Phase Space Structure of Generalized Gaussian Cat States

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We analyze generalized Gaussian cat states obtained by superposing arbitrary Gaussian states, e.g., a coherent state and a squeezed state. The Wigner functions of such states exhibit the typical pair of Gaussian hills plus an interference term which presents a novel structure, as compared with the standard superposition of coherent states (degenerate case). We prove that, in any dimensions, the structure of the interference term is characterized by a particular quadratic form; in one degree of freedom the phase is hyperbolic. This phase-space structure survives the action of a thermal reservoir. We also discuss certain superpositions of mixed Gaussian states generated by conditional Gaussian operations or Kerr-type dynamics on thermal states.

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I. INTRODUCTION

The creation of quantum superpositions of living organisms, what used to be merely a ridiculous possibility, may become feasible in a near future. Of course, the center-of-mass states of a microbe, discussed by Romero-Isart et al., are still very distant from the fantastic dead/alive Schrödinger cat. In any case, for the time being, one will have to be satisfied with inorganic quantum superpositions which, however, may involve a large number of particles (“cat states”). At present they can be generated in the laboratory in a variety of systems, e.g., optical cavities, superconducting devices, Bose-Einstein condensates, free-propagating light beams, etc. Not only are such states crucial for testing the nonlocality of quantum mechanics outside the microscopic domain, but they also play an important role in some applications of quantum physics like information processing and quantum metrology.

The nonlocal quality of a cat state is most clearly displayed when it is looked through the Wigner representation. For instance, in the case of a superposition of two coherent states, the Wigner function exhibits two Gaussian “hills”, corresponding to a classical superposition, and a nonclassical third Gaussian modulated by interference fringes halfway between the pair. This interference pattern is composed of alternating positive and negative straight bands parallel to the line joining the two Gaussian peaks.

The distributive property of the Wigner function allows one to construct the interference pattern of the superposition of several (or many) coherent states from the corresponding Wigner function for each pair. In fact, as the set of coherent states is complete, any given pure state can be expressed as a coherent-state superposition: its Wigner function becomes a sum of cat-state Wigner functions. This observation is sometimes enough to understand qualitative properties of general superpositions, like sensitivity to perturbations, or to the action of the environment (decoherence). In some proposed experiments the generated states consist of superpositions of coherent states lying on a manifold in phase space. This is the case, e.g., of the so-called arc states. The Wigner function of such a state is characterized by a banana-shaped positive region decorated by interference fringes. Reciprocally, a state whose Wigner function is localized in the vicinity of a phase-space manifold, can be fitted by a combination of coherent states concentrated on that manifold.

Consider now the evolution of a superposition of coherent states under a nonlinear unitary dynamics. In semiclassical regimes, for short times, an individual coherent state moves along a classical trajectory while suffering (approximately) a linear distortion. So, the wavepacket remains Gaussian but may become squeezed and rotated. In general, different coherent states of the superposition will be distorted in different ways. This takes us to the following basic question: what happens to the Wigner function of a superposition of two coherent states when they are linearly, but independently, distorted? For instance, what is the interference structure of a superposition of a coherent state and a squeezed state?

This paper is dedicated to the study of the coherence structure of generalized Gaussian cats in the Wigner representation. In the pure case, a Gaussian cat is, by definition, just a superposition of pure Gaussian states. Even though there is no natural way of defining what a superposition of mixed states would be, there are certain dynamics or protocols which produce what may be called mixed Gaussian cat states, given that the Wigner function of such states are sums of Gaussians. We analyze two classes of mixed cats: (i) pure cats that suffered the decohering action of a thermal environment, and (ii) cats...
generated either by conditional Gaussian operations or by a Kerr-like dynamics on input thermal states. In all cases we focus on the geometric structure of the interference fringes.

The paper is organized as follows. We start by presenting our original dynamical motivation for studying generic superpositions of Gaussian states (Sect. II). In Sect. III we summarize the geometrical formalism behind the Wigner representation which will be instrumental for deriving most of the results of this paper. In Sect. IV we analyze in detail the problem of a superposition of arbitrary pure Gaussian states. We show that for one-degree of freedom (two dimensional phase space) the interference pattern is in general hyperbolic (instead of the linear pattern exhibited by superposition of coherent states). For higher dimensions we derive a general normal form linear effects can be clearly seen after two pulses (kicks).

In both cases the dynamics generated by a Kerr-like dynamics on input thermal states. In all cases we make the approximation that, in a suitable canonical coordinate system, the fringe pattern is still hyperbolic in each canonical plane. Curiously, this pattern survives the action of a classical trajectory while preserving its shape (frozen semiclasical nature [18]).

Our approximate description of the final state starts by decomposing the initial squeezed state \( |\psi_0\rangle \) into a suitable superposition of coherent states which are themselves propagated approximately by taking advantage of their quasiclassical nature [18].

In the crudest scheme, each coherent state moves along a classical trajectory while preserving its shape (frozen Gaussian approximation). In the next level of improvement, an initial coherent state stays Gaussian during its evolution but may become squeezed and rotated (thawed Gaussian approximation) [19]. Both schemes correspond to approximating the exact Hamiltonian by its Taylor expansion around the instantaneous center of the wavepacket \( (q(t), p(t)) = x(t) \):

\[
H(\hat{x}, t) = H(x(t)) + (\hat{x} - x(t)) \left[ \frac{\partial H}{\partial \hat{x}} \right]_{x(t)} + (\hat{x} - x(t)) \left[ \frac{\partial^2 H}{\partial \hat{x}^2} \right]_{x(t)} (\hat{x} - x(t)) + \ldots \tag{1}
\]

Here \( \hat{x} \) represents the pair of canonical operators \( (\hat{q}, \hat{p}) \).

The frozen/thawed schemes are obtained by truncating the Taylor expansion to first/second order in \( \hat{x} \) [17, 19]. In both cases the dynamics generated by \( H(\hat{x}, t) \) is linear.

