High-Dimensional Menger-Type Curvatures - Part II: d-Separation and a Menagerie of Curvatures

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Abstract

We estimate $d$-dimensional least squares approximations of an arbitrary $d$-regular measure $\mu$ via discrete curvatures of $d + 2$ variables. The main result bounds the least squares error of approximating $\mu$ (or its restrictions to balls) with a $d$-plane by an average of the discrete Menger-type curvature over a restricted set of simplices. Its proof is constructive and even suggests an algorithm for an approximate least squares $d$-plane. A consequent result bounds a multiscale error term (used for quantifying the approximation of $\mu$ with a sufficiently regular surface) by an integral of the discrete Menger-type curvature over all simplices. The preceding paper (part I) provided the opposite inequalities of these two results. This paper also demonstrates the use of a few other discrete curvatures which are different than the Menger-type curvature. Furthermore, it shows that a curvature suggested by Léger (Annals of Math, 149(3), p. 831-869, 1999) does not fit within our framework.

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1 Introduction

We propose an approximate construction for the least squares $d$-plane of a $d$-regular measure (defined below) and an estimate for the corresponding least squares error via Menger-type curvatures as well as other curvatures.

There are two different kinds of motivation for this investigation. The first one concerns clustering of data points sampled around intersecting $d$-planes [2, 8] (and even more general $d$-dimensional manifolds [1]). Here a possible approach is to assign local discrete curvatures (of at least $d + 2$ points) that distinguish the different clusters (i.e., they are “sufficiently small” within each cluster and “large” for points of mixed clusters). Indeed, some of the results of this paper and the preceding one show that within each cluster the discrete curvatures discussed here (or its variants [3, 17]) are tightly controlled on average by the $d$-dimensional least squares error of the cluster.

Another use of the current study is in the constructive approximation of sufficiently regular $d$-dimensional surfaces for $d$-regular measures [11] [5] [6], and we will expand on it later.

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Basic Setting

Our current setting includes a real separable Hilbert space $H$ of possibly infinite dimension, an intrinsic dimension $d \in \mathbb{N}$, where $d < \dim(H)$, and a $d$-regular measure $\mu$ on $H$ (or equivalently, a $d$-dimensional Ahlfors regular measure). That is, a locally finite Borel measure $\mu$ with a constant $C \geq 1$ such that for all $x \in \text{supp}(\mu)$ and $0 < t \leq \text{diam}(\text{supp}(\mu))$ with the corresponding closed ball $B(x, t)$:

$$C^{-1} \cdot t^d \leq \mu(B(x, t)) \leq C \cdot t^d. \quad (1)$$

The smallest constant $C$ satisfying equation (1) is denoted by $C_{\mu}$ and is called the regularity constant of $\mu$.

This abstract setting allows estimates that are independent of the ambient dimension. In Section 7 we discuss possible extensions of some of the estimates below to other settings.

Simplices and their Menger-type Curvature

We often work with $(d + 1)$-simplices in $H$. We represent such a simplex with vertices $x_0, \ldots, x_{d+1}$ by $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$. We usually do not distinguish between the $(d + 1)$-simplex and its representation $X$. The following functions of $X$ will be frequently used. The largest and smallest edge lengths of $X$ are denoted by $\text{diam}(X)$ and $\text{min}(X)$ respectively. The $d$-content $M_{d+1}(X)$ is the multiplication of the $(d + 1)$-volume of the simplex $X$ by $(d + 1)!$. Equivalently, it is the $(d + 1)$-volume of any parallelotope generated by the vertices of $X$. The polar sine of $X = (x_0, \ldots, x_{d+1})$ at the vertex $x_i$, $0 \leq i \leq d + 1$, is the function

$$p_{d \sin x_i}(X) = \prod_{0 \leq j \leq d + 1, j \neq i} \|x_j - x_i\|,$$

where it is zero if the denominator is zero. When $d = 1$, the polar sine (hereafter abbreviated to p-sine) reduces to the absolute value of the ordinary sine of the angle between two vectors.

The primary discrete curvature we have worked with is

$$c_{\text{MT}}(X) = \sqrt{\frac{1}{d + 2} \cdot \frac{1}{\text{diam}(X)^{d(d+1)}} \sum_{i=0}^{d+1} p_{d \sin x_i}^2(X)},$$

where it is zero if the denominator is zero. Other possible curvatures are introduced in Section 6 as well as in [17].

Least Squares $d$-Planes via Curvatures

We fix a location $x \in \text{supp}(\mu)$ and a scale $0 < t \leq \text{diam}(\text{supp}(\mu))$ with the corresponding ball $B = B(x, t)$ in $H$ (by a ball we always mean the closed ball). The scaled least squares error of approximating $\mu$ by $d$-planes at $B$ has the form

$$\beta_2(B) = \sqrt{\inf_{d\text{-planes}} B \int_B \left(\text{dist}(y, L) / \text{diam}(B)\right)^2 \frac{d\mu(y)}{\mu(B)}},$$

where it is zero if $\mu(B) = 0$. If $B = B(x, t)$, then we often denote $\beta_2(B)$ by $\beta_2(x, t)$. Any $d$-plane $L$ that minimizes the above infimum is referred to as a least squares $d$-plane for the restriction of
μ onto B, i.e., μ|B. Our main result concerns the estimation of β2(B) by Menger-type curvatures. Before stating it we develop the specific integrals of curvatures to be used and review a previous result.

We arbitrarily fix 0 < λ < 2 and define the following set of simplices in B = B(x, t) with sufficiently large edge lengths

\[ U_λ(B) = \left\{ X \in B^{d+2} : \min(X) \geq λ \cdot t \right\}. \]  

(2)

We then define the following squared curvature of μ at B = B(x, t) as a function of x and t:

\[ c_{MT}^2(x, t, λ) = \int_{U_λ(B(x, t))} c_{MT}^2(X) dμ^{d+2}(X). \]

In [16] we controlled from below the local least squares approximation error for μ as follows.

**Proposition 1.1.** There exists a constant \( C_0 = C_0(d, C_μ) \geq 1 \) such that

\[ c_{MT}^2(x, t, λ) \leq \frac{C_0}{λ^{d(d+1)+4}} \cdot \frac{β^2_2(x, t) \cdot μ(B(x, t))}{t} \]

for all \( λ > 0 \), \( x \in \text{supp}(μ) \), and \( 0 < t \leq \text{diam}(\text{supp}(μ)) \).

Here we establish the following opposite inequality. Its constructive proof also suggests an algorithm for an approximate least squares d-plane.

**Theorem 1.1.** There exist constants \( 0 < λ_0 = λ_0(d, C_μ) < 2 \) and \( C_1 = C_1(d, C_μ) \geq 1 \) such that

\[ β^2_2(x, t) \cdot μ(B(x, t)) \leq C_1 \cdot c_{MT}^2(x, t, λ_0), \]

for all \( x \in \text{supp}(μ) \) and \( 0 < t \leq \text{diam}(\text{supp}(μ)) \).

In [17] we extend both Proposition 1.1 and Theorem 1.1 to more general measures (e.g., truncated Gaussian distributions around Lipschitz graphs). We actually use there different scalings of both the discrete Menger-type curvature and the underlying integral, which allow us to replace \( U_λ(B) \) by \( B^{d+2} \).

**Underlying Regular Surfaces via Curvatures**

The problem of approximating μ by a d-plane and estimating the corresponding least squares error extends to approximating μ by a “sufficiently regular d-dimensional surface” and estimating the “cumulative error” of multiscale least squares approximations. We demonstrate those notions for an arbitrarily fixed ball \( B \subseteq H \).

The cumulative error of multiscale least squares approximation, or equivalently the Jones-type flatness [11, 5, 6, 16], is defined for the restriction of μ to B as follows:

\[ J_2(μ|B) = \int_B \int_0^{\text{diam}(B)} β^2_2(x, t) \frac{dt}{t} dμ(x). \]

The technical notion of a sufficiently smooth d-dimensional surface, or more precisely an \( ω \)-regular surface for an \( A_1 \) weight \( ω \), is presented in [5, 6] (see also [16, Section 6]).

David and Semmes [5, 6] showed that there exists such a surface containing the restriction of the support of μ to B if and only if there is a finite uniform bound on the quantities \( \{ J_2(μ|B')/μ(B') \} \) \( B' \subseteq B \).

If any of the two equivalent conditions hold, then \( μ|B \) is called uniformly rectifiable.
Uniform rectifiability thus asks for a nice parametrization of $\mu|_B$ (by a sufficiently regular $d$-dimensional surface). In fact, the supporting theory [3, 6] suggests a multiscale construction of such a parametrization by least squares approximations at different balls in $B$ (in the spirit of [11]). Furthermore the smoothness of the underlying surface, i.e., the sizes of its parameters, can be controlled by the least uniform bound of $\{J_2(\mu|_{B'})/\mu(B')\}_{B' \subseteq B}$. This is analogous to characterizing the smoothness of functions by quantities based on wavelet coefficients or similar Littlewood-Paley estimates as demonstrated in [4] and [6, Subsection 1.3].

While uniform rectifiability is tailored for the restricted setting of $d$-regular measures, it is a rich ground for various notions of quantitative geometry, which can be further extended and applied in more general and practical settings. For example, insights of curve construction of uniform rectifiability have been used in [13, 14, 15] for rather practical measures. The idea was to avoid approximation of the whole support of the given measure by a curve (as done in uniform rectifiability), but parametrize only a large fraction of the support [13], i.e., allowing an outlier component, and moreover constructing a strip around the main curve [14, 15] in order to “cover” a noisy component around it.

Our current work extends uniform rectifiability by using discrete curvatures instead of least squares approximations of different scales and locations. For this purpose we use the following squared Menger-type curvature of $\mu|_B$:

$$c^2_{\text{MT}}(\mu|_B) = \int_{B^{d+2}} c^2_{\text{MT}}(X) \, d\mu^{d+2}(X).$$

We also extend this definition as follows. We arbitrarily fix $0 < \lambda \leq 1$ and define the following set of well scaled simplices in $B$:

$$W_\lambda(B) = \{ X \in B^{d+2} : \min(X) \geq \lambda \cdot \text{diam}(X) > 0 \}. \quad (3)$$

The squared Menger-type curvature of $\mu|_B$ with respect to the parameter $\lambda$ has the form

$$c^2_{\text{MT}}(\mu|_B, \lambda) = \int_{W_\lambda(B)} c^2_{\text{MT}}(X) \, d\mu^{d+2}(X).$$

Clearly, for any $\lambda > 0$ we have that

$$c^2_{\text{MT}}(\mu|_B, \lambda) \leq c^2_{\text{MT}}(\mu|_B).$$

In [16] we established the following result.

**Theorem 1.2.** There exists a constant $C_2 = C_2(d, C_\mu) \geq 1$ such that

$$c^2_{\text{MT}}(\mu|_B) \leq C_2 \cdot J_2(\mu|_{6B})$$

for all balls $B \subseteq H$.

Using the constant $\lambda_0$ of Theorem [11] we establish here an opposite inequality to Theorem [12] as follows.

**Theorem 1.3.** There exists a constant $C_3 = C_3(d, C_\mu) \geq 1$ such that

$$J_2(\mu|_B) \leq C_3 \cdot c^2_{\text{MT}}(\mu|_{3B}, \lambda_0/2)$$

for all balls $B \subseteq H$ with $\text{diam}(B) \leq \text{diam}(\text{supp}(\mu))$.

We thus obtain that the Jones-type flatness of $\mu|_B$ can be replaced by the squared Menger-type curvature of $\mu|_B$. That is, multiscale least squares approximations for uniform rectifiability, in particular for surface reconstruction, could be replaced by using Menger-type curvatures of various kinds of simplices.
A Curvature of Léger and Related Methodology

A previous curvature for studying the rectifiability and uniform rectifiability of measures for \( d \geq 1 \) was proposed by Léger [12]. However, his analysis was done only for \( d = 1 \) and when we try to generalize it to \( d > 1 \) we find that his curvature controls a quantity which characterizes a property weaker than uniform rectifiability (see Subsections 6.2 and 6.3).

Despite the fact that Léger’s curvature is unsuitable for our purposes, his basic analysis is instrumental in this paper. Similar to his work in [12], the main ingredients of our analysis include repeated applications of both Fubini’s Theorem and Chebychev’s inequality as well as various metric inequalities and identities. However, we needed to develop additional analytic and combinatoric propositions for the case where \( d > 1 \). In particular, we have generalized the separation of points by pairwise distances employed in [12] to a \( d \)-dimensional separation of simplices (see Section 3). This \( d \)-separation plays a fundamental role in the proof of Theorem 1.1, which is the main result of this paper. Weaker notions of \( d \)-separation have been applied earlier by David and Semmes [5, Lemma 5.8] and Tolsa [25, Lemma 8.2].

