Brane singularities and their avoidance

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Abstract

The singularity structure and the corresponding asymptotic behavior of a 3-brane coupled to a scalar field or to a perfect fluid in a five-dimensional bulk is analyzed in full generality using the method of asymptotic splittings. In the case of the scalar field, it is shown that the collapse singularity at a finite distance from the brane can be avoided only at the expense of making the brane world-volume positively or negatively curved. In the case where the bulk field content is parametrized by an analog of perfect fluid with an arbitrary equation of state $P = \gamma \rho$ between the ‘pressure’ $P$ and the ‘density’ $\rho$, our results depend crucially on the constant fluid parameter $\gamma$.
(i) For $\gamma > -1/2$, the flat brane solution suffers from a collapse singularity at a finite distance that disappears in the curved case.
(ii) For $\gamma < -1$, the singularity cannot be avoided and it becomes of the big rip type for a flat brane.
(iii) For $-1 < \gamma \leq -1/2$, the surprising result is found that while the curved brane solution is singular, the flat brane is not, opening the possibility for a revival of the self-tuning proposal.

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1. Introduction

Some time ago, an interesting idea to address the cosmological constant problem was proposed, based on the so-called self-tuning mechanism [1, 2]. The simplest model consists of a 3-brane embedded in a five-dimensional bulk, in the presence of a scalar field. The latter is coupled to the brane in a particular way, motivated by string theory, that allows a flat brane world-volume solution independently of the brane tension value. It was, however, realized that a singularity appears in the bulk at some finite distance from the brane, which can also be thought of as a reservoir through which the vacuum energy decays.

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An obvious question is then whether the development of such a singularity is a generic feature of these models, or under what conditions may be avoided. In this paper, we investigate this question in a generalized class of models. Since in this case a general solution cannot be found analytically, we use a powerful tool developed a few years ago, called the method of asymptotic splittings, that allows one to compute all possible asymptotic behaviors of solutions to the equations of motion around the assumed location of a singularity [3].

As a first step in our analysis, we consider an extended version of the simplest model allowing for curved brane world-volume. We then show that the emergence of the finite-distance singularity is the only possible asymptotic behavior for a flat brane, whereas for a curved brane the singularity is shifted to an infinite distance. We also provide a detailed study of the asymptotics of this model using the method of asymptotic splittings expounded in [3]. A preliminary version of these results was published in [4].

Next, we extend the previous analysis to the case where the bulk matter is described by an analog of a perfect fluid with an arbitrary equation of state \( P = \gamma \rho \). In fact, the case of a massless bulk scalar is a particular case of such a fluid, corresponding to the value \( \gamma = 1 \). Here, on the one hand, we are interested in the general dynamics of ‘evolution’ of such a brane-world analog to cosmology for arbitrary \( \gamma \), in order to reveal the various types of singularities that may develop within a finite distance from the original position of the brane, and on the other hand we seek to determine conditions that may lead to the avoidance of the singularities shifting them to an infinite distance away from the brane.

In particular, we shall show that the existence of a perfect fluid in the bulk enhances the dynamical possibilities of brane evolution in the fluid bulk. Such possibilities stem from the different possible behaviors of the fluid density and the derivative of the warp factor with respect to the extra dimension. The result depends crucially on the values of the parameter \( \gamma \).

In general, we find three regions of \( \gamma \) leading to qualitatively different results.

• In the region \( \gamma > -1/2 \), the situation is very similar to the case of a massless bulk scalar field. Here, the main result we establish is twofold.
  – The existence of a singularity at a finite distance is unavoidable in all solutions with a flat brane. This confirms and extends the results of earlier works that made similar investigations in different models, using other methods [5, 6].
  – The singularity can be avoided (e.g. moved at an infinite distance) when the brane becomes curved, either positively or negatively. Thus, requiring the absence of singularity brings back the cosmological constant problem, since the brane curvature depends on its tension that receives quartically divergent quantum corrections.

• The situation changes drastically in the region \(-1 < \gamma < -1/2 \). The curved brane solution becomes singular while the flat brane is regular. Thus, this region seems to avoid the main obstruction of the self-tuning proposal: any value of the brane tension is absorbed in the solution and the brane remains flat. The main question is then whether there is a field theory realization of such a fluid producing naturally an effective equation of state of this type.

• Finally, in the region \( \gamma < -1 \), corresponding to the analog of a phantom equation of state, we show that it is possible for the brane to be ripped apart in as much the same way as in a big rip singularity. This happens only in the flat case, while curved brane solutions develop ‘standard’ collapse singularities. No regular solution is found in this region.

Besides the above regions, the values \( \gamma = -1/2, -1 \) are of special significance: when \( \gamma = -1/2 \), we again find a regular flat brane solution with the so-called sudden behavior [7], as well as a non-singular curved brane, while for \( \gamma = -1 \) there is only a singular curved solution.
As mentioned above, it would be very interesting to understand whether there are field theory representations reproducing the ‘exotic’ regions of \( \gamma \leq -1/2 \). Obviously the analogy of the perfect fluid concerning the positivity energy conditions does not seem to apply in this case where time is replaced by an additional space coordinate. However, some restrictions may be applied from usual field theory axioms. Also, the formation of singularities discussed here is better understood in a dynamical rather than the usual geometric sense met in general relativity. In the latter case, cosmological singularities are formed together with conjugate (or focal) points in spacetime, and for this it is necessary that there exists at least one timelike dimension (and any number of spacelike ones, greater than 2). The timelike dimension then forces the geodesics to focus along it rather than along any of the spacelike dimensions. In the problem discussed in this paper, the timelike dimension is on the brane while the singularities are formed along spacelike dimensions in the bulk. As we show below these singularities are real in the dynamical sense that some component of the solution vector \((a, a', \rho)\) diverges there. Therefore, we abandon the usual interpretation according to which the universe comes to an end in a finite time possibly through geodesic refocusing, and instead we study how dynamical effects guide our brane systems to extreme behaviors.

The structure of this paper is as follows: in section 2, we analyze the case of a bulk scalar field. We first choose appropriate variables and rewrite the basic field equations in the form of a dynamical system; secondly, we introduce some convenient terminology for the different types of singularity to be met later in our analysis. Then, in subsections 2.1 and 2.2, we give the asymptotics of the models consisting of flat and curved branes respectively. In section 3, we study the case in which the bulk is filled with an analog of a perfect fluid. We first derive the form of the dynamical system and single out the possible dominant balances, organizing centers of all the evolutionary behaviors that fully characterize this case. In subsections 3.1 and 3.2, we study carefully the asymptotics around collapse singularities of two types, that we call type I and type II singularities, respectively. In subsection 3.3, we explore the dynamics as the brane approaches a big rip singularity, while in subsection 3.4 we look at a milder singularity that resembles in many ways the so-called sudden (non-singular) behavior introduced in [7]. In subsection 3.5, we analyze the possibility of avoiding finite-distance singularities leading to the existence of regular brane evolution in the bulk. Finally, in section 4 we conclude and also comment on the possible future work in various directions, considering for instance other forms of matter in the bulk. In the appendix, we briefly outline the basic steps of the method of asymptotic splittings.

2. Dynamics of scalar field–brane configuration

In this section, we study dynamical aspects of a braneworld model \((V_4 \equiv \mathbb{R} \times M_3, g_4)\) consisting of a 3-brane \((M_3, g_3)\) embedded in a five-dimensional bulk space \((V_5, g_5)\) with a scalar field minimally coupled to the bulk. The total action \(S_{\text{total}}\) splits in two parts, namely the bulk action \(S_{\text{bulk}}\) and the brane action \(S_{\text{brane}}\):

\[
S_{\text{total}} = S_{\text{bulk}} + S_{\text{brane}},
\]

with

\[
S_{\text{bulk}} = \int \left( \frac{R}{2k_5} - \frac{\lambda}{2} (\nabla \phi)^2 \right) \mathrm{d}\mu_{g_5}, \quad \text{at} \quad Y = Y_*,
\]

\[
S_{\text{brane}} = -\int f(\phi) \mathrm{d}\mu_{g_4}, \quad \text{at} \quad Y = Y_*. \]
where the measures \( d\mu_{g_5} = d^5x \sqrt{\det g_5} \), and \( d\mu_{g_4} = d^4x \sqrt{\det g_4} \), \( Y \) denotes the fifth bulk dimension, \( Y_0 \) is the assumed initial position of the brane, \( \lambda \) is a parameter defining the type of scalar field \( \phi \), \( \kappa_5^2 = M_5^{-3} \), \( M_5 \) being the five-dimensional Planck mass and \( f(\phi) \) denotes the tension of the brane as a function of the scalar field.

