ABSTRACT. We study statistical inference for small-noise-perturbed multiscale dynamical systems under the assumption that we observe a single time series from the slow process. We study both averaging and homogenization regimes, constructing statistical estimators which we prove to be consistent, asymptotically normal (with explicit characterization of the limiting variance), and, in certain cases, asymptotically normal estimates, making the results readily applicable. For high-frequency observations, we prove consistency and asymptotic normality under a condition restricting the rate at which the number of observations may grow vis-à-vis the separation of scales. The estimators are based on an appropriate misspecified model motivated by a second-order stochastic Taylor expansion of the slow component with respect to a function of the time-scale separation parameter. Numerical simulations illustrate the theoretical results.

1 Introduction

Let us consider a family of $m + (d - m)$-dimensional processes $(X^\varepsilon, Y^\varepsilon)_T = \{(X^\varepsilon_t, Y^\varepsilon_t)\}_{0 \leq t \leq T}$ satisfying the stochastic differential equations (SDEs)

\begin{align}
    dX^\varepsilon_t &= \frac{\varepsilon}{\delta} b(X^\varepsilon_t, Y^\varepsilon_t)dt + c_0(X^\varepsilon_t, Y^\varepsilon_t)dt + \sqrt{\varepsilon} \sigma(X^\varepsilon_t, Y^\varepsilon_t)dB_t,
    \\
    dY^\varepsilon_t &= \frac{\varepsilon}{\delta^2} f(X^\varepsilon_t, Y^\varepsilon_t)dt + \frac{1}{\delta} g(X^\varepsilon_t, Y^\varepsilon_t)dt + \frac{\sqrt{\varepsilon}}{\delta} \tau_1(X^\varepsilon_t, Y^\varepsilon_t)dB_t + \frac{\sqrt{\varepsilon}}{\delta} \tau_2(X^\varepsilon_t, Y^\varepsilon_t)dB_t,
    \\
    X^\varepsilon_0 &= x_0 \in \mathcal{X} = \mathbb{R}^m, Y^\varepsilon_0 = y_0 \in \mathcal{Y} = \mathbb{R}^{d - m}.
\end{align}

Here, $W$ and $B$ are independent Wiener processes and $\varepsilon = (\varepsilon, \delta) \in \mathbb{R}^2_+$ satisfy $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that $\delta$ is the time-scale separation parameter while $\varepsilon$ dictates the size of the noise. The unknown vector $\theta \in \Theta \subset \mathbb{R}^k$ is to be estimated.

Data from physical dynamical systems commonly exhibit multiple characteristic space- or time-scales. In addition, stochastic noise may be introduced to account for uncertainty or as an essential part of a particular modeling problem. Consequently, multiscale stochastic differential equation models like (1) are widely employed in applied fields including physics, chemistry, and biology [5, 17, 35], neuroscience [18], meteorology [22], and econometrics and mathematical finance [11, 34] to describe stochastically-perturbed dynamical systems with two or more different space- or time-scales.

The goal of this paper is to study the problem of statistical inference for the unknown parameter vector $\theta \in \Theta \subset \mathbb{R}^k$ based on a fixed discrete-time sample $\{x_{tk}\}_{k=1}^n$ from an observation of the slow component $X^\varepsilon$. We consider this problem in the following two regimes:

1. (‘$\infty$ regime,’ or ‘homogenization regime’) $\lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = \infty$.
2. (‘$\gamma$ regime,’ or ‘averaging regime’) $\lim_{\varepsilon \to 0} \frac{\delta}{\varepsilon} = \gamma \in (0, \infty)$.

Statistical inference for diffusions without multiple scales (i.e., $\delta \equiv 1$) has been very well studied in the literature; see for example [4, 20, 21, 28]. In [4, 21, 28], $\epsilon \equiv \delta \equiv 1$ and the asymptotic behavior of the maximum likelihood estimator (MLE) based on continuous data is studied in the time horizon limit $T \to \infty$. This is directly analogous to the limit $n \to \infty$ in the classical setting of i.i.d. observations. The case of fixed time horizon $T$ but $\epsilon \to 0$ has been studied in [20], likewise based on continuous data and without multiple scales.

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scales. Estimation based on discrete data without multiple scales has also been addressed in the literature; see for example [10, 51, 62].

Maximum likelihood estimation from continuous data for multiscale models with noise of order one has been studied in [1, 2, 19, 23, 26]. The authors of [23, 26] prove that in the averaging regime, the MLE induced by the (nondeterministic) limit of the slow process $X^\varepsilon$ in (1) with $\varepsilon \equiv 1$ as $\delta \to 0$ is consistent under the assumption that coefficients are bounded and that the fast process $Y^\varepsilon$ takes values in a torus. It is important to point out that the regime $\varepsilon \to 0$ which we study in this paper is different in that the diffusion coefficient $\sqrt{\varepsilon} \sigma$ vanishes in the limit and, as described precisely by Theorem 1, $X^\varepsilon$ converges to the solution of an ODE rather than an SDE; the (deterministic) limit does not induce a well-defined likelihood. Meanwhile, it is shown also in [23] that direct application of the principle of maximum likelihood with discretely-sampled data via Euler-Maruyama approximation produces consistent estimates only if the data is first appropriately subsampled. Most closely related to the present work are [13, 30], wherein the authors prove consistency and asymptotic normality of the continuous-data MLE for special cases of (1).

Our focus in this paper is different. We address estimation from discrete data for multiscale diffusion models like (1). We assume that we are given only a discrete-time sample $\{x_{tk}\}_{k=1}^n$ from a single observation of the slow process $X^\varepsilon$; we assume that no data are available from the fast process $Y^\varepsilon$. Motivated by a second-order stochastic Taylor expansion with respect to $\sqrt{\varepsilon}$, we construct a minimum contrast estimator (hereinafter referred to as the MCE) based on an appropriate misspecified model. Firstly, we prove that for any given fixed value of $n$, in either the averaging or the homogenization regime, this estimator is consistent and asymptotically normal as $\varepsilon + \delta \to 0$, with a limiting variance which we calculate explicitly. Knowing the limiting variance is important for statistical inference as it allows one to control the error, construct confidence intervals, and develop appropriate hypothesis tests. Going a step further, we demonstrate that the inverse of the limiting variance converges as $n \to \infty$ to the Fisher information for the continuous model; that is, one attains the Cramer-Rao lower bound, which is to say that the estimator is asymptotically statistically efficient. Secondly, we study a simplified minimum contrast estimator (hereinafter referred to as the SMCE) that can be considerably easier to apply and presents improved robustness in numerical studies. We show that the simplified estimator is also consistent and asymptotically normal, although it is not asymptotically efficient. Thirdly, we study the behavior of the two estimators in the joint limit $\varepsilon + \delta \to 0$ and $n \to \infty$ (the high-frequency regime), showing that consistency and asymptotic normality (and, for the MCE, asymptotic efficiency) are retained provided that the sampling interval $\Delta := T/n$ does not decrease too quickly relative to $\varepsilon$ (or equivalently, depending on the regime, to $\delta$). In particular, we prove in the high-frequency regime (a) consistency of both MCE and SMCE under the assumption that $\varepsilon = o(\Delta)$, and (b) asymptotic normality of the MCE under the stronger assumption $\varepsilon = o(\Delta^2)$ but asymptotic normality of SMCE under the assumption $\varepsilon = o(\Delta)$ only. It is clear that these conditions can be written also in terms of $\delta$ and $\Delta$. Despite our best efforts we have not managed to relax these assumptions, which are reminiscent of the subsampling prescribed in [2, 23]; nevertheless, numerical simulations suggest that the estimators remain well behaved, even when $\varepsilon$ is not of lower order than $\Delta$ (see Tables 5 and 6 and Figures 9-12 in Section 7).

To the best of our knowledge, this is the first paper on discrete-time estimation for multiscale models to describe estimators that are consistent, asymptotically normal, and asymptotically statistically efficient for both the averaging and homogenization regimes. We characterize the limiting variance of the estimators explicitly. For high-frequency observations, we require that $\Delta$ not decrease too quickly relative to $\varepsilon$; this is reminiscent of the subsampling prescribed in [2, 23], although our case is different in that we take $T$ to be fixed, and so also the relationship between $\Delta := T/n$ and $n$ is fixed. We emphasize that our estimators can be applied without precise knowledge of $\delta$ (which is tricky to estimate in practice) or $\varepsilon$.

Let us now briefly discuss the approach that we will take. The main idea is to establish a second-order stochastic Taylor expansion of $X^\varepsilon$ of the form

$$X^\varepsilon = \bar{X} + \sqrt{\varepsilon} \varphi^\varepsilon,$$

where $\bar{X}$ is the deterministic law-of-large-numbers limit of $X^\varepsilon$ and $\varphi^\varepsilon$ converges in distribution to a Gaussian process as $\varepsilon \to 0$. This representation motivates a misspecified model in which the conditional distribution of $X^\varepsilon_t$ given $X^\varepsilon_{tk-1} = x_{tk-1}$ is approximated by a Gaussian random variable having a specified mean and variance. The principle of maximum likelihood applied to this discrete, approximate process leads us to a certain contrast function, the minimizer of which we take to be our estimator. Naturally, appropriate identifiability conditions must be assumed; these conditions are however typically satisfied provided that
the coefficients in the original model are sufficiently regular. We mention here that we have restricted the dependence of $f, \tau$ on the parameter $\theta$ to the coefficients $b$ and $c$ for purposes of presentation only; as inspection of the proofs will make apparent, the results of the paper continue to hold unchanged if one allows the other coefficients $\sigma, f, g, \tau_1, \tau_2$ also to depend on $\theta$.

The rest of this paper is organized as follows. Section 2 presents the main assumptions of the paper and introduces helpful notation. Section 3 develops a second-order stochastic Taylor expansion of $X^t$; the representation thus obtained motivates the methods of statistical inference presented in later sections. Section 4 develops the proposed estimator assuming a fixed number $n$ of data points. We prove that the estimator is consistent and asymptotically normal as $n \to \infty$. Similarly, asymptotic normality holds for the SMCE provided that $\epsilon = o(\Delta)$ and for the MCE if $\epsilon = o(\Delta^2)$. Section 7 presents the results of numerical simulations to illustrate the theoretical results; simulations are described also that deliberately violate the relations $\epsilon = o(\Delta)$ and $\epsilon = o(\Delta^2)$ that we impose for the high-frequency theory, in order to substantiate our conjecture that it may yet be possible to weaken these assumptions. Section 8 summarizes our conclusions and discusses future directions of the research. Finally, an Appendix collects technical estimates used throughout the paper.

2 Preliminaries and Assumptions

Let us begin with a discussion of the main assumptions that we carry throughout the paper. We work with a canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions (namely, $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets). Recall from (1) that $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^{d-m}$ are, respectively, the state spaces of the slow and fast components of the dynamics.

To avoid ambiguity, we will always write $|\cdot|$ for the Frobenius (Euclidean) norm, and $||\cdot||$ for the operator norm of a matrix.

To guarantee that (1) is well posed and that our limit results are valid, we impose the following regularity and growth conditions:

**Condition 1.** (Regularity of Coefficients)

**Conditions on $c_0$**

1. $\exists K > 0, q > 0, r \in [0, 1); \forall \theta \in \Theta, |c_0(x, y)| \leq K(1 + |x|^r)(1 + |y|^q).
2. $\exists K > 0, q > 0; \forall \theta \in \Theta, |\nabla_x c_0(x, y)| + |\nabla_y c_0(x, y)| \leq K(1 + |y|^q).
3. $\forall \theta \in \Theta, c_0$ has two continuous derivatives in $x$, Hölder continuous in $y$ uniformly in $x$.
4. $\forall \theta \in \Theta, \nabla_x \nabla_y c_0(x, y)$ is jointly continuous in $x$ and $y$.
5. $c_0(x, y)$ has two locally bounded derivatives in $\theta$ with at most polynomial growth in $x$ and $y$.

**Conditions on $\sigma$**

1. $\forall N > 0, \exists C(N); \forall x_1, x_2 \in \mathcal{X}, \forall y \in \mathcal{Y}$ with $|y| \leq N, |\sigma(x_1, y) - \sigma(x_2, y)| \leq C(N)|x_1 - x_2|.
2. $\exists K > 0, q > 0; |\sigma(x, y)| \leq K(1 + |x|^{1/2})(1 + |y|^q).
3. $\sigma \sigma^T$ is uniformly nondegenerate.

**Conditions on $f, \tau_1, \tau_2$**

1. $f, \tau_1 \tau_1^T,$ and $\tau_2 \tau_2^T$ are twice differentiable in $x$ and $y$, the first and second derivatives in $x$ being bounded, and all partial derivatives up to second order being Hölder continuous in $y$ uniformly in $x$. 

2. \( \tau_2 \tau_2^T \) is uniformly nondegenerate.

**Conditions on \( b_\theta, g \)**

1. \( b_\theta \) satisfies the same smoothness conditions as \( c_\theta \).

2. In the \( \infty \) regime, \( b_\theta \) and its derivatives are bounded uniformly in the first variable by polynomials in the second variable.

3. In the \( \infty \) regime, \( g \) satisfies the same conditions as \( c_\theta \); in the \( \gamma \) regime, \( g \) satisfies the same conditions as \( f \).

In the limit of infinite scale separation, the slow process appears from the perspective of the fast to become ‘frozen.’ To guarantee that the fast process has an invariant distribution when the slow process is ‘frozen,’ we impose the following recurrence condition:

**Condition 2. (Recurrence Condition)**

1. In the \( \infty \) regime, \( \lim_{|y| \to \infty} \sup_{x \in X} \left( f(x, y) \cdot y \right) = -\infty. \)

2. In the \( \gamma \) regime, \( \lim_{|y| \to \infty} \sup_{x \in X} \left( (\gamma f + g)(x, y) \cdot y \right) = -\infty. \)

The conditions on \( f, \tau_1, \tau_2 \) in Condition 1 and Condition 2 guarantee that for each fixed \( x \in X \) one has on \( Y \), in the \( \infty \) regime, a unique invariant measure \( \mu_{\infty, x} \) associated with the operator

\[
L_{\infty, x} := f(x, \cdot) \cdot \nabla_y + \frac{1}{2} (\tau_1 \tau_1^T + \tau_2 \tau_2^T)(x, \cdot) \cdot \nabla_y^2,
\]

and in the \( \gamma \) regime, a unique invariant measure \( \mu_{\gamma, x} \) associated with the operator

\[
L_{\gamma, x} := (\gamma f + g)(x, \cdot) \cdot \nabla_y + \frac{\gamma}{2} (\tau_1 \tau_1^T + \tau_2 \tau_2^T)(x, \cdot) \cdot \nabla_y^2.
\]

For existence of an invariant measure, the interested reader may see for example [33]; uniqueness is a consequence of nondegenerate diffusion, as for example in [25].

In the \( \infty \) regime, we assume a standard centering condition on the asymptotically singular coefficient \( \frac{\nu}{\delta} \int_0^\infty b_\theta(X^*, Y^*) ds \) in order to obtain a homogenization limit.

**Condition 3. (Centering Condition for the \( \infty \) Regime)** For each fixed \( x \in X \), \( \int_Y b(x, y) \mu_{\infty, x}(dy) = 0. \)

We also need a condition on the relative rates at which \( \epsilon \) and \( \delta \) vanish in order to derive a suitable second-order approximation of the slow component \( X^\epsilon \).

**Condition 4.**

1. In the \( \infty \) regime, \( \frac{\epsilon^{3/2}}{\delta} \to \ell_\infty \) as \( \epsilon \to 0 \) for some \( \ell_\infty \in (0, \infty] \).

