SOLVING SEQUENTIAL LINEAR M-FRACTIONAL DIFFERENTIAL EQUATIONS WITH CONSTANTS COEFFICIENTS

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Abstract. Fractional calculus is a powerful and effective tool for modelling nonlinear systems. The M-derivative is the generalisation of alternative fractional derivative introduced by Katugampola[6]. This M-derivative obey the properties of integer calculus. In this paper, we present the method for solving M-fractional sequential linear differential equations with constant coefficients for $\alpha \geq 0$ and $\beta > 0$. Existence and Uniqueness of the solutions for the $n^{th}$ order sequential linear M-fractional differential equations are discussed in detail. We have present illustration for homogeneous and non homogeneous case.

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1. Introduction

While L’Hospital has proposed the idea of fractional derivative in the 17th century, several researchers concerted fractional derivative in the recent centuries. Riemann-Liouville, Caputo and other fractional derivatives are defined on the basis of fractional integral form [8, 12, 13].

Recently, Khalil et al.[7] and Katugampola [6] proposed fractional derivatives in the limit form as in usual derivative such as conformable fractional derivative and alternative fractional derivative. Based on these derivative, Sousa and Oliveira [15] introduced M-fractional derivatives which satisfies properties of integer-order calculus.

Theory and applications of the sequential linear fractional differential equations involving Hadamard, Riemann-Liouville, Caputo and Conformable derivatives have been investigated in [1, 2, 3, 4, 9, 10, 11].

Lately, Gokdogan et al [5] have proved existence and uniqueness theorems for solving sequential linear conformable fractional differential equations. Unal et al [14] provide method to solve sequential linear conformable fractional differential equations with constants coefficients. In this work, We present Existence

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and Uniqueness theorems and solutions of sequential linear $M$-fractional differential equations.

The arrangement of this paper is as following: In section 2 we present the concept of $M$-fractional derivative. In section 3 we provide existence and uniqueness theorems for sequential Linear $M$-fractional differential equations. In section 4 we propose the solutions of sequential Linear $M$-fractional differential equations. In section 5 we present solutions of Non-homogeneous case. Finally, Conclusion is present in section 6.

2. $M$-FRACTIONAL CALCULUS

In this section, we give some necessary definitions and theorems of $M$ derivative which are explained in [15].

**Definition 2.1.** Let $f : [0, \infty) \to \mathbb{R}$ be a function and $t > 0$. Then for $0 < \alpha < 1$, the $M$-fractional derivative of $f$ of order $\alpha$ is defined as

$$D_M^{\alpha,\beta} f(t) = \lim_{\epsilon \to 0} \frac{f(tE_\beta(\epsilon t^{-\alpha}) - f(t)}{\epsilon}$$

Where $E_\beta(\cdot), \beta > 0$ is Mittag-Leffler function with one parameter.

If $f$ is $M$-differentiable in some interval $(0, a), a > 0$ and

$$\lim_{t \to 0^+} D_M^{\alpha,\beta} f(t)$$

exists, then we have

$$D_M^{\alpha,\beta} f(0) = \lim_{t \to 0^+} D_M^{\alpha,\beta} f(t)$$

**Theorem 2.1.** Let $0 < \alpha \leq 1, \beta > 0, a, b \in \mathbb{R}$ and $f, g$ be $M$-differentiable at a point $t > 0$. Then

1. $D_M^{\alpha,\beta}(af + bg)(t) = aD_M^{\alpha,\beta} f(t) + bD_M^{\alpha,\beta} g(t)$ for all $a, b \in \mathbb{R}$.
2. $D_M^{\alpha,\beta}(f \cdot g)(t) = f(t)D_M^{\alpha,\beta} g(t) + g(t)D_M^{\alpha,\beta} f(t)
3. D_M^{\alpha,\beta}(\{f\})(t) = \frac{g(t)D_M^{\alpha,\beta}(f(t)-f(t))D_M^{\alpha,\beta} g(t)}{|g(t)|^2}
4. D_M^{\alpha,\beta}(c) = 0$, where $f(t) = c$ is a constant.
5. $D_M^{\alpha,\beta} t^a = \frac{a}{\Gamma(\beta+1)} t^{a-\alpha}
6. Moreover, $f$ is differentiable, then $D_M^{\alpha,\beta} f(t) = \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{df(t)}{dt}$

Additionally, $M$-derivatives of certain functions as follows:

1. $D_M^{\alpha,\beta} \sin(\frac{1}{\alpha} t^\alpha) = \frac{\cos(\frac{1}{\alpha} t^\alpha)}{\Gamma(\beta+1)}
2. D_M^{\alpha,\beta} \cos(\frac{1}{\alpha} t^\alpha) = \frac{-\sin(\frac{1}{\alpha} t^\alpha)}{\Gamma(\beta+1)}
3. D_M^{\alpha,\beta} e^{\frac{t^\alpha}{\alpha}} = \frac{e^{\frac{t^\alpha}{\alpha}}}{\Gamma(\beta+1)}$
**Theorem 2.2.** Let \( a > 0 \) and \( t \geq a \). Then, the \( M \)-integral of order \( \alpha \) of a function \( f \) is defined by

\[
M I_{a}^{\alpha} f(t) = \Gamma(\beta + 1) \int_{a}^{t} \frac{f(x)}{x^{1-\alpha}} dx
\]

**Theorem 2.3.** Let \( a \geq 0 \) and \( 0 < \alpha < 1 \). Also let \( f \) be a continuous function such that there exists \( M I_{a}^{\alpha} f \). Then

\[
D_{M}^{\alpha, \beta}(M I_{a}^{\alpha} f(t)) = f(t)
\]

**Theorem 2.4.** Let \( f, g : [a, b] \to \mathbb{R} \) be two functions such that \( f, g \) are differentiable and \( 0 < \alpha < 1 \). Then

\[
\int_{a}^{b} f(x) D_{M}^{\alpha, \beta} g(x) dx = f(x) g(x)|_{a}^{b} - \int_{a}^{b} g(x) f(x) dx
\]

where \( d_{a} x = \frac{\Gamma(\beta+1)}{x^{1-\alpha}} dx \)

### 3. Existence and Uniqueness Theorem

Let linear sequential \( M \)-fractional differential equation of order \( n \alpha \)

\[
nD_{M}^{\alpha, \beta} y + p_{n-1}(t) nD_{M}^{\alpha, \beta} y + ... + p_{2}(t)^{2} D_{M}^{\alpha, \beta} y + p_{1}(t) D_{M}^{\alpha, \beta} y + p_{0}(t) y = 0 \quad (1)
\]

where \( nD_{M}^{\alpha, \beta} y = D_{M}^{\alpha, \beta} D_{M}^{\alpha, \beta} ... D_{M}^{\alpha, \beta} y \), \( n \) times