Let us illustrate the semiclassical schemes above with a numerical example. Consider the propagation of a squeezed state in the Kicked Harmonic Oscillator (KHO). In suitable units the KHO Hamiltonian reads [21]:

\[
H(\hat{q}, \hat{p}, t) = \frac{1}{2} (\hat{p}^2 + \hat{q}^2) + K \cos(\hat{q}) \sum_{n=-\infty}^{\infty} \delta(t - n\tau). \tag{2}
\]

We choose an initial state \( |\psi_0\rangle \) which is centered at the phase-space origin and squeezed along the \( q \)-axis. By appropriate choice of the parameters \( \tau \) and \( K \), strong nonlinear effects can be clearly seen after two pulses (kicks). See Fig. I.

![Figure 1](image.png)

**FIG. 1:** (color online) Nonlinear dynamics in the kicked harmonic oscillator. Shown are Wigner functions of (a) the squeezed state \( |\psi_0\rangle \) at \( t = 2 \); (b) the initial squeezed state \( |\psi_0\rangle (t = 0) \), which is centered at the origin but has been displaced for clarity; (c) a coherent state, displayed for reference. Parameters are \( K = 2, \tau = \pi/3, \hbar = 0.0128 \).

Our approximate description of the final state starts by decomposing the initial squeezed state \( |\psi_0\rangle \) in terms of a one-parameter family of coherent states \( |\phi(q')\rangle \) with centers lying on the line \( p = 0 \):

\[
|\psi_0\rangle = \int_{-\infty}^{+\infty} dq' \, C(q') \, |\phi(q')\rangle, \tag{3}
\]

where \( \langle q|\psi_0\rangle, \langle q'|\phi(q')\rangle \) and \( C(q') \) are real Gaussians [21] (in numerical calculations the integral is substituted by a finite sum). Then we propagate each \( |\phi(q')\rangle \) using the instantaneous local quadratic Hamiltonian, i.e., in Eq. (2) we make the approximation

\[
\cos(\hat{q}) \approx a(t) + b(t) \hat{q} + c(t) \hat{q}^2, \tag{4}
\]

with \( a(t) = \cos(q(t)), \) etc. In this way the final state becomes a superposition of Gaussian states, their centers threaded by a curved manifold in phase space (see Fig. 2).

Figure 3 presents a numerical comparison between approximate and exact Wigner functions. The excellent...
agreement indicates that the global nonlinear dynamics of the initial state can be understood, both qualitatively and quantitatively, as the collective effect of a swarm of localized states evolving linearly along classical trajectories.

The interference pattern of the Wigner function of Fig. 1 can now be viewed as arising from the superposition of the elementary patterns for each pair of final Gaussian states. In principle each wavepacket suffers a different distortion, so, the cat states obtained by superposing pairs of wavepackets are in general nonstandard in the sense that their interference fringes are not parallel. Inspection of Fig. 2 suggests that in the general case the interference fringes may be hyperbolic.

In Sect. IV below we formulate and prove a precise statement about the hyperbolicity of the interference pattern of the superposition of two arbitrary pure Gaussian states. But first we need to review some mathematical results and notations which are extremely useful when dealing with Wigner functions and their related characteristic functions.

### III. MATHEMATICAL FORMALISM

Most of the formulae of this section are widespread in the literature, sometimes under different notations [17, 22–24]. For the sake of self-completeness we present them compactly using a unified notation. The idea of the present formalism is to exploit the geometry of translations and reflections behind the Wigner representation [22]. In this way, many of the cumbersome integrals usually needed in Wigner calculus are reduced to simple algebra.

#### A. Translations and Reflections

Given a quantum state described by the density operator $\hat{\rho}$, its Wigner function is defined by

$$W(x) = \frac{1}{(2\pi\hbar)^n} \int_{-\infty}^{+\infty} dq' \langle q + \frac{1}{2}q' | \hat{\rho} | q - \frac{1}{2}q'\rangle e^{-ip\cdot q'/\hbar}. \quad (5)$$

We assume that the system possesses $n$ degrees of freedom. Canonical coordinates will be represented collectively by the column vectors $q = (q_1, \ldots, q_n)^T$ and $p = (p_1, \ldots, p_n)^T$ (of course, $\top$ means “transposed”). A point in phase space is characterized by a $2n$-dimensional column vector $x = (q_1, \ldots, p_n)^T$. A similar notation is used for operators $\hat{q}$, $\hat{p}$, and $\hat{x}$, e.g., $\hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)^T$. (As an alternative to the mechanical point of view, one may adopt an optical perspective: If $\hat{\rho}$ represents the state of an $n$-mode quantum radiation field, then $q$ and $p$ are conjugate quadrature-vectors of the field.)

Closely related to the Wigner function, the (symmetric ordered) characteristic function $\chi(\xi)$ is defined as the expectation value of the phase-space translation operator:

$$\chi(\xi) = \frac{1}{(2\pi\hbar)^n} \text{tr} \left( \hat{\rho} \hat{T}_{\xi} \right), \quad (6)$$

with $\xi = (\xi_q, \xi_p)^T$, and

$$\hat{T}_{\xi} = e^{i\xi\cdot \hat{x}/\hbar}. \quad (7)$$
In the last equation we have introduced the wedge (symplectic) product:
\[
\xi \wedge \dot{\xi} = \xi_p \cdot \dot{\xi}_q - \xi_q \cdot \dot{\xi}_p,
\] (8)
which can be turned into an ordinary scalar product (or matrix product) with the help of the symplectic matrix \(J\),
\[
\xi \wedge \dot{\xi} = (J\xi) \cdot \dot{\xi} = \xi^T J^T \dot{\xi},
\] (9)
where \(J\) is given by
\[
J = \left(\begin{array}{cc}
0_n & I_n \\
-I_n & 0_n
\end{array}\right).
\] (10)

Here \(I_n\) and \(0_n\) are the \(n\)-dimensional identity and null matrix respectively. A symplectic matrix \(S\) is defined by the property of preserving the wedge product, what amounts to \(SJ^TS^T = J\).

