TOROIDAL ACTIONS ON LEVEL 1 MODULES OF $U_q(\widehat{\mathfrak{sl}}_n)$.

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ABSTRACT. Recently Varagnolo and Vasserot established that the $q$-deformed Fock spaces due to Hayashi, and Kashiwara, Miwa and Stern, admit actions of the quantum toroidal algebra $U'_q(\mathfrak{sl}_n,\text{tor})$ ($n \geq 3$) with the level $(0,1)$. In the present article we propose a more detailed proof of this fact than the one given by Varagnolo and Vasserot. The proof is based on certain non-trivial properties of Cherednik’s commuting difference operators. The quantum toroidal action on the Fock space depends on a certain parameter $\kappa$. We find that with a specific choice of this parameter the action on the Fock spaces gives rise to the toroidal action on irreducible level-1 highest weight modules of the affine quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$. Similarly, by a specific choice of the parameter, the level $(1,0)$ vertex representation of the quantum toroidal algebra gives rise to a $U'_q(\mathfrak{sl}_n,\text{tor})$-module structure on irreducible level-1 highest weight $U_q(\widehat{\mathfrak{sl}}_n)$-modules.

1. Introduction

Recently a new algebraic object – the quantum toroidal algebra $U'_q(\mathfrak{sl}_n,\text{tor})$ has been introduced in [5], [13]. This quantum algebra is a $q$-deformation of the enveloping algebra of a central extension of the Lie algebra $\mathfrak{sl}_n[s^{\pm 1}, t^{\pm 1}]$. Several results concerning representations of the quantum toroidal algebra were obtained [13], [10], [14]. One of these results is the Schur-type duality between representations of the toroidal Hecke algebra and representations of $U'_q(\mathfrak{sl}_n,\text{tor})$ established by Varagnolo and Vasserot [13]. This duality is analogous to the duality between affine Hecke algebra and the quantum affine algebra $U'_q(\widehat{\mathfrak{sl}}_n)$ given by Chari and Pressley [2]. It is known that $U'_q(\mathfrak{sl}_n,\text{tor})$ contains two subalgebras $U^{(1)'}_q(\mathfrak{sl}_n)$ and $U^{(2)'}_q(\mathfrak{sl}_n)$ isomorphic to $U'_q(\widehat{\mathfrak{sl}}_n)$. A module of $U'_q(\mathfrak{sl}_n,\text{tor})$ is said to have level $(k,l)$ if the $U^{(1)'}_q(\mathfrak{sl}_n)$-action on this module has level $k$ and the $U^{(2)'}_q(\mathfrak{sl}_n)$-action has level $l$.

Representations of $U'_q(\mathfrak{sl}_n,\text{tor})$ obtained by the Varangolo-Vasserot duality have the level $(0,0)$. They are analogues of level-0 modules of affine quantum algebras.

The first example of a toroidal module with a non-trivial level was constructed in [10]. This vertex operator construction is an analogue of the Frenkel-Jing bosonization for level-1 $U'_q(\mathfrak{sl}_n)$-modules. It gives a toroidal module with level $(1,0)$. We summarize this bosonic construction in section 3.
A $q$-fermionic construction of toroidal modules with non-trivial level has recently been proposed in [14]. Origins of this construction lay in the theory of integrable models with long-range interaction. It has been known for some time, starting with the work [1], that level-1 highest weight modules of $U'_q(\hat{\mathfrak{sl}}_n)$ admit a level-0 action of the same quantum affine algebra $U'_q(\hat{\mathfrak{sl}}_n)$ \[7,11]. In particular in [11] it was shown that the Fock space module of $U'_q(\hat{\mathfrak{sl}}_n)$ \[6,8] is simultaneously a level-0 $U'_q(\hat{\mathfrak{sl}}_n)$-module.

The main result of the work [14] is that for $n \geq 3$ the level-0 and level-1 actions of $U'_q(\hat{\mathfrak{sl}}_n)$ on the Fock space are exactly actions of the subalgebras $U^{(1)}_q(\hat{\mathfrak{sl}}_n)$ and $U^{(2)}_q(\hat{\mathfrak{sl}}_n)$ in the toroidal algebra, so that the Fock space may be regarded as a level $(0,1)$ module of $U'_q(\mathfrak{sl}_{n,\text{tor}})$.

From our viewpoint the proof of this fact given in the Theorem of section 12 in the paper [14] omits certain technical details – notably in the proof of the relations (6.9) - (6.14) of the present work. One of our main objectives in this paper is to supply these details by giving a complete proof, based on Lemmas 6 and 7, that the Fock space is a module of the toroidal quantum algebra (Theorem [13]). Let us emphasize that the general idea of our proof is not new methodologically compared with the idea of the proof in [14]. We arrived at this idea before the appearance of the work [14]. However, at the time when the paper [14] became available to us we did not know a complete proof of Theorem [13], specifically a proof of the relations (6.12) - (6.14) in Lemma [13] was missing.

In this paper we will also show that the vertex representation of [10] is isomorphic to the Fock space as a level-1 $U'_q(\hat{\mathfrak{sl}}_n)$-module. This means that on the Fock space we have two actions of the toroidal algebra – one action of level $(1,0)$ and another action of level $(0,1)$ such that the action of the $U^{(1)}_q(\hat{\mathfrak{sl}}_n)$-subalgebra in the former coincides with the action of the $U^{(2)}_q(\hat{\mathfrak{sl}}_n)$-subalgebra in the latter. We are not aware of any good explanation of this phenomenon.

Apart from the matters discussed above we give a proof of irreducibility of the Fock space as a $U'_q(\mathfrak{sl}_{n,\text{tor}})$-module. We also demonstrate that irreducible highest weight level-1 modules of $U'_q(\hat{\mathfrak{sl}}_n)$ admit actions of toroidal algebra with levels $(1,0)$ and $(0,1)$ induced from the corresponding actions on the Fock space.

2. Definition of quantum toroidal algebras

2.1. Let $\mathfrak{sl}_n$ be the semisimple Lie algebra of type $A_{n-1}$ and $\hat{\mathfrak{sl}}_n$ the affine Kac-Moody Lie algebra of type $A^{(1)}_{n-1}$. We denote their Cartan subalgebras by $\mathfrak{h}$ and $\hat{\mathfrak{h}}$. We denote by $\alpha_0, \ldots, \alpha_{n-1}$ the simple roots and by $h_0, \ldots, h_{n-1}$ the simple coroots of $\hat{\mathfrak{sl}}_n$. Let $P = \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ be the weight lattice. Here $\delta$ is the null root. Let $Q = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i$ be the root lattice. Note that $\alpha_0 = -\Lambda_{n-1} + 2\Lambda_0 - \Lambda_1 + \delta$, $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$ ($1 \leq i \leq n-1$). Here the indices are extended cyclically such as $\Lambda_i = \Lambda_{i+n}$. Let
\[ \mathcal{P} = \oplus_{i=1}^{n-1} \mathbb{Z} \Lambda_i \] be the classical weight lattice and \( \overline{Q} = \oplus_{i=1}^{n-1} \mathbb{Z} \alpha_i \) the classical root lattice. The inclusion of \( \mathcal{P} \) into \( P \) is given by \( \Lambda_i = \Lambda_1 - \Lambda_0 \). We also set \( \overline{\mathcal{P}} = 0 \).

We denote the pairing of \( \mathfrak{h} \) and \( \mathfrak{h}^* \) (resp. \( \hat{\mathfrak{h}} \) and \( \hat{\mathfrak{h}}^* \)) by \( \langle , \rangle \). The invariant bilinear form on \( P \) is given by \( \langle \alpha_i | \alpha_j \rangle = -\delta_{ij-1} + 2\delta_{ij} - \delta_{ij+1} \) and \( \langle \delta | \delta \rangle = 0 \).

Let \( [k] = (q^k - q^{-k})/(q - q^{-1}) \) be the q-integers and denote \( [n]! = \prod_{k=1}^{n} [k] \), \( \left[ \begin{array}{c} m \\ r \end{array} \right] = [m]!/[r]![m - r]! \).

2.2. \( U_q(\mathfrak{sl}_n) \) is a \( \mathbb{Q}(q) \)-algebra generated by the symbols \( E_i, F_i, K_i^\pm (i = 0, \cdots, n-1) \) \( q^{\pm d} \) which satisfy the following defining relations:

\[ K_i^\pm K_j^\pm = K_j^\pm K_i^\pm, \quad K_i^+ E_j K_j^- = q^{(\alpha_i, \alpha_j)} E_j, \quad K_i^+ F_j K_j^- = q^{-\langle \alpha_i, \alpha_j \rangle} F_j, \]

\[ K_i^+ q^{\pm d} = q^{\pm d} K_i^+, \quad q^d E_j q^{-d} = q^{\delta_{ij}} E_j, \quad q^d F_j q^{-d} = q^{-\delta_{ij}} F_j, \]

\[ [E_i, F_j] = \delta_{ij} \frac{K_i^+ - K_i^-}{q - q^{-1}}, \]

for \( i \neq j \),

\[ \sum_{r=0}^{m} (-1)^r \left[ \begin{array}{c} m \\ r \end{array} \right] E_i^r E_j E_i^{m-r}, \quad \sum_{r=0}^{m} (-1)^r \left[ \begin{array}{c} m \\ r \end{array} \right] F_i^r F_j F_i^{m-r}. \]

Here \( m = \langle h_i, \alpha_j \rangle \).

Let \( U_q'(\mathfrak{sl}_n) \) be the subalgebra generated by \( E_i, F_i, K_i^\pm (i = 0, \cdots, n-1) \).

The coproduct \( \Delta \) of \( U_q(\mathfrak{sl}_n) \) is given as follows:

\[ \Delta(K_i^\pm) = K_i^\pm \otimes K_i^\pm, \quad \Delta(q^{\pm d}) = q^{\pm d} \otimes q^{\pm d}, \]

\[ \Delta(E_i) = E_i \otimes K_i^+ + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^- \otimes F_i. \]

2.3. We will give the definition of the quantum toroidal algebra \( U_q(\mathfrak{sl}_{n,\text{tor}}) \). Fix an integer \( n \geq 3 \) and \( \kappa \in \mathbb{Q}(q)^\times \).

**Definition 1.** The quantum toroidal algebra \( U_q(\mathfrak{sl}_{n,\text{tor}}) \) is an associative algebra over \( \mathbb{Q}(q) \) with generators:

\[ E_{i,k}, \quad F_{i,k}, \quad H_{i,l}, \quad K_i^\pm, \quad q^{\pm d_1}, \quad q^{\pm d_2}, \]

for \( k \in \mathbb{Z}, l \in \mathbb{Z}\setminus\{0\} \) and \( i = 0, 1, \cdots, n-1 \).

The relations are expressed in terms of the formal series

\[ E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \]

\[ F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k}, \]

\[ K_i^\pm(z) = K_i^\pm \exp(\pm(q - q^{-1}) \sum_{k \geq 1} H_{i,\pm k} z^{\pm k}), \]
as follows:

\[ q^{\pm i^c} \text{ are central,} \]

\[ K_i^+ K_i^- = K_i^- K_i^+ = 1, \]  \hspace{1cm} (2.2)

\[ K_i^+(z) K_j^+(w) = K_j^+(w) K_i^+(z) \]  \hspace{1cm} (2.3)

\[ \theta_{\pm (h_i, \alpha_j)}(q^{-c m_{ij}} \frac{z}{w}) K_i^-(z) K_j^+(w) = \theta_{\pm (h_i, \alpha_j)}(q^{c m_{ij}} \frac{z}{w}) K_j^+(w) K_i^-(z) \]  \hspace{1cm} (2.4)

\[ q^{d_1} K_i^\pm(z) q^{-d_1} = K_i^\pm(q^{-1}z), \quad q^{d_1} E_i(z) q^{-d_1} = E_i(q^{-1}z), \quad q^{d_1} F_i(z) q^{-d_1} = F_i(q^{-1}z), \]  \hspace{1cm} (2.5)

\[ [q^{d_2}, K_i^\pm(z)] = 0, \quad q^{d_2} E_i(z) q^{-d_2} = q^{\delta_{00}} E_i(z), \quad q^{d_2} F_i(z) q^{-d_2} = q^{-\delta_{00}} F_i(z), \]

\[ K_i^+(z) E_j(w) = \theta_{\pm (h_i, \alpha_j)}(q^{\frac{i^c}{2}} \kappa^{m_{ij}} w^{\pm z^\mp}) E_j(w) K_i^+(z) \]  \hspace{1cm} (2.6)

\[ K_i^+(z) F_j(w) = \theta_{\pm (h_i, \alpha_j)}(q^{\frac{i^c}{2}} \kappa^{m_{ij}} w^{\pm z^\mp}) F_j(w) K_i^+(z) \]

\[ [E_i(z), F_j(w)] = \delta_{i,j} \frac{1}{q - q^{-1}} \{ \delta(q^{c \frac{w}{z}}) K_i^+(z) q^{\frac{i^c}{2}} w - \delta(q^{c \frac{z}{w}}) K_i^-(z) q^{\frac{i^c}{2}} z \} \]  \hspace{1cm} (2.7)

\[ (\kappa^{m_{ij}} z - q^{(h_i, \alpha_j)} w) E_i(z) E_j(w) = (q^{(h_i, \alpha_j)} \kappa^{m_{ij}} z - w) E_j(w) E_i(z) \]  \hspace{1cm} (2.8)

\[ (\kappa^{m_{ij}} z - q^{-(h_i, \alpha_j)} w) F_i(z) F_j(w) = (q^{- (h_i, \alpha_j)} \kappa^{m_{ij}} z - w) F_j(w) F_i(z) \]

\[ \sum_{\sigma \in S_m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} E_i(z_{\sigma(1)}) \cdots E_i(z_{\sigma(r)}) E_j(w) E_i(z_{\sigma(r+1)}) \cdots E_i(z_{\sigma(m)}) = 0 \]  \hspace{1cm} (2.9)

where \( i \neq j \) and \( m = 1 - \langle h_i, \alpha_j \rangle \).

