Hypergeometric polynomials are optimal

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Received: 15 December 2017 / Accepted: 13 October 2019 / Published online: 7 December 2019
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Abstract
With any integer convex polytope \( P \subset \mathbb{R}^n \) we associate a multivariate hypergeometric polynomial whose set of exponents is \( \mathbb{Z}^n \cap P \). This polynomial is defined uniquely up to a constant multiple and satisfies a holonomic system of partial differential equations of Horn’s type. We prove that under certain nondegeneracy conditions any such polynomial is optimal in the sense of [7], i.e., that the topology of its amoeba [13] is as complex as it could possibly be. Using this, we derive optimal properties of several classical families of multivariate hypergeometric polynomials.

1 Introduction

Zeros of hypergeometric functions are known to exhibit highly complicated behavior. The univariate case has been extensively studied both classically (see, e.g., [12,15]) and recently (see [3,4,24] and the references therein). Already the distribution of zeros of polynomial instances of the simplest non-elementary hypergeometric function \( _2F_1(a, b; c; x) \) is far from being clear. When one of the parameters \( a, b \) equals a nonpositive integer, say \( a = -d \), the series representing \( _2F_1 \) terminates and the hypergeometric function is a polynomial of degree \( d \) in \( x \) (see [3]). By letting the parameters \( a, b, c \) assume values in various ranges, one can obtain a wide variety of shapes. Some of them are highly regular (see, e.g., Fig. 1) while other are nearly chaotic.

In the present paper, we introduce a definition of a multivariate hypergeometric polynomial in \( n \geq 2 \) complex variables that is coherent with the properties of classical hypergeometric polynomials. This polynomial is defined by an integer convex polytope \( P \subset \mathbb{R}^n \), its set of exponents is \( \mathbb{Z}^n \cap P \). For this polynomial to be “truly hypergeometric” in the sense made precise below, we need to assume that any pair of points in \( \mathbb{Z}^n \cap P \) can be connected by a polygonal line with unit sides and integer vertices. This assumption does not affect the
generality of the results since any polytope that does not satisfy this condition gives rise to a finite number of hypergeometric polynomials that can be considered independently.

The following definition is central in the paper and brings together the intrinsic properties of the classical families of hypergeometric polynomials: the denseness, convexity, and irreducibility of the support, as well as the property of being a solution to a suitable system of linear differential equations with polynomial coefficients.

**Definition 1.1** For \( n \geq 2 \) let \( P \subset \mathbb{R}^n \) be an integer convex polytope such that any two points in \( P \cap \mathbb{Z}^n \) can be connected by a polygonal line with unit sides. Let \( \langle B_i, s \rangle + c_i = 0, \ i = 1, \ldots, q \) be the equations of the hyperplanes containing the faces of \( P \) with \( B_i \) being the outer normal to \( P \) at the respective face with integer relatively prime components.

The polynomial

\[
\sum_{s \in P \cap \mathbb{Z}^n} x^{s} \prod_{i=1}^{q} \frac{1}{\Gamma(1 - \langle B_i, s \rangle - c_i)}
\]

will be called the hypergeometric polynomial defined by the polytope \( P \).

The hypergeometric polynomial associated with the polytope \( P \) is defined uniquely up to a constant multiple and satisfies a holonomic system of partial differential equations of Horn’s type [19,21]. We prove that under certain nondegeneracy conditions (see Theorem 3.8) any such polynomial is optimal in the sense of [7]. Generally speaking, this means that the topology of the amoeba [7,13] of such a polynomial is as complicated as it could possibly be (see Definition 2.8). This property is the multivariate counterpart of the property of having different absolute values of the roots for a polynomial in a single variable. We show various families of classically known multivariate polynomials to be optimal (possibly after a monomial change of variables): a biorthogonal basis in the unit ball, certain polynomial instances of the Appel \( F_1 \) function, bivariate Chebyshev polynomials of the second kind etc.

Pictures of amoebas in the paper have been created with Matlab R2018b. The authors are thankful to L. Lang for valuable comments on the paper and to the referee for the careful reading, deep insight, and numerous helpful suggestions.
2 Hypergeometric systems and amoebas

Throughout the paper, we denote by \( n \) the number of \( x \in \mathbb{C}^n \) variables. For \( \alpha = (\alpha_1, \ldots, \alpha_n) \), we use the notation \( |\alpha| = \sum_{j=1}^{n} \alpha_j \) and \( \alpha! = \alpha_1! \cdots \alpha_n! \). For \( x = (x_1, \ldots, x_n) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), we denote by \( x^\alpha \) the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

**Definition 2.1** A formal Laurent series

\[
\sum_{s \in \mathbb{Z}^n} \varphi(s) x^s
\]  

is called *hypergeometric* if for any \( j = 1, \ldots, n \) the quotient \( \varphi(s + e_j)/\varphi(s) \) is a rational function in \( s = (s_1, \ldots, s_n) \). Throughout the paper we denote this rational function by \( P_j(s)/Q_j(s + e_j) \). Here \( \{e_j\}_{j=1}^n \) is the standard basis of the lattice \( \mathbb{Z}^n \). By the *support* of this series we mean the subset of \( \mathbb{Z}^n \) on which \( \varphi(s) \neq 0 \).

By a *hypergeometric function* we mean a (typically multi-valued) analytic function obtained by means of analytic continuation of a hypergeometric series with a nonempty domain of convergence along all possible paths in \( \mathbb{C}^n \).

