Quantum Fisher Information Flow in Non-Markovian Processes of Open Systems

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We establish an information-theoretic approach for quantitatively characterizing the Non-Markovianity of open quantum processes. Here, the quantum Fisher information (QFI) flow provides a measure to statistically distinguish Markovian and non-Markovian processes. A basic relation between the QFI flow and non-Markovianity is unveiled for quantum dynamics of open systems. For a class of time-local master equations, the exactly-analytic solution shows that for each fixed time the QFI flow is decomposed into additive sub-flows according to different dissipative channels.

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I. INTRODUCTION

Any system in the realistic world is open since it inevitably interacts with its environment. The time evolutions of open systems are simply classified into Markovian and non-Markovian ones according to the ways to lose energy or information [1]. In most situations, Markovian process uniquely determines its final steady state as an thermal equilibrium [2], which is independent of its initial one. In this sense a Markovian process is essentially an information erasure process, thus tends to continuously reduce the distinguishability between any two initial states [6].

However, Markovian description for an open quantum system is only an approximation to most of realistic processes, which are of non-Markovian. With many recent investigations about non-Markovian dynamics by making use of various analytical approaches and numerical simulations, a computable measure of “Markovianity” for quantum channels was introduced in Ref. [3]. Most recently, it was also recognized that difference between them can be measured through the continuous increment of the state distinguishability [8]. Then this increment is intuitively interpreted as the revival of information flow between the bath and the system though there no quantitative information measure is utilized. Based on this measure of the non-Markovianity, a method for direct measurement of the non-Markovian character was proposed [20]. Another approach based on entanglement is proposed in Ref. [5]. In this paper, the quantum Fisher information (QFI) flow are introduced to directly characterize the non-Markovianity of the quantum dynamics of open systems.

Actually, in the system-plus-bath approach for open quantum systems, the effective dynamics of the reduced density matrix $\rho$ is induced by tracing over environment [1]. The simplest reduced dynamics is the quantum Markovian process described by dynamical semi-groups [6]. There, the reduced density matrix $\rho$ at time $t + dt$ is determined completely by the one at time $t$. Contrarily, the general reduced dynamics may be of non-Markovian when the surrounding environment may retain a memory of the information about states at earlier times, and transfer it back to the system to affect its evolution. In this sense the Markovian process only happens when the environmental correlation time is relatively short so that memory effects can be neglected. These memory-based considerations for the Markovian approximation also mean that the information-theoretical characterizing of the non-Markovianity is a quite natural fashion. However, it is still an open question that how to treat the information flow in open quantum systems based on a solid information-theoretic foundation.

In this paper, we establish such a framework by adopting the QFI flow as the quantitative measure for the information flow. The QFI characterizes the statistical distinguishability of reduced density matrix [7, 8]. An intuitive picture of the memory effect of a non-Markovian behavior then immediately follows from the dynamic return of the QFI, which is depicted by the inward QFI flow. For a class of the non-Markovian master equations in time-local forms, we exactly calculate the information flows. The analytic results show that the total QFI flow can be decomposed into the split contributions from different dissipative channels for each fixed time. On the other hand, the QFI plays an essential role in quantum metrology [6], where the highest precision of estimating an unknown parameter we may achieve is related to inverse of the QFI. We point out this QFI flow approach is feasible to work for understanding the problems of quantum metrology.

II. QUANTUM FISHER INFORMATION IN NON-MARKOVIAN DYNAMICS

We consider the quantum processes described by the following time-local master equation [6, 10]

$$\frac{\partial}{\partial t} \rho(t) = \mathcal{K}(t)\rho(t), \quad (1)$$

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where $\mathcal{K}(t)$ is a super-operator acting on the reduced density matrix $\rho(t)$ as

$$
\mathcal{K}(t)\rho = -i[H, \rho] + \sum_i \gamma_i \left[ A_i \rho,A_i^\dagger - \frac{1}{2} \{ A_i^\dagger A_i, \rho \} \right],
$$

with $H(t)$ the Hermitian Hamiltonian for the open quantum system without the couplings to the bath. $\{\cdot,\cdot\}$ denotes the anti-commutator. If all $\gamma_i$ and $A_i$ are time independent, and all $\gamma_i$ are positive, equation (2) is the conventional master equation of the Lindblad form [11], which describes the conventional Markovian process. However, by making use of a variety of methods, such as the time-convolutionless projection operator technique [13], Feynman-Vervon influence functional methods, such as the time-convolutionless projection operator technique [13], Feynman-Vervon influence functional theory [14] and some others [15], the parameters $\gamma_i = \gamma_i(t)$ and $A_i = A_i(t)$ in the time-local master equation may explicitly depend on time, and $\gamma_i$ even may become negative sometimes. Obviously, the non-Markovian character resides in these time-dependent coefficients.

