Parity sequences of the 3x+1 map on the 2-adic integers and Euclidean embedding

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Abstract

In this paper, we consider the one-to-one correspondence between a 2-adic integer and its parity sequence under iteration of the so-called “3x + 1” map. First, we prove a new formula for the inverse transform. Next, we briefly review what is known about the induced automorphism and study its dynamics on the 2-adic integers. We find that it is ergodic on many small odd invariant sets, and that it has two odd cycles of period 2 in addition to its two odd fixed points. Finally, a plane embedding is presented, for which we establish affine self-similarity by using functional equations.

1 Introduction

It is an unsolved problem [15, 19] to prove that the repeated iteration of the famous “3x + 1” map acting on the positive integers and defined by

\[ T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{otherwise,} \end{cases} \] (1)

always leads to the value 1, whatever the starting integer of the sequence. And it is not even known whether the orbits \((n, T(n), T(T(n)), \ldots)\) are bounded for all \(n\), nor is it known if there exists any non-trivial cycle. This problem, whose origin remains unclear (cf. History and Background section in [19, p. 5]), has received a great variety of names like the 3x + 1 problem, the Collatz conjecture, the Syracuse problem, la conjecture tchèque ...

Its intrinsic hardness is frequently attributed to the unpredictability of the successive parities of the iterates in most sequences, before 1 is reached [11, 10]. Therefore, it seems relevant to study the properties of the parity vectors defined as follows.
**Definition 1.1.** For any two positive integers \( j \) and \( n \), we call *parity vector* of \( n \) and length \( j \), the vector

\[
V_j(n) = (n, T(n), \ldots, T^{j-1}(n)) \mod 2,
\]

where \( T^k \) denotes the \( k \)-th iterate of \( T \).

This notion was introduced independently by Everett and Terras, and named *parity vector* by Lagarias [17].

It was quite easy to state [13, 27] that any two integers have same parity vectors of length \( j \) if and only if they belong to the same congruence class modulo \( 2^j \). From this property, we derive that each function \( V_j \) sends with a one-to-one correspondence any set of \( 2^j \) consecutive integers to the set of all parity vectors of length \( j \). There is consequently an infinite class of integers producing exactly any finite sequence of parities under iteration of \( T \).

## 2 Two formulae for the inverse transform

In this part, we may freely extend the definition of the functions \( T \) and \( V_j \) to the ring \( \mathbb{Z} \) of rational integers, as in [17].

The calculation of a parity vector \( V_j(n) \) is straightforward by applying the map \( T \) repeatedly \( j \) times from the initial integer \( n \). Conversely, one may use the forthcoming Lemma 2.2 to obtain all the integers that have any given parity vector. In fact, this lemma is a well known expression with various formulations [7, 23, 27] and generalizations [6, 22], further studied in [21].

**Definition 2.1.** Let \( j \) a positive integer. We say that a vector \( S = (s_0, s_1, \ldots, s_{j-1}) \) of length \( j \) is a finite *binary sequence* if \( s_k = 0 \) or \( 1 \) for all \( 0 \leq k \leq j - 1 \).

We further define the *partial sum* functions \( \sigma_k \) applying on \( S \) by

\[
\sigma_k(S) = \sum_{i=0}^{k} s_i \quad \text{for each } k \leq j - 1.
\]

One may notice that the above functions \( \sigma_k \) are essentially the same as the functions \( \text{pop}_k \) introduced in [6] and used in a similar way. These functions frequently appear in various forms within the literature on the \( 3x+1 \) problem.
Lemma 2.2. (First formulation of the inverse transform) Let $S$ be a finite binary sequence $(s_0, s_1, \ldots, s_{j-1})$ of length $j$. The set of integers $n$ for which $V_j(n) = S$ is given by the congruence class

$$n \equiv -\sum_{k=0}^{j-1} s_k 2^k 3^{-\sigma_k(S)} \pmod{2^j}. \quad (4)$$

Proof. Suppose that $V_j(n) = S$. Then equation (4) follows from the formula

$$2^j T^j(n) = 3^{\sigma_{j-1}(S)} \left( n + \sum_{k=0}^{j-1} s_k 2^k 3^{-\sigma_k(S)} \right),$$

which is easy to state by induction on $j$ (see [27]). \hfill \Box

Example 2.3. By Lemma 2.2, the odd integers $n$ leading to sequences where every odd term is followed by two even terms on the first $j$ iterations of the map $T$, are such that

$$n \equiv -\left\lfloor \frac{j-2}{3} \right\rfloor 3^{-(k+1)} - \frac{1}{3} \left( \frac{8}{3} \right) \left\lceil \frac{j+2}{3} \right\rceil - 1 \equiv \frac{1}{5} \pmod{2^j}.$$

For the increasing lengths $j = 3, 6, 9, \ldots$, the smallest positive values of $n$ are $5, 13, 205, \ldots$ respectively.

The discovery of a second formulation of the inverse transform came after studying the particular case of sequences where all terms but one are odd [26] (see also Example 2.6). It can be stated in different ways. Here we give a very short proof based on the conjugate function below.

Definition 2.4. Let us consider the function

$$U : \mathbb{Z} \rightarrow \mathbb{Z},$$

$$n \mapsto \begin{cases} 
\frac{n+1}{2} & \text{if } n \text{ is odd}, \\
\frac{3n}{2} & \text{otherwise}.
\end{cases}$$

There holds the conjugacy relationship

$$U(n + 1) = T(n) + 1 \quad \text{for all } n. \quad (5)$$

Furthermore, for any binary sequence $S = (s_0, s_1, \ldots, s_{j-1})$ of length $j$ and any integer $n$ such that $U^k(n) \equiv s_k \pmod{2}$ for each $k$, one has

$$n \equiv -\sum_{k=0}^{j-1} s_k 2^k 3^{\sigma_k(S)-k-1} \pmod{2^j}. \quad (6)$$
The last congruence (6) is derived from the equation

\[ 2^j U^j(n) = 3^j - \sigma_j(S) \left( n + \sum_{k=0}^{j-1} s_k 2^k 3^{\sigma_k(S)-k-1} \right), \]

which may be easily proved by induction on \( j \), exactly as in Lemma 2.2.

We are now able to provide a new formulation of the inverse transform of the functions \( V_j \), which has little difference with the previous one in Lemma 2.2. It turns out to be practical for sequences of \( T \) iterations that contain many odd terms, because the corresponding terms in formula (7) vanish.

**Theorem 2.5.** (Second formulation of the inverse transform) Let \( S \) be a finite binary sequence \((s_0, s_1, \ldots, s_{j-1})\). The set of integers \( n \) for which \( V_j(n) = S \) is given by the congruence class

\[ n \equiv -1 - \sum_{k=0}^{j-1} (1 - s_k) 2^k 3^{\sigma_k(S)-k-1} \pmod{2^j}. \]  

**Proof.** Let \( n \) such that \( V_j(n) = S \), and consider the binary sequence \( S_U = (U^k(n+1) \mod 2^j)_{k=0}^{j-1} \). The conjugacy (5) gives \( U^k(n+1) = T^k(n) + 1 \), so that \( S_U = (1 - s_0, \ldots, 1 - s_{j-1}) \) and \( \sigma_k(S_U) = k + 1 - \sigma_k(S) \) for every \( k \).