The Weyl-Heisenberg translation operator \(\hat{T}_\xi\) is equivalent to Glauber’s optical displacement operator \(\hat{D}(\alpha)\), which is expressed in terms of a complex parameter \(\alpha\) and the annihilation/creation operators \(\hat{a}/\hat{a}^\dagger\):
\[
\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}).
\] (11)
The correspondence is established by making the identifications \(\alpha = (\xi_0 + i\xi_q)/\sqrt{2\hbar}\) and \(\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2\hbar}\).

Though not so widely known, the Wigner function can also be written as an expectation value:
\[
W(x) = \frac{1}{(\pi\hbar)^n} \text{tr}\left(\hat{\rho} \hat{R}_x\right).
\] (12)
The (Grossman-Royer) operators \(\hat{R}_x\) are both unitary and hermitian. In particular, \(\hat{R}_{x=0}\) coincides with the usual parity operator, i.e., \(\hat{R}_0|q\rangle = |-q\rangle\). Given that the parity operator corresponds to a geometrical reflection through the phase-space origin, and taking into account the property
\[
\hat{T}_\xi \hat{R}_0 \hat{T}_\xi^\dagger = \hat{R}_x,
\] (13)
the operator \(\hat{R}_x\) is interpreted naturally as the quantum version of the phase-space reflection through the point \(x\).

Both sets of reflections and translations, \(\{\hat{R}_x\}\) and \(\{\hat{T}_\xi\}\), constitute orthogonal bases in operator space (with respect to the Hilbert-Schmidt product). Thus any operator \(\hat{A}\) can be written as a linear combination of reflections or translations:
\[
\hat{A} = \frac{1}{(2\pi\hbar)^n} \int_{-\infty}^{+\infty} dx \ A(x) \hat{R}_x,
\] (14)
\[
= \frac{1}{(2\pi\hbar)^n} \int_{-\infty}^{+\infty} d\xi \ \hat{A}(\xi) \hat{T}_\xi.
\] (15)

where the reflection and translation symbols, \(A(x)\) and \(\hat{A}(\xi)\) respectively, are given by:
\[
A(x) = 2^n \text{tr}\left(\hat{A} \hat{R}_x\right),
\] (16)
\[
\hat{A}(\xi) = \text{tr}\left(\hat{A} \hat{T}_\xi\right).
\] (17)
The symbols \(\hat{A}(\xi)\) and \(A(x)\) are related via a (symplectic) Fourier transformation
\[
A(x) = \frac{1}{(2\pi\hbar)^n} \int d\xi \ \hat{A}(\xi) e^{i\xi x/\hbar}.
\] (18)

The relation between the Wigner and characteristic functions is a particular case \((\hat{A} = \hat{\rho})\) of the general expression above:
\[
W(x) = \frac{1}{(2\pi\hbar)^n} \int d\xi \ \chi(\xi) e^{i\xi x/\hbar}.
\] (19)

### B. Metaplectic transformations

The quantum analog of a linear canonical (symplectic) transformation \(S\) is a metaplectic unitary operator \(\hat{M}_S\). We have already seen an example: the parity operator \(\hat{R}_0\) is the quantum version of the classical reflection \(\hat{S} = -1_{2n}\). A metaplectic operator can be thought as the quantum propagator associated with a Hamiltonian that is purely quadratic in the canonical operators \(\hat{q}\) and \(\hat{p}\). The metaplectic operators together with the translation operators (associated with Hamiltonians that are purely linear in \(\hat{q}\) and \(\hat{p}\)), constitute the set of unitary Gaussian operations, i.e., operations that leave the set of Gaussian states invariant.

Metaplectic operators respect the classical group composition law, i.e., if \(S \) and \(S'\) are symplectic transformations, then
\[
\hat{M}_{SS'} = \pm \hat{M}_S \hat{M}_{S'},\n\] (20)
where the \(\pm\) sign (unessential for our purposes) depends on both \(S, S'\).

Metaplectic transformations interact according to the following formulas:
\[
\hat{T}_\xi \hat{R}_x \hat{T}_\xi^\dagger = \hat{R}_{x+\xi},
\] (21)
\[
\hat{M}_S \hat{T}_\xi \hat{M}_S^T = \hat{T}_{S\xi},
\] (22)
\[
\hat{M}_S \hat{R}_x \hat{M}_{S'} = \hat{R}_{Sx}.\n\] (23)

These expressions entail the covariance of the Wigner function with respect to both translations and metaplectic transformations. When a state is translated, \(\hat{\rho}' = \hat{T}_\xi \hat{\rho} \hat{T}_\xi^\dagger\), one deduces from Eq. (21) that its Wigner function is also translated:
\[
W_{\rho'}(x) = W_{\rho}(x - \xi),
\] (24)
using that $\hat{T}_\xi = \hat{T}_{-\xi}$, and the definition \cite{12}. Analogously, for a metaplectically deformed state, $\rho' = M_{-1}\hat{\rho}M^\dagger_{-1}$, one easily proves the metaplectic covariance of the Wigner and the characteristic functions
\begin{align}
W_{\rho'}(x) &= W_{\rho}(S^{-1}x), \\
\chi_{\rho'}(\xi) &= \chi_{\rho}(S^{-1}\xi),
\end{align}
[using $M^\dagger_0 = M_{-1}$, and Eqs. \cite{22,23}].

The reflection symbol $M_{0}(x)$ is an essential ingredient in our calculations of Sec. \ref{sec:4}. It is obtained from \cite{10} and given by \cite{22,24}:
\begin{equation}
M_{0}(x) = \frac{2^n i^\nu}{\sqrt{|\det(S + 1_{2n})|}} \exp \left( \frac{i}{\hbar} x \cdot C_S x \right),
\end{equation}
where the symmetric matrix $C_S$ is the Cayley transform of the symplectic matrix $S$ \cite{22,24}:
\begin{equation}
C_S = J(S - I_{2n})(S + I_{2n})^{-1}.
\end{equation}

An explicit expression for the integer $\nu$ can be found in Ref. \cite{24}. Like the $\pm$ sign in Eq. \cite{20}, $\nu$ is a delicate object, and not really necessary for our discussions. For these reasons we shall omit all details about this phase.

