ZARISKI-DENSE SURFACE GROUPS IN NON-UNIFORM LATTICES OF SPLIT REAL LIE GROUPS

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Abstract. For $\text{SL}(n, \mathbb{R})$ ($n \geq 3$), $\text{SO}(n+1, n)$ ($n \geq 2$), $\text{Sp}(2n, \mathbb{R})$ ($n \geq 2$) and for the adjoint real split form of the exceptional group $G_2$, we exhibit non-uniform lattices in which we construct thin Hitchin representations by arithmetic methods. These representations give infinitely many orbits under the action of the mapping class group (except maybe for $G_2$). In particular, we show that when $p \neq 2$ is prime every non-uniform lattice of $\text{SL}(p, \mathbb{R})$ contains thin Hitchin representations.

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Introduction

Let $G$ be a semi-simple algebraic Lie group. A lattice in $G$ is a discrete subgroup $\Gamma$ such that $G/\Gamma$ has finite volume with respect to the Haar measure. We say that $\Gamma$ is uniform if $G/\Gamma$ is compact, non-uniform otherwise. Let $\Gamma$ be a lattice in $G$. We say that a subgroup $\Pi < \Gamma$ is thin if it is of infinite index in $\Gamma$ and Zariski dense in $G$. Thin groups have been shown to share many properties with lattices and are an active field of research (see Sarnak [Sar14] and [KLLR19]). The main purpose of this paper is to find thin surface groups in non-uniform lattices of the classical split groups and the adjoint split form of $G_2$.

Denote by $S_g$ a closed connected orientable surface of genus $g$ at least 2, $\pi_1(S_g)$ its fundamental group and $\text{MCG}(S_g) = \text{Aut}(\pi_1(S_g))/\text{Int}(\pi_1(S_g))$ its mapping class group.

The surface groups we find are images of representations that lie in the Hitchin component of the corresponding split group $G$ (see Hitchin [Hit92]). This is a special connected component of the character variety $\mathcal{X}(\pi_1(S_g), G) = \text{Hom}(\pi_1(S_g), G)/G$, which for $G = \text{PSL}(n, \mathbb{R})$ can be defined as follows.

For any $n \geq 2$ denote by $\tau_n : \text{SL}(2, \mathbb{C}) \to \text{SL}(n, \mathbb{C})$ the representation where 

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \text{SL}(2, \mathbb{C})
$$

acts on the space of homogeneous polynomials in two variables $X$ and $Y$ of degree $n - 1$ as

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} X^{n-i-1}Y^i = (aX + cY)^{n-i-1}(bX + dY)^i
$$

for every $0 \leq i \leq n-1$. We call $\tau_n$ the irreducible representation. Denote also by $\tau_n : \text{PSL}(2, \mathbb{C}) \to \text{PSL}(n, \mathbb{C})$ the induced representation. The Hitchin component is the connected component of the character variety $\mathcal{X}(\pi_1(S_g), \text{PSL}(n, \mathbb{R}))$ that contains the equivalence class of a representation of the form $\tau_n \circ j$ with $j : \pi_1(S_g) \to \text{PSL}(2, \mathbb{R})$ faithful and discrete. We call Hitchin representation a representation whose equivalence class is in the Hitchin component. An important feature of Hitchin representations is the following result.

**Theorem 0.1** (Labourie [Lab06], Fock and Goncharov [FG03]). Hitchin representations are discrete and faithful.

Faithfulness of a Hitchin representation plays a crucial role in our construction. We say that a Hitchin representation is thin if its image is a thin subgroup of a lattice of $G$. The question we address is: which lattices of $G$ contains the image of a thin Hitchin representation?

The lattices we will consider are $\mathbb{Q}$-arithmetic subgroups (we introduce this notation to simplify the exposition). A $\mathbb{R}/\mathbb{Q}$-form of $G$ is a $\mathbb{Q}$-algebraic

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1We are not considering its conjugacy class: $\tau_n$ is fixed.
group that is isomorphic to G as an $\mathbb{R}$-algebraic group. We say that two subgroups $\Gamma_1$ and $\Gamma_2$ of $G(\mathbb{R})$ are commensurable if $\Gamma_1 \cap \Gamma_2$ has finite index in both $\Gamma_1$ and $\Gamma_2$. If $H$ is a $\mathbb{Q}$-algebraic subgroup of $GL_n$, we define its $\mathbb{Z}$-points as $H(\mathbb{Q}) \cap GL_n(\mathbb{Z})$. This latter depends on the embedding of $H$ in $GL_n$, but two different embeddings give rise to commensurable subgroups.

**Definition 0.2.** A $\mathbb{Q}$-arithmetic subgroup is a subgroup of G that is conjugate to a subgroup commensurable with the integer points of a $\mathbb{Q}$-form of $G^2$.

Such groups are lattices of G as shown by Harish-Chandra. See Section 1 for more details.

In this paper we find thin Hitchin representations with image in some $\mathbb{Q}$-arithmetic lattices of the Lie groups $SL(n, \mathbb{R})$, $SO(k+1,k)$, $Sp(2n, \mathbb{R})$ and $G_2$. We start by fixing a cocompact $\mathbb{Q}$-arithmetic lattice of $SL(2, \mathbb{R})$. Let $a, b \in \mathbb{N}^*$ and denote by

$$
\Gamma_{a,b} = \left\{ \left( \begin{array}{cccc}
    x_0 + \sqrt{ax_1} & \sqrt{bx_2} + \sqrt{abx_3} \\
    \sqrt{bx_2} - \sqrt{abx_3} & x_0 - \sqrt{ax_1}
\end{array} \right) \mid x_i \in \mathbb{Z}, \ Det = 1 \right\}.
$$

This is a cocompact lattice of $SL(2, \mathbb{R})$ if and only if $ax^2 + by^2 = 1$ has no solution $(x, y) \in \mathbb{Q}^2$. If it is cocompact, then up to finite cover the quotient $H^2/\Gamma_{a,b}$ is a closed surface of genus at least 2. For some $a, b \in \mathbb{N}^*$ $\Gamma_{a,b}/\{\pm I_2\}$ is torsion-free (e.g. $a = b \neq 1$). In this case we construct representations of $\Gamma_{a,b}$. Otherwise we construct representations of a torsion-free finite-index subgroup of $\Gamma_{a,b}$.

We begin by classifying the $\mathbb{Q}$-arithmetic lattices $\Lambda$ of $G$ that contain $\tau_n(\Gamma_{a,b})$ up to finite index, where $G$ is one of the Lie groups above. This is the core of the proof and uses nonabelian Galois cohomology. The group $\tau_n(\Gamma_{a,b})$ is not Zariski-dense in $G$ since it lies in $\tau_n(SL(2, \mathbb{R}))$. We then "bend" the representation enough so that it still lies in $\Lambda$ but is Zariski-dense. The bending technique was introduced by Johnson and Millson [JM86] and has already been used to construct thin subgroups, for example in Long and Thistlethwaite [LT20] or Ballas and Long [BL20]. To bend, we need to find a simple closed curve on the surface which has a big enough centralizer in $\Lambda$. For $\Lambda = SL(n, \mathbb{Z})$, this is not possible. Using nonabelian Galois cohomology, we are able to compute the centralizer of this curve in $\Lambda$ explicitly. Since the deformed representations lie in the Hitchin component, we know that they are still faithful. Their image will be thus a thin surface subgroup of $\Lambda$.

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2Note that some arithmetic groups are not $\mathbb{Q}$-arithmetic groups. Indeed we only consider the $\mathbb{Q}$-forms of $G$ while arithmetic groups also refer to the integer points of a $\mathbb{Q}$-forms of $G \times K$ with $K$ a compact group. However if $G$ is simple and not compact, every non-uniform arithmetic group is $\mathbb{Q}$-arithmetic (see Morris’s book [Mor15] corollary 5.3.2).

3We know that the surface subgroups we constructed have to be of infinite index in the lattices because the virtual cohomological dimension of a surface group is 2 and the virtual cohomological dimension of the lattices considered here is at least 3 (see Aramayona, Degrijse, Martínez-Pérez and Souto [ADMPS17]).
We first state our results for $n$ odd. For $d \in \mathbb{N}$ not a square and $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial denote by

$$\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}]) = \{ M \in \text{SL}(n, \mathbb{Z}[\sqrt{d}]) \mid \sigma(M)^\top M = I_n \}$$

where $\sigma(M)$ consists of applying $\sigma$ to all entries of $M$. This is a lattice of $\text{SL}(n, \mathbb{R})$. Here is a description of the $\mathbb{Q}$-arithmetic lattices of $\text{SL}(n, \mathbb{R})$ that contain $\tau_n(\Gamma_{a,b})$.

**Proposition A.1.** Let $\Gamma$ be a $\mathbb{Q}$-arithmetic subgroup of $\text{SL}(2, \mathbb{R})$ and $n \geq 3$ be odd. Then $\tau_n(\Gamma)$ lies in a subgroup of $\text{SL}(n, \mathbb{R})$ commensurable with a conjugate of $\text{SL}(n, \mathbb{Z})$ and in a subgroup commensurable with a conjugate of $\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$ for every $d \in \mathbb{N}$ not a square with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial.

Furthermore these are the only $\mathbb{Q}$-arithmetic subgroups of $\text{SL}(n, \mathbb{R})$ that contain $\tau_n(\Gamma)$ up to commensurability.

**Remark 0.3.** As a consequence, even though $\tau_n(\Gamma_{a,b})$ has coefficients in $\mathbb{Z}[\sqrt{a}, \sqrt{b}]$ it can be conjugated to a subgroup of $\text{SL}(n, \mathbb{Z})$.

The proposition implies that the corresponding locally symmetric spaces

$$\text{SO}(n) \backslash \text{SL}(n, \mathbb{R}) / \Lambda$$

where $\Lambda$ is (a torsion free finite index subgroup of) one of the arithmetic subgroups listed above, contain a totally geodesic surface of irreducible type\(^5\). Except when $\Lambda = \text{SL}(n, \mathbb{Z})$, we manage to bend the representation $\tau_n$ to make it Zariski-dense.

**Theorem A.1.** Let $n \geq 3$ be odd. Let $d \in \mathbb{N}$ be not a square and $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial. There exists $g \geq 2$ such that the $\mathbb{Q}$-arithmetic subgroup $\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$ of $\text{SL}(n, \mathbb{R})$ contains infinitely many $\text{MCG}(S_g)$-orbits of thin Hitchin representations of $\pi_1(S_g)$.

See Theorem 5.15 for a slightly stronger version of this result. Theorem A.1 should be compare to the result of Borel which states that there is only finitely many arithmetic surfaces of a given genus up to isometry (see Theorem 11.3.1 in [MR03]). Note that the group $\text{SL}(n, \mathbb{Z})$ for $n$ odd is not part of our list but has been shown to contain infinitely many conjugacy classes of thin Hitchin representations by Long and Thistlethwaite [LT20]. This result was also announced\(^6\) by Burger, Labourie and Wienhard (see Theorem 24 in [Wie18] and [Bur15]). Together with Proposition 1.3 below, this implies the following.

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\(^4\)M\(^\top\) is the transpose of the matrix M.

\(^5\)i.e. such that the copy of the hyperbolic plane in the universal cover of the locally symmetric space comes from an irreducible embedding of $\text{SL}(2, \mathbb{R})$ in $\text{SL}(n, \mathbb{R})$

\(^6\)They announced that for $n \geq 5$ odd, $\text{SL}(n, \mathbb{Z})$ contains infinitely many MCG-orbits of thin Hitchin representations.
Corollary A.1. Let \( p \neq 2 \) be a prime. All non-uniform lattices of \( \text{SL}(p, \mathbb{R}) \) contain a thin Hitchin representation.

They moreover contain infinitely many MCG-orbits of thin Hitchin representations except maybe the lattices commensurable with a conjugate of \( \text{SL}(p, \mathbb{Z}) \). For \( p = 3 \), Long, Reid and Thistlethwaite proved in [LRT11] and [LR14] that all non-uniform lattice of \( \text{SL}(3, \mathbb{R}) \) contain infinitely many conjugacy classes of thin Hitchin representations.

We also construct Zariski-dense surface groups in \( \mathbb{Q} \)-arithmetic subgroups of other split Lie groups. For \( p, q \in \mathbb{N} \) denote by \( I_{p,q} = (I_p - I_q) \).

Theorem A.2. Let \( k \geq 2 \). For each of the following \( \mathbb{Q} \)-arithmetic subgroups \( \Lambda \) of \( \text{SO}(I_{k+1,k+1}, \mathbb{R}) \), there exists \( g \geq 2 \) such that \( \Lambda \) contains infinitely many \( \text{MCG}(S_g) \)-orbits of thin Hitchin representations of \( \pi_1(S_g) \):

- if \( k \equiv 0, 3[4] \) : \( \text{SO}(I_{k+1,k}, \mathbb{Z}) \)
- if \( k \equiv 1, 2[4] \) : \( \text{SO}(Q, \mathbb{Z}) \) for \( Q \in \text{SL}(2k+1, \mathbb{Q}) \) a symmetric matrix of signature \((k+1,k)\) which is not equivalent to \( I_{k+1,k} \) over \( \mathbb{Q} \).

In particular, for \( k \equiv 1, 2[4] \), every non-uniform lattice of \( \text{SO}(I_{k+1,k}, \mathbb{R}) \) not commensurable with a conjugate of \( \text{SO}(I_{k+1,k}, \mathbb{Z}) \), contains infinitely many \( \text{MCG}(S_g) \)-orbits of thin Hitchin representations of \( \pi_1(S_g) \) for some \( g \geq 2 \).

This is shown by establishing an analogue of Proposition A.1 in the case of \( \text{SO}(I_{k+1,k}, \mathbb{R}) \).

Theorem A.3. There exists \( g \geq 2 \) such that the group \( G_2(\mathbb{Z}) \), which is a \( \mathbb{Q} \)-arithmetic subgroup of \( G_2(\mathbb{R}) \), contains infinitely many non-conjugate thin Hitchin representations of \( \pi_1(S_g) \).

Here \( G_2(\mathbb{R}) \) is the adjoint connected split real Lie group of type \( G_2 \) (see Definition 3.6). We expect \( G_2(\mathbb{Z}) \) to contain infinitely many MCG-orbits of thin Hitchin representations (see Remark 5.16). It appears that \( G_2(\mathbb{Z}) \) is the only non-uniform lattice of \( G_2(\mathbb{R}) \) up to commensurability and conjugation (see Remark 3.8).

Proposition A.1 shows that for \( n \) odd, the list of \( \mathbb{Q} \)-arithmetic subgroups of \( \text{SL}(n, \mathbb{R}) \) that contain the group \( \tau_n(\Gamma) \) up to commensurability is the same for all \( \Gamma \). For \( n \) even, the list depends on the group \( \Gamma \). To be more precise we need to define more concepts.

The \( \mathbb{Q} \)-arithmetic subgroups of \( \text{SL}(2, \mathbb{R}) \) can be described in terms of quaternion algebras. We say that a unital associative algebra \( A \) over a field \( k \) is a quaternion algebra if there exists \( i \) and \( j \) in \( A \) such that \((1, i, j, ij)\) is a basis of \( A \) with \( i^2 = a \in k^\times \), \( j^2 = b \in k^\times \) and \( ij = -ji \). We denote by \( A = (a, b)_k \).

Note that \((a, b)_k \simeq (b, a)_k \simeq (a, -ab)_k\) and if \( x \in k^\times \) then \((ax^2, b)_k \simeq (a, b)_k\). We say that \( A \) is a division algebra if every non-zero element of \( A \) is
invertible. Quaternion algebras are endowed with an involution called the conjugation

\[ A \to A, \quad x = x_0 + x_1 i + x_2 j + x_3 i j \mapsto \overline{x} = x_0 - x_1 i - x_2 j - x_3 i j, \]

which allows us to define a norm \( N_{\text{red}} : A \to k, \quad x \mapsto x \overline{x} \). Suppose that \( k \) is a number field and denote \( O_k \) its ring of integers. An order of \( A \) is a finitely generated \( O_k \)-submodule containing a basis of \( A \) over \( k \), containing \( 1 \) and which is a subring of \( A \). For example if \( a, b \in O_k \) then \( O = O_k[1, i, j, ij] \) is an order of \( A \).

We say that \( A \) splits over a field extension \( K \) if \( A \otimes_k K \cong M_2(K) \). For example, a quaternion algebra over \( \mathbb{Q} \) splits over \( \mathbb{R} \) if and only if \( a \) or \( b \) is positive. The \( \mathbb{R}/\mathbb{Q} \)-forms of \( SL(2) \) are algebraic groups defined by

\[ H(K) = \{ x \in A \otimes \mathbb{Q} K \mid N_{\text{red}}(x) = 1 \} \]

for any field extension \( K \) of \( \mathbb{Q} \), with \( A \) a quaternion algebra over \( \mathbb{Q} \) that splits over \( \mathbb{R} \). Its integer points are commensurable with \( O \), for \( O \) an order of \( A \). We can embed \( O^1 \) in \( SL(2, \mathbb{R}) \) using

\[ O^1 \hookrightarrow A^1 \hookrightarrow (A \otimes \mathbb{Q} \mathbb{R})^1 \cong SL(2, \mathbb{R}) \]

and the image will be commensurable with \( \Gamma_{a,b} \), defined above. Thus \( \mathbb{Q} \)-arithmetic subgroups of \( SL(2, \mathbb{R}) \) are subgroups commensurable to conjugates of \( O^1 \), for \( O \) an order of a quaternion algebra \( A \) that splits over \( \mathbb{R} \). Furthermore \( O^1 \) is cocompact if and only if \( A \) is a quaternion division algebra, which is equivalent to \( ax^2 + by^2 = 1 \) having no solutions \( (x, y) \in \mathbb{Q}^2 \).

We can now state our result for \( n \) even. If \( A \) splits over \( \mathbb{R} \) we can embed \( M_{2n}(O) \) in \( M_n(\mathbb{R}) \). This allows us to define \( SL(\frac{n}{2}, O) \) as the matrices of \( M_{2n}(O) \) which have determinant \( 1 \) in \( M_n(\mathbb{R}) \). For \( \sigma \in \text{Gal}(\sqrt{d}/\mathbb{Q}) \) define

\[ \text{SU}(I_{\frac{n}{2}}, \sigma; O \otimes \mathbb{Z}[\sqrt{d}]) = \{ M \in SL(\frac{n}{2}, O \otimes \mathbb{Z}[\sqrt{d}]) \mid \partial(M)^T M = I_{\frac{n}{2}} \} \]

where \( \partial : A \otimes \mathbb{Q} \mathbb{Q}(\sqrt{d}) \to A \otimes \mathbb{Q} \mathbb{Q}(\sqrt{d}), \quad x \otimes t \mapsto x \otimes \sigma(t) \) and \( \partial(M) \) consists of applying \( \partial \) to all entries of \( M \). Here is the list of \( \mathbb{Q} \)-arithmetic groups that contain \( \tau_n(\Gamma_{a,b}) \).

**Proposition A.2.** Let \( \Gamma \) be a \( \mathbb{Q} \)-arithmetic subgroup of \( SL(2, \mathbb{R}) \) and \( n \geq 4 \) be even. Let \( O \) be an order of a quaternion algebra \( A \) over \( \mathbb{Q} \) such that \( \Gamma \) is commensurable with \( O^1 \).

Then \( \tau_n(\Gamma) \) lies in a subgroup of \( SL(n, \mathbb{R}) \) commensurable with a conjugate of

- \( SL(\frac{n}{2}, O) \),
- \( \text{SU}(I_{\frac{n}{2}}, \sigma; \mathbb{Z}[\sqrt{d}]) \) for any \( d \in \mathbb{N} \) not a square such that \( A \) splits over \( \mathbb{Q}(\sqrt{d}) \) with \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \) non-trivial,
- \( \text{SU}(I_{\frac{n}{2}}, -\sigma; O \otimes \mathbb{Z}[\sqrt{d}]) \) for any \( d \in \mathbb{N} \) not a square such that \( A \) does not split over \( \mathbb{Q}(\sqrt{d}) \) with \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \) non-trivial.