Varying the total action (2.1) with respect to \( g_5 \), we find the five-dimensional Einstein field equations in the form [4]

\[
R_{AB} - \frac{1}{2} g_{AB} R = \lambda \kappa_5^2 \left( \nabla_A \phi \nabla_B \phi - \frac{1}{2} g_{AB} (\nabla \phi)^2 \right) + \frac{2 \kappa_5^2}{\sqrt{\det g_5}} \frac{\delta(\sqrt{\det g_4} f(\phi))}{\delta \phi} \delta_{AB} \delta(Y), \tag{2.4}
\]

while the scalar field equation is obtained by variation of action (2.1) with respect to \( \phi \) [4] and it is

\[
\lambda \Box_5 \phi = - \frac{1}{\sqrt{\det g_5}} \frac{\delta(\sqrt{\det g_4} f(\phi))}{\delta \phi} \delta(Y), \tag{2.5}
\]

where \( A, B = 1, 2, 3, 4, 5 \) and \( \alpha, \beta = 1, 2, 3, 4 \) while \( \delta(Y) = 1 \) at \( Y = Y_0 \) and vanishing everywhere else, and

\[
\Box_5 \phi = \frac{1}{\sqrt{\det g_5}} \nabla_A(\sqrt{\det g_5} g^{AB} \nabla_B \phi). \tag{2.6}
\]

In the following, we assume a bulk metric of the form

\[
g_5 = a^2(Y) g_4 + dY^2, \tag{2.7}
\]

where \( g_4 \) is the four-dimensional flat, de Sitter or anti de Sitter metric, i.e.

\[
g_4 = -dt^2 + f_k^2 g_3, \tag{2.8}
\]

with

\[
g_3 = dr^2 + h_k^2 g_2, \tag{2.9}
\]

and

\[
g_2 = d\theta^2 + \sin^2 \theta \ d\phi^2. \tag{2.10}
\]

Here \( f_k = 1, \cosh(Ht)/H, \cos(Ht)/H \) (\( H^{-1} \) is the de Sitter curvature radius) and \( h_k = r, \sin r, \sinh r \), respectively.

The field equations (2.4) and (2.5) then take the form

\[
a^2 \frac{\partial^2 \phi}{a^2} = \frac{\lambda \kappa_5^2 \phi^2}{12} + \frac{kH^2}{a^2}, \tag{2.11}
\]

\[
a \frac{a''}{a} = - \frac{\lambda \kappa_5^2 \phi^2}{4}, \tag{2.12}
\]

\[
\phi'' + 4 a' \phi' = 0, \tag{2.13}
\]

where the prime (‘) denotes differentiation with respect to \( Y \), and \( k = 0, \pm 1 \). The variables to be determined are \( a, a' \) and \( \phi' \). These three equations are not independent, since equation (2.12) was derived after substitution of equation (2.11) in the field equation \( G_{aa} = \kappa_5^2 T_{aa}, a = 1, 2, 3, 4 \):

\[
a'' + a' \frac{a^2}{a^2} - \frac{kH^2}{a^2} = - \frac{\lambda \kappa_5^2 \phi^2}{6}. \tag{2.14}
\]
In our analysis below we use the independent equations (2.12) and (2.13) to determine the unknown variables \( a, a' \) and \( \phi' \), while equation (2.11) will then play the role of a constraint equation for our system.

Assuming a \( Y \rightarrow -Y \) symmetry and solving equations (2.4) (the \(-\alpha \alpha\)- component, \( \alpha = 1, 2, 3, 4 \)) and (2.5) on the brane we get

\[
a'(Y_*) = -\frac{\kappa_5^2}{6} f(\phi(Y_*)) a(Y_*), \tag{2.15}
\]

\[
\phi'(Y_*) = \frac{f'(\phi(Y_*))}{2\lambda}. \tag{2.16}
\]

The particular coupling used in [1] allows only for flat solutions to exist. This easily follows by using equations (2.15) and (2.16) and solving the FRW equation (2.11) on the brane for \( kH^2 \):

\[
kH^2 = \frac{a^2(Y_*) \kappa_5^2}{12} \left( \frac{\kappa_5^2}{3} f^2(\phi(Y_*)) - \frac{f'^2(\phi(Y_*))}{4\lambda} \right).
\]

Clearly, \( k \) is identically zero if and only if

\[
\frac{f'(\phi)}{f(\phi)} = 2 \sqrt{\frac{\lambda}{3\kappa_5}},
\]

or equivalently, if and only if \( f(\phi) \propto e^{2\sqrt{\lambda/3\kappa_5} \phi} \) (the authors of [1] have set \( \lambda = 3 \) and hence the appropriate choice for the brane tension in that case is \( f(\phi) \propto e^{\kappa_5 \phi} \)). In our more general problem, the coupling function cannot be fixed this way. By working with other couplings we can allow for non-flat, maximally symmetric solutions to exist and avoid having the singularity at a finite distance away from the position of the brane.

Our purpose is to find all possible asymptotic behaviors around the assumed position of a singularity, denoted by \( Y_* \), emerging from general or particular solutions of the system (2.11)–(2.13). The most useful tool for this analysis is the method of asymptotic splittings [3] (see the appendix for a brief introduction), in which we start by setting

\[
x = a, \quad y = a', \quad z = \phi' . \tag{2.17}
\]

The field equations (2.12) and (2.13) become the following system of ordinary differential equations:

\[
x' = y \tag{2.18}
\]

\[
y' = -\lambda A z^2 x \tag{2.19}
\]

\[
z' = -4y \frac{z}{x} . \tag{2.20}
\]

where \( A = \kappa_5^2/4 \). Hence, we have a dynamical system determined by the non-polynomial vector field

\[
f = \left( y, -\lambda A z^2 x, -4y \frac{z}{x} \right)^\top . \tag{2.21}
\]

Equation (2.11) does not include any terms containing derivatives with respect to \( Y \); it is a constraint equation which in terms of the new variables takes the form

\[
\frac{y^2}{x^2} = \frac{A \lambda}{3} z^2 + \frac{kH^2}{x^2} . \tag{2.22}
\]
Equations (2.18)–(2.20) and (2.22) constitute the basic dynamical system of our study in this section.

Before we proceed with the analysis of the above system, we introduce the following terminology, which we use in subsequent paragraphs, for the possible singularities to occur at a finite distance from the brane. Specifically, we call a state where

(i) $a \to 0, a' \to \infty$ and $\rho \to \infty$: a singularity of collapse type I.
(ii) $a \to 0, a' \to a'_s$ and $\rho \to \rho_s$: a singularity of collapse type IIa,
(iii) $a \to 0, a' \to a'_s$ and $\rho \to \rho_s$: a singularity of collapse type IIb,
(iv) $a \to 0, a' \to a'_s$ and $\rho \to \infty$: a singularity of collapse type IIc,
where $a'_s, \rho_s$ are non-vanishing finite constants.

We denote the density of the matter component in the bulk that is considered each time. For the case of interest in this section, $\rho = \frac{\lambda \phi'^2}{2}$ and we show that there are two major cases to be treated, the first is when we choose $k = 0$ in (2.22) and corresponds to a brane being flat, while in the second case $k \neq 0$, giving constant curvature to the brane. We treat these two cases independently. One important result of our analysis of this system will be that the inclusion of nonzero curvature for the brane moves the singularity (that is of the collapse type I class) an infinite distance away from the brane.

2.1. Collapse type I singularity: flat brane

In this subsection we take $k = 0$ in the constraint equation (2.22):

$$\frac{y^2}{x^2} = \frac{A\lambda}{3} z^2. \quad (2.23)$$

We show that the only possible asymptotic behavior of the solutions of this system (flat brane) is that $a \to 0, a' \to \infty$ and $\phi' \to \infty$, as $Y \to Y_s$.

