A NOTE ON SAXL CONJECTURE

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Abstract. We discuss the effectiveness of Bessenrodt, Ikenmeyer and Pak’s results about Saxl conjecture. These results are based on dominance order and nonvanishing irreducible characters. We show that the probability that a partition is comparable in dominance order to the staircase partition tends to zero as the staircase partition grows. Vanishing conditions for irreducible characters are discussed. Moreover, we show that the occurrence of irreducible representations corresponding to partitions with Durfee size less than 3 can be reduced to the first 26 staircase partitions.

1. Introduction

In representation theory and related fields, the Kronecker coefficients play a crucial role. For partitions $\lambda, \mu \vdash n$, let $[\lambda]$ and $[\mu]$ be two irreducible representations of $S_n$. The tensor product $[\lambda] \otimes [\mu]$ is an $S_n$-representation via the diagonal embedding $\pi \mapsto (\pi, \pi)$, $\pi \in S_n$. This $S_n$-representation decomposes as follows

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu)[\nu],$$

where the coefficients $g(\lambda, \mu, \nu)$ are called the Kronecker coefficients. In spite of their importance, little is known about the Kronecker coefficients, leaving some fundamental questions unanswered. For example, no combinatorial description akin to the Littlewood-Richardson rule is known for the Kronecker coefficients. Another important question is to determine how difficult it is, algorithmically, to compute Kronecker coefficients, or to decide their positivity.

In 2012, J. Saxl conjectured that all irreducible representations of the symmetric group occur in the decomposition of the tensor square of the irreducible representation corresponding to the staircase partition [16]. Let $\rho_m$ ($m \in \mathbb{N}$) denote the staircase partitions. So the Saxl Conjecture claims that $g(\rho_m, \rho_m, \lambda) > 0$ for all $\lambda \vdash m(m + 1)/2$. Many progresses have been made on this conjecture, see for example [4, 9, 12, 16].

For $\lambda \vdash m(m + 1)/2$, we say $\lambda$ satisfies Saxl conjecture if $g(\rho_m, \rho_m, \lambda) > 0$. In [9, Thm. 2.1], Ikenmeyer showed that if a partition $\nu \vdash m(m + 1)/2$ is comparable in the dominance order to $\rho_m$ then $\nu$ satisfies Saxl conjecture. Inspired by his result, we want to know the proportion of partitions that are comparable to $\rho_m$ in all partitions of $m(m + 1)/2$. By the result of [18], we show that the proportion tends to zero as $m \to \infty$ (see Corollary 3.4). Thus the probability that a partition is comparable to the staircase partition tends to zero as $m \to \infty$. Moreover, we discuss how to construct incomparable pairs and the geometry of permutohedron.

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Another criterion to find partitions satisfying Saxl conjecture is based on nonvanishing irreducible characters. For a partition \( \lambda \), let \( \hat{\lambda} \) denote its principal hook partition. In [16, Lemma 1.3], Pak et al. showed that if \( \chi^\lambda(\rho_m) \neq 0 \), then \( \lambda \) satisfies Saxl conjecture. Recently, Bessenrodt showed that if \( \chi^\lambda(\rho_m) \neq 0 \) then \( \lambda \) and \( \hat{\lambda} \) satisfy Saxl conjecture (see more in [4, Cor. 4.4]). Moreover, Bessenrodt showed that all double hooks (i.e. partitions with Durfee size 2) satisfy Saxl conjecture. However, if \( \chi^\lambda(\rho_m) = 0 \) or \( \chi^\lambda(\rho_m) = 0 \) we can’t decide whether \( \lambda \) satisfies Saxl conjecture by their criterions. Thus vanishing conditions on \( \rho_m \) and \( \rho_m \) reflect the effectiveness of their results. In this paper, we discuss vanishing properties of irreducible characters on a fixed conjugacy class, especially on \( \rho_m \) and \( \rho_m \). For \( \mu = m(m+1)/2 \) with Durfee size 3, we show that if \( g(\rho_m, \rho_m, \mu) > 0 \) for \( 1 \leq m \leq 26 \), then \( g(\rho_m, \rho_m, \mu) > 0 \) for all \( m \in \mathbb{N} \). Thus the occurrence of irreducible representations corresponding to partitions with Durfee size less than 3 can be reduced to the first 26 staircase partitions. Moreover, we define staircase-like partitions for each \( n \) and provide possible candidates for tensor square conjectures [16, Conj. 1.1].

The paper is organized as follows. In Section 2, we summarize basic definitions and results used in this paper. In Section 3, we show that the probability that a partition is comparable to \( \rho_m \) tends to zero as \( m \to \infty \). Related properties are also discussed such as the construction of incomparable pairs and geometry of permutohedron. By cores and permutohedron, we discuss vanishing conditions of irreducible characters in Section 4. In Section 5, we show that the occurrence of irreducible representations corresponding to partitions with Durfee size 3 can be reduced to the first 26 staircase partitions. A general Saxl conjecture and related problems are raised in Section 6.

2. Preliminaries

If \( A \) is a set, the cardinality of \( A \) is denoted by \(|A|\). A partition \( \lambda \) of \( n \), denoted by \( \lambda \vdash n \), is defined to be a weakly decreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of non-negative integers such that the sum \( \lambda_1 + \lambda_2 + \cdots + \lambda_k = n \). We call \( n \) the size of \( \lambda \) and denote it by \(|\lambda|\). The length of a partition \( \lambda \) is the number of its nonzero entries and denoted by \( \ell(\lambda) \). The set of all partitions of \( n \) is denoted by \( P(n) \). To a partition \( \lambda \) we associate its Young diagram, which is a top-aligned and left-aligned array of boxes such that in row \( i \) we have \( \lambda_i \) boxes. Thus for \( \lambda \vdash n \) the corresponding Young diagram has \( n \) boxes. For example, for \( \lambda = (6, 5, 3) \) the corresponding Young diagram is

```
+---+---+---+---+---+
|   |   |   |   |   |
+---+---+---+---+---+
|   |   |   |   |
+---+---+---+
|   |   |
+---+---+
|   |
+---+
```

