DYNAMIC OPTIMAL DECISION MAKING FOR MANUFACTURERS WITH LIMITED ATTENTION BASED ON SPARSE DYNAMIC PROGRAMMING

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Abstract. In a fully competitive industry, the market demand is changing rapidly. Thus, it is important for manufacturers to manage their inventory effectively as well as to determine the best order quantity and optimal production strategy. In this paper, our concern is how shall a manufacturer with limited attention determine his optimal order quantity and optimal production strategy in an environment when many factors are volatile, such as the price of raw materials (respectively, finished goods) and attrition rate of inventory of raw materials (respectively, finished product). Under this environment, it is observed, according to various empirical studies, that decision makers tend to focus their attention on factors with major changes. Taking all these into account, our problem is formulated as a discrete-time stochastic dynamic programming. We propose a general approach based on the sparse dynamic programming method to solve this multidimensional dynamic programming problem. From the numerical examples solved using the proposed method, it is interesting to observe that decision makers with limited attention do not adjust their final decision when the volatility is small.

1. Introduction. It is observed, according to a survey published in Efficio Consulting magazine, that more than 55% of purchasing managers have regarded the fluctuation of products price as one of the biggest challenges (Jenkinson [17]). Sebastian et al. [29] have found that the financial loss incurred by the manufacturing industry due to excessive price changes is as high as five hundred million euros. Thus, uncertain price fluctuation is clearly a critical factor which is required to be dealt with during the formulation of both the pricing and ordering strategies. Moreover, many products will deteriorate though with different rates. For some, their deterioration effect is negligible, but for others, their deterioration rates are

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much too significant to be ignored (Goyal and Giri [16]). So, when determining optimal orders quantity and production strategy, a manufacturer should not only consider the fluctuation of price, but also the deterioration of inventory and other factors. However, there are cognition costs when he considers those factors and his attention are scared. In this paper, we will study the decision problem of a manufacturer with limited attention, i.e., the manufacturer will allocate his attention to salience factors and ignore the factors with small fluctuation when making decision.

The optimal inventory control problem has been studied in different settings by many researchers over the years. Taking into consideration of random fluctuation nature of the market price, Zheng [36] studies the inventory system under the assumption that the demand is a random variable. Kalymon [19] models the price fluctuation as a Markov process and Gavirneni [15] obtains the optimal order type production strategy under the same assumption. Zipkin [37], Wu and Chen [34] study the optimal inventory system under the situation when both the demand and price are uncertain. Berling and Martinez-de-Albeniz [3] investigate the optimal inventory policy, where the price fluctuation is assumed to follow geometric Brownian motion or Ornstein Uhlenbeck process. Devalkar et al. [9] study a multi-period optimization problem taking into consideration of raw materials procurement, production strategy and product sales. For this multi-period optimization problem, the optimal solution is obtained. Akella [2], Gaur and Seshadri [14], Berling and Rosling [4] investigate the optimal inventory strategy from the perspective of risk aversion. In their opinion, decision makers should consider hedging inventory risk for products with short life cycle, such as seasonal products, and they should deal with this risk through effective inventory management. Also, they believe that it is a good strategy for firms to avoid risk by ordering goods from multiple suppliers. For example, Fu [11] studies the benefits of multi-channel ordering for a supply chain in single period. For some other researchers, they study the optimal inventory strategy under the situation for which the random nature of the products deterioration rates in stock is being taken into consideration. Zhang, Bai and Tang [35] consider the problem of simultaneously determining the price and inventory control strategies for deteriorating items. Bouras and Tadj [7] assume that new and remanufactured items are subject to deterioration dynamic demands and obtain the explicit expressions of optimal manufacturing rate and inventory levels. Lashgari, Taleizadeh and Sana [20] study an inventory control problem for deteriorating items with back-ordering and financial considerations. Whitin [33] appears to be the first paper devoted to the study of the inventory of perishable goods. Nahmias and Demmy [25], Raafat [26], Shah and Shah [30] are three survey articles reviewing the progress in the study of the inventory management of perishable goods in the 1980s and 1990s, respectively. Chen and Lin [8] study the inventory problem of perishable goods under the situation when the life cycle is normally distributed. Sivakumar [32] studies related problems for which customers demand is taken into consideration.

In real life, there is a cost incurred for obtaining and processing information (hereinafter referred to as “information cost”). Thus, decision makers would rationally choose to ignore some less important information. Kahneman [18] regards attention as a scarce resource. Sims [31] introduces for the first time the concept of limited attention (i.e. rational inattention), where it is pointed out that the decision maker’s ability to process information is limited. Thus, when a large amount
of information is received, the decision maker would selectively focus on those information which are important, while ignoring those which are less important. Liu et al. [21] consider dynamic pricing policies for two firms producing products that display network effects in duopoly markets where consumers are bounded rational. Bi et al. [5] find the effect of the first price on the memory window and long-term profits decreases as the length of memory window increases considering that consumers memories are limited and their recall of previous prices obeys a first-order Markov stochastic process. In Duffie and Sun [10], Reis [27], Abel and Panageas [1], and Schwarzstein [28], no information received are ignored, where the information cost is modeled as a fixed information cost. Although these models are meaningful, the number of variables involved is huge and hence they are difficult to be solved effectively. More importantly, many of these information can be ignored without causing noticeable concern. In Sims [31], the entropy theory is used to model the information cost, and since then, this model is used by many researchers to study rational inattention, see, for example, (Mackowiak and Wiederholt [22], [23]). In this model, the decision made by the decision maker is regarded as random. Thus, it is hard to work out the analytical solution (Matejka and Sims [24]). In Gabaix [12], the assumption of fully rational consumers is being relaxed and the corresponding dynamic programming is formulated and solved. In Bi et al. [6], the sparse max operator proposed by Gabaix [12] is extended and a new approach to deal with dynamic programming under stochastic terms under the assumption of the agents limited attention is proposed.

However, it appears that the influence of the fluctuations of various external factors on the optimal inventory control has not been seriously investigated in the literature from the perspective of decision makers with limited attention. Based on the work by Gabaix [12], we study the optimal inventory strategy under the situation when the decision makers attention is limited. The one or two-periods model considered in Gabaix [13] is extended into an infinite-period model in this paper. Based on the empirical evidence in the oil industry and dynamic market price (Wu and Chen [34]), we shall uncover critical inter-relationship between product price and inventory across two dimensions: time and product manufacturing. We use two-echelon inventory system to study the dynamic pricing and inventory strategies for enterprises. The first level of inventory is for the storage of raw materials while the second level of inventory is for the storage of finished products. The aim of this paper is on the analysis of the influence of the volatility factors on the optimal decision when the manufacturer’s attention is limited.

2. Model assumptions and setup.

2.1. Model assumptions. Consider the situation for which there is only one manufacturer. It sells a finished product in the market. This manufacturer decides on both the quantities of products to be ordered and to be produced. The aim is to manage the inventory effectively such that the long-term profit is maximized. We assume that the manufacturers attention is limited, meaning that the manufactures attention will only focus on important factors during decision-making. Other less important factors will only be taken into consideration when their fluctuations exceed a certain threshold. Assume that the production of one unit of the finished goods requires only one unit of one kind of raw material. As shown in Figure 1, the manufacturers production inventory system consists of three decision-making parts:
1) The storage of raw materials; 2) The control of raw material consumption rate; and 3) The storage of finished goods.

