Neumann spectral-Galerkin method for the inverse scattering Steklov eigenvalues and applications

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Abstract
In this paper, we consider the numerical approximation of the Steklov eigenvalue problem that arises in inverse acoustic scattering. The underlying scattering problem is for an inhomogeneous isotropic medium. These eigenvalues have been proposed to be used as a target signature since they can be recovered from the scattering data. A spectral-Galerkin method is studied where the basis functions are the Neumann eigenfunctions of the Laplacian. Error estimates for the eigenvalues are proven by appealing to Weyl’s Law. We will test this method against separation of variables in order to validate the theoretical convergence. We also consider the inverse spectral problem of estimating/recovering the refractive index from the knowledge of the Steklov eigenvalues. Since the eigenvalues are monotone with respect to a real-valued refractive index this implies that they can be used for non-destructive testing. Some numerical examples are provided for the inverse spectral problem.

Keywords: Steklov Eigenvalues · Inverse Scattering · Spectral-Galerkin Method · Error Estimates · Parameter Estimation

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1 Introduction
This manuscript focuses on the numerical approximation of a non-selfadjoint Steklov eigenvalue problem that arises in inverse acoustic scattering. A similar eigenvalue problem has been analyzed for the electromagnetic scattering problem in [12]. The numerical method employed here is a spectral-Galerkin method where the basis functions are finitely many Neumann eigenfunctions of the Laplacian. In [25] we see that the Neumann eigenfunctions of the Laplacian form a basis for the Sobolev Space $H^1(D)$. Our convergence analysis of the spectral-Galerkin method will use
the Weyl’s asymptotic for the Neumann eigenvalues. In [18] a similar method was used to approximate the zero-index transmission eigenvalues with a conductivity condition where finitely many Dirichlet eigenfunctions of the Laplacian are used as the approximation space. We will also numerically investigate the inverse spectral problem of estimating the refractive index from the Steklov eigenvalues.

The Steklov eigenvalue problem we consider here is associated with the direct scattering problem: find the total field $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ for $d = 2, 3$ such that

$$\Delta u + k^2 n u = 0 \quad \text{in} \quad \mathbb{R}^d$$

with $u = u^s + u^i$. The incident field is given by $u^i = e^{ikx \cdot \hat{y}}$ with the incident direction $\hat{y}$ is a point on the unit circle/sphere. Here, we let $n \in L^\infty(\mathbb{R}^d)$ denote the refractive index with $\text{supp}(n - 1) = \Omega$. We assume that the scatterer $\Omega \subset \mathbb{R}^d$ is a bounded simply connected open set. The scattered field $u^s$ satisfies the Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x|^{(d-1)/2} \left( \frac{\partial u^s}{\partial |x|} - ik u^s \right) = 0$$

which is satisfied uniformly with respect to $\hat{x} = x/|x|$. The scattered field $u^s$ satisfying (1) and (2) has the asymptotic expansion as $|x| \to \infty$ (see for e.g. [8])

$$u^s(x, \hat{y}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left\{ u^\infty(\hat{x}, \hat{y}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\}.$$ 

Now, we define the far-field operator $F : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S})$ by

$$Fg(\hat{x}) = \int_\mathbb{S} u^\infty(\hat{x}, \hat{y}) g(\hat{y}) \, ds(\hat{y})$$

where $\mathbb{S}$ denotes the boundary of the unit circle/sphere. To continue, we will now define an auxiliary total field $u_\lambda \in H^1_{\text{loc}}(\mathbb{R}^d)$ with $\text{Im}(\lambda) \geq 0$ satisfying

$$\Delta u_\lambda + k^2 u_\lambda = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \quad \text{and} \quad \partial_\nu u_\lambda + \lambda u_\lambda = 0 \quad \text{on} \quad \partial D.$$ 

The auxiliary total field is given by $u_\lambda = u_\lambda^s + u^i$ and the auxiliary scattered field $u_\lambda^s$ also satisfies the Sommerfeld radiation condition (2). Here, the region $D$ is taken to be any bounded simply connected open set with a smooth boundary such that $\Omega \subseteq D$. Similarly, the auxiliary scattered field $u_\lambda^s$ gives rise to the auxiliary far-field operator $F_\lambda : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S})$ given by

$$F_\lambda g(\hat{x}) = \int_\mathbb{S} u_\lambda^\infty(\hat{x}, \hat{y}) g(\hat{y}) \, ds(\hat{y}).$$
It is shown in [3] that the modified far-field operator $F - F_\lambda$ is injective with a dense range if and only if $\lambda \in \mathbb{C}$ is not a Steklov eigenvalue for the scattering problem (1). In [9] it is shown that the knowledge of the modified far-field operator $F - F_\lambda$ can be used to recover the Steklov eigenvalues. Since $F$ is given by physical measurements and $F_\lambda$ can be computed numerically/analytically gives that the Steklov eigenvalues can be determined by the far-field data. In [9, 21] it is shown that the largest positive Steklov eigenvalue depends monotonically on the refractive index provided $n$ is real-valued implying the eigenvalue can be used as a target signature.

