Interacting scalar field theory in $\kappa$-Minkowski spacetime

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We construct an complex scalar field theory in $\kappa$-Minkowski spacetime, which respects $\kappa$-deformed Poincaré symmetry. One-loop calculation shows that the theory is finite and needs finite renormalization to be compatible with the $\kappa \to \infty$ limit. The loop result also has an imaginary valued correction due to the complex poles present in the propagator.

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I. INTRODUCTION

Poincaré symmetry has been a primary geometric notion for the Minkowski spacetime (MST) and played the guiding role of construction of quantum field theory. At the short distance of Planck length scale, however, the spacetime itself may change its concept due to the quantum gravity effect. On this purpose, Poincaré algebra in momentum space is deformed [1] and a new scale parameter $\kappa$ is introduced, which will be an order of Planck length. The $\kappa$-deformed Poincaré algebra (KPA) can have many different forms. Based on bicrossproduct basis [2] where the four momenta are commuting each other but the boost relation is deformed, the dual picture of the KPA is realized in terms of non-commuting spacetime [2, 3]. This non-commuting spacetime is called $\kappa$-Minkowski spacetime (KMST), where the rotational symmetry is preserved but time and space coordinates do not commute each other,

$$\left[\hat{x}^0, \hat{x}^i\right] = \frac{i}{\kappa} \hat{x}^i, \quad \left[\hat{x}^i, \hat{x}^j\right] = 0, \quad i, j = 1, 2, 3.$$  \hspace{1cm} (1)

The Planck scale parameter $\kappa$ has the role of a deformation parameter. When $\kappa$ approaches infinity, the deformed Poincaré algebra in momentum space reduces to the ordinary Poincaré algebra and therefore, the ordinary Poincaré symmetry is recovered in Minkowski spacetime. The $\kappa$-deformed realization implies that the special relativity is deformed and the energy momentum relation also has a new form. This will result in a change of the group velocity of photon. In this respect, KPA implies the doubly special relativity [4] and the deformation parameter $\kappa$ reflect the Planck scale physics.

After the appearance of the KPA, it is soon realized that the differential structure of the KMST of 4 spacetime dimension is not realized in 4 dimensional spacetime but needs to be constructed in 5 dimensional spacetime [5, 6]. This reminds of the Snyder’s approach where non-commutative coordinates are realized in 4 dimensional De Sitter space [7].
differential calculus is realized in exponential operator \( \{ e^{-ip\hat{x}} \} \) with an appropriate ordering of \( \hat{t} \) and \( \hat{x} \).

\[
d\{ e^{-ip\hat{x}} \} \equiv \hat{t}^A \partial_A \{ e^{-ip\hat{x}} \} \tag{2}
\]

where \( A = 0, 1, 2, 3, 5 \) and \( \hat{t}^A \) is the differential element, \( \hat{t}^\mu = d\hat{x}^\mu \) with \( \mu = 0, 1, 2, 3 \) and \( \hat{t}^5 \) is the new differential element. The momentum realization of the derivatives is given as

\[
\partial_A \{ e^{-ip\hat{x}} \} = \chi_A(p) \{ e^{-ip\hat{x}} \} \tag{3}
\]

and \( \chi_\mu(p) \) behaves as a 4-vector element and \( \chi_5 \) as an invariant in KPA [10],

\[
\begin{align*}
[N_i, \chi_j] &= i\delta_{ij}\chi_0, & [M_i, \chi_j] &= i\epsilon_{ijk}\chi_k \\
[N_i, \chi_0] &= i\chi_i, & [M_i, \chi_0] &= 0 \\
[N_i, \chi_5] &= 0 = [M_i, \chi_5].
\end{align*}
\tag{4}
\]

Here \( i, j = 1, 2, 3 \) and \( M_i \) and \( N_i \) are the rotation and boost generators of KPA, respectively. It is worth to mention that the corresponding derivative is realized in the 4-dimensional De Sitter space:

\[
(P_0)^2 - (P_i)^2 - (P_5)^2 = -\kappa^2, \tag{5}
\]

with \( P_\mu = \chi_\mu \) and \( P_5 = 4\chi_5 - \kappa \).

The invariant property of \( \chi_5 \) leads one to construct 4-dimensional system without invoking the fifth dimensional tangential direction if one requires physical system to respect the \( \kappa \)-deformed Poincaré symmetry (KPS). Based on this 4-dimensional differential structure, one requires the on-shell condition to be

\[
\chi_\mu\chi^\mu = m^2 \tag{6}
\]

where \( m \) is the particle mass and constructs the scalar field theory with KPS [8, 9, 10]. Still, an interacting (field) theory needs more elaboration since it is not clear how to construct the many particle states from the right choice of vacuum since the many particle states constructed so far show the non-local nature. (See for example [8, 12] and references there in).

To understand the physical effects of the \( \kappa \)-deformation, one may study black-body radiation [10] and Casimir energy [11] using the mass-shell condition only, which uses essentially the free field theory only. It turns out that the thermal energy of the blackbody radiation due to the massless mode of the KMST (\( m^2 = 0 \) in (6)) reduces to the Stephan-Boltzmann law (proportional to \( T^4 \)) when \( \kappa \to \infty \) limit is taken if one takes care of ordinary modes (OM) from the mass-shell condition (6) which reduces to the one from Einstein’s special relativity.