Taking some real number $\theta$ in the reduced density matrix $\rho(\theta; t)$ as the inference parameter, we write down the QFI [10]

$$
\mathcal{F}(\theta; t) := \text{Tr} \left[ L^2(\theta; t)\rho(\theta; t) \right],
$$

where $L(\theta; t)$ is the so-called symmetric logarithmic derivative (SLD), which are Hermitian operators determined by

$$
\frac{\partial}{\partial \theta} \rho(\theta; t) = \frac{1}{2} \left[ L(\theta; t)\rho(\theta; t) + \rho(\theta; t)L(\theta; t) \right].
$$

An important essential feature of the QFI is that we can obtain the achievable lower bound of the mean-square error of unbiased estimators for the parameter $\theta$ through the quantum Cramér-Rao (QCR) theorem

$$
\text{Var}(\theta; t) \geq \frac{1}{M} \mathcal{F}(\theta; t),
$$

where $M$ represents the times of the independent measurements [10]. From the QCR theorem, we can see that the QFI is indeed a measure of a certain kind of information with respect to the precision of estimating the inference parameter. The relations between the QFI and the statistical distinguishability of $\rho(\theta; t)$ and its neighbor has been pointed out in some previous works [4, 8, 10].

Flow of the QFI and its decomposition. — Here we use the QFI to characterize the non-Markovianity of the open quantum system by introducing the QFI flow, which is defined as the change rate $I := \partial \mathcal{F}/\partial t$ of the QFI. As a central result in this paper, a proposition about the decomposition of the QFI flow is given as follows:

**Proposition.** For an open quantum system described by the time local master equation [4], the QFI flow $I = \sum_i I_i$ is explicitly written as a sum of subflows $I_i = \gamma_i J_i$ with

$$
J_i := -\text{Tr} \left\{ \rho [L, A_i]^\dagger [L, A_i] \right\} \leq 0.
$$

**Proof.** From the differential of Eq. (4) with respect to time $t$, we have

$$
\frac{\partial}{\partial \theta} \rho(\theta; t) = \frac{1}{2} \left[ \dot{\rho} L + L \dot{\rho} + \rho L + \dot{\rho} L \right].
$$

It gives

$$
\text{Tr} \left[ \dot{\rho} LL + \rho L L \right] = \text{Tr} \left[ 2L \partial_\theta \rho(\theta) \right] = \text{Tr} \left[ 2\rho L^2 \right].
$$

From the differential of Eq. (3) with respect to time $t$, we obtain the QFI flow as

$$
I = \text{Tr} \left[ L \left( \frac{\partial \rho}{\partial \theta} \right) \right].
$$

where the operator $L := L(2\partial/\partial \theta - L)$ is defined. By using the concrete expression of the master equation [4], we split the QFI flow into those individuals corresponding to the different dissipative channels as $I = \text{Tr} \left[ L \mathcal{K}(t)\rho(t) \right]$ or $I = \sum_i \gamma_i \text{Tr} \left[ L(A_i A_i^\dagger) - \frac{1}{2} L(\{ A_i^\dagger A_i, \theta \}) \right]$. After some algebra, we get the decomposition $I = \sum_i \gamma_i J_i$, where $J_i$ is just given in Eq. (6). It finally proves the proposition.

The above proposition and its proof contain rich implications in physics. Firstly, the decomposition of the QFI flow corresponding to the different dissipative channels is due to the linearity of QFI flow equation (9) with respect to $\partial \rho/\partial t$ and the concrete form of the time-local master equation [4]. This is not a simple decomposition since each subflow depends on the whole SLD $L(\theta; t)$, meanwhile, $L(\theta; t)$ is deduced from $\rho(\theta; t)$, whose evolution depends on every dissipative channel and the unitary part of the master equation. So this kind of decomposition does not mean different dissipative channels are separable to influence the change the QFI for a period of time. However, for each fixed time $t > 0$, the QFI flow at the present moment are decomposed into the split contributions from different dissipative channels. In this sense, we interpret $I_i(t) = \gamma_i(t) J_i(t)$ as a subflow of the QFI at time $t$ caused by the dissipative channel described by $A_i(t)$ and $\gamma_i(t)$. The magnitude of the QFI subflow is determined by a state-independent factor $\gamma_i$ and a state-dependent factor $J_i$.