\(^7\text{this is not a norm in the usual sense}\)
Furthermore these are the only $\mathbb{Q}$-arithmetic subgroups of $\text{SL}(n, \mathbb{R})$ that contain $\tau_n(\Gamma)$ up to commensurability.

For some arithmetic groups above, we manage to bend the representation $\tau_n$ to make it thin.

**Theorem A.4.** Let $n \geq 4$ be even. For the following $\mathbb{Q}$-arithmetic subgroups $\Lambda$ of $\text{SL}(n, \mathbb{R})$, there exists $g \geq 2$ such that $\Lambda$ contains infinitely many $\text{MCG}(S_g)$-orbits of thin Hitchin representations of $\pi_1(S_g)$:

- $\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$ for any $d \in \mathbb{N}$ not a square with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial
- $\text{SU}(I_n^2, \sigma; \mathbb{O} \otimes \mathbb{Z}[\sqrt{d}])$ for any $d \in \mathbb{N}$ not a square and $\mathbb{O}$ an order of a quaternion algebra over $\mathbb{Q}$ that does not split over $\mathbb{Q}(\sqrt{d})$ with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial.

See Theorem 5.15 for a slightly stronger version of this result. Note that the subgroup $\text{SL}(4, \mathbb{Z})$ is not part of our list but has been shown to contain infinitely many non-conjugate thin Hitchin representations by Long and Thistlethwaite [LT18]. We also construct thin Hitchin representations in $\mathbb{Q}$-arithmetic subgroups of $\text{Sp}(n, \mathbb{R})$.

**Theorem A.5.** Let $n \geq 4$ be even. Let $\mathbb{O}$ be an order of a division quaternion algebra over $\mathbb{Q}$ with conjugation $\overline{-}$. There exists $g \geq 2$ such that the $\mathbb{Q}$-arithmetic subgroup $\text{SU}(I_n^2, \overline{-}; \mathbb{O})$ of $\text{Sp}(n, \mathbb{R})$ contains infinitely many $\text{MCG}(S_g)$-orbits of thin Hitchin representations of $\pi_1(S_g)$.

Together with Proposition 1.5 below, the theorem implies the following.

**Corollary A.2.** Let $n \geq 4$ be even. For every non-uniform lattice $\Lambda$ of $\text{Sp}(n, \mathbb{R})$ not commensurable with a conjugate of $\text{Sp}(n, \mathbb{Z})$, there exists $g \geq 2$ such that $\Lambda$ contains infinitely many $\text{MCG}(S_g)$-orbits of thin Hitchin representations of $\pi_1(S_g)$.

Again, the group $\text{Sp}(4, \mathbb{Z})$ is not part of our list but has been shown to contain infinitely many non-conjugate thin Hitchin representations in [LT18]. Together with [LT18], the corollary shows that every non-uniform lattice of $\text{Sp}(4, \mathbb{R})$ contains a thin Hitchin representation.

Constructing surface subgroups in lattices of Lie groups (not necessarily Zariski-dense) has been done for uniform lattices of many semisimple Lie groups. In [Ham15], Hamenstädt proves that any cocompact lattice of a rank one simple Lie group of non-compact type not isomorphic to $\text{SO}(2n, 1)$ contains surface subgroups. Kahn, Labourie and Mozes [KLM18] prove the same result for many other semi-simple Lie groups, notably the complex ones. It is expected that their construction provides Zariski-dense surface groups. As it is explained in §1.2 of [KLM18] their proof does not work in setting we are dealing with, namely split real Lie groups.
Organisation of the paper. In Section 1 we give some background on Q-arithmetic subgroups in semi-simple Lie groups and describe their classification for the relevant Lie groups. In Section 2 we consider the relations between the irreducible representation \( \tau_n \) and the \( \mathbb{R}/\mathbb{Q} \)-forms of \( \text{SL}_n \). We prove there Propositions A. In section 3 we give the corresponding results for \( \text{Sp}(2n, \mathbb{R}) \), \( \text{SO}(I_{k+1,k}, \mathbb{R}) \) and \( G_2(\mathbb{R}) \). In Section 4 we describe the “bending” construction and show some technical lemmas. Finally in Section 5 we give the proof of Theorems A.

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1. Background on Q-arithmetic groups

Let \( k \) be an algebraic number field with ring of integers \( \mathcal{O}_k \). A central simple algebra over \( k \) is a unital associative algebra \( D \) which is simple and whose center is exactly \( k \). Quaternion algebras are a particular case of central simple algebras. If \( D \otimes_k \mathbb{R} \simeq \mathbb{M}_d(\mathbb{R}) \), then there is an isomorphism \( \phi : M_n(D) \otimes_k \mathbb{R} \rightarrow M_{dn}(\mathbb{R}) \). It induces a norm on \( M_n(D) \) which is \( N_{\text{red}} : M_n(D) \rightarrow k, M \mapsto \det(\phi(M)) \). This is independent of \( \phi \). We can thus define

\[
\text{SL}(n,D) = \{ M \in M_n(D) | N_{\text{red}}(M) = 1 \}.
\]

If \( B \in M_n(D) \) and \( \partial : D \rightarrow D \) is an involution we denote by

\[
\text{SU}(B, \partial; D) = \{ M \in \text{SL}(n,D) | \partial(M^\top)BM = B \}.
\]

If \( R \) is a subring of \( D \) then we define \( \text{SL}(n,R) = \text{SL}(n,D) \cap M_n(R) \) and \( \text{SU}(B, \partial; R) = \text{SU}(B, \partial; D) \cap M_n(R) \). Those notations will be used to define Q-arithmetic subgroups of \( \text{SL}(n,R) \).

An anti-involution of \( D \) is a map \( \partial : D \rightarrow D \) such that for all \( a, b \in D \)

\[
\partial(a + b) = \partial(a) + \partial(b), \quad \partial(ab) = \partial(b)\partial(a) \quad \text{and} \quad \partial^2(a) = a.
\]

For example, the conjugation \( \overline{ } \) of a quaternion algebra is an anti-involution. An anti-involution induces an involution of \( k \) that can be trivial or not. If \( A \) is a quaternion algebra over \( k \), with \( L \) a quadratic extension of \( k \) and \( \sigma \) the non-trivial element of \( \text{Gal}(L/k) \) then \( \overline{ } \otimes \sigma \) is an anti-involution on \( A \otimes_k L \) which takes \( x \otimes l \) to \( \overline{x} \otimes \sigma(l) \) and thus induces a non-trivial involution of \( L \).

We say that \( D \) is a division algebra if every non-zero element of \( D \) is invertible. The degree of a division algebra is the square root of its dimension.

\[\text{this is not a norm in the usual sense}\]
An order of $D$ is a finitely generated $O_k$-submodule of $D$ containing a basis of $D$ over $k$, which is a subring of $D$ and contains the unit element.

Let $\partial$ be an anti-involution on $D$ and let $f : D^n \times D^n \to D$ satisfy
\[
\begin{align*}
f(\lambda u + v, w) &= \lambda f(u, w) + f(v, w) \\
f(u, \lambda v + w) &= \partial(\lambda)f(u, v) + f(u, w) \\
f(u, v) &= \partial(f(v, u))
\end{align*}
\]
for all $u, v, w \in D^n$ and $\lambda \in D$. Then $x \mapsto f(x, x)$ is called a $\partial$-Hermitian form. If $\partial$ is trivial, we call it a quadratic form. The discriminant of a $\partial$-Hermitian form is the determinant of a matrix representing $f$ in any basis. It is well-defined up elements of the form $\lambda\partial(\lambda)$, $\lambda \in D$. The rank of a form is the rank of a matrix representing $f$ in any basis. The form is said to be non-degenerate if it has full rank. Here are the result we will need about $\partial$-Hermitian forms.

**Proposition 1.1** (See §4 in Lewis [Lew82] or example 5 of §1 in Milnor [Mil69]). Let $d \in \mathbb{N}$ be not a square and let $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ be non-trivial. Then $\sigma$-Hermitian forms on $\mathbb{Q}(\sqrt{d})$ are classified by their rank and their discriminant.$^9$

**Proposition 1.2** (See §5 in [Lew82]). Let $A$ be a quaternion division algebra over $\mathbb{Q}$ that splits over $\mathbb{R}$ with conjugation $\overline{\cdot}$. Then $\overline{\cdot}$-Hermitian forms on $A$ are classified by their rank.

We can state the classification of $\mathbb{Q}$-arithmetic groups in the cases of $\text{SL}(n, \mathbb{R})$, $\text{Sp}(2n, \mathbb{R})$ and $\text{SO}(I_{k+1,k}, \mathbb{R})$. It is based on the classification of arithmetic subgroups in classical Lie groups, see Morris §18.5 [Mor15]. See also Remark 3.8 for the classification of $\mathbb{Q}$-arithmetic subgroups in the case of $G_2$.

The $\mathbb{Q}$-arithmetic subgroups of $\text{SL}(n, \mathbb{R})$ are the subgroups commensurable with conjugates of:

- $\text{SL}(m, \mathcal{O})$ where $\mathcal{O}$ is an order of a division algebra $D$ over $\mathbb{Q}$ of degree $d$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_d(\mathbb{R})$ and with $n = dm$.
- $\text{SU}(B, \partial; \mathcal{O})$ where $L$ is a quadratic real extension of $\mathbb{Q}$, $D$ is a division algebra over $L$ of degree $d$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_d(\mathbb{R}) \oplus M_d(\mathbb{R})$, $\mathcal{O}$ is an order in $D$, $\partial$ is an anti-involution of $D$ whose restriction to $L$ is the non-trivial automorphism of $L$ over $\mathbb{Q}$, $B$ is an invertible matrix in $M_m(D)$ satisfying $\partial(B^\top) = B$ with $n = md$.

The following proposition will be used in the proof of Corollary A.1.

**Proposition 1.3.** Let $p \neq 2$ be prime. Non-uniform lattices of $\text{SL}(p, \mathbb{R})$ are subgroups commensurable with conjugates of $\text{SL}(p, \mathbb{Z})$ or of $\text{SU}(I_p, \sigma; \mathbb{Z}(\sqrt{d}))$ for $d \in \mathbb{N}$ not a square and for the non-trivial $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$.

\[^9\text{since } d > 0 \text{ we have that } \mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R} \oplus \mathbb{R} \text{ and thus there is no local invariant at the infinite place of } \mathbb{Q} \]
Proof. By Margulis Arithmeticity Theorem (Theorem 16.3.1 in [Mor15]) all non-uniform lattices of $SL(p, \mathbb{R})$ are arithmetic subgroups. As can be seen in [Mor15], Corollary 5.3.2, non-uniform arithmetic subgroups are $\mathbb{Q}$-arithmetic. Thus they are commensurable with conjugates of the $\mathbb{Z}$-points of a $\mathbb{R}/\mathbb{Q}$-form of $SL_p$. As shown in §18.5 of [Mor15], their $\mathbb{Q}$-points can only be of two kinds.

First they can be of the form $SL(m, D)$ for $D$ a division algebra of degree $r$ over $\mathbb{Q}$ such that $D \otimes \mathbb{Q} \mathbb{R} \cong M_r(\mathbb{R})$. Here we must have $rm = p$. If $(r, m) = (1, p)$ the arithmetic subgroup is of the form $SL(p, \mathbb{Z})$. If $(r, m) = (p, 1)$ the arithmetic subgroup is of the form $SL(1, \mathcal{O})$ for $\mathcal{O}$ an order of $D$, but this is a uniform lattice (see the classification in §18.5 in [Mor15]).

Secondly they can be of the form $SU(B, \partial; D)$ for $D$ a division algebra of degree $r$ over a real quadratic extension $L$ of $\mathbb{Q}$ such that $D \otimes \mathbb{Q} L \cong M_r(\mathbb{R}) \oplus M_r(\mathbb{R}), \partial$ an anti-involution of $D$ whose restriction to $L$ is the non-trivial $\sigma \in \text{Gal}(L/\mathbb{Q})$ and $B \in M_m(D)$ invertible satisfying $\partial(B^\top) = B$. Once again we must have $mr = p$. If $(r, m) = (1, p)$ the arithmetic subgroup is of the form $SU(B, \sigma; \mathbb{Z}[\sqrt{d}])$ where $L = \mathbb{Q}(\sqrt{d})$. If $(r, m) = (p, 1)$ the arithmetic subgroup is of the form $SU(B, \partial; \mathcal{O})$ where $\mathcal{O}$ is an order of $D$, but this is a uniform lattice (see the classification in §18.5 in [Mor15]).

It remains to show that all subgroups of the form $SU(B, \sigma; \mathbb{Z}[\sqrt{d}])$ are conjugates of $SU(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$. By Proposition 1.1, up to scalar multiplication and conjugation there is only one non-degenerate hermitian form on $\mathbb{Q}(\sqrt{d})^p$. Hence $SU(B, \sigma; \mathbb{Z}[\sqrt{d}])$ is conjugate to $SU(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$. \qed

Proposition 1.4. Let $k \geq 2$. The $\mathbb{Q}$-arithmetic subgroups of $SO(I_{k+1, k}, \mathbb{R})$ are the subgroups commensurable with conjugates of $SO(B, \mathbb{Z})$ for $B \in SL(2k+1, \mathbb{Q})$ symmetric of signature $(k + 1, k)$.

Proof. This follows form the classification in §18.5 in [Mor15] when we restrict to $F = \mathbb{Q}$. Note that we can always assume that $B$ has determinant 1 up to multiplying $B$ by a scalar and up to congruence. \qed

The $\mathbb{Q}$-arithmetic subgroups of $Sp(2n, \mathbb{R})$ are the subgroups commensurable with conjugates of:

- $Sp(2n, \mathbb{Z})$
- $SU(B, ^\top; \mathcal{O})$ where $A$ is a division quaternion algebra over $\mathbb{Q}$ such that $A \otimes \mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R}), \mathcal{O}$ is an order in $A$, $\bar{\cdot}$ is the conjugation of $A$ and $B$ is an invertible matrix in $M_n(A)$ satisfying $\overline{B^\top} = B$.

Proposition 1.5. Let $n \geq 2$. Non-uniform lattices of $Sp(2n, \mathbb{R})$ are subgroups commensurable with conjugates of $Sp(2n, \mathbb{Z})$ or of $SU(I_{n, -}^\top; \mathcal{O})$ for $\mathcal{O}$ an order of a division quaternion algebra over $\mathbb{Q}$ that splits over $\mathbb{R}$ with conjugation $\bar{\cdot}$.

Proof. By Margulis Arithmeticity Theorem (Theorem 16.3.1 in [Mor15]) all non-uniform lattices of $Sp(2n, \mathbb{R})$ are arithmetic subgroups. As can be
seen in [Mor15], Corollary 5.3.2, all non-uniform arithmetic groups are \( \mathbb{Q} \)-arithmetic. From the classification those can only be commensurable to conjugates of \( \text{Sp}(2n, \mathbb{Z}) \) or of \( \text{SU}(B, \mathcal{O}) \) with notations as above. By Proposition 1.2 there is only one nondegenerate \( \overline{\cdot} \)-Hermitian form on a division quaternion algebra over \( \mathbb{Q} \) that splits over \( \mathbb{R} \). Thus \( \text{SU}(B, \mathcal{O}) \) is conjugate to \( \text{SU}(I_n, \mathcal{O}) \).

\[ \square \]

2. \( \mathbb{R}/\mathbb{Q} \)-forms of \( \text{SL}_n \)

In this section we study \( \mathbb{R}/\mathbb{Q} \)-forms of \( \text{SL}_n \) using cohomological tools.

2.1. Non-abelian cohomology. Let \( G \) be a topological group acting continuously on a topological group \( H \) by automorphism. For \( g \in G \) and \( \phi \in \text{Aut}(H) \) we denote by

\[ g\phi : x \mapsto g\phi(g^{-1}x). \]

A 1-cocycle is a map \( \xi : G \to \text{Aut}(H) \) that satisfies

\[ \xi(g_1g_2) = \xi(g_1) \circ g_1\xi(g_2) \]

for all \( g_1, g_2 \in G \). We say that two 1-cocyles \( \xi \) and \( \zeta \) are equivalent if there exists \( \phi \in \text{Aut}(H) \) such that for all \( g \in G \) we have

\[ \xi(g) = \phi^{-1} \circ \zeta(g) \circ g\phi. \]

**Definition 2.1.** The first Galois cohomology set of \( G \) (with coefficient in \( H \)) is the set of equivalence classes of continuous 1-cocycles. It is denoted by \( H^1(G, H) \). It is not a group in general.

2.2. \( \mathbb{R}/\mathbb{Q} \)-forms and nonabelian Galois cohomology. We want to describe \( \mathbb{R}/\mathbb{Q} \)-forms of \( \text{SL}_n \). Unfortunately, the extension \( \mathbb{R}/\mathbb{Q} \) is not Galois, which prevents us from using the cohomological description of forms of algebraic groups. However, thanks to Proposition 2 of III.§1 in [Ser02], \( \mathbb{R}/\mathbb{Q} \)-forms of \( \text{SL}_n \) are \( \overline{\mathbb{Q}}/\mathbb{Q} \)-forms of \( \text{SL}_n \) where \( \overline{\mathbb{Q}} \) is the subfield of \( \mathbb{C} \) of algebraic numbers over \( \mathbb{Q} \).

Let \( k \) be an algebraic number field. We start by recalling the cohomological description of \( \overline{\mathbb{Q}}/k \)-forms of an algebraic group. The main references for the material covered here are [Mor15] chapter 18 and Platonov and Rapinchuk’s book [PR94] section 1.3.

Let \( G \) be a \( k \)-algebraic subgroup of \( \text{GL}_n \). A \( \overline{\mathbb{Q}}/k \)-form of \( G \) is an algebraic group \( H \) over \( k \) such that \( H \cong G \) over \( \overline{\mathbb{Q}} \). We endow \( \text{Gal}(\overline{\mathbb{Q}}/k) \) with its profinite topology (see [PR94] page 22) and \( \text{Aut}(G(\overline{\mathbb{Q}})) \) with the discrete topology. An element \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/k) \) acts on \( G(\overline{\mathbb{Q}}) \) by applying \( \sigma \) to all entries of the matrix \( g \in G(\overline{\mathbb{Q}}) \). This allows us to define \( H^1(\text{Gal}(\overline{\mathbb{Q}}/k), \text{Aut}(G(\overline{\mathbb{Q}}))) \), the first Galois cohomology set of \( G \).

\[ ^{10}\text{We do not suppose } H \text{ abelian.} \]
Let $\xi : \text{Gal}(\overline{Q}/k) \to \text{Aut}(G(\overline{Q}))$ be a 1-cocycle and denote $\overline{Q}[G]$ the coordinate ring of $G$. We define

$$\xi_k[G] = \{ f \in \overline{Q}[G] \mid \sigma \circ f \circ \sigma^{-1} \circ \xi(\sigma)^{-1} = f \ \forall \sigma \in \text{Gal}(\overline{Q}/k) \}.$$  

This is a $k$-algebra and we let $\xi G$ be the $k$-algebraic subgroup of $\text{GL}_n$ that has $\xi_k[G]$ as coordinate ring. It is a $\overline{Q}/k$-form of $G$ since $\overline{Q} \otimes_k \xi_k[G] \simeq \overline{Q}[G]$.

Concretely, its $k$-points are

$$\xi G(k) = \{ M \in G(\overline{Q}) \mid \xi(\sigma) \circ \sigma(M) = M \ \forall \sigma \in \text{Gal}(\overline{Q}/k) \}.$$  

Reciprocally if $H$ is a $\overline{Q}/k$-form of $G$ then denote by $\Phi : H(\overline{Q}) \to G(\overline{Q})$ an isomorphism and let

$$\xi : \text{Gal}(\overline{Q}/k) \to \text{Aut}(G(\overline{Q})), \sigma \mapsto \Phi \circ \sigma \circ \Phi^{-1} \circ \sigma^{-1}.$$  

This is a 1-cocycle. In this way, $H^1(\text{Gal}(\overline{Q}/k), \text{Aut}(G(\overline{Q})))$ classifies $\overline{Q}/k$-forms of $G$ (see [PR94] Theorem 2.9 and §2.2.3):

**Proposition 2.2.** Denote by $\mathcal{F}$ the set of isomorphism classes of $\overline{Q}/k$-forms of $G$. The map

$$H^1(\text{Gal}(\overline{Q}/k), \text{Aut}(G(\overline{Q}))) \to \mathcal{F}, \ [\xi] \mapsto [\xi G]$$  

is a bijection.

