A New Family of Fractional Renewal Processes

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Abstract

Fractional renewal processes as a generalization of Poisson process are already in the literature. In this paper, by introducing a new concept of generalized density function, the authors construct new fractional renewal processes in the $\alpha$-fractional space and show that it is another interesting and useful generalization of Poisson process.

Keywords: Lévy density function, fractional renewal process, fractional calculus, Mittag-Leffler function,

AMS Subject Classifications: 26A33, 33E12, 60G22, 60k05;

1 Introduction

In [6, p.16], the authors states “By time change via the inverse stable subordinator the standard Poisson process is transformed to the fractional Poisson process.”

In physics, the area of Beck-Cohen superstatistics is growing fast as can be seen in [1, 12, 19, 24], which belongs to Bayesian world in Statistics. In [1] and their other papers, the authors treat them under the two categories, one of which is associated with a non-density function to find useful information and interpretation of some physical systems in nature. In statistical aspects, the pathway idea is one of areas growing dramatically in this direction as shown in [12, 19]. In mathematics, we have similar situations as the Dirac delta generalized function shows up in many places of pure mathematics even though it violates the true definition of a function.

In the theory of renewal process, Poisson process plays an enormous role since it is the only one that comes from Bernoulli trial by the limit process. Many similar generalizations have been attempted. In [5], the authors provide a fractional renewal process, called a fractional Poisson process, and it appears as a good generalization of the original Poisson process and Erlang process.

In [10], the author insists that with respect to $\alpha$ the fractional world has to be categorized and classified under the condition that its Lévy structure be preserved where $\alpha$ is fixed and lies between 0 and 1.

In this paper, motivated by and combination of these ideas, the authors define a generalized random variable along with its generalized distribution and apply it to the world of the $\alpha$-fractional space. Furthermore the authors introduce a family of new fractional renewal processes.

To see the role of Lévy structure, the correspondence via J-transformation from the author’s paper [10] provides the following

$$e^{-x} \longleftrightarrow t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha).$$ (1.1)
In this paper, there is an analogous correspondence

\[
\text{Poisson process} \leftrightarrow \alpha\text{-fractional Poisson process. (1.2)}
\]

\section{\(\alpha\)-Fractional Space}

Note that \(\alpha\) varies in \((0, 1)\) throughout this paper and \(\alpha\) is fixed always.

\textbf{Definition 2.1} Let \(f(x)\) have \(H\)-function representation and be convergent \([20, 21]\). Then \(f(x)\) is said to be a function with Lévy structure if its \(H\)-function representation has the factor \(\frac{\Gamma(-\frac{s}{\alpha} + \frac{1}{\alpha})}{\alpha \Gamma(1-s)}\) in the integrand. In short, \(\frac{\Gamma(-\frac{s}{\alpha} + \frac{1}{\alpha})}{\alpha \Gamma(1-s)}\) will be called Lévy structure.

Note that the Laplace transform of \(\frac{1}{2\pi i} \int \frac{\Gamma\left(\frac{1}{\alpha} - \frac{s}{\alpha}\right)}{\alpha \Gamma(1-s)} x^{-s} ds, \ 1 > \text{Re}(s) > 0, \ \alpha > 0\) is \(e^{-s^\alpha}\), which is shown in \([18]\). Lévy structure is named from this relation.

In \([18]\), the author shows a process to lift a gamma density to a generalized Mittag-Leffler density function by using statistical techniques and it is in Example 2.1.

\textbf{Example 2.1} Let \(x\) be a simple exponential random variable with the density function \(f(x) = e^{-x}\). We attach the Lévy structure to \(E[x^{\frac{1}{\alpha}}]^{s-1} = \int_0^{\infty} (x^{\frac{1}{\alpha}})^{s-1} e^{-x} dx\). Then

\[
x^{\alpha-1} E(\alpha, \alpha)(-x^\alpha) = \frac{1}{2\pi i} \int \frac{\Gamma\left(\frac{1}{\alpha} - \frac{s}{\alpha}\right)}{\alpha \Gamma(1-s)} x^{-s} ds \tag{2.1}
\]

by the residue theorem.

In \([10]\), the author shows using Mellin transformation property how to lift a function in the ordinary space to a corresponding function in the \(\alpha\)-fractional(or \(\alpha\)-level) space in an analytic way including statistical ones.

Let \(f(x)\) be a function, which does not have Levy structure and lives in the ordinary space and \(h(x)\) be the Lévy density function as a kernel.

\textbf{Definition 2.2} Define the J-transform of \(f(x)\) by

\[
J(f)(x) = \lim_{\gamma \to \infty} \int_0^x \left(\frac{t}{x}\right)^\alpha f_2 \left(\left(\frac{x}{t}\right)^\gamma E(\alpha, \alpha \gamma)(-\gamma t^\alpha)\right) dt \tag{2.2}
\]

where \(\alpha\) fixed in \(0 < \alpha < 1, x \geq 0\) \(f_2(x) = xf(x)\) and \(f(x)\) is integrable and continuous on the interval.

\textbf{Example 2.2} Let \(f(x) = e^{-x}\). Then

\[
J(f)(x) = x^{\alpha-1} E(\alpha, \alpha)(-x^\alpha).
\]
Ordinary space and $\alpha$-fractional space

The concept or term of ordinary space looks strange. But we want the ordinary space to be generated by $\{0, 1, x, x^2, \ldots\}$, namely the exponents of the variables being non-negative integers only. Then we obtain the corresponding $\alpha$-fractional space generated by the elements spawned as J-transforms of $\{0, 1, x, x^2, \ldots\}$. The basis for $\alpha$-fractional space seems inherited from that of the ordinary space. But from the point of view of Probability theory, the following correspondence looks natural:

$$
\left\{0, 1, x, x^2, \ldots\right\} \longleftrightarrow \left\{0, \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \frac{x^{2\alpha-1}}{\Gamma(2\alpha)}, \frac{x^{3\alpha-1}}{\Gamma(3\alpha)}, \ldots\right\} \quad (2.3)
$$

Note that the factorial coefficients in front of variables must appear as statistical quantities since Mellin transformation and its family preserve and carry statistical measures. Therefore every function will have the Lévy structure in their $H$-function representations in the sense of the author [10]. In the $\alpha$-fractional world, we can change to the alternative definition of the Riemann-Liouville fractional derivative without the ordinary derivative as follows:

$$
(D_0^\alpha \phi_\alpha)(t) = \int_0^t \frac{(t-x)^{-\alpha-1}}{\Gamma(-\alpha)} \phi_\alpha(x) dx. \quad (2.4)
$$

We emphasise that the main function $\phi_\alpha(t)$ is the solution of the Reimann-Liouville fractional integral equation $\phi_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} = - (D_0^{-\alpha} \phi_\alpha)(t)$ whereas $g_1(t) = e^{-t}$ is the solution of $g_1(t) - 1 = - \int_0^t g_1(x) dx$.

3 Standard Poisson Process and CTRW

In [5], the authors describe the underlying theory and setting of renewal processes (Poisson process is a special case of renewal process) and continuous time random walk (CTRW).

