Quantum Duality in Mathematical Finance

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Abstract

Mathematical finance explores the consistency relationships between the prices of securities imposed by elementary economic principles. Commonplace among these are the absence of arbitrage and the equivalence of expectation and price, both essentially algebraic constraints on the valuation map. The principles that govern pricing are here reviewed in the context of the stochastic and functional calculus of quantum processes. Framed in terms of the duality between states, the arbitrage-free valuation maps, and observables, the contractual settlements of securities, quantum groups are central to the approach. Translating the economic principles into this framework, a link is made between option pricing and von Neumann algebras that is illuminating in both directions. The essay concludes with the construction of an interest rate model from the irreducible representations of a semisimple Lie algebra, demonstrating its application in the pricing of European and Bermudan swaptions.

The data associated with a system is presented in this essay as a pair of complementary $\ast$-algebras, comprising the test states and observables that are combined in the investigation of the system. The state space $U$ and the observable space $\Omega$ are paired by a bilinear map:

$$\bullet : U \times \Omega \to \mathbb{C}$$

(1)

that sends the test state $z \in U$ and the test observable $a \in \Omega$ to their valuation $z \bullet a \in \mathbb{C}$. With the emphasis on empirical determination, these spaces are completed as locally convex topological spaces that include all the experiments uniquely identified by their test valuations.

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Figure 1: Visualise the pairing $z \cdot a$ of the state $z$ with the observable $a$ as the stacking of two boxes, where the lower box represents the state and the upper box represents the observable. The dual interpretation of the information model is generated by reversing this diagram.

While the definition is symmetric between these dual spaces, their naming indicates the distinct roles they play in the model of information. On one side, the $\ast$-algebra models the accumulation of states, and is a prerequisite for the development of the stochastic calculus. On the opposite side, the $\ast$-algebra models the product of observables, and is a prerequisite for the development of the functional calculus. The operations of the $\ast$-algebra – the unit, product and involution – are then interpreted according to their application.

| Observable: | Unit | Product | Involution | Antipode |
|-------------|------|---------|------------|----------|
|             | Unit Constant | Multiply | Complex Conjugate | $\ast$ = Reverse |
| State:      | Stasis | Accumulate | Conjugate-Reverse | |

These are the building blocks for the string diagrams of valuations, symbolising operations that act upwards on states and downwards on observables. Duality is the statement that these interpretations can be reversed, furnishing two inequivalent information models from the same pairing of $\ast$-algebras.

In the application to finance, the economy is empirically investigated by price testing: state is the economic model used to assess the present value of future cashflows and observable is the contractual termsheet that prescribes the settlement of the derivative security. Reducing the economic data to a single underlying price, the Black-Scholes model combines log-Brownian diffusion with the terminal payoff of the European call option. Extensions of this seminal model follow the same pattern, adding complexity in the variables of the economic state and the terms of the settlement observable. In each case, the state generates the equilibrium equation satisfied by the derivative price and the observable provides its boundary conditions, completing the specification of the price problem. Stochastic calculus accommodates the evolution of the economy over time and functional calculus supports the convex relationships among derivative payoffs. Understanding the duality between these two structures, and how this is utilised in pricing, is the main objective of this essay.
The prototypical example is the classical group, wherein test states are elements of the group and test observables are subsets of the group, respectively identified with economic events and the Arrow-Debreu securities contingent on them. Abstracting this structure, the rules of the quantum group extract the essential properties of classical groups that enable the stochastic and functional calculus. Applied to economies, founding principles can then be developed as axioms within a model of information established on purely algebraic grounds, encompassing the elementary economic assumptions that govern pricing such as the absence of arbitrage and the equivalence of expectation and price.

**Classical stochastic calculus**

Taking the states to be elements of a classical group, the operations that facilitate the stochastic calculus are defined as the following maps on states:

| Observable: | Unit | Product | Antipode |
|-------------|------|---------|----------|
| x           |      |         |          |
| y           |      |         |          |
| Delete      |      |         |          |
| Copy        |      |         |          |

| State:                      |
|-----------------------------|
| 1                           |
| Group Identity              |
| xy                          |
| Group Product               |
| $x^{-1}$                    |
| Group Inverse               |

where the unit state is the group identity, the product of two states is the group product, and the antipode of the state is the group inverse. In the finance application, the element of the group represents an economic event. The operations then enable the accumulation of events over time.

Stochastic calculus relies on the ability to switch between integral and differential perspectives for the evolution of the system. Over a discrete schedule, the evolution is equivalently described by its endpoints $(x_1, \ldots, x_n)$ or its increments $(y_1, \ldots, y_n)$, related by:

$$x_i = y_1 \cdots y_i \quad y_i = x_{i-1}^{-1} x_i$$

for $i = 1, \ldots, n$, where the path is initialised with $x_0 = 1$. The first map integrates the increments to generate the endpoints and the second map differentiates the endpoints to generate the increments. The operations of the group define the integrator and differentiator maps:

$$(y_1, y_2) \mapsto (y_1, y_1 y_2)$$

$$(x_1, x_2) \mapsto (x_1, x_1^{-1} x_2)$$
These maps are inverse to each other, a property that is derived from the group axioms in the string diagrams:

Of the three steps in this derivation, the first exploits associativity and the last exploits unitality. The derivation is then completed by using the inverse property in the middle step.

Integration and differentiation intermediate between observables modelled on the endpoints and states modelled on the increments, repeated over multiple time steps as the operations:

\[(y_i) \mapsto (y_1 \ldots y_i)\] \[\quad (x_i) \mapsto (x_{i-1}^{-1}x_i)\]

Valuation proceeds as the composition of operations that implement the settlement observable of the derivative with the action of the pricing state that computes its present value, interchanging along the way between the endpoint and increment representations for the evolution.

**Classical functional calculus**

Taking the observables to be scalar-valued functions on a classical set, the operations that facilitate the functional calculus are defined as the following maps on observables:

| Observable: | Unit | Product | Involution |
|-------------|------|---------|------------|
|             | 1 Unit Constant | ab Multiply | a Complex Conjugate |
where the unit observable is the unit constant, the product of two observables is
the pointwise product, and the involution of the observable is the pointwise com-
plex conjugate. In the finance application, the scalar-valued function represents
the value of a derivative security whose settlement is contingent on the events
in the set. The operations then enable convex relationships between derivative
payoffs.

Functional calculus expands the space of observables beyond polynomial
combinations via topological closure, utilising a background state to measure
convergence. Two remarkable theorems from functional analysis underpin this
expansion: the Gelfand-Naimark-Segal construction represents the observables
as operators; and the Radon-Nikodym theorem represents the states as opera-
tors that commute with the observables. Topological closure then generates the
empirical states and observables respectively as the commutant $U'$ and double-
commutant $U''$ of the test observables $U$. The $*$-algebras of states and observ-
ables are in this way identified with dual von Neumann algebras acting on a
Hilbert space.

These constructions have natural interpretation in mathematical finance,
emerging from the relationship between expectation and price. Guiding eco-
nomic principles relate the pricing state $z_c$, mapping observables to their present
values in currency $c$, to the economic state $z_e$, mapping observables to their ex-
pected values. Both these states are positive maps on observables: positivity
of the economic state ensures that the probabilities of events are positive, and
positivity of the pricing state ensures that the state prices of events are positive.
They are furthermore assumed to be equivalent, in the sense that the price of
an Arrow-Debreu security is non-zero if and only if the probability of the event
it indicates is non-zero. These properties of the pricing state are asserted in the
following principles.

**Principle of No-Arbitrage:** The pricing state does not permit arbitrage.

$$ z_c \cdot aa^* \geq 0 $$

(3)

**Principle of Equivalence:** The pricing and economic states are equivalent.

$$ z_e \cdot aa^* \neq 0 \iff z_c \cdot aa^* \neq 0 $$

(4)

A stronger principle that applies equivalence uniformly across observables is
considered in this essay:

$$ m_-(z_e \cdot aa^*) \leq z_c \cdot aa^* \leq m_+(z_e \cdot aa^*) $$

(5)

for scalars $m_-$ and $m_+$ satisfying $0 < m_- \leq m_+ < \infty$. State prices are thus
confined within a finite range of the corresponding probabilities. Implicit in
these expressions is a concept of positivity defined algebraically using only the
product and involution: the observable $b$ is positive if it can be written in the
form $b = aa^*$ for an observable $a$.

The economic state creates an inner product on the observables $a$ and $b$:

$$ \langle a|b \rangle := z_e \cdot ab^* $$

(6)

and the observable $b$ acts as an operator $[b]$ on the Hilbert space completion
of the resulting inner product space via multiplication on the right:

$$ (a|[b] := \langle ab |) $$

(7)
These definitions, which are rigidified by removing degeneracies from the observables and completing the operators, are the content of the Gelfand-Naimark-Segal construction. Strong equivalence of the economic and pricing states then derives the Radon-Nikodym theorem:

\[ z_c \bullet a = \langle 1 | [z_c] | a | 1 \rangle \]  

(8)

representing the state \( z_c \) as an operator \([z_c]\) on the Hilbert space that commutes with all the observables:

\[ [z_c][a] = [a][z_c] \]  

(9)

In this picture, the economic state is uniquely identified by its representation as the identity operator, \([z_e] = 1\). The operator \([z_c]\) that represents the pricing state scales probabilities to Arrow-Debreu prices, and positivity of this operator is sufficient to ensure the absence of arbitrage.

With these economic principles, states and observables are generated from a von Neumann algebra. The economic model, assumed a priori, creates a Hilbert space via the Gelfand-Naimark-Segal construction. Empirical states and observables are then identified, thanks to the Radon-Nikodym theorem, with operators in the commutant and double-commutant of the test observables. Representing the economic and pricing states in this way enables operator methods in mathematical finance.

**Noncommutative economics**

None of the constructions in the stochastic and functional calculus require commutativity of the product for either states or observables, and the novel contribution of this presentation is in the extension to noncommutativity. The founding principles of mathematical finance effectively identify the theories of arbitrage-free pricing models and von Neumann algebras, an observation that is instructive for both disciplines as noncommutativity impacts the price of convexity in ways that characterise the algebra.

The bulk of this essay is devoted to the development of the quantum framework for mathematical finance, justifying the approach on the grounds that it both contains and extends the classical framework. That the extension presents novel and useful behaviour is easily demonstrated in the discrete model, in which the observable is represented as a self-adjoint complex matrix with a finite set of real eigenvalues. The classical variant assumes that all observables commute, and without loss of generality are represented as diagonal matrices with their eigenvalues on the diagonal. The quantum variant drops this commutativity condition. Since the observable is evaluated by its eigenvalues – the roots of its characteristic polynomial – the transition from classical to quantum introduces complexity and thereby enriches the phenomenology.

