Nonbinary Quantum Cyclic and Subsystem Codes Over Asymmetrically-decohered Quantum Channels

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Abstract—Quantum computers theoretically are able to solve certain problems more quickly than any deterministic or probabilistic computers. A quantum computer exploits the rules of quantum mechanics to speed up computations. However, one has to mitigate the resulting noise and decoherence effects to avoid computational errors in order to successfully build quantum computers.

In this paper, we construct asymmetric quantum codes to protect quantum information over asymmetric quantum channels, \( \Pr Z \leq \Pr X \). Two generic methods are presented to derive asymmetric quantum cyclic codes using the generator polynomials and defining sets of classical cyclic codes. Consequently, the methods allow us to construct several families of quantum BCH, RS, and RM codes over asymmetric quantum channels. Finally, the methods are used to construct families of asymmetric subsystem codes.

I. INTRODUCTION

Quantum computers theoretically are able to solve certain problems more quickly than any deterministic or probabilistic computers. An example of such problems is the factorization of large integers in polynomial time. The novel idea is that a quantum computer exploits the rules of quantum mechanics to speed up computations. However, one has to mitigate the resulting noise and decoherence effects to avoid computational errors in order to successfully build quantum computers. Recently, the theory of quantum error-correcting codes is extended to include construction of such codes over asymmetric quantum channels — qubit-flip and phase-shift errors may have equal or different probabilities, \( \Pr Z \leq \Pr X \), the terminology is explained later. Asymmetric quantum error control codes (AQEC) are quantum codes defined over biased quantum channels. Construction of such codes first appeared in [7], [10], [13]. The code construction of AQEC is the CSS construction of QEC based on two classical cyclic codes. For more details on the CSS constructions of QEC see for example [5], [6], [14]–[17].

There have been several attempts to characterize the noise error model in quantum information [12]. In [16] the CSS construction of a quantum code that corrects the errors separately was stated. However, the percentage between the qubit-flip and phase-shift error probabilities was not known for certain physical realization. Recently, quantum error correction has been extended over amplitude-damping channels [5].

We expand the construction of quantum error correction by designing stabilizer codes that can correct phase-flip and qubit-flip errors separately. Assume that the quantum noise operators occur independently and with different probabilities in quantum states. Our goal is to adapt the constructed quantum codes to more realistic noise models based on an appropriate physical phenomena.

Motivated by their classical counterparts, the asymmetric quantum cyclic codes that we derive have online simple encoding and decoding circuits that can be implemented using shift-registers with feedback connections. Also, their algebraic structure makes it easy to derive their code parameters. Furthermore, their stabilizer can be defined easily using generator polynomials of classical cyclic codes, in addition, it is simple to derive self-orthogonal nested-code conditions for these cyclic classes of codes.

In this paper we construct quantum error-correcting codes that correct quantum errors that may destroy quantum information with different probabilities. We derive two generic framework methods that can be applied to any classical cyclic codes in order to derive asymmetric quantum cyclic codes. The methods are used to derive Asymmetric quantum BCH, RM, RS codes. In addition, they are used to derive families of asymmetric subsystem codes over finite fields. Several classes of asymmetric quantum codes are also shown in [1], [10], [13].

Notation: Let \( q \) be a power of a prime integer \( p \). We denote by \( \mathbb{F}_q \) the finite field with \( q \) elements. We define the Euclidean inner product \( \langle x | y \rangle = \sum_{i=1}^{n} x_i y_i \) and the Euclidean dual of a code \( C \subseteq \mathbb{F}_q^n \) as

\[
C^\perp = \{ x \in \mathbb{F}_q^n \mid \langle x | y \rangle = 0 \text{ for all } y \in C \}.
\]

We also define the Hermitian inner product for vectors \( x, y \) in \( \mathbb{F}_q^n \) as \( \langle x | y \rangle_h = \sum_{i=1}^{n} x_i^q y_i \) and the Hermitian dual of \( C \subseteq \mathbb{F}_q^n \) as

\[
C^{\perp_h} = \{ x \in \mathbb{F}_q^n \mid \langle x | y \rangle_h = 0 \text{ for all } y \in C \}.
\]

An \([n, k, d]_q\) denotes a classical code \( C \) with length \( n \), dimension \( k \), and minimum distance \( d \) over \( \mathbb{F}_q \). A quantum code \( Q \) is denoted by \([[[n, k, d]]_q]\).