In these formulas we denote \( \theta_m(z) = \frac{z^m - 1}{z^m} \) for \( m \in \mathbb{Z} \), the \( \theta_m(z) \) is to be regarded as the expansion of the right-hand side above in non-negative powers of the argument.
z; \delta(z) = \sum_{k \in \mathbb{Z}} z^k, m_{ij} are the entries of the following n x n-matrix

\[ M = \begin{pmatrix}
  0 & -1 & 0 & \ldots & 0 & 1 \\
  1 & 0 & -1 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & 0 & -1 \\
  -1 & 0 & 0 & \ldots & 1 & 0 \\
\end{pmatrix}. \]

Let \( U'_q(\mathfrak{sl}_{n,tor}) \) be the subalgebra of \( U_q(\mathfrak{sl}_{n,tor}) \) generated by \( E_{i,k}, F_{i,k}, K_i^\pm, H_{i,l}, q^{\pm \frac{1}{2}c} \).

2.4. Let \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) be the subalgebra of \( U_q(\mathfrak{sl}_{n,tor}) \) generated by \( E_{i,k}, F_{i,k}, K_i^\pm, H_{i,l}, q^{\pm \frac{1}{2}c} \) \((1 \leq i \leq n-1, k \in \mathbb{Z}, l \in \mathbb{Z}\setminus\{0\})\) and \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) the subalgebra generated by \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) and \( q^{\pm d_1} \). Let \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \) the subalgebra of \( U_q(\mathfrak{sl}_{n,tor}) \) generated by \( E_{i,0}, F_{i,0}, K_i^+ \) \((0 \leq i \leq n-1)\) and \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \) the subalgebra generated by \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \) and \( q^{\pm d_2} \).

The following lemma is already known \[3, 13].

**Lemma 1.** Both \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) and \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \) are isomorphic to \( U_q(\hat{\mathfrak{sl}}_n) \).

Following \[14\] we will call the \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) the vertical subalgebra of \( U_q(\mathfrak{sl}_{n,tor}) \) and denote its Chevalley generators by \( e_i, f_i, k_i^{\pm 1} \) \((i \in \{0, \ldots, n-1\})\). The \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \) will accordingly be called the horizontal subalgebra and for its Chevalley generators \( E_{i,0}, F_{i,0}, K_i^+ \) we will use the notations \( E_i, F_i, K_i^+ \) \((i \in \{0, \ldots, n-1\})\) respectively.

**Lemma 2.** \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) and \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \) generate \( U_q(\mathfrak{sl}_{n,tor}) \).

Proof. Let \( \mathcal{U} \) be the subalgebra of \( U_q(\mathfrak{sl}_{n,tor}) \) which is generated by \( U_q^{(1)}(\hat{\mathfrak{sl}}_n) \) and \( U_q^{(2)}(\hat{\mathfrak{sl}}_n) \). It is enough to show that \( E_{0,k}, F_{0,k}, H_{0,l} \in \mathcal{U} \). By the defining relations we have

\[ [H_{1,l}, E_{0,k}] = -\frac{1}{l} [l] q^{-\frac{1}{2}l[l]l} c_{K_i} E_{0,k+l}. \]

Since \( E_{0,0} \) and \( H_{1,l} \) \((l \in \mathbb{Z}\setminus\{0\})\) are the elements of \( \mathcal{U} \) we have \( E_{0,k} \in \mathcal{U} \) for any \( k \). Similarly we have \( F_{0,k}, H_{0,l} \in \mathcal{U} \).

**Definition 2.** Let \( V \) be a \( U_q(\mathfrak{sl}_{n,tor}) \)-module. We say that \( V \) has level \( (l_1, l_2) \) if \( V \) has level \( l_2 \) as \( U_q^{(j)}(\hat{\mathfrak{sl}}_n) \)-module \((j = 1, 2)\).

On a level \( (l_1, l_2) \) module the generator \( q^{\frac{1}{2}c} \) acts as the multiplication by \( q^{\frac{1}{2}l_2} \) and the element \( K_0K_1 \ldots K_{n-1} \) acts as the multiplication by \( q^{l_1} \).
3. The vertex representation

3.1. In this section we assume $c = 1$. Let us recall the results on the vertex representation of the quantum toroidal algebra \([10]\).

Let $S_n$ be the subalgebra of $U_q(\mathfrak{sl}_{n,\text{tor}})$, generated by the $H_{i,l}$ ($0 \leq i \leq n - 1$, $l \in \mathbb{Z}\setminus\{0\}$). By definition the commutation relations of $\{H_{i,l}\}$ are following:

$$[H_{i,k}, H_{j,l}] = \delta_{k+l,0} \frac{1}{k} [k \langle h_i, \alpha_j \rangle] \frac{q^k - q^{-k}}{q - q^{-1}} \kappa_{-km_{ij}}.$$ 

Let $\mathcal{F}_n$ be the Fock space of $S_n$. That is, $\mathcal{F}_n$ is generated by the vacuum vector $v_0$ and the defining relations are $H_{i,l}v_0 = 0$ for $l > 0$.

3.2. We introduce a twisted group algebra $Q(q)\{\mathcal{T}\}$ defined as the $Q(q)$-algebra generated by symbols $e^{\pm \alpha_2}, e^{\pm \alpha_3}, \ldots, e^{\pm \alpha_{n-1}}, e^{\pm \Lambda_{n-1}}$ which satisfy the following relations:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i}, \quad e^{\alpha_i} e^{\Lambda_{n-1}} = (-1)^{\delta_{i,n-1}} e^{\Lambda_{n-1}} e^{\alpha_i}.$$ 

For $\alpha = \sum_{i=2}^{n-1} m_i \alpha_i + m_n \Lambda_{n-1}$ we denote $e^{\alpha} = (e^{\alpha_2})^{m_2} (e^{\alpha_3})^{m_3} \cdots (e^{\alpha_{n-1}})^{m_{n-1}} (e^{\Lambda_{n-1}})^{m_n}$.

We denote by $Q(q)\{\mathcal{T}\}$ the subalgebra of $Q(q)\{\mathcal{T}\}$ generated by $e^{\alpha_i}$ ($1 \leq i \leq n-1$). Set

$$W(M) = \mathcal{F}_n \otimes Q(q)\{\mathcal{T}\} e^{\Lambda_M} \quad \text{for} \quad 0 \leq M \leq n - 1,$$

here we denote $\Lambda_0 = 0$.

For $i = 0, 1, \ldots, n - 1$ we denote

$$\overline{\alpha}_i = \begin{cases} -\sum_{j=1}^{n-1} \alpha_j, & i = 0, \\ \alpha_i, & i \neq 0, \end{cases} \quad \overline{h}_i = \begin{cases} -\sum_{j=1}^{n-1} h_j, & i = 0, \\ h_i, & i \neq 0, \end{cases}$$

We define the operators on $W(M)$, $H_{i,l}$, $e^\alpha$, $(\alpha \in \mathcal{T})$, $\partial_{\overline{\alpha}_i}$, $z^{H_{i,0}}$ ($i = 0, 1, \ldots, n - 1$) and $d$ as follows:

For $v \otimes e^\beta = H_{i_1,-k_1} \cdots H_{i_N,-k_N} v_0 \otimes e^{\sum_{j=1}^{n-1} m_j \alpha_j + \Lambda_M} \in W(M)$,

$$H_{i,l} (v \otimes e^\beta) = (H_{i,l} v) \otimes e^\beta,$$

$$e^\alpha (v \otimes e^\beta) = v \otimes e^\alpha e^\beta,$$

$$\partial_{\overline{\alpha}_i} (v \otimes e^\beta) = \langle \overline{h}_i, \beta \rangle v \otimes e^\beta,$$

$$z^{H_{i,0}} (v \otimes e^\beta) = z^{\langle \overline{\alpha}_i, \beta \rangle} k_{\frac{1}{2} \sum_{k=1}^{n-1} \langle \overline{\alpha}_i, \alpha_k \rangle m_k} (v \otimes e^\beta),$$

$$d(v \otimes e^\beta) = (-\sum_{s=1}^{N} k_s - \frac{\langle \beta, \beta \rangle}{2} + \frac{\langle \Lambda_M, \Lambda_M \rangle}{2}) v \otimes e^\beta.$$ 

Let us denote by $U^+_q(\mathfrak{sl}_{n,\text{tor}})$ the subalgebra of $U_q(\mathfrak{sl}_{n,\text{tor}})$ generated by $E_{i,k}, F_{i,k}, K_1^\pm, H_{i,l}$, $q^{\pm \overline{\epsilon}}$ and $q^{\pm d_i}$. 
Proposition 1. \[\text{Let } c = 1. \text{ Then for each } M \text{ and } n, \text{ the following action gives a } U_q^*(\mathfrak{s}l_{n,\text{tor}})\text{-module structure on } W(M):\]

\[
q_{i,c}^k \mapsto q^k, \\
q_{d_1} \mapsto q^d,
\]

\[
E_i(z) \mapsto \exp(\sum_{k \geq 1} \frac{H_{i,-k}}{k} (q^{-1/2}z)^k) \exp(\sum_{k \geq 1} -\frac{H_{i,k}}{k} (q^{1/2}z)^{-k}) e^{\alpha_i^* z} \delta_{\alpha_i^* + 1},
\]

\[
F_i(z) \mapsto \exp(\sum_{k \geq 1} -\frac{H_{i,k}}{k} (q^{1/2}z)^k) \exp(\sum_{k \geq 1} H_{i,-k} (q^{-1/2}z)^{-k}) e^{-\alpha_i^* z} \delta_{\alpha_i^* + 1},
\]

\[
K_i^+(z) \mapsto \exp((q - q^{-1}) \sum_{k \geq 1} H_{i,k} z^{-k}) q^{\alpha_i^*},
\]

\[
K_i^-(z) \mapsto \exp(-(q - q^{-1}) \sum_{k \geq 1} H_{i,-k} z^k) q^{-\alpha_i^*}
\]

for \(0 \leq i \leq n - 1\). Moreover \(W(M)\) has level \((1,0)\) as a \(U_q^*(\mathfrak{s}l_{n,\text{tor}})\)-module.