As a toy example, consider the binomial model of two underlying payoffs \( P \) and \( R \) that both have eigenvalues 0 and 1. The option to receive one unit of \( R \) in exchange for \( k \geq 0 \) units of \( P \) has price:

\[ o[k] := \text{tr}[(R - kP)^+] \]  

(10)

in this model, computed as the sum of the positive eigenvalues of the bracketed matrix. The trace acts as the expectation operator, assuming for convenience
that the weights of the pricing distribution are absorbed in the payoff matrices. Classicality imposes commutativity of the matrices, leaving only two possible cases. The first is:

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

for which \( o[k] = (1 - k)^+ \), and the second is:

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

for which \( o[k] = 1 \). These are respectively the minimum and maximum possible prices for the options. The quantum model discovers prices between these bounds by rotating the diagonalising basis of \( R \):

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} \cos[\theta]^2 & \cos[\theta]\sin[\theta] \\ \cos[\theta]\sin[\theta] & \sin[\theta]^2 \end{bmatrix}
\]

for angles in the range \( 0 \leq \theta \leq \pi/2 \), leading to the option price:

\[
o[k] = \frac{1}{2}(1 - k) + \frac{1}{2}\sqrt{1 - 2k\cos[2\theta] + k^2}
\]

given by the only positive eigenvalue. The price is recreated as the expected value of the option expressed as a function of the swap rate \( s \) with the definition:

\[
\int_{s=0}^{\infty} (s - k)^+ \text{pdf}[s] \, ds := o[k]
\]

for the implied probability density \( \text{pdf}[s] \), deriving the expression:

\[
\text{cdlf}[s] = \frac{1}{2} \left( 1 + \frac{s - \cos[2\theta]}{\sqrt{1 - 2s\cos[2\theta] + s^2}} \right)
\]

for the implied cumulative density \( \text{cdlf}[s] \) on the half line \( 0 \leq s < \infty \).

As these graphs show for the case \( \theta = \pi/10 \), the quantum extension magically creates an implied pricing distribution with continuous support from a model that has only two discrete eigenstates.

Increasing the number of available eigenstates rapidly expands the range of implied pricing distributions generated by the quantum multinomial model. The examples on the next page calibrate the model with five eigenstates to four cases of the SABR model, and the graphs below are calibrations to the Black-Scholes
Figure 2: The implied cumulative density and (non-annualised) lognormal volatility in the quantum multinomial model with five eigenstates calibrated to four examples of the SABR model with parameters \((t, \sigma, \alpha, \beta, \rho)\) given by \((4, 0.25, 0.4, 1, 0), (4, 0.25, 0.6, 1, 0), (4, 0.25, 0.4, 0.5, 0), (4, 0.25, 0.4, 1, 0.5).
model. These examples demonstrate the novel and useful phenomenology that is unlocked by admitting noncommutativity in the information model.

![Cumulative Density (σ=0.5)](image1)

![Implied Volatility (σ=0.5)](image2)

![Cumulative Density (σ=1)](image3)

![Implied Volatility (σ=1)](image4)

| P Eigenvalue | R Eigenvalue | R Rotation |
|--------------|--------------|------------|
| 0.4986       | 0.7259       | -0.0324    |
| 0.4664       | 0.1523       | -0.0145    |
| 0.9112       | 0.1408       | -0.3284    |
| 0.0035       | 0.0076       | -0.2284    |
| 0.0033       | 0.0013       | -0.3284    |

All the components of the information model are located in the complementary *-algebraic structures of states and observables and their empirical pairing. In the following, these components are woven into a coherent whole in the definition of the quantum group, with duality as the unifying principle.

1 The information category

At its most abstract, the information model resides within a symmetric monoidal category whose objects represent systems and whose morphisms represent the processes that act between them. The category offers a convenient mathematical shorthand for the combinatorics of process building, capturing the commonalities of systems across many contexts. It is beyond the scope of this essay to explore their coherence rules in detail; instead, a diagrammatic language is presented, noting that sensible transformations performed on string diagrams identify with well-defined relations within the category.

In this setting, the principal lesson of category theory is that systems are best understood in terms of the algebraic properties of their connecting processes. As a minimal base for empirical investigation, the system is here associated with a collection of experiments, decomposing as the pairing of state and observable, whose measurements are the only source of knowledge regarding the system. This framework supports all the algebraic operations needed in the developments that follow, and includes within its dominion important dynamical systems such as statistical and quantum mechanics.
1.1 Process building

The binary operands of the symmetric monoidal category, concatenation $\otimes$ and composition $\circ$, are respectively the parallel and serial combination of processes, and the rules they adhere to mirror the properties of string diagrams. This is sufficient to support a model that includes a natural notion of state and observable along with a consistent definition of valuation.

The systems $K$ and $L$ are concatenated in the combined system $K \otimes L$, a method of system construction that is assumed to be unital and associative up to natural isomorphisms:

\[
1 \otimes K \cong K \cong K \otimes 1 \\
(K \otimes L) \otimes M \cong K \otimes (L \otimes M)
\]

where $1$ is the empty system, the monoidal unit of the category. The identifications in these expressions are implemented by the unitor and associator natural isomorphisms of the category, which for notational convenience are omitted in the following where they are understood from the context.

The variant of the category considered in this essay assumes that the hom-set of processes between the systems $K$ and $L$ decomposes as the (not necessarily disjoint) union of hom-subsets:

\[
\text{Hom}[K, L] = \bigcup_{\alpha \in \Gamma} \text{Hom}^\alpha[K, L] 
\]

parametrised by a signature group $\Gamma$ associated with the category. In this decomposition, the process $e$ in the hom-set $\text{Hom}^\alpha[K, L]$ is tagged with domain $\text{dom}[e] = K$, codomain $\text{cod}[e] = L$ and signature $\text{sig}[e] = \alpha$. This metadata determines the compatibility of processes for concatenation and composition, the two methods of process building in the category.

Introducing the diagrammatic language, the process $e$ with signature $\alpha$ between the systems $K$ and $L$ is visualised as a box:

\[
e \in \text{Hom}^\alpha[K, L]
\]

Systems are represented in the diagram as legs attached to the box; by convention, the leg is omitted when it represents the empty system. The primitive processes of the symmetric monoidal category are the identity $i$ and braid $\tau$, which enable the rearrangement of concatenated systems:

\[
\begin{array}{c|c}
\text{Identity} & \text{Braid} \\
\hline
1 \in \text{Hom}^1[K, K] & \tau \in \text{Hom}^1[K \otimes L, L \otimes K]
\end{array}
\]
Concatenation is the parallel combination of compatible processes, visualised in string diagrams as the horizontal stacking of boxes:

\[
\begin{array}{c}
  e \quad f \\
  K \\
  M
\end{array}
\]

\[ e \in \text{Hom}^\alpha[K, L], f \in \text{Hom}^\alpha[M, N] \mapsto e \otimes f \in \text{Hom}^\alpha[K \otimes M, L \otimes N] \quad (20) \]

Composition is the serial combination of compatible processes, visualised in string diagrams as the vertical stacking of boxes:

\[
\begin{array}{c}
  f \\
  e \\
  K \\
  M
\end{array}
\]

\[ e \in \text{Hom}^\alpha[K, L], f \in \text{Hom}^\beta[L, M] \mapsto e \circ f \in \text{Hom}^{\alpha \beta}[K, M] \quad (21) \]

Well-formed diagrams built from these combinations are determined, up to the natural isomorphisms of the category, according to the associativity rules of concatenation and composition and the interchange rule between them:

\[ (e \otimes f) \otimes g = e \otimes (f \otimes g) \]
\[ (e \circ f) \circ g = e \circ (f \circ g) \]
\[ (e \otimes f) \circ (g \otimes h) = (e \circ g) \otimes (f \circ h) \]
These stacking rules, encoding the universal characteristics of process building, are validated by the coherence axioms of the category. Combinations of the identity and braid generate arbitrary permutations of concatenated systems. Beyond these primitive processes, additional processes in the hom-sets express structural properties of the systems they connect.

Completing the diagrammatic language of the category, states and observables are identified respectively as the processes from and to the empty system:

\[
\begin{array}{ccc}
K & \rightarrow & z \\
\downarrow & & \downarrow \\
\rightarrow & a & \rightarrow \\
\downarrow & & \downarrow \\
z & \rightarrow & K
\end{array}
\]

\[z \in \text{Hom}^\alpha[1, K] \quad z \circ a \in \text{Hom}^\beta[1, 1] \quad a \in \text{Hom}^\beta[K, 1]\]

With these identifications, valuation becomes the composition of state and observable, interpreting the hom-set Hom[1, 1] as the scalars for the information model. Processes can then be interpreted as maps on states (the Schrödinger picture) or as maps on observables (the Heisenberg picture). These interpretations are consistent with valuation thanks to the associativity of composition:

\[
\begin{array}{ccc}
a & \rightarrow & e \\
\downarrow & & \downarrow \\
z & \rightarrow & z \\
\end{array}
\]

\[(z \circ e) \circ a = z \circ (e \circ a) \quad (22)\]

Any monoidal category generates in this way two models of information: one obtained by reading diagrams upwards, and one obtained by reading diagrams downwards. These dual interpretations exchange the roles of state and observable but evaluate within the same set of scalars. Additional processes in the hom-sets then dictate the nature of the scalars and the method of evaluation for states and observables.

### 1.2 Empirical processes

In proposing the category used throughout this essay as the locale for the information model, the foremost consideration is its empirical interpretation. To this end, the information category is defined to be the set of all bilinear pairings, with morphisms expressed as adjointable semilinear maps. The system is represented as a pair of vector spaces, the test states and test observables, combined in experiments whose measurements are given by the pairing. With this setup, a process is equivalently defined as a semilinear map between the state or observable spaces of its domain and codomain systems. Adhering to the empirical principle, two processes are then identified if they are indistinguishable by test experiments.
The information category is created from the field \( \mathbb{F} \) that provides its space of measurements. The automorphism group \( \text{aut}[\mathbb{F}] \) acts as signature group for the processes, in recognition of the equivalences in the measurement system presented by the automorphisms. Semilinearity as a concept is then defined for maps between the vector spaces over the field.

The system \( K \) is associated with the data \((\mathbb{U}[K], \mathbb{O}[K], \bullet)\), where the state space \( \mathbb{U}[K] \) and the observable space \( \mathbb{O}[K] \) are vector spaces, and the pairing:

\[
\bullet : \mathbb{U}[K] \times \mathbb{O}[K] \to \mathbb{F}
\]

is a bilinear map that combines the test state \( z \in \mathbb{U}[K] \) with the test observable \( a \in \mathbb{O}[K] \) to generate their valuation \( z \bullet a \in \mathbb{F} \). In this perspective, systems are identified with the experiments that can be performed on them, and all further constructions are defined in terms of these measurements.

The empty system \( 1 \) has data given by the scalars:

\[
\mathbb{U}[1] := \mathbb{F} \\
\mathbb{O}[1] := \mathbb{F}
\]

with the pairing on the empty system defined by:

\[
\lambda \bullet \mu := \lambda \mu
\]

for the test state \( \lambda \in \mathbb{U}[1] \) and the test observable \( \mu \in \mathbb{O}[1] \). This trivial definition for the pairing is, up to a scale factor, the only one supported by the state and observable spaces of the empty system. Valuations of this system have no information content.

