Koszul-Tate Cohomology as Lowest-Energy Modules of Non-Centrally Extended Diffeomorphism Algebras

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Abstract

Fock modules for multi-dimensional Virasoro algebras (non-central extensions of the diffeomorphism algebra \textit{vect}(N)) have recently been reported. Using ideas from the antifield formalism, I construct new classes of lowest-energy modules, as cohomology groups of a certain Fock complex. The Fock construction involves a passage to $p$-jets prior to normal ordering, but the abelian charges usually diverge in the limit $p \to \infty$. The requirement of a finite limit imposes severe restrictions on the number of spacetime dimensions and on the order of the Euler-Lagrange (EL) equations. Under some natural assumptions (the EL equations are first order for fermions and second order for bosons, and no reducible gauge symmetries appear), finiteness is only possible when the number of spacetime dimensions $N = 4$.

1 Introduction

In a recent paper [10], I constructed Fock modules of non-centrally extended diffeomorphism and current algebras in $N$-dimensional spacetime, i.e. the higher-dimensional generalizations of Virasoro and affine algebras. More precisely, I considered the \textit{DGRO (Diffeomorphism, Gauge, Reparameterization, Observer)} algebra \textit{DGRO}(N, \mathfrak{g}), where $\mathfrak{g}$ is a finite-dimensional Lie algebra. The crucial idea was to first expand all fields in a multi-dimensional Taylor series around the points on a one-dimensional curve (“the observer’s trajectory”), and then to truncate at some finite order $p$. We thus obtain a realization of $\textit{vect}(N) \ltimes \textit{map}(N, \mathfrak{g})$ (semi-direct product of diffeomorphism
and current algebras on the space of trajectories in the space of tensor-valued $p$-jets. This space consists of finitely many functions of a single variable, which is precisely the situation where the normal ordering prescription works. After normal ordering, a Fock representation of the DGRO algebra is obtained. Related work can be found in [1, 2, 3, 9, 10, 11, 16, 17, 18]. Cocycles of the diffeomorphism algebra were classified by Dzumadil’daev [4] and reviewed in [12].

To progress further, one now wants to construct more interesting modules of lowest-energy type. A natural idea is to consider a complex of Fock modules:

\[
\begin{array}{ccccccc}
\vdots & \overset{Q}{\leftarrow} & J^pF^{-1} & \overset{Q}{\leftarrow} & J^pF^0 & \overset{Q}{\leftarrow} & J^pF^1 & \overset{Q}{\leftarrow} & J^pF^2 & \overset{Q}{\leftarrow} & \cdots \\
\downarrow L & & \downarrow L & & \downarrow L & & \downarrow L & & \downarrow L & & \vdots \\
\vdots & \overset{Q}{\leftarrow} & J^pF^{-1} & \overset{Q}{\leftarrow} & J^pF^0 & \overset{Q}{\leftarrow} & J^pF^1 & \overset{Q}{\leftarrow} & J^pF^2 & \overset{Q}{\leftarrow} & \cdots \\
\end{array}
\]

(1.1)

Here $J^pF^g$ are $DGRO(N,g)$ Fock modules, the vertical maps denote the module action, $Q^2 = 0$, and all squares commute. In this situation, $DGRO(N,g)$ will act in a well-defined manner on the cohomology groups $H^g(Q)$, which thus acquire a module structure.

The problem is now to find such a complex. A natural candidate is found in the physics of gauge theories, as formulated cohomologically in the anti-field formalism [7]. The goal of classical physics is to find the stationary surface $\Sigma$, i.e. the set of solutions to the Euler-Lagrange (EL) equations, viewed as a submanifold embedded in configuration space $Q$. Dually, one wants to construct the function algebra $C(\Sigma) = C(Q)/I$, where $I$ is the ideal generated by the EL equations. For each field $\phi_\alpha$ and EL equation $E^\alpha = 0$, introduce an anti-field $\phi^{*\alpha}$ of opposite Grassmann parity. The extended configuration space $C(Q^{*})$ can be decomposed into subspaces $C^g(Q^{*})$ of fixed antifield number $g$, where $afn \phi_\alpha = 0$, $afn \phi^{*\alpha} = 1$. As is well known, the Koszul-Tate (KT) complex

\[
0 \overset{\delta}{\leftarrow} C^0(Q^{*}) \overset{\delta}{\leftarrow} C^1(Q^{*}) \overset{\delta}{\leftarrow} C^2(Q^{*}) \overset{\delta}{\leftarrow} \cdots,
\]

(1.2)

where $\delta \phi_\alpha = 0$ and $\delta \phi^{*\alpha} = E^\alpha$, yields a resolution of $C(\Sigma)$; the cohomology groups $H^g(\delta) = 0$ unless $g = 0$, and $H^0(\delta) = C(Q)/I$ [7].

The idea in this paper is to consider not just functions on the stationary surface, but all differential operators on it. The KT differential $\delta$ can then

\footnote{In previous writings, I have denoted the diffeomorphism algebra, or algebra of vector fields, $vect(N)$ by $diff(N)$.}
be written as a bracket: \( \delta F = [Q, F] \), where the KT charge \( Q = \int \mathcal{E}^\alpha \pi^\alpha_\alpha \) and \( \pi^\alpha_\alpha \) is the canonical momentum corresponding to \( \phi^{*\alpha} \). If we pass to the space of \( p \)-jets before momenta are introduced, the Fock construction applies. Since the KT charge consists of commuting operators, it does not need to be normal ordered, and we recover precisely the situation in \( \square \); the cohomology groups are well-defined \( DGRO(N, \mathfrak{g}) \) modules of lowest-energy type.

An outstanding problem is to take the jet order \( p \) to infinity, because infinite jets essentially contain the same information as the original fields. This limit is problematic, because the abelian charges diverge with \( p \). However, it was noted in \cite{13} that if we have several independent jets, of order \( p, p - 1, ..., p - r \), we can arrange so that the leading terms cancel, and the abelian charges are finite in \( N = r \) dimensions (they vanish in \( N < r \) dimensions). This situation applies here, because the anti-fields correspond to lower-order jets; the order depends on the order of the EL equations. A set of consistency conditions can therefore be formulated. These conditions are very restrictive, and natural solutions exist in four dimensions only.

Hence quantum diffeomorphism symmetry is only possible provided that spacetime is four-dimensional.

\section{DGRO algebra}

Let \( \xi = \xi^\mu(x) \partial_\mu, \ x \in \mathbb{R}^N, \ \partial_\mu = \partial/\partial x^\mu \), be a vector field, with commutator \([\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\mu \partial_\mu \xi^\nu \partial_\nu\). Greek indices \( \mu, \nu = 1, 2, ..., N \) label the spacetime coordinates and the summation convention is used on all kinds of indices. The diffeomorphism algebra (algebra of vector fields, Witt algebra) \( \text{vect}(N) \) is generated by Lie derivatives \( L_\xi \). In particular, we refer to diffeomorphisms on the circle as reparametrizations. They form an additional \( \text{vect}(1) \) algebra with generators \( L_f \), where \( f = f(t) dt/dt, \ t \in S^1 \), is a vector field on the circle. Let \( \text{map}(N, \mathfrak{g}) \) be the current algebra corresponding to the finite-dimensional semisimple Lie algebra \( \mathfrak{g} \) with basis \( J^a \), structure constants \( f^{ab}_c \), and Killing metric \( \delta^{ab} \). The brackets in \( \mathfrak{g} \) are \([J^a, J^b] = if^{ab}_c J^c \). A basis for \( \text{map}(N, \mathfrak{g}) \) is given by \( \mathfrak{g} \)-valued functions \( X = X_a(x) J^a \) with commutator \([X, Y] = if^{ab}_c X_a Y_b J^c \). Finally, let \( \text{Obs}(N) \) be the space of local functionals of the observer’s trajectory \( q^\mu(t) \), i.e. polynomial functions of \( q^\mu(t), \dot{q}^\mu(t), ... \ d^k q^\mu(t)/dt^k, \ k \) finite, regarded as a commutative algebra. \( \text{Obs}(N) \) is a \( \text{vect}(N) \) module in a natural manner.

The DGRO algebra \( DGRO(N, \mathfrak{g}) \) is an abelian but non-central Lie al-
gebra extension of $\text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1)$ by $\text{Obs}(N)$:

$$0 \rightarrow \text{Obs}(N) \rightarrow \text{DGRO}(N, g) \rightarrow \text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1) \rightarrow 0.$$ 

The brackets are given by

\[
\begin{align*}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \, \dot{q}^\mu(t) \left\{ c_1 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) + c_2 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) \right\}, \\
[\mathcal{L}_\xi, \mathcal{J}_X] &= \mathcal{J}_\xi X, \\
[\mathcal{J}_X, \mathcal{J}_Y] &= \mathcal{J}_{[X, Y]} - \frac{c_5}{2\pi i} \delta^{ab} \int dt \, \dot{q}^\rho(t) \partial_\rho X_a(q(t)) Y_b(q(t)), \\
[L_f, \mathcal{L}_\xi] &= \frac{c_3}{4\pi i} \int dt \, (\ddot{f}(t) - i \dot{f}(t)) \partial_\mu \xi^\mu(q(t)), \\
[L_f, \mathcal{J}_X] &= 0, \\
[L_f, L_g] &= L_{[f, g]} + \frac{c_4}{24\pi i} \int dt (\dddot{f}(t) \dot{g}(t) - \dot{f}(t) \dddot{g}(t)), \\
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[L_f, q^\mu(t)] &= -f(t) \dot{q}^\mu(t), \\
[\mathcal{J}_X, q^\mu(t)] &= [q^\mu(s), q^\mu(t)] = 0,
\end{align*}
\]

extended to all of $\text{Obs}(N)$ by Leibniz’ rule and linearity. The numbers $c_1 - c_5$ are called abelian charges. In previous papers, I considered a slightly more complicated extension which depends on three additional abelian charges $c_6 - c_8$, but they vanish automatically when $g$ is semisimple. The DGRO algebra is the natural higher-dimensional generalization of the Virasoro and affine Kac-Moody algebras.

