Mod $p$ Hecke algebras and dual equivariant cohomology I: the case of $GL_2$

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Abstract

Let $F$ be a $p$-adic local field and $G = GL_2$ over $F$. Let $\mathcal{H}^{(1)}$ be the pro-$p$ Iwahori-Hecke algebra of the group $G(F)$ with coefficients in the algebraic closure $\mathbb{F}_p$. We show that the supersingular irreducible $\mathcal{H}^{(1)}$-modules of dimension 2 can be realized through the equivariant cohomology of the flag variety of the Langlands dual group $\hat{G}$ over $\mathbb{F}_p$.

Contents

1 Introduction

2 The pro-$p$-Iwahori-Hecke algebra

3 The non-regular case and dual equivariant $K$-theory

4 The regular case and dual equivariant intersection theory

5 Tame Galois representations and supersingular modules

1 Introduction

Let $F$ be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_p$ and let $G$ be a connected split reductive group over $F$. Let $\mathcal{H} = R/I \setminus G(F)/I$ be the Iwahori-Hecke algebra associated to an Iwahori subgroup $I \subset G(F)$, with coefficients in an algebraically closed field $R$. On the other hand, let $\hat{G}$ be the Langlands dual group of $G$ over $R$, and $\hat{B}$ the flag variety of Borel subgroups of $\hat{G}$ over $R$. 


When $R = \mathbb{C}$, the irreducible $\mathcal{H}$-modules appear as subquotients of the Grothendieck group $K^G(\hat{B})_\mathbb{C}$ of $G$-equivariant coherent sheaves on $\hat{B}$. As such they can be parametrized by the isomorphism classes of irreducible tame $\hat{G}(\mathbb{C})$-representations of the absolute Galois group $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ of $\mathbb{F}$, thereby realizing the tame local Langlands correspondence (in this setting also called the Deligne-Lusztig conjecture for Hecke modules): Kazhdan-Lusztig [KL87], Ginzburg [GZ97]. The idea of studying various cohomological invariants of the flag variety by means of Hecke operators (nowadays called Demazure operators) goes back to earlier work of Demazure [D73, D74].

The approach to the Deligne-Lusztig conjecture is based on the construction of a natural $\mathcal{H}$-action on the whole $K$-group $K^G(\hat{B})_\mathbb{C}$ which identifies the center of $\mathcal{H}$ with the $K$-group of the base point $K^G(\text{pt})_\mathbb{C}$. The finite part of $\mathcal{H}$ acts thereby via appropriate $q$-deformations of Demazure operators.

When $R = \overline{\mathbb{F}}_q$ any irreducible $\hat{G}(\overline{\mathbb{F}}_q)$-representation of $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ is tame and the Iwahori-Hecke algebra needs to be replaced by the bigger pro-$p$-Iwahori-Hecke algebra

$$\mathcal{H}^{(1)} = \overline{\mathbb{F}}_q[I^{(1)} \setminus G(F)/I^{(1)}].$$

Here, $I^{(1)} \subset I$ is the unique pro-$p$ Sylow subgroup of $I$. The algebra $\mathcal{H}^{(1)}$ was introduced by Vignéras and its structure theory developed in a series of papers [V04, V05, V06, V14, V15, V16, V17]. The class of so-called supersingular irreducible $\mathcal{H}^{(1)}$-modules figures prominently among all irreducible $\mathcal{H}^{(1)}$-modules, since it is expected to be related to the arithmetic over the field $\mathbb{F}$. For $G = GL_n$, there is a distinguished correspondence between supersingular irreducible $\mathcal{H}^{(1)}$-modules of dimension $n$ and irreducible $GL_n(\overline{\mathbb{F}}_q)$-representations of $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$: Breuil [B03], Vignéras [V04, V05], Colmez [C10], Grosse-Klönne [GK16, GK18].

Our aim is to show that the supersingular irreducible $\mathcal{H}^{(1)}$-modules of dimension $n$ can again be realized as subquotients of some $\hat{G}$-equivariant cohomology theory of the flag variety $\hat{B}$ over $\overline{\mathbb{F}}_q$, although in a way different from the $\mathbb{C}$-coefficient case. Here we discuss the case $n = 2$, and we will treat the case of general $n$ in a subsequent article [PS2].

From now on, let $R = \overline{\mathbb{F}}_q$ and $G = GL_2$. The algebra $\mathcal{H}^{(1)}$ splits as a direct product of subalgebras $\mathcal{H}^{(1)}_\gamma$ indexed by the orbits $\gamma$ of $\mathcal{O}_2$ in the set of characters of $(\overline{\mathbb{F}}_q)^2$, namely the Iwahori components corresponding to trivial orbits, and the regular components. Accordingly, the category of $\mathcal{H}^{(1)}$-modules decomposes as the product of the module categories for the component algebras. In each component sits a unique supersingular module of dimension $2$ with given central character. On the dual side, we have the projective line $\hat{B} = \mathbb{P}_{\overline{\mathbb{F}}_q}^1$ over $\overline{\mathbb{F}}_q$ with its natural action by fractional transformations of the algebraic group $\hat{G} = GL_2(\overline{\mathbb{F}}_q)$.

For a non-regular orbit $\gamma$, the component algebra $\mathcal{H}^{\gamma}$ is isomorphic to the mod $p$ Iwahori-Hecke algebra $\mathcal{H} = \overline{\mathbb{F}}_q[I \setminus GL_2(F)/I]$ and the quadratic relations in $\mathcal{H}$ are idempotent of type $T_s^2 = -T_s$. The $\hat{G}$-equivariant $K$-theory $K^G(\hat{B})_{\overline{\mathbb{F}}_q}$ of $\hat{B}$ comes with an action of the classical Demazure operator at $q = 0$. Our first result is that this action extends uniquely to an action of the full algebra $\mathcal{H}$ on $K^G(\hat{B})_{\overline{\mathbb{F}}_q}$, which is faithful and which identifies the center $Z(\mathcal{H})$ of $\mathcal{H}$ with the base ring $K^G(\text{pt})_{\overline{\mathbb{F}}_q}$. It is constructed from natural presentations of the algebras $\mathcal{H}$ and $Z(\mathcal{H})$ [V04] and through the characteristic homomorphism

$$Z(\Lambda) \to K^G(\hat{B})$$

which identifies the equivariant $K$-ring with the group ring of characters $\Lambda$ of a maximal torus in $\hat{G}$. In particular, everything is explicit. We finally show that, given a supersingular central character $\theta : Z(\mathcal{H}) \to \overline{\mathbb{F}}_q$, the central reduction $K^G(\hat{B})_{\theta}$ is isomorphic to the unique supersingular $\mathcal{H}$-module of dimension $2$ with central character $\theta$.

For a regular orbit $\gamma$, the component algebra $\mathcal{H}^{\gamma}$ is isomorphic to Vignéras second Iwahori-Hecke algebra $\mathcal{H}_2$ [V04]. It can be viewed as a certain twisted version of two copies of the mod $p$ nil Hecke ring $\mathcal{H}^{\text{nil}}$ (introduced over the complex numbers by Kostant-Kumar [KK80]). In particular, the quadratic relations are nilpotent of type $T_s^2 = 0$. The $\hat{G}$-equivariant intersection theory $CH^G(\hat{B})_{\overline{\mathbb{F}}_q}$ of $\hat{B}$ comes with an action of the classical Demazure operator at $q = 0$. We show that this action extends to a faithful action of $\mathcal{H}^{\text{nil}}$ on $CH^G(\hat{B})_{\overline{\mathbb{F}}_q}$. To incorporate the twisting, we
then pass to the square $\hat{B}^2$ of $\hat{B}$ and extend the action to a faithful action of $\mathcal{H}_2$ on $CH^{\hat{G}}(\hat{B}^2)_{\mathbb{P}_q}$. The action identifies a large part $Z^0(\mathcal{H}_2)$ of the center $Z(\mathcal{H}_2)$ with the base ring $CH^{\hat{G}}(\text{pt})_{\mathbb{P}_q}$. As a technical point, one actually has to pass to a certain localization of the Chow groups to realize these actions, but we do not go into this in the introduction. As in the non-regular case, the action is constructed from natural presentations of the algebras $\mathcal{H}_2$ and $Z(\mathcal{H}_2)$ \cite{V04} and through the characteristic homomorphism

$$\text{Sym}(\Lambda) \xrightarrow{\cong} CH^{\hat{G}}(\hat{B})$$

which identifies the equivariant Chow ring with the symmetric algebra on the character group $\Lambda$. So again, everything is explicit. We finally show that, given a supersingular central character $\theta : Z(\mathcal{H}_2) \to \mathbb{F}_q$, the semisimplification of the $Z^0(\mathcal{H}_2)$-reduction of (the localization of) $CH^{\hat{G}}(\hat{B}^2)_{\mathbb{P}_q}$ equals a direct sum of four copies of the unique supersingular $\mathcal{H}_2$-module of dimension 2 with central character $\theta$.

In a final section we discuss the aforementioned bijection between supersingular irreducible $\mathcal{H}_q^{(1)}$-modules of dimension 2 and irreducible smooth $GL_2(\mathbb{F}_q)$-representations of $\text{Gal}(\mathbb{F}/F)$ in the light of our geometric language.

\textit{Notation:} In general, the letter $F$ denotes a locally compact complete non-archimedean field with ring of integers $\mathcal{O}_F$. Let $\mathbb{F}_q$ be its residue field, of characteristic $p$ and cardinality $q$. We denote by $G$ the algebraic group $GL_2$ over $F$ and by $B := G(F)$ its group of $F$-rational points. Let $T \subset G$ be the torus of diagonal matrices. Finally, $I \subset G$ denotes the upper triangular standard Iwahori subgroup and $I^{(1)} \subset I$ denotes the unique pro-$p$ Sylow subgroup of $I$. Without further mentioning, all modules will be left modules.

## 2 The pro-$p$-Iwahori-Hecke algebra

Let $R$ be any commutative ring. The \textit{pro-$p$ Iwahori Hecke algebra of $G$ with coefficients in $R$} is defined to be the convolution algebra $\mathcal{H}_R^{(1)}(q) := (R[I^{(1)}]\backslash G/I^{(1)}], \ast)$ generated by the $I^{(1)}$-double cosets in $G$. In the sequel, \textit{we will assume that $R$ is an algebra over the ring}

$$\mathbb{Z}[\frac{1}{q-1}, \mu_{q-1}].$$

The first examples we have in mind are $R = \mathbb{F}_q$ or its algebraic closure $R = \overline{\mathbb{F}_q}$.

### 2.1 Weyl groups and cocharacters

#### 2.1.1. We denote by

$$\Lambda = \text{Hom}(G_m, T) = \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2 \simeq \mathbb{Z} \oplus \mathbb{Z}$$

the lattice of cocharacters of $T$ with standard basis $\eta_1(x) = \text{diag}(x, 1)$ and $\eta_2(x) = \text{diag}(1, x)$. Then $\alpha = (1, -1) \in \Lambda$ is a root and

$$s = s_\alpha = s_{(1,-1)} : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$(n_1, n_2) \mapsto (n_2, -n_1)$$

is the associated reflection generating the Weyl group $W_0 = \{1, s\}$. The element $s$ acts on $\Lambda$ and hence also on the group ring $\mathbb{Z}[\Lambda]$. The two invariant elements

$$\xi_1 := e^{(1,0)} + e^{(0,1)} \quad \text{and} \quad \xi_2 := e^{(1,1)}$$

in $\mathbb{Z}[\Lambda]^s$ define a ring isomorphism

$$\xi^+ : \mathbb{Z}[\Lambda^+] = \mathbb{Z}[e^{(1,0)}, (e^{(1,1)})^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}[\Lambda]^s$$

$$e^{(1,0)} \mapsto \xi_1$$

$$e^{(1,1)} \mapsto \xi_2$$
where $\Lambda^+ := \mathbb{Z}_{\geq 0}(1, 0) \oplus \mathbb{Z}(1, 1)$ is the monoid of dominant cocharacters.

2.1.2. We introduce the affine Weyl group $W_{\text{aff}}$ and the Iwahori-Weyl group $W$ of $G$:

$$W_{\text{aff}} := e^\mathbb{Z}(1,-1) \times W_0 \subset W := e^\Lambda \rtimes W_0.$$ 

With 

$$u := e^{(1,0)}s = se^{(0,1)}$$

one has $W = W_{\text{aff}} \rtimes \Omega$ where $\Omega = u^2 \simeq \mathbb{Z}$. Let $s_0 = e^{(1,-1)}s = se^{(-1,1)} = usu^{-1}$. Recall that the pair $(W_{\text{aff}}, \{s_0, s\})$ is a Coxeter group and its length function $\ell$ can be inflated to $W$ via $\ell|_{\Omega} = 0$.

2.2 Idempotents and component algebras

2.2.1. We have the finite diagonal torus $T := T(\mathbb{F}_q)$ and its group ring $R[T]$. As $q - 1$ is invertible in $R$, so is $|T| = (q - 1)^2$ and hence $R[T]$ is a semisimple ring. The canonical isomorphism $T \simeq I/I(1)$ induces an inclusion 

$$R[T] \subset H_R^{(1)}(q).$$

We denote by $T^\vee$ the set of characters 

$$\lambda : T \to \mathbb{F}_q^*$$

of $T$, with its natural $W_0$-action given by 

$$^s\lambda(t_1, t_2) = \lambda(t_2, t_1)$$

for $(t_1, t_2) \in T$. The number of $W_0$-orbits in $T^\vee$ equals $\frac{q^2 - q}{2}$. Also $W$ acts on $T^\vee$ through the canonical quotient map $W \to W_0$.

2.2.2. Definition. For all $\lambda \in T^\vee$, define 

$$\varepsilon_\lambda := |T|^{-1} \sum_{t \in T} \lambda^{-1}(t)T_t \in R[T]$$

and for all $\gamma \in T^\vee/W_0$,

$$\varepsilon_\gamma := \sum_{\lambda \in \gamma} \varepsilon_\lambda \in R[T].$$

Following the terminology of [V04], we call $|\gamma| = 1$ the Iwahori case or non-regular case and $|\gamma| = 2$ the regular case.

