THE COMMUTATOR SUBGROUP OF THE HECKE GROUP $G_5$ IS NOT CONGRUENCE

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Abstract. Let $q \geq 3$ be an integer and let $G_q$ be the Hecke group associated with $q$. We prove that the power subgroup $G_q^2$ and the commutator subgroup $G_q'$ are not congruence.

1. Introduction

1.1. Let $q \geq 3$ be a a fixed integer. The (homogeneous) Hecke group $H_q$ is defined to be the maximal discrete subgroup of $SL_2(\mathbb{R})$ generated by $S$ and $T$, where $\lambda_q = 2\cos(\pi/q)$,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}. \quad (1.1)$$

Let $A$ be an ideal of $\mathbb{Z}[\lambda_q]$. The principal congruence subgroup of $H_q$ of level $A$ is defined to be

$$H(q, A) = \{(a_{ij}) \in H_q : a_{11} - 1, a_{22} - 1, a_{12}, a_{21} \in A\}. \quad (1.2)$$

Let $Z = \langle \pm 1 \rangle$. The (inhomogeneous) Hecke group and its principal congruence subgroup are defined as $G_q = H_q/Z$ and $G(q, A) = H(q, A)Z/Z$. A subgroup $K$ of $G_q$ is congruence if $G(q, A) \subseteq K$ for some $A$. Whether subgroups of finite indices are congruence have been studied extensively (see [F], [Lu], [S]). In the case $q = 3$, it is known that not every subgroup of finite index of the modular group $G_3$ is congruence and that the commutator subgroup $G_3'$ is congruence of level 6. We suspect that $q = 3$ is the only case that $G_q'$ is congruence (see Discussion 5.3). The main purpose of the present article is to show that

**Proposition 5.2.** The subgroups $G_5^2$ and $G_5'$ of the Hecke group $G_5$ are not congruence.

Note that $G_q^2$ is the subgroup of $G_q$ generated by all the elements of the form $x^n \in G_q$. Note also that in the case $q \geq 3$ is a prime, these two groups $G_q^2$ and $G_q'$ are special in the sense that they are the only normal torsion subgroups of $G_q$.

1.2. Our proof of the above proposition is elementary and requires some basic facts about the fundamental domains of certain subgroups of $G_5$. The following two facts about $G_5$ are essential in our proof as well.

(i) If $G_5^2$ is congruence, then $G(5, 5) \subseteq G_5^2$ (Lemma 5.1).

(ii) $G_5/G(5, 5) \cong E_{20}$/PSL$(2, 5)$ does not possess subgroups of index 5 (Proposition 5.2), where $E_{20}$ is an elementary abelian 5-group of order $5^3$. Note that the indices of $G_5^2$ and $G_5'$ in $G_q$ are $q$ and $2q$ respectively (Lemmata 3.3).

1.3. The rest of the article is organised as follows. In Sections 2 and 3, we study the geometric aspects of the Hecke group $G_q$, such study allows us to give the geometric invariants (index, number of elliptic elements, number of cusps, genius) of $G_5^2$ and $G_5'$. Section 4 lists all the known results which is necessary for our study of $G_5^2$ and $G_5'$. They are mainly results on the indices of the principal congruence subgroups of $G_5$. Section 5 gives us the main result of the present article. The present article is part of our project on $G_q$. We have determined the normalisers (see [L]) and the indices (see [LL1], [LLT2]) for some congruence subgroups of $G_5$. We are currently working on the index formula for $G(q, \pi)$, where $q \geq 7$.

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2. Geometric invariants

In [K], Kulkarni applied a combination of geometric and arithmetic methods to show that one can produce a set of independent generators in the sense of Rademacher for the congruence subgroups of the modular group, in fact for all subgroups of finite indices. His method can be generalised to all subgroups of finite indices of the Hecke groups \( G_q \), where \( q \) is a prime. See [LLT1] for detail [Propositions 8-10 and section 3 of [LLT1]]. In short, for each subgroup \( V \) of finite index of \( G_q \), one can associate to \( V \) a set of Hecke-Farey symbols \( \{ -\infty, x_0, x_1, \cdots, x_n, \infty \} \), a special polygon (fundamental domain) \( \Phi \), and an additional structure on each consecutive pair of \( x_i \)'s of the three types described below:

\[
x_{i-\frac{a}{u}}, x_{i+1}, x_{i-\frac{v}{u}}, x_{i+1}, x_{i-\frac{x}{u}}, x_{i+1}.
\]

where \( a \) is a nature number. Each nature number \( a \) occurs exactly twice or not at all. Similar to the modular group, the actual values of the \( a \)'s is unimportant: it is the pairing induced on the consecutive pairs that matters.