In the asymmetric ordering, the ordinary massless mode is explicitly given as

\[
\Omega^+_p = -\kappa \log(1 - \frac{|p|}{\kappa}), \quad \Omega^-_p = \kappa \log(1 + \frac{|p|}{\kappa}). \tag{7}
\]

In fact, the mass-shell condition (6) also allows high momentum mode (HM) which exist only when its momentum is greater than \( \kappa \).

\[
\Omega^H_p = -\kappa \log\left(\frac{|p|}{\kappa} - 1\right). \tag{8}
\]
It is shown in [10] that the Stephan-Boltzmann law would be spoiled if the HM were to be included in the thermal distribution, whose contribution turns out to be proportional to $T^2$ or $T^3$ depending on how one treats the negative energy part of HM. Thus, one needs to eliminate the HM from the on-shell. The same thing applies to the symmetric ordering case.

On the other hand, the study of Casimir energy on a spherical shell shows that in the asymmetric ordering case the vacuum can break particle and anti-particle symmetry at Planck scale: The Casimir energy of the negative mode (anti-particle) in [11] is not the same as the one due to the positive mode (particle) if the HM is not included. Thus, if one requires the vacuum to respect the particle and antiparticle symmetry at Planck scale, one cannot adopt the asymmetric ordering dispersion relation. This reasoning forces us to adopt the symmetric ordering only to have the particle and anti-particle symmetry at the Planck scale.

In this paper we are going to construct an interacting complex scalar field theory imposing the KPS. We present the essential element for the free field theory in Sec. II and construct its interacting scalar field theory in Sec. III. We evaluate the one-loop correction of propagator in Sec. IV and one-loop correction of vertex in Sec. V. Sec. VI is the summary and discussion.

II. FREE SCALAR FIELD THEORY

To construct the free field theory with KSP one defines a field variable in momentum space,

$$\phi(x) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \varphi(p).$$  

Here both the coordinate variable $x$ and momenta $p$ are treated as commuting variables. The non-commuting nature of KMST is encoded in $\kappa$- product between field variables: The product of exponential element is required to satisfy the composition rule [13]

$$e^{-ip \cdot x} \ast e^{-iq \cdot x} = e^{-i(v(p,q) \cdot x)}.$$

In this paper, we will adopt the composition law corresponding to the symmetric ordering;

$$v(p, q) = (p^0 + q^0, \mathbf{p} e^{-q_0^2/2\kappa} + \mathbf{q} e^{p_0^2/2\kappa}).$$

The homomorphism of the product of field variables reproduces the KMST effect and this way, one can avoid various conceptual difficulties of non-commuting spacetime geometry.

The KPS is the guiding principle to construct the field theory and is applied to the free scalar action explicitly in [10]. The free analogue of massive complex scalar theory is given as

$$S_2 = \int d^4x \phi^c(x) \ast \left[-\partial_\mu \ast \partial^\mu \ast -m^2\right] \phi(x).$$

$\phi^c(x)$ is the conjugate of the scalar field and is expressed just as the complex conjugate of the field in this symmetric ordering case:

$$\phi^c(x) = \int_p e^{ip \cdot x} \varphi^\dagger(p),$$

In momentum space, the action in (12) is given as

$$S_2 = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu \varphi^\dagger(p)} \left(\delta^{\mu\nu} \chi_\nu(p) - m^2\right) \varphi(p).$$
where $\kappa$-Poincaré invariance sets $\alpha = 3/(2\kappa)$ (see below [18]). Explicit form of $\chi_\mu$ is given as

$$
\chi_0(p) = \kappa \left[ \sinh \frac{p_0}{\kappa} + \frac{p^2}{2\kappa^2} \right], \quad \chi_\mu(p) = p_\mu e^{i\phi_\mu}.
$$

$\chi_\mu$ is the 4-vector $\chi[4]$ and $\chi^\mu\chi_\mu(p)$ and $\chi_5$ are invariants in KPA

$$
\chi^\mu\chi_\mu(p) = \chi_\mu\chi_\mu(-p) = M^2_s(p) \left( 1 + \frac{M^2_s(p)}{4\kappa^2} \right)
$$

(15)

$$
\chi_5 = -\frac{M^2_s(p)}{8\kappa}
$$

(16)

where $M^2_s(p)$ is the Casimir invariant

$$
M^2_s(p) = M^2_s(-p) = \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - p^2.
$$

(17)

One notes that the the integration measure given in (14) is invariant under the KPS:

$$
d^4p e^{i\phi_0} = \frac{\kappa}{4} \frac{d\chi_0 d\chi_1 d\chi_2 d\chi_3}{\chi_5}.
$$

(18)

