On dual 3-brane actions with partially broken $N = 2$ supersymmetry

F.Gonzalez-Rey 1, I.Y. Park 2 and M. Roček 3

Institute for Theoretical Physics
State University of New York
Stony Brook, N. Y. 11794-3840

Abstract

The $N = 1$ superspace generalization of the 3-brane action in 6 dimensions with partially broken $N = 2$ supersymmetry can be constructed using $N = 1$ chiral, complex linear, or real linear superfields. The physical scalars of these multiplets give equivalent descriptions of the two transverse coordinates. The second supersymmetry is realized nonlinearly in all these actions. We derive the superspace brane actions and their nonlinear supersymmetry for both kinds of linear superfields when we break $N = 2$ supersymmetry spontaneously. This breaking is realized in the free action of hypermultiplets that live in $N = 2$ projective superspace by constraining the $N = 2$ multiplets to reduce them to a pure Goldstone multiplet. For the chiral superfield, the superspace brane action and nonlinear supersymmetry can be deduced by dualizing the brane action of either linear superfield. We find that the dual action is unique up to field redefinitions that introduce arbitrariness in the dependence on the auxiliary fields.

1email: glezrey@insti.physics.sunysb.edu
2email: ipark@insti.physics.sunysb.edu
3email: rocek@insti.physics.sunysb.edu
1 Introduction

The manifestly \( N = 1 \) supersymmetric form of 3-brane actions in six dimensions \([1]\) with a nonlinearily realized extended supersymmetry has been recently proposed by different authors \([2, 3]\). The origin of this additional symmetry can be traced to the spontaneous partial breaking of a linearly realized \( N = 2 \) supersymmetry \([4]\) in the action of hypermultiplets that live in projective superspace\([5]\). A similar partial supersymmetry breaking has also been studied in the \( N = 2 \) super Maxwell action \([6, 7, 8, 3]\). Imposing the additional constraint that the \( N = 1 \) chiral superfields contained in these \( N=2 \) multiplets are nilpotent, the \( N = 1 \) superspace action is uniquely determined for the \( N = 1 \) Maxwell and tensor multiplets \([3]\).

In this article, we study the partial breaking of \( N = 2 \) supersymmetry in the action of another off-shell description of the hypermultiplet (the \( O(4) \) projective multiplet). This description involves a chiral, a complex linear \([2]\) and a real auxiliary \( N = 1 \) superfield. Imposing the nilpotency constraint in the chiral component we find the corresponding \( N = 1 \) action of the complex linear multiplet with extended supersymmetry.

These actions with partially broken \( N = 2 \) supersymmetry are nonlinear functionals of real or complex linear superfields in the case of \( O(2) \) and \( O(4) \) hypermultiplets respectively. These fields can be dualized by introducing chiral Lagrange multipliers that impose the corresponding linearity constraint. Using the duality equations that relate the linear superfields and their chiral duals we derive the nonlinear action of the latter. This action is essentially the functional proposed in \([2, 3]\) up to terms depending on auxiliary fields that can be eliminated by a field redefinition.

We also compute the x-space components of the \( N = 1 \) complex linear superfield action with partially broken supersymmetry. We find that they are of the same form as the bosonic action computed for the dual chiral superfield. This is consistent with the fact that in this case the duality only involves the auxiliary components of the complex linear superfield: its physical degrees of freedom are the same as those of the chiral dual\([3]\).

Finally we study the effect of the superpotential deformation on the duality transformation of the \( O(4) \) multiplet. We find that the dual action is unchanged after rescaling the dual chiral field by a constant that depends on the deformation parameter.

2 Partial breaking of \( N = 2 \) supersymmetry in the hypermultiplet action

In this section we review the partial breaking of \( N = 2 \) supersymmetry in the free action of \( O(2) \) hypermultiplets \([3]\). Then we derive a similar partial breaking in the \( O(4) \) case. This last case can be straightforwardly generalized to the generic \( O(2p) \) multiplet but the qualitative features of the symmetry breaking are the same and therefore we will restrict our analysis to the \( O(4) \) multiplet.

\(^1\)For a review of Projective superspace see \([5]\) and references therein.
\(^2\) The complex linear multiplet was first discussed in \([9]\). The on-shell description of a \( N = 2 \) hypermultiplet involving a chiral and a complex linear multiplet was discussed in \([10]\).
\(^3\) In contrast, the real linear superfield has a bosonic action that can be expressed in terms of dual gauge tensor fields \([3]\).
The $O(2p)$ hypermultiplet can be parameterized using a complex coordinate $\zeta$

$$\eta = \sum_{n=-p}^{+p} \eta_n \zeta^n .$$  \hspace{1cm} (1)

The component superfields obey the constraints

$$Q_\alpha \eta_n = -D_\alpha \eta_{n+1} , \quad \bar{Q}_\dot{\alpha} \eta_n = \bar{D}_{\dot{\alpha}} \eta_{n-1} , \quad (2)$$

and these guarantee that any action of the type

$$S_{O(2p)} = (-)^p \frac{1}{2} \int d^4x \, D^2\bar{D}^2 \int \frac{d\zeta}{2\pi i \zeta} F(\eta, \zeta) \quad \hspace{1cm} (3)$$

is off-shell $N = 2$ supersymmetric. In particular for the $O(2)$ multiplet

$$\eta = \bar{\Phi} \zeta + G = \Phi \zeta$$  \hspace{1cm} (4)

the constraints imply $\bar{D}_{\dot{\alpha}} \Phi = 0 = D^2 G = \bar{D}^2 G$, i.e., $\eta$ contains an $N = 1$ chiral scalar superfield, its conjugate, and an $N = 1$ real linear superfield (also known as tensor multiplet). We explicitly break the second supersymmetry while we preserve $N = 1$ and Lorentz invariance by giving $\bar{\Phi}$ and $G$ a nonzero v.e.v.

$$\bar{\Phi} \equiv \bar{\Phi} + (\theta^2)^2 , \quad \langle \bar{\Phi} \rangle = 0$$
$$G \equiv G - \theta^{1_a} \theta^{2_\dot{\alpha}} - \bar{\theta}^{1_\dot{\alpha}} \bar{\theta}^{2_\dot{\alpha}} , \quad \langle G \rangle = 0 . \quad \hspace{1cm} (5)$$

The constraints (4) can be rewritten in terms of the shifted $N = 1$ chiral and tensor superfields\footnote{For notational simplicity we write the supercovariant derivatives of both supersymmetries as follows $D_{1\dot{\alpha}} = D_\alpha , \quad D_{2\dot{\alpha}} = Q_\dot{\alpha}$.}

$$Q_\alpha \bar{\Phi} = -D_\alpha G$$
$$Q_\alpha G = \theta^{1_\dot{\alpha}} + D_\alpha \bar{\Phi}$$
$$Q^2 \bar{\Phi} - 1 = -D^2 \bar{\Phi} . \quad \hspace{1cm} (6)$$

If we impose the additional constraint $\Phi^2 = 0 = \bar{\Phi}^2$ we find

$$Q^2 \left( \frac{1}{2} \bar{\Phi}^2 \right) = 0 \implies \bar{\Phi} = D^2 \left( \bar{\Phi} \Phi - \frac{1}{2} G^2 \right) \implies \bar{\Phi} = -\frac{1}{2} \frac{(D^2 G)(D_\alpha G)}{1 - D^2 \bar{\Phi}} \quad \hspace{1cm} (7)$$

and the corresponding conjugate. This implies that any action of the form $S_{O(2)}$ (3), including the free action $\int d^4x \, d^4 \theta (\Phi \Phi - \frac{1}{2} G^2)$, is proportional to $\int d^4x \, d^2 \theta \Phi (G)$. An illuminating form of the action is

\footnote{We follow the superspace conventions of [11]. In particular $D_\alpha \theta^2 = \delta_\alpha^\beta$.}
\[ S_{O(2)} = \int d^4x \, D^2 \bar{D}^2 \left( \frac{1}{4} \frac{D^\alpha G D_\alpha G \bar{D}^\dot{\alpha} G \bar{D}_\dot{\alpha} G}{(1 - D^2 \Phi)(1 - D^2 \bar{\Phi})} - \frac{1}{2} G^2 \right). \]  

(8)

Since the first term in the action (8) contains the maximum number of fermionic superfields \( \Psi = DG, \bar{\Psi} = \bar{D}G \), we can replace \( \bar{D}^2 \bar{\Phi} \) and \( D^2 \Phi \) in the denominator by the solution of the system of equations

\[ \bar{D}^2 \bar{\Phi} = \frac{1}{2} \frac{(D^\alpha D^\alpha G)(D_\alpha \bar{D}_\alpha G)}{1 - D^2 \Phi} + O(\Psi) \]
\[ D^2 \Phi = \frac{1}{2} \frac{(D^\alpha D^\dot{\alpha} G)(D_\alpha \bar{D}_\dot{\alpha} G)}{1 - D^2 \Phi} + O(\bar{\Psi}), \]  