**Standard Poisson Process and Erlang Process**

Poisson process is suitable for some situations like modelling of counting the number of arrivals of customers in a particular place in a given amount of time or some physical system showing random behaviors such as a jump to the next position. The probability for $n$ arrivals in the given time interval can be calculated with $p(n, t_1)$ for fixed $t_1$. It is assumed that $\lambda = 1$. Poisson process can be defined as an infinite sequence $0 = t_0 < t_1 < t_2 \cdots$ of events separated by i.i.d. (independent and identically distributed) random waiting times $T_j = t_j - t_{j-1}$ of exponential distributions. The defining characteristics of the Poisson process are time homogeneity, independence for different waiting time random variables and infinitesimal interval probabilities. These assumptions make $N(t)$ be a Poisson process with the Poisson distribution with the parameter $t$ and the expectation of $N(t)$ (the average number of arrivals in the given time interval)

$$
p(n, t) = \frac{t^n}{n!} e^{-t}, \quad m(t) = <N(t)> = t. \quad (3.1)
$$

$m(t)$ is known as a renewal function.

Its inverse process, which is called Erlang process, is designed to calculate the probability for
the time when the $n$-th arrival has just happened with its Erlang density function $q_n(t) = \frac{t^{n-1}}{(n-1)!}e^{-t}, t \geq 0$ and its cumulative distribution $Q_n(t)$.

In the probabilistic language, the following property is called memoryless:

$$P\{X > t + x\} = P\{X > t\}P\{X > x\}. \tag{3.2}$$

Since the exponential function satisfies this property only, Poisson process is expected to have memoryless property. But the Mittag-Leffler density function, which is the main function in this paper, does not have this property but the property of long-term-memory type.

**Continuous Time Random Walk**

In the theory of CTRW, by using Dirac delta functions, one is able to develop random walk model in discrete space as a special case of CTRW which is known as a compound Poisson process associated with random walk. It can be considered that time variable and space variable vary in the positive real line: choose $w(x) = \delta(x - 1), \phi_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}), x \geq 0, t > 0$. Notice that as indicated in [5], we can ignore the possible delta peak at the origin of the time line by taking $t > 0$. So the Cox-Weiss series for CTRW can be used and the Laplace-Laplace solution shall be taken to be:

$$\tilde{p}_\alpha(k, s) := \tilde{\Psi}_\alpha(s) \sum_{n=0}^{\infty} (\tilde{\phi}_\alpha(s)\tilde{w}(k))^n = \tilde{\Psi}_\alpha(s) \frac{1}{1 - \tilde{\phi}_\alpha(s)\tilde{w}(k)}, \quad p_\alpha(x, t) := \sum_{n=0}^{\infty} P_{\alpha,\alpha}(t)\delta(x - n) \tag{3.3}$$

where $\Psi_\alpha(t)$ is the marginal distribution of the density function $\phi_\alpha(t)$, which is defined in section 5.

In section 5 we shall find out some analogues of the integral equation of the CTRW and the Kolmogorov-Feller equation (the master equation of the compound Poisson process).

**4 Abel-Volterra Equation of the Second Kind**

$$f(x) - g(x) = -c(D_{\alpha}^{-\alpha}f)(x) \tag{4.1}$$

where $g(x)$ is a general integrable function on the finite interval $[0, b]$ and $0 < \alpha < 1$. For detailed history, theory and applications, see [8]. This integral equation is known as Abel-Volterra integral equation of the second kind. The solution of the Abel-Volterra integral equation is

$$f(x) = g(x) - c \int_{0}^{x} (x - t)^{\alpha-1}E_{\alpha,\alpha}(-c(x - t)^{\alpha})g(t)dt. \tag{4.2}$$

The following table from the author’s paper [10] gives certain densities and their Lévy structures.

**functions without Levy structure**
\[
g(x) \rightarrow f(x)
\]

\[
\begin{array}{|c|c|}
\hline
1 & E_\alpha(-cx) \\
\hline
x & xE_{\alpha,2}(-cx) \\
\hline
\sum_{k=0}^{\infty}(-x)^k E_{\alpha,k+1}(-cx) & x^{-1}E_{\alpha,2}(-cx) \\
\hline
x^{-1}E_{\alpha,1}^\gamma(-cx) & x^{-1}E_{\alpha,2}^\gamma(-cx) \\
\hline
\end{array}
\]

functions with Levy structure

So when \( c = (1 - \tilde{w}(k)) \) and \( g(x) = 1 \), the Abel-Volterra integral equation becomes

\[
f(t) - 1 = -(1 - \tilde{w}(k))(D_0^{-\alpha} f)(t). \tag{4.3}
\]

Then \( E_\alpha(-(1 - \tilde{w}(k))t^\alpha) \), which appears in [3], becomes the solution of the above Abel-Volterra integral equation. As suggested in [10], we consider an Abel-Volterra integral equation with Levy structure preserved as follows: Set \( g(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) and \( N_0 = 1 \) where \( c = (1 - \tilde{w}(k)) \). Then the Abel-Volterra integral equation becomes

\[
f(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} = -c(D_0^{-\alpha} f)(t). \tag{4.4}
\]

We have the following solution

\[
f(t) = t^{\alpha-1}E_{\alpha,\alpha}(-ct^\alpha). \tag{4.5}
\]

Define \( \tilde{p}_\alpha(k, t) := t^{\alpha-1}E_{\alpha,\alpha}(-(1 - w(k))t^\alpha) \). From the Laplace-pair table, we have

\[
\tilde{p}_\alpha(k, s) = \frac{1}{s^\alpha} \left( 1 + \frac{(1 - \tilde{w}(k))}{s^\alpha} \right)^{-1} = \frac{1}{(1 + s^\alpha)} \left( \frac{1}{1 + \frac{w(k)}{1 + s^\alpha}} \right) \tag{4.6}
\]

\[
= \sum_{n=0}^{\infty} \tilde{\phi}_{\alpha}^{1+n}(s) \tilde{w}(k)^n \tag{4.7}
\]

When \( \tilde{w}(k) = e^{-k} \), we have

\[
p_\alpha(x, t) = \sum_{n=0}^{\infty} (\phi_{\alpha}^{s(1+n)}) (t) \delta(x - n). \tag{4.8}
\]

### 5 A Fractional Renewal Process of Order \( \alpha \)

**Definition 5.1** By an asymptotic limit, we mean that if a density is asymptotically equivalent to a power function of the form \( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) for a fixed \( \alpha \) with \(|\alpha| < 1\) when its parameter or variable approaches to infinity, then the power function is said to be the asymptotic limit of the density function.
Example 5.1  The Gaussian normal density has the asymptotic limit as \( t \to \infty \)

\[
\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \exp\left( -\frac{x^2}{t} \right) \to \frac{t^{1/2-1}}{\Gamma(\frac{1}{2})}.
\]  

(5.1)

Example 5.2  The Lévy density function of order \( \frac{1}{2} \) has the asymptotic limit

\[
\frac{at^{-\frac{3}{2}}}{2\Gamma(\frac{1}{2})} \exp\left( -\frac{a^2}{t} \right) = \frac{at^{-\frac{3}{2}}}{-\Gamma(-\frac{1}{2})} \exp\left( -\frac{a^2}{t} \right) \to \frac{t^{-\frac{1}{2}-1}}{-\Gamma(-\frac{1}{2})}
\]  

as \( a \) goes to \(-1\) and \( t \) tends to infinity.

Note that for the Green functions appearing in the above examples, we refer to [17], also refer to [3] for the systematic treatment of Lévy density functions and the author provides the explicit formulae of Lévy stable density functions of the type \( (p/q)^{\frac{\alpha}{n}} \) in [11].

Definition 5.2  An asymptotic limit is said to be a generalized density function if the limit is from a density function and it makes sense when the function is inside an integral. In fact, it is not a density function in general and it may be called a supplementary density function following the spiritual concept of Dirac delta generalized function. Accordingly a generalized random variable is to be defined with its generalized density function.