II. CLASSICAL CYCLIC CODES

Cyclic codes are of greater interest because they have efficient encoding and decoding algorithms. In addition, they
have well-studied algebraic structure. Let \( n \) be a positive integer and \( \mathbb{F}_q \) be a finite field with \( q \) elements. A cyclic code \( C \) is a principle ideal of

\[
R_n = \mathbb{F}_q[x]/(x^n - 1),
\]

where \( \mathbb{F}_q[x] \) is the ring of polynomials in invariant \( x \). Every cyclic code \( C \) is generated by either a generator polynomial \( g(x) \) or generator matrix \( G \). Furthermore, every cyclic code is a linear code that has dimension \( k = n - \deg(g(x)) \). Let \( c(x) \) be a codeword in \( \mathbb{F}_q^n \) then \( c(x) = m(x)g(x) \), where \( m(x) \) is the message to be encoded. Consequently, every codeword can be written uniquely using a polynomial in \( \mathbb{F}_q^n \). Also, a codeword \( c(x) \) in \( \mathbb{F}_q^n \) is in \( C \) with defining set \( T \) if and only if \( c(x^\alpha) = 0 \) for all \( i \in T \). Every cyclic code generated by a generator polynomial \( g(x) \) has a parity check polynomial \( h(x) \) and is defined by the generator polynomial \( g(x) \) divides \( x^k h(1/x)/h(0) \) where \( h(x) = (x^n - 1)/g(x) \). Clearly, the parity check polynomial \( h(x) \) can be used to define the dual code \( C^\perp \) such that \( g(x) h(x) \mod (x^n - 1) = 0 \). Recall that the dual cyclic code \( C^\perp \) is defined by the generator polynomial \( g^\perp(x) = x^k h(x^{-1})/h(0) \). Let \( \alpha \) be an element in \( \mathbb{F}_q \). Then, sometimes, the code is defined by the roots of the generator polynomial \( g(x) \). Let \( T \) be the set of roots of \( g(x) \), \( T \) is the defining set of \( C \), then

\[
g(x) = \prod_{\alpha \in T} (x - x^\alpha).
\]

The set \( T \) is the union of cyclotomic cosets modulo \( n \) that has \( x^\alpha \) as a root. More details in cyclic codes can be found in \([9],[11]\). The following Lemma is needed to derive cyclic AQEC.

**Lemma 1**: Let \( C_i \) be cyclic codes of length \( n \) over \( \mathbb{F}_q \) with defining set \( T_i \) for \( i = 1, 2 \). Then

i) \( C_1 \cap C_2 \) has defining set \( T_1 \cup T_2 \).

ii) \( C_1 + C_2 \) has defining set \( T_1 \cap T_2 \).

iii) \( C_1 \subseteq C_2 \) if and only if \( T_2 \subseteq T_1 \).

iv) \( C_1^\perp \subseteq C_1+ \mod(2) \) if and only if \( C_1^\perp \mod(2) \subseteq C_1 \).

We will provide an analytical method not a computer search method to derive such codes. The benefit of this method is that it is much easier to derive families of AQEC. We define the classical cyclic code using the defining set and generator polynomial \([8],[9]\). The following lemma establishes conditions when \( C_2 \subseteq C_1 \).

**Lemma 2**: Let \( C_{T_i} \) and \( g_i(x) \) be the defining set and generator polynomial of a cyclic code \( C_i \) for \( i = 1, 2 \). If one of the following conditions

i) \( T_{C_1} \subseteq T_{C_2} \),

ii) \( g_1(x) \) divides \( g_2(x) \),

iii) \( h_2(x) \) divides \( h_1(x) \),

then \( C_2 \subseteq C_1 \).

**Proof**: The proof is straightforward from the definition of the codes \( C_1 \) and \( C_2 \) and by using Lemma \([1]\).

### III. Deriving Asymmetric Quantum Codes

We will show how to derive asymmetric quantum cyclic codes based on a given classical cyclic code using the CSS construction as follows.

