THE $L^2$ VOLUME OF THE SPACE OF HOLOMORPHIC MAPS FROM KÄHLER RIEMANN SURFACES TO $\mathbb{CP}^k$

CHIH-CHUNG LIU

Abstract. We prove the conjectural formula for the $L^2$ volume of the space of degree $r$ holomorphic maps from a compact Kähler Riemann surface of genus $b$ to $\mathbb{CP}^k$. This formula was posed in [B] and rigorously verified in [Sp] for a special case using independent techniques.

1. Introduction

Given a closed Riemann surface $\Sigma$ of genus $b$ with compatible Riemannian metric with Kähler form $\omega$, we study the space $\mathcal{H}_{r,k}$, or the space of degree $r$ holomorphic maps from $\Sigma$ to $\mathbb{CP}^k$. This space has generated great interest in topological field theory, in which $\mathcal{H}_{r,k}$ is the moduli space of charge $r$ $\mathbb{CP}^k$ lumps on $\Sigma$, or the static solitons to field equations of $\mathbb{CP}^k$ model on $\Sigma$ within the topological sector $r$. From the variational point of view, holomorphic maps are precisely the minimizers to the $L^2$ energy functional

$$E(\phi) = \int_{\Sigma} \|d\phi\|^2 dv\text{ol}_\Sigma,$$

where the norm above will be defined in section 2. The volume of $\mathcal{H}_{r,k}$ with respect to the $L^2$ metric defined by the energy functional is a vital quantity in studying the dynamics of solitons. With a clear understanding of the volume growth, the equations of classical gas of lumps may be deduced and motions of solitons at low energy may be approximated. See [M], [S], or [Sp] for further details.

A plausible formula for the $L^2$ volume of $\mathcal{H}_{r,k}$ has been conjectured in [B] and rigorously verified in [Sp] for the case $(r, b) = (1, 0)$ (i.e. degree 1 maps on $S^2$). In this paper, we confirm the validity of the formula in general.

Theorem 1.1 (Main Theorem). The $L^2$ volume of $\mathcal{H}_{r,k}$ is

$$\frac{(k + 1)^b}{q!} \left( \frac{Vol_\Sigma}{2} \right)^q,$$

where $q = b + (k + 1)(r + 1 - b) - 1$ and $b$ is the genus of $\Sigma$.

Throughout this paper, we make the universal assumption $r > 2 - 2b$ that ensures $\mathcal{H}_{r,k}$ to be a nonempty complex manifold.
2. Preliminaries

In this section we aim to provide a concrete description of the space $\mathcal{H}_{r,k}$ as a complex manifold of finite dimension, equipped with the standard $L^2$ metrics on maps. Theorem 2.6, the main result of this section, is an easy consequence of results from [Mi], [Gr], and [K-M]. To ensure that the $L^2$ metric is meaningful, we assume that the images of the holomorphic maps discussed here are non-degenerate, that is, they do not lie in any hyperplane.

For clarity, we first list several basic and well known definitions of the objects used in this article. Readers familiar with these notions may skip to Theorem 2.6 and consequences following it. The elementary building block for various moduli spaces is the $r$ symmetric product of $\Sigma$, or the space of unordered $r$-tuple of points on $\Sigma$, or the space of effective divisors of degree $r$.

**Definition 2.1 (Symmetric Product).** Given the $r$-fold direct product of $\Sigma$, we define

$$Sym^r\Sigma := \Sigma \times \ldots \times \Sigma / \sim,$$

where the equivalent relation is given by

$$(p_1, \ldots, p_r) \sim (p_{\sigma(1)}, \ldots, p_{\sigma(r)}),$$

with $\sigma \in S_r$, the symmetric group.

Elements of $Sym^r\Sigma$ are denoted by $E = \Sigma_{i=1}^r p_i$, where $p_i$’s need not be distinct. It is a standard fact that $Sym^r\Sigma$ is a complex manifold of complex dimension $r$. The local holomorphic coordinates are given by elementary symmetric functions. (See, for example, [Gr]).

The space $Sym^r\Sigma$ is closely related to the space of nonzero meromorphic functions on $\Sigma$ of degree $r$, or $K^*_r(\Sigma)$. Since $\Sigma$ is compact, any $f \in K^*_r(\Sigma)$ has $r$ poles and $r$ zeros, counting multiplicities. Each meromorphic function is then associated with a pair of effective divisors of degree $r$:

$$(f) = (f)_0 - (f)_{\infty},$$

where $(f)_0, (f)_{\infty} \in Sym^r\Sigma$ are disjoint. The converse statement is the classical Abel’s Theorem, that a pair of effective divisors of the same degree gives rise to a meromorphic map as above if they have the same value under the Abel-Jacobi map. Precisely (cf. Chapter V of [Gr]),

**Definition 2.2 (Abel-Jacobi Map).** Given $\{\omega_i\}_{i=1}^b$, the $\mathbb{C}$-basis of the space of holomorphic one forms on $\Sigma$, we define

$$\mu : Div(\Sigma) \to J(\Sigma),$$

as follows. For $E = \Sigma_{i=1}^l p_i \in Div(\Sigma)$, we have
Theorem 2.3

\[
\mu(D) = \left( \sum_{i=1}^{l} \int_{q}^{p_i} \omega_1 \right) \cdots \left( \sum_{i=1}^{l} \int_{q}^{p_i} \omega_b \right) .
\]  

(2.1)

Here, \( q \) is any point outside the support of \( D \), and \( J(\Sigma) \) is the Jacobian torus of \( \Sigma \), or the quotient of \( \mathbb{C}^b \) by the \( \mathbb{Z} \)-span of row vectors of the \( b \times 3b \) period matrix of \( \Sigma \), \( (\omega_i(\gamma_j)) \), where \( \{\gamma_j\} \) is the basis for the holomorphic one forms of \( \Sigma \).

One notes that the quotient space eliminates the ambiguities of the definition of \( \mu \) in (2.1).

A few well-known properties of \( \mu \) are relevant for the discussions of this section (see [G1] for details):

- If \( E \sim D' \), that is, \( E - E' = (f) \) for some \( f \in K(\Sigma) \), then \( \mu(E) = \mu(E') \).
- \( \mu \) is holomorphic on \( \text{Sym}^r \Sigma \).
- \( \mu^{-1}(\mu(E)) = |E| := \{ F \in \text{Div}(\Sigma) \mid F \sim E \} \).

The third fact in particular implies that the fiber of \( \mu \) at each \( \mu(E) \in J(\Sigma) \) is the projectivization of the vector space

\[
\mathcal{L}(E) := \{ f \in K(\Sigma) \mid (f) + E \geq 0 \}.
\]

For \( r > 2b - 2 \), Riemann-Roch Theorem implies that for all \( E \in \text{Sym}^r \Sigma \subset \text{Div}(\Sigma) \), the fiber of \( \mu \) at \( \mu(E) \) is a projective vector space of dimension \( r - b \). Therefore, \( \text{Sym}^r \Sigma \) is a vector bundle of rank \( r - b \) over the Jacobian torus \( J(\Sigma) \) with the projection map \( \mu \).

We can now state the sufficient condition for the difference of a pair of divisors to be principal.

Theorem 2.3 (Abel’s Theorem). Given a pair \( (E, E') \in \text{Sym}^1 \Sigma \times \text{Sym}^1 \Sigma \) with \( \mu(E) = \mu(E') \), there exists \( f \in K^*_1(\Sigma) \) so that \( (f) = E - E' \).

The map \( f \) is unique up to multiplication of a non-vanishing holomorphic map. For compact Riemann surface, it is then unique up to a nonzero multiple.

We are now equipped with sufficient language to describe the space \( \mathcal{H}_{r,k} \). In [K-M], a complete description was provided, without explicit proof, for the space of based holomorphic maps from \( \Sigma \) to \( \mathbb{CP}^k \).

Definition 2.4. Given \( p_0 \in \Sigma \) and \( q_0 \in \mathbb{CP}^k \), we define the space of based holomorphic maps, \( \mathcal{H}^*_{r,k} \), to be the subset of \( \mathcal{H}_{r,k} \) of all maps sending \( p_0 \) to \( q_0 \).

Proposition 2.5. The space \( \mathcal{H}^*_{r,k}(\Sigma, \mathbb{CP}^k) \) can be identified with an open set subset of \( k + 1 \) direct product of \( \text{Sym}^r \Sigma \) given by

\[
\mathcal{H}^*_{r,k} := \{ (E_0, \ldots, E_k) \in \text{Sym}^r \Sigma \times \cdots \times \text{Sym}^r \Sigma \mid \cap_j E_j = \emptyset \text{ and } \mu(E_0) = \mu(E_j) \forall j \}.
\]

Proof. Given \( f \in \text{Hol}^0_r(\Sigma, \mathbb{CP}^k) \), since \( f(\Sigma) \) is nondegenerate, it is a curve of degree \( k \) in \( \mathbb{CP}^k \). Let \( H_1 \) be the hyperplane of \( \mathbb{CP}^k \) defined by the canonical
section $s_1$ of $O(1)$. In local coordinate, points of $H_1$ is of the form $[0 : z_1 : \ldots : z_k]$. Then, counting multiplicities, we have

$$H_1 \cap f(\Sigma) = \{q_1, \ldots, q_k\}.$$  

Since $f$ is nondegenerate, $q_1, \ldots, q_k$ are in general position in $\mathbb{CP}^{k-1}$, and we may define coordinate patches of $\mathbb{CP}^k$ so that each $q_j$ has homogeneous coordinate $[\delta_{ij}]_{i=0}^k$, $1 \leq j \leq k$.