2.2. Model setup. Let $p_1(t)$ and $p_2(t)$ denote, respectively, the price of raw material and the price of finished goods in $t$ period, and these two prices change dynamically as given below:

$$p_1(t) = \bar{p}_1 + \hat{p}_1(t).$$  \hspace{1cm} (1)

$$p_2(t) = \bar{p}_2 + \hat{p}_2(t).$$  \hspace{1cm} (2)

where $\bar{p}_1$ and $\bar{p}_2$ denote their mean values, while $\hat{p}_1(t)$ and $\hat{p}_2(t)$ denote their respective random fluctuations. We assume that these random fluctuations of prices are AR(1) processes, that is,

$$\hat{p}_1(t + 1) = \rho_{\hat{p}_1}\hat{p}_1(t) + \varepsilon_{\hat{p}_1,t+1}.$$  \hspace{1cm} (3)

and

$$\hat{p}_2(t + 1) = \rho_{\hat{p}_2}\hat{p}_2(t) + \varepsilon_{\hat{p}_2,t+1}.$$  \hspace{1cm} (4)

where $\rho_{\hat{p}_1}$ and $\rho_{\hat{p}_2}$ denote, respectively, the correlation coefficients of the respective fluctuations, and are time independent, while $\varepsilon_{\hat{p}_1,t+1}$ and $\varepsilon_{\hat{p}_2,t+1}$ are mean-zero shocks. Now, let $x_1(t)$ and $x_2(t)$ denote, respectively, the inventory of raw materials and the inventory of finished goods in $t$ period. They are given by

$$x_1(t + 1) = (1 - \theta_1(t))x_1(t) + \lambda(t) - q(t).$$  \hspace{1cm} (5)

and

$$x_2(t + 1) = (1 - \theta_2(t))x_2(t) + q(t) - s(t).$$  \hspace{1cm} (6)

where $\lambda(t)$, $q(t)$ and $s(t)$ denote, respectively, the purchase rate of raw materials, production rate, and sales rate of products. $\theta_1(t)$ and $\theta_2(t)$ represent the deterioration rates of raw materials and finished goods in $t$ period, respectively. Since the external environment (such as temperature) is never deterministic, $\theta_1(t)$ and $\theta_2(t)$ are changing dynamically, that is,

$$\theta_1(t) = \bar{\theta}_1 + \hat{\theta}_1(t).$$  \hspace{1cm} (7)

$$\theta_2(t) = \bar{\theta}_2 + \hat{\theta}_2(t).$$  \hspace{1cm} (8)

where $\bar{\theta}_1$ and $\bar{\theta}_2$ are the respective mean values of the deterioration rates, and $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ denote their random fluctuation parts in $t$ period. Assume that the random fluctuations of these deterioration rates follow AR(1) processes, that is,

$$\hat{\theta}_1(t + 1) = \rho_{\hat{\theta}_1}\hat{\theta}_1(t) + \varepsilon_{\hat{\theta}_1,t+1}.$$  \hspace{1cm} (9)
\[ \theta_2(t + 1) = \rho_{\theta_2} \theta_2(t) + \varepsilon_{\theta_2,t+1}. \]  
(10)

where \( \rho_{\theta_1}, \rho_{\theta_2}, \varepsilon_{\theta_1,t+1} \) and \( \varepsilon_{\theta_2,t+1} \) are defined similarly as those of \( \rho_{p_1}, \rho_{p_2}, \varepsilon_{p_1,t+1} \) and \( \varepsilon_{p_2,t+1} \).

The manufacturer also needs to consider the operating cost and holding cost of both the raw materials and finished goods. Operating cost includes all costs, except holding cost (such as labor cost for cargo handling and mechanical manufacturing cost). In general, operating cost will be high when holding cost is very low (Wu and Chen [34]). For example, low inventory of finished goods may induce an increase in stockout cost, or additional transportation cost, as the manufacturer will tend to get the delivery of finished goods from elsewhere for meeting the demand. Let \( g(q, x_1, x_2) \) and \( h(x_1(t), x_2(t)) \) denote the operating cost and holding cost in \( t \) period, respectively. They can be written (see Wu and Chen [34]) as follows:

\[ g(q(t), x_1(t), x_2(t)) = \alpha q(t) + \delta q^2(t) - \gamma (x_1(t) + x_2(t)). \]  
(11)

and

\[ h(x_1(t), x_2(t)) = \eta (x_1^2(t) + x_2^2(t)). \]  
(12)

where \( \alpha, \delta, \gamma \) and \( \eta \) are constants. To continue, the following assumptions are imposed.

**Assumption 1.** The operating cost \( g(q, x_1, x_2) \) is increasing with respect to \( x_1(t) \) and \( x_2(t) \), while it is decreasing with respect to \( q(t) \).

**Assumption 2.** The holding cost \( h(x_1(t), x_2(t)) \) is a strictly increasing function with respect to \( x_1(t) \) and \( x_2(t) \). The long-term profit of fully rational manufacturer can be expressed as:

\[ V(x_1, x_2, \hat{p}_1, \hat{p}_2, \hat{\theta}_1, \hat{\theta}_2) \]
\[ = \max_{a_t} \sum_{t=0}^{\infty} \beta^t [s(t)p_2(t) - \lambda(t)p_1(t) - g(q(t), x_1(t), x_2(t)) - h(x_1(t), x_2(t))]. \]  
(13)

subject to

\[ p_1(t) = \hat{p}_1 \hat{p}_1(t) + \varepsilon_{\hat{p}_1,t+1} \]
\[ p_2(t) = \hat{p}_2 \hat{p}_2(t) + \varepsilon_{\hat{p}_2,t+1} \]
\[ x_1(t + 1) = (1 - \theta_1(t))x_1(t) + \lambda(t) - q(t), \]
\[ \theta_1(t) = \tilde{\theta}_1 + \tilde{\theta}_1(t) + \varepsilon_{\tilde{\theta}_1,t+1} \]
\[ x_2(t + 1) = (1 - \theta_2(t))x_2(t) + q(t) - s(t), \]
\[ \theta_2(t) = \tilde{\theta}_2 + \tilde{\theta}_2(t) + \varepsilon_{\tilde{\theta}_2,t+1}. \]

where \( \beta \) is the discount factor, and \( a_t = \{ \lambda t, p_1, s_1 \} \) are decision variables. We assume that the long-term profit is twice continuously differentiable.

For simplicity, hereinafter, we will omit the time marking variable (namely, \( t \) or \( t + 1 \) ) including its brackets, and add a single quotation mark to each of the variables in the time period \( t + 1 \) so as to distinguish these variables from those in the time period \( t \). For example, \( x_1 \) and \( x_1' \) denote, respectively, \( x_1(t) \) and \( x_1(t+1) \).

Let \( V^*(x_1, x_2, \hat{p}_1, \hat{p}_2, \hat{\theta}_1, \hat{\theta}_2) \) be the value function, which represents the maximal long-term profit starting from the state \( (x_1, x_2, \hat{p}_1, \hat{p}_2, \hat{\theta}_1, \hat{\theta}_2) \) all the way to infinite
time. Then, by Bellman equation for the long term profit of the manufacturer with limited attention, it follows that

\[
V^\pi(x_1, x_2, \hat{p}_1, \hat{p}_2, \hat{\theta}_1, \hat{\theta}_2) = \max_{s, \lambda, q} \left( sp_2 - \lambda p_1 - g(q, x_1, x_2) - h(x_1, x_2) \right) + \beta V^\pi(x'_1, x'_2, \hat{p}'_1, \hat{p}'_2, \hat{\theta}'_1, \hat{\theta}'_2) \tag{14}
\]

subject to

\[
p_1 = \bar{p}_1 + \hat{p}_1. \tag{15}
\]

\[
\hat{p}'_1 = \rho \hat{p}_1 + \varepsilon \hat{p}'_1. \tag{16}
\]

\[
p_2 = \bar{p}_2 + \hat{p}_2. \tag{17}
\]

\[
\hat{p}'_2 = \rho \hat{p}_2 + \varepsilon \hat{p}'_2. \tag{18}
\]

\[
x'_1 = (1 - \theta_1)x_1 + \lambda - q. \tag{19}
\]

\[
\theta_1 = \bar{\theta}_1 + \hat{\theta}_1. \tag{20}
\]

\[
\hat{\theta}'_1 = \rho \hat{\theta}_1 + \varepsilon \hat{\theta}'_1. \tag{21}
\]

\[
x'_2 = (1 - \theta_2)x_2 + q - s. \tag{22}
\]

\[
\theta_2 = \bar{\theta}_2 + \hat{\theta}_2. \tag{23}
\]

\[
\hat{\theta}'_2 = \rho \hat{\theta}_2 + \varepsilon \hat{\theta}'_2. \tag{24}
\]

Let this dynamic optimization programming problem be referred to as Problem (P).

