$T^3$ deformations and $\beta$-deformed geometries

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Abstract

We discuss $\beta$-deformed geometries on two types of $T^3$'s where the direction along the third coordinate is not orthogonal to the direction along the second coordinate or the direction along the first coordinate. We show that the intersection angle between the direction along the third coordinate and the direction along the second coordinate corresponds to the parameter of the S-duality of the $\beta$-deformation while the intersection angle between the direction along the third coordinate and the direction along the first coordinate generalizes the $\beta$-deformed geometry.

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1 Introduction

Type IIB supergravity on a two-torus has an $E_{3,3} = SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry in the classical low-energy limit. It is shown that a specific $SL(3, \mathbb{R})$ transformation generates the gravity dual [1] of the marginal deformation [2]. The $SL(3, \mathbb{R})$ is geometric from the eleven dimensional viewpoint. In type IIB supergravity, it consists of an $SL(2, \mathbb{R})$ acting on the Kähler structure modulus and an S-duality transformation $SL(2, \mathbb{R})_s$. A particular $SL(2, \mathbb{R})$ transformation on a two-torus produces a non-singular geometry if the original geometry is non-singular, which is the gravity description of the $\gamma$-deformation of the gauge theory to introduce phases in the superpotential. This corresponds to a TsT (T-duality, shift, T-duality) transformation. It breaks the supersymmetries which depend on the coordinates of the two-torus. The $\beta$-deformation of the gauge theory is obtained by complexifying the parameter in the superpotential. In the gravity side, the deformation corresponds to a particular $SL(3, \mathbb{R})$ transformation, which is formed by the $SL(2, \mathbb{R})$ and an S-duality transformation $SL(2, \mathbb{R})_s$ or equivalently STsTS (S-duality, T-duality, shift, T-duality, S-duality). Since the $SL(2, \mathbb{R})$ is the nongeometric part of $SO(2, 2, \mathbb{R}) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, which is a subgroup of the type IIB T-duality group, the $SL(2, \mathbb{R})$ can be viewed as $O(2, 2, \mathbb{R})$ acting on the background matrix $E = g + B$ [3]. The corresponding $O(2, 2, \mathbb{R})$ matrices for the deformations discussed in [1, 4] are identified in [5]. Multiparameter deformations of them are discussed in [4, 6]. General formulae for TsT transformations and examples of the deformed gauge theories and their gravity duals are discussed in [7]. Various aspects of marginal deformations of gauge theories and the gravity duals of them are investigated in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

In [1], the type IIB supergravity solution is obtained from eleven dimensional supergravity on a rectangular three-torus $(\varphi^1, \varphi^2, \varphi^3)$ by a dimensional reduction along $\varphi^3$ and a T-duality transformation along $\varphi^1$. The new geometry [1] is generated by the $SL(2, \mathbb{R})$ or $SL(3, \mathbb{R})$ transformation of the type IIB supergravity solution.

We study the $SL(2, \mathbb{R})$ or $SL(3, \mathbb{R})$ transformation on deformed $T^3$'s. We deform a rectangular three-torus by two types of $SL(3, \mathbb{R})$ transformations so that the direction along the third coordinate is not orthogonal to the direction along the second coordinate or the direction along the first coordinate. By a dimensional reduction along the direction along the third coordinate and a T-duality transformation along the direction along the first coordinate after the transformations, we obtain geometries with the intersection angles of the directions as parameters. We discuss the role of the parameters under the $SL(2, \mathbb{R})$ or $SL(3, \mathbb{R})$ transformation which generates the $\beta$-deformed geometry [1].

In section 2, we review Lunin and Maldacena’s solution generating technique and discuss the two types of $SL(3, \mathbb{R})$ transformations. In section 3, we study the type IIB supergravity solution obtained by a dimensional reduction along the direction along the third coordinate, which is not orthogonal to the direction along the second coordinate, followed by a T-duality transformation along the direction along the first coordinate. We show that the intersection angle forms the S-duality transformation of [1] under the TsT transformation. In section 4, we study the type IIB supergravity solution obtained by a dimensional reduction along the direction along the third
coordinate, which is not orthogonal to the direction along the first coordinate, followed by a T-duality transformation along the direction along the first coordinate. We show that the intersection angle generalizes the $\beta$-deformed geometry. In section 5, we summarize our results.

