The rank of a bimatrix game is the matrix rank of the sum of the two payoff matrices. This paper comprehensively analyzes games of rank one, and shows the following: (1) For a game of rank $r$, the set of its Nash equilibria is the intersection of a generically one-dimensional set of equilibria of parameterized games of rank $r - 1$ with a hyperplane. (2) One equilibrium of a rank-1 game can be found in polynomial time. (3) All equilibria of a rank-1 game can be found by following a piecewise linear path. In contrast, such a path-following method finds only one equilibrium of a bimatrix game. (4) The number of equilibria of a rank-1 game may be exponential. (5) There is a homeomorphism between the space of bimatrix games and their equilibrium correspondence that preserves rank. It is a variation of the homeomorphism used for the concept of strategic stability of an equilibrium component.

KEYWORDS: bimatrix game, Nash equilibrium, rank-1 game, polynomial-time algorithm, homeomorphism.

1. INTRODUCTION

Non-cooperative games are basic economic models. The main concept to analyze them is Nash equilibrium, which recommends to each player a (typically randomized) strategy that is optimal for that player if the other players follow their recommendations. In order to give such a recommendation, a Nash equilibrium must be found by some method (including any adjustment process). For larger games this requires computer algorithms. We consider bimatrix games, which are two-player games in strategic form. The algorithm by Lemke and Howson (1964) finds one equilibrium of a bimatrix game. Finding all equilibria is feasible only for small games because of the exponential number of mixed strategies that typically need to be checked for the equilibrium property (Avis, Rosenberg, Savani, and von Stengel, 2010).

Kannan and Theobald (2010) introduced a hierarchy of bimatrix games based on the matrix rank of the sum of the two payoff matrices. Games of rank 0 are
zero-sum games, which can be solved by linear programming. This paper comprehensively studies games of rank 1. Rank-1 games are economically more interesting than zero-sum games, by allowing a “multiplicative” interaction in addition to an arbitrary zero-sum component (discussed further in Section 10). We will show that, like general bimatrix games, they can have exponentially many disjoint equilibria. On the other hand, as our main results show, they are computationally tractable: One equilibrium of a rank-1 game can be found fast (in polynomial time), and finding all equilibria takes comparable time to finding a single equilibrium of a general bimatrix game. Large rank-1 games are therefore attractive as detailed models of interaction, on a similar scale to, but more general than, zero-sum games. Rank-1 bimatrix games and their computational analysis should therefore become a new tool in economic modeling.

The computational complexity (required running time) of computing a Nash equilibrium of a game has received substantial interest in the last two decades. A computational problem is considered tractable if it can be solved in polynomial time. Savani and von Stengel (2006) showed that the algorithm by Lemke and Howson (1964) may have exponential running time. (Their examples require carefully constructed matrices, comparable to linear programs where the simplex algorithm, which otherwise works well in practice, has exponential running time, see Klee and Minty, 1972.) The path-following Lemke–Howson algorithm implies that finding an equilibrium of a bimatrix game belongs to the complexity class PPAD defined by Papadimitriou (1994). PPAD describes certain computational problems where the existence of a solution is known, and the problem is to find one explicit solution. (In contrast, the better known complexity class NP applies to decision problems, which are problems that have a “yes” or “no” answer.) Other problems in PPAD include the computation of an approximate Brouwer fixed point, related problems in economics such as market equilibria (Vazirani and Yannakakis, 2011), and the computation of an approximate Nash equilibrium of a game with many players. (In games with three or more players, unlike in two-player games, the mixed strategy probabilities in a Nash equilibrium may be irrational numbers. A suitable concept for such games is approximate Nash equilibrium, and finding an exact Nash equilibrium is an even harder computational problem, see Etessami and Yannakakis, 2010.) A celebrated result is that all problems in PPAD can be reduced to finding a Nash equilibrium in a bimatrix game, which makes this problem “PPAD-complete” (Chen and Deng, 2006; Chen, Deng, and Teng, 2009; Daskalakis, Goldberg, and Papadimitriou, 2009). No polynomial-time algorithm for finding a Nash equilibrium of a general bimatrix game is known.

Kannan and Theobald (2010) describe an algorithm to find ε-approximate Nash
equilibria in games of fixed rank, with running time that is polynomial in $1/\varepsilon$ and the input length, but exponential in the rank. In the present paper, we prove that an exact Nash equilibrium of a rank-1 game can be found in polynomial time. However, we also show that a rank-1 game may have exponentially many equilibria. Moreover, games of higher fixed rank $r$ are PPAD-hard and thus as computationally difficult as general bimatrix games; this has been shown by Mehta (2018) for $r \geq 3$ and is claimed to hold for $r = 2$ (Chen and Paparas, 2019). In the context of the “rank” hierarchy, rank-1 games are therefore the most complex type of games that are expected to be computationally tractable.

Section 2 states the notation and preliminary results used in this paper, and compares our approach with the work of Theobald (2009). In Section 3, we show that the set of equilibria of a game of rank $r$ is the intersection of a hyperplane with a set of equilibria of parameterized games of rank $r - 1$. When $r = 1$, these are parameterized zero-sum games whose equilibria are the solutions to a parameterized linear program (LP). In order to deal with possibly degenerate games which are awkward to handle with pivoting methods, we recall relevant results from Adler and Monteiro (1992) in Section 4. The intersection with the hyperplane gives rise to a polynomial-time binary search for one equilibrium of a rank-1 game, explained in Section 5. In Section 6, we describe completely the set of all Nash equilibria of a rank-1 game, and outline a corresponding equilibrium enumeration method.

Section 7 describes an example (which may be useful to consult in between) that illustrates our main results, and a second example that shows that binary search fails in general for games of rank 2 or higher. A construction of rank-1 games with exponentially many equilibria is shown in Section 8. In Section 9, we describe a variant of the structure theorem of Kohlberg and Mertens (1986), which is important for the concept of strategic stability of an equilibrium component. We introduce a new homeomorphism between the space of bimatrix games and its equilibrium correspondence. This homeomorphism preserves the sum of the payoff matrices, and hence the rank of the games. In the concluding Section 10, we present a tentative example of an economic model based on rank-1 games, and note some open questions.

A preliminary version of our work was published in STOC 2011 (Adsul, Garg, Mehta, and Sohoni, 2011), and the result of Section 8 in von Stengel (2012). The mathematical development in the present paper is almost entirely new in all parts.
In this section we state our notation for bimatrix games and recall the “complementarity” characterization of Nash equilibria in terms of suitable polyhedra. We also briefly compare our approach with Theobald (2009).

We use the following notation. The transpose of a matrix $C$ is written $C^\top$. All vectors are column vectors, so if $x \in \mathbb{R}^m$ then $x$ is an $m \times 1$ matrix and $x^\top$ is the corresponding row vector in $\mathbb{R}^{1 \times m}$. In matrix products, scalars are treated like $1 \times 1$ matrices. Let $0$ and $1$ be vectors with all components equal to 0 and 1, respectively, their dimension depending on the context. Inequalities like $x \geq 0$ hold for all components. The components of a vector $x \in \mathbb{R}^m$ are $x_1, \ldots, x_m$.

For $c \in \mathbb{R}^k$ and $\gamma \in \mathbb{R}$, a hyperplane is of the form $\{z \in \mathbb{R}^k \mid c^\top z = \gamma\}$, and a halfspace of the form $\{z \in \mathbb{R}^k \mid c^\top z \leq \gamma\}$. A polyhedron is an intersection of finitely many halfspaces, and called a polytope if it is bounded. A face of a polyhedron $P$ is of the form $P \cap \{z \in \mathbb{R}^k \mid c^\top z = \gamma\}$ where $P \subseteq \{z \in \mathbb{R}^k \mid c^\top z \leq \gamma\}$. It can be shown that any face of $P$ can be obtained by turning some of the inequalities that define $P$ into equalities (Schrijver, 1986, Section 8.3). If a face of $P$ consists of a single point, it is called a vertex of $P$. If $S \subseteq X \times Y$ for sets $S, X, Y$, then $\{x \in X \mid (x, y) \in S \text{ for some } y \in Y\}$ is called the projection of $S$ on $X$, also written as $\{x \mid (x, y) \in S\}$.

A bimatrix game is a pair $(A, B)$ of $m \times n$ matrices with rows as pure strategies of player 1 and columns as pure strategies of player 2. The players simultaneously choose their pure strategies, with the corresponding entry of $A$ as payoff to player 1 and of $B$ to player 2. The sets $X$ and $Y$ of mixed (that is, randomized) strategies of player 1 and player 2 are given by

(1) \quad X = \{x \in \mathbb{R}^m \mid x \geq 0, \mathbb{1}^\top x = 1\}, \quad Y = \{y \in \mathbb{R}^n \mid y \geq 0, \mathbb{1}^\top y = 1\}.

For the mixed strategy pair $(x, y) \in X \times Y$, the expected payoffs to the two players are $x^\top A y$ and $x^\top B y$, respectively. A best response $x$ of player 1 against $y$ maximizes his expected payoff $x^\top A y$, and a best response $y$ of player 2 against $x$ maximizes her expected payoff $x^\top B y$. A Nash equilibrium (NE) is a pair of mutual best responses.

Consider mixed strategies $x \in X$ and $y \in Y$. If $x$ is a best response to $y$, then its expected payoff $x^\top A y$ is clearly at least the payoff $(A y)_i$ for any pure strategy $i$ of player 1. Moreover, $x$ is a best response to $y$ if and only if any pure strategy $i$ in the support of $x$ (that is, where $x_i > 0$) is a pure best response to $y$ (Nash, 1951). The following lemma, due to Mangasarian (1964), states this “best-response condition” in terms of suitable polyhedra.
Lemma 1. Let \((A, B)\) be an \(m \times n\) bimatrix game. Consider the polyhedra

\[
\overline{P} = \{ (x, v) \in X \times \mathbb{R} \mid B^\top x \leq v \}, \\
\overline{Q} = \{ (y, u) \in Y \times \mathbb{R} \mid Ay \leq u \}.
\]

Let \((x, y) \in X \times Y\). Then \(x\) is a best response to \(y\) if and only if \((y, u) \in \overline{Q}\) and for all rows \(i\)

\(x_i = 0 \text{ or } (Ay)_i = u \quad (1 \leq i \leq m),
\)

and \(y\) is a best response to \(x\) if and only if \((x, v) \in \overline{P}\) and for all columns \(j\)

\(y_j = 0 \text{ or } (B^\top x)_j = v \quad (1 \leq j \leq n).
\)

If both conditions hold, then \(u\) and \(v\) are the unique payoffs to player 1 and 2 in the Nash equilibrium \((x, y)\).

A bimatrix game is degenerate if there is a mixed strategy that has more pure best responses than the size of its support (von Stengel, 2002). A degenerate game may have infinite sets of equilibria. They can be described by suitable faces of \(\overline{P}\) and \(\overline{Q}\), as explained further in Section 6. Our analysis applies to general games that may be degenerate.

The object of study of our paper are bimatrix games of fixed rank, introduced by Kannan and Theobald (2010). They generalize zero-sum games, which are games of rank zero.

Definition 2. The rank of a bimatrix game \((A, B)\) is the matrix rank of \(A + B\).

For comparison of our approach with Theobald (2009), we consider a quadratic program, due to Mangasarian and Stone (1964), that captures the NE of \((A, B)\).

Lemma 3. The strategy pair \((x, y)\) is a Nash equilibrium of \((A, B)\) if and only if it is a solution to

\[
\text{maximize } x^\top (A + B)y - u - v \text{ subject to } (x, v) \in \overline{P}, \quad (y, u) \in \overline{Q}.
\]

The optimum value of (5) is zero, with \(u = x^\top Ay\) and \(v = x^\top By\).

Proof: Consider any solution to (5). Then \(v\) is at least the best-response payoff of player 2 against \(x\) because \((x, v) \in \overline{P}\), and \(u\) is at least the best-response payoff of player 1 against \(y\) because \((y, u) \in \overline{Q}\). Hence, \(x^\top (A + B)y - u - v \leq 0\). Furthermore, (3) and (4) imply that \(x^\top (A + B)y - u - v\) is zero if and only if \((x, y)\) is a NE, in which case \(u = x^\top Ay\) and \(v = x^\top By\). □
The quadratic program (5) shows the importance of the rank of the matrix \( A + B \). For zero-sum games, the rank of \( A + B \) is zero and (5) is a linear program, a well-known fact (Dantzig, 1963). For a rank-1 game \((A, B)\) with \( A + B = ab^\top \), the bilinear term \( x^\top (A + B)y \) in the objective function becomes the product \( (x^\top a)(b^\top y) \) of two linear terms. The resulting optimization problem is called a linear multiplicative program. Solving a general linear multiplicative program is NP-hard (Matsui, 1996).

Consider a rank-1 game \((A, B)\) where \( A + B = ab^\top \). Similar to parametric simplex methods for solving linear multiplicative programs (Konno, Yajima, and Matsui, 1991), Theobald (2009) describes an algorithm to enumerate all equilibria of \((A, B)\). For a real parameter \( \xi \), he considers the parameterized LP

\[
\text{maximize } x^\top a \xi - u - v \quad \text{subject to } (x, v) \in \mathcal{P}, \quad (y, u) \in \mathcal{Q}, \quad b^\top y = \xi.
\]

In any solution to (6), \( x^\top a \xi = x^\top ab^\top y = x^\top (A + B)y \). Hence, by Lemma 3, any optimal solution to (6) is an equilibrium of \((A, B)\) if and only its optimum is zero. Moreover, \( b^\top y = \xi \) implies that \( \xi \) is a convex combination of the components \( b_1, \ldots, b_n \) of \( b \), so that one can restrict \( \xi \) to the interval \([\min\{b_1, \ldots, b_n\}, \max\{b_1, \ldots, b_n\}]\). By partitioning this interval into segments where (6) uses the same basic variables, Theobald (2009) obtains an enumeration of all NE of \((A, B)\).