For each $j \in \{1, \ldots, k\}$, let $E_j := f^{-1}(q_j)$. Since $f$ is of degree $r$, it follows that $E_j \in \text{Sym}^r \Sigma$. For $j = 0$, we let $E_0 := f^{-1}(1 : 0 : \ldots : 0) \in \text{Sym}^r \Sigma$. We must show that the tuple $(E_0, \ldots, E_k)$ satisfies the condition of the open set given above. It is clear that $E_0 \cap_{j=1}^k E_j = \emptyset$ since $f(E_j) \subset H_1$ for all $j \geq 1$ and therefore $\cap_{j=0}^k E_j = \emptyset$. For the equalities of the values of these divisors under Abel-Jacobi map, it suffices to observe that all divisors given above are linear equivalent. Indeed, define $\pi_{ij} \in K(\mathbb{CP}^k)$ by

$$\pi_{ ij}(\mathbb{[z_0 : \ldots : z_k]}) := \frac{z_j}{z_i}.$$  

Each $\pi_{ij}$ is globally defined on $\mathbb{CP}^k$ and therefore $\pi_{ij} \circ f \in K^*(\Sigma)$. Moreover, with the choice of coordinates associated to $q_j$’s, it is clear that for all $j$,

$$(\pi_{j0} \circ f) = E_0 - E_j.$$  

It then follows that all divisors chosen above are linearly equivalent.

Conversely, given $(E_0, \ldots, E_k)$ satisfying the conditions of the open subset. Abel’s Theorem ensures that for each $j \geq 1$, there exists $f_j \in K^*(\Sigma)$ such that $(f_j) = E_j - E_0$, unique up to a nonzero multiple $\lambda_j \in \mathbb{C}^*$. We define the corresponding map $f$ coordinate-wise by

$$f := [1 : f_1 : \ldots : f_k].$$

This map is holomorphic away from $\cup_{j \geq 1} E_j$. However, near each point of $E_j$, we may holomorphically extend $f$ across the singularities by multiplying the poles of minimum order of vanishing (See [Mi] for details), obtaining a holomorphic map from $\Sigma$ to $\mathbb{CP}^k$. $f$ is clearly of degree $r$, since $f^{-1}(0 : 1 : \ldots : 1) = E_0 \in \text{Sym}^r \Sigma$. Finally, since $f$ is a based map, the constant multiples $\lambda_j$ mentioned above are completely determined, and the two associations constitute a one-to-one correspondence between $H^*_{r,k}$ and $\mathcal{H}_{r,k}^*$. 

\[\square\]

Lifting the restriction on based maps corresponds to the freedom of choosing $(\lambda_j)_{j=1}^k \in (\mathbb{C}^*)^{\oplus k}$ and we have established

**Theorem 2.6.** $\mathcal{H}_{r,k}$ is diffeomorphic to $H \oplus \mathbb{CP}^k$, where $\mathcal{H}_{r,k}^*$ is defined in Proposition 2.5.

**Proof.** In the proof of Proposition 2.5, the function $f$ defined by $(E_j) \in \mathcal{H}_{r,k}^*$ is unique up to a choice of $k + 1$ nonzero constants (including 1). Moreover, the
choice of these $k + 1$ constants define the same $f$ if they lie on the same line in $\mathbb{C}^{k+1}$.

The complex dimension of $H_{r,k}$ can be readily computed.

**Lemma 2.7.** $H_{r,k}$ is a complex manifold of complex dimension

$$(k + 1)r - k(b - 1).$$

**Proof.** It suffices to show that the space $\mathcal{H}_{r,k}^*$ is of dimension $(k + 1)r - kb$, since the remaining summand in the conclusion of Theorem 2.6 contributes $k$ to the dimension. Note that $\mathcal{H}_{r,k}^*$ is the submanifold of a $(k + 1)r$-dimensional manifold defined by the linear map $s : (\text{Sym}^r \Sigma)^{(k+1)} \to (J(\Sigma))^k$ given as,

$$s(E_0, E_1, \ldots, E_k) := (\mu(E_1) - \mu(E_0), \ldots, \mu(E_k) - \mu(E_0)).$$

It suffices to show that $s$ is of rank $kb$ at every point on the inverse image of 0. Indeed, consider the linear map $\Delta : \text{Sym}^r \Sigma \times \text{Sym}^r \Sigma \to J(\Sigma)$ defined by

$$\Delta(E, E') = \mu(E) - \mu(E').$$

From the third property of the Abel-Jacobi map listed immediately after Definition 2.2, we see that the inverse image of 0 is precisely

$$\{(E, E') \mid E' \in |E|\},$$

which is of dimension $r + \dim |E| = 2r - b$ by Riemann-Roch Theorem (note again $r > 2 - 2b$). The rank of $\Delta$ is then $2r - (2r - b) = b$ and therefore each component of $s$ contributes rank $b$ to the entire map. This completes the proof. □

The space $H_{r,k}$ is not compact and is potentially problematic when discussing convergence of integrals over it. As such, we introduce the well known Uhlenbeck compactification of $H_{r,k}$ and will later discuss the convergence of integrals over this compact set. The following description is summarized from [B-D-W]:

**Definition 2.8.** The *Uhlenbeck compactification* of $H_{r,k}$ is defined by

$$\mathcal{H}_{r,k} := \bigsqcup_{l=0}^r \left[ \text{Sym}^l \Sigma \times \text{Hol}_{r-l}(\Sigma, \mathbb{C}\mathbb{P}^k) \right].$$

The topology of $\mathcal{H}_{r,k}$ is given by weak* topology on $\text{Sym}^l \Sigma$ and $C_0^\infty$ topology on $\text{Hol}_{r-l}(\Sigma, \mathbb{C}\mathbb{P}^k)$. Precisely, for each $(E, f) \in \mathcal{H}_{r,k}$, we associate the distribution $e(f) + \delta_E$, where $e(f) = |df|^2$ the energy density and $\delta_E$ is the Dirac distribution supported in $E$. Around $(E, f)$, there are two local bases of neighbourhoods of topologies: one basis $W$ of $e(f) + \delta_E$ in the weak* topology and one basis $N$ of $f$ in $C_0^\infty(\Sigma, E)$. Collecting them together, we form a basis of neighbourhoods

$$V(E, f) = \{(E', f') \in \mathcal{H}_{r,k} \mid E' \in W \text{ and } f' \in N\}.$$
One notes that for \( l = 0 \), this topology degenerates into \( C_0^\infty \) topology of functions, which coincide with the subspace topology inherited from the direct sums of symmetric products. Indeed, the locations of zeros determines the smooth topology as well as the energy densities. Moreover, the smooth structure of symmetric spaces provides a natural smooth structure for \( \overline{\mathcal{H}}_{r,k} \), namely, the \( C^\infty \) topology controlled by the locations of the zeros. In [S-U], it was shown that \( \mathcal{H}_{r,k} \) with this topology is compact. Since \( \mathcal{H}_{r,k} \) is open and dense subset of \( \overline{\mathcal{H}}_{r,k} \), it is indeed a smooth compactification of \( \mathcal{H}_{r,k} \). In our main result, we will derive the convergence statement of integrals on \( \overline{\mathcal{H}}_{r,k} \) and ensure that the complement does not contribute the integral.

We may describe \( \overline{\mathcal{H}}_{r,k} \) with explicit coordinates. We observe that \( \overline{\mathcal{H}}_{r,k} \) is a closed submanifold of \( (\text{Sym}^r \Sigma)^{\times k+1} \times \mathbb{C}P^k \). The component from symmetric products determines the zeros from \( k+1 \) meromorphic functions on \( \Sigma \), a total \( k+1 \) unordered \( r \)-tuples of points from \( \Sigma \). With the \( k \) constraint equations by Abel-Jacobi maps of rank \( b \), it follows that there are \( m = r(k+1) - kb \) points from \( (\text{Sym}^r \Sigma)^{\times k+1} \) that determines the correct zeros of each meromorphic function up to a free choice of point in \( \mathbb{C}P^k \). \( \overline{\mathcal{H}}_{r,k} \) therefore possesses local coordinate description

\[
(w_1, \ldots w_m, w_{m+1}, \ldots, w_m),
\]

where the coordinates \( (w_1, \ldots w_m) \) is the local coordinate of the submanifold of \( (\text{Sym}^r \Sigma)^{\times k+1} \) defined by Abel-Jacobi equations and \( (w_{m+1}, \ldots, w_m) \) is the local coordinate for \( \mathbb{C}P^k \). Here, \( m = (k+1)r - kb \) as in Lemma 2.7. (2.2) is also a local coordinate for the open subset \( \mathcal{H}_{r,k} \), with the additional open condition that \( (w_1, \ldots w_m) \) does not yield meromorphic functions with common zeros.

Lastly, we define the \( L^2 \) metrics on \( \overline{\mathcal{H}}_{r,k} \) with respect to the Kähler metric \( \omega \) on \( \Sigma \) and Fubini-Study metric on \( \mathbb{C}P^k \). Given \( f \in \text{Hol}_r(\Sigma, \mathbb{C}P^{k-1}) \), the tangent space of \( \mathcal{H}_{r,k} \) at \( f \) can be identified with the space of sections of the pullback bundle of \( T\mathbb{C}P^{k-1} \) via \( f \):

\[
T_f \mathcal{H}_{r,k} \simeq \Gamma(f^*T\mathbb{C}P^{k-1}).
\]

Given \( u, v \in T_f \mathcal{H}_{r,k} \), they can be viewed as a pullbacked sections on \( \Sigma \), which can be pushed forward by \( f \) to be tangent vectors on \( \mathbb{C}P^{k-1} \), on which Fubini-Study metric can be applied. We define

**Definition 2.9.** \( L^2 \) Metrics on \( \mathcal{H}_{r,k} \)

The \( L^2 \) metric on \( \mathcal{H}_{r,k} \) is given by

\[
\langle u, v \rangle_{L^2} := \int_{\Sigma} \langle f_* u, f_* v \rangle_{FS} \text{vol}_\Sigma.
\]

Here, the \( f_* \) denotes the pushforward of \( f \).