3. Model analysis and solution method. Problem (P) is hard to be analyzed and solved because there are too many state variables and some of them fluctuate stochastically. However, we know that during the time of making decision, the decision maker may ignore those states if the fluctuations of these states are small enough. Thus, we propose to analyze Problem (P) based on the sparse max operator (Gabaix [12]).

3.1. Sparse max operator (Gabaix (2014)). An agent faces a maximization problem \( \max_a v(a, x) \), where \( a \) is a decision variable and \( x = (x_1, x_2, ..., x_n) \) is a state variable. Let \( m = (m_1, m_2, ..., m_n) \) be an attention vector, and let \( v(a, x, m) \) be a utility function. To make clear the ideas, take the following quadratic example

\[
v(a, x, m) := -\frac{1}{2} \left( a - \sum_{i=1}^{n} m_i x_i \right)^2. \tag{25}
\]

If \( m_i = 1 \), the agent pay full attention to \( x_i \); when \( m_i = 0 \), the agent is fully inattentive to it. We define a default action

\[
a^d := \arg \max_a v(a, x, m^d). \tag{26}
\]

where \( m^d \) is a default attention vector (\( m^d = 0 \) in most applications). We assume that agents have thinking cost \( \kappa \). When \( \kappa = 0 \), the agent is the traditional agent. The \( x_i, i = 1, 2, ..., n \) are random variables drawn from a distribution with mean \( \mu_i \) and standard deviation \( \sigma_i \).

For example, suppose the optimal consumption is chosen as \( a \), then the decision maker should consider not only his wealth, \( x_1 \), but also the interest rate, \( x_2 \), demographic trends in China, \( x_3 \), etc. There may be \( n > 100 \) factors that should in
principle be taken into account. However, most of them have a small impact on his
decision, i.e., their impact \( m_i \) is small in absolute value.

**Definition 3.1.** (Sparse max operator (Gabaix [12])) The sparse max,
\( \text{smax}_{a} m v(a, x, m) \), is defined according to the following procedure.

**Step 1.** Choose the attention vector \( m^* \):
\[
m^* = \arg \min_{m \in [0,1]^n} (E[v(a, x, m)] - C(m)). \tag{27}
\]
where \( C(m) \) is a thinking cost function (for example, \( C(m) = \kappa \sum_{i=1}^{n} m_i^a \)).

**Step 2.** Choose the action
\[
a^s = \arg \max_a v(a, x, m^*). \tag{28}
\]
and set the resulting utility to be \( v^s = v(a^s, x) \).

With these considerations, the definition of the attention function is given by
\[
A(\sigma^2) := \arg \min_{m \in [0,1]^n} (E[v(a, x, m)] - C(m)). \tag{29}
\]
It meaning that the optimal attention is given to a state variable with variance \( \sigma^2 \), and other factors are normalized to 1. Figure 2 plots the shape of attention function. The intuition is that the \( x_i \)'s are truncated. If \( |\sigma^2| \) is small enough, then \( x_i \) shouldnt be matter much anyway. Thus, \( m_{i}^* = 0 \), and the agent doesnt pay attention to \( x_i \) (if \( m_d^i = 0 \)). By performing a Taylor expansion of \( v(a, x, m) \) in a small neighbor of \( x \), we can define the truncation function (see Figure 3) as given below:
\[
\tau(b, \kappa) := bA\left(\frac{b^2}{\kappa^2}\right). \tag{30}
\]
where \( b \) is the coefficient associated with the state variable \( x \) in a linear function. The following lemma gives a more explicit version of the action.

**Lemma 3.2.** (Gabaix [12]) Suppose that the rational action is
\[
a^r(x) = a^d + \sum_i b_i x_i + O(||x||^2). \tag{31}
\]
Then the sparse action is
\[
a^s(x) = a^d + \sum_i \tau(b_i, \frac{\kappa_a}{\sigma_i}) x_i + O(||x||^2). \tag{32}
\]
where \( \kappa_a := (\kappa/|v_{aa}|)^{1/2} \).

Attention and Truncation Functions. In Gabaix [12], it is recommended that the attention function \( A(\sigma^2) \) be given by
\[
A(\sigma^2) = \max(1 - \frac{1}{\sigma^2}, 0). \tag{33}
\]
Then, the corresponding truncation function \( \tau_\alpha(b, \kappa) \) is (using (30)):
\[
\tau(b, \kappa) = b \max(1 - \frac{\kappa^2}{b^2}, 0). \tag{34}
\]
Figure 2 plots the attention function, and Figure 3 plots the corresponding truncation function.
3.2. Model analysis. Based on the sparse max operator defined earlier, we analyze Problem (P) in two steps: (1) define default action of Problem (P); and (2) obtain the sparse optimal decision by Taylor expansion and truncation function.

Let $x = \{x_1, x_2\}$ denote the state variable for which the decision maker must consider and $y = \{y_1, y_2, y_3, y_4\} = \{\hat{p}_1, \hat{p}_2, \hat{\theta}_1, \hat{\theta}_2\}$ be the state variable with small fluctuation and may be ignored.

First, assume that there is no random fluctuation in $y$, i.e. $\hat{p}_1 = 0$, $\hat{p}_2 = 0$, $\hat{\theta}_1 = 0$, $\hat{\theta}_2 = 0$. Then, we define the default solution of Problem (P) as follows:

$$a^d = \arg \max \left\{ u(t) + \beta V(x_1, x_2, 0, 0, 0, 0) \right\}. \quad (35)$$

where $u(t) = sp_2 - \lambda p_1 - g(q, x_1, x_2) - h(x_1, x_2)$. Then, using Taylor expansion, it is shown in equation (31) (Gabaix [12]) that the optimal decision by a fully rational decision maker is:

$$a^r(y) = a^d + \sum_i a_i y_i + o(\|y\|^2). \quad (36)$$

For the decision maker with limited attention, his optimal decision can be expressed as:

$$a^s(y) = a^d + \sum_i \tau(a_{yi}, \frac{\kappa \sigma_y}{\sigma_a}) y_i + o(\|y\|^2), \quad (37)$$

where $s$ denotes the sparse or limited attention. $\kappa$ is the thinking cost, $\kappa \in [0, 1]$; $\sigma_a$ and $\sigma_y$ denote, respectively, the standard deviations of $a$ and $y$, and $a_{yi} = \frac{V_y}{V_a}$, while $\tau(b, k) = \max(1 - \frac{b^2}{2k}, 0)$. 

**Figure 2. Attention function**

**Figure 3. Truncation function**
Lemma 3.3. For a fully rational manufacturer, the optimal ordering rate, production rate and sales rate are, respectively:

$$\lambda^* = \lambda^d + \lambda_{\hat{p}_1} \hat{p}_1 + \lambda_{\hat{p}_2} \hat{p}_2 + \lambda_{\hat{\theta}_1} \hat{\theta}_1 + \lambda_{\hat{\theta}_2} \hat{\theta}_2,$$

and

$$q^* = q^d + q_{\hat{p}_1} \hat{p}_1 + q_{\hat{p}_2} \hat{p}_2 + q_{\hat{\theta}_1} \hat{\theta}_1 + q_{\hat{\theta}_2} \hat{\theta}_2,$$

$$s^* = s^d + s_{\hat{p}_1} \hat{p}_1 + s_{\hat{p}_2} \hat{p}_2 + s_{\hat{\theta}_1} \hat{\theta}_1 + s_{\hat{\theta}_2} \hat{\theta}_2;$$

Where \( r \) denotes fully rational and \( d \) denotes the default value. Therefore, \( \lambda^d, q^d \) and \( s^d \) are the optimal control variables when the fluctuations of the less important factors are zero, that is, \( \hat{p}_1 = 0, \hat{p}_2 = 0, \hat{\theta}_1 = 0, \hat{\theta}_2 = 0 \). They are referred to as the default decision variables for the manufacturer.