Our approach is somewhat similar, with a parameter \( \lambda \) and the equality \( x^\top a = \lambda \). However, we consider a different LP which is parameterized by \( \lambda \) and involves only the payoff matrix \( A \) and the vector \( b \) used in \( A + B = ab^\top \). That LP, given in (19) below, has \( x \) as primal and \( y \) as dual variables, whereas in (6) both \( x \) and \( y \) are primal with less closely related constraints. We consider the hyperplane defined by \( x^\top a = \lambda \) separately from the parameterized LP. The intersection of the hyperplane with the solutions to the parameterized LP defines the equilibria of the rank-1 game. This structural insight can be used both for finding an exact NE in polynomial time by binary search (see Section 5) and for enumerating all equilibria (see Section 6). As a topic for further research, it may be interesting if this approach can be extended to more general linear multiplicative programs.

3. RANK REDUCTION

The central result of this short section is Theorem 7. It states that the set of Nash equilibria of a game of rank \( r \) is the intersection of a set \( \mathcal{N} \) of equilibria of parameterized games of rank \( r - 1 \) with a suitable hyperplane. In subsequent sections, we show how to exploit this property algorithmically when \( r = 1 \).

The following lemma states the well-known fact that the equilibria of a bimatrix game are unchanged when subtracting a separate constant \( b_j \) from each column \( j \).
of the row player’s payoff matrix. Call two games \textit{strategically equivalent} if they have the same Nash equilibria.

**Lemma 4** If \( b \in \mathbb{R}^n \), then the \( m \times n \) game \((A, B)\) is strategically equivalent to the game \((A - 1b^\top, B)\).

**Proof:** This holds by Lemma 1, because the equilibrium payoff \( u \) to player 1 in the game \((A, B)\) changes to \( u - b^\top y \) in \((A - 1b^\top, B)\): Clearly, \( Ay \leq 1u \) is equivalent to \((A - 1b^\top)y \leq (u - b^\top y)\), and \((Ay)_i = u \) is equivalent to \((A - 1b^\top)y)_i = u - b^\top y\). \( \square \)

**Lemma 5** An \( m \times n \) bimatrix game of positive rank \( r \) can be written as \((A, C + ab^\top)\) for suitable \( a \in \mathbb{R}^m \), \( b \in \mathbb{R}^n \), and a game \((A, C)\) of rank \( r - 1 \).

**Proof:** An \( m \times n \) matrix is of rank at most \( r \) if and only if it can be written as the sum of \( r \) rank-1 matrices, that is, as \( a_1b_1^\top + \cdots + a rb_r^\top \) for suitable \( a_q \in \mathbb{R}^m \) and \( b_q \in \mathbb{R}^n \) for \( 1 \leq q \leq r \). This is easily seen by writing the \( j \)th column of the matrix as \( \sum_{q=1}^r a_q b_q^\top \) and letting \( b_q^\top = (b_{q1}, \ldots, b_{qn}) \) (see also Wardlaw (2005)). Suppose \((A, B)\) is of rank \( r \), with \( A + B = \sum_{q=1}^r a_q b_q^\top \) and therefore \( B = -A + \sum_{q=1}^r a_q b_q^\top \). Let \( C = -A + \sum_{q=1}^{r-1} a_q b_q^\top \) and \( a = a_r \), \( b = b_r \), so that \( B = C + ab^\top \); obviously, \( A + C \) is of rank \( r - 1 \). \( \square \)

The following is a simple but central lemma.

**Lemma 6** Let \( A, C \in \mathbb{R}^{m \times n} \), \( x \in X \), \( y \in Y \), \( a \in \mathbb{R}^m \), \( b \in \mathbb{R}^n \), \( \lambda \in \mathbb{R} \). The following are equivalent:

(a) \((x, y)\) is an equilibrium of \((A, C + ab^\top)\),

(b) \((x, y)\) is an equilibrium of \((A, C + 1\lambda b^\top)\) and \( x^\top a = \lambda \),

(c) \((x, y)\) is an equilibrium of \((A - 1\lambda b^\top, C + 1\lambda b^\top)\) and \( x^\top a = \lambda \).

**Proof:** The equivalence of (a) and (b) holds because the players get in both games the same expected payoffs for their pure strategies: this is immediate for player 1, and if \( x^\top a = \lambda \), then the column payoffs are given by

\[
(7) \quad x^\top (C + ab^\top) = x^\top C + \lambda b^\top = x^\top C + x^\top 1\lambda b^\top = x^\top (C + 1\lambda b^\top) .
\]

The games in (b) and (c) are strategically equivalent by Lemma 4. \( \square \)

Consider a game \((A, B)\) of positive rank \( r \) where \( B = C + ab^\top \) so that \((A, C)\) is a game of rank \( r - 1 \) according to Lemma 5. Then the game \((A - 1\lambda b^\top, C + 1\lambda b^\top)\) in Lemma 6(c) has the same sum \( A + C \) of its payoff matrices and hence also
rank $r - 1$, for any choice of the parameter $\lambda$. Let $\mathcal{N}$ be the set of Nash equilibria together with $\lambda$ of these parameterized games,

$$(8) \quad \mathcal{N} = \{(\lambda, x, y) \in \mathbb{R} \times X \times Y \mid (x, y) \text{ is a NE of } (A - 1\lambda b^\top, C + 1\lambda b^\top)\}$$

where by Lemma 6(b)

$$(9) \quad \mathcal{N} = \{(\lambda, x, y) \in \mathbb{R} \times X \times Y \mid (x, y) \text{ is a NE of } (A, C + 1\lambda b^\top)\}.$$

These considerations imply the following main result of this section.

**Theorem 7** Given a bimatrix game $(A, C + ab^\top)$, its set of Nash equilibria is exactly the projection on $X \times Y$ of the intersection of $\mathcal{N}$ and the hyperplane $H$ defined by

$$(10) \quad H = \{(\lambda, x, y) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \mid x^\top a = \lambda\}.$$

Theorem 7 asserts that for any rank-$r$ game of the form $(A, C + ab^\top)$, every Nash equilibrium of the game is captured by the set $\mathcal{N}$ in (8) of games of rank $r - 1$ which are parameterized by $\lambda$, intersected with the hyperplane $H$ in (10). Can this rank reduction be leveraged to get an efficient algorithm to find a Nash equilibrium for a game of arbitrary constant rank? As will be discussed in Section 7, this does not work in general. However, it does work for rank-1 games.

4. **PARAMETERIZED LINEAR PROGRAMS**

Our aim is to describe the equilibria of rank-1 games $(A, -A + ab^\top)$ using the rank reduction of the previous section. For this, we consider the set $\mathcal{N}$ in (9) for $C = -A$,

$$(11) \quad \mathcal{N} = \{(\lambda, x, y) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \mid (x, y) \text{ is a NE of } (A, -A + 1\lambda b^\top)\},$$

where by (8)

$$ (12) \quad \mathcal{N} = \{(\lambda, x, y) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \mid (x, y) \text{ is a NE of } (A - 1\lambda b^\top, -A + 1\lambda b^\top)\},$$

which is the set of equilibria of zero-sum games parameterized by $\lambda$. These correspond to the solutions of a parameterized linear program (LP). In this section, we review the structure of such parameterized LPs with a particular view towards nongeneric cases and polynomial-time algorithms as studied by Adler and Monteiro (1992). In essence, such parameterized LPs have finitely many special values of the parameter $\lambda$ called *breakpoints*. These separate the set $\mathcal{N}$ into a connected sequence of polyhedral segments (which generically are line segments). They are described in Theorem 16 in the next section, where we will present a polynomial-time algorithm for finding one equilibrium of a rank-1 game. In the subsequent section we present another algorithm for finding all equilibria.
We assume familiarity with notions of linear programming such as LP duality and complementary slackness; see, for example, Schrijver (1986). The following well-known lemma (Dantzig, 1963, p. 286) states that the equilibria of a zero-sum game are the primal and dual solutions to an LP.

**Lemma 8** Consider an $m \times n$ zero-sum game $(M, -M)$. In any equilibrium $(x, y)$ of this game, $y$ is a minmax strategy of player 2, which is a solution to the LP with variables $y$ in $\mathbb{R}^n$ and $u$ in $\mathbb{R}$:

\[(13) \quad \text{maximize } u \quad \text{subject to} \quad My + 1u \leq 0, \quad y \in Y,\]

and $x$ is a maxmin strategy of player 1, which is a solution to the dual LP to $(13)$. For the optimal value of $u$ in $(13)$, the maxmin payoff to player 1 and minmax cost to player 2 and hence value of the game is $-u$.

**Proof:** The dual LP to $(13)$ has variables $x \in \mathbb{R}^m$ and $v \in \mathbb{R}$ and states

\[(14) \quad \text{minimize } v \quad \text{subject to} \quad x^\top M + v1^\top \geq 0^\top, \quad x \in X.\]

Both LPs are feasible (with sufficiently small $u$ and large $v$). Let $(y, u)$ be an optimal solution to $(13)$ and $(x, v)$ to $(14)$. Then $u = v$ by LP duality, and $(13)$ and $(14)$ state $My \leq 1(-u)$, that is, player 2 pays no more than $-u$ for any row, and $x^\top M \geq (-v)1^\top$, that is, player 1 gets at least $-v$ in every column, where $-u = -v$ which is therefore the value of the game.

With the dual constraints written as $x^\top (-M) \leq v1^\top$, the complementary slackness conditions between the primal and the dual are exactly the Nash equilibrium conditions (3) and (4) of Lemma 1 (except for the changed sign of $u$ so that we do not have to write $x \in X$ in (14) as $-1^\top x = -1$ and $x \geq 0$). Hence, $(x, y)$ is a Nash equilibrium. \qed

Applied to $M = A - 1\lambda b^\top$ in (12), the LP (13) in Lemma 8 says:

\[(15) \quad \text{maximize } u \quad \text{subject to} \quad (A - 1\lambda b^\top)y + 1u \leq 0, \quad y \in Y.\]

In (15), the matrix $A$ is parameterized. The substitution $u = \lambda b^\top y + t$ gives the equivalent LP where only the objective function is parameterized:

\[(16) \quad \text{maximize } \lambda b^\top y + t \quad \text{subject to} \quad Ay + 1t \leq 0, \quad y \in Y.\]

This is a standard parameterized linear programming problem. We stay close to the notation of Adler and Monteiro (1992) who consider a primal LP with minimization subject to equality constraints, variables $x$, and a parameterized right
hand side, of which (16) is the dual, a maximization problem subject to inequalities, with variables \(y\), and a parameterized objective function. We write (16) as

\[
D_\lambda : \text{ maximize } \lambda b^T y + t \quad \text{subject to } (y, t) \in D
\]

with the fixed polyhedron

\[
D = \{ (y, t) \in \mathbb{R}^n \times \mathbb{R} \mid Ay + t \leq 0 \}
\]

\[
\begin{align*}
\mathbf{1}^T y &= 1 \\
y &\geq 0
\end{align*}
\]

The LP \(D_\lambda\) is the dual of the following LP \(P_\lambda\) with a parameterized right hand side, where we use slack variables \(s \in \mathbb{R}^n\) to express the inequality \(A^T x + 1v \geq b\lambda\) as an equality, in line with Adler and Monteiro (1992):

\[
P_\lambda : \text{ minimize } v \quad \text{subject to } A^T x + 1v - s = b\lambda
\]

\[
\begin{align*}
\mathbf{1}^T x &= 1 \\
x, \quad s &\geq 0
\end{align*}
\]

For optimal solutions \((y, t)\) to \(D_\lambda\) and \((x, v, s)\) to \(P_\lambda\) we have \(\lambda b^T y + t = v\). The next lemma (essentially a corollary to Lemma 6 and Lemma 8) states that \(-t\) and \(v\) can be interpreted as the player’s payoffs for the games in Lemma 6(a) and (b), and asserts that \(t, v, s\) are uniquely determined by \((\lambda, x, y)\) (that is, a point on \(N\)).

**Lemma 9**  Let \(\lambda \in \mathbb{R}\). Then \((x, y)\) is an equilibrium of the game \((A, -A + \mathbf{1}\lambda b^T)\) if and only if \((y, t)\) is an optimal solution to \(D_\lambda\) in (17) for some \(t\) which is uniquely determined by \(y\), and \((x, v, s)\) is an optimal solution to \(P_\lambda\) in (19) for some \(v\) and \(s\) which are uniquely determined by \(\lambda\) and \(x\). The equilibrium payoffs are \(-t\) to player 1 and \(v\) to player 2. If \(x^T a = \lambda\), these are also the payoffs in the game \((A, -A + ab^T)\), and \((x, y)\) is an equilibrium of that game.

**Proof:**  By Lemma 6 with \(C = -A\), the game \((A, -A + ab^T)\) has the same equilibria \((x, y)\) and, by (7), payoffs as the game \((A, -A + \mathbf{1}\lambda b^T)\) if \(x^T a = \lambda\). Consider any optimal solutions \((y, t)\) to \(D_\lambda\) and \((x, v, s)\) to \(P_\lambda\). Then \(Ay + t \leq 0\) states for each row \(i\) of \(A\) the inequality \((Ay)_i \leq -t\). Complementary slackness, equivalent to LP optimality, states that \((Ay)_i = -t\) whenever \(x_i > 0\). This is the equilibrium condition in (3) that states that \(x\) is a best response to \(y\). Because it holds for at least one \(i\), it uniquely determines \(-t\), which is the equilibrium payoff to player 1 in the above games.
Similarly, the constraint \( s = A^T x - b\lambda + \mathbb{1} v \) in (19) means that \( s \) is determined by \((x, \lambda, v)\), and states \( s_j = (A^T x - b\lambda)_j + v \geq 0 \) for all \( j \), or equivalently \((-A^T + b\lambda 1^T)x)_j \leq v\). Complementary slackness, equivalent to LP optimality, states that this inequality is tight whenever \( y_j > 0 \). This is the condition (4) that states that \( y \) is a best response to \( x \) in the game \((A, -A + 1\lambda b^T)\), and it uniquely determines \( v \) as the equilibrium payoff to player 2. \( \Box \)

Primal-dual pairs \( P_\lambda, D_\lambda \) of LPs with a parameter \( \lambda \) have been studied since Gass and Saaty (1955). The next result is well known, which we show following Jansen, De Jong, Roos, and Terlaky (1997).