With respect to this metric, we aim to verify the conjectured formula for the volume of \( \mathcal{H}_{r,k} \).
3. VORTEX EQUATIONS AND ITS MODULI SPACES

We hereby provide a brief summary of the $s$-vortex equations and its moduli spaces. Standard texts and papers for this subject include [L-T], [S], [R], [M] (of more physical interest), [Br], [Br1], [G], and [B] (of more mathematical aspects). The conclusion below is drawn directly from [L], which contains a comprehensive summary the standard texts.

On the same Riemann surface defined above, we consider a Hermitian line bundle $(L, H)$ with degree $r > 2 - 2b$ possessing smooth global sections. For each $k \in \mathbb{N}$ and $s \in \mathbb{R}$, the $s$-vortex equations are defined on $(\mathcal{D}, \phi_0, \ldots, \phi_k) \in \mathcal{A}(H) \times (\Omega^0(L))^n(k+1)$, the product of the space of $H$-unitary connections and $k + 1$ tuples of smooth sections. We have

$$
\begin{align*}
F_D^{(0,2)} &= 0 \\
D^{(0,1)} \phi &= 0 \\
\sqrt{-1} \Lambda F_D + \frac{s^2}{2} (\sum_{i=0}^k |\phi_i|^2_H - 1) &= 0.
\end{align*}
$$

Here, $F_D$ is the curvature form of the connection $D$. We use $\phi$ to denote a $(k + 1)$-tuple of sections $(\phi_i)_{i=0}^k$ and the second equation is an abbreviation that all sections are holomorphic with respect to $D$, namely, $D^{(0,1)} \phi_i = 0 \ \forall i$. Via Bogomolny-type arguments, the set of equations are precisely the minimizing equations the $s$-Yang-Mills-Higgs energy functional.

The solution space to (3.1) is invariant under unitary gauge action, and we let $\nu_{k+1}(s)$ be the $U(1)$ gauge classes of solutions to (3.1). Elements of this space are called vortices. For a line bundle $L$, it is well known that the stability condition, or the non-emptyness of $\nu_{k+1}(s)$, is simply

$$
\frac{4\pi r}{\text{vol } \Sigma} \geq s^2.
$$

Note that we intentionally define $k + 1$ sections for vortex equations to accommodate the main theme of this article, the space of holomorphic maps to $\mathbb{CP}^k$. It has been noted in [B-D-W] and [B] that $\mathcal{H}_{r,k}$ is diffeomorphic to the open subset of $\nu_{k+1}(s)$, denoted by $\nu_{k+1,0}(s)$, consisting of vortices with $k + 1$ sections that do not vanish simultaneously:

$$
\nu_{k+1,0}(s) := \{[D, (\phi_i)_{i=0}^k] \in \nu_k(s) \mid \cap_i \phi_i^{-1}(0) = \emptyset\}.
$$

We have,

**Theorem 3.1.** For all $s$ such that $s^2 \in \left[\frac{4\pi r}{\text{vol } \Sigma}, \infty\right)$, there is a diffeomorphism

$$
\Phi_s : \mathcal{H}_{r,k} \rightarrow \nu_{k+1,0}(s).
$$

**Proof.** (Sketch) We provide only the definition of $\Phi_s$ as a reference for later construction. For complete proofs and justifications, see [Br] for the case $s = 1$ and [L] for generalization to all $s$.

The inverse direction of $\Phi_s$ is more obvious. For $[D, \phi] \in \nu_{k+1,0}(s)$, we define
\[
\Phi^{-1}_s([D, \phi])(z) := \tilde{\phi}(z) = [\phi_0(z), \ldots, \phi_k(z)].
\]

The map is well defined as \( \phi_i(z) \)'s are never all zero. Moreover, on a \( U(1) \) bundle, the transition map multiplies each section by a uniform nonzero scalar, having no effect on the definition of \( \tilde{\phi} \). Since each component is holomorphic near every \( z \in \Sigma \) and has order of vanishing \( r \), \( \Phi^{-1}_s([D, \phi]) \) is indeed an element of \( \mathcal{H}_{r,k} \).

The forward direction is also a standard construction in algebraic geometry. We start with a holomorphic map \( \tilde{\phi} \in \mathcal{H}_{r,k} \). Consider \( \mathcal{O}_{\mathbb{CP}^k}(1) \), the anti-tautological line bundle over \( \mathbb{CP}^k \) with hyperplane sections \( s_0, \ldots, s_k \). Each \( s_j \) vanishes precisely on the hyperplane defined by \( z_j = 0 \). Let \( L = \tilde{\phi}^* \mathcal{O}_{\mathbb{CP}^k}(1) \) be the pullbacked line bundle on \( \Sigma \) endowed with sections \( \phi = (\phi_0, \ldots, \phi_k) \in \Omega^0(L)^{k+1} \) by pulling back \( s_1, \ldots, s_k \) via \( \tilde{\phi} \). The map \( \tilde{\phi} \) also endows a holomorphic structure \( \bar{\partial}_L \) and a background metric \( H \) on \( L \) by pulling back the standard complex structure and Fubini-Study metric on \( \mathcal{O}_{\mathbb{CP}^k}(1) \), providing a corresponding background unitary connection \( D \). We then find a suitable complex gauge transformation turning \( (D, \phi) \) into a pair that solves the vortex equations (3.1). On a line bundle, such a transformation is given by a nonzero real smooth function \( g_s = e^{u_s} \), acting on \( (D, \phi) \) in the standard ways:

\[
g_s \cdot D = D \left( e^{u_s}(\bar{\partial}_Le^{-u_s}) \right) := D_s,
\]

and

\[
g_s \cdot \phi_i = e^{u_s} \phi_i := \phi_{i,s}.
\]

Let \( \phi_s = (\phi_{i,s})_{i=0}^k \). Plugging \( (D_s, \phi_s) \) into (3.1), we see that the function \( u_s \) must satisfies the following Kazdan-Warner type equation:

\[
- \Delta_\omega u_s + \frac{s^2}{2} \sum_{i=0}^k |\phi_i|_H^2 e^{2u_s} + \left( \sqrt{-1} \Lambda F_H - \frac{s^2}{2} \right) = 0.
\] (3.2)

Here, \( \Delta_\omega \) is the positive definite Laplacian determined by \( \omega \). It has been shown in [Br] (for \( s = 1 \)) and [L] (for general \( s \)) that equation (3.2) has a unique smooth solution provided that \( s \) is large enough. The analytic tool used to solve \( u_s \) is developed in [K-W] using the method of super and sub solutions. We will mention more details in the next section. That the two correspondences are smooth inverses to each other is verified in [Br] and [L] via straightforward arguments.

With this correspondence, the entire vortex moduli space can be identified without difficulty. For a general vortex \( [D, \phi] \in \nu_k(s) \), we first locate the common zeros of the \( \phi_i \)'s, an element \( E \in \text{Sym}^a \Sigma \) for some \( a \in \{0, \ldots, r\} \). The \( k+1 \) sections define a holomorphic map from \( E \) to \( \mathbb{CP}^k \) in identical way as \( \Phi^{-1}_s \). This is map is bounded away from \( E \), and can therefore be uniquely extended to the entire \( \Sigma \) by locally dividing out the zeros of minimum order of vanishing near
each common zero. The holomorphic map defined is then of degree \( r - a \). See [B-D-W] and [Mi] for details. We have

**Proposition 3.2.** The vortex moduli space \( \nu_k(s) \) is diffeomorphic to the Uhlenbeck compactification of \( \mathcal{H}_{r,k} \):

\[
\mathcal{H}_{r,k} = \bigcup_{a=0}^{r} \left( \text{Sym}^i \Sigma \times \text{Hol}_{r-a}(\Sigma, \mathbb{C}P^k) \right).
\]

The \( L^2 \) metrics for \( \nu_k(s) \) is also defined naturally. At \( (D_s, \phi_s) \in \mathcal{A}(H) \times \Omega^0(L)^{\times k+1} \), we define,

**Definition 3.3** (\( L^2 \) Metrics on \( \nu_{k+1}(s) \)).

\[
g_s((\dot{A}_s, \dot{\phi}_s), (\dot{A}_s, \dot{\phi}_s)) = \int_{\Sigma} \frac{1}{2s^2} \dot{A}_s \wedge \dot{\bar{A}}_s + <\dot{\phi}_s, \dot{\phi}_s>_{H} \text{vol}_\Sigma. \tag{3.3}
\]

Here, \((\dot{A}_s, \dot{\phi}_s)\) denotes a tangent vector in \( T_{(D_s, \phi_s)}(\mathcal{A}(H) \times \Omega^0(L)^{\times k+1}) \simeq \mathcal{A}_1(\Sigma) \times \Omega^0(L)^{\times k+1} \), and \( \langle \dot{\phi}_s, \dot{\phi}_s \rangle = \sum_{i=0}^{k} \langle \dot{\phi}_{i,s}, \dot{\phi}_{i,s} \rangle \). The identification is justified by the fact that \( \Omega^0(L) \) is a vector space and \( \mathcal{A}(H) \) is an affine space modeled on the vector space \( \mathcal{A}_1(\Sigma) \), the space of complex valued one forms on \( \Sigma \). (cf. Chapter V of [R]). This identification also justifies the applications of Hodge star \( \ast \) and \(<\cdot, \cdot>_{H} \) in the integrand of (3.3), since \((\dot{A}_s, \dot{\phi}_s)\) lies in essentially isomorphic spaces as \((D_s, \phi_s)\) does. By choosing tangent vectors orthogonal to \( \mathcal{G} \)-gauge transformations, (3.3) descends to a well defined metric on the quotient space \((\mathcal{A}(H) \times \Omega^0(L)^{\times (k+1)}) / \mathcal{G}) \) and restricts to the open subset \( \nu_{k,0}(s) \) (cf. [L]). With respect to this metric, Baptista has explicitly computed the formula for the volume of \( \nu_k(s) \) in [B], generalized directly from [M-N]:

**Theorem 3.4.** For a degree \( r \) Hermitian line bundle \( L \) over a closed Kähler Riemann surface \( \Sigma \) of genus \( b \), where \( r > 2b - 2 \), the \( L^2 \) volume of the vortex moduli space \( \nu_{k+1}(s) \) is given by

\[
\text{Vol}_{\nu_{k+1}}(s) = \sum_{i=0}^{b} \frac{b!(k+1)^{b-i}}{i!(q-i)!(b-i)!} \left( \frac{4\pi}{s^2} \right)^i \left( \text{Vol}_\Sigma - \frac{4\pi}{s^2} r \right)^{q-i}, \tag{3.4}
\]

where

\[
q = b + (k+1)(r+1-b) - 1.
\]

The theory for the construction of this formula is to realize \( \nu_{k+1}(s) \) as a fiber bundle over the moduli space of holomorphic structures, which are identified with the Jacobian torus. The Kähler form of the \( L^2 \) metric (3.3) can be decomposed into parallel and horizontal components and fiberwise integration is performed to obtain the formula.