(9)

where we have dropped terms linear or quadratic in the fermionic fields. The solution to the resulting quadratic equation must be chosen to avoid giving a vacuum expectation value to the bosonic superfield \( D^2 \Phi \), as that would break \( N = 1 \) supersymmetry. With such a choice the resulting nonlinear action is [2, 3]

\[ S_G = \int d^4x \, d^4 \theta \left( -\frac{1}{2} G^2 + 2 \Psi^2 \bar{\Psi}^2 f \right). \]  

(10)

We follow the notation in [3, 4]

\[ \Psi^2 \equiv \frac{1}{2} D^\alpha G D_\alpha G \]  
\[ f \equiv \frac{1}{1 - A + \sqrt{1 - 2A + B^2}} \]
\[ A \equiv \frac{1}{2} \left( (D \bar{D} G)^2 + (\bar{D} D G)^2 \right) \]
\[ B \equiv \frac{1}{2} \left( (D \bar{D} G)^2 - (\bar{D} D G)^2 \right). \]

This action is by construction invariant under the nonlinear supersymmetry transformations

\[ \delta G = (\epsilon^\alpha Q_\alpha + \epsilon^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}})G = \epsilon^\alpha (\theta_\alpha + D_\alpha \Phi(G)) + \epsilon^{\dot{\alpha}} (\bar{\theta}_{\dot{\alpha}} + \bar{D}_{\dot{\alpha}} \bar{\Phi}(G)). \]  

(12)

Now we turn our attention to other off-shell representations of the \( N = 2 \) algebra describing the same degrees of freedom, and try to implement a similar partial breaking of \( N = 2 \) supersymmetry on their free action. The simplest of these representations is the projective \( O(4) \) multiplet. The projective constraints (2) tell us that this multiplet contains a \( N = 1 \) chiral superfield and its conjugate, a \( N = 1 \) complex linear superfield [3, 10] and its conjugate, and a real unconstrained superfield

\[ \eta = \frac{\bar{\Phi}}{\zeta^2} + \frac{\bar{\Sigma}}{\zeta} + X - \Sigma \zeta + \Phi \zeta^2, \quad D_\alpha \bar{\Phi} = 0 = D^2 \Sigma, \quad \bar{X} = X. \]  

(13)
A vacuum expectation value for the chiral field and the complex linear field induces modified constraints as before

\[ \bar{\Phi} = \bar{\Phi} + (\theta^2)^2 \quad \Rightarrow \quad Q_\alpha \bar{\Phi} = -D_\alpha \bar{\Sigma} \]
\[ \bar{\Sigma} = \bar{\Sigma} - \theta^\alpha \theta^\beta \quad \Rightarrow \quad Q_\alpha \bar{\Sigma} = \theta^\alpha - D_\alpha X \]
\[ Q^2 \bar{\Phi} = 1 + D^2 X \, . \quad (14) \]

The additional constraint \( \bar{\Phi}^2 = 0 \) once again gives a solution in terms of the fermionic superfields \( \tilde{\Psi}_\alpha = D_\alpha \bar{\Sigma}, \tilde{\bar{\Psi}}_\dot{\alpha} = \bar{D}_{\dot{\alpha}} \Sigma \) and the auxiliary field \( X \)

\[ \bar{\Phi} = -\frac{1}{2} \tilde{\Psi}_\alpha \tilde{\bar{\Psi}}_\dot{\alpha} \, . \quad (15) \]

In this case we have the auxiliary superfield \( X \) in the denominator and we cannot eliminate it as easily as before. We may, however, substitute (15) into the free action of the \( O(4) \) multiplet

\[ S_{O(4)} = \int d^4x d^4\theta \left( \Phi \bar{\Phi} - \Sigma \bar{\Sigma} + \frac{1}{2} X^2 \right) \]
\[ = \int d^4x d^4\theta \left( -\Sigma \bar{\Sigma} + \frac{1}{4(1 + D^2 X)(1 + \bar{D}^2 X)} \right) + \frac{1}{2} X^2 \, . \quad (16) \]

and then obtain the algebraic field equation of the auxiliary \( X \)

\[ X = \frac{-1}{4(1 + D^2 X)(1 + \bar{D}^2 X)} \left( \tilde{\Psi}_\alpha \tilde{\bar{\Psi}}_\dot{\alpha} \frac{(\bar{D} D \Sigma)^2}{1 + D^2 X} + \tilde{\bar{\Psi}}_\dot{\alpha} \tilde{\Psi}_\alpha \frac{(D D \Sigma)^2}{1 + \bar{D}^2 X} \right) + O(\tilde{\Psi}^2 \Psi, \tilde{\bar{\Psi}}^2 \bar{\Psi}) \, . \quad (17) \]

We have dropped terms with more than two fermionic superfields because they do not contribute to the term \( X^2 \) in the action. For the same reason the term with four fermionic fields picks up the purely bosonic part of \( D^2 X \) and \( \bar{D}^2 X \)

\[ D^2 X = \frac{1}{4(1 + D^2 X)(1 + \bar{D}^2 X)} \frac{(\bar{D} D \Sigma)^2}{1 + D^2 X} + O(\tilde{\Psi}, \bar{\Psi}) \]
\[ \bar{D}^2 X = \frac{1}{4(1 + D^2 X)(1 + \bar{D}^2 X)} \frac{(D D \Sigma)^2}{1 + \bar{D}^2 X} + O(\tilde{\bar{\Psi}}, \tilde{\bar{\Psi}}) \, . \quad (18) \]

From this identity we learn that the purely bosonic part of this superfield is real \( D^2 X = \bar{D}^2 X \).

For notational simplicity we work from now on with the shifted real field

\[ Y = 1 + D^2 X \, , \quad (19) \]

which obeys

\[ Y^3(Y + 1) = \frac{1}{4}(\bar{D} D \Sigma)^2(\bar{D} D \Sigma)^2 \, . \quad (20) \]
It is possible to solve this equation for \( Y \) and choose the root with unit \( v.e.v. \)

\[
Y = \frac{1}{4} + \frac{\sqrt{\mathcal{Z}}}{12} + \frac{\sqrt{6}}{12} \left( \frac{3\mathcal{Z}^2 - \mathcal{Z}^2 - 12(D\bar{D}\Sigma)^2(D\bar{D}\Sigma)^2}{\mathcal{Z}^2} \right) \left( \frac{\mathcal{Z}^2 + 9\mathcal{Z}}{\mathcal{Z}^2} \right)
\]

where

\[
\mathcal{Z} = -27(D\bar{D}\Sigma)^2(D\bar{D}\Sigma)^2 + 3\sqrt{81 ((D\bar{D}\Sigma)^2(D\bar{D}\Sigma)^2)^3 + 192 ((D\bar{D}\Sigma)^2(D\bar{D}\Sigma)^2)^3}
\]

\[
\mathcal{V} = 9 + 6\mathcal{Z} + 72\frac{(D\bar{D}\Sigma)^2(D\bar{D}\Sigma)^2}{\mathcal{Z}^2} .
\]

However, we find it more useful to maintain the dependence of the action on the real superfield \( Y \). The resulting nonlinear action is

\[
S_{O(4)} = \int d^4x \ d^4\theta \left( -\Sigma\Sigma + \bar{\Psi}^2\bar{\Psi}2Y + \frac{1}{2}Y^3 \right) ,
\]

and by construction it is also invariant under the nonlinear supersymmetric transformations

\[
\delta\Sigma = (e^\alpha Q_\alpha + \bar{e}^\dot{\alpha} \bar{Q}_{\dot{\alpha}})\Sigma = e^\alpha D_\alpha \Phi(\Sigma, \bar{\Sigma}) + \bar{e}^{\dot{\alpha}} \left( \bar{\theta}_{\dot{\alpha}} - \bar{D}_{\dot{\alpha}} X(\Sigma, \bar{\Sigma}) \right) .
\]

It is interesting to note that in this case we can add a superpotential deformation proportional to the constrained chiral superfield, which is still \( N = 2 \) supersymmetric but not the same as the nonlinear action

\[
S_\Phi = \frac{\beta}{2} \int d^4x \ d^4\theta \int \frac{d\zeta}{2\pi i\zeta} \eta^2 \left( \zeta^2 + \frac{1}{\zeta^2} \right) = \beta \int d^4x D^2 \bar{D}^2 \left( \Phi X + \frac{1}{2} \Sigma\Sigma + c.c. \right)
\]

\[
= \beta \int d^4x D^2 \left( \Phi \bar{D}^2 X + \bar{\Psi}^2 \right) + c.c. = \beta \int d^4x D^2 \left( \Phi \bar{D}^2 X - \Phi(1 + \bar{D}^2 X) ) + c.c. ) \right.
\]

\[
= -\beta \int d^4x D^2 \Phi .
\]

In the last section we will prove that adding this superpotential deformation only amounts to a change in the normalization of the dual chiral Lagrange multiplier and the appearance of an overall factor multiplying the dual action. Such rescaling is well defined for any value of the parameter \( \beta \) except \( \beta = \pm 1 \) where it becomes singular.