Lemma 3.4. For a manufacturer with limited attention, the optimal ordering rate, production rate and sales rate are, respectively:

$$\lambda^* = \lambda^d + \tau(\lambda_{\hat{p}_1}, \frac{\kappa \sigma_{\lambda}}{\sigma_{p_1}}) \hat{p}_1 + \tau(\lambda_{\hat{p}_2}, \frac{\kappa \sigma_{\lambda}}{\sigma_{p_2}}) \hat{p}_2 + \tau(\lambda_{\hat{\theta}_1}, \frac{\kappa \sigma_{\lambda}}{\sigma_{\theta_1}}) \hat{\theta}_1 + \tau(\lambda_{\hat{\theta}_2}, \frac{\kappa \sigma_{\lambda}}{\sigma_{\theta_2}}) \hat{\theta}_2,$$

$$q^* = q^d + \tau(q_{\hat{p}_1}, \frac{\kappa \sigma}{\sigma_{p_1}}) \hat{p}_1 + \tau(q_{\hat{p}_2}, \frac{\kappa \sigma}{\sigma_{p_2}}) \hat{p}_2 + \tau(q_{\hat{\theta}_1}, \frac{\kappa \sigma}{\sigma_{\theta_1}}) \hat{\theta}_1 + \tau(q_{\hat{\theta}_2}, \frac{\kappa \sigma}{\sigma_{\theta_2}}) \hat{\theta}_2,$$

$$s^* = s^d + \tau(s_{\hat{p}_1}, \frac{\kappa \sigma_{\lambda}}{\sigma_{p_1}}) \hat{p}_1 + \tau(s_{\hat{p}_2}, \frac{\kappa \sigma_{\lambda}}{\sigma_{p_2}}) \hat{p}_2 + \tau(s_{\hat{\theta}_1}, \frac{\kappa \sigma_{\lambda}}{\sigma_{\theta_1}}) \hat{\theta}_1 + \tau(s_{\hat{\theta}_2}, \frac{\kappa \sigma_{\lambda}}{\sigma_{\theta_2}}) \hat{\theta}_2;$$

Where \( \lambda_{\hat{p}_1} \) denotes the first order derivative of the control variable \( \lambda \) with respect to the volatile variable \( \hat{p}_1 \), and \( \sigma_{\lambda} \) denotes the standard deviation of \( \lambda \). Other parameters carry similar meanings.

Proposition 1. The manufacturers default control variables \( \lambda^d, q^d \) and \( s^d \) are, respectively, given by:

$$\lambda^d = \hat{\theta}_1 x_1 + q = \theta_1 x_1 + \frac{\beta}{2\delta} \left[ -\frac{\gamma - 2\eta x_1}{1 - \beta(1 - \theta_1)} + \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \theta_2)} \right] - \frac{\alpha}{2\delta},$$

(38)

$$q^d = \frac{\beta}{2\delta} \left[ -\frac{\gamma - 2\eta x_1}{1 - \beta(1 - \theta_1)} + \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \theta_2)} \right] - \frac{\alpha}{2\delta},$$

(39)

$$s^d = q - \hat{\theta}_2 x_2 = \frac{\beta}{2\delta} \left[ -\frac{\gamma - 2\eta x_1}{1 - \beta(1 - \theta_1)} + \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \theta_2)} \right] - \frac{\alpha}{2\delta} - \hat{\theta}_2 x_2.$$

(40)

Proof. See Appendix.

Based on the implicit function theorem, we can show the validity of the results presented in the following proposition.
Proposition 2. 1) The influence coefficients of the volatile variables $\hat{p}_1$, $\hat{p}_2$, $\hat{\theta}_1$ and $\hat{\theta}_2$ on the ordering rate of the raw materials $\lambda$ are expressed, respectively, by

$$\lambda_{\hat{p}_1} = \frac{-1 + \beta \rho_{p1} V_{x_1' p_1'}}{\beta V_{x_1' x_1'}} \quad (41)$$

$$\lambda_{\hat{p}_2} = \frac{\rho_{p2} V_{x_2' p_2'}}{V_{x_1' x_1'}} \quad (42)$$

$$\lambda_{\hat{\theta}_1} = \frac{-x_1 V_{x_1' x_1'} + \rho_{\theta_1} V_{x_1' \theta_1}}{V_{x_1' x_1'}} \quad (43)$$

$$\lambda_{\hat{\theta}_2} = \frac{-\rho_{\theta_2} V_{x_2' \theta_2}}{V_{x_1' x_1'}} \quad (44)$$

2) Similarly, the influence coefficients of the volatile variables $\hat{p}_1$, $\hat{p}_2$, $\hat{\theta}_1$ and $\hat{\theta}_2$ on the production rate $q$ can be expressed, respectively, by

$$q_{\hat{p}_1} = \frac{-\beta \rho_{p1} V_{x_1' p_1'} + \beta \rho_{p3} V_{x_2' p_2'}}{-2b + \beta V_{x_1' x_1'} + \beta V_{x_2' x_2'}} \quad (45)$$

$$q_{\hat{p}_2} = \frac{-\beta \rho_{p2} V_{x_2' p_2'} + \beta \rho_{p3} V_{x_2' p_2'}}{-2b + \beta V_{x_1' x_1'} + \beta V_{x_2' x_2'}} \quad (46)$$

$$q_{\hat{\theta}_1} = \frac{\beta x_1 V_{x_1' x_1'} - \beta \rho_{\theta_1} V_{x_1' \theta_1} + \beta \rho_{\theta_1} V_{x_2' \theta_1}}{-2b + \beta V_{x_1' x_1'} + \beta V_{x_2' x_2'}} \quad (47)$$

$$q_{\hat{\theta}_2} = \frac{-\beta x_2 V_{x_2' x_2'} - \beta \rho_{\theta_2} V_{x_2' \theta_2} + \beta \rho_{\theta_2} V_{x_2' \theta_2}}{-2b + \beta V_{x_1' x_1'} + \beta V_{x_2' x_2'}} \quad (48)$$

3) The influence coefficients of the volatile variables $\hat{p}_1$, $\hat{p}_2$, $\hat{\theta}_1$ and $\hat{\theta}_2$ on the sales rate $s$ can be expressed, respectively, by

$$s_{\hat{p}_1} = \frac{\rho_{p1} V_{x_1' p_1'}}{V_{x_2' x_2'}} \quad (49)$$

$$s_{\hat{p}_2} = \frac{1 - \beta \rho_{p3} V_{x_2' p_2'}}{-\beta V_{x_1' x_1'}} \quad (50)$$

$$s_{\hat{\theta}_1} = \frac{\rho_{\theta_1} V_{x_1' \theta_1}}{V_{x_2' x_2'}} \quad (51)$$

$$s_{\hat{\theta}_2} = \frac{-x_2 V_{x_2' x_2'} - \rho_{\theta_2} V_{x_2' \theta_2}}{-\beta V_{x_1' x_1'}} \quad (52)$$

where

$$V_{x_1' p_1'} = \frac{-\beta \eta + \delta \hat{\theta}_1 (1 - \beta (1 - \hat{\theta}_1))}{\delta (1 - \beta \rho_{p1} \rho_{p3} (1 - \beta (1 - \hat{\theta}_1))},$$

