The mixed scalar curvature flow and harmonic foliations

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Abstract

We introduce and study the flow of metrics on a foliated Riemannian manifold \((M, g)\), whose velocity along the orthogonal distribution is proportional to the mixed scalar curvature, \(S_{\text{mix}}\). The flow is used to examine the question: When a foliation admits a metric with a given property of \(S_{\text{mix}}\) (e.g., positive or negative)? We observe that the flow preserves harmonicity of foliations and yields the Burgers type equation along the leaves for the mean curvature vector \(H\) of orthogonal distribution. If \(H\) is leaf-wise conservative, then its potential obeys the non-linear heat equation
\[
\partial_t u = n \Delta_F u + (n \beta_F + \Phi) u + \Psi_F^1 u^{-1} - \Psi_F^2 u^{-3}
\]
with a leaf-wise constant \(\Phi\) and known functions \(\beta_F \geq 0\) and \(\Psi_F^i \geq 0\). We study the asymptotic behavior of its solutions and prove that under certain conditions (in terms of spectral parameters of leaf-wise Schrödinger operator \(H_F = -\Delta_F - \beta_F \text{id}\)) there exists a unique global solution \(g_t\), whose \(S_{\text{mix}}\) converges exponentially as \(t \to \infty\) to a leaf-wise constant. The metrics are smooth on \(M\) when all leaves are compact and have finite holonomy group. Hence, in certain cases, there exist \(\mathcal{D}\)-conformal to \(g\) metrics, whose \(S_{\text{mix}}\) is negative or positive.

Keywords: foliation; flow of metrics; conformal; mixed scalar curvature; mean curvature; holonomy; Burgers equation; leaf-wise Schrödinger operator

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Introduction

In Introduction we discuss the question on prescribing the mixed scalar curvature of a foliation and define the flow of leaf-wise conformal metrics depending on this kind of curvature.

1. Geometry of foliations. Let \((M^{n+p}, g)\) be a connected closed (i.e., compact without a boundary) Riemannian manifold, endowed with a \(p\)-dimensional foliation \(\mathcal{F}\), i.e., a partition into submanifolds (called leaves) of the same dimension \(p\), and \(\nabla\) the Levi-Civita connection of \(g\). The tangent bundle to \(M\) is decomposed orthogonally as \(T(M) = D_{\mathcal{F}} \oplus D\), where the distribution \(D_{\mathcal{F}}\) is tangent to \(\mathcal{F}\). Denote by \((\cdot)^F\) and \((\cdot)^\perp\) projections onto \(D_{\mathcal{F}}\) and \(D\), respectively. The second fundamental tensor and the mean curvature vector field of \(\mathcal{F}\) are given by
\[
h_{\mathcal{F}}(X,Y) := (\nabla_X Y)^\perp, \quad H_{\mathcal{F}} := \text{Tr}_g h \quad (X, Y \in D_{\mathcal{F}}).
\]
A Riemannian manifold \((M, g)\) may admit many kinds of geometrically interesting foliations. Totally geodesic (i.e. \(h_{\mathcal{F}} = 0\)) and harmonic (i.e. \(H_{\mathcal{F}} = 0\)) foliations are among these kinds that enjoyed a lot of investigation of many geometers (see [1], and a survey in [2]). Simple examples are parallel circles or winding lines on a flat torus, and a Hopf family of great circles on the 3-sphere. The second fundamental tensor \(h\) and the integrability tensor \(T\) of the distribution \(\mathcal{D}\) are defined by
\[
h(X,Y) := (1/2) (\nabla_X Y + \nabla_Y X)^F, \quad T(X,Y) := (1/2) [X, Y]^F \quad (X, Y \in D). \tag{1}
\]
The mean curvature vector field of \(\mathcal{D}\) is given by \(H = \text{Tr}_g h\). A foliation \(\mathcal{F}\) is said to be Riemannian, or transversely harmonic, if, respectively, \(h = 0\), or \(H = 0\). Conformal foliations (i.e., \(h = \)
(1/n)H · g|D) were introduced by Vaisman \cite{15} as foliations admitting a transversal conformal structure. Such foliations extend the class of Riemannian foliations.

One of the principal problems of geometry of foliations reads as follows, see \cite{12}:

**Given a foliation \( \mathcal{F} \) on a manifold \( M \) and a geometric property \( (P) \), does there exist a Riemannian metric \( g \) on \( M \) such that \( \mathcal{F} \) enjoys \( (P) \) with respect to \( g \)?**

Such problems of the existence and classification of metrics on foliations (first posed explicitly by H. Gluck for geodesic foliations) have been studied intensively by many geometers in the 1970’s.

A foliation is geometrically taut, if there is a Riemannian metric making \( \mathcal{F} \) harmonic. H. Rummel characterized such foliations by existence of an \( \mathcal{F} \)-closed \( p \)-form on \( M \) that is transverse to \( \mathcal{F} \). D. Sullivan provided a topological tautness condition for geometric tautness. By the Novikov Theorem (see \cite{3}) and Sullivan’s results, the sphere \( S^3 \) has no 2-dimensional taut foliations. In recent decades, several tools for proving results of this sort have been developed. Among them, one may find Sullivan’s foliated cycles and new integral formulae, see \cite{16} and a survey in \cite{12}.

2. **The mixed scalar curvature.** There are three kinds of Riemannian curvature for a foliation: tangential, transversal, and mixed (a plane that contains a tangent vector to the foliation and a vector orthogonal to it is said to be mixed). The geometrical sense of the mixed curvature follows from the fact that for a totally geodesic foliation, certain components of the curvature tensor, see \cite{9}, regulate the deviation of leaves along the leaf geodesics. In general relativity, the geodesic deviation equation is an equation involving the Riemann curvature tensor, which measures the change in separation of neighboring geodesics or, equivalently, the tidal force experienced by a rigid body moving along a geodesic. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics.

Let \( \{E_i, \mathcal{E}_a\}_{1 \leq n, a \leq p} \) be a local orthonormal frame on \( TM \) adapted to \( \mathcal{D} \) and \( \mathcal{D}_F \). The mixed scalar curvature is the following function:

\[
\text{Sc}_{\text{mix}}(g) = \sum_{i=1}^n \sum_{a=1}^p R(\mathcal{E}_a, E_i, E_i, E_i),
\]

Recall the formula, see \cite{16}:

\[
\text{Sc}_{\text{mix}}(g) = \text{div}(H + H_\mathcal{F}) + \|H\|^2 + \|H_\mathcal{F}\|^2 + \|T\|^2 - \|h\|^2 - \|h_\mathcal{F}\|^2.
\]  

Integrating (2) over a closed manifold and using the Divergence Theorem, we obtain the integral formula with the total \( \text{Sc}_{\text{mix}}(g) \). Thus, (2) yields decomposition criteria for foliated manifolds under constraints on the sign of \( \text{Sc}_{\text{mix}}(g) \), see \cite{16} and a survey in \cite{9}.

The basic question that we address in the paper is the following: **Which foliations admit a metric with a given property of \( \text{Sc}_{\text{mix}} \) (e.g., positive or negative)?**

**Example 1.** (a) If a distribution either \( \mathcal{D} \) or \( \mathcal{D}_\mathcal{F} \) is one-dimensional and a unit vector field \( N \) is tangent to it, then the mixed scalar curvature is simply the Ricci curvature \( \text{Ric}_g(N, N) \). On a foliated surface \( (M^2, g) \) this coincides with the Gaussian curvature: \( \text{Sc}_{\text{mix}}(g) = \text{Ric}_g(N, N) = K(g) \).

(b) For any \( n \geq 2 \) and \( p \geq 1 \) there exists a fibre bundle with a closed \( (n + p) \)-dimensional total space and compact \( p \)-dimensional totally geodesic fibers, having constant mixed scalar curvature. To show this, consider the Hopf fibration \( \pi : S^3 \to S^2 \) of a unit sphere \( S^3 \) by great circles (closed geodesics). Let \( (\tilde{F}, g_1) \) and \( (\tilde{B}, g_2) \) be closed Riemannian manifolds with dimensions, respectively, \( p - 1 \) and \( n - 2 \). Let \( M = \tilde{F} \times S^3 \times \tilde{B} \) has the metric product \( g = g_1 \times g_2 \). Then \( \pi : M \to S^2 \times \tilde{B} \) is a fibration with a totally geodesic fiber \( \tilde{F} \times S^3 \). Certainly, \( \text{Sc}_{\text{mix}}(g) \equiv 2 > 0 \).

3. **Flows of metrics on foliations.** We shall examine the basic question using evolution equations. A flow of metrics on a manifold is a solution \( g_t \) of a differential equation \( \partial_t g = S(g) \), where the geometric functional \( S(g) \) is a symmetric \((0,2)\)-tensor usually related to some kind of curvature. This corresponds to a dynamical system in the infinite-dimensional space of all appropriate geometric structures on a given manifold. Denote by \( \mathcal{M}(M, \mathcal{D}, \mathcal{D}_\mathcal{F}) \) the space of smooth Riemannian metrics on \( M \) such that \( \mathcal{D}_\mathcal{F} \) is orthogonal to \( \mathcal{D} \). Elements of \( \mathcal{M}(M, \mathcal{D}, \mathcal{D}_\mathcal{F}) \) are called adapted metrics to the pair \( (\mathcal{D}, \mathcal{D}_\mathcal{F}) \). The notion of the \( \mathcal{D} \)-truncated \((r,2)\)-tensor field \( S^1 \) (where \( r = 0, 1 \)) will be helpful: \( S^1(X_1, X_2) = S(X_1^\perp, X_2^\perp) \). The \( \mathcal{D} \)-truncated metric tensor \( g^\perp \) is given by...
$g^\perp(X_1, X_2) = g(X_1, X_2)$ and $g^\perp(Y, \cdot) = 0$ for all $X_i \in \mathcal{D}$, $Y \in \mathcal{D}_X$. For $\mathcal{D}$-conformal adapted metrics we have $S^\perp(g) = s(g) g^\perp$ where $s(g)$ is a smooth function on the space of metrics on $M$.

Rovenski and Walczak \cite{12} (see also \cite{10}) studied flows of metrics that depend on the extrinsic geometry of codimension-one foliations, and posed the question:

*Given a geometric property $(P)$, can one find an $\mathcal{F}$-truncated flow $\partial_t g = S^\perp(g)$ on a foliation $(M, \mathcal{F})$ such that the solution metrics $g_t$ ($t \geq 0$) converge to a metric for which $\mathcal{F}$ enjoys $(P)$?*

Rovenski and Wolak \cite{13} studied $\mathcal{D}$-conformal flows of metrics on a foliation in order to prescribe the mean curvature vector $H$ of $\mathcal{D}$. In aim to prescribe the sign of $Sc_{\text{mix}}(g)$, we study the following *mixed scalar curvature flow* of metrics $g_t$, see also \cite{11}:

$$\partial_t g = -2(Sc_{\text{mix}}(g) - \Phi) g^\perp.$$  

(3)

Here $\Phi : M \to \mathbb{R}$ is a leaf-wise constant function, its value is clarified in what follows. By Lemma \ref{2} (Section 2.2), the flow (3) preserves harmonic (in particular, totally geodesic) foliations.

One may ask the question: *Given a Riemannian manifold $(M, g)$ with a harmonic foliation $\mathcal{F}$, when do solution metrics $g_t$ of (3) converge to the limit metric $\bar{g}$ with $Sc_{\text{mix}}(\bar{g})$ positive or negative?*

**Example 2.** (a) Let $(M^2, g_0)$ be a surface of Gaussian curvature $K$, endowed with a unit geodesic vector field $N$. Certainly, \ref{4} reduces to the following view:

$$\partial_t g = -2(K(g) - \Phi) g^\perp,$$  

(4)

that looks like the normalized Ricci flow on surfaces, but uses the truncated metric $g^\perp$ instead of $g$.

Let $k \in C^2(M)$ be the geodesic curvature of curves orthogonal to $N$. From \ref{4} we obtain the PDE $\partial_t k = K_x$ (along a trajectory $\gamma(x)$ of $N$). The above yields the *Burgers equation*

$$\partial_t k = k_{xx} - (k^2)_x,$$

which is the prototype for advection-diffusion processes in gas and fluid dynamics, and acoustics. When $k$ and $K$ are known, the metrics may be recovered as $g^\perp_t = g^\perp_0 \exp(-2 \int_0^t (K(s, t) - \Phi) \, ds)$.