The concatenated system \( K \otimes L \) that combines the systems \( K \) and \( L \) has data given by the algebraic tensor products:

\[
\mathbb{U}[K \otimes L] := \mathbb{U}[K] \otimes \mathbb{U}[L] \\
\mathbb{O}[K \otimes L] := \mathbb{O}[K] \otimes \mathbb{O}[L]
\]

with the pairing on the concatenated system defined by:

\[
(z \otimes y) \bullet (a \otimes b) := (z \bullet a)(y \bullet b)
\]

for the test states \( z \in \mathbb{U}[K] \) and \( y \in \mathbb{U}[L] \) and the test observables \( a \in \mathbb{O}[K] \) and \( b \in \mathbb{O}[L] \). Experiments in the concatenated system are linear combinations of products of experiments available in the component subsystems.

Imposing the empirical principle, the process is identified by its contractions with test states and observables, encapsulated in its valuation maps.

**Schrödinger picture:** The process \( \varepsilon \) acts on the state \( z \) and is paired with the observable \( a \) to generate its valuation \( z \circ \varepsilon \bullet a \).

**Heisenberg picture:** The process \( \varepsilon \) acts on the observable \( a \) and is paired with the state \( z \) to generate its valuation \( z \bullet \varepsilon \circ a \).

The process is defined by its association with the state map \( z \mapsto z \circ \varepsilon \) and the observable map \( a \mapsto \varepsilon \circ a \). For a process with signature automorphism \( \alpha \), the state map is assumed to be \( \alpha \)-linear and the observable map is assumed to be
\( \alpha^{-1} \)-linear, meaning that both maps commute with addition, and commutation with scalar multiplication takes the form:

\[
(\lambda z) \circ e = \lambda^{\alpha} (z \circ e) \quad \quad e \circ (\lambda a) = \lambda^{\alpha^{-1}} (e \circ a)
\]

(28)

where the action of the automorphism on scalars is denoted by the superscript. Consistency of the Schrödinger and Heisenberg pictures is then imposed by the following condition on the valuation maps:

\[
z \circ e \bullet a = (z \bullet e \circ a)^\alpha
\]

(29)

asserting that the state and observable maps are mutually adjoint.

Two constructions are considered for the state and observable maps. In the algebraic construction, available for any field \( F \), the maps are defined directly on the spaces of test states and observables. In the topological construction, available for the complex field \( \mathbb{C} \), the maps are defined on the spaces of empirical states and observables obtained as the topological completions of their test counterparts. While the second construction introduces technical challenges, discussed but not fully explored here, the expansion it presents covers a wider range of useful models.

**Algebraic construction**

The simplest way to generate the valuation maps of the process is by defining the state and observable maps on the test states and observables.

**Definition 1** (Process). *The process \( e \in \text{hom}^\alpha [K, L] \) is associated with the data \((U[e], \Omega[e])\), where the state map \( U[e] \) and the observable map \( \Omega[e] \) respectively \( \alpha \)-linear and \( \alpha^{-1} \)-linear maps:

\[
U[e] : z \in U[K] \mapsto z \circ e \in U[L]
\]

(30)

\[
\Omega[e] : a \in \Omega[L] \mapsto e \circ a \in \Omega[K]
\]

These semilinear maps are mutually adjoint, satisfying:

\[
z \circ e \bullet a = (z \bullet e \circ a)^\alpha
\]

(31)

for the test state \( z \in U[K] \) and the test observable \( a \in \Omega[L] \).

The universal constructions of the category – the identity and braid processes and the binary operands of concatenation and composition – are defined with respect to the test states and observables:

| State                  | Observable                  |
|------------------------|-----------------------------|
| **Identity:**          |                             |
| \( z \circ 1 := z \)   | \( 1 \circ a := a \)        |
| **Braid:**             |                             |
| \( (z \otimes y) \circ \tau := y \otimes z \) | \( \tau \circ (a \otimes b) := b \otimes a \) |
| **Concatenation:**     |                             |
| \( (z \otimes y) \circ (e \otimes f) := (e \otimes f) \circ (a \otimes b) := (z \otimes e) \otimes (y \otimes f) \) | \( (e \circ a) \otimes (f \circ b) \) |
| **Composition:**       |                             |
| \( z \circ (e \circ f) := (z \circ e) \circ f \) | \( (e \circ f) \circ a := e \circ (f \circ a) \) |

Direct application verifies these constructions satisfy the coherence rules of the symmetric monoidal category.
Automorphisms in the signature group and the data associated with systems are located in the hom-spaces. The automorphism $\alpha \in \Gamma$ is identified with the process $\alpha \in \text{hom}^\alpha[1, 1]$ via the definition:

$$\lambda \circ \alpha := \lambda^\alpha \quad \alpha \circ \lambda := \lambda^{\alpha^{-1}}$$

(32)

States and observables are identified with processes via bijections:

$$z \in U[K] \mapsto z \in \text{hom}^1[1, K]$$

(33)

$$a \in \mathfrak{n}[K] \mapsto a \in \text{hom}^1[K, 1]$$

implemented by the linear maps:

$$\lambda \circ z := \lambda z$$

$$z \circ a := z \bullet a =: z \circ a$$

$$\lambda a := a \circ \lambda$$

(34)

with the inverse identifications $z = 1 \circ z$ and $a = a \circ 1$. The spaces of states and observables of the system $K$ are thus identified respectively as the hom-spaces $U[K] \cong \text{hom}^1[1, K]$ and $\mathfrak{n}[K] \cong \text{hom}^1[K, 1]$.

**Topological construction**

Empirical distinctiveness is the core design principle behind the information category. States or observables that cannot be distinguished by their valuations are identified, an objective achieved by factoring out their empirical equivalence classes. If they are furthermore endowed with a sense of proximity, the definition can be expanded to include sequences whose valuations converge. The spaces associated with the system are thus extended topologically by including convergent sequences and reduced algebraically by quotienting with the empirical degeneracies.

Topology is introduced to the information category with the assignments:

- The field of scalars is $F = \mathbb{C}$, the complex scalars.
- The signature group is $\Gamma = \{1, *\}$, the identity and complex conjugation.

assumed from hereon. This field is a $C^*$-algebra in the norm topology generated by the modulus, and the signature group is its $\mathbb{Z}_2$-group of continuous automorphisms. The pairing then generates families of seminorms $\{p_a : a \in \mathfrak{n}[K]\}$ and $\{q_z : z \in U[K]\}$ that implement locally convex topologies on the spaces of states and observables associated with the system $K$:

$$p_a[z] := |z \bullet a| =: q_z[a]$$

(35)

The following notions of convergence for sequences of test states and observables are defined.

**Convergence (states):** The sequence of test states $(z_n)$ converges if the sequence of valuations $(z_n \bullet a)$ converges for all test observables $a$.

**Convergence (observables):** The sequence of test observables $(a_n)$ converges if the sequence of valuations $(z \bullet a_n)$ converges for all test states $z$. 

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Completing the test spaces in their locally convex topologies, the empirical state space \( \mathcal{U}[K] \) is defined to be the space of convergent sequences of test states modulo the sequences whose valuations converge to zero, and the empirical observable space \( \mathcal{P}[K] \) is defined to be the space of convergent sequences of test observables modulo the sequences whose valuations converge to zero.

The pairing of test states and observables does not necessarily extend to a pairing of empirical states and observables. There are, however, two partial extensions in the bilinear maps:

\[
\bullet : (z, a) \in \mathcal{U}[K] \times \mathcal{P}[K] \mapsto z \bullet a := \lim_{n \to \infty} z_n \bullet a_n \in \mathbb{C} \quad (36)
\]

\[
\bullet : (z, a) \in \mathcal{U}[K] \times \mathcal{P}[K] \mapsto z \bullet a := \lim_{n \to \infty} z \bullet a_n \in \mathbb{C} \quad (37)
\]

The limits on the right exist and are independent of the choice of representing sequences by construction for the empirical states and observables.

The process is associated with its state and observable maps. Topology enables an alternative definition for the process, extending these maps to the completed spaces of states and observables and imposing continuity.

**Definition 2 (Process).** The process \( \varepsilon \in \text{Hom}^{\alpha}[K, L] \) is associated with the data \((\mathcal{U}[\varepsilon], \mathcal{P}[\varepsilon])\), where the state map \( \mathcal{U}[\varepsilon] \) and the observable map \( \mathcal{P}[\varepsilon] \) are respectively \( \alpha \)-linear and \( \alpha^{-1} \)-linear continuous maps:

\[
\mathcal{U}[\varepsilon] : z \in \mathcal{U}[K] \mapsto z \circ \varepsilon \in \mathcal{U}[L] \\
\mathcal{P}[\varepsilon] : a \in \mathcal{P}[L] \mapsto \varepsilon \circ a \in \mathcal{P}[K] 
\]

These semilinear continuous maps are mutually adjoint, satisfying:

\[
z \circ \varepsilon \bullet a = (z \bullet \varepsilon \circ a)^\alpha \quad (38)
\]

for the test state \( z \in \mathcal{U}[K] \) and the test observable \( a \in \mathcal{P}[L] \).

Extending the action of the process reflects that, in practice, measurements extrapolate from refinements of the experimental apparatus, an operation that only makes sense if the process is continuous.

The following developments assume that the universal constructions of the category are well defined and satisfy the coherence rules of the symmetric monoidal category; in particular, that the extension from test to empirical states and observables is natural, as expressed in the commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{U}[L] \otimes \mathcal{U}[N] & \rightarrow & \mathcal{U}[L \otimes N] \\
\mathcal{U}[\varepsilon] \otimes \mathcal{P}[f] & \downarrow & \mathcal{U}[\varepsilon \otimes f] \\
\mathcal{U}[K] \otimes \mathcal{U}[M] & \rightarrow & \mathcal{U}[K \otimes M] \\
\mathcal{P}[\varepsilon] \otimes \mathcal{P}[f] & \downarrow & \mathcal{P}[\varepsilon \otimes f] \\
\mathcal{P}[K] \otimes \mathcal{P}[M] & \rightarrow & \mathcal{P}[K \otimes M]
\end{array}
\]

Technical challenges emerge from the gap between the tensor product of completed topological spaces and the completed tensor product of topological spaces. Continuity uniquely and naturally extends the tensor product of process maps, filling the gap with appropriate limits. Confirmation of this statement is beyond the scope of this essay, here noting only that topological completion is trivial when the state and observable spaces are finite dimensional.
1.3 Quantum groups

With the apparatus of the information category in place, the quantum group is defined by its extended lexicon of processes and the grammar they satisfy. The construction is motivated by the properties of classical groups, and is essentially equivalent to the definition of the classical group in the case when the product of observables is commutative. Removing this ugly asymmetry from the otherwise pristine duality between state and observable is the principal abstract achievement of the quantum group. The variance this generates has numerous applications, including as the origin of indeterminism in quantum mechanics and as a source of value for convexity in the pricing of derivative securities.