**Lemma 10** For \( \lambda \in \mathbb{R} \), let \( \phi(\lambda) \) be the optimum value of \( P_\lambda \) and hence of \( D_\lambda \). Then \( \phi : \mathbb{R} \to \mathbb{R} \) is the pointwise maximum of a finite number of affine functions on \( \mathbb{R} \) and therefore piecewise linear and convex.

**Proof:** The optimum of \( D_\lambda \) exists for any \( \lambda \) and is taken at a vertex of the polyhedron \( D \) in (18). Let \( V \) be the set of vertices of \( D \), which is finite. Hence,

\[
\phi(\lambda) = \max \{ \lambda (b^T y) + t \mid (y, t) \in V \}
\]

where for each of the finitely many \((y, t)\) in \( V \) the function \( \lambda \mapsto \lambda (b^T y) + t \) is affine. Hence, \( \phi \) is the pointwise maximum of a finite number of affine functions as claimed. The epigraph of \( \phi \) given by \( E = \{ (\lambda, \theta) \mid \theta \geq \phi(\lambda) \} \) is the intersection of the convex epigraphs of these affine functions, so \( E \) is convex and \( \phi \) is a convex function. \( \Box \)

By (20), the function \( \phi(\lambda) \) is the “upper envelope” of the affine functions \( \lambda \mapsto \lambda (b^T y) + t \) defined by the vertices \((y, t)\) of \( D \). A breakpoint is any \( \lambda^* \) so that \( \phi(\lambda) \) has different left and right derivatives when \( \lambda \) approaches \( \lambda^* \) from below or above, denoted by \( \phi'_-(\lambda^*) \) and \( \phi'_+(\lambda^*) \), respectively.

For any LP \( L \), say, let \( \text{OptFace}(L) \) be the face of the domain of \( L \) where its optimum is attained. For any \( \lambda \) we denote \( \text{OptFace}(D_\lambda) \) by \( Y(\lambda) \), that is,

\[
Y(\lambda) = \{ (y, t) \in D \mid \lambda b^T y + t = \phi(\lambda) \}.
\]

Then the left and right derivatives of \( \phi \) at \( \lambda \) are characterized as follows (obvious from (20), also Prop. 2.4 of Adler and Monteiro (1992)):

\[
\phi'_-(\lambda) = \min \{ b^T y \mid (y, t) \in Y(\lambda) \},
\phi'_+(\lambda) = \max \{ b^T y \mid (y, t) \in Y(\lambda) \}.
\]
which are the optima of the two LPs

\[
\begin{align*}
SL^{\min}(\lambda) : & \quad \text{minimize } b^T y \text{ subject to } (y, t) \in Y(\lambda), \\
SL^{\max}(\lambda) : & \quad \text{maximize } b^T y \text{ subject to } (y, t) \in Y(\lambda).
\end{align*}
\]

That is, \(\lambda^*\) is a breakpoint if and only if \(\phi'_-(\lambda^*) < \phi'_+(\lambda^*)\). Clearly, in that case there are at least two vertices \((y, t)\) and \((\hat{y}, \hat{t})\) of \(D\) that define two different affine functions \(\lambda \mapsto \lambda(b^T y) + t\) and \(\lambda \mapsto \lambda(b^T \hat{y}) + \hat{t}\) that meet at \(\lambda = \lambda^*\) to define the maximum \(\phi(\lambda^*)\) in (20). These are also vertices of \(Y(\lambda^*)\), which is then a higher-dimensional face (such as an edge) of \(D\). The following central observation shows that the breakpoints give all the information about the optimal faces \(Y(\lambda)\) of \(D_\lambda\) for any \(\lambda\) between these breakpoints.

**Theorem 11** (Adler and Monteiro, 1992, Theorem 4.1) Let \(\lambda_1, \ldots, \lambda_K\) be the breakpoints, in increasing order, for the parameterized LPs \(P_\lambda\) and \(D_\lambda\), and let \(\lambda_0 = -\infty\) and \(\lambda_{K+1} = \infty\). For \(0 \leq k \leq K\), consider any \(\lambda'_k \in (\lambda_k, \lambda_{k+1})\). Then \(Y(\lambda'_k) = \text{OptFace}(SL^{\max}(\lambda_k))\) for \(1 \leq k \leq K\), and \(Y(\lambda'_K) = \text{OptFace}(SL^{\min}(\lambda_{K+1}))\) for \(0 \leq k \leq K - 1\).

For finding the solutions to \(P_\lambda\) as a function of \(\lambda\), the nondegenerate case is straightforward, where \(Y(\lambda)\) is a vertex of \(D_\lambda\) unless \(\lambda\) is a breakpoint, in which case \(Y(\lambda)\) is an edge of \(D_\lambda\). Then these vertices uniquely describe the pieces of the piecewise linear function \(\phi(\lambda)\), and can be traversed by a parameterized simplex algorithm Gass and Saaty (1955). An example is shown in the right diagram of Figure 4 below with the constraints (44) for \(Ay + \mathbf{1}t \leq 0\) in \(D\), with the additional constraints \(0 \leq y_2 \leq 1\) to represent \(y \in Y\), and objective function \(\lambda b^T y + t\) given by \(\lambda(1 - 2y_2) + t\). The three linear parts of \(\phi(\lambda)\) are

\[
\phi(\lambda) = \begin{cases} 
-\lambda - 1 & \text{for } \lambda \leq -\frac{1}{2} \\
-\frac{1}{2} & \text{for } -\frac{1}{2} \leq \lambda \leq \frac{1}{2} \\
\lambda - 1 & \text{for } \frac{1}{2} \leq \lambda
\end{cases}
\]

which correspond to the optimal vertices \((y_2, t)\) of \(D\) given by \((1, -1), (\frac{1}{2}, -\frac{1}{2})\), and \((0, -1)\). The two breakpoints are \(\lambda_1 = -\frac{1}{2}\) and \(\lambda_2 = \frac{1}{2}\) which correspond to the two edges of \(D\).

In the degenerate case, one typically does not get polynomial-time algorithms by considering vertices and corresponding basic solutions to the LP \(P_\lambda\) as in a parameterized simplex algorithm. Instead of partitioning the variables of \(P_\lambda\) into basic and nonbasic variables, Adler and Monteiro (1992) consider “optimal partitions”; we use here only the partition part that replaces the nonbasic variables,
which we denote by \( M(\lambda) \cup N(\lambda) \) in (26) below (called \( N(\lambda) \) in Adler and Monteiro (1992)). This is the set of variables of the dual LP \( D_\lambda \) that may be strictly positive in an optimal solution, which represent the “true inequalities” of \( Y(\lambda) \).

**Definition 12**  For some \( A, b, C, d \) suppose that the constraints in \( x \)

\[
(25) \quad Ax \leq b, \quadCx = d
\]

are feasible. Then any row \( i \) of \( Ax \leq b \) so that \((b - Ax)_i > 0 \) for some feasible \( x \) is called a true inequality of (25).

If there are solutions \( x \) and \( \hat{x} \) to (25) so that \((b - Ax)_i > 0 \) and \((b - A\hat{x})_i > 0 \) then both inequalities are true for \( x_{1i}^\dagger + \hat{x}_{1i}^\dagger \), so there is a unique largest set of true inequalities with some feasible solution where all these strict inequalities hold simultaneously. These define the relative interior of the polyhedron defined by (25).

Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^n \). Let \( M(\lambda) \cup N(\lambda) \) be the set of true inequalities of the optimal face \( Y(\lambda) \) of \( D_\lambda \) in (17), that is,

\[
(26) \quad M(\lambda) = \{i \in \{1, \ldots, m\} \mid (Ay)_i + t < 0 \text{ for some } (y, t) \in Y(\lambda)\}, \quad \quad N(\lambda) = \{j \in \{1, \ldots, n\} \mid y_j > 0 \text{ for some } (y, t) \in Y(\lambda)\}.
\]

Any non-true inequality of \( Y(\lambda) \) is always tight, that is, \((Ay)_i + t = 0 \) if \( i \not\in M(\lambda) \) and \( y_j = 0 \) if \( j \not\in N(\lambda) \). It can be shown that for such \( i \) and \( j \) there are optimal solutions \((x, v, s)\) to \( P_\lambda \) where \( x_i > 0 \) and \( s_j > 0 \), so these are the true inequalities of \( \text{OptFace}(P_\lambda) \). This is also known as “strict complementary slackness” (Schrijver, 1986, Section 7.9). Consider the polyhedron \( P \) of the constraints for \( P_\lambda \) in (19) where \( \lambda \) is allowed to vary,

\[
(27) \quad P = \{(\lambda, x, v, s) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \mid A^\top x + v - s = b\lambda, \ x \in X, \ s \geq 0 \}.
\]

The following lemma considers the face of \( P \) defined by the equations \( x_i = 0 \) for \( i \in M(\lambda) \) and \( s_j = 0 \) for \( j \in N(\lambda) \), which are necessary and sufficient for a feasible solution to \( P_\lambda \) to be optimal. This is immediate from the standard complementary slackness condition.

**Lemma 13**  Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^n \). For \( M \subseteq \{1, \ldots, m\} \) and \( N \subseteq \{1, \ldots, n\} \), with \( x_M = (x_i)_{i \in M} \) and \( s_N = (s_j)_{j \in N} \), define

\[
(28) \quad P(M, N) = \{(\lambda, x, v, s) \in P \mid x_M = 0, \ s_N = 0 \}.
\]

Then any feasible solution \((x, v, s)\) to \( P_\lambda \) is optimal if and only if \((\lambda, x, v, s) \in P(M(\lambda), N(\lambda)) \).
Crucially, according to Theorem 11, for any $\lambda$ in an open interval $(\lambda_k, \lambda_{k+1})$ (for $0 \leq k \leq K$) the optimal face $Y(\lambda)$ is constant in $\lambda$. Hence, for all $\lambda \in (\lambda_k, \lambda_{k+1})$ the true inequalities $(M(\lambda), N(\lambda))$ of $Y(\lambda)$ are equal to some fixed $(M, N)$, and for the points $(\lambda, x, v, s)$ in $P(M, N)$ the value of $\lambda$ can be any real in the closed interval $[\lambda_k, \lambda_{k+1}]$. Namely, with the LPs

$$BR^{\text{max}}(M, N) : \begin{array}{ll}
\text{maximize} & \lambda \\
\text{subject to} & (\lambda, x, v, s) \in P(M, N),
\end{array}$$

$$BR^{\text{min}}(M, N) : \begin{array}{ll}
\text{minimize} & \lambda \\
\text{subject to} & (\lambda, x, v, s) \in P(M, N),
\end{array}$$

the following holds.

**Lemma 14** Consider $\lambda_0, \lambda_1, \ldots, \lambda_K, \lambda_{k+1}$ and $\lambda'_k \in (\lambda_k, \lambda_{k+1})$ for $0 \leq k \leq K$ as in Theorem 11. Let $M^k = M(\lambda'_k)$ and $N^k = N(\lambda'_k)$ (which do not depend on the choice of $\lambda'_k$). Then for $1 \leq k \leq K$,

(a) the breakpoint $\lambda_k$ is the optimum of the LP $BR^{\text{max}}(M^{k-1}, N^{k-1})$ and of the LP $BR^{\text{min}}(M^k, N^k)$;

(b) if $(\lambda, x, v, s) \in P(M(\lambda_k), N(\lambda_k))$ then $\lambda = \lambda_k$.

**Proof:** See Adler and Monteiro (1992), p. 171 for (a), and Theorem 3.1(a) and Lemma 3.1(b) for (b). \qed

Lemma 14(a) implies that for any $\lambda$ in the open interval $(\lambda_k, \lambda_{k+1})$, for $1 \leq k \leq K - 1$, the endpoints of the closed interval $[\lambda_k, \lambda_{k+1}]$ are given by the minimum and maximum of $\lambda$ for $(\lambda, x, v, s) \in P(M, N)$ where $M = M(\lambda)$ and $N = N(\lambda)$. Lemma 14(b) and Lemma 13 imply that if $\lambda$ is itself a breakpoint, then $P(M, N) = \{\lambda\} \times \text{OptFace}(P_\lambda)$.

As we will describe in detail in the next section, Theorem 11 and Lemma 14 lead to a description of the set of optimal solutions to $P_\lambda$ and $D_\lambda$ for all $\lambda$ with the help of the breakpoints $\lambda_1, \ldots, \lambda_K$ in the form of $2K+1$ polyhedral segments (which are lines in the nondegenerate case). Any solution $(x, v, s)$ to $P_\lambda$ is optimal if and only if $(\lambda, x, v, s)$ belongs to $P(M(\lambda), N(\lambda))$, which is a face of $P$, and any solution to $D_\lambda$ is optimal if and only if it belongs to $Y(\lambda)$, which is a face of $D$. For $\lambda$ between two breakpoints, these faces do not change (but $x$ typically varies with $\lambda$), and their Cartesian product defines $K+1$ of the segments. If $\lambda$ is equal to a breakpoint, the set $P(M(\lambda), N(\lambda))$ is a subset of the two adjoining faces $P(M(\lambda'), N(\lambda'))$ for $\lambda'$ near $\lambda$, whereas $Y(\lambda)$ is a superset of the adjoining faces $Y(\lambda')$, as described in Theorem 11. This defines the other $K$ segments. Using this we will give a precise description of the set $\mathcal{N}$ in Theorem 16 below.

Adler and Monteiro (1992) describe how to generate the breakpoints of $P_\lambda, D_\lambda$ in polynomial time per breakpoint, with a polynomial-time algorithm applied to
the LPs (17), (23), (29), which we will adapt to our purpose. (However, the number of breakpoints may be exponential, see Murty (1980).) The true inequalities in Definition 12 can also be found with an LP, according to the following lemma (Prop. 4.1 of Adler and Monteiro (1992)), due to Freund, Roundy, and Todd (1985); for an alternative polynomial-time algorithm see Mehrotra and Ye (1993).

Lemma 15 For \(A, b, C, d\) and the constraints (25) consider the LP

\[
\begin{align*}
\text{maximize} & \quad 1^T u \\
\text{subject to} & \quad Ax + u - b\alpha \leq 0, \\
& \quad Cx - d\alpha = 0, \\
& \quad 0 \leq u \leq 1, \\
& \quad \alpha \geq 1.
\end{align*}
\]

Then (25) is feasible if and only if (30) is feasible and bounded, and any optimal solution \((x^*, u^*, \alpha^*)\) to (30) satisfies \(u_i^* = 1\) (and \(u_i^* = 0\) otherwise) if and only if \(i\) is a true inequality of (25). For such an optimal solution \((x^*, u^*, \alpha^*)\) to (30), \(x = x^*(1/\alpha^*)\) is a solution to (25) where \((b - Ax)_i > 0\) for all true inequalities \(i\).

