CHARACTERIZING NILPOTENT LIE ALGEBRAS RELY ON THE DIMENSION OF THEIR 2-NILPOTENT MULTIPLIERS

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ABSTRACT. There are some results on nilpotent Lie algebras $L$ investigate the structure of $L$ rely on the study of its 2-nilpotent multiplier. It is showed that the dimension of the 2-nilpotent multiplier of $L$ is equal to $\frac{1}{2}n(n - 2)(n - 1) + 3 - s_2(L)$. Characterizing the structure of all nilpotent Lie algebras has been obtained for the case $s_2(L) = 0$. This paper is devoted to the characterization of all nilpotent Lie algebras when $0 \leq s_2(L) \leq 6$. Moreover, we show that which of them are 2-capable.

1. Introduction

For an $n$-dimensional nilpotent non-abelian Lie algebra $L$, it is well-know that the dimension of its Schur multiplier is equal to $\frac{1}{2}(n - 1)(n - 2) + 1 - s(L)$ for some $s(L) \geq 0$, by a result of [7, Theorem 3.1]. There are several papers devoted to investigation of the structure of an $n$-dimensional nilpotent non-abelian Lie algebra $L$ rely on $s(L)$. The structure of all nilpotent non-abelian Lie algebras $L$ is obtain when $s(L) = 0, 1, 2, 3$ in [7, 8, 12]. These results not only characterize a nilpotent Lie algebra in terms of $s(L)$ but also they can help to shorten the processes of finding the structure of a nilpotent Lie algebra $L$ in terms of $t(L) = \frac{1}{2}n(n - 1) - \dim M(L)$ (see [2, 8]).

Let $L$ be a Lie algebra presented as the quotient of a free Lie algebra $F$ by an ideal $R$. Then the 2-nilpotent multiplier of $L$, $M^{(2)}(L)$, is isomorphic to $\frac{R \cap F^3}{[R, F, F]}$. It is a less extent the $c$-nilpotent multiplier $M^{(c)}(L)$ for $c = 2$ (see [10]). The study of the 2-nilpotent multiplier of Lie algebras can lead to the classification of algebras Lie algebra into the equivalence classes as in the group theory case (see [3]). It also gives a criterion for detecting the 2-capability of Lie algebras. Recall that a Lie algebra $L$ is said to be 2-capable provided that $L \cong H/Z_2(H)$ for a Lie algebra $H$.

In [10], the second author showed that the dimension of the 2-nilpotent multiplier of an $n$-dimensional non-abelian nilpotent Lie algebra $L$ with the derived subalgebra of dimension $m$ is bounded by $\frac{1}{4}(n - m)((n + 2m - 2)(n - m - 1) + 3(m - 1)) + 3$. Then $\dim M^{(2)}(L) \leq \frac{1}{4}n(n - 2)(n - 1) + 3$ and so we have $\dim M^{(2)}(L) = \frac{1}{4}n(n - 2)(n - 1) + 3 - s_2(L)$ for some $s_2(L) \geq 0$. The structure of all non-abelian nilpotent Lie algebras is obtained when $s_2(L) = 0$ in [10]. The current paper is devoted to
obtain the structure of all nilpotent non-abelian Lie algebras \( L \) when \( 1 \leq s_2(L) \leq 6 \). Moreover, we specify which of them are capable.

2. Preliminaries

Following to Shirshov in [15], for a free Lie algebra \( L \) on the set \( X = \{x_1, x_2, \ldots\} \).

(i) The generators \( x_1, x_2, \ldots, x_n \) are basic commutators of length one and ordered by setting \( x_i < x_j \) if \( i < j \).

(ii) If all the basic commutators \( d_i \) of length less than \( t \) have been defined and ordered, then we may define the basic commutators of length \( t \) to be all commutators of the form \([d_i, d_j]\) such that the sum of lengths of \( d_i \) and \( d_j \) is \( t \), \( d_i > d_j \), and if \( d_i = [d_s, d_t] \), then \( d_j \geq d_t \). The basic commutators of length \( t \) follow those of lengths less than \( t \). The basic commutators of the same length can be ordered in any way, but usually the lexicographical order is used.

