Finding Fair and Efficient Allocations

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Abstract

We study the problem of allocating indivisible goods fairly and efficiently among a set of agents who have additive valuations for the goods. Here, an allocation is said to be fair if it is envy-free up to one good (EF1), which means that each agent prefers its own bundle over the bundle of any other agent, up to the removal of one good. In addition, an allocation is deemed to be efficient if it satisfies Pareto efficiency. A notable result of Caragiannis et al. (2016) shows that—under additive valuations—there is no need to trade efficiency for fairness; specifically, an allocation that maximizes the Nash social welfare (NSW) objective simultaneously satisfies these two seemingly incompatible properties. However, this approach does not directly provide an efficient algorithm for finding fair and efficient allocations, since maximizing NSW is an NP-hard problem.

In this paper, we bypass this barrier, and develop a pseudo-polynomial time algorithm for finding allocations which are EF1 and Pareto efficient; in particular, when the valuations are bounded, our algorithm finds such an allocation in polynomial time. Furthermore, we establish a stronger existence result compared to Caragiannis et al. (2016): For additive valuations, there always exists an allocation that is EF1 and fractionally Pareto efficient. The approach developed in the paper leads to a polynomial-time 1.45-approximation for maximizing the Nash social welfare objective. This improves upon the best known approximation ratio for this problem. Our results are based on constructing Fisher markets wherein specific equilibria are not only efficient, but also fair.

1 Introduction

The theory of fair division addresses the fundamental problem of allocating goods or resources among agents in a fair and efficient manner. Such problems arise in many real-world settings such as government auctions, divorce settlements and border disputes. Starting with the work of Steinhaus, Banach, and Knaster [Ste48], a vast literature in economics and mathematics has developed to formally address fair division [BT96; Mou14; BCE +16]. In particular, solution concepts such as envy-freeness, proportionality, and equitability have been proposed to formally capture fairness. Many interesting connections have also been found between fair division and fields such as topology, measure theory, combinatorics, and algorithms.

A majority of prior work, though, has focused on the fair division of divisible goods,1 and questions related to fairly allocating indivisible goods remain relatively underexplored. In fact, most of

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The first author was supported in part by a Ramanujan Fellowship (SERB - SB/S2/RJN-128/2015).
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1These are goods that can be fractionally assigned and model resources such as land. By contrast, indivisible goods model discrete resources that must be assigned integrally.
the classical solution concepts and algorithms that have been designed for divisible goods are not directly applicable to indivisible goods. For example, while an envy-free allocation\(^2\) of divisible goods is guaranteed to exist [Str80], the same cannot be said about indivisible goods; indeed, when allocating a single indivisible good between two agents, any allocation is bound to make the losing agent envious. We note that in the case of additive valuations,\(^3\) the well-known result of Varian [Var74] establishes a stronger result for divisible goods: There always exists an allocation which is not only envy-free (i.e., fair), but also Pareto efficient—a standard notion of efficiency in economics. Furthermore, such an allocation can be computed in polynomial time by solving the Eisenberg-Gale convex program [EG59].\(^4\) These results, however, do not directly provide fairness or efficiency guarantees for settings such as allocating courses at universities [OSB10] or dividing inherited artwork that entail allocation of indivisible goods.

These considerations have motivated recent work on developing relevant notions of fairness, along with existence and algorithmic results for the problem of fair division of indivisible goods [Bud11; PW14; BL16]. Our paper contributes to this line of work and shows that—in terms of a natural and necessary relaxation of envy-freeness—guarantees analogous to the fundamental result of Varian [Var74] hold even for indivisible goods: Under additive valuations, fair and efficient allocations always exist,\(^5\) and such allocations can be computed in (pseudo)-polynomial time.

We consider an allocation of indivisible goods to be fair if it is envy-free up to one good (EF1). This notion was defined by Budish [Bud11], and provides a compelling relaxation of the envy-freeness property. Specifically, an allocation is said to be EF1 if each agent prefers its own bundle over the bundle of any other agent, up to the removal of the most valuable good from the other agent’s bundle. Even though the existence of envy-free allocations is not guaranteed in the context of indivisible goods, EF1 allocations always exist—even under general, combinatorial valuations—and can be found in polynomial time [LMMS04].

With this notion of fairness in hand, it is relevant to ask whether we can (and if so, to what extent) achieve efficiency along with fairness while allocating indivisible goods.\(^6\) This question was recently studied by Caragiannis et al. [CKM+16], who showed a striking result that there is no need to trade efficiency for fairness: If the valuations are additive, then an allocation which maximizes the Nash social welfare—defined to be the geometric mean of the agents’ valuations [AI81, Volume 2, Chapter 14]—is both fair (EF1) and (Pareto) efficient. However, maximizing the Nash social welfare over integral allocations is an NP-hard problem [NNRR14], and hence this existence result does not automatically provide an efficient algorithm for finding fair and efficient allocations of indivisible goods.

In this paper, we bypass this barrier, and develop a pseudo-polynomial time algorithm for finding EF1 and Pareto efficient allocations of indivisible goods under additive valuations (Theorem 1). In particular, when the valuations are bounded, our algorithm finds such an allocation in polynomial time (Theorem 2). Furthermore, we establish a stronger existence result compared to Caragiannis et al. [CKM+16]: For additive valuations, there always exists an allocation that is EF1 and fractionally Pareto efficient (Theorem 3).

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\(^2\)That is, an allocation in which no agent envies (or prefers) the bundle of any other agent over its own.

\(^3\)We say that the valuations are additive if the valuation of an agent for any set of goods is the sum of its valuations for the individual goods in that set.

\(^4\)In fact, there exists a combinatorial algorithm for computing a market equilibrium [DPSV08].

\(^5\)Our existence result strengthens the one established by Caragiannis et al. [CKM+16].

\(^6\)Fairness, by itself, does not guarantee efficiency; in fact, an EF1 allocation can be highly inefficient (Section B.5).
Interestingly, our approach also leads to a polynomial-time 1.45-approximation algorithm for the Nash social welfare maximization (NSW) problem. Approximating NSW with indivisible goods is an interesting problem in its own right [CG15; AGMV16; AGSS17; CDG+17], and our result, in fact, improves upon the best-known approximation ratio for this problem.

**Our techniques** It is well-known from the fundamental theorems of welfare economics that markets tend towards efficiency. Intuitively, our results are based on establishing a complementary result that markets can be fair as well. In particular, we construct a market along with an underlying equilibrium which is integral (i.e., corresponds to an allocation of the indivisible goods) and EF1. The fact that this allocation is a market equilibrium ensures, via the first welfare theorem, that it is Pareto efficient as well.

In our construction, we start with a (fractionally) Pareto efficient allocation, and iteratively modify the allocation by exchanging goods between the agents. The goal of the exchange step is to locally move towards a fair allocation. Additionally, throughout these exchanges, we maintain a set of prices which ensure that the current allocation corresponds to an equilibrium outcome for the existing market. We stop when the equilibrium of the market (i.e., the allocation at hand) satisfies *price envy-freeness up to one good* (refer to Section 4.1 for a formal definition). Essentially, this property ensures that under the given market prices, the spending of an agent cannot exceed that of any other by more than its spending on some individual good. Requiring the spendings to be balanced in this manner implies the desired EF1 property for the corresponding fair division instance; see Section 4.2 for a detailed description of this construction.

At a high level, our framework is a constructive extension of the one used by Varian [Var74], which shows that for divisible goods, a fair and efficient allocation can be obtained from an equilibrium of a market in which all agents are endowed with equal budgets. While this competitive equilibrium with equal incomes (CEEI) solution is quite appealing in the case of divisible goods, it is not particularly applicable when addressing indivisible goods. In particular, a CEEI wherein the goods are integrally allocated in not guaranteed to exist, and there are settings wherein every rounding of a CEEI solution violates EF1 [CKM+16].

**Other related work** As mentioned previously, NSW is a well-studied objective in context of fair division. The problem of maximizing NSW with indivisible goods is APX-hard [Lee17], and a recent line of work has focussed on developing constant-factor approximation algorithms for this problem [CG15; AGMV16; AGSS17; CDG+17]. A relevant contribution of this paper is to improve upon the best-known approximation ratio for NSW maximization.

While the problem of approximating NSW is interesting in its own right, it is relevant to note that, in and of itself, an allocation that approximates this objective is not guaranteed to be EF1 or Pareto efficient (see Section B.7 for an example). We also provide a counterexample (Section B.6) in which every rounding of the “spending restricted outcome”—a market equilibrium notion used in the design of approximation algorithms for NSW [CG15; AGMV16; CDG+17]—will violate EF1.

Another well-studied fairness criterion in the context of indivisible goods is that of *maximin share* [Bud11]. The maximin share of an agent refers to the minimum utility that it can guarantee for itself when asked to partition the set of goods into \( n \) bundles such that the remaining \( n - 1 \) agents pick their bundles adversarially. An allocation is said to be maximin fair if the utility of each agent is at least its maximin share. While a maximin fair allocation is not guaranteed to exist [PW14; KPW16], approximation algorithms that provide each agent a constant fraction of its maximin share are known [PW14; AMNS15; BK17; GHS+17]. We show that in conjunction with fractional Pareto efficiency,
however, no algorithm can provide each agent with a better than \( \frac{1}{n} \) fraction of its maximin value (Section B.3). We further show that our algorithm achieves this bound (Lemma 19).

2 Preliminaries

2.1 The Fair Division Model

Problem instance An instance of the fair division problem is a tuple \( ([n], [m], \mathcal{V}) \), where \([n] = \{1, 2, \ldots, n\}\) denotes the set of \( n \in \mathbb{N} \) agents, \([m] = \{1, 2, \ldots, m\}\) denotes the set of \( m \in \mathbb{N} \) goods, and the valuation profile \( \mathcal{V} = \{v_1, v_2, \ldots, v_n\}\) specifies the preferences of each agent \( i \in [n] \) over the set of goods \([m]\) via a valuation function \( v_i : 2^m \rightarrow \mathbb{Z}_+ \cup \{0\} \). We will assume throughout that the valuation functions are additive, that is, for any agent \( i \in [n] \) and any set of goods \( G \subseteq [m] \), \( v_i(G) := \sum_{j \in G} v_i(\{j\}) \).\(^7\) For simplicity, we will write \( v_{i,j} \) instead of \( v_i(\{j\}) \) for a singleton good \( j \in [m] \). We will assume that \( v_{i,j} \) is non-negative and integral for each agent \( i \in [n] \) and each good \( j \in [m] \). We will also assume that for each good \( j \in [m] \), there exists some agent \( i \in [n] \) with a non-zero valuation for it, i.e., \( v_{i,j} > 0 \).

Allocation An allocation \( x \in \{0, 1\}^{n \times m} \) refers to an \( n \)-partition \((x_1, \ldots, x_n)\) of \([m]\), where \( x_i \subseteq [m] \) is the bundle allocated to the agent \( i \). We let \( \mathcal{X} \) denote the set of all \( n \) partitions of \([m]\). Given an allocation \( x \), the valuation of an agent \( i \in [n] \) for the bundle \( x_i \) is \( v_i(x_i) = \sum_{j \in x_i} v_{i,j} \).

Another useful notion is that of a fractional allocation. A fractional allocation \( x \in \{0, 1\}^{n \times m} \) refers to a (possibly) fractional assignment of the goods to the agents such that no more than one unit of each good is allocated, i.e., for all \( j \in [m] \), we have \( \sum_{i \in [n]} x_{i,j} \leq 1 \). Throughout the paper, we will use the term allocation to refer to an integral allocation, and explicitly write fractional allocation otherwise.

2.2 Fairness Notions

Envy-freeness and its variants Given an instance \(([n], [m], \mathcal{V})\) and an allocation \( x \), we say that an agent \( i \in [n] \) envies another agent \( k \in [n] \) if \( i \) prefers the bundle of \( k \) over its own bundle, i.e., \( v_i(x_k) > v_i(x_i) \). An allocation \( x \) is said to be envy-free (EF) if each agent prefers its own bundle over that of any other agent, i.e., for every pair of agents \( i, k \in [n] \), we have \( v_i(x_i) \geq v_i(x_k) \).

An allocation \( x \) is said to be envy-free up to one good (EF1) if for every pair of agents \( i, k \in [n] \), there exists a good \( j \in x_k \) such that \( v_i(x_i) \geq v_i(x_{k \setminus \{j\}}) \). Given any \( \varepsilon > 0 \), an allocation \( x \) is said to be \( \varepsilon \)-approximately envy-free up to one good (\( \varepsilon \)-EF1) if for every pair of agents \( i, k \in [n] \), there exists a good \( j \in x_k \) such that \((1 + \varepsilon)v_i(x_i) \geq v_i(x_{k \setminus \{j\}}) \). Note that an allocation is 0-EF1 if and only if it is EF1. The notion of EF1 first appeared in the work of Budish [Bud11].

Nash Welfare Given a (fractional) allocation \( x \), write \( \text{NW}(x) := \left( \prod_{i \in [n]} v_i(x_i) \right)^{\frac{1}{n}} \) to denote the Nash social welfare of \( x \). An allocation \( x^* \) said to be Nash optimal if \( x^* \in \arg \max_{x \in \mathcal{X}} \text{NW}(x) \).

2.3 Efficiency Notions

Pareto efficiency Given an instance \(([n], [m], \mathcal{V})\) and an allocation \( x \), we say that \( x \) is Pareto dominated by another allocation \( y \) if \( v_i(y_i) \geq v_i(x_i) \) for every agent \( i \in [n] \), and \( v_k(y_k) > v_k(x_k) \) for some

\(^7\)We assume that \( v_i(\emptyset) = 0 \) for each agent \( i \in [n] \).
agent $k \in [n]$. An allocation is said to be Pareto efficient (PO) if it is not Pareto dominated by any other allocation.

Some of our results use a generalization of Pareto efficiency, which we call fractional Pareto efficiency. An allocation is said to be fractionally Pareto efficient ($fPO$) if it not Pareto dominated by any fractional allocation. Thus, a fractionally Pareto efficient allocation is also Pareto efficient (i.e., $fPO \Rightarrow PO$), but the converse is not necessarily true (Section B.4 provides an example).

3 Main Results

Algorithmic Results:

**Theorem 1.** Given any fair division instance with additive valuations, an allocation that is envy-free up to one good (EF1) and Pareto efficient (PO) can be found in $O(poly(m, n, \max_{i,j} v_{i,j}))$ time.

We say that the bounded valuations assumption holds if—for a fixed constant $c$—we have $v_{i,j} \in \{0, 1, 2, \ldots, c\}$ for each agent $i \in [n]$ and each good $j \in [m]$.

**Theorem 2.** Given any fair division instance with additive and bounded valuations, an allocation that is envy-free up to one good (EF1) and Pareto efficient (PO) can be found in polynomial time.

Existence Result:

**Theorem 3.** Given any fair division instance with additive valuations, there always exists an allocation that is envy-free up to one good (EF1) and fractionally Pareto efficient ($fPO$).

Approximating Nash social welfare:

**Theorem 4.** There exists a polynomial-time $1.45$-approximation algorithm for the Nash social welfare maximization problem.

4 Proofs of Main Results

4.1 Market Terminology

**Fisher market** The Fisher market is a fundamental model in the economics of resource allocation [BS00]. It captures the setting where a set of buyers enter the market with prespecified budgets, and use it to buy goods that provide maximum utility per unit of money spent. Specifically, a Fisher market consists of a set $[n] = \{1, 2, \ldots, n\}$ of $n$ buyers, a set $[m] = \{1, 2, \ldots, m\}$ of $m$ divisible goods (exactly one unit of each good is available), and a valuation profile $V = \{v_1, v_2, \ldots, v_n\}$. Each buyer $i \in [n]$ has an initial endowment (or budget) $e_i > 0$. The endowment holds no intrinsic value for a buyer and is only used for buying the goods. We call $e = (e_1, \ldots, e_n)$ the endowment vector, and denote a market instance by $([n], [m], V, e)$.

A market outcome is given by the pair $(x, p)$, where the allocation vector $x = (x_1, \ldots, x_n)$ is a fractional allocation of the $m$ goods, and the price vector $p = (p_1, \ldots, p_m)$ associates a price $p_j \geq 0$ with each good $j \in [m]$. The spending of buyer $i$ under the market outcome $(x, p)$ is given by $p(x_i) = \sum_{j=1}^{m} x_{i,j} p_j$. The valuation derived by the buyer $i$ under the market outcome $(x, p)$ is given by $v_i(x_i) = \sum_{j=1}^{m} x_{i,j} v_{i,j}$. 