We do not distinguish between a partition \( \lambda \) and its Young diagram. If we transpose a Young diagram at the main diagonal we obtain another Young diagram, which is called the conjugate partition of \( \lambda \) and denoted by \( \lambda' \). The row lengths of \( \lambda' \) are the column lengths of \( \lambda \). In the example above we have \( \lambda' = (3, 3, 3, 2, 2, 1) \). A partition \( \lambda \) is called self-conjugate if \( \lambda = \lambda' \). Sometimes we use the notation which indicates the number of times each integer occurs as a part for a partition. For example, we write \( \lambda = (3, 3, 3, 2, 2, 1) \) as \( (3^3, 2^2, 1) \) which means that 3 parts of \( \lambda \) are equal to 3, and so on. If the boxes arranged using matrix coordinates, the hook of box \((i, j)\) in a Young diagram is given by the box itself, the boxes to its right and below and is denoted by \( h_{i,j} \). The hook length is the number of boxes in a hook and denoted by \( |h_{i,j}| \). We denote by \( d(\lambda) \) the Durfee size of \( \lambda \), i.e. the number of boxes in the main diagonal of \( \lambda \). Define the principal hook partition by \( \lambda \hat{=} = (h_{1,1}, \ldots, h_{s,s}) \), where \( s = d(\lambda) \). So for \( \lambda = (6, 5, 3) \) above we have \( d(\lambda) = 3 \) and \( \lambda \hat{=} = (8, 5, 1) \). For \( m \geq 1 \), we call \( \rho_m = (m, m-1, \ldots, 2, 1) \) the staircase partition which is a partition of \( \binom{m+1}{2} \).
For $n \in \mathbb{N}$ let $S_n$ denote the symmetric group on $n$ symbols. For a partition $\lambda \vdash n$ let $[\lambda]$ denote the irreducible $S_n$-representation of type $\lambda$. The corresponding irreducible character is denoted by $\chi^\lambda$. For $\nu \vdash n$, let $\chi^{\lambda} (\nu)$ denote the value of $\chi^\lambda$ on the conjugacy class of cycle type $\nu$ of the symmetric group $S_n$.

A partition $\lambda$ dominates another partition $\mu$, denoted by $\lambda \succeq \mu$ if for all $k$ we have $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$. If $\lambda$ dominates $\mu$ or $\mu$ dominates $\lambda$, we say that $\lambda$ and $\mu$ are comparable in the dominance order. For $\lambda \in P(n)$, let $C(\lambda) \subseteq P(n)$ be the set of partitions which are comparable to $\lambda$. Let $\Lambda(\lambda) \subseteq C(\lambda)$ (resp. $V(\lambda)$) be the set of partitions which are less (resp. greater) than $\lambda$ in dominance order. In literatures (see e.g. [20, Sec. 3.1]), $\Lambda(\lambda)$ (resp. $V(\lambda)$) is called the principal order ideal generated by $\lambda$ (resp. principal dual order ideal generated by $\lambda$). If $\lambda \succeq \mu$, then we have $\lambda' \succeq \mu'$ [10, Lem. 1.4.11].

For two partitions $\lambda$ and $\mu$, let $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, ...) \text{ and } \lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, ...)$. For each partition $\lambda \in P(n)$, there are at least $2^{n-1}$ partitions less than $\lambda$, $\mu$ such that $\lambda \succeq \mu$. For each $\lambda$, let $\Phi(\lambda)$ be the set of partitions comparable to $\lambda$. It is well known that $\Phi(\lambda)$ is a semigroup generated by $\lambda$. For example, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then $\Phi(\lambda)$ is the semigroup generated by $\lambda$.

### Proposition 3.1.

For $m \geq 3$ there exist at least $2^m$ partitions that are comparable to $\rho_m$. Particularly, we have $|\Phi(\rho_m)| > 2^m$.

**Proof.** We will show by induction that there are $2^{m-1}$ partitions less than $\rho_m$. Then by taking transpose we obtain another $2^{m-1}$ partitions that are greater than $\rho_m$.

There are five partitions less than $\rho_3$: $(2,2,1,1), (2,2,2), (2,1^4), (1^4), (1^6)$. Assume that there are at least $2^{m-2}$ partitions less than $\rho_{m-1}$. For each $\lambda \in \Lambda(\rho_{m-1})$ define two partitions by $(m, \lambda)$ and $(\lambda, 1^m)$. Then it is not hard to see that they belong to $\Lambda(\rho_m)$. Moreover, we can see that for $\lambda, \mu \in \Lambda(\rho_{m-1})$ if $\lambda \neq \mu$ then $(m, \lambda), (\lambda, 1^m), (m, \mu)$ and $(\mu, 1^m)$ are pairwise different. Thus for each partition in $\Lambda(\rho_{m-1})$ we obtain two new partitions which are pairwise different. So by induction there are at least $2^{m-1}$ partitions less than $\rho_m$.

The lower bound $|\Phi(\rho_m)| > 2^m$ is followed by Theorem 2.1 of [9].

For $\lambda \in P(n)$, if $\sum_{j=1}^{\ell} \lambda'_j \geq \sum_{j=1}^{\ell} \lambda_j + 1$ then $\lambda$ is said to be graphical [18]. If $\lambda \succeq \lambda'$ then $\lambda$ is said to be conjugate-upward. Let $G(n)$ and $U(n)$ denote the set of all graphical and conjugate-upward partitions, respectively. The following theorem gives an upper bound for $|G(n)|/|P(n)|$ which is also suitable for $|U(n)|/|P(n)|$ (see the discussion in [18, Sect. 1]).