Remark. We have generalized random variables of order \( \alpha \) with \( |\alpha| < 1 \). When \( \alpha \) is equal to zero, then the power function become the Dirac delta function in the sense of Gel’fond and Shilov [1]. Here we assume that we are able to obtain asymptotic limits of all of \( \alpha \). The generalized density functions of the form \( t^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \), \( 0 < \alpha < 1 \), plays a role as the Riemann-Liouville fractional integral and the generalized density functions of the form \( t^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(-\alpha)} \), \( 0 < \alpha < 1 \), plays a role as the Riemann-Liouville fractional derivative in the \( \alpha \)-fractional world.

Let \( X_1 \) be the generalized random variable of order \( \alpha \) fixed with \( 0 < \alpha < 1 \) and \( S_1 \) be the Mittag-Leffler random variable of the same order \( \alpha \). Then \( X_1 \) has the corresponding generalized density function of the form \( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) and \( S_1 \) does the density function of the form \( \phi_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) \).

We construct a world \( \Omega \) of random variables with marginal distributions obtained by considering \( X_1 + S_1 \) from the density function. Then we have the following relations:

\[
\Phi_\alpha(t) = (\alpha t^{\alpha-1} \phi_\alpha)(t) = \int_0^t \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)} \phi_\alpha(t_1)dt_1,
\]

(5.3)

\[
\Psi_\alpha(t) = (t W^\alpha \phi_\alpha)(t) = \int_t^\infty \frac{(t_1-t)^{\alpha-1}}{\Gamma(\alpha)} \phi_\alpha(t_1)dt_1 = \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \Phi_\alpha(t).
\]

(5.4)

It can be called \( \Phi_\alpha(t) := P(T \leq t) \) the failure marginal probability and \( \Psi_\alpha(t) := P(T > t) \) the survival marginal Probability. Note that the subscript \( \alpha \) is used to indicate that the function lives in the \( \alpha \)-fractional space.
Lemma 5.1 The following equalities are true:
\[ \phi_\alpha(t) = -(\alpha D_t^\alpha \Psi_\alpha)(t), \quad \Psi_\alpha(t) = \phi_\alpha(t). \] (5.5)

Proof Use the alternative definition for \( D_t^\alpha \). \( \square \)

This lemma enables us to define a generalization of the Poisson process in the \( \alpha \)-fractional space. Note that the process \( \alpha D_t^\alpha \Psi_\alpha(t) \) is the reverse process from the marginal distribution to the density function.

In this section, we frequently calculate \( \phi_\alpha^n \) for \( n \in \mathbb{N} \). So we present the following formula: it is known that \( \frac{(n)_k}{k!} \), \( k = 0, 1, 2, 3, \ldots \), generates the sequence of counting numbers when \( n = 2 \), that of triangular numbers when \( n = 3 \), that of tetrahedral numbers when \( n = 4 \) and so on, which all come from the diagonal sequences of Pascal’s triangle. The following equalities are necessary:

\[
\sum_{k=0}^{\infty} t^k \sum_{n=0}^{\infty} t^n = \sum_{k=0}^{\infty} (\sum_{n=0}^{k} t^k) = \sum_{k=0}^{\infty} (k+1)t^k \tag{5.6}
\]

\[
\sum_{k=0}^{\infty} \frac{(k+1)t^k}{2} \sum_{n=0}^{\infty} t^n = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} t^k \tag{5.7}
\]

\[
\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} t^k \sum_{n=0}^{\infty} t^n = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)(k+3)}{3!} t^k \tag{5.8}
\]

\[
\cdots \tag{5.9}
\]

The coefficients of the above series \( (5.6), (5.7), (5.8) \) and so on also generate the diagonal sequences of Pascal’s triangle. It can be verified using the properties of the diagonal sequences of Pascal’s triangle. Therefore the sequences generated by \( \frac{(n)_k}{k!} \), \( k = 0, 1, 2, 3, \ldots \) are equal to those obtained from the coefficients of the above series with the correspondence that \( n = 2 \leftrightarrow (5.6), n = 3 \leftrightarrow (5.7), n = 4 \leftrightarrow (5.8) \) and so on. By using these properties, we obtain

\[
\phi_\alpha^n = t^{n-1} E_{\alpha,n}(\alpha t^{\alpha}) = t^{\alpha-1} E_{\alpha,n}(\alpha(1-w(k))) t^{\alpha} \tag{5.10}
\]

5.1 Mittag-Leffler function \( t^{\alpha-1} E_{\alpha,n}(\alpha(1-w(k))) t^{\alpha} \)

Cox-Weiss Series and Analogue of Montroll-Weiss Equation

We have the following Cox-Weiss series

\[
\tilde{p}_\alpha(k, s) = \tilde{\Psi}_\alpha(s) \sum_{n=0}^{\infty} \phi_\alpha(s)^n w(k)^n \tag{5.11}
\]

Since \( \tilde{\Psi}_\alpha(s) = \frac{1 - \phi_\alpha(s)}{s^\alpha} \), we obtain the analogue of the famous Montroll-Weiss equation from the Cox-Weiss series.

\[
\tilde{p}_\alpha(k, s) = \frac{1 - \phi_\alpha(s)}{s^\alpha} \sum_{n=0}^{\infty} \phi_\alpha(s)^n w(k)^n \tag{5.12}
\]
The Laplace transform of $\phi_\alpha(s)$ is $\frac{1}{1+s^\alpha}$. Hence we get

$$\widetilde{p}_\alpha(k, s) = \phi_\alpha(s) \sum_{n=0}^{\infty} \phi_\alpha(s) \ w(k) \ n.$$  

(5.13)

which is equal to (4.7).

**Renewal Function**

Following the procedure in [5], when $\tilde{w}(k) = e^{-k}$, we get

$$\overline{p}_\alpha(x, t) = \sum_{n=0}^{\infty} P_\alpha(t) \delta(x - n) = \sum_{n=0}^{\infty} (\Psi_\alpha * \phi_\alpha^n)(t) \delta(x - n)$$  

(5.14)

and

$$m_\alpha(t) = -\frac{\partial}{\partial k} \overline{p}_\alpha(k, t)|_{k=0} = \left( \sum_{n=0}^{\infty} nP_\alpha(t) e^{-nk} \right) |_{k=0} = \sum_{n=0}^{\infty} nP_\alpha(t).$$  

(5.15)

By applying this and $\sum_{n=0}^{\infty} n z^n = \frac{z}{(1-z)^2}$, $|z| < 1$, we obtain

$$\overline{m}_\alpha(s) = \overline{\Psi}_\alpha(s) \sum_{n=0}^{\infty} n \phi_\alpha(s) \ n = \frac{1 - \phi_\alpha(s)}{s^\alpha} \left( \frac{\phi_\alpha(s)}{(1 - \phi_\alpha(s))^2} \right) = \frac{\overline{\phi}_\alpha(s)}{s^\alpha(1 - \phi_\alpha(s))}.$$  

(5.16)

Therefore we have derived the reciprocal pair of the relationships in the Laplace domain

$$\overline{m}_\alpha(s) = \frac{\overline{\phi}_\alpha(s)}{s^\alpha(1 - \phi_\alpha(s))}, \quad \overline{\phi}_\alpha(s) = \frac{s^\alpha \overline{m}_\alpha(s)}{1 + s^\alpha \overline{m}_\alpha(s)}.$$  

(5.17)

Hence the renewal equation is

$$m_\alpha(t) = (D_0^{-\alpha} \phi_\alpha)(t) + (m_\alpha * \phi_\alpha)(t)$$  

(5.18)

On the other hand, we have the expression

$$\overline{p}_\alpha(k, t) = t^{\alpha-1} E_{(\alpha, \alpha)}(-(1-e^{-k})t^\alpha).$$  

(5.19)

From this, we derive the following

$$m_\alpha(t) = -\frac{\partial}{\partial k} \overline{p}_\alpha(k, t)|_{k=0} = \frac{t^{2\alpha-1}}{\Gamma(2\alpha)}, \quad 0 < \alpha < 1.$$  

(5.20)

We have a well-known infinite system of differential-difference equations for the Poisson process with intensity $\lambda > 0$ and $t \geq 0$,

$$P_0(t) = e^{-\lambda t}, \quad \frac{d}{dt} P_n(t) = \lambda (P_{n-1}(t) - P_n(t)), \quad n \geq 1.$$  

(5.21)
with initial conditions \( P_n(0) = 0, n = 1, 2, 3, \ldots \), which can be used to define the Poisson process. We give an analogous system of fractional differential-difference equations for the fractional poisson process in the \( \alpha \)-fractional space.