**Theorem 2.2** (Ore, Sato, see [10]) *The coefficients of a hypergeometric series are given by the formula*

\[
\varphi(s) = t^s U(s) \prod_{i=1}^{m} \Gamma(\langle A_i, s \rangle + c_i),
\]

where \( t^s = t_1^{s_1} \cdots t_n^{s_n} \), \( t_j, c_i \in \mathbb{C}, A_i = (A_{i,1}, \ldots, A_{i,n}) \in \mathbb{Z}^n, i = 1, \ldots, m, j = 1, \ldots, n \), and \( U(s) \) is the product of a certain rational function and a periodic function \( \phi(s) \) such that \( \phi(s + e_j) \equiv \phi(s) \) for every \( j = 1, \ldots, n \).

Given the above data \( (t_j, U(s), A_i, c_i) \) that determines the coefficient of a hypergeometric series, it is straightforward to compute the rational functions \( P_j(s)/Q_j(s + e_j) \) using the \( \Gamma \)-function identity. The converse requires solving a system of difference equations which is only solvable under some compatibility conditions on \( P_j, Q_j \). A careful analysis of this system of difference equations has been performed in [19].

We call any function of the form (2.2) *the Ore–Sato coefficient of a hypergeometric series*. In this paper the Ore–Sato coefficient (2.2) plays the role of a primary object which generates everything else: the series, the hypergeometric system of differential equations, its polynomial solution (if any) and its amoeba. We also assume that \( m \geq n \) since otherwise the corresponding hypergeometric series (2.1) is just a linear combination of hypergeometric series in fewer variables (times an arbitrary function in remaining variables that makes the system non-holonomic) and \( n \) can be reduced to meet the inequality.

**Definition 2.3** *The Horn system of an Ore–Sato coefficient*. A (formal) Laurent series \( \sum_{s \in \mathbb{Z}^n} \varphi(s)x^s \) whose coefficient satisfies the relations \( \varphi(s + e_j)/\varphi(s) = P_j(s)/Q_j(s + e_j) \) is a (formal) solution to the following system of partial differential equations of hypergeometric type

\[
x_j P_j(\theta) f(x) = Q_j(\theta) f(x), \quad j = 1, \ldots, n.
\]

Here \( \theta = (\theta_1, \ldots, \theta_n), \theta_j = x_j \frac{\partial}{\partial x_j} \). The system (2.3) will be referred to as *the Horn hypergeometric system defined by the Ore–Sato coefficient \( \varphi(s) \)* (see [10]) and denoted by \( \text{Horn}(\varphi) \). In this paper we only treat holonomic Horn hypergeometric systems, i.e. \( \text{rank}(\text{Horn}(\varphi)) \) is always assumed to be finite.
We will often be dealing with the important special case of an Ore–Sato coefficient \( (2.2) \) where \( t_j = 1 \) for any \( j = 1, \ldots, n \) and \( U(s) \equiv 1 \). In this case the following operators \( P_j(\theta) \) and \( Q_j(\theta) \) explicitly determine the system \((2.3)\):

\[
P_j(s) = \prod_{i:A_{i,j}>0}^{A_{i,j}-1} \prod_{\ell_j^{(i)}=0} |A_i, s| + c_i + \ell_j^{(i)},
\]

\[
Q_j(s) = \prod_{i:A_{i,j}>0}^{A_{i,j}-1} \prod_{\ell_j^{(i)}=0} |A_i, s| + c_i + \ell_j^{(i)}.
\]

**Definition 2.4** The support of a series solution to \((2.3)\) is called \textit{irreducible} if there exists no series solution to \((2.3)\) supported in its proper nonempty subset.

**Definition 2.5** The \textit{amoeba} \( A_f \) of a Laurent polynomial \( f(x) \) (or of the algebraic hypersurface \( \{ f(x) = 0 \} \)) is defined to be the image of the hypersurface \( \{ f(x) = 0 \} \) under the map \( \log : (x_1, \ldots, x_n) \mapsto (\log |x_1|, \ldots, log |x_n|) \).

Despite losing \( n \) real dimensions, the amoeba of an algebraic hypersurface encodes several intrinsic properties \([7, 8, 17, 22]\). The main results of the paper describe topological properties of the amoebas of hypergeometric polynomials. The next lemma shows that certain transformations of a polynomial do not affect the topology of its amoeba.

**Lemma 2.6** The number of connected components of the amoeba complement of a (Laurent) polynomial \( p(x_1, \ldots, x_n) \) is the same as that of the polynomial \( x^a p(t_1 x^{v_1}, \ldots, t_n x^{v_n})^\ell \) for any \( \ell \in \mathbb{N}, a = (a_1, \ldots, a_n) \in \mathbb{Z}^n, t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n \) and any nondegenerate integer matrix \( v \) with the rows \( v_1, \ldots, v_n \). That is, there is a bijection between the connected components of the complements of the two amoebas; moreover, the orders \([7]\) and the recession cones \([16]\) of the corresponding components are transformed into each other by the linear map with the matrix \( v \).

**Proof** A monomial factor can only vanish in the union of the coordinate hyperplanes that is mapped to infinity by the logarithmic map. The amoeba does not reflect the multiplicities of the zeros of a polynomial and it can therefore be raised to any positive power. The map \( (x_1, \ldots, x_n) \mapsto (t_1 x^{v_1}, \ldots, t_n x^{v_n}) \) corresponds to the shift of the amoeba space with respect to the vector \( \log |t_1|, \ldots, \log |t_n| \). Finally, a nondegenerate monomial change of variables in the complex torus \((\mathbb{C}^*)^n\) corresponds to the linear transformation of the amoeba space defined by the matrix of exponents of the monomials. Clearly a nondegenerate linear map preserves the topology of amoebas and provides a bijection for the recession cones of the component components. The last statement of the lemma is an immediate consequence of the definition of the order of a component in the amoeba complement \([7]\). \( \square \)

Recall that the \textit{Newton polytope} \( \mathcal{N}_{p(x)} \) of a Laurent polynomial \( p(x) \) is defined to be the convex hull in \( \mathbb{R}^n \) of the support of \( p(x) \). The following result shows that the Newton polytope \( \mathcal{N}_{p(x)} \) reflects the structure of the amoeba \( A_{p(x)} \) \([7, \text{ Theorem 2.8 and Proposition 2.6}]\).

