Spectrum of Charged Black Holes – The Big Fix
Mechanism Revisited
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Abstract

Following an earlier suggestion of the authors [gr-qc/9607030], we use some basic properties of Euclidean black hole thermodynamics and the quantum mechanics of systems with periodic phase space coordinate to derive the discrete two-parameter area spectrum of generic charged spherically symmetric black holes in any dimension. For the Reissner-Nordstrom black hole we get
\[ A/4\sqrt{G\hbar} = \pi (2n + p + 1), \]
where the integer \(p = 0, 1, 2, \ldots\) gives the charge spectrum, with \(Q = \pm \sqrt{\hbar p}.\) The quantity \(\pi (2n + 1), n = 0, 1, \ldots\) gives a measure of the excess of the mass/energy over the critical minimum (i.e. extremal) value allowed for a given fixed charge \(Q.\) The classical critical bound cannot be saturated due to vacuum fluctuations of the horizon, so that generically extremal black holes do not appear in the physical spectrum. Consistency also requires the black hole charge to be an integer multiple of any fundamental elementary particle charge: \(Q = \pm me, m = 0, 1, 2, \ldots\) As a by-product this yields a relation between the fine structure constant and integer parameters of the black hole – a kind of the Coleman big fix mechanism induced by black holes. In four dimensions, this relationship is \(\alpha^2/\hbar = p/m^2\) and requires the fine structure constant to be a rational number. Finally, we prove that the horizon area is an adiabatic invariant, as has been conjectured previously.

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1 Introduction

One of the most important unsolved problems in theoretical physics concerns the synthesis of quantum mechanics and general relativity. By the very nature of the problem, it is difficult, if not impossible, to find experimental clues as to what form such a synthesis might take. Candidate theories like string theory and quantum geometry do not as yet have experimentally verifiable predictions. It is therefore useful to examine theoretical arguments about what to expect generically from a quantum theory of gravity. In this context, black holes provide an ideal theoretical laboratory.

There is a great deal of evidence for the existence of black holes in binary systems and at the center of many galaxies, including our own. The work of Bekenstein and Hawking in the mid-seventies has shown that black holes behave as thermodynamic systems, where the surface gravity and horizon area represent the temperature and entropy, respectively. Moreover, black holes emit thermal radiation via quantum processes near the horizon. The Bekenstein-Hawking \[1\] entropy \(S_{BH}\) for a black hole is proportional to the area \(A\) of its (outer) horizon:

\[
S_{BH} = \frac{A}{4G\hbar}. \tag{1}
\]

This formula is assumed to be generically valid for black holes in any space-time dimension. As a specific example, in spherically symmetric Einstein-Maxwell theory in 4 dimensions we have the Reissner-Nordström black hole. In this case (we work in units where \(c = 1\)):

\[
A = 4\pi \left( GM + \sqrt{G^2 M^2 - GQ^2} \right)^2. \tag{2}
\]

The microscopic origin of this thermodynamic behavior is yet to be understood in general, and it is commonly believed that such an understanding can only be achieved in the context of a quantum theory of gravity. One fundamental question that has been asked in recent years is: what is the quantum spectrum of the fundamental observables, namely mass and charge? The answer to this question will determine the transition rates between quantum states and hence will have observable consequences for the Hawking radiation spectrum. Bekenstein and Mukhanov \[2, 3\] have argued from very general
grounds that the area of quantum black holes (and hence the entropy) should have a uniformly spaced spectrum, of the form:

\[ A \propto n, \quad n = 0, 1, 2, \ldots \]  

(3)

Bekenstein’s arguments are based in part on a conjectured relationship between horizon area and adiabatic invariants \( \mathcal{A} \), which by the Bohr-Sommerfeld quantization rule, always have a discrete spectrum. In Bekenstein’s own words, these arguments involve “a mixture of classical hints and quantum ideas”.

The purpose of this paper is to derive a precise form of Eq.(3) from essentially one important assumption which encodes the semi-classical thermodynamic content of black hole dynamics, following the analysis developed in \cite{4}. In particular, we will assume that \( P_M \), the variable conjugate to the black hole mass \( M \) is periodic, with period equal to the inverse Hawking temperature associated with the black hole. This is strictly true only in the Euclidean (imaginary time) sector of the theory. However, this single assumption, plus a natural periodicity condition on the \( U(1) \) phase associated with the electromagnetic potential allow us to derive rigorously both the area and charge quantization conditions from standard quantum mechanics. Other derivations of spectra similar to Eq.(3) exist, but these are tied to specific models \cite{5}, in particular theories of gravity \cite{6} and periodicity assumptions in Lorentzian time (imposed by hand in \cite{7} or attempted to be justified in \cite{8} on account of bounded motion of the Einstein-Rosen bridge throat in Kruskal spacetime). To the best of our knowledge, though, no previous work has utilized the periodicity of the \( U(1) \) phase to obtain a charge quantization condition in addition to the area spectrum. Moreover, it is the interplay between the periodicity of the imaginary time and that of the \( U(1) \) phase that ultimately yields the interesting constraint on the fine structure constant. One important consequence of our analysis is that near extremal black holes appear in the spectrum as highly quantum (low quantum number) objects. Another direct outcome is a proof of Bekenstein’s conjecture that area (and entropy) is an adiabatic invariant associated with black hole dynamics.

It is important to note that our analysis is very general: it is valid for a large class of charged, or uncharged black holes. This class includes spherically symmetric black holes in Einstein gravity (with or without cosmological
constant) in any dimension as well as the rotating BTZ black hole, with the modification that the “charge” actually corresponds to the angular momentum of the black hole.

The paper is organized as follows: Section 2 defines what we mean by “generic black holes”, and reviews their dynamical and thermodynamic properties in the context of generic 2-D dilaton graviton. Section 3 presents the quantization of two well known toy models in order to motivate the methodology that we apply in Section 4 to derive the charge and area spectrum of generic black holes. In Section 5, some consequences of our quantization are derived, while Section 6 contains a summary and conclusions.

