The Fourier Decomposition Method for nonlinear and nonstationary time series analysis

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Abstract—In this paper, we propose the Fourier Decomposition Method (FDM) for nonlinear (i.e. data generated by nonlinear systems) and nonstationary time series analysis. The proposed FDM decomposes any data into a small number of ‘Fourier intrinsic band functions’ (FIBFs). The FDM present a generalized Fourier expansion with variable amplitudes and frequencies of a time series by the Fourier method itself. We derive the idea of zero-phase filter bank based multivariate FDM (MFDM) algorithm, for the analysis of multivariate nonlinear and nonstationary time series, from the FDM. We also present an algorithm to obtain cutoff frequencies for MFDM. The MFDM algorithm is generating finite number of band limited multivariate FIBFs (MFIBFs). The MFDM preserves some intrinsic physical properties of the multivariate data, such as scale alignment, trend and instantaneous frequency. The proposed methods produce the results in a time-frequency-energy distribution that reveals the intrinsic structures of a data. Simulations have been carried out and comparison is made with the Empirical Mode Decomposition (EMD) methods in the analysis of various simulated as well as real life time series, and results show that the proposed methods are powerful tools for analyzing and obtaining the time-frequency-energy representation of any data.

Index Terms—The Fourier decomposition method (FDM); Fourier intrinsic band functions (FIBFs) and analytic FIBFs (AFIBFs); zero-phase filter bank based multivariate FDM (MFDM); Empirical Mode Decomposition (EMD).

I. INTRODUCTION

The time-frequency representation (TFR) is a well established powerful tool for the analysis of time series signals. There exist many types of time-frequency (TF) analysis methods, e.g. linear (the short-time Fourier transform), quadratic (the Wigner-Ville distribution) and Wavelet transforms. The TFR is achieved by formulation often referred as time-frequency-distribution (TFD) and provides insight into the complex structure of a signal consisting of several components. The analysis of nonstationary signals are not well presented by these methods.

Recently developed Empirical Mode Decomposition (EMD) has provided a general method for examining the TFD, and has been applied to all kinds of data. The EMD is an adaptive signal analysis algorithm for the analysis of nonstationary and nonlinear signals (i.e. signals generated from nonlinear systems). The EMD has become an established method for the signal analysis in various applications, e.g. medical studies and image analysis. The EMD decomposes any given data into a finite number of narrow band intrinsic mode functions (IMFs) which are derived directly from the data, whereas other signal decomposition techniques (like the Fourier, Wavelets, etc.) incorporate predefined fixed basis for signal modeling and analysis. The Ensemble EMD (EEMD) is a noise-assisted data analysis method developed in to overcome the timescale separation problem of EMD. The Multivariate EMD (MEMD) developed in is a generalization of the EMD for multichannel data analysis. The Compact EMD (CEMD) algorithm is proposed in to reduce mode mixing, end effect, and detrend uncertainty present in EMD and to reduce computation complexity of EEMD as well. The IMFs generated by EMD are dependent on distribution of local extrema of signal and the type of spline used for upper and lower envelope interpolation and the traditional EMD uses cubic spline for upper and lower envelope interpolation. The EMD algorithm proposed in to reduce mode mixing and detrend uncertainty, uses nonpolynomial cubic spline interpolation to obtain upper and lower envelopes, and have shown in that it improves orthogonality among IMFs.

The energy preserving property is important for any kind of transformation, and it is obtained by the orthogonal decomposition of signal in various transforms like the Fourier, Wavelet, Fourier-Bessel, etc. The energy preserving property is especially important for the accurate and faithful analysis of three dimensional time-frequency-energy distribution of a signal. The EMD algorithms, proposed in , ensure orthogonality or energy preserving property or both in decomposition of signal into IMFs and refereed to as energy preserving EMD (EPEMD).

In spite of considerable success of EMD, all of the EMD algorithms are based on empirical, heuristic and ad hoc procedure that make them hard to analyze mathematically, and EMD may suffer from mode mixing, detrend uncertainty, aliasing and end effect artefacts. There is a lack of mathematical understanding of the EMD algorithm, e.g. IMFs dependence on the number of shifting and the stopping criteria, convergence property and stability to noise perturbation. In spite of all these limitations, EMD is the widely used nonstationary data analysis method, therefore, in this paper, EMD is used as a reference to establish the validity, reliability and calibration of the proposed method.

There is an understanding in literature (e.g. ) that the Fourier methods can not be used for nonlinear systems and non stationary data analysis, and various reasons (e.g. linearity, periodicity or stationarity) are provided to support

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it. The Fourier transform is valid under very general Dirichlet conditions (i.e. signal is absolutely integrable with finite number of maxima and minima, and finite number of finite discontinuities in any interval) thus include nonlinear and nonstationary signals as well. Therefore, in this study, we explore and provide algorithms to analyze nonlinear system and nonstationary data by the Fourier method termed as the Fourier Decomposition Method (FDM), which generates small number of Fourier intrinsic band functions (FIBFs), and we show that the Fourier method is superior tool for spectrum as well as time-frequency analysis of any signal.

In this paper, we also propose a method, which captures the features of the MEMD, using a zero-phase filter bank (ZPFB) approach to construct the multivariate FIBFs and residue components. This multivariate FDM (MFDM) algorithm generates matched multivariate FIBFs and residue through zero-phase filtering. Thus, we propose a adaptive data-driven ZPFB based time-frequency analysis method.