2.2.3. Proposition. For all $\lambda \in T^\vee$, the element $\varepsilon_\lambda$ is an idempotent. For all $\gamma \in T^\vee/W_0$, the element $\varepsilon_\gamma$ is a central idempotent in $H_R^{(1)}(q)$. The $R$-algebra $H_R^{(1)}(q)$ is the direct product of its sub-$R$-algebras $H_R^{(1)}(q)\varepsilon_\gamma$, i.e.

$$H_R^{(1)}(q) = \prod_{\gamma \in T^\vee/W_0} H_R^{(1)}(q)\varepsilon_\gamma.$$ 

Proof. This follows from [V04] Prop. 3.1 and its proof. \qed

The proposition implies that the category of $H_R^{(1)}(q)$-modules decomposes into a finite product of the module categories for the individual component rings $H_R^{(1)}(q)\varepsilon_\gamma$. 

4
2.3 The Iwahori-Hecke algebra

Our reference for the following is [V04 1.1/2].

2.3.1. Definition. Let \( q \) be an indeterminate. The generic Iwahori-Hecke algebra is the \( \mathbb{Z}[q] \)-algebra \( \mathcal{H}(q) \) defined by generators

\[
\mathcal{H}(q) := \bigoplus_{(n_1, n_2) \in \mathbb{Z}^2} \mathbb{Z}[q]T_{c(n_1, n_2)} \oplus \mathbb{Z}[q]T_{e(n_1, n_2)}
\]

and relations:

- **braid relations**
  \[
  T_wT_{w'} = T_{ww'} \quad \text{for } w, w' \in W \text{ if } \ell(w) + \ell(w') = \ell(ww')
  \]

- **quadratic relations**
  \[
  \begin{cases}
  T_2^2 = (q - 1)T_q + q \\
  T_{w_0}^2 = (q - 1)T_{w_0} + q.
  \end{cases}
  \]

2.3.2. Setting \( S := T_\text{aff} \) and \( U := T_u \), one can check that

\[
\mathcal{H}(q) = \mathbb{Z}[q][S,U^{\pm 1}], \quad S^2 = (q - 1)S + q, \quad U^2 = SU^2
\]

is a presentation of \( \mathcal{H}(q) \). For example, \( S_0 := T_{w_0} = USU^{-1} \). We also have the generic finite and affine Hecke algebras

\[
\mathcal{H}_0(q) = \mathbb{Z}[q][S] \subset \mathcal{H}_\text{aff}(q) = \mathbb{Z}[q][S_0, S].
\]

The algebra \( \mathcal{H}_0(q) \) has two characters corresponding to \( S \mapsto 0 \) and \( S \mapsto -1 \). Similarly, \( \mathcal{H}_\text{aff}(q) \) has four characters. The two characters different from the trivial character \( S_0, S \mapsto 0 \) and the sign character \( S_0, S \mapsto -1 \) are called supersingular.

2.3.3. The center \( Z(\mathcal{H}(q)) \) of the algebra \( \mathcal{H}(q) \) admits the explicit description via the algebra isomorphism

\[
\mathcal{Z}(q) : \mathbb{Z}[q][\Lambda^+] = \mathbb{Z}[q][e^{(1,0)}, (e^{(1,1)})^{\pm 1}] \xrightarrow{\cong} Z(\mathcal{H}(q))
\]

\[
\begin{align*}
e^{(1,0)} &\mapsto \zeta_1 := U(S - (q - 1)) + SU \\
e^{(1,1)} &\mapsto \zeta_2 := U^2.
\end{align*}
\]

In particular,

\[
Z(\mathcal{H}(q)) = \mathbb{Z}[q][US + (1 - q)U + SU, U^{\pm 2}] \subset \mathbb{Z}[q][S,U^{\pm 1}] = \mathcal{H}(q).
\]

2.3.4. Now let \( \gamma \in T'/W_0 \) such that \( |\gamma| = 1 \), say \( \gamma = \{\lambda\} \). The ring homomorphism \( \mathbb{Z}[q] \rightarrow R, q \mapsto q \), induces an isomorphism of \( R \)-algebras

\[
\mathcal{H}(q) \otimes_{\mathbb{Z}[q]} R \xrightarrow{\cong} \mathcal{H}_R^{(1)}(q) \varepsilon_{\gamma}, \quad T_w \mapsto \varepsilon_{\lambda}T_w.
\]

2.4 The second Iwahori-Hecke algebra

Our reference for the following is [V04 2.2], as well as [KK06] for the basic theory of the nil Hecke algebra. We keep the notation introduced above.

2.4.1. Definition. The generic nil Hecke algebra is the \( \mathbb{Z}[q] \)-algebra \( \mathcal{H}_\text{nil}(q) \) defined by generators

\[
\mathcal{H}_\text{nil}(q) := \bigoplus_{(n_1, n_2) \in \mathbb{Z}^2} \mathbb{Z}[q]T_{e(n_1, n_2)} \oplus \mathbb{Z}[q]T_{e(n_1, n_2)}
\]

and relations:
- braid relations

\[ T_w T_{w'} = T_{ww'} \text{ for } w, w' \in W \text{ if } \ell(w) + \ell(w') = \ell(ww') \]

- quadratic relations

\[
\begin{cases}
T_x^2 = q \\
T^2_{s_0} = q
\end{cases}
\]

2.4.2. Setting \( S := T_s \) and \( U := T_u \), one can check that

\[ H^{\nil}(q) = \mathbb{Z}[q][S,U, U^{\pm 1}], \quad S^2 = q, \quad U^2 S = SU^2 \]

is a presentation of \( H^{\nil}(q) \). Again, \( S_0 := T_{s_0} = USU^{-1} \). The center \( \mathbb{Z}(H^{\nil}(q)) \) admits the explicit description via the algebra isomorphism

\[ \mathcal{Z}^{\nil}(q) : \mathbb{Z}[q][\Lambda^+] = \mathbb{Z}[q][\epsilon(1,0), (\epsilon(1,1))^\mp 1] \xrightarrow{\cong} \mathbb{Z}(H^{\nil}(q)) \]

\[ \epsilon(1,0) \mapsto \zeta_1 := US + SU \]

\[ \epsilon(1,1) \mapsto \zeta_2 := U^2. \]

In particular,

\[ \mathbb{Z}(H^{\nil}(q)) = \mathbb{Z}[q][US + SU, U^{\pm 2}] \subset \mathbb{Z}[q][S, U^{\pm 1}] = H^{\nil}(q). \]

2.4.3. Form the twisted tensor product algebra

\[ H_2(q) := (\mathbb{Z}[q] \times \mathbb{Z}[q]) \otimes_{\mathbb{Z}[q]} H^{\nil}(q). \]

With the formal symbols \( \epsilon_1 = (1,0) \) and \( \epsilon_2 = (0,1) \), the ring multiplication is given by

\[ (\epsilon_1 \otimes T_w) \cdot (\epsilon_{i'} \otimes T_{w'}) = (\epsilon_1 \epsilon_{i'} \otimes T_w T_{w'}) \]

for all \( 1 \leq i, i' \leq 2 \). Here, \( W \) acts through its quotient \( W_0 \) and \( s \in W_0 \) acts on the set \( \{1, 2\} \) by interchanging the two elements. The multiplicative unit element in the ring \( \mathbb{Z}[q] \times \mathbb{Z}[q] \) is \( (1,1) = \epsilon_1 + \epsilon_2 \) and the multiplicative unit element in the ring \( H_2(q) \) is \( (1,1) \otimes 1 \). We identify the rings \( \mathbb{Z}[q] \times \mathbb{Z}[q] \) and \( H^{\nil}(q) \) with subrings of \( H_2(q) \) via the maps \( (a,b) \mapsto (a,b) \otimes 1 \) and \( a \mapsto (1,1) \otimes a \) respectively. In particular, we will write \( \epsilon_1, \epsilon_2, S_0, S, U \in H_2(q) \) etc.

We also introduce the generic affine Hecke algebra

\[ H_{2, \text{aff}}(q) = (\mathbb{Z}[q] \times \mathbb{Z}[q]) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q][S_0, S]. \]

It is a subalgebra of \( H_2(q) \) and has two supersingular characters \( \chi_1 \) and \( \chi_2 \), namely \( \chi_1(\epsilon_1) = 1 \) and \( \chi_1(\epsilon_2) = 0 \) and \( \chi_1(S_0) = \chi_1(S) = 0 \). Similarly for \( \chi_2 \).

2.4.4. The structure of \( H_2(q) \) as an algebra over its center can be made explicit. In fact, there is an algebra isomorphism with an algebra of \( 2 \times 2 \)-matrices

\[ H_2(q) \cong M(2, \mathbb{Z}(q)), \quad \mathbb{Z}(q) := \mathbb{Z}[q][X, Y, z^{\pm 1}, (XY)] \]

which maps the center \( \mathbb{Z}(H_2(q)) \) to the scalar matrices \( \mathbb{Z}(q) \). Under this isomorphism, we have

\[ S \mapsto \begin{pmatrix} 0 & Y \\ z_2^{-1}X & 0 \end{pmatrix}, \quad U \mapsto \begin{pmatrix} 0 & z_2 \\ 1 & 0 \end{pmatrix}, \]

\[ \varepsilon_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

The induced map \( Z(H_2(q)) \rightarrow \mathbb{Z}(q) \) satisfies

\[ \zeta_1 \mapsto \begin{pmatrix} X + Y & 0 \\ 0 & X + Y \end{pmatrix}, \quad \zeta_2 \mapsto \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix}. \]

In particular, the subring

\[ Z^\circ(H_2(q)) := \mathbb{Z}[q][\zeta_1, \zeta_2^{\pm 1}] = Z(H^{\nil}(q)) \subset H^{\nil}(q) \subset H_2(q) \]

lies in fact in the center \( Z(H_2(q)) \) of \( H_2(q) \).
2.4.5. Now let $\gamma \in T^\vee / W_0$ such that $|\gamma| = 2$, say $\gamma = \{\lambda, \lambda\}$. The ring homomorphism $\mathbb{Z}[q] \to R$, $q \mapsto q$, induces an isomorphism of $R$-algebras

$$\mathcal{H}_2(q) \otimes_{\mathbb{Z}[q]} R \xrightarrow{\sim} \mathcal{H}_2^{(1)}(q) \varepsilon_\gamma, \quad \varepsilon_1 \otimes T_w \mapsto \varepsilon_\lambda T_w, \quad \varepsilon_2 \otimes T_w \mapsto \varepsilon \varepsilon_\lambda T_w.$$

2.4.6. Remark. We have used the same letters $S_0, S, U, \zeta_1, \zeta_2$ for the corresponding Hecke operators in the Iwahori Hecke algebra and in the second Iwahori Hecke algebra. This should not lead to confusion, as we will always treat non-regular components and regular components separately in our discussion.

3 The non-regular case and dual equivariant $K$-theory

3.1 Recollections from algebraic $K\hat{G}$-theory

For basic notions from equivariant algebraic $K$-theory we refer to [Th87]. A useful introduction may also be found in [CG97, chap. 5].

3.1.1. We let

$$\hat{G} := \text{GL}_2 / \mathbb{F}_q$$

be the Langlands dual group of $G$ over the algebraic closure $\mathbb{F}_q$ of $\mathbb{F}_q$. The dual torus

$$\hat{T} := \text{Spec} \mathbb{F}_q[A] \subset \hat{G}$$

identifies with the torus of diagonal matrices in $\hat{G}$. A basic object is

$$R(\hat{G}) := \text{the representation ring of } \hat{G},$$

i.e. the Grothendieck ring of the abelian tensor category of all finite dimensional $\hat{G}$-representations. It can be viewed as the equivariant $K$-theory $K\hat{G}(pt)$ of the base point $pt = \text{Spec} \mathbb{F}_q$. To compute it, we introduce the representation ring $R(\hat{T})$ of $\hat{T}$ which identifies canonically, as a ring with $W_0$-action, with the group ring of $\Lambda$, i.e.

$$R(\hat{T}) = \mathbb{Z}[\Lambda].$$

The formal character $\chi_V \in \mathbb{Z}[\Lambda]^*$ of a representation $V$ is an invariant function and is defined by

$$\chi_V(e^\lambda) = \dim_{\mathbb{F}_q} V_\lambda$$

for all $\lambda \in \Lambda$ where $V_\lambda$ is the $\lambda$-weight space of $V$. The map $V \mapsto \chi_V$ induces a ring isomorphism

$$\chi^* : R(\hat{G}) \xrightarrow{\sim} \mathbb{Z}[\Lambda]^*.$$

The $\mathbb{Z}[\Lambda]^*$-module $\mathbb{Z}[\Lambda]$ is free of rank 2, with basis $\{1, e^{(-1,0)}\}$,

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[\Lambda]^* \oplus \mathbb{Z}[\Lambda]^* e^{(-1,0)}.$$

3.1.2. We let

$$\hat{B} := \mathbb{P}_q^1$$

be the projective line over $\mathbb{F}_q$ endowed with its left $\hat{G}$-action by fractional transformations

$$\left(\begin{array}{cc}a & b \\c & d \end{array}\right) \cdot (x) = \frac{ax + b}{cx + d}.$$

Here, $x$ is a local coordinate on $\mathbb{P}_q^1$. The stabilizer of the point $x = \infty$ is the Borel subgroup $\hat{B}$ of upper triangular matrices and we may thus write $\hat{B} = \hat{G} / \hat{B}$. We denote by

$$K^\hat{G}(\hat{B}) := \text{the Grothendieck group of all } \hat{G}\text{-equivariant coherent } \mathcal{O}_G\text{-modules.}$$
Given a representation \( V \) and an equivariant coherent sheaf \( \mathcal{F} \), the diagonal action of \( \hat{G} \) makes \( \mathcal{F} \otimes_{\mathcal{T}_q} V \) an equivariant coherent sheaf. In this way, \( K^{\hat{G}}(\hat{B}) \) becomes a module over the ring \( R(\hat{G}) \).