A $\overline{Q}/k$-form of $G$ is said to be inner if its associated 1-cocycle has values in $\text{Inn}(G(\overline{Q}))$, the group of inner automorphisms. Since $\text{Aut}(\text{SL}(2, \overline{Q})) = \text{Inn}(\text{SL}(2, \overline{Q})), \overline{Q}/k$-form of $\text{SL}_2$ is inner.

2.3. **Compatible cocycles.** Let $a, b \in \mathbb{N}$. Denote by\footnote{It is well defined since $\text{PSL}(n, \overline{Q}) = \text{PGL}(n, \overline{Q})$.}

$$T_{a,b} : \text{Gal}(\overline{Q}/Q) \to \text{PSL}(2, \overline{Q})$$

$$\sigma \mapsto \begin{cases} I_2 & \text{if } \sigma(\sqrt{a}) = \sqrt{a} \text{ and } \sigma(\sqrt{b}) = \sqrt{b} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = \sqrt{a} \text{ and } \sigma(\sqrt{b}) = -\sqrt{b} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = -\sqrt{a} \text{ and } \sigma(\sqrt{b}) = \sqrt{b} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \sigma(\sqrt{a}) = -\sqrt{a} \text{ and } \sigma(\sqrt{b}) = -\sqrt{b} \end{cases}$$

For simplicity we will often write $T_{\sigma}$ or $T_{a,b}^\sigma$ instead of $T_{a,b}^\sigma(\sigma)$.

**Lemma 2.3.** For every continuous 1-cocycle $\xi : \text{Gal}(\overline{Q}/Q) \to \text{Aut}(\text{SL}_2(\overline{Q}))$ such that $\xi\text{SL}_2(\mathbb{R}) \simeq \text{SL}(2, \mathbb{R})$ there exist $a, b \in \mathbb{N}$ and $P \in \text{SL}(2, \overline{Q})$ such that $\xi(\sigma) = \text{Int}(P^{-1} T_{a,b}^\sigma P)$ for every $\sigma \in \text{Gal}(\overline{Q}/Q)$\footnote{We use the convention $\text{Int}(M)(X) = MXM^{-1}$.}.
For all $\xi$ with $\xi : \text{Gal}(\mathbb{Q}/Q) \rightarrow \text{Aut}(SL_2(Q))$ a quadratic field extension (see Proposition 2.17 in [PR94]). If $a$ and $b$ are both negative then $\xi SL_2(R) \simeq SU(2)$. Thus $a$ or $b$ is positive and up to a change of basis we can assume that $a$ and $b$ are both positive integers. Denote by

$$SL(1,A) = \left\{ \left( \begin{array}{ccc} x_0 + \sqrt{a}x_1 & \sqrt{b}x_2 + \sqrt{ab}x_3 \\ \sqrt{b}x_2 - \sqrt{ab}x_3 & x_0 - \sqrt{a}x_1 \end{array} \right) \mid x_i \in Q, \text{ Det} = 1 \right\}$$

the elements of norm 1 of $A$. When the $x_i$ are taken in $\mathbb{Q}$ we actually get $SL(2,\mathbb{Q})$. Denote by $\Phi$ the identity embedding in $SL(2,\mathbb{Q})$. Direct computation shows that the 1-cocycle associated to $\Phi$ is $\sigma \mapsto \text{Int}(T_{a,b}^\sigma)$. The $\mathbb{Q}/Q$-form of $SL_2$ associated to $\xi$ is actually a conjugate of $SL(1,A)$ whence the result.

**Definition 2.4.** Let $n \geq 3$ and let $G$ be a $Q$-algebraic subgroup of $SL_n$ such that $\tau_n(SL(2,\mathbb{Q})) < G(\mathbb{Q})$. Let $\xi : \text{Gal}(\mathbb{Q}/Q) \rightarrow \text{Aut}(SL_2(\mathbb{Q}))$ and $\zeta : \text{Gal}(\mathbb{Q}/Q) \rightarrow \text{Aut}(G(\mathbb{Q}))$ be 1-cocyles. We say that $\zeta$ is $\tau_n$-compatible with $\xi$ if $\tau_n(\zeta SL_2(Q)) < \zeta G(Q)$.

**Definition 2.5.** For all $n \geq 2$ define $J_n$ as

$$J_n = \left( \begin{array}{cccc} (n-1)! \\ \vdots \\ (-1)^{i-1}(n-i)!(i-1)! \\ \ldots \\ \ldots \\ (-1)^{n-1}(n-1)! \end{array} \right)$$

Note that for any $M \in SL(2,\mathbb{C})$, $\tau_n(M)^\top J_n \tau_n(M) = J_n$ (see for example McGarraghy [McG05]).

**Proposition 2.6.** Let $n \geq 3$. Let $\xi : \text{Gal}(\mathbb{Q}/Q) \rightarrow \text{Aut}(SL_2(\mathbb{Q}))$ and $\zeta : \text{Gal}(\mathbb{Q}/Q) \rightarrow \text{Aut}(SL_n(\mathbb{Q}))$ be continuous 1-cocycles. Suppose that $\xi SL_2(R) \simeq SL(2,R)$. Write $\xi(\sigma) = \text{Int}(P^{-1}T_{a,b}^\sigma \sigma(P))$ for all $\sigma$ with $P \in PSL(2,\mathbb{R})$. Then $\zeta$ is $\tau_n$-compatible with $\xi$ if and only if either

1. $\zeta : \sigma \mapsto \text{Int}(\tau_n(P^{-1}T_{a,b}^\sigma \sigma(P)))$

2. $\zeta : \sigma \mapsto \begin{cases} \text{Int}(\tau_n(P^{-1}T_{a,b}^\sigma \sigma(P))) & \text{if } \sigma(\sqrt{a}) = \sqrt{a} \\ \text{Int}(\tau_n(P^{-1}T_{a,b}^\sigma J_n^{-1}) \circ \omega \circ \text{Int}(\tau_n \circ \sigma(P))) & \text{if } \sigma(\sqrt{a}) = -\sqrt{a} \end{cases}$

where $\omega(M) = (M^\top)^{-1}$.

**Proof.** Up to conjugation we can assume that $P = I_2$. Suppose that $\tau_n(\zeta SL_2(Q)) < \zeta SL_n(Q)$. From now on, we fix $\sigma \in \text{Gal}(\mathbb{Q}/Q)$. Since $\text{Aut}(SL_n(\mathbb{Q})) = \text{Inn}(SL_n(\mathbb{Q})) \rtimes \langle \omega \rangle$ (see Theorem 2.8 in [PR94]), $\zeta(\sigma)$ is either inner or the composition of an inner automorphism with $\omega$. 
Suppose that $\zeta(\sigma) = \text{Int}(A_\sigma)$ for some $A_\sigma \in \text{PSL}(n, \overline{Q})$. Then, by definition, all $M \in \xi\text{SL}_2(\overline{Q})$ satisfy

$$
\tau_n(T_\sigma \sigma(M) T_\sigma^{-1}) = \tau_n(M) = A_\sigma \sigma(\tau_n(M)) A_\sigma^{-1}
$$

$$
\implies A_\sigma \tau_n(T_\sigma^{-1}) \tau_n(M) = \tau_n(M) A_\sigma \tau_n(T_\sigma)^{-1}
$$
since $\tau_n \circ \sigma = \sigma \circ \tau_n$. Hence $A_\sigma \tau_n(T_\sigma)^{-1}$ commutes with $\tau_n(\xi\text{SL}_2(\overline{Q}))$ thus with its Zariski-closure which is $\tau_n(\text{SL}_2(\overline{Q}))$. Schur’s lemma (see Lemma 1.7 in [FH91]\(^{13}\)) shows that $A_\sigma = \tau_n(T_\sigma)$. It follows that if $\zeta$ is inner, it is of the form of equation (1).

Suppose that $\zeta(\sigma) = \text{Int}(A_\sigma) \circ \omega$ for some fix $\sigma$ and some $A_\sigma \in \text{PSL}(n, \overline{Q})$. Then for all $M \in \xi\text{SL}_2(\overline{Q})$

$$
\tau_n(T_\sigma \sigma(M) T_\sigma^{-1}) = \tau_n(M) = A_\sigma \omega(\sigma \circ \tau_n(M)) A_\sigma^{-1}
$$

$$
\implies \tau_n(T_\sigma \sigma(M) T_\sigma^{-1}) = \tau_n(M) = A_\sigma J_n \sigma(\tau_n(M)) J_n^{-1} A_\sigma^{-1}
$$

$$
\implies A_\sigma J_n \tau_n(T_\sigma)^{-1} \tau_n(M) = \tau_n(M) A_\sigma J_n \tau_n(T_\sigma)^{-1}.
$$

By the argument as above we get $A_\sigma = \tau_n(T_\sigma) J_n^{-1}$.

Since $J_n$ is in $\text{PGL}(n, \overline{Q})$ and $T_\sigma$ has a representative in $\text{PSL}(2, \overline{Q})$, for all $\sigma$ the automorphism $\zeta(\sigma)$ is fixed under the action of $\text{Gal}(\overline{Q}/Q)$. Applying the 1-cocycle formula we see that $\zeta(\sigma \tau) = \zeta(\sigma)\zeta(\tau)$ for all $\sigma, \tau \in \text{Gal}(\overline{Q}/Q)$ which means that $\zeta$ is group homomorphism. Suppose $\zeta$ is not inner. Then $W = \zeta^{-1}((\text{Im}(\text{SL}_n(\overline{Q})))$ is of index 2 in $\text{Gal}(\overline{Q}/Q)$. It is closed and open because $\zeta$ is continuous. Thus its fixed field is a quadratic extension of $\overline{Q}$ which we denote $\overline{Q}(\sqrt{d})$ for some $d \in \mathbb{Z}$. Finally we have shown that $\zeta$ is of the form of equation (2).

Reciprocally, all maps defined by (1) or (2) are 1-cocyles as shown by computation using the fact that $\tau_n(T_\sigma)$ and $J_n$ commute. \(\square\)

### 2.4. Determining the $\overline{Q}$-form associated to a $\tau_n$-compatible cocycle.

We now explain how to determine the $\overline{Q}/Q$-form $\xi\text{SL}_n$ for a $\tau_n$-compatible 1-cocyle $\xi$.

Let $k$ be a field. Wedderburn’s Theorem (see [MR03] Theorem 2.9.6) states that every central simple algebra over $k$ is isomorphic to $M_n(D)$ for some $n$ and some division algebra $D$. We say that two central simple algebras $A_1$ and $A_2$ over $k$ are equivalent if $A_1 \simeq M_n(D)$ and $A_2 \simeq M_m(D)$ for a division algebra $D$ over $k$. The set of equivalence classes of central simple algebras over $k$ is called the Brauer group denoted $\text{Br}(k)$. It has a group structure induced by the tensor product, the inverse of a given central simple algebra being its opposite algebra.

Suppose $k$ is an algebraic number field and consider $\overline{Q}$ endowed with the discrete topology. A factor set or a 2-cocyle is a map $f : \text{Gal}(\overline{Q}/k) \times \text{Gal}(\overline{Q}/k) \to \overline{Q}$ that satisfies

$$
\sigma_1 f(\sigma_2, \sigma_3) f(\sigma_1 \sigma_2, \sigma_3)^{-1} f(\sigma_1, \sigma_2 \sigma_3) f(\sigma_1, \sigma_2)^{-1} = 1
$$

\(^{13}\)Lemma 1.7 of [FH91] is true even if the group is not finite.
for any $\sigma_1, \sigma_2, \sigma_3 \in \text{Gal}(\overline{Q}/k)$. We will often denote $f_{\sigma, \tau}$ for $f(\sigma, \tau)$. The abelian group structure of $\overline{Q}^*$ induces a group structure on the set of factor sets. A factor set $f$ is said to be trivial if there exists $\phi : \text{Gal}(\overline{Q}/k) \to \overline{Q}^*$ such that

$$f(\sigma_1, \sigma_2) = \phi(\sigma_1) \phi(\sigma_2)^{-1} \phi(\sigma_1)^{-1}$$

for all $\sigma_1, \sigma_2 \in \text{Gal}(\overline{Q}/k)$. We denote $f_{\sigma, \tau} = f(\sigma, \tau)$. The abelian group structure of $\overline{Q}^*$ induces a group structure on the set of factor sets. A factor set $f$ is said to be trivial if there exists $\phi : \text{Gal}(\overline{Q}/k) \to \overline{Q}^*$ such that $f(\sigma_1, \sigma_2) = \phi(\sigma_1) \phi(\sigma_2)^{-1} \phi(\sigma_1)^{-1}$ for all $\sigma_1, \sigma_2 \in \text{Gal}(\overline{Q}/k)$. We denote by $H_2(\text{Gal}(\overline{Q}/k), \overline{Q}^*)$ the group of continuous factor sets modulo the subgroup consisting of trivial factor sets. Since $\overline{Q}^*$ is abelian, this is traditional group cohomology.

From an equivalence class $[(a_{\sigma, \tau})_{\sigma, \tau}] \in H_2(\text{Gal}(\overline{Q}/k), \overline{Q}^*)$ we construct an associated central simple algebra as follows. Since $H_2(\text{Gal}(\overline{Q}/k), \overline{Q}^*) = \lim\limits_{\rightarrow} H_2(\text{Gal}(K/k), K^*)$ where $K$ runs through all finite Galois extensions of $k$, there exists such a $K$ and a class of factor set $[(b_{\sigma, \tau})_{\sigma, \tau}] \in H_2(\text{Gal}(K/k), K^*)$ such that $[(a_{\sigma, \tau})_{\sigma, \tau}]$ is the image of $[(b_{\sigma, \tau})_{\sigma, \tau}]$. We then consider $(K, k, (b_{\sigma, \tau})_{\sigma, \tau})$ the central simple algebra over $k$ whose underlying vector space is $\bigoplus_{\sigma \in \text{Gal}(K/k)} K v_\sigma$ and multiplication defined by

$$(x v_\sigma)(y v_\tau) = x \sigma(y) b_{\sigma, \tau} v_{\sigma \tau}$$

for all $\sigma, \tau \in \text{Gal}(K, k)$.

The equivalence class of central simple algebras corresponding to $[(a_{\sigma, \tau})_{\sigma, \tau}]$ in the Brauer group is the class of $(K, k, (b_{\sigma, \tau})_{\sigma, \tau})$. It follows that (see Theorem 4.4.7 in [GS17]):

**Proposition 2.7.** The map that associates an equivalence class of central simple algebras to a class of factor set induces an isomorphism

$$H^2(\text{Gal}(\overline{Q}/k), \overline{Q}^*) \cong \text{Br}(k).$$

The exact sequence

$$1 \to \overline{Q}^* \to \text{GL}_n(\overline{Q}) \overset{\pi}{\to} \text{PSL}_n(\overline{Q}) \to 1$$

gives rise to a map

$$\delta : H^1(\text{Gal}(\overline{Q}/k), \text{PSL}_n(\overline{Q})) \to H^2(\text{Gal}(\overline{Q}/k), \overline{Q}^*) \cong \text{Br}(k)$$

such that if $[\zeta] \in H^1(\text{Gal}(\overline{Q}/k), \text{PSL}_n(\overline{Q}))$ gives rise to the inner $\overline{Q}/k$-form $^{14}$

$$H(K) = \text{SL}(n, D \otimes_k K)$$

for any field extension $K$ of $k$ and with $D$ a central simple algebra over $k$, then $\delta(\zeta) = D$. We can calculate $\delta([\zeta])$ explicitly. First for every $\sigma \in \text{Gal}(\overline{Q}/k)$ choose an element $M_\sigma \in \text{GL}_n(\overline{Q})$ such that $\pi(M_\sigma) = \zeta(\sigma)$. Then define $a_{\sigma, \tau} = M_\sigma \sigma(M_\tau) M_\tau^{-1} \in \overline{Q}^*$ for every $\sigma, \tau \in \text{Gal}(\overline{Q}/k)$. This is a factor set. Under the isomorphism of Proposition 2.7, it corresponds to the equivalence class of $D$. 

$^{14}$All inner $\overline{Q}/k$-forms of $\text{SL}_n$ are of this form (see Proposition 2.17 in [PR94]).
If $\xi : \text{Gal}(\overline{Q}/Q) \to \text{PSL}(2, \overline{Q})$ is a cocycle we denote by
$$\xi M_2(Q) = \{x \in M_2(\overline{Q}) \mid \xi(\sigma)(\sigma(x))\xi(\sigma)^{-1} = x\}.$$  
It is a quaternion algebra over $Q$.

**Proposition 2.8.** Let $\xi : \text{Gal}(\overline{Q}/Q) \to \text{Aut}(\text{SL}_2(\overline{Q}))$ be a 1-cocycle such that $\xi \text{SL}_2(R) \simeq \text{SL}(2, R)$. Let $\xi : \text{Gal}(\overline{Q}/Q) \to \text{Aut}(\text{SL}_n(\overline{Q}))$ be a 1-cocycle $\tau_n$-compatible with $\xi$ such that $\xi \text{SL}_n(R) \simeq \text{SL}(n, R)$. If $\xi$ is inner, then
$$\xi \text{SL}_n(Q) \simeq \begin{cases} 
\text{SL}(n, Q) & \text{if } n \text{ is odd} \\
\text{SL}(\frac{n}{2}, \xi M_2(Q)) & \text{if } n \text{ is even}.
\end{cases}$$

If $\xi$ is not inner, denote $Q(\sqrt{d})$ the associated quadratic extension. Then
$$\xi \text{SL}_n(Q) \simeq \begin{cases} 
\text{SU}(I_n, \sigma; Q(\sqrt{d})) & \text{if } n \text{ is odd} \\
\text{SU}(I_\frac{n}{2}, -\psi; \xi M_2(Q) \otimes Q(\sqrt{d})) & \text{if } n \text{ is even}.
\end{cases}$$

**Proof.** Let $k$ denote either $Q$ or $Q(\sqrt{d})$ depending on wether $\xi$ is inner or not and $\xi_k$ the restriction of $\xi$ to $\text{Gal}(\overline{Q}/Q)$. Restricted to $\text{Gal}(\overline{Q}/k)$, $\xi$ has values in $\text{PSL}(2, \overline{Q})$. For every $\sigma \in \text{Gal}(\overline{Q}/k)$ choose $M_{\sigma} \in \text{GL}_2(\overline{Q})$ such that $\pi(M_{\sigma}) = \xi_k(\sigma)$. Define $a_{\sigma, \tau} = M_{\sigma} \sigma(M_{\tau}) M_{\sigma}^{-1}$ for every $\sigma, \tau \in \text{Gal}(\overline{Q}/k)$. Note that $\xi_k(\sigma) = \tau_n(\xi_k(\sigma))$ for all $\sigma$. Then
$$\xi_k(\sigma) = \tau_n(\xi_k(\sigma)) = \tau_n(\pi(M_{\sigma})) = \pi(\tau_n(M_{\sigma})).$$

for every $\sigma \in \text{Gal}(\overline{Q}/k)$. Thus
$$\tau_n(M_{\sigma})\sigma(\tau_n(M_{\tau}))\tau_n(M_{\sigma}^{-1}) = \tau_n(M_{\sigma})\sigma(M_{\tau}) M_{\sigma}^{-1}$$
for every $\sigma, \tau \in \text{Gal}(\overline{Q}/k)$ so $\delta(\xi)$ is the 2-cocycle given by $(a_{\sigma, \tau})^{n-1}$. Define
$$\psi : \text{Gal}(\overline{Q}/k) \to \overline{Q}^*$$
$$\sigma \mapsto \text{Det}(M_{\sigma})$$
and note that
$$\psi(\sigma)\sigma(\psi(\tau))\psi(\sigma^{-1})^{-1} = \text{Det}(M_{\sigma})\sigma(\text{Det}(M_{\tau})) \text{Det}(M_{\sigma\tau})^{-1}$$
$$= \text{Det}(M_{\sigma}) \sigma(M_{\tau}) M_{\sigma\tau}^{-1})$$
$$= \text{Det}(a_{\sigma, \tau} I_2) = (a_{\sigma, \tau})^2$$
for all $\sigma, \tau \in \text{Gal}(\overline{Q}/k)$. If $n$ is odd then $(a_{\sigma, \tau})^{n-1} = \psi(\sigma)^l \sigma(\psi(\tau))^l \psi(\sigma^{-1})^{-l}$ where $l = \frac{n-1}{2}$ so $(a_{\sigma, \tau})_{\sigma, \tau}$ is actually trivial in $H^2(\text{Gal}(\overline{Q}/k, \overline{Q}))$. If $n$ is even then $(a_{\sigma, \tau})^{n-1} = \psi(\sigma)^l \sigma(\psi(\tau))^l \psi(\sigma^{-1})^{-l} a_{\sigma, \tau}$ where $l = \frac{n-2}{2}$ so $(a_{\sigma, \tau})^{n-1}$ is equivalent to $a_{\sigma, \tau}$ in $H^2(\text{Gal}(\overline{Q}/Q, \overline{Q}))$. Thus we have determined $\xi_k \text{SL}_n(k)$.