### 3 Koszul-Tate cohomology

#### 3.1 Classical representations of $\text{vect}(N) \ltimes \text{map}(N, g)$

Let $J^a = (J^a_\beta)$ and $T^\mu_\nu = (T^\mu_\nu)$ be matrices satisfying $g$ and $\text{gl}(N)$, respectively, where the brackets in $\text{gl}(N)$ are

\[
[T^\mu_\nu, T^\rho_\sigma] = \delta^\mu_\sigma T^\rho_\nu - \delta^\rho_\nu T^\mu_\sigma.
\]  

(3.1)

It is straightforward to verify that $\mathcal{L}_\xi = \xi^\mu \partial_\mu + \partial_\nu \xi^\mu T^\nu_\mu$ and $\mathcal{J}_X = X_a(x)J^a$ satisfy $\text{vect}(N) \ltimes \text{map}(N, g)$. This implies that its modules are tensor densi-
ties valued in \( g \) modules. The \( \text{vect}(N) \ltimes \text{map}(N, g) \) action is given by

\[
[\mathcal{L}_\xi, \phi_\alpha(x)] = -\xi^\mu(x) \partial_\mu \phi_\alpha(x) - \partial_\nu \xi^\mu(x) T^\nu_{\alpha\beta} \phi_\beta(x),
\]

\[
[J_X, \phi_\alpha(x)] = -X_a(x) J^a_{\beta\alpha} \phi_\beta(x),
\]

(3.2)

Let \( Q = Q(\phi) \) denote the module spanned by all \( \phi_\alpha(x), x \in \mathbb{R}^N \). In physics terms, \( Q \) is our configuration space.

### 3.2 KT complex for functions of \( x \)

Clearly, \( \text{vect}(N) \ltimes \text{map}(N, g) \) acts not only on \( Q \) but also on the space of local functionals on \( Q \); denote this space \( C(Q) \). This module is highly reducible; e.g., \( C(Q) = \bigoplus_{n=0}^{\infty} C_n(Q) \), where \( C_n(Q) \) consists of functionals that are homogeneous of degree \( n \) in \( \phi \).

An interesting submodule of \( C(Q) \) can be constructed as follows. Let \( S = \int d^N x \mathcal{L}(\phi) \) be an invariant action (in the sense of physics) and \( \mathcal{L}(\phi) \) the associated Lagrangian. The Lagrangian is a local functional of \( \phi \), i.e. a function of \( \phi_\alpha(x) \) and its derivatives \( \partial_\mu \phi_\alpha(x), \partial_\mu \partial_\nu \phi_\alpha(x), \) etc., up to some finite order, all evaluated at the same point \( x \). In practice, the Lagrangian only depends on first-order derivatives. The Euler-Lagrange (EL) equations,

\[
\mathcal{E}_\alpha(x) \equiv \frac{\delta S}{\delta \phi_\alpha(x)} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha}(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_\alpha}(x) = 0,
\]

(3.3)

generate an ideal \( \mathcal{I} \subset C(Q) \), and the factor space \( C(Q)/\mathcal{I} \) is still a \( \text{vect}(N) \ltimes \text{map}(N, g) \) module due to the invariance assumption. This factor space is most conveniently described as a resolution of a certain Koszul-Tate (KT) complex. For each field \( \phi_\alpha(x) \), introduce an antifield \( \phi^{*\alpha}(x) \) transforming as the corresponding EL equation \( \mathcal{E}_\alpha(x) \). We then consider the extended configuration space \( Q^* \) as the span of \( \phi_\alpha(x) \) and \( \phi^{*\alpha}(x) \). Now consider the space of local functionals on \( Q^* \): \( C(Q^*) = C(Q) \otimes C(\phi^*) \), where \( C(\phi^*) \) denotes the space of local functionals of \( \phi^*(x) \). If \( \phi \) is bosonic (\( C(Q) \) consists of symmetric functionals), then \( \phi^* \) is fermionic (\( C(\phi^*) \) consists of anti-symmetric functionals), and vice versa.

Define the anti-field number by \( \text{afn} \phi_\alpha = 0 \), \( \text{afn} \phi^{*\alpha} = 1 \). \( C(Q^*) \) can be decomposed into subspaces \( C^g(Q^*) \) of fixed antifield number \( g \):

\[
C(Q^*) = \bigoplus_{g=0}^{\infty} C(Q) \otimes C^g(\phi^*) \equiv \bigoplus_{g=0}^{\infty} C^g(Q^*).
\]

(3.4)

The KT complex takes the form

\[
0 \leftarrow \delta C^0(Q^*) \leftarrow \delta C^1(Q^*) \leftarrow \delta C^2(Q^*) \leftarrow \delta \ldots
\]

(3.5)
where the KT differential $\delta$ is defined by

$$
\delta \phi_\alpha(x) = 0, \quad \delta \phi^{*\alpha}(x) = \mathcal{E}^\alpha(x).
$$

(3.6)

By a standard argument [7], the cohomology groups $H^g(\delta) = 0$ unless $g = 0$, and $H^0(\delta) = C(Q)/\mathcal{I}$. $H^0(\delta)$ can be thought of as the space $C(\Sigma)$ of functions on the stationary surface $\Sigma$, i.e. the set of solutions to the EL equations, embedded as a submanifold in $Q$.

Introduce canonical momenta $\pi_\alpha(x) = \delta/\delta \phi_\alpha(x)$ and $\pi^{*\alpha}(x) = \delta/\delta \phi^{*\alpha}(x)$ satisfying

$$
[\pi_\alpha(x), \phi_\beta(y)] = [\pi^{*\alpha}(x), \phi^{*\beta}(y)] = \delta_\alpha^\beta \delta(x - y),
$$

(3.7)

and all other brackets vanish. The antifield number is $\text{afn} \pi_\alpha = 0$, $\text{afn} \pi^{*\alpha} = -1$. The KT differential can then be written as a bracket: $\delta F = [Q, F]$, where

$$
Q = \int d^N x \mathcal{E}^\alpha(x) \pi^{*\alpha}(x).
$$

(3.8)

Let $\mathcal{P}$ be the phase space corresponding to $Q$, i.e. the span of $\phi_\alpha(x)$ and $\pi^\alpha(x)$, and let $\mathcal{P}^*$ be the enlarged phase space, i.e. the span of $\phi_\alpha(x)$, $\phi^{*\alpha}(x)$, $\pi^\alpha(x)$ and $\pi^{*\alpha}(x)$. The expression (3.8) defines a differential, also denoted by $Q$, which acts on the space $C(\mathcal{P}^*)$ of local functionals on $\mathcal{P}^*$. Note that $C(\mathcal{P}^*)$ is a non-commutative algebra, which can be thought of as the algebra of differential operators on $\mathcal{P}^*$. The decomposition into subspaces of fixed antifield number now extends indefinitely in both directions:

$$
C(\mathcal{P}^*) = \bigoplus_{g = -\infty}^{\infty} C^g(\mathcal{P}^*).
$$

(3.9)

Accordingly, we obtain the two-sided complex

$$
\ldots \xleftarrow{Q} C^{-1}(\mathcal{P}^*) \xleftarrow{Q} C^{0}(\mathcal{P}^*) \xleftarrow{Q} \ldots \xleftarrow{Q} C^1(\mathcal{P}^*) \xleftarrow{Q} \ldots
$$

(3.10)

The cohomology group $H^{0}(Q)$ can be thought of as the space of differential operators on the stationary surface $\Sigma$. However, I do not know if (3.10) is a resolution, i.e. if the other cohomology groups vanish.

There is a problem: the EL equations may be dependent, i.e. there may be relations of the form

$$
r^\alpha(x) = r_0^\alpha(x) \mathcal{E}^\alpha(x) \equiv 0,
$$

(3.11)

where $r_0^\alpha(x)$ is some functional of $\phi_\alpha(x)$. Then $H^1(Q) \neq 0$, because $r_0^\alpha(x) \phi^{*\alpha}(x)$ is KT closed: $[Q, r_0^\alpha(x) \phi^{*\alpha}(x)] = 0$. The standard way to kill this unwanted
cohomology is to introduce a second-order antifield $b^a(x)$. Let $[Q, b^a(x)] = r^a_\alpha(x)\phi^{*\alpha}(x)$, which makes the latter expression exact and thus makes it vanish in cohomology. To obtain the explicit expression for $Q$, introduce the second-order antifield momentum $c_a(x)$, with the non-zero bracket $[c_a(x), b^b(x)] = \delta_b^a \delta(x - y)$. The full KT differential is now

$$Q = \int d^N x \ (E^\alpha(x)\pi^*_\alpha(x) + r^a_\alpha(x)\phi^{*\alpha}(x)c_a(x)). \quad (3.12)$$

There can in principle be relations also among the $r^a_\alpha(x)$ of the form $Z^A(x) = Z^A_a(x)r^a_\alpha(x) \equiv 0$. If so, it is necessary to introduce higher-order antifields to eliminate the unwanted cohomology. However, we will assume that the gauge symmetries are irreducible, i.e. that no non-trivial higher-order relations exist, since this is the case in all experimentally established theories of physics.