(i) The side pairing \( \circ \) is an elliptic element of order 2 that pairs the even line \((x_i, x_{i+1})\) with itself. The trace of such an element is 0.

(ii) The side pairing \( \bullet \) is an elliptic element of order \( q \) that pairs the odd line \((x_i, x_{i+1})\). The absolute value of the trace of such an element is \( \lambda_q \).

(iii) The two sides with the label \( u \) are paired together by an element of infinite order.

(iv) The special polygon associated to the HFS is a fundamental domain of \( V \) and the side pairings \( I = \{ \sigma_1, \sigma_2, \cdots, \sigma_m \} \) associated to the HFS is a set of independent generators of \( V \) (Theorem 7, Propositions 8-10 of [LLT1]).

(v) The number \( d \) of special triangles (a special triangle is a fundamental domain of \( G_q \)) of special polygons is the index of the subgroup.

(vi) The set of independent generators consists of \( r \) matrices of infinite order, where \( r \) is the number of the nature number \( a \)'s in the Hecke-Farey symbols.

(vii) The subgroup has \( v_2 \) (the number of the circles \( \circ \) in HFS) inequivalent classes of elliptic elements of order 2. Each class has exactly one representative in \( I \).

(viii) The subgroup has \( v_q \) (the number of the bullets \( \bullet \) in HFS) inequivalent classes of elliptic elements of order \( q \). Each class has exactly one representative in \( I \).

(ix) The Hecke-Farey symbols can be partitioned into \( v_\infty \) classes under the action of the set of independent generators, which gives the number of cusps of the subgroup.

(x) The genus \( g \) can be determined by the Riemann-Hurwitz formula.

\[
(q - 2)d = qv_2 + 2(q - 1)v_q + 4qg + 2qv_\infty - 4q.
\]

(xi) The width of a cusp \( x \), denoted by \( w(x) \), is the number of even lines in \( \Phi \) that comes into \( x \). Algebraically, it is the smallest positive integer \( m \) such that \( \pm T_q^m \) is conjugate in \( G_q \) to an element of \( K \) fixing \( x \) (keep in mind that a matrix is identified with its negative in \( G_q \)). The least common multiple \( N \) of the cusp widths of \( V \) is called the geometric width of \( V \).

Discussion 2.1. The vertices of the Hecke-Farey symbols can be obtained by applying Lemma 3 of [LLT1] and the side pairings in (i)-(iii) of the above can be obtained by Propositions 8-10 of [LLT1].

3. Subgroups of small indices, Power subgroups

Let \( q \geq 3 \) be a prime and let \( K \) be a subgroup of \( G_q \). It is clear that if \( K \) is of index 2, then the only possible Hecke-Farey symbols for \( K \) is \( \{ -\infty, 0, \infty \} \) with the set of independent generators \( \{ ST^{-1}, T^{-1}S \} \). The invariants of \( K \) is given by

\[
d = 2, v_2 = 0, v_q = 2, v_\infty = 1, g = 0.
\]
It is not clear that $G_q$ cannot possess subgroups of indices between 3 and $q - 1$ from algebraic point of view. However, it is clear that there is no such Hecke-Farey symbols. As a consequence, we have the following:

**Proposition 3.1.** Let $K$ be a subgroup of $G_q$ of index at most $q - 1$. Then $K$ is generated by the set of independent generators $\{ST^{-1}, T^{-1}S\}$, where $o(ST^{-1}) = o(T^{-1}S) = q$. The invariants of $K$ are $d = 2, v_2 = 0, v_q = 2, v_\infty = 1, g = 0$. Further, $[K : K'] = q^2$.