The number of all basic commutators on a set \( X = \{x_1, x_2, \ldots, x_d\} \) of length \( n \) is denoted by \( l_d(n) \). Thanks to [3], we have

\[
l_d(n) = \frac{1}{n} \sum_{m|n} \mu(m)d^m,
\]

where \( \mu(m) \) is the Möbius function, defined by \( \mu(1) = 1, \mu(k) = 0 \) if \( k \) is divisible by a square, and \( \mu(p_1 \ldots p_s) = (-1)^s \) if \( p_1, \ldots, p_s \) are distinct prime numbers. Using the the topside statement and looking [13, Lemma 1.1] and [15], we have the following.

**Theorem 2.1.** Let \( F \) be a free Lie algebra on set \( X \), then \( F^c/F^{c+i} \) is an abelian Lie algebra with the basis of all basic commutators on \( X \) of lengths \( c, c+1, \ldots, c+i-1 \) for all \( 0 \leq i \leq c \). In particular, \( F^c/F^{c+1} \) is an abelian Lie algebra of dimension \( l_d(c) \), where \( F^c \) is the \( c \)-th term of the lower central series of \( F \).

The following theorem improves the result of [1] Theorem 2.5] for \( c = 2 \) when \( L \) is a non-abelian nilpotent Lie algebra.

**Theorem 2.2.** [10, Theorem 2.14] Let \( L \) be an \( n \)-dimensional nilpotent Lie algebra with the derived subalgebra of dimension \( m \) \( (m \geq 1) \). Then \( \dim M^{(2)}(L) \leq \frac{1}{3}(n-m)((n+2m-2)(n-m-1)+3(m-1)) + 3\). If \( m = 1 \), then \( \dim M^{(2)}(L) = \frac{1}{3}n(n-1)(n-2) + 3 \) if and only if \( L \cong H(1) \oplus A(n-3) \).

3. Main Results

This section is devoted to obtain new result on the dimension of the 2-nilpotent multiplier of a non-abelian nilpotent Lie algebra. We are going to obtain the structure of all Lie algebras \( L \) such that \( 1 \leq s_2(L) \leq 6 \). We need the following two easy lemmas for the next investigation.

**Lemma 3.1.** Let \( L \) be an \( n \)-dimensional nilpotent Lie algebra with the derived subalgebra of dimension \( m \) \( (m \geq 3) \). Then \( \dim M^{(2)}(L) \leq \frac{1}{4}n(n-2)(n-1)-2 \).
Proof. By using Theorem 2.2 and our assumption, we have
\[
\dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3} \left( n - m \right) \left( n + 2m - 2 \right) \left( n - m - 1 \right) + 3(m - 1) + 3 \leq \frac{1}{3} (n - 3) \left( n + 4 \right) \left( n - 4 \right) + 3(3 - 1) + 3 \\
= \frac{1}{3} (n - 3) \left( n + 4 \right) \left( n - 4 \right) + 3 = \frac{1}{3} (n^3 - 3n^2) - \frac{10n}{3} + 10 + 3 - 2 + 2 \\
= \frac{1}{3} (n^3 - 3n^2) - 3(\frac{2n}{3} - 3) - 2 \leq \frac{1}{3} n(n - 2)(n - 1) - 2.
\]
The result is obtained.

Lemma 3.2. Let \( L \) be an \( n \)-dimensional nilpotent Lie algebra with the derived subalgebra of dimension 2. Then \( \dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3} n(n - 2)(n - 1) + 1 \).

Proof. By invoking Theorem 2.2 we have
\[
\dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3} (n - 2) \left( n + 2 \right) (n - 3) + 3 \leq \frac{1}{3} (n - 2) (n^2 - n - 3) + 3 \\
= \frac{1}{3} (n^3 - 3n^2) - (\frac{n}{3} - 5) \leq \frac{1}{3} (n^3 - 3n^2) + \frac{2n}{3} + 1 = \frac{1}{3} n(n - 2)(n - 1) + 1,
\]
as required.