It remains to compute $\xi \text{SL}_n(Q)$ in the case where $\xi$ is not inner. Note that since $\xi \text{SL}_n(R) \simeq \text{SL}_n(R)$ we must have $d > 0$. For all $\sigma \in \text{Gal}(\overline{Q}/Q(\sqrt{d}))$ we have $\xi(\sigma) \circ \sigma(M) = M$. This means that $M \in \xi_{Q(\sqrt{d})} \text{SL}_n(Q(\sqrt{d}))$. Using Lemma 2.3 there exists $a, b \in \mathbb{N}$ and $P \in \text{SL}(2, \overline{Q})$ such that $\xi(\sigma) = \ldots$
\[ \text{Int}(P^{-1} T_{\sigma, b}^a \sigma(P)) \] for every \( \sigma \in \text{Gal}(\bar{Q}/\mathbb{Q}) \). Up to conjugation we can assume that \( P = I_2 \).

For any \( \sigma \in \text{Gal}(\bar{Q}/\mathbb{Q}) \) such that \( \sigma(\sqrt{d}) = -\sqrt{d} \) the equations becomes
\[
\sigma(M)^\top J_n \tau_n(T_{\sigma, b}^a)^{-1} M = J_n \tau_n(T_{\sigma, b}^a)^{-1}
\]
It remains to understand the Hermitian form \( J_n \tau_n(T_{\sigma, b}^a)^{-1} \).

Let \( n \) be odd. By Proposition 1.1 \( \sigma \)-Hermitian forms over \( \mathbb{Q}(\sqrt{d}) \) are classified by its rank and its discriminant. Direct computations show that \( J_n \tau_n(T_{\sigma, b}^a)^{-1} \) has rank \( n \) and its discriminant is a square or minus a square. Up to multiplying the matrix by \(-1\), which does not affect the unitary group, we see that it is congruent to \( I_n \) and the corresponding unitary groups are conjugate.

Let \( n \) be even. An element of \( \zeta_{\mathbb{Q}(\sqrt{d})} \text{SL}_n(\mathbb{Q}(\sqrt{d})) \) satisfies equation (3) for one automorphism \( \sigma \in \text{Gal}(\bar{Q}/\mathbb{Q}) \) such that \( \sigma(\sqrt{d}) = -\sqrt{d} \) if and only if it satisfies it for all such \( \sigma \). Up to change of basis, we can suppose that neither \( a \) nor \( b \) is a square. Under this assumption, \( \sigma \) induces on \( \zeta_{\mathbb{Q}(\sqrt{d})} \text{M}_2(\mathbb{Q}(\sqrt{d})) \) the anti-involution \(-\otimes \tau\) where \( \tau \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \) is non-trivial and \(-\) is the conjugation of \( \zeta_{\mathbb{Q}(\sqrt{d})} \text{M}_2(\mathbb{Q}(\sqrt{d})) \). From §7 in [Lew82], \(-\otimes \tau\)-Hermitian form over a quaternion algebra over \( \mathbb{Q}(\sqrt{d}) \) is classified by its discriminant and its rank. Again, direct computations show that \( J_n \tau_n(T_{\sigma, b}^a)^{-1} \) has rank \( n \), and its discriminant is a square up to sign. Thus it is congruent to \( I_n \) and the corresponding unitary groups are conjugate. \( \square \)

### 2.5. Proof of Proposition A.1 and A.2.

**Definition 2.9.** Let \( G \) be a linear \( \mathbb{Q} \)-algebraic group. Let \( \zeta : \text{Gal}(\bar{Q}/\mathbb{Q}) \to G(\bar{Q}) \) a continuous 1-cocycle. We denote by \( \zeta \text{G}(\mathbb{Z}) \) any subgroup of \( \zeta \text{G}(\mathbb{Q}) \) commensurable with the \( \mathbb{Z} \)-points of \( \zeta \text{G}(\mathbb{Q}) \).

**Proof of Propositions A.1 and A.2.** If \( \phi : G \to G' \) is a \( \mathbb{Q} \)-homomorphism of algebraic groups over \( \mathbb{Q} \) then for any \( \mathbb{Q} \)-arithmetic subgroup \( \Gamma \) of \( \text{G}(\mathbb{Q}) \), \( \phi(\Gamma) \) is contained in an \( \mathbb{Q} \)-arithmetic subgroup of \( G'(\mathbb{Q}) \) (see Milne Proposition 5.2 of Appendix A [Mil13]). It implies that if \( \xi \) and \( \zeta \) are 1-cocyles which are \( \tau_n \)-compatible, any \( \mathbb{Q} \)-arithmetic subgroup of \( \zeta \text{SL}_2(\mathbb{Q}) \) has image under \( \tau_n \) which lies in a \( \mathbb{Q} \)-arithmetic subgroup of \( \zeta \text{SL}_n(\mathbb{Q}) \) with \( \zeta \) a 1-cocycle \( \tau_n \)-compatible with \( \xi \). The \( \mathbb{Q} \)-arithmetic subgroups of \( \zeta \text{SL}_n(\mathbb{Q}) \) are commensurable with groups isomorphic to

- \( \text{SL}(n, \mathbb{Z}) \) if \( n \) is odd and \( \zeta \) is inner
- \( \text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}]) \) if \( n \) is odd and \( \zeta \) is not inner
- \( \text{SL}(\frac{n}{2}, \mathcal{O}) \) if \( n \) is even and \( \zeta \) is inner

\[^{15}\text{Note that } \text{M}_2(\mathbb{Q}) \simeq (5, 5)_Q.\]
\begin{itemize}
\item $\text{SU}(I_2, - \otimes \sigma; \mathcal{O} \otimes \mathbb{Z}[\sqrt{d}])$ if $n$ is even, $\zeta$ is not inner and $\zeta M_2(\mathbb{Q}) \otimes \mathbb{Q} \mathbb{Q}(\sqrt{d}) \neq M_2(\mathbb{Q}(\sqrt{d}))$
\item $\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$ if $n$ is even, $\zeta$ is not inner and $\zeta M_2(\mathbb{Q}) \otimes \mathbb{Q}(\sqrt{d}) \cong M_2(\mathbb{Q}(\sqrt{d}))$
\end{itemize}

$\mathcal{O}$ is an order of the quaternion algebra $\zeta M_2(\mathbb{Q})$ and $d \in \mathbb{N}$ square free. We get arithmetic subgroups of $\text{SL}(n, \mathbb{R})$ exactly when $\zeta$ is inner or when $\zeta$ is not inner and $d > 0$.

Conversely let $\Gamma$ be a $\mathbb{Q}$-arithmetic subgroup of $\text{SL}(2, \mathbb{R})$. Since every $\mathbb{R}/\mathbb{Q}$-form of $\text{SL}_2$ is a $\overline{\mathbb{Q}}/\mathbb{Q}$-form of $\text{SL}_2$ (see Proposition 2 of III.§1 in [Ser02]) there exists

$$\tilde{\xi} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(\text{SL}_2(\mathbb{Q}))$$

a 1-cocycle such that $\tilde{\xi} \text{SL}_2(\mathbb{Z})$ is commensurable with $\Gamma$. Let $\Lambda$ be a $\mathbb{Q}$-arithmetic subgroup of $\text{SL}(n, \mathbb{R})$ and

$$\xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(\text{SL}_n(\overline{\mathbb{Q}}))$$

be a 1-cocycle such that $\xi \text{SL}_n(\mathbb{Z})$ is commensurable with $\Lambda$. We assume that $\tau_n(\Gamma) < \Lambda$ and we want to show that $\tau_n(\xi \text{SL}_2(\mathbb{Q})) < \xi \text{SL}_n(\mathbb{Q})$. For any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote

$$\tau_n^\sigma : \text{SL}_2(\overline{\mathbb{Q}}) \rightarrow \text{SL}_n(\overline{\mathbb{Q}}), \ g \mapsto \xi(\sigma) \circ \sigma \circ \tau_n \circ (\xi(\sigma) \circ \sigma)^{-1}(g).$$

This is an algebraic morphism that coincides with $\tau_n$ on a finite index subgroup of $\xi \text{SL}_2(\mathbb{Z})$. For any $\sigma$ consider

$$(\tau_n, \tau_n^\sigma) : \text{SL}_2(\overline{\mathbb{Q}}) \rightarrow \text{SL}_n(\overline{\mathbb{Q}}) \times \text{SL}_n(\overline{\mathbb{Q}}).$$

Since $\{(g, h) \in \text{SL}_n(\overline{\mathbb{Q}}) \times \text{SL}_n(\overline{\mathbb{Q}}) | h = g\}$ is closed in the product for the Zariski topology, its inverse image by $(\tau_n, \tau_n^\sigma)$ has to be closed in $\text{SL}_2(\overline{\mathbb{Q}})$ so contains the Zariski-closure of $\xi \text{SL}_2(\mathbb{Z})$. Since any finite index subgroup of $\text{SL}_2(\mathbb{Z})$ is Zariski-dense in $\text{SL}_2(\overline{\mathbb{Q}})$, we conclude that $\tau_n$ and $\tau_n^\sigma$ coincide on $\text{SL}_2(\overline{\mathbb{Q}})$.

For every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $g \in \xi \text{SL}_2(\mathbb{Q})$ we have $\tau_n(g) = \tau_n^\sigma(g)$. It means that $\tau_n(\xi \text{SL}_2(\mathbb{Q})) < \xi \text{SL}_n(\mathbb{Q})$. Thus $\xi$ and $\zeta$ are $\tau_n$-compatible and Proposition 2.8 concludes the proof.

\section{R/\Q-forms of Other Split Real Lie Groups}

In this section we establish analogues of Propositions A.1 and A.2 in the cases of $\text{Sp}(2n, \mathbb{R})$, $\text{SO}(J_n, \mathbb{R})$ and $G_2$.

\subsection{R/\Q-forms of $\text{Sp}(2n, \mathbb{R})$}

Let $n \geq 2$.

\begin{lemma}
Let $\tilde{\xi} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(\text{SL}_2(\overline{\mathbb{Q}}))$ be a 1-cocycle such that $\tilde{\xi} \text{SL}_2(\mathbb{R}) \cong \text{SL}(2, \mathbb{R})$. Let $\xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(\text{Sp}_{2n}(\overline{\mathbb{Q}}))$ be a 1-cocycle $\tau_{2n}$-compatible with $\xi$. Then

$$\xi \text{Sp}_{2n}(\mathbb{Q}) \cong \text{SU}(I_{2n}, - \xi M_2(\mathbb{Q})).$$
\end{lemma}
Proof. Using the embedding 
\[ \text{Aut}(\text{Sp}_{2n}(\overline{Q})) \simeq \text{PSp}(2n, \overline{Q}) \hookrightarrow \text{Aut}(\text{SL}_{2n}(\overline{Q})) \]
we see that \( \zeta \text{Sp}_{2n}(Q) < \zeta \text{SL}_{2n}(Q) \). All \( \overline{Q}/Q \)-forms of \( \text{Sp}_{2n} \) are isomorphic to \( \text{Sp}_{2n} \) or to \( \text{H}(K) = \text{SU}(I_n, A \otimes_Q K) \) for any field extension \( K \) of \( Q \) and with a quaternion division algebra \( A \) over \( Q \) (since there is only one non-degenerate \( - \)-Hermitian form on \( A \), see Proposition 1.2). There can only be one 1-cocycle \( \tau_{2n} \)-compatible with \( \xi \) since all automorphisms of \( \text{Sp}_{2n}(\overline{Q}) \) are inner (see Proposition 2.6). Using Proposition 2.8 we see that the only possibility is that \( A \simeq \xi \text{M}_2(Q) \). \( \square \)

Proposition 3.2. Let \( \Gamma \) be an arithmetic subgroup of \( \text{SL}(2, \mathbb{R}) \). Suppose \( \Gamma \) is commensurable with \( \mathcal{O}^1 \) where \( \mathcal{O} \) is an order of a quaternion division algebra over \( Q \).

Then \( \tau_{2n}(\Gamma) \) lies in a subgroup of \( \text{Sp}(2n, \mathbb{R}) \) commensurable with a conjugate of \( \text{SU}(I_n, -; \mathcal{O}) \).

Furthermore this is the only \( Q \)-arithmetic subgroup of \( \text{Sp}(2n, \mathbb{R}) \) that contains \( \tau_n(\Gamma) \) up to commensurability.

Proof. Thanks to Proposition 5.2 of Appendix A in Milne [Mil13] \( \mathcal{O} \)-arithmetic subgroups of \( \zeta \text{SL}_2(Q) \) have image under \( \tau_{2n} \) which lies in a \( \mathcal{O} \)-arithmetic subgroup of \( \zeta \text{Sp}_{2n}(\overline{Q}) \) with \( \zeta \) a 1-cocycle \( \tau_{2n} \)-compatible with \( \xi \). Lemma 3.1 shows in which arithmetic subgroup \( \tau_n(\Gamma) \) lie in. They are all arithmetic subgroups of \( \text{Sp}(J_{2n}, \mathbb{R}) \).

The converse is proven the same way as in the proof of Proposition A.1 and A.2. \( \square \)

3.2. \( \mathbb{R}/Q \)-forms of \( \text{SO}(k + 1, k) \). Let \( n = 2k + 1 \geq 3 \) be odd. Theorem 2.8 of [PR94] implies that \( \text{Aut}(\text{SO}(J_n, \overline{Q})) \simeq \text{SO}(J_n, \overline{Q}) \). Hence isomorphism classes of \( \overline{Q}/Q \)-forms of \( \text{SO}(J_n) \) are classified by \( H^1(\text{Gal}(\overline{Q}/Q), \text{SO}(J_n, \overline{Q})) \), as explained in Section 2. Proposition 2.8 in [PR94] states that elements of \( H^1(\text{Gal}(\overline{Q}/Q), \text{SO}(J_n, \overline{Q})) \) are in one-to-one correspondence with \( \mathbb{Q} \)-equivalence classes of quadratic forms over \( \mathbb{Q}^n \) that have discriminant 1.

The correspondence works as follows. If \( B \in \text{SL}(n, \mathbb{Q}) \) is a symmetric matrix, then the associated equivalence class of 1-cocycle is the one defined by the \( \overline{Q}/Q \)-form \( \text{SO}(B) \). Reciprocally let \( \xi \) is a 1-cocycle. The embedding \( \text{SO}(J_n, \overline{Q}) \hookrightarrow \text{GL}(n, \overline{Q}) \) induces a map

\[ H^1(\text{Gal}(\overline{Q}/Q), \text{SO}(J_n, \overline{Q})) \xrightarrow{\phi} H^1(\text{Gal}(\overline{Q}/Q), \text{GL}(n, \overline{Q})). \]

The latter is trivial by Hilbert’s Theorem 90 (see Lemma 2.2 in [PR94]). Thus there exists \( S \in \text{GL}(n, \overline{Q}) \) such that for all \( \sigma \in \text{Gal}(\overline{Q}/Q) \) we have \( \phi(\xi)(\sigma) = S^{-1} \sigma(S) \). The associated symmetric matrix is \( S^{-1} J_n S^{-1} \).

Here is the way to determine the matrix \( S \) (it comes from an examination of the proof of Hilbert’s Theorem 90). Since \( \xi \in \varprojlim \bigcap H^1(\text{Gal}(K/Q), \text{SO}(J_n, K)) \),

\[ \text{SO}(J_n, \overline{Q}) = \{ M \in \text{SL}(n, \overline{Q}) \mid M^T J_n M = J_n \}. \]
where $K$ runs through all finite extensions of $Q$, there exists such a $K$ with $ξ ∈ H^1(\text{Gal}(K/Q), \text{SO}(J_n, K))$. Define

$$f : K^n → K^n, \ x ↦ \sum_{σ ∈ \text{Gal}(K/Q)} ξ(σ)x.$$

Pick $v_1, ..., v_n ∈ K^n$ such that $(f(v_1), ..., f(v_n))$ is a basis of $K^n$. Then $S = (f(v_1)|...|f(v_n))^{-1}$.

Quadratic forms over $Q$ are classified up to equivalence by global and local invariants. The main reference for this subject is Serre’s book [Ser73]. If $q(x) = \sum a_ix_i^2$ is a non-degenerate quadratic form over $Q$ and $p$ is a prime number, define its Hasse invariant in $Q_p$ as

$$\mathcal{E}_p(q) = \bigotimes_{i<j}(a_i, a_j)_{Q_p} ∈ \text{Br}(Q_p).$$

The Hass-Minkowski Theorem (Theorem 9 in [Ser73]) now states that two quadratic forms over $Q$ are equivalent if and only if they have the same discriminant, the same signature and the same Hasse invariant in $Q_p$ for all primes $p$.

We can compute the invariants of $J_n$. Its discriminant is a square. Its signature is $(k + 1, k)$ if $n ≡ 1[4]$ and $(k, k + 1)$ if $n ≡ 3[4]$. Using the fact that

$$\mathcal{E}(q ⊥ q') = \mathcal{E}_p(q) ⊗ \mathcal{E}_p(q') ⊗ (\text{disc}(q), \text{disc}(q'))_{Q_p}$$

for all primes $p$, we can show that $\mathcal{E}_p(J_n) = 1$ for $p \neq 2$ and that $\mathcal{E}_2(J_n) = 1$ if $n ≡ ±1[8]$ and $\mathcal{E}_2(J_n) = (-1, -1)_{Q_2}$ if $n ≡ ±3[8]$.

**Lemma 3.3.** Let $ξ : \text{Gal}(\overline{Q}/Q) → \text{Aut}(\text{SL}_2(\overline{Q}))$ be a 1-cocycle such that $ξ(σ) = \text{Int}(P^{-1}T^a^bσ(P))$ for all $σ$ with $a, b ∈ N$. Let $n = 2k + 1 ≥ 3$ be odd and $ξ : \text{Gal}(\overline{Q}/Q) → \text{Aut}(\text{SO}(J_n, \overline{Q}))$ be a 1-cocycle $τ_n$-compatible with $ξ$. Then

$$ξ_{\text{SO}(J_n)}(Q) ≃ \begin{cases} \text{SO}(J_n, Q) & \text{if } n ≡ ±1[8] \\ \text{SO}(Q, Q) & \text{if } n ≡ ±3[8] \end{cases}$$

where $Q ∈ \text{SL}(n, Q)$ is a symmetric matrix of signature equal to the signature of $J_n$ and of Hasse invariant $\mathcal{E}_p(Q) = (a, b)_{Q_p} ⊗ (-1, -1)_{Q_p}$ for all primes $p$.