*Proof.* Since $\{ST^{-1}, T^{-1}S\}$ is a set of independent generators and $o(ST^{-1}) = o(T^{-1}S) = q$, one must have $[K : K'] = q^2$. The rest of the proposition is clear. □

**Remark.** Note that unlike $G_q (q \geq 5), G_3 = PSL_2(\mathbb{Z})$ does possess subgroups of all possible indices, which can be proved by investigation of the Hecke-Farey symbols.

3.1. Power subgroups of $G_q$. Denoted by $G_q^a$ the subgroup of $G_q$ generated by all the elements of the form $x^n$, where $x \in G_q$. It is clear that $G_q^a$ is a characteristic subgroup of $G_q$. Since $G_q$ is a free product of two elliptic elements of orders 2 and $q$ respectively, $G_q^a$ is a proper subgroup of $G_q$ if and only if $\gcd(n, 2q) \neq 1$. The following are well known.

**Lemma 3.2.** Let $q$ be an odd prime. Then $G_q^2$ is the only subgroup of $G_q$ of index 2. $G_q^2$ is a free product of two elliptic elements of order $q$. In particular, $[G_q^2 : [G_q^2, G_q^2]] = q^2$.

*Proof.* Since $\{S, ST^{-1}\}$ is a set of independent generators of $G_q$, $o(S) = 2, o(ST^{-1}) = q$, one has $ST^{-1}, T^{-1}S \in G_q^2, S \notin G_q^2$. We may now complete the proof of the lemma by applying Proposition 3.1. □

**Lemma 3.3.** Let $q$ be an odd prime. Then $G_q^q$ is the only normal subgroup of $G_q$ of index $q$. Further, $G_q^q$ is a free product of $q$ elliptic elements of order 2 and $[G_q^q : [G_q^q, G_q^q]] = 2^q$. The invariants of $G_q^q$ are given by $d = q, v_2 = q, v_q = 0, v_\infty = 1, g = 0$.

*Proof.* It is clear that $S \in G_q^q, ST^{-1} \notin G_q^q$. Hence $G_q^q$ is a proper subgroup that contains all the elliptic elements of order 2 ($G_q^q$ is normal). Let $K$ be the subgroup of $G_q$ with Hecke-Farey symbols

\[ \{-\infty = x_0 \xrightarrow{q} x_1 \xrightarrow{1} x_2 \cdots x_{q-1/2} \xrightarrow{q+1/2} \cdots x_{q-2} \xrightarrow{q} x_{q-1} \xrightarrow{q} x_q = \infty\}, \]

where the $x_i^q$ are the vertices of an ideal $q$-gon of depth 1 (see Discussion 2.1 of Section 2). It follows that $[G_q^q : K] = q$ and that a set of independent generators of $K$ is given by $\{g_1, g_2, \cdots, g_q\}$, where $o(g_i) = 2$ for all $i$. Since $G_q^q$ contains all the elliptic elements of order 2, we conclude that $K$ is a subgroup of $G_q^q$. An easy study of the indices implies that $G_q^q = K$.

Since $G_q^q = K$ is generated by $q$ independent generators of order 2, $[G_q^q : [G_q^q, G_q^q]] = 2^q$. Let $I$ be a normal subgroup of index $q$ of $G_q$. Since $q$ is an odd prime, $S = S^q \in I$. Since $I$ is normal, $I$ contains all the elliptic elements of order 2. Hence $G_q^q \subseteq I$. Since they have the same index, one must have $I = G_q^q$. This implies that $G_q^q$ is the only normal subgroup of index $q$ of $G_q$.

**Example 3.4.** The side pairings associated with the Hecke-Farey symbols of $G_q^q$ is given by

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\lambda & -1 \\
\lambda + 2 & -\lambda
\end{pmatrix}
\begin{pmatrix}
2\lambda + 1 & -2\lambda - 2 \\
\lambda + 2 & -\lambda
\end{pmatrix}
\begin{pmatrix}
2\lambda + 1 & -\lambda - 2 \\
2\lambda + 2 & -2\lambda - 1
\end{pmatrix}
\begin{pmatrix}
\lambda & -\lambda - 2 \\
1 & 1
\end{pmatrix}.
\]

4. Known results about $G_5$

Applying the main results in [LL1] and [LL2] (Section 7 of [LL1] and Theorem 4.1 of [LL2]), we have the following.