Following the basic steps of the method of asymptotic splittings expounded in the appendix, we start our asymptotic analysis by inserting in the system (2.18)–(2.20) the forms

$$(x, y, z) = (\alpha \Upsilon^p, \beta \Upsilon^q, \zeta \Upsilon^m), \quad (2.24)$$

where $\Upsilon = Y - Y_s$ and

$$(p, q, m) \in \mathbb{Q}^3 \quad \text{and} \quad (\alpha, \beta, \zeta) \in \mathbb{C}^3 \setminus \{0\}. \quad (2.25)$$

We find that in the neighborhood of the singularity the only possible dominant balance, that is pairs of the form

$$B = [a, p], \quad \text{where} \quad a = (\alpha, \beta, \delta), \quad p = (p, q, r), \quad (2.26)$$

determining the dominant asymptotics as we approach the singularity, is the following:

$$B_1 = \{(\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda})), (1/4, -3/4, -1)\}. \quad (2.27)$$

Note that a second balance $B_2$ becomes only possible when we allow for nonzero curvature, $k \neq 0$, and will be analyzed in the next subsection. There are no other acceptable balances; hence, all the possible asymptotic behaviors for a flat and curved brane can be described uniquely by the balances $B_1$ and $B_2$ respectively. Our purpose is then to construct asymptotic expansions of solutions in the form of a series defined by

$$x = \Upsilon^p (a + \sum_{j=1}^{\infty} c_j \Upsilon^{j/s}), \quad (2.28)$$

where $x = (x, y, z), c_j = (c_{j1}, c_{j2}, c_{j3})$ and $s$ is the least common multiple of the denominators of the positive $K$-exponents (cf [3, 8]).
First we calculate the Kowalevskaya matrix (K-matrix in short) given by
\[ K = Df(a) - \text{diag}(p), \] (2.29)
where \( Df(a) \) is the Jacobian matrix of \( f \), which in our case reads
\[
Df(x, y, z) = \begin{pmatrix}
0 & 1 & 0 \\
-\lambda x z & 0 & -2\lambda x z \\
\frac{4 y z}{x^2} & -\frac{4z}{x} & -\frac{4y}{x}
\end{pmatrix},
\] (2.30)
to be evaluated on \( a \). For the \( B_1 \) balance we have that \( a = (\alpha, \alpha/4, \sqrt{3}/(4\sqrt{A\lambda})) \), and \( p = (1/4, -3/4, -1) \), thus giving
\[
K_1 = \begin{pmatrix}
-\frac{1}{4} & 1 & 0 \\
-\frac{3}{16} & \frac{3}{4} & -\frac{\sqrt{3}A\lambda\alpha}{2} \\
\sqrt{\frac{3}{4\alpha\sqrt{A\lambda}}} & -\sqrt{\frac{3}{4\alpha\sqrt{A\lambda}}} & 0
\end{pmatrix}.
\] (2.31)

The next step is to calculate the \( K \)-exponents for this balance. These exponents are the eigenvalues of the \( K_1 \) matrix and constitute its spectrum, \( \text{spec}(K_1) \). The arbitrary constants of any (particular or general) solution first appear in those terms in the series (2.28) whose coefficients \( c_k \) have indices \( k = \varrho s \), where \( \varrho \) is a non-negative \( K \)-exponent. The number of non-negative \( K \)-exponents therefore equals the number of arbitrary constants that appear in the series expansions (2.28). There is always the \( -1 \) exponent that corresponds to an arbitrary constant that is the position of the singularity, \( Y_s \). The balance \( B_1 \) corresponds thus to a general solution in our case if and only if it possesses two non-negative \( K \)-exponents (the third arbitrary constant is the position of the singularity, \( Y_s \)). Here we find
\[
\text{spec}(K_1) = \{-1, 0, 3/2\}
\] (2.32)
so that \( B_1 \) indeed corresponds to a general solution. After substituting in the system (2.18)–(2.20) the series expansions
\[
x = \sum_{j=0}^{\infty} c_{j1} \gamma^{j/2+1/4}, \quad y = \sum_{j=0}^{\infty} c_{j2} \gamma^{j/2-3/4}, \quad z = \sum_{j=0}^{\infty} c_{j3} \gamma^{j/2-1},
\] (2.33)
we arrive at the following asymptotic solution around the singularity:
\[
x = \alpha \gamma^{1/4} + \frac{4}{7} c_{32} \gamma^{7/4} + \cdots
\] (2.34)
\[
y = \frac{\alpha}{4} \gamma^{-3/4} + c_{32} \gamma^{3/4} + \cdots
\] (2.35)
\[
z = \frac{\sqrt{3}}{4\sqrt{A\lambda}} \gamma^{-1} - \frac{4\sqrt{3}}{7\alpha\sqrt{A\lambda}} c_{32} \gamma^{1/2} + \cdots.
\] (2.36)

The last step is to check whether for each \( j \) satisfying \( j/2 = \varrho \) with \( \varrho \) a positive eigenvalue, the corresponding eigenvector \( v \) of the \( K_1 \) matrix is such that the compatibility conditions hold, namely
\[
v^\top \cdot P_j = 0,
\] (2.37)
where \( P_j \) are the polynomials in \( c_1, \ldots, c_{j-1} \) given by
\[
K_1 c_j - (j/s) c_j = P_j.
\] (2.38)
Here the relation \( j/2 = 3/2 \) is valid only for \( j = 3 \) and the associated eigenvector is

\[
v^\top = \left( -\frac{\alpha \sqrt{A\lambda}}{\sqrt{3}}, -\frac{7\alpha \sqrt{A\lambda}}{4\sqrt{3}}, 1 \right).
\]  
(2.39)

The compatibility condition

\[
v^\top \cdot (K_1 - (3/2)I_3)c_3 = 0
\]  
(2.40)

therefore indeed holds, since

\[
(K_1 - (3/2)I_3)c_3 = c_{32}\begin{pmatrix}
-7/4 & 1 & 0 \\
-3/16 & -3/4 & -\frac{\alpha \sqrt{3A\lambda}}{2} \\
-4\alpha \sqrt{A\lambda} & -\frac{\sqrt{3}}{\alpha \sqrt{A\lambda}} & -\frac{\sqrt{3}}{2}
\end{pmatrix}\begin{pmatrix}
\frac{4}{7} \\
1 \\
\frac{4\sqrt{3}}{7\alpha \sqrt{A\lambda}}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]  
(2.41)

This shows that a representation of the solution asymptotically by a Puiseux series as given in equations (2.34)–(2.36) is valid. Hence, we conclude that at a finite distance from the brane, a collapse type I singularity develops, i.e. as \( Y \to Y_s \) the asymptotic forms of the variables are

\[
a \to 0, \quad a' \to \infty, \quad \phi' \to \infty.
\]  
(2.42)

This is exactly the asymptotic behavior of the solution found previously by Arkani-Hamed et al in [1]. Our analysis shows that this is the only possible asymptotic behavior for a flat brane, since there exist no other dominant balances in this case.

2.2. Behavior at infinity: curved brane

In this subsection we show that the collapse type I singularity that necessarily arises in the case of a flat brane is avoided (or shifted at an infinite distance away from the brane) when we consider a curved brane instead.

The new asymptotics follow from the study of a second balance that results from the substitution of (2.24) in (2.18)–(2.20). We calculate this new balance to be

\[
B_2 = \{ (\alpha, \alpha, 0), (1, 0, -1) \}.
\]  
(2.43)

It corresponds to a particular solution for a curved brane, since it satisfies equation (2.22) for \( k \neq 0 \) and \( \alpha^2 = kH^2 \) (here we have to sacrifice one arbitrary constant by setting it equal to \( kH^2 \), \( k = \pm 1 \)). The \( \mathcal{K} \)-matrix of \( B_2 \) is

\[
\mathcal{K}_2 = Df(\alpha, \alpha, 0) - \text{diag}(1, 0, -1) = \begin{pmatrix}
-1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -3
\end{pmatrix},
\]  
(2.44)

with eigenvalues

\[
\text{spec}(\mathcal{K}_2) = \{-1, 0, -3\}.
\]  
(2.45)

Thus for the balance \( B_2 \) we find two distinct, negative, integer \( \mathcal{K} \)-exponents and an infinite expansion in negative powers of a particular solution (recall that we had to sacrifice one arbitrary constant) around the presumed singularity at \( Y_s \), with the negative \( \mathcal{K} \)-exponents signaling the positions where the arbitrary constants first appear [9]. We therefore expand the variables in series with descending powers of \( Y \) in order to meet the two arbitrary constants occurring for \( j = -1 \) and \( j = -3 \), i.e.

\[
x = \sum_{j=0}^{\infty} c_j j^j Y^{j+1}, \quad y = \sum_{j=0}^{\infty} c_j j^j Y^j, \quad z = \sum_{j=0}^{\infty} c_j j^j Y^{j-1}.
\]  
(2.46)
Substituting these series expansions back in the system (2.18)–(2.20) and after some manipulation, we find the following asymptotic behavior:

\[ x = \alpha \Upsilon + c_{-11} + \cdots \]  
\[ y = \alpha + \cdots \]  
\[ z = c_{-33} \Upsilon^{-4} + \cdots . \]  

It is easy to check the compatibility conditions for \( j = -1 \) and \( j = -3 \). We find that

\[ (K_2 + I_3) c^\prime \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} c_{-11} \\ 0 \\ 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]  
\[ (K_2 + 3I_3) c_{-3} \quad = \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{-11} \\ 0 \\ c_{33} \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]  

so that the compatibility conditions are indeed satisfied. The expansions given by equations (2.47)–(2.49) are therefore valid, and we can say that as \( \Upsilon \to 0 \), or equivalently as \( S \equiv 1/\Upsilon \to \infty \), we have that

\[ a \to \infty, \quad a' \to \infty, \quad \phi' \to \infty. \]  

Therefore for a curved brane we find that there can be no finite-distance singularities. The only possible asymptotic behavior is the one given in (2.52) which is only valid at an infinite distance from the brane.