The characteristic homomorphism in algebraic \( K^{\hat{G}} \)-theory is a ring isomorphism
\[
e^G : \mathbb{Z}[\Lambda] \xrightarrow{\sim} K^{\hat{G}}(\hat{B}).
\]
It maps \( e^\lambda \) with \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \) to the class of the \( \hat{G} \)-equivariant line bundle \( \mathcal{O}_{\mathbb{P}^1}(\lambda_1 - \lambda_2) \otimes \text{det}^\lambda \) where \( \text{det} \) is the determinant character of \( \hat{G} \). The characteristic homomorphism is compatible with the character morphism \( \chi_* \), i.e. \( e^G \) is \( \mathbb{Z}[\Lambda]^* \cong R(\hat{G}) \)-linear.

3.1.3. For the definition of the classical Demazure operators on algebraic \( K \)-theory we refer to [Dr2] [Dr4]. The Demazure operators
\[
D_s, D'_s \in \text{End}_{R(\hat{T})}(R(\hat{T}))
\]
are defined by:
\[
D_s(a) = \frac{a - s(a)}{1 - e^{(1, -1)}} \quad \text{and} \quad D'_s(a) = \frac{a - s(a)e^{(1, -1)}}{1 - e^{(1, -1)}}
\]
for \( a \in R(\hat{T}) \). They are the projectors on \( R(\hat{T})^*e^{(-1, 0)} \) along \( R(\hat{T})^* \), and on \( R(\hat{T})^* \) along \( R(\hat{T})^*e^{(1, 0)} \), respectively. In particular \( D_s^2 = D_s \) and \( D'_s^2 = D'_s \). One sets
\[
D_s(q) := D_s - q D'_s \in \text{End}_{R(\hat{T})^*}[q](R(\hat{T})[q])
\]
and checks by direct calculation that
\[
D_s(q)^2 = q - (q - 1) D_s(q).
\]
In particular, we obtain a well-defined \( \mathbb{Z}[q] \)-algebra homomorphism
\[
\mathcal{A}_0(q) : \mathcal{H}_0(q) = \mathbb{Z}[q][S] \longrightarrow \text{End}_{R(\hat{T})^*}[q](R(\hat{T})[q]), \quad S \longmapsto -D_s(q)
\]
which we call the Demazure representation.

3.2 The morphism from \( R(\hat{G})[q] \) to the center of \( \mathcal{H}(q) \)
In the following we identify the rings
\[
R(\hat{G})[q] \simeq \mathbb{Z}[q][\Lambda]^* = \mathbb{Z}[q][\xi_1, \xi_2^{\pm 1}]
\]
via the character isomorphism \( \chi_* \). We have the \( \mathbb{Z}[q] \)-algebra isomorphism coming via base change from the isomorphism \( \xi^+ \), cf. [2.1.1]
\[
\xi^+ : \mathbb{Z}[q][e^{(1,0)}, (e^{(1,1)})^{\pm 1}] \xrightarrow{\sim} \mathbb{Z}[q][\xi_1, \xi_2^{\pm 1}]
\]
\[
e^{(1,0)} \longmapsto \xi_1 \quad e^{(1,1)} \longmapsto \xi_2.
\]
On the other hand, the source of \( \xi^+ \) is isomorphic to the center \( Z(\mathcal{H}(q)) \) of \( \mathcal{H}(q) \) via the isomorphism \( \mathcal{Z}'(q) \), cf. [2.3.2]. The composition
\[
\mathcal{Z}'(q) \circ (\xi^+)^{-1} : R(\hat{G})[q] \xrightarrow{\sim} Z(\mathcal{H}(q))
\]
\[
\xi_1 \longmapsto \zeta_1 = U(S - (q - 1)) + SU \quad \xi_2 \longmapsto \zeta_2 = U^2
\]
is then a ring isomorphism.
3.3 The extended Demazure representation \( \mathcal{A}(q) \)

Recall the Demazure representation \( \mathcal{A}_0(q) \) of the finite algebra \( \mathcal{H}_0(q) \) by \( R(\widehat{G})[q] \)-linear operators on the \( K \)-theory \( K^G(\widehat{B}) \), cf. \[ \text{3.1.3} \] We have the following first main result.

3.3.1. Theorem. There is a unique ring homomorphism

\[
\mathcal{A}(q) : \mathcal{H}(q) \to \text{End}_{R(\widehat{G})[q]}(K^G(\widehat{B})[q])
\]

which extends the ring homomorphism \( \mathcal{A}_0(q) \) and coincides on \( Z(\mathcal{H}(q)) \) with the isomorphism

\[
Z(\mathcal{H}(q)) \xrightarrow{\approx} R(\widehat{G})[q]
\]

\[
\zeta_1 \mapsto \xi_1
\]

\[
\zeta_2 \mapsto \xi_2.
\]

The homomorphism \( \mathcal{A}(q) \) is injective.

Proof : Such an extension exists if and only if there exists

\[
\mathcal{A}(q)(U) \in \text{End}_{R(\widehat{G})[q]}(K^G(\widehat{B})[q])
\]

satisfying

1. \( \mathcal{A}(q)(U) \) is invertible ;
2. \( \mathcal{A}(q)(U)^2 = \mathcal{A}(q)(U^2) = \mathcal{A}(q)(\zeta_2) = \xi_2 \text{Id} ;
3. \( \mathcal{A}(q)(U)\mathcal{A}_0(q)(S) + (1-q)\mathcal{A}(q)(U) + \mathcal{A}_0(q)(S)\mathcal{A}(q)(U) = \mathcal{A}(q)(US + (1-q)U + SU) = \mathcal{A}(q)(\zeta_1) = \xi_1 \text{Id}.

To find such an operator \( \mathcal{A}(q)(U) \), we write

\[
K^G(\widehat{B})[q] = R(\widehat{T})[q] = R(\widehat{T})^+[q] \oplus R(\widehat{T})^-[q][e^{(-1,0)}],
\]

and use the \( R(\widehat{T})^+[q] \)-basis \( \{1, e^{(-1,0)}\} \) to identify \( \text{End}_{R(\widehat{G})[q]}(K^G(\widehat{B})[q]) \) with the algebra of \( 2 \times 2 \) matrices over the ring \( R(\widehat{T})^+[q] \). Then, by definition,

\[
\mathcal{A}_0(q)(S) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + q \begin{pmatrix} 1 & \xi_1 e^{(-1,-1)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & \xi_1 e^{(-1,-1)} \\ 0 & -1 \end{pmatrix}.
\]

Hence, if we set

\[
\mathcal{A}(q)(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},
\]

we get

\[
\mathcal{A}(q)(U)^2 = e^{(1,1)} \text{Id} \iff \begin{pmatrix} a^2 + bc & c(a + d) \\ b(a + d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} e^{(1,1)} & 0 \\ 0 & e^{(1,1)} \end{pmatrix}
\]

and

\[
\mathcal{A}(q)(U)\mathcal{A}_0(q)(S) + (1-q)\mathcal{A}(q)(U) + \mathcal{A}_0(q)(S)\mathcal{A}(q)(U) = \xi_1 \text{Id} \iff \begin{pmatrix} (q + 1)a + q\xi_1 e^{(-1,-1)}b & q\xi_1 e^{(-1,-1)}(a + d) \\ 0 & -(q + 1)d + q\xi_1 e^{(-1,-1)}b \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1 \end{pmatrix}.
\]

These two conditions together are in turn equivalent to

\[
\begin{cases}
  a & = -d \\
  bc & = e^{(1,1)} - a^2 \\
  (q + 1)a & = \xi_1 - q\xi_1 e^{(-1,-1)}b.
\end{cases}
\]
Moreover, in this case, the determinant
\[ ad - bc = -a^2 - (e^{(1,1)} - a^2) = -e^{(1,1)} \]
is invertible. Specialising to \( q = 0 \), we find that there is exactly one \( R(\hat{G})[q] \)-algebra homomorphism
\[ \mathcal{A}(q) : \mathcal{H}(q) \longrightarrow \text{End}_{R(\hat{G})[q]}(K\hat{G}(\hat{B})[q]), \]
extending the ring homomorphism \( \mathcal{A}_0(q) \), corresponding to the matrix
\[ \mathcal{A}(q)(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} := \begin{pmatrix} \xi_1 & e^{(-1,-1)}\xi_1^2 - 1 \\ -e^{(1,1)} & -\xi_1 \end{pmatrix}. \]

Note that \( a, b, c, d \in R(\hat{T})^* \subset R(\hat{T})^*[q]. \) The injectivity of the map \( \mathcal{A}(q) \) will be proved in the next subsection. \( \square \)

### 3.4 Faithfulness of \( \mathcal{A}(q) \)

Let us show that the map \( \mathcal{A}(q) \) is injective. It follows from \( \ref{3.3} \) that the ring \( \mathcal{H}(q) \) is generated by the elements
\[ 1, \ S, \ U, \ SU \]
over its center \( Z(\mathcal{H}(q)) = \mathbb{Z}[\zeta_1, \zeta_2^\pm 1][q] \). As the latter is mapped isomorphically to the center \( R(\hat{G})[q] = \mathbb{Z}[\xi_1, \xi_2^\pm 1][q] \) of the matrix algebra \( \text{End}_{R(\hat{G})[q]}(K\hat{G}(\hat{B})[q]) \) by \( \mathcal{A}(q) \), it suffices to check that the images
\[ 1, \ \mathcal{A}_0(q)(S), \ \mathcal{A}(q)(U), \ \mathcal{A}_0(q)(S)\mathcal{A}(q)(U) \]
of \( 1, S, U, SU \) by \( \mathcal{A}(q) \) are free over \( R(\hat{G})[q] \). To ease notation, we will write \( \xi \) instead of \( \xi_1 \) in the following calculation. So let \( \alpha, \beta, \gamma, \delta \in R(\hat{T})^*[q] \) (which is an integral domain) be such that
\[ \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} q & q\xi e^{(-1,-1)} \\ 0 & -1 \end{pmatrix} + \gamma \begin{pmatrix} a & c \\ b & -a \end{pmatrix} + \delta \begin{pmatrix} q & q\xi e^{(-1,-1)} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = 0. \]
This is equivalent to the expression
\[ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \beta q \begin{pmatrix} q & \xi e^{(-1,-1)} \\ 0 & -1 \end{pmatrix} + \gamma \begin{pmatrix} a & c \\ b & -a \end{pmatrix} + \delta \begin{pmatrix} q & \xi e^{(-1,-1)}b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = 0. \]
being zero, i.e. to the identity
\[ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \beta q \begin{pmatrix} q & \xi e^{(-1,-1)} \\ 0 & -1 \end{pmatrix} + \gamma \begin{pmatrix} a & c \\ b & -a \end{pmatrix} + \delta \begin{pmatrix} (\xi - a) \end{pmatrix} = 0. \]
Then
\[ \begin{cases} \alpha + \beta q + \gamma a + \delta (\xi - a) = 0 \\ (\gamma - \delta)b = 0 \\ \beta q\xi e^{(-1,-1)} + \gamma c + \delta q(c - a\xi e^{(-1,-1)}) = 0 \\ \alpha - \beta + (\delta - \gamma)a = 0. \end{cases} \]
As \( b \neq 0 \), we obtain \( \delta = \gamma \) and
\[ \begin{cases} \alpha + \beta q + \gamma \xi = 0 \\ \beta q\xi e^{(-1,-1)} + \gamma((q + 1)c - q\xi e^{(-1,-1)}a) = 0 \\ \alpha - \beta = 0. \end{cases} \]
Hence \( \alpha = \beta \) and
\[ \begin{cases} \alpha(q + 1) + \gamma \xi = 0 \\ \alpha q\xi e^{(-1,-1)} + \gamma((q + 1)c - q\xi e^{(-1,-1)}a) = 0. \end{cases} \]
The latter system has determinant
\[(q + 1)((q + 1)c - q\xi e^{(-1,-1)}a) - q\xi^2 e^{(-1,-1)},\]
which is nonzero (its specialization at \( q = 0 \) is equal to \( c \neq 0 \)), whence \( \alpha = \gamma = 0 = \beta = \delta \).
This concludes the proof and shows that the map \( \mathcal{A}(q) \) is injective. We record the following two corollaries of the proof.

3.4.1. Corollary. The ring \( \mathcal{H}(q) \) is a free \( \mathbb{Z}(\mathcal{H}(q)) \)-module on the basis \( 1, S, U, SU \).

3.4.2. Corollary. The representation \( \mathcal{A}(0) \) is injective.

3.5 Supersingular modules

In this section we work at \( q = 0 \) and over the algebraic closure \( \overline{\mathbb{F}}_q \) of the field \( \mathbb{F}_q \).

3.5.1. Consider the ring homomorphism \( \mathbb{Z}[q] \to \overline{\mathbb{F}}_q, \, q \mapsto q = 0 \), and let
\[ \mathcal{H}_{\mathcal{F}} = \mathcal{H}(q) \otimes_{\mathbb{Z}[q]} \mathbb{F}_q = \mathbb{F}_q[S, U^\pm]. \]

The characters of \( \mathcal{H}_{\mathcal{F}} \) are parametrised by the set \( \{0, -1\} \times \overline{\mathbb{F}}_q^\times \) via evaluation on the elements \( S \) and \( U \). Let \( (\tau_1, \tau_2) \in \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q \). A standard module over \( \mathcal{H}_{\mathcal{F}} \) of dimension 2 is defined to be a module of type
\[ M_2(\tau_1, \tau_2) := \mathbb{F}_q^m \oplus \mathbb{F}_q U^m, \quad Sm = -m, \quad SUm = \tau_1 m, \quad U^2m = \tau_2 m. \]
The center \( Z(\mathcal{H}_{\mathcal{F}}) = \overline{\mathbb{F}}_q[\zeta_1, \zeta_2 \pm 1] \) acts on the module \( M_2(\tau_1, \tau_2) \) via the character \( \zeta_1 \mapsto \tau_1, \zeta_2 \mapsto \tau_2 \).
The module \( M_2(\tau_1, \tau_2) \) is reducible if and only if \( \tau_1 = \tau_2 \). It is called supersingular if \( \tau_1 = 0 \). A supersingular module is irreducible.
Any simple finite dimensional \( \mathcal{H}_{\mathcal{F}} \)-module is either a character or a standard module [V04, 1.4].