2 Generic Charged Black Holes

Consider a spherically symmetric metric in \(d\) spacetime dimensions of the form:

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + r^2(x, t) d\Omega^{(d-2)},
\]

where \(x^\alpha\) denotes the coordinates of the radial and time parts of the metric, while, \(d\Omega^{(d-2)}\) is the metric on the unit \((d-2)\)-sphere, and \(r\) is the invariant radius of the \((d-2)\)-sphere running through the point labeled by coordinates \(x, t\). We would like to consider generic charged black holes, so we will not restrict ourselves at this stage to any particular gravity theory in \(d\)-dimensions. We assume only that the spherically symmetric, vacuum sector has a Birkhoff-type theorem which states that all such solutions are static (have a timelike Killing vector) and can be parametrized by two coordinate invariant parameters, which we choose to be the mass, \(M\) and charge, \(Q\). In this case there is always a coordinate system in which, locally at least, the metric takes the form:

\[
ds^2 = -f(x; M, Q) dt^2 + \frac{dx^2}{f(x; M, Q)} + r^2(x) d\Omega^{(d-2)}.
\]

This ‘Schwarzschild-like’ coordinate system is essentially unique. The associated time coordinate \(t\) we call the “Schwarzschild time” for future reference. The corresponding functions \(f(x; M, Q)\) and \(r(x)\) are essentially uniquely determined by the dynamical equations of the particular theory under consideration. We assume the existence of at least one event horizon, whose
location \( x_h(M, Q) \) is given implicitly as a function of the mass and charge by:

\[
 f(x_h; M, Q) = 0.
\] (6)

In the case that multiple horizons exist, \( x_h \) will refer to the outermost horizon. We stress that we do not need to assume any particular form for the gravitational Lagrangian, only that a Birkhoff-type theorem exists in the spherically symmetric sector. However, as a specific example, we can again consider the Reissner-Nordström solution, in which \( r = x \) and \( f = (1 - 2GM/r + GQ^2/r^2) \), so that there are horizons at \( r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2} \). In this case \( r_h = r_+ \).

We now examine the thermodynamic behavior of generic charged black holes. The following derivation of the black hole temperature is most useful for the subsequent analysis. The basic idea is to Euclideanize the solution (5) by defining \( t_E = -it \), and requiring the resulting solution exterior to the horizon to be regular. To this end, we define new (Euclidean) coordinates:

\[
 R^2(x) = a^2 f(x; M, Q), \\
 \alpha = t_E/a,
\] (7)

where the constant \( a \) will determine the temperature of the black hole. Note that the horizon is located at \( R = 0 \). In these coordinates, the radial part of the metric reads:

\[
 ds^2_E = R^2 d\alpha^2 + N(R) dR^2.
\] (8)

This geometry will be regular for \( R \geq 0 \), free from conical singularities, providing that \( \alpha \) is an angular coordinate, whose period we assume to be \( 2\pi \), and the function \( N(R) \) goes to unity at \( R = 0 \). These conditions determine the constant \( a \) to be:

\[
 a(M, Q) = \frac{1}{f'(x_h; M, Q)},
\] (9)

where the prime denotes differentiation with respect to \( x \), and hence requires the Euclidean time coordinate to be periodic, with range:

\[
 0 \leq t_E \leq 4\pi/f'(x_h; M, Q).
\] (10)
In the imaginary time formulation of finite temperature quantum field theory the periodicity of the Euclidean time is proportional to inverse temperature. Applying this principle to the present calculation yields the correct Hawking temperature:

\[ T_H(M, Q) = \frac{\hbar f'(x_h; M, Q)}{4\pi}. \]  

This expression agrees in all known cases with semi-classical quantum field theoretic calculations of the temperature of the thermal radiation emitted by black holes. Note that we wish to interpret Eq.\((11)\) as the temperature of an exterior horizon. It must therefore by non-negative. For the Reissner-Nordström black hole this gives the condition \(GQ^2 \leq M^2\). More generally, this condition restricts the charge and mass to have values for which \(T_H\) as given in Eq.\((11)\) is non-negative. It will play an important role below.

Once the Hawking temperature is determined, it is straightforward to deduce the expression for the Bekenstein-Hawking entropy of the black hole \(S_{BH}(M, Q)\), which obeys the generalized first law of thermodynamics:

\[ \delta M = T_H(M, Q)\delta S_{BH}(M, Q) + \Phi(M, Q)\delta Q. \]  

The second term is the contribution to the energy from the work required to insert charge \(\delta Q\) into the black hole, where \(\Phi(M, Q)\) is the electrostatic potential at the horizon. (Strictly this assumes that the electrostatic potential vanishes at infinity.)

For the Reissner-Nordström solution, the Hawking temperature is \(T_{BH} = \hbar(r_+ - r_-)/4\pi r_+^2\), while the electrostatic potential is \(\Phi = Q/r_+\). It can easily be verified that these quantities imply that the Bekenstein Hawking entropy as given in Eq.\((11)\) satisfies the generalized first law Eq.\((12)\).

Recall that we are assuming that the theory under consideration, whatever it is, admits a Birkhoff theorem, i.e. that \(M\) and \(Q\) are the only diffeomorphism invariant parameters in the solution space. In this case it is straightforward to deduce the general form of the reduced action that describes the dynamics of the spherically symmetric sector of an isolated, generic charged black hole:

\[ I^{\text{red}} = \int dt \left( P_M \dot{M} + P_Q \dot{Q} - H(M, Q) \right), \]  

(13)
where $P_M$ and $P_Q$ are the conjugates to $M$ and $Q$, respectively. The specific form of the Hamiltonian is irrelevant, except that it is independent of $P_M$ and $P_Q$. This guarantees that $M$ and $Q$ are constants of motion.