Similarly, non-homogeneous fractional differential equation with \( M \)-derivative is

\[
nD_{M}^{\alpha, \beta} y + p_{n-1}(t) nD_{M}^{\alpha, \beta} y + ... + p_{2}(t)^{2} D_{M}^{\alpha, \beta} y + p_{1}(t) D_{M}^{\alpha, \beta} y + p_{0}(t) y = f(t) \quad (2)
\]

We define an \( n \)-th order differential operator for eqn. (1) as following

\[
L_{\alpha, \beta}[y] = n D_{M}^{\alpha, \beta} y + p_{n-1}(t) nD_{M}^{\alpha, \beta} y + ... + p_{2}(t)^{2} D_{M}^{\alpha, \beta} y + p_{1}(t) D_{M}^{\alpha, \beta} y + p_{0}(t) y = 0 \quad (3)
\]

**Theorem 3.1.** Let \( \Gamma(\beta+1) t^{\alpha-1} p(t), \Gamma(\beta+1) t^{\alpha-1} f(t) \in C(a, b) \) and let \( y \) be \( M \)-differentiable for \( 0 < \alpha \leq 1 \) and \( \beta > 0 \). Then the initial value problem

\[
D_{M}^{\alpha, \beta} y + p(t) y = f(t) \quad (4)
\]

\[
y(t_{0}) = y_{0} \quad (5)
\]

has exactly one solution on the interval \( (a, b) \) where \( t_{0} \in (a, b) \)

**Proof.** Using property (6) in Theorem 2.1, we have

\[
D_{M}^{\alpha, \beta} y + p(t) y = f(t)
\]

\[
\frac{t^{1-\alpha}}{\Gamma(\beta+1)} y' + p(t) y = f(t)
\]

\[
y' + \Gamma(\beta+1) t^{\alpha-1} p(t) y = \Gamma(\beta+1) t^{\alpha-1} f(t)
\]
The proof is clear from classical linear fundamental theorem existence and uniqueness. □

**Theorem 3.2.** If \( \Gamma(\beta+1)t^{\alpha-1}p_{n-1}(t), \ldots, \Gamma(\beta+1)t^{\alpha-1}p_1(t), \Gamma(\beta+1)t^{\alpha-1}p_0(t), \Gamma(\beta+1) \) \( f(t) \in C(a, b) \) and \( y \) be \( n \) times \( M \)-differentiable function, then a solution \( y(t) \) of the initial value problem

\[
^nD_M^{\alpha,\beta}y + p_{n-1}(t)y + \ldots + p_2(t)^2D_M^{\alpha,\beta}y + p_1(t)D_M^{\alpha,\beta}y + p_0(t)y = f(t) \quad (6)
\]

\[
y(t_0) = y_0, D_M^{\alpha,\beta}y(t_0) = y_1, \ldots, n^{-1}D_M^{\alpha,\beta}y(t_0) = y_{n-1}, a < t_0 < b \quad (7)
\]

**Proof.** The existence of a local solution is obtained by transform our problem into the first order system of differential equations. So, we introduce new variables

\[
x_1 = y, x_2 = D_M^{\alpha,\beta}y, x_3 = D_M^{\alpha,\beta}y, \ldots, x_n = n^{-1}D_M^{\alpha,\beta}y
\]

In this, we have

\[
D_M^{\alpha,\beta}x_1 = x_2
\]

\[
D_M^{\alpha,\beta}x_2 = x_3
\]

\[
\vdots
\]

\[
D_M^{\alpha,\beta}x_{n-1} = x_n
\]

\[
D_M^{\alpha,\beta}x_n = -p_{n-1}x_n - \ldots - p_2x_3 - p_1x_2 - p_0x_1 + f(t)
\]

The above equations can be written as the following

\[
D_M^{\alpha,\beta}X(t) + P(t)X(t) = F(t)
\]

\[
X'(t) + \Gamma(\beta+1)t^{\alpha-1}P(t)X(t) = \Gamma(\beta+1)t^{\alpha-1}F(t)
\]

The existence and uniqueness of solution (6)-(7) follows from classical theorems on existence and uniqueness for system equation. □

**Theorem 3.3.** If \( y_1 \) and \( y_2 \) are \( n \) times \( M \)-differentiable functions and \( c_1, c_2 \) are arbitrary numbers, then \( L_{\alpha,\beta} \) is linear.

\[
i.e., L_{\alpha,\beta}[c_1y_1 + c_2y_2] = c_1L_{\alpha,\beta}[y_1] + c_2L_{\alpha,\beta}[y_2]
\]

**Proof.** We can easily derived the proof of this theorem by applying same procedure in Theorem-4.3 [5] to \( M \)-derivative. □
Theorem 3.4. If \( y_1, y_2, \ldots, y_n \) are the solutions of equation \( L_{\alpha,\beta}[y] = 0 \) and \( c_1, c_2, \ldots, c_n \) are arbitrary constants, then the linear combination \( y(t) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \) is also solution of \( L_{\alpha,\beta}[y] = 0 \).

Proof. We can easily derived the proof of this theorem by applying same procedure in Theorem-4.4[5] to \( M \)- derivative.

Definition 3.1. For \( n \) functions \( y_1, y_2, \ldots, y_n \), we define the \( M \)-Wronskain of these function to be the determinant

\[
W_{\alpha,\beta}(t) = \begin{vmatrix}
y_1 & y_2 & \cdots & y_n \\
D_M^{\alpha,\beta} y_1 & D_M^{\alpha,\beta} y_2 & \cdots & D_M^{\alpha,\beta} y_n \\
\vdots & \vdots & \ddots & \vdots \\
n^{-1} D_M^{\alpha,\beta} y_1 & n^{-1} D_M^{\alpha,\beta} y_2 & \cdots & n^{-1} D_M^{\alpha,\beta} y_n
\end{vmatrix}
\]

Theorem 3.5. Let \( y_1, y_2, \ldots, y_n \) be \( n \) solutions of \( L_{\alpha,\beta}[y] = 0 \). If there is a \( t_0 \in (a, b) \) such that \( W_{\alpha,\beta}(t_0) \neq 0 \), then \( y_1, y_2, \ldots, y_n \) is a fundamental set of solutions.