3.5.2. Now consider the base change of the representation \( \mathcal{A} := \mathcal{A}(0) \) to \( \mathbb{F}_q \)
\[ \mathcal{A} : \mathcal{H}_{\mathcal{F}} \longrightarrow \text{End}_{R(G)_{\mathcal{F}}} (K^G(\mathcal{B})_{\mathcal{F}}) = \text{End}_{\mathcal{F}_q}[\zeta_1, \zeta_2 \pm 1](\mathbb{F}_q[e^\pm m, e^\pm m]). \]
Recall that the image of \( Z(\mathcal{H}_{\mathcal{F}}) = \overline{\mathbb{F}}_q[\zeta_1, \zeta_2 \pm 1] \) is \( R(\tilde{G})_{\mathcal{F}} = \overline{\mathbb{F}}_q[\zeta_1, \zeta_2 \pm 1] \).

Let us fix a character \( \theta : Z(\mathcal{H}_{\mathcal{F}}) \to \overline{\mathbb{F}}_q \). Following [V04], we call \( \theta \) supersingular if \( \theta(\zeta_1) = 0 \). Consider the base change of \( \mathcal{A} \) along \( \theta \)
\[ \mathcal{A}_\theta := \mathcal{A} \otimes_{Z(\mathcal{H}_{\mathcal{F}})} \overline{\mathbb{F}}_q, \quad K^G(\mathcal{B})_{\theta} := K^G(\mathcal{B})_{\mathcal{F}} \otimes_{Z(\mathcal{H}_{\mathcal{F}})} \overline{\mathbb{F}}_q = K^G(\mathcal{B})_{\mathcal{F}} \otimes_{R(G)_{\mathcal{F}}} (R(\tilde{G})_{\mathcal{F}} \otimes_{Z(\mathcal{H}_{\mathcal{F}})} \overline{\mathbb{F}}_q) \]
\[ \mathcal{A}_\theta : \mathcal{H}_\theta \longrightarrow \text{End}_{\mathcal{F}_q}(K^G(\mathcal{B})_{\theta}). \]

3.5.3. Proposition. The representation \( \mathcal{A}_\theta \) is faithful if and only if \( \theta(\zeta_1)^2 \neq \theta(\zeta_2) \). In this case, \( \mathcal{A}_\theta \) is an algebra isomorphism
\[ \mathcal{A}_\theta : \mathcal{H}_\theta \xrightarrow{\cong} \text{End}_{\mathcal{F}_q}(K^G(\mathcal{B})_{\theta}). \]

Proof: The discussion in the preceding section [52] shows that \( \mathcal{H}_\theta \) has \( \mathbb{F}_q \)-basis given by \( 1, S, U, SU \). Moreover, their images
\[ 1, \mathcal{A}_\theta(S), \mathcal{A}_\theta(U), \mathcal{A}_\theta(S) \mathcal{A}_\theta(U) \]
by \( \mathcal{A}_\theta \) are linearly independent over \( \mathbb{F}_q \) if and only if the scalar \( c = e^{(-1,-1)} \xi_2^2 - 1 \in R(\tilde{G})_{\mathcal{F}} \) does not reduce to zero via \( \theta \), i.e. if and only if \( \xi_2^2 \xi_1^2 - 1 \notin \ker \theta \). In this case, the map \( \mathcal{A}_\theta \) is injective and then bijective since \( \dim_{\mathbb{F}_q} K^G(\mathcal{B})_{\theta} = 2 \). □
3.5.4. Corollary. The $\mathcal{H}_{\gamma}$-module $K^G(\hat{B}_\theta)$ is isomorphic to the standard module $M_2(\tau_1, \tau_2)$ where $\tau_1 = \theta(\zeta_1)$ and $\tau_2 = \theta(\zeta_2)$. In particular, if $\theta$ is supersingular, then $K^G(\hat{B}_\theta)$ is isomorphic to the unique supersingular $\mathcal{H}_{\gamma}$-module with central character $\theta$.

Proof: In the case $\tau_1 \neq \tau_2$, the module $K^G(\hat{B}_\theta)$ is irreducible by the preceding proposition and hence is standard. In general, it suffices to find $m \in K^G(\hat{B}_\theta)$ with $Sm = -m$ and to verify that $\{m, Um\}$ are linearly independent. For example, $m = e^n$ is a possible choice, cf. below. \square

A "standard basis" for the module $K^G(\hat{B}_\theta)$ comes from the so-called Pittie-Steinberg basis \cite{St75} of $\mathbb{F}_q[e^{\pm n}, e^{\pm m}]$ over $\mathbb{F}_q[\zeta_1, \zeta_2^{\pm 1}]$. It is given by

$$z_e = 1, \quad z_s = e^{n_2}.$$ 

It induces a basis of $\mathbb{F}_q[e^{\pm n}, e^{\pm m}] \otimes_{\mathbb{F}_q[\zeta_1, \zeta_2^{\pm 1}]} \mathbb{F}_q$ over $\mathbb{F}_q$ for any character $\theta$ of $\mathbb{F}_q[\zeta_1, \zeta_2^{\pm 1}]$. Let $\tau_2 = \theta(\zeta_2)$. The matrices of $S$, $U$ and $S_0 = USU^{-1}$ in the latter basis are

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -\tau_2 \\ -1 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

The two characters of $\mathcal{H}_{\gamma, \theta} = \mathbb{F}_q[S]$ corresponding to $S \mapsto 0$ and $S \mapsto -1$ are realized by $z_e$ and $z_s$. From the matrix of $S_0$, we see in fact that the whole affine algebra $\mathcal{H}_{aff, \gamma} := \mathbb{F}_q[S_0, S]$ acts on $z_e$ and $z_s$ via the two supersingular characters of $\mathcal{H}_{aff, \gamma}$, cf. 3.3.2.

3.5.5. We extend this discussion of the component $\gamma = 1$ to any other non-regular component as follows. Consider the quotient map

$$\mathbb{T}' \to \mathbb{T}' / W_0.$$

For any $\gamma \in \mathbb{T}' / W_0$ define the $\mathbb{F}_q$-variety

$$\tilde{B}_\gamma := \hat{B} \times \pi^{-1}(\gamma).$$

Suppose $|\gamma| = 1$. We have the algebra isomorphism $\mathcal{H}_{\gamma, \theta} \cong \mathcal{H}_{\gamma, \theta}^{(1)} \varepsilon_\gamma$ from 2.3.4. It identifies the center $Z(\mathcal{H}_{\gamma, \theta})$ with the center of $\mathcal{H}_{\gamma, \theta}^{(1)} \varepsilon_\gamma$. In this way, we let the component algebra $\mathcal{H}_{\gamma, \theta}^{(1)} \varepsilon_\gamma$ act on $K^G(\hat{B}_\gamma)$, and we denote this representation by $K^G(\hat{B}_\gamma)_{\gamma, \theta}$. We may then state, in obvious terminology, that any supersingular character $\theta$ of the center of $\mathcal{H}_{\gamma, \theta}^{(1)} \varepsilon_\gamma$ gives rise to the supersingular irreducible $\mathcal{H}_{\gamma, \theta}^{(1)} \varepsilon_\gamma$-module $K^G(\hat{B}_\gamma)_{\theta}$.

4 The regular case and dual equivariant intersection theory

4.1 Recollections from algebraic $CH^G$-theory

For basic notions from equivariant algebraic intersection theory we refer to \cite{EG96} and \cite{Br97}. As in the case of equivariant $K$-theory, the characteristic homomorphism will make everything explicit.

4.1.1. We denote by Sym($\Lambda$) the symmetric algebra of the lattice $\Lambda$ endowed with its natural action of the reflection $s$. The equivariant intersection theory of the base point $pt = \text{Spec} \, \mathbb{F}_q$ canonically identifies with the ring of invariants

$$\text{Sym}(\Lambda)^s \simeq CH^G(pt),$$

cf. \cite[sec. 3.2]{EG96}. Recall our basis elements $\eta_1 := (1, 0)$ and $\eta_2 := (0, 1)$ of $\Lambda$, so that $\text{Sym}(\Lambda) = \mathbb{Z} [\eta_1, \eta_2]$. We define the invariant elements

$$\xi_1 := \eta_1 + \eta_2 \quad \text{and} \quad \xi_2 := \eta_1 \eta_2.$$
in Sym(Λ)*. Then
\[ \text{Sym}(\Lambda)^* = \mathbb{Z}[\zeta_1', \zeta_2'] \]
and, after inverting the prime 2, the Sym(Λ)*-module Sym(Λ) is free of rank 2, on the basis \( \{1, \frac{a+\eta_2}{\eta_1-\eta_2}\} \).

4.1.2. The equivariant Chern class of line bundles in the algebraic \( CH^G \)-theory of \( \hat{B} \) is a map
\[ c_1^G : \text{Pic}^G(\hat{B}) \longrightarrow CH^G(\hat{B}) \]
which is a group homomorphism. Then, the corresponding characteristic homomorphism is a ring isomorphism
\[ c^G : \text{Sym}(\Lambda) \longrightarrow CH^G(\hat{B}), \]
which maps \( \lambda = (\lambda_1, \lambda_2) \in \Lambda \) to the equivariant Chern class of the line bundle \( \mathcal{O}_{\pi^!}(\lambda_1 - \lambda_2) \otimes \text{det}^{\lambda_2} \) on \( \hat{B} = \mathbb{P}^1_{\mathbb{Z}_2} \), i.e.
\[ c^G(\lambda) = c_1^G(\mathcal{O}_{\pi^!}(\lambda_1 - \lambda_2) \otimes \text{det}^{\lambda_2}). \]

Note here that the algebraic group \( \hat{G} = G \times \mathbb{Z}_2 \) is special (in the sense of [EG96, 6.3]) and the map \( c^G \) is therefore already bijective at the integral level \([Br97, \text{sec. 6.6}]\). The homomorphism \( c^G \) is \( \text{Sym}(\Lambda)^* \simeq CH^G(\text{pt}) \)-linear.

To emphasize the duality and the analogy with the case of \( K \)-theory (and to ease notation), we abbreviate from now on
\[ S(\hat{T}) := \text{Sym}(\Lambda) \quad \text{and} \quad S(\hat{G}) := \text{Sym}(\Lambda)^*. \]

4.1.3. For the definition of the classical Demazure operators on algebraic intersection theory, we refer to [D73]. The Demazure operators
\[ D_s, D_s' \in \text{End}_{S(\hat{T})}(S(\hat{T})) \]
are defined by:
\[ D_s(a) = \frac{a - s(a)}{\eta_1 - \eta_2} \quad \text{and} \quad D'_s(a) = \frac{a - s(a)(1 - (\eta_1 - \eta_2))}{\eta_1 - \eta_2} \]
for \( a \in S(\hat{T}) \). Then \( D_s \) is the projector on \( S(\hat{T})^a \eta_1 - \eta_2 \) along \( S(\hat{T})^r \), and \((-D_s) + D'_s = s\). In particular, \( D_s^2 = 0 \) and \( D'_s^2 = \text{id} \). One sets
\[ D_s(q) := D_s - qD'_s \in \text{End}_{S(\hat{T})^q}(S(\hat{T})) \]
and checks by direct calculation that \( D_s(q)^2 = q^2 \). We obtain thus a well-defined \( \mathbb{Z} \)-algebra homomorphism
\[ \omega_{0}^{\text{nil}}(q) : H_{0}^{\text{nil}}(q) = \mathbb{Z}[q][S] \longrightarrow \text{End}_{S(\hat{T})^q}(S(\hat{T})) \quad \text{q} \longmapsto q^2 \], \( S \longmapsto -D_s(q) \)
which we call the Demazure representation.

4.2 The morphism from \( S(\hat{G})[q] \) to the center of \( \mathcal{H}^{\text{nil}}(q) \)
The version of the homomorphism \((\zeta^*)^{-1}\) in the regular case is the \( \mathbb{Z}[q] \)-algebra homomorphism
\[ S(\hat{G})[q] = \mathbb{Z}[q][\zeta_1', \zeta_2'] \longrightarrow \mathbb{Z}[q][e^{(1,0)}, \{e^{(1,1)}\}^{\pm 1}] \]
\[ \zeta_1' \longmapsto e^{(1,0)} \]
\[ \zeta_2' \longmapsto e^{(1,1)} \]
which becomes an isomorphism after inverting \( \zeta_2' \). Its composition with \( \mathcal{Z}^{\text{nil}}(q) \), cf. 2.4.2 therefore gives a ring isomorphism
\[
\begin{align*}
S(\hat{G})[q][\zeta_2'^{-1}] & \cong \mathcal{Z}(\mathcal{H}^{\text{nil}}(q)) \\
\zeta_1' & \longmapsto \zeta_1 = US + SU \\
\zeta_2' & \longmapsto \zeta_2 = U^2.
\end{align*}
\]
4.3 The extended Demazure representation $\mathscr{L}_p^{\text{nil}}(q)$ at $q = 0$

Recall the Demazure representation $\mathscr{A}_0^{\text{nil}}(q)$ of the finite algebra $H_0^{\text{nil}}(q)$ by $S(\hat{G})[q]$-linear operators on the intersection theory $CH^G(\hat{B})$, cf. 4.1.3. In this section we work at $q = 0$. We write $\mathscr{A}_0^{\text{nil}}$ for the specialization of $\mathscr{A}_0^{\text{nil}}(q)$ at $q = 0$.