$$V_{x_1' p_2'} = \frac{-\beta \eta}{\delta (1 - \beta \rho_{p2} (1 - \beta (1 - \hat{\theta}_1))},$$

$$V_{x_1' \theta_1' = 0},$$

$$V_{x_1' \theta_2' = 0},$$

$$V_{x_2' p_1'} = \frac{-\beta \eta}{\delta (1 - \beta \rho_{p1} (1 - \beta (1 - \hat{\theta}_2)))},$$

$$V_{x_2' p_2'} = \frac{-\beta \eta + \delta \hat{\theta}_2 (1 - \beta (1 - \hat{\theta}_2))}{\delta (1 - \beta \rho_{p2} (1 - \beta (1 - \hat{\theta}_2)))}.$$
Let \( \sigma \) be the thinking cost parameter, \( \lambda \) the ordering rate, \( \kappa \) the volatility parameters on the ordering rate, and \( \theta \) the variables on the increase of decision making when \( \kappa \) is increasing. In the case when \( \kappa = 0 \), meaning that the manufacturer is fully rational, we have \( \lambda^s_1 = -3.7809 \), \( \lambda^s_2 = 0.412 \) and \( \lambda^s_{\theta_1} = 14.2348 \). These results show that the change of the deterioration rate has a greater impact on the ordering rate of the raw materials compared to the change of the

\[
V_{x_1,\theta_1} = 0, \quad V_{x_2,\theta_2} = -\beta [V_2 + (1 - \theta_2)x_2V_2^2 - x_2^2] \\
V_2 = \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \theta_2)}, \quad V_2^2 = \frac{-2\eta}{1 - \beta(1 - \theta_2)}.
\]

Proof. See Appendix.

From the results obtained above, it is clearly observed that, \( \lambda_{\theta_2} = 0 \), and \( s_{\theta_1} = 0 \). This shows that the changing rate of inventory of the finished goods is unrelated to the optimal ordering rate when the volatile variables are independent of each other, and the changing rate of the inventory of the raw materials is also unrelated to the optimal sales.

4. Numerical examples. For illustration, numerical examples are considered in this section. As described in Wu and Chen [34], Berling and Martinez-de-Albeniz [3], we consider a chemical manufacturer, such as BASF for which its raw materials are crude oil and its finished goods are petroleum products. \( p_1, p_2 \) and \( \theta_1, \theta_2 \) represent the prices and the deterioration rates of crude oil and petroleum, respectively.

We next explain the impact of the volatile variables \( \theta_1, \theta_2, \theta_1 \) and \( \theta_2 \) on the optimal ordering rate \( \lambda \), the optimal production rate, \( q \), and the optimal sales rate, \( s \), in the case when the thinking cost parameter, \( \kappa \), is allowed to change.

First, we define the influence coefficients, \( \hat{p}_1, \hat{p}_2, \hat{\theta}_1 \) and \( \hat{\theta}_2 \), of the volatile variables on the ordering rates, \( \lambda^s_{\hat{p}_1}, \lambda^s_{\hat{p}_2}, \lambda^s_{\hat{\theta}_1} \) and \( \lambda^s_{\hat{\theta}_2} \), of the raw materials as given below:

\[
\lambda^s_{\hat{p}_1} = \tau(\lambda_{\hat{p}_1}, \frac{\kappa \sigma_\lambda}{\sigma_{\hat{p}_1}}), \quad \lambda^s_{\hat{p}_2} = \tau(\lambda_{\hat{p}_2}, \frac{\kappa \sigma_\lambda}{\sigma_{\hat{p}_2}}), \quad \lambda^s_{\hat{\theta}_1} = \tau(\lambda_{\hat{\theta}_1}, \frac{\kappa \sigma_\lambda}{\sigma_{\hat{\theta}_1}}), \quad \lambda^s_{\hat{\theta}_2} = \tau(\lambda_{\hat{\theta}_2}, \frac{\kappa \sigma_\lambda}{\sigma_{\hat{\theta}_2}}).
\]

Let \( \sigma_\lambda = 0.1, \sigma_{\hat{p}_1} = 0.5, \sigma_{\hat{p}_2} = 0.11, \sigma_{\hat{\theta}_1} = 0.75, \sigma_{\hat{\theta}_2} = 1.54, \rho_{\hat{p}_1} = 0.4, \rho_{\hat{p}_2} = 0.2, \rho_{\hat{\theta}_1} = 0.5, \rho_{\hat{\theta}_2} = 0.6, \beta = 0.85, \alpha = 2, \delta = 0.25, \gamma = 10, \eta = 0.5, \theta_1 = 0.3, \theta_2 = 10.\)

From Figures 4-6, it is observed that all the influence coefficients of the volatile variables on the ordering rate decrease with respect to the increase of \( \kappa \), indicating that the manufacturer tends to pay less attention to those volatile parts during decision making when \( \kappa \) is increasing. In the case when \( \kappa = 0 \), meaning that the manufacturer is fully rational, we have \( \lambda^s_{\hat{p}_1} = -3.7809, \lambda^s_{\hat{p}_2} = 0.412 \) and \( \lambda^s_{\hat{\theta}_1} = 14.2348 \). These results show that the change of the deterioration rate has a greater impact on the ordering rate of the raw materials compared to the change of the
external price. Therefore, the manufacturer tends to pay a closer attention to the storage condition of the raw materials when deciding whether or not to purchase them. Alternatively, the manufacturer may try to improve the storage conditions (such as temperature and ventilation) so as to minimize the loss of the raw materials during the storage process. No matter how large the value of $\kappa$ is, we have $\lambda_{\hat{p}_1}^s < 0$, $\lambda_{\hat{\theta}_1}^s > 0$. This indicates that the price volatility of the raw materials is strictly negatively correlated to the ordering rate. When the price fluctuates more sharply, the manufacturer tends to purchase fewer raw materials so as to avoid the loss incurred by price reduction. However, the rate of the inventory loss of the raw materials is strictly positively correlated to the ordering rate. This means that the manufacturer prefers to order more raw materials to guarantee the normal production although there is a higher risk of inventory loss, because the manufacturer needs to provide enough finished goods to meet the demand of the consumers.

When $\kappa \in [0.45, 1]$, we have $\lambda_{\hat{p}_2}^s = 0$, meaning that the manufacturer orders the raw materials without considering the price volatility of the finished goods (or it is considered as less important factor) when the thinking cost is large (for example, it needs high labor cost or large material resources when making decision). Next, we will study the impact of these volatile variables on the production rate. Let $\sigma_q = 0.1, \bar{\theta}_2 = 0.1, x_2 = 5$. Then, we have Figures 7-10.

From Figures 7-10, we observe that the impact of each of the volatile variables on the optimal control variables decreases with the increase of $\kappa$. It can be seen from Figure 5 and Figure 6 that the price fluctuations of both the raw materials and finished goods have little impact on the optimal production rate, especially the impact of $\hat{p}_2$ on the optimal production rate, $\left| q_{\hat{p}_2} \right| < 0.38$ no matter what the value of the $\kappa$ is, and $q_{\hat{p}_2} = 0$ when $\kappa \in [0.41, 1]$. From Figure 7 and Figure 8, it is observed that the impact of the change rate of both the inventories of the raw materials and finished goods on the optimal production rate is small and remains steady at 7.8. This indicates that the impact of the change rate of the raw materials inventory or finished goods inventory on the optimal production rate is limited whether the manufacturer is fully rational or inattentive. It is obvious that $\left| q_{\hat{p}_2} \right| < \left| q_{\hat{p}_1} \right|$, meaning that the impact of the price fluctuation of the raw materials on the production rate is greater than the impact of the price fluctuation of the
finished goods on the production rate. Hence, the manufacturer should pay more attention to the fluctuation of the raw materials price when making a decision on the production rate. Let $\sigma_s = 0.2$. Then, we have Figures 11-12.

