We study the supersymmetric quantum mechanical systems that arise from discrete light cone quantization of theories with minimal supersymmetry in various dimensions. These systems have previously arisen in the study of black hole moduli spaces, are distinguished by having fewer fermionic fields than the familiar Kähler and hyper-Kähler models. This is a contribution to the Yuri Golfand memorial volume.

1 Preface

In the three decades since the pioneering works, supersymmetry has steadily grown both in importance and in depth. It is the most widely anticipated form of new physics to be seen near the weak scale; it is central to the structure of string theory; it is the key to the recent understanding of nonperturbative physics both in field theory and string theory; it is the focal point of many connections between mathematics and physics; and, it appears to be the principle that is responsible for cancellation of quantum corrections, stabilizing both the weak symmetry breaking scale and spacetime itself. It seems likely that we still have much to learn in these and other directions.

Even in quantum mechanics, supersymmetry has many applications. Most recently, through discrete light cone quantization, supersymmetric quantum mechanics has been proposed as a nonperturbative formulation of M theory and other exotic systems. Supersymmetric quantum mechanics is in some ways less constrained than supersymmetric field theory, due to the reduced spacetime symmetry. For example, there is no direct relation between the number of bosonic and fermionic fields (quantum mechanical coordinates). In this paper we study some supersymmetric quantum mechanical systems that arise from the discrete light cone quantization of theories with minimal supersymmetry.
in various dimensions. These are distinguished by having fewer fermionic fields than the familiar Kähler and hyper-Kähler models.

2 Introduction

Light-cone quantization has played an important role in field and string theory. In this description, free quanta satisfy nonrelativistic kinematics, with the positive ‘spatial’ momentum \( p_- \) playing the role of the mass. Of particular interest is discrete light-cone quantization (DLCQ) in which the light-like spatial direction is compact. In this case \( p_- \) is quantized as well as positive, so that a sector of given total \( p_- \) contains only a finite number of nonrelativistic particles. This has been used for numerical and analytic study of quantum field theories, and recently has been used to provide a nonperturbative definition of M theory.

In recent work we have attempted to employ DLCQ to study weak/strong duality in supersymmetric gauge theories. This has required a version of DLCQ that preserves this duality, the so-called light-like limit (LLL). This is more complicated than the usual DLCQ, in that one must in many cases solve a dynamical problem just to obtain the DLCQ Hamiltonian: it is an effective Hamiltonian in the Wilsonian sense, rather than a simple reduction of the original Hamiltonian.

Given this complication, it is useful to take maximum advantage of supersymmetry to restrict the form of the Hamiltonian. We are thus interested in various supersymmetric quantum mechanical systems. In this paper we focus on field theories with minimal (unextended) supersymmetry in various dimensions, because these lead to supersymmetric QM systems that are somewhat unfamiliar. The more general context of DLCQ and LLL will be discussed in a forthcoming paper.

Let us begin with some simple counting. We start with a field theory in \( d \) spacetime dimensions with \( \mathcal{N} \) real supersymmetry charges. The nonrelativistic particles move in the \( d - 2 \) transverse dimensions. A sector with \( k \) particles is then described by quantum mechanics with \( N_B = k(d - 2) \) real bosonic coordinates.

The supersymmetries separate into \( \frac{1}{2} \mathcal{N} \) dynamical charges \( Q_\alpha \) and \( \frac{1}{2} \mathcal{N} \) kinematical charges \( q_\alpha \), with the algebra

\[
\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}p_+ \equiv 2\delta_{\alpha\beta}H, \\
\{q_\sigma, Q_\beta\} = 2\Gamma^i_{\sigma\beta}p_i, \\
\{q_\sigma, q_\rho\} = 2\delta_{\sigma\rho}p_-. 
\]  
(1)

We can work in an eigenbasis for \( p_- \) so that the \( q_\sigma \) are essentially oscillator
Table 1: Number of real bosonic and fermionic coordinates per particle.