**Proof.** Using

$$\text{Aut}(\text{SO}(J_n, \overline{Q})) ≃ \text{SO}(J_n, \overline{Q}) \hookrightarrow \text{PSL}(n, \overline{Q})$$

we consider $i∗ξ$ as 1-cocycle with value in $\text{Aut}(\text{SL}_n(\overline{Q}))$ which is $τ_n$-compatible with $ξ$. Proposition 2.6 implies that for all $σ$ the automorphism

$$i(ξ(σ)) = \text{Int}(τ_n(P^{-1}T^a^bσ(P))).$$

Let $\tilde{T}^a^bσ$ be a lift of $T^a^bσ$ to $\text{SL}_2(\overline{Q})$. Then $ξ(σ) = τ_n(P^{-1}T^a^bσ(P))$ for all $σ ∈ \text{Gal}(\overline{Q}/Q)$. Up to conjugation, we can assume that $P = I_2$. 
Let $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Then $\zeta \in H^1(\text{Gal}(K/\mathbb{Q}), \text{SO}(J_n, K))$. We define

$$f : K^n \to K^n, \quad x \mapsto \sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} T^{a,b}_\sigma \sigma(x).$$

If $K$ is a degree 2 extension of $\mathbb{Q}$ let $\sigma \in \text{Gal}(K/\mathbb{Q})$ be the non-trivial element. If $K$ is a degree 4 extension of $\mathbb{Q}$, let $\sigma \in \text{Gal}(K/\mathbb{Q})$ be such that $\sigma(\sqrt{a}) = -\sqrt{a}$ and $\sigma(\sqrt{b}) = \sqrt{b}$ and let $\tau \in \text{Gal}(K/\mathbb{Q})$ be such that $\tau(\sqrt{a}) = \sqrt{a}$ and $\tau(\sqrt{b}) = -\sqrt{b}$. We note $(e_1, ..., e_n)$ the canonical basis of $K^n$.

Suppose $n \equiv 1 \pmod{4}$. If $K = \mathbb{Q}$ then the symmetric matrix associated to $\zeta$ is $J_n$.

We suppose that $K$ is a degree 2 extension of $\mathbb{Q}$. If $a$ or $b$ is a square, then $(a, b)_\mathbb{Q} = 1$ and up to change of basis we can assume both of them is a square. This has been treated in the previous case. Suppose that $a$ and $b$ are not squares. We have

$$f : K^n \to K^n, \quad x \mapsto x + \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \sigma(x).$$

We then define for all $1 \leq i \leq \frac{k}{2}$

$$v_{2i-1} = \frac{e_{2i-1} + e_{2k-2i+3}}{2}, \quad v_{2i} = \frac{e_{2i} - e_{2k-2i+2}}{2},$$

$$v_{2k-2i+2} = \frac{\sqrt{ae_{2i}} + \sqrt{ae_{2k-2i+2}}}{2}, \quad v_{2k-2i+3} = \frac{\sqrt{ae_{2i-1}} - \sqrt{ae_{2k-2i+3}}}{2}$$

and $v_{k+1} = \frac{e_{k+1}}{2}$. Denote

$$S^{-1} = (f(v_1), ..., f(v_n)) = \begin{pmatrix} 1 & \sqrt{a} & \cdots & \cdots & \cdots & \cdots & \sqrt{a} \\ \sqrt{a} & 1 & \cdots & \cdots & \cdots & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \sqrt{a} \end{pmatrix}.$$  

The symmetric matrix associated to $\zeta$ is $S^{-\top} J_n S^{-1} =

$$
\begin{pmatrix}
2(n-1)! \\
2(n-2)! \\
\vdots \\
(k!k)! \\
\vdots \\
-2a(n-2)! \\
-2a(n-1)!
\end{pmatrix}
$$

which has for Hasse invariant for any prime $p$

$$(-1, -1)_{Q_p}^{\frac{n-1}{2}} \otimes \bigotimes_{j=1,3,...,k-1} (a, -j(n-j))_{Q_p} \simeq (-1, -1)_{Q_p}^{\frac{n-1}{2}} \otimes (a, (-1)^{n-1})_{Q_p}.$$
since $\prod_{j=1,3,..,k-1} j(n-j)$ is a square as can be shown by induction. Thus if $n \equiv 1[8]$ then $S^{-\top} J_n S^{-1}$ is equivalent to $J_n$. If $n \equiv -3[8]$ then $S^{-\top} J_n S^{-1}$ has for Hasse invariant $(-1, -1)_{\mathbb{Q}_p} \otimes (a, a)_{\mathbb{Q}_p}$ for all primes $p$.

We suppose that $K$ is a degree 4 extension of $\mathbb{Q}$. Then $f : K^n \to K^n$

$$x \mapsto x + \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \tau(x) + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \sigma(x) + \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ -1 \end{pmatrix} \sigma(\tau(x)).$$

For all $1 \leq i \leq k$ we let

$$v_{2i-1} = \frac{\sqrt{a} e_{2i-1}}{2}, \quad v_{2i} = \frac{\sqrt{b} e_{2i}}{2},$$

$$v_{2k-2i+2} = -\frac{\sqrt{ab}}{2} e_{2k-2i+2}, \quad v_{2k-2i+3} = \frac{e_{2k-2i+3}}{2}$$

and $v_{k+1} = \frac{e_{k+1}}{4}$. Denote

$$S^{-1} = (f(v_1), ..., f(v_n)) = \begin{pmatrix} \sqrt{a} & \sqrt{b} & \cdots & \sqrt{ab} \\ \sqrt{b} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -\sqrt{a} & \cdots & \cdots & 1 \end{pmatrix}.$$
We then define for all $1 \leq i \leq \frac{k+1}{2}$

$$v_{2i-1} = e_{2i-1}, \quad v_{2i} = e_{2i},$$

$$v_{2k-2i+2} = \sqrt{ae_{2k-2i+2}}, \quad v_{2k-2i+3} = -\sqrt{ae_{2k-2i+3}}$$

and $v_{k+1} = \frac{\sqrt{ae_{k+1}}}{2}$. Denote

$$S^{-1} = (f(v_1), \ldots, f(v_n)) = \begin{pmatrix} 1 & \sqrt{a} & \sqrt{a} \\ 1 & \sqrt{a} & \sqrt{a} \\ \vdots & \vdots & \vdots \\ -1 & \sqrt{a} & -\sqrt{a} \\ 1 & \sqrt{a} & -\sqrt{a} \end{pmatrix}.$$

The symmetric matrix associated to $\zeta$ is $S^{-\top} J_n S^{-1} =$

$$\begin{pmatrix} 2(n-1)! \\ 2(n-2)! \\ \vdots \\ -ak!k! \\ \vdots \\ -2a(n-2)! \\ -2a(n-1)! \end{pmatrix}$$

which has Hasse invariant

$$(-1, -1)^{n+1}_{Q_p} \otimes (a, -2a)_{Q_p} \otimes \prod_{j=1,3,\ldots,k} (a, -j(n-j))_{Q_p} \simeq (-1, -1)^{n+1}_{Q_p} \otimes (a, (-1)^{n-3}a)_{Q_p}$$

for all primes $p$ since $2 \prod_{j=1,3,\ldots,k} j(n-j)$ is a square as can be shown by induction. Thus if $n \equiv -1[8]$ then $S^{-\top} J_n S^{-1}$ is equivalent to $J_n$. If $n \equiv 3[8]$ then $S^{-\top} J_n S^{-1}$ has Hasse invariant $(-1, -1)_{Q_p} \otimes (a, a)_{Q_p}$.

We suppose that $K$ is a degree 4 extension of $Q$. Then $f : K^n \rightarrow K^n$

$$x \mapsto x + \begin{pmatrix} -1 \\ \vdots \\ 1 \end{pmatrix} \sigma(x) + \begin{pmatrix} 1 \\ \vdots \\ -1 \end{pmatrix} \tau(x).$$

We then define for all $1 \leq i \leq \frac{k+1}{2}$

$$v_{2i-1} = \frac{\sqrt{be_{2i-1}}}{2}, \quad v_{2i} = \frac{e_{2i}}{2},$$

$$v_{2k-2i+2} = \frac{\sqrt{ae_{2k-2i+2}}}{2}, \quad v_{2k-2i+3} = \frac{\sqrt{be_{2k-2i+3}}}{2}$$
and \( v_{k+1} = \frac{\sqrt{a} v_{k+1}}{4} \). Denote
\[
S^{-1} = (f(v_1), \ldots, f(v_n)) = \begin{pmatrix}
\sqrt{b} & 1 & \sqrt{ab} \\
& \ddots & \ddots \\
& & \sqrt{a} \\
-\sqrt{b} & \sqrt{a} & \ddots & \ddots \\
& & & \sqrt{a} & \sqrt{ab}
\end{pmatrix}.
\]
The symmetric matrix associated to \( \zeta \) is \( S^{-\top} J_n S^{-1} = \)
\[
\begin{pmatrix}
-2b(n-1)! & 2(n-2)! & \cdots & \cdots & -a(n-2)! & 2ab(n-1)!
\end{pmatrix}
\]
which has Hasse invariant
\[
(-1, -1)_{Q_p}^{\frac{n+1}{2}} \otimes (a, 2)_{Q_p} \otimes \bigotimes_{j=1,3,\ldots,k} (a, bj(n-j))_{Q_p} \simeq (-1, -1)_{Q_p}^{\frac{n+1}{2}} \otimes (a, b^{\frac{n+1}{2}})_{Q_p}
\]
for all primes \( p \). Thus if \( n \equiv -1[8] \), \( S^{-\top} J_n S^{-1} \) is equivalent to \( J_n \). If \( n \equiv 3[8] \), \( S^{-\top} J_n S^{-1} \) has for Hasse invariant \( (-1, -1)_{Q_p} \otimes (a, b)_{Q_p} \).

**Proposition 3.4.** Let \( \Gamma \) be a \( Q \)-arithmetic subgroup of \( SL(2, R) \) and \( n = 2k + 1 \geq 3 \). Suppose \( \Gamma \) is commensurable with the norm 1 elements of an order of a quaternion algebra \( (a, b)_Q \).

If \( n \equiv \pm 1[8] \), then \( \tau_n(\Gamma) \) lies in a subgroup of \( SO(J_n, R) \) commensurable with a conjugate of \( SO(J_n, Z) \).

If \( n \equiv \pm 3[8] \), then \( \tau_n(\Gamma) \) lies in a subgroup of \( SO(J_n, R) \) commensurable with a conjugate of \( SO(Q, Z) \) for \( Q \in SL(n, Q) \) a symmetric matrix of signature equal to the signature of \( J_n \) and of Hasse invariant \( E_p(Q) = (a, b)_{Q_p} \otimes (-1, -1)_{Q_p} \) for every prime \( p \).

Further more these are the only \( Q \)-arithmetic subgroups of \( SO(J_n, R) \) that contain \( \tau_n(\Gamma) \) up to commensurability.

**Proof.** Thanks to Proposition 5.2. in Milne [Mil13] \( Q \)-arithmetic subgroups of \( SL_2(Q) \) have image under \( \tau_n \) which lies in a \( Q \)-arithmetic subgroup of \( SO(J_n, Q) \) with \( \zeta \) a 1-cocycle \( \tau_n \)-compatible with \( \xi \). Lemma 3.3 shows in which \( Q \)-arithmetic subgroup \( \tau_n(\Gamma) \) lie in. They are all arithmetic subgroups of \( SO(J_n, R) \).

The converse is proven the same way as in the proof of Proposition A.1. \( \Box \)
3.3. \( \mathbb{R}/\mathbb{Q} \)-forms of \( G_2 \). We now give the corresponding result for \( G_2 \).

**Definition 3.5.** An octonion algebra over a field \( F \) of characteristic not equal to 2 is a unital but non associative algebra of dimension 8 over \( F \) such that there exists a nondegenerate quadratic form \( N \) on the algebra satisfying \( N(xy) = N(x)N(y) \) for all \( x, y \). We say that the octonion algebra is split if \( N \) is isotropic.

**Definition 3.6.** Let \( R \) be a ring. Let us denote \( \times : R^7 \times R^7 \rightarrow R^7 \)

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5 \\
    x_6 \\
    x_7
\end{pmatrix},
\begin{pmatrix}
    y_1 \\
    y_2 \\
    y_3 \\
    y_4 \\
    y_5 \\
    y_6 \\
    y_7
\end{pmatrix} \mapsto \begin{pmatrix}
6(x_1y_4 - x_4y_1) - 4(x_2y_3 - x_3y_2) \\
24(x_1y_5 - x_5y_1) - 6(x_2y_4 - x_4y_2) \\
60(x_1y_6 - x_6y_1) - 6(x_3y_4 - x_4y_3) \\
120(x_1y_7 - x_7y_1) + 20(x_2y_6 - x_6y_2) - 8(x_3y_5 - x_5y_3) \\
60(x_2y_7 - x_7y_2) - 6(x_4y_5 - x_5y_4) \\
24(x_3y_7 - x_7y_3) - 6(x_4y_6 - x_6y_4) \\
6(x_4y_7 - x_7y_4) - 4(x_5y_6 - x_6y_5)
\end{pmatrix}.
\]

Define \( G_2(R) = \{ M \in \text{SO}(J_7, R) \mid M(x \times y) = Mx \times My, \forall x, y \in R^7 \} \).

It appears that \( G_2(R) \) is a Lie group isomorphic to the connected centerless real split form of type \( G_2 \). To prove this we should prove that \( G_2(R) \) is the automorphism group of the unique split octonion algebra over \( R \) (see Theorem 1.8.1 in Springer and Veldkamp’s book [SV00]). Denote \( O = R \oplus R^7 \). We define the product on \( O \) by

\[
(t, v)(s, w) = (ts - v^\top J_7 w, tw + sv + v \times w)
\]

for all \( t, s \in R \) and \( v, w \in R^7 \). With this product, \( O \) has a structure of a unital algebra over \( R \). Denote \( N : O \rightarrow R \)

\[
N(t, v) = t^2 + v^\top J_7 v.
\]

This is a nondegenerate quadratic form on \( O \). We can check that

\[
v^\top J_7(v \times w) = 0
\]

and

\[
(v \times w)^\top J_7(v \times w) = (v^\top J_7 v)(w^\top J_7 w) - (v^\top J_7 w)^2
\]

for all \( v, w \in R^7 \). Those two properties together imply that \( N(xy) = N(x)N(y) \). Since \( N \) is isotropic, \( O \) is isomorphic to the unique split octonion algebra over \( R \).

Let us show that \( G_2(R) \cong \text{Aut}(O) \) to conclude that \( G_2(R) \) is a Lie group of type \( G_2 \). Pick \( g \in G_2(R) \) and define its action on \( O \) by \( g \cdot (t, v) = (g(t), g(v)) \) for all \( (t, v) \in O \). Then for all \( x, y \in O \) \( g(xy) = g(x)g(y) \). Thus \( g \) defines an automorphism of \( O \). Suppose \( \phi \in \text{Aut}(O) \). Then \( \phi(1, 0) = (1, 0) \) and thus \( \phi \) fixes \( R \oplus \{0\} \). Since

\[
N(t, v) = (t, v)(t, -v)
\]
\( N(\phi(x)) = N(x) \) for all \( x \in \mathfrak{O} \). This implies that \( \phi \) has to preserve the orthogonal of \( \mathbb{R} \oplus \{0\} \) for \( N \), i.e. \( \phi \) preserves \( \mathbb{R}^7 \). For any \( v, w \in \mathbb{R}^7 \) we have

\[
\phi(0, v)\phi(0, w) = \phi((0, v)(0, w)) \\
\implies (0, \phi|_{\mathbb{R}^7}(v))(0, \phi|_{\mathbb{R}^7}(w)) = \phi(-v^\top J_7 w, v \times w) \\
\implies (-\phi|_{\mathbb{R}^7}(v)^\top J_7 \phi|_{\mathbb{R}^7}(w), \phi|_{\mathbb{R}^7}(v) \times \phi|_{\mathbb{R}^7}(w)) = (-v^\top J_7 w, \phi|_{\mathbb{R}^7}(v \times w))
\]

which imply that \( \phi|_{\mathbb{R}^7} \in G_2(\mathbb{R}) \).

**Remark 3.7.** We can show that \( \tau_7(\text{PSL}(2, \mathbb{R})) \subset G_2(\mathbb{R}) \) by checking it only for \( \text{PSL}(2, \mathbb{Z}) \), since the latter is Zariski-dense in \( \text{PSL}(2, \mathbb{R}) \).

Denote \( O_{\overline{Q}} \) the unique octonion algebra over \( \overline{Q} \) (see §1.10 in [SV00]). Since \( \text{Aut}(G_2(\overline{Q})) \simeq G_2(\overline{Q}) \) (see Theorem 2.8 in [PR94])

\[
H^1(\text{Gal}(\overline{Q}/Q), \text{Aut}(G_2(\overline{Q}))) \simeq H^1(\text{Gal}((\overline{Q}/Q), G_2(\overline{Q}))) \\
\simeq H^1(\text{Gal}(\overline{Q}/Q), \text{Aut}(O_{\overline{Q}})).
\]

The latter has only two elements for there is only two octonions algebras over \( Q \), as stated in §1.10 of [SV00]. Since there is two connected centerless simple real Lie groups of type \( G_2 \), the split one and the compact one, it implies that \( G_2 \) has only one \( \mathbb{R}/Q \)-form which is itself.

**Remark 3.8.** Since \( G_2 \) has only one \( \mathbb{R}/Q \)-form, all its \( Q \)-arithmetic subgroups are commensurable up to conjugation. In particular, all its non-uniform lattices are commensurable up to conjugation (see Corollary 5.3.2 in [Mor15]).

**Proposition 3.9.** Let \( \Gamma \) be a \( Q \)-arithmetic subgroup of \( \text{SL}(2, \mathbb{R}) \). Then \( \tau_7(\Gamma) \) lies in a subgroup of \( G_2(\mathbb{R}) \) commensurable with a conjugate of \( G_2(\mathbb{Z}) \).

Furthermore this is the only \( Q \)-arithmetic subgroups of \( G_2(\mathbb{R}) \) that contain \( \tau_7(\Gamma) \) up to commensurability.

**Proof.** Suppose \( \Gamma \) is the \( \mathbb{Z} \)-points of a \( Q \)-form of \( \text{SL}_2(\mathbb{R}) \) associated to the 1-cocycle \( \xi \). Thanks to Proposition 2.6, there is only one 1-cocycle

\[
\zeta : \text{Gal}(\overline{Q}/Q) \to G_2(\overline{Q})
\]

which is \( \tau_7 \)-compatible with \( \xi \). We have \( \zeta G_2(Q) \simeq G_2(Q) \). Furthermore \( \tau_7(\Gamma) \subset G_2(\mathbb{Z}) \).

The converse is proven the same way as in the proof of A.1. \( \square \)

### 4. Generalities on bending

Let \( \Gamma \) be a cocompact \( Q \)-arithmetic subgroup of \( \text{SL}(2, \mathbb{R}) \). Propositions A.1 and A.2 tell us that there exists a \( Q \)-arithmetic subgroup of \( \text{SL}(n, \mathbb{R}) \) \( \Lambda \) such that \( \tau_n(\Gamma) \subset \Lambda \). This implies that \( \tau_n(\Gamma/\{\pm 1_2\}) \subset \Lambda/\{\pm 1_n\} \) when \( n \) is odd and \( \tau_n(\Gamma/\{\pm 1_2\}) \subset \Lambda/\{\pm 1_n\} \) when \( n \) is even. There exists a finite-index subgroup of \( \Gamma/\{\pm 1_2\} \) which is torsion free and hence a surface group.
Let $\tau_n$ induces a representation of a surface group into an arithmetic subgroup of $\text{PSL}(n, \mathbb{R})$ which, by definition, is a Hitchin representation. The image of this representation is not Zariski-dense in $\text{PSL}(n, \mathbb{R})$ since it lies in $\tau_n(\text{PSL}(2, \mathbb{R}))$. We will deform it so that it becomes Zariski-dense. The technique used is called bending, as introduced by Johnson and Millson [JMS6].