For better readability we make a slight abuse of notation and denote the elements $\xi_i'$ by $\xi_i$ in this and the following sections. Moreover, $p$ will always be an odd prime.

4.3.1. A ring homomorphism

$$\mathscr{A}^{\text{nil}} : H^{\text{nil}} \longrightarrow \text{End}_{S(\hat{G})}(CH^G(\hat{B}))$$

which extends $\mathscr{A}_0^{\text{nil}}$ and which is linear with respect to the above ring homomorphism $S(\hat{G}) \rightarrow Z(H^{\text{nil}})$ does not exist, even after inverting $\xi_2$. However, there exists a natural good approximation (after inverting the prime 2). We will explain these points in the following.

4.3.2. An extension of $\mathscr{A}_0^{\text{nil}}$, linear with respect to $S(\hat{G}) \rightarrow Z(H^{\text{nil}})$, exists if and only if there is an operator

$$\mathscr{A}^{\text{nil}}(U) \in \text{End}_{S(\hat{G})[\xi_2^{-1}]}(CH^G(\hat{B})[\xi_2^{-1}])$$

satisfying

1. $\mathscr{A}^{\text{nil}}(U)$ is invertible;
2. $\mathscr{A}^{\text{nil}}(U)^2 = \mathscr{A}^{\text{nil}}(U^2) = \xi_2 \text{Id}$, i.e. $\mathscr{A}^{\text{nil}}(U)^2 = \xi_2 \text{Id}$;
3. $\mathscr{A}^{\text{nil}}(U)\mathscr{A}^{\text{nil}}(S) + \mathscr{A}^{\text{nil}}(S)\mathscr{A}^{\text{nil}}(U) = \mathscr{A}^{\text{nil}}(US + SU) = \xi_1 \text{Id}$.

Tensoring by $F_p$, we may write

$$CH^G(\hat{B})_{F_p} = S(\hat{G})_{F_p} \oplus S(\hat{G})_{F_p} \eta_1 - \eta_2 \over 2,$$

and identify $\text{End}_{S(\hat{G})_{F_p}}(CH^G(\hat{B})_{F_p})$ with the algebra of $2 \times 2$-matrices over the ring $S(\hat{G})_{F_p}$. The analogous statements hold after inverting $\xi_2$.

Then, by definition,

$$\mathscr{A}^{\text{nil}}_{0,F_p}(S) = -D_s = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Hence, if we set

$$\mathscr{A}^{\text{nil}}_{F_p}(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

we obtain

$$\mathscr{A}^{\text{nil}}_{F_p}(U)^2 = \xi_2 \text{Id} \iff \begin{pmatrix} a^2 + bc & c(a + d) \\ b(a + d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} \xi_2 & 0 \\ 0 & \xi_2 \end{pmatrix}$$

and

$$\mathscr{A}^{\text{nil}}_{F_p}(-D_s) + (-D_s)\mathscr{A}^{\text{nil}}_{F_p}(U) = \xi_1 \text{Id} \iff \begin{cases} a = -d \\ b = -\xi_1, \end{cases}$$

and then the first system becomes equivalent to the equation

$$a^2 - \xi_1 c = \xi_2 \in F_p[\xi_1, \xi_2^\pm 1].$$

However, since $\xi_2$ has no square root in the ring $F_p[\xi_2^\pm 1]$, this latter equation has no solution (take $\xi_1 = 0$ !). Consequently, there does not exist any matrix $\mathscr{A}^{\text{nil}}_{F_p}(U)$ with coefficients in $S(\hat{G})_{F_p}[\xi_2^{-1}]$ satisfying conditions 1, 2, 3, above.
The injectivity part of the theorem will be shown in the next subsection. The discussion preceding the theorem shows that the matrix

\[ \mathfrak{A}_{nil}^{nil} : \mathcal{H}_{F_p}^{nil} \to \text{End}_{S(\mathcal{G})_F_p}[\mathcal{S}^{\pm 1}(\hat{\mathcal{G}})] \]

which extends the ring homomorphism \( \mathfrak{A}_0^{nil} \) and coincides on Z(\( \mathcal{H}_F^{nil} \)) with the homomorphism

\[ Z(\mathcal{H}_F^{nil}) \xrightarrow{\sim} F_p[\xi_1, \xi_2^{\pm 2}] \subset S(\hat{\mathcal{G}})_{F_p}[\xi_2^{-1}] \]

\[ \xi_1 \mapsto -\xi_1 \]

\[ \xi_2 \mapsto \xi_2. \]

The homomorphism \( \mathfrak{A}_{nil}^{nil} \) is injective.

**Proof.** The discussion preceding the theorem shows that the matrix

\[ \mathfrak{A}_{nil}^{nil}(U) := \left( \begin{array}{cc} \frac{\xi_2}{2} - \xi_2 & -\xi_1 \left( \frac{\xi_2^2}{4} - \xi_2 \right) \\ \xi_1 & -\left( \frac{\xi_2}{2} - \xi_2 \right) \end{array} \right) \]

does satisfy the three conditions

1. \( \mathfrak{A}_{nil}^{nil}(U) \) is invertible;
2. \( \mathfrak{A}_{nil}^{nil}(U)^2 = (\xi_2)^2 \text{Id} \);
3. \( \mathfrak{A}_{nil}^{nil}(US + SU) = -\xi_1 \text{Id} \).

The injectivity part of the theorem will be shown in the next subsection.

**4.3.4. Remark.** The minus sign before \( \xi_1 \) appearing in the value of \( \mathfrak{A}_{nil}^{nil} \) on \( \xi_1 = US + SU \) could be avoided by setting \( \mathfrak{A}_0^{nil}(S) := D_s \) instead of \( -D_s \) in the Demazure representation. But we will not do this.

**4.3.5. Remark.** In the Iwahori case, one can check that the action of \( U \) coincides with the action of the Weyl element \( e^m s \). In the regular case, the action of the element \( \eta_1 s \) does not satisfy the conditions 1-3 appearing in the above proof. However, the action of \( \eta_1 s \) does and, in fact, its matrix is given by matrix \( \mathfrak{A}_{nil}^{nil}(U) \). So the choice of the matrix \( \mathfrak{A}_{nil}^{nil}(U) \) is in close analogy with the Iwahori case. Our chosen extension \( \mathfrak{A}_{nil}^{nil} \) of \( \mathfrak{A}_0^{nil} \) seems to be distinguished for at least this reason. This observation also shows that the action of \( U \) can actually be defined integrally, i.e. before inverting the prime 2.

**4.4 Faithfulness of \( \mathfrak{A}_{nil}^{nil} \)**

Let us show that the map \( \mathfrak{A}_{nil}^{nil} \) is injective. It follows from 2.4.2 that the ring \( \mathcal{H}_F^{nil} \) is generated by the elements

\[ 1, S, U, SU \]

over its center \( Z(\mathcal{H}_F^{nil}) = F_p[\xi_1, \xi_2^{\pm 1}] \). The latter is mapped isomorphically to the subring

\[ F_p[\xi_1, \xi_2^{\pm 2}] \subset S(\hat{\mathcal{G}})_{F_p}[\xi_2^{-1}] \]

of the matrix algebra \( \text{End}_{S(\mathcal{G})_F_p}[\mathcal{S}^{\pm 1}(\hat{\mathcal{G}})] \) by \( \mathfrak{A}_{nil}^{nil} \). For injectivity, it therefore suffices to show that the images

\[ 1, \mathfrak{A}_0^{nil}(S), \mathfrak{A}_{nil}^{nil}(U), \mathfrak{A}_0^{nil}(S)\mathfrak{A}_{nil}^{nil}(U) \]

are distinct.
of $1, S, U, SU$ under $\mathcal{A}_p^{\text{nil}}$ are free over $S(\hat{G})[\xi_2^{-1}]$. To this end, let $\alpha, \beta, \gamma, \delta \in S(\hat{G})[\xi_2^{-1}]$ (which is an integral domain) be such that

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} a & c \\ b & -a \end{pmatrix} + \delta \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = 0,$$

i.e.

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & -\beta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \gamma a & \gamma c \\ \gamma b & -\gamma a \end{pmatrix} + \begin{pmatrix} -\delta b & \delta a \\ 0 & 0 \end{pmatrix} = 0.$$

Then

$$\begin{cases} \alpha + \gamma a - \delta b &= 0 \\ \gamma b &= 0 \\ -\beta + \gamma c + \delta a &= 0 \\ \alpha - \gamma a &= 0, \end{cases}$$

with $\alpha, \beta, \gamma, \delta \in S(\hat{G})[\xi_2^{-1}]$. Now recall our choice

$$\mathcal{A}_p^{\text{nil}}(U) = \begin{pmatrix} a & c \\ b & -a \end{pmatrix} : = \begin{pmatrix} \xi_2^2 - \xi_2 \\ -\xi_1^2 (\xi_2^2 - \xi_2) \\ \xi_1 \\ -(\xi_2^2 - \xi_2) \end{pmatrix}.$$ 

In particular, $b = \xi_1$ implies $\gamma = 0$, and then $\alpha = 0$, $\delta = 0$ and $\beta = 0$. This shows that the map $\mathcal{A}_p^{\text{nil}}$ is injective and concludes the proof. We record the following corollary of the proof.

4.4.1. Corollary. The ring $\mathcal{H}_p^{\text{nil}}$ is a free $Z(\mathcal{H}_p^{\text{nil}})$-module on the basis $1, S, U, SU$.

4.5 The twisted representation $\mathcal{A}_{2, p}$

4.5.1. In the algebra

$$\mathcal{H}_2 := \mathcal{H}_2(0) = (\mathbb{Z} \otimes \mathbb{Z}) \otimes Z \mathcal{H}_{\text{nil}}$$

we have the two subrings $\mathcal{H}_{\text{nil}}$ and $\mathbb{Z} \otimes \mathbb{Z}$. The aim of this section is to extend the representation $\mathcal{A}_{2, p}$ from $\mathcal{H}_{\text{nil}}$ to the whole algebra $\mathcal{H}_{2, p} := \mathcal{H}_2 \otimes \mathbb{F}_p$. To this end, we consider the $\mathbb{F}_q$-variety

$$\hat{B}^2 := \hat{B}_1 \coprod \hat{B}_2,$$

where $\hat{B}_1$ and $\hat{B}_2$ are two copies of $\hat{B}$. We have

$$CH^G(\hat{B}^2) = CH^G(\hat{B}_1) \times CH^G(\hat{B}_2).$$

After base change to $\mathbb{F}_p$, the ring $\mathcal{H}_{\text{nil}}$ acts $S(\hat{G})[\xi_2^{-1}]$-linearly on $CH^G(\hat{B})[\xi_2^{-1}]$ through the map $\mathcal{A}_{2, p}^{\text{nil}}$. We extend this action diagonally to $CH^G(\hat{B}^2)[\xi_2^{-1}]$, thus defining a ring homomorphism

$$\text{diag}(\mathcal{A}_{2, p}^{\text{nil}}) : \mathcal{H}_{p^{\text{nil}}} \longrightarrow \text{End}_{S(\hat{G})[\xi_2^{-1}]}(CH^G(\hat{B}^2)[\xi_2^{-1}]).$$

Because of the twisted multiplication in the algebra $\mathcal{H}_2$, we need to introduce the permutation action of $W$

$$\text{perm} : W \longrightarrow W_0 \longrightarrow \text{Aut}_{S(\hat{G})}(CH^G(\hat{B}^2))$$

which permutes the two factors of $CH^G(\hat{B}^2)$.

On the other hand, we can consider the projection $p_i$ from $CH^G(\hat{B}^2)$ to $CH^G(\hat{B}_i)$ as an $S(\hat{G})$-linear endomorphism of $CH^G(\hat{B}_i)$, for $i = 1, 2$. The rule $\varepsilon_i \mapsto p_i$ defines a ring homomorphism

$$\text{proj} : \mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2 \longrightarrow \text{End}_{S(\hat{G})}(CH^G(\hat{B}^2)).$$
4.5.2. Proposition. There exists a unique ring homomorphism

\[ \mathcal{A}_{2,F_p} : \mathcal{H}_{2,F_p} \rightarrow \text{End}_{S(G)_{F_p}[G^{-1}]}(CH\tilde{G}(B^2)_{F_p}[G^{-1}]) \]

such that

- \( \mathcal{A}_{2,F_p}(\pi_{\mathcal{H}_{2,F_p}})(T_w) = \text{diag}(\mathcal{A}_{2,F_p}(T_w)) \circ \text{perm}(w) \) for all \( w \in W \),
- \( \mathcal{A}_{2,F_p}(\pi_{\mathcal{H}_{2,F_p}})(w) = \text{proj} \).

The homomorphism \( \mathcal{A}_{2,F_p} \) is injective.

**Proof.** Recall that \( W_0 \) acts on the set \( \{1, 2\} \) by interchanging the two elements and then \( W \) acts via its projection to \( W_0 \). As \( \{\varepsilon, T_w, i, w \in \{1, 2\} \times W \} \) is a \( F_p \)-basis of \( \mathcal{H}_{2,F_p} \), such a ring homomorphism is uniquely determined by the formula

\[ \mathcal{A}_{2,F_p}(\varepsilon, T_w) = p_i \circ \text{diag}(\mathcal{A}_{2,F_p}(T_w)) \circ \text{perm}(w). \]

Conversely, taking this formula as a definition of \( \mathcal{A}_{2,F_p} \), we need to check that the resulting \( F_p \)-linear map is a ring homomorphism, i.e.

\[ \mathcal{A}_{2,F_p}((1, 1)) = \text{Id} \]

and

\[ \mathcal{A}_{2,F_p}(\varepsilon_i T_w \cdot \varepsilon_i T_w) = \mathcal{A}_{2,F_p}(\varepsilon_i T_w) \circ \mathcal{A}_{2,F_p}(\varepsilon_i T_w). \]

The first condition is clear because \( (1, 1) = \varepsilon_1 + \varepsilon_2 \) and \( p_i + p_{\bar{i}} = \text{Id} \). Let us check the second condition.