Proposition 2. \[\text{As a } U_q^{(1)}(\mathfrak{sl}_n)\text{-module}\]

\[\text{ch}_{W(M)} = \frac{ch_{L(\Lambda_M)}}{\varphi(e^{-\delta})},\]

where \(L(\Lambda_M)\) is the irreducible highest weight \(U_q(\mathfrak{sl}_n)\text{-module with the highest weight } \Lambda_M \text{ and } \varphi(x) = \prod_{k>0}(1 - x^k).\)

Definition 3. \[\text{We say that } \kappa \in \mathbb{Q}(q)^\times \text{ is generic if for any } k \in \mathbb{Z}_{>0} \text{ the } n \times n\text{-matrix } G(n, k, \kappa) = ([k\langle h_i, \alpha_j \rangle] \kappa^{-km_{ij}}) \text{ is invertible.}\]

Theorem 3. \[\text{If } \kappa \text{ is generic then } W(M) \text{ is irreducible } U_q^*(\mathfrak{s}l_{n,\text{tor}})\text{-module.}\]

3.3. Non-generic case. \[\text{We shall start the following lemma.}\]

Lemma 3. \[\text{If } \kappa \text{ is not generic then } \kappa \in \{\pm q, \pm q^{-1}\}.\]

Proof. \[\text{We assume } \det G(n, k, \kappa) = 0 \text{ for fixed } k. \text{ By an easy calculation we have } \det G(n, k, \kappa) = q^{-nk}[k](q^{nk} - \kappa^n)(q^{nk} - \kappa^{-nk}). \text{ Therefore if } nk \text{ is even then } \kappa = \pm q^{\pm 1} \text{ and if } nk \text{ is odd then } \kappa = q^{\pm 1}. \text{ Thus we have the statement.}\]

Until the end of this subsection we assume \(\kappa = q^{\pm 1}\). Let \(\hat{B}_k = \sum_{i=0}^{n-1} H_{i,k} \text{ and } \hat{S} \text{ be the algebra generated by } \hat{B}_k \text{ (}k \in \mathbb{Z}\backslash\{0\}). \text{ It is easy to see the following lemma.}\]

Lemma 4. \[\text{If } \kappa = q^{\pm 1} \text{ then } [H_{i,l}, \hat{B}_k] = 0 \text{ for any } i, k \text{ and } l. \text{ Moreover } \hat{S} \text{ is an abelian subalgebra of } U_q(\mathfrak{sl}_{n,\text{tor}}).\]

\[\text{By the above lemma and the definition of the action on } W(M) \text{ we have the following lemma immediately.}\]
Lemma 5. \([\hat{S}, U'_q(\mathfrak{sl}_{n,\text{tor}})] = 0\) on \(W(M)\).

Let \(\hat{S} = \bigoplus_{k>0} \mathbb{Q}(q)\{X \in \hat{S}| q^{d_i} X q^{-d_i} = q^k X\}\). Set \(W(M)^{(1)} = \hat{S}^{<0} W(M)\).

Proposition 4. If \(\kappa = q^{\pm 1}\) then

1. \(W(M)^{(1)}\) is a proper \(U_q^{*}(\mathfrak{sl}_{n,\text{tor}})\)-submodule of \(W(M)\) and
2. \(W(M)/W(M)^{(1)}\) is an irreducible \(U_q^{*}(\mathfrak{sl}_{n,\text{tor}})\)-module.

Proof. (1) By Lemma 5, \(W(M)^{(1)}\) has a \(U_q^{*}(\mathfrak{sl}_{n,\text{tor}})\)-module structure. Since \(v_0 \otimes e \check{X}_M\) does not belong to \(W(M)^{(1)}\), it is a proper submodule.

(2) As \(U_q^{(1)}(\hat{\mathfrak{sl}}_n)\)-module the character of \(W(M)\) is already known (See Proposition 2). By the definition we have

\[
\text{ch}_{W(M)/W(M)^{(1)}} = \text{ch}_{L(\Lambda_M)}
\]

as \(U_q^{(1)}(\mathfrak{sl}_n)\)-module. Thus \(W(M)/W(M)^{(1)} \cong L(\Lambda_M)\) as \(U_q^{(1)}(\mathfrak{sl}_n)\)-module. Therefore \(W(M)/W(M)^{(1)}\) is irreducible. \(\square\)

Corollary 1. (1) If \(\kappa = q^{\pm 1}\) then \(L(\Lambda_M)\) has \(U_q(\mathfrak{sl}_{n,\text{tor}})\)-module structure.

(2) Therefore we have another \(U_q(\mathfrak{sl}_n)\)-module structure on \(L(\Lambda_M)\) with level 0.

4. Action of \(U'_q(\mathfrak{sl}_{n,\text{tor}})\) on the Finite Wedge Product

4.1. Toroidal Hecke algebra. Let \(q \in \mathbb{C}^{\times}\). The toroidal Hecke algebra of type \(\mathfrak{gl}_N\), \(H_{\text{tor}}\) is a unital associative algebra over \(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]\) with generators \(T_i^\pm, X_j^\pm, Y_j^\pm\), \(i = 1, 2, \ldots, N - 1\), \(j = 1, 2, \ldots, N\) and relations

\[
T_i T_i^{-1} = T_i^{-1} T_i = 1, \quad (T_i + 1)(T_i - q^2) = 0,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},
\]

\[
T_i T_j = T_j T_i \quad \text{if} \quad |j - i| > 1,
\]

\[
X_0 Y_i = x Y_i X_0, \quad X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i,
\]

\[
X_j T_i = T_i X_j, \quad Y_j T_i = T_i Y_j \quad \text{if} \quad j \neq i, i + 1 > 1,
\]

\[
T_i X_i T_i = q^2 X_{i+1}, \quad T_i^{-1} Y_i T_i^{-1} = q^{-2} Y_{i+1},
\]

\[
X_0 Y_{i+1} X_2^{-1} Y_1 = q^{-2} y T_i^2,
\]

where \(X_0 = X_1 X_2 \cdots X_N\). In \(H_{\text{tor}}\) the subalgebras \(\hat{H}_N(q)^{(1)}\) generated by \(T_i^\pm, Y_j^\pm\), and \(\hat{H}_N(q)^{(2)}\) generated by \(T_i^\pm, X_j^\pm\) are both isomorphic to the affine Hecke algebra. And the subalgebra \(H_N(q)\) generated by \(T_i^\pm\) is isomorphic to the Hecke algebra of type \(\mathfrak{gl}_N\).
Let $p \in \mathbb{C}^n$ and consider the following operators in $\text{End}(\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}])$

- Multiplication operator $z_i$, $i = 1, 2, \ldots, N$,
- $g_{i,j} = \frac{q^{-1}z_i - qz_j}{z_i - z_j}(K_{i,j} - 1) + q$, $1 \leq i \neq j \leq N$,
- The family of $N$ commutative Cherednik’s operators:
  
  $Y_i^{(N)} = g_{i,i+1}^{-1}K_{i,i+1} \cdots g_{i,N}^{-1}K_{i,N}p^{D_i}K_{1,i}g_{1,i} \cdots K_{i-1,i}g_{i-1,i}$, $i = 1, 2, \ldots, N$,

where $K_{i,j}$ acts on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ by permuting variables $z_i, z_j$ and $p^{D_i}$ is the difference operator

$$p^{D_i}f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, p z_i, \ldots, z_N), \quad f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}].$$

The following result is due to I. Cherednik [3, 4]:

**Proposition 5.** The map

$$T_i \mapsto \tilde{T}_i = -q g_{i,i+1}^{-1}, \quad X_i \mapsto z_i, \quad Y_i \mapsto q^{1-N}Y_i^{(N)}, \quad x \mapsto p, \quad y \mapsto 1$$

(4.1)

defines a right action of $H_{tor}$ on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$.

The commuting difference operators $Y_1^{(N)}, \ldots, Y_N^{(N)}$ are called Cherednik’s operators.

Let $V = \mathbb{C}^n$, with basis $\{v_1, \ldots, v_n\}$. Then $\otimes^N V$ admits a left $H_N(q)$-action given by

$$T_i \mapsto \tilde{T}_i = 1^{\otimes^{i-1}} \otimes \tilde{T} \otimes 1^{\otimes^{N-i-1}}, \text{ where } \tilde{T} \in \text{End}(\otimes^2 V) \quad (4.2)$$

and

$$\tilde{T}(v_{\epsilon_1} \otimes v_{\epsilon_2}) = \begin{cases} 
q^2 v_{\epsilon_1} \otimes v_{\epsilon_2} & \text{if } \epsilon_1 = \epsilon_2, \\
q v_{\epsilon_2} \otimes v_{\epsilon_1} & \text{if } \epsilon_1 < \epsilon_2, \\
v_{\epsilon_2} \otimes (q^2 - 1)v_{\epsilon_1} & \text{if } \epsilon_1 > \epsilon_2.
\end{cases} \quad (4.3)$$

### 4.2. $q$-wedge product.

Let $V(z) = \mathbb{C}[z^{\pm 1}] \otimes V$, with basis $\{z^m \otimes v_\epsilon\}$, $m \in \mathbb{Z}$, $\epsilon \in \{1, 2, \ldots, n\}$. Often it will be convenient to set $k = \epsilon - nm$ and $u_k = z^m \otimes v_\epsilon$. Then $\{u_k\}, k \in \mathbb{Z}$ is a basis of $V(z)$. In what follows we will write $z^m v_\epsilon$ as a short-hand for $z^m \otimes v_\epsilon$, and use both notations: $u_k$ and $z^m v_\epsilon$ switching between them according to convenience.

The two actions of the Hecke algebra are naturally extended on the tensor product $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ so that $T_i$ acts trivially on $\otimes^N V$ and $\tilde{T}_i$ acts trivially on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. The vector space $\otimes^N V(z)$ is identified with $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ and the $q$-wedge product [5] is defined as the following quotient space:

$$\wedge^N V(z) = \otimes^N V(z)/\sum_{i=1}^{N-1} \text{Ker} \left( \tilde{T}_i + q^2(T_i)^{-1} \right). \quad (4.4)$$
Since for any $i = 1, 2, \ldots, N - 1$ we have

$$\text{Ker}\left( \hat{T}_i + q^2 (T_i)^{-1} \right) = \text{Im}\left( \hat{T}_i - T_i \right) \quad (4.5)$$

the definition (4.4) is equivalent to

$$\wedge^N V(z) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes_{H_N(q)} (\otimes^N V). \quad (4.6)$$

Let $\Lambda : \otimes^N V(z) \to \wedge^N V(z)$ be the quotient map specified by (4.4). The image of a pure tensor $u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}$ under this map is called a wedge and is denoted by

$$u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_N} := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}). \quad (4.7)$$

A wedge is normally ordered if $k_1 > k_2 > \cdots > k_N$. In [8] it is proven that normally ordered wedges form a basis in $\wedge^N V(z)$, and that any wedge is expressed as a linear combination of normally ordered wedges by using the normal ordering rules:

$$u_l \wedge u_m = -u_m \wedge u_l, \quad \text{for} \ l = m \mod n, \quad (4.8)$$

$$u_l \wedge u_m = -qu_m \wedge u_l + (q^2 - 1)(u_{m-i} \wedge u_{l+i} - qu_{m-n} \wedge u_{l+n} +$$

$$+q^2u_{m-n-i} \wedge u_{l+n+i} + \cdots), \quad \text{for} \ l < m, m - l = i \mod n, 0 < i < n. \quad (4.9)$$

The sum above continues as long as the wedges in the right-hand side are normally ordered.

4.3. Action of the quantum toroidal algebra on the wedge product. In the paper [13] the following result is proven

**Theorem 6** (Varagnolo and Vasserot). Suppose that $x = \kappa^{-n}q^n$ and $y = 1$. Then for any right $H_{tor}$-module $R$ the vector space $R \otimes_{H_N(q)} (\otimes^N V)$ is a left $U_q'(\widehat{\mathfrak{sl}_n})$-module with central charge $(0, 0)$.

Moreover the action of the vertical subalgebra $U_q^{(1)'}(\widehat{\mathfrak{sl}_n})$ generated by $e_i, f_i, k_i^{\pm 1}$ and the action of the horizontal subalgebra $U_q^{(2)'}(\widehat{\mathfrak{sl}_n})$ generated by $E_i, F_i, K_i^{\pm 1}$ ($i = 0, 1, \ldots, n - 1$) are given on the space $R \otimes_{H_N(q)} (\otimes^N V)$ as follows: Let $m \in R$ and
v \in \otimes^N V$, then

$$E_i(m \otimes v) = e_i(m \otimes v) = \sum_{j=1}^{N} m \otimes E_j^{i,j+1} K_{j+1} \cdots K_N v,$$

$$F_i(m \otimes v) = f_i(m \otimes v) = \sum_{j=1}^{N} m \otimes (K_j^{i})^{-1} \cdots (K_{j-1}^{i})^{-1} E_j^{1,n} v,$$

$$K_i(m \otimes v) = k_i(m \otimes v) = m \otimes K_1^{i} K_2^{i} \cdots K_N^{i} v, \quad (i = 1, 2, \ldots, n - 1)$$

$$e_0(m \otimes v) = \sum_{j=1}^{N} mY_{j}^{-1} \otimes E_j^{0,1} K_{j+1} \cdots K_N v,$$

$$f_0(m \otimes v) = \sum_{j=1}^{N} mY_{j} \otimes (K_1^{0})^{-1} \cdots (K_{j-1}^{0})^{-1} E_j^{1,n} v,$$

$$E_0(m \otimes v) = \sum_{j=1}^{N} mX_{j} \otimes E_j^{n,1} K_{j+1} \cdots K_N v,$$

$$F_0(m \otimes v) = \sum_{j=1}^{N} mX_{j}^{-1} \otimes (K_1^{0})^{-1} \cdots (K_{j-1}^{0})^{-1} E_j^{1,n} v,$$

$$K_0 = k_0 = (K_1 K_2 \cdots K_{n-1})^{-1}.$$

Here $E_j^{i,k} = 1^{\otimes j-i} \otimes E_j^{i,k} \otimes 1^{\otimes N-j}$, where $E_j^{i,k} \in \text{End}(V)$ is the matrix unit in the basis $v_1, \ldots, v_n$, and $K_j = q^{E_j^{i,j}-E_j^{j+1,i+1}}, \quad K_j^{0} = (K_1^{j} K_2^{j} \cdots K_N^{j-1})^{-1}$.

