A graphon approach to limiting spectral distributions of Wigner-type matrices

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Abstract
We present a new approach, based on graphon theory, to finding the limiting spectral distributions of general Wigner-type matrices. This approach determines the moments of the limiting measures and the equations of their Stieltjes transforms explicitly with weaker assumptions on the convergence of variance profiles than previous results. As applications, we give a new proof of the semicircle law for generalized Wigner matrices and determine the limiting spectral distributions for three sparse inhomogeneous random graph models with sparsity $\omega(1/n)$: inhomogeneous random graphs with roughly equal expected degrees, $W$-random graphs and stochastic block models with a growing number of blocks. Furthermore, we show our theorems can be applied to random Gram matrices with a variance profile for which we can find the limiting spectral distributions under weaker assumptions than previous results.

KEYWORDS
graphon, homomorphism density, inhomogeneous random graph, spectral distribution, Wigner-type matrix

1 | INTRODUCTION

1.1 | Eigenvalue statistics of random matrices

Random matrix theory is a central topic in probability and statistical physics with many connections to various areas such as combinatorics, numerical analysis, statistics, and theoretical computer science. One of the primary goals of random matrix theory is to study the limiting laws for eigenvalues of $(n \times n)$ Hermitian random matrices as $n \to \infty$. 

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Classically, a Wigner matrix is a Hermitian random matrix whose entries are independent and identically distributed random variables up to the symmetry constraint, and have zero expectation and variance 1. As has been known since Wigner’s seminal paper [53] in various formats, for Wigner matrices, the empirical spectral distribution converges almost surely to the semicircle law. The i.i.d. requirement and the constant variance condition are not essential for proving the semicircle law, as can be seen from the fact that generalized Wigner matrices, whose entries have different variances but each column of the variance profile is stochastic, turned out to obey the semicircle law [10, 32, 36], under various conditions as well. Beyond the semicircle law, the Wigner matrices exhibit universality [31, 51], a phenomenon that has been recently shown to hold for other models, including generalized Wigner matrices [32], adjacency matrices of Erdős-Rényi random graphs [28, 29, 42, 52] and general Wigner-type matrices [3].

A slightly different direction of research is to investigate structured random matrix models whose limiting spectral distribution is not the semicircle law. One such example is random block matrices, whose limiting spectral distribution has been found in [34, 50] using free probability. Ding [26] used moment methods to derive the limiting spectral distribution of random block matrices for a fixed number of blocks (a claim in [26] that the method extends to the growing number of blocks case is unfortunately incorrect). Recently Alt and coworkers [8] provided a unified way to study the global law for a general class of non-Hermitian random block matrices including Wigner-type matrices.

1.2 | Graphons and convergence of graph sequences

Understanding large networks is a fundamental problem in modern graph theory and to properly define a limit object, an important issue is to have good definitions of convergence for graph sequences. Graphons, introduced in 2006 by Lovász and Szegedy [45] as limits of dense graph sequences, aim to provide a solution to this question. Roughly speaking, the set of finite graphs endowed with the cut metric (see Definition 2.3) gives rise to a metric space, and the completion of this space is the space of graphons. These objects may be realized as symmetric, Lebesgue measurable functions from $[0, 1]^2$ to $\mathbb{R}$. They also characterize the convergence of graph sequences based on graph homomorphism densities [20, 21]. Recently, graphon theory has been generalized for sparse graph sequences [18, 19, 35, 43].

The most relevant results for our endeavor are the connections between two types of convergences: left convergence in the sense of homomorphism densities and convergence in cut metric. In our approach, for the general Wigner-type matrices, we will regard the variance profile matrices $S_n$ as a graphon sequence. The convergence of empirical spectral distributions is connected to the convergence of this graphon sequence associated with $S_n$ in either left convergence sense or in cut metric.

1.3 | Random graph models

One of the most basic models for random graphs is the Erdős-Rényi random graph. The scaled adjacency matrix $\frac{A_n}{\sqrt{np}}$ of Erdős-Rényi random graph $\mathcal{G}(n, p)$ has the semicircle law as limiting spectral distribution [27, 52] when $np \to \infty$.

Random graphs generated from an inhomogeneous Erdős-Rényi model $\mathcal{G}(n, (p_{ij}))$, where edges exist independently with given probabilities $p_{ij}$ is a generalization of the classical Erdős-Rényi model $\mathcal{G}(n, p)$. Recently, there are some results on the largest eigenvalue [13, 14] and the spectrum of the Laplacian matrices [22] of inhomogeneous Erdős-Rényi model random graphs. Many popular graph
models arise as special cases of $G(n, (p_{ij}))$ such as random graphs with given expected degrees [24], stochastic block models (SBMs) [41], and $W$-random graphs [18, 45].

The SBM is a random graph model with planted clusters. It is widely used as a canonical model to study clustering and community detection in network and data sciences [1]. Here one assumes that a random graph was generated by first partitioning vertices into unknown $d$ groups, and then connecting two vertices with a probability that depends on their assigned groups. Specifically, suppose we have a partition of $[n] = V_1 \cup V_2 \cup \cdots \cup V_d$ for some integer $d$, and that $|V_i| = n_i$ for $i = 1, \ldots, d$. Suppose that for any pair $(k, l) \in [d] \times [d]$ there is a $p_{kl} \in [0, 1]$ such that for any $i \in V_k, j \in V_l$,

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{kl}, \\ 0, & \text{otherwise.} \end{cases}$$

Also, if $k = l$, there is a $p_{kk}$ such that $a_{ii} = 0$ for $i \in V_k$ and for any $i \neq j, i, j \in V_k$,

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{kk}, \\ 0, & \text{otherwise.} \end{cases}$$

The task for community detection is to find the unknown partition of a random graph sampled from the SBM. In this article, we will consider the limiting spectral distribution of the adjacency matrix of an SBM. Since permuting the adjacency matrix does not change its spectrum, we may assume its adjacency matrix has a block structure by a proper permutation.

As the number of vertices grows, the network might not be well described by an SBM with a fixed number of blocks. Instead, we might consider the case where the number of blocks grows as well [23] (see Section 7). A different model that generates nonparametric random graphs is called $W$-random graphs and is achieved by sampling points uniformly from a graphon $W$. We will define a sparse version of $W$-random graphs in Section 5 for which one can obtain a limiting spectral distribution when the sparsity $\rho_n = o(1/n)$.

For inhomogeneous graphs with bounded expected degree introduced by Bollobás, Janson and Riordan [16], their graphon limits will be 0 and our main result will not cover this regime. This is because the graphon limit is only suitable for graph sequences with unbounded degrees. Instead, the spectrum of random graphs with bounded expected degrees was studied in [17] by local weak convergence [5, 15], a graph limit theory for graph sequences with bounded degrees.

### 1.4 Random Gram matrices

Let $X$ be a $m \times n$ random matrix with independent, centered entries with unit variance, where $\frac{m}{n}$ converges to some positive constant as $n \to \infty$. It is known that the empirical spectral distribution converges to the Marčenko-Pastur law [48]. However, some applications in wireless communication require understanding the spectrum of $\frac{1}{n}XX^*$ where $X$ has a variance profile [25, 39]. Such matrices are called random Gram matrices. The limiting spectral distribution of a random Gram matrix with noncentered diagonal entries and a variance profile was obtained in [38] under the assumptions that the $(4 + \epsilon)$-th moments of entries in $X$ are bounded and the variance profile comes from a continuous function. The local law and singularities of the density of states of random Gram matrices were analyzed in [6, 7].

We use the symmetrization trick to connect the eigenvalues of $\frac{1}{n}XX^*$ to eigenvalues of a Hermitian matrix $H := \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$. As a corollary from our main theorem in Section 3, when $EX = 0$, we obtain...
the moments and Stieltjes transforms of the limiting spectral distributions under weaker assumptions than [38]. In particular, we only need entries in $X$ to have finite second moments, and the variance profile of $H_n$ converges in terms of homomorphism densities.

### 1.5 Contributions of this article

We obtained a formula to compute the moments of limiting spectral distributions of general Wigner-type matrices from graph homomorphism densities, and we derived quadratic vector equations as in [2] from this formula.

Previous approaches to the problem require the variance profiles to converge to a function whose set of discontinuities has measure zero [10, 38, 50], we make no such requirement here. The method in [50] is based on free probability theory, and it is assumed that all entries of the matrix are Gaussian, while our Theorems 3.2 and 3.4 work for non-i.i.d. entries with general distributions. Especially, we cover a variety of sparse matrix models (see Sections 4-7). The argument in [10] is based on a sophisticated moment method for band matrix models, and our method proof based on graphon theory is much simpler and can be applied to many different models including random Gram matrices. For random Gram matrices, in [38], it is assumed that all entries have $(4 + \varepsilon)$ moments and the variance profile is continuous. The continuity assumption is used to show the Stieltjes transform of the empirical measure converges to the Stieltjes transform of the limiting measure. We remove the technical higher moments and the continuity assumptions since our combinatorial approach requires less regularity.

All the three previous results above assume the limiting variance profile exists and is continuous. This assumption is used to have an error control under $L^\infty$-norm between the $n$-step variance profile and the limiting variance profile, which will guarantee that either the moments of the empirical measure converge or the Stieltjes transform the empirical measure converges. However, this $L^\infty$-convergence is only a stronger sufficient condition compared to our condition in Theorems 3.2 and 3.4. The key observation in our approach is that permuting a random matrix does not change its spectrum, but the continuity of the variance is destroyed. The cut metric in the graphon theory is a suitable tool to exploit the permutation invariant property of the spectrum (see Theorem 3.4).

Moreover, we realize that to make the moments of the empirical measure converge, we do not need to assume the moments of the limiting measure is an integral in terms of the limiting variance profile. All we need is the convergence of homomorphism density from trees. We show two examples in Section 4 where we do not have a limiting variance profile but the moments of the empirical measure still converge: generalized Wigner matrices and inhomogeneous random graphs with roughly equal expected degrees.

Besides, if the limiting distribution is not the semicircle law, previous results only implicitly characterize the Stieltjes transform of the limiting measure by the quadratic vector equations (see (3.2) and (3.3)), which are not easy to solve. Our combinatorial approach explicitly determines the moments of the limiting distributions in terms of sums of graphon integrals. Our convergence condition (see Theorem 3.2 (1)) is the weakest so far for the existence of limiting spectral distributions and covers a variety of models like generalized Wigner matrices, adjacency matrices of sparse SBMs with a growing number of blocks, and random Gram matrices.