A system \( K \) is a quantum group if its hom-spaces include two families of processes acting as unit, product and involution on the states and observables:

| Unit | Product | Involution |
|------|---------|------------|
| \( \eta \in \text{Hom}^1[K,1] \) | \( \nabla \in \text{Hom}^1[K,K \otimes K] \) | \( * \in \text{Hom}^*[K,K] \) |
| \( u \in \text{Hom}^1[1,K] \) | \( \Delta \in \text{Hom}^1[K \otimes K,K] \) | \( * \in \text{Hom}^*[K,K] \) |

The axioms that relate these structural processes of the quantum group, presented in the figures on the next page, fall into three categories. The first set of axioms implies that the states and observables separately have the structure of \( * \)-algebras. By requiring them to commute, the second set of axioms combines the two \( * \)-algebras into a \( * \)-bialgebra. The final axiom then enables the creation of an antipode that generates the structure of a Hopf \( * \)-algebra on the system:

\[
\text{Antipode} = \quad =
\]

\[
s = * * \in \text{Hom}^1[K,K]
\]

These processes, with the rules they satisfy, are sufficient for the development of stochastic and functional calculus on the system.

The standard notation for the \( * \)-algebra expresses the product by juxtaposition and the involution by the \( \ast \) superscript:

| State | Observable |
|-------|------------|
| Unit: | 1 := 1 \circ u \quad 1 := \eta \circ 1 |
| Product: | ay := (a \otimes y) \circ \Delta \quad ab := \nabla \circ (a \otimes b) |
| Involution: | a^* := a \circ a \quad a^* := a \otimes a |

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Figure 3: The axioms of \(*\)-algebra: The states and observables separately support unit, product and involution processes that are unital, associative, antiautomorphic and involutive.

\[
\begin{align*}
\Delta &\circ \eta = \eta \\
\Delta &\circ \Delta = (\Delta \otimes \Delta) \circ \tau \\
\tau &\circ \Delta = \Delta \circ \tau
\end{align*}
\]

Figure 4: The axioms of \(*\)-bialgebra: Complementarity of the \(*\)-algebras of states and observables is expressed by these commutation relations between their structural processes.

\[
\begin{align*}
\Delta &\circ \eta = \eta \otimes \eta \\
\Delta &\circ \Delta = (\Delta \otimes \Delta) \circ \Delta \\
\tau &\circ \Delta = \Delta \circ \tau
\end{align*}
\]

Figure 5: The axiom of Hopf \(*\)-algebra: This axiom relates all the structural processes of the \(*\)-bialgebra. The antipode, defined as the composition of the two involutions, then satisfies the Hopf axiom.

\[
\Delta \circ (\Delta \otimes \Delta) = \eta \circ (\Delta \otimes \Delta) = \Delta \circ (\Delta \otimes \Delta)
\]
With this notation, the axioms of $*$-algebra are more familiarly expressed as:

| Axiom                  | State | Observable |
|------------------------|-------|------------|
| **Unital:**            | $1z = z = 1$ | $1a = a = a1$ |
| **Associative:**       | $(zy)x = z(yx)$ | $(ab)c = a(bc)$ |
| **Antiautomorphic:**   | $y^*z^* = (zy)^*$ | $b^*a^* = (ab)^*$ |
| **Involutive:**        | $z^{**} = z$ | $a^{**} = a$ |

While the unit, product and involution are intuitively denoted, the dual counit, coproduct and coinvolution are less easy to express by their actions on states and observables. The Sweedler notation, which can be effective in finite dimensions, does not extend well to the topologically completed case and will not be used in this essay.

Quantum groups are closed under concatenation with the definitions:

| Axiom                  | State | Observable |
|------------------------|-------|------------|
| **Unit:**              | $\eta := \eta \otimes \eta$ | $\eta := \eta \otimes \eta$ |
| **Product:**           | $\Delta := (\iota \otimes \tau \otimes \iota) \circ (\Delta \otimes \Delta)$ | $\nabla := (\nabla \otimes \nabla) \circ (\iota \otimes \tau \otimes \iota)$ |
| **Involution:**        | $* := * \otimes *$ | $* := * \otimes *$ |

where the process for the concatenated system, appearing on the left in these expressions, is defined on the right in terms of the corresponding processes for its component subsystems. Concatenated quantum groups thus inherit the structural processes from their component subsystems, enabling the extension of the functional calculus to combinations of independent systems, and supporting the development of the stochastic calculus on discrete schedules.

## 2 Mathematical finance

Mathematical finance studies the relationship between the economic state $z_e$ and the pricing state $z_c$, respectively quantifying the expected and present values of economic variables. Applied to undetermined observables, the economic state is inherently subjective, providing an assessment of economic conditions yet to be revealed. In contrast, the pricing state is marked to observed market prices, incorporating the market consensus and external factors such as liquidity and funding, and relegating subjective expectations to the unhedged convexities.

A sound platform for mathematical finance thus needs to establish the essential links between the economic and pricing states without being overly constraining. The constraints considered here prevent arbitrage and impose a loose relationship that allows both subjective and market expectations to influence price. Markets are not obliged to obey these principles, but they are satisfied to a reasonable approximation in normal conditions, and are a point of attraction even when markets are stressed.

These principles assert that the economic and pricing states are linear and positive, and are equivalent in the sense that they have the same space of null observables. Implemented in the information category, the structural processes of the quantum group are leveraged for their algebraic properties: linearity and positivity are expressed as algebraic relationships applied separately to the states; equivalence then relates the two states.
**Linearity:** The economic and pricing states are linear.

\[
z_e \bullet (\lambda a + \mu b) = \lambda (z_e \bullet a) + \mu (z_e \bullet b)
\]

\[
z_c \bullet (\lambda a + \mu b) = \lambda (z_c \bullet a) + \mu (z_c \bullet b)
\]

**Positivity:** The economic and pricing states are real and positive.

\[
z_e \bullet a^* = (z_e \bullet a)^*
\]

\[
z_e \bullet aa^* \geq 0
\]

\[
z_c \bullet a^* = (z_c \bullet a)^*
\]

\[
z_c \bullet aa^* \geq 0
\]

**Equivalence:** The economic and pricing states are equivalent.

\[
z_e \bullet aa^* \neq 0 \iff z_c \bullet aa^* \neq 0
\]

The quantum group supports these principles by furnishing the notions of linearity and positivity. For the economic state, these are the defining properties of expectation, enforcing positivity of variance. Supplemented with the normalisation \(z_e \bullet 1 = 1\), the principles imply the state acts as a probability measure on families of commuting observables. For the pricing state, linearity means that the price of a portfolio is the weighted sum of the prices of its constituents, and positivity means that positive observables have positive prices, thereby preventing arbitrage. Equivalence then requires that a positive observable has non-zero price if and only if it has non-zero expectation, so that financial value is not attributed to economic events deemed to be impossible.

### 2.1 Positivity and the functional calculus

In the context of the information category, positivity immediately leads to the Cauchy-Schwarz inequality, which in turn enables operator representations of the states and observables. Defining a copositive state to be a state that satisfies the linearity and positivity principles, the inequality strengthens the condition of positivity, and in so doing opens a bridge from algebra to topology.

**Theorem 1 (Cauchy-Schwarz inequality).** The copositive state \(z\) satisfies the inequality:

\[
|z \bullet ab^*|^2 \leq (z \bullet aa^*)(z \bullet bb^*)
\]

for the observables \(a\) and \(b\).

**Proof.** This elementary result is a consequence of the following inequality applied with appropriate choices for the scalars \(\alpha\) and \(\beta\):

\[
0 \leq z \bullet (\alpha a - \beta b)(\alpha a - \beta b)^*
\]

\[
= \alpha \alpha^* z \bullet aa^* - \alpha\beta^* z \bullet ab^* - \beta\alpha^* z \bullet ba^* + \beta \beta^* z \bullet bb^*
\]

exploiting the reality and positivity properties of the copositive state.

The Cauchy-Schwarz inequality initiates a chain of foundational results, taking in the Gelfand-Naimark-Segal construction and the Radon-Nikodym theorem, that terminates with the representation of states and observables as dual von Neumann algebras on a Hilbert space.
The Gelfand-Naimark-Segal construction

The observable $a$ is, by definition, a null observable for the copositive state $z_e$ if it satisfies $z_e \cdot aa^* = 0$. Thanks to the Cauchy-Schwarz inequality, this is equivalent to the condition $z_e \cdot ab^* = 0$ for all observables $b$. From this property, the set of null observables:

$$N_e = \{ a : z_e \cdot aa^* = 0 \}$$

is a right ideal of the space $\Omega$ of observables. The quotient space $\Omega/N_e$ is then equipped with an inner product:

$$\langle a + N_e | b + N_e \rangle := z_e \cdot ab^*$$

and the observable $b$ is represented as an operator $[b]$ on the quotient space with the definition:

$$\langle a + N_e | [b] := (ab + N_e)$$

In this string diagram, the inner product of the observables $a$ and $b$ in the state $z_e$ is given by:

$$z_e \cdot \nabla \circ ((a \circ \ast) \otimes (\ast \circ b))$$

using the identification of states and observables with processes.

The representation on the inner product space respects the $*$-algebras of observables and operators, with the $*$-homomorphism properties:

$$[1] = 1$$
$$[ab] = [a][b]$$
$$[a^*] = [a]^*$$

The first two properties are trivial, and the final property follows from:

$$\langle b + N_e | [a]c + N_e \rangle^* = (z_e \cdot bca^*)^* = z_e \cdot ca^*b^* = \langle c + N_e | [a^*] |b + N_e \rangle$$

The operator $[a]$ is thus adjointable with adjoint $[a^*]$. The vector $(1 + N_e)$ is generating for the representation, and the corresponding pure state satisfies:

$$z_e \cdot a = (1 + N_e) [a]1 + N_e$$

This expression reconstructs the copositive state from the representation.

**Topological completion.** This purely algebraic presentation is the algebraic content of the Gelfand-Naimark-Segal construction. Completing the inner product space $\Omega/N_e$ to the Hilbert space $H_e$, the operator $[a] : \Omega/N_e \to \Omega/N_e$ extends to an operator $[a] : H_e \to H_e$ if and only if it is bounded in the operator norm:

$$\| [a] \| = \sqrt{\sup \{ z_e \cdot caa^*c^* : z_e \cdot cc^* = 1 \}} < \infty$$

otherwise it is unbounded. Observables are represented on the Hilbert space as operators that may be unbounded but which are always defined on the common dense subspace $\Omega/N_e \subset H_e$. 