| $d$ | $\mathcal{N}$ | $N_B$ | $N_F/k = N_Q$ |
|-----|---------------|-------|--------------|
| 3   | 2             | 1     | 1            |
| 4   | 4             | 2     | 2            |
| 5   | 8             | 3     | 4            |
| 6   | 8             | 4     | 4            |
| 7   | 16            | 5     | 8            |
| 8   | 16            | 6     | 8            |
| 9   | 16            | 7     | 8            |
| 10  | 16            | 8     | 8            |

variables. The system can be separated into center-of-mass variables $(q_\sigma, p_i)$ and the rest. The algebra of the $q_\sigma$ determines how the center-of-mass variables enter into $Q_\alpha$ and $H$ but does not otherwise constrain the theory. The essential problem is then supersymmetric quantum mechanics with $N_Q = \frac{1}{2}N$ supercharges $Q_\alpha$. To count the fermionic coordinates, consider first a single particle. It has $2^{N/4}$ spin states, which can be thought of as generated by the $q_A$. A state with $k$ particles thus has $2^{kN/4}$ states. These would be described by quantizing $N_F = \frac{1}{4}kN$ real fermionic coordinates, with action first order in time. In table 1 we give the count of real bosonic and fermionic coordinates in $3 \leq d \leq 10$, for the minimal supersymmetry algebra in each dimension.

Consider the case of $d = 6$, where there are four supercharges. Supersymmetric QM with $N_Q = 4$ can be obtained by dimensional reduction from the familiar $D = 4$, $N = 1$ nonlinear sigma models giving QM on a Kähler space. A space with $4k$ bosonic coordinates can be described with $2k$ chiral superfields, each having complex scalar. Each chiral superfield also contains two complex fermion fields, giving a total of $8k$ real fermion fields. Under dimensional reduction to QM, each field becomes a coordinate. From table 1, the number of fermionic coordinates in the Kähler models is twice that in the DLCQ theory, so the latter are not of the familiar type. Note from the table that in dimensions 3, 4, 6, and 10 the numbers of bosonic and fermion coordinates are equal. In other dimensions the number of fermionic coordinates is greater, but always less than would be obtained with $D = 4$, $N = 1$ superfields.

After some work on this problem, we learned that the quantum mechanical systems with $N_B = N_F$ (as in $d = 3, 4, 6, 10$) had already been constructed in

\[^a\text{We are interested in particles that are massless (or BPS) in } d\text{ dimensions, and so transform in small multiplets.}\]
some detail by Gibbons, Papadopoulos, and Stelle, following earlier work of Coles and Papadopoulos, Gibbons, Rietdijk, and van Holten, and De Jonghe, Peeters, and Sfetsos; they have been further analyzed in a recent paper by Michelson and Strominger. The physical motivation was different, namely the study of black hole moduli spaces. The QM systems for \( d = 4, 6, \) and 10 are respectively the 2A, 4A, and 8A models of those authors (for \( d = 3, \) where \( N_Q = 1, \) the models are constructed with standard superfields). The supersymmetry multiplets cannot be obtained by dimensional reduction from four dimensions, but they can be obtained dimensional reduction of the \((2, 0)\), \((4, 0)\), and \((8, 0)\) multiplets in two dimensions. Not all quantum mechanical Lagrangians can be obtained by dimensional reduction, because the one-dimensional theory has less spacetime symmetry.

In this paper we will describe some small extensions of the earlier work on these models. First, we show how the the 4A theories can be constructed with \( D = 1, N = 4 \) superfields (the reduction of the familiar \( D = 4, N = 1 \) superfields) with an extra constraint; the previous work used \( D = 1, N = 1 \) superfields. Second, we consider the addition of a gauge field on moduli space. Finally, we show how models with \( N_B < N_F \) can be obtained by a reduction of those with \( N_B = N_F \); a new feature here is that a potential energy term can arise.