For the adaptive data analysis approach, the most difficult challenge has been to establish a general adaptive decomposition method of analysis without a priori basis. In this study, we propose the FDM and MFDM general adaptive data analysis methods that are inspired by the EMD methods and their filter bank properties [15], [16]. This paper is organized as follows: In section II the EMD algorithm is briefly presented. We propose the Fourier decomposition method (FDM) in section III. We propose the ZPFB based multivariate FDM algorithm in section IV. Section V has simulation results. Section VI presents conclusions.

II. BRIEF OVERVIEW OF THE EMD ALGORITHM

There are various methods of nonstationary data processing like, the spectrogram; the wavelet analysis; the Wigner-Ville distribution; evolutionary spectrum [17]; the empirical orthogonal function expansion (EOF) (or principal component analysis or singular value decomposition); Synchrosqueezed wavelet transforms [18]; the EMD, etc.

The EMD is a well established multiresolution method that decomposes nonlinear and nonstationary data into a set of finite band-limited IMFs and residue through the sifting process. The decomposed signal $x(t)$ is expressed as the sum of $\ell$ IMF components plus the final residue as

$$
x(t) = \sum_{i=1}^{\ell} y_i(t) + r(t) = \sum_{i=1}^{\ell+1} y_i(t),
$$

where $y_i(t)$ is the $i^{th}$ IMF and $r(t) = y_{\ell+1}(t)$ is final residue. The IMFs admit amplitude-frequency modulated (AM-FM) representation [19] (i.e. $y_i(t) \approx a_i(t) \cos(\phi_i(t))$, with $a_i(t), \frac{d\phi_i(t)}{dt} \neq 0, \forall t$) and well-behaved Hilbert transforms [20]. For any IMF $y_i(t)$, its Hilbert transform $\hat{y}_i(t)$ is defined as convolution of $y_i(t)$ and $1/\pi t$, i.e. $\hat{y}_i(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y_i(t)}{t-x} dx$ and the Hilbert transform emphasizes the local properties of $y_i(t)$. An analytic signal $z_i(t)$ can be represented by $z_i(t) = y_i(t) + j\hat{y}_i(t) = a_i(t) \exp(j\phi_i(t))$, where $a_i(t) = \sqrt{y_i^2(t) + \hat{y}_i^2(t)}$ and $\phi_i(t) = \tan^{-1}(\hat{y}_i(t)/y_i(t))$ are instantaneous amplitude and phase of $y_i(t)$. The instantaneous frequency (IF) of $y_i(t)$ is defined as: $\omega_i(t) = \phi_i'(t) = \frac{y_i'(t)y_j(t) - y_i(t)y_j'(t)}{y_i^2(t) + \hat{y}_i^2(t)}$. The physical meaning of IF $\omega_i(t)$ constrains that $\phi_i(t)$ must be a mono-component function of time, the Bedrosian and Nuttall theorems [19], [20] impose non-overlapping spectra constraints on the pair [$a_i(t), \cos(\phi_i(t))$].

All IMFs must satisfy two basic conditions: (1) In the complete range of time series, the number of extrema (i.e. maxima and minima) and the number of zero crossings are equal or differ at most by one. (2) At any point of time in the complete range of time series, the average of the values of upper and lower envelopes, obtained by the interpolation of local maxima and the local minima, is zero. The first condition ensure that IMFs are narrow band signals and the second condition is necessary to ensure that the IF does not have redundant fluctuations because of asymmetric waveforms.

III. THE FOURIER DECOMPOSITION METHOD

We propose a class of functions termed as the Fourier intrinsic band functions (FIBFs), belong to $C^\infty[a, b]$, here with the following formal definition. The Fourier intrinsic band functions (FIBFs) are functions that satisfy the following conditions: (1) The FIBFs are zero mean functions (i.e. $\int_a^b y_i(t) dt = 0$). (2) The FIBFs are orthogonal functions (i.e. $\int_a^b y_i(t)y_j(t) dt = 0$, for $i \neq j$). (3) The FIBFs form the analytic FIBFs (AFIBFs) with IF and amplitude always greater than zero, i.e., $y_i(t) + j\hat{y}_i(t) = a_i(t) \exp(j\phi_i(t))$, with $a_i(t), \frac{d\phi_i(t)}{dt} \geq 0, \forall t$. Thus, the AFIBFs are monocomponent signals and, physically, the IF has meaning only for monocomponent signals, i.e., signal has only one frequency or a narrow range of frequencies varying as a function of time [24]. Thus, the FIBF is sum of zero mean sinusoidal functions of consecutive frequency band.

The main objective of this study is to obtain unique representation of multicomponent signal as a sum of constant and monocomponent signals, i.e. signals which can be represented by the following model [24]:

$$
x(t) = \sum_{i=1}^{M} y_i(t) + n(t),
$$

where $n(t)$ is a noise representing any residue (constant or trend) components and the $y_i(t)$ are $M$ single component nonstationary signals, i.e., $y_i(t)$ are FIBFs as we defined above.

The necessary conditions [1], for a basis to represent a nonlinear and nonstationary time series, are completeness, orthogonality, locality, and adaptiveness. The FIBFs, intrinsically, follow all the necessary conditions by virtue of the decomposition.