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The Radon-Nikodym theorem
When there are two copositive states \( z_e \) and \( z_c \), there are two inner product spaces \( \mathbb{N}/N_e \) and \( \mathbb{N}/N_c \) with inner products:

\[
\langle a + N_e | b + N_e \rangle = z_e \cdot ab^* \quad (52)
\]

\[
\langle a + N_c | b + N_c \rangle = z_c \cdot ab^* \quad (53)
\]

Now consider copositive states that satisfy the order relation:

\[
z_e \cdot aa^* \neq 0 \implies z_c \cdot aa^* \neq 0
\]

In words, if an observable is null for \( z_e \) then it is null for \( z_c \). The right ideals \( N_e \) and \( N_c \) satisfy \( N_e \subset N_c \) in this case, and there is a surjective operator \( m_c: \mathbb{N}/N_e \rightarrow \mathbb{N}/N_c \) defined by:

\[
\langle a + N_e | m_c := \langle a + N_c \rangle \quad (54)
\]

Assuming for now that this operator has an adjoint \( m_c^*: \mathbb{N}/N_c \rightarrow \mathbb{N}/N_e \), define the positive operator:

\[
[z_c] = m_c m_c^* \quad (55)
\]

on \( \mathbb{N}/N_e \). The operator satisfies:

\[
\langle b + N_e | [z_c][a] | c + N_e \rangle = z_c \cdot bac^* = \langle b + N_e | [z_c][a] | c + N_e \rangle \quad (56)
\]

The matrix elements of \([z_c][a]\) thus match the matrix elements of \([a][z_c]\). Both copositive states are reconstructed using the pure state associated with the unit in this representation:

\[
z_e \cdot a = (1 + N_e|[a]|1 + N_e)
\]

\[
z_c \cdot a = (1 + N_c|[z_c]|a|1 + N_c) = (1 + N_e|[a]|1 + N_e)
\]

where the operator \([z_c]\), the Radon-Nikodym weight of the copositive state \( z_c \) over the copositive state \( z_e \), commutes with the operators \([a]\) associated with observables, in the sense that the matrix elements of the commutator vanish.

Topological completion. This purely algebraic presentation is the algebraic content of the Radon-Nikodym theorem. The weak link in the argument is the assumption of adjointability, which relies on the existence of orthogonals. Completing the inner product spaces \( \mathbb{N}/N_e \) and \( \mathbb{N}/N_c \) respectively to the Hilbert spaces \( \mathcal{H}_e \) and \( \mathcal{H}_c \), the operator \( m_c: \mathbb{N}/N_e \rightarrow \mathbb{N}/N_c \) extends to an operator \( m_c: \mathcal{H}_e \rightarrow \mathcal{H}_c \) if and only if it is bounded in the operator norm:

\[
\|m_c\| = \sqrt{\sup \{z_c \cdot aa^*: z_e \cdot aa^* = 1\}} < \infty \quad (58)
\]

in which case the adjoint \( m_c^*: \mathcal{H}_e \rightarrow \mathcal{H}_e \) exists and the corresponding weight \([z_c]: \mathcal{H}_e \rightarrow \mathcal{H}_e \) is a well-defined bounded operator. The Radon-Nikodym theorem is thus satisfied on the completion if the order relation between the copositive states is strengthened to:

\[
z_e \cdot aa^* \leq \|m_c\|^2 (z_c \cdot aa^*) \quad (59)
\]
for the finite scalar $\|m_c\|$, guaranteeing the existence of the operator representation for the copositive state by imposing the order relation uniformly across observables.

Existence of the adjoint is the main reason for considering the Hilbert space completion. While the orthogonal may not exist in an inner product space, it is possible to get arbitrarily close to the orthogonal in the norm defined by the inner product. The completed space thus always contains orthogonals, a property that is exploited via the Riesz representation theorem to construct the adjoint of a bounded operator.

### 2.2 Time and the stochastic calculus

The progression from static to dynamic systems introduces the notion of time as a partially-ordered set. A discrete schedule is defined to be a finite ordered sequence of times:

$$P = (p_0, \ldots, p_n)$$

with $p_{i-1} \leq p_i$, where $n$ is the number of intervals in the schedule. Introduce the following notation for the set, length, start and end of the schedule:

$$\{P\} = \{p_0, \ldots, p_n\}$$

$$|P| = n$$

$$P_- = p_0$$

$$P_+ = p_n$$

Two consecutive schedules $P$ and $Q$ satisfying $P_+ = Q_-$ are concatenated to create a new schedule $P \lor Q$:

$$P \lor Q = (p_0, \ldots, p_{|P|} = q_0, \ldots, q_{|Q|})$$

with:

$$\{P \lor Q\} = \{P\} \cup \{Q\}$$

$$|P \lor Q| = |P| + |Q|$$

$$(P \lor Q)_- = P_-$$

$$(P \lor Q)_+ = Q_+$$

Refinement is then the order relation $P \supset Q$ for schedules $P$ and $Q$ satisfying:

$$\{P\} \supset \{Q\}$$

$$P_- = Q_-$$

$$P_+ = Q_+$$

The two schedules start and end at the same times, but the schedule $P$ refines the schedule $Q$ as all the intervals from $Q$ are mergers of intervals from $P$.

These definitions establish a bicategory structure on time, with 0-cells given by the times, 1-cells given by the schedules and 2-cells given by the refinement relations. Concatenation and composition are implemented on compatible refinements with the definitions:

$$(P \supset Q) \lor (R \supset S) := (P \lor R) \supset (Q \lor S)$$

$$(P \supset Q) \circ (Q \supset R) := P \supset R$$
In the following, the time category is restricted to refinements with coarse schedules $Q$ satisfying $|Q| \geq 1$. These refinements are generated via concatenation from the total refinements $P \supset (P, P)$ for each schedule $P$, equivalently via concatenation and composition from the elementary refinements:

\[
(p) \supset (p, p) \quad (p, q, r) \supset (p, r)
\]

and the trivial refinements $P \supset P$ for each schedule $P$. This removes refinements such as $(p, p) \supset (p)$ that do not map to algebraic operations.

The string diagram for the refinement links each interval in the coarse schedule with the intervals it merges in the refined schedule. As examples, this string diagram presents the concatenation of two refinements:

\[
(0, 1) \supset (0, 1, 1) \lor (1, 2, 3, 4) \supset (1, 4) = \end{equation}
\[
0, 1 \supset 0, 1, 1 \lor 1, 2, 3, 4 \supset 1, 4 = \end{equation}
\]

and this string diagram presents the composition of two refinements:

\[
((0, 1) \supset (0, 1, 2, 3, 4) \lor (1, 2, 3, 4) \supset (1, 4)) = ((0, 1, 2, 3, 4) \supset (0, 1, 1, 4))
\]

The resemblance with string diagrams of the state algebra is intentional. Refinement is a generalisation of algebra, and the dynamic system is implemented as a functor from this generalised algebra.

**Dynamic systems**

A dynamic system is defined to be a monoidal functor from the time category to the information category. The functors are themselves objects of a monoidal category whose morphisms are the natural transformations between them. In this category, the monoidal unit $1$ is the functor that maps all schedules to the empty system; a state of the dynamic system $K$ is then a natural transformation $z \in \text{Hom}^1[1, K]$ between the functors.

Expanding the definitions, the dynamic model is implemented by an accumulation functor $K$ from the time category to the information category and a natural transformation $z$ from the unit functor to the accumulation functor. The schedule $P$ maps to the system $K[P]$ and the state $z[P] \in \text{Hom}^1[1, K[P]]$, consistent with concatenation in the form:

\[
K[P] \otimes K[Q] = K[P \lor Q]
\]

\[
z[P] \otimes z[Q] = z[P \lor Q]
\]
for compatible schedules. The refinement $P \supset Q$ then maps to the process:

$$K[P \supset Q] \in \text{Hom}^1[K[P], K[Q]] \quad (68)$$

that accumulates the state for each merger of intervals in the refinement. Functoriality is expressed in the properties:

$$K[P \supset Q] \otimes K[R \supset S] = K[(P \lor R) \supset (Q \lor S)] \quad (69)$$

$$K[P \supset Q] \circ K[Q \supset R] = K[P \supset R]$$

for compatible refinements, and naturality is expressed in the property:

$$z[P] \circ K[P \supset Q] = z[Q] \quad (70)$$

for each refinement.

Functoriality means that string diagrams in the time category are replicated as string diagrams in the information category. The following string diagram is an example of the compatibility of state with concatenation:

applied to the concatenation $(0, 1) \lor (1, 2, 3, 4) = (0, 1, 2, 3, 4)$, and the next string diagram is an example of the naturality property:

applied to the refinement $(0, 1, 2, 3, 4) \supset (0, 1, 1, 4) = (0, 1, 1, 4)$.

**Accumulation and algebra**

The functorial properties of the accumulation processes associated with refinements generalise the algebraic properties of the state algebra. If the dynamic system is stationary, in the sense that the static system is the same for all intervals, the link between accumulation and algebra is explicit. Taking this static system to be a quantum group $K$, the dynamic system is implemented with the definition:

$$K[P] := \otimes^{\lvert P \rvert} K \quad (71)$$
for the schedule and the definitions:

\[
K[(p) \supset (p, p)] := u \\
K[(p, q, r) \supset (p, r)] := \Delta
\]

for the elementary refinements. The definition is extended to other refinements by concatenation and composition, assuming functoriality and mapping trivial refinements to the identity. Consistency in this definition is assured by the algebraic properties of the unit and product processes.

The consistency conditions imply the states form a semigroup:

\[
1 = z[(p, p)] \\
z[(p, q)] z[(q, r)] = z[(p, r)]
\]

These semigroup properties are expressed in the following string diagrams.

String diagrams of the dynamic system constructed from these elementary diagrams express the compatibility of the state with the partially ordered nature of time.

### Integration and differentiation

In the transition from discrete to continuous dynamics, the stochastic calculus approximates the path-dependent observable as a discrete integral and attempts to construct a well-defined continuous integral by refining the discrete schedule to the continuous schedule. The technical challenge in this construction lies in the differing perspectives for states, defined on the increments of the schedule, and observables, defined on the endpoints of the schedule.

For the schedule \( P = (p_0, \ldots, p_n) \), these perspectives differ in the way they deconstruct the schedule into intervals:

\[
(p_0, p_1), (p_1, p_2), \ldots, (p_{n-1}, p_n) \\
(p_0, p_1), (p_0, p_2), \ldots, (p_0, p_n)
\]

With the first deconstruction, the state evaluates on the accumulated change over each increment of the schedule. With the second deconstruction, the observable depends on the accumulated change to each endpoint of the schedule. While the increment states \( \{z[(p_{i-1}, p_i)]\} \) are independent by construction, the endpoint states \( \{z[(p_0, p_i)]\} \) are not. This needs to be factored into the discrete integral for the observables.
The perspectives are interchanged by the integrator and differentiator:

\[
\mathcal{I} := (\nabla \otimes \mathbf{i}) \circ (\mathbf{i} \otimes \Delta)
\]

\[
\mathcal{D} := (\nabla \otimes \mathbf{i}) \circ (\mathbf{i} \otimes s \otimes \mathbf{i}) \circ (\mathbf{i} \otimes \Delta)
\]

Thanks to the quantum group properties, these processes are mutually inverse:

\[
\mathcal{D} \circ \mathcal{I} = \mathbf{i} \otimes \mathbf{i} = \mathcal{I} \circ \mathcal{D}
\]

implying the two perspectives have the same information content. Using concatenation and composition, the processes integrate and differentiate over the discrete schedule:

Continuous integration is then obtained in the limit of refinement.