The organization of this article is as follows: In Section 2, we introduce definitions and facts that will be used in our proofs. In Section 3, we state and prove the main theorems for general Wigner-type matrices and then specialize our results to different models in Sections 4-7. In Section 8, we extend our results to random Gram matrices with a variance profile.
2 | PRELIMINARY

2.1 | Random matrix theory

We recall some basic definitions in random matrix theory. For any $n \times n$ Hermitian matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, the empirical spectral distribution (ESD) of $A$ is defined by

$$ F_A(x) := \frac{1}{n} \sum_{i=1}^{n} 1_{\{\lambda_i \leq x\}}. $$

Our main task in this article is to investigate the convergence of the sequence of empirical spectral distribution $\{F_A^n\}$ to the limiting spectral distribution for a given sequence of structured random matrices. A useful tool to study the convergence of measure is the Stieltjes transform.

Let $\mu$ be a probability measure on $\mathbb{R}$. The Stieltjes transform of $\mu$ is a function $s(z)$ defined on the upper half plane $\mathbb{C}^+$ by the formula:

$$ s(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x), \quad z \in \mathbb{C}^+. $$

Suppose that $\mu$ is compactly supported, and denote $r := \sup \{|t| \mid t \in \text{supp}(\mu)\}$. We then have a power series expansion

$$ s(z) = \sum_{k=0}^{\infty} \frac{\beta_k}{z^{k+1}}, \quad |z| \geq r, \quad (2.1) $$

where $\beta_k := \int_{\mathbb{R}} x^k d\mu(x)$ is the $k$-th moment of $\mu$ for $k \geq 0$.

We recall some combinatorial objects related to random matrix theory.

Definition 2.1. The rooted planar tree is a planar graph with no cycles, with one distinguished vertex as a root, and with a choice of ordering at each vertex. The ordering defines a way to explore the tree starting at the root. Depth-first search is an algorithm for traversing rooted planar trees. One starts at the root and explores as far as possible along each branch before backtracking. An enumeration of the vertices of a tree is said to have depth-first search order if it is the output of the depth-first search.

The Dyck paths of length $2k$ are bijective to rooted planar trees of $k+1$ vertices by the depth-first search (see Lem. 2.1.6 in [9]). Hence the number of rooted planar trees with $k+1$ vertices is the $k$-th Catalan number $C_k := \frac{1}{k+1} \binom{2k}{k}$.

2.2 | Graphon theory

We introduce definitions from graphon theory. For more details, see [44].

Definition 2.2. A graphon is a symmetric, integrable function $W : [0, 1]^2 \to \mathbb{R}$.

Here symmetric means $W(x, y) = W(y, x)$ for all $x, y \in [0, 1]$. Every weighted graph $G$ has an associated graphon $W^G$ constructed as follows. First divide the interval $[0, 1]$ into intervals $I_1, \ldots, I_{|V(G)|}$ of length $\frac{1}{|V(G)|}$, then give the edge weight $\beta_{ij}$ on $I_i \times I_j$, for all $i, j \in V(G)$. In this way, every finite weighted graph gives rise to a graphon (see Figure 1).
The most important metric on the space of graphons is the cut metric. The space that contains all graphons taking values in \([0, 1]\) endowed with the cut metric is a compact metric space.

**Definition 2.3.** For a graphon \(W : [0, 1]^2 \to \mathbb{R}\), the cut norm is defined by

\[
\|W\|_\Box := \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W(x, y) \, dx \, dy \right|
\]

where \(S, T\) range over all measurable subsets of \([0, 1]\). Given two graphons \(W, W' : [0, 1]^2 \to \mathbb{R}\), define \(d_\Box(W, W') := \|W - W'\|_\Box\) and the cut metric \(\delta_\Box\) is defined by

\[
\delta_\Box(W, W') := \inf_\sigma d_\Box(W\sigma, W'),
\]

where \(\sigma\) ranges over all measure-preserving bijections \([0, 1] \to [0, 1]\) and \(W\sigma(x, y) := W(\sigma(x), \sigma(y))\).

Using the cut metric, we can compare two graphs with different sizes and measure their similarity, which defines a type of convergence of graph sequences whose limiting object is the graphon we introduced. Another way of defining the convergence of graphs is to consider graph homomorphisms.

**Definition 2.4.** For any graphon \(W\) and multigraph \(F = (V, E)\) (without loops), define the homomorphism density from \(F\) to \(W\) as

\[
t(F, W) := \int_{[0, 1]^{|V|}} \prod_{\{i, j\} \in E} W(x_i, x_j) \prod_{i \in V} dx_i.
\]

One may define homomorphism density from partially labeled graphs to graphons, as follows:

**Definition 2.5.** Let \(F = (V, E)\) be a \(k\)-labeled multigraph. Let \(V_0 = V \setminus [k]\) be the set of unlabeled vertices. For any graphon \(W\), and \(x_1, \ldots, x_k \in [0, 1]\), define

\[
t_{x_1, \ldots, x_k}(F, W) := \int_{x \in [0, 1]^{|V|}} \prod_{\{i, j\} \in E} W(x_i, x_j) \prod_{i \in V_0} dx_i.
\]  \(\text{(2.2)}\)

This is a function of \(x_1, \ldots, x_k\).

It is natural to think two graphons \(W\) and \(W'\) are similar if they have similar homomorphism densities from any finite graph \(G\). This leads to the following definition of left convergence.

**Definition 2.6.** Let \(W_n\) be a sequence of graphons. We say \(W_n\) is convergent from the left if \(t(F, W_n)\) converges for any finite simple (no loops, no multiedges, no directions) graph \(F\).
The importance of homomorphism densities is that they characterize convergence under the cut metric. Let $\mathcal{W}_0$ be the set of all graphons such that $0 \leq W \leq 1$. The following is a characterization of convergence in the space $\mathcal{W}_0$, known as Thm. 11.5 in [44].

**Theorem 2.7.** Let $\{W_n\}$ be a sequence of graphons in $\mathcal{W}_0$ and let $W \in \mathcal{W}_0$. Then $t(F, W_n) \to t(F, W)$ for all finite simple graphs if and only if $\delta_{\square}(W_n, W) \to 0$.

## 3 | MAIN RESULTS FOR GENERAL WIGNER-TYPE MATRICES

### 3.1 | Set-up and main results

Let $A_n$ be a Hermitian random matrix whose entries above and on the diagonal of $A_n$ are independent. Assume a *general Wigner-type matrix* $A_n$ with a variance profile matrix $S_n$ satisfies the following conditions:

1. $Ea_{ij} = 0, E|a_{ij}|^2 = s_{ij}$.
2. (Lindeberg’s condition) for any constant $\eta > 0$,
   \[
   \lim_{n \to \infty} \frac{1}{n^2} \sum_{1 \leq i, j \leq n} E[|a_{ij}|^2 1(|a_{ij}| \geq \eta \sqrt{n})] = 0. \tag{3.1}
   \]
3. $\sup_{ij} s_{ij} \leq C$ for some constant $C \geq 0$.

**Remark 3.1.** If we assume entries of $A_n$ are of the form $a_{ij} = s_{ij}\xi_{ij}$ where the $\xi_{ij}$’s have mean $0$, variance $1$ and are i.i.d. up to symmetry, then the Lindeberg’s condition (3.1) holds by the dominated convergence theorem.

To begin with, we associate a graphon $W_n$ to the matrix $S_n$ in the following way. Consider $S_n$ as the adjacency matrix of a weighted graph $G_n$ on $[n]$ such that the weight of the edge $(i, j)$ is $s_{ij}$, then $W_n$ is defined as the corresponding graphon to $G_n$. We say $W_n$ is a *graphon representation of $S_n$*. We define $M_n := \frac{1}{\sqrt{n}}A_n$ and denote all rooted planar tree with $k + 1$ vertices as $T_j^{k+1}, 1 \leq j \leq C_k$. Now we are ready to state our main results for the limiting spectral distributions of general Wigner-type matrices.

**Theorem 3.2.** Let $A_n$ be a general Wigner-type matrix and $W_n$ be the corresponding graphon of $S_n$. The following holds:

1. If for any finite tree $T$, $t(T, W_n)$ converges as $n \to \infty$, the empirical spectral distribution of $M_n$ converges almost surely to a probability measure $\mu$ such that for $k \geq 0$,
   \[
   \int x^{2k} d\mu = \sum_{j=1}^{C_k} \lim_{n \to \infty} t(T_j^{k+1}, W_n), \quad \int x^{2k+1} d\mu = 0.
   \]
2. If $\delta_{\square}(W_n, W) \to 0$ for some graphon $W$ as $n \to \infty$, then for all $k \geq 0$,
   \[
   \int x^{2k} d\mu = \sum_{j=1}^{C_k} t(T_j^{k+1}, W), \quad \int x^{2k+1} d\mu = 0.
   \]
Remark 3.3. Similar moment formulas appear in the study of traffic distributions in free probability theory [46, 47].

Using the connection between the moments of the limiting spectral distribution and its Stieltjes transform described in (2.1), we can derive the equations for the Stieltjes transform of the limiting measure by the following theorem.

**Theorem 3.4.** Let $A_n$ be a general Wigner-type matrix and $W_n$ be the corresponding graphon of $S_n$. If $\delta(W_n, W) \to 0$ for some graphon $W$, then the empirical spectral distribution of $M_n := A_n / \sqrt{n}$ converges almost surely to a probability measure $\mu$ whose Stieltjes transform $s(z)$ is an analytic solution defined on $\mathbb{C}^+$ by the following equations:

$$ s(z) = \int_0^1 a(z, x) dx, \quad (3.2) $$

$$ a(z, x)^{-1} = z - \int_0^1 W(x, y) a(z, y) dy, \quad x \in [0, 1], \quad (3.3) $$

where $a(z, x)$ is the unique analytic solution of (3.3) defined on $\mathbb{C}^+ \times [0, 1]$.

Moreover, for $|z| > 2\|W\|_{\infty}^{1/2}$,

$$ a(z, x) = \sum_{k=0}^{\infty} \frac{\beta_{2k}(x)}{z^{2k+1}}, \quad \beta_{2k}(x) := \sum_{j=1}^{C_k} t_3(T_j^{k+1}, W), \quad (3.4) $$

where $t_3(T_j^{k+1}, W) := \int_{[0,1]^k} \prod_{uv \in E(T_j^{k+1})} W(x_u, x_v) \prod_{i=2}^{k+1} dx_i. \quad (3.5)$

Remark 3.5. In (3.5), $t_3(T_j^{k+1}, W)$ is a function of $x_1$, and in (3.4) $t_3(T_j^{k+1}, W)$ is the function evaluated at $x_1 = x$.