Organization of the Paper

This paper is organized as follows. Section 2 provides the preliminary notation and some related elementary propositions. Section 3 states and proves a geometric proposition regarding the \( d \)-dimensional separation of points in the support of an arbitrary \( d \)-regular measure \( \mu \) on \( H \). Sections 4 and 5 contain the proofs of Theorem 1.1 and Theorem 1.3 respectively. Section 6 discusses other possible curvatures and their relation to the curvature \( c_{MT} \), as well as some problems with the curvature suggested by Léger [12]. Finally, Section 7 suggests new and mostly open directions to extend this work.

2 Basic Notation and Definitions

2.1 Main Context and Notational Conventions

We fix a real separable Hilbert space \( H \), and denote its inner product, induced norm, and the dimension (possibly infinite) by \( \langle \cdot , \cdot \rangle \), \( \| \cdot \| \), and \( \dim(H) \) respectively. For \( m \in \mathbb{N} \), we denote the Cartesian product of \( m \) copies of \( H \) by \( H^m \).

If \( A \subset H \), then we denote its diameter by \( \text{diam}(A) \). If \( \mu \) is a measure on \( H \), then its support is denoted by \( \text{supp}(\mu) \). For \( \mu \) and \( A \subset H \), we denote the restriction of \( \mu \) to \( A \) by \( \mu|_A \).

We denote the closed ball in \( H \), centered at \( x \in H \) and of radius \( t \), by \( B(x,t) \). If both the center and radius are indeterminate, then we use the notation \( B \).

We summarize some notational conventions as follows. We typically denote scalars with values at least 1 by upper-case plain letters, e.g., \( C \); arbitrary integers by lower case letter, e.g., \( i, j \) and large integers by \( M \) and \( N \); and arbitrary real numbers by lower-case Greek or lower-case letters, e.g., \( \rho, r \).

We reserve \( x, y, \) and \( z \) to denote elements of \( H \); \( X \) to denote elements of \( H^m \) for \( m \geq 3 \); \( L \) for a complete affine subspace of \( H \) (possibly a linear subspace); \( V \) to denote a complete linear subspace of \( H \); \( B \) to denote closed balls in \( H \); and \( t \) for arbitrary length scales, in particular, radii of balls (we always assume that \( t \in \mathbb{R} \), even when writing \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \) and when \( \text{supp}(\mu) \) is unbounded).

We say that a real-valued function \( f \) is controlled by a real-valued function \( g \), which we denote by \( f \preceq g \), if there exists a positive constant \( C \) such that \( f \leq C \cdot g \). Similarly, \( f \) is comparable to \( g \),
denoted by \( f \approx g \), if \( f \lesssim g \) and \( g \lesssim f \). The constants of control or comparability may depend on some arguments of \( f \) and \( g \), which we make sure to indicate if unclear from the context.

More specific notation and definitions commonly used throughout the paper, as well as related propositions, are described in the following subsections according to topic.

### 2.2 Elements of \( H^{n+1} \) and Corresponding Notation

Fixing \( n \geq 1 \), we denote an element of \( H^{n+1} \) by \( X = (x_0, \ldots, x_n) \). For \( 0 \leq i \leq n \), we let \( (X)_i = x_i \) denote the projection of \( X \) onto its \( i \)th \( H \)-valued coordinate. The \( 0 \)th coordinate \( (X)_0 = x_0 \) is special in many of our calculations.

For \( 0 \leq i \leq n \) and \( X = (x_0, \ldots, x_n) \in H^{n+1} \), let \( X(i) \) be the following element of \( H^n \):

\[
X(i) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),
\]

that is, \( X(i) \) is the projection of \( X \) onto \( H^n \) that eliminates its \( i \)th coordinate. Furthermore, for \( n \geq 2 \) and \( 0 \leq i < j \leq n \), let \( X(i; j) \) be the following element of \( H^{n-1} \):

\[
X(i; j) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).
\]

If \( 1 \leq i \leq n \), \( X(i) \in H^n \), and \( y \in H \), we form \( X(y, i) \in H^{n+1} \) as follows:

\[
X(y, i) = (x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n).
\]

Finally, if \( y \in H \) and \( z \in H \), \( n \geq 2 \), and \( 1 \leq i < j \leq n \), then we define the elements \( X(y, i; j), X(i; z, j) \in H^n \) and \( X(y, i; z, j) \in H^{n+1} \) by the following formulas:

\[
X(y, i; j) = (x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),
\]

\[
X(i; z, j) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n),
\]

and

\[
X(y, i; z, j) = (x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n).
\]

**Remark 1.** We usually take \( n = d + 1 \), and without referring directly to \( X \in H^{d+2} \) we often denote elements of \( H^{d+1} \) by \( X(i) \) for some \( 1 \leq i \leq d + 1 \) and elements of \( H^d \) by \( X(i; j) \) where \( 0 \leq i < j \leq d \).

### 2.3 Notation for Subsets of \( H \)

For a ball \( B(x, r) \) and \( \gamma > 0 \), let \( \gamma \cdot B(x, r) = B(x, \gamma \cdot r) \). If \( x \in H \) and \( r_1, r_2 > 0 \), then we define the annulus

\[
A(x, r_1, r_2) = \{ y \in H : r_1 < \| x - y \| \leq r_2 \} = B(x, r_2) \setminus B(x, r_1).
\]

For \( n \geq 1 \) and \( X \in H^{n+1} \), we say that \( X \) is non-degenerate if the set \( \{ x_1 - x_0, \ldots, x_n - x_0 \} \) is linearly independent, and we say that \( X \) is degenerate otherwise. For \( X \in H^{d+1} \) let \( L[X] \) denote the affine subspace of \( H \) of minimal dimension containing the vertices of \( X \), i.e., the coordinates of \( X \), and let \( V[X] \) be the linear subspace parallel to \( L[X] \).

If \( V \) is a linear subspace of \( H \), we denote its orthogonal complement by \( V^\perp \). If \( L \) is a complete affine subspace of \( H \) and \( x \in H \), we denote the distance between \( x \) and \( L \) by \( \text{dist}(x, L) \). If \( n \leq \dim(H) \), we use the phrase \( n \)-plane to refer to an \( n \)-dimensional affine subspace of \( H \).

If \( \eta \geq 0 \) and \( L \) is a complete affine subspace of \( H \), we define the tube of height \( \eta \) on \( L \) to be

\[
T_{\text{tube}}(L, \eta) = \{ y \in H : \text{dist}(y, L) \leq \eta \}.
\]
2.4 Elementary Properties of \(d\)-regular Measures

We describe here basic properties of the \(d\)-regular measure \(\mu\). The following bound (extending the upper bound of \(d\)-regularity for all radii) is straightforward:

\[
\mu(B(x, t)) \leq C_\mu \cdot t^d \quad \text{for all } x \in \text{supp}(\mu) \text{ and } t > 0. \tag{9}
\]

The following lemma is derived immediately from equation (11).

Lemma 2.1. If \(x \in \text{supp}(\mu), 0 < t \leq \text{diam}(\text{supp}(\mu)), \text{and } 0 < s < 1/C_\mu^{2/d}, \text{then}

\[
\frac{\mu(A(x, s \cdot t, t))}{\mu(B(x, t))} \geq \left(1 - s^d \cdot C_\mu^2 \right) > 0.
\]

The following proposition requires a little more work and is established in [18].

Proposition 2.1. If \(m \in \mathbb{N}\) is such that \(1 \leq m < d\), \(\mu\) is a \(d\)-regular measure on \(H\) with regularity constant \(C_\mu\), \(0 \leq \epsilon \leq 1\), and \(L\) an \(m\)-dimensional affine subspace of \(H\), then for all \(x \in \text{supp}(\mu) \cap L\) and \(0 < t \leq \text{diam}(\text{supp}(\mu))\)

\[
\mu(T_{\text{ube}}(L, \epsilon \cdot t) \cap B(x, t)) \leq 2^{m+\frac{3d}{2}} \cdot C_\mu \cdot \epsilon^{d-m} \cdot t^d.
\]

2.5 Elementary Properties of the Polar Sine and the Menger-Type Curvature

If \(n \geq 1\), \(X \in H^{n+2}\), and \(1 \leq i \leq n+1\), then let \(\theta_i(X)\) denote the elevation angle of \(x_i - x_0\) with respect to \(V[X(i)]\). We note that if \(\min(X) > 0\), then

\[
\sin(\theta_i(X)) = \frac{\text{dist}(x_i, L[X(i)])}{\|x_i - x_0\|}. \tag{10}
\]

The \(d\)-dimensional p-sine satisfies the following product formula [18]:

Proposition 2.2. If \(X = (x_0, \ldots, x_{d+1}) \in H^{d+2}\) and \(1 \leq i \leq d+1\), then

\[
p_{d\sin x_0}(X) = \sin(\theta_i(X)) \cdot p_{d\sin x_0}(X(i)).
\]

Proposition 2.2 and equation (10) imply the following lower bound for the p-sine.

Lemma 2.2. If \(x \in H\), \(0 < t < \infty\), and \(X \in B(x, t)^{d+2}\) is such that

\[
\text{M}_d(X(i)) \geq \omega \cdot t^d \quad \text{for some } 0 < \omega \leq 1 \text{ and } 1 \leq i \leq d+1,
\]

then

\[
p_{d\sin x_0}(X) \geq \frac{\omega}{2^{d+1}} \cdot \frac{\text{dist}(x_i, L[X(i)])}{t}.
\]

The definition of the p-sine implies the following generalization of the one-dimensional law of sines: If \(X = (x_0, \ldots, x_{d+1}) \in H^{d+2}\) is such that \(\min(X) > 0\), then

\[
\prod_{0 \leq s < r \leq d+1 \atop s, r \neq i} \|x_s - x_r\| = \prod_{0 \leq \ell < q \leq d+1 \atop \ell, q \neq j} \|x_\ell - x_q\| \quad \text{for all } 0 \leq i < j \leq d+1. \tag{11}
\]

Finally, we recall the following expression for \(c_{MT}^2(\mu|_B)\) which was established in [16]:

\[
c_{MT}^2(\mu|_B) = \int_{B^{d+2}} \frac{p_{d\sin x_0}(X)}{\text{diam}(X)^{d(d+1)}} \, d\mu^{d+2}(X).
\]
2.6 Jones-Type Flatness for $1 \leq p < \infty$

For any fixed $1 \leq p < \infty$, $x \in H$ and $0 < t < \infty$, we define the $d$-dimensional Jones’ numbers [5] as follows:

$$\beta_p(x, t) = \begin{cases} 
\inf_{d:\text{-planes } L} \left( \int_{B(x,t)} \left( \frac{\text{dist}(y, L)}{2 \cdot t} \right)^p \frac{d\mu(y)}{\mu(B(x, t))} \right)^{1/p}, & \text{if } \mu(B(x, t)) > 0; \\
0, & \text{if } \mu(B(x, t)) = 0.
\end{cases}$$

For any fixed $1 \leq p < \infty$ and any ball $B$ in $H$, we define the continuous local Jones-type flatness as follows:

$$J_p(\mu|_B) = \int_B \int_0^{\text{diam}(B)} \beta^2_p(x, t) \frac{dt}{t} \, d\mu(x).$$

3 On $d$-Separation of $d$-Regular Measures

We introduce here a notion of $d$-dimensional separation of $(d + 1)$-simplices and show that there are many such separated simplices in $\text{supp}(\mu)^{d+2}$. Specifically, we show that independently of $x \in \text{supp}(\mu)$ and $0 < t \leq \text{diam}(\text{supp}(\mu))$ there exists a “sufficiently large” amount of $(d + 1)$-simplices, $X \in [B(x, t) \cap \text{supp}(\mu)]^{d+2}$, whose $d$-dimensional faces, $\{X(i)\}_{i=0}^{d+1}$, are “sufficiently large”. We refer to this property as $d$-separation of the measure $\mu$ and also refer to the corresponding simplices as $d$-separated. Similar notions were already applied by David and Semmes [5, Lemma 5.8] and Tolsa [25, Lemma 8.2].

3.1 $n$-Separated Simplices

Let $X \in H^{d+2}$ with $\text{diam}(X) > 0$. We say that $X$ is $1$-separated for $\omega > 0$ if

$$\frac{\min(X)}{\text{diam}(X)} \geq \omega.$$ 

We say that $X$ is $d$-separated for $\omega > 0$ if

$$\frac{\min_{0 < i \leq d+1} M_d(X(i))}{\text{diam}^d(X)} \geq \omega.$$ 

More generally, we say that $X$ is $n$-separated for $\omega > 0$ and $1 < n < d$ if the minimal $n$-content through its vertices scaled by $\text{diam}^n(X)$ is larger than $\omega$. We typically do not mention the constant $\omega$, and we just say $n$-separated if $\omega$ is clear from the context.