Proof: If the LP (30) is feasible then it is also bounded because \(u \leq 1\). Let \(I\) be the set of true inequalities of (25), that is, \((b - Ax)_i = \epsilon_i > 0\) for \(i \in I\) for some \(x\) with \(Cx = d\). Choose \(\alpha^* \geq 1\) so that \(\alpha^* \geq 1/\epsilon_i\) for all \(i \in I\). Then \((b\alpha^* - A(x\alpha^*))_i = (b - Ax)_i\alpha^* = \epsilon_i\alpha^* \geq 1\) for \(i \in I\). Hence, \(x^* = x\alpha^*\) and \(u^*\) defined by \(u_i^* = 1\) if \(i \in I\), and \(u_i^* = 0\) otherwise, give a feasible solution \((x^*, u^*, \alpha^*)\) to the LP (30). This solution is also optimal because any solution \((\hat{x}, \hat{u}, \hat{\alpha})\) to (30) where \(\hat{u}_i > 0\) would give a solution \(x = \hat{x}(1/\hat{\alpha})\) to (25) with \((b - A\hat{x})_i > 0\) and thus \(i \in I\), so for any feasible solution \((x, u, \alpha)\) to (30) we have \(u_i = 0\) whenever \(i \not\in I\). This proves the claim. \(\square\)

5. FINDING ONE EQUILIBRIUM OF A RANK-1 GAME BY BINARY SEARCH

We use the results of the previous section to present a polynomial-time algorithm for finding one equilibrium of a rank-1 game \((A, -A + ab^T)\), using binary search for a suitable value of the parameter \(\lambda\) in Theorem 7. The search maintains a pair of successively closer parameter values and corresponding equilibria of the game \((A, -A + \lambda b^T)\) that are on opposite sides of the hyperplane \(H\) in (10). Generically, the set \(\mathcal{N}\) in (11) is a piecewise linear path which has to intersect \(H\) between these two parameter values. In general, the segments of that “path” are products of certain faces of the polyhedra \(D\) in (17) and \(P\) in (27) described in Theorem 11 and Lemma 14 using the breakpoints \(\lambda_1, \ldots, \lambda_K\) of the LPs \(P_\lambda\) and \(D_\lambda\).

We give a complete description of \(\mathcal{N}\) in terms of these faces of \(P\) and \(D\), which we project to \(\mathbb{R} \times X\) (for the possible values of \((\lambda, x)\)) and \(Y\). Namely, consider
\( \lambda_0, \lambda_1, \ldots, \lambda_K, \lambda_{K+1} \) and \( \lambda'_k \in (\lambda_k, \lambda_{k+1}) \) for \( 0 \leq k \leq K \) as in Theorem 11. For \( 0 \leq k \leq K \), define

\begin{equation}
X'_k = \{(\lambda, x) \mid (\lambda, x, v, s) \in P(M(\lambda'_k), N(\lambda'_k)) \}.
\end{equation}

Note that for any \( (\lambda, x, v, s) \in P(M(\lambda'), N(\lambda')) \) (for any \( \lambda' \in \mathbb{R} \)) the components \( v \) and \( s \) are uniquely determined by \( (\lambda, x) \) by Lemma 9. Similarly, let

\begin{equation}
Y'_k = \{y \mid (y, t) \in Y(\lambda'_k) \}
\end{equation}

where again \( t \) in \((y, t)\) is uniquely determined by \( y \). Recall that the choice of \( \lambda'_k \in (\lambda_k, \lambda_{k+1}) \) does not matter for the definitions of \( X'_k \) and \( Y'_k \). The polyhedra \( X'_k \times Y'_k \) for \( 0 \leq k \leq K \) (which for \( k = 0 \) and \( k = K + 1 \) are infinite, otherwise bounded) represent \( K + 1 \) of the segments that constitute \( N \) between any two breakpoints \( \lambda_k \) and \( \lambda_{k+1} \). They are successively connected by \( K \) further segments, which are polytopes \( X_k \times Y_k \) that correspond to the breakpoints themselves. These are for \( 1 \leq k \leq K \) defined by

\begin{equation}
X_k = \{(\lambda, x) \mid (\lambda, x, v, s) \in P(M(\lambda_k), N(\lambda_k)) \}
\end{equation}

and

\begin{equation}
Y_k = \{y \mid (y, t) \in Y(\lambda_k) \}.
\end{equation}

**Theorem 16**  The set \( N \) in (11) is given by

\begin{equation}
N = (X'_0 \times Y'_0) \cup \bigcup_{k=1}^{K} ((X_k \times Y_k) \cup (X'_k \times Y'_k)),
\end{equation}

where for \( 1 \leq k \leq K \) we have

\begin{equation}
Y_k \supseteq Y'_{k-1} \cup Y'_k
\end{equation}

and

\begin{equation}
X_k \subseteq X'_{k-1} \cap X'_k.
\end{equation}

**Proof:** This follows from Lemma 9, Lemma 13, and Theorem 11. By Theorem 11, \( Y(\lambda'_k) \) is the optimal face of \( SL^{\max}(\lambda_k) \) which is a subset of \( Y(\lambda_k) \). Hence, \( Y'_k \subseteq Y_k \), and similarly \( Y'_{k-1} \subseteq Y_k \), which implies (36). In addition, we have \( M(\lambda'_k) \subseteq M(\lambda_k) \) and \( N(\lambda'_k) \subseteq N(\lambda_k) \) and thus \( X_k \subseteq X'_k \) because of the additional tight constraints in \( P(M(\lambda_k), N(\lambda_k)) \). Similarly, \( X'_k \subseteq X'_{k-1} \). This shows (37). \( \square \)

The preceding characterization of \( N \) is used in the following lemma.
LEMMA 17  Let $\underline{\lambda} \leq \overline{\lambda}$ and $x, \overline{x} \in X$ and $y, \overline{y} \in Y$ so that for $\mathcal{N}$ in (11)

$$(38) \quad (\underline{\lambda}, x, y) \in \mathcal{N}, \quad \underline{\lambda} \leq x^\top a, \quad (\overline{\lambda}, \overline{x}, \overline{y}) \in \mathcal{N}, \quad x^\top a \leq \overline{\lambda}.$$ 

Then $x^\top a = \lambda$ for some $(\lambda, x, y) \in \mathcal{N}$ with $\lambda \in [\underline{\lambda}, \overline{\lambda}]$.

PROOF: Consider the largest $\lambda^*$ so that $\lambda^* \in [\underline{\lambda}, \overline{\lambda}]$ and there are $x^*, y^*$ with $(\lambda^*, x^*, y^*) \in \mathcal{N}$ and $\lambda^* \leq x^\top a$, which exists since $\underline{\lambda}$ fulfills this property and $\mathcal{N}$ is closed by Theorem 16. If $\lambda^* = \overline{\lambda}$ then both $(\lambda^*, x^*)$ and $(\lambda^*, x^*)$ belong to the same set $X_k$ or $X'_k$ which is convex, where since $\overline{x}^\top a \leq \lambda^*$ and $\lambda^* \leq x^\top a$ we have $x^\top a = \lambda^*$ for a suitable convex combination $x$ of $\overline{x}$ and $x^*$, and $(\lambda^*, x, y^*) \in \mathcal{N}$, as claimed.

Hence, we can assume $\lambda^* < \overline{\lambda}$. Suppose $\lambda^*$ is a breakpoint $\lambda_k$, so that $(\lambda^*, x^*) \in X_k$. Consider $\lambda' \in (\lambda_k, \min(\lambda_{k+1}, \overline{\lambda})$ and $(\lambda', x', y') \in X'_k \times Y'_k$ where $\lambda' > x^\top a$ by maximality of $\lambda^*$. By (37), we have $(\lambda^*, x^*) \in X'_k$ and hence $(\lambda^*, x^*, y^*) \in X'_k \times Y'_k$. Because $\lambda^* \leq x^\top a$ and $\lambda' > x^\top a$, a suitable convex combination $(\lambda, x, y')$ of $(\lambda^*, x^*, y^*)$ and $(\lambda', x', y')$ belongs to $\mathcal{N}$ and fulfills $\lambda = x^\top a$ as claimed (in fact, $(\lambda, x, y') = (\lambda^*, x^*, y^*)$ does by maximality of $\lambda^*$). If $\lambda^*$ is not a breakpoint, we directly have $(\lambda^*, x^*, y^*) \in X'_k \times Y'_k$ for some $k$ and can choose $(\lambda', x', y^*) \in X'_k \times Y'_k$ with $\lambda^* < \lambda' \leq \overline{\lambda}$ and apply the same argument. \qed

The binary search algorithm will maintain (38) as an invariant while halving the length of the interval $[\underline{\lambda}, \overline{\lambda}]$ in each iteration.

Lemma 17 ensures that the interval contains some $\lambda$ with $(\lambda, x, y) \in \mathcal{N}$ and $x^\top a = \lambda$ (which is not true when applied to games of higher rank, as shown in the example in Figure 5 below). Let $\lambda' = (\underline{\lambda} + \overline{\lambda})/2$ and let $x'$ be the strategy of player 1 in an equilibrium $(x', y')$ of the game $(A, -A + \lambda'b^\top)$, which is found as a solution $(x', v', s')$ to $P_{\lambda'}$. If $\lambda' \leq x^\top a$, it is natural to proceed with $\underline{\lambda}$ set to $\lambda'$ (written as $\underline{\lambda} \leftarrow \lambda'$), otherwise with $\overline{\lambda} \leftarrow \lambda'$. The binary search should terminate once $\underline{\lambda}$ and $\overline{\lambda}$ are in the same interval $[\lambda_k, \lambda_{k+1}]$ between two breakpoints, with the desired equilibrium found in $(X'_k \times Y'_k) \cap H$.

However, this straightforward approach has the following problems:

(i) the search may converge to an equilibrium $(x, y)$ with $x^\top a = \lambda$ where $\lambda$ is a breakpoint $\lambda_k$, so that $\underline{\lambda}$ and $\overline{\lambda}$ are always in different intervals $(\lambda_{k-1}, \lambda_k]$ and $[\lambda_k, \lambda_{k+1})$ and the described termination condition fails;

(ii) the number of digits to describe $\underline{\lambda}$ and $\overline{\lambda}$ may pile up, which slows down solving $P_{\lambda'}$.

We address these problems as follows. First, we identify with $M = M(\lambda')$, $N = N(\lambda')$ the face $P(M, N)$ of $P$ that contains $(\lambda', x', v', s')$. We then check if that face contains some $(\lambda, x, v, s)$ with $x^\top a = \lambda$. Depending on whether $\lambda' \leq x^\top a$ or $x^\top a \leq \lambda'$, this is achieved by one of the following variations of the LPs in
Suppose these variations will also be used for the enumeration of all equilibria in Section 6):

\[
P_{\text{max}}(M, N, a, \lambda') : \begin{align*}
\text{maximize} & \quad \lambda - x^\top a \\
\text{subject to} & \quad (\lambda, x, v, s) \in P(M, N), \\
& \quad x^\top a \geq \lambda \geq \lambda',
\end{align*}
\]

(39)

\[
P_{\text{min}}(M, N, a, \lambda') : \begin{align*}
\text{minimize} & \quad \lambda - x^\top a \\
\text{subject to} & \quad (\lambda, x, v, s) \in P(M, N), \\
& \quad x^\top a \leq \lambda \leq \lambda'.
\end{align*}
\]

Figure 1 illustrates \(P_{\text{max}}(M, N, a, \lambda')\) where \(\lambda' < x^\top a\), and \(\lambda'\) is between two breakpoints \(\lambda_{k-1}\) and \(\lambda_k\) (but \(\lambda'\) could also be a breakpoint itself), so that \(P(M, N)\) is projected to \(X'_{k-1}\). Here the optimal solution \(x'\) is not unique, but always fulfills \(\lambda' < x^\top a\). Moreover, \(X'_{k-1} \times Y'_{k-1}\) and \(H\) intersect. In the left diagram in Figure 1, \(P(M, N)\) is not just a line segment but a higher-dimensional polytope. It contains some \((\lambda, x, v, s)\) and \((\lambda, \hat{x}, \hat{v}, \hat{s})\) with \(x^\top a < \lambda < \hat{x}^\top a\), for example for \(\lambda = \hat{\lambda}\), but not for \(\lambda = \lambda'\) nor \(\lambda = \lambda_k\). In the right diagram of Figure 1, we always have \(\lambda < x^\top a\), and \(P_{\text{max}}(M, N, a, \lambda')\) attains its optimum \(\lambda^*\) at \(\lambda'\) because for the corresponding \((x^*, \lambda^*)\), shown as a dot, \(\lambda^* - x^* \top a\) is least negative. Here, the solution \(\lambda^* = \lambda_k\) would be more useful for proceeding because it is the next breakpoint. We will introduce an extra computation step to achieve this, as we discuss further below.

The next lemma states that the appropriate LP in (39) identifies if there is an equilibrium \((x, y)\) of the game \((A, -A + \mathbb{1} \lambda b^\top)\) with \(x^\top a = \lambda\) for some \(\lambda\) between \(\lambda'\) and the next breakpoint \(\lambda_k\).

**Lemma 18** Let \(\lambda_k\) be a breakpoint of \(P_\lambda\) and \(D_\lambda\) as in Theorem 11, \(1 \leq k \leq K\). Let \(\lambda' \in \mathbb{R}\), let \((x', v', s')\) be an optimal solution to \(P_{\lambda'}\), and let \((M, N) = (M(\lambda'), N(\lambda'))\) as in (26).