Theorem 3.4 holds under a stronger condition compared to Theorem 3.2. We provide two examples in Section 4 to show that it is possible to have tree densities converge but the empirical graphon does not converge under the cut metric. We show that the limiting spectral distribution can still exist. However, to have the Equations (3.2) and (3.3), we need a well-defined measurable function $W$ that $W_n$ converges to, therefore we need the condition of graphon convergence under the cut metric.

(3.2) and (3.3) have been known as quadratic vector equations in [2, 4], where the properties of the solution are discussed under more assumptions on variance profiles to prove local law and universality. A similar expansion as (3.4) and (3.5) has been derived in [30]. The central role of (3.3) in the context of random matrices has been recognized by many authors, see [37, 40, 50].

Wigner-type matrices is a special case for the Kronecker random matrices introduced in [8], and the global law has been proved in Theorem 2.7 of [8], which states the following: let $H_n$ be a Kronecker random matrix and $\mu_n^H$ be its empirical spectral distribution, then there exists a deterministic sequence of probability measure $\mu_n$ such that $\mu_n^H - \mu_n$ converges weakly to the zero measure as $n \to \infty$. In particular, for Wigner-type matrices, the global law holds under the assumptions of bounded variances and bounded moments. Our Theorems 3.2 and 3.4 give a moment method proof of the global law in [8] for Wigner-type matrices under bounded variances and Lindeberg’s condition. Our new contribution is a weaker condition for the convergence of the empirical spectral distribution $\mu_n^M$ of $M_n$. 
In Sections 3.2 and 3.3, we provide the proofs for Theorems 3.2 and 3.4, respectively. We briefly summarize the proof ideas here. In the proof of Theorem 3.2, we revisit the standard path-counting moment method proof for the semicircle law (see, e.g., [12]). Since our matrix model has a variance profile, we encode different variances as weights on the paths and represent the moments of the empirical measure as a sum of homomorphism densities. Then if the tree homomorphism densities converge, the limiting spectral distribution exists.

For the proof of Theorem 3.4, since we assume that the variance profile converges under the cut norm, we can obtain a limiting graphon \( W \). To obtain (3.3) we expand \( a(z, x) \) in (3.3) as a power series of homomorphism density from partially labeled trees to graphon \( W \) denoted by \( \beta_{2k}(x) \) in (3.4). Then we prove a graphon version of the Catalan number recursion formula for \( \beta_{2k}(x) \) in (3.11) and show that this essentially implies the quadratic vector equations (3.2) and (3.3). This recursion formula (3.11) for tree homomorphism densities to a graphon could be of independent interest.

### 3.2 | Proof of Theorem 3.2

Using the truncation argument as in [12, 26], we can first apply moment methods to a general Wigner-type matrix with bounded entries in the following lemma.

**Lemma 3.6.** Assume a Hermitian random matrix \( A_n \) with a variance profile \( S_n \) satisfies

1. \( \mathbb{E}[a_{ij}] = 0, \mathbb{E}[|a_{ij}|^2] = s_{ij} \). \( \{a_{ij}\}_{1 \leq i, j \leq n} \) are independent up to symmetry.
2. \( |a_{ij}| \leq \eta_n \sqrt{n} \) for some positive decreasing sequence \( \eta_n \) such that \( \eta_n \to 0 \).
3. \( \sup_{ij} s_{ij} \leq C \) for a constant \( C \geq 0 \).

Let \( W_n \) be the graphon representation of \( S_n \). Then for every fixed integer \( k \geq 0 \), we have the following asymptotic formulas:

\[
\frac{1}{n} \mathbb{E}[\text{tr} M_n^{2k}] = \sum_{j=1}^{C_k} t(T_j^{k+1}, W_n) + o(1), \tag{3.6}
\]

\[
\frac{1}{n} \mathbb{E}[\text{tr} M_n^{2k+1}] = o(1), \tag{3.7}
\]

where \( \{T_j^{k+1}, 1 \leq j \leq C_k\} \) are all rooted planar trees of \( k + 1 \) vertices.

**Proof.** We start with expanding the expected normalized trace. For any integer \( h \geq 0 \),

\[
\frac{1}{n} \mathbb{E}[\text{tr} M_n^h] = \frac{1}{n^{h/2+1}} \text{tr}(A_n^h) = \frac{1}{n^{h/2+1}} \sum_{1 \leq i_1, \ldots, i_h \leq n} \mathbb{E}[a_{i_1i_2}a_{i_2i_3} \cdots a_{i_hi_1}].
\]

Each term in the above sum corresponds to a closed walk (with possible self-loops) \( (i_1, i_2, \ldots, i_h) \) of length \( h \) in the complete graph \( K_n \) on vertices \( \{1, \ldots, n\} \). Any closed walk can be classified into one of the following three categories:

- \( C_1 \): All closed walks such that each edge appears exactly twice.
- \( C_2 \): All closed walks that have at least one edge which appears only once.
- \( C_3 \): All other closed walks.
By independence, it is easy to see that every term corresponding to a walk in $C_2$ is zero. We call a walk that is not in $C_2$ a good walk. Consider a good walk that uses $p$ different edges $e_1, \ldots, e_p$ with corresponding multiplicity $t_1, \ldots, t_p$, and each $t_i \geq 2$, such that $t_1 + \cdots + t_p = h$. Now the term corresponding to a good walk has the form

$$\mathbb{E}[a_{e_1}^{t_1} \cdots a_{e_p}^{t_p}].$$

Such a walk uses at most $p+1$ vertices and an upper bound for the number of good walks of this type is $np$. Since $|a_{ij}| \leq n\sqrt{n}$, and $\sup_{ij} \text{Var}(a_{ij}) = \sup_{ij} s_{ij} \leq C$, we have

$$\mathbb{E}[a_{e_1}^{t_1} \cdots a_{e_p}^{t_p}] \leq \mathbb{E}[a_{e_1}^{2}] \cdots \mathbb{E}[a_{e_p}^{2}] (n\sqrt{n})^{t_1+\cdots+t_p-2p} \leq n^{h-2p} n^{h/2-p} C^p.$$

When $h = 2k + 1$, we have

$$\frac{1}{n} \mathbb{E}[\text{tr} M_{2k+1}^n] = \frac{1}{n^{h/2+1}} \sum_{p=1}^{k} \sum \mathbb{E}[a_{e_1}^{t_1} \cdots a_{e_p}^{t_p}] \leq \frac{1}{n^{k+3/2}} \sum_{p=1}^{k} n^{p+1} p^h (n\sqrt{n})^{t_1+\cdots+t_p-2p} \leq n^{h-2p} n^{h/2-p} C^p$$

$$= \sum_{p=1}^{k} p^h n^{h-2p} C^p = O(n^h) = o(1).$$

When $h = 2k$, let $S_i$ denote the sum of all terms in $C_i$, $1 \leq i \leq 3$. By independence, we have $S_2 = 0$. Each walk in $C_3$ uses $p$ different edges with $p \leq k - 1$. We then have

$$S_3 = \frac{1}{n^{h/2+1}} \sum_{p=1}^{k-1} \sum \mathbb{E}[a_{e_1}^{t_1} \cdots a_{e_p}^{t_p}] \leq \frac{1}{n^{k+1}} \sum_{p=1}^{k-1} n^{p+1} p^h (n\sqrt{n})^{t_1+\cdots+t_p-2p} \left( \sup_{ij} s_{ij} \right)^p$$

$$= \sum_{p=1}^{k-1} p^h n^{h-2p} C^p = o(1).$$

Now it remains to compute $S_1$. For the closed walk that contains a self-loop, the number of distinct vertices is at most $k$, which implies the total contribution of such closed walks is $O(n^k)$, hence such terms are negligible in the limit of $S_1$. We only need to consider closed walks that use $k+1$ distinct vertices. Each closed walk in $C_1$ with $k+1$ distinct vertices in $\{1, \ldots, n\}$ is a closed walk on a tree of $k+1$ vertices that visits each edge twice. Given an unlabeled rooted planar tree $T$ and a depth-first search closed walk with vertices chosen from $[n]$, there is a one-to-one correspondence between such walk and a labeling of $T$ (see Figure 2). There are $C_k$ many rooted planar trees with $k+1$ vertices and for
each rooted planar tree $T_{j}^{k+1}$, the ordering of the vertices from 1 to $k+1$ is fixed by its depth-first search. Let $T_{l_{j}}^{k+1}$ be any labeled tree with the unlabeled rooted tree $T_{l_{j}}^{k+1}$ and a labeling $l = (l_{1}, \ldots, l_{k+1}), 1 \leq l_{i} \leq n, 1 \leq i \leq k+1$ for its vertices from 1 to $k+1$. For terms in $C_{1}$, any possible labeling $l$ must satisfy that $l_{1}, \ldots, l_{k+1}$ are distinct. Let $E(T_{l_{j}}^{k+1})$ be the edge set of $T_{l_{j}}^{k+1}$. Then $S_{1}$ can be written as

$$S_{1} = \frac{1}{n^{k+1}} \sum_{j=1}^{c_{l}} \sum_{l=1}^{l_{k+1}} \sum_{l_{k+1}}^{l_{k+1}} E_{e \in E(T_{l_{j}}^{k+1})} a_{e}^{2} = \frac{1}{n^{k+1}} \sum_{j=1}^{c_{l}} \sum_{l=1}^{l_{k+1}} \sum_{l_{k+1}}^{l_{k+1}} s_{e}.$$  

(3.8)

Consider

$$S_{1} = \frac{1}{n^{k+1}} \sum_{j=1}^{c_{l}} \sum_{l=1}^{l_{k+1}} \sum_{l_{k+1}}^{l_{k+1}} E_{e \in E(T_{l_{j}}^{k+1})} s_{e}.$$  

where $l$ now stands for every possible labeling which allows some of $l_{1}, \ldots, l_{k+1}$ to coincide, then we have

$$|S_{1} - S_{1}'| \leq \frac{1}{n^{k+1}} C_{k}(k + 1)n^{k}(\sup_{ij} s_{ij})^{k} = O\left(\frac{1}{n}\right).$$

On the other hand,

$$t(T_{j}^{k+1}, W_{n}) = \int_{[0,1]^{k+1}} \prod_{e \in E(T_{j}^{k+1})} W_{n}(x_{u}, x_{v})dx_{1} \cdots dx_{k+1}$$

$$= \frac{1}{n^{k+1}} \sum_{l_{k+1}}^{l_{k+1}} \sum_{l_{1}}^{l_{1}} \sum_{l_{k+1}}^{l_{k+1}} \prod_{e \in E(T_{l_{j}}^{k+1})} s_{e}.$$  

(3.9)

Note that $S_{1} = \sum_{j=1}^{c_{l}} t(T_{j}^{k+1}, W_{n})$. From (3.8) and (3.9), we get $S_{1} = \sum_{j=1}^{c_{l}} t(T_{j}^{k+1}, W_{n}) + o(1)$. Combining the estimates of $S_{1}, S_{2}$ and $S_{3}$, the conclusion of Lemma 3.6 follows.