The available data are usually of finite duration, nonstationary and generated from the systems that are generally nonlinear. Let $x(t)$ be a time limited $[t_1, t_1 + T_0]$ real valued signal which follows the Dirichlet conditions. We construct the periodic signal as $x_{T_0}(t) = \sum_{s=-\infty}^{\infty} x(t - sT_0)$ such that $x(t) = x_{T_0}(t)u(t)$, where $u(t) = 1$, for $t_1 \leq t \leq t_1 + T_0$ and zero otherwise. The Fourier series expansion of $x_{T_0}(t)$ is given by

$$
x_{T_0}(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)],
$$
where frequency (rad/s) \( \omega_0 = \frac{2\pi}{T_0} \), \( a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t_0) \, dt \), 
\[ a_k = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} x(t_0) \cos(k \omega_0 t) \, dt \quad \text{and} \quad b_k = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} x(t_0) \sin(k \omega_0 t) \, dt. \]
We write (3) as 
\[ x(t_0) = a_0 + \frac{1}{2} \sum_{k=1}^{\infty} c_k \exp(jk \omega_0 t) + c_k^* \exp(-jk \omega_0 t), \quad (4) \]
where \( c_k = (a_k - jb_k) \) and \( c_k^* = (a_k + jb_k) \). From (4), it is clear that 
\[ x(t_0) = a_0 + \text{Re}\{z(t_0)\}, \quad (5) \]
where analytic function 
\[ z(t_0) = \sum_{k=1}^{\infty} c_k \exp(jk \omega_0 t) \quad (6) \]
is complex conjugate of \( z(t_0) = \sum_{k=1}^{\infty} c_k^* \exp(-jk \omega_0 t) \) and \( \text{Re}\{z(t_0)\} \) denotes the real part of \( z(t_0) \). We write \( z(t_0) \) as 
\[ z(t_0) = \sum_{i=1}^{M} a_i(t) \exp(j\phi_i(t)), \quad (7) \]
where, in forward (i.e. low to high frequency scan) search of AFIBFs, \( a_1(t) \exp(j\phi_1(t)) = \sum_{k=1}^{N_0} c_k \exp(jk \omega_0 t), \) 
\( a_2(t) \exp(j\phi_2(t)) = \sum_{k=1}^{(N_0+1)} c_k \exp(jk \omega_0 t), \) , ..., 
\( a_M(t) \exp(j\phi_M(t)) = \sum_{k=1}^{(N_0+1)} c_k \exp(jk \omega_0 t), \) i.e. 
\[ a_i(t) \exp(j\phi_i(t)) = \sum_{k=N_{i-1}+1}^{N_i} c_k \exp(jk \omega_0 t), \quad (8) \]
with \( N_0 = 0 \) and \( N_M = \infty \). The FIBFs are the real part of AFIBFs presented in Eq. (8). To obtain minimum number of AFIBFs in low to high frequency scan (LTH-FS), for each \( i \), start with \((N_{i-1}+1)\), increase and select the maximum value of \( N_i \) such that \((N_{i-1}+1) \leq N_i \leq \infty \) and 
\[ a_i(t), \omega_i(t) = \frac{d\phi_i(t)}{dt} \geq 0, \forall t. \quad (9) \]

Similarly, in reverse (i.e. high to low frequency scan (HTL-FS)) search of AFIBFs, we obtain 
\[ a_1(t) \exp(j\phi_1(t)) = \sum_{k=N_1}^{\infty} c_k \exp(jk \omega_0 t), \]
\[ a_2(t) \exp(j\phi_2(t)) = \sum_{k=N_1+1}^{(N_1+1)} c_k \exp(jk \omega_0 t), \] , ..., 
\[ a_M(t) \exp(j\phi_M(t)) = \sum_{k=N_1+1}^{(N_1+1)} c_k \exp(jk \omega_0 t), \] and the lower and upper limits of sum in Eq. (8) will change to \( k = N_i \) to \((N_{i-1}+1)\), respectively, with \( N_0 = \infty, N_M = 1 \). Here, we start with \((N_{i-1}+1)\), decrease and select minimum value of \( N_i \) such that \( 1 \leq N_i \leq (N_{i-1}+1) \) and Eq. (9) is satisfied for \( i = 1, \cdots, M \). 

Thus, we have obtained a generalized Fourier expansion of a time series in Eq. (7) by the Fourier method itself. The variable amplitude and the IF have improved the efficiency of the expansion by expanding the signal into finite number of analytic FIBFs, in (7), and enabled the expansion to accommodate nonstationary data. Thus, we have obtained a variable amplitude and frequency representation, whereas, the classical Fourier expansion provides the constant amplitude and fixed-frequency representation.

For each FIBFs, the amplitude \( a_i(t) \) and IF \( f_i(t) \) are functions of time, therefore, we define the three dimensional \( \{t, f_i(t), a_i(t)\} \) time-frequency distribution of amplitude as the Fourier-Hilbert spectrum (FHS) \( H(f, t) \). The marginal Hilbert spectrum (MHS), derived from Hilbert spectrum, is defined in (11), similarly, here we derive the marginal Fourier-Hilbert spectrum from the FHS and define as: 
\[ h(f) = \int_0^{T_0} H(f, t) \, dt. \quad (10) \]
The marginal Fourier-Hilbert spectrum (MFHS) offers a measure of total amplitude (or energy) contribution from each value of frequency in a probabilistic sense. The frequency in either \( H(f, t) \) or \( h(f) \) has a totally different meaning from the Fourier spectral analysis (11). The presence of energy at each frequency in MFHS \( h(f) \) means only that, in the total duration of the signal, there is a higher likelihood for such a wave (FIBF) to have appeared locally. The frequency in the MFHS indicates only the likelihood that an oscillation with such a frequency exists. The exact occurrence time of that oscillation is given in the full Fourier-Hilbert spectrum. We can also define the instantaneous energy density, which can be used to measure the fluctuation of energy with time, as 
\[ E(t) = \int_0^{f_M} H^2(f, t) \, df, \quad (11) \]
where \( f_M \) is a maximum frequency of signal. From Eq. (3), we obtain the energy of signal \( x(t) \) (or power of signal \( x(t) \)) by the Parseval’s theorem as \( E_x = a_0^2 + \sum_{i=1}^{N} |a_i|^2 \) and from Eq. (6) energy of the analytic signal (or power of signal \( z(t) \)) as \( E_z = \sum_{i=1}^{N} |a_i|^2 + \frac{b_i^2}{2} \), therefore, relation between \( E_x \) and \( E_z \) is given by 
\[ E_z = a_0^2 + \frac{E_x}{2}. \quad (12) \]

Hence, energy of zero mean signal is half of the energy of its analytic signal.