2. In the \( \gamma \) regime, \( \frac{\sqrt{\epsilon}}{\sqrt{\delta}} \to \ell_\gamma \) as \( \epsilon \to 0 \) for some \( \ell_\gamma \in (0, \infty] \).

Let us conclude this section with an introduction of notational conventions. When convenient, we will use an asterisk in place of the regime signifier (writing \( L_{\gamma, x}, \mu_{\gamma, x}, \) etc.) if a statement is to be understood for both regimes. We will in many instances wish to integrate functions of two variables \( x \) and \( y \) in the second variable over the invariant measures \( \mu_{\gamma, x} \) to obtain ‘averaged’ functions of \( x \) only. We will use a bar to distinguish an averaged function from the original function; that is, for a generic function \( h \) of \( x \) and \( y \),

\[
\bar{h}(x) := \int_Y h(x, y) \mu_{\gamma, x}(dy).
\]

This bar notation should not be confused with \( \bar{X} \), which is the first-order limit of the stochastic process \( X^\epsilon \), nor with \( \bar{\theta} \), which is an estimator. We will denote by \( A : B \) the Frobenius inner product \( \Sigma_{i,j} [a_{i,j} \cdot b_{i,j}] \) of matrices \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \).
3 Asymptotic Behavior of $X^\varepsilon$

In this section, we develop a stochastic Taylor expansion of $X^\varepsilon$, establishing in particular a representation

$$X^\varepsilon = \bar{X}_* + \sqrt{\varepsilon} \varphi^\varepsilon_*,$$

in which $\bar{X}_*$ is deterministic and $\varphi^\varepsilon_*$ converges in distribution to a Gaussian process as $\varepsilon \to 0$. This representation is key, as it allows us to define estimators in terms of an appropriate misspecified model and prove consistency and asymptotic normality (see Sections 4, 5 and 6). We continue to suppress the parameter $\theta$, as the results of this section hold independently of (and indeed uniformly over) $\theta \in \Theta$.

Intuitively speaking, as $\delta \to 0$, the fast dynamics accelerate relative to the slow and $Y^\varepsilon_t$ tends to its invariant distribution at each ‘frozen’ value $X^\varepsilon_t$. At the same time, the driving noise $\sqrt{\varepsilon} \sigma dW$ of the slow process tends to 0. It is therefore reasonable to anticipate that, in the limit, $X^\varepsilon$ should hew closely to the solution $\bar{X}_*$ of an ODE involving coefficients averaged over the fast dynamics.

Recall that in the $\infty$ regime, $\varepsilon \delta \to \infty$; to describe the asymptotic behavior of $X^\varepsilon$ in this regime, we need to find the limiting contribution of the singular coefficient $\varepsilon \delta \int_0^t b(X^\varepsilon_s, Y^\varepsilon_s) ds$. It turns out that under Condition 3, the limiting contribution may be captured in terms of the solution of a certain Poisson equation. By Theorem 3 in [25], there is a unique solution $\chi$ in the class of functions that grow at most polynomially in $|y|$ as $y \to \infty$ of the equation

$$L_{\infty,x} \chi(x, y) = -b(x, y), \quad \int_Y \chi(x, y) \mu_{\infty, x}(dy) = 0.$$  \hfill (3)

Theorem 1 establishes that the averaged coefficient in the ODE for $\bar{X}_\infty$ is $\bar{\lambda}_\infty$, where

$$\lambda_\infty(x, y) := (\nabla_y \chi \cdot g + c)(x, y).$$  \hfill (4)

In the $\gamma$ regime, there is no singular coefficient with which to contend, and the corresponding averaged coefficient is $\bar{\lambda}_\gamma$, where

$$\lambda_\gamma(x, y) := (\gamma b + c)(x, y).$$  \hfill (5)

Let us state this law-of-large-numbers approximation precisely in the form of a theorem. This gives a first-order approximation to $X^\varepsilon$ when $\varepsilon, \delta$ are small.

**Theorem 1.** Assume Conditions 2 and 3 and, in the $\infty$ regime, Condition 4; let $*$ denote the regime. For any initial condition $(x_0, y_0) \in X \times Y$ and $0 \leq p < \infty$, there is a constant $\tilde{K}$ such that for $\varepsilon$ sufficiently small,

$$E \sup_{0 \leq t \leq T} |X^\varepsilon_t - \bar{X}_{*, t}|^p \leq \tilde{K} \cdot \varepsilon^{p/2},$$

where $\bar{X}_*$ is the (deterministic) solution of the integral equation

$$\bar{X}_{*, t} := x_0 + \int_0^t \bar{\lambda}_*(\bar{X}_{*, s}) ds,$$

where $\bar{\lambda}_*$ is obtained, depending on the regime, by averaging (4) or (5) over the invariant measures $\mu_{*, x}$.

Theorem 1 extends Theorem 1 of [13]. The proof relies on Lemma 7 in the Appendix; we omit the details, as given Lemma 7 the argument follows nearly verbatim the proof of Theorem 1 of [13].

The essential content of Theorem 1 is that $X^\varepsilon$ tends to a deterministic first-order limit. The minimum contrast estimators that we study in this paper exploit the structure of the random fluctuations of $X^\varepsilon$ about this limit; to do this, we must obtain a description of the asymptotic behavior to higher order.

This, in turn, turns on quantifying more precisely the difference between the true drift $\frac{\varepsilon}{\delta} b(X^\varepsilon_t, Y^\varepsilon_t) + c(X^\varepsilon_t, Y^\varepsilon_t)$ and the approximate drift $\bar{\lambda}_*(X_t)$. The limiting coefficient $\bar{\lambda}_*$ plays the role of intermediary. By
Theorem 3 in [25], there is a unique solution $\Phi_*$ in the class of functions that grow at most polynomially in $|y|$ as $y \to \infty$ of the equation

$$L_{t,x} \Phi_*(x,y) = -\left(\lambda_* - \bar{\lambda}_x\right)(x,y), \quad \int_y \Phi_*(x,y) \mu_{t,x}(dy) = 0. \quad (6)$$

The function $\Phi$ will feature in our description of the fluctuations.

We need to define one more quantity before stating the next result. For any $0 \leq s \leq t \leq T$, let $Z_s(t,s)$ denote the matrix-valued solution to the equation

$$\frac{dZ_s(t,s)}{dt} = (\nabla_x \bar{\lambda}_s)(\bar{X}_{s,t})Z_s(t,s), \quad Z_s(s,s) = 1_m,$$

where $1_m$ denotes the $m \times m$ identity matrix. By Proposition 2.14 of [6], the continuity of $P \int_0^\infty (1_m - Z_s(t,s))ds$ guarantees the semi-group relations $Z_s(t,s) = Z_s(t,\rho)Z_s(\rho,s)$ and the invertibility of $Z_s(t,s)$.

**Theorem 2.** Assume Conditions 1, 2, and 4 and, in the infinite regime, Condition 3.

In the infinite regime, we have a representation

$$\frac{1}{\sqrt{\epsilon}}(X_{\epsilon,t}^\gamma - \bar{X}_{\infty,t}) = \frac{1}{\sqrt{\epsilon}} \int_0^t Z_{\infty}(t,s)\nabla_y \Phi_{\infty} g(\bar{X}_{\infty,s})ds + \int_0^t Z_{\infty}(t,s)(\sigma + \nabla_y \chi \cdot \tau_1)(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma)dW_s$$

$$+ \int_0^t Z_{\infty}(t,s)\nabla_y \chi \cdot \tau_2(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma)dB_s + \mathcal{R}_{\infty}^{\gamma,t},$$

such that for any $\eta > 0$, we have $\lim_{\epsilon \to 0} P \left( \sup_{0 \leq t \leq T} |\mathcal{R}_{\infty}^{\gamma,t}| > \eta \right) = 0.$

In the finite regime, we have a representation

$$\frac{1}{\sqrt{\epsilon}}(X_{\epsilon,t}^\gamma - \bar{X}_{\epsilon,t}) = \frac{1}{\sqrt{\epsilon}} \int_0^t Z_{\gamma}(t,s)\gamma b - \nabla_y \Phi_{\gamma} g(\bar{X}_{\epsilon,s})ds + \int_0^t Z_{\gamma}(t,s)(\sigma + \nabla_y \Phi_{\gamma} \cdot \tau_1)(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma)dW_s$$

$$+ \int_0^t Z_{\gamma}(t,s)\nabla_y \Phi_{\gamma} \cdot \tau_2(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma)dB_s + \mathcal{R}_{\gamma,t}^{\epsilon},$$

such that for any $\eta > 0$, we have $\lim_{\epsilon \to 0} P \left( \sup_{0 \leq t \leq T} |\mathcal{R}_{\gamma,t}^{\epsilon}| > \eta \right) = 0.$

**Remark 1.** Before proceeding to the proof, we mention that a similar result, albeit one insufficient for our purposes, is established in [25]. It is proven there that the fluctuations process $\eta_t := \frac{X_{\epsilon,t}^\gamma - \bar{X}_{\epsilon,t}}{\sqrt{\epsilon}}$ converges weakly in the space of continuous functions $C([0,T],\mathbb{R}^m)$ to the solution of a certain Ornstein-Uhlenbeck-type process $\eta_t$ (with different dynamics for the different regimes). Unfortunately, this result is of limited use in the statistical setting because the limit is in distribution, which means that the limit and limit do not live necessarily, as such, on the same space; hence the need for Theorem 2, which establishes a representation in path space.

**Proof of Theorem 3.** Let us suppress the regime subscript and suppose at first that we are in the infinite regime. We write

$$\frac{1}{\sqrt{\epsilon}}(X_{\epsilon,t}^\gamma - \bar{X}_{\epsilon,t}) = I + II + III,$$

where

$$I := \frac{1}{\sqrt{\epsilon}} \int_0^t (\bar{\lambda}(X_{\epsilon,s}^\gamma) - \bar{\lambda}(\bar{X}_{s}))ds,$$

$$II := \frac{1}{\sqrt{\epsilon}} \int_0^t (\lambda(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma) - \bar{\lambda}(X_{\epsilon,s}^\gamma))ds,$$

$$III := \int_0^t \sigma(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma)dW_s + \frac{1}{\sqrt{\epsilon}} \int_0^t \left(\frac{1}{\delta} b - \nabla_y \chi \cdot g\right)(X_{\epsilon,s}^\gamma, Y_{\epsilon,s}^\gamma)ds.$$
We begin with an approximation of each part.

\[ I = \int_0^t (\nabla_x \lambda)(\bar{X}_s) \cdot \frac{1}{\sqrt{\epsilon}} (X_s^\epsilon - \bar{X}_s)ds + \int_0^t \left[ (\nabla_x \lambda)(X_s^\epsilon) - (\nabla_x \lambda)(\bar{X}_s) \right] \cdot \frac{1}{\sqrt{\epsilon}} (X_s^\epsilon - \bar{X}_s)ds \]

\[ = \int_0^t (\nabla_x \lambda)(\bar{X}_s) \cdot \frac{1}{\sqrt{\epsilon}} (X_s^\epsilon - \bar{X}_s)ds + R^\epsilon_{I,t}. \]

where \( X_s^\epsilon \) is an appropriately-chosen point on the segment connecting \( X_s^\epsilon \) with \( \bar{X}_s \).

\[ R^\epsilon_{I,t} := \int_0^t \left[ (\nabla_x \lambda)(X_s^\epsilon) - (\nabla_x \lambda)(\bar{X}_s) \right] \cdot \frac{1}{\sqrt{\epsilon}} (X_s^\epsilon - \bar{X}_s)ds \]

vanishes in probability uniformly in \( t \) as \( \epsilon \to 0 \); to see this, note firstly that \( \int_0^t \left[ (\nabla_x \lambda)(X_s^\epsilon) - (\nabla_x \lambda)(\bar{X}_s) \right] ds \) vanishes in probability by compactness of \( [0,T] \), continuity of \( \nabla_x \lambda \), and Theorem 1, and secondly that

\[ \sup_{0 \leq t \leq T} \left| \frac{1}{\sqrt{\epsilon}} (X_s^\epsilon - \bar{X}_t) \right| \]

is bounded in probability by Theorem 1.

Letting \( \Phi \) be as in [6], applying the Itô formula to \( \Phi(x,y) \) with \( (x,y) = (X_t^\epsilon, Y_t^\epsilon) \), and rearranging terms, we have

\[ II = \frac{\delta}{\sqrt{\epsilon}} \int_0^t \nabla_y \Phi \cdot g(X_s^\epsilon, Y_s^\epsilon)ds \]

\[ + \frac{\delta}{\sqrt{\epsilon}} \int_0^t \nabla_x \Phi \cdot b(X_s^\epsilon, Y_s^\epsilon)ds + \frac{\delta^2}{\epsilon} \int_0^t \nabla_x \Phi \cdot c(X_s^\epsilon, Y_s^\epsilon)ds + \frac{\delta^2}{\sqrt{\epsilon}} \int_0^t \sigma \sigma^T : \nabla_x^2 \Phi(X_s^\epsilon, Y_s^\epsilon)ds \]

\[ + \frac{\delta}{\sqrt{\epsilon}} \int_0^t \nabla_y \Phi \cdot \tau_1 \nabla_x \Phi(X_s^\epsilon, Y_s^\epsilon)ds + \frac{\delta^2}{\epsilon} \int_0^t \nabla_x \Phi \cdot \sigma(X_s^\epsilon, Y_s^\epsilon)ds \]

\[ = \frac{1}{\sqrt{\epsilon}} \int_0^t \nabla_y \Phi \cdot g(\bar{X}_s)ds - R^\epsilon_{II,t}. \]

\[ R^\epsilon_{II} \]

vanishes in probability uniformly in \( t \) as \( \epsilon \to 0 \); to see this, note firstly that

\[ \frac{1}{\sqrt{\epsilon}} \int_0^t \left[ \nabla_y \Phi \cdot g(X_s^\epsilon, Y_s^\epsilon) - \nabla_y \Phi \cdot g(\bar{X}_s) \right] ds \]

vanishes in probability by Lemma 4, secondly that \( (\delta/\sqrt{\epsilon})(\Phi(X_t^\epsilon, Y_t^\epsilon) - \Phi(x_0, y_0)) \) vanishes in probability by the argument of Corollary 1 of [24], and thirdly that the remaining integrals are bounded in probability by Lemma 5 while their prefactors vanish.

Letting \( \chi \) be as in [6], applying the Itô formula to \( \chi(x,y) \) with \( (x,y) = (X_t^\epsilon, Y_t^\epsilon) \), and rearranging terms, we have

\[ III = \int_0^t (\sigma + \nabla_x \chi \cdot \tau_1)(X_s^\epsilon, Y_s^\epsilon)dW_s + \int_0^t \nabla_x \chi \cdot \tau_2(X_s^\epsilon, Y_s^\epsilon)dB_s \]

\[ + \sqrt{\epsilon} \int_0^t \nabla_x \Phi \cdot b(X_s^\epsilon, Y_s^\epsilon)ds + \frac{\delta}{\sqrt{\epsilon}} \int_0^t \nabla_x \chi \cdot c(X_s^\epsilon, Y_s^\epsilon)ds + \sqrt{\epsilon} \int_0^t \sigma \sigma^T : \nabla_x^2 \chi(X_s^\epsilon, Y_s^\epsilon)ds \]

\[ + \sqrt{\epsilon} \int_0^t \sigma \tau_1^T : \nabla_y \nabla_x \Phi(X_s^\epsilon, Y_s^\epsilon)ds + \int_0^t \nabla_x \chi \cdot \sigma(X_s^\epsilon, Y_s^\epsilon)dW_s - (\delta/\sqrt{\epsilon}) (\chi(X_t^\epsilon, Y_t^\epsilon) - \chi(x_0, y_0)) \]

\[ = \int_0^t (\sigma + \nabla_x \chi \cdot \tau_1)(X_s^\epsilon, Y_s^\epsilon)dW_s + \int_0^t \nabla_y \chi \cdot \tau_2(X_s^\epsilon, Y_s^\epsilon)dB_s + R^\epsilon_{III,t}. \]
\( \mathcal{R}_{II} \) vanishes in probability uniformly in \( t \) as \( \epsilon \to 0 \); to see this, note firstly that 
\( (\delta/\sqrt{\epsilon}) (\chi(X^\epsilon_t, Y^\epsilon_t) - \chi(x_0, y_0)) \) vanishes in probability by the argument of Corollary 1 of [24], and secondly 
that the remaining integrals are bounded in probability by Lemma 5 while their prefactors vanish. 