Let’s introduce a notation for the propagator function $\Delta^{-1}(p) = \chi_\mu\chi^\mu - m^2$ which is explicitly written as

$$
\Delta(p) = \frac{4\kappa^2}{\left( 2\kappa^2 \cosh(p^0/\kappa) - p^2 - m^2 \right) \left( 2\kappa^2 \cosh(p^0/\kappa) - p^2 + m^2 \right)}
$$

$$
m^2 = 2\kappa^2 \sqrt{1 + m^2/\kappa^2}.
$$

(19)

The on-shell dispersion relation is given as $\Delta^{-1}(p) = 0$;

$$
2\kappa^2 \cosh(p^0/\kappa) = p^2 + m^2
$$

(20)

$$
2\kappa^2 \cosh(p^0/\kappa) = p^2 - m^2.
$$

(21)

The dispersion relation in (20) corresponds to the ordinary mode (OM) since this reduces to the ordinary particle and antiparticle dispersion relation as $\kappa \rightarrow \infty$. The second one (21) corresponds to the tachyon mode since the mode is obtained by putting $m_\kappa \rightarrow im_\kappa$ in (20). The tachyon mode, when its momentum is sufficiently large $p^2 - m^2 > 0$, becomes a real mode corresponding to the high momentum mode (HM). This HM should not be included in on-shell mode since HM will spoil the blackbody radiation law at $\kappa \rightarrow \infty$ limit [10].

The propagator function $\Delta(p)$ has the periodic property

$$
\Delta(p_0 + i2\kappa\pi, p) = \Delta(p_0, p)
$$

(22)

and thus possesses an infinite number of poles on the complex plane of $p^0$. It is convenient for later use to separate the OM and TM contribution, each satisfying the periodicity relation [22];

$$
\Delta(p) = \Delta_P(p) - \Delta_T(p)
$$

$$
\Delta_P(p) = \frac{2\kappa^2/m_\kappa^2}{2\kappa^2 \cosh(p^0/\kappa) - p^2 - m^2}
$$

$$
\Delta_T(p) = \frac{2\kappa^2/m_\kappa^2}{2\kappa^2 \cosh(p^0/\kappa) - p^2 + m^2}.
$$

(23)
III. INTERACTING SCALAR FIELD THEORY

We will assume there is one complex scalar field in this paper. Extension to many fields is straight-forward. To find an interaction which respects KPS, one notices that KPS is preserved in the $\kappa$-product interaction

$$\int d^4x \phi_2^*(x) \ast \phi_2(x)$$

(24)

where $\phi_2(x)$ represents composite two fields. There are two ways to represent $\phi_2$;

$$\phi_2^{(A)}(x) = \phi(x)\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \varphi_2^{(A)}(p)$$

$$\phi_2^{(B)}(x) = \phi^c(x)\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \varphi_2^{(B)}(p)$$

(25)

where

$$\varphi_2^{(A)}(p) = \int \frac{d^4q}{(2\pi)^4} \varphi_2(p-q)\varphi_2(q)$$

$$\varphi_2^{(B)}(p) = \int \frac{d^4q}{(2\pi)^4} \varphi_2^\dagger(p-q)\varphi_2(q)$$

(26)

This allows two types of interactions.

$$S_4^{(A)} = \lambda_A \int \frac{d^4p}{(2\pi)^4} e^{\alpha \rho\varphi_2^{(A)\dagger}(p)\varphi_2^{(A)}(p)}$$

$$= \lambda_A \int \frac{d^4p_1d^4p_2d^4p_3d^4p_4}{(2\pi)^{12}} e^{\alpha(p_1^0 + p_2^0)} \varphi_2(p_1)\varphi_2^\dagger(p_2)\varphi_2(p_3)\varphi_2(p_4)\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

$$S_4^{(B)} = \lambda_A \int \frac{d^4p}{(2\pi)^4} e^{\alpha \rho\varphi_2^{(B)\dagger}(p)\varphi_2^{(B)}(p)}$$

$$= \int \frac{d^4p_1d^4p_2d^4p_3d^4p_4}{(2\pi)^{12}} \cosh \left( \frac{\alpha(p_1 - p_2)}{2} \right) \cosh \left( \frac{\alpha(p_3 - p_4)}{2} \right) \varphi_2(p_1)\varphi_2^\dagger(p_2)\varphi_2(p_3)\varphi_2(p_4)\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

(27)

(28)

where the bosonic permutation symmetry of the scalar field is used in the last identity. It turns out that the B-type interaction, however, spoils the KPS after loop correction. Thus we will consider A-type interaction only (27).