Lemma 3.6 connects the moments of the trace of $M_{n}$ to homomorphism densities from trees to the graphon $W_{n}$. To proceed with the proof of Theorem 3.2, we need the following lemma.

Lemma 3.7. In order to prove the conclusion of Theorem 3.2, it suffices to prove it under the following conditions:

1. $\mathbb{E}a_{ij} = 0, \mathbb{E}|a_{ij}|^{2} = s_{ij}$ and $\{a_{ij}\}_{1 \leq i, j \leq n}$ are independent up to symmetry.
2. $|a_{ij}| \leq \eta_{n}\sqrt{n}$ for some positive decreasing sequence $\eta_{n}$ such that $\eta_{n} \to 0$.
3. $\sup_{ij} s_{ij} \leq C$, for some constant $C \geq 0$.

The proof of Lemma 3.7 follows verbatim as the proof of Thm. 2.9 in [12], so we do not give it here. The followings are two results that are used in the proof and will be used elsewhere in the article, so we give them here. See section A in [12] for further details.

Lemma 3.8 (Rank Inequality). Let $A_{n}, B_{n}$ be two $n \times n$ Hermitian matrices. Let $F_{A_{n}}, F_{B_{n}}$ be the empirical spectral distributions of $A_{n}$ and $B_{n}$, then

$$\|F_{A_{n}} - F_{B_{n}}\| \leq \frac{\text{rank}(A_{n} - B_{n})}{n},$$

where $\| \cdot \|$ is the $L^{\infty}$-norm.
Lemma 3.9 (Lévy Distance Bound). Let $L$ be the Lévy distance between two distribution functions, we have for any $n \times n$ Hermitian matrices $A_n$ and $B_n$,

$$L^3(F^{A_n}, F^{B_n}) \leq \frac{1}{n} \text{tr}[(A_n - B_n)(A_n - B_n)^*].$$

With Lemma 3.7, we will prove Theorem 3.2 under assumptions in Lemma 3.7.

Proof of Theorem 3.2. By Lemma 3.7, it suffices to prove Theorem 3.2 under the conditions (1)-(3) in Lemma 3.7. We now assume these conditions hold. Then (3.6) and (3.7) in Lemma 3.6 can be applied here.

(1) Since for any finite tree $T$, $t(T, W_n)$ converges as $n \to \infty$, we can define

$$\beta_{2k} := \lim_{n \to \infty} \frac{1}{n} \text{tr}[M_n^{2k}] = \lim_{n \to \infty} \sum_{j=1}^{C_k} t(T_{j+1}^{2k}, W_n), \quad \beta_{2k+1} := \lim_{n \to \infty} \frac{1}{n} \text{tr}[M_n^{2k+1}] = 0.$$ 

With Carleman’s lemma (Lems. B.1 and B.3 in [12]), in order to show the limiting spectral distribution of $M_n$ is uniquely determined by the moments, it suffices to show that for each integer $k \geq 0$, almost surely we have

$$\lim_{n \to \infty} \frac{1}{n} \text{tr}M_n^k = \beta_k, \quad \text{and} \quad \liminf_{k \to \infty} \frac{1}{k^{1/2k}} \beta_{2k} < \infty.$$ 

The remaining of the proof is similar to proof of Theorem 2.9 in [12], and we include it here for completeness. Let $G(i)$ be the graph induced by the closed walk $i = (i_1, \ldots, i_k)$. Define $A(G(i)) := a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$. Then

$$\mathbb{E} \left| \frac{1}{n} \text{tr}M_n^k - \frac{1}{n} \mathbb{E}[\text{tr}M_n^k] \right|^4 = \frac{1}{n^{k+2k}} \sum_{i_1, i_2 \leq 4} \mathbb{E} \prod_{j=1}^4 [A(G(i_j)) - \mathbb{E}A(G(i_j))]$$

Consider a quadruple closed walk $i_j, 1 \leq j \leq 4$. By independence, for the nonzero term, the graph $\cup_{j=1}^4 G(i_j)$ has at most two connected components. Assume there are $q$ edges in $\cup_{j=1}^4 G(i_j)$ with multiplicity $v_1, \ldots, v_q$, then $v_1 + \cdots + v_q = 4k$. The number of vertices in $\cup_{j=1}^4 G(i_j)$ is at most $q + 2$. To make every term in the expansion of $\mathbb{E} \prod_{j=1}^4 (A(G(i_j)) - \mathbb{E}A(G(i_j)))$ nonzero, the multiplicity of each edge is at least 2, so $q \leq 2k$ and the corresponding term satisfies

$$\mathbb{E} \prod_{j=1}^4 [A(G(i_j)) - \mathbb{E}A(G(i_j))] \leq C^q(n \sqrt{n})^{4k-2q}. \quad (3.10)$$

If $q = 2k$, we have $v_1 = \cdots = v_q = 2$. Since the graph $\cup_{j=1}^4 G(i_j)$ has at most two connected components with at most $2k + 1$ vertices, there must be a cycle in $\cup_{j=1}^4 G(i_j)$. So the number of such graphs is at most $n^{2k+1}$. Therefore from (3.10),

$$\mathbb{E} \left| \frac{1}{n} \text{tr}M_n^k - \frac{1}{n} \mathbb{E}[\text{tr}M_n^k] \right|^4 \leq \frac{1}{n^{4+2k}} \sum_{i_1, i_2 \leq 4} \mathbb{E} \prod_{j=1}^4 [A(G(i_j)) - \mathbb{E}A(G(i_j))]$$

$$\leq \frac{1}{n^{4+2k}} \left( C^2 k n^{2k+1} + \sum_{q < 2k} C^q n^{|q|+2} (n \sqrt{n})^{4k-2q} \right) = o \left( \frac{1}{n^2} \right).$$
Then by Borel-Cantelli lemma,

$$\lim_{n \to \infty} \frac{1}{n} \text{tr} M^k_n = \beta_k \ a.s.$$ 

Moreover, since we have

$$\beta_{2k} = \lim_{n \to \infty} \sum_{j=1}^{C_k} t(T^{k+1}_j, W_n) \leq C_k C^k,$$

which implies \( \liminf_{k \to \infty} \frac{1}{k} \beta_{2k}^{1/2k} = 0. \)

(2) Since \( \delta_{\overline{G}}(W_n, W) \to 0, \) by Theorem 2.7, we have

$$\lim_{n \to \infty} t(T^{k+1}_j, W_n) = t(T^{k+1}_j, W)$$

for any rooted planar tree \( T^{k+1}_j \) with \( k \geq 1, 1 \leq j \leq C_k. \) Therefore for all \( k \geq 0, \)

$$\lim_{n \to \infty} \frac{1}{n} \text{tr} M^{2k}_n = \sum_{j=1}^{C_k} t(T^{k+1}_j, W), \quad \lim_{n \to \infty} \frac{1}{n} \text{tr} M^{2k+1}_n = 0 \ a.s.$$ 

This completes the proof. ■

### 3.3 Proof of Theorem 3.4

**Proof.** Since

$$\limsup_{k \to \infty} (\beta_{2k}(x))^{1/(2k+1)} \leq 2 \|W\|^{1/2}_\infty$$

for all \( x \in [0, 1], \) we have for \( |z| > 2 \|W\|^{1/2}_\infty, \) \( a(z, x) = \sum_{k=0}^{\infty} \beta_{2k}(x) \) converges. Note that

$$\int_0^1 \beta_{2k}(x)dx = \sum_{j=1}^{C_k} \int_0^1 t_s(T^{k+1}_j, W)dx = \sum_{j=1}^{C_k} t(T^{k+1}_j, W) = \beta_{2k},$$

which implies for \( |z| > 2 \|W\|^{1/2}_\infty, \) \( s(z) = \sum_{k=0}^{\infty} \frac{\beta_{2k}}{z^{2k+1}} = \int_0^1 a(z, x)dx. \)

Next we show (3.3) holds for \( |z| > 2 \|W\|^{1/2}_\infty, \) which is equivalent to show

$$a(z, x) \int_0^1 W(x, y)a(z, y)dy = za(z, x) - 1, \quad \forall x \in [0, 1]. \quad (3.11)$$

We order the vertices in each rooted planar tree \( T^{k+1}_j \) from 1 to \( k + 1 \) by depth-first search order (the root for each \( T^{k+1}_j \) is always denoted by 1). Define a function

$$f_{j,k}(x_1, x_2, \ldots, x_{k+1}) = \prod_{uv \in E(T^{k+1}_j)} W(x_u, x_v).$$
Now we expand $a(z, x)$ as follows:

$$a(z, x) = \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \sum_{j=1}^{C_k} t_k(T_j^{k+1}, W) = \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \sum_{j=1}^{C_k} \int_{[0,1]^k} f_{j,k}(x, x_2, \ldots, x_{k+1}) \prod_{i=2}^{k+1} dx_i.$$

Then we can write $\int_0^1 W(x, y)a(z, y)dy$ as

$$\sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \sum_{j=1}^{C_k} \int_{[0,1]^k} W(x, y)f_{j,k}(y, x_2, \ldots, x_{k+1})dy \prod_{i=2}^{k+1} dx_i. \quad (3.12)$$

Denote

$$B_{j,k}(x) := \int_{[0,1]^k} W(x, y)f_{j,k}(y, x_2, \ldots, x_{k+1})dy \prod_{i=2}^{k+1} dx_i.$$

Let $T_j^{k+1*}$ be the rooted planar tree $T_j^{k+1}$ with a new edge attached to the root and the new vertex ordered $k + 2$ (see Figure 3). Let $t_k(T_j^{k+1*}, W)$ be the homomorphism density from partially labeled graph $T_j^{k+1*}$ to $W$ with the new vertex labeled $x$.

