Probabilistically Robust PAC Learning
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Abstract

Recently, Robey et al. propose a notion of probabilistic robustness, which, at a high-level, requires a classifier to be robust to most but not all perturbations. They show that for certain hypothesis classes where proper learning under worst-case robustness is not possible, proper learning under probabilistic robustness is possible with sample complexity exponentially smaller than in the worst-case robustness setting. This motivates the question of whether proper learning under probabilistic robustness is always possible. In this paper, we show that this is not the case. We exhibit examples of hypothesis classes \(H\) with finite VC dimension that are not probabilistically PAC learnable with any proper learning rule. However, if we compare the output of the learner to the best hypothesis for a slightly stronger level of probabilistic robustness, we show that not only is proper learning always possible, but it is possible via empirical risk minimization.

1 Introduction

As deep neural networks become increasingly ubiquitous, their susceptibility to test-time adversarial attacks has become more and more apparent. Designing learning algorithms that are robust to these test-time adversarial perturbations has garnered increasing attention by machine learning researchers and practitioners alike. Prior work on adversarially robust learning has mainly focused on learnability under the worst-case robust risk \([\text{MHS19, AKM19, CBM18}]\), defined as

\[
R_U(h; \mathcal{D}) := \mathbb{E}_{(x, y) \sim \mathcal{D}} \left[ \sup_{z \in \mathcal{U}(x)} \mathbb{1}\{h(z) \neq y\} \right],
\]

where \(\mathcal{U}(x) \subset \mathcal{X}\) is an arbitrary but fixed perturbation set (for example \(L_p\) balls). In practice, worst-case adversarial robustness is commonly achieved via Empirical Risk Minimization (ERM) of some convex surrogate of the robust loss \([\text{BSS20}]\). However, a seminal result by \([\text{MHS19}]\) shows that ERM-based learners, and more generally proper learning rules, does not always work. Furthermore, several empirical studies have shown that classifiers trained to achieve worst-case adversarial robustness also exhibit degraded nominal performance \([\text{DHHR20, RXY+19, SZC+18, TSE+18, YRZ+20, ZYJ+19, RCPH22}]\).

In light of these difficulties, several works have started to study when proper learning, and more specifically, when ERM is possible for achieving adversarial robustness. In this vain, \([\text{APU22}]\) and \([\text{BHK+22}]\) consider adversarial robust learning in the tolerant setting, where the error of the learner is compared with the best achievable error with respect to a slightly larger perturbation set. Here, they show that the sample complexity of tolerant robust learning can be significantly lower than the current known sample complexity for adversarially robust learning and that proper learning via ERM can be possible under certain assumptions. However, these works focus on metric spaces, while we keep our space \(\mathcal{X}\) general. In a different direction, several works have considered relaxing the worst-case nature of the adversarial robust risk \(R_G(h; \mathcal{D})\) \([\text{RCPH22, LBSS20, LBSS21, LF19, RBZK21}]\). However, the PAC learnability under these more relaxed notions of adversarial robust risk has not been well studied.

Recently, \([\text{RCPH22}]\) consider a probabilistic relaxation of adversarial robustness when the instance space \(\mathcal{X}\) is \(\mathbb{R}^d\). In this setting, the goal is to find a hypothesis \(h \in \mathcal{H}\) with low probabilistic robust risk, defined as,

\[
R_{\Delta, \mu}^{\rho}(h; \mathcal{D}) := \mathbb{E}_{(x, y) \sim \mathcal{D}} \left[ \mathbb{1}\{P_{z \sim \mu} (h(x + z) \neq y) > \rho\} \right],
\]
where $\Delta \subset \mathbb{R}^d$ is a set of perturbations, $\mu$ is some fixed measure over $\Delta$, and $\rho$ is some pre-specified robustness level. They show that for certain hypothesis classes where proper learning under adversarial robustness is not possible, proper learning under probabilistic robustness is possible, with sample complexity that can be exponentially smaller than what is required for (improper) learning under adversarial robustness.

In this work, we extend the notion of probabilistic robustness to general instance spaces $X$ and study the probabilistically robust PAC learnability of arbitrary hypothesis classes. In order to define the probabilistic robust risk for an arbitrary instance space $X$, we alternatively represent $\mathcal{U}$ as an index set $\mathcal{G}$ of perturbation functions $g : X \to X$ such that $\mathcal{G}(x) = \{g(x) : g \in \mathcal{G}\} = \mathcal{U}(x)$. This representation still preserves the arbitrary nature of $\mathcal{U}$. Indeed, in Appendix A, we prove that these two models are actually equivalent - every perturbation set $\mathcal{U}$ can be represented by a set of perturbation functions $\mathcal{G}$, and vice versa. Under this reparameterization, we can (re)define the probabilistic robust risk as

$$R_G^{\rho, \mu}(h ; \mathcal{D}) := \mathbb{E}_{(x,y) \sim \mathcal{D}} \left[ \mathbb{P}_{g \sim \mu} \{ h(g(x)) \neq y \} \right],$$

where $\mathcal{G}$ is an arbitrary set of perturbations functions, $\mu$ is a fixed but arbitrary probability measure over $\mathcal{G}$, and $\rho$ is a pre-specified robustness level. Note that we can also rewrite the adversarial robust risk as $R_G(h ; \mathcal{D}) = \mathbb{E}_{(x,y) \sim \mathcal{D}} \sup_{g \in \mathcal{G}} \mathbb{P}_{g \sim \mu} \{ h(g(x)) \neq y \}$.

Roughly speaking, learning under probabilistic robustness asks to find a hypothesis $h \in \mathcal{H}$ that is robust to most, but not all, perturbations for each example in the support of the data distribution $\mathcal{D}$. As highlighted in [RCPH22], this notion of robustness is desirable as it nicely interpolates between worst and average case robustness via an interpretable parameter $\rho$, while being more computationally tractable compared to existing relaxations. We note that probabilistic robustness is a strict relaxation of worst-case robustness. While $R_G(h ; \mathcal{D}) \leq \epsilon$ implies $R_G^{\rho, \mu}(h ; \mathcal{D}, \rho) \leq \epsilon$ for every $\rho \in [0,1]$ and every hypothesis $h$, the converse is not true even for $\rho = 0$. Indeed, when $\rho = 0$, there exists problems where a classifier that is non-robust to a countably infinite number of perturbations for every $x \in X$ still achieves $R_G^{\rho, \mu}(h ; \mathcal{D}) = 0$, assuming a continuous measure $\mu$.

Which hypothesis classes are probabilistically robustly learnable, and that so using proper learning rules which output predictors in $\mathcal{H}$? Concretely, for a hypothesis class $\mathcal{H} \subset \mathcal{Y}^X$, and an adversary $(\mathcal{G}, \mu)$, is it possible to identify, based on an i.i.d. sample, a hypothesis $h \in \mathcal{H}$ whose probabilistic robust risk is close to the best possible risk amongst those hypotheses in $\mathcal{H}$? Our main results, summarized below, aim to answer these questions.

- We show that for any robustness level $\rho$, there exists an adversary $(\mathcal{G}, \mu)$ and a hypothesis class $\mathcal{H}$ with finite VC dimension that cannot be learned under probabilistic robustness using any proper learning rule.

- We show that for every robustness level $\rho$, hypothesis class $\mathcal{H}$, and adversary $(\mathcal{G}, \mu)$, there exists a proper learning rule that can output a hypothesis $h \in \mathcal{H}$ whose probabilistic robust risk at level $\rho$ is comparable to the probabilistic robust risk of the best hypothesis at a level $\rho^* < \rho$. Moreover, the proper learning rule is an ERM-based learner.

- We show that for every $\rho$, $\mathcal{H}$, and adversary $(\mathcal{G}, \mu)$, running ERM over the adversarial robust risk is sufficient to find a predictor $h \in \mathcal{H}$ whose probabilistic robust risk at the level $\rho$ is comparable to the adversarial robust risk of the best hypothesis in $\mathcal{H}$.

- We show a more general property, termed Sandwich Uniform Convergence, which unifies our results in probabilistically robust learning and allows us to study Robust PAC Learning with Tolerance.

Our results indicate that although proper learning is not always possible for adversarially and probabilistically robust PAC learning, the story changes if we compare the output of the learner to the best hypothesis of a stronger notion of robustness. In this sense, our results suggest that practitioners need not disregard ERM-based approaches to achieve test-time robustness, but should rather change how they interpret the generalization guarantees of the resulting classifier.
2 Notation, Preliminaries, and Problem Setup

2.1 Notation
Throughout this paper we will let $|k|$ denote the set of integers $\{1, \ldots, k\}$, $\mathcal{X}$ denote an instance space, $\mathcal{Y} = \{-1, 1\}$ denote our label space, and $\mathcal{D}$ be any distribution over $\mathcal{X} \times \mathcal{Y}$. $\mathcal{H} \subset \mathcal{Y}^\mathcal{X}$ will denote a hypothesis class mapping examples in $\mathcal{X}$ to labels in $\mathcal{Y}$.

2.2 Problem Setting
In the adversarially robust setting, there exists an adversary who picks an arbitrary index set $\mathcal{G}$ of perturbation functions $g : \mathcal{X} \rightarrow \mathcal{X}$. At test time, the adversary intercepts a test example $(x, y) \sim \mathcal{D}$, picks a perturbation function $g \in \mathcal{G}$, and passes $g(x)$ to the learner. In the probabilistic setting, the adversary additionally selects a probability measure $\mu$ over $\mathcal{G}$. At test time, the adversary samples $g \sim \mu$, and passes $g(x)$ to the learner. In this work, we assume that $\mu$ is fixed and does not depend on $x \in \mathcal{X}$. We show that even in this setting, (proper) learning can be difficult. We leave the case where even $\mu$ might depend on $x$ for future work. We define the adversarial and probabilistic robust loss as

$$\ell_g(h, (x, y)) := \sup_{g \in \mathcal{G}} \mathbb{1}\{h(g(x)) \neq y\},$$

and

$$\ell^\rho_{\mathcal{G}, \mu}(h, (x, y)) := \mathbb{1}\{|\mathbb{P}_{g \sim \mu}(h(g(x))) \neq y| > \rho\},$$

respectively, where $0 \leq \rho < 1$ is a robustness-level selected apriori. The adversarial and probabilistic robust risk can now be defined as $R_{\mathcal{G}}(h; \mathcal{D}) = \mathbb{E}_{(x, y) \sim \mathcal{D}}[\ell_g(h, (x, y))]$ and $R^\rho_{\mathcal{G}, \mu}(h; \mathcal{D}) = \mathbb{E}_{(x, y) \sim \mathcal{D}}[\ell^\rho_{\mathcal{G}, \mu}(h, (x, y))]$, respectively.

In this paper, we are primarily interested in the sample complexity of probabilistic robust learning. Given a hypothesis class $\mathcal{H}$, and labelled samples from an unknown distribution $\mathcal{D}$, our goal is to design a learning algorithm $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{Y}^{\mathcal{X}}$ such that for any distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, $A$ finds a hypothesis with low probabilistic risk using a number of samples that is independent from $\mathcal{D}$. The following definition formalizes the notion probabilistically robust PAC learning:

**Definition 1** ($\rho$-Probabilistically Robust PAC Learning). For any $\epsilon, \delta \in (0, 1)$ and any $\rho \in [0, 1)$, the sample complexity of $\rho$-probabilistically robust $(\epsilon, \delta)$-PAC learning of $\mathcal{H}$ with respect to adversary $(\mathcal{G}, \mu)$, denoted $n(\epsilon, \delta, \rho; \mathcal{H}, \mathcal{G}, \mu)$, is the smallest number $m \in \mathbb{N}$ for which there exists a learning rule $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{Y}^\mathcal{X}$ such that for every distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, with probability at least $1 - \delta$ over $\mathcal{S} \sim \mathcal{D}^m$,

$$R^\rho_{\mathcal{G}, \mu}(A(S); \mathcal{D}) \leq \inf_{h \in \mathcal{H}} R^\rho_{\mathcal{G}, \mu}(h; \mathcal{D}) + \epsilon.$$

We say that $\mathcal{H}$ is probabilistically robustly PAC learnable with respect to adversary $(\mathcal{G}, \mu)$ at a level of $\rho$, if for all $\epsilon, \delta \in (0, 1)$, $n(\epsilon, \delta, \rho; \mathcal{H}, \mathcal{G}, \mu)$ is finite.

In addition, in this work we focus on understanding when proper learning is possible. A learning algorithm $A$ is proper if it always outputs a hypothesis in $\mathcal{H}$.

**Definition 2** (Proper Learning). A hypothesis class $\mathcal{H}$ is properly $\rho$-probabilistically robustly learnable if it can be learned according to Definition 1 using a learning rule $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{H}$ that always outputs a hypothesis in $\mathcal{H}$. Any learning algorithm $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{Y}^\mathcal{X}$ is an improper learner.

The ability to achieve test-time robustness via proper learning rules is important from a practical standpoint. It aligns better with the current approaches used in practice and proper learning algorithms are often more simpler to implement than improper ones. In fact, the common attempt to achieve robustness in practice is to learn through empirical risk minimization (ERM). That is, given a hypothesis class $\mathcal{H}$, and sample $S = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, compute either

$$\hat{h} \in \text{ERM}(S; \mathcal{G}) := \arg \min_{h \in \mathcal{H}} \hat{R}_g(h; S)$$

or

$$\hat{h} \in \text{PRERM}(S; \mathcal{G}, \mu, \rho) := \arg \min_{h \in \mathcal{H}} \hat{R}^\rho_{\mathcal{G}, \mu}(h; S, \rho)$$

where $\hat{R}_g(h; S) = \frac{1}{m} \sum_{i=1}^{m} \ell_g(h, (x_1, y_1))$ and $\hat{R}^\rho_{\mathcal{G}, \mu}(h; S) := \frac{1}{m} \sum_{i=1}^{m} \ell^\rho_{\mathcal{G}, \mu}(h, (x_i, y_i))$. 