Our action is written as

$$S = \int d^4x \left( \phi^c(x) \ast [-\partial_{\mu} \ast \partial^\mu \ast m^2] \phi(x) - \frac{\lambda}{4} \phi_2^2(x) \ast \phi_2(x) \right)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{\alpha \rho\varphi^\dagger(p)\varphi(p)\Delta^{-1}(p)}$$

$$- \frac{\lambda}{4} \int \frac{d^4p_1d^4p_2d^4p_3d^4p_4}{(2\pi)^{12}} e^{\alpha(p_1^0 + p_2^0)} \varphi^\dagger(p_1)\varphi^\dagger(p_2)\varphi(p_3)\varphi(p_4)\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

(29)
From this action the Feynman rule follows. The propagator is given as (see Fig. 1 for notation)

\[ S_F^{(0)}(p_1, p_2) = S_F(p_1) \, \delta(4)(p_1 - p_2) \]  

(30)

where \( S_F(p) = e^{-\alpha p^0} \Delta(p) \). Four point vertex is given as \( \Gamma_4^{(0)}(p_1, p_2; p_3, p_4) \) where

\[ \Gamma_4^{(0)}(p_1, p_2; p_3, p_4) = \lambda \frac{1}{(2\pi)^4} e^{\alpha(p_1^0 + p_2^0)} \delta(4)(p_1 + p_2 - p_3 - p_4). \]  

(31)

FIG. 1: Feynman rule for two and four point function

IV. ONE LOOP CORRECTION OF THE PROPAGATOR

One loop correction of the propagator is given as

\[ S_F^{(1)}(p) = \lambda \int \frac{d^4q}{(2\pi)^4} e^{\alpha(p^0 + q_0)} e^{-\alpha q^0} \Delta(q) \equiv e^{\alpha p^0} \Delta \Gamma_2^{(1)}(p) \]  

(32)

where \( \Delta \Gamma_2^{(1)}(p) \) is independent of the external momentum \( p \)

\[ \Delta \Gamma_2^{(1)}(p) = \lambda \int \frac{d^4q}{(2\pi)^4} \Delta(q). \]  

(33)

Using the explicit form of \( \Delta(p) \) in (19) one may put this two point function correction as

\[ \Delta \Gamma_2^{(1)}(p) = \lambda \int \frac{d^4q}{(2\pi)^4} \left\{ \Delta_F(q) - \Delta_T(q) \right\} \]

\[ = \lambda \frac{\kappa}{m_\kappa^2} \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{dt}{2\pi} \left\{ \frac{1}{(t-\beta)(t-1/\beta)} - \frac{1}{(t-\gamma)(t-1/\gamma)} \right\} \]  

(34)

where \( t = e^{q^0/\kappa} \) is used in the integration and

\[ \beta + \frac{1}{\beta} = 2a, \quad a = \frac{q^2 + m_\kappa^2}{2\kappa^2}; \quad \gamma + \frac{1}{\gamma} = 2b, \quad b = \frac{q^2 - m_\kappa^2}{2\kappa^2}. \]  

(35)
Here $a > 1$ and $\beta$ and $1/\beta$ are positive real for the whole range of the 3-momentum integration. Therefore, the integrand has two simple poles at $t = \beta$ and $1/\beta$ where we set
\[
\beta = a + \sqrt{a^2 - 1}, \quad 1/\beta = a - \sqrt{a^2 - 1}.
\] (36)

To avoid this singularity, one employs the small $\epsilon$ prescription so that the pole lies at $t = \beta - i\epsilon, 1/\beta + i\epsilon$. The $\epsilon$ prescription ($t = \beta - i\epsilon, t = 1/\beta + i\epsilon$) amounts to put ($p^0 = \sqrt{m^2 + p^2} - i\epsilon, p^0 = -\sqrt{m^2 + p^2} + i\epsilon$) in the ordinary field theory when $\kappa \to \infty$.

After this prescription, one can evaluate the integration using the contour given in Fig. (2). The contribution over the quarter circles at infinity is neglected since it is canceled by the tachyon contribution. This prescription results in the integration
\[
I_P(\beta) = \int_0^\infty \frac{dt}{2\pi} \frac{1}{(t - \beta + i\epsilon)(t - 1/\beta - i\epsilon)} = I_P^{(W)}(\beta) + I_P^{(C)}(\beta)
\]
\[
I_P^{(W)}(\beta) = \frac{-i}{\beta - 1/\beta}
\]
\[
I_P^{(C)}(\beta) = i \int_0^\infty \frac{d\tau}{2\pi} \frac{1}{(i\tau - \beta)(i\tau - 1/\beta)} = -\frac{1}{\beta - 1/\beta} \left( \frac{\ln(\beta^2)}{2\pi} \right)
\] (37)

where $I_P^{(W)}(\beta)$ is the pole contribution and $I_P^{(C)}(\beta)$ is the integrated value along the imaginary axis.

---

**FIG. 2:** contour integration of $t$.

About the tachyon contribution, the real part of $\gamma$ changes its sign depending on the 3-momentum range. In addition, $\gamma$ becomes complex when $0 < b^2 < 1$. Noting that $\gamma$ and $1/\gamma$ poles correspond to the tachyon poles ($m_\kappa \to im_\kappa$ from the $\beta$ and $1/\beta$ pole) one can prescribe $\gamma$ to lie on the lower half plane of the complex $t$-plane for the whole range of the 3-momentum and $1/\gamma$ on the upper half plane.

\[
\gamma = \begin{cases} 
  b + \sqrt{b^2 - 1} - i\epsilon & \text{when } b^2 > 1 \\
  b - i\sqrt{1 - b^2} \equiv e^{-i\Theta_\gamma} & \text{when } -1 < b < 1
\end{cases}
\]
\[
\frac{1}{\gamma} = \begin{cases} 
  b - \sqrt{b^2 - 1} + i\epsilon & \text{when } b^2 > 1 \\
  b + i\sqrt{1 - b^2} \equiv e^{i\Theta_\gamma} & \text{when } -1 < b < 1
\end{cases}
\] (38)
where \( \cos \Theta_{\gamma} = b \) and \( 0 < \Theta_{\gamma} < \pi \). The schematic flow is seen in the Fig. 3.