### Theorem 3.2. [18, Thm. 3.1]

For $G(n)$, $U(n)$ and $n$ large enough, we have

$$\frac{|U(n)|}{|P(n)|}, \frac{|G(n)|}{|P(n)|} \leq \exp \left( -\frac{0.11 \log n}{\log \log n} \right).$$

For $C(\lambda)$ defined in Section 2, by Theorem 3.2 we have the following theorem.
Theorem 3.3. Suppose that $\lambda \in P(n)$ is self-conjugate. Then $\lim_{n \to +\infty} \frac{|C(\lambda)|}{|P(n)|} = 0.$

Proof. By definition we have $C(\lambda) = V(\lambda) \cup \Lambda(\lambda).$ For any $\mu \in \Lambda(\lambda),$ we have that $\mu \subseteq \lambda$ and therefore $\mu' \preceq \lambda'.$ Since $\lambda' = \lambda,$ we have $\mu \preceq \lambda.$ Thus, $\Lambda(\lambda) \subseteq U(n)$ and there is a bijection between $V(\lambda)$ and $\Lambda(\lambda)$ by taking transpose. So we have that $|C(\lambda)| = 2|\Lambda(\lambda)|.$ By Theorem 3.2 we have $\lim_{n \to +\infty} \frac{|U(n)|}{|P(n)|} = 0.$ Since $\Lambda(\lambda) \subseteq U(n),$ we have $\lim_{n \to +\infty} \frac{|\Lambda(\lambda)|}{|P(n)|} = 0$ and

$$\lim_{n \to +\infty} \frac{|C(\lambda)|}{|P(n)|} = \lim_{n \to +\infty} 2 \frac{|\Lambda(\lambda)|}{|P(n)|} = 0.$$ 

\[ \blacksquare \]

In Theorem 3.3 if we let $\lambda = \rho_m,$ we have the following corollary.

Corollary 3.4. For $\rho_m,$ we have $\lim_{m \to +\infty} \frac{|C(\rho_m)|}{|P(m)|} = 0.$ That is, the probability that a partition is comparable to $\rho_m$ is zero as $m \to \infty.$

It is natural to ask the complexity of deciding when two partitions are comparable. Since the positivity of Kostka number is equivalent to the comparability of partitions, Proposition 1 of [14] can be reformulated as the following.

Proposition 3.5. [14, Prop. 1] Given $\lambda$ and $\mu,$ whether or not $\lambda$ and $\mu$ are comparable can be answered in polynomial time.

By Theorem 4.1 of [18], we know that almost all partitions are incomparable. Given a partition we want to know which partitions are incomparable to it and how many partitions are comparable to it. In the following, we give some related characterizations.

By Theorem 1.4.10 of [10], we can see that $\lambda \succeq \mu$ (resp. $\lambda \preceq \mu$) if and only if $\lambda$ can be obtained from $\mu$ by moving some boxes upward (resp. downward). In contrary to this, we get two typical movements which can produce incomparable partitions. Let $a$, $b$ be two boxes which lie in row $i$ and $j$, respectively. Simultaneously, we move one upward and another downward such that the resulting is still a partition. If $i < j$ and $a$ (resp. $b$) moves upward (resp. downward), we call it outer movement. If $i < j$ and $a$ (resp. $b$) moves downward to row $i'$ (resp. upward to $j'$) such that $i' < j'$, we call it inner movement. The following proposition is straightforward.

Proposition 3.6. For $\lambda$, $\mu \vdash n$ if $\lambda$ can be obtained from $\mu$ by an outer movement or inner movement, then $\lambda$ and $\mu$ are incomparable.

Example 3.7. Let $\lambda = (4, 3, 2, 1)$ and $\mu = (5, 2, 1, 1, 1).$ Then we have that $\lambda$ and $\mu$ are incomparable. They can be transferred to each other by outer and inner movements as follows.

Since self-conjugate partitions can only be transferred to each other by a series of outer and inner movements, we have the following proposition.

Proposition 3.8. Let $\lambda, \mu \in P(n)$ be self-conjugate. Then $\lambda$ and $\mu$ are incomparable.
For \( \mu \vdash n \), we have \( \Lambda(\mu) = \{ \nu \mid \nu \subseteq \mu, \nu \in P(n) \} \). For \( k \geq \ell(\mu) \) write \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) where \( \mu_i = 0 \) for \( i > \ell(\mu) \). Then \( \mu \) can be viewed as a vector in \( \mathbb{R}^k \). For each \( k \geq \ell(\mu) \), define the \( k \)-th permutohedron \( P_k(\mu) \subseteq \mathbb{R}^k \) by

\[
P_k(\mu) := \text{ConvexHull}( (\mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(k)}) | \sigma \in S_k ) .
\]

It is well known that \( \Lambda(\mu) \subseteq P_k(\mu) \), the \( n \)-th permutohedron \( P_n(\mu) \) associated to \( \mu \) (see e. g. [19]). Next, we show that \( \Lambda(\mu) \) lies in a convex subset of \( P_n(\mu) \). In fact, define the hyperplanes in \( \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R}, \ i = 1, 2, \ldots, n \} \) by

\[ H_i := \{ x_i - x_{i+1} = 0 \}, \]

where \( 1, 2, \ldots, n-1 \). In literatures (see e. g. [6, Def. 2.6]), \( H_i \) is called the braid arrangement fan. Let

\[ H_i^+ = \{ x_i - x_{i+1} \geq 0 \} . \]

Then \( H_i^+ \) is convex. For each partition \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \), we should have \( \nu_i \geq \nu_{i+1} \). Then we can see that

\[
\Lambda(\mu) = P_n(\mu) \cap H_1^+ \cap H_2^+ \cap \ldots \cap H_n^+ \cap \mathbb{Z}^n .
\]

By discussions above, we have the following proposition.

**Proposition 3.9.** Suppose that \( \mu \vdash n \). Then \( \Lambda(\mu) \) consists of the lattice points of a convex polytope which is formed by cutting \( P_n(\mu) \) through the braid arrangement fan.

**Problem 3.10.** \( P_m(\rho_{m-1}) \) is called the regular permutohedron in [19]. It was shown that \( P_m(\rho_{m-1}) \) is a graphical zonotope. It is interesting to decide whether \( P_m(\rho_{m-1}) \) is a graphical zonotope or not. Moreover, using the symmetry of the braid arrangement fan, can we give an estimation of \( \Lambda(\mu) \) by the results in [19]? It is nontrivial, since when \( \mu = (n) \), we have \( \Lambda((n)) = |P(n)| \) whose asymptotic estimation was given by Hardy and Ramanujan [2].

### 4. Vanishing conditions for irreducible characters and Saxl conjecture

If \( \chi^t(\rho_m) \) and \( \chi^t(\rho_{m-1}) \) are zeroes on \( \rho_m \) and \( \rho_{m-1} \), by Pak and Bessenrodt’s criterions we can’t decide whether \([1] \) appears in \( [\rho_m] \otimes [\rho_{m-1}] \) or not. Thus the number of irreducible characters which take zeroes on \( \rho_m \) and \( \rho_{m-1} \) reflects the effectiveness of their results. Motivated by this, in this section we discuss vanishing conditions of irreducible characters.