If \( i' \neq w^{-1} i \), then both sides of the claimed equality vanish. Now assume that \( i = w i' \). On the left hand side we find

\[ \mathcal{A}_{2,F_p}(\varepsilon_i T_w \cdot \varepsilon_i T_w) = \mathcal{A}_{2,F_p}(\varepsilon_i T_w T_w), \]

while on the right hand side, we find

\[ \mathcal{A}_{2,F_p}(\varepsilon_i T_w) \circ \mathcal{A}_{2,F_p}(\varepsilon_{w^{-1}} T_w) \]

\[ = p_i \circ \text{diag}(\mathcal{A}_{2,F_p}(T_w)) \circ \text{perm}(w) \circ p_{\bar{w}^{-1} i} \circ \text{diag}(\mathcal{A}_{2,F_p}(T_w)) \circ \text{perm}(w') \]

\[ = p_i \circ \text{diag}(\mathcal{A}_{2,F_p}(T_w)) \circ \text{diag}(\mathcal{A}_{2,F_p}(T_w)) \circ p_{\bar{w}^{-1} i} \circ (w^{-1} i) \]

\[ = p_i \circ \text{diag}(\mathcal{A}_{2,F_p}(T_w) T_w) \circ p_{\bar{w}^{-1} i}. \]

If \( \ell(w i') \neq \ell(w) + \ell(w') \), then \( T_w T_w' = 0 \) and both sides vanish. Otherwise \( T_w T_w' = T_{ww'} \), so that the left hand side becomes

\[ \mathcal{A}_{2,F_p}(\varepsilon_i T_{ww'}) = p_i \circ \text{diag}(\mathcal{A}_{2,F_p}(T_{ww'})) \circ \text{perm}(ww'), \]

and the right hand side

\[ p_i \circ \text{diag}(\mathcal{A}_{2,F_p}(T_{ww'})) \circ p_{\bar{w}^{-1} i}. \]

These two operators are equal. This proves the existence and the uniqueness of the extension \( \mathcal{A}_{2,F_p} \). Its injectivity will be shown in the next subsection. \( \square \)

4.6 Faithfulness of \( \mathcal{A}_{2,F_p} \)

Let us show that the map \( \mathcal{A}_{2,F_p} \) is injective. This is equivalent to show that the family

\[ \{ \mathcal{A}_{2,F_p}(\varepsilon_i T_w), (i, w) \in \{1, 2\} \times W \} \]

is free over \( F_p \). So let \( \{n_{i,w} \in F_p(1, 2) \times W) \) such that

\[ \sum_{i, w} n_{i,w} \mathcal{A}_{2,F_p}(\varepsilon_i T_w) = 0. \]
Let us fix $i_0 \in \{1, 2\}$. Composing by $p_{i_0}$ on the left, we get
\[
\sum_w n_{i_0,w} \mathcal{A}_F(T_w) \xi_{i_0} = 0.
\]
The left hand side can be rewritten as
\[
\sum_w n_{i_0,w} p_{i_0} \circ \text{diag}(\mathcal{A}_F(T_w)) \circ \text{perm}(w) = \sum_w n_{i_0,w} p_{i_0} \circ \text{diag}(\mathcal{A}_F(T_w)) \circ p_{w^{-1} i_0}.
\]
Now let us fix $w_0 \in W_0$. Composing by $p_{w_0^{-1} i_0}$ on the right, we get
\[
\sum_{w \in \Lambda w_0} n_{i_0,w} p_{i_0} \circ \text{diag}(\mathcal{A}_F(T_w)) \circ p_{w_0^{-1} i_0} = 0.
\]
Then, for each $w \in \Lambda w_0$, remark that
\[
p_{i_0} \circ \text{diag}(\mathcal{A}_F(T_w)) \circ p_{w_0^{-1} i_0} = \xi_{i_0} \circ \text{diag}(\mathcal{A}_F(T_w)) \circ \xi_{w_0^{-1} i_0}
\]
in $\text{End} (CH \hat{G}(\tilde{B}_1)[\xi_2^{-1}] \times CH \hat{G}(\tilde{B}_2)[\xi_2^{-1}])$, where $\xi_{i_0} \circ \xi_{w_0^{-1} i_0}$ is the canonical map
\[
\xi_{i_0} \circ \xi_{w_0^{-1} i_0} : CH \hat{G}(\tilde{B}_1)[\xi_2^{-1}] \to CH \hat{G}(\tilde{B}_2)[\xi_2^{-1}]
\]
As the latter is injective, we get
\[
0 = \sum_{w \in \Lambda w_0} n_{i_0,w} \mathcal{A}_F(T_w) \circ p_{w_0^{-1} i_0} = \mathcal{A}_F(T_w) \circ p_{w_0^{-1} i_0}.
\]
Finally, as $p_{w_0^{-1} i_0} : CH \hat{G}(\tilde{B}_2)[\xi_2^{-1}] \to CH \hat{G}(\tilde{B}_1)[\xi_2^{-1}]$ is surjective, and as $\mathcal{A}_F$ is injective, cf. 4.41 we get $n_{i_0,w} = 0$ for all $w \in \Lambda w_0$. This concludes the proof that $\mathcal{A}_F$ is injective.

### 4.7 Supersingular modules

In this section we work over the algebraic closure $\overline{\mathbb{F}}_q$ of the field $\mathbb{F}_q$.

#### 4.7.1. Recall from 2.4.4 that
\[
\mathcal{H}_2(\overline{\mathbb{F}}_q) = \mathcal{H}_2 \otimes \overline{\mathbb{F}}_q \cong (\mathbb{F}_q \times \overline{\mathbb{F}}_q) \otimes_{\mathbb{F}_q} \mathbb{F}_q[S, U^{\pm 1}]
\]
has the structure of a $2 \times 2$-matrix algebra over its center $Z(\mathcal{H}_2(\overline{\mathbb{F}}_q))$. Since $\overline{\mathbb{F}}_q$ is algebraically closed, $Z(\mathcal{H}_2(\overline{\mathbb{F}}_q))$ acts on any finite-dimensional irreducible $\mathcal{H}_2(\overline{\mathbb{F}}_q)$-module by a character (Schur’s lemma).

Let $\theta$ be a character of $Z(\mathcal{H}_2(\overline{\mathbb{F}}_q))$. Then
\[
\mathcal{H}_{2,\theta} := \mathcal{H}_2 \otimes Z(\mathcal{H}_2(\overline{\mathbb{F}}_q)), \theta \overline{\mathbb{F}}_q
\]
is isomorphic to the matrix algebra $M(2, \overline{\mathbb{F}}_q)$. In particular, it is a semisimple (even simple) ring.

#### 4.7.2. The unique irreducible $\mathcal{H}_2(\overline{\mathbb{F}}_q)$-module with central character $\theta$ is called the standard module with character $\theta$. Its $\overline{\mathbb{F}}_q$-dimension is 2 and it is isomorphic to the standard representation $\overline{\mathbb{F}}_q^{2 \times 2}$ of the matrix algebra $M(2, \overline{\mathbb{F}}_q)$. The image of the basis $\{(1, 0), (0, 1)\}$ of $\overline{\mathbb{F}}_q^{2 \times 2}$ is called a standard basis.

A central character $\theta$ is called supersingular if $\theta(X) = \theta(Y) = 0$ (or, equivalently, if $\theta(\zeta_1) = 0$). If $\theta$ is supersingular, then the affine algebra $\mathcal{H}_{2, \text{aff}}(\overline{\mathbb{F}}_q)$ acts on the standard basis of the module via the characters $\chi_1$, respectively $\chi_2$ and the action of $U$ interchanges the two, cf. 2.4.3 and 2.4.4.

For more details we refer to [VO03] 2.3.
Recall that the image under the map $\mathcal{A}_{2,\mathcal{F}_q}$ of the central subring

$$Z^\circ(H_{2,\mathcal{F}_q}) = \mathcal{F}_q[\xi_1, \xi_2^\pm 1] \subset Z(H_{2,\mathcal{F}_q})$$

is the subring of scalars

$$\mathcal{F}_q[\xi_1, \xi_2^\pm 1] \subset \mathcal{F}_q[\xi_1, \xi_2^\pm 1] = S(\mathcal{G})_{\mathcal{F}_q}[\xi_2^{-1}].$$

4.7.4. Let us fix a supersingular central character $\theta$ and denote its restriction to $Z^\circ := Z^\circ(H_{2,\mathcal{F}_q})$ by $\theta$, too. Then consider the $H_{2,\mathcal{F}_q}$-action on the base change

$$CH^G(\mathcal{B})[\xi_2^{-1}]_{\theta} := CH^G(\mathcal{B})_{\mathcal{F}_q}[\xi_2^{-1}] \otimes Z^\circ \mathcal{F}_q = CH^G(\mathcal{B})_{\mathcal{F}_q}[\xi_2^{-1}] \otimes S(\mathcal{G})_{\mathcal{F}_q}[\xi_2^{-1}](S(\mathcal{G})_{\mathcal{F}_q}[\xi_2^{-1}] \otimes Z^\circ \mathcal{F}_q).$$

For the base ring, we have

$$S(\mathcal{G})_{\mathcal{F}_q}[\xi_2^{-1}] \otimes Z^\circ \mathcal{F}_q = \mathcal{F}_q[\xi_1, \xi_2^\pm 1] \otimes \mathcal{A}_{2,\mathcal{F}_q}[\xi_1, \xi_2^\pm 1] \otimes \mathcal{F}_q$$

where $\mathcal{A}_{2,\mathcal{F}_q}(\xi_1) = -\xi_1$ and $\mathcal{A}_{2,\mathcal{F}_q}(\xi_2) = \xi_2^2$. Now put $\theta(\xi_2) = b \in \mathcal{F}_q^\times$. Then

$$S(\mathcal{G})_{\mathcal{F}_q}[\xi_2^{-1}] \otimes Z^\circ \mathcal{F}_q = \mathcal{F}_q[\xi_1, \xi_2^\pm 1]/(\xi_1, \xi_2^2 - b) = \mathcal{F}_q[\xi_2]/(\xi_2^2 - b) =: A$$

and so

$$CH^G(\mathcal{B})[\xi_2^{-1}]_{\theta} = \mathcal{F}_q[\eta_1^{\pm 1}, \eta_2^{\pm 2}] \otimes \mathcal{F}_q[\xi_1, \xi_2^\pm 1] \mathcal{F}_q[\xi_2]/(\xi_2^2 - b) = \mathcal{F}_q[\eta_1^{\pm 1}, \eta_2^{\pm 2}] \otimes \mathcal{A}_{2,\mathcal{F}_q}(\xi_1, \xi_2^\pm 1) A.$$

Note that the $\mathcal{F}_q$-algebra $A$ is isomorphic to the direct product $\mathcal{F}_q \times \mathcal{F}_q$ (the isomorphism depending on the choice of a square root of $b$ in $\mathcal{F}_q$). An $A$-basis of $CH^G(\mathcal{B})[\xi_2^{-1}]_{\theta}$ is given by the four elements

$$\{1_i, \frac{\eta_1 - \eta_2}{2} 1_i\}_{i=1,2}$$

where

$$1_i \in CH^G(\mathcal{B}_i) \subset CH^G(\mathcal{B}_1) \times CH^G(\mathcal{B}_2) = CH^G(\mathcal{B}).$$

is the equivariant Chern class of the structure sheaf on $\mathcal{B}_i$, for $i = 1, 2$. The $\mathcal{F}_q$-dimension of $CH^G(\mathcal{B})[\xi_2^{-1}]_{\theta}$ is therefore $8$ and $H_{2,\mathcal{F}_q}$ acts $A$-linearly. The length of the $H_{2,\mathcal{F}_q}$-module $CH^G(\mathcal{B})[\xi_2^{-1}]_{\theta}$ is $4$ and the central character of any irreducible subquotient is necessarily equal to $\theta$, since this is true by construction after restriction to $Z^\circ$. In the following, we compute explicitly a composition series.

4.7.5. Proposition. The algebra $H_{2,\text{aff},\mathcal{F}_q}$ acts on $1_i \in CH^G(\mathcal{B})[\xi_2^{-1}]_{\theta}$ by the supersingular character $\chi_i$, for $i = 1, 2$.

Proof: The action of $H_{2,\text{aff},\mathcal{F}_q}$ on $CH^G(\mathcal{B})_{\mathcal{F}_q}[\xi_2^{-1}]$ is defined by the map $\mathcal{A}_{2,\mathcal{F}_q}$. Hence, by definition,

$$\varepsilon_{i'} \cdot 1_i = \begin{cases} 1_i & \text{if } i' = i \\ 0 & \text{otherwise.} \end{cases}$$

We calculate

$$S \cdot 1_i = \text{diag}(-D_s) \circ \text{perm}(s)(1_i) = \text{diag}(-D_s)1_i = 0.$$ 

Moreover,

$$U^{-1} \cdot 1_i = \text{diag}(U^{-1}) \circ \text{perm}(u^{-1})(1_i) = \text{diag}(U^{-1})1_i = s(\eta_1^{-2})1_i = \eta_2^{-2}1_i$$

and

$$D_s(\eta_2^{-2}) = \frac{\eta_2^{-2} - \eta_1^{-2}}{\eta_1 - \eta_2} = (\eta_1 \eta_2)^{-2} \eta_1^{-2} - \eta_2^{-2} = (\eta_1 \eta_2)^{-2}(\eta_1 + \eta_2) = \frac{\xi_1}{\xi_2^2}.$$
Therefore,
\[ SU^{-1} \cdot 1_i = \text{diag}(-D_s) \circ \text{perm}(s)(\eta_2^{-2}1_{s,i}) = -D_s(\eta_2^{-2})1_i = -\frac{\xi_1}{\xi_2}1_i = 0 \]
since \(\xi_1 = 0\) in \(CH^G(\hat{B}_i)\)[\(\xi_2^{-1}\)]\(_g\). It follows that \(S_0 \cdot 1_i = USU^{-1} \cdot 1_i = 0\). \(\square\)

**4.7.6. Proposition.** A composition series with simple subquotients of the \(H_{2\mathbb{F}_q}\)-module

\[ CH^G(\hat{B}^2)\)[\(\xi_2^{-1}\)]\(_g\)

is given by

\[
\{0\} \\
\subset \mathbb{F}_q1_1 \oplus \mathbb{F}_q(U \cdot 1_i) \\
\subset A_{1_1} \oplus A(U \cdot 1_i) = A_{1_1} \oplus A_{1_i} \\
\subset A_{1_1} \oplus A_{1_i} \oplus \mathbb{F}_q(\frac{\eta_1 - \eta_2}{2}) \subset \mathbb{F}_q(U \cdot \frac{\eta_1 - \eta_2}{2}) \\
\subset CH^G(\hat{B}^2)\)[\(\xi_2^{-1}\)]\(_g\).
\]

Here the direct sums \(\oplus\) are taken in the sense of \(\mathbb{F}_q\)-vector spaces.