2 $T^3$ deformation

The type IIB supergravity solution with $U(1) \times U(1)$ symmetry [1] is derived from an eleven-dimensional supergravity solution with $U(1) \times U(1) \times U(1)$ symmetry. The coordinates of the three-torus are $(\varphi^1, \varphi^2, \varphi^3)$. The type IIB supergravity solution is obtained by a dimensional reduction along $\varphi^3$ and a T-duality transformation along $\varphi^1$ [1]:

$$
\begin{align*}
\text{ds}_{IIB}^2 &= F \left[ \frac{1}{\sqrt{\Delta}} \left( D\varphi^1 - C D\varphi^2 \right)^2 + \sqrt{\Delta} (D\varphi^2)^2 \right] + \frac{\tau_{-2/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu, \\
B^{(2)} &= B_{12} D\varphi^1 \wedge D\varphi^2 + (B_{1\mu} D\varphi^1 + B_{2\mu} D\varphi^2) \wedge dx^\mu - \frac{1}{2} \left( A^m_{\mu} B_{mv} - \tilde{b}_{\mu v} \right) dx^\mu \wedge dx^v, \\
C^{(0)} &= \tau_1, \\
C^{(2)} &= C_{12} D\varphi^1 \wedge D\varphi^2 + (C_{1\mu} D\varphi^1 + C_{2\mu} D\varphi^2) \wedge dx^\mu - \frac{1}{2} \left( A^m_{\mu} C_{mv} - \tilde{c}_{\mu v} \right) dx^\mu \wedge dx^v, \\
C^{(4)} &= -\frac{1}{2} \left( \tilde{d}_{\mu} + B_{12} \tilde{c}_{\mu v} - \epsilon^{m\nu} B_{m\mu} C_{n\nu} - B_{12} A^m_{\mu} C_{mv} \right) dx^\mu \wedge dx^v \wedge D\varphi^1 \wedge D\varphi^2 \\
&\quad + \frac{1}{6} \left[ C_{\mu\nu\lambda} + 3 \left( \tilde{b}_{\mu v} + A^1_{\mu} B_{1v} - A^2_{\mu} B_{2v} \right) C_{1\lambda} \right] dx^\mu \wedge dx^v \wedge dx^\lambda \wedge D\varphi^1 \\
&\quad + \hat{d}_{\mu_1 \mu_2 \mu_3 \mu_4} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} + \hat{d}_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge D\varphi^2,
\end{align*}
$$

$$
D\varphi^1 \equiv d\varphi^1 + A^1_{\mu} dx^\mu, \quad D\varphi^2 \equiv d\varphi^2 + A^2_{\mu} dx^\mu, \quad (2.1)
$$

$$
M = gg^T, \\
g^T = \begin{pmatrix}
\tau_1^{-1/3} F^{-1/3} & 0 & 0 \\
0 & \tau_2^{-1/3} F^{2/3} & 0 \\
0 & 0 & \tau_2^{-2/3} F^{-1/3}
\end{pmatrix} \begin{pmatrix}
1 & B_{12} & 0 \\
0 & 1 & 0 \\
\tau_1 & -C_{12} + \tau_1 B_{12} & 1
\end{pmatrix}. \quad (2.2)
$$