The situation is summarized in the following table:

| $g$ | Field       | Momentum | Ideal              |
|-----|-------------|-----------|--------------------|
| 0   | $\phi_\alpha(x)$ | $\pi^\alpha(x)$ | $-$                |
| 1   | $\phi^{*\alpha}(x)$ | $\pi^*_\alpha(x)$ | $E^\alpha(x) \approx 0$ |
| 2   | $b^a(x)$    | $c_a(x)$  | $r^a_\alpha(x)\phi^{*\alpha}(x) \approx 0$ |

### 3.3 KT complex for functions of $x$ and $t$

In [10] Fock representations of $DGRO(N, g)$ were constructed. Its classical modules consist of $g$-valued tensor fields which also transform as densities under reparametrizations:

$$\begin{align*}
[\mathcal{L}_\xi, \phi_\alpha(x, t)] &= -\xi^\mu(x)\partial_\mu \phi_\alpha(x, t) - \partial_\mu \xi^\mu(x)T^\beta_{\alpha\mu}\phi_\beta(x, t), \\
[\mathcal{J}_X, \phi_\alpha(x, t)] &= -X_a(x)J^\beta_a\phi_\beta(x, t), \\
[\mathcal{L}_f, \phi_\alpha(x, t)] &= -f(t)\partial_t \phi_\alpha(x, t) - \lambda(\dot{f}(t) - if(t))\phi_\alpha(x, t).
\end{align*} \quad (3.14)$$

Denote the linear span of $\phi_\alpha(x, t)$ by $Q(t)$ and the corresponding phase space by $P(t)$. The EL equations now read $E^\alpha(x, t) = 0$; they are obtained from (3.3) by replacing $\phi_\alpha(x)$ by $\phi_\alpha(x, t)$ everywhere. The KT charge (3.12) is replaced by

$$Q = \int d^N x dt \ (E^\alpha(x, t)\pi^*_\alpha(x, t) + r^a_\alpha(x, t)\phi^{*\alpha}(x, t)c_a(x, t)). \quad (3.15)$$
Since the space $C(Q(t))$ is larger than $C(Q)$, we must factor out a larger ideal to obtain a resolution of the same space $C(\Sigma)$. It is easy to see that the necessary additional requirement is $\partial_t \phi_\alpha(x,t) \approx 0$; to implement this constraint in cohomology, we introduce the antifield $\overline{\phi}_\alpha(x,t)$ with canonical momentum $\overline{\pi}_\alpha(x,t)$. Since $E^\alpha(x,t)$ depends on $\phi_\alpha(x,t)$ only, we now have $\partial_t E^\alpha(x,t) = 0$, which generates unwanted cohomology. This is eliminated by introducing a second-order antifield $\overline{\phi}^\alpha(x,t)$. Finally, the other second-order antifield $b^\alpha(x,t)$, associated with the gauge symmetry, is now reducible.

Correct this by introducing a third-order antifield $b^\alpha(x,t)$. The situation is summarized in the following table:

| g | Field       | Momentum     | Ideal            |
|---|-------------|--------------|------------------|
| 0 | $\phi_\alpha(x,t)$ | $\pi^\alpha(x,t)$ | $-$               |
| 1 | $\phi^{*\alpha}(x,t)$ | $\pi^{*}_\alpha(x,t)$ | $E^\alpha(x,t) \approx 0$ |
| 1 | $\overline{\phi}_\alpha(x,t)$ | $\overline{\pi}_\alpha(x,t)$ | $\partial_t \phi_\alpha(x,t) \approx 0$ |
| 2 | $b^\alpha(x,t)$ | $c_\alpha(x,t)$ | $r^\alpha_\alpha(x,t)\phi^{*\alpha}(x,t) \approx 0$ |
| 2 | $\overline{\phi}^{*\alpha}(x,t)$ | $\overline{\pi}^{*}_\alpha(x,t)$ | $\partial_t \phi^{*\alpha}(x,t) \approx 0$ |
| 3 | $b^\alpha(x,t)$ | $\overline{\pi}^{*}_\alpha(x,t)$ | $\partial_t b^\alpha(x,t) \approx 0$ |

The full KT differential becomes

$$Q = \int d^N x dt \left( E^\alpha(x,t) \pi^{*\alpha}_\alpha(x,t) + r^\alpha_\alpha(x,t)\phi^{*\alpha}_\alpha(x,t)c_\alpha(x,t) \right) \partial_t \phi_\alpha(x,t) \overline{\pi}_\alpha(x,t) + \partial_t \phi^{*\alpha}_\alpha(x,t) \overline{\pi}^{*\alpha}_\alpha(x,t) + \partial_t b^\alpha(x,t) \overline{\pi}^{*}_\alpha(x,t).$$

As before, we obtain resolutions of $C(Q(t))/\mathcal{I}(t) = C(\Sigma)$ and $C(P(t))/\mathcal{I}(t)$, where $\mathcal{I}(t)$ is the totality of all relevant ideals.

### 3.4 KT complex in jet space

The crucial idea in [10] is to introduce a privileged curve $q^\mu(t) \in \mathbb{R}^N$ in spacetime ("the observer’s trajectory") and to expand all fields and antifields in a Taylor series around this curve, before introducing canonical momenta. Hence e.g.,

$$\phi_\alpha(x,t) = \sum_{|m| \leq p} \frac{1}{m!} \phi_{\alpha,m}(t)(x - q(t))^m,$$

where $m = (m_1, m_2, ..., m_N)$, all $m_\mu \geq 0$, is a multi-index of length $|m| = \sum_{\mu=1}^N m_\mu$ and $m! = m_1!m_2!...m_N!$. Denote by $\mu$ a unit vector in the $\mu^{th}$
direction, so that $\mathbf{m} + \mu = (m_1, ..., m_\mu + 1, ..., m_N)$, and let

$$\phi_{\alpha, \mathbf{m}}(t) = \partial_\mathbf{m} \phi_\alpha(q(t), t) = \partial_{m_1} \cdots \partial_{m_N} \phi_\alpha(q(t), t)$$

be the $|\mathbf{m}|$th order derivative of $\phi_\alpha(x, t)$ on the observer’s trajectory $q^\mu(t)$. Such objects transform as

$$[\mathcal{L}_\xi, \phi_{\alpha, \mathbf{m}}(t)] = \partial_\mathbf{m}([\mathcal{L}_\xi, \phi_\alpha(q(t), t)]) + [\mathcal{L}_\xi, q^\mu(t)] \partial_\mu \partial_\mathbf{m} \phi_\alpha(q(t), t)$$

$$\equiv - \sum_{|\mathbf{n}| \leq |\mathbf{m}|} T^{\beta \mathbf{n}}_{\alpha \mathbf{m}}(\xi(q(t))) \phi_{\beta, \mathbf{n}}(t),$$

$$[J_X, \phi_{\alpha, \mathbf{m}}(t)] = \partial_\mathbf{m}([J_X, \phi_\alpha(q(t), t)])$$

$$\equiv - \sum_{|\mathbf{n}| \leq |\mathbf{m}|} J^{\beta \mathbf{n}}_{\alpha \mathbf{m}}(X(q(t))) \phi_{\beta, \mathbf{n}}(t),$$

$$[L_f, \phi_{\alpha, \mathbf{m}}(t)] = - f(t) \phi_{\alpha, \mathbf{m}}(t) - \lambda(f(t) - i f(t)) \phi_{\alpha, \mathbf{m}}(t),$$

where

$$T^\mathbf{m}_\mathbf{n}(\xi) \equiv (T^\alpha_{\beta \mathbf{m}}(\xi))$$

$$= \binom{\mathbf{n}}{\mathbf{m}} \partial_{\mathbf{n} - \mathbf{m} + \nu} \xi^\nu T^\nu_{\mu} + \binom{\mathbf{n}}{\mathbf{m} - \mu} \partial_{\mathbf{n} - \mathbf{m} + \mu} \xi^\mu = \delta^\mathbf{m}_{\mathbf{n}} T^\nu_{\mu},$$

$$J^\mathbf{m}_\mathbf{n}(X) \equiv (J^\alpha_{\beta \mathbf{m}}(X)) = \binom{\mathbf{n}}{\mathbf{m}} \partial_{\mathbf{n} - \mathbf{m}} X_\alpha J^\alpha, \quad (3.21)$$

and

$$\binom{\mathbf{m}}{\mathbf{n}} = \frac{\mathbf{m}!}{\mathbf{n}!(\mathbf{m} - \mathbf{n})!} = \binom{m_1}{n_1} \binom{m_2}{n_2} \cdots \binom{m_N}{n_N}. \quad (3.22)$$

Here and henceforth we use the convention that a sum over a multi-index runs over all values of length at most $p$. Since $T^\mathbf{m}_\mathbf{n}(\xi)$ and $J^\mathbf{m}_\mathbf{n}(X)$ vanish whenever $|\mathbf{n}| > |\mathbf{m}|$, the sums over $\mathbf{n}$ in (3.20) are in fact further restricted.