### 3 Dual nonlinear actions

It is well known that the \( N = 1 \) supersymmetric actions of real linear and complex linear superfields can be dualized into actions of chiral fields describing the same on-shell degrees of freedom. For the real linear superfield this is a true T-duality that relates different physical x-space fields of the corresponding nonlinear sigma models. For the complex linear superfield

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\( ^6 \) The pure free \( N = 1 \) complex linear superfield action including a quadratic term \( \beta(\Sigma^2 + \bar{\Sigma}^2)/2 \), was first considered in [10], where it was observed that the value \( \beta = \pm 1 \) is critical and needs special treatment.
it is a pseudo-duality that merely changes the auxiliary fields without affecting the physical ones.

The duality transformation is performed by relaxing the linearity constraint on the linear superfield. Simultaneously we add to its action chiral Lagrange multipliers that enforce the constraint. When we impose the field equations of the unconstrained superfield, we obtain the dual action of the chiral Lagrange multiplier.

If we apply this transformation to nonlinear actions of linear superfields, the corresponding duality equations are also nonlinear. In general it is difficult to find the dual nonlinear action explicitly. However, for the actions with partially broken $N = 2$ supersymmetry that we described above, the presence of fermionic superfields in the nonlinear term simplifies the problem.

3.1 Tensor multiplet

We begin studying the duality transformation on the real linear superfield. We relax the linearity constraint and we add a Lagrange multiplier that in this case is necessarily real

$$
\begin{align*}
S &= \int d^4x d^4\theta \left( -\frac{1}{2} G^2 + G(\phi + \bar{\phi}) + 2\Psi^2\bar{\Psi}^2 f \right) \\
&= \int d^4x d^4\theta \left( \phi\bar{\phi} - \frac{1}{2}(G - \phi - \bar{\phi})^2 + 2\Psi^2\bar{\Psi}^2 f \right)
\end{align*}
$$

(26)

Note that once we have relaxed the linearity constraint on $G$, there is large class of actions which are equivalent modulo terms containing $D^2G$ and $\bar{D}^2G$. They all reduce to the same nonlinear action when we eliminate the Lagrange multiplier. Such terms can be reabsorbed in to a redefinition of the chiral Lagrange multiplier

$$
G(\phi + \bar{\phi}) + F(G)D^2G + \bar{F}(G)\bar{D}^2G \longrightarrow G \left( \phi + \bar{D}^2F + \bar{\phi} + D^2\bar{F} \right).
$$

(27)

Later we will make use of this freedom to redefine the dual chiral field and simplify the dual action. The duality equations that allow us to find $G(\phi, \bar{\phi})$ are derived from

$$
\frac{\delta S}{\delta G} = 0 \Rightarrow \phi + \bar{\phi} = G + \Psi^2D^2GK + \bar{\Psi}^2\bar{D}^2G\bar{K}
\text{+} \Psi^\alpha\bar{\Psi}^\dot{\alpha} \left( (\bar{D}_\dot{\alpha}D_\alpha G)H - (D_\alpha\bar{D}_\dot{\alpha} G)\bar{H} \right) + O(\Psi^2\bar{\Psi}, \Psi\bar{\Psi}^2),
$$

(28)

where the functionals $K$ and $H$ depend on the bosonic superfields $A, B$, (see (12)) and

$$
\begin{align*}
Q &\equiv (D^2G)(\bar{D}^2G) \\
P &\equiv (D_\alpha\bar{D}^\alpha)(\bar{D}_\dot{\alpha}D_\alpha G).
\end{align*}
$$

(29)

$K$ and $H$ are given explicitly in the appendix A. We have dropped terms with more than two fermionic fields because they do not contribute to the action.
\((G - \phi - \bar{\phi})^2 = \Psi^2 \bar{\Psi}^2 \left(2QK\bar{K} + \frac{1}{2}PH\bar{H} - \frac{1}{4}(A + B)\bar{H}^2 - \frac{1}{4}(A - B)H^2 \right). \tag{30}\)

In the action (29) we have two terms with the maximum number of fermionic fields, multiplied by a function of the bosonic fields in (29).

\[S = \int d^4x \ d^4\theta \left(\phi \bar{\phi} + \Psi^2 \bar{\Psi}^2 \left[2f - QK - \frac{1}{4}PH\bar{H} + \frac{1}{8}(A + B)H^2 + \frac{1}{8}(A - B)H^2 \right] \right). \tag{31}\]

Acting with \(N = 1\) spinor derivatives on equation (28) we find the duality equations relating the fermionic field \(\psi \equiv D\phi\) to \(\Psi\) and its conjugate

\[
\psi_\alpha = \Psi_\alpha \left(1 - QK + \frac{1}{2}(A + B)\bar{H} \right) - \Psi_\beta (\bar{D}^\dot{\alpha}D^\alpha G)(D_\dot{\alpha}D_\alpha G)H \\
+ \bar{\Psi}^\alpha D^2 G \left((D_\dot{\alpha}D_\alpha G)(\bar{H} - \bar{K}) - (\bar{D}_\dot{\alpha}D_\alpha G)\bar{H}\right) + O(\Psi^2, \bar{\Psi}^2, \Psi \bar{\Psi}) \tag{32}\]

We have again dropped terms with too many fermionic fields which do not contribute to the four fermion factor in (31). Using the fermionic duality equation (32) this factor can be rewritten as the product of four dual fermionic fields multiplying some complicated function \(1/M\) (see (96)) of the bosonic superfields

\[\Psi^2 \bar{\Psi}^2 = \frac{\psi^2 \bar{\psi}^2}{M(P, Q, A, B)}. \tag{33}\]

Acting once more with spinor derivatives on (32) and dropping all terms with fermionic fields we obtain nonlinear equations

\[
D^2\phi = (D^2 G \left(1 - 4Qf - 2Pf \right) + O(\Psi, \bar{\Psi}) \tag{34}\]
\[
\bar{D}_\dot{\alpha}D_\alpha \phi = (\bar{D}_\dot{\alpha}D_\alpha G \left(1 + Q(H - K) + \frac{1}{2}(A + B)\bar{H} \right) \\
- (D_\dot{\alpha}D_\alpha G \left(\bar{Q}(\bar{H} - \bar{K}) + \frac{1}{2}(A - B)H \right) + O(\Psi, \bar{\Psi})
\]

and their corresponding complex conjugates. Squaring these equations and taking the product of the last one and its conjugate we find the relation between the bosonic fields \(P, Q, (A + B), (A - B)\) and the corresponding duals

\[
q = D^2\phi \bar{D}^2\bar{\phi} \\
p = -\partial\phi \partial\bar{\phi} \\
a + b = (i\partial\phi)^2 \\
a - b = (i\partial\bar{\phi})^2 \tag{35}\]

7
up to fermionic superfields which again do not contribute to $\psi^2 \bar{\psi}^2 / M(P, Q, A, B)$. These relations are remarkably simple for $p$ and $q$

$$A + P = a + p$$

$$q = Q \left(1 - 2P f - 4Q f\right)^2,$$

while for $a + b$ they are more complicated nonlinear equations (we omit the obvious conjugate equation for $a - b$)

$$a + b = (i\partial\phi)^2$$

$$= (A - B) \left[1 + Q(H - K) + \frac{(A + B)}{2} H\right]^2 + (A + B) \left[Q(\bar{H} - \bar{K}) + \frac{(A - B)}{2} H\right]^2$$

$$- 2P \left[1 + Q(H - K) + \frac{(A + B)}{2} H\right] \left[Q(\bar{H} - \bar{K}) + \frac{(A - B)}{2} H\right].$$

We can eliminate the variable $P$ using the first equation, and we are left a system of three coupled nonlinear equations in $Q, A$ and $B$. Finding a closed solution is a difficult task since we do not know in general how to invert this change of variables. One thing we can do is to solve these equations iteratively to find the expansion of $Q, A, B$ to any given order in $p, q, a, b$.