(b) For the Hopf fibration $\pi : (S^{2m+1}, g_{\text{can}}) \to \mathbb{C}P^m$ of a unit sphere with fiber $S^1$, the orthogonal distribution $\mathcal{D}$ is non-integrable while $h = 0$. By \ref{2}, $Sc_{\text{mix}} = 2m$. Thus, $g_{\text{can}}$ on $S^{2m+1}$ is a fixed point in $\mathcal{M}(S^{2m+1}, \mathcal{D}, \mathcal{D}_\mathcal{F})$ of the flow (3) with $\Phi = 2m$.

4. **The nonlinear heat equation.** The solution strategy is based on deducing from \ref{3} the forced Burgers type equation

$$\partial_t H + \nabla F \|H\|^2 = n \nabla F (\text{div}_F H) + X,$$

for certain vector field $X$, see Proposition \ref{2}. If $H$ is leaf-wise conservative, i.e., $H = -n \nabla F \log u$ for a leaf-wise smooth function $u(x, t) > 0$, this and \ref{2} yield the non-linear heat equation

$$(1/n) \partial_t u = \Delta_F u + (\beta \Phi + \Phi/n) u + (\Psi^F_1/n) u^{-1} - (\Psi^F_2/n) u^{-3}, \quad u(\cdot, 0) = u_0,$$  

(5)

where functions $\beta \Phi(x) \geq 0$ and $\Psi^F_i(x) \geq 0$ are known, and $\Delta_F$ is the leaf-wise Laplacian, see \ref{3}. We study the asymptotic behavior of its solutions and prove that under certain conditions (in terms of spectral parameters of leaf-wise Schrödinger operator $H_F = -\Delta_F - \beta \Phi \text{id}$) the flow (4) has a unique global solution $g_t$, whose $Sc_{\text{mix}}$ converges exponentially to a leaf-wise constant. The metrics are smooth on $M$ when all leaves are compact and have finite holonomy group. Thus, in certain cases, there exist $\mathcal{D}$-conformal to $g$ metrics, whose $Sc_{\text{mix}}$ is negative or positive.

5. **The structure of the paper.** Section 1 contains main results (Proposition 1, Theorem 2 and Corollaries 1-4), their proofs and examples for one-dimensional case and for twisted products. These are supported by results of Section 3 (Theorems 3-4) about non-linear PDE (4) on a closed Riemannian manifold. Throughout the paper everything (manifolds, foliations, etc.) is assumed to be smooth (i.e., $C^\infty$-differentiable) and oriented. We also assume that all the leaves of a foliation $\mathcal{F}$ are compact minimal submanifolds.
1 Main results

Based on the “linear algebra” inequality $n \|h\|^2 \geq \|H\|^2$ with the equality when $D$ is totally umbilical, we introduce the following function (a measure of “non-umbilicity” of $D$):

$$\beta_D := \frac{1}{n} \left(n \|h\|^2 - \|H\|^2\right) \geq 0.$$  

(6)

For $p = 1$, let $k_i$ be the principal curvatures of $\mathcal{D}$. Then $\beta_D = n^{-2} (n \tau_2 - \tau_1^2) = n^{-2} \sum_{i<j} (k_i - k_j)^2$. Next lemma allows us to reduce $\textbf{(3)}$ to the leaf-wise PDE (with space derivatives along $\mathcal{F}$ only).

Lemma 1 (see also [11]). Let $\mathcal{F}$ be a harmonic foliation on $(M, g)$. Then $\textbf{(2)}$ reads as

$$\text{Sc}_{\text{mix}}(g) = \text{div}_\mathcal{F} H - \|H\|^2_\mathcal{F}/n + \|T\|^2_\mathcal{F} - \|h_\mathcal{F}\|^2_\mathcal{F} - n \beta_D.$$  

(7)

Proof. From $\textbf{(2)}$, using $H_\mathcal{F} = 0$ and identity $\text{div} H = \text{div}_\mathcal{F} H - \|H\|^2_\mathcal{F}$, we obtain

$$\text{Sc}_{\text{mix}}(g) = \text{div}_\mathcal{F} H - \|h\|^2_\mathcal{F} + \|T\|^2_\mathcal{F} - \|h_\mathcal{F}\|^2_\mathcal{F}.$$  

Substituting $\|h\|^2_\mathcal{F} = n \beta_D + \|H\|^2_\mathcal{F}/n$ due to $\textbf{(3)}$, we get $\textbf{(7)}$. \hfill $\square$

We denote $\nabla^\mathcal{F} f := (\nabla f)^\mathcal{F}$. Given a vector field $X$ and a function $u$ on $M$, define the functions using the leaf-wise derivatives: the divergence $\text{div}^\mathcal{F} X = \sum_{i=1}^n g(\nabla^\mathcal{F}_e X, e_i)$ and the Laplacian $\Delta^\mathcal{F} u = \text{div}_\mathcal{F} (\nabla^\mathcal{F} u)$. Notice that operators $\nabla^\mathcal{F}$, $\nabla^\mathcal{F}$, and $\Delta^\mathcal{F}$ (i.e., on the leaves) are $t$-independent.

The Schrödinger operator is central to all of quantum mechanics. By Proposition 4 (in Section 222), the flow of metrics $\Phi$ preserves the leaf-wise Schrödinger operator $\mathcal{H}$, given by

$$\mathcal{H}(u) = -\Delta^\mathcal{F} u - \beta_D u.$$  

(8)

The spectrum of $\mathcal{H}_\mathcal{F}$ on any compact leaf $F$ is an infinite sequence of isolated real eigenvalues $\lambda_0^F \leq \lambda_1^F \leq \ldots \leq \lambda_j^F \leq \ldots$ counting their multiplicities, and $\lim_{j \to \infty} \lambda_j^F = \infty$. One may fix in $L^2(F)$ an orthonormal basis of corresponding eigenfunctions $\{e_j\}$, i.e., $\mathcal{H}_\mathcal{F}(e_j) = \lambda_j^F e_j$. If all leaves are compact then $\lambda_j^F$ are leaf-wise constant functions and $\{e_j\}$ are leaf-wise smooth functions on $M$.

If the leaf $F(x)$ through $x \in M$ is compact then $\lambda_0^F \leq 0$ (since $\beta_D \geq 0$) and the eigenfunction $e_0$ (called the ground state) may be chosen positive. The fundamental gap $\lambda_0^F - \lambda_1^F > 0$ of $\mathcal{H}_\mathcal{F}$ has mathematical and physical implications (e.g., in refinements of Poincaré inequality and a priori estimates), it also is used to control the rate of convergence in numerical methods of computation. Note that the least eigenvalue of operator $-\Delta^\mathcal{F} u - (\beta_D + \frac{\lambda_j^F}{\lambda_0^F}) u$ is $\lambda_0^F - \frac{\lambda_j^F}{\lambda_0^F}$.

An important step in the study of evolutionary PDEs is to show short-time existence/uniqueness.

Proposition 1. Let $\mathcal{F}$ be a harmonic foliation on a closed Riemannian manifold $(M, g_0)$. Then the linearization of $\textbf{(3)}$ is the leaf-wise parabolic PDE, hence $\textbf{(3)}$ has a unique solution $g_t$ defined on a positive time interval $[0, t_0)$ and smooth on the leaves.

We shall say that a smooth function $f(t, x)$ on $(0, \infty) \times X$ converges to $\bar{f}(x)$ as $t \to \infty$ in $C^\infty$, if it converges in $C^k$-norm for any $k \geq 0$. It converges exponentially fast if there exists $\omega > 0$ (called the exponential rate) such that $\lim_{t \to \infty} e^{\omega t} \|f(t, \cdot) - \bar{f}\|_{C^k} = 0$ for any $k \geq 0$.

Define the domain $U := \{x \in M : \Psi^\mathcal{F}_1, \Psi^\mathcal{F}_2 \neq 0\}$ and the functions

$$\Psi^\mathcal{F}_1 := u_0^2 \|h_\mathcal{F}\|^2_{g_0}, \quad \Psi^\mathcal{F}_2 := u_0^4 \|T\|^2_{g_0}.$$  

(9)

Proposition 2. Let $\mathcal{F}$ be a harmonic foliation on a Riemannian manifold $(M, g_0)$ and a family of metrics $g_t (0 \leq t < t_0)$ solve $\textbf{(3)}$. Then

$$\partial_t H + \nabla^\mathcal{F} \|H\|^2_{\mathcal{F}} = n \nabla^\mathcal{F} (\text{div}_\mathcal{F} H) + n \nabla^\mathcal{F} (\|T\|^2_{\mathcal{F}} - \|h_\mathcal{F}\|^2_{\mathcal{F}} - n \beta_D).$$  

(10)

Suppose that $H_0 = -n \nabla^\mathcal{F} \log u_0$ for a function $u_0 > 0$, then $H_t = -n \nabla^\mathcal{F} \log u$ for some positive function $u : M \times [0, t_0)$, moreover,
(i) if \( \Psi^F_2 \neq 0 \) then \( u = (\Psi^F_2)^{1/4} \|T\|^{1/2}_g, \) and the non-linear PDE (15) is satisfied.
(ii) if \( \Psi^F_1 \equiv 0 \equiv \Psi^F_2 \) then the potential function \( u \) may be chosen as a solution of the linear PDE

\[
(1/n) \partial_t u = \Delta_F u + \beta_D u, \quad u(\cdot, 0) = u_0. \tag{11}
\]

Under certain conditions, (10) and (11) have single-point exponential attractors. Kirsch-Simon [7] studied the forced Burgers PDE on \( \mathbb{R}^n \) and proved the polynomial convergence of a solution.

Based on Proposition (2(ii) and methods of (11), we obtain the following.

**Theorem 1.** Let \( F \) be a totally geodesic compact foliation with integrable orthogonal distribution on a Riemannian manifold \((M, g_0)\), and \( H_0 = -n \nabla^F \log u_0 \) for a function \( u_0 > 0 \). Then (3) has a unique global solution \( g_t (t \geq 0) \) smooth on the leaves. If \( \Phi = n \lambda_0^F \) then, as \( t \to \infty \), the metrics \( g_t \) converge in \( C^\infty \) with the exponential rate \( n(\lambda_1^F - \lambda_0^F) \) to the limit metric \( \bar{g} \) and

\[
\text{Sc}_{\text{mix}}(\bar{g}) = n \lambda_0^F, \quad \bar{H} = -n \nabla^F \log e_0.
\]

Moreover, if the leaves have finite holonomy group, then all \( g_t \) and \( \bar{g} \) are smooth on \( M \).

**Example 3.** (a) (Example 2a continued). Let \((T^2, g_0)\) be a torus with Gaussian curvature \( K \) and a vector field \( N \), whose trajectories are closed geodesics. Suppose that the curvature \( k \) of orthogonal (to \( N \)) curves obeys \( k = N(\psi_0) \) for a smooth function \( \psi_0 \) on \( T^2 \). In this case, \( -\mathcal{H} \) in (8) coincides with the leaf-wise Laplacian, hence \( \lambda_0^F = 0 \) and \( e_0 = \text{const.} \) We also have \( \Psi = T = \beta_D = 0 \) by Theorem 1. The flow of metrics (11) on \( T^2 \) admits a unique solution \( g_t (t \geq 0) \).

If \( \Phi = 0 \) then, as \( t \to \infty \), the metrics converge to a flat metric, and \( N \)-curves compose a rational linear foliation.

(b) If \( \beta_D = 0 \) then \( \lambda_0^F = 0 \), see Theorem 1. This appear for a family \((M, g_t) = M_1 \times_f M_2 \) of twisted products, i.e., the manifold \( M = M_1 \times M_2 \) with the metrics \( g_t = f_t^2 g_1 + g_2 \) \((t \geq 0)\), where \( f_t \in C^\infty (M_1 \times M_2) \) are positive functions. The submanifolds \( \{x\} \times M_2 \) of a twisted product compose a totally geodesic foliation \( F \), while \( M_1 \times \{y\} \) are totally umbilical with the leaf-wise conservative mean curvature vector \( H = -n \nabla^F f \) see [8]. By (7), we have \( \text{Sc}_{\text{mix}}(g) = \text{div}_F H - \|H\|^2/n \).

If the metrics \( g_t \) solve (3) then \( H \) obeys the Burgers type equation (see (10) with \( \beta_D = 0 \))

\[
\partial_t H + \nabla^F \|H\|^2 = n \nabla^F (\text{div}_F H). \tag{12}
\]

In this case, \( \partial_t f = n \Delta_f f + \Phi f \) and the function \( f := e^{-\Phi t} f \) obeys the heat equation \( \partial_t f = n \Delta_f f \).

