How instanton combinatorics solves Painlevé VI, V and III

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Abstract
We elaborate on a recently conjectured relation of Painlevé transcendents and 2D conformal field theory. General solutions of Painlevé VI, V and III are expressed in terms of $c = 1$ conformal blocks and their irregular limits, AGT related to instanton partition functions in $\mathcal{N} = 2$ supersymmetric gauge theories with $N_f = 0, 1, 2, 3, 4$. The resulting combinatorial series representations of Painlevé functions provide an efficient tool for their numerical computation at finite values of the argument. The series involves sums over bipartitions which, in the simplest cases, coincide with Gessel expansions of certain Toeplitz determinants. Considered applications include Fredholm determinants of classical integrable kernels, scaled gap probability in the bulk of the Gaussian Unitary Ensemble, and all-order conformal perturbation theory expansions of correlation functions in the sine–Gordon field theory at the free-fermion point.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Painlevé transcendents [13] are nowadays widely recognized as important special functions with a broad range of applications including integrable models, combinatorics and random matrix theory. Many aspects of Painlevé equations, such as their analytic and geometric properties, asymptotic problems, special solutions and discretization, have been extensively studied in the last four decades.

From the point of view of the theory of classical special functions [60], the surprising feature of these developments is the absence of a transparent connection to the representation theory. Instead, the Riemann–Hilbert approach [22] is typically used. It is well-known that
Painlevé equations emerge most naturally in the study of monodromy preserving deformations of linear ordinary differential equations (ODEs). Thus, by analogy with the solution of classical integrable systems by the inverse scattering method, the questions on nonlinear Painlevé functions may be asked in terms of linear monodromy. In particular, one may attempt to realize the following program:

- label different Painlevé functions by monodromy data of the auxiliary linear problem,
- express their asymptotics near the critical points in terms of monodromy,
- construct the solution using the asymptotic behaviour as the initial condition.

Starting from the foundational work of Jimbo [32], there are many results available on the first two points, but the lack of algebraic structure makes the last one difficult to tackle. In other words, the question is

... how does one combine asymptotic information about the solutions obtained from the Riemann–Hilbert problem, together with efficient numerical codes in order to compute the solution $u(x)$ at finite values of $x$? [14, Painlevé Project Problem].

In [27], a solution to this problem was suggested for the sixth Painlevé equation. It was shown that Painlevé VI tau function $\tau_{VI}(t)$ can be thought of as a correlation function of primary fields in 2D conformal field theory (CFT) [5] with a central charge $c = 1$. Under natural minimal assumptions on the primary content of the theory and fusion rules, $\tau_{VI}(t)$ may then be written as a linear combination of Virasoro conformal blocks. Being purely representation-theoretic quantities, these CFT special functions can be computed in several ways. In particular, the recently proven [1] AGT conjecture [2] relates them to instanton partition functions in $\mathcal{N} = 2$ SUSY 4D Yang–Mills theories [6, 21, 45, 46], which can be expressed as sums over tuples of partitions. This results in combinatorial series representations of $\tau_{VI}(t)$ around the critical points 0, 1, $\infty$.

The aim of this paper is to extend the results of [27] to Painlevé V and Painlevé III, and to make them accessible to a wider audience. With this purpose in mind, we deliberately include some background material and illustrate our claims with a number of explicit examples and applications to random matrix theory and integrable quantum field theory (QFT).

The plan of the paper is as follows: section 2 sets the notation and explains the relation between different Painlevé equations and their various forms. In section 3, we recall some basics on conformal blocks and AGT correspondence. Conjectural general solutions of Painlevé VI, V and III are presented and discussed in section 4. In particular, it is shown that our combinatorial expansions can be seen as a generalization of Gessel’s theorem representation of classical Toeplitz determinant solutions. Section 5 is devoted to applications, which includes Fredholm determinants of classical integrable kernels (hypergeometric, Whittaker, confluent hypergeometric and modified Bessel), scaled Gaussian Unitary Ensemble (GUE) bulk gap probability and correlators of exponential fields in the sine–Gordon model at the free-fermion point.

2. Painlevé equations

2.1. Canonical form

Painlevé VI, V, and III ($P_{VI}$, $P_{V}$, $P_{III}$) first appeared as a part of the classification of second-order ODEs without movable branch points, having the form $q^{''} = F(t, q, q')$, where $F$ is rational in $q$ and $q'$, and locally analytic in $t$. In this context, they are usually written as follows [30]:

$\tau_{VI}(t)$ can be thought of as a correlation function of primary fields in 2D CFT with a central charge $c = 1$. Under natural minimal assumptions on the primary content of the theory and fusion rules, $\tau_{VI}(t)$ may then be written as a linear combination of Virasoro conformal blocks. Being purely representation-theoretic quantities, these CFT special functions can be computed in several ways. In particular, the recently proven [1] AGT conjecture [2] relates them to instanton partition functions in $\mathcal{N} = 2$ SUSY 4D Yang–Mills theories [6, 21, 45, 46], which can be expressed as sums over tuples of partitions. This results in combinatorial series representations of $\tau_{VI}(t)$ around the critical points 0, 1, $\infty$.

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The plan of the paper is as follows: section 2 sets the notation and explains the relation between different Painlevé equations and their various forms. In section 3, we recall some basics on conformal blocks and AGT correspondence. Conjectural general solutions of Painlevé VI, V and III are presented and discussed in section 4. In particular, it is shown that our combinatorial expansions can be seen as a generalization of Gessel’s theorem representation of classical Toeplitz determinant solutions. Section 5 is devoted to applications, which includes Fredholm determinants of classical integrable kernels (hypergeometric, Whittaker, confluent hypergeometric and modified Bessel), scaled GUE bulk gap probability and correlators of exponential fields in the sine–Gordon model at the free-fermion point.

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Painlevé VI:
\[
\frac{d^2 q}{dr^2} = \frac{1}{2} \left( \frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \left( \frac{dq}{dr} \right)^2 - \left( \frac{1}{r} + \frac{1}{r-1} + \frac{1}{r-t} \right) \frac{dq}{dr}
\]
\[+ \frac{2(q-1)(q-t)}{r^2(r-1)^2} \left( \alpha + \beta t + \frac{\gamma (t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-1)^2} \right),
\] (2.1)

Painlevé V:
\[
\frac{d^2 q}{dr^2} = \left( \frac{1}{2q} + \frac{1}{q-1} \right) \left( \frac{dq}{dr} \right)^2 - \frac{1}{q} \frac{dq}{dr} + \frac{(q-1)^2}{r^2} \left( \alpha q + \frac{\beta}{q} \right) + \frac{\gamma q}{r} + \frac{\delta q(q+1)}{q-1},
\] (2.2)

Painlevé III:
\[
\frac{d^2 q}{dr^2} = \frac{1}{q} \left( \frac{dq}{dr} \right)^2 - \frac{1}{q} \frac{dq}{dr} + \frac{\alpha q^2 \beta}{r} + \frac{\gamma q^3}{r} + \frac{\delta}{r q}. \tag{2.3}
\]

Painlevé III’:
\[
\frac{d^2 q}{dr^2} = \frac{1}{q} \left( \frac{dq}{dr} \right)^2 - \frac{1}{q} \frac{dq}{dr} + \frac{q^2 (\alpha + \gamma q)}{4r^2} + \frac{\beta}{4r} + \frac{\delta}{4q}, \tag{2.4}
\]

which reduces to \( P_{III} \) by setting \( q_{tr} = q_{tr}^0 \), \( q_{tr} = q_{tr}^0 q_{tr} \).

2.2. Parameterization

We write four \( P_{VI} \) parameters as
\[
(\alpha, \beta, \gamma, \delta)_{VI} = (\theta_1 + \frac{1}{2}, -\theta_0^2, 0, \frac{1}{4} - \theta_0^2). \tag{2.5}
\]

If \( \delta \neq 0 \) in \( P_V \), then one can set \( \delta = -\frac{1}{2} \) by rescaling the independent variable. \( P_V \) with \( \delta = 0 \) is reducible to \( P_{III} \) (see e.g. transformations (1.24)–(1.26) in [12]) which will be treated separately. Hence we may set
\[
(\alpha, \beta, \gamma, \delta)_{V} = (2\theta_0^2, -2\theta_0^2, 2\theta_* - 1, -\frac{1}{4}). \tag{2.6}
\]

The case of \( P_{III} \) is slightly more involved. In the generic situation, when \( \gamma \delta \neq 0 \), one can assume that \( \gamma = -\delta = 4 \) by rescaling \( t \) and \( q \). We will then write
\[
(\alpha, \beta, \gamma, \delta)_{III} = (8\theta_0^4, 4 - 8\theta_0^4, 4, -4). \tag{2.7}
\]

The variable change \( q \to q^{-1} \) maps \( P_{III} \) with \( \delta = 0 \) to \( P_{III} \) with \( \gamma = 0 \). Assume that \( \gamma = 0 \) and \( \alpha \delta \neq 0 \), then the scaling freedom can be used to set
\[
(\alpha, \beta, \gamma, \delta)_{III} = (8, 4 - 8\theta_0^4, 0, -4). \tag{2.8}
\]

For \( \gamma = \delta = 0, \alpha \beta \neq 0 \) we can set
\[
(\alpha, \beta, \gamma, \delta)_{III} = (8, -8, 0, 0). \tag{2.9}
\]

Finally, for \( \alpha = \gamma = 0 \) (and, similarly, for \( \beta = \delta = 0 \) by \( q \to q^{-1} \)), the general (two-parameter) solution of \( P_{III} \) is known [38]. It reads
\[
q(t) = \mu t^{1-v} + \frac{\beta}{\nu^2} t + \frac{\beta^2 + \nu^2}{4\mu^4} t^{1+v},
\]
where \( \mu, \nu \) are two arbitrary integration constants. Excluding this last solvable case, there remain three inequivalent \( P_{III} \)s with two, one and zero parameters.
2.3. Hamiltonian form

Painlevé equations can be written as non-autonomous Hamiltonian systems [39]. In this approach, (2.1), (2.2) and (2.4) are obtained by eliminating momentum \( p \) from the equations

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad J = \text{VI, V, III}_{1,2,3},
\]

where the Hamiltonians are given by

\[
t(t - 1)H_{\text{VI}} = q(q - 1)(q - t)p\left(p - \frac{2\theta_0}{q} - \frac{2\theta_1}{q - 1} - \frac{2\theta_2}{q - t}\right) + (\theta_0 + \theta_1 + \theta_\infty)(\theta_0 + \theta_1 + \theta_\infty - 1)q, \tag{2.10}
\]

\[
tH_V = (q - 1)(pq - 2\theta_t)(pq - p + 2\theta_\infty) - tpq + ((\theta_\sigma + \theta_\infty)^2 - \theta_0^2)q + \left(\theta_\sigma - \frac{\theta_0}{2}\right)t - 2\left(\theta_\sigma + \frac{\theta_0}{2}\right)^2, \tag{2.11}
\]

\[
tH_{\text{III}} = (pq + \theta_\infty)^2 + tp - \theta_\infty q - \frac{q^2}{4}, \tag{2.12}
\]

\[
tH_{\text{III}} = (pq + \theta_\infty)^2 + tp - q, \tag{2.13}
\]

\[
tH_{\text{III}} = p^2q^2 - q - \frac{t}{q}. \tag{2.14}
\]

The Hamiltonian structure provides one of the ways of construction of Okamoto–Bäcklund transformations [47], generating an infinite number of Painlevé solutions from a given one.