Theorem 3.3. Let \( L \) be an \( n \)-dimensional nilpotent Lie algebra and \( \dim L^2 = 1 \). Then \( L \cong H(k) \oplus A(n - 2k - 1) \) and
\[(i) \quad \mathcal{M}^{(2)}(L) \cong A(\frac{1}{3} n(n - 1)(n - 2) + 3), \text{ if } k = 1.
(ii) \quad \mathcal{M}^{(2)}(L) \cong A(\frac{1}{3} n(n - 1)(n - 2)), \text{ for all } k \geq 2.
\]

Corollary 3.4. There is no \( n \)-dimensional nilpotent Lie algebra \( L \) with the derived subalgebra of dimension \( m \geq 1 \) such that \( \dim \mathcal{M}^{(2)}(L) = \frac{1}{3} n(n - 2)(n - 1) + 2 \) or equally \( s_2(L) = 1 \).

Proof. The result follows from Lemmas 3.1, 3.2 and Theorem 3.3.

By using the notation and terminology of 4.6, we have

Proposition 3.5. The 2-nilpotent multiplier of the Lie algebras
\[
L_{4,3} = \langle x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle,
L_{5,8} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle
\]
and
\[
L_{5,5} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle
\]
is abelian of dimension 6, 18 and 17, respectively.

Proof. Let \( L \cong L_{4,3} \) and \( F \) be a free Lie algebra on the set \( \{x_1, x_2\} \) and \( R = \langle [x_1, x_2, x_3] \rangle \). Since \( L_{4,3} \) is of class 3, \( F^4 \subseteq R \) and so
\[
\mathcal{M}^{(2)}(L_{4,3}) \cong \frac{\langle [x_1, x_2, x_3] \rangle + F^4/F^6}{\langle [x_1, x_2, x_3] \rangle + F^4/F^6}.
\]

Theorem 2.1 implies \( \dim F^4/F^6 = l_2(4) + l_2(5) = 3 + 6 = 9 \). It is easy to see that
\[
\langle [x_1, x_2, x_3], F, F \rangle/F^6 = \langle [x_1, x_2, x_3], F, F \rangle/F^6/F^6 = \langle [x_1, x_2, x_3] + F^6, [x_1, x_2, x_3], [x_1, x_2, x_3] + F^6 \rangle
\]
and so \( \dim [R, F, F]/F^6 = 4 \).

It follows \( \dim \mathcal{M}^{(2)}(L_{4,3}) = 10 - 4 = 6 \).

Now, let \( L \cong L_{5,8} \). Clearly, \( L = \langle x_1, x_2, x_3 | [x_2, x_3] = x_5, 0 \leq i, j, k \leq 3 \rangle \).
Proof. \(\dim M_{4 F} \).} JoHari and P. Niroomand

Therefore \([R, F, F]/F^5 = \langle [x_2, x_3, x_1, x_1] + F^5, [x_2, x_3, x_2, x_1] + F^5, [x_2, x_3, x_2, x_2] + F^5, [x_2, x_3, x_3, x_1] + F^5, [x_2, x_3, x_3, x_2] + F^5, [x_2, x_3, x_3, x_3] + F^5, [x_1, x_2, [x_1, x_3]] + F^5\) and so \(\dim [R, F, F]/F^5 = 8\). It follows \(\dim M^{(2)}(L_{5,8}) = 26 - 8 = 18\).

Let \(L \cong L_{5,5}\) and \(F\) be a free Lie algebra on the set \(\{x_1, x_2, x_4\}\) and \(R = \langle [x_1, x_2, x_3], [x_2, x_4, x_1], [x_2, x_4, x_2], [x_2, x_4, x_4], [x_1, x_4, x_1], [x_1, x_1, x_2], [x_2, x_4, x_4], [x_1, x_1, x_1] + F^4\) so \(R/F^6 \cong F^3/\langle [x_1, x_1, x_1]\rangle/\langle [x_1, x_2, x_1]\rangle + F^6\). Since \(L_{5,5}\) is of class 3, \(F^4 \subseteq R\) and so

\[
M^{(2)}(L_{5,5}) \cong \frac{F^3/\langle [x_1, x_2, x_1]\rangle + F^6}{\langle [x_1, x_4]\rangle, [F, F] + \langle [x_1, x_2, x_1]\rangle + F^5/\langle [x_1, x_2, x_1]\rangle + F^6}.
\]