We note that the $n$-separation of an element $X$ implies the $j$-separation for all $1 \leq j < n$. For example, given a $d$-separated element $X$, using the product formula for contents we have that

$$\omega \cdot \text{diam}^d(X) \leq \min_{0 \leq i \leq d+1} M_d(X(i)) \leq \min(X) \cdot \text{diam}^{d-1}(X).$$

Hence, $X$ is 1-separated for $\omega$. 
3.2 n-Separated Balls and Measures

Let \( B(x,t) \subseteq H, m, n \in \mathbb{N}, m \geq n \geq 1, \) and \( \omega > 0 \). We say that a collection of \( m+1 \) balls, \( \{B_i\}_{i=0}^m \), is \( n \)-separated in \( B(x,t) \) for \( \omega \) if

\[
\bigcup_{0 \leq i \leq m} B_i \subseteq B(x,t),
\]

and any \( n+1 \) points drawn without repetition from any sub-collection of \( n+1 \) distinct balls is \( n \)-separated for \( \omega \). That is,

\[
\min_{\tilde{X} \in \prod_{i \in I} B_i} \mathcal{M}_\mu(\tilde{X}) \geq \omega \cdot t^n, \quad \text{for each set } I \text{ of } n+1 \text{ distinct indices in } \{0, \ldots, m\}.
\]

We extend this definition to \( d \)-regular measures in the following way. For \( x \in \text{supp}(\mu) \) and \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \), we say that \( \mu \) is \( n \)-separated in \( B(x,t) \) (for \( 0 < \delta < 1 \) and \( \omega > 0 \)) if there exist \( (n+2) \) balls, \( \{B_i\}_{i=0}^{n+1} \), which are \( n \)-separated (for \( \omega \)) in \( B(x,t) \), centered on \( \text{supp}(\mu) \) and satisfy

\[
\min_{0 \leq i \leq n+1} \frac{\text{diam}(B_i)}{2 \cdot t} \geq \delta.
\]

We show here that \( \mu \) is \( d \)-separated at all scales and locations in the following sense.

**Proposition 3.1.** There exist \( 0 < \delta_\mu = \delta_\mu(d,C_\mu) < 1 \) and \( \omega_\mu = \omega_\mu(d,C_\mu) > 0 \) such that for any ball \( B(x,t) \subseteq H \) with \( x \in \text{supp}(\mu) \) and \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \), the following property is satisfied: There exists a \( d \)-separated collection of \( d + 2 \) balls, \( \{B(x_i,\delta_\mu \cdot t)\}_{i=0}^{d+1} \), contained in \( B(x,t) \) as well as centered on \( \text{supp}(\mu) \).

\[ \hat{\mu} \]

3.3 Proof of Proposition 3.1

For simplicity, we look at the ball \( B(x,2 \cdot t) \) (instead of \( B(x,t) \)) and reduce Proposition 3.1 to the following two parts.

**Part I:** There exist constants \( 0 < \delta_d = \delta_d(d,C_\mu) \leq 1/2 \) and \( \omega_d = \omega_d(d,C_\mu) > 0 \) such that for every \( x \in \text{supp}(\mu) \) and \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \), there is a collection of \( d+1 \) balls, \( \{B(x_i,\delta_d \cdot t)\}_{i=0}^d \), that are \( d \)-separated for \( \omega_d \) in \( B(x,2 \cdot t) \) and whose centers \( \{x_i\}_{i=0}^d \) are in \( B(x,t) \cap \text{supp}(\mu) \).

**Part II:** Given the \( d \)-separated balls for \( B(x,2 \cdot t), \{B(x_i,\delta_d \cdot t)\}_{i=0}^d \), constructed in Part I, there is a point \( x_{d+1} \in B(x,t) \cap \text{supp}(\mu) \) and constants

\[
0 < \tilde{\delta}_\mu = \tilde{\delta}_\mu(d,C_\mu) \leq \delta_d \text{ and } \tilde{\omega}_\mu = \tilde{\omega}_\mu(d,C_\mu) > 0
\]

such that the collection of \( (d + 2) \) balls, \( \{B(x_i,\tilde{\delta}_\mu \cdot t)\}_{i=0}^{d+1} \), is also \( d \)-separated in the ball \( B(x,2 \cdot t) \).

Parts I and II imply the desired proposition for the ball \( B(x,t) \) with \( \delta_\mu = \tilde{\delta}_\mu/2 \) and \( \omega_\mu = \tilde{\omega}_\mu/2^d > 0 \).

We establish Parts I and II in Subsections 3.3.1 and 3.3.2 respectively. An elementary lemma used in Subsection 3.3.2 is proved separately in Subsection 3.3.3.

**Remark 2.** The statement of Part I is practically equivalent to the statement of [25, Lemma 8.2], which was stated without a proof. In fact, the formulation of [25, Lemma 8.2] shows how to slightly extend our statements beyond Ahlfors regularity.
3.3.1 Part I of the Proof

Our proof is inductive on $n$. If $n = 1$, then let $x_0 = x$ and $\delta_0 = \sqrt[3]{\frac{1}{2} \cdot C_\mu^2}$. By Lemma 2.1 we have the inequality
\[
\mu(A(x_0, \delta_0 \cdot t, t)) \geq \frac{1}{2} \cdot \mu(B(x_0, t)) > 0.
\]
Then, we arbitrarily fix $x_1 \in A(x_0, \delta_0 \cdot t, t) \cap \text{supp}(\mu)$ and set $\delta_1 = \delta_0 / 3$. For any $\bar{x}_0 \in B(x_0, \delta_1 \cdot t)$ and $\bar{x}_1 \in B(x_1, \delta_1 \cdot t)$, let $X_1 = (\bar{x}_0, \bar{x}_1)$. Clearly we have
\[
M_1(X_1) = \|\bar{x}_0 - \bar{x}_1\| \geq \delta_1 \cdot t,
\]
and thus the statement holds for $n = 1$, where
\[
\omega_1 = \delta_1 = \frac{1}{3} \left(\frac{1}{2 \cdot C_\mu^2}\right)^{1/d} \leq \frac{1}{2}.
\]

Now, for some $1 \leq n < d$, we take the induction hypothesis to be the existence of $n + 1$ points $\{x_0, \ldots, x_n\} \subseteq B(x, t) \cap \text{supp}(\mu)$, and constants $0 < \delta_n \leq \frac{1}{2}$ and $\omega_n > 0$ such that the collection of $n + 1$ balls $\{B(x_i, \delta_n \cdot t)\}_{i=0}^n$ is $n$-separated (for $\omega_n$) in $B(x_2 \cdot t)$. We further assume that $x_0 = x$ (which was satisfied for $n = 1$). We will construct a point $x_{n+1} \in B(x, t) \cap \text{supp}(\mu)$ and constants $0 < \delta_{n+1} \leq \delta_n$ and $\omega_{n+1} > 0$ such that the collection of balls $\{B(x_i, \delta_{n+1} \cdot t)\}_{i=0}^{n+1}$ is $(n+1)$-separated (for $\omega_{n+1}$) in $B(x, 2 \cdot t)$ for $\omega_{n+1}$. For the set of balls of the induction hypothesis, $\{B(x_i, \delta_n \cdot t)\}_{i=0}^n$, let $X_n = (x_0, \ldots, x_n)$ denote the non-degenerate simplex generated by their centers, and furthermore let $P$ denote the orthogonal projection of $H$ onto the $n$-plane $L[X_n]$. Let $\delta$ be an arbitrary constant with $0 < \delta \leq \delta_n \leq 1/2$, where we will eventually specify a choice for $\delta$, i.e., the constant $\delta_{n+1}$ mentioned above.

We take an arbitrary element
\[
\bar{X}_n = (\bar{x}_0, \ldots, \bar{x}_n) \in \prod_{i=0}^n B(x_i, \delta \cdot t),
\]
(12)
and for such $\bar{X}_n$, we note that $\{\bar{x}_0, \ldots, \bar{x}_n\} \subseteq B(x_0, \frac{3}{2} \cdot t)$, and thus
\[
\text{diam}(\bar{X}_n) \leq 3 \cdot t.
\]
(13)
Let $\bar{P}_\delta$ denote the orthogonal projection of $H$ onto the $n$-plane $L[\bar{X}_n]$. For convenience, we suppress the dependence of $P$ and $\bar{P}_\delta$ on the elements $X_n$ and $\bar{X}_n$ respectively.

The induction step consists of three parts. The first is the existence of a constant $\epsilon_n > 0$ (independent of $x$ and $t$) and an element $x_{n+1} \in B(x_0, t) \cap \text{supp}(\mu)$ such that
\[
\|x_{n+1} - P(x_{n+1})\| \geq \epsilon_n \cdot t.
\]
(14)
The second part is the existence of a constant $0 < \delta_{n+1} = \delta_{n+1}(n, \delta_n, \omega_n, \epsilon_n) \leq \delta_n$ such that
\[
\|x_{n+1} - \bar{P}_{\delta_{n+1}}(x_{n+1})\| \geq \frac{2}{3} \cdot \epsilon_n \cdot t.
\]
(15)
The last part of the induction proof is showing that for any $\bar{x}_{n+1} \in B(x_{n+1}, \delta_{n+1} \cdot t)$, we have the lower bound
\[
\|\bar{x}_{n+1} - \bar{P}_{\delta_{n+1}}(\bar{x}_{n+1})\| \geq \frac{\epsilon_n}{3} \cdot t.
\]
(16)
Then, we conclude the proof of part I by combining equation (16) with the induction hypothesis and the product formula for contents. That is, we obtain that for any \(1 \leq n \leq d\) the family of balls \(\{B(x_i, \delta_{n+1} \cdot t)\}_{i=0}^{n+1}\) is \((n+1)\)-separated in \(B(x, 2 \cdot t)\) for the constant 

\[
\omega_{n+1} = \frac{\epsilon_n \cdot \omega_n}{3}.
\]

Now, to prove equation (14) for \(1 \leq n < d\), let 

\[
\epsilon_n = \left(\frac{1}{2^{\frac{n}{2}+n+1} \cdot C_n^2}\right)^{1/(d-n)}.
\]

Noting that \(\text{dim}(L[X_n]) = n < d\), Proposition 2.1 implies that:

\[
\mu(B(x, t) \setminus \text{Tube}(L[X_n], \epsilon_n \cdot t)) > \frac{1}{2} \cdot \mu(B(x, t)) > 0,
\]

in particular,

\[
[B(x, t) \cap \text{supp}(\mu)] \setminus \text{Tube}(L[X_n], \epsilon_n \cdot t) \neq \emptyset.
\]

We arbitrarily fix \(x_{n+1} \in [B(x, t) \cap \text{supp}(\mu)] \setminus \text{Tube}(L[x_n], \epsilon_n \cdot t)\), and we immediately obtain equation (14). We also note that

\[
\|x_{n+1} - P(\tilde{P}_\delta(x_{n+1}))\| \geq \epsilon_n \cdot t.
\]

This follows from equation (14) and the fact that \(P(x_{n+1})\) is the closest point to \(x_{n+1}\) in the \(n\)-plane \(L[X_n]\).

To establish equation (15), we will first show that there exists a constant \(C_4 = C_4(n, \omega_n) > 0\) such that for any \(0 < \delta < \delta_n\) we have the uniform upper bound

\[
\|P(\tilde{P}_\delta(y)) - \tilde{P}_\delta(y)\| \leq C_4 \cdot \delta \cdot t, \text{ for all } y \in B(x_0, t).
\]

Then, imposing the following restriction on \(\delta\):

\[
C_4 \cdot \delta \leq \frac{\epsilon_n}{3},
\]

and applying equations (17)-(19), we derive equation (15) as follows

\[
\|x_{n+1} - \tilde{P}_\delta(x_{n+1})\| \geq \|x_{n+1} - P(\tilde{P}_\delta(x_{n+1}))\| - \|P(\tilde{P}_\delta(x_{n+1})) - \tilde{P}_\delta(x_{n+1})\| \geq \frac{2}{3} \cdot \epsilon_n \cdot t.
\]

