ON THE CLASSIFICATION OF TORIC FANO 4-FOLDS

VICTOR V. BATYREV

Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
e-mail: batyrev@bastau.mathematik.uni-tuebingen.de

Abstract

The biregular classification of smooth $d$-dimensional toric Fano varieties of dimension $d$ is equivalent to the classification of special simplicial polyhedra $P$ in $\mathbb{R}^d$, so called Fano polyhedra, up to an isomorphism of the standard lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. In this paper we explain the complete biregular classification of all 4-dimensional smooth toric Fano varieties. The main result states that there exist exactly 123 different types of toric Fano 4-folds up to isomorphism.

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1 Introduction

A smooth projective $d$-dimensional algebraic variety $V$ over $\mathbb{C}$ is called a Fano manifold if the anticanonical sheaf of $V$ is ample. In the case $d = 2$ the surface $V$ is also called a Del Pezzo surface. It is a classical result that all Del Pezzo surfaces can be obtained from $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing ups of respectively $r \leq 8$ and $r \leq 7$ points in general position (see e.g. [23]). This shows that there exist exactly 10 different types of Del Pezzo surfaces up to deformation.

First fundamental results towards the complete classification of Fano 3-folds up to deformation were received by Iskovskih [17, 18] more than 20 years ago. The complete classification of Fano 3-folds has been obtained later in the papers of Mori and Mukai [23, 27, 28] (see also [29, 11]). It has been proved by Kollár, Miyaoka and Mori [20, 21] that there exists only finitely many smooth Fano manifolds of fixed dimension $d$ up to deformation. For Fano $d$-folds with Picard number one this statement was independently obtained by Nadel [30]).

The complete classification of all Fano 4-folds up to deformation still remains an open problem. One expects that the complete list of Fano 4-folds up to deformation could include many hundreds of different types. For this reason, it looks more meaningful to restrict the classification of Fano manifolds of dimension $d \geq 4$ to some special classes for which the complete list is no so large.

In the present paper we consider Fano $d$-folds which are simultaneously toric varieties, i.e., admitting a regular effective action of a $d$-dimensional algebraic torus $(\mathbb{C}^*)^d$ [12, 15, 34]. Since toric varieties are determined by some discrete combinatorial data, they do not admit nontrivial algebraic deformations. For this reason, it makes sense to classify toric Fano $d$-folds up to biregular isomorphism. Toric Fano manifolds of dimension $d \leq 3$ have been classified in the author’s paper [1] and independently in the paper of K.Watanabe & M.Watanabe [37]. There exist exactly 5 different toric Del Pezzo surfaces and exactly 18 different toric Fano 3-folds. Some special classes of $d$-dimensional Fano manifolds of arbitrary dimension $d$ have been classified by Voskresenskiî & Klyachko [36] and Ewald [14].

It is important to stress that the classification problem of toric Fano $d$-folds up to isomorphism can be reformulated purely into a combinatorial classification problem of special convex polyhedra $P \subset \mathbb{R}^d$, called Fano polyhedra, up to linear unimodular transformation from $GL(d, \mathbb{Z})$. Every $d$-dimensional Fano polyhedron $P$ determines a toric Fano $d$-fold $V(P)$. Fano polyhedra form a special subclass of reflexive polyhedra which were introduced in [6]. We want to remark that an explicit classification of higher dimensional Fano polyhedra $P$ up to unimodular transformations needs a special method for describing such polyhedra. For instance, if we described a $d$-dimensional Fano polyhedron $P$ with $n$ vertices just by the $d \times n$-matrix $M(P)$ consisting of the coordinates of the vertices, then we would meet the following two difficulties:

i) the $d \times n$-matrix $M(P)$ doesn’t show the combinatorial structure of $P$ which contain a lot of information about the geometry of the corresponding toric Fano $d$-fold $V(P)$;

ii) having two $d \times n$-matrices $M(P_1)$ and $M(P_2)$ which describe given Fano
polyhedra $P_1$ and $P_2$, it is not easy to decide whether $P_1$ and $P_2$ are equivalent up to an unimodular linear transformation or not.

Both these difficulties disappear if one uses the language of primitive collections and primitive relations for describing toric Fano $d$-folds. This language was introduced in [5] and developed in [7] in the connection with [10].

The main result of the paper is the complete classification of toric Fano 4-folds up to isomorphism. This classification was obtained in the author’s PhD thesis [3]. In Diplomarbeit [13] S. Evertz independently has classified 4-dimensional Fano having 8 vertices using some computer calculations. Both papers [3] and [13] have many common ideas and both use the combinatorial classification of simplicial 4-dimensional polyhedra with 8 vertices due to Grunbaum and Streedharan [16]. Comparing the list of 4-dimensional Fano polyhedra with 8 vertices in [3] and [13], some discrepancy in results has been observed: only 3 polyhedra of the combinatorial type $P^8_{26}$ (instead of 5 [3]) were found in [13], and 2 Fano polyhedra of the combinatorial type $P^8_{28}$ from [13] were missing in the list of [3] (the combinatorial type $P^8_{28}$ was excluded from considerations in [3] by some error).

The classification of toric Fano 4-folds was used by Y. Nakagawa in [32, 33] for finding all toric Fano 4-folds which admit Einstein-Kähler metrics. This result of Nakagawa extends the one of Mabuchi [22] for toric Fano 3-folds.

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2 Toric Fano manifolds of dimension $d$

2.1 Fano polyhedra

Definition 2.1.1 Let $P$ be a convex polyhedron in $\mathbb{R}^d$, $V(P) = \{v_1, \ldots, v_n\}$ the set of all vertices of $P$. We call $P$ a Fano polyhedron if the following conditions are satisfied:

(i) the elements of $V(P)$ belong to the standard integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$;
(ii) $P$ contains the lattice element $(0, \ldots, 0) \in \mathbb{Z}^d$ in its interior;
(iii) $P$ is a simplicial polyhedron, i.e., each face of $P$ is a simplex;
(iv) vertices $v_{i_1}, \ldots, v_{i_d}$ of any $(d - 1)$-dimensional face $F = [v_{i_1}, \ldots, v_{i_d}]$ of $P$ form a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^d$, i.e., the coordinates of $v_{i_1}, \ldots, v_{i_d}$ form a matrix $A$ with $\det A = \pm 1$.

Definition 2.1.2 Let $P$ be a Fano polyhedron and $\{v_{i_1}, \ldots, v_{i_m}\}$ be the set of vertices of a $(k - 1)$-dimensional face of $F \subset P$ ($1 \leq m \leq d$). We denote by $\sigma(F)$ the $m$-dimensional cone consisting of all nonnegative $\mathbb{R}$-linear combinations of $v_{i_1}, \ldots, v_{i_m}$, i.e.,

$$\sigma(F) := \{\lambda_1 v_{i_1} + \ldots + \lambda_m v_{i_m} \in \mathbb{R}^n : \lambda_i \geq 0 (1 \leq i \leq m)\}.$$
The system of cones
\[ \Sigma(P) = \{0\} \cup \{\sigma(F)\}_{F \subseteq P}, \]
where \( F \) runs over all proper faces of \( P \), we call the \textbf{polyhedral fan} associated with the Fano polyhedron \( P \).

**Definition 2.1.3** Let \( P \) be a Fano polyhedron. A subset
\[ \mathcal{P} = \{v_i, \ldots, v_k\} \subset V(P) \]
will be called a \textbf{primitive collection} if it satisfies the conditions
(i) \( \mathcal{P} \) does not belong to a cone from \( \Sigma(P) \);
(ii) any proper subset of \( \mathcal{P} \) is contained in some cone from \( \Sigma(P) \).

**Definition 2.1.4** Let \( P \) be a Fano polyhedron and \( \mathcal{P} = \{v_i, \ldots, v_k\} \) be a primitive collection of its vertices. Denote by \( \sigma(\mathcal{P}) \) the cone of minimal dimension in \( \Sigma(P) \) containing the integral point \( s(\mathcal{P}) = v_i + \ldots + v_k \) (Since the cones of the fan \( \Sigma(P) \) cover the whole space \( \mathbb{R}^d \), the point \( s(\mathcal{P}) \) must belong to at least one cone \( \sigma(F) \subset \Sigma(P) \) for some face \( F \subset P \).) Let \( v_{j_1}, \ldots, v_{j_m} \) are generators of \( \sigma(\mathcal{P}) \). Then the element \( s(\mathcal{P}) \) has a unique representation as a positive integral linear combination of \( v_{j_1}, \ldots, v_{j_m} \):
\[ s(\mathcal{P}) = c_1 v_{j_1} + \ldots + c_m v_{j_m}, \quad c_i > 0, \quad c_i \in \mathbb{Z}. \]
(If \( s(\mathcal{P}) = 0 \), then the set \( \{v_{j_1}, \ldots, v_{j_m}\} \) is assumed to be empty.) We will call the linear relation
\[ R(\mathcal{P}) : v_i + \ldots + v_k - c_1 v_{j_1} - \ldots - c_m y_{j_m} = 0 \]
among the vertices of \( P \) the \textbf{primitive relation} corresponding to the primitive collection \( \mathcal{P} \). The positive integers \( c_1, \ldots, c_m \) we call the \textbf{coefficients} of the primitive relation \( R(\mathcal{P}) \). The integer
\[ \Delta(\mathcal{P}) := k - \sum_{i=1}^{m} c_i \]
we call the \textbf{degree} of the primitive collection \( \mathcal{P} \).

Our next purpose is to show that primitive relations are useful for combinatorial descriptions of Fano polyhedra \( P \).

**Definition 2.1.5** Let \( P_1 \) and \( P_2 \) be two Fano polyhedra. Then \( P_1 \) and \( P_2 \) are called \textbf{combinatorially equivalent} if and only if there exists a bijective mapping
\[ \rho : V(P_1) \to V(P_2) \]
which respects the face-relation, i.e. , \( v_{i_1}, \ldots, v_{i_k} \) are vertices of a \((k-1)\)-dimensional face of \( P_1 \) if and only if \( \rho(v_{i_1}), \ldots, \rho(v_{i_k}) \) are vertices of a \((k-1)\)-dimensional face of \( P_2 \).
By Definition 2.1.3, one immediately obtains the following characterization of the combinatorial equivalence class of $P$ using primitive collections:

**Proposition 2.1.6** Let $P_1$ and $P_2$ be two Fano polyhedra. Then $P_1$ and $P_2$ are combinatorically equivalent if and only if there exists a bijective mapping

$$\rho : \mathcal{V}(P_1) \to \mathcal{V}(P_2)$$

which induces a one-to-one correspondence between primitive collections of vertices in $\mathcal{V}(P_1)$ and $\mathcal{V}(P_2)$.

**Definition 2.1.7** Two Fano polyhedra $P_1$ and $P_2$ are called **isomorphic** if there exists an automorphism $\rho \in \text{GL}(d, \mathbb{Z})$ of the lattice $\mathbb{Z}^d$ such that $\rho(P_1) = P_2$.

Assume that two Fano polyhedra $P_1$ and $P_2$ are isomorphic. Then the induced bijection $\rho : \mathcal{V}(P_1) \to \mathcal{V}(P_2)$ between vertices obviously respects linear relations among them. It is less evident to see that the opposite statement holds, so that we have the following:

**Proposition 2.1.8** Two Fano polyhedra $P_1$ and $P_2$ are isomorphic if and only if there exist a bijective mapping between $\mathcal{V}(P_1)$ and $\mathcal{V}(P_2)$ which respects not only primitive collections of vertices in $\mathcal{V}(P_1)$ and $\mathcal{V}(P_2)$, but also the corresponding primitive relations.

The less evident part ”if” in 2.1.8 follows immediately from the following statement which we formulate without proof:

**Lemma 2.1.9** Let $P$ be a $d$-dimensional Fano polyhedron having $n$ vertices. Denote by $L(P)$ sublattice of rank $n - d$ in $\mathbb{Z}^n$ consisting of all linear relations with integral coefficients among elements of $\mathcal{V}(P)$. Then $L(P)$ is generated by primitive relations.

**Proposition 2.1.10** Let $P$ be a Fano polyhedron and $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_k}\} \subset \mathcal{V}(P)$ is a primitive collection. Then $\Delta(\mathcal{P}) > 0$. In particular, exist only finitely many possibilities for the coefficients $c_i$ in the primitive relation $R(\mathcal{P})$ if $P$ has a fixed dimension $d$.

**Proof.** It follows from 2.1.3 that the rational point

$$r(\mathcal{P}) = \frac{1}{k} (v_{i_1} + \ldots + v_{i_k})$$

belongs to the interior of $P$. Therefore, the corresponding primitive relation $R(\mathcal{P})$ has positive integer coefficients $\{c_i\}$ satisfying the inequality

$$c_1 + \ldots + c_m < k.$$

Obviously, $k \leq \text{dim } P + 1 = d + 1$. This implies that there exists only a finite number of possibilities for positive integers $c_i$ if the dimension $d$ is fixed. □
Proposition 2.1.11  The number of vertices \( n = n(P) \) of any \( d \)-dimensional Fano polyhedron \( P \) is not greater than \( 2(2^d - 1) \).

Proof. Consider the canonical surjective homomorphism
\[
\alpha : \mathbb{Z}^d \to \mathbb{Z}^d/(2\mathbb{Z})^d.
\]

First we remark that no vertex \( v \) of \( P \) belongs to the kernel of \( \alpha \). Otherwise all coordinates of \( v \) would be divisible by 2. This contradicts to 2.1.1(iv).

Assume now that \( \alpha(v_i) = \alpha(v_j) \neq 0 \) for two different vertices \( v_i, v_j \in P \). Then \((v_i + v_j)/2\) is an integral point belonging to \( P \). Since the only lattice points belonging to the boundary of \( P \) are its vertices and \((v_i + v_j)/2\) can not be a vertex of \( P \) (see 2.1.1(iv)), we conclude that \((v_i + v_j)/2 = 0\). Thus \( v_i \) and \( v_j \) are centrally symmetric vertices of \( P \). This implies that preimage \( \alpha^{-1}(x) \) of any nonzero element \( x \in \mathbb{Z}^d/(2\mathbb{Z})^d \) contains at most two different vertices of \( P \). Since \( \mathbb{Z}^d/(2\mathbb{Z})^d \) contains \( 2^d - 1 \) nonzero elements, we obtain the required upper bound for the number \( n(P) \).

\[\blacksquare\]

Remark 2.1.12 The above exponential upper bound for the number of vertices of \( P \) is sharp only for \( d = 1, 2 \). A better polynomial upper bound
\[n(P) \leq d^2 + 1, \text{ for } d > 2\]
has been obtained by Klyachko and Voskreshenskiï in [30].

Theorem 2.1.13 Let \( d \) be a fixed positive integer. Then there exist only finitely many \( d \)-dimensional Fano polyhedra up to isomorphism.

Proof. By 2.1.6 and 2.1.11, there exist only finitely many different combinatorial types of \( d \)-dimensional Fano polyhedra. On the other hand, for a fixed combinatorial type of \( P \) and for a fixed primitive collection \( \mathcal{P} \subset \mathcal{V}(P) \) there exist only finitely many possibilities for the corresponding primitive relation \( R(\mathcal{P}) \) (see 2.1.10). Therefore there exist only finitely many possibilities for primitive relations among vertices of a Fano polyhedron of a fixed dimension \( d \). It remains to apply 2.1.8. \(\blacksquare\)

Remark 2.1.14 Theorem 2.1.13 was proved in [3, 5, 30] by other methods.

2.2 Toric Fano \( d \)-folds

Let \( P \) be a \( d \)-dimensional Fano polyhedron. It follows from the theory of toric varieties [12, 13, 83] that the fan \( \Sigma(P) \) defines a smooth projective toric variety \( V(P) \) having ample anticanonical sheaf, i.e., a toric Fano \( d \)-fold. For convenience, we give an explicit geometric description of the toric Fano \( d \)-fold \( V(P) \) using our language of primitive collections and primitive relations:
**Definition 2.2.1** Let $\mathbb{A}^n$ be $n$-dimensional affine space over $\mathbb{C}$ with the complex coordinates $z_1, \ldots, z_n$, where $n$ is the number of vertices of $P$. We establish the one-to-one correspondence $z_i \leftrightarrow v_i$ between the coordinates $z_1, \ldots, z_n$ and the vertices $v_1, \ldots, v_n$. If $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_k}\} \subset \mathcal{V}(P)$ be a primitive collection, then we define an affine subspace $\mathbb{A}(\mathcal{P}) \subset \mathbb{A}^n$:

$$\mathbb{A}(\mathcal{P}) := \{(z_1, \ldots, z_n) \in \mathbb{A}^n : z_{i_1} = \cdots = z_{i_k} = 0\}.$$ 

Denote by $U(\mathcal{P})$ the Zariski open subset

$$U(\mathcal{P}) := \mathbb{A}^n \setminus \bigcup_{\mathcal{P} \subset \mathcal{V}(P)} \mathbb{A}(\mathcal{P}).$$

**Definition 2.2.2** Let $L(\mathcal{P}) \subset \mathbb{Z}^n$ be the $(n - d)$-dimensional sublattice consisting of integral linear relations among vertices of $P$, i.e.,

$$L(\mathcal{P}) := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 v_1 + \cdots + \lambda_n v_n = 0\}.$$ 

For every element $\lambda = (\lambda_1, \ldots, \lambda_n)$ we define a 1-parameter subgroup $T_\lambda$ in $(\mathbb{C}^*)^n$ acting canonically on $\mathbb{A}^n$ as follows

$$t(z_1, \ldots, z_n) = (t^{\lambda_1} z_1, \ldots, t^{\lambda_n} z_n), \ t \in \mathbb{C}^*.$$ 

Denote by $T(\mathcal{P})$ the $(n - d)$-dimensional algebraic subgroup in $(\mathbb{C}^*)^n$ generated by all 1-parameter subgroups $T_\lambda, \ \lambda \in L(\mathcal{P})$.