A different strategy that has proved more successful is to notice that the system of equations can be solved exactly if we set $q = 0 = Q$

$$A_0 = -\frac{a(1 - p) + a^2 - b^2}{(1 - p)^2 - (a^2 - b^2)}$$

$$B_0 = \frac{b}{\sqrt{(1 - p - a) [(1 - p)^2 - (a^2 - b^2)]}}.$$  

We also notice from the duality equation (37) that the bosonic field $Q$ is always at least linear in its dual $q$

$$Q = \frac{q}{(1 - 2(a + p - A f - 4fQ))^2}.$$  

Assuming that the solutions $A(q, p, a, b), B(q, p, a, b)$ and $Q$ have a Taylor series expansion in $q$

$$A = \sum_{i=0}^{+\infty} q^i A_i(p, a, b)$$

$$B = \sum_{i=0}^{+\infty} q^i B_i(p, a, b)$$

$$Q = \sum_{i=1}^{+\infty} q^i Q_i(p, a, b),$$

(41)
we may solve for $Q$ iteratively

$$Q_1 = \frac{q}{(1 - 2(a + p - A_0)f_0)^2}, \quad (42)$$

where $f_0 = f(A_0, B_0)$. We can substitute the Taylor series expansion of $Q$, $A$, and $B$ on the r.h.s. of (38). Since the l.h.s. is $q$-independent, each order $n$ in $q$ on the r.h.s. gives equations for $A_{i\leq n}, B_{j\leq n}$ that can be solved.

Finally we replace the Taylor expansions $Q = \sum_{i=0}^{\infty} q^i Q_i(p,a,b), A = \sum_{i=0}^{\infty} q^i A_i(p,a,b)$, and $B = \sum_{i=0}^{\infty} q^i B_i(p,a,b)$ in the action (31) to find

$$S_{\text{dual}} = \int d^4x d^4\theta \left( \phi \bar{\phi} + \psi^2 \bar{\psi}^2 \left[ L_0(p,a,b) + q L_1(p,a,b) + q^2 L_2(p,a,b) + \ldots \right] \right). \quad (43)$$

We are able to derive the dual action as a Taylor series in the auxiliary superfield $q = (D^2 \phi)(\bar{D}^2 \bar{\phi})$. Its coefficients are analytic functions of the dual bosonic field $p$ and the real combination $a^2 - b^2$. The $q$-independent part of the action exactly agrees with the form proposed in [3]

$$S_{\text{dual}}(q = 0) = \int d^4x d^4\theta \left( \phi \bar{\phi} + \frac{2\psi^2 \bar{\psi}^2}{1 + \partial \phi \partial \bar{\phi} + \sqrt{(1 + \partial \phi \partial \bar{\phi})^2 - (\partial \phi)^2 (\partial \bar{\phi})^2}} \right). \quad (44)$$

The coefficient linear in $q$ is also remarkably simple

$$L_1 = \frac{2}{(1-p+r)^2} \frac{(1+r)^2}{2r^2 + r - 1 + p}, \quad (45)$$

where $r$ is the square root appearing in $L_0$

$$r = \sqrt{(1-p)^2 - (a^2 - b^2)} = \sqrt{(1 + \partial \phi \partial \bar{\phi})^2 - (\partial \phi)^2 (\partial \bar{\phi})^2}. \quad (46)$$

Higher order coefficients become increasingly complicated, but the key feature is that they remain nonsingular around the zero v.e.v. of the bosonic superfields. That will prove crucial when we perform a field redefinition of $\phi$ to eliminate the $q$ dependent terms in (43).

It is natural to expect that their presence in the dual $N = 1$ nonlinear action is somewhat arbitrary, because they are completely irrelevant in the computation of the physical $x$-space bosonic action. They only contribute auxiliary field terms that are always quadratic or higher in the auxiliary field, and they can therefore be set to zero as a solution to their algebraic field equations.

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7 In the appendix B we explain how this is done in some detail, when we solve similar duality equations for the complex linear multiplet.
3.2 Complex linear multiplet

We have found a nonlinear \( N = 1 \) action of a chiral field by dualizing the real linear superfield. It has an on-shell nonlinear \( N = 2 \) supersymmetry that can be derived by defining compensating transformation laws for \( \phi \) in (23): The variation of \( \phi \) and its conjugate must cancel the additional terms appearing in \( \delta S \) when we relax the constraint on the tensor superfield.

Now we want to derive the nonlinear action of a chiral \( N = 1 \) multiplet dual to the complex linear superfield of (23). We follow a very similar argument, though the algebra and the equations are somewhat simplified by the fact that we have kept some of the \( \Sigma \)-dependence implicit in the function \( Y \). As before, we relax the constraint on the linear superfield and add complex Lagrange multipliers to (23)

\[
S = \int d^4x d^4\theta \left( -\Sigma \bar{\Sigma} + \Sigma \phi + \bar{\Sigma} \bar{\phi} + \bar{\Psi} \bar{\bar{\Psi}} \frac{2Y + 1}{Y^3} \right)
\]

\[
= \int d^4x d^4\theta \left( \phi \bar{\phi} - (\Sigma - \bar{\phi})(\bar{\Sigma} - \phi) + \bar{\Psi} \bar{\bar{\Psi}} \frac{2Y + 1}{Y^3} \right).
\]

(47)

The basic duality relations are now

\[
\frac{\delta S}{\delta \Sigma} = 0 \Rightarrow \phi = \bar{\Sigma} + \bar{\Psi} \bar{\bar{D}} \Sigma \frac{2}{Y^2} + \bar{\Psi} \bar{\bar{D}} \Sigma \left( \frac{(\bar{D} D \Sigma)^2 (D^a \bar{D}^a \Sigma) (\bar{D} a D_a \Sigma)}{2Y^6} \right)
\]

\[+ \bar{\Psi} \bar{\bar{D}} \bar{\Sigma} \frac{2}{Y^2} + (D a D a \Sigma) \frac{\bar{D} D \Sigma \bar{D} D \Sigma D D \Sigma}{2Y^6} \right] + O(\bar{\Psi} \bar{\bar{D}} \psi, \bar{\Psi} \bar{\Sigma}^2),
\]

where we have used (20) implicitly. To simplify the notation we define bosonic superfields analogous to those in (29)

\[
Q = (D^2 \Sigma)(\bar{D}^2 \Sigma)
\]

\[
P = (D^a \bar{D}^a \Sigma)(\bar{D} a D_a \Sigma)
\]

\[
A = \frac{1}{2} \left( (D \bar{D} \Sigma)^2 + (\bar{D} D \Sigma)^2 \right)
\]

\[
B = \frac{1}{2} \left( (D \bar{D} \Sigma)^2 - (\bar{D} D \Sigma)^2 \right),
\]

(49)

although in fact the bosonic fields \( A \) and \( B \) always appear in the real combination \( A^2 - B^2 = 4(Y^4 + Y^3) \). The presence of fermionic superfields in the duality equations simplifies again our calculation of the Lagrange multiplier term

\[
(S - \bar{\phi})(\bar{\Sigma} - \phi) = \bar{\Psi} \bar{\bar{D}} \bar{\Sigma} \left( 4Q \frac{2Y + 1}{Y^5} + \frac{P}{Y^4} + QP(Q + P) \frac{Y + 1}{Y^9} \right).
\]

(50)

Combining the last two terms in (47) we obtain

\[
S = \int d^4x d^4\theta \left( \phi \bar{\phi} + \bar{\Psi} \bar{\bar{D}} \bar{\Sigma} \left[ \frac{2Y + 1}{Y^3} \left( 1 - \frac{4Q}{Y^2} \right) \right. \right. \left. \left. - \frac{P}{Y^4} - QP(Q + P) \frac{Y + 1}{Y^9} \right] \right).
\]

(51)
The relevant fermionic fields are again linear in their duals, as follows from acting with spinor derivatives on \( \psi_\beta \equiv D_\beta \phi = \left( 1 - Q \frac{3Y + 1}{Y^3} \right) \bar{\Psi}_\beta - \frac{1}{Y^2} (D^\alpha \bar{D^\alpha} \Sigma) (D_\beta \bar{D}_\alpha \Sigma) \bar{\Psi}_\alpha \) \( \psi_\beta \equiv D_\beta \phi = \left( 1 - Q \frac{3Y + 1}{Y^3} \right) \bar{\Psi}_\beta - \frac{1}{Y^2} (D^\alpha \bar{D^\alpha} \Sigma) (D_\beta \bar{D}_\alpha \Sigma) \bar{\Psi}_\alpha \) (52)

\[ -D^2 \Sigma \left( (D \bar{D} \Sigma)^2 \frac{Q + P}{2Y^6} (D_\beta \bar{D}_\alpha \Sigma) + \frac{1}{Y^2} (D_\alpha \bar{D}_\beta \Sigma) \right) \bar{\Psi}^\alpha + O(\bar{\Psi}^2, \bar{\psi}^2, \bar{\psi}) \].