Secondly, one of the advantages of such decomposition comes from the link between the direction of each QFI subflow $I_i$ and the sign of the decay rate $\gamma_i$. Because $J_i$ is non-positive, we conclude that a negative $\gamma_i(t)$ implies an inward QFI subflow ($I_i > 0$), except the trivial case of $J_i = 0$. The temporary appearance of negative decay rates is already considered as the essential feature of the non-Markovian behaviors [17], here this is justifiﬁed through the return of the QFI. For the case that all $\gamma_i(t)$ are positive, the master equation [2] describes a so-called time-dependent Markovian quantum process [4, 12, 13, 18, 19], in which cases, $I$ always decreases. If the total QFI flow $I(t)$ is positive at time $t$, it signiﬁes at least one of $\gamma_i(t)$ is negative. In such cases, the QFI flows back to the open system and the non-Markovian behavior emerges.
Actually, like the trace distance used in Ref. [3], the dynamical return of the QFI is linked to the divisibility property of the dynamical map of quantum processes. If the master equation is of the form [2], the corresponding dynamical map is infinitely divisible provided that all $\gamma_i$ are positive [21]. In such cases, for arbitrary time $t > 0$, the dynamical map from time $t$ to $t + dt$ is a completely positive and trace-preserving map. Thus the QFI decreases during this time interval since the QFI is monotonically with respect to a completely positive and trace-preserving map [21].

Thirdly, there should be some restriction on the evolution of the QFI. It is seen from the above proof of the proposition that the coherent part of Eq. (2), i.e. $-i[H(t), \rho(t)]$, does not contribute to the total QFI flow directly. This observation directly leads to the no-cloning theorem in quantum information, which states that we cannot use unitary operations to evolve the states $|\psi(\theta)\rangle \otimes |0\rangle$ into $|\psi(\theta)\rangle \otimes |\psi(\theta)\rangle$ as a quantum copy [22]. This is because the QFI of the target states is twice as the one of the source states, due to the additivity of the QFI for the product states. Besides, if the total system (system plus environment) is assumed closed and the QFI is only distributed in the system initially, then the QFI of the reduced density matrix during evolution should not be greater than the one at the initial time, for the invariance of the QFI under unitary evolution and the non-increasing of the QFI under partial trace operation. This restriction should be reflected in the QFI flow obtained from a proper master equation.

III. TWO-LEVEL SYSTEM

Now we use an example of two-level system (qubit) to explicitly illustrate our discovery about the intrinsic relation between the QFI flow and the non-Markovianity of the open quantum system and its impaction to parameter estimation. In the quantum metrology context, the QFI gives a theoretical-achievable limit on the precision when estimating an unknown parameter, according to the QCR theorem [5]. To estimate the parameter as precisely as possible, we should optimize input states to maximize the QFI, and then optimize measurements to achieve the Cramér-Rao bound [5]. However, due to the interaction with environment, the QFI will change and affect the precision of the parameter estimation.

Here, the QFI-based parameter is assumed to be induced by a single-qubit phase gate $U_\phi := |g\rangle \langle g| + \exp(i\phi)|e\rangle \langle e|$ acting on the qubit, where $\phi = \theta$ is some inference parameters (see Fig. 1). To estimate the unknown parameter $\phi$ as precisely as possible, the optimal input state may be chosen as $|\psi_{\text{opt}}\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, which maximizes the QFI of the output state $U_\phi|\psi_{\text{opt}}\rangle$, see Ref. [9]. In the following model, after the phase gate operation and before the measurement performed, the qubit is assumed as an atom coupled to a reservoir consisting of harmonic oscillators in the vacuum. The total Hamiltonian of this typical model [1, 23, 24] reads