**Definition 2.2.3** We define the toric manifold $V(\mathcal{P})$ as the quotient of $U(\mathcal{P})$ modulo the canonical linear action of $T(\mathcal{P})$ on $U(\mathcal{P})$ (it is easy to see that $T(\mathcal{P})$ acts free on $U(\mathcal{P})$).

**Theorem 2.2.4** The toric manifold $V(\mathcal{P})$ is a smooth projective $d$-dimensional toric variety with ample anticanonical sheaf. Any smooth compact $d$-dimensional toric variety $V$ having ample anticanonical sheaf, i.e., a toric Fano manifold, is isomorphic to $V(\mathcal{P})$ for some $d$-dimensional Fano polyhedron $P$. Moreover, two Fano varieties $V(P_1)$ and $V(P_2)$ corresponding to two $d$-dimensional Fano polyhedra $P_1$ and $P_2$ are biregular isomorphic (as abstract algebraic varieties) if and only if $P_1$ and $P_2$ are isomorphic (as Fano polyhedra).

**Proof.** We explain only the way how an isomorphism of $V(P_1)$ and $V(P_2)$ as abstract complex manifolds implies isomorphism of the corresponding polyhedra $P_1$ and $P_2$. The rest part of the proof is an easy consequence of the standard theory of toric varieties $[12, 13, 34]$.

Let $\psi : V(P_1) \to V(P_2)$ be a biregular isomorphism of two toric Fano $d$-folds. Let $\mathbb{G}_i := Aut(V(P_i)) (i = 1, 2)$ be the algebraic group of biregular authomorphisms
of $V(P_i)$. Denote by $T_i \subset G_i (i = 1, 2)$ the maximal $d$-dimensional torus canonically embedded as open subset in $V(P_i)$. The isomorphism $\psi$ induces an isomorphism
\[ \tilde{\psi} : G_1 \rightarrow G_2 \]
\[ \gamma \mapsto \psi \circ \gamma \circ \psi^{-1}, \quad \gamma \in G_1. \]

Now we show that there exists an isomorphism $\phi : V(P_1) \rightarrow V(P_2)$ such that the corresponding isomorphism $\varphi : G_1 \rightarrow G_2$ has the property $\varphi(T_1) = T_2$. Indeed, $\tilde{\psi}(T_1)$ and $T_2$ are two maximal tori in the linear algebraic group $G_2$. By the well-known theorem of Borel [3], $T_2$ and $\tilde{\psi}(T_1)$ are conjugate by some element $\gamma_0 \in G_2$, i.e., $\tilde{\psi}(T_1) = \gamma_0^{-1} T_2 \gamma_0$. Now we define $\varphi := \gamma_0 \circ \psi$. By our definition, we have
\[ \varphi(T_1) = \gamma_0(\tilde{\psi}(T_1)) \gamma_0^{-1} = T_2. \]

Therefore $\varphi$ is an isomorphism of toric varieties $V(P_1)$ and $V(P_2)$ which respects the torus actions, i.e., an equivariant isomorphism. It follows from the theory of toric varieties that any equivariant isomorphism of toric varieties is determined by an isomorphism of fans $\Sigma(P_1)$ and $\Sigma(P_2)$. This isomorphism determines an isomorphism of Fano polyhedra $P_1$ and $P_2$. \hfill \Box

2.3 Primitive relations and extremal rays of Mori

The theory of extremal rays introduced by Mori [24, 25] was the main technical tool in the classification of Fano 3-folds [26, 27, 28]. It is natural to expect a similar role of extremal rays in the classification of toric Fano $d$-folds. The language of primitive relations turns out to be very convenient for this purpose:

**Theorem 2.3.1** The group $A_1(V(P))$ of 1-dimensional cycles on $V(P)$ relative to numerical equivalence is canonically isomorphic to the $(n - d)$-dimensional lattice $L(P)$ of integral relations among vertices of $P$. Moreover, the cone of Mori $\overline{NE}(V(P)) \subset A_1(V(P)) \otimes \mathbb{R}$ is canonically isomorphic to the cone generated by all primitive relations
\[ v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_m v_{j_m} = 0 \]
which we consider as elements of $L(P) \subset \mathbb{Z}^n$.

**Remark 2.3.2** By theorem [2.3.1], every 1-dimensional face of $\overline{NE}(V(P))$, e.g., an extremal ray, is generated by a primitive relation. However, the converse is not true in general. There might be primitive relations which corresponds interior lattice points on faces of dimension $\geq 2$ in $\overline{NE}(V(P))$. 8
Theorem 2.3.3 Assume that a primitive relation

\[ R(P) : v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_m v_{j_m} = 0 \]

defines a generator \( \lambda_P \) of some 1-dimensional face of \( \text{NE}(V(P)) \) (e.g., this condition is always satisfied if \( \Delta(P) = 1 \)). Then the following statements hold:

(i) The vertices \( v_{j_1}, \ldots, v_{j_m} \) together with any \( k-1 \) vertices from \( \{v_{i_1}, \ldots, v_{i_k}\} \) are generators of a \( (m+k-1) \)-dimensional cone in \( \Sigma(P) \).

(ii) A 1-dimensional cycle on \( V(P) \) representing the class of \( R(P) \) in \( A_1(V(P)) \) is a smooth projective rational curve with the normal bundle

\[
\bigoplus_{2}^{k-2} \mathcal{O}(1) \oplus \bigoplus_{d+1-k-m} \mathcal{O} \oplus \bigoplus (-c_1) \oplus \cdots \oplus \mathcal{O}(-c_m).
\]

In particular, the anticanonical degree of \( \lambda_P \in A_1(V(P)) \) equals \( \Delta(P) \).

(iii) If \( m = 1 \) (i.e., \( v_{i_1} + \cdots + v_{i_k} - c v_j \) is a primitive relation), then the exceptional locus of the extremal contraction corresponding to \( \lambda_P \) is a divisor \( D_j \) on \( V(P) \) which is a \( \mathbb{P}^{k-1} \)-bundle over a smooth \( (d-k) \)-dimensional toric variety \( W \).

Corollary 2.3.4 Let \( P \) be a Fano \( d \)-polyhedron, and let \( V \) be the corresponding toric Fano \( d \)-fold. Assume that there exists a primitive relation \( R(P) \) among vertices of \( P \) as follows

\[ v_i + v_j - v_l = 0. \]

Then \( R(P) \) gives rise to an extremal ray with the extremal contraction \( V \to V' \) to a smooth projective toric variety \( V' \) (\( V' \) is not necessary a Fano \( d \)-fold in general).

Remark 2.3.5 The statements of 2.3.1 and 2.3.3 can be obtained from the Mori’s theory for projective toric varieties (see [33] or [34], 2.5).

The language of primitive collections and primitive relations among generators of 1-dimensional cones in \( \Sigma \) can be used in description of arbitrary (not necessary Fano) smooth compact toric varieties \( \mathbb{P}_\Sigma \) (see [3]):

Proposition 2.3.6 Let \( \mathbb{P}_\Sigma \) be a compact smooth \( d \)-dimensional toric variety corresponding to a complete \( d \)-dimensional regular fan \( \Sigma \) with generators \( \{v_1, \ldots, v_n\} \). Then the anticanonical class of \( \mathbb{P}_\Sigma \) is ample (resp. numerically effective) if and only if for every primitive relation

\[ v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_m v_{j_m} = 0 \]

one has \( k - \sum_{i=1}^{m} c_i > 0 \) (resp. \( k - \sum_{i=1}^{m} c_i \geq 0 \) ).
There exist another combinatorial method of description of smooth compact \( d \)-dimensional toric manifolds via so called weighted triangulations of \((d - 1)\)-dimensional sphere \( S^{d-1} \) \cite{34}. This method gives an explicit information about all 1-dimensional toric strata and their normal bundles. For toric Fano manifolds the degrees of normal bundles to all 1-dimensional strata must be at least \(-1\). This requirement puts some restrictions of the number of faces of Fano polyhedra \( P \):

**Theorem 2.3.7** Let \( f_i(P) \) be the number of \( i \)-dimensional faces of a \( d \)-dimensional Fano polyhedron \( P \). Then

\[
12f_{d-3}(P) \geq (3d - 4)f_{d-2}(P).
\]

Moreover, the equality holds if and only if all 1-dimensional torus invariant strata on \( V(P) \) are rational curves having the anticanonical degree 1, i.e., the degree of the normal bundle to every 1-dimensional stratum on \( V(P) \) is \(-1\).

**Proof.** Let \( w(P) \) be the sum of all weights in the weighted triangulation of \( S^{d-1} \) defining the fan \( \Sigma(P) \). It is know that a sequence \( w_1, \ldots , w_s \) of integers in a weighted circular graph defining a 2-dimensional smooth projective variety satisfies the condition (see \cite{34}, p.45):

\[
\sum_{j=1}^{s} a_j = 12 - 3s.
\]

Using this formula for each 2-dimensional toric stratum in \( V(P) \) corresponding to a \((d - 3)\)-dimensional face of \( P \), we obtain

\[
w(P) = 12f_{d-3}(P) - 3(d - 1)f_{d-2}(P).
\]

On the other hand, all 1-dimensional toric strata in \( V(P) \) are parametrized by \((d - 2)\)-dimensional faces of \( P \). Since the sum of weights by every \((d - 2)\)-dimensional face of \( P \) equals the degree of the normal bundle, we conclude

\[
w(P) \geq -f_{d-2}(P).
\]

Moreover, the last inequality becomes equality iff the degree of the normal bundle to every 1-dimensional stratum on \( V(P) \) is \(-1\). This implies the statement of theorem. \( \square \)

**Remark 2.3.8** In \cite{13} the number \( w(P) = 12f_{d-3}(P) - 3(d - 1)f_{d-2}(P) \) was called the total weight of \( P \).

### 2.4 Projections of Fano polyhedra

**Definition 2.4.1** Let \( P \) be a \( d \)-dimensional Fano polyhedron, \( v_i \in \mathcal{V}(P) \) a vertex of \( P \), \( \mathbb{R}\langle v_i \rangle \) the 1-dimensional subspace in \( \mathbb{R}^n \) generated by \( v_i \). We denote by

\[
\pi_i : \mathbb{R}^n \to \mathbb{R}^n / \mathbb{R}\langle v_i \rangle
\]

the canonical epimorphism. The image \( P_i = \pi_i(P) \) we call a \( \pi_i \)-projection of \( P \).
Remark 2.4.2 It would be perfect for the classification of \(d\)-dimensional Fano polyhedra by induction on \(d\), if the polyhedra \(P_i\) were again Fano polyhedra of dimension \(d - 1\). Unfortunately, this is not true in general. However, the polyhedron \(P_i\) is always very close to a Fano polyhedron.

**Proposition 2.4.3** The polyhedron \(P_i\) is the convex hull of all points \(\pi_i(v_j)\) such that the segment \([v_i, v_j]\) is an edge of \(P\). Moreover, \(0 \in \mathbb{R}^n/\mathbb{R}(v_i)\) is the unique interior lattice point of \(P_i\).

**Proof.** By definition, \(P_i\) is the convex hull of \(\pi_i\)-images of vertices of \(P\). Let \(P'_i \subset P_i\) be the convex hull of \(\pi_i\)-images of all vertices \(v_j\) such that \([v_i, v_j]\) is a face of \(P\). It remains to prove that \(P_i \subset P'_i\).

Assume that \([v_i, v_k]\) is not an edge of \(P\). Then \(\{v_i, v_k\}\) is a primitive collection. There are two possibilities for the corresponding primitive relation:

- **Case I.** \(v_i + v_k = 0\). This implies that \(\pi_i(v_k) = 0\). Obviously, \(0 \in P'_i\).
- **Case II.** \(v_i + v_k = v_l\). By 2.3.3(iii), \([v_i, v_l]\) is a face of \(P\). So \(\pi_i(v_k) = \pi_i(v_l) \in P'_i\).

It follows from 2.1.3(iii) that \(P_i\) is covered by \(\pi_i\)-images of all \((d - 1)\)-faces of \(P\) containing \(v_i\). Hence, by 2.1.4(iv), \(0\) is the unique interior lattice point of \(P_i\). \(\square\)

**Proposition 2.4.4** The anticanonical class of toric divisor \(D_i \subset V(P)\) corresponding to the vertex \(v_i\) is always numerically effective. In particular, \(P_i\) is always a reflexive polyhedron.

**Proof.** Let \(\alpha_i : \text{Pic} V(P) \to \text{Pic} D_i\) be the natural surjective mapping induced by restriction of Cartier divisors. This induces the injective mapping of dual lattices \(\alpha_i^* : A_1(D_i) \to A_1(V(P))\). Since the cone of Mori of any toric variety is generated by classes 1-strata, \(\alpha_i^*(\text{NE}(D_i))\) is a face of the cone \(\text{NE}(V(P))\). So it is sufficient to prove that for any primitive relation \(R(\mathcal{P})\) such that \(\lambda_\mathcal{P} \in \alpha_i^*(\text{NE}(D_i))\). Thus, for a primitive relation

\[
R(\mathcal{P}) : v_{i_1} + \ldots + v_{i_k} - c_1 v_{j_1} - \ldots - c_m v_{j_m} = 0
\]

representing a 1-staturn in \(D_i\) (see 2.3.1), one has \((-K_{D_i}, \lambda_\mathcal{P}) \geq 0\). By the adjunction formula, we have

\[
(-K_{D_i}, \lambda_\mathcal{P}) = (-K_{V(P)}, \lambda_\mathcal{P}) - (D_i, \lambda_\mathcal{P}).
\]

Since \(-K_{V(P)}\) is ample, \((-K_{V(P)}, \lambda_\mathcal{P}) \geq 1\). By 2.3.3(ii), we obtain

\[
(D_i, \lambda_\mathcal{P}) = \begin{cases} 1 & \text{if } i \in \{i_1, \ldots, i_k\}, \\ -c_i & \text{if } i = j_l, \\ 0 & \text{otherwise}, \end{cases}
\]

i.e., \((D_i, \lambda_\mathcal{P}) \geq -1\). Hence, \((-K_{D_i}, \lambda_\mathcal{P}) \geq 0\). \(\square\)

**Corollary 2.4.5** Assume that all primitive collection containing \(v_i\) have degree \(\Delta \geq 2\). Then \(D_i\) is a \((d - 1)\)-dimensional toric Fano manifold, i.e., \(P_i\) is a \((d - 1)\)-dimensional Fano polyhedron.
Corollary 2.4.6 Let $\Sigma_i(P) \subset \mathbb{R}^n/\mathbb{R}(v_i)$ be the $(d-1)$-dimensional complete regular fan defining $D_i$, $\Gamma$ a $(d-2)$-dimensional face of $P_i$. Then there exist only the following possibilities for $\Gamma$:

(i) $\Gamma$ contains $(d-1)$ lattice points which generate a $(d-1)$-dimensional cone from $\Sigma_i(P)$;

(ii) There exist a primitive collection $\mathcal{P} = \{v, v_{i_1}, \ldots, v_{i_k}\}$ of degree 1, i.e., a primitive relation

$$v_i + v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_m v_{j_m} = 0, \quad k = \sum_{s=1}^{m} c_s,$$

such that $\Gamma$ contains exactly $d$ lattice points satisfying the uniquely determined linear relation

$$\pi_i(v_{i_1}) + \cdots + \pi_i(v_{i_k}) - c_1 \pi_i(v_{j_1}) - \cdots - c_m \pi_i(v_{j_m}) = 0.$$

Definition 2.4.7 A nonzero lattice point $v \in P_i$ is called double point if there exist two different vertices $v_j, v_k \in P$ such that $\pi_i(v_j) = \pi_i(v_k) = v$.

Proposition 2.4.8 Let $\Gamma$ be a $(d-2)$-dimensional face of $P_i$.

(i) If $\Gamma$ contains $d$ lattice points, then no one of these point is double.

(ii) If $\Gamma$ contains $(d-1)$ lattice points, then $\Gamma$ contains at most 1 double point.

Proof. It is clear that $P$ has a supporting $(d-1)$-dimensional affine hyperplane $H \subset \mathbb{R}^d$ such that $\pi_i(H)$ is a $(d-2)$-dimensional affine hyperplane in $\mathbb{R}^n/\mathbb{R}(v_i)$ with $\pi_i(H) \cap P_i = \Gamma$. Now both statements (i) and (ii) follow from the fact that $H$ can not contain more than $d$ vertices of $P$. \hfill $\square$

Proposition 2.4.9 Assume that $P$ contains two centrally symmetric vertices $v_i$ and $v_j$, i.e., $v_i + v_j = 0$ is a primitive relation. If

$$R(\mathcal{P}) : v_i + v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_m v_{j_m} = 0$$

is a primitive relation and $k = \sum_{s=1}^{m} c_s$, then $c_1 = \cdots = c_m = 1$, $k = m$ and

$$v_j + v_{j_1} + \cdots + v_{j_m} - v_{i_1} - \cdots - v_{i_k} = 0$$

is again a primitive relation.