With this notation, $B_{j,k}(x_{k+2})$ can be written as

$$\int_{[0,1]^k} W(x_{k+2}, x_1)f_{j,k}(x_1, x_2, \ldots, x_{k+1}) \prod_{i=1}^{k+1} dx_i = \prod_{i \in E(T_j^{k+1*})} W(x_{i}, x_i) \prod_{i=1}^{k+1} dx_i = t_{x_{k+2}}(T_j^{k+1*}, W). \quad (3.13)$$

So (3.12) and (3.13) implies $\int_0^1 W(x, y)a(z, y)dy = \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \sum_{j=1}^{C_k} t_k(T_j^{k+1*}, W)$.

Therefore

$$a(z, x)\int_0^1 W(x, y)a(z, y)dy = \left( \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \sum_{i=1}^{C_k} t_i(T_i^{k+1*}, W) \right) \left( \sum_{l=0}^{\infty} \frac{1}{z^{2l+1}} \sum_{j=1}^{C_l} t_j(T_j^{l+1*}, W) \right)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{z^{2(k+l)+2}} \sum_{i=1}^{C_k} \sum_{j=1}^{C_l} t_i(T_i^{k+1*}, W)t_j(T_j^{l+1*}, W). \quad (3.14)$$

Let $\{T_i^{k+l+2}, 1 \leq i \leq C_k, 1 \leq j \leq C_l \}$ be all rooted planar trees with $k + l + 2$ vertices generated by combining $T_i^{k+1}$ and $T_j^{l+1*}$ in the following way.
FIGURE 4 Combining $T_j^{k+1}$ with $T_j^{l+1*}$ yields a new rooted planar tree of $k + l + 2$ vertices [Color figure can be viewed at wileyonlinelibrary.com]

(1) First of all, by attaching the new labeled vertex of $T_l^{k+1*}$ to the root of $T_k^{l+1}$, we get a new tree $T$ of $k + l + 2$ vertices.

(2) Choose the root of $T$ to be the root of $T_k^{l+1}$. Order all vertices coming from $T_k^{l+1}$ with $1, 2, \ldots, k + 1$ and order vertices coming from $T_l^{k+1*}$ with $k + 2, k + 3, \ldots, k + l + 2$ both in depth-first search order. Then $T$ becomes a rooted planar tree $T_{i,j}^{k+l+2}$ of $k + l + 2$ vertices (see Figure 4).

Let $t_x(T_{i,j}^{k+l+2}, W)$ be the homomorphism density from partially labeled tree $T_{i,j}^{k+l+2}$ to $W$ with the root labeled $x$. Using our notation, we have

$$t_x(T_{i,j}^{k+1}, W)t_x(T_{j}^{l+1*}, W) = t_x(T_{i,j}^{k+l+2}, W).$$

Now let $s = k + l + 1$, then (3.14) can be written as

$$\sum_{s=1}^{\infty} \frac{1}{z^{2s}} \sum_{k,l \geq 0} C_k C_l \sum_{i=1}^{s} \sum_{j=1}^{s} t_x(T_{i,j}^{s+1}, W). \quad (3.15)$$

Since all rooted planar trees in the set $\{T_{i,j}^{s+1} \mid 1 \leq i \leq C_l, 1 \leq j \leq C_k\}$ are different, from the Catalan number recurrence, there are

$$\sum_{k+l \leq s-1} C_k C_l = \sum_{k=0}^{s-1} C_k C_{s-1-k} = C_s$$

many, which implies $\{T_{i,j}^{s+1} \mid 1 \leq i \leq C_l, 1 \leq j \leq C_k\}$ are all rooted planar trees of $s + 1$ vertices. Now (3.15) can be written as

$$\sum_{s=1}^{\infty} \frac{1}{z^{2s}} \sum_{i=1}^{s} t_x(T_{i}^{s+1}, W) = z a(z, x) - 1.$$ 

Therefore (3.11) holds for $|z| > 2\|W\|_{\infty}^{1/2}$. Since (3.11) has a unique analytic solution on $\mathbb{C}^+$ (see Thm. 2.1 in [2]), by analytic continuation, $a(z, x)$ has a unique extension on $\mathbb{C}^+ \times [0, 1]$ such that (3.11) holds for all $z \in \mathbb{C}^+$. This completes the proof.

4 | GENERALIZED WIGNER MATRICES

The semicircle law for generalized Wigner matrices whose variance profile is doubly stochastic and comes from discretizing a function with zero-measure discontinuities was proved in [10,49]. The local
semicircle law and universality of generalized Wigner matrices have been studied in [32, 33] with a lower bound on the variance profile and conditions on the distributions of entries. With Theorem 3.2, we can have a quick proof of the semicircle law for generalized Wigner matrices under Lindeberg’s condition. Compared to [10, 49], where the $L^\infty$-convergence of the variance profile is assumed, we do not even need to assume the variance profile converges under the cut metric. We will only need the weaker condition: the convergence of $t(T, W_n)$ for any finite tree $T$. In this section, we will show that the condition in Theorem 3.2, the convergence of tree integrals, is indeed a weaker condition than the convergence of the variance profile under the cut metric. Below we provide two examples where assumptions in [10, 50] fail, but our Theorem 3.2 holds.

We make the following assumptions for our generalized Wigner matrices. Let $A_n$ be a random Hermitian matrix such that entries are independent up to symmetry, and satisfies the following conditions:

1. $E[a_{ij}] = 0$, $E[|a_{ij}|^2] = s_{ij}$.
2. $\frac{1}{n} \sum_{j=1}^{n} s_{ij} = 1 + o(1)$ for all $1 \leq i \leq n$.
3. For any constant $\eta > 0$, $\lim_{n \to \infty} \frac{1}{n} \sum_{1 \leq i,j \leq n} E[|a_{ij}|^2 1(|a_{ij}| \geq \eta \sqrt{n})] = 0$.
4. $\sup_{ij} s_{ij} \leq C$ for a constant $C > 0$.

We use our general formula in Theorem 3.2 to get the semicircle law. An important observation is, when the variance profile is almost stochastic, the homomorphism densities in Theorem 3.2 are easy to compute, as shown in the following lemma. The main idea is that we can start computing the homomorphism density integral from leaves on the tree.

**Lemma 4.1.** Let $\{W_n\}_{n \geq 1}$ be any sequence of graphons such that $0 \leq W_n(x, y) \leq C$ almost everywhere for some constant $C > 0$. If for $x \in [0, 1]$ almost everywhere,

$$\lim_{n \to \infty} \int_{0}^{1} W_n(x, y) dy = 1,$$

then $\lim_{n \to \infty} t(T, W_n) = 1$ for any finite tree $T$.

**Proof.** We induct on the number of vertices of a tree. Let $k = |V|$. For $k = 2$, by dominated convergence theorem,

$$\lim_{n \to \infty} t(T, W_n) = \int_{0}^{1} W_n(x, y) dx dy = 1. \quad (4.1)$$

Assume for any trees with $k - 1$ vertices the statement holds. For any tree $T$ with $k$ vertices, we order the vertices in $T$ by depth-first search. Then the vertex with label $k$ is a leaf. Note that

$$t(T, W_n) = \int_{[0,1]^k} \prod_{j \in E} W_n(x_i, x_j) dx_1 \ldots dx_k$$

$$= \int_{[0,1]^k} W_n(x_{k-1}, x_k) \prod_{j \in E \setminus \{k-1,k\}} W_n(x_i, x_j) dx_1 \ldots dx_k$$

$$= \int_{[0,1]^{k-1}} \left( \int_{[0,1]} W_n(x_{k-1}, x_k) dx_k \right) \prod_{j \in E \setminus \{k-1,k\}} W_n(x_i, x_j) dx_1 \ldots dx_{k-1}$$
Let $T'$ be the tree $T$ with the edge $\{k - 1, k\}$ removed, then we have

$$t(T', W_n) = \int_{[0,1]^{k-1}} \prod_{ij \in E \setminus \{k-1,k\}} W_n(x_i, x_j) dx_1 \ldots dx_{k-1},$$

$$t(T, W_n) - t(T', W_n) = \int_{[0,1]^{k-1}} \left( \int_{[0,1]} W_n(x_{k-1}, x_k) dx_k - 1 \right) \prod_{ij \in E \setminus \{k-1,k\}} W_n(x_i, x_j) dx_1 \ldots dx_{k-1}.$$

By dominated convergence theorem and (4.1) we obtain

$$\lim_{n \to \infty} |t(T, W_n) - t(T', W_n)| = 0.$$

Moreover, by our assumption of the induction, $\lim_{n \to \infty} t(T', W_n) = 1$, therefore $\lim_{n \to \infty} t(T, W_n) = 1$. This completes the proof.

Now we can give a quick proof of the semicircle law for generalized Wigner matrices in the following theorem, which is a quick consequence of Lemma 4.1 and Theorem 3.2.

**Theorem 4.2.** Let $A_n$ be a generalized Wigner matrix with assumptions above. The limiting spectral distribution of $M_n := \frac{A_n}{\sqrt{n}}$ converges weakly almost surely to the semicircle law.

**Proof.** Let $W_n$ be the graphon representation of the variance profile for $A_n$. From condition (2), we have

$$\lim_{n \to \infty} \int_{[0,1]} W_n(x, y) dy = 1$$

for $x \in [0,1]$ almost everywhere. Then by Lemma 4.1, $\lim_{n \to \infty} t(T, W_n) = 1$ for any finite tree $T$.

By part (1) in Theorem 3.2, the empirical spectral distribution of $M_n$ converges almost surely to a probability measure $\nu$ such that for all $k \geq 0$.

$$\int x^{2k} d\nu = C_k, \quad \int x^{2k+1} d\nu = 0.$$  \hspace{1cm} (4.2)

It is known that the semicircle law is uniquely determined by its moments, therefore the limiting spectral distribution for $M_n$ is the semicircle law.

Theorem 4.2 can be applied to study the spectrum of inhomogeneous random graphs with roughly equal expected degrees. This is a sparse random graph model where no limiting variance profile is assumed, so the theorems in [10, 50] do not apply here. Consider the inhomogeneous Erdős-Rényi model $G(n, (p_{ij}))$ with adjacency matrix $A_n$, where edges exist independently with given probabilities $p_{ij}$ such that $p_{ij} = p_{ji}$. Assume

$$\sum_{i=1}^{n} p_{ij} = (1 + o(1))n\alpha \quad \text{for all } j \in [n]$$  \hspace{1cm} (4.3)

with some $\alpha \to 0$, $\alpha = o\left(\frac{1}{n}\right)$, and

$$\max_{ij} p_{ij} \leq C\alpha \quad \text{for some constant } C \geq 1.$$  \hspace{1cm} (4.4)
Corollary 4.3. Under the assumptions (4.3) and (4.4), the empirical spectral distribution of the scaled adjacency matrix $\frac{A_n}{\sqrt{na}}$ converges almost surely to the semicircle law.