The practically available signals are, generally, discrete and for discrete signal, \( x[n] \), of length \( N \), through the discrete Fourier transform (DFT), we can write \( x[n] \) as 
\[ x[n] = \sum_{k=0}^{N-1} X[k] \exp\left(\frac{j2\pi kn}{N}\right), \quad (13) \]
where \( X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\left(-\frac{j2\pi kn}{N}\right) \) is the DFT of signal \( x[n] \). Let \( N \) be an even number (similarly, we can proceed when \( N \) is an odd number), then \( X[0] \) and \( X[\frac{N}{2}] \) are real numbers; and we can write \( x[n] \) as 
\[ x[n] = x[0] + \sum_{k=1}^{\frac{N}{2}-1} X[k] \exp\left(\frac{j2\pi kn}{N}\right) + X[\frac{N}{2}] \exp(j\pi n) \]
\[ + \sum_{k=\frac{N}{2}+1}^{N-1} X[k] \exp\left(\frac{j2\pi kn}{N}\right). \quad (14) \]
Since \( x[n] \) is real, therefore, \( z_1[n] = \sum_{k=1}^{\frac{N}{2}-1} X[k] \exp\left(\frac{j2\pi kn}{N}\right) \) is complex conjugate of \( z_2[n] = \sum_{k=\frac{N}{2}+1}^{N-1} X[k] \exp\left(\frac{j2\pi kn}{N}\right) \) and we can write (14) as 
\[ x[n] = x[0] + 2\text{Re}\{z_1[n]\} + X[\frac{N}{2}](-1)^n, \quad (15) \]
where $Re\{z_1[n]\}$ denote the real part of $z_1[n]$. Now, we write analytic signal $z_1[n]$ as

$$\sum_{k=1}^{N-1} X[k] \exp\left(\frac{j2\pi kn}{N}\right) = \sum_{i=1}^{M} a_i[n] \exp(j\varphi_i[n]), \quad (16)$$

where, in forward (low to high frequency scan) search of AFIBFs, we obtain $a_1[n] \exp(j\varphi_1[n]) = \sum_{k=(N_i+1)}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, $a_2[n] \exp(j\varphi_2[n]) = \sum_{k=(N_i+1)}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, ..., $a_M[n] \exp(j\varphi_M[n]) = \sum_{k=(N_i+1)}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, i.e.

$$a_i[n] \exp(j\varphi_i[n]) = \sum_{k=N_i+1}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right), \quad (17)$$

with $N_0 = 0$ and $N_M = (N - 1)$. To obtain minimum number of AFIBFs in LTH-FS, for each $i$, we scan from $(N_i+1)$ to $(N\to N_i)$, obtain maximum value of $N_i$ such that $(N_i+1) \leq N_i \leq (N - 1)$ and phase $\varphi_i[n]$ is a monotonically increasing function, i.e. $\omega_i[n] = (\varphi_i[n+1] - \varphi_i[n]) \geq 0$ or

$$\omega_i[n] = \left(\frac{\varphi_i[n+1] - \varphi_i[n]}{2}\right) \geq 0, \quad \forall n \quad (18)$$

and $a_i[n] \geq 0$ for $i = 1, \ldots, M$.

Similarly, in HTL-FS for FIBFs, we obtain $a_1[n] \exp(j\varphi_1[n]) = \sum_{k=N_i+1}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, $a_2[n] \exp(j\varphi_2[n]) = \sum_{k=N_i+1}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, ..., $a_M[n] \exp(j\varphi_M[n]) = \sum_{k=N_i+1}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, and the lower and upper limits of sum in Eq. (17) will change to $k = N_i$ to $(N - 1)$, respectively, with $N_0 = 1$, $N_M = 1$.

Thus, FDM provide two view, low to high frequency and high to low frequency view, of the signal and generate two set of time-frequency-energy distribution. Depending on the signal, both view may be same or sometimes they reveal two different features of the signal. The FDM is summarized in Algorithm A and B. The FDM with Fourier transform (FT) and discrete time Fourier transform (DTFT) is summarized in appendix.

Algorithm A: The FDM algorithm (LTH-FS) to obtain AFIBFs, for $i = 1, \ldots, M$ with $N_0 = 0$ and $N_M = \frac{N}{2}$.

STEP 1. Obtain $X[k] = FFT\{x[n]\}$.

STEP 2. Set $AFIBF_i = \sum_{k=(N_i+1)}^{N_i} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, obtain maximum value of $N_i$ such that $(N_i+1) \leq N_i \leq (N\to N_i)$ and phase $\varphi_i[n]$ of $AFIBF_i$ is a monotonically increasing function, i.e. $\omega_i[n] = (\varphi_i[n+1] - \varphi_i[n]) \geq 0$.

Algorithm B: The FDM algorithm (HTL-FS) to obtain AFIBFs, for $i = 1, \ldots, M$ with $N_0 = 2$ and $N_M = 1$.

STEP 1. Obtain $X[k] = FFT\{x[n]\}$.