To establish equation (18) and calculate the constant \(C_4\), we first express the projection of any \(y \in H\) onto \(L[X_n]\) as 

\[
\tilde{P}_\delta(y) = \tilde{x}_0 + \sum_{i=1}^{n} \tilde{s}_i(y) \cdot (\tilde{x}_i - \tilde{x}_0),
\]

with \(\tilde{s}_i(y) \in \mathbb{R}, 1 \leq i \leq n\). For \(0 \leq i \leq n\), the points \(\tilde{x}_i\) have the decomposition 

\[
\tilde{x}_i = x_i + \tilde{z}_i + \tilde{e}_i,
\]

where \(\tilde{z}_i \in \text{Span}\{x_1 - x_0, \ldots, x_n - x_0\}\) and \(\tilde{e}_i\) is orthogonal to \(\text{Span}\{x_1 - x_0, \ldots, x_n - x_0\}\). Therefore, we have the following equality for all \(y \in H\): 

\[
P(\tilde{P}_\delta(y)) - \tilde{P}_\delta(y) = -\left(\tilde{e}_0 + \sum_{i=1}^{n} \tilde{s}_i(y) \cdot (\tilde{e}_i - \tilde{e}_0)\right).
\]
Furthermore, equations (12) and (20) imply the following inequality for all $0 \leq i \leq n$
\[
\|\tilde{e}_i\|^2 \leq \|\tilde{z}_i\|^2 + \|\tilde{z}_i\|^2 \leq (\delta \cdot t)^2.
\tag{22}
\]
Thus, applying equation (22) and the triangle inequality to the RHS of equation (21), we have that
\[
\|P(\tilde{P}_\delta(y)) - \tilde{P}_\delta(y)\| \leq \left(1 + \sum_{i=1}^{n} 2 \cdot |s_i(y)|\right) \cdot \delta \cdot t.
\tag{23}
\]
Now, to bound the RHS of equation (23) for $y \in B(x_0, t)$ (and thereby calculate an upper bound for the constant $C_4$ of equation (18)), we calculate a uniform bound for the quantities $\{|s_i(y)|\}_{i=1}^n$. In fact, we will establish the following inequality for all $y \in B(x_0, t)$:
\[
\max_{1 \leq i \leq n} |\tilde{s}_i(y)| \leq \frac{2 \cdot 3^{n-1}}{\omega_n}.
\tag{24}
\]
The combination of such a bound with equation (23) clearly implies equation (18), where
\[
C_4 = \left(1 + n \cdot \frac{4 \cdot 3^{n-1}}{\omega_n}\right)
\tag{25}
\]
We first note that the coefficients $\tilde{s}_i(y)$ satisfy the following equation for all $1 \leq i \leq n$:
\[
\sin(\theta_i(\tilde{X}_n)) \cdot |\tilde{s}_i(y)| \cdot \|\tilde{x}_i - \tilde{x}_0\| = \text{dist}(\tilde{P}_\delta(y), L[X_n(i)]).
\tag{26}
\]
Obtaining an upper bound on the RHS of equation (26) as well as a lower bound on the quantity $\sin(\theta_i(\tilde{X}_n)) \cdot \|\tilde{x}_i - \tilde{x}_0\|$, will then establish equation (24).
We determine an upper bound by noting that $\tilde{x}_0 \in B(x_0, t) \cap L[X(i)]$, and thus for any $y \in B(x_0, t)$
\[
\text{dist}(\tilde{P}_\delta(y), L[X_n(i)]) \leq \left\| \tilde{P}_\delta(y) - \tilde{x}_0 \right\| \leq \|y - \tilde{x}_0\| \leq 2 \cdot t.
\tag{27}
\]
In order to obtain the lower bound, we apply the product formula for contents as well as equation (13), and get that for any $0 < \delta \leq \delta_n$ and all $1 \leq i \leq n$
\[
M_n(\tilde{X}_n) = \sin(\theta_i(\tilde{X}_n)) \cdot \|\tilde{x}_i - \tilde{x}_0\| \cdot M_{n-1}(\tilde{x}_n(i)) \leq \sin(\theta_i(\tilde{X}_n)) \cdot \|\tilde{x}_i - \tilde{x}_0\| \cdot 3^{n-1} \cdot t^{n-1}.
\]
Combining this with the induction hypothesis, i.e., $M_n(\tilde{X}_n) \geq \omega_n \cdot t^n$, we obtain the inequality
\[
\min_{1 \leq i \leq n} \sin(\theta_i(\tilde{X}_n)) \cdot \|\tilde{x}_i - \tilde{x}_0\| \geq \frac{\omega_n}{3^{n-1}} \cdot t.
\tag{28}
\]
Applying the bounds of equations (28) and (27) to equation (26), we conclude equation (24), and consequently equations (18) and (25). We note that the constant $\delta = \delta_{n+1}$ needs to satisfy equation (19) and the requirement $0 < \delta_{n+1} \leq \delta_n$. We thus set its value in the following way:
\[
\delta_{n+1} = \min \left\{ \frac{\epsilon_n}{3 \left(1 + n \cdot \frac{4 \cdot 3^{n-1}}{\omega_n}\right)}, \delta_n \right\}.
\tag{29}
\]
To prove the final part of the induction argument, i.e., equation (16), we apply the triangle inequality and equations (12) (with $\delta = \delta_{n+1}$), (15) and (29), obtaining that for any $\tilde{x}_{n+1} \in B(x_{n+1}, \delta_{n+1} \cdot t)$
\[
\left\| \tilde{x}_{n+1} - \tilde{P}_{\delta_{n+1}}(\tilde{x}_{n+1}) \right\| \geq \left\| x_{n+1} - \tilde{P}_{\delta_{n+1}}(\tilde{x}_{n+1}) \right\| - \left\| \tilde{x}_{n+1} - x_{n+1} \right\| \geq
\]
\[
\left\| x_{n+1} - \tilde{P}_{\delta_{n+1}}(x_{n+1}) \right\| - \left\| \tilde{x}_{n+1} - x_{n+1} \right\| \geq \frac{\epsilon_n}{3} \cdot t.
\]
3.3.2 Part II of Proof

Using the set of $d$-separated balls of Part I, $\{B(x_i, \delta_d \cdot t)\}_{i=0}^d$, we take the element $X_d = (x_0, \ldots, x_d)$, and for $0 < \rho < 1$ we define the constant

$$
\epsilon_\rho = \frac{1 - \rho}{2 \frac{\delta_d}{T-1} \cdot \rho^2}.
$$

We note that by Proposition 2.1

$$
\min_{0 \leq i \leq d} \mu \left( B(x, t) \setminus \text{Tube} \left( L[X_d(i)], \epsilon_\rho \cdot t \right) \right) \geq \rho \cdot \mu \left( B(x, t) \right). \quad (30)
$$

Hence, imposing the restriction $\rho > d/(d + 1)$, and applying Lemma 3.1 (presented in Subsection 3.3.3 below) with $\nu$ being the restricted and scaled measure $\mu|_{B(x,t)}/\mu(B(x,t))$, $\xi = \rho$, $A_i = B(x, t) \setminus \text{Tube}(L[X_d(i)], \epsilon_\rho \cdot t)$ for $0 \leq i \leq d$, and $k = d$, we get the following lower bound:

$$
\mu \left( B(x, t) \setminus \bigcup_{i=0}^d \text{Tube} \left( L[X_d(i)], \epsilon_\rho \cdot t \right) \right) > 0. \quad (31)
$$

Therefore, for such $\rho$ there exists a point $x_{d+1} \in B(x, t) \cap \text{supp}(\mu)$ so that

$$
\min_{0 \leq i \leq d} \text{dist} \left( x_{d+1}, L[X_d(i)] \right) > \epsilon_\rho \cdot t. \quad (32)
$$

To choose the constants $\tilde{\delta}_\mu = \tilde{\delta}_\mu(d, C_\mu) > 0$ and $\tilde{\omega}_\mu = \tilde{\omega}_\mu(d, C_\mu) > 0$, as well as verify the claim of $d$-separation, we use practically the same arguments as those for proving equations (14)-(16). We arbitrarily fix $0 < \delta \leq \delta_d$, while later specifying its value, and an element

$$
\tilde{X}_d = (\tilde{x}_0, \ldots, \tilde{x}_d) \in \prod_{i=0}^d B(x_i, \delta \cdot t).
$$

By the conclusion of Part I of the proof, we have that

$$
M_d(\tilde{X}_d) \geq \omega_d \cdot t^d.
$$

Furthermore, $\text{diam}(\tilde{X}_d) \leq 3 \cdot t$. Combining these with the product formula for contents, we obtain the inequality

$$
\min_{0 \leq i \leq d} M_{d-1}(\tilde{X}_d(i)) \geq \frac{\omega_d}{3} \cdot t^{d-1}. \quad (33)
$$

For $0 \leq i \leq d$, let $P_i$ and $\tilde{P}_{\delta,i}$ denote the orthogonal projections of $H$ onto $L[X_d(i)]$ and $L[\tilde{X}_d(i)]$, respectively. By virtually the same argument producing equation (18), while applying equation (33), we have that for all $y \in B(x, t)$,

$$
\max_{0 \leq i \leq d} \left\| P_i \left( \tilde{P}_{\delta,i}(y) \right) - \tilde{P}_{\delta,i}(y) \right\| \leq \left( 1 + (d - 1) \cdot \frac{4 \cdot 3^{d-1}}{\omega_d} \right) \cdot \delta \cdot t.
$$

Next, we impose the further restriction $\rho_0 = \frac{d+0.5}{d+1}$, and for this value of $\rho$ we set

$$
\tilde{\delta}_\mu = \min \left\{ \frac{\epsilon_\rho_0}{3 \left( 1 + (d - 1) \cdot \frac{4 \cdot 3^{d-1}}{\omega_d} \right)}, \delta_d \right\}.
$$
By the same calculations producing equation (16), we get that
\[
\min_{0 \leq i \leq d} \| \tilde{x}_{d+1} - \tilde{P}_{\delta_{\mu}},_i (\tilde{x}_{d+1}) \| \geq \frac{\epsilon_{\rho_0}}{3} \cdot t \quad \text{for all } \tilde{x}_{d+1} \in B(x_{d+1}, \delta_{\mu}, t). \tag{34}
\]

Finally, combining Part I and equations (33) and (34) along with the product formula for contents, we have that for any \( \tilde{X}_{d+1} = (\tilde{x}_0, \ldots, \tilde{x}_{d+1}) \in \prod_{i=0}^{d+1} B(x_i, \delta \cdot t) \),

the following inequality is satisfied
\[
\min_{0 \leq i \leq d+1} M_d (\tilde{X}_{d+1}(i)) \geq \omega_d \cdot \epsilon_{\rho_0} \cdot t^d.
\]

Therefore, taking
\[
\tilde{\omega}_{\mu} = \frac{\epsilon_{\rho_0} \cdot \omega_d}{9},
\]

the collection of balls \( \{ B(x_i, \delta_{\mu} \cdot t) \}_{i=0}^{d+1} \) is \( d \)-separated in \( B(x, 2 \cdot t) \) for \( \tilde{\omega}_{\mu} \).

\[\square\]

### 3.3.3 An Elementary Lemma

We establish the following elementary proposition which was used in Subsection 3.3.2 and will also be used later in Subsection 4.3.

**Lemma 3.1.** If \( \nu \) is a Borel probability measure, \( A_0, A_1, \ldots, A_d \) are measurable sets (w.r.t. \( \nu \)), \( 0 < \xi < 1 \), and
\[
\min_{0 \leq i \leq d} \nu(A_i) \geq \xi, \tag{35}
\]

then for any \( 0 \leq k \leq d \) the following inequality holds
\[
\nu \left( \bigcap_{i=0}^{k} A_i \right) \geq (k + 1) \cdot \xi - k. \tag{36}
\]

**Proof.** The proof is by induction. Equation (35) clearly implies the inequality of equation (36) when \( k = 0 \). Supposing that equation (36) holds for some \( 0 \leq k < d \), we note that
\[
1 \geq \nu \left( \bigcup_{i=0}^{k} A_i \right) \cup A_{k+1} = \nu \left( \bigcup_{i=0}^{k} A_i \right) + \nu(A_{k+1}) - \nu \left( \bigcap_{i=0}^{k+1} A_i \right). \tag{37}
\]

Thus, by the induction hypothesis and equation (37) we have that
\[
\nu \left( \bigcap_{i=0}^{k+1} A_i \right) \geq \nu \left( \bigcap_{i=0}^{k} A_i \right) + \nu(A_{k+1}) - 1 \geq (k + 1) \cdot \xi - k + \xi - 1 = (k + 2) \cdot \xi - (k + 1).
\]

\[\square\]
4 The Proof of Theorem 1.1

In order to prove Theorem 1.1 we will establish the existence of constants \( \lambda_0 = \lambda_0(d, C_\mu) \) and \( C_1 = C_1(d, C_\mu) \) such that there exists a \( d \)-plane \( L_{(x,t)} \) with

\[
\int_{B(x,t)} \left( \frac{\text{dist} (y, L_{(x,t)})}{2 \cdot t} \right)^2 \, d\mu(y) \leq C_1 \cdot c_{MT}^2(x, t, \lambda_0), \tag{38}
\]

for any \( x \in \text{supp}(\mu) \) and \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \). Applying the definition of the \( \beta_2 \) numbers to equation (38) then proves Theorem 1.1.

Our approach for establishing equation (38) generalizes the proof of Léger [12, Lemma 2] for the case \( d = 1 \). In that case, constructing the line \( L_{(x,t)} \) is relatively straightforward and short. However, for \( d \geq 2 \) there are combinatorial and geometric issues that do not manifest themselves when \( d = 1 \), e.g., the proofs of Proposition 4.1 and Lemma 4.1 below, and the notion of \( d \)-separation for \( d \geq 2 \) (Section 3 above). We present the overall argument in Subsection 4.2, and we leave the details to Subsections 4.3 and 4.4. Preliminary notation and observations are provided in Subsection 4.1.