In view of the Theorem and the Proposition the wedge product $\wedge^N V(z)$ is a left $U_q(\mathfrak{sl}_{n,tor})$-module with $q = qp^{-1}$ such that the action of the generators of the vertical and the horizontal subalgebras is given by the formulas (4.10 - 4.17) where for $m \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ and $v \in \otimes^N V$ we identify $m \otimes v$ with $\Lambda(m \otimes v)$, and use the maps in the Proposition to define a right action of $H_{tor}$ on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$.

Sometimes we will denote the action of the vertical (horizontal) subalgebra on the wedge product $\wedge^N V(z)$ by $U_0^{(N)}(U_1^{(N)})$.

The actions of $E_{i,k}, F_{i,k}, H_{i,l}$ $(1 \leq i \leq n - 1, \quad k \in \mathbb{Z}, \quad l \in \mathbb{Z} \setminus \{0\})$ are determined by the actions of the Chevalley generators of $U_q^{(1)}(\mathfrak{sl}_n)$.

To prove the Theorem Varagnolo and Vasserot defined the following operator $\psi$:

$$\psi(m \otimes v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N}) = mX_1^{-\delta_{n,\epsilon_1}} X_2^{-\delta_{n,\epsilon_2}} \cdots X_N^{-\delta_{n,\epsilon_N}} \otimes v_{\epsilon_1+1} \otimes v_{\epsilon_2+1} \otimes \cdots \otimes v_{\epsilon_N+1}.$$
In this notation we identify \( v_{n+1} = v_1 \). By taking into account the relations of the toroidal Hecke algebra, one can confirm that the action of \( \psi \) is well-defined on \( R \otimes_{H_N(q)} (\otimes^N V) \).

The following proposition is proved in \cite{13}.

**Proposition 7.** Let \( E_i(z), F_i(z), K^\pm_i(z) \) be the series that specify actions of the generators of \( U'_q(sl_{n,tor}) \) on \( R \otimes_{H_N(q)} (\otimes^N V) \). Then one has

\[
\begin{align*}
\psi^{-1}E_i(z)\psi &= E_{i-1}(q^{-1}kz), \\
\psi^{-2}E_i(z)\psi^2 &= E_{n-1}(x^{-1}q^{n-2}k^2-nz), \\
\psi^{-1}F_i(z)\psi &= F_{i-1}(q^{-1}kz), \\
\psi^{-2}F_i(z)\psi^2 &= F_{n-1}(x^{-1}q^{n-2}k^2-nz), \\
\psi^{-1}K^\pm_i(z)\psi &= K^\pm_{i-1}(q^{-1}kz), \\
\psi^{-2}K^\pm_i(z)\psi^2 &= K^\pm_{n-1}(x^{-1}q^{n-2}k^2-nz),
\end{align*}
\]

where \( 2 \leq i \leq n-1 \).

On the other hand, one can prove the following proposition \cite{13} easily.

**Proposition 8.** Let \( W \) be a \( U'_q(sl_{n}) \)-module such that actions of the generators are given by the series \( E_i(z), F_i(z), H_i(z) \) \( (1 \leq i \leq n-1) \) \cite{4.2 – 2.9}. If there exists \( \psi \in \text{End}(W) \) and \( \kappa \in \mathbb{C} \) such that

\[
\begin{align*}
\tilde{\psi}^{-1}E_i(z)\tilde{\psi} &= E_{i-1}(q^{-1}kz), \\
\tilde{\psi}^{-2}E_i(z)\tilde{\psi}^2 &= E_{n-1}(q^{-2}k^2z), \\
\tilde{\psi}^{-1}F_i(z)\tilde{\psi} &= F_{i-1}(q^{-1}kz), \\
\tilde{\psi}^{-2}F_i(z)\tilde{\psi}^2 &= F_{n-1}(q^{-2}k^2z), \\
\tilde{\psi}^{-1}K^\pm_i(z)\tilde{\psi} &= K^\pm_{i-1}(q^{-1}kz), \\
\tilde{\psi}^{-2}K^\pm_i(z)\tilde{\psi}^2 &= K^\pm_{n-1}(q^{-2}k^2z),
\end{align*}
\]

where \( 2 \leq i \leq n-1 \), then the \( W \) is an \( U'_q(sl_{n,tor}) \)-module such that the actions of \( E_0(z), F_0(z), H_0(z) \) are given as follows:

\[
\begin{align*}
E_0(z) &= \tilde{\psi}^{-1}E_0(qk^{-1}z)\tilde{\psi}, \\
F_0(z) &= \tilde{\psi}^{-1}F_0(qk^{-1}z)\tilde{\psi}, \\
K^\pm_0(z) &= \tilde{\psi}^{-1}K^\pm_0(qk^{-1}z)\tilde{\psi}.
\end{align*}
\]

4.4 Semi-infinite wedge product. In the subsection \cite{4.2}, we defined the space \( V(z) \), its basis \( \{ u_k \} \), \( k \in \mathbb{Z} \) and the space \( \wedge^N V(z) \). In this section we define the semi-infinite wedge product \( \wedge^\infty V(z) \) and for any integer \( M \) its subspace \( F_M[\mathbb{F}] \). Later we will define a representation of \( U'_q(sl_{n,tor}) \) on this space.

Let \( \otimes^\mathbb{F} V(z) \) be the space spanned by the vectors \( u_{k_1} \otimes u_{k_2} \otimes \ldots \), \( k_{i+1} = k_i - 1, \ i >> 1 \). We define the space \( \wedge^\infty V(z) \) by the quotient of \( \otimes^\mathbb{F} V(z) \):

\[
\wedge^\infty V(z) := \otimes^\mathbb{F} V(z)/\sum_{i=1}^\infty \text{Ker} \left( \tilde{T}_i + q^2(T_i)^{-1} \right).
\]

Let \( \Lambda : \otimes^\mathbb{F} V(z) \to \wedge^\infty V(z) \) be the quotient map specified by \((4.28)\). The image of a pure tensor \( u_{k_1} \otimes u_{k_2} \otimes \ldots \) under this map is called a semi-infinite wedge and is denoted by

\[
u_{k_1} \wedge u_{k_2} \wedge \cdots := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \ldots). \]
A semi-infinite wedge is normally ordered if \( k_1 > k_2 > \cdots \) and \( k_{i+1} = k_i - 1 \) \((i >> 1)\). In [8] it is proven that normally ordered semi-infinite wedges form a basis in \( \wedge \mathcal{F} V(z) \).

Let \( U_M \) be the subspace of \( \otimes \mathcal{F} V(z) \) spanned by the vectors \( u_{k_1} \otimes u_{k_2} \otimes \cdots \), \((k_i = M - i + 1, i >> 1)\). Let \( F_M \) be the quotient space of \( U_M \) defined by the map \((4.29)\). Then \( F_M \) is a subspace of \( \wedge \mathcal{F} V(z) \), and the vectors \( u_{k_1} \wedge u_{k_2} \wedge \cdots \), \((k_1 > k_2 > \cdots, k_i = M - i + 1, i >> 1)\) form the basis of \( F_M \). We will call the space \( F_M \) the Fock space.

5. The two actions of \( U'_q(\hat{\mathfrak{sl}}_n) \) on the Fock space

5.1. Level-zero action of \( U'_q(\hat{\mathfrak{sl}}_n) \) on the Fock space. Here we define a level-0 action of \( U'_q(\hat{\mathfrak{sl}}_n) \) on \( F_M \) \((M \in \mathbb{Z})\) following the paper [11]. The definition we give below is equivalent to the one given in [11]. However, compared to [11], we change slightly the precise wording so as to make the idea of this definition more transparent.

Let \( e := (\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \) where \( \epsilon_i \in \{1, 2, \ldots, n\} \). For a sequence \( e \) we set

\[
v_e := v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N} \quad (\in \otimes^N \mathbb{C}^n).
\]

A sequence \( m := (m_1, m_2, \ldots, m_N) \) from \( \mathbb{Z}^N \) is called \( n \)-strict if it contains no more than \( n \) equal elements of any given value. Let us define the sets \( \mathcal{M}_N^n \) and \( \mathcal{E}(m) \) by

\[
\mathcal{M}_N^n := \{ m = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N \mid m_1 \leq m_2 \leq \cdots \leq m_N, \ m \text{ is \( n \)-strict} \},
\]

and for \( m \in \mathcal{M}_N^n \)

\[
\mathcal{E}(m) := \{ e = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \in \{1, 2, \ldots, n\}^N \mid \epsilon_i > \epsilon_{i+1} \text{ for all } i \text{ s.t. } m_i = m_{i+1} \}.
\]

In these notations the set

\[
\{ w(m, e) := \Lambda(z^m \otimes v_e) \equiv z^{m_1}v_{\epsilon_1} \wedge z^{m_2}v_{\epsilon_2} \wedge \cdots \wedge z^{m_N}v_{\epsilon_N} \mid m \in \mathcal{M}_N^n, e \in \mathcal{E}(m) \}.
\]

is nothing but the base of the normally ordered wedges in \( \wedge^N V(z) \). We will use the notation \( w(m, e) \) exclusively for normally ordered wedges.

Similarly for a semi-infinite wedge \( w = u_{k_1} \wedge u_{k_2} \wedge \cdots = z^{m_1}v_{\epsilon_1} \wedge z^{m_2}v_{\epsilon_2} \wedge \cdots \), such that \( w \in F_M \), the semi-infinite sequences \( m = (m_1, m_2, \ldots) \) and \( e = (\epsilon_1, \epsilon_2, \ldots) \) are defined by \( k_i = \epsilon_i - nm_i, \epsilon_i \in \{1, 2, \ldots, n\}, m_i \in \mathbb{Z} \). In particular the \( m \)- and \( e \)-sequences of the vacuum vector in \( F_M \) will be denoted by \( m^0 \) and \( e^0 \):

\[
| M \rangle = u_M \wedge u_{M-1} \wedge u_{M-2} \wedge \cdots = z^{m_1}v_{\epsilon_1}^0 \wedge z^{m_2}v_{\epsilon_2}^0 \wedge z^{m_3}v_{\epsilon_3}^0 \wedge \cdots.
\]
The Fock space $F_M$ is $\mathbb{Z}_{\geq 0}$-graded. For any semi-infinite wedge $w = u_{k_1} \wedge u_{k_2} \wedge \ldots \equiv z^{m_1}u_{e_1} \wedge z^{m_2}u_{e_2} \wedge \ldots \in F_M$ the degree $|w|$ is defined by

$$|w| = \sum_{i \geq 1} m_i^0 - m_i. \quad (5.6)$$

Later we will see that the $F_M$ is a toroidal module. In this module the degree generator $d_2$ acts as the grading operator whose eigenvalue on a normally ordered wedge $w$ is equal to the degree $(5.6)$. Clearly the degree is a finite non-negative integer for any wedge $u_{k_1} \wedge u_{k_2} \wedge \ldots$ in $F_M$ because of the asymptotic condition $k_i = M - i + 1, i >> 1$. Let us denote by $F_M^k \subset F_M$ the homogeneous component of degree $k$.

We will define a level-zero action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the Fock space $F_M$ in such a way that each homogeneous component $F_M^k$ will be invariant with respect to this action. Throughout this section we fix an integer $M$ and $s \in \{0, 1, 2, \ldots, n - 1\}$ such that $M = s \mod n$.