We now define the inverse scattering Steklov eigenvalue problem associated with (1). These are defined as the values $\lambda \in \mathbb{C}$ with $\text{Im}(\lambda) \geq 0$ such that there is a nontrivial solution $w \in H^1(D)$ satisfying

$$\Delta w + k^2 nw = 0 \text{ in } D \quad \text{and} \quad \partial_\nu w + \lambda w = 0 \text{ on } \partial D. \quad (4)$$

Recall, that $n \in L^\infty(D)$ such that $\text{supp}(n - 1) = \Omega$ where the scatterer $\Omega \subseteq D$. Here $k > 0$ denotes the wavenumber and for the non-selfadjoint case of an absorbing media (i.e. $n$ is complex-valued) we have that

$$n = n_R + \frac{n_I}{k} \quad \text{with} \quad n_R > 0 \text{ and } n_I \geq 0.$$

This eigenvalue problem was introduced and studied in [3, 9] to overcome the shortcomings of the transmission eigenvalue problem that is obtained by only considering the injectivity of the far-field operator $F$. See [16] for a numerical method with the transmission eigenvalues for the inverse spectral problem. In [20] a continuous finite element method with a spectral indicator is used to approximate the Steklov eigenvalues. Whereas in [22] a discontinuous finite element method is used as an approximation scheme for this problem. See [1, 23, 26] for applications of other Galerkin methods applied to the selfadjoint Steklov eigenvalue problems. We also mention that this idea of augmenting the far-field operator has been employed in [4, 13] to obtain new eigenvalue problems associated with the scattering problem (1).

The remainder of the paper is ordered as follows. We begin our investigation in the next section by defining the associated source problem for the Steklov eigenvalue problem. Next, we consider the approximation properties for the Neumann spectral-Galerkin method’s approximation space. Here we take our spectral basis to be finitely many Neumann eigenfunctions for the Laplacian. Then we will study the convergence and prove error estimates for computing the Steklov eigenvalue and eigenfunctions via our approximation space. We will then provide some numerical examples in two dimensions to study the convergence rate. This will show that the proposed spectral method is effective for computing the eigenvalues for a modest size discretized system.
Lastly, we consider the inverse spectral problem of estimating the refractive index from the knowledge of the eigenvalues.

2 The Steklov Eigenvalue Problem

In this section, we will consider the variational formulation of the inverse scattering Steklov eigenvalue problem (4). The analysis here will be used to prove the convergence of our approximation method. To begin, recall that the Sobolev Space

\[ H^1(D) = \{ \varphi \in L^2(D) : \partial_{x_i} \varphi \in L^2(D) \text{ for } i = 1, \ldots, d \}. \]

Now, by appealing to Green’s first Theorem it is clear that the variational formulation of (4) is given by

\[ a(w, \varphi) = -\lambda b(w, \varphi) \text{ for all } \varphi \in H^1(D) \] (5)

where the bounded sesquilinear forms are defined by

\[ a(w, \varphi) = \int_D \nabla w \cdot \nabla \varphi - k^2 n w \varphi \, dx \quad \text{and} \quad b(w, \varphi) = \int_{\partial D} w \varphi \, ds. \] (6)

Since \( w \) is nontrivial we will assume that it is normalized with \( \| w \|_{L^2(\partial D)} = 1 \). Note that \( w \neq 0 \text{ a.e. on } \partial D \) due to the impedance condition in (4).

As in [20, 22] we will define the associated Neumann-to-Dirichlet (NtD) operator for the source problem associated with (5). To this end, define the source problem: find \( w \in H^1(D) \) such that for any \( f \in L^2(\partial D) \)

\[ a(w, \varphi) = b(f, \varphi) \text{ for all } \varphi \in H^1(D). \] (7)

It is clear that \( w \in H^1(D) \) satisfies the boundary value problem

\[ \Delta w + k^2 n w = 0 \text{ in } D \quad \text{and} \quad \partial_{\nu} w = f \text{ on } \partial D. \]

Assuming that \( k \) is not an associated Neumann eigenvalue for the differential operator \( \Delta + k^2 n \) in \( D \) then we have that the source problem (7) is well-posed (see for e.g. [20]). Therefore, we can define the NtD operator associated with source problem (7) as \( T : L^2(\partial D) \hookrightarrow L^2(\partial D) \) such that

\[ Tf = w|_{\partial D} \text{ where } w \in H^1(D) \text{ solves (7) for } f \in L^2(\partial D). \] (8)
By the Trace Theorem (see for e.g. [14]) we have that \( \text{Range}(T) \subseteq H^{1/2}(\partial D) \) and the compact embedding of \( H^{1/2}(\partial D) \) into \( L^2(\partial D) \) (see for e.g. [8]) implies that \( T \) is a compact operator. Now, let \( \mu \in \mathbb{C} \) be an eigenvalue of \( T \) with corresponding eigenfunction \( w \), then by (7) we have that
\[
Tw|_{\partial D} = -\lambda^{-1}w|_{\partial D} \quad \text{which implies that} \quad \mu = -\lambda^{-1}.
\]
Note, that \( \mu \neq 0 \) provided that \( k \) is not a Dirichlet eigenvalue for the differential operator \( \Delta + k^2 n \) in \( D \).

**Assumption 2.1.** The wave number \( k \in \mathbb{R} \) is not a Dirichlet or Neumann eigenvalue for the differential operator \( \Delta + k^2 n \) in \( D \).

Notice, that assumption 2.1 is not restrictive since the set of Dirichlet or Neumann eigenvalues is discrete which gives that any choice of wavenumber \( k \) is almost surely not an associated eigenvalue. Also, if \( n \) is complex-valued then there are no real Dirichlet or Neumann eigenvalues.

### 3 Analysis of the Approximation

Here we analyze the proposed approximation method of the variational formulation (5) of the inverse scattering Steklov eigenvalue problem. The method proposed here will be referred to as a Neumann spectral-Galerkin method. This is a spectral method where the basis functions are taken to be a finite number of the Neumann eigenfunctions for the Laplacian. The basis functions are denoted \( \phi_j \in H^1(D) \) with the corresponding Neumann eigenvalues \( \sigma_j \in \mathbb{R}_{\geq 0} \) that satisfy
\[
-\Delta \phi_j = \sigma_j \phi_j \quad \text{in} \quad D, \quad \partial_\nu \phi_j = 0 \quad \text{on} \quad \partial D \quad \text{where} \quad \|\phi_j\|_{L^2(D)} = 1.
\]

Here, we assume that the sequence \( \sigma_j \) is arranged in increasing order.

