), but here the independent variable $x$ and dependent variable $y(x)$ of (32) change into $t$ and $v_{up}(t)$ respectively, then we thus find :

$$v_{up}(t)=\frac{\sqrt{|D|}}{2a}\tan\left(\frac{\sqrt{|D|}}{2}(t-t_0)\right)+\tan^{-1}\left(\frac{2av_0}{\sqrt{|D|}}\right). \tag{39}$$

Suppose at $t_0=0$ the initial velocity is $v_0$, then after taking $a=-k$ and $\sqrt{|D|}=2\sqrt{kg}$ into (39), we find :

$$v_{up}(t)=-\frac{g}{k}\tan\left(\sqrt{kt}-\tan^{-1}\left(v_0\sqrt{\frac{k}{g}}\right)\right). \tag{40}$$

Here we have applied the odd properties of arctangent function i.e., $\tan^{-1}\left(-v_0\sqrt{\frac{k}{g}}\right)=-\tan^{-1}\left(v_0\sqrt{\frac{k}{g}}\right)$. As conventioned previously that the velocity of body leads to positive $y$-axis, therefore we must write (40) as positive quantity. This can be governed by applying the odd properties of tangent function, so that (40) becomes :

$$\vec{v}_{up}(t)=\frac{g}{k}\tan\left(\tan^{-1}\left(v_0\sqrt{\frac{k}{g}}\right)-\sqrt{kt}\right). \tag{41}$$

The physical meaning of (41) corresponds to reducing the magnitude of velocity until zero when the body reaches its maximum height before falling back to the ground.

After obtaining velocity of body, further our task is go on to determine the position of body $y_{up}(t)$ from definition of velocity $\vec{v}_{up}(t)=\frac{dy_{up}}{dt}\hat{j}$. We can obtain the position of body by integrating (41) respect to time as following, 

$$y_{up}(t)-y_0=\int_0^ty(v)dt, \tag{42}$$
here, \( y_{up}(0) = y_0 \) is the initial position at the ground. Further, after inserting the velocity of (41) into (42) and setting \( y_0 = 0 \), thus we find the position of body as,

\[
y_{up}(t) = \int_0^t g \frac{1}{k} \tan^{-1} \left( \frac{v_0 \sqrt{k}}{g} - \sqrt{k g \tau} \right) d\tau.
\]  

(43)

To carried out the integration of (43) we use the common indefinite integral \( \int \tan(z) dz = -\ln(\cos(z)) + C \) that gives,

\[
y_{up}(t) = \frac{1}{k} \ln \cos \left( \tan^{-1} \left( v_0 \sqrt{k} \frac{1}{g} - \sqrt{g k t} \right) \right) \frac{1}{k} \ln \cos \left( \tan^{-1} \left( v_0 \sqrt{k} \frac{1}{g} \right) \right).
\]  

(44)

The second term in the R.H.S of (44) can be simplified by using the following identity [1],

\[
\cos(\tan^{-1} \alpha) = \frac{1}{\sqrt{1 + \alpha^2}}.
\]  

(45)

Hence we find the position of body above ground as function of time is in the form,

\[
y_{up}(t) = \frac{1}{k} \ln \left\{ \frac{1}{1 + \left( v_0 \sqrt{k \frac{1}{g}} \right)^2} \cos \left( \tan^{-1} \left( v_0 \sqrt{k} \frac{1}{g} - \sqrt{g k t} \right) \right) \right\}.
\]  

(46)

3.2 Downward vertical body motion

For a body moving in downward direction or leads to negative \( y \)-axis, the gravitational force still leads to negative \( y \)-axis i.e., \( m g \left( - \hat{j} \right) \). Meanwhile the quadratic resistance force leads to upward that is \( f_r = k m \left( - v_{down} \right)^2 \left( \hat{j} \right) \). Here \( - v_{down} \) corresponds to the direction of body motion that leads to negative \( y \)-axis.

According to the second Newton’s law, the corresponding equation of downward motion is of form

\[
m g \left( - \hat{j} \right) + k m \left( - v_{down} \right)^2 \left( \hat{j} \right) = m \frac{d \vec{v}_{down}}{dt},
\]  

(47)

where the direction of velocity \( \vec{v}_{down} \) leads to negative \( y \)-axis. After removing the vector symbol, then (47) reduces to the form,

\[
\frac{d v_{down}}{dt} = k v_{down}^2 - g.
\]  

(48)

Appear the form of (48) also satisfies the RDE of certain form in (26), with two coefficients \( a = k \) and \( c = -g \). Because the RDE having positive discriminant i.e., \( D = 0^2 - 4(k)(-g) = 4kg \), then we solve (48) by using (27). But here the independent variable \( x \) and dependent variable \( y(x) \) of (27) must be changed into \( t \) and \( v_{down}(t) \) respectively. After taking the initial velocity of body falling as zero, we find the solution of (48) in the form,

\[
v_{down}(t) = -\frac{\sqrt{4kg}}{2k} \tanh \left( \frac{\sqrt{4kg}}{2} t - \tan^{-1} \left( \frac{2k(0)}{\sqrt{4kg}} \right) \right) - \sqrt{\frac{g}{k}} \tanh \left( \sqrt{k} t \right).
\]  