Proof. By 2.3.3(i), the set of vertices $\{v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_m}\}$ is a set of vertices of a $(k + m)$-dimensional face $F$ of $P$. In particular, one has $k + m \leq d$. Since the lattice points $\pi_i(v_{i_1}), \ldots, \pi_i(v_{i_k}), \pi_i(v_{j_1}), \ldots, \pi_i(v_{j_m})$ satisfy the linear relation

$$\pi_i(v_{i_1}) + \cdots + \pi_i(v_{i_k}) - c_1 \pi_i(v_{j_1}) - \cdots - c_m \pi_i(v_{j_m}) = 0,$$
there exists a \((d-2)\)-dimensional face \(\Gamma\) of \(P\). Since \(v_i\) and \(v_j\) are centrally symmetric, we can identify two projections \(\pi_i(P)\) and \(\pi_j(P)\). Applying 2.4.6 for the \((d-2)\)-dimensional face \(\Gamma\) of \(P_j = \pi_j(P)\), we conclude that there must be a primitive relation

\[ R(P') : v_j + v_1 + \cdots + v_r - d_1 v_{m_1} - \cdots - d_s v_{m_s} = 0, \]

where \(\pi_j(v_1), \ldots, \pi_j(v_r), \pi_j(v_{m_1}), \ldots, \pi_j(v_{m_s})\) are lattice points in \(\Gamma\) satisfying the linear relation

\[ \pi_j(v_1) + \cdots + \pi_j(v_r) - d_1 \pi_j(v_{m_1}) - \cdots - d_s \pi_j(v_{m_s}) = 0. \]

By 2.4.6, there exists a unique linear relation among all \(d\) lattice points of \(\Gamma\). This implies that either \(k = r, m = s\) and \(c_i = d_i\) \((i = 1, \ldots, m)\), or \(k = s, m = r\) and \(c_i = 1\) \((i = 1, \ldots, m)\). The first case is impossible, because the sum of \(R(P)\) and \(R(P')\) would be a nontrivial linear relation among vertices of \(F\). The second case implies exactly the required statement.

\[ \Box \]

**Proposition 2.4.10** Let \(v \in P_i\) be a nonzero double point, i.e., \(0 \neq v = \pi_i(v_j) = \pi_i(v_k)\) for some two different vertices \(v_j, v_k \in P\). Then exactly one of the following two situations holds:

(i) \(v_i + v_k = v_j\) is a primitive relation;
(ii) \(v_i + v_j = v_k\) is a primitive relation.

**Proof.** Since \(\pi_i(v_j) = \pi_i(v_k)\), we have \(v_j - v_k = av_i\) for some \(a \in \mathbb{Z}\). Obviously, \(a \neq 0\). Assume that \(a > 0\). Then the point \(v_j = v_k + av_i\) belongs to the relative interior of the cone generated by \(v_i\) and \(v_k\). Therefore, by 2.1.1(iii,iv), the set \(P := \{v_i, v_k\}\) must be a primitive collection. The corresponding primitive relation can not be \(v_i + v_k = 0\), because otherwise \(\pi_i(v_k)\) would be 0. By 2.1.10, the only possibility for the primitive relation \(R(P)\) is \(v_i + v_k = v_j\). This implies that \(\pi_i(v_i) = \pi_i(v_k) = v\). By 2.1.1(iv), \(\pi_i^{-1}(v)\) can’t contain more than 2 vertices of \(P\). Hence, \(v_i = v_j\) and \(a = 1\). So we come to the situation (i). Analogously, if \(a < 0\), then one obtains the second case (ii).

\[ \Box \]

**Definition 2.4.11** Let \(v \in P_i = \pi_i(P)\) be a nonzero double point \((0 \neq v = \pi_i(v_j) = \pi_i(v_k))\). If \(v_i + v_k = v_j\) is a primitive relation, then we call the vertex \(v_j \in P\) a \(\pi_i\)-link of \(v_k\). If \(v_i + v_j = v_k\) is a primitive relation, then we call the vertex \(v_k \in P\) a \(\pi_i\)-link of \(v_j\).

**Proposition 2.4.12** Let \(v_k\) be a vertex in \(V(P) \setminus \{v_i, v_j\}\). Then \(\pi_i(v_k)\) is a double point of \(P_i\) if and only if exactly one of two sets \(\{v_i, v_k\}\) and \(\{v_j, v_k\}\) is a primitive collection.

**Proof.** Assume that \(\pi_i(v_k)\) is a double point of \(P_i\), i.e., there exists a vertex \(v_l \in V(P) \setminus \{v_i, v_j, v_k\}\) such that \(\pi_i(v_l) = \pi_i(v_k)\). By 2.4.10, exactly one of the two subsets \(\{v_i, v_k\}\) and \(\{v_i, v_l\}\) is a primitive collection. Analogously, by 2.4.10, exactly
one of the two subsets \( \{v_j, v_k\} \) and \( \{v_j, v_l\} \) is a primitive collection. If \( \{v_j, v_l\} \) and \( \{v_i, v_l\} \) were primitive collections, then, by 2.4.10, we would have \( v_j + v_l = v_k \) and \( v_i + v_l = v_k \). This implies a contradiction to \( v_i \neq v_j \). If both \( \{v_j, v_k\} \) and \( \{v_i, v_k\} \) were primitive collections, then, by 2.4.10, we would have \( v_j + v_k = v_l \) and \( v_i + v_k = v_l \). This contradicts \( v_i \neq v_j \).

Now assume that \( \pi_i(v_k) \) is not a double point of \( P_i \). If e.g. \( \{v_k, v_i\} \) is a primitive collection, then the corresponding primitive relation can’t be of the form \( v_k + v_i = 0 \) (otherwise we would have \( v_i = v_j \)). Hence, the only possibility for the primitive collection is \( v_k + v_i = v_l \) where \( v_l \in \mathcal{V}(P) \setminus \{v_i, v_j, v_k\} \). This implies that \( \pi_i(v_k) = \pi_i(v_l) \), i.e., \( \pi_i(v_k) \) is a double point of \( P_i \). Contradiction.

\[ \blacksquare \]

### 2.5 Toric Fano 3-folds

The purpose of this section is to illustrate for toric Fano 3-folds the method which will be the main tool for our classification of 4-dimensional Fano manifolds. Our purpose is the following theorem which was proved in [1] and [37]:

**Theorem 2.5.1** There exist exactly 18 different types of 3-dimensional smooth toric Fano varieties. The maximum of the Picard number of such varieties is equal to 5.

This statement can be equivalently reformulated as follows:

**Theorem 2.5.2** There exist exactly 18 different types of 3-dimensional Fano polyhedra \( P \) up to isomorphism. The maximal number of vertices of \( P \) is 8:

| the number of vertices | 4 | 5 | 6 | 7 | 8 |
|------------------------|---|---|---|---|---|
| the number of polyhedra | 1 | 4 | 7 | 4 | 2 |

**Proposition 2.5.3** Assume that a 3-dimensional Fano polyhedron satisfies the property: for every primitive collection \( \{v_i, v_j\} \subset \mathcal{V}(P) \), the corresponding primitive relation is \( v_i + v_j = 0 \). Then the number \( n = n(P) \) of vertices of \( P \) is not greater than 6.

**Proof.** Let \( k \) be the number of primitive collections in \( \mathcal{V}(P) \) consisting of 2 vertices. Since \( f_1(P) = 3n - 6 \), we have

\[ k = \binom{n}{2} - (3n - 6) = \frac{(n - 3)(n - 4)}{2}. \]

On the other hand, our assumption on \( P \) implies that any two different primitive collections consisting of 2 vertices can not have common elements, i.e., \( 2k \leq n \). Therefore, \( n^2 - 8n + 12 \leq 0 \). This implies \( n \leq 6 \).

**Proposition 2.5.4** Any 3-dimensional Fano polyhedron has at most 8 vertices.
Proof. Using 2.5.3, we can assume that there exists a primitive relation of the type
\[ v_1 + v_2 - v_3 = 0. \]

By 2.3.4, \( D_3 \) is a ruled toric surface, i.e., \([v_3, v_1]\) and \([v_3, v_2]\) are faces of \( P \) and there are at most 2 more vertices of \( P \), e.g., \( v_4 \) and \( v_5 \), which are connected by edges \([v_3, v_4]\) and \([v_3, v_5]\) with \( v_3 \). By 2.4.8, among \( \{\pi_3(v_1), \pi_3(v_2), \pi_3(v_4), \pi_3(v_5)\} \subset P_3 \) there exist at most two double points. Thus, there might be at most three vertices of \( P \) which are not joined with \( v_3 \) by an edge: one centrally symmetric vertex \( v_6 = -v_3 \), and two vertices \( v_7 \) and \( v_8 \) such that \( \pi_3(v_7) \) and \( \pi_3(v_8) \) are double vertices of \( P_3 \) (the latter implies that \( \{v_3, v_7\} \) and \( \{v_3, v_8\} \) are primitive collections of degree 1). \( \square \)

**Proposition 2.5.5** There exist exactly 4 different 3-dimensional Fano polyhedra with 5 vertices.

**Proof.** It is known that the combinatorial type of 3-dimensional simplicial polyhedron having 5 vertices is unique: it is defined by two primitive collections \( \{v_1, v_2\} \) and \( \{v_3, v_4, v_5\} \). Let \( B \) denotes this combinatorial type. Then, up to change of indices in the numeration of vertices, the corresponding primitive relations are

| Fano polyhedron \( P \) | \( B_1 \) | \( B_2 \) | \( B_3 \) | \( B_4 \) |
|-------------------------|---------|---------|---------|---------|
| \( v_1 + v_2 = 0 \)     | 0       | 0       | \( v_3 \) | 0       |
| \( v_3 + v_4 + v_5 = 0 \) | \( 2v_1 \) | \( v_1 \) | 0       | \( v_1 \) |

\( \square \)

**Proposition 2.5.6** There exist exactly 7 different 3-dimensional Fano polyhedra with 6 vertices.

**Proof.** It is easy to show that there exist exactly two different combinatorial types \( C \) and \( D \) of 3-dimensional Fano polyhedra having 6 vertices.

Using explicit description of arbitrary smooth projective toric varieties with the Picard number 3 \cite{5} together with 2.3.6, we obtain that, up to renumeration of vertices of \( P \), the primitive relations are

1) the type \( C \): \( v_1 + v_2 = 0 \) and

| Fano polyhedron \( P \) | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) | \( C_5 \) |
|-------------------------|---------|---------|---------|---------|---------|
| \( v_3 + v_5 = v_1 \)   | \( v_1 \) | \( v_1 \) | 0       | 0       | \( v_1 \) |
| \( v_4 + v_6 = v_1 \)   | \( v_1 \) | \( v_1 \) | 0       | \( v_3 \) | \( v_2 \) |

2) for the type \( D \): \( v_3 + v_6 = 0 \), \( v_4 + v_6 = v_5 \), \( v_3 + v_5 = v_4 \), and

| Fano polyhedron \( P \) | \( D_1 \) | \( D_2 \) |
|-------------------------|---------|---------|
| \( v_1 + v_2 + v_4 = 2v_3 \) | \( v_3 \) | \( v_3 \) |
| \( v_1 + v_2 + v_5 = v_3 \) | \( v_3 \) | 0       |
Proposition 2.5.7 Assume that a 3-dimensional Fano polyhedron $P$ contains two centrally symmetric vertices $v_i, v_j \in \mathcal{V}(P)$: $v_i + v_j = 0$. Then $P_i = \pi_i(P)$ is a 2-dimensional Fano polyhedron. Moreover, if $v + v' = v''$ holds for some vertices $v, v', v'' \in \mathcal{V}(P_i)$, then no one of two vertices $v$ and $v'$ is a double point of $P_i$.

Proof. By 2.4.6, if $P_i$ is not a Fano polyhedron, then there exists a face $\Gamma$ of $P_i$ containing exactly 3 lattice points. The only possible linear relation (obtained from centrally symmetric vertices) corresponding primitive relation is $v_i = v' = v''$. Assume that $a, b \in \mathbb{Z}_{\geq 0}$, $v_k \in \mathcal{V}(P) \setminus \{v_k\}$ such that $v_i = v_k = v$. The linear relation $v + v' = v''$ implies that at least one of the following two equalities holds:

$$v_k + v_l = v_k + av_j,$$  

$$v_k + v_l = v_k + bv_j.$$  

Assume, for instance, that the first equality holds. We want to show that in this case $\{v_k, v_l\} \subset \mathcal{V}(P)$ is a primitive collection. If it were not the case, then we would have $a \neq 0$ and $[v_m, v_l]$ couldn’t be an edge of $P$ (otherwise two cones $\sigma([v_k, v_l])$ and $\sigma([v_m, v_l])$ would have a common point in their relative interior). So, $\{v_m, v_l\}$ is a primitive collection. On the other hand, the single possibility for the corresponding primitive relation is $v_m + v_l = v_k$, i.e., $v''$ is another double point. This contradicts 2.4.8. Hence, $\{v_k, v_l\}$ must be a primitive collection. The single possibility for corresponding primitive relation is $v_k + v_l = v_m$. On the other hand, the same arguments applied to the set of vertices $\{v_s, v_l, v_m\}$ instead of $\{v_k, v_l, v_m\}$ show that $v_s + v_l = v_m$ must be a primitive relation too. Both above relations imply $v_k = v_s$.

\[\square\]

Proposition 2.5.8 There exist exactly 2 different 3-dimensional Fano polyhedra with 8 vertices.

Proof. By 2.5.3, there exists a primitive relation of the type $v_1 + v_3 = v_2$. By 2.3.3(i), $[v_1, v_2]$ and $[v_3, v_2]$ are edges of $P$. It follows from 2.3.3(iii) that the divisor $D_2$ has Picard number 2, i.e., that $v_2$ is joined by edges of $P$ with exactly two more vertices, e.g., $v_7$ and $v_8$. By 2.4.3, $P_2 = \pi_2(P)$ contains exactly 5 lattice points. By 2.4.8, at most 3 lattice points in $P$ could be double points. Since $P$ contains 8 points, $P_2$ must contain exactly 3 double points. By 2.4.8 these double points could be either $\{0, \pi_2(v_1), \pi_2(v_3)\}$, or $\{0, \pi_2(v_7), \pi_2(v_8)\}$, i.e., there always exists a
centrally symmetric to \( v_2 \) vertex \( v_5 = -v_2 \) and there always exist two vertices \( v_4, v_6 \) such that \( \{v_2, v_4\} \) and \( \{v_2, v_6\} \) are primitive collections. By 2.3.3(i), \( P_2 \) must be a Fano polygon and two double vertices of \( P_2 \) must be centrally symmetric. Without loss of generality we can assume that \( v_1 \) and \( v_3 \) are \( \pi_2 \)-links of \( v_4 \) and \( v_6 \). Therefore the set \( \{v_1, v_2, v_3, v_4, v_5, v_6\} \) is contained in a 2-dimensional vector subspace. These 6 vertices of \( P \) lying in this subspace form a 2-dimensional Fano polyhedron with 6 vertices and the combinatorial type of \( P \) is determined uniquely (we denote it by \( \mathcal{F} \)). Now one sees that the primitive relations are \( v_3 + v_6 = 0 \), \( v_1 + v_3 = v_2 \), \( v_2 + v_6 = v_1 \), \( v_2 + v_4 = v_3 \), \( v_2 + v_5 = 0 \), \( v_1 + v_4 = 0 \), \( v_4 + v_6 = v_5 \), \( v_1 + v_5 = v_6 \), \( v_3 + v_5 = v_4 \) and \( v_7 + v_8 = 0 \).

| Fano polyhedron | \( \mathcal{F}_1 \) | \( \mathcal{F}_2 \) |
|-----------------|----------------|----------------|
| \( v_7 + v_8 = 0 \) | \( v_1 \) | \( v_1 \) |

\[\square\]

**Proposition 2.5.9** There exist exactly 4 different 3-dimensional Fano polyhedra with 7 vertices.

*Proof.* By 2.5.3, there exists a primitive relation of the type \( v_1 + v_3 = v_2 \). By 2.3.3(i), \( \{v_1, v_2\} \) and \( \{v_3, v_2\} \) are edges of \( P \). It follows from 2.3.3(iii) that the divisor \( D_2 \) has Picard number 2, i.e., that \( v_2 \) is joined by edges of \( P \) with exactly two more vertices, e.g., \( v_6 \) and \( v_7 \). By 2.4.8, we have the following 3 cases:

**Case 1.** One of two vertices \( v_4 \) or \( v_5 \) is centrally symmetric to \( v_2 \), e.g., \( v_2 + v_4 = 0 \). Then we use 2.3.4 to obtain that \( P_2 \) is a Fano polygon whose single double vertex \( \pi_2(v_5) \) belongs to a pair of centrally symmetric vertices of \( P_2 \). So, one of two sets \( \{v_1, v_2, v_3, v_4, v_5\} \) or \( \{v_2, v_3, v_5, v_6, v_7\} \) must be contained in a 2-dimensional vector subspace. Therefore, vertices of \( P \) lying in this subspace form a Fano polygon \( P_2 \) with 5 vertices. We assume that vertices \( \{v_1, v_2, v_3, v_4, v_5\} \) form such a polygon. Then the combinatorial type of \( P \) is determined uniquely (we denote it by \( \mathcal{E} \)). This shows that the primitive relations up to a numeration of vertices are \( v_2 + v_4 = 0 \), \( v_3 + v_5 = 0 \), \( v_1 + v_3 = v_2 \), \( v_2 + v_5 = v_1 \), \( v_1 + v_4 = v_5 \) and \( v_6 + v_7 = 0 \).

| Fano polyhedron | \( \mathcal{E}_1 \) | \( \mathcal{E}_2 \) | \( \mathcal{E}_3 \) | \( \mathcal{E}_4 \) |
|-----------------|----------------|----------------|----------------|----------------|
| \( v_6 + v_7 = 0 \) | \( v_1 \) | \( v_2 \) | \( 0 \) | \( v_3 \) |

**Case 2.** \( \pi_2(v_1) \) and \( \pi_2(v_3) \) are double vertices of \( P_2 \), i.e., we have two primitive relations \( \{v_1, v_2, v_3, v_4, v_5\} \) \( v_2 + v_4 = v_1 \) and \( v_2 + v_5 = v_2 \). Since \( \pi_2(v_1) \) and \( \pi_2(v_3) \) are centrally symmetric, the set \( \{v_1, v_2, v_3, v_4, v_5\} \) is contained in a 2-dimensional vector subspace and at least one of two vertices \( v_1 \) and \( v_2 \) must possess a centrally symmetric one, we come to the situation considered in Case 1.