Let \( H_1 \) and \( G_1 \) be the parity check and generator matrices of a classical code \( C_i \) with parameters \([n, k_i, d_i]\) for \( i \in \{1, 2\} \). The commutativity condition of \( H_1 \) and \( H_2 \) is stated as

\[
H_1 H_2^T + H_2 H_1^T = 0. \tag{1}
\]

Without loss of generality, we will assume that one of these two classical codes controls the phase-shift errors, while the other codes controls the bit-flip errors. Hence the CSS construction of a binary AQEC can be stated as follows. Hence the codes \( C_1 \) and \( C_2 \) are mapped to \( H_x \) and \( H_z \), respectively.

**Definition 3**: Given two classical binary codes \( C_1 \) and \( C_2 \) such that \( C_2^\perp \subseteq C_1 \). If we form \( G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \), and \( H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \), then

\[
H_1 H_2^T - H_2 H_1^T = 0 \tag{2}
\]

Let \( d_1 = \min \{ \text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp) \} \) and \( d_2 = \max \{ \text{wt}(C_2 \setminus C_1^\perp), \text{wt}(C_1 \setminus C_2^\perp) \} \), such that \( k_1 + k_2 > n \). If we assume that \( C_1 \) corrects the qubit-flip errors and \( C_2 \) corrects the phase-shift errors, then there exists AQEC with parameters

\[
\left[\begin{array}{c} \lfloor n, k_1 + k_2 - n, d_2/d_1 \rfloor \end{array} \right]. \tag{3}
\]

The following theorem shows the CSS construction of asymmetric quantum error control codes over \( \mathbb{F}_q \).

**Theorem 4 (CSS AQEC)**: Let \( C_1 \) and \( C_2 \) be two classical codes with parameters \([n, k_1, d_1]_q\) and \([n, k_2, d_2]_q\) respectively, and \( d_x = \min \{ \text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp) \} \), and \( d_2 = \max \{ \text{wt}(C_2 \setminus C_1^\perp), \text{wt}(C_1 \setminus C_2^\perp) \} \). If \( C_2^\perp \subseteq C_1 \), then

i) there exists an AQEC with parameters \([n, \text{dim} C_1 - \text{dim} C_2^\perp, d_2/d_1]_q\) that is \([n, k_1 + k_2 - n, d_2/d_1]_q\).

ii) there exists an asymmetric subsystem code with parameters \([n, k_1 + k_2 - n - r, r, d_1/d_2]_q\) for \( 0 \leq r \leq k_1 + k_2 - n \).

Furthermore, all constructed codes are pure to their minimum distances.

Therefore, it is straightforward to derive asymmetric quantum control codes from two classical codes as shown in Lemma \([8]\) as well as a subsystem code. Of course, one wishes to increase the values of \( d_z \) vers. \( d_x \) for the same code length and dimension.

If the AQEC has minimum distances \( d_z \) and \( d_x \) with \( d_z \geq d_x \), then it can correct all qubit-flip errors \( \leq \lfloor (d_z - 1)/2 \rfloor \) and all phase-shift errors \( \leq \lfloor (d_z - 1)/2 \rfloor \), respectively, as shown in the following result.

**Lemma 5**: An \([n, k, d_z/d_x]_q\) asymmetric quantum code corrects all qubit-flip errors up to \( \lfloor (d_z - 1)/2 \rfloor \) and all phase-shift errors up to \( \lfloor (d_z - 1)/2 \rfloor \).
The codes derived in [3] for primitive and nonprimitive quantum BCH codes assume that qubit-flip errors, phase-shift errors, and their combination occur with equal probability, where $\Pr Z = \Pr X = \Pr Y = p/3$, $\Pr I = 1 - p$, and $\{X, Z, Y, I\}$ are the binary Pauli operators $P$, see [6], [14]. We aim to generalize these quantum BCH codes over asymmetric quantum channels. Furthermore, we will derive a much larger class of AQEC based on any two cyclic codes. Such codes include RS, RM, and Hamming codes.

IV. ASYMMETRIC QUANTUM CYCLIC CODES

Recently the theory of quantum error-correcting codes (QEC) has been extended to asymmetric quantum error-correcting codes (AQEC), in which the quantum errors have biased probabilities. In this section we will give two methods to derive asymmetric quantum cyclic codes. One method is based on the generator polynomial of a cyclic code, while the other is directly from the defining set of cyclic code.