We can rewrite the four fermion numerator as before

\[ \bar{\psi}^2 \psi^2 = \psi^2 \bar{\psi}^2 \frac{1}{N(Q, P, Y)} \],

where the function \( N \) is defined in the appendix A (53). Acting once more with spinor derivatives on (52)

\[ D^2 \phi = D^2 \Sigma \left( 1 - \frac{P Y^2 + 1}{Y^3} - 2Q \frac{2Y + 1}{Y^3} \right) + O(\bar{\Psi}, \bar{\psi}) \] (54)

\[ \bar{D}_\alpha D_\alpha \phi = (\bar{D}_\alpha D_\alpha \Sigma) \left( 1 - \frac{Q}{2Y^3} - 2Q \frac{2Y + 1}{Y^3} \right) - (D_\alpha \bar{D}_\alpha \Sigma) \left( \frac{D \bar{D} \Sigma}{2Y^2} \right) \left( 1 - \frac{Q + P}{Y^4} \right) + O(\bar{\Psi}, \bar{\psi}) \]

we square these equations and compute the product of the last one and its conjugate to find the duality equations relating the bosonic superfields \( Q, P \) and \( Y \) to the duals \( q, p \) and \( a^2 - b^2 \). The relations we find are very different because of the complex nature of the linear superfield, but the equation defining the auxiliary superfield \( q \) has the same structure

\[ q = Q \left( 1 - \frac{P}{Y^3} - 2Q \frac{2Y + 1}{Y^3} \right)^2 \].

This is important because we may once again solve for \( Q = Q(q) \) iteratively and apply the same procedure as in the \( O(2) \) multiplet case to solve the nonlinear equations of \( P(q, p, a^2 - b^2) \) and \( Y(q, p, a^2 - b^2) \) order by order in \( q \) (see appendix B). Substituting this solution on the action (51) we find once more its Taylor series expansion in \( q \)

\[ S_{\text{dual}} = \int d^4 x d^4 \theta \left( \phi \bar{\phi} + \frac{2\psi^2 \bar{\psi}^2}{1 + \partial \phi \bar{\partial} \phi + \sqrt{(1 + \partial \phi \bar{\partial} \phi)^2 - (\partial \phi)^2 (\partial \bar{\phi})^2}} \right) \]

\[ + q \frac{2\psi^2 \bar{\psi}^2}{(1 - p + r)^2} \frac{p^2 + p - 2 - 2(1 - p) r^2 + (3p - 4) r}{3(1 - p)^2 + (4p - 5) r^2 - 2(1 - p)r} + O(q^2) \].

The fact that the \( q \)-independent part of the action is the same as in the tensor multiplet case does not come as a surprise. Both the \( O(2) \) multiplet and the \( O(4) \) multiplet free action are dual to the same free action of two chiral fields that realizes \( N = 2 \) supersymmetry on-shell. The disagreement on the \( q \)-dependent terms is surprising and might appear disturbing; however, as we see below, the \( q \)-dependent terms are irrelevant since they can be removed by a field redefinition. This is the subject of our next section.
4 Redefinition of the dual Chiral field

We have found the nonlinear actions of a chiral multiplet dual to a tensor multiplet and a complex linear multiplet. They are both of the form

\[ S_{\text{dual}} = \int d^4x d^4\theta \left( \phi \bar{\phi} + \psi^2 \bar{\psi}^2 \left[ \mathcal{L}_0 + q\mathcal{L}_1 + q^2\mathcal{L}_2 + \ldots \right] \right). \]  

(57)

Can we find a field redefinition of the dual chiral field that preserves the first two terms in this action while we change the rest? A suitable redefinition is the following

\[ \phi \equiv \varphi + D^2 \left( \bar{\psi}^2 \psi^2 \left( D^2 \varphi \right) h(\hat{q}, \hat{p}, \hat{c}) \right), \]  

(58)

where \( \varphi \) is a new chiral field and we define the superfields

\[ \hat{\psi}_\alpha \equiv D_\alpha \varphi \]
\[ \hat{q} \equiv D^2 \varphi \bar{D}^2 \bar{\varphi} \]
\[ \hat{p} \equiv -\partial \varphi \partial \bar{\varphi} \]
\[ \hat{c} \equiv \hat{a}^2 - \hat{b}^2 \equiv (\partial \varphi)^2 (\partial \bar{\varphi})^2. \]

(59)

The real function \( h \) is assumed to have a Taylor series expansion in \( \hat{q} \) with coefficients which are analytic functions of \( \hat{p} \) and \( \hat{c} \) around their vanishing v.e.v.

\[ h(\hat{q}, \hat{p}, \hat{c}) = \sum_{i=0}^{+\infty} \hat{q}^i h_i(\hat{p}, \hat{c}). \]  

(60)

We must now rewrite the action (57) in terms of the new chiral field. The nice property of this peculiar field redefinition is that it involves fermionic superfields and we can therefore apply the same simplifying arguments that have proved so useful throughout. The kinetic term transforms into

\[ \phi \bar{\phi} = \varphi \bar{\varphi} + \bar{\psi}^2 \psi^2 \hat{q} \left[ 2h + h^2 \left( \hat{q}(\hat{q} + \hat{p}) + \frac{\hat{c}}{4} \right) \right]. \]  

(61)

Note that the transformation of the kinetic term does not introduce unwanted \( \hat{q} \)-independent terms. As before, when we compute \( \psi \) in terms of \( \hat{\psi} \) we only keep terms which are linear in the redefined fermionic field

\[ \psi_\alpha \equiv (1 + \hat{q}^2 h) \hat{\psi}_\alpha + \hat{q} h (\bar{D}_{\dot{\beta}} D^\beta \varphi)(D_\alpha \bar{D}_{\dot{\beta}} \bar{\varphi}) \hat{\psi}_{\dot{\beta}} \]

\[ + D^2 \varphi h \left( \hat{q}(\bar{D}_\alpha D_\varphi) - \frac{1}{2}(\partial \varphi)^2 (D_\alpha \bar{D}_\varphi) \right) \bar{\psi}^\dot{\alpha} + O(\bar{\psi}^2, \psi^2, \bar{\psi} \psi), \]  

(62)

because we are only interested in computing

\[ \psi^2 \bar{\psi}^2 = \bar{\psi}^2 \psi^2 (1 + \hat{q} \mathcal{M}(h, \hat{q}, \hat{p}, \hat{c})). \]  

(63)

\[ 8 \text{This guarantees there exists a nonsingular inverse redefinition.} \]
Here the function $\mathcal{M}$ is a finite polynomial on the real variables $h, \hat{q}, \hat{p}$ and $\hat{c}$ (see appendix A (99)). Acting with an additional spinor derivative on $\psi$ it is easy to see that

$$\bar{D} \tilde{\alpha} D \alpha \phi = \bar{D} \tilde{\alpha} D \alpha \varphi .$$

Therefore the bosonic superfields $(a^2 - b^2)$, and $p$ are the same as their redefined partners $\hat{c}$ and $\hat{p}$ up to irrelevant terms containing fermionic fields. This guarantees that the piece of the action without auxiliary superfields remains unchanged

$$\hat{\psi}^2 \bar{\hat{\psi}} L_0(p, a^2 - b^2) = \hat{\psi}^2 \bar{\hat{\psi}} L_0(\hat{p}, \hat{c}) .$$

On the other hand the auxiliary bosonic superfield transforms nontrivially

$$D^2 \phi = D^2 \varphi \left[ 1 + \frac{1}{4} \hat{c}^4 \left( 1 + \hat{p} L_0 + \hat{c} L_1 \right) \right] + O(\hat{\psi}, \bar{\hat{\psi}}) .$$

As a result we obtain the following nonlinear relation

$$q = \hat{q} \left[ 1 + h \left( \hat{q} \hat{p} + \hat{c} \right) \right]^2 .$$

Collecting all these results the redefined dual action may be written as

$$S_{\text{dual}} = \int d^4 x d^4 \theta \left( \varphi \bar{\varphi} + \hat{\psi}^2 \bar{\hat{\psi}} \hat{q} \left[ 2h + h^2 \left( \hat{q} \hat{p} + \hat{c} \right) \right] \right) + \hat{\psi}^2 \bar{\hat{\psi}} \left( 1 + \frac{1}{4} \hat{c}^4 \left( 1 + \hat{p} L_0 + \hat{c} L_1 \right) \right) .$$