**Case 3.** \( \pi_2(v_6) \) and \( \pi_2(v_7) \) are double vertices of \( P_2 \), i.e., we have two primitive relations \( v_2 + v_4 = v_6 \) and \( v_2 + v_5 = v_7 \). If \( \pi_2(v_6) \) and \( \pi_2(v_7) \) are centrally symmetric, then the set \( \{v_2, v_4, v_5, v_6, v_7\} \) is contained in a 2-dimensional vector subspace and at least one of two vertices \( v_6 \) and \( v_7 \) must possess a centrally symmetric one, i.e., we come to the situation considered in Case 1. It remains to exclude the situation, when
\( \pi_2(v_6) \) and \( \pi_2(v_7) \) are not centrally symmetric vertices of \( P_2 \). Indeed, by \( \ref{2.3.3}(i) \), both vertices \( v_6 \) and \( v_7 \) have valence 4. Moreover, \( \{v_6, v_5\}, \{v_6, v_7\} \) and \( \{v_4, v_7\} \) are primitive collections. Since

\[
\pi_2(v_6) + \pi_2(v_5) = \pi_2(v_6) + \pi_2(v_7) = \pi_2(v_4) + \pi_2(v_7) = v \neq 0
\]

where \( v \) is not a double vertex of \( P_2 \), we obtain the single possibility for the corresponding primitive relations:

\[
v_6 + v_5 = v_6 + v_7 = v_4 + v_7 = v_i,
\]

where \( v_i \) is a vertex of \( P \). The last equalities are impossible, since they imply \( v_5 = v_7 \) and \( v_6 = v_4 \).

\[\square\]

**Remark 2.5.10** In the table below we give the list of 3-dimensional toric Fano varieties \( V = V(P) \). We denote by \( S_i \) the Del Pezzo surface obtained by the blow up of \( i \) points on \( \mathbb{P}^2 \). The number \( a(V) \) denotes the dimension of the group \( \text{Aut} (V) \) of biregular automorphisms of \( V \).
| $n^0$ | $c_1^3$ | $b_2$ | $h^0$ | $a(V)$ | the type of Fano polytope $P$ and the geometry of $V$ |
|-------|--------|-------|-------|--------|-----------------------------------------------|
| 1     | 64     | 1     | 35    | 15 | $\mathbb{P}^3$ |
| 2     | 62     | 2     | 34    | 15 | $\mathcal{B}_1$, $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$ |
| 3     | 56     | 2     | 31    | 12 | $\mathcal{B}_2$, $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ |
| 4     | 54     | 2     | 30    | 11 | $\mathcal{B}_3$, $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ |
| 5     | 54     | 2     | 30    | 11 | $\mathcal{B}_4$, $\mathbb{P}^2 \times \mathbb{P}^1$ |
| 6     | 52     | 3     | 29    | 11 | $\mathcal{C}_1$, $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1))$ |
| 7     | 50     | 3     | 28    | 10 | $\mathcal{C}_2$, $\mathbb{P}_{S_1}(\mathcal{O} \oplus \mathcal{O}(l))$, where $l^2 = 1$ on $S_1$ |
| 8     | 48     | 3     | 27    | 9  | $\mathcal{C}_3$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ |
| 9     | 48     | 3     | 27    | 9  | $\mathcal{C}_4$, $S_1 \times \mathbb{P}^1$ |
| 10    | 44     | 3     | 25    | 7  | $\mathcal{C}_5$, $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, -1))$ |
| 11    | 50     | 3     | 28    | 10 | $\mathcal{D}_1$, the blow up of $\mathbb{P}^1$ on $n^0 3$ |
| 12    | 46     | 3     | 26    | 8  | $\mathcal{D}_2$, the blow up of $\mathbb{P}^1$ on $n^0 5$ |
| 13    | 46     | 4     | 26    | 9  | $\mathcal{E}_1$, $S_2$-bundle over $\mathbb{P}^1$ |
| 14    | 44     | 4     | 25    | 8  | $\mathcal{E}_2$, $S_2$-bundle over $\mathbb{P}^1$ |
| 15    | 42     | 4     | 24    | 7  | $\mathcal{E}_3$, $S_2 \times \mathbb{P}^1$ |
| 16    | 40     | 4     | 23    | 6  | $\mathcal{E}_4$, $S_2$-bundle over $\mathbb{P}^1$ |
| 17    | 36     | 5     | 21    | 5  | $\mathcal{F}_1$, $S_3 \times \mathbb{P}^1$ |
| 18    | 36     | 5     | 21    | 5  | $\mathcal{F}_2$, $S_3$-bundle over $\mathbb{P}^1$ |

### 3 Classification of 4-dimensional Fano polyhedra
3.1 Fano 4-polyhedra with \( \leq 7 \) vertices

There there exists a unique toric 4-fold, 4-dimensional projective space \( \mathbb{P}^4 \), having the Picard number 1. The corresponding Fano 4-polyhedron is a simplex.

**Proposition 3.1.1** There exist exactly 9 different 4-dimensional Fano polyhedra with 6 vertices having two possible combinatorial types \( B \) and \( C \). These 4-polyhedra are defined by the following primitive relations

\[
\begin{array}{c|ccccc}
\text{Fano polyhedron} P & B_1 & B_2 & B_3 & B_4 & B_5 \\
v_1 + v_2 + v_3 + v_4 = & 3v_5 & 2v_5 & v_5 & 0 & 0 \\
v_5 + v_6 = & 0 & 0 & 0 & 0 & v_1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{Fano polyhedron} P & C_1 & C_2 & C_3 & C_4 \\
v_1 + v_2 + v_3 = & 0 & 0 & 0 & 0 \\
v_4 + v_5 + v_6 = & 2v_1 & v_1 & v_1 + v_2 & 0 \\
\end{array}
\]

Proof. The statement immediately follows from the general description of toric Fano manifolds with the Picard number 2 (see [19]) and from 2.3.6. \( \square \)

**Proposition 3.1.2** There exist exactly 28 different 4-dimensional Fano polyhedra with 7 vertices having three possible combinatorial types \( D \), \( E \), and \( G \). These 4-polyhedra are defined by the following primitive relations

(i) Type \( D \) :

\[
\begin{array}{c|ccccccc}
\text{Fano polyhedron} P & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 \\
v_1 + v_2 + v_3 = & 2v_6 & 2v_4 & v_4 + v_6 & 2v_6 & 2v_4 & v_6 & 0 \\
v_4 + v_5 = & v_6 & v_6 & v_6 & v_1 & 0 & v_6 & v_1 \\
v_6 + v_7 = & 0 & 0 & 0 & 0 & 0 & 0 & v_1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{Fano polyhedron} P & D_8 & D_9 & D_{10} & D_{11} \\
v_1 + v_2 + v_3 = & v_4 & v_4 + v_6 & v_6 & 0 \\
v_4 + v_5 = & v_6 & 0 & v_1 & v_1 & 0 & 0 \\
v_6 + v_7 = & 0 & 0 & 0 & v_4 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{Fano polyhedron} P & D_{14} & D_{15} & D_{16} & D_{17} \\
v_1 + v_2 + v_3 = & 0 & 0 & v_4 + v_7 & 0 \\
v_4 + v_5 = & 0 & 0 & v_6 & v_1 & v_6 & v_6 \\
v_6 + v_7 = & v_1 & v_4 & 0 & v_2 & 0 & 0 \\
\end{array}
\]

(ii) Type \( E \) : \( v_1 + v_7 = 0, \ v_1 + v_2 = v_6, \ v_6 + v_7 = v_2 \) and
(iii) Type $G$:

| Fano polyhedron $P$ | $E_1$ | $E_2$ | $E_3$ |
|--------------------|-------|-------|-------|
| $v_2 + v_3 + v_4 + v_5 = 2v_1$ | $v_1$ | $0$   |       |
| $v_3 + v_4 + v_5 + v_6 = 3v_1$ | $2v_1$ | $v_1$ |       |

Proof. There are 4 different possible combinatorial types of simplicial fans with 7 vertices defining smooth projective toric 4-folds. These combinatorial types are described by the following Gale diagrams:

Using the general description of toric Fano manifolds with the Picard number 3 and from 2.3.6, it is easy to show that only first 3 combinatorial types admit a realization by a Fano 4-polyhedron.

$\square$

3.2 Fano 4-polyhedra having a vertex of valence 5

Proposition 3.2.1 Let $P$ be a Fano 4-polyhedron, $v_0$ an a vertex of $P$, $P_0 = \pi_0(P)$ the projection of the polyhedron $P$ in $\mathbb{R}^3/\mathbb{R}\langle v_0 \rangle$. Then there exist only 4 possibilities for 2-dimensional faces of the polyhedron $P_0$:
Moreover, if $P$ contains a centrally symmetric vertex $v_i = -v_0$, then only $F(1)$ and $F(4)$ can be 2-dimensional faces of $P_0$.

**Proof.** We apply 2.4.6(ii) to the 4-dimensional case. It is sufficient to find all possible primitive relations $R(P)$ with $\Delta(P) = 1$ giving rise to faces $\Gamma \subset P_i$ which are different from the standard triangle $F(1)$. Using 2.3.3(i), we find the following types of primitive relations:

$$v_0 + v_1 + v_2 + v_3 = 3v_4 \rightarrow F(2), \quad v_0 + v_1 + v_2 = 2v_3 \rightarrow F(3),$$

and

$$v_0 + v_1 + v_2 = v_3 + v_4 \rightarrow F(4).$$

By 2.4.9, if there exists a vertex $v_i = -v_0$, then only the primitive relation $v_0 + v_1 + v_2 = v_3 + v_4$ is possible.

It turns out that the existence of two centrally symmetric vertices $v_i$ and $v_j$ of a Fano 4-polyhedron $P$ allows completely describe the combinatorial structure of $P$ via its $\pi_i$-projection $P_i$:

**Theorem 3.2.2** Assume that $P$ contains two centrally symmetric vertices $v_i$ and $v_j$, i.e., $v_i + v_j = 0$ is a primitive relation. Then a set $\{v_1, v_2, v_3, v_4\} \subset V(P)$ generate a 4-dimensional cone $\sigma(F)$ in $\Sigma(P)$ if and only if we have one of the following situations:

(i) the set $\{v_1, v_2, v_3, v_4\}$ contains $v_i$ (or resp. $v_j$), the other 3 vertices are connected by edges with $v_i$ (or resp. with $v_j$), and the convex hull of their $\pi_i$-projections is a 2-dimensional simplicial face of $P_i$ of the type $F(1)$;

(ii) $\{v_1, v_2, v_3, v_4\} \cap \{v_i, v_j\} = \emptyset$ and there exist two different vertices $v_k, v_l \in \{v_1, v_2, v_3, v_4\}$ such that $\pi_i(v_k) = \pi_i(v_l)$, and the convex hull of

$$\{\pi_i(v_1), \pi_i(v_2), \pi_i(v_3), \pi_i(v_4)\}$$

is a 2-dimensional simplicial face $\Gamma$ of $P_i$ of the type $F(1)$;

(iii) the set $\{v_1, v_2, v_3, v_4\}$ contains $v_i$ (or resp. $v_j$), the convex hull of the $\pi_i$-projections of the other 3 vertices is a 2-dimensional simplex $\Theta$ (of the type $F(1)$) which is a half of a face $\Gamma \subset P_i$ of the type $F(4)$, and the cone over $\Theta$ belongs to $\Sigma_i(P)$ (resp. to $\Sigma_j(P)$);
(iv) \( \{v_1, v_2, v_3, v_4\} \cap \{v_i, v_j\} = \emptyset \), each vertex from \( \{v_1, v_2, v_3, v_4\} \) is connected by an edge with both vertices \( v_i, v_j \), and the convex hull of

\[
\{\pi_i(v_1), \pi_i(v_2), \pi_i(v_3), \pi_i(v_4)\}
\]
is a 2-dimensional face \( \Gamma \subset P_i \) of the type \( F(4) \), i.e., its vertices satisfy a relation of the form

\[
\pi_j(v_{k_1}) + \pi_j(v_{k_2}) = \pi_j(v_{l_1}) + \pi_j(v_{l_2})
\]
where \( \{v_{k_1}, v_{k_2}\} \) and \( \{v_{l_1}, v_{l_2}\} \) are two disjoint subsets in \( \{v_1, v_2, v_3, v_4\} \).

**Corollary 3.2.3** Let \( P \) be a Fano 4-polyhedron containing two centrally symmetric vertices \( v_i, v_j \in V(P) \). Then the combinatorial type is uniquely determined by the following data:

(i) the combinatorial type of the 3-dimensional fan \( \Sigma_i(P) \) defining the toric divisor \( D_i \subset V(P) \);
(ii) the convex polyhedron \( P_i = \pi_i(P) \);
(iii) the set \( \delta(P_i) \subset V(P_i) \) of all double vertices of \( P_i \).

In particular, the number \( f_3(P) \) of 3-dimensional faces of \( P \) equals

\[
4\alpha_i + \beta_i - 8 + \sum_{v \in \delta(P)} \gamma(v),
\]
where \( \alpha_i \) is the valence of \( v_i \) in \( P \), \( \beta_i \) is the number of 2-dimensional faces of \( P_i \) of the type \( F(4) \), and \( \gamma(v) \) is the valence of a double vertex \( v \in P_i \).

**Definition 3.2.4** We use the notation \( b, c_1, c_2 \) and \( d_2 \) from [16] for the combinatorial types of some simplicial triangulations of the 2-dimensional sphere \( S^2 \) with 5, 6 and 7 vertices \( n_i \). These combinatorial types are described by the following stereographic projections of \( S^2 \) from the vertex \( n_1 = \infty \):

Using the restriction on the set of double points (see 2.4.5), one obtains:
**Theorem 3.2.5** Let \( P \) be a Fano 4-polyhedron with at least 8 vertices and \( v_0 \in P \) is a vertex of valence 5. Assume that that vertices \( n_j \ (1 \leq j \leq 5) \) of the diagram \( \text{(3.2.4)(b)} \) correspond to vertices \( v_j \ (1 \leq j \leq 5) \) of the polyhedron \( P \). Then \( f_0(P) \leq 9 \), and one of the following 4 situations holds:

(i) \( f_0(P) = 8 \), and one has the primitive relations

\[
    v_0 + v_7 = v_1, \quad v_0 + v_6 = 0;
\]

(ii) \( f_0(P) = 8 \), and one has the primitive relations

\[
    v_0 + v_7 = v_3, \quad v_0 + v_6 = 0;
\]

(iii) \( f_0(P) = 8 \), and one has the primitive relations

\[
    v_0 + v_7 = v_1, \quad v_0 + v_6 = v_2;
\]

(iv) \( f_0(P) = 9 \), and one has the primitive relations

\[
    v_0 + v_7 = v_1, \quad v_0 + v_6 = v_2, \quad v_0 + v_8 = 0.
\]

We consider separately all cases 3.2.5(i)-(iv).