**Theorem 2.7** (See \([7]\)) \( p(x) \) be a Laurent polynomial and let \( \{ M \} \) denote the family of connected components of the amoeba complement \( cA_{p(x)} \). There exists an injective function \( v : \{ M \} \rightarrow \mathbb{Z}^n \cap \mathcal{N}_{p(x)} \) such that the cone which is dual to \( \mathcal{N}_{p(x)} \) at the point \( v(M) \) coincides with the recession cone \([16]\) of \( M \). In particular, the number of connected components of \( cA_{p(x)} \) cannot be smaller than the number of vertices of \( \mathcal{N}_{p(x)} \) and cannot exceed the number of integer points in \( \mathcal{N}_{p(x)} \).
The two extreme values for the number of connected components of the complement of an amoeba are of particular interest [16].

**Definition 2.8** (Cf. [7, Definition 2.9]) A Laurent polynomial \( p(x) \) is called **optimal** if the number of connected components of its amoeba complement \( \mathcal{A}_{p(x)} \) equals the number of integer points in the Newton polytope \( \mathcal{N}_{p(x)} \). We will say that an algebraic hypersurface \( \mathcal{H} \subseteq (\mathbb{C}^*)^n \), \( n \geq 2 \), is optimal if the generator of the vanishing ideal \( I(\mathcal{H}) \) is an optimal polynomial.

Since the amoeba of a polynomial does not carry any information on the multiplicities of its roots, any one-dimensional amoeba (which is just a finite set of distinct points in \( a_1, \ldots, a_k \in \mathbb{R} \)) can be treated as the amoeba of the optimal polynomial \( \prod_{j=1}^{k} (x - e^{a_j}) \) all of whose roots are positive and distinct. We remark that a real-rooted univariate polynomial with positive coefficients is not necessarily optimal, but its vanishing locus is.

**Example 2.9** In accordance with Definition 2.3 the Ore–Sato coefficient

\[
\varphi(s, t) = \Gamma(-s - t + 1)\Gamma(2s - t - 2)\Gamma(-s + 2t - 2)
\]

yields the polynomials

\[
\begin{align*}
P_1(s, t) &= (2s - t - 2)(2s - t - 1), \\
P_2(s, t) &= (s - 2t + 1)(s - 2t + 2), \quad Q_1(s, t) = (s - 2t + 2)(s + t - 1), \\
Q_2(s, t) &= -(2s - t - 2)(s + t - 1).
\end{align*}
\]

The corresponding Horn hypergeometric system is given by the linear differential operators

\[
\begin{bmatrix}
x(2\theta_x - \theta_y - 2)(2\theta_x - \theta_y - 1) - (\theta_x - 2\theta_y + 2)(\theta_x + \theta_y - 1), \\
y(\theta_x - 2\theta_y + 1)(\theta_x - 2\theta_y + 2) + (2\theta_x - \theta_y - 2)(\theta_x + \theta_y - 1).
\end{bmatrix}
\] (2.4)

It is straightforward to check that (2.4) is satisfied by the polynomial \( p(x, y) = x + y + 6xy + x^2y^2 \). We remark that (any constant multiple of) the monomial \( x^\alpha y^\beta \) alone is also a solution to (2.4) and therefore \( p(x, y) \) is a linear combination of polynomial solutions to this system of differential equations. In what follows we will focus on “truly hypergeometric” polynomials, i.e., on polynomial solutions to hypergeometric systems which do not admit such a decomposition.

The amoeba of \( p(x, y) \) together with its compactified version [13] are depicted in Fig. 2.

An algebraic hypersurface is optimal if the topology of its amoeba is as complicated as it could possibly be in the view of Theorem 2.7 (that is, the number of connected components in the amoeba complement is maximal). A bivariate polynomial is optimal if and only if its amoeba has the maximal possible number of bounded connected components in its complement and the maximal number of parallel tentacles.

In the view of Lemma 2.6 we do not distinguish polynomials whose zero loci in \( (\mathbb{C}^*)^n \) can be transformed into each other by a nondegenerate monomial change of variables. The reason for this is illustrated by the following example.

**Example 2.10** A typical example of a family of optimal (after a suitable monomial change of variables) polynomials arising in a different theory is given by the biorthogonal family in the unit ball \( \{V_\alpha(x)\}_{\alpha \in \mathbb{N}_0^n} \), \( x \in \mathbb{C}^n \) defined through their generating function (see [5, Section 2.3]):

\[
(1 - 2\langle a, x \rangle + \|a\|^2)^{-\frac{1-n}{2}} = \sum_{\alpha \in \mathbb{N}_0^n} a^\alpha V_\alpha(x).
\]
Neglecting an inessential monomial factor of $V_\alpha(x)$ (whose zero locus is contained in the union of the coordinate hyperplanes and therefore does not affect the amoeba, see Lemma 2.6) one can represent $V_\alpha(x) = \tilde{V}_\alpha(\xi)$ with $\tilde{V}_\alpha$ being a polynomial in $\xi_j = x_j^2$. One can check that the polynomial $\tilde{V}_\alpha(\xi)$ is optimal. The Newton polygon and the amoeba of the bivariate polynomial $\tilde{V}_{(6,10)}$ are depicted in Fig. 3.