Partitions of \( n \) whose set of hook lengths do not contain \( t \) for some fixed integer \( t \) are called \( t \)-core partitions. Let \( c_t(n) \) denote the number of \( t \)-cores in partitions of \( n \). It has been shown that \( c_t(n) > 0 \) for all positive integers \( n \) and \( t \geq 4 \) (see [15] and its references). Moreover, if \( t_1 \) and \( t_2 \) are fixed integers satisfying \( 4 \leq t_1 < t_2 \), then \( c_{t_1}(n) < c_{t_2}(n) \) for sufficiently large \( n \).

By the Murnaghan-Nakayama rule (see for example [16] and [10, Sec. 2.4]), we have the following proposition.

**Proposition 4.1.** Let \( \mu = (\mu_1, \mu_2, \ldots) \). If \( \lambda \) is a \( \mu_i \)-core for some \( i = 1, 2, \ldots \), then \( \chi^t(\mu) = 0 \).

Particularly, for \( \rho_m = (m, m-1, \ldots, 1) \) and \( \rho_{m-1} = (2m - 1, 2m - 5, 2m - 9, \ldots) \) we have the following corollary.

**Corollary 4.2.** Suppose that \( \lambda = \frac{m(m+1)}{2} \) and \( \chi^t \) is the corresponding irreducible character.

1. If \( \lambda \) is a \( i \)-core for some \( i = 1, 2, \ldots, m \), then \( \chi^t(\rho_m) = 0 \).
2. If \( \lambda \) is a \( i \)-core for some \( i = 2m - 1, 2m - 5, 2m - 9, \ldots \), then \( \chi^t(\rho_{m-1}) = 0 \).
Since $\rho_m$ are (the only) 2-cores and 2 is one part of $\rho_m$, we have $\chi^{p_m}(\rho_m) = 0$. In [16, Prop. 4.16], Pak et al. gave an estimation of partitions satisfying $\chi^i(\rho_m) = 0$. From their proof we know that those partitions are $(2m - 1)$-cores. Thus, $\ell$-cores provide an important family of vanishing conditions for irreducible characters. Partitions in the following proposition are also $(2m - 1)$-cores.

**Proposition 4.3.** For $\lambda \vdash m(m + 1)/2$, if $\lambda_1$ and $\ell(\lambda) \leq m - 1$, then $\chi^i(\rho_m) = 0$ and $\rho_m, \lambda$ are incomparable.

**Proof.** By the assumption, we have that $\lambda_1 \leq 2m - 3$ which is the largest hook length of $\lambda$. Thus, $\lambda$ is a $(2m - 1)$-core. So we have $\chi^i(\rho_m) = 0$ by Corollary 4.2.

If $\rho_m$ and $\lambda$ are comparable, by $\lambda_1 \leq m - 1$ we should have $\rho_m \succeq \lambda$. On the other hand, by $\ell(\lambda) \leq m - 1 < m$ we have $\rho_m$ is strictly less than $\lambda$ in dominance order, which is a contradiction. \(\square\)

For partitions in Proposition 4.3, we can’t decide whether they satisfy Saxl conjecture by Ikenmeyer and Pak’s criterions. In [1, Thm. 2], the authors gave an estimation of such partitions. Similar result can be found in [18, Lemma 2.2]. Intuitively, if we want to obtain $\lambda$ by moving boxes from $\rho_m$, the inner movement described in Section 3 should be taken.

It was shown in [17, Thm 7.1] that for any $\lambda, \nu$ deciding whether $\chi^i(\nu) = 0$ is NP-hard. It can be reduced to the classical NP-complete Knapsack problem. Expanding the power sum function $p_\mu$ into Schur functions $s_\lambda$ (see [20, Cor. 7.17.4]) we have

$$p_\mu = \sum_{\lambda \in P(n)} \chi^i(\mu)s_\lambda.$$  

Hence, the character values on $\mu$ are the coefficients in the expansion above. In [3, Cor. 4.3], the authors showed that there exist probabilistic polynomial time algorithms for computing an expansion of a given power sum $p_\mu$. Thus the result there can be reformulated as the following. Similar results can be found in [11].

**Proposition 4.4.** There exist probabilistic polynomial time algorithms for computing $\chi^i(\mu)$. In particular, deciding whether $\chi^i(\mu) = 0$ can be done in probabilistic polynomial time.

In the following, for a partition $\mu$ we let $N(\mu)$ denote the number of irreducible characters $\chi^i$ such that $\chi^i(\mu) = 0$.

**Proposition 4.5.** For $\mu \vdash n$ and $i = 1, 2, \ldots, n$, if $i$ appears $a_i$ times as the part of $\mu$, then $N(\mu) \geq |P(n)| - 1^{a_1}a_1!2^{a_2}a_2! \cdots n^{a_n}a_n!$.

**Proof.** For $g \in S_n$, suppose that the conjugacy class that contains $g$ has cycle type $\mu$ such that the length of cycle $i$ appears $a_i$ times. Then by Theorem 4 of [7] we have

$$N(\mu) \geq |P(n)| - |C_{S_n}(g)|,$$

where $C_{S_n}(g)$ is the centralizer group of $g$. Moreover, by Lemma 1.2.15 of [10] we have

$$|C_{S_n}(g)| = 1^{a_1}a_1!2^{a_2}a_2! \cdots n^{a_n}a_n!$$

which completes the proof. \(\square\)

**Remark 4.6.** In Proposition 4.5, if $\mu$ have many small parts, then the lower bound $|P(n)| - 1^{a_1}a_1!2^{a_2}a_2! \cdots n^{a_n}a_n!$ can be less than 0. Thus in this condition the lower bound is trivial. For example, if we let $\mu = \rho_m$ then $a_i = 1$ for $i = 1, 2, \ldots, m$, and for $g \in S_{\rho_m}$ with cycle type $\rho_m$ we have $|C_{S_{\rho_m}}(g)| = m!$. By the asymptotic formula $|P(n)| \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right)$. 
it is not hard to see that there exists $k$ such that $|P(m(m+1)/2)| - m! < 0$ for all $m \geq k$. In fact, by direct computation we have $k = 7$. However, it is not hard to see that the lower bound is useful when $\mu$ consists of large parts.