Although Eq. (13) is motivated by completely general arguments, it is reassuring that we can arrive at it in a more standard way. Start with Einstein-Maxwell action in $d-$dimensions

$$I = -\int d^d x \sqrt{-g} \left[ \frac{R^{(d)}}{16\pi G_d} + \frac{F_{AB} F^{AB}}{4} \right], \quad (14)$$

where $A, B = 0, ..., d - 1$ and $F_{AB} = \partial_A A_B - \partial_B A_A$ is a $d$-dimensional abelian field strength. If one substitutes into Eq.(14) the generic spherically symmetric form of the metric (4) and restricts the vector potential $A$ to be spherically symmetric as well, one obtains a dimensionally reduced ‘dilaton gravity’ action in two spacetime dimensions, the generic form of which is:

$$I^{\text{eff}} = \frac{1}{2G} \int d^2 x \left[ D(r) R(g) + U(r) |\nabla r|^2 + V(r) - \frac{1}{4} W(r) F^{\mu\nu} F_{\mu\nu} \right], \quad (15)$$

where $\mu, \nu = 0, 1$. In this context $r(x,t)$ plays the role of a dilaton field, and $D(r), U(r), V(r)$ and $W(r)$ are arbitrary functions of $r$. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu$ is the field strength associated with the spherically symmetric components of the gauge potential $A_\mu$. For future reference we note that under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$ where $\lambda(x,t)$ is an arbitrary function of $x,t$. The boundary values of the gauge function $\lambda$ will play an important role in the subsequent analysis. For specific choices of the functions $D, U, V$ and $W$ this action correctly describes the dynamics of the spherically symmetric sector of a very large class of higher dimensional black holes. For details see [9] and [10].

The most general solution to the equations derived from Eq.(15) does have a timelike Killing vector and is of the form (5). The details of the Hamiltonian analysis can be found in [11]. Here we will simply summarize the results. One proceeds as usual by parametrizing the metric:

$$ds^2 = e^{2\rho} \left[-\mu^2 dt^2 + (dx + \nu dt)^2\right], \quad (16)$$

where $\mu$ and $\nu$ are the lapse and shift functions, which play the role of Lagrange multipliers enforcing the two constraints associated with diffeomorphism invariance in two spacetime dimensions. $A_0$ is also a Lagrange
multiplier that enforces the Gauss law constraint. The only physical fields at this stage are therefore the spatial metric $g_{11} = e^{2\rho}$, the dilaton field, $r$, and the spatial component $A_1$ of the vector potential. As with all diffeomorphism invariant theories, the canonical Hamiltonian of the gravitational sector includes a linear combination of constraints. In two spacetime dimensions, this is also true of the electromagnetic sector. The canonical Hamiltonian is therefore of the form:

$$H_c = \int dx \left[ \nu \mathcal{F} + \frac{\mu}{2G} \mathcal{G} + A_0 \mathcal{J} \right] + H_{ADM},$$

(17)

where $H_{ADM}$ is the ADM surface integral yielding the Hamiltonian in the reduced action Eq.(13), $\mathcal{F}$ and $\mathcal{J}$ generate spatial diffeomorphisms and U(1) gauge transformations, respectively, while $\mathcal{G}$ is the Hamiltonian constraint generating time reparametrizations. All three are functions of $\rho, r, A_1$ and their canonically conjugate momenta. Once the constraints are imposed, the Hamiltonian reduces to the surface term $H_{ADM}$, which depends on only two gauge invariant parameters, namely the mass $M$ and charge $Q$. Finally, we arrive at the reduced action Eq.(13).

The specific form of $H_{ADM}$ depends on the boundary conditions that are imposed. The choice of boundary conditions is in turn dictated by the physical circumstances. For the present purposes, the most useful boundary conditions are those first considered by Louko and Whiting [11] in the context of Schwarzschild black holes, and then later generalized to both uncharged [12] and charged [10] black holes in generic two-dimensional dilaton gravity. We consider eternal black holes with at least one bifurcative horizon, so that there is an exterior region whose Kruskal diagram is as shown in Fig. [1]. Note that Fig.[1] shows only the exterior wedge of the Kruskal diagram, since that is all we need to consider for our analysis. In order to analyze the thermodynamic behaviour of the black holes, we would like to consider solutions that can be analytically continued to the regular Euclidean solutions described above. We therefore restrict to spatial slices that lie entirely in this exterior wedge. In particular, we follow Louko and Whiting [11] and require the left hand side of the slice to approach the bifurcation point along a static Schwarzschild slice, while the right side of the slice ends on a line of constant $r = r_B$ (i.e. a box of fixed radius $r_B$). Sample slices are shown in Fig.[1]. These boundary conditions yield a boundary term in the Hamiltonian that depends on the mass and charge on the black hole, and on several external
Figure 1: Kruskal diagram of exterior region of eternal Reissner-Nordström-like black hole with bifurcation horizon at $r = r_+$. Dashed lines denote surfaces of constant Schwarzschild time $t$. A slice of given $P_M = t_1 - t_0$ has a fixed difference between the Schwarzschild time $t$, which it approaches at infinity (or at radius $r_B$), and the Schwarzschild time $t_0$, which it approaches at the bifurcation point.
variables, such as the radius of the box, the value of the electromagnetic gauge potential at the box. (For details of the boundary conditions, see [10].) Moreover, on analytic continuation to periodic Euclidean time, the exterior wedge in Fig.[1] is mapped onto a “cigar tube”. The closed end of the cigar tube corresponds to the bifurcative horizon, and is regular providing the period of the Euclidean time coordinate is chosen as described above.

Note that we started with a phase space consisting of three fields and their conjugates. Since the Hamiltonian contains three first class constraints that each generate a local gauge transformation, the standard Dirac Hamiltonian analysis leads us to conclude that there are no field theoretic physical modes in this class of theories. This is consistent with our assumption that the theory admits a generalized Birkhoff theorem: the only diffeomorphism invariant parameters in the solution space are the mass and charge. It also agrees with our intuition about spherically symmetric gravity and electromagnetism: there is no monopole radiation in either theory.