Putting everything together and setting \( \mathcal{R}^\varepsilon := \mathcal{R}^\varepsilon_I + \mathcal{R}^\varepsilon_{II} + \mathcal{R}^\varepsilon_{III} \), we have 
\[
\frac{1}{\sqrt{\epsilon}} (X^\varepsilon_t - \bar{X}_t) = \int_0^t (\nabla_x \bar{\lambda})(\bar{X}_s) \cdot \frac{1}{\sqrt{\epsilon}} (X^\varepsilon_s - \bar{X}_s) ds + \int_0^t \nabla_y \bar{\Phi} \cdot g(\bar{X}_s) ds \\
+ \int_0^t (\sigma + \nabla_y \chi \cdot \tau_1)(X^\varepsilon_s, Y^\varepsilon_s) dW_s + \int_0^t \nabla_y \chi \cdot \tau_2(X^\varepsilon_s, Y^\varepsilon_s) dB_s + \mathcal{R}^\varepsilon_t. 
\]

The desired representation follows; \( \mathcal{R}^\varepsilon \) vanishes in probability by the same arguments as did \( \mathcal{R}^\varepsilon \).

The proof for the \( \gamma \) regime is similar and is therefore omitted. This concludes the proof of the Theorem. \( \square \)

**Remark 2.** It is not difficult to see that if, for example, \( \nabla_x \bar{\lambda} \) is Lipschitz, then in fact \( \mathcal{R}^\varepsilon \) and \( \mathcal{R}^\varepsilon \) vanish in \( \mathcal{L}^p \). This stronger convergence is however not necessary for our purposes.

### 4 Statistical Inference for a fixed number of observations

Let us now use Theorem 2 to motivate the statistical estimators for the unknown parameter vector \( \theta \). Suppose that the drift coefficients in the model are known functions of an unknown parameter \( \theta \). Suppose further that the drift coefficients in the model are known functions of an unknown parameter \( \theta \). Theorem 2 suggests the minimum contrast estimator (MCE) of \( \theta \) established in Theorem 2. Set, in the \( \infty \) regime,
\[
J^\theta_\infty (x, y) := \frac{1}{\ell_\infty} (\nabla \Phi_\infty \cdot g)^\theta (x, y),
\]
\[
q^\theta_\infty (x, y) := \left[ (\sigma + \nabla_y \chi \cdot \tau_1)(x, y) \right] \left[ (\sigma + \nabla_y \chi \cdot \tau_2)(x, y) \right]^T \theta (x, y)
\]
and, in the \( \gamma \) regime,
\[
J^\theta_\gamma (x, y) := \frac{1}{\ell_\gamma} (\gamma b - \nabla_y \Phi_\gamma \cdot g)^\theta (x, y),
\]
\[
q^\theta_\gamma (x, y) := \left[ (\sigma + \nabla_y \Phi_\gamma \cdot \tau_1)(x, y) \right] \left[ (\sigma + \nabla_y \Phi_\gamma \cdot \tau_2)(x, y) \right]^T \theta (x, y).
\]

Omitting the regime subscript and denoting by \( Z^\theta(t, s) \) the \( \theta \)-dependent solution to [7], Theorem 2 suggests heuristically that
\[
X^\varepsilon_t - \bar{X}^\theta = \sqrt{\epsilon} \int_0^t Z^\theta(t, s) \cdot J^\theta(X^\varepsilon_s) ds = \sqrt{\epsilon} \int_0^t Z^\theta(t, s) \cdot (q^\theta)^{1/2}(X^\varepsilon_s) d\tilde{W}_s,
\]
where \( \tilde{W} \) is a Wiener process. We emphasize that this approximation is not mathematically correct. Nevertheless, continuing the heuristic argument, it appears that for each \( k \), the conditional distribution of \( X^\varepsilon_t \) given \( X^\varepsilon_{tk-1} = x_{tk-1} \) is approximated by a Gaussian distribution with mean
\[
\bar{X}_{tk} + Z^\theta(t, k, \bar{X}_{tk-1}) = \frac{1}{\sqrt{\epsilon}} \int_0^{tk} Z^\theta(t, s) \cdot J^\theta(X^\varepsilon_s) ds
\]
and variance \( \epsilon \int_{tk-1}^{tk} Z^\theta(t, s) \cdot q^\theta(\bar{X}^\theta_s) (Z^\theta)^2(t, k, s) ds \).

Essentially, the heuristic approximation produces a misspecified model, the principle of maximum likelihood applied to which suggests the minimum contrast estimator (MCE)
\[
\hat{\theta}^\varepsilon(x_{tk}) \equiv \arg\min_{\theta \in \Theta} U^\varepsilon(\theta; x_{tk})
\]
(9)
with contrast function

$$U^\epsilon(\theta; \{x_{tk}\}_{k=1}^n) := \sum_{k=1}^n \left[ (F_k^\epsilon)^T(\theta; \{x_{tk}\}_{k=1}^n) \cdot Q_k^{-1}(\theta) \cdot F_k^\epsilon(\theta; \{x_{tk}\}_{k=1}^n) \right], \quad (10)$$

where

$$F_k^\epsilon(\theta; \{x_{tk}\}_{k=1}^n) := \left[ x_{tk} - \bar{X}_{tk}^\theta - Z^\theta(t_k, t_{k-1}) [x_{tk-1} - \bar{X}_{tk-1}^\theta] \right]$$

$$- \sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s) \cdot J^\theta(\bar{X}_s^\theta) ds,$$

$$Q_k(\theta) := \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s) \cdot \phi^\theta(\bar{X}_s^\theta) \cdot (Z^\theta)^T(t_k, s) ds.$$

By Lemma 11 we have that the inverse matrices $Q_k^{-1}$ exist.

In Theorem 3, we prove that for each fixed $n$, the minimum contrast estimator is a consistent estimator of the true value $\theta_0$. In addition, Theorem 3 establishes for each fixed $n$ asymptotic normality and identifies the limiting variance as $\epsilon + \delta \to 0$. If one then takes $n \to \infty$ the limiting variance converges to the variance of the estimator when one has continuous time observations, which establishes that the estimator is also asymptotically efficient in that sense (see Lemma 1).

Before developing the theory, we need to define the limiting contrast function

$$\tilde{U}(\theta; \{x_{tk}\}_{k=1}^n) := \sum_{k=1}^n \left[ \tilde{F}_k^T(\theta; \{x_{tk}\}_{k=1}^n) \cdot Q_k^{-1}(\theta) \cdot \tilde{F}_k(\theta; \{x_{tk}\}_{k=1}^n) \right],$$

where

$$\tilde{F}_k(\theta; \{x_{tk}\}_{k=1}^n) := \left[ x_{tk} - \bar{X}_{tk}^\theta - Z^\theta(t_k, t_{k-1}) [x_{tk-1} - \bar{X}_{tk-1}^\theta] \right]. \quad (12)$$

By Lemma 12 in the Appendix, we have that

$$\lim_{\epsilon \to 0} P \left( \sup_{\theta_1, \theta_2 \in \Theta} |U^\epsilon(\theta_2; \{X_{tk}^{\epsilon, \theta_1}\}_{k=1}^n) - \tilde{U}(\theta_2; \{X_{tk}^{\theta_1}\}_{k=1}^n)| > \eta \right) = 0.$$

Denoting by $\theta_0$ the true value of the unknown parameter, we must assume an appropriate identifiability condition to guarantee that $\tilde{U}(\theta; \{\bar{X}_{tk}^{\theta_0}\}_{k=1}^n)$ is uniquely minimized at $\theta = \theta_0$.

**Condition 5. (Identifiability Condition 1)** For any $\theta_0 \in \Theta$, $\bar{X}_{tk}^\theta = \bar{X}_{tk}^{\theta_0}$ for $k = 1, 2, \ldots, n$ if and only if $\theta = \theta_0$.

**Remark 3.** Condition 6 implies that $\tilde{F}_k(\theta; \{\bar{X}_{tk}^{\theta_0}\}_{k=1}^n) = 0$ for $k = 1, 2, \ldots, n$ if and only if $\theta = \theta_0$, and hence that $\tilde{U}(\theta; \{\bar{X}_{tk}^{\theta_0}\}_{k=1}^n)$ is uniquely minimized at $\theta = \theta_0$.

**Theorem 3. (Consistency of the MCE)** Assume Conditions 4, 5, and 7 and, in the $\infty$ regime, Condition 3. For any $\theta_0 \in \Theta$ and $\eta > 0$,

$$\lim_{\epsilon \to 0} P \left( |\tilde{U}(\theta_0; \{X_{tk}^{\epsilon, \theta_0}\}_{k=1}^n) - \theta_0| > \eta \right) = 0.$$

**Proof.** Consider the modulus of continuity

$$w^\epsilon(\{x_{tk}\}_{k=1}^n, \phi) := \sup_{\theta_1, \theta_2 \in \Theta; |\theta_1 - \theta_2| \leq \phi} |U^\epsilon(\theta_1; \{x_{tk}\}_{k=1}^n) - U^\epsilon(\theta_2; \{x_{tk}\}_{k=1}^n)|.$$

By Theorem 3.2.8 of [2], it suffices to show that $w^\epsilon(\{X_{tk}^{\epsilon, \theta_0}\}_{k=1}^n, \phi)$ converges in probability uniformly in $\phi$ to a function of $\phi$ that tends to 0 as $\phi \to 0$. In fact,

$$\lim_{\epsilon \to 0} P \left( \sup_{\theta \in \Theta} \sup_{\phi > 0} |w^\epsilon(\{X_{tk}^{\epsilon, \theta_0}\}_{k=1}^n, \phi) - w^\theta(\phi)| > \eta \right) = 0.$$
Proof. Let us suppress the data where exists, we may re-express this as

$$
\tilde{\theta} := \sup_{\theta_1, \theta_2 \in \Theta; |\theta_1 - \theta_2| \leq \phi} \left| \tilde{U}(\theta_1; \{X^n_{\theta_{k}}\}_{k=1}^n) - \tilde{U}(\theta_2; \{X^n_{\theta_{k}}\}_{k=1}^n) \right|
$$

this is immediate by Lemma 12 upon writing

$$
\left( U^\varepsilon(\theta_1; \{X^n_{\theta_{k}}\}_{k=1}^n) - U^\varepsilon(\theta_2; \{X^n_{\theta_{k}}\}_{k=1}^n) \right) = \left( \tilde{U}(\theta_1; \{X^n_{\theta_{k}}\}_{k=1}^n) - \tilde{U}(\theta_2; \{X^n_{\theta_{k}}\}_{k=1}^n) \right)
$$

That $$\lim_{\varepsilon \to 0} \tilde{\theta}(\phi) = 0$$ is a corollary of Lemma 13 in the Appendix, concluding the proof of the theorem.

Before proceeding to a central limit theorem for the MCE, we must assume an appropriate identifiability condition to guarantee that the variance is well behaved in the limit.

**Condition 6. (Identifiability Condition 2)** For any $$\theta \in \Theta$$, the weighted difference

$$
Z^\theta(t_k, t_{k-1}) \cdot \nabla^\theta \tilde{X}_{k-1}^\theta - \nabla^\theta \tilde{X}_k^\theta
$$

is nonzero for at least one value of $$k = 1, 2, \ldots, n$$.

**Theorem 4. (Asymptotic Normality of the MCE)** Assume Conditions 4, 5, and 6, and, in the infinite regime, Condition 3. For any given $$\theta_0 \in \Theta$$, $$\frac{1}{\sqrt{\varepsilon}} (\tilde{\theta}(\{X^n_{\theta_0}\}_{k=1}^n) - \theta_0)$$ converges in distribution as $$\varepsilon \to 0$$ to the normal distribution $$N(0, M(\theta_0))$$, where the covariance is given by the formula

$$
M(\theta) := \sum_{k=1}^n \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla^\theta X_{k-1}^\theta - \nabla^\theta X_k^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla^\theta X_{k-1}^\theta - \nabla^\theta X_k^\theta \right) \right]^{-1}
$$

**Proof.** Let us suppress the data $$\{X^n_{\theta_k}\}_{k=1}^n$$. By Taylor’s theorem,

$$
0 = \frac{1}{\sqrt{\varepsilon}}(\tilde{\theta}^\varepsilon - \theta_0) = \frac{1}{\sqrt{\varepsilon}}(\nabla^\theta U^\varepsilon)(\theta_0) + \frac{1}{\sqrt{\varepsilon}}(\nabla^2_\theta U^\varepsilon)(\theta^\varepsilon; \cdot) \cdot (\tilde{\theta}^\varepsilon - \theta_0),
$$

where $$\theta^\varepsilon; \cdot$$ is an appropriately chosen point on the segment connecting $$\tilde{\theta}^\varepsilon$$ with $$\theta_0$$. Assuming the inverse exists, we may re-express this as

$$
\frac{1}{\sqrt{\varepsilon}}(\tilde{\theta}^\varepsilon - \theta_0) = (\nabla^2_\theta U^\varepsilon)^{-1}(\theta^\varepsilon; \cdot) \cdot \frac{1}{\sqrt{\varepsilon}}(\nabla^\theta U^\varepsilon)(\theta_0). \quad (13)
$$

Thus, it suffices to establish a limit in distribution of $$\frac{1}{\sqrt{\varepsilon}}(\nabla^\theta U^\varepsilon)(\theta_0)$$ and an invertible limit in probability of $$(\nabla^2_\theta U^\varepsilon)(\theta^\varepsilon; \cdot)$$ (see also Section 3.3.4 of [2] for a rigorous justification of this now-classical approach).