### 3 Superfield Description of the 4A Theories

To deduce the field content we consider the states of a free massless particle in six dimensions. The 6-dimensional supersymmetry algebra has an \( SU(2) \) \( R \)-symmetry. Under \( SO(5, 1) \times SU(2)_R \) the supercharge transforms as \((4, 2)\). The DLCQ breaks this to \( SO(4) \times SU(2)_R = SU(2)_1 \times SU(2)_2 \times SU(2)_R \). The supercharge decomposes

\[
\begin{align*}
(\frac{1}{2}, 2, 1, 2) &: \quad Q_{AX} \\
(-\frac{1}{2}, 1, 2, 2) &: \quad q_{MX},
\end{align*}
\]

where the \( \pm \frac{1}{2} \) is the transformation under the longitudinal \( SO(1, 1) \). Indices \( A, ..., M, ..., \) and \( X, ... \) are used to label doublets of the three \( SU(2) \)'s. The supercharges satisfy a reality condition

\[
Q_{AX}^* = \epsilon^{AB} \epsilon^{XY} Q_{BY}, \quad q_{MX}^* = \epsilon^{MN} \epsilon^{XY} q_{NY}.
\]

On a massless particle with \( p_+ = 0 \), the \( Q_{AX} \) vanish while the \( q_{MX} \) generate the spin states. In the quantum mechanics, the latter role is played by the quantized fermionic coordinates \( \psi \), so we can identify these as transforming as
\(\psi_{MX}\) with the same reality condition as the \(q_{MX}\). The bosonic coordinates are vectors of \(SO(4)\) and so transform as \(X_{AM}\), again with a reality condition:

\[
X^*_A = \epsilon^{AB} \epsilon^{MN} X_{BN} , \quad \psi^*_M = \epsilon^{MN} \epsilon^{XY} \psi_{NY} .
\] (4)

We can readily write down the supersymmetry algebra for a free particle,

\[
\begin{align*}
[Q_{AX}, X_B^M] &= \sqrt{2} \epsilon_{AB} \psi^M_X , \\
\{Q_{AX}, \psi_Y^M\} &= i \sqrt{2} \epsilon_{XY} \dot{X}_A^M .
\end{align*}
\] (5)

We use conventions of Wess and Bagger. For example, \(\epsilon^{12} = \epsilon^{21} = +1\), and indices are raised and lowered by left-multiplication with \(\epsilon\). Now let us compare with the superfields obtained by dimensional reduction from \(D = 4\). We should emphasize that dimensional reduction and DLCQ, while they both lead to supersymmetric QM, are completely different. The DLCQ takes a quantum field theory in \(d\) dimensions to QM in \(d - 2\) spatial dimensions. Reduction takes quantum field theory in \(D\) dimensions to quantum field theory in 1 dimension (time), where the fields are reinterpreted as coordinates.

To get the right number of bosons we take two chiral superfields \(\Phi^i\). In addition to the doublet index \(i\) there is a doublet index \(\alpha\) on the superspace coordinates \(\theta\) and a doublet index \(\dot{\alpha}\) on their conjugates. Because the reduction breaks \(SO(3,1)\) to \(SO(3)\), indices \(\alpha\) and \(\dot{\alpha}\) transform in the same way. The superderivative algebra reduces to

\[
\{D_\alpha, D_\dot{\beta}\} = 2i \delta_{\alpha\dot{\beta}} \frac{\partial}{\partial t} .
\] (6)

The supersymmetry transformations of the scalar components are

\[
\begin{align*}
[Q_\alpha, A_i] &= \sqrt{2} \psi_i^\alpha , \\
[Q_\alpha, A_i^\ast] &= 0 ,
\end{align*}
\] (7)

and the conjugate relations. The DLCQ and superfield transformations are of the same form provided we identify

\[
Q_\alpha = Q_{2X}|_{X=\alpha} , \quad A_i = X_i^M|_{M=i} , \quad \psi_i^\alpha = \psi^M_X|_{X=\alpha, M=i} .
\] (8)

Notice in particular that \(A_i^\ast\) is then \(-\epsilon_{ij} X_j^\alpha\).