(49)

where the \((-\) sign means that the velocity of body leads to negative \( y \)-axis as appear in the following vectorial form,

\[
\vec{v}_{down}(t) = \sqrt{\frac{g}{k}} \tanh \left( \sqrt{k} t \right) \hat{j}.
\]  

(50)
Next, we go to determine the position of body any time after falling from the highest position \( y_0 = H \) measured from the ground. The position of body \( y_{\text{down}}(t) \) is determined by integrating the velocity of (50), here we use the indefinite integral \( \int \tanh(z) dz = \ln \cosh(z) + C \) and we thus find,

\[
y_{\text{down}}(t) - H = \int_0^t \left( -\frac{g}{k} \tanh\left(\sqrt{gk} \tau \right) \right) d\tau = -\frac{1}{k} \left[ \ln \cosh\left(\sqrt{gkt}\right) - \ln \cosh(0) \right]
\]

Because \( \ln \cosh(0) = 0 \) then the position of body above ground is of form,

\[
y_{\text{down}}(t) = H - \frac{1}{k} \ln \cosh\left(\sqrt{gkt}\right)
\]

3.3 Discussion

In this section we analyse the result of applying the closed-form solution of RDE in solving the equation of vertical body motion due to the quadratic resistance and gravitational forces. We have utilised the closed-form of RDE for obtaining the velocity of body required to determine both travel time and position of the body. Our task further is to verify the validity of the RDE’s solution. Here the verification is performed to prove that the round trip time from ground-to-ground in the quadratic resistive medium is different to the round trip time in the zero air drag.

We begin our analysing with the vertical body motion in upward direction. The main physical quantities of this upward motion those are the highest position that can be reached by body \( y_{\text{max}} = H \), and the travel time \( t_{y_{\text{max}}} \) needed to reach the highest position. The travel time to reach the highest position commonly is determined by setting the value of velocity of body at the highest position of (41) as zero. But here in order to check the validity of applying the closed form solution of RDE, we determine the velocity from the derivative of position of (46) respect to time, i.e.,

\[
\frac{dy_{\text{up}}}{dt} = \frac{1}{k} \left[ 1 + \left( \frac{v_0}{\sqrt{g}} \right) \right] - \frac{\sin^{-1}\left( \frac{v_0}{\sqrt{g}} \right) - \sqrt{gkt}}{\cos^{-1}\left( \frac{v_0}{\sqrt{g}} \right) - \sqrt{gkt}}
\]

Next, we set the velocity (53) to be zero at the travel time reaching the highest position, that is

\[
\left. \frac{dy_{\text{up}}}{dt} \right|_{t = t_{y_{\text{max}}}} = 0; \quad \sin^{-1}\left( \frac{v_0}{\sqrt{g}} \right) - \sqrt{gkt_{y_{\text{max}}}} = 0
\]

From (54) we obtain the travel time to reach the highest position as the form,

\[
t_{y_{\text{max}}} = \frac{1}{\sqrt{gk}} \sin^{-1}\left( \frac{v_0}{\sqrt{g}} \right)
\]

By using the travel time of (55) thus we get the highest position of body measured from the ground as following form,

\[
H = y(t_{y_{\text{max}}}) = \frac{1}{k} \ln \left[ 1 + \left( \frac{v_0}{\sqrt{g}} \right)^2 \cos^{-1}\left( \frac{v_0}{\sqrt{g}} \right) - \sqrt{gkt_{y_{\text{max}}}} \right]
\]

after inserting \( t_{y_{\text{max}}} \) of (55) we get
\[ H = \frac{1}{k} \ln \left\{ 1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2 \right\} \]  

(56)

Further, we formulate the average velocity of body along its motion path from the ground until the highest position viz the definition \( \vec{v}_{\text{avg,up}} = \frac{(H - 0) \hat{j}}{t_{\text{ymax}} - 0} \) i.e., and we get

\[ \vec{v}_{\text{avg,up}} = \ln \left\{ 1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2 \right\} - \frac{k \tan^{-1}\left( \frac{k}{v_0 \sqrt{g}} \right)}{g} \hat{j} \]  

(57)

Now, we analyse the motion of body in downward direction. The travel time of body from the highest position to ground \( (t_{\text{down}}) \) is obtained from expression of the position of body from the ground, that is from (52). But here we denote \( y(t) = y_{\text{down}}(t) \),

\[ y_{\text{down}}(t) = H - \frac{1}{k} \ln \cosh \left( \sqrt{gkt} \right) \]  