It is interesting that apart from the balances (2.27) and (2.43), a third balance which initially arises from the substitution of (2.24) in (2.18)–(2.20), namely the form

\[ B_3 = \{ (\alpha, 0, 0), (0, -1, -1) \}, \]

is not acceptable in this case of the bulk scalar field, since it does not give the necessary \(-1 \) \( K \)-exponent. If this balance were acceptable it would yield a finite-distance behavior of the type \( a \to a, \quad a' \to a', \quad \phi' \to \phi' \), where \( a \) is the constant appearing in the balance \( B_3 \) and \( a', \phi' \) are also constants. This would be similar to the sudden behavior met in classical four-dimensional cosmologies with a perfect fluid [7], in the sense that the warp factor, in analogy with the scale factor, its derivative and the density all remain finite. Below, we find that such a balance does become possible when we replace the scalar field with a perfect fluid.

### 3. Dynamics in a perfect fluid bulk

In this section, we rewrite the brane model living in a bulk, which we now consider to be filled with a perfect fluid, as a dynamical system in three basic variables and completely identify the principal modes of approach to its singularities; that is, we find all the dominant balances of the system. In this case, bulk space is filled with a perfect fluid with the equation of state \( P = \gamma \rho \), where the pressure \( P \) and the density \( \rho \) are functions only of the fifth dimension, \( Y \). We assume again a bulk metric of the form (2.7)–(2.10) and an energy–momentum tensor of the form \( T_{AB} = (\rho + P)u_A u_B - P g_{AB} \), where \( A, B = 1, 2, 3, 4, 5 \) and \( u_A = (0, 0, 0, 0, 1) \), with the fifth coordinate corresponding to \( Y \).

The five-dimensional Einstein equations,

\[ G_{AB} = \kappa_5^2 T_{AB}, \]  

where \( \kappa_5 = \sqrt{5/6} \kappa \), and \( \kappa \) is the five-dimensional gravitational constant. In terms of the bulk scalar field, this equation is equivalent to the following

\[ \kappa_5^2 T_{AB} = \kappa_5^2 \left( u_A u_B - \gamma \rho \delta_{AB} \right), \]  

where \( \kappa_5^2 \) is the five-dimensional gravitational constant and \( \gamma \) is the pressure

can be written in the following form:

\[
\frac{a''}{a} = -\kappa^2 \left(1 + 2\gamma\right) \frac{1}{6} \rho,
\]

(3.2)

\[
\frac{a'}{a} = \frac{\kappa^2}{6} \rho + \frac{kH^2}{a^2},
\]

(3.3)

where as in the case treated previously \(k = 0, \pm 1\), and the prime (') denotes differentiation with respect to \(Y\). The equation of conservation,

\[
\nabla_B T^{AB} = 0,
\]

(3.4)

becomes

\[
\rho' + 4(1 + \gamma) \frac{a'}{a} \rho = 0.
\]

(3.5)

Introducing the new variables

\[x = a, \quad y = a', \quad w = \rho,\]

(6.6)

equations (3.2) and (3.5) take the form

\[
x' = y,
\]

(3.7)

\[
y' = -2A \left(1 + 2\gamma\right) \frac{w}{3},
\]

(3.8)

\[
w' = -4(1 + \gamma) \frac{y}{x} w,
\]

(3.9)

while equation (3.3) reads

\[
\frac{y^2}{x^2} = \frac{2A}{3} \rho + \frac{kH^2}{x^3}, \quad A = \frac{\kappa^2}{4}.
\]

(3.10)

Since this last equation does not contain derivatives with respect to \(Y\), it is a velocity-independent constraint equation for the system (3.7)–(3.9).

The next step is to apply the method of asymptotic splittings in an effort to find all possible asymptotic behaviors of the dynamical system (3.7)–(3.9) with the constraint (3.10), by building series expansions of the solutions around the presumed position of the singularity at \(Y_s\). We note that the system (3.7)–(3.9) is a weight homogeneous system determined by the vector field

\[
f = \left(y, -2A \left(1 + 2\gamma\right) \frac{w}{3}, -4(1 + \gamma) \frac{y}{x} w\right)\top.
\]

(3.11)

In order to compute all possible dominant balances that describe the principal asymptotics of the system,

\[
(x, y, w) = (\alpha \Upsilon^p, \beta \Upsilon^q, \xi \Upsilon^m),
\]

(3.12)

where \(\Upsilon = Y - Y_s\), we look for pairs of the form \((2.26)\) with \((2.25)\), as we did in section 2 for the case of the bulk scalar field. We find after some calculation the following list of all possible balances for our basic system (3.7)–(3.10):

\[
\gamma B_1 = \left\{ (\alpha, \alpha p, \frac{3}{2A} p^2) \right\}, \quad (p, p - 1, -2),
\]

(3.13)

\[
\gamma B_2 = \{(\alpha, \alpha, 0), (1, 0, -2)\}, \quad \gamma \neq -1/2,
\]

(3.14)
Jacobian matrix of $f$ geometry that we must consider. The balances $\gamma B_i$ represented by each one of the balances above and determines uniquely the type of spatial ranges of $\gamma$ satisfy the constraint equation (3.10). This fact alters the presumed generality of the solution represented by each one of the balances above and determines uniquely the type of spatial geometry that we must consider. The balances $\gamma B_1$ and $-1/2B_2$ are found when we set $k = 0$, and describe a (potentially general) solution corresponding to a flat brane, while the balances $\gamma B_2$ and $-1/2B_3$ were found when $k \neq 0$ and describe particular solutions of curved branes (since we already have to sacrifice the arbitrary constant $\alpha$ by imposing $a^2 = kH^2$). For the balance $-1/2B_2$, on the other hand, $k$ is not specified and hence it describes a particular solution for a curved or flat brane (particularly since we have to set $\delta = (3/(2A))(1 - kH^2/a^2)$ to satisfy equation (3.10)).

Each one of these balances are analyzed in detail in the following subsections according to the nature of asymptotic behaviors they imply.

### 3.1. Collapse type I singularity

We focus in this subsection exclusively on a study of the balance $\gamma B_1$ and show that for certain ranges of $\gamma$ it gives the generic asymptotic behavior of a flat brane to a singularity of collapse type I. Our analysis implies that such behavior in the case of a fluid bulk can only result from a $\gamma B_1$ type of balance.

As a first step we calculate the $\mathcal{K}$-matrix, $\mathcal{K} = Df(a) - \text{diag}(p)$, where $Df(a)$ is the Jacobian matrix of $f$.

$$
Df(x, y, w) = \begin{pmatrix}
0 & 1 & 0 \\
-\frac{2}{3}(1 + 2\gamma)Ax & 0 & -\frac{2}{3}(1 + 2\gamma)Ax \\
\frac{4(1 + \gamma)}{x^2}yw & -4(1 + \gamma)w^2 & -4(1 + \gamma)\frac{w}{x}
\end{pmatrix},
$$

(3.18)

to be evaluated on $a$. The balance $\gamma B_1$ has $a = (\alpha, \alpha p, 3p^2/2A)$, and $p = (p, p - 1, -2)$, with $p = 1/(2(\gamma + 1))$. Thus the $\mathcal{K}$-matrix for this balance is

$$
\gamma \mathcal{K}_1 = Df \begin{pmatrix}
\alpha, \alpha p, \\
\frac{3}{2A}p^2
\end{pmatrix} - \text{diag}(p, p - 1, -2)
= Df \begin{pmatrix}
\frac{1}{2(1 + \gamma)}, \\
\frac{3}{8A(1 + \gamma)^2}
\end{pmatrix} - \text{diag}\left(\frac{1}{2(1 + \gamma)}, \frac{1}{2(1 + \gamma)}, -2\right)
= \begin{pmatrix}
-\frac{1}{2(1 + \gamma)} & 1 & 0 \\
1 + 2\gamma & \frac{1 + 2\gamma}{2(1 + \gamma)} & -\frac{2}{3}(1 + 2\gamma)Aa \\
\frac{3}{4(1 + \gamma)^2Aa} & \frac{3}{2(1 + \gamma)Aa} & 0
\end{pmatrix}.
$$

(3.19)

$\text{f}$ is the vector field resulting from the dynamical system (3.7)–(3.9) and $[a, p]$ is the balance $\gamma B_1$. 