For the weighted gradient of the contrast function, we have that

$$
\frac{1}{\sqrt{\varepsilon}}(\nabla^\theta U^\varepsilon)(\theta_0) = [I + II + III + IV]_{\theta = \theta_0}, \quad (14)
$$

where

$$
I := \sqrt{\varepsilon} \sum_{k=1}^n \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla^\theta X_{k-1}^\theta - \nabla^\theta X_k^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot F_k(\theta) \right],
$$

$$
II := -\sqrt{\varepsilon} \sum_{k=1}^n \left[ \left( \nabla^\theta Z^\theta(t_k, t_{k-1}) \cdot \left( X_{k-1}^\theta - \tilde{X}_{k-1}^\theta \right) \right)^T \cdot Q_k^{-1}(\theta) \cdot F_k(\theta) \right],
$$

$$
III := -2\sqrt{\varepsilon} \sum_{k=1}^n \left[ \left( \nabla^\theta \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s) \cdot \tilde{X}_s^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot F_k(\theta) \right],
$$

$$
IV := \sqrt{\varepsilon} \sum_{k=1}^n \left[ (F_k)^T(\theta) \cdot \nabla^\theta (Q_k^{-1}(\theta)) \cdot F_k(\theta) \right].
$$
Note that $\|\nabla_\theta Z^\theta(t_k, t_{k-1})\|$ is uniformly bounded, that \(\frac{1}{\sqrt{\epsilon}} E|X^\epsilon_{t_k} - \bar{X}^\theta_{t_k}|\) is bounded for \(\epsilon\) sufficiently small by Theorem 11, that $||Q_k^{-1}(\theta)||$ is bounded by Lemma 8, and that $E|F_k^\epsilon(\theta)|$ vanishes at the rate of $\sqrt{\epsilon}$ by Lemma 8. Putting these together, we deduce that $II|_{\theta = \theta_0}$ vanishes in probability. Meanwhile, $I|_{\theta = \theta_0}$ and $IV|_{\theta = \theta_0}$ similarly vanish in probability. Meanwhile, $I|_{\theta = \theta_0}$ converges in distribution to $\mathcal{N}(0, \Gamma(\theta_0))$ with

$$
\Gamma(\theta) = 4\Sigma_{k=1}^n \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right) \right].
$$

To see this, let $\psi$ stand for $\chi$ in the $\infty$ regime and for $\Phi$ in the $\gamma$ regime. Recalling the definition of $F_k^\epsilon$ we write, with $\tilde{R}^\epsilon$ as in Theorem 2

$$
F_k^\epsilon(\theta) = \sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s)(\sigma + \nabla_y \psi \cdot \tau_1)(X_s^\epsilon, Y_s^\epsilon) dW_s + \sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s)\nabla_y \psi \cdot \tau_2(X_s^\epsilon, Y_s^\epsilon) dB_s + \sqrt{\epsilon} \left[ \tilde{R}^\epsilon_{t_k} - Z^\theta(t_k, t_{k-1})\tilde{R}^\epsilon(t_{k-1}) \right].
$$

Thus, we see that the sequence \(\frac{1}{\sqrt{\epsilon}} F_k^\epsilon(\theta)\) converges in distribution, as a sequence, to the sequence \(\left\{ \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s)\tilde{\epsilon}_n^{1/2}(\tilde{X}_s^\theta) d(W, B)_s \right\}_{k=1}^n\) of independent Gaussian random variables, whence $I$ converges in distribution to

$$
2\Sigma_{k=1}^n \int_{t_{k-1}}^{t_k} \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot Z^\theta(t_k, s)\tilde{\epsilon}_n^{1/2}(\tilde{X}_s^\theta) d(W, B)_s,
$$

which is of course a centered Gaussian random variable with covariance matrix

$$
4\Sigma_{k=1}^n \int_{t_{k-1}}^{t_k} \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot Z^\theta(t_k, s)\tilde{\epsilon}_n^{1/2}(\tilde{X}_s^\theta) \right] ds
$$

$$
= 4\Sigma_{k=1}^n \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot Z^\theta(t_k, s)\tilde{\epsilon}_n^{1/2}(\tilde{X}_s^\theta) \right] \cdot Q_k(\theta)
$$

$$
\times \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right)^T \cdot Q_k^{-1}(\theta) \cdot Z^\theta(t_k, s)\tilde{\epsilon}_n^{1/2}(\tilde{X}_s^\theta) \right]^T,
$$

which is exactly $\Gamma(\theta)$. For the Hessian of the contrast function, we have, in the notation of (14),

$$
(\nabla_\theta^2 U^\epsilon)(\theta, \epsilon) = \sqrt{\epsilon} \left[ \nabla_\theta (I + II + III + IV) \right] \bigg|_{\theta = \theta_0}^\epsilon.
$$

$$
\sqrt{\epsilon} \nabla_\theta II \bigg|_{\theta = \theta_0}^\epsilon, \sqrt{\epsilon} \nabla_\theta III \bigg|_{\theta = \theta_0}^\epsilon, \text{ and } \sqrt{\epsilon} \nabla_\theta IV \bigg|_{\theta = \theta_0}^\epsilon \text{ vanish in probability. Meanwhile, the first term may be}
$$
rewritten as

\[ \sqrt{\epsilon} \nabla_\theta I \bigg|_{\theta,0} = 2\sum_{k=1}^{\infty} \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right)^T \cdot Q_K^{-1}(\theta) \cdot \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right) \right] \bigg|_{\theta,0} \]

\[ - 2\sum_{k=1}^{\infty} \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right)^T \cdot Q_K^{-1}(\theta) \cdot \left( \nabla_\theta Z^\theta(t_k, t_{k-1}) : X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right) \right] \bigg|_{\theta,0} \]

\[ - 2\sum_{k=1}^{\infty} \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right)^T \cdot Q_K^{-1}(\theta) \cdot \nabla_\theta \int_{t_{k-1}}^{t_k} Z^\theta(t_k, s) \cdot Q^0(\mathbf{X}^\theta, s) ds \bigg|_{\theta,0} \right) \]

\[ + 2\sum_{k=1}^{\infty} \left[ \nabla_\theta \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right)^T \cdot Q_K^{-1}(\theta) \cdot F^0(\theta) \right] \bigg|_{\theta,0} \]

which is invertible by Lemma 11 and Condition 6.

Recalling (13) and (15), we conclude that \( \frac{1}{\sqrt{\epsilon}}(\theta^\epsilon(\{X^\theta_{k}\}_{k=1}^n) - \theta_0) \) converges in distribution as \( \epsilon \to 0 \) to \( N(0, M(\theta_0)) \), with

\[ M(\theta) = \left[ \sum_{k=1}^{\infty} \left[ \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right)^T \cdot Q_K^{-1}(\theta) \cdot \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta \right) \right] \right]^{-1}, \]

concluding the proof of the Theorem.

Let us conclude this section by computing the limit \( \lim_{n \to \infty} M^{-1}(\theta; n) \), where we write \( M^{-1}(\theta; n) = M^{-1}(\theta) \) to emphasize the dependence on the number of data points \( n \). Recall that in the \( \infty \) regime with \( b \neq 0 \) (the homogenization regime), the continuous-time MLE is biased in general and thus a central limit theorem in the traditional sense is impossible. In the case that \( b \equiv 0 \), however, the central limit theorem is derived in [13][20]. Lemma 11 states that \( \lim_{n \to \infty} M^{-1}(\theta; n) \) agrees with the limiting Fisher information matrix established therein; hence, the estimator considered in this paper is also asymptotically statistically efficient as first \( \epsilon + \delta \to 0 \) and then \( n \to \infty \).

**Lemma 1.** Assume Conditions [7][8][9][10] and [11] and, in the \( \infty \) regime, Condition [3]. For any given \( \theta \in \Theta \),

\[ \lim_{n \to \infty} M^{-1}(\theta; n) = \int_0^T (\nabla_\theta \lambda^\theta)^T (\lambda^\theta) (q^\theta)^{-1}(\lambda^\theta) (\nabla_\theta \lambda^\theta) (\lambda^\theta) ds, \]

where \( M^{-1}(\theta; n) \) is the limiting Fisher information with \( n \) samples as per Theorem [4].

**Proof.** Notice that

\[ Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta X_{t_{k-1}}^\theta - \nabla_\theta X_{t_k}^\theta = \left( Z^\theta(t_k, t_{k-1}) - 1 - \int_{t_{k-1}}^{t_k} (\nabla_\theta \lambda^\theta)(\lambda^\theta) du \right) \cdot \nabla_\theta X_{t_k}^\theta \]

\[ + \int_{t_{k-1}}^{t_k} (\nabla_\theta \lambda^\theta)(\lambda^\theta) du \cdot \nabla_\theta X_{t_k}^\theta - \nabla_\theta (X_{t_k}^\theta - X_{t_{k-1}}^\theta) \]

\[ = I + II + III, \]
where, with $1_m$ being the $m \times m$ identity matrix,

\[
I := \left(Z^\theta(t_k, t_{k-1}) - 1_m - \int_{t_{k-1}}^{t_k} (\nabla_x \bar{X}^\theta)(\bar{X}^\theta_{u}) du \right) \cdot \nabla_\theta \bar{X}^\theta_{t_k-1},
\]

\[
II := \int_{t_{k-1}}^{t_k} (\nabla_x \bar{X}^\theta)(\bar{X}^\theta_{u}) \cdot \nabla_\theta \left( \bar{X}_{t_k-1}^\theta - \bar{X}_u^\theta \right) du,
\]

\[
III := - \int_{t_{k-1}}^{t_k} (\nabla_\theta \bar{X}^\theta)(\bar{X}^\theta_{u}) du.
\]

By Taylor’s theorem, there is a constant $K$ such that $|I| \leq K \cdot \Delta^2$; likewise, there is a constant $K$ such that $|II| \leq K \cdot \Delta^2$. Meanwhile, the exponential terms in the definition of $Q$ converge uniformly to the identity. Finally, $III := - \int_{t_{k-1}}^{t_k} (\nabla_\theta \bar{X}^\theta)(\bar{X}^\theta_{u}) du$ behaves like $-\Delta \cdot (\nabla_\theta \bar{X}^\theta)(\bar{X}_{t_k-1}^\theta)$, whence we conclude that $M^{-1}(\theta; n)$ behaves asymptotically as the Riemann sum

\[
\Delta \cdot \sum_{k=1}^{n} \left[(\nabla_\theta \bar{X}^\theta)^T \cdot (\bar{q}^\theta)^{-1} \cdot (\nabla_\theta \bar{X}^\theta)\right] (\bar{X}_{t_k-1}^\theta),
\]

whose limit is of course $\int_0^T \left[(\nabla_\theta \bar{X}^\theta)^T \cdot (\bar{q}^\theta)^{-1} \cdot (\nabla_\theta \bar{X}^\theta)\right] (\bar{X}_s^\theta) ds$, concluding the proof of the lemma.

\[\square\]

## 5 A Simplified Estimator

In this section we show that consistency and asymptotic normality are still achieved, even if one omits the covariance weights $Q_{ij}^{-1}$ in the contrast function. On the one hand, one incurs by this omission an increase in the limiting variance, and asymptotic efficiency is lost (see Lemma 3). On the other hand, the simplification affords certain advantages - for instance, one may compute the simplified contrast even if $\sigma$ is unknown. The simplified estimation procedure also presents enhanced robustness in numerical simulations (see Section 7), showing less sensitivity to the values of $\epsilon$ and $\delta$, which are commonly unknown in practice.

With the omission of the weights, one essentially imposes the additional simplification of constant diffusion on the misspecified model that gave rise to the first contrast estimator [9]. Accordingly, the proofs of the results presented in this section follow near verbatim the arguments of Section 4 and will not be presented separately in this section. The identifiability conditions remain unchanged. We also substitute from the outset the $\tilde{F}_k$ of equation (12) for the $F_k^\theta$ of equation (11); it is not hard to see that the integral terms lost in doing so are in any event asymptotically negligible.

Let us therefore define the simplified minimum contrast estimator (SMCE)

\[
\tilde{\theta}(\{x_{t_k}\}_{k=1}^{n}) := \arg\min_{\theta \in \Theta} \sum_{k=1}^{n} |\tilde{F}_k(\theta; \{x_{t_k}\}_{k=1}^{n})|^2
\]

\(\text{(19)}\)

**Theorem 5.** (Consistency of the SMCE) Assume Conditions 1, 2, 4, 5, and 6 and, in the $\infty$ regime, Condition 3. For any $\theta_0 \in \Theta$ and $\eta > 0$,

\[
\lim_{\epsilon \to 0} P \left( |\tilde{\theta}(\{X_{t_k}^{\epsilon, \theta_0}\}_{k=1}^{n}) - \theta_0| > \eta \right) = 0.
\]

**Theorem 6.** (Asymptotic Normality of the SMCE) Assume Conditions 1, 2, 4, 5, and 6 and, in the $\infty$ regime, Condition 3. For any given $\theta_0 \in \Theta$, \(\sqrt{\epsilon} (\tilde{\theta}(\{X_{t_k}^{\epsilon, \theta_0}\}_{k=1}^{n}) - \theta_0)\) converges in distribution as $\epsilon \to 0$ to the normal distribution $N(0, \tilde{M}(\theta_0))$, where the covariance is given by the formula

\[
\tilde{M}(\theta) := \Psi^{-1}(\theta) \cdot \Xi(\theta) \cdot \Psi^{-1}(\theta),
\]

where

\[
\Psi(\theta) := \sum_{k=1}^{n} \left[ Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k-1}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right]^T \cdot \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k-1}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right),
\]

\[
\Xi(\theta) := \sum_{k=1}^{n} \left[ Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k-1}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right]^T \cdot Q_k(\theta) \cdot \left( Z^\theta(t_k, t_{k-1}) \cdot \nabla_\theta \bar{X}_{t_k-1}^\theta - \nabla_\theta \bar{X}_{t_k}^\theta \right).
\]
Lemma 2. Assume Conditions [1] [2] [3] [4] and [6] and, in the $\infty$ regime, Condition [8]. For any given $\theta \in \Theta$, 
\[
\lim_{n \to \infty} \tilde{M}^{-1}(\theta; n) = \tilde{\Psi}(\theta) \cdot \tilde{\Xi}^{-1}(\theta) \cdot \tilde{\Psi}(\theta),
\]
where
\[
\tilde{\Psi}(\theta) := \int_0^T \left[ (\nabla_{\theta} \tilde{\lambda}^0)^T \cdot (\nabla_{\theta} \tilde{\lambda}^0) \right] (\tilde{X}_s^\theta) ds,
\]
\[
\tilde{\Xi}(\theta) := \int_0^T \left[ (\nabla_{\theta} \tilde{\lambda}^0)^T \cdot \tilde{q}^0 \cdot (\nabla_{\theta} \tilde{\lambda}^0) \right] (\tilde{X}_s^\theta) ds.
\]

The next lemma establishes that the variance of the SMCE is bounded below by that of the MCE in the sense that the difference of the covariance matrices is positive semidefinite. In general, one loses asymptotic statistical efficiency when one moves from the MCE to the SMCE.

Lemma 3. Assume Conditions [1] [2] [3] [4] [5] and [6] and, in the $\infty$ regime, Condition [8]. Let $M(\theta; n)$ and $\bar{M}(\theta; n)$ be as in Theorems [4] and [6] respectively. For any $\theta \in \Theta$ and $n \geq 1$, the $k \times k$ matrix
\[
\bar{M}(\theta; n) - M(\theta; n)
\]
is positive semidefinite. In particular, for any $\theta \in \Theta$, the difference of limits
\[
\lim_{n \to \infty} \bar{M}(\theta; n) - \lim_{n \to \infty} M(\theta; n)
\]
is positive semidefinite.