Proof. We need to show that if \( y(t) \) is a solution of \( L_{\alpha,\beta}[y] = 0 \), then we can write \( y(t) \) as a linear combination of \( y_1, y_2, \ldots, y_n \).

\[
i.e, y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n
\]

so the problem reduces to finding the constants \( c_1, c_2, \ldots, c_n \). These constants are found by solving the following linear system of \( n \) equations

\[
\begin{align*}
c_1 y_1(t_0) + c_2 y_2(t_0) + \ldots + c_n y_n(t_0) &= y(t_0) \\
c_1 D_M^{\alpha,\beta} y_1(t_0) + c_2 D_M^{\alpha,\beta} y_2(t_0) + \ldots + c_n D_M^{\alpha,\beta} y_n(t_0) &= D_M^{\alpha,\beta} y(t_0) \\
\vdots \\
c_1^{n-1} D_M^{\alpha,\beta} y_1(t_0) + c_2^{n-1} D_M^{\alpha,\beta} y_2(t_0) + \ldots + c_n^{n-1} D_M^{\alpha,\beta} y_n(t_0) &= n^{-1} D_M^{\alpha,\beta} y(t_0)
\end{align*}
\]

Using Cramers rule, we can find

\[
c_i = \frac{W_{\alpha,\beta}^i(t_0)}{W_{\alpha,\beta}(t_0)}, 1 \leq i \leq n
\]

Since \( W_{\alpha,\beta}(t_0) \neq 0 \), it follows that \( c_1, c_2, \ldots, c_n \) exist. \( \square \)

Theorem 3.6. Let \( y_1, y_2, \ldots, y_n \) be \( n \) solutions of \( L_{\alpha,\beta}[y] = 0 \). Then

1. \( W_{\alpha,\beta}(t) \) satisfies the differential equation \( D_M^{\alpha,\beta} W_{\alpha,\beta} + p(n-1) W_{\alpha,\beta} = 0 \)
2. If \( t_0 \) is any point in \( (a, b) \), then

\[
W_{\alpha,\beta}(t) = W_{\alpha,\beta}(t_0) e^{-\Gamma(\beta+1) \int_{t_0}^{t} x^{\alpha-1} p_{n-1}(x) dx}
\]

Further, if \( W_{\alpha,\beta}(t_0) \neq 0 \) then \( W_{\alpha,\beta}(t) \neq 0 \) for all \( t \in (a, b) \)
Proof. (1) Let us introduce new variables

\[ x_1 = y, x_2 = D_M^{\alpha,\beta} y, x_3 = 2D_M^{\alpha,\beta} y, \ldots, x_n = n-1D_M^{\alpha,\beta} y \]

From this, we have

\[ D_M^{\alpha,\beta} x_1 = x_2 \]
\[ D_M^{\alpha,\beta} x_2 = x_3 \]
\[ \vdots \]
\[ D_M^{\alpha,\beta} x_{n-1} = x_n \]

\[ D_M^{\alpha,\beta} X(t) = P(t)X(t) \]

We have

\[ D_M^{\alpha,\beta} W_{\alpha,\beta}(t) = (a_{11} + a_{22} + \ldots + a_{nn})W_{\alpha,\beta}(t) \]

In our case

\[ a_{11} + a_{22} + \ldots + a_{nn} = -p_{n-1}(t) \]

So,

\[ D_M^{\alpha,\beta} W_{\alpha,\beta}(t) + p_{n-1}(t)W_{\alpha,\beta}(t) = 0 \]

(2) The above differential equation can be solved by the method of integrating factor, we have

\[ W_{\alpha,\beta}(t) = W_{\alpha,\beta}(t_0)e^{-\Gamma(\beta+1)\int_{t_0}^{t} x^{\alpha-1}p_{n-1}(x)dx} \]

Thus the proof of theorem is completed. \( \square \)

**Theorem 3.7.** If \( \{y_1, y_2, \ldots, y_n\} \) is a fundamental set of solutions of \( L_{\alpha,\beta}[y] = 0 \) where \( \Gamma(\beta+1)t^{\alpha-1}p_{n-1}(t) \ldots \Gamma(\beta+1)t^{\alpha-1}p_1(t), \Gamma(\beta+1)t^{\alpha-1}p_0(t) \in C(a, b) \), then \( W_{\alpha,\beta}(t) \neq 0 \) for all \( t \in (a, b) \).

**Proof.** By applying procedure in Theorem-4.8 [5] to \( M \)-derivative, we can easily prove this theorem. \( \square \)

**Theorem 3.8.** Let \( \Gamma(\beta+1)t^{\alpha-1}p_{n-1}(t), \ldots, \Gamma(\beta+1)t^{\alpha-1}p_1(t), \Gamma(\beta+1)t^{\alpha-1}p_0(t) \in C(a, b) \). The solution set \( \{y_1, y_2, \ldots, y_n\} \) is a fundamental set of solutions to the equation \( L_{\alpha,\beta}[y] = 0 \) if and only if the functions \( y_1, y_2, \ldots, y_n \) are linearly independent.
Proof. By applying procedure in Theorem-4.9 [5] to $M$-derivative, we can easily prove this theorem. □

**Theorem 3.9.** Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the equation (1) and $y_p$ be any particular solution of the non homogeneous equation (2). Then the general solution of the equation is $y = c_1y_1 + c_2y_2 + ... + c_ny_n + y_p$

Proof. Let $L_{\alpha,\beta}$ be the differential operator and $y(t)$ and $y_p(t)$ be the solutions of the non homogeneous equation $L_{\alpha,\beta}[y] = f(t)$. If we take $u(t) = y(t) - y_p(t)$, then by linearity of $L_{\alpha,\beta}$ we have,

$$L_{\alpha,\beta}[u] = L_{\alpha,\beta}[y(t) - y_p(t)] = L_{\alpha,\beta}[y(t)] - L_{\alpha,\beta}[y_p(t)] = f(t) - f(t) = 0$$

Then $u(t)$ is a solution of the homogenous equation $L_{\alpha,\beta}[y] = 0$. Then by Theorem 3.4

$$u(t) = c_1y_1(t) + c_2y_2(t) + ... + c_ny_n(t)$$

i.e,

$$y(t) - y_p(t) = c_1y_1(t) + c_2y_2(t) + ... + c_ny_n(t)$$

Then

$$y(t) = c_1y_1(t) + c_2y_2(t) + ... + c_ny_n(t) + y_p(t)$$

□

4. Solution of Homogeneous Case

Consider the $n$ times $M$-differentiable function $y$ for $\alpha \in (0,1]$ and $\beta > 0$. The homogeneous sequential linear fractional differential equation with $M$-derivative is

$$^{n}D_{\alpha}^{\beta}y + p_{n-1}(t)^{n-1}D_{\alpha}^{\beta}y + ... + p_2(t)^2D_{\alpha}^{\beta}y + p_1(t)D_{\alpha}^{\beta}y + p_0(t)y = 0 \quad (8)$$

where $^{n}D_{\alpha}^{\beta}y = D_{\alpha}^{\beta}D_{\alpha}^{\beta} ... D_{\alpha}^{\beta}y$ $n$ times, and the coefficients $p_0, p_1, ..., p_{n-1}$ are real constants.