The asymptotic behaviours of \( \nu_{k+1}(s) \), (3.3), and (3.4) as \( s \to \infty \) are our main interest. One observes that in \( s \)-vortex equations (3.1), the only \( s \) dependence
lies in the third equation. As \( s \) increase, the curvature term in the third equation becomes negligible compared to the section terms and we may formally define the vortex moduli space at infinity, \( \nu_{k+1}(\infty) \), to be the unitary gauge class of solutions to

\[
\begin{aligned}
F^{(0,2)}_D &= 0 \\
D^{(0,1)}\phi &= 0 \\
\sum_{i=0}^k |\phi_i|^2_H - 1 &= 0.
\end{aligned}
\]  

(3.5)

It is then immediately clear from the third equation in (3.5) that no \( k+1 \) tuples of sections in \( \nu_{k+1}(\infty) \) vanish simultaneously. We have \( \nu_{k+1}(\infty) = \nu_{k+1,0}(\infty) \). Moreover, (3.5) indicates that solutions to these equations consist of precisely a unitary connection \( D \) and \( k+1 \) holomorphic sections with image in the unit sphere \( S^{2k+1} \). Modulo gauge transformation, each vortex at \( s = \infty \) then defines a holomorphic map from \( \Sigma \) to \( \mathbb{C}P^k \) in exactly identical way provided in the proof of Theorem 3.1. We have a diffeomorphism

\[
\Phi_\infty : \mathcal{H}_{r,k} \to \nu_{k+1}(\infty)
\]

defined with precisely the same way as \( \Phi_s \). A naturally plausible attempt to compute the \( L^2 \) volume of \( \mathcal{H}_{r,k} \), as conjectured in [B], is then to let \( s \to \infty \) in (3.4). With the established result in [L] that the \( L^2 \) metric (3.3) converges to the \( L^2 \) metric (2.3) in the sense of Cheeger-Gromov, we aim to further establish the fact that formula (3.4) is true at \( s = \infty \).

4. Convergence of Moduli Spaces and Their Metrics

We provide a brief conclusion of solving \( u_s \) using techniques developed in [K-W], which are important tools for the main constructions of this article. We recall equation (3.2), the PDE characterizing solutions to vortex equations:

\[
-\Delta_\omega u_s + \frac{s^2}{2} \sum_{i=0}^k |\phi_i|^2_H e^{2u_s} + \left( \sqrt{-1}\Lambda F_H - \frac{s^2}{2} \right) = 0.
\]  

(4.1)

Solving this equation is equivalent to the Kazdan-Warner equations in [K-W] in the following ways. Normalizing the Kähler metric \( \omega \) so that \( Vol_\omega \Sigma = 2 \), consider

\[
c(s) = 2 \int_\Sigma \left( \sqrt{-1}\Lambda F_H - \frac{s^2}{2} \right) dvol_\omega = 2 \int_\Sigma \sqrt{-1}\Lambda F_H \omega^n - \frac{s^2}{2} dvol_\omega = 2 \int_\Sigma \sqrt{-1}\Lambda F_H \omega^n - s^2 = 2c_1 - s^2,
\]

where \( c_1 = \int_\Sigma \sqrt{-1}\Lambda F_H \) is independent of \( s \) and \( H \). Consider \( \psi \), a solution to:

\[
\Delta_\omega \psi = \left( \sqrt{-1}\Lambda F_H - \frac{s^2}{2} \right) - \frac{c(s)}{2} = \sqrt{-1}\Lambda F_H - c_1,
\]  

(4.2)
which is clearly independent of \( s \). Here, \( \Delta_{\omega} \) is the Laplacian operator with respect to the Kähler form \( \omega \). Setting \( \varphi_s := 2(u_s - \psi) \), one can readily see that \( u_s \) satisfies (3.2) if and only if \( \varphi_s \) satisfies:

\[
\Delta \varphi_s - \frac{s^2}{2} \left( \sum_{i=0}^{k} |\phi_i|_H^2 e^{2\psi} \right) e^{\varphi_s} - c(s) = 0.
\] (4.3)

This is of the form:

\[
\Delta \varphi = - \left( \frac{s^2}{2} h \right) e^{\varphi_s} + c(s),
\] (4.4)

with the strictly negative norm function

\[
h = - \sum_{i=0}^{k} |\phi_i|_H^2 e^{2\psi}
\] (4.5)

and \( c(s) < 0 \) for \( s \) in appropriate range. Lemma 9.3 in \[K-W\] guarantees the unique solution to exist using the method of super and sub solutions, a consequence of the maximum principle applied to the operators \( L_s := \Delta_{\omega} - s^2 kI \), where \( k > 0 \) is a constant determined by \( h \). In \[K-W\], sufficient conditions for solutions for Kazdan-Warner equations to exist are merely \( h \leq 0 \). However, to obtain further uniformity and convergent behaviors of \( u_s \) over \( s \) requires \( h < 0 \), which depends crucially on the non-simultaneous vanishing of the \( k+1 \) sections. In \[L\], the author has proved

**Theorem 4.1.** On a compact Riemannian manifold \( M \) without boundary, let \( c_1 \) be any constant, \( c_2 \) any positive constant, and \( h \) any negative smooth function. Let \( c(s) = c_1 - c_2 s^2 \), for each \( s \) large enough, the unique solutions \( \varphi_s \in C^\infty \) for the equations

\[
\Delta \varphi_s = c(s) - s^2 h e^{\varphi_s}.
\]

are uniformly bounded in all Sobolev spaces. Moreover, in the limit \( s \to \infty \), \( \varphi_s \) converges smoothly to

\[
\varphi_\infty = \log \left( \frac{c_2}{-h} \right),
\]

the unique solution to

\[
h e^{\varphi_\infty} + c_2 = 0.
\]

The theorem is an independent analytic result with the additional benefit that it holds on general compact Riemannian manifolds of any dimension and complex structure is not necessary. In particular, the uniformity of Kazdan-Warner equations can be deduced from this theorem with \( c_1 = \int_{\Sigma} \sqrt{-1} \Lambda F_H d\text{vol}_H \) and \( c_2 = 1 \). Theorem 4.1 is proved by explicitly constructing the family of approximated solutions \( v_s \) that approach \( \varphi_\infty \) smoothly and moreover, the difference of \( v_s \) and \( \varphi_s \)
approaches zero in any Sobolev norm. The family of approximated solutions is given by

$$v_s := \log \left( \frac{\Delta \omega (- \log(-h)) - c(s)}{-s^2 h} \right)$$

(4.6)

with the family of error terms

$$E_s := \Delta \omega \left( \frac{\Delta \omega (- \log(-h)) - c(s)}{s^2} \right)$$

(4.7)

so that

$$\Delta \omega v_s = c(s) = s^2 h e^{v_s} + E_s.$$ 

The two families $v_s$ and $\varphi_s$ satisfy the following lemma:

**Lemma 4.2.** For all $l \in \mathbb{N}$, the family of actual and approximated solutions to the Kazdan-Warner equations (4.4), $\varphi_s$ and $v_s$, we have

$$\lim_{s \to \infty} \|\varphi_s - v_s\|_{H^l,\infty} = 0.$$ 

The lemma will again be a crucial part of the main result of this paper. The complete proof is summarized in the next section.

The convergence result and its proof may be directly applied to prove a conjecture on convergence of $L^2$ metrics of vortex moduli spaces (3.3) to the ordinary $L^2$ metric on maps (2.3), posed in [B] and proved in [L]. The convergence holds in the sense of Cheeger-Gromov:

**Definition 4.3** (Cheeger-Gromov Convergence). For all $l \in \mathbb{N}$ and $p \geq 1$, a family of $n$-dimensional Riemannian manifolds $(M_s, g_s)$ is said to converge to a fixed Riemannian manifold $(M, g)$ in $H^{l,p}$, in the sense of Cheeger-Gromov, if there is a covering chart $\{U_k, (x^k_i)\}$ on $M$ and a family of diffeomorphisms $F_s : M \to M_s$, such that

$$\left\| F_s^*(g_s) \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) - g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \right\|_{H^{l,p}(U_k)} \to 0.$$ 

(4.8)

as $s \to \infty$, for all $k$ and $i, j \in \{1, \ldots, n\}$.

We have

**Theorem 4.4** (Baptista’s Conjecture). Equipping $\mathbb{CP}^k$ with the Fubini-Study metric, the sequence of metrics $g_s$ on $\nu_{k+1,0}(s)$ given by (3.3) converges in all Sobolev spaces, in the sense of Cheeger-Gromov, to the $L^2$ metric $\langle \cdot, \cdot \rangle_{L^2}$ on $H_{r,k}$ given by (2.3). The family of diffeomorphisms are precisely $\Phi_s$, as constructed in Theorem 3.1.