From the introduction we know that the superfield description has twice as many fermion fields as the DLCQ. Here this arises because the latter satisfy the reality condition,

\[
\psi_i^{\ast\alpha} = -\epsilon_{ij} \epsilon^{\alpha\beta} \psi_j^{\beta} .
\] (9)

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This can be incorporated into the superfield formalism by the new constraint

\[ \bar{D}_\alpha \Phi^{i*} = -\delta_{ij} \epsilon^{\alpha \beta} D_\beta \Phi^j, \quad (10) \]

which is in addition to the usual chiral constraint \( \bar{D}_\beta \Phi^j = 0 \). This has no effect on the lowest, bosonic, components \( A^i \), while eliminating half of the fermionic components. It also eliminates the auxiliary components in terms of the velocities.

To describe \( k \) free particles one would use \( k \) pairs of superfields, with the constraint Eq. (10) on each. To obtain a general interacting theory we must consider the generalization

\[ \bar{D}_\alpha \Phi^{i*} = J_{i j}^{\dot{\alpha}} \bar{\Phi}^j \quad (11) \]

This nonlinear constraint must be consistent, meaning that it represents only the same number of constraints as the linearized version Eq. (10), eliminating the auxiliary fields and half the fermions, and leaving the lowest components unconstrained. First, from the conjugate of Eq. (11) we obtain

\[ J_{i j}^{\dot{\alpha}} (J_{k \beta}^{\dot{\gamma}})^* = \delta_{ik} \delta_{\dot{\alpha} \dot{\gamma}} \quad (12) \]

Second, acting on both sides with \( \bar{D}_{\dot{\gamma}} \) gives

\[ \bar{D}_{\dot{\gamma}} \bar{D}_\alpha \Phi^{i*} = J_{j i}^{\dot{\alpha} \dot{\beta}} (\Phi^\beta) D_\beta \Phi^j + 2i J_{j i}^{\dot{\alpha} \dot{\beta} \gamma} \delta_{\dot{\beta} \dot{\gamma}} \partial_\gamma \Phi^j \quad (13) \]

Focus on the lowest component of this equation. The LHS is antisymmetric in \( \dot{\alpha} \dot{\gamma} \); in order that this equation not constrain the velocities, but just eliminate the auxiliary field, the second term on the RHS must also be antisymmetric in \( \dot{\alpha} \dot{\gamma} \). Thus,

\[ J_{j i}^{\dot{\alpha} \dot{\beta} \gamma} (\Phi^\beta) = J_{j i}^{\dot{\alpha} \dot{\beta}} (\Phi^\beta) \delta_{\dot{\alpha} \dot{\gamma}} \epsilon^{\alpha \beta}, \quad J_{j i}^{\dot{\alpha} \dot{\beta}} J_{k j}^{\dot{\gamma}} = -\delta_{i k} \quad (14) \]

where \( J_{j k}^{\dot{\gamma}} \equiv (J_{j k})^{\gamma} \). The part of Eq. (14) which is symmetric in \( \dot{\alpha} \dot{\gamma} \) comes only from the first term on the RHS, and can now be written

\[ 0 = J_{j i}^{\dot{\alpha} \dot{\beta} \gamma} (D^\alpha \Phi^j D^\beta \Phi^i + D^\alpha \Phi^j D^\gamma \Phi^i) \quad (15) \]

The expression in parentheses is antisymmetric in \( i j \). In order that Eq. (14) not represent new constraints on the fields we need that it hold identically, and so

\[ J_{j i}^{\dot{\alpha} \dot{\beta} \gamma} J_{i j}^{\dot{\alpha} \dot{\beta} \gamma} = 0 \quad (15) \]

Finally, acting on both sides of Eq. (14) with \( D_{\dot{\gamma}} \) and using the constraints gives

\[ 2i \epsilon_{\dot{\gamma} \dot{\delta}} \partial_\dot{\gamma} \Phi^{i*} = J_{j i}^{\dot{\alpha} \dot{\beta} \gamma} D_{\dot{\beta}} \Phi^j D_{\dot{\alpha}} \Phi^i + J_{j i}^{\dot{\alpha} \dot{\gamma} \delta} D_{\dot{\gamma}} D_{\dot{\delta}} \Phi^j \quad (17) \]
The antisymmetric part of this again relates the auxiliary field to the velocities. Only the first term on the RHS has a symmetric part, and its vanishing identically implies

\[ J_{[j,k]} = 0 \]  