Even when $S + I_{2n}$ is singular, Eq. \cite{27} is still meaningful if we interpret that formula as a limit (in which one of the eigenvalues of $S$ tends to $-1$) \cite{17}.

In principle we could have worked in the $q$-representation. However, the expression for $M_{0}(q,q')$ requires the explicit splitting of $S$ into four blocks \cite{17}. This would lead to extremely cumbersome expressions when considering pairs of metaplectic transformations, like in Sec. \ref{sec:4} below.

\section{C. Gaussian States}

According to the standard definition, a state is said Gaussian if its Wigner function is a Gaussian. Any pure Gaussian state can be obtained from a fiducial Gaussian state by the combined action of a translation and a metaplectic transformation \cite{24}.

For simplicity, we take as fiducial state the ground state of the $n$-dimensional isotropic harmonic oscillator. It will be denoted $|0\rangle$. In a system of units where frequency and mass are unity, $\omega = m = 1$, the position wavefunction reads:
\begin{equation}
\langle q|0\rangle = \frac{1}{(\pi\hbar)^{n/4}} e^{-q^2/2\hbar}.
\end{equation}

A general pure Gaussian state can be obtained from $|0\rangle$ by the successive application of a metaplectic operator and a translation:
\begin{equation}
|S,\zeta\rangle = \hat{T}_\xi \hat{M}_0 |0\rangle.
\end{equation}

For the sake of compactness of forthcoming formulae, let us define a normalized Gaussian function:
\begin{equation}
\hat{G}(x;M,\zeta) = \frac{\sqrt{|\det M|}}{(\pi\hbar)^n} \exp \left[ - (x - \zeta) \cdot M (x - \zeta)/\hbar \right],
\end{equation}
where $x$ and $\zeta$ are $2n$-dimensional real vectors, and $M$ is a symmetric $2n \times 2n$ complex matrix with positive real part, $\Re M > 0$. The branch of square root is chosen in such a way that it reduces continuously to the positive root in the case of real $M$ \cite{28}. Note that if $M$ is symplectic (a case we shall meet several times), then $\det M = 1$ \cite{29}.

When $\hat{G}(x;M,\zeta)$ represents a probability distribution $M^{-1}$ is the covariance matrix. Sometimes we shall abuse of the language and use such denomination for arbitrary Gaussians; in that cases “covariance matrix” is to be understood as a shorthand for “the inverse of the matrix of the quadratic form $...$”.

The Wigner function of the fiducial state \cite{29} is readily calculated:
\begin{equation}
W_0(x) = \hat{G}(x;1,0).
\end{equation}

The Wigner function of the general Gaussian state $|S,\zeta\rangle$ is obtained from the fiducial one by using the covariance properties \cite{24,25}. The result is
\begin{equation}
W_{|S,\zeta\rangle}(x) = \hat{G} \left[ x; (SS^\top)^{-1}, \zeta \right].
\end{equation}

In this case, the covariance matrix $SS^\top$ is symplectic. It is worth showing also the corresponding characteristic function:
\begin{equation}
\chi_{|S,\zeta\rangle}(\xi) = 2^{-n} \hat{G} \left[ \xi/2; (SS^\top)^{-1}, 0 \right] e^{i\xi \cdot \zeta/\hbar}.
\end{equation}

Note that, for pure Gaussian states, $W(x)$ coincides with $\chi(2x)$, except for a factor and a shift of origin.

\section{IV. GAUSSIAN CAT STATES}

We shall call “Gaussian cat state” any superposition of two pure Gaussian states, i.e.,
\begin{equation}
|\Psi\rangle = a |U,\psi\rangle + b |V,\upsilon\rangle,
\end{equation}
where $a$ and $b$ are complex amplitudes, arbitrary to the extent that the state remains normalized. Its Wigner function reads
\begin{equation}
W_{\Psi}(x) = |a|^2 W_{|U,\psi\rangle}(x) + |b|^2 W_{|V,\upsilon\rangle}(x) + |ab| \mathcal{I}(x).
\end{equation}

The Wigner function of the superposition is the sum of the individual Wigner functions (two Gaussian “hills” centered at $U$ and $V$) plus an interference term, given by
\begin{equation}
\mathcal{I}(x) = \frac{2}{(\pi\hbar)^{n}} \Re \left[ e^{i\varphi} \langle U,\psi | \hat{R}_x | V,\upsilon \rangle \right],
\end{equation}
with $\varphi = \arg(a^*b)$. A typical example of the superposition \cite{30} was depicted in Fig. 2.
The purpose of this section is to explore the structure of the interference term. We start by writing the matrix element in the last equation as a vacuum expectation value:

\[ \langle U, u | \hat{R}_v | V, v \rangle = \langle 0 | \hat{M}_{\mu} \hat{T}_u \hat{R}_x \hat{T}_v | \hat{M}_G | 0 \rangle. \]  

(38)

First we use the fact that the composition of a translation and a reflection is also a reflection, \[ \hat{R}_x \hat{R}_\xi = e^{i \xi \wedge x / \hbar} \hat{R}_x - \hat{R}_\xi / 2, \]  

(39)

\[ \hat{T}_\xi \hat{T}_x = e^{i \xi \wedge x / \hbar} \hat{R}_{\xi + \xi / 2}, \]  

(40)

in order to transform \[ \hat{T}_u \hat{R}_x \hat{T}_v \]  

into a single reflection times a phase. After invoking Eqs. (20, 23) we arrive at

\[ \langle U, u | \hat{R}_x | V, v \rangle = e^{i x \wedge \zeta / \hbar + i \zeta \wedge \eta / 2} \langle 0 | \hat{R}_{U^{-1}(x - \eta)} \hat{M}_{U^{-1}} | 0 \rangle, \]  

(41)

with

\[ \zeta = u - v, \]  

(42)

\[ \eta = (u + v) / 2. \]  

(43)

We sketch the next steps skipping the details: (i) Expand \[ \hat{M}_{U^{-1}V} \] into reflection operators with the help of Eqs. (14, 27). (ii) Turn the resulting product of reflections into a translation using the composition formula \[ \hat{R}_x \hat{R}_\zeta = e^{2 i y \wedge \zeta / \hbar} \hat{T}_{2(x - y)} \].