Let $l$ be a non-negative integer and define $V_M^{s+nl} \subset \wedge^{s+nl}V(z)$ as follows:

$$V_M^{s+nl} = \bigoplus_{m \in M_{s+nl}, e \in E(m)} Cw(m, e). \quad (5.7)$$

Notice that the condition $m_{s+nl} \leq m_{s+nl}^0$ in this definition is equivalent to the condition

$$m_i \leq m_i^0 \quad \text{for all } i = 1, 2, \ldots, s + nl \quad (5.8)$$

since the sequence $m$ is $n$-strict and non-decreasing.

The vector space $V_M^{s+nl}$ has a grading similar to the grading of the Fock space $F_M$. In this case the degree $|w|$ of a wedge $w = u_{k_1} \wedge u_{k_2} \wedge \ldots \wedge u_{k_{s+nl}} \equiv z^{m_1}u_{e_1} \wedge z^{m_2}u_{e_2} \wedge \ldots \wedge z^{m_{s+nl}}u_{e_{s+nl}} \in V_M^{s+nl}$ is defined by

$$|w| = \sum_{i = 1}^{s+nl} m_i^0 - m_i. \quad (5.9)$$

Due to $(5.8)$ the degree is a non-negative integer, and for $k \in \mathbb{Z}_{\geq 0}$ we denote by $V_M^{s+nl,k}$ the homogeneous component of degree $k$.

The following result is contained in the paper [11]:

**Proposition 9.** For each $k \in \mathbb{Z}_{\geq 0}$ the homogeneous component $V_M^{s+nl,k} \subset \wedge^{s+nl}V(z)$ is invariant under the $U_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(s+nl)}$ defined in section 13.

**Definition 4.** For each $k \in \mathbb{Z}_{\geq 0}$ define a map $\overline{\rho}_l^{M,k} : V_M^{s+nl,k} \to F_M$ by setting for $w \in V_M^{s+nl,k}$

$$\overline{\rho}_l^{M,k}(w) = w \wedge |M - s - nl|. \quad (5.10)$$

Clearly we have $|\overline{\rho}_l^{M,k}(w)| = |w|$ and hence $\overline{\rho}_l^{M,k} : V_M^{s+nl,k} \to F_M^k$ for all $k \in \mathbb{Z}_{\geq 0}$.
Proposition 10. When \( l \geq k \) the map \( \rho_{l}^{M,k} \) is an isomorphism of vector spaces.

Proof. (Surjectivity) If a normally ordered wedge \( w = u_{k_{1}} \wedge u_{k_{2}} \wedge \ldots \) belongs to \( F^{k}_{M} \) then \( k_{i} = M - i + 1 \) for all \( i \geq s + nk + 1 \). For otherwise, the degree of \( w \) must be greater or equal to \( k + 1 \) (Cf. Proposition 5(ii) in [11]). Thus when \( l \geq k \) we have \( w = w_{(s+nl)} \wedge |M - s - nl| \), and \( w_{(s+nl)} \in V^{+nl,k}_{M} \). Since a basis of \( F^{k}_{M} \) is formed by normally ordered wedges, the surjectivity follows.

(Injectivity) If \( w, w' \in V^{s+nl,k}_{M} \) are two distinct normally ordered wedges, then \( w \wedge |M - s - nl|, w' \wedge |M - s - nl| \) are distinct and, as implied by the definition (5.7), normally ordered wedges in \( F^{k}_{M} \). Thus the injectivity follows.

This proposition has an immediate corollary:

Corollary 2. For each triple of non-negative integers \( k, l, m \) such that \( k \leq l < m \) the map \( \rho_{l,m}^{M,k} : V^{s+nl,k}_{M} \to V^{s+nm,k}_{M} \), defined for any \( w \in V^{s+nl,k}_{M} \) by

\[
\rho_{l,m}^{M,k}(w) = w \wedge u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge \cdots \wedge u_{M-s-nm+1},
\]

is an isomorphism of vector spaces.

Moreover we have a stronger statement:

Proposition 11. For each triple of non-negative integers \( k, l, m \) such that \( k \leq l < m \) the map \( \rho_{l,m}^{M,k} : V^{s+nl,k}_{M} \to V^{s+nm,k}_{M} \) is an isomorphism of the \( U_{q}(\hat{sl}_{n}) \)-modules.

Proof. In the proof of this and some of the subsequent propositions the following lemma concerning the Cherednik's operators and proved in the paper [11] plays an essential role:

Lemma 6. Let \( m = (m_{1}, m_{2}, \ldots, m_{N}) \in \mathbb{Z}^{N} \) be a sequence such that:

\[
m_{1}, m_{2}, \ldots, m_{N-k} < m_{N-k+1} = m_{N-k+2} = \cdots = m_{N} \equiv t; \quad 1 \leq k \leq N.
\]

Then the following relations hold:

for \( 0 \leq l \leq k - 1 \)

\[
z^{m}(Y^{(N)}_{N-l})^{\pm 1} = p^{\pm t} q^{\pm (2k-2l-N-1)} z^{m} + [\ldots],
\]

(5.13)

for \( 1 \leq i \leq N - k \)

\[
z^{m}(Y^{(N)}_{i})^{\pm 1} = q^{\pm k} z^{m}(Y^{(N-k)}_{i})^{\pm 1} + [\ldots],
\]

(5.14)

where [\ldots] signifies a linear combination of monomials \( z^{n} \equiv z_{1}^{n_{1}} z_{2}^{n_{2}} \cdots z_{N}^{n_{N}} \) such that:

\[
n_{1}, n_{2}, \ldots, n_{N} \leq t, \quad \text{and} \quad \#\{n_{i}|n_{i} = t\} < k.
\]

(5.15)

To prove the proposition it is sufficient to assume that \( m \) is equal to \( l+1 \). And since the isomorphism of the linear spaces has been already established in the Corollary
Therefore we have
\[ a^{(s+nl+n)}M,k_{li}^{l+1}(w) = \overline{a}^{M,k}_{li}a^{(s+nl)}(w) \quad (k \leq l). \]  
(5.16)

Here \( a^{(N)} \) represents the action of the generator \( a \) in \( T_{i}^{(N)} \).

Let \( m' \in M_{s+nl}^{n} \) and \( e' \in E(\mathcal{E}(m')) \) be such that \( w(m', e') \in V_{M}^{s+nl,k} \) and take in (5.16) \( w = w(m', e') \) and \( a = f_{0} \). And let \( m \in M_{s+nl+n}^{n} \) and \( e \in E(\mathcal{E}(m)) \) be such that
\[ w(m, e) = \wedge(z^{m} \otimes v_{e}) = w(m', e') \wedge u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge \cdots \wedge u_{M-s-nl-n+1}, \]
so that
\[ w(m, e) = \frac{M,k}{li}^{l+1}(w(m', e')). \]
(5.17)

Now act with \( f_{0}^{(s+nl+n)} \) on \( w(m, e) \):
\[ f_{0}^{(s+nl+n)}w(m, e) = \Lambda(\sum_{j=1}^{s+nl+n} q^{1-(s+nl+n)}z^{m}y_{j}^{(s+nl+n)} \otimes (k_{0}^{0})^{-1} \cdots (k_{j-1}^{0})^{-1}e_{j}^{1}v_{e}). \]
(5.19)

Lemma 6 where we take \( N = s + nl + n, k = n, t = m_{s+nl+n}^{0} \) and the normal ordering rules allow us to write
\[ f_{0}^{(s+nl+n)}w(m, e) = (f_{0}^{(s+nl)}w(m', e')) \wedge u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge \cdots \wedge u_{M-s-nl-n+1} + p_{s+nl+n}m_{0}^{0}q^{2(1-nl-n)-s}((k_{0}^{(s+nl)})^{-1}w(m', e')) \wedge \]
\[ \wedge u_{M-s-nl-n+1} \wedge u_{M-s-nl-1} \wedge u_{M-s-nl-2} \wedge \cdots \wedge u_{M-s-nl-n+1} + \tilde{w}, \]
(5.20)

where the \( \tilde{w} \) is a linear combination of normally ordered wedges \( w(n, \tau), n \in M_{s+nl+n}^{n}, \tau \in E(n) \) such that for the sequence \( n = (n_{1}, n_{2}, \ldots, n_{s+nl+n}) \) we have
\[ n_{1}, n_{2}, \ldots, n_{s+nl+n} \leq m_{s+nl+n}^{0}, \]
\[ \#\{n_{i} | n_{i} = m_{s+nl+n}^{0}\} < n. \]
(5.21)
(5.22)

The inequality (5.22) implies, in particular, that \( n_{s+nl+n} < m_{s+nl+n}^{0} = m_{s+nl+n}^{0} \). And from the last inequality it follows that degree of the wedge \( w(n, \tau) \) is greater or equal to \( l + 1 \) (Cf. Proposition 5(ii) in [4]). Since the degree of \( f_{0}^{(s+nl+n)}w(m, e) \) equals to the degree of \( w(m, e) \) and equals to \( k \), and since the degrees of the first two summands in (5.20) are equal to \( k \) as well, taking the condition \( k \leq l \) into account we find that \( \tilde{w} \) equals to zero.

Now consider the second summand in (5.20). Lemma 2.2 in [3] shows that
\[ u_{M-s-nl-n+1} \wedge u_{M-s-nl-1} \wedge u_{M-s-nl-2} \wedge \cdots \wedge u_{M-s-nl-n+1} = 0. \]
(5.23)

Therefore we have
\[ f_{0}^{(s+nl+n)}w(m, e) = (f_{0}^{(s+nl)}w(m', e')) \wedge u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge \cdots \wedge u_{M-s-nl-n+1}; \]
(5.24)
which proves (5.16) for $a = f_0$.

The proof of (5.16) for the rest of the $U_q(\hat{\mathfrak{sl}}_n)$-generators is carried out in the same way.

Now we make the final step and define on the vector space $F^k_M$ a level-zero action of $U'_q(\hat{\mathfrak{sl}}_n)$ by using Propositions 10 and 11:

**Definition 5.** The vector space $F^k_M$ is a level-0 module of $U'_q(\hat{\mathfrak{sl}}_n)$ with the action $U_0$ defined by

$$U_0 = \mathcal{P}^{M,k}_l U'_0 \mathcal{P}^{M,k-1}_l$$

where $l \geq k$. (5.25)

Due to proposition 11 this definition does not depend on the choice of $l$ as long as $l$ is greater or equal to $k$. Since we have

$$F_M = \bigoplus_{k \geq 0} F^k_M$$

the level-0 action $U_0$ extends to the entire Fock space $F_M$.

5.2. **Level-one action of $U'_q(\hat{\mathfrak{sl}}_n)$ on the Fock space.** In this section we review the level-one action of $U'_q(\hat{\mathfrak{sl}}_n)$ on the Fock space $F_M$.

First we define the action of $U'_q(\hat{\mathfrak{sl}}_n)$ (generated by $E_i$, $F_i$, $K_i$, $i = 0, \ldots, n - 1$) on the vector $|M'angle$ as follows.

$$E_i |M'angle = 0,$$

$$F_i |M'angle = \begin{cases} u_{M' + 1} \wedge u_{M' - 1} \wedge u_{M' - 2} \wedge \cdots & \text{if } i \equiv M' \mod n; \\
0 & \text{otherwise}, \end{cases}$$

$$K_i |M'angle = \begin{cases} q |M'angle & \text{if } i \equiv M' \mod n; \\
|M'angle & \text{otherwise}. \end{cases}$$

For every element $v \in F_M$, there exist $N$ such that

$$v = v^{(N)} \wedge |M - N\rangle, \quad v^{(N)} \in \wedge^N V(z).$$

We define the action of $E_i$, $F_i$, $K_i$, $i = 0, \ldots, n - 1$ on the vector $v$ as follows.

$$E_i v := E_i v^{(N)} \wedge K_i |M - N\rangle + v^{(N)} \wedge E_i |M - N\rangle,$$

$$F_i v := F_i v^{(N)} \wedge |M - N\rangle + K_i^{-1} v^{(N)} \wedge F_i |M - N\rangle,$$

$$K_i v := K_i v^{(N)} \wedge K_i |M - N\rangle.$$ (5.33)

The actions of $E_i$, $F_i$, $K_i$, $i = 0, \ldots, n - 1$ on $v^{(N)}$ are determined in the section 4.3. The definition of the actions on $v$ does not depend on $N$ and is well-defined, and we can easily check that the $U'_q(\hat{\mathfrak{sl}}_n)$-module defined in this section is level-1. We will use the notation $U_1$ for this $U'_q(\hat{\mathfrak{sl}}_n)$-action on the Fock space.
6. Action of $U'_q(\mathfrak{sl}_{n,\text{tor}})$ on the Fock space $F_M$

6.1. Action of $U'_q(\mathfrak{sl}_{n,\text{tor}})$ on the Fock space $F_M$. On the module defined in [13], the map (1.18) can be written in the wedge notation as follows

$$\psi^N(u_k_1 \land u_k_2 \land \cdots \land u_k_N) = u_{k_1+1} \land u_{k_2+1} \land \cdots \land u_{k_N+1} \quad (6.1)$$

For the toroidal action on $\wedge^N(z) = \otimes^N V(z)/\sum_{i=1}^{N-1} \text{Ker} \left( \mathbf{T}_i + q^2(\mathbf{T}_i)^{-1} \right)$, we get the following relations

$$\psi^{-1}_N E_i(z) \psi_N = E_{i-1}(q^{-1}z), \quad \psi^{-2}_N E_1(z) \psi^2_N = E_{n-1}(q^{-2}z), \quad \psi^{-1}_N F_i(z) \psi_N = F_{i-1}(q^{-1}z), \quad \psi^{-2}_N F_1(z) \psi^2_N = F_{n-1}(q^{-2}z), \quad \psi^{-1}_N K^+_i(z) \psi_N = K^+_{i-1}(q^{-1}z), \quad \psi^{-2}_N K^+_1(z) \psi^2_N = K^+_{n-1}(q^{-2}z), \quad \psi^{-1}_N K^-_i(z) \psi_N = K^-_{i-1}(q^{-1}z), \quad \psi^{-2}_N K^-_1(z) \psi^2_N = K^-_{n-1}(q^{-2}z), \quad (6.2)$$

where $i \in \{2, \ldots, n-1\}$ and $\kappa = p^{-1/n}q$. 