Theorem 2.1 implies \(\dim F^3/F^6 = l_3(3) + l_3(4) + l_3(5) = 8 + 18 + l_3(5)\). It is easy to see that \([R, F, F]/F^6 = \langle [x_1, x_4, x_1, x_1], [x_1, x_4, x_2, x_1], [x_1, x_4, x_4, x_1], [x_1, x_4, x_4, x_2], [x_1, x_4, x_4, x_4], [x_1, x_2, [x_1, x_4]] + F^5/F^6\) and so \(\dim [R, F, F]/F^6 = 8 + l_3(5)\). Therefore \(\dim M^{(3)}(L_{5,5}) = l_3(3) + l_3(4) + l_3(5) - 1 - l_3(5) - 8 = 17\), as required.

A Lie algebra \(L\) is called capable if \(L \cong H/Z(H)\) for a Lie algebra \(H\). See 9 for more information on this topic.

**Proposition 3.6.** Let \(L\) be a non-capable \(n\)-dimensional nilpotent Lie algebra of class \(3\) with the derived subalgebra of dimension \(2\) and \(n \geq 6\). Then \(\dim M^{(2)}(L) = \frac{1}{6}(n - 1)(n - 2)(n - 3) + 2\).

**Proof.** By 11 Lemma 4.5, Corollary 4.11 and Theorem 5.1, \(Z^*(L) = L^3 \cong A(1)\) and so \(L/L^3 \cong H(1) \oplus A(n - 4)\). Since \(L\) is not 2-capable, we have \(\dim M^{(2)}(L) = \dim M^{(2)}(L/L^3) - 1 = \frac{1}{6}(n - 1)(n - 2)(n - 3) + 2\), by using Theorem 5.3 and 10 Lemma 2.2 and Theorem 3.2.

**Lemma 3.7.** There is no \(n\)-dimensional nilpotent Lie algebra \(L\) with the derived subalgebra of dimension \(2\) such that \(\dim M^{(2)}(L) = \frac{1}{6}n(n - 2)(n - 1) + 1\) or equally \(s_2(L) = 2\).

**Proof.** By contrary, let there be an \(n\)-dimensional nilpotent Lie algebra \(L\) with the derived subalgebra of dimension \(2\) such that \(\dim M^{(2)}(L) = \frac{1}{6}n(n - 2)(n - 1) + 1\). Let \(B\) be a one dimensional central ideal of \(L\) is contained in \(L^2\). Since \(\dim (L/B)^2 = 1\), we have \(\dim M^{(2)}(L/B) \leq \frac{1}{6}(n - 1)(n - 2)(n - 3) + 3\) by using Theorem 5.3.
Theorem 3.8. There is no $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension 2 such that $\dim \mathcal{M}(2)(L) = \frac{1}{3}n(n-2)(n-1)$ or equally $s_2(L) = 3$.

Proof. By contrary, let there be an $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension 2 such that $\dim \mathcal{M}(2)(L) = \frac{1}{3}n(n-2)(n-1)$. Let $B$ be a one dimensional central ideal of $L$ is contained in $L^2$. Since $\dim(L/B)^2 = 1$, we have $\dim \mathcal{M}(2)(L/B) \leq \frac{1}{3}(n-1)(n-2)(n-3) + 3$, by using Theorem 3.3. Now Theorem 2.4 implies

$$\frac{1}{3}(n-2)(n^2-n) + 1 = \frac{1}{3}n(n-1)(n-2) + 1 = \dim \mathcal{M}(2)(L) \leq \dim \mathcal{M}(2)(L/B) + \dim(L/L^2 \otimes L/L^2 \otimes B) \leq \frac{1}{3}(n-1)(n-2)(n-3) + 3 + (n-2)^2 - \dim L^3 \cap B$$