Pak and Bessenrodt’s criterions lead us to study the non-zero character values on a fixed conjugacy class which correspond to the non-zero elements on columns of the character table. It can also be seen to connect to work on nonvanishing conjugacy classes, that is, conjugacy classes on which no irreducible character vanishes. A partition is a nonvanishing partition if it labels a nonvanishing conjugacy class of a symmetric group. It has been shown that any nonvanishing partition should be of the form $(3^a, 2^b, 1^c)$ for some $a, b, c \geq 0$ [8, 13]. The following proposition gives a necessary nonvanishing condition for irreducible characters.

**Proposition 4.7.** [10, Cor.2.4.9] If $\chi^\lambda(\mu) \neq 0$, then $\lambda \in V(\mu)$.

By Proposition 4.7 we have the following corollary which provides vanishing conditions on $\rho_m$. Hence we can obtain irreducible representations that can’t be decided by Pak et al.’s criterion.

**Corollary 4.8.** Suppose that $\lambda \vdash n$. If $\lambda$ is strictly less than or incomparable with $\rho_m$ in dominance order, then $\chi^\lambda(\rho_m) = 0$.

However, by the following proposition we can see that Proposition 4.7 tells us nothing about $N(\rho_m)$.

**Proposition 4.9.** If $\mu \leq \rho_m$, then we have $\lambda \in V(\mu)$ for all $\lambda \vdash m(m+1)/2$. Particularly, we have $\lambda \in V(\rho_m)$ for all $\lambda \vdash m(m+1)/2$.

**Proof.** It suffices to show that $\rho_m \leq \lambda$ for each $\lambda \vdash m(m+1)/2$. Let $\lambda = (\lambda_1, \lambda_2, ...)$ and $\rho_m = (\rho_{m,1}, \rho_{m,2}, ..., \rho_{m,m})$. Then we have that $\sum_i \lambda_i - \sum_i \lambda_{i+1} \geq 2$ and $\rho_{m,i} - \rho_{m,i+1} = 1$ for $i = 1, 2, ...$.

If $\rho_m \geq \lambda$ or they are incomparable, then we have that $\sum_i \lambda_i \geq \sum_i \rho_{m,i}$ for some $i$. If $\lambda_{i+1} \geq \rho_{m,i+1}$, then combined with $\lambda_i - \lambda_{i+1} \geq 2$ for all $i = 1, 2, ...$, we have $\lambda_k > \rho_{m,k}$ for $k = 1, 2, ...$ which is a contradiction. If $\lambda_{i+1} < \rho_{m,i+1}$, we have $\ell(\lambda) \leq m$ and $\ell(\lambda) = \sum_i \lambda_i < \sum_i \rho_{m,i} = m(m+1)/2$ which is also a contradiction.

Combined with Proposition 4.7, Corollary 4.8 and Proposition 4.9 we can see that Bessenrodt’s criterion may be more effective than Pak’s (see also Remark 4.5 in [4]). So we have the following problem.

**Problem 4.10.** Does it hold that $N(\rho_m) < N(\rho_m)$ for $m \geq 3$?

The following problem gives an upper bound of non-zero elements on columns of the character table of $S_n$. It was mentioned in [13] and raised by A. Evseev.

**Problem 4.11.** [13] Let $\pi \in S_n$. Does it always hold that the number of irreducible characters of $S_n$ not vanishing on $\pi$ is at most equal to the number of irreducible characters of $C_{S_n}(\pi)$?

5. **Triple-hooks in tensor squares**

In [9, Cor. 6.1] and [4, Thm. 4.10], Ikenmeyer and Bessenrodt showed that irreducible representations corresponding to hooks and double-hooks appear in $[\rho_m] \otimes [\rho_m]$, respectively. In this section, we discuss the appearance of triple-hooks, that is, partitions with
Durfee size 3. Our main result is that if the tensor squares of the first 26 staircase partitions contain irreducible representations corresponding to triple-hooks then all tensor squares contain triple-hooks (see Theorem 5.8).

**Lemma 5.1.** [12, Thm. 9.1] For partitions \( \mu, \nu \vdash n \), if \( \mu \) has distinct row lengths and \( \mu \leq \nu \), then \( g(\mu, \mu, \nu) > 0 \).

It is well known that the Kronecker coefficient is invariant when two of its partitions take transpose (see e.g. Lemma 2.2 and 2.3 in [12]).

**Lemma 5.2.** For Kronecker coefficient \( g(\lambda, \mu, \nu) \), we have \( g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu') = g(\lambda', \mu', \nu) \).

By Lemma 5.2, Corollary 1.9 of [21] can be reformulated as follows.

**Lemma 5.3.** [21, Cor. 1.9] Suppose that \( \lambda = (m, m - 1) \). Then \( \lambda' = (2^{m-1}, 1) \) and \( g(\lambda', \lambda', \mu) > 0 \) for \( \ell(\mu) \leq 4 \).

**Lemma 5.4.** For \( m \geq 3 \), let \( \nu = (2m - 3, 2m - 3) \) and \( \mu = (m, m - 1, m - 2, m - 3) \). Then \( \mu' = (4^{m-3}, 3, 2, 1) \) and \( g(\mu', \mu', \nu) > 0 \).

**Proof.** By definition we have \( \mu' = (4^{m-3}, 3, 2, 1) \). For \( m \geq 3 \), we have \( \nu \geq \mu \). Since \( \mu \) has distinct row lengths, by Lemma 5.1 we have \( g(\mu, \mu, \nu) > 0 \). Thus by Lemma 5.2 we have \( g(\mu', \mu', \nu) > 0 \). \( \square \)

**Lemma 5.5.** If \( \lambda = (m, m - 1, m - 2) \) and \( \mu = (m - 1, m - 1, m - 1) \), then \( \lambda' = (3^{m-2}, 2, 1) \) and \( g(\lambda', \lambda', \mu) > 0 \).