2.4. Sigma form and tau functions

The time-dependent Hamiltonians (2.10)–(2.14) themselves satisfy nonlinear second-order ODEs. To write them, introduce auxiliary functions

\[
\sigma_{\text{VI}} = t(t - 1)H_{\text{VI}} - q(q - 1)p + (\theta_0 + \theta_1 + \theta_\infty)q - (\theta_0 + \theta_1)^2t + \frac{\theta_0^2 + \theta_\infty^2 - \theta_0^2 - \theta_\infty^2 - 4\theta_0\theta_1}{2}, \tag{2.15}
\]

\[
\sigma_1 = tH_1, \quad J = \text{V, III}_{1,2,3}. \tag{2.16}
\]

They satisfy the so-called \( \sigma \)-form of Painlevé equations [23, 31, 47]:

\[
P_{\text{VI}} : \quad (\sigma')^2 = (\sigma - t\sigma' + 2(\sigma')^2 - \frac{1}{4}(2\sigma' - \theta_\infty)^2 - 4\theta_0^2)((2\sigma' + \theta_\infty)^2 - 4\theta_1^2), \tag{2.17}
\]

\[
P_{\text{V}} : \quad (\sigma''^2) = (\sigma - t\sigma' - 2(\sigma')^2 - \frac{1}{4}(2\sigma' - \theta_0^2)((2\sigma' + \theta_\infty)^2 - 4\theta_1^2), \tag{2.18}
\]

\[
P_{\text{III}} : \quad (\sigma''^2) = (4(\sigma')^2 - 1)(\sigma - t\sigma') - 4\theta_0\theta_\infty, \tag{2.19}
\]

\[
P_{\text{III}} : \quad (\sigma''^2) = (4(\sigma')^2 - 1)(\sigma - t\sigma') - 4\theta_0\theta_\infty + 1, \tag{2.20}
\]

\[
P_{\text{III}} : \quad (\sigma''^2) = (4(\sigma')^2 - 1)(\sigma - t\sigma') - 4\sigma', \tag{2.21}
\]

which also appear in the classification of second order, second degree ODEs with Painlevé property [12].
The solutions of (2.1)–(2.4) can thus be mapped to solutions of (2.17)–(2.21). Conversely, one can recover conventional Painlevé functions from the solutions of \( \sigma \)-Painlevé equations using the following formulas:

\[
P_{VI} : \quad q = \frac{1}{q - t} + \frac{1}{2} \left( \frac{1}{t + 1 - t} \right) = \frac{2t(1 - t)\sigma'' + (\sigma' + \theta_2^2 - \theta_2^\infty)(2t - 1)\sigma' - 2\sigma + \theta_0^2 - \theta_0^2 + 4\theta_0^2(\theta_0^2 - \theta_1^2)}{2(1 - t)(\sigma' + (\theta_1 - \theta_\infty)^2)(\sigma' + (\theta_1 + \theta_\infty)^2)},
\]

\[
P_V : \quad q = \frac{2t(\sigma'' + \sigma - t\sigma' + 2(\sigma')^2)}{(2\sigma' - \theta_\infty)^2 - 4\theta_0^2},
\]

\[
P_{III} : \quad q = \frac{2t\sigma'' + 4\theta_\ast \sigma' - 2\theta_\ast}{4(\sigma')^2 - 1},
\]

\[
P_{III} : \quad q = -\frac{1}{2(\sigma')^2},
\]

\[
P_{III} : \quad q = -\frac{1}{\sigma'}.
\]

Finally, define the tau functions of \( P_{VI}, P_V \) and \( P_{III} \) by

\[
\sigma_{VI}(t) = t(t - 1) \frac{d}{dt} \ln \left( \frac{\theta_0^2 - \theta_1^2 - \theta_\infty^2}{\theta_0^2 - \theta_1^2 - \theta_\infty^2} \tau_{VI}(t) \right),
\]

\[
\sigma_V(t) = t \frac{d}{dt} \ln \left( e^{-\theta_0^2 t} - e^{-\theta_1^2 t} - e^{-\theta_\infty^2 t} \tau_V(t) \right),
\]

\[
\sigma_J(t) = t \frac{d}{dt} \ln \tau_J(t), \quad J = III_{1,2,3}.
\]

Our solution below is formulated in terms of combinatorial expansions of these tau functions in powers of \( t \). Expansions of \( \sigma \) and \( q \) can then be obtained from the relations (2.22)–(2.26) and (2.27)–(2.29).

2.5. Coalescence

As is well-known, Painlevé VI produces all other Painlevé equations in certain scaling limits. The equations considered in the present paper form the first line of the coalescence cascade. Every step to the right or to the bottom of the diagram decreases the number of parameters in the corresponding equation by 1. There exist more limits than shown in figure 1; the interested reader may consult [34, 49] for details.
Let us now describe the scaling limits we need below (first line):

- \( P_{\nu} \to P_{\nu}^{I} \): set in \( P_{\nu}^{I} \)

\[
\begin{align*}
\theta_{1} &= \frac{\Lambda + \theta_{*}}{2}, \\
\theta_{\infty} &= \frac{\Lambda - \theta_{*}}{2},
\end{align*}
\]  
(2.30)

then solutions of \( P_{\nu} \) can be obtained as the limits

\[
\begin{align*}
1 - q_{\nu} (t) &= \lim_{t/\Lambda \to \infty} \frac{t/\Lambda}{q_{\nu}^{I}(t/\Lambda)}, \\
\sigma_{\nu} (t) &= \lim_{t/\Lambda \to \infty} \left( \frac{\Lambda^2 - \Lambda t - \theta_{*}^2 - 2\theta_{0}^2 - 2\theta_{1}^2}{4} - \sigma_{\nu}^{I}(t/\Lambda) \right), \\
\tau_{\nu} (t) &= \lim_{t/\Lambda \to \infty} (t/\Lambda)^{\theta_{0}^2 + \theta_{1}^2} \tau_{\nu}^{I}(t/\Lambda).
\end{align*}
\]  
(2.31)

- \( P_{\nu} \to P_{\nu}^{III} \): this limiting transition is described by

\[
\begin{align*}
\theta_{0} &= \frac{\Lambda - \theta_{*}}{2}, \\
\theta_{1} &= \frac{\Lambda + \theta_{*}}{2},
\end{align*}
\]  
(2.34)

\[
\begin{align*}
q_{\nu}^{III} (t) &= \lim_{t/\Lambda \to \infty} \Lambda (1 - q_{\nu} (t/\Lambda)), \\
\sigma_{\nu}^{III} (t) &= \lim_{t/\Lambda \to \infty} \left( \frac{\Lambda^2 + \theta_{0}^2 + \theta_{1}^2}{2} + \sigma_{\nu}^{I}(t/\Lambda) \right), \\
\tau_{\nu}^{III} (t) &= \lim_{t/\Lambda \to \infty} \tau_{\nu}^{I}(t/\Lambda).
\end{align*}
\]  
(2.35)

- \( P_{\nu}^{I} \to P_{\nu}^{III} \): similarly,

\[
\begin{align*}
q_{\nu}^{III} (t) &= \lim_{t/\theta_{*} \to \infty} \theta_{*} q_{\nu}^{III}(t/\theta_{*}), \\
\sigma_{\nu}^{III} (t) &= \lim_{t/\theta_{*} \to \infty} \sigma_{\nu}^{III}(t/\theta_{*}), \\
\tau_{\nu}^{III} (t) &= \lim_{t/\theta_{*} \to \infty} \tau_{\nu}^{III}(t/\theta_{*}).
\end{align*}
\]  
(2.38)

- \( P_{\nu}^{III} \to P_{\nu}^{I} \):

\[
\begin{align*}
q_{\nu}^{III} (t) &= \lim_{t/\theta_{*} \to \infty} q_{\nu}^{I}(t/\theta_{*}), \\
\sigma_{\nu}^{III} (t) &= \lim_{t/\theta_{*} \to \infty} \sigma_{\nu}^{III}(t/\theta_{*}), \\
\tau_{\nu}^{III} (t) &= \lim_{t/\theta_{*} \to \infty} \tau_{\nu}^{III}(t/\theta_{*}).
\end{align*}
\]  
(2.41)

2.6. Analytic properties

The Painlevé property [30] implies that the only branch points of \( P_{\nu}^{I} \) and \( P_{\nu,III_{1,2,3}} \) transcendents in the extended complex \( t \)-plane are 0, 1, \( \infty \) and 0, \( \infty \), respectively. The corresponding tau functions are holomorphic on the universal covers of \( \mathbb{P}^{1} \setminus [0,1, \infty] \) and \( \mathbb{P}^{1} \setminus [0, \infty] \). The functions \( q \) and \( \sigma \) may also have movable poles, e.g., associated to zeros of \( \tau \). We introduce the branch cuts \( (-\infty,0] \cup [1, \infty) \) (for \( P_{\nu}^{I} \)) and \( (-\infty,0] \) (for \( P_{\nu,III_{1,2,3}} \)) and adopt the principal branch convention for all fractional powers of \( t \) and \( 1-t \).
3. Conformal blocks and instanton partition functions

3.1. Conformal blocks

Here we review basic notions about conformal blocks in 2D CFT [5]. For the sake of brevity, simplicity and relevance for the rest of the paper, we will concentrate on conformal blocks for the four-point correlator on $\mathbb{P}^1$ and Virasoro algebra

$$[L_m, L_n] = (n - m) L_{m+n} + \frac{c}{12} (n^3 - n) \delta_{m+n,0}.$$ 

Only chiral primary fields $O$ will be considered, i.e. their antiholomorphic conformal dimensions $\Delta_O = 0$. We do not require invariance of correlators under the braid group action on the positions of fields to avoid constraints on dimensions.

The three-point correlator of primary fields is fixed by conformal symmetry up to a constant factor,

$$\langle O_3(z_3) O_2(z_2) O_1(z_1) \rangle = C(\Delta_3, \Delta_2, \Delta_1) z_{21}^{\Delta_3-\Delta_1} z_{32}^{\Delta_3-\Delta_2} z_{31}^{\Delta_3-\Delta_2-\Delta_1},$$

where $z_{ij} = z_i - z_j$ and $\Delta_j$ stand for holomorphic dimensions. Thanks to conformal invariance, it suffices to consider more special coordinate dependence, namely, we may set $z_1 = 0, z_2 = t, z_3 = R$ with $R \gg t$. It is also customary to define $\langle O(\infty) \ldots \rangle = \lim_{R \to \infty} R^{2 \Delta_O} \langle O(R) \ldots \rangle$, so that, for instance, $\langle O_3(\infty) O_2(t) O_1(0) \rangle = C(\Delta_3, \Delta_2, \Delta_1) t^{\Delta_3-\Delta_1-\Delta_2}$.

Besides primary fields, conformal field theory also contains their descendants $L_{-\lambda} O = L_{-\lambda_1} \ldots L_{-\lambda_N} O$, naturally labelled by partitions $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N > 0\}$. Partitions can be identified in the obvious way with Young diagrams. As they play an important role in the rest of the paper, we take the opportunity to fix some notation for later purposes. The set of all Young diagrams will be denoted by $\mathcal{Y}$. For $\lambda \in \mathcal{Y}$, $\lambda'$ denotes the transposed diagram, $\lambda_i$ and $\lambda'_j$ the number of boxes in $i$th row and $j$th column of $\lambda$, and $|\lambda|$ the total number of boxes. Given a box $(i, j) \in \lambda$, its hook length is defined as $h_{\lambda}(i, j) = \lambda_i + \lambda'_j - i - j + 1$ (see figure 2).

Conformal symmetry also allows us to compute in explicit form the three-point functions involving one descendant:

$$\langle O_3(\infty) O_2(t) L_{-\lambda} O_1(0) \rangle = C(\Delta_3, \Delta_2, \Delta_1) y_{\lambda}(\Delta_1, \Delta_2, \Delta_3) t^{\Delta_3-\Delta_1-\Delta_2-|\lambda|},$$

$$\langle L_{-\lambda} O_3(\infty) O_2(t) O_1(0) \rangle = C(\Delta_3, \Delta_2, \Delta_1) y_{\lambda}(\Delta_3, \Delta_2, \Delta_1) t^{\Delta_3-\Delta_1-\Delta_2+|\lambda|},$$

(3.1) (3.2)
where [40]
\[
\gamma_x(\Delta_1, \Delta_2, \Delta_3) = \prod_{j=1}^{N} \left( \Delta_1 - \Delta_3 + \lambda_j \Delta_2 + \sum_{k=1}^{j-1} \lambda_k \right).
\] (3.3)

The action of \( L_{-\lambda} \) on the field at infinity in (3.2) should be understood as a result of successive contour integration with the energy–momentum tensor. This action can be transferred to the fields at 0 and \( t \) by deformation of the contour.