Let $S$ be a closed orientable surface of genus at least 2 and $\gamma$ be a simple closed curve on $S$. We say that $\gamma$ is separating if the complement of its image has two connected components, otherwise it is non-separating. Let $j : \pi_1(S) \to \text{PSL}(2, \mathbb{R})$ be a discrete and faithful representation. Let $\rho = \tau_n \circ j$. Choose $B \in \text{PSL}(n, \mathbb{R})$ which commutes with $\rho(\gamma)$.

Suppose $\gamma$ is separating and denote by $C$ and $D$ the two connected components of the complement of its image. Van Kampen’s theorem tells us that the fundamental group of $S$ is an amalgamated product:

$$\pi_1(S) = \pi_1(C) *_{[\gamma]} \pi_1(D)$$

Denote by

$$\rho^1 : \pi_1(C) \to \text{PSL}(n, \mathbb{R}) \ g \mapsto \rho(g)$$
$$\rho^2 : \pi_1(D) \to \text{PSL}(n, \mathbb{R}) \ g \mapsto B \rho(g) B^{-1}$$

Together they induce a new representation

$$\rho_B : \pi_1(S) \to \text{PSL}(n, \mathbb{R}).$$

Suppose $\gamma$ is non-separating and denote by $C$ the complement of its image. Then the fundamental group of $S$ is an HNN-extension of the fundamental group of $C$. More precisely let $T$ be a tubular neighborhood of the image of $\gamma$ in $S$. The curve $\gamma$ separates $T$ into two connected components which we denote by $T_1$ and $T_2$. Denote by $i : T_1 \to C$ the inclusion. Pick $p_1 \in T_1$, $p_2 \in T_2$ and $\eta : [0; 1] \to S$ a path from $p_1$ to $p_2$ that does not intersect $\gamma$. Let

$$\phi : \pi_1(T_2, p_2) \xrightarrow{\sim} \pi_1(T_1, p_1), \ [\gamma] \mapsto [\eta^{-1} \circ \gamma \circ \eta].$$

Suppose $\pi_1(C, p_1) = \langle g_1, \ldots, g_k \rangle$. Then

$$\pi_1(S) = \langle g_1, \ldots, g_k, s \mid i_*(\phi(g)) = s^{-1}i_*(g)s \ \forall g \in \pi_1(T_1, p_1) \rangle.$$

Denote by

$$\rho^1 : \pi_1(C, p_1) \to \text{PSL}(n, \mathbb{R}), \ g \mapsto \rho(g)$$
$$\rho^2 : \langle s \rangle \to \text{PSL}(n, \mathbb{R}), \ s^k \mapsto (B \rho(s))^k$$

Together they induce a representation

$$\rho_B : \pi_1(S) \to \text{PSL}(n, \mathbb{R})$$

since elements of $i_*(\pi_1(T_1, p_1))$ are actually powers of $\gamma$.

We call $\rho_B$ the bending of $\rho$ using $B$.

**Lemma 4.1.** Let $B_0, B_1 \in \text{PSL}(n, \mathbb{R})$ which commute to $\rho([\gamma])$. The representations $\rho_{B_0}$ and $\rho_{B_1}$ are conjugate if and only if $B_0 = B_1$. 
Proof. Suppose that there exists $P \in \text{PSL}(n, \mathbb{R})$ such that $P \rho_B \circ P^{-1} = \rho_{B_1}$. Then
\[ P \rho_{\pi_1(C)} \circ P^{-1} = \rho_{\pi_1(C)} \]
and since $j(\pi_1(C))$ is Zariski-dense in $\text{PSL}(2, \mathbb{R})$, $P$ commutes with all matrices in $\tau_n(\text{PSL}(2, \mathbb{R}))$. The latter is an absolutely irreducible group, see for example Theorem 2.5 in [Tit71]. Thus Schur’s lemma implies that $P = I_n$.

Suppose $\gamma$ is separating. Then
\[ B_0 \rho_{\pi_1(D)} \circ B_0^{-1} = B_1 \rho_{\pi_1(D)} \circ B_1^{-1}. \]
From this we deduce that $B_1^{-1} B_0 = I_n$ by the same argument as above. Suppose $\gamma$ is non-separating. Then $B_0 \rho(s) = B_1 \rho(s)$ which shows that $B_0 = B_1$. □

Lemma 4.2. The group $\rho_B(\pi_1(S))$ lies in a conjugate of $\tau_n(\text{PSL}(2, \mathbb{R}))$ if and only if $B \in \tau_n(\text{PGL}(2, \mathbb{R}))$.

Proof. Suppose $\rho_B(\pi_1(S))$ lies in $P \tau_n(\text{PSL}(2, \mathbb{R})) \circ P^{-1}$ for some $P \in \text{PSL}(n, \mathbb{R})$. The group $\rho_B(\pi_1(C))$ is Zariski-dense in $\tau_n(\text{PSL}(2, \mathbb{R}))$, since $j(\pi_1(C))$ is Zariski-dense in $\text{PSL}(2, \mathbb{R})$. Hence the equality
\[ \tau_n(\text{PSL}(2, \mathbb{R})) = P \tau_n(\text{PSL}(2, \mathbb{R})) \circ P^{-1}. \]
Thus $\text{Int}(P)$ induces an automorphism of $\tau_n(\text{PSL}(2, \mathbb{R}))$. We deduce that there exists $X \in \text{PGL}(2, \mathbb{R})$ such that $\tau_n(X) \circ X^{-1}$ commutes with all elements of $\tau_n(\text{PSL}(2, \mathbb{R}))$ which is an absolutely irreducible subgroup of $\text{PSL}(n, \mathbb{R})$. Schur’s lemma implies that $P = \tau_n(X)$. Consequently $\rho_B(\pi_1(S))$ lies in $\tau_n(\text{PSL}(2, \mathbb{R}))$.

Suppose now $\gamma$ is separating. Then $\rho_B(\pi_1(D)) = B \rho(\pi_1(D)) \circ B^{-1}$ lies in $B \tau_n(\text{PSL}(2, \mathbb{R})) \circ B^{-1}$ which is thus equal to $\tau_n(\text{PSL}(2, \mathbb{R}))$. By the above argument we get that $B \in \tau_n(\text{PGL}(2, \mathbb{R}))$. Suppose $\gamma$ is non-separating, then $B \rho(s) \in \tau_n(\text{PSL}(2, \mathbb{R}))$ so $B \in \tau_n(\text{PSL}(2, \mathbb{R}))$. □

Lemma 4.3. The group $\rho_B$ preserves a bilinear form represented by a matrix $J \in \text{PGL}(n, \mathbb{R})$ if and only if $J = B^\top J_n B$ in $\text{PGL}(n, \mathbb{R})$ and in that case $J = J_n$.

Proof. Suppose $\rho_B$ preserves $J$. Then for all $s \in \pi_1(C)$
\[ J^{-1} J_n = \rho(s)^{-1} J_n \rho(s) \]
so $J^{-1} J_n$ commutes with $\rho(\pi_1(C))$, and so with $\tau_n(\text{PSL}(2, \mathbb{R}))$. The latter is absolutely irreducible, so Schur’s lemma implies that $J$ is a scalar multiple of $J_n$. Finally $\rho_B$ preserves $J_n$.

Suppose $\gamma$ is separating. For all $s \in \pi_1(D)$ we have
\[ (B \rho(s) \circ B^{-1})^\top J_n (B \rho(s) \circ B^{-1}) = J_n \]
\[ \Rightarrow \rho(s)^\top B^\top J_n B \rho(s) = B^\top J_n B \]
\[ ^{17}A \text{ subgroup } G < \text{PGL}(n, \mathbb{R}) \text{ is said absolutely irreducible if } G < \text{PGL}(n, \mathbb{C}) \text{ is irreducible.} \]
so $\rho(\pi_1(D))$ preserves $J_n$ and $B^\top J_n B$ which thus must be equal in $\text{PGL}(n, \mathbb{R})$ by the same argument as above. Suppose $\gamma$ is non-separating. Then
\[(B \rho(s))^\top J_n (B \rho(s)) = J_n \implies B^\top J_n B = J_n.\]

\[\square\]

**Lemma 4.4.** The group $\rho_B(\pi_1(S))$ lies in a conjugate of $G_2(\mathbb{R})$ if and only if $B \in G_2(\mathbb{R})$.

**Proof.** Suppose $\rho_B(\pi_1(S)) \subset P G_2(\mathbb{R}) P^{-1}$ for some $P \in \text{PSL}(n, \mathbb{R})$. Then $\rho(\pi_1(C))$ lies in $P G_2(\mathbb{R}) P^{-1}$ and so does $\tau_7(\text{PSL}(2, \mathbb{R}))$. Thus
\[P^{-1} \tau_7(\text{PSL}(2, \mathbb{R})) P\]
is a Lie subgroup of $G_2(\mathbb{R})$. Denote by $\mathfrak{h} = d_1 \tau_7(\mathfrak{sl}_2(\mathbb{R}))$ and $\mathfrak{g}_2$ the Lie algebra of $G_2(\mathbb{R})$. The algebra $\mathfrak{h}$ is a principal $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}_2$ in the sense of Kostant §5 [Kos59] since it acts irreducibly\(^{18}\) on $\mathbb{R}^7$. There exists $g \in G_2(\mathbb{R})$ such that
\[P^{-1} \mathfrak{h} P = g \mathfrak{h} g^{-1}\]
since all automorphisms of $\mathfrak{g}_2$ are inner, as can be seen in Gündogan [Gü10]. We can deduce that $X \mapsto P gX(P g)^{-1}$ induces an inner automorphism of $\mathfrak{h}$. There exists $M \in \text{PSL}(2, \mathbb{R})$ such that for all $X \in \mathfrak{h}$ we have $g P X(g P)^{-1} = \tau_7(M) X \tau_7(M)^{-1}$ which implies that $\tau_7(M)^{-1} g P$ commutes with all elements of $\mathfrak{h}$. We conclude that $g^{-1} \tau(M\tau_7) = P$ since $\mathfrak{h}$ acts absolutely irreducibly on $\mathbb{R}^7$ and $P \in G_2(\mathbb{R})$. We have shown that $\rho_B(\pi_1(S)) \subset G_2(\mathbb{R})$. Suppose $\gamma$ is separating. Then $B \rho(\pi_1(D)) B^{-1} \subset G_2(\mathbb{R})$ and its Zariski closure, which is $B \tau_7(\text{PSL}(2, \mathbb{R})) B^{-1}$, is also in $G_2(\mathbb{R})$. By the same argument as above, we show that $B \in G_2(\mathbb{R})$. If $\gamma$ is non-separating, then $B \rho(s) \in G_2(\mathbb{R})$ so $B \in G_2(\mathbb{R})$. \[\square\]

5. **Construction of Zariski dense surface groups**

Let $O^1$ be a torsion-free $\mathbb{Q}$-arithmetic cocompact lattice of $\text{SL}(2, \mathbb{R})$. As shown in Section 2, the irreducible embedding $\tau_n$ provides a representation of $O^1$ that lies in the Hitchin component of the surface $\mathbb{H}^2/O^1$ with image in a lattice $\Lambda$. We will bend this representation along a specific simple closed curve using a matrix in $\Lambda$ so that the image of the new representation will be Zariski-dense.

5.1. **The simple closed curve used to bend.** Let $A$ be a quaternion division algebra over $\mathbb{Q}$ that splits over $\mathbb{R}$. Up to isomorphism, it is of the form $(a, b)_\mathbb{Q}$ with $a, b \in \mathbb{N}$ not squares. Denote $\{1, i, j, ij\}$ a basis of $A$ such

\[^{18}\text{we can show that a } \mathfrak{sl}_2\text{-subalgebra that acts irreducibly on } \mathbb{R}^7 \text{ is principal by using Theorem 5.3 in [Kos59] and using the Hasse diagram of the representation of } G_2(\mathbb{R}) \text{ for the fundamental weight associated to the short root (see for example Figure 1 in [Sam20])}\]
that \(i^2 = a, j^2 = b\) and \(ij = -ji\). We embed \(A\) in \(\mathbb{M}_2(\mathbb{Q})\) using the map defined by

\[
(4) \\
1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ i \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \ j \mapsto \begin{pmatrix} 0 & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}, \ ij \mapsto \begin{pmatrix} 0 & \sqrt{ab} \\ -\sqrt{ab} & 0 \end{pmatrix}.
\]

Thus \(\text{SL}(1, A)\) corresponds to the obtained matrices which have determinant 1.

Let \(\mathcal{O}\) be the order \(\mathbb{Z}[1, i, j, k]\). Then

\[
\text{SL}(1, \mathcal{O}) \simeq \left\{ \begin{pmatrix} x_0 + \sqrt{ax_1} & \sqrt{bx_2} + \sqrt{abx_3} \\ \sqrt{bx_2} - \sqrt{abx_3} & x_0 - \sqrt{ax_1} \end{pmatrix} \mid x_i \in \mathbb{Z}, \text{ Det} = 1 \right\}.
\]

Diagonal elements of this group are in bijective correspondence with the solutions of the Pell equation \(x_0^2 - ax_1^2 = 1\) and are thus infinite in number by Dirichlet Unit’s Theorem (see Theorem 0.4.2. in [MR03]).

Let \(S\) be a closed orientable surface of genus at least 2 and suppose there exists a faithful representation \(j : \pi_1(S) \to \text{PSL}(1, \mathcal{O}) = \text{SL}(1, \mathcal{O})/\{\pm I_2\}\).

**Lemma 5.1.** There exists a simple closed curve on \(S\) which image under \(j\) is diagonal for the canonical basis.

**Proof.** First of all, there exists diagonal elements in \(j(\pi_1(S))\) because it is a finite index subgroup of \(\text{PSL}(1, \mathcal{O})\). Diagonal elements in \(j(\pi(S))\) form a cyclic group. Let \(D\) be a generator and denote by \(\gamma\) the closed curve it represents. Then \(\gamma\) is simple if and only if its lifts to \(\hat{H}^2\) are disjoints from each other. We identify \(\partial \hat{H}^2 = \mathbb{P}(\mathbb{R}^2) \simeq \mathbb{R}_1, [x : y] \mapsto \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ \infty & \text{otherwise.} \end{cases}\)

The curve \(\gamma\) has a lift \(\alpha\) which is the geodesic joining 0 to \(\infty\) since \(D\) is diagonal. To check if \(\gamma\) is simple we only need to check that the images of this lift under \(j(\pi(S))\) are either \(\alpha\) or do not intersect \(\alpha\). Let

\[
g = \begin{pmatrix} x_0 + \sqrt{ax_1} & \sqrt{bx_2} + \sqrt{abx_3} \\ \sqrt{bx_2} - \sqrt{abx_3} & x_0 - \sqrt{ax_1} \end{pmatrix}
\]

be an element of \(j(\pi_1(S))\) which is not diagonal. The image of \(\alpha\) under \(g\) is the geodesic joining \(g(0)\) to \(g(\infty)\). Note that neither of those can be 0 or \(\infty\) since \(j(\pi_1(S))\) is a discrete subgroup. We need to prove that the product \(g(0)g(\infty)\) is positive; computation gives

\[
g(0)g(\infty) = \frac{(x_0 + \sqrt{ax_1})(\sqrt{bx_2} + \sqrt{abx_3})}{(\sqrt{bx_2} - \sqrt{abx_3})(x_0 - \sqrt{ax_1})} > 0
\]

\[
\iff (x_0^2 - ax_1^2)(bx_2^2 - abx_3^2) > 0
\]

\[
\iff (x_0^2 - ax_1^2)(x_0^2 - ax_1^2 - 1) > 0
\]

because \(\text{Det}(g) = 1\), and the latter is positive because the product of two consecutive integers is always positive or null. \(\square\)
From now on, we denote by $\gamma$ the simple closed curve on $S$ whose image under $j$ is diagonal. For every $n \geq 3$, $\tau_n(j(\gamma))$ is diagonal. Our bending construction requires to find matrices that commute with $\tau_n(j(\gamma))$, i.e. which are diagonal, this is the goal of next part.

5.2. Computation of the centralizer in $\Lambda$. Since the algebraic group defined by $H(K) = \text{SL}(1, A \otimes_{\mathbb{Q}} K)$, for every field extension $K$ of $\mathbb{Q}$, is a $\overline{\mathbb{Q}}/\mathbb{Q}$ form of $\text{SL}_2$, we can calculate the associated 1-cocycle using the embedding described in (4). Direct computations show that the 1-cocycle associated to $H$ is

$$\xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{PSL}(2, \overline{\mathbb{Q}}), \ \sigma \mapsto \text{Int}(T_{\sigma}^{a,b}).$$

Lemma 2.3 tells us that all continuous 1-cocycles $\eta : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\text{SL}_2(\overline{\mathbb{Q}}))$ such that $\eta_{\text{SL}_2}(R) \cong \text{SL}(2, R)$ are equivalent to $\xi$. Denote by $\zeta : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\text{SL}_n(\overline{\mathbb{Q}}))$ a 1-cocycle $\tau_n$-compatible with $\xi$.

Let

$$(5) \quad \begin{pmatrix} w_1 & \cdots & w_n \\ & & \\
& & \end{pmatrix}, \quad \prod_i w_i = 1$$

be a diagonal matrix.

**Lemma 5.2.** If $\zeta$ is inner, the diagonal elements of $\zeta_{\text{SL}_n}(\mathbb{Q})$ are of the form (5) with $w_i \in \mathbb{Q}(\sqrt{a})$ and $\sigma(w_i) = w_{n-i+1}$ for all $i$ with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$ non-trivial.

If $\zeta$ is not inner let $\mathbb{Q}(\sqrt{d})$ be the associated extension of $\mathbb{Q}$.

i) If $\sqrt{a} \in \mathbb{Q}(\sqrt{d})$ then the diagonal elements of $\zeta_{\text{SL}_n}(\mathbb{Q})$ are of the form (5) with $w_i \in \mathbb{Q}(\sqrt{a})$ and $\sigma(w_i) = 1$ for all $i$ with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$ non-trivial.

ii) If $\sqrt{a} \notin \mathbb{Q}(\sqrt{d})$ then the diagonal elements of $\zeta_{\text{SL}_n}(\mathbb{Q})$ are of the form (5) with $w_i \in \mathbb{Q}(\sqrt{a}, \sqrt{d})$, $\sigma(w_i) = w_{n-i+1}$ for all $i$ with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{a}, \sqrt{d})/\mathbb{Q}(\sqrt{d}))$ non-trivial and $w_i \tau(w_{n-i+1}) = 1$ for all $i$ with $\tau \in \text{Gal}(\mathbb{Q}(\sqrt{a}, \sqrt{d})/\mathbb{Q}(\sqrt{a}))$ non-trivial.

**Proof.** Since $\zeta$ is $\tau_n$-compatible with $\xi$, Proposition 2.6 describes $\zeta$ explicitly.\footnote{Note that in the notations of Proposition 2.6, $P = I_2$ because of the definition of $\xi$.}

First suppose $\zeta$ is inner. We are looking for diagonal matrices $D \in \text{SL}(n, \overline{\mathbb{Q}})$ such that $\zeta(\theta) \circ \theta(D) = D$ for all $\theta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, i.e.

$$\tau_n(T_{\theta})\theta(D)\tau_n(T_{\theta})^{-1} = D.$$

If $\theta$ fixes $\sqrt{a}$ the previous equation tells us that $D$ has coefficients in $\mathbb{Q}(\sqrt{a})$. If $\theta$ doesn’t fix $\sqrt{a}$ then the equation tells us that $\theta(w_{n-i+1}) = w_i$ for all $i$. Hence the result.
Suppose now that $\zeta$ is not inner. We are looking for diagonal matrices $D \in \text{SL}(n, \mathbb{Q})$ such that $\zeta(\theta) \circ \theta(D) = D$ for all $\theta \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, i.e.
\[
\tau_n(T_\theta) \theta(D) \tau_n(T_\theta)^{-1} = D
\]
whenever $\theta$ fixes $\sqrt{d}$ and
\[
\tau_n(T_\theta) J_n^{-1} \theta(D) J_n \tau_n(T_\theta)^{-1} = D
\]
whenever $\theta$ doesn’t fix $\sqrt{d}$.