The two diffeomorphism invariant observables $M$ and $Q$ can be written explicitly as functions of the phase space variables. When the constraints are satisfied, these functions are spatially constant, and by virtue of Hamilton’s equations they are also time independent. Their canonical conjugates $P_M$ and $P_Q$ are functionals of the phase space observables: i.e. integrals over spatial slices. These phase space observables are not invariant under arbitrary diffeomorphisms and gauge transformations. Their invariance would contradict the generalized Birkhoff theorem which allows only two invariant observables. However, $P_M$ and $P_Q$ are physical observables in the Hamiltonian sense because they are invariant under local diffeomorphisms and gauge transformations that vanish on the boundaries of the system. As shown in [13], [14] for spherically symmetric gravity and in [15] for the generic theory, for the assumed boundary conditions in which the metric approaches its Schwarzschild form at either end of the spatial slice, the momentum $P_M$ conjugate to $M$ is proportional to the difference between the Schwarzschild times at either end of the slice (see Fig.[1]). It is therefore invariant only under local diffeomorphism, i.e. those that vanish at the boundaries of the system. Similarly, the momentum $P_Q$ conjugate to $Q$ is not invariant under all gauge transformations, only those that vanish on the boundaries. Specifically, explicit calculation shows that the following relationship holds between
$P_Q$, the momentum conjugate to $Q$ and $P_M$:
\[
\delta P_Q = -\Phi \delta P_M + \delta \lambda,
\] (18)
where the variations refer to variations under a change in boundary conditions, $\Phi$ is the electrostatic potential at the boundary under consideration, and $\delta \lambda$ is the variation in $U(1)$ gauge transformation $\lambda$ at the boundary. This relationship will be important in what follows.

3 Quantization: Toy Models

We now have some information about the classical reduced phase space of black holes. However, an action of the form (13), given in terms of constants of motion, is not a traditional starting point for quantization, and it is in general difficult to know how to proceed. Before considering the physical system at hand, namely black holes, we will therefore look at a couple of standard toy models that illustrate the utility, and validity of our method.

3.1 The Simple Harmonic Oscillator

Consider the following action:
\[
I = \int dt (P_M \dot{M} - M)
\] (19)
where $M \geq 0$. Clearly this system is analogous to (13). $M$ is a constant of motion, whose classical values is bounded below by zero. The equations of motion imply that $P_M = t + \text{constant}$ is the time variable in the problem. It is possible to proceed once the boundary conditions on the conjugate momentum $P_M$ are known. In particular, let us suppose that the dynamics for this system is known to be periodic, with period $\frac{2\pi}{\omega_0}$, for some angular frequency $\omega_0$. One could try to construct self adjoint operators for $P_M$ and $M$ incorporating these boundary conditions, but it is significantly easier to first transform at the classical level to variables in which the global structure of the phase space is easier to deal with. Therefore consider the following transformation:
\[
X = \sqrt{B(M)} \cos(P_M \omega_0)
\] (20)
\[
P_X = \sqrt{B(M)} \sin(P_M \omega_0)
\] (21)
In terms of the new variables \((X, P_X)\), the periodicity of \(P_M\) is manifest. This transformation is canonical if and only if \(B(M) = 2M/\omega_0 + a_0\), where \(a_0\) is an arbitrary constant. If we choose \(a_0 = 0\), then the transformation is well-defined for all \(M \geq 0\) as required. In this case, we have succeeded in mapping the phase space \(M, P_M\), which has the topology of a half cylinder (since \(M \geq 0\) and \(P_M\) is periodic, to the space \(X, P_X\) which has the topology of the complete plane \(R^2\), providing we include the origin, which corresponds to \(M = 0\). Quantization is now straightforward. In particular note that in terms of the new variables:

\[
M = \frac{\omega_0}{2} (X^2 + P_X^2) \tag{22}
\]

so that \(M\) is effectively the energy of a harmonic oscillator, with unit mass, and fundamental frequency \(\omega_0\). By choosing the usual measure and factor ordering one obtains the harmonic oscillator spectrum for \(M\):

\[
M_n = \hbar \omega_0 (n + 1/2) \tag{23}
\]

This is of course no surprise since the boundary conditions that we imposed on \(M\) and \(P_M\) were precisely those of the harmonic oscillator. \(B(M) = 2M/\omega_0\) in our canonical transformation is the action variable associated with the angular coordinate \(\alpha = \omega P_M\), and as expected by the Bohr-Sommerfeld relation, it has an equally spaced spectrum. What this example is meant to illustrate is that, starting from variables that corresponded to a constant of motion and its canonical conjugate, the boundary/periodicity conditions lead, via the canonical transformation, to the correct harmonic oscillator spectrum. If this procedure were only valid for the harmonic oscillator, it would not be of any use, so we illustrate its utility in one more toy example before going on to the black hole case.

### 3.2 Bouncing Ball

We now consider precisely the same action \((19)\), again with the condition \(M \geq 0\), but with a different periodicity condition on its conjugate. Suppose that we know physical solutions to be periodic in \(P_M\) with period \(\Delta P_M = L \sqrt{2m/\lambda}\). We again try a canonical transformation that accurately reflects this
periodicity:

\[
X = \sqrt{B(M)} \cos \left( \frac{2\pi P_M \sqrt{M}}{L\sqrt{2m}} \right)
\]  

(24)

\[
P_X = \sqrt{B(M)} \sin \left( \frac{2\pi P_M \sqrt{M}}{L\sqrt{2m}} \right)
\]  

(25)

In this case, the condition that the transformation be canonical is satisfied if

\[B(M) = \frac{2L}{\pi} \sqrt{2Mm}.\]

In terms of the new variables, \(B(M)\) again looks like the Hamiltonian for a harmonic oscillator (note however that \(B(M)\) is not the Hamiltonian for the physical system). Its spectrum, up to possible factor ordering ambiguities is therefore:

\[B_n = \hbar(n + 1/2)\]  

(26)

The corresponding spectrum for the Hamiltonian, \(M\) is:

\[M_n = \frac{1}{2m} \left( \frac{\hbar \pi}{2L} (n + 1/2) \right)^2\]  

(27)

The physical interpretation of this example can be made explicit by going to new coordinates \((q,p)\) defined by

\[
M = \frac{p^2}{2m}
\]  

(28)

\[
P_M = \frac{mq}{p}
\]  

(29)

We therefore see that \(M\) and \(P_M\) are the Hamiltonian and time corresponding to a free particle with position \(q\) and momentum \(p\). The periodicity given above is that associated with a free particle “bouncing” between two walls (infinite potential barriers) a distance \(L\) apart. The motion repeats when \(q\) goes through \(2L\), which, using the above transformations gives the correct periodicity in terms of the time variable \(P_M\). Again we find that \(B(M) = (2L/\pi)|p|\) is the adiabatic invariant for this system. The spectrum we obtained for \(|p|\), namely \(|p|_n = \hbar\pi(n + 1/2)/2L, n = 0, 1, 2..\) is not quite the same as that derived in most text books on quantum mechanics. The spectrum derived by more traditional methods is:

\[|p|_n = \frac{\hbar\pi n}{2L}, \quad n = 1, 2, \ldots\]  

(30)
This difference between these two spectra is simply one of factor ordering: our method can be made to yield the standard spectrum by choosing a different factor ordering for the operators $X$ and $P_X$ in $B$. This example shows again that by knowing the range/periodicity of the variables in (13) it is possible to deduce the spectrum of the corresponding adiabatic invariant and energy, at least up to factor ordering.