The relation (3.2) is extremely important as it allows us to determine the coefficients of the operator product expansion (OPE) of primary fields
\[
\mathcal{O}_2(t)\mathcal{O}_1(0) = \sum_{\mu} \sum_{\mu' \in \mathcal{Y}} C(\Delta_a, \Delta_2, \Delta_1) \beta_{\mu}^\ast (\Delta_a, \Delta_2, \Delta_1) t^{\Delta_a - \Delta_1 - \Delta_2 + |\mu|} L_{-\mu} \mathcal{O}_a(0). \] (3.4)

Indeed, assuming orthonormality of the basis of primaries, \( \langle \mathcal{O}_a(\infty) \mathcal{O}_b(0) \rangle = \delta_{ab} \), and considering the correlator of both sides of the last relation with the descendant \( L_{-\lambda} \mathcal{O}_a(\infty) \), one finds that
\[
\beta_{\lambda}^\ast (\Delta_a, \Delta_2, \Delta_1) = \sum_{\mu \in \mathcal{Y}} [Q(\Delta_a)]_{\lambda \mu} \gamma_{\mu} (\Delta_a, \Delta_2, \Delta_1),
\] (3.5)

where \( Q_{\mu}(\Delta_a) = \langle L_{-\lambda} \mathcal{O}_a(\infty) L_{-\mu} \mathcal{O}_a(0) \rangle \) is the Kac–Shapovalov matrix. It can be computed algebraically as the matrix element of descendant states
\[
Q_{\mu}(\Delta) = \langle \Delta | L_{\lambda_1} \cdots L_{\lambda_m} L_{-\mu} \cdots L_{-\mu_1} | \Delta \rangle,
\] (3.6)

where \( |\Delta\rangle \) and \( \langle \Delta | \) denote the highest weight vectors annihilated by all \( L_{\nu} \), and, respectively, all \( L_{\nu} \), satisfying \( L_0 |\Delta\rangle = \Delta |\Delta\rangle \), \( \langle \Delta | L_0 = \langle \Delta | \Delta \) and normalized as \( \langle \Delta | \Delta \rangle = 1 \). It is easy to understand that \( Q(\Delta) \) has a block-diagonal structure: \( Q_{\mu}(\Delta) \sim \delta_{\mu,|\Delta|} \).

We can now finally calculate the four-point correlator \( \langle \mathcal{O}_4(\infty) \mathcal{O}_3(1) \mathcal{O}_2(t) \mathcal{O}_1(0) \rangle \). Replace therein the product of fields \( \mathcal{O}_2(t)\mathcal{O}_1(0) \) by the OPE (3.4) and then use (3.1) and (3.5). The result is
\[
\langle \mathcal{O}_4(\infty) \mathcal{O}_3(1) \mathcal{O}_2(t) \mathcal{O}_1(0) \rangle = \sum_{\mu} C(\Delta_4, \Delta_3, \Delta_a) C(\Delta_a, \Delta_2, \Delta_1) t^{\Delta_a - \Delta_1 - \Delta_2}
\times F_{\mu}(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t), \] (3.7)

where we have introduced the notation
\[
F_{\mu}(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t) = \sum_{\lambda, \mu} \gamma_{\lambda}(\Delta, \Delta_3, \Delta_4) [Q(\Delta)]_{\lambda \mu} \gamma_{\mu}(\Delta, \Delta_2, \Delta_1) t^{\mu |\Delta|}.
\] (3.8)

The representation (3.7) separates model-dependent information (three-point functions \( C(\Delta_1, \Delta_j, \Delta_k) \)) from the universal pieces fixed solely by Virasoro symmetry.

The function (3.8) is called four-point conformal block. It is a power series in \( t \) with coefficients depending on four external dimensions \( \Delta_{1,2,3,4} \), one intermediate dimension \( \Delta \), and the central charge \( c \) which enters via the Kac–Shapovalov matrix. These coefficients can in principle be calculated using (3.3) and (3.6). For the reader’s convenience, we reproduce below several first terms of the series:
\[
F_{\mu}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; \Delta; t) = 1 + \frac{(\Delta - \Delta_1 + \Delta_2)(\Delta - \Delta_4 + \Delta_3)}{\Delta} t + \frac{(\Delta - \Delta_1 + \Delta_2)(\Delta - \Delta_1 + 1)(\Delta - \Delta_4 + \Delta_3)(\Delta - \Delta_4 + \Delta_3)}{2 \Delta(1 + 2 \Delta)} t^2 + \ldots
\]

Direct (i.e. based on (3.8)) computation of conformal block coefficients becomes rather complicated at higher levels. An explicit representation for the arbitrary level was found only recently in a surprisingly different framework.
3.2. $\mathcal{N} = 2$ SUSY theories

The AGT correspondence [2] relates conformal blocks of 2D CFT to Nekrasov functions [45, 46]. These functions represent the instanton parts of $\epsilon_1, \epsilon_2$-regularized partition functions in 4D $\mathcal{N} = 2$ SUSY quiver gauge theories.

The simplest case of AGT correspondence deals with $SU(2)$ gauge theory with extra $N_f = 2N_c = 4$ fundamental (i.e. transforming in the spin-$\frac{1}{2}$ representation of the gauge group) matter hypermultiplets with masses $\mu_1, \ldots, \mu_4$. Parameters of this theory are related to those of a four-point conformal block on the sphere by

$$
\mu_1 = \alpha_3 - \alpha_4 + \frac{\epsilon}{2}, \quad \mu_2 = \alpha_1 - \alpha_2 + \frac{\epsilon}{2}, \\
\mu_3 = \alpha_1 + \alpha_2 - \frac{\epsilon}{2}, \quad \mu_4 = \alpha_3 + \alpha_4 - \frac{\epsilon}{2}, \\
c = 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}, \quad \epsilon = \epsilon_1 + \epsilon_2, \\
\Delta_v = \frac{\alpha_v (\epsilon - \alpha_v)}{\epsilon_1\epsilon_2}, \quad v = 1, 2, 3, 4.
$$

The intermediate dimension $\Delta$ of the conformal block is expressed via the eigenvalues $\pm a$ of the vacuum expectation value of the scalar field in the gauge multiplet:

$$
\Delta = \frac{a (\epsilon - \alpha)}{\epsilon_1\epsilon_2}, \quad \alpha = \frac{\epsilon}{2} + a.
$$

The parameter $t$ (anharmonic ratio of four points on the sphere) in the conformal block expansion is related to the bare complex coupling constant $\tau_{UV}$ on the gauge side by

$$
t = \exp 2\pi i \tau_{UV}, \quad \tau_{UV} = \frac{4\pi i}{\theta_{UV}} + \frac{\theta_{UV}}{2\pi}.
$$

The partition function in the regularized theory is an integral over a compactified moduli space $\mathcal{M}$ of instantons. The integral is given by a sum of explicitly computable contributions coming from fixed points of a torus action on $\mathcal{M}$, which are labelled by pairs of partitions. On the CFT side, this is interpreted as an existence of a geometrically distinguished basis of states in the highest weight representations of the Virasoro algebra. For more details, generalizations and further references, the reader is referred to [4].

When all four masses $\mu_{1,2,3,4} \to \infty$, the fundamental hypermultiplets decouple and we get pure gauge theory. Decoupling only some of them yields asymptotically free theories with $N_f < 2N_c$. From the gauge theory point of view, the parameter $t_{N_f}$ can be considered as a dynamically generated scale. Taking into account the renormalization group dependence of the coupling constant, one finds that in the decoupling process this scale should transform in the appropriate way: $\mu_{N_f} \to \infty, t_{N_f} \to 0, t_{N_f-1} = \mu_{N_f} t_{N_f}$, fixed [16, 54]. The corresponding Nekrasov functions are related to irregular conformal blocks [25, 26, 41].

Conformal blocks relevant to Painlevé VI equation [27] are characterized by the central charge $c = 1$ and external dimensions $\theta^v_\pm (v = 0, 1, \infty)$ so that we can set $\epsilon_1 = -\epsilon_2 = 1$, $\alpha_1 = \theta_0$, $\alpha_2 = \theta_1$, $\alpha_3 = \theta_0$, $\alpha_4 = \theta_\infty$ and

$$
\mu_1 = \theta_1 - \theta_\infty, \quad \mu_2 = \theta_0 - \theta_1, \quad \mu_3 = \theta_0 + \theta_1, \quad \mu_4 = \theta_1 + \theta_\infty.
$$

Under such identification of parameters, the scaling limits corresponding to the first line of the coalescence scheme in figure 1 describe successive decoupling of the matter hypermultiplets (see figure 3).

This observation will be used in the next section for the construction of the combinatorial series for $P_V$ and $P_{\Pi_{1,1}}$ tau functions.
Let us first recall the main result of [27] as well as some motivation and evidence for it.

4. Solutions

4.1. Painlevé VI

Let us first recall the main result of [27] as well as some motivation and evidence for it.

**Conjecture 1.** Generic $P_{VI}$ tau function can be written in the form of conformal expansion around the critical point $t = 0$:

$$\tau_{VI}(t) = \sum_{n \in \mathbb{Z}} C_{VI}(\theta_0, \theta_1, \theta_2, \sigma; n) s_{VI}^n \exp \left( n^2 - \theta_1^2 - \theta_2^2 \right) B_{VI}(\theta_0, \theta_1, \theta_2, \sigma; t). \quad (4.1)$$

The parameters $\sigma$ and $s_{VI}$ play the role of two integration constants. $B_{VI}(\theta_0, \theta_1, \theta_2, \sigma; t)$ coincides with conformal block function $F_{vi}(\theta_1^2, \theta_2^2, \theta_3^2, \sigma; t)$ and is explicitly given by the combinatorial series

$$B_{VI}(\theta_0, \theta_1, \theta_2, \sigma; t) = (1 - t)^{2 \theta_0} \sum_{\lambda, \mu, \nu \in \mathbb{Y}} B_{VI, \lambda, \mu, \nu} (\theta_0, \theta_1, \theta_2, \sigma) t^{\lambda + \mu + \nu}, \quad (4.2)$$

where $G^{[\mu_1, \ldots, \mu_n]}_{[\alpha_1, \ldots, \alpha_n]} = \prod_{i=1}^n G(\alpha_i) \prod_{i=1}^n G(z)$ and $G(z)$ denotes the Barnes function (see appendix A).

The above claim was obtained in [27] by identifying $\tau_{VI}(t)$ with a chiral correlator $\langle O_{L_v}(\infty) O_{L_v}(1) O_{L_v}(0) \rangle$ of Virasoro primary fields indexed by matrices $L_v \in sl(2, \mathbb{C})$ which are related to monodromy matrices of the auxiliary linear problem for $P_{VI}$ by $M_v = \exp 2 \pi i L_v \in SL(2, \mathbb{C})$. The dimensions of $O_{L_v}$ are equal to $\Delta_v = \frac{1}{2} \text{Tr} L_v^2 = \theta_v^2$.

Our main assumption is that this set of primaries closes under OPE algebra. The conservation of monodromy then implies that the dimension spectrum of primary fields appearing in the OPE of $O_{L_v}(t)$ is discrete and has the form $(\sigma_\ell + \mathbb{Z})^2$, where $2 \cos 2 \pi \sigma_\ell = \text{Tr} M_\ell M_\ell$. This fixes the structure of the $s$-channel expansion (4.1) upon identification $\sigma = \sigma_\ell$. The constants $C_{VI}$ are obtained from Jimbo’s asymptotic formula [32] interpreted as a recursion relation, whereas (4.2)–(4.3) is nothing but the AGT representation for $c = 1$ conformal block rewritten in terms of $P_{VI}$ parameters.

In fact, Jimbo’s formula also expresses the second integration constant $s_{VI}$ in terms of monodromy. To give an explicit relation, we need to introduce monodromy invariants

$$p_v = 2 \cos \pi \theta_v = \text{Tr}_v M_v, \quad v = 0, 1, \infty, \quad (4.5)$$

$$p_{\mu \nu} = 2 \cos 2 \pi \sigma_{\mu \nu} = \text{Tr}_{\mu \nu} M_{\mu \nu}, \quad \mu, \nu = 0, 1. \quad (4.6)$$

**Figure 3.** Decoupling of matter hypermultiplets.
Similarly to the above, the quantities $\sigma_{\ell}$ and $\sigma_{01}$ determine the spectrum of intermediate states in the $t$- and $u$-channel. The triple $\tilde{\sigma} = (\sigma_{0\ell}, \sigma_{1\ell}, \sigma_{01})$ provides the most symmetric way to label $P_{VI}$ transcendentals with the same $\tilde{\theta} = (\theta_{0\ell}, \theta_{1}, \theta_{01})$. The elements of this triple are not independent: they satisfy a constraint

$$p_{\ell 0} p_{11} p_{01} + p_{\ell 0}^2 + p_{11}^2 + p_{01}^2 - \omega_{0\ell} p_{\ell 0} - \omega_{1\ell} p_{11} - \omega_{01} p_{01} + \omega_{4} = 4,$$

where

$$\omega_{0\ell} = p_{0\ell} p_{\ell 0} + p_{1\ell} p_{\ell 1} + p_{01} p_{01},$$
$$\omega_{1\ell} = p_{1\ell} p_{\ell 1} + p_{0\ell} p_{\ell 0},$$
$$\omega_{01} = p_{01} p_{11} + p_{0\ell} p_{\ell 0},$$
$$\omega_{4} = p_{01}^2 + p_{11}^2 + p_{0\ell}^2 + p_{\ell 0}^2 + p_{0\ell} p_{\ell 1} p_{1\ell}.$$

Hence, for fixed $\sigma_{0\ell}, \sigma_{1\ell}$, there are at most two possible values for $p_{01}$.