$$= \frac{1}{3}(n-2)(n^2-n-3) + 3 - \dim L^3 \cap B.$$ 

If $\dim(L) = 2$, then $L^3 = 0$ so $n \leq 5$. If $\dim(L) = 3$, then since $B = L^2 \cap Z(L) = L^3 \cong A(1)$, we have $n \leq 4$. Let $\dim(L) = 2$. Hence, our assumption and looking at the classification of all nilpotent Lie algebras listed in [9] show that $L \cong L_{5,8}$. By Proposition 3.5 we have $\dim \mathcal{M}(2)(L_{5,8}) = 18$. It contradicts our assumption that $\dim \mathcal{M}(2)(L_{5,8}) = 20$. Now, let $\dim(L) = 3$. By a similar way, we have $L \cong L_{4,3}$. Using Proposition 3.5 we have $\dim \mathcal{M}(2)(L_{4,3}) = 6$. It contradicts our assumption that $\dim \mathcal{M}(2)(L_{4,3}) = 8$. Hence, the supposition is false and the statement is true.

Theorem 3.9. There is no $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension $m = 2$ such that $\dim \mathcal{M}(2)(L) = \frac{1}{3}n(n-2)(n-1)-1$ or equally $s_2(L) = 4$.

Proof. By contrary, let there be an $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension 2 such that $\dim \mathcal{M}(2)(L) = \frac{1}{3}n(n-2)(n-1)-1$ and $B$ be a one dimensional central ideal of $L$ is contained in $L^2$. Since $\dim(L/B)^2 = 1$, we have $\dim \mathcal{M}(2)(L/B) \leq \frac{1}{3}(n-1)(n-2)(n-3) + 3$, by using Theorem 3.3. Now
Theorem 2.4] implies
\[
\frac{1}{3}(n-2)(n^2-n) - 1 = \frac{1}{3}n(n-1)(n-2) - 1 = \dim \mathcal{M}^{(2)}(L) \\
\dim \mathcal{M}^{(2)}(L/B) + \dim(L/L^2 \otimes L/L^2 \otimes B) - \dim L^3 \cap B \\
\frac{1}{3}(n-1)(n-2)(n-3) + 3 + (n-2)^2 - \dim L^3 \cap B \\
= \frac{1}{3}(n-2)(n^2 - n-3) + 3 - \dim L^3 \cap B.
\]

If \(cl(L) = 2\), then \(L^3 = 0\) so \(n \leq 6\). If \(cl(L) = 3\), then since \(B = L^2 \cap Z(L) = L^3 \cong A(1), n \leq 5\). Let \(cl(L) = 2\). Hence, our assumption and looking at the classification of all nilpotent Lie algebras listed in [4, 6], we obtain \(L \cong L_{5,5} \oplus A(1), L \cong L_{5,22}(\epsilon) \) or \(L \cong L_{6,7}(\eta)\). By Proposition 3.5 and [10, Theorem 2.5], we have \(\dim(M(2)(L_{5,5})) = 18\) and \(\dim(M(2)(L_{5,8} \oplus A(1))) = 30\). It contradicts our assumption. Now, let \(L \cong L_{5,22}(\epsilon)\) and \(B\) be a one dimensional central ideal of \(L_{5,22}(\epsilon)\) is contained in \(L_{6,22}(\epsilon)^2\). Since \(\dim(L_{6,22}(\epsilon)/B)^2 = 1\) and \(L_{6,22}(\epsilon)/B \cong H(2)\), we have \(\dim(M(2)(H(2))) = 20\), by using Theorem 3.5. Now [11, Theorem 2.4] implies \(\dim(M(2)(L_{6,22}(\epsilon))) \leq \dim(M(2)(H(2))) + \dim(H(2)/H(2)^2 \otimes H(2)/H(2)^2 \otimes B) = 20 + 16 = 36\). Similarly, we have \(\dim(M(2)(L_{6,7}(\eta))) \leq 36\). They contradict our assumption that \(\dim(M(2)(L_{6,22}(\epsilon))) = 39 = \dim(M(2)(L_{6,7}(\eta)))\). Now let \(cl(L) = 3\). Hence, by looking at the classification of all nilpotent Lie algebras listed in [6], we obtain \(L \cong L_{4,3}, L \cong L_{4,3} \oplus A(1)\) or \(L \cong L_{5,5}\). By Proposition 3.5 and [10, Theorem 2.5], \(\dim(M(2)(L_{4,3})) = 6\), \(\dim(M(2)(L_{5,5})) = 17\) and \(\dim(M(2)(L_{4,3} \oplus A(1))) = 12\). They contradict our assumption that \(s_2(L) = 4\). Hence the result is obtained. \(\square\)