Under an $SL(3, \mathbb{R})$ transformation

$$
\begin{pmatrix}
\varphi^1 \\
\varphi^2 \\
\varphi^3
\end{pmatrix}' = (A^T)^{-1} \begin{pmatrix}
\varphi^1 \\
\varphi^2 \\
\varphi^3
\end{pmatrix}, \quad (2.3)
$$

the metric transforms as

$$
M \rightarrow \Lambda M \Lambda^T = g' g'^T, \quad (2.4)
$$
\[ g'^T = \left( \begin{array}{ccc} \frac{1}{\sqrt{G\tau_2}} \frac{1}{3} F^{-1/3} & 0 & 0 \\ 0 & \sqrt{G\tau_2} \frac{1}{3} F^{2/3} & 0 \\ 0 & 0 & \sqrt{H\tau_2}^{-2/3} F^{-1/3} \end{array} \right) \left( \begin{array}{ccc} 1 & B'_{12} & 0 \\ 0 & 1 & 0 \\ \chi' & -C'_{12} + \chi' B'_{12} & 1 \end{array} \right), \] (2.5)

and the fields

\[ V^{(1)}(\mu) = \left( \begin{array}{c} -B_2^\mu \\ A_1^\mu \\ C_2^\mu \end{array} \right), \quad V^{(2)}(\mu) = \left( \begin{array}{c} B_1^\mu \\ A_2^\mu \\ -C_1^\mu \end{array} \right), \quad W_{\mu\nu} = \left( \begin{array}{c} \tilde{c}_{\mu\nu} \\ \tilde{d}_{\mu\nu} \\ \tilde{b}_{\mu\nu} \end{array} \right), \] (2.6)

transform as vectors

\[ V^{(i)}(\mu) \rightarrow (A^T)^{-1} V^{(i)}(\mu), \quad W_{\mu\nu} \rightarrow A W_{\mu\nu}. \] (2.7)

The scalars \( \Delta, C \) and the three form \( C_{\mu\nu}\lambda \) stay invariant. The \( SL(3, \mathbb{R}) \) matrix [1], which generates the gravity duals of the \( \beta \)-deformations is

\[ A^T_{LM} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ \gamma & 1 & \sigma \\ 0 & 0 & 1 \end{array} \right). \] (2.8)

We investigate two types of type IIB supergravity solutions. One is a type IIB supergravity solution which is obtained from an eleven dimensional supergravity solution living on a slanted three-torus where the direction along the third coordinate is not orthogonal to the direction along the second coordinate. The other is a type IIB supergravity solution which is obtained from an eleven dimensional supergravity solution living on a slanted three-torus where the direction along the third coordinate is not orthogonal to the direction along the first coordinate. The torus deformation can be done by an \( SL(3, \mathbb{R}) \) transformation

\[ \left( \begin{array}{c} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{array} \right)' = (L)^{-1} \left( \begin{array}{c} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{array} \right), \quad L = L_1, \ L_2 \in SL(3, \mathbb{R}), \] (2.9)

with

\[ L_1 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & (r_3/R_2) \cos \xi \\ 0 & 0 & 1 \end{array} \right), \quad L_2 = \left( \begin{array}{ccc} 1 & 0 & (r_3/R_1) \cos \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \] (2.10)

which are constrained by

\[ r_3 = \frac{R_3}{\sin \xi}. \] (2.11)
$R_i, (i = 1, 2, 3)$ are the radii of the $T^3$ before the transformations and $r_3$ is the radius of the third direction after the transformations. We study the type IIB supergravity solutions produced by a dimensional reduction along the direction along the third coordinate and a T-duality transformation along the direction along the first coordinate by using (2.5) and (2.7). We assume that only the metric, complex field $\tau = \tau_1 + i\tau_2$ and $\tilde{d}_{\mu\nu}$ are excited and the other fields are zero.

3 $T^3$ deformed by $L_1$

We deform a rectangular three-torus by $L_1$ in (2.10)

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & L_3^2 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$L_3^2 = \frac{r_3}{R_2} \cos \xi = k^{-1} \cot \xi, \quad k = \frac{R_2}{R_3}. \quad (3.2)$$

For fixed $R_2$ and $\xi$, the torus radius $r_3$ can be chosen so that $L_3^2 \sim L_3^2 + 1$.