**Proof.** We will show it by \( m = 3k, 3k - 1, 3k - 2 \) for \( k \in \mathbb{N} \), respectively.

(1) Suppose that \( m = 3k \). Then \( m - 3 = 3(k - 1) \). Let \( \tau = (m - 3, m - 3, m - 3) \). By Theorem 4.6 of [16], we have \( g((3^3), (3^3), (3^3)) > 0 \). Thus stretching it we have \( g(\tau, \tau, \tau) = g(((3k - 3)^3), ((3k - 3)^3), ((3k - 3)^3)) > 0 \) by the semigroup property. Moreover, we have \( g(\rho_2, \rho_3, (2^3)) > 0 \). Thus, by semigroup property we have \( g(\lambda, \lambda, \mu) = g(\tau + \rho_3, \tau + \rho_3, \tau + (2^3)) > 0 \).

(2) Suppose that \( m = 3k - 1 \). Then \( m - 2 = 3(k - 1) \). Let \( \sigma = (m - 2, m - 2, m - 2) \). Just like the proof in (1) we have \( g(\sigma, \sigma, \sigma) > 0 \). Since \( g(\rho_2, \rho_2, (1^3)) > 0 \), we have \( g(\lambda, \lambda, \mu) = g(\sigma + \rho_2, \sigma + \rho_2, \sigma + (1^3)) > 0 \).

(3) Suppose that \( m = 3k - 2 \). Then \( m - 4 = 3(k - 2) \). Let \( \nu = (m - 4, m - 4, m - 4) \). Just like the proof in (1) we have \( g(\nu, \nu, \nu) > 0 \). Let \( \gamma = (2^3) \). By [21, Thm. 1.6], we have \( g(\gamma, \gamma, \gamma) = g((3, 3), (3, 3), (2^3)) > 0 \). Moreover, \( g(\rho_2, \rho_2, (1^3)) > 0 \). So we have \( g(\lambda, \lambda, \mu) = g(\nu + \gamma + \rho_2, \nu + \gamma + \rho_2, \nu + \gamma + (1^3)) > 0 \).

By definition and Lemma 5.2 we have \( \lambda' = (3^{m-2}, 2, 1) \) and \( g(\lambda', \lambda', \mu) = g(\lambda, \lambda, \mu) > 0 \) which completes the proof. \( \square \)

In the following, we give a proof of Corollary 6.1 in [9] without using its Theorem 2.1. Similar method is used in the proof of Proposition 5.7 and Theorem 5.8.

**Proposition 5.6.** For every \( \nu \vdash m(m + 1)/2 \), if \( d(\nu) = 1 \) (i.e. \( \nu \) is a hook), then we have \( g(\rho_m, \rho_m, \nu) > 0 \).
Proof. If \(d(v) = 1\), we let \(v = (a, 1, \ldots)\) and \(v' = (x, 1, \ldots)\). Then we have \(|v| = a + x - 1 = m(m + 1)/2\). Suppose that \(g(\rho_k, \rho_k, \tau) > 0\) for \(1 \leq k \leq m-1\) and \(\tau + k(k+1)/2\) with \(d(\tau) = 1\). We will show that \(g(\rho_m, \rho_m, \nu) > 0\) by induction.

(1) If \(a \geq m + 1\), then \(\tau = v - (m)\) is a partition of \((m-1)m/2\) which is also a hook. Then by induction we have \(g(\rho_{m-1}, \rho_{m-1}, \tau) > 0\). Since \(g((1^m), (1^m), (m)) > 0\), by semigroup property we have
\[
g(\rho_m, \rho_m, \nu) = g(\rho_{m-1} + (1^m), \rho_{m-1} + (1^m), \tau + (m)) > 0.
\]

(2) If \(x \geq m + 1\), then by Lemma 5.2 the positivity of \(g(\rho_m, \rho_m, \nu)\) is equivalent to \(g(\rho_m, \rho_m, \nu') > 0\) which follows from the proof in (1).

(3) Suppose that both \(a\) and \(x\) are less than \(m\). Then we have
\[
\frac{m(m + 1)}{2} = |v| = a + x - 1 \\
\leq m + m - 1 \\
= 2m - 1,
\]
which implies that \(m \leq 2\). It is well known that for all \(v + 3\) we have \(g(\rho_2, \rho_2, v) > 0\). □

In [4], Bessenrod showed that all double-hooks appear in \([\rho_m] \otimes [\rho_m]\). In the following, we show a weaker result in a different way.

Proposition 5.7. Suppose that \(\mu + (m+1)/2\) and \(d(\mu) = 2\). If \(g(\rho_m, \rho_m, \mu) > 0\) for \(1 \leq m \leq 13\), then \(g(\rho_m, \rho_m, \mu) > 0\) for all \(m \geq 1\).

Proof. If \(d(\mu) = 2\), we let \(\mu = (a, b, \ldots)\) and \(\mu' = (x, y, \ldots)\). Then we have \(|\mu| = a + b + x + y - 4 = m(m + 1)/2\). Suppose that \(g(\rho_k, \rho_k, \lambda) > 0\) for \(1 \leq k \leq m - 1\) and \(d(\lambda) = 2\). Under the assumption \(g(\rho_m, \rho_m, \mu) > 0\) for \(1 \leq m \leq 13\) we will show that \(g(\rho_m, \rho_m, \mu) > 0\) for all \(m \geq 1\) by induction.

(1) Assume that \(a + b - 4 \geq 4m - 6\).

If \(a - b \geq m\), then \(\mu - (m)\) is a partition. Then by assumption we have \(g(\rho_{m-1}, \rho_{m-1}, \mu - (m)) > 0\). Since \(g((1^m), (1^m), (m)) > 0\), by semigroup property we have
\[
g(\rho_m, \rho_m, \mu) = g(\rho_{m-1} + (1^m), \rho_{m-1} + (1^m), \mu - (m) + (m)) > 0.
\]

In the following, we assume that \(a < b < m\).