Proof. Without loss of generality, it is enough to prove that $\bar{M}^{-1}(\theta; n) - M^{-1}(\theta; n)$ is negative semidefinite. We suppress the parameters $\theta$ and $n$. Recall the notation $\Psi$ and $\Xi$ from Theorem [6] and define
\[
d_k := Z(t_k, t_{k-1}) \cdot \nabla_{\theta} \tilde{X}_{t_{k-1}} - \nabla_{\theta} \tilde{X}_t
\]
so that
\[
\Psi = \Sigma_k d_k^T d_k, \\
\Xi = \Sigma_k d_k^T Q_k d_k.
\]
Let us prove the lemma by showing that the matrix difference $\bar{M}^{-1}(\theta; n) - M^{-1}(\theta; n)$ has no negative eigenvalues. Let $\xi \in \mathbb{R}^k \setminus \{0\}$ be given.
\[
|\xi^T \bar{M}^{-1} \xi|^2 = |\Sigma_k \langle d_k \Xi^{-1} \Psi \xi, d_k \xi \rangle|^2 \\
= |\Sigma_k ((Q^{1/2})^T d_k \Xi^{-1} \Psi \xi, (Q^{1/2})^{-1} d_k \xi)|^2 \\
\leq \left[ \Sigma_k ((Q^{1/2})^T d_k \Xi^{-1} \Psi \xi)^2 \right] \cdot \left[ \Sigma_k ((Q^{1/2})^{-1} d_k \xi)^2 \right] \\
= |\Sigma_k \langle d_k^T Q_k d_k \Xi^{-1} \Psi \xi, \Xi^{-1} \Psi \xi \rangle | \cdot |\Sigma_k \langle d_k^T Q_k d_k \xi, \xi \rangle | \\
= |\xi^T \bar{M}^{-1} \xi | \cdot |\xi^T M^{-1} \xi |.
\]
Dividing the first and last by $\xi^T \bar{M}^{-1} \xi$, which is positive, one obtains
\[
\xi^T \left[ \bar{M}^{-1} - M^{-1} \right] \xi \geq 0,
\]
which proves [20] since $\xi$ was an arbitrary nonzero vector. The limit statement [21] is immediate by continuity. \qed
6 Limit of High-Frequency Observation

In Section 4, we considered the important case in which the number of observations \( n \) is fixed, obtaining consistent and asymptotically-normal estimators. The question naturally arises as to whether these properties carry over into the asymptotic regime of high-frequency observation, in which \( n \) is taken arbitrarily large at the same time that \( \epsilon \) is taken to vanish; that is, recalling our notation \( \Delta := T/n \) for the sampling interval, we wish to consider the behavior of the estimators in the joint limit \( \epsilon + \Delta \to 0 \).

It turns out that if \( \epsilon \) is \( o(\Delta) \) as \( \Delta \to 0 \), then the theory of Sections 4 and 5 carries over with only minor adjustments.

**Condition 7.** *(Identifiability Condition 3)* For any \( \theta_0 \in \Theta, \) the integral

\[
\Lambda_t(\theta, \theta_0) := \tilde{\lambda}^\theta(\tilde{X}^\theta_t) - \tilde{\lambda}^\theta(\tilde{X}^\theta_t) - (\nabla_x \tilde{\lambda}^\theta)(\tilde{X}^\theta_t) \cdot [\tilde{X}^\theta_t - \tilde{X}^\theta_t]
\]

vanishes for all \( t \in [0, T] \) if and only if \( \theta = \theta_0 \).

**Theorem 7.** *(Consistency of the MCE and SMCE as \( \epsilon + \Delta \to 0) \)* Assume Conditions 2, 3, 4, and 7 and, in the \( \infty \) regime, Condition 3. Assume that \( \Delta \) does not decrease too quickly relative to \( \epsilon \), so that \( \epsilon \) is \( o(\Delta) \) as \( \Delta \to 0 \). For any \( \theta_0 \in \Theta \) and \( \eta > 0 \),

\[
\lim_{(\epsilon + \Delta) \to 0} P \left( |\hat{\theta}^\epsilon(\{X_{t_k}^{\theta_0}\}_{k=1}^\infty) - \theta_0| > \eta \right) = 0,
\]

\[
\lim_{(\epsilon + \Delta) \to 0} P \left( |\hat{\theta}^\epsilon(\{X_{t_k}^{\theta_0}\}_{k=1}^\infty) - \theta_0| > \eta \right) = 0.
\]

**Proof.** The proof is similar to those of Theorems 3 and 5 with Lemma 10 in place of Lemma 12.

The following identifiability condition is the limiting analogue of Condition 6.

**Condition 8.** *(Identifiability Condition 4)* For any \( \theta \in \Theta, \) the matrix-valued integral

\[
\int_0^T [(\nabla_\theta \tilde{\lambda}^\theta)^T (\nabla_\theta \tilde{\lambda}^\theta)](\tilde{X}^\theta_s)ds
\]

is invertible.

We wish to mimic the proof of Theorem 4. When one is considering the joint limit \( \epsilon + \Delta \to 0 \), the contribution of the covariance corrections \( Q_{k^-1} \) to the \( \theta \) derivatives of the contrast function becomes a rather delicate quantity to handle. We therefore state and prove asymptotic normality of the MCE in this joint limit, only under the stronger assumption that \( \epsilon \) is \( o(\Delta^2) \), rather than merely \( o(\Delta) \), as \( \Delta \to 0 \). This is Theorem 8.

Notably, the stronger assumption is not needed to adapt with little modification the proof of asymptotic normality of the SMCE for the same joint limit. With the omission of the weights, one essentially imposes the additional simplification of constant diffusion on the misspecified model that motivates the estimator - in particular, the corrections \( Q_{k^-1} \) are essentially replaced with \( 1/\Delta \), and the delicate \( \theta \) dependence is avoided altogether. Note that Theorem 9 assumes only that \( \epsilon \) is \( o(\Delta) \).

**Theorem 8.** *(Asymptotic Normality of the MCE as \( \epsilon + \Delta \to 0) \)* Assume Conditions 1, 2, 3, 4, 7, and 8 and, in the \( \infty \) regime, Condition 3. Assume that \( \Delta \) decreases slowly relative to \( \epsilon \), so that \( \epsilon \) is \( o(\Delta^2) \) as \( \Delta \to 0 \). For any given \( \theta_0 \in \Theta, \)

\[
\frac{1}{\epsilon \sqrt{T}} (\hat{\theta}^\epsilon(\{X_{t_k}^{\theta_0}\}_{k=1}^\infty) - \theta_0)
\]

converges in distribution as \( \epsilon + \Delta \to 0 \) to the normal distribution \( \mathcal{N}(0, M(\theta_0)) \), where the covariance is given by the formula

\[
M(\theta) := \left[ \int_0^T (\nabla_\theta \tilde{\lambda}^\theta)^T (\tilde{X}^\theta_s) \cdot (\sigma^\theta)^{-1}(\tilde{X}^\theta_s) \cdot (\nabla_\theta \tilde{\lambda}^\theta)(\tilde{X}^\theta_s)ds \right]^{-1}.
\]

**Proof.** Let us examine the proof of Theorem 4. Consider, in the notation of (14), the term II. It is easy to see that \( \nabla_\theta Z^\theta(t_k, t_{k-1}) \) is uniformly bounded by a constant times \( \Delta \), and Lemma 11 establishes that
\[ ||Q_k^{-1}(\theta)|| \] is bounded by a constant times \(1/\Delta\). Meanwhile, \(E\left[\frac{1}{\sqrt{n}} \left( X_{t_n}^{\epsilon,\theta_0} - \bar{X}_{n-1}^{\theta} \right) \right]^2 \) is uniformly bounded by Theorem 1. Thus, by an application of Hölder’s inequality, \(II\) will vanish even as \(n \to \infty\), provided that the general term \(E[F_k(\theta)]^2\) vanishes faster than \(\Delta^2\); but this is immediate from Lemma 14, since we have assumed that \(\epsilon\) tends to 0 faster than does \(\Delta\). \(III\) vanishes similarly. \(IV\) vanishes by Lemma 14 and the fact that \(||\nabla_\theta(Q_k^{-1}(\theta))||\) is bounded by a constant times \(1/\Delta\) (it is in order to accommodate this somewhat naïve bound on the derivative of the covariance correction that we have assumed for this theorem that \(\epsilon\) is \(o(\Delta^2)\), rather than merely \(o(\Delta)\)). The convergence in distribution of \(I\) is as before, although in the regime of joint asymptotics \(\epsilon + \Delta \to 0\), one must of course combine the arguments of Theorem 1 and Lemma 1.

The convergence in probability of the Hessian (17) even as \(n \to \infty\) is likewise.

\[ \square \]

**Theorem 9.** (Asymptotic Normality of the SMCE as \(\epsilon + \Delta \to 0\)) Assume Conditions 1, 2, 4, 7, and 8 and, in the \(\infty\) regime, Condition 3. Assume that \(\Delta\) does not decrease too quickly relative to \(\epsilon\), so that \(\epsilon\) is \(o(\Delta)\) as \(\Delta \to 0\). For any given \(\theta_0 \in \Theta\), \(\frac{1}{\sqrt{n}}(\hat{\theta}^\epsilon\{X_{t_n}^{\epsilon,\theta_0}\}_{n=1}^\infty - \theta_0)\) converges in distribution as \(\epsilon + \Delta \to 0\) to the normal distribution \(N(0, \hat{M}(\theta_0))\), where the covariance is given by the formula

\[
\hat{M}(\theta) := \hat{\Psi}^{-1}(\theta) \cdot \hat{\Xi}(\theta) \cdot \hat{\Psi}^{-1}(\theta),
\]

where

\[
\hat{\Psi}(\theta) := \int_0^T \left( (\nabla_\theta \bar{\lambda}^\theta)^T \cdot (\nabla_\theta \bar{\lambda}^\theta) \right) (\bar{X}_s^\theta) ds,
\]

\[
\hat{\Xi}(\theta) := \int_0^T \left( (\nabla_\theta \bar{\lambda}^\theta)^T \cdot \bar{q}^\theta \cdot (\nabla_\theta \bar{\lambda}^\theta) \right) (\bar{X}_s^\theta) ds.
\]

**Proof.** As before, the arguments for the SMCE follow nearly verbatim those for the MCE. The weaker assumption \(\epsilon = o(\Delta)\) suffices in this case because the term for which it was needed in the proof of Theorem 8 i.e., term IV in (14), does not appear if one is using the simplified contrast function in place of the original contrast function.

\[ \square \]

**Remark 4.** It is clear that one can re-express the conditions \(\epsilon = o(\Delta)\) and \(\epsilon = o(\Delta^2)\) in terms of \(\delta\) and \(\Delta\). We have deliberately chosen instead to present the results in terms of \(\epsilon\) and \(\Delta\) because \(\Delta\) is known in general (via \(T\) and \(n\)) whereas \(\epsilon\) can in principle be estimated via the magnitude of the quadratic variation of the process \(X^\epsilon\). We emphasize that in contrast to the rest of the subsampling literature, e.g. 23, 23, we do not (in fact, cannot) impose conditions on how \(1/\Delta\) grows relative to \(n\), due simply to the fact that we have fixed \(T = n \cdot \Delta\) to study the regime of small noise.

### 7 Numerical Examples

We now present data from numerical simulations to supplement and illustrate the theory. We begin by considering the system

\[
\begin{align*}
\epsilon X_t^\epsilon &= \frac{\epsilon}{6} (\sin(Y_t^\epsilon) - \cos(Y_t^\epsilon)) dt - \theta_0 X_t^\epsilon dt + \sqrt{\epsilon} dW_t \\
Y_t^\epsilon &= X_t^\epsilon/\delta
\end{align*}
\]  

(23)

for \(t \in [0, T = 1]\) with \(X_0^\delta = Y_0^\delta = 1 \in \mathbb{R}\). This system was studied as an example in 23. We suppose that we are in the \(\infty\) regime with \(\delta = \epsilon^{3/2}\). The results of Sections 1, 5 and 6 on consistency and asymptotic normality are illustrated in the numerical data that follow.

The limit \(\bar{X}\) of the slow process \(X^\epsilon\) in (23) is

\[
\bar{X}_t = e^{-t - \theta_0 \left( \frac{\epsilon}{4} \right)^2},
\]
where \( L := \int_0^{2\pi} e^{2(\sin(y) + \cos(y))} \, dy \).

We fix \( \theta_0 = 1 \) and simulate trajectories using an Euler scheme with \( 10^8 \) evenly-spaced discrete time steps. We simulate both estimators, \( \hat{\theta}^\varepsilon \) and \( \tilde{\theta}^\varepsilon \), 1000 times for each combination of \( \epsilon = 10^{-2}, 10^{-3} \) and \( n = 10, 10^2, 10^3 \). Tables 1 and 2 present in each case the mean estimate, a normal-based confidence interval using the empirical standard deviation, and the theoretical standard deviation as per Theorems 4 and 6. The histograms in Figures 1-4 compare the empirical distribution of the estimates with the theoretical density curve.

| \( \epsilon \) | \( n \) | Mean Estimator | 68% Confidence Interval | 95% Confidence Interval | Theoretical SD |
|---|---|---|---|---|---|
| \( 10^{-2} \) | 10 | 1.0173 | (0.5483, 1.4862) | (0.0794, 1.9552) | 0.4370 |
| \( 10^{-2} \) | \( 10^2 \) | 1.0430 | (0.5688, 1.5173) | (0.0945, 1.9916) | 0.4370 |
| \( 10^{-2} \) | \( 10^3 \) | 1.0686 | (0.6104, 1.5267) | (0.1523, 1.9848) | 0.4370 |
| \( 10^{-3} \) | 10 | 1.0005 | (0.8610, 1.1401) | (0.7215, 1.2794) | 0.1382 |
| \( 10^{-3} \) | \( 10^2 \) | 0.9992 | (0.8571, 1.1413) | (0.7150, 1.2834) | 0.1382 |
| \( 10^{-3} \) | \( 10^3 \) | 1.0068 | (0.8643, 1.1494) | (0.7218, 1.2919) | 0.1382 |

Table 1: Example 1 - MCE of \( \theta_0 = 1 \) with empirical confidence intervals and theoretical standard deviations.

| \( \epsilon \) | \( n \) | Mean Estimator | 68% Confidence Interval | 95% Confidence Interval | Theoretical SD |
|---|---|---|---|---|---|
| \( 10^{-2} \) | 10 | 1.0710 | (0.5902, 1.5518) | (0.1095, 2.0326) | 0.4370 |
| \( 10^{-2} \) | \( 10^2 \) | 1.0649 | (0.5845, 1.5453) | (0.1041, 2.0256) | 0.4370 |
| \( 10^{-2} \) | \( 10^3 \) | 1.0764 | (0.5958, 1.5570) | (0.1151, 2.0377) | 0.4370 |
| \( 10^{-3} \) | 10 | 1.0115 | (0.8732, 1.1498) | (0.7349, 1.2880) | 0.1382 |
| \( 10^{-3} \) | \( 10^2 \) | 1.0074 | (0.8598, 1.1549) | (0.7122, 1.3025) | 0.1382 |
| \( 10^{-3} \) | \( 10^3 \) | 1.0108 | (0.8690, 1.1525) | (0.7272, 1.2943) | 0.1382 |

Table 2: Example 1 - SMCE of \( \theta_0 = 1 \) with empirical confidence intervals and theoretical standard deviations.

Figure 1: Example 1, MCE, \( \epsilon = 10^{-3}, n = 10 \)  
Figure 2: Example 1, MCE, \( \epsilon = 10^{-3}, n = 10^3 \)

Tables 1 and 2 and Figures 1-4 illustrate that even in the homogenization regime and even with a small number of data points, the minimum contrast estimators are consistent and asymptotically normal. We point out that although the empirical variance of \( \tilde{\theta}^\varepsilon \) was larger than that of \( \hat{\theta}^\varepsilon \), the difference was rather small, especially for small values of \( \epsilon \) - this is consistent with the fact that the theoretical limiting difference in this example was rather negligible. It is also worth point out that coefficients depending on \( \theta \) appear in both slow and fast components.
Notice that the dependence on the fast process in (23) is periodic and that one may therefore interpret the fast process as taking values in a torus. We now consider a second example to illustrate the case where the fast dynamics are not restricted to a compact space.

\[
\begin{align*}
\frac{dX^\varepsilon_t}{\varepsilon} &= \frac{\varepsilon}{\delta} \theta_0 Y^\varepsilon_t dt + \theta_0 X^\varepsilon_t (Y^\varepsilon_t)^2 dt + \sqrt{\varepsilon} dW_t \\
\frac{dY^\varepsilon_t}{\varepsilon} &= -\frac{1}{\delta^2} \theta_0 Y^\varepsilon_t dt + \frac{\sqrt{\varepsilon}}{\delta} dB_t
\end{align*}
\]

for \( t \in [0, T = 1] \) with \( X^\varepsilon_0 = Y^\varepsilon_0 = 1 \in \mathbb{R} \). Again, we suppose that we are in the \( \infty \) regime with \( \delta = \varepsilon^{3/2} \) (this is the more challenging regime, relative to the \( \gamma \) regime). As in the first example, coefficients depending on \( \theta \) appear in both slow and fast components.