Numerous other families of optimal multivariate polynomials can be found in [5, Chapter 2].
3 Hypergeometric polynomials in several variables

Throughout the rest of the paper we will only consider polynomials in \( n \) variables whose Newton polytopes have nonzero \( n \)-dimensional volume. If the volume of such a Newton polytope is zero, a suitable monomial change of variables can be used to reduce the number of variables.

From now on we adopt the following definitions.

**Definition 3.1** A set \( S \subseteq \mathbb{Z}^n \) is called \( \mathbb{Z}^n \)-convex if the condition \( \{ \lambda s^{(0)} + (1 - \lambda)s^{(1)} : \lambda \in [0, 1] \} \cap \mathbb{Z}^n \subseteq S \) holds for any \( s^{(0)}, s^{(1)} \in S \).

**Definition 3.2** A set \( S \subseteq \mathbb{Z}^n \) is said to be \( \mathbb{Z}^n \)-connected if any two points of this set can be connected by a polygonal line with unit sides and vertices in \( S \).

For instance, the support of the bivariate hypergeometric polynomial \( x + y + 6xy + x^2y^2 \) in Example 2.9 is a \( \mathbb{Z}^2 \)-convex but not a \( \mathbb{Z}^2 \)-connected set. In fact, it consists of two \( \mathbb{Z}^2 \)-connected components: \( \{ (1, 0), (0, 1), (1, 1) \} \) and \( \{ (2, 2) \} \).

Recall that the support \( S \) of a solution to the hypergeometric system (2.3) is called irreducible if there is no nonzero solution to (2.3) supported in a proper nonempty subset of \( S \). Any irreducible support of a solution to (2.3) is always a \( \mathbb{Z}^n \)-connected set. The following statement has been established in [20].

**Lemma 3.3** (See [20]) If the support \( S \) of a polynomial solution to the system (2.3) is irreducible, then \( S \) is a \( \mathbb{Z}^n \)-convex set.

We next show that any convex integer polytope supports an irreducible solution to a suitable instance of the hypergeometric system (2.3).

**Lemma 3.4** For any convex integer polytope \( P \subseteq \mathbb{R}^n \) such that \( P \cap \mathbb{Z}^n \) is \( \mathbb{Z}^n \)-connected, there exists a hypergeometric system of the form (2.3) and its polynomial solution \( p(x) \) with irreducible support such that \( N_p(x) = P \).

**Proof** Let \( \langle B_i, s \rangle + c_i = 0 \), \( i = 1, \ldots, q \) be the equations of the hyperplanes containing the faces of \( P \) with \( B_i \) being the outer normal to \( P \) at the respective face. Since \( P \) is an integer polytope, we may without loss of generality assume the components of the vector \( B_i \) to be integer and relatively prime.

Consider the Ore–Sato coefficient

\[
\varphi(s) = \prod_{i=1}^{q} \Gamma(\langle B_i, s \rangle + c_i).
\]

By Definition 2.3 the hypergeometric system defined by \( \varphi(s) \) only depends on the quotients \( R_j(s) := \varphi(s + e_j)/\varphi(s) \) that are rational functions in \( s \). Using the \( \Gamma \)-function identity

\[
\Gamma(z) = \frac{\pi}{\sin(\pi z)}
\]

together with the fact that the meromorphic function \( \frac{\sqrt{-1}\pi}{\sin(\pi z)} \) is periodic with the period 1, we conclude that the quotients \( R_j(s) \) coincide with those for the entire function

\[
\exp\left( \sqrt{-1}\pi \left( \sum_{i=1}^{q} \langle B_i, s \rangle + c_i \right) \right) \prod_{i=1}^{q} \Gamma(1 - \langle B_i, s \rangle - c_i).
\]
The support of the polynomial with the coefficient (3.1) does not change if we replace the numerator of (3.1) by 1. In fact, this numerator is the exponential part \( r^s \) of the Ore–Sato coefficient (2.2) and, by Lemma 2.6, it affects neither the support of the polynomial solutions to the corresponding hypergeometric system nor the topological properties of their amoebas. For these reasons we define the coefficient of the hypergeometric polynomial under construction to be

\[
\psi_P(s) := \frac{1}{q \prod_{i=1}^q \Gamma(1 - \langle B_i, s \rangle - c_i)}.
\] (3.2)

The function \( \psi_P(s) \) is completely defined by the integer polytope \( P \). By the construction the function \( \psi_P(s) \) vanishes at any lattice point that does not belong to \( P \). Moreover, it is positive in \( P \). Define the polynomial \( p(x) \) to be

\[
p(x) = \sum_{s \in P \cap \mathbb{Z}^n} \psi_P(s)x^s.
\]

By the explicit construction, the polynomial \( p(x) \) is supported in \( P \cap \mathbb{Z}^n \) and satisfies the hypergeometric system \( \text{Horn}(\psi_P(s)) \). The support is irreducible since \( P \cap \mathbb{Z}^n \) is \( \mathbb{Z}^n \)-connected and since \( \psi_P(s) \) does not vanish in \( P \).