Let \( M_1 \) and \( M_2 \) be closed and \( g_t = f_t^2 g_1 + g_2 \) solve (3) with \( \Phi = 0 \), then \( g_t \) converge as \( t \to \infty \) in \( C^\infty \) with the exponential rate \( \lambda_1^F \) to the metric \( \bar{g} = f_0^2 g_1 + g_2 \), where \( f_0 = \int_{M_2} f(0, \cdot, y) dy \), see (11).

Define \( d_{u_0, e_0} := \min_F (u_0/e_0)/\max_F (u_0/e_0) > 0 \).

The central result of the work is the following.

**Theorem 2.** Let \( F \) be a harmonic compact foliation on a closed Riemannian manifold \((M, g_0)\) and \( H_0 = -n \nabla^F \log u_0 \) for a smooth function \( u_0 > 0 \). If \( \Phi \) obeys the inequality

\[
\Phi \geq n \lambda_0^F + d_{u_0, e_0}^{-4} \max_F \|T\|^2_{g_0}, \tag{13}
\]

then (3) admits a unique global solution \( g_t (t \geq 0) \) smooth on any leaf \( F \), moreover, if all leaves have finite holonomy group, then \( g_t \) are smooth on \( M \), and for any \( \alpha \in (0, \min\{\lambda_1^F - \lambda_0^F, 2(\Phi/n - \lambda_0^F)\}) \) we have the leaf-wise convergence in \( C^\infty \), as \( t \to \infty \), with the exponential rate \( n\alpha \):

\[
\text{Sc}_{\text{mix}}(g_t) \to n \lambda_0^F - \Phi \leq 0, \quad H_t \to -n \nabla^F \log e_0, \quad h_F(g_t) \to 0.
\]

For \( T = 0 \), condition (13) becomes \( \Phi \geq n \lambda_0^F \), and we have the following.

**Corollary 1.** Let \( F \) be a harmonic compact foliation on a Riemannian manifold \((M, g_0)\) with integrable normal distribution and \( H_0 = -n \nabla^F \log u_0 \) for a function \( u_0 > 0 \). If \( \Phi \geq n \lambda_0^F \) then the claim of Theorem 2 holds.
The above results are summarized (due to the basic question) in the following.

**Corollary 2.** Let $\mathcal{F}$ be a harmonic compact foliation on a closed Riemannian manifold $(M, g)$ and $H = -n \nabla^F \log u_0$ for a smooth function $u_0 > 0$.

(i) Then for any $c > d_{u_0}^4 \max_F ||T||_g^2$ there exists a $\mathcal{D}$-conformal to $g$ metric $\tilde{g}$ with $\text{Sc}_{\text{mix}}(\tilde{g}) \leq -c$.

(ii) If $\xi^4 ||T||_g^2 \leq c^4 + d_{u_0}^4 \max_F ||T||_g^2$, where $\xi = u_0/(\tilde{u}_0^4)$ and $\tilde{u}_0$ is defined in Section 3.3, then there exists a $\mathcal{D}$-conformal to $g$ metric $\tilde{g}$ with $\text{Sc}_{\text{mix}}(\tilde{g}) > 0$.

We consider applications to a Riemannian manifold $(M, g)$ with a unit vector field $N$ (i.e., $p = 1$ or/and $n = 1$). In this case, $\text{Sc}_{\text{mix}}(g)$ is the Ricci curvature $\text{Ric}_g(N, N)$ in the $N$-direction.

**Case** $p = 1$. Let $N$ be tangent to a geodesic foliation $\mathcal{F}, h$ the scalar second fundamental form and $H = \text{Tr}_g h$ the mean curvature of $\mathcal{D} = N^\perp$. We have $h_F = 0$, and (4) reads $\text{Ric}(N, N) = ||T||^2 + N(H) - H^2/n - n \beta_D$. Let the metric evolves as, see (5),

$$\partial_t g = -2 (\text{Ric}_g(N, N) - \Phi) g^\perp,$$

then $H$ obeys the PDE along $N$-curves, see (10), $\partial_t H + N(H^2) = n N(N(H)) + n (||T||^2 - n \beta_D)$. Suppose that $H = -n N(\log u_0)$ for a leaf-wise smooth function $u_0 > 0$ on $M$, then we assume $H = -n N(\log u)$ for a positive function $u : M \times [0, t_0) \to \mathbb{R}$, see Proposition 2.

If $\mathcal{D}$ is integrable, then the function $u(\cdot , t) > 0$ may be chosen as a solution of the following linear heat equation, see (11), $\partial_t u = n N(N(u)) + n \beta_D u$, where $u(\cdot , 0) = u_0$. By Theorem 2, the flow (14) admits a unique solution $g_t (t \geq 0)$. If $\lambda_0^F - \Phi/n < 0$ then we have exponential convergence as $t \to \infty$ of $g_t \to \tilde{g}$, $H \to -n N(\log e_0)$ and $\text{Ric}_g(N, N) \to n \lambda_0^F - \Phi$.

If $\mathcal{D}$ is nowhere integrable, then $u = \Psi_2^T/\Psi_2^T ||T||_{g_0}^{-1/2}$ (with $\Psi_2^T := u_0^4 ||T||^2_{g_0} > 0$), moreover, the potential function $u > 0$ solves the non-linear heat equation, see (5),

$$(1/n) \partial_t u = N(N(u)) + (\beta_D + \Phi/n) u - (\Psi_2^F/n) u^{-3}, \quad u(\cdot , 0) = u_0.$$ 

If (13) are satisfied, then (14) admits a unique solution $g_t (t \geq 0)$. We have exponential convergence as $t \to \infty$ of functions $H \to -n N(\log e_0)$ and $\text{Ric}_g(N, N) \to n \lambda_0^F - \Phi$. By Theorem 2 we have

**Corollary 3.** Let $N$ be a unit vector field tangent to a geodesic foliation $\mathcal{F}$ on $(M, g)$.

(i) Then for any $c > d_{u_0}^4 \max_F ||T||_g^2$ there is $\mathcal{D}$-conformal to $g$ metric $\tilde{g}$ with the property $\text{Ric}_\tilde{g}(N, N) \leq -c < 0$.

(ii) If $\xi^4 ||T||_g^2 \leq c^4 + d_{u_0}^4 \max_F ||T||_g^2$, where $\xi = u_0/(\tilde{u}_0^4)$ and $\tilde{u}_0$ is defined in Section 3.3, then there exists $\mathcal{D}$-conformal to $g$ metric $\tilde{g}$ such that $\text{Ric}_\tilde{g}(N, N) > 0$.

**Case** $n = 1$. Let $N$ be orthogonal to a compact harmonic foliation $\mathcal{F}$ of codimension one. Then $\beta_D = H_F = 0$, $H = \nabla_N N$, $\Psi_2^F = 0$, $\Psi_1^F = u_0^4 ||h_F||^2_{g_0}$, the operator (8) coincides with $-\Delta_F$ (hence, $\lambda_0^F = 0$ and $e_0 = \text{const}$), and (14) reads $\text{Ric}(N, N) = \text{div}_F H - ||H||^2 + ||h_F||^2$. By (10) we have

$$\partial_t H + \nabla^F ||H||_{g_0}^2 = \nabla^F (\text{div}_F H) - \nabla^F (||h_F||^2_{g_0}).$$

Suppose the condition $H_0 = -\nabla^F \log u_0$ for a leaf-wise smooth function $u_0 > 0$ on $M$. Then

$$H = -\nabla^F \log u, \text{ where, see (5),}$$

$$\partial_t u = \Delta_F u + \Phi u + \Psi_1^F u^{-1}, \quad u(\cdot , 0) = u_0.$$ 

If $\Phi > 0$ then (13) holds and, by Theorem 2, the flow (14) admits a unique global solution $g_t (t \geq 0)$. As $t \to \infty$, we have convergence $H \to 0$, $\text{Ric}_g(N, N) \to -\Phi$, $h_F(g_t) \to 0$ with the exponential rate $\alpha$ for any $\alpha \in (0, \min\{\lambda_0^F, 2 \Phi\})$. By Theorem 2, we have the following.

**Corollary 4.** Let $\mathcal{F}$ be a codimension one harmonic foliation with a unit normal vector field $N$. Then for any $c > d_{u_0}^4 \max_F ||T||_g^2$ there is $\mathcal{D}$-conformal to $g$ metric $\tilde{g}$ with $\text{Ric}_\tilde{g}(N, N) \leq -c$.  


For a totally geodesic foliation $\mathcal{F}$, i.e., $h_\mathcal{F} \equiv 0$, (7) reads $\text{Ric}(N,N) = \text{div}_\mathcal{F} H - \|H\|^2 = \text{div} H$.

Let the metric evolves by (14). By Proposition 2 $H$ obeys the homogeneous Burgers equation $\partial_t H + \nabla^F H = \nabla^F (\text{div}_\mathcal{F} H)$. Suppose that the curvature vector $H$ of $N$-curves is leaf-wise conservative: $H = -\nabla^F \log u$ for a function $u > 0$. This yields the heat equation $\partial_t u = \Delta_F u$.

Solution of above PDE satisfies on the leaves $\bar{u} := \lim_{t \to \infty} u(t,x) = \int_{E_x} u_0(x) \, dx / \text{Vol}(F_x, g)$. Since $\nabla^F e_0 = 0$, we have $\bar{H} = \lim_{t \to \infty} H(t, \cdot) = 0$. Then $\text{Ric}_\bar{g}(N,N) = 0$, where $\bar{g} = \lim_{t \to \infty} g_t$.

**Surfaces:** $n = p = 1$. Let $(M^2, g)$ be a surface with a geodesic unit vector field $N$. The metric in biregular foliated coordinates $(x, \theta)$ is $g = dx^2 + \rho^2 d\theta^2$, where $\rho$ is a positive function and $\partial_\theta$ is the $N$-derivative. Let $M^2 \subset \mathbb{R}^3 : r = |\rho(x) \cos \theta, \rho(x) \sin \theta, h(x)|$ be a rotational surface, where $0 \leq x < l$, $|\theta| \leq \pi$, $\rho \geq 0$ and $(\rho')^2 + (h')^2 = 1$. Its metric belongs to warped products, see Example 3. The profile curves $\theta = \text{const}$ are geodesics tangent to $N$.

Let $M^2_1 \subset \mathbb{R}^3$ be a one-parameter family of surfaces of revolution (foliated by profile curves) such that the induced metric $g_t$ obeys (1). The profile of $M^2_1$ (parameterized as above) is $XZ$-plane curve $\gamma_0 = r(\cdot, 0)$, and $\theta$-curves are circles in $\mathbb{R}^3$. Thus $N = r_x$ is the unit normal to $\theta$-curves on $M^2_1$. Since the geodesic curvature of parallels is $k = -(\log \rho)_x$, we have $\partial_\theta \rho = \rho_{xx} - \Phi \rho$.

When $\Phi = 0$, the flow of metrics $\partial_t g = -2K g^{+}$ reduces to $\rho_{,t} = -K \rho$. Since $(\rho, x)_t = (\rho, x)_x$ by the maximum principle, we have the inequality $|\rho_{,x}| < 1$ for all $t \geq 0$. When such a solution $\rho(x,t)$ ($t \geq 0$) is known, we find $h = \int \sqrt{1 - (\rho_{,x})^2} \, dx$. Suppose that the boundary conditions are $\rho(0,t) = \rho_1$, $\rho(l,t) = \rho_2$ and $h(t) = h_1$, where $\rho_2 \geq \rho_1 \geq 0$ and $t \geq 0$. By the heat equation theory, the solution $\rho$ approaches as $t \to \infty$ to a linear function $\tilde{\rho} = x \rho_1 + (l - x) \rho_2$. Also, $h$ approaches as $t \to \infty$ to a linear function $\tilde{h} = x h_1 + (l - x) h_2$, where $h_2$ may be determined from the equality $(\rho_2 - \rho_1)^2 + (h_2 - h_1)^2 = \tilde{l}^2$. The curves $\gamma_\ell$ are isometric one to another for all $t$ (with the same arc-length parameter $x$). The limit curve $\lim_{t \to \infty} \gamma_\ell = \tilde{\gamma} = [\tilde{\rho}, \tilde{h}]$ is a line segment of length $l$. Thus, $M_t$ approach as $t \to \infty$ to the flat surface of revolution $\bar{M} = \text{the patch of a cone or a cylinder generated by } \tilde{\gamma}$.