Now $s_{VI}$ can be written as

$$s_{VI} = \frac{(p'_{\ell 0} - p_{\ell 0}) - (p'_{01} - p_{01}) e^{2 \pi i \sigma_{4}}}{(2 \cos 2 \pi (\theta_{0\ell} - \sigma_{0\ell}) - p_{0\ell})(2 \cos 2 \pi (\theta_{1} - \sigma_{1\ell}) - p_{1\ell})},$$

where we have introduced the notation

$$p'_{\ell 0} = \omega_{0\ell} - p_{0\ell} - p_{\ell 0} p_{01},$$
$$p'_{1\ell} = \omega_{1\ell} - p_{1\ell} - p_{0\ell} p_{01},$$
$$p'_{01} = \omega_{01} - p_{01} - p_{0\ell} p_{\ell 1}.$$

**Remark 2.** Combinatorial expansions of type (4.1) can also be found around the two remaining critical points $t = 1, \infty$, as their role is completely analogous to that of $t = 0$. For instance, the series around $t = 1$ is obtained by the exchange $[32, 37]$

$$t \leftrightarrow 1 - t, \quad \theta_{0} \leftrightarrow \theta_{1}, \quad \sigma_{0\ell} \leftrightarrow \sigma_{1\ell}, \quad p_{01} \leftrightarrow p'_{01}.$$ (4.9)

This gives

$$\chi_{0\ell}(\tilde{\theta}, \tilde{\sigma}) t_{VI}(t) = \sum_{n \in \mathbb{Z}} C_{VI}(\theta_{1}, \theta_{0}, \theta_{\infty}, \sigma_{1\ell} + n) \tilde{s}_{VI}^{n}(1 - t)^{(\sigma_{0\ell} + n) - \sigma_{1\ell} - \sigma_{4}}$$

$$\times B_{VI}(\theta_{1}, \theta_{0}, \theta_{\infty}, \sigma_{1\ell} + n; 1 - t),$$

with $B_{VI}$ and $C_{VI}$ defined in (4.2)–(4.4) and

$$\tilde{s}_{VI} = \frac{(p'_{\ell 0} - p_{\ell 0}) - (p'_{01} - p_{01}) e^{-2 \pi i \sigma_{4}}}{(2 \cos 2 \pi (\theta_{0\ell} - \sigma_{0\ell}) - p_{0\ell})(2 \cos 2 \pi (\theta_{1} - \sigma_{1\ell}) - p_{1\ell})}.$$ (4.11)

Since the normalization of $t_{VI}(t)$ is already implicitly fixed by (4.1), the expansion (4.10) contains an additional overall constant factor $[\chi_{0\ell}(\theta, \sigma)]^{-1}$. Finding the explicit form of this connection coefficient is an important open problem which will be treated in a separate paper.

Note, however, that $\chi_{0\ell}(\theta, \sigma)$ disappears from $P_{VI}$ functions $\sigma_{VI}(t)$ and $q_{VI}(t)$.

Painlevé VI equation (2.17) allows us to compute the tau function expansions near the critical points recursively, order by order, starting from the leading asymptotic terms determined by Jimbo’s formula. Comparing the result with conjecture 1 provides the most straightforward and convincing test of the latter. Keeping all $\theta, \sigma$ arbitrary, we have checked in this way (see section 3 of [27] for the details) about 30 first terms of the asymptotic expansion. Also, in a few special cases where $P_{VI}$ solutions are known explicitly, the check can be carried out to arbitrary order. This includes Picard elliptic solutions and the simplest solutions of Riccati/Chazy type, which correspond to Ashkin–Teller conformal blocks [63] and correlators.
involving low-level degenerate fields. More complicated Riccati solutions are discussed in subsection 4.3 of the present paper.

The numerical efficiency of the expansions (4.1), (4.10) is illustrated in figure 4. For random complex $\vec{\theta}, \vec{\sigma}$ we plot on the same graph the series for $\sigma_{VI}(t)$ around $t = 0$ and $t = 1$ keeping the terms up to $O(tL)$ and $O((1-t)L)$ with $L = 5, 15$. Zooming near the endpoints $t = 0, 1$ would display oscillations of rapidly increasing frequency and decreasing amplitude due to non-zero imaginary parts of $\sigma_0, \sigma_1$.

4.2. Painlevé V and IIIIs

Next we consider the scaling limit $P_{VI} \to P_V$ given by (2.30)–(2.33). Conformal block function $B_{VI}(\theta_0, \theta_1, \Delta-\mu, \Delta-\nu, \sigma; \frac{1}{\Lambda})$ has a well-defined limit as $\Lambda \to \infty$, which can be calculated termwise in (4.2) and (4.3). The asymptotics of the structure constants $C_{VI}(\theta_0, \theta_1, \Delta-\mu, \Delta-\nu, \sigma)$ ensures consistency of the expansion (4.1) with the limit (2.33). More precisely, using the estimate (A.1) from appendix A, it is easy to check that

$$
\lim_{\Lambda \to \infty} \Lambda^{-\sigma^2} C_{VI}(\theta_0, \theta_1, \frac{\Delta-\mu}{2}, \frac{\Delta-\nu}{2}, \sigma) G^2(1 + \Lambda) = C_V(\theta_0, \theta_1, \sigma),
$$

(4.12)

where

$$
C_V(\theta_0, \theta_1, \sigma) = \prod_{\epsilon = \pm} G^{1 + \theta_1 + \epsilon \sigma, 1 + \theta_0 + \epsilon \sigma, 1 + \theta_1 - \theta_0 + \epsilon \sigma, 1 + 2 \epsilon \sigma}. \quad (4.13)
$$
One could even completely get rid of the denominator in the l.h.s of (4.12) by modifying the normalization of $P_V$ tau function in (4.1) (e.g. by dividing all structure constants in (4.4) by a $\sigma$-independent factor $G^2 (1 + \theta_1 + \theta_\infty)$).

Altogether, this leads to

**Conjecture 3.** $P_V$ is solved by the following tau function expansion at $t = 0$:

$$\tau_V(t) = \sum_{n \in \mathbb{Z}} C_V(\theta_0, \theta_1, \theta_*, \sigma + n) \; \phi^{(n)}_{0,\sigma} B_V(\theta_0, \theta_1, \theta_*, \sigma + n; t).$$  \hfill (4.14)

Here again $\sigma$ and $s_V$ are arbitrary parameters, irregular conformal block $B_V(\theta_0, \theta_1, \theta_*, \sigma; t)$ is a power series defined by

$$B_V(\theta_0, \theta_1, \theta_*, \sigma; t) = e^{-\theta_j t} \sum_{\lambda, \mu \in \mathbb{Y}} B^{(V)}_{\lambda, \mu}(\theta_0, \theta_1, \theta_*, \sigma) \; t^{k|\lambda| + |\mu|},$$  \hfill (4.15)

where the structure constants $C_V(\theta_0, \theta_1, \theta_*, \sigma)$ are given by (4.13).

The second $P_V$ critical point $t = \infty$ corresponds to irregular singularity of the associated $2 \times 2$ linear system and is obtained by the fusion of two $P_V$ critical points 1, $\infty$. The expansion around this point cannot be extracted from $P_V$ series and requires the knowledge of complete irregular OPEs. For the same reason, we are so far unable to treat $P_V$, $P_I$ and $P_I$. However, long-distance expansions of this kind are available in a few special cases where the solutions of $P_{V,III}$ can be expressed in terms of Fredholm determinants, see section 5.

Because of the presence of irregular singular points, monodromy data for $P_V$ involves Stokes multipliers. The expression for the integration constants $\sigma, s_V$ of conjecture 3 in terms of nontrivial $P_{III}$ equations:

**Conjecture 4.** The expansion of $\tau_{III}(t)$ at $t = 0$ can be written as

$$\tau_{III}(t) = \sum_{n \in \mathbb{Z}} C_{III}(\theta_0, \theta_1, \theta_*, \sigma + n) \; \phi^{(n)}_{0,\sigma} \; B_{III}(\theta_0, \theta_1, \theta_*, \sigma + n; t),$$  \hfill (4.17)

where the irregular conformal block $B_{III}(\theta_0, \theta_1, \theta_*, \sigma; t)$ is given by

$$B_{III}(\theta_0, \theta_1, \theta_*, \sigma; t) = e^{-\theta_j t} \sum_{\lambda, \mu \in \mathbb{Y}} B^{(III)}_{\lambda, \mu}(\theta_0, \theta_1, \theta_*, \sigma) \; t^{k|\lambda| + |\mu|},$$  \hfill (4.18)

where the structure constants $C_{III}(\theta_0, \theta_1, \theta_*, \sigma)$ can be written as

$$C_{III}(\theta_0, \theta_1, \theta_*, \sigma) = \prod_{\epsilon = \pm} G \left[ 1 + \theta_1 + \epsilon \sigma, 1 + \theta_1 + \epsilon \sigma \right] \left. \right| 1 + 2 \epsilon \sigma.$$  \hfill (4.20)
Conjecture 5. The expansion of $\tau_{III}^2(t)$ at $t = 0$ is given by

$$\tau_{III}^2(t) = \sum_{n \in \mathbb{Z}} C_{III}^2 (\theta_a \sigma + n) s^0_{III} t^{(\sigma+n)^2} B_{III} (\theta_a \sigma + n ; t),$$  \hspace{1cm} (4.21)

with arbitrary $\sigma, s_{III}$ and

$$B_{III} (\theta_a \sigma ; t) = \sum_{\lambda, \mu \in \mathbb{Y}} B^{|\lambda|+|\mu|}_{\lambda, \mu} (\theta_a \sigma \mu) t^{|\lambda|+|\mu|}.$$  \hspace{1cm} (4.22)

$$B^{|\lambda|+|\mu|}_{\lambda, \mu} (\theta_a \sigma \mu) = \prod_{(i,j) \in \lambda} \frac{\theta_a + \sigma + i - j}{h^*_{\lambda} (i,j) (\lambda^*_{j} + \mu_i - i - j + 1 + 2\sigma)^2} \times \prod_{(i,j) \in \mu} \frac{\theta_a - \sigma + i - j}{h^*_{\mu} (i,j) (\lambda_i + \mu^*_{j} - i - j + 1 - 2\sigma)^2}.$$  \hspace{1cm} (4.23)

$$C_{III}^2 (\theta_a \sigma) = \prod_{\epsilon = \pm} G(1 + \theta_a + \epsilon \sigma) G(1 + 2\epsilon \sigma)^{-1}.$$  \hspace{1cm} (4.24)

Conjecture 6. The expansion of $P_{III}^1$ tau function at $t = 0$ is

$$\tau_{III}^2(t) = \sum_{n \in \mathbb{Z}} C_{III}^2 (\sigma + n) s^0_{III} t^{(\sigma+n)^2} B_{III} (\sigma + n ; t),$$  \hspace{1cm} (4.25)

where

$$B_{III} (\sigma ; t) = \sum_{\lambda, \mu \in \mathbb{Y}} B^{|\lambda|+|\mu|}_{\lambda, \mu} (\sigma \mu) t^{|\lambda|+|\mu|}.$$  \hspace{1cm} (4.26)

$$B^{|\lambda|+|\mu|}_{\lambda, \mu} (\sigma \mu) = \left[ \prod_{(i,j) \in \lambda} h_{\lambda} (i,j) (\lambda^*_{j} + \mu_i - i - j + 1 + 2\sigma) \times \prod_{(i,j) \in \mu} h_{\mu} (i,j) (\lambda_i + \mu^*_{j} - i - j + 1 - 2\sigma) \right]^{-1}.$$  \hspace{1cm} (4.27)

$$C_{III} (\sigma) = [G(1 + 2\sigma) G(1 - 2\sigma)]^{-1}.$$  \hspace{1cm} (4.28)

As in the $P_{VI}$ case, conjectures 3–6 can be verified by iterative reconstruction of the tau function expansions from the leading asymptotic terms using the equations $P_{V,III,1,2,3}$.