(i) $G_5/G(5, 5) \cong G(5, \lambda + 2)/G(5, 5) \cong G_5/G(5, \lambda + 2) \cong E_5 \rtimes PSL(2, 5)$, where $G(5, \lambda + 2)/G(5, 5) \cong E_{5r} \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ is the elementary abelian group of order $5^3$ and $G_5/G(5, \lambda + 2) \cong PSL(2, 5) \cong A_5$.

(ii) Let $V$ be a congruence subgroup of $G_5$. Suppose that the geometric level of $V$ is $r$ where $r$ is odd (see (xi) for the definition of the geometric level), then $G(5, r) \subseteq V$. 


5. \(G_5^2\) AND \(G_5'\) ARE NOT CONGRUENCE

It is well known that the commutator subgroup of \(\Gamma = G_3\) is congruence. The main purpose of this section is to show that the commutator subgroup \(G_5'\) of \(G_5\) is not congruence.

**Lemma 5.1.** If \(G_5^2\) is congruence, then \(G(5, 5) \subseteq G_5^2\).

**Proof.** By Lemma 3.3, the geometric level (see (xi) for the definition of the geometric level) of \(G_5^2\) is 5. By (ii) of Section 4, \(G(5, 5) \subseteq G_5^2\). \(\square\)

5.1. The group structure of \(G(5, \lambda + 2)/G(5, 5)\). Recall first that \(5 = \lambda^2(\lambda + 2)^2\). By Example 3 of [LTL1],

\[
a = \begin{pmatrix} -11\lambda - 6 & 10\lambda + 5 \\ 4\lambda + 3 & -4\lambda - 2 \end{pmatrix} = T^{-2} \begin{pmatrix} 3\lambda + 2 & -2\lambda - 3 \\ 4\lambda + 3 & -4\lambda - 2 \end{pmatrix} \in G(5, \lambda + 2) - G(5, 5) \tag{5.1}
\]

By (i) of Section 4, \(G(5, \lambda + 2)/G(5, 5)\) is elementary abelian of order \(5^3\). It follows that \(G(5, \lambda + 2)/G(5, 5)\) can be generated by \(\Delta = \{a, b = SaS^{-1}, c = JaJ^{-1}\}\) (see (5.3) for the definition of \(\Delta\)). Note that \(\Delta\) modulo \(G(5, 5)\) is given by

\[
a \equiv I + (\lambda + 2) \begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix}, \quad b \equiv I + (\lambda + 2) \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \quad c \equiv I + (\lambda + 2) \begin{pmatrix} 1 & 4 \\ 0 & 4 \end{pmatrix}. \tag{5.2}
\]

**Proposition 5.2.** \(G_5^2\) and \(G_5'\) are not congruence.

**Proof.** Since \(G_5'/G_5^2\) is abelian of order 10 and \(G_5^2\) is the only normal subgroup of \(G_5\) of index 5 (Lemma 3.3), \(G_5' \subseteq G_5^2\). To prove our assertion, it suffices to show that \(G_5^2\) is not congruence. Suppose that \(G_5^2\) is congruence. By Lemma 5.1, \(G(5, 5) \subseteq G_5^2\). Since \(G_5/G(5, \lambda + 2) \cong A_5\) has no normal subgroup of index 5 and \(G_5^2\) has index 5 in \(G_5\), \(G(5, \lambda + 2)\) is not a subgroup of \(G_5^2\). This implies that \(G_5^2(5, \lambda + 2) = G_5\). By Second Isomorphism Theorem, \([G(5, \lambda + 2)/G_5^2 \cap G(5, \lambda + 2)] = 5\) and \([G_5^2 \cap G(5, \lambda + 2)/G(5, 5)] = 5^2\). Note that \(E_{5^2}A_5 \cong G_5/G(5, 5)\) acts on \(D = [G_5^2 \cap G(5, \lambda + 2)]/G(5, 5) \cong \mathbb{Z}_5 \times \mathbb{Z}_5\) by conjugation. Note also that \(D\) is a subgroup of \((\Delta)\). Recall that

\[
J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Aut } G_5. \tag{5.3}
\]