Proof. Consider the matrix $M_n = \frac{A_n - E A_n}{\sqrt{na}}$. Then by (4.3) and (4.4), one can check that $M_n$ satisfies the assumptions (1)-(4) above for the generalized Wigner matrices. By Theorem 4.2, the empirical spectral distribution of $\frac{A_n - E A_n}{\sqrt{na}}$ converges to the semicircle law almost surely. By Lemma 3.9, we have almost surely

$$L^3 \left( \frac{A_n}{\sqrt{na}}, \frac{A_n - E A_n}{\sqrt{na}} \right) \leq \frac{1}{n} \text{tr} \left[ \left( \frac{E A_n}{\sqrt{na}} \right)^2 \right] = \frac{1}{n^2 \alpha} \sum_{i,j=1}^{n} (E a_{ij})^2$$

$$= \frac{\sum_{i,j=1}^{n} p_{ij}^2}{n^2 \alpha} \leq \frac{n^2 C^2 \alpha^2}{n^2 \alpha} = C^2 \alpha = o(1),$$

where the last line of inequalities are from (4.4). Then $\frac{A_n}{\sqrt{na}}$ and $\frac{A_n - E A_n}{\sqrt{na}}$ have the same limiting spectral distribution almost surely. This completes the proof.

5 | SPARSE W-RANDOM GRAPHS

Given a graphon $W : [0, 1]^2 \to [0, 1]$, following the definitions in [18], one can generate a sequence of sparse random graphs $G_n$ in the following way. We choose a sparsity parameter $\rho_n$ such that

$$\sup_n \rho_n < 1 \text{ with } \rho_n \to 0 \text{ and } n \rho_n \to \infty.$$

Let $x_1, \ldots, x_n$ be i.i.d. chosen uniformly from $[0, 1]$. For a graph $G_n$, $i$ and $j$ are connected with probability $\rho_n W(x_i, x_j)$ independently for all $i \neq j$. We define $G_n$ to be a sparse W-random graph, and the sequence $\{G_n\}$ is denoted by $G(n, W, \rho_n)$. Note that we use the same i.i.d. sequence $x_1, \ldots, x_n$ when constructing $G_n$ for different values of $n$ without resampling the $x_i$'s. We determine the limiting spectral distributions for the adjacency matrices of sparse W-random graphs in the following theorem. This is a novel application of our theorem that cannot be covered by any previous results, since $W$ can be any bounded measurable function.

Theorem 5.1. Let $G(n, W, \rho_n)$ be a sequence of sparse W-random graphs with adjacency matrices $\{A_n\}_{n \geq 1}$. The limiting spectral distribution of $\frac{A_n}{\sqrt{np_n}}$ converges almost surely to a probability measure $\mu$ such that

$$\int_{\mathbb{R}} x^{2k} d\mu = \sum_{j=1}^{G_i} \text{tr}(T_j^{k+1}, W), \quad \int_{\mathbb{R}} x^{2k+1} d\mu = 0.$$

Moreover, its Stieltjes transform $s(z)$ satisfies the following equation:

$$s(z) = \int_{0}^{1} a(z, x) dx, \quad a(z, x) = z - \int_{0}^{1} W(x, y) d\mu,$$

$$\forall x \in [0, 1].$$

Proof. Let

$$B_n := \frac{A_n - \mathbb{E}[A_n|x_1, \ldots, x_n]}{\sqrt{\rho_n}} = (b_{ij})_{1 \leq i, j \leq n}.$$
Note that $B_n$ is now a function of $x_1, \ldots, x_n$. Since $n\eta_n \to \infty$ and $|b_{ij}| \leq \frac{2}{\sqrt{n}}$, we have that for any constant $\eta > 0$,

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbb{E} \left[ |b_{ij}|^2 1(|b_{ij}| \geq \eta \sqrt{n}) | x_1, \ldots, x_n \right] = 0,$$

then the Lindeberg’s condition (8.1) holds for $B_n$. Let $S_n$ be the variance profile matrix of $B_n$. Then we have $s_{ii} = 0, 1 \leq i \leq n$ and for all $i \neq j$,

$$s_{ij} = \frac{\rho_n W(x_i, x_j)(1 - \rho_n W(x_i, x_j))}{\rho_n} = W(x_i, x_j) + o(1).$$

Let $W_n$ be the graphon representation of the matrix $S_n$ and let $\tilde{W}_n$ be the graphon of a weighted complete graph on $[n]$ with edge weights $W(x_i, x_j)$ for each edge $ij$. It implies that

$$W_n(x, y) = \tilde{W}_n(x, y) + o(1), \quad \forall (x, y) \in [0, 1]^2.$$

By dominated convergence theorem, we get $\lim_{n \to \infty} \delta_{\square}(\tilde{W}_n, W_n) = 0$. From Theorem 4.5 (a) in [20], we have $\lim_{n \to \infty} \delta_{\square}(\tilde{W}_n, W) = 0$ almost surely, which implies $\lim_{n \to \infty} \delta_{\square}(W_n, W) = 0$ almost surely. Therefore from Theorem 3.2 (2), the limiting spectral distribution of $\frac{B_n}{\sqrt{n}}$ exists almost surely and its moments and Stieltjes transform are given by Theorems 3.2 and 3.4. Next we show $\frac{B_n}{\sqrt{n}}$ and $\frac{A_n}{\sqrt{\rho_n}}$ have the same limiting spectral distribution.

By Lemma 3.9, we have almost surely

$$L^3(F_{\frac{\rho_n}{\sqrt{n}}}, F_{\frac{n_n}{\sqrt{n}}}) \leq \frac{1}{n} \text{tr} \left[ \left( \frac{A_n}{\sqrt{n\rho_n}} - \frac{B_n}{\sqrt{n}} \right)^2 \right] = \frac{1}{n^2\rho_n} \text{tr} (\mathbb{E}[A_n | x_1, \ldots, x_n])^2. \quad (5.1)$$

By the way we generate our $W$-random graphs, we have for all $i \neq j$,

$$\mathbb{E}[(A_n)_{ij} | x_1, \ldots, x_n] = \rho_n W(x_i, x_j).$$

Therefore the right hand side in (5.1) is almost surely bounded by

$$\frac{\rho_n}{n^2} \sum_{i \neq j} W^2(x_i, x_j) \leq \rho_n = o(1),$$

which implies $\lim_{n \to \infty} L^3(F_{\frac{\rho_n}{\sqrt{n}}}, F_{\frac{n_n}{\sqrt{n}}}) = 0$ almost surely. This completes the proof.

## 6 RANDOM BLOCK MATRICES

Consider an $n \times n$ random Hermitian matrix $A_n$ composed of $d^2$ many rectangular blocks as follows. We can write $A_n$ as $A_n := \sum_{k,l=1}^{d} E_{kl} \otimes A^{(k,l)}$, where $\otimes$ denotes the Kronecker product of matrices, $E_{kl}$ are the elementary $d \times d$ matrices having 1 at entry $(k, l)$ and 0 otherwise. The blocks $A^{(k,l)}$, $1 \leq k \leq l \leq d$ are of size $n_k \times n_l$ and consist of independent entries subject to symmetry. To summarize, we consider a random block matrix $A_n$ with the following assumptions:
We partition the \( n^2 \) in (2).

From the definition, we have
\[
\text{Proof.}
\]
\( \alpha_k \) follow from Theorem 3.2. The existence and uniqueness of \( \mu \) surely to a probability measure and its Stieltjes transform \( s \).

Let \( \text{An} \) be a random block matrix satisfying the assumptions above. Let \( M_n \) to address this model.

Note that \( \text{An} \) profile for everywhere. Hence
\[
\text{Wn} \quad \text{is a step function defined on} \quad [0, 1]^2.
\]
Below is a version of Theorem 3.2, written specifically to address this model.

**Theorem 6.1.** Let \( \text{An} \) be a random block matrix satisfying the assumptions above. Let \( M_n = \frac{\text{An}}{\sqrt{n}} \) and \( \text{Wn} \) be the graphon defined in (6.1). Then the limiting spectral distribution of \( M_n \) converges almost surely to a probability measure \( \mu \) such that
\[
\int_{\mathbb{R}} x^{2k} d\mu(x) = \sum_{j=1}^{C} t(T_j^{k+1}, W), \quad \int_{\mathbb{R}} x^{2k+1} d\mu(x) = 0, 
\]
and its Stieltjes transform \( s(z) \) satisfies \( s(z) = \sum_{k=1}^{d} \alpha_k \alpha_k(z) \), where for all \( 1 \leq k \leq d \)
\[
a_k(z) = z - \sum_{i=1}^{d} \alpha_i s_{ik} \alpha_i(z).
\]

**Proof.** From the definition, we have \( \text{Wn}(x, y) \to \text{W(x, y)} \) as \( n \to \infty \) for \( (x, y) \in [0, 1]^2 \) almost everywhere. Hence
\[
\| W_n - W \| = \sup_{S, T} \int_{S \times T} | W_n(x, y) - W(x, y) | dx dy
\]
\[
\leq \int_{[0, 1]^2} | W_n(x, y) - W(x, y) | dx dy.
\]
Since \( | W_n(x, y) | \leq C \), by the dominated convergence theorem, we have \( \| W_n - W \| \to 0 \) as \( n \to \infty \).
(6.2) follow from Theorem 3.2. The existence and uniqueness of \( a_k(z) \), \( 1 \leq k \leq d \) follows from Thm.
2.1 in [2].

Now we consider the case where the number of blocks \( d \) depends on \( n \) such that \( d \to \infty \) as \( n \to \infty \).