4 Black Hole Quantization

As shown by the examples above, the first step towards quantization in terms of the present variables is to determine the ranges of the various phase space observables. It is reasonable to keep $M$ non-negative, while $Q$ must be a real number satisfying the condition $T_H(M, Q) \geq 0$. It can be shown that this condition can generically be expressed as a bound on the entropy in terms of the charge, namely:

$$S_{BH}(M, Q) \geq S_0(Q),$$

where the function $S_0(Q)$ depends on the theory under consideration. For example, it can be verified by examining Eq.(2) that in the case of Reissner-Nordstrom black holes the condition, $G^2 M^2 \geq G Q^2$ requires Eq.(31) with $S_0(Q) = \pi Q^2/\hbar$. For dimensionally reduced Einstein-Maxwell theory in $d$ dimensions with $S^{(d-2)}$ spherical symmetry, one has

$$S_0(Q) = K(d) Q^{(d-2)/(d-3)},$$

where

$$K(d) = (1/4)(A_{d-2}/G_d)^{(d-4)/(d-3)}(8\pi/((d-2)(d-3))^{(d-4)/(d-3)},$$

and $A_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2))$ is the area of the unit $d-2$ sphere. It is interesting to note that in all cases except $d = 4$, the entropy bound depends explicitly on the gravitational constant $G_d$.

As discussed in Section 2, in black hole geometrodynamics $P_M$ gives the difference of the Schwarzschild times at the ends of the spacelike slice running across the Kruskal diagram. When analytically continued to the Euclidean spacetime, this variable becomes imaginary and, as motivated from semi-classical thermodynamics, periodic, with period given by the inverse
Hawking temperature \((10)\). We will therefore henceforth make the assumption that we must identify:

\[
P_M \sim P_M + \frac{1}{T_H(M, Q)} . \tag{34}
\]

We are now in a situation familiar in classical mechanics, where there exists a periodic, angular variable. Akin to the action-angle formulation of the harmonic oscillator, we ‘unwrap’ our gravitational phase space, by transforming to a set of unrestricted variables. Consider the following transformation \((M, Q, P_M, P_Q) \rightarrow (X, Q, \Pi_X, \Pi_Q)\):

\[
X = \sqrt{\frac{\hbar B(M, Q)}{\pi}} \cos(2\pi P_M T_H(M, Q)/\hbar) , \tag{35}
\]

\[
\Pi_X = \sqrt{\frac{\hbar B(M, Q)}{\pi}} \sin(2\pi P_M T_H(M, Q)/\hbar) ,
\]

\[
Q = Q , \tag{36}
\]

\[
\Pi_Q = \Pi_Q(M, P_M, Q, P_Q) ,
\]

where \(B(M, Q)\) and \(\Pi_Q(M, P_M, Q, P_Q)\) are functions that will be determined by the condition that the transformation be canonical. Transformations Eq.\((35)\) yield a pair of non-periodic variables in a way that incorporates directly the correct periodicity of \(P_M\).

A straightforward calculation reveals that, up to a total variation in independent variables \(M\) and \(Q\):

\[
\Pi_X \Delta X + \Pi_Q \Delta Q = P_M \left( T_H \frac{\partial B}{\partial M} \right) \Delta M + \left( \Pi_Q + P_M T_H \frac{\partial B}{\partial Q} \right) \Delta Q , \tag{37}
\]

so that this transformation is canonical when

\[
\frac{\partial B}{\partial M} = \frac{1}{T_H(M, Q)} , \tag{38}
\]

\[
P_Q = \Pi_Q + P_M T_H \frac{\partial B}{\partial Q} . \tag{39}
\]

From the first law Eq.\((12)\) we see that

\[
\frac{\partial B}{\partial M} = \frac{\partial S_{BH}}{\partial M} \tag{40}
\]
It therefore follows that

\[ B(M, Q) = S_{BH}(M, Q) + F(Q), \]  

(41)

where \( S_{BH} \) is the Bekenstein-Hawking entropy associated with a black hole of mass \( M \) and charge \( Q \), while \( F(Q) \) is an arbitrary function of the charge. This function can be fixed by noting that the transformation (Eq. (35)) maps the fundamental domain of the initial phase space variables \((M, P_M)\), \(0 \leq 2\pi P_M T_H(M, Q)/\hbar < 2\pi\), to the exterior of a disc of finite radius, such that 

\[ B(M, Q) \geq S_0(Q) + F(Q), \]

(42)

Here \( S_0(Q) \) is the function that determines the minimum entropy in terms of charge for the generic theory. To avoid ambiguity in quantization caused by the necessity of imposing boundary conditions at the minimal radius of \( S_{BH}(M, Q) = S_0(Q) \), it is natural to remove this round “hole” in phase space plane. We, therefore, demand that \( F(Q) = -S_0(Q) \). This achieves two crucial things: first that the phase space topology in the new variables is trivial – a complete two-dimensional plane – and secondly that Eq. (31) is identically satisfied. Moreover, the precise extremal limit corresponds to the origin of this plane. With this choice, \( \Pi_Q \) is uniquely determined to be:

\[ \Pi_Q = \frac{\hbar}{e^\chi} + \frac{\hbar}{2\pi} S'_0(Q) \alpha, \]

(43)

where ‘ denotes differentiation with respect to \( Q \) and we have defined the variables: \( \chi = \frac{\hbar}{2\pi}(P_Q + \Phi P_M) \) and \( \alpha = 2\pi P_M T_H(M, Q) \).