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We then calculate what the $K$-exponents for this balance actually are. Recall that these exponents are the eigenvalues of the matrix $\gamma K_1$ and constitute its spectrum, $\text{spec}(\gamma K_1)$. The balance $\gamma B_1$ corresponds to a general solution in our case if and only if it possesses two non-negative $K$-exponents (the third arbitrary constant is the position of the singularity, $Y_s$).

Here we find

$$\text{spec}(\gamma K_1) = \{-1, 0, \frac{1 + 2\gamma}{1 + \gamma}\}.$$  \hspace{1cm} (3.20)

The last eigenvalue is a function of the $\gamma$ parameter and it is positive when either $\gamma < -1$, or $\gamma > -1/2$. We consider here the case $\gamma > -1/2$, since, as it will soon follow, this range of $\gamma$ is adequate for the occurrence of a collapse type I singularity. The case of $\gamma < -1$ leads to a big rip singularity and will be examined in subsection 3.3.

Let us assume $\gamma = -1/4$ for concreteness. Then

$$\text{spec}_{-1/4}(\gamma K_1) = \{-1, 0, 2/3\}.$$  \hspace{1cm} (3.22)

Substituting in the system (3.7)–(3.9) the particular value $\gamma = -1/4$ and the forms

$$x = \sum_{j=0}^{\infty} c_j \gamma^{j/3}, \quad y = \sum_{j=0}^{\infty} c_{j2} \gamma^{j/3-1/3}, \quad w = \sum_{j=0}^{\infty} c_{j3} \gamma^{j/3-2},$$

we arrive at the following asymptotic expansions:

$$x = \alpha \gamma^{2/3} - \frac{A\alpha}{2} c_{23} \gamma^{4/3} + \cdots,$$  \hspace{1cm} (3.24)

$$y = 2/3 \alpha \gamma^{-1/3} - \frac{2}{3} A\alpha c_{23} \gamma^{1/3} + \cdots,$$  \hspace{1cm} (3.25)

$$w = \frac{2}{3A} \gamma^{-2} + c_{23} \gamma^{-4/3} + \cdots.$$ \hspace{1cm} (3.26)

For this to be a valid solution we need to check whether the compatibility condition holds true for each $j$ satisfying $j/3 = \rho$ with $\rho$ a positive eigenvalue. Here the corresponding relation $j/3 = 2/3$ is valid only for $j = 2$ and the compatibility condition indeed holds since

$$(-1/4 K_1 - (2/3)I_3)c_2 = \begin{pmatrix} 2/3 & 1 & 0 \\ -2 & 1 & -A\alpha \\ 2 & -A\alpha & -3 \end{pmatrix} c_{23} \begin{pmatrix} A\alpha \\ -2A\alpha \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$  \hspace{1cm} (3.27)

Equations (3.24)–(3.26) then imply that as $\gamma \to 0$,

$$a \to 0, \quad a' \to \infty, \quad \rho \to \infty.$$  \hspace{1cm} (3.28)

This asymptotic behavior corresponds to a general solution of a flat brane that is valid around a collapse type I singularity. We thus regain a behavior similar to the one met in subsection 2.1 for the case of a flat brane in a scalar field bulk.

#### 3.2. Collapse type II singularities

In this subsection, we show that for a curved brane ($k = \pm 1$) the long-term (distance) behavior of all solutions which depend on the asymptotics near finite-distance singularities turn out to be of a very different nature. In particular, we show that the balances $\gamma B_2$ for $\gamma < -1/2,$
We note that the third arbitrary constant appears at the value $j = -\frac{3}{4}(1 + 2\gamma)$, $\gamma < -\frac{1}{2}$. After substituting the forms,

$$x = \sum_{j=0}^{\infty} c_j \gamma^j, \quad y = \sum_{j=0}^{\infty} c_j^2 \gamma^j', \quad w = \sum_{j=0}^{\infty} c_j^3 \gamma^{j-2},$$

in the system (3.7)–(3.9), to proceed we may try giving different values to $\gamma$: Inserting the value $\gamma = -\frac{3}{4}$ in the system for concreteness we meet a third arbitrary constant at $j = 1$ (spec($-\frac{3}{4}K_2)$ = {$-1$, 0, 1}). We then arrive at the following asymptotic forms of the solution:

$$x = \alpha \gamma + \frac{A\alpha}{6} c_{13} \gamma^2 + \cdots,$$

$$y = \alpha + \frac{A\alpha}{3} c_{13} \gamma + \cdots,$$

$$w = c_{13} \gamma^{-1} + \cdots,$$ (3.34)

where $c_{13} \neq 0$.\(^5\) We need to check the validity of the compatibility condition for $j = 1$. But this is trivially satisfied since

$$\left( -\frac{3}{4}K_2 - I_3 \right) c_{13} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & A\alpha/3 \\ 0 & 0 & 0 \end{pmatrix} c_{13} \begin{pmatrix} A\alpha/6 \\ A\alpha/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ (3.35)

The series expansions in equations (3.32)–(3.34) are therefore valid and we conclude that as $\gamma \to 0$,

$$a \to 0, \quad a' \to \alpha, \quad \rho \to \infty, \quad \alpha \neq 0.$$ (3.36)

This is a collapse type IIc singularity. It will follow from the analysis below that the behavior of $\rho$ depends on our choice of $\gamma$ (thus giving rise to three possible subcases of a type II singularity). Indeed, choosing for instance $\gamma = -1$ (spec($-1K_2$) = {$-1$, 0, 2}), we find that the solution is given by the forms

$$x = \alpha \gamma + \frac{A\alpha}{9} c_{23} \gamma^3 + \cdots,$$ (3.37)

$$y = \alpha + \frac{A\alpha}{3} c_{23} \gamma^2 + \cdots,$$ (3.38)

\(^5\) If we do not set from the beginning $\gamma = -\frac{3}{4}$ but instead we let $\gamma$ be arbitrary, then in the last step of the calculations at the $j = 1$ level we find that either $c_{13} = 0$ or $\gamma = -\frac{3}{4}$.
\begin{equation}
w = c_{23} + \cdots, \tag{3.39}
\end{equation}

where \(c_{23} \neq 0\). Note that the compatibility condition is satisfied here as well since

\begin{equation}
(-1K_2 - 2I_3)\mathbf{c}_2 = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -2 & 2Aa/3 \\ 0 & 0 & 0 \end{pmatrix} c_{23} \begin{pmatrix} Aa/9 \\ Aa/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.40}
\end{equation}

We see that as \(Y \to 0\),

\begin{equation}
a \to 0, \quad a' \to \alpha, \quad \rho \to c_{23}, \quad \alpha \neq 0. \tag{3.41}
\end{equation}

This is a collapse type IIb singularity in our terminology and is clearly different from (3.36).

A yet different behavior is met if we choose for instance \(\gamma = -\frac{5}{4}\). The \(K\)-exponents are given by \(\text{spec}(-\frac{5}{4}K_2) = \{-1, 0, 3\}\), and the series expansions become

\begin{equation}
x = \alpha Y + \frac{Aa}{12} c_{33} Y^4 + \cdots, \tag{3.42}
\end{equation}

\begin{equation}
y = \alpha + \frac{Aa}{3} c_{33} Y^2 + \cdots, \tag{3.43}
\end{equation}

\begin{equation}
w = c_{33} Y + \cdots, \tag{3.44}
\end{equation}

where \(c_{33} \neq 0\). These expansions are valid locally around the singularity, since the compatibility condition holds true because

\begin{equation}
(-\frac{5}{4}K_2 - 3I_3)\mathbf{c}_1 = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -3 & Aa \\ 0 & 0 & 0 \end{pmatrix} c_{32} \begin{pmatrix} Aa/12 \\ Aa/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.45}
\end{equation}

For \(Y \to 0\), we have that

\begin{equation}
a \to 0, \quad a' \to \alpha, \quad \rho \to 0, \quad \alpha \neq 0, \tag{3.46}
\end{equation}

which means that this is a collapse type IIa singularity. This balance therefore leads to the asymptotic behavior of a particular solution describing a curved brane approaching a collapse type II singularity, i.e. \(a \to 0\) and \(a' \to \alpha\). The behavior of the density of the perfect fluid varies dramatically: we can have an infinite density, a constant density, or even no flow of ‘energy’ at all as we approach the finite-distance singularity into the extra dimension at \(Y_s\), depending on the values of the \(\gamma\) parameter.

