A PREDICTABLE ROGUE WAVE AND GENERATING MECHANISMS

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\textbf{Abstract.} Due to the widely applications in almost all branches of science, high dimensional KP equation is selected as universal model to describe rogue wave phenomenon. A lump is an algebraically localized wave decayed in all space directions and exists in all time. Starting from a special lump containing seven arbitrary independent parameters and four constraint conditions with all the physical properties shown, an invisible lump is found with the combination of lump part and exponential part. Because of the domination of the exponential part, the lump will be invisible in some special area, or the lump is cutoff by the induced visible soliton. While the lump part remains invariant, lump will keep its positions, path and amplitude before it is invisible. Furthermore, as a rogue wave/instanton is a localized wave decayed in all space and time directions, a rogue wave / instanton can also be produced by cutting a lump between two visible solitons. The special dispersive for the visible soliton(s) shows the soliton(s) are completely determined by the lump or the visible soliton(s) are induced by the lumps. Because the induced soliton(s) is visible, it is possible to give a prediction of the positions, the wave height and even the path for such kind of rogue waves.

I. Introduction

Rogue waves correspond to large-amplitude waves are extreme events that seldom appear on the ocean surfaces. Such waves can be accompanied by deep troughs (holes), which occur before and/or after the largest crest. In the last decades, a large interest including theories and experiments has spread from oceanography into several other areas of research, such as nonlinear optical systems [1, 2, 3], plasmas [4, 5], fluid dynamics and atmosphere [6, 7, 8], Bose-Einstein condensations (BECs) [9], financial system [10], microwave oscillators [11], etc. Various physical models of the rogue wave phenomenon have been intensively developed and many laboratory experiments [12, 13, 14] conducted. These investigations makes scientists to understand the physics of the huge wave appearance and its relation to environmental conditions. Though the origin of rogue waves is still a matter of debate [15], there are two important and common features characterizing rogue wave phenomena which have been known and accepted: 1) its amplitude is more than twice (or 2.5 times) that of the average amplitude of the significant wave height; 2) a rogue wave “appears from nowhere and disappears without a trace”, in other words, rogue wave is believed to be unpredictable.

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By using the numerical simulations and approximate stability analysis methods, most of the work and interpretation of the theories and experimental results are based on the nonlinear Schrödinger equation [16], the Davey-Stewartson system, the Korteweg-de Vries equation [17, 18] that rules the dynamics of weakly nonlinear, narrow band surface gravity waves. However, these equations may fail to catch the very steep profile that characterizes the extreme events because they are one-dimensional.

In this manuscript, we select the well known high dimensional Kadomtsev-Petviashivili (KP) equation that can be found almost in all physical fields as our model to illustrate the predictable rogue wave phenomenon. The paper is organized as following. We first establish a general form of the lump solutions for the $(2 + 1)$-dimensional KP equation, then extend the general form to a more general one with seven arbitrary independent parameters and four constraint conditions which is shown in Sec. II with all the physical properties of the lump being provided. By combining the lump part and exponential part, it is found in Sec. III that a lump will be invisible in some special area. Thus the lump is cutoff by the visible soliton which is totally determined by the lump. Because the lump part is invariant, the lump will keep its physical properties until it meets the induced soliton. In Sec. IV, a rogue wave/instanton is produced by cutting a lump between two solitons. Once the lump reaches a huge amplitude, it will become a rouge wave, otherwise an instanton for general amplitudes. The understanding of the physics of this kind of rogue wave phenomenon is very significant because the visible two solitons are induced by the lumps with the lumps becoming rogue waves/instantons which contains enough information for the lump, it is possible to give a prediction for such kind of rogue waves in some senses. The positions, the path, the wave height and even the emerge time may be predict. Sec. V is a short summary and discussion.

II. LUMP SOLUTION TO KP EQUATION

In further consideration, we chose the Kadomtsev-Petviashivili (KP) equation

$$u_t + u_{xxx} + 6uu_x + \sigma^2 \int u_{yy} dx = 0,$$

(1)

where $\sigma^2 = \pm 1$ as our model because the KP equation Eq. (2) is widely accepted in almost all physical fields, such as surface and internal water waves [19], nonlinear optics [20, 21], shallow water waves [22], ion-acoustic waves in plasmas [23, 24, 25, 26, 27, 28], ferromagnetics [29], Bose-Einstein condensation [30] and string theory [31]. Studies shows that KP equation is one of the most important integrable equations in high dimensions in the sense of allowing a Lax pair, bilinear form [32], an infinity of conservation laws, soliton and multisoliton solutions, etc [33, 34].