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Given a price vector \( p = (p_1, \ldots, p_m) \), define the bang-per-buck ratio of buyer \( i \) for good \( j \) as \( \alpha_{i,j} = v_{i,j}/p_j \), and its maximum bang-per-buck ratio as \( \alpha_i = \max_j \alpha_{i,j} \).\(^8\) Let \( \text{MBB}_i = \{ j \in [m] : v_{i,j}/p_j = \alpha_i \} \) denote the set of all goods that maximize the bang-per-buck ratio for buyer \( i \) at the price vector \( p \). We call \( \text{MBB}_i \) the maximum bang-per-buck set (or MBB set) of buyer \( i \) at price vector \( p \). Given a set \( S \subseteq [n] \) of agents, we write \( \text{MBB}_S \) to refer to \( \bigcup_{i \in S} \text{MBB}_i \).

An outcome \( \langle x, p \rangle \) is said to be a Fisher market equilibrium if it satisfies the following conditions:

- **Market clearing**: Each good is either priced at zero or is completely allocated. That is, for each good \( j \in [m] \), either \( p_j = 0 \) or \( \sum_{i=1}^n x_{i,j} = 1 \).

- **Budget exhaustion**: Buyers spend their endowments completely. That is, \( p(x_i) = e_i \) for each agent \( i \in [n] \).

- **Maximum bang-per-buck allocation**: Each buyer’s allocation is a subset of its MBB set. That is, for any buyer \( i \in [n] \) and any good \( j \in [m] \), \( x_{i,j} > 0 \Rightarrow j \in \text{MBB}_i \). Stated differently, each buyer only spends on its maximum bang-per-buck goods. Notice that a consequence of spending only on MBB goods is that each buyer maximizes its utility at the given prices \( p \) under the budget constraints. One might expect that the utility maximization condition is, in fact, equivalent to the MBB allocation condition. Section B.1 shows that this is not the case; indeed, MBB allocation is a strictly stronger requirement than utility maximization.

Refer to Section A.5 for additional market preliminaries.

**Proposition 1** (First Welfare Theorem). For a Fisher market with linear utilities, any equilibrium outcome is fractionally Pareto efficient.

**Price envy-freeness and its variants** Several of our results rely on constructing market outcomes with a property called price envy-freeness—a notion we consider to be of independent interest. Specifically, let \( x \) be an allocation and let \( p \) be a price vector for a given Fisher market. We say that \( x \) is price envy-free (pEF) with respect to \( p \) if for every pair of buyers \( i, k \in [n] \), we have \( p(x_i) \geq p(x_k) \). Similarly, \( x \) is said to be price envy-free up to one good (pEF1) with respect to \( p \) if for every pair of buyers \( i, k \in [n] \), there exists a good \( j \in x_k \) such that \( p(x_i) \geq p(x_k \setminus \{j\}) \). Finally, given any \( \varepsilon > 0 \), we say that an allocation \( x \) is \( \varepsilon \)-approximately price envy-free up to one good (\( \varepsilon \)-pEF1) with respect to \( p \) if for every pair of buyers \( i, k \in [n] \), there exists a good \( j \in x_k \) such that \( (1 + \varepsilon)p(x_i) \geq p(x_k \setminus \{j\}) \). Notice that an allocation is 0-pEF1 if and only if it is pEF1.

**MBB graph and alternating paths** The MBB graph of a Fisher market instance with a price vector \( p \) is defined as a bipartite graph \( G \) whose vertex set consists of the set of agents \( [n] \) and the set of goods \( [m] \), and there is an edge between an agent \( i \in [n] \) and a good \( j \in [m] \) if \( j \in \text{MBB}_i \) (called an MBB edge). Additionally, given an allocation \( x \), we can augment the MBB graph by adding allocation edges, i.e., an edge between an agent \( i \in [n] \) and a good \( j \in [m] \) such that \( j \in x_i \). For an augmented MBB graph, we define an alternating path \( P = (i, j_1, i_1, j_2, i_2, \ldots, i_{\ell-1}, j_\ell, k) \) from agent \( i \) to agent \( k \) (and involving the agents \( i_1, i_2, \ldots, i_{\ell-1} \) and the goods \( j_1, j_2, \ldots, j_\ell \)) as a series of alternating MBB and allocation edges such that \( j_1 \in \text{MBB}_{i_1} \cap x_{i_1}, j_2 \in \text{MBB}_{i_1} \cap x_{i_2}, \ldots, j_\ell \in \text{MBB}_{i_{\ell-1}} \cap x_k \). If such a path exists, we say that the agent \( k \) is reachable from agent \( i \) via an alternating path. Notice that no agent or good is allowed to repeat in an alternating path. We say that the path \( P \) is of length \( 2\ell \) since it consists of \( \ell \) MBB edges and \( \ell \) allocation edges.

\(^8\)If \( v_{i,j} = 0 \) and \( p_j = 0 \), then we let \( \alpha_{i,j} = 0 \).
Hierarchical structure. Let $G$ denote the augmented MBB graph for a Fisher market instance with the market outcome $(x, p)$. Fix a source agent $i \in [n]$ in $G$. Define the level of an agent $k \in [n]$ as half the length of the shortest alternating path from $i$ to $k$ (if one exists). The level of the source agent $i$ is defined to be zero. If there is no alternating path from $i$ to some agent $k$ in $G$ (i.e., if $k$ is not reachable from $i$), then the level of $k$ is set to be $n$. The hierarchy structure $H_i$ of agent $i$ is defined as a level-wise collection of all agents that are reachable from $i$, i.e., $H_i = \{H^0_i, H^1_i, H^2_i, \ldots, \}$, where $H^\ell_i$ denotes the set of agents that are at level $\ell$ with respect to the agent $i$. Section A.1 provides a polynomial time subroutine called BuildHierarchy for constructing the hierarchy.

We remark that given a hierarchy $H_i$, we will overload the term alternating path to refer to a series of alternating MBB and allocation edges connecting agents at a lower level to those at a higher level. That is, a path $P = (i, j_1, i_1, j_2, i_2, \ldots, i_{\ell-1}, j_{\ell}, k)$ involving agents from the hierarchy $H_i$ is said to be an alternating path if (1) $j_1 \in \text{MBB}_i \cap x_i$, $j_2 \in \text{MBB}_{i_1} \cap x_{i_1}, \ldots$, $j_{\ell} \in \text{MBB}_{i_{\ell-1}} \cap x_{i_{\ell-1}}$, and (2) $\text{level}(i) < \text{level}(i_1) < \text{level}(i_2) < \cdots < \text{level}(i_{\ell-1}) < \text{level}(k)$. In particular, an alternating path in a hierarchy cannot have edges between agents at the same level.

Violators and path-violators. Given a Fisher market instance and a market outcome $(x, p)$, an agent $i \in [n]$ with the smallest spending among all the agents is called the least spender, i.e., $i \in \arg\min_{k \in [n]} p(x_k)$ (ties are broken according to a pre-specified ordering over the agents). An agent $k \in [n]$ is said to be a violator if for every good $j \in x_k$, we have that $p(x_k \setminus \{j\}) > p(x_i)$. Similarly, agent $k \in [n]$ is said to be an $\varepsilon$-violator if for every good $j \in x_k$, we have that $p(x_k \setminus \{j\}) > (1 + \varepsilon) \cdot p(x_i)$. Notice that if no agent is a violator ($\varepsilon$-violator), then the allocation $x$ is $\text{pEF1}$ ($\varepsilon$-pEF1) with respect to $p$.

A closely related notion is that of a path-violator. Given a hierarchy $H_i$, an agent $k \in H_i$ is said to be a path-violator with respect to the alternating path $P = (i, j_1, i_1, j_2, i_2, \ldots, i_{\ell-1}, j_{\ell}, k)$ if $p(x_k \setminus \{j_{\ell}\}) > p(x_i)$. Observe that a path-violator (along a path $P$) need not be a violator, since there can be a good $j \in x_k$ not on the path $P$ such that $p(x_k \setminus \{j\}) \leq p(x_i)$. Similarly, an agent $k \in H_i$ is said to be an $\varepsilon$-path-violator with respect to the alternating path $P = (i, j_1, i_1, j_2, i_2, \ldots, i_{\ell-1}, j_{\ell}, k)$ if $p(x_k \setminus \{j_{\ell}\}) > (1 + \varepsilon) \cdot p(x_i)$.

4.2. Description of the Algorithm.

Given any fair division instance $I = ([n], [m], V)$ as input and parameter $\varepsilon > 0$, our algorithm (Algorithm 1), referred to as ALG, constructs a market equilibrium $(x, p)$ with respect to a Fisher market instance $([n], [m], V, e)$ (for a suitable endowment vector $e$) with the properties that (i) $x$ is an integral allocation, and (ii) $x$ is $\varepsilon$-pEF1 with respect to $p$. The second property allows us to show that the allocation $x$ is $\varepsilon$-EF1 for the corresponding fair division instance $I$ (Lemma 1). Furthermore, by Proposition 1, the allocation $x$ is also guaranteed to be FPO for the underlying Fisher market, and this implication transfers to the fair division instance $I$ as well.

In order to construct the desired Fisher market equilibrium, our algorithm starts with a welfare-maximizing allocation $x$ and a price vector $p$ such that $x$ is FPO and each agent gets a subset of its MBB goods (this is Phase 1 of ALG). If the allocation $x$ is also $\text{pEF1}$ with respect to $p$, then the algorithm terminates with the output $(x, p)$. Otherwise, the algorithm proceeds to the next phase.

In Phase 2, the algorithm works with the hierarchy of the least spending agent (refer to Section 4.1 for relevant definitions), and performs a series of exchanges (or swaps) of goods between the agents in the hierarchy (without changing the prices). The swaps are aimed at ensuring that at the end of
Phase 2, no agent in the hierarchy is \( pEF1 \) envied by the least spender. Furthermore, all exchanges in Phase 2 happen only along the MBB edges, thus maintaining at each stage, the condition that \( x \) is an equilibrium allocation, and hence, fPO.

If, at the end of Phase 2, the current allocation \( x \) is still not \( \varepsilon \)-\( pEF1 \) with respect to the price vector \( p \), the algorithm moves to Phase 3. This phase consists of uniformly raising the prices of the goods owned by the members of the hierarchy. The price-rise is continued until either the allocation \( x \) becomes \( pEF1 \) with respect to the new price vector \( p \), or a new agent gets added to the hierarchy. In the latter case, the algorithm goes back to the start of Phase 2.

**Lemma 1.** Let \( \varepsilon \geq 0 \), and let \( x \) be an allocation for a market instance \( \langle [n], [m], V, e \rangle \) such that (1) \( x \) is \( \varepsilon \)-approximately price-envy-free up to one good (\( \varepsilon \)-\( pEF1 \)) with respect to a price vector \( p \), and (2) \( x_i \subseteq MBB_i \) for each buyer \( i \in [n] \). Then, \( x \) is \( \varepsilon \)-approximately envy-free up to one good (\( \varepsilon \)-\( EF1 \)) for the associated fair division instance \( \langle [n], [m], V \rangle \).

**Proof.** Since \( x \) is \( \varepsilon \)-\( pEF1 \) with respect to the price vector \( p \), for any pair of buyers \( i, k \in [n] \), there exists a good \( g \in x_k \) such that \((1 + \varepsilon) \cdot p(x_i) \geq p(x_k \setminus \{j\})\). Multiplying both sides by the maximum bang-per-buck ratio of agent \( i \) (namely \( \alpha_i \)), we get

\[
\alpha_i \cdot (1 + \varepsilon) \cdot p(x_i) \geq \alpha_i \cdot p(x_k \setminus \{j\})
\]

\[
\Rightarrow \quad (1 + \varepsilon) \cdot v_i(x_i) \geq \alpha_i \cdot p(x_k \setminus \{j\}) \quad \text{(since } x_i \subseteq MBB_i) 
\]

which is the \( \varepsilon \)-\( EF1 \) guarantee for the allocation \( x \).

### 4.3 Analysis of ALG When the Valuations are powers-of-\( r \)

In this section, we will analyze \( ALG \) under the assumption that all valuations are powers-of-\( r \), i.e., there exists a number \( r > 1 \) such that for each agent \( i \in [n] \) and each good \( j \in [m] \), we have \( v_{i,j} \in \{0, r^a\} \) for some natural number \( a \) (possibly depending on \( i \) and \( j \)).

**Time steps and events** We will start by defining the notion of a *time step* that will be useful in the subsequent analysis. Notice that the execution of \( ALG \) can be described in terms of the following four events: (1) *Swap* operation in Phase 2, (2) *Change* in the identity of least spender in Phase 2, (3) *Raise* prices by \( \alpha = \alpha_1 \) or \( \alpha = \alpha_3 \) in Phase 3, and (4) *Terminate* and return the current allocation and price vector. We use the term *time step* (or simply a *step*) to denote the indexing of any execution of \( ALG \), e.g., \( ALG \) performs a swap operation on the first and second time steps, followed by a price-rise in the third time step, and so on. We will use the phrase “at time step \( t \)” to denote the state of the algorithm before the event for time step \( t \) takes place. Notice that each event stated above runs in polynomial time, and therefore it suffices analyze the running time of \( ALG \) in terms of the total number of events (or time steps).

**Lemma 2 (Correctness of \( ALG \)).** Given any \( \varepsilon > 0 \), the allocation returned by \( ALG \) is \( \varepsilon \)-approximately envy-free up to one good (\( \varepsilon \)-\( EF1 \)) and fractionally Pareto efficient (fPO).

**Proof.** Let the output of \( ALG \) be \((x, p)\). The fact that \( x \) is fPO follows from the observation that at each step of the algorithm, the allocation of any agent is a subset of its MBB goods, i.e., at each time step, we have \( x_i \subseteq MBB_i \) for each agent \( i \in [n] \). This is certainly true at the end of Phase 1.
Algorithm 1: ALG

Input: An instance $\mathcal{I} = ([n], [m], \mathcal{V})$ such that valuations are powers-of-$r$.
Output: An integral allocation $z$ and a price vector $q$.
Parameter: $\varepsilon, r$ such that $\varepsilon > r - 1$.

// --------------------------Phase 1: Initialization--------------------------
1 $x \leftarrow$ Welfare-maximizing allocation (allocate each good $j$ to agent $i \in \arg \max_{k \in [n]} v_{k,j}$)
2 $p \leftarrow$ For every good $j \in [m]$, set $p_j = v_{i,j}$ if $j \in x_i$.
3 if $(x, p)$ is $\varepsilon$-pEF1 then return $(x, p)$

// ---------------Phase 2: Removing price-envy within hierarchy-------------------
4 $i \leftarrow$ least spender under $(x, p)$ /* break ties lexicographically */
5 $\mathcal{H}_i \leftarrow \text{BUILDHIERARCHY}(i, x, p)$
6 $\ell \leftarrow 1$
7 if $\mathcal{H}_i^\ell$ is non-empty and $(x, p)$ is not $\varepsilon$-pEF1 then
8  if $h \in \mathcal{H}_i^\ell$ is an $\varepsilon$-path-violator along the alternating path $P = \{i, j_1, h_1, \ldots, j_{\ell-1}, h_{\ell-1}, j, h\}$ then
9    $x_h \leftarrow x_h \setminus \{j\}$ and $x_{h_{\ell-1}} \leftarrow x_{h_{\ell-1}} \cup \{j\}$ /* Swap operation */
10   Repeat Phase 2 starting from Line 4
11 else
12  $\ell \leftarrow \ell + 1$
13 end

else if $(x, p)$ is $\varepsilon$-pEF1 then
14  return $(x, p)$
else
15  Move to Phase 3 starting from Line 19
16 end

// --------------------------Phase 3: Price-rise--------------------------
19 $\alpha_1 \leftarrow \min_{h \in \mathcal{H}_i, j \in [m] \setminus x_{\mathcal{H}_i}, \mathcal{V}_{h,j}/p_j}$ $\text{MBB}_h$
20 $\alpha_2 \leftarrow \frac{1}{p(x_i)} \max_{k \in [n] \setminus \mathcal{H}_i, j \in x_k} \min_{j \in [m] \setminus \mathcal{H}_i, j \in x_k} p(x_k \setminus \{j\})$
21 $\alpha_3 \leftarrow r^s$, where $s$ is the smallest integral power of $r$ such that $r^s > \frac{p(x_h)}{p(x_i)}$, where $i$ is the least spender and $h \in \arg \min_{k \in [n] \setminus \mathcal{H}_i} p(x_k)$.
22 $\alpha \leftarrow \min(\alpha_1, \alpha_2, \alpha_3)$
23 for each good $j \in x_{\mathcal{H}_i}$ do
24  $p_j \leftarrow \alpha \cdot p_j$
25 endfor
26 if $\alpha = \alpha_2$ then
27  return $(x, p)$
28 else
29  Repeat Phase 2 starting from Line 4
30 end
by way of setting the prices. In Phase 2, each swap operation only happens along an alternating MBB-allocation edge, which maintains the MBB condition. Phase 3 involves raising the prices of the goods owned by the members of the hierarchy \( H_i \) without changing the allocation. We will argue that for each agent \( k \in [n] \), if \( x_k \subseteq MBB_k \) before the price-rise, then the same continues to hold after the price-rise. Indeed, for any agent \( k \notin H_i \), we have \( x_k \cap x_{H_i} = \emptyset \) by construction of the hierarchy. As a result, raising the prices of the goods in \( x_{H_i} \) does not affect the bang-per-buck ratio of agent \( k \) for the goods in \( x_k \) (and can only reduce its bang-per-buck ratio for the goods outside of \( x_k \)), thus maintaining the above condition. For any agent \( k \in H_i \), we have \( MBB_k \subseteq x_{H_i} \) by construction of the hierarchy. Raising the prices of the goods in \( x_{H_i} \) therefore corresponds to lowering the MBB ratios for the agents in \( H_i \). By choice of \( \alpha_1 \), the price-rise stops as soon as a new MBB-edge appears between an agent \( k \in H_i \) and a good \( j \notin x_{H_i} \). This ensures that the new maximum bang-per-buck ratio for any agent \( k \in H_i \) does not fall below its second-largest bang-per-buck ratio prior to the price-rise, thus guaranteeing \( x_k \subseteq MBB_k \).