2 Proof of main results

2.1 Holonomy of a compact foliation

The notion of holonomy uses that of a germ of a locally defined diffeomorphism (i.e., an equivalence class of certain maps). The germs of diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with fixed origin form a group, denoted by $\text{Diff}_0(\mathbb{R}^n)$. We denote by $\text{Diff}_0^k(\mathbb{R}^n)$ the subgroup of germs of diffeomorphisms which preserve orientation of $\mathbb{R}^n$. Let $(M, \mathcal{F})$ be a foliated manifold, $x, y$ be two points on a leaf $F$ and $Q_x, Q_y$ be transversal sections (diffeomorphic to $\mathbb{R}^n$). To any path $\alpha$ from $x$ to $y$ in $F$ we associate a germ of a diffeomorphism $h(\alpha) : (Q_x, x) \to (Q_y, y)$ called the holonomy of the path, for $x = y$, this is a generalization of a first return map. We obtain a homomorphism of groups $h : \pi_1(F) \to \text{Diff}_0^k(\mathbb{R}^n)$. The image $\text{Hol}(F) := h(\pi_1(F))$ is called the holonomy group of $F$. If $\mathcal{F}$ is transversally orientable then $\text{Hol}(F)$ is a subgroup of $\text{Diff}_0^k(\mathbb{R}^n)$.

Certainly, $\text{Hol}(F)$ is finite when the first fundamental group, $\pi_1(F)$, is finite. Note that a foliation whose leaves are the fibers of a fibre bundle has trivial holonomy group.

**Definition 1.** A foliated bundle is a fibre bundle $p : E \to F$ admitting a foliation $\mathcal{F}$ whose leaves meet transversely all the fibers $E_x = p^{-1}(x)$ ($x \in F$), and the bundle projection restricted to each leaf $F' \in \mathcal{F}$ is a covering map $p : F' \to F$. There is a representation $h_x : \pi_1(F, x) \to \text{Diff}(E_x)$ of the first fundamental group $\pi_1(F, x)$ in the group of diffeomorphisms of a fiber, called the total holonomy homomorphism for the foliated bundle.

**Local Reeb stability Theorem** (see [3, Vol. I]). Let $F$ be a compact leaf of a foliated manifold $(M, \mathcal{F})$ and $\text{Hol}(F)$ is finite. Then there is a normal neighborhood $pr : V \to F$ of $F$ in $M$ such that $(V, \mathcal{F}_V, pr)$ is a foliated bundle with all leaves compact (and transversal to the fibers). Furthermore,
each leaf $F' \subset V$ has finite holonomy group of order at most the order of $\text{Hol}(F)$ and the covering $\text{pr}vert_{F'} : F' \to F$ has $k$ sheets, where $k \leq \text{order of } \text{Hol}(F)$.

In other words, for a compact leaf $F$ with finite holonomy group, there exists a saturated neighborhood $V$ of $F$ in $M$ and a diffeomorphism from $E := \hat{F} \times_{\text{Hol}(F)} \mathbb{R}^n$ under which $F|_V$ corresponds to the bundle foliation on $E$. Here $\hat{F}$ is a covering space of $F$ associated with $\text{Hol}(F)$.

The following method construction foliations is related to local Reeb stability Theorem. Suppose that a group $G$ acts freely and properly discontinuously on a connected manifold $\hat{F}$ such that $\hat{F}/G = F$. Suppose also that $G$ acts on a manifold $Q$. Now form the quotient space $E := \hat{F} \times G Q$, obtained from the product space $\tilde{E} := \hat{F} \times Q$ by identifying $(g y, z)$ with $(y, gz)$ for any $y \in \hat{F}$, $g \in G$ and $z \in Q$. Thus $E$ is the orbit space of $\tilde{E}$ w.r.t. a properly discontinuous action of $G$. It is also Hausdorff, so it is a manifold. The projection $\text{pr}_1 : E \to \hat{F}$ induces a submersion $\text{pr} : E \to F$, so we have the commutative diagram $(\tilde{F} \to F) \circ \text{pr}_1 = \text{pr} \circ (\tilde{E} \to E)$. The map $\text{pr}$ has the structure of a fibre bundle over $F$ with vertical fiber $Q$. (Fibre bundles which can be obtained in this way are exactly those with discrete structure group). We claim that $E$ admits also horizontal leaves, so that $\text{pr}$ maps each leaf to $F$ as a covering projection. Indeed, the foliation $\tilde{F}$ on $\tilde{E}$, which is given by the submersion $\text{pr}_2 : \tilde{E} \to Q$, is invariant under the action of $G$, and hence we obtain the quotient foliation $\mathcal{F} = \tilde{F}/G$ on $E$. If $z \in Q$ and $G_z \subset G$ is the isotropy group at $z$ of the action by $G$ on $Q$, then the leaf of $E$ obtained from the leaf $\tilde{F} \times \{z\}$ is naturally diffeomorphic to $\tilde{F}/G_z$, and $\text{pr}$ restricted to this leaf is the covering $\tilde{F}/G_z \to F$.

2.2 $\mathcal{D}$-conformal adapted variations of metrics

The Levi-Civita connection $\nabla$ of a metric $g$ on $M$ is given by well-known formula

$$2 g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \quad (X, Y, Z \in TM).$$

(15)

Let $g_t$ be a smooth family of metrics on $(M, \mathcal{F})$ and $S = \partial_t g$. Since the difference of two connections is a tensor, $\partial_t \nabla^t$ is a $(1, 2)$-tensor on $(M, g_t)$. Differentiating (15) with respect to $t$ yields, see [12],

$$2 g_t((\partial_t \nabla^t)(X, Y), Z) = (\nabla^t_X S)(Y, Z) + (\nabla^t_Y S)(X, Z) - (\nabla^t_Z S)(X, Y)$$

(16)

for all $t$-independent vector fields $X, Y, Z \in \Gamma(TM)$. If $S = s(g)g^\perp$, for short we write

$$\partial_t g = s g^\perp$$

(17)

for a certain $t$-dependent function $s$ on $M$. In this case, the volume form $d \text{vol}$ evolves as [4]

$$(d/dt)(d \text{vol}_t) = (n/2) s_t d \text{vol}_t.$$  

(18)

Lemma 2. For $\mathcal{D}$-conformal adapted variations (17) of metrics we have

$$\partial_t h_{\mathcal{F}} = -s h_{\mathcal{F}}, \quad \partial_t H_{\mathcal{F}} = -s H_{\mathcal{F}}.$$  

(19)

Hence, variations (17) of metrics preserve harmonic and totally geodesic foliations.

Proof. Let $g_t$ $(t \geq 0)$ be a family of metrics on $(M, \mathcal{F})$ such that $\partial_t g_t = S(g)$, where the tensor $S(g)$ is $\mathcal{D}$-truncated. Using (16), we find for $X \in \mathcal{D}$ and $\xi, \eta \in \mathcal{D}_F$,

$$2 g_t(\partial_t h_{\mathcal{F}}(\xi, \eta), X) = g_t(\partial_t (\nabla^t_\xi \eta) + \partial_t (\nabla^t_\eta \xi), X) = (\nabla^t_\xi S)(X, \eta) + (\nabla^t_\eta S)(X, \xi) - (\nabla^t_X S)(\xi, \eta) = -S(\nabla^t_\xi \eta, X) - S(\nabla^t_\eta \xi, X) = -2 S(h_{\mathcal{F}}(\xi, \eta), X).$$

Assuming $S(g) = s(g)g^\perp$, we have (19). Tracing this we have (19). By the theory of ODEs, if $H_{\mathcal{F}} = 0$ or $h_{\mathcal{F}} = 0$ at $t = 0$ then respectively $H_{\mathcal{F}} = 0$ or $h_{\mathcal{F}} = 0$ for all $t > 0$. □
The co-nullity operator is defined by \( C_N(X) = -\langle \nabla_X N \rangle \), for \( X \in \mathcal{D}, \ N \in \mathcal{D}_F \). One may decompose \( C \) into symmetric and skew-symmetric parts as \( C = A_N + T_N^2 \). The Weingarten operator \( A_N \) of \( \mathcal{D} \) and the operator \( T_N^2 \) are related with tensors \( h \) and \( T \), see (1), by

\[
g(A_N(X), Y) = g(h(X, Y), N), \quad g(T_N^2(X), Y) = g(T(X, Y), N), \quad X, Y \in \mathcal{D}.
\]

The proof of the next lemma is based on (16) with \( S = s g^\perp \).

**Lemma 3** (see [11] and [13]). For \( \mathcal{D} \)-conformal adapted variations \( \mathcal{D} \) of metrics we have

\[
\partial_t A_N = -\frac{1}{2} N(s) \delta \mathcal{D}, \quad \partial_t T_N^2 = -s T_N^2 (N \in \mathcal{D}_F), \quad \partial_t H = -\frac{n}{2} \nabla^F s, \quad \partial_t (\text{div}_F H) = -\frac{n}{2} \Delta_F s. \tag{20}
\]

By (20)\(1,2\), the variations \( \mathcal{D} \) preserve conformal foliations, i.e., the property \( \beta_\mathcal{D} \equiv 0 \).

**Lemma 4** (Conservation laws). Let \( g_t \ (t \geq 0) \) be \( \mathcal{D} \)-conformal metrics \( \mathcal{D} \) on a foliated manifold \((M, \mathcal{F}, \mathcal{D})\) such that \( H_0 = -n \nabla^F \log u_0 \) for a positive function \( u_0 \in C^\infty(M) \). Then the following two functions and two vector fields on \( U \) are \( t \)-independent:

\[
\beta_\mathcal{D}, \quad \| h_F \|^2 / \| T \|^2, \quad H - (n/2) \nabla^F \log \| T \|, \quad H - n \nabla^F \log \| h_F \|.
\]

**Proof.** Using Lemma 3 and \( g^\perp(H, \cdot) = 0 \), we calculate

\[
\partial_t \| h \|^2 = \partial_t \sum_\alpha \text{Tr} (A^2_{\alpha}) = 2 \sum_\alpha \text{Tr} (A_{\alpha}^2 H_{\alpha}) = -2 \sum_\alpha A_{\alpha} (s) \text{Tr} A_{\alpha} = -g(\nabla s, H),
\]

\[
\partial_t g(H, H) = s g^\perp(H, H) + 2 g(\partial_t H, H) = -n g(\nabla s, H).
\]

Hence, \( n \partial_t \beta_\mathcal{D} = \partial_t \| h \|^2 - \frac{1}{2} \partial_t g(H, H) = 0 \), that is the function \( \beta_\mathcal{D} \) doesn’t depend on \( t \).

For any function \( f \in C^1_0(M) \) and a vector \( N \in \mathcal{D}_F \), using \( (\partial_t g)(\cdot, N) = 0 \), we find

\[
g(\nabla^F (\partial_t f), N) = N(\partial_t f) = \partial_t N(f) = \partial_t g(\nabla^F f, N) = g(\partial_t (\nabla^F f), N).
\]

Hence \( \nabla^F (\partial_t f) = \partial_t (\nabla^F f) \). By Lemma 3 we find

\[
\partial_t \| T \|^2 = -\partial_t \sum_\alpha \text{Tr} \left( (T_{\alpha}^2)^2 \right) = -2 \sum_\alpha \text{Tr} (T_{\alpha}^2 (\partial_t T_{\alpha}^2)) = 2s \sum_\alpha \text{Tr} \left( (T_{\alpha}^2)^2 \right) = -2s \| T \|^2. \tag{21}
\]

Similarly, by Lemma 4 and using the proof of Lemma 2, we obtain

\[
\partial_t \| h_F \|^2 = -s \| h_F \|^2. \tag{22}
\]

By the above, \( h_F \neq 0 \neq T \) on \( U \), and we have \( \partial_t \log \| T \|^2_{g_t} = -2s \) and \( \partial_t \log \| h_F \|^2_{g_t} = -s \). Using \( \nabla^F \partial_t H = \partial_t (\nabla^F) \), we obtain \( \partial_t H_t = (n/2) \partial_t \nabla^F \log \| T \|^2_{g_t} + \partial_t H_t = n \partial_t \nabla^F \log \| h_F \|^2_{g_t} \), moreover, \( \partial_t (\| h_F \|^2 / \| T \|) = 0 \). From the above the claim follows.