2.3 Complexity Measures

Under the standard 0-1 risk, the Vapnik-Chervonenkis dimension (VC dimension) plays an important role in characterizing PAC learnability, and more specifically, when ERM is possible. A hypothesis class \( \mathcal{H} \) is PAC learnable if and only if its VC dimension is finite [VC71].

**Definition 3 (VC Dimension).** A set \( \{x_1, ..., x_n\} \in \mathcal{X} \) is shattered by \( \mathcal{H} \), if \( \forall y_1, ..., y_n \in \mathcal{Y}, \exists h \in \mathcal{H} \), s.t. \( \forall i \in [n], h(x_i) = y_i. \) The VC dimension of \( \mathcal{H} \), denoted \( \text{VC}(\mathcal{H}) \), is defined as the largest natural number \( n \in \mathbb{N} \) such that there exists a set \( \{x_1, ..., x_n\} \in \mathcal{X} \) that is shattered by \( \mathcal{H} \). When the VC dimension is finite, \( \mathcal{H} \) is properly learnable via an Empirical Risk Minimization (ERM) oracle.

Is finite VC dimension also sufficient for proper probabilistic robust PAC learning against an arbitrary adversary? Note that in the adversarially robust setting, this is not the case, as shown by [MHS19]. One sufficient condition for proper learning, based on Vapnik’s “General Learning” [Vap06], is the finiteness of the VC dimension of the probabilistic robust loss class \( \mathcal{L}_{\mathcal{H}}^{\mu, \rho} \):

\[
\mathcal{L}_{\mathcal{H}}^{\mu, \rho} = \{(x, y) \mapsto \mathbb{I}(\mathbb{P}_{\mathcal{D} \sim \mu}(h(g(x)) \neq y) > \rho) : h \in \mathcal{H}\}.
\]

In particular, if the VC dimension of the probabilistic robust loss class \( \mathcal{L}_{\mathcal{H}}^{\mu, \rho} \) is finite, then \( \mathcal{H} \) is probabilistically robustly PAC learnable via oracle access to a Probabilistic Robust Empirical Risk Minimizer (PRERM) with sample complexity that scales linearly with \( \text{VC}(\mathcal{H}) \). In this sense, if one can upper bound \( \text{VC}(\mathcal{L}_{\mathcal{H}}^{\mu, \rho}) \) in terms of \( \text{VC}(\mathcal{H}) \), then finite VC dimension is sufficient for proper learnability. Unfortunately, in the next section we will show that there can be an arbitrarily large gap between these two quantities even when the measure \( \mu \) is fixed, and proper learning overall might also not be possible for probabilistically robust PAC learning.

In addition to the VC dimension, the fat shattering dimension at scale of \( \gamma \) of a real-valued function class \( \mathcal{F} \), denoted \( \text{fat}(\mathcal{F}, \gamma) \), is another complexity measure that will be useful to us.

**Definition 4 (Fat Shattering Dimension).** Let \( \mathcal{F} \subseteq \{0, 1\}^\mathcal{X} \) and \( \gamma > 0 \). We say that \( S = \{x_1, ..., x_m\} \subseteq \mathcal{X} \) is \( \gamma \)-shattered by \( \mathcal{F} \) if there exists a witness \( r \in \{r_1, ..., r_m\} \in [0, 1]^m \) such that for each \( (\sigma_1, ..., \sigma_m) \in \{-1, 1\}^m \) there is a function \( f_\sigma \in \mathcal{F} \) such that

\[
\forall i \in [m] \quad \begin{cases} f_\sigma(x_i) \geq r_i + \gamma, & \text{if } \sigma_i = 1 \\ f_\sigma(x_i) \leq r_i - \gamma, & \text{if } \sigma_i = -1. \end{cases}
\]

The fat-shattering dimension of \( \mathcal{F} \) at scale \( \gamma \), denoted as \( \text{fat}(\mathcal{F}, \gamma) \), is the largest natural number \( n \in \mathbb{N} \) such that \( n \) points in \( \mathcal{X} \) can be \( \gamma \)-shattered by \( \mathcal{F} \). We say that \( \mathcal{F} \) has finite fat-shattering dimension if \( \text{fat}(\mathcal{F}, \gamma) < \infty \) for all \( \gamma > 0 \).

Analogously to VC Dimension, the Fat Shattering Dimension plays a key role in characterizing the PAC learnability of real-valued function classes. More specifically, Theorem 1 shows that for any Lipschitz loss function, finite Fat Shattering Dimension implies uniform convergence.

**Theorem 1 (Fat Shattering-based Uniform Convergence).** If \( \mathcal{F} \subseteq \mathbb{R}^\mathcal{X} \) is a real-valued function class with finite fat shattering dimension, then for any \( L \)-Lipschitz loss function \( \ell : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1] \), with probability \( 1 - \delta \) over \( S \sim \mathcal{D}^m \) of size \( m = O\left(\frac{\text{fat}(\mathcal{F}, \gamma)}{\ell^2} + \ln\frac{1}{\delta} \gamma \right) \), for all functions \( f \in \mathcal{F} \) simultaneously,

\[
|\mathbb{E}_\mathcal{D}[\ell(f(x), y)] - \mathbb{E}_S[\ell(f(x), y)]| \leq \epsilon
\]

where \( K \) is some universal constant.

In this paper, the Fat Shattering Dimension will be useful in characterizing the complexity of the \((\mathcal{G}, \mu)\)-smoothed hypothesis class \( \mathcal{H} \), defined hereinafter as

\[
\mathcal{F}_{\mathcal{H}, \mathcal{G}, \mu} := \{\mathbb{E}_{\mathcal{G} \sim \mu}[h(g(x))] : h \in \mathcal{H}\}.
\]

Lastly, we introduce the empirical Rademacher complexity, another complexity measure useful for showing uniform convergence.
Definition 5 (Empirical Rademacher Complexity). Let $D$ be a distribution over $\mathcal{X} \times \mathcal{Y}$, and let $\ell(h, (x, y)) \leq c$ be a bounded loss function. Let $S = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ be a set of examples drawn i.i.d from $D$ and $F = \{(x, y) \mapsto \ell(h, (x, y)) : h \in \mathcal{H}\}$ be a loss class. The empirical Rademacher complexity of $F$ is defined as

$$\hat{\mathcal{R}}_m(F) = \mathbb{E}_\sigma \left[ \sup_{f \in F} \left( \frac{1}{m} \sum_{i=1}^{m} \sigma_i f(x_i, y_i) \right) \right]$$

where $\sigma_1, \ldots, \sigma_m$ are independent Rademacher random variables, random variables uniformly distributed over $\{-1, 1\}$.

A standard result relates the empirical Rademacher complexity to the generalization error of hypotheses in $\mathcal{H}$.

Theorem 2 (Rademacher-based Uniform Convergence). With probability at least $1 - \delta$ over the sample $S \sim D^m$, for all $h \in \mathcal{H}$ simultaneously,

$$\left| \mathbb{E}_D[\ell(h, (x, y))] - \hat{\mathbb{E}}_S[\ell(h, (x, y))] \right| \leq 2\hat{\mathcal{R}}_m(F) + O\left( c \sqrt{\frac{\ln(\frac{1}{\delta})}{m}} \right)$$

where $\hat{\mathbb{E}}_S[\ell(h, (x, y))] = \frac{1}{|S|} \sum_{(x, y) \in S} \ell(h, (x, y))$ is the empirical average of the loss over $S$.

3 Proportion Probabilistically Robust Learning Is Not Always Possible

In this section, we show that even for hypothesis classes with finite VC dimension, $\rho$-probabilistically robust PAC learning might not be possible using any proper learning rule. In particular, this implies that even if there is a probabilistic robust hypothesis in $\mathcal{H}$, and even with arbitrarily large number of samples, the PRERL may not guarantee low risk. That is, finite VC dimension is not sufficient for proper $\rho$-probabilistically robust PAC learning.

For the proofs in this section we fix $\mathcal{X} = \mathbb{R}^d$, $G = \{g_\delta : \delta \in \mathbb{R}^d, ||\delta||_\mu < \gamma\}$ s.t. $g_\delta(x) = x + \delta$ for all $x \in \mathcal{X}$ for some $\gamma > 0$, and $\mu$ to be the uniform measure over $G$. In other words, we are picking our perturbation sets to be $L_p$ balls of radius $\gamma$ and our perturbation measures to be uniform over each perturbation set. Note that by construction of $G$, a uniform measure $\mu$ over $G$ also induces a uniform measure $\mu'$ over $G(x) = \{g_\delta(x) : g_\delta \in G\} \subset \mathbb{R}^d$. For this section, we will overload notation by letting the uniform measure $\mu$ over $G$ also represent its induced measure over $G(x)$.

We start by showing that for every $\rho \in [0, 1)$, there can be an arbitrary gap between the VC dimension of the loss class and the VC dimension of the hypothesis class.

Lemma 3. For every $\rho \in [0, 1)$ and $m \in \mathbb{N}$, there exists a hypothesis class $\mathcal{H} \subset \mathcal{Y}^{\mathcal{X}}$ s.t. $VC(\mathcal{H}) \leq 1$ but $VC(L_{\mathcal{H}}^{m, \rho}) \geq m$.

Proof. Fix a $\rho \in [0, 1)$. Let $m \in \mathbb{N}$. Pick $m$ center points $c_1, \ldots, c_m$ in $\mathcal{X}$ such that for all $i, j \in [m]$, $G(c_i) \cap G(c_j) = 0$. For each center $c_i$, consider $2m^{-1} + 1$ disjoint subsets of its perturbation set $G(c_i)$ which do not contain $c_i$. Label $2m^{-1}$ of these subsets with a unique bitstring $b \in \{0, 1\}^m$ fixing $b_i = 1$. Let $\mathcal{B}_i^b$ denote the subset labelled by bitstring $b$ and let $\mathcal{B}_i$ denote the single remaining subset that was not labelled. Let $\mu(\mathcal{B}_i) = \rho$ and $0 < \mu(\mathcal{B}_i^b) < \frac{1 - \rho}{m - 1}$ for every $b \in \{\{0, 1\}^m | b_i = 1\}$. If $\rho = 0$, let $\mathcal{B}_i = \emptyset$ for all $i \in [m]$. Observe that indeed $\mu\left(\mathcal{B}_i \cup \left(\bigcup_{b} \mathcal{B}_i^b\right)\right) < 1$. For bitstring $b \in \{0, 1\}^m$, define the hypothesis $h_b$ as

$$h_b(z) = \begin{cases} 0 & \text{if } z \in \bigcup_{i=1}^{m} \mathcal{B}_i^b \cup \mathcal{B}_i \\ 1 & \text{otherwise} \end{cases}$$

and consider the hypothesis class $\mathcal{H} = \{h_b | b \in \{0, 1\}^m\}$ which consists of all $2^m$ hypotheses, one for each bitstring. Finally, define $\mathcal{B} = \bigcup_{i=1}^{m} \bigcup_{b \in \{\{0, 1\}^m | b_i = 1\}} \mathcal{B}_i^b \cup \mathcal{B}_i$ as the union of all the subsets. We first show that $\mathcal{H}$ has VC dimension at most 1. Consider two points $x_1, x_2 \in \mathcal{X}$. We will show case by case that every possible pair of points cannot be shattered by $\mathcal{H}$. First, consider the case where,
wlog, \( x_1 \notin B \). Then, \( \forall h \in \mathcal{H}, h(x_1) = 1 \), and thus shattering is not possible. Now, consider the case where both \( x_1 \in B \) and \( x_2 \in B \). If either \( x_1 \) or \( x_2 \) is in \( \bigcup_{i=1}^{m} B_i \), then every hypothesis \( h \in \mathcal{H} \) will label it as 0, and thus these two points cannot be shattered. If \( x_1 \in B_i^0 \) and \( x_2 \in B_j^0 \) for \( i \neq j \), then \( h_0(x_1) = h_0(x_2) = 0 \), but \( \forall h \in \mathcal{H} \) s.t. \( h \neq h_0 \), \( h(x_1) = h(x_2) = 1 \). If \( x_1 \in B_i^1 \) and \( x_2 \in B_j^1 \) for \( i \neq j \), then there exists no hypothesis in \( \mathcal{H} \) that can label \((x_1, x_2)\) as \((0, 0)\). Thus, overall, no two points \( x_1, x_2 \in \mathcal{X} \) can be shattered by \( \mathcal{H} \) implying that \( VC(\mathcal{H}) \leq 1 \).