To explore the lump solutions to the KP equation, we rewrite the KP equation Eq. (1) as

$$(u_t + u_{xxx} + 6uu_{xxx})_x + \sigma^2 u_{yy} = 0.$$  

(2)
It is well known that under the transformation
\[ u = 2(\ln f)_{xx}, \]
the KP equation Eq. (2) is transformed into the bilinear form
\[ (D_x D_t + D_x^4 + \sigma^2 D_x^2)f \cdot f = 0, \]
where the operators \( D_t, D_x \) and \( D_y \) are defined as
\[ D_m D^n D^k f \cdot g = \lim_{x'=x, y'=y, t'=t} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^n \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^k f(x', y', t'), \]
which were first introduced by Hirota [32]. It is easy to see that the differential equation Eq. (4) is equivalent to the following form
\[ ff_{xt} - f_t f_x + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 + \sigma^2 (f_{yy} f - f_y^2) = 0. \]
Once the solution \( f \) to the bilinear equation Eq. (4) is found, the solution to the KP equation Eq. (2) or Eq. (1) is also obtained by the transformation \( u = 2(\ln f)_{xx} \). Now we will start from the equivalent equation Eq. (5) to construct the general lump solutions to KP equation.

II.1. General Single Lump solution to KP Equation. To solve the bilinear form of the \((2 + 1)\)-dimensional KP equation Eq. (5) in single lump form, we take the ansatz that \( f \) is
\[ f = \sum_{i\leq j=0}^{3} a_{ij} x_i x_j + f_0, \]
with \( x_1 = x, x_2 = y, x_3 = t, \) and \( x_0 = 1, \) and \( a_{ij}, i \leq j \), where \( i, j = 0, 1, 2, 3 \) and \( f_0 \) being real constants to be determined. Thus \( f \) is read as
\[ f = a_{11} x^2 + 2a_{12} xy + 2a_{13} xt + a_{22} y^2 + 2a_{23} yt + a_{33} t^2 + 2a_{01} x \]
\[ + 2a_{02} y + 2a_{03} t + a_0 + f_0. \]
which contains eleven parameters to be determined.

Substituting Eq. (7) into the bilinear form of KP equation Eq. (5) and collecting the coefficients of \( \{x, y, t\} \) yields ten equations reading as:
\[ \sigma^2 a_{12} a_{22} + a_{11} a_{23} = 0, \]
\[ \sigma^2 a_{22} a_{23} + a_{12} a_{33} = 0, \]
\[ \sigma^2 a_{11} a_{22} - 2\sigma^2 a_{12}^2 - a_{11} a_{13} = 0, \]
\[ \sigma^2 a_{22}^2 + 2a_{12} a_{23} - a_{13} a_{22} = 0, \]
\[ \sigma^2 a_{22} a_{33} - 2\sigma^2 a_{23}^2 - a_{13} a_{33} = 0, \]
\[ \sigma^2 a_{01} a_{22} - 2\sigma^2 a_{02} a_{12} - a_{03} a_{11} = 0, \]
\[ 2\sigma^2 a_{02} a_{33} - \sigma^2 a_{03} a_{22} + a_{01} a_{33} = 0. \]
Solving Eqs. (8) - (17), it is easy to find out the ten determining equations only need five solutions for \(a_{i3}, i = 0, 1, 2, 3\) and \(f_0\). The results are

\[
a_{03} = \frac{\sigma^2(a_{11}a_{22} - 2a_{02}a_{12})}{a_{11}},
\]

\[
a_{13} = \frac{\sigma^2(a_{11}a_{22} - 2a_{12}^2)}{a_{11}},
\]

\[
a_{23} = -\frac{\sigma^2a_{12}a_{22}}{a_{11}},
\]

\[
a_{33} = \frac{a_{22}^2}{a_{11}},
\]

\[
f_0 = -a_{00} + \frac{a_{01}^2a_{22} - 2a_{01}a_{02}a_{12} + a_{02}^2a_{11}}{a_{11}a_{22} - a_{12}^2} - \frac{3\sigma^2a_{11}^3}{a_{11}a_{22} - a_{12}^2},
\]

where the other six parameters for \(a_{ij}, i \leq j = 0, 1, 2\) are arbitrary constants.