#### 3.1 Analysis of the Approximation Space

Now, we will analyze the approximation space given by
\[
V_N(D) = \text{span}\{\phi_j\}_{j=1}^N \quad \text{for some fixed} \quad N \in \mathbb{N}.
\]

We begin by studying the approximation properties of the finite dimensional subspace \( V_N(D) \subset H^1(D) \). It is well-known that the eigenfunctions \( \{\phi_j\}_{j=1}^\infty \) form an
orthonormal basis of $L^2(D)$ and that for any $f \in H^1(D)$

$$f = \sum_{j=1}^{\infty} (f, \phi_j)_{L^2(D)} \phi_j$$

such that

$$\|f\|_{H^1(D)}^2 = \sum_{j=1}^{\infty} (1 + \sigma_j) \| (f, \phi_j)_{L^2(D)} \|^2$$

(10)

by the results in Chapter 9 of [25]. The Fourier-Series representation (10) along

with Weyl’s law will be used to show the approximation rates for the space $V_N(D)$. Recall, Weyl’s asymptotic formula (see for e.g. [2, 19, 27]) gives that there exists two

c_1, c_2 > 0 independent of $j$ such that

$$c_1 j^{2/d} \leq \sigma_j \leq c_2 j^{2/d} \quad \text{for} \quad j \gg 1$$

where again the dimension $d = 2, 3$. Now, we define the $L^2(D)$ projection onto the

approximation space $V_N(D)$ denote by $\Pi_N : L^2(D) \mapsto V_N(D)$ such that

$$\Pi_N f = \sum_{j=1}^{N} (f, \phi_j)_{L^2(D)} \phi_j$$

for some fixed $N \in \mathbb{N}$

for all $f \in L^2(D)$. By (10) we have the point-wise convergence

$$\|(I - \Pi_N)f\|_{H^1(D)}^2 = \sum_{j=N+1}^{\infty} (1 + \sigma_j) \| (f, \phi_j)_{L^2(D)} \|^2 \to 0 \quad \text{as} \quad N \to \infty.$$

We now prove some convergence rates in the approximation space $V_N(D)$. This will give that the approximation space has sufficient approximation properties for our Galerkin method. For the rest of the paper $C$ will be an arbitrary positive constant

that does not depend on parameter $N \in \mathbb{N}$.

**Theorem 3.1.** Assume that $f \in H^1(D)$ then

$$\|(I - \Pi_N)f\|_{L^2(D)} \leq \frac{C}{(N + 1)^{1/d}} \|f\|_{H^1(D)} \quad \text{as} \quad N \to \infty.$$

**Proof.** It is clear that by the definition of the projection operator $\Pi_N$ we have that

$$\|(I - \Pi_N)f\|_{L^2(D)}^2 = \sum_{j=N+1}^{\infty} \| (f, \phi_j)_{L^2(D)} \|^2$$

$$\leq \sigma_{N+1}^{-1} \sum_{j=N+1}^{\infty} \sigma_j \| (f, \phi_j)_{L^2(D)} \|^2$$

$$\leq \frac{C}{(N + 1)^{2/d}} \sum_{j=1}^{\infty} (1 + \sigma_j) \| (f, \phi_j)_{L^2(D)} \|^2$$

6
provide that \( N \) is large enough. Note that we have used Weyl’s law for the Neumann eigenvalues. This proves the result by \([10]\).

**Theorem 3.2.** Assume that \( f \in H^2(D) \) such that \( \partial_v f = 0 \) on \( \partial D \) then

\[
\|(I - \Pi_N) f\|_{H^1(D)} \leq \frac{C}{(N + 1)^{1/d}} \|f\|_{H^2(D)} \quad \text{as} \quad N \to \infty.
\]

**Proof.** To begin, we notice that \( \Delta f \in L^2(D) \) and since \( \{\phi_j\}_{j=1}^{\infty} \) is an orthonormal basis of \( L^2(D) \) we have that

\[
\Delta f = \sum_{j=1}^{\infty} (\Delta f, \phi_j)_{L^2(D)} \phi_j.
\]

By Green’s second Theorem we derive that

\[
(\Delta f, \phi_j)_{L^2(D)} = (f, \Delta \phi_j)_{L^2(D)} = -\sigma_j (f, \phi_j)_{L^2(D)}
\]

where we have used \([9]\) as well as the zero Neumann condition for \( f \). Therefore, we can conclude that

\[
\|\Delta f\|_{L^2(D)}^2 = \sum_{j=1}^{\infty} \sigma_j^2 |(f, \phi_j)_{L^2(D)}|^2 < \infty.
\]

Now, as in the previous result we will use Weyl’s law for the Neumann eigenvalues. To this end, we estimate

\[
\|(I - \Pi_N) f\|_{H^1(D)}^2 \leq 2 \sum_{j=N+1}^{\infty} \sigma_j |(f, \phi_j)_{L^2(D)}|^2
\]

\[
\leq 2\sigma_{N+1}^{-1} \sum_{j=N+1}^{\infty} \sigma_j^2 |(f, \phi_j)_{L^2(D)}|^2
\]

\[
\leq \frac{C}{(N + 1)^{2/d}} \sum_{j=1}^{\infty} \sigma_j^2 |(f, \phi_j)_{L^2(D)}|^2
\]

provide that \( N \) is large enough. This proves the claim. \( \square \)

Being motivated by the proof of Theorem 3.2 we define a subset of \( L^2(D) \) denoted by \( \mathcal{D}(\Delta^m) \) for some \( m \in \mathbb{R}_{\geq 0} \) such that

\[
\|f\|_{\mathcal{D}(\Delta^m)}^2 = \sum_{j=1}^{\infty} \sigma_j^{2m} |(f, \phi_j)_{L^2(D)}|^2 < \infty.
\] (11)
It is clear that $\mathcal{D}(\Delta^m)$ is a Hilbert space with norm given by equation (11). If $m$ is a positive integer this subspace of $L^2(D)$ can be seen as the space of functions where the $m$th Laplacian applied to the Fourier-Series (10) is a convergent in $L^2(D)$. We now prove a convergence rate for $f \in H^1(D) \cap \mathcal{D}(\Delta^m)$ for $m > 1/2$.