Now we are ready to show that the VC dimension of the loss class is at least \( m \). Specifically, given the sample of labelled points \( S = \{(c_1, 1), \ldots, (c_m, 1)\} \), we will show that the loss behavior corresponding to hypothesis \( h_0 \) on the sample \( S \) is exactly \( b \). Since \( \mathcal{H} \) contains all the hypothesis corresponding to every single bitstring \( b \in \{0, 1\}^m \), the loss class of \( \mathcal{H} \) will shatter \( S \). In order to prove that the loss behavior of \( h_0 \) on the sample \( S \) is exactly \( b \), it suffices to show that the probabilistic loss of \( h_0 \) on example \((c_i, 1)\) is \( b_i \), where \( b_i \) denotes the \( i \)th bit of \( b \). By definition,

\[
\ell_{\mathcal{G}, \mu}^b(h_0) = \mathbb{1}\{\mathbb{P}_{z \sim \mu}(h_0(z) \neq 1) > \rho\}
\]

\[
= \mathbb{1}\{\mathbb{P}_{z \sim \mu}(h_0(z) = 0) > \rho\}
\]

\[
= \mathbb{1}\{\mathbb{P}_{z \sim \mu}(z \in B_i^1 \cup B_i) > \rho\}
\]

\[
= \mathbb{1}\{\mu(B_i^1 \cup B_i) > \rho\}
\]

\[
= b_i.
\]

Thus, the loss behavior of \( h_0 \) on \( S \) is \( b \), and the total number of distinct loss behaviors over each hypothesis in \( \mathcal{H} \) on \( S \) is \( 2^m \), implying that the VC dimension of the loss class is at least \( m \). This completes the construction and proof of the claim.

Next, we show that the hypothesis class construction in Lemma 3 can be used to show the existence of a hypothesis class that cannot be learnt properly. Specifically, Lemma 4 below follows exactly from Lemma 3 in [MHS19]. We include the full proof of Lemma 4 in the Appendix.

**Lemma 4.** Let \( m \in \mathbb{N} \). For every \( \rho \in [0, 1) \) there exists \( \mathcal{H} \subset \mathcal{Y}^X \) with \( VC(\mathcal{H}) \leq 1 \) such that for any proper learner \( A \colon (\mathcal{X} \times \mathcal{Y})^* \to \mathcal{H} \): (1) there is a distribution \( \mathcal{D} \) over \( \mathcal{X} \times \mathcal{Y} \) and a hypothesis \( h^* \in \mathcal{H} \) where \( R_{\mathcal{G}, \mu}(h^*; \mathcal{D}) = 0 \) and (2) with probability at least \( 1/7 \) over \( S \sim \mathcal{D}^m \), \( R_{\mathcal{G}, \mu}(A(S); \mathcal{D}) > 1/8 \).

Finally, we state our main theorem indicating that proper learning is not possible for any \( \rho \in [0, 1) \), even for \( \rho \) arbitrarily close to 1. Again, the proof of Theorem 5 closely follows that in [MHS19], however, since our hypothesis class construction in Lemma 3 is different, we include a complete proof below.

**Theorem 5.** Fix \( \rho \in [0, 1] \). There exists a hypothesis class \( \mathcal{H} \subset \mathcal{Y}^X \) with \( VC(\mathcal{H}) \leq 1 \) and an adversary \( (\mathcal{G}, \mu) \) such that \( \mathcal{H} \) is not properly \( \rho \)-probabilistically robustly PAC learnable.

**Proof.** Fix \( \rho \in [0, 1] \). Let \( (C_m)_{m \in \mathbb{N}} \) be an infinite sequence of disjoint sets such that each set \( C_m \) contains \( 3m \) distinct center points from \( \mathcal{X} \), where for any \( c_i, c_j \in \bigcup_{m=1}^{\infty} C_m \) such that \( c_i \neq c_j \), we have \( \mathcal{G}(c_i) \cap \mathcal{G}(c_j) = \emptyset \). For every \( m \in \mathbb{N} \), construct \( \mathcal{H}_m \) on \( C_m \) as in Lemma 3. In addition, a key part of this proof is to ensure that the hypothesis in \( \mathcal{H}_m \) are non-robust to points in \( C_m \), for all \( m' \neq m \). To do so, we will need to adjust each hypothesis \( h_b \in \mathcal{H}_m \) carefully. By definition, for every \( m \in \mathbb{N} \), \( \mathcal{H}_m \) consists of \( 2^{3m} \) hypothesis of the form

\[
h_b(z) = \begin{cases} 0 & \text{if } z \in \bigcup_{i=1}^{3m} B_i^b \cup B_i \\ 1 & \text{otherwise} \end{cases}
\]

for each bitstring \( b \in \{0, 1\}^{3m} \). Note that the same set \( \bigcup_{i=1}^{3m} B_i \) is shared across every hypothesis \( h_b \in \mathcal{H}_m \). For each \( m \in \mathbb{N} \), let \( B^m = \bigcup_{i=1}^{3m} B_i \) be exactly the union of these \( 3m \) sets. Next, from the construction in Lemma 3, for every center \( c_i \in C_m \), \( \mu(B_i \cup (\bigcup_b B_i^b)) < 1 \). Thus, there exists a set \( \tilde{B}_i \subset \mathcal{G}(c_i) \) s.t. \( \mu(\tilde{B}_i) > 0 \) and \( \tilde{B}_i \cap (B_i \cup (\bigcup_b B_i^b)) = \emptyset \). Consider one such subset \( \tilde{B}_i \) from each of the \( 3m \) centers in \( C_m \) and let \( \tilde{B}^m = \bigcup_{i=1}^{3m} \tilde{B}_i \). Finally, make the following adjustment to each \( h_b \in \mathcal{H}_m \),

\[
h_b(z) = \begin{cases} 0 & \text{if } z \in \bigcup_{i=1}^{3m} B_i^b \cup B_i \text{ or } z \in B^{m'} \cup \tilde{B}^m \text{ for } m' \neq m \\ 1 & \text{otherwise} \end{cases}
\]
One can verify that every hypothesis in $H_m$ has a non-robust region (i.e. $B^{m'} \cup \tilde{B}^{m'}$ for $m' \neq m$) with mass strictly bigger than $\rho$ in every center in $C_{m'}$ for every $m' \neq m$. Thus, the hypotheses in $H_m$ are non-robust to points in $C_{m'}$ for all $m' \neq m$. Finally, as we did in Lemma 4, for each $m$, we only keep the subset of hypothesis $H'_m = \{ h_b \in H_m : \sum_{i=1}^{3n} b_i = m \}$. Note that for each $m \in \mathbb{N}$, the hypothesis class $H'_m$ behaves exactly like the hypothesis class from Lemma 4 on $C_m$.

Let $H = \bigcup_{m=1}^{\infty} H'_m$ and $G(C_m) = \bigcup_{i=1}^{3n} G(c_i)$. Since we have modified the hypothesis class, we need to reprove that its VC dimension is still at most 1. Consider two points $x_1, x_2 \in \mathcal{X}$. If either $x_1$ or $x_2$ is not in $\bigcup_{m=1}^{\infty} G(C_m)$ and not in $\bigcup_{m=1}^{\infty} B^m \cup \tilde{B}^m$, then all hypothesis predict $x_1$ or $x_2$ as 1. If both $x_1$ and $x_2$ are in $B^m \cup \tilde{B}^m$ for some $m \in \mathbb{N}$, then:

- if either $x_1$ or $x_2$ are in $B^m$, every hypothesis in $H$ labels either $x_1$ or $x_2$ as 0.
- if both $x_1$ and $x_2$ are in $\tilde{B}^m$, we can only get the labeling $(1, 1)$ from hypotheses in $H_m$ and the labelling $(0, 0)$ from the hypotheses in $H'_m$ for $m' \neq m$.

In the case both $x_1$ and $x_2$ are in $G(C_m) \setminus (B^m \cup \tilde{B}^m)$, then, they cannot be shattered by Lemma 3. In the case $x_1 \in B^m \cup \tilde{B}^m$ and $x_2 \in G(C_m) \setminus (B^m \cup \tilde{B}^m)$:

- if $x_1$ is in $B^m$, every hypothesis in $H$ labels $x_1$ as 0.
- if $x_1$ is in $\tilde{B}^m$ then, we can never get the labelling $(0, 0)$.

If $x_1 \in B^i \cup \tilde{B}^i$ and $x_2 \in B^j \cup \tilde{B}^j$ for $i \neq j$, then:

- if either $x_1$ or $x_2$ are in $B^i$ or $B^j$ respectively, every hypothesis in $H$ labels either $x_1$ or $x_2$ as 0.
- if both $x_1$ and $x_2$ are in $\tilde{B}^i$ and $\tilde{B}^j$ respectively, we can never get the labelling $(1, 1)$.

In the case $x_1 \in B^i \cup \tilde{B}^i$ and $x_2 \in G(C_j) \setminus (B^i \cup \tilde{B}^i)$ for $j \neq i$, then we cannot obtain the labelling $(1, 0)$. If $x_1 \in G(C_j) \setminus (B^i \cup \tilde{B}^i)$ and $x_2 \in G(C_j) \setminus (B^i \cup \tilde{B}^i)$ for $i \neq j$, then we cannot obtain the labelling $(0, 0)$. Since we shown that for all possible $x_1$ and $x_2$, $H$ cannot shatter them, $\text{VC}(H) \leq 1$.

We now use the same reasoning in [MHS19], to show that no proper learning rule works. By Lemma 4, for any proper learning rule $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow H$ and for any $m \in \mathbb{N}$, we can construct a distribution $D$ over $C_m$ (which has $3m$ points from $\mathcal{X}$) where there exists a hypothesis $h^* \in H'_m$ that achieves $R^\rho_{\mathcal{G}}(h^*; D, \rho) = 0$, but with probability at least 1/7 over $S \sim D^m$, $R^\rho_{\mathcal{G}}(A(S); D, \rho) > 1/8$. Note that it suffices to only consider hypothesis in $H'_m$ because, by construction, all hypothesis in $H'_m$ for $m' \neq m$ are not probabilistically robust on $C_m$, and thus always achieve loss 1 on all points in $C_m$. Thus, rule $A$ will do worse if it picks hypotheses from these classes. This shows that the sample complexity of properly probabilistically robust PAC learning $H$ is arbitrarily large, allowing us to conclude that $H$ is not properly learnable.

\section{($\rho, \rho^*$)-Probabilistically Robust PAC Learning}

In light of the hardness result of Section 3, we are interested in understanding when VC dimension is sufficient for proper learning, and more specifically, ERM. To this end, we slightly tweak the learning setup of Definition 1 by allowing $A$ to compete against the hypothesis minimizing the probabilistic robust risk at a level $\rho^* < \rho$. Definition 6 formally introduces this new learning setting which we denote as $(\rho, \rho^*)$-probabilistically robust PAC learning.

\begin{definition}[$(\rho, \rho^*)$-Probabilistically Robust PAC Learning] For any $\epsilon, \delta \in (0, 1)$ and any $0 \leq \rho^* < \rho < 1$, the sample complexity of $(\rho, \rho^*)$-probabilistically robust ($\epsilon, \delta$)-PAC learning of $H$ with respect to adversary $(\mathcal{G}, \mu)$, denoted $n(\epsilon, \delta, \rho, \rho^*; H, \mathcal{G}, \mu)$, is the smallest number $m \in \mathbb{N}$ for which there exists a learning rule $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{Y}^\mathcal{X}$ such that for every distribution $D$ over $\mathcal{X} \times \mathcal{Y}$, with probability at least $1 - \delta$ over $S \sim D^m$, $R^\rho_{\mathcal{G}, \mu}(A(S); D) \leq \inf_{h \in H} R^\rho_{\mathcal{G}, \mu}(h; D) + \epsilon$.

We say that $H$ is $(\rho, \rho^*)$-probabilistically robustly PAC learnable with respect to adversary $(\mathcal{G}, \mu)$ at a level of $\rho$, if $\forall \epsilon, \delta \in (0, 1)$, $n(\epsilon, \delta, \rho; H, \mathcal{G}, \mu)$ is finite.
\end{definition}
The first result of this section shows that while VC dimension is not sufficient for proper \( \rho \)-probabilistically robust PAC learning, VC Dimension is sufficient for proper \((\rho, \rho^*)\)-probabilistically robust PAC learning. Our second result shows that while VC Dimension is sufficient, it is not necessary for \((\rho, \rho^*)\)-probabilistically robust pAC Learning. This naturally leads to an interesting question of what dimension characterizes \((\rho, \rho^*)\)-probabilistically robust PAC learning.

4.1 VC Classes are \((\rho, \rho^*)\)-Probabilistically Robust PAC Learnable

The main result of this subsection, given by Theorem 6 below, states that VC classes are \((\rho, \rho^*)\)-probabilistically robustly learnable using proper learning rules.

**Theorem 6** (Proper \((\rho, \rho^*)\)-Probabilistically Robust PAC Learning). For every hypothesis class \( \mathcal{H} \) and adversary \((G, \mu)\), there exists a proper learning rule \( \mathcal{A}: (X \times Y)^n \rightarrow \mathcal{H} \) such that for every distribution \( \mathcal{D} \) over \( X \times Y \), with probability at least \( 1 - \delta \) over \( S \sim \mathcal{D}^n \), algorithm \( \mathcal{A} \) achieves

\[
R^\rho_{G,\mu}(A(S); \mathcal{D}) \leq \inf_{h \in \mathcal{H}} R^\rho_{G,\mu}(h; \mathcal{D}) + \epsilon
\]

with

\[
n(\epsilon, \delta, \rho, \rho^*; \mathcal{H}, G, \mu) = O \left( \frac{VC(\mathcal{H})}{(\rho - \rho^*)^2} \ln\left(\frac{1}{\rho - \rho^*}\right) + \ln\left(\frac{1}{\delta}\right) \right)
\]

samples.

Our main technique to prove Theorem 6 is to consider a different probabilistic robust loss function, which we call the probabilistic robust ramp loss. For this new loss function, we can show the uniform convergence property. Finally, we can upper and lower bound the ramp risk by the probabilistic robust risk immediately gives rise to a proper learning algorithm: ERM over the probabilistic robust ramp loss. This is exactly the proper learning algorithm referred to in Theorem 6. We now formally sketch this idea below. The full proof of Theorem 6 can be found in Appendix C.