We can now define a Fisher market where the endowment of each player is its spending under \( x \). Since \( (x, p) \) is an equilibrium for this market, we have from the First Welfare Theorem (Proposition 1) that \( x \) is \( \epsilon \)-PO.

Next, we will argue that \( x \) is \( \epsilon \)-EF1. Note that \( Alg \) terminates only if one of the following conditions holds: (i) the current outcome \( (x, p) \) is \( \epsilon \)-pEF1, or (ii) when \( \alpha = \alpha_2 \) (in Line 26). In the first case, via Lemma 1, we get that \( x \) is \( \epsilon \)-EF1 for the underlying fair division instance. Therefore, we only need to analyze the second case, which happens when \( \alpha_2 \leq \min \{\alpha_1, \alpha_3\} \).

Say the termination happens at time step \( t \), and let \( q \) be the price vector maintained by \( Alg \) just before the price-rise step that leads to termination. Also, at time step \( t \), let \( i \) be the least spender, \( H_i \) be the hierarchy of agent \( i \), and \( h \) be the agent with the smallest spending in \( [n] \setminus H_i \), breaking ties lexicographically. By construction, at the end of (any) execution of Phase 2, no agent in the least spender’s hierarchy is an \( \epsilon \)-path-violator. Hence, in particular, \( i \) does not \( \epsilon \)-price envy any agent in \( H_i \) with respect to the price vector \( q \).

After the time step \( t \), \( Alg \) terminates with, say, the allocation \( x \) and price vector \( p \). Since Phase 3 only affects the price vector, the allocation maintained by \( Alg \) just before termination is \( x \) as well. Moreover, Phase 3 only affects the prices in \( x_{H_i} \), hence, for all agents \( k \in [n] \setminus H_i \), we have \( p(x_k) = q(x_k) \).

Scaling the prices by \( \alpha = \alpha_2 \) ensures that the following inequality holds for all agents \( k \in [n] \setminus H_i \):
\[
(1 + \epsilon) p(x_i) \geq \min_{j \in x_k} p(x_k \setminus \{j\}).
\]
In addition, the price-rise maintains such an inequality for every agent \( k \in H_i \). Overall, we get that the \( \epsilon \)-pEF1 condition is satisfied for agent \( i \).

Now, if agent \( i \) is a least spender under \( (x, p) \) (i.e., \( i \) continues to a least spender after the price rise), then \( x \) is \( \epsilon \)-pEF1 with respect to \( p \), and we are done. Otherwise, if agent \( i \) is not a least spender under \( (x, p) \), then agent \( h \) must be a least spender under \( (x, p) \).

The fact that \( \alpha = \alpha_2 \) implies that \( \alpha_2 \leq \alpha_3 \). Also, the definition of \( \alpha_3 \) ensures that \( r \frac{q(x_h)}{q(x_i)} \geq \alpha_3 \). Therefore,
\[
\frac{r q(x_h)}{q(x_i)} \geq \alpha_2 \\
\Rightarrow r q(x_h) \geq p(x_i) \quad \text{(since } p(x_i) = \alpha_2 q(x_i)) \\
\Rightarrow r p(x_h) \geq p(x_i) \quad \text{(since } p(x_h) = q(x_h)) \\
\Rightarrow r (1 + \epsilon) p(x_h) \geq \min_{j \in x_k} p(x_k \setminus \{j\}) \quad \text{for all } k \in [n],
\]
where the last inequality follows from the fact that the $\epsilon$-pEF1 condition is satisfied for agent $i$. Since $r < (1 + \varepsilon)$, the new least spender (agent $h$) satisfies $(1 + 3\varepsilon) p(x_h) \geq \min_{j \in X_i} p(x_k \setminus \{j\})$ for all $k \in [n]$. This implies that $(x, p)$ is $(3\varepsilon)$-pEF1. With an initial choice of $\varepsilon$ that is a third of the current choice, we can show that $x$ is in fact $\varepsilon$-pEF1.

The stated claim now follows from the observation in Lemma 1 that an equilibrium allocation that is $\varepsilon$-pEF1 in the market instance is $\varepsilon$-EF1 for the corresponding fair division instance. \qed

**Lemma 3 (Running time bound for Alg).** Given any $\varepsilon > 0$, Alg terminates in $O\left(\text{poly}(m, n, \frac{1}{\varepsilon}) \cdot \frac{1}{\log r} \cdot \log^2 \max_{k \in [n]} v_k([m])\right)$ steps when all valuations are powers-of-$r$.

**Proof.** The proof of this lemma follows from Corollary 1 and Lemma 9, which are proved next in Section 4.3.1. \qed

### 4.3.1 Proof of Lemma 3

This section provides the proofs of Corollary 1 and Lemma 9, which are used to establish Lemma 3. These proofs rely on several intermediate results (Lemmas 4 to 8). The proofs of Lemmas 4 to 6 below are deferred to the appendix (Sections A.2.1 to A.2.3).

Let $i_t \in [n]$ denote the least spender at time step $t$, and let $x^t$ and $p^t$ denote the corresponding allocation and the price vector respectively.

**Lemma 4.** The spending of the least spender cannot decrease, i.e., for each time step $t$, $p^t(x^t_i) \leq p^{t+1}(x^t_{i_{t+1}})$.

Let $E_t \subset [n]$ denote the set of all $\varepsilon$-violators at time $t$. That is, $E_t = \{h \in [n] : (1 + \varepsilon) \cdot p^t(x^t_i) < p^t(x^t_i \setminus \{j\})\}$ for every good $j \in x^t_h$.

**Lemma 5.** At the beginning of each Phase 3 step, say at time $t$, $p^t(x^t_i) \leq \max_{k \in [n]} v_k([m])$.

**Lemma 6.** Alg can perform at most $\text{poly}(n, m)$ number of consecutive swap operations before either the identity of the least spender changes or a Phase 3 step occurs.

**Lemma 7.** Consider a series of consecutive time steps consisting entirely of Phase 2 operations, i.e., either swap operations or change in the identity of the least spender. Let $t$ be a time step at which an agent $i$ ceases to be the least spender, and let $t' > t$ be the first time step after $t$ at which $i$ once again becomes the least spender. Let $(x, p)$ and $(x', p')$ denote the corresponding allocation and price vectors. Then, either $x_i \subset x'_i$ or $p'(x'_i) \geq (1 + \varepsilon)p(x_i)$, or both.

**Proof.** Observe that any change in the identity of the least spender during Phase 2 happens when the previous least spender receives a good via a swap operation. This means that agent $i$ must receive a good at time $t$. If, in addition, agent $i$ does not lose any good during the time interval between $t$ and $t'$, then we already have that $x_i \subset x'_i$ and the claim follows. Therefore, for the rest of the proof, we will assume that agent $i$ loses one or more goods between $t$ and $t'$.

Among all time steps between $t$ and $t'$ at which agent $i$ loses a good, let $\overline{t}$ denote the last one. Let $j \in [m]$ denote the good lost by agent $i$ at time $\overline{t}$, and let $k$ denote the least spender at that time. Also, let $(\overline{x}, \overline{p})$ denote the allocation and price vector just before $i$ loses the good $j$.

We know from Lemma 4 that the spending of the least spender cannot decrease with time. Thus, $\overline{p}(\overline{x}_k) \geq p(x_i)$. (1)
Since agent $i$ loses the good $j$ at time $t$, it must be an $\varepsilon$-path-violator with respect to the allocation $\mathbf{x}$ and $\mathbf{p}$. Hence,

$$p(\mathbf{x} \setminus \{j\}) > (1 + \varepsilon) \cdot p(\mathbf{x}_k).$$

Finally, since $j$ is the last good lost by $i$ before the time step $t'$, we have that

$$p'(\mathbf{x}_i) \geq p(\mathbf{x}_i \setminus \{j\}).$$

Equations (1) to (3) together give us the desired result.

**Lemma 8.** The identity of the least spender can change at most \(\text{poly}(n, m, \frac{1}{\varepsilon}) \cdot \log \max_{k \in [n]} v_k([m])\) number of times during Phase 2 before a Phase 3 step occurs.

**Proof.** Recall from Lemma 7 that every time \(\text{Alg}\) cycles back to an agent (say agent $i$) as the least spender, either the allocation of $i$ strictly grows by at least one good, or its spending grows at least by a multiplicative factor of \((1 + \varepsilon)\). By pigeonhole principle, for every $n + 1$ identity-change events, \(\text{Alg}\) must cycle back to some agent. Therefore, for every $n + 1$ consecutive identity-change events (possibly interspersed with swap operations), either an agent obtains an extra good (without losing any) or its spending grows by a factor of \((1 + \varepsilon)\). This observation, along with the fact that the spending of the least spender can never decrease with time (Lemma 4), implies that for every $m \cdot (n + 1)$ identity-change operations, the spending of the least spender must increase by a factor of \((1 + \varepsilon)\). Furthermore, we know from Lemma 5 that the spending of the least spender at the beginning of each Phase 3 step is at most $\max_{k \in [n]} v_k([m])$. Hence, assuming that the initial spending of the least spender is at least 1 (refer to Section A.4 for explanation of why this assumption is without loss of generality), there can be at most \(\text{poly}(m, n) \cdot \log_{1+\varepsilon} \max_{k \in [n]} v_k([m])\) identity-change events during Phase 2 before a Phase 3 step occurs. By using $\log(1 + \varepsilon) \geq \varepsilon - \varepsilon^2$, we obtain the desired result.

**Corollary 1.** Phase 2 of \(\text{Alg}\) can continue for at most \(\text{poly}(n, m, \frac{1}{\varepsilon}) \cdot \log \max_{k \in [n]} v_k([m])\) consecutive time steps before a Phase 3 step occurs.

**Proof.** Follows from Lemmas 6 and 8.

**Lemma 9.** \(\text{Alg}\) can perform at most $n \cdot \log_\alpha \max_{k \in [n]} v_k([m])$ number of Phase 3 steps.

**Proof.** Recall that each Phase 3 step increments the prices of the goods owned by the members of the hierarchy by a multiplicative factor of $\alpha$, where $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$. Since the algorithm terminates if $\alpha = \alpha_2$, we will assume throughout that $\alpha$ is always equal to either $\alpha_1$ or $\alpha_3$.

**Price-rise by $\alpha = \alpha_1$:** In order to bound the number of price-rise steps with $\alpha = \alpha_1$, we will show that each such step multiplicatively increments the spending of the least spender by a positive integral power of $r$. Then, by using Lemmas 4 and 5, we can conclude that there can be at most $\log_\alpha \max_{k \in [n]} v_k([m])$ price-rise steps with $\alpha = \alpha_1$.

Notice that since the valuations are powers-of-$r$, the initial prices of all the goods at the end of Phase 1 are also powers-of-$r$. We claim that if all prices are powers-of-$r$ prior to the price-rise step with $\alpha = \alpha_1$, then the same continues to hold after the price-rise. Indeed, if all prices are powers-of-$r$, then all bang-per-buck ratios must also be (possibly negative) powers-of-$r$ (since all valuations are powers-of-$r$). Moreover, $\alpha_1$ is itself a ratio of two such bang-per-buck ratios, and thus it must also be

\[\text{We are assuming here that the initial spending of the least spender is at least 1; refer to Section A.4 for explanation of why this assumption is without loss of generality.}\]
a power-of-$r$. Raising the prices by $\alpha_1$ preserves the powers-of-$r$ property. Therefore, each price-rise by $\alpha = \alpha_1$ multiplicatively increments the prices by a power of $r$. Furthermore, this increment must be strict since the price-rise step introduces an MBB edge between a member of the hierarchy and a good outside the hierarchy that was not present earlier. Hence, the spending of the least spender grows by a positive integral power of $r$, as desired.

Price-rise by $\alpha = \alpha_3$: We will now bound the number of price-rise steps with $\alpha = \alpha_3$. While the price-increments in this case are also in powers-of-$r$ (this follows from the choice of $\alpha_3$), the spending of the least spender may not increment by the same factor after each such step due to a change in the identity of the least spender. However, by pigeonhole principle, the algorithm must cycle back to the same agent after $n$ steps, and therefore the spending of the least spender must grow multiplicatively by a positive integral power of $r$ after every $n$ price-rise steps with $\alpha = \alpha_3$. By a similar reasoning as for the case $\alpha = \alpha_1$, we get the desired result.

\[\square\]

4.4 Analysis of \textsc{Alg} for General Valuations: Proofs of Theorems 1 and 2

In this section, we show that for any given fair division instance $I = \langle [n], [m], V \rangle$, an allocation that is envy-free up to one good (EF1) and Pareto efficient (PO) can be found in pseudo-polynomial time (Theorem 1). We will prove Theorem 1 by running \textsc{Alg} on a $\delta$-rounded version $I' = \langle [n], [m], V' \rangle$ of the given instance $I$. In particular, the instance $I'$ is a powers-of-$(1 + \delta)$ instance constructed by rounding up the valuations to the nearest integer power of $(1 + \delta)$. That is, for each agent $i \in [n]$ and each good $j \in [m]$, the valuation $v'_{i,j}$ for the instance $I'$ is defined as

$$v'_{i,j} := \begin{cases} (1 + \delta)^{\lceil \log_2 v_{i,j} \rceil} & \text{if } v_{i,j} > 0, \\ 0 & \text{if } v_{i,j} = 0. \end{cases}$$

Notice that $v_{i,j} \leq v'_{i,j} \leq (1 + \delta)v_{i,j}$ for each agent $i$ and each good $j$.

The remainder of this section is organised as follows: For the $\delta$-rounded instance $I'$, we will show in Lemma 10 that \textsc{Alg} returns (in polynomial time) an allocation $x$ that is $\varepsilon$-EF1 and fPO (for $I'$). Then, with the help of Lemmas 11 and 12, we will argue that for appropriately small $\varepsilon$ and $\delta$, the allocation $x$ is in fact EF1 and PO with respect to the original instance $I$ (Theorem 1).

**Lemma 10.** There exists an algorithm which, for any given fair division instance $I = \langle [n], [m], V \rangle$, runs in $O(\text{poly}(n, m, \frac{1}{\varepsilon}, \log \max_k v_k([m])))$ time, and finds a $\varepsilon$-EF1 and fPO allocation for $I'$, the $\varepsilon$-rounded version of $I$.

**Proof.** Let $I' = \langle [n], [m], V' \rangle$ be the $\varepsilon$-rounded version of $I$. Since $I'$ is a powers-of-$(1 + \varepsilon)$ instance, the result follows from Lemmas 2 and 3. \[\square\]

**Lemma 11.** Let $I = \langle [n], [m], V \rangle$ be a fair division instance, and let $\varepsilon \leq \frac{1}{2 \max_k v_k([m])}$. Then, an allocation $x$ is $\varepsilon$-EF1 with respect to the instance $I$ if and only if it is EF1 with respect to $I$.