It suffices to write the inverse formula (6) applied to \( n + 1 \),

\[ n + 1 \equiv - \sum_{k=0}^{j-1} (1 - s_k) 2^k 3^{\sigma_k(S_U)-k-1} \pmod{2^j}, \]

to conclude the proof.

**Example 2.6.** Let \( j \) a positive integer. Suppose we want to find the integers \( n \) for which the parity vector \( V_j(n) \) contains exactly once the value 0. Then we can write that

\[ n \equiv -1 - \left( \frac{2}{3} \right)^k \pmod{2^j} \]

where \( k \) is the only integer lower than \( j \) such that \( T^k(n) \) is even. See [26, §6] for a brief study of those integers in \( \mathbb{Z}^+ \).
In fact, we obtain from Lemma 2.2 and Theorem 2.5 infinitely many formulations by considering linear combinations of (4) and (7). E. g., a simple addition gives

\[ 2n + 1 \equiv - \sum_{k=0}^{j-1} 2^k 3^{-\sigma_k(V_j(n))} \pmod{2^j} \quad \text{for all integers } j > 0 \text{ and } n. \]

From a subtraction, there yields a congruence that does not involve the “3x+1” map. It may be regarded as a rather remarkable connection between binary sequences and modular arithmetic.

**Corollary 2.7.** Let \( j \) be a positive integer. For any finite binary sequence \( S = (s_0, s_1, \ldots, s_{j-1}) \),

\[ \sum_{k=0}^{j-1} (-1)^{s_k} 2^k 3^{-\sigma_k(S)} \equiv -1 \pmod{2^j}. \] (8)

**Proof.** Let \( n \) a positive integer such that \( V_j(n) = S \). Subtracting the second formulation of the inverse transform \([7]\) from the first formulation \([4]\) gives the desired result, by writing \((-1)^{s_k} = (1 - s_k) - s_k\) for any \( k \).

This corollary can also be proved directly by induction on \( j \), then leading to an alternate proof of Theorem 2.5 which is left to the reader.

## 3 Ultrametric extension

### 3.1 The space of 2-adic integers

It was early suggested \([21, 22]\) to extend the definition of the map \( T \) to the ring \( \mathbb{Z}_2 \) of 2-adic integers, that is, numbers of the form \( \sum_{k=0}^{\infty} a_k 2^k \) with \( a_k = 0 \) or 1 for all \( k \). The standard shorthand notation \((\ldots a_2 a_1 a_0)_2\) from right to left\(^1\) may be used for the sake of conciseness, and the parentheses are most often omitted. A periodic expansion is usually indicated by an upper bar. E. g., one may write

\[ (\ldots 010101)_2 = \overline{01} = \sum_{k=0}^{\infty} (k \pmod{2}) 2^k = \sum_{k=0}^{\infty} 2^{2k} = -\frac{1}{3}. \]

\(^1\)Some authors prefer to write the 2-adic “digits” from left to right.
Recall that all rational number with an odd denominator has an eventually periodic 2-adic expansion in \( \mathbb{Z}_2 \).

Besides, a metric can be derived from the 2-adic norm

\[
\left| \sum_{k=0}^{\infty} a_k 2^k \right|_2 = 2^{-l} \quad \text{with} \quad l = \min \{ k \geq 0 : a_k \neq 0 \}, \quad \text{and} \quad |0|_2 = 0.
\]

The space \( \mathbb{Z}_2 \) is said to be ultrametric, due to the strong triangle inequality

\[
|x + z|_2 \leq \max (|x + y|_2, |y + z|_2)
\]

for all \( x, y \) and \( z \). Therefore it is not Euclidean.

When needed, we apply the usual Haar measure on \( \mathbb{Z}_2 \), here noted \( \mu \), such that \( \mu(\mathbb{Z}_2) = 1 \), and refer to it as the 2-adic measure.

The function \( T \) remains well-defined on \( \mathbb{Z}_2 \), where it is known to be continuous and measure-preserving \([21]\). As was observed many times \([1,17,23]\), iterating \( T \) on \( \mathbb{Z}_2 \) is leading to a much greater variety of behaviours, due to its ergodic \([21]\) and strongly mixing \([17]\) dynamics, and interesting properties are thus arising.

### 3.2 Parity sequences

Let first introduce the notion of parity sequence.

**Definition 3.1.** For all 2-adic integer \( x \), the infinite binary sequence

\[
V_\infty(x) = (x, T(x), T^2(x), \ldots) \mod 2
\]

is called the parity sequence of \( x \).

It is remarkable, as mentioned in \([17]\), that the \( V_\infty \) function is a one-to-one and onto transform from \( \mathbb{Z}_2 \) to \( \{0,1\}^\infty \). Every infinite binary sequence is the parity sequence, via \( T \) iteration, of exactly one 2-adic integer. As a consequence, there exist 2-adic cycles of every period. A complete list of the 23 cycles of period at most 6 is given in \([18]\). Since eventually periodic sequences have density zero in \( \{0,1\}^\infty \), we infer that almost all orbits in \( \mathbb{Z}_2 \) do not contain a cycle.

From Lemma \([2,2]\) and Theorem \([2,5]\) one immediately derives two formulae to express the inverse transform \( V_\infty^{-1} \).
Corollary 3.2. Let \( S \) be an infinite binary sequence \((s_0, s_1, s_2, \ldots)\). The 2-adic integer \( x \) such that \( V_\infty(x) = S \) is given by any of the 2-adically convergent expansions
\[
x = -\sum_{k=0}^{\infty} s_k 2^k 3^{-\sigma_k(S)}
\]
and
\[
x = -1 - \sum_{k=0}^{\infty} (1 - s_k) 2^k 3^{-\sigma_k(S)}.
\]
where \( \sigma_k \) denotes the partial sum function \( \sigma_k(S) = \sum_{i=0}^{k} s_i \).

Example 3.3. Let \( S = (s_0, s_1, s_2, \ldots) \) with \( s_k = 1 \) for all \( k \). Applying the inverse transform \((11)\), we get \( V_{-1}(S) = -1 = (\ldots111111)_2 = \overline{1}_2 \).

The question whether the inverse formula \((10)\) leads to a convergent series when evaluated in the set of real numbers has been investigated in various papers \([12, 20, 21]\). Note that the sum of the series is negative or zero, when it exists. Interestingly, both series on the right hand side of \((10)\) and \((11)\) are expected to be divergent (resp. convergent) for the parity sequences of positive (resp. negative) rational integers.

One may also remark that the equation \((10)\) is quite similar to the expression of the real function \( \theta \) mentioned at the end of Coquet’s paper \([9, \S 7]\), which is convergent and turns out to be fractal.

Yet, we do not further discuss this issue in the present paper.

3.3 Automorphism

It is convenient to encode parity sequences as 2-adic integers, so as to give it a rational value when it is eventually periodic, as done by Lagarias in \([17]\). This yields an automorphism in \( \mathbb{Z}_2 \).