**Theorem 3.10.** Let \(L\) be an \(n\)-dimensional nilpotent Lie algebra with the derived subalgebra of dimension \(m \geq 1\). Then

(i) \(\dim \mathcal{M}^{(2)}(L) = \frac{1}{3}n(n-2)(n-1) + 3\) or equally \(s_2(L) = 0\) if and only if \(L \cong H(1) \oplus A(n-3)\).

(ii) There is no \(n\)-dimensional nilpotent Lie algebra \(L\) with the derived subalgebra of dimension \(m \geq 1\) such that \(\dim \mathcal{M}^{(2)}(L) = \frac{1}{3}n(n-2)(n-1) + 2\) or equally \(s_2(L) = 1\).

(iii) There is no \(n\)-dimensional nilpotent Lie algebra \(L\) with the derived subalgebra of dimension \(m \geq 1\) such that \(\dim \mathcal{M}^{(2)}(L) = \frac{1}{3}n(n-2)(n-1) + 1\) or equally \(s_2(L) = 2\).

(iv) There is no \(n\)-dimensional nilpotent Lie algebra \(L\) with the derived subalgebra of dimension \(m \geq 2\) such that \(\dim \mathcal{M}^{(2)}(L) = \frac{1}{3}n(n-2)(n-1)\) or equally \(s_2(L) = 3\).

(v) There is no \(n\)-dimensional nilpotent Lie algebra \(L\) with the derived subalgebra of dimension \(m \geq 1\) such that \(\dim \mathcal{M}^{(2)}(L) = \frac{1}{3}n(n-2)(n-1) - 1\) or equally \(s_2(L) = 4\).

**Proof.** The result follows from Theorem 2.2, Lemma 3.1, Theorem 3.3, Corollary 3.4, Lemma 3.7, Theorems 3.8 and 3.9. \(\square\)

**Corollary 3.11.** Let \(L\) be an \(n\)-dimensional nilpotent Lie algebra with the derived subalgebra of dimension \(m \geq 2\). Then \(\dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3}n(n-2)(n-1) - 2\).
Proof. The result follows from Theorem 3.10.

Theorem 3.12. There is no n-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension $m \geq 3$ such that $\dim \mathcal{M}^{(2)}(L) = \frac{1}{3} n(n - 2)(n - 1) - 2$ or $\dim \mathcal{M}^{(2)}(L) = \frac{4}{3} n(n - 2)(n - 1) - 3$.

Proof. By contrary, let there be an $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension $m \geq 3$ such that $\dim \mathcal{M}^{(2)}(L) = \frac{1}{3} n(n - 2)(n - 1) - 2$. Let $B$ be a one dimensional central ideal of $L$ is contained in $L^2$. Since $\dim(L/B)^2 \geq 2$, we have $\dim \mathcal{M}^{(2)}(L/B) \leq \frac{1}{3}(n - 1)(n - 2)(n - 3) - 2$, by using Corollary 3.11. Now [1] Theorem 2.4] implies

\[
\frac{1}{3}(n - 2)(n^2 - n) - 2 = \frac{1}{3} n(n - 1)(n - 2) - 2 = \dim \mathcal{M}^{(2)}(L) \leq \dim \mathcal{M}^{(2)}(L) + \\
\dim L^3 \cap B \leq \dim \mathcal{M}^{(2)}(L/B) + \dim(L/L^2 \otimes L/L^2 \otimes B) \leq \\
\frac{1}{3}(n - 1)(n - 2)(n - 3) - 2 + (n - 3)^2,
\]

and so $n \leq 3$, which is a contradiction. By a similar way, we can see that there is no $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension $m \geq 3$ such that $\dim \mathcal{M}^{(2)}(L) = \frac{4}{3} n(n - 2)(n - 1) - 3$. The result follows.