**Proof.** Since $x$ is $\varepsilon$-EF1, we have that for every pair of agents $i, k \in [n]$, there exists a good $j \in x_k$ such that $(1 + \varepsilon) \cdot v_i(x_k) \geq v_i(x_k \setminus \{j\})$. By choice of $\varepsilon$, this means that $v_i(x_k \setminus \{j\}) - v_i(x_k) \leq \frac{1}{2}$. Since all valuations are integral under the instance $I$, it must be that $v_i(x_k \setminus \{j\}) - v_i(x_k) \leq 0$, which is the desired EF1 condition. \[\square\]

**Lemma 12.** Let $I = \langle [n], [m], V \rangle$ be a fair division instance, and let $\delta \leq \frac{1}{2 \max_k v_k([m])}$. Then, an allocation $x$ that is fPO for the instance $I'$ (the $\delta$-rounded version of $I$) must also be PO for the original instance $I$.
Proof. Let $\mathcal{I}' = ([n], [m], \mathcal{V}')$ denote the $\delta$-rounded version of $\mathcal{I}$. Since the allocation $x$ is fPO for the instance $\mathcal{I}'$, we know from the Second Welfare Theorem for Fisher markets (Theorem 5) that there exists a price vector $p = (p_1, \ldots, p_m)$ and an endowment vector $e = (e_1, \ldots, e_n)$ such that $(x, p)$ is a market equilibrium for the market instance $([n], [m], \mathcal{V}', e)$. That is, for each agent $i \in [n]$, $e_i = p(x_i)$ and $x_i \subseteq \text{MBB}'_i$, where $\text{MBB}'_i = \{j \in [m] : v_{i,j}^p / p_j = \alpha_i'\}$ denotes the maximum bang-per-buck ratio of agent $i$ under the instance $\mathcal{I}'$, and $\alpha_i'$ denotes its maximum bang-per-buck ratio. As usual, we use $\alpha_i$ to denote the maximum bang-per-buck ratio of agent $i$ in the instance $\mathcal{I}$.

Under the endowments $e$ and prices $p$, let $o_i$ denote the maximum utility of agent $i$. Since $\alpha_i$ is an integer (since the valuations are integral), and it satisfies the following chain of inequalities:

1. $o_i \leq \alpha_i \cdot e_i$ (from the definition of $\alpha_i$ and additivity of valuations)
2. $o_i \leq \alpha_i' \cdot e_i$ (since $v_{i,j} \leq v_{i,j}'$ for each good $j \in [m]$)
3. $o_i \leq v_i'(x_i)$ (since $x_i \subseteq \text{MBB}'_i$ and $e_i = p(x_i)$)
4. $o_i \leq (1 + \delta) \cdot v_i(x_i)$ (since $v_{i,j}' \leq (1 + \delta) \cdot v_{i,j}$ for each good $j \in [m]$)
5. $o_i - v_i(x_i) \leq \delta \cdot v_i(x_i)$
6. $o_i - v_i(x_i) \leq 1/2$ (by choice of $\delta$)
7. $o_i - v_i(x_i) \leq 0$ (because of integrality).

Hence, $o_i \leq v_i(x_i)$. Besides, since $p(x_i) = e_i$, the bundle $x_i$ satisfies $x_i \in \arg \max_{S \subseteq [m]: p(S) \leq e_i} v_i(S)$. In other words, $x_i$ is a utility-maximizing bundle for agent $i$ under an integral market\footnote{We say that $(y, q)$ is an integral market equilibrium with respect to the market $([n], [m], \mathcal{V}, e)$ if (1) the market clearing conditions hold, and (2) for each agent $i \in [n]$, the bundle $y_i \subseteq [m]$ is a utility-maximizing bundle under the budget $e_i$ and price vector $q$. That is, $y_i \in \arg \max_{S \subseteq [m]: q(S) \leq e_i} v_i(S)$ for each agent $i \in [n]$.} with the price vector $p$ and budget $e_i$. Therefore, $(x, p)$ is an integral market equilibrium for the market instance $([n], [m], \mathcal{V}, e)$, and thus, by the First Welfare Theorem of integral markets (Theorem 2.3 in \cite{BNTC17}), $x$ is PO for the instance $\mathcal{I}$. 

\[\square\]

**Theorem 1.** Given any fair division instance with additive valuations, an allocation that is envy-free up to one good (EF1) and Pareto efficient (PO) can be found in $O(\text{poly}(m, n, \max_{i,j} v_{i,j}))$ time.

**Proof.** Let $\mathcal{I} = ([n], [m], \mathcal{V})$ be the given fair division instance, and let $\mathcal{I}' = ([n], [m], \mathcal{V}')$ be the $\varepsilon$-rounded version of $\mathcal{I}$. Let $\varepsilon = \frac{1}{6\max_{i,k} v_k([m])}$.

From Lemma 10, we know that an allocation $x$ that is $\varepsilon$-EF1 and fPO for $\mathcal{I}'$ can be found in $O(\text{poly}(m, n, \max_{i \in [n]} v_k([m])))$ time. Additionally, we know from Lemma 12 that $x$ is PO and Pareto efficient for $\mathcal{I}$. Therefore, it suffices to show that the allocation $x$ is EF1 for the original instance $\mathcal{I}$. The rest of the proof will show that $x$ is $3\varepsilon$-EF1 for $\mathcal{I}$. Then, from Lemma 11, it will follow that $x$ is EF1 for $\mathcal{I}$ since $3\varepsilon \leq \frac{1}{2\max_{i,k} v_k([m])}$.

Since $x$ is $\varepsilon$-EF1 for $\mathcal{I}'$, for each pair of agents $i, k \in [n]$, there exists a good $j \in x_k$ such that

$$(1 + \varepsilon) \cdot v_i'(x_i) \geq v_i'(x_i \setminus \{j\}).$$

Since $\mathcal{I}'$ is a $\varepsilon$-rounded version of $\mathcal{I}$, we have that $v_{i,j}' \leq (1 + \varepsilon) \cdot v_{i,j}$ for each good $j \in [m]$, i.e.,

$$(1 + \varepsilon)^2 \cdot v_i(x_i) \geq v_i'(x_i \setminus \{j\}).$$
Finally, since the valuations in $I'$ are a rounded up version of that in $I$, we have that $v_{i,j} \leq v'_{i,j}$ for each good $j \in [m]$, and thus

$$(1 + \varepsilon)^2 \cdot v_i(x_i) \geq v_i(x_k \setminus \{j\})$$

$$\Rightarrow (1 + 3\varepsilon) \cdot v_i(x_i) \geq v_i(x_k \setminus \{j\}),$$

as desired. The running time bound in the statement of the theorem can be obtained by observing that, by additivity of valuations, $\max_k v_k([m]) \leq m \cdot \max_{i,j} v_{i,j}$. \hfill $\square$

**Theorem 2.** Given any fair division instance with additive and bounded valuations, an allocation that is envy-free up to one good (EF1) and Pareto efficient (PO) can be found in polynomial time.

**Proof.** Instantiate the running time bound in Theorem 1 with $\max_{i,j} v_{i,j} \leq c$. \hfill $\square$

### 4.5 Existence Result: Proof of Theorem 3

**Theorem 3.** Given any fair division instance with additive valuations, there always exists an allocation that is envy-free up to one good (EF1) and fractionally Pareto efficient (fPO).

**Proof.** Given any fair division instance $I = \langle [n], [m], \mathcal{V} \rangle$, define $\delta_z = \frac{1}{m \cdot \max_k v_k([m])}$ for any natural number $z \in \mathbb{N}$. Write $I^z = \langle [n], [m], \mathcal{V}^z \rangle$ to denote the $\delta_z$-rounded version of $I$, with $\mathcal{V}^z = \{v_1^z, v_2^z, \ldots, v_n^z\}$ being the set of rounded valuations. Lemma 10 ensures that each rounded version $I^z$ admits an allocation $x^z$ which is $\delta_z$-EF1 and fPO.

First, we note that for all $z \geq 1$, allocation $x^z$ is an EF1 allocation for the instance $I$. This follows from the analysis of Theorem 1, wherein we showed that if an allocation $x$ is $\delta_z$-EF1 with respect to $I^z$ (the $\delta_z$-rounded version of $I$), then $x$ is EF1 for $I$, as long as $\delta_z \leq \frac{1}{6 \max_k v_k([m])}$.

Next, to complete the proof, we will show that in the sequence of EF1 allocations $(x^z)_{z \in \mathbb{N}}$ there exists one which is also fPO for $I$. In particular, by applying the Second Welfare Theorem of Fisher markets (Theorem 5), we get that for every $z$, there exists a price vector $p^z$ such that $x^z$ satisfies the MBB condition with respect to $p^z$ and the valuations in $I^z$. That is, $x^z_i \subseteq \text{MBB}_i^z = \left\{ j \in [m] : \frac{v_i^z(j)}{p_j^z} = \alpha_i^z \right\}$ for each agent $i \in [n]$; here, $\alpha_i^z$ is the maximum bang-per-buck ratio of agent $i$ with respect to the price vector $p^z$, i.e., $\alpha_i^z := \max_j \frac{v_i^z(j)}{p_j^z}$.

Note that if the outcome $(x^z, I^z, p^z)$ satisfies the MBB condition, then so does $(x^z, I^z, c \cdot p^z)$, for any positive $c$. In particular, by scaling $p^z$ by a factor of $c = \frac{1}{\max_j \frac{1}{p_j^z}}$, we ensure that the price vector $c \cdot p^z$ lies in $[0, 1]^m$ for all $z$. Therefore, without loss of generality, we will assume that $p^z$ is bounded for all $z$. In addition, since the number of allocations are finite, there exists an infinite sequence $z_1, z_2, \ldots$ such that $x^{z_a} = x^{z_b}$ for all $a, b \in \mathbb{N}$. Let $x$ denote the allocation $x^{z_a}$, i.e., $x := x^{z_a}$ for all $a \in \mathbb{N}$. We know from the Bolzano-Weierstrass theorem that there exists a subsequence of $(p^{z_1}, p^{z_2}, \ldots)$ and a price vector $p^*$ such that the subsequence converges to $p^*$, i.e., $\lim_{a \to \infty} p^{z_a} = p^*$. Let this subsequence be $(p^{z_1}, p^{z_2}, \ldots)$.

Overall, by a relabeling of the indices, we can guarantee the existence of a sequence of natural numbers $(z_1, z_2, \ldots)$ such that $x^{z_a} = x$ for all $a \in \mathbb{N}$ and $\lim_{a \to \infty} p^{z_a} = p^*$. In particular, we have that $x$ is an fPO allocation for each instance in the sequence $I^{z_1}, I^{z_2}, \ldots$ and hence, it satisfies the MBB condition in these instances as well.

Since $I^{z_a}$ is a $\delta_{z_a}$-rounded version of $I$, for any agent $i \in [n]$ and good $j \in [m]$ we have that $v_i(j) \leq v_i^{z_a}(j) \leq (1 + \delta_{z_a})v_i(j)$. Hence, from the sandwich theorem, we get that $v_i^{z_a}(j)$ converges to $v_i(j)$ as $a$ tends to infinity, $\lim_{a \to \infty} v_i^{z_a}(j) = v_i(j)$. 

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Using the fact that \( x \) satisfies the MBB condition with respect \((p^{z_a},I^{z_a})\), we will show that it satisfies the MBB condition for \((p^*,I)\) as well. Specifically, consider any agent \( i \in [n] \) along with any pair of goods \( j \in [m] \) and \( j' \in x_i \) such that \( v_i^{z_a}(j') \) is non-zero and, hence, \( v_i(j) \) is non-zero.\(^\text{11}\) Now, the MBB condition in \( I^{z_a} \) implies that

\[
\frac{p_j^{z_a}}{v_i^{z_a}(j')} \leq \frac{p_j^{z_a}}{v_i^{z_a}(j)}
\]

Taking limit \( a \) tending to infinity on both sides of the inequality, we get

\[
\lim_{a \to \infty} \frac{p_j^{z_a}}{v_i^{z_a}(j')} \leq \lim_{a \to \infty} \frac{p_j^{z_a}}{v_i^{z_a}(j)} \Rightarrow \frac{p_j^{z_a}}{v_i^{z_a}(j')} \leq \frac{p_j^{z_a}}{v_i^{z_a}(j)}
\]

Therefore, the allocation \( x \) satisfies the MBB condition with respect to the price vector \( p^* \) and the fair division instance \( I \). Hence, using the First Welfare Theorem (Proposition 1), we get that \( x \) is fPO for \( I \). This completes the proof of the theorem. \( \square \)

4.6 Nash Social Welfare Approximation: Proof of Theorem 4

This section proves that ALG provides a 1.45-approximation for the Nash social welfare maximization problem in polynomial time.

We begin by showing (in Lemma 13) that if the valuations of all the agents are identical, then any EF1 allocation satisfies the stated approximation bound. We will then use this result to prove Theorem 4.

**Lemma 13.** If all the agents in a fair division instance \((|n|,|m|,V)\) have identical, additive valuations (i.e., \( v_i = v_k \) for all \( i,k \in [n] \)), then any \( \varepsilon \)-EF1 allocation of the instance achieves an approximation factor of 1.45 for the corresponding Nash social welfare maximization problem.

**Proof.** Write \( x = (x_1, x_2, \ldots, x_n) \) to denote an \( \varepsilon \)-EF1 allocation of the given instance \( I \) in which \( v \) is the (identical and additive) valuation function of all the agents. Let \( \ell \) denote the value of the least valued bundle in \( x \), i.e., \( \ell := \min_{i \in [n]} v(x_i) \). By reindexing, we have that \( v(x_1) \geq v(x_2) \geq \ldots \geq v(x_n) = \ell \).

We will use \( g_i \) to denote a largest valued good in each bundle \( i \), i.e., for all \( i \in [n] \) write \( g_i \in \arg\max_{g \in x_i} v(g) \). Let \( B \) denote the set \( \{g_1, g_2, \ldots, g_{n-1}\} \). The fact that \( x \) is an \( \varepsilon \)-EF1 allocation implies that \( v(x_i \setminus \{g_i\}) \leq (1 + \varepsilon)\ell \) for all \( i \in [n] \). For a small enough constant \( \varepsilon \), the multiplicative factor of \((1 + \varepsilon)\) degrades the overall approximation by a small factor. Hence, for ease of presentation, we ignore it from further consideration and instead work with the following inequality

\[
v(x_i \setminus \{g_i\}) \leq \ell \quad \text{for all } i \in [n]. \quad (4)
\]

We will now consider partially-fractional allocations wherein only the goods in \( B \) have to be allocated integrally, and the remaining goods can be fractionally allocated. Formally, we will be considering fractional allocations \( y \in [0,1]^{n \times m} \) in which for each \( j \in B \) we have \( y_{i,j} \in \{0,1\} \) (for all \( i \)) and for the remaining goods (i.e., for \( j \in [m] \setminus B \) we have \( y_{i,j} \in [0,1] \), for all \( i \). Write \( \mathcal{F} \) to denote the set of all such partially-fractional allocations.

\(^{11}\)Since \( I^{z_a} \) is a \( \delta_{z_a} \)-rounded version of \( I \), \( v_i^{z_a}(j) \) is zero if and only if \( v_i(j) \) is zero.
Let \( \omega \) be an allocation \( \omega = (\omega_1, \ldots, \omega_n) \) which maximizes Nash social welfare over \( \mathcal{F} \). From Proposition 2 in Section A.3, we have an optimal \( \omega \) such that \( g_i \in \omega_i \) for all \( i \in [n-1] \) (by reindexing the bundles in \( \omega \) and property (i) in Proposition 2), and if \( v(\omega_i) > \min_{k \in [n]} v(\omega_k) \), then \( \omega_1 = 1 \) (property (ii) in Proposition 2). Note that all the integral allocations belong to \( \mathcal{F} \), hence \( \text{NW}(\omega) \geq \text{NW}(x^*) \), where \( x^* \) denotes the Nash optimal over integral allocations. Hence, to prove the lemma, it suffices to show that \( \frac{\text{NW}(x)}{1.43} \text{NW}(\omega) \).

Define \( \alpha := \min_{k \in [n]} \frac{v(\omega_k)}{\ell} \) and let \( H = \{ k \in [n] : v(x_k) > \alpha \ell \} \). We will now consider partially-fractional allocations wherein only (and all) the goods in \( x_H \) have to be allocated integrally, and the remaining goods can be fractionally allocated. Write \( \mathcal{F}' \) to denote the set of all such partially-fractional allocations.

We begin by constructing an allocation \( x' \in \mathcal{F}' \) (from the allocation \( x \)) such that \( \text{NW}(x') < \text{NW}(x) \). This helps in the analysis, since the ratio \( \frac{\text{NW}(x')}{\text{NW}(\omega)} \) turns out to be easier to estimate in terms of \( \alpha, \ell, \) and \( n \).

We initialize our allocation \( x' \leftarrow x \). While there exist two agents \( i, k \in [n] \) such that \( l < v(x'_i) < v(x'_k) < \alpha \ell \), we transfer goods of value \( \Delta := \min\{v(x'_i) - \ell, \alpha \ell - v(x'_k)\} \) from \( x'_i \) (the lesser valued bundle) to \( x'_k \) (the larger valued bundle). In particular, this transfer of goods ensures that the Nash social welfare does not increase. Also, this process terminates, because after every iteration of the while loop either \( v(x'_i') = \ell \) or \( v(x'_k') = \alpha \ell \), and hence at least one of these agents does not participate in future iterations of the while loop. Moreover, note that, \( x'_k = x_k \) for all \( k \in H \) (as agents in \( H \) do not participate in the while loop). This proves that \( x' \in \mathcal{F}' \).