The limit \( \bar{X} \) of the slow process \( X^\varepsilon \) in (24) is

\[
\bar{X}_t = e^{t \theta_0^2/2}
\]

We fix \( \theta_0 = 1 \) and simulate trajectories using an Euler scheme with \( 10^8 \) evenly-spaced discrete time steps. We simulate \( \bar{\theta}^\varepsilon \) 1000 times for each of \( \varepsilon = 10^{-2}, 10^{-3} \) with \( n = 10 \); we simulate \( \bar{\theta}^\varepsilon \) 1000 times for each combination of \( \varepsilon = 10^{-2}, 10^{-3} \) and \( n = 10, 10^2, 10^3 \). Tables 3 and 4 present in each case the mean estimate, a normal-based confidence interval using the empirical standard deviation, and the theoretical standard deviation as per Theorem 4. The histograms in Figures 5-8 compare the empirical distribution of the estimates with the theoretical density curve.

| \( \varepsilon \) | \( n \) | Mean Estimator | 68% Confidence Interval | 95% Confidence Interval | Theoretical SD |
|----------------|-----|----------------|------------------------|------------------------|----------------|
| 10^{-2} | 10 | 1.1042 | (0.9961, 1.2123) | (0.8879, 1.3205) | 0.1079 |
| 10^{-3} | 10 | 1.0129 | (0.9800, 1.0458) | (0.9472, 1.0787) | 0.0341 |

Table 3: Example 2 - MCE of \( \theta_0 = 1 \) with empirical confidence intervals and theoretical standard deviations.
| $\epsilon$ | $n$ | Mean Estimator | 68% Confidence Interval | 95% Confidence Interval | Theoretical SD |
|-----------|-----|----------------|------------------------|------------------------|--------------|
| $10^{-2}$ | 10  | 0.9840         | (0.8704, 1.0977)       | (0.7567, 1.2114)       | 0.1079       |
| $10^{-2}$ | $10^2$ | 0.9773       | (0.8561, 1.0985)       | (0.7349, 1.2197)       | 0.1079       |
| $10^{-2}$ | $10^3$ | 0.9754       | (0.8544, 1.0963)       | (0.7335, 1.2173)       | 0.1079       |
| $10^{-3}$ | 10  | 1.0008        | (0.9662, 1.0354)       | (0.9316, 1.0699)       | 0.0341       |
| $10^{-3}$ | $10^2$ | 0.9994       | (0.9654, 1.0333)       | (0.9314, 1.0673)       | 0.0341       |
| $10^{-3}$ | $10^3$ | 1.0003       | (0.9659, 1.0347)       | (0.9314, 1.0692)       | 0.0341       |

Table 4: Example 2 - SMCE of $\theta_0 = 1$ with empirical confidence intervals and theoretical standard deviations.

Figure 5: Example 2, MCE, $\epsilon = 10^{-3}$, $n = 10$
Figure 6: Example 2, SMCE, $\epsilon = 10^{-3}$, $n = 10$

Figure 7: Example 2, SMCE, $\epsilon = 10^{-3}$, $n = 10^2$
Figure 8: Example 2, SMCE, $\epsilon = 10^{-3}$, $n = 10^3$

Lastly, we revisit Examples 1 and 2 to see what happens when we deliberately violate the assumption that $\epsilon$ is of lower order than $\Delta := T/n$. We repeat some of the simulations above with $n = 10^6$; the results are presented in Tables 5 and 6 and Figures 9-12. One sees that the estimators continue to behave well, suggesting that it may be possible to relax our assumptions about the relationship between $\epsilon$ and $\Delta$ in the theory of Section 6 concerning estimation based on high-frequency observations.
Table 5: Example 1 revisited - MCE vs SMCE with $\epsilon = 10^{-2}$ and large $n$ ($n = 10^{6}$).

| Estimator | Mean Estimator | 68% Confidence Interval | 95% Confidence Interval | Theoretical SD |
|-----------|---------------|------------------------|------------------------|----------------|
| MCE       | 1.0427        | (0.5630, 1.5225)       | (0.0833, 2.0022)       | 0.437          |
| SMCE      | 1.1553        | (0.6839, 1.6268)       | (0.2125, 2.0982)       | 0.437          |

Remark 5. Data for the MCE $\bar{\theta}^\epsilon$ for the first example [24] are presented for $n = 10$ only. We have run simulations also with $n = 10^2, 10^3$, etc., but the estimators then began to exhibit a strong positive bias. In contrast, a glance at Table 4 reveals that the SMCE $\tilde{\theta}^\epsilon$ for the very same model produced robust estimates across all combinations of $\epsilon$ and $n$.

Let us attempt to offer some insight into the reason for this. It turns out that the typical simplified
contrast curve in this model increases more steeply as one moves rightward from the minimum \( \tilde{\theta} \) than as one moves leftward. Meanwhile, the effect of introducing the covariance weights \( Q_k^{-1} \) in this example is to rescale the contrast by a factor proportional to \( \frac{\theta^2}{(1+\theta^4)^{1/2}} \), which has the potential to balance the leftward and rightward gradients about the new minimum \( \bar{\theta} \), but at the same time may lead to numerical instabilities when \( \Delta \) is small. However, for a given value of \( \epsilon \), the convexity of the simplified contrast about its minimum tends to diminish, whence the introduction of the covariance weights causes the minimum to move too far to the right due to the form that it takes as \( \epsilon \) gets smaller.

We would also like to remark here that how small \( \epsilon \) should be for a given problem for the theorems to apply is problem dependent. The fact that SMCE appears to be more robust than MCE and at the same time easier to compute (one need not compute the weights \( Q_k^{-1} \)) can be viewed as a practical advantage despite the increase in the variance.

8 Conclusions

We have presented in this paper discrete-data estimators for unknown parameters in multiscale diffusion models with small noise. The estimators are defined as minimizers of certain contrast functions motivated by a general second-order stochastic Taylor expansion of the slow process; we have shown them to be consistent, asymptotically normal, and, in certain cases, asymptotically statistically efficient.

We also considered the case of high-frequency observation, in which the number of samples ‘n’ increases to infinity concurrently with the vanishing of the small noise. We showed that the asymptotic properties established for the fixed-n case are still valid provided that the sampling interval \( \Delta := T/n \) does not vanish too quickly relative to the small noise. Such conditions are reminiscent of the subsampling prescribed in prior literature on statistical estimation for multiscale models. Despite our best efforts we have not managed to relax them, as has already been done for the case of small noise without multiple scales (see [16, 31, 32]). Nevertheless, the numerical simulations of Section 7 suggest that improvement is possible, posing an important question for future research to investigate. The crux is the development of an optimal characterization of the rate at which the approximation error \( \sqrt{\epsilon} \left[ \tilde{R}_{\epsilon,\theta} \right. \left. Z^0(t_k, t_{k-1}) \bar{R}_{\epsilon,\theta} \right] \) vanishes with respect to \( \epsilon = (\epsilon, \delta) \) and \( \Delta \); the \( \delta \) dependence in particular presents an analytic challenge of unusual delicacy.

Appendix

We gather here technical results to which we appeal in the proofs of the main results of this paper.

A.1 Bounds Extended from [13]

The SDEs for \( X^\epsilon \) and \( Y^\epsilon \) that we consider in this paper extend those in [13]. Certain fundamental auxiliary bounds established in [13] are also valid in our case. We gather the necessary bounds in this section. Because the proofs are similar, we do not present them here; the interested reader is referred to [13] for details.

Lemma 4. Assume Conditions 2 and 3 and, in the \( \infty \) regime, Condition 3. For any \( p > 0 \) there is a constant \( K \) such that uniformly in \( \epsilon \) (and hence also \( \delta = \delta(\epsilon) \)) sufficiently small,

\[
E \sup_{0 \leq t \leq T} |X^\epsilon_t|^p \leq \tilde{K},
\]

\[
E \sup_{0 \leq t \leq T} |X^\epsilon_t - \tilde{X}_t|^p \leq \tilde{K}.
\]

This is similar to Lemma 6 of [13].

Sketch of Proof. The only significant difference relative to Lemma 6 of [13] is the introduction of the asymptotically singular coefficient \( \frac{\epsilon}{\delta} b \) in the \( \infty \) regime. Letting \( \chi \) be as in [3], applying the Itô formula to
where \( \bar{\chi}(x, y) \), and rearranging terms, we obtain
\[
\int_0^t \frac{\epsilon}{\delta} b(X^\varepsilon_s, Y^\varepsilon_s) ds = \int_0^t \nabla_y \chi \cdot g(X^\varepsilon_s, Y^\varepsilon_s) ds + \sqrt{\epsilon} \int_0^t (\nabla_y \chi \cdot \tau_1, \nabla_y \chi \cdot \tau_2)(X^\varepsilon_s, Y^\varepsilon_s)d(W, B)_s + \epsilon \int_0^t \nabla_x \chi \cdot b(X^\varepsilon_s, Y^\varepsilon_s) ds + \delta \int_0^t \nabla_x \chi \cdot c(X^\varepsilon_s, Y^\varepsilon_s) ds + \epsilon \delta \int_0^t \sigma \tau^T_1 : \nabla_x^2 \chi(X^\varepsilon_s, Y^\varepsilon_s) ds \\
+ \epsilon \int_0^t \sigma \tau^T_1 : \nabla_y \nabla_x \chi(X^\varepsilon_s, Y^\varepsilon_s) ds + \sqrt{\epsilon \delta} \int_0^t \nabla_x \chi \cdot \sigma(X^\varepsilon_s, Y^\varepsilon_s) dW_s - \delta (\chi(X^\varepsilon_t, Y^\varepsilon_t) - \chi(x_0, y_0)) \\
=: \int_0^t \nabla_y \chi \cdot g(X^\varepsilon_s, Y^\varepsilon_s) ds + R^\varepsilon_t; \\
\]
hence we can write
\[
X^\varepsilon_t = x_0 + \int_0^t \lambda_\infty(X^\varepsilon_s, Y^\varepsilon_s) ds + \sqrt{\epsilon} \int_0^t \sigma(X^\varepsilon_s, Y^\varepsilon_s) dW_s + R^\varepsilon_t.
\]

The function \( \chi \) and its various derivatives are bounded uniformly in the first variable by polynomials in the second variable (see Chapters 2 and 3 of [15]). Noting the vanishing prefactors, we conclude that \( R^\varepsilon_t \) vanishes. Thus, the proof may proceed as that of Lemma 6 of [13] with \( \lambda_\infty \) taking the place of \( c \); we omit the details.

Lemma 5. Assume Conditions 2 and 3 and, in the \( \infty \) regime, Condition 3. Let \( h \) be a function of \( x \) and \( y \) satisfying \( |h(x, y)| \leq K(1 + |x|^r)(1 + |y|^q) \) for some fixed positive constants \( K, q, r \) and let \( V \) be either of the Wiener processes \( W, B \). For any \( 0 \leq p \leq \infty \), there is a constant \( \tilde{K} \) such that for epsilon (and hence also \( \delta = \delta(\epsilon) \)) sufficiently small,
\[
E \int_0^T |h(X^\varepsilon_t, Y^\varepsilon_s)|^p dt \leq \tilde{K},
\]
\[
E \sup_{0 \leq t \leq T} \left| \int_0^t h(X^\varepsilon_s, Y^\varepsilon_s) dV_s \right|^p \leq \tilde{K}.
\]

Given Lemma 4, the proof of Lemma 5 is similar to that of Lemma 7 of [13]; we omit the details.

Lemma 6. Assume Conditions 2 and 3 and, in the \( \infty \) regime, Condition 3. Let \( h(x, y) \) be a function such that \( |h(x, y)| \leq K(1 + |x|^r)(1 + |y|^q) \) for some fixed positive constants \( K, q, r \), and which, along with each of its derivatives up to second order, is H"older continuous in \( y \) uniformly in \( x \) with absolute value growing at most polynomially in \( |y| \) as \( y \to \infty \). For any \( 0 \leq p < \infty \), there is a constant \( \tilde{K} \) such that for \( \epsilon \) (and hence also \( \delta = \delta(\epsilon) \)) sufficiently small,
\[
E \sup_{0 \leq t \leq T} \left| \int_0^t \left( h(X^\varepsilon_s, Y^\varepsilon_s) - \bar{h}(X^\varepsilon_s) \right) ds \right|^p \leq \tilde{K}(\sqrt{\epsilon} + \sqrt{\delta})^p,
\]
where \( \bar{h}(x) \) is the averaged function \( \int_Y h(x, y) \mu_x(dy) \).

Given Lemma 4, the proof of Lemma 5 is similar to that of Theorem 4 of [13]; we omit the details.

Lemma 7. Assume Conditions 2 and 3 and, in the \( \infty \) regime, Condition 3. Let \( h(x, y) \) be a function such that \( |h(x, y)| \leq K(1 + |x|^r)(1 + |y|^q) \) for some fixed positive constants \( K, q, r \), and which, along with each of its derivatives up to second order, is H"older continuous in \( y \) uniformly in \( x \) with absolute value growing at most polynomially in \( |y| \) as \( y \to \infty \). For any \( 0 \leq p < \infty \), there is a constant \( \tilde{K} \) such that for \( \epsilon \) (and hence also \( \delta = \delta(\epsilon) \)) sufficiently small,
\[
E \sup_{0 \leq t \leq T} \left| \int_0^t \left( h(X^\varepsilon_s, Y^\varepsilon_s) - \bar{h}(X^\varepsilon_s) \right) ds \right|^p \leq \tilde{K}(\sqrt{\epsilon} + \sqrt{\delta})^p,
\]
where \( \bar{h}(x) \) is the averaged function \( \int_Y h(x, y) \mu_x(dy) \).

Given Theorem 4 and Lemma 6, the proof of Lemma 7 is similar to that of Lemma 10 of [13]; we omit the details.
A.2 Lemmata on F and Q

**Lemma 8.** Assume Conditions 1 and 2 and, in the $\infty$ regime, Condition 3 There is a constant $K$ such that for $\epsilon$ (and hence also $\delta = \delta(\epsilon)$) sufficiently small,

$$E|F_k^\epsilon(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n)|^p \leq \tilde{K} \cdot \epsilon^{p/2},$$  \hspace{1cm} (26)

$$E|\tilde{F}_k^\epsilon(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n)|^p \leq \tilde{K} \cdot \epsilon^{p/2}.$$  \hspace{1cm} (27)

*Proof.* Recalling that

$$F_k^\epsilon(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n) = \left[|X_{t_k}^\epsilon - \bar{X}_{t_k}^\epsilon| - Z^\theta(t_k, t_{k-1})|X_{t_{k-1}}^\epsilon - \bar{X}_{t_{k-1}}^\epsilon|\right] - \sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^\theta(t, s) \cdot J^\theta(\bar{X}_s^\theta)ds,$$  \hspace{1cm} (28)

The first statement, (26), follows by the triangle inequality and Theorem 1.