We partition the \( n \) vertices into \( d \) classes: \( [n] = V_1 \cup V_2 \cup \cdots \cup V_d \). Let \( m_0 = 0, m_i = \sum_{j=1}^{i} n_j \) and
\[
V_i = \{ m_{i-1} + 1, m_{i-1} + 2, \ldots, m_i \}
\]
for \( i = 1, \ldots, d \). We say the class \( V_i \) is small if \( \frac{n_i}{n} \to \alpha_i = 0 \), and \( V_i \) is big if \( \frac{n_i}{n} \to \alpha_i > 0 \).
The adjacency matrix $\mathbf{A}_n$ of an SBM with a growing number of classes is a random block matrix. A new issue here is $\mathbb{E}\mathbf{A}_n \neq 0$, which does not fit our assumptions in Section 6. However some perturbation...
analysis of the empirical measures can be applied to address this issue. In this section, we consider the adjacency matrix $A_n$ for both sparse and dense SBMs with the following assumptions:

1. $\frac{\alpha_k}{n} \rightarrow \alpha_k \in [0, \infty), 1 \leq k \leq d$, where $d$ depends on $n$.
2. Diagonal elements in $A_n$ are 0. Entries in the block $V_i \times V_i$ are independent Bernoulli random variables with parameter $p_{ii}$ depending on $n$ up to symmetry. Entries in the block $V_k \times V_i, k \neq l$ are independent Bernoulli random variables with parameter $p_{kl}$ depending on $n$.
3. Let $p = \sup_i p_{ii}$. Assume $p = o(\frac{1}{n})$ and $\sup_n p < 1$.
4. Denote $\sigma^2 := p(1 - p_{ii})$, and assume
   \[
   \lim_{n \rightarrow \infty} \frac{p_{ii}(1 - p_{ii})}{\sigma^2} = s_{ij} \in [0, 1] \text{ for some constant } s_{ij}.
   \]

If $p \rightarrow 0$ (the sparse case), by the same argument in (4.5), $\frac{A_n - \mathbb{E}A_n}{\sigma \sqrt{n}}$ and $\frac{A_n}{\sigma \sqrt{n}}$ have the same limiting spectral distribution, we then have the following corollary from Theorem 6.2.

**Corollary 7.1.** Let $A_n$ be the adjacency matrix of a sparse SBM with $p \rightarrow 0$, $d \rightarrow \infty$ as $n \rightarrow \infty$. The empirical spectral distribution of $\frac{A_n}{\sqrt{n}}$ converges almost surely to a probability measure $\mu$ if one of the extra conditions below holds:

1. $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, or
2. $\sum_{i=1}^{\infty} \alpha_i = \alpha < 1, \alpha_1 \geq \alpha_2 \geq \cdots \geq 0$; also, for any two small classes $V_k, V_l, k \neq l, s_{kl} = s_0$ for some constant $s_0$. For any large class $V_k$ and small class $V_l$, $s_{kl} = s_{l0}$ for some constant $s_{l0}$.

If $p \not\rightarrow 0$ (the dense case), to get the limiting spectral distribution of the noncentered matrix $A_n$, we need to consider the effect of $\mathbb{E}A_n$. If $\mathbb{E}A_n$ is of relatively low rank, we can still do a perturbation analysis from Lemma 3.8. The following theorem is a statement for the dense case.

**Corollary 7.2.** The empirical spectral distribution of the adjacent matrix $\frac{A_n}{\sigma \sqrt{n}}$ for a SBM with $p > c$ for a constant $c > 0$ converges almost surely if $d = o(n)$ and one of the following holds:

1. $\sum_{i=1}^{\infty} \alpha_i = 1, \alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, or
2. $\sum_{i=1}^{\infty} \alpha_i = \alpha < 1, \alpha_1 \geq \alpha_2 \geq \cdots \geq 0$. For any two small classes $V_k, V_l, k \neq l, s_{kl} = s_0$ for some constant $s_0$. For any large class $V_k$ and small class $V_l$, $s_{kl} = s_{l0}$ for some constant $s_{l0}$.

**Proof.** Let $\tilde{A}_n$ be a random block matrix such that $\tilde{a}_{ij} = a_{ij}$ for $i \neq j$ and $\{\tilde{a}_{ii}\}_{i \leq i \leq n}$ be independent Bernoulli random variables with parameter $p_{kk}$ if $i \in V_k$. Then $\text{rank}(\mathbb{E}\tilde{A}_n) = d$.

Let $L\left(F_{\frac{A_n}{\sigma \sqrt{n}}}, F_{\frac{\tilde{A}_n}{\sigma \sqrt{n}}}ight)$ be the Lévy distance between the empirical spectral measures of $\frac{A_n}{\sigma \sqrt{n}}$ and $\frac{\tilde{A}_n}{\sigma \sqrt{n}}$, then by Lemma 3.9,

\[
L^3\left(F_{\frac{A_n}{\sigma \sqrt{n}}}, F_{\frac{\tilde{A}_n}{\sigma \sqrt{n}}}ight) \leq \frac{1}{\sigma^2 n^2} \text{tr} \left(\tilde{A}_n - A_n\right)^2 = \frac{1}{\sigma^2 n^2} \sum_{i=1}^{n} \tilde{a}_{ii}^2.
\]  

(7.1)

The right hand side of (7.1) is bounded by $\frac{1}{n \sigma^2} = o(1)$ almost surely. So we have almost surely

\[
\lim_{n \rightarrow \infty} L^3\left(F_{\frac{A_n}{\sigma \sqrt{n}}}, F_{\frac{\tilde{A}_n}{\sigma \sqrt{n}}}ight) = 0.
\]  

(7.2)
Recall that the limiting distribution of $\frac{\tilde{\lambda}_n - \tilde{E}\tilde{\lambda}_n}{\sigma\sqrt{n}}$ exists from Theorem 6.2 for random block matrices. By the rank inequality (Lemma 3.8), we have almost surely

$$\|F_{\tilde{\lambda}_n - \tilde{E}\tilde{\lambda}_n} - F_{\tilde{\lambda}_n} - 1\| \leq \frac{\text{rank}(\tilde{\lambda}_n - \tilde{E}\tilde{\lambda}_n - \tilde{\lambda}_n)}{\sqrt{n}} = \frac{\text{rank}(\tilde{E}\tilde{\lambda}_n)}{n} = \frac{d}{n} = o(1).$$  

Then combining (7.2) and (7.3), almost surely $\tilde{\lambda}_n - E\tilde{\lambda}_n$ has the same limiting spectral distribution as $\tilde{\lambda}_n - E\tilde{\lambda}_n$. The conclusion then follows.

Below, we give an example showing how to construct dense SBMs with a growing number of blocks which satisfies one of the assumptions in Corollary 7.2. Below is a lemma to justify that our two examples work.

**Lemma 7.3.** Assume $\sum_{i=1}^{\infty} \alpha_i = \alpha \leq 1$ and $1 \geq \alpha_1 \geq \alpha_2 \geq \cdots > 0$. Let $k(n) := \sup \{ k : \alpha_k \geq \frac{1}{n} \}$, then $\frac{k(n)}{n} = o(1)$.

**Proof.** If not, there exists a subsequence $\{n_l\}$ such that $\frac{k(n_l)}{n_l} \geq \varepsilon > 0$ for some $\varepsilon$. Then

$$\frac{1}{n_l} \leq \alpha_{k(n_l)} \quad \text{and} \quad \frac{k(n_l) - k(n_{l-1})}{n_l} \leq \sum_{j=k(n_{l+1})+1}^{k(n_l)} \alpha_i.$$

Hence

$$\sum_{i=1}^{\infty} \frac{k(n_l) - k(n_{l-1})}{n_l} \leq \sum_{i=1}^{\infty} \alpha_i = \alpha,$$

$$\sum_{i=1}^{\infty} \frac{k(n_{l+1}) - k(n_l)}{k(n_{l+1})} \leq \frac{\alpha}{\varepsilon} < \infty.$$  

This implies $\frac{k(n_{l+1}) - k(n_l)}{k(n_{l+1})} \to 0$, so $\frac{k(n_{l+1})}{k(n_l)} \to 1$ as $n \to \infty$, therefore (7.4) implies

$$\sum_{l=1}^{\infty} \frac{k(n_{l+1}) - k(n_l)}{k(n_l)} < \infty.$$  

However,

$$\sum_{l=1}^{\infty} \frac{k(n_{l+1}) - k(n_l)}{k(n_l)} \geq \int_{k(n_l)}^{\infty} \frac{1}{x} dx = \infty,$$

which is a contradiction to (7.5). Lemma 7.3 is then proved.

**Example 7.4.** Let $\alpha_1 \geq \alpha_2 \geq \cdots > 0$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. For each $n$, we generate the class $V_i$ with size $n_i = \lfloor n \alpha_i \rfloor$ for $i = 1, 2, \ldots$ until $n_i = 0$. Then we generate the last class $V_d$ with size $n_d = n - \sum_{i=1}^{d-1} n_i$. Note that for every fixed $i, \frac{n_i}{n} \to \alpha_i$. From Lemma 7.3, the number of blocks satisfies $d \leq k(n) + 1 = o(n)$. In particular, we have the following examples for the choice of $\alpha_i$'s:
\( \alpha_i = \frac{C}{\gamma^i} \) for some constant \( C, \gamma > 0 \) with \( \sum_{i=1}^{\infty} \alpha_i = 1 \).

(2) \( \alpha_i = \frac{C}{\beta^i} \) for some \( C > 0, \beta > 1 \) with \( \sum_{i=1}^{\infty} \alpha_i = 1 \).

Example 7.5. Let \( \alpha_1 \geq \alpha_2 \geq \cdots > 0 \) and \( \sum_{i=1}^{\infty} \alpha_i = \alpha < 1 \). For each \( n \), we can generate a class \( V_i \) with size \( n_i = \lfloor n\alpha_i \rfloor \) for \( i = 1, 2, \ldots \), until \( n_i = 0 \). Then generate \( o(n) \) many small classes of size \( o(n) \). By Lemma 7.3, \( d = o(n) \).

8 | RANDOM GRAM MATRICES

In the last section, we present an example beyond general Wigner-type matrices to which our main result can apply. Let \( X_n \) be a \( m \times n \) complex random matrix whose entries are independent. Consider a random Gram matrix \( M_n := \frac{1}{n} X_n X_n^* \) with a variance profile matrix \( S_n = (s_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) satisfies the following conditions:

(1) \( \mathbb{E} x_{ij} = 0, \mathbb{E} |x_{ij}|^2 = s_{ij}, \) for all \( 1 \leq i \leq m, 1 \leq j \leq n \).

(2) (Lindeberg’s condition) for any constant \( \eta > 0 \),

\[
\lim_{n \to \infty} \frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{E} [|x_{ij}|^2 1(|x_{ij}| \geq \eta \sqrt{n})] = 0. \tag{8.1}
\]

(3) \( \sup_{ij} s_{ij} \leq C \) for some constant \( C \geq 0 \).

(4) \( \lim_{n \to \infty} \frac{m}{n} = y \in (0, \infty) \).

Let

\[
H_n := \begin{bmatrix} 0 & X_n \\ X_n^* & 0 \end{bmatrix}. \tag{8.2}
\]

We first find the relation between the trace of \( M_n \) and the trace of \( H_n \) in the following lemma.