On the space $\wedge^\infty V(z)$, we introduce the following map

$$\psi_\infty(u_k_1 \land u_k_2 \land \cdots) := u_{k_1+1} \land u_{k_2+1} \land \cdots. \quad (6.5)$$

Note that $\psi_\infty(F_M) = F_{M+1}$.

Proposition 12. For each vector $v \in \wedge^\infty V(z)$ and $i \in \{2, \ldots, n-1\}$ we have

$$ \psi^{-1}_\infty E_i(z) \psi_\infty v = E_{i-1}(q^{-1}z)v, \quad \psi^{-2}_\infty E_1(z) \psi^2_\infty v = E_{n-1}(q^{-2}z)v, \quad (6.6)$$

$$ \psi^{-1}_\infty F_i(z) \psi_\infty v = F_{i-1}(q^{-1}z)v, \quad \psi^{-2}_\infty F_1(z) \psi^2_\infty v = F_{n-1}(q^{-2}z)v, \quad (6.7)$$

$$ \psi^{-1}_\infty K^+_i(z) \psi_\infty v = K^+_{i-1}(q^{-1}z)v, \quad \psi^{-2}_\infty K^+_1(z) \psi^2_\infty v = K^+_{n-1}(q^{-2}z)v. \quad (6.8)$$

Here the action of $U'^{(1)'}_q(\hat{\mathfrak{sl}}_n)$ is the action $U_0$ defined in section [5.7] and $\kappa = p^{-1/n}q$.

Proof. To prove this proposition we need the following lemma.

Lemma 7. Let $k, l, N$ be integers satisfying $l \geq k$ and $N = s + ln$. Here $s$ is the integer such that $s \equiv M \mod n$, $0 \leq s \leq n-1$. Assume $v \in V^s_{M} \otimes^{ln}$ then we have

$$E_i(z)(v \land u_{M-N}) = (E_i(z) \cdot v) \land u_{M-N}, \quad (6.9)$$

$$F_i(z)(v \land u_{M-N}) = (F_i(z) \cdot v) \land u_{M-N}, \quad (6.10)$$

$$K^+_i(z)(v \land u_{M-N}) = (K^+_i(z) \cdot v) \land u_{M-N}, \quad (6.11)$$

$$E_{n-1}(z)(v \land u_{M-N} \land u_{M-N-1}) = (E_{n-1}(z) \cdot v) \land u_{M-N} \land u_{M-N-1}, \quad (6.12)$$

$$F_{n-1}(z)(v \land u_{M-N} \land u_{M-N-1}) = (F_{n-1}(z) \cdot v) \land u_{M-N} \land u_{M-N-1}, \quad (6.13)$$

$$K^+_n(z)(v \land u_{M-N} \land u_{M-N-1}) = (K^+_n(z) \cdot v) \land u_{M-N} \land u_{M-N-1}, \quad (6.14)$$

where $1 \leq i \leq n-2$.

Proof. By the relations (2.4), (2.7), we can show that for each $i$ ($1 \leq i \leq n-1$) the subalgebra in $U'^{(1)'}_q(\hat{\mathfrak{sl}}_n)$ generated by $E_i, F_i, H_i, K^\pm_i$ ($l' \in \mathbb{Z}$, $m' \in \mathbb{Z} \setminus \{0\}$) is, in fact, generated by only the elements $E_{i,0}, F_{i,0}, K^\pm_{i,1}$ and $E_{i,1}$. By the definition of the representation, every generator of $U'^{(1)'}_q(\hat{\mathfrak{sl}}_n)$ preserves the degree in the sense
of (5.6). So it is sufficient to show that the actions of $E_{i,0}, F_{i,0}, K_i^\pm, F_{i,1}$ and $F_{i,-1}$ satisfy the relations (6.9–6.14). For the case $E_{i,0}, F_{i,0}, K_i^\pm$, by the definition of the actions (4.10–4.12) we can check that that $E_{i,0}, F_{i,0}, K_i^\pm$ satisfy the relations (6.9–6.14) directly. Let us show that

$$F_{i,\pm 1}(v \wedge z^m v_n) = (F_{i,\pm 1} \cdot v) \wedge z^m v_n;$$  \hfill (6.15)

$$F_{n-1,\pm 1}(v \wedge z^m v_n \wedge z^m v_{n-1}) = (F_{n-1,\pm 1} \cdot v) \wedge z^m v_n \wedge z^m v_{n-1},$$  \hfill (6.16)

where $v \in V_M^{i+2,n,k}$ ($1 \leq i \leq n-2$) and $m$ is such that $u_{M-(s+n)} = z^m v_n$.

We will prove (6.16). In the proof we will use the two different notations:

$$u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_{N+2}} \quad \text{or} \quad \Lambda(z^m \otimes e_i)$$  \hfill (6.17)

where $k_i = \epsilon_i - Nm_i$ and $m = (m_1, \ldots, m_{N+2}), \quad e = (\epsilon_1, \ldots, \epsilon_{N+2})$  \hfill (6.18)

for an element from $\Lambda^{N+2}V(z) = \mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N+2}^{\pm 1}] \otimes (\otimes^{N+2}V)/\sum_{i=1}^{N+1} \text{Im}(T_i - T_{i}^{-1})$.

For any $M', M'' (1 \leq M' \leq M'' \leq N + 2)$, we define the $U_q'(\mathfrak{g}_n)$–action on the space $\mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N+2}^{\pm 1}] \otimes (\otimes^{N+2}V)$ in terms of the Chevalley generators:

$$e_i(P(z) \otimes w) = \sum_{j=M'}^{M''} P(z) \otimes E_j^{i+1} K_{j+1}^{0} \cdots K_{M'}^{0} w;$$  \hfill (6.19)

$$f_i(P(z) \otimes w) = \sum_{j=M'}^{M''} P(z) \otimes (K_{M'}^{i})^{-1} \cdots (K_{j-1}^{i})^{-1} E_j^{i+1} w;$$  \hfill (6.20)

$$k_i(P(z) \otimes w) = P(z) \otimes (K_{M'}^{i}K_{M'+1}^{0} \cdots K_{M''}^{0}) w;$$  \hfill (6.21)

$$e_0(P(z) \otimes w) = \sum_{j=M'}^{M''} P(z) \cdot (q^{-N-1} Y_j^{(N+2)})^{-1} \otimes E_j^{0,1} K_{j+1}^{0} \cdots K_{M'}^{0} w;$$  \hfill (6.22)

$$f_0(P(z) \otimes w) = \sum_{j=M'}^{M''} P(z) \cdot (q^{-N-1} Y_j^{(N+2)}) \otimes (K_{M'}^{0})^{-1} \cdots (K_{j-1}^{0})^{-1} E_j^{1,n} w;$$  \hfill (6.23)

$$k_0(P(z) \otimes w) = P(z) \otimes (K_{M'}^{0}K_{M'+1}^{0} \cdots K_{M''}^{0}) w.$$  \hfill (6.24)

Here $i = 1, \ldots, n-1$, $E_j^{1,i'} = 1^{\otimes -i} \otimes E_j^{i,i'} \otimes 1^{\otimes N+2-i}$, $K_j^{i} = q^{E_j^{i,i}-E_j^{i+1,i+1}}$, $K_j^{0} = (K_j^{1}K_j^{2} \cdots K_j^{N})^{-1}$, $P(z) \in \mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N+2}^{\pm 1}]$ and $w \in \otimes^{N+2}V$. This action is well-defined because of the commutativity of $Y_i^{(N+2)}$ ($i = 1, \ldots, N + 2$). The actions of the Drinfel’d generators are determined by the actions of the Chevalley generators.

Let $X$ be an element of $U_q'(\mathfrak{g}_n)$, then we define the action $X^{(x)}$ on the space $\mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N+2}^{\pm 1}] \otimes (\otimes^{N+2}V)$ by (6.19–6.24) and $M' = 1$, $M'' = N$.

We define the action $X^{(**)}$ on the space $\mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N+2}^{\pm 1}] \otimes (\otimes^{N+2}V)$ by (6.19–6.24) and $M' = N + 1$, $M'' = N + 2$. 
We also define the action $X^{(j)}$ ($j = 1, \ldots, N + 2$) on the space $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^{N+2}V)$ by (6.19–6.24) and $M' = j$, $M'' = j$.

With these definitions, for any two elements $X$ and $Y$ from $U'_q(\widehat{sl}_n)$, the operators $X^{(s)}$ and $Y^{(**)}$ commute. This shows that if we have $\Delta X = \sum \nu Y_\nu \otimes Z_\nu$ then $X(P(z) \otimes w) = \sum \nu Y_\nu^{(s)} Z_\nu^{(**)}(P(z) \otimes w)$.

The following equations are satisfied modulo $\Lambda(U_{N+}^{(s)} \cdot (U_{N+}^{(s)}))$. Here $U_{N+}$, $U_{N+}^{(s)}$ are the left ideals generated by $\{E_{i,k'}\}$, $\{F_{i,k'}, F_{j,p}\}$.

\[
\begin{align*}
F_{n-1,1} & \Lambda(P(z) \otimes w) \equiv \Lambda((K_{n-1}^{(*)}F_{n-1,1}^{(**)} + F_{n-1,1}^{(*)})(P(z) \otimes w)), \\
F_{n-1,-1} & \equiv \Lambda((K_{n-1}^{(*)})^{-1}F_{n-1,-1}^{(**)} + F_{n-1,-1}^{(*)}) \\
& + (q^{-1} - q)(K_{n-1}^{(*)})^{-1}H_{n-1,-1}^{(**)}(P(z) \otimes w)), \\
P(z) \otimes w & \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^{N+2}V).
\end{align*}
\]

These equations follow from the the coproduct formulas which have been obtained in \cite{[I]} Prop. 3.2.A:

\[
\begin{align*}
\Delta(F_{i,1}) & \equiv K_i \otimes F_{i,1} + F_{i,1} \otimes 1 \quad \text{mod } U_{N+} \otimes U_{N+}^{2}, \\
\Delta(F_{i,-1}) & \equiv K_i^{-1} \otimes F_{i,-1} + F_{i,-1} \otimes 1 \\
& + (q^{-1} - q)K_i^{-1}H_{i,-1} \otimes F_{i,0} \quad \text{mod } U_{N+} \otimes U_{N+}^{2}.
\end{align*}
\]

Now we will show the equality

\[
\Lambda(F_{n-1,\pm 1}(P(z) \otimes w)) = \Lambda(F_{n-1,\pm 1}^{(*)}(P(z) \otimes w)).
\]

where

\[
\begin{align*}
P(z) & = z_1^{m_1}z_2^{m_2} \cdots z_N^{m_N} = v_{i_1} \otimes \cdots \otimes v_{i_{N+2}}, \text{ and} \\
m_{N+1} & = m_{N+2} = m, \quad m_i < m \ (i = 1, \ldots, N), \\
|\Lambda(P(z) \otimes w)| & = k, \quad \epsilon_{N+1} = n, \quad \epsilon_{N+2} = n - 1.
\end{align*}
\]

First let us prove that any element in $U_{N+}^{(*)} \cdot (U_{N+}^{(*)})^{(**)}$ annihilates a vector $P(z) \otimes w$ that satisfies (6.30). It is enough to show that

\[
(F_{i',k'}^{(**)}F_{j',p'}^{(*)})(z_{N+1}^{m}z_{N+2}^{m} \bar{v} \otimes (v_n \otimes v_{n-1})) = 0,
\]

for $\bar{v} \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^2V)$.

