Abstract

We define the Artinian and Noetherian algebra which consist of formal series involving exponents which are not necessarily integers. All of the usual operations are defined here and characterized. As an application, we compute the algebra of symmetric functions with nonnegative real exponents. The applications to logarithmic series and the Umbral calculus are deferred to another paper.[9]

Les Séries à Exposants Quelconques

On définit ici les algèbres Artinienne et Noetherienne comme étant des algèbres constituées des séries formelles à exposants pas nécessairement entiers. On définit sur ces algèbres toutes les opérations classiques et on les caractérise. Comme exemple d’exploitation de cette théorie, on s’intéresse à algèbre de fonctions symétriques à exposants réels en non-négatifs. Une autre publication [9] est consacrée aux applications aux séries logarithmiques et au calcul ombral.

Dedicated to
Hélène

Contents

1 Introduction 2

1.1 The Artinian Algebra and Noetherian Algebra 3
1.2 Topology 6
1.3 Complex Numbers Raised to a Real Power 8
1.4 Artinian Series to a Real Power 11
1.5 Composition of Series 16

2 Derivative 19

3 LaGrange Inversion 20

4 Symmetric Artinian and Noetherian Series 21
1 Introduction

The simplest type of series is the polynomial. However, it is common to occasionally consider other more general series formal power series, Laurent series, inverse formal power series and so on. Nevertheless, little consideration has been given to series whose exponents are not necessarily integers. We would like to derive a “continuous” analog of Laurent or inverse Laurent series in which exponents are chosen from the real numbers (or any other poset) instead of merely from the integers. Here for any choice of coefficients and exponents we define two sets of formal series: Artinian and Noetherian.

These series are not merely of academic interest, since the Noetherian series in infinitely many variables represent asymptotic expansions of real functions with respect to the ladder of comparison

\[ x^a \log x^b \log \log x^c \cdots \]

where \( a, b, c, \ldots \) are real numbers, and the Artinian series in the derivative represent the logarithmic analog of shift-invariant operators. We defer this application to another article [9].

However, as typical applications of this theory, we will compute the Artinian and Noetherian algebras of symmetric functions with nonnegative real exponents, and derive the Lagrange inversion formula for Artinian and Noetherian series.

For these applications, it must be proven that one can manipulate formal power series with real exponents as easily as one does polynomials. To that end, we define Artinian series in which for all real numbers \( a \) there are finitely many terms of degree \( b \) with \( b < a \), and Noetherian series which have the dual condition.

They are equipped with a topology and operations which make them a topological algebra over the complex numbers. Moreover, both types of series actually form a field.

Even further operations can be defined. We define \( f(x) \) raised to any real exponent \( a \). This definition involves the choice of any arbitrary integer \( n \), so we write \( f(x)^{a/n} \). The operation is then characterized in terms of its algebraic properties. We further define the composition of one series with another \( f(g; n) \) and characterize this. This composition is not always associative; however it is associative in the cases relevant to [9].
1.1 The Artinian Algebra and Noetherian Algebra

**Definition 1.1** (Artinian Sequence) Let $R$ be a poset (usually the real numbers). Define an Artinian poset to be any subposet $S \subseteq R$ such that for all $a \in S$ there are only finitely many $b \leq a$ such that $b \in S$. That is, all principal ideals of the subposet $S$ are finite.

Let $K$ be an additive group (usually the complex numbers). We say a sequence $(c_a)_{a \in R}$ of group elements indexed by the poset is Artinian if its support is Artinian. That is, if for all $a \in R$ there are only finitely many $b \leq a$ such that $c_b \neq 0$.

Dually, we say $S$ is a Noetherian poset if for all $a \in S$ there are only finitely many $b \geq a$ such that $b \in S$. That is, all principal filters of the subposet $S$ are finite. We say a sequence is Noetherian if its support is Noetherian. That is, if for all $a \in R$ there are only finitely many $b \geq a$ such that $c_b \neq 0$.

We note a few trivial observations. Nonempty Artinian (resp. Noetherian) posets have minimal (resp. maximal) elements, so nonzero Artinian (resp. Noetherian) sequences have lowest nonzero terms.

A poset is both Artinian and Noetherian if and only if it is finite, so a sequence is both Artinian and Noetherian if and only if it has finite support.

**Definition 1.2** (Artinian Algebra) Define the Artinian algebra $K(x)^R$ to be the set of all formal sums $f(x) = \sum_{a \in R} c_a x^a$ such that the $c_a$ form an Artinian sequence. $f(x)$ is called a Artinian series.

Define the degree of $f(x)$ to be

$$
\text{deg}(f(x)) = \begin{cases} 
\max \{a : c_a \neq 0\} & \text{if } f(x) \neq 0, \text{ and } \\
-\infty & \text{if } f(x) = 0.
\end{cases}
$$

Similarly, the Noetherian algebra $K(x)_R$ is the collection of all formal sums $f(x) = \sum_{a \in R} c_a x^a$ where the $c_a$ form a Noetherian sequence. $f(x)$ is called a Noetherian series Define its degree to be

$$
\text{deg}(f(x)) = \begin{cases} 
\min \{a : c_a \neq 0\} & \text{if } f(x) \neq 0, \text{ and } \\
+\infty & \text{if } f(x) = 0.
\end{cases}
$$

In either case, we denote the coefficient of $x^a$ in $f(x)$ by $[x^a] f(x)$. For $f(x) \neq 0$, the coefficient $[x^{\text{deg}(f(x))}] f(x)$ is called the leading coefficient of $f(x)$, and $(\binom{\text{deg}(f(x))}{f(x)}) x^{\text{deg}(f(x))}$ is called the leading term of $f(x)$. A series of degree one is called a delta series.
PROPOSITION 1.3 Suppose $K$ is a ring, and $R$ is an ordered monoid. Then the Artinian Algebra and the Noetherian Algebra are $K$-algebras when they are equipped with the operations:

\[
\left( \sum_a c_a x^a \right) + \left( \sum_a d_a x^a \right) = \sum_a (c_a + d_a)x^a \tag{1}
\]

\[
\left( \sum_a c_a x^a \right) \left( \sum_a d_a x^a \right) = \sum_c \left( \sum_{a+b=c} c_a d_b \right) x^c \tag{2}
\]

\[
z \left( \sum_a c_a x^a \right) = \sum_a (zc_a)x^a. \tag{3}
\]

Proof: It must be shown that all these operations are well defined, and that they obey the axioms of $K$-algebras. For brevity, we only show that multiplication is well defined in the Noetherian Algebra. To do this, we must first show that the coefficients of a product are well defined, and then show that the coefficients form a Noetherian sequence.

Let $e_c = \sum_{a+b=c} c_a d_b$. To show that $e_c$ is well defined, let $a_0$ be the maximum $a$ such that $c_a \neq 0$. Now, there are only finitely many $b \geq -a - a_0$ such that $d_b \neq 0$. Thus, $e_c$ is well defined.

Let $b_0$ be the maximum $b$ such that $d_b \neq 0$. Chose some $c' \geq c$ such that $e_{c'} \neq 0$. Then there are some $a$ and $b$ which sum to $c'$ and such that $c_a \neq 0$ and $d_b \neq 0$. Moreover, $a > c - b_0$ and $b > c - a_0$, so there is only finitely many such $a$ and $b$. Thus, there are only finitely many such $c'$. Hence, the sequence $e_c$ is Noetherian.

COROLLARY 1.4 Suppose $K$ is an integral domain, and $R$ is an ordered monoid. Then the Artinian algebra and Noetherian algebra are integral domains.