![Flow schematic](image)

**FIG. 3:** Flow of \( \gamma \) and \( 1/\gamma \) during the 3-momentum integration when magnitude of momentum increases.

This prescription results in the \( t \)-integration

\[
\mathcal{I}_T(\gamma) = \int_0^\infty \frac{dt}{2\pi} \frac{1}{(t - \gamma - i\epsilon)(t - 1/\gamma - i\epsilon)} = \mathcal{I}_T^{(W)}(\gamma) + \mathcal{I}_T^{(C)}(\gamma)
\]

\[
\mathcal{I}_T^{(W)}(\gamma) = \frac{-i}{\gamma - 1/\gamma} \theta(b)
\]

\[
\mathcal{I}_T^{(C)}(\gamma) = i \int_0^\infty \frac{d\tau}{2\pi} \frac{1}{(i\tau - \gamma)(i\tau - 1/\gamma)} = -\frac{1}{\gamma - 1/\gamma} \left( \frac{\ln \gamma^2}{2\pi} \right)
\]

where \( \theta(b) \) is the Heavyside step function so that the pole contribution is not vanishing only when \( b > 1 \).

After the \( t \)-integration, one is left with the form,

\[
\Delta \Gamma_2^{(1)}(p) = \lambda \kappa^2 \frac{F(x)}{x}.
\]

Here, \( F(x) \) is a function of \( x \equiv m^2_\kappa/(2\kappa^2) = \sqrt{1 + m^2/\kappa^2} \) with \( x \geq 1 \) and is given in terms of the 3-momentum integration:

\[
F(x) = -\frac{\sqrt{2}}{\pi^2} \int_0^\infty r^2 dr \left( \frac{1}{B - 1/B} \left( i + \frac{\ln(B^2)}{2\pi} \right) - \frac{1}{C - 1/C} \left( i\theta(r - r_2) + \frac{\ln(C^2)}{2\pi} \right) \right)
\]

\[
B = r^2 + x + \sqrt{(r^2 + x)^2 - 1}
\]

\[
C = \begin{cases} r^2 - x + \sqrt{(r^2 - x)^2 - 1} & \text{when } 0 < r < r_1 \text{ or } r > r_3 \\ r^2 - x - i\sqrt{1 - (r^2 - x)^2} & \text{when } r_1 < r < r_3 \end{cases}
\]

where \( r_1 = \sqrt{x - 1} \), \( r_2 = \sqrt{x} \), and \( r_3 = \sqrt{x + 1} \).
Noting that the one-loop correction $S_F^{(1)}(p)$ is proportional to the measure factor $e^{\alpha p^0}$, one can shift the propagator function

$$\Delta^{-1}(p) \rightarrow \Delta^{-1}(p) - \Delta\Gamma_2^{(1)}(p),$$

which will shift the mass $m_\kappa^4 \rightarrow m_\kappa^4 + 4\kappa^4 \lambda F(x)/x$ or equivalently

$$m^2 \rightarrow m^2 + \lambda\kappa^2 \frac{F(x)}{x}.$$  

(44)

It turns out that $F(x)$ is finite but has imaginary part,

$$\frac{\pi^2}{\sqrt{2}} \text{Re} (F(x)) = - \int_0^\infty \frac{r^2 \ln(B^2)}{4\pi \sqrt{(r^2 + x)^2 - 1}} + \left( \int_0^{r_1} + \int_{r_3}^\infty \right) \frac{r^2 \ln(C^2)}{4\pi \sqrt{(r^2 - x)^2 - 1}}$$

$$+ \int_{r_1}^{r_3} \frac{r^2 \cos^{-1}(r^2 - x)}{2\pi \sqrt{1 - (r^2 - x)^2}} - \int_{r_2}^{r_3} \frac{r^2}{2\sqrt{1 - (r^2 - x)^2}}$$

(45)

$$\frac{\pi^2}{\sqrt{2}} \text{Im} (F(x)) = - \int_0^\infty \frac{r^2}{2\sqrt{(r^2 + x)^2 - 1}} + \int_{r_3}^\infty \frac{r^2}{2\sqrt{(r^2 - x)^2 - 1}}.$$  

(46)

The quadratic divergence in the particle and anti-particle contribution at large momentum is compensated by the tachyon contribution. Explicit evaluation is given as

$$F(x) = 0.01803 + i/\pi^2 + O(x - 1).$$

(47)

The price for this finiteness is that $F(x)$ is not real. The imaginary contribution arises from the complex poles present in the propagator, which have the role in the off-shell loop correction.