STEP 2. Set $AFIBF_i = \sum_{k=N_i}^{(N_N-1)} X[k] \exp\left(\frac{j2\pi kn}{N}\right)$, obtain minimum value of $N_i$ such that $1 \leq N_i \leq (N - 1)$ and phase $\varphi_i[n]$ of $AFIBF_i$ is a monotonically increasing function, i.e. $\omega_i[n] = (\varphi_i[n+1] - \varphi_i[n]) \geq 0$.

IV. MULTIVARIATE FOURIER DECOMPOSITION METHOD

From (8) and (17), we observe that the operation that generates the FIBFs is nothing but the Fourier based zero-phase filtering (ZPF). This is another motivation, apart from the FB properties of IMFs, to use the Fourier or other methods of zero-phase filtering to decompose any data into a set of FIBFs. The ZPF of a real valued signal $x[n]$ by zero-phase filter ($h_z[n]$) can be obtained by two methods: (1) Convolution method, i.e. $y[n] = x[n] * (h_z[n]) \Rightarrow Y[k] = X[k]H_z[k]$, where $h_z[n] = h[n] * h[-n]$ and $H_z[k] = |H[k]|^2$. (2) The Fourier method, i.e. set $H_z[k] = 1$ at desired frequency band and $H_z[k] = 0$ otherwise, obtain output by the inverse DFT, i.e. $y[n] = \sum_{k=0}^{N-1} X[k]H_z[k] \exp(j2\pi kn/N)$, where $X[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n] \exp(-j2\pi kn/N)$.

We use ZPF that does not shift the features of the signal and propose multivariate FDM (MFDM) algorithm to generate multivariate FIBFs (MFIBFs) and residue as follows. Apply zero-phase high pass filtering (ZP-HPF) with cutoff frequency $f_{c1}$ to P-variate (P-channel) time series $(x_p(t))_{p=1}^P$ and obtain first MFIBF $y_{p1}(t)$. The first residue is obtained as

$$r_{p1}(t) = x_p(t) - y_{p1}(t), \quad p = 1, 2, \ldots, P. \quad (19)$$

Apply ZP-HPF with cutoff frequency $f_{c2}$ to residue $r_{p1}(t)$ and obtain second MFIBF $y_{p2}(t)$. The second residue is obtained as

$$r_{p2}(t) = r_{p1}(t) - y_{p2}(t), \quad p = 1, 2, \ldots, P. \quad (20)$$

We can repeat this ZP-HPF procedure $t$ times and obtain final MFIBF $y_{pt}(t)$ and residue (with cutoff frequency $f_{ct}$)

$$r_{pt}(t) = r_{p(t-1)}(t) - y_{pt}(t), \quad p = 1, 2, \ldots, P. \quad (21)$$

Through the addition of (19), (20) and (21) we obtain expression similar to (11), i.e.

$$x_p(t) = \sum_{i=1}^{t} y_{pi}(t) + r_{pt}(t), \quad p = 1, 2, \ldots, P. \quad (22)$$

When we use the Fourier based zero-phase filtering, as in Eq. (8), to obtain MFIBFs, first two conditions of FIBFs are fully satisfied and the third one is approximately satisfied (i.e. satisfied in all practical sense), obviously, it can not be guaranteed simultaneously to all P-channel data. This is similar to MEMD algorithm problem where in derivation of multivariate IMFs, first condition of IMF is not imposed [9].

The question is, how to obtain cutoff frequencies (CFs) $f_{c1}, f_{c2}, \ldots, f_{ct}$ corresponding to zero-phase high pass filters $h_{1z}(t), h_{2z}(t), \ldots, h_{zt}(t)$? There is lot of flexibility and are various ways to select CFs, e.g. dyadic (i.e. $f_{c1} = \frac{f_M}{2}, f_{c2} = \frac{f_M}{4}, \ldots, f_{ct} = \frac{f_M}{2^t}$, where $f_M$ is the maximum frequency of a signal $x(t)$ and for the sampled signal, maximum frequency is $(\frac{N}{2})$ half of the sampling frequency), non-dyadic, uniform and non-uniform CFs. We can take the Fourier transform of signal $x(t)$ to obtain its spectrum details and make strategy to decide CFs.

For narrowband signal, we define ratio of center frequency ($f_C$) to bandwidth (BW) as

$$m = \frac{f_C}{f_H - f_L}, \quad f_C = (f_H + f_L)/2, \quad (23)$$
where \( f_H \) is the highest frequency and \( f_L \) is the lowest frequency of a narrowband signal. From (23) we obtain

\[
f_L = [(2m - 1)/(2m + 1)]f_H, \quad m > 1/2.
\] (24)

From (23) and (24), we observe that the ratios, for the consecutive \( i \)th and \( (i + 1) \)th bands, of center frequencies \( f_{Ci} \), CFs \( f_{ci} \) and BWs \( (f_{Hi} - f_{Li}) \) are constant, i.e.

\[
f_{Ci}/f_{Ci+1} = f_{ci}/f_{ci+1} = (f_{Hi} - f_{Li})/(f_{Hi+1} - f_{Li+1}) = l.
\] (25)

From (23), (24) and (25), we obtain \( l = (2m + 1)/(2m - 1) \) or \( m = (1/2)(l + 1)/(l - 1) \) with \( l > 1 \), and as \([m \to \infty, \quad l \to 1, \quad [m \to 1/2, \quad l \to \infty] \). Here, we have liberty to select any suitable value of \( l \) or \( m \), and greater the value of \( m \) (or lesser the value of \( l \)) narrow the band, whereas in the case of dyadic FB \( l = 2 \) and \( m = 1.5 \) are fixed values. If required, we can vary the value of \( m \) (or \( l \)) for each band rather than taking the fixed value. Thus, we here propose the compact and elegant way to decide CFs as summarized in Algorithm C.