PROPOSITION 1.5 Suppose $K$ is a field, and $R$ is an Archimedean group. Then the Artinian Algebra and the Noetherian Algebra are fields.

Proof: Given Proposition 1.3, it suffices only to calculate inverses. We only treat the Noetherian case, since the Artinian case is similar.

Let $f(x)$ be a nonzero Noetherian series of degree $a$ with coefficients given by $c_b = [x^b]f(x)$. We calculate the coefficients of its inverse. Let $d_{-a} = c_a^{-1}$, and for $b < -a$, let

\[
d_b = -c_a^{-1} \sum_{b+e \leq -a} d_b c_{a+b-e}.
\]

$d_b$ is well defined, since there are finitely many nonzero terms being summed over in each step of the recursion. We claim that $d_b$ is Noetherian. If so, it is easy to check that $\sum_b d_b x^b$ is the multiplicative inverse of $f(x)$. 

Suppose \( d_b \) is not Noetherian. Then there is a sequence \( b_1 < b_2 < \cdots < -a \) such that \( d_{b_i} \neq 0 \) for all \( i \). Since the reals are Archimedean, \( b_i - b_{i+1} \) must tend to zero (in the order topology). However \( d_b \neq 0 \) for \( b < -a \) only if \( d_{a+b-e} \neq 0 \) where \( e \) is one of the finitely many \( e \geq a - b \) such that \( c_e \neq 0 \). The set of differences among the various \( e \) is finite so it has a lower bound. Contradiction.

Note that Theorem 1.22 provides an easier proof of a more sweeping result under slightly stronger conditions. Conversely, the following Porism strips the requirements for the existence of inverses to the bone.

**Porism 1.6** Suppose \( K \) is a ring, and \( R \) is an Archimedean monoid. Then a nonzero Artinian or Noetherian series has a multiplicative inverse if and only if its degree has an additive inverse and its leading coefficient has a multiplicative inverse.

Some of the most common types are series are Noetherian or Artinian series:

**Example 1.1** \( K(\mathbb{N}) \) is merely the ring of polynomials in the variable \( x \). The only invertible series are invertible constants.

**Example 1.2** \( K(\mathbb{Z}) \) (where \( K \) is a field) is the field of Laurent series in the variable \( x \).

**Example 1.3** \( K(\mathbb{N}) \) is the ring of formal power series in the variable \( x \). The only invertible series are ones with an invertible constant term.

We wish to define series in several variables. However, we must be careful, since the Artinian algebra \( (K(\mathbb{N}))_{R} \) is not necessarily equal to the Artinian algebra \( (K(\mathbb{Z}))_{R} \). For example,

\[
\sum_{n \geq 0} x^n y^{-n} \in \left( K(\mathbb{Z}) \right)_{(y)} \subsetneq \left( K(\mathbb{Z}) \right)_{(x)}.
\]

**Definition 1.7** (Multivariate Noetherian Algebra) Let \( \mathcal{R} = \{R_1, \ldots, R_n\} \) be a collection of posets and let \( X = \{x_1, \ldots, x_n\} \) be a collection of variables. Given a group \( K \) we recursively define the multivariate Noetherian algebra in the variables \( X \) indexed by \( \mathcal{R} \) by the recursion

\[
K(x_1; R_1, \ldots, x_n; R_n) = \begin{cases} 
(K(x_n; R_n)) (x_1; R_1, \ldots, x_{n-1}; R_{n-1}) & \text{for } n > 0, \text{ and} \\
K & \text{for } n = 0.
\end{cases}
\]
Let $\mathcal{R} = \{R_1, R_2, \ldots\}$ be an infinite collection of posets and let $\mathcal{X} = \{x_1, x_2, \ldots\}$ be an infinite collection of variables. The infinite multivariate Noetherian algebra in the variables $\mathcal{X}$ indexed by $\mathcal{R}$ is defined to be the direct limit (ie: union) of the finite multivariate Noetherian algebras $K(x_1; R_1, x_2; R_2, \ldots) = \bigcup_{n \geq 0} K(x_1; R_1, \ldots, x_n; R_n)$.

When $R_n = R$ for all $n$, we write $K(x_1, \ldots, x_n)_R$ for the finite multivariate Noetherian algebra, and $K(x_1, x_2, \ldots)_R$ for the infinite multivariate Noetherian algebra.

We chose this particular definition for the multivariate Noetherian algebra because in \textsuperscript{4} the continuous Logarithmic Algebra $I = C(x, \log x, \log \log x, \ldots)_R$ defined in terms of it contains only asymptotic expansions as $x$ tends towards infinity whereas not all members of $C(\log \log x, \log x, x)_R$ represent any sort of asymptotic expansion.

An Artinian algebra of several variables $K(x_1, \ldots, x_n)^R$ or $K(x_1, x_2, \ldots)^R$ could be defined similarly, for there is a strong duality between the Noetherian and Artinian algebra:

**Proposition 1.8** When $K$ is a ring, and $R$ is an ordered group, there is a canonical isomorphism $\iota : K(x)^R \rightarrow K(x)_R$ given by $\iota : x^a \mapsto x^{-a}$.

Under the topology to be defined in the next section, this isomorphism is actually a topological algebra isomorphism.

### 1.2 Topology

We now give an alternate definition of the Artinian and Noetherian algebras as the topological completion of a simpler algebra. This algebra called the finite algebra is actually the intersection of the Artinian and Noetherian algebras.

**Definition 1.9** (Finite Algebra) Suppose $K$ is a topological ring, and $R$ is a ordered monoid. Define the finite algebra $K[x; R]$ to be the set of all finite sums $\sum_{a \in R} c_a x^a$. That is $c_a \in K$ is nonzero only for finitely many values of $a$. Members of the finite algebra are called finite series.

The Noetherian (resp. Artinian) topology for $K[x; R]$ is the finest topology such that

1. $c_n x^{a_n}$ is a Cauchy sequence whenever
   - (a) $a_n$ is a Cauchy sequence in the order topology of $R$, and
   - (b) $c_n$ is a Cauchy sequence in the topology of $K$.
2. $c_n x^{a_n}$ converges to zero whenever $a_n$ decreases (resp. increases) without bound.
3. Finite sums of the above two items are also Cauchy sequences.

**Proposition 1.10** The finite algebra is a topological $K$-algebra when equipped with either of the above topologies and the operations defined by the equations (1–3).

**Proof:** By the proof of Proposition 1.3, it suffices to observe that addition and multiplication are continuous. That is, the sum and product of two Cauchy sequences should themselves be Cauchy sequences.

Suppose that for $1 \leq m \leq j + k$, $(c_{nm}x^{anm})_{n \geq 0}$ is a Cauchy sequence indexed by $n$. Let $f_n(x) = \sum_{m=1}^{j} c_{nm}x^{anm}$, and $g_n(x) = \sum_{m=j+1}^{j+k} c_{nm}x^{anm}$. These are arbitrary Cauchy sequences. Their sum $\sum_{m=1}^{j+k} c_{nm}x^{anm}$ is also Cauchy. Their product is a finite sum of sequences $c_{nm}c_{n,m'}x^{anm+a_{nm'}}$. Now, if either $a_{nm}$ or $a_{nm'}$ decreases (resp. increases) without bound while the other either does the same or is Cauchy, then their sum decreases without bound. Hence, the sequence is Cauchy. Whereas, if both $a_{nm}$ and $a_{nm'}$ are Cauchy, then so is their sum. Moreover, $c_{nm}$ and $c_{nm'}$ would then both be Cauchy, so their product is. Hence, the sequence in question is Cauchy.