Since \([G(5, 5) \cap G(5, \lambda + 5)]/G(5, 5) = D\) is invariant under the conjugation of \(E_{5^2}A_5 \cong G_5/G(5, 5)\), \(D\) is invariant under the conjugation of \(J\) and every element of \(G_5\) (in particular, \(S\) and \(T\)). However, one sees by direct calculation that the only subgroup of \((\Delta)\) invariant under \(J\), \(S\), and \(T\) is \((\Delta)\) itself (see Appendix A). A contradiction. Hence \(G_5^2\) is not congruence. \(\square\)

**Discussion 5.3.** A key step in the proof of \(G_5'\) is congruence is that \(G_3/G(3, 3) \cong A_4 \cong E_4 \mathbb{Z}_4\) has a normal subgroup of index 3 (see Lemma 3.7). This fact is no longer true if \(q = 5\) as \(G_5/G(5, 5)\) possesses no normal subgroups of index 5. As this may be true for all \(q \geq 5\), we therefore suggest that \(G_5'\) is not congruence if \(q \geq 5\).

6. Appendix A

**Lemma A1.** Let \(\pi = \lambda + 2\) and let \(\Delta = \{a, b, c\}\), where \(a, b, c\) are given as in (5.2). Then the only nontrivial subgroup of \((\Delta)\) invariant under the action of \(S, T\) and \(J\) is \((\Delta)\).

**Proof.** Since \((I + \pi U)(I + \pi V) \equiv I + \pi(U + V) \mod 5\), multiplication of \((I + \pi U)(I + \pi V)\) can be transformed into addition of \(U\) and \(V\). This makes the multiplication of matrices \(a, b, \) and \(c\) easy. Consequently, one has

\[
r = (ac)(ab) \equiv I + \pi \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad s = (ac)(ab)^{-1} \equiv I + \pi \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \quad t = bc \equiv I + \pi \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}.
\]

It is clear that \((\Delta) = \{a, b, c\} = \{r, s, t\}\). Let \(A, B \in G_5\). Set \(A^B = BAB^{-1}\). Direct calculation shows that

\[
r^S = s^{-1}, r^T = rs^{-1}, t^2, r^J = s, s^S = r^{-1}, s^T = s, s^J = r, t^S = t^{-1}, t^T = st, t^J = t^{-1}. \tag{A1}
\]
Denoted by $M$ a nontrivial subgroup of $\langle r, s, t \rangle$ that is invariant under the conjugation of $J$, $S$ and $T$. Let $1 \neq \sigma = r^i s^j t^k \in M$. One sees easily that

(i) If $k \not\equiv 0 \pmod{5}$, without loss of generality, we may assume that $k = 1$. Then $\sigma^4 \sigma^5 = t^{i-2} \in M$. It follows that $t \in M$. Hence $t^T = st \in M$. Consequently, $s \in M$. This implies $s^3 = r^{-1} \in M$. In summary, $r, s, t \in M$.

(ii) If $k \equiv 0 \pmod{5}$, then $\sigma$ takes the form $r^i s^j$. Suppose that $i \equiv 0 \pmod{5}$. Then $1 \neq s^j \in M$. It follows that $s \in M$. Consequently, $r = s^T \in M$. Hence $r s^{-1} t^2 = r^T \in M$. As a consequence, $t \in M$. In summary, $r, s, t \in M$. In the case $i \not\equiv 0 \pmod{5}$, we may assume that $i = 1$. Hence $r s^j \in M$. It follows that $(r s^j)^T (r s^j)^{-1} = s^{-1} t^2 \in M$. Consequently, $(s^{-1} t^2)^T = st^2 \in M$. This implies that $(s^{-1} t^2)(st^2) = t^4 \in M$. Hence $t \in M$. One now sees easily that $r, s, t \in M$.

Hence the only nontrivial subgroup of $\langle \Delta \rangle$ invariant under $J$, $S$ and $T$ is $\langle \Delta \rangle$. $\square$

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