4.3. Classical solutions: AGT versus Gessel’s theorem

In 2002, Forrester and Witte [24] proved a remarkable determinant representation for a family of Riccati solutions of $P_{VI}$. Their result can be restated as follows. Define a five-parameter family of $N \times N$ Toeplitz determinants

$$D_{N}^{(\nu, \nu', \eta, \xi)} (t) = \det \left[ A_{j-k}^{(\nu, \nu', \eta, \xi)} (t) \right]_{j,k=0}^{N-1},$$  \hspace{1cm} (4.29)

$$A_{m}^{(\nu, \nu', \eta, \xi)} (t) = \frac{\Gamma (1 + \nu) \Gamma (1 + \nu') t^{-\nu'} (1 - t)^{\nu'}}{\Gamma (1 + \eta - m) \Gamma (1 - \eta + m + \nu')} \binom{-\nu, 1 + \nu'}{1 + \eta - m} t \binom{-\nu', 1 + \nu}{1 + \eta - m} t^{-1} + \frac{\xi \Gamma (1 + \nu) \Gamma (1 + \nu') t^{-\nu'} (1 - t)^{\nu'}}{\Gamma (1 - \eta + m) \Gamma (1 + \eta - m + \nu')} \binom{-\nu, 1 + \nu'}{1 - \eta + m} t \binom{-\nu', 1 + \nu}{1 - \eta - m} t^{-1}.$$  \hspace{1cm} (4.30)
Then the function
\[ \tau_N^{(\nu, \nu', \eta, \xi)}(t) = (1 - t)^{-\frac{N(N + 1)}{2}} D_N^{(\nu, \nu', \eta, \xi)}(t) \] (4.31)
is a tau function of \( P_{VI} \) with parameters \( (\theta_0, \theta_1, \theta_2, \theta_3)_{VI} = \frac{1}{2}(\eta, N - N - \nu - \nu', \nu + \nu' + \eta) \).

Looking at the asymptotic expansions of \( D_N^{(\nu, \nu', \eta, \xi)}(t) \) at 0 and 1, one can also identify the monodromy exponents
\[ (\sigma_0, \sigma_1, \sigma_2)_{VI} = \frac{1}{2}(N + \eta, v + \nu', N + \nu - \nu' + \eta). \]

Almost all structure constants in (4.1) vanish because of the relations \( \theta_i = \frac{N}{2}, \sigma_0 = \theta_0 + \theta_i \) (recall that Barnes G-function has zeros at negative integer values of the argument). The only non-zero constants correspond to \( n = 0, -1, \ldots, -N \), so that there remain only \( N + 1 \) conformal blocks. The parameter \( s_{\lambda}^V \) in (4.1) is related to \( \xi \) in (4.30) by
\[ \xi_{s_{\lambda}^V} = \frac{\sin \pi \nu \sin \pi (\eta - \nu')}{\sin \pi \nu' \sin \pi (\eta + \nu')}. \] (4.32)

Let us now consider in more detail the case \( \xi \to 0 \). Then (4.32) implies that \( s_{\lambda}^V \to \infty \), which means that the expansion (4.1) at \( t = 0 \) contains only one \((n = 0)\) conformal block \( B_{VI}(\frac{N}{2}, \frac{N}{2}, \frac{N + 1 + \nu}{2}, \frac{N + 1 + \nu'}{2}, t) \). The product over boxes of \( \lambda \) in the AGT representation (4.3) contains a factor \( i - j \) due to the relation \( \sigma_0 = \theta_0 + \theta_i \). Since this expression vanishes for the box (1, 1), the quantity \( B_{VI}^{(\nu, \nu', \eta)} \) does so for any non-empty \( \mu \). Moreover, the factor \( i - j \) in the product over boxes of \( \lambda \) reduces the summation in (4.2) to Young’s diagrams with \( \lambda_1 \leq N \) (i.e. with the length of their first row not exceeding \( N \)). Therefore, conjecture 1 for the above parameters is equivalent to the following identity:
\[ D_N^{(\nu, \nu', \eta, 0)}(t) = C_N \sum_{\lambda \vdash \eta, \lambda_1 \leq N} t^{\frac{\nu_1}{2}} \prod_{(i, j) \in \lambda} \frac{i - j + N}{i - j + N + \eta} \frac{(i - j - \nu)(i - j - \nu' + \eta)}{h_{\lambda}(i, j)}, \] (4.33)
where the constant prefactor
\[ C_N = G\left[ 1 + N, 1 + \nu + \nu', 1 + \eta, 1 - \nu + \nu', 1 + \nu', 1 + \nu' \right] \] (4.34)
can be computed using pure Fisher–Hartwig determinant.

In the limit \( \eta \to 0 \), the left-hand side of (4.33) reduces to \( N \times N \) Toeplitz determinant with the symbol
\[ A(\xi) = (1 + \sqrt{\xi})^\nu (1 + \sqrt{\xi}^{-1})^{\nu'}. \] (4.35)
Also, \( C_N = 1 \) and the first factor in the product on the right disappears so that the rhs coincides with the length distribution function of the first row of a random Young’s diagram distributed according to the so-called z-measure [8]. The equality (4.33) can then be rigorously demonstrated using a dual version of Gessel’s theorem [29, 59].

**Remark 7.** We draw the reader’s attention to the fact that the Toeplitz determinant with the symbol (4.35) with \( \nu = -\nu' = \frac{1}{2} \) coincides with a diagonal two-point Ising spin correlation function on the infinite square lattice. Its relation to \( P_{VI} \) is rather well-known [33]. It is intriguing, however, that this lattice correlator is equal to (a particular limit of) a conformal block in continuous 2D CFT with \( c = 1 \).

Analogous results for \( P_V \) and \( P_{VI}^c \) can be obtained by successively sending \( \nu' \) and \( \nu \) to infinity. For instance, consider instead of \( A^{(\nu, \nu', \eta, \xi)}(t) \) and \( \tau_N^{(\nu, \nu', \eta, \xi)}(t) \) the quantities
\[ A_m^{(\nu, \eta, \xi)}(t) = \frac{t^{\xi/v}}{\Gamma(1 + \eta - m)} \frac{\Gamma(1 + \eta - m, -t)}{\Gamma(1 + \eta - m)} I_1(-v, 1 + \eta - m, -t) + \frac{\xi \Gamma(1 + \nu) e^{-\xi v}}{\Gamma(1 + \eta - m) \Gamma(1 + m + v)} I_1(1 + v, 1 + \eta + m, t). \] (4.36)
\[ \tau^\xi_{N}(\nu,\eta,\xi)(t) = t^{\xi^2} e^{\xi t} \det[A_{j-k}^{(\nu,\eta,\xi)}(t)]_{j,k=0}^{N=1} \] (4.37)

then \( \tau^\xi_{N}(\nu,\eta,\xi)(t) \) is a tau function of \( P_N \) with \((\theta_0, \theta_1, \theta_\ast)_{\nu} = \frac{1}{2} (\eta, N, N + \eta + 2\nu) \). Similarly, if we define

\[ A^{\xi}_{m}(\xi) = I_{m-n}(2\sqrt{t}) + \xi I_{m-\eta}(2\sqrt{t}), \] (4.38)

\[ \tau^\xi_{N}(\nu,\eta,\xi)(t) = t^{\xi^2} e^{\xi t} \det[A_{j-k}^{(\nu,\eta,\xi)}(t)]_{j,k=0}^{N=1} \] (4.39)

then \( \tau^\xi_{N}(\nu,\eta,\xi)(t) \) is a \( P_{N\xi} \) tau function with \( \theta_\ast = \frac{N+\eta}{2}, \theta_\ast = \frac{N-\eta}{2} \). For \( \xi = 0 \) and \( \eta \to 0 \) the symbols of Toeplitz determinants \( (4.37), (4.39) \) are smooth and can be written as \((1 + \sqrt{t} \xi)^{\nu} e^{\sqrt{t} \xi^{-1}}\) and \( e^{\sqrt{t}(\xi + \xi^{-1})} \). Gessel representations of these determinants coincide with the results derived from conjectures 3 and 4.

In the general case \( \xi \neq 0 \), the function \( \tau^{(\nu,\nu',\eta,\xi)}(t) \) is a polynomial of degree \( N \) in \( \xi \). The coefficients of \( N + 1 \) different powers of \( \xi \) are \( s \)-channel conformal blocks with internal dimensions \((\theta_k + \theta_\ast - k)^2\), where \( k = 0, \ldots, N \). Alternatively, one can first transform hypergeometric functions to make them depend on \( 1-t \) and then expand the determinant in powers of \( \xi \), the analogue of parameter \( \xi \). The result has the form \( (4.10) \) of a sum of \( s \)-channel conformal blocks with internal dimensions \((\theta_k + \theta_\ast - k)^2\), again with \( k = 0, \ldots, N \). The relations between the expansion parameters are given by \( (4.32), \xi s_{\nu l} = \xi s_{\nu l} = k, (1-s_{\nu l})(1-s_{\nu l}) = 1+k \).

The CFT interpretation of this picture is as follows. The tau function \( (4.31) \) is a four-point correlator of primaries which involves a level \( N+1 \) degenerate field (here \( O_{\nu l}(t) \)). Its expansions at \( t = 0 \) and \( t = 1 \) incorporates all allowed intermediate dimensions. Determinant representation \( (4.29) \) can in fact be used to compute the fusion matrix for the corresponding two sets of conformal blocks. This task simplifies in the case \( \xi = 0 \), where we are left with one \( s \)-channel block transforming into a linear combination of \( N+1 \) \( s \)-channel ones.

5. Examples and applications

5.1. Integrable kernels

In many applications of Painlevé equations the relevant tau functions can be written as Fredholm determinants of scalar integral operators of the form \( \det(1 - K_{jl}) \), where \( K_{jl} \) denotes the restriction of the kernel \( K(x,y) \) to some interval \( I \subset \mathbb{R} \). These kernels usually have integrable form, that is

\[ K(x,y) = \lambda \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y}, \quad \lambda \in \mathbb{C}. \] (5.1)

As is well-known, given \( I = \bigcup_{j=1}^{2n} (a_{2j-1}, a_{2j}) \) and \( \varphi, \psi \) verifying the differentiation formulas

\[ \left( \begin{array}{c} \varphi'(x) \\ \psi'(x) \end{array} \right) = A(x) \left( \begin{array}{c} \varphi(x) \\ \psi(x) \end{array} \right), \]

with some rational matrix \( A(x) \), the corresponding Fredholm determinant satisfies a system of PDEs with respect to \( |a_j| \) [58]. For \( \varphi, \psi \) given by classical special functions and sufficiently simple \( I \), this system can often be solved in terms of Painlevé functions [23, 37, 56–58, 61].
5.1.1. Hypergeometric kernel. The most general known example corresponds to the choice
\[ \psi_G(x) = \Gamma \left[ \frac{1 + v + \eta, 1 + v' + \eta'}{1 + \eta} \right] \frac{x^{\frac{1 + v + \eta'}{1 + \eta} - 1}}{(1 - x)^{\frac{1 + v + \eta'}{1 + \eta} - 1}} F_2 \left[ \frac{1 + v + \eta', 1 + v' + \eta'}{2 + v + v' + \eta + \eta'} \frac{x}{|x - 1|} \right], \]
\[ \psi_G(x) = \Gamma \left[ \frac{1 + v + \eta, 1 + v' + \eta'}{1 + \eta} \right] \frac{x^{\frac{1 + v + \eta'}{1 + \eta} - 1}}{(1 - x)^{\frac{1 + v + \eta'}{1 + \eta} - 1}} F_2 \left[ \frac{v + \eta', v' + \eta'}{v + v' + \eta + \eta'} \frac{x}{|x - 1|} \right], \]
with \( \lambda = \pi^{-2} \sin \nu \sin \nu' \). The kernel \( K_G(x, y) \) contains four parameters \( v, v', \eta, \eta' \in \mathbb{C} \) chosen so that the Fredholm determinant
\[ D_G(t) = \det(1 - K_G(t, t)), \quad t \in (0, 1). \]  
(5.2)
is well-defined. We will not try to determine the set of all possible values of \( v, v', \eta, \eta' \); the interested reader may find examples of admissible domains in [10].