Note that $K[x; R_1][y; R_2] = K[y; R_2][x; R_1]$ as an algebra but not as a topological algebra under either topology. The expression $K[x; y; R_1, R_2]$ denotes $K[y; R_2][x; R_1]$ (and $K[x; y; R]$ denotes $K[y; R][x; R]$). Expressions like $K[x_1, \ldots, x_n; R_1, \ldots, R_n]$ (and $K[x_1, \ldots, x_n; R]$) are defined in a similar manner. Finally, we define

$$K[x_1, x_2, \ldots; R_1, R_2, \ldots] = \bigcup_{n \geq 0} K[x_1, \ldots, x_n; R_1, \ldots, R_n]$$

and

$$K[x_1, x_2, \ldots; R] = \bigcup_{n \geq 0} K[x_1, \ldots, x_n; R].$$

Now, we can give an alternate definition of the Noetherian (resp. Artinian) algebra.

**Theorem 1.11** In the Noetherian (resp. Artinian) topology, the topological completion of the finite algebra $K[x; R]$ is the Noetherian (resp. Artinian) algebra $\hat{K}[x]_R$ (resp. $\hat{K}[x]^R$) where $\hat{K}$ and $\hat{R}$ are the completions of $K$ and $R$ respectively.

**Corollary 1.12** Suppose that $K$ and $R$ are complete. Then in the Noetherian (resp. Artinian) topology the completion of $K[x; R]$ is $K(x)_R$ (resp. $K(x)^R$).

**Corollary 1.13** In the Noetherian topology:
1. The completion of \( K[x_1, \ldots, x_n; R_1, \ldots, R_n] \) is \( \widehat{K}(x_1; \widehat{R}_1, \ldots, x_n; \widehat{R}_n) \), and

2. The completion of \( K[x_1, x_2, \ldots; R_1, R_2, \ldots] \) is \( \widehat{K}(x_1; \widehat{R}_1, x_2; \widehat{R}_2, \ldots) \)

where \( \widehat{K} \) and \( \widehat{R}_i \) are the completions of \( K \) and \( R_i \) respectively.

**Example 1.4** The completion of \( (Q[i])[x; Q] \) in the Noetherian (resp. Artinian) topology is \( C(x)R \) (resp. \( C(x)R \)) where \( i \) is the imaginary unit.

Note that \( K(x^R) \) can only be defined in this manner if \( K \) and \( R \) are complete. Since Proposition 1.5 supposes that \( R \) is Archimedean, and the real numbers are the only complete Archimedean field \( \mathbb{R} \), we may maintain the assumption that \( R \) is the field of real numbers.

### 1.3 Complex Numbers Raised to a Real Power

We need to define the exponentiation operation for real exponents. Note that the expression \( a^t \) is already well defined when \( a \) is a positive real and \( t \) is an arbitrary real. It is given by \( a^t = e^{t \log a} \).

However, there is no unique way to do this when \( a \) is a arbitrary complex number. For example, \( \sqrt{-4} = \pm 2i \). In fact, given any complex number \( z \) and real number \( t \) we can define a whole family of values for \( z^t \) indexed by integer \( n \).

**Definition 1.14** (Exponentiation of a Complex Number) Given a nonzero complex number \( z \). Recall that \( z \) can be uniquely written in the form \( ae^{i\theta} \) where \( i \) is the imaginary unit \( \sqrt{-1} \), \( a \) is a positive real number, and \( 0 \leq \theta < 2\pi \). \( a \) is called the modulus of \( z \) denoted \( |z| \) and \( \theta \) is called the argument of \( z \) denoted \( \arg z \). The modulus and argument of 0 are both defined to be 0.

Define the \( n \)th value of \( z \) to the \( t \) power to be

\[
z^{t;n} = a^t e^{it(\theta + 2\pi n)}.
\]

For \( z = 0 \), we define \( z^{t;n} = 0 \) whenever \( t \) is positive, and leave it undefined when \( t \) is nonpositive.

Notice that for all \( z, a, n, \) and \( m \), the exponentials \( z^{a;n} \) and \( z^{a;m} \) differ by at most a factor of equal to a root of unity. Thus, we can say that \( \sim z^a \) is well defined up to multiplication by factor of modulus 1.

**Theorem 1.15** (Characterization of Exponentiation) \( f \) is a topological group homomorphism from the group of real numbers under addition to the group of nonzero complex number under multiplication if and only if for some nonzero complex number \( z \) and some integer \( n \), \( f(t) = z^{t;n} \).
1.3 Complex Numbers Raised to a Real Power

Proof: \((f(t) = z^{t/n} \rightarrow \textbf{Homomorphism})\) Let \(s\) and \(t\) be real numbers. Set \(a = |z|, \text{ and } \theta = \arg(z)\).

\[
\begin{align*}
  f(s)f(t) &= z^{s/n}z^{t/n} \\
  &= a^te^{i(t+2n\pi)}a^se^{i(s+2n\pi)} \\
  &= a^{s+t}e^{i(s+t)(\theta+2n\pi)} \\
  &= z^{s+t/n} \\
  &= f(s+t).
\end{align*}
\]

\((f(t) = z^{t/n} \rightarrow \textbf{Continuous})\) Since \(z^{t+\delta;n} = z^{t:n}z^{\delta;n}\), it suffices to show

\[
\lim_{\delta \to 0} z^{\delta;n} = 1.
\]

However, this is immediately obvious.

\((\text{Only If})\) We first characterize the set of such homomorphisms from the rational numbers to the nonzero complex numbers. Since this set is closely related to an extension of the ring of integers which is of independent interest we pause and discuss this ring briefly. Then we determine which homomorphisms are continuous. These functions then have a unique continuation to the real numbers.

Let \(f(1) = z = ae^{i\theta}\) with \(a \geq 0\) and \(0 \leq \theta < 2\pi\). By induction we have, \(f(na) = f(a)^n\) for all integers \(n\) and real numbers \(a\). Thus, \(f(n/m)^m = f(n) = z^n\). Hence, \(f(n/m)\) is one of the \(m\) distinct \(m^{th}\) roots of \(z^n\). Thus,

\[
f(n/m) = a^{n/m}e^{ni(\theta+2k_{n,m}\pi)/m} = z^{n/m;k}
\]

for some \(k_{nm}\). However, in general this \(k_{nm}\) might depend on \(n\) and \(m\).

Now,

\[
\begin{align*}
  f(n/m) &= f(1/m)^n \\
  &= \left(z^{1/m:k_{1,m}}\right)^n \\
  &= z^{1/m:k_{1,m}}
\end{align*}
\]

by the first part of this proof. Hence, we may assume without loss of generality that \(k_{n1,m} = k_{n2,m}\) for all integers \(n_1, n_2, \text{ and } m\). We therefore denote \(k_{nm}\) by \(k_m\).

Next, observe that the only the value of \(k_m\) modulo \(m\) is relevant, so we assume that \(0 \leq k_m < m\).

Finally, note that

\[
\begin{align*}
  z^{1/m:k_m} &= f(1/m) \\
  &= f(1/nm)^n \\
  &= z^{n/m:k_{nm}},
\end{align*}
\]

so we must have \(k_{(nm)} \equiv k_m\) modulo \(n\). Let \(Z\) be the set of such sequences.

\[
Z = \{(k_n)_{n > 0} : 0 \leq k_n < n, \text{ and } k_{nm} \equiv k_m \text{ modulo } n\}\]
We now digress to consider the algebraic properties of \( \mathbb{Z} \) before continuing on to complete the proof.