The conclusion of the above lemma still holds even without the condition of \( \mathbb{Z}^n \)-connectedness of the set of integer points in the defining polynomial. However, the support of the polynomial produced by the construction in the proof of the lemma will no longer be irreducible. Such a polynomial cannot be considered as “truly hypergeometric” since it is a linear combination of two or more polynomials satisfying the same hypergeometric system, see Example 2.9. The properties of the amoeba of such a polynomial are in general heavily dependent on the coefficients of this linear combination. Yet, it is always possible to choose these coefficients in such a way that the hypergeometric polynomial with the (reducible) support \( P \cap \mathbb{Z}^n \) is optimal. Thus we may and will without loss of generality assume throughout the rest of the paper that the set \( P \cap \mathbb{Z}^n \) is \( \mathbb{Z}^n \)-connected.

**Remark 3.5** One can still not define a hypergeometric polynomial to be a polynomial solution to (2.3) with a \( \mathbb{Z}^n \)-convex irreducible support \( S \) since any polynomial supported in \( S \) will satisfy this condition. This can be seen by introducing more factors into the Ore–Sato coefficient that will affect the coefficients of the polynomial solution but will not corrupt its support. Instead, we will distinguish the only polynomial that has support \( S \) and satisfies the hypergeometric system of the smallest possible holonomic rank.

In accordance with Definition 1.1, by a multivariate hypergeometric polynomial supported in a convex integer polytope \( P \subset \mathbb{R}^n, n \geq 2 \) we will mean the polynomial

\[
\sum_{s \in P \cap \mathbb{Z}^n} \psi_P(s)x^s
\]

with \( \psi_P(s) \) defined by (3.2). This polynomial is indeed hypergeometric since by Lemma 3.4 it satisfies a hypergeometric system of partial differential equations and cannot be represented as the sum of two or more linearly independent solutions to this system.

By construction, a translation of the defining polytope \( P \) by an integer vector results in multiplication of the corresponding hypergeometric polynomial with a monomial. Thanks to Lemma 2.6 this does not affect the amoeba of (3.3). Throughout the rest of the paper we identify polytopes that are translations of each other with respect to an integer vector.
Remark 3.6  The hypergeometric polynomial introduced in Definition 1.1 satisfies the hyper-
geometric system of the smallest possible holonomic rank among all hypergeometric systems
that admit an irreducible polynomial solution with the support $P$. This property can be used
as the definition of a $P$-supported hypergeometric polynomial.

In one dimension, Definition 1.1 yields a class of polynomials that is far too small to be inter-
esting. Namely, for the segment $[a, b] \subset \mathbb{R}$ with the integer endpoints $a, b$ the corresponding
polynomial is given by $x^a(x + 1)^{b-a}$. It is hypergeometric in the sense of Definition 1.1
and satisfies a hypergeometric differential equation (that is a special instance of (2.3)) of the
smallest possible holonomic rank 1. In what follows we focus on the multivariate case.

Example 3.7  Here we compute hypergeometric polynomials associated with certain families
of integer convex polytopes and investigate their properties.

1. Direct product of segments  This is a multivariate example of a hypergeometric polynomial
which is not optimal, even though its vanishing locus is. If the polytope $P$ in Definition 1.1
is the direct product of segments, we may without loss of generality assume it to be
$P = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$ for $a_j \in \mathbb{N}$. In this case the Ore–Sato coefficient (3.2)
is given by

$$
\prod_{j=1}^{n} \Gamma(s_j + 1)^{-1}.
$$

The corresponding hypergeometric polynomial is a constant multiple of $\prod_{j=1}^{n} (x_j + 1)^{a_j}$.
Its amoeba is the union of the coordinate hyperplanes in $\mathbb{R}^n$. The zero locus of this
polynomial is optimal in the sense of Definition 2.8.

2. A simplex.  Let now the polytope $P \subset \mathbb{R}^n$ be defined as the convex hull of the origin and
the points $(0, \ldots, k, \ldots, 0)$ ($k$ in the $j$-th position) for $j = 1, \ldots, n$. The corresponding
Ore–Sato coefficient is given by

$$
\left( \frac{\prod_{\varepsilon_j = \pm 1}^{n} \Gamma(\varepsilon_1 s_1 + \ldots + \varepsilon_n s_n + 2)}{\Gamma(2-s-t) \Gamma(2-s+t) \Gamma(2+s-t) \Gamma(2+s+t)} \right)^{-1}.
$$

The hypergeometric polynomial defined by this coefficient is a constant multiple of $(x_1 + \ldots + x_n + 1)^k$. Its zero locus is optimal in the sense of Definition 2.8, its amoeba
being just a hyperplane amoeba [7].

3. Cross-polytopes.  Recall that the $n$-dimensional cross-polytope is the convex polytope
whose vertices are all the permutations of $(\pm 1, 0, \ldots, 0) \in \mathbb{R}^n$. The only integer point
of a cross-polytope that is not its vertex is the origin. The Ore–Sato coefficient defined
by the $n$-dimensional cross-polytope is given by

$$
\left( \prod_{\varepsilon_j = \pm 1}^{n} \Gamma(\varepsilon_1 s_1 + \ldots + \varepsilon_n s_n + 2) \right)^{-1}.
$$

The hypergeometric polynomial supported in a cross-polytope is not necessarily optimal.
For instance, let $n = 2$ and the polytope $P$ be the two-dimensional cross-polytope with
the vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$. The Ore–Sato coefficient associated with this
polytope is

$$
\left( \frac{\prod_{\varepsilon_j = \pm 1}^{n} \Gamma(\varepsilon_1 s_1 + \ldots + \varepsilon_n s_n + 2)}{\Gamma(2-s-t) \Gamma(2-s+t) \Gamma(2+s-t) \Gamma(2+s+t)} \right)^{-1}.
$$
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and the corresponding hypergeometric polynomial is given (up to a monomial multiple that is unimportant due to Lemma 2.6) by \( p(x, y) = x + y + 4xy + x^2y + xy^2 \). This polynomial is not optimal. This can be seen by the direct computation of its amoeba. An alternative way to prove this is to observe that the origin must belong to the component of order \((1, 1)\), if any, in the complement of the amoeba of the polynomial \( x + y + axy + x^2y + xy^2 \). However, \( p(-1, -1) = 0 \) while the point \((-1, -1)\) is mapped to the origin by the map Log. In fact, \( p(x, y) \) is “on the boundary” of the set of optimal polynomials supported in \( P \).