The following string diagrams demonstrate the use of these processes in the evaluation of a path-dependent observable.
The path-dependent observable $a$ is contingent on the state accumulated from time $p$ to times $q$, $r$ and $s$, and the three examples demonstrate ways the observable can be evaluated by the incremental states of the valuation model $z$. The first example is conditional valuation:

$$(t \otimes ((t \otimes z[(r, s)]) \circ \mathcal{I})) \circ a$$  

(76)

applying the state on the interval $(r, s)$ to generate an observable contingent on the state accumulated to the times $q$ and $r$. The second example completes the valuation:

$$z[(p, q)] \bullet ((t \otimes z[(q, r)]) \circ \mathcal{I}) \circ ((t \otimes (t \otimes z[(r, s)]) \circ \mathcal{I})) \circ a$$  

(77)

by iterating the action of the state on the third, second and first intervals. In the third example, this sequential application of the state is interlaced:

$$z[(p, q)] \bullet T_q[(t \otimes z[(q, r)]) \circ \mathcal{I}] \circ T_r[(t \otimes (t \otimes z[(r, s)]) \circ \mathcal{I})) \circ T_s[a]]$$  

(78)

with additional transformations acting on the observable at each stage.

As demonstrated in these examples, conditional valuation over the final interval in the schedule $P \vee (p, q)$ is the composition of integrator and state:

$$(\otimes^{[P]^{-1}}) \otimes ((t \otimes z[(p, q)]) \circ \mathcal{I})$$  

(79)

Generalising the examples, a valuation scheme is defined that applies the transformation, the integrator and the state to create a sequence of observables $(a_0, \ldots, a_n)$ on the schedule $(p_0, \ldots, p_n)$ by iterating the actions:

$$a_{i-1} = ((t \otimes z[(p_{i-1}, p_i)]) \circ \mathcal{I}) \circ T_i[a_i]$$  

(80)

for $i = 2, \ldots, n$, terminating the scheme with the valuation:

$$a_0 = z[(p_0, p_1)] \bullet T_1[a_1]$$  

(81)

Transformations are drawn from the functional calculus, polynomially combining the unit and product of the observable algebra, and extending beyond this using convergent limits in the operator representation of the Gelfand-Naimark-Segal construction. The integrator and state then combine to roll the valuation to the preceding time.

This is a typical sequence in the pricing of financial derivatives, where the termsheet prescribes both the terminal payoff and additional payments or termination clauses at earlier times. The structural processes of the quantum group are essential for this construction, consistently capturing the accumulation of states and the convexities between observables, and equipping the valuation model with a notion of integral and differential. The quantum group thus provisions the valuation model with all the operations necessary to express and evaluate the financial derivative.

The formalism simplifies when the observable depends only on the accumulated state from the start to the end of the interval, and does not depend on the path between these times. There are many useful problems where this assumption can be made, and the reduction in dimensionality it presents dramatically improves the effectiveness of numerical methods, as will be demonstrated in the
discrete model considered later in this essay. The conditional valuation \( z \triangleleft a \) of the observable \( a \) by the state \( z \) is defined by:

\[
z \triangleleft a := (1 \otimes z) \circ \Delta \circ a
\]  

(82)

using the product to express the observable as being contingent on two intervals and then contracting with the state on the second interval. This action of states on observables can be compounded, and the compounded action is generated by the product of states:

\[
y \triangleleft (z \triangleleft a) = yz \triangleleft a
\]  

(83)

thanks to associativity. Conditional valuation thus defines an action of the semigroup of states on the observables. As in the path-dependent case, this action of states is interlaced with transformations on the observable to generate a valuation scheme that can be used in pricing.

3 Discrete states and observables

The most general valuation model with finite states and observables is developed from its representation as operators on a finite-dimensional space. This model is easy to evaluate and avoids technical pitfalls, and can be embedded within common algorithms that discretise the states and observables. None of these statements carry merit, though, if the resulting model does not exhibit novel properties useful for real applications. Fortunately, the transition from classical to quantum significantly extends the phenomenology of the model, even in the finite-dimensional case. As an example of the approach, option pricing – the most elementary challenge of mathematical finance – is enriched by the quantum extension in ways that characterise the underlying von Neumann algebra.
3.1 Finite-dimensional von Neumann algebras

The mutually-commutant von Neumann algebras of states and observables decompose as direct sums of factors with trivial centres, each isomorphic to a matrix algebra. A convenient representation, used in the following, decomposes the Hilbert space:

\[ H = \bigoplus_{i=1}^{n} H_i \]  

where \( H_i = \mathbb{C}[M_i, N_i] \) is the space of complex matrices with \( M_i \) rows and \( N_i \) columns, equipped with the Hilbert-Schmidt inner product:

\[ \langle \phi_i | \psi_i \rangle := \text{tr}[\phi_i \psi_i^*] \]  

for the matrices \( \phi_i, \psi_i \in H_i \). The von Neumann algebras \( U \) of states and \( \Omega \) of observables then decompose:

\[ U = \bigoplus_{i=1}^{n} U_i \quad \Omega = \bigoplus_{i=1}^{n} \Omega_i \]  

where \( U_i = \mathbb{C}[M_i] \) is the algebra of complex square matrices with \( M_i \) rows and columns, and \( \Omega_i = \mathbb{C}[N_i] \) is the algebra of complex square matrices with \( N_i \) rows and columns. These algebras are represented on the Hilbert space via left and right multiplication:

\[ \langle \phi_i | [z] | a_i \rangle := \langle z^t_i \phi_i | a_i \rangle = \langle \phi_i | [a_i] | z_i \rangle \]  

for the state \( z = z_1 \oplus \cdots \oplus z_n \in U \) and the observable \( a = a_1 \oplus \cdots \oplus a_n \in \Omega \). The representations commute:

\[ \langle [z] | z_i | a_i \rangle = \langle [z] | a_i | z_i \rangle = \langle [a_i] | [z_i] \rangle \]  

so that the states and observables are mutually commutant, \( U' = U \) and \( \Omega' = \Omega \).

With this representation, the valuation model is expressed as:

\[ z \cdot a = \sum_{i=1}^{n} \langle \omega_i | [z_i] | a_i \rangle | \omega_i \rangle = \sum_{i=1}^{n} \text{tr}[\omega_i a_i \omega_i^*] \]  

in terms of its weight matrix \( \omega = \omega_1 \oplus \cdots \oplus \omega_n \in H \). In the application to finance, the weight \( \omega \) describes the economic state:

\[ z_e \cdot a = \sum_{i=1}^{n} \langle \omega_i | [a_i] | \omega_i \rangle = \sum_{i=1}^{n} \text{tr}[\omega_i a_i \omega_i^*] \]  

which is positive by construction. A pricing state that is equivalent to the economic state is obtained by inserting its Radon-Nikodym matrix \( z_c \) in the trace; the pricing state is then absent of arbitrage when this matrix is positive definite.

3.2 Quantum option pricing

The structural parameters of the valuation model are the pairs of strictly positive integers \( (M_1, N_1), \ldots, (M_n, N_n) \) that dimension its von Neumann factors. Classical valuation sets each of these integers to one, reducing the model to evaluation against a discrete distribution with weights \( (z_1 | \omega_1 |^2, \ldots, z_n | \omega_n |^2) \). If there is a novel contribution from quantum valuation, it must therefore emerge
from the trace of noncommuting matrices in dimensions greater than one. In this section, the focus is restricted to the single-factor case \(n = 1\), and the subscripts indexing the factor are dropped from expressions.

The multiple settlements associated with a typical financial derivative may occur at different times and in different currencies, and the present value of each of these settlements is evaluated using the state associated with its settlement conditions. In this general case, the derivative has present value:

\[
\begin{align*}
    u := \sum_c z_c \cdot a_c &= \text{tr} \left[ \sum_c z_c^t \omega a_c \omega^* \right] \\
    \text{(91)}
\end{align*}
\]

where the observable \(a_c\) is the payoff with settlement conditions \(c\) and the state \(z_c\) is the corresponding pricing state, accounting for discounting and the exchange rate of the settlement.

Overlaying this underlying with optionality, if the holder has the right but not the obligation to settle, the exercise decision needs to be incorporated into the option price:

\[
\begin{align*}
    o_p := \sum_c z_c \cdot p a_c p &= \text{tr} \left[ \sum_c z_c^t \omega p a_c p \omega^* \right] \\
    \text{(92)}
\end{align*}
\]

Exercise is indicated by the projection observable \(p\), a self-adjoint matrix with eigenvalues in \(\{0, 1\}\), expressed algebraically as the properties \(p^* = p = p^2\). Optimal exercise maximises the price of the option:

\[
\begin{align*}
    o := \sup_p \left\{ \sum_c z_c \cdot p a_c p \right\} = \sup_p \left\{ \text{tr} \left[ \sum_c z_c^t \omega p a_c p \omega^* \right] \right\} \\
    \text{(93)}
\end{align*}
\]

taking the supremum over all projections. The projection that achieves this supremum represents the optimal exercise strategy for the option. Noncommutativity introduces complexity in this calculation, preventing the simultaneous diagonalisation of the matrices in the sum, which leads to an option price that cannot be replicated in a classical discrete model.

As an example where this supremum, and the projection that achieves it, can be calculated explicitly, consider the case where the settlement amounts are known in advance. In this case, the settlement observables are proportional to the identity, \(a_c = a_c 1\), and cyclicity of trace derives the price:

\[
\begin{align*}
    o_p &= \text{tr} \left[ p \left( \sum_c a_c Z_c \right) p \right] \\
    \text{(94)}
\end{align*}
\]

where:

\[
Z_c = \omega^* z_c^t \omega
\]

The exercise decision that maximises this expression is the projection onto the direct sum of the eigenspaces with positive eigenvalues of the bracketed matrix, and the option price becomes:

\[
\begin{align*}
    o &= \text{tr} \left[ \left( \sum_c a_c Z_c \right)^+ \right] \\
    \text{(96)}
\end{align*}
\]

where the trace on the right is the sum of the positive eigenvalues of the bracketed matrix. This evaluation of the option price thus involves the roots of the characteristic polynomial of a \(N\)-dimensional matrix.
By the absence of arbitrage for the state \( z_c \), the matrix \( Z_c \) is positive. The option price is thus evaluated as the sum of the positive eigenvalues for the difference of two positive matrices:

\[
o = \text{tr}[(R - P)^+] \tag{97}
\]

paying the matrix \( P \) and receiving the matrix \( R \) defined by:

\[
P = - \sum_{a_c < 0} a_c Z_c \tag{98}
\]
\[
R = \sum_{a_c > 0} a_c Z_c
\]

The underlying has the structure of a swap, exchanging the settlements with \( a_c < 0 \) for the settlements with \( a_c > 0 \), and the option confers the right but not the obligation to enter the swap. This expression can be used to model the prices of common derivatives, such as foreign exchange options and interest rate swaptions.