\( \mathbb{Z} \) is actually a ring since it is the inverse limit (along the lattice of integers ordered by divisibility) of the rings \( \mathbb{Z}_n \) with respect to the projections:

\[
\phi_{nm,n} : \mathbb{Z}_{nm} \rightarrow \mathbb{Z}_n
\]

\[
[k]_{nm} \mapsto [k]_n
\]

where \([k]_n = k + n\mathbb{Z}\) is the equivalence class of \(k\) modulo \(n\).

Note that \( \mathbb{Z} \) can be embedded in \( \mathbb{Z} \) by representing \(k \in \mathbb{Z}\) by the sequence \((k_n)_{n \in \mathbb{P}}\) where \(k_n \equiv k \mod n\).

However, \( \mathbb{Z} \) and \( \mathbb{Z} \) are not identified by this embedding. For example, if we let \(k_n \equiv \sum_{j=1}^{n} j!\) modulo \(n\) then \((k_n)_{n \geq 1} \in \mathbb{Z}\) but \((k_n)_{n \geq 1}\) does not correspond to the sequence of remainders of any integer.

\( \mathbb{Z} \) can be described as a set of “pseudointegers” which are characterized by their remainders when divided by integers. The only requirement on these remainders is that they be pairwise consistent. End of Digression.

To achieve any further results, we now must assume that \(f\) is continuous. Thus,

\[
1 = f(0) = \lim_{m \to +\infty} f(1/m) = \lim_{m \to +\infty} z^{1/m} k_m = \left( \lim_{m \to +\infty} a^{1/m} \right) \exp \left( \lim_{m \to +\infty} i\theta/m \right) \lim_{m \to +\infty} \exp (2k_m i\pi/m) = \lim_{m \to +\infty} \exp (2k_m i\pi/m).
\]

Hence, the only limit points of the sequence \(k_m/m\) are 0 and 1. Without loss of generality,

\[
\lim_{m \to +\infty} k_m/m = 0.
\]

Otherwise, eliminate terms from the sequence or multiply by the image of \(-1\) in \(\mathbb{Z}\).

Thus, for all positive integers \(j\), \(k_m/m\) is eventually less than \(1/j\). In particular, \(k_m\) is eventually equal to \(k_{jm}\). In fact, \(k_m\) is eventually constant. Thus, the sequence \(k_m\) is the representation of some positive integer \(k\), and hence \(f(t) = z^{t\cdot k}\). ■

Proposition 1.16 Let \(z_1\) and \(z_2\) be complex numbers. Let \(t\) be a real number and let \(n\) and \(m\) be integers. Then

\[
(z_1 z_2)^{t \cdot k} = z_1^{t \cdot n} z_2^{t \cdot m}
\]

where

\[
k = \begin{cases} 
   n + m & \text{if } \arg z_1 + \arg z_2 < 2\pi, \text{ and} \\
   n + m + 1 & \text{if } \arg z_1 + \arg z_2 \geq 2\pi.
\end{cases}
\]
Proof: Let \( z_1 = ae^{i\theta} \) and \( z_2 = be^{i\phi} \) with \( 0 \leq a, b < 2\pi \) and \( a, b \geq 0 \). Then
\[
\begin{align*}
z_1^{ti} &= a^te^{iti(\theta+2n\pi)} \\
z_2^{tj} &= b^te^{iti(\phi+2m\pi)} \\
(z_1 z_2)^{tk} &= \begin{cases} (ab)^t e^{iti(\theta+\phi+2k\pi)} & \text{if } \theta + \phi < 2\pi, \\ (ab)^t e^{iti(\theta+\phi+2(k-1)\pi)} & \text{if } \theta + \phi \geq 2\pi. \end{cases}
\end{align*}
\]

**Corollary 1.17** The map \( z \mapsto z^{tn} \) is continuous at all points \( z \) which are not nonnegative real numbers.

Proof: For \( z_0 \not\in \mathbb{R}^+ \), \( \arg z_0 \neq 0 \), so for \( h \in \mathbb{C} \) near 1, \( \arg(z_0h) < 2\pi \). The result now follows from Proposition 1.16 since
\[
\begin{align*}
\lim_{z \to z_0} z^{tn} &= \lim_{h \to 1} (z_0h)^{tn} \\
&= \lim_{h \to 1} z_0^{cn} h^{tn} \\
&= z_0^{tn}. \quad \square
\end{align*}
\]

**Proposition 1.18** Let \( z \) be a complex number. Let \( s \) and \( t \) be real numbers and let \( n \) be an integer. Then
\[
z^{st; n} = (z^{sn})^{tn+k}
\]
where \( k \) is the greatest integer less than or equal to \( s \arg z/2\pi \).

Proof: Let \( z = ae^{i\theta} \) where \( 0 \leq \theta < 2\pi \), and \( a \geq 0 \). Then
\[
\begin{align*}
z^{st; n} &= a^s e^{sti(\theta+2n\pi)} \\
(z^{s;i})^{tj} &= a^s e^{ti(\phi+2j\pi)}
\end{align*}
\]
where \( 0 \leq \phi < 2\pi \) and \( \phi \equiv s(\theta + 2n\pi) \mod 2\pi. \quad \square
\]

### 1.4 Artinian Series to a Real Power

In this section, we define the exponentiation of any nonzero Artinian or Noetherian series whose coefficients possess a well defined exponentiation.

As we have seen, these could be series whose coefficients are chosen from some subset of the complex numbers via Definition 1.14. Alternatively, the coefficients might themselves be Artinian or Noetherian series of some sort whose exponentiation is defined below.
INTRODUCTION

First, we must define a few preliminaries: A multiset $M$ (on a set $S$) is merely a function (from the set $S$) to the nonnegative integers. The multiset is finite if the sum of its values is finite. This sum is denoted $|M|$. We use the notation $\sum_{a \in M} f(a)$ to denote $\sum_{a \in S} M(a)f(a)$. In other words, this is a sum with multiplicities. Similarly, we calculate products with multiplicities. $\prod_{a \in M} f(a) = \prod_{a \in S} f(a)^{M(a)}$.

Next, define the multinomial coefficient $\binom{x}{M}$ by
\[
\binom{x}{M} = \frac{x(x-1) \cdots (x+1-|M|)}{\prod_{j \in S} M(j)!}.
\]

Finally, define the argument of an Artinian or Noetherian series to be the argument of its leading coefficient.

DEFINITION 1.19 (Exponentiation of a Series) Let $g(x)$ be a nonzero Artinian (resp. Noetherian) series whose non-zero coefficients have well defined exponentials. For example, suppose that $g(x)$ is a series with complex coefficients, or that $g(x)$ is a series whose coefficients are series with complex coefficients, or that $g(x)$ is a series whose coefficients are series with complex coefficients and so on.

Let $d = \deg(g(x))$, and $c_a = [x^a]g(x)$. Then for all integers $n$ and real numbers $t$ define the $g(x)$ to the power $t$ indexed by $n$ by the sum
\[
g(x)^{t;n} = c_d^{t;n} x^{dt} \sum_{M} \left( \prod_{a \in M} \frac{c_a + d}{c_d} x^a \right)
\]
over all multisets $M$ over the set $\mathbb{R} - \{0\}$ of nonzero real numbers.

For the exponentiation of $g(x) = 0$, we follow the same convention as in Definition 1.14.

PROPOSITION 1.20 Let $g(x)$ be a nonzero Artinian (resp. Noetherian) series. Let $t$ be a real number, and $n$ an integer. Then $g(x)^{t;n}$ is a well defined Artinian (resp. Noetherian) series.