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger)$$

with $B = \sum_k g_k b_k$, where $\omega_0$ denotes the transition frequency of the atom with ground and excited states $|g\rangle$ and $|e\rangle$, and $\sigma_\pm$ the raising and lowering operators of atom; $b_k^\dagger$ and $b_k$ are respectively the creating and annihilation operators of the bath mode of frequencies $\omega_k$. $g_k$ denote the coupling constants. Then we consider Lorentzian spectral density $J(\omega) = \lambda W^2/\{\pi[(\omega_0 - \omega)^2 + \lambda^2]\}$, where $W$ is the transition strength, and $\lambda$ defines the spectral width of the coupling, which is related to the reservoir correlation time scale $\tau_B$ by $\tau_B = \lambda^{-1}$ [1, 23]. The Lorentzian spectral density describes the reservoir composed of lossy cavity, see Ref. [1]. The time-local master equation of the form (2) can be obtained exactly as follows [1]

$$\frac{\partial}{\partial t}\rho_S(t) = \gamma(t) \left( \sigma_- \rho_S(t) \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho_S(t)\} \right)$$

where $\gamma(t) = -2 \dot{h}(t)/h(t)$ with a crucial characteristic function [1]:

$$h(t) = \begin{cases} e^{-\lambda t/2} \left[ \cosh \left( \frac{d}{2} \right) + \frac{d}{2} \sinh \left( \frac{d}{2} \right) \right], & W \leq \frac{d}{2}, \\ e^{-\lambda t/2} \left[ \cos \left( \frac{d}{2} \right) + \frac{d}{2} \sin \left( \frac{d}{2} \right) \right], & W > \frac{d}{2}, \end{cases}$$

where $d = \sqrt{\lambda^2 - 4W^2}$.

Taking the initial state $U_\phi|\psi_{\text{opt}}\rangle$, the reduced density matrix of the atom obeys the master equation (11). Its solution is $\rho_S(t) = (I + B \sigma)/2$, where $B = (h(t) \cos \phi, -h(t) \sin \phi, h(t)^2 - 1)$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. In order to calculate the QFI flow, we first diagonalize this reduced density matrix as $\rho_S(t) = \sum_j p_j(t)|\psi_j(t)\rangle \langle \psi_j(t)|$. In this diagonal representation, the SLD with matrix elements $L_{ij} = 2\langle \psi_i | \partial_\phi \rho_S | \psi_j \rangle / (p_i + p_j)$ is obtained explicitly as

$$L(t) = ih(t) \left[ |\psi_1(t)\rangle \langle \psi_2(t)| - |\psi_2(t)\rangle \langle \psi_1(t)| \right].$$

Further, we have $J = -\text{Tr}(\rho [L, \sigma_-] \cdot [L, \sigma_-]) = -h(t)^2$. Then the exact solution for the QFI flow

$$I_\phi(t) = \gamma(t) J(t) = 2h(t) \dot{h}(t).$$
is obtained, which leads to $\mathcal{F}_\phi = h(t)^2$.

Therefore, the characteristic of the QFI flow is determined by the function $h(t)$, which has two very different kinds of behaviors. The corresponding properties of the QFI flow are shown in Fig. 2. In the weak coupling regime ($W < \lambda/2$), the function $\gamma(t)$ is always positive, thus the QFI is always lost during the time evolution of the open system. In the strong coupling regime ($W > \lambda/2$), the function $\gamma(t)$ takes on negative values within certain intervals of time, see Fig. 2 (d), which displays the non-Markovianity. Obviously in these time intervals, the QFI flow is inward. It is remarkable that although $\gamma(t)$ diverges at certain time, the QFI flow does not, see Figs. 2 (c) and (d). This is because the QFI flow is determined by two factors, $\gamma(t)$ and $\mathcal{J}(t)$.

**IV. CONCLUSION**

In summary, based on the QFI flow, we have proposed an information-theoretical approach for characterizing the time-dependent memory effect of environment for its surrounding open quantum systems. In this approach the Markovian process is considered as a QFI erasure process, and the return of the QFI, i.e. an inward QFI flow, clearly signatures the non-Markovian process. Using the time-local master equations, we have showed that for each fixed time $t > 0$ the QFI flow is decomposable according to different dissipative channels, and the direction of each sub-flow is determined by the sign of decay rates. With this decomposition form, the relationship between the temporary appearance of negative decay rate and the non-Markovian characteristic is justified. Although in the present work, the analysis of the QFI flow is based on a time-local master equation, the concept of the QFI flow may be still available in more general cases.

The present approach is associated with the current development of quantum metrology, which concerns on finding an optimal fashion to make high-resolution and highly sensitive measurement of physical parameters. Due to the interaction with environment in experiments, like the photon losses in the optical interferometry or the presence of quantum noise, the QFI will change and affect the precision of the parameter estimation. Therefore, it is worthy to study the dynamical evolution of the QFI in the context of quantum metrology, especially for non-Markovian processes.

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