Suppose that \(a > b\). If \(a - b\) is odd, then \(2m - 1 - (a - b)\) is even. Let \(\tau = (2m-1-(a-b))/2 + a - b, (2m-1-(a-b))/2 + 2m - 1\). By \(a + b + 4 \geq 4m - 6\) and \(\ell(\tau) \leq 2\), we have that \(\mu - \tau\) is still a partition. By assumption and Lemma 5.3 we have \(g(\rho_{m-2}, \rho_{m-2}, \mu - \tau) > 0\) and \(g((2m-1, 1), (2m-1, 1), \tau) > 0\). Hence, by semigroup property we have
\[
g(\rho_m, \rho_m, \mu) = g(\rho_{m-2} + (2m-1, 1), \rho_{m-2} + (2m-1, 1), \mu - \tau + \tau) > 0.
\]

If \(a - b\) is even, then \(a - b \geq 2\) and \(2m - 1 - (a - b)\) is odd. Define \(\nu = (2m-1-(a-b))/2 + a - b, (2m-1-(a-b))/2 + 2m - 1\). Then \(\nu\) is a partition of \(2m - 1\). By \(a + b + 4 \geq 4m - 6\) and \(\ell(\nu) = 2\), we have that \(\mu - \nu\) is still a partition. Then, similarly we have
\[
g(\rho_m, \rho_m, \mu) = g(\rho_{m-2} + (2m-1, 1), \rho_{m-2} + (2m-1, 1), \mu - \nu + \nu) > 0.
\]

Suppose that \(a = b\). Let \(\sigma = (2m - 3, 2m - 3) + 4m - 6\). Then by \(a + b - 4 \geq 4m - 6\), we have \(\mu - \sigma\) is a partition. By assumption we have \(g(\rho_{m-4}, \rho_{m-4}, \mu - \sigma) > 0\). By Lemma 5.4, we have \(g((4m-3, 3, 2, 1), (4m-3, 3, 2, 1), \sigma) > 0\). Hence, by semigroup property we have
\[
g(\rho_m, \rho_m, \mu) = g(\rho_{m-4} + (4m-3, 3, 2, 1), \rho_{m-4} + (4m-3, 3, 2, 1), \mu - \sigma + \sigma) > 0.
\]
(2) Similarly, if \( x + y - 4 \geq 4m - 6 \), by taking transpose and the same proof as above we also have \( g(\rho_m, \rho_m, \mu) > 0 \).

(3) If both \( a + b - 4 < 4m - 6 \) and \( x + y - 4 < 4m - 6 \), then we have that

\[
\frac{m(m + 1)}{2} = |\mu| = a + b + x + y - 4
\]

\[
< 4m - 6 + 4 + 4m - 6
\]

\[
= 8m - 8,
\]

which implies that \( m \leq 13 \).

Thus, if \( g(\rho_m, \rho_m, \mu) > 0 \) for \( 1 \leq m \leq 13 \), we have \( g(\rho_m, \rho_m, \mu) > 0 \) for all \( m \geq 1 \). \( \square \)

Following the proof of Proposition 5.7, we obtain the main result of this section.

**Theorem 5.8.** Suppose that \( \mu + m(m + 1)/2 \) and \( d(\mu) = 3 \). If \( g(\rho_m, \rho_m, \mu) > 0 \) for \( 1 \leq m \leq 26 \), then \( g(\rho_m, \rho_m, \mu) > 0 \) for all \( m \geq 1 \).

**Proof.** If \( d(\mu) = 3 \), we let \( \mu = (a, b, c,...) \) and \( \mu' = (x, y, z,...) \). Then we have \( |\mu| = a + b + c + x + y + z - 9 = m(m + 1)/2 \). Suppose that \( g(\rho_{k+1}, \rho_{k+1}, \lambda) > 0 \) for \( 1 \leq k \leq m - 1 \) and \( d(\lambda) = 3 \). Under the assumption \( g(\rho_m, \rho_m, \mu) > 0 \) for \( 1 \leq m \leq 26 \), we will show \( g(\rho_m, \rho_m, \mu) > 0 \) by induction.

(1) Assume that \( a - b + 2(b - c) \geq 4m - 6 \). Following the proof of Proposition 5.7, we can find partitions \( \tau + m, 2m - 1 \) or \( 4m - 6 \) such that \( \ell(\tau) \leq 2 \) and \( \mu - \tau \) is still a partition. Then by induction we have \( g(\rho_m, \rho_m, \mu) > 0 \).

(2) Assume that \( c - 3 \geq m - 1 \). Let \( \tau = (m - 1, m - 1, m - 1) \). Then \( \mu - \tau \) is also a partition. By assumption we have \( g(\rho_{m-3}, \rho_{m-3}, \mu - \tau) > 0 \). By Lemma 5.5, we have \( g((3^{m-2}, 2, 1), (3^{m-2}, 2, 1), \tau) > 0 \). Hence, by semigroup property we have

\[
g(\rho_m, \rho_m, \mu) = g(\rho_{m-3} + (3^{m-2}, 2, 1), \rho_{m-3} + (3^{m-2}, 2, 1), \mu - \tau + \tau) > 0.
\]

(3) Similarly, if \( x - y + 2(y - z) \geq 4m - 6 \) or \( z - 3 \geq m - 1 \), by taking transpose and the same proof as above we also have \( g(\rho_m, \rho_m, \mu) > 0 \).

(4) Suppose that

- \( a - b + 2(b - c) < 4m - 6 \) and \( c - 3 < m - 1 \);

- \( x - y + 2(y - z) < 4m - 6 \) and \( z - 3 < m - 1 \).

Then we have that

\[
\frac{m(m + 1)}{2} = |\mu| = 9 + a - b + 2(b - c) + 3(c - 3) + x - y + 2(y - z) + 3(z - 3)
\]

\[
< 9 + 4m - 6 + 3(m - 1) + 4m - 6 + 3(m - 1)
\]

\[
= 14m - 9,
\]

which implies that \( m \leq 26 \).

Thus, if \( g(\rho_m, \rho_m, \mu) > 0 \) for \( 1 \leq m \leq 26 \), then we have \( g(\rho_m, \rho_m, \mu) > 0 \) for all \( m \geq 1 \). \( \square \)

**Remark 5.9.** It has been verified that Saxl conjecture is true for \( \rho_m \) when \( 1 \leq m \leq 9 \) [12, Sec. 7]. It is interesting to find a simple way to verify that \( g(\rho_m, \rho_m, \mu) > 0 \) for \( 1 \leq m \leq 26 \) and \( \mu + m(m + 1)/2 \) with \( d(\mu) = 3 \).
6. Final remarks

For each $n$, we consider self-conjugate partitions that are close to staircase partitions as follows. For each $n \in \mathbb{N}$, there exist $m, k$ such that $n = \frac{m(m+1)}{2} + k$ where $0 \leq k \leq m$. It is not hard to verify the following conditions.