(a) Suppose \(\lambda' \in (\lambda_{k-1}, \lambda_k]\) and \(\lambda' \leq x'^\top a\). Let \((\lambda^*, x^*, v^*, s^*)\) be an optimal solution to \(P_{\text{max}}(M, N, a, \lambda')\). Then \(\lambda^* \in [\lambda', \lambda_k]\), and the game \((A, -A + \mathbb{1} \lambda b^\top)\) has an equilibrium \((x, y)\) with \(x^\top a = \lambda\) for some \(\lambda \in [\lambda', \lambda_k]\) if and only if this holds for \(\lambda = \lambda^*\) and \(x = x^*\).

(b) Suppose \(\lambda' \in [\lambda_k, \lambda_{k+1})\) and \(\lambda' \geq x'^\top a\). Let \((\lambda^*, x^*, v^*, s^*)\) be an optimal solution to \(P_{\text{min}}(M, N, a, \lambda')\). Then \(\lambda^* \in [\lambda_k, \lambda']\), and the game \((A, -A + \mathbb{1} \lambda b^\top)\) has an equilibrium \((x, y)\) with \(x^\top a = \lambda\) for some \(\lambda \in [\lambda_k, \lambda']\) if and only if this holds for \(\lambda = \lambda^*\) and \(x = x^*\).
Figure 1.— (Color online) Illustration of $P_{\max}(M, N, a, \lambda')$ in (39) for $\lambda' \in (\lambda_{k-1}, \lambda_k)$, with $M = M(\lambda')$, $N = N(\lambda')$, and $P(M, N)$ as a polytope (left) or line segment (right).

Proof: We prove (a), where (b) is entirely analogous. By Lemma 13, $(\lambda', x', v', s')$ is feasible for $P_{\max}(M, N, a, \lambda')$. Clearly $\lambda' \leq \lambda^*$, and Lemma 14 implies $\lambda^* \leq \lambda_k$. Because $\lambda \leq x^\top a$ for any feasible solution $(\lambda, x, v, s)$, the objective function $\lambda - x^\top a$ is nonpositive, and zero and hence optimal if and only if $\lambda = x^\top a$, in which case $x$ is part of the described equilibrium $(x, y)$. □

We now describe the BINSEARCH algorithm in Figure 2, where we will return to the LPs in (39). The conditions $x^\top a = \lambda$ and $x \in X$ mean that $\lambda$ is a convex combination of the components $a_1, \ldots, a_m$ of $a$, so that we can initialize $\underline{\lambda}$ and $\bar{\lambda}$ as their minimum and maximum in line 3 of the algorithm. The main loop of the algorithm is between lines 4 and 18. The candidate value for $\lambda$ (called $\lambda'$ in the above explanations) is the midpoint between $\underline{\lambda}$ and $\bar{\lambda}$ in line 5. Line 6 computes some optimal solution $(x, v, s)$ of the LP $P_{\lambda}$ in (19), where the dual LP $D_{\lambda}$ in (17) is typically solved alongside $P_{\lambda}$. The optimum $\phi(\lambda)$ of $P_{\lambda}$ and $D_{\lambda}$ determines the optimal face $Y(\lambda)$ of $D_{\lambda}$ in (21). The true inequalities $M, N$ of $Y(\lambda)$ in line 7 are determined according to (26), for example with the help of the LP in Lemma 15.

Lines 8 to 12, and symmetrically 13 to 17, use the LPs in (39). In order to match the notation in the discussion before Lemma 18, let $\lambda' = \lambda$. Consider the case $\lambda' \leq x^\top a$, handled in lines 8 to 12. Line 9 invokes the LP $P_{\max}(M, N, a, \lambda')$. By
BinSearch

1 Input: $A \in \mathbb{R}^{m \times n}$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$
2 Output: one Nash equilibrium of the game $(A, -A + ab^\top)$
3 $\lambda \leftarrow \min\{a_1, \ldots, a_m\}$, $\overline{\lambda} \leftarrow \max\{a_1, \ldots, a_m\}$
4 repeat
5 $\lambda \leftarrow (\lambda + \overline{\lambda})/2$
6 $(x, v, s) \leftarrow$ solution of $P_\lambda$
7 $(M, N) \leftarrow (M(\lambda), N(\lambda))$
8 if $\lambda \leq x^\top a$ then
9 $$(\lambda^*, x^*, v^*, s^*) \leftarrow$ solution of $P_{\max}(M, N, a, \lambda)$
10 if $\lambda^* < x^* \cdot a$ then
11 $$(\lambda^*, x^*, v^*, s^*) \leftarrow$ solution of $BR_{\max}(M, N)$
12 $\lambda \leftarrow \lambda^*$
13 else [ know: $x^\top a < \lambda$ ]
14 $$(\lambda^*, x^*, v^*, s^*) \leftarrow$ solution of $P_{\min}(M, N, a, \lambda)$
15 if $x^* \cdot a < \lambda^*$ then
16 $$(\lambda^*, x^*, v^*, s^*) \leftarrow$ solution of $BR_{\min}(M, N)$
17 $\overline{\lambda} \leftarrow \lambda^*$
18 until $x^* \cdot a = \lambda^*$
19 $(y^*, t^*) \leftarrow$ solution of $D_{\lambda^*}$
20 output $(x^*, y^*)$

Figure 2.— The BinSearch algorithm for finding one Nash equilibrium of a rank-1 game $(A, -A + ab^\top)$

Lemma 18, the optimum $(\lambda^*, x^*, v^*, s^*)$ to this LP will find the desired equilibrium with $\lambda^* = x^* \cdot a$ if there is one for some $\lambda^*$ up to the next breakpoint $\lambda_k$, that is, for $\lambda^* \in [\lambda', \lambda_k]$. Suppose this is not the case, that is, $\lambda^* < x^* \cdot a$ and the optimum $\lambda^* = x^* \cdot a$ of $P_{\max}(M, N, a, \lambda')$ is negative. By Lemma 18, in this case the next breakpoint $\lambda_k$ does not define an equilibrium, so that problem (i) above does not occur. However, as shown in the right diagram in Figure 1, this may result in $\lambda^* = \lambda'$. We could simply continue with $\lambda \leftarrow \lambda^*$ as in line 12, but if $\lambda^* = \lambda'$ this increases the description size of $\lambda$ which we would like to keep bounded to avoid problem (ii) (the description size of $\lambda$ probably increases only by one bit per main iteration, but it is useful to keep it independent of the number of iter-
lations both for the computation and for the analysis). In line 10, the condition $\lambda^* < x^*{\top}a$ recognizes that the current segment of $\mathcal{N}$ contains no equilibrium, and then $BR_{\max}(M,N)$ in line 11 computes $\lambda^*$ as the next breakpoint $\lambda_k$ according to Lemma 14(a); the LP in line 11 can be solved by starting from the current solution to $P_{\max}(M,N,a,a')$. The left diagram in Figure 1 shows that we cannot simply replace the objective function $\lambda - x^*{\top}a$ of $P_{\max}(M,N,a,a')$ by $\lambda$: While this would compute the next breakpoint $\lambda_k$, it may overlook that the current segment of $\mathcal{N}$ defined by $P(M,N)$ intersects the hyperplane $H$; this could possibly miss the equilibrium altogether, for example if $\overline{\lambda} = \hat{\lambda}$ as shown in the diagram (in particular if $\overline{\lambda}$ still has its initial value, which is not checked in the algorithm as to whether it produces an equilibrium).

In summary: lines 8 to 11 find $\lambda^*$ and $x^*$ so that either (a) $x^*{\top}a = \lambda^*$, or (b) $\lambda^* < x^*{\top}a$ and $\lambda^*$ is a breakpoint and $(\lambda + \overline{\lambda})/2 = \lambda \leq \lambda^* < \overline{\lambda}$, which implies $\overline{\lambda} - \lambda^* \leq (\overline{\lambda} - \lambda)/2$. The next value of $\lambda$ is set to $\lambda^*$ in line 12. In case (a), the loop terminates in line 18. In case (b), the loop continues, and in the next iteration the difference $\overline{\lambda} - \lambda$ has shrunk by at least one half. The analogous statements hold for lines 13–17. The following theorem states the correctness and polynomial running time of the algorithm.

**Theorem 19** Algorithm BinSearch finds one equilibrium of the rank-1 game $(A, -A + ab{\top})$. Assume that the entries of $A, a, b$ are rational numbers with combined bit length $L$, and that LPs are solved with polynomial-time solvers that return extreme LP solutions obtained from linear equations derived from $A, a, b$. Then BinSearch runs in polynomial time in $L$.

**Proof:** During the main loop, the invariant (38) is preserved, and the length of the interval $[\underline{\lambda}, \overline{\lambda}]$ shrinks by at least a factor of two per iteration. By Lemma 17, a solution $(\lambda, x, y) \in \mathcal{N}$ with $x^*{\top}a = \lambda$ and $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ is guaranteed to exist. The termination condition $x^*{\top}a = \lambda^*$ in line 18 holds once $\lambda$ reaches a segment of $\mathcal{N}$ that intersects $H$, which is identified with one of the LPs in line 9 or 14 by Lemma 18. Because the length of the search interval $[\underline{\lambda}, \overline{\lambda}]$ shrinks by at least half in each iteration, the search interval eventually contains at most one breakpoint $\lambda_k$. If there is no breakpoint in $[\underline{\lambda}, \overline{\lambda}]$, then $(M(\lambda), N(\lambda)) = (M(\overline{\lambda}), N(\overline{\lambda})) = (M(\lambda), N(\lambda))$ for $\lambda = (\lambda + \overline{\lambda})/2$. Hence, a solution $(\lambda^*, x^*, v^*, s^*)$ to $P_{\max}(M(\lambda), N(\lambda), a, a')$ or to $P_{\min}(M(\lambda), N(\lambda), a, \lambda)$ determines an equilibrium $(x^*, y^*)$ to $(A, -A + ab{\top})$ by Lemma 18 and Lemma 6. This holds also if there is a single breakpoint $\lambda_k$ in $[\underline{\lambda}, \overline{\lambda}]$. Hence, as claimed, the algorithm computes an equilibrium $(x^*, y^*)$ of $(A, -A + ab{\top})$.

The number of overall iterations is polynomial for the following reason. Any breakpoint $\lambda$ is part of a vertex $(\lambda, x, v, s)$ of $P$ by Lemma 14(a). This vertex is a so-
ution to a linear system of equations where each component (such as \( \lambda \)) is a fraction with an integer determinant obtained from \( A, b \) in the denominator (which has a polynomial of bits), and distinct fractions for different breakpoints \( \lambda \). Hence, any two breakpoints have minimum distance \( 1/2^p(L) \) for some polynomial \( p \) (see also (Schrijver, 1986, Section 10.2)). Therefore, there will be at most \( O(p(L)) \) binary search iterations until the search interval contains at most one breakpoint and the search terminates.

Each iteration of the algorithm solves three or four LPs. The first is \( P_\lambda \) in line 6. Using the optimum \( \phi(\lambda) \) of that LP, in line 7 the true inequalities in (26) of \( Y(\lambda) \) in (21) are found with another LP as in Lemma 15. The third LP is either \( P^{\text{max}}(M, N, a, \lambda) \) in line 9 or \( P^{\text{min}}(M, N, a, \lambda) \) in line 14. The fourth LP is either \( BR^{\text{max}}(M, N) \) or \( BR^{\text{min}}(M, N) \) in line 11 or 16, respectively (which just relaxes the extra constraints of the previous LP in (39) and has a different objective function). In all cases, the output \( \lambda^* \) is described in terms of \( A, a, b \) and found in polynomial time in the bit size \( L \), and \( \lambda^* \) itself has polynomial bit size (Schrijver, 1986, Corollary 10.2a(iii)). In the next iteration, \( \lambda^* \) determines with the constant arithmetic expression in line 5 the next parameter \( \lambda \) for \( P_\lambda \) in line 6 and for \( (M, N) \) in line 7 so that the bit size of \( \lambda \) remains polynomial in \( L \). Hence, each main iteration takes polynomial time, and the overall running time is polynomial.

In practice, as observed in (Adler and Monteiro, 1992, Section 5), in the nondegenerate case the segments of \( N \) are line segments. Then the LP in line 9 or 14 is solved starting from the current solution to \( P_\lambda \) in line 6 with a single pivot, and similarly the next LP in line 11 or 16.

6. ENUMERATING ALL EQUILIBRIA OF A RANK-1 GAME

In this section, we show how to obtain a complete description of all Nash equilibria of a rank-1 game with the help of Theorem 7 and Theorem 16.

A degenerate bimatrix game may have infinite sets of Nash equilibria. They can be described via maximal Nash subsets (Jansen, 1981), called “sub-solutions” by Nash (1951). A Nash subset for \( (A, B) \) is a nonempty product set \( S \times T \) where \( S \subseteq X \) and \( T \subseteq Y \) so that every \( (x, y) \) in \( S \times T \) is an equilibrium of \( (A, B) \); in other words, any two equilibrium strategies \( x \in S \) and \( y \in T \) are “exchangeable”. Using the “best response polyhedra” \( \overline{P} \) and \( \overline{Q} \) in (2), it can be shown that any maximal Nash subset \( S \times T \) is a polytope, with \( S \) as a suitable face of \( \overline{P} \) projected to \( X \), and \( T \) as a suitable face of \( \overline{Q} \) projected to \( Y \) (Avis, Rosenberg, Savani, and von Stengel, 2010). These faces are defined by converting some inequalities in (2) to equations, which have to fulfill the equilibrium conditions (3) and (4). The usual output for “enumerating” all equilibria consists of listing all maximal Nash subsets \( S \times T \) via the vertices of \( S \) and \( T \). These are vertices of \( \overline{P} \) and \( \overline{Q} \), respectively (projected
to $X$ and $Y$) that define the “extreme” Nash equilibria of $(A, B)$, with maximal Nash subsets obtained as maximally exchangeable sets of such vertices (Avis, Rosenberg, Savani, and von Stengel, 2010, Prop. 4). Maximal Nash subsets may intersect, in which case their vertex sets intersect. In a nondegenerate game, all maximal Nash subsets are singletons.

For a rank-1 game $(A, -A + ab^\top)$, its set of Nash equilibria is $N \cap H$ projected to $X \times Y$ by Theorem 7, with $N$ in (11) and $H$ in (10). By (35), $N$ is the union of polyhedra, whose nonempty intersections with $H$ give almost directly the maximal Nash subsets.