(58)

Setting \( y_{\text{down}}(t) = 0 \) we find the relation between travel time to ground \( t_{\text{down}} \) with the highest position \( H \) as following,

\[ t_{\text{down}} = \frac{1}{\sqrt{gk}} \cosh^{-1}(e^{kH}) \]  

(59)

Inserting \( H \) of (56) into (59) gives,

\[ t_{\text{down}} = \frac{1}{\sqrt{gk}} \cosh^{-1} \left\{ \frac{k}{\sqrt{g}} \ln \left( 1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2 \right) \right\} = \frac{1}{\sqrt{gk}} \cosh^{-1} \left( \sqrt{1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2} \right) \]  

(60)

Next, by using (60) we define the average velocity of body along its motion path from highest position to ground,

\[ \vec{v}_{\text{avg,down}} = \frac{(0 - H) \hat{j}}{t_{\text{down}} - 0} = \ln \left\{ 1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2 \right\} = \ln \left\{ \frac{1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2}{\sqrt{1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2}} \right\} \]  

(61)

And defining the resultant of average velocity along the round trip as \( \vec{v}_{\text{avg}} = \vec{v}_{\text{avg,up}} + \vec{v}_{\text{avg,down}} \) then we find,

\[ \vec{v}_{\text{avg}} = \left[ \ln \left\{ 1 + \left( \frac{k}{v_0 \sqrt{g}} \right)^2 \right\} - \frac{k}{\sqrt{g}} \cosh^{-1} \left( \frac{k}{v_0 \sqrt{g}} \right) \right] \hat{j} \]  

(62)
As appear in the equation of (62) because \( \cosh^{-1} \left( \sqrt{1 + \left( \frac{v_0}{g} \right)^2} \right) \neq \cos^{-1} \left( \sqrt{1 + \left( \frac{v_0}{g} \right)^2} \right) \) for \( v_0 \neq 0 \) then the resultant of average velocity of the body along the round trip is not equal to zero (\( v_{\text{avg}} \neq 0 \)). Of course this is caused by the existency of quadratic resistance force in the resistive medium that affected the body movement along vertical path.

Now we formulate the total of round trip time as summation of travel time of upward motion from ground to highest position of (55) that is \( t_{\text{up}} = t_{\text{max}} \), and travel time of downward motion from highest position to ground \( (t_{\text{down}}) \) of (60). For this purpose the equation (60) is written in the new form according to [16] so that consisting the travel time of upward motion. The new form of \( t_{\text{down}} \) is in the form,

\[
t_{\text{down}} = \frac{1}{\sqrt{gk}} \ln \left( \frac{k}{g} \right) + \frac{1}{\sqrt{gk}} \ln \left( 1 + \left( \frac{1}{\sqrt{v_0 g}} \right)^2 \right)
\]

that can be expand to the form,

\[
t_{\text{down}} = \frac{1}{\sqrt{gk}} \ln \left( \frac{k}{g} \right) + \frac{1}{\sqrt{gk}} \ln \left( 1 + \left( \frac{1}{\sqrt{v_0 g}} \right)^2 \right)
\]

Finally from (55) and (64) we find the round trip time of vertical body motion in the resistive medium that excerts quadratic resistance force as the form,

\[
t_{\text{up}} + t_{\text{down}} = \frac{2}{\sqrt{gk}} \tan^{-1} \left( \frac{v_0}{\sqrt{gk}} \right) + \frac{1}{\sqrt{gk}} \ln \left( 1 + \left( \frac{1}{\sqrt{v_0 g}} \right)^2 \right)
\]

Because first term of R.H.S of (65) can be approaced as \( \frac{2v_0}{g} \) then we take a conclusion that the round trip time of vertical body motion in the resistive medium is more than \( \frac{2v_0}{g} \) that we have known as the round trip in the medium of zero air drag. By inspection to the equations (62) and (65) we also conclude that \( \sqrt{\frac{g}{k}} \) is the main quantity as the caused discrepancy the average velocity and round trip time in the resistive medium to the medium with zero air drag. According to diminsion, as appear in the equation (50), the \( \sqrt{\frac{g}{k}} \) being as the quantity of terminal velocity of body that moving in downward direction.

4. Conclusion

After implementing the closed-form solution for Riccati differential equation of constant coefficients (RDE) by discriminant criteria, we take the conclusion: the solution form of RDE which having negative discriminant \( (D < 0) \) are in tangent function, while the solution form of RDE which having positive discriminant \( (D > 0) \) are in hyperbolic tangent function; the vertical body motion in upward direction corresponds to the negative discriminant, otherwise the downward vertical body motion corresponds to the positive discriminant; and the quantity causing the discrepancy of the round trip time and average velocity of vertical body motion compared to the medium of zero air drag is terminal velocity that consists the gravitational acceleration and the proportional constant of quadratic resistance force.
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