**Proof.** (Sketch) We provide a brief outline of the proof in [L] with some adjustments to the notations. Fix $\phi \in H_{r,k}$. Pick canonical hyperplane sections $s_0, \ldots, s_k$ with
constant Fubini-Study norm 1. On the pullback line bundle \( L = \tilde{\phi}^*\mathcal{O}(1) \) with pullback metric \( H \) locally given by

\[
H(z) = \tilde{\phi}^*H_{FS}(z) = \frac{1}{\sum_{i=0}^{k} |\tilde{\phi}_i(z)|}
\]

the pullback sections \( (\phi_i = \tilde{\phi}^*s_i)_{i=0}^{k} \) are of constant \( H \)-norm 1. As in section 3 these initial data give rise to the norm function \( h \) and the special gauge function \( u \) that leads to the solution of vortex equations. Infinitesimally, given a tangent vector \( \xi_\alpha \in T_{\tilde{\phi}^*H_{r,k}} \cong \Gamma(\tilde{\phi}^*\mathbb{C}P^k) \), there are corresponding infinitesimal variations of metric and gauge function, denoted by \( H^\alpha \) and \( u^\alpha \). Moreover, there are \( k+1 \) sections \( \phi^\alpha := (\phi^\alpha_i) \), identified with \( \xi_\alpha \) via the pullback of the Euler sequence on \( \mathbb{C}P^k \) via \( \tilde{\phi} \). With these, we identify the pushforward of \( \psi_\alpha \) via \( \Phi_s \):

\[
\Phi_{s,\ast}(\xi_i) = (A^\alpha_i, \phi^\alpha_i) = (A^\alpha + 2\frac{\partial u^\alpha}{\partial z} dz, e^{u^\alpha} \phi^\alpha + e^{u^\alpha} u^\alpha \phi) \in T_{[D_s,\phi_s]}\nu_{k+1,0}(s),
\]

(4.9)

where \( A^\alpha \) is the initial tangent vector of the connection form induced by \( \xi_i \). Given \( \xi_\alpha, \xi_\beta \in T_{\tilde{\phi}^*H_{r,k}} \), the pullback metric \( g^\ast_s \) at \( \phi \) are then determined by an \( m \times m \) matrix of smooth functions:

\[
g^\ast_s(\xi_\alpha, \xi_\beta) := g_s(\Phi_{s,\ast}(\xi_\alpha), \Phi_{s,\ast}(\xi_\beta)) = \int_{\Sigma} \frac{(A^\alpha + 2\frac{\partial u^\alpha}{\partial z})}{2s^2} (A^\beta + 2\frac{\partial u^\beta}{\partial z}) \text{vol}_\Sigma + \int_{\Sigma} \left( (\phi^\alpha, \phi^\beta)_H e^{2u_s} + (u^\alpha u^\beta e^{2u_s}) \right) \text{vol}_\Sigma.
\]

(4.10)

A fact implicitly used in this formula is that the sections \( \phi \) and the metric \( H \) are chosen so that the sum of the \( H \) norms of \( \phi \)'s are constant, and therefore \( <\phi, \phi^\alpha>_H = 0 \forall \alpha \).

Recall that \( \varphi_s = 2(u_s - \psi) \), where \( \psi \) is the solution to

\[
\Delta_\omega \psi = \sqrt{-1}\Lambda F_H - c_1,
\]

and the background sections are chosen so that \( e^{2u_s} = -he^\varphi \), (4.10) can be rewritten as
\[ g_s^*(\xi_\alpha, \xi_\beta) = \int_\Sigma \left( \frac{A^\alpha + \partial \phi_s^\alpha}{\partial z} + 2 \frac{\partial \psi_s^\alpha}{\partial z} \right) \left( \frac{A^\beta + \partial \phi_s^\beta}{\partial z} + 2 \frac{\partial \psi_s^\beta}{\partial z} \right) 2s^2 \ dvol_\Sigma \]

\[ + \int_\Sigma \left( \langle \phi^\alpha, \phi^\beta \rangle_H (-he^\varphi) - (he^\varphi)^\alpha u_s^\beta \right) dvol_\Sigma. \]

(4.11)

The pointwise definition of \( g_s^* \) extends naturally to a smooth function near \( \hat{\phi} \). Take an open neighborhood \( U \) near \( \hat{\phi} \) with local coordinate \( \{w_1, \ldots, w_m\} \) centered at \( \hat{\phi} \) as described in (2.2). Let \( \{\hat{\xi}_1, \ldots, \hat{\xi}_m\} \) be the corresponding frame of \( T\mathcal{H}_{r,k} \) over \( U \). Each holomorphic function \( \hat{\eta} \in U \) yields a unique set of background metric, sections, and gauge function depending smoothly on variation of holomorphic functions. The functions \( H, \psi, h, \varphi_s \) and \( u_s \) in the integrand of (4.11) are therefore understood to be smooth functions defined on \( U \times \Sigma \), and we amend these notations to \( \hat{\tilde{A}}, \hat{\tilde{\psi}}, \hat{\tilde{h}}, \hat{\tilde{\varphi}}_s \) and \( \hat{\tilde{u}}_s \) to reflect their dependencies on \( w_1, \ldots, w_m \).

The formula (4.11) then defines smooth functions on \( U \):

\[ g_{s,\alpha,\beta}^* := g_s^* (\hat{\xi}_\alpha, \hat{\xi}_\beta) = \int_\Sigma \left[ \hat{\tilde{A}}^\alpha + \partial \hat{\tilde{\psi}}^\alpha \right] \left[ \hat{\tilde{A}}^\beta + \partial \hat{\tilde{\psi}}^\beta \right] 2s^2 \ dvol_\Sigma \]

\[ + \int_\Sigma \left[ \langle \phi^\alpha, \phi^\beta \rangle_H (-he^{\hat{\varphi}}) - (he^{\hat{\varphi}})^\alpha \hat{\tilde{u}}_s^\beta \right] dvol_\Sigma. \]

(4.12)

The proof of Theorem 4.1 is then mimicked to show that the first and third terms in the integrand of (4.12) converge to zero and \( he^{\hat{\varphi}} \to 1 \) smoothly as \( s \to \infty \), at all points on \( U \).

\[ \square \]

To this end, we have established that \( g_s^* \) is locally represented by a finite collection of smooth functions, each converging smoothly, as functions on \( U \), to the finite collection of smooth functions representing the ordinary \( L^2 \) metrics. To establish the convergence of volumes, we must show that the volume forms induced by each \( g_s^* \) converges to the volume form induced by the \( L^2 \) metric in \( L^1(U) \) as \( s \to \infty \). An obvious challenge comes from the fact that singularities for \( \varphi_s \) may develop near the boundary of the non-compact set \( U \) corresponding to vortices with common zeros. In such situations, pointwise convergence of volume forms does not necessarily imply that integrals, namely the volume, converge to the integral of the limiting volume form. However, the next section assures that the convergence in integral continues to hold.
5. Main Constructions

In this section we establish the main theorem of this article.

**Theorem 5.1** (Main Theorem). The $L^2$ volume of $\mathcal{H}_{r,k}$ is

$$\frac{b(k + 1)^b}{q!b!},$$

where $q = b + (k + 1)(r + 1 - b) - 1$ and $b$ is the genus of $\Sigma$.

The theorem follows from several technical computations. Since the spaces discussed in the Main Theorem are open, where boundedness conditions required for the convergence of integrals are more difficult to achieve, we wish to establish the result on the compactifications of these spaces, noting that compactifications do not affect the $L^2$ volumes.

**Proposition 5.2.** With respect to the $L^2$ metric defined in (3.3), we have

$$\operatorname{Vol}s\nu_{k+1}(s) = \operatorname{Vol}s\nu_{k+1,0}(s),$$

for all $s$ large enough.

**Proof.** The proposition is an immediate consequence of Propositions 3.2 and Lemma 2.7. We recall the diffeomorphic correspondence

$$\Phi_s : \nu_{k+1}(s) \sim H = \bigsqcup_{i=0}^r \text{Sym}^i \Sigma \times \text{Hol}_{r-i}(\Sigma, \mathbb{C}P^k).$$

Let $\nu_{k+1,a}(s)$ be the class of vortices with common zeros a divisor of degree $a$, which corresponds to the $a$th stratification above. Lemma 2.7 then implies that $\nu_{k+1,a}(s)$ is of dimension

$$k(r - b + 1) + r - ka,$$

and therefore $\nu_{k+1,a}(s)$ if of strictly lower dimension than $\nu_{k+1,0}(s)$ for all $a > 0$. The arguments are valid for all $s$ large enough, including $\infty$, and the proposition is therefore established. \(\Box\)

This proposition implies, together with (3.4), that

$$\operatorname{Vol}\nu_{k+1,0}(s) = \sum_{i=0}^b \frac{b!(k + 1)^{b-i}}{(q - i)!(b - i)!} \left(\frac{4\pi}{s^2}\right)^i \left(\operatorname{Vol}\Sigma - \frac{4\pi}{s^2}r\right)^{q-i}, \quad (5.1)$$

where $q = b + (k + 1)(r + 1 - b) - 1$, which implies that

$$\operatorname{Vol}\mathcal{H} = \operatorname{Vol}\mathcal{H}_{r,k}. \quad (5.2)$$

Therefore, we may compute the volume of the compactified spaces, where $L^p$ convergence is more feasible. To discuss convergence on the compactified domain, we first construct smooth extensions of the basic functions and forms that give rise to metric functions $g_{\alpha, \beta, s}$. Throughout the entire constructions below, we reserve
\( \alpha, \beta \in \{1, \ldots, m\} \) as indices coordinates of the finite dimensional manifold \( \overline{H}_{r,k} \) described in \( (2.2) \). Moreover, we adopt the notation "\( f^{\alpha} \)" to denote the derivative, or the infinitesimal variation of an object \( f \), induced from the \( \alpha^{th} \) component of the local frame of \( T \overline{H}_{r,k} \).

**Proposition 5.3.** The curvature \( \tilde{F} \) given by the pullback metric \( \tilde{H} \) can be smoothly extended to a smooth \((1,1)\) form on \( \overline{H}_{r,k} \).