We now turn to an analysis of the balances \(-1/2B_3\), for \(r < -2\), and \(-1/2B_4\). The \(K\)-matrix for \(-1/2B_3\) is

\begin{equation}
-1/2K_3 = DI (\alpha, \alpha, 0) - \text{diag}(1, 0, r) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 - r \end{pmatrix}. \tag{3.47}
\end{equation}

and hence

\begin{equation}
\text{spec}(-\frac{1}{2}K_3) = \{-1, 0, -2 - r\}. \tag{3.48}
\end{equation}

Taking \(-2 - r > 0\), we have two non-negative \(K\)-exponents. (The case \(-2 - r < 0\) is considered later, in subsection 3.5, since it is quite different, and it does not imply the existence of a finite-distance singularity.) For \(r = -3\) as an example, we substitute the forms

\begin{equation}
x = \Sigma_{j=0}^{\infty} c_{j1} Y^{j+1}, \quad y = \Sigma_{j=0}^{\infty} c_{j2} Y^j, \quad w = \Sigma_{j=0}^{\infty} c_{j3} Y^{j-3}. \tag{3.49}
\end{equation}

6 Had we let \(\gamma\) be arbitrary we would have found that in the step \(j = 2\) of the procedure either \(c_{23} = 0\) or \(\gamma = -1\).

7 Here again, had we let \(\gamma\) be arbitrary we would have found that in the step \(j = 3\) of the procedure, either \(c_{33} = 0\), or \(\gamma = -5/4\).
and arrive at the expansions

\begin{align}
  x &= \alpha \Upsilon + \cdots, \\
  y &= \alpha + \cdots, \\
  w &= c_{13} \Upsilon^{-2} + \cdots.
\end{align}

The compatibility condition is satisfied because

\begin{align}
  (-1/2 K_3 - I_3) c_1 &= \begin{pmatrix}
  -2 & 1 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 0
  \end{pmatrix} \begin{pmatrix}
  0 \\
  0 \\
  1
  \end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0
  \end{pmatrix},
\end{align}

and so the expansions (3.50)–(3.52) are valid ones in the vicinity of the singularity. The general behavior of the solution is then characterized by the asymptotic forms

\begin{align}
  a &\to 0, \quad a' \to \alpha, \quad \rho \to \infty, \quad \alpha \neq 0.
\end{align}

The balance \(-1/2 B_3\) for \(r < -2\) therefore implies the existence of a collapse type IIc singularity during the dynamical evolution of the curved brane living (and moving) in this specific perfect fluid bulk.

The balance \(-1/2 B_4\), on the other hand, is one with

\begin{align}
  -1/2 K_4 = D \Phi (\alpha, \alpha, \delta) - \text{diag}(1, 0, -2) = \begin{pmatrix}
  -1 & 1 & 0 \\
  0 & 0 & 0 \\
  28/\alpha & 28/\alpha & 0
  \end{pmatrix},
\end{align}

and

\begin{align}
  \text{spec}(-1/2 K_4) = \{-1, 0, 0\}.
\end{align}

We note that the double multiplicity of the zero eigenvalue reflects the fact that there were already two arbitrary constants \(\alpha\) and \(\delta\) in this balance (recall though that \(\delta\) had to be sacrificed in order for this balance to satisfy the constraint (3.10)). We can thus write

\begin{align}
  x &= \alpha \Upsilon + \cdots, \\
  y &= \alpha + \cdots, \\
  w &= \delta \Upsilon^{-2} + \cdots,
\end{align}

so that as \(\Upsilon \to 0\), a collapse type IIc singularity develops, i.e.

\begin{align}
  a &\to 0, \quad a' \to \alpha, \quad \rho \to \infty, \quad \alpha \neq 0.
\end{align}

### 3.3. Big rip singularities

In this subsection we return to the study of the balance \(\gamma B_1\) but focus on different \(\gamma\) values. In particular, we show that when \(\gamma < -1\), a flat brane develops a big rip singularity at a finite distance. This new asymptotic behavior implied by the balance \(\gamma B_1\) (when \(\gamma < -1\)) is equally general to the one found in subsection 3.1.

For purposes of illustration, let us take \(\gamma = -2\). Then the balance \(-2 B_1\) and the \(-2 K_{11}\)-exponents read, respectively,

\begin{align}
  -2 B_1 &= \{(\alpha, -\alpha/2, 3/(8A)), (-1/2, -3/2, -2)\},
\end{align}

where \(A\) is the angular momentum of the brane.

The balance \(-2 K_{11}\) can be written as

\begin{align}
  -2 K_{11} = \begin{pmatrix}
  -2 & 1 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 0
  \end{pmatrix} \begin{pmatrix}
  0 \\
  0 \\
  1
  \end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0
  \end{pmatrix},
\end{align}

and so the expansions (3.50)–(3.52) are valid ones in the vicinity of the singularity. The general behavior of the solution is then characterized by the asymptotic forms

\begin{align}
  a &\to 0, \quad a' \to \alpha, \quad \rho \to \infty, \quad \alpha \neq 0.
\end{align}

The balance \(-1/2 B_3\) for \(r < -2\) therefore implies the existence of a collapse type IIc singularity during the dynamical evolution of the curved brane living (and moving) in this specific perfect fluid bulk.

The balance \(-1/2 B_4\), on the other hand, is one with

\begin{align}
  -1/2 K_4 = D \Phi (\alpha, \alpha, \delta) - \text{diag}(1, 0, -2) = \begin{pmatrix}
  -1 & 1 & 0 \\
  0 & 0 & 0 \\
  28/\alpha & 28/\alpha & 0
  \end{pmatrix},
\end{align}

and

\begin{align}
  \text{spec}(-1/2 K_4) = \{-1, 0, 0\}.
\end{align}

We note that the double multiplicity of the zero eigenvalue reflects the fact that there were already two arbitrary constants \(\alpha\) and \(\delta\) in this balance (recall though that \(\delta\) had to be sacrificed in order for this balance to satisfy the constraint (3.10)). We can thus write

\begin{align}
  x &= \alpha \Upsilon + \cdots, \\
  y &= \alpha + \cdots, \\
  w &= \delta \Upsilon^{-2} + \cdots,
\end{align}

so that as \(\Upsilon \to 0\), a collapse type IIc singularity develops, i.e.

\begin{align}
  a &\to 0, \quad a' \to \alpha, \quad \rho \to \infty, \quad \alpha \neq 0.
\end{align}

### 3.3. Big rip singularities

In this subsection we return to the study of the balance \(\gamma B_1\) but focus on different \(\gamma\) values. In particular, we show that when \(\gamma < -1\), a flat brane develops a big rip singularity at a finite distance. This new asymptotic behavior implied by the balance \(\gamma B_1\) (when \(\gamma < -1\)) is equally general to the one found in subsection 3.1.

For purposes of illustration, let us take \(\gamma = -2\). Then the balance \(-2 B_1\) and the \(-2 K_{11}\)-exponents read, respectively,

\begin{align}
  -2 B_1 &= \{(\alpha, -\alpha/2, 3/(8A)), (-1/2, -3/2, -2)\},
\end{align}

where \(A\) is the angular momentum of the brane.
is also characterized by all quantities $a$ can say that this singularity bares many similarities to the one studied in [10–12], since it
between the warp factor of our braneworld and the scale factor of an expanding universe, we
of the present paper, wherein the same brane moving in a
\[ \gamma > 1 \] leads to a general solution in which a flat brane develops a big rip singularity. This implements the behavior found in subsection 3.1
\[ \gamma < -1 \] so we have to set $r = 1$ in order to have the necessary $-1$ eigenvalue corresponding to the arbitrary position of the ‘singularity’, $Y_s$. After substitution of the forms
\[ x = \Sigma_{j=0}^{\infty} c_j \gamma^j, \quad y = \Sigma_{j=0}^{\infty} c_{j+1} \gamma^{j+1}, \quad w = \Sigma_{j=0}^{\infty} c_{j+3} \gamma^{j+3}, \]
we find that the solution reads
\[ x = \alpha + c_1 \tau + \cdots, \]
(3.72)
\[ y = c_{11} + \cdots, \]
(3.73)
\[ w = 0 + \cdots. \]
(3.74)

The compatibility condition is satisfied, since
\[ \left. \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)_{c_1} = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right), \]
(3.75)
and we see that as \( \tau \to 0, \)
\[ a \to \alpha, \quad a' \to c_{11}, \quad \rho \to 0, \quad \alpha \neq 0. \]
(3.76)
This clearly indicates that the brane experiences the so-called sudden behavior (cf [7]).

3.5. Behavior at infinity

A qualitatively different picture than what we have already encountered in our analysis of brane singularities in a fluid bulk is attained by exploiting either the balance \( B_1 \) with \(-1 < \gamma < -1/2, \) or the balance \( B_2 \) with \( \gamma > -1/2, \) or the balance \( B_3 \) with \( r > -2. \)
We show in this subsection that these three balances and only these offer the possibility of avoiding the finite-distance singularities met before and may describe the behavior of our model at infinity.