4.1 Notation and Preliminary Observations

For any \( x \in \text{supp}(\mu) \), \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \), \( 0 < \lambda < 2 \), and \( 0 \leq i < j \leq d + 1 \) we define the following slices of the set \( U_\lambda(B(x,t)) \) of equation (22):

\[
U_\lambda (x, t \mid X(i)) = \{ y \in B(x, t) : X(y, i) \in U_\lambda(B(x, t)) \} = \},
\]

\[
U_\lambda (x, t \mid X(i; j)) = \{ (y, z) \in B(x, t)^2 : X(y, i; z, j) \in U_\lambda(B(x, t)) \} = \},
\]

\[
U_\lambda (x, t \mid X(y; i)) = \{ z \in B(x, t) : X(y, i; z, j) \in U_\lambda(B(x, t)) \} = \},
\]

\[
U_\lambda (x, t \mid X(i; y, j)) = \{ z \in B(x, t) : X(z, i; y, j) \in U_\lambda(B(x, t)) \} = \}.
\]

In addition, we fix the following constant of 1-separation

\[
\lambda_0 = \frac{\delta_\mu}{2}, \tag{39}
\]

where \( \delta_\mu \) is the constant suggested by Proposition 3.1.

For the remainder of the proof (i.e., the whole section) we arbitrarily fix \( x \in \text{supp}(\mu) \) and \( 0 < t \leq \text{diam}(\text{supp}(\mu)) \), and some \( d \)-separated collection of balls \( \{B(x_i, \delta_\mu \cdot t)\}_{i=0}^{d+1} \) in \( B(x, t) \) for the constant \( \omega_\mu \) (see Proposition 3.1). We denote \( B_i = B(x_i, \delta_\mu \cdot t) \) for \( 0 \leq i \leq d + 1 \). Restricting our attention to only the first \( (d + 1) \) balls, we also form an arbitrary element

\[
\tilde{X}(d + 1) = (\tilde{x}_0, \ldots, \tilde{x}_d) \in \prod_{i=0}^{d} \frac{1}{2} \cdot B_i.
\]

We note that

\[
B(x, t) \setminus \bigcup_{i=0}^{d} B_i \subseteq U_{\lambda_0}(x, t \mid \tilde{X}(d + 1)) \tag{40}
\]

and

\[
B_i \nsubseteq U_{\lambda_0}(x, t \mid \tilde{X}(d + 1)), \text{ for each } 0 \leq i \leq d. \tag{41}
\]
4.2 The Essence of the Proof of Theorem 1.1

For $0 < \rho < \infty$ let

$$
\mathcal{E}(\rho) = \left\{ \tilde{X}(d+1) \in \prod_{i=0}^{d} \frac{1}{2} \cdot B_i : \int_{U_{\lambda_0}(x,t | \tilde{X}(d+1))} \frac{p_0 \sin^{2} \frac{\omega}{C_{\mu}} \left( \tilde{X}(y,d+1) \right)}{\text{diam} \left( \tilde{X}(y,d+1) \right) d(\tilde{X}(y,d+1))} \, d\mu(y) \leq \rho \cdot \frac{\omega^{2}_{\mu}}{\mu} \cdot \frac{c_{MT}^{2}(x,t,\lambda)}{\mu(d+1)} \right\}. \quad (42)
$$

We will show that $\mu^{d+1}(\mathcal{E}(\rho))$ is sufficiently large for some $0 < \rho < \infty$.

First, applying Chebychev’s inequality to equation (42) we obtain that

$$
\mu^{d+1} \left( \prod_{i=0}^{d} \frac{1}{2} \cdot B_i \setminus \mathcal{E}(\rho) \right) \leq \frac{d^{d+1}}{\rho}. \quad (43)
$$

Next, we note that the $d$-regularity of $\mu$ implies that

$$
\mu^{d+1} \left( \prod_{i=0}^{d} \frac{1}{2} \cdot B_i \right) \geq \frac{1}{C_{d+1}^{d+1}} \cdot (\lambda_0 \cdot t)^{d(d+1)}. \quad (44)
$$

Thus, combining equations (43) and (44), and taking

$$
\rho_1 = \rho_1(d, C_{\mu}) > \frac{2}{\lambda_0^{d(d+1)} \cdot C_{d+1}}, \quad (45)
$$

we obtain the lower bound

$$
\mu^{d+1}(\mathcal{E}(\rho_1)) > \frac{1}{2} \cdot \mu^{d+1} \left( \prod_{i=0}^{d} \frac{1}{2} \cdot B_i \right) > 0. \quad (46)
$$

We will show that the desired $d$-plane, $L(x,t)$, of equation (38) is obtained by $L[\tilde{X}(d+1)]$ for some $\tilde{X}(d+1) \in \mathcal{E}(\rho_1)$. In fact, for any such $\tilde{X}(d+1)$ we immediately obtain control on a part of the integral on the LHS of equation (38) as follows. Since $\tilde{X}(d+1) \in \mathcal{E}(\rho_1)$ is $d$-separated, by Lemma 2.2 the following lower bound holds for all $y \in H$

$$
\frac{p_0 \sin^{2} \frac{\omega}{C_{\mu}} \left( \tilde{X}(y,d+1) \right)}{\text{diam} \left( \tilde{X}(y,d+1) \right) d(\tilde{X}(y,d+1))} \geq \frac{\omega^{2}_{\mu}}{2(d+1)(d+2)} \cdot \left( \frac{\text{dist}(y,L[\tilde{X}(d+1)])}{t} \right)^{2} \cdot \frac{1}{\mu(d+1)}. \quad (47)
$$

Thus, by equations (42) and (47) we have that for any $\tilde{X}(d+1) \in \mathcal{E}(\rho_1)$

$$
\int_{U_{\lambda_0}(x,t | \tilde{X}(d+1))} \left( \frac{\text{dist}(y,L[\tilde{X}(d+1)])}{t} \right)^{2} \, d\mu(y) \leq \frac{2^{(d+1)(d+2)}}{\omega^{2}_{\mu}} \cdot \rho_1 \cdot c_{MT}^{2}(x,t,\lambda_0). \quad (48)
$$

Combining this with the set inclusion of equation (40) implies that

$$
\int_{B(x,t) \setminus \bigcup_{i=0}^{d} B_i} \left( \frac{\text{dist}(y,L[\tilde{X}(d+1)])}{t} \right)^{2} \, d\mu(y) \leq \frac{2^{(d+1)(d+2)}}{\omega^{2}_{\mu}} \cdot \rho_1 \cdot c_{MT}^{2}(x,t,\lambda_0). \quad (49)
$$
Despite the upper bound of equation (49), the condition of the set $E(\rho_1)$ does not help us to obtain a bound for the integrals over the individual balls $B_i$, $0 \leq i \leq d$. This incompleteness follows from equation (41). In order to obtain such an upper bound (thus concluding equation (38)), we must impose further restrictions on the element $\tilde{X}(d+1)$.

For $0 < \rho < \infty$, let

$$A(\rho) = \left\{ \tilde{X}(d+1) \in \prod_{i=0}^{d} \frac{1}{2} \cdot B_i : \max_{0 \leq i \leq d} \int_{U_{\lambda_0}} \left( \frac{p_d \sin^2(\tilde{X}(y, i; z, d+1))}{\text{diam}(\tilde{X}(y, i; z, d+1))} \right)^{d(d+1)} d\mu^2(y, z) \leq \rho \cdot c^2_{MT}(x, t, \lambda_0) \right\}. \quad (50)$$

Below in Subsection 4.3 we prove the following lemma.

**Lemma 4.1.** There exists a constant $\rho_2 = \rho_2(d, C_\mu) > 0$ such that

$$\mu^{d+1}(A(\rho_2)) > \frac{1}{2} \cdot \mu^{d+1} \left( \prod_{i=0}^{d} \frac{1}{2} \cdot B_i \right) > 0,$$

for any $x \in \text{supp}(\mu)$ and $0 < t \leq \text{diam}(\text{supp}(\mu))$.

The condition imposed by $A(\rho_2)$ yields the following estimate which is proved in Subsection 4.4.

**Proposition 4.1.** There exists a constant $C_5 = C_5(d, C_\mu)$ such that for any element $\tilde{X}(d+1) \in A(\rho_2)$:

$$\max_{0 \leq i \leq d} \int_{B_i} \left( \frac{\text{dist}(y, L[\tilde{X}(d+1)])}{t} \right)^2 \, d\mu(y) \leq C_5 \cdot c^2_{MT}(x, t, \lambda_0).$$

Finally, equation (46) and Lemma 4.1 imply that $E(\rho_1) \cap A(\rho_2) \neq \emptyset$. Thus, fixing an arbitrary $\tilde{X}(d+1) \in E(\rho_1) \cap A(\rho_2)$, equation (38) is deduced from equation (49) and Proposition 4.1.

### 4.3 The Proof of Lemma 4.1

For each $0 \leq i \leq d$ we define the following cartesian product

$$A_i = \prod_{0 \leq j \neq i \leq d} \frac{1}{2} \cdot B_j.$$

We note that the $d$-regularity of $\mu$ and equation (39) trivially imply the following estimate for each $0 \leq i \leq d$:

$$\mu^d(A_i) \geq \frac{1}{C_\mu} \cdot (\lambda_0 \cdot t)^{d^2}. \quad (51)$$

Then, for $0 < \rho < \infty$ and $0 \leq i \leq d$, we define the set

$$A_i(\rho) = \left\{ \tilde{X}(i; d+1) \in A_i : \int_{U_{\lambda_0}} \left( \frac{p_d \sin^2(\tilde{X}(y, i; z, d+1))}{\text{diam}(\tilde{X}(y, i; z, d+1))} \right)^{d(d+1)} d\mu^2(y, z) \leq \rho \cdot c^2_{MT}(x, t, \lambda_0) \right\}, \quad (52)$$
and we embed it in the product $\prod_{j=0}^{d} \frac{1}{2} \cdot B_j$ by defining the set

$$A_i(\rho) = \left\{ \tilde{X}(y, i; d+1) : \tilde{X}(i, d+1) \in A_i(\rho) \text{ and } y \in \frac{1}{2} \cdot B_i \right\}. $$

From this definition we see that

$$\mu^{d+1}(A_i(\rho)) = \mu^d(A_i(\rho)) \cdot \mu \left( \frac{1}{2} \cdot B_i \right). \quad (53)$$

Furthermore, for the set $A(\rho)$ of equation (50), we note the inclusion

$$\bigcap_{i=0}^{d} A_i(\rho) \subseteq A(\rho) \subseteq \prod_{i=0}^{d} \frac{1}{2} \cdot B_i. \quad (54)$$

We next find $\rho$ such that $\mu^{d+1}(A(\rho))$ is sufficiently large. We do this by first using equation (53) to find $\rho$ such that the individual $\mu^{d+1}(A_i(\rho))$, $0 \leq i \leq d$, are sufficiently large, and then applying equation (54) to get the desired conclusion about $A(\rho)$.

Applying Chebychev’s inequality to equation (52) implies that for all $0 \leq i \leq d$

$$\mu^d(A_i(\rho)) \geq \mu^d(A_i) - \frac{\mu^d}{\rho}. \quad (55)$$

In order to choose $\rho$, for any $0 < \xi < 1$ we define

$$\rho(\xi) = \frac{C \mu^d}{1 - \xi} \cdot \left( \frac{1}{\lambda_0} \right)^{d^2},$$

and by applying the estimates of equations (51) and (55) we obtain that

$$\mu^d(A_i(\rho(\xi))) \geq \xi \cdot \mu^d(A_i), \text{ for each } 0 \leq i \leq d. \quad (56)$$

Hence, by equations (53) and (50) we have the lower bound

$$\mu^{d+1}(A_i(\rho(\xi))) \geq \xi \cdot \mu^{d+1} \left( \prod_{j=0}^{d} \frac{1}{2} \cdot B_j \right), \text{ for all } 0 \leq i \leq d. \quad (57)$$

Therefore, letting $\rho_2 = \rho_2(\xi)$ where

$$\xi > \frac{d + 1/2}{d + 1}, \quad (58)$$

and applying Lemma 3.1 (with $\nu$ being the measure $\mu^{d+1}$ restricted to the set $\prod_{j=0}^{d} \frac{1}{2} \cdot B_j$ and scaled to 1 on that set, $A_i = A_i(\rho(\xi))$ for $0 \leq i \leq d$, and $k = d$) we get the following lower bound:

$$\mu^{d+1} \left( \bigcap_{i=0}^{d} A_i(\rho_2(\xi)) \right) \geq ((d + 1) \cdot \xi - d) \cdot \mu^{d+1} \left( \prod_{j=0}^{d} \frac{1}{2} \cdot B_j \right) > \frac{1}{2} \cdot \mu^{d+1} \left( \prod_{j=0}^{d} \frac{1}{2} \cdot B_j \right). \quad (59)$$

Lemma 4.1 thus follows from equations (54) and (59). \qed
4.4 The Proof of Proposition 4.1

Up until this point, we have not used the full statement of Proposition 3.1. We have only used the first \((d+1)\) balls, \(B_0, \ldots, B_d\), in the definitions of the sets \(E(\rho_1)\) and \(A(\rho_2)\), and we have completely ignored the \((d+2)\)-nd ball of the \(d\)-separated collection \(\{B_j\}_{j=0}^{d+1}\). The proof of Proposition 4.1 requires the use of this final ball, which we have denoted by \(B_{d+1}\). We use this ball to formulate the following lemma (whose proof is given in Subsection 4.4.1).