We define an $r^n$-order differential operator for eqn. (1) as following

$$L_{\alpha,\beta}[y] = ^nD_{\alpha}^{\beta}y + p_{n-1}^{n-1}D_{\alpha}^{\beta}y + ... + p_2^{2}D_{\alpha}^{\beta}y + p_1D_{\alpha}^{\beta}y + p_0y = 0 \quad (9)$$

If $y_1(t), y_2(t), ..., y_n(t)$ are linearly independent solutions of Eqn.(1), then general solution is

$$y = c_1y_1(t) + c_2y_2(t) + ... + c_ny_n(t)$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

**Lemma 4.1.** Suppose that $L_{\alpha,\beta}[]$ is a linear operator with constant coefficients and $\alpha \in (0,1]$ and $\beta > 0$, then for $t > 0$

$$L_{\alpha,\beta}[e^{\frac{\Gamma(\beta+1)}{\alpha}r^\alpha}] = P_n(r)[e^{\frac{\Gamma(\beta+1)}{\alpha}r^\alpha}]$$

Where $P_n(r) = r^n + p_{n-1}r^{n-1} + ... + p_0$ and $r$ is a real or complex constant.
**Lemma 4.2.**

Let \( D_M^{\alpha, \beta} y = r e^{r \Gamma(\beta+1)} t^\alpha \) be the characteristic equation. From Theorem 3.3 it follows that

\[
D_M^{\alpha, \beta} y = r e^{r \Gamma(\beta+1)} t^\alpha, \quad \ldots, \quad D_M^{n, \beta} y = r^n e^{r \Gamma(\beta+1)} t^\alpha
\]  

(10)

Proof.  

\( M \)-derivatives of \( y = e^{r \Gamma(\beta+1)} t^\alpha \) are

\[
D_M^{\alpha, \beta} y = r e^{r \Gamma(\beta+1)} t^\alpha, \quad D_M^{\alpha, \beta} y = r^2 e^{r \Gamma(\beta+1)} t^\alpha, \quad \ldots, \quad D_M^{n, \beta} y = r^n e^{r \Gamma(\beta+1)} t^\alpha
\]

We substitute

\[
y = e^{r \Gamma(\beta+1)} t^\alpha, \quad \text{and Eqn.}(10) \text{ in } L_{\alpha, \beta}[y]
\]

\[
L_{\alpha, \beta}[e^{r \Gamma(\beta+1)} t^\alpha] = \left( r^n D_M^{\alpha, \beta} + p_{n-1} D_M^{\alpha, \beta} + \ldots + p_2 D_M^{\alpha, \beta} + p_1 D_M^{\alpha, \beta} + p_0 \right) e^{r \Gamma(\beta+1)} t^\alpha
\]

\[
= (r^n + P_{n-1} r^{n-1} + \ldots + P_0) e^{r \Gamma(\beta+1)} t^\alpha
\]

Hence, the proof is completed. \( \square \)

The solution to the equation (8) is \( y = e^{r \Gamma(\beta+1)} t^\alpha \).

It follows from Eqn.\((9)\) and Lemma 3.1 that

\[
L_{\alpha, \beta}[e^{r \Gamma(\beta+1)} t^\alpha] = p_n(r) e^{r \Gamma(\beta+1)} t^\alpha = 0
\]

Where \( P_n(r) = r^n + P_{n-1} r^{n-1} + \ldots + P_0 \) is called as the characteristic polynomial. For all \( r \), we have \( e^{r \Gamma(\beta+1)} t^\alpha \neq 0 \). Hence \( P_n(r) = 0 \).

Here

\[
r^n + P_{n-1} r^{n-1} + \ldots + P_0 = 0
\]

(11)

is called as the characteristic equation.

**Lemma 4.2.** Let \( r \) be a root of the characteristic equation \((11)\), then

\[
\frac{\partial}{\partial r} \left( L_{\alpha, \beta}[e^{r \Gamma(\beta+1)} t^\alpha] \right) = L_{\alpha, \beta}[e^{r \Gamma(\beta+1)} t^\alpha]
\]

and

\[
\frac{\partial^l}{\partial r^l} e^{r \Gamma(\beta+1)} t^\alpha = \left( \Gamma(\beta+1) t^\alpha \right)^l e^{r \Gamma(\beta+1)} t^\alpha \]

\( l \) is integer.

Proof. From Theorem 3.3 it follows that \( L_{\alpha, \beta}[\cdot] \) is linear and also \( \frac{\partial}{\partial r} \) is linear by property of classical derivative. Hence

\[
\frac{\partial}{\partial r} \left( L_{\alpha, \beta}[e^{r \Gamma(\beta+1)} t^\alpha] \right) = L_{\alpha, \beta}[e^{r \Gamma(\beta+1)} t^\alpha]
\]

Additionally, from classical derivative, it follows that

\[
\frac{\partial^l}{\partial r^l} e^{r \Gamma(\beta+1)} t^\alpha = \left( \Gamma(\beta+1) t^\alpha \right)^l e^{r \Gamma(\beta+1)} t^\alpha
\]

\( \square \)