It is necessary to point out that in order to solve the ten determining equations Eqs. (8) - (17), we use two non-zero conditions for \(a_{11} \neq 0\) and \(a_{11}a_{22} - a_{12}^2 \neq 0\). If \(a_{11} = 0\) and \(a_{11}a_{22} - a_{12}^2 = 0\), one can find under the transformation of \(u = 2(\ln f)_{xx}\), the KP equation possesses no lump solutions.

The results denote we need only five constraint conditions and two non-zero conditions for ten determining equations with five arbitrary independent parameters for \(a_{11}, a_{12}, a_{22}, a_{01},\) and \(a_{02}\). (According to Eq. (22), \(a_{00}\) being combined into \(f_0\) results in the numbers of independent parameters decreasing to ten.)

Substituting the results Eqs. (18) - (22) into Eq. (7), then using the transformation of \(u = 2(\ln f)_{xx}\), the lump solution to the KP equation Eq. (2) is thus obtained

\[
u = \frac{4a_{11}}{f_1} - \frac{8[a_{11}x + a_{12}y + \frac{\sigma^2}{a_{11}}(a_{11}a_{22} - 2a_{12}^2)t + a_{01}]}{f_1^2},
\]

where

\[
f_1 = a_{11}x^2 + 2(a_{12}y + a_{01})x + a_{22}y^2 + 2a_{02}y + \frac{a_{22}^2t^2}{a_{11}} + \frac{2\sigma^2}{a_{11}}[(a_{11}a_{22} - 2a_{12}^2) - 2a_{12}^2]x - a_{12}a_{22}y + (a_{01}a_{22} - 2a_{02}a_{12})]t - \frac{3\sigma^2a_{11}^3}{a_{11}a_{22} - a_{12}^2} + a_{01}^2a_{22} - 2a_{01}a_{02}a_{12} + a_{02}^2a_{11}}{a_{11}a_{22} - a_{12}^2},
\]

with five parameters \(a_{11}, a_{12}, a_{22}, a_{01},\) and \(a_{02}\) being arbitrary constants.
We can conclude the solutions Eqs. (23) - (24), containing ten parameters with five constraints and two non-zero conditions are general lump solutions to KP equation Eq. (2). The known lump solutions in [35, 36, 38], which contain nine parameters with three constraints and two non-zero conditions, and [39, 40] including two parameters can all be considered as special cases of our results.

It is interesting the ten determining equations for ten parameters $a_{ij}$ need only five constraints for the parameters $a_{i3}$, $i = 0$, 1, 2, 3 and $f_0$. Then it is natural to ask does it exist a lump solution including more free parameters? Is it possible to solve the ten equations with less constraints? Can we construct a more general lump solution with more arbitrary independent parameters?

Fortunately, we successfully find out less constraints for the ten determining equations Eqs. (8) - (17) and construct a more general lump solution to the KP equation which includes seven arbitrary independent parameters and four constraint conditions.

II.2. A More General Single Lump Solution to KP Equation. In order to construct the lump solutions with more freedoms, we still take the ansatz

$$f = \sum_{i \leq j=0}^{3} a_{ij} x_i x_j + f_0,$$

with $x_1 = x$, $x_2 = y$, $x_3 = t$, and $x_0 = 1$, and $a_{ij}$, $f_0$ being real constants to be determined. But different to Eq. (6), $a_{ij}$ is redefined as

$$a_{ij} = \langle A_i | A_j \rangle = \sum_{m=1}^{n} A_{im} A_{jm},$$

where

$$|A_1\rangle = \overrightarrow{k} = |k\rangle, \quad |A_2\rangle = \overrightarrow{p} = |p\rangle, \quad |A_3\rangle = \overrightarrow{\omega} = |\omega\rangle, \quad |A_0\rangle = \overrightarrow{\alpha} = |\alpha\rangle,$$

are vectors and $k_m$, $p_m$, $\omega_m$ and $\alpha_m$ are real constants to be determined.

Substituting Eqs. (25) - (27) into the bilinear form of the KP equation and eliminating the coefficients of $\{x, y, t\}$, it is quite natural to find the same ten equations in determinations of $a_{ij}$ as Eqs. (8) - (17) with the five solutions Eqs. (18) - (22). But further more, one can easily check that the following two constraints

$$\omega_m = {\sigma^2(a_{22}k_m - 2a_{12}p_m)}{a_{11}},$$

$$f_0 = -a_{00} + \frac{a_{01}^2 a_{22} - 2a_{01}a_{02}a_{12} + a_{02}^2 a_{11}}{a_{11}a_{22} - a_{12}^2} - \frac{3\sigma^2 a_{11}^2}{a_{11}a_{22} - a_{12}^2},$$

solve the five constraint conditions Eqs. (18) - (22)!