3.2.6 Consider the case 3.2.5(i). Without loss of generality we can that there exists the primitive relation \( v_1 + v_2 = v_0 \) (otherwise we could start with \( v_1 \) instead of \( v_0 \)). Thus, \( D_0 \) is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \):

\[
    \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(a)).
\]

By 3.2.1, the 3-polyhedron \( P_0 \) can not contain a 2-dimensional face of the type \( F(2) \). Therefore, \( |a| \leq 2 \), i.e., \( P_0 \) is a Fano 3-polyhedron. By 3.2.3, the combinatorial type of \( P \) is uniquely determined. We denote it by \( H \) (this type correspond to \( P_{13}^8 \) from \([16]\)). Now it is easy to find the primitive relations defining \( P \):

\[
    v_1 + v_2 = v_0, \quad v_0 + v_7 = v_1, \quad v_1 + v_6 = v_7, \quad v_2 + v_7 = 0, \quad v_0 + v_6 = 0
\]

\[
    \begin{array}{|c|c|c|c|c|}
    \hline
    \text{Fano polyhedron} & P & H_1 & H_2 & H_3 & H_4 & H_5 \\
    \hline
    v_3 + v_4 + v_5 = & 2v_1 & v_0 + v_1 & 2v_0 & v_1 & v_0 \\
    \hline
    \end{array}
\]

\[
    \begin{array}{|c|c|c|c|c|}
    \hline
    \text{Fano polyhedron} & P & H_6 & H_7 & H_8 & H_9 & H_{10} \\
    \hline
    v_3 + v_4 + v_5 = & v_0 + v_2 & 2v_2 & 0 & v_2 & v_2 + v_6 \\
    \hline
    \end{array}
\]

3.2.7 Consider the case 3.2.5(ii). By 2.4.4 and 3.2.1, one obtains that either \( D_0 \) is a Fano 3-fold, or \( D_0 \) is isomorphic to

\[
    \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)).
\]

If \( D_0 \) is a Fano 3-fold, then the combinatorial type of \( P \) is uniquely determined by 3.2.3. We denote this combinatorial type by \( I \) (this type correspond to \( P_{17}^8 \) from \([16]\)). The primitive relations are

\[
    v_0 + v_7 = v_3, \quad v_3 + v_6 = v_7, \quad v_0 + v_6 = 0, \quad v_0 + v_8
\]

and
If $D_0$ is not a Fano 3-fold, i.e., $D_0 \cong \mathbb{P}^n(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$, then we determine uniquely the combinatorial type of $P$ using 3.2.3 and denote this combinatorial type by $J$ (this type correspond to $P_{21}^8$ from [10]). The primitive relations defining $P$ are

\[
\begin{align*}
    v_3 + v_6 &= v_7, \\
    v_0 + v_1 + v_2 &= v_4 + v_5, \\
    v_4 + v_5 + v_6 &= v_1 + v_2, \\
    v_0 + v_7 &= v_3, \\
    v_0 + v_6 &= 0
\end{align*}
\]

and

\[
\begin{align*}
    v_3 + v_4 + v_5 &= 0, \\
    v_4 + v_5 + v_7 &= v_6, \\
    v_1 + v_2 + v_3 &= v_6, \\
    v_1 + v_2 + v_7 &= 2v_6
\end{align*}
\]

3.2.8 Consider the case 3.2.5(iii). It this case the vertex $v_1$ satisfies the same conditions as the vertex $v_0$ in 3.2.5(i) and we do not obtain new Fano 4-polyhedra.

3.2.9 Consider the case 3.2.5(iv). By 3.2.1, $D_0$ must be a Fano 3-fold, otherwise $P_0$ could have at most one double vertex (see 2.4.8). On the other hand, all 6 vertices $v_0, v_1, v_2, v_6, v_7, v_8$ are contained in a 2-dimensional linear subspace of $\mathbb{R}^4$. Thus, $P$ defines a locally trivial toric bundle over $\mathbb{P}^2$ whose fiber is a Del Pezzo surface with Picard rank 4. If we denote the combinatorial type of $P$ by $K$, then the primitive relations defining $P$ are $v_0 + v_7 = v_1, v_1 + v_8 = v_7, v_0 + v_8 = 0, v_8 + v_2 = v_6, v_7 + v_6 = v_8, v_1 + v_6 = 0, v_0 + v_6 = v_2, v_1 + v_2 = v_0, v_7 + v_2 = 0$ and

\[
\begin{align*}
    v_3 + v_4 + v_5 &= 2v_0, \\
    v_1 + v_2 + v_3 &= v_0
\end{align*}
\]

Thus, we come to the following theorem.

**Theorem 3.2.10** There exist exactly 31 different Fano 4-polyhedra $P$ such that $f_0(P) \geq 8$ and $P$ has a vertex of valence 5.
Corollary 3.2.11 Among the combinatorial types \( P_8^i \), \( i = 1, \ldots, 21 \), there exist exactly 3 types \( P_{13}^8 \), \( P_{17}^8 \), and \( P_{21}^8 \) which admit a realization by Fano 4-polyhedra.

Proof. According to the table in [10], for every combinatorial type from \( P_8^i \), \( i = 1, \ldots, 12 \), there always exists a vertex \( v_i \) of valence 4. By [2, 4.8], \( P_1 \) contains at most 1 double vertex. This implies that the number of vertices of \( P \) could be at most 7. Contradiction to \( f_0(P) = 8 \). For every combinatorial type from \( P_{13}, \ldots, P_{25} \), there always exists a vertex of valence 5. Thus, the statement follows from 3.2.6-3.2.9.

3.3 Fano 4-polyhedra with 8 vertices

Proposition 3.3.1 Let \( V' \) (resp. \( V'' \)) be a 3-dimensional smooth projective toric variety defined by a fan \( \Sigma' \) (resp. by \( \Sigma'' \)) having the combinatorial type \( c_1 \) (resp. \( c_2 \)). Assume that both \( V' \) and \( V'' \) have numerically effective anticanonical class and both \( V' \) and \( V'' \) are not Fano 3-folds. Let \( P' \) (resp. \( P'' \)) be the convex hull of generators of all 1-dimensional cones in \( \Sigma' \) (resp. in \( \Sigma'' \)). If both polyhedra \( P' \) and \( p'' \) have only 2-dimensional faces the of the types \( F(1) \) and \( F(4) \), then the fans \( \Sigma' \) and \( \Sigma'' \) are defined by the following primitive relations among vertices \( n_i \) (\( 1 \leq i \leq 6 \)):

(i) Type \( c_1 \) : \( n_6 + n_3 = n_1 + n_2, n_1 + n_2 + n_5 = n_6, n_1 + n_2 + n_4 = \lambda n_3, \) where \( \lambda = 0, 1, 2 \);

(ii) Type \( c_2 \) : \( n_1 + n_2 = n_3 + n_6, n_3 + n_5 = 0, n_4 + n_6 = (\mu - 1)n_3, \) where \( \mu = 0, 1, 2 \).

Proof. The statement follows from the explicit description of 3-dimensional smooth projective toric varieties with the Picard number 3 (see [1]). We remark also that two reflexive 4-polyhedra \( P'(\lambda) \) and \( P''(\mu) \) corresponding to different values of \( \lambda \) and \( \mu \) are isomorphic if and only if \( \lambda = \mu \).

Proposition 3.3.2 Let \( P \) be a Fano 4-polyhedron such that among its vertices there exists the primitive relation \( v_1 + v_2 = v_3 \), where \( D_0 \) is a toric \( \mathbb{P}^1 \)-bundle over a toric Del Pezzo surface with the Picard number 3 or 4. Then the toric variety \( V' \) obtained from the contraction of the exceptional divisor \( D_0 \subset V(P) \) is again a toric Fano 4-fold.

Proof. Denote by \( D_{[1,2]} \) the toric Del Pezzo surface on \( V' \) obtained as the image of \( D_0 \). It is sufficient to prove that for any effective 1-cycle \( C \) on \( D_{[1,2]} \) the intersection number \( K_{V'} \cdot C \) is positive. Note that the cone of effective 1-cycles on the Del Pezzo surface \( D_{[1,2]} \) is always generated by exceptional rational curves of first kind. Thus, it is sufficient to assume that \( C \) is a 1-dimensional toric stratum on \( D_{[1,2]} \) such that \( C \cdot C = -1 \).

Let \( \mathbb{R}_{\geq 0}(v_1, v_2, v_3) \in \Sigma' \) be the 3-dimensional cone of the fan \( \Sigma' \) corresponding to the stratum \( C \subset V' \). Consider two 4-dimensional cones \( \mathbb{R}_{\geq 0}(v_1, v_2, v_3, v_4) \) and \( \mathbb{R}_{\geq 0}(v_1, v_2, v_3, v_5) \) in \( \Sigma' \) having \( \mathbb{R}_{\geq 0}(v_1, v_2, v_3) \) as a common face. It suffices to prove that in the linear relation

\[ v_5 + v_4 + x_1v_1 + x_2v_2 + x_3v_3 = 0 \]
the coefficients satisfy the inequality \( x_1 + x_2 + x_3 \leq -1 \). Since \( C \) is an exceptional curve, we get \( x_3 = -1 \). Since the anticanonical divisor is ample on \( V(P) \), we obtain that the coefficients of two linear relations

\[
v_5 + v_4 + y_1v_0 + y_2v_2 + y_3v_3 = 0
\]

and

\[
v_5 + v_4 + z_1v_1 + z_2v_0 + z_3v_3 = 0,
\]

satisfy the inequalities \( y_1 + y_2 + y_3 \leq -1 \) and \( z_1 + z_2 + z_3 \leq -1 \). Combining these two relations with \( v_1 + v_2 = v_0 \), we obtain

\[(y_1, y_2, y_3) = (x_1, x_2 - x_1, x_3) \quad \text{and} \quad (z_1, z_2, z_3) = (x_1 - x_2, x_2, x_3).\]

Thus, using \( x_3 = -1 \), we obtain two inequalities \( x_1 \leq 0, x_2 \leq 0 \).

\[\square\]

**Proposition 3.3.3** Let \( P \) be a Fano 4-polyhedron with 8 vertices such that there exists a primitive relation of the type \( v_i + v_j = 0 \). Then the following statements hold:

(i) If \( \Sigma_i(P) \) has the combinatorial type \( c_1 \) and \( \Sigma_j(P) \) has the combinatorial type \( c_2 \), then \( P \) has the combinatorial type \( P_{26}^8 \);

(ii) If both \( \Sigma_i(P) \) and \( \Sigma_j(P) \) have the combinatorial type \( c_1 \), then \( P \) has the combinatorial type \( P_{17}^8 \);

(iii) If both \( \Sigma_i(P) \) and \( \Sigma_j(P) \) have the combinatorial type \( c_2 \), then \( P \) has the combinatorial type \( P_{34}^8 \).

**Proof.** The statements follow from [3.2.3].

\[\square\]

**Proposition 3.3.4** Among the combinatorial types \( P_{27}^8, \ldots, P_{33}^8 \) only the type \( P_{28}^8 \) admits a realization by a Fano 4-polyhedron.

**Proof.** We shall use the same numeration of vertices as in [16].

Assume that \( P \) has the combinatorial type \( P_{27}^8 \). Then \( \{v_6, v_8\} \) is a primitive collection. The combinatorial type of \( \Sigma_6(P) \) is \( c_2 \). The combinatorial type of \( \Sigma_8(P) \) is \( c_1 \). Since \( V(P) \) contains no vertex \( v_i \neq v_6 \) such that \( \Sigma_i(P) \) has the combinatorial type \( c_2 \) or \( d_2 \), it follows from [2.3.4] that \( v_6 + v_8 = 0 \) is the single possibility for the corresponding primitive relation. This contradicts [3.3.3(i)].

By similar method we obtain contradictions to [3.3.3(ii)] for the combinatorial types \( P_{29}^8, P_{31}^8, P_{32}^8, P_{33}^8 \), because for these 4 combinatorial types there is no any vertex \( v_i \) such that \( \Sigma_i(P) \) has the combinatorial type \( c_2 \) or \( d_2 \).

Finally, assume that \( P \) has the combinatorial type \( P_{30}^8 \). Then \( V(P) \) contains only one primitive collection \( \{v_1, v_8\} \) consisting of 2 elements. We note that both \( \Sigma_1(P) \), \( \Sigma_8(P) \) have the combinatorial type \( c_1 \), there is no vertex \( v_i \in V(P) \) such that \( \Sigma_i(P) \) has the combinatorial type \( c_2 \), and only \( \Sigma_2(P) \) has the combinatorial type \( d_2 \). By [3.3.3(ii)] and [2.3.4] the single possibility for the primitive relation is \( v_1 + v_8 = v_2 \).

By [3.3.2] the contraction of the divisor \( D_2 \) yields a toric Fano 4-fold \( V' \). The Fano 4-polyhedron \( P' \) corresponding to \( V' \) has 7 vertices, and any two vertices \( v_i, v_j \in P' \) are connected by the edge \([v_i, v_j]\) of \( P' \). This contradicts [3.1.2].

\[\square\]
Proposition 3.3.5 There exist exactly 2 different Fano 4-polyhedra having the combinatorial type \( Z := \text{P}^{8}_{28} \).

Proof. We shall use the same numeration of vertices of \( P \) as in [16]. Then \( \{v_1, v_8\} \) and \( \{v_5, v_7\} \) are primitive collections. By 3.3.3 \( \Delta(\{v_1, v_8\}) = \Delta(\{v_5, v_7\}) = 1 \). By 2.3.4 only \( v_4 \) and \( v_6 \) could be values for the sums \( v_1 + v_8 \) and \( v_5 + v_7 \). Since each edge \([v_1, v_6], [v_5, v_4]\) is contained in exactly four 3-dimensional faces of \( P \) and each edge \([v_1, v_4], [v_5, v_6]\) is contained in exactly five 3-dimensional faces of \( P \), the corresponding primitive relations are \( v_1 + v_8 = v_4 \), \( v_5 + v_7 = v_6 \). In order to find possibilities for other primitive relations, we apply 3.3.2 and obtain that the contraction \( V' \) of \( D_4 \) or \( D_6 \) is again a Fano 4-fold corresponding to a Fano 4-polyhedron \( P' \) with 7 vertices (only 4-polyhedra \( G_4, G_6 \) are possible). Using our classification in 3.1, we come to the following primitive relations:

| Fano polyhedron \( P \) | \( Z_1 \) | \( Z_2 \) |
|--------------------------|---------|---------|
| \( v_3 + v_8 + v_7 = \)  | \( 0 \)  | \( v_2 \) |
| \( v_3 + v_4 + v_6 = \)  | \( v_1 + v_5 \) | \( 0 \)  |
| \( v_3 + v_4 + v_7 = \)  | \( v_1 \)  | \( v_1 + v_2 \) |
| \( v_3 + v_6 + v_8 = \)  | \( v_5 \)  | \( v_2 + v_5 \) |

\( \square \)

Proposition 3.3.6 There exist exactly 13 different Fano 4-polyhedra having the combinatorial type \( L := \text{P}^{8}_{34} \). The primitive relations are \( v_1 + v_8 = 0 \) and

| Fano polyhedron \( P \) | \( L_8 \) | \( L_9 \) | \( L_{10} \) | \( L_{11} \) | \( L_{12} \) | \( L_{13} \) |
|--------------------------|---------|---------|---------|---------|---------|---------|
| \( v_2 + v_4 = \)       | \( v_1 \)  | \( v_1 \)  | \( v_1 \)  | \( 0 \)    | \( 0 \)    | \( v_1 \)  |
| \( v_4 + v_5 = \)       | \( v_1 \)  | \( v_3 \)  | \( v_1 \)  | \( v_3 \)  | \( v_3 \)  | \( 0 \)    |
| \( v_6 + v_7 = \)       | \( v_1 \)  | \( v_3 \)  | \( v_4 \)  | \( v_3 \)  | \( v_4 \)  | \( v_4 \)  |

\( \square \)

Proposition 3.3.7 There exist exactly 4 different Fano 4-polyhedra having the combinatorial type \( M := \text{P}^{8}_{26} \) and a pair of centrally symmetric vertices \( v_1, v_8 \).
Proof. By 3.2.2 and 3.2.3, if \( v_1 \) and \( v_8 \) are centrally symmetric vertices, then \( P_1 := \pi_1(P) \) is not a Fano polyhedron. Moreover, \( P_1 \) contains exactly one 2-dimensional face of the type \( F(4) \). Therefore all possibilities for \( P_1 \) are described in 3.3.1. Using these descriptions of \( P_1 \), we come to the following primitive relations: \( v_1 + v_8 = 0 \), \( v_1 + v_2 + v_3 = v_4 + v_6 \), \( v_4 + v_6 + v_8 = v_1 + v_2 \) and

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Fano polyhedron } P & M_1 & M_2 & M_3 & M_4 \\
\hline
v_4 + v_5 = & 0 & v_1 & v_1 & 0 \\
v_6 + v_7 = & 0 & v_1 & v_5 & 0 \\
v_2 + v_3 + v_5 = & v_6 + v_8 & v_6 & v_6 & 0 \\
v_2 + v_3 + v_7 = & v_4 + v_8 & v_4 & 0 & v_4 + v_8 \\
\hline
\end{array}
\]

\[\square\]

Proposition 3.3.8 There exists exactly one Fano 4-polyhedron having the combinatorial type \( M := P_{26}^S \) and containing no pair of centrally symmetric vertices.

Proof. Using the numeration of vertices of \( P \) as in [16], we obtain the following primitive collections

\[\{v_1, v_8\}, \{v_4, v_5\}, \{v_6, v_7\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_5\}, \{v_2, v_3, v_7\}, \{v_4, v_6, v_8\}.\]

The fans \( \Sigma_4(P) \), \( \Sigma_6(P) \), \( \Sigma_8(P) \) have the combinatorial type \( c_1 \). The fans \( \Sigma_1(P) \), \( \Sigma_5(P) \), \( \Sigma_7(P) \) have the combinatorial type \( c_2 \). The fans \( \Sigma_2(P) \) and \( \Sigma_3(P) \) have the combinatorial type \( d_5 \) (see the notations in [16]).