This follows immediately from the observation that $wt(v_n) + wt(v_{n-1}) - \alpha_{i'} - \alpha_{j'}$ is not a weight of $\otimes^2V$.

Next we will show that $\Lambda(F_{n-1,\pm 1}^{(*)}(P(z) \otimes w)) = 0$, ($P(z)$ and $w$ satisfy (6.30)). By the formulas (6.27) and (6.28), we have the following identities modulo
Λ(UN_+^{(N+1)}(UN_2^{(N+2)})(P(z) \otimes w)):

Λ(F_{n-1,1}^{(N+2)}(P(z) \otimes w)) \equiv Λ((K_{n-1}^{(N+1)} F_{n-1,1}^{(N+2)} + F_{n-1,1}^{(N+1)})(P(z) \otimes w)), \quad (6.32)

F_{n-1,-1}Λ(P(z) \otimes w) \equiv Λ(((K_{n-1}^{(N+1)})^{-1} F_{n-1,-1}^{(N+2)} + F_{n-1,-1}^{(N+1)}

+ (q^{-1} - q)(K_{n-1}^{(N+1)})^{-1} H_{n-1,-1}^{(N+1)} F_{n-1,0}^{(N+2)})(P(z) \otimes w)), \quad (6.33)

\equiv Λ(((K_{n-1}^{(N+1)})^{-1} F_{n-1,-1}^{(N+2)} + F_{n-1,-1}^{(N+1)}

+ (q^{-1} - q)[E_{i,0}^{(N+1)}, F_{i,-1}^{(N+1)}] F_{n-1,0}^{(N+2)})(P(z) \otimes w)),

here we used the relation [E_{i,0}, F_{i,-1}] = K_{i,0} H_{i,-1} which is proved by (2.7). The following formula is essentially written in \( [3] \) Prop. 3.2.B:

\[ F_{i,l}^{(l)}(P(z) \otimes (\otimes_{j=1}^{l} v_{i,j})) = P'(z)(q^{-N-1} Y_{i}^{(N+2)})^{-1} \otimes (\otimes_{j=1}^{l-1} v_{i,j}) \otimes \delta_{i,l+1} v_{i,l+1} \otimes (\otimes_{j=l+1}^{N+2} v_{i,j}), \quad (6.34) \]

where \( P'(z) \in \mathbb{C}[z_{i}^{\pm 1}, \ldots, z_{N}^{\pm 1}] \) and \( v_{j} \in \mathbb{C}^{n} \).

By the formula (6.34), we have \( (UN_+^{(N+1)}(UN_2^{(N+2)})(P(z) \otimes w)) = 0 \), and by the formula (6.34), Lemma 6 (5.13) and the normal ordering rules, we get

\[ Λ(F_{n-1,1,n}^{(s)}(P(z) \otimes w)) \quad (6.35) \]

\[ = Λ(\alpha_{\pm 1} z_{N+1}^{m} z_{N+2}^{m} \bar{v} \otimes v_{n} \otimes v_{n}) + \bar{w}. \]

Here \( \alpha_{\pm 1} \) are certain coefficients, \( \bar{v} \in \mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N}^{\pm 1}] \otimes (\otimes^{N} V) \) and \( \bar{w} \) is a linear combination of normally ordered wedges \( w(\mathbf{n}, \tau), \mathbf{n} \in \mathcal{M}_{s+n+2}^{m}, \tau \in \mathcal{E}(\mathbf{n}) \) (see subsection \( [5.1] \)) such that for the sequence \( n = (n_{1}, n_{2}, \ldots, n_{s+n+2}) \) we have

\[ n_{1}, n_{2}, \ldots, n_{s+n+2} \leq m, \text{ and } \# \{ n_{i}, n_{i} = m \} < 2. \quad (6.36) \]

The inequality (6.36) implies, in particular, that \( n_{s+n+1} < m \). By the definition of the space \( \Lambda^{N+2} V(z) \), we have \( Λ(z_{N+1}^{m} z_{N+2}^{m} \bar{v} \otimes v_{n} \otimes v_{n}) = 0 \).

Assume \( \bar{w} \neq 0 \). From the inequality \( n_{s+n+1} < m \) it follows that \( |\bar{w}| \geq l + 1 \) (Cf. Proposition 5(ii) in \( [11] \)). On the other hand we have \( |\bar{w}| = |Λ(P(z) \otimes w)| = k \). By the condition \( k \leq l \) this is a contradiction. Therefore we conclude that \( \bar{w} = 0 \), and hence \( Λ(F_{n-1,1,n}^{(s)}(P(z) \otimes w)) = 0 \).

Let us prove \( Λ((K_{n-1}^{(s)})^{-1} H_{n-1,-1}^{(s)} F_{n-1,0}^{(s)}(P(z) \otimes w)) = 0 \).

We have

\[ Λ((K_{n-1}^{(s)})^{-1} H_{n-1,-1}^{(s)} F_{n-1,0}^{(s)}(z_{N+1}^{m} z_{N+2}^{m} \bar{v} \otimes (v_{n} \otimes v_{n-1}))) \quad (6.37) \]

\[ = Λ((K_{n-1}^{(s)})^{-1} H_{n-1,-1}^{(s)}(z_{N+1}^{m} z_{N+2}^{m} \bar{v} \otimes (v_{n} \otimes v_{n}))), \]

here \( \bar{v} \in \mathbb{C}[z_{1}^{\pm 1}, \ldots, z_{N}^{\pm 1}] \otimes (\otimes^{N} V) \). Since \( H_{n-1,-1}^{(s)} \) belongs to the algebra generated by the operators \( e_{i}^{(s)}, f_{i}^{(s)}, (k_{i}^{(s)})^{\pm} (i = 0, \ldots, n - 1) \), \( H_{n-1,-1}^{(s)} \) belongs to the algebra
generated by the operators $\left( Y_j^{(N+2)} \right)^{\pm 1}, E_j^{l,l'} (1 \leq j \leq N, 1 \leq l, l' \leq n)$. By Lemma 6 (5.14) and the normal ordering rules, we have

$$\Lambda((K_{n-1}^{(s)})^{-1}H_{n-1,-1}(z_{n+1}^{m}z_{n+2}^{m}\bar{v} \otimes (v_n \otimes v_n)))$$

$$= \Lambda(z_{n+1}^{m}z_{n+2}^{m}\bar{v} \otimes (v_n \otimes v_n)) + \tilde{w}. \quad (6.38)$$

Here $\bar{v} \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ and $\tilde{w}$ is the element which has the property written after the relation (6.35). By the previous discussion we have $\tilde{w} = 0$.

Thus we have shown (6.29). To prove (6.16) we must show that in the right-hand side of the last equation we can replace $q_1^{-(N+2)}Y_i^{(N+2)}$ by $q_1^{-N}Y_i^{(N)}$ $(1 \leq i \leq N)$. By Lemma 6 (5.14), for $1 \leq i \leq N$ and a sequence $\mathbf{m} = (m_1, m_2, \ldots, m_{N+2}) \in \mathbb{Z}^N$ such that $m_1, m_2, \ldots, m_N < m_{N+1} = m_{N+2} = m$ we have:

$$z^{\mathbf{m}}(Y_i^{(N+2)})^{\pm 1} = q^{\pm 2}z^{\mathbf{m}}(Y_i^{(N)})^{\pm 1} + [\ldots], \quad (6.39)$$

where $[\ldots]$ signifies a linear combination of monomials $z^{\mathbf{n}} \equiv z_1^{n_1}z_2^{n_2}\ldots z_{N+2}^{n_{N+2}}$ such that $n_1, n_2, \ldots, n_{N+2} \leq m$ and $\#\{n_i | n_i = m\} < 2$. By the normal ordering rules, we can write

$$\Lambda(q^{-1-(N+2)}Y_i^{(N+2)}(z_{n+1}^{m}z_{n+2}^{m}\bar{v} \otimes (v_n \otimes v_{n-1})))$$

$$= \Lambda(q^{-1-N}Y_i^{(N)}(z_{n+1}^{m}z_{n+2}^{m}\bar{v} \otimes (v_n \otimes v_{n-1}))) + \tilde{w}, \quad (6.40)$$

where $\bar{v} \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes (\otimes^N V)$ and the $\tilde{w}$ again has the same meaning as the $\tilde{w}$ in relation (6.35). Repeating the discussion after (6.35) we can show that $\tilde{w} = 0$. Hence we get (6.16).

To prove (6.15), consider the tensor product $\mathbb{C}[z_1^{\pm 1}, \ldots, z_{N+1}^{\pm 1}] \otimes (\otimes^N V) \otimes V$, use the formulas (6.27), (6.28) and continue the proof in a way that is completely analogous to the proof of (6.16).

Now we proceed with the proof of Proposition 12. It is sufficient to show the statement of the proposition for the vector $v$ such that $v \in F^k_M$. We put $v = v_{(N)} \wedge |M-N\rangle$, where $v_{(N)} \in V_{M}^{N,k}$, $N = s + ln$ as in Lemma 7. By Proposition 8, Lemma 7 (6.9) - (6.11) and the relation (6.5), we can show the following relations ($X = E, F$ or $K^{\pm 1}, 2 \leq i \leq n - 1$):
\[ X_{i-1}(q^{-1} \kappa z) \cdot v \]
\[ = (X_{i-1}(q^{-1} \kappa z)v(N)) \wedge u_{M-N} \wedge |M - N - 1) \]
\[ = X_{i-1}(q^{-1} \kappa z)(v(N) \wedge u_{M-N}) \wedge |M - N - 1) \]
\[ = (\psi_{N+1}^{-1}X_i(z)\psi_{N+1}(v(N) \wedge u_{M-N})) \wedge |M - N - 1) \]
\[ = \psi_{\infty}^{-1}X_i(z)(\psi_{N+1}(v(N) \wedge u_{M-N}) \wedge |M - N)) \]
\[ = \psi_{\infty}^{-1}X_i(z)(\psi_{\infty}(v(N) \wedge u_{M-N} \wedge |M - N - 1)), \]

This is exactly the statement of the Proposition \[ (i \neq n - 1) \]. For the case \( i = n - 1 \) we use Lemma \[ (6.12) - (6.14) \] and a straightforward modification of (6.41).

In view of Propositions \[ 8 \] and \[ 12 \] we can now define a \( U_q'(\mathfrak{sl}_{n,tor}) \)-action on the space \( \wedge \mathfrak{g}V(z) \). By this definition each subspace \( F_M \) is invariant with respect to the \( U_q'(\mathfrak{sl}_{n,tor}) \)-action. Thus we arrive at

**Theorem 13** (Varagnolo and Vasserot \[ 14 \]). The \( q \)-deformed Fock space \( F_M \) is a \( U_q'(\mathfrak{sl}_{n,tor}) \)-module. The actions of \( X_i(z) (X = E, F \) or \( K^\pm, 1 \leq i \leq n - 1) \) are determined in subsection \[ 7.2 \] by the Chevalley generators. The actions of \( X_0(z) (X = E, F \) or \( K^\pm) \) are determined by \( \psi_{\infty}^{-1}X_1(p^{1/n}z)\psi_{\infty} \) (in this notation \( X_1(p^{1/n}z) \) act on \( F_{M+1} \), but \( X_0(z) \) act on \( F_M \)).

**Remark** One can verify that the action of the subalgebra \( U_q'(\widehat{\mathfrak{sl}}_n) \) coincides with the level-1 action of \( U_q'(\mathfrak{sl}_n) \) defined in Section \[ 5.2 \] because we have \( X_0 = \psi_{\infty}^{-1}X_1\psi_{\infty} \) (\( X_i \) is the Chevalley generator of the level-1 action of \( U_q'(\mathfrak{sl}_n) \) defined in Section \[ 7.2 \]). Hence the action of \( U_q'(\mathfrak{sl}_{n,tor}) \) on the Fock space has the level \( (0,1) \).