For any \( 0 \leq \rho^* < \rho < 1 \), define the probabilistic ramp loss,

\[
\ell_{G,\mu}^{\rho,\rho^*}(h, (x, y)) := \min(1, \max(0, \frac{\mathbb{P}_{y \sim \mu}[h(g(x)) \neq y] - \rho^*}{\rho - \rho^*})).
\]

Note that unlike the probabilistic robust loss, the ramp loss takes values in \([0, 1]\) and is \(\frac{1}{\rho - \rho^*}\)-Lipschitz with respect to \(\mathbb{P}_{y \sim \mu}[h(g(x)) \neq y]\). Crucially, observe that the probabilistic robust losses at the level of \(\rho^*\) and \(\rho\) sandwich the ramp loss. Indeed, for all \(h \in \mathcal{H}, (x, y) \in X \times Y, (G, \mu)\), and \(\rho^* < \rho\), we have that

\[
\ell_{G,\mu}^{\rho^*}(h, (x, y)) \leq \ell_{G,\mu}^{\rho,\rho^*}(h, (x, y)) \leq \ell_{G,\mu}^{\rho^*}(h, (x, y)),
\]

which allows us to relate the ramp risk to the probabilistic robust risk via

\[
R^\rho_{G,\mu}(h; \mathcal{D}) \leq \mathbb{E}_{(x,y) \sim \mathcal{D}} \left[ \ell_{G,\mu}^{\rho,\rho^*}(h, (x, y)) \right] \leq R^\rho^*_{G,\mu}(h; \mathcal{D}).
\]

Figure 1 visually showcases how the probabilistic robust losses at \(\rho\) and \(\rho^*\) sandwich the probabilistic ramp loss at \(\rho, \rho^*\). Now we show that if \(\mathcal{H}\) has finite VC dimension, then the probabilistic robust ramp loss enjoys the uniform convergence property. Roughly speaking, uniform convergence implies that for a sufficiently large sample, with high probability, the empirical ramp loss over the sample is close to the population ramp loss for every hypothesis simultaneously. In the standard PAC learning setting, finite VC dimension implies uniform convergence of the 0-1 loss. The theorem below shows that finite VC dimension also implies uniform convergence of the probabilistic robust ramp loss.

**Theorem 7** (Uniform Convergence of Ramp Loss). Let \(\mathcal{H}\) be a hypothesis class with finite VC dimension and \((G, \mu)\) be an adversary. With probability at least \(1 - \delta\) over a sample \(S \sim \mathcal{D}^n\) of size

\[
n = O \left( \frac{VC(\mathcal{H})}{(\rho - \rho^*)^2} \ln\left(\frac{1}{\rho - \rho^*}\right) + \ln\left(\frac{1}{\delta}\right) \right),
\]

for all \(h \in \mathcal{H}\) simultaneously,

\[
\mathbb{E}_\mathcal{D} \left[ \ell_{G,\mu}^{\rho,\rho^*}(h, (x, y)) \right] - \mathbb{E}_S \left[ \ell_{G,\mu}^{\rho,\rho^*}(h, (x, y)) \right] \leq \epsilon.
\]
in terms of the VC dimension of $H$.

For every hypothesis class $H$ and adversary $(\mathcal{G}, \mu)$, the learning rule $A(S) = \text{PRRERM}(S; \mathcal{G}, \mu, \rho, 0)$ achieves, for any distribution $D$ over $X \times Y$, with probability at least $1 - \delta$ over $S \sim D^n$, $$R^\rho_{G, \mu}(A(S); D) \leq \inf_{h \in H} R^\rho_{G, \mu}(h; D) + \epsilon$$

Figure 1: Comparison of probabilistic robust ramp loss to probabilistic robust losses of hypothesis $h$ on example $(x, y)$. The probabilistic robust losses at $\rho$ and $\rho^*$ sandwich the probabilistic robust ramp loss at $\rho, \rho^*$. 

At a high-level, our proof for Theorem 7 bounds the Rademacher complexity of the ramp loss class in terms of the VC dimension of $H$ by exploiting the fact that the ramp loss is $(\frac{1}{\rho - \rho^*})$-Lipschitz with respect to $\mathbb{P}_{g-\mu}[h(g(x)) \neq y]$. We defer the full details of the proof to Appendix C.

With uniform convergence of the ramp loss in hand, we can now give a proper learning algorithm to Definition 2. By Theorem 6, for a sample $S$ and adversary $(\mathcal{G}, \mu)$, outputs the empirical ramp loss minimizer over $H$; $(\mathcal{G}, \mu, \rho, \rho^*)$ immediately get the following corollary by setting $\rho = 0$. 

We defer the full details of the proof to Appendix C.

From here, we use the fact that the probabilistic robust loss sandwiches the ramp loss. More specifically, since $\mathbb{E}_D \left[ \ell_{G, \mu}^{\rho^*}(A(S), (x, y)) \right] \geq R^\rho_{G, \mu}(A(S); D)$ and $\inf_{h \in H} \mathbb{E}_D \left[ \ell_{G, \mu}^{\rho^*}(h, (x, y)) \right] \leq \inf_{h \in H} R^\rho_{G, \mu}(h; D)$, we have that $$R^\rho_{G, \mu}(A(S); D) \leq \inf_{h \in H} R^\rho_{G, \mu}(h; D) + \epsilon.$$ matching the generalization bound in Theorem 6. Thus, in stark contrast to Section 3, where proper learning is not always possible, here we have shown that if we compare our learner to the best hypothesis matching the generalization bound in Theorem 6. Thus, in stark contrast to Section 3, where proper learning is not always possible, here we have shown that if we compare our learner to the best hypothesis

stronger level of probabilistic robustness, then not only is proper learning possible for VC classes, but it is possible via an ERM-based learner. Since Theorem 6 holds for any $0 \leq \rho^* < \rho$, we immediately get the following corollary by setting $\rho^* = 0$. 

Corollary 8. For every hypothesis class $H$ and adversary $(\mathcal{G}, \mu)$, the learning rule $A(S) = \text{PRRERM}(S; \mathcal{G}, \mu, \rho, 0)$ achieves, for any distribution $D$ over $X \times Y$, with probability at least $1 - \delta$ over $S \sim D^n$, $$R^\rho_{G, \mu}(A(S); D) \leq \inf_{h \in H} R^\rho_{G, \mu}(h; D) + \epsilon$$

with sample complexity, $$n(\epsilon, \delta, \rho; H, \mathcal{G}, \mu) = O \left( \frac{\text{VC}(H)}{\rho^2} \ln \left( \frac{1}{\epsilon^2} \right) + \ln \left( \frac{1}{\delta} \right) \right).$$
We conclude this subsection with a remark on Theorem 6. Solving for the error rate in the sample complexity, we have that the excess error $\epsilon$ scales approximately according to $O(\frac{1}{\rho - \rho^*} \sqrt{\frac{\text{VC}(\mathcal{H})}{n}})$ ignoring the dependence on $\delta$ and other log factors. Fixing the sample size $n$, $\rho$, $\delta$, we find that $\rho^*$ acts as a tuneable parameter that controls a tradeoff between the excess error $\epsilon$ and the approximation error $\inf_{h \in \mathcal{H}} R_{\rho,\mu}^*(h; D)$. Indeed, as $\rho^*$ increases, $\epsilon$ increases, but $\inf_{h \in \mathcal{H}} R_{\rho,\mu}^*(h; D)$ decreases. Likewise, as $\rho^*$ decreases, we have that $\epsilon$ decreases, but $\inf_{h \in \mathcal{H}} R_{\rho,\mu}^*(h; D)$ increases. In this sense, there exists a notion of an optimal $\rho^*$, defined by $\inf_{\rho^* \in (0,\rho)} \left( \inf_{h \in \mathcal{H}} R_{\rho,\mu}^*(h; D) + O(\frac{1}{\rho - \rho^*} \sqrt{\frac{\text{VC}(\mathcal{H})}{n}}) \right)$.

4.2 Finite VC Dimension is Not Necessary for $(\rho, \rho^*)$-Probabilistically Robust PAC Learning

While finite VC dimension is sufficient for proper $(\rho, \rho^*)$-probabilistically robust PAC learning, we show that it is not necessary. Theorem 9 below makes this claim more precise.

**Theorem 9.** There exists $(\mathcal{H}, \mathcal{G}, \mu)$ s.t. $\text{VC}(\mathcal{H}) = \infty$ but $\mathcal{H}$ is still still (properly) $(\rho, \rho^*)$-probabistically robust PAC learning with respect to $(\mathcal{G}, \mu)$.

**Proof.** Fix $\rho > \rho^* \geq 0$. Let $X = \mathbb{R}$ and $\mathcal{H} = \{ \text{sign} (\sin (\omega x)) : \omega \in \mathbb{R} \}$. Without loss of generality, assume $\text{sign} (\sin (0)) = 1$. For every $x \in X$ and $c \in [-1, 1]$, define $g_c(x) = cx$. Then, let $\mathcal{G} = \{ g_c : c \in [-1, 1] \}$ and $\mu$ be uniform over $\mathcal{G}$. First, $\text{VC}(\mathcal{H}) = \infty$ as desired. Next, to show learnability, by the proof sketch of Theorem 6, it suffices to show that the ramp loss

$$L^p_{\mathcal{G},\mu}(h, (x, y)) = \min(1, \max(0, \frac{\mathbb{P}_{g \sim \mu} [h(g(x)) \neq y] - \rho^*}{\rho - \rho^*})).$$

enjoys the uniform convergence property despite $\text{VC}(\mathcal{H}) = \infty$. Observe that we can write,

$$\mathbb{P}_{g \sim \mu} [h(g(x)) \neq y] = \mathbb{E}_{g \sim \mu} [\mathbb{I} \{ h(g(x)) \neq y \}] = \mathbb{E}_{g \sim \mu} \left[ \frac{1 - h(g(x))y}{2} \right] = \frac{1 - y \mathbb{E}_{g \sim \mu} [h(g(x))]}{2}$$

from which we have that,

$$L^p_{\mathcal{G},\mu}(h, (x, y)) = \min(1, \max(0, \frac{1 - y \mathbb{E}_{g \sim \mu} [h(g(x))] - 2\rho^*}{2(\rho - \rho^*)}).$$

Note that $L^p_{\mathcal{G},\mu}(h, (x, y))$ is a $\frac{1}{\rho - \rho^*}$-Lipschitz loss function with respect to the real-valued function $\mathbb{E}_{g \sim \mu} [h(g(x))]$. Therefore, by Theorem 1, $L^p_{\mathcal{G},\mu}(h, (x, y))$ enjoys the uniform convergence property if the loss class $\mathcal{F}_{\mathcal{G},\mu}^p = \{ \mathbb{E}_{g \sim \mu} [h_\omega(g(x))] : h_\omega \in \mathcal{H} \}$ has a finite Fat Shattering Dimension. But for every $h_\omega \in H$,

$$\mathbb{E}_{g \sim \mu} [h_\omega(g(x))] = \mathbb{E}_{c \sim \text{Unif}(-1, 1)} [\text{sign}(\sin(c(x)))] = \frac{1}{2} \int_{-1}^{1} \text{sign}(\sin(c(x))) dc.$$

Since $\text{sign}(ax)$ is an odd function, $\text{sign}(\sin(ax))$ is also odd, from which it follows that for all $h_\omega \in H$:

$$\mathbb{E}_{g \sim \mu} [h_\omega(g(x))] = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{otherwise}. \end{cases}$$

Therefore, $\mathcal{F}_{\mathcal{G},\mu}^p = \{ f \}$ where $f(x) = 1$ if $x = 0$ and $f(x) = 0$ if $x \neq 0$. Since $\mathcal{F}_{\mathcal{G},\mu}^p$ is a singleton, its Fat Shattering Dimension at every scale $\gamma > 0$ is trivially finite. Thus, using Theorem 1, and following the same steps in the proof sketch of Theorem 6, we have that $(\mathcal{H}, \mathcal{G}, \mu)$ is $(\rho, \rho^*)$-robustly learnable by $\text{PRRERM}(S; \mathcal{G}, \mu, \rho, \rho^*)$ with sample complexity that scales according to $O\left( \frac{\text{fat}(\mathcal{F}_{\mathcal{G},\mu}^p, \rho, \rho^*)}{\epsilon^2} \ln \frac{\ln(1/\epsilon)}{\ln(\ln(1/\epsilon))} \right)$, where $K$ is some universal constant. \qed
Theorem 9 shows that by carefully choosing $(\mathcal{G}, \mu)$ the complexity of $\mathcal{H}$ can be essentially smoothed out. This idea closely follows the intuition of Chapelle et al. in their work on Vicinal Risk Minimization [CWBV00]. One immediate insight from the proof is that finiteness of the Fat Shattering Dimension of $\mathcal{F}_{\mathcal{G}, \mu}^{\mathcal{H}}$ is necessary and sufficient for the learnability of the ramp loss. Clearly, Theorem 9 also shows that finite Fat Shattering Dimension of $\mathcal{F}_{\mathcal{G}, \mu}^{\mathcal{H}}$ is sufficient for $(\rho, \rho^*)$-probabilistically robust PAC learning. Whether finiteness of the Fat Shattering Dimension of $\mathcal{F}_{\mathcal{G}, \mu}^{\mathcal{H}}$ is also necessary is an interesting future direction.