Lemma 8.1. For any integer \( k \geq 1 \), the following holds:

\[
\frac{1}{m} \text{tr} M_n^k = \frac{(m+n)^k}{2mn^k} \text{tr} \left( \frac{H_n}{\sqrt{n+m}} \right)^{2k}. \tag{8.3}
\]

Proof. It is a simple linear algebra result that nonzero eigenvalues of \( H \) come in pairs \( \{-\sqrt{\lambda}, \sqrt{\lambda}\} \) where \( \lambda > 0 \) is a nonzero eigenvalue of \( X_n X_n^* \). Therefore for \( k \geq 1 \),

\[
\text{tr}(H_n^{2k}) = 2 \text{tr}(X_n X_n^*)^k. \tag{8.4}
\]

We then have for \( k \geq 1 \),

\[
\frac{1}{m} \text{tr} M_n^k = \frac{1}{m} \text{tr} \left( \frac{1}{n} X_n X_n^* \right)^k = \frac{1}{2n^k m} \cdot 2 \text{tr}(X_n X_n^*)^k = \frac{(m+n)^k}{2mn^k} \text{tr} \left( \frac{H_n}{\sqrt{n+m}} \right)^{2k}. \tag{8.5}
\]
Since $H_n$ is a $(n + m) \times (n + m)$ general Wigner-type matrix with a variance profile
\begin{equation}
\Sigma_n := \begin{bmatrix} 0 & S_n \\ S_n^T & 0 \end{bmatrix},
\end{equation}
we can decide the moments of the limiting spectral distribution of $M_n$ from Theorem 3.2 and Lemma 8.1 in the following theorem.

**Theorem 8.2.** Let $M_n$ be a random Gram matrix with the assumptions above and $W_n$ be the corresponding graphon of $\Sigma_n$. If for any finite tree $T$, $t(T, W_n)$ converges as $n \to \infty$, then the empirical spectral distribution of $M_n$ converges almost surely to a probability measure $\mu$ such that for $k \geq 1$,
\begin{equation}
\int x^k d\mu = \frac{(1 + y)^{k+1}}{2y} C_k \lim_{n \to \infty} t(T_j^{k+1}, W_n).
\end{equation}

**Proof.** From Lemma 8.1, for $k \geq 1$,
\begin{equation}
\frac{1}{m} \text{tr} M_n^k = \frac{(m + n)^k}{2mn^k} \cdot \frac{1}{n + m} \text{tr} \left( \frac{H_n}{\sqrt{n + m}} \right)^{2k}.
\end{equation}

From Theorem 3.2, almost surely
\begin{equation}
\lim_{n \to \infty} \frac{1}{n + m} \text{tr} \left( \frac{H_n}{\sqrt{n + m}} \right)^{2k} = \sum_{j=1}^{C_k} \lim_{n \to \infty} t(T_j^{k+1}, W_n).
\end{equation}

Since $\lim_{n \to \infty} \frac{m}{n} = y > 0$, The result follows from (8.7).

Finally we derive the Stieltjes transform of the limiting spectral distribution from Theorem 3.4.

**Theorem 8.3.** Let $M_n$ be a random Gram matrix with a variance profile $S_n$ and $W_n$ be the corresponding graphon of $\Sigma_n$ defined in (8.6). If $\delta(W_n, W) \to 0$ for some graphon $W$, then the empirical spectral distribution of $\frac{M_n}{\sqrt{n}}$ converges almost surely to a probability measure $\mu$ whose Stieltjes transform $s(z)$ is an analytic solution defined on $\mathbb{C}^+$ by the following equations:
\begin{align}
s(z) &= \frac{1 + y}{y} \int_0^\frac{1}{1+y} b(z, u) du, \\
b(z, u)^{-1} &= z - \int_\frac{1}{1+y}^1 \frac{W(u, v)}{(1 + y)^{-1} - \int_0^\frac{1}{1+y} W(u, t) b(z, t) dt} dv.
\end{align}

where $b(z, u)$ is an analytic function defined on $\mathbb{C}^+ \times \left[0, \frac{y}{1+y}\right]$.

**Remark 8.4.** Up to notational differences, (8.8), (8.9) are the centered case ($\mathbb{E}M_n = 0$) of the equations in [38] (see section 5.1 in [38]), where a noncentered form of the equations were also derived under the assumptions of $(4 + \varepsilon)$-bounded moments and the continuity of the variance profile. Recently, (8.8), (8.9) were also studied in [6, 7], where the local law for the centered case was proved under
stronger assumptions including bounded $k$-moments of each entry for each $k$ and irreducibility condition on the variance profile. Our Theorems 8.2 and 8.3 give the weakest assumption so far for the existence of the limiting distribution and the quadratic vector equations only for the centered case.

**Proof.** Let $s(z)$ be the Stieltjes transform of the limiting spectral distribution of $\frac{M_n}{\sqrt{n}}$. Let

$$\gamma_k := \int x^k d\mu, \quad m_{2k} := \sum_{j=1}^C t(T_j^{k+1}, W), \quad \text{and} \quad m(z) := \sum_{k=0}^\infty \frac{m_{2k}}{z^{2k+1}}.$$ 

By Theorem 8.2, for $k \geq 1$,

$$\gamma_k = \frac{(1+y)^{k+1}}{2y} m_{2k}.$$ 

Note that $m_0 = \gamma_0 = 1$, we have for $|z|$ sufficiently large,

$$s(z) = \sum_{k=0}^\infty \frac{\gamma_k}{z^{k+1}} = \frac{1}{z} + \sum_{k=1}^\infty \frac{m_{2k}}{2z^{k+1}} (1+y)^{k+1}$$

$$= \sum_{k=0}^\infty \frac{m_{2k}}{2z^{k+1}} (1+y)^{k+1} + \frac{y-1}{2yz} = \frac{1}{2y} \sqrt{\frac{1+y}{z}} m(\sqrt{\frac{z}{1+y}}) + \frac{y-1}{2yz}. \quad (8.10)$$

From Theorems 3.2 and (2.1), we know $m(z)$ is the Stieltjes transform of the limiting spectral distribution of $\frac{H_n}{\sqrt{n+m}}$. Moreover, from Theorem 3.4, we have

$$m(z) = \int_0^1 a(z, u) du, \quad (8.11)$$

$$a(z, u)^{-1} = z - \int_0^1 W(u, v) a(z, v) dv, \quad (8.12)$$

for some analytic function $a(z, u)$ defined on $\mathbb{C}^+ \times [0, 1]$. It remains to translate the equations above to an equation for $s(z)$. Let

$$a_1(z, x) := a(z, x), \quad \text{for} \ x \in \left[0, \frac{y}{1+y}\right],$$

$$a_2(z, x) := a(z, x), \quad \text{for} \ x \in \left[\frac{y}{1+y}, 1\right].$$

Since $\frac{m}{n} \to y \in (0, \infty)$, and $W_n$ is the corresponding graphon of $\Sigma_n$, its limit $W$ will have a bipartite structure, that is, $W(u, v) = 0$ for $(u, v) \in \left[0, \frac{y}{1+y}\right] \cup \left[\frac{y}{1+y}, 1\right]$. Then we have the following equations from (8.12):

$$a_1(z, u)^{-1} = z - \int_{\frac{y}{1+y}}^1 W(u, v) a_2(z, v) dv, \quad (8.13)$$

$$a_2(z, u)^{-1} = z - \int_0^{\frac{y}{1+y}} W(u, v) a_1(z, v) dv. \quad (8.14)$$
From (8.11) and (8.19), we have the following relation between $a_1(z, u)$:

$$a_1(z, u)^{-1} = z - \int_{\frac{y}{1+y}}^{1} \frac{W(u, v)}{z - \int_{0}^{\frac{y}{1+y}} W(u, t)a_1(z, t)dt} dv. \tag{8.15}$$

Let $b(z, u) := \frac{a_1(\sqrt{\frac{y}{1+y}})}{\sqrt{z(1+y)}}$. Then $b(z, u)$ is an analytic function defined on $C^+ \times \left[0, \frac{y}{1+y}\right]$. From (8.15), we can substitute $a_1(z, u)$ with $b(z, u)$ and get

$$b(z, u)^{-1} = z - \int_{\frac{y}{1+y}}^{1} \frac{W(u, v)}{(1 + y)^{-1} - \int_{0}^{\frac{y}{1+y}} W(u, t)b(z, t)dt} dv. \tag{8.16}$$

By multiplying with $a_1(z, u)$, $a_2(z, u)$ on both sides in (8.13) and (8.14) respectively, we have

$$1 = za_1(z, u) - a_1(z, u) \int_{\frac{y}{1+y}}^{1} W(u, v)a_2(z, v)dv, \tag{8.17}$$

$$1 = za_2(z, u) - a_2(z, u) \int_{\frac{y}{1+y}}^{1} W(u, v)a_1(z, v)dv. \tag{8.18}$$

From (8.17) and (8.18), by integration with respect to $u$, we have

$$\frac{y}{1+y} = z \int_{0}^{\frac{y}{1+y}} a_1(z, u)du - \int_{0}^{\frac{y}{1+y}} \int_{\frac{y}{1+y}}^{1} W(u, v)a_1(z, u)a_2(z, v)dudv,$$

$$\frac{1}{1+y} = z \int_{\frac{y}{1+y}}^{1} a_1(z, u)du - \int_{\frac{y}{1+y}}^{1} \int_{0}^{\frac{y}{1+y}} W(u, v)a_2(z, u)a_1(z, v)dudv.$$

Therefore we have

$$\int_{0}^{\frac{y}{1+y}} a_1(z, u)du - \int_{\frac{y}{1+y}}^{1} a_2(z, u)du = \frac{y - 1}{z(1+y)}. \tag{8.19}$$

From (8.11) and (8.19), we have the following relation between $m(z)$ and $a_1(z, u)$:

$$m(z) = \int_{0}^{\frac{y}{1+y}} a_1(z, u)du + \int_{\frac{y}{1+y}}^{1} a_2(z, u)du = 2 \int_{0}^{\frac{y}{1+y}} a_1(z, u)du - \frac{y - 1}{z(1+y)}. \tag{8.20}$$

With (8.10) and (8.20), we obtain the following equation for $s(z)$:

$$s(z) = \frac{1+y}{y} \int_{0}^{\frac{y}{1+y}} b(z, u)du,$$

where $b(z, u)$ satisfies the Equation (8.16). This completes the proof. \hfill \blacksquare

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