The above \( F_1 \) kernel first appeared in the harmonic analysis on the infinite-dimensional unitary group \([10, 11]\). Later it was shown \([37]\) that the determinant \((5.2)\) coincides with a correlator of twist fields in the massive Dirac theory on the hyperbolic disk \([17, 36, 51]\). From the point of view of the present paper, the most interesting feature of \( D_G(t) \) is that it is a Painlevé VI tau function, see \([10]\) and also section 5 of \([37]\) for a simpler proof. The corresponding \( F_{VI} \) parameters are
\[ (\theta_0, \theta_1, \theta_2, \theta_\infty)_{VI} = \frac{1}{2} (v + v' + \eta + \eta', 0, v - v', \eta - \eta'). \]

Monodromy characterizing this particular solution is determined by \([37]\)
\[ \sigma_{0t} = \frac{v + v' + \eta + \eta'}{2}, \quad \sigma_{1t} = \frac{v + v'}{2}, \]
\[ \cos 2\pi \sigma_{0t} = 2 e^{-\pi i (\eta + \eta' + v + v')} \sin \pi v \sin \pi v' + \cos \pi (\eta - \eta'). \]

Substituting these parameters into \((4.11)\), it can be easily checked that \( \tilde{v}_{VI} = 1. \) Remark 2 then implies that the large gap \((t \to 1)\) expansion of the \( F_1 \) kernel determinant is given by
\[ D_G(t) = \chi_G^{-1} \sum_{n \in \mathbb{N}} C_G(v + n, v' + n, \eta - n, \eta' - n) (1 - t)^{(v+n)(v'+n)} \times B_G(v + n, v' + n, \eta - n, \eta' - n; 1 - t), \]  
(5.3)
where
\[ C_G(v, v', \eta, \eta') = G[1 + \eta, 1 + \eta', 1 + \eta + v + v', 1 + \eta' + v + v'] \Gamma \prod_{\epsilon = \pm} \left[ 1 + \epsilon v, 1 + \epsilon v' \right], \]
\[ B_G(v, v', \eta, \eta'; 1 - t) = B_{VI}(v + v', 0, \frac{v + v' + \eta + \eta'}{2}, \frac{\eta - \eta' + v + v'}{2}; 1 - t), \]
and \( B_{VI} \) is given by \((4.2)\) and \((4.3)\). Also, \([37, \text{conjecture 8}]\) suggests that the constant \( \chi_G \) is equal to
\[ \chi_G = G[1 + \eta + v, 1 + \eta + v', 1 + \eta' + v, 1 + \eta' + v']. \]  
(5.4)

Constructing the expansion at \( t = 0 \) is less straightforward. It is of course possible to compute a few first terms in the small gap asymptotics directly by expanding \( D_G(t) \) into Fredholm series. This yields, for instance,
\[ D_G(t) = 1 - \kappa_G t^{1+\eta+\eta'+v+v'} (1 + o(1)), \]
with
\[ \kappa_G = \lambda \Gamma \left[ \frac{1 + \eta + v, 1 + \eta' + v, 1 + \eta + v', 1 + \eta' + v'}{2 + \eta + \eta' + v + v', 2 + \eta + \eta' + v + v'} \right]. \]
On the other hand, direct application of conjecture 1 is ambiguous because of special parameter values. First, Barnes functions \( \lambda, \mu \) vanish for \( n \geq 0 \). At the same time the quantity \( s_{\nu, ij} \) diverges due to a zero denominator. The right way to handle this is to fix the values of \( \theta \) and \( \sigma_\mu \), and then consider the limit \( \sigma_\mu \to \theta_0 \) with the help of the formulas (A.2)–(A.3) from appendix A. The result is that only the terms with \( n \geq 0 \) survive in the sum over \( n \) and the structure constants reduce to

\[
\tilde{C}_G(v, v', \eta, \eta', n) = (-\lambda)^n G \left[ \frac{1 + n, 1 + \eta + \eta' + v + v' + n^2}{1 + \eta + \eta' + v + v' + 2n} \right] \\
\times G \left[ \frac{1 + \eta + v + n, 1 + \eta' + v + n, 1 + \eta + v' + n, 1 + \eta' + v' + n}{1 + \eta + v + \eta' + v, 1 + \eta + v' + \eta' + v'} \right].
\]

(5.5)

In addition, because of the factors \( \eta, \eta' \) in the products over boxes of \( \lambda, \mu \in \mathcal{Y} \) combinatorial summation in conformal blocks can be restricted to Young’s diagrams with \( \lambda_1 \leq n, \mu_1 \leq n \).

This leads to the following expansion of \( D_G(t) \) near \( t = 0 \):

\[
D_G(t) = \sum_{n=0}^{\infty} \tilde{C}_G(v, v', \eta, \eta', n) t^{m(\eta + \eta' + v + v')} \sum_{\lambda, \mu \in \mathcal{Y} | \lambda_1 \leq n, \mu_1 \leq n} B_{\lambda, \mu}^G(v, v', \eta, \eta', n) t^{|\lambda| + |\mu|},
\]

(5.6)

where \( \tilde{C}_G(v, v', \eta, \eta', n) \) is given by (5.5) and

\[
B_{\lambda, \mu}^G(v, v', \eta, \eta', n) = \prod_{(i, j) \in \lambda} \frac{(i - j - n)(i - j + n + \eta + \eta' + v + v')(i - j + n + \eta + v)(i - j + n + \eta' + v)}{h^2_{\lambda'}(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2n + \eta + \eta' + v + v')^2} \\
\times \prod_{(i, j) \in \mu} \frac{(i - j - n)(i - j + n - \eta - \eta' - v - v')(i - j - n - \eta - v')(i - j - n - \eta' - v')}{h^2_{\mu'}(i, j)(\lambda'_j + \mu_i - i - j + 1 + 2n - \eta - \eta' - v - v')^2}.
\]

Note that individual conformal blocks in the sum over \( n \) in (5.6) give the corresponding terms in the Fredholm series of the \( \psi \) kernel determinant. Numerical checks for randomly chosen \( \eta, \eta', v, v' \) show that the expansions (5.3) and (5.6) perfectly match for intermediate values of \( t \). In particular, this confirms the conjectural expression (5.4).

5.1.2. Whittaker kernel. The Whittaker kernel [9, 10] emerges in the limit

\[
K_W(x, y) = \lim_{\eta' \to \infty} \frac{1}{\eta'} K_G \left( 1 - \frac{x}{\eta'}, 1 - \frac{y}{\eta'} \right).
\]

It contains three parameters \( v, v', \eta \) and has integrable form (5.1), \( \lambda \) is the same as above and

\[
\varphi_W(x) = \Gamma(1 + \eta + v)x^{\frac{1}{2}}W_{-\frac{\eta + v}{2} + 1, -\frac{1}{2}}(x),
\]

\[
\psi_W(x) = \Gamma(1 + \eta + v')x^{\frac{1}{2}}W_{-\frac{\eta + v'}{2} + 1, -\frac{1}{2}}(x),
\]

where \( W_{k, \nu}(x) \) denote the Whittaker functions.

Fredholm determinant

\[
D_W(t) = \det(1 - K_W|_{t, \infty}), \quad t \in (0, \infty),
\]

is related to a particular Painlevé V tau function by

\[
D_W(t) = t^{-\frac{v - v'^2}{4}} \tau_V(t),
\]

(5.7)

\[
(\theta_0, \theta, \theta_*) = \frac{1}{2} (v - v', 0, 2\eta + v + v').
\]

(5.8)
Its expansion around $t = 0$ may be found from
\[
D_W(t) = \lim_{\eta' \to \infty} D_G \left( 1 - \frac{t}{\eta'} \right).
\]
Namely, the appropriate termwise limit of (5.3) gives
\[
D_W(t) = \sum_{n \in \mathbb{Z}} C_W(v + n, \nu, \eta - n) t^{v(n\eta + \nu + \eta)} B_W(v + n, \nu, \eta - n; t),
\]
where the limits of structure constants and conformal blocks are
\[
C_W(v, \nu, \eta) = G \left[ \begin{array}{c} 1 + v, 1 - v, 1 + \nu, 1 - \nu, 1 + \eta, 1 + \nu + \eta \end{array} \right],
\]
\[
B_W(v, \nu, \eta; t) = B_V \left( \begin{array}{c} v - \nu, 0, \frac{v + \nu}{2} \end{array} \right).
\]
and $B_V$ was defined in (4.15) and (4.16). Although we are not able to write a similar combinatorial expansion at $t = \infty$, in the latter case $D_W(t)$ can still be expanded into the Fredholm series. Hence, for example,
\[
D_W(t \to \infty) = 1 - \lambda(1 + \eta + v) \sum_{n = 0}^{\infty} \frac{(\nu + v + \eta)^n}{n!} t^n [1 + O(t^{-1})].
\]

5.1.3. Confluent hypergeometric kernel. Another interesting scaling limit of the $2F_1$ kernel corresponds to setting
\[
v' = v_0' - i\Lambda, \quad \eta = \eta_0 + i\Lambda,
\]
and then considering
\[
K_{F}(x, y) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} K_G \left( \frac{x}{\Lambda}, \frac{y}{\Lambda} \right).
\]
The result is the so-called confluent hypergeometric kernel [10, 15]. It depends on three parameters
\[
r_+ = v + \eta', \quad r_- = v_0' + \eta_0, \quad \xi = \frac{1 - e^{2\pi i \nu}}{2\pi} e^{\frac{\pi(i\nu - 1)}{\nu}},
\]
and is given by (5.1) with
\[
\lambda = \xi \Gamma \left[ \begin{array}{c} 1 + r_+, 1 + r_- \end{array} \right],
\]
\[
\psi_\nu(x) = x^{r_+ - r_-} e^{-\frac{\pi}{4} i} F_1(r_+ + 1, r_+ + r_- + 2, i\nu),
\]
\[
\psi_\nu(x) = x^{r_- - r_+} e^{-\frac{\pi}{4} i} F_1(r_- + 1, r_- + r_+ + 2, i\nu).
\]
Similarly to (5.7) and (5.8), the $2F_1$ kernel determinant
\[
D_{F}(t) = \det(1 - K_{F}\mid_{(0,t)}), \quad t \in (0, \infty),
\]
can be expressed [10] in terms of a Painlevé V tau function:
\[
D_{F}(t) = \frac{1}{\sin(\theta_0 - \theta_+ - \theta_+)} \tau_V(\xi), \quad t \in (0, \infty),
\]
and is given by (5.10) with
\[
(\theta_0, \theta_+, \theta_+) = \frac{1}{2} (r_+ + r_-, 0, r_-).
\]
Note that $D_{F}(t) = \lim_{\lambda \to \infty} D_G \left( \frac{t}{\lambda} \right)$. Applying this termwise to (5.6) and using the properties (A.1) and (A.2) of the Barnes function, we derive the expansion of $D_{F}(t)$ at $t = 0$:
\[
D_{F}(t) = \sum_{n=0}^{\infty} C_F(r_+, r_-, n) (-\xi)^n t^{r_+ r_- + n} \sum_{\lambda, \mu \in \mathbb{Z} \setminus \lambda} B_{\lambda, \mu}^n (r_+, r_-, n; \xi) \tau_V(\xi),
\]
where
\[ C_F(r_+, r_-, n) = G \left[ 1 + n, 1 + r_+ + r_+ - n \over 1 + r_+ + r_- + 2n \right]^2 G \left[ 1 + r_+ + n, 1 + r_+ + n \over 1 + r_+, 1 + r_- \right], \]
\[ B_{\lambda,\mu}^F(r_+, r_-, n) = \prod_{(i,j) \in \lambda} \frac{(i-j+n)(i-j+n+r_+)(i-j+n+r_-)}{h_I^2(i,j)(\lambda'_j + \mu_j - i - j + 2n + r_+ + r_-)^2} \]
\times \prod_{(i,j) \in \mu} \frac{(i-j+n)(i-j+n-r_+)(i-j+n-r_-)}{h_I^2(i,j)(\lambda'_j + \mu'_j - i - j + 2n + r_+ - r_-)^2}. \]