We begin with the balance \( B_1 \) when \(-1 < \gamma < -1/2. \) Choosing for instance \( \gamma = -4/5, \)
we find \( \text{spec}(-4/5K_1) = \{-1, 0, -3\}, \) and hence we may expand \((x, y, w)\) in descending powers in order to meet the arbitrary constants appearing at \( j = -1 \) and \( j = -3, i.e.
\[ x = \sum_{j=0}^{\infty} c_{j1} \tau^{j+5/2}, \quad y = \sum_{j=0}^{\infty} c_{j2} \tau^{j+3/2}, \quad w = \sum_{j=0}^{\infty} c_{j3} \tau^{j-2}. \]
(3.77)
We find
\[ x = \alpha \tau^{5/2} + c_{-11} \tau^{3/2} + 3/(10\alpha) c_{-11}^2 \tau^{1/2} + c_{-31} \tau^{-1/2} + \cdots, \]
(3.78)
\[ y = 5\alpha/2 \tau^{3/2} + 3/2 c_{-11} \tau^{1/2} + 3/(20\alpha) c_{-11}^2 \tau^{-1/2} - 1/2 c_{-31} \tau^{-3/2} + \cdots, \]
(3.79)
\[ w = 75/(8\alpha) \tau^{-2} - 15/(2\alpha) c_{-11} \tau^{-3} + 9/(4\alpha^2) c_{-11}^2 \tau^{-4} \]
\[ + (-15/(2\alpha) c_{-31} - 9/(4\alpha^2) c_{-11}^2) \tau^{-5} + \cdots. \]
(3.80)
The compatibility conditions at \( j = -1 \) are satisfied, since
\[ \left. \begin{pmatrix} -3/2 \\ 15/4 \\ 75/(4\alpha) \end{pmatrix} \right)_{c_{-11}} = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right), \]
(3.81)
But for \( j = -3 \) we find
\[ \left. \begin{pmatrix} 1/2 \\ 15/4 \\ 75/(4\alpha) \end{pmatrix} \right)_{c_{-31}} = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).
(3.81)
\[
x \times \left( \begin{array}{c}
c_{-31} \\
-1/2c_{-31} \\
-15/(2A\alpha)c_{-31} - 9/(4A\alpha^3)c_{-11} \\
0 \\
-9/(10\alpha^2)c_{-11} \\
-27/(4A\alpha^3)c_{-11}
\end{array} \right) = P_{-3}.
\]

An eigenvector corresponding to the eigenvalue \( j = -3 \) is \( v^\top = (-2A\alpha/15, A\alpha/15, 1) \), and hence we have

\[
v^\top \cdot P_{-3} \neq 0,
\]

unless \( c_{-11} = 0 \). In order to satisfy the compatibility condition at \( j = -3 \) we set \( c_{-11} = 0 \). The solution (3.78)–(3.80) with \( c_{-11} = 0 \) reads

\[
x = \alpha\gamma^{5/2} + c_{-31} \gamma^{-1/2} + \cdots, \tag{3.84}
\]

\[
y = 5\alpha/2\gamma^{3/2} - 1/2c_{-31} \gamma^{-3/2} + \cdots, \tag{3.85}
\]

\[
w = 75/(8A)\gamma^{-2} - 15/(2A\alpha)c_{-31} \gamma^{-5} + \cdots, \tag{3.86}
\]

and it is a particular solution containing two arbitrary constants. As \( S \equiv 1/\gamma \to \infty \), we conclude that

\[
a \to \infty, \quad a' \to \infty, \quad \rho \to \infty, \tag{3.87}
\]

and we can therefore avoid the finite-distance singularity in this case.

Next we examine the balance \( rB_2 \) when \( \gamma > -1/2 \). For \( \gamma = 0 \), we have that \( \text{spec}(\mathcal{K}_2) = \{ -1, 0, -2 \} \), and hence we substitute

\[
x = \Sigma_{j=0}^\infty c_{j1} \gamma^j, \quad y = \Sigma_{j=0}^\infty c_{j2} \gamma^j, \quad w = \Sigma_{j=0}^\infty c_{j3} \gamma^j + \cdots, \tag{3.88}
\]

and find

\[
x = \alpha\gamma + c_{-11} - A\alpha/3c_{-23} \gamma^{-1} + \cdots, \tag{3.89}
\]

\[
y = \alpha + A\alpha/3c_{-23} \gamma^{-2} + \cdots, \tag{3.90}
\]

\[
w = c_{-23} \gamma^{-4} + \cdots. \tag{3.91}
\]

The compatibility conditions at \( j = -1 \) and \( j = -2 \) are indeed satisfied, since

\[
(\mathcal{K}_2 + \mathcal{I}_3)c_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -2A\alpha/3 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.92}
\]

and

\[
(\mathcal{K}_2 + 2\mathcal{I}_3)c_{-2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -2A\alpha/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -A\alpha/3 \\ A\alpha/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.93}
\]

As \( S \equiv 1/\gamma \to \infty \), we conclude that

\[
a \to \infty, \quad a' \to \infty, \quad \rho \to \infty, \tag{3.94}
\]

and the finite-distance singularity is shifted at an infinite distance.

We now move on to the balance \( -1/2B_3, r > -2 \). In this case we have two negative \( \mathcal{K} \)-exponents. If we choose the value \( r = 0 \), then the spectrum is found to be

\[
\text{spec}(-1/2\mathcal{K}_3) = \{ -1, 0, -2 \}. \tag{3.95}
\]
and so inserting the forms
\[ x = \sum_{j=-\infty}^{\infty} c_j \psi_j, \quad y = \sum_{j=0}^{\infty} c_j \psi_j, \quad w = \sum_{j=0}^{\infty} c_j \psi_j, \]
we obtain
\[ x = \alpha \psi + c_{-11}, \quad (3.97) \]
\[ y = \alpha, \quad (3.98) \]
\[ w = c_{-23} \psi^{-2} + \cdots, \quad (3.99) \]
which validates the compatibility conditions at \( j = -1 \) and \( j = -2 \), since
\[ (-1/3 \bar{K}_3 + I_3)c_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.100) \]
and
\[ (-1/3 \bar{K}_3 + 2I_3)c_{-2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.101) \]
We see that as \( S \equiv 1/\psi \to \infty, \)
\[ a \to c_{-11}, \quad a' \to \alpha, \quad \rho \to \infty, \quad \alpha \neq 0, \quad (3.102) \]
so that the balance \(-1/3 \bar{B}_3\) for \( r > -2 \) also offers the possibility of escaping the finite-distance singularities. Hence in such cases we find a regular (singularity-free) evolution of the brane as it travels in the bulk filled with the type of matter considered above.

### 4. Conclusions

In this paper we studied a braneworld consisting of a 3-brane embedded in a five-dimensional bulk space, filled either with a scalar field or with an analog of perfect fluid, giving special emphasis in the possible formation of finite-distance singularities away from the brane into the bulk.

For the case of a scalar field in the bulk, we have shown that the dynamical behavior of the model strongly depends on the spatial geometry of the brane, in particular whether it is flat or not. For a flat brane the model experiences a finite-distance singularity, that we call a collapse type I singularity \((a \to 0, a' \to \infty, \phi' \to \infty, \text{as} \ Y \to Ys)\), toward which all the vacuum energy decays, whereas for a curved brane the model avoids the singularity which is now located at an infinite distance. Note that in this case, the collapse type I singularity is the only possible one that can develop at a finite distance from the (flat) brane.

In the second part of this paper, we studied the dynamical ‘evolution’ of a braneworld where we replaced the scalar field with a ‘perfect fluid’ possessing a general equation of state \( P = \gamma \rho \), characterized by the constant parameter \( \gamma \). For a flat brane, we find that it is possible to have within finite distance from the brane a collapse type I singularity met previously in the case of a scalar field (where this singularity was the only type possible, as we already mentioned above). In the fluid case, we showed that in addition to that singularity which appears inevitably in all flat brane solutions with \( \gamma > -1/2 \), there are two other new types (for a flat brane): the first one is the very distinct big rip singularity which occurs with \( a \to \infty, a' \to -\infty \) and \( \rho \to \infty \), only when a phantom-type equation of state with \( \gamma < -1 \) is considered. The second one is a collapse type IIc singularity which may be described by
the behavior $a \to 0$, $a' \to \alpha$ and $\rho \to \infty$. This is less general than the collapse type I and the big rip singularities and it arises only when $\gamma = -1/2$. Besides these singular solutions, we found the surprising result of flat branes without finite-distance singularities in the region $-1 < \gamma \leq -1/2$. Moreover, for $\gamma = -1/2$, there is also a solution with sudden behavior having $a$ and $a'$ finite and vanishing density $\rho \to 0$ [7].