(6.1) If $m$ is even, then for each $k$ there are self-conjugate partitions $\lambda \vdash n$ such that $\rho_m \subseteq \lambda \subseteq \rho_{m+1}$.

(6.2) If $m$ is odd and $k$ even, then for each $k$ there are self-conjugate partitions $\lambda \vdash n$ such that $\rho_m \subseteq \lambda \subseteq \rho_{m+1}$.

(6.3) If $m$ and $k$ are odd, then no self-conjugate partitions of $n$ lie between $\rho_m$ and $\rho_{m+1}$.

But we can find self-conjugate partitions $\lambda \vdash n$ such that $\rho_{m-1} \subseteq \lambda \subseteq \rho_{m+2}$. In fact, if $k = 1$ we can find self-conjugate partitions $\lambda \vdash n$ such that $\rho_{m-1} \subseteq \lambda \subseteq \rho_{m+1}$. For example, if $n = 7$, then $m = 3$ and $k = 1$. We let $\lambda = (4, 1, 1, 1)$. If $k = 3, 5, 7$... we can find self-conjugate partitions $\lambda \vdash n$ such that $\rho_m \subseteq \lambda \subseteq \rho_{m+2}$. For example, if $n = 18$, then $m = 5$ and $k = 3$. We let $\lambda = (5, 4, 4, 4, 1)$.

Definition 6.1. Self-conjugate partitions satisfying conditions (6.1), (6.2), (6.3) above are called staircase-like partitions.

There exist staircase-like partitions for each $n \geq 3$. If $n = m(m+1)/2$ is a triangular number, then the corresponding staircase-like partition is just $\rho_m$. Comparing with Conjecture 1.1 of [16], we give a general Saxl conjecture as follows.

Conjecture 1 (General Saxl conjecture). For every $n$ except 2, 4, 9, suppose that $\lambda \vdash n$ is a staircase-like partition. Then $[\lambda] \otimes [\lambda]$ contains every irreducible representation of $S_n$ as a constituent.

If $\lambda$ is self-conjugate, then in several ways we can add 1 or 2 boxes on $\lambda$ to make it become another self-conjugate partition. For example, adding a box on $\rho_4 = (4, 3, 2, 1)$ we get $(4, 3, 3, 1)$ which is self-conjugate. Adding 2 boxes on $\rho_4 = (4, 3, 2, 1)$ we get a self-conjugate partition $(5, 3, 2, 1, 1)$. We want to know the growth behavior of Kronecker coefficient as the growth of partitions. We raise the following problem. Related discussions can be found in [5].

Problem 6.2. For $\lambda \vdash n$, suppose that $[\lambda] \otimes [\lambda]$ contains every irreducible representation of $S_n$ as a constituent. By adding at most 2 boxes on $\lambda$ we get another self-conjugate partition $\mu$ (not uniquely) such that $\lambda \subseteq \mu$ and $|\mu/\lambda| = 2$ (or 1). Does there always exists some $\mu$ such that $[\mu] \otimes [\mu]$ also contains every irreducible representation of $S_{n+2}$ (or $S_{n+1}$) as a constituent?

Remark 6.3. If Conjecture 1 is true, then it can be viewed as a special case of Problem 6.2. In fact, we can get $\rho_n$ by adding 1 or 2 boxes on $\rho_{n-1}$ consecutively and keeping the partitions staircase-like.

References

1. G. Almkvist, G. E. Andrews, A Hardy-Ramanujan Formula for Restricted Partitions, Journal of Number Theory, 38 (1991), 135-144.
2. G. E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, 1976.
3. A. Barvinok, S.V. Fomin, Sparse interpolation of symmetric polynomials, Advances in Applied Mathematics, 18 (1997), 271-285.
4. C. Bessenrodt, Critical classes, Kronecker products of spin characters, and the Saxl conjecture, Algebraic Combinatorics, 1 (2018), 353-369.
5. E. Briand, A. Rattany, and M. Rosasz, On the growth of Kronecker coefficients, Seminaire Lotharingien de Combinatoire, 78B (2017) Article 70, 12 pp.
6. F. Castillo, F. Liu, Berline-Vergne Valuation and Generalized Permutahedra, arXiv:1509.07884v2. To appear in: Discrete & Computational Geometry.

7. P. X. Gallagher, Group characters and commutators, Math. Z. 79(1962), 122-126.

8. A. Giambruno, G. Leal, Central Units, Class Sums and Characters of the Symmetric Group, Communications in Algebra, 38(2010), 3889-3896.

9. C. Ikenmeyer, The Saxl conjecture and the dominance order, Discrete Mathematics 338(2015) 1970-1975.

10. G. James, A. Kerber, The Representation Theory of the Symmetric Group, Addison-Wesley, London, 1981.

11. C.T. Hepler, On the complexity of computing characters of finite groups, thesis, University of Calgary, 1994, available at https://dspace.ucalgary.ca/handle/1880/45530.

12. S. Luo, M. Sellke, The Saxl conjecture for fourth powers via the semigroup property, J. Algebr. Comb. 45(2017) 33-80.

13. L. Morotti, On the number of non-zero character values in generalized blocks of symmetric groups, J. Algebr. Comb. 47(2018) 233-239.

14. H. Narayanan, On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients, J. Algebr. Comb. 24(2006) 347-354.

15. R. Nath, Advances in the Theory of Cores and Simultaneous Core Partitions, The American Mathematical Monthly, 124(2017), 844-861.

16. I. Pak, G. Panova, E. Vallejo, Kronecker products, characters, partitions, and the tensor square conjectures, Adv. Math. 288(2016) 702-731.

17. I. Pak, G. Panova, On the complexity of computing Kronecker coefficients, Comput. Complex. 26(2017), 1-36.

18. B. Pittel, Asymptotic joint distribution of the extremities of a random Young diagram and enumeration of graphical partitions, Adv. Math. 330(2018), 280-306.

19. A. Postnikov, Permutahedra, Associahedra, and Beyond, International Mathematics Research Notices, Vol. 2009, No. 6, 1026-1106.

20. R.P. Stanley, Enumerative Combinatorics, Vol. 1 and Vol. 2, Cambridge Univ. Press, Cambridge, 2012, 1999.

21. V. V. Tewari, Kronecker coefficients for some near-rectangular partitions, Journal of Algebra 429(2015) 287-317.

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