**Proof.** First of all, \(U \cdot 1_i := \text{diag}(U) \circ \text{perm}(u)(1_{s,i}) = \text{diag}(U)1_{s,i} = \eta_1^21_{s,i} = -\xi_21_{s,i} \in A^\times 1_{s,i}\)

because \(0 = \xi_1 = \eta_1 + \eta_2\) and \(0 = \xi_2^2 = \eta_1^2 + \eta_2^2 + 2\xi_2\) in \(CH^G(\hat{B})\)[\(\xi_2^{-1}\)]\(_g\). Hence the three first \(\oplus\) appearing in the statement of the proposition are indeed direct sums. These three sums are \(U\)-stable by construction. Moreover, by the preceding proposition, \(H_{2\mathbb{F}_q}\) acts by the character \(\chi_i\) on \(1_i\), hence by the character \(\chi_{-1}\) on \(U \cdot 1_i\). It follows that \(\mathbb{F}_q1_i \oplus \mathbb{F}_q(U \cdot 1_i)\) realizes the standard \(H_{2\mathbb{F}_q}\)-module with central character \(\theta\), and that \(A_{1_1} \oplus A(U \cdot 1_i)\) is an \(H_{2\mathbb{F}_q}\)-submodule of \(CH^G(\hat{B}^2)\)[\(\xi_2^{-1}\)]\(_g\) of dimension 4 over \(\mathbb{F}_q\). In fact, if \(L \subset A\) is any \(\mathbb{F}_q\)-line, the same arguments show that \(L_{1_1} \oplus L(U \cdot 1_i)\) realizes the standard \(H_{2\mathbb{F}_q}\)-module with central character \(\theta\). In particular, the module \(A_{1_1} \oplus A(U \cdot 1_i)\) is semisimple.

Now let us compute the action of \(H_{2\mathbb{F}_q}\) on the element \(\frac{\eta_1 - \eta_2}{2}1_i\), for \(i = 1, 2\). We have

\[ \varepsilon_{i'} \cdot \frac{\eta_1 - \eta_2}{2}1_i = \begin{cases} \frac{\eta_1 - \eta_2}{2}1_i & \text{if } i' = i \\ 0 & \text{otherwise.} \end{cases} \]

Next

\[ S \cdot \frac{\eta_1 - \eta_2}{2}1_i := \text{diag}(S) \circ \text{perm}(s)(\frac{\eta_1 - \eta_2}{2}1_i) = \text{diag}(S)(\frac{\eta_1 - \eta_2}{2}1_i) = -1_{s,i}, \]

\[ U^{-1} \cdot \frac{\eta_1 - \eta_2}{2}1_i := \text{diag}(U^{-1}) \circ \text{perm}(u^{-1})(\frac{\eta_1 - \eta_2}{2}1_i) = \text{diag}(U^{-1})(\frac{\eta_1 - \eta_2}{2}1_i) = \eta_2^{-2}\eta_2 - \eta_1^{-2}1_{s,i}, \]

\[ D_s(\eta_2^{-2}\eta_2 - \eta_1^{-2}) = \frac{1}{\eta_1 - \eta_2}(\eta_2^{-2}\eta_2 - \eta_1^{-2}) = \frac{\xi_1^2 - 2\xi_2}{2\xi_2^2}, \]

\[ SU^{-1} \cdot \frac{\eta_1 - \eta_2}{2}1_i = \text{diag}(S) \circ \text{perm}(s)(\eta_2^{-2}\eta_2 - \eta_1^{-2}1_{s,i}) = \frac{\xi_1^2 - 2\xi_2}{2\xi_2^2}1_i, \]

\[ S_0 \cdot \frac{\eta_1 - \eta_2}{2}1_i := USU^{-1} \cdot \frac{\eta_1 - \eta_2}{2}1_i = \text{diag}(U) \circ \text{perm}(u)(\frac{\xi_1^2 - 2\xi_2}{2\xi_2^2}1_i) = \eta_1^2\frac{\xi_1^2 - 2\xi_2}{2\xi_2^2}1_{s,i} = 1_{s,i}, \]

because \(\xi_1 = 0\) and (hence) \(\eta_1^2 = -\xi_2\) in \(CH^G(\hat{B})\)[\(\xi_2^{-1}\)]\(_g\), and finally

\[ U \cdot \frac{\eta_1 - \eta_2}{2}1_i = \text{diag}(U) \circ \text{perm}(u)(\frac{\eta_1 - \eta_2}{2}1_i) = \text{diag}(U)(\frac{\eta_1 - \eta_2}{2}1_i) = \xi_2\frac{\eta_1 - \eta_2}{2}1_{s,i}, \]

20
which lies in $A^\times(\frac{m-n}{2}1_{1_1})$. Neither of the two elements $\frac{m-n}{2}1_{1_1}$ and $U \cdot \frac{m-n}{2}1_{1_1}$ lies in the (semisimple) module $A1_1 \oplus A(U \cdot 1_1)$. Hence the three last $\oplus$ appearing in the statement are indeed direct and they form a sub-$H_{\text{aff}}q$-module of dimension 6 over $\mathbb{F}_q$. So the series appearing in the statement is indeed a composition series with irreducible subquotients.

4.7.7. Remark. We see from the proof of the preceding proposition that the characters of $H_{\text{aff}}q$ in the sub-$H_{\text{aff}}q$-module

$$A1_1 \oplus A1_1 \oplus \mathbb{F}_q(\frac{m-n}{2}1_{1_1}) \oplus \mathbb{F}_q(U \cdot \frac{m-n}{2}1_{1_1})$$

are contained in $A1_1 \oplus A1_1$. Hence this submodule is not semi-simple. A fortiori the whole module $CH^G(\hat{B}^2)([\xi_2^{-1}])_p$ is not semisimple and, hence, has no central character.

4.7.8. Now we transfer this discussion to any regular component of the algebra $H^{(1)}q$ as follows. Let $\gamma = \{\lambda, \lambda\} \in T'/W_0$ be a regular orbit and form the $\mathbb{F}_q$-variety

$$\hat{B}^\gamma = \hat{B} \times \pi^{-1}(\gamma) = \hat{B} \prod \hat{B}_\lambda,$$

where $\hat{B}_\lambda$ and $\hat{B}_\lambda$ are two copies of $\hat{B}$. We have the algebra isomorphism $H_{\text{aff}}q \cong H^{(1)}q_{\epsilon\gamma}$ from 2.4.5. In this way, the representation $\mathcal{M}_{\text{aff}}q$ induces a representation

$$\mathcal{M}_{\text{aff}}q : H^{(1)}q_{\epsilon\gamma} \rightarrow \text{End}_{\Sigma(G)}(CH^G(\hat{B}^\gamma)([\xi^{-1}_2])).$$

We may then state, in obvious terminology, that any supersingular character $\theta$ of the center of $H^{(1)}q_{\epsilon\gamma}$ gives rise to the $H^{(1)}q_{\epsilon\gamma}$-module $CH^G(\hat{B}^\gamma)([\xi^{-1}_2])_p$ and that the semisimplification of the latter module equals a direct sum of four copies of the unique supersingular $H^{(1)}q_{\epsilon\gamma}$-module with central character $\theta$.

5 Tame Galois representations and supersingular modules

Our reference for basic results on tame Galois representations is [V94].

5.1. Let $\varpi \in \mathcal{O}_F$ be a uniformizer and let $f$ be the degree of the residue field extension $\mathbb{F}_q/\mathbb{F}_p$, i.e. $q = p^f$. Let $\text{Gal}(\overline{F}/F)$ denote the absolute Galois group of $F$. Let $I \subset \text{Gal}(\overline{F}/F)$ be its inertia subgroup. We fix an element $\varphi \in \text{Gal}(\overline{F}/F)$ lifting the Frobenius $x \mapsto x^q$ on $\text{Gal}(\overline{F}/F)/I$. The unique pro-$p$-Sylow subgroup of $I$ is denoted by $P$ (the wild inertia subgroup) and the quotient $I/P$ is pro-cyclic with pro-order prime to $p$. We choose a lift $v \in I$ of a topological generator for $I/P$. Let $W \subset \text{Gal}(\overline{F}/F)$ denote the Weil group of $F$. The quotient group $W/P$ is topologically generated by (the images of) $\varphi$ and $v$ and the only relation between these two generators is $\varphi v \varphi^{-1} = v^q$. There is a topological isomorphism

$$W/P \simeq \lim_{\rightarrow \mu_p} \mathbb{F}_p^\times$$

where the projective limit is taken with respect to the norm maps $\mathbb{F}_p^\times \rightarrow \mathbb{F}_p^\times$. We denote by $\omega_n$ the projection map $W/P \rightarrow \mathbb{F}_p^\times$ followed by the inclusion $\mathbb{F}_p^\times \subseteq \mathbb{F}_q^\times$. We shall only be concerned with the characters $\omega_n$ and $\omega_2$. The character $\omega_2$ extends from $W$ to $\text{Gal}(\overline{F}/F)$ by choosing a root $\sqrt{-q}$ and letting $\text{Gal}(\overline{F}/F)$ act as

$$g \mapsto g \frac{\sqrt{q}}{\sqrt{q}} \in \mu_{q-1}(F)$$

followed by reduction mod $\varpi$. The character

$$\omega_2 : \text{Gal}(\overline{F}/F) \rightarrow \mathbb{F}_q^\times$$

21
depends on the choice of $\varpi$ (but not on the choice of $\sqrt{-\varpi}$) and equals the reduction mod $\varpi F$ of the Lubin-Tate character $\chi_L : \text{Gal}(\overline{F} / F) \to \mathcal{O}_F^\times$ associated to the uniformizer $\varpi$. By changing $\varphi$ by an element of $I$, if necessary, we may assume $\omega_f(\varphi) = 1$. We normalize local class field theory $W_{ab} \simeq F^\times$ by sending the geometric Frobenius $\varphi^{-1}$ to $\varpi$. We view the restriction of $\omega_f$ to $W$ as a character of $F^\times$.

5.2. The set of isomorphism classes of irreducible smooth Galois representations

$$\rho : \text{Gal}(\overline{F} / F) \to \hat{G} = \text{GL}_2(\mathbb{F}_q)$$

is in bijection with the set of equivalence classes of pairs $(s, t) \in \hat{G}^2$ such that

$$s = \begin{pmatrix} 0 & 1 \\ -b & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} y & 0 \\ 0 & y^q \end{pmatrix}$$

with $b \in \mathbb{F}_q^\times$ and $y \in \mathbb{F}_q^\times \setminus \mathbb{F}_q$. Here, two pairs $(s, t)$ and $(s', t')$ are equivalent if $s = s'$ and $t, t'$ are $\text{Gal}(\mathbb{F}_q^\times / \mathbb{F}_q)$-conjugate. Note that $\det(s) = b$ and $\det(t) = y^q$. The bijection is induced by the map $\rho \mapsto (\rho(\varphi), \rho(v))$. The number of equivalence classes of such pairs $(s, t)$ equals $\frac{q^2 - q}{2}$ and hence coincides with the number of $W_0$-orbits in $T^\vee$.

5.3. By the above numerical coincidence (the ”miracle” from [V94]), there exist (many) bijections between the isomorphism classes of irreducible smooth two-dimensional Galois representations and the isomorphism classes of supersingular two-dimensional $\mathcal{H}_{\mathbb{F}_q}^{(1)}$-modules. In the following we discuss a a certain example of such a bijection in our geometric language.

Let $\rho$ be a two-dimensional irreducible smooth Galois representation with parameters $(s, t)$. Since the element $\omega_{2f}(v)$ generates $\mathbb{F}_{q^2}^\times$, the element $t$ uniquely determines an exponent $1 \leq h \leq q^2 - 1$, such that

$$\omega_{2f}(v)^h = y.$$

Replacing $\rho$ by an isomorphic representation $\rho'$ which replaces $y$ by its Galois conjugate $y^q$, replaces $h$ by the rest of the euclidian division of $qh$ by $q^2 - 1$. We call either of the two numbers an exponent of $\rho$.

5.4. Lemma. There is $0 \leq i \leq q - 2$ such that $\rho \otimes \omega_f^{-i}$ has an exponent $\leq q - 1$.

Proof : This is implicit in the discussion in [V94]. Let $\omega_{2f}(v)^h = y$. Then $h \leq q^2 - 2$ since $y \neq 1$. Moreover, $q^2 - 2 - (q - 2)(q + 1) = q$. Since $\omega_{2f}^{q+1} = \omega_f$, twisting with $\omega_f$ reduces to the case $h \leq q$. Replacing $y$ by its Galois conjugate $y^q$, if necessary, reduces then further to $h \leq q - 1$.

By the lemma, we may associate two numbers $1 \leq h \leq q - 1$ and $0 \leq i \leq q - 2$ to the representation $\rho$. We form the character

$$\omega_f^{h-1+i} \otimes \omega_f^i : (F^\times)^2 \to \mathbb{F}_q^\times, \quad (t_1, t_2) \mapsto \omega_f^{h-1+i}(t_1)\omega_f^i(t_2)$$

and restrict to $\mu_{q-1}(F)^2$. This gives rise to an element $\lambda(\rho)$ of $T^\vee$ and we take its $W_0$-orbit $\gamma_{\rho}$.