By using (2.5) and (2.7), we find the geometry in ten dimensions with the intersection angle $\xi$ as a parameter:

$$\bar{G}^{-1} = 1 + (\tau_1^2 + \tau_2^2) k^{-2} \cot^2 \xi F^2, \quad H = 1 + \tau_2^2 k^{-2} \cot^2 \xi F^2,$$

$$d\bar{s}^2 = \bar{F} \left[ \frac{1}{\sqrt{\Delta}} \left( D\varphi^1 - CD\varphi^2 \right)^2 + \sqrt{\Delta} \left( D\varphi^2 \right)^2 \right] + \frac{e^{2\Phi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu,$$

$$\bar{F} = FG\sqrt{H}, \quad e^\Phi = \sqrt{GH} \tau_2^{-1}, \quad \bar{\chi} = H^{-1} \tau_1,$$

$$\bar{B}^{(2)} = -GF^2 \tau_1 k^{-1} \cot \xi D\varphi^1 \wedge D\varphi^2 + \frac{1}{2} \tilde{d}_{\mu\nu} k^{-1} \cot \xi dx^\mu \wedge dx^\nu,$$

$$\bar{C}^{(2)} = -GF^2 (\tau_1^2 + \tau_2^2) k^{-1} \cot \xi D\varphi^1 \wedge D\varphi^2,$$

$$\bar{C}^{(4)} = -\frac{1}{2} \tilde{d}_{\mu\nu} D\varphi^1 \wedge D\varphi^2 \wedge dx^\mu \wedge dx^\nu. \quad (3.3)$$

We show that the intersection angle $\xi$ forms the representation of the S-duality $SL(2, \mathbb{R})_s$ with the parameter $\gamma$ of the $SL(2, \mathbb{R})$ transformation considered in [1]. It can be guessed since the
matrix (2.8) is factorized as

$$\Lambda^T_{LM} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 1 & 0 \end{pmatrix}.$$ (3.4)

The component $L^2_3$ of (3.1) corresponds to $\sigma$. We discuss the $\beta$-deformed geometry. The $SL(2, \mathbb{R})$ transformation can be realized as the TzT transformation [1, 4] or $O(2, 2, \mathbb{R})$ [1, 5] acting on the background matrix. We present the TzT transformation. By a T-duality transformation along $\varphi^1$, a shift transformation $\varphi^2 \rightarrow \varphi^2 + \gamma \varphi^1$ and a T-duality transformation along $\varphi^1$, we obtain a geometry

$$ds^2 = \frac{\tilde{F}}{(1 + \gamma B_{12})^2 + \gamma^2 F^2} \left[ \frac{1}{2\sqrt{\Delta}} (D\varphi^1 - CD\varphi^2)^2 + \sqrt{\Delta} (D\varphi^2)^2 \right] + \frac{e^{2\Phi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu,$$

$$e^{2\varphi^1} = \frac{(1 + \gamma B_{12})^2 + \gamma^2 F^2}{(1 + \gamma B_{12})^2 + \gamma^2 F^2},$$

$$\tilde{B}^{(2)}\varphi = \frac{B_{12} + \gamma \left( B_{12}^2 + \tilde{F}^2 \right) + \gamma^2 F^2}{(1 + \gamma B_{12})^2 + \gamma^2 F^2} D\varphi^1 \wedge D\varphi^2 + \frac{1}{2} \tilde{d}_{\mu\nu} \gamma^{-1} \cot \xi dx^\mu \wedge dx^\nu,$$

$$\check{\varphi} = \check{\varphi} - \gamma (\check{C}_{12} - \check{\varphi} B_{12}),$$

$$\check{C}^{(2)}\varphi = \frac{C_{12}(1 + \gamma B_{12}) + \gamma \check{\varphi} F^2}{(1 + \gamma B_{12})^2 + \gamma^2 F^2} D\varphi^1 \wedge D\varphi^2 + \frac{1}{2} \gamma^{-1} \tilde{d}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$\check{C}^{(4)}\varphi = -\frac{1}{2} \tilde{d}_{\mu\nu} \frac{1 + \gamma B_{12}}{(1 + \gamma B_{12})^2 + \gamma^2 F^2} D\varphi^1 \wedge D\varphi^2 \wedge dx^\mu \wedge dx^\nu.$$ (3.5)