**Lemma 4.2.** There exist constants \(C_6 = C_6(d, C_\mu, \lambda_0)\) and \(C_7 = C_7(d, C_\mu, \lambda_0)\) such that for any fixed \(\tilde{X}(d+1) \in E(\rho_1) \cap A(\rho_2)\) and fixed \(0 \leq i \leq d\), the following property is satisfied: There exists a point

\[
\tilde{x}_{d+1} \in \frac{1}{2} \cdot B_{d+1} \cap \text{supp}(\mu)
\]

with

\[
\int_{U_{\lambda_0}} \left| \tilde{x}(i; \tilde{x}_{d+1}, d+1) \right| \frac{p_d \sin^2 \left( \tilde{X}(y, i; \tilde{x}_{d+1}, d) \right)}{diam \left( \tilde{X}(y, i; \tilde{x}_{d+1}, d+1) \right)} \, d\mu(y) \leq C_6 \cdot \frac{c_{\text{MT}}^2(x, t, \lambda_0)}{\rho^d + d}, \quad (60)
\]

and

\[
\left( \frac{\text{dist}(\tilde{x}_{d+1}, L[\tilde{X}(d+1)])}{t} \right)^2 \leq C_7 \cdot \frac{c_{\text{MT}}^2(x, t, \lambda_0)}{t^d}. \quad (61)
\]

We will prove this lemma in Subsection 4.4.1 and will then use it in Subsection 4.4.2 to prove Proposition 4.1.

4.4.1 Proof of Lemma 4.2

To construct the point \(\tilde{x}_{d+1}\), for any fixed \(\tilde{X}(d+1) \in A(\rho_2)\) and any \(0 \leq i \leq d\) we define the following two sets for \(\rho_2\) of Lemma 4.1 and any \(0 < \tau < \infty\):

\[
Q(\tau) = \left\{ z \in \frac{1}{2} \cdot B_{d+1} : \int_{U_{\lambda_0}} \frac{p_d \sin^2 \left( \tilde{X}(y, i; z, d+1) \right)}{diam \left( \tilde{X}(y, i; z, d+1) \right)} \, d\mu(y) \leq \frac{\tau}{t^d} \cdot \rho_2 \cdot \frac{c_{\text{MT}}^2(x, t, \lambda_0)}{t^d} \right\},
\]

and

\[
G(\tau) = \left\{ z \in \frac{1}{2} \cdot B_{d+1} : \left( \frac{\text{dist}(z, L[\tilde{X}(d+1)])}{t} \right)^2 \leq \frac{\tau}{t^d} \cdot \rho_1 \cdot \frac{2^{(d+1)(d+2)}}{\omega_\mu^2} \cdot \frac{c_{\text{MT}}^2(x, t, \lambda_0)}{t^d} \right\}.
\]

The idea is to find a large enough \(\tau\) so that the intersection of these two sets is non-empty.

We first focus on the set \(Q(\tau)\) and specify a value for \(\tau\) such that \(\mu(Q(\tau))\) is sufficiently large.
Since $\tilde{X}(d + 1) \in \mathcal{A}(\rho_2)$, by equation (50) we get that

$$
\int_{\frac{1}{2} : B_{d+1}} \left( \int_{U_{\lambda_0}(x, t | \tilde{X}(i; z, d+1))} p \mu \sin^2 (\tilde{X}(y; i, z, d+1)) \left( \frac{\tilde{X}(y; i, z, d+1)}{\text{diam} (\tilde{X}(y, i; z, d+1))} d\mu(y) \right) \right) d\mu(z) \leq \frac{\rho_2}{t^d} \cdot c_{MT}^2(x, t, \lambda_0).
$$

Hence, by Chebychev’s inequality we obtain

$$
\mu(\mathcal{Q}(\tau)) \geq \mu \left( \frac{1}{2} \cdot B_{d+1} \right) - \frac{t^d}{\tau}.
$$

We thus fix

$$
\tau_0 = \tau_0(d, C_\mu) > \frac{2 \cdot C_\mu}{\lambda_0^d},
$$

and by the $d$-regularity of $\mu$ we have the lower bound

$$
\mu(\mathcal{Q}(\tau_0)) > \frac{1}{2} \cdot \mu \left( \frac{1}{2} \cdot B_{d+1} \right). \quad (62)
$$

Clearly, one can choose any $\tilde{x}_{d+1}$ in $\mathcal{Q}(\tau_0) \neq \emptyset$ and it will satisfy equation (60) with $C_6 = \tau_0 \cdot \rho_2$.

Next, to choose $\tau$ such that $\mu(\mathcal{G}(\tau))$ is also sufficiently large, i.e., to find a point $\tilde{x}_{d+1}$ which satisfies equation (61) as well, we apply equation (48) and Chebychev’s inequality to obtain

$$
\mu(\mathcal{G}(\tau)) \geq \mu \left( \frac{1}{2} \cdot B_{d+1} \right) - \frac{t^d}{\tau}.
$$

Hence, for $\tau = \tau_0$, by the $d$-regularity of $\mu$ we get that

$$
\mu(\mathcal{G}(\tau_0)) > \frac{1}{2} \cdot \mu \left( \frac{1}{2} \cdot B_{d+1} \right). \quad (63)
$$

Finally, the combination of equations (62) and (63) results in the inequality

$$
\mu(\mathcal{Q}(\tau_0) \cap \mathcal{G}(\tau_0)) > 0,
$$

and therefore the lemma is established with $C_6$ (as specified above) and

$$
C_7 = \tau_0 \cdot \rho_1 \cdot \frac{2^{(d+1)(d+2)}}{\omega_\mu}.
$$

4.4.2 Deriving Proposition 4.1 from Lemma 4.2

We arbitrarily fix an index $0 \leq i \leq d$ and prove Proposition 4.1 by specifying a constant $C_5 = C_5(d, C_\mu, \lambda_0)$ such that

$$
\int_{B_i} \left( \frac{\text{dist}(y, L[\tilde{X}(d+1)])}{t} \right)^2 d\mu(y) \leq C_5 \cdot c_{MT}^2(x, t, \lambda_0), \text{ for all } 0 \leq i \leq d. \quad (64)
$$
Our strategy for proving equation (64) is to first show that for any point $\tilde{x}_{d+1}$ satisfying Lemma 4.2 the following inequality holds (for the fixed index $i$ and $C_6$ as in Lemma 4.2):

$$\int_{B_i} \left( \frac{\text{dist}(y, L[\tilde{X}(i; \tilde{x}_{d+1}, d + 1)])}{t} \right)^2 \, d\mu(y) \leq \frac{2^{(d+1)(d+2)}}{\omega^2_\mu} \cdot \left( \frac{2}{\lambda_0} \right)^{(d+1)} \cdot C_6 \cdot c_{MT}^2(x, t, \lambda_0). \quad (65)$$

Then, a basic geometric argument shows that equation (65) implies equation (64).

Now, if $\tilde{x}_{d+1}$ is a point satisfying Lemma 4.2 then since $\tilde{x}_{d+1} \in B_{d+1}$ and the collection of balls $\{B_i\}_{i=0}^{d+1}$ is $d$-separated we have that

$$B_i \subseteq U_{\lambda_0}(x, t | \tilde{X}(i; \tilde{x}_{d+1}, d + 1)) \quad (66)$$

and

$$M_d(\tilde{X}(i; \tilde{x}_{d+1}, d + 1)) \geq \omega_\mu \cdot t^d. \quad (67)$$

If $1 \leq i \leq d$, then Lemma 2.2 implies that for any $y \in B_i$

$$\frac{p_d \sin^2_{\tilde{x}_0} \left( \tilde{X}(y, i; \tilde{x}_{d+1}, d + 1) \right)}{\text{diam} \left( \tilde{X}(y, i; \tilde{x}_{d+1}, d + 1) \right)^{(d+1)}} \geq \frac{\omega^2_\mu}{2^{(d+1)(d+2)}} \cdot \left( \frac{\text{dist}(y, L[\tilde{X}(i; \tilde{x}_{d+1}, d + 1)])}{t} \right)^2 \cdot \frac{1}{t^{d(d+1)}}. \quad (68)$$

Combining this inequality with equations (60) and (65), we conclude equation (65) in this case.

However, if $i = 0$, then we cannot directly apply Lemma 2.2. Instead, we first note that for all $y \in B_0$

$$\frac{\delta_\mu}{2} \cdot t = \lambda_0 \cdot t \leq \min(\tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1)) \leq \text{diam}(\tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1)) \leq 2 \cdot t. \quad (69)$$

Then, using these bounds we apply the law of sines for the polar sine (see equation (11)) to obtain the lower bound

$$p_d \sin_{\tilde{x}_0} \left( \tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1) \right) \geq$$

$$\left( \frac{\min(\tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1))}{\text{diam}(\tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1))} \right)^{d(d+1)} \cdot p_d \sin_{\tilde{x}_1} \left( \tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1) \right) \geq$$

$$\left( \frac{\lambda_0}{2} \right)^{d(d+1)} \cdot p_d \sin_{\tilde{x}_1} \left( \tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1) \right). \quad (70)$$

Applying Lemma 2.2 to the RHS of equation (70), and then applying the RHS inequality of equation (69) to the resulting equation gives the inequality

$$\frac{p_d \sin^2_{\tilde{x}_0} \left( \tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1) \right)}{\text{diam} \left( \tilde{X}(y, 0; \tilde{x}_{d+1}, d + 1) \right)^{(d+1)}} \geq$$

$$\frac{\omega^2_\mu}{2^{(d+1)(d+2)}} \cdot \left( \frac{\lambda_0}{2} \right)^{d(d+1)} \cdot \left( \frac{\text{dist}(y, L[\tilde{X}(0; \tilde{x}_{d+1}, d + 1)])}{t} \right)^2 \cdot \frac{1}{t^{d(d+1)}}. \quad (71)$$
We note that
\[
\sin(\phi) \leq \frac{\omega^2}{2(d+1)(d+2)} \cdot \left( \frac{\lambda_0}{2} \right)^{d(d+1)} \cdot \left( \frac{\text{dist}(y, L[\overline{X}(i; d+1)])}{t} \right)^2 \cdot \frac{1}{t^{d(d+1)}}.
\]  

Combining equation (72) with equations (60) and (66) implies equation (65) for the fixed index \(i\).

Next, equation (65) implies equation (64) via the following argument. We first note that the elements \(\overline{X}(d+1), \overline{X}(i; d+1), d + 1\), and \(\overline{X}(i; d + 1)\) are all non-degenerate, and we define the orthogonal projections
\[
P_{d+1} : H \to L[\overline{X}(d+1)],
\]
\[
P_i : H \to L[\overline{X}(i; d+1), d + 1],
\]
\[
P_{i,d+1} : H \to L[\overline{X}(i; d + 1)].
\]

Using these projections we can reduce the situation to two cases.

The first is when \(P_{d+1} = P_i\), that is, \(L[\overline{X}(d+1)] = L[\overline{X}(i; d+1)]\). In this case equation (64) holds trivially by equation (65) with
\[
C_5 \geq 2^{(d+1)(d+2)} \frac{2^{d(d+1)}}{\omega^2} \cdot \left( \frac{2}{\lambda_0} \right)^{d(d+1)} \cdot C_6.
\]

The second is when \(P_{d+1} \neq P_i\). In this case, we rely on the following inequality.
\[
\left( \frac{\text{dist}(y, P_{d+1}(y))}{t} \right)^2 \leq \left( \frac{\text{dist}(y, P_{d+1}(P_i(y)))}{t} \right)^2 \leq 2 \cdot \left[ \left( \frac{\text{dist}(y, P_i(y))}{t} \right)^2 + \left( \frac{\text{dist}(P_i(y), P_{d+1}(P_i(y)))}{t} \right)^2 \right].
\]  

Integrating the inequality of equation (73) over the ball \(B_i\) and applying the inequality of equation (65), we obtain the bound
\[
\int_{B_i} \left( \frac{\text{dist}(y, P_{d+1}(y))}{t} \right)^2 \text{d}\mu(y) \leq 2 \cdot 2^{(d+1)(d+2)} \frac{2^{d(d+1)}}{\omega^2} \cdot \left( \frac{2}{\lambda_0} \right)^{d(d+1)} \cdot C_6 \cdot c^2_{\text{MT}}(x, t, \lambda_0) + 2 \int_{B_i} \left( \frac{\text{dist}(P_i(y), P_{d+1}(P_i(y)))}{t} \right)^2 \text{d}\mu(y).
\]

The only thing remaining is to bound the second term on the RHS of equation (74).

Since \(P_{d+1} \neq P_i\), the \(d\)-planes \(L[\overline{X}(d+1)]\) and \(L[\overline{X}(i; d+1), d + 1]\) are distinct. Let \(\alpha\) denote the dihedral angle between these two \(d\)-planes along their intersection, the \((d-1)\)-plane \(L[\overline{X}(i; d + 1)]\).