A. AQEC Based on Generator Polynomials of Cyclic Codes

Let $C_1$ be a cyclic code with parameters $[n, k, d]_q$ defined by a generator polynomial $g_1(x)$. Let $S = \{1, 2, \ldots, \delta_1 - 1\}$, for some integer $\delta_1 < n$, be the set of roots of the polynomial $g_1(x)$ such that

$$g_1(x) = \prod_{i \in S} (x - \alpha^i) \tag{4}$$

It is a well-known fact that the dimension of the code $C_1$ is given by

$$k_1 = n - \deg(g_1(x)) \tag{5}$$

We also know that the dimension of the dual code $C_1^\perp$ is given by $k_1^\perp = n - k_1 = \deg(g_1(x))$.

The idea that we propose is simple. Let $f(x) = (x^b - 1)$ be a polynomial such that $1 \leq \deg(f(x)) \leq n - k$. We extend the polynomial $g_1(x)$ to the polynomial $g_2(x)$ such that

$$g_2^\perp(x) = f(x)g_1(x) \tag{6}$$

Now, let $g_2^\perp(x)$ be the generator polynomial of the code $C_2^\perp$ that has dimension $k_2^\perp = n - \deg(f(x)g_1(x)) < k_1$. From the cyclic structure of the codes $C_1$ and $C_2^\perp$, we can see that $C_2^\perp \subseteq C_1$, therefore $C_1^\perp \subseteq C_2$. Let $d_1 = \wt(C_1\setminus C_2^\perp)$ and $d_2 = \wt(C_2\setminus C_1^\perp)$ then we have the following theorem. We can also change the rules of the code $C_1$ and $C_2$ to make sure that $d_2 > d_1$.

**Theorem 6:** Let $C_1$ be a cyclic code with parameters $[n, k_1, d_1]_q$ and a generator polynomial $g_1(x)$. Let $C_2^\perp$ be a cyclic code defined by the polynomial $f(x)g_1(x)$ such that $b = \deg(f(x)) \geq 1$, then there exists AQEC with parameters $[[n, 2k_1 - b - n, d_z/d_z]]_q$, where $d_z = \min\{\wt(C_1\setminus C_2^\perp), \wt(C_2\setminus C_1^\perp)\}$ and $d_2 = \max\{\wt(C_1\setminus C_2^\perp), \wt(C_2\setminus C_1^\perp)\}$. Furthermore the code can correct $[(d_z - 1)/2]$ qubit-flip errors and $[(d_2 - 1)/2]$ phase-shift errors.

**Proof:** We proceed the proof as follows.

i) We know that the dual code $C_1^\perp$ has dimension $k_1^\perp = \deg(g_1(x))$. Also, $C_1^\perp$ has a generator polynomial $h_1(x) = x^{n-k}h_1'(1/x)$ where $h_1'(x) = (x^n - 1)/g_1(x)$. Let $f(x)$ be a nonzero polynomial such that $f(x)g_1(x)$ defines a code $C_2^\perp$. Now the code $C_2^\perp$ has dimension $k_2^\perp = n - \deg(f(x)g_1(x)) = n - (k_1 + b) < k_1$.

ii) We notice that the polynomial $g_1(x)$ is a factor of the polynomial $f(x)g_1(x)$, therefore the code generated by later is a subcode of the code generated by the former. Then we have $C_2^\perp \subseteq C_1$. Hence, the code $C_2^\perp$ has dimension $k_2^\perp = n - (k_1 + b)$.

iii) Also, the code $C_2^\perp$ has dimension $k_1 + b$ and generator polynomial given by $g_2(x) = (x^n - 1)/(f(x)g_1(x)) = h_1(x)/f(x)$. Hence the $g_2(x)$ is a factor of $h_1(x)$, therefore $C_2^\perp$ is a subcode in $C_2$, $C_1^\perp \subseteq C_2$. There exists asymmetric quantum cyclic code with parameters

- $\dim C_1 - \dim C_2^\perp = k_1 - (n - k_1 - b)$.
- $d_z = \min\{\wt(C_2\setminus C_1^\perp), \wt(C_1\setminus C_2^\perp)\}$ and $d_2 = \max\{\wt(C_2\setminus C_1^\perp), \wt(C_1\setminus C_2^\perp)\}$.