We remark that the generic polynomial

\[
p(x_1, \ldots, x_n) = c + \sum_{j=1}^{n} \left( a_j x_j + \frac{b_j}{x_j} \right)
\]

in the \( n \)-dimensional cross-polytope and having positive coefficients \( a_j, b_j, c \) is optimal if and only if \( \sum_{j=1}^{n} \sqrt{a_j b_j} < \frac{c}{2} \). Thus the hypergeometric polynomial defined by the \( n \)-dimensional cross-polytope is optimal if and only if \( n > 2 \).

4. The Hirzebruch surface. Recall that the Hirzebruch surface \( \mathbb{F}_1 \) is defined by the fan generated by \((1, 0), (0, 1), (-1, 1)\) and \((0, -1)\). The hypergeometric polynomial supported in (a translation of) the convex hull of these vectors is defined by the Ore–Sato coefficient

\[
\frac{\Gamma(3 - t)\Gamma(4 - s - t)\Gamma(t - s + 2)\Gamma(2s + t - 1)}{(2s + t - 1)}.
\]

This polynomial is a constant multiple of \( h(x, y) = 3x + 12xy + 2x^2y + 2y^2 + 3xy^2 \) and is optimal.

The hypergeometric polynomials introduced in Definition 1.1 enjoy properties that are parallel to the properties of classical hypergeometric polynomials. Indeed, for any integer convex polytope there is only one (up to a constant multiple) hypergeometric polynomial of the form (3.3) supported in this polytope; this polynomial is dense; it satisfies the hypergeometric system \( \text{Horn}(\psi_P(s)) \). The counterpart of the properties of the roots of (3.3) is given by the next theorem which is the main result of the paper.

**Theorem 3.8** Let

\[
p(x) = \sum_{s \in P \cap \mathbb{Z}^n} \psi_P(s)x^s
\]

be the hypergeometric polynomial supported in the convex integer polytope \( P \subset \mathbb{R}^n, n \geq 2 \) (see Definition 1.1). Assume that the subdivision of \( P \) which is dual to the tropical hypersurface associated with the tropical polynomial \( p_{\text{tr trop}} \) defined by

\[
\sum_{s \in P \cap \mathbb{Z}^n} \psi_P(s)x^s
\]
Hypergeometric polynomials are optimal

\[ p_{\text{trop}}(\zeta) := \max_{s \in P \cap \mathbb{Z}^n} \{ \log |\psi_P(s)| + \langle s, \zeta \rangle \}, \tag{3.4} \]

is a triangulation. Then the hypergeometric polynomial \( p(x) \) is optimal.

**Proof** Multiplication of a polynomial with a nonzero constant does not affect its amoeba. We may thus without loss of generality assume that (possibly after a suitable normalization) \(|\psi_P(s)| \geq 1\) for every vertex \( s \in \text{vert}(P) \).

By definition, the function \( \psi_P(s) \) is well-defined, finite and positive on \( P \). Furthermore, by the Bohr-Mollerup theorem \( \psi_P(s) \) is a strictly logarithmically concave function on \( P \).

We will need the notion of the weighted compactified amoeba of a polynomial [14].

**Definition 3.9** (See [23] and [14].) The weighted moment map associated with the algebraic hypersurface \( \{ x \in \mathbb{C}^n : f(x) := \sum_{s \in S} a_s x^s = 0 \} \) is defined to be

\[ \mu_f(x) := \frac{\sum_{s \in S} s \cdot |a_s| |x^s|}{\sum_{s \in S} |a_s| |x^s|}. \]

It follows from the general theory of moment maps [11] that \( \mu_f(\mathbb{C}^n) \subseteq \mathcal{N}_f \).

**Definition 3.10** By the weighted compactified amoeba of a Laurent polynomial \( f(x) \) we mean the set \( \mu_f(\{ f(x) = 0 \}) \). We denote it by \( \text{WCA}(f) \).

Recall that the Hadamard power of order \( r \in \mathbb{R} \) of a polynomial \( f(x) = \sum_{s \in S} a_s x^s \) is defined to be \( f^{[r]}(x) := \sum_{s \in S} a_s^r x^s \). By [14, Theorem 3.3] the set-theoretical limit

\[ \mathcal{P}^\infty_{p(x)} := \lim_{r \to \infty} WCA(p^{[r]}(x)) \tag{3.5} \]

is a polyhedral complex such that its complement in \( P \) has the same topology as the complement of the amoeba \( A_{p(x)} \), i.e., \( \pi_0(\mathbb{R}^n \setminus A_{p(x)}) = \pi_0 \left( P \setminus \mathcal{P}^\infty_{p(x)} \right) \). The polyhedral complex \( \mathcal{P}^\infty_{p(x)} \) for the Hirzebruch polynomial is depicted in Fig. 4 (c) inside the Newton polygon of that polynomial. An approximation of the polyhedral complex \( \mathcal{P}^\infty_{p(x,y)} \) is shown in Fig. 9c.