We note that \(\sin(\alpha) > 0\). Furthermore, for all \(y \in B(x, t)\) we have that
\[
\text{dist}(P_i(y), P_{d+1}(P_i(y))) = \sin(\alpha) \cdot \text{dist}(P_i(y), P_{i,d+1}(P_i(y))).
\]
We can bound the RHS of equation (75) by bounding each of the factors separately. For any
\( j \neq i, d+1 \), we have the inclusion
\[
\tilde{x}_j = (\tilde{X}(d+1))_j \in B(x, t) \cap L[\tilde{X}(i; d+1)] \subseteq L[\tilde{X}(i; \tilde{x}_{d+1}, d+1)].
\]
Hence we obtain that for all \( y \in B(x, t) \)
\[
\text{dist}(P_i(y), P_{i,d+1}(P_i(y))) \leq \|P_i(y) - \tilde{x}_j\| = \|P_i(y) - \tilde{x}_j\| \leq \|y - \tilde{x}_j\| \leq 2 \cdot t. \tag{76}
\]
To bound \( \sin(\alpha) \), we observe that
\[
\sin(\alpha) = \frac{\text{dist}(\tilde{x}_{d+1}, P_{d+1}(\tilde{x}_{d+1}))}{\text{dist}(\tilde{x}_{d+1}, P_{i,d+1}(\tilde{x}_{d+1}))}. \tag{77}
\]
By equation (67) and the product formula for contents we have that
\[
\text{dist}(\tilde{x}_{d+1}, P_{i,d+1}(\tilde{x}_{d+1})) \geq \frac{\omega \mu}{2^{d-1}} \cdot t. \tag{78}
\]
Applying equations (76) and (78) to the RHS of equation (75), we have the following uniform upper
bound for all \( y \in B(x, t) \)
\[
\text{dist}(P_i(y), P_{d+1}(P_i(y))) \leq \frac{2^d}{\omega \mu} \cdot \text{dist}(\tilde{x}_{d+1}, L[\tilde{X}(d+1)]) \mu(B_i) \leq \frac{2^d \cdot C \mu \cdot C_2}{\omega_2^2} \cdot c_{\text{MT}}(x, t, \lambda_0). \tag{80}
\]
Finally, applying equation (80) to the RHS of equation (74) finishes the proof of equation (64), and
thus concludes Proposition 4.1.

5 The Proof of Theorem 1.3

Theorem 1.3 is an easy consequence of Theorem 1.1 and the following proposition, which actually
holds for any non-negative function on \( H^{d+2} \) instead of \( c_{\text{MT}} \).

Proposition 5.1. If \( \lambda > 0 \), then
\[
\int_B \int_0^{\text{diam}(B)} c_{\text{MT}}^2(x, t, \lambda) \frac{dt}{d+1} \mu(x) \leq \left(\frac{2}{\lambda}\right)^d \cdot \frac{C \mu}{d} \cdot c_{\text{MT}}^2(\mu|3.B, \lambda/2), \tag{81}
\]
for any ball \( B \subseteq H \).
Indeed, Theorem 1.1 and the \( d \)-regularity of \( \mu \) imply that

\[
\beta^2_2(x, t) \leq \frac{c^2_{\text{MT}}(x, t, \lambda_0)}{t^d}, \text{ for all } x \in \text{supp}(\mu) \text{ and } 0 < t \leq \text{diam}(\text{supp}(\mu)),
\]

where the comparison depends only on \( d \) and \( C_{\mu} \). Thus, by Proposition 5.1 we obtain the following estimate for all balls \( B \) such that \( \text{diam}(B) \leq \text{diam}(\text{supp}(\mu)) \):

\[
J_2(\mu|_B) = \int_B \left( \int_0^{\text{diam}(B)} \beta^2_2(x, t) \frac{dt}{t} \right) d\mu(x) \leq \frac{c^2_{\text{MT}}(\mu|_{3B}, \lambda_0/2)}{c^2_{\text{MT}}(\mu|_{3B})},
\]

where again the comparison depends only on \( d \) and \( C_{\mu} \).

The rest of this section proves Proposition 5.1.

5.1 Proof of Proposition 5.1

5.1.1 Preliminary Notation and Observations

For any \( x \in H \) and \( 0 < t < \infty \) we note the following trivial inclusion

\[
U_\lambda(B(x, t)) \subseteq W_{\lambda/2}(B(x, t)).
\]

If \( B \) is a ball of finite diameter in \( H \), let

\[
U_\lambda(B) = \left\{ (x, X, t) \in B \times H^{d+2} \times (0, \text{diam}(B)) : X \in U_\lambda(B(x, t)) \right\}.
\]

For fixed \( (x, X) \in B \times H^{d+2} \), we define the slice of \( U_\lambda(B) \) corresponding to \( (x, X) \):

\[
U_\lambda(B \mid x, X) = \left\{ t > 0 : (x, X, t) \in U_\lambda(B) \right\},
\]

and we note that

\[
U_\lambda(B \mid x, X) = \left[ \max_{0 \leq i \leq d+1} \|x_i - x\|, \frac{\min(X)}{\lambda} \right] := [u_1(x, X), u_2(x, X)].
\]

We define the following two projections. Let \( P_{1,2} : H \times H^{d+2} \times (0, \infty) \to H \times H^{d+2} \) be such that \( P_{1,2}(x, X, t) = (x, X) \), and let \( P_2 : H \times H^{d+2} \to H^{d+2} \) be the projection such that \( P_2(x, X) = X \).

We adopt the harmless convention of taking \( P_2(B, X, t) = P_2(x, X) = X \).

At last we note that the combination of equation (82) and the definition of \( U_\lambda(B) \) implies the inclusion

\[
P_2(U_\lambda(B)) \subseteq W_{\lambda/2}(3 \cdot B). \tag{84}
\]

5.1.2 Details of the Proof

We first apply Fubini’s Theorem and the definition of \( c^2_{\text{MT}}(x, t, \lambda) \) to rewrite the integral on the LHS of equation 81 in the following form

\[
\int_B \int_0^{\text{diam}(B)} c^2_{\text{MT}}(x, t, \lambda) \frac{dt}{t^{d+1}} d\mu(x) = \int_{P_{1,2}(U_\lambda(B))} c^2_{\text{MT}}(X) \left( \int_{U_\lambda(B \mid x, X)} \frac{dt}{t^{d+1}} \right) d\mu^{d+3}(x, X). \tag{85}
\]
Then, for \((x, X) \in P_{1,2} (\mathcal{U}_\lambda(B))\) such that \(\mathcal{U}_\lambda (B \mid x, X) \neq \emptyset\), let
\[
I(x, X) = \int_{\mathcal{U}_\lambda(B \mid x, X)} \frac{dt}{t^{d+1}}.
\] (86)

In view of equation [83] we get that
\[
I(x, X) = \int_{u_1(x, X)}^{u_2(x, X)} \frac{dt}{t^{d+1}}.
\]

We note that \(u_1(x, X) > 0\) a.e. on \(P_{1,2} (\mathcal{U}_\lambda(B))\) (w.r.t. \(\mu^{d+3}\)), and thus \(I(x, X) < \infty\) a.e. on \(P_{1,2} (\mathcal{U}_\lambda(B))\). This yields the inequality
\[
I(x, X) = \frac{1}{d} \left( \frac{1}{u_1(x, X)^d} - \frac{1}{u_2(x, X)^d} \right) \leq \frac{1}{d} \max_{0 \leq i \leq d+1} \frac{1}{\|x_i - x\|^d} \text{ a.e. on } P_{1,2} (\mathcal{U}_\lambda(B)).
\] (87)

Moreover, we can restrict our attention to \((x, X)\) such that \(I(x, X) > 0\). Defining the set
\[
I^{-1}(0, \infty) = \{(x, X) \in P_{1,2} (\mathcal{U}_\lambda(B)) : 0 < I(x, X) < \infty\},
\] (88)

and combining equations [85]-[87] we obtain the inequality
\[
\int_B \frac{\alpha_{\text{MT}}(x, t, \lambda)}{t^{d+1}} d\mu(x) \leq \frac{1}{d} \max_{0 \leq i \leq d+1} \frac{\alpha_{\text{MT}}(X)}{\|x_i - x\|^d} d\mu^{d+3}(x, X). \tag{89}
\]

In order to estimate the RHS of equation (89) we again apply Fubini’s Theorem. More specifically, for any \(X \in P_2 (I^{-1}(0, \infty))\) we define
\[
\overline{I}^{-1}(0, \infty) | X = \{ x \in H : (x, X) \in I^{-1}(0, \infty) \},
\]
and thus rewrite equation (89) as follows.
\[
\int_B \int_0^{\text{diam}(B)} \frac{\alpha_{\text{MT}}(x, t)}{t^{d+1}} d\mu(x) \leq \frac{1}{d} \int_{P_2(I^{-1}(0, \infty))} \alpha_{\text{MT}}(X) \left[ \int_{I^{-1}(0, \infty)} \frac{d\mu(x)}{\max_{0 \leq i \leq d+1} \|x_i - x\|^d} \right] d\mu^{d+2}(X). \tag{90}
\]

We next bound the inner integral of the above equation for a.e. \(X \in P_2 (I^{-1}(0, \infty))\), that is, the integral
\[
\int_{I^{-1}(0, \infty)} \frac{d\mu(x)}{\max_{0 \leq i \leq d+1} \|x_i - x\|^d}.
\] (91)

If \(X \in P_2 (I^{-1}(0, \infty))\) and \(x \in I^{-1}(0, \infty) | X\), then
\[
\frac{\min(X)}{\lambda} > \max_{0 \leq i \leq d+1} \|x_i - x\| > 0.
\]

Hence, for fixed \(X \in P_2 (I^{-1}(0, \infty))\), with \(x_0 = (X)_0\), we have the set inclusion
\[
I^{-1}(0, \infty) | X \subseteq B \left( x_0, \lambda^{-1} \cdot \min(X) \right).
\]
Thus the integral of equation (91) is bounded by
\[\int_{B(x_0, \lambda^{-1}\min(X))} \frac{d\mu(x)}{\max_{0 \leq i \leq d+1} \|x_i - x\|^d}.\]

Furthermore, \(\min(X) > 0\) a.e. on \(H^{d+2}\), and by the triangle inequality we obtain
\[\min(X) \leq 2 \cdot \max_{0 \leq i \leq d+1} \|x_i - x\|.\]

Combining these observations with the upper bound of equation (9), we have the following inequality for a.e. \(X \in P_2(I^{-1}(0, \infty))\):
\[\int_{I^{-1}(0, \infty)} X \frac{d\mu(x)}{\max_{0 \leq i \leq d+1} \|x_i - x\|^d} \leq 2^d \int_{B(x_0, \lambda^{-1}\min(X))} \frac{d\mu(x)}{\min(X)^d} \leq \left(\frac{2}{\lambda}\right)^d C.\]

Applying this uniform bound to the RHS of equation (90) we get that
\[\int_{B} \int_{0}^{\text{diam}(B)} c_{MT}(x, t) \frac{dt}{t^{d+1}} d\mu(x) \leq \left(\frac{2}{\lambda}\right)^d \cdot C \int_{P_2(I^{-1}(0, \infty))} c_{MT}^2(X) d\mu^{d+2}(X). \quad (92)\]

Further application of equations (84) and (88) bounds the integral on the RHS of equation (92) by \(c_{MT}^2(\mu|_{\lambda} B, \lambda/2)\) and thus concludes the proof.

6 A Menagerie of Curvatures

Here we discuss a variety of curvatures for \(d\)-regular measures on \(H\). In Subsection 6.1 we describe some curvatures that can be used to characterize uniform rectifiability, while indicating two levels of information needed for this purpose. In Subsection 6.2 we briefly give an example of continuous curvatures that can be used to quantify the \((p, p)\)-geometric property \(1 \leq p < \infty\) of David and Semmes [6]. Finally, in Subsection 6.3 we discuss our doubts about the utility of a previously suggested curvature for the purposes of implying the rectifiability of \(\mu\) [12, Theorem 0.3].

6.1 Curvatures Characterizing Uniform Rectifiability

We start with a few continuous curvatures that are completely equivalent to the Jones-type flatness (in the sense of Theorems 1.2 and 1.3). They thus characterize the uniform rectifiability of \(\mu\) by the criterion that the ratios between the curvatures of \(\mu|_B\) and the corresponding measures \(\mu(B)\) are uniformly bounded for all balls \(B\) in \(H\). We remark that here the continuous curvature of \(\mu|_B\) is obtained by integrating a corresponding discrete curvature over all \((d+1)\)-simplices in \(B^{d+2}\).