Definition 3.4. Let \( Q \) denote the function
\[
Q : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2
\]
\[
x \mapsto \sum_{k=0}^{\infty} s_k 2^k
\]
where \((s_0, s_1, s_2, \ldots)\) is the parity sequence of \( x \), as defined in 3.1.
The function $Q$ is a one-to-one and onto morphism \[6, 17\]. It is also non-expandable\[2\] with respect to the 2-adic norm, since it satisfies the 1-Lipschitz condition

$$|Q(x) - Q(y)|_2 \leq |x - y|_2$$  for all $x$ and $y$, or, equivalently,

$$x \equiv y \pmod{2^n} \implies Q(x) \equiv Q(y) \pmod{2^n}.$$  \hfill (12)

The fact that $Q$ is one-to-one further implies (see \[2\]) the reciprocal

$$x \equiv y \pmod{2^n} \iff Q(x) \equiv Q(y) \pmod{2^n},$$  \hfill (13)

which makes $Q$ a 2-adic isometry \[6\].

For convenience purposes, we prefer to use the simple notation $Q$, as in \[1\], rather than the original notation $Q_\infty$. Its inverse $Q^{-1}$, named the 3$x$+1 \textit{conjugacy map} and noted $\Phi$ in \[6, 23\], is known \[5\] to conjugate the map $T$ with the \textit{shift map} $S$, whose definition follows. For the sake of clarity, we rephrase all known and conjectured properties related to $\Phi$ in term of the function $Q$.

\textbf{Definition 3.5.} Let the \textit{shift map} $S$ denote the function

$$S : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$$

$$x \mapsto \begin{cases} 
\frac{x-1}{2} & \text{for } x \text{ odd}, \\
\frac{x}{2} & \text{otherwise}.
\end{cases}$$

There holds the conjugacy

$$T = Q^{-1} \circ S \circ Q.$$  \hfill (14)

In the context of the 3$x$+1 problem, it is crucial to determine which values of $Q$ are rational. Indeed, we have the well-known statements holding for all 2-adic integer $x$:

\begin{itemize}
  \item the sequence $x, T(x), T^2(x), \ldots$ is leading to a cycle if and only if $Q(x)$ is rational \[1\]. It is an immediate consequence of the conjugacy \hfill (14) as noted by Monks \[23\];
\end{itemize}
• $Q(x)$ is rational $\implies x$ is rational [5] (see also [1], Theorem 5).

It results that all cycles within the dynamics of $T$ on $\mathbb{Z}_2$ are rational [18]. Lagarias’ Periodicity Conjecture [17] asserts that the reciprocal of the second statement above also holds. This would prove that all rational points in $\mathbb{Z}_2$ are preperiodic.

**Conjecture 3.6.** (Periodicity Conjecture) *For any 2-adic integer $x$, $Q(x)$ is rational if and only if $x$ is rational.*

Furthermore, the function $Q$ allows to formulate differently the $3x + 1$ problem [5, 17].

**Conjecture 3.7.** *(3x + 1 problem)*

$$Q(\mathbb{Z}^+) \subset \frac{1}{3} \mathbb{Z} \quad \text{or, equivalently,} \quad \mathbb{Z}^+ \subset Q^{-1}\left(\frac{1}{3} \mathbb{Z}\right).$$

It asserts that every positive integer has an eventually periodic parity sequence of period 2, ending with an infinite alternation of 0 and 1 (the case of a fixed parity is trivially excluded), as do all and only sequences of iterations of the map $T$ on $\mathbb{Z}^+$ that reach the cycle $(1,2)$. Note that the reverse inclusion in Conjecture 3.7 does not hold, since $Q^{-1}(1) = -1/3$, by formula (10).

### 3.4 Functional equations

As a semiring, the set $\mathbb{N}$ of natural integers is completely generated by all finite compositions of the two functions $x \mapsto 2x$ and $x \mapsto 2x + 1$ starting from 0, thus reversing the action of the shift map $S$. Therefore, it is tempting to search for functional equations that express $Q(2x)$ and $Q(2x + 1)$ from $Q(x)$. Such equations exist for $x$ in $\mathbb{Z}_2$ or in a subset of $\mathbb{Z}_2$, as shown in Theorem 3.8. It turns out that equation (18) is a sort of 2-adic extension of previous results by Andaloro [4] and Garner [14]. We also establish similar equations for the inverse transform $Q^{-1}$ (see [12] for a generalization).

**Theorem 3.8.** *The functions $Q^{-1}$ and $Q$ are solution to the functional equations*

\begin{align*}
Q^{-1}(2x) &= 2Q^{-1}(x), \quad (15) \\
Q^{-1}(2x + 1) &= \frac{2Q^{-1}(x) - 1}{3}, \quad (16) \\
Q(2x) &= 2Q(x) \quad (17)
\end{align*}
for all 2-adic integer \( x \), and

\[
Q(2x + 1) = 2Q(x) - 2^k + 1
\]

(18)

for \( x \equiv -1 - (-2)^{k-2} \pmod{2^k} \) and \( k \geq 2 \).

Proof of (15) and (16). First, one may rewrite equation (14) as

\[
T \circ Q^{-1} = Q^{-1} \circ S.
\]

(19)

Consider a 2-adic integer \( x \). Putting together (19) with the fact that

\[
S(2x + 1) = S(2x) = x
\]

and that

\[
Q^{-1}(y) \equiv y \pmod{2} \quad \text{for all } y,
\]

we obtain

\[
Q^{-1}(x) = Q^{-1} \circ S(2x) = T \circ Q^{-1}(2x) = \frac{Q^{-1}(2x)}{2}
\]

and

\[
Q^{-1}(x) = Q^{-1} \circ S(2x + 1) = T \circ Q^{-1}(2x + 1) = \frac{1 + 2 + \ldots + 2^{k-1} + 2^kQ(2 + 3z)}{2}.
\]

Proof of (17). Replacing \( x \) by \( Q(x) \) in (15) gives \( Q^{-1}(2Q(x)) = 2x \), leading to \( 2Q(x) = Q(2x) \).