Thus we have shown that is possible to make the dependence of the dual action on auxiliary superfields $\hat{q}$ completely arbitrary, while the physical part of the action remains unchanged. In particular we may now replace the function $h$ by its Taylor series expansion (70) in the auxiliary superfield $\hat{q}$ and solve for the coefficients $h_i$ that eliminate the $\hat{q}$ dependence of the action at each order. We begin imposing the condition that the linear piece of the redefined action vanishes

$$0 = h_0^2 \hat{c} \left( 1 + \hat{p} L_0 + \frac{1}{4} \hat{c} L_1 \right) + 2h_0 \left( 1 + \hat{p} L_0 + \frac{1}{4} \hat{c} L_1 \right) + L_1 .$$

This gives a quadratic equation for the coefficient $h_0$. To have an invertible field redefinition we choose the nonsingular root

$$h_0 = -\frac{4}{\hat{c}} \left( 1 - \sqrt{1 - \frac{\hat{c} L_1}{4(1 + 4 \hat{p} L_0 + \hat{c} L_1)}} \right) .$$

Next we impose that the term quadratic in $\hat{q}$ vanishes. The result is a linear equation on $h_1$, with coefficients depending on $h_0, L_0, L_1, L_2$ and the bosonic superfields $\hat{p}, \hat{c}$. We can continue this process and we find at every order $\hat{q}^n$ a linear equation on $h_{n-1}$ with the same linear coefficient.
\[ 0 = 2h_n \left( (1 + h_0 \hat{c}) (1 + \hat{p} \mathcal{L}_0) + \frac{\hat{c}}{4} \mathcal{L}_1 \right) + C_n (h_{i < n}, \mathcal{L}_{j < n+1}, \hat{p}, \hat{c}) \]  \tag{71}

Thus we can eliminate the auxiliary superfield dependence at each order in the Taylor series expansion of the action. The last consistency check we must perform is to make sure that the solution to each equation

\[ 2h_n = - \frac{C_n}{(1 + h_0 \hat{c}) (1 + \hat{p} \mathcal{L}_0) + \frac{\hat{c}}{4} \mathcal{L}_1} \]  \tag{72}

is nonsingular. This is automatically guaranteed because the linear coefficient of all these equations has a non-vanishing v.e.v., and \( C_n \) is a function of nonsingular quantities.

5 Nonlinear Bosonic action in x-space

In this section we integrate the Grassmann coordinates on the nonlinear action (23) of the complex linear superfield. We only analyze the bosonic part; this reproduces the dependence of the 3-brane transverse coordinates on the longitudinal ones. The supersymmetric fermionic partner coordinates can be trivially derived from the additional terms. Since the duality between the complex linear superfield and its dual chiral superfield does not exchange the physical x-space components, the x-space action of the bosonic fields must be the same as that obtained from (14). Historically the supersymmetric action (14) was actually guessed \( \text{[3]} \) (see also \( \text{[2]} \)) from the bosonic one

\[ S_{\text{bos}} = \int d^4x \sqrt{1 + 2(\partial \phi_o \partial \bar{\phi}_o) + (\partial \phi_o \partial \bar{\phi}_o)^2 - (\partial \phi_o)^2(\partial \bar{\phi}_o)^2}, \]  \tag{73}

where the field \( \phi_o = X_4 + iX_5 \) denotes the physical scalar in the chiral superfield \( \phi \) and we integrate over the longitudinal coordinates \( d^4x = dx_0 dx_1 dx_2 dx_3 \).

Before we start our calculation let us define a useful notation for the components of the complex linear superfield

\[
\begin{align*}
\sigma &\equiv \Sigma|_{\theta=0} \\
\chi_\alpha &\equiv D_\alpha \Sigma|_{\theta=0} \\
\lambda_\dot{\alpha} &\equiv \bar{D}_{\dot{\alpha}} \Sigma|_{\theta=0} \\
F_{\alpha\dot{\alpha}} &\equiv D_\alpha \bar{D}_{\dot{\alpha}} \Sigma|_{\theta=0}
\end{align*}
\]  \tag{74}

Since we are studying only the bosonic part of the x-space action, when we integrate the Grassmann coordinates in (23) we drop all terms containing the fermionic fields \( \chi_\alpha, \lambda_\dot{\alpha} \) and their conjugates. In addition, the auxiliary field \( D^2 \Sigma|_{\theta=0} \) and its conjugate always enter quadratically and they can be set to zero using their algebraic field equations. Thus the bosonic action is

\[ S = \int d^4x \left( iF \cdot \partial \sigma + i\bar{F} \cdot \partial \bar{\sigma} - F \cdot \bar{F} + \frac{1}{2} \partial \sigma \cdot \partial \bar{\sigma} + \frac{1}{4} F^2 \bar{F}^2 \frac{2y + 1}{y^3} \right), \]  \tag{75}
where the field \( y \) represents now its lowest component of the superfield \( Y \). It is defined by the identity \( 4(y^1 + y^3) = F^2 \bar{F}^2 \). The auxiliary field \( F_{\dot{a}a} \) does not enter quadratically in the bosonic action, but it is not dynamical and it may be eliminated imposing its algebraic field equation

\[
F_{\dot{a}a} = i\partial_{\dot{a}a}\sigma + F_{\dot{a}a} \frac{F^2}{2y^2} ,
\]

and substituting \( \bar{F}_{\dot{a}a} \) by the corresponding conjugate field equation. The solution is

\[
F_{\dot{a}a} = i(\partial_{\dot{a}a}\sigma)y + i(\partial_{\dot{a}a}\bar{\sigma})\frac{F^2}{2y^2} .
\]

Squaring this relation and the corresponding conjugate we find after some algebraic manipulations

\[
(\partial\sigma)^2 F^2 = (\partial\bar{\sigma})^2 F^2 = -(1 + \partial\sigma \cdot \partial\bar{\sigma}) \frac{4(y^3 + y^2)}{2y + 1}
\]

\[
y = \frac{1}{2} \left( 1 + \frac{1 + \partial\sigma \cdot \partial\bar{\sigma}}{\sqrt{(1 + \partial\sigma \cdot \partial\bar{\sigma})^2 - (\partial\sigma)^2(\partial\bar{\sigma})^2}} \right)
\]

Finally we substitute these solutions in the action (73) we find the same type of 3-brane action as in (73)

\[
S_{bos} = \int d^4x \sqrt{1 + 2(\partial\sigma\partial\bar{\sigma}) + (\partial\sigma\partial\bar{\sigma})^2 - (\partial\sigma)^2(\partial\bar{\sigma})^2} .
\]

This exactly the bosonic action we expect in agreement with the fact that the physical fields \( \phi_o = \bar{\sigma} \) are not affected by the \( N = 1 \) duality transformation between the superfields \( \phi \) and \( \Sigma \).

### 6 Superpotential deformation on the \( O(4) \) nonlinear action

In this last section we study the effect of adding a superpotential deformation (25) to the complex linear multiplet action (16) with partially broken supersymmetry

\[
S_{O(4)} = \int d^4x d^4\theta \left( \Phi \Phi - \Sigma\Sigma + \frac{1}{2}X^2 + \beta X(\Phi + \bar{\Phi}) + \frac{\beta}{2}(\Sigma^2 + \bar{\Sigma}^2) \right) .
\]

It is possible to replace the constrained chiral field \( \Phi \) by its solution (13) and obtain the equation of motion for the unconstrained superfield \( X \). The bosonic x-space action can then be computed as in the previous section. The counting of fermionic fields again simplifies our calculations, but the implicit dependence of \( D^2X \) on the scalar component of \( P \) and its conjugate cannot be solved exactly for a generic value of the parameter \( \beta \). For the special values \( \beta = \pm 1 \) there is a solution but the x-space action vanishes.
A more illuminating analysis is provided by the dualization of the action (80) with partially broken supersymmetry to a nonlinear action of a chiral Lagrange multiplier. Our task is slightly simplified if we redefine the auxiliary superfield in (80) by a shift

\[ X 
\]

\[ \rightarrow X = X + \beta(\Phi + \bar{\Phi}) \]  

(81)

The action (80) can be written

\[ S_{O(4)} = \int d^4x d^4\theta \left( (1 - \beta^2)\Phi\bar{\Phi} + \frac{1}{2} X^2 - \Sigma \bar{\Sigma} + \frac{\beta}{2}(\Sigma^2 + \bar{\Sigma}^2) \right) \]

(82)

where the nilpotent antichiral field (15) can be expressed in terms of this redefined \( \chi \)

\[ \bar{\Phi} = -\frac{1}{2} \bar{\Psi}_\alpha \bar{\Psi} \]

(83)