Notice that there exists at most one agent \( s \) such that \( v(x'_s) \in (\ell, \alpha \ell) \). This is because if there were more than one such agent, the while loop would not have terminated. For every other agent \( k \in [n] \setminus H \), \( v(x_k) \) is either \( \ell \) or \( \alpha \ell \). For large enough \( n \), we can assume \( v(x'_k) = \ell \).

Let \( h = |\{ k \in [n] : v(x_k) > \alpha \ell \}| \) denote the cardinality of the set \( H \) (i.e., \( h = |H| \)). Let \( t = |\{ k \in [n] : v(x'_k) \geq \alpha \ell \}| \) denote the number of agents in the allocation \( x' \) with a valuation at least \( \alpha \ell \). This structure of \( x' \) gives us the following lower bound

\[
\text{NW}(x') \geq \left( \prod_{i=1}^h v(x_i) \times (\alpha \ell)^{(t-h)} \times \ell^{(n-t)} \right)^{1/n}.
\]

Using Proposition 2 for all \( k \in H \), we will argue that \( v(\omega_k) \leq v(x_k) \). Recall that the bundles in \( x \) and \( \omega \) are indexed such that \( g_k \in x_k \) and \( g_k \in \omega_k \). Suppose, for contradiction, that there exists a \( k \in H \) such that \( v(\omega_k) > v(x_k) \). In such a case, we have \( \{g_k\} \subseteq \omega_k \) and \( v(\omega_k) > \alpha \ell \); since, for all \( k \in H \), \( v(x_k) > \alpha \ell \). This contradicts property (ii) in Proposition 2, which requires that \( \omega_k = \{g_k\} \), since \( \omega_k \) is of value strictly greater than \( \alpha \ell = \min \omega v(\omega_a) \).

Once again using Proposition 2, we can show that \( v(\omega_k) = \alpha \ell \) for all \( k \in [n] \setminus H \). This follows from the fact that if \( v(\omega_k) > \alpha \ell = \min v(\omega_a) \), then—by property (ii) of Proposition 2—\( \omega_k \) must be a singleton; specifically, \( \omega_k = \{g_k\} \). Since \( \{g_k\} \subseteq x_k \) and \( v(x_k) \leq \alpha \ell \) for \( k \in [n] \setminus H \), we get a contradiction that \( v(\omega_k) \leq \alpha \ell \).

These observations imply that Nash social welfare of \( \omega \) is upper bounded as follows

\[
\text{NW}(\omega) \leq \left( \prod_{i=1}^h v(x_i) \times (\alpha \ell)^{(n-h)} \right)^{1/n}.
\]
From Equation (4) and from the fact that \( x'_k = x_k \) for all \( k \in H \), we have \( v(x'_k \setminus \{g_k\}) = v(x_k \setminus \{g_k\}) \leq \ell \) for all \( k \in H \). Hence, we can upper bound the value of all goods, besides the \( h \) goods in \( \{g_k \mid k \in H\} \):

\[
\sum_{i \in H} v(x'_i \setminus \{g_i\}) + \sum_{i \in [n] \setminus H} v(x'_i) \leq h\ell + \alpha\ell(t - h) + \ell(n - t).
\]

Next we derive a lower bound for this total value by considering \( \omega \). Note that, all the goods in \( \{g_k \mid k \in H\} \) are integrally allocated in \( \omega \). Therefore, there are at least \( n - h \) bundles in \( \omega \) that do not contain any good from \( \{g_k \mid k \in H\} \). The cumulative value of these bundles is at least \( \alpha\ell(n - h) \); since, \( \min_{a \in [n]} v(\omega_a) = \alpha\ell \). Using Equation (5) we get

\[
\alpha\ell(n - h) \leq h\ell + \alpha\ell(t - h) + \ell(n - t).
\]

Therefore, \( t \geq (1 - \frac{1}{\alpha}) n \), i.e., \( (n - t) \leq \frac{n}{\alpha} \). Recall that \( \text{NW}(x) \geq \text{NW}(x') \). Hence,

\[
\frac{\text{NW}(x)}{\text{NW}(\omega)} \geq \frac{\text{NW}(x')}{\text{NW}(\omega)} \geq \left( \frac{\prod_{i=1}^{h} v(x_i) \times (\alpha\ell)^{(t-h)} \times e^{(n-t)}}{(\prod_{i=1}^{n} v(x_i) \times (\alpha\ell)^{n-h})} \right)^{1/n} = \alpha^{-\frac{n-t}{n}} \geq \alpha^{-\frac{1}{\alpha}} \geq e^{-1/\varepsilon} \approx 1.45.^{12}
\]

This completes the proof of the lemma.

The proof of Theorem 4 relies on transforming a general fair division instance into one which has identical valuations and showing that the Nash social welfare of the allocation returned by \( \text{ALG} \) is preserved in this transformation.

**Theorem 4.** There exists a polynomial-time 1.45-approximation algorithm for the Nash social welfare maximization problem.

**Proof.** For an instance \( \langle [n], [m], \mathcal{V} \rangle \), write \( z(\mathbf{q}) \) to denote the allocation (the price vector) returned by \( \text{ALG} \) executed over the \( \varepsilon \)-rounded version of \( \mathcal{I} \); here, parameter \( \varepsilon \) is set to a small constant. We will establish the stated approximation guarantee by showing that allocation \( z \) satisfies \( \text{NW}(z) \geq \frac{1}{1.45} \max_{y \in \mathcal{X}} \text{NW}(y) \).

As shown in Lemma 10, the allocation \( z \) is \( \varepsilon \)-pEF1 with respect to \( \mathbf{q} \), and it satisfies \( z_i \subseteq \text{MBB}_i \) for each agent \( i \in [n] \). Moreover, such an allocation can be found in polynomial time.

Let \( \alpha_i \) denote the maximum bang-per-buck ratio of agent \( i \) with respect to \( \mathbf{q} \). Note that if we scale each agent’s valuation by \( \frac{1}{\alpha_i} \neq 0 \)—i.e., set \( v^*_i,j = \frac{1}{\alpha_i} v_i,j \) for all \( i \) and \( j \)—then the Nash social welfare of any allocation, \( y \), in the scaled instance is \( \left( \prod_{i=1}^{n} v_i(y_i) \right)^{1/n} \) times the Nash social welfare of \( y \) in the original instance. We will show that \( z \) achieves an approximation factor of 1.45 in the scaled instance, and hence obtain the stated approximation guarantee.

Write \( \omega \) to denote an (integral) allocation that maximizes the Nash social welfare (i.e., \( \omega \in \arg \max_{y \in \mathcal{X}} (\prod_{i=1}^{n} v_i(y_i))^{1/n} \)) in the given instance. Note that \( \omega \) is Nash optimal in the scaled instance as well. Also, for each agent \( i \), the scaled valuations \( v^*_i,j \) satisfy the following inequalities: \( v^*_i,j = q_j \) for all \( j \in \text{MBB}_i \), and \( v^*_i,j < q_j \) for all \( j \notin \text{MBB}_i \). Therefore, for any agent \( i \), we have \( v'(z_i) = q(z_i) \) (since

\[\text{Note that the function } z^{1/s} \text{ with } z \geq 0 \text{ is maximized at } z = c.\]

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12Note that the function \( z^{1/s} \) with \( z \geq 0 \) is maximized at \( z = c \).
$z_i \subseteq \text{MBB}_i$ and $v'_i(\omega_i) \leq q(\omega_i)$. Consequently, the Nash social welfare of the computed allocation $z$ and the optimal allocation $\omega$ in the scaled instance satisfy

$$\left( \prod_{i=1}^{n} v'_i(z_i) \right)^{1/n} = \left( \prod_{i=1}^{n} q(z_i) \right)^{1/n}$$

(6)

and

$$\left( \prod_{i=1}^{n} v'_i(\omega_i) \right)^{1/n} \leq \left( \prod_{i=1}^{n} q(\omega_i) \right)^{1/n}.$$  

(7)

We will further transform the valuations to get a fair division instance in which the valuations of all the agents are identical: specifically, we consider an instance in which the valuation of every agent for a good $j$ is the price $q_j$, i.e., the valuation of any bundle is the price of the bundle.

Note that $z$ is $\varepsilon$-pEF1 (with respect to $q$) and, hence, it is $\varepsilon$-EF1 in this identical valuations instance. Therefore, Lemma 13 implies that

$$\left( \prod_{i=1}^{n} q(z_i) \right)^{1/n} \geq \frac{1}{1.45} \max_{y \in X} \left( \prod_{i=1}^{n} q(y_i) \right)^{1/n} \geq \frac{1}{1.45} \left( \prod_{i=1}^{n} q(\omega_i) \right)^{1/n} \geq \frac{1}{1.45} \left( \prod_{i=1}^{n} v'_i(\omega_i) \right)^{1/n}.$$  

(using Equation (7)).

The previous inequality and Equation (6) gives us an approximation factor of 1.45 under valuations $v'_i$:s:

$$\left( \prod_{i=1}^{n} v'_i(z_i) \right)^{1/n} \geq \frac{1}{1.45} \left( \prod_{i=1}^{n} v'_i(\omega_i) \right)^{1/n}.$$  

This establishes the stated approximation guarantee.

\[\square\]

References

[AGMV16] Nima Anari, Shayan Oveis Gharan, Tung Mai, and Vijay V Vazirani. Nash Social Welfare for Indivisible Items under Separable, Piecewise-Linear Concave Utilities. arXiv preprint arXiv:1612.05191, 2016. (Cited on page 3)

[AGSS17] Nima Anari, Shayan Oveis Gharan, Amin Saberi, and Mohit Singh. Nash Social Welfare, Matrix Permanent, and Stable Polynomials. In Proceedings of the 8th Conference on Innovations in Theoretical Computer Science (ITCS), 2017. (Cited on page 3)

[AI81] Kenneth Joseph Arrow and Michael D Intriligator. Handbook of Mathematical Economics. 1981. (Cited on page 2)
[AMNS15] Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi. Approximation Algorithms for Computing Maximin Share Allocations. In *International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 39–51. Springer, 2015. (Cited on page 3)

[BCE+16] Felix Brandt, Vincent Conitzer, Ulle Endriss, Ariel D Procaccia, and Jérôme Lang. *Handbook of Computational Social Choice*. Cambridge University Press, 2016. (Cited on page 1)

[BK17] Siddharth Barman and Sanath Kumar Krishnamurthy. Approximation Algorithms for Maximin Fair Division. In *Proceedings of the 2017 ACM Conference on Economics and Computation (EC)*, pages 647–664. ACM, 2017. (Cited on page 3)

[BL16] Sylvain Bouveret and Michel Lemaître. Characterizing Conflicts in Fair Division of Indivisible Goods Using a Scale of Criteria. *Autonomous Agents and Multi-Agent Systems*, 30(2):259–290, 2016. (Cited on page 2)

[BNTC17] Moshe Babaioff, Noam Nisan, and Inbal Talgam-Cohen. Competitive Equilibria with Indivisible Goods and Generic Budgets. *arXiv preprint arXiv:1703.08150*, 2017. (Cited on page 14)

[BS00] William C Brainard and Herbert Scarf. How to Compute Equilibrium Prices in 1891. Technical report, Cowles Foundation for Research in Economics, Yale University, 2000. (Cited on page 5)

[BT96] Steven J Brams and Alan D Taylor. *Fair Division: From Cake-cutting to Dispute Resolution*. Cambridge University Press, 1996. (Cited on page 1)

[Bud11] Eric Budish. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011. (Cited on pages 2, 3, and 4)

[CDG+17] Richard Cole, Nikhil Devanur, Vasilis Gkatzelis, Kamal Jain, Tung Mai, Vijay V Vazirani, and Sadra Yazdanbod. Convex Program Duality, Fisher Markets, and Nash Social Welfare. In *Proceedings of the 2017 ACM Conference on Economics and Computation (EC)*, pages 459–460. ACM, 2017. (Cited on page 3)

[CG15] Richard Cole and Vasilis Gkatzelis. Approximating the Nash Social Welfare with Indivisible Items. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing (STOC)*, pages 371–380. ACM, 2015. (Cited on pages 3 and 33)

[CKM+16] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. The Unreasonable Fairness of Maximum Nash Welfare. In *Proceedings of the 2016 ACM Conference on Economics and Computation (EC)*, pages 305–322. ACM, 2016. (Cited on pages 2, 3, and 33)

[DPSV08] Nikhil R Devanur, Christos H Papadimitriou, Amin Saberi, and Vijay V Vazirani. Market Equilibrium via a Primal–Dual Algorithm for a Convex Program. *Journal of the ACM (JACM)*, 55(5):22, 2008. (Cited on page 2)
[EG59] Edmund Eisenberg and David Gale. Consensus of Subjective Probabilities: The Parimutuel Method. *The Annals of Mathematical Statistics*, 30(1):165–168, 1959. (Cited on pages 2 and 28)

[GHS+17] Mohammad Ghodsi, MohammadTaghi HajiAghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair Allocation of Indivisible Goods: Improvement and Generalization. arXiv preprint arXiv:1704.00222, 2017. (Cited on page 3)

[Hal35] Philip Hall. On Representatives of Subsets. *Journal of the London Mathematical Society*, 1(1):26–30, 1935. (Cited on page 27)

[KPW16] David Kurokawa, Ariel D Procaccia, and Junxing Wang. When can the Maximin Share Guarantee be Guaranteed? In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence*, pages 523–529. AAAI Press, 2016. (Cited on pages 3 and 31)

[Lee17] Euiwoong Lee. APX-hardness of Maximizing Nash Social Welfare with Indivisible Items. *Information Processing Letters*, 122:17–20, 2017. (Cited on page 3)

[LMMS04] Richard J Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On Approximately Fair Allocations of Indivisible Goods. In *Proceedings of the 5th ACM conference on Electronic commerce (EC)*, pages 125–131. ACM, 2004. (Cited on page 2)

[Mou14] Hervé Moulin. *Cooperative Microeconomics: A Game-theoretic Introduction*. Princeton University Press, 2014. (Cited on page 1)

[NNRR14] Nhan-Tam Nguyen, Trung Thanh Nguyen, Magnus Roos, and Jörg Rothe. Computational Complexity and Approximability of Social Welfare Optimization in Multiagent Resource Allocation. *Autonomous agents and multi-agent systems*, 28(2):256–289, 2014. (Cited on page 2)

[OSB10] Abraham Othman, Tuomas Sandholm, and Eric Budish. Finding Approximate Competitive Equilibria: Efficient and Fair Course Allocation. In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 873–880, 2010. (Cited on page 2)

[PW14] Ariel D Procaccia and Junxing Wang. Fair Enough: Guaranteeing Approximate Maximin Shares. In *Proceedings of the fifteenth ACM conference on Economics and computation (EC)*, pages 675–692. ACM, 2014. (Cited on pages 2, 3, and 31)

[Ste48] Hugo Steinhaus. The Problem of Fair Division. *Econometrica*, 16:101–104, 1948. (Cited on page 1)

[Str80] Walter Stromquist. How to Cut a Cake Fairly. *The American Mathematical Monthly*, 87(8):640–644, 1980. (Cited on page 2)

[Var74] Hal R Varian. Equity, Envy, and Efficiency. *Journal of economic theory*, 9(1):63–91, 1974. (Cited on pages 2 and 3)
A Appendix-I

A.1 BuildHierarchy subroutine

This section provides a polynomial time subroutine for constructing the hierarchy of agent $i$.

Given an allocation $x$, we use $x^{-1}(j)$ to refer to the agent that owns the good $j$, i.e., $x^{-1}(j) = i$ if $j \in x_i$. Similarly, for a set of goods $G \subseteq [m]$, we write $x^{-1}(G)$ to refer to the set of all agents that own one or more goods in the set $G$.

\begin{algorithm}[H]
\centering
\textbf{Algorithm 2: BuildHierarchy}
\begin{algorithmic}
\STATE \textbf{Input:} An agent $i$, an allocation $x$, and a price vector $p$.
\STATE \textbf{Output:} A hierarchy structure $H_i = \{H^0_i, H^1_i, \ldots\}$ for agent $i$.
\STATE \\
\STATE // Initialization
\STATE 1 $H^0_i \leftarrow \{i\}$ \hfill /* $i$ is the Level-0 agent */
\STATE 2 $\ell \leftarrow 0$
\STATE \\
\STATE // Build the hierarchy level-wise
\STATE 3 \textbf{while} $H^\ell_i$ is non-empty \textbf{do}
\STATE 4 \hspace{1em} $H^{\ell+1}_i \leftarrow x^{-1}(\text{MBB}_H^\ell_i) \setminus \cup_{k=0}^{\ell} H^k_i$
\hspace{1em} /* Add to $H^{\ell+1}_i$ any agent that is not currently in the hierarchy and is reachable from some member of $H^\ell_i$ via an MBB-allocation edge. */
\STATE 5 \hspace{1em} $\ell \leftarrow \ell + 1$
\STATE 6 \textbf{end}
\STATE 7 \textbf{return} $H_i = \{H^0_i, H^1_i, \ldots\}$.
\end{algorithmic}
\end{algorithm}

A.2 Omitted Proofs from Section 4.3

A.2.1 Proof of Lemma 4

\textbf{Lemma 4.} The spending of the least spender cannot decrease, i.e., for each time step $t$, $p^t(x^t_i) \leq p^{t+1}(x^{t+1}_i)$.