An important reservation regarding the canonical transformations of the above type is that they are performed for the Euclidean theory in which only the periodicity of the variable \( \alpha = 2\pi P_M T_M \) makes sense. The possibility of such canonical transformations and subsequent quantization altogether can be called in question because of their Euclidean status. The justification of this procedure, however, follows from the important fact that the Euclideanization of the canonical action Eq. (13) is not just the analytic continuation to the imaginary range of the time \( t \), but simultaneously the same continuation for the momenta variables. In contrast to the usual situation this leaves us with the same (up to an overall \( i \)-factor) real canonical action in
Euclidean variables (for the usual Wick rotation procedure the kinetic term of the canonical action acquires an extra imaginary unit factor). This special type of Euclideanization is not universal, because it is possible only for cyclic momenta not entering the Hamiltonian as in Eq. Eq.(13). This explains why the canonical equations of motion retain the same form in Euclidean regime as in the Lorentzian one and why they can be rewritten in terms of the same Poisson bracket \( \{ M, P_M \} = 1 \) (under the usual Wick rotation such a bracket becomes imaginary for Euclidean variables). The further quantization as a promotion of \( (M, P_M) \) (or canonically related \( (X, \Pi_X) \)) to the operator level subject to canonical commutation relations, \( [X, \Pi_X] = i \{ X, \Pi_X \} = i \), is straightforward and runs as follows.

From Eq.(41) and Eq.(42) we see that in the gravitational sector of \( (X, \Pi_X) \) for fixed charge \( Q \)

\[
S_{BH} - S_0(Q) = \frac{2\pi}{\hbar} \left( \frac{X^2}{2} + \frac{\Pi_X^2}{2} \right) .
\]  

(44)

This operator is the Hamiltonian of a harmonic oscillator with the mass and frequency both equal to \( \hbar/2\pi \). Since the domain of variables \( X \) and \( \Pi_X \) is an entire two-dimensional plane, their quantization becomes trivial. To be precise, with zero boundary conditions at infinity of this plane one can define self-adjoint operators in the space of square-integrable wavefunctions and thus obtain the quantum mechanical spectrum for the entropy of generic charged black holes:

\[
S_{BH} = 2\pi \left( n + \frac{1}{2} \right) + S_0(Q) \quad n = 0, 1, 2, ...
\]  

(45)

It is important to note that due to vacuum fluctuations the limit of extremal black hole, \( S_{BH} = S_0(Q) \), corresponding classically to the origin of the phase-space plane \( (X, \Pi_X) \), cannot be achieved at the quantum level. Eq.(43) also implies that near extremal black holes, despite being potentially macroscopic objects (i.e. large \( Q \), and horizon area) are necessarily highly quantum objects\(^1\) in the sense of corresponding to small quantum number, \( n \).

To complete the analysis we now need to quantize the electromagnetic sector and derive the spectrum for the operator \( Q \). This requires knowledge of

\[^1\text{We are grateful to Valeri Frolov for raising this issue.}\]
the boundary conditions on its conjugate $\Pi_Q$. For compact gauge group $U(1)$ Eq. (18) suggests that the linear combination $\chi = e\lambda/\bar{\hbar} = e(P_Q + \Phi P_M)/\hbar$ is an “angular coordinate” with period $2\pi$, where $e$ is the electromagnetic coupling. In particular, suppose that there exists a charged scalar field $\psi$ minimally coupled to the vector potential via the covariant derivative

$$D_\mu \psi = \left(\partial_\mu - ie\frac{e}{\bar{\hbar}}A_\mu\right)\psi,$$

(46)

where $e$ gives the strength of the electromagnetic coupling. Under a gauge transformation, $A_\mu \to A_\mu + \partial_\mu \lambda$, $D_\mu$ is invariant providing that $\psi \to e^{(ie\lambda/\bar{\hbar})}\psi$. Thus, $\lambda$ has the period claimed. Examining Eq. (43) we now see that $\Pi_Q$ is a function of two angular coordinates $\chi$ and $\alpha$ which, according to arguments given above are both periodic with period $2\pi$

$$\chi \sim \chi + 2\pi n_1, \quad \alpha \sim \alpha + 2\pi n_2.$$

(47)

It is therefore necessary to identify the phase space points

$$(Q, \Pi_Q) \sim (Q, \Pi_Q + 2\pi n_1 \frac{\hbar}{e} + n_2 \hbar S'_0(Q)).$$

(48)

In the coordinate representation, $\hat{Q} = -i\hbar \partial/\partial \Pi_Q$, the wave functions for charge eigenstates take the form

$$\psi_Q(\Pi_Q) = (\text{const}) \times \exp(iQ\Pi_Q/\bar{\hbar}).$$

(49)

This wave function is single valued under the identification Eq. (48) when for any integer $n_1$ and $n_2$ the following combination

$$n_1 \frac{Q}{e} + n_2 \frac{Q}{2\pi} S'_0(Q) = n_3$$

(50)

is also given by an integer number $n_3$. This immediately results in two quantization conditions with two integer numbers $m$ and $p$

$$\frac{Q}{e} = m,$$

(51)

$$\frac{Q}{2\pi} S'_0(Q) = p,$$

(52)
which together imply not only the black hole charge \( Q \) is integer multiple of the \( U(1) \) charge \( e \), but also that the latter is subject to a quantization condition: the allowed value of \( e \) is constrained in terms of \( m \) and \( p \).

In order to determine the implications of these quantization conditions one must know the specific form of \( S_0(Q) \). For concreteness, take first Reissner-Nordström black holes with \( S_0(Q) = \pi Q^2/\hbar \). The charged black hole itself is characterized by two integer numbers \( n \) and \( p \) which determine its horizon area (entropy) and charge

\[
S_{BH} = 2\pi n + \pi (p+1),
\]

\[
Q^2 = \hbar p.
\]

The quantum number \( p \) determines the charge of the quantum black and hence its minimal entropy \( S_0 = \pi (p+1) \). The quantum number \( n \) determines the excited level of the black hole over the "vacuum", \( n = 0 \) for which the entropy achieves its minimum value. Classically this vacuum would correspond to an extremal black hole with minimal admissible value of its mass \( M = Q/\sqrt{G} \) and entropy \( S_0(Q) \) for a given charge \( Q \). It is a remarkable feature of our analysis that this classical lower bound on the mass is never actually saturated due to vacuum fluctuations of the horizon – the \( +\pi \) contribution in Eq.(53) survives in the critical limit \( n = 0 \) of a charged quantum black hole. Thus, extremal black holes are not in the physical spectrum (at least for our Weyl type quantization).