**Lemma 4.3.** If \( r_1 \) is a root of multiplicity of \( \mu_1 \) of the characteristic equation \((11)\), then the functions \( y_{1,l}(t) \), where \( l = 0, 1, \ldots, \mu_1 - 1 \) such that

\[
y_{1,l} = \left( \Gamma(\beta+1) t^\alpha \right)^l e^{r_1 \Gamma(\beta+1)} t^\alpha
\]

are solutions of Eq.\((8)\).
Proof. Consider \( L_{\alpha,\beta} \left[ \frac{e^{\Gamma(\beta+1)} t^\alpha}{\alpha} \right] \) = \( p_n(r) e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \). From Lemma 4.2 and applying classical Leibniz rule it follows that

\[
\begin{align*}
\left\{ L_{\alpha,\beta} \left[ \frac{\partial^j}{\partial r^j} e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right] \right\}_{r=r_1} = & \left\{ \frac{\partial^j}{\partial r^j} \left[ L_{\alpha,\beta} \left[ e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right] \right] \right\}_{r=r_1} = \left\{ \frac{\partial^j}{\partial r^j} \left[ p_n(r) e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right] \right\}_{r=r_1} \\
\left\{ L_{\alpha,\beta} \left[ \frac{\partial^j}{\partial r^j} e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right] \right\}_{r=r_1} = & \sum_{j=0}^{l} \binom{l}{j} \left[ \frac{\partial^{l-j}}{\partial r^{l-j}} e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right]_{r=r_1} \frac{\partial^j}{\partial r^j} \left[ P_n(r) \right]_{r=r_1}
\end{align*}
\]

Since \( \frac{\partial^j}{\partial r^j} \left[ P_n(r) \right]_{r=r_1} = 0 \) for \( j = 0, 1, ..., \mu_1 - 1 \)

\[
\left\{ L_{\alpha,\beta} \left[ \frac{\partial^j}{\partial r^j} e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right] \right\}_{r=r_1} = 0
\]

From Lemma 4.2

\[
\left\{ L_{\alpha,\beta} \left[ \left( \frac{\Gamma(\beta+1)}{\alpha} t^\alpha \right)^l e^{\frac{r(\beta+1)}{\alpha} t^\alpha} \right] \right\}_{r=r_1} = 0
\]

\[
L_{\alpha,\beta} \left[ y_{1,l}(t) \right] = 0
\]

Hence \( y_{1,l}(t) \) are solutions of Eq.(8). □

**Corollary 4.1.** Let \( r_j, j = 1, 2, ..., k \) be distinct roots of multiplicity \( \mu_j, j = 1, 2, ..., k \) of the characteristic Eq.(5). Then the following functions

\[
\bigcup_{j=1}^{k} \left\{ \left( \frac{\Gamma(\beta+1)}{\alpha} t^\alpha \right)^l e^{\frac{r_j(\beta+1)}{\alpha} t^\alpha} \right\}_{l=0}^{\mu_j-1}
\]

are linearly independent solutions of Eq.(8).

**Proof.** Corollary 4.1 follows from Lemma 4.3 and Theorem 3.5. □

**Lemma 4.4.** If \( r_1 \) and \( \bar{r}_1 \) (\( r_1 = a + ib, b \neq 0 \)) are complex roots of multiplicity \( \sigma_1 \) of the characteristic equation (11), then for \( l = 0, 1, ..., \sigma_1 - 1 \), the functions

\[
y_{1,l}(t) = \left( \frac{\Gamma(\beta+1)}{\alpha} t^\alpha \right)^l e^{\frac{r_1(\beta+1)}{\alpha} t^\alpha} \left[ \cos \left( \frac{b\Gamma(\beta+1)}{\alpha} t^\alpha \right) + isin \left( \frac{b\Gamma(\beta+1)}{\alpha} t^\alpha \right) \right]
\]

and

\[
y_{2,l}(t) = \left( \frac{\Gamma(\beta+1)}{\alpha} t^\alpha \right)^l e^{\frac{r_1(\beta+1)}{\alpha} t^\alpha} \left[ \cos \left( \frac{b\Gamma(\beta+1)}{\alpha} t^\alpha \right) - isin \left( \frac{b\Gamma(\beta+1)}{\alpha} t^\alpha \right) \right]
\]

are linearly independent solutions of Eq.(8).

**Proof.** Since \( r_1 = a + ib \) is a root of multiplicity \( \sigma_1 \) of the characteristic equation (11), From Lemma 4.3 and using Eulers identity it follows that, the functions

\[
y_{1,l}(t) = \left( \frac{\Gamma(\beta+1)}{\alpha} t^\alpha \right)^l e^{\frac{(a+ib)(\beta+1)}{\alpha} t^\alpha}
\]
\[ y_{1,l}(t) = \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{\frac{\alpha r_l^{(\beta+1)}}{\alpha} t^\alpha} \left[ \cos \left( \frac{b \Gamma(\beta + 1)}{\alpha} t^\alpha \right) + i \sin \left( \frac{b \Gamma(\beta + 1)}{\alpha} t^\alpha \right) \right] \]

are solutions of the Eq. (8). Similarly, for \( \tilde{r}_l = a - ib \), the functions

\[ y_{2,l}(t) = \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{\frac{(a - ib) \Gamma(\beta+1)}{\alpha} t^\alpha} \]

are solutions of the Eq. (8). Hence proof is completed. \( \square \)

**Corollary 4.2.** If \( \{ r_j, \tilde{r}_j \}_{j=1}^m \), \( r_j = a_j + ib_j, b_j \neq 0 \) distinct 2m roots of multiplicity \( \{ \sigma_j \}_{j=1}^m \) of the characteristic equation (11), then, the functions

\[ \bigcup_{j=1}^m \left\{ \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{\frac{a_j r_j^{(\beta+1)}}{\alpha} t^\alpha} \cos \left( \frac{b_j \Gamma(\beta + 1)}{\alpha} t^\alpha \right) + \sin \left( \frac{b_j \Gamma(\beta + 1)}{\alpha} t^\alpha \right) \right\} \]

and

\[ \bigcup_{j=1}^m \left\{ \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{\frac{a_j \tilde{r}_j^{(\beta+1)}}{\alpha} t^\alpha} \cos \left( \frac{b_j \Gamma(\beta + 1)}{\alpha} t^\alpha \right) - \sin \left( \frac{b_j \Gamma(\beta + 1)}{\alpha} t^\alpha \right) \right\} \]

are solutions of the Eq. (8). Hence proof is completed. \( \square \)