\textbf{Proof.} The spending of the least spender can vary between any two consecutive time steps either due to a swap operation in Phase 2 or a price-rise in Phase 3. In Phase 2, the spending of the least spender can be affected via a swap operation only if it receives a good, which results in an increase in its spending. At the same time, the agent losing the good cannot become the new least spender due to the $\varepsilon$-path-violator condition. Similarly, in Phase 3, the spending of the least spender cannot decrease since the prices of the goods can only increase. Finally, any identity change, either in Phase 2 or in Phase 3, occurs only when the spending of the old least spender grows beyond that of the new one, once again implying that the spending of the least spender cannot decrease. 

A.2.2 Proof of Lemma 5

The proof of Lemma 5 relies on a series of intermediate results (Lemmas 14 to 17), which we state and prove below.

\textbf{Lemma 14.} Let $t$ and $t'$ be two time steps at which ALG performs a price-rise in Phase 3 such that $t' > t$. Then, $E_{t'} \subseteq E_t$. 

Proof. Recall that \( E_t \) denotes the set of all \( \epsilon \)-violators at time \( t \). Suppose, for contradiction, that there exists an agent \( k \in E_{t'} \setminus E_t \). Our proof will consist of two main arguments: First, that \( k \) can become an \( \epsilon \)-violator only during Phase 2. Second, that if there is a swap operation that turns \( k \) into an \( \epsilon \)-violator, then there is a subsequent swap operation that turns it back into a non-\( \epsilon \)-violator, contradicting the assumption that \( k \) is an \( \epsilon \)-violator at the beginning of the price-rise step at time \( t' \).

We will start by showing that a non-\( \epsilon \)-violator cannot turn into an \( \epsilon \)-violator during Phase 3. Recall that Phase 3 involves uniformly raising the prices of the goods owned by the members of the hierarchy. As a result, the spending of any agent outside the hierarchy remains unchanged, while the spending of any agent in the hierarchy grows multiplicatively by the same factor as that of the least spender. Thus, a non-\( \epsilon \)-violator prior to the price-rise continues to be so after it.

Next, let us assume that a non-\( \epsilon \)-violator \( k \) at level \( \ell \) in the hierarchy) becomes a \( \epsilon \)-violator after receiving a good \( j \) via a swap at time step \( t \) (where \( t < t' \)). Recall that a swap operation involves transferring a good from an agent at a higher level to another agent at a lower level in the hierarchy. Furthermore, \( \text{ALG} \) performs a swap for an agent at level \( \ell + 1 \) only if no agent in the levels 1, 2, \ldots, \( \ell \) is an \( \epsilon \)-path violator. Therefore, \( k \) cannot be an \( \epsilon \)-path violator before the swap, i.e., there exists a good \( j' \) on an alternating path of length \( 2\ell \) from \( i_\tau \) to \( k \) such that
\[
(1 + \epsilon) \cdot p^T(x^T_{k'}) \geq p^T(x^T_k \setminus \{j'\}).
\] (8)

Moreover, since \( k \) becomes an \( \epsilon \)-violator (and hence, an \( \epsilon \)-path violator) after receiving the good \( j \), we have that
\[
(1 + \epsilon) \cdot p^{T+1}(x^{T+1}_{\tau t+1}) < p^{T+1}(x^T_k \cup \{j\} \setminus \{j'\}).
\] (9)

Since neither the identity of the least spender nor the price-vector changes in this process, Equation (9) can be rewritten as
\[
(1 + \epsilon) \cdot p^T(x^T_{k'}) < p^T(x^T_k \cup \{j\} \setminus \{j'\}).
\] (10)

Notice that the swap involving the good \( j \) does not affect the alternating path from \( i_\tau \) to \( k \) via the good \( j' \), and therefore \( k \) continues to be at level \( \ell \). In fact, \( k \) is the only agent on level \( \ell \) or below that is an \( \epsilon \)-path-violator. Therefore, in a subsequent swap operation, the good \( j' \) will be taken away from \( k \), resulting in the allocation \( (x^T_k \cup \{j\} \setminus \{j'\}) \) for \( k \). After this step, agent \( k \) once again becomes a non-\( \epsilon \)-violator with respect to the good \( j \), providing the desired contradiction. \( \square \)

Lemma 15. Let \( t \) and \( t' \) be two time steps at which \( \text{ALG} \) performs a price-rise in Phase 3 such that \( t' > t \). Then, for any agent \( k \in E_{t'} \), \( x^T_{k'} \subseteq x^T_k \).

Proof. (Sketch) The proof is very similar to that of Lemma 14. Suppose, for contradiction, that there exists a good \( j \in x^T_{k'} \setminus x^T_k \) for some agent \( k \in E_{t'} \). Then, agent \( k \) must have acquired the good \( j \) via a swap operation at time \( t \) (between \( t \) and \( t' \)). This means that agent \( k \) cannot be an \( \epsilon \)-path violator at time \( t \), and thus cannot be an \( \epsilon \)-violator. By an argument similar to the one in the proof of Lemma 14, we can argue that \( k \) cannot be an \( \epsilon \)-violator at any subsequent price-rise event, contradicting the fact that \( k \in E_{t'} \). \( \square \)

Lemma 16. At the beginning of each price-rise step at time \( t \), \( E_t \cap \mathcal{H}_t = \emptyset \).

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Proof. Suppose, for contradiction, that there exists an agent $k \in E_t \cap \mathcal{H}_{i_t}$. Since $k \in E_t$, we have $(1 + \varepsilon) \cdot p^t(x_{i_t}^k) < p^t(x_{k}^t \setminus \{j\})$ for every good $j \in x_{k}^t$. Furthermore, since $k \in \mathcal{H}_{i_t}$, there must exist an alternating path from the least spender $i_t$ to $k$ that involves some good $j' \in x_{k}^t$. Thus, $k$ is also an $\varepsilon$-path violator, which means that ALG will perform a swap operation in Phase 2 at time $t$, as opposed to a price-rise operation.

Lemma 17. Let $t$ and $t'$ be two time steps at which ALG performs a price-rise such that $t' > t$. Then, for any agent $k \in E_{t'}$, we have $p^{t'}(x_{k}^{t'}) \leq p^t(x_{k}^t)$.

Proof. We already know from Lemma 15 that for each agent $k \in E_{t'}$, $x_{k}^{t'} \subseteq x_{k}^t$. Let $x_{E_{t'}} = \bigcup_{k \in E_t} x_{k}^{t'}$ denote the set of goods owned by the members of the set $E_{t'}$. Then, it suffices to show that the prices of the goods in $x_{E_{t'}}$ stay unchanged between $t$ and $t'$.

Suppose, for contradiction, that the price of some good $j \in x_{E_{t'}}$ increases as a result of a price-rise event at time $t$ (between $t$ and $t'$). Since price-rise only affects the goods owned by the members of the hierarchy, there must exist an agent $h \in \mathcal{H}_{i_t}$ such that $j \in x_{h,t}^t$. Then, by Lemma 16, we have that $h \notin E_T$. In other words, the good $j$ cannot be owned by an $\varepsilon$-violator at time $t$, i.e., $j \notin x_{E_{t'}}$. Along with Lemma 15, this means that the good $j$ cannot be owned by an $\varepsilon$-violator going forward (in particular, at time step $t'$), and hence $j \notin x_{E_{t'}}$, which is a contradiction.

We are now ready to prove Lemma 5.

Lemma 5. At the beginning of each Phase 3 step, say at time $t$, $p^t(x_{i_t}^t) \leq \max_{k \in [n]} v_k([m])$.

Proof. Notice that the set $E_t$ must be non-empty at the beginning of each price-rise step, otherwise the pair $(x, p)$ is already pEF1 and ALG will terminate before entering Phase 3. By definition of the set $E_t$, the spending of the least spender $i_t$ is at most the spending of any agent $h \in E_t$, i.e., $p^t(x_{i_t}^t) \leq p^t(x_{h}^t)$ for every $h \in E_t$. From Lemma 17, we have that the spending of any agent $h \in E_t$ at the beginning of the price-rise step at time $t$ is at most its initial spending (i.e., spending at the time of the first price-rise event). We already know that Phase 2 does not alter the prices of the goods, and that agent $h$ does not receive any good during the first run of Phase 2. Therefore, by way of setting the prices in Phase 1, the initial spending of agent $h$ is at most its valuation for the allocation immediately after Phase 1, i.e., $p^t(x_{h}^t) \leq v_h(x_{h}^0) \leq v_h([m])$, where $x^0$ denotes the allocation immediately after Phase 1. Therefore, we have $p^t(x_{i_t}^t) \leq \max_{k \in [n]} v_k([m])$, as desired.

A.2.3 Proof of Lemma 6

Let $\mathcal{H}_{i_t}$ denote the hierarchy at time step $t$. The level of an agent $h \in [n]$ at time $t$ is defined as

$$\text{level}(h, t) = \begin{cases} 
\ell & \text{if } h \in \mathcal{H}_{i_t}^{\ell} \\
n & \text{if } h \notin \mathcal{H}_{i_t}.
\end{cases}$$

Notice that for any agent $h \in \mathcal{H}_{i_t}$, $\text{level}(h, t) \leq n - 1$, since there are $n$ agents overall, and the least spender $i_t$ is assumed to be at level 0.

We say that a good $j \in [m]$ is critical for an agent $h \in \mathcal{H}_{i_t}^{\ell} \setminus \{i_t\}$ at time $t$ if $j \in x_{h}^t$ and there is an alternating path of length $2\ell$ from the least spender $i_t$ to $h$ that includes the good $j$. We denote the set of all critical goods for an agent $h$ at time $t$ by $G_{h,t}$. It is easy to see that the set $G_{h,t}$ is non-empty if and only if $h \in \mathcal{H}_{i_t}^{\ell} \setminus \{i_t\}$ for some $\ell \in \{1, 2, \ldots \}$. 

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Lemma 6. Alg can perform at most poly\((n, m)\) number of consecutive swap operations before either the identity of the least spender changes or a Phase 3 step occurs.

Proof. We will show via a potential function argument that each swap operation causes a drop of at least one in the value of a non-negative function that is bounded above by poly\((m, n)\), implying that Alg can perform at most poly\((m, n)\) such swaps.

Suppose that Alg performs a swap operation at time step \(t\) that involves transferring a good \(j \in [m]\) from an agent \(h_\ell\) at level \(\ell\) to another agent \(h_{\ell-1}\) at level \(\ell - 1\). Consider the function \(f(t)\) defined as below:

\[
f(t) = \sum_{h \in [n] \setminus \{i_t\}} m(n - \text{level}(h, t)) + |G_{h, t}|.
\]

Notice that \(f(t)\) is always non-negative, and its maximum value is at most poly\((n, m)\), since \(\text{level}(h, t) \geq 0\) and \(|G_{h, t}| \leq m\) for every agent \(h \in [n] \setminus \{i_t\}\). The rest of the proof will separately analyze the contribution of the agents \(h_\ell, h_{\ell-1}\), and any other agent \(h \in [n] \setminus \{i_t, h_{\ell-1}, h_\ell\}\) to the term \(f(t)\).

First, consider the agent \(h_\ell\). We will argue that either \(\text{level}(h_\ell, t + 1) = \text{level}(h_\ell, t)\) and \(G_{h_\ell, t+1} \subseteq G_{h_\ell, t}\), or \(\text{level}(h_\ell, t + 1) > \text{level}(h_\ell, t)\). Both conditions cause the potential term to drop by at least one (all else being constant): the first condition simply causes a drop of at least one in the term \(|G_{h, t}|\) without a change in the level term; the second term causes a drop of at least one in the level term, which is scaled by a factor of \(m\), and thus cannot be offset by any possible increase in the critical goods term. The reason why the two conditions hold is as follows: Notice that \(j\) is a critical good for \(h_\ell\) before the swap takes place, i.e., \(j \in G_{h, t}\). If \(j\) is the only good in the set \(G_{h, t}\), then, after the swap operation, either \(h_\ell\) moves to a higher level (than \(\ell\)) in the hierarchy, or it ceases to be a member of the hierarchy. Either way, we have that \(\text{level}(h_\ell, t + 1) > \text{level}(h_\ell, t)\). On the other hand, if \(j\) is not the only good in the set \(G_{h, t}\), then the level of \(h_\ell\) does not change upon the loss of the good \(j\). However, since \(h_\ell\) does not receive any good during the swap operation, and none of the previously non-critical goods become critical, we have that \(G_{h_\ell, t+1} \subseteq G_{h_\ell, t}\).

Next, consider the agent \(h_{\ell-1}\). We will show that \(\text{level}(h_{\ell-1}, t + 1) = \text{level}(h_{\ell-1}, t)\) and \(G_{h_{\ell-1}, t+1} = G_{h_{\ell-1}, t}\), implying that all else being constant, the potential cannot increase. Notice that a swap operation can (possibly) create one or more new alternating paths from the least spender \(i\) to the agent \(h_{\ell-1}\). We will argue that any such path must be of length at least \(2\ell\), and therefore it can neither lower the level nor make \(j\) a critical good for \(h_{\ell-1}\). Indeed, any new path to \(h_{\ell-1}\) must involve the good \(j\) and another agent \(h'\) such that \(j \in \text{MBB}_{h'}\). Since \(j\) was previously owned by \(h_\ell\), there must already exist an alternating path from \(i\) to \(h_\ell\) via \(h'\) before the swap takes place. Since the level of \(h_\ell\) before the swap is \(\ell\), any such path must be of length at least \(2(\ell - 1)\), implying that \(\text{level}(h', t) \geq \ell - 1\). This, in turn, means that any new path to \(h_{\ell-1}\) (via \(h'\)) must be of length at least \(2\ell\), which is strictly longer than the existing path of length \(2(\ell - 1)\) that defines its level.

Finally, we will show that for any agent \(h \in [n] \setminus \{i, h_{\ell-1}, h_\ell\}\), \(\text{level}(h, t + 1) \geq \text{level}(h, t)\) and \(G_{h, t+1} \subseteq G_{h, t}\). Observe that the only way in which the level of an agent \(h\) can decrease is via a newly created alternating path (at time \(t + 1\)) from \(i\) to \(h\) that involves the good \(j\) (this is because any path from \(i\) to \(h\) that does not involve the good \(j\) must also have existed prior to the swap operation). Any such path must consist of three parts: (1) a path from \(i\) to an agent \(h' \neq h_{\ell-1}\) such that \(j \in \text{MBB}_{h'}\), (2) an MBB-allocation edge from \(h'\) to \(h_{\ell-1}\), and (3) a path from \(h_{\ell-1}\) to \(h\). By a similar argument as in the previous paragraph, we know that parts (1) and (2) together constitute a path of length at least \(2\ell\) from \(i\) to \(h_{\ell-1}\). By contrast, since the level of \(h_{\ell-1}\) does not change, there must exist another path of
length $2(\ell - 1)$ from $i$ to $h_{\ell-1}$ before the swap operation. Along with part (3), this provides a strictly shorter route than any newly created path, implying that $\text{level}(h, t + 1) \geq \text{level}(h, t)$. Similarly, since the allocation of agent $h$ does not change at time $t$, and none of the previously non-critical goods become critical, we have $G_{h,t+1} \subseteq G_{h,t}$.

Thus, we have shown that for each agent $h \in [n] \setminus \{i\}$, $\text{level}(h, t + 1) \geq \text{level}(h, t)$ and $G_{h,t-1,t+1} \subseteq G_{h,t-1,t}$, and one of these implications holds strictly for the agent $h_{\ell}$. This forces the potential function to drop by at least one at each swap operation, giving us the desired result. □

A.3 A Result used in the Proof of Theorem 4

**Proposition 2.** Given $n$ agents with identical, additive valuation $v$ over a set of $m \geq n$ goods, along with a subset of goods $B \subset [m]$ of cardinality less than $n$, i.e., $|B| < n$. Let $F$ denote the set of partially-fractional allocations in which the goods in $B$ have to be integrally allocated and the remaining goods in $[m] \setminus B$ can be fractionally allocated. Then, there exists a fractional allocation $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ which maximizes the Nash social welfare over $F$ and satisfies the following properties:

(i) Each bundle in $\omega$ contains at most one good from $B$, i.e., for each $i \in [n]$ we have $|\{j \mid \omega_{i,j} > 0\} \cap B| \leq 1$.