By introducing

$$G^{-1} = 1 + (\gamma - \tau_1 \kappa^{-1} \cot \xi)^2 F^2 + \tau_1^2 k^2 \cot^2 \xi F^2,$$ (3.6)

and using the relations in (3.3), the geometry (3.5) can be rewritten as

$$ds^2 = \tilde{F}' \left[ \frac{1}{2\sqrt{\Delta}} (D\varphi^1 - CD\varphi^2)^2 + \sqrt{\Delta} (D\varphi^2)^2 \right] + \frac{e^{2\Phi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu,$$

$$\tilde{F}' = FG\sqrt{H},$$

$$e^{\Phi'} = \sqrt{GH} \tau_2^{-1},$$

$$\tilde{B}^{(2)}\varphi = GF^2 (\gamma - \tau_1 \kappa^{-1} \cot \xi) D\varphi^1 \wedge D\varphi^2 + \frac{1}{2} \tilde{d}_{\mu\nu} \gamma^{-1} \cot \xi dx^\mu \wedge dx^\nu,$$

$$\check{\varphi} = H^{-1}(\tau_1 + \gamma \tau_2 k^{-1} \cot \xi F^2),$$

$$\check{C}^{(2)}\varphi = GF^2 [\tau_1 \gamma - (\tau_1^2 + \tau_2^2) \kappa^{-1} \cot \xi] D\varphi^1 \wedge D\varphi^2 + \frac{1}{2} \gamma \tilde{d}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$F^{(5)} = \tilde{F}(5) + \ast \tilde{F}(5).$$ (3.7)
where the star is taken with the new metric.

This is the $\beta$-deformed geometry \cite{1} with $\sigma = k^{-1} \cot \xi$. For $\xi = \frac{\pi}{2}$, the geometry (3.7) is the gravity dual of the $\gamma$-deformation.

4 $T^3$ deformed by $L_2$

We have shown that the intersection angle between the direction along the third coordinate and the direction along the second coordinate is the parameter for the S-duality of (2.8). It is therefore expected that the intersection angle between the direction along the third coordinate and the direction along the first coordinate is a parameter which generalizes the $SL(3, \mathbb{R})$ transformation (2.8). We deform a rectangular three-torus by $L_2$ in (2.10)

$$L_2 = \begin{pmatrix} 1 & 0 & L_3^1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(4.1)

where

$$L_3^1 = \frac{r_3}{R_1} \cos \xi = l^{-1} \cot \xi, \quad l = \frac{R_1}{R_3}.$$  

(4.2)

By using (2.5) and (2.7), we find the geometry with the intersection angle $\xi$ as a parameter:

$$\tilde{G}^{-1} = 1,$$

$$\tilde{H} = 1 + 2l^{-1} \cot \xi \tau_1 + l^{-2} \cot^2 \xi |\tau|^2, \quad (|\tau|^2 = \tau_1^2 + \tau_2^2),$$

$$d\tilde{s}^2 = \tilde{F} \left[ \frac{1}{\sqrt{\Delta}} (D \varphi^1 - CD \varphi^2)^2 + \sqrt{\Delta} (D \varphi^2)^2 \right] + \frac{e^{2\Phi}/3}{F^{1/3}} g_{\mu \nu} dx^\mu dx^\nu,$$

$$\tilde{F} = F \sqrt{H},$$

$$e^\Phi = H \tau_2^{-1},$$

$$\tilde{\chi} = H^{-1} \left( \tau_1 + l^{-1} \cot \xi |\tau|^2 \right),$$

$$\tilde{B}^{(2)} = 0,$$

$$\tilde{C}^{(2)} = 0,$$

$$\tilde{C}^{(4)} = -\frac{1}{2} \tilde{d}_{\mu \nu} D \varphi^1 \wedge D \varphi^2 \wedge dx^\mu \wedge dx^\nu.$$  

(4.3)

The axion-dilaton field $\tau = \tau_1 + i \tau_2$ transforms as

$$\tilde{\tau} = \tilde{\chi} + ie^{-\Phi} = \frac{\tau}{l^{-1} \cot \xi \tau + 1}.$$  