5.1.4. Sine kernel. Certain specializations of the \( 1F_1 \) kernel play an important role in the random matrix theory. In particular, for \( r_+ = r_- = r \) it coincides with the Bessel kernel \([42, 61]\)
\[ K_B(x, y) = \frac{\pi \xi \sqrt{x^2 y^2}}{2} \frac{J_{r+\frac{3}{2}}(\frac{x}{2}) J_{r-\frac{3}{2}}(\frac{y}{2}) - J_{r-\frac{3}{2}}(\frac{x}{2}) J_{r+\frac{3}{2}}(\frac{y}{2})}{x - y}, \]
which in the case \( r = 0 \) reduces to the celebrated sine kernel
\[ K_{\text{sine}}(x, y) = \frac{2\pi \sin \frac{x - y}{2}}{x - y}. \]

It is well-known that the determinant
\[ D_{\text{sine}}(t) = \det(1 - K_{\text{sine}}(0,t)) \]
for \( \xi = \frac{1}{2\pi} \) coincides with the scaled gap probability in the bulk of the GUE \([23]\). The expansion (5.12) thus gives a complete series for this quantity:
\[ D_{\text{sine}}(t) = \sum_{n=0}^{\infty} \frac{G^n(1 + n)}{G^2(1 + 2n)} (-\xi)^n t^n \sum_{\lambda,\mu \in \mathcal{P}(\lambda,\mu \leq n)} B_{\lambda,\mu}^{\text{sine}}(n)(t)^{|\lambda| + |\mu|}, \tag{5.13} \]
where
\[ B_{\lambda,\mu}^{\text{sine}}(n) = \prod_{(i,j) \in \lambda} \frac{(i-j+n)^3}{h_I^2(i,j)(\lambda'_j + \mu_j - i - j + 1 + 2n)^2} \]
\times \prod_{(i,j) \in \mu} \frac{(i-j+n)^3}{h_I^2(i,j)(\lambda'_j + \mu'_j - i - j + 1 - 2n)^2}. \]

The first terms of the series (5.13) are recorded in appendix B. In particular, they reproduce the results obtained by an iterative expansion of the corresponding Painlevé V solution, see equation (8.114) in \([23]\). Note that our \( t = 2\sqrt{\pi}t_{(23)} \), \( \xi = \frac{6t_{(23)}}{\sqrt{\pi}r} \). We have also checked the agreement of (5.13) with the known large gap \( t \to \infty \) asymptotics \([18]\)
\[ D_{\text{sine}}(4t) \bigg|_{\xi = \frac{6t}{2\pi}} = \sqrt{\pi} G^2 \left( \frac{1}{2} \right) t^{-\frac{3}{2}} e^{-\frac{r^2}{2}} \left[ 1 + \frac{1}{32} t^{-2} + \frac{81}{2048} t^{-4} + O(t^{-6}) \right]. \]

5.1.5. Modified Bessel kernel. One may also study a further scaling limit of the \( 1F_1 \) kernel
by setting
\[ r_+ = \frac{r}{2} + i\Lambda, \quad \xi = \frac{r e^{\pi \Lambda}}{2\pi}, \]
and defining
\[ K_{B2}(x, y) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} K_F \left( \frac{x}{\Lambda}, \frac{y}{\Lambda} \right). \]
Asymptotic properties of the confluent hypergeometric function imply that
\[ K_{B2}(x, y) = \xi_{B2} \sqrt{\frac{2}{\pi}} \frac{L_{r+1}(2\sqrt{y})L_{r-1}(2\sqrt{x}) - L_{r-1}(2\sqrt{y})L_{r+1}(2\sqrt{x})}{x - y}. \]

Fredholm determinant \( D_{B2}(t) = \det(1 - K_{B2}|_{(0,t)}) \) is related to a tau function of Painlevé III, with \( \theta_* = -\theta^* = \frac{\pi}{2} \) by
\[ D_{B2}(t) = t^{-\frac{\pi}{2}} e^{\frac{\pi}{2} \tau_{III}(t)}. \]

Its small gap expansion can be calculated using \( D_{B2}(t) \sim \lim_{\lambda \to \infty} D_{F}(\frac{t}{\lambda}) \). We find
\[ D_{B2}(t) = \sum_{n=0}^{\infty} G \left[ 1 + n, 1 + r + n \right]^{2} (-\xi_{B2}r)^{n} \rho^{(s+r)} \sum_{\lambda, \mu \in \mathbb{V} | \lambda |, \mu | \mu} B_{B2}^{B2}(r, n)|l|^{2} |l|^{2} \]
\[ B_{B2}^{B2}(r, n) = \prod_{(i,j) \in \lambda} \frac{(i - j + n)(i - j + n + r)}{h_{\lambda}(i, j)(\lambda_{j} + \mu - i - j + 2n + r)^{2}} \]
\[ \quad \times \prod_{(i,j) \in \mu} \frac{(i - j - n)(i - j - n - r)}{h_{\mu}(i, j)(\lambda_{j} + \mu_{j} - i - j + 2n - r)^{2}}. \]

5.2. Sine–Gordon exponential fields

A well-known example of an appearance of Painlevé transcendent in integrable QFT is provided by the two-point correlation function of exponential fields \( Q_{O_\nu}(mr) = \langle O_\nu(0) O_\nu(r) \rangle \) in the sine–Gordon model at the free-fermion point \([7, 52]\). The spectrum of this model consists of fermionic excitations of mass \( m \), parameterized by the topological charge \( \epsilon = \pm 1 \) and rapidity \( \theta \in \mathbb{R} \). Lattice counterparts of the exponential fields have been introduced and studied in \([28, 50]\).

5.2.1. From form factors to Macdonald kernel. Under normalization \( \langle O_\nu \rangle = 1 \), the exponential fields are completely determined by their two-particle form factors \([53]\)
\[ F_{\nu}(\theta, \theta') = \langle \theta; \theta' | O_\nu(0) | \text{vac} \rangle = \frac{i \sin \pi \nu}{2\pi \cosh \frac{\theta - \theta'}{2}}. \]

Multiparticle form factors can be written as determinants of two-particle ones. This allows us to sum up the form factor expansion
\[ Q_{mr} = \sum_{m=0}^{\infty} \sum_{\epsilon_{1}, ..., \epsilon_{m}= \pm 1} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\theta_{1} \cdots d\theta_{n} e^{-mr \sum_{a=1}^{n} \cosh \theta_{a}} \]
\[ \times \langle \text{vac} | O_\nu(0) | \theta_{1}, ..., \theta_{n} | | O_\nu(0) | \text{vac} \rangle \]

\[ \text{to the Fredholm determinant } Q_{mr} = \det(1 - K_{SG}). \] The corresponding kernel acts on \( L^{2}(\mathbb{R}) \) and is expressed in terms of dressed two-particle form factors:
\[ K_{SG}(\theta, \theta') = \int_{-\infty}^{\infty} F_{\nu}(\theta'', \theta) F_{\nu}(\theta''', \theta') e^{-\frac{\pi}{2} (\cosh \theta + 2 \cosh \theta'' + \cosh \theta''') d\theta''}. \]

Let us show that \( K_{SG}(\theta, \theta') \) is equivalent to a more familiar classical integrable kernel \( K_{M}(x, y) \) on \( L^{2}(\mathbb{R}), \infty) \). The latter is defined by \( (5.1) \) with \( \lambda = \pi^{-2} \sin \pi \nu \sin \pi \nu' \) and \( \varphi, \psi \) given by Macdonald functions
\[ \varphi_{M}(x) = 2\sqrt{2} K_{\nu', \nu+1}(2\sqrt{x}), \quad \psi_{M}(x) = 2K_{\nu', \nu}(2\sqrt{x}). \]
This kernel can be seen as a further scaling limit of the Whittaker kernel from the previous subsection. Indeed, one may check that

\[ K_M(x, y) = \lim_{\eta \to \infty} \frac{1}{\eta} K_W \left( \frac{x}{\eta}, \frac{y}{\eta} \right). \]

By equivalence of \( K_{SG} \) and \( K_M \) we mean that \( \text{Tr} K_{SG}^n = \text{Tr} K_M^n \) for any \( n \in \mathbb{Z}_{\geq 0} \).

Here is a proof. First note that the Macdonald kernel admits an alternative simple form

\[ K_M(x, y) = \lambda \int_1^\infty \psi_M(xt) \psi_M(yt) \, dt. \quad (5.17) \]

This representation results from the identity

\[ \frac{d}{dt} \left[ \varphi_M(xt) \psi_M(yt) - \varphi_M(yt) \psi_M(xt) \right] = -(x - y) \psi_M(xt) \psi_M(yt), \]

which is itself an easy consequence of the differentiation formulas

\[ \frac{d}{dx} \left( \psi_M(x) \psi_M(x) \right) = \left( \begin{array}{c} \nu - \nu' \\ 2 \\ \nu - \nu \\ -1 \end{array} \right) \left( \psi_M(x) \psi_M(x) \right). \]

On the other hand, parameterizing the rapidities as \( u = e^\theta \), one can write \( \kappa_n = \text{Tr} K_{SG}^n \) as

\[ \kappa_n = \lambda^n \int_0^\infty \ldots \int_0^\infty \frac{d^n u_{2n}}{u_1} \frac{d^{n-1} u_{2n+1}}{u_1} \exp \left\{ - \frac{m r}{\nu} (u_j + u_{j+1}) \right\} \quad (5.18) \]

with \( u_{2n+1} = u_1 \). Now make in (5.18) the following replacements:

\[ e^{-\frac{\nu}{2} (u_{2j-1} + u_{2j})} = \int_0^\infty e^{-t_{2j-1} (u_{2j-1} + u_{2j})} \, dt_{2j-1}, \]

\[ e^{-\frac{\nu}{2} (u_{2j-1}^+ + u_{2j}^+)} = u_{2j-1} u_{2j+1} \int_0^\infty e^{-t_{2j-1} (u_{2j-1}^+ + u_{2j}^+)} \, dt_{2j-1}, \]

where \( j = 1, \ldots, n \). This yields a \( 4n \)-fold integral

\[ \int_0^\infty \frac{d u_1}{u_1} \ldots \int_0^\infty \frac{d u_{2n}}{u_1} \int_0^\infty \frac{d u_{2n+1}}{u_1} \ldots \int_0^\infty \frac{d u_{2n}}{u_1} \prod_{j=1}^{2n-1} u_{2j-1}^0 u_{2j+1}^{-1} e^{-t_{2j-1} (u_{2j-1}^0 + u_{2j+1}^{-1})}, \]

with \( t_0 = t_{2n} \). The variables \( u_1, \ldots, u_{2n} \) are now decoupled. Integrating them out with the help of the standard integral representation of the Macdonald function

\[ \int_0^\infty u^{-1+\nu} e^{-t u^{-1}} \, du = (t/t')^\frac{\nu}{2} \psi_M(t'), \quad t, t' > 0, \]

we finally obtain

\[ \kappa_n = \lambda^n \int_0^\infty \frac{d t_1}{t_1} \ldots \int_0^\infty \frac{d t_{2n}}{t_1} \int_0^\infty \frac{d t_{2n+1}}{t_1} \ldots \int_0^\infty \frac{d t_{2n}}{t_1} \psi_M(t_{j-1} - t_j). \]

After the change of variables \( t_{2j-1} \mapsto \frac{m}{t_{2j-1}}, t_{2j} \mapsto \frac{2}{m} \) the last expression can obviously be written as \( \text{Tr} K_M^n \) with \( K_M \) given by (5.17).
For instance, taking into account the contribution of two-particle states, we find corrections to this long-distance behaviour can be obtained from the form factor expansion.