Finally, there exists a third quantum number \( m \) which shows that the charge \( Q \) is multiple of the \( U(1) \) coupling constant \( e \). However, this coupling constant is not completely arbitrary, since Eq.(51) and Eq.(52) fix the value of fine structure constant in terms of integer numbers \( m \) and \( p \):

\[
e^2/\hbar = \frac{p}{m^2}.
\]

Thus, \( e^2/\hbar \) must be a rational number

We would like to stress that the essential qualitative features of the spectrum are generic to all spherically symmetric black holes, and not specific to the Reissner-Nordström case. For completeness, we consider spherically symmetric black holes in \( d \) dimensions, for which \( S_0(Q) \) is given by Eq.(52). In this case, the condition Eq.(52) generalizes to

\[
e^2/\hbar = \frac{p^{2(d-2)/(d-3)}}{m^2} (d-3)^2 \left( \frac{8\pi G_d d - 3}{A_{d-2} d - 2} \right)^{(d-4)/(d-2)}.
\]
Interestingly, the fine structure constant spectrum for dimensionalities other than four depends on the d-dimensional gravitational coupling constant $G_d$.

5 Adiabatic Invariants and the Black Hole Emission Spectrum

With the above transformations, one can prove Bekenstein’s conjecture that the horizon area is an adiabatic invariant [3]. We can express (44) in terms of the area variable as follows:

$$A - 4G\hbar S_0(Q) = 8\pi G \left( \frac{X^2}{2} + \frac{\Pi^2_X}{2} \right).$$

(57)

Now, consider the integral

$$J_X = \oint \Pi_X dX$$

where the integration is over one complete period and $J_X$ is the angle variable for the oscillator in the action-angle formalism. Now, it is well known that for a periodic system, under an adiabatic (slow) perturbation with a time dependent parameter $\lambda(t)$, satisfying the condition $\delta\lambda/\lambda \ll \delta t/T$ for a time interval $\delta t$ ($T$ is the time period of oscillation), $J_X$ remains invariant, although both the energy and the fundamental frequency can change considerably over a period of time [17]. For the harmonic oscillator Hamiltonian under consideration, the above integral is simply the area of a circle of radius squared $(A - 4G\hbar S_0(Q))/4G$. Thus, it follows that

$$J_X = \pi \frac{A - 4G\hbar S_0(Q)}{4G}$$

(58)

is an adiabatic invariant. In addition, note that the assumed periodicity of the phase space involving $Q$ and $\Pi_Q$ implies that

$$J_Q = \oint \Pi_Q dQ$$

(59)

is an adiabatic invariant as well. Consequently, from the expression for $\Pi_Q$ in (13), it follows that the $U(1)$ charge $Q$ (and thus any function of $Q$) is an
adiabatic invariant as well. Thus, we conclude that the area observable is itself an adiabatic invariant, thus proving the conjecture made in [3] that the horizon area is an adiabatic invariant in quantum gravity. The effect upon quantization becomes evident as well. According to the Bohr-Sommerfeld quantization rules, action variables are always quantized [18]. Thus the area operator of quantum gravity must be quantized as seen in the preceding calculations. Note however, that the appearance of the ‘ground state entropy’ of $\pi$ could not have been guessed from old quantum theory, without solving the Schrödinger equation.

Exponentiating the entropy, we obtain the actual degeneracy of the black hole in the level $n$:

$$g(n) = e^{2\pi(n+1/2)+S_0(Q)}$$

(60)

It is interesting to note that the since $g(0) \neq 1$, the ground state is degenerate. The fact that $g(0)$ is not an integer should not be considered too disturbing at this stage. We have taken the semi-classical expression for the Hawking temperature to be exact when determining the periodicity of the Euclidean time coordinate. In this context it is amusing and perhaps relevant that $g(0) = 23$ (i.e. is an integer) if one simply replaces $T_H$ by $0.998T_H$ in the canonical transformations Eq. (35), and everywhere in the subsequent analysis.

Now let us consider a physical process in which the black hole emits a photon by making a quantum jump from one level to the next lower level. The exact mechanism of such a jump is irrelevant in the current analysis. To calculate the frequency of the emitted photon, we go back to the entropy formula (1) applied to a Reissner-Nordström black hole (2). Assuming that the black hole decays by emitting just one photon with the lowest allowed frequency $\omega_0$ (for simplicity we assume uncharged particle emission and four dimensions), its initial and final masses are $M + \hbar\omega_0$ and $M$ respectively, and the following relation holds:

$$S(M + \hbar\omega_0, Q) - S(M, Q) = S(n+1) - S(n) = \pi$$

from where it follows that

$$\omega_0 = \frac{(r_+ - r_-)\pi}{A}.$$  

(61)
This frequency for the $Q \rightarrow 0$ limit agrees with that found in [3] up to factors of order unity. However, unlike there, here the fundamental frequency can be made arbitrarily small by going near the extremal limit (i.e. small quantum number $n$), while keeping the charge, and mass large. So, although the emission spectrum is in general discrete, in the near extremal limit, the Hawking spectrum will become almost continuous, as predicted by semiclassical analyses and also by the area spectrum of loop quantum gravity [19].

Finally, our quantization procedure indicates that at the end stage of radiation form the black hole, there is a Planck size remnant which is left behind, corresponding to the ‘zero point area’ which inevitably results from the uncertainty principle. It is tempting to speculate that this remnant will contain any information that may have entered the black hole before and during its evaporation process. It would be an interesting exercise to construct an explicit model of radiation by introducing an interaction Hamiltonian and compute the transition amplitudes between the levels.

6 Summary and Conclusions

In this paper we have quantized the charged black hole sector of gravity, and derived both the area (entropy) and the charge spectrum by first deriving the reduced action for the spherically symmetric sector of generic charged black holes, and then providing input about the black hole thermodynamics by assuming periodicity of the phase space variable conjugate to the black hole mass. Although the periodicity is motivated by the Euclideanized black hole solutions, so that its relevance to Minkowskian black holes is not beyond doubt, it is not altogether unnatural to associate a fundamental time scale with physical black holes (see [8], [7] and [20]). Moreover, the beauty and simplicity of the resulting analysis, as well as its remarkable generality, suggests that it must have something to do with the real world at the quantum level.