By 2.3.4, the sum \( v_1 + v_8 \) must be equal to either \( v_5 \) or \( v_7 \). Without loss of generality, we assume that \( v_1 + v_8 = v_5 \). Applying again 2.3.4, we see that the sum \( v_4 + v_5 \) must be equal to either \( v_1 \) or \( v_7 \). If \( v_1 + v_5 = v_1 \), then \( v_4 + v_8 = 0 \), contradiction to our assumption on \( P \). Thus, the single possibility is \( v_4 + v_5 = v_7 \). By similar method, we obtain \( v_6 + v_7 = v_1 \). This implies that \( v_4 + v_6 + v_8 = 0 \). Using already known primitive relations, we can restrict possibilities for the fan \( \Sigma_2(P) \) having the type \( d_5 \) such that the convex hull of the generators of all 1-dimensional cones in \( \Sigma_2(P) \) is a reflexive polyhedron whose 2-dimensional faces isomorphic to \( F(1) \) or \( F(4) \). As a result, we obtain the required primitive relations which are uniquely defined up to the cyclic permutation of the primitive pairs of vertices

\[\{v_1, v_8\} \rightarrow \{v_4, v_5\} \rightarrow \{v_6, v_7\} \rightarrow \{v_1, v_8\}.\]

So we obtain the following primitive relations defining the polyhedron \( M_5 \): \( v_1 + v_8 = v_5 \), \( v_4 + v_5 = v_7 \), \( v_6 + v_7 = v_1 \), \( v_1 + v_2 + v_3 = v_6 \), \( v_2 + v_3 + v_5 = v_6 + v_8 \), \( v_2 + v_3 + v_7 = 0 \), \( v_4 + v_6 + v_8 = 0 \).

\[\square\]

Proposition 3.3.9 The combinatorial types \( P_{35}^S \), \( P_{36}^S \), \( P_{37}^S \) do not admit a realization by a Fano 4-polyhedron \( P \).
Proof. For combinatorial types $P^8_{35}$, $P^8_{36}$, $P^8_{37}$ all vertices have valence 7. Therefore, $\mathcal{V}(P)$ does not contain a primitive collection consisting of exactly two vertices. This contradicts 3.5.3 \hfill \Box

Collecting together all our results, we come to the following:

**Theorem 3.3.10** There exist exactly 47 different Fano 4-polyhedra with 8 vertices. Among 37 all possible combinatorial types of simplicial convex 4-polyhedra with 8 vertices only 6 ones $P^8_{13}$, $P^8_{17}$, $P^8_{21}$, $P^8_{26}$, $P^8_{28}$, $P^8_{34}$ admit a realization by a Fano 4-polyhedron.

### 3.4 Fano 4-polyhedra having a vertex of valence 6

**Proposition 3.4.1** Let $P$ be a Fano 4-polyhedron with at least 9 vertices which contains a vertex $v_i$ of valence 6 and does not contain vertices of valence $\leq 5$. Then either $P$ contains a vertex $v_j$ such that the fan $\Sigma_j(P)$ has the combinatorial type $c_2$, or $f_0(P) = 9$ and $P$ is isomorphic to the following Fano 4-polyhedron defined by the primitive relations: $v_0 + v_7 = 0$, $v_0 + v_8 = v_1$, $v_3 + v_5 = v_4$, $v_4 + v_6 = v_5$, $v_1 + v_7 = v_8$, $v_3 + v_6 = 0$, $v_1 + v_2 + v_5 = v_0 + v_6$, $v_1 + v_2 + v_4 = v_0$, $v_2 + v_5 + v_8 = v_6$, $v_1 + v_2 + v_8 = 0$.

**Proof.** There exist exactly two different possibilities for the combinatorial type of $\Sigma_i(P)$: $c_1$ and $c_2$. In the second case we can put $j = i$. Now assume that $\Sigma_i(P)$ has the type $c_1$. Putting $i = 0$ and using the numeration for the vertices joined with $v_0$ as in 3.2.4, we obtain at least two primitive collections $\{v_0, v_7\}$ and $\{v_0, v_8\}$. So, at least one of two points $\pi_0(v_7), \pi_0(v_8) \in P_0 = \pi_0(P)$ is a double vertex of $P_0$. Assume that $\pi_0(v_8)$ is a double vertex. Then, by 2.4.10, we have a primitive relation $v_0 + v_8 = v_8$ for some vertex $v_k \in P$ ($1 \leq k \leq 6$). If the corresponding vertex $n_k$ in the graph $c_1$ has valence 3 (i.e., $k \in \{3, 6\}$), then $v_k$ has valence 5 (contradiction). If $n_k$ in the graph $c_1$ has valence 4 (i.e., $k \in \{4, 5\}$), then $v_k$ has valence 6. By 2.3.3(iii), $\Sigma_k(P)$ has the combinatorial type $c_2$. The single remaining possibility is $k \in \{1, 2\}$.

Assume that $k = 1$. By 2.4.8, $\pi_0(v_1)$ is the unique double vertex of $P_0$. Therefore, we obtain that $f_0(P) \leq 9$, i.e., $f_0(P) = 9$. Moreover, by 3.3.2, the contraction of $D_1$ is again a Fano 4-fold $V'$ corresponding to a Fano 4-polyhedron $P'$ having 8 vertices. Using our results in 3.2 and 3.3, we obtain that the combinatorial type of $P'$ equals $I$ and $P'$ must be isomorphic to $I_{13}$. The last fact implies the above primitive relations describing $P$. \hfill \Box

**Proposition 3.4.2** There exist exactly 20 different Fano 4-polyhedra $P$ having 9 vertices and containing a vertex $v_i$ such that the fan $\Sigma_i(P)$ has the combinatorial type $c_2$.

**Proof.** We assume that $i = 0$, i.e., $v_0 \in P$ is the vertex of valence 6. We choose a numeration of vertices $v_j$ ($1 \leq j \leq 6$) of $P$ in such a way that it corresponds to the numeration of the vertices $n_j$ ($1 \leq j \leq 6$) in the graph $c_2$. Then $\{v_0, v_7\}$ and $\{v_0, v_8\}$
are primitive collections. There exist the following three cases for the corresponding primitive relations:

**Case I:** $P_0$ is a Fano 3-polyhedron and the primitive relations are

$$v_0 + v_8 = 0, \ v_0 + v_7 = v_k \ (1 \leq k \leq 6).$$

Without loss of generality, we can assume $k = 1$. Then the combinatorial type of $P$ is determined by $3.2.3$ and $P$ is determined by the primitive relations $v_1 + v_2 = v_0$, $v_1 + v_8 = v_7$, $v_2 + v_7 = 0$ and

$$\begin{array}{c|cccccc}
\text{Fano polyhedron } P & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
v_3 + v_5 = & v_1 & v_1 & v_0 & v_0 & 0 & \\
v_4 + v_6 = & v_1 & v_3 & v_0 & v_0 & v_1 & v_1 \\
\end{array}$$

Then

$$\begin{array}{c|cccccc}
\text{Fano polyhedron } P & Q_7 & Q_8 & Q_9 & Q_{10} & Q_{11} & Q_{12} \\
v_3 + v_5 = & v_0 & 0 & v_0 & 0 & 0 & v_1 \\
v_4 + v_6 = & v_7 & v_0 & v_2 & v_3 & 0 & v_2 \\
\end{array}$$

**Case II:** $P_0$ is not a Fano 3-polyhedron and the primitive relations are

$$v_0 + v_8 = 0, \ v_0 + v_7 = v_k \ (1 \leq k \leq 6).$$

Then $P_0$ is isomorphic to one of 3 polyhedra from $3.3.1$ and the combinatorial type of $P$ is uniquely determined by $3.2.3$. By $2.4.8$, only $\pi_0(v_4)$ or $\pi_0(v_5)$ can be double a vertex of $P_0$. Assume that $v_4$ is a double vertex, i.e., $k = 4$. Then we obtain the primitive relations $v_0 + v_1 + v_2 = v_3 + v_6$, $v_3 + v_6 + v_8 = v_1 + v_2$, $v_0 + v_7 = v_4$, $v_4 + v_8 = v_7$, $v_0 + v_8 = 0$. By $3.2.3$, $\{v_7, v_6\}$ and $\{v_4, v_6\}$ are primitive collections. Therefore, $\pi_0(v_4)$ and $\pi_0(v_5)$ must be centrally symmetric. So, the lattice points $v_0, v_4, v_6, v_7, v_8$ are vertices of a 2-dimensional Fano polyhedron and one of two pairs $\{v_4, v_6\}$ or $\{v_7, v_6\}$ consists of centrally symmetric vertices. Without loss of generality we shall assume that $v_7 + v_6 = 0$. Then, using the classification of Fano 4-polyhedra of the combinatorial type $P^8_{26}$, we obtain the primitive relations: $v_1 + v_2 + v_7 = v_3 + v_8$, $v_1 + v_2 + v_4 = v_3$ and

$$\begin{array}{c|ccc}
\text{Fano polyhedron } P & R_1 & R_2 & R_3 \\
v_3 + v_5 = & v_4 & v_0 & 0 \\
v_1 + v_2 + v_5 = & 0 & v_6 & v_6 + v_8 \\
\end{array}$$
Case III: The primitive relations are

\[ v_0 + v_8 = v_l, \quad v_0 + v_7 = v_k \quad (1 \leq l \neq k \leq 6). \]

By 2.4.8(ii), the pair \( \{v_k, v_l\} \) must coincide with one of the pairs \( \{v_1, v_2\}, \{v_3, v_5\}, \{v_4, v_6\} \). Assume that \( k = 1 \) and \( l = 2 \). We note that \( P_0 \) can contain only faces of the type \( F(1) \) or \( F(4) \) (otherwise there would exist a primitive collection of the type \( v_0 + v_1 + v_2 = 2v_j \) or \( v_0 + v_1 + v_2 + v_3 = 3v_j \), and this would imply, by 2.3.3, that \( v_i \) is a vertex of valence \( \leq 5 \)). Furthermore, by 3.3.1, we obtain that \( P_0 \) must a Fano 3-polyhedron.

If \( \pi_0(v_1) + \pi_0(v_2) = \pi_0(v_i) \neq 0 \) for some vertex \( v_i \in \mathcal{V}(P) \), then we obtain two primitive relations \( v_1 + v_8 = v_i \) and \( v_2 + v_7 = v_i \). On the other hand, by 2.3.4, \( \Sigma_1(P) \) and \( \Sigma_2(P) \) have the combinatorial type \( c_2 \). This implies that \( \{v_1, v_2\} \) is also a primitive collection. Considering the \( \pi_0 \)-projection, we obtain the unique possibility: \( v_1 + v_2 = v_i \). But the last equation contradicts \( v_1 + v_8 = v_i \) and \( v_2 + v_7 = v_i \).

It remains the single possibility \( \pi_0(v_1) + \pi_0(v_2) = 0 \). Then all 5 vertices \( v_1, v_2, v_0, v_7, v_8 \) are contained in a 2-dimensional linear subspace in \( \mathbb{R}^4 \) and form a 2-dimensional Fano polyhedron. Thus, the vertex \( v_1 \) satisfies the same conditions as the vertex \( v_0 \) in Case I. We have shown that the considered case reduces to the case I, so we obtain no new Fano 4-polyhedra.

**Proposition 3.4.3** There exist exactly 8 different Fano 4-polyhedra \( P \) having at least 10 vertices and containing a vertex \( v_i \) such that \( \Sigma_i(P) \) has the combinatorial type \( c_2 \).

**Proof.** The primitive relations are \( v_1 + v_3 = v_2, v_2 + v_4 = v_3, v_1 + v_4 = 0, v_3 + v_5 = v_4, v_4 + v_6 = v_5, v_2 + v_5 = 0, v_1 + v_5 = v_6, v_2 + v_6 = v_1, v_3 + v_6 = 0 \)

| Fano polyhedron \( P \) | \( U_1 \) | \( U_2 \) | \( U_3 \) | \( U_4 \) | \( U_5 \) | \( U_6 \) | \( U_7 \) | \( U_8 \) |
|------------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( v_8 + v_7 = \)      | \( v_1 \) | \( v_1 \) | \( 0 \) | \( 0 \) | \( v_1 \) | \( v_1 \) | \( v_1 \) | \( v_1 \) |
| \( v_9 + v_{10} = \)   | \( v_1 \) | \( v_8 \) | \( v_2 \) | \( 0 \) | \( 0 \) | \( v_3 \) | \( v_4 \) | \( v_4 \) |

\[ \square \]

### 3.5 Fano 4-polyhedra whose all vertices have valence \( \geq 7 \)

**Proposition 3.5.1** For a Fano 4-polyhedron \( P \), one has

\[ 4f_0(P) \geq f_1(P). \]

**Proof.** The statement follows from the general inequality in 2.3.7 together with the Dehn-Sommerville relation \( f_2(P) = 2(f_1(P) - f_0(P)) \).

\[ \square \]

**Corollary 3.5.2** Assume that any two vertices of a Fano 4-polyhedron \( P \) are joined by a 1-dimensional face, then \( f_0(P) \leq 9 \).
Proof. Note that

\[ f_1(P) = \frac{f_0(P)(f_0(P) - 1)}{2}. \]

By 3.5.1, one has \( f_0(P) \leq 9 \). \( \square \)

**Theorem 3.5.3** Let \( P \) be a Fano 4-polyhedron such that all vertices of \( P \) have valence \( \geq 7 \). Then there always exists a primitive collection containing exactly two vertices of \( P \).

**Lemma 3.5.4** Assume that any two vertices of a Fano 4-polyhedron \( P \) are joined by a 1-dimensional face, then all primitive relations corresponding to primitive collections of degree 1 have the form

\[ v_{i_1} + v_{i_2} + v_{i_3} = v_{j_1} + v_{j_2}. \]

**Proof.** Assume that there exists a primitive relation of the type

\[ v_{i_1} + v_{i_2} + v_{i_3} + v_{i_4} = 3v_j. \]

Then, by 2.3.3(iii), \( D_j \) is isomorphic to \( \mathbb{P}^3 \), i.e., \( v_j \) has valence 4. Contradiction.

Assume that there exists a primitive relation of the type

\[ v_{i_1} + v_{i_2} + v_{i_3} = 2v_j, \]

Then, by 2.3.3(iii), \( D_j \) is a toric \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}^1 \), i.e., \( v_j \) has valence 5. Contradiction. \( \square \)

**Lemma 3.5.5** Assume that any two vertices of a Fano 4-polyhedron are joined by a 1-dimensional face and

\[ v_1 + v_2 + v_3 = v_4 + v_5 \]

is a primitive relation. Then for any other primitive relations of the type

\[ v_{i_1} + v_{i_2} + v_{i_3} = v_{j_1} + v_{j_2}, \]

one has

\[ \{v_{j_1}, v_{j_2}\} \not\subset \{v_1, v_2, v_3, v_4, v_5\}. \]
Proof. Assume that \( \{v_{j_1}, v_{j_2}\} \subset \{v_1, v_2, v_3, v_4, v_5\} \). Then there exists a vertex \( v_i \in \{v_1, v_2, v_3\} \) which is not contained in \( \{v_{j_1}, v_{j_2}\} \). Without loss of generality we assume that \( v_i = v_3 \). Then, by 2.3.3(i), the edge \([v_4, v_5] \subset P\) is contained in the 3-dimensional face \([v_1, v_2, v_4, v_5]\). On the other hand, by 2.3.3(i), the edge \([v_{j_1}, v_{j_2}]\) is contained in exactly three 3-dimensional faces of \( P \):

\[
[v_{i_1}, v_{i_2}, v_{j_1}, v_{j_2}], [v_{i_1}, v_{i_3}, v_{j_1}, v_{j_2}], [v_{i_2}, v_{i_3}, v_{j_1}, v_{j_2}].
\]

This shows that \([v_1, v_2, v_4, v_5]\) must coincide with one of these 3 faces. Assume that

\[
[v_{i_1}, v_{i_2}, v_{j_1}, v_{j_2}] = [v_1, v_2, v_4, v_5].
\]

Then our two primitive relations imply \( v_3 = v_{i_3} \). Contradiction.

\[\Box\]

Lemma 3.5.6 Assume that any two vertices of a Fano 4-polyhedron are joined by a 1-dimensional edge. Then the number of different primitive relations of depth 1 is not more than \( f_3(P)/3 \).

Proof. Let \( v_1, v_2, v_3, v_4 \) be vertices of a 3-dimensional face of \( P \). By 3.5.5, there exists at most one primitive relation of the type

\[
v_{i_1} + v_{i_2} + v_{i_3} = v_{j_1} + v_{j_2}
\]

such that

\[
\{v_{j_1}, v_{j_2}\} \subset \{v_1, v_2, v_3, v_4\}.
\]

On the other hand, for any primitive relation of depth 1 as above, there exist exactly three 3-dimensional faces of \( P \) containing \([v_{j_1}, v_{j_2}]\). \[\Box\]

Lemma 3.5.7 Assume that any two vertices of a Fano 4-polyhedron are joined by a 1-dimensional face. Then the number of 2-dimensional faces of \( P \) having weight \(-1\) is not more than 18 (resp. 27) if \( f_0(P) = 8 \) (resp. if \( f_0(P) = 9 \)).

Proof. If \( f_0(P) = 8 \), then \( f_3(P) = 20 \). Thus, according to 3.5.6 the number of primitive relations of depth 1 is not more than \([20/3] = 6 \). On the other hand, by 2.3.3 and by arguments in the proof of 2.3.7, every 2-dimensional face of \( P \) having weight \(-1\) appears from a primitive relation of degree 1. Moreover, every primitive relation of the type

\[
v_{i_1} + v_{i_2} + v_{i_3} = v_{j_1} + v_{j_2}
\]

gives rise to exactly three 2-dimensional faces of \( P \)

\[
[v_{i_1}, v_{j_1}, v_{j_2}], [v_{i_2}, v_{j_1}, v_{j_2}], [v_{i_3}, v_{j_1}, v_{j_2}]
\]

having the weight \(-1\). Thus, the number of these 2-dimensional faces \( \leq 18 \).
If \( f_0(P) = 9 \), then \( f_3(P) = 27 \). By similar arguments, we obtain the number of 2-dimensional faces of \( P \) having weight \(-1\) is not more than 27.

\[ \square \]

**Proof of theorem 3.5.3** Assume that any two vertices of a Fano polyhedron \( P \) are joined by a 1-dimensional face. Using Dehn-Sommerville equalities and arguments in the proof of 2.3.7, we obtain that the total weight of the Fano polyhedron \( P \) equals

\[ w(P) = 18f_0(P) - 6f_1(P). \]

Denote by \( f^*_2(P) \) the number of 2-dimensional faces of \( P \) having the weight \(-1\).