In [23], the following Laplace pair is provided:

\[
\frac{t^{\alpha(n+1)-1}}{n!} E^{(n)}_{(\alpha,\alpha)}(-t^\alpha) \longleftrightarrow \frac{1}{(1 + s^\alpha)^{n+1}}
\]

So we get

\[
\tilde{P}_\alpha(k, t) = \sum_{n=0}^{\infty} \frac{t^{\alpha-1} t \alpha n}{n!} E^{(n)}_{(\alpha,\alpha)}(-t^\alpha) \tilde{w}^n(k).
\]

**Counting Probabilities**

Therefore the counting probabilities are

\[
P_{n,\alpha}(t) = P\{N(t) = n\} = \frac{t^{\alpha-1} t \alpha n}{n!} E^{(n)}_{(\alpha,\alpha)}(-t^\alpha).
\]

Or directly

\[
P_{n,\alpha}(t) = \phi^*(n+1)(t) = t^{(n+1)\alpha-1} E^{n+1}_{\alpha,(n+1)\alpha}(-t^\alpha) = \frac{t(n+1)\alpha-1}{n!} \sum_{k=0}^{\infty} \frac{\Gamma(n + 1 + k) (-1)^k t^\alpha}{\Gamma(\alpha n + \alpha + ak) k!}
\]

**Fractional Differential-Difference Equations**

Then we have the following analogous differential-difference Equations

\[
\tilde{P}_{n,\alpha}(s) = \frac{1}{(1 + s^\alpha)^{n+1}}
\]

(5.26)

\[
(1 + s^\alpha)\tilde{P}_{n,\alpha}(s) = \tilde{P}_{n-1,\alpha}(s)
\]

(5.27)

\[
s^\alpha \tilde{P}_{n,\alpha}(s) = \tilde{P}_{n-1,\alpha}(s) - \tilde{P}_{n,\alpha}(s)
\]

(5.28)

\[
P_{0,\alpha}(t) = t^\alpha - \tilde{P}_{\alpha,\alpha}(-t^\alpha), (D_{0+}^\alpha P_{n,\alpha})(t) = P_{n-1,\alpha}(t) - P_{n,\alpha}(t).
\]

(5.29)

where \( D_{0+}^\alpha \) is defined in the appendix.

**Erlang Densities**

Since \( q_{n,\alpha}(t) = \phi^\alpha_n(t) \), it can be calculated directly. From the formula (5.10), it can be easily checked that

\[
q_{n,\alpha}(t) = t^{\alpha-1} E^n_{\alpha,\alpha}(-t^\alpha) = \frac{t^{\alpha-1} t \alpha n}{(n-1)!} \sum_{k=0}^{\infty} \frac{\Gamma(n + k) (-1)^k t^\alpha}{\Gamma(\alpha n + \alpha + ak) k!}.
\]

(5.30)

**Fractional Integral Equation of the Cotinuous Time Random Walk**

We can derive the fractional integral equation of the CTRW as follows:

\[
\tilde{p}_\alpha(k, s) = \frac{\tilde{p}_\alpha(s)}{1 - \tilde{\phi}_\alpha(s) \tilde{w}(k)}
\]

\[
\tilde{p}_\alpha(k, s) - \tilde{\phi}_\alpha(s) \tilde{w}(k) \tilde{p}_\alpha(k, s) = \tilde{\phi}_\alpha(s)
\]

\[
\tilde{p}_\alpha(k, s) = \tilde{\phi}_\alpha(s) + \tilde{\phi}_\alpha(s) \tilde{w}(k) \tilde{p}_\alpha(k, s)
\]

\[
p_\alpha(x, t) = \Psi_\alpha(t) \delta(x) + \int_0^x \phi_\alpha(t - t_1) dt_1 \int_0^x w(x - x_1) p_\alpha(x_1, t_1) dx_1.
\]
Fractional Version of Kolmogorov-Feller Equation

\[
\tilde{p}_\alpha(k, s) = \frac{\tilde{\phi}_\alpha(s)}{1 - \tilde{\phi}_\alpha(s)\tilde{w}(k)} \quad (5.31)
\]

\[
\tilde{p}_\alpha(k, s) = \frac{1}{1 + s^\alpha} \frac{1}{1 - \tilde{\phi}_\alpha(s)\tilde{w}(k)} \quad (5.32)
\]

\[
1 + s^\alpha \tilde{p}_\alpha(k, s) = \frac{1}{1 - \tilde{\phi}_\alpha(s)\tilde{w}(k)} \quad (5.33)
\]

\[
s^\alpha \tilde{p}_\alpha(k, s) = -\tilde{p}_\alpha(k, s) + \frac{1}{1 - \tilde{\phi}_\alpha(s)\tilde{w}(k)} - 1 + 1 \quad (5.34)
\]

\[
s^\alpha \tilde{p}_\alpha(k, s) = -\tilde{p}_\alpha(k, s) + \tilde{w}(k)\tilde{p}_\alpha(k, s) + 1 \quad (5.35)
\]

\[
(D^\alpha_{0^+}p_\alpha(x, t))(t) = -p_\alpha(x, t) + \int_0^x w(x - x_1)p_\alpha(x_1, t)dx_1 + \delta(x)\delta(t) \quad (5.36)
\]

\[
(D^\alpha_{0^+}p_\alpha(x, t))(t) = -p_\alpha(x, t) + \int_0^x w(x - x_1)p_\alpha(x_1, t)dx_1 \quad (5.37)
\]

\[
(D^\alpha_{0^+}p_\alpha(x, t))(t) = -p_\alpha(x, t) + \int_0^x \delta(x - x_1)p_\alpha(x_1, t)dx_1. \quad (5.38)
\]

Notice that we have taken \( t > 0 \) to avoid the problem at \( t = 0 \). Therefore \( \delta(t) = 0 \) which explains the above equation.

5.2 Another Interesting Density of Mittag-Leffler Type

Take a look at \( \phi_{**}\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) \). The \( H \)-function representation of this function is

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\Gamma(l - \frac{1}{\alpha} + \frac{s}{\alpha})}{\Gamma(l)} \frac{\Gamma\left(\frac{1}{\alpha} - \frac{s}{\alpha}\right)}{\alpha \Gamma(1 - s)} t^{-s}ds \]

which has the Lévy structure. This function appears in the literature and is called a generalized Mittag-Leffler density, see [9, p.28]. And its Laplace transform is \( \frac{1}{1 + s^\alpha} \). The most interesting part is its relationship with the fractional Poisson process which we have developed in section 5.1.