![FIG. 4: Contour integration along $t = 1$](image-url)
This imaginary contribution can be seen using a different contour integration of $t$ as in Fig. [4] where
\[
\int_0^\infty \frac{dt}{(t - \beta + i\epsilon)(t - 1/\beta - i\epsilon)} = \int_{-\infty}^\infty \frac{id\tau}{(1 + i\tau - \beta)(1 + i\tau - 1/\beta)} - \int_0^\infty \frac{id\tau}{(i\tau - \beta)(i\tau - 1/\beta)} = \frac{1}{\beta - 1/\beta} \left( \ln \left( \frac{1 - \beta + i\tau}{1 - \beta + i\tau} \right) \right|_{\tau = 0}^{\tau = \infty} - \ln \left( \frac{i\tau - \beta}{i\tau - 1/\beta} \right) \right|_{\tau = 0}^{\tau = \infty} \]  
(48)

which is the same as the one in (37). The contour in Fig. [4] on the other hand, can be regarded as the Wick rotation $p^0 \rightarrow ip^0$ at $\kappa \rightarrow \infty$, since $t - 1 \approx p^0/\kappa$. This shows that one cannot do the Wick rotation without including the poles at the complex $p^0$-plane due to the periodicity of $\Delta(p)$ in (22) in this KMST theory.

In addition, as $\kappa \rightarrow \infty$ the one loop correction becomes infinite since $\Delta\Gamma_2^{(1)}(p)$ is quadratic in $\kappa$. This forces one to renormalize away the imaginary mass correction as well as the quadratic $\kappa$ term to have the proper theory at $\kappa \rightarrow \infty$ limit; $m^2 + \Delta m^2$ with
\[
\left( \Delta m^2 \right)^{(1)} = -\lambda\kappa^2 \frac{F(x)}{x}. \]  
(49)

**V. ONE LOOP CORRECTION OF THE VERTEX**

The one-loop correction of the four point function is given as
\[
\Gamma_4^{(1)}(p_1, p_2; p_3, p_4) = i\lambda^2 e^{\alpha(p_3^0 + p_4^0)} \int \frac{d^4q}{(2\pi)^4} \Delta \left( q + \frac{p_1 - p_3}{2} \right) \Delta \left( q - \frac{p_1 - p_3}{2} \right) \frac{1}{2} \Delta \left( \frac{p_1 + p_2}{2} + q \right) \left( \frac{p_1 + p_2}{2} - q \right) + p_3 \leftrightarrow p_4 \]  
(50)

At $p_i = 0$ one has the one-loop correction, $\Gamma_4^{(1)}$ as a function of $x = m_\kappa^2/(2\kappa^2)$

\[
\Gamma_4^{(1)}(0; x) = \lambda^2 \int \frac{d^4q}{(2\pi)^4} A^2(q) = \frac{3\sqrt{2}}{\pi^2} \lambda^2 \int_0^\infty r^2 dr I(r; x) \]  
(51)

where \( r = |q|/(\sqrt{2}\kappa^2) \) and $B$ and $C$ are defined in (42). $I_C(r; x)$ is the imaginary axis contribution

\[
I_C(r; x) = i \int_0^\infty \frac{dt}{2\pi t} \left( \frac{4t^2}{(t - B)(t - 1/B)(t - C)(t - 1/C)} \right)^2 
\]  
(52)
FIG. 5: One loop vertex correction

and $I_W(r; x)$ is the pole contribution,

$$I_W(r; x) = -\theta(r) \left( \frac{B^2(1 + B^2)}{(B^2 - 1)^3} + \frac{1}{2x B^2 - 1} \right) - \theta(r - 1) \left( \frac{C^2(1 + C^2)}{(C^2 - 1)^3} - \frac{1}{2x C^2 - 1} \right)$$

(53)

Let us consider the value at the imaginary axis. One may put $I_C(r)$ in a more convenient form,

$$I_C(r; x) = i \int_0^\infty \frac{d\tau}{2\pi\tau} \frac{1}{(r^2 + x - 1 + a_0^2)(r^2 + x - 1 - a_0^*)^2}$$

(54)

where $a_0 = 1 - i(\tau - 1/\tau)/2$ and $a_0^*$ is the complex conjugate of $a_0$. Introducing $\tau = e^\theta$, one has $a_0 = \cosh \theta e^{-i\xi}$ and $\tan \xi \equiv \sinh \theta$. Integration of $I_C(r; x)$ over $r$ is given as

$$C(x) = \int_0^\infty r^2 dr I_C(r; x)$$

$$= i \int_{-\infty}^\infty \frac{d\theta}{2\pi} \int_0^\infty \frac{dr}{(r^2 + x - 1)^2 + (a_0 - a_0^*)(r^2 + x - 1 - 1)^2}.$$  

(55)