Proof: It suffices to show that $[x^b](g(x)^{t;n})$ is well defined for all real numbers $b$, and that there are finitely many $a \leq b$ (resp. $a \geq b$) such that $[x^a](g(x)^{t;n})$ nonzero.

Now, there are finitely many $a \leq b - dt$ (resp. $a \geq b - dt$) such that $c_a \neq 0$. Denote them $a_1, \ldots, a_k$.

We claim that the summation yields only finitely many terms of degree less (resp. greater) than or equal to $b$. Thus, $[x^b](g(x)^{t;n})$ is well defined, and there are only finitely many $a \leq b$ (resp. $a \geq b$) such that $[x^a](g(x)^{t;n})$ is nonzero.

The claim is true since for the only summands which contribute terms of interest are those indexed by a multiset $M$ such that
1.4 Artinian Series to a Real Power

1. When \( a \in \{a_1, \ldots, a_k\} \), we have \( M(a) = 0 \).

2. Conversely, \( M(a_j) \leq \frac{b - dt}{a_j} \).

Obviously, there are only finitely many such multisets.

Let \( t \) be a real number, and let \( n \) and \( m \) be integers. Then \( g(x)^{t/n} \) and \( g(x)^{t/m} \) differ only by a factor which is a root of unity. Thus “\( g(x)^t \)” is well defined up to multiplication by complex numbers of modulus one.

In particular, \( g(x)^{-1/n} = g(x)^{-1/m} \) for all \( m \) and \( n \). It will be shown (Proposition 1.22) that the exponentiation \( g(x)^{-1/m} \) gives an explicit formulation of the reciprocal of \( g(x) \) which had been shown to exist via Proposition 1.5.

**Proposition 1.21** The isomorphism \( \iota \) defined in Proposition 1.8 preserves exponentiation. That is, informally,

\[
g^{t/n}(1/x) = g(1/x)^{t/n}
\]

for all Noetherian or Artinian series \( g(x) \), and all integers \( n \) and real numbers \( t \).

**Theorem 1.22** (Characterization of Exponentiation) \( \sigma \) is a topological group homomorphism from the group of real numbers under addition to the group of nonzero Artinian (resp. Noetherian) series under multiplication if and only if for some nonzero series \( g(x) \) and some integer \( n \), \( \sigma(t) = g(x)^{t/n} \).

**Proof:** (If) The function \( e^{t/n}_d \) is continuous by Theorem 1.15, and polynomials are continuous. So for this direction of need only show

\[
f(x)^{a+b/n} = (f(x)^{a/n}) (f(x)^{b/n}) .
\]

This is true for \( f(x) \) a monomial by Theorem 1.15, so it suffices to show for \( f(x) = 1 + \sum_{a>0} c_a x^a \).

\[
f(x)^{a_0} f(x)^{b_0} = \left( \sum_M \left( a \atop M \right) \prod_{a \in M} c_a x^a \right) \left( \sum_N \left( b \atop P \right) \prod_{b \in M} c_b x^b \right) = \sum_P \left( \sum_{M+N=P} \left( a \atop M \right) \left( b \atop M \right) \right) \prod_{a \in P} c_a x^a ,
\]

and this is equal to

\[
\sum_P \left( a + b \atop P \right) \sum_{a \in P} c_a x^a
\]

by the analytic version of the multivariate Van der Monde convolution.

(Only If) Up to a constant, every series has exactly one \( n^{\text{th}} \) root. Since there are \( n \) choices of constant by Theorem 1.13, there is a choice of \( k \) such that \( \sigma(a) = \sigma(1)^{a/k} \) for all rational numbers \( a \). By continuity, this holds for all real numbers \( a \).
**Proposition 1.23** Let \( f(x) \) and \( g(x) \) be Artinian (resp. Noetherian) series. Then for all integers \( j \) and \( k \) and real numbers \( t \),

\[
(f(x)g(x))^{t;n} = (f(x)^{t;k}) (g(x)^{t;j})
\]

where

\[
n = \begin{cases} 
  j + k & \text{if } \arg f(x) + \arg g(x) < 2\pi, \\
  j + k + 1 & \text{if } \arg f(x) + \arg g(x) \geq 2\pi.
\end{cases}
\]

**Proof:** By Proposition [1.16](#), this holds for constants, so we may assume without loss of generality that \( f(x) \) and \( g(x) \) have leading term 1. Let

\[
c_a = [x^a]f(x), \quad d_b = [x^b]g(x),
\]

and \( e_b = \sum_{a_1 + a_2 = b} c_{a_1} d_{a_2} \), so that \( f(x)g(x) = 1 + \sum_{b > 0} e_b x^b \) (resp. \( f(x)g(x) = 1 + \sum_{b < 0} e_b x^b \)).

Now,

\[
\sum_P \left( \begin{array}{c} t \\ P \end{array} \right) \sum_{b \in P} c_b x^b = \sum_P \left( \begin{array}{c} t \\ P \end{array} \right) \sum_{b \in P, a_1 + a_2 = b} c_{a_1} d_{a_2} x^b
\]

\[
= \sum_P \left( \begin{array}{c} t \\ P \end{array} \right) \sum_{M+N=P} \left( \prod_a \left( \frac{P(a)}{M(a)} \right) \right) \left( \prod_{a_1 \in M} c_{a_1} x^{a_1} \right) \left( \prod_{a_2 \in N} c_{a_2} x^{a_2} \right)
\]

\[
= \sum_{M,N} \frac{t(t-1) \cdots (t+1-|M|-|N|)}{M!N!(t+1-M-N)!} \prod_{a_1 \in M} c_{a_1} x^{a_1} \prod_{a_2 \in N} d_{a_2} x^{a_2}
\]

\[
= f(x)^{t;i} g(x)^{t;j}
\]

since \( t(t-1) \cdots (t+1-m) \) is a sequence of polynomials of binomial type. [13]  

The following Lemma demonstrates that equation (4) is much more general than previously indicated.

**Lemma 1.24** Let \( f(x) \) be an Artinian (resp. Noetherian) series of degree \( d \). Suppose \( f(x) \) is given by the following convergent expansion

\[
f(x) = \sum_{j \geq 0} c_j x^{a_j}
\]

where \((a_j)_{j \geq 0}\) is a sequence of not necessarily distinct real numbers such that \( a_0 = d \) and for \( j \) positive, \( a_j > d \) (resp. \( a_j < d \)). Then \( f(x)^{t;n} \) is given by the sum

\[
f(x)^{t;n} = (c_d)^{t;n} x^{a_d} \sum_M \left( \begin{array}{c} t \\ M \end{array} \right) \prod_{j \in M} \frac{c_j}{c_0} x^{a_j-a_0}
\]

over all finite multisets of positive integers.
1.4 Artinian Series to a Real Power

Proof: Suppose that \( a_i, a_j, a_k, \ldots \) are all equal to \( a \). Then any selection of \( i, j, k, \ldots \) for the multiset \( M \) contributes a factor of \( x^a \) just as a selection of \( a \) normally would. By the multinomial theorem,

\[
\sum_{|M|=n} \binom{n}{M} \prod_{l \in M} \frac{c_l}{c_0} = \left( \frac{c_i + c_j + c_k + \cdots}{c_0} \right)^n
\]

where \( n \) is a nonnegative integer and the sum ranges over all \( n \)-element multiset over \( \{i, j, k, \ldots \} \). Now, multiply both sides by \( t(t-1)\ldots(t+1-n)/n! \). □

Proposition 1.25 Let \( t \) be a real number and \( n \) an integer. Then the map \( f(x) \mapsto f(x)^{t,n} \) is continuous at all nonzero Artinian (resp. Noetherian) series \( f(x) \) such that \( \arg f(x) \neq 0 \). □

Proposition 1.26 Let \( f(x) \) and \( g(x) \) be an Artinian (resp. Noetherian) series. Then for all integers \( n \) and real numbers \( s \) and \( t \)

\[
f(x)^{st,n} = (f(x)^{s,n})^{t,n+k}
\]

where \( k \) is the greatest integer less than or equal to \( s \arg f(x)/2\pi \).