Slightly generalising the presentation, consider the option to receive one unit of the settlements with \( a_c > 0 \) in exchange for \( k \) units of the settlements with \( a_c < 0 \), where \( k \geq 0 \) is the strike of the option. For strikes near zero, the received settlements dominate and the option is always exercised. For high strikes, the paid settlements dominate and the option is never exercised. In between, there is a regime where it may or may not be optimal to exercise.

By introducing the strike, the option price can be expressed in terms of an implied probability density \( \text{pdf}[s] \) for the swap rate \( s \):

\[
\int_{s=0}^{\infty} (s - k)^+ \text{pdf}[s] \, ds := \text{tr}[(R - kP)^+] / \text{tr}[P] \tag{99}
\]

From this definition, the implied cumulative density \( \text{cdf}[s] \) is derived as:

\[
\text{cdf}[s] := 1 + \frac{d}{dk} \left. \frac{\text{tr}[(R - kP)^+] / \text{tr}[P]}{s} \right|_{k=s} \tag{100}
\]

In the classical discrete model, the probability density is discrete, supported on at most \( N \) points, and the cumulative density is a step function. As will be demonstrated below, the probability density of the quantum discrete model has both discrete and continuous components.

To see how this happens, further simplify to the two-dimensional binomial model, \( N = 2 \). Without loss of generality, the two matrices \( P \) and \( R \) are expressed as:

\[
P = \begin{bmatrix} p_+ & 0 \\ 0 & p_- \end{bmatrix} \tag{101}
\]
\[
R = \begin{bmatrix} r_+ \cos[\theta]^2 + r_- \sin[\theta]^2 & (r_+ - r_-)e^{-i\phi} \cos[\theta] \sin[\theta] \\ (r_+ - r_-)e^{i\phi} \cos[\theta] \sin[\theta] & r_+ \sin[\theta]^2 + r_- \cos[\theta]^2 \end{bmatrix}
\]

where the eigenvalues \( \{p_-, p_+\} \) of the matrix \( P \) and the eigenvalues \( \{r_-, r_+\} \) of the matrix \( R \) are assumed to satisfy \( 0 \leq p_- \leq p_+ \) and \( 0 \leq r_- \leq r_+ \). The coordinate frame is chosen to diagonalise \( P \), and the angles \( \theta \) and \( \phi \) then rotate
to the coordinate frame that diagonalises $R$. It is these rotations that create the quantum features of the model.

The eigenvalues \( \{ u_-[k], u_+[k] \} \) of the matrix $R - kP$ are computed as the roots of its characteristic binomial:

$$ u_\pm[k] = \frac{1}{2}(r - kp) \pm \frac{1}{2} \sqrt{r^2 - 2k\bar{p}\bar{r}\cos[2\theta] + k^2\bar{p}^2} \quad (102) $$

where:

$$ \bar{p} = p_+ + p_- \quad \bar{p} = p_+ - p_- \quad (103) $$

$$ \bar{r} = r_+ + r_- \quad \bar{r} = r_+ - r_- $$

The option price sums only the positive eigenvalues. Three regimes for the option price are delimited by the strikes:

$$ k_\pm = \frac{\bar{r}\bar{p} - \bar{r}\bar{r}\cos[2\theta]}{\bar{p}^2 - \bar{p}^2} \quad (104) $$

satisfying $0 \leq k_- \leq k_+$. If $p_- = 0$, these delimiting strikes are:

$$ k_- = \frac{1}{2} \frac{r^2 - \bar{r}^2}{\bar{r} - \bar{r}\cos[2\theta]} \quad k_+ = \infty \quad (105) $$

and if $r_- = 0$, these delimiting strikes are:

$$ k_- = 0 \quad k_+ = 2\bar{r}\frac{\bar{r}\bar{p} - \bar{r}\bar{r}\cos[2\theta]}{\bar{p}^2 - \bar{p}^2} \quad (106) $$

In the low-strike regime $k \leq k_-$, both eigenvalues are positive and the option is always exercised. In the mid-strike regime $k_- \leq k \leq k_+$, the eigenvalues satisfy $u_-[k] \leq 0 \leq u_+[k]$ and the option price is the positive eigenvalue. In the high-strike regime $k_+ \leq k$, both eigenvalues are negative and the option is never exercised. The option price is thus:

$$ o[k] = \bar{r} - k\bar{p} \quad k < k_- \quad (107) $$

$$ o[k] = \frac{1}{2}(r - kp) + \frac{1}{2} \sqrt{r^2 - 2k\bar{p}\bar{r}\cos[2\theta] + k^2\bar{p}^2} \quad k_- \leq k < k_+ $$

$$ o[k] = 0 \quad k_+ \leq k $$

Differentiating the option price with respect to the strike generates the implied cumulative density:

$$ \text{cdf}[s] = 0 \quad s < k_- \quad (108) $$

$$ \text{cdf}[s] = \frac{1}{2} \left( 1 + \frac{\bar{p}}{\bar{r}} \frac{s\bar{p} - \bar{r}\cos[2\theta]}{\sqrt{s^2 - 2s\bar{r}\bar{p}\cos[2\theta] + s^2\bar{p}^2}} \right) \quad k_- \leq s < k_+ $$

$$ \text{cdf}[s] = 1 \quad k_+ \leq s $$

This is clearly not the cumulative density of a discrete distribution. The distribution has discrete probability density at the boundary points $s = k_\pm$, but also has continuous probability density inbetween. No classical discrete model is able to generate this distribution.
Figure 6: The implied probability and cumulative densities for option pricing in the quantum binomial model with eigenvalues 0.2 and 0.8 for the pay and receive matrices. The distribution changes as the angle $\theta$ between the diagonalising bases for the two matrices varies. In the cases $\theta = 0$ and $\theta = \pi / 2$, the matrices are simultaneously diagonalisable and the result is a classical binomial model. As $\theta$ varies between these extremes, quantum effects create a continuous density between the two points of the discrete density.
In the binomial model that generates the graphs on the previous page, the matrices $P$ and $R$ both have eigenvalues $\{0.2, 0.8\}$, and the graphs present the implied probability and cumulative densities for six different angles of separation $\theta$ between the diagonalising bases of the matrices. Classical pricing corresponds to the cases $\theta = 0$ and $\theta = \pi/2$: in the case $\theta = 0$, the matrices positively correlate and their simultaneous eigenvalues are $\{(0.2, 0.2), (0.8, 0.8)\}$, so the swap rate has a single eigenvalue $\{1\}$; in the case $\theta = \pi/2$, the matrices negatively correlate and their simultaneous eigenvalues are $\{(0.2, 0.8), (0.8, 0.2)\}$, so the swap rate has two eigenvalues $\{1/4, 4\}$. For other values of $\theta$, the distribution has finite probabilities at two discrete points and a continuous density between these boundary points.

The quantum binomial model has particularly interesting behaviour when the matrices have zero eigenvalues, as the implied distribution is in this case supported on the upper half line, and the resulting option price is similar to that produced by popular stochastic volatility models. In the next example, both the matrices $P$ and $R$ are assumed to have eigenvalues $\{0, 1\}$.

The last set of graphs in this series represents the distribution by its implied volatility smile, which for each strike expresses the option price as the lognormal volatility used in the Black-Scholes formula to reproduce the price.

Quantum tunneling effectively leaks the discrete probabilities into the interior, implying a distribution with continuous support from a binomial model whose observables have at most two discrete eigenvalues. Higher dimensions generate more complex phenomenology, with eigenvalues computed as the roots of the higher-order characteristic polynomial of the matrices. As with the two-
dimensional case, options are then priced using a combination of a discrete
distribution and a continuous distribution on its convex support.

Beyond finite dimensions, the same method is applied to any von Neumann
algebra that has a trace, extending the model to all type I and II factors. With
some effort, the model can even be developed on type III factors by replacing the
trace with a Kubo-Martin-Schwinger state. In each case, exercise is optimised
over all available projections, making a direct connection between the option
price model and the family of projections in the von Neumann algebra.

4 Lie algebras and interest rates

In this section, a dynamic system is constructed from a semisimple Lie algebra
$K$ over the complex scalars. The construction exploits the weight lattice of its
set of irreducible representations, wherein each positive weight $p$ is associated
with the simple module $K[p]$ generated by the action of the universal enveloping
algebra on the eigenvector with highest weight. Option pricing in this model is
rich in features and numerically efficient, reducing to the eigenvalue problem as
outlined in the previous section. Remarkably, everything needed by the dynamic
system, including time and the states and observables, can be uniquely identified
from the Lie algebra.

**Definition 3 (Lie dynamic system).** The dynamic system generated by the
semisimple Lie algebra $K$ has time given by its weight lattice of irreducible rep-
resentations, partially ordered by a choice of simple roots. For the schedule
$P = (p_0, \ldots, p_n)$, define the module:

$$K[P] := K[p_1 - p_0] \otimes \cdots \otimes K[p_n - p_{n-1}]$$ (109)

where $K[p]$ is the simple module with highest weight $p$. The states and observ-
bles associated with the schedule are the endomorphisms on this module:

$$U[K[P]] := \text{End}[K[P]] =: \mathfrak{U}[K[P]]$$ (110)

with pairing defined by the trace:

$$z \bullet a := \text{tr}[za]$$ (111)

for the state $z \in U[K[P]]$ and the observable $a \in \mathfrak{U}[K[P]]$.

Clebsch-Gordan decomposition expresses the tensor product of simple mod-
ules $K[p_1] \otimes \cdots \otimes K[p_n]$ as the direct sum of simple modules. For the dynamic
system, the key property needed from this highly non-trivial decomposition is
the inclusion of the simple submodule:

$$K[p_1 + \cdots + p_n] \subset K[p_1] \otimes \cdots \otimes K[p_n]$$ (112)

with multiplicity one, separating the data for the total weight from the addi-
tional data that describes the weight path. Use the decomposition to define the
isometry and coisometry:

$$\Delta : K[p_1 + \cdots + p_n] \to K[p_1] \otimes \cdots \otimes K[p_n]$$ (113)

$$\Delta^* : K[p_1] \otimes \cdots \otimes K[p_n] \to K[p_1 + \cdots + p_n]$$
satisfying the property:
\[ \Delta \Delta^* = 1 \] (114)

The existence of a unique contribution with highest weight to the decomposition of the tensor product supports the definition of accumulation in the dynamic system, which provides the conditions for the consistent expression of the state.