**Proof.** We first extend the domain of the pullback metric \( \tilde{H} \) to \( \overline{H}_{r,k} \times \Sigma \). One notes that \( \overline{H}_{r,k} \) is identified with \( \overline{H}_{r,k} := \{ (E_0, \ldots, E_k) \in \text{Sym}^r \Sigma \times \cdots \times \text{Sym}^r \Sigma \mid \mu(E_0) = \mu(E_j) \forall j \} \times \mathbb{CP}^k \). (5.3)

Namely, it is \( \mathcal{H}_{r,k} \) without the open condition \( \cap_j E_j = \emptyset \). Each \( (E_0, \ldots, E_k) \in \overline{H}_{r,k} \) is associated to an element in \( \overline{H}_{r,k} \) as follows. Let \( E := \cap_j E_j = \sum_{a=1}^{n_E} d_a p_a \) with \( \sum_{a=1}^{n_E} d_a = l \).

Evidently, \( E \in \text{Sym}^l \Sigma \) for some \( l \). For each \( j \), let \( E_j' = E_j - E \). The tuple \( (E_0', \ldots, E_k') \) then have no point in common and still satisfies the defining equations for \( \mathcal{H} \) (and \( \mathcal{H}_{r,k} \)). It then defines a holomorphic map \( \tilde{\phi}_E : \Sigma \to \mathbb{CP}^k \) of degree \( r - l \) as in Proposition 2.5. We then consider the pullback Fubini-Study metric via \( \tilde{\phi}_E \) with the locally defined function:

\[
\tilde{H}'(E_0, \ldots, E_k)(z) = \frac{1}{\sum_{i=0}^{k} |\tilde{\phi}_{E,i}(z)|^2}.
\] (5.4)

This metric is defined on the line bundle \( L = \tilde{\phi}_E^*(\mathcal{O}(1)) \) over \( \Sigma \) with degree \( r - l \).

The family \( \tilde{H}' \) clearly agrees with \( \tilde{H} \) originally defined on \( \mathcal{H}_{r,k} \times \Sigma \) with \( E = \emptyset \). The corresponding family of curvatures is then locally given by

\[
F_{\tilde{H}'} := \overline{\partial} \partial \log \tilde{H}'.
\] (5.5)

\( F_{\tilde{H}'} \) is clearly smooth and we claim that it is a smooth extension of \( F_{\tilde{H}} \). Indeed, for \( E = \sum_{a=1}^{n_E} n_a p_a \), pick a neighborhood \( U \) so that \( U \cap E = \{p_a\} \). On \( U - \{p_a\} \), we re-write
\[ F_{\tilde{H}} = -\bar{\partial} \partial \log \left( |z-p_a|^{2d_a} \sum_{i=0}^{k} \left| \frac{\hat{\phi}_i(z)}{(z-p_a)^{d_a}} \right|^2 \right) \]

\[ = -d_a \bar{\partial} \partial \log |z-p_a|^2 - \bar{\partial} \partial \log \left( \sum_{i=0}^{k} \left| \frac{\hat{\phi}_i(z)}{(z-p_a)^{d_a}} \right|^2 \right) \]

\[ = -\bar{\partial} \partial \log \left( \sum_{i=0}^{k} \frac{\hat{\phi}_i(z)}{(z-p_a)^{d_a}} \right)^2 . \]

(5.6)

The third equality is true because \( z \neq p_a \) and therefore \( \bar{\partial} \partial \log |z-p_a|^2 = 0 \). The third line in (5.6) is defined on the entire \( \mathcal{U} \), which also agree with the curvature given by the extended metric function \( \tilde{H}' \) when divisors accumulate a \( p_a \), and the proof is complete.

\[ \square \]

**Proposition 5.4.** The norm function \( h \) defined in (4.5), amended in Theorem 4.1, can be smoothly extended to a nonzero smooth function on \( \overline{\mathcal{H}_{r,k}} \times \Sigma \).

**Proof.** The proof follows easily from Proposition 5.3. With the definition of pull-back Fubini-Study metric in mind, we examine the definition of \( \tilde{h} : \overline{\mathcal{H}_{r,k}} \times \Sigma \rightarrow \mathbb{R} \)

\[ \tilde{h}(\tilde{\phi}, z) = -e^{2\tilde{\psi}} \sum_{i=0}^{k} \left| \tilde{\phi}^* (s_i) \right|^2 \tilde{H} \]

\[ = -e^{2\tilde{\psi}} \frac{\sum_{i=0}^{k} \left| s_i (\tilde{\phi}(z)) \right|^2}{\sum_{j=0}^{k} \left| \tilde{\phi}(z) \right|^2} . \]

(5.7)

where \( \tilde{\psi} \) solves the equation

\[ \Delta_{\omega} \tilde{\psi} = \sqrt{-1} \Lambda F_{\tilde{H}} - c_1 . \]

One notes that in the second line of the definition of \( \tilde{h} \), the fraction is in fact a well defined smooth function on \( \overline{\mathcal{H}_{r,k}} \), as accumulations of common zeros appear simultaneously on the numerator and denominator with the same order. For the exponential term, we observe that since the curvature form can be smoothly extended to \( \overline{\mathcal{H}_{r,k}} \), so can \( \tilde{\psi} \). Indeed, the solution to the Laplacian equation

\[ \Delta_{\omega} \tilde{\psi}' = \sqrt{-1} \Lambda F_{\tilde{H}'} - \sqrt{-1} \Lambda F_{\tilde{H}'} \]

satisfies the condition and therefore \( \tilde{h} \) is smoothly extended to a nonzero smooth function on \( \overline{\mathcal{H}_{r,k}} \times \Sigma \).
Propositions 5.3 and 5.4 allow us to establish the convergence of metric functions of $g^*$ to $g_{L^2}$ in all $L^p(\mathcal{H}_{r,k})$. For each $\alpha, \beta \in \{1, \ldots, m\}$, we reexamine the formula of $g^*_{\alpha, \beta, s}$:

$$g^*_{\alpha, \beta, s} = X_{\alpha, \beta, s} + Y_{\alpha, \beta, s} + Z_{\alpha, \beta, s} \in C^\infty(\mathcal{U}),$$

where

$$X_{\alpha, \beta, s} = \int_{\Sigma} \left[ \frac{\tilde{A}^\alpha + \frac{\partial \tilde{\phi}^\alpha}{\partial z} + 2\frac{\partial \tilde{\psi}^\alpha}{\partial z}}{2s^2} \right] \frac{1}{2s^2} d\text{vol}_\Sigma \quad (5.8)$$

$$Y_{\alpha, \beta, s} = -\int_{\Sigma} \left( \tilde{h}e^{\tilde{\phi}^s} \right)^\alpha \tilde{u}_\beta^s d\text{vol}_\Sigma \quad (5.9)$$

and

$$Z_{\alpha, \beta, s} = \int_{\Sigma} \langle \phi^\alpha, \phi^\beta \rangle_H \left( -\tilde{h}e^{\tilde{\phi}^s} \right) d\text{vol}_\Sigma \quad (5.10)$$

We now prove their expected convergence in $L^p(\mathcal{H}_{r,k})$. The domains of all the functions and forms considered below are now defined on $\mathcal{H}_{r,k} \times \Sigma$ unless otherwise specified.

**Proposition 5.5.** For all $\alpha, \beta$, the functions

$$Z_{\alpha, \beta, s} = \int_{\Sigma} \langle \phi^\alpha, \phi^\beta \rangle_H \left( -\tilde{h}e^{\tilde{\phi}^s} \right) d\text{vol}_\Sigma$$

converge to

$$\int_{\Sigma} \langle \phi^\alpha, \phi^\beta \rangle_H d\text{vol}_\Sigma$$

as $s \to \infty$ in $L^p(\mathcal{H}_{r,k})$ for all $p$.

**Proof.** By the Main Theorem in [L], the function $-\tilde{h}e^{\tilde{\phi}^s}$ converges to 1 pointwise on $\mathcal{H}_{r,k}$. (Note that the Kähler form of $\Sigma$ is renormalized so that $\text{Vol}(\Sigma) = 2$.) Therefore,

$$Z_{\alpha, \beta, s} \to \int_{\Sigma} \langle \phi^\alpha, \phi^\beta \rangle_H = \int_{\Sigma} \langle \tilde{\phi}^s(\xi^\alpha), \tilde{\phi}^s(\xi^\beta) \rangle_{H_{FS}},$$

pointwise as $s \to \infty$ on $\mathcal{H}_{r,k}$.

To ensure the convergence in $L^p(\mathcal{H}_{r,k})$, we need appropriate bounds to apply dominated convergence theorem. The bound is obtained from uniform estimates of $\tilde{h}e^{\tilde{\phi}^s}$ with the approximated solutions to Kazdan-Warner equations, constructed in [L].
\[ \bar{c}(s) := 2 \int_{\Sigma} \sqrt{-1} \Lambda F_{\bar{H}} - s^2 \]

we define the approximated solutions

\[ \tilde{\psi}_s := \log \left( \frac{-\Delta_{\omega} \left( \log(-\tilde{h}) \right) - \bar{c}(s)}{-s^2 \tilde{h}} \right) \]  

(5.11)

with error functions

\[ \tilde{E}_s := \Delta_{\omega} \log \left( \frac{\Delta \left( -\log(-\tilde{h}) \right) - \bar{c}(s)}{s^2} \right) \].  

(5.12)

so that

\[ \Delta_{\omega} \tilde{\psi}_s = -s^2 \tilde{h} \tilde{\psi}_s + \tilde{E}(s)_s \]  

(5.13)

By Proposition [5.4], \( \tilde{\psi}_s \) and \( \tilde{E}(s) \) are smooth on \( \overline{H_{r,k}} \times \Sigma \). \( \bar{c}(s) \) is constant on each connected component of \( \overline{H_{r,k}} \) and may be treated as constant when differentiating with respect to any variable. These functions provide natural bounds for \( Z_{\alpha,\beta,s} \).