B. Cyclic AQEC using the Defining Sets Extension

We can give a general construction for a cyclic AQEC over $\mathbb{F}_q$ if the defining sets of the classical cyclic codes are known.

**Theorem 7:** Let $C_1$ be a $k$-dimensional cyclic code of length $n$ over $\mathbb{F}_q$. Let $T_{C_1}$ and $T_{C_2}$ respectively denote the defining sets of $C_1$ and $C_2$. If $T$ is a subset of $T_{C_1} \setminus T_{C_2}$, that is the union of cyclotomic cosets, then one can define a cyclic code $C_2$ of length $n$ over $\mathbb{F}_q$ by the defining set $T_{C_2} = T_{C_1} \setminus (T \cup T^{-1})$. If $b = |T \cup T^{-1}|$ is in the range $0 \leq b < 2k - n$ then there exists asymmetric quantum code with parameters

$$[[n, 2k - b - n, d_z/d_z]]_q,$$

where $d_z = \min\{\wt(C_2 \setminus C_1^\perp), \wt(C_1\setminus C_2^\perp)\}$ and $d_2 = \max\{\wt(C_2\setminus C_1^\perp), \wt(C_1\setminus C_2^\perp)\}$.
TABLE I
FAMILIES OF ASYMMETRIC QUANTUM CYCLIC CODES

| q | $C_1$ BCH Code | $C_2$ BCH Code | AQEC |
|---|---|---|---|
| 2 | [15, 11, 3] | [15, 7, 5] | [15, 3, 5/3] |
| 2 | [15, 8, 4] | [15, 7, 5] | [15, 6, 5/2] |
| 2 | [31, 21, 5] | [31, 16, 7] | [31, 6, 7/3] |
| 2 | [31, 26, 3] | [31, 16, 7] | [31, 10, 8/3] |
| 2 | [31, 26, 3] | [31, 11, 11] | [31, 6, 13/3] |
| 2 | [31, 26, 3] | [31, 16, 15] | [31, 1, 15/3] |
| 2 | [127, 113, 5] | [127, 78, 15] | [127, 64, 15/5] |
| 2 | [127, 106, 7] | [127, 77, 27] | [127, 56, 25/7] |

**Proof:** Observe that if $s$ is an element of the set $S = T_{C_1} \setminus C_1 = T_{C_2} \setminus (N \cup T_{C_2})$, then $s$ is an element of $S$ as well. In particular, $T^{-1}$ is a subset of $T_{C_1} \setminus C_1$.

By definition, the cyclic code $C_2$ has the defining set $T_{C_2} = T_{C_1} \setminus (T \cup T^{-1})$; thus, the dual code $C_2^\perp$ has the defining set $T_{C_2} = N \setminus T_{C_1} = C_1 \cup (T \cup T^{-1})$.

Since $n - k = |T_{C_1}|$ and $b = |T \cup T^{-1}|$, we have $\dim_q C_1 = n - |T_{C_1}| = k$ and $\dim_q C_2 = n - |T_{C_2}| = k + b$.

Thus, there exists an $F_q$-linear asymmetric quantum code $Q$ with parameters $[n, k_0, d_z/d_z/q]$, where

i) $k_0 = \dim_q C_1 - \dim_q C_2^\perp = k - (n - (k + b)) = 2k + b - n$,

ii) $d_z = \min\{\wt(C_2 \setminus C_1^\perp), \wt(C_1 \cap C_2^\perp)\}$ and $d_z = \max\{\wt(C_2 \setminus C_1^\perp), \wt(C_1 \setminus C_2^\perp)\}$,

as claimed.

The usefulness of the previous theorem is that one can directly derive asymmetric quantum codes from the set of roots (defining set) of a cyclic code. We also notice that the integer $b$ represents a size of a cyclotomic coset (set of roots), in other words, it does not represent one root in $T_{C_1}$.

V. AQEC AND CONNECTION WITH SUBSYSTEM CODES

In this section we establish the connection between AQEC and subsystem codes. Furthermore we derive a larger class of quantum codes called asymmetric subsystem codes (ASSC). We derive families of subsystem BCH codes and cyclic subsystem codes over $F_q$. In [2] we construct several families of subsystem cyclic, BCH, RS and MDS codes over $F_{q^2}$ with much more details.