That is to say, two constraint conditions Eqs. (28) - (29) solve the ten determining equations Eqs. (8) - (17) with $a_{ij}$ being related to arbitrary constants of $k_m$, $p_m$ and $\alpha_m$. It indicates that ten determining equations Eqs. (8) - (17) need two constraint conditions which successfully decreases the number of constraint conditions from five to two.
Consequently, the corresponding lump solution of \( u \) to KP equation is obtained by substituting the results of Eqs. (25) - (27) with the constraint conditions Eqs. (28) - (29) into the transformation \( u = 2(\ln f)_{xx} \).

Now it seems that the results for Eqs. (25) - (27) contains infinitely many arbitrary parameters. Next we will check whether the solutions have a maximum numbers of free parameters.

If \( n = 2 \) in Eq. (26), Eq. (25) will have nine parameters for \( k_n, p_m, \omega_i m \) and \( \alpha_m, m = 1, 2, \) and \( f_0 \). Among all the parameters, \( \omega_1 \) and \( \omega_2 \) are determined by Eq. (28) reading as

\[
\omega_1 = -\frac{\sigma^2[k_1(p_1^2 - p_2^2) + 2k_2p_1p_2]}{k_1^2 + k_2^2}, \quad \omega_2 = -\frac{\sigma^2[k_2(-p_1^2 + p_2^2) + 2k_1p_1p_2]}{k_1^2 + k_2^2},
\]

and \( f_0 \) is restricted by Eq. (29)

\[
f_0 = \frac{-3\sigma^2(k_1^2 + k_2^2)^3}{(k_1p_2 - k_2p_1)^2},
\]

with \( k_n, p_m \) and \( \alpha_m m = 1, 2 \) being arbitrary constants. The solution possesses nine parameters with six arbitrary independent constants and three constraint conditions.

If we take \( n = 3 \), though Eq. (26) contains twelve parameters for \( k_n, p_m, \omega_m \) and \( \alpha_m, m = 1, 2, 3 \), there are eleven independent parameters indeed according to the expression of \( f \) in Eq. (25). The constraint conditions of \( \omega_m, m = 1, 2, 3 \) and \( f_0 \) are

\[
\omega_1 = -\frac{\sigma^2[k_1(p_1^2 - p_2^2 - p_3^2) + 2p_1(k_2p_2 + k_3p_3)]}{k_1^2 + k_2^2 + k_3^2},
\]

\[
\omega_2 = -\frac{\sigma^2[k_2(-p_1^2 + p_2^2 - p_3^2) + 2p_2(k_1p_1 + k_3p_3)]}{k_1^2 + k_2^2 + k_3^2},
\]

\[
\omega_3 = -\frac{\sigma^2[k_3(-p_1^2 - p_2^2 + p_3^2) + 2p_3(k_1p_1 + k_2p_2)]}{k_1^2 + k_2^2 + k_3^2},
\]

\[
f_0 = \frac{-[\alpha_1(k_2p_3 - k_3p_2) + \alpha_2(k_3p_1 - k_1p_3) + \alpha_3(k_1p_2 - k_2p_1)]^2 + 3\sigma^2(k_1^2 + k_2^2 + k_3^2)^3}{(k_1p_2 - k_2p_1)^2 + (k_2p_3 - k_3p_2)^2 + (k_1p_3 - k_3p_1)^2}.
\]

That means we have eleven independent parameters with seven arbitrary constants and four constraint conditions in this case.

For \( n \geq 4 \) case, the number of constraint conditions for \( \omega_m, m = 1, 2, \ldots, n \) and \( f_0 \) is \( \geq 5 \) while the number of independent parameters presented by Eq. (25) are still eleven. Compared with \( n = 3 \) case, the number of constraint conditions increases but the number of arbitrary parameters decreases. Thus it is impossible for us to find more arbitrary free parameters in the lump solutions for \( n \geq 4 \).