Indeed, one observes that

\[ \tilde{h} \tilde{\psi}_s = \frac{-\Delta_{\omega} \left( \log(-\tilde{h}) \right) - \bar{c}(s)}{-s^2} \]

and therefore

\[ \int_{\Sigma} \tilde{h} \tilde{\psi}_s vol_{\Sigma} = \frac{\bar{c}(s)}{s^2} Vol(\Sigma) \leq K_1, \]

where \( K_1 \) is a uniform constant over \( \overline{H_{r,k}} \). The difference between \( \tilde{h} \tilde{\psi}_s \) and \( \tilde{h} \tilde{\psi}_s \) is estimated uniformly using the maximum principal of \( \Delta_{\omega} \). For each \( s \) and \( R \in \overline{H_{r,k}} \), since \( \tilde{\varphi}_s(R, \cdot) - \tilde{\psi}_s(R, \cdot) \) is smooth on \( \Sigma \), we may pick \( x_s^R, y_s^R \in \Sigma \) so that

\[ \tilde{\varphi}_s(x_s^R) - \tilde{\psi}_s(x_s^R) = \sup_{z \in \Sigma} \{ \tilde{\varphi}_s(z) - \tilde{\psi}_s(z) \}, \]

and

\[ \tilde{\varphi}_s(y_s^R) - \tilde{\psi}_s(y_s^R) = \inf_{z \in \Sigma} \{ \tilde{\varphi}_s(z) - \tilde{\psi}_s(z) \}. \]

The maximum principle of \( \Delta_{\omega} \) then implies that

\[ \Delta_{\omega} (\tilde{\varphi}_s - \tilde{\psi}_s) (x_s^R) \leq 0, \]  

(5.14)

and

\[ \Delta_{\omega} (\tilde{\varphi}_s - \tilde{\psi}_s) (y_s^R) \geq 0. \]  

(5.15)

Subtracting (5.13) from the Kazdan-Warner equations for \( \tilde{\varphi}_s \), we have
\[ \Delta_\omega \tilde{\varphi}_s = -s^2 \tilde{h} \tilde{e}^{\tilde{v}_s} + \tilde{c}(s). \]

(5.14), (5.15), and the choices of \( x_R^s, y_R^s \) together then yield the estimates

\[ \left( \tilde{h} e^{\tilde{v}_s} - \tilde{h} e^{\tilde{v}_s} \right)(R, z) \leq \frac{\tilde{E}_s}{(-s^2 \hbar)|_{(R,x_R^s)}} \left( \tilde{h} e^{\tilde{v}_s} \right)(R, z), \tag{5.16} \]

and

\[ \left( \tilde{h} e^{\tilde{v}_s} - \tilde{h} e^{\tilde{v}_s} \right)(R, z) \geq \frac{\tilde{E}_s}{(-s^2 \hbar)|_{(R,y_R^s)}} \left( \tilde{h} e^{\tilde{v}_s} \right)(R, z), \tag{5.17} \]

for all \( R \in \overline{\mathcal{H}}_{r,k} \). (See the proof of Theorem 3.4 in [L] for complete process.)

Therefore, we have

\[ \left| \tilde{h} e^{\tilde{v}_s} - \tilde{h} e^{\tilde{v}_s} \right| \leq \frac{1}{s^2} \sup_{H \times \Sigma} \left| \tilde{h} e^{\tilde{v}_s} \right| \sup_{H \times \Sigma} \left| \tilde{E}_s \right| \]

\[ \leq K_2, \tag{5.18} \]

where \( K_2 \) is again a uniform constant. This is possible since for \( s \) large enough, \( \tilde{h} e^{\tilde{v}_s} \) is a smooth nonzero function uniformly bounded function on \( \overline{\mathcal{H}}_{r,k} \) and \( \tilde{E}_s \) is smooth and uniformly bounded as well, as can be readily observed at the beginning of the proof. We now possess sufficient estimates for the \( L^p \) bound.

For all \( p \geq 1 \),

\[ \left| \int_\Sigma \tilde{h} e^{\tilde{v}_s} dvol_\Sigma \right|^p = \left| \int_\Sigma \tilde{h} e^{\tilde{v}_s} dvol_\Sigma + \int_\Sigma \left( \tilde{h} e^{\tilde{v}_s} - \tilde{h} e^{\tilde{v}_s} \right) dvol_\Sigma \right|^p \]

\[ \leq \left( \int_\Sigma \left| \tilde{h} e^{\tilde{v}_s} \right| dvol_\Sigma + \int_\Sigma \left| \tilde{h} e^{\tilde{v}_s} - \tilde{h} e^{\tilde{v}_s} \right| dvol_\Sigma \right)^p \]

\[ \leq (K_1 + K_2)^p \in L^1(\overline{\mathcal{H}}_{r,k}), \tag{5.19} \]

since \( \overline{\mathcal{H}}_{r,k} \) is compact. It is now clear that \( |Z_{\alpha,\beta,s}|^p \) are uniformly bounded on \( \overline{\mathcal{H}}_{r,k} \). Since the integral

\[ \int_\Sigma \langle \phi^\alpha, \phi^\beta \rangle_H dvol_\Sigma \]

depends smoothly on variation of holomorphic maps and locations of zeros, it is uniformly bounded on \( \overline{\mathcal{H}}_{r,k} \). Together with (5.19), we conclude that \( |Z_{\alpha,\beta,s}|^p \) are uniformly bounded by a constant, and therefore an \( L^1 \) function on \( \overline{\mathcal{H}}_{r,k} \). Dominated convergence theorem then applies to yield the conclusion of the proposition. \( \square \)

The next desired convergence is
Proposition 5.6. For each $\alpha, \beta$, the smooth functions

$$Y_{\alpha,\beta,s} = - \int_{\Sigma} \left( \tilde{h} e^{\tilde{\varphi}_s} \right)^\alpha \tilde{u}^\beta_s d\text{vol}_\Sigma$$

converge to 0 as $s \to \infty$ in $L^p(\mathcal{H}_{r,k})$ for all $p$.

**Proof.** We begin by differentiating Kazdan-Warner equation as well as the approximated equation with respect to the $\beta^{th}$ coordinate of $\mathcal{H}_{r,k}$. The $\beta$-differentiation clearly commutes with $\Delta_\omega$ since they are defined on different spaces. We obtain the following linearizations:

$$\Delta_\omega \tilde{\varphi}_s^\beta = -s^2 \left( \tilde{h}^\alpha + \tilde{h} \tilde{\varphi}_s^\beta \right) e^{\tilde{\varphi}_s}, \quad (5.20)$$

and

$$\Delta_\omega \tilde{v}_s^\beta = -s^2 \left( \tilde{h}^\alpha + \tilde{h} \tilde{v}_s^\beta \right) e^{\tilde{v}_s} + \tilde{E}_s^\alpha. \quad (5.21)$$

The $\beta$-derivatives of $\tilde{v}_s$ and $\tilde{E}_s$ can be readily computed:

$$\tilde{v}_s^\beta = \frac{1}{s^2} \frac{-s^2 \tilde{h}}{\Delta_\omega \left( -\log(-\tilde{h}) \right) - \tilde{c}(s)} \left[ \Delta_\omega \left( -\log(-\tilde{h}) \right) - \tilde{c}(s) \right]^\beta, \quad (5.22)$$

$$\tilde{E}_s^\beta = \frac{1}{\tilde{c}(s)} \Delta_\omega \left[ \frac{1}{\Delta_\omega \left( -\log(-\tilde{h}) \right) - \tilde{c}(s)} - 1 \right] \left[ \Delta_\omega \left( -\log(-\tilde{h}) \right) \right]^\beta. \quad (5.23)$$

One can readily verify that for $s$ large enough, the fact that $\tilde{h}$ is a smooth positive function on $\mathcal{H}_{r,k} \times \Sigma$ implies that both $\tilde{v}_s^\beta$ and $\tilde{E}_s^\beta$ are $\frac{1}{s^2}$ times functions that are uniformly bounded on $\mathcal{H}_{r,k} \times \Sigma$, and therefore uniformly converge to 0 on $\mathcal{H}_{r,k} \times \Sigma$.

We then repeat the arguments of maximum principle to the difference of equations (5.20) and (5.21) to obtain uniformly decaying bounds for $\tilde{\varphi}_s^\beta$ and $\tilde{v}_s^\beta$. We find

$$|\tilde{\varphi}_s^\beta - \tilde{v}_s^\beta| \leq \sup_{\mathcal{H}_{r,k} \times \Sigma} \left| \tilde{E}_s^\beta \left( \frac{\tilde{h}^\beta (e^{\tilde{\varphi}_s} - e^{\tilde{v}_s}) + (\tilde{h} e^{\tilde{\varphi}_s} - \tilde{h} e^{\tilde{v}_s}) \tilde{v}_s^\beta}{\tilde{h} e^{\tilde{\varphi}_s}} \right) \right|. \quad (5.24)$$

Since $\tilde{h} < 0$ is smooth, $\tilde{h}^\beta$ is uniformly bounded on $\mathcal{H}_{r,k} \times \Sigma$. By (5.13), we conclude that the fraction in the right hand side of (5.24) is uniformly bounded. Since $\tilde{E}_s^\beta$ decays to 0 uniformly on $\mathcal{H}_{r,k} \times \Sigma$ as $s \to \infty$, so does $\tilde{\varphi}_s^\beta - \tilde{v}_s^\beta$, which implies that $\tilde{\varphi}_s^\beta \to 0$ uniformly as $s \to \infty$ since $\tilde{v}_s^\beta$ does.