\section{$(\rho, \mathcal{G})$-Probabilistically Robust PAC Learning}

Can stronger, measure-independent, learning guarantees be achieved if we instead compare the learner’s probabilistic robust risk to the best adversarially robust risk over $\mathcal{H}$? In this section, we answer this in the affirmative. We show that if one specifically wants to compete against the best hypothesis for the worst-case adversarial robust risk, then it is sufficient to just run RERM over the training sample $S$. Note that unlike the previous two sections, here we want to devise learning algorithms that do not have access to or use $\mu$. Definition 7 formalizes this setting.

**Definition 7** ($(\rho, \mathcal{G})$-Probabilistically Robust PAC Learning). For any $\epsilon, \delta \in (0, 1)$ and any $\rho > 0$, the sample complexity of $(\rho, \mathcal{G})$-probabilistically robust $(\epsilon, \delta)$-PAC learning of $\mathcal{H}$ with respect to adversary $\mathcal{G}$, denoted $n(\epsilon, \delta, \rho; \mathcal{H}, \mathcal{G})$, is the smallest number $m \in \mathbb{N}$ for which there exists a learning rule $A : (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{Y}^\mathcal{X}$ such that for every distribution $D$ over $\mathcal{X} \times \mathcal{Y}$ and any measure $\mu$ over $\mathcal{G}$, with probability at least $1 - \delta$ over $S \sim D^n$,

$$R^\rho_{\mathcal{G}, \mu}(A(S); D) \leq \inf_{h \in \mathcal{H}} R^\rho_{\mathcal{G}}(h; D) + \epsilon.$$

We say that $\mathcal{H}$ is $(\rho, \mathcal{G})$-probabilistically robustly PAC learnable with respect to adversary $\mathcal{G}$ at a level of $\rho$, if for all $\epsilon, \delta \in (0, 1)$, $n(\epsilon, \delta, \rho; \mathcal{H}, \mathcal{G})$ is finite.

Perhaps surprisingly, the main result of this section shows that not only is proper, measure-independent, learning possible for VC classes, but it is possible by just running RERM.

**Theorem 10** (Proper $(\rho, \mathcal{G})$-Probabilistically Robust PAC Learning). For every hypothesis class $\mathcal{H}$ and adversary $\mathcal{G}$, the learning rule $A(S) = \text{RERM}(S; \mathcal{G})$ achieves, for any measure $\mu$ over $\mathcal{G}$ and any distribution $D$ over $\mathcal{X} \times \mathcal{Y}$, with probability at least $1 - \delta$ over $S \sim D^n$,

$$R^\rho_{\mathcal{G}, \mu}(A(S); D) \leq \inf_{h \in \mathcal{H}} R^\rho_{\mathcal{G}}(h; D) + \epsilon$$

using

$$n(\epsilon, \delta, \rho; \mathcal{H}, \mathcal{G}) = O \left( \frac{\text{VC} (\mathcal{H})}{\rho^2} \ln (\frac{1}{\rho \epsilon}) + \ln (\frac{1}{\delta}) \right).$$

number of samples.

We make a few remarks about the practical importance of Theorem 10. Theorem 10 implies that for any pre-specified perturbation function class $\mathcal{G}$ (for example $L_p$ balls), running RERM is sufficient to obtain a hypothesis that is probabilistically robust with respect to any fixed measure $\mu$ over $\mathcal{G}$. Moreover, the level of probabilistic robustness of the predictor output by RERM, as measured by $1 - \rho$, scales directly with the sample size - the more samples one has, the smaller $\rho$ can be made. Alternatively, for a fixed sample size $m$, desired error $\epsilon$ and confidence $\delta$, one can use the sample complexity guarantee in Theorem 10 to back-solve the robustness guarantee $\rho$ of the hypothesis RERM$(S; \mathcal{G})$. Thus, from a practical standpoint, one has the ability to quantify the robustness guarantees of the output classifier at an example-level basis. We highlight that the generalization bound in Theorem 10, is a measure-independent guarantee. This means that $\rho$ does not quantify the level of robustness of the output hypothesis with respect to any one particular measure, but for all measures. This is desirable, as in contrast to the previous section, the $\rho$ here more succinctly quantifies the level of robustness achieved by the output classifier. Lastly, we highlight that while the sample complexity of adversarially robust PAC learning can be exponential in the VC dimension of $\mathcal{H}$ [MHS19], this is not the case for $(\rho, \mathcal{G})$-probabilistically robust PAC learning, where we only get a linear dependence on $d$.  

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We now provide the proof of Theorem 10 below. The key idea is to exploit the fact that uniform convergence holds for ramp loss for any values of $\rho > \rho^* \geq 0$.

**Proof.** (of Theorem 10) Fix $0 < \rho$ and let $\mathcal{H}$ be a hypothesis class with $\text{VC}(\mathcal{H}) = d$. Let $\mathcal{G}$ be an arbitrary adversary. Let $\mathcal{D}$ be an arbitrary distribution over $\mathcal{X} \times \mathcal{Y}$ and $S = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ an i.i.d. sample of size $m$.

Let $\hat{h} = \mathcal{A}(S) = \text{RERM}(S; \mathcal{G})$ denote the hypothesis output by $\mathcal{A}$ and $h^* = \inf_{h \in \mathcal{H}} R_{\mathcal{G}}(h; \mathcal{D})$. Fix a measure $\mu$ over $\mathcal{G}$. By Theorem 7, for a sample $S \sim \mathcal{D}^m$ of size $m = O \left( \frac{d \ln(\frac{1}{\delta}) + \ln(\frac{1}{\epsilon})}{\epsilon^2} \right)$, with probability $1 - \frac{\delta}{2}$, for all $h \in \mathcal{H}$ simultaneously,

$$\left| \mathbb{E}_\mathcal{D} \left[ \ell_{\mathcal{G},\mu}^0(h, (x, y)) \right] - \mathbb{E}_S \left[ \ell_{\mathcal{G},\mu}^0(h, (x, y)) \right] \right| \leq \frac{\epsilon}{2}.$$

In particular, this means that

$$\mathbb{E}_\mathcal{D} \left[ \ell_{\mathcal{G},\mu}^0(\hat{h}, (x, y)) \right] - \mathbb{E}_S \left[ \ell_{\mathcal{G},\mu}^0(\hat{h}, (x, y)) \right] \leq \frac{\epsilon}{2}.$$

Noting that $R_{\mathcal{G},\mu}^0(\hat{h}; \mathcal{D}) \leq \mathbb{E}_\mathcal{D} \left[ \ell_{\mathcal{G},\mu}^0(\hat{h}, (x, y)) \right]$ and $\mathbb{E}_S \left[ \ell_{\mathcal{G},\mu}^0(\hat{h}, (x, y)) \right] \leq \hat{R}_{\mathcal{G}}(\hat{h}; S) \leq \hat{R}_{\mathcal{G}}(h^*; S)$, we have that with probability $1 - \frac{\delta}{2}$,

$$R_{\mathcal{G},\mu}^0(\hat{h}; \mathcal{D}) \leq \hat{R}_{\mathcal{G}}(h^*; S) + \frac{\epsilon}{2}.$$

Thus, it suffices to upper bound $\hat{R}_{\mathcal{G}}(h^*; S)$. By Hoeffding’s Inequality, we have that with probability $1 - \frac{\delta}{2}$,

$$\hat{R}_{\mathcal{G}}(h^*; S) \leq R_{\mathcal{G}}(h^*; \mathcal{D}) + \frac{\epsilon}{2}.$$

Putting things together and using the union bound, we have that for $m = O \left( \frac{d \ln(\frac{1}{\delta}) + \ln(\frac{1}{\epsilon})}{\epsilon^2} \right)$ samples, for any measure $\mu$ over $\mathcal{G}$, with probability $1 - \delta$ over the draw $S \sim \mathcal{D}^m$,

$$R_{\mathcal{G},\mu}^0(\hat{h}; \mathcal{D}) \leq R_{\mathcal{G}}(h^*; \mathcal{D}) + \epsilon,$$

matching the generalization bound in the Theorem statement. \qed

## 6 Beyond Probabilistic Robustness

The key idea powering the results in Sections 4 and 5 is the ability to find a loss function enjoying uniform convergence that lies between the loss we use to evaluate our learner and the loss we use to evaluate the “best” hypothesis. In this section, we start by showing that this idea is useful more generally.

**Lemma 11** (Sandwich Uniform Convergence). Let $\ell_1(h, (x, y))$ and $\ell_2(h, (x, y))$ be bounded, non-negative loss functions s.t. for all $h \in \mathcal{H}$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we have $\ell_1(h, (x, y)) \leq \ell_2(h, (x, y)) \leq 1$. If there exists a loss function $\ell(h, (x, y))$ s.t. $\ell_1(h, (x, y)) \leq \ell(h, (x, y)) \leq \ell_2(h, (x, y))$ and $\ell(h, (x, y))$ enjoys the uniform convergence property with sample complexity $n(\epsilon, \delta)$, then the learning rule $\mathcal{A}(S) = \inf_{h \in \mathcal{H}} \mathbb{E}_S \left[ \ell_2(h, (x, y)) \right]$ achieves with probability $1 - \delta$ over a draw $S \sim \mathcal{D}^m$,

$$\mathbb{E}_\mathcal{D} \left[ \ell_1(\mathcal{A}(S), (x, y)) \right] - \inf_{h \in \mathcal{H}} \mathbb{E}_\mathcal{D} \left[ \ell_2(h, (x, y)) \right] \leq \epsilon$$

with sample complexity,

$$m(\epsilon, \delta, \mathcal{H}) = n(\epsilon, \delta) + O \left( \frac{\ln(\frac{1}{\delta})}{\epsilon^2} \right).$$
Proof. Let $A(S) = \inf_{h \in \mathcal{H}} \mathbb{E}_S [\ell_2(h, (x, y))]$. By uniform convergence of $\hat{\ell}(h, (x, y))$, we have that for sample size $m = n(\frac{\epsilon}{2}, \frac{1}{2})$, with probability at least $1 - \frac{\delta}{2}$, over a sample $S \sim D^m$, for every hypothesis $h \in \mathcal{H}$ simultaneously,

$$\mathbb{E}_{\mathcal{D}} \left[ \hat{\ell}(h, (x, y)) \right] \leq \mathbb{E}_S \left[ \hat{\ell}(h, (x, y)) \right] + \frac{\epsilon}{2}.$$ 

In particular, this implies that for $\hat{h} = A(S)$, we have

$$\mathbb{E}_{\mathcal{D}} \left[ \hat{\ell}(\hat{h}, (x, y)) \right] \leq \mathbb{E}_S \left[ \hat{\ell}(\hat{h}, (x, y)) \right] + \frac{\epsilon}{2}.$$ 

Since, $\ell_1(h, (x, y)) \leq \hat{\ell}(h, (x, y)) \leq \ell_2(h, (x, y))$, we can write

$$\mathbb{E}_{\mathcal{D}} \left[ \ell_1(h, (x, y)) \right] \leq \mathbb{E}_{\mathcal{D}} \left[ \hat{\ell}(h, (x, y)) \right] \leq \mathbb{E}_S \left[ \hat{\ell}(h, (x, y)) \right] + \frac{\epsilon}{2} \leq \mathbb{E}_S \left[ \ell_2(h, (x, y)) \right] + \frac{\epsilon}{2}\mathcal{E}_S \left[ \ell_2(h^*, (x, y)) \right] + \frac{\epsilon}{2}.$$ 

where $h^* = \inf_{h \in \mathcal{H}} \mathbb{E}_{\mathcal{D}} [\ell_2(h, (x, y))]$.

It now only remains to upper bound $\mathbb{E}_S [\ell_2(h^*, (x, y))]$ with high probability. However, a standard Hoeffding bound tells us that with probability $1 - \frac{\delta}{2}$ over a sample $S$ of size $O(\frac{\ln(\frac{1}{\delta})}{\epsilon^2})$, $\mathbb{E}_S [\ell_2(h^*, (x, y))] \leq \mathbb{E}_{\mathcal{D}} [\ell_2(h^*, (x, y))] + \frac{\epsilon}{2}$. 

Thus, putting things together, we get that with probability at least $1 - \delta$ over a sample of size $n(\epsilon, \delta) + O(\frac{\ln(\frac{1}{\delta})}{\epsilon^2})$, we have

$$\mathbb{E}_{\mathcal{D}} \left[ \ell_1(\hat{h}, (x, y)) \right] \leq \mathbb{E}_{\mathcal{D}} \left[ \ell_2(h^*, (x, y)) \right] + \epsilon$$ 

as desired. \hfill \Box

Lemma 11 immediately implies Theorem 6 and 10. To see this, fix $\ell_1(h, (x, y)) = \ell_{G, \rho}^\epsilon (h, (x, y))$ and $\hat{\ell}(h, (x, y)) = \ell_{G, \rho}^\epsilon (h, (x, y))$. Then, it suffices to let $\ell_2(h, (x, y)) = \ell_{G, \rho}^\epsilon (h, (x, y))$ or $\ell_2(h, (x, y)) = \ell_2(h, (x, y)) = \ell_2(h, (x, y))$ respectively to get the results from Section 4 and 5. In fact, Lemma 11 can also be used to provide upper bounds on the probabilistic robust risk that might be sharper than those in Section 3. See Appendix E for more details.