Counting the number of fermionic superfields in \( \Phi\bar{\Phi} \) we realize once more that only the bosonic part of \((D^2\Phi)\) and \(\bar{D}^2\bar{\Phi}\) contributes to its denominator. We can therefore solve for this superfields after acting with spinor derivatives on (83) and its complex conjugate

\[ (D^2\Phi)_{\text{bos}} = \frac{1}{2} \frac{\mathcal{A} + \mathcal{B}}{1 + D^2\chi - \beta(D^2\Phi)_{\text{bos}}} \]

\[ (\bar{D}^2\bar{\Phi})_{\text{bos}} = \frac{1}{2} \frac{\mathcal{A} - \mathcal{B}}{1 + D^2\chi - \beta(D^2\Phi)_{\text{bos}}} \]  

(84)

We substitute the solution of this system of quadratic equations in the action (82) and we find

\[ S_{O(4)} = \int d^4x d^4\theta \left( (1 - \beta^2)\mathcal{F}\bar{\Psi}_\alpha \bar{\Psi}_\alpha \bar{\Psi} + \frac{1}{2} \chi^2 - \Sigma \bar{\Sigma} + \frac{\beta}{2}(\Sigma^2 + \bar{\Sigma}^2) \right) \]

(85)

where

\[ \mathcal{F} = \frac{1}{2} \left( \mathcal{Y}\bar{\mathcal{Y}} - \beta \mathcal{A} + \sqrt{(\mathcal{Y}\bar{\mathcal{Y}})^2 - 2\beta \mathcal{A}\mathcal{Y}\bar{\mathcal{Y}} + \beta^2 \mathcal{B}^2} \right)^{-1}, \quad \mathcal{Y} = 1 + D^2\chi \]  

(86)

The field equation of the unconstrained superfield \( \chi \) reveals that it is at least quadratic in \( \bar{\Psi} \) and in \( \bar{\Psi} \)

\[ \chi = -\left( \bar{\Psi}_\alpha \bar{\Psi}_\alpha \mathcal{Y}(\mathcal{A} - \mathcal{B}) + \bar{\Psi}_\alpha \bar{\Psi}_\alpha \mathcal{Y}(\mathcal{A} + \mathcal{B}) \right) \frac{(1 - \beta^2)\mathcal{F}}{\sqrt{(\mathcal{Y}\bar{\mathcal{Y}})^2 - 2\beta \mathcal{A}\mathcal{Y}\bar{\mathcal{Y}} + \beta^2 \mathcal{B}^2}} + O(\bar{\Psi}\bar{\Psi}, \bar{\Psi}\bar{\Psi}) \]  

(87)

Therefore \( \chi^2 \) contains again the maximal number of fermionic superfields and only the bosonic part of \( \mathcal{Y} \) contributes to the action. Acting with spinor derivatives on (87) and keeping only terms without fermionic superfields we find as in the \( \beta = 0 \) case that \( (\mathcal{Y})_{\text{bos}} = (\mathcal{Y})_{\text{bos}} \). The algebraic equation defining \( (\mathcal{Y})_{\text{bos}} \) in terms of \( \mathcal{A} \) and \( \mathcal{B} \) is now a higher order polynomial that we cannot solve exactly (to simplify the notation we drop from now on the bosonic subscript)
\[
\mathcal{Y} - 1 = \left(1 - \beta^2\right) \frac{(A^2 - B^2) \mathcal{Y} \mathcal{F}}{\sqrt{(\mathcal{Y}^4 - 2\beta A \mathcal{Y}^2 + \beta^2 B^2)}}.
\]  

(88)

Keeping the dependence of the action on \( \mathcal{Y} \) explicit does not help us solve the duality equations in this case. We may however solve for \( \mathcal{Y} = \mathcal{Y}(A, B) \) iteratively and that in turn allows us to solve the duality equations also iteratively.

Substituting \( X^2 \) by its field equation in the action, we introduce Lagrange multipliers that enforce the linearity constraint on a relaxed \( \Sigma \)

\[
S = \int d^4x d^4\theta \left( \bar{\Psi}_\alpha \tilde{\Psi}_\alpha \Psi_\alpha \Psi_\alpha (1 - \beta^2) \mathcal{F} \left( 1 + \frac{(1 - \beta^2) \mathcal{F} (A^2 - B^2)}{\sqrt{(\mathcal{Y}^4 - 2\beta A \mathcal{Y}^2 + \beta^2 B^2)}} \right) \right.
\]

\[
- \Sigma \bar{\Sigma} + \frac{\beta}{2} (\Sigma^2 + \bar{\Sigma}^2) + \Sigma \phi + \bar{\Sigma} \bar{\phi}
\]

(89)

To manipulate the duality equations that we obtain from the functional differentiation with respect to the unconstrained \( \Sigma \) and \( \bar{\Sigma} \), rewrite the last line in matrix form

\[
S = S_0 + \int d^4x d^4\theta \frac{1}{2} \left[ (\Sigma, \bar{\Sigma}) + (\phi, \bar{\phi}) M^{-1} \right] \left[ \left( \frac{\partial}{\partial \Sigma} \right) + M^{-1} \left( \frac{\partial}{\partial \phi} \right) \right] - \frac{1}{2} (\phi, \bar{\phi}) M^{-1} \left( \frac{\partial}{\partial \phi} \right),
\]

where

\[
M = \begin{pmatrix}
\beta & -1 \\
-1 & \beta
\end{pmatrix}
\]

(90)

Obtaining the duality equations is now a straightforward calculation very similar to what we did before

\[
M \left[ \left( \frac{\partial}{\partial \Sigma} \right) + M^{-1} \left( \frac{\partial}{\partial \phi} \right) \right] = \left( \frac{\delta S_0}{\delta \Sigma} \right), \frac{\delta S_0}{\delta \bar{\Sigma}}
\]

(92)

The fields \( \delta S_0/\delta \Sigma \) and its conjugate are again quadratic in fermionic superfields and the analysis resembles closely that of the \( \beta = 0 \) case. Since we know that the dependence on the auxiliary superfield \( Q \) decouples when we go to x-space components and we are mostly interested in finding out if the superpotential deformation introduces any new qualitative features, we will simplify our calculation by setting \( q = 0 = Q \) in the duality equations.

We follow the same steps as in the \( \beta = 0 \) case and we find similar duality equations with a more complicated dependence on the superfields \( \mathcal{Y}, A, B, P \). In this case it is not possible to solve the equations exactly even after setting \( Q = 0 = q \). However, the bosonic variables \( a, b, p \) are at least linear on their duals \( P, A, B \) and we can therefore solve the duality equations iteratively to find the Taylor series expansion of \( P(a, b, p), A(a, b, p), B(a, b, p) \).

Expanding the action to a given order \( n \) on \( P, A, B \) and substituting the iterative solution up to that order we obtain the expansion of the dual action to \( n \)-th order in \( a, b, p \). What
we find after this tedious but straightforward calculation is that the dual chiral field gets rescaled and the action is multiplied by an overall constant

\[ S_{\text{dual}}(q = 0) = (1 - \beta^2) \int d^4x d^4\theta \left( \frac{\phi \bar{\phi}}{(1 - \beta^2)^2} + \frac{\psi^2 \bar{\psi}^2}{(1 - \beta^2)^2} \mathcal{L}_0 \left( \frac{p}{(1 - \beta^2)^2}, \frac{a^2 - b^2}{(1 - \beta^2)^4} \right) \right) \quad (93) \]

The rescaling of the dual chiral field becomes singular when \( \beta = \pm 1 \). For other values of the deformation parameter \( \beta \) the superpotential does not seem to add any significant new feature (see also [10]).