Define \( \tilde{p}_{\alpha, l}(k, t) := t^{\alpha-1}E_{\alpha,\alpha}(-1 - w(k)t^\alpha) \). Then we get

\[
\tilde{p}_{\alpha, l}(k, s) := \tilde{p}_{\alpha}(k, s) = \left( \frac{\tilde{\phi}_\alpha(s)}{1 - \tilde{\phi}_\alpha(s)\tilde{w}(k)} \right)^t \quad (5.39)
\]

\[
\tilde{p}_{\alpha, l}(k, s) = \tilde{\phi}_\alpha(s) \sum_{n=0}^{\infty} \frac{(l)_n \tilde{\phi}_\alpha^n \tilde{w}(k)^n}{n!} \tilde{\phi}_\alpha(s) \sum_{n=0}^{\infty} \binom{n + l - 1}{n} \tilde{\phi}_\alpha^n \tilde{w}(k)^n. \quad (5.40)
\]

The corresponding analogous Cox-Weiss series is:

\[
p_{\alpha, l}(x, t) = p_{\alpha, l}(x, t) = \sum_{n=0}^{\infty} \frac{(l)_n}{n!} (\Psi_{**\alpha} * \phi_{**\alpha}^n)(t)w^{*n}(x). \quad (5.41)
\]

And we obtain

\[
\tilde{p}_{\alpha, l}(k, t) = \sum_{n=0}^{\infty} \frac{(l)_n}{n!} \left( \frac{t^{\alpha-1}l^{\alpha}}{(n + l - 1)!} E_{\alpha,\alpha}^{(n + l - 1)}(-t^\alpha) \right) \tilde{w}(k). \quad (5.42)
\]
Counting Probabilities

Therefore the counting probabilities are

\[ P_{n,a,l}(t) = P\{N_1(t) + N_2(t) + \cdots + N_l(t) = n\} = \frac{(l)_n}{n!} \frac{t^{al-1}t^{\alpha n}}{(n + l - 1)!} E^{(n+l-1)}_{(a,\alpha)}(-t^\alpha) \] (5.43)

where \( N_j(t) \) is I.I.D for \( j = 1 \ldots l \). Or directly

\[ P_{n,a,l}(t) = \frac{(l)_n t^{(n+l)\alpha-1} E^{n+l}_{a,(n+l)\alpha}(-t^\alpha)}{n!} \]

(5.44)

\[ = \frac{n!}{(n + l - 1)!} \frac{1}{n} \sum_{k=0}^{\infty} \frac{\Gamma(n + l + k)}{\Gamma(\alpha(n + l) + ak)} (-1)^k t^{ak} \] (5.45)

Renewal Function

For the renewal function and the density function, we obtain

\[ \tilde{m}_{a,l}(s) = \Psi_{a}(s) \sum_{n=0}^{\infty} \frac{(l)_n}{n!} n \phi_a(s)^n \]

\[ = \frac{l \phi_a(s)}{s^\alpha (1 - \phi_a(s))} \]

\[ \phi_a(s) = \frac{s^a \tilde{m}_{a,l}(s)}{(l + s^a \tilde{m}_{a,l}(s))} \] (5.46)

since \( \sum_{n=0}^{\infty} \frac{(l)_n z^n}{n!} = \frac{lz}{(1 - z)^{l+1}}, |z| < 1 \). And the renewal equation is given by

\[ m_{a,l}(t) = l(D_{0+}^{-a} \phi_a)(t) + (m_{a,l} * \phi_a)(t) \] (5.47)

We can derive the following from the expression \( \tilde{p}_{a,l}(k,t) = t^{al-1} E^l_{a,\alpha l}(-(1 - e^{-k})t^\alpha) \):

\[ m_{a,l}(t) = -\frac{\partial}{\partial k} \tilde{p}_{a,l}(k,t)|_{k=0} = \frac{t^{la-\alpha-1}}{\Gamma((l+1)\alpha)} \] (5.48)

Fractional Differential-Difference Equations

And we also get the following analogous differential-difference Equations

\[ \tilde{P}_{n,a,l}(s) = \frac{(l)_n}{n!(1 + s^\alpha)^{l+n}} \] (5.49)

\[ \frac{n s^\alpha}{n + l - 1} \tilde{P}_{n,a,l}(s) = \tilde{P}_{n-1,a,l}(s) - \frac{n}{n + l - 1} \tilde{P}_{n,a,l}(s) \] (5.50)

\[ P_{0,a,l}(t) = t^{al-1} E^l_{a,\alpha l}(-t^\alpha) \] (5.51)

\[ \frac{n}{n + l - 1} D_{0+}^{a} P_{n,a,l}(t) = P_{n-1,a,l}(t) - \frac{n}{n + l - 1} P_{n,a,l}(t) \] (5.52)

Erlang Densities

For the corresponding Erlang densities, we have

\[ q_{n,a,l}(t) = q_{n,a,l}^{al}(t) = t^{n\alpha-1} E^{lna}_{a,lan}(-t^\alpha) \] (5.53)

\[ = \frac{t^{n\alpha-1}}{(ln - 1)!} \sum_{k=0}^{\infty} \frac{\Gamma(ln + k)}{\Gamma(\alpha(ln + k))} \frac{(-1)^k t^{ak}}{k!} \] (5.54)
Fractional Integral Equation of the Continuous Time Random Walk

The fractional integral equation of the CTRW is as follows:

$$\tilde{p}_{\alpha,l}(k, s) = \left( \frac{\Psi_{\alpha}(s)}{1 - \tilde{\phi}_{\alpha}(s)\tilde{\omega}(k)} \right)^l$$

$$\left( \sum_{j=0}^{l} \binom{l}{j} (-1)^j (\tilde{\phi}_{\alpha}(s)\tilde{\omega}(k))^j \right)\tilde{p}_{\alpha,l}(k, s) = (\tilde{\Psi}_{\alpha}(s))^l$$

$$\sum_{j=0}^{l} \binom{l}{j} (-1)^j \tilde{\tilde{\phi}}_{\alpha}^j(s)\tilde{\omega}^j(k)\tilde{p}_{\alpha,l}(k, s) = (\tilde{\tilde{\Psi}}_{\alpha}(s))^l$$

$$p_{\alpha,l}(x, t) = \Psi_{\alpha}^l(t)\delta(x) - \sum_{j=1}^{l} \left( \binom{l}{j} (-1)^j \phi_{\alpha}^j(t) * p_{\alpha,l}(x, t) * w^j(x) \right).$$