Denote the space spanned by $q^\mu(t)$ and $\{\phi_{\alpha, \mathbf{m}}(t)\}_{|\mathbf{m}| \leq p}$ by $J^p \mathcal{Q}$. $\phi_{\alpha, \mathbf{m}}(t)$ will be referred to as a $p$-jet, where $p$ is the truncation order. This space is not a $\text{DGRO}(N, g)$ module, because diffeomorphisms act non-linearly on jets. $p$-jets are usually defined as an equivalence class of functions: two functions are equivalent if all derivatives up to order $p$, evaluated at $q^\mu$, agree. However, each class has a unique representative which is a polynomial of order at most $p$, namely the Taylor expansion around $q^\mu$, so we may canonically identify jets with Taylor series. Since $q^\mu(t)$ depends on a parameter $t$, we deal in fact with trajectories in jet space, but these will also be called jets for brevity.
on the trajectory, as can be seen in (2.1). However, the space \( C(J^pQ) \) of functionals on \( J^pQ \) (local in \( t \)) is a module, because the action on a \( p \)-jet can never produce a jet of order higher than \( p \). Equivalently, there is a non-linear realization of the DGRO algebra on the jet space \( J^pQ \).

Expand also the EL equations and the anti-fields in multi-dimensional Taylor series. Set \( \mathcal{E}^\alpha_{,m}(t) = \partial_{r} \mathcal{E}^\alpha(q(t), t) \) and \( \phi^{*, \alpha}_{,m}(t) = \partial_{r} \phi^{*, \alpha}(q(t), t) \). What must be noted is that we can only define \( \mathcal{E}^\alpha_{,m}(t) \) for \( |m| \leq p - o_\alpha \), where \( o_\alpha \) is the order of the EL equation \( \mathcal{E}^\alpha(x) \). This is because \( \mathcal{E}^\alpha_{,m}(t) \) is a function of \( \phi_{\alpha, n}(t) \) for all \( |n| \leq |m| + o_\alpha \), and \( \phi_{\alpha, n}(t) \) is undefined for \( |n| > p \). Similarly, the relations (3.11) and the corresponding second-order anti-fields \( b^a_{,m}(t) \) give rise to the jets \( r^a_{,m}(t) = \partial_{r} (r^a_{,m}(q(t), t) \phi^{*, \alpha}(q(t), t)) \) and \( b^a_{,m}(t) = \partial_{r} b^a(q(t), t) \), respectively. If the relations \( r^a_{,m} \) are of order \( \varsigma_\alpha \) in the derivatives, \( r^a_{,m}(t) \) and \( b^a_{,m}(t) \) is only defined for \( |m| \leq p - \varsigma_\alpha \).

The conditions of type \( \partial_t \phi_{\alpha}(x, t) \) give rise to additional constraints:

\[
\begin{align*}
D_t \phi_{\alpha, m}(t) &\equiv \phi_{\alpha, m}(t) - \dot{\phi}_{\alpha, m+\mu}(t) \approx 0, \\
D_t \phi^{*, \alpha}_{,m}(t) &\equiv \phi^{*, \alpha}_{,m}(t) - \dot{\phi}^{*, \alpha}_{,m+\mu}(t) \approx 0, \\
D_t b^a_{,m}(t) &\equiv b^a_{,m}(t) - \dot{b}^a_{,m+\mu}(t) \approx 0.
\end{align*}
\]

These conditions are eliminated in cohomology by the introduction of further (second and third order) anti-fields \( \bar{\phi}_{\alpha, m}(t) \), \( \bar{\phi}^{*, \alpha}_{,m}(t) \) and \( \bar{b}^a_{,m}(t) \). The conditions in (3.23), and hence the barred antifields, are only defined for one order less than the corresponding unbarred antifield, since \( |m + \mu| = |m| + 1 \).

Add dual coordinates (jet momenta) \( p^\mu(t) \), \( \pi^{a,m}(t) \), \( \pi^{s,m}_{,\alpha}(t) \), \( c^m_{,\alpha}(t) \), \( \pi^{a,m}_{,\alpha}(t) \), \( \pi^{s,m}_{,\alpha}(t) \), \( \pi^{m}_{,\alpha}(t) \), \( \pi^{m}_{,\alpha}(t) \) and \( \pi^{m}_{,\alpha}(t) \), which satisfy

\[
\begin{align*}
[p^\mu(s), q^\nu(t)] &= \delta^\nu_\mu \delta(s - t), \\
[\pi^{a,m}(s), \phi_{\beta,n}(t)] &= \delta^a_\beta \delta_{mn} \delta(s - t), \\
[\pi^{s,m}_{,\alpha}(s), \phi^{*, \beta}_{,n}(t)] &= \delta^\beta_\alpha \delta_{mn} \delta(s - t), \\
[c^m_{,\alpha}(s), b^b_{,n}(t)] &= \delta^b_\alpha \delta_{mn} \delta(s - t), \\
[\pi^{a,m}_{,\alpha}(s), \bar{\phi}_{\beta,n}(t)] &= \delta^a_\beta \delta_{mn} \delta(s - t), \\
[\pi^{s,m}_{,\alpha}(s), \bar{\phi}^{*, \beta}_{,n}(t)] &= \delta^\beta_\alpha \delta_{mn} \delta(s - t), \\
[\pi^{m}_{,\alpha}(s), \bar{b}^b_{,n}(t)] &= \delta^b_\alpha \delta_{mn} \delta(s - t),
\end{align*}
\]

and all other brackets vanish. Denote the phase space spanned by all jets and jet momenta by \( J^pP^* \) and the space of local functionals on \( J^pP^* \) by \( C(J^pP^*) \); alternatively, this space may be considered as the differential oper-
ators on $J^pQ^*$. The KT differential acting on the space of $C(J^pP^*)$ becomes

\[
Q = \int dt \left( \sum_{|m| \leq p-o} \mathcal{L}^a_{\alpha}(t) \pi^a_{\alpha}(t) + \sum_{|m| \leq p-o} r^a_{\alpha} \right) + \sum_{|m| \leq p-o} D_t \phi_{\alpha,m}(t) \pi^a_{\alpha}(t) + \sum_{|m| \leq p-o} D_t \phi^*_{\alpha,m}(t) \pi^a_{\alpha}(t) \\
+ \sum_{|m| \leq p-o} D_t b^a_{\alpha,m}(x,t) \pi^a_{\alpha}(x,t). \tag{3.25}
\]

The situation is summarized in the following table:

| $g$ | Field | Momentum | Order | Ideal |
|-----|-------|----------|-------|-------|
| 0   | $\phi_{\alpha,m}(t)$ | $\pi^{a}_{\alpha,m}(t)$ | $p$   | $-$   |
| 1   | $\phi^*_{\alpha,m}(t)$ | $\pi^{a}_{\alpha,m}(t)$ | $p-o$ | $\mathcal{L}^a_{\alpha}(t) \approx 0$ |
| 1   | $\pi^{a}_{\alpha,m}(t)$ | $\pi^{a}_{\alpha,m}(t)$ | $p-1$ | $D_t \phi_{\alpha,m}(t) \approx 0$ |
| 2   | $b^a_{\alpha,m}(t)$ | $c^a_{\alpha,m}(t)$ | $p-s$ | $r^a_{\alpha} \approx 0$ |
| 2   | $\phi^*_{\alpha,m}(t)$ | $\pi^{a}_{\alpha,m}(t)$ | $p-o-s$ | $D_t \phi^*_{\alpha,m}(t) \approx 0$ |
| 3   | $b^a_{\alpha,m}(t)$ | $c^a_{\alpha,m}(t)$ | $p-s-o$ | $D_t b^a_{\alpha,m}(t) \approx 0$ |

### 3.5 KT complex for the observer’s trajectory

Until now we have not been very explicit about the set of fields. In this subsection we will assume that among the fields $\phi_{\alpha}(x)$ is a metric $g_{\mu\nu}(x)$, either as a fundamental field or expressed in terms of vielbeine. We can then construct the following derived quantities:

1. The Levi-Civita connection $\Gamma^\nu_{\sigma\tau}(x,t) = \frac{1}{2} g^{\nu\rho}(x,t) \left( \partial_\sigma g_{\rho\tau}(x,t) - \partial_\tau g_{\rho\sigma}(x,t) \right) + \partial_\tau g_{\rho\sigma}(x,t) - \partial_\rho g_{\sigma\tau}(x,t)$.

2. The einbein $e(t) = \sqrt{g_{\mu\nu}(q(t),t)q^\nu(t)q^\mu(t)}$.

3. The reparametrization connection $\Gamma(t) = -e^{-1}(t) \dot{e}(t)$.

Consider the geodesic operator

\[
\mathcal{G}_\mu(t) = e^{-1}(t) g_{\mu\nu}(t) \left( \dot{q}^\nu(t) + \Gamma(t) q^\nu(t) + \Gamma^\nu_{\sigma\tau}(t) q^\sigma(t) \dot{q}_\tau(t) \right), \tag{3.27}
\]
where $g_{\mu\nu}(t)$ and $\Gamma^\nu_{\sigma\tau}(t)$ are the zero-jets corresponding to the metric and Levi-Civita connection, respectively. If we define the proper time derivative by
\[
\frac{d\phi}{d\tau}(t) = e^{-1}(t)\frac{d}{dt}(e(t)\phi(t)),
\]
(3.28) takes on the suggestive form
\[
G^\mu(t) = g^\mu_{\nu}(t)\left(\frac{d^2q^\nu}{d\tau^2}(t) + \Gamma^\nu_{\sigma\tau}(t)\frac{dq^\sigma}{d\tau}(t)\frac{dq^\tau}{d\tau}(t)\right).
\]
(3.29)

It is straightforward to check that the geodesic equation $G^\mu(t) = 0$ transforms homogeneously under $DGRO(N, g)$:
\[
[L_\xi, G^\nu(t)] = -\partial^\nu \xi^\mu(q(t))G^\mu(t),
\]
\[
[J_X, G^\nu(t)] = 0,
\]
\[
[L_f, G^\nu(t)] = -f(t)\dot{G}^\nu(t) - \dot{f}(t)\dot{G}^\nu(t).
\]
(3.30)