(ii) If a bundle has value strictly greater than $\min_k v(\omega_k)$, then it contains exactly one integral good, i.e., if $v(\omega_i) > \min_k v(\omega_k)$, then $\omega_{i,j} = 1$ for some $j \in B$, and $\omega_{i,j'} = 0$ for all $j' \neq j$.

**Proof.** We will consider a fractional allocation $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ which maximizes the Nash social welfare over $P$, and show that it can be transformed (without decreasing NSW) into an allocation in $F$ which satisfies the stated properties.

Note that if a bundle of $\omega$, say $\omega_i$, contains a fractionally-allocatable good—i.e., $\omega_{i,j'} > 0$ for some $j' \notin B$—then its value is equal to $\mu := \min_k v(\omega_k)$. This observation follows from the fact that if $\omega_i$ contains a fractionally-allocatable good $j'$ and $v(\omega_i) > \mu$ then we can “redistribute” $j'$ between $\omega_i$ and $\arg \min_k v(\omega_k)$ to obtain another fractional allocation in $P$ with NSW strictly greater than that of $\omega$. This contradicts the optimality of $\omega$.

Hence, any bundle of value strictly greater than $\mu$ can only contain goods from $B$. We will show that such bundles are necessarily singletons, and hence establish property (ii). Say, for contradiction, that some bundle $\omega_i$ contains goods $a, b \in B$ (i.e., $\omega_{i,a} = \omega_{i,b} = 1$) and $v(\omega_i) > \mu$. Since $|B| < n$, there exists a bundle—say, $\omega_f$—in $\omega$ which does not contain any good from $B$. Given that $\omega_f$ consists only fractionally-allocatable goods $v(\omega_f) = \mu$. Now, we can assign good, say, $a$ to $\omega_f$ and redistribute the fractional goods from $\omega_f$ between this bundle and $\omega$ to obtain two new bundles, $\omega'_i$ and $\omega'_f$ of value strictly greater than $\mu$. This gives us a new fractional allocation in $P$ with NSW strictly greater than $\text{NSW}(\omega)$, thereby contradicting the optimality of $\omega$. Hence, by way of contradiction, we get that any bundle of value strictly greater than $\mu$ has to be a singleton.

Next we address property (i). Note that for bundles with value strictly greater than $\mu$ this property already holds. Hence, we need to consider bundles of value exactly equal to $\min_k v(\omega_k)$. Write $x_a$ to denote such a bundle which contains two goods from $B$. Since $|B| < n$, there exists a bundle, say $x_f$, which contains only fractionally-allocatable goods. Note that we can swap a good $g \in B$ from $x_a$ to $x_f$, and (fractionally) allocate goods of total value $v(g)$ back into $x_a$ to obtain another allocation in $F$ with the same NSW as $\omega$. Such a swap is always possible, since $v(g) \leq v(x_a) = \mu = v(x_f)$. Repeating this process at most $n$ times we can obtain a NSW maximizer which satisfies property (i). □
A.4 Corner Cases

In Section 4.3, we showed two results—namely, Lemmas 8 and 9—that provide running time bounds for Phase 2 and Phase 3 by showing that the spending of the least spender increases by some multiplicative factor in every polynomial number of steps. These results implicitly assume that the spending of the least spender is non-zero to begin with. In this section, we show that this assumption holds without loss of generality.

Our reasoning will depend on whether or not a given fair division instance is a Hall’s violator [Hal35]. We will show that if an instance satisfies Hall’s condition (i.e., is not a Hall’s violator), then the spending of the least spender (in Alg) becomes non-zero in $O(n^2)$ steps. If, on the other hand, the instance is a Hall’s violator, then we can break down its analysis into analysing a smaller instance that satisfies Hall’s condition and a trivial instance.

Consider any fair division instance $I = ([n], [m], V)$. Write $G = ([n], [m], E)$ to denote the unweighted bipartite graph between the set of agents and the set of goods such that $E = \{(i, j)|v_{i,j} > 0\}$. The instance $I$ is said to be a Hall’s violator if there exists a set of agents $T \subseteq [n]$ that together (positively) value at most $|T| - 1$ goods, i.e., the set $T$ violates Hall’s condition in the graph $G$. We call the set $T$ a maximal Hall’s violator if no strict superset of $T$ is a Hall’s violator.

We will first show that an instance can be checked for Hall’s condition in $O(\text{poly}(n, m))$ time.

Checking an instance for Hall’s condition Let $M$ be a maximum (unweighted) matching of $G$. A given instance $I$ is a Hall’s violator if and only if there exists an agent that is unmatched under $M$. The matching $M$ can be computed in $O(\text{poly}(n, m))$ time.

Instances that satisfy Hall’s condition Our first result in this section (Lemma 18) pertains to instances that satisfy Hall’s condition.

**Lemma 18.** Let $I = ([n], [m], V)$ be an input instance to Alg that satisfies powers-of-$r$ and Hall’s conditions. Then, at each time step after the first $O(n^2)$ steps, the spending of each agent under Alg is strictly greater than zero.

**Proof.** Observe that once the spending of an agent becomes non-zero during the run of Alg, it can never become zero again. This is because for the spending of an agent to drop back to zero, it must lose its last good via a swap operation in Phase 2. However, such an exchange is disallowed by the $\varepsilon$-path-violator condition. Therefore, it suffices to show that the spending of each agent under Alg becomes non-zero after the first $O(n^2)$ steps.

Let agent $i$ be the least spender at the end of Phase 1 of Alg. Assume, without loss of generality, that the spending of agent $i$ at the end of Phase 1 is zero (otherwise the lemma follows immediately). We will show that after $O(n)$ steps, the spending of agent $i$ strictly exceeds zero. The desired running time bound of $O(n^2)$ then follows via a similar argument for the other agents.

Our proof for the above claim consists of case analysis for whether or not at the end of Phase 1, there exists some agent in the hierarchy $H_i$ that owns two or more goods. Suppose there exists an agent $k \in H_i$ that owns two or more goods (if there are multiple such agents, tie-break in favor of agents at a lower level in $H_i$, and then according to a pre-specified lexicographic ordering). Then, $k$ must be an $\varepsilon$-violator, and therefore also an $\varepsilon$-path-violator (along some alternating path $P$). Additionally, by the choice of agent $k$, no agent at a lower level is an $\varepsilon$-path-violator. As a result, $k$ must lose a good under a swap operation in Phase 2 to its predecessor along path $P$, who acquires
two goods as a result, and becomes the new \( \varepsilon \)-path-violator. This series of swaps continues for \( O(n) \) steps, and ends with the least spender receiving a new good.

Next, suppose that each agent in \( \mathcal{H}_i \) owns exactly one good. Since the given instance \( \mathcal{I} \) satisfies Hall’s condition, and agent \( i \) does not yet own any good, there must exist an agent \( k \notin \mathcal{H}_i \) that owns two or more goods. We will show that such an agent must get added to the hierarchy in \( O(n) \) steps. Then, by the above argument, in further \( O(n) \) steps, agent \( i \) must receive a good that takes its spending strictly above zero.

Since each agent in the hierarchy owns exactly one good, there are no \( \varepsilon \)-path-violators in \( \mathcal{H}_i \), and \( \text{ALG} \) proceeds directly to Phase 3. Once again, since the least spender does not own any good, its spending cannot change as a result of price-rise, and therefore a new agent gets added to the hierarchy. If this agent has more than one good, then the lemma follows. Otherwise, the price-rise step is repeated. Therefore, after \( O(n) \) such steps, an agent with two or more goods must get added to the hierarchy, as desired.

Instances that violate Hall’s condition If the given instance \( \mathcal{I} = \langle [n], [m], V \rangle \) is a Hall’s violator, then we will first find a maximum (cardinality) matching \( M \) in \( \mathcal{G} \). Write \( A \subseteq [n] \) to denote the set of agents that get matched in \( M \) and note that the instance \( \mathcal{I}' = \langle A, [m], V' \rangle \) satisfies Hall’s condition; here \( V' \) is the set of valuations of the agents in \( A \). Also, the maximality of \( M \) ensures that for every good \( j \in [m] \), there exists \( i \in A \) such that \( v_{i,j} > 0 \). We will show that the allocation obtained by applying \( \text{ALG} \) on \( \mathcal{I}' \) gives us an allocation which is EF1 and \( \text{fPO} \) for \( \mathcal{I} \).

In particular, write \( x \) and \( p \) to denote the allocation and price vector returned by \( \text{ALG} \) on \( \mathcal{I}' \). For the remaining agents \( i \in [n] \setminus A \), we will set \( x_i = \emptyset \). By construction, \( x \) satisfies the EF1 condition for the set of agents \( A \). Hence, to show that \( x \) is EF1 for \( \mathcal{I} \) we need to show that this condition also holds for the agents in \( [n] \setminus A \). Say, for contradiction, that there exists an agent \( b \in [n] \setminus A \) that EF1 envies \( a \in A \). This happens if and only if \( a \) is allocated two or more goods valued by \( b \), i.e., iff \( |x_a \cap \{ g \mid v_{b,g} > 0 \}| \geq 2 \). We will show that this contradicts the fact that \( M \) is a maximum matching, and hence prove that \( x \) is an EF1 allocation.

Since \( \mathcal{I}' \) satisfies Hall’s condition, via Lemma 18, we have that each agent in \( A \) has non-zero spending under \( (x, p) \), i.e., for each agent \( i \in A \) we have \( |x_i| \geq 1 \). Consider a new matching \( M' \) (in \( \mathcal{G} \)) wherein \( i \in A \setminus \{ a \} \) is matched to a good in \( x_i \), \( a \) is matched to good, say \( g_a \in x_a \), and \( b \) is also matched to a good from the non-empty set \( (x_a \setminus \{ a \}) \cap \{ g \mid v_{b,g} > 0 \} \). We get a contradiction that the size of the matching \( M' \) is strictly greater than the size of \( M \). Hence, \( x \) is guaranteed to be an EF1 allocation for \( \mathcal{I} \).

Finally, to show that \( x \) is \( \text{fPO} \) for \( \mathcal{I} \) note that we can set the endowments of all the agents in \( [n] \setminus A \) to be zero and get that the market equilibrium conditions are satisfied by \( (x, p) \) for \( \mathcal{I} \). That is, \( x \) can be shown to be the equilibrium outcome of some Fisher market, and thus, from Proposition 1, \( x \) must be \( \text{fPO} \) for \( \mathcal{I} \).

A.5 Additional Market Preliminaries

Eisenberg-Gale program The convex program of Eisenberg and Gale [EG59] is known to characterize the equilibria of the Fisher market.
maximize \( \sum_{i=1}^{n} e_i \cdot \log(u_i) \)
subject to \( u_i = \sum_{j=1}^{m} v_{i,j} x_{i,j} \quad \forall i \in [n] \),
\( \sum_{i=1}^{n} x_{i,j} \leq 1 \quad \forall j \in [m] \), and
\( x_{i,j} \geq 0 \quad \forall i \in [n] \) and \( j \in [m] \).

**KKT conditions and maximum bang per buck** We will now describe the KKT conditions for the Eisenberg-Gale program. Let \( p_j \) denote the Lagrangian variable corresponding to the constraint \( \sum_{i=1}^{n} x_{i,j} \leq 1 \). Then,

1. **Dual feasibility**: \( p_j \geq 0 \) for each \( j \in [m] \).

2. **Complementary slackness**:
   - (a) For each \( j \in [m] \),
     \[ p_j > 0 \implies \sum_{i=1}^{n} x_{i,j} = 1. \]
   - (b) For each \( i \in [n] \) and \( j \in [m] \),
     \[ \frac{v_{i,j}}{p_j} \leq \frac{\sum_{j=1}^{m} v_{i,j} x_{i,j}}{e_i}. \]
   - (c) For each \( i \in [n] \) and \( j \in [m] \),
     \[ x_{i,j} > 0 \implies \frac{v_{i,j}}{p_j} = \frac{\sum_{j=1}^{m} v_{i,j} x_{i,j}}{e_i}. \]

Under the assumption that each good has an interested buyer (i.e., for each good \( j \), \( v_{i,j} > 0 \) for some buyer \( i \in [n] \)) and each buyer is interested in some good (i.e., for each buyer \( i \in [n] \), \( v_{i,j} > 0 \) for some good \( j \in [m] \)), it can be verified that optimal solutions to the Eisenberg-Gale program characterize the market equilibria of the corresponding Fisher market.

### A.6 Second Welfare Theorem for Fisher Markets

**Theorem 5.** Let \( \mathcal{I} = ([n], [m], \mathcal{V}) \) be an instance of the fair division problem, and let \( \mathbf{x} \) be a fractionally Pareto efficient (fPO) allocation for \( \mathcal{I} \). Then, there is a price vector \( \mathbf{p} = (p_1, \ldots, p_m) \) and an endowment vector \( \mathbf{e} = (e_1, \ldots, e_n) \) such that \( (\mathbf{x}, \mathbf{p}) \) is a market equilibrium for the market instance \( ([n], [m], \mathcal{V}, \mathbf{e}) \).

**Proof.** Our proof relies on the formulation of a linear program that characterizes the set of all fPO allocations with respect to a given set of utilities. The market equilibrium conditions then follow from linear programming duality and complementary slackness conditions.

Let \( \mathbf{u} = (u_1, \ldots, u_n) \) denote the utility vector induced by the given fPO allocation \( \mathbf{x} \). That is,
\[ u_i = \sum_{j=1}^{m} v_{i,j} x_{i,j} \] for each \( i \in [n] \). Consider the linear program given by Equation (11):

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{n} \sum_{j=1}^{m} v_{i,j} y_{i,j} \\
\text{subject to} & \quad \sum_{j=1}^{m} v_{i,j} y_{i,j} \geq u_i \quad \forall \ i \in [n], \\
& \quad \sum_{i=1}^{n} y_{i,j} \leq 1 \quad \forall \ j \in [m], \text{ and} \\
& \quad y_{i,j} \geq 0 \quad \forall \ i \in [n] \text{ and } j \in [m].
\end{align*}
\]

(11)

A feasible solution to Equation (11) is a fractional allocation \( y \in \mathcal{F}_{\text{Frac}} \) that provides each buyer \( i \in [n] \) with utility at least \( u_i \). Notice that the objective in Equation (11) is precisely the sum of utilities of all buyers under \( y \). Since the allocation \( x \) is \( \text{fPO} \), we know that the optimal objective value must be equal to \( \sum_{i=1}^{n} u_i \). Hence, \( x \) is a primal optimal solution.

The dual of the program in Equation (11) is given by Equation (12) below:

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} u_i d_i \\
\text{subject to} & \quad p_j - d_i v_{i,j} \geq v_{i,j} \quad \forall \ i \in [n] \text{ and } j \in [m], \\
& \quad d_i \geq 0 \quad \forall \ i \in [n], \text{ and} \\
& \quad p_j \geq 0 \quad \forall \ j \in [m].
\end{align*}
\]

(12)

The dual variables \( d_i \) and \( p_j \) correspond to the first and second primal constraints respectively.

Let \((x, p^*, d^*)\) be a tuple of optimal primal and dual solutions given a utility vector \( u \) induced by the \( \text{fPO} \) allocation \( x \). As discussed above, the optimal primal objective must be \( \sum_{i=1}^{n} u_i \). Therefore, by strong duality,

\[
\sum_{i=1}^{n} u_i = \sum_{j=1}^{m} p_j^* - \sum_{i=1}^{n} u_i d_i^*. \tag{13}
\]

Furthermore, dual feasibility implies that for every \( i \in [n] \) and \( j \in [m] \),

\[
\begin{align*}
p_j^* - d_i^* v_{i,j} \geq v_{i,j} & \quad \Rightarrow \quad p_j^* \geq (1 + d_i^*) v_{i,j} \\
& \quad \Rightarrow \quad 1 + d_i^* \leq \frac{p_j^*}{v_{i,j}} \\
& \quad \Rightarrow \quad 1 + d_i^* \leq \min_k \frac{p_{j,k}^*}{v_{i,k}}. \tag{14}
\end{align*}
\]

The final inequality in Equation (14) in fact holds with an equality (otherwise the dual objective can be improved without violating feasibility). Thus, for each \( i \in [n] \),

\[
1 + d_i^* = \min_k \frac{p_{j,k}^*}{v_{i,k}}. \tag{15}
\]

Let \( e = (e_1, \ldots, e_n) \) be an endowment vector defined as \( e_i = \sum_{j=1}^{m} p_j^* x_{i,j} \) for every \( i \in [n] \). We claim that \((x, p^*)\) is a market equilibrium for the market instance \( (\mathcal{N}, \mathcal{M}, V, e) \). The proof of the claim follows from checking the market equilibrium conditions, as below:

- **Market clearing**: Recall that \( p_j \) is the dual variable corresponding to the primal constraint \( \sum_{i=1}^{n} y_{i,j} \leq 1 \). Hence, by complementary slackness, for each \( j \in [m] \), either \( p_j^* = 0 \) or \( \sum_{i=1}^{n} x_{i,j} = 1 \), which is precisely the market clearing condition.
• **Budget exhaustion**: This follows from the choice of the endowment vector \( e \).