**Theorem 3.3.** Assume that $f \in H^1(D) \cap \mathcal{D}(\Delta^m)$ for some $m > 1/2$ then

$$
\| (I - \Pi_N) f \|_{H^1(D)} \leq \frac{C}{(N + 1)^{(2m-1)/d}} \| f \|_{\mathcal{D}(\Delta^m)} \quad \text{as} \quad N \to \infty.
$$

**Proof.** To prove the claim recall we have that for $N$ sufficiently large

$$
\| (I - \Pi_N) f \|_{H^1(D)}^2 \leq 2 \sum_{j=N+1}^{\infty} \sigma_j |(f, \phi_j)_{L^2(D)}|^2 \leq \frac{2}{\sigma_{N+1}^{2m-1}} \sum_{j=N+1}^{\infty} \sigma_j^{2m} |(f, \phi_j)_{L^2(D)}|^2
$$

where we have used the Fourier-Series representation. Now, by again appealing to Weyl’s law we conclude that

$$
\| (I - \Pi_N) f \|_{H^1(D)}^2 \leq \frac{C}{(N + 1)^{2(2m-1)/d}} \sum_{j=1}^{\infty} \sigma_j^{2m} |(f, \phi_j)_{L^2(D)}|^2
$$

proving the estimate by the definition of $\mathcal{D}(\Delta^m)$.

### 3.2 Analysis of the Spectral Approximation

Here we prove the convergence and error estimates for the Neumann spectral-Galerkin method for computing the inverse scattering Steklov eigenvalues. The analysis in this section uses the approximation properties of the space $V_N(D)$. We will employ similar techniques as in [20] where this eigenvalue problem was studied using a conforming finite element approximation.

To begin, let the trace space of $V_N(D)$ be denoted

$$
V_N(\partial D) = \{ f_N \in L^2(\partial D) : f_N = w_N|_{\partial D} \text{ where } w_N \in V_N(D) \} \subset L^2(\partial D).
$$

We now define the Neumann spectral-Galerkin approximation of the NtD mapping as the operator $T_N : L^2(\partial D) \mapsto V_N(\partial D)$ such that $w_N \in V_N(D)$ satisfies

$$
a(w_N, \varphi_N) = b(f, \varphi_N) \quad \text{for all} \quad \varphi_N \in V_N(D) \quad \text{where} \quad T_N f = w_N|_{\partial D}.
$$

It is clear that if (12) is well-posed then $T_N$ is a well defined compact operator. The goal now is to prove the well-posedness of the discrete source problem (12).
In [20] it is shown that the sesquilinear form \( a(\cdot, \cdot) + \alpha (\cdot, \cdot) \) is coercive on \( H^1(D) \) for \( \alpha > 0 \) sufficiently large. This implies that [12] is Fredholm of index zero and therefore uniqueness implies well-posedness. We now use a duality argument to prove uniqueness. To this end, we define \( u \in H^1(D) \) to be the unique solution to
\[
\Delta u + k^2 \bar{n} u = (w_N - w) \quad \text{in} \quad D \quad \text{and} \quad \partial_n u = 0 \quad \text{on} \quad \partial D
\]
where \( w \in H^1(D) \) is the solution to (7) and \( w \in V_N(D) \) is a solution to (12). By elliptic regularity (see for e.g. [14]) we have that the solution \( u \in H^2(D) \) and satisfies the regularity estimate
\[
\| u \|_{H^2(D)} \leq C \| w - w_N \|_{L^2(D)}.
\]
Therefore, by appealing to Green’s first Theorem we have that
\[
\| w - w_N \|_{L^2(D)}^2 = a(w - w_N, u) = a(w - w_N, u - \Pi_N u) \quad \text{by Galerkin orthogonality}
\leq C \| w - w_N \|_{H^1(D)} \| u - \Pi_N u \|_{H^1(D)}.
\]
By using Theorem 3.2 and the regularity estimate we have that
\[
\| w - w_N \|_{L^2(D)} \leq \frac{C}{(N + 1)^{1/d}} \| w - w_N \|_{H^1(D)}. \tag{13}
\]
Now, using the fact that \( a(\cdot, \cdot) + \alpha (\cdot, \cdot) \) is coercive on \( H^1(D) \) along with the Galerkin orthogonality and inequality (13) we have the estimates
\[
\| w - w_N \|_{H^1(D)} \leq C \left| a(w - w_N, w - w_N) + \alpha \| w - w_N \|_{L^2(D)}^2 \right|
\leq C \| w - w_N \|_{H^1(D)} \| w - \Pi_N w \|_{H^1(D)} + \frac{C}{(N + 1)^{2/d}} \| w - w_N \|_{H^1(D)}.
\]
This implies that for \( N \) sufficiently large we have the estimate
\[
\| w - w_N \|_{H^1(D)} \leq \| w - \Pi_N w \|_{H^1(D)}. \tag{14}
\]
Therefore, by inequality (14) and the well-posedness of (7) we can conclude that (12) is well-posed. This implies that the spectral approximation of the NtD mapping \( T_N \) is a well define compact operator.
Now, define the Neumann spectral-Galerkin approximation of the inverse scattering Steklov eigenvalue problem (5) to be given by: find the values $\lambda_N \in \mathbb{C}$ and nontrivial $w_N \in V_N(D)$ that satisfies the variational equality

$$a(w_N, \varphi_N) = -\lambda_N b(w_N, \varphi_N) \quad \text{for all } \varphi_N \in V_N(D). \quad (15)$$

Therefore, just as in Section 2 we have that $\lambda_N \neq 0$ satisfies (15) provided that

$$T_N w_N \big|_{\partial D} = -\lambda_N^{-1} w_N \big|_{\partial D} \quad \text{where } \|w_N\|_{L^2(\partial D)} = 1.$$ 

In order to prove the convergence of the Neumann spectral-Galerkin approximation we can use the results in [6, 24]. To this end, we are now ready to prove that the spectral approximation $T_N$ converges to $T$ in norm.