(18)

Eqs. 14, 16, and 18 imply consistency of the constraints: for example, one can regard them as determining the derivatives of \( \Phi_i \) with respect to \( \bar{\theta}^1, \bar{\theta}^2 \), and \( \theta^2 \), and these are integrable. They imply

\[
\psi_{i^*} = J^i_j(A, A^*) \psi_j^2 \\
F^i = -iJ^i_j \dot{\bar{A}}^j + J^i_j J^j_k \theta^k \psi_1 \psi_2^i, 
\]

(19)

for the components of \( \Phi \)

\[
\Phi^i = A^i(y) + \sqrt{2} \theta \psi^i(y) + \theta\theta F^i(y), 
\]

(20)

where \( y = t - i\theta^\alpha \bar{\theta}^\dot{\alpha} \delta_{\alpha\dot{\alpha}} \).

The geometric interpretation of the constraint Eqs. 14, 16, and 18 is as follows. Eq. 14 implies that the tensor \( I_2 \) defined by

\[
I_2 \left[ \begin{array}{c} \Phi^i \\ \Phi^{i*} \end{array} \right] = \left[ \begin{array}{c} J^i_j \Phi^{j*} \\ J^j_i \Phi^j \end{array} \right] 
\]

(21)

is an almost complex structure. This is in addition to the usual complex structure

\[
I_1 \left[ \begin{array}{c} \Phi^i \\ \Phi^{i*} \end{array} \right] = \left[ \begin{array}{c} i\Phi^i \\ -i\Phi^{i*} \end{array} \right] 
\]

(22)

of the superfield formalism. Eqs. 16 and 18 are then the vanishing of the Nijenhuis tensor for \( I_2 \), written in the complex coordinates associated to \( I_1 \). Thus \( I_1 \) is a complex structure, as is \( I_3 = I_1 I_2 = -I_2 I_1 \), in agreement with the conditions found previously.

An invariant action is given by the superspace invariant

\[
\int dt d^4 \theta K(\Phi, \Phi^*) 
\]

(23)

This automatically satisfies the geometric conditions for supersymmetry from refs. 12, though we have not shown that all solutions are of this form. The bosonic kinetic term is

\[
K_{i\dot{j}}(A, A^*)(\dot{A}^i \dot{A}^{j*} + F^i F^{j*}) = K_{i\dot{j}}(A, A^*)(\dot{A}^i \dot{A}^{j*} + J^i_{\dot{k}} J^{j*}_{\dot{l}} \dot{A}^\dot{k} \dot{A}^{\dot{l}}) 
\]

(24)

\textit{b} The superspace formalism is not essential here; once the transformation laws are determined one can take the fourth variation of a scalar function.
Introducing general real coordinates $x^a$, the metric on moduli space can be written symmetrically among the three complex structures,

$$g_{ab} = \sum_{r=0}^{3} I_r^c I_r^d K_{cd},$$  \hspace{1cm} (25)

where $I_0$ is the identity.

4 Gauge Fields on Moduli Space

A small generalization of previous work is to add a gauge field on moduli space. We will do this both with $D = 1$, $N = 1$ superfields, following ref. 8, and the $D = 1$, $N = 4$ superfields above.

In $N = 1$ superfields we add a term

$$S' = \int dt d\theta A_a DX^a$$ \hspace{1cm} (26)

where the lowest component of $X^a$ is the real coordinate $x^a$. Under the variation

$$\delta X^a = \epsilon I^a_{rb} DX^b,$$ \hspace{1cm} (27)

the variation of the action is

$$\delta S' = \int dt d\theta (A_{a,b} \delta X^b DX^a + A_a D\delta X^a)$$

$$= \int dt d\theta (A_{a,b} - A_{b,a}) DX^b \delta X^a$$

$$= \epsilon \int dt d\theta F_{ab} I^a_{rc} DX^b DX^c.$$ \hspace{1cm} (28)

By antisymmetry of $DX^b DX^c$,

$$F_{ab} I^a_{rc} + F_{ca} I^a_{rb} = 0.$$ \hspace{1cm} (29)

Thus the condition for 4A supersymmetry is that the indices of $F_{ab}$ be mixed --- $F$ be a (1,1)-form --- in all three complex structures.