It can be concluded that the solution of Eqs. (25) - (27) for \( n = 3 \) is a more general solution which contains seven arbitrary independent parameters and four constraint conditions. By
taking $n = 3$, the solution $f$ to the bilinear KP equation Eq. (5) is

$$f_{\text{lump}} \equiv a_{11}x^2 + 2a_{12}xy + 2a_{13}xt + a_{22}y^2 + 2a_{23}yt + a_{33}t^2 + 2a_{01}x + 2a_{02}y + 2a_{03}t + 2\sigma^2 \frac{a^2_{11}}{a_{11}a_{22} - a_{12}^2}, \quad (36)$$

where $a_{ij}$ is related to arbitrary constants for $k_m, p_m, \alpha_m$ and $\omega_m$ by Eqs. (26) - (27), or equivalently Eq. (25) and Eqs. (32) - (34). Then the corresponding more general single lump solution to KP equation Eq. (1) is

$$u_{\text{lump}} = \frac{4a_{11}}{f_{\text{lump}}} - \frac{8(a_{11}x + a_{12}y + a_{13}t + a_{01})^2}{f_{\text{lump}}}^2. \quad (37)$$

According to the transformation $u = 2(ln f)_{xx}$, we have to put a constraint

$$f > 0,$$

which leads to the requirement of $f_0 > 0$ to insure $u$ analytical. From the special expressions of $f_0$ given by Eq. (31) for $n = 2$ and Eq. (35) for $n = 3$, we know $\sigma^2$ have to be fixed as $-1$.

To see the physical properties of a lump more concretely, we present the positions of the lump described by $[x, y]$ at any time. Generally, the positions of a lump can be provided by using a curve related to time $t$. By some simple differential calculations of $u_x = u_y = 0$, the positions of a lump is proved to be

$$x = -\frac{\sigma^2 a_{12}t}{a_{11}} - \frac{a_{01}a_{22} - a_{12}a_{02}}{a_{11}a_{22} - a_{12}^2}, \quad (38)$$

$$y = \frac{2\sigma^2 a_{12}t}{a_{11}} + \frac{a_{01}a_{12} - a_{11}a_{02}}{a_{11}a_{22} - a_{12}^2},$$

or the lump moves along the straight line

$$y = -\frac{2a_{12}x}{a_{22}} - \frac{a_{01}a_{12}a_{22} + a_{11}a_{02}a_{22} - 2a_{12}a_{02}}{a_{22}(a_{11}a_{22} - a_{12}^2)}. \quad (39)$$

The amplitude of a lump is also considered which is defined by the difference between the minimum and maximum values a lump could have. The positions of the minimum and maximum value of a lump can be obtained by solving the system $\{u_x = 0, u_y = 0\}$. It is not difficult to verify that the amplitude of a lump is

$$A_{\text{lump}} = \left| -\frac{3\sigma^2(a_{11}a_{22} - a_{12}^2)}{2a_{11}^2} \right|, \quad (40)$$

which reveals that the amplitude of a lump is also a constant related to the arbitrary constants of $a_{11}$ and $a_{22}$.

A special lump Eq. (37) is shown in FIG. 1 with the arbitrary constants selected as

$$k_1 = 1, \quad k_2 = 1, \quad k_3 = -\frac{1}{2}, \quad p_1 = 1,$$

$$p_2 = 1, \quad p_3 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1. \quad (41)$$
Figure 1. The exhibition of the lump determined by Eqs. (37) with (41). (a) is the structure at $t = 0$; (b) is the corresponding density plot; the red line in (c) is $y = -x - 1$ according to Eq. (39) showing the positions of the lump at time $-20$, $0$ and $20$, respectively.

III. An Invisible Lump

A lump will be invisible in some special area. For example, if we take

$$f_{\text{lumpoff}} \equiv f_{\text{lump}} + a_0 e^{k_0 x + p_0 y + \omega_0 t + x_0},$$

(42)

which consists of lump part and exponential part, the lump part will become invisible at the special area of $k_0 x + p_0 y + \omega_0 t + x_0 > 0$ because of the domination of the exponentiation part. That is to say, the lump only exists at the special area of $k_0 x + p_0 y + \omega_0 t + x_0 < 0$.

Substituting Eq. (42) into the KP equation Eq. (5) with a direct calculation, we find $k_0$, $p_0$ and $\omega_0$ are

$$k_0^2 = -\frac{\sigma^2(a_{11}a_{22} - a_{12}^2)}{3a_{11}^2},$$

(43)
A PREDICTABLE ROGUE WAVE AND GENERATING MECHANISMS

\[ p_0 = \frac{k_0 a_{12}}{a_{11}} \]

\[ \omega_0 = -\frac{k_0^4 + \sigma^2 p_0^2}{k_0} \]

where \( a_{ij} \) is related to arbitrary constants of \( k_m, p_m \) and \( \alpha_m \) with \( a_0 \) and \( x_0 \) being arbitrary constants.