**Theorem 3.4.** Let the operators $T : L^2(\partial D) \mapsto L^2(\partial D)$ be as defined in (8) and $T_N : L^2(\partial D) \mapsto V_N(\partial D)$ be as defined in (12). Then

$$\|T - T_N\|_{L^2(\partial D) \mapsto L^2(\partial D)} \leq \frac{C}{(N + 1)^{1/2d}} \quad \text{as } N \to \infty.$$ 

**Proof.** To prove the claim, we have that for any $f \in L^2(\partial D)$ that

$$\|T - T_N\|_{L^2(\partial D) \mapsto L^2(\partial D)} \leq \frac{C}{(N + 1)^{1/2d}} \quad \text{by Theorem 1.6.6 in [7]}$$

$$\leq \frac{C}{(N + 1)^{1/2d}} \|w - w_N\|_{H^1(D)} \quad \text{by inequality (13)}$$

$$\leq \frac{C}{(N + 1)^{1/2d}} \|w - \Pi_N w\|_{H^1(D)} \quad \text{by inequality (14)}.$$ 

Now, by the uniform boundedness principle (see for e.g. [5]) we have that the operator norm of $I - \Pi_N : H^1(D) \mapsto H^1(D)$ is bounded uniformly with respect to $N$. Therefore, we have that

$$\|T - T_N\|_{L^2(\partial D) \mapsto L^2(\partial D)} \leq \frac{C}{(N + 1)^{1/2d}} \|w\|_{H^1(D)} \leq \frac{C}{(N + 1)^{1/2d}} \|f\|_{L^2(\partial D)}$$

by the well-posedness of [7], proving the claim. \hfill \Box

By the norm convergence of the spectral approximation of the NtD mapping we can conclude that convergence of the approximation of the Steklov eigenvalues. The following is a consequence of the results in [24] and Theorem 3.4.
Theorem 3.5. Let \((\lambda_N, w_N) \in \mathbb{C} \times V_N(D)\) be an eigenpair for (15). Then there is an eigenpair \((\lambda, w) \in \mathbb{C} \times H^1(D)\) for (5) such that

\[
|\lambda - \lambda_N| \leq \frac{C}{(N + 1)^{1/2d}} \quad \text{and} \quad \|w - w_N\|_{L^2(\partial D)} \leq \frac{C}{(N + 1)^{1/2d}} \quad \text{as} \quad N \to \infty.
\]

Theorem 3.5 gives the convergence of our approximation for the inverse scattering Steklov eigenvalues. We are now interested in determining a spectral convergence rate for our approximation. To do so, we denote the eigenspace associated with \(\lambda\) as \(E(\lambda)\) and recall the space \(D(\Delta^m)\) defined by the Fourier-Series constraint (11).

Theorem 3.6. Assume the eigenspace \(E(\lambda) \subset D(\Delta^m)\) for some \(m > 1/2\). Then for every eigenvalue \(\lambda_N\) for (15) there is an eigenvalue \(\lambda\) for (5) such that

\[
|\lambda - \lambda_N| \leq \frac{C}{(N + 1)^{(4m-1)/2d}} \sup_{w \in E(\lambda): \|w\|_{L^2(\partial D)} = 1} \|w\|_{D(\Delta^m)} \quad \text{as} \quad N \to \infty.
\]

Proof. To prove the estimate we use Theorem 7.3 in [6]. From this we have that we need to estimate

\[
\|T - T_N\|_{E(\lambda) \rightarrow L^2(\partial D)} = \sup_{w \in E(\lambda): \|w\|_{L^2(\partial D)} = 1} \|T - T_N\|_{L^2(\partial D)}
\]

in order to obtain the convergence rate for the eigenvalues. For any \(w \in E(\lambda)\) we have that \(\partial_\nu w = -\lambda w\) on \(\partial D\). Therefore, we have the estimates

\[
\|T - T_N\|_{L^2(\partial D)} \leq \frac{C}{(N + 1)^{1/2d}} \|I - \Pi_N w\|_{H^1(D)} \quad \text{by the proof of Theorem 3.4}
\]

\[
\leq \frac{C}{(N + 1)^{(4m-1)/2d}} \|w\|_{D(\Delta^m)} \quad \text{by Theorem 3.3}
\]

Now, by taking the supremum over \(E(\lambda)\) such that \(\|w\|_{L^2(\partial D)} = 1\) gives that

\[
\|T - T_N\|_{E(\lambda) \rightarrow L^2(\partial D)} \leq \frac{C}{(N + 1)^{(4m-1)/2d}} \sup_{w \in E(\lambda): \|w\|_{L^2(\partial D)} = 1} \|w\|_{D(\Delta^m)}
\]

which proves the claim. \(\square\)

Even though our main focus is on computing the eigenvalues we will consider the convergence of the eigenfunctions in the region \(D\). The following result gives the convergence of the eigenfunctions in the \(H^1(D)\) norm.
Theorem 3.7. Let \( w_N \in V_N(D) \) be an eigenfunction for (15). Then there is an eigenfunction \( w \in H^1(D) \) for (5) such that

\[
\| w - w_N \|_{H^1(D)} \leq \frac{C}{(N+1)^{1/4d}} \quad \text{as} \quad N \to \infty.
\]