Fractional Version of Kolmogorov-Feller Equation

$$\tilde{p}_{\alpha,l}(k, s) = \left( \frac{\tilde{\Psi}_{\alpha}(s)}{1 - \tilde{\phi}_{\alpha}(s)\tilde{\omega}(k)} \right)^l$$

$$(1 + s^{\alpha})\tilde{p}_{\alpha,l}(k, s) = \frac{\tilde{\Psi}_{\alpha}^{l-1}(s)}{(1 - \tilde{\phi}_{\alpha}(s)\tilde{\omega}(k))^l}$$

$$s^{\alpha}\tilde{p}_{\alpha,l}(k, s) = -\tilde{p}_{\alpha,l}(k, s) + \tilde{\Psi}_{\alpha}(s)\tilde{p}_{\alpha,l}(k, s)$$

$$(D^{\alpha}_{0+}p_{\alpha,l}(x, t))(t) = -p_{\alpha,l}(x, t) + \int_{0}^{t} \Psi_{\alpha}(t - t_1)p_{\alpha,l}(x, t_1)dt_1. \quad (5.58)$$

$$\tilde{p}_{\alpha,l}(k, s) = \left( \frac{\tilde{\phi}_{\alpha}(s)}{1 - \tilde{\phi}_{\alpha}(s)\tilde{\omega}(k)} \right)^l$$

$$(1 + s^{\alpha})\tilde{p}_{\alpha,l}(k, s) = \frac{1}{(1 - \tilde{\phi}_{\alpha}(s)\tilde{\omega}(k))}\tilde{p}_{\alpha}^{l-1}(k, s)$$

$$(1 + s^{\alpha})\tilde{p}_{\alpha,l}(k, s) = \left( \frac{1}{(1 - \tilde{\phi}_{\alpha}(s)\tilde{\omega}(k))} - 1 + 1 \right)\tilde{p}_{\alpha}^{l-1}(k, s)$$

$$(1 + s^{\alpha})\tilde{p}_{\alpha,l}(k, s) = \tilde{\omega}(k)\tilde{p}_{\alpha}^{l}(k, s) + \tilde{\phi}_{\alpha}^{l-1}(k, s)$$

$$s^{\alpha}\tilde{p}_{\alpha,l}(k, s) = -\tilde{p}_{\alpha,l}(k, s) + \tilde{\omega}(k)\tilde{p}_{\alpha,l}(k, s) + \tilde{p}_{\alpha}^{l-1}(k, s)$$

$$(D^{\alpha}_{0+}p_{\alpha,l}(x, t))(t) = -p_{\alpha,l}(x, t) + \int_{0}^{t} w(x - x_1)p_{\alpha,l}(x_1, t)dx_1 + p^{s(l-1)}_{\alpha}(x, t). \quad (5.64)$$

6 Operational Time and Subordination Integrals

The concept of operational time is introduced in [6], which is an analogous concept of that in operational calculus. Let

$$p_{1}(y, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}e^{-t}\delta(y - n). \quad (6.1)$$
By using the equality \( \int_0^\infty e^{-at} dt = \frac{1}{a} \) for \( a = s^\alpha + 1 - e^{-k} \), we can get the following named as the subordination integral, which connects Poisson process to \( \alpha \)-fractional Poisson process when the time \( t_* \) evolves into \( t \) with respect to the Lévy law via Laplace transformation,

\[
\tilde{p}_\alpha(k, s) = \frac{1}{1 + s^\alpha - e^{-k}} \int_0^\infty e^{-t_*(s^\alpha + 1 - e^{-k})} dt_* = \int_0^\infty e^{-t_*(1 - e^{-k})} e^{-t_* s^\alpha} dt_*
\]

\[
p_\alpha(x, t) = \int_0^\infty p_1(x, t_*) p_2(t_*, t) dt_* = \int_0^\infty \sum_{n=0}^\infty \frac{t_*^n}{n!} e^{-t_*} \delta(x - n) \alpha t_* \mathcal{M}_\alpha \left( \frac{t_*}{\alpha} \right) dt_*. \tag{6.4}
\]

**Evolution equation for the density \( p_2(t_*, t) \) of \( t = t(t_*) \)**

In [7], the evolution equation for \( p_2(t_*, t) \) is defined by

\[
\tilde{p}_2(s_*, s) = \frac{1}{s^\alpha + s_*}, \quad \tilde{p}_2(t_*, s) = \exp(-t_* s^\alpha), \quad \tilde{p}_2(s_*, t) = t^\alpha - 1 E_{\alpha, \alpha}(-s_* t^\alpha) \tag{6.5}
\]

\[
s_* \tilde{p}_2(s_*, s) - 1 = -s^\alpha \tilde{p}_2(s_*, s) \tag{6.6}
\]

\[
\frac{\partial}{\partial t_*} p_2(t_*, t) = -(D_{\alpha} p_2(t_*, t))(t), \quad p_2(0+, t) = \delta(t) \tag{6.7}
\]

In [17], the famous initial value problem, known as signalling problem, with the partial differential equation \( \frac{\partial}{\partial t} p(t_*, t) = \frac{\partial^2}{\partial t^2} p(t_*, t) \) and the initial conditions, \( \lim_{t_* \to 0^+} p(t_*, t) = \delta(t), \quad t > 0, \quad \lim_{t_* \to \infty} p(t_*, t) = 0 \), has the following Laplace transform of the solution and the solution

\[
\tilde{p}(t_*, s) = e^{-t_* s^\alpha} \quad \text{and} \quad p(t_*, t) = \frac{t_*}{2 \sqrt{\pi t}} \exp \left( -\frac{t_*^2}{4t} \right), \quad 0 < t < \infty. \tag{6.8}
\]

Therefore this partial differential equation can describe the behavior of the function \( p(t_*, t) \), which is the integrand of (6.4), or the relationship between two different time lines for the case when \( \alpha = 0.5 \). In [6], the authors have found \( q_\alpha(t_*, t) \) for their processes and gave a relation with \( p_2(t_*, t) \).

\[
q_\alpha(t_*, t) = t^{-\alpha} M_\alpha(t_* t^{-\alpha}) \longleftrightarrow \tilde{q}_\alpha(t_*, s) = s^\alpha - 1 e^{-t_* s^\alpha} \tag{6.9}
\]

\[
p_2(t_*, t) = \frac{\alpha t_*}{t_1 + \alpha} M_\alpha \left( \frac{t_*}{t^\alpha} \right) \longleftrightarrow \tilde{p}_2(t_*, s) = e^{-t_* s^\alpha} \tag{6.10}
\]

\[
q_\alpha(t_*, t) = (I_0^{-\alpha} p_2(t_*, t_1))(t) \tag{6.11}
\]

From the above relation, the processes developed here evolve in a different time line compared with the processes doing in a certain time fashion in [6].
Furthermore, we have similar results for the cases in section 5.2.

\[
\tilde{\rho}_{\alpha,l}(k, s) = \left( \frac{\phi_{\alpha}(s)}{1 - \phi_{\alpha}(s)w(k)} \right) \cdots \left( \frac{\phi_{\alpha}(s)}{1 - \phi_{\alpha}(s)w(k)} \right) \tag{6.12}
\]

\[
= \left( \frac{1}{1 + s^\alpha \cdot e^{-k}} \right) \cdots \left( \frac{1}{1 + s^\alpha \cdot e^{-k}} \right) \tag{6.13}
\]

\[
= \int_0^\infty e^{-t_{*,1}(1-e^{-k})} e^{-t_{*,1} s^\alpha} dt_{*,1} \cdots \int_0^\infty e^{-t_{*,l}(1-e^{-k})} e^{-t_{*,l} s^\alpha} dt_{*,l} \tag{6.14}
\]

\[
= \int_0^\infty \cdots \int_0^\infty \exp\left(- (1 - e^{-k})(t_{*,1} + t_{*,2} + \cdots + t_{*,l})\right)
\times \exp\left(- s^\alpha (t_{*,1} + t_{*,2} + \cdots + t_{*,l})\right) dt_{*,1} \cdots dt_{*,l} \tag{6.15}
\]

\[
= \int_0^\infty \cdots \int_0^\infty \sum_{n=0}^\infty \frac{(t_{*,1} + t_{*,2} + \cdots + t_{*,l})^n e^{-nk}}{n!} \exp\left(- (t_{*,1} + t_{*,2} + \cdots + t_{*,l})\right)
\times \exp\left(- s^\alpha (t_{*,1} + t_{*,2} + \cdots + t_{*,l})\right) dt_{*,1} \cdots dt_{*,l} \tag{6.16}
\]

\[
p_2(s_{*,1}, s_{*,2}, \ldots, s_{*,l}, t) = \frac{\alpha (t_{*,1} + t_{*,2} + \cdots + t_{*,l})}{t_{*,1} + t_{*,2} + \cdots + t_{*,l}} M_{\alpha} \left( \frac{(t_{*,1} + t_{*,2} + \cdots + t_{*,l})}{t^\alpha} \right) \tag{6.17}
\]

### 7 Statistical Analysis and Some Remarks

**A Random Variable as a Product of Two Independent Random Variables**

In [9, 18], the Mittag-Leffler random variable is well examined. To support our theory, we will examine the statistical techniques used there in detail and $p_2(t_*, s)$ in the subordination integral will be reinterpreted.