**Definition 4** (Lie dynamic state). The accumulation functor associated with the semisimple Lie algebra \( K \) is defined on the refinement \( P \supset (P_-, P_+) \) by:
\[
K[P \supset (P_-, P_+)] \circ a := \Delta^* a \Delta
\]
(115)
\[
z \circ K[P \supset (P_-, P_+)] := \Delta z \Delta^*
\]
for the state \( z \in U[K[P]] \) and the observable \( a \in \mathfrak{f}[K((P_-, P_+))] \), where:
\[
\Delta : K[(P_-, P_+)] \to K[P]
\]
(116)
\[
\Delta^* : K[P] \to K[(P_-, P_+)]
\]
are the isometry and coisometry that identify the highest-weight submodule in the tensor product. The definition is extended to other refinements by concatenation.

The state \( z \in U[K] \) associates the state \( z[P] \in U[K[P]] \) with the schedule \( P \).

This association satisfies the naturality condition:
\[
\Delta(z[[p_0, p_1]) \otimes \cdots \otimes z[[p_{n-1}, p_n]]) \Delta^* = z[[p_0, p_n]]
\]
(117)
for the schedule \( P = (p_0, \ldots, p_n) \), enforcing consistency of the state with refinement in the dynamic system.

**4.1 Interest rate modelling**

By adding time to the information model, the dynamic system generated by the semisimple Lie algebra is able to price financial derivatives contingent on the state of the economy at multiple times, generating credible implied volatility smiles from a dynamic model with finite discrete eigenstates.

The interest rate model is simplified with the assumption that the floating rate matches the discount rate over its accrual period, so that the floating accrual can be replaced by notional exchange. This simplification is easily removed, but is retained here to avoid obscuring the contribution that the Lie algebra makes to the price of convexity.

Discounting in the interest rate model is marked to the price of the discount bond \( \mathfrak{d}[p] \) that returns the unit payoff at time \( p \). In the pricing state \( z \), the discount factor at earlier time \( e \) is:
\[
\mathfrak{d}[p][e] = z[[e, p]] \circ 1
\]
(118)
The discount curve \( \mathfrak{d} \) at time zero maps the positive time \( p \) to its discount factor \( \mathfrak{d}[p] \), and the model is calibrated to this curve with the condition:
\[
\mathfrak{d}[p] = \text{tr}[z[[0, p]]]
\]
(119)
mapping the trace of the pricing state with the discount curve. With this normalisation, the model consistently prices financial derivatives constructed as linear combinations of discount bonds.
The basic interest rate derivative is the swap \( u[P, \delta, k] \) parametrised by its accrual schedule \( P = (p_0, \ldots, p_n) \), its accrual fractions \( \delta = (\delta_1, \ldots, \delta_n) \) and its fixed rate \( k \). The price of the swap at earlier time \( e \) is:

\[
\begin{align*}
\mathbb{u}[P, \delta, k][e] &= (d[p_0][e] - d[p_n][e]) - k \sum_{i=1}^{n} d[p_i][e] \delta_i \\
\end{align*}
\]

where the first two terms price the floating leg, collapsed to initial and final notional exchange, and the last term prices the fixed leg. The model is calibrated to the par swap rate \( u[P, \delta, k] \) at time zero with the condition:

\[
\begin{align*}
\mathbb{u}[P, \delta, k][0] &= d[p_0] - d[p_n] \\
\end{align*}
\]

that derives the par swap rate from the discount curve.

Interest rate options are priced in this model by incorporating the contribution from the exercise decision. The European swaption \( \mathcal{o}[e, P, \delta, k] \) is the option to enter the swap \( u[P, \delta, k] \) exercisable at a single time \( e \) prior to the schedule of the swap. The price at time zero for the European swaption is then:

\[
\begin{align*}
\mathcal{o}[e, P, \delta, k] &= \text{tr}[z((0, e)] \mathbb{u}[P, \delta, k][e]^+] \\
\end{align*}
\]

The Bermudan swaption \( \mathcal{b}[P, \delta, k] \) is the option to enter the swap \( u[P, \delta, k] \) exercisable at each time \( p_i = p_0, \ldots, p_{n-1} \) during the schedule of the swap, where the price \( u[P, \delta, k][p_i] \) is assumed to include only the accruals on the remainder \( (p_i, \ldots, p_n) \) of the schedule. The price of the Bermudan swaption thus satisfies the valuation scheme:

\[
\begin{align*}
\mathcal{b}[P, \delta, k][p_n] &= 0 \\
\mathcal{b}[P, \delta, k][p_i] &= \mathcal{z}(p_i, p_{i+1}] \cdot \mathcal{b}[P, \delta, k][p_{i+1}] \\
&+ (u[P, \delta, k][p_i] - \mathcal{z}(p_i, p_{i+1}] \cdot \mathcal{b}[P, \delta, k][p_{i+1}])^+ \\
\end{align*}
\]

for \( i = 0, \ldots, n - 1 \), terminating with the price:

\[
\begin{align*}
\mathcal{b}[P, \delta, k] &= \text{tr}[z((0, p_0)] \mathcal{b}[P, \delta, k][p_0]] \\
\end{align*}
\]

at time zero. The first term in the iteration is the continuation value of the Bermudan swaption, and the second term is the additional value from the option to replace this with the underlying swap.

Both these types of interest rate option are evaluated from the weighted trace of the positive component of the payoff matrix, in a single step for the European case and embedded in the valuation scheme for the Bermudan case. Volatility emerges from the lack of simultaneous diagonalisability of the fixed and floating legs in the swap, creating a wide variety of realistic implied volatility smiles from the eigenvalue algorithm for the option price.

The price expressions involve the operation of conditional valuation constructed from the accumulation functor of the dynamic system, in turn derived from the irreducible representations of the Lie algebra. Semisimplicity ensures this construction remains within the confines of finite dimensions, and the weight lattice imposes a recombining structure that generalises the classical multinomial model. This is demonstrated below for the case of the two-dimensional special linear Lie algebra.
4.2 Special linear Lie algebra in two dimensions

Constructed from the representations of the semisimple Lie algebra, the dynamic system depends on the identification of the submodule with highest weight in the Clebsch-Gordan decomposition of the tensor product. While there are algorithms for determining this decomposition, it is highly non-trivial and has no simple expression in general.

In the following the Lie algebra \( K = \text{sl}[2] \), the special linear Lie algebra in two dimensions, is considered in depth. The decomposition is straightforward in this case, and the resulting dynamic system highlights the main attributes of representation theory that have utility in pricing. This discrete model combines the numerical efficiency of the classical binomial model with the more realistic marginal distributions implied by its quantum extension.

The Lie algebra \( K \) is spanned by the standard basis \( \{h, e, f\} \) satisfying the commutation relations:

\[
\begin{align*}
[h, e] &= 2e \\
[f, h] &= -2f \\
[f, e] &= h
\end{align*}
\]

The weight lattice of the Lie algebra is the set of integers, and the simple module \( K[p] \) generated by the positive weight \( p \) is \((p + 1)\)-dimensional. There is a basis \( \{\langle p, q \rangle : q = -p, -p + 2, \ldots, p - 2, p\} \) of eigenvectors of \( h \) with \( e \) acting as raising operator and \( f \) acting as lowering operator:

\[
\begin{align*}
\langle p, q \rangle | h = q \langle p, q \rangle \\
\langle p, q \rangle | e &= \frac{1}{2} \sqrt{(p - q)(p + q + 2)} \langle p, q + 2 \rangle \\
\langle p, q \rangle | f &= \frac{1}{2} \sqrt{(p + q)(p - q + 2)} \langle p, q - 2 \rangle
\end{align*}
\]

The Clebsch-Gordan decomposition is derived from these relations. Separating out the contribution \( K[p] \otimes K[1] \) from the tensor product \( K[p] \otimes K[1] \), the dynamic system is generated from the mutually-adjoint linear maps:

\[
\begin{align*}
\Delta : K[p] \otimes K[1] \rightarrow K[p + 1] \\
\Delta^* : K[p + 1] \rightarrow K[p] \otimes K[1]
\end{align*}
\]

where the isometry is defined by:

\[
\langle p + 1, q \rangle \Delta = \sqrt{\frac{p + q + 1}{2(p + 1)}} \langle p, q - 1 \rangle \otimes \langle 1, 1 \rangle + \sqrt{\frac{p - q + 1}{2(p + 1)}} \langle p, q + 1 \rangle \otimes \langle 1, -1 \rangle
\]

and the coisometry is defined by:

\[
\begin{align*}
\langle (p + 1, q) \Delta^* &= \sqrt{\frac{p + q + 2}{2(p + 1)}} \langle p + 1, q + 1 \rangle \\
\langle (p, q) \otimes (1, -1) \Delta^* &= \sqrt{\frac{p - q + 2}{2(p + 1)}} \langle p + 1, q - 1 \rangle
\end{align*}
\]
For the weights $e$ and $p$ satisfying $e \leq p$, conditional valuation of the observable $a$ on the interval $(e, p+1)$ by the state $z$ on the interval $(p, p+1)$ generates the observable $z \triangleleft a$ on the interval $(e,p)$:

$$z \triangleleft a = \text{tr}_{K[1]}[(1 \otimes z)\Delta^* a \Delta] \quad (130)$$

This expression is evaluated by decomposing the accumulation map into its raising and lowering components:

$$\Delta_\pm : K[p+1] \rightarrow K[p] \quad (131)$$

$$\Delta^*_\pm : K[p] \rightarrow K[p+1]$$

defined by:

$$\langle p+1, q | \Delta_\pm = \sqrt{\frac{p \pm q + 1}{2(p+1)}} \langle p, q \mp 1 | \quad (132)$$

$$\langle p, q | \Delta^*_\pm = \sqrt{\frac{p \pm q + 2}{2(p+1)}} \langle p+1, q \pm 1 |$$

With these maps, the action of the state on the observable resolves to:

$$z \triangleleft a = z_{++} \Delta^*_+ a \Delta_+ + z_{+-} \Delta^*_- a \Delta_+ + z_{-+} \Delta^*_+ a \Delta_- + z_{--} \Delta^*_- a \Delta_- \quad (133)$$

where:

$$z = \begin{bmatrix} z_{++} & z_{+-} \\ z_{-+} & z_{--} \end{bmatrix} \quad (134)$$

This explicit expression for conditional valuation in the case of the special linear Lie algebra in two dimensions combines with the earlier price expressions to create an efficient scheme for pricing interest rate options.

A comparable scheme is derived from the representations of any semisimple Lie algebra, albeit at the expense of increasing complexity in the Clebsch-Gordan decomposition of the tensor product. More generally, the approach relies on the recombining structure of the weight space of irreducible representations, and any algebraic construction that has a similar lattice of representations could be used in place. This includes Kac-Moody algebras, Lie superalgebras and quantum deformations of Lie algebras, all cases where the universal enveloping algebra can be structured as a quantum group. Characterising features of the source algebra are then related to the prices of options in the corresponding interest rate model.
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