**Theorem 4.1.** If \( \{ r_j \}_{j=1}^k \) are distinct k roots of multiplicity \( \{ \mu_j \}_{j=1}^k \) and \( \{ \lambda_j, \bar{\lambda}_j \}_{j=1}^m \), \( \lambda_j = a_j + ib_j, b_j \neq 0 \) are distinct 2m roots of multiplicity \( \{ \sigma_j \}_{j=1}^m \) of the characteristic equation (11) such that \( \sum_{j=1}^k \mu_j = 2 \sum_{j=1}^m \sigma_j = n \), then the functions

\[ \bigcup_{j=1}^k \left\{ \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{r_j^{(\beta+1)}} t^\alpha \right\}^{\mu_j-1} \]

\[ \bigcup_{j=1}^m \left\{ \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{a_j \tilde{r}_j^{(\beta+1)}} t^\alpha \right\}^{\sigma_j-1} \]

and

\[ \bigcup_{j=1}^m \left\{ \left( \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right)^l e^{a_j \tilde{r}_j^{(\beta+1)}} t^\alpha \right\}^{\sigma_j-1} \]

are the fundamental set of solutions of the equation (8).
Proof. The proof of the Theorem 4.1 is follows from Corollary 4.1, Corollary 4.2 and Theorem 3.5. □

Example 4.1.

$$2D_M^{\alpha,\beta}y + 4D_M^{\alpha,\beta}y + 3y = 0 \quad (12)$$

The characteristic equation of (12) is

$$r^2 + 4r + 3 = 0$$

Therefore, the roots are $$r = -3$$ and $$r = -1$$

Hence, the general solution is

$$y(t) = c_1e^{-3\Gamma(\beta+1)/\alpha} + c_2e^{-\Gamma(\beta+1)/\alpha}$$

Example 4.2.

$$2D_M^{\alpha,\beta}y - 4D_M^{\alpha,\beta}y + 4y = 0 \quad (13)$$

The characteristic equation of (13) is

$$r^2 - 4r + 4 = 0$$

The roots are $$r_1, r_2 = 2$$

Hence, the general solution is

$$y(t) = (c_1 + c_2 \Gamma(\beta+1)/\alpha) e^{2\Gamma(\beta+1)/\alpha}$$

Example 4.3.

$$2D_M^{\alpha,\beta}y + 4D_M^{\alpha,\beta}y + 5y = 0 \quad (14)$$

The characteristic equation of (14) is

$$r^2 + 4r + 5 = 0$$

The roots are $$r_1 = -2 + i$$ and $$r_2 = -2 - i$$

Hence, the general solution is

$$y(t) = e^{-2\Gamma(\beta+1)/\alpha} \left[ \cos \left( \frac{\Gamma(\beta+1)\alpha}{\alpha} t^\alpha \right) + i \sin \left( \frac{\Gamma(\beta+1)\alpha}{\alpha} t^\alpha \right) \right]$$

5. Solution of Non-Homogeneous Case

In this section, Method of variation of parameters is applied to derive the particular solution of the equation.

$$nD_M^{\alpha,\beta}y + p_{n-1}(t)D_M^{\alpha,\beta}y + \ldots + p_2(t)D_M^{\alpha,\beta}y + p_1(t)D_M^{\alpha,\beta}y + p_0(t)y = f(t) \quad (15)$$

where $$y$$ is $$n$$ times $$M$$-differentiable function for $$\alpha \in (0,1]$$ and $$\beta > 0.$$
Theorem 5.1. If \( u(t) \) is a solution of homogeneous case of the equation (15) such that

\[
u(t) = \sum_{i=1}^{n} c_i y_i(t)
\]

then particular solution of the equation (15) is

\[
v(t) = \sum_{i=1}^{n} c_i(t) y_i(t)
\]

Where \( c_1(t), c_2(t), ..., c_n(t) \) provide following system of equations

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)y_i(t) = 0
\]

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)D_{M}^{\alpha,\beta} y_i(t) = 0
\]

\[
\vdots
\]

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)n^{-2}D_{M}^{\alpha,\beta} y_i(t) = 0
\]

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)n^{-1}D_{M}^{\alpha,\beta} y_i(t) = f(t)
\]

Proof. The solution of the equation (15) is in the form

\[
v(t) = \sum_{i=1}^{n} c_i(t) y_i(t)
\]

The \( M \)-derivative of \( v(t) \) for \( \alpha \in (0, 1] \) and \( \beta > 0 \) will be

\[
D_{M}^{\alpha,\beta} v(t) = \sum_{i=1}^{n} c_i(t)D_{M}^{\alpha,\beta} y_i(t) + \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)y_i(t)
\]

Applying the first condition \( \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)y_i(t) = 0 \), we obtain

\[
D_{M}^{\alpha,\beta} v(t) = \sum_{i=1}^{n} c_i(t)D_{M}^{\alpha,\beta} y_i(t)
\]

If we calculate the \( M \)-derivative of \( D_{M}^{\alpha,\beta} v(t) \) for \( \alpha \in (0, 1] \) and \( \beta > 0 \), then we get

\[
2 D_{M}^{\alpha,\beta} v(t) = \sum_{i=1}^{n} c_i(t)D_{M}^{\alpha,\beta} y_i(t) + \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)D_{M}^{\alpha,\beta} y_i(t)
\]
Apply second condition \( \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t) D_{M}^{\alpha,\beta} y_i(t) = 0 \), we obtain

\[
2 D_{M}^{\alpha,\beta} v(t) = \sum_{i=1}^{n} c_i(t)^2 D_{M}^{\alpha,\beta} y_i(t)
\]

By continuing in this way, we get

\[
n^{-1} D_{M}^{\alpha,\beta} v(t) = \sum_{i=1}^{n} c_i(t)^{n-1} D_{M}^{\alpha,\beta} y_i(t) + \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)^{n-2} D_{M}^{\alpha,\beta} y_i(t)
\]

We substitute \( v(t), 2 D_{M}^{\alpha,\beta} v(t), ..., n D_{M}^{\alpha,\beta} v(t) \) in the equation (15), we have

\[
\sum_{i=1}^{n} c_i(t)^n D_{M}^{\alpha,\beta} y_i(t) + \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)^{n-1} D_{M}^{\alpha,\beta} y_i(t) + p_{n-1} \sum_{i=1}^{n} c_i(t)^{n-1} D_{M}^{\alpha,\beta} y_i(t) + ... + p_1 D_{M}^{\alpha,\beta} y_i(t) + p_0 y_i(t) = f(t)
\]

Since \( y_1(t), y_2(t), ..., y_n(t) \) are solutions of homogeneous case of equation (8), then

\[
\sum_{i=1}^{n} c_i(t)^{n-1} D_{M}^{\alpha,\beta} y_i(t) = f(t)
\]

We obtain \( n^{th} \) condition as

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)^{n-1} D_{M}^{\alpha,\beta} y_i(t) = f(t)
\]

Hence we obtain the following system

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)y_i(t) = 0
\]