After this, one may interchange the integration of $r$ and $\tau$. Integration over $r$ at the massless limit ($x = 1$) is given as

$$\int_0^\infty \frac{dr}{(r^2 + a_0^2)(r^2 - a_0^*)^2} = \frac{1}{(\cosh \theta)^{5/2}} \int_0^\infty \frac{ds}{(s^4 - 2i \sin \xi)(s^2 + 1)^2}$$

$$= \begin{cases} \frac{\pi e^{3i\xi/2}}{4(e^{2\xi} + 1)^3} & \text{when } \theta > 0 \\ \frac{\pi e^{3i\xi/2}}{4(e^{2\xi} - 1)^3} & \text{when } \theta < 0 \end{cases}$$

(56)
And the integration over \( \tau \) gives

\[
C(x = 1) = i \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{1}{(\cosh \theta)^{5/2}} \left( \frac{\theta(\theta) \pi e^{3i\xi/2}}{4(e^{i\xi} + i)^3} + \theta(-\theta) \frac{\pi e^{3i\xi/2}}{4(e^{i\xi} - i)^3} \right)
\]

\[
= i \frac{1}{8} \int_{-\infty}^{\infty} \frac{d\theta}{\cosh \theta} \left( \frac{(1 + i \sinh \theta)^{3/2}}{1 + (\sinh \theta + \cosh \theta)^3} + \frac{(1 + i \sinh \theta)^{3/2}}{1 + (\sinh \theta - \cosh \theta)^3} \right)
\]

\[
= \left( \int_{0}^{\infty} \frac{d\theta}{\cosh \theta} \frac{(1 + i \sinh \theta)^{3/2}}{(1 + i e^{\theta})^3} + \text{c.c.} \right)
\]

\[
= -0.033257 i
\]

which is finite but imaginary, absent in the ordinary field theory at \( \kappa \to \infty \). \( x \) dependence of the integration is given in powers of \((x - 1)\) or \((m/2\kappa)^2\).

\( \mathcal{I}_W \) is conveniently rewritten in terms of \( a = r^2 + x \) and \( b = r^2 - x = a - 2x \) using (42):

\[
\mathcal{I}_W(r; x) = -\theta(a - x) \left( \frac{a}{4(a^2 - 1)^{3/2}} + \frac{1}{4x(a^2 - 1)^{1/2}} \right)
\]

\[
- \theta(b) \left( \frac{b}{4(b^2 - 1)^{3/2}} - \frac{1}{4x(b^2 - 1)^{1/2}} \right)
\]

Integrating \( \mathcal{I}_W(r; x) \) over \( r \), one has

\[
\int_{0}^{\infty} dr r^2 \mathcal{I}_W(r; x) = -\frac{1}{8} \int_{0}^{\infty} da \sqrt{a - x} \left( \frac{a}{(a^2 - 1)^{3/2}} + \frac{1}{x(a^2 - 1)^{1/2}} \right)
\]

\[
- \frac{1}{8} \int_{0}^{\infty} db \sqrt{b + x} \left( \frac{b}{(b^2 - 1)^{3/2}} - \frac{1}{x(b^2 - 1)^{1/2}} \right)
\]

In this way, one meets the role of branch cut at \( b = \pm 1 \) given in Fig. 6. For example, one may evaluate the integration of \( b \) avoiding the branch cut along the upper half unit circle in Fig. 6

\[
\int_{0}^{2\pi} \frac{db}{(b - 1)^{n/2}} = \int_{0}^{1-2} \frac{db}{(e^{-i\pi(x - 1)})^{n/2}} + \int_{0}^{2} \frac{db}{(b - 1)^{n/2}} - \int_{0}^{\pi} \frac{i e^{i\phi} d\phi}{(e^{i\phi})^{n/2}}
\]

\[
= i \int_{-\pi}^{0} d\phi e^{i(1-n/2)\phi} = \frac{2}{n-2} (-e^{i\pi n/2} - 1)
\]

which has no \( \epsilon \) dependence in the final result for integer \( n \).

One can conveniently divide the integration (59) as follows:

\[
\int_{0}^{\infty} dr r^2 \mathcal{I}_W(r; x) = \mathcal{W}_1(x) + \mathcal{W}_2(x) + \mathcal{W}_3(x).
\]

\( \mathcal{W}_1(x) \) is the ultraviolet contribution

\[
\mathcal{W}_1(x) = -\frac{1}{8} \int_{0}^{\infty} dc \left( \frac{c\sqrt{c - x} + \sqrt{c + x}}{(c^2 - 1)^{3/2}} + \frac{\sqrt{c - x} - \sqrt{c + x}}{x(c^2 - 1)^{1/2}} \right)
\]

\[
= -0.199907 + O(x - 1)
\]

(62)
which turns out to be finite. $\mathcal{W}_2$ and $\mathcal{W}_3$ are infrared parts. $\mathcal{W}_2$ involves the branch cut at $b = \pm 1$ in Fig. 6.

\[
W_2(x) = -\frac{1}{8} \int_0^2 db \frac{\sqrt{b + x}}{(b^2 - 1)^{3/2}} \left( b - \frac{b^2 - 1}{x} \right). \tag{63}
\]