Proof: This is true for constants by Proposition 1.18, so without loss of generality, the leading term of \( f(x) \) is 1 and \( n \) and \( k \) are zero. Let \( c_a = [x^a]f(x) \). By Lemma 1.24,

\[
(f(x)^{s,0})^{t,0} = \sum_M \binom{t}{M} \prod_{M \in M} \binom{s}{M} \prod_{a \in M} c_a x^a
\]

where the sum is over finite multisets \( M \) of finite nonempty multisets \( M \) of nonzero real numbers. We invert the order of summation:

\[
(f(x)^{s,0})^{t,0} = \sum_N \left( \sum_M \binom{t}{M} \prod_{M \in M} \binom{s}{M} \right) \prod_{a \in N} c_a x^a
\]

where the inner sum is over multisets \( M \) as above such that

\[
N = \sum_{M \in M} M.
\]

It remains now to show that

\[
\sum_M \binom{t}{M} \prod_{M \in M} \binom{s}{M} = \binom{st}{N}.
\]

When \( s \) and \( t \) are nonnegative integers, this holds by combinatorial reasoning; both sides count the number of partitions of an \( st \)-set whose multiset of block sizes is \( N \). Moreover, since this is an identity of polynomials we now know that it holds for all \( s \) and \( t \). □
Proposition 1.27 Let \( f(x) \) be an Artinian (resp. Noetherian) series of degree \( d \). Let \( t \) be a real number, and \( n \) be an integer. Then \( \deg(f(x)^{1/n}) = td \).

Corollary 1.28 Let \( f(x) \) be an Artinian (resp. Noetherian) series whose degree is nonzero. Let \( n \) be an integer. Then the set \( \{f(x)^{1/n} : t \in \mathbb{R}\} \) is a \( K \)-pseudobasis for the Artinian algebra (resp. Noetherian algebra).

Actually, much more is true of these pseudobases as we see in the next section.

1.5 Composition of Series

Definition 1.29 (Composition of Series) Given two Artinian (resp. Noetherian) series \( f(x) \) and \( g(x) \) such that \( g(x) \) has positive degree, and an integer \( n \). Suppose that \( f(x) = \sum a_c x^a \). Then define the \( n \)th composition of \( f(x) \) with \( g(x) \) to be

\[
f(g; n) = \sum a_c g(x)^{a,n}.
\]

Note that \( g(x) \) must have positive degree regardless of whether the series is Artinian or Noetherian. In other words, the isomorphism \( \iota \) from Propositions 1.8 and 1.21 does not preserve composition.

Theorem 1.30 (Characterization of Composition) Let \( g(x) \) be an Artinian (resp. Noetherian) series positive degree. Then for all integers \( n \), \( f(x) \to f(g; n) \) is a continuous field automorphism of the Artinian (resp. Noetherian) algebra which fixes all constants.

Conversely, any continuous field automorphism of the Artinian (resp. Noetherian) algebra which fixes all constants is of the form \( f(x) \to f(g; n) \) for some Artinian (resp. Noetherian) series of positive degree \( g(x) \) and some integer \( n \).

Proof: We must show the following for both Artinian and Noetherian series:

- \( f(g; n) \) is a well defined Artinian (resp. Noetherian) series.
- Composition is continuous.
- Composition preserves addition.
- Composition preserves constants.
- Composition preserves multiplication.
1.5 Composition of Series

- The Converse.

**Well Defined** Suppose \( \deg(g(x)) = d > 0 \), and \( f(x) = \sum_a c_a x^a \). Then

\[
[x^a](f(g; n)) = \sum_{b \leq a/d} c_b[x^a](g(x)^{b/n})
\]
(resp. \( \sum_{b \geq a/d} c_b[x^a](g(x)^{b/n}) \)) is a finite sum since \( c_b \) is an Artinian (resp. Noetherian) sequence.

The sequence \( [x^a](f(g; n)) \) is itself Artinian (resp. Noetherian) since there are only finitely many \( b \leq a/d \) (resp. \( b \geq a/d \)) such that \( c_b \neq 0 \) and each \( g(x)^{b/n} \) contributes only finitely many terms of order less (resp. greater) than \( x^a \).

**Continuous** It is a convergent sum of continuous functions by Theorem 1.22.

**Addition** Obvious.

**Constants** If \( f(x) = c \), then \( f(g; n) = cg(x)^{b/n} = c \).

**Multiplication** Immediately follows from Theorem 1.22.

**Converse** Such an automorphism \( \sigma \) could be composed with the group automorphism \( \theta : a \mapsto x^a \).

The resulting map \( \theta \circ \sigma \) satisfies the conditions of Theorem 1.22, so we must have \( \sigma x^a = g(x)^{a/n} \) for some series \( g(x) \) and integer \( n \). By continuity, we know \( g(x)^{a/n} \) tends to zero as \( a \) tends to \( +\infty \) (resp. \( -\infty \)). Thus, \( \deg(g(x)) \geq 0 \) and \( \sigma f(x) = f(g; n) \).

**Proposition 1.31** For all Artinian (resp. Noetherian) series \( f(x) \) and \( g(x) \) and integers \( n \),

\[
\deg(f(g; n)) = \deg(f(x)) \deg(g(x)).
\]

**Proof:** Proposition 1.27.

As opposed to the composition of functions or of the usual sort of formal power series, this composition is not necessarily associative.

**Example 1.5** Let \( b, \) and \( c \) be small positive real numbers. Set \( f(x) = x \), \( g(x) = x^b \), and \( h(x) = x^c \). Then

\[
(f(g(1); h; 1)) = x^{bc} e^{2\pi i(1+c(b+1))}
\]

and

\[
f(g(h(1); 1)) = x^{bc} e^{2\pi i(1+2b)}.
\]

Their quotient is \( e^{2\pi i(1-b)} \) which is not equal to one.
However, all is not lost; indeed, composition by is associative in the case we are most interested in: delta series; for in [9], the relevant computations involve compositions with delta series.

**THEOREM 1.32** Let m be an integer, and let f(x), g(x), and h(x) be Artinian (resp. Noetherian) series such that g(x) and h(x) are delta series and \( \arg g(x) + \arg h(x) < 2\pi \). Then

\[
(f(g; 0))(h; m) = f(g(h; m); 0).
\]

**Proof:** By linearity and continuity, it suffices to consider \( f(x) = x^a \). Suppose \( c_b = [x^b]f(x) \) and \( d_b = [x^b]g(x) \). Then

\[
(f(g; 0))(h; m) = (g(x)^{a_0})(h; m) = \left( \sum_M \frac{M^t}{\prod_{b \in M} c_b} \right) (h; m)
\]

\[
= \left( \sum_M \frac{M^t}{c_1} \prod_{b \in M} c_{b+1}h(x)^{b_m} \right) \sum_M \frac{M^t}{\prod_{b \in M} c_1}.
\]

Whereas,

\[
f(g(h; m); 0) = f \left( c_1h(x) + \sum_{b>1} c_bh(x)^{b_m}; 0 \right)
\]

\[
= (c_1h(x)^{a_0}) \sum_M \frac{M^t}{\prod_{b \in M} c_{b+1}h(x)^{b_m}}
\]

by Lemma 1.24. Finally, \( c_1^{a_0}h(x)^{a_m} = (c_1h(x))^{a_m} \) by Proposition 1.23.