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**Appendix**

**A  Explicit formulas**

In this appendix we include the explicit form of various expressions appearing in sections 3 and 4. The basic duality duality equation (28) is defined by

\[
\phi + \bar{\phi} = G + \Psi^2(\bar{D}^2G)K + \bar{\Psi}^2(D^2G)\bar{K} + \Psi^a\bar{\Psi}^\alpha \left( (\bar{D}_\alpha D_a G)H - (D_\alpha \bar{D}_a G)\bar{H} \right)
\]

\[
K = 4f + 2(A + B)(f_A + f_B) - 2P(f_A - f_B)
\]

\[
H = 2f + 2Q(f_A - f_B) + (A - B)(f_A + f_B) . \quad (94)
\]

The product of four fermionic superfields derived from (32) is

\[
\psi^2 \bar{\psi}^2 = \Psi^2 \bar{\Psi}^2 M
\]

\[
M = g_1 \bar{g}_1 + Q^2 g_2 \bar{g}_2 + QP(g_3 \bar{g}_3 + g_4 \bar{g}_4) - Q(A + B)g_3 \bar{g}_4 - Q(A - B) \bar{g}_3 g_4
\]

\[
+ QP \left( g_3(A + B) \bar{H}^2 + \bar{g}_3(A - B)H^2 \right) - QP \left( g_4(A - B) \bar{H}^2 + g_4(A + B)H^2 \right)
\]

\[
+ QP \left( \frac{3}{4}(A^2 - B^2) - P^2 \right) H^2 \bar{H}^2 ,
\]

where

\[
g_1 = \left( 1 - QK + \frac{A + B}{2} \bar{H} \right)^2 + \frac{A^2 - B^2}{4} \bar{H}^2 - P \left( 1 - QK + \frac{A + B}{2} \bar{H} \right) H
\]
\[ g_2 = \frac{A+B}{2} (\bar{H} - \bar{K})^2 + \frac{A-B}{2} H^2 - P(\bar{H} - \bar{K})H \]
\[ g_3 = 2(\bar{H} - \bar{K}) \left( 1 - QK + \frac{A+B}{2} \bar{H} \right) \]
\[ g_4 = 2H \left( 1 - QK + \frac{A+B}{2} \bar{H} \right) + (A+B)(\bar{H} - \bar{K}). \] (96)

Similarly, in the complex linear superfield action the product of four fermionic superfields (53) is defined by
\[ \psi \bar{\psi}^2 = \Psi \bar{\Psi}^2 N \]
\[ N = e_1 + \mathcal{Q} e_2 + \frac{1}{4} \mathcal{P} (e_3 + \tilde{e}_4) - \frac{1}{4} \mathcal{Q} e_3 \left( e_4 (D \bar{D} \Sigma)^2 + \bar{e}_4 (D \bar{D} \Sigma)^2 \right) \]
\[ + \frac{Q}{Y^4} e_3 \left( \mathcal{P}^2 - 2(Y^4 + Y^3) \right) - \frac{Q}{2Y^4} \mathcal{P} \left( e_4 (D \bar{D} \Sigma)^2 + \bar{e}_4 (D \bar{D} \Sigma)^2 \right) \]
\[ + \frac{Q}{Y^8} \left( \mathcal{P}^2 - 3(Y^4 + Y^3) \right), \] (97)
where
\[ e_1 = \left( 1 - \frac{3Y+1}{Y^3} \right)^2 - \frac{\mathcal{P}}{Y^2} \left( 1 - \frac{3Y+1}{Y^3} \right) + \frac{Y+1}{Y} \]
\[ e_2 = \frac{Y - 1}{Y^5} \left[ 1 + \frac{\mathcal{P}}{Y^4} (\mathcal{Q} + \mathcal{P}) + \frac{Y - 1}{Y^5} (\mathcal{Q} + \mathcal{P})^2 \right]^2 \]
\[ e_3 = \frac{2}{Y^2} \left[ \frac{Y+1}{Y^3} (\mathcal{Q} + \mathcal{P}) - \left( 1 - \frac{3Y+1}{Y^3} \right) \right] \]
\[ e_4 = \frac{(D \bar{D} \Sigma)^2}{Y^6} (\mathcal{Q} + \mathcal{P}) \left( 1 - \frac{3Y+1}{Y^3} \right). \] (98)

Finally, the product of four dual fermionic superfields (53) is after the redefinition (58)
\[ \psi^2 \bar{\psi}^2 = \tilde{\psi}^2 \bar{\tilde{\psi}}^2 (1 + \tilde{q} \mathcal{M}(h, \tilde{q}, \tilde{p}, \tilde{c})) \]
\[ (1 + \tilde{q} \mathcal{M}(h, \tilde{q}, \tilde{p}, \tilde{c})) = d_1 + \tilde{q}^2 d_2 + \frac{1}{4} \tilde{q} \tilde{p} (d_3^2 + d_4 \bar{d}_4) + \frac{1}{4} \tilde{q} d_3 \left( \bar{d}_4 (\partial \varphi)^2 + d_4 (\partial \varphi)^2 \right) \]
\[ + \tilde{q}^3 h^2 d_3 \left( \tilde{p}^2 - \frac{\tilde{c}}{2} \right) + \tilde{q}^3 \tilde{p} \left( \bar{d}_4 (\partial \varphi)^2 + d_4 (\partial \varphi)^2 \right) \]
\[ + \tilde{q}^5 h^4 \tilde{p} \left( \tilde{p}^2 - \frac{3}{4} \tilde{c} \right) \] (99)
where
\[ d_1 = \frac{1}{4} \left[ 2(1 + h \tilde{q}^2)^2 + \tilde{q}^2 h^2 \frac{\tilde{c}}{2} + 2 \tilde{q} h (1 + h \tilde{q}^2) \tilde{p} \right]^2 \]
\[ d_2 = \frac{1}{4} h^4 \left( q^2 + \hat{q} \hat{p} + \frac{\hat{c}}{4} \right)^2 \]

\[ d_3 = 2 \hat{q} h \left( 1 + \hat{q}^2 h + \frac{\hat{c}}{4} \right) \]

\[ d_4 = -h (1 + \hat{q}^2 h) (\partial \phi)^2 \]

\[ (100) \]

## B Solution to nonlinear equations

We must still provide a derivation of the solution to the system of nonlinear equations relating bosonic superfields in the complex linear multiplet and the corresponding duals. The analysis of the tensor multiplet is very similar and can be quickly reproduced.

First we note that the system of equations has an exact solution when \( q = 0 = Q \) (we disregard the other two solutions for \( Q \) since we are basically expanding around the zero v.e.v. of this field)

\[ Y_0 = \frac{1}{2} \left( 1 + \frac{1 - p}{\sqrt{(1-p)^2 - (a^2 - b^2)}} \right) \]

\[ \mathcal{P}_0 = \frac{4(Y_0^3 + Y_0^2) + pY_0}{2Y_0 + 1} \]

\[ (101) \]

As we mentioned before, solving \( Q \) iteratively we find

\[ Q = \frac{q}{(1 - \frac{2Y + 1}{\mathcal{P}})^2} + \sum_{n=2}^{+\infty} q^n Q_n(\mathcal{P}, Y) \]

\[ (102) \]

and substituting it into our duality equations we obtain the expansion

\[ p(Q, \mathcal{P}, Y) = \sum_{i=0}^{+\infty} q^i p_i(\mathcal{P}, Y) \]

\[ (\partial \phi)^2 (\partial \bar{\phi})^2 \equiv c = \sum_{i=0}^{+\infty} q^i c_i(\mathcal{P}, Y) \]

\[ (103) \]

The bosonic superfields \( \mathcal{P}, Y \) that solve the duality equations are functions of the dual variables \( \mathcal{P} = \mathcal{P}(q, p, c), Y = Y(q, p, c) \). Assuming that these solutions have a Taylor series expansion in the auxiliary superfield

\[ \mathcal{P} = \mathcal{P}_0(p, c) + q \mathcal{P}_1(p, c) + q^2 \mathcal{P}_2(p, c) + \ldots \]

\[ Y = Y_0(p, c) + q Y_1(p, c) + q^2 Y_2(p, c) \]

\[ (104) \]

we may replace this formal solution in (103)
\[ p = p_0(P_0, Y_0) + q P_1 \frac{\partial p_0}{\partial P} \bigg|_{P=P_0} + q Y_1 \frac{\partial p_0}{\partial Y} \bigg|_{Y=Y_0} + q p_1(P_0, Y_0) + O(q^2) \]
\[ c = c_0(P_0, Y_0) + q P_1 \frac{\partial c_0}{\partial P} \bigg|_{P=P_0} + q Y_1 \frac{\partial c_0}{\partial Y} \bigg|_{Y=Y_0} + q c_1(P_0, Y_0) + O(q^2) \]  

(105)

The variables \( p \) and \( c \) are independent of \( q \) and therefore only the homogeneous term on the r.h.s. can be nonzero. At linear order in \( q \) we find
\[
\begin{pmatrix}
\frac{\partial p_0}{\partial P} \bigg|_{P=P_0} & \frac{\partial p_0}{\partial Y} \bigg|_{Y=Y_0} \\
\frac{\partial c_0}{\partial P} \bigg|_{P=P_0} & \frac{\partial c_0}{\partial Y} \bigg|_{Y=Y_0}
\end{pmatrix}
\begin{pmatrix}
P_1 \\
Y_1
\end{pmatrix}
= - \begin{pmatrix}
p_1 \\
c_1
\end{pmatrix}.
\]

(106)

Inverting the Jacobian it is very straightforward to find the linear coefficient of the solution (104). The equations obtained from higher orders in \( q \) provide the higher coefficients.

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