In addition to Lemma 8, we need a lemma to establish that $F_k^\epsilon(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n)$ and $E|\tilde{F}_k^\epsilon(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n)|$ are uniformly bounded in $\epsilon$ sufficiently small even as $\theta_1$ and $\theta_2$ vary independently. Of course, with two distinct values of $\theta$, one should not expect the functions to vanish in the limit.

**Lemma 9.** Assume Conditions 1 and 2 and, in the $\infty$ regime, Condition 3 There is a constant $\tilde{K}$ such that for $\epsilon$ (and hence also $\delta = \delta(\epsilon)$) sufficiently small,

$$\sup_{(\theta_1, \theta_2) \in \Theta^2} E|\tilde{F}_k^\epsilon(\theta_2; \{X_{t_k}^{\epsilon, \theta_1}\}_{k=1}^n)| \leq \tilde{K},$$  \hspace{1cm} (29)

$$\sup_{(\theta_1, \theta_2) \in \Theta^2} E|\tilde{F}_k^\epsilon(\theta_2; \{X_{t_k}^{\epsilon, \theta_1}\}_{k=1}^n)| \leq \tilde{K}.$$  \hspace{1cm} (30)

*Proof.* Notice that

$$\sup_{(\theta_1, \theta_2) \in \Theta^2} |F_k^\epsilon(\theta_2; \{X_{t_k}^{\epsilon, \theta_1}\}_{k=1}^n)| \leq I + II + III + IV + V,$$  \hspace{1cm} (31)

where

$$I := \sup_{\theta \in \Theta} |X_{t_k}^{\epsilon, \theta} - X_{t_{k-1}}^{\epsilon, \theta}|;$$  \hspace{1cm} (32)

$$II := \sup_{\theta \in \Theta} \left|\bar{X}_{t_k}^{\theta} - \bar{X}_{t_{k-1}}^{\theta}\right| \leq \sup_{\theta \in \Theta, 0 \leq t \leq T} \left|[(\nabla_x \bar{X}^\theta)(\bar{X}_t^\theta)]\right| \cdot \Delta;$$

$$III := \sup_{\theta \in \Theta} \left|1_m - Z^\theta(t_k, t_{k-1})\right| \cdot \sup_{\theta \in \Theta} |X_{t_{k-1}}^{\epsilon, \theta}|$$

$$\leq \left[\sup_{\theta \in \Theta, 0 \leq t \leq T} \left|Z^\theta(t, 0) \cdot (\nabla_x \bar{X}^\theta)(\bar{X}_t^\theta)\right|\right] \cdot \left[\sup_{\theta \in \Theta, 0 \leq t \leq T} |X_{t_{k-1}}^{\epsilon, \theta}|\right] \cdot \Delta;$$

$$IV := \sup_{\theta \in \Theta} \left|1_m - Z^\theta(t_k, t_{k-1})\right| \cdot \sup_{\theta \in \Theta} |\bar{X}_{t_{k-1}}^{\theta}|$$

$$\leq \left[\sup_{\theta \in \Theta, 0 \leq t \leq T} \left|Z^\theta(t, 0) \cdot (\nabla_x \bar{X}^\theta)(\bar{X}_t^\theta)\right|\right] \cdot \left[\sup_{\theta \in \Theta, 0 \leq t \leq T} |\bar{X}_{t_{k-1}}^{\theta}|\right] \cdot \Delta;$$

$$V := \sup_{\theta \in \Theta} \sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^\theta(t, s) \cdot J^\theta(\bar{X}_s^\theta)ds \leq \sqrt{\epsilon} \cdot \left[\sup_{\theta \in \Theta, 0 \leq t \leq T} |Z^\theta(t, 0) \cdot J^\theta(\bar{X}_t^\theta)|\right] \cdot \Delta.$$
By Lemma 4, there is a constant $K$ such that for $\epsilon$ sufficiently small, $EI \leq K$. Similarly, there is a (perhaps larger) constant $\tilde{K}$ such that for $\epsilon$ sufficiently small, $II + EIII + IV + V \leq K\Delta$. The first statement, (29), follows easily.

Lemma 10. Assume Conditions 1 and 2 and, in the $\infty$ regime, Condition 3. There is a constant $\tilde{K}$ such that for all $n > 0$, $k = 1, 2, \ldots, n$, and $\epsilon$ (and hence also $\delta = \delta(\epsilon)$) sufficiently small,

\[
\sup_{(\theta_1, \theta_2) \in \Theta^2} |F_k^\epsilon(\theta_2; \{\bar{X}_{t_k}^\theta\}_{k=1}^n)| \leq \Delta \cdot \tilde{K}, \tag{33}
\]

\[
\sup_{(\theta_1, \theta_2) \in \Theta^2} |\tilde{F}_k^\epsilon(\theta_2; \{\bar{X}_{t_k}^\theta\}_{k=1}^n)| \leq \Delta \cdot \tilde{K}. \tag{34}
\]

Proof. We write

\[
\sup_{(\theta_1, \theta_2) \in \Theta^2} |F_k^\epsilon(\theta_2; \{\bar{X}_{t_k}^\theta\}_{k=1}^n)| \leq 2 \cdot I + 2 \cdot II + III, \tag{35}
\]

where

\[
I := \sup_{\theta \in \Theta} |\bar{X}_{t_k}^\theta - \bar{X}_{t_{k-1}}^\theta| \leq \left[ \sup_{\theta \in \Theta, 0 \leq t \leq T} |(\nabla_x \bar{X}^\theta)(\bar{X}_t^\theta)| \right] \cdot \Delta;
\]

\[
II := \sup_{\theta \in \Theta} |1_m - Z^\theta(t_k, t_{k-1})| \cdot \sup_{\theta \in \Theta} |\bar{X}_{t_{k-1}}^\theta| \leq \left[ \sup_{\theta \in \Theta, 0 \leq t \leq T} |Z^\theta(t, 0) \cdot (\nabla_x \bar{X}^\theta)(\bar{X}_t^\theta)| \right] \cdot \left[ \sup_{\theta \in \Theta, 0 \leq t \leq T} |\bar{X}_t^\theta| \right] \cdot \Delta;
\]

\[
III := \sup_{\theta \in \Theta} \sqrt{t} \int_{t_{k-1}}^{t_k} \left| Z^\theta(t, s) \cdot \tilde{J}^\theta(\bar{X}_s^\theta) \right| ds \leq \sqrt{t} \cdot \left[ \sup_{\theta \in \Theta, 0 \leq t \leq T} |Z^\theta(t, 0) \cdot \tilde{J}^\theta(\bar{X}_t^\theta)| \right] \cdot \Delta.
\]

The first statement, (33), follows easily. (34) may be obtained in the same way by omitting the last term in (31).

Lemma 11. Assume Conditions 1 and 2 and, in the $\infty$ regime, Condition 3. There is a positive constant $\tilde{K}$ such that for all $n > 0$ and $k = 1, 2, \ldots, n$,

\[
\inf_{\theta \in \Theta} ||Q_k(\theta)|| \geq \Delta \cdot \tilde{K};
\]

in particular, the inverse matrices $Q_k^{-1}$ exist and $\sup_{\theta \in \Theta} ||Q_k^{-1}(\theta)|| \leq \Delta^{-1} \cdot \tilde{K}^{-1}$.

Proof. Recalling our nondegeneracy assumptions, $q^\theta$ is uniformly nondegenerate by the argument of Theorem 11.3 of [27]. The result follows readily upon noting that the norms of the exponential terms in the definition of $Q$ are bounded uniformly in all parameters away from zero.

A.3 Lemmata on the Contrast Function

Lemma 12. (Limit of the Contrast Function) Assume Conditions 1 and 2 and, in the $\infty$ regime, Condition 3. Fix $n > 0$. For any $\eta > 0$,

\[
\lim_{\epsilon \to 0} P \left( \sup_{\theta_1, \theta_2 \in \Theta} \left| U^\epsilon(\theta_2; \{X_{t_k}^{r, \theta_1}\}_{k=1}^n) - \bar{U}(\theta_2; \{\bar{X}_{t_k}^{\theta_1}\}_{k=1}^n) \right| > \eta \right) = 0.
\]
Proof. We begin by writing
\[
U^\varepsilon(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) - \bar{U}(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n) = \left[U^\varepsilon(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) - U^\varepsilon(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right]
+ \left[U^\varepsilon(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n) - \bar{U}(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right]
= \left[\sum_{k=1}^n A^\varepsilon_k(\theta_1, \theta_2)\right] + \left[\sum_{k=1}^n B^\varepsilon_k(\theta_1, \theta_2)\right],
\]
where
\[
A^\varepsilon_k(\theta_1, \theta_2) := \left[F^\varepsilon_k(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) - F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right]^T Q^\varepsilon_k^{-1}(\theta_2) \left[F^\varepsilon_k(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) + F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right],
\]
\[
B^\varepsilon_k(\theta_1, \theta_2) := \left[F^\varepsilon_k(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) - F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right]^T Q^\varepsilon_k^{-1}(\theta_2) \left[F^\varepsilon_k(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) + F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right].
\]
As we have taken \(n\) to be fixed, it suffices to show that each term vanishes in probability uniformly in \(\theta_1, \theta_2\) as \(\epsilon \to 0\).

For \(A^\varepsilon_k\), we have that
\[
Q^\varepsilon_k^{-1}(\theta_2) \left[F^\varepsilon_k(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) + F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right] (36)
\]
is bounded by Lemmata 9, 10 and 11 while
\[
F^\varepsilon_k(\theta_2; \{X^\varepsilon_{t_k}\}_{k=1}^n) - F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n) = [X^\varepsilon_{t_k} - \bar{X}^\varepsilon_{t_k}] - [Z^{\theta_2}(t_k, t_{k-1}) \cdot [X^\varepsilon_{t_{k-1}} - \bar{X}^\varepsilon_{t_{k-1}}]] (37)
\]
vansishes by Theorem 1 and the fact that the Frobenius norm of \(Z^{\theta_2}(t_k, t_{k-1})\) is bounded.

For \(B^\varepsilon_k\), we have that
\[
Q^\varepsilon_k^{-1}(\theta_2) \left[F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n) + \bar{F}_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n)\right] (38)
\]
is bounded by Lemmata 10 and 11 while
\[
F^\varepsilon_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n) - \bar{F}_k(\theta_2; \{\bar{X}^\varepsilon_{t_k}\}_{k=1}^n) = -\sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^{\theta_2}(t_k, s) \cdot \tilde{J}^{\theta_2}(\bar{X}^\varepsilon_{t_k}) ds (39)
\]
vansishes by the vanishing prefactor \(-\sqrt{\epsilon}\) and the fact that the Euclidean norm of \(\int_{t_{k-1}}^{t_k} Z^{\theta_2}(t_k, s) \cdot \tilde{J}^{\theta_2}(\bar{X}^\varepsilon_{t_k}) ds\) is bounded (indeed, proportionally to \(\Delta\)).

\[\square\]

**Lemma 13.** Assume Conditions 1 and 2 and, in the \(\infty\) regime, Condition 3. There is a constant \(\tilde{K}\) such that for any \(\theta \in \Theta\) and \(n > 0\),
\[
\sup_{\theta \in \Theta} \bar{w}^\theta(\phi) \leq \tilde{K} \cdot \phi,
\]
where
\[
\bar{w}^\theta(\phi) := \sup_{\theta_1, \theta_2 \in \Theta, \|\theta_1 - \theta_2\|_2 \leq \phi} \left|\bar{U}(\theta_1; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) - \bar{U}(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n)\right|.
\]

**Proof.** We write
\[
\bar{U}(\theta_1; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) - \bar{U}(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) = \left[\sum_{k=1}^n C_k(\theta_1, \theta_2)\right] + \left[\sum_{k=1}^n D_k(\theta_1, \theta_2)\right],
\]
where
\[
C_k(\theta_1, \theta_2) := \left[\tilde{F}_k(\theta_1; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) - \tilde{F}_k(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n)\right]^T Q_k^{-1}(\theta_1) \left[\tilde{F}_k(\theta_1; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) + \tilde{F}_k(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n)\right],
\]
\[
D_k(\theta_1, \theta_2) := \tilde{F}_k^T(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) \cdot \left[Q_k^{-1}(\theta_1) - Q_k^{-1}(\theta_2)\right] \cdot \tilde{F}_k(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n)
= \tilde{F}_k^T(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n) \cdot Q_k^{-1}(\theta_1) \cdot \left[Q_k(\theta_2) - Q_k(\theta_1)\right] \cdot Q_k^{-1}(\theta_2) \cdot \tilde{F}_k(\theta_2; \{\bar{X}^\theta_{t_k}\}_{k=1}^n).
\]

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It suffices to show that there is a constant $K$ such that the absolute value of each term is bounded by $K \cdot \Delta \cdot |\theta_1 - \theta_2|$. 

For $C_k$, the terms $Q_k^{-1}(\theta_1) \left[ \hat{F}_k(\theta_1; \{X^\theta_{t_k}\}_{k=1}^n) + \hat{F}_k(\theta_2; \{X^\theta_{t_k}\}_{k=1}^n) \right]$ are bounded by Lemmata 10 and 11, so it will suffice to bound the terms $|\hat{F}_k(\theta_1; \{X^\theta_{t_k}\}_{k=1}^n) - \hat{F}_k(\theta_2; \{X^\theta_{t_k}\}_{k=1}^n)|$, individually, by a certain constant times $\Delta \cdot |\theta_1 - \theta_2|$. 