We expand our understanding of the theory of quantum error control codes by correcting the quantum errors $X$ and $Z$ separately using two different classical codes, in addition to correcting only errors in a small subspace. Subsystem codes are a generalization of the theory of quantum error control codes, in which errors can be corrected as well as avoided (isolated).

Let $Q$ be a quantum code such that $H = Q \oplus Q^\perp$, where $Q^\perp$ is the orthogonal complement of $Q$. We can define the subsystem code $Q = A \otimes B$, see Fig[1] as follows

**Definition 8 (Subsystem Codes):** An $[n, k, r, d]_q$ subsystem code is a decomposition of the subspace $Q$ into a tensor product of two vector spaces $A$ and $B$ such that $Q = A \otimes B$, where $\dim A = q^k$ and $\dim B = q^r$. The code $Q$ is able to detect all errors of weight less than $d$ on subsystem $A$.

Subsystem codes can be constructed from the classical codes over $F_q$ and $F_{q^2}$. Such codes do not need the classical codes to be self-orthogonal (or dual-containing) as shown in the Euclidean construction. We have given general constructions of subsystem codes in [4] known as the subsystem CSS and Hermitian Constructions. We provide a proof for the following special case of the CSS construction.

**Theorem 9 (ASSC Euclidean Construction):** If $C_1$ is a $k_1$-dimensional $F_q$-linear code of length $n$ that has a $k_2$-dimensional subcode $C_2 = C_1 \cap C_1^\perp$ and $k_1 + k_2 < n$, then there exist

$$[n, n - (k_1 + k_2), k_1 - k_2, d_z/d_z]_q,$$

$$[n, k_1 - k_2, n - (k_1 + k_2), d_z/d_z]_q,$$

subsystem codes, where $d_z = \max\{\wt(C_2^\perp \setminus C_1), \wt(C_1 \setminus C_2)\}$ and $d_z = \min\{\wt(C_2^\perp \setminus C_1), \wt(C_1 \setminus C_2)\}$.

**Proof:** The proof can be proceeded by defining pairs of codes as follows. Let us define the code $X = C_1 \times C_1 \setminus \mathbb{F}_q$, therefore $X^\perp = (C_1 \times C_1)^\perp = C_1^\perp \times C_1^\perp$. Hence $Y = X \cap X^\perp = (C_1 \times C_1) \cap (C_1^\perp \times C_1^\perp) = C_2 \times C_2$. Thus, $\dim_q Y = 2k_2$. Hence $|X|/|Y| = q^{2(k_1 + k_2)}$. By Theorem 1, there exists a subsystem code $Q = A \otimes B$ with parameters $[n, \log_q \dim A, \log_q \dim B, d_z/d_z]_q$ such that

i) $\dim A = q^{n/(|X|/|Y|)} = q^{n - k_1 - k_2}$,

ii) $\dim B = (|X|/|Y|) = q^{k_1 - k_2}$,

iii) $d_z = \max\{\swt(Y^\perp \setminus X), \swt(X^\perp \setminus Y)\} = \min\{\wt(C_2^\perp \setminus C_1), \wt(C_1 \setminus C_2)\}$, and $d_z = \min\{\swt(Y^\perp \setminus X), \swt(X^\perp \setminus Y)\} = \max\{\wt(C_2^\perp \setminus C_1), \wt(C_1 \setminus C_2)\}$.

Exchanging the rules of the codes $C_1$ and $C_1^\perp$ gives us the other subsystem code with the given parameters.

Subsystem codes (SCC) require the code $C_2$ to be self-orthogonal, $C_2 \subseteq C_2^\perp$. AQEC and SCC are both can be constructed from the pair-nested classical codes, as we call them. From this result, we can see that any two classical codes $C_1$ and $C_2$ such that $C_2 = C_1 \cap C_1^\perp \subseteq C_2^\perp$, in which they can be used to construct a subsystem code (SCC), can be also used to construct asymmetric quantum code (AQEC). Asymmetric subsystem codes (ASSC) are much larger class than the class of symmetric subsystem codes, in which the quantum errors occur with different probabilities in the former one and have equal probabilities in the later one. In short, AQEC does not require the intersection code to be self-orthogonal.

The construction in Lemma 3 can be generalized to ASSC CSS construction in a similar way. This means that we can look at an AQEC with parameters $[n, k, r, d_z/d_z]_q$, as subsystem code with parameters $[n, k, 0, d_z/d_z]_q$. Therefore all results shown in [2], [4] are a direct consequence by just fixing the minimum distance condition.