It is also possible to use a coarser level of information by introducing the parameter \(\lambda\) and modifying the continuous curvature of \(\mu|_B\) by integrating over the well-scaled set of simplices \(W_\lambda(B)\) (see equation (3)). In the case of the Menger-type curvatures, both types of continuous curvatures are comparable (up to possible blow ups of the ball \(B\)). We thus say that the Menger-type curvature is stable (when \(\lambda\) approaches zero). In Subsection 6.1.2 we present a discrete curvature for which we can easily compare the Jones-type flatness with the latter type of continuous curvature (with parameter \(\lambda\)). Currently, we cannot decide if this curvature is stable and thus cannot use the former version of continuous curvature to characterize uniform rectifiability.
6.1.1 Additional Stable Curvatures

We define the following discrete curvatures

\[
c_{\min}(X) = \sqrt{\min_{0 \leq i \leq d+1} \frac{p_d \sin^2 x_i(X)}{\text{diam}(X)^{d(d+1)}}},
\]

\[
c_{\text{vol}}(X) = \sqrt{\frac{M_{d+1}^2(X)}{\text{diam}(X)^{(d+1)(d+2)}}},
\]

and

\[
c_{\max}(X) = \sqrt{\max_{0 \leq i \leq d+1} \frac{p_d \sin^2 x_i(X)}{\text{diam}(X)^{d(d+1)}}}.
\]

We note that for all \( X \in H^{d+2} \) with \( \text{min}(X) > 0 \):

\[
c_{\text{MT}}^2(X) \approx c_{\max}^2(X) \geq c_{\min}^2(X) \geq c_{\text{vol}}^2(X).
\]

Furthermore, we note that for \( \lambda > 0 \) and any 1-separated element \( X \),

\[
c_{\text{vol}}^2(X) \geq \lambda^{2(d+1)} \cdot c_{\text{MT}}^2(X).
\]

As such, analogous estimates to those of Theorems 1.2 and 1.3 hold for the curvatures \( c_{\min}, c_{\text{vol}}, \) and \( c_{\max} \). In particular, the continuous curvatures (integrated over all simplices in corresponding products of balls) \( c_{\min}, c_{\text{MT}}, c_{\text{vol}} \) and \( c_{\max} \) are comparable (up to possible blow ups of the underlying balls).

We can further extend this collection of stable curvatures. For example, we may include any order statistics of the \( p \)-sines of vertices of a given simplex (replacing the maximum or the minimum, which are used in \( c_{\max} \) and \( c_{\min} \) respectively).

6.1.2 An Algebraic Curvature with Questionable Stability

For \( X \in H^{d+2} \), let

\[
c_{\text{alg}}(X) = \begin{cases} 
\frac{p_d \sin x_0(X)}{\prod_{1 \leq i < j \leq d+1} \|x_i - x_j\|}, & \text{if } \text{min}(X) > 0, \\
0, & \text{otherwise}.
\end{cases}
\]

We see that unlike the curvatures of Subsection 6.1.1 this one is algebraic. In fact, it is the invariant ratio of the law of polar sines expressed in equation (11).

We trivially have the inequality

\[
c_{\text{alg}}^2(X) \geq c_{\text{MT}}^2(X), \text{ for all } X \in H^{d+2}.
\]

Furthermore, if \( X \) is 1-separated for \( \lambda > 0 \), then we also have the opposite inequality:

\[
c_{\text{MT}}^2(X) \geq \lambda^{d(d+1)} \cdot c_{\text{alg}}^2(X).
\]

Hence, for any \( 0 < \lambda \leq \lambda_0 \) (where \( \lambda_0 \) is the constant suggested by Theorem 1.1) and all balls \( B \subseteq H \) we have the estimate (with constants depending on \( d, C_\mu, \) and \( \lambda \)):

\[
J_2(\mu|\frac{1}{2}B) \lesssim c_{\text{alg}}^2(\mu|B, \lambda/2) \lesssim J_2(\mu|6B).
\]

(93)
Thus, fixing such $\lambda$ one can use the curvatures $\{c^2_{\text{alg}}(\mu|B, \lambda/2)\}_{B \subseteq H}$ to characterize uniform rectifiability.

We are not sure whether one can replace the curvature $c^2_{\text{alg}}(\mu|B, \lambda/2)$ in equation (93) by the curvature $c^2_{\text{alg}}(\mu|B)$ (even with the introduction of blow ups of the ball $B$ in that equation). That is, we cannot decide at this point if the algebraic curvature is stable or not. It is clear though that our methods for controlling $c^2_{\text{MT}}(\mu|B)$ by $J_2(\mu|B)$ (as expressed in Theorem 1.2 and established in [16]) are insufficient for controlling $c^2_{\text{alg}}(\mu|B)$ by $J_2(\mu|B)$ for some $C > 1$ (independent of $B$).

### 6.2 Curvatures Characterizing the $(p, p)$-Geometric Property $(1 \leq p < \infty)$

For $1 \leq p < \infty$, let
\[
\widetilde{J}_p(\mu|B) = \int_B \int_0^{\text{diam}(B)} \beta_p^p(x, t) \frac{d\mu(x)}{t} dt.
\]

We note that if $p \neq 2$, then $\widetilde{J}_p(\mu|B)$ differs from $J_p(\mu|B)$ (defined in equation (2.6)) in the power of $\beta_p(x, t)$. A $d$-regular measure $\mu$ on $H$ satisfies the $(p, p)$-geometric property [6, Part IV] if there exists a constant $C = C(\mu)$ such that
\[
\widetilde{J}_p(\mu|B) \leq C \cdot \mu(B), \text{ for all balls } B \subseteq H.
\]

The methods of this paper and [16] extend to comparing $\widetilde{J}_p$ with a different kind of continuous curvature as follows.

**Theorem 6.1.** If $\mu$ is a $d$-regular measure on $H$ and $1 \leq p < \infty$, then there exists a constant $C_8 = C_8(d, C_\mu, p)$ such that
\[
\frac{1}{C_8} \cdot \widetilde{J}_p(\mu|\frac{1}{2}B) \leq \int_{B^{d+2}} \frac{p \text{sin}^p(x_0)(X)}{\text{diam}(X)^{d+1}} \, d\mu^{d+2}(X) \leq C_8 \cdot \widetilde{J}_p(\mu|6B),
\]
for all balls $B \subseteq H$.

We thus obtain the following characterization of the $(p, p)$-geometric property.

**Corollary 6.1.** If $1 \leq p < \infty$ and $\mu$ is a $d$-regular measure on $H$, then $\mu$ satisfies the $(p, p)$-geometric property if and only if there exists a constant $C = C(d, C_\mu)$ such that
\[
\int_{B^{d+2}} \frac{p \text{sin}^p(x_0)(X)}{\text{diam}(X)^{d+1}} \, d\mu^{d+2}(X) \leq C \cdot \mu(B)
\]
for all balls $B \subseteq H$.

If $p = 2$ then Theorem 6.1 coincides with the combination of Theorems 1.2 and 1.3. Similarly, in that case Corollary 6.1 coincides with [16, Theorem 1.3].

### 6.3 A Previously Suggested Curvature

Léger [12] proposed the following discrete curvature for $(d+1)$-simplices $X = (x_0, \ldots, x_{d+1}) \in H^{d+2}$ where $d \geq 1$:
\[
c^d_{L}(X) = \frac{\text{dist}(x_0, L[X(0)])^{d+1}}{\prod_{i=1}^{d+1} ||x_i - x_0||^{d+1}},
\]

28
and the corresponding continuous curvature for $\mu$ restricted to any ball $B \subseteq H$

$$c_{L}^{d+1}(\mu|B) = \int_{B^{d+2}} c_{L}^{d+1}(X) \, d\mu^{d+2}(X).$$

If $d = 1$, then his curvature coincides with the Menger curvature \[19, 20\] (up to multiplication by a constant). He showed how to use his curvature in that case to infer rectifiability properties of $\mu$. In particular, he established Theorem 1.3 when $d = 1$.

Léger’s approach for proving the same type of results for $d \geq 2$ (while using the curvature $c_{L}^{d+1}(X)$) ostensibly requires a bound of the form

$$J_{2}(\mu|B) \lesssim c_{L}^{d+1}(\mu|\frac{1}{3}B),$$

thus generalizing \[12, Lemma 2.5\]. However, any analysis or adaptation of the proof of \[12, Lemma 2.5\] to the curvature $c_{L}^{d+1}(X)$, where $d > 1$, seems to give at best the following lower bound.

**Proposition 6.1.** There exists a constant $C_{9} = C_{9}(d, C_{\mu})$ such that

$$\tilde{J}_{d+1}\left(\mu|\frac{1}{3}B\right) \leq C_{9} \cdot c_{L}^{d+1}(\mu|B)$$

for any ball $B \subseteq H$.

If $d > 1$, then the function $\tilde{J}_{d+1}$ (discussed in the previous subsection) can be significantly smaller than the required quantity $J_{2}$ (especially for very large $d$). This function also characterizes the $(d + 1, d + 1)$-geometric property (see Corollary 6.1), which includes measures that are not uniformly rectifiable whenever $d > 1$.

**7 Open and New Directions**

We conclude the work presented here as well as in \[16\] by suggesting possible directions for extending it. Most of them are wide open.

$L_{2}(\mu)$ boundedness of $d$-dimensional Riesz transform

Mattila, Melnikov and Verdera \[19\] used the one-dimensional Menger curvature to show that a one-regular measure $\mu$ is uniformly rectifiable if and only if the one-dimensional Riesz transform is bounded in $L_{2}(\mu)$ (they actually showed it for the Cauchy kernel where $H = \mathbb{C}$, but their analysis extends easily to the former case). Farag \[7\] showed that direct generalization of their arguments to higher-dimensional Riesz transforms are not possible. Nevertheless, one might suggest alternative strategies to study whether $L_{2}(\mu)$ boundedness of the Riesz kernel implies that for all balls $B \subseteq H$ the quantity $c_{MT}(\mu|B)/\mu(B)$ is finite. In \[8\] numerical experiments have been performed in order to test some heuristic strategies for such study. However, at this stage they have not advanced our theoretical understanding of the problem.

**Calculus of curvatures**

Our work suggests various computational techniques for obtaining careful estimates of the high-dimensional Menger-type curvatures. Our writing indicates the lack of some very basic machinery for those curvatures. Indeed, while our analysis was based on very elementary ideas, its writing...
required substantial development. Nevertheless, the techniques established here could be taken for granted in subsequent papers. Still more technical tools need to be developed, and in particular we are interested in tools leading to solutions of the following straightforward questions: 1. determination of stability of the algebraic curvature $c_{\text{alg}}$ when $d > 1$ (see Subsection 6.1.2); 2. characterization of the $(p, q)$ geometric property [6, Part IV] by the Menger-type curvature for special cases where $p \neq q$; 3. relating the Menger-type curvature with the functions $J_p$ for all $1 \leq p < 2 \cdot d/(d - 2)$ (we also wonder if such a relation is different for the two cases: $p < 2$ and $p > 2$).

**Rectifiability by Menger-type curvatures**

When $d = 1$ Léger [12] formulated a modified version of Theorem 1.3 for more general measures and used it to establish a criterion for rectifiability for another large class of measures. His analysis immediately extends to our setting, in particular, it implies that finiteness of the $d$-dimensional Menger-type curvatures of certain measures is a sufficient condition for $d$-dimensional rectifiability. We ask about a necessary condition for rectifiability (as sharp as possible) formulated in terms of the Menger-type curvatures.

**Extensions to noisy setting**

Ahlfors regular measures (i.e., $d$-regular) are rather synthetic for real applications. In [17] we extend both Proposition 1.1 and Theorem 1.1 to a wide class of probability distributions, in particular, distributions with “additive noise” around $d$-dimensional surfaces. An application of this result appears in [2].

**Discrete Curvatures of General Metric Spaces**

Immo Hahlomaa [26] has formed a Menger-type curvature of one-regular measures in metric spaces using the Gromov product. He [27] and also Raanan Schul [21] have used it to characterize uniform rectifiability of such measures, i.e., the existence of a sufficiently regular curve containing their support. The Gromov product is indeed a natural quantity for this purpose since it is quasi-isometric to distances to “geodesic curves” in any fixed ball [28]. We inquire about a similar quantity that can be used to form $d$-dimensional Menger-type curvatures, where $d > 1$, for characterizing uniform rectifiability in some non-Euclidean metric spaces. In this case uniform rectifiability can be defined by a parametrizing regular surface [24] or alternatively by big pieces of bi-Lipschitz images at all relevant scales and locations [22, 23].

**Multi-manifold data modeling and applications**

Insights of Proposition 1.1 and Theorem 1.1 are used in [2, 3] to solve the problem of hybrid linear modeling. In this setting, data is sampled from a mixture of affine subspaces (with additive noise and outliers) and one needs to cluster the data appropriately. This problem generalizes to the setting of multi-manifold modeling, where affine subspaces take the form of manifolds. It also further extends to the case where the data is embedded in metric space and not necessarily a Euclidean space (here affine subspaces are replaced by geodesic surfaces). We believe that any extension of the theory presented in this paper and in [16] to those general settings could be used to enhance the methods for solving such problems (see e.g., [1]).
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