**2.3 Proofs of Propositions 1, 2, Theorem 2 and Corollary 2**

**Proof of Proposition 1** Let \( g_t = g_0 + s g_0^\perp \ (0 \leq t < \varepsilon) \) be \( \mathcal{D} \)-conformal metrics on a foliated manifold \((M, \mathcal{F})\), where \( s : M \times [0, \varepsilon) \to \mathbb{R} \) is a smooth function. By Lemma 2 \( \mathcal{F} \) is harmonic with respect to all \( g_t \). We differentiate (17) by \( t \), and apply Lemmas 3 and 4 to obtain

\[
\partial_t \text{Sc}_{\text{mix}}(g_t) = -(n/2) \Delta_F s + g(\nabla s, H) + s \left( \| h_F \|^2_{g_t} - 2 \| T \|^2_{g_t} \right).
\]

Hence, the linearization of (3) at \( g_0 \) is the following linear PDE for \( s \) on the leaves:

\[
\partial_t s = n \Delta_F s - 2 g_0(\nabla s, H_0) - 2 \left( \text{Sc}_{\text{mix}}(g_0) + \| h_F \|^2_{g_0} - 2 \| T \|^2_{g_0} \right) s,
\]
The result follows from the theory of linear parabolic PDEs (see [2]) and the finite holonomy assumption (i.e., the local Reeb stability Theorem in Section 2.1). □

**Proof of Proposition 2** By Theorem [1], (3) admits a unit local leaf-wise smooth solution \( g_t \) \((0 \leq t < t_0)\). The functions \( Sc_{mix}(g_t) \), \( H_t \), \( \|T\|_{g_t} \) and \( \|h_F\|_{g_t} \) etc. are then uniquely determined for \( 0 \leq t < t_0 \). From (20) with \( s = -2(Sc_{mix}(g_t) - \Phi) \) and using (7) we obtain (10).

(i) By Lemma (ii), \( H_t = -(n/4)\nabla F \log \|T\|_{g_t} = X \) for some vector field \( X \) on \( M \). Since \( H_0 \) is conservative, \( X = -(n/4)\nabla F \log \psi \) for some leaf-wise smooth function \( \psi > 0 \) on \( M \). Hence, \( H_t = -n\nabla F \log \psi(X) \) and, by condition \( H_0 = -n\nabla F \log u_0 \), one may take \( \psi = u_0^4 \|T\|_{g_0}^2 \).

Define a leaf-wise smooth function \( u := (\Psi_F^2)^{1/4}\|T\|_{g_t}^{-1/2} \) on \( U \times [0, t_0) \) and calculate
\[
\partial_t (\log \|T\|_{g_t}^2) = -4\partial_t \log (\|T\|_{g_t}^{-1/2}) = -4\partial_t \log ((\Psi_F^2)^{-1/4}u) = -4\partial_t \log u.
\]
By Lemma 4 and (10) \( \|h_F\|_{g_t}^2/\|T\|_{g_t} = \Psi_1 F (\Psi_F^2)^{-1/2} \), thus, \( u = (\Psi_1 F)^{1/2}\|h_F\|_{g_t}^{-1} \) on \( U \times [0, t_0) \) and
\[
\partial_t (\log \|h_F\|_{g_t}^2) = -2\partial_t \log (\|h_F\|_{g_t}^{-1}) = -2\partial_t \log (\Psi_1^2)^{-1/2} = -2\partial_t \log u.
\]
From the above and (7) we then obtain
\[
\partial_t \log u = -(1/4) \partial_t (\log \|T\|_{g_t}^2) = s/2 = -Sc_{mix}(g_t) + \Phi = n \Delta_F \log u + n g(\nabla F \log u, \nabla F \log u) + n \beta_D + \Phi + \Psi_1 F u^2 - \Psi_2 F u^4.
\]
Substituting \( \partial_t \log u = u^{-1} \partial_t u, \nabla F \log u = u^{-1} \nabla F u \) and \( \Delta_F \log u = u^{-1} \Delta_F u - u^{-2} g(\nabla F u, \nabla F u) \), we find that \( u \) solves the non-linear heat equation (5).

(ii) Note that \( H \) obeys a forced leaf-wise Burgers equation (a consequence of (10))
\[
\partial_t H + \nabla F \|H\|_{g_t}^2 = n \nabla F (\text{div}_F H) - n^2 \nabla F \beta_D,
\]
The rest of the proof see in [11] Proposition 2. □

**Proof of Theorem 2** (i) By Theorem 1 there exists a unique local solution \( g_t \) on \( M \times [0, t_0) \). By Proposition 2(ii), \( H \) obeys (10), and \( H = -n\nabla F \log u \) for some positive function \( u \) satisfying (5) with \( u(\cdot, 0) = u_0 \). Note that conditions (13) yield \( (u_0^4)^{1/2} \geq (\Psi_F^2)^{1/2}/(\Phi - n\lambda_0^F) \), see (28) with \( \beta = \beta_D + \Phi/n \) and \( \lambda_0^F - \Phi/n < 0 \) and definitions (9) and (24). By Theorem 3 one may leaf-wise smoothly extend a solution of (5) on \( M \times [0, \infty) \), hence \( H_t(x) \) is defined for \( t \geq 0 \) and is smooth on the leaves. By Theorem (i), \( u \to \infty \) as \( t \to \infty \) with exponential rate \( na \) for \( a \in (0, \min\{\lambda_0^F - \lambda_0^F, 2(\Phi/n - \lambda_0^F)\}) \). Hence, \( \Psi_F^2 u^{-4} \) is leaf-wise smooth, moreover, \( \|T\|_{g_t} \to 0 \) and \( h_F(g_t) \to 0 \) as \( t \to \infty \). By Theorem (ii), \( H_t = -n\nabla F \log u \) approaches in \( C^\infty \), as \( t \to \infty \), to the vector field \( H = -n\nabla F \log e_0 \), hence \( \text{div}_F H_t \) approaches to the leaf-wise smooth function \( -n \Delta_F \log e_0 \). Since \( -\Delta_F e_0 = (\beta_D + \Phi/n)e_0 = \lambda_0^F e_0 \), we have, as \( t \to \infty \),
\[
\text{div}_F H_t - \|H\|_{g_t}^2/n \to -n(\Delta_F e_0)/e_0 = n(\lambda_0^F + \beta_D) - \Phi.
\]
By (7), \( Sc_{mix}(\cdot, t) \) approaches exponentially to \( n\lambda_0^F - \Phi \) as \( t \to \infty \). Then a smooth solution to (3) is \( g_t = g_0 \exp(-\int_0^t (Sc_{mix}(\cdot, \tau) - \Phi) \ d\tau) \), where \( t \geq 0 \), see also Section 3.3.

(ii) The smoothness of \( g_t \) on \( M \) follows from the finite holonomy assumption and results of Section 3.3 (see also [11]). Indeed, let \( F \) be a leaf. By the local Reeb stability Theorem (see Section 2.1), there is a normal neighborhood \( \text{pr} : V \to F \) of \( F \) (with a smooth normal section \( Q \) – an open \( n \)-dimensional disk) such that \( (V, F, \text{pr}) \) is a foliated bundle diffeomorphic to \( F \times_{\text{Hol}(F)} Q \). There is a regular covering \( \tilde{\text{pr}} : \tilde{F} \to F \) with covering group \( \text{Hol}(F) \), since this group is finite, \( \tilde{F} \) is a compact manifold. The normal neighborhood \( \tilde{V} \), as a fiber bundle over \( F \), can be pulled back via \( \tilde{\text{pr}} \) to a bundle \( \tilde{V} \) over \( \tilde{F} \) with the same fiber. The standard pull-back construction yields a canonical covering \( \tilde{\text{pr}} : \tilde{V} \to V \) (a submersion), enabling us to lift the foliation \( \mathcal{F}_{\tilde{V}} \) to a foliation \( \tilde{\mathcal{F}} \) of \( \tilde{V} \), transverse to the fibers and having \( \tilde{F} \) as a leaf. Since the covering group
is exactly the holonomy group of $F$, $\tilde{\pi}_t : \pi_1(\tilde{F}, \tilde{x}) \to \pi_1(F, x)$ injects $\pi_1(\tilde{F}, \tilde{x})$ onto the subgroup of $\pi_1(F, x)$, and the leaf $\tilde{F}$ of $F$ has trivial holonomy. By [3] Theorem 2.4.1], $\tilde{F}$ has a neighborhood in $\tilde{V}$ that is a foliated disk bundle with all leaves diffeomorphic to $\tilde{F}$. Let $\tilde{g}_t$ be the lifts of the metrics $g_t$ from $V$ onto the product $\tilde{V}$. The corresponding foliation on $\tilde{V}$ is harmonic, the lift of the laplacian and potential function, $\tilde{\beta}_D$, smoothly depend on $q$ on $\tilde{V}$. (In case of totally geodesic foliation the leaves are isometric one to another, see for example [3]. Hence the leaf-wise laplacian on $\tilde{V}$ doesn’t depend on $q \in Q$). The functions $\tilde{e}_0$ and solution to (13), $u_t$ $(t > 0)$, are smooth on $\tilde{V}$ (where $Q$ can be replaced by smaller normal section), and they are lifts of leaf-wise smooth functions $e_0$ and $u_t$ $(t > 0)$ on $V$ or on a smaller neighborhood $W$ of $F$. The vector fields $\tilde{H}_t$ and $\tilde{H}$ are smooth on $\tilde{V}$, and they are lifts of smooth vector fields $H_t$ and $\tilde{H}$ on a neighborhood of $M$.

**Proof of Corollary 2.** The claim (i) follows directly from Theorem 2. The metrics $g_t$ $(g_0 = g)$ of Theorem 2 diverge as $t \to \infty$ with the exponential rate $\mu = \Phi - n\lambda_0^2$:

$$\exists C > 1, \forall X \in D, \forall t \geq 0 : C^{-1}e^{2\mu t}g(X, X) \leq g_t(X, X) \leq Ce^{2\mu t}g(X, X).$$

Consider $D$-conformal metrics $\tilde{g}_t = g_F + e^{-2\mu t}(g_0)$. By [2], $\tilde{H}_t = H_t$. Let $(\cdot, \cdot)_0$ be the inner product and the norm in $L_2(F)$ for any leaf $F$. The function $v = e^{-\mu t}u$ converges as $t \to \infty$ to $\tilde{u}_0\tilde{e}_0$, where $\tilde{u}_0 = (\tilde{u}, \tilde{e}_0)_0 = u_0^2 + \int_0^{\infty} q_0(\tau) d\tau$, see Theorem 4 and (11). For $t \to \infty$ we have

$$\|h_F\|^2 \|h_F\|_{\tilde{g}} = C\psi_1^2/\nu^2 \to \xi^2\|h_F\|^2_{\tilde{g}}, \quad \|T\|^2_{\tilde{g}} = C\psi_1^2/\nu^4 \to \xi^4\|T\|^2_{\tilde{g}},$$

and the metrics $\tilde{g}_t$ converge as $t \to \infty$ to the metric $\tilde{g}_\infty = g_F + \xi^{-2}g_{\tilde{g}}$. By (7), we find

$$S_{\text{mix}}(\tilde{g}_\infty) = n\lambda_0^2 - \Phi + \xi^4\|T\|^2_{\tilde{g}} - \xi^2\|h_F\|^2_{\tilde{g}}.$$  

Comparing with (13) completes the proof of (ii). 

**3 Results for PDEs**

The section plays an important role in proofs of main results (see Section 2).

Let $(F, g)$ be a closed $p$-dimensional Riemannian manifold, e.g., a leaf of a foliation $F$. Functional spaces over $F$ will be denoted without writing $(F)$, for example, $L_2$ instead of $L_2(F)$.

Let $H^l$ be the Hilbert space of differentiable by Sobolev real functions on $F$, with the inner product $(\cdot, \cdot)_l$ and the norm $\|\cdot\|_l$. In particular, $H^0 = L_2$ with the product $(\cdot, \cdot)_0$ and the norm $\|\cdot\|_0$.

If $E$ is a Banach space, we denote by $\|\cdot\|_E$ the norm of vectors in this space. If $B$ and $C$ are real Banach spaces, we denote by $B^r(B, C)$ the Banach space of all bounded $r$-linear operators

$$A : \prod_{i=1}^r B \to C$$

with the norm $\|A\|_{B^r(B, C)} = \sup_{v_1, \ldots, v_r \in B^0} \|A(v_1, \ldots, v_r)\|_C$. If $r = 1$, we shall write $B(B, C)$ and $A(\cdot)$, and if $B = C$ we shall write $B^r(B)$ and $B(B)$, respectively.