Proof of (18). Let \( k \geq 2 \) and let \( x, y \) be 2-adic integers such that \( x = -1 - (-2)^{k-2} + 2^k y \). Starting from \( x \) and applying repeatedly the map \( T \), it is easily seen that the first \( k - 3 \) iterates are odd, while the next one is even:

\[
T^{k-2}(x) = -1 - (-3)^{k-2} + 3^{k-2}(4y) \equiv 2 \pmod{4}.
\]

Putting \( T^{k-2}(x) = 2 + 4z \), we get \( T^{k-1}(x) = 1 + 2z \) and \( T^k(x) = 2 + 3z \). Since \( x \) has parity vector \( V_k(x) = (1, 1, \ldots, 1, 0, 1) \), one may write

\[
Q(x) = 1 + 2 + \ldots + 2^{k-3} + 2^{k-1} + 2^kQ(2 + 3z)
\]

for \( k \geq 3 \). In the case \( k = 2 \), the above expression simplifies to \( Q(x) = 2 + 4Q(2 + 3z) \).
On the other hand, starting from \(2x + 1\) and applying \(k - 1\) times the map \(T\), we get after \((k - 2)\) odd iterates the even value \(T^{k-1}(2x + 1) = -1 + (-3)^{k-1} + 3^{k-1}(4y) = 3T^{k-2}(x) + 2 = 8 + 12z\). The next two iterates are \(T^k(2x + 1) = 4 + 6z\) and \(T^{k+1}(2x + 1) = 2 + 3z\). It follows that \(2x + 1\) has parity vector \(V_{k+1}(2x + 1) = (1, 1, \ldots, 1, 0, 0)\) and we get

\[
Q(2x + 1) = 1 + 2 + \ldots + 2^{k-2} + 2^{k+1}Q(2 + 3z) = 2Q(x) - 2^k + 1.
\]

One may ask whether Theorem 3.8 provides a general algorithm to calculate in a finite number of steps the exact value of the function \(Q\) applied to an arbitrary positive integer. Unfortunately, the answer appears to be negative, since equation (18) only applies to a subset of \(\mathbb{Z}_2\) of 2-adic measure

\[
2^{-2} + 2^{-3} + 2^{-4} + \ldots = 1/2.
\]

The case \(k = 2\) can be restated as

\[
Q(4x + 1) = 4Q(x) - 3 \quad \text{for} \quad x \equiv 1 \pmod{2}.
\]  

(20)

by replacing \(x\) by \(2x\) in (18) and using (17). One may further combine (18) and (20) to produce the functional equations

\[
Q(8x + 5) = 4Q(2x + 1) - 3 = 8Q(x) - 2^{k+2} + 1
\]

for \(x \equiv -1 - (-2)^{k-2} \pmod{2^k}\) and \(k \geq 2\).

The function \(Q\) satisfies many other functional equations that are not combinations of (17) and (18) like

\[
Q(3x + 1) = Q(x) - 1 \quad \text{for} \quad x \equiv 1 \pmod{2}.
\]  

(21)

Such equations are always related to the phenomenon of coalescence within the dynamics of \(T\). E.g., the equation (21) derives directly from the equality \(T(3x+1) = T(x)\) for \(x \equiv 1 \pmod{2}\). See [4, 14] for other examples of generic coalescences.

### 3.5 Ergodicity

The ergodic dynamics of the \(3x + 1\) map \(T\) on \(\mathbb{Z}_2\) is quite well understood, and, paradoxically, it does not provide any indication on the validity of the \(3x + 1\) Conjecture, as is discussed in [1].
Yet, in view of the Periodicity Conjecture, it could be helpful to better specify the dynamics of $Q$, which appears to be more complicated.

In what follows, we refer to [2] for the ergodicity of a measure-preserving function on the 2-adic integers.

Since $Q$ is isometric, it induces in the finite set $\mathbb{Z}/2^n\mathbb{Z}$ a permutation $Q_n$, whose dynamics is easier to study.

**Definition 3.9.** For all integer $n \geq 0$, let $Q_n$ denote the function

$$Q_n : \mathbb{Z}/2^n\mathbb{Z} \rightarrow \mathbb{Z}/2^n\mathbb{Z}$$

$$x \mapsto Q(x) \mod 2^n.$$  

It was proved in [17] that the order of $Q_n$ is always a power of 2, and the following theorem was finally stated in [6].

**Theorem 3.10.** (Bernstein, Lagarias) For every integer $n \geq 6$, the order of the permutation $Q_n$ is equal to $2^{n-4}$. Moreover, the length of any cycle in $Q_n$ is a power of 2.

It is known that, when lifting from $\mathbb{Z}/2^n\mathbb{Z}$ to $\mathbb{Z}/2^{n+1}\mathbb{Z}$, any cycle of the permutation $Q_n$ either splits into two cycles whose period is unchanged, or undergoes a period-doubling.

**Definition 3.11.** Let $m \geq k \geq 0$ and let $C = (c_1, \ldots, c_{2^k})$ be a cycle of the permutation $Q_m$ of length $2^k$. We say that $C$ has an ever-doubling period if, for all $n \geq m$, the elements $c_1, \ldots, c_{2^k}$ of $C$ are all included in a single cycle of $Q_n$ of length $2^{n-m+k}$.

The proof of Theorem 3.10 is based on the fact that the cycle (5,17) of $Q_5$ has an ever-doubling period, as proved in [6].

Now, we can use this result to study the dynamics of $Q$ and $Q^{-1}$ on the topological space $\mathbb{Z}_2$. To this aim, we need the notion of 2-adic ball.

**Definition 3.12.** For any $y \in \mathbb{Z}_2$ and $r \geq 0$, let $B(y, r)$ denote the (closed) ball \( \{ x \in \mathbb{Z}_2 : |x - y|_2 \leq r \} \) with center $y$ and radius $r$. Equivalently, one has $B(y, 2^{-k}) = \{ x \in \mathbb{Z}_2 : x \equiv y \pmod{2^k} \}$ for all integer $k \geq 0$. Its 2-adic measure is given by its radius: $\mu(B(y, 2^{-k})) = 2^{-k}$.

Recall that the function $Q$ is measure-preserving [6, 17] and, unlike the map $T$, it is not ergodic, since it preserves the parity.
One may at first observe that all forward and backward orbits remain close to the initial point, and that the 2-adic distance is even smaller when the number of iterations is highly divisible by 2. This fact is illustrated in the table below that gives some of the iterates of the 2-adic integer $01101_2 = 1/5$.

| $j$ | $Q^j (\frac{1}{5})$ | $|Q^j (\frac{1}{5}) - \frac{1}{5}|_2$ | $Q^{-j} (\frac{1}{5})$ | $|Q^{-j} (\frac{1}{5}) - \frac{1}{5}|_2$ |
|-----|---------------------|----------------------------------|---------------------|----------------------------------|
| 1   | $-\frac{1}{5} = 001_2$ | $2^{-2}$                       | $\frac{13}{21} = 00110012_2$ | $2^{-2}$ |
| 2   | $\frac{17}{5} = 001101101_2$ | $2^{-4}$                       | $-\frac{1}{11} = 00010111012_2$ | $2^{-4}$ |
| 3   | $\frac{1863}{31} = \ldots 1001_2$ | $2^{-2}$                       | $\frac{373}{751} = \ldots 1001_2$ | $2^{-2}$ |
| 4   | $\ldots 00001101_2$ | $2^{-6}$                       | $\ldots 10001101_2$ | $2^{-6}$ |

The previous observation is due to the congruences (22) and (23), which follow from Theorem 3.10.

**Corollary 3.13.** For all 2-adic integer $x$ and all $k \geq 2$,

$$Q^2(x) \equiv x \pmod{2^4},$$

$$Q^{2^k}(x) \equiv x \pmod{2^{k+4}},$$

or equivalently, $Q^2(x) \in B(x, 2^{-4})$ and $Q^{2^k}(x) \in B(x, 2^{-k-4})$.