It can therefore be used to eliminate the observer’s trajectory, apart from initial conditions. To implement this constraint in cohomology, we introduce the trajectory antifield $q^*_{\mu}(t)$, with momentum $p^*_{\mu}(t)$. They obey the non-zero anticommutation relation
\[
[p^*_{\mu}(s), q^*_{\nu}(t)] = \delta^\nu_\mu \delta(s - t),
\]
(3.31)

which is fermionic since $G^\nu(t)$ is bosonic. The contribution to the KT differential is
\[
Q = \int dt \ G^\mu(t)p^*_{\mu}(t).
\]
(3.32)

4 Quantization

The KT complexes constructed in the previous section were all classical in the sense that the abelian charges of the DGRO algebra vanish. To quantize the theory, we introduce a Fock vacuum annihilated by all negative Fourier modes; see [10] for an explicit description on how this is carried out. To avoid ill defined expressions acting on the Fock vacuum, all expressions must be normal ordered with respect to frequency; this is denoted by double dots ($\cdots$). It follows immediately from (3.20) that the following operators define
a realization of $DGRO(N,g)$ in Fock space:

$$L_\xi = \int dt \left\{ :\xi^\mu(q(t))p_\mu(t): + \sum_{|n|\leq|m|} T_{\alpha m}^{\beta n}(\xi(q(t))):\phi_\beta,n(t)\pi^{\alpha,m}(t): \right\}$$

$$= \int dt \left\{ :\xi^\mu(q(t))p_\mu(t): - \xi^\mu(q(t))P_\mu(t) + \sum_{|n|\leq|m|\leq|p|} \left( \begin{array}{c|c} m & n \end{array} \right) \partial_{m-n}\xi^\mu(q(t))E_{n+\mu}^m(t) \right\} + T_{d\xi},$$

$$T_{d\xi} = \int dt \sum_{|n|\leq|m|\leq|p|} \left( \begin{array}{c|c} m & n \end{array} \right) \partial_{m-n+\nu}\xi^\mu(q(t))T_{\mu n}^{m\nu}(t), \quad (4.1)$$

$$J_X = \int dt \sum_{|n|\leq|m|\leq|p|} J_{\alpha m}^{\beta n}(X(q(t))):\phi_\beta,n(t)\pi^{\alpha,m}(t):$$

$$= \int dt \sum_{|n|\leq|m|\leq|p|} \left( \begin{array}{c|c} m & n \end{array} \right) \partial_{m-n}X_\alpha(q(t))J_{n}^{m\alpha}(t),$$

$$L_f = \int dt f(t)L(t) + \lambda(\dot{f}(t) - if(t))E(t),$$

where

$$P_\mu(t) = \sum_{|m|\leq|p|} E_{m+\mu}^m(t),$$

$$L(t) = - :q^\mu(t)p_\mu(t): + F(t),$$

$$E_{n}^m(t) = :\pi^{\alpha,m}(t)\phi_{\alpha,n}(t):,$$

$$J_{n}^{m\alpha}(t) = J_{\alpha \beta}^{\alpha m}(t)\phi_{\beta,n}(t):,$$

$$T_{\mu n}^{m\nu}(t) = T_{\alpha \beta}^{\alpha \mu}(t)\phi_{\beta,n}(t):,$$

$$F(t) = \sum_{|m|\leq|p|} :\pi^{\alpha,m}(t)\phi_{\alpha,m}(t):$$

(4.2)
and $T_\nu^\mu = (T^{\alpha\mu}_{\beta\nu})$ and $J^a = (J^{\alpha a}_{\beta a})$ are matrices that generate $gl(N)$ and $g$, respectively. The currents in (4.2) satisfy an algebra of the form

\[
[T^{m\mu}_{n\sigma}(s), T^{r\rho}_{s\tau}(t)] = \cdots + \frac{1}{2\pi i} (k_1 \delta^\mu_{\sigma} \delta^\rho_{\nu} + k_2 \delta^\mu_{\nu} \delta^\rho_{\sigma}) \delta^r_m \delta^r_n (s - t),
\]

\[
[T^{m\mu}_{n\sigma}(s), E^r_s(t)] = \cdots + \frac{k_3}{2\pi i} \delta^\mu_{\sigma} \delta^r_m \delta^r_n (s - t),
\]

\[
[T^{m\mu}_{n\sigma}(s), J^{rb}_s(t)] = \cdots + 0,
\]

\[
[J^{ma}_m(s), T^{r\rho}_{s\tau}(t)] = \cdots + k_5 \delta^a_b \delta^m_m \delta^r_n \delta^r_n (s - t),
\]

\[
[J^{ma}_m(s), E^r_s(t)] = \cdots + 0,
\]

\[
[J^{ma}_m(s), J^{rb}_s(t)] = \cdots + k_6 \delta^a_b \delta^m_m \delta^r_n \delta^r_n (s - t),
\]

\[
[E^m_n(s), E^r_s(t)] = \cdots + \frac{k_4}{2\pi i} \delta^m_m \delta^r_n \delta^r_n (s - t),
\]

\[
[F(s), F(t)] = \cdots + \frac{c}{24\pi i} \left( N + \frac{p}{N} \right) (\delta^r (s - t) + \dot{\delta} (s - t)),
\]

\[
[F(s), E^m_n(t)] = \cdots + \frac{d_0}{4\pi i} \delta^m_m (\delta (s - t) + i \dot{\delta} (s - t)),
\]

\[
[F(s), J^{ma}_m(t)] = \cdots + 0,
\]

\[
[F(s), T^{m\mu}_{n\sigma}(t)] = \cdots + \frac{d_1}{4\pi i} \delta^m_m (\delta (s - t) + i \dot{\delta} (s - t)).
\]

Here I have not written down regular terms explicitly; they form an open algebra which is described in [13].

Let $\mathfrak{g}$ be a $gl(N)$ representation and $M$ a $g$ representation. Define numbers $u, v, w, x, y$ by

\[
\text{tr} 1 = x, \quad \text{tr} T^\mu_\tau T^\sigma_\tau = u \delta^\mu_\nu \delta^\sigma_\nu + v \delta^\mu_\nu \delta^\sigma_\tau, \quad \text{tr} T^\mu_\nu = w \delta^\mu_\nu, \quad \text{tr} J^a J^b = y \delta^{ab},
\]

where the trace is taken in the $\mathfrak{g} \oplus gl(N)$ representation $M \oplus \mathfrak{g}$. The relation to the numbers $k_0(\mathfrak{g})$, $k_1(\mathfrak{g})$, $k_2(\mathfrak{g})$, and $y_M$ defined in [13] is

\[
u = k_2(\mathfrak{g}) \dim M, \quad y = \dim \mathfrak{g} y_M,
\]

provided that $\mathfrak{g}$ and $M$ are irreducible. The values of the abelian charges were given in [10], Theorems 1 and 3, and again in [13], Theorem 1. They

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depend on the central extensions in (4.3):

\[
\begin{align*}
c_1 &= 1 - k_1 \binom{N + p}{N} - k_4 \binom{N + p + 1}{N + 2}, \\
c_2 &= -k_2 \binom{N + p}{N} - 2k_3 \binom{N + p}{N + 1} - k_4 \binom{N + p}{N + 2}, \\
c_3 &= 1 + d_1 \binom{N + p}{N} + d_0 \binom{N + p}{N + 1}, \\
c_4 &= 2N - c \binom{N + p}{N}, \\
c_5 &= k_5 \binom{N + p}{N}.
\end{align*}
\]

For the Fock module, we have

\[
\begin{align*}
k_1 &= \mp u, & k_2 &= \mp v, & k_3 &= \mp w, & k_4 &= \mp x, \\
k_5 &= \mp y, & d_0 &= \mp x, & d_1 &= \mp w, & c &= \mp x.
\end{align*}
\]

The upper signs apply to bosons and the lower signs to fermions. Note that I have used the assumption that \( g \) is semisimple to put \( k_6 = k_7 = k_8 = 0 \).

The \( DGRO(N, g) \) representations obtained in this fashion are well defined for all finite values of the jet order \( p \). In order to reconstruct the original field by means of the Taylor series (3.18), one must take the limit \( p \to \infty \). A necessary condition for taking this limit is that the abelian charges have a finite limit. Taken at face value, the prospects for succeeding appear bleak. When \( p \) is large, \( \binom{m+p}{n} \approx p^n/n! \), so the abelian charges (4.6) diverge; the worst case is \( c_1 \approx c_2 \approx p^{N+2}/(N + 2)! \), which diverges in all dimensions \( N > -2 \). In [13] I devised a way out of this problem: consider a more general realization by taking the direct sum of operators corresponding to different values of the jet order \( p \). Take the sum of \( r + 1 \) terms like those in (4.2), with \( p \) replaced by \( p, p − 1, ..., p − r \), respectively, and with \( g \) and \( M \) replaced by \( g^{(i)} \) and \( M^{(i)} \) in the \( p − i \) term.