If $M$ is a $k$-regular manifold or an open neighborhood of the origin in a real Banach space, and $N$ is a real Banach space, we denote by $C^k(M, N)$ $(k \geq 1)$ the Banach space of all $C^k$-regular functions $f : M \to N$, for which the following norm is finite:

$$\|f\|_{C^k(M, N)} = \sup_{x \in M} \max\{|f(x)|_N, \max_{1 \leq |\alpha| \leq k} \|d^\alpha f(x)\|_{B^1(T_x, M, N)}\}.$$  

Denote by $\|\cdot\|_{C^k}$, where $0 \leq k < \infty$, the norm in the Banach space $C^k$; certainly, $\|\cdot\|_C$ when $k = 0$. In coordinates $(x_1, \ldots, x_p)$ on $F$, we have $\|f\|_{C^k} = \max_{x \in F} \max_{1 \leq |\alpha| \leq k} |d^\alpha f(x)|$, where $\alpha \geq 0$ is the multi-index of order $|\alpha| = \sum_{i=1} p \alpha_i$, and $d^\alpha$ is the partial derivative.
3.1 The Schrödinger operator

For a smooth (non-constant in general) function $\beta$ on $(F,g)$, the Schrödinger operator, see [8],

$$\mathcal{H}(u) := -\Delta u - \beta u$$

(23)

is self-adjoint and bounded from below (but it is unbounded). The domain of definition of $\mathcal{H}$ is $H^2$. The spectrum $\sigma(\mathcal{H})$ of $\mathcal{H}$ consists of an infinite sequence of isolated real eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \lambda_j \leq \ldots$ counting their multiplicities, and $\lambda_j \to \infty$ as $j \to \infty$. If we fix in $L_2$ an orthonormal basis of corresponding eigenfunctions $\{\phi_j\}$ (i.e., $\mathcal{H}(\phi_j) = \lambda_j \phi_j$) then any function $u \in L_2$ is expanded into the series (converging to $u$ in the $L_2$-norm)

$$u(x) = \sum_{j=0}^{\infty} c_j \phi_j(x), \quad c_j = (u, \phi_j)_0 = \int_F u(x) \phi_j(x) \, dx.$$  

(24)

The proof of (24) is based on the following facts. Since by the Elliptic regularity Theorem with $k=0$, we have $\mathcal{H}^{-1} : L_2 \to H^2$ when and the embedding of $H^2$ into $L_2$ is continuous and compact, see [2], then the operator $\mathcal{H}^{-1} : L_2 \to L_2$ is compact. This means that the spectrum $\sigma(\mathcal{H})$ is discrete, hence by the spectral expansion theorem for compact self-adjoint operators, $\{\phi_j\}_{j \geq 0}$ form an orthonormal basis in $L_2$, see [5 I, Ch. VII, Sect. 4; and II, Ch. XII, Sect. 2]. One can add a constant to $\beta$ such that $\mathcal{H}$ becomes invertible in $L_2$ (e.g., $\lambda_0 > 0$) and $\mathcal{H}^{-1}$ is bounded in $L_2$.

**Proposition 3** (see [11]). Let $\beta$ be a smooth function on a closed Riemannian manifold $(F,g)$. Then the eigenspace of operator $\mathcal{H}$, corresponding to the least eigenvalue, $\lambda_0$, is one-dimensional, and it contains a positive smooth eigenfunction, $\phi_0$.

The following facts will be used.

**Sobolev embedding Theorem** (see [2]). If a nonnegative $k \in \mathbb{Z}$ and $l \in \mathbb{N}$ are such that $2l > p + 2k$, then $H^l$ is continuously embedded into $C^k$.

**Elliptic regularity Theorem** (see [2]). If $\mathcal{H}$ is given by (23) and $0 \notin \sigma(\mathcal{H})$, then for any nonnegative $k \in \mathbb{Z}$ we have $\mathcal{H}^{-1} : H^k \to H^{k+2}$.

The Cauchy’s problem for the heat equation with a linear reaction term, see [5], has a form

$$\partial_t u = \Delta u + \beta u, \quad u(x,0) = u_0(x).$$

(25)

After scaling the time and replacements of functions

$$t/n \to t, \quad \Psi^F/n \to \Psi, \quad \beta F/n \to \beta, \quad \lambda_0 F/n \to \lambda_0,$$

the problem [5] reads as the following Cauchy’s problem for the non-linear heat equation on $(F,g)$:

$$\partial_t u = \Delta u + \beta u + \Psi(x) u^{-1} - \Psi_2(x) u^{-3}, \quad u(x,0) = u_0(x).$$

(26)

By [2] Theorem 4.51], the parabolic PDE (26) has a unique smooth solution $u(\cdot,t)$ for $t \in [0,t_0]$. Denote by $C_t = F \times [0,t]$ the cylinder with the base $F$. Define the quantities

$$\Psi_i^+ = \max_F (\Psi_i/e_i^2), \quad \Psi_i^- = \min_F (\Psi_i/e_i^2), \quad i = 1, 2,$$

$$\tilde{u}_i^+ = \max_F (u_0/e_0), \quad \tilde{u}_i^- = \min_F (u_0/e_0), \quad \beta^- = \min_F |\beta|. $$

(27)

We shall use the following scalar maximum principle [1 Theorem 4.4].

**Proposition 4.** Suppose that $X(t)$ is a smooth family of vector field on a closed Riemannian manifold $(F,g)$, and $f \in C^\infty(\mathbb{R} \times [0,T))$. Let $u : F \times [0,T) \to \mathbb{R}$ be a $C^\infty$ supersolution to

$$\partial_t u \geq \Delta_g u + (X(t), \nabla u) + f(u,t).$$

Suppose that $\varphi : [0,T] \to \mathbb{R}$ solves the Cauchy’s problem for ODEs $(d/dt)\varphi = f(\varphi(t),t), \quad \varphi(0) = C$. If $u(\cdot,0) \geq C$ then $u(\cdot,t) \geq \varphi(t)$ for all $t \in [0,T]$. (Claim also holds with the sense of all three inequalities reversed).
3.2 Long-time solution

Lemma 5. Let \( \lambda_0 < 0 \) for \((F,g)\) and \( u(x,t) > 0 \) be a solution in \( C_{t_0} \) of (26) with the condition
\[
(u_0^-)^4 \geq \frac{\Psi_2^+}{|\lambda_0|},
\] (28)
see [13]. Then the following a priori estimates are valid:
\[
w_-(t) \leq u(x,t)/e_0(x) \leq w_+(t), \quad (x,t) \in C_{t_0},
\] (29)
where \( \lambda_0 < 0 \) and
\[
w_-(t) = e^{-\lambda_0 t} \left( (u_0^-)^4 + \frac{\Psi_2^+}{\lambda_0} - e^{4\lambda_0 t} \frac{\Psi_2^+}{\lambda_0} \right), \quad w_+(t) = e^{-\lambda_0 t} \left( (u_0^+)^4 - \frac{\Psi_1^+}{\lambda_0} + e^{2\lambda_0 t} \frac{\Psi_1^+}{\lambda_0} \right).
\] (30)

Proof. Since \( e_0(x) > 0 \) on \( F \), we can change the unknown function in (26):
\[
u(x,t) = e_0(x) w(x,t).
\]
Substituting into (26) and using \( \Delta e_0 + \beta e_0 = -\lambda e_0 e_0 \), we obtain the Cauchy’s problem for \( w(x,t) \):
\[
\partial_t w = \Delta w - \lambda_0 w + 2 g(\nabla \log e_0, \nabla w) + \Psi_0 e^{-2}(x) \Psi_1(x) w^{-1} - e_0^{-4}(x) \Psi_2(x) w^{-3}, \quad w(\cdot,0) = u_0/e_0. \] (31)
Then, using (27) and (29) holds, where \( w_-(t) \) and \( w_+(t) \) are solutions of the following Cauchy’s problems for ODEs
\[
\frac{d}{dt} w_- = -\lambda_0 w_+ - \Psi_2^+ w^{-3}, \quad w_-(0) = u_0^-; \quad \frac{d}{dt} w_+ = -\lambda_0 w_+ + \Psi_1^+ w^{-1}, \quad w_+(0) = u_0^+.
\]
One may check that these solutions are expressed by (30) and \( w_-(t) < w_+(t) \) for all \( t \geq 0 \). \( \square \)

Note that if \( \Psi_i^+ = 0 \) (i.e., \( \Psi_i \equiv 0 \)) then (29) reads \( u_0^+ e^{-\lambda_0 t} \leq u(\cdot,0)/e_0 \leq u_0^+ e^{-\lambda_0 t} \). Define
\[
v(x,t) = e^{\lambda_0 t} u(x,t),
\]
see (26), and obtain the Cauchy’s problem
\[
\partial_t v = \Delta v + (\beta + \lambda_0) v + Q, \quad v(x,0) = u_0(x),
\] (32)
where \( Q := \sum_{i=1}^2 (-1)^{i+1} \Psi_i(x) v^{1-2i}(x,t) e^{2\lambda_0 t} \). Certainly, \( Q = Q_1 - Q_2 \), where
\[
Q_1(x,t) = \Psi_1(x) v^{-1}(x,t) e^{2\lambda_0 t}, \quad Q_2(x,t) = \Psi_2(x) v^{-3}(x,t) e^{4\lambda_0 t}.
\]

Lemma 6. Let \( v(x,t) \) be a positive solution of (32) in \( C_{t_0} = F \times [0, t_0) \), where \( \lambda_0 < 0 \), the functions \( u_0 > 0 \) and \( \Psi_i \geq 0 \) belong to \( C^{\infty} \) and (28) is satisfied. Then
(i) for any multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \) there exists a real \( C_\alpha \geq 0 \) such that
\[
|\partial_x^\alpha v(x,t)| \leq C_\alpha (1+t)^{|\alpha|}, \quad (x,t) \in C_{t_0}.
\]
(ii) for any multi-index \( \alpha \) there exist real \( \bar{Q}_{ia} \geq 0 \) (i = 1, 2) such that
\[
|\partial_x^\alpha Q_i(x,t)| \leq \bar{Q}_{ia} (1+t)^{|\alpha|} e^{2\lambda_0 t}, \quad (x,t) \in C_{t_0}, \quad i = 1, 2.
\] (33)
Proof. Using (30) and (31), we estimate the solution \( v(x,t) \) of (32) when (28) holds:

\[
v_\ast \leq v(x,t)/e_0(x) \leq v_+, \quad (x,t) \in F \times [0,\infty),
\]

where the constants are given by \( v_\ast = (\frac{(n_0)^4}{\Psi_j^2}/|\lambda_0|)^{\frac{1}{2}} \) and \( v_+ = (\frac{(n_0)^2}{\Psi_j^2}/|\lambda_0|)^{\frac{1}{2}} \).