5.2.2. Painlevé III and asymptotics. Painlevé representations of the two-point function of exponential fields [7, 52] can now be rederived by applying the standard random matrix theory techniques [58] to the Macdonald kernel. The final result is that \( Q_{\text{mr}} \) coincides, up to a simple prefactor, with a tau function of Painlevé III equation with parameters \( \theta_\ast = -\theta_\ast = \sigma \):}

\[
Q(2\sqrt{t}) = t^{-\frac{\nu-\nu'}{2}} e^{\frac{1}{2} \tau_{\text{III}}(t)}.
\]

The integration constants specifying this tau function are [32]

\[
\sigma = \frac{\nu + \nu'}{2}, \quad s_{\text{III}} = 1.
\]

In general, the tau function is defined up to multiplication by a constant, which in the case at hand is fixed by the normalization of the VEVs: \( Q_{\text{mr}} \) \( \approx 1 \) as \( r \to \infty \). Subleading corrections to this long-distance behaviour can be obtained from the form factor expansion. For instance, taking into account the contribution of two-particle states, we find

\[
1 - Q(2\sqrt{t}) = \lambda \int_{t}^{\infty} \frac{d\nu}{Q_{\text{mr}}^{-1/2} e^{-4\nu^2}} (\varphi_M'(x) - \varphi_M(x)) \, dx + O(t^{-1} e^{-8\nu^2}).
\]

Short-distance asymptotics of \( Q_{\text{mr}} \) is also known. Assume that \( |\text{Re}(\nu + \nu')| < 1 \), then, as \( t \to 0 \),

\[
Q(2\sqrt{t}) \approx C_{\text{SG}}(\nu, \nu') e^{\nu \nu'}.
\]

The value of \( \sigma \) in (5.19) is determined by the exponent \( \nu \nu' \), found in [32]. The coefficient \( C_{\text{SG}}(\nu, \nu') \) was calculated by Basor and Tracy in [3]:

\[
C_{\text{SG}}(\nu, \nu') = G \left[ 1 + \nu, 1 - \nu, 1 + \nu', 1 - \nu'; 1 + \nu + \nu', 1 - \nu - \nu' \right].
\]

Note that the last expression coincides with \( C_{\text{III}}(\nu, \nu', \nu, \nu') \) defined by (4.20). This simply means that the normalization of \( \tau_{\text{III}}(t) \) in conjecture 4 corresponds to setting \( \langle O_{\nu} \rangle = 1 \) in the sine–Gordon case.

5.2.3. Short-distance expansion of \( \langle O_{\nu}(0)O_{\nu}(r) \rangle \). We are now ready to write a complete short-distance expansion of the two-point correlator \( Q_{\text{mr}} \). Combining the above with conjecture 4 gives the following series:

\[
Q_{\text{mr}} = \sum_{n \in \mathbb{Z}} C_{\text{SG}}(\nu + n, \nu' + n) \sum_{\lambda, \mu \in Y} B_{\text{SG}}^{\nu, \nu'}(\nu + n, \nu' + n) \left( \frac{m^2 r^2}{4} \right)^{(\nu+n)(\nu'+n)+|\lambda|+|\mu|}.
\]

where \( C_{\text{SG}}(\nu, \nu') \) is defined by (5.20) and

\[
B_{\text{SG}}^{\nu, \nu'}(\nu, \nu') = \prod_{(i, j) \in \lambda} \frac{(i - j + v)(i - j + v')}{K^2(i, j)(\lambda_i + \mu_i - i - j + 1 + v + v')^2}
\times \prod_{(i, j) \in \mu} \frac{(i - j - v)(i - j - v')}{K^2(i, j)(\lambda_j + \mu_j - i - j - v - v')^2}.
\]

The series (5.21) has a familiar structure of conformal perturbation expansion [19, 64]. The non-analytic factors \( m^2 e^{(\nu+n)(\nu'+n)} \) correspond to non-perturbative VEVs of the primary fields.
which appear in the operator product expansion $O_{\nu}(0) O_{\nu'}(r)$. All other corrections, including the VEVs of descendant fields and CPT, come in integer powers of the coupling $m^2$.

Figure 5 illustrates how well the series (5.21) fits the form factor expansion to give all-distance behaviour of the correlator. We fix $\nu = 0.3, \nu' = 0.45$ and compute the expansion $Q_{15}(mr)$ taking into account the terms with $n = -4, \ldots, 4$ up to descendant level 15, as we did before for $P_{VI}$. Plots (A), (B), (C), (D) correspond to the logarithms of $-\sum_{j=1}^{l-1} \frac{1}{j} \text{Tr} K_j S_G - \ln Q_{15}(mr)$ (solid lines) and $\frac{1}{l} \text{Tr} K_{SG}^l$ (dotted lines) for $l = 1, 2, 3, 4$. Hence $Q_{15}(mr)$ correctly accounts for the two-particle form factor contribution to long-distance asymptotics up to $mr \approx 3.2$, four-particle contribution up to $mr \approx 2.6$, six-particle and eight-particle ones up to $mr \approx 2.1$ and $mr \approx 1.7$.

Let us stress that we are dealing here with a correlation function of massive theory. Its description by holomorphic conformal blocks therefore looks rather surprising and is presumably related to the affine $\hat{sl}(2)$ symmetry of the free-fermion sine–Gordon theory [35].

5.2.4. Painlevé III and 2D polymers. The change of variables $q_{\text{III}}(t) = \frac{r}{4} \exp \psi(r), r = 4 \sqrt{t}$ maps $P_{\text{III}}$ with $\theta_0 = 0, \theta_\nu = \frac{1}{2}$ to the radial sinh–Gordon equation

$$\psi'' + \frac{1}{r} \psi' = \frac{1}{2} \sinh 2\psi.$$ (5.22)

A particular solution of this equation describes universal scaling functions of 2D polymers [20, 65]. It is characterized by the boundary conditions

$$\psi(r \to 0) \sim -\frac{1}{3} \ln r - \frac{1}{2} \ln \frac{\mu}{4} + O(r^{4/3}), \quad \mu = \frac{\Gamma^2(1/3)}{\Gamma^2(2/3)}.$$ (5.23)

which in our notation correspond to integration constants $\sigma_{\text{III}_i} = \frac{1}{2}, s_{\text{III}_i} = 1$. In fact the relevant $P_{\text{III}_i}$ solution is a Bäcklund transform of a solution associated to the tau
function considered in the previous subsection. The precise relation between the two quantities is
\[
\sinh^2 \psi(r) = -\left[ (\ln Q(r))'' + r^{-1} (\ln Q(r))' \right] \bigg|_{v=v'=-\frac{1}{2}}.
\]

On the other hand, it is known [48] that \( P_{II} \) with \( \theta_* = 0, \theta_* = \frac{1}{4} \) (and hence the radial sinh–Gordon equation!) is equivalent to \( P_{III} \). Namely, if we consider the following quadratic transformation
\[
t_{III} = \frac{t^2}{16}, \quad q_{III}(t_{III}) = \frac{q^2(t)}{4},
\]
then \( q(t) \) satisfies appropriate \( P_{III} \). This allows us to give an alternative characterization of the solution (5.23) via the expansion (4.25)–(4.28):
\[
e^{-2\psi(r)} = -4r^{-1} \frac{d}{dr} \frac{d}{dr} \ln \tau_{III} (2^{-12} r^4) \bigg|_{t_{III} = 1, \sigma = \frac{1}{6}},
\]
or, in yet another form,
\[
e^{\psi(r)} = \frac{\tau_{III} (2^{-12} r^4) |_{t_{III} = 1, \sigma = \frac{1}{6}}}{\tau_{III} (2^{-12} r^4) |_{t_{III} = 1, \sigma = \frac{1}{6}}},
\]
where the normalization of both tau functions in the last formula is precisely the same as in conjecture 6.

6. Discussion

We believe that by explaining the title of this paper we have partially answered P Deift’s question from the introduction. Besides the obvious need for rigorous proofs of our claims in section 4, many other questions beg to be addressed. Why instantons? Is there a way to obtain irregular ‘form factor’ expansions at \( \infty \) for general solutions of \( P_V \) and \( P_{III} \)? What about \( P_{IV} \), \( P_{II} \) and \( P_I \)?

A particularly interesting problem, already mentioned above, concerns the computation of connection coefficients of Painlevé tau functions (akin to Dyson–Widom constants in random matrix theory). In the \( P_{VI} \) case this is very much related to determining the fusion matrix for \( c = 1 \) generic conformal blocks.

Another intriguing issue is the quantization of Painlevé equations [43, 62]. The existing paradigm usually associates isomonodromic deformations to the semiclassical limit of CFT [55]. For instance, the scalar Lax pairs for \( P_{VI-1} \) emerge in the \( c \to \infty \) limit of two BPZ-type differential operators [44]. The results presented here and in [27] suggest a completely different, \( c = 1 \) point of view. In this picture, classical Riccati solutions of Painlevé equations may be naturally deformed to Coulomb \( \beta \)-integrals. It would be nice to understand whether the general case allows for a similar \( \beta \)-deformation.

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Appendix A. Barnes function

Barnes G-function satisfies the functional equation
\[ G(1 + z) = (2\pi)^z \exp \left( -\frac{z + z^2 (1 + \gamma)}{2} \right) \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right) \exp \left( \frac{z^2}{2k} - z \right), \]
where \( \gamma \) is the Euler’s constant, or via the integral representation
\[ G(1 + z) = (2\pi)^z \exp \left[ \int_0^\infty \frac{dt}{t} \left( 1 - \frac{e^{-zt}}{4 \sinh^2 \frac{t}{2}} - \frac{z}{t} + \frac{z^2}{2} e^{-t} \right) \right], \quad \text{Re } z > -1. \]

It is analytic in the whole complex plane and has the following asymptotic expansion as \( |z| \to \infty, \arg z \neq \pi \):
\[ \ln G(1 + z) = \left( \frac{z^2}{2} - \frac{1}{12} \right) \ln z - \frac{3z^2}{4} + \frac{z}{2} \ln 2\pi + \zeta'(1) + O(z^{-2}). \]

One of the consequences of this asymptotic behaviour is the formula
\[ G\left[ 1 + z + \alpha, 1 + z - \alpha \right] = G\left[ 1 + z + \beta, 1 + z - \beta \right] \quad (A.1) \]

Another useful relation is
\[ G\left[ 1 + z + n, 1 - z - n \right] = (-1)^{n+1} \left( \frac{\pi}{\sin \pi z} \right)^n, \quad n \in \mathbb{Z}. \]

It is easy to deduce from it that, as \( \epsilon \to 0 \),
\[ G(1 + \epsilon - n) \sim \epsilon^n (-1)^{n+1} G(1 + n), \quad n \in \mathbb{Z}_{\geq 0}. \]

Appendix B. Sine kernel conformal blocks

Consider the functions
\[ B_{\text{sine}}(n; t) = \sum_{\lambda, \mu \in Y, |\lambda|, |\mu| \leq n} B_{\lambda, \mu}^{\text{sine}}(n) (it)^{|\lambda|+|\mu|}, \]
which appear in the expansion (5.13) of the GUE gap probability. Below we record the terms contributing to \( D_{\text{sine}}(t) \) as at least \( t^{30} \):
\[ B_{\text{sine}}(0; t) = B_{\text{sine}}(1; t) = 1, \]
\[ B_{\text{sine}}(2; t) = 1 - \frac{t^2}{75} + \frac{t^4}{7840} - \frac{t^6}{1134000} + \frac{t^8}{219542400} - \frac{t^{10}}{55091836800} + \frac{t^{12}}{1743565824000} - \frac{t^{14}}{680252206233600} + \frac{t^{16}}{321007903856640000} \]
\[ + \frac{t^{18}}{180308450089912320000} + \frac{t^{20}}{118919276998708925440000} - \frac{t^{22}}{9102064226815752192000000} + \frac{t^{24}}{80033190460574888381644800000} \]
\[ - \frac{t^{26}}{8012846806826604896305152000000} + O(t^{28}). \]
\[ B_{\text{sine}}(3, t) = 1 - \frac{18t^2}{1225} + \frac{t^4}{94.789t^{12}} - \frac{2293t^6}{3922033500} + \frac{3581t^8}{1616027212800} - \frac{71t^{10}}{76477t^{14}} + \frac{6178831567324416000}{407221t^{16}} - \frac{2570452778021883955200}{10908183686400} \]
\[ + \frac{84123909089802035200000}{245265109t^{18}} - \frac{36550090093738750000000}{40956413t^{20}} + \frac{510254748374093327507340928000000}{174931t^{22}} + O(t^{22}), \]
\[ B_{\text{sine}}(4, t) = 1 - \frac{20t^2}{1323} + \frac{83t^4}{4585051t^{10}} - \frac{3267581821768}{174931t^{12}} + \frac{3926946127104}{5892151877t^{14}} - \frac{572172412994582400}{10556078423502470838189840} + O(t^{16}), \]
\[ B_{\text{sine}}(5, t) = 1 - \frac{50t^2}{3267} + \frac{475t^4}{4008004} + O(t^6). \]

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