Since the weight of every 2-dimensional face of \( P \) is at least \(-1\), we obtain

\[ w(P) + f^*_2(P) \geq 0. \]

If \( f_0(P) = 8 \), then \( w(P) = 18 \cdot 8 - 6 \cdot 28 = -24 \). By 3.5.4, \( w(P) + f^*_2(P) \leq -6 \). Contradiction.

If \( f_0(P) = 9 \), then \( w(P) = 18 \cdot 9 - 6 \cdot 36 = -54 \). By 3.5.4, \( w(P) + f^*_2(P) \leq -27 \). Contradiction. \[ \square \]

**Proposition 3.5.8** Let \( P \) be a Fano 4-polyhedron. Assume that for any primitive collection \( \{v_i, v_j\} \subset V(P) \) the corresponding primitive relation has form \( v_i + v_j = 0 \).

Then the following statements hold:

(i) \( f_0(P) \leq 10 \);

(ii) if \( f_0(P) = 10 \), then \( P \) is isomorphic to the convex hull of the vectors

\[ \pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm (e_1 + e_2 + e_3 + e_4). \]

(iii) if \( f_0(P) = 9 \), then \( P \) is isomorphic to the convex hull of the vectors

\[ \pm e_1, \pm e_2, \pm e_3, \pm e_4, (e_1 + e_2 + e_3 + e_4). \]

**Proof.** (i) Since every vertex \( v_i \in V(P) \) is joined by a 1-dimensional face with at least \( f_0(P) - 2 \) other vertices of \( P \), one has

\[ 2f_1(P) \geq f_0(P)(f_0(P) - 2). \]

On the other hand, \( 2f_1(P) \leq 8f_0(P) \) (see 3.5.1). Thus, \( f_0 \leq 10 \).

(ii) By 3.5.3, there exists at least one primitive collection consisting of two vertices, e.g., \( v_9 \) and \( v_{10} \). We shall assume that the corresponding primitive relation is \( v_9 + v_{10} = 0 \). Let \( P_{10} \) be the 3-dimensional projection of \( P \) into \( \mathbb{R}^4/\mathbb{R} < v_{10} > \). By 3.2.3, the combinatorial type of \( \Sigma_{10}(P) \) and the polyhedron \( P_{10} \) allow us to recover the combinatorial structure of the Fano 4-polyhedron \( P \). Since \( \Sigma_{10}(P) \) consists of 12 cones of dimension 4, the polyhedron \( P_{10} \) has at most six 2-dimensional faces of the type \( F(4) \). On the other hand, there are 10 pairs \( \{v'_i, v'_j\} \) of vertices of \( P_{10} \) such
that the cone $\mathbb{R}_{\geq 0}v'_i + \mathbb{R}_{\geq 0}v'_j$ does not belong to the fan $\Sigma_{10}(P)$. By 3.2.3, any such a pair $\{v'_i, v'_j\}$ gives rise to a primitive collection $\{v_i, v_j, v_{10}\} \subset \mathcal{V}(P)$ unless $v'_i, v'_j$ are opposite vertices of some 2-dimensional face of $P_{10}$ having the combinatorial type $F(4)$. Thus, among vertices $\{v_1, \ldots, v_8\}$ there exists at least $10 - 6 = 4$ primitive collections containing only two vertices. This shows that vertices $\{v_1, \ldots, v_8\}$ can be divided into 4 pairs of centrally symmetric vectors. Thus, $P$ is centrally symmetric Fano 4-polyhedron having 10 vertices. By results of Ewald and Voskresenskii & Klyachko [14, 36], there exists a unique possibility for $P$.

(iii) By 3.5.3, there exists at least one primitive collection containing exactly two vertices, e.g., $v_8$ and $v_9$. Therefore, we have $v_8 + v_9 = 0$. Let $P_9$ be the 3-dimensional projection of $P$ into $\mathbb{R}^4/\mathbb{R} < v_9 >$. Let us prove by the same method as in (ii) that among $\{v_1, \ldots, v_7\}$ there exist 3 pairs of centrally symmetric vertices. By our assumption about $P$, it suffices to prove that there exist at least 3 primitive collections in $\{v_1, \ldots, v_7\}$ containing exactly 2 vertices.

Denote by $f(P_9)$ the number of 2-dimensional faces of $P$ having the type $F(4)$. Since $\Sigma_9(P)$ contains exactly 10 cones of dimension 3, we obtain $f(P_9) \leq 5$. On the other hand, there exist 6 pairs $\{v'_i, v'_j\}$ of vertices of $P_9$ such that $\mathbb{R}_{\geq 0}v'_i + \mathbb{R}_{\geq 0}v'_j$ does not belong to the fan $\Sigma_9(P)$. By 3.2.3, any such a pair $\{v'_i, v'_j\}$ gives rise to a primitive collection $\{v_i, v_j, v_9\}$ in $\mathcal{V}(P)$ unless $v'_i, v'_j$ are opposite vertices of some 2-dimensional face of $P_9$ having the combinatorial type $F(4)$. Thus, among vertices $\{v_1, \ldots, v_7\}$ there exist at least $6 - f(P_9) \geq 1$ primitive collections.

Let us prove that $f(P_9) \neq 5, 4$.

Assume that $f(P_9) = 5$. Then $f(P_9) = f_2(P_9)$, and $f_1(P_9) = 10$. On the other hand, every vertex $v'_i \in P_9$ is contained in at least three 1-dimensional faces of $P_9$. Therefore, $f_1(P_9) \geq 7 \cdot 3/2$. Contradiction.

Assume that $f(P_9) = 4$. Then $f_2(P_9) = 6$, and $f_1(P_9) = 11$. For every vertex $v_i \in \{v_1, \ldots, v_7\}$, we denote by $r_i$ the number of 1-dimensional faces of $P_9$ containing $\pi_9(v_i)$, and denote by $c_i$ the number of 2-dimensional faces of $P_9$ having the type $F(4)$ and containing $\pi_9(v_i)$. By 3.2.3, $6 - c_i - r_i$ is the number of primitive collections in $\mathcal{V}(P)$ of the type $\{v_i, v_j\}$ where $v_j \in \{v_1, \ldots, v_7\} \setminus v_i$. By our assumption on $P$, $c_i + r_i$ must be 5 or 6. On the other hand, we have

$$\sum_{i=1}^{7} r_i = 2f_1(P_9) = 22, \quad \sum_{i=1}^{7} c_i = 4f(P_9) = 16.$$

There exists a unique representation of 38 as a sum of 7 numbers which are equal to 5 or 6:

$$38 = 5 + 5 + 5 + 5 + 6 + 6 + 6 = \sum_{i=1}^{7} (r_i + c_i).$$

This representation determines the combinatorial type of $P_9$ uniquely. In particular, there always exists a vertex $\pi_9(v_7) \in P_9$ which is a common vertex of two triangles at the boundary of $P_9$. Since $f(P_9) = 4$, there exist two pairs of centrally symmetric vertices among $\{v_1, \ldots, v_6\}$. We assume, for instance, that $v_1 + v_2 = v_3 + v_4 =$.
0. Then, applying an automorphism of 3-dimensional integral lattice, we can transform \( v'_1, v'_2, v'_3, v'_4, v'_7 \) to the form
\[
v'_1 = (1, 0, 0) = -v'_2, \quad v'_3 = (0, 1, 0) = -v'_4, \quad v'_7 = (0, 0, 1).
\]
For vertices \( v'_6 \) and \( v'_5 \) we have only the following 4 possibilities
\[(1, 1, -1), (-1, 1, -1), (1, -1, -1), (-1, -1, -1).\]
By direct checking all possible case for \( v'_6 \) and \( v'_5 \), one concludes that it is impossible to get a reflexive 3-polyhedron \( P_0 \) having exactly 4 faces of the type \( F(4) \).

Thus, vertices \( \{v_1, \ldots, v_7\} \) of \( P \) can be always divided into 3 pairs of centrally symmetric vectors. We assume that these centrally pairs are \( \{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\} \). Together with \( \{v_8, v_9\} \). Since there exist a 4-dimensional cone in \( \Sigma(P) \) which does not contain \( v_7 \), we can assume that \( v_2, v_4, v_6, v_8 \) is a basis of \( \mathbb{Z}^4 \). Now it is easy to check directly that \( v_7 \in \{\pm v_2, \pm v_4, \pm v_6, \pm v_8\} \).

**Proposition 3.5.9** Let \( P \) be a Fano 4-polyhedron. Then the following statements hold

(i) \( f_0(P) \leq 12 \);
(ii) if \( f_0(P) = 12 \), then \( V(P) \) is the product of two Del Pezzo surfaces \( S_3 \) with the Picard rank 4;
(iii) if \( f_0(P) = 11 \), then \( V(P) \) is the product of a Del Pezzo surface \( S_3 \) with the Picard rank 4 and a Del Pezzo surface \( S_2 \) with the Picard rank 3.

**Proof.** (i) By 3.5.3, we can assume that there exists a primitive collection containing exactly 2 vertices \( \{v_1, v_2\} \). Moreover, by 3.5.5, we can assume that the corresponding primitive relation is \( v_1 + v_2 = v_0 \). By 2.3.4, \( D_0 \) is a \( \mathbb{P}^1 \)-bundle over a toric surface \( S \). We know that the canonical class of \( D_0 \) is numerically effective (see 2.4.3) and \( P_0 = \pi_0(P) \) can contain only faces of the types \( F(1) \) and \( F(4) \) if \( f_0(P) \geq 10 \) (see 3.2-3.4). So the toric surface \( S \) can have at most Picard number 4, i.e., the valence of \( D_0 \) is at most 8. Let \( \{v_3, \ldots, v_k\} \) \((k \leq 8)\) be the set of vertices which are joined with \( v_0 \) by an edge of \( P \). If one vertex \( \pi_0(v_i) \in \{\pi_0(v_3), \ldots, \pi_0(v_k)\} \) is a double vertex, then by 2.3.4, \( \Sigma_i(P) \) would have the combinatorial type \( v_2 \) (since the cone \( \mathbb{R}_{\geq 0}\pi_0(v_i) \) is contained in exactly 4 cones of \( \Sigma_i(P) \)). By 3.4.2 and 3.4.3, this would imply that \( f_0(P) \leq 10 \). If \( \pi_0(v_1) \) and \( \pi_0(v_2) \) are double points of \( P_0 \), then there exists at most 3 vertices which are not not joined with \( v_0 \): at most 2 vertices projecting to \( \pi_0(v_1) \) and \( \pi_0(v_2) \) and at most one centrally symmetric to \( v_0 \) vertex. This implies that \( f_0(P) \leq 12 \).

(ii) It follows from the arguments in (i) that if \( f_0(P) = 12 \), then \( D_0 \) is a \( \mathbb{P}^1 \)-bundle over a Del Pezzo surface \( S_3 \) obtained by blow ups of 3-points in \( \mathbb{P}^2 \). Moreover, there exist primitive relations \( v_0 + v_9 = v_1, v_0 + v_{10} = v_2 \) and \( v_0 + v_{11} = 0 \), i.e., 6 vertices \( v_0, v_1, v_2, v_9, v_{10}, v_{11} \) are contained in a 2-dimensional linear subspace in \( \mathbb{R}^4 \) and their convex hull is isomorphic to a single Fano 2-polyhedron with 6 vertices. By 3.2.3, the combinatorial type of \( P \) is determined by \( P_0 \), in particular,
\{v_3, v_4, v_5, v_6, v_7, v_8\} contains 9 primitive collections consisting of 2 vertices. Since, \(P_0\) contains only faces of the type \(F(1)\) and \(F(4)\) the latter implies that the convex hull of \(\{v_3, v_4, v_5, v_6, v_7, v_8\}\) again must be isomorphic to a single Fano 2-polyhedron with 6 vertices.

(iii) Let \(f_0(P) = 11\). Our previous arguments in (i) and (ii) show that we have a primitive relation \(v_1 + v_2 = v_0\). It remains to consider the following two cases:

**Case 1**: Valence of \(v_0\) equals 7. Then there exist 3 vertices which are not joined with \(v_0\): 2 vertices \(v_8, v_9\) projecting to \(\pi_0(v_1), \pi_0(v_2)\) and one centrally symmetric to \(v_0\) vertex \(v_{10}\). As above we obtain that \(v_0, v_1, v_2, v_8, v_9, v_{10}\) are contained in a 2-dimensional linear subspace in \(\mathbb{R}^4\) and their convex hull is isomorphic to a single Fano 2-polyhedron with 6 vertices and the convex hull of \(\{v_3, v_4, v_5, v_6, v_7\}\) is isomorphic to a single Fano 2-polyhedron with 5 vertices. Therefore \(V\) is isomorphic to \(S_2 \times S_3\).

**Case 2**: Valence of \(v_0\) equals 8. Then the vertices \(v_0, v_1, v_2, v_9, v_{10}\) are contained in a 2-dimensional linear subspace in \(\mathbb{R}^4\) and their convex hull is isomorphic to a single Fano 2-polyhedron with 5 vertices. Without loss of generality we can assume that \(v_0\) and \(v_{10}\) are centrally symmetric (otherwise we could replace \(v_0\) by \(v_1\) or by \(v_2\)). It remains to show that the convex hull of \(\{v_3, v_4, v_5, v_6, v_7, v_8\}\) is isomorphic a single Fano 2-polyhedron with 6 vertices. The last property follows from that fact that the combinatorial type of \(P\) is determined by \(P_0\) (see 3.2.3) and the faces of \(P_0\) are of the type \(F(1)\) or \(F(4)\).

\[\square\]

**Proposition 3.5.10** Let \(P\) be a Fano 4-polyhedron such that \(f_0(P) \in \{9, 10\}\) and all vertices of \(P\) have valence \(\geq 7\). Then the following statements hold

(i) if \(f_0(P) = 9\), then \(P\) is isomorphic to the polyhedron from 3.5.8(iii);

(ii) if \(f_0(P) = 10\), then \(P\) is isomorphic to the polyhedron from 3.5.8(ii) or \(V(P)\) is the product of two Del Pezzo surfaces \(S_2\) with the Picard rank 3.

**Proof.** We can assume that there always exists a primitive relation of the type \(v_1 + v_2 = v_3\). By 2.3.4, using the contraction of the exceptional divisor \(D_3\), we obtain a smooth projective toric variety \(V'\). We remark that \(V\) must be a toric Fano 4-fold coresponding to a Fano 4-polyhedron \(P'\), because vertices \(v_1, v_2 \in P'\) have valence at least 7, and the surface \(D_{[1,2]}\) has the Picard number at least 3 (see 3.3.2).

To prove (i), we check all possibilities for the polyhedron \(P'\) using the classification of Fano 4-polyhedra with 8 vertices. As a result, this completes the classification of Fano 4-polyhedra with 9 vertices.