Recall the relation $\tilde{u}_s = \frac{1}{2} \tilde{\varphi}_s + \tilde{\psi}$, we then have

$$\tilde{u}_s^\beta \to \tilde{\psi}^\beta \quad \text{uniformly on } \mathcal{H}_{r,k} \times \Sigma \text{ as } s \to \infty, \quad (5.25)$$
where $\tilde{\psi}^\beta$ is the solution to

$$\Delta_\omega \tilde{\psi}^\beta = (\sqrt{-1} \Lambda F_{\tilde{H}})^\beta.$$ 

Since $F_{\tilde{H}}$ is smooth on $\overline{\mathcal{H}_{r,k} \times \Sigma}$, elliptic regularity ensures that $\tilde{\psi}^\beta$ is smooth and therefore uniformly bounded, and so is $\tilde{u}^\beta$. Combining this fact with equation (4.28) in [L], we have shown that $Y_{\alpha,\beta,s} \to 0$ as $s \to \infty$ pointwise on $\overline{\mathcal{H}_{r,k}}$. It remains to construct a uniform bound to apply the dominated convergence theorem to establish convergence in $L^p(\overline{\mathcal{H}_{r,k}})$, which follows clearly from straightforward computations:

$$(\tilde{h} e^{\tilde{\phi}_s})^\alpha = (\tilde{h}^\alpha + \tilde{\phi}^\alpha_s) e^{\tilde{\phi}_s} \to (\tilde{h}^\alpha + \tilde{v}^\alpha_s) e^{\tilde{v}_s},$$

uniformly on $\overline{\mathcal{H}_{r,k} \times \Sigma}$ as $s \to \infty$. Since every term in the limit is uniformly bounded over $s$, so is $(\tilde{h} e^{\tilde{\phi}_s})^\alpha$. It follows that $Y_{\alpha,\beta,s}$ are uniformly bounded on $\overline{\mathcal{H}_{r,k}}$ and dominated convergence theorem is applicable.

Finally we show

**Proposition 5.7.** For each $\alpha, \beta$, the smooth functions

$$X_{\alpha,\beta,s} = \int_\Sigma \left[ \tilde{A}^\alpha \wedge \bar{\ast} (\tilde{A}^\beta + \left( \frac{\partial \phi^\alpha_s}{\partial z} + 2 \frac{\partial \tilde{\psi}^\beta}{\partial z} \right) dz) \right]$$

converge to 0 as $s \to \infty$ in $L^p(\mathcal{H})$ for all $p$.

**Proof.** By Lemma 4.4 in [L] and the smooth extension of $\tilde{h}$, $\frac{\partial \phi^\alpha_s}{\partial z}$ and $\frac{\partial \tilde{\psi}^\beta}{\partial z}$ are uniformly bounded. $\frac{\partial \tilde{\phi}^\alpha_s}{\partial z}$ and $\frac{\partial \tilde{\psi}^\beta}{\partial z}$ are smooth on $\overline{\mathcal{H}_{r,k} \times \Sigma}$, independent of $s$, and therefore uniformly bounded. We need to, however, ensure that the integrals

$$I^1_{\alpha,\beta,s} := \int_\Sigma \tilde{A}^\alpha \wedge \bar{\ast} \tilde{A}^\beta$$

and

$$I^2_{\alpha,\beta,s} := \int_\Sigma \tilde{A}^\alpha \wedge \bar{\ast} \left( \frac{\partial \tilde{\phi}^\alpha_s}{\partial z} + 2 \frac{\partial \tilde{\psi}^\beta}{\partial z} \right) dz$$

are uniformly bounded on $\overline{\mathcal{H}_{r,k}}$. The boundedness conditions ensure that the numerator of the integrand of $X_{\alpha,\beta,s}$ are uniformly bounded and therefore decays to 0 uniformly as $s \to \infty$.

Recall that, for $\tilde{\phi} \in \mathcal{H}_{r,k}$, $\tilde{A}(\tilde{\phi})$ is the unitary connection form with respect to the metric $\tilde{\phi}^s H_{FS}$. It is locally given by the (1,0) form

$$\tilde{A}(\tilde{\phi}) = d' \left( \log \tilde{H}(\tilde{\phi}) \right),$$

where
\[ \tilde{H}(\tilde{\phi}) = \frac{1}{\sum_{i=0}^{k} |\tilde{\phi}_i|^2}, \]
is a smooth real function defined over a precompact coordinate neighborhood \( U \subset \Sigma \). Using partition of unity to piece together these local expressions, we then have,

\[
I_{\alpha, \beta, s}^1 = \int_{\Sigma} \left[ d'(\log \tilde{H}) \right]^\alpha \wedge \left[ d''(\log \tilde{H}) \right]^\beta \\
= \int_{\Sigma} d'(\log \tilde{H})^\alpha \wedge d''(\log \tilde{H})^\beta \\
= -\int_{\Sigma} (\log \tilde{H})^\alpha d'd''(\log \tilde{H})^\beta \\
= -\int_{\Sigma} (\log \tilde{H})^\alpha \tilde{F}^\beta,
\]
where \( \tilde{F}^\beta \) is the \( \beta \)-derivative of the curvature (1,1)-form of \( \tilde{A} \). The indices \( \alpha, \beta \) commute with integration and exterior derivatives since they are operators defined on different spaces. The third equality follows from Stokes’ theorem. This equality holds true on \( \mathcal{H}_{r,k} \), except \( \tilde{H} \) blows up near the boundary of \( \mathcal{H}_{r,k} \). We however will show that the uniform boundedness continues to hold despite this setback.

\( \tilde{F}, \) and \( \tilde{F}^\beta, \) are smooth, and therefore uniformly bounded, two forms on \( \mathcal{H}_{r,k} \times \Sigma \) by Proposition 5.3. Furthermore, we claim that the function

\[
\int_{\Sigma} \left| (\log \tilde{H})^\alpha \right| vol_{\Sigma}
\]
is uniformly bounded on \( \mathcal{H}_{r,k} \). Indeed, the singularities of the function

\[
\log \left( \frac{1}{\sum_{i=0}^{k} |\phi_i|^2} \right)
\]
are precisely the common zeros of the \( \tilde{\phi}_i \)'s that develop near the boundary of \( \mathcal{H}_{r,k} \). In the coordinate description of \( \mathcal{H}_{r,k} \) given in (2.2), it corresponds to condition that for some \( \alpha \), every \( \tilde{\phi}_i \) vanishes at \( w_\alpha \) with order \( m_\alpha^i > 0 \). Let \( m_\alpha \) be the minimum of these orders. Fix a open neighborhood \( U \subset \Sigma \) around \( w_\alpha \) such that for all \( i \), \( |\tilde{\phi}_i| = |z - w_\alpha|^m f_i \), where \( f_i \) is a nonvanishing smooth function on \( U \).

We then rewrite

\[
\log \tilde{H} = \log \left( \frac{1}{\sum_{i=0}^{k} |\phi_i|^2} \right) = m_\alpha \log \left( \frac{1}{|z - w_\alpha|^2} \right) + \tilde{G},
\]
where \( \tilde{G} \) is a smooth function on \( \mathcal{H}_{r,k} \times U \). Differentiating with respect to \( w_\alpha \) and taking the norm, we have
\[
\left| \left( \log \hat{H} \right)^\alpha \right| = \frac{1}{|z - w_\alpha|} + \tilde{G}^\alpha. \tag{5.29}
\]

Since \( \tilde{G}^\alpha \) is smooth on \( \overline{\mathcal{H}_{r,k}} \times U \), its integral over \( \Sigma \) is smooth and therefore uniformly bounded over \( \overline{\mathcal{H}_{r,k}} \). The singular part of \( closref{5.29} \) is also integrable over \( \Sigma \), using polar coordinate, and the integral depends smoothly on \( w_\alpha \), the location of common zeros for \( \hat{\phi}_i \)'s. We conclude that there is a constant \( K_U \) such that

\[
\int_U \left| \left( \log \hat{H} \right)^\alpha \right| \, dvol_\Sigma \leq K_U \tag{5.30}
\]

holds on all \( \overline{\mathcal{H}_{r,k}} \). Since \( \Sigma \) is compact, we conclude that

\[
\int_\Sigma \left| \left( \log \hat{H} \right)^\alpha \right| \, dvol_\Sigma
\]

is uniformly bounded over \( \overline{\mathcal{H}_{r,k}} \). This proves the claim.

From the claim and the uniform bound of \( \tilde{F}^\beta \), we conclude that for all \( \alpha, \beta, I^{1}_{\alpha,\beta,s} \) are uniformly bounded, over \( s \), on \( \overline{\mathcal{H}_{r,k}} \times \Sigma \). In fact, since

\[
\left( \frac{\partial \hat{\psi}^\beta}{\partial z} + 2 \frac{\partial \hat{\psi}^\beta}{\partial z} \right)
\]

are uniformly bounded on \( \overline{\mathcal{H}_{r,k}} \times \Sigma \), estimate on \( \int_\Sigma \left| \left( \log \hat{H} \right)^\alpha \right| \, dvol_\Sigma \) show that \( I^{2}_{\alpha,\beta,s} \) are uniformly bounded for all \( \alpha, \beta, s \).

\[\square\]

The main theorem is now an immediate consequence of the analytic results we have established.

**Proof. (of the Main Theorem 5.1)**

Covering \( \mathcal{H}_{r,k} \) by local coordinate patches \( \{ \mathcal{U} \} \), there is a smooth partition of unity \( \{ \psi_\mathcal{U} \} \) subordinate to this covering. The \( L^2 \) volume of \( \mathcal{H}_{r,k} \) is then
In terms of local representations of metrics on coordinates \((w_1, \ldots, w_m)\) on \(U\), the volume forms for each \(s\) is

\[
dvol_s^* = \sqrt{\det(g_{\alpha,\beta,s}^*)} \, dw_1 \wedge \cdots \wedge dw_m.
\]

From Propositions 5.5-5.7 it follows that \(\det(g_{\alpha,\beta,s}^*)\) converge to \(\det(g_{L^2,\alpha,\beta})\) in all \(L^p(\mathcal{H}_{r,k})\), and therefore \(L^p(\mathcal{H}_{r,k})\). (5.31) allows us to take the limit of the volume of \(\nu_{k+1,0}(s)\) given by (5.1) as \(s \to \infty\), hence proving the main theorem.

\[\square\]

One might notices that the extension of the metrics and curvature forms over the boundary of \(\mathcal{H}_{r,k}\) lowers the topological degree. In fact, we have experimented the bubbling phenomenon in the elementary settings. The author is eager to explore generalization of this result to more general cases, such as the one posed in [B1].

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