**Theorem 20** Let $(A, -A + ab^\top)$ be a rank-1 bimatrix game, and let $\lambda_0, \lambda_1, \ldots, \lambda_K, \lambda_{K+1}$ and $\lambda_k \in (\lambda_k, \lambda_{k+1})$ for $0 \leq k \leq K$ as in Theorem 11. With (31), (32), (33), (34), let

\[
\begin{align*}
S_k &= \{ x \mid (\lambda, x) \in X_k, \, x^\top a = \lambda \} \quad (1 \leq k \leq K), \\
L_k &= \{ \lambda \mid (\lambda, x) \in X_k, \, x^\top a = \lambda \} \quad (1 \leq k \leq K), \\
S'_k &= \{ x \mid (\lambda, x) \in X'_k, \, x^\top a = \lambda \} \quad (0 \leq k \leq K), \\
L'_k &= \{ \lambda \mid (\lambda, x) \in X'_k, \, x^\top a = \lambda \} \quad (0 \leq k \leq K).
\end{align*}
\]

Then the maximal Nash subsets of $(A, -A + ab^\top)$ are the sets $S_k \times Y_k$ if $S_k \neq \emptyset$, and $S'_k \times Y'_k$ if $S'_k \neq \emptyset$ and $L'_k$ is not equal to $\{\lambda_k\}$ or $\{\lambda_{k+1}\}$.

**Proof:** Each set $S_k$ is the projection of $(X_k \times Y_k) \cap H$ on $X$, and $S'_k$ is the projection of $(X'_k \times Y'_k) \cap H$ on $X$, with $L_k$ and $L'_k$ containing the corresponding set of $\lambda$’s. Hence, by Theorem 16, if $S_k \neq \emptyset$ then $S_k \times Y_k$ is a Nash subset, and if $S'_k \neq \emptyset$ then $S'_k \times Y'_k$ is a Nash subset, and the union of these is the set of all equilibria which is the projection of $N \cap H$ on $X \times Y$ by Theorem 7. The only question is which of these Nash subsets are inclusion-maximal. By Corollary 3.2 of Adler and Monteiro (1992), $Y_k \cap Y_{k+1} = Y'_k$ where $Y_k$ and $Y_{k+1}$ contain $Y'_k$ properly, $Y_k \cap Y_{\ell} = \emptyset$ whenever $|k - \ell| \geq 2$, and $Y'_k \cap Y'_\ell = \emptyset$ whenever $k \neq \ell$, and Lemma 14 implies $L_k = \{\lambda_k\} = L_{k-1} \cap L_k$. So the only possible inclusions are that $S'_k \times Y'_k$ is a subset of $S_k \times Y_k$ or of $S_{k+1} \times Y_{k+1}$. Suppose $x \in S'_k$, that is, $(\lambda, x) \in X'_k$ and $x^\top a = \lambda$. If this implies $\lambda = \lambda_k$ then $L'_k = \{\lambda_k\}$. By Lemma 13, this means $x$ is part of an optimal solution $(x, v, s)$ to $P_{\lambda_k}$ and hence $x \in S_k$, which shows the proper inclusion $S'_k \times Y'_k \subset S_k \times Y_k$ because $Y'_k \subset Y_k$. Similarly, $L'_k = \{\lambda_{k+1}\}$ implies $S'_k \times Y'_k \subset S_k \times Y_{k+1}$. These are the only possible inclusions because if $x \in S'_k$ with $(\lambda, x) \in X'_k$ so that $x^\top a = \lambda \notin \{\lambda_k, \lambda_{k+1}\}$ we clearly cannot have $x \in S_k$, say, where $x^\top a = \lambda_k$.

This proves the theorem. We also note that the described sets $S_k$ and $S'_k$ are defined in terms of the game $(A, -A + ab^\top)$ independently of the parameter $\lambda$. 
Namely, the condition $x^\top a = a^\top x = \lambda$ implies that the polyhedron $\mathcal{P}$ in (2) for $B = -A + ab^\top$ is given by

$$
\mathcal{P} = \{ (x,v) \in X \times \mathbb{R} \mid (-A + ab^\top)^\top x \leq \mathbb{1}v \} = \{ (x,v) \in X \times \mathbb{R} \mid -A^\top x + b\lambda \leq \mathbb{1}v \},
$$

so $S_k$ and $S'_{k}$ are projections of certain faces of $\mathcal{P}$. \hfill \Box

A suitable algorithm that enumerates all Nash equilibria can be adapted from the algorithm by Adler and Monteiro (1992, p. 173) that proceeds from breakpoint to breakpoint using Theorem 11. The corresponding segments of $N$ can then be checked for nonempty intersections with $H$, which are then output as maximal Nash subsets if they meet the conditions of Theorem 20.

We give an outline of this algorithm. Suppose $\lambda$ is equal to a breakpoint $\lambda_{k}$. Then $Y_{k}$ in (34) is the projection of $Y(\lambda_{k}) = \text{OptFace}(D_{\lambda_{k}})$, and $X_{k}$ in (33) is the projection of $\text{OptFace}(P_{\lambda_{k}})$ by Lemma 14(b) and Lemma 13. If $(X_{k} \times Y) \cap H$ is not empty, its projection to $X \times Y$ is a maximal Nash subset $S_{k} \times Y_{k}$. Start from some $(\lambda,x) \in X_{k}$. If $\lambda = x^\top a$ then $x \in S_{k}$, which is a suitable starting point for the vertex enumeration of the polytope $S_{k}$, for example with the program lrs (Avis, 2000). If $\lambda < x^\top a$ or $\lambda > x^\top a$ then the condition $(X_{k} \times Y) \cap H \neq \emptyset$ is checked with one of the LPs in (39) by Lemma 18 which then have optimal value zero, with optimum $(\lambda^*,x^*,v^*,s^*)$; then $\lambda^* = x^*\top a$, and $x^* \in S_{k}$ is a new starting point to enumerate the vertices of $S_{k}$.

The next segment to be tested for its intersection with $H$ is $X'_{k} \times Y'_{k}$ in (31) and (32). For that purpose it is not necessary to find some $\lambda' \in (\lambda_{k},\lambda_{k+1})$, because $Y(\lambda') = \text{OptFace}(\text{SL}^{\text{max}}(\lambda_{k}))$ by Theorem 11, and the true inequalities $M \cup N$ of that face are found by Lemma 15, so that one obtains $X'_{k}$ as the projection of $P(M,N)$. Moreover, we have $x \in X_{k} \subseteq X'_{k}$. If $\lambda = x^\top a$ then $x$ is also a starting point for the enumeration of the vertices of $S'_{k}$, which gives the Nash subset $S'_{k} \times Y'_{k}$ (which is, however, not maximal if $S'_{k} \subseteq S_{k}$, see Theorem 20). If $\lambda < x^\top a$ then we solve $P^{\text{max}}(M,N,a,\lambda_{k})$ in (39) to find out if $H$ intersects the current segment $X'_{k} \times Y'_{k}$, and similarly $P^{\text{min}}(M,N,a,\lambda_{k})$ if $\lambda > x^\top a$. Finally, the next breakpoint $\lambda_{k+1}$ is found as the solution to $BR^{\text{max}}(M,N)$ in (29) by Lemma 14(a).

For initialization and termination of this algorithm, we use that the possible values of $\lambda$ can be restricted to $[\underline{\lambda},\bar{\lambda}]$ with $\underline{\lambda}$ and $\bar{\lambda}$ as minimum and maximum of $\{a_{1},\ldots,a_{m}\}$. The initialization is $\lambda = \underline{\lambda}$, which is decided to be a breakpoint or not as described after (23). The constraint $\lambda \leq \bar{\lambda}$ is added to the step of finding the next breakpoint, which terminates the algorithm when it is found to hold as equality.
This algorithm, based on Theorem 20, for enumerating all Nash equilibria of a rank-1 game has the following noteworthy features. First, it works for all games (degenerate or not), and its characterization of maximal Nash subsets is simpler than for general bimatrix games (Avis, Rosenberg, Savani, and von Stengel, 2010), and could even be adapted to easily represent these Nash subsets in terms of their inequalities rather than their vertices (which would be of interest if they are high-dimensional). Secondly, the algorithm in effect traverses $N$ which is generically a path. Rather than by solving a succession of LPs, it can also be implemented by a variant of the algorithm by Lemke (1965) with the additional linear constraints $\lambda \geq x^\top a$ or $\lambda \leq x^\top a$, depending on the current sign of $\lambda - x^\top a$. Here, traversing this path gives all Nash equilibria, whereas for general bimatrix games Lemke’s algorithm (as in von Stengel, van den Elzen, and Talman, 2002 or Govindan and Wilson, 2003) only finds one Nash equilibrium.

7. TWO EXAMPLES

In this section, we illustrate the results of the previous sections with an example of a rank-1 game. After that we will give an example that shows that binary search will in general not work for a game of rank 2 or higher, even though Lemma 6 suggests the possibility of finding a Nash equilibrium of such a game via a recursive rank reduction.

Consider the following rank-1 game $(A, B)$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}, \quad A + B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = ab^\top,$$

where $a^\top = (2, -1)$ and $b^\top = (1, -1)$. This game has the two pure equilibria $((1, 0), (1, 0))$ and $((0, 1), (0, 1))$, and the mixed equilibrium $((\frac{1}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{1}{2}))$. By Theorem 7(b), these are the equilibria $(x, y)$ of the game $(A, -A + \lambda b^\top)$ so that $x^\top a = \lambda$. For $x = (1, 0), (\frac{1}{3}, \frac{2}{3}), (0, 1)$, this means $\lambda = 2, -\frac{1}{4}, -1$.

Figure 3 shows the set $\mathcal{N}$ in (11) where $(x, y)$ is an equilibrium of the parameterized game $(A, -A + \lambda b^\top)$, where

$$-A + \lambda b^\top = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}.$$

These equilibria are pure except when $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$, when the unique mixed strategy $(1 - x_2, x_2)$ of player 1 is given by equalizing the column payoffs, $-(1 - x_2) + \lambda = -x_2 - \lambda$, that is, $\lambda = \frac{1}{2} - x_2$. The white dots indicate the intersection of $\mathcal{N}$ with the hyperplane $H$ in (10), which is defined by the equation $\lambda = x^\top a = 2(1 - x_2) - x_2 = 2 - 3x_2$, and no constraints on $y$. 
Figure 3.— (Color online) The path $\mathcal{N}$ in (11) for the game (43), for $x = (1 - x_2, x_2) \in X$ and $y = (1 - y_2, y_2) \in Y$, and the hyperplane $H$ in (10).

Figure 4.— (Color online) The LP $P_\lambda$ in (19) and the polyhedron $D$ in (18) with the objective function of the LP $D_\lambda$ in (17) for $\lambda = -\frac{1}{4}$, for the game (43).

Figure 4 shows the domains of the LPs $P_\lambda$ in (19) and $D_\lambda$ in (17) for $\lambda = -\frac{1}{4}$. Again we show $x$ in $X$ as $(1 - x_2, x_2)$ and $y$ in $Y$ as $(1 - y_2, y_2)$. The constraints of $P_\lambda$ are then $1 - x_2 + v \geq \lambda$ and $x_2 + v \geq -\lambda$, which for $\lambda = -\frac{1}{4}$ are $v \geq -\frac{5}{4} + x_2$ and $v \geq \frac{1}{4} - x_2$. The constraints $Ay + 1t \leq 0$ of $D_\lambda$ are

\begin{equation}
1 - y_2 + t \leq 0 \quad \text{and} \quad y_2 + t \leq 0,
\end{equation}
and the objective function $\lambda b^T y + t$ is $\lambda (1 - y_2 - y_2) + t$, with gradient $(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial t}) = (-2\lambda, 1)$ for $\lambda = -\frac{1}{4}$. For $\lambda > \frac{1}{2}$, the optimum of $D_\lambda$ is attained at the vertex $(y_1, y_2, t) = (1, 0, -1)$ of $D$, for $\frac{1}{2} > \lambda > -\frac{1}{2}$ at the vertex $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, and for $-\frac{1}{2} > \lambda$ at the vertex $(0, 1, -1)$. For $\lambda_2 = \frac{1}{2}$ and $\lambda_1 = -\frac{1}{2}$, the optimal face of $D_\lambda$ is an edge of $D$. These are the two breakpoints $\lambda_1$ and $\lambda_2$ in Theorem 11.

Figure 3 also demonstrates the characterization of the path $\mathcal{N}$ in Theorem 16. The left diagram shows (from left to right) the three pieces $X'_2$, $X'_1$, $X'_0$, each of which happen to intersect $H$. In the central diagram, the vertical parts of the path are $Y'_2$, $Y'_1$, $Y'_0$, and the horizontal parts (for the breakpoints) are $Y_2$ and $Y_1$. This corresponds to the following, more elementary game-theoretic explanation. Except when $\lambda = -\frac{1}{2}$ or $\lambda = \frac{1}{2}$, player 2’s equilibrium strategy $y$ in the game $(A, -A + \lambda b^T)$ is constant in $\lambda$, which holds because player 1’s payoff matrix $A$ does not change with $\lambda$ and $y$ is chosen so as to make player 1 indifferent between the pure strategies in the support of his equilibrium strategy. When $\lambda = -\frac{1}{2}$ or $\lambda = \frac{1}{2}$, the game is degenerate, and player 2’s equilibrium strategies form a line segment, which allows the change of support of her equilibrium strategy $y$.