Notice that Lemma 11 only requires the existence of such a sandwiched loss function that enjoys uniform convergence - we do not actually require it to be computable. We will now exploit this fact to give learning guarantees in Adversarially Robust PAC learning with Tolerance, studied by [BHK+22] and [APU22]. In Adversarially Robust Learning with Tolerance, the learner’s adversarial robust risk under a perturbation set $G$ is compared with the best achievable adversarial robust risk for a larger perturbation set $G'$ of $G$. [APU22] study the setting where both $G$ and $G'$ induce $L_p$ balls with radius $r$ and $(1 + \gamma) r$ respectively. In [BHK+22], $G$ is arbitrary, but $G'$ is constructed such that it induces perturbation sets that are the union of balls with radius $\gamma$ that cover $G$. Critically, [BHK+22] show that, under certain assumptions, running RERM over a larger perturbation set $G'$ suffices tolerant Robust PAC learning. In this section, we take a slightly different approach to Robust Learning with Tolerance. Instead of having the learner compete against the best possible risk for a larger perturbation set, we have the learner compete against the best possible adversarial robust risk for $G$, but evaluate the learner’s adversarial robust risk using a smaller perturbation set $G' \subset G$. That is, we reverse the role of $G'$ and $G$.

For what $G' \subset G$ is Tolerant Robust PAC learning possible via RERM? As an immediate result of Lemma 11 and Vapnik’s General Theory of Learning, finite VC dimension of the loss class

$$\mathcal{L}_{\mathcal{H}}^{G'} = \{ (x, y) \mapsto \sup_{g \in g'} I \{ h(g(x)) \neq y \} : h \in \mathcal{H} \}$$
is sufficient. Note that finite VC dimension of \( \mathcal{L}_H^G \) implies that the loss function \( \ell_G(h, (x, y)) \) enjoys the uniform convergence property with sample complexity \( O\left(\frac{\text{VC}(\mathcal{L}_H^G)}{\epsilon^2} \ln\left(\frac{1}{\delta}\right) + \ln\left(\frac{1}{\delta}\right)\right) \). Thus, taking \( \ell_1(h, (x, y)) = \ell(h, (x, y)) = \ell_G(h, (x, y)) \) and \( \ell_2(h, (x, y)) = \ell_G(h, (x, y)) \) in Lemma 11, we have that if there exists a \( \mathcal{G}' \subset \mathcal{G} \) s.t. \( \text{VC}(\mathcal{L}_H^G) < \infty \), then with probability \( 1 - \delta \) over a sample \( S \sim \mathcal{D}^n \) of size \( n = O\left(\frac{\text{VC}(\mathcal{L}_H^G)}{\epsilon^2} \ln\left(\frac{1}{\delta}\right) + \ln\left(\frac{1}{\delta}\right)\right) \), we have

\[
R_G'(\mathcal{A}(S); \mathcal{D}) \leq \inf_{h \in \mathcal{H}} R_G(h; \mathcal{D}) + \epsilon,
\]

where \( \mathcal{A}(S) = \text{RERM}(S; \mathcal{G}) \). Note that this must be also true for the largest such \( \mathcal{G}' \subset \mathcal{G} \).

Alternatively, if \( \mathcal{G}' \subset \mathcal{G} \) such that there exists a finite subset \( \mathcal{G}^* \subset \mathcal{G} \) where \( \ell_{\mathcal{G}^*}(h, (x, y)) \leq \ell_{\mathcal{G}}(h, (x, y)) \), then Tolerantly Robust PAC learning via RERM is possible with sample complexity that scales according to \( O\left(\frac{\text{VC}(\mathcal{H}) \log(|\mathcal{G}^*|)}{\epsilon^2} \ln\left(\frac{1}{\delta}\right) + \ln\left(\frac{1}{\delta}\right)\right) \). This result essentially comes from the fact that the VC dimension of the loss class for any finite perturbation set \( \mathcal{G}^* \) incurs only a \( \log(|\mathcal{G}^*|) \) blow-up from the VC dimension of \( \mathcal{H} \) (see Lemma 1.1 in [AKM19]). Thus, finite VC dimension of \( \mathcal{H} \) implies finite VC dimension of the loss class \( \mathcal{L}_H^G \) which implies uniform convergence of the loss \( \ell_{\mathcal{G}^*}(h, (x, y)) \), as needed for Lemma 11 to hold. We now give an example where such a finite approximation of \( \mathcal{G}' \) is possible. In order to do so, we will need to consider a metric space of perturbation functions \((\mathcal{G}, d)\) and define a notion of “nice” perturbation sets, a concept similar to “regular” hypothesis classes from [BHK+22].

**Definition 8** (r-Nice Perturbation Set). Let \( \mathcal{H} \) be an arbitrary hypothesis class and \((\mathcal{G}, d)\) be a metric space of perturbation functions. Let \( B_r(g) \) denote a closed ball of radius \( r \) centered around \( g \in \mathcal{G} \). We say that \( \mathcal{G}' \subset \mathcal{G} \) is \( r \)-Nice with respect to \( \mathcal{H} \), if for all \( x \in \mathcal{X}, h \in \mathcal{H}, \) and \( g \in \mathcal{G}' \), there exists a \( g^* \in \mathcal{G} \), such that \( g \in B_r(g^*) \) and for all \( g' \in B_r(g^*) \), we have \( h(g(x)) = h(g'(x)) \).

At a high level, Definition 8 prevents a situation where a hypothesis \( h \in \mathcal{H} \) is non-robust to an isolated perturbation function \( g \in \mathcal{G}' \) for any given labelled example \((x, y) \in \mathcal{X} \times \mathcal{Y}\). If a hypothesis \( h \) is non-robust to a perturbation \( g \in \mathcal{G}' \), then Definition 8 asserts that there must exist a small ball of perturbation functions in \( \mathcal{G} \) over which \( h \) is also non-robust. Next, we define the external covering number, which is useful for constructing a finite approximation of a potentially infinite set of perturbation functions.

**Definition 9.** Let \((\mathcal{M}, d)\) be a metric space, let \( \mathcal{K} \subset \mathcal{M} \) be a subset, and \( r > 0 \). Let \( B_r(x) \) denote the ball of radius \( r \) centered around \( x \in \mathcal{M} \). A subset \( \mathcal{C} \subset \mathcal{M} \) is an \( r \)-external covering of \( \mathcal{K} \) if \( \mathcal{K} \subset \bigcup_{c \in \mathcal{C}} B_r(c) \). The external covering number of \( \mathcal{K} \), denoted \( N_r(\mathcal{K}) \), is the smallest cardinality of any \( r \)-external covering of \( \mathcal{K} \).

Finally, let \( \mathcal{G}'_r = \bigcup_{g \in \mathcal{G}'} B_r(g) \) denote the union over all balls of radius \( r \) with centers in \( \mathcal{G}' \). We are now ready to present our main Theorem along with its proof. Theorem 12 states that if there exists a set \( \mathcal{G}' \subset \mathcal{G} \) that is \( r \)-Nice with respect to \( \mathcal{H} \), then Tolerant Robust PAC learning is possible via RERM with sample complexity that scales logarithmically with \( N_{\frac{r}{2}}(\mathcal{G}'_r) \).

**Theorem 12** (Tolerant Robust PAC learning under Nice Perturbations). Let \( \mathcal{H} \subset \mathcal{Y}^\mathcal{X} \) be an arbitrary hypothesis class. Let \((\mathcal{G}, d)\) be a metric space of perturbation functions. If there exists subsets \( \mathcal{G}', \mathcal{G}'_r \subset \mathcal{G} \) such that \( \mathcal{G}' \) is \( r \)-Nice with respect to \( \mathcal{H} \), then the learning rule \( \mathcal{A}(S) = \text{RERM}(S; \mathcal{G}) \) achieves, for any distribution \( \mathcal{D} \) over \( \mathcal{X} \times \mathcal{Y} \), with probability at least \( 1 - \delta \) over \( S \sim \mathcal{D}^n \),

\[
R_G'(\mathcal{A}(S); \mathcal{D}) \leq \inf_{h \in \mathcal{H}} R_G(h; \mathcal{D}) + \epsilon,
\]

using

\[
n(\epsilon, \delta; \mathcal{H}, \mathcal{G}) = O\left(\frac{\text{VC}(\mathcal{H}) \log(N_{\frac{r}{2}}(\mathcal{G}'_r)) \ln\left(\frac{1}{\delta}\right) + \ln\left(\frac{1}{\delta}\right)}{\epsilon^2}\right).
\]

number of samples, where \( N_{\frac{r}{2}}(\mathcal{G}'_r) \) is the \( \frac{r}{2} \)-external covering number of \( \mathcal{G}'_r \).
Proof. (of Theorem 12) Assume that there exists subsets $G', G'_r \subseteq G$ such that $G'$ is $r$-Nice with respect to $H$. By Lemma 11, it is sufficient to find a perturbation set $\tilde{G}$ s.t. (1) $\ell_{G'}(h, (x, y)) \leq \ell_{\tilde{G}}(h, (x, y)) \leq \ell_{G}(h, (x, y))$ and (2) $\ell_{\tilde{G}}(h, (x, y))$ enjoys the uniform convergence property with sample complexity $O\left(\frac{\log(N_{2r}(G'_r))}{\epsilon^2} + \ln(\frac{1}{\delta})\right)$. Let $\tilde{G} \subseteq G$ be the minimal $\frac{r}{2}$-cover of $G'_r$ with cardinality $N_{2r}(G'_r)$. By Lemma 1.1 of [AKM19], the loss class $L_{H,2}^s$ has VC dimension at most $O(\log(|G'_r|) = O(\log(N_{2r}(G'_r)))$, implying that $\ell_{\tilde{G}}(h, (x, y))$ enjoys the uniform convergence property with sample complexity $O\left(\frac{\log(N_{2r}(G'_r))}{\epsilon^2} + \ln(\frac{1}{\delta})\right)$. It now remains to show that for our choice of $\tilde{G}$, we have $\ell_{G'}(h, (x, y)) \leq \ell_{\tilde{G}}(h, (x, y)) \leq \ell_{G}(h, (x, y))$. Since, $\tilde{G} \subseteq G$, the upper-bound is trivial. Thus, we only focus on proving the lower-bound, $\ell_{G'}(h, (x, y)) \leq \ell_{\tilde{G}}(h, (x, y))$ for all $h \in H$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Fix $h \in H$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$. If $\ell_{G'}(h, (x, y)) = 1$, then there exists a $g \in G'$ s.t. $h(g(x)) \neq y$. Let $g$ denote one such perturbation function. By the $r$-Niceness property of $G'$ with respect to $H$, there must exist $B_r(g^*)$ centered at some $g^* \in G$ such that $g \in B_r(g^*)$ and $h(g(x)) = h(g^*(x))$ for all $g' \in B_r(g^*)$. This implies that $h(g'(x)) \neq y$ for all $g' \in B_r(g^*)$. Next, since $d(g, g^*) \leq r$, there must exist a ball $B_{r/2}(\hat{g})$ s.t. $B_{r/2}(\hat{g}) \subseteq B_r(g) \cap B_r(g^*)$. But if this is the case, then $B_{r/2}(\hat{g}) \subseteq G'_r$, which means that if $\tilde{G}$ is a $r/2$-cover of $G'_r$ it must contain at least one function $\hat{g}$ from $B_{r/2}(\hat{g})$. Since $B_{r/2}(\hat{g}) \subseteq B_r(g^*)$, we also have that $h(\hat{g}(x)) \neq y$. Therefore, we have that there exists a perturbation function $\hat{g} \in \tilde{G}$ s.t. $h(\hat{g}(x)) \neq y$ as desired. This concludes the proof. \hfill \Box

7 Discussion and Future Directions

In this work, we further the study of probabilistically robust PAC learning by characterizing which hypothesis classes are properly learnable. Perhaps surprisingly, we show that finite VC dimension is still not sufficient for proper probabilistically robust PAC learning, and that more complicated, improper learning rules are needed. However, if we compare the learner to the best hypothesis for a slightly stronger notion of robustness, then not only does proper learning become possible, but an ERM-based learner is sufficient.

There are several interesting open directions in this line of work. First, what dimension characterizes (proper) $(\rho, \rho^*)$-probabilistically robust PAC learning? We showed that finite VC dimension is sufficient but not necessary. What dimension characterizes proper $\rho$-probabilistically robust PAC learning? Finally, in this paper we mainly focused on proper learning rules. Improper learning rules for $\rho$-robustly learnable PAC learning is an open and interesting future direction.

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A Equivalence between Adversarial Robustness Models

In this section, we show that the perturbation set and perturbation function models are equivalent.

**Theorem 13** (Equivalence between $\mathcal{G}$ and $\mathcal{U}$). Let $\mathcal{X}$ be an arbitrary domain. There exists a perturbation set $\mathcal{U} : \mathcal{X} \to 2^\mathcal{X}$ if and only if there exists a set of perturbation functions $\mathcal{G}$ s.t. $\mathcal{G}(x) = \{g(x) : g \in \mathcal{G}\} = \mathcal{U}(x)$ for all $x \in \mathcal{X}$.
Proof. We first show that every set of perturbation functions \( \mathcal{G} \) induces a perturbation set \( \mathcal{U} \). Let \( \mathcal{G} \) be an arbitrary set of perturbation functions \( g: \mathcal{X} \rightarrow \mathcal{X} \). Then, for each \( x \in \mathcal{X} \), define \( \mathcal{U}(x) := \{ g(x) : g \in \mathcal{G} \} \), which completes the proof of this direction.