The result is quite interesting because it demonstrates a special soliton induced by lump. Eq. (45), considered as a special dispersive relation, indicates \( \omega_0 \) is related to \( k_0 \) and \( p_0 \), while \( k_0 \) and \( p_0 \) are completely determined by the lump according to Eqs. (43) - (44). Thus the soliton (the exponential part) is induced by the lump. The existence of the soliton is based on the existence of the lump. If the lump does not exist, the soliton will also disappear. Once the soliton is induced, due to the domination of the exponentiation part, the lump will be invisible. The lump becomes a lumpoff which is cutoff by the soliton induced by itself.

By the transformation \( u = 2(\ln f)_{xx} \), the invisible lump solution to the KP equation Eq. (2) is thus obtained.

To show the invisible lump clearly, we take a special example with the arbitrary constants chosen as

\[ k_1 = 1, \quad k_2 = \frac{1}{2}, \quad k_3 = 1, \quad p_1 = 1, \quad p_2 = \frac{1}{2}, \quad p_3 = -\frac{1}{2}, \]

\[ \alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = 0, \quad x_0 = 1, \quad a_0 = 3, \]

and according to Eqs. (43) - (45), the parameters of the induced soliton are

\[ k_0 = \pm \frac{\sqrt{15}}{9}, \quad p_0 = \pm \frac{\sqrt{15}}{27}, \quad \omega_0 = \mp \frac{2\sqrt{15}}{243}, \]

then the lumpoff solution is obtained with two different cases. If \( k_0 > 0 \), the lump is visible at the area of \( k_0 x + p_0 y + \omega_0 t + x_0 < 0 \), which is cutoff by the induced soliton and eventually disappears. Otherwise, a revers process for \( k_0 < 0 \) showing the interaction between the invisible lump and the induced soliton with the lump emerging at \( k_0 x + p_0 y + \omega_0 t + x_0 > 0 \).

Because we take the combination of unchanged lump part and exponential part, it is easy to find the corresponding positions of the lump are derived from Eq. (38), say

\[ x = \frac{2}{3} t - \frac{2}{15}, \quad y = -\frac{2}{3} t - \frac{4}{15}. \]

FIG. 2 is the evolution plot of the interaction of the visible lump and the induced soliton with the parameter selections in Eqs. (46) - (47) for \( \{+k_0, +p_0, -\omega_0\} \), showing the lump is visible at first and becomes disappears after being cutoff by the induced soliton. FIG. 3 is the corresponding contour plot with the blue line being \( y = -x - \frac{2}{5} \). The lump remains its path until it is cutoff by the induced soliton.

If the parameters is selected as \( \{-k_0, -p_0, +\omega_0\} \), the invisible lump is obtained and ultimately separates our from the induced soliton after a special time with the density plot being shown in FIG. 4.
Figure 2. A lump and the induced soliton with parameters selected in Eqs. (46) - (47) with $k_0 > 0$ at (a) $t = -50$, (b) $t = 25$, (c) $t = 80$ and (d) $t = 250$, respectively. The lump emerges at $k_0x + p_0y + \omega_0t + x_0 < 0$ area and eventually disappears.

It is interesting the lump being cutoff by the induced soliton result in the invisible of the lump in some special area. An algebraic rogue wave/instanton can also be produced by cutting a lump between two solitons. That is to say, the lump appears at a special area for $k_0x + p_0y + \omega_0t + x_0 \sim 0$. Thus, the lump becomes a special rogue wave / instanton.

IV. Predictable Rogue Wave/Instanton

A rogue wave/instanton is a localized wave decayed in all space and time directions. In fact, in quantum field theories the instantons studies allow scientists to see the previously hidden logarithmic structure of the states and operators [41, 42]. Physicists believe that instantons are the key to explore the interactions principle in the standard model. Studies exhibits that instantons have been shown in integrable systems, such as DS equation [43] and many (2 + 1)-dimensional models [44] via the multiple linear variable separation approach.
Figure 3. The corresponding contour plot of the visible lump with parameters selected in Eqs. (46) and (47) at (a) $t = -60$, (b) $t = -20$, (c) $t = 0$ and (d) $t = 250$, respectively. The blue line is $y = -x - \frac{2}{5}$ which reveals the positions of the lump during the whole process.