From Figure 11 and Figure 12, it is observed that $\hat{p}_1$ has a greater impact on the optimal sales rate compared to $\hat{p}_2$, that is, $|s_{\hat{p}_1}| > |s_{\hat{p}_2}|$. $|s_{\hat{p}_1}|$ decreases with the increase of $\kappa$ but it is still above zero. This means that this factor is always under the consideration of the manufacturer. However, $s_{\hat{p}_2} = 0$ when $\kappa \geq 0.38$, which indicates that the manufacturer will not take this factor into account when making a decision on the optimal production rate if the thinking cost is beyond a threshold. It is found that the fluctuation of the raw materials price has a greater impact on the optimal sales rate compared to the fluctuation of the finished goods price under uncertainty, because the price of the raw materials determines the production cost directly, and the manufacturer can adjust its production strategy accordingly when the production cost is too high. From Figure 13, it is found that the value of $s_{\hat{\theta}_2}$ remains steady at $-4.528$ with the change of $\kappa$, and the absolute value of this number is larger compared to other influence coefficients. These results indicate that the manufacturer often pays special attention to the inventory of the finished
goods no matter what the thinking cost is during the selling period, because he can sell to the consumers directly. A great loss will be incurred by the manufacturer if the finished products are damaged in the storage. Therefore, the manufacturer should focus his attention mainly on the inventory of the finished goods and try to reduce the loss to the minimum.

5. Conclusions. In this study, it is found that the manufacturer should consider the price fluctuation and wastage of both the raw materials and finished goods when a decision is required to be made on the ordering and production. In this paper, we assume that there is a thinking cost incurred for the manufacturer when considering these factors. Thus, the manufacturer should try to trade off between the thinking cost (if these factors are taken into account) and the irrational loss (if some of these factors are ignored). Hence, we build an optimal inventory control model for which the manufacturer is required to deal with these volatile variables. We have developed a solution method to solve the dynamic programming problem in discrete time when the decision maker is inattentive. The numerical results obtained indicate that: 1) the deterioration rate of the finished goods inventory has little influence on the optimal ordering rate and the loss rate of the raw materials inventory also has little influence on the optimal sales rate; 2) the price volatility of the raw materials influences the raw materials order quantity, and the price volatility of the finished goods has little impact on the raw materials order quantity and is independent of the manufacturers thinking cost; 3) the price volatility of both the raw materials and finished goods has little influence on the optimal production rate, and the deterioration rate of both the raw materials inventory and finished goods inventory has great impact on the optimal production rate and the corresponding influence coefficients change with the change of the thinking cost, which indicates that the manufacturer should pay more attention to the change rate of the inventory when making a decision on production: 4) the volatility of the raw materials price has greater impact on the sales rate compared to the volatility of the finished goods price, this is probably because the manufacturer sells the finished goods in a fully competitive market and he can only accept the price rather than change it in real world. Therefore, the manufacturer should pay more attention to how to avoid the loss due to the sharp change of the raw materials price.
Appendix.

Proof of Proposition 1.

Proof. Consider the case of \( \hat{p}_1 = 0, \hat{p}_2 = 0, \hat{\theta}_1 = 0, \hat{\theta}_2 = 0 \). Then, we have the default optimal model of Problem (P) (see equation (14)), which is given by

\[
V(x_1, x_2, 0, 0, 0, 0) = \max_{s, \lambda, q} \left[ s \bar{p}_2 - \lambda \bar{p}_1 - aq - bq^2 + c(x_1 + x_2) - d(x_1^2 + x_2^2) \right] + \beta V(x_1', x_2', 0, 0, 0, 0) \tag{A.1}
\]

The state transition functions are:

\[
x_1' = (1 - \bar{\theta}_1)x_1 + \lambda t - q t
\]

\[
x_2' = (1 - \bar{\theta}_2)x_2 + q t - s t
\]

Differentiating (A.1) with respect to the state variable \( x_1 \) gives:

\[
V_{x_1} = \gamma - 2\eta x_1 + \beta V_{x_1'} \frac{\partial x_1'}{\partial x_1} \tag{A.2}
\]

Since \( \frac{\partial x_1'}{\partial x_1} = 1 - \bar{\theta}_1 \) and \( V_{x_1} = V_{x_1'} \), we have,

\[
V_{x_1} = \frac{\gamma - 2\eta x_1}{1 - \beta(1 - \bar{\theta}_1)}. \tag{A.3}
\]

Differentiating (A.1) with respect to the state variable \( x_2 \) gives:

\[
V_{x_2} = \gamma - 2\eta x_2 + \beta V_{x_2'} \frac{\partial x_2'}{\partial x_2} \tag{A.4}
\]

Since \( \frac{\partial x_2'}{\partial x_2} = 1 - \bar{\theta}_2 \) and \( V_{x_2} = V_{x_2'} \), we have,

\[
V_{x_2} = \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \bar{\theta}_2)}. \tag{A.5}
\]

Differentiating (A.1) with respect to the control variable \( q \) yields:

\[
0 = -\alpha - 2\delta q + \beta V_{x_1'} \frac{\partial x_1'}{\partial q} + \beta V_{x_2'} \frac{\partial x_2'}{\partial q}. \tag{A.6}
\]

Clearly, \( \frac{\partial x_1'}{\partial q} = -1 \) and \( \frac{\partial x_2'}{\partial q} = 1 \). Thus, by substituting them into (A.6) and combining the result obtained with (A.3) and (A.5), we obtain

\[
q^d = \frac{\beta}{2\delta} \left[ \frac{\gamma - 2\eta x_1}{1 - \beta(1 - \bar{\theta}_1)} + \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \bar{\theta}_2)} \right] - \frac{\alpha}{2\delta}. \tag{A.7}
\]

In the steady state, we have \( x_1' = x_1 \) and \( x_2' = x_2 \). Thus, by substituting them into the state transition functions, it gives

\[
\lambda = \bar{\theta}_1 x_1 + q, \tag{A.8}
\]

\[
s = q - \theta_2 x_2. \tag{A.9}
\]

Combining them with (A.7), we obtain

\[
\lambda^d = \bar{\theta}_1 x_1 + \frac{\beta}{2\delta} \left[ \frac{\gamma - 2\eta x_1}{1 - \beta(1 - \bar{\theta}_1)} + \frac{\gamma - 2\eta x_2}{1 - \beta(1 - \bar{\theta}_2)} \right] - \frac{\alpha}{2\delta}. \tag{A.10}
\]
Differentiating (A.12) with respect to the control variable $\lambda$

Similarly, we have

Thus, it follows that

Using the implicit function theorem, we obtain

Differentiating (A.12) with respect to the control variable $\lambda$ and volatile factors $\hat{p}_1$, $\hat{p}_2$, $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively, we obtain

\[\phi(x_1, x_2, \hat{p}_1, \hat{p}_2, \hat{\theta}_1, \hat{\theta}_2, \lambda) = -(\hat{p}_1 + \hat{p}_1) + \beta Vx_1. \tag{A.12}\]
Now the unknown terms are \( V_{x_1'} \), \( V_{x_1'\hat{p}_1} \), \( V_{x_1'\hat{p}_2} \), \( V_{x_1'\hat{\theta}_1} \) and \( V_{x_1'\hat{\theta}_2} \). Differentiating (A.3) with respect to \( x_1 \), we have

\[
V_{x_1'} = \frac{-2\eta}{1 - \beta(1 - \theta_1)}.
\]

Clearly, \( V_{x_1'x_1} = V_{x_1x_1} \) in the steady state, and hence

\[
V_{x_1'x_1} = \frac{-2\eta}{1 - \beta(1 - \theta_1)}.
\]