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)D_{M}^{\alpha,\beta} y_i(t) = 0
\]

\[
\vdots
\]

\[
\sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)^{n-2} D_{M}^{\alpha,\beta} y_i(t) = 0
\]
\[ \sum_{i=1}^{n} D_{M}^{\alpha,\beta} c_i(t)^{n-1} D_{M}^{\alpha,\beta} y_i(t) = f(t) \]

Solving the above system (17) provides \( D_{M}^{\alpha,\beta} c_i(t), i = 1, 2, ..., n \). Therefore we can write the particular solution of equation (15) as

\[ v(t) = \sum_{i=1}^{n} c_i(t)y_i(t) \]

**Example 5.1.**

\[ 2D_{M}^{\alpha,\beta} y + 4D_{M}^{\alpha,\beta} y + 3y = f(t) \]

(a) Let \( f(t) = e^{2\alpha} \). For \( v(t) = c_1(t)e^{-\frac{3\Gamma(\beta+1)}{\alpha} t} + c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha} t} \), the system of equations are built by the conditions as following

\[ D_{M}^{\alpha,\beta} c_1(t)e^{-\frac{3\Gamma(\beta+1)}{\alpha} t} + D_{M}^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha} t} = 0 \]

\[ -3D_{M}^{\alpha,\beta} c_1(t)e^{-\frac{3\Gamma(\beta+1)}{\alpha} t} - D_{M}^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha} t} = e^{2\alpha} \]

Solving the above system of equations and using M-integral we obtain

\[ c_1(t) = \frac{\Gamma(\beta+1) e^{\frac{2\alpha + 3\Gamma(\beta+1)}{\alpha} t}}{4\alpha + 3\alpha t} \]

\[ c_2(t) = \frac{\Gamma(\beta+1) e^{\frac{2\alpha + 3\Gamma(\beta+1)}{\alpha} t}}{4\alpha + 2\alpha t} \]

Then particular solution \( v(t) \) is

\[ v(t) = \frac{\Gamma(\beta+1)^2}{4\alpha^2 + 8\alpha \Gamma(\beta+1) + 3\alpha^2 (\beta+1)^2} e^{2\alpha} \]

(b) Let \( f(t) = 2t^{2\alpha} + t^\alpha - 3. \) The system of equations for this case is

\[ D_{M}^{\alpha,\beta} c_1(t)e^{-\frac{3\Gamma(\beta+1)}{\alpha} t} + D_{M}^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha} t} = 0 \]

\[ -3D_{M}^{\alpha,\beta} c_1(t)e^{-\frac{3\Gamma(\beta+1)}{\alpha} t} - D_{M}^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha} t} = 2t^{2\alpha} + t^\alpha - 3 \]

Solve this system of equations, we have

\[ c_1(t) = \frac{-1}{3} t^{2\alpha} e^{\frac{3\Gamma(\beta+1)}{\alpha} t} + \left( \frac{4\alpha - 3\Gamma(\beta+1)}{18\Gamma(\beta+1)} \right) t^\alpha e^{\frac{3\Gamma(\beta+1)}{\alpha} t} \]

\[ + \left( \frac{-4\alpha^2 + 3\alpha \Gamma(\beta+1) + 27\Gamma(\beta+1)^2}{54\Gamma(\beta+1)^2} \right) e^{\frac{3\Gamma(\beta+1)}{\alpha} t} \]

\[ c_2(t) = t^{2\alpha} e^{\frac{\Gamma(\beta+1)}{\alpha} t} + \left( \frac{\Gamma(\beta+1) - 4\alpha}{2\Gamma(\beta+1)} \right) t^\alpha e^{\frac{\Gamma(\beta+1)}{\alpha} t} \]

\[ + \left( \frac{4\alpha^2 - (\alpha \Gamma(\beta+1) + 3\Gamma(\beta+1)^2)}{2\Gamma(\beta+1)^2} \right) e^{\frac{\Gamma(\beta+1)}{\alpha} t} \]

Hence, particular solution \( v(t) \) is obtained by

\[ v(t) = \frac{2}{3} t^{2\alpha} + \left( \frac{3\Gamma(\beta+1) - 16\alpha}{9\Gamma(\beta+1)} \right) t^\alpha e^{\frac{\Gamma(\beta+1)}{\alpha} t} + \left( \frac{52\alpha^2 - 12\alpha \Gamma(\beta+1) - 27\Gamma(\beta+1)^2}{27\Gamma(\beta+1)^2} \right) e^{\frac{\Gamma(\beta+1)}{\alpha} t} \]
(c) Let \( f(t) = \sin(2t^\alpha) \). The system of equations for this case is

\[
D_M^{\alpha,\beta} c_1(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} + D_M^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} = 0
\]

\[
-3D_M^{\alpha,\beta} c_1(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} - D_M^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} = \sin(2t^\alpha)
\]

Solve this system of equations, we have

\[
c_1(t) = \frac{-3\Gamma(\beta+1)^2}{8\alpha^2 + 18\Gamma(\beta+1)^2} e^{\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} \sin(2t^\alpha) + \frac{\alpha\Gamma(\beta+1)}{4\alpha^2 + 9\Gamma(\beta+1)^2} e^{\frac{3\Gamma(\beta+1)}{\alpha}t^\alpha} \cos(2t^\alpha)
\]

\[
c_2(t) = \frac{\Gamma(\beta+1)^2}{8\alpha^2 + 2\Gamma(\beta+1)^2} e^{\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} \sin(2t^\alpha) - \frac{\alpha\Gamma(\beta+1)}{4\alpha^2 + \Gamma(\beta+1)^2} e^{\frac{3\Gamma(\beta+1)}{\alpha}t^\alpha} \cos(2t^\alpha)
\]

Hence, particular solution \( v(t) \) is obtained by

\[
v(t) = \frac{-4\alpha^2\Gamma(\beta+1)^2 + 3\Gamma(\beta+1)^4}{16\alpha^4 + 40\alpha^2\Gamma(\beta+1)^2 + 9\Gamma(\beta+1)^4} \sin(2t^\alpha)
\]

\[
-\frac{8\alpha\Gamma(\beta+1)^3}{16\alpha^4 + 40\alpha^2\Gamma(\beta+1)^2 + 9\Gamma(\beta+1)^4} \cos(2t^\alpha)
\]

(d) Let \( f(t) = e^{2t^\alpha} t^\alpha \). The system of equations for this case is

\[
D_M^{\alpha,\beta} c_1(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} + D_M^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} = 0
\]

\[
-3D_M^{\alpha,\beta} c_1(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} - D_M^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} = e^{2t^\alpha} t^\alpha
\]