• **MBB allocation**: Notice that \( y_{i,j} \) is the primal variable corresponding to the dual constraint \( p_j - d_i v_{i,j} \geq v_{i,j} \). Therefore, by complementary slackness, we have that

\[
\begin{align*}
    x_{i,j} > 0 \Rightarrow & \quad p_j^* - d_i^* v_{i,j} = v_{i,j} \\
\Rightarrow & \quad p_j^* = (1 + d_i^*) v_{i,j} \\
\Rightarrow & \quad \frac{v_{i,j}}{p_j^*} = \max_k \frac{v_{i,k}}{p_k} \quad \text{(from Equation (15))},
\end{align*}
\]

which gives the MBB allocation condition.

We have therefore shown the existence of a price vector \( p^* \) and an endowment vector \( e^* \) such that \((x, p)\) is a market equilibrium for the market instance \( \langle [n], [m], V, e \rangle \), as desired. \( \square \)

# B Appendix-II

## B.1 Utility Maximization Does Not Imply MBB Condition in Fisher Markets

Consider a Fisher market instance with two buyers and three goods. The initial endowments are 130 for buyer 1 and 50 for buyer 2. The valuations of the buyers are as follows: \( v_{1,1} = 100, v_{1,2} = 50, v_{1,3} = 1, v_{2,1} = 1, v_{2,2} = 99, \) and \( v_{2,3} = 100 \).

Let \( p = (p_1, p_2, p_3) \) be a price vector, where \( p_1 = 70, p_2 = 60, \) and \( p_3 = 50 \). The bang per buck ratios are given by \( \alpha_{1,1} = \frac{100}{70}, \alpha_{1,2} = \frac{50}{60}, \alpha_{1,3} = \frac{1}{70}, \alpha_{2,1} = \frac{50}{99}, \) and \( \alpha_{2,2} = \frac{60}{99}, \) and \( \alpha_{2,3} = \frac{100}{99} \).

Thus, MBB\(_1\) = \{\( g_1 \)\} and MBB\(_2\) = \{\( g_3 \)\}. Consider an allocation \( x \in \mathcal{X}_{\text{Frac}} \) given by \( x_1 = \{g_1, g_2\} \) and \( x_2 = \{g_3\} \). Notice that \( x \) does not satisfy the maximum bang per buck allocation condition. However, the pair \((x, p)\) is utility maximizing under the given budget constraints.

## B.2 Any EF1 Allocation is \( \frac{1}{n} \)-approximately Maximin Fair

**Maximin share** Let \( S \subseteq [m] \) be any subset of goods, and let \( \mathcal{X}_S \) denote the set of all \( n \)-partitions of \( S \). Then, the \( n \)-maximin share of an agent \( i \in [n] \) is defined as

\[
\mu^n_i(S) = \max_{(x_1, \ldots, x_n) \in \mathcal{X}_S} \min_{k \in [n]} v_i(x_k).
\]

In words, the \( n \)-maximin share of an agent \( i \) is the minimum utility that it can guarantee for itself when it is allowed to partition the goods into \( n \) bundles, and it receives the least preferred bundle in that partition. We will use \( \mu_i \) instead of \( \mu^n_i(S) \) to denote the \( n \)-maximin share of agent \( i \) when the set \( S \) and the size of the partition is clear from the context. Further, we will use MMS to abbreviate the \( n \)-maximin share.

An allocation \( x \) is said to be maximin fair if the utility of each agent under \( x \) is at least its MMS value. That is, for each agent \( i \in [n], v_i(x_i) \geq \mu_i \). It is known that a maximin fair allocation may not always exist [PW14; KPW16]. As a result, approximate notions of MMS have been studied in the literature. An allocation \( x \) is said to be \( \alpha \)-approximately maximin fair (for \( 0 \leq \alpha \leq 1 \)) if for each agent \( i \in [n], v_i(x_i) \geq \alpha \cdot \mu_i \). We abbreviate this notion using \( \alpha \)-MMS.

**Lemma 19.** An EF1 allocation is \( \frac{1}{n} \)-approximately maximin fair.
Proof. Let \( \mathbf{x} \) be an EF1 allocation for a given instance \( \mathcal{I} = ([n], [m], \mathcal{V}) \). We will show that \( \nu_n(\mathbf{x}_n) \geq \frac{1}{n} \mu_n \), where \( \mu_n \) denotes the maximin share of agent \( n \) (the lemma will follow from a similar argument for the other agents). Let \( \mathbf{y} \) be an allocation such that \( \nu_n(\mathbf{y}_i) \geq \mu_n \) for all \( i \in [n] \) (such an allocation exists by definition of \( \mu_n \)). For each \( k \in [n-1] \), let \( j_k \) denote the favorite good of agent \( n \) in the bundle \( \mathbf{x}_k \), i.e., \( \nu_{n,j_k} = \max_{j \in \mathbf{x}_k} \nu_{n,j} \). Since \( \mathbf{x} \) is EF1, we have \( \nu_n(\mathbf{x}_k \setminus \{j_k\}) \leq \nu_n(\mathbf{x}_n) \). Further, by pigeonhole principle, there must exist a bundle \( \mathbf{y}_k \) in the allocation \( \mathbf{y} \) such that \( \mathbf{y}_k \cap \{j_1, j_2, \ldots, j_{n-1}\} = \emptyset \). Thus,

\[
\mu_n \leq \nu_n(\mathbf{y}_k) \\
\leq \nu_n([m] \setminus \{j_1, j_2, \ldots, j_{n-1}\}) \\
\leq \nu_n(\mathbf{x}_n) + \sum_{i=1}^{n-1} \nu_n(\mathbf{x}_k \setminus \{j_k\}) \\
\leq n \cdot \nu_n(\mathbf{x}_n),
\]

where the first inequality follows from the choice of \( \mathbf{y} \), the second follows from pigeonhole principle, the third follows from rewriting the set of goods, and fourth follows from EF1 property.

\( \square \)

### B.3 An Instance where a \( \left( \frac{1}{n} + \varepsilon \right) \)-MMS and fPO Allocation Does Not Exist

This section provides an example of a fair division instance where no fractionally Pareto efficient (fPO) allocation can be \( \left( \frac{1}{n} + \varepsilon \right) \)-approximately maximin fair for any \( \varepsilon > 0 \).

We define an instance \( \mathcal{I}^{n,c} = ([n], [m], \mathcal{V}) \) with \( n \) agents and \( m = 2n - 1 \) goods. The set of goods \([m]\) is divided into \((n-1)\) high-valued goods \( h_1, \ldots, h_{n-1} \) and \( n \) signature goods \( g_1, \ldots, g_n \). Each agent \( i \in [n] \) values a high-valued good \( h_j \) at \( v_{i,j} = c \). Each signature good \( g_j \) is valued by the agent with the same index \( j \in [n] \) at \( v_{i,j} = \frac{1}{n} - \delta \), where \( \delta \in \left(0, \frac{n\varepsilon}{(n-1)(1+n\varepsilon)}\right) \) is a pre-specified constant.

Let \( \mathbf{x} \) be any integral fPO allocation. The claim below asserts that there must exist an agent \( i \in [n] \) such that \( \mathbf{x}_i \subseteq \{g_i\} \).

**Claim 1.** Let \( \mathbf{x} \) be any integral fPO allocation with respect to the instance \( \mathcal{I}^{n,c} \). Then, there exists an agent \( i \in [n] \) such that \( \mathbf{x}_i \subseteq \{g_i\} \).

**Proof.** Observe that the number of high-valued goods is strictly smaller than the number of agents. Hence, some agent (say \( i_1 \)) must miss out on a high-valued good under the allocation \( \mathbf{x} \). If \( \mathbf{x}_{i_1} \subseteq \{g_i\} \), then the claim follows. So, let us assume that \( i_1 \) gets the signature good of some other agent \( i_2 \), i.e., \( g_i \in \mathbf{x}_{i_1} \). We will now show that the claim must hold for the agent \( i_2 \).

Since \( \mathbf{x} \) is fPO, we have \( g_{i_1} \in \text{MBB}_{i_1} \). In addition, from Theorem 5, we know that there exists a price vector \( \mathbf{p} \) such that \((\mathbf{x}, \mathbf{p})\) is a market equilibrium. Hence, the bang per buck ratio of agent \( i_1 \) for the good \( g_{i_2} \) is at least that for any other good \( j \). That is,

\[
\alpha_{i_1,i_2} \geq \alpha_{i_1,j} \text{ for every } j \in [m]. \quad (16)
\]

Furthermore, since \( v_{i_1,j} = v_{i_2,j} \) for every \( j \in [m] \setminus \{g_{i_1}, g_{i_2}\} \), we have that

\[
\alpha_{i_1,j} = \alpha_{i_2,j} \text{ for every } j \in [m] \setminus \{g_{i_1}, g_{i_2}\}. \quad (17)
\]

Next, since \( v_{i_2,i_2} > v_{i_1,i_2} \) by construction, we also have that \( \alpha_{i_2,i_2} > \alpha_{i_1,i_2} \). Along with Equations (16) and (17), this gives

\[
\alpha_{i_2,i_2} > \alpha_{i_1,i_2}
\]
\[ \geq \alpha_{i_1,j} \text{ for every } j \in [m] \quad \text{(Equation (16))} \]
\[ = \alpha_{i_2,j} \text{ for every } j \in [m] \setminus \{g_1, g_2\} \quad \text{(Equation (17))}. \]

Similarly, since \( v_{i_1,i_1} > v_{i_2,i_1} \), we have that \( \alpha_{i_1,i_1} > \alpha_{i_2,i_1} \). Combining this with the above implications, we have that \( \alpha_{i_1,j} > \alpha_{i_2,j} \text{ for every } j \in [m] \setminus \{g_1, g_2\} \). In other words, the MBB set of agent \( i_2 \) consists only of the good \( g_{i_2} \). Since \( x \) is an fPO allocation, we have that \( x_{i_2} \subseteq \text{MBB}_{i_2} = \{g_{i_2}\} \). Hence, the claim holds for the agent \( i_2 \).

Let us now consider the instance \( I^{n,n} \). Notice that the maximin share of any agent \( i \) for this instance is given by \( \mu_i = 1 - (n - 1)\delta \), and is realised for the partition where \( i \) receives the signature goods of all agents, while each of the remaining \( (n - 1) \) agents gets a high-valued good. Therefore, any allocation \( x \) that is \((\frac{1}{n} + \varepsilon)\)-approximately maximin fair must ensure that \( v_i(x_i) \geq (\frac{1}{n} + \varepsilon) \cdot (1 - (n - 1)\delta) \) for each \( i \in [n] \). By our choice of \( \delta \), this means that \( v_i(x_i) > \frac{1}{n} \) for each \( i \in [n] \). Claim 1 shows that under any fPO allocation, there is always some agent \( i \) such that \( v_i(x_i) \leq \frac{1}{n} \), leading to the desired implication.

### B.4 An Instance where a 1.44-approximate NSW and fPO Allocation Does Not Exist

Consider the fair division instance \( I^{3,3} \) as defined in Section B.3 with \( n = 3 \) and \( c \geq 1 \). Among all allocations that are fPO for this instance, let \( x \) denote the one with the largest Nash social welfare. From Claim 1 in Section B.3, we know that there exists some agent, say agent 3, such that \( x_3 \subseteq \{g_3\} \).

If \( x_3 = \emptyset \), then \( NW(x) = 0 \). However, since there exists an fPO allocation with non-zero Nash social welfare (namely the welfare maximizing allocation), we must have that \( x_3 \subseteq \{g_3\} \). Indeed, \( NW(x) = \left((c + \frac{1}{2}) \cdot (c + \frac{3}{2}) \cdot \frac{1}{3}\right)^\frac{1}{3} \) for the allocation \( x = (\{h_1, g_1\}, \{h_2, g_2\}, \{g_3\}) \).

It is easy to check that the allocation \( y = (\{h_1\}, \{h_2\}, \{g_1, g_2, g_3\}) \) is the Nash optimal allocation (without the fPO constraint), and \( NW(y) = (c^2 \cdot (1 - 2\delta))^\frac{1}{3} \approx c^{2/3} \) for small \( \delta \). Therefore,

\[
\frac{NW(y)}{NW(x)} \geq \left( \frac{c^2}{\left(\frac{1}{3}c + \frac{1}{3}\right)^2} \right)^\frac{1}{3} = 3^\frac{1}{3} \left( \frac{3c}{3c + 1} \right)^\frac{1}{3} \geq 1.44 \text{ for large } c.
\]

An allocation that is PO but not fPO Notice that the allocation \( y \) in the above example is Nash optimal, and hence, by the result of Caragiannis et al. [CKM⁺16], is Pareto efficient (PO). However, there is no agent that receives only a subset of its signature goods under \( y \). Therefore, from Claim 1 in Section B.3, \( y \) cannot be fPO.

### B.5 EF1 Allocations can be Highly Inefficient

Define an instance \( \langle [n], [m], V \rangle \) with \( m = n \) such that \( v_{i,j} = 1 \) if \( i = j \) and 0 otherwise. Consider an allocation \( x \) such that \( x_i = \{g_{i+1}\} \) for all \( i \in \{n - 1\} \) and \( x_n = \{g_1\} \). Clearly, \( x \) is an EF1 allocation. However, \( x \) is highly inefficient since each agent gets a valuation of zero.

### B.6 Every Rounding of Spending Restricted Outcome Violates EF1

In this section, we provide an example of a fair division instance where every rounding of the spending restricted equilibrium [CG15] violates EF1 condition. Our example doubles up as a counterexample for the rounding of CEEI outcomes as well.
Consider an instance with \( n = 5 \) agents and \( m = 7 \) goods. Let \( \{g_1, g_2, \ldots, g_7\} \) denote the set of goods, and let \( v_1, v_2, \ldots, v_5 \) denote the valuations.

|      | \( g_1 \) | \( g_2 \) | \( g_3 \) | \( g_4 \) | \( g_5 \) | \( g_6 \) | \( g_7 \) |
|------|---------|---------|---------|---------|---------|---------|---------|
| Agent 1 | 3/4     | 0       | 0       | 3/4     | 0       | 0       | 0       |
| Agent 2 | 0       | 3/4     | 0       | 3/4     | 0       | 0       | 0       |
| Agent 3 | 0       | 0       | 3/4     | 3/4     | 0       | 0       | 0       |
| Agent 4 | 0.7     | 0.7     | 0.7     | 2/3     | 0       |   2/3   |         |
| Agent 5 | 0.7     | 0.7     | 0.7     | 0.7     | 0       |   2/3   |   2/3   |

The unique CEEI price vector is given by \( p = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \). Moreover, there is a unique CEEI fractional allocation for this instance, namely \( y = (\{g_1, g_4\}, \{g_2, g_4\}, \{g_3, g_4\}, \{g_5, g_7\}, \{g_6, g_7\}) \). Since the price of each good is strictly less than 1, \( y \) is also the unique spending restricted outcome. Let \( x \) be any rounding of the fractional allocation \( y \). Then, there exists an agent \( i \in \{1, 2, 3\} \) and an agent \( k \in \{4, 5\} \) such that \( x_i = \{g_i, g_4\} \) and \( x_k = \{g_{k+1}\} \). Hence \( v_k(x_k) = \frac{2}{3} < 0.7 = v_k(x_i \setminus \{j\}) \) for any \( j \in x_i \), i.e., the allocation \( x \) is not EF1.

**B.7 Approximate NSW may not be EF1 or PO**

**Example 1.** Consider an instance \( \mathcal{I} = ([n], [m], V) \) with \( m = 2n \). For each agent \( i \in [n-1], v_{i,j} = 1/2 \) for each \( j \in [2n-2] \), and \( v_{i,(2n-1)} = v_{i,2n} = 0 \). Also, \( v_{n,j} = 1/2 \) for each good \( j \in [2n-2] \). Finally, let \( v_{n,(2n-1)} = 2^{-n} \) and \( v_{n,2n} = 1 - 2^{-n} \). The allocation \( x = (\{1,2\}, \{3,4\}, \ldots, \{(2n-1),2n\}) \) is Nash optimal, and \( NW(x) = 1 \). The allocation \( y = ((1,2,2n), \{3,4\}, \{5,6\}, \ldots, \{(2n-1)\}) \) is a 2-approximation to Nash social welfare since \( NW(y) = 1/2 \). However, the allocation \( y \) is not EF1, since agent \( n \) envies every other agent by more than up to one good. Similarly, \( y \) is not PO since the allocation \( x \) Pareto dominates \( y \).