**Theorem 7.1 (The Law of Total Expectation)**

\[
E(x) = E[E(x|y)] \tag{7.1}
\]

where $x$ and $y$ are random variables having a joint distribution, all the expectations do exist and in the marginal space of $y$, the outside expectation is calculated.
In [18], the author statistically describes the structure of Mittag-Leffler random variable as a product of two independent random variables by using the lemma 7.1. The method is as follows: Let \( t^* \) be a exponential random variable with the density function \( g_1(t^*) = e^{-t^*}, \ 0 \leq t^* < \infty \) and \( y \) be a positive Lévy random variable with the density function \( g_2(y) = \frac{1}{2\pi i} \oint_{L} \frac{\Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha \Gamma(1 - s)} y^{-s} ds, \ 1 > \text{Re}(s) > 0, \ \alpha > 0, \ y > 0 \). Then the Mittag-Leffler random variable \( t \) has the form of the product of two independently distributed random variables \( t_{\alpha}^* y \). The proof goes as follows using expectations and their properties below:

- \( E[e^{-sg}] = e^{-s^\alpha} \)
- \( E[\exp(-\{st^*_\alpha\}y)|t^*] = e^{-s^\alpha t^*} \)
- \( E[E[\exp(-\{st^*_\alpha\}y)|t^*]] = E[e^{-s^\alpha t^*}] = \int_0^\infty e^{-t^*} e^{-s^\alpha t^*} dt^* = \frac{1}{1 + s^\alpha} = E[e^{-st}] \).

Assume that there exists the random variable 1 of the constant function \( g(x) = 1 \) in the ordinary space. As analysed above, the Mittag-Leffler variable \( u \) of \( \phi(u) = u^{\alpha - 1} E_{\alpha,\alpha}(-u^\alpha) \) can be considered as the product of two independent random variables \( t_1 \) exponentially distributed and \( y \) Lévy-distributed with index \( \alpha \), namely \( u = yt_1^\alpha \). Note that this technique is well known for real density functions and J-transform will take the place for non-density functions which is magic and a different recipe of Mellin convolution technique. We shall analyse the relationship between a density function \( f(t_1) \) and its cumulative distribution \( \Phi(t_2) \) statistically. Look at the mathematical relation between them in the ordinary space in the below form

\[
\Phi(t_2) = \int_0^{t_2} 1 \cdot f(t_1) dt_1 = \int_0^{t_2} g(t_2 - t_1) \cdot f(t_1) dt_1. \quad (7.2)
\]

This relation can be interpreted in statistics as follows. We can consider \( \Phi(t_2) \) as the density function of the random variable \( t_2 = t_1 + 1 \) which is the Laplace convolution of two functions as in (7.2). With the help of J-transformation, we have the corresponding generalized random variable \( y_1 = y 1^{\frac{1}{\alpha}} \) of 1 in the \( \alpha \)-fractional space with the generalized density function \( \frac{y_1^{\alpha - 1}}{\Gamma(\alpha)} \). Because there are no statistical methods for this correspondence except J-transformation. From the above analysis, we take the random variable \( t_3 = yt_1^\frac{1}{\alpha} + y 1^{\frac{1}{\alpha}} \) for the marginal cumulative distribution of the density function \( \phi_\alpha(u) \) of the Mittag-Leffler random variable \( u = yt_1^\frac{1}{\alpha} \). That leads to the following relation between a density function and its marginal cumulative distribution in the \( \alpha \)-fractional space

\[
\Phi_\alpha(t_3) = \int_0^{t_3} \frac{\frac{1}{\alpha} \cdot \Gamma(\frac{1}{\alpha} - \frac{s}{\alpha})}{\alpha \Gamma(1 - s)} u^{\alpha - 1} E_{\alpha,\alpha}(-u^\alpha) du \quad (7.3)
\]

which supports the theory developed in section 5.

**The Subordination Integral**
Corollory 7.1 Let $h_1(x)$ be a density function of a random variable $x$. Let $h_2(y|x)$ be a conditional density function of a variable $y$ given $x$. Define

$$h(x, y) = \begin{cases} h_2(y|x)h_1(x) & \text{if } h_1(x) > 0 \text{ for all } x \\ 0 & \text{if } h_1(x) = 0. \end{cases} \quad (7.4)$$

Then $h(x, y)$ is a joint density function and $h_1(x)$ is the marginal density function of $x$.

We have the subordination integral by

$$\tilde{p}_\alpha(k, s) = \int_0^\infty e^{-t^\alpha} e^{-ts} dt_*, \quad p_\alpha(x, t) = \int_0^\infty p_1(x, t_*)p_2(t_*, t)dt_* \quad (7.5)$$

From the Corollory 7.1, $p_3(t_*, t) = e^{-t^\alpha}p_2(t_*, t)$ is a joint distribution of $t_*$ and $t$. Therefore we get, via statistical paths,

$$\int_0^\infty e^{-t^\alpha} e^{-s^\alpha} dt_* = \frac{1}{1 + s^\alpha} = \tilde{\phi}_\alpha(s) \to \phi(t) : \text{the density in the marginal space of } t \quad (7.6)$$

$$\int_0^\infty e^{-t^\alpha} p_2(t_*, t)dt_* = e^{-t^\alpha} : \text{the density in the marginal space of } t_* \quad (7.7)$$

The evolution equation for $p_3(t_*, t)$ can be derived to

$$p_3(t_*, t) = -\frac{\partial}{\partial t_*}p_3(t_*, t) - (D_{0^+}^\alpha p_3(t_*, t))(t), \quad p_3(0^+, t) = \delta(t). \quad (7.8)$$

Stochastic Analysis and Pathways

Let $f(t_*) = e^{-t^\alpha}$ and $\phi_\alpha(t)$ be density functions of $t_*$ and $t$. Then we get the following stochastic processes:

$$f(t_*) = e^{-t^\alpha} \implies \lim_{t \to \infty} f^{stl}(t_*) = \delta(t_* - 1), \quad \mathcal{L}\{\delta(t_* - 1)\}(s) = e^{-s} \quad (7.9)$$

$$\phi_\alpha(t) = t^\alpha E_{\alpha, \alpha}(-t^\alpha) \implies \lim_{t \to \infty} t^\alpha \phi_\alpha^{stl}(t^\alpha t) = \delta(t - 1), \quad \mathcal{L}\{\delta(t - 1)\}(s) = e^{-s^\alpha} \quad (7.10)$$

$$F(t) = f^{sn}(t_*) \implies \lim_{t \to \infty} f^{stl}(t_*) = \delta(t_* - n), \quad \mathcal{L}\{\delta(t_* - n)\}(s) = e^{-ns} \quad (7.11)$$