(i) Denote for brevity by \( D_j = \partial_{x_j} \) \( (j = 1,2,\ldots,p) \) and \( D_\alpha = \partial^\alpha = D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_p} \). Differentiating (32) by \( x_1,\ldots,x_p \), we obtain the following PDEs for the functions \( p_\alpha(x,t) := \partial^\alpha v(x,t) \):

\[
\partial_t p_j = (\Delta + (\lambda_0 + \beta) \text{id})p_j + D_j(\beta)v + D_j(Q),
\]

\[
\partial_t p_{jk} = (\Delta + (\lambda_0 + \beta) \text{id})p_{jk} + D_j(\beta)p_k + D_k(\beta)p_j + D_{jk}(\beta)v + D_{jk}(Q),
\]

and so on, where \( 1 \leq j,k \leq p \) and

\[
D_j(Q) = \sum_i (-1)^{i+1} e^{2i\lambda_0 t} v^{-2i}(D_j(\Psi_i) v + (1 - 2i) \Psi_i p_j),
\]

\[
D_{jk}(Q) = \sum_i (-1)^{i+1} e^{2i\lambda_0 t} v^{-2i}(D_{jk}(\Psi_i) v + (1 - 2i) D_j(\Psi_i)p_k + (1 - 2i) D_k(\Psi_i)p_j
- 2i(1 - 2i)v^{-1}(\Psi_i p_j p_k + (1 - 2i)\Psi_i p_{jk})),
\]

and so on. Let us change unknown functions in equations (35), and so on:

\[
p_j = \tilde{p}_j e_0, \quad p_{jk} = \tilde{p}_{jk} e_0, \ldots
\]

Then in the same manner, as (31) have been obtained from (26), we get for \( j,k = 1,2,\ldots,p \)

\[
\partial_t \tilde{p}_j = \Delta \tilde{p}_j + 2g(\nabla \log e_0, \nabla \tilde{p}_j) + b_j + b_j/e_0 + D_j(\beta)v/e_0,
\]

\[
\partial_t \tilde{p}_{jk} = \Delta \tilde{p}_{jk} + 2g(\nabla \log e_0, \nabla \tilde{p}_{jk}) + \tilde{a}_{jk} + b_j/e_0 + D_j(\beta)v/e_0,
\]

and so on, where

\[
a = \sum_i (-1)^{i+1} (1 - 2i) \Psi_i v^{-2i} e^{2i\lambda_0 t}, \quad b_j = \sum_i (-1)^{i+1} D_j(\Psi_i) v^{-1-2i} e^{2i\lambda_0 t},
\]

\[
b_{jk} = \sum_i (-1)^{i+1} e^{2i\lambda_0 t} v^{-2i}(D_{jk}(\Psi_i) v^{-1} + (1 - 2i) D_j(\Psi_i)\tilde{p}_k + D_k(\Psi_i)\tilde{p}_j - 2i \Psi_i e_0/v \tilde{p}_j \tilde{p}_k)).
\]

From (31) and (36) – (38) we get the differential inequalities

\[
-a^+(t)|\tilde{p}_j| - b^+_j - \beta^+_j v_+ \leq \partial_t \tilde{p}_j - \Delta \tilde{p}_j - 2g(\nabla \log e_0, \nabla \tilde{p}_j) \leq a^+(t)|\tilde{p}_j| + b^+_j + \beta^+_j v_+,
\]

\[
-a^+(t)|\tilde{p}_{jk}| - b^+_j - \beta^+_j v_+ \leq \partial_t \tilde{p}_{jk} - \Delta \tilde{p}_{jk} - 2g(\nabla \log e_0, \nabla \tilde{p}_{jk}) \leq a^+(t)|\tilde{p}_{jk}| + b^+_j + \beta^+_j v_+
\]

for \( j = 1,2,\ldots,p \), where

\[
a^+(t) = \sum_i (2i - 1) \Psi_i^+(v_-)^{-2i} e^{2i\lambda_0 t}, \quad b^+_j = \sum_i ((v_-)^{-1-2i} \max \left| D_j(\Psi_i) \right|), \quad \beta^+_j = \max \left| D_j(\beta) \right|,
\]

\[
b^+_{jk} = \sum_i e^{2i\lambda_0 t} (v_-)^{-2i} \left( \max \left| D_{jk}(\Psi_i) / e_0^{2i} \right| v_+ + (2i - 1) \left( \max \left| D_j(\Psi_i) / e_0^{2i} \right| \tilde{p}_k + 2i (v_-)^{-1} \Psi_j \tilde{p}_j \tilde{p}_k \right) \right), \quad \beta^+_{jk} = \max \left| D_{jk}(\beta) \right|.
\]

By the maximum principle of Proposition 4 the estimate \( |\tilde{p}_j(x,t)| \leq \tilde{p}_j^+(t) \) is valid for any \( (x,t) \in C_\infty = F \times [0,\infty) \), where \( p_j^+(t) \) solves the Cauchy’s problem for the ODE:

\[
\frac{d}{dt} p_j^+ = a^+(t)|p_j^+| + b^+_j + \beta^+_j v_+, \quad p_j^+(0) = \tilde{p}_j^0 := \max_F |\tilde{p}_j(\cdot,0)|.
\]

As is known,

\[
p_j^+(t) = \tilde{p}_j^0 \exp \left( \int_0^t a^+(\tau) d\tau \right) + \int_0^t (b^+_j + \beta^+_j v_+) \exp \left( \int_s^t a^+(\tau) d\tau \right) ds .
\]
In view of (39), we have
\[ \int_0^\infty a^+(\tau) \, d\tau < \infty, \]
the above yield that for any \( j \in \{1, 2, \ldots, p\} \) there exists a real \( \tilde{C}_j > 0 \) such that
\[ |\tilde{p}_j(x,t)| \leq \tilde{C}_j(1+t), \quad (x,t) \in C_\infty. \]

In view of (37), this completes the proof of (i) for \(|\alpha| = 1\).

Similarly we obtain that for any \( j, k \in \{1, 2, \ldots, p\} \) there exists a real \( \tilde{C}_{j,k} \geq 0 \) such that
\[ |\tilde{p}_{j,k}(x,t)| \leq \tilde{C}_{j,k}(1+t)^2 \text{ for } (x,t) \in C_\infty. \]
By (37), we obtain claim (i) for \(|\alpha| = 2\). By induction with respect to \(|\alpha|\) we prove (i) for any \( \alpha \).

(ii) Estimates (33) for \(|\alpha| = 0\), \(|Q_1(x,t)| \leq (\max F \Psi_1)(e^{-t}+e^{2i\lambda_0 t})\), follow immediately from (34). Estimates (33) for \(|\alpha| = 1\) follow from claim (i), estimates (34) and equalities (35). By induction with respect to \(|\alpha|\) we prove (ii) for any \( \alpha \).

Theorem 3. The Cauchy’s problem (26) on \( F \), with \( \lambda_0 < 0 \) and the initial value \( u_0(x) \) satisfying (28), admits a unique smooth solution \( u(x,t) > 0 \) in the cylinder \( C_\infty = F \times [0, \infty) \).

Proof. The positive solution \( u(x,t) \) of (26) satisfies a priori estimates (29) on any cylinder \( C_t \), where it exists. By standard arguments, using the local theorem of the existence and uniqueness for semi-flows, we obtain that the solution can be uniquely prolonged on the cylinder \( C_\infty \). Then, by Lemma 6 all partial derivatives by \( x \) of \( u(x,t) \) exist in \( C_\infty \). Hence, \( u \) is smooth on \( C_\infty \).

3.3 Asymptotic behavior of solutions

Recall that \( \lambda_0 \) and \( e_0 > 0 \) are the least eigenvalue and the ground state of the operator (23).

Theorem 4. Let \( u > 0 \) be a smooth solution on \( C_\infty \) of (26) with \( \lambda_0 < 0 \) and the initial value \( u_0(x) \) satisfying (28) (see Theorem 3). Then there exists a solution \( \tilde{u} \) on \( C_\infty \) of the linear PDE
\[ \partial_t \tilde{u} = \Delta \tilde{u} + (\beta(x) + \lambda_0) \tilde{u} \]
(40)
such that for any \( \alpha \in (0, \min \{\lambda_1 - \lambda_0, 2|\lambda_0|\}) \) and any \( k \in \mathbb{N} \)
\[ (i) \quad u = e^{-\lambda_0 t}(\tilde{u} + \theta(x,t)), \quad (ii) \quad \nabla \log u = \nabla \log e_0 + \theta_1(x,t), \]
where \( \|\theta(\cdot, t)\| \infty = O(e^{-at}) \) and \( \|\theta_1(\cdot, t)\| \infty = O(e^{-at}) \) as \( t \to \infty \).

Proof. (i) Let \( G_0(t,x,y) \) be the fundamental solution of (11), called the heat kernel. As is known, \( G_0(t,x,y) = \sum_j e^{(\lambda_0 - \lambda_j) t} e_j(x) e_j(y) \). Due to the Duhamel’s principle, the solution \( v = e^{\lambda_0 t}u \) of the Cauchy’s problem (32) satisfies the nonlinear integral equation
\[ v(x,t) = \int_F G_0(t,x,y) u_0(y) \, dy + \int_0^t \left( \int_F G_0(t-\tau,x,y) Q(y,\tau) \, dy \right) \, d\tau. \]
(41)
Expand \( v \), \( u_0 \) and \( Q \) into Fourier series by eigensystem \( \{e_j\}\):
\[ v(x,t) = \sum_{j=0}^\infty v_j(t) e_j(x), \quad v_j(t) = (v(\cdot, t), e_j)_0 = \int_F v(y,t) e_j(y) \, dy, \]
\[ u_0(x) = \sum_{j=0}^\infty u_j^0 e_j(x), \quad u_j^0 = (u_0, e_j)_0 = \int_F u_0(y) e_j(y) \, dy, \]
\[ Q(x,t) = \sum_{j=0}^\infty q_j(t) e_j(x), \quad q_j(t) = (Q(\cdot, t), e_j)_0 = \int_F Q(y,t) e_j(y) \, dy. \]
(42)
Then we obtain from (41):
\[ v_j(t) = u_j^0 e^{(\lambda_0 - \lambda_j) t} + \int_0^t e^{(\lambda_0 - \lambda_j)(t-\tau)} q_j(\tau) \, d\tau \quad (j = 0, 1, \ldots). \]
(43)
Substituting $v_j(t)$ of (43) into (42), we represent $v$ in the form $v = \tilde{u} + \theta$, where

$$
\tilde{u} = \tilde{u}_0 e_0 + \sum_{j=1}^{\infty} u_j^0 e^{(\lambda_0 - \lambda_j)t} e_j, \quad \tilde{u}_0 = u_0 + \int_0^\infty q_0(\tau) \, d\tau,
$$

$$
\theta = - \left( \int_0^\infty q_0(\tau) \, d\tau \right) e_0 + \sum_{j=1}^{\infty} \tilde{v}_j e_j, \quad \tilde{v}_j = \int_0^t e^{(\lambda_0 - \lambda_j)(t-\tau)} q_j(\tau) \, d\tau.
$$

Observe that $\tilde{u}$ solves (40) with the initial condition $\tilde{u}(\cdot, 0) = u_0 + \left( \int_0^\infty q_0(\tau) \, d\tau \right) e_0$.

Let us take $k \in \mathbb{N}, \ l = \left\lceil \frac{p}{4} + k/2 \right\rceil + 1$ and $\gamma < \lambda_0$. Using assumption $u_0 \in C^\infty(F)$ and the fact that $Q(\cdot, t) \in C^\infty(F)$ for any $t \geq 0$, we may consider the functions $w_0 := (H - \gamma \text{id})_t u_0$ and $P(\cdot, t) := (H - \gamma \text{id})_t Q(\cdot, t)$, which have the same properties: $w_0 \in C^\infty(F)$ and $P(\cdot, t) \in C^\infty(F)$ for any $t \geq 0$. Let us represent

$$
(u_0, e_j)_0 e_j = ((H - \gamma \text{id})_t - l w_0, e_j)_0 e_j = (w_0, (H - \gamma \text{id})_t - l e_j)_0 e_j
$$

$$
= (w_0, e_j)_0 \frac{e_j}{\lambda_j - \gamma} = (H - \gamma \text{id})_t - l ((w_0, e_j)_0 e_j).
$$

Similarly, we obtain

$$(Q(\cdot, t), e_j)_0 e_j = (H - \gamma \text{id})_t - l ((P(\cdot, t), e_j)_0 e_j).$$

Using (15) and taking into account that the operator $(H - \gamma \text{id})_t$ acts continuously in $L_2$ and that the series in (44) and (45) converge in $L_2$, we obtain the representations:

$$
\sum_{j=1}^{\infty} u_j^0 e^{(\lambda_0 - \lambda_j)t} e_j = (H - \gamma \text{id})_t - l \sum_{j=1}^{\infty} e^{(\lambda_0 - \lambda_j)t} (w_0, e_j)_0 e_j,
$$

$$
\sum_{j=1}^{\infty} \tilde{v}_j(t)e_j = (H - \gamma \text{id})_t - l \int_0^t \left( \sum_{j=1}^{\infty} e^{(\lambda_0 - \lambda_j)(t-\tau)} (P(\cdot, t), e_j)_0 e_j \right) d\tau.
$$