(44)

(iii) At this point one must calculate the average \[ \langle 0 | \hat{T}_\xi | 0 \rangle \], for a certain \( \xi \). But this is essentially the characteristic function of the vacuum, which can be obtained from Eq. (38) by setting \( S = 1 \) and \( \xi = 0 \). (iv) Calculate the remaining Gaussian integral. The final result is

\[ \langle 0 | \hat{R}_{U^{-1}(x - \eta)} \hat{M}_{U^{-1}V} | 0 \rangle = (\pi \hbar)^n \left| K \right| G(x; \mathbf{G}, \eta), \]  

(45)

with

\[ K = \frac{2^n i^\mu}{\sqrt{\det[(U + V) + i(U - V) J]}}, \]  

(46)

\[ G = (UU^T + VV^T)^{-1} \left[ 2 - i \left( 2UU^T - 2VV^T \right) J \right], \]  

(47)

and \( \mu \) an integer.

Summing up, the interference term may be written as

\[ \mathcal{I}(x) = 2 \left| K \right| \text{Re} \left[ e^{i x \wedge \zeta / \hbar + i \phi} G(x; \mathbf{G}, \eta) \right]. \]  

(48)

Here \( \phi \) stands for a phase that does not depend on \( x \). Its precise value is not relevant for the forthcoming analysis.

It can be checked that \( G \) is not only symmetric but is also complex symplectic, i.e., \( G \) is not only symmetric but is also complex symplectic. Then, one has \( \det G = 1 \). \[ \mathbf{22} \]

A. Geometry of the Wigner Function

The interference pattern \[ \mathcal{I} \] can be factored into a positive Gaussian envelope times an oscillatory function. The envelope is given by:

\[ \mathcal{I}_{\text{env}}(x) = \frac{2 |K|}{(\pi \hbar)^n} \exp \left[ -(x - \eta) \cdot \text{Re} \mathbf{G}(x - \eta) / \hbar \right]. \]  

(49)

This Gaussian is centered at \( x = \eta \), the midpoint between the centers of the individual Wigner distributions \( W_{U(u)}(x) \) and \( W_{V(v)}(x) \). Its covariance matrix, \( \left( \text{Re} \mathbf{G} \right)^{-1} \), is an average of the individual covariance matrices:

\[ \left( \text{Re} \mathbf{G} \right)^{-1} = (UU^T + VV^T) / 2. \]  

(50)

If \( U = V \) (equally distorted wavepackets), then the shape (covariance matrix) of the Gaussian envelope is equal to the individual Wigner functions. The height of the envelope depends on the amplitudes \( a \) and \( b \) of the Gaussian states [see Eq. (35)], e.g., for equal amplitude superpositions, \( |a| = |b| \), the Gaussian envelope \[ \mathcal{I}_{\text{env}} \] is twice as high as the individual Wigner functions.

The oscillatory part reads

\[ \mathcal{I}_{\text{osc}}(x) = \cos \left[ \phi + x \cdot \zeta / \hbar + (x - \eta) \cdot \text{Im} \mathbf{G}(x - \eta) / \hbar \right], \]  

(51)

with

\[ \text{Im} \mathbf{G} = - (UU^T + VV^T)^{-1} (UU^T - VV^T) J. \]  

(52)

The first observation is that when \( U = V \) the quadratic term vanishes, and the phase becomes purely linear. In this well known case, the interference pattern is composed of straight lines parallel to the vector \( \zeta \) joining the centers \( u \) and \( v \) (see Fig. 4).

Now we switch to the general case \( U \neq V \). We want to characterize the quadratic form \( x \cdot \text{Im} \mathbf{G} x \), which determines the shape of the oscillations of the Wigner function. This shape will be clearly identifiable when, by a suitable canonical coordinate transformation, we reduce \( \text{Im} \mathbf{G} \) to a normal form.

Without loss of generality, we shall assume \( V = I_{2n} \). The general case can be recovered by setting \( U \rightarrow V^{-1} U \), then making the canonical transformation \( x \rightarrow Vx \), etc.

Our starting point is the Euler decomposition \[ \mathbf{30} \]: any symplectic matrix may be put in a specific diagonal form by the use of two orthogonal symplectic transformations:

\[ S \in \text{Sp}(2n, \mathbb{R}) \Rightarrow S = O \Lambda S O', \]  

(53)

with \( O, O' \in \text{Sp}(2n, \mathbb{R}) \cap \text{SO}(2n) \) and

\[ \Lambda_5 = \text{diag}(s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}), \quad s_i \geq 1. \]  

(54)

We obtain as a corollary that \( SS^T \) is diagonalized by an orthogonal symplectic matrix:

\[ SS^T = O \Lambda_5 S^T O^T, \]  

(55)
with the matrix of eigenvalues

$$\Lambda_{SS} = \text{diag}(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}),$$

where $\lambda_i = s_i^2 \geq 1$. We are now in the position of calculating the eigenvalues of $\text{Im} G$. With the choice $V = I_{2n}$, Eq. (52) reduces to

$$\text{Im} G' = -\left(UU^\top + I_{2n}\right)^{-1}\left(UU^\top - I_{2n}\right)J.$$  (57)

Using the corollary (55) as applied to $UU^\top$ we get

$$-O^\top \text{Im} G' O = \frac{\Lambda_{UUT} - I_{2n}}{\Lambda_{UUT} + I_{2n}} J = \left(\begin{array}{cc} \Xi & 0_n \\ 0_n & -\Xi \end{array}\right) J = \left(\begin{array}{cc} 0_n & \Xi \\ \Xi & 0_n \end{array}\right).$$  (58)

where $\Xi$ is a diagonal matrix with positive entries:

$$\Xi_{ii} = \frac{\lambda_i - 1}{\lambda_i + 1} \equiv \theta_i.$$  (59)

The anti-diagonal matrix $O^\top \text{Im} G' O$ can be diagonalized by the symplectic orthogonal matrix

$$H = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} I_n & I_n \\ -I_n & I_n \end{array}\right)$$

(a $\pi/4$ rotation in each coordinate plane), that is,

$$-HO^\top \text{Im} G' OH^\top = \left(\begin{array}{cc} \Xi & 0_n \\ 0_n & -\Xi \end{array}\right).$$  (61)

So, we have shown that for any superposition there exists a canonical coordinate system $(Q, P)$ where the quadratic part of the phase of oscillatory term reduces to the normal form

$$\frac{1}{\hbar} \sum_{i=1}^n \theta_i \left(Q_i^2 - P_i^2\right),$$

with $\theta_i > 0$. In one degree of freedom, except for the degenerate case $U = V$, the pattern is always hyperbolic (see Fig. 4).