**Porism 1.33** Let m and n be integers, and let f(x), g(x), and h(x) be nonzero Artinian (resp. Noetherian) series with \( \deg(g(x)), \deg(h(x)) > 0 \). Then the series \( (f(g; 0))(h; m) \) and \( f(g(h; m); 0) \) only differ by a factor which is a root of unity.

**Corollary 1.34** The set of Artinian (resp. Noetherian) delta series is a monoid with respect to 0-composition.

**Proposition 1.35** For all integers n and Artinian (resp. Noetherian) series f(x), the map \( g(x) \mapsto f(g; n) \) is continuous at all series \( g(x) \) of positive degree such that \( \arg g(x) \neq 0 \).

**Proof:** Proposition 1.25.
2 Derivative

**Definition 2.1** (Derivative) We define the derivative (with respect to \(x\)) to be the continuous linear map on the Artinian (resp. Noetherian) algebra denoted \(D\) or \(\frac{d}{dx}\) such that

\[Dx^a = ax^{a-1}.\]

The derivative of \(f(x)\) is denoted \(f'(x)\).

**Proposition 2.2** (Derivation) The derivative is a derivation. That is,

\[D(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).\]

**Proof:** Let \(c_a = [x^a]f(x)\) and \(d_a = [x^a]g(x)\). Then

\[f'(x)g(x) + f(x)g'(x) = \sum_b \sum_{a_1 + a_2 = b} (a_1 + a_2)c_{a_1}d_{a_2}x^{b-1} = D(f(x)g(x)).\]

**Proposition 2.3** Let \(f(x)\) be a nonzero Artinian (resp. Noetherian) series. Let \(t\) be a real number and \(n\) be an integer. Then

\[D(f(x)^{t,n}) = tf'(x)f(x)^{t-1,n}\]

**Proof:** Let \(f(x) = \sum_b c_b x^b\). Suppose \(f(x)\) has degree \(d\). Then

\[tf'(x)f(x)^{t-1,n} = t \left( \sum_a ac_a x^{a-1} \right) \left( c_d^{t-1,n} x^{d(t-1)} \right) \sum_M \left( \begin{array}{c} t-1 \\ M \end{array} \right) \prod_{a \in M} \frac{c_a + d}{c_d} x^a\]

\[= c_d^{t,n} \left( dt x^{dt-1} + tx^{dt} \sum_{a \neq 0} \frac{(a + d)c_a + d}{c_d} x^{a-1} \right) \sum_M \left( \begin{array}{c} t-1 \\ M \end{array} \right) \prod_{a \in M} \frac{c_a + d}{c_d} x^a\]

\[= (Dc_d^{t,n}x^{dt}) \left( \sum_M \left( \begin{array}{c} t \\ M \end{array} \right) \prod_{a \in M} \frac{c_a + d}{c_d} x^a \right) + c_d^{t,n} x^{dt} D \left( \sum_M \left( \begin{array}{c} t \\ M \end{array} \right) \prod_{a \in M} \frac{c_a + d}{c_d} x^a \right)\]

\[= Df(x)^{a,n}.\]

**Proposition 2.4** (Chain Rule) For all valid \(n\)-compositions of series \(f(x)\) and \(g(x)\),

\[D(f(g; n)) = f'(g; n)g'(x)\]

**Proof:** By linearity and continuity, it suffices to consider the case \(f(x) = x^a\). However, this case was treated by Proposition 2.3.
3 LaGrange Inversion

In this section, we show the existence of compositional inverses to nonzero Artinian (resp. Noetherian) series of positive degree, and we give an explicit formula for their coefficients.

Proposition 3.1 Let \( f(x) \) be a Artinian (resp. Noetherian) series whose leading coefficient is a positive real number, and whose degree is positive. Then \( f(x) \) possesses a left (resp. right) 0-compositional inverse denoted \( f^{(-1;0)}(x) \) for all integers \( n \).

Thus, the set of such series is a semigroup under 0-composition.

Proof: We construct \( f^{(-1;0)}(x) \) from its coefficients \( c_a = \left[ x^a \right] f^{(-1;0)}(x) \).

Let \( b = \deg(f(x)) \). For \( a < 1/b \), set \( c_a = 0 \). Then set \( c_{1/b} = d_{b;0} \) where \( \sum_{e < a} c_e x^e \).

As in the proof of Proposition 3.1, \( c_a \) is an Artinian series since \( \mathbb{R} \) is Archimedean.

Let \( g(x) = \sum_a c_a x^a \). We now immediately have \( x = f(g; n) \).

Theorem 3.2 (LaGrange Inversion) Let \( f(x) \) be a delta series with leading coefficient a positive real number. Then

\[
\left[ x^a \right] \left( f^{(-1;0)}(x) \right)^{b;0} = b \left[ x^{a-b} \right] \left( \frac{x}{f(x)} \right)^{n;0}
\]

for all real numbers \( a \) and \( b \).

Note that for \( b = 1 \), equation (3) immediately gives the coefficients of \( f^{(-1;0)}(x) \).

Proof: First, observe that \( [x^{-1}] Dg(x) = 0 \) for any series \( g(x) \).

Now, let \( d_a = [x^a] \left( f^{(-1;0)}(x) \right)^{b;0} \). Then

\[
x^b = \sum_{c \geq b} d_c f(x)^{c;0}
\]

\[
bx^{b-1} = \sum_{c \geq b} cd_c f(x)^{c-1;0} f'(x)
\]

\[
bx^{b-1} / f(x)^{a;n} = \sum_{c \geq b} cd_c f(x)^{c-a-1;0} f'(x).
\]

Now, for \( a \neq c \), \( f(x)^{c-a-1;0} f'(x) \) is the derivative of \( \frac{1}{c-a} Df(x)^{c-a-1;0} \). Hence, by the above observation

\[
\left[ x^{-1} \right] \frac{kx^{k-1}}{f(x)^{a;0}} = \left[ x^{-1} \right] ad_{a} \frac{Df(x)}{f(x)}
\]

\[
= ad_{a}
\]

\[
= a \left[ x^a \right] \left( f^{(-1;0)}(x) \right)^{b;0}.
\]
4 Symmetric Artinian and Noetherian Series

The Symmetric Algebras

As a final direct application of theory of Artinian (resp. Noetherian) series, we now demonstrate that symmetric Artinian (resp. Noetherian) series can be studied in the same way as symmetric functions with integral exponents.

A multivariate Artinian (resp. Noetherian) series is symmetric if it is invariant under exchanges of variables. Let the $\Lambda(n)$ denote the symmetric part of the multivariate Noetherian algebra $K(x_1, x_2, \ldots, x_n)^{R^+}$, and let $\Lambda^*(n)$ denote the symmetric part of the multivariate Artinian algebra $K(x_1, x_2, \ldots, x_n)^{R^+}$ where $R^+$ is the set of nonnegative real numbers.

Now, let $\Lambda_a(n)$ and $\Lambda^*_a(n)$ consist of those series in $\Lambda$ and $\Lambda^*$ respectively which are homogeneous of degree $a$. Thus, $\Lambda(n)$ and $\Lambda^*(n)$ are graded by the modules $\Lambda_a(n)$ and $\Lambda^*_a(n)$.