To prove (ii), we check all possibilities for the polyhedron \(P'\) using the just received classification of Fano 4-polyhedra with 9 vertices. \[\square\]

**Theorem 3.5.11** There exist exactly 123 different types of 4-dimensional Fano polyhedra \(P\). The maximal number of vertices of \(P\) is 12:

| the number of vertices | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|------------------------|---|---|---|---|---|----|----|----|
| the number of polyhedra | 1 | 9 | 28 | 47 | 26 | 10 | 1  | 1  |
4 Table of toric Fano 4-folds

In the following table below we give the list of all toric Fano 4-folds $V$ with their numerical characteristics: the products of characteristic classes $c_1^4, c_1^2c_2$; the Betti numbers $b_2, b_4$; the dimensions $h^0 = h^0(V, -K_V)$ and $a(V) := \dim \text{Aut}(V)$.

| $n^0$ | $c_1^4$ | $c_1^2c_2$ | $b_2$ | $b_4$ | $a(V)$ | $h^0$ | the type of Fano polytope |
|-------|---------|------------|-------|-------|--------|-------|--------------------------|
| 1     | 625     | 250        | 1     | 1     | 24     | 126   | $\mathbb{P}^4$            |
| 2     | 800     | 296        | 2     | 2     | 36     | 159   | $B_1, \mathbb{P}^3(\mathcal{O} \oplus \mathcal{O}(3))$ |
| 3     | 640     | 256        | 2     | 2     | 26     | 129   | $B_2, \mathbb{P}^3(\mathcal{O} \oplus \mathcal{O}(2))$ |
| 4     | 544     | 232        | 2     | 2     | 20     | 111   | $B_3, \mathbb{P}^3\mathcal{O} \oplus \mathcal{O}(1)$ |
| 5     | 512     | 224        | 2     | 2     | 18     | 105   | $B_4, \mathbb{P}^1 \times \mathbb{P}^3$ |
| 6     | 512     | 224        | 2     | 2     | 18     | 105   | $B_5, \mathbb{P}^3(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ |
| 7     | 594     | 240        | 2     | 3     | 24     | 120   | $C_1, \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ |
| 8     | 513     | 222        | 2     | 3     | 18     | 105   | $C_2, \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ |
| 9     | 513     | 222        | 2     | 3     | 18     | 105   | $C_3, \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$ |
| 10    | 486     | 216        | 2     | 3     | 16     | 100   | $C_4, \mathbb{P}^2 \times \mathbb{P}^2$ |
| 11    | 605     | 254        | 3     | 3     | 23     | 123   | $E_1$, blow up of $\mathbb{P}^2$ on $n^0_5$ |
| 12    | 489     | 222        | 3     | 3     | 17     | 101   | $E_2$, blow up of $\mathbb{P}^2$ on $n^0_4$ |
| 13    | 431     | 206        | 3     | 3     | 14     | 90    | $E_3$, blow up of $\mathbb{P}^2$ on $n^0_5$ |
| 14    | 592     | 244        | 3     | 4     | 24     | 120   | $D_1, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1, 2))$ |
| 15    | 576     | 240        | 3     | 4     | 23     | 117   | $D_2, \mathbb{P}^1$-bundle over $V(B_1)$ |
| 16    | 560     | 236        | 3     | 4     | 22     | 114   | $D_3, \mathbb{P}^1$-bundle over $V(B_2)$ |
| 17    | 560     | 236        | 3     | 4     | 22     | 114   | $D_4, \mathbb{P}^1$-bundle over $V(B_3)$ |
| $n^0$ | $c_1^4$ | $c_2^2$ | $b_2$ | $b_4$ | $a(V)$ | $h^0$ | the type of Fano polytope |
|-------|--------|--------|-------|-------|--------|-------|--------------------------|
| 18    | 496    | 220    | 3     | 4     | 18     | 102   | $D_5, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(2))$ |
| 19    | 496    | 220    | 3     | 4     | 18     | 102   | $D_6, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1, 1))$ |
| 20    | 486    | 216    | 3     | 4     | 18     | 100   | $D_7, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1, 1))$ |
| 21    | 432    | 204    | 3     | 4     | 14     | 90    | $D_8, \mathbb{P}^1$-bundle over $V(\mathcal{B}_2)$ |
| 22    | 464    | 212    | 3     | 4     | 16     | 96    | $D_9, \mathbb{P}^1$-bundle over $V(\mathcal{B}_2)$ |
| 23    | 464    | 212    | 3     | 4     | 16     | 96    | $D_{10}, \mathbb{P}^1$-bundle over $V(\mathcal{B}_3)$ |
| 24    | 459    | 210    | 3     | 4     | 15     | 95    | $D_{11}, \mathbb{P}^1(F_1(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(l)))$ $l$ is a curve of index 1 on $F_1$ |
| 25    | 448    | 208    | 3     | 4     | 15     | 93    | $D_{12}, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1))$ |
| 26    | 432    | 204    | 3     | 4     | 14     | 90    | $D_{13}, \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ |
| 27    | 432    | 204    | 3     | 4     | 14     | 90    | $D_{14}, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ |
| 28    | 432    | 204    | 3     | 4     | 14     | 90    | $D_{15}, F^1 \times \mathbb{P}^2$ |
| 29    | 432    | 204    | 3     | 4     | 14     | 90    | $D_{16}, \mathbb{P}^1$-bundle over $V(\mathcal{B}_1)$ |
| 30    | 405    | 198    | 3     | 4     | 12     | 85    | $D_{17}, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(0, 0) \oplus \mathcal{O}(0, 1))$ |
| 31    | 400    | 196    | 3     | 4     | 12     | 84    | $D_{18}, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(-1, 2))$ |
| 32    | 400    | 196    | 3     | 4     | 12     | 84    | $D_{19}, \mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(-1, 1))$ |
| 33    | 529    | 226    | 3     | 5     | 20     | 108   | $G_1$, is not contractible smoothly |
| 34    | 450    | 204    | 3     | 5     | 16     | 93    | $G_2$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n_8^{09}$ |
| 35    | 433    | 202    | 3     | 5     | 14     | 90    | $G_3$, blow up of a curve $\mathbb{P}^1$ on $n_9^{09}$ |
| 36    | 417    | 198    | 3     | 5     | 13     | 87    | $G_4$, blow up of a surface $F_1$ on $n_8^{09}$ |
| 37    | 406    | 196    | 3     | 5     | 12     | 85    | $G_5$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n_9^{09}$ |
| $n^0$ | $c_1^4$ | $c_2^2c_2$ | $b_2$ | $b_4$ | $a(V)$ | $h^0$ | the type of Fano polytope |
|-------|---------|------------|-------|-------|--------|-------|----------------------------|
| 38    | 401     | 194        | 3     | 5     | 12     | 84    | $G_6$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{010}$ |
| 39    | 588     | 240        | 4     | 5     | 22     | 114   | $H_1$, blow up of two surfaces $\mathbb{P}^2$ on $n^{003}$ |
| 40    | 505     | 226        | 4     | 5     | 19     | 104   | $H_2$, blow up of a surface $\mathbb{P}^2$ on $n^{016}$ |
| 41    | 478     | 220        | 4     | 5     | 17     | 99    | $H_3$, blow up of a surface $\mathbb{P}^2$ on $n^{014}$ |
| 42    | 447     | 210        | 4     | 5     | 16     | 93    | $H_4$, blow up of two surfaces $\mathbb{P}^2$ on $n^{008}$ |
| 43    | 415     | 202        | 4     | 5     | 14     | 87    | $H_5$, blow up of a surface $\mathbb{P}^2$ on $n^{019}$ |
| 44    | 409     | 202        | 4     | 5     | 13     | 96    | $H_6$, blow up of a surface $\mathbb{P}^2$ on $n^{022}$ |
| 45    | 382     | 196        | 4     | 5     | 11     | 81    | $H_7$, blow up of two surfaces $\mathbb{P}^2$ on $n^{003}$ |
| 46    | 378     | 192        | 4     | 5     | 12     | 80    | $H_8$, $\mathbb{P}^2 \times S_2$ |
| 47    | 367     | 190        | 4     | 5     | 11     | 78    | $H_9$, blow up of two surfaces $\mathbb{P}^2$ on $n^{008}$ |
| 48    | 351     | 186        | 4     | 5     | 10     | 75    | $H_{10}$, blow up of a surface $\mathbb{P}^2$ on $n^{019}$ |
| 49    | 480     | 216        | 4     | 6     | 18     | 99    | $L_1$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O}(1,1,1))$ |
| 50    | 464     | 212        | 4     | 6     | 17     | 96    | $L_2$, $\mathbb{P}^1$-bundle over $\mathbb{V}(\mathcal{C}_1)$ |
| 51    | 448     | 208        | 4     | 6     | 16     | 93    | $L_3$, $\mathbb{P}^1 \times F_1 (\mathcal{O} \oplus \mathcal{O}(1) \otimes \mathcal{O}(l))$ $l$ is a curve of index 1 on $F_1$ |
| 52    | 432     | 204        | 4     | 6     | 15     | 90    | $L_4$, $\mathbb{P}^1$-bundle over $\mathbb{V}(\mathcal{C}_3)$ |
| 53    | 416     | 200        | 4     | 6     | 14     | 87    | $L_5$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O}(1,1,1))$ |
| 54    | 400     | 196        | 4     | 6     | 13     | 84    | $L_6$, $\mathbb{P}^1 \times F_1 (\mathcal{O} \oplus \mathcal{O}(l))$ $l$ is a curve of index 1 on $F_1$ |
| 55    | 384     | 192        | 4     | 6     | 12     | 81    | $L_7$, $F_1 \times F_1$ |
| 56    | 384     | 192        | 4     | 6     | 12     | 81    | $L_8$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}$ |
| 57    | 384     | 192        | 4     | 6     | 12     | 81    | $L_9$, $\mathbb{P}^1 \times \mathbb{P}^1 \times F_1$ |
| \(n^0\) | \(c_1^2\) | \(c_2^2\) | \(b_2\) | \(b_4\) | \(a(V)\) | \(h^0\) | Type of Fano polytope |
|---|---|---|---|---|---|---|---|
| 58 | 384 | 192 | 4 | 6 | 12 | 81 | \(L_{10}, \mathbb{P}^1\)-bundle over \(V(D_2)\) |
| 59 | 352 | 184 | 4 | 6 | 10 | 75 | \(L_{11}, \mathbb{P}^1 \times V(D_2)\) |
| 60 | 352 | 184 | 4 | 6 | 10 | 75 | \(L_{12}, \mathbb{P}_{\mathbb{P}^1 \times \mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(1) \otimes \mathcal{O}(-l))\) \(l\) is a curve of index 1 on \(F_1\) |
| 61 | 352 | 184 | 4 | 6 | 10 | 75 | \(L_{13}, \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1, -1))\) |
| 62 | 496 | 220 | 4 | 6 | 19 | 102 | \(I_1\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,17}\) |
| 63 | 463 | 214 | 4 | 6 | 16 | 96 | \(I_2\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,19}\) |
| 64 | 442 | 208 | 4 | 6 | 15 | 92 | \(I_3\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,21}\) |
| 65 | 433 | 202 | 4 | 6 | 13 | 87 | \(I_4\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,23}\) |
| 66 | 415 | 202 | 4 | 6 | 13 | 87 | \(I_5\), blow up of a surface \(\mathbb{P}^2\) on \(n^{0,23}\) |
| 67 | 411 | 198 | 4 | 6 | 14 | 86 | \(I_6\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,23}\) |
| 68 | 400 | 196 | 4 | 6 | 13 | 84 | \(I_7, \mathbb{P}^1 \times V(D_1)\) |
| 69 | 384 | 192 | 4 | 6 | 12 | 81 | \(I_8\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,21}\) |
| 70 | 390 | 192 | 4 | 6 | 13 | 82 | \(I_9\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,25}\) |
| 71 | 389 | 194 | 4 | 6 | 12 | 82 | \(I_{10}\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,28}\) |
| 72 | 384 | 192 | 4 | 6 | 12 | 81 | \(I_{11}\), \(\mathbb{P}^1\)-bundle over \(V(D_1)\) |
| 73 | 347 | 182 | 4 | 6 | 10 | 74 | \(I_{12}\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,28}\) |
| 74 | 368 | 188 | 4 | 6 | 11 | 78 | \(I_{13}, \mathbb{P}^1 \times V(D_2)\) |
| 75 | 357 | 186 | 4 | 6 | 10 | 76 | \(I_{14}\), blow up of a surface \(\mathbb{F}_1\) on \(n^{0,27}\) |
| 76 | 337 | 178 | 4 | 6 | 10 | 72 | \(I_{15}\), blow up of a surface \(\mathbb{P}^1 \times \mathbb{P}^1\) on \(n^{0,32}\) |
| 77 | 385 | 190 | 4 | 7 | 12 | 81 | \(M_1\), is not contractible smoothly |
| $n^0$ | $a(V)$ | $h^0$ | $c_1^4$ | $c_1^2c_2$ | $b_2$ | $b_4$ | $a(V)$ | $h^0$ | the type of Fano polytope |
|------|--------|------|--------|--------|------|------|--------|------|----------------------------|
| 78   | 14     | 87   | 417    | 198    | 4    | 7    |        |      | $M_2$, is not contractible smoothly |
| 79   | 11     | 78   | 369    | 186    | 4    | 7    |        |      | $M_3$, blow up of a surface $F_1$ on $n^{035}$ |
| 80   | 11     | 78   | 369    | 186    | 4    | 7    |        |      | $M_4$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{035}$ |
| 81   | 11     | 77   | 364    | 184    | 4    | 7    |        |      | $M_5$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $\mathbb{P}^2 \times \mathbb{P}^2$, and then blow up of $F_1$ |
| 82   | 10     | 78   | 368    | 188    | 4    | 7    |        |      | $J_1$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{035}$ |
| 83   | 8      | 70   | 326    | 176    | 4    | 7    |        |      | $J_2$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{035}$ |
| 84   | 16     | 92   | 422    | 208    | 5    | 8    |        |      | $Q_1$, blow up of two surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{020}$ |
| 85   | 14     | 85   | 405    | 198    | 5    | 8    |        |      | $Q_2$, blow up of two surfaces $F_1$ on $n^{028}$ |
| 86   | 13     | 83   | 394    | 196    | 5    | 8    |        |      | $Q_3$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{049}$ |
| 87   | 14     | 85   | 405    | 198    | 5    | 8    |        |      | $Q_4$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{051}$ |
| 88   | 12     | 79   | 373    | 190    | 5    | 8    |        |      | $Q_5$, blow up of a surface $F_1$ on $n^{051}$ |
| 89   | 12     | 78   | 368    | 188    | 5    | 8    |        |      | $Q_6$, $\mathbb{P}^1 \times V(\mathcal{E}_1)$ |
| 90   | 12     | 77   | 363    | 186    | 5    | 8    |        |      | $Q_7$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{055}$ |
| 91   | 11     | 75   | 352    | 184    | 5    | 8    |        |      | $Q_8$, $\mathbb{P}^1 \times V(\mathcal{E}_2)$ |
| 92   | 10     | 73   | 341    | 182    | 5    | 8    |        |      | $Q_9$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{051}$ |
| 93   | 10     | 72   | 336    | 180    | 5    | 8    |        |      | $Q_{10}$, $F_1 \times S_2$ |
| 94   | 10     | 72   | 336    | 180    | 5    | 8    |        |      | $Q_{11}$, $\mathbb{P}^1 \times \mathbb{P}^1 \times S_2$ |
| 95   | 10     | 71   | 331    | 188    | 5    | 8    |        |      | $Q_{12}$, blow up of two surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{030}$ |
| 96   | 9      | 71   | 330    | 180    | 5    | 8    |        |      | $Q_{13}$, blow up of two surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{020}$ |
| 97   | 9      | 70   | 325    | 178    | 5    | 8    |        |      | $Q_{14}$, blow up of two surfaces $F_1$ on $n^{024}$ |
| $n^0$ | $c_1^4$ | $c_1^2c_2$ | $b_2$ | $b_4$ | $a(V)$ | $h^0$ | the type of Fano polytope |
|-------|---------|------------|-------|-------|--------|-------|--------------------------|
| 98    | 320     | 176        | 5     | 8     | 9      | 69    | $Q_{15}, \mathbb{P}^1 \times V(\mathcal{E}_4)$ |
| 99    | 310     | 172        | 5     | 8     | 9      | 67    | $Q_{16}$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{061}$ |
| 100   | 299     | 170        | 5     | 8     | 8      | 65    | $Q_{17}$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{055}$ |
| 101   | 364     | 196        | 5     | 6     | 10     | 78    | $K_1$, blow up of three surfaces $\mathbb{P}^2$ on $n^{07}$ |
| 102   | 354     | 192        | 5     | 6     | 10     | 76    | $K_2$, blow up of two surfaces $\mathbb{P}^2$ on $n^{022}$ |
| 103   | 334     | 184        | 5     | 6     | 10     | 72    | $K_3$, blow up of three surfaces $\mathbb{P}^2$ on $n^{08}$ |
| 104   | 324     | 180        | 5     | 6     | 10     | 70    | $K_4$, $\mathbb{P}^2 \times S_3$ |
| 105   | 332     | 176        | 5     | 9     | 10     | 71    | $R_1$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{078}$ |
| 106   | 321     | 174        | 5     | 9     | 9      | 69    | $R_2$, blow up of a surface $S_1$ on $n^{078}$ |
| 107   | 305     | 170        | 5     | 9     | 8      | 66    | $R_3$, blow up of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{079}$ |
| 108   | 310     | 172        | 5     | 9     | 8      | 67    | blow up of a surface $S_2$ on $\mathbb{P}^1 \times V(D_2)$ see 3.4.1 |
| 109   | 308     | 176        | 6     | 10    | 8      | 67    | $U_1$, blow up of three surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{020}$ |
| 110   | 298     | 172        | 6     | 10    | 8      | 65    | $U_2$, blow up of three surfaces $F_1$ on $n^{024}$ |
| 111   | 298     | 172        | 6     | 10    | 8      | 65    | $U_3$, blow up of two surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $S_1 \times S_1$ |
| 112   | 288     | 168        | 6     | 10    | 8      | 63    | $U_4$, $S_1 \times S_3$ |
| 113   | 288     | 168        | 6     | 10    | 8      | 63    | $U_5$, $\mathbb{P}^1 \times \mathbb{P}^1 \times S_3$ |
| 114   | 288     | 168        | 6     | 10    | 8      | 63    | $U_6$, $\mathbb{P}^1 \times V(F_1)$ |
| 115   | 278     | 164        | 6     | 10    | 8      | 61    | $U_7$, blow up of three surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $n^{030}$ |
| 116   | 268     | 160        | 6     | 10    | 8      | 59    | $U_8$, blow up of two surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ on $\mathbb{P}^1 \times V(\mathcal{E}_5)$ |
| 117   | 307     | 166        | 5     | 11    | 8      | 66    | see 3.5.8(ii) |

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Using the generalized Euler exact sequence for smooth projective toric varieties [4], one obtains the following formula for the dimension of the group of biregular automorphisms of a toric Fano 4-fold $V$:

$$a(V) = 22 + 3b_2 - b_4 + \frac{2c_1^4 - 5c_1c_2}{12}.$$  

If $W \subset V$ is a smooth Calabi-Yau 3-fold, then the Hodge number $h^{2,1}(W)$ is determined by the formula

$$h^{2,1}(W) = h^0(V, -K_V) - a(V) - 1 = \frac{c_1^2c_2}{2} + b_4 - 3b_2 - 22,$$

where $h^0(V, -K_V) = 1 + \frac{1}{12}(c_1^2c_2 + 2c_1^4)$.

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