Solve this system of equations, we have

\[
c_1(t) = \frac{-\Gamma(\beta+1)}{4\alpha + 6\Gamma(\beta+1)} t^\alpha e^{\frac{2\alpha + 3\Gamma(\beta+1)}{\alpha}t^\alpha} + \frac{\alpha\Gamma(\beta+1)}{2(2\alpha + 3\Gamma(\beta+1))^2} e^{\frac{2\alpha + 3\Gamma(\beta+1)}{\alpha}t^\alpha}
\]

\[
c_2(t) = \frac{-\Gamma(\beta+1)}{4\alpha + 2\Gamma(\beta+1)} t^\alpha e^{\frac{2\alpha + \Gamma(\beta+1)}{\alpha}t^\alpha} - \frac{\alpha\Gamma(\beta+1)}{2(2\alpha + \Gamma(\beta+1))^2} e^{\frac{2\alpha + \Gamma(\beta+1)}{\alpha}t^\alpha}
\]

Hence, particular solution \( v(t) \) is obtained by

\[
v(t) = \frac{\Gamma(\beta+1)^2}{4\alpha^2 + 8\alpha\Gamma(\beta+1) + 3\Gamma(\beta+1)^2} t^\alpha e^{2t^\alpha} - \frac{4\alpha^2\Gamma(\beta+1)^2 + 4\alpha\Gamma(\beta+1)^3}{(4\alpha^2 + 8\alpha\Gamma(\beta+1) + 3\Gamma(\beta+1)^2)^2} e^{2t^\alpha}
\]

(e) Let \( f(t) = e^{-4t^\alpha} \). Take \( \alpha \neq \frac{3}{4} \) and \( \alpha \neq \frac{1}{4} \), the system of equations for this case is

\[
D_M^{\alpha,\beta} c_1(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} + D_M^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} = 0
\]

\[
-3D_M^{\alpha,\beta} c_1(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} - D_M^{\alpha,\beta} c_2(t)e^{-\frac{\Gamma(\beta+1)}{\alpha}t^\alpha} = e^{-4t^\alpha}
\]

Solve this system of equations, we have

\[
c_1(t) = \frac{\Gamma(\beta+1)}{8\alpha - 6\Gamma(\beta+1)} e^{\frac{\Gamma(\beta+1) - 4\alpha}{\alpha}t^\alpha}
\]
\[ c_2(t) = \frac{\Gamma(\beta + 1)}{2\Gamma(\beta + 1) - 8\alpha} e^{\frac{\Gamma(\beta + 1) - 4\alpha}{\alpha}} t^\alpha \]

Hence, we obtain particular solution \( v(t) \) as following:

\[ v(t) = \frac{\Gamma(\beta + 1)^2}{16\alpha^2 - 16\alpha\Gamma(\beta + 1) + 3\Gamma(\beta + 1)^2} e^{-4t^\alpha} \]

Take \( \alpha = \frac{3}{4} \) and \( v(t) = c_1(t)e^{-4\Gamma(\beta + 1)t^{\frac{3}{4}}} + c_2(t)e^{-\frac{4\Gamma(\beta + 1)}{3}t^{\frac{3}{4}}} \)

The system of equations is

\[ D^\frac{3}{4}_M c_1(t)e^{-4\Gamma(\beta + 1)t^{\frac{3}{4}}} + D^\frac{3}{4}_M c_2(t)e^{-\frac{4\Gamma(\beta + 1)}{3}t^{\frac{3}{4}}} = 0 \]

\[ -3D^\frac{3}{4}_M c_1(t)e^{-4\Gamma(\beta + 1)t^{\frac{3}{4}}} - D^\frac{3}{4}_M c_2(t)e^{-\frac{4\Gamma(\beta + 1)}{3}t^{\frac{3}{4}}} = e^{-4t^\frac{3}{4}} \]

We solve the above equation, \( c_1(t) = -\frac{\Gamma(\beta + 1)}{6 + 6\Gamma(\beta + 1)} e^{(-4+4\Gamma(\beta + 1))t^{\frac{3}{4}}} \) and \( c_2(t) = \frac{\Gamma(\beta + 1)}{6 + 2(\beta + 1)} e^{(-12+4\Gamma(\beta + 1))t^{\frac{3}{4}}} \) is obtained.

The particular solution is \( v(t) = -\frac{\Gamma(\beta + 1)^2}{9 - 12\Gamma(\beta + 1) + 3\Gamma(\beta + 1)^2} e^{-4t^\frac{3}{4}} \)

Take \( \alpha = \frac{1}{4} \) and \( v(t) = c_1(t)e^{-12\Gamma(\beta + 1)t^{\frac{1}{4}}} + c_2(t)e^{-4\Gamma(\beta + 1)t^{\frac{1}{4}}} \)

The system of equations is

\[ D^\frac{1}{4}_M c_1(t)e^{-12\Gamma(\beta + 1)t^{\frac{1}{4}}} + D^\frac{1}{4}_M c_2(t)e^{-4\Gamma(\beta + 1)t^{\frac{1}{4}}} = 0 \]

\[ -3D^\frac{1}{4}_M c_1(t)e^{-12\Gamma(\beta + 1)t^{\frac{1}{4}}} - D^\frac{1}{4}_M c_2(t)e^{-4\Gamma(\beta + 1)t^{\frac{1}{4}}} = e^{-4t^\frac{1}{4}} \]

We solve the above equation, \( c_1(t) = -\frac{\Gamma(\beta + 1)}{2 + 6\Gamma(\beta + 1)} e^{(-4+12\Gamma(\beta + 1))t^{\frac{1}{4}}} \) and \( c_2(t) = \frac{\Gamma(\beta + 1)}{2 + 2(\beta + 1)} e^{(-4+4\Gamma(\beta + 1))t^{\frac{1}{4}}} \) is obtained.

The particular solution is \( v(t) = -\frac{4\Gamma(\beta + 1)^2}{4 - 6\Gamma(\beta + 1) + 12\Gamma(\beta + 1)^2} e^{-4t^\frac{1}{4}} \)

6. Conclusion

In this paper, Existences and Uniqueness theorems for sequential linear \( M \)-fractional differential equations are presented. We give solution of \( M \)-fractional differential equations with constants for homogeneous case using fractional exponential function and for non homogeneous case, we applied method of variation of parameters.

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