The nonlinear interference patterns we described for continuous-variable systems may also be observed in discrete Wigner functions of finite-dimensional states. Consider, for instance, the phase-space representation of Grover’s search algorithm for an $N$-qubit system. The computer starts in a pure momentum state (an equal superposition of all basis states) and, after some iterations, evolves into a position eigenstate corresponding to the searched item. At intermediate times the computer state is a weighted superposition of a momentum state and a position state. The Wigner function of such states (see Fig. 2 in Ref. [31]) are very similar to the hyperbolic squeezed-state superposition in our Fig. 4.

V. DECOHERED GAUSSIAN CAT STATES

This section discusses what happens to the Wigner function of a pure Gaussian cat state when it evolves under a general linear dynamics, i.e., a dynamics that preserves Gaussian states. In the unitary case such dynamics are generated by quadratic Hamiltonians. We have seen that this corresponds to a metaplectic evolution operator which transforms the Wigner function according to the covariance rule (if the Hamiltonian contains a term linear in $\hat{x}$ there will be some additional translation of the Wigner function).

The most general linear evolution of a density operator $\hat{\rho}$ is described by the master equation (written in the
Lindblad form):\[33\]
\[
\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} \left[ \hat{H}, \rho \right] - \frac{1}{2\hbar} \sum_{k=1}^{M} \left( \hat{L}_k \hat{L}_k \rho + \hat{\rho} \hat{L}_k \hat{L}_k - 2 \hat{L}_k \hat{L}_k \hat{\rho} \right),
\]
with a quadratic Hamiltonian $\hat{H}$ and linear Lindblad operators $\hat{L}_k$.

Let us now consider the evolution of the Wigner function. The corresponding evolution equation for the Wigner function $W(x,t)$ is obtained by using the standard recipes for transforming master equations into partial differential equations \[26, 34\]. In the case of a quadratic (in $\hat{x}$) Lindblad equation we arrive at a so-called linear Fokker-Plank equation:

\[
\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x^T} (\mathbf{A} W) + \frac{1}{2} \frac{\partial}{\partial x} \mathbf{D} \frac{\partial}{\partial x} W.
\]

The drift and diffusion matrices, $\mathbf{A}$ and $\mathbf{D}$, respectively, can be compactly written as

\[
\mathbf{A} = \hbar \text{Re } \mathbf{T},
\]
\[
\mathbf{D} = \frac{1}{2\hbar} \mathbf{L} \mathbf{L}^\top,
\]

where

\[
\mathbf{L} = \sum_{k=1}^{M} \lambda_k \lambda_k^\top.
\]

Note that $\mathbf{D}$ is real symmetric nonnegative, while, in general, $\mathbf{A}$ is just real (Im$\mathbf{T}$ is antisymmetric). In spite of some differences in the notation, the results above coincide with those in Ref. \[33\] (see also \[35\]).

Let us now consider the evolution of the Wigner function of an initially pure Gaussian cat state. The expression for $W(x,t)$ can be obtained by convolving the initial distribution $W(x_0,0)$ [given by Eq. \[39\]] with the Fokker-Plank propagator:

\[
W(x,t) = \int_0^t dx_0 P(x,t|x_0,0) W(x_0,0).
\]

The propagator $P(x,t|x_0,0)$, i.e., the Green function for Eq. \[60\], is a real Gaussian function of both variables $x$ and $x_0$. If $W(x_0,0)$ is a sum of Gaussians, then $W(x,t)$ will also be.

For an initial Wigner function like that in Eq. \[60\], the integral above can be computed explicitly to get the desired solution at any given time $t$. However, we are only interested in the general structure of the Wigner function of the evolved Gaussian cat state, which is determined by the purely quadratic part of the evolving Gaussians. For this reason we shall focus on the evolution of the covariance matrices of the individual Gaussians that compose the cat state; there are four of them: two real ones and a complex-conjugate pair.

Given a Wigner function satisfying the Fokker-Plank equation \[60\], its associated covariance matrix obeys the following equation of motion \[39\]:

\[
\frac{d\mathbf{C}}{dt} = \mathbf{A} \mathbf{C} + \mathbf{C} \mathbf{A}^\top + \mathbf{D}.
\]

We remark that here and below, the “covariance” $\mathbf{C}$ stands for the inverse matrix of the quadratic form of any of the four Gaussians composing the cat-state Wigner function. At $t=0$ these matrices are symplectic but, when the dynamics \[71\] sets in, the symplectic symmetry is lost.

The solution satisfying the initial condition $\mathbf{C}(t=0) = \mathbf{C}_0$ is given by \[51\]:

\[
\mathbf{C}(t) = e^{\mathbf{A} t} \mathbf{C}_0 e^{\mathbf{A}^\top t} + \int_0^t dt' e^{\mathbf{A}(t-t')} D e^{\mathbf{A}^\top(t-t')}.
\]

This equation allows us to evolve one-by-one the covariance matrices of the individual Gaussian terms of the cat state \[39\].