Such a sum of contributions arises naturally from the KT complex, because the antifields are only defined up to an order smaller than \( p \) (e.g. \( p − o_n \) or \( p − σ_a \)). Summing the contributions from the various entries in
subsections 3.4 and 3.5, (4.1) is replaced by

\[
\mathcal{L}_\xi = \int dt \left\{ \xi^\mu(q(t)) p_\mu(t) : - \partial_\nu \xi^\mu(q(t)) : q^\nu(t) + \nu \xi^\mu(q(t)) : p^\nu(t) : + \right. \\
\sum_{|n| \leq |m| \leq p} T^{\beta n}_{\alpha m}(\xi(q(t))) : \phi_{\beta n}(t) \pi^{\alpha m}(t) : + \\
\left. - \sum_{|n| \leq |m| \leq p - o} T^{\beta n}_{\alpha m}(\xi(q(t))) : \phi^*_{\beta n}(t) \pi^{* \alpha m}(t) : + \right. \\
\sum_{|n| \leq |m| \leq p - \varsigma} T^{\beta n}_{\alpha m}(\xi(q(t))) : b_n(t) c^m(t) : + \\
\sum_{|n| \leq |m| \leq p - 1} T^{\beta n}_{\alpha m}(\xi(q(t))) : \bar{b}_n(t) \bar{c}^m(t) : \\
- \sum_{|n| \leq |m| \leq p - o - 1} T^{\beta n}_{\alpha m}(\xi(q(t))) : \bar{\phi}_{\beta n}(t) \pi^{* \alpha m}(t) : + \\
\sum_{|n| \leq |m| \leq p - \varsigma - 1} T^{\beta n}_{\alpha m}(\xi(q(t))) : \bar{b}_n(t) \bar{c}^m(t) : \right\},
\]

(4.8)

and similar contributions to \( J_X \) and \( L_f \). Here \( T^{\beta n}_{\alpha m}(\xi) \) and \( T^{\beta n}_{\alpha m}(\xi) \) are matrices in the two different \( gl(N) \) representations acting on fields and second-order antifields, respectively; the action on first order antifields is dual to the field action, because this is how the EL equations transform.

Denote the numbers \( u, v, w, x, y \) in the modules \( \phi^{(i)} \) and \( M^{(i)} \), defined as in (4.4), by \( u_i, v_i, w_i, x_i, y_i \), respectively. Of course, there is only one contribution from the observer's trajectory. Then it was shown in [13], Theorem 3, that

\[
c_1 = -U \begin{pmatrix} N + p - r \\ N - r \end{pmatrix}, \quad c_2 = -V \begin{pmatrix} N + p - r \\ N - r \end{pmatrix},
\]

\[
c_3 = W \begin{pmatrix} N + p - r \\ N - r \end{pmatrix}, \quad c_4 = X \begin{pmatrix} N + p - r \\ N - r \end{pmatrix}, \quad c_5 = Y \begin{pmatrix} N + p - r \\ N - r \end{pmatrix},
\]

(4.9)

where \( u_0 = U, v_0 = V, w_0 = W, x_0 = X \) and \( y_0 = Y \), provided that the
following conditions hold:

\[ i \quad u_i + \sum_{j=0}^{i-2} \sum_{\ell=0}^{j} x_\ell = (-)^i \binom{r}{i} U, \]

\[ ii \quad v_i + \sum_{j=0}^{i-1} (2w_j + \sum_{\ell=0}^{j-1} x_\ell) = (-)^i \binom{r}{i} V, \]

\[ iii \quad w_i + \sum_{j=0}^{i-1} x_j = (-)^i \binom{r}{i} W, \]

\[ iv \quad x_i = (-)^i \binom{r}{i} X, \]

\[ v \quad y_i = (-)^i \binom{r}{i} Y, \]

\[ vi \quad \sum_{i=0}^{r} (2w_j + \sum_{\ell=0}^{i-1} x_\ell) = 0, \]

\[ vii \quad \sum_{i=0}^{r} x_i = 0, \]

\[ viii \quad \sum_{i=0}^{r-1} \sum_{j=0}^{i} x_j = 0. \]

The abelian charges diverge if \( N > r \) and vanish if \( N < r \). When \( N = r \), they are independent of \( p \) and in general non-zero.

Define

\[ \alpha_i = \sum_{j=0}^{i-2} \sum_{\ell=0}^{j} (-)^\ell \binom{r}{\ell}, \]

\[ \beta_i = \sum_{j=0}^{i-1} (-)^j \binom{r}{j}, \]

\[ \gamma_i = \sum_{j=0}^{i-1} \sum_{\ell=0}^{j-1} (-)^\ell \binom{r}{\ell}. \]

Using the recurrence formula

\[ \binom{n}{i} = \binom{n}{i-1} + \binom{n-1}{i-1} \] (4.12)
and a straightforward induction argument, it can be shown that
\[ \alpha_i = \gamma_i = (-i)^i \binom{r-2}{i-2}, \quad \beta_i = -(-i)^i \binom{r-1}{i-1}. \] (4.13)

The conditions (4.10) become
\[ i \quad u_i + \alpha_i X = (-i)^i \binom{r}{i} U, \]
\[ ii \quad v_i + 2\beta_i W - \gamma_i X = (-i)^i \binom{r}{i} V, \]
\[ iii \quad w_i + \beta_i X = (-i)^i \binom{r}{i} W, \]
\[ iv \quad x_i = (-i)^i \binom{r}{i} X, \]
\[ v \quad y_i = (-i)^i \binom{r}{i} Y. \] (4.14)

The remaining conditions follow immediately from the identities \( \alpha_{r+1} = \beta_{r+1} = 0. \)

There is another, simpler way to arrive at (4.14). Ignoring the finite contributions from the observer’s trajectory, (4.14) can be rewritten as
\[ c_1 = uA(p) + xC(p-1), \]
\[ c_2 = vA(p) + 2wB(p-1) + xC(p-2), \]
\[ c_3 = -wA(p) + xB(p-1), \]
\[ c_4 = xA(p), \]
\[ c_5 = -yA(p), \] (4.15)

where
\[ A(p) = \sum_{|m| \leq p} 1 = \binom{N+p}{N}, \]
\[ B(p) = \sum_{|m| \leq p} (m_1 + 1) = \binom{N+p+1}{N+1}, \] (4.16)
\[ C(p) = \sum_{|m| \leq p} (m_1 + 1)(m_2 + 1) = \binom{N+p+2}{N+2}. \]

As discussed in [13], the numbers arise from sums over multi-indices length \( |m| \leq p \) of certain components. The restriction to finite length can be viewed
as a regularization, with the nice property that diffeomorphism invariance is preserved. Another possible regularization is to introduce the fugacity \( \zeta = (\zeta_1, ..., \zeta_N) \), and consider the expressions

\[
A(\zeta) = \sum_{|m|} \zeta^m = \prod_{i=1}^N \frac{1}{1 - \zeta^i},
\]

\[
B(\zeta) = \sum_{|m|} (m_1 + 1)\zeta^m = (\zeta_1 \frac{\partial}{\partial \zeta_1} + 1)A(\zeta) = \frac{1}{1 - \zeta_1}A(\zeta),
\]

\[
C(\zeta) = \sum_{|m|} (m_1 + 1)(m_2 + 1)\zeta^m = (\zeta_1 \frac{\partial}{\partial \zeta_1} + 1)(\zeta_2 \frac{\partial}{\partial \zeta_2} + 1)A(\zeta)
\]

\[
= \frac{1}{1 - \zeta_1} \frac{1}{1 - \zeta_2}A(\zeta).
\]

If we now put all \( \zeta_i = \zeta \), we obtain

\[
A(\zeta) = \frac{1}{(1 - \zeta)^N}, \quad B(\zeta) = \frac{1}{(1 - \zeta)^{N+1}}, \quad C(\zeta) = \frac{1}{(1 - \zeta)^{N+2}}.
\]

The limit \( p \to \infty \) is replaced by \( \zeta \to 1 \).

However, it is not \( B(p) \) and \( C(p) \) that appear in (4.15), but rather \( B(p-1), C(p-1), \) and \( C(p-2) \). If

\[
A(p) \sim \sum_{k=0}^p a_k \zeta^k \to A(\zeta)
\]

then

\[
A(p-1) \sim \sum_{k=0}^{p-1} a_{k-1} \zeta^k = \sum_{\ell=0}^{p-1} a_{\ell} \zeta^{\ell+1} \to \zeta A(\zeta).
\]

More generally, \( A(p-k) \sim \zeta^k A(\zeta) \).

Equation (4.15) contains the contributions from a single, fermionic jet of order \( p \). With several jets of order \( p, \ldots, p-r \), the term \( uA(p) \) becomes \( u_0A(p)+u_1A(p-1)+\ldots+u_rA(p-r) \), so we must replace the parameter \( u \) with the function \( u(\zeta) = \sum_{i=0}^r u_i \zeta^i \). Taking this into account, (4.15) corresponds to

\[
c_1 = u(\zeta)A(\zeta) + x(\zeta)\zeta C(\zeta),
\]

\[
c_2 = v(\zeta)A(\zeta) + 2w(\zeta)\zeta B(\zeta) + x(\zeta)\zeta^2 C(\zeta),
\]

\[
c_3 = -w(\zeta)A(\zeta) + x(\zeta)\zeta B(\zeta),
\]

\[
c_4 = x(\zeta)A(\zeta),
\]

\[
c_5 = -y(\zeta)A(\zeta).
\]
Now demand that
\[ c_1 = U, \quad c_2 = V, \quad c_3 = -W, \quad c_4 = X, \quad c_5 = -Y, \quad (4.22) \]
in the limit \( \zeta \to 1 \). The simplest way to achieve this is to require that (4.22) holds for all \( \zeta \). This leads to
\[
\begin{align*}
    u(\zeta) &= U(1 - \zeta)^N - X\zeta(1 - \zeta)^{N-2}, \\
v(\zeta) &= V(1 - \zeta)^N + 2W\zeta(1 - \zeta)^{N-1} - X\zeta^2(1 - \zeta)^{N-2}, \\
w(\zeta) &= W(1 - \zeta)^N - X\zeta(1 - \zeta)^{N-1}, \\
x(\zeta) &= X(1 - \zeta)^N, \\
y(\zeta) &= Y(1 - \zeta)^N.
\end{align*}
\]
Equation (4.14) is recovered after an expansion in \( \zeta \). E.g.,
\[
x(\zeta) = \sum_{i=0}^{r} x_i \zeta^i = X(1 - \zeta)^N = \sum_{i=0}^{N} (-i \binom{N}{i}) X \zeta^i, \quad (4.24)
\]
leading to \( x_i = (-)^i \binom{N}{i} X \) and \( r = N \).