In $N = 4$ superfields consider the half-superspace integral

$$S'' = \int dt d^2\theta W(\Phi, \Phi^*) + \text{c.c.}$$

$$= -iW_{ij} J^i \bar{J}^j + \text{fermionic + c.c.}.$$ \hspace{1cm} (30)
so that
\[ A_j = iW_{\bar{j}}^j J_{\bar{j}}^j , \quad A_{\bar{j}} = -iW^{*}_{\bar{j}} J_{\bar{j}}^j . \] (31)

We do not assume \( W \) to be holomorphic, but will see how it is constrained by supersymmetry. The supersymmetry variation of \( S'' \) is the integral of a total derivative except for the \( \bar{\theta} \) derivative
\[
\int dt \, d^2 \theta W_{\bar{j}} J_{\bar{j}}^j + c.c. = -i \int dt \, d^2 \theta A_{\bar{j}} \Phi^j + c.c. \] (32)
using constraint Eq. 11. In order that this integrate to zero, we must have
\[ A_j = \lambda_j , \quad \text{or equivalently} \]
\[ F_{jk} = F_{\bar{k}\bar{j}} = 0 . \] (33)

Thus \( F \) is \( (1,1) \) with respect to \( I_1 \). Eq. 11 is equivalent to
\[ 0 = \partial_{\bar{j}} (J_{\bar{j}}^j A_j) = J_{\bar{j}}^j (\partial_{\bar{j}} A_j) , \] (34)
using Eq. 18. Together with its conjugate this implies
\[ J_{[i} \bar{j}^i F_{\bar{k}k]} = 0 , \] (35)
which is the statement that \( F \) is \( (1,1) \) with respect to both \( I_2 \) and \( I_3 \). Thus we recover the same models as with \( N = 1 \) superfields.

5 Dimensional Reduction

Systems with \( N_B < N_F \) can be obtained from those with \( N_B = N_F \) by a process of reduction. This requires that the original system have one or more isometries. With a single isometry, for example, we can choose coordinates in which it takes the form \( \delta \xi = \epsilon \). Then \( \xi \) does not appear in the action undifferentiated, and so \( \dot{\xi} \) can be regarded as an independent auxiliary field \( F \).

In \( N = 1 \) superfield language, the corresponding superfield \( \Xi \) does not appear in the action but \( D\Xi \) may. The lower component of \( D\Xi \) is a fermion and the upper is \( \xi \to F \). Such superfields, with a fermion and an auxiliary field, were considered in ref. 9, where they were denoted \( \psi \). They can be used to construct models with \( N_F > N_B \). However, it appears that not all such models can be obtained by dimensional reduction: we can work backwards, introducing a superfield \( \Xi \) and replacing \( \psi \) everywhere with \( D\Xi \), only if the
\[ ^* \text{Eq. 34 would appear to be stronger than Eq. 33, but the difference is a gauge choice. The earlier Eq. 31 implies that } A_j = \partial_j \lambda \text{ and } A_{\bar{j}} = \partial_{\bar{j}} \lambda^* . \text{ The real part of } \lambda \text{ can be set to zero by a gauge transformation, and then Eqs. 34 and 33 are equivalent.} \]
supersymmetry variations of $\psi$ are total spinor derivatives. For $D = 1$, $N = 1$ systems this is always possible, but for higher supersymmetries it is nontrivial.

This reduction process is suggested by the application to DLCQ: the systems in table 1 having $N_B < N_F$ can be obtained from those with $N_B = N_F$ by dimensional reduction. For example we can go from the $d = 6$ system to the $d = 5$ system by Kaluza-Klein reduction on $x^4$. In a $k$-particle system, all wavefunctions will be independent of $x^4$, so there are actually $k$ isometries.