Theorem 3.13. Let $L$ be an $n$-dimensional nilpotent Lie algebra with the derived subalgebra of dimension 2. Then

(i) $\dim \mathcal{M}^{(2)}(L) = \frac{1}{3} n(n - 1) = 2$ or equally $s_2(L) = 5$ if and only if $L \cong L_{5,8}$ or $L \cong L_{4,3}$.

(ii) $\dim \mathcal{M}^{(2)}(L) = \frac{1}{3} n(n - 1) - 3$ or equally $s_2(L) = 6$ if and only if $L \cong L_{5,5}$.

Proof. (i) Let there be an $n$-dimensional nilpotent Lie algebra $L$ with the derived subalgebra of dimension 2 such that $\dim \mathcal{M}^{(2)}(L) = \frac{1}{3} n(n - 1) - 2$ and $B$ be a one dimensional central ideal of $L$ in contained $L^2$. Since $\dim(L/B)^2 = 1$, we have $\dim \mathcal{M}^{(2)}(L/B) \leq \frac{1}{3}(n - 1)(n - 2)(n - 3) + 3$ by using Theorem 3.3. Now [1] Theorem 2.4] implies

\[
\frac{1}{3}(n - 2)(n^2 - n) - 2 = \frac{1}{3} n(n - 1)(n - 2) - 2 = \dim \mathcal{M}^{(2)}(L) \leq \\
\dim \mathcal{M}^{(2)}(L/B) + \dim(L/L^2 \otimes L/L^2 \otimes B) - \dim L^3 \cap B \leq \\
\frac{1}{3}(n - 1)(n - 2)(n - 3) + 3 + (n - 2)^2 - \dim L^3 \cap B
\]

\[
= \frac{1}{3}(n - 2)(n^2 - n) + 3 - \dim L^3 \cap B.
\]

If $cl(L) = 2$, then $L^3 = 0$ so $n \leq 7$. If $cl(L) = 3$, then since $B = L^2 \cap Z(L) = L^3 \cong A(1)$, $n \leq 6$. Let $cl(L) = 2$. Hence, by looking at the classification of all nilpotent Lie algebras listed in [4, 6, 11], we obtain $L \cong L_{5,8}, L \cong L_{5,8} \oplus A(1), L \cong L_{5,8} \oplus A(2), L \cong L_{6,22}(\epsilon)$, $L \cong L_{6,22}(\epsilon) \oplus A(1), L \cong L_{6,22}(\epsilon) \oplus A(1)$, $L \cong L_{6,7}(\eta), L \cong L_{6,7}(\eta) \oplus A(1), L \cong L_1$ or $L \cong L_2$. By Proposition 3.3 and [10] Theorem 2.5], $\dim(\mathcal{M}^{(2)}(L_{5,8})) = 18$ and so $\dim(\mathcal{M}^{(2)}(L_{5,8} \oplus A(1))) = 30$ and $\dim(\mathcal{M}^{(2)}(L_{5,8} \oplus A(2))) = 50$. It contradicts our assumption that $s_2(L) = 5$.