Proof. To prove the claim we first note that by Theorem 3.4 we have that

\[
\| w - w_N \|_{L^2(\partial D)} \leq \frac{C}{(N+1)^{1/2d}} \quad \text{and} \quad |\lambda - \lambda_N| \leq \frac{C}{(N+1)^{1/2d}}.
\]

Since, \( w_N \) satisfies (12) with \( f = -\lambda_N w_N \). The well-posedness of (12) and converges estimates above implies that \( w_N \) is a bound sequence in \( H^1(D) \). Now, to prove the convergence we again use a duality argument. Therefore, we let \( u \in H^1(D) \) be the unique solution to

\[
\Delta u + k^2 \bar{\nu}u = (w_N - w) \quad \text{in} \quad D \quad \text{and} \quad \partial \nu u = 0 \quad \text{on} \quad \partial D.
\]

By elliptic regularity we have that \( u \in H^3(D) \) and is bounded with respect to \( N \). Green’s first Theorem and some simple calculations using (5) and (15) gives that

\[
\| w - w_N \|^2_{L^2(D)} = a(w - w_N, u)
\]

\[
= (\lambda_N - \lambda)b(w, u) + \lambda_N b(w - w_N, u)
\]

\[
- \lambda_N b(w_N, (I - \Pi_N)u) - a(w_N, (I - \Pi_N)u).
\]

Notice, that by the convergence rate of the eigenfunctions on \( \partial D \) and eigenvalues we have the estimate

\[
| (\lambda_N - \lambda)b(w, u) | \leq \frac{C}{(N+1)^{1/2d}} \quad \text{and} \quad | \lambda_N b(w - w_N, u) | \leq \frac{C}{(N+1)^{1/2d}}
\]

where we have also used the fact that \( u \) is bound in \( H^2(D) \). By appealing to the approximation rate in Theorem 3.2

\[
| \lambda_N b(w_N, (I - \Pi_N)u) | \leq \frac{C}{(N+1)^{1/4d}}\quad \text{and} \quad | a(w_N, (I - \Pi_N)u) | \leq \frac{C}{(N+1)^{1/4d}}
\]

where we have used the Trace Theorem and the fact that \( w_N \) is a bound sequence in \( H^1(D) \). This implies that

\[
\| w - w_N \|^2_{L^2(D)} \leq \frac{C}{(N+1)^{1/2d}}.
\]
Simple calculations give that for \((\lambda, w)\) and \((\lambda_N, w_N)\) eigenpairs for (5) and (15) then
\[
a(w_N - w, w_N - w) + \lambda b(w_N - w, w_N - w) = (\lambda - \lambda_N) b(w_N, w_N).
\]
Recall, that the sesquilinear form \(a(\cdot, \cdot) + \alpha (\cdot, \cdot)_{L^2(D)}\) is coercive on \(H^1(D)\) for \(\alpha > 0\) sufficiently large. Therefore, we have that
\[
\|w - w_N\|^2_{H^1(D)} \leq C \left| a(w - w_N, w - w_N) + \alpha \|w - w_N\|^2_{L^2(D)} \right|
\leq C \left\{ |\lambda - \lambda_N| + |\lambda b(w_N - w, w_N - w)| + \|w - w_N\|^2_{L^2(D)} \right\}.
\]
By combining the above estimates proves the claim.

4 Numerical Examples

This section is dedicated to providing numerical examples of our Neumann spectral-Galerkin method for computing the inverse scattering Steklov eigenvalues. The convergence rate will be numerically studied as well as examples for constant and variable refractive index \(n\). We also consider the inverse spectral problem of estimating/recovering the refractive index from the knowledge of the eigenvalues. This problem is also considered in \([21]\) where a Bayesian approach is used. Here we will use the monotonicity (see for e.g. \([3, 21]\)) of the largest positive eigenvalue denoted \(\lambda_1\) to estimate a positive refractive index.

We take the domain \(D\) to be given by the unit disk in \(\mathbb{R}^2\). Note that \(D\) can always be chosen to be a disk with radius sufficiently large such that the scatterer \(\Omega \subseteq D\). In the following examples, the approximation space is given by the span of finitely many Neumann eigenfunctions
\[
\phi_j(r, \vartheta) = J_p \left( \sqrt{\sigma_{p,q}} r \right) \cos(p \vartheta) \quad \text{with index } \ j = j(p, q) \in \mathbb{N}.
\]
The square root of the Neumann eigenvalues \(\sqrt{\sigma_{p,q}}\) corresponds to the \(q\)th non-negative root of the \(p\)th first kind Bessel function’s derivative denoted \(J'_p\) for all \(p \in \mathbb{N} \cup \{0\}\) and \(q \in \mathbb{N}\). Some of the values of \(\sqrt{\sigma_{p,q}}\) can be found in \([28]\).

We will use 25 basis functions where \(0 \leq p \leq 4\) and \(1 \leq q \leq 5\). In the following sections we take the approximation space
\[
V_N(D) \subseteq \text{Span}\left\{ \phi_j(p, q)(r, \vartheta) \right\}_{p=0, q=1}^{p=4, q=5} \quad \text{giving that } \ w_N(x) = \sum_{j=1}^{N} c_j \phi_j(x)
\]
for some constants $c_j$. By substitution $w_N(x)$ into (15) and taking $\varphi_N = \phi_i(x)$ we obtain that the eigenvalues $\lambda_N$ satisfying (15) correspond to the eigenvalues for the matrix equation

$$(\mathbf{A} + \lambda_N \mathbf{B}) \mathbf{c} = 0 \quad \text{where} \quad A_{i,j} = a(\phi_j, \phi_i) \quad \text{and} \quad B_{i,j} = b(\phi_j, \phi_i).$$

(16)

For the examples presented this method is implemented in MATLAB where the ‘eig’ command is used to solve 16. To compute the Galerkin matrices we employ a 2d Gaussian quadrature scheme. We also note that the orthogonality of the basis functions is used in our implementation.