1) Solving \( V_{x_1'\hat{p}_1} \). Differentiating the Bellman equation (14) with respect to the state variable \( \hat{p}_1 \) gives:

\[
V_{\hat{p}_1} = -\lambda + \beta V_{\hat{p}_1'} \rho_{\hat{p}_1}.
\]

Since \( V_{\hat{p}_1} = V_{\hat{p}_1'} \), we have

\[
V_{\hat{p}_1} = \frac{-\lambda}{1 - \beta \rho_{\hat{p}_1}}.
\]

Differentiating this term with respect to \( x_1 \) gives:

\[
V_{\hat{p}_1x_1} = -\frac{\partial \lambda}{\partial x_1} \frac{1}{1 - \beta \rho_{\hat{p}_1}}.
\]

From (A.10), we obtain the concrete expression of \( \lambda \). Substituting it into

\[
V_{\hat{p}_1x_1} = -\frac{\partial \lambda}{\partial x_1} \frac{1}{1 - \beta \rho_{\hat{p}_1}},
\]

gives

\[
V_{\hat{p}_1x_1} = -\frac{\eta + \bar{\theta}_1 \delta (1 - \bar{\theta}_2)}{\delta (1 - \beta \rho_{\hat{p}_1})(1 - \theta_2)}.
\]

It is also known that \( V_{\hat{p}_1x_1} = V_{x_1\hat{p}_1} \) and \( V_{x_1\hat{p}_1} = V_{x_1\hat{p}_1'} \). Thus,

\[
V_{x_1\hat{p}_1} = -\frac{\eta + \bar{\theta}_1 \delta (1 - \bar{\theta}_2)}{\delta (1 - \beta \rho_{\hat{p}_1})(1 - \theta_2)}.
\]

2) Solving \( V_{x_1'\hat{p}_2} \). Differentiating the Bellman equation (14) with respect to the state variable \( \hat{p}_2 \) gives:

\[
V_{\hat{p}_2} = s + \beta V_{\hat{p}_2'} \rho_{\hat{p}_2}.
\]

Since \( V_{\hat{p}_2} = V_{\hat{p}_2'} \), we have

\[
V_{\hat{p}_2} = \frac{s}{1 - \beta \rho_{\hat{p}_2}}.
\]

Differentiating this term with respect to \( x_1 \) gives:

\[
V_{\hat{p}_2x_1} = -\frac{\partial s}{\partial x_1} \frac{1}{1 - \beta \rho_{\hat{p}_2}}.
\]

From (A.11), we obtain the concrete expression of \( s \). Substituting it into

\[
V_{\hat{p}_2x_1} = -\frac{\partial s}{\partial x_1} \frac{1}{1 - \beta \rho_{\hat{p}_2}},
\]

gives

\[
V_{\hat{p}_2x_1} = \frac{\beta \eta}{\delta (1 - \beta \rho_{\hat{p}_2})(1 - \beta(1 - \theta_1))}.
\]
We also know that $V_{\bar{p}_2x_1} = V_{x_1\bar{p}_2}$ and $V_{x_1\bar{p}_2} = V_{x_1'\bar{p}_2'}$. Thus,

$$V_{x_1'\bar{p}_2'} = \frac{\beta\eta}{\delta(1 - \beta\rho_{p_2})(1 - \beta(1 - \theta_1))}.$$

3) Solving $V_{x_1'\bar{p}_1'}$. Differentiating the Bellman equation (14) with respect to the state variable $\theta_1$ gives:

$$V_{\theta_1} = \beta V_{x_1'} \frac{\partial x_1'}{\partial \theta_1} + \beta V_{\theta_1'} \frac{\partial \theta_1'}{\partial \theta_1},$$

that is,

$$V_{\theta_1} = \beta V_{x_1'}(-x_1) + \beta V_{\theta_1'} \rho_{\bar{p}_1}.$$

Differentiating this formula with respect to the control variable $x_1$ and letting $\bar{p}_1 = 0, \bar{p}_2 = 0, \theta_1 = 0$ and $\theta_2 = 0$, we obtain

$$V_{\bar{p}_1x_1} = -\beta[V_{x_1'} + V_{x_1'x_1'}(1 - \bar{\theta}_1)x_1] + \beta V_{\bar{p}_1x_1'} \rho_{\bar{p}_1}.$$

Similarly, based on Proposition 1 and $V_{\bar{p}_1x_1} = V_{x_1',\theta_1}, V_{x_1',\theta_1} = V_{x_1',\theta_1'}$, it follows that

$$V_{x_1'\theta_1'} = -\beta[V_{x_1'} + V_{x_1'x_1'}(1 - \bar{\theta}_1)x_1]$$

$$\frac{1}{1 - \beta \rho_{\bar{p}_1}}.$$

4) Solving $V_{x_1'\bar{p}_2'}$. Differentiating the Bellman equation (14) with respect to the state variable $\theta_2$ gives:

$$V_{\theta_2} = \beta V_{x_2'} \frac{\partial x_2'}{\partial \theta_2} + \beta V_{\theta_2'} \frac{\partial \theta_2'}{\partial \theta_2},$$

that is,

$$V_{\theta_2} = \beta V_{x_2'}(-x_2) + \beta V_{\theta_2'} \rho_{\bar{p}_2}.$$

Differentiating this term with respect to $x_1$ and letting $\bar{p}_1 = 0, \bar{p}_2 = 0, \theta_1 = 0$ and $\theta_2 = 0$, we obtain

$$V_{\bar{p}_2x_1} = 0.$$

Similarly, $V_{\bar{p}_1x_1} = V_{x_1',\theta_1'}, V_{x_1',\theta_1'} = V_{x_1',\theta_1'}$. Thus, we have

$$V_{x_1'\theta_1'} = 0.$$

Therefore,

$$\lambda_{\bar{p}_1} = -1 + \beta \rho_{\bar{p}_1} \frac{V_{x_1'p_1'}}{V_{x_1'x_1'}}, \quad \lambda_{\bar{p}_2} = -\rho_{\bar{p}_2} \frac{V_{x_1'\bar{p}_2'}}{V_{x_1'x_1'}},$$

$$\lambda_{\theta_1} = -x_1 V_{x_1'x_1'} + \rho_{\bar{p}_1} \frac{V_{x_1'\bar{p}_1'}}{V_{x_1'x_1'}}, \quad \lambda_{\theta_2} = -\rho_{\bar{p}_2} \frac{V_{x_1'\bar{p}_2'}}{V_{x_1'x_1'}}.$$

where

$$V_{x_1'p_1'} = -\beta\eta + \delta \theta_1(1 - \beta(1 - \theta_1)) \frac{\delta(1 - \beta\rho_{p_1})(1 - \beta(1 - \theta_1))}{\delta(1 - \beta\rho_{p_2})(1 - \beta(1 - \theta_1))}, \quad V_{x_1'\bar{p}_2'} = \frac{\beta\eta}{\delta(1 - \beta\rho_{p_2})(1 - \beta(1 - \theta_1))},$$