Algorithm C: An algorithm to obtain cutoff frequencies

1. Select suitable value of \( m \) and set \( f_H = F_s/2 \).
2. Set \( f_{ci} = [(2m - 1)/(2m + 1)]f_H \).
3. Set \( f_H = f_{ci} \).
4. Repeat step 2 to 3 for \( i = 1, 2, \ldots, \ell \).

In MFDM, we can use zero-phase low pass filtering (ZPLPF) in place of ZP-HPF to decompose signal in order of residue to first MFIBFs, i.e. \( r_p(x(t)), y_p(t), \ldots, y_{p1}(t) \). We use zero-phase filtering as it preserve features (e.g. maxima, minima, etc.) in the filtered time waveform exactly where those features occur in the unfiltered waveform, whereas conventional (non zero-phase) filtering shifts the features in the signal and hence cannot be used. The zero-phase filtering of time series can be obtained through the finite impulse response (FIR) or infinite impulse response (IIR) filters.

Similar to the MEMD and noise-assisted MEMD (NA-MEMD) [16], this MFDM algorithm produces the equal number of scale-aligned MFIBFs for all channels and preserving joint channel properties that make it suitable for direct multichannel modelling. The FDM does not suffer from mode mixing, detrend uncertainty and end effect artefacts as extraction of FIBFs does not depend on distribution of local extrema across the range of signal.

V. SIMULATION RESULTS

The online available MATLAB software of MEMD [21], EMD and EEMD [22] have been used in simulation results.

A. Multivariate data decomposition

We used quadri-variate time series signal, which is summation of sinusoids (with combination of frequencies \( f_1 = 4Hz, f_2 = 8Hz, f_3 = 16Hz, f_4 = 32Hz \)) and Gaussian white noise of zero mean and standard deviation of 0.2, i.e.

\[ x_j(t) = \sum_{i=1}^{4} \sin(2\pi f_i t) + n_j(t) \] (for \( j = 1, \ldots, 4 \)). A 32Hz sinusoid is present to first, second and fourth channels; a 16Hz sinusoid is present to first, second and third channels; a 8Hz sinusoid is common to all channels; a 4Hz sinusoid is present to first, third and fourth channels. On the same machine, computation time for MFDM is 0.45 sec. and for MEMD is 69.5 sec. in this simulation. The MFDM algorithm, similar to MEMD, generating perfectly aligned intrinsic bands, as shown in Figure 1 in all the four channels, whereas, MFDM is computationally more efficient.

B. Intermittency and mode mixing

The intermittency, in the time series, is a main cause of mode mixing [e.g. signal \( x(t) \) in Figure 2(a)] and mode splitting in EMD algorithm. These issues are mitigated by EEMD and NA-MEMD. The decompositions of a signal \( x(t) \) through FDM algorithm is shown in Figure 2(b) without end effect artefacts. The MFDM is able to localize the mono-component sinusoid within a single FIBF and outperforming EMD [2(c) and EEMD [2(d)]. On the same machine, computation time for MFDM, EMD and EEMD are 0.56 sec., 0.21 sec. and 77.18 sec., respectively. The ensemble size for EEMD was \( N = 500 \) with the 16.94 dB signal-to-noise power ratio.

C. Time-Frequency-Energy Analysis

Figure 3 shows time-frequency-energy (TFE) estimates for a nonstationary signal mixture of a linear chirp and frequency modulated (FM) sinusoid, obtained using the FDM and EMD. There is an enhanced TFE tracking when using FDM with low to high frequency scan and other one (HTL-FS) is similar to EMD.

D. Intrawave frequency modulation

First, we decompose the following signal that has intrawave frequency modulation and it is considered challenging because the instantaneous frequency itself has very high frequency modulation [23]

\[
x(t) = \frac{1}{1.2 + \cos(2\pi t)} + \frac{\cos(32\pi t + 0.2 \cos(64\pi t))}{1.5 + \sin(2\pi t)}
\] (26)

Second, we consider a model wave

\[
x(t) = \cos(\omega t + \epsilon \sin \omega t)
\] (27)
that satisfies the following highly nonlinear differential equation [11]
\[
\frac{d^2x(t)}{dt^2} + [\omega + \epsilon \omega \cos(\omega t)]^2 x(t) - [\epsilon \omega^2 \sin(\omega t)] \sqrt{1 - x^2(t)} = 0
\]
with \( \omega = 1 \) and \( \epsilon = 0.5 \). We demonstrate that, Figure [5] and [6] our method well applies to these challenging cases with good accuracy. These examples clearly demonstrate that the FDM can indeed analyze nonlinear signals and it is a nonlinear decomposition method.

E. Analysis of a white Gaussian noise

Figure [8] shows the TFE analysis of a white Gaussian noise (with zero mean, unit variance, 1024 samples and sampling frequency \( F_s = 100 \) Hz) obtained from the FDM and EMD algorithm. Clearly, both LTH-FS and HTL-FS views of TFE is similar and complete data is decomposed in FIBFs. Figure [9] shows the power spectral density (PSD) plot of same white Gaussian noise with the FDM and EMD algorithm. The FDM has dived the complete data in narrowband and orthogonal FIBFs. Both LTH-FS and HTL-FS views of PSD looking similar but FIBFs have different frequency band, e.g. approximately 40 Hz is cutoff frequency for one of the band in PSD (LTH-FS), whereas, it is mid frequency of the one band in other PSD (HTL-FS) view. There are enhanced TFE and PSD tracking when using FDM.