We close by attempting to interpret some of the more intriguing implications of the spectrum that we have derived. The interpretation of the first quantization condition Eq.(51) is obvious – the black hole can absorb and emit integer number of particles with fundamental charge $e$. The second condition Eq.(55) implies that the value of this fundamental constant must
be related to the integer parameters of the black hole $m$ and $p$. This can be interpreted in one of two ways. If one considers the black hole states to be fundamental, then the presence of a charged black hole in the universe would fix the value of the fine structure constant and hence the fundamental unit of electric charge. This is somewhat analogous to the way the presence of a single Dirac magnetic monopole of magnetic charge $g$ requires all electric charges in its vicinity to be quantized in units of $2\pi\hbar/g$. This is also reminiscent of Coleman’s old idea [21], that wormhole physics may fix the conventional fundamental constants of nature.

It would seem that the big fix mechanism of [21] based upon sharply peaked distribution functions of Euclidean quantum gravity is conceptually different from our mechanism of single valued wavefunctions in the space of periodic variables. However, under a closer examination these concepts might turn out more closely related than one could have anticipated. Indeed, we do not know yet the correct interpretation of the black hole thermodynamics input – the periodicity in Euclidean time with the inverse Hawking temperature period. The formalism of Euclidean quantum field theory, as is well known, can originate from two distinctively different physical situations – from the description of thermodynamical ensemble (statistical, i.e. not pure, state) or from the description of classically forbidden transitions between pure states – quantum mechanical underbarrier tunneling. Quite amazingly, in quantum gravity these two functions of the Euclidean formalism are not yet clearly separated. That is why this field of science was designed to have a special name – Euclidean quantum gravity – very ambiguous and flexible for possible interpretations. Indeed, the Euclidean section of the Schwarzschild solution can, on one hand, be regarded as a saddle point of the path integral for the statistical partition function and, on the other hand, can be viewed as a classical configuration interpolating in the imaginary time between the two causally disconnected spacetime domains – right and left wedges of the Kruskal diagram. Our requirement of periodicity in the imaginary time apparently can be viewed as a kind of consistency of quantum states in these two domains, or the finiteness of the semiclassical underbarrier transition amplitude between them (remember that the Hawking periodicity requirement is based on the absence of conical singularity which is, in its turn, motivated by the regularity of the semiclassical distribution). So the amplitudes not satisfying this requirement can be regarded as suppressed to zero, just like in the Coleman big fix paradigm, the vanishing probability of having funda-
mental constants that violate the superselection rules imposed by coherent states of baby universes. In this respect, our derivation of restrictions on the fine structure constant can be regarded as a big fix mechanism revisited in context of black hole physics.

The physical significance of our result and its foundation of the above type is still not clear, and at this stage one might raise a number of objections to this conclusion. In particular, one might ask how this picture stands the coexistence of different black hole with different charges each demanding its own fundamental value of $e$, one might call in question the very interpretation of the restriction on $e$, which could be related not to the quantization consistency requirement, but rather indicate to the structure of interferometric patterns caused by the black hole scattering of quantum electrons.

Alternatively, one can take the viewpoint that the charged fields are fundamental, so that our quantum black hole states must somehow be created out of these fundamental fields. In this case, the condition Eq.(55) simply means that the two charge quantum numbers $m$ and $p$ are not independent. However, Eq.(55) only yields consistent solutions for $m$ and $p$ if $e^2/\hbar$ is a rational number. So we are still in the position that the black hole quantum mechanics places a non-trivial condition on the fine structure constant.

Clearly, the actual experimental value of the fine structure constant, known with incredibly high accuracy to be $4\pi\hbar/e^2 = 137.03608...$, cannot be approximated by the rational number Eq.(55) with reasonably small integers $p$ and $m$. So whatever viewpoint is adopted, this mechanism remains very speculative. It does, however, deserve further study, not least because of the fascinating consequence that a fundamental constant of nature is restricted by the consistency of the black hole quantum mechanics.

After the original version of this paper was written we learned about two other derivations of spectra for charged black holes [22] and [23]. Both the principles of derivation and the quantitative results of these works differ essentially from our approach. These two works bear in common the fact that within the full or partial Hamiltonian reduction one of the variables – describing the proper time in a spacelike slicing of the Reissner-Nordstrom or Kerr-Newman geometry between the two horizons at $r_\pm$ – has a finite range related to the mass of black hole. The periodic extension of this Lorentzian time variable from this range yields a quantization condition similar to the one proposed in [7]. It should be emphasised, however, that these works suffer either from a rather wishful handling of the Wheeler-DeWitt equa-
tion [22] or from a very restrictive type of spacetime foliation that does not cover the entire spacelike infinity [23]. (The latter paper includes angular momentum as well as charge, but the actual Hamiltonian reduction is conjectured rather than explicitly performed). The different foundations of these methods from ours result in quantitatively different conclusions. Interestingly, the quantities that acquire the equally spaced spectra in [22, 23] and in our case Eq.(45) are respectively 
\[ S_+ - S_-, \quad S_+ + S_- \quad \text{and} \quad S_0 = \pi Q^2/\hbar \]
where \( S_{\pm} \equiv \pi r_{\pm}^2/G\hbar \) are the entropies associated with the outer and inner horizons, while \( S_0 = \pi Q^2/\hbar \) is the intermediate quantity – the BPS bound Eq.(31). It is important, however, that despite this qualitative resemblance, references [22] and [23] do not incorporate the second quantization condition (39) and therefore do not predict restrictions on the fine structure constant.

The fact that these methods yield only qualitatively similar results to ours is indicative of their conceptually different foundations. In our opinion, the assumption of periodicity in Lorentzian time, whether it is introduced by hand as in [7] or effectively induced by spacetime foliations pinched between inner and outer horizons as in [22, 23], does not seem convincing. On the other hand, the appeal to Euclidean quantum gravity in the form of the underbarrier dynamics in imaginary time interpolating between the wedges of the Kruskal diagram looks more promising. However, its ultimate justification might require knowledge of the as yet unfinished chapter of Hawking’s virtual black holes theory [24].

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