### 4.1 Comparison to Separation Variables

In this section, we will compare our Neumann spectral-Galerkin approximation to the analytically computed eigenvalue for the unit disk. To due so, assuming that $\Omega = D$ is given by the unit disk in $\mathbb{R}^2$ then in [9] we have that for $n$ constant that the eigenvalues can be determined by separation of variables. This gives that

$$\lambda = -k\sqrt{n}J'_m(k\sqrt{n}) J_m(k\sqrt{n}) \quad \text{for any} \ m \geq 0.$$ (17)

We will test the accuracy of the Spectral approximation by comparing it to the values given by (17). In our examples, we will take $n$ to be real and complex-valued to show that the approximation is valid for either case. Also, for all our numerical experiments in this and the following section, we will take the wavenumber $k = 1$ for simplicity. In Tables 1 and 2 we present the approximated eigenvalue $\lambda_{1,N}$ for various degrees of freedom $N$ as well as the relative error. In Figure 1 the log-log convergence plots for the eigenvalues are presented. From the convergence plot we see that the convergence seems to be $O(N^{-1})$ in the two examples.

| $N$ | $\lambda_{1,N}$ | $\text{Rel. Error}$ |
|-----|------------------|---------------------|
| 10  | 1.1872162        | 0.1378900           |
| 15  | 1.2500365        | 0.0922724           |
| 20  | 1.2816379        | 0.0693246           |
| 25  | 1.3007182        | 0.0554693           |

Table 1: The first approximated eigenvalue for various $N$ with $n = 2$ to demonstrate the convergence to $\lambda_1 = 1.3771053$ as $N \to \infty$. The approximate convergence rate for this example is computed to be 0.993592.
Table 2: The first approximated eigenvalue for various $N$ with $n = 2 + i$ to demonstrate the convergence to $\lambda_1 = 1.17422 + 0.92123i$ as $N \to \infty$. The approximate convergence rate is computed to be 1.0036874.

| $N$ | $\lambda_{1,N}$ | Rel. Error |
|-----|------------------|------------|
| 10  | 1.10178          | 0.151973   |
| 15  | 1.12973          | 0.1011953  |
| 20  | 1.14240          | 0.0758263  |
| 25  | 1.14957          | 0.0605793  |

We will now show that our numerical scheme is valid for a piecewise constant refractive index in $D$. To this end, assume that $D$ is the unit disk and the scatterer $\Omega$ is given by the disk with radius $\rho < 1$. Now, define the refractive index

$$n = \begin{cases} 
1, & \rho < r \\
(1 + n_1)^2, & r \leq \rho
\end{cases}$$

where $n_1$ is a positive constant. In [9] it is shown using separation of variables and the asymptotic expansions of the Bessel function’s that the first eigenvalue is given by the following expansion for $k\rho \ll 1$

$$\lambda_1 = -k \frac{J'_0(k)}{J_0(k)} + \frac{1}{2} n_1 (2 + n_1) (k\rho)^2 + \mathcal{O}((k\rho)^4).$$

Using this expansion for the first eigenvalue we will compare with our numerically approximated eigenvalue. One would expect that the values be close provided $k\rho$
is sufficiently small. In Table 3 we report the eigenvalues computed for $\rho = 1/2^p$ by the Spectral approximation as well as the values from the first two terms in the asymptotic expansion \([15]\) for various values of $p$.

| Approximation $\lambda_{1,N}$ | Asymptotic Formula |
|--------------------------------|---------------------|
| $p = 1$                        | 0.780984210069194   |
| $p = 2$                        | 0.617530179557115   |
| $p = 3$                        | 0.581111365230462   |
| $p = 4$                        | 0.565820243626941   |

Table 3: Comparison with the asymptotic formula \([18]\) for $n = 2$ where the scatterer is given by the disk with $\rho = 1/2^p$ for $p = 1, 2, 3, 4$.

### 4.2 Parameter Estimation

In this section, we provide a simple method for estimating the refractive index from the knowledge of the inverse scattering Steklov eigenvalues. It has been shown in \([3,9]\) that the Steklov eigenvalues can be recovered from the knowledge of the far-field data via the Linear Sampling Method and Generalized Linear Sampling Method. In \([21]\) the eigenvalues are recovered from near-field measurement by using the Reciprocity Gap Method. Therefore, for simplicity, we will use the eigenvalues computed by the Neumann spectral-Galerkin method as a stand-in for the eigenvalues computed from the data and we wish to estimate $n$.

To begin, we present the numerical approximation of the Steklov eigenvalues and eigenfunctions for a variable refractive index $n$. The eigenvalues presented here will be used in the approximation of $n$. In Table 4 we report the first three eigenvalues where the scatterer is either the unit disk or disk with radius $\rho = 1/2$ where the refractive index is given by $n = 2 + r(\sin \theta - \cos \theta)$. Since we have also proven the convergence of the eigenfunctions, we give the contour plots for the first three Steklov eigenfunctions associated with the eigenvalues in Figure 2.

| Disk w/ radius $\rho$ | 1st eigenvalue $\lambda_{1,N}$ | 2nd eigenvalue $\lambda_{2,N}$ | 3rd eigenvalue $\lambda_{3,N}$ |
|----------------------|--------------------------------|--------------------------------|-------------------------------|
| $\rho = 1$           | 1.33947280348                  | -0.47739381775                 | -1.75712435055                |
| $\rho = 1/2$         | 0.78174886356                  | -0.74001156781                 | -1.95378594455                |

Table 4: The first three eigenvalues for two different scatterers where the refractive index is given by $n = 2 + r(\sin \theta - \cos \theta)$. 