F. TFE Analysis of unit sample sequence

The unit sample sequence defined as \( \delta[n - n_0] = 1 \) at \( n = n_0 \) and zero otherwise. By using the relation \( z[n] = \)
where real part of corresponds to half of the Nyquist frequency, i.e. Figure 10 shows the plots of real, imaginary part and absolute value of (26).

\[ \frac{1}{T_0} \int_0^T X(\omega) \exp(j\omega n) \, d\omega, \]

we obtain the analytic representation of \( x[n] = \delta[n-n_0] \iff X(\omega) = \exp(-j\omega n_0) \)

\[
\begin{align*}
z[n] &= \sin(\pi(n-n_0)) + j[1 - \cos(\pi(n-n_0))] \\
&= a[n] \exp(j\phi[n]),
\end{align*}
\]

(29)

where real part of \( z[n] \) is \( \delta[n-n_0] = \frac{\sin(\pi(n-n_0))}{\pi(n-n_0)}, \)

\( a[n] = \frac{\sin(\frac{\pi(n-n_0)}{2})}{\frac{\pi}{2}(n-n_0)} \)

\( \phi[n] = \frac{\pi}{2}(n-n_0) \)

and hence \( \omega[n] = \frac{\pi}{T} \) which corresponds to half of the Nyquist frequency, i.e. \( \frac{F_s}{2} \) Hz. Figure 10 shows the plots of real, imaginary part and absolute value of \( z[n] \) with \( n_0 = 199 \), sampling frequency \( F_s = 100 \) Hz, length \( N = 400 \). Theoretically, this clearly indicate that most of the energy of signal \( \delta[n-n_0] \) is concentrated at time \( t = 1.99 \text{ sec.} \) (\( n_0 = 199 \) ) and frequency \( f = 25 \text{ Hz} \). Figure 12 shows the TFE analysis of unit sample sequence \( \delta[n-n_0] \) from the FDM, EEMD and continuous wavelet transform (CWT) methods. This clearly indicate that the TFE plot obtained by the FDM method is same as theoretical estimation, whereas there is energy spread over a range of frequencies and lack of accuracy in TFE plot by the EEMD and CWT methods.
G. Application to a earthquake signal analysis

Earthquake time series signal is nonlinear and nonstationary data. The Elcentro Earthquake data has been taken from [25] and is shown in Figure 13. The critical frequency range that matter in the structural design is less than $10Hz$, and from the Fourier based power spectral density (PSD), marginal spectrum by FDM and marginal spectrum by EMD in Figure 13 show that almost all the energy in this data is within $10Hz$. The instantaneous energy fluctuations by the FMD and EMD methods, as shown in Figure 14 are similar in nature. The TFE distribution by the FMD, EMD and CWT methods are shown in Figure 15, and all three methods indicate maximum energy concentration around $1.7Hz$ and 2 second. There is enhanced TFE tracking by FDM methods as it provide better details of how the different waves arrive from the epical center to the recording station, e.g. the compression waves of small amplitude but higher frequency range of 10 to $20Hz$, the shear and surface waves of strongest amplitude and lower frequency range of below $5Hz$ which does most of the damage, and other body shear waves which are present over the full duration of the data span.

VI. Conclusion

In this paper, we have proposed: (1) The Fourier Decomposition Method (FDM), for nonlinear and nonstationary time series analysis, which decomposes any data into a small number of ‘Fourier intrinsic band functions’ (FIBFs). The FDM is a generalized Fourier expansion with variable amplitudes and frequencies of a time series by the Fourier method itself. (2) The zero-phase filter bank based multivariate
FDM (MFDM) algorithm, for the analysis of multivariate nonlinear and nonstationary time series, which is generating finite number of band limited multivariate FIBFs (MFIBFs).

(3) An algorithm to obtain cutoff frequencies required in MFDM algorithm for zero-phase high or low pass filtering of multivariate signals.

The fundamental and important conceptual innovations of this study are the Fourier method (i.e. FDM) and the introduction of the FIBFs. The FIBFs form the basis of the decomposition that are complete, orthogonal, local and adaptive. The instantaneous frequencies of the FIBFs produce a time-frequency-energy distribution of any signal. A time-frequency-energy distribution of a signal is used in various fields of science and engineering for analysis of physical phenomena and engineering systems. The proposed methods produce the final presentation of the results in a time-frequency-energy distribution that reveals the imbedded structures of a signal. Unlike the various EMD algorithms, the FDM and MFDM are mathematically well defined, supported by the well established theories of filter and the Fourier transforms. The FDM and MFDM methods do not suffer from mode mixing, detrend uncertainty and end effect artefacts as extraction of FIBFs does not depend on distribution of local extrema across the range of data. Simulation results demonstrate the power of the proposed methods.

APPENDIX

A. The FDM for continuous time real function

Let \( x(t) \) be a non-periodic, real function of time and follow the Dirichlet conditions, then the Fourier transform (FT) of \( x(t) \) is defined as

\[
X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) \, dt \tag{30}
\]

and inverse Fourier transform is defined as

\[
x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft) \, df \tag{31}
\]

It is easy to show that

\[
\int_{0}^{\infty} X(f) \exp(j2\pi ft) \, df = \int_{-\infty}^{\infty} X(-f) \exp(-j2\pi ft) \, df. \tag{32}
\]

From Eq. (30) \( X(-f) = X^*(f) \) and, hence, we rewrite Eq. (31) as

\[
x(t) = \int_{0}^{\infty} [X(f) \exp(j2\pi ft) + X^*(f) \exp(-j2\pi ft)] \, df \tag{32}
\]