By the Elliptic Regularity Theorem and the Sobolev Embedding Theorem (see Section 3.1), the operator $(H - \gamma \text{id})_t$ acts continuously from $L_2$ into $C^k$. Then we have

$$
\| \sum_{j=1}^{\infty} u_j^0 e^{(\lambda_0 - \lambda_j)t} e_j \|_{C^k} \leq \|(H - \gamma \text{id})_t - l\|_{B(L_2, C^k)} \cdot \| \sum_{j=1}^{\infty} e^{(\lambda_0 - \lambda_j)t} (w_0, e_j)_0 e_j \|_0
$$

$$
= \|(H - \gamma \text{id})_t - l\|_{B(L_2, C^k)} \cdot \| \sum_{j=1}^{\infty} e^{2(\lambda_0 - \lambda_j)t} (w_0, e_j)_0^2 \|_{0}^{1/2} \leq \|(H - \gamma \text{id})_t - l\|_{B(L_2, C^k)} \cdot \| \sum_{j=1}^{\infty} e^{(\lambda_0 - \lambda_j)(t-\tau)} (P(\cdot, t), e_j)_0 e_j \|_0
$$

$$
\leq \|(H - \gamma \text{id})_t - l\|_{B(L_2, C^k)} \int_0^t \| \sum_{j=1}^{\infty} e^{(\lambda_0 - \lambda_j)(t-\tau)} (P(\cdot, t), e_j)_0 e_j \|_0 \, d\tau
$$

$$
\leq \|(H - \gamma \text{id})_t - l\|_{B(L_2, C^k)} \int_0^t \| \sum_{j=1}^{\infty} e^{2(\lambda_0 - \lambda_j)(t-\tau)} (P(\cdot, t), e_j)_0^2 \|_{0}^{1/2} \, d\tau
$$

$$
\leq \|(H - \gamma \text{id})_t - l\|_{B(L_2, C^k)} \int_0^t e^{(\lambda_0 - \lambda_j)(t-\tau)} \| P(\cdot, t) \|_0 \, d\tau.
$$

On the other hand, by Lemma (ii),

$$
\| P(\cdot, t) \|_0 \leq \sqrt{\text{Vol}(F, g)} \| (H - \gamma \text{id})_t^1 (Q(\cdot, t)) \|_{C^0} \leq \tilde{Q}(1 + t)^{2l} e^{2\lambda_0 t}
$$

for some $\tilde{Q} \geq 0$. Then, continuing (47), we find

$$
\int_0^t e^{(\lambda_0 - \lambda_1)(t-\tau)} \| P(\cdot, t) \|_0 \, d\tau \leq \tilde{Q} \int_0^t e^{(\lambda_0 - \lambda_1)(t-\tau)} (1 + \tau)^{2l} e^{2\lambda_0 t} \, d\tau
$$

$$
< \tilde{Q} e^{(\lambda_0 - \lambda_1)t} (1 + t)^{2l} \int_0^t e^{(\lambda_1 - \lambda_0 + 2\lambda_0) \tau} \, d\tau = \tilde{Q}(1 + t)^{2l} \left\{ \begin{array}{ll}
\frac{e^{2\lambda_0 t} - e^{(\lambda_0 - \lambda_1)t}}{\lambda_1 - \lambda_0 + 2\lambda_0} & \text{if } 2\lambda_0 \neq \lambda_0 - \lambda_1, \\
e^{(\lambda_0 - \lambda_1)t} & \text{if } 2\lambda_0 = \lambda_0 - \lambda_1.
\end{array} \right.
$$
From (14) – (18) we get claim (i).

(ii) From (14) and (14) we obtain

\[ u = e^{-\lambda t}(\tilde{u}_0^0 e_0 + \tilde{\theta}(\cdot, t)), \quad \nabla u = e^{-\lambda t}(\tilde{u}_0^0 \nabla e_0 + \nabla \tilde{\theta}(\cdot, t)), \]

where \( \tilde{\theta}(\cdot, t) = \theta(\cdot, t) + \sum_{j=1}^{\infty} u_j^0 e^{(\lambda_j - \lambda)t} e_j \).

In view of (15), \( \| \tilde{\theta}(\cdot, t) \|_{L^\infty} = O(e^{-\alpha t}) \) for any \( k \in \mathbb{N} \). Furthermore, since \( \tilde{u}(\cdot, 0) > 0 \) on \( F \), then \( \tilde{u}_0^0 = (\tilde{u}(\cdot, 0), e_0) > 0 \). Using

\[ w(\cdot, t, \tau) := \tau u(\cdot, t) + (1 - \tau) \tilde{u}_0^0 e^{-\lambda t} e_0 = e^{-\lambda t}(\tilde{u}_0^0 e_0 + \tau \tilde{\theta}(\cdot, t)), \]

we have

\[
\theta_1(\cdot, t) = \nabla \log u(\cdot, t) - \nabla \log e_0 = \int_0^1 \frac{\partial}{\partial \tau} \left( \nabla \log w(\cdot, t, \tau) \right) d\tau = \int_0^1 \left( \frac{\nabla \tilde{\theta}(\cdot, t)}{\tilde{u}_0^0 e_0 + \tau \tilde{\theta}(\cdot, t)} - \frac{\tilde{\theta}(\cdot, t)(\tilde{u}_0^0 \nabla e_0 + \tau \nabla \tilde{\theta}(\cdot, t))}{(\tilde{u}_0^0 e_0 + \tau \tilde{\theta}(\cdot, t))^2} \right) d\tau.
\]

By the above, and the fact that \( \inf\{ |\tilde{u}_0^0 e_0 + \tau \tilde{\theta}(\cdot, t)| : \ x \in F, \ t \in [t_0, \infty), \ \tau \in [0, 1] \} > 0 \) holds for \( t_0 > 0 \) large enough, follows claim (ii).

### 3.4 Nonlinear heat equation with parameter

Let the metric \( g \), the connection \( \nabla \) and the laplacian \( \Delta \) smoothly depend on \( q \), which belongs to an open subset \( Q \) of \( \mathbb{R}^n \). Consider the Cauchy’s problem on a closed Riemannian manifold \( (F, g) \):

\[
\partial_t u = \Delta u + f(x, u, q), \quad u(x, 0, q) = u_0(x, q).
\]

(49)

Here, \( f \) is defined in the domain \( D = F \times I \times Q \), where \( I \subseteq \mathbb{R} \) is an interval, and \( u_0 \) is defined in the domain \( \tilde{D} = F \times Q \) and satisfies the condition: \( u_0(x, q) \in I \) for any \( x \in F \) and \( q \in Q \).

**Proposition 5.** Suppose that \( f \in C^\infty(D) \), \( u_0 \in C^\infty(\tilde{D}) \), all partial derivatives of \( f \) and \( u_0 \) by \( x \), \( u \) and \( q \) are bounded in \( D \) and \( \tilde{D} \), and for any \( q \in Q \) there exists an unique solution \( u : F \times [0, T] \times Q \rightarrow \mathbb{R} \) of the Cauchy’s problem (14) such that all its partial derivatives by \( x \) are bounded in \( F \times [0, T] \times Q \). Then \( u(\cdot, t, \cdot) \in C^\infty(F \times Q) \) for any \( t \in [0, T] \).

**Proof** is standard, we give it for the convenience of a reader. As is known, \( u(\cdot, t, q) \in C^\infty(F) \) for any \( q \in Q \), \( t \in [0, T] \). We should prove the smooth dependence on \( q \) of the solution \( u(x, t, q) \) and of all its partial derivatives by \( x \) for any fixed \( t \in [0, T] \). We shall divide the proof into several steps.

**Step 1:** the continuous dependence of \( u(x, t, q) \) in \( q \). To show this, take \( q_0 \in Q \) and denote by \( v(x, t, q) = u(x, t, q) - u(x, t, q_0) \) and \( v_0(x, q) = u_0(x, q) - u_0(x, q_0) \). Let us represent:

\[
f(x, u(x, t, q), q) - f(x, u(x, t, q_0), q_0) = F(x, t, q)v(x, t, q) + G(x, t, q) \cdot (q - q_0),
\]

where

\[
F(x, t, q) = \int_0^1 \partial_{q} f(x, u(x, t, q_0) + \tau v(x, t, q), q_0 + \tau(q - q_0)) d\tau,
\]

\[
G(x, t, q) = \int_0^1 \text{grad}_q f(x, u(x, t, q_0) + \tau v(x, t, q), q_0 + \tau(q - q_0)) d\tau.
\]

Then the function \( v(x, t, q) \) is a solution of the Cauchy’s problem:

\[
\partial_t v = \Delta v + F(x, t, q)v + G(x, t, q) \cdot (q - q_0), \quad v|_{t=0} = G_0(x, q) \cdot (q - q_0),
\]

where \( G_0(x, q) = \int_0^1 \text{grad}_q v_0(x, q_0 + \tau(q - q_0)) d\tau \). Then by the maximum principle of Proposition [14]

\[
|v(x, t, q)| \leq w(t, q) \quad \forall (x, t, q) \in F \times [0, T] \times Q,
\]

(50)
where \( w(t,q) \) is the solution of the following Cauchy’s problem for the ODE:

\[
\partial_t w = \tilde{F}|w| + \tilde{G} |q - q_0|, \quad w(0,q) = \tilde{G}_0 |q - q_0|
\]  (51)

with \( \tilde{F} = \sup_{F \times [0,T] \times Q} |F|, \tilde{G} = \sup_{F \times [0,T] \times Q} |G| \) and \( \tilde{G}_0 = \sup_{F \times Q} |G_0| \). Then, from (50) and (51) we get

\[
|v(x,t,q)| \leq \left( \tilde{G}_0 e^{\tilde{F}t} + (e^{\tilde{F}t} - 1)(\tilde{G}/\tilde{F}) \right) |q - q_0|, \quad (x,t,q) \in F \times [0,T] \times Q,
\]

which implies the claim of Step 1.

**Step 2**: all the partial derivatives of \( u(x,t,q) \) by \( x \) are continuous in \( q \). Differentiating subsequently by \( x \) both sides of the equation and of the initial condition in (49), we have the following Cauchy’s problems for \( p_\alpha = \partial_x^\alpha u \) (\( \alpha \) is the multi-index):

\[
\partial_t p_i = \Delta p_i + \partial_x f(x,u(x,t,q),q) p_i + \partial^2_x f(x,u(x,t,q),q) p_i, \quad p_i|_{t=0} = \delta_x u_0(x,q),
\]

\[
\partial_t p_{i,j} = \Delta p_{i,j} + \partial_x f(x,u(x,t,q),q) p_{i,j} + \partial^2_x f(x,u(x,t,q),q) p_i p_j + \partial^3_x f(x,u(x,t,q),q) p_{i,j} + \partial^4_x f(x,u(x,t,q),q) p_{i,j}, \quad p_{i,j}|_{t=0} = \delta_x^i u_0(x,q),
\]

and so on. Applying the claim of Step 1 to these Cauchy’s problems, we prove the claim of Step 2.

**Step 3**: \( u(x,t,q) \) is smooth with respect to \( q \). Take \( q_0 \in Q \) and consider the divided difference

\[
\delta_y(x,t,s) = \frac{1}{s} \left( u(x,t,q_0 + sy) - u(x,t,q_0) \right) \quad (y \in \mathbb{R}^n, s \in \mathbb{R}).
\]

Denote by \( \delta^0_y(x,s) = \frac{u(x,q_0 + sy) - u(x,q_0)}{s} \). As in the Step 1, we obtain the Cauchy’s problem for \( \delta_y \)

\[
\partial_t \delta_y = \Delta \delta_y + \tilde{F}(x,t,s) \delta_y + \tilde{G}(x,t,s) \cdot y, \quad \delta_y|_{t=0} = \delta^0_y(x,s),
\]

\[
\tilde{F}(x,t,s) = \int_0^1 \partial_x f(x,u(x,t,q_0) + \tau (u(x,t,q_0 + sy) - u(x,t,q_0))) \, d\tau, \quad q_0 + sy \in \mathbb{R}^n.
\]

Applying to the Cauchy’s problem (53), the claim of Step 1, we conclude that \( \delta_y(x,t,s) \) is continuous by \( s \) at the point \( s = 0 \), that is there exists the directional derivative \( d_y(x,t,q_0) = \lim_{s \to 0} \delta_y(x,t,s) \cdot y \). Moreover, \( d_y(x,t,q) \) is the solution of Cauchy’s problem

\[
\partial_t d_y = \Delta d_y + \partial_x f(x,u(x,t,q),q)d_y + \operatorname{grad}_q f(x,u(x,t,q)) \cdot y, \quad d_y|_{t=0} = \operatorname{grad}_q u_0(x,q) \cdot y.
\]

Applying the claim of Step 1 to this Cauchy’s problem, we find that \( d_y(x,t,q) = \operatorname{grad}_q u(x,t,q) \cdot y \) continuously depends on \( q \) for any \( y \in \mathbb{R}^n \). Thus, \( u(x,t,q) \) is \( C^1 \)-regular in \( q \). Applying the above arguments to the Cauchy’s problem (54), we conclude that \( u(x,t,q) \) belongs to \( C^2 \) with respect to \( q \). Finally, we prove by induction that \( u(x,t,q) \) is smooth in \( q \).

**Step 4**: Applying all the arguments of Step 3 to the Cauchy’s problems (52) and so on, we prove that all derivatives of \( u(x,t,q) \) in \( x \) smoothly depend on \( q \). \( \square \)

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