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Figure 2: Plots of the first three eigenfunctions for the unit disk and disk with radius 1/2 scatterers with refractive index $n = 2 + r(\sin \theta - \cos \theta)$. The dotted line is the boundary of the scatterer.

Now, we report the eigenvalues for a constant refractive index where the scatterer $\Omega \neq D$. We consider the boundary of the scatterer to be given in polar coordinates such that $\partial \Omega = \rho(\theta)(\cos \theta, \sin \theta)$ where $0 < \rho(\theta) < 1$ is a $2\pi$-periodic function. Here we consider a pear, elliptical, and rounded-square shaped scatterer given by

\[
\rho(\theta) = \begin{cases} 
0.3(2 + 0.3\cos(3\theta)), \\
0.35(2 + 0.3\sin(2\theta)) \quad \text{and} \\
0.75(|\sin(\theta)|^{5} + |\cos(\theta)|^{5})^{-1/5}
\end{cases}
\]

respectively. The eigenvalues are reported in Table 5 and the associated eigenfunctions are plotted in Figure 3.

| Scatterer      | 1st eigenvalue $\lambda_{1,N}$ | 2nd eigenvalue $\lambda_{2,N}$ | 3rd eigenvalue $\lambda_{3,N}$ |
|----------------|-------------------------------|-------------------------------|-------------------------------|
| Pear-Shaped    | 0.89339093521                 | -0.70841945488                | -1.94018366846                |
| Elliptical-Shaped | 0.97880829577                | -0.67854111485                | -1.93207985011                |
| Rounded-Square | 1.11759427187                | -0.60744622788                | -1.90328635229                |

Table 5: The first three eigenvalues for three different scatterers with $n = 2$. 

Lastly, we turn our attention to estimating the refractive index. To this end, we will assume that the scatterer $\Omega$ is known and begin with the case when $D = \Omega$. The method proposed here is to approximate $n$ by a constant value. Therefore, in order to approximate $n(x)$ we find the unique value $n_{\text{approx}}$ that is the solution to

$$\lambda_1(n_{\text{approx}}) = \lambda_{1,N}(n)$$

(19)

where $\lambda_1$ given by equation (17) for $m = 0$ is the largest positive inverse scattering Steklov eigenvalue. To solve the above transcendental equation (19) we use the ‘fzero’ command in MATLAB. As we see in our examples the approximation $n_{\text{approx}}$ seems to be the average value of $n$ in $D$ just as the case for the transmission eigenvalues [11, 18]. Therefore, if we will assume that the solution to (19) approximates
the average value of $n$ over $D$. Since we know a priori that $n = 1$ in $D \setminus \overline{\Omega}$ we can use a two step process to estimate $n$.

- Step 1: Solve (19) to determine an initial $n_{\text{approx}}$.
- Step 2: Define the new approximation $n_{\text{approx},1}$ such that $n = 1$ for $x \in D \setminus \overline{\Omega}$ and $n = n_{\text{approx},1}$ for $x \in \Omega$ where the constant $n_{\text{approx},1}$ is given by

$$n_{\text{approx},1} = \frac{n_{\text{approx}}|D| - |D \setminus \overline{\Omega}|}{|\Omega|}. \quad (20)$$

Here $|\cdot|$ denotes the area of a Lebesgue measurable set in $\mathbb{R}^2$. Equation (20) is obtained by the assumption that the initial estimate $n_{\text{approx}}$ is the average value of $n$ in $D$. This method is implemented for the eigenvalues presented in Tables 4 and 5 where the approximations of the refractive index $n$ are reported in Table 6.

| Scatterer                  | Refractive Index $n$                                      | Approximation of $n$ |
|----------------------------|----------------------------------------------------------|----------------------|
| Disk w/ $\rho = 1$        | $n = 2 + r(\sin \theta - \cos \theta)$                 | 1.961032             |
| Disk w/ $\rho = 1/2$      | $n = 2 + r(\sin \theta - \cos \theta)$                 | 2.181511             |
| Disk w/ $\rho = 1$        | $n = 2$                                                  | 1.920193             |
| Pear-Shaped               | $n = 2$                                                  | 1.894312             |
| Elliptical-Shaped         | $n = 2$                                                  | 2.111828             |
| Rounded-Square            | $n = 2$                                                  | 2.053623             |

Table 6: Here we approximate the refractive index $n$ by a constant in the scatterer $\Omega$ for multiple shaped scatterers.

5 Summary and Conclusions

In conclusion, we have provided a numerical method for computing the inverse acoustic scattering Steklov eigenvalues via the Neumann spectral-Galerkin approximation method. The approximation space is taken to be the span of the first $N$ Neumann eigenfunctions for the Laplacian in $D$. The analysis presented here is valid for any chosen auxiliary domain $D$ were the sufficiently smooth boundary $\partial D$ for $d = 2, 3$. One needs to have computed the Neumann eigenfunction for the domain D to employ this method. In the application of inverse scattering that is the focus of this
paper, the domain $D$ can be chosen to be a disk that contains the scatterer. Since the Neumann eigenfunctions for a disk are well known via separation for variables this method can be always be applied for this problem. We have presented numerical examples to validate the theoretical results as well as investigated estimating the refractive index from the first eigenvalue. Another possible application of this method is to use the Neumann spectral-Galerkin method to compute the inverse scattering Trace Class Stekloff eigenvalues studied in [13]. This is a new modified Stekloff eigenvalue problem whose numerical approximation by Galerkin methods has not been rigorously analyzed. Also, for multiple scatterers, one can try and augment the method presented in [16] to recover the refractive index.

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