Based on the known results, we construct the rogue wave/instanton solutions to the KP equation Eq. (1) by assuming

$$f_{\text{Instanton}} \equiv f_{\text{lump}} + a_1 \cosh(k_0 x + p_0 y + \omega_0 t + x_0) + g_0,$$  

(49)

where $f_{\text{lump}}$ is shown in Eq. (36), and $k_0$, $p_0$, $\omega_0$ satisfy the constraint conditions of Eqs. (43) - (45), with $a_1$ and $g_0$ being two arbitrary constants to be determined. As the visible lump being confined in a special area for $k_0 x + p_0 y + \omega_0 t + x_0 < 0$, the lump should be confined in a twin-soliton induced by itself. In other words, the lump’s emerge for the area $k_0 x + p_0 y + \omega_0 t + x_0 \sim 0$ for a special time leads to the lump becomes a rogue wave/instanton.

Substituting Eq. (49) into the bilinear form of the KP equation Eq. (5) and eliminating all the coefficients of the polynomials of $\{x, y, t, \cosh \text{ and/or sinh}\}$ by using the known results
FIGURE 4. The density plot of the visible lump with $k_0 < 0$ at (a) $t = -250$, (b) $t = -25$, (c) $t = 0$ and (d) $t = 50$, respectively according to Eqs. (46 - (47). The lump is invisible and ultimately separation out from the induced soliton.

and constraints, one can easily find the only possible choice for $g_0$ is

$$g_0 = -\frac{\sigma^2 a_1^2 (a_{11} a_{22} - a_{12}^2)}{12 a_{11}^3},$$

with $a_1$ being an arbitrary constant. It is straightforward to find out the instanton/rouge wave solution to the KP equation through the transformation $u = 2(\ln f)_{xx}$.

The rouge wave/instanton is obtained by cutting the lump between two solitons with the lump being visible in the area $k_0 x + p_0 y + \omega_0 t + x_0 \sim 0$ for a special time. Due to the existence of the lump, a twin-soliton is induced according to the special dispersion relation Eq. (45) which is visible all time because of the domination of the cosh part. The visible of solitons leads to the invisible of the lump, thus the lump is visible only when it moves to the line $k_0 x + p_0 y + \omega_0 t + x_0 \sim 0$. Once the lump reaches a large amplitude, it will become a rouge wave, or be an instanton for general amplitudes.
Our results shows novel generating and prediction mechanism for this kind of rouge waves. Because the twin-soliton includes enough information \((k_0, p_0\) and \(\omega_0)\) of the invisible lump (algebraic) part \((a_{11}, a_{22}\) and \(a_{12})\). The position, the path and the wave height, even the emerge time of the rouge wave may be predict.

Due to the unchanged lump part in solution Eq. (49), the invisible lump will move along the path Eq. (38) or Eq. (39). Once the invisible lump comes to \(k_0 x + p_0 y + \omega_0 t + x_0 \sim 0\), it will appear until it reaches peak at the cross point of the centerline \(k_0 x + p_0 y + \omega_0 t + x_0 = 0\) of the two solitons. That means the rouge wave will appear or reach its peak at the time

\[
t = -\frac{9k_0a_{11}^3(k_0a_{10} - a_{11}x_0)}{2(a_{11}a_{22} - a_{12}^2)^2},
\]

with the place

\[
x = -\frac{a_{10}a_{22} - a_{12}a_{20}}{a_{11}a_{22} - a_{12}^2} - \frac{9k_0a_{12}^2a_{11}(k_0a_{10} - a_{11}x_0)}{\sigma^2(a_{11}a_{22} - a_{12}^2)^2},
\]
\[
y = a_{01}a_{12} - a_{11}a_{02} \frac{9k_0a_{12}a_{11}^2(k_0a_{10} - a_{11}x_0)}{\sigma^2(a_{11}a_{22} - a_{12}^2)^2}.
\]

Moreover, the maximum value of wave hight of the instanton/rouge wave is predicted

\[
A_{\text{instanton}} = \left| \frac{8a_{11}(a_{11}a_{22} - a_{12}^2)}{a_{1}(a_{11}a_{22} - a_{12}^2) + 6a_{11}^3} \right|,
\]

which indicates that the amplitude is related to the soliton’s parameter \(a_1\) and the lump part of \(a_{11}, a_{12}, a_{22}\) by calculating the value of \(u\) when the lump arrive at the center of the twin-soliton.