By Porism 1.6, a symmetric series is invertible if and only if its constant term is zero.

Also note that given any Artinian (resp. Noetherian) symmetric series $p(x_1, x_2, \ldots, x_n)$ without constant term, any Artinian (resp. Noetherian) series $q(y)$, and any integer $k$; the composition $q(p(x_1, x_2, \ldots, x_n); k)$ is well defined.

Let us say that the ring $K$ is an exponential ring over the monoid $R$ if it is associated with a family of ring isomorphisms indexed by pairs $(a; k)$ where $a \in R$, and $k$ is an integer; the image of a point $x \in K$ under the morphism indexed by $(a; k)$ is $x^{a;k}$. We further require that $x^{a;k}x^{b;k} = x^{a+b;k}$. If $K$ is endowed with a topology, an algebra structure over another exponential ring, or a grading, then these maps must be continuous with respect to that topology, respect the scalar multiplication, and preserve the grading.

Above we first showed that the complex numbers formed an exponential ring over the real numbers. Next, we showed that if $K$ was an exponential ring over the real numbers, then so is $K(x)^{R}$ and $K(x)^{R^*}$. This implies that the corresponding multivariate and even infinite multivariate Noetherian and Artinian algebras are exponential rings over the real numbers. Clearly, by the above remarks the algebras of Artinian and Noetherian symmetric series are graded exponential algebras over the nonnegative real numbers.

Before continuing to consider symmetric series, we pose the following question:

**Open Problem 4.1** Is it possible to work with Artinian and Noetherian transcendence bases in the same manner as one works with algebraic transcendence bases?
Now, the map $\Lambda(n) \to \Lambda(n-1)$ (resp. $\Lambda(n) \to \Lambda(n-1)$) formed by setting the $n^{th}$ variable equal to zero is obviously an exponential ring homomorphism. (That is, it is an algebra homomorphism and it respects the exponential maps.) Thus, we can proceed to invoke inverse limits along the category of graded exponential algebras and thus define the Artinian and Noetherian symmetric algebras in infinitely many variables.

$$\Lambda = \lim_{\leftarrow n} \Lambda(n)$$

$$\Lambda^* = \lim_{\leftarrow n} \Lambda^*(n).$$

Note that this inverse limit must be taken as a graded exponential algebra. Otherwise, one would unintentionally include series of arbitrarily large degree such as the infinite product $\prod_{i=0}^{\infty}(1 + x_i)$.

We may not define a symmetric algebra which is an exponential ring over all of $\mathbb{Z}$—at least not in an infinite number of variables. Consider the product of the symmetric series $\sum x_i$ and $\sum x_i^{-1}$. Their product in a finite number of variables $n$ is $n + 2 \sum_{1 \leq i < j \leq n} x_i x_j$. Thus, the product is not well behaved under changes in the number of variables. In particular, the product is not well defined when the set of variables is infinite.

The Monomial Symmetric Series

Normally, the monomial symmetric function is indexed by an integer partition $\lambda$; that is, a nonincreasing vector with finite support. (The nonzero entries of $\lambda$ are called its parts, and $\ell(\lambda)$ is the number of parts of $\lambda$). Here this is not appropriate, so we define a real partition to be a vector $\beta$ of nonnegative real numbers such that $\beta$ is nonincreasing and has finite support.

**Definition 4.2** (Monomial Symmetric Series) For each real partition $\beta$, define the monomial symmetric function by the sum

$$m_\beta(x) = \sum_\alpha x^\alpha$$

over all distinct permutations $\alpha$ of the partition $\beta$ where $x = \{x_1, x_2, \cdots\}$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

Clearly, the monomial symmetric series are well defined, they are indeed symmetric, and they generalize the classical monomial symmetric functions. Moreover, the $m_\beta(x)$ where the $\beta_*$ sum to $a$ form a basis for the $\Lambda_a$ (resp. $\Lambda^*_a$). Thus, the collection of all $m_\beta(x)$ form a basis for $\Lambda$ (resp. $\Lambda^*$).
Transcendence Bases

Recall the definition of the elementary symmetric function \( e_n(x) \) and complete symmetric function \( h_n(x) \). They are defined explicitly by the sums

\[
e_n(x) = \sum_{\mu \in \mathcal{P}^*} m_\mu(y)
\]
\[
h_n(x) = \sum_{\lambda \in \mathcal{P}} m_\lambda(y)
\]

over integer partitions \( \mu \) with distinct parts and over all integer partitions \( \lambda \), and they are defined implicitly by the generating functions

\[
\prod_{n \geq 1} (1 + x_n y) = \sum_{n \geq 0} e_n(x) y^n
\]
\[
\prod_{n \geq 1} (1 - x_n y)^{-1} = \sum_{n \geq 0} h_n(x) y^n.
\]

Now, generalize to Artinian and Noetherian series.

**Definition 4.3** (Elementary and Complete Symmetric Series) Let \( \beta \) be a real partition. Define the elementary or complete symmetric series by the product

\[
e_\beta(x) = e_1(x)^{\beta_2 - \beta_1} e_2(x)^{\beta_3 - \beta_2} \ldots
\]
\[
h_\beta(x) = h_1(x)^{\beta_2 - \beta_1} h_2(x)^{\beta_3 - \beta_2} \ldots.
\]

These series are well defined and symmetric, and is a generalization of the definition of \( e_\lambda(x) \) and \( h_\lambda(x) \). Each series can be expressed as a sum

\[
m_\beta(x) + \sum_{\gamma > \beta} a_{\gamma, \beta} m_\gamma(x)
\]

over real partitions \( \gamma \) occurring after \( \beta \) in reverse lexicographical order. Thus, the \( e_\beta(x) \) and the \( h_\beta(x) \) are bases for \( \Lambda \) and \( \Lambda^* \).

In general, we say that a subset \( S \) of an exponential ring \( K \) over the monoid \( R \) is an Artinian (resp. Noetherian) transcendence basis if for a given \( n \), every element \( x \in R \) is represented by a unique Artinian (resp. Noetherian) series over the elements of \( S \).

By the above reasoning, the elementary and complete symmetric functions \( e_n(x) \) and \( h_n(x) \) each form an Artinian transcendence basis for \( \Lambda^* \), and a Noetherian transcendence basis for \( \Lambda \). Thus, the classical
involution of the algebra of symmetric functions \( \omega : h_n(x) \leftrightarrow e_n(x) \) extends to a well defined involution of the exponential algebras of symmetric series \( \Lambda \) or \( \Lambda^* \).

Thus, we can define the forgotten symmetric series via the identity \( f_\beta(x) = \omega m_\beta(x) \).

Define the power sum symmetric function \( p_n(x) = \sum_i x_i^n \). As in [12], we can uniquely express the \( p_n(x) \) in terms of the \( e_n(x) \) and visa-versa. Thus, the power sum symmetric functions constitute yet another Noetherian transcendence basis for \( \Lambda \) and Artinian transcendence basis for \( \Lambda^* \).

Thus, the power sum symmetric series which are products of the power sum symmetric functions

\[
p_\beta(x) = p_1(x)^{\beta_1 - \beta_2} p_1(x)^{\beta_1 - \beta_2} \ldots
\]

form a basis for \( \Lambda \) and \( \Lambda^* \). To compute the action of \( \omega \) with respect to this basis, note that \( \omega p_n(x) = (-1)^{n-1} p_n(x) \). Thus,

\[
\omega p_\beta(x) = (-1)^{\beta_1 + \beta_2 + \cdots} p_\beta(x).
\]
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