**Proof.** Theorem [3.10] implies that, for an integer $k \geq 2$, the length of every cycle of $Q_{k+4}$ divides $2^k$. From this, we infer the congruence (23).

Likewise, the permutation $Q_4$ has ten fixed points and three cycles of length 2, which are (1,5), (2,10) and (9,13). Hence, the equation (22). \qed

Consequently, whatever the 2-adic integer $x$, its forward and backward orbits under iteration of $Q$ have elements arbitrarily close to $x$.

**Corollary 3.14.** For all 2-adic integer $x$,

$$\lim_{k \to \infty} Q^{2^k}(x) = \lim_{k \to \infty} Q^{-2^k}(x) = x.$$

Though the dynamics of $Q$ is not truly ergodic on $\mathbb{Z}_2$, this may occur on some invariant subsets.

Let us recall a known criterion for the ergodicity of non-expandable functions, by Anashin (Proposition 4.1 in [2]). See also [3].

---

3In [2], the term compatible is used instead of non-expandable for the same meaning.
Theorem 3.15. (Anashin) A non-expandable function $F : \mathbb{Z}_2 \to \mathbb{Z}_2$ is ergodic if and only if $F$ induces modulo $2^n$ a permutation with a single cycle for all positive integer $n$.

The next theorem shows that $Q$ is ergodic in a neighbourhood of each cycle of $Q_m$ having an ever-doubling period, for any $m \geq 0$.

Theorem 3.16. Let $Q_m$ denotes the permutation induced by $Q$ in $\mathbb{Z}/2^m\mathbb{Z}$. For all $m \geq k \geq 0$ and all cycle $C = (c_1, \ldots, c_{2k})$ of $Q_m$ having an ever-doubling period, the restriction of $Q$ to $B(c_1, 2^{-m}) \cup \ldots \cup B(c_{2k}, 2^{-m})$ is ergodic.

Proof. Let $n \geq m$. Put $K = B(c_1, 2^{-m}) \cup \ldots \cup B(c_{2k}, 2^{-m})$ and $K_n = K \mod 2^n$. Since $Q$ is isometric, the sets $K$ and $K_n$ are left invariant by $Q$ and $Q_n$ respectively.

Let $C_n$ be the cycle of the permutation $Q_n$ that contains all the elements $c_1, \ldots, c_{2k}$ of $C$. Its length is equal to $2^{n-m+k}$. Moreover, it is included in $K_n$ whose cardinality is equal to $2^{n-m+k}$. Therefore, the restriction of $Q_n$ to the set $K_n$ is a permutation with a single cycle $C_n$.

From Theorem 3.15, we deduce that the restriction of $Q$ to the set $K$ is ergodic. For completeness, it is not difficult to find a suitable bijection $F : \mathbb{Z}_2 \to K$ for which the conjugated function $F^{-1} \circ Q \circ F$ acting on $\mathbb{Z}_2$ is non-expandable and ergodic. E. g., one may use the function

$$F : \mathbb{Z}_2 \to K,$$

$$x \mapsto c_{i+1} + 2^{m-k}(x - i), \quad \text{where } i = x \mod 2^k,$$

which is one-to-one and onto. \hfill \square

Let us point out that the set where ergodicity occurs in Theorem 3.16 is closed, hence it is the $\omega$-limit set in $\mathbb{Z}_2$ of any point of the cycle $C$. For convenience, we write $\omega(C)$ to refer to this set.

Definition 3.17. Whenever $Q$ is ergodic on an invariant set, we call it an ergodic set. Moreover, we call ergodic domain the union of all the ergodic sets.

The fact that $Q$ is bijective further implies that $Q^{-1}$, namely, the $3x + 1$ conjugacy map, has the same ergodic domain as $Q$. 14
Table 1: Odd cycles $C$ of $Q_m$ of length $2^k$ and having an ever-doubling period for $0 \leq m - k \leq 6$. The last column gives the $2$-adic measure, equal to $2^{k-m}$, of their $\omega$-limit sets for the function $Q$.

| $m$ | $k$ | $C$ | $\mu(\omega(C))$ |
|-----|-----|-----|------------------|
| 5   | 1   | (5,17) | $2^{-4}$         |
| 6   | 2   | (9, 29, 25, 13) | $2^{-4}$         |
| 6   | 2   | (41, 61, 57, 45) | $2^{-4}$         |
| 8   | 2   | (27, 251, 219, 59) | $2^{-6}$         |
| 8   | 2   | (91, 187, 155, 123) | $2^{-6}$         |

In order to identify the cycles having an ever-doubling period, it is convenient to use the following criterion (see Theorem 3.1 in [6]), whose original formulation and vocabulary have been significantly modified.

**Theorem 3.18.** (Bernstein, Lagarias) Let $m \geq k \geq 2$ and let $C$ be a cycle of $Q_m$ of length $2^k$. If $C$ is part of a cycle of $Q_{m+2}$ of length $2^{k+2}$, then $C$ has an ever-doubling period.

After some straightforward numerical computations, we find, by applying Theorems 3.16 and 3.18 above, that $Q$ is ergodic on the $\omega$-limit sets of any of the cycles of $Q_m$ in Table 1. Though it is not the case for any other odd set of $2$-adic measure at least $2^{-6}$, there are many smaller odd ergodic sets. In Table 2, we provide their respective numbers when sorted by size (see also Table 2.2 in [6]). We obtain that the total measure of the odd ergodic sets exceeds 0.48, whereas it is trivially upper bounded by $\mu(B(1, 1/2)) = 1/2$.

For each odd ergodic set of measure $2^{-k}$ and each $m \geq 1$, we easily get, by applying the autoconjugacy (17) repeatedly $m$ times, an even ergodic set of measure $2^{-k-m}$. It yields that the ergodic domain has a measure greater than 0.96.

**Conjecture 3.19.** (Ergodicity Conjecture) The ergodic domain of $Q$ has full $2$-adic measure.
Table 2: Numbers \( N_k \) of odd ergodic sets of 2-adic measure \( 2^{-k} \), \( k \leq 16 \).

| \( k \) | \( N_k \) | \( N_k \times 2^{-k} \) | \( k \) | \( N_k \) | \( N_k \times 2^{-k} \) |
|---|---|---|---|---|---|
| 1 | 0 | 0.000 | 9 | 11 | 0.021 |
| 2 | 0 | 0.000 | 10 | 29 | 0.028 |
| 3 | 0 | 0.000 | 11 | 54 | 0.026 |
| 4 | 3 | 0.187 | 12 | 91 | 0.022 |
| 5 | 0 | 0.000 | 13 | 118 | 0.014 |
| 6 | 2 | 0.031 | 14 | 213 | 0.013 |
| 7 | 10 | 0.078 | 15 | 282 | 0.008 |
| 8 | 11 | 0.042 | 16 | 436 | 0.006 |

We expect this conjecture to be closely related to the distribution of periodic orbits, about which little is known.

### 3.6 Cycles

The search of the periodic points of the function \( Q \) is far from trivial due to the fact that it is nowhere differentiable, as proved by Müller in [25] (see also [5] for a short proof).