5 Solutions to the constraint equations

5.1 Original constraint equations

Let us now consider the solutions of (4.10) for the numbers \( x_i \), which can be interpreted as the number of fields and anti-fields. First assume that the field \( \phi_{\alpha, m}(t) \) is fermionic with \( x_F \) components, which gives \( x_0 = x_F \). We may assume, by the spin-statistics theorem, that the EL equations are first order, so the bosonic antifields \( \phi^*_{\alpha, m}(t) \) contribute \(-x_F\) to \( x_1 \). The barred antifields \( \phi^*_{\alpha, m}(t) \) are also defined up to order \( p - 1 \), and so give \( x_1 = -x_F \), and the barred second-order antifields \( \phi^{*\alpha}_{\alpha, m}(t) \) give \( x_2 = x_F \). Further assume that the fermionic EL equations have \( x_S \) gauge symmetries, i.e. the second-order antifields \( b^a_{\alpha, m}(t) \) give \( x_2 = x_S \). In established theories, \( x_S = 0 \), but we will need a non-zero value for \( x_S \). Finally, the corresponding barred antifields give \( x_3 = -x_S \).

For bosons the situation is analogous, with two exceptions: all signs are reversed, and the EL equations are assumed to be second order. Hence \( \phi^*_{\alpha, m}(t) \) yields \( x_2 = x_B \) and the gauge antifields \( b^a_{\alpha, m}(t) \) give \( x_3 = -x_G \). Accordingly, the barred antifields are one order higher.
The situation is summarized in the following tables, where the upper half is valid if the original field is fermionic and the lower half if it is bosonic:

| $g$ | Field                  | Order | $x$     |
|-----|------------------------|-------|---------|
| 0   | $\phi_{\alpha,m}(t)$ | $p$   | $x_F$  |
| 1   | $\Phi_{\alpha,m}(t)$  | $p-1$ | $-x_F$ |
| 1   | $\phi^{\alpha}_{\beta,m}(t)$ | $p-1$ | $-x_F$ |
| 2   | $\phi_{\alpha,m}(t)$  | $p-2$ | $x_F$  |
| 2   | $b^{\alpha}_{\beta,m}(t)$ | $p-2$ | $x_S$  |
| 3   | $\Phi_{\alpha,m}(t)$  | $p-3$ | $-x_S$ |

If we add all contributions of the same order, we see that relation iv in (4.14) can only be satisfied provided that

$$
p : \quad x_F - x_B = X
$$

$$
p - 1 : \quad -2x_F + x_B = -rX,
$$

$$
p - 2 : \quad x_B + x_F + x_S = \left(\frac{r}{2}\right)X,
$$

$$
p - 3 : \quad -x_B - x_S - x_G = -\left(\frac{r}{3}\right)X,
$$

$$
p - 4 : \quad x_G = \left(\frac{r}{4}\right)X,
$$

$$
p - 5 : \quad 0 = -\left(\frac{r}{5}\right)X,
$$

(5.1)

The last equation holds only if $r \leq 4$ (or trivially if $X = 0$). On the other hand, if we demand that there is at least one bosonic gauge condition, the $p - 4$ equation yields $r \geq 4$. Such a demand is natural, because both the
Maxwell/Yang-Mills and the Einstein equations have this property. Therefore, we are unambiguously guided to consider \( r = 4 \) (and thus \( N = 4 \)). The specialization of (5.2) to four dimensions reads

\[
p \ : \quad x_F - x_B = X \\
p - 1 \ : \quad -2x_F + x_B = -4X, \\
p - 2 \ : \quad x_B + x_F + x_S = 6X, \\
p - 3 \ : \quad -x_B - x_S - x_G = -4X, \\
p - 4 \ : \quad x_G = X.
\]

Clearly, the unique solution to these equations is

\[
x_F = 3X, \quad x_B = 2X, \quad x_S = X, \quad x_G = X.
\]

### 5.2 Reduced constraint equations

The barred antifields, associated with the constraints (3.23), can conveniently be eliminated first. For each field or antifield of order \( p - \alpha \), there is a corresponding barred antifield of opposite Grassmann parity and one order lower. E.g., for the field \( \phi_{\alpha,m}(t) \) at order \( p \), we have the antifield \( \bar{\phi}_{\alpha,m}(t) \) at order \( p - 1 \), and for \( \phi^{*\alpha}_{m}(t) \) at order \( p - \alpha \), we have \( \bar{\phi}^{*\alpha}_{m}(t) \) at order \( p - \alpha - 1 \). Thus, if \( x_i \) has a contribution \( x'_i \) from an unbarred (anti-)field, then \( x_{i+1} \) has the contribution \(-x'_i\). In particular, \( x'_r = 0 \). This means that the total value for \( c_4 \) in (4.4) becomes

\[
c_4 = \sum_{i=0}^{r} x_i \binom{N + p - i}{N} \\
= \sum_{i=0}^{r-1} x'_i \binom{N + p - i}{N} - \sum_{i=1}^{r} x'_{i-1} \binom{N + p - i}{N} \\
= \sum_{i=0}^{r-1} x'_i \left\{ \binom{N + p - i}{N} - \binom{N + p - i - 1}{N} \right\} \\
= \sum_{i=0}^{r-1} x'_i \binom{N - 1 + p - i}{N - 1}.
\]

We recognize that this expression is of the same form as the original expression, with the replacements \( x_i \to x'_i \), \( N \to N - 1 \), and \( r \to r - 1 \). The finiteness conditions for the original parameters \( x_i \) are thus equivalent to
the same conditions for the reduced parameters $x'_i$ in one dimension less. Henceforth we only consider the reduced parameters, and skip the primes to avoid unnecessarily cumbersome notation.

The tables (5.1) are replaced by (fermions first, bosons second)

| $g$ | Field          | Order | $x$   |
|-----|----------------|-------|-------|
| 0   | $\phi_{\alpha,m}(t)$ | $p$  | $x_F$ |
| 1   | $\phi^*_{\alpha,m}(t)$ | $p - 1$ | $-x_F$ |
| 2   | $b^\alpha_m(t)$ | $p - 2$ | $x_S$ |

| $g$ | Field          | Order | $x$   |
|-----|----------------|-------|-------|
| 0   | $\phi_{\alpha,m}(t)$ | $p$  | $-x_B$ |
| 1   | $\phi^*_{\alpha,m}(t)$ | $p - 2$ | $x_B$ |
| 2   | $b^\alpha_m(t)$ | $p - 3$ | $-x_G$ |

The reduced version of (5.2) becomes

\[
p : \quad x_F - x_B = X
\]
\[
p - 1 : \quad -x_F = -rX,
\]
\[
p - 2 : \quad x_B + x_S = \binom{r}{2} X,
\]
\[
p - 3 : \quad -x_G = \binom{r}{3} X,
\]
\[
p - 4 : \quad 0 = \binom{r}{4} X, \ldots
\]

If we sum the first three equations, we obtain $x_S = (r^2 - 3r + 2)X/2$. Thus the only case where we can avoid the fermionic gauge symmetries is if $r = 2$, i.e. $N = 3$. However, if we put $r = 2$, we get from the fourth equation $x_G = 0$, which means that there are no bosonic gauge symmetries either. But this can not be the case, assuming that the Einstein equation is included among our EL equations, and thus $r > 2$. However, the last equation is clearly impossible to satisfy if $r \geq 4$. For $r = 3$, i.e. $N = 4$, (5.7) becomes

\[
p : \quad x_F - x_B = X
\]
\[
p - 1 : \quad -x_F = -3X,
\]
\[
p - 2 : \quad x_B + x_S = 3X,
\]
\[
p - 3 : \quad -x_G = -X.
\]
The solution to these equations is of course still given by (5.4).