It is convenient to split Eq. \[72\] into real and imaginary parts:

\[
\text{Re}\mathbf{C}(t) = e^{\mathbf{A} t} \text{Re}\mathbf{C}_0 e^{\mathbf{A}^\top t} + \int_0^t dt' e^{\mathbf{A}(t-t')} \text{D} e^{\mathbf{A}^\top(t-t')}\]
\[
\text{Im}\mathbf{C}(t) = e^{\mathbf{A} t} \text{Im}\mathbf{C}_0 e^{\mathbf{A}^\top t}.
\]

If $\text{Re}\mathbf{C}_0$ is nonsingular, then, evidently, $\text{Im}\mathbf{C}(t)$ remains nonsingular for all times. Moreover, Eq. \[74\] says that $\text{Im}\mathbf{C}(t)$ is congruent to $\text{Im}\mathbf{C}_0$, so, the full signature of $\text{Im}\mathbf{C}(t)$ remains constant in time (this is Sylvester’s law of inertia \[38\]: “signature” indicates a triplet of integers: the number of positive, negative and zero-valued eigenvalues). Similar congruence arguments, and the fact that $\text{Re}\mathbf{C}_0$ is positive definite and $\mathbf{D}$ is nonnegative, imply that $\text{Re}\mathbf{C}(t)$ remains positive for all times.

Now we have collected the basic ingredients to prove the main result of this section: the equation of motion \[72\] preserves the signatures of both $\text{Re}\mathbf{C}^{-1}(t)$ and $\text{Im}\mathbf{C}^{-1}(t)$, i.e.,

\[
\text{Re}\mathbf{C}^{-1}(t) \sim \text{Re}\left(\mathbf{C}_0^{-1}\right)
\]
\[
\text{Im}\mathbf{C}^{-1}(t) \sim \text{Im}\left(\mathbf{C}_0^{-1}\right),
\]

the symbol $\sim$ meaning “congruent to”.

This result leads to the following implications for the structure of the Wigner function that evolves from \[60\]. Obviously, the terms that evolve from $W_{(u,u)}(x)$ and $W_{(v,v)}(x)$, remain positive Gaussians for all times, their covariances evolving according to Eq. \[73\] in these cases one has $\text{Im}\mathbf{C}(t) = 0$, $\forall t$. Concerning the interference term $\mathcal{E}(x)$, the preservation of the signature of $\text{Im}\mathbf{C}^{-1}(t)$
means that the oscillation pattern will still be represented by the normal form \([62]\) in some coordinate system. However, differently from the pure case, the new coordinates will not be canonical in general. Of course, the oscillation pattern is multiplied by a Gaussian window.

The proof of \((75,76)\) for the interference term starts by expressing the inverse of \(C(t)\) in terms of its imaginary and real parts (omitting the time dependence):

\[
\begin{align*}
\text{Re} \left( C^{-1} \right) & = \left[ \text{Re} C + \text{Im} C (\text{Re} C)^{-1} \text{Im} C \right]^{-1}, \\
\text{Im} \left( C^{-1} \right) & = \left[ \text{Im} C + \text{Re} C (\text{Im} C)^{-1} \text{Re} C \right]^{-1}.
\end{align*}
\]

(77)

Let us analyze the first line \((77)\): \(\text{Re} C > 0\), then \((\text{Re} C)^{-1} > 0\), then \(\text{Im} C (\text{Re} C)^{-1} \text{Im} C > 0\) (Sylvester’s law). As the sum of positive matrices is positive, then \(\text{Re} \left( C^{-1} \right)\) is positive for all times.

In the second line \((78)\) we use that, \(\text{Re} C\) being positive, it has a positive square root: \(\text{Re} C = M^2\). Then, defining \(\text{Im} C' = M^{-1}(\text{Im} C)M^{-1}\), which has the same signature as \(\text{Im} C\), we get

\[
\text{Im} \left( C^{-1} \right) = M^{-1} \left[ \text{Im} C' + (\text{Im} C')^{-1} \right]^{-1} M^{-1}.
\]

(79)

Noting that \(\text{Im} C' + (\text{Im} C')^{-1} \sim \text{Im} C'\), we obtain

\[
\text{Im} \left( C^{-1} \right) \sim \text{Im} C' \sim \text{Im} C \sim \text{Im} C_0 \sim \text{Im} \left( C_0^{-1} \right).
\]

(80)

The last congruence is a consequence of the initial correlation matrix \(C_0\) being symmetric symplectic: \(C_0^{-1} = JC_0J^T\). \(\square\)

### VI. Superposition of Mixed Gaussian States

In the previous section we considered a pure cat state which was subjected to the decohering action of a linear reservoir. Now we study the opposite situation: a pure Gaussian state is first decohered and afterwards used as input for a cat-generating protocol. In what respects the Wigner-function interference pattern, we shall see that the latter case is more general, as elliptical structures may also appear.

Consider, for instance, the cat-generating scheme of Fig. 5 \([3, 39]\).

If the initial Gaussian state \(\hat{\rho}\) is pure, i.e., \(\hat{\rho} = |\psi\rangle \langle \psi|\), then the final state is also pure, \(\rho' = |\psi'\rangle \langle \psi'|\), with

\[
|\psi'\rangle = N_{\pm} (1 \pm \hat{U}) |\psi\rangle,
\]

(81)

where the sign \(\pm\) depends on the result of the qubit measurement, and \(N_{\pm}\) is a state-dependent normalization factor. When \(\hat{U}\) is a linear operation (a combination of a translation and a metaplectic unitary), the output state \((81)\) becomes a pure Gaussian cat state, exactly like those described in Sec. IV. However, if the initial state is Gaussian mixed, then the final state,

\[
\rho' = N_{\pm} (1 \pm \hat{U}) \rho_0 (1 \pm \hat{U})^\dagger,
\]

(82)

may be called a mixed Gaussian cat \([40]\). A possible implementation of the circuit above for generating mixed superpositions uses two free-propagating quantum optical modes interacting via a cross-Kerr nonlinear crystal \([40]\).