Hereafter, we call \( Q \)-cycle (resp. \( T \)-cycle) a periodic orbit of the function \( Q \) (resp. \( T \)). Unlike for the map \( T \), it is not known whether the \( Q \)-cycles are all rational. Note that the set of even \( Q \)-cycles may be easily deduced from the odd \( Q \)-cycles by using the functional equation (17). As a consequence of Theorem 3.10, the period of any \( Q \)-cycle is always a power of 2. In contrast with the cycles having an ever-doubling period introduced in 3.11, a \( Q \)-cycle corresponds to some cycle of \( Q_n \) whose period remains unchanged for all sufficiently large \( n \), and that systematically splits into two cycles of \( Q_{n+1} \), one of which splits again, and so on as \( n \) increases. From a heuristic point of view, it is much like an infinite branching process that allows one to estimate the number of short cycles of \( Q_n \) for large \( n \), as in [6, §6].
Table 3: Known odd $Q$-cycles and their corresponding $T$-cycles.

| Period | $Q$-cycle | $T$-cycles |
|--------|------------|------------|
| 1      | $(-1) = (\overline{1}_2)$ | $(-1)$ |
|        | $\left(\frac{1}{3}\right) = (\overline{0}\overline{1}_2)$ | $(1, 2)$ |
| 2      | $\left(-\frac{1}{5}, 1\right) = (0\overline{1}_2, 0\overline{1}_2)$ | $(0)$ and $(1, 2)$ |
|        | $\left(-\frac{1}{5}, \frac{5}{7}\right) = (00\overline{1}_2, 00\overline{1}_2)$ | $(\left(\frac{1}{5}, \frac{4}{5}, \frac{5}{7}\right)$ and $(\frac{5}{7}, \frac{11}{7}, \frac{20}{7}, \frac{10}{7})$ |

First, one observes in Example 3.3 that $-1$ is a fixed point for the function $Q$, as for $T$. In fact, there are infinitely many, since $-2^k$ and $2^k/3$ are fixed points for all integer $k \geq 0$, in addition to the trivial fixed point $0$. It is conjectured that $-1$ and $1/3$ are the only odd ones (Fixed Point Conjecture, in [6]). Numerically, it is easy to verify that any such point is necessarily very close, if not equal, to $-1$ or $1/3$.

In the same paper, Bernstein and Lagarias also mentioned the existence of the odd rational cycle $\left(-1/3, 1\right)$ of period 2, and conjectured that there are finitely many odd cycles for any given period $2^j$ (3x+1 Conjugacy Finiteness Conjecture).

Lately, I found that $\left(-1/5, 5/7\right)$ is another odd rational cycle of period 2. Indeed,

$$Q\left(-\frac{1}{5}\right) = 00\overline{1}_2 = \frac{5}{7} \quad \text{and} \quad Q\left(\frac{5}{7}\right) = 00\overline{1}_2 = -\frac{1}{5}.$$ 

Next, I conducted a numerical verification up to period 16 on the set of rationals of the form $p/q$ where $p, q$ are odd coprime integers lower than 1000 in absolute value. Working modulo $2^{40}$ was enough to rule out the candidates that are not parts of known $Q$-cycles.

In Table 3, we list the known odd $Q$-cycles, and, for each rational element, the $T$-cycle appearing in its orbit of $T$ iterates. So far, no $Q$-cycle was found having a prime period strictly greater than 2. This leads one to think that there is none.

**Conjecture 3.20.** (Odd Cycles Conjecture) *The function $Q$ has exactly two*
odd fixed points, \(-1\) and \(\frac{1}{3}\), and two odd cycles of prime period 2, \((-\frac{1}{3}, 1)\) and \((-\frac{1}{5}, \frac{5}{2})\). There exists no other odd cycle, rational or not.

4 The 3x + 1 set

4.1 Euclidean embedding

Overall, the automorphism \(Q\) and its inverse remain somewhat mysterious. Yet, it seems possible to somehow visualize its action by applying a continuous function \(M\) that sends \(\mathbb{Z}_2\) to the real interval [0, 2]. Note that \(M\), as defined in [4.1], is very similar to the Monna\(^4\) map [24]. We obtain thereby on Figure 1 a graphical representation of the function \(Q\) in a part of the plane. We call it the 3x + 1 set, as it fully encodes the dynamics of the 3x + 1 map. This method has been already used, e.g. in [16], to represent the graph of the map\(^5\) \(T\) acting on \(\mathbb{Z}_2\). Recall that the parameter space \(\mathbb{Z}_2\) is not Euclidean and totally disconnected, which makes it difficult to visualize [8, 11].

Definition 4.1. Let \(M\) denote the continuous 2-Lipschitz map

\[
M : \mathbb{Z}_2 \longrightarrow [0, 2]
\]

\[
M(r) = \sum_{k=0}^{\infty} r_k 2^{-k}.
\]

As in definition [3.4] we also consider the map

\[
Q : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2
\]

\[
x \longmapsto \sum_{k=0}^{\infty} (T^k(x) \mod 2) 2^k.
\]

Putting \(X = M\) and \(Y = M \circ Q\), the parametric set \((X, Y)(\mathbb{Z}_2)\) of the plane \(\mathbb{R}^2\) is called the 3x + 1 set.

The definition of the 3x + 1 set involves a 2-adic parametrization. However, one can visualize it by taking only positive integers. This is due to the density of \(\mathbb{Z}^+\) in \(\mathbb{Z}_2\), and to the fact that \(Q\) is non-expandable. Practically, it suffices to calculate the parity vectors of length \(k\) for every positive integer up to \(2^k\) for some \(k\) reasonably large, and apply \(M\) on the resulting binary expansions. We took \(k = 12\) in Figure 1.

\(^4\)The original Monna map sends \(\mathbb{Z}_2\) to [0, 1].

\(^5\)Most often, a variant of the map \(T\) is considered, leading to slower or faster dynamics.
Figure 1: Representation of the $3x + 1$ set. The line segments indicate the points from Table 4 along with their respective 2-adic parameter value.
Table 4: Some rational points in the $3x+1$ set associated with an odd rational value of the $2$-adic parameter $r$ and sorted by increasing abscissa.

| $r$ | $Q(r)$ | $X(r)$ | $Y(r)$ | $r$ | $Q(r)$ | $X(r)$ | $Y(r)$ |
|-----|--------|--------|--------|-----|--------|--------|--------|
| 1   | $-\frac{1}{3}$ | 1      | $\frac{4}{3}$ | 3   | $-\frac{23}{3}$ | $\frac{3}{2}$ | $\frac{37}{24}$ |
| 17  | $-\frac{401}{3}$ | $\frac{17}{16}$ | $\frac{493}{384}$ | 5   | $\frac{5}{7}$ | $-\frac{1}{5}$ | $\frac{11}{7}$ |
| 9   | $-\frac{6377}{3}$ | $\frac{9}{8}$ | $\frac{8941}{6144}$ | $-\frac{1}{5}$ | $\frac{5}{7}$ | $\frac{8}{5}$ | $\frac{11}{7}$ |
| $-7$ | $-\frac{5}{7}$ | $\frac{5}{4}$ | $\frac{10}{7}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{5}{3}$ | $\frac{5}{3}$ |
| 5   | $-\frac{13}{3}$ | $\frac{5}{4}$ | $\frac{13}{12}$ | $-5$ | $-\frac{3}{7}$ | $\frac{1}{4}$ | $\frac{12}{7}$ |
| $-\frac{1}{3}$ | 1      | $\frac{4}{3}$ | 1      | $7$   | $-\frac{1595}{3}$ | $\frac{7}{4}$ | $\frac{2797}{1536}$ |
| $-3$ | $-7$   | $\frac{3}{2}$ | $\frac{5}{4}$ | $-1$  | $-1$   | 2      | 2      |