In contrast to the bulk scalar field case where all curved brane solutions were regular, in the case of a perfect fluid in the bulk we found also singular such solutions. The possible corresponding finite-distance singularities are the ones comprising the collapse type II class. These are singularities with $a \to 0$, $a' \to \alpha$ and $\rho, \rho_s, \rho_s, \rho_s, \rho_s \to 0, \rho_s, \rho_s, \rho_s, \rho_s, \rho_s$ (corresponding to types IIa, b and c, respectively). The interesting feature of this class of singularities is that it allows the ‘energy’ leak into the extra dimension to vary and be monitored each time by the $\gamma$ parameter that defines the type of fluid; they all arise in the region $\gamma \leq -1/2$. On the other hand, we showed that for a curved brane the possibility of avoiding the finite-distance singularities that was offered in the scalar field case is still valid here, but only in the region $\gamma \geq -1/2$.

For illustration, we present a summary of all different behaviors we found for flat and curved branes in the table above, using the notation for the various singularities introduced in section 2 after equation (2.22) and the balances (3.13)–(3.17).

An open question is whether there exist any physical constraints on $\gamma$ analogous to the weak and strong energy conditions of matter in ordinary perfect fluid cosmology. A related question is to find possible field theory realizations of the ‘exotic’ regions of $\gamma \leq -1/2$, where interesting solutions with unexpected behavior were found. The most important issue of course is to clarify the possibility of singularity avoidance at finite distance in flat brane solutions. There is no reason why the non-singular behavior for flat branes discovered here should not persist for arbitrary values of the brane tension and, indeed, it is to be expected that only particular asymptotic modes of behavior, i.e. specific detailed forms of asymptotic solutions, would depend on such values. Thus, the self-tuning mechanism appears to be a property of a general (non-singular) flat brane solution that depends on two arbitrary constants in the region $-1 < \gamma < -1/2$ (three for the general solution with sudden behavior when $\gamma = -1/2$). Similarly, as we have shown here, the existence of singular curved brane solutions in some regions of $\gamma$ is independent of the sign of the scalar curvature (as long as the latter remains nonzero for curved branes), but the particular way of asymptotic approach to the singularity is sensitive to that sign and it may therefore change with different values of the brane tension.

It would also be interesting to further investigate whether the properties of finite-distance singularities (and their possible avoidance) encountered here continue to emerge in more general systems, such as the case in which a scalar field coexists with a perfect fluid in the bulk [13]. The analysis of this more involved case that allows for fluid interactions may also shed
light to the factors that control how these two bulk matter components compete on approach to the singularity, or even predict new types of singularities that might then become feasible, as well as possible situations where they can be avoided.

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Appendix A. The method of asymptotic splittings

We refer briefly here to the basic steps of the method of asymptotic splittings. A detailed analysis can be found in [3].

Consider a system of \( n \) first-order ordinary differential equations

\[
\dot{x} = f(x),
\]

where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, f(x) = (f_1(x), \ldots, f_n(x)) \) and \( Y \) being the independent variable. In this paper, we refrain from calling \( Y \) a time variable and giving it the interpretation of time. Since we are interested in singularities located at a distance from the brane and into the bulk, it seems more appropriate to talk about finite-distance singularities and give to the \( Y \) variable a spatial interpretation. The general solution of the above system contains \( n \) arbitrary constants and describes all possible behaviors of the system starting from arbitrary initial data. Any particular solution of (A.1), on the other hand, contains less than \( n \) arbitrary constants and describes a possible behavior of the system emerging from a proper subset of initial data space.

We say that a solution of the dynamical system (A.1) exhibits a finite-distance singularity if there exists a \( Y_s \in \mathbb{R} \) and a \( x_0 \in \mathbb{R}^n \) such that

\[
\lim_{Y \to Y_s} \|x(Y; x_0)\| = \infty,
\]

where \( \| \cdot \| \) is any \( L^p \) norm. The purpose of singularity analysis (cf [3, 8]) is to build series expansions of solutions around the presumed position of a singularity at \( Y_s \) in order to study the different asymptotic behaviors of the solutions of the system (A.1) as one approaches this singularity. In particular, we look for series expansions of solutions that take the form of a Puiseux series (any log terms absent), namely a series of the form

\[
x = \Upsilon^p (a + \sum_{j=1}^{\infty} c_j \Upsilon^{q(j)}),
\]

where \( \Upsilon = Y - Y_s, p \in \mathbb{Q}^n, s \in \mathbb{N} \).

The method of asymptotic splittings for any system of the form (A.1) is realized by taking the following steps.

1. First, we find all the possible weight-homogeneous decompositions of the vector field \( f \) by splitting it into components \( f^{(j)} \): 

\[
f = f^{(0)} + f^{(1)} + \cdots + f^{(k)},
\]

with each of these components being weight homogeneous, that is to say

\[
f^{(j)}(a \Upsilon^p) = \tau^{p^{q^{(j)}} - 1} f^{(j)}(a) \quad j = 0, \ldots, k,
\]

where \( a \in \mathbb{R}^n \) and \( q^{(j)} \) are the positive non-dominant exponents that are defined by (A.7) below.
We substitute the forms $x = a \Upsilon^p$ in the system $x' = f^{(0)}(x)$ in order to find all possible dominant balances, i.e. finite sets of the form $\{a, p\}$. The order of each balance is defined as the number of the nonzero components of $a$.

For each of these balances we check the validity of the following dominance condition:

$$
\lim_{\Upsilon \to 0} \frac{\sum_{j=1}^{k} f^{(j)}(a \Upsilon^p)}{\Upsilon^{p-1}} = 0,
$$

and define the non-dominant exponents $q^{(j)}$, $j = 1, \ldots, k$, by the requirement that

$$
\frac{\sum_{j=1}^{k} f^{(j)}(\Upsilon^p)}{\Upsilon^{p-1}} \sim \Upsilon^{q^{(j)}}.
$$

The balances that cannot satisfy condition (A.6) are then discarded.

We compute the Kovalevskaya matrix $\mathcal{K}$ defined by

$$
\mathcal{K} = Df^{(0)}(a) - \text{diag} \, p,
$$

where $Df^{(0)}(a)$ is the Jacobian matrix of $f^{(0)}$ evaluated at $a$.

We calculate the spectrum of the $\mathcal{K}$-matrix, $\text{spec}(\mathcal{K})$, that is the set of its $n$ eigenvalues also called the $\mathcal{K}$-exponents. The arbitrary constants of any particular or general solution first appear in those terms in the series (A.3) whose coefficients $c_k$ have indices $k = \varrho s$, where $\varrho$ is a non-negative $\mathcal{K}$-exponent and $s$ is the least common multiple of the denominators of the set consisting of the non-dominant exponents $q^{(j)}$ and of the positive $\mathcal{K}$-exponents (cf [3, 8]). The number of non-negative $\mathcal{K}$-exponents therefore equals the number of arbitrary constants that appear in the series expansions of (A.3). There is always the $-1$ exponent that corresponds to the position of the singularity, $Y_s$. A dominant balance corresponds thus to a general solution if it possesses $n-1$ non-negative $\mathcal{K}$-exponents (the $n$th arbitrary constant is the position of the singularity, $Y_s$).

We substitute the Puiseux series:

$$
x_i = \sum_{j=0}^{\infty} c_{ji} \Upsilon^{p j/s}, \quad i = 1, \ldots, n,
$$

in the system (A.1).

We find the coefficients $c_{ji}$ by solving the recursion relations

$$
\mathcal{K} c_j - \frac{j}{s} c_j = P_j(c_1, \ldots, c_{j-1}),
$$

where $P_j$ are polynomials that are read off from the original system.

We verify that for every $j = \varrho s$, with $\varrho$ a positive $\mathcal{K}$-exponent, the following compatibility conditions hold:

$$
u^\top \cdot P_j = 0,
$$

where $\nu$ is an eigenvector associated with the positive $\mathcal{K}$-exponent $\varrho$.

We repeat the procedure for each possible decomposition.

We note that if the compatibility condition above (equation (A.11)) is violated at some eigenvalue in the $\text{spec}(\mathcal{K})$, then the original Puiseux series representation of the solution cannot be admitted and instead we have to use a $\psi$-series for each one of the eigenvalues with this property. This is a series that includes log terms of the form

$$
x = \Upsilon^p(a + \sum_{i=1}^{\infty} c_{ij} \Upsilon^{j/s} (\Upsilon^0 \log \Upsilon)^{j/s}),
$$

where $\varrho$ is the $\mathcal{K}$-exponent for which the compatibility condition is violated. The rest of the procedure in this case is the same as before.
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