Let \( \lambda \) be a positive real number. It follows from the definition (3.4) of the tropical polynomial \( p_{\text{trop}}(\xi) \) that \( p_{\text{trop}}(\lambda \xi) = \lambda p_{\text{trop}}(\xi) \) and hence for any \( \lambda > 0 \) the polynomial \( p^{[\lambda]}(x) \) satisfies the assumptions of Theorem 3.3 in [14]. Using this theorem we conclude that \( \pi_0 \left( \mathbb{R}^n \setminus A_{p^{[\lambda]}(x)} \right) = \pi_0 \left( P \setminus \mathcal{P}^\infty_{p^{[\lambda]}(x)} \right) \). Furthermore, by the definition of \( \mathcal{P}^\infty_{p(x)} \) we have \( \mathcal{P}^\infty_{p(x)} = \mathcal{P}^\infty_{p^{[\lambda]}(x)} \).

We now follow the arguments in the proof of Theorem 2 in [18] to show that for sufficiently big \( \lambda \) the polynomial \( p^{[\lambda]}(x) \) is optimal. Given a polynomial \( f(x) = \sum_{s \in S} a_s x^s \), we define (cf. [18, p. 357])

\[ m_s(f) := \inf_{x \in \mathbb{R}^n} \frac{\sum_{s' \neq s} |a_{s'}| \exp(x, s')}{|a_s| \exp(x, s)}. \]

It follows from the argument principle that the amoeba of the polynomial \( f \) has a component of order \( s \) in its complement whenever \( m_s(f) < 1 \). A polynomial is optimal if the complement of its amoeba has components of all possible orders. Using [18, p. 358] we conclude that
$m_s(p^{[λ]}) \to 0$ as $λ \to +∞$ for any $s ∈ P \cap \mathbb{Z}^n$. Hence the polynomial $p^{[λ]}(x)$ is optimal provided that $λ$ is sufficiently big. The above arguments yield the equalities

$$
\pi_0(\mathbb{R}^n \setminus A_{p(λ)}) = \pi_0(P \setminus \mathcal{P}_∞) = \pi_0(P \setminus \mathcal{P}_∞^{[λ]}(x)) = \pi_0(\mathbb{R}^n \setminus A_{p^{[λ]}(x)})
$$

and hence for all such values of $λ$ components of all orders are present in the complement of the amoeba $A_{p(λ)}$, i.e., the polynomial $p(x)$ is optimal. The proof is complete.

We observe that the hypergeometric polynomial in Example 3.7 3) does not satisfy the assumption of Theorem 3.8 since the tropical polynomial (3.4) is identically zero in this case. Numerous computer experiments suggest that the non-degeneracy condition may only fail for a few hypergeometric polynomials of low degree provided that the combinatorial structure of the defining polytope is rich enough. All examined hypergeometric polynomials of sufficiently high degree were found to be optimal. However, to provide an exhaustive list of all degenerate polytopes in all dimensions which might lead to nonoptimal hypergeometric polynomials appears to be a task of formidable combinatorial complexity.

The class of dense optimal multivariate polynomials with $\mathbb{Z}^n$-convex supports is of course much wider than the class of hypergeometric polynomials. By [18] the coefficient of a polynomial only has to be “logarithmically concave enough” for the polynomial itself to be optimal.

**Example 3.11** The bivariate Ore–Sato coefficient

$$
φ(s, t) = (Γ(t + 1)Γ(1 + 6s − 3t)Γ(31 − 6s − 2t))^{-1}
$$

defines a confluent holonomic hypergeometric system with the polynomial solution

$$
p_0(x, y) = 1 + 593775x + 86493225x^2 + 86493225x^3 + 593775x^4 + x^5 + 39331656000xy + 349363434420000x^2y + 558981495072000x^3y + 2163241080000x^4y + 54513675216000x^5y^2 + 211295005137216000x^2y^2 + 6867087666959520000x^3y^2 + 10357598291040000x^4y^2 + 15382276373989324800000x^5y^2 + 16920504011388257280000x^3y^3 + 33807200821954560000x^4y^3 + 3045690722049886310400000x^5y^3 + 639595051630476125184000000x^3y^4 + 18420337486957712405299200000x^3y^5 + 36840674973915424810598400000x^3y^6.
$$

The support of this optimal hypergeometric polynomial is $\mathbb{Z}^2$-convex and has a triangular convex hull. The amoeba of $p_0(x, y)$ is depicted in Fig. 5.
**Remark 3.12** Recall that the Bergman kernel of a complex ellipsoidal domain is given by a rational hypergeometric function [16]. One can check that the numerators of such rational functions are not necessarily optimal polynomials. The singular divisors of the GKZ-hypergeometric functions [2] are known to be solid [16]. Thus the optimal property of the divisors of hypergeometric polynomials cannot be extended to the classes of rational or algebraic hypergeometric functions.

### 4 Alternative definitions of a hypergeometric polynomial

Polynomial instances of hypergeometric functions in one and several variables are very diverse. They comprise the classical Chebyshev polynomials of the first and the second kind, the Gegenbauer, Hermite, Jacobi, Laguerre and Legendre polynomials as well as their numerous multivariate analogues [5].