Suppose that $\sqrt{a} \in \mathbb{Q}(\sqrt{d})$. The first equation tells us that $D$ has coefficients in $\mathbb{Q}(\sqrt{a})$ and the second that $w_i \theta(w_i) = 1$ for $\theta \in \text{Gal}(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$ non-trivial.

We assume that $\sqrt{a} \notin \mathbb{Q}(\sqrt{d})$. If $\theta$ fixes $\sqrt{a}$ then we conclude as in the previous case that $D$ has coefficients in $\mathbb{Q}(\sqrt{a}, \sqrt{d})$. If $\theta$ doesn’t fix $\sqrt{a}$ and doesn’t fix $\sqrt{d}$ then the equation we get is $\theta(w_{n-i+1})^{-1} = w_i$ for all $i$. The case where $\theta$ fixes $\sqrt{a}$ but not $\sqrt{d}$ can be deduced from the previous equation. Hence the result. □

5.3. Arithmetic properties of the trace. Let $\Pi$ be a subgroup of $\text{GL}(n, \mathbb{R})$.

Definition 5.3. Denote by
\[
\text{Tr}(\Pi) = \{\text{Tr}(\gamma)\mid \gamma \in \Pi\}.
\]
If $\rho : \Pi \to \text{GL}(n, \mathbb{R})$ is a representation we denote by $\text{Tr}(\rho) = \text{Tr}(\rho(\Pi))$.

In order to prove that two representations $\rho_1$ and $\rho_2$ of a surface group are not in the same orbit under the Mapping Class Group it suffices to show that $\text{Tr}(\rho_1) \neq \text{Tr}(\rho_2)$. This is how we will prove in the next part that there is infinitely many MCG-orbits of thin Hitchin representations in a given lattice. For this we will study the arithmetic properties of $\text{Tr}(\Pi)$, where $\Pi$ is a thin subgroup of a lattice, using the Strong-approximation Theorem which we now recall.

Let $G$ be an algebraic subgroup of $\text{GL}_n$ over $\mathbb{Q}$. Denote by $I$ the ideal of $\mathbb{Q}[X_{11}, X_{12}, ..., X_{nn}, \text{Det}^{-1}]$ of polynomials vanishing on $G$ and by
\[
I_0 = I \cap \mathbb{Z}[X_{11}, X_{12}, ..., X_{nn}, \text{Det}^{-1}].
\]
Let $p$ be a prime. We define an algebraic group over the finite field with $p$ elements $\mathbb{F}_p$ by
\[
\mathbb{G}_p(K) = \text{Hom}(\mathbb{F}_p[X_{11}, X_{12}, ..., X_{nn}, \text{Det}^{-1}]/I_0, K).
\]

Definition 5.4. The algebraic group $\mathbb{G}_p$ is called the reduction of $G$ modulo $p$.

The latter depends on the embedding of $G$ into $\text{GL}_n$. In fact, the algebraic group $\mathbb{G}_p$ is canonically defined for almost all primes $p$. See Section 3.3 in [PR94] for more details.

\[20\text{“almost all” means “all except finitely many”}\]
Let $G$ be a connected\textsuperscript{21} semisimple $\mathbb{Q}$-algebraic group. We say that $G$ is 
\textit{simply connected} if for any connected $\mathbb{Q}$-algebraic group $H$ and any surjective
morphism $f : G \rightarrow H$ of $\mathbb{Q}$-algebraic groups with finite kernel, $f$ is an
isomorphism. We say that $G$ is \textit{absolutely almost simple} if $G(\mathbb{Q})$ is not
commutative and has no non-trivial connected normal subgroup.

\textbf{Theorem 5.5} (Strong-approximation, see [MVW84]). \textit{Let $G$ be a connected
simply-connected absolutely almost simple algebraic group defined over $\mathbb{Q}$. Let $\Pi$
be a Zariski-dense finitely generated subgroup of $G(\mathbb{Q})$. Then for
almost all primes $p$ the reduction modulo $p$ of $\Pi$ equals $G_p(\mathbb{F}_p)$.}

Let $n \geq 2$. Denote by $SU(I_n, \mathbb{F}_p) = \{ M \in SL(n, \mathbb{F}_p^2) | \overline{\sigma}(M)^\top M = I_n \}$ with $\sigma \in \text{Gal}(\mathbb{F}_p^2/\mathbb{F}_p)$ non-trivial. Let $d \in \mathbb{N}$ be not a square and $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ non-trivial. For $p$ sufficiently large, the reduction of $SU(I_n, \sigma; \mathbb{Q}(\sqrt{d}))$ modulo $p$ has for $\mathbb{F}_p$-points $SU(I_n, \sigma; \mathbb{Q}(\sqrt{d}))_p(\mathbb{F}_p) = \begin{cases} SL_n(\mathbb{F}_p) & \text{if } d \text{ is a square modulo } p \\ SU(I_n, \mathbb{F}_p) & \text{if } d \text{ is not a square modulo } p \end{cases}$

To see this, identify $\mathbb{Q}(\sqrt{d})$ with $\mathbb{Q}[X]/(X^2 - d)$. Then elements of $SU(I_n, \sigma; \mathbb{Q}(\sqrt{d}))$ are matrices $M$ with coefficients in $\mathbb{Q}[X]/(X^2 - d)$ that satisfy

$$\partial(M)^\top M = I_n$$

where $\partial$ is the $\mathbb{Q}$-linear map that sends $X$ to $-X$. Note that

$$\mathbb{F}_p[X]/(X^2 - d) \simeq \begin{cases} \mathbb{F}_p \times \mathbb{F}_p & \text{if } d \text{ is a square modulo } p \\ \mathbb{F}_p^2 & \text{if } d \text{ is not a square modulo } p \end{cases}$$

In the first case, $\partial$ induces the map

$$(x, y) \mapsto (y, x).$$

Hence the reduction of $SU(I_n, \sigma; \mathbb{Q}(\sqrt{d}))$ is

$$\{(M, N) \in SL_n(\mathbb{F}_p) \times SL_n(\mathbb{F}_p) | (M, N)(N^\top, M^\top) = (I_n, I_n)\} \simeq SL_n(\mathbb{F}_p)$$

by projection onto the first coordinate.

In the second case, $\partial$ induces the non-trivial field automorphism of $\mathbb{F}_p^2$. Thus the group we get is $SU(I_n, \mathbb{F}_p)$.

\textbf{Lemma 5.6.} \textit{For any prime $p$ and $n \geq 2$ $\text{Tr}(SL(n, \mathbb{F}_p)) = \mathbb{F}_p$.}

\textit{Proof.} Let $a \in \mathbb{F}_p$. The matrix

$$\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

\textsuperscript{21}in the Zariski topology
is in $\text{SL}(2, \mathbb{F}_p)$ and has trace equal to $a$. The result follows for any $n \geq 2$ since we can embed $\text{SL}(2, \mathbb{F}_p)$ in $\text{SL}(n, \mathbb{F}_p)$ using

$$M \mapsto \begin{pmatrix} M & \mathbb{I}_{n-2} \end{pmatrix}.$$ 

$\square$

**Lemma 5.7.** For any prime $p$ and $n \geq 3$ $\text{Tr}(\text{SU}(I_n, \mathbb{F}_p)) = \mathbb{F}_p^2$.

**Proof.** We can embed $\text{SU}(I_3, \mathbb{F}_p)$ in $\text{SU}(I_n, \mathbb{F}_p)$ for any $n \geq 3$ using

$$M \mapsto \begin{pmatrix} M & \mathbb{I}_{n-2} \end{pmatrix}.$$ 

Hence it suffices to prove the result for $n = 3$. Let $a \in \mathbb{F}_p^2$. Since the map

$$N : \mathbb{F}_p^2 \to \mathbb{F}_p, \; x \mapsto x \sigma(x)$$

is surjective$^{22}$, there exists $b \in \mathbb{F}_p^2$ such that $N(b) = -a - \sigma(a)$. Let

$$M = \begin{pmatrix} a & b & 1 \\ \sigma(b) & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Then $\text{Det}(M) = 1$ and $\sigma(M)^{\top} J M = J$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Thus a conjugate of $M$ lies in $\text{SU}(I_3, \mathbb{F}_p)$. Since $\text{Tr}(M) = a - 1$, $\text{Tr}(\text{SU}(I_3, \mathbb{F}_p)) = \mathbb{F}_p^2$. $\square$

Let $n \geq 3$ and $A \in \text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$. Then $\text{Tr}(A) \in \mathbb{Z}[\sqrt{d}]$. The reduction of $A$ modulo $p$ can be described in the following way. Identify $\mathbb{Z}[\sqrt{d}] = \mathbb{Z}[X]/(X^2 - d)$. Then $A$ becomes a matrix whose entries are equivalence classes of polynomials. Since

$$\mathbb{Z}[X]/(X^2 - d)/(p) \simeq \mathbb{Z}_p[X]/(X^2 - d) \simeq \mathbb{F}_p[X]/(X^2 - d),$$

the entries of the reduction of $A$ modulo $p$ are elements of

$$\begin{cases} \mathbb{F}_p \times \mathbb{F}_p & \text{if } d \text{ is a square modulo } p \\ \mathbb{F}_p^2 & \text{if } d \text{ is not a square modulo } p. \end{cases}$$

If $d$ is a square modulo $p$ we consider the trace of the reduction of $A$ to be an element of $\mathbb{F}_p$ using the projection on the first coordinate.

$^{22}$pick an element that generates the cyclic group $\mathbb{F}_p^\times$. It maps under $N$ to a generator of $\mathbb{F}_p^\times$. 
Proposition 5.8. Let \( n \geq 3 \) and \( \Pi \) be a subgroup of \( \text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}]) \) which is Zariski-dense in \( \text{SL}(n, \mathbb{R}) \). Then the reduction of \( \text{Tr}(\Pi) \subset \mathbb{Z}[\sqrt{d}] \) is equal to
\[
\begin{cases}
  F_p & \text{if } d \text{ is a square modulo } p \\
  F_{p^2} & \text{if } d \text{ is not a square modulo } p
\end{cases}
\]
for almost all primes \( p \).

Proof. By Strong-approximation (Theorem 5.5), for almost all primes \( p \), the reduction of \( \Pi \) is equal to \( \text{SL}(n, F_p) \) or to \( \text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}]) \) depending on \( p \). Lemma 5.6 and 5.7 prove the result. \( \square \)

Let \( n \) be even. Let \( A \) be a quaternion algebra over \( \mathbb{Q} \), its conjugation and \( \mathcal{O} \) an order of \( A \). For \( p \) sufficiently large, the reduction of \( \text{SU}(I_n^2, \sigma; \mathcal{O} \otimes \mathbb{Q}(\sqrt{d})) \) is
\[
\begin{cases}
  \text{SL}(n, F_p) & \text{if } d \text{ is a square modulo } p \\
  \text{SU}(I_n, F_p) & \text{if } d \text{ is not a square modulo } p
\end{cases}
\]

Proposition 5.9. Let \( n \) be even and \( \Pi \) be a subgroup of \( \text{SU}(I_n^2, \sigma; \mathcal{O} \otimes \mathbb{Z}_p \mathbb{Z}[\sqrt{d}]) \) which is Zariski-dense in \( \text{SL}(n, \mathbb{R}) \). Then the reduction of \( \text{Tr}(\Pi) \subset \mathbb{Z}[\sqrt{d}] \) is equal to
\[
\begin{cases}
  F_p & \text{if } d \text{ is a square modulo } p \\
  F_{p^2} & \text{if } d \text{ is not a square modulo } p
\end{cases}
\]
for almost all primes \( p \).

Proof. The proof is the same as Proposition 5.8. \( \square \)

Denote by \( \text{Sp}(n, F_p) = \{ M \in \text{SL}(n, F_p) | M^T J M = J \} \) where \( J \) is a block-diagonal matrix with \( n \) block on the diagonal all equal to
\[
\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}.
\]

For \( p \) sufficiently large, the reduction of \( \text{SU}(I_n^2, \sigma; A) \) has for \( F_p \)-points \( \text{Sp}(n, F_p) \).

To see this, recall that elements of
\[
\text{SU}(I_n^2, \sigma; A) = \{ M \in \text{SL}(\frac{n}{2}, A) | \overline{M}^T M = I_{\frac{n}{2}} \}.
\]

If we embed \( A \) in \( M_2(\mathbb{C}) \) using \( A \leftrightarrow A \otimes \mathbb{Q} \mathbb{C} \simeq M_2(\mathbb{C}) \) then the conjugation on \( A \) agrees with
\[
M_2(\mathbb{C}) \to M_2(\mathbb{C}), \, X \mapsto \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} X^T \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix}.
\]

Hence the equation defining \( \text{SU}(I_n^2, \sigma; A) \) becomes \( M^T J M = J \).

Lemma 5.10. For any prime \( p \) and \( n \geq 2 \) even \( \text{Tr}(\text{Sp}(n, F_p)) = F_p \).
Proof. The proof follows from Lemma 5.6 and from the embedding of $\text{SL}(2, \mathbb{F}_p)$ in $\text{Sp}(n, \mathbb{F}_p)$ using

$$M \mapsto \begin{pmatrix} M & \mathbf{I}_{n-2} \end{pmatrix}.$$

\[\square\]

**Proposition 5.11.** Let $n \geq 2$ be even and $\Pi$ be a subgroup of $\text{SU}(I_n^2, \mathbb{O})$ which is Zariski-dense in $\text{Sp}(n, \mathbb{R})$. Then the reduction of $\text{Tr}(\Pi) \subset \mathbb{Z}$ modulo $p$ is equal to $\mathbb{F}_p$ for almost all primes $p$.

**Proof.** By the Strong-approximation Theorem, for almost all primes the reduction of $\Pi$ is equal to $\text{Sp}(n, \mathbb{F}_p)$. Lemma 5.10 proves that the reduction of $\text{Tr}(\Pi)$ is all of $\mathbb{F}_p$ for almost all primes $\square$

Let $n \geq 4$ and $p \neq 2$ be prime. Denote by

$$\text{SO}(I_n, \mathbb{F}_p) = \{M \in \text{SL}(n, \mathbb{F}_p) | M^\top M = I_n\}$$

and by $\Omega(I_n, \mathbb{F}_p)$ the commutator subgroup of $\text{SO}(I_n, \mathbb{F}_p)$. Suppose $n$ is odd and let $Q \in \text{SL}(n, \mathbb{Q})$ be a symmetric matrix. For $p$ sufficiently large, the reduction of $\text{SO}(Q, \mathbb{Q})$ modulo $p$ has for $\mathbb{F}_p$-points $\text{SO}(I_n, \mathbb{F}_p)$.

**Lemma 5.12.** For any prime $p \neq 2$ and $n \geq 4$ $\text{Tr}(\Omega(I_n, \mathbb{F}_p)) = \mathbb{F}_p$.

**Proof.** We can embed $\Omega(I_4, \mathbb{F}_p)$ in $\Omega(I_n, \mathbb{F}_p)$ using

$$M \mapsto \begin{pmatrix} M & \mathbf{I}_{n-4} \end{pmatrix}.$$

Hence is suffices to prove the result for $n = 4$. Pick $a \in \mathbb{F}_p^\times$. Let

$$M = \begin{pmatrix} 0 & 0 & -a & a \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

We can check that $\text{Det}(M) = 1$ and that $M^\top J M = J$ where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\text{Det}(J) = 1$, $J$ is congruent to $I_4$. It follows that a conjugate of $M$ lies in $\text{SO}(I_4, \mathbb{F}_p)$. As we can see in [Suz82] paragraph 5 chapter 3, $\Omega(I_4, \mathbb{F}_p)$ is of index 2 in $\text{SO}(I_4, \mathbb{F}_p)$. Hence a conjugate of $M^2$ lies in $\Omega(I_4, \mathbb{F}_p)$. Since $\text{Tr}(M^2) = -2a + 4$ and $\text{Tr}(I_4) = 4$, $\text{Tr}(\Omega(I_4, \mathbb{F}_p)) = \mathbb{F}_p$. $\square$

**Proposition 5.13.** Let $n \geq 5$ be odd and $Q \in \text{SL}(n, \mathbb{Q})$ a symmetric matrix. Let $\Pi$ be a subgroup of $\text{SO}(Q, \mathbb{Z})$ which is Zariski-dense in $\text{SO}(Q, \mathbb{R})$. Then the reduction of $\text{Tr}(\Pi) \subset \mathbb{Z}$ modulo $p$ is equal to $\mathbb{F}_p$ for almost all primes $p$. 
5.4. Certifying the Zariski-density of the bending. We can now prove Theorems A.1 and A.4. Let $A$ be a quaternion division algebra over $\mathbb{Q}$ with basis $\{1, i, j, k\}$ (i.e. $i^2 = a$ and $j^2 = b$) that splits over $\mathbb{R}$ and $\mathcal{O} = \mathbb{Z}[1, i, j, k]$. We embed $\mathcal{O}$ in $M_2(\mathbb{Q})$ using the map defined in (4). Then $\mathcal{O}^1$ is commensurable with $\xi SL_2(\mathbb{Z})$ for $\xi : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to PSL(2, \mathbb{Q})$ a 1-cocycle. Let $S$ be a closed orientable surface of genus at least 2 and suppose there exists a faithful representation $\rho : \pi_1(S) \to PSL(1, \mathcal{O})$. Denote by $\gamma$ the simple closed curve on $S$ which image under $j$ is diagonal, as provided by Lemma 5.1. Let $\rho = \tau_n \circ j$. Denote $\zeta : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \text{Aut}(SL_n(\mathbb{Q}))$ a 1-cocycle $\tau_n$-compatible with $\xi$ (see Definition 2.4).

We bend $\rho$ along the curve $\gamma$ (see Section 4 for the description of bending). When it’s possible, we exhibit a diagonal matrix $B$ with positive coefficients which is in $\xi SL_n(\mathbb{Z})$ so that $\rho_B(\pi_1(S)) \subset P(\xi SL_n(\mathbb{Z}))$ but such that $\rho_B(\pi_1(S))$ is Zariski dense in $PSL(n, \mathbb{R})$. Since $B$ has positive coefficients, there is a continuous path from $I_n$ to $B$ inside the centralizer of $\gamma$ and thus there is a continuous path from $\rho$ to $\rho_B$ which guaranteas that $\rho_B$ is a Hitchin representation. We can thus use the classification of the Zariski density of Hitchin representations.

**Theorem 5.14** (Guichard [Gui], see also Sambarino [Sam20]). Let $\rho : \pi_1(S) \to PSL(n, \mathbb{R})$ be a Hitchin representation$^{24}$. Then the Zariski closure of $\rho$ is either $PSL(n, \mathbb{R})$ or conjugated to one of the following:

- $\tau_n(PGL(2, \mathbb{R})) \cap PSL(n, \mathbb{R})$
- $\text{PSp}(2k, \mathbb{R})$ if $n = 2k$ for all $k \geq 1$,
- $\text{SO}(I_{k+1}, \mathbb{R})$ if $n = 2k + 1$ for all $k \geq 1$,
- $G_2(\mathbb{R})$ if $n = 7$.

We could also use a weaker version of this theorem. Since our representations are constructed via bending, it is easy to see that their Zariski-closures contain a principal $SL_2(\mathbb{R})$. The classification of their Zariski-closures then follows from a paper by Dynkin (see tables 13 and 14 in [Dyn57], [Sam20] also gives a proof of this).

Since $a$ is not a square, $\mathbb{Z}[\sqrt{a}]^\times \simeq (\pm 1) \times \langle \omega \rangle$, see Dirichlet’s Unit Theorem (Theorem 0.4.2 in [MR03]), where $\omega$ is the fundamental unit of $\mathbb{Z}[\sqrt{a}]$. Note that $\omega \sigma(\omega) = \pm 1$.