$$\psi_\alpha(t) = \phi_\alpha^{sn}(t) \implies \lim_{t \to \infty} t^\alpha \psi_\alpha^{stl}(t^\alpha t) = \delta(t - n^\frac{1}{\alpha}), \quad \mathcal{L}\{\delta(t - n^\frac{1}{\alpha})\}(s) = e^{-ns^\alpha} \quad (7.12)$$

$$f(t_*) = e^{-t^\alpha} \implies \lim_{t \to \infty} n f^{stl}(\frac{1}{n} t_*) = \delta(t_* - n), \quad \mathcal{L}\{\delta(t_* - n)\}(s) = e^{-ns} \quad (7.13)$$

$$\phi_\alpha(t) = t^\alpha E_{\alpha, \alpha}(-t^\alpha) \implies \lim_{t \to \infty} \left(\frac{1}{t_*}\right)^\frac{1}{\alpha} \phi_\alpha^{stl}\left(\frac{1}{t_*}\right)^\frac{1}{\alpha} t = \delta(t - (t_* \frac{1}{\alpha})), \quad \mathcal{L}\{\delta(t - (t_* \frac{1}{\alpha}))\}(s) = e^{-t_* s^\alpha} \quad (7.14)$$

$$\mathcal{L}\{\delta(t - (t_* \frac{1}{\alpha}))\}(s) = e^{-t_* s^\alpha}. \quad (7.15)$$

The positive-oriented (extreme) stable process $t = t(t_*)$ with $p_2(t_*, t)$ is described with $t_* = n\tau_*$ and $\tau_*$ as a step-size, $\tau_* > 0$, in [7], which is (7.15) with $t_* = n\tau_*$, $n = 1, 2, 3, \ldots$ (7.10) is treated as a stochastic process by applying Laplace technique without the explicit form in [22] and appears
explicitly in [10]. The explicit form in [10] appears firstly in the literature. But we emphasise that in [22], the process was not treated in the \(\alpha\)-fractional space.

In the last remark of [22], the author describes the Mittag-Leffler process in [22] as a stochastic process subordinated to a stable process by directing the gamma process, or randomizing the parameter \(s\) with gamma distribution. For the each \(l\)-th event of the time line of the Poisson process in this paper,

\[
p_4(t_s, t) = f^{sl}(t_s)p_2(t_s, t) \tag{7.16}
\]

\[
\int_0^\infty f^{sl}(t_s)e^{-s^\alpha t_s}dt_s = \left(\frac{1}{1 + s^\alpha}\right)^l = \bar{\phi}_\alpha(l) \quad \Rightarrow \quad \bar{\phi}_\alpha(t) : \text{the density in the marginal space of } t
\]

\[
\int_0^\infty f^{sl}(t_s)p_2(t_s, t)dt = f^{sl}(t_s) : \text{the density in the marginal space of } t_s.
\]

**J-transformation Correspondence**

The following table from the author’s paper [10] supplies correspondence between ordinary space and \(\alpha\)-fractional space via J-transformation:

| \(\text{ordinary space} \) | \(\alpha\)-level\((\text{fractional})\) \(\text{space}\) |
|-----------------------------|--------------------------------------------------|
| 1                           | \(t^{\alpha-1}\Gamma(\alpha)\)                    |
| \(t\)                       | \(t^{\alpha-1}\Gamma(2\alpha)\)                  |
| \(e^{-t}\)                  | \(t^{\alpha-1}E_{(\alpha,\alpha)}(-t^\alpha)\)  |
| \(\frac{t}{(l-1)!}e^{-t}\) | \(t^{\alpha-1}E_{(\alpha,\alpha)}^l(-t^\alpha)\) |

In this paper, we develop the theory of \(\alpha\)-fractional Poisson process. As stated in section 3, the renewal function of the Poisson process is \(m(t) = t\), but that of the \(\alpha\)-fractional Poisson process is \(m(t) = \frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\), \(0 < \alpha < 1\). The renewal function in the \(\alpha\)-fractional world looks peculiar. On the other hand, we get the same function corresponding from the renewal function of ordinary Poisson case in the J-transformation correspondence table above, namely \(t \longleftrightarrow \frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\). It is interesting that the J-transformation correspondence table provides the densities of two processes as well as their renewal functions.

8 Remark

We think that our theory is related to the theory of infinite field extention and a view of irrationality in mathematics and the theory of long-term-memory type in statistics. We also recognise the fact, which should be overcome by some logical and practical argument, that the equation (5.4) gives some wierd difference from the usual statistical approach.
9 Appendix

Pochammer symbol is defined as

\[(b)_k = b(b+1) \cdots (b+k-1), \quad (b)_0 = 1. \tag{9.1}\]

**Definition 9.1** Let \( f(x) \in L(a,b), \alpha \in \mathbb{C}, Re(\alpha) > 0, \) then

\[(I_0^+ f)(x) = (D_0^- f)(x) = (D_x^- f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0, \tag{9.2}\]

which is called the Riemann-Liouville left-sided fractional integral of order \( \alpha \) in \([21]\) or Abel integral operator for \( D_x^- \) in \([8]\).

**Definition 9.2** The Weyl integral of order \( \alpha \) is defined as follows:

\[ (x W_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt, \quad (0 < x < \infty) \tag{9.3} \]

where \( \alpha \in \mathbb{C}, Re(\alpha) > 0. \)

**Definition 9.3** The Mainardi function is defined as

\[ M_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\alpha n + (1-\alpha)]} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)! \Gamma(\alpha n) \sin(\pi \alpha n)} \tag{9.4} \]

with \( z \in \mathbb{C} \) and \( 0 < \alpha < 1. \) For detailed information, we refer to \([14]\).

**Definition 9.4** Mittag-Leffler function of 3 parameters is defined by

\[ E_{\alpha,\beta}^\gamma(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma) x^k}{\Gamma(\alpha k + \beta) k!} (\alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha) > 0, Re(\beta) > 0). \tag{9.5} \]

The importance and usage of the Mittag-Leffler function have been enlarged immensely, especially in the area of Fractional Calculus. For theoretical approaches and its applications, there are many good books and papers dealing with functions of Mittag-Leffler type as can be found in the reference.
### Laplace Transformation Correspondence

| function | Laplace transform |
|----------|-------------------|
| $E_\alpha(at^\alpha)$ | $\frac{s^{\alpha-1}}{a + \frac{s^\alpha}{a}}$ |
| $\frac{t^{\alpha n}}{n!}E_\alpha^{(n)}(-t^\alpha)$ | $\frac{s^{\alpha-1}}{(1 + s^\alpha)^{n+1}}$ |
| $\frac{x^{-\alpha n-1}}{\Gamma(-n\alpha)}$ | $s^{n\alpha}$ |
| $\lim_{\gamma \to \infty} t^{\alpha \gamma - 1} \left( \frac{\gamma}{n} \right)^{\gamma} E_{(a,\alpha \gamma)}^{\gamma} \left( -\frac{\gamma t^\alpha}{n} \right)$ in [5, 14] | $e^{-ns^\alpha}$ |
| $t^{\alpha k + \beta - 1}E_{(\alpha,\beta)}^{(k)}(-at^\alpha), E_{\alpha,\beta}^{(k)}(y) = \frac{d^k}{dy^k} E_{(\alpha,\beta)}^{(k)}(y)$ | $\frac{k!s^{\alpha-\beta}}{(a + s^\alpha)^{(k+1)}}, \text{Re}(s) > 1$ |
| $t^{\beta-1}E_{(\alpha,\beta)}^{(0)}(-ct^\alpha)$ | $\frac{s^{\alpha-\beta}}{c + s^\alpha}$ |

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