We have shown in [2] that All stabilizer codes (pure and impure) can be reduced to subsystem codes as shown in the
following result.

**Theorem 10 (Trading Dimensions of ASSC and Co-SCC):**
Let \( q \) be a power of a prime \( p \). If there exists an \( F_q \)-linear \( [[n, k, r, d_x/d_z]]_q \) asymmetric subsystem code (stabilizer code if \( r = 0 \)) with \( k > 1 \) that is pure to \( d' \), then there exists an \( F_q \)-linear \( [[n, k−1, r+1, \geq d_x/d_z]]_q \) subsystem code that is pure to \( \min\{d_x, d'\} \). If a pure \( (F_q \text{-linear}) [[n, k, r, d_x/d_z]]_q \) asymmetric subsystem code exists, then a pure \( (F_q \text{-linear}) [[n, k+r, d_x/d_z]]_q \) stabilizer code exists.

VI. AQEC based on Two Cyclic Codes

In this section we can also derive asymmetric quantum codes based on two cyclic codes and their intersections. We do not necessarily assume that the code \( C_1 \) is an extension of the code \( C_2^1 \). However, we assume that \( C_2^1 \subset C_1 \). The benefit of designing AQEC based on two different classical codes is that we guarantee the minimum distance \( d_z \) to be large in comparison to \( d_x \). In this case we can assume that \( C_1 \) is a binary BCH code with small minimum distance, while \( C_2 \) is an LDPC code with large minimum distance.

The only requirement one needs to satisfy is that \( C_1 \subseteq C_{1+i(\mod 2)} \). There have been many families that satisfy this condition. For example (15, 7) BCH code turns out to be an LDPC code. We will show an example to illustrate our theory.

A. Illustrative Examples

The following example illustrates the previous constructions. It gives a family of asymmetric quantum codes derived from the Hamming code with fixed minimum distance, and a BCH code with various designed distance.

**Example 11:** Let \( C_1 \) be the Hamming code with parameters \([n, k, 3]\) where \( n = 2^m − 1 \) and \( k = 2^m − m − 1 \). Consider \( C_2 \) to be a BCH code with parameters \( n \) and designed distance \( \delta \geq 5 \). Clearly the \( d_z = wt(C_2) > d_x = wt(C_1) = 3 \). Let \( k_2 \) be the dimension of \( C_2 \), then one can derive asymmetric quantum code with parameters \([n, k_1 + k_2 − n, d_z/3]]_q \). In fact, one can short the columns of the parity check matrix of the Hamming code \( C_1 \) to obtain a cyclic code with less dimension and large minimum distance, in which it can be used as \( C_2 \).

VII. Conclusion and Discussion

We presented two generic methods to derive asymmetric quantum error control codes based on two classical cyclic codes over finite fields. We showed that one can always start by a cyclic code with arbitrary dimension and minimum distance, and will be able to derive AQEC using the CSS construction. The method is also used to derive a family of subsystem codes.

Based on the generic methods that we develop, all classical cyclic codes can be used to construct asymmetric quantum cyclic codes and subsystem codes. In a quantum computer that utilizes asymmetric quantum cyclic codes to protection quantum information, such codes are superior in a sense that online encoding and decoding circuits will be used. In addition quantum shirt registers can be implemented. Our future will include bounds on the minimum distance and dimension of such codes. Furthermore such work will include the best optimal and perfect asymmetric quantum codes.

Such asymmetric quantum error control codes aim to correct the phase-shift errors that occur more frequently than qubit-flip errors. An attempt to address the fault tolerant operations and quantum circuits of such codes are given in [13], where an analysis for Becan-Shor asymmetric subsystem code is analyzed and a fault-tolerant circuit is given.

S. A. A. dedicates this paper to Dr. Moustafa Mahmoud who passed away in 10/31/2009 at the age of 88. Dr. Mahmoud was an Egyptian scientist and a prolific author, who boarded the ship of natural science, medicine, physics, knowledge, philosophy, and religion. He authored books and presented more than 400 TV video lectures to deeply explain the earth, sun, time, life, death, space, Holy scriptures, quantum theory and A. Einstein’s work.

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