The action of the map $M$ is not easy to figure to oneself, as it does not preserve the usual order between the rational numbers. For instance, the images of positive and negative rational numbers are deeply intertwined. But, conveniently, $M$ sends $2$-adic balls with radius $2^{-k}$ onto real intervals of length $2^{1-k}$, so that the set of odd $2$-adic integers is entirely mapped on the interval $[1, 2]$. We call it the “odd” side. Moreover, rationality is always preserved, which leads to an immediate reformulation of the Periodicity Conjecture.

**Conjecture 4.2. (Rational Points Conjecture) All points in the $3x + 1$ set have coordinates that are either both rational or both irrational.**

In Table 4 we provide the coordinates of various rational points from the $3x + 1$ set that are represented in Figure 1. Among them, six points are associated with one of the odd $Q$-cycles in Table 3, whose respective parameter values are $-1$, $-1/3$, $-1/5$, $1/3$, $5/7$ and $1$.

Let us point out that $M$, as defined in 4.1, is not one-to-one, since

$$M(1) = M(-2) = 1$$

and, more generally,

$$M(n + 2^k) = M(n - 2^{k+1}) = M(n) + 2^{-k} \text{ for } 0 \leq n \leq 2^k - 1.$$
As a result, the mapping $M$ is not truly an embedding, although, when restricted to $\mathbb{Z}_2 \setminus \mathbb{Z}$, $M$ is one-to-one. Besides, it is easily seen that the set $M(\mathbb{Z})$ coincides with the set of dyadic numbers, namely, rationals whose denominator is a power of 2, from the interval $[0, 2]$. This explains, thereby, the presence of infinitely many discontinuities in the $3x + 1$ set, at each point whose abscissa is dyadic.

**Lemma 4.3.** The function $(X, Y) : \mathbb{Z}_2 \to [0, 2]^2$ is one-to-one.

**Proof.** Suppose there exist two different 2-adic integers $a$ and $b$ such that $X(a) = X(b)$ and $Y(a) = Y(b)$. We infer that $a, b, Q(a)$ and $Q(b)$ are all in $\mathbb{Z}$, and $Q(a)$ or $Q(b)$ is positive. E. g., say $Q(a)$ is a positive integer, so it has a finite 2-adic expansion. From the inverse formula (10), it follows that $a$ is rational with denominator a power of 3 strictly greater than 1, and numerator coprime to 3. Since $a$ is a rational integer, there yields a contradiction. \qed

We conclude from Lemma 4.3 that the transformation $(X, Y)$ is an embedding from the parameter space $\mathbb{Z}_2$ to the Euclidean space $\mathbb{R}^2$. Each point in the $3x + 1$ set corresponds to a unique parity sequence.

Let us observe in Figure 1 that it has a rather symmetric aspect with respect to the first diagonal. This is mainly due to the congruence (22) in Corollary 3.13. However, it turns out that few points are effectively symmetric. Despite the non-injectivity of the map $M$, it is most likely that only those points whose parameter values are part of a $Q$-cycle of period at most 2, are symmetric. We obtain thereby a symmetric subset that is expected to contain exactly six points in the upper right quarter of the $3x + 1$ set, two of them being on the diagonal (see Conjecture 3.20 and Table 4).

One may further notice a number of affine self-similarities, some of which are explicated in the next part.

### 4.2 Self-similarity

As a result of the functional equations (17) and (18) satisfied by $Q$, it is possible to delimit regions of the $3x + 1$ set that are identical through an affine transformation. To this aim, we first introduce two infinite families of real intervals, which realize a covering of the half-open interval $[0, 2)$.

**Definition 4.4.** For all integer $k \geq 2$, let

$$
\alpha_k = -1 - (-2)^{k-2} \mod 2^k, \quad m_k = 2^{k-2} - 1, \quad n_k = 3 \times 2^{k-2} - 1,
$$

21
so that $\alpha_k = m_k$ if $k$ is odd, and $\alpha_k = n_k$ otherwise. Then define the real intervals

$$I_k = [M(m_k), M(n_k)] = [2 - 2^{3-k}, 2 - 3 \times 2^{1-k}]$$

and

$$J_k = [M(n_k), M(m_{k+1})] = [2 - 3 \times 2^{1-k}, 2 - 2^{2-k}]$$

of length $2^{1-k}$. The mapping $M$ sends the 2-adic ball $B(\alpha_k, 2^{-k})$ onto $I_k$ or $J_k$ alternatively, according to the parity of $k$.

The next lemma, along with Corollary 4.6, will prove useful to delimit in the $3x + 1$ set all parts corresponding to parametric values in the same congruence class as $m_k$ or $n_k$ modulo $2^k$.

**Lemma 4.5.** The integers $(m_k)_{k \geq 2}$ and $(n_k)_{k \geq 2}$ have the properties

$$Q(m_k) \equiv m_k \pmod{2^k} \quad \text{and} \quad Q(n_k) \equiv n_k \pmod{2^k} \quad \text{for } k \text{ even}, \quad (24)$$

$$Q(m_k) \equiv n_k \pmod{2^k} \quad \text{and} \quad Q(n_k) \equiv m_k \pmod{2^k} \quad \text{for } k \text{ odd}. \quad (25)$$

**Proof.** The function $Q$ induces a permutation on $\mathbb{Z}/2^k\mathbb{Z}$. Thus, we can reason on $Q^{-1}$ instead of $Q$.

Let us write, first, the binary representations

$$m_k = \underbrace{0011 \ldots 1}_{k} \quad \text{and} \quad n_k = \underbrace{1011 \ldots 1}_{k}.$$ 

Then, applying formula (11) from Corollary 3.2, we get

$$Q^{-1}(m_k) \equiv -1 - 2^{k-2}3^{-k} - 2^{k-1}3^{2-k} \pmod{2^k}$$

$$\equiv -1 - 2^{k-2}3^{1-k} \pmod{2^k}$$

$$\equiv -1 + (-2)^{k-2} \pmod{2^k}$$

since $3 \equiv -1 \pmod{4}$, and similarly,

$$Q^{-1}(n_k) \equiv -1 - 2^{k-2}3^{2-k} \pmod{2^k}$$

$$\equiv -1 - (-2)^{k-2} \pmod{2^k}.$$ 

The properties (24) and (25) follow, by considering the parity of $k$. \qed
Corollary 4.6. For all \( k \geq 2 \),

\[
Y(B(\alpha_k, 2^{-k})) = J_k \quad \text{and} \quad Y(B(2\alpha_k + 1, 2^{-k-1})) = I_{k+1},
\]

where \( Y = M \circ Q \) and \( B(\alpha_k, 2^{-k}) \) is the closed ball with center \( \alpha_k \) and radius \( 2^{-k} \) in \( \mathbb{Z}_2 \), as defined in 3.12.