Putting $b = 1 + e^{i\phi}$ one has

\[
W_2(x) = -\frac{i}{8} \int_{-\pi}^0 d\phi \frac{e^{-i\phi/2} \sqrt{1 + x + e^{i\phi}}}{(2 + e^{i\phi})^{3/2}} \left( 1 + e^{i\phi} - \frac{e^{i\phi}(2 + e^{i\phi})}{x} \right).
\]

Putting $b = 1 + e^{i\phi}$ one has

\[
W_2(x) = -\frac{i}{8} \int_{0}^{\pi} d\phi \frac{e^{-i\phi/2} \sqrt{1 + x + e^{i\phi}}}{(2 + e^{i\phi})^{3/2}} \left( 1 + e^{i\phi} - \frac{e^{i\phi}(2 + e^{i\phi})}{x} \right)
\]

\[
= 0.320599 + 0.0470968i + O(x - 1). \tag{64}
\]

$\mathcal{W}_3(x)$, on the other hand, is infrared sensitive

\[
W_3(x) = -\frac{1}{8} \int_x^2 da \sqrt{a - x} \left( \frac{a}{(a^2 - 1)^{3/2}} + \frac{1}{x(a^2 - 1)^{1/2}} \right)
\]

\[
= \frac{1}{16\sqrt{2}} \ln(x - 1) - 0.086266 + O(x - 1). \tag{65}
\]

Note that the logarithmic term is divergent when $x \to 1$ limit, which can be considered either as the massless limit or as $\kappa \to \infty$ limit. Combining all the terms, one has

\[
\Gamma_4^{(1)}(0; x) = \frac{3\sqrt{2}}{\pi^2} \lambda^2 \left( \frac{1}{16\sqrt{2}} \ln(x - 1) + 0.034426 + 0.013840i + O(x - 1) \right). \tag{66}
\]

VI. SUMMARY AND DISCUSSION

We considered an interacting complex scalar field theory in the $\kappa$-Minkowski spacetime. The theory is given in momentum space representation based on the symmetric ordering of the $\kappa$-deformation of Poincaré algebra.

Explicit calculation shows that the one loop correction is finite, which has been the old dream of non-commutative theory since Snyder’s work [7]. Even though the theory is
finite, the theory does show the divergent behavior as $\kappa \to \infty$ limit since the propagator correction is order of $\kappa^2$ and the vertex correction is order of $\ln(m/\kappa)$. In addition, the loop correction inevitably induces the imaginary correction due to the presence of the complex poles present in the propagator. Thus, one needs to make a finite renormalization to have the ordinary complex field theory at the $\kappa \to \infty$ limit. In this way, one can see a renormalization group flow of the theory in terms of the new scale $\kappa$. It is worth to mention that the logarithmic dependence of $\kappa$ appears in the infrared sensitive way through the ratio of Planck scale $\kappa$ and the infrared scale $m$.

Finally, one can confirm that $\kappa$-deformed Poincaré symmetry is respected in this complex scalar theory even after the loop correction since the exponential measure factor is maintained. This is because in this A-type interaction [27] the exponential measure term in the vertex and the one in the internal propagator compensate each other as far as the internal momentum is concerned. If one considered a real scalar theory, then the Bose symmetry requires the measure factor to be a cosh function rather than an exponential as in $B$-type interaction [28] and this would spoil the KPS after the loop correction.

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[1] J. Lukierski, A. Nowicki, H. Ruegg, and V. N. Tolstoy, Phys. Lett. B264, 331 (1991); J. Lukierski and H. Ruegg, Phys. Lett. B329, 189 (1994) [arXiv:hep-th/9310117].
[2] S. Majid and H. Ruegg, Phys. Lett. B334, 348 (1994) [arXiv:hep-th/9405107].
[3] S. Zakrzewski, J. Phys A27, 2075 (1994).
[4] G. Amelino-Camelia, Phys. Lett. B510, 255 (2001) [arXiv:hep-th/0012238]; Int. J. Mod. Phys. D11, 35 (2002) [arXiv:gr-qc/0012051].
[5] A. Sitarz, Phys. Lett. B349, 42 (1995) [arXiv:hep-th/9409014].
[6] C. Gonera, P. Kosiński, and P. Maślanka, J. Math. Phys. 37, 5820 (1996) [arXiv:q-alg/9602007].
[7] H. Snyder, Phys. Rev. 71, 38 (1947).
[8] P. Kosiński, J. Lukierski, and P. Maślanka, Phys. Rev. D62, 025004 (2000) [arXiv:hep-th/9902037].
[9] H.-C. Kim, J. H. Yee, and C. Rim, Phys. Rev. D75, 045017 (2007) [arXiv:hep-th/0701054].
[10] H.-C. Kim, C. Rim, and J. H. Yee, Phys. Rev. D76, 105012 (2007) [arXiv:0705.4628].
[11] H.-C. Kim, C. Rim, and J. H. Yee, [arXiv:0710.5633].
[12] M. Daszkiewicz, J. Lukierski and M. Woronowicz, [arXiv:0708.1561].
[13] L. Freidel, J. Kowalski-Glikman and S. Nowak, Phys. Lett. B648, 70 (2007) [arXiv:hep-th/0612170].