(4.4)
The torus deformation provides a generalized $\beta$-deformed geometry under the $SL(3, \mathbb{R})$ transformation (2.8) of [1], since the matrix

$$\Lambda^T = \begin{pmatrix} 1 & 0 & l^{-1} \cot \xi \\ \gamma & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix},$$

(4.5)
is factorized as

$$\Lambda^T = L_2 \Lambda_{LM}^T.$$  

(4.6)

The $SL(3, \mathbb{R})$ transformation (2.8) of the geometry (4.3) is

$$G^{-1} = 1 + (\gamma - \sigma \chi)^2 \bar{F}^2 + \sigma^2 e^{-2\bar{\Phi}} \bar{F}^2,$$

$$\bar{H}' = 1 + \sigma^2 e^{-2\bar{\Phi}} \bar{F}^2,$$

d$s^2 = \bar{F}' \left[ \frac{1}{\sqrt{\Delta}} (D\varphi^1 - CD\varphi^2)^2 + \sqrt{\Delta} (D\varphi^2)^2 \right] + \frac{e^{2\Psi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu,$$

$$F' = FG\sqrt{H'},$$

$$e^\Phi' = \sqrt{GH'} e^\Phi,$$

$$\bar{\chi}' = H^{-1} (\bar{\chi} + \gamma \sigma e^{-2\bar{\Phi}} \bar{F}^2),$$

$$\bar{B}'(2) = G\bar{F}^2 (\gamma - \sigma \chi) D\varphi^1 \wedge D\varphi^2 + \frac{\sigma}{2} \bar{a}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$\bar{C}'(2) = G\bar{F}^2 (\gamma \bar{\chi} - \sigma \bar{\varphi}^2) D\varphi^1 \wedge D\varphi^2 + \frac{\gamma}{2} \bar{a}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$F(5) = \bar{F}(5) + \ast \bar{F}(5) .$$

(4.7)

The star is taken with the new metric. By using (4.3), the geometry (4.7) can be rewritten as

$$G^{-1} = 1 + (\gamma^2 f - 2\gamma \sigma h + \sigma^2 g) \bar{F}^2,$$

$$H = f + \tau_2^2 \sigma^2 \bar{F}^2,$$

d$s^2 = \bar{F}' \left[ \frac{1}{\sqrt{\Delta}} (D\varphi^1 - CD\varphi^2)^2 + \sqrt{\Delta} (D\varphi^2)^2 \right] + \frac{e^{2\Psi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu,$$

$$F' = FG\sqrt{H},$$

$$e^\Phi' = \sqrt{GH} \tau_2^{-1},$$

$$\bar{\chi}' = H^{-1} (h + \gamma \sigma \tau_2^2 F^2),$$

$$\bar{B}'(2) = GF^2 (\gamma f - \sigma h) D\varphi^1 \wedge D\varphi^2 + \frac{\sigma}{2} \bar{a}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$\bar{C}'(2) = GF^2 (\gamma h - \sigma g) D\varphi^1 \wedge D\varphi^2 + \frac{\gamma}{2} \bar{a}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

(4.8)
where

\[
\begin{align*}
f &= (1 + l^{-1}\cot \xi \tau_1)^2 + l^{-2}\cot^2 \xi \tau_2^2, \\
g &= |\tau_1|^2, \\
h &= \tau_1 + l^{-1}\cot \xi |\tau|^2.
\end{align*}
\]

The geometry (4.8) with \( f, g \) and \( h \) is valid for any \( SL(3, \mathbb{R}) \) transformation of the form

\[
L = \begin{pmatrix} L_1^1 & 0 & L_3^1 \\ 0 & 1 & 0 \\ L_1^3 & 0 & L_3^3 \end{pmatrix}, \quad \det L = 1,
\]

\[
\Lambda^T = L \Lambda^T_{LM} = \begin{pmatrix} L_1^1 & 0 & L_3^1 \\ \gamma & 1 & \sigma \\ L_1^3 & 0 & L_3^3 \end{pmatrix},
\]

with

\[
\begin{align*}
f &= (L_3^3 + L_3^1 \tau_1)^2 + (L_3^1)^2 \tau_2^2, \\
g &= (L_1^3 + L_1^1 \tau_1)^2 + (L_1^1)^2 \tau_2^2, \\
h &= (L_3^3 + L_3^1 \tau_1) (L_3^1 + L_1^1 \tau_1) + L_1^1 L_3^1 \tau_2^2.
\end{align*}
\]