For instance, if we choose the arbitrary constants as

\[
k_1 = \frac{1}{4}, \quad k_2 = 1, \quad k_3 = \frac{1}{2}, \quad p_1 = \frac{3}{2}, \quad p_2 = -1, \quad p_3 = 1,
\]
\[
\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = \frac{1}{2}, \quad x_0 = 1, \quad a_1 = \frac{1}{20}, \quad \sigma^2 = -1,
\]

then the result shows a special rouge wave with the corresponding positions directly obtained from Eq. (38)

\[
x = \frac{68}{21}t - \frac{38}{89}, \quad y = \frac{4}{21}t - \frac{43}{89},
\]

or alteratively the path is

\[
y = \frac{x}{17} - \frac{693}{1513}.
\]

The rouge wave will reach its peak at about \(t = -0.27\) in \(\{x = -1.30, y = -0.53\}\) with the amplitude being \(A = \frac{508800}{1411763}\) according to Eqs. (51) - (53), respectively.

The evolution plot of the rouge wave is exhibited in FIG. 5 (a) - (e), and (f) shows the change of wave height in the plane of \(y = 0\). It is obviously seen that the rouge wave emerges inside the twin-soliton which is the special area for \(k_0 x + p_0 y + \omega_0 t + x_0 \sim 0\). The visible two solitons lead to the invisible of the lump. The wave height of the lump is far more than
that of the twin-soliton’s. FIG. 6 is the corresponding contour plot with the blue line being Eq. (56) that reveals the path of the rogue wave, and (f) is the density plot at $t = 0$.

**Figure 5.** The evolution plot of the rogue wave with parameters selections of Eq. (54) at time (a) $t = -50$, (b) $t = -5$, (c) $t = 0$, (d) $t = 5$ and (e) $t = 50$, respectively. (f) is the corresponding wave height in $y = 0$ plane at $t = -50$ in blue, $t = 0$ in green and $t = 50$ in red.

Except for rogue wave, it is possible to find the instanton with small amplitude under suitable parameters selections from Eq. (53). If the arbitrary parameters are selected as

\[
\begin{align*}
  k_1 &= \frac{1}{4}, & k_2 &= 1, & k_3 &= \frac{1}{4}, & p_1 &= \frac{3}{2}, & p_2 &= -1, & p_3 &= \frac{1}{2},\\
  \alpha_1 &= 1, & \alpha_2 &= 0, & \alpha_3 &= \frac{1}{2}, & x_0 &= 1, & a_1 &= 10, & \sigma^2 &= -1,
\end{align*}
\]

then we have an instanton with the amplitude being only about $A = \frac{8496}{11627}$ which can be verified by substituting the parameters into the amplitude expression. FIG. (7) exhibits the evolution of the instanton in (a) - (e) with the wave shape in the plane of $y = 0$ in (f). Compared with rogue wave shown in FIG. (5), the wave height of the instanton is similar to that of the twin-soliton.
Figure 6. The corresponding contour plots of rogue wave with parameters selections Eq. (54) at (a) \( t = -50 \), (b) \( t = -5 \), (c) \( t = 0 \), (d) \( t = 5 \) (e) and (e) \( t = 50 \), respectively. The blue line is Eq. (56). (f) is the density plot at \( t = 0 \).

V. Summary and Discussion

In summary, we choose the KP equation as our universal model to present rogue wave phenomenon not only because it is widely used in almost all branches of science, but also it is high dimensional model. The widely applications and generality of KP equation makes it possible to explain almost all the rogue wave phenomenon. The progress in the understanding of the physics of the rogue wave phenomenon and development of adequate mathematical models is very significant.

Though lump solutions to KP equation have been presented by many authors, we find a special lump containing seven independent parameters and four constraint conditions which is believed to be a more general form. The positions, the path and the amplitude of the lump are also provided.

By using the combination of lump part and exponential function, lumpoff and instanton/rouge wave solutions to the KP equation are successfully illustrated. Lump is localized
wave decayed in all space directions and exists in all time, but it will be invisible in a special area due to the domination of the exponential function. It is interesting that a special dispersive relation for the soliton(s) is found showing that the soliton(s) is totally determined by the lump. That means the lump may induce soliton(s). We should emphasize that the soliton(s) is induced by the lump. If there is no lump, there is no soliton(s) with the special dispersion relation. Furthermore, whence soliton(s) is induced by the lump, the lump will be cutoff by the induced soliton(s) and become invisible. When a soliton is induced by the lump, the lump will be cutoff and become invisible to a lumpoff. When two solitons are induced by the lump, the lump will become a rogue wave (or instanton) and can only be visible at an instant time. Because the induced visible solitons contains enough information of the lump, it is possible to predict the emerge time and the locations for this kind of rogue wave in some senses.
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