![Figure 5](image.png)

**Figure 5.**— (Color online) The path $\mathcal{N}$ of equilibria of the games in (46) where the binary search method fails

Our second example shows that the binary search algorithm no longer works for rank-$r$ games with $r > 1$. Consider the following game $(A, B)$ of rank 2:

$$ A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = C + ab^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, $$

(45)
where \( a^\top = (-3, 2) \) and \( b^\top = (1, 0) \). Here, \((A, B)\) is of rank 2 and \((A, C)\) is of rank 1. The only equilibrium of \((A, B)\) is the pure equilibrium \(((1, 0), (1, 0))\). The parameterized game \((A, C + \lambda b^\top)\) has payoff matrices

\[
A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad C + \lambda b^\top = \begin{bmatrix} 4 + \lambda & 0 \\ \lambda & 0 \end{bmatrix}.
\]

It has the following equilibria \((x, y)\) depending on \(\lambda\), which define the set \(\mathcal{N}\) in (9), shown in Figure 5: The pure equilibrium \(((1, 0), (1, 0))\) for \(\lambda \geq -4\); the pure equilibrium \(((0, 1), (0, 1))\) for \(\lambda \leq 0\); the mixed equilibrium \(((-\frac{1}{2}, 1 + \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))\) for \(-4 < \lambda < 0\), and two further components \(((1, 0), (1 - y_2, y_2))\) when \(\lambda = -4\) and \(((0, 1), (1 - y_2, y_2))\) when \(y_2 \in [0, \frac{1}{2}]\) when \(\lambda = 0\) where the game in (46) is degenerate. These are multiple disjoint equilibrium components for \(-4 \leq \lambda \leq 0\), which cannot happen for a parameterized zero-sum game. As a result, \(\lambda\) may change non-monotonically along the path \(\mathcal{N}\), which in general causes a binary search to fail, as we show next.

We describe a suitably adapted binary search method for this example, where instead of solving parameterized LPs we find equilibria of the parameterized game (46) of lower rank. The smallest and largest components of \(a\) as in line 3 of the BinSEARCH algorithm are \(\underline{\lambda} = -3\) and \(\overline{\lambda} = 2\). For \(\lambda = \overline{\lambda}\), the only equilibrium of the game in (46) is \((\overline{x}, y) = ((1, 0), (1, 0))\), but for \(\lambda = \underline{\lambda}\) there are multiple equilibria, where we choose \((x, y) = ((0, 1), (0, 1))\). Then \(\underline{\lambda} = -3 < \overline{x}^\top a = 2\) and \(\overline{x}^\top a = -3 < \overline{\lambda} = 2\), so we next consider the midpoint \(\lambda = (\underline{\lambda} + \overline{\lambda})/2 = -1/2\) as in line 5 of BinSEARCH, and compute a new equilibrium of this parameterized game. Suppose this is again \((x, y) = ((0, 1), (0, 1))\), so that because \(\lambda < x^\top a\) the assignment \((\underline{\lambda}, x, y) \leftarrow (\lambda, x, y)\) takes place for the binary search to continue. This is the situation shown in Figure 5. At this point, the method will no longer succeed in finding a suitable value of \(\lambda\) because the search interval \([\underline{\lambda}, \overline{\lambda}] = [-\frac{1}{2}, 2]\) no longer contains the only possible value for \(\lambda\), namely \(-3\). The problem is that in that interval, the set \(\mathcal{N}\) consists of two disconnected parts where \(\lambda < x^\top a\) and \(\lambda > x^\top a\) on opposite sides of the hyperplane \(H\), so that \(\mathcal{N}\) no longer intersects with \(H\). Hence, even though the values of \(\lambda\) converge, the corresponding equilibria \((x, y)\) on the two sides of \(H\) will not converge.

This example shows that because of the non-monotonicity of \(\lambda\) along the path \(\mathcal{N}\), there is no equivalent statement to Lemma 17 that would guarantee that a binary search will succeed.

8. RANK-1 GAMES WITH EXPONENTIALLY MANY EQUILIBRIA

Kannan and Theobald (2010, Open Problem 9) asked if the number of Nash equilibria of a nondegenerate rank-1 game is polynomially bounded. This is not
the case, because our next result shows that this number may be exponential.

**Theorem 21**  Let $p > 2$ and let $(A, B)$ be the $n \times n$ bimatrix game with entries of $A$

\[
    a_{ij} = \begin{cases} 
        2p^{i+j} & \text{if } j > i \\
        p^{2i} & \text{if } j = i \\
        0 & \text{if } j < i
    \end{cases}
\]

for $1 \leq i, j \leq n$, and $B = A^T$. Then $A + B$ is of rank 1, and $(A, B)$ is a nondegenerate bimatrix game with $2^n - 1$ many Nash equilibria.

**Proof:**  By (47), $A + B = ab^T$ with the $n$ components of $a$ and $b$ defined by $a_i = p^i$ and $b_j = 2p^j$ for $1 \leq i, j \leq n$, so $A + B$ is of rank 1.

Let $y \in Y$ with support $S$. Consider a row $i$ and let $T = \{j \in S \mid j > i\}$. Because $A$ is upper triangular, the expected payoff against $y$ in row $i$ is

\[
    (Ay)_i = a_{ii}y_i + \sum_{j \in T} a_{ij}y_j.
\]

Suppose $i \notin S$. If $T$ is empty, then $(Ay)_i = 0 < (Ay)_1$, otherwise let $t = \min T$ and note that for $j \in T$ we have $a_{ij} = 2p^{i+j} < p^{1+i+j} \leq p^{t+j} \leq a_{ij}$, so $(Ay)_i < (Ay)_1$. Hence, no row $i$ outside $S$ is a best response to $y$. Similarly, because the game is symmetric, any column that is a best response to $x$ in $X$ belongs to the support of $x$. This shows that the game is nondegenerate. Moreover, if $(x, y)$ is an equilibrium of $(A, B)$, then $x$ and $y$ have equal supports.

For any nonempty subset $S$ of $\{1, \ldots, n\}$, we construct a mixed strategy $y$ with support $S$ so that $(y, y)$ is an equilibrium of $(A, B)$. This implies that the game has $2^n - 1$ many equilibria, one for each support set $S$. The equilibrium condition holds if $(Ay)_i = u$ for $i \in S$ with equilibrium payoff $u$, because then $(Ay)_i < u$ for $i \notin S$ as shown above. We start with $s = \max S$, where $(Ay)_s = a_{ss}y_s = u$, by fixing $u$ as some positive constant (e.g., $u = 1$), which determines $y_s$. Once $y_i$ is known for all $i \in S$ (and $y_i = 0$ for $i \notin S$), we scale $y$ and $u$ by multiplication with $1/1^T y$ so that $y$ becomes a mixed strategy. Assume that $i \in S$ and $T = \{j \in S \mid j > i\} \neq \emptyset$ and assume that $y_k$ has been found for all $k$ in $T$ so that $(Ay)_k = u$ for all $k$ in $T$, which is true for $T = \{s\}$. Then, as shown above, $\sum_{j \in T} a_{ij}y_j < \sum_{j \in T} a_{ij}y_j = (Ay)_i = u$ for $t = \min T$, so $y_i$ is determined by $(Ay)_i = u$ in (48), and $y_i > 0$. By induction, this determines $y_i$ for all $i$ in $S$, and after re-scaling gives the desired equilibrium strategy $y$. $\square$

By Theorem 7, the equilibria $(x, y)$ of a rank-1 game are the intersection of the path $N$ in (11) with the hyperplane $H$ in (10). The exponential number of Nash equilibria of the game in Theorem 21 shows that $N$ has exponentially many line
segments. Murty (1980) describes a parameterized LP with such an exponentially long path of length $2^n$. The payoffs for the game in Theorem 21 have been inspired by Murty’s example, but are not systematically constructed from it, which would be interesting. See von Stengel (2012) for further discussions and related work on the maximal number of Nash equilibria in bimatrix games, such as von Stengel (1999).

9. A RANK-PRESERVING STRUCTURE THEOREM

Nash equilibria of games are in general not unique, which has led to a large literature on equilibrium refinements (van Damme, 1991) that impose additional conditions on equilibria, such as stability against small changes in the game parameters, as proposed in the seminal paper by Kohlberg and Mertens (1986) (KM). They showed that stability has to apply to equilibrium components, that is, maximal sets of equilibria that are topologically connected (which for bimatrix games are unions of intersecting maximal Nash subsets, see Section 6). That is, an equilibrium component is stable if every perturbed game has an equilibrium near that component (although possibly in different positions depending on the perturbation, which is why any single equilibrium may fail to be stable). KM proved the existence of stable equilibrium components with the help of a structure theorem (Kohlberg and Mertens, 1986, Theorem 1) which states that the equilibrium correspondence $E$ over the set $\Gamma$ of strategic-form games with a given number of players and numbers of strategies is homeomorphic to $\Gamma$ itself.

In this section, we present in Theorem 23 a similar structure theorem with a new homeomorphism for bimatrix games that preserves rank. In analogy to Kohlberg and Mertens (1986, Appendix B), one consequence of this new structure theorem is the existence of an equilibrium component in a game $(A, B)$ that is stable with respect to small perturbations that preserve the sum $A + B$ of the payoff matrices. This is not interesting for zero-sum games which always have only one component, but it is for games of higher rank and applies, for example, to perturbations of the matrix $A$ in a rank-1 game given as $(A, -A + ab^\top)$. Furthermore, a number of equilibrium-finding algorithms can be interpreted as following a path on the equilibrium correspondence $E$ via the KM homeomorphism and suitable projections (Wilson, 1992; Govindan and Wilson, 2003). As a topic for further research, it may be interesting to study our new homeomorphism in this context, or, similar to Jansen and Vermeulen (2001), the computation of equilibrium components that are stable with respect to small perturbations that preserve the sum $A + B$ of the payoff matrices.

We first recall the KM homeomorphism from Kohlberg and Mertens (1986). Let $\Gamma$ be the set of $m \times n$ bimatrix games $(A, B)$ and $E \subseteq \Gamma \times X \times Y$ be its equilibrium
correspondence,

\[(49) \quad E = \{(A, B, x, y) \mid (A, B) \in \Gamma, (x, y) \text{ is a NE of } (A, B)\} \]

To distinguish the dimensions of the all-zero and all-one vectors we write them as \(0, 1 \in \mathbb{R}^m\) and \(0, 1 \in \mathbb{R}^n\). Let \(a\) and \(b\) be the vectors of row and column averages of \(A\) and \(B\),

\[(50) \quad a = A \frac{1}{n}, \quad b = B^\top 1 \frac{1}{m}. \]

Then \(A\) and \(B\) correspond uniquely to pairs \((\tilde{A}, a)\) and \((\tilde{B}, b)\) with

\[(51) \quad A = \tilde{A} + a 1^\top, \quad B = \tilde{B} + b 1^\top, \quad \tilde{A} 1 = 0, \quad 1^\top \tilde{B} = 0^\top, \]

with \(a\) and \(b\) as in (50). That is, \((A, B)\) is parameterized by a “base game” \((\tilde{A}, \tilde{B})\) where each row of player 1 and each column of player 2 gets payoff zero when the other player randomizes uniformly (as in \(\tilde{\phi}\) where each row of \(A\) and each column of \(B\) gets payoff zero when \(a\) and \(b\) are the vectors of row and column averages of \(A\) and \(B\),

\[(52) \quad C = \tilde{A} + (Ay + x) 1^\top, \quad D = \tilde{B} + 1(x^\top B + y^\top). \]

That is, \((C, D)\) has the same “base game” \((\tilde{A}, \tilde{B})\) as \((A, B)\) but different parameters \((Ay + x) \in \mathbb{R}^m\) and \((B^\top x + y) \in \mathbb{R}^n\). The fact that \((x, y)\) is an equilibrium of \((A, B)\) implies that \(\tilde{\phi}^{-1}\) is injective (and therefore \(\phi\) well-defined), by the following intuition. Because \(x\) is a best response to \(y\), each row of the vector \(Ay\) of expected payoffs in the support of \(x\) has maximal and equal value \(u\) among all components of \(Ay\), by (3). This condition allows us to re-construct \(x\) from the sum \(c = Ay + x\), which is used in the definition of \(C\) in (52) and which can be obtained from \(C\). Suppose the components \(c_i\) of \(c\) are heights of \(m\) “poles in the water” of which a certain amount \(x_i\) is “above the waterline” depending on the “water level” \(w\), where

\[(53) \quad x_i = \max(c_i - w, 0), \]

so \(x_i \geq 0\) and if \(c_i < w\) then \(x_i = 0\). For any \(c \in \mathbb{R}^m\), there is a unique choice of \(w \in \mathbb{R}\) in (53) so that \(\sum_{i=1}^m x_i = 1\) and therefore \(x \in X\). By this construction of \(w\) and \(x\), all components \(p_i\) of the vector \(p = c - x\) fulfill (a) \(w = \max_k p_k\), and (b) \(x_i > 0\) implies \(p_i = w\), as when \(p = Ay\) and \(x\) is a best response to \(y\). In a similar way, \(y\) is a best response to \(x\) and the sum \(x^\top B + y^\top\) used to define \(D\) in (52) is
special because it allows us first to obtain a vector \( d \in \mathbb{R}^n \) from \( D \), and second to obtain the original \( y \in Y \) and \( q \in \mathbb{R}^n \) so that \( d = q + y \) and \( q^\top = x^\top B \). The following lemma states this construction, which we apply afterwards to define the KM homeomorphism, and will later use again for our new homeomorphism.

**Lemma 22** Given \( c \in \mathbb{R}^n \) and \( d \in \mathbb{R}^n \), there are unique \( x \in X, y \in Y, p \in \mathbb{R}^m \) and \( q \in \mathbb{R}^n \) so that

\[
\begin{align*}
  c &= p + x, & d &= q + y, \\
  x_i &= 0 & p_i &= \max_{1 \leq k \leq m} p_k & (1 \leq i \leq m), \\
  y_j &= 0 & q_j &= \max_{1 \leq l \leq n} q_l & (1 \leq j \leq n).
\end{align*}
\]

**Proof:** For \( t \in \mathbb{R} \), let \( t^+ = \max(t, 0) \), and

\[
\begin{align*}
  u &= \min \{ w \in \mathbb{R} \mid \sum_{i=1}^{m} (c_i - w)^+ \leq 1 \}, \\
  v &= \min \{ w \in \mathbb{R} \mid \sum_{j=1}^{n} (d_j - w)^+ \leq 1 \},
\end{align*}
\]

where \( u \) (and similarly \( v \)) is the unique lowest “water level” \( w \) so that the “heights” of the components \( c_i \) of \( c \) that are “above the waterline” sum up to (at most) one. Then

\[
\begin{align*}
  x_i &= (c_i - u)^+ & (1 \leq i \leq m), & y_j &= (d_j - v)^+ & (1 \leq j \leq n),
\end{align*}
\]

and \( p = c - x \) and \( q = d - y \) fulfill (54), and \( x, y, p, q \) are uniquely determined by the conditions \( x \in X, y \in Y, \) and (54).