The solutions to the other conditions in (4.10) follow by analogous considerations. Assume that there are \( x_F \) fermions that contribute \( x_F, y_F, u_F, v_F, w_F \) to parameters \( x, y, u, v, w \), respectively. The contributions from the \( x_B \) bosons, \( x_S \) fermionic gauge conditions, and \( x_G \) bosonic gauge conditions are denoted analogously. Since we have already excluded \( N \neq 4 \), we limit ourselves to this case. We also use the reduced parameters and hence we set \( r = 3 \). The numbers \( \alpha_i \) and \( \beta_i \) in (4.11) are

\[
\begin{array}{cccc}
  i & \alpha_i & \beta_i & \gamma_i \\
  0 & 0 & 0 & 0 \\
  1 & 0 & 1 & 0 \\
  2 & 1 & -2 & 1 \\
  3 & -1 & 1 & -1 \\
\end{array}
\]  

(5.9)

By following exactly the same arguments as for \( x_i \) above, we see that the reduced version of (4.14) becomes

\[
p : \quad \begin{align*}
  x_F - x_B &= X \\
  y_F - y_B &= Y \\
  -x_F &= -3X \\
  -y_F &= -3Y \\
  x_B + x_S &= 3X \\
  y_B + y_S &= 3Y \\
  -x_G &= -X \\
  -y_G &= -Y \\
  u_F - u_B &= U \\
  v_F - v_B &= V \\
  -u_F &= -3U \\
  -v_F &= -3V - 2W \\
  u_B + u_S &= 3U - X \\
  v_B + v_S &= 3V + 4W + X \\
  -u_G &= -U + X \\
  -v_G &= -V - W - X
\end{align*}
\]  

(5.10)

\[
p : \quad \begin{align*}
  w_F - w_B &= W \\
  -w_F &= -3W - X \\
  w_B + w_S &= 3W + 2X \\
  -w_G &= -W - X
\end{align*}
\]
The solutions are given by

\begin{align*}
x_B &= 2X \\
y_B &= 2Y \\
x_F &= 3X \\
y_F &= 3Y \\
x_S &= X \\y_S &= Y \\
x_G &= X \\y_G &= Y \\
\end{align*}

\begin{align*}
u_B &= 2U \\
v_B &= 2V + 2W \\
u_F &= 3U \\
v_F &= 3V + 2W \\
u_S &= U - X \\
v_S &= V + 2W + X \\
u_G &= U - X \\
v_G &= V + 2W + X \\
\end{align*}

\[ (5.11) \]

\[ w_B = 2W + X \]

\[ w_F = 3W + X \]

\[ w_S = W + X \]

\[ w_G = W + X \]

This is our main result. It expresses the twenty parameters \( x_B - w_G \) in terms of the five parameters \( X, Y, U, V, W \). For this particular choice of parameters, the abelian charges in (4.9) are given by (4.22), independent of \( p \). Hence there is no manifest obstruction to the limit \( p \to \infty \).

### 5.3 Comparison with known physics

All experimentally known physics is well described by quantum theory, gravity, and the standard model in four dimensions. We have already seen that quantum general covariance more or less dictates that spacetime has \( N = 4 \) dimensions (5.2). It is therefore interesting to investigate to what extent the particle content matches (5.4); recall that \( x = \text{tr} \, 1 \) equals the number of field components.

The bosonic content of the theory is given by the following table. Standard notation for the fields is used, and one must remember that it is the naïve number of components that enters the equation, not the gauge-invariant physical content. E.g., the photon is described by the four com-
ponents $A_\mu$ rather than the two physical transverse components.

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Field & Name & EL equation & $x_B$ \\
\hline
$A_\mu^a$ & Gauge bosons & $D_\nu F^{a\mu} = j^{a\mu}$ & $12 \times 4 = 48$ \\
g_{\mu\nu} & Metric & $G^{\mu\nu} = \frac{1}{8\pi} T^{\mu\nu}$ & $10$ \\
$H$ & Higgs field & $g^{\mu\nu} \partial_\mu \partial_\nu H = V(H)$ & $2$ \\
\hline
\end{tabular}
\end{center}

(5.12)

The total number of bosons in the theory is thus $x_B = 48 + 10 + 2 = 60$, which implies $X = 30$ by (5.4). The number of gauge conditions is $x_G = 16$, which implies $X = 16$. There is certainly a discrepancy here.

The fermionic content in the first generation is given by

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Field & Name & EL equation & $x_F$ \\
\hline
$u$ & Up quark & $\not{D} u = \ldots$ & $2 \times 3 = 6$ \\
d & Down quark & $\not{D} d = \ldots$ & $2 \times 3 = 6$ \\
e & Electron & $\not{D} e = \ldots$ & $2$ \\
$\nu_L$ & Left-handed neutrino & $\not{D} \nu_L = \ldots$ & $1$ \\
\hline
\end{tabular}
\end{center}

(5.13)

The number of fermions in the first generation is thus $x_F = 6 + 6 + 2 + 1 = 15$. Counting all three generations and anti-particles, we find that the total number of fermions is $x_F = 2 \times 3 \times 15 = 90$, which implies $X = 30$. There are no fermionic gauge conditions, so $x_S = 0$, which implies $X = 0$.

It is clear that the predictions for $X$ (30, 16, 30, 0) are not mutually consistent. However, to cancel the leading terms, of order $p$ and $p - 1$, it is only necessary that $2x_F = 3x_B$, which is indeed the case in known physics. It is therefore tempting to speculate that known physics is a first approximation of a more elegant theory, which has the same field content but more gauge conditions, including fermionic ones. This issue will be addressed in the discussion.

It is important to check that the results remain the same if the same physical situation is described with a different, but equivalent, set of fields. Typically, such spurious degrees of freedom have algebraic EL equations.
Denote the original (bosonic, say) $x_B$ fields by $\phi_{\alpha,m}(t)$ and let $\psi_{i,m}(t)$ be $x_A$ spurious fields, defined for $|m| \leq p$. The contribution to $x_0$ from the bosonic fields is thus $x_B + x_A$. There are also $x_A$ new EL equations $E_{i,m}^i(t)$, defined for $|n| \leq |m|$, but not of higher order. The corresponding anti-fields $\psi^{*i}_{j,m}(t)$ add $-x_A$ to $x_0$. The total result is $x_0 = x_B + x_A - x_A = x_B$, as before.

An example is given by the gravitational field in vielbein formalism. Instead of the 10 components of the metric $g_{\mu\nu} = g_{\nu\mu}$ we have the 16 vielbeine $e^i_\mu$. However, the requirement that the metric $g_{\mu\nu} = \epsilon^i_\mu \epsilon^{i\nu}$ be symmetric gives rise to 6 algebraic conditions, so the contribution to $x_0$ is still 10.

6 Discussion

There are two key lessons to be learnt from twentieth century physics:

- General relativity teaches us the importance of diffeomorphism invariance. Physics is fully relational; there is no background stage over which physics takes place. Rather, geometry itself participates actively in the dynamics. Note that this is very different from mere coordinate invariance, because there is no compensating background metric.

- Quantum theory teaches us the importance of projective lowest-energy representations; the passage from Poisson brackets to commutators makes normal ordering necessary. E.g., to study angular momentum from a quantum perspective, it is not sufficient to limit oneself to proper (integer spin) representations of the rotation group $SO(3)$; one must also include the projective (half-integer spin) representations. For finite-dimensional groups such as $SO(3)$, projectivity only manifests itself on the group level, but in the infinite-dimensional case already the Lie algebra is modified. Algebras of linear growth acquire central extensions, e.g. the Virasoro and affine Kac-Moody algebras, whereas algebras of polynomial but non-linear growth acquire abelian but non-central extensions.

The successful construction of a quantum theory of gravity will probably combine these two insights. It seems obvious that the correct way to combine diffeomorphism invariance and projective representations is to consider projective representations of the diffeomorphism group, which on the Lie algebra level gives rise to the DGRO algebra. To even think about quantum gravity without understanding $DGRO(N, \mathbf{g})$ seems to be a doomed
project. It would be like doing classical gravity without tensor fields (= proper representations of the diffeomorphism group), or like doing quantum theory without spinors (= projective representations of $SO(3)$). Hence it seems appropriate to refer to DGRO algebra symmetry as *quantum general covariance*. In fact, the Fock modules considered in [10] automatically solve some of the outstanding problems in quantum gravity. By definition, it clarifies the role of diffeomorphisms, and there are no causality problems, because the theory only involves events on the observer’s trajectory and such events are always causally related [14].

With the introduction of the Koszul-Tate cohomology in the present paper, dynamics has entered representation theory, presumably for the first time. Since the classical KT cohomology is equivalent to standard formulations of classical physics, and the presence of Virasoro-like cocycles signals quantization, the construction in the present paper can be regarded as a novel quantization method, although the relation to other quantization schemes is unclear. An important feature is that abelian extensions pose no problem, as long as they are finite in the $p \to \infty$ limit.

The existence of this limit may be viewed as a requirement on objective reality; the $p$-jets living on the observer’s trajectory can be extended to fields defined throughout spacetime by means of a Taylor expansion, only if the limit $p \to \infty$ is well defined. This imposes severe constraints on the field content. As we saw in Section 5.3, most of these conditions are in qualitative agreement with established theories of physics, in particular with the standard model. We find both fermions and bosons, with EL equations of first and second order, respectively, and bosonic gauge constraints of third order. Moreover, spacetime must have four dimensions provided that there are no reducible gauge conditions.

However, there is also sharp disagreement on the number of gauge conditions. Quantum general covariance predicts the existence of fermionic gauge freedom of second order, and additional bosonic gauge freedom at third order. This points toward some kind of modification, maybe involving superalgebras. An interesting possibility is to consider a gauge theory based on the exceptional Lie superalgebra $\mathfrak{mb}(3|8)$, which is the simple vectorial superalgebra of maximal depth 3 [8, 19]. Classically, the corresponding gauge algebra $\text{map}(N, \mathfrak{mb}(3|8))$ acts on functions $\phi_\alpha(x,y)$, valued in modules of the grade zero subalgebra $sl(3) \oplus sl(2) \oplus gl(1)$, i.e. the non-compact form of the symmetries of the standard model. Here $x = (x^\mu) \in \mathbb{R}^N$ is a spacetime coordinate and $y = (y^i) \in \mathbb{C}^{3|8}$ is a coordinate in internal space. I have recently attempted to generalize the standard model to a gauge theory with $\mathfrak{mb}(3|8)$ symmetry, and extra conditions on the fermionic fields do indeed
arise [15]. However, the considerations in that paper are purely classical. It is clear that quantization in the spirit of the present paper can be carried out \((\text{map}(N, \mathfrak{m} \mathfrak{b}(3|8)) \subset \text{vect}(N + 3|8))\), but this task has not yet been undertaken.

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