Now, let $L \cong L_{6,22}(\epsilon)$ and $B$ be a one dimensional central ideal of $L_{6,22}(\epsilon)$ is contained in $L_{6,22}(\epsilon)^2$. Since $\dim(L_{6,22}(\epsilon)/B)^2 = 1$ and $L_{6,22}(\epsilon)/B \cong H(2)$,
we have dim $\mathcal{M}(2)(H(2)) = 20$, by using Theorem 3.3. Now Theorem 2.4 implies dim $\mathcal{M}(2)(L_{6,22}(\epsilon)) \leq \dim \mathcal{M}(2)(H(2)) + \dim(H(2)/H(2)^2 \otimes H(2)/H(2)^2 \otimes B) = 20 + 16 = 36$ and hence $\dim \mathcal{M}(2)(L_{6,22}(\epsilon) \oplus A(1)) \leq 66$. Similarly, we have $\dim \mathcal{M}(2)(L_{6,7}(\eta)) \leq 36$ and hence $\dim \mathcal{M}(2)(L_{6,7}(\eta) \oplus A(1)) \leq 66$. They cannot happen because of our assumption that $s_2(L) = 5$. Also, if $L \cong L_1$ or $L \cong L_2$, then let $B$ be a one dimensional central ideal of $L$ contained in $L^2$. Since $\dim(L/B)^2 = 1$ and $L/B \cong H(2) \oplus A(1)$, we have $\dim \mathcal{M}(2)(H(2) \oplus A(1)) = 40$, by using Theorem 3.3. Now Theorem 2.4 implies $\dim \mathcal{M}(2)(L) \leq \dim \mathcal{M}(2)(H(2) \oplus A(1)) + \dim(L/L^2 \otimes L/L^2 \otimes B) = 40 + 25 = 65$, which contradicts our assumption that $s_2(L) = 5$. Hence we should have $L \cong L_{5,8}$. In the case that $\mathfrak{cl}(L) = 3$. Hence, by looking the classification of all nilpotent Lie algebras of dimension 4 listed in [6], we obtain $L \cong L_{4,3}$. Proposition 3.5 implies $\dim(\mathcal{M}(2)(L_{4,3})) = 6$ so $S_2(L_{4,3}) = 5$. By a similar way, there is no a Lie algebra such that $s_2(L) = 5$ when $\dim L \geq 5$.

(ii) By a similar technique is used in the proof of part (i), we conclude that $L \cong L_{5,5}$. The converse holds by Proposition 3.5.

\begin{proof}
The result is obtained by using Theorem 3.3, Corollary 3.11, Theorems 3.12 and 3.13.
\end{proof}

Recall from [10], a Lie algebra $L$ is said to be 2-capable if $L \cong H/Z_2(H)$ for a Lie algebra $H$. In the following corollary, we specify which ones of Lie algebras with $0 \leq s_2(L) \leq 6$ are capable.

\begin{corollary}
Let $L$ be an $n$-dimensional nilpotent Lie algebra with the derived subalgebra of dimension $m \geq 1$. Then

(a) $s_2(L) = 0$ if and only if $L \cong H(1) \oplus A(n - 3)$.

(b) There is no $n$-dimensional nilpotent Lie algebra $L$ such that $s_2(L) = 1, 2, 4$.

(c) $s_2(L) = 3$ if and only if $L \cong H(k) \oplus A(n - 2k - 1)$ for all $k \geq 2$.

(d) $s_2(L) = 5$ if and only if $L \cong L_{4,3}$ or $L \cong L_{5,8}$.

(e) $s_2(L) = 6$ if and only if $L \cong L_{5,5}$.

\end{corollary}

\begin{proof}
By using Theorem 3.14 $L$ is isomorphic to one of the Lie algebras $H(k) \oplus A(n - 3)$, for all $k \geq 1$, $L_{4,3}$, $L_{5,5}$ or $L_{5,8}$. By invoking [10] Theorem 3.3, $H(1) \oplus A(n - 3)$ is 2-capable. Let $L \cong L_{4,3}$ and $B$ be a one dimensional central ideal of $L$ contained in $L^2$. Since $\dim(L/B)^2 = 1$, we have $\dim \mathcal{M}(2)(L/B) \leq 3$, by using Theorem 3.3. Since $\dim \mathcal{M}(2)(L/B) \leq \dim \mathcal{M}(2)(L) = 5$, [10] Theorem 3.2 implies $L_{4,3}$ is 2-capable. By a similar way, $L \cong L_{5,5}$ and $L \cong L_{5,8}$ are 2-capable. Hence the result follows.
\end{proof}

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