Now we will show the converse - every perturbation set \( \mathcal{U} \) induces a point-wise equivalent set \( \mathcal{G} \) of perturbation functions. Let \( \mathcal{U} \) be an arbitrary perturbation set mapping points in \( \mathcal{X} \) to subsets in \( \mathcal{X} \). Assume that \( \mathcal{U}(x) \) is not empty for all \( x \in \mathcal{X} \). Let \( \tilde{z}_x \) denote an arbitrary perturbation from \( \mathcal{U}(x) \). For every \( x \in \mathcal{X} \), and every \( z \in \mathcal{U}(x) \), define the perturbation function \( g^z_x(t) = z \mathbb{I}\{t = x\} + \tilde{z}_t \mathbb{I}\{t \neq x\} \) for \( t \in \mathcal{X} \). Observe that \( g^z_x(x) = z \in \mathcal{U}(x) \) and \( g^z_x(x') = \tilde{z}_{x'} \in \mathcal{U}(x') \). Finally, let \( \mathcal{G} = \bigcup_{x \in \mathcal{X}} \bigcup_{z \in \mathcal{U}(x)} \{ g^z_x \} \). To verify that \( \mathcal{G} = \mathcal{U} \), consider an arbitrary point \( x' \in \mathcal{X} \). Observe that

\[
\mathcal{G}(x') = \bigcup_{x \in \mathcal{X}} \bigcup_{z \in \mathcal{U}(x)} \{ g^z_x(x') \}
\]

\[
= \left( \bigcup_{z \in \mathcal{U}(x')} \{ g^z_x(x') \} \right) \cup \left( \bigcup_{x \in \mathcal{X} \setminus x'} \bigcup_{z \in \mathcal{U}(x)} \{ g^z_x(x') \} \right)
\]

\[
= \left( \bigcup_{z \in \mathcal{U}(x')} \{ z \} \right) \cup \left( \bigcup_{x \in \mathcal{X} \setminus x'} \bigcup_{z \in \mathcal{U}(x)} \{ \tilde{z}_x \} \right)
\]

\[
= \mathcal{U}(x') \cup \tilde{z}_{x'}
\]

\[
= \mathcal{U}(x').
\]

This completes the proof. \( \square \)

### B Proof of Lemma 4

**Proof.** [of Lemma 4] This proof closely follows Lemma 3 from [MHS19]. In fact, the only difference is in the construction of the hypothesis class, which we will describe below.

Fix \( \rho \in [0, 1] \). Let \( m \in \mathbb{N} \). Construct a hypothesis class \( \mathcal{H}_0 \) as in Lemma 3 on 3m centers \( c_1, \ldots, c_{3m} \) based on \( \rho \). By the construction in Lemma 3, we know that \( \mathcal{L}^\rho_{\mathcal{H}_0} \) satters the sample \( C = \{(c_1, 1), \ldots, (c_{3m}, 1)\} \). Instead of keeping all of \( \mathcal{H}_0 \), we will only keep a subset \( \mathcal{H} \) of \( \mathcal{H}_0 \), namely those classifiers that are probabilistically robustly correct on subsets of size 2m of \( C \). More specifically, recall from the construction in Lemma 3, that each hypothesis \( h_b \in \mathcal{H}_0 \) is parameterized by a bitstring \( b \in \{0, 1\}^{3m} \) where if \( b_i = 1 \), then \( h_b \) is not robust to example \( (c_i, 1) \). Therefore, \( \mathcal{H} = \{h_b \in \mathcal{H}_0 : \sum_{i=1}^{3m} b_i = m\} \). Now, let \( \mathcal{A}: (\mathcal{X} \times \mathcal{Y})^* \rightarrow \mathcal{H} \) be an arbitrary proper learning rule. Consider a set of distributions \( \mathcal{D}_1, \ldots, \mathcal{D}_L \) where \( L = \binom{3m}{2m} \). Each distribution \( \mathcal{D}_i \) is uniform over exactly 2m centers in \( C \). Critically, note that by our construction of \( \mathcal{H} \), every distribution \( \mathcal{D}_i \) is probabilistically robustly realizable by a hypothesis in \( \mathcal{H} \). That is, for all \( \mathcal{D}_i \), there exists a hypothesis \( h^* \in \mathcal{H} \) s.t. \( R^\rho_{\mathcal{G}, \mu}(h^*; \mathcal{D}) = 0 \). Observe that this satisfies the first condition in Lemma 4. For the second condition, at a high-level, the idea is to use the probabilistic method to show that there exists a distribution \( \mathcal{D}_i \) where \( \mathbb{E}_{S \sim \mathcal{D}^m} \left[ R^\rho_{\mathcal{G}, \mu}(\mathcal{A}(S); \mathcal{D}_i) \right] \geq \frac{1}{4} \) and then use a variant of Markov’s inequality to show that with probability at least 1/7 over \( S \sim \mathcal{D}^m \), \( R^\rho_{\mathcal{G}, \mu}(\mathcal{A}(S); \mathcal{D}) > 1/8 \).

Let \( S \in \mathcal{C}^m \) be an arbitrary set of \( m \) points. Let \( \mathcal{C} \) be a uniform distribution over \( C \). Let \( \mathcal{P} \) be a uniform distribution over \( \mathcal{D}_1, \ldots, \mathcal{D}_L \). Let \( \mathcal{E}_S \) denote the event that \( S \subseteq \text{supp}(\mathcal{D}_i) \) for \( \mathcal{D}_i \sim \mathcal{P} \). Given the event \( \mathcal{E}_S \), we will lower bound the expected probabilistic robust loss of the hypothesis the proper learning rule \( \mathcal{A} \) outputs,

\[
\mathbb{E}_{\mathcal{D}_i \sim \mathcal{P}} \left[ R^\rho_{\mathcal{G}, \mu}(\mathcal{A}(S); \mathcal{D}_i) | \mathcal{E}_S \right] = \mathbb{E}_{\mathcal{D}_i \sim \mathcal{P}} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_i} \left[ \mathbb{I}\{ \mathbb{P}_{g \sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho \} \right] | \mathcal{E}_S \right].
\]

Conditioning on the event that \( (x, y) \notin S \), denoted, \( \mathcal{E}_{(x,y) \notin S} \),

\[
\mathbb{E}_{(x,y) \sim \mathcal{D}_i} \left[ \mathbb{I}\{ \mathbb{P}_{g \sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho \} \right] > \mathbb{P}_{(x,y) \sim \mathcal{D}_i} \left[ \mathbb{E}_{(x,y) \notin S} \left[ \mathbb{I}\{ \mathbb{P}_{g \sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho \} | \mathcal{E}_{(x,y) \notin S} \right] \right]
\]

\[
\times \mathbb{E}_{(x,y) \sim \mathcal{D}_i} \left[ \mathbb{I}\{ \mathbb{P}_{g \sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho \} | \mathcal{E}_{(x,y) \notin S} \right].
\]
Since $D_i$ is supported over $2m$ points and $|S| = m$, $\mathbb{P}_{(x,y)\sim D_i} [E_{(x,y)\notin S}] \geq \frac{1}{2}$ since in the worst-case $S \subseteq \text{supp}(D_i)$. Thus, we obtain the lower bound,

$$\mathbb{E}_{D_i \sim \mathcal{D}} \left[ R^p_{g,\mu}(\mathcal{A}(S); D_i) | E_S \right] \geq \frac{1}{2} \mathbb{E}_{D_i \sim \mathcal{D}} \left[ \mathcal{E}_{(x,y)\sim D_i} \left[ \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\}|E_{(x,y)\notin S}\right] \mathcal{E}_{(x,y)\sim D_i} \left[ \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\}|E_{(x,y)\notin S}\right] \right].$$

Unravelling the expectation over the draw from $D_i$, we have,

$$\mathbb{E}_{(x,y)\sim D_i} \left[ \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\}|E_{(x,y)\notin S}\right] \geq \frac{1}{m} \sum_{(x,y)\in \text{supp}(D_i) \setminus S} \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\}$$

Observing that $\mathbb{E}_{D_i \sim \mathcal{D}} \left[ \mathbb{1}\{(x,y) \in \text{supp}(D_i)\}|E_S \right] \geq \frac{1}{2}$ yields,

$$\mathbb{E}_{D_i \sim \mathcal{D}} \left[ \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\}|E_{(x,y)\notin S}\right] \geq \frac{1}{2m} \sum_{(x,y)\notin S} \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\}.$$

Since $\mathcal{A}(S) \in \mathcal{H}$, by construction of $\mathcal{H}$, there are at least $m$ points in $C$ where $\mathcal{A}$ is not probabilistically robustly correct. Therefore,

$$\frac{1}{2m} \sum_{(x,y)\notin S} \mathbb{1}\{\mathbb{P}_{g\sim \mu}(\mathcal{A}(S)(g(x)) \neq y) > \rho\} \geq \frac{1}{2},$$

from which we have that, $\mathbb{E}_{D_i \sim \mathcal{D}} \left[ R^p_{g,\mu}(\mathcal{A}(S); D_i) | E_S \right] \geq \frac{1}{4}$. By the law of total expectation, we have that

$$\mathbb{E}_{D_i \sim \mathcal{D}} \left[ E_{S \sim \mathcal{D}_i} \left[ R^p_{g,\mu}(\mathcal{A}(S); D_i) \right] \right] = \mathbb{E}_{S \sim \mathcal{C}} \left[ E_{D_i \sim \mathcal{D} | E_S} \left[ R^p_{g,\mu}(\mathcal{A}(S); D_i) \right] \right] \geq \frac{1}{4}$$

Since the expectation over $D_1, ..., D_T$ is at least $1/4$, there must exist a distribution $D_i$ where $\mathbb{E}_{S \sim \mathcal{D}_i} \left[ R^p_{g,\mu}(\mathcal{A}(S); D_i) \right] \geq 1/4$. Using a variant of Markov’s inequality, we have that

$$\mathbb{P}_{S \sim \mathcal{D}_i} \left[ R^p_{g,\mu}(\mathcal{A}(S); D_i) > 1/8 \right] \geq 1/7$$

which completes the proof. $\square$

### C Proof of Theorem 6

**Proof.** (of Theorem 6) Fix $0 \leq \rho^* < \rho$ and let $\mathcal{H}$ be a hypothesis class with $\text{VC}(\mathcal{H}) = d$. Let $(\mathcal{G}, \mu)$ be an arbitrary adversary. Let $\mathcal{D}$ be an arbitrary distribution over $\mathcal{X} \times \mathcal{Y}$ and $S = \{(x_1, y_1), ..., (x_m, y_m)\}$ an i.i.d. sample of size $m$.

Consider the learning algorithm $\mathcal{A}$ that computes $\text{PRERM}(S; \mathcal{G}, \mu, \rho, \rho^*)$ on a sample $S$. That is,

$$\mathcal{A}(S) = \text{PRERM}(S; \mathcal{G}, \mu, \rho, \rho^*) = \arg\min_{h \in \mathcal{H}} \mathbb{E}_{S \sim \mathcal{C}} \left[ \ell_{\mu}(g_{\rho^*}(h, (x, y))) \right].$$

Note that $\mathcal{A}$ is indeed a proper learning algorithm according to Definition 2. Let $\hat{h} = \mathcal{A}(S)$ denote hypothesis output by $\mathcal{A}$ and $h^* = \inf_{h \in \mathcal{H}} \mathbb{E}_{D} \left[ \ell_{\mu}(g_{\rho^*}(h, (x, y))) \right]$. It remains to show that if the sample size $m = O \left( \frac{d^2}{\epsilon^2 \rho \rho^* \ln(1/\delta)} \right), \rho \rho^* \leq \frac{1}{2}$, then $\hat{h}$ achieves the stated generalization bound with probability $1 - \delta$. By Theorem 7, if $m = O \left( \frac{d^2}{\epsilon^2 \rho \rho^* \ln(1/\delta)} \right), \rho \rho^* \leq \frac{1}{2}$, we have that with probability $1 - \delta$, for all $h \in \mathcal{H}$ simultaneously,

$$\left| \mathbb{E}_{D} \left[ \ell_{\mu}(g_{\rho^*}(h, (x, y))) \right] - \mathbb{E}_{S} \left[ \ell_{\mu}(g_{\rho^*}(h, (x, y))) \right] \right| \leq \frac{\epsilon}{2}.$$
This means that both $\mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] - \mathbb{E}_S \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] \leq \frac{\epsilon}{2}$ and $\mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] - \mathbb{E}_S \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] \leq \frac{\epsilon}{2}$. By definition of $\hat{h}$, note that $\mathbb{E}_S \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] \leq \mathbb{E}_S \left[ \ell_{\rho}^{\rho,\rho}(h^*, (x,y)) \right]$. Putting things together, we have that

$$
\mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] - (\mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(h^*, (x,y)) \right] + \frac{\epsilon}{2}) \leq \mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] - \mathbb{E}_S \left[ \ell_{\rho}^{\rho,\rho}(h^*, (x,y)) \right] \leq \mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] - \mathbb{E}_S \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] \leq \frac{\epsilon}{2}
$$

from which we can deduce that

$$
\mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] - \inf_{h \in \mathcal{H}} \mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(h, (x,y)) \right] \leq \epsilon.
$$

Finally, by noting that $\mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(\hat{h}, (x,y)) \right] \geq R_{\rho,\rho}^{\rho}(\hat{h}; \mathcal{D})$ and $\inf_{h \in \mathcal{H}} \mathbb{E}_D \left[ \ell_{\rho}^{\rho,\rho}(h, (x,y)) \right] \leq \inf_{h \in \mathcal{H}} R_{\rho,\rho}^{\rho}(h; \mathcal{D})$, it’s not too hard to see that,

$$
R_{\rho,\rho}^{\rho}(\hat{h}; \mathcal{D}) \leq \inf_{h \in \mathcal{H}} R_{\rho,\rho}^{\rho}(h; \mathcal{D}) + \epsilon.
$$