In this Eq., second term is complex conjugate of first term. As \( x(t) \) is a real function, we can write

\[
x(t) = \text{Re}\{z(t)\} \tag{33}
\]

where analytic function \( z(t) = 2 \int_{0}^{\infty} X(f) \exp(j2\pi ft) \, df \) and \( \text{Re}\{z(t)\} \) denote real part of function \( z(t) \). We write Eq. (33) as

\[
2 \int_{0}^{\infty} X(f) \exp(j2\pi ft) \, df = \sum_{i=1}^{M} a_i(t) \exp(j\phi_i(t)) \tag{34}
\]
where (with $f_0 = 0, f_M = \infty$)

$$a_i(t) \exp(j\phi_i(t)) = 2 \int_{f_{i-1}}^{f_i} X(f) \exp(j2\pi ft) \, df, \quad (35)$$

for $i = 1, \cdots, M$. To obtain minimum number of AFIBFs in low to high frequency scan, for each $i$, start with $f_{i-1}$, increase and select the maximum value of $f_i$ such that $f_{i-1} \leq f_i \leq \infty$ and

$$a_i(t), f_i(t) = \frac{1}{2\pi} \frac{d\phi_i(t)}{dt} \geq 0, \forall t. \quad (36)$$

Similarly, in high to low frequency scan, the lower and upper limits of integration in (35) will change to $f_i$ to $f_{i-1}$, respectively, with $f_0 = \infty, f_M = 0$, and we can obtain minimum number of AFIBFs by selecting the minimum value of $f_i$ such that $0 \leq f_i \leq f_{i-1}$ and Eq. (36) is satisfied.

From Eq. (32) and (33), it is easy to obtain relationship between the energy of original signal $x(t)$ and energy of its analytic signal $z(t)$ as

$$E_x = \frac{E_z}{2}. \quad (37)$$

that is, the energy of analytic signal is twice of the energy of original signal.

The IF characterizes a local frequency behavior as a function of time. In a dual way, the local time behavior as a function of frequency is characterized by the group delay (GD) : $t_i(f) = -\frac{1}{2\pi} \frac{d\phi_i(f)}{df}$. The GD measures the average time arrival of the frequency $f$. In general, the IF and GD define two different curves in the time-frequency plane. Similar to the IF process, for causal signal $x(t)$, we obtain

$$a_i(f) \exp(-j\phi_i(f)) = \int_{t_{i-1}}^{t_i} x(t) \exp(-j2\pi ft) \, dt, \quad (38)$$

with $t_0 = 0, t_M = \infty$ such that $a_i(f), t_i(f) = -\frac{1}{2\pi} \frac{d\phi_i(f)}{df} \geq 0, \forall f, \text{ for } i = 1, \cdots, M$.

**B. The FDM for discrete time real function**

Let $x[n]$ be a non-periodic, real function of time and follow the Dirichlet conditions, then the discrete time Fourier transform (DTFT) of $x[n]$ is defined as

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] \exp(-j\omega n) \quad (39)$$
and inverse discrete time Fourier transform (IDTFT) is defined as

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \exp(j\omega n) \, d\omega \quad (40) \]

It is easy to show that \( \int_{-\pi}^{0} X(\omega) \exp(j\omega n) \, d\omega = \int_{0}^{\pi} X(-\omega) \exp(-j\omega n) \, d\omega \). From Eq. (39), \( X(-\omega) = X^*(\omega) \) and, hence, we rewrite Eq. (40) as

\[ x[n] = \frac{1}{2\pi} \left[ \int_{0}^{\pi} X(\omega) \exp(j\omega n) \, d\omega + \int_{0}^{\pi} X^*(\omega) \exp(-j\omega n) \, d\omega \right]. \quad (41) \]

In this Eq., second term is complex conjugate of first term. As \( x[n] \) is real function, we can write

\[ x[n] = \text{Re}\{z[n]\} \quad (42) \]

where analytic signal \( z[n] = \frac{1}{\pi} \int_{0}^{\pi} X(\omega) \exp(j\omega n) \, d\omega \) and \( \text{Re}\{z[n]\} \) denote real part of function \( z[n] \). We write Eq. (42) as

\[ \frac{1}{\pi} \int_{0}^{\pi} X(\omega) \exp(j\omega n) \, d\omega = \sum_{i=1}^{M} a_i(n) \exp(j\phi_i[n]) \quad (43) \]

where (with \( \omega_0 = 0, \omega_M = \pi \))

\[ a_i[n] \exp(j\phi_i[n]) = \frac{1}{\pi} \int_{\omega_{i-1}}^{\omega_i} X(\omega) \exp(j\omega n) \, d\omega, \quad (44) \]

for \( i = 1, \ldots, M \). To obtain minimum number of AFIBFs in low to high frequency scan, for each \( i \), start with \( \omega_{i-1} \), increase and select the maximum value of \( \omega_i \) such that \( \omega_{i-1} \leq \omega_i \leq \pi \) and phase \( \phi_i[n] \) is a monotonically increasing function, i.e.

\[ a_i[n], \omega_i[n] = (\phi_i[n + 1] - \phi_i[n]) \geq 0, \forall n. \quad (45) \]

Similarly, in high to low frequency scan, the lower and upper limits of integration in Eq. (44) will change to \( \omega_i \) to \( \omega_{i-1} \), respectively, with \( \omega_0 = \pi, \omega_M = 0 \), and we can obtain minimum number of AFIBFs by selecting the minimum value of \( \omega_i \) such that \( 0 \leq \omega_i \leq \omega_{i-1} \) and Eq. (45) is satisfied.

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