The generalization of the \( \beta \)-deformed geometry corresponds to the \( SL(2, \mathbb{R}) \) symmetry of type IIB theory as we have seen in (4.4). It is also the symmetry of the toroidal compactification \([18, 19]\). We consider a case in which \( \tau_1 = 0 \) by using the shift symmetry of the type IIB supergravity and \( \tau_2 = \frac{R_1}{R_3} \) by equating the axion-dilaton field with the torus modulus of the rectangular torus before the transformation as it is done in \([18]\). Then \( \tau = i \frac{R_1}{R_3} \) transforms under (4.4) as

\[
\bar{\tau} = \frac{R_1}{R_3} e^{i \xi}.
\]

This is the torus moduli of the two-torus \((\varphi^1, \varphi^3)\) deformed by (4.1) as expected.

We identify the complex scalar field \( \tau \) with the torus modulus of the rectangular torus before
the deformation (4.1). Then (4.8) becomes simpler. The geometry is

\[ G^{-1} = 1 + \gamma^2 F^2 + (\sigma l - \gamma \cot \xi)^2 F^2, \]
\[ H = \csc^2 \xi + \sigma^2 l^2 F^2, \]
\[ ds^2 = \tilde{F}' \left[ \frac{1}{\sqrt{\Delta}} (D\varphi_1 - CD\varphi_2)^2 + \sqrt{\Delta} (D\varphi_2)^2 \right] + \frac{e^{2\Phi/3}}{F^{n/3}} g_{\mu\nu} dx^\mu dx^\nu, \]
\[ \tilde{F}' = FG\sqrt{H}, \]
\[ e^\Phi' = \sqrt{GHI^{-1}}, \]
\[ \tilde{\chi}' = \frac{1}{H} (l \cot \xi + \gamma \sigma l^2 F^2), \]
\[ \tilde{B}'^{(2)} = GF^2 (\gamma \csc^2 \xi - \sigma l \cot \xi) D\varphi_1 \wedge D\varphi_2 + \frac{\sigma}{2} \tilde{d}_{\mu\nu} dx^\mu \wedge dx^\nu, \]
\[ \tilde{C}'^{(2)} = GF^2 (-\sigma l^2 + \gamma l \cot \xi) D\varphi_1 \wedge D\varphi_2 + \frac{\gamma}{2} \tilde{d}_{\mu\nu} dx^\mu \wedge dx^\nu. \]

This geometry is generally applicable when the axion field \( \tau_1 \) is pure gauge. For \( \xi = \frac{\pi}{2} \) and \( l = 1 \), (4.13) becomes the geometry obtained in the appendix of [1].

5 Discussion

We have studied the gravity duals of the marginal deformation with a complex parameter on two types of slanted \( T^3 \)'s where the direction along the third coordinate is not orthogonal to the direction along the second coordinate or the direction along the first coordinate. We have shown that in the supergravity solution derived from eleven dimensional supergravity on the slanted three-torus where the direction along the third coordinate is not orthogonal to the direction along the second coordinate, the intersection angle between them corresponds to the parameter of the S-duality \( SL(2, \mathbb{R})_s \) of the \( \beta \)-deformation. In the supergravity solution derived from eleven dimensional supergravity on the slanted three-torus where the direction along the third coordinate is not orthogonal to the direction along the first coordinate, the intersection angle between them corresponds to the parameter of the \( SL(2, \mathbb{R}) \) symmetry of type IIB supergravity which is the symmetry of the toroidal compactification. Therefore the intersection angle generalizes the \( \beta \)-deformed geometry. We have proposed a simpler form which is applicable when the axion field is pure gauge.

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