---

$^{23}$Indeed Remark 3.6 in [Nor87] shows that $A(F_p)^+$ is of index 2 in $SO(I_n, F_p)$. Note also that $\Omega(I_n, F_p)$ is of index 2 in $SO(I_n, F_p)$, as we can see in [Suz82] paragraph 5 chapter 3. Being the commutator subgroup, $\Omega(I_n, F_p)$ is the only index 2 subgroup of $SO(I_n, F_p)$.

$^{24}$by which we mean a representation whose equivalence class is in the Hitchin component of $X(\pi_1(S), PSL(n, \mathbb{R}))$.
Proof of Theorem A.1. Let $n \geq 3$ be odd and assume $\zeta$ is not inner. Let $\mathbb{Q}(\sqrt{d})$ be the quadratic extension of $\mathbb{Q}$ associated to $\zeta$ with $d \in \mathbb{N}$. Proposition 2.8 implies that $\zeta \text{SL}_n(\mathbb{Z})$ is commensurable with a conjugate of the group $\text{SU}(I_n, \sigma, \mathbb{Z}[\sqrt{d}])$.

Suppose that $\sqrt{a} \in \mathbb{Q}(\sqrt{d})$. To construct thin Hitchin representations in $\zeta \text{SL}_n(\mathbb{Z})$ we bend $\rho$ using a diagonal matrix with positive coefficients $B \in \zeta \text{SL}_n(\mathbb{Z})$ which is not in $\text{SO}(J_n, \mathbb{R})$. Then Theorem 5.14 and Lemma 4.3 show that $\rho B$ is Zariski-dense in $\text{SL}(n, \mathbb{R})$. By Lemma 5.2, diagonal elements of $\zeta \text{SL}_n(\mathbb{Z})$ are of the form

$$
\begin{pmatrix}
    w_1 \\
    \vdots \\
    w_n
\end{pmatrix}
$$

with $w_i \in \mathbb{Z}[\sqrt{a}]$ such that $w_i \sigma(w_i) = 1$ with $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$ non-trivial. For instance, we can use the matrix

$$B_0 = \text{Diag}(\omega^4, 1, \ldots, 1, \omega^{-2}, \omega^{-2})$$

The sequence of bendings $(\rho B^n)_n$ gives infinitely many MCG-orbits of thin Hitchins representations. Indeed, suppose that there are only finitely many such orbits. By Theorem 5.5 and Proposition 5.8, there exists a finite set of primes $\mathcal{V}$ such that for every prime $p$ not in $\mathcal{V}$ and any $B$ as above, the reduction of $\text{Tr}(\rho B)$ modulo $p$ is equal to

$$
\begin{cases}
    \mathbb{F}_p & \text{if } d \text{ is a square modulo } p \\
    \mathbb{F}_{p^2} & \text{if } d \text{ is not a square modulo } p.
\end{cases}
$$

There exists a polynomial $P \in \mathbb{Z}[X]$ such that $\text{Tr}(\tau_n(A)) = P(\text{Tr}(A))$. For any prime $p$ the polynomial $P$ induces a map

$$
\mathbb{F}_p \to \mathbb{F}_{p^2}, \ x \mapsto P(x)
$$

and this map is not surjective for infinitely many primes (see Corollary 1.8 in [Sha12] for example). Pick a prime $p \notin \mathcal{V}$ for which the map induced by $P$ is not surjective. Pick $n \in \mathbb{N}$ such that $B^n$ is not in $\text{SO}(J_n, \mathbb{R})$ but is trivial modulo $p$, which exists since every element of a finite group has finite order. Then modulo $p$, $\text{Tr}(\rho B) \equiv \text{Tr}(\rho) \equiv P(\text{Tr}(j))[p]$ which is not $\mathbb{F}_p$. This is a contradiction. We have thus constructed infinitely many MCG-orbits of thin Hitchin representations in $\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$ whenever $\sqrt{a} \in \mathbb{Q}(\sqrt{d})$.

Suppose $\sqrt{a} \notin \mathbb{Q}(\sqrt{d})$. Proposition 2.8 implies that $\zeta \text{SL}_n(\mathbb{Z})$ is commensurable with a conjugate of $\text{SU}(I_n, \sigma; \mathbb{Z}[\sqrt{d}])$.

Let $\sigma, \tau \in \text{Gal}(\mathbb{Q}(\sqrt{a}, \sqrt{d})/\mathbb{Q})$ be such that

$$
\sigma(\sqrt{a}) = -\sqrt{a}, \quad \sigma(\sqrt{d}) = \sqrt{d}
$$
$$
\tau(\sqrt{a}) = \sqrt{a}, \quad \tau(\sqrt{d}) = -\sqrt{d}.
$$

Denote $\theta \in \mathbb{Z}[\sqrt{a}]^\times$ a fundamental unit. Since $\theta \sigma(\theta) \tau(\theta) \sigma(\tau(\theta)) = \pm 1$ and $\sigma(\theta) = \theta$, we deduce that $\theta^2 \tau(\theta)^2 = 1$ and $\theta^2 \sigma(\theta^2) = \theta^4 \neq 1$. 


To construct thin Hitchin representations in \( \varsigma SL_n(\mathbb{Z}) \) we bend \( \rho \) using a diagonal matrix with positive entries \( B \in \varsigma SL_n(\mathbb{Z}) \) which is not in \( SO(J_n, \mathbb{R}) \).

Then Theorem 5.14 and Lemma 4.3 prove that \( \rho_B(\pi_1(S)) \) is Zariski-dense in \( SL(n, \mathbb{R}) \). By Lemma 5.2, diagonal elements of \( \varsigma SL_n(\mathbb{Z}) \) are of the form

\[
\begin{pmatrix}
  w_1 \\
  \vdots \\
  w_n
\end{pmatrix}
\]

with \( w_i \in \mathbb{Z}[\sqrt{a}, \sqrt{d}] \), \( \sigma(w_i) = w_{n-i+1} \) and \( w_i \tau(w_{n-i+1}) = 1 \) for all \( i \). For instance, we can use

\[
B_0 = \text{Diag}(\theta^2, 1, \ldots, 1, \theta^{-4}, 1, \ldots, 1, \theta^2).
\]

To obtain infinitely many MCG-orbits of thin Hitchin representations one proceeds as in the proof of the previous case. □

**Proof of corollary A.1.** It is a consequence of Theorem A.1, Theorem 1.1 in [LT20] and Proposition 1.3. □

**Proof of Theorem A.4.** Let \( n \geq 4 \) be even and suppose \( \zeta \) is not inner. Let \( Q(\sqrt{d}) \) be the quadratic extension of \( Q \) associated to \( \zeta \) with \( d \in \mathbb{N} \).

Suppose that \( \sqrt{a} \in Q(\sqrt{d}) \). Proposition 2.8 implies that \( \varsigma SL_n(\mathbb{Z}) \) is commensurable with a conjugate of \( SU(I_n, \sigma; \mathbb{Z}[\sqrt{d}]) \). To construct thin Hitchin representations in \( \varsigma SL_n(\mathbb{Z}) \) we bend \( \rho \) using a diagonal matrix with positive entries \( B \in \varsigma SL_n(\mathbb{Z}) \) which is not in \( Sp(J_n, \mathbb{R}) \). Then Theorem 5.14 and Lemma 4.3 show that \( \rho_B(\pi_1(S)) \) is Zariski-dense in \( PSL(n, \mathbb{R}) \). By Lemma 5.2, diagonal elements of \( \varsigma SL_n(\mathbb{Z}) \) are of the form

\[
\begin{pmatrix}
  w_1 \\
  \vdots \\
  w_n
\end{pmatrix}
\]

with \( w_i \in \mathbb{Z}[\sqrt{a}] \) such that \( w_i \sigma(w_i) = 1 \) with \( \sigma \in \text{Gal}(Q(\sqrt{a})/Q) \) non-trivial. For instance, we can use the matrix

\[
B_0 = \text{Diag}(\omega^4, 1, \ldots, 1, \omega^{-2}, \omega^{-2}).
\]

To obtain infinitely many MCG-orbits of thin Hitchin representations one proceeds as in proof of Theorem A.1.

Suppose that \( \sqrt{a} \notin Q(\sqrt{d}) \). Proposition 2.8 implies that \( \varsigma SL_n(\mathbb{Z}) \) is commensurable with a conjugate of \( SU(I_n, \tau \otimes \sigma; \mathbb{Z}[\sqrt{d}]) \).

Let \( \sigma, \tau \in \text{Gal}(Q(\sqrt{a}, \sqrt{d})/Q) \) be such that

\[
\sigma(\sqrt{a}) = -\sqrt{a}, \quad \sigma(\sqrt{d}) = \sqrt{d}
\]

\[
\tau(\sqrt{a}) = \sqrt{a}, \quad \tau(\sqrt{d}) = -\sqrt{d}.
\]

Denote \( \theta \in \mathbb{Z}[\sqrt{d}]^* \) a fundamental unit. Since \( \theta \sigma(\theta) \tau(\theta) \sigma(\tau(\theta)) \) is \( \pm 1 \) and \( \sigma(\theta) = \theta \), we deduce that \( \theta^2 \tau(\theta)^2 = 1 \) and \( \theta^2 \sigma(\theta^2) = \theta^4 \neq 1 \).
To construct thin Hitchin representations in $\zeta \text{SL}_n(\mathbb{Z})$ we bend $\rho$ using a diagonal matrix with positive entries $B \in \zeta \text{SL}_n(\mathbb{Z})$ which is not in $\text{Sp}(J_n, \mathbb{R})$. Then Theorem 5.14 and Lemma 4.3 prove that $\rho_B(\pi_1(S))$ is Zariski-dense in $\text{PSL}(n, \mathbb{R})$. By Lemma 5.2, diagonal elements of $\zeta \text{SL}_n(\mathbb{Z})$ are of the form

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

with $w_i \in \mathbb{Z}[\sqrt{a}, \sqrt{d}]$, $\sigma(w_i) = w_{n-i+1}$ and $w_i \tau(w_{n-i+1}) = 1$ for all $i$. For instance, we can use

$$B_0 = \text{Diag}(\theta^2, \theta^{-2}, 1, \ldots, 1, \theta^{-2}, \theta^2).$$

To obtain infinitely many MCG-orbits of thin Hitchin representations one proceeds as in the proof of Theorem A.1.

To conclude it remains to show that Hitchin representations lift to $\text{SL}(n, \mathbb{R})$. If one Hitchin representation lifts, all its connected component lift (this follows from the presentation of a surface group). Since representations in the Teichmüller component of $\text{PSL}(2, \mathbb{R})$ lift to $\text{SL}(2, \mathbb{R})$, all Hitchin representations lift to $\text{SL}(n, \mathbb{R})$. \hfill \Box

We actually proved a slightly stronger version of Theorems A.1 and A.4.

**Theorem 5.15.** Let $n \geq 3$ and suppose $\zeta$ is not inner. There exists a matrix $B \in \zeta \text{SL}_n(\mathbb{R})$ with positive coefficients such that for all $n \geq 1$ $\rho_B(\pi_1(S)) < \zeta \text{SL}_n(\mathbb{Z})$ and is Zariski-dense in $\text{SL}(n, \mathbb{R})$. Further more this sequence of representations correspond to infinitely many MCG(S)-orbits.

We can now prove Theorems A.2, A.3 and A.5. We use the same strategy as above. For some $\mathbb{R}/\mathbb{Q}$-forms $\zeta G$ of a given algebraic split real Lie group $G$ we exhibit a matrix

$$B \in \zeta G(\mathbb{Z}) = \zeta \text{SL}_n(\mathbb{Z}) \cap G(\mathbb{R})$$

which is diagonal (so that it commutes with $\rho(\gamma)$) with positive entries and such that $\rho_B(\pi_1(S))$ is Zariski-dense in the corresponding Lie group. Even thought there is several 1-cocyle with value in $\text{Aut}(\text{SL}_n(\mathbb{Q}))$ which are $\tau_n$-compatible with $\xi$, they all induce the same 1-cocyle on $\text{SO}(J_n, \mathbb{Q})$, $G_2(\mathbb{Q})$ and $\text{Sp}(J_n, \mathbb{Q})$.

**Proof of Theorem A.2.** Suppose $n \neq 7$ is odd. To construct thin Hitchin representations in $\zeta \text{SO}(J_n)(\mathbb{Z})$ we bend $\rho$ using a diagonal matrix $B \in \zeta \text{SO}(J_n)(\mathbb{Z})$ which is not in $\tau_n(\text{PGL}(2, \mathbb{R}))$. Then Theorem 5.14 and Lemma 4.2 show that $\rho_B(\pi_1(S))$ is Zariski-dense in $\text{SO}(J_n, \mathbb{R})$. For instance, we can use the matrix

$$B_0 = \text{Diag}(\omega^2, 1, \ldots, 1, \sigma(\omega)^2).$$

Since it preserves $J_n$ but is not in $\tau_n(\text{PGL}(2, \mathbb{R}))$, lemma 4.2 shows that $\rho_B(\pi_1(S))$ is Zariski dense in $\text{SO}(J_n, \mathbb{R})$. 
Suppose \( n = 7 \). To construct thin Hitchin representations in \( \text{SO}(J_7, \mathbb{Z}) \) we bend \( \rho \) using a diagonal matrix \( B \in \mathfrak{z}\text{SO}(J_7)(\mathbb{Z}) \) which is not in \( G_2(\mathbb{Z}) \). Then Theorem 5.14 and Lemma 4.4 show that \( \rho_B(\pi_1(S)) \) is Zariski-dense in \( \text{SO}(J_7, \mathbb{R}) \). For instance, we can use the matrix

\[
B_0 = \text{Diag}(\omega^2, \omega^2, 1, 1, 1, \omega, \sigma(\omega)^2).
\]

Let \( n \geq 5 \) be odd. To prove that this gives infinitely many MCG-orbits of thin Hitchin representations, we suppose that there is only finitely many such orbits. By Theorem 5.13, there exists a finite set of primes \( \mathcal{V} \) such that for every prime \( p \) not in \( \mathcal{V} \) and any \( B \) as above, the reduction of \( \text{Tr}(\rho_B) \) modulo \( p \) is equal to \( F_p \). There exists a polynomial \( P \in \mathbb{Z}[X] \) such that \( \text{Tr}(\tau_n(A)) = P(\text{Tr}(A)) \). For any prime \( p \) the polynomial \( P \) induces a map

\[
F_p \to F_p, \ x \mapsto P(x)
\]

and this map is not surjective for infinitely many primes (see Corollary 1.8 in [Shal12] for example). Pick a prime \( p \not\in \mathcal{V} \) for which the map induced by \( P \) is not surjective. Choose \( B \in \mathfrak{z}\text{SO}(J_n)(\mathbb{Z}) \) which is not in \( \tau_n(\text{PGL}(2, \mathbb{R})) \) and not in \( G_2(\mathbb{Z}) \) if \( n = 7 \) but is trivial modulo \( p \). For instance a suitable power of \( B_0 \). Then modulo \( p \), \( \text{Tr}(\rho_B) \equiv \text{Tr}(\rho) \equiv P(\text{Tr}(j))[p] \) which is not \( F_p \). This is a contradiction.

Let \( n \equiv \pm 3[8] \). We still have to prove that we constructed thin Hitchin representations in all \( \text{SO}(Q, \mathbb{Z}) \) for \( Q \in \text{SL}(2k + 1, \mathbb{Q}) \) of signature \((k + 1,k)\) not equivalent to \( I_{k+1,k} \). The construction above can be done for any quaternion division algebra \( A \) satisfying \( A \otimes \mathbb{Q} \mathbb{R} \simeq \mathbb{R} \). For any finite set \( \Omega \) of places of \( Q \) of even cardinality containing the real place, there is a quaternion algebra \( (a, b)_Q \) such that \( (a, b)_Q \otimes \mathbb{Q} \mathbb{R} \simeq M_2(\mathbb{R}) \) and

\[
(a, b)_Q v \otimes (-1, -1)_Q v \not\equiv M_2(\mathbb{Q}_v)
\]

for all \( v \in \Omega \) and only those (see Theorem 7.3.6 in [MR03]). The quaternion algebra \( (a, b)_Q \) is a division algebra except when \( (a, b)_Q \simeq M_2(\mathbb{Q}) \), i.e. \( \Omega = \{2, \infty\} \). The quadratic form having non-trivial Hasse invariant exactly at \( \Omega = \{2, \infty\} \) is \( J_n \). Since \( J_n \) is equivalent to \( \pm I_{k+1,k} \), we constructed thin Hitchin representations in all \( \text{SO}(Q, \mathbb{Z}) \) with \( Q \) of signature \((k + 1,k)\) except when \( Q \) is equivalent to \( \pm I_{k+1,k} \).

\[\square\]

**Proof of Theorem 4.3.** Suppose \( n = 7 \). To construct thin Hitchin representations in \( G_2(\mathbb{Z}) \) (see definition 3.6 for \( G_2 \)) we bend \( \rho \) using a diagonal matrix \( B \) which is in \( G_2(\mathbb{Z}) \) but not in \( \tau_7(\text{PGL}(2, \mathbb{R})) \). Then Theorem 5.14 and Lemma 4.2 show that \( \rho_B(\pi_1(S)) \) is Zariski-dense in \( G_2(\mathbb{R}) \). For instance, we can use the matrix

\[
B_0 = \text{Diag}(\omega^2, \omega^2, 1, 1, 1, \sigma(\omega)^2, \sigma(\omega)^2).
\]

Since \( B \not\in \tau_7(\text{PGL}(2, \mathbb{R})) \), Lemma 4.1 implies that the representations \( \rho_{Bk} \) are not conjugate.

\[\square\]

**Remark 5.16.** If one can guarantee that \( \text{Tr}(G_2(F_p)) = F_p \) for all primes \( p \) except maybe finitely many, then the same proof as in the other cases shows
that $G_2(Z)$ contains infinitely many MCG-orbits of thin Hitchin representations.

**Proof of Theorem A.5.** Let $n \geq 4$ be even. Then $\zeta\text{Sp}(J_n)(Z)$ is commensurable with $\text{SU}(I_n,\mathcal{O})$. To construct thin Hitchin representations in $\zeta\text{Sp}(J_n)(Z)$ we bend $\rho$ with a diagonal matrix $B$ that is in $\zeta\text{Sp}(J_n)(Z)$ and is not in $\tau_n(PGL(2,R))$. Then Theorem 5.14 and Lemma 4.2 show that $\rho_B(\pi_1(S))$ is Zariski dense in $\text{PSp}(J_n,R)$. For instance, we can use the matrix

$$B_0 = \text{Diag}(\omega^2, ..., \omega^2, \sigma(\omega)^2, ..., \sigma(\omega)^2).$$

Since all Hitchin representations in $\text{PSp}(n,R)$ lift to $Sp(n,R)$ we have constructed thin Hitchin representations in $\zeta\text{Sp}(n,Z)$.

To prove that this gives infinitely many MCG-orbits of thin Hitchin representations, we suppose that there is only finitely many such orbits. By Theorem 5.11, there exists a finite set of primes $V$ such that for every prime $p$ not in $V$ and any B as above, the reduction of $\text{Tr}(\rho_B)$ modulo $p$ is equal to $F_p$. There exists a polynomial $P \in Z[X]$ such that $\text{Tr}(\tau_n(A)) = P(\text{Tr}(A))$. For any prime $p$ the polynomial $P$ induces a map

$$F_p \to F_p, \ x \mapsto P(x)$$

and this map is not surjective for infinitely many primes (see Corollary 1.8 in [Sha12] for example). Pick a prime $p \notin V$ for which the map induced by $P$ is not surjective. Choose $B \in \zeta\text{Sp}(J_n)(Z)$ which is not in $\tau_n(PGL(2,R))$ but is trivial modulo $p$. Then modulo $p$, $\text{Tr}(\rho_B) \equiv \text{Tr}(\rho) \equiv P(\text{Tr}(j))[p]$ which is not $F_p$. This is a contradiction. □

**Proof of corollary A.2.** It is a consequence of Theorem A.5 and Proposition 1.5. □

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