5.1 Examples

Let us illustrate the reduction for the 4A theory with a single isometry. The general bosonic action with an isometry is

$$\frac{1}{2}g_{mn}\dot{x}^m\dot{x}^n + \frac{1}{2}g_{\xi\xi}(\dot{\xi} + A'_m\dot{x}^m)^2 + A_m\dot{x}^m + A_\xi\dot{\xi}. \quad (36)$$

Here $m, n$ run only over the $4k - 1$ non-cyclic coordinates. Note the Kaluza-Klein gauge field on moduli space $A'_m$, which is in addition to the gauge field $(A_m, A_4)$ introduced in section 4. Replacing $\dot{\xi} \to F$ and solving for $F$ gives

$$\frac{1}{2}g_{mn}\dot{x}^m\dot{x}^n - \frac{1}{2g_{\xi\xi}}A^2 + (A_m - A_\xi A'_m)\dot{x}^m. \quad (37)$$

Thus a potential energy $A^2/g_{\xi\xi}$ is introduced. The net reduced gauge field $A_m - A_\xi A'_m$ is invariant under reparameterization of $\xi \to \xi + \lambda(x^m)$.

Now let us begin with the simplest example, $k = 1$ with the linear constraint Eq. 10 and no gauge field. The bosonic action in Eq. 24 takes the form

$$(K_{11} + K_{22})(\dot{A}^1 \dot{A}^{1*} + \dot{A}^2 \dot{A}^{2*}). \quad (38)$$

In real coordinates this is

$$\frac{1}{2}H(x^a)\dot{x}^b\dot{x}^b, \quad (39)$$

where $\nabla^2 K$ is a generic function of $x^1, x^2, x^3, x^4$. This example appears in refs. [14, 8].

Now reduce under translation in $x^4$. To make the result more interesting add a constant potential $A_4 = C_1$. Then $H$ depends only on $x^m = x^1, x^2, x^3$ and the reduced bosonic action is

$$\frac{1}{2}H(x^m)\dot{x}^m\dot{x}^m - \frac{C_1^2}{2H(x^m)}. \quad (40)$$
Focusing now on systems with spherical symmetry we have \( H = H(r) \). An obvious generalization is to add a spherically symmetric magnetic field

\[
\mathcal{B}_m = \frac{C_2 x^m}{r^3}, \quad \mathcal{A}_4 = C_1 + \frac{C_2}{r}.
\]  

The form of \( \mathcal{A}_4 \) follows from the \((1,1)\) condition on \( \mathcal{F} \), which here reduces to self-duality. The reduced action is then

\[
\frac{1}{2} H(r) \dot{x}^m \dot{x}^m - \frac{1}{2H(r)} \left( C_1 + \frac{C_2}{r} \right)^2 + \mathcal{A}_m \dot{x}^m.
\]  

An independent search for spherically symmetric quantum mechanical systems produced a further one-parameter generalization of these models. The bosonic actions are all of the form Eq. 42; the generalization appears in the supersymmetry transformations and consequently the fermionic terms. These models can be obtained from reduction as follows. For \( k = 1 \) there is a one-parameter generalization of the complex structures, given by Taub-NUT space, with the constant complex structure as a limit. This space has an \( SU(2) \times U(1) \) isometry, so reducing on the \( U(1) \) leaves a theory with \( SU(2) \) rotational symmetry.

5.2 Taub-NUT Reduction

In the remainder of this paper we work out some of the (rather tedious) details of this reduction, beginning with the bosonic action. In coordinates \((x^1, x^2, x^3, \xi)\) with \( \xi \) cyclic, the three complex structures are

\[
I_{1b}^a = \frac{1}{S} \begin{bmatrix}
-a_1 & -a_2 & 0 & -1 \\
0 & 0 & -S & 0 \\
0 & S & 0 & 0 \\
S^2 + a_1^2 & a_1a_2 & Sa_2 & a_1
\end{bmatrix},
\]

\[
I_{2b}^a = \frac{1}{S} \begin{bmatrix}
0 & 0 & S & 0 \\
-a_1 & -a_2 & 0 & -1 \\
-S & 0 & 0 & 0 \\
a_1a_2 & S^2 + a_2^2 & -Sa_1 & a_2
\end{bmatrix},
\]

\[
I_{3b}^a = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -S^{-1}a_2 & 0 & -S^{-1} \\
-a_2 & a_1 & S & 0
\end{bmatrix}, \tag{43}
\]
where \( m \) is the Taub-NUT parameter and \( a_n \) is a magnetic monopole field of unit strength,
\[
S = \frac{1}{2m} + \frac{1}{r}, \quad a_\phi = \cos \theta .
\] (44)