Proof. From Lemma 4.5, we get

\[
Q(\alpha_k) \equiv n_k \pmod{2^k} \quad \text{and} \quad Q(2\alpha_k + 1) \equiv m_{k+1} \pmod{2^{k+1}},
\]

whatever the parity of \( k \). Since \( Q \) is an isometry, it yields

\[
Q(B(\alpha_k, 2^{-k})) = B(n_k, 2^{-k}) \quad \text{and} \quad Q(B(2\alpha_k + 1, 2^{-k-1})) = B(m_{k+1}, 2^{-k-1}).
\]

Hence the result.

We now establish the existence of infinitely many affine relationships within the \( 3x+1 \) set and give their analytic expressions.

Theorem 4.7. The set \((X,Y)(\mathbb{Z}_2)\), namely, the \( 3x+1 \) set, admits the affine relationships

\[
(X,Y)(2r) = \frac{1}{2}(X,Y)(r) \tag{26}
\]

for \( r \in \mathbb{Z}_2 \), and

\[
(X,Y)(2r + 1) = \frac{1}{2}(X,Y)(r) + (1, 1 - 2^{-k}) \tag{27}
\]

for \( r \equiv \alpha_k \pmod{2^k} \) and \( k \geq 2 \), where \( X = M \) and \( Y = M \circ Q \) as in definition 4.4.

Proof of (26). Let \( r \) be a 2-adic integer. Then,

\[
M(2r) = \frac{1}{2}M(r)
\]

and, by the functional equation (17),

\[
M(Q(2r)) = M(2Q(r)) = \frac{1}{2}M(Q(r)).
\]

\[\square\]
Table 5: Pairs of boxes covering two parts of the $3x + 1$ set that coincide modulo an affine transformation with scaling factor $\frac{1}{2}$. Note that $k \geq 2$.

| Parity | Box 1         | Box 2         | Affine transformation |
|--------|---------------|---------------|-----------------------|
| -      | $[0, 2]^2$    | $[0, 1]^2$    | $(x, y) \mapsto (\frac{x}{2}, \frac{y}{2})$ |
| $k$ even | $J_k^2$      | $J_{k+1} \times I_{k+1}$ | $(x, y) \mapsto (1 + \frac{x}{2}, 1 + \frac{y}{2} - 2^{-k})$ |
| $k$ odd | $I_k \times J_k$ | $I_{k+1}^2$ | $(x, y) \mapsto (1 + \frac{x}{2}, 1 + \frac{y}{2} - 2^{-k})$ |

Proof of (27). Let $r \equiv \alpha_k \pmod{2^k}$. It is easily seen that

$$M(2r + 1) = 1 + \frac{1}{2}M(r).$$

Now, recall the functional equation (18):

$$Q(2r + 1) = 2Q(r) + 1 - 2^k.$$

Lemma 4.5 gives $Q(r) \equiv n_k \pmod{2^k}$, yielding the 2-adic expansion

$$Q(r) = 1 + 2 + 2^2 + \ldots + 2^{k-3} + 2^{k-1} + \ldots.$$

Hence, we obtain

$$M(Q(2r + 1)) = 1 + \frac{1}{2}M(Q(r)) - 2^{-k}.$$

It follows from Theorem 4.7 together with Corollary 4.6 that some parts of the $3x + 1$ set, described in Table 5, are identical through the affine transformations represented on Figures 2a and 2b. The presence of the nested boxes $[0, 2]^2$ and $[0, 1]^2$ on the first line of Table 5, related to the autoconjugacy (17), is leading to an infinite descent on the “even” side of the $3x + 1$ set, towards the origin. On the other hand, the functional equation (18) only applies to the odd 2-adic integers, except when $k = 2$. From the corresponding pairs of boxes in Table 5, we obtain a covering of the “odd” side of the
Figure 2: (a-b) Identical parts of the $3x + 1$ set through the affine transformations (26) and (27) in (a) and (b) respectively. (c) Enlarged parts of the $3x + 1$ set delimited by some of the boxes in (b), namely, $J_k^2$ for $k$ even, and $I_k \times J_k$ for $k$ odd.
$3x + 1$ set, which takes the form in Figure 2b of an infinite “cascade” along the first diagonal.

Other functional equations like (16) and (21) do not imply self-similarity on our plane representation in Figure 1 due to the value 3 which is not a power of 2.

On Figure 2c, we show different parts of the $3x + 1$ set, mainly from the “odd” side. They are included in some of the boxes from the second column of Table 5 for $2 \leq k \leq 7$. Putting together all affine similarities, it yields that the content of the first square $J_2$ is made entirely of small copies of $J_k^2$ for even $k \geq 4$, and $I_k \times J_k$ for odd $k \geq 3$, plus an extra point at $(\frac{1}{2}, \frac{1}{2})$. It is a puzzling question whether the other squares on Figure 2c are also made of small pieces taken from the odd side of the set.

Yet, one observes some relative diversity of patterns. Unlike for the Cantor ternary set, there seems to be no simple geometric scheme that would reproduce the $3x + 1$ set.

Finally, the unveiling of self-similarity at all scales raises the question of the Hausdorff dimension, to which the following theorem answers without much difficulty.

**Theorem 4.8.** The $3x + 1$ set has Hausdorff dimension 1.

**Proof.** First, observe that the Hausdorff dimension of the $3x + 1$ set is at least 1, as it contains at least one point for each abscissa taken in the interval $[0, 2]$.

Now, we construct a minimal covering of the $3x + 1$ set with boxes of a given size.

Let $k \geq 0$. For every integer $n$ such that $0 \leq n \leq 2^k - 1$, the points $(X, Y)(r)$ with $r \in B(n, 2^{-k})$ are included in the square box

$$X(B(n, 2^{-k})) \times Y(B(n, 2^{-k}))$$

$$= M(B(n, 2^{-k})) \times M(B(Q(n), 2^{-k}))$$

$$= [M(n), M(n) + 2^{1-k}] \times [M(q(n, k)), M(q(n, k)) + 2^{1-k}]$$

where $q(n, k) = Q(n) \mod 2^k$.

It yields a covering of the $3x + 1$ set with $2^k$ boxes of side-length $2^{1-k}$. The number of boxes is minimal because the number of intervals of length $2^{1-k}$ required to cover $[0, 2]$ is at least $2^k$.

Therefore, the “box-counting” dimension of the $3x + 1$ set is equal to

$$\lim_{k \to \infty} \frac{\log(2^k)}{\log \left( \frac{1}{2^{1-k}} \right)} = 1,$$

26
which is an upper bound for the Hausdorff dimension.

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