6.2. **Action of \( U_q'(\mathfrak{sl}_{n,tor}) \) on level-1 irreducible \( U_q'(\widehat{\mathfrak{sl}}_n) \)-modules.** In the paper \[ 8 \] it was demonstrated that the Fock space \( F_M \) admits an action of the Heisenberg algebra \( H \) which commutes with the level-1 action \( U_1 \) of the algebra \( U_q'(\widehat{\mathfrak{sl}}_n) \). The Heisenberg algebra is a unital \( \mathbb{C} \)-algebra generated by elements \( 1, B_a \) with \( a \in \mathbb{Z}_{\neq 0} \) which are subject to relations

\[ [B_a, B_b] = \delta_{a+b,0}a \frac{1 - q^{2na}}{1 - q^{2a}}, \]

\[ (6.42) \]

The Fock space \( F_M \) is an \( H \)-module with the action of the generators given by \[ 8 \]

\[ B_a = \sum_{i=1}^{\infty} z_i^a. \]

\[ (6.43) \]

Let \( \mathbb{C}[H_{-}] \) be the Fock space of \( H \), i.e., \( \mathbb{C}[H_{-}] = \mathbb{C}[B_{-1}, B_{-2}, \ldots] \). The element \( B_{-a} \) \((a = 1, 2, \ldots)\) acts on \( \mathbb{C}[H_{-}] \) by multiplication. The action of \( B_a \) \((a = 1, 2, \ldots)\)
is given by (6.42) together with the relation
\[ B_a \cdot 1 = 0 \quad \text{for } a \geq 1. \] (6.44)

Let \( \Lambda_i (i \in \{0, 1, \ldots, n-1\}) \) be the fundamental weights of \( \widehat{\mathfrak{sl}}_n' \). And let \( V(\Lambda_i) \) be the irreducible (level-1) highest weight module of \( U_q'(\widehat{\mathfrak{sl}}_n) \) with highest weight vector \( V(\Lambda_i) \) and highest weight \( \Lambda_i \).

The following results are proven in [8]:

- The action of the Heisenberg algebra on \( F_M \) and the action \( U_1 \) of \( U_q'(\widehat{\mathfrak{sl}}_n) \) commute.
- There is an isomorphism
\[ \iota_M : F_M \cong V(\Lambda_i) \otimes \mathbb{C} [H_+] \quad (M = i \mod n) \] (6.45)

of \( U_q'(\widehat{\mathfrak{sl}}_n) \otimes H \)-modules normalized so that \( \iota_M (|M\rangle) = V(\Lambda_i) \otimes 1 \).

In general the level-0 \( U_q'(\widehat{\mathfrak{sl}}_n) \)-action \( U_0 \) does not commute with the Heisenberg algebra. However if we choose the parameter \( p \) in \( U_0 \) in a special way, then the following result holds:

**Proposition 14.** At \( p = 1 \) we have
\[ [U_0, H_] = 0. \] (6.46)

**Proof.** Let \( w \in F_M^k \). Then by Proposition 10 for any \( l \geq k \) there is a unique \( w_{(s+nl)} \in V_M^{s+nl,k} \) such that
\[ w = w_{(s+nl)} \wedge |M - s - nl\rangle. \] (6.47)

Let \( m \geq 1 \) and let us act with \( B_{-m} \) on the \( w \):
\[ B_{-m} w = (B_{-m}^{(s+nl)} w_{(s+nl)}) \wedge |M - s - nl\rangle + w_{(s+nl)} \wedge B_{-m} |M - s - nl\rangle, \] (6.48)

where \( B_{-m}^{(N)} = \sum_{i=1}^{N} z_i^{-m} \) (\( N = 1, 2, \ldots \)). In view of (6.47) and (6.43) in the second summand above we may write
\[ w_{(s+nl)} = w_{(s+mk)} \wedge u_{M-s-nk} \wedge u_{M-s-nk-1} \wedge \ldots u_{M-s-nl+1}, \] (6.49)
\[ B_{-m} |M - s - nl\rangle = u_{M-s-nl+nm} \wedge u_{M-s-nl-1} \wedge u_{M-s-nl-2} \wedge \ldots + \] (6.50)
\[ + u_{M-s-nl} \wedge u_{M-s-nl+nm-1} \wedge u_{M-s-nl-2} \wedge \ldots + \]
\[ + u_{M-s-nl} \wedge u_{M-s-nl-1} \wedge u_{M-s-nl+nm-2} \wedge \ldots + \]
\[ + \ldots. \]
Now let us choose \( l \) to be greater or equal to \( k + m \). Then by Lemma 2.2 in [8] the second summand in (6.48) vanishes and we have
\[
B_{-m}w = (B_{-m}^{(s+nl)} w_{(s+nl)}) \wedge |M - s - nl|.
\] (6.51)
The degree of \( B_{-m}w \) is equal to \( k + m \). Note that since \( m \geq 1 \) we have also
\[
B_{-m}^{(s+nl)} w_{(s+nl)} \in V^{s+nl,k+m}_M.
\] (6.52)
Therefore by definition (5.25) of the action \( U_0 \) for any element \( a \) of \( U'_q(\hat{s}l_n) \) we have (note that \( l \geq k + m \)):
\[
aB_{-m}w = \left(a^{(s+nl)}B_{-m}^{(s+nl)} w_{(s+nl)}\right) \wedge |M - s - nl|.
\] (6.53)
At \( p = 1 \) the operators \( a^{(N)} \) and \( B_{a}^{(N)} \) commute for all finite \( N \) and non-zero \( a \) since \( B_{a}^{(N)} \) are symmetric in \( z_1, \ldots, z_N \) and thereby commute with the operators \( Y_1^{(N)}, \ldots, Y_N^{(N)} \).
Thus we have
\[
aB_{-m}w = \left(B_{-m}^{(s+nl)} a^{(s+nl)} w_{(s+nl)}\right) \wedge |M - s - nl|.
\] (6.54)
Taking into account that degree of \( aw \) is equal to the degree of \( w \) and is equal to \( k \) we may repeat the discussion leading to (6.51) and find that
\[
aB_{-m}w = B_{-m}aw.
\] (6.55)

**Remark** Note that it is not true that \( U_0 \) at \( p = 1 \) commutes with the subalgebra \( H_+ \) generated by \( 1, B_1, B_2, \ldots \). This fact is known from consideration of the Yangian limit in [12]. If we attempt to repeat the discussion in the proof of the preceding proposition for \( B_{-m} \) with negative \( m \), we find that the inclusion (6.52) fails to hold in general.

Let \( H'_- \) be the non-unital subalgebra in \( H \) generated by \( B_{-1}, B_{-2}, \ldots \). Proposition 14 allows us to define a level-0 \( U'_q(\hat{s}l_n) \)-module structure on the irreducible level-1 module \( V(\Lambda_i) \) \((i \in \{0, 1, \ldots, n - 1\})\). Indeed from this proposition it follows that the subspace
\[
H'_- F_M \subset F_M
\] (6.56)
is invariant with respect to the action \( U_0 \) at \( p = 1 \) and therefore a level-0 action of \( U'_q(\hat{s}l_n) \) is defined on the quotient space
\[
F_M / (H'_- F_M)
\] (6.57)
which in view of (5.43) is isomorphic to \( V(\Lambda_i) \) with \( i = M \mod n \). We do not know whether this level-0 action coincides with the level-0 action defined in the paper [7]. However the Yangian limit considered in [12] suggests an affirmative answer to this question. The results of [12] also lead to the following conjecture:
Conjecture 1. At $p = q^{2n}$ we have
\[ [U_0, H_+] = 0. \] (6.58)

According to Lemma 3 the subalgebras $U_1$ and $U_0$ generate the action of $U''(\mathfrak{sl}_{n,tor})$ on the Fock space, hence combining Proposition 14 with the fact that $U_1$ commutes with the Heisenberg algebra, we find that the toroidal action at $p = 1(\kappa = q)$ commutes with $H_-$. Repeating the discussion leading to the equation (6.57) we conclude that

Proposition 15. The highest weight irreducible $U''(\hat{\mathfrak{sl}}_n)$ module $V(\Lambda_i)$ ($i = 0, \ldots, n-1$) is a level $(0,1)$ toroidal module with $\kappa = q$. In this module the action of the subalgebra $U''_q(\hat{\mathfrak{sl}}_n)$ coincides with the level 1 action of $U''_q(\hat{\mathfrak{sl}}_n)$. And the action of the subalgebra $U''_q(\hat{\mathfrak{sl}}_n)$ is induced from the level 0 $U''_q(\hat{\mathfrak{sl}}_n)$-action $U_0$ on the Fock space.

6.3. Irreducibility of the Fock space as a toroidal module. At $q = 1$ both $U''_q(\hat{\mathfrak{sl}}_n)$ and $U''_q(\hat{\mathfrak{sl}}_n)$ are isomorphic to $U''(\hat{\mathfrak{sl}}_n)$. Action of these subalgebras on the Fock space $F_M$ is now given by the generators

\[ E_i = e_i = \sum_{j=1}^{\infty} E_j^{i,i+1}, \quad F_i = f_i = \sum_{j=1}^{\infty} E_j^{i+1,i}, \] (6.59)

\[ H_i = h_i = \sum_{j=1}^{\infty} E_j^{i,i} - E_j^{i+1,i+1}, \quad (i = 1, 2, \ldots, n-1) \] (6.60)

\[ e_0 = \sum_{j=1}^{\infty} p^{-D_j} E_j^{n,1}, \quad f_0 = \sum_{j=1}^{\infty} p^{D_j} E_j^{1,n}, \] (6.61)

\[ E_0 = \sum_{j=1}^{\infty} z_j E_j^{n,1}, \quad F_0 = \sum_{j=1}^{\infty} z_j^{-1} E_j^{1,n}, \] (6.62)

\[ h_0 = -h_1 - h_2 - \cdots - h_{n-1}, \] (6.63)

\[ H_0 = 1 - H_1 - H_2 - \cdots - H_{n-1}. \] (6.64)

It is straightforward to verify that these generators are well-defined on the Fock space provided we specify normalization of the Cartan subalgebra in $\mathfrak{sl}_n$ as

\[ H_i|M\rangle = h_i|M\rangle = \delta(M \equiv i \text{ mod } n)|M\rangle \quad (i = 1, 2, \ldots, n-1). \] (6.65)

The affine Lie algebra $\hat{\mathfrak{sl}}_n'$ is realized as the central extension of the loop algebra $\mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}]$ by the center $\mathbb{C}c$, so that if $x(m) = x \otimes t^m$ for $x \in \mathfrak{sl}_n$, then a system of generators for $U''(\hat{\mathfrak{sl}}_n)$ is provided by $E_{i,j}(m)$ ($i \neq j \in \{1, \ldots, n\}$), $H_i(m)$ ($i \in \{1, \ldots, n-1\}$, $m \in \mathbb{Z}$) and $c$, where $E_{i,j}$ are matrix units regarded as generators of $\mathfrak{sl}_n$ and $H_i = E_{i,i} - E_{i+1,i+1}$. 
In terms of these generators the action of $U_q^{(2)\prime}(\hat{\mathfrak{sl}}_n)$ is given by

$$E_{i,j}(m) = \sum_{k=1}^{\infty} z^m_k E_{i,j}^{k}, \quad H_i(m) = \sum_{k=1}^{\infty} z^m_k (E_{i,i}^{k} - E_{i+1,i+1}^{k}), \quad c = 1.$$  

(6.66)

**Proposition 16.** Let $q = 1$ and $\vert p \vert \neq 1$. Then the Fock space $F_M$ is an irreducible module of the algebra $U_q^{(2)\prime}(\hat{\mathfrak{sl}}_n)$.  

**Proof.** Let $B_m (m \in \mathbb{Z} \setminus \{0\})$ be the generators (6.43) of the $H$-action on $F_M$. A computation shows that we have

$$B_m = \sum_{i=1}^{n-1} (i + \frac{pm}{1-pm}) H_i(m) + \frac{np^m}{p^{2m} - 1} [[E_{1,n}(0), [E_{n,1}(m), f_0]], e_0]$$  

(6.67)

and therefore the Heisenberg action is included into the action of toroidal algebra. This implies that the $U_q^{(2)\prime}(\hat{\mathfrak{sl}}_n) \otimes H$ action is included into the toroidal action as well. However the former action is irreducible in view of the decomposition (6.45).

**Corollary 3.** Let $\vert p \vert \neq 1$. Then there is $\epsilon > 0$ such that the Fock space $F_M$ is an irreducible module of the algebra $U_q^{(2)\prime}(\hat{\mathfrak{sl}}_n)$ for all $q$ with $\vert q - 1 \vert < \epsilon$.

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