$$V_{x_1'\bar{p}_1'} = -\beta[V_{x_1'} + (1 - \bar{\theta}_1)x_1 V_{x_1'x_1'}] \frac{1}{1 - \beta \rho_{\bar{p}_1}(1 - \theta_1)}, \quad V_{x_1'\theta_1'} = 0, \quad V_{x_1'x_1'} = \frac{-2\eta}{1 - \beta(1 - \theta_1)}.$$
Similarly,
\[
q_{\tilde{p}_1} = -\beta \rho_{\tilde{p}_1} V_{x_1 x_1'} + \beta \rho_{\tilde{p}_2} V_{x_2 x_2'},
q_{\tilde{p}_2} = -\beta \rho_{\tilde{p}_2} V_{x_1 x_1'} + \beta \rho_{\tilde{p}_2} V_{x_2 x_2'},
\]
\[
q_{\tilde{\theta}_1} = -\beta \rho_{\tilde{\theta}_1} V_{x_1 x_1'} - \beta \rho_{\tilde{\theta}_1} V_{x_2 x_2'},
q_{\tilde{\theta}_2} = -\beta \rho_{\tilde{\theta}_2} V_{x_1 x_1'} - \beta \rho_{\tilde{\theta}_2} V_{x_2 x_2'},
\]
\[
s_{\tilde{p}_1} = \frac{\rho_{\tilde{p}_1} V_{x_1 x_1'}}{V_{x_2 x_2'}},
s_{\tilde{p}_2} = \frac{1 - \beta \rho_{\tilde{p}_2} V_{x_2 x_2'}}{V_{x_1 x_1'}},
\]
\[
s_{\tilde{\theta}_1} = \frac{x_2 V_{x_2 x_2'} - \rho_{\tilde{\theta}_1} V_{x_2 x_2'}}{V_{x_1 x_1'}},
s_{\tilde{\theta}_2} = \frac{x_2 V_{x_2 x_2'} - \rho_{\tilde{\theta}_2} V_{x_2 x_2'}}{V_{x_1 x_1'}}.
\]
where,
\[
V_{x_2 x_2'} = \frac{\beta \eta}{\delta(1 - \beta \rho_{\tilde{p}_1})(1 - \beta (1 - \tilde{\theta}_2))},
V_{x_2 x_2'} = \frac{-\beta \eta}{\delta(1 - \beta \rho_{\tilde{p}_1})(1 - \beta (1 - \tilde{\theta}_2))},
V_{x_2 x_2'} = \frac{-\beta [V_{x_2 x_2'} + (1 - \tilde{\theta}_2) x_2 V_{x_2 x_2'}]}{1 - \beta \rho_{\tilde{\theta}_1}(1 - \tilde{\theta}_2)},
V_{x_2 x_2'} = \frac{-2 \eta}{1 - \beta (1 - \tilde{\theta}_2)}.
\]

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REFERENCES

[1] A. B. Abel, J. C. Eberly and S. Panageas, Optimal inattention to the stock market with information costs and transactions costs, Econometrica, 81 (2013), 1455–1481.
[2] R. Akella, V. F. Araman and J. Kleinheinz, B2B Markets: Procurement and Supplier Risk Management in E-Business, in Supply chain management: models, applications, and research directions, Springer, (2005), 33–66.
[3] P. Berling and V. Martínez-de-Albéniz, Optimal inventory policies when purchase price and demand are stochastic, Operations Research, 59 (2011), 109–124.
[4] P. Berling and K. Rosling, The effects of financial risks on inventory policy, Management Science, 51 (2002), 1804–1815.
[5] W. Bi, G. Li and M. Liu, Dynamic pricing with stochastic reference effects based on a finite memory window, International Journal of Production Research, 55 (2017), 3331–3348.
[6] W. Bi, L. Tian, H. Liu and X. Chen, A stochastic dynamic programming approach based on bounded rationality and application to dynamic portfolio choice, Discrete Dynamics in Nature and Society, 2014 (2014), Article ID 840725, 11 pages.
[7] A. Bouras and L. Tadj, Production planning in a three-stock reverse-logistics system with deteriorating items under a continuous review policy, Journal of Industrial and Management Optimization, 11 (2015), 1041–1058.
[8] J.-M. Chen and C.-S. Lin, An optimal replenishment model for inventory items with normally distributed deterioration, Production Planning and Control, 13 (2002), 470–480.
[9] S. K. Devalkar, R. Anupindi and A. Sinha, Integrated optimization of procurement, processing, and trade of commodities, Operations Research, 59 (2011), 1369–1381.
[10] D. Duffie and T. Sun, Transactions costs and portfolio choice in a discrete-continuous-time setting, Journal of Economic Dynamics and Control, 14 (1990), 35–51.
[11] Q. Fu, C. Y. Lee and C. P. Teo, Procurement management using option contracts: Random spot price and the portfolio effect, IIE Transactions, 42 (2010), 793–811.
[12] X. Gabaix, A sparsity-based model of bounded rationality, Quarterly Journal of Economics, 129 (2014), 1661–1710.
[13] X. Gabaix, Sparse Dynamic Programming and Aggregate Fluctuations, Working Paper, New York University, 2016.
[14] V. Gaur and S. Seshadri, Hedging inventory risk through market instruments, Manufacturing and Service Operations Management, 7 (2005), 103–120.
[15] S. Gavirneni, Periodic review inventory control with fluctuating purchasing costs, Operations Research Letters, 32 (2004), 374–379.
[16] S. Goyal and B. C. Giri, Recent trends in modeling of deteriorating inventory, European Journal of Operational Research, 134 (2001), 1–16.
[17] J. Jenkinson, Procurement in action, the efficio grassroots procurement survey 2011, Efficio Consulting, 2011.
[18] D. Kahneman, Attention and Effort, Englewood Cliffs, NJ: Prentice-Hall, 1973.
[19] B. A. Kalymon, Stochastic prices in a single-item inventory purchasing model, Operations Research, 19 (1971), 1434–1458.
[20] M. Lashgari, A. A. Taleizadeh and S. S. Sana, An inventory control problem for deteriorating items with back-ordering and financial considerations under two levels of trade credit linked to order quantity, Journal of Industrial and Management Optimization, 12 (2016), 1091–1119.
[21] H. Liu, X. Luo, W. Bi, Y. Man and K. L. Teo, Dynamic pricing of network goods in duopoly markets with boundedly rational consumers, J. Ind. Manag. Optim, 13 (2017), 427–445.
[22] B. Mackowiak and M Wiederholt, Information processing and limited liability, The American Economic Review, 102 (2012), 30–34.
[23] B. Mackowiak and M. Wiederholt, Inattention to Rare Events, 2015. Available at SSRN 2477548: https://ssrn.com/abstract=2650452.
[24] F. Matejka and C. A. Sims, Discrete actions in information-constrained tracking problems, Princeton University Manuscript, (2011).
[25] S. Nahmias and W. S. Demmy, Operating characteristics of an inventory system with rationing, Management Science, 27 (1981), 1236–1245.
[26] F. Raafat, Survey of literature on continuously deteriorating inventory models, Journal of the Operational Research Society, 42 (1991), 27–37.
[27] R. Reis, Inattentive consumers, Journal of Monetary Economics, 53 (2006), 1761–1800.
[28] J. Schwartzstein, Selective attention and learning, Journal of the European Economic Association, 12 (2014), 1423–1452.
[29] K. Sebastian, A. Maessen and S. Strasmann, Mastering the Uniqueness of Commodity Pricing: How to Guide, Set and Control Prices, Simon-Kucher-Whitepaper, 2010.
[30] N. H. Shah and Y. Shah, Literature survey on inventory models for deteriorating items, Ekonomski Analiz, 44 (2000), 221–237.
[31] C. A. Sims, Implications of rational inattention, Journal of Monetary Economics, 50 (2003), 665–690.
[32] B. Sivakumar, A perishable inventory system with retrial demands and a finite population, Journal of Computational and Applied Mathematics, 224 (2009), 29–38.
[33] T. M. Whitin, Inventory control and price theory, Management Science, 2 (1955), 61–68.
[34] Q. Wu and H. Chen, Optimal control and equilibrium behavior of production-inventory systems, Management Science, 56 (2010), 1362–1379.
[35] J. X. Zhang, Z. Y. Bai and W. S. Tang, Optimal pricing policy for deteriorating items with preservation technology investment, Journal of Industrial and Management Optimization, 10 (2014), 1261–1277.
[36] Y.-S. Zheng, Optimal control policy for stochastic inventory systems with Markovian discount opportunities, Operations Research, 42 (1994), 721–738.
[37] P. Zipkin, Critical number policies for inventory models with periodic data, Management Science, 35 (1989), 71–80.

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