A formally similar result is produced when a single mode propagates in a medium exhibiting a Kerr nonlinearity. In the interaction representation, the evolution is governed by the Hamiltonian \([41]\)

\[
\hat{H} = \gamma \hat{n}^2,
\]

(83)

where \(\hat{n} = \hat{a}^\dagger \hat{a}\) is the number operator, and \(\gamma\) an energy scale. At the revival time

\[
T = \frac{2\pi \hbar}{\gamma},
\]

(84)

an initial coherent state \(|\alpha_0\rangle\) is perfectly reconstructed. For times equal to \((\mu/\nu)T\), with \(\mu/\nu\) an irreducible fraction, the evolved state consists of a superposition of \(\nu\) coherent states lying on a circle of radius \(|\alpha_0|\) (fractional revivals) \([42]\). However, this phenomenon is not restricted to coherent states, because at fractional-revival times the Kerr propagator becomes a sum of harmonic oscillator propagators, i.e.,

\[
e^{-2\pi i \mu \hat{n}^2/\nu} = \sum_{k=1}^{\nu} c_k e^{-2\pi i k \hat{n}^2/\nu} = \sum_{k=1}^{\nu} c_k \hat{M}_k,
\]

(85)

where \(c_k\) are complex Fourier coefficients having equal moduli \([13]\). The notation \(\hat{M}_k\) emphasizes the metaplectic nature of the harmonic oscillator propagators, which correspond to phase-space rotations of angles \(2\pi k/\nu\). Equation \(85\) shows that at revival times any state will be transformed into a superposition of \(\nu\) (rotated) replicas of itself. If the initial state \(\rho_0\) is Gaussian mixed, say
a displaced thermal state, it will evolve into the mixed Gaussian superposition

\[ \hat{\rho}' = \sum_{k,j=1}^{\nu} c_k c_j^* \hat{M}_k \hat{\rho}_0 \hat{M}^\dagger_j. \]  

(86)

Figure 6(a) exhibits an example for \( \nu = 4 \) (a mixed “compass” state). Two kind of interference patterns can be identified. The patterns corresponding to opposite replicas are linear, while those corresponding to contiguous replicas are elliptical (circular). In all cases, the extension of the interference regions are smaller than those corresponding to the diagonal terms \( \hat{M}_k \hat{\rho}_0 \hat{M}^\dagger_k \). Had we started with a suitable thermal squeezed state we should have obtained hyperbolic fringes instead of elliptical ones (graphics not shown).

Note the formal similarity between both cat-generating schemes described above: The interference terms generated with the Kerr Hamiltonian [Eq. (82)] can also be obtained from the scheme of Fig. 5 [Eq. (88)], by using alternatively \( \pi/4 \) or \( \pi/2 \) controlled rotations. So, in general, both schemes produce similar mixed cat states. Such cat states are structurally different from the decohered cats of Sect. VII which can only exhibit hyperbolic fringes in their Wigner functions.

We shall not present a detailed analytical description of the Wigner functions of the mixed cat states (82,86). Explicit expressions can be worked out along the lines of Sect. IV (see also Ref. 45). However, we would like to exhibit a simple analytical explanation of the shrinking of interference patterns (see Fig. 6).

Consider for simplicity the binary cat produced by the Kerr dynamics at half the revival time (\( \nu = 2 \)):

\[ \hat{\rho}' = \frac{1}{2} (1 + i\hat{M}_x) \hat{\rho}_0 (1 - i\hat{M}^\dagger_x) = \frac{1}{2} (1 + i\hat{R}_0) \hat{\rho}_0 (1 - i\hat{R}_0). \]  

(87)

Here \( \hat{M}_x \) denotes the half-period harmonic evolution, which is equivalent to the parity operation \( \hat{R}_0 \), i.e., the reflection through the phase-space origin. The Wigner function is calculated from Eq. (12):

\[ 2\pi \hbar W'(x) = \text{tr} \hat{\rho}_0 \hat{R}_x + \text{tr} \hat{R}_0 \hat{\rho}_0 \hat{R}_0 \hat{R}_x + 2 \text{Re} i \text{tr} \hat{R}_0 \hat{\rho}_0 \hat{R}_x. \]  

(88)

Using the cyclic property of the trace, and the composition formulae for reflections and translations (39,40,44), we obtain the cat Wigner function,

\[ 2W'(x) = W_0(x) + W_0(-x) + 4 \text{Re} i \chi_0 (-2x), \]  

(89)

in terms of the Wigner and characteristic functions of the initial state \( \hat{\rho}_0 \) [11]. If \( \hat{\rho}_0 \) is a displaced thermal state, \( \hat{\rho}_0 = \hat{T}_\eta \hat{\rho}_0 \hat{T}^\dagger_\eta \), then we arrive at

\[ 2W'(x) = W_{th}(x + \eta) + W_{th}(x - \eta) - 4 \sin(x \wedge 2\eta) \chi_{th}(2x). \]  

(90)

Remarkably the interference pattern of the cat Wigner function is described by the characteristic function of the thermal state. When temperature grows \( W_{th}(x) \) becomes wider. Then, the characteristic function must shrink, because Wigner and characteristic functions are related by a Fourier transform. So, by increasing the temperature one can make the interference pattern as small as desired [40].

VII. CONCLUDING REMARKS

We have studied the Wigner functions of general superpositions of Gaussian states. For the pure case, we showed that the structure of the interference pattern is hyperbolic in general. This structure is robust against the action of a linear environment. We also analyzed two families of mixed Gaussian cat states which may also exhibit elliptic fringes.

Our approach was geometric, qualitative. Anyway, the analytical tools presented here can in principle be used for quantitative purposes, like describing how cat states lose coherence in linear environments (39,44,45), or to determine the order of nonclassicality (40) of the generalized cats.

As far as we know, generalized cat states, i.e., showing nonlinear interference patterns in their Wigner functions, have not yet been created in the laboratory. Some squeezed superpositions have already been produced (4), others may become reality soon (49,51). However, such squeezed cats are still degenerate: as there is no relative squeezing between the superposed states, the Wigner interference pattern is linear.

We would like to conclude by mentioning some theoretical generation of “hyperbolic” cats (52). A set of quantum states localized on the classical periodic orbits of a
chaotic map, can be used as a basis in which the description of the eigenstates of its quantum version is greatly simplified. This set can be improved with the inclusion of short time propagation along the stable and unstable manifolds of the periodic orbits. These “scar functions” when viewed through a phase space representation look very much like the hyperbolic cat in our Fig. 4(c).

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