The KM homeomorphism \( \phi : (C, D) \mapsto (A, B, x, y) \) is then defined as follows.

(a) Let \( c = C \mathbb{1} \mathbb{1}^\top_m, d = D^\top \mathbb{1} \mathbb{1}^\top_m, \tilde{A} = C - c \mathbb{1}^\top \), and \( \tilde{B} = D - d \mathbb{1}^\top \).

(b) Apply Lemma 22 to get \( x, y, p, q \) so that (54) holds.

(c) Let \( a = c - x - \tilde{A} y \) and \( b = d - y - \tilde{B} x \), and define \( A \) and \( B \) by (51).

Then \( \phi \) is continuous because it is defined by continuous linear mappings and (55) and (56) for (b). We show that \( (A, B, x, y) \in E \). We have \( Ay = (\tilde{A} + a \mathbb{1}^\top) y = \tilde{A} y + a = \tilde{A} y + c - x - \tilde{A} y = p \), and similarly \( x^\top B = x^\top \tilde{B} + b^\top = d^\top - y^\top = q^\top \). Then the conditions (54) are equivalent to the best-response conditions (3) and (4), that is, \( (x, y) \) is indeed an equilibrium of \( (A, B) \). Moreover, \( c = p + x = Ay + x \) and \( d = B^\top x + y \), which shows that the (continuous) function \( (A, B, x, y) \mapsto (C, D) \) in (52) is indeed the inverse of \( \phi \) (so \( \phi \) is injective), and also that \( \phi \) is surjective, because we can start in (52) from any \( (A, B, x, y) \in E \).

The KM homeomorphism does not operate within a subset of games of fixed rank (for example, the zero-sum games). Our new homeomorphism \( \psi : \Gamma \to E \) has this property. Consider a fixed matrix \( M \in \mathbb{R}^{m \times n} \), the set \( \Gamma_M \) bimatrix games
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\((A, B)\) with \(A + B = M\), and \(E_M\) as the equilibrium correspondence \(E\) in (49) restricted to these games,

\[\Gamma_M = \{(A, B) \in \Gamma | A + B = M\}, \quad E_M = \{(A, B, x, y) \in E | (A, B) \in \Gamma_M\}.\]

The following theorem states we can restrict \(\psi\) to a homeomorphism \(\Gamma_M \to E_M\) for any \(M\) (for example, the all-zero matrix \(M\)). Also, \(\psi\) is continuous in \(M\) and therefore a homeomorphism \(\Gamma \to E\) like the KM homeomorphism.

**THEOREM 23** Let \(M \in \mathbb{R}^{m \times n}\). There is a homeomorphism \(\psi : \Gamma_M \to E_M\), \((C, D) \mapsto (A, B, x, y)\), that is, \(A + B = M\) for all \((C, D) \in \Gamma_M\).

**PROOF:** We will use a new parameterization of any matrix \(A\) in \(\mathbb{R}^{m \times n}\), which corresponds uniquely to a quadruple \((\hat{A}, \gamma, a, b)\) with \(\hat{A} \in \mathbb{R}^{m \times n}\), \(\gamma \in \mathbb{R}\), \(a \in \mathbb{R}^m\), and \(b \in \mathbb{R}^n\) according to

\[A = \hat{A} + 1_1 \gamma 1_1^\top + a 1_1^\top + b 1_1^\top\]

so that

\[1_1^\top \hat{A} = 0^\top, \quad \hat{A} 1_1 = 0, \quad 1_1^\top a = 0, \quad b 1_1^\top = 0.\]

It is easy to see that \(\hat{A}, \gamma, a,\) and \(b\) are uniquely given by \(A\), (58), and

\[\gamma = \frac{1}{m} 1_1^\top A 1_1 1_1, \quad a = A 1_1 1_1 - 1_1 \gamma, \quad b 1_1^\top = \frac{1}{m} 1_1^\top A - \gamma 1_1^\top.\]

The homeomorphism \(\psi : \Gamma_M \to E_M\), \((C, D) \mapsto (A, B, x, y)\) uses this parameterization of \(C\) and only changes the vectors \(a\) and \(b\), and maintains the sum \(M\) of the payoff matrices, that is, \(A + B = C + D = M\). Like for the KM homeomorphism, we first describe its inverse \(\psi^{-1}\), which maps \((A, B, x, y)\) in \(E_M\) to \((C, D)\) in \(\Gamma_M\). Let \(A + B = M\) and \((x, y)\) be an equilibrium of \((A, B)\). Let \(A\) be represented as in (58) so that (59) holds, and let

\[C = \hat{A} + 1_1 \gamma 1_1^\top + c 1_1^\top + d 1_1^\top\]

with \(c\) and \(d\) given by

\[c = \rho(Ay + x), \quad d = \sigma(B^\top x + y)\]

where \(\rho : \mathbb{R}^m \to \mathbb{R}^m\) and \(\sigma : \mathbb{R}^n \to \mathbb{R}^n\) are the linear projections on the hyperplane through the origin with normal vector \(1\) respectively \(1_1\),

\[\rho(x) = x - 1\left(\frac{1}{m} 1_1^\top x\right), \quad \sigma(y) = y - 1\left(\frac{1}{n} 1_1^\top y\right)\]
which achieves $1^\top \rho(x) = 0$ and $1^\top \sigma(y) = 0$ for any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, as required for a parameterization of the payoff matrix $C$ like it is done for $A$ in (59). With $C$ thus encoded, we let $D = M - C$.

The homeomorphism $\psi : (C, D) \mapsto (A, B, x, y)$ itself is obtained as follows. Let $(C, D) \in \Gamma_M$. Similar to (58) we represent $C$ by (61) where as in (60)

\begin{equation}
\gamma = \frac{1}{m} 1^\top C 1 \frac{1}{n}, \quad c = C 1 \frac{1}{n} - 1 \gamma, \quad d^\top = \frac{1}{m} 1^\top C - \gamma 1^\top,
\end{equation}

which implies

\begin{equation}
1^\top \hat{A} = 0^\top, \quad \hat{A} 1 = 0, \quad 1^\top c = 0, \quad d^\top 1 = 0.
\end{equation}

Given $c$ and $d$, we determine $x \in X, y \in Y, p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ by Lemma 22 so that (54) holds. Then, let

\begin{equation}
a = c - \rho(\hat{A} y + x), \quad b = \sigma((M - \hat{A})^\top x + y) - d
\end{equation}

so that $a$ and $b$ fulfill (59), define $A$ by (58), and let $B = M - A$. Like $\phi$ before, $\psi$ is defined by linear maps and the continuous operations in (55) and (56) and is therefore continuous.

We show that $\psi(C, D) = (A, B, x, y) \in E_M$. Because $A + B = M$, we only need to show the equilibrium property. Using (58), $1^\top y = 1$, (66), $c = p + x$, and the definition of $\rho$ in (63),

\begin{equation}
Ay = \hat{A} y + 1 \gamma 1^\top y + a 1^\top y + 1 b^\top y \\
= \hat{A} y + 1 \gamma + a + 1 b^\top y \\
= \hat{A} y + 1 \gamma + c - \rho(\hat{A} y + x) + 1 b^\top y \\
= \hat{A} y + 1 \gamma + p + x - (\hat{A} y + x) + 1(\frac{1}{m} 1^\top (\hat{A} y + x)) + 1 b^\top y \\
= p + 1(\gamma + \frac{1}{m} 1^\top (\hat{A} y + x) + b^\top y)
\end{equation}

for some $\alpha \in \mathbb{R}$ which means that $(Ay)_i = p_i + \alpha$ for $1 \leq i \leq m$ and therefore by (54) the best-response condition (3) holds (which is unaffected by a constant shift), that is, $x$ is a best response to $y$. Similarly, using $1^\top x = 1$, (66), the definition
of $\sigma$ in (63), and $d = q + y$,

$$B^T x = (M - A)^T x$$

$$= (M - \hat{A} - 1\gamma \mathbb{1}^T x - a \mathbb{1}^T x - b \mathbb{1}^T x)$$

$$= (M - \hat{A})^T x - 1\gamma \mathbb{1}^T x - a \mathbb{1}^T x - b \mathbb{1}^T x$$

$$= (M - \hat{A})^T x - 1\gamma - a \mathbb{1}^T x - b$$

$$= (M - \hat{A})^T x - 1\gamma - a \mathbb{1}^T x - \sigma((M - \hat{A})^T x + y) + d$$

$$= -1\gamma - a \mathbb{1}^T x - y + 1 \frac{1}{n} \mathbb{1}^T ((M - \hat{A})^T x + y) + q + y$$

$$= 1\beta + q$$

for some $\beta \in \mathbb{R}$ which means that $(B^T x)_j = q_j + \beta$ for $1 \leq j \leq n$ and therefore by (54) the best-response condition (4) holds, that is, $y$ is a best response to $x$. Hence, $(x, y)$ is indeed an equilibrium of $(A, B)$.

To show that $\psi$ has the inverse described in (61) and (62), note that $\rho$ and $\sigma$ in (63) are linear and $\rho(1) = 0$ and $\sigma(1) = 0$. Therefore, for $\psi(C, D) = (A, B, x, y)$ with $C$ as in (61), we have by (67) and (68) and because $1^T c = 0$ and $1^T d = 0$,

$$\rho(Ay + x) = \rho(p + 1a + x) = \rho(p + x) = \rho(c) = c,$$

$$\sigma(B^T x + y) = \sigma(1\beta + q + y) = \sigma(q + y) = \sigma(d) = d,$$

that is, $\psi$ has indeed the (continuous) inverse described in (62) and $\psi$ is both injective and surjective. This shows that $\psi$ is indeed a homeomorphism from $\Gamma_M$ to $E_M$. $\square$

10. CONCLUSIONS

We conclude with some open questions. Our analysis shows that rank-1 games are computationally easy to analyze: One Nash equilibrium can be found in polynomial time, and enumerating all equilibria can be performed by following a piecewise linear path, similar to finding a single Nash equilibrium of a bimatrix game (which is in general a PPAD-hard problem).

As described in Section 6, the path of solutions to the parameterized LP consists in general of polyhedral segments whose intersections with the hyperplane $H$ define the sets of Nash equilibria of the rank-1 game. This set-up suggests the application of smoothed analysis as pioneered by Spielman and Teng (2004) for the “shadow vertex algorithm” for parameterized LPs. This analysis has been subsequently improved and simplified; for recent developments see Dadush and Huiberts (2018). In smoothed analysis, the LP data are perturbed by some moderate Gaussian noise which cancels “pathological” cases that lead to exponential
worst-case examples, like the game constructed in Section 8. Applied to our parameterized LP, it would imply that in expectation there is a polynomial number of segments in Theorem 16. If this holds, the number of Nash equilibria is similarly polynomially bounded by Theorem 20 (the Nash subsets are all single equilibria because the perturbed game is generic and therefore nondegenerate with probability one). However, the standard framework of smoothed analysis (as in e.g. Dadush and Huiberts, 2018) assumes that the LP constraints are of the form $Ax \leq 1$, which is not the case for the LP (16) that we consider, so combining this with our approach requires a careful study that we leave for future work. For a general bimatrix game, finding one equilibrium is PPAD-hard even under smoothed analysis (Chen, Deng, and Teng, 2009). However, it is not known if a perturbed game may have exponentially long Lemke–Howson paths; the long paths in Savani and von Stengel (2006) do not persist due to exponential size differences in the payoffs.

In Section 8 we described rank-1 games with exponentially many equilibria (also with exponential size differences in the payoffs). This raises the following question: Can all equilibria of a rank-1 game be computed in running time that is polynomial in the size of the input and output? Such an algorithm is called “output efficient”. For example, the algorithm by Adler and Monteiro (1992) that computes all segments of a parameterized LP is output efficient. We have extended this algorithm in Section 6. For general bimatrix games, an output efficient algorithm that finds all Nash equilibria would imply $P = NP$ because it is NP-hard to decide if a game has more than one Nash equilibrium (Gilboa and Zemel, 1989). Our binary search algorithm gives no information about the existence of a second equilibrium, so it is conceivable that finding a second Nash equilibrium of a rank-1 game is also NP-hard. The existence of an output efficient algorithm to find all Nash equilibria of a rank-1 game is an open question.

General bimatrix games are computationally difficult, but rank-1 games are computationally easy. One should therefore investigate economic applications of large rank-1 games, also as approximate economic models that can serve as fast-solvable benchmarks. As a possible starting point, we describe here a simple “trade game”, which suggests that rank-1 games are much more versatile and economically interesting than zero-sum games. Let player 1 be a seller of a product who can choose possible quality levels $a_i$ for $i = 1, \ldots, m$, and let player 2 be a buyer who can decide on possible quantity levels $b_j$ for $j = 1, \ldots, n$ that she buys from the seller. A price $p_{ij}$ that is paid from buyer to seller can be chosen arbitrarily for each $i$ and $j$. Suppose there are further parameters $\alpha, \beta, \gamma_j$, and $\delta_i$ so that
the payoffs to the players are

\begin{align}
\text{payoff to player 1:} & \quad p_{ij} - \alpha a_i b_j + \gamma_j \\
\text{payoff to player 2:} & \quad -p_{ij} + \beta a_i b_j + \delta_i.
\end{align}

We further assume that $\beta > \alpha > 0$, which reflects that high quality is costly to produce for player 1 and beneficial for player 2, with $\beta - \alpha$ representing the benefits from trade. The additional parameter $\gamma_j$ (increasing with $b_j$) is an additional benefit to player 1 for higher amounts of sold quantities, and similarly $\delta_i$ to player 2 for higher quality. Neither $\gamma_j$ nor $\delta_i$ affect the players’ best responses and can therefore assumed to be zero. This gives a strategically equivalent game whose sums of payoffs are $(\beta - \alpha)a_ib_j$ and therefore of rank one. Because rank-1 games can be analyzed very fast, this “trade game” can be studied for large values of $m$ and $n$, and in particular for its possibly many price levels. The concrete economic interpretation of such games and their equilibria remains to be investigated. Bulow and Levin (2006) consider a “multiplication game” which is a matching game between $n$ workers and $n$ firms where the suitability of a worker for a firm is described by a matrix of rank one. However, it is a game with $2n$ players, not two players.

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