Notice that 

$$\hat{F}_k(\theta_1; \{X^\theta_{t_k}\}_{k=1}^n) - \hat{F}_k(\theta_2; \{X^\theta_{t_k}\}_{k=1}^n) = I + II + III,$$

where 

$$I := [X^\theta_{t_k} - X^\theta_{t_{k-1}}] - [X^\theta_{t_{k-1}} - X^\theta_{t_k}],$$

$$II := [Z^{\theta_1}(t_k, t_{k-1}) - 1_m] \cdot [X^\theta_{t_{k-1}} - X^\theta_{t_k}],$$

$$III := [Z^{\theta_2}(t_k, t_{k-1}) - Z^{\theta_1}(t_k, t_{k-1})] \cdot [X^\theta_{t_{k-1}} - X^\theta_{t_k}].$$

By the triangle inequality, 

$$|\hat{F}_k(\theta_1; \{X^\theta_{t_k}\}_{k=1}^n) - \hat{F}_k(\theta_2; \{X^\theta_{t_k}\}_{k=1}^n)| \leq |I| + |II| + |III|.$$ 

Firstly, 

$$|I| \leq \int_{t_{k-1}}^{t_k} |\tilde{\lambda}^\theta(X^\theta_{s}) - \tilde{\lambda}^\theta(X^\theta_{s})|ds \leq \left( \sup_{\tilde{\theta} \in \theta, 0 \leq t \leq T} |\nabla_{\tilde{\theta}}(\tilde{\lambda}(X^\theta_{s}))| \right) \cdot \Delta \cdot |\theta_1 - \theta_2|;$$

secondly, 

$$|II| \leq \left( \sup_{\tilde{\theta} \in \theta, 0 \leq t \leq T} |(\nabla_{\tilde{\theta}X}(\tilde{\lambda})(X^\theta_{s}))| \right) \cdot \Delta \cdot \int_0^{t_k} |\tilde{\lambda}^\theta(X^\theta_{s}) - \tilde{\lambda}^\theta(X^\theta_{s})|ds \leq \left( \sup_{\tilde{\theta} \in \theta, 0 \leq t \leq T} |(\nabla_{\tilde{\theta}X}(\tilde{\lambda})(X^\theta_{s}))| \right) \cdot T \cdot \Delta \cdot |\theta_1 - \theta_2|;$$

thirdly, 

$$|III| \leq \left| Z^{\theta_1}(t_k, t_{k-1}) \right| \cdot \left| \int_{t_{k-1}}^{t_k} (\nabla_{\tilde{\theta}X} \tilde{\lambda}^\theta)(X^\theta_{s}) - (\nabla_{\tilde{\theta}X} \tilde{\lambda}^\theta)(X^\theta_{s}) ds \right| \cdot |X^\theta_{t_{k-1}} - X^\theta_{t_k}| \leq \left( \sup_{\tilde{\theta} \in \theta, 0 \leq t \leq T} |\tilde{\theta}(t, 0)| \right) \left( \sup_{\tilde{\theta} \in \theta, 0 \leq t \leq T} |\nabla_{\tilde{\theta}} (\nabla_{\tilde{\theta}X} \tilde{\lambda}^\theta)(X^\theta_{s})|\right) \cdot \left( \sup_{\tilde{\theta} \in \theta, 0 \leq t \leq T} 2|\tilde{X}^\theta_{s}| \right) \cdot \Delta \cdot |\theta_1 - \theta_2|.$$ 

hence indeed there is a constant $K$ such that 

$$|\hat{F}_k(\theta_1; \{X^\theta_{t_k}\}_{k=1}^n) - \hat{F}_k(\theta_2; \{X^\theta_{t_k}\}_{k=1}^n)| \leq K \cdot \Delta \cdot |\theta_1 - \theta_2|.$$ 

For $D_k$, the terms $Q_k^{-1}(\theta_1) \cdot \hat{F}_k(\theta_2; \{X^\theta_{t_k}\}_{k=1}^n)$ are bounded by Lemmata 10 and 11, so using the fact that the Frobenius norm is equivalent to the operator norm, it will suffice to bound the terms $|Q_k(\theta_2) - Q_k(\theta_1)|$, individually, by a certain constant times $\Delta \cdot |\theta_1 - \theta_2|$. 

We have 

$$Q_k(\theta_2) - Q_k(\theta_1) = IV + V,$$
where

\[ IV := \int_{t_{k-1}}^{t_k} \left[ Z^\theta_2(t_k, s) \cdot \left( q^\theta_2(\tilde{X}^\theta_2) - q^\theta_1(\tilde{X}^\theta_1) \right) \cdot e^{\int_{s}^{t_k} (\nabla, \tilde{X}^\theta_2)^T (\tilde{X}^\theta_2) du} ds \right] \]

\[ V := \int_{t_{k-1}}^{t_k} \left( Z^\theta_2(t_k, s) - Z^\theta_1(t_k, s) \right) \cdot q^\theta_1(\tilde{X}^\theta_1) \cdot \left( Z^\theta_2(t_k, s) + Z^\theta_1(t_k, s) \right) ; \]

by the triangle inequality,

\[ |Q_k(\theta_2) - Q_k(\theta_1)| \leq |IV| + |V|. \]

Firstly,

\[ |IV| \leq \left[ \sup_{\tilde{\theta} \in \Theta, 0 \leq t \leq T} \left| Z^{\tilde{\theta}}(t, 0)(Z^{\tilde{\theta}})'(t, 0) \right| \right] \cdot \left[ \sup_{\tilde{\theta} \in \Theta, 0 \leq t \leq T} \| \nabla_{\tilde{\theta}} \left( q^\theta(\tilde{X}^\theta_t) \right) \| \right] \cdot \Delta \cdot |\theta_1 - \theta_2| ; \]

secondly,

\[ |V| \leq \left[ \sup_{\tilde{\theta} \in \Theta, 0 \leq t \leq T} \left| Z^{\tilde{\theta}}(t, 0) \right| \right] \cdot \left[ \sup_{\tilde{\theta} \in \Theta, 0 \leq t \leq T} \| \nabla_{\tilde{\theta}} \left( (\nabla, \tilde{X}^\theta_t) (\tilde{X}^\theta_t) \right) \| \right] \cdot \left[ \sup_{\tilde{\theta} \in \Theta, 0 \leq t \leq T} \| q^\theta(\tilde{X}^\theta_t) \| \right] \cdot \Delta \cdot |\theta_1 - \theta_2| , \]

where the difference of exponentials has been handled exactly as in the bound above for |III|; hence indeed there is a (perhaps larger) constant \( K \) such that

\[ |Q_k(\theta_2) - Q_k(\theta_1)| \leq K \cdot \Delta \cdot |\theta_1 - \theta_2| , \]

completing the proof of the lemma.

\[ \square \]

### A.4 Lemmata for Frequent Sampling

The contrast functions upon which our estimators are founded are defined as sums over a number of indices equal to the number of sampled data \( n \). When \( n \) is fixed, termwise convergence as \( \epsilon \to 0 \) is enough to deduce convergence of the sums. When, however, \( n \) is permitted to increase simultaneously as \( \epsilon \to 0 \), one must examine the convergence of sums over increasingly many indices. Convergence is easily deduced if statements of termwise convergence can be strengthened so that they uniformly outpace the increasing number of indices. The strengthened bounds that allow us to take this approach are established in this section.

**Lemma 14.** Assume Conditions 4 and 5 and, in the \( \infty \) regime, Condition 3. Assume that the sampling interval \( \Delta := T/n \) does not decrease too quickly relative to \( \epsilon \); that is, suppose that there is a finite constant \( G \) such that always \( \epsilon \cdot n \leq G \). There is a constant \( \tilde{K} \) such that for \( \epsilon \) (and hence also \( \delta = \delta(\epsilon) \)) sufficiently small, independently of \( (n, k) \),

\[ E[F_k^\infty(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n)]^2 \leq \tilde{K} \cdot \epsilon \cdot \Delta , \]

\[ E[\tilde{F}_k(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n)]^2 \leq \tilde{K} \cdot \epsilon \cdot \Delta . \]

**Proof.** Recall that

\[ F_k^\infty(\theta; \{X_{t_k}^\epsilon\}_{k=1}^n) = \left[ X_{t_k}^\epsilon - \tilde{X}_{t_k}^\epsilon \right] - Z^{\theta}(t_k, t_{k-1})[X_{t_{k-1}}^\epsilon - \tilde{X}_{t_{k-1}}^\epsilon] \]

\[ - \sqrt{\epsilon} \int_{t_{k-1}}^{t_k} Z^{\theta}(t, s) \cdot J^\theta(\tilde{X}^\theta_s) ds \]

\[ = I + II + III , \]

where

\[ I := \int_{t_{k-1}}^{t_k} \left[ X_{t_k}^\epsilon(\tilde{X}^\theta_2) - \tilde{X}_{t_k}^\epsilon(\tilde{X}^\theta_2) \right] \cdot e^{\int_{s}^{t_k} (\nabla, \tilde{X}^\theta_2)^T (\tilde{X}^\theta_2) du} ds \]

\[ II := \int_{t_{k-1}}^{t_k} \left( X_{t_k}^\epsilon(\tilde{X}^\theta_2) - \tilde{X}_{t_k}^\epsilon(\tilde{X}^\theta_2) \right) \cdot \left( X_{t_k}^\epsilon(\tilde{X}^\theta_1) - \tilde{X}_{t_k}^\epsilon(\tilde{X}^\theta_1) \right) ds \]

\[ III := \int_{t_{k-1}}^{t_k} \left( X_{t_k}^\epsilon(\tilde{X}^\theta_2) - \tilde{X}_{t_k}^\epsilon(\tilde{X}^\theta_2) \right) \cdot \left( X_{t_k}^\epsilon(\tilde{X}^\theta_1) + \tilde{X}_{t_k}^\epsilon(\tilde{X}^\theta_1) \right) ds \]

\]; that is, suppose that there is a finite constant \( K \) such that

\[ |Q_k(\theta_2) - Q_k(\theta_1)| \leq K \cdot \Delta \cdot |\theta_1 - \theta_2| , \]

completing the proof of the lemma.

\[ \square \]
where

\[ I := \int_{t_{k-1}}^{t_k} \left[ \frac{\epsilon}{\delta} b(X^{\epsilon,\theta}_s, Y^{\epsilon,\theta}_s) + c(X^{\epsilon,\theta}_s, Y^{\epsilon,\theta}_s) - \lambda(X^\theta_s) \right] ds, \]

\[ II := -\sqrt{\epsilon} \int_{t_{k-1}}^{t_k} \sigma(X^{\epsilon,\theta}_s, Y^{\epsilon,\theta}_s)dW_s, \]

\[ III := (1 - Z^\theta(t_k, t_{k-1})) \cdot [X^{\epsilon,\theta}_{t_{k-1}} - \bar{X}_t^\theta]; \]

it suffices to consider each of these separately.

\( I \) is handled by arguments similar to those presented in the proofs of Theorem 2 and Lemma 4. It follows by the calculations [5, 25] and, in the \( \infty \) regime, [28], that there is a constant \( K \), which may be chosen independently of \((n, k)\), such that

\[ E|I|^2 \leq \delta^2 \cdot K \leq \epsilon \cdot \Delta \cdot K, \]

where we have used the fact that \( \delta = O(\epsilon) \) and the assumption that \( \epsilon = O(\Delta) \).

Meanwhile, it is clear that there is a (perhaps larger) constant \( K \) such that

\[ E|II|^2 \leq \epsilon \cdot \Delta \cdot K. \]

Finally, in light of Theorem 1 and the fact that \( ||1_m - Z^\theta_2(t_k, t_{k-1})|| \) is bounded by a finite constant times \( \Delta \) as \( \Delta \to 0 \), it is clear that for a (perhaps larger) constant \( K \),

\[ E|III|^2 \leq \epsilon \cdot \Delta^2 \cdot K \leq \epsilon \cdot \Delta \cdot T \cdot K. \]

The gives the first statement, [40], in the \( \infty \) regime. The proof for the \( \gamma \) regime is similar and easier. [41] may be obtained in the same way by omitting the last term in [42].

**Lemma 15.** Assume Conditions 2 and 3 and, in the \( \infty \) regime, Condition 5. Assume that the sampling interval \( \Delta := T/n \) does not decrease too quickly relative to \( \epsilon \); that is, suppose that there is a finite constant \( G \) such that always \( \epsilon \cdot n \leq G \). There is a constant \( \tilde{K} \) such that for \( \epsilon \) (and hence also \( \delta = \delta(\epsilon) \)) sufficiently small, independently of \((n, k)\),

\[ \sup_{(\theta_1, \theta_2) \in \mathcal{E}} E|F_k(\theta_2; \{X^{\epsilon,\theta_1}_{t_k}\}_{k=1}^n)| \leq \Delta \cdot \tilde{K}, \]

\[ \sup_{(\theta_1, \theta_2) \in \mathcal{E}} E|\tilde{F}_k(\theta_2; \{X^{\epsilon,\theta_1}_{t_k}\}_{k=1}^n)| \leq \Delta \cdot \tilde{K}. \]

**Proof.** Let us examine the proof of Lemma 4 with particular attention to (32). It is not difficult to see that a constant \( K \) may be found such that for \( \epsilon \) sufficiently small,

\[ EI \leq K(\sqrt{\epsilon} + \sqrt{\Delta}) \sqrt{\Delta} \leq K(\sqrt{\epsilon \cdot n} + 1) \Delta. \]

Thus one sees that with the new assumption that \( \epsilon \cdot n \leq G \), \( EI \) is bounded for \( \epsilon \) sufficiently small, independently of \( n \), by a finite constant times \( \Delta \). The same having been noted for \( II + EIII + IV + V \), the desired bounds follow. 

**Lemma 16.** (Limit of the Contrast Function as \( \epsilon + \Delta \to 0 \)) Assume Conditions 2 and 3 and, in the \( \infty \) regime, Condition 5. Assume that the sampling interval \( \Delta := T/n \) does not decrease too quickly relative to \( \epsilon \), so that \( \epsilon \) is \( o(\Delta) \) as \( \Delta \to 0 \). For any \( \eta > 0 \),

\[ \lim_{(\epsilon + \Delta) \to 0} \mathbb{P} \left( \sup_{\theta_1, \theta_2 \in \mathcal{E}} |U^\epsilon(\theta_2; \{X^{\epsilon,\theta_1}_{t_k}\}_{k=1}^n) - \tilde{U}(\theta_2; \{\bar{X}^{\theta_1}_{t_k}\}_{k=1}^n)| > \eta \right) = 0. \]
Proof. Let us examine the proof of Lemma 12. It will suffice to show that the terms $A_k^ε, B_k^ε$ vanish faster than $\Delta$ in probability uniformly in $θ_1, θ_2$ as $ε + Δ \to 0$. Recall that Lemmata 10 and 11 apply uniformly in $n$. Thus, (38) is bounded uniformly in $n$ and, with the new assumption that $ε \cdot n \leq G$, one may replace Lemma 9 with Lemma 15 to establish the same for (36). Meanwhile, it is clear that (39) vanishes faster than $Δ$. Thus, if we establish that (37) also vanishes faster than $Δ$, we will have proven the lemma.

Splitting the right-hand side of (37), we obtain

$$F_ε^k(θ_2; \{X_{t_k}^{ε,θ_1}\}_{k=1}^n) - F_ε^k(θ_2; \{\bar{X}_{t_k}^{θ_1}\}_{k=1}^n) = I + II + III,$$

where

$$I := \int_{t_{k-1}}^{t_k} \left[ \frac{ε}{δ} b(X_s^{ε,θ_1}, Y_s^{ε,θ_1}) + c(X_s^{ε,θ_1}, Y_s^{ε,θ_1}) - \bar{λ}^{θ_1}(\bar{X}_s^{θ_1}) \right] ds,$$

$$II := \sqrt{ε} \int_{t_{k-1}}^{t_k} σ(X_s^{ε,θ_1}, Y_s^{ε,θ_1}) dW_s,$$

$$III := (1_m - Z^{θ_2}(t_k, t_{k-1})) \cdot [X_{t_{k-1}}^{ε,θ_1} - \bar{X}_{t_{k-1}}^{θ_1}].$$

$I$ is handled by an argument similar to the proof of Lemma 4 - letting $χ$ be as in (3), one applies the Itô formula to $χ(X_t^ε, Y_t^ε)$ and rearranges the terms to see that there is a constant $K$, which may be chosen independently of $(n, k)$, such that

$$E|I| \leq δ \cdot K \leq ε \cdot K,$$

where we have used the fact that $δ$ is $O(ε)$. Meanwhile, it is clear that there is likewise a (perhaps larger) constant $K$ such that

$$E|II| \leq \sqrt{ε} Δ \cdot K.$$

Finally, in light of Theorem 1 and the fact that $(1_m - Z^{θ_2}(t_k, t_{k-1}))$ is bounded by a finite constant times $Δ$, it is clear that for a (perhaps larger) constant $K$,

$$E|III| \leq \sqrt{ε} Δ \cdot K.$$

Hence,

$$E|F_ε^k(θ_2; \{X_{t_k}^{ε,θ_1}\}_{k=1}^n) - F_ε^k(θ_2; \{\bar{X}_{t_k}^{θ_1}\}_{k=1}^n)| \leq (ε + \sqrt{ε} Δ) \cdot 2^k \cdot K,$$

which is enough since we have assumed $ε$ is $o(Δ)$.

\[\square\]

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