After some computation, one finds from Eq. 25 that the general metric with a \( U(1) \) isometry is
\[
ds^2 = H(x^m) [dx^n dx^n + S^{-2}(dx + a_n dx^n)^2] ,
\] (45)

where \( H = \partial_m \partial_m K \) and \( K \) is independent of \( \xi \). For \( H = 2mS \) this is the hyper-Kähler Taub-NUT metric, so these metrics are conformal to Taub-NUT.

We now add a gauge field \( A_a \). We find it convenient to impose rotational invariance before the \((1,1)\) condition. The condition that the gauge field term \( A_a \partial_a \) be invariant up to a time derivative under a general Killing vector \( L^a \) is
\[
L^a F_{bc,a} = L^a F_{ca} - L^c F_{ba} .
\] (46)

In this case the Killing vectors are
\[
L_1 = x^2 \partial_3 - x^3 \partial_2 + \frac{rx^1}{(x^1)^2 + (x^2)^2} \partial_\xi , \\
L_2 = x^3 \partial_1 - x^1 \partial_3 + \frac{rx^2}{(x^1)^2 + (x^2)^2} \partial_\xi , \\
L_3 = x^1 \partial_2 - x^2 \partial_1 , \quad L_\xi = \partial_\xi .
\] (47)

Rotational invariance then implies
\[
F_{m\xi} = x^m f(r), \quad F_{12} = x^3 g(r), \quad F_{23} = x^1 h(r, x^3), \quad F_{31} = x^2 h(r, x^3) ,
\] (48)

where \( h(r, x^3) = g(r) - rf(r)/(r^2 - (x^3)^2) \). The Bianchi identity becomes
\[
f + rf' = 3rg + r^2 g'
\] (49)

and the \((1,1)\) conditions reduce to
\[
g = -f/2m .
\] (50)

Integrating the Bianchi identity gives
\[
f = \frac{a}{r(r + 2m)^2}
\] (51)

with \( a \) an integration constant. The gauge field is then
\[
A_\phi = \frac{az}{2m(r + 2m)} , \quad A_\xi = -\frac{a}{r + 2m} + b .
\] (52)
Upon reduction, the bosonic action is the same as in Eq. 42 with
\[ C_1 = \frac{b}{2m}, \quad C_2 = \frac{2mb - a}{2m}. \] (53)

Taking the limit \( m \to 0 \) with \( C_1 \) and \( C_2 \) fixed gives the earlier models. In the bosonic action the three parameters \((m, a, b)\) collapse into two, \((C_1, C_2)\). However, the third parameter appears in the supersymmetry transformation, Eq. 27, since this depends on the complex structure and therefore on \( m \) separately. The fermionic terms in the action are therefore also affected. The general Lagrangian can be written
\[
\frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{i}{2} \psi^a \left[ g_{ab} \psi^b + (\Gamma_{abc} - c_{abc}) \psi^b \psi^c \right] - \frac{1}{6} c_{abc,d} \psi^a \psi^b \psi^c \psi^d + A_a \dot{x}^a + \frac{1}{2} F_{ab} \psi^a \psi^b, \quad (54)
\]
where \( c \) is a torsion form. The torsion form here is
\[
c = \left( H'(r) + \frac{2m}{r^2} \right) \sin \theta d\phi \wedge d\theta \wedge d\xi \quad (55)
\]
(see the appendix to ref. 14 or Eq. 4.14 of ref. 8). Thus the fermionic terms depend separately on \( m \), through \( g_{ab} \) and \( c_{abc} \). For the Taub-NUT metric the torsion vanishes, as it must.

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