Thus, we have given a proper learning algorithm achieving the stated generalization bound with sample complexity $m = O \left( \frac{d}{\epsilon^2} \ln \left( \frac{1}{d\epsilon^2} \right) \right)$.

\[\square\]

D Proof of Theorem 7

Proof. (of Theorem 7) Fix $0 \leq \rho^* < \rho$ and let $\mathcal{H}$ be a hypothesis class with $\text{VC}(\mathcal{H}) = d$. Let $(\mathcal{G}, \mu)$ be an arbitrary adversary. Recall the probabilistic robust ramp loss:

$$
\ell_{\rho}^{\rho,\rho}(h, (x,y)) := \min \{ 1, \max (0, \frac{\mathbb{P}_{g \sim \mu} [h(g(x)) \neq y] - \rho^*}{\rho - \rho^*}) \}.
$$

Note that $\ell_{\rho}^{\rho,\rho}(h, (x,y))$ is bounded above by 1. Let $S = \{(x_1, y_1), ..., (x_m, y_m)\}$ be a set of examples drawn i.i.d from $\mathcal{D}$ and define the probabilistic robust ramp loss class $\mathcal{F} = \{(x,y) \mapsto \ell_{\rho}^{\rho,\rho}(h(x,y)) : h \in \mathcal{H}\}$. Observe that we can reparameterize $\mathcal{F}$ as the composition of a $\frac{1}{\rho - \rho^*}$-Lipschitz function $b(x) = \min(1, \max (0, \frac{x - \rho^*}{\rho - \rho^*}))$ and the function class $\mathcal{F}_{\rho,\rho}^{\rho} = \{(x,y) \mapsto \mathbb{P}_{g \sim \mu} [h(g(x)) \neq y] : h \in \mathcal{H}\}$. That is, $\mathcal{F} = b \circ \mathcal{F}_{\rho,\rho}^{\rho} = \{ b \circ f : f \in \mathcal{F}_{\rho,\rho}^{\rho} \}$. By Theorem 2, to show the uniform convergence property of the probabilistic robust ramp loss, it suffices to upper bound $\mathcal{R}_m(\mathcal{F}) = \mathcal{R}_m(b \circ \mathcal{F}_{\rho,\rho}^{\rho})$, the empirical Rademacher complexity of the probabilistic robust ramp loss class. Since $b$ is $\frac{1}{\rho - \rho^*}$-Lipschitz, by Talagrand’s contraction principle, it follows that

$$
\mathcal{R}_m(\mathcal{F}) = \mathcal{R}_m(b \circ \mathcal{F}_{\rho,\rho}^{\rho}) \leq \frac{1}{\rho - \rho^*} \cdot \mathcal{R}_m(\mathcal{F}_{\rho,\rho}^{\rho}).
$$
Thus, it actually suffices to upperbound $\mathcal{R}_m(\mathcal{F}^\mathcal{H}_{G\mu})$ instead. Starting with the definition of the empirical Rademacher complexity:

$$\mathcal{R}_m(\mathcal{F}^\mathcal{H}_{G\mu}) = \mathbb{E}_{x \sim \{\pm 1\}^m} \left[ \sup_{f \in \mathcal{F}^\mathcal{H}_{G\mu}} \left( \frac{1}{m} \sum_{i=1}^m \sigma_i f(x_i, y_i) \right) \right]$$

$$= \frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i \mathbb{E}_{g \sim \mu} [h(g(x_i)) \neq y_i] \right) \right]$$

$$= \frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i \mathbb{E}_{g \sim \mu} [\mathbb{I} \{h(g(x_i)) \neq y_i\}] \right) \right]$$

$$= \frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i \mathbb{E}_{g \sim \mu} \left[ 1 - h(g(x_i))y_i \right] \right) \right]$$

$$= \frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i \mathbb{E}_{g \sim \mu} \left[ 1 - h(g(x_i))y_i \right] \right) \right]$$

$$= \frac{1}{2m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i \mathbb{E}_{g \sim \mu} [h(g(x_i))y_i] \right) \right]$$

$$= \frac{1}{2m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i y_i \mathbb{E}_{g \sim \mu} [h(g(x_i))] \right) \right]$$

$$= \frac{1}{2m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \sum_{i=1}^m \sigma_i \mathbb{E}_{g \sim \mu} [h(g(x_i))] \right) \right]$$

$$= \frac{1}{2m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \left( \mathbb{E}_{g \sim \mu} \left[ \sum_{i=1}^m \sigma_i h(g(x_i)) \right] \right) \right]$$

Fix some sequence of Rademacher random variables $\sigma_1, ..., \sigma_m$. Then, observe that for all $g \in \mathcal{G}$ and $h \in \mathcal{H}$, $\sum_{i=1}^m \sigma_i h(g(x_i)) \leq \sup_{h \in \mathcal{H}} \sum_{i=1}^m \sigma_i h(g(x_i))$ which implies $\mathbb{E}_{g \sim \mu} \left[ \sum_{i=1}^m \sigma_i h(g(x_i)) \right] \leq \mathbb{E}_{g \sim \mu} \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^m \sigma_i h(g(x_i)) \right]$. Finally, taking the supremum over $\mathcal{H}$ of both sides yields

$$\sup_{h \in \mathcal{H}} \mathbb{E}_{g \sim \mu} \left[ \sum_{i=1}^m \sigma_i h(g(x_i)) \right] \leq \mathbb{E}_{g \sim \mu} \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^m \sigma_i h(g(x_i)) \right].$$

Substituting back in, we find that

$$\mathcal{R}_m(\mathcal{F}^\mathcal{H}_{G\mu}) \leq \frac{1}{2m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \mathbb{E}_{g \sim \mu} \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^m \sigma_i h(g(x_i)) \right] \right]$$

$$= \frac{1}{2} \mathbb{E}_{g \sim \mu} \left[ \frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^m \sigma_i h(g(x_i)) \right] \right].$$

Note that the quantity $\frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^m \sigma_i h(g(x_i)) \right]$ is the empirical Rademacher complexity of the hypothesis class $\mathcal{H}$ over the sample $\{g(x_1), ..., g(x_m)\}$ drawn i.i.d from the distribution defined by first sampling from the marginal data distribution, $x \sim D_X$, and then applying the transformation $g(x)$. By standard VC arguments (see [cite]), $\mathcal{R}_m(\mathcal{H}) \leq \sqrt{\frac{2d \ln \left( \frac{m}{4} \right)}{m}}$, which implies that

$$\mathcal{R}_m(\mathcal{F}^\mathcal{H}_{G\mu}) \leq \frac{1}{2} \mathbb{E}_{g \sim \mu} \left[ \mathcal{R}_m(\mathcal{H}) \right] \leq \frac{1}{2} \sqrt{\frac{2d \ln \left( \frac{m}{4} \right)}{m}},$$

from which we have,

$$\mathcal{R}_m(\mathcal{F}) = \mathcal{R}_m(b \circ \mathcal{F}^\mathcal{H}_{G\mu}) \leq \frac{1}{\rho - \rho^*} \sqrt{\frac{d \ln \left( \frac{m}{4} \right)}{2m}} = \sqrt{\frac{d \ln \left( \frac{m}{4} \right)}{2m \left( \rho - \rho^* \right)^2}}.$$
Theorem 2 then implies that with probability $1 - \delta$, for all $h \in \mathcal{H}$ simultaneously, we have

$$\mathbb{E}_D \left[ \ell_{G,h}^*(h, (x, y)) - \hat{E}_S \left[ \ell_{G,h}^*(h, (x, y)) \right] \right] \leq 2\mathfrak{R}_m(\mathcal{F}) + O \left( \sqrt{\frac{\ln(\frac{1}{\delta})}{m}} \right)$$

\[= \sqrt{\frac{2d\ln \left( \frac{m}{\epsilon} \right)}{m\rho - \rho^*}} + O \left( \sqrt{\frac{\ln(\frac{1}{\delta})}{m}} \right) \]

\[= O \left( \sqrt{\frac{d}{(\rho - \rho^*)^2} \ln \left( \frac{m}{\epsilon} \right)} + \ln \left( \frac{1}{\delta} \right) \right) \]

Letting $\epsilon \geq O \left( \sqrt{\frac{d}{(\rho - \rho^*)^2} \ln \left( \frac{m}{\epsilon} \right) + \ln \left( \frac{1}{\delta} \right)} \right)$ and inverting to solve for $m$, we get that taking $m = O \left( \frac{d}{(\rho - \rho^*)^2} \ln \left( \frac{m}{\epsilon} \right) + \ln \left( \frac{1}{\delta} \right) \right)$ samples is sufficient to achieve with probability $1 - \delta$,

$$\mathbb{E}_D \left[ \ell_{G,h}^*(h, (x, y)) - \hat{E}_S \left[ \ell_{G,h}^*(h, (x, y)) \right] \right] \leq \epsilon$$

for all $h \in \mathcal{H}$ simultaneously. This completes the proof.

\[\square\]

### E More Bounds on the Probabilistic Robust Risk

Theorem 14 states that proper learning via ERM is possible for any Lipschitz loss function that upper bounds the indicator $1 \{ \mathbb{P}_{g \sim \mu} (h(g(x)) \neq y) > \rho \}$.

**Theorem 14.** Let $\ell(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a $L$-Lipschitz function in $x$ s.t. $1 \{ x > t \} \leq \ell(x, t)$ for all $x, t \in [0, 1]$. Then, with probability $1 - \delta$, the hypothesis $\hat{h} = \arg \min_{h \in \mathcal{H}} \mathbb{E}_S \left[ \ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho) \right]$ achieves

$$\mathcal{R}_{G,h}^\rho(\hat{h}; \mathcal{D}) \leq \inf_{h \in \mathcal{H}} \mathbb{E}_D \left[ \ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho) \right] + \epsilon$$

with sample complexity $n(\epsilon, \delta; G, \mu) = O \left( \frac{\text{VC}(\mathcal{H}) L^2 \ln \left( \frac{4}{\epsilon} \right) + \ln \left( \frac{1}{\delta} \right)}{\epsilon^2} \right)$.

**Proof.** Let $\ell(x, t)$ be a $L$-Lipschitz function in $x$ s.t. $1 \{ x > t \} \leq \ell(x, t)$. Then, $\ell_{G,h}^*(h, (x, y)) = 1 \{ \mathbb{P}_{g \sim \mu}(h(g(x)) \neq y) > \rho \} \leq \ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho)$ By Lemma 11, it now suffices to show that the loss function $\ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho)$ enjoys the uniform convergence property with sample complexity $O \left( \frac{\text{VC}(\mathcal{H}) L^2 \ln \left( \frac{4}{\epsilon} \right) + \ln \left( \frac{1}{\delta} \right)}{\epsilon^2} \right)$. To show uniform convergence, by Theorem 2, it suffices to bound the empirical Rademacher complexity of the loss class $\mathcal{L} = \{ (x, y) \to \ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho) \}$. Since $\ell(x, t)$ is $L$-Lipschitz, by Talagrand’s Contraction Lemma, we have that $\hat{\mathfrak{R}}_{m}(\mathcal{L}) \leq L \mathfrak{R}_{m}(\mathcal{F}_{G,h}^H)$ where recall $\mathcal{F}_{G,h}^H = \{ (x, y) \to \mathbb{P}_{g \sim \mu}(h(g(x)) \neq y) \}$. However, by the proof of Theorem 7 in Appendix D, we know that $\hat{\mathfrak{R}}_{m}(\mathcal{F}_{G,h}^H) \leq O \left( \sqrt{\frac{\text{VC}(\mathcal{H}) L^2 \ln \left( \frac{4}{\epsilon} \right)}{m}} \right)$. Thus, we have that $\hat{\mathfrak{R}}_{m}(\mathcal{L}) \leq O \left( L \sqrt{\frac{\text{VC}(\mathcal{H}) L^2 \ln \left( \frac{4}{\epsilon} \right)}{m}} \right)$. Theorem 2 and standard sample complexity inversion gives that the loss $\ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho)$ enjoys the uniform convergence property with sample complexity $O \left( \frac{\text{VC}(\mathcal{H}) L^2 \ln \left( \frac{4}{\epsilon} \right) + \ln \left( \frac{1}{\delta} \right)}{\epsilon^2} \right)$, as required by Lemma 11. Using Lemma 11 with $\ell_1(h, (x, y)) = \ell_{G,h}^*$ and $\ell_2(h, (x, y)) = \ell(\mathbb{P}_{g \sim \mu}(h(g(x)) \neq y), \rho)$ completes the proof.

\[\square\]
Since \( \ell(x, t) = \frac{x}{t} \) is \( \frac{1}{t} \)-Lipschitz, we get by Theorem 14, with probability \( 1 - \delta \), the hypothesis \( \hat{h} = \arg \min_{h \in \mathcal{H}} \mathbb{E}_S [\mathbb{P}_{g \sim \mu} (h(g(x)) \neq y)] \) achieves

\[
\mathcal{R}^\rho_{\mathcal{G}, \mu} (\hat{h}; \mathcal{D}) \leq \inf_{h \in \mathcal{H}} \mathbb{E}_D \left[ \frac{\mathbb{P}_{g \sim \mu} (h(g(x)) \neq y)}{\rho} \right] + \epsilon
\]

with sample complexity \( n(\epsilon, \delta; \rho, \mathcal{G}, \mu) = O \left( \frac{\text{VC}(\mathcal{H})}{\epsilon^2 \rho^2} \ln \left( \frac{1}{\epsilon \rho} \right) \right) \).