5.5. Lemma. The orbit $\gamma_{\rho}$ depends only on the isomorphism class of $\rho$.

Proof : Suppose $\rho' \simeq \rho$ with

$$\rho'(v) = t' = \begin{pmatrix} y^q & 0 \\ 0 & y \end{pmatrix}.$$

By the preceding lemma, there is $0 \leq i \leq q - 2$ and an exponent $1 \leq h \leq q - 1$ of $\rho \otimes \omega_f^{-i}$. If $1 < h$, then by definition $\omega_{2f}^h(v) = y_0 \omega_f^{-i}(v)$, so that $\omega_{2f}^h(v) = y^q \omega_f^{-i}(v)$, and hence $\omega_{2f}^{h-1+i}(v) = y^q \omega_f^{-(h-1+i)}(v)$, using $qh = q - (h - 1) + (h - 1)(q + 1)$. Then $1 \leq h' := q - (h - 1) \leq q - 1$ and taking $0 \leq i' \leq q - 2$ congruent to $h - 1 + i$ mod $q - 1$, we obtain that $h'$ is an exponent for $\rho' \otimes \omega_f^{-i'}$. In particular, $\lambda(\rho') := \omega_f^{h'+1+i'} \otimes \omega_f^{-i'}$, which is $s$-conjugate to $\lambda(\rho)$. If $h = 1$, then by definition $\omega_{2f}(v) = y^q \omega_f^{-i}(v)$, which implies $\lambda(\rho') = \lambda(\rho)$ in this case. □
We call $\rho$ (non-)regular if the orbit $\gamma_\rho$ is (non-)regular. On the other hand, we view the element $s = \rho(\varphi)$ as a supersingular character $\theta_s$ of the center $Z(\mathcal{H}_{\mathbb{F}_q}^{(1)})$, i.e. $\theta_s(\zeta_1) = 0$ and $\theta_s(\zeta_2) = b$. Finally, we have the $\mathbb{F}_q$-variety

$$\tilde{B}^\gamma = \tilde{B} \times \pi^{-1}(\gamma)$$

coming from the quotient map $\mathbb{T}^\gamma \to \mathbb{T}^\gamma/W_0$. These data give rise to the supersingular $\mathcal{H}_{\mathbb{F}_q}^{(1)}$-module

$$\mathcal{M}(\rho) := \begin{cases} K^G(\tilde{B}^\gamma)\theta_s & \text{if } \rho \text{ non-regular} \\ CH^G(\tilde{B}^\gamma)[\xi_s^{-1}]\theta_s & \text{if } \rho \text{ regular.} \end{cases}$$

Recall that $\mathcal{H}_{\mathbb{F}_q}^{(1)}$ acts on $\mathcal{M}(\rho)$ via the projection onto $\mathcal{H}_{\mathbb{F}_q}^{(1)}\gamma_\rho$ followed by the extended Demazure representation $\omega^{[\rho]}$. Recall also that the semisimplification of $\mathcal{M}(\rho)$ is a direct sum of four copies of the supersingular standard module, if $\rho$ is regular. By abuse of notation, we denote a simple subquotient of $\mathcal{M}(\rho)$ again by $\mathcal{M}(\rho)$.

**5.6. Proposition.** The map $\rho \mapsto \mathcal{M}(\rho)$ gives a bijection between the isomorphism classes of two-dimensional irreducible smooth $\mathbb{F}_q$-representations of $\text{Gal}(\overline{F}/F)$ and the isomorphism classes of two-dimensional supersingular $\mathcal{H}_{\mathbb{F}_q}^{(1)}$-modules.

**Proof:** By construction, the restriction of $\omega_i^{h-1}$ to $\mu_{q-1}(F) \simeq \mathbb{F}_q^\times$ is given by the exponentiation $x \mapsto x^{b_i-1}$. Given $0 \leq i \leq q-2$ and $1 \leq h \leq q-1$, and $b \in \mathbb{F}_q^\times$, the parameter $y := \omega_2f(i)b$ lies in $\mathbb{F}_q^\times \setminus \mathbb{F}_q$ and the pair $(s, t)$ determines a Galois representation $\rho$ having $h$ common exponent. Hence, $\rho \otimes \omega_i^{h-1}$ gives rise to the character $\omega_i^{h-1}1^{\rho}$. The elements of type $\gamma_\rho$ exhaust all orbits in $\mathbb{T}^\gamma/W_0$. Since a two-dimensional supersingular $\mathcal{H}_{\mathbb{F}_q}^{(1)}$-module is determined by its $\gamma$-component and its central character, the map $\rho \mapsto \mathcal{M}(\rho)$ is seen to be surjective. It is then bijective, since source and target have the same cardinality. \hfill \Box

**5.7.** Let $F$ be a finite extension of $\mathbb{Q}_p$. A distinguished natural bijection between irreducible two-dimensional $\text{Gal}(\overline{F}/F)$-representations and supersingular two-dimensional $\mathcal{H}_{\mathbb{F}_q}^{(1)}$-modules is established by Breuil [Br03] for $F = \mathbb{Q}_p$ (see [Be11] for its relation to the $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$) and by Grosse-Klönne [GK15] for general $F/\mathbb{Q}_p$. In this final paragraph we will show that the bijection $\rho \mapsto \mathcal{M}(\rho)$ from 5.6 coincides in this case with the bijections [Br03] and [GK15].

The case $F = \mathbb{Q}_p$ follows directly from the explicit formulae given in [Be11], 1.3]. For the general case, we briefly recall the main construction from [GK15] in the case of standard supersingular modules of dimension 2. Let $F_\phi$ be the special Lubin-Tate group with Frobenius power series $\phi(t) = \omega t + t^q$. Let $F_\infty/F$ be the extension generated by all torsion points of $F_\phi$ and let $\Gamma = \text{Gal}(F_\infty/F)$. We identify in the following $\Gamma \simeq \mathfrak{o}_{\mathbb{F}_q}^\phi$ via the character $\chi_L$.

Let $k/\mathbb{F}_q$ be a finite extension and let $\mathcal{H}^{(1)}_{\mathbb{F}_q} := \mathcal{H}^{(1)}(\mathbb{F}_q) \otimes_{\mathbb{Z}[\mathfrak{q}]} k$ via $q \mapsto q = 0$. Let $M$ be a two-dimensional standard supersingular $\mathcal{H}^{(1)}_{\mathbb{F}_q}$-module, arising from a supersingular character $\chi : \mathcal{H}^{(1)}_{\mathbb{F}_q} \to k$. Let $e_0 \in M$ such that $\mathcal{H}^{(1)}_{\mathbb{F}_q}$ acts on $e_0$ via $\chi$ and put $e_1 = T^{-1}e_0$ (where $\omega = u^{-1}$ in our notation). The character $\chi$ determines two numbers $0 \leq k_0, k_1 \leq q - 1$ with $(k_0, k_1) \neq (0,0), (q-1,q-1)$. One considers $M$ a $k[[t]]$-module with $t = 0$ on $M$. Let $\Gamma = \mathfrak{o}_{\mathbb{F}_q}^\phi$ act on $M$ via

$$\gamma(m) = T_{\eta(\mathfrak{f})^{-1}}(m)$$

for $\gamma \in \mathfrak{o}_{\mathbb{F}_q}^\phi$ with reduction $\mathfrak{f} \in \mathbb{F}_q^\times$ and $\eta_{\mathfrak{f}}(\mathfrak{f})^{-1} = \text{diag}(\mathfrak{f}^{-1}, 1) \in \mathfrak{t}$. The $k[[t]][[\mathfrak{c}}]-submodule $\nabla(M)$ of

$$k[[t]][[\mathfrak{c}}]\otimes_{k[[t]][[\mathfrak{c}}] M \simeq \nabla(M)$$

\footnote{For example, if $M$ is an $H_{\mathbb{F}_q}$-module on which $U^2 = \zeta_2$ acts via the scalar $\theta(\zeta_2) = \tau_2$, then $U = U_{\mathbb{F}_q}$ acts via the scalar $\theta(\zeta_2) = \tau_2$.}
is then generated by the two elements $h(e_j) = t^{b_j} \varphi \otimes T_{\omega}^{-1}(e_j) + 1 \otimes e_j$ thereby defining the relation between the Frobenius $\varphi$ and the Hecke action of $T_\omega$. Note that in the case of $GL_2$, the cocharacter $e^* \circ [GK18, 2.1]$ is equal to $\eta_1$.

The module $\nabla(M)$ is stable under the $\Gamma$-action and thus the quotient

$$\Delta(M) = (k[[t]][\varphi] \otimes k[[t]] M)/\nabla(M)$$

defines a $k[[t]][\varphi, \Gamma]$-module. It is torsion standard cyclic with weights $(k_0, k_1)$ in the sense of [GK18, 1.3], according to [GK18, Lemma 5.1]. Let $\Delta(M)^* = \text{Hom}_k(\Delta(M), k)$. By a general construction (which goes back to Colmez and Emerton in the case $F = Q_p$ and $\phi(t) = (1 + t)^p - 1$, as recalled in [BrTe 2.6]) the $k((t))$-vector space

$$\Delta(M)^* \otimes k[[t]] k((t))$$

is in a natural way an étale Lubin-Tate $(\varphi, \Gamma)$-module of dimension 2. The correspondence $M \mapsto \Delta(M)^* \otimes k[[t]] k((t))$ extends in fact to a fully faithful functor from a suitable category of supersingular $H^{(1)}$-modules to the category of étale $(\varphi, \Gamma)$-modules over $k((t))$. The composite functor to the category of continuous $\text{Gal}((F)/F)$-representations over $k$ is denoted by $M \mapsto V(M)$. It induces the aforementioned bijection between irreducible two-dimensional $\text{Gal}((F)/F)$-representations and supersingular two-dimensional $H^{(1)}$-modules.

5.8. Proposition. The inverse map to the bijection $M \mapsto V(M)$ is given by the map $\rho \mapsto M(\rho)$.

Proof: The correspondence $M \mapsto V(M)$ is compatible with the twist by a character of $F^\times$ and local class field theory, such that the determinant corresponds to the central character restricted to $F^\times$. By its very construction, the same is true for the correspondence $\rho \mapsto M(\rho)$. It therefore suffices to compare them on irreducible Galois representations having parameters $b = 1$ and $i_0 = 0$. Let $k = F_{q^2}$ in the following. Let $\omega^h_{2j}$ be the Galois representation with exponent $1 \leq h \leq q - 1$ and $b = 1$ and $i_0 = 0$. Let $D = (\varphi, \Gamma)$-module associated to $\rho := \omega^h_{2j}$ and let $M$ be a supersingular $H^{(1)}_k$-module such that $\Delta(M)^* \otimes k[[t]] k((t)) \simeq D$. According to the main result of [PS3] for $n = 2$, the module $D$ has a basis $\{g_0, g_1\}$ such that

$$\gamma(g_j) = \frac{1}{t_j}(h^{(q - 1)}) g_j$$

for all $\gamma \in \Gamma$ and $\varphi(g_0) = g_1$ and $\varphi(g_1) = -t^{h(q - 1)} g_0$. Here, $\frac{1}{t_j}(h(t)) = \omega_f(t) / \gamma(t) \in k[[t]]^\times$. Define the triple $(k_0, k_1, k_2) = (h - 1, q - h, 1)$ and $i_j := q - 1 - k_{2-j}$, so that $i_0 = i_2 = q - h$ and $i_1 = 2q - h - 1$. Define the triple $(h_0, h_1, h_2) = (0, i_1, i_0 + i_1)$. Note that $h_2 = h(q - 1)$. Put $f_j = t^{b_j} g_j$ for $j = 0, 1$ and let $D^j \subset D$ be the $k[[t]]$-submodule generated by $\{f_0, f_1\}$. Let $(D^j)^*$ be the $k$-linear dual. Define $e'_i \in (D^j)^*$ via $e'_i(f_j) = \delta_{ij}$ and $e'_i = 0$ on $tD^j$. Using the explicit formulæ for the $\psi$-operator on $k((t))$ as described in [GK18, Lemma 1.1] one may follow the argument of [GK16, Lemma 6.4] and show that $D^j$ is a $\psi$-stable lattice in $D$ and that $\{e'_0, e'_1\}$ is a $k$-basis of the $t$-torsion part of $(D^j)^*$ satisfying

$$t^{k_1} \varphi(e'_0) = e'_1 \quad \text{and} \quad t^{k_0} \varphi(e'_1) = -e'_0.$$

But according to [GK18, 1.15] there is only one $\psi$-stable lattice in $\Delta(M)^* \otimes k[[t]] k((t))$, namely $\Delta(M)^*$. It follows that $\Delta(M) \simeq (D^j)^*$ and so the weights of the torsion standard cyclic $k[[t]][\varphi, \Gamma]$-module $\Delta(M)$ are $(k_0, k_1)$. Since $k_0 = h - 1$, one deduces from [GK18, Lemma 4.1/5.1] that $e_1 \equiv h - 1 \mod (q - 1)$. This means $\lambda \circ \alpha^\vee(x)^{-1} = x^{h-1}$ for the character $\lambda \in \mathbb{T}^\vee$ of $M$. Since $i = 0$ and hence $a = 0$ (in the notation of [GK16, 2.2]), we arrive therefore at

$$\lambda(\text{diag}(x_1, x_2)) = \lambda(e^*(x_1 x_2)\alpha^\vee(x_2)^{-1}) = e^*(x_1 x_2)^*x_2^{h-1} = x_2^{h-1}.$$

Hence the image of $\lambda$ in $\mathbb{T}^\vee / W_0$ coincides with $\gamma_\rho$. This implies $M \simeq M(\rho)$, as claimed. \qed

24
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