Despite the diversity of families of hypergeometric polynomials, most of them share the following key properties that justify the usage of the term “hypergeometric”:

1. The polynomials are dense (possibly after a suitable monomial change of variables). That is, such a polynomial contains all the monomials whose exponent vectors are the integer points in its Newton polytope.
2. The coefficients of a hypergeometric polynomial are related through a linear recursion with polynomial coefficients.
3. For univariate polynomials, there is typically a single representative (up to a suitable normalization) of a given degree within a family of hypergeometric polynomials. This applies, in particular, to the Chebyshev, Hermite, and Jacobi polynomials as well as to several other families of polynomials in one complex variable.
4. All polynomials in the family satisfy a linear homogeneous differential equation of low fixed order with polynomial coefficients and fixed principal symbol (or a system of such equations) whose parameters encode the degree of a polynomial. For instance, the Chebyshev polynomials of both kinds, the Gegenbauer, Jacobi, and Legendre polynomials all satisfy second-order linear homogeneous differential equations.
5. In the case of one dimension, the absolute values of the roots of a classical hypergeometric polynomial are all different (possibly after a suitable monomial change of variables).
6. Many of hypergeometric polynomials enjoy various extremal properties. For instance, the Chebyshev polynomial of the first kind can be defined as the degree $n$ polynomial with the leading coefficient $2^{n-1}$ and least deviation from zero on $[-1,1]$.

Most of the well-known families of hypergeometric polynomials [5,24] satisfy certain linear differential equations with polynomial coefficients that are special instances of (2.3). However, the family of all holonomic systems of partial differential equations of the form (2.3) is far too vast to serve as a definition of a hypergeometric polynomial. In fact, from the point of view of the general Definition 2.1 given above, any polynomial in any number of variables is a hypergeometric function. This is made precise in the following statement.

**Theorem 4.1** For any polynomial $p(x) \in \mathbb{C}[x_1, \ldots, x_n]$ there exists a nonconfluent [16] holonomic hypergeometric system of the form (2.3) having $p(x)$ as one of its solutions. Moreover, it can be chosen in such a way that the hypergeometric ideal in the Weyl algebra defining this system admits a basis that consists of a commutative family of differential operators.
Proof In fact, any given polynomial is annihilated by a family of hypergeometric ideals that contains continuously many elements. To prove the theorem, we present an explicit representative with desired properties.

Let \( p(x) = \sum_{\alpha \in S} c_{\alpha} x^{\alpha} \) be a polynomial with the support \( S \). Here and throughout the proof we assume \( S \) to be finite. We denote by \( \#S \) the cardinality of \( S \) and let \( |\alpha| = \alpha_1 + \ldots + \alpha_n \). For \( s \in \mathbb{C}^n \) define the Ore–Sato coefficient \( \varphi(s) \) by

\[
\varphi(s) = \prod_{\alpha \in S} (s_1 + \ldots + s_n - |\alpha|) \prod_{\alpha \in S} \prod_{j=1}^n (s_j - \alpha_j).
\]

By Definition 2.3, the action of the \( j \)-th hypergeometric differential operator in the system defined by this Ore–Sato coefficient on \( p(x) \) is given by

\[
\left( x_j \prod_{\alpha \in S} (\theta_1 + \ldots + \theta_n - |\alpha|) - \prod_{\alpha \in S} (\theta_j - \alpha_j) \right) \sum_{\beta \in S} c_{\beta} x^{\beta} = \sum_{\beta \in S} c_{\beta} \left( \left( x_j \prod_{\alpha \in S} (\theta_1 + \ldots + \theta_n - |\alpha|) \right) x^{\beta} - \left( \prod_{\alpha \in S} (\theta_j - \alpha_j) \right) x^{\beta} \right) \equiv 0
\]

since \( \mathbb{C}[\theta_1, \ldots, \theta_n] \) is a commutative subring in the Weyl algebra and \( (\theta_1 + \ldots + \theta_n - |\alpha|) x^{\alpha} = (\theta_j - \alpha_j) x^{\alpha} \equiv 0 \) for any \( j = 1, \ldots, n \).

The hypergeometric system defined by the Ore–Sato coefficient (4.1) is nonconfluent by definition. By Theorem 2.8 in [20] this system is holonomic with the holonomic rank \( \#S \). The fact that it is generated by a commutative family of hypergeometric operators follows from Lemma 2.5 in [20]. □

Since every monomial in \( p(x) \) is annihilated by each operator in the hypergeometric system defined by (4.1), the same argument works for any Puiseux polynomial with arbitrary exponents in \( \mathbb{C}^n \). Although \( p(x) \) is in the kernels of the differential operators that form a holonomic hypergeometric system, the monomials in \( p(x) \) are in no way related to each other. Thus a meaningful definition of a hypergeometric polynomial based on (2.3) requires further assumptions on the Ore–Sato coefficient that defines the system.

Despite varying terminology, the classical hypergeometric series \( F_1, \ldots, F_4, G_1, \ldots, G_3, H_1, \ldots, H_7 \) as well as other entries of the Horn list [6] are universally considered to be intrinsically hypergeometric. For resonant parameters [21], many of these series terminate and turn out to be bivariate hypergeometric polynomials.

Appell’s \( F_1 \) is one of the most important classical hypergeometric series since by the results of [6] any bivariate hypergeometric system of second-order equations and holonomic rank 3 can be transformed into the system for \( F_1 \) or a particular limiting case of this system. The following statement follows from Theorem 3.8.

Corollary 4.2 The polynomial instances of the Appell \( F_1(a, b_1, b_2, c; x, y) \) hypergeometric function are optimal for \( a, b_1, b_2, -c < 0 \) and \( a > b_1 + b_2 \).

Proof The imposed conditions on the parameters of \( F_1(a, b_1, b_2, c; x, y) \) yield a one-to-one correspondence between the \( \Gamma \)-factors in the coefficient of the power series expansion of \( F_1 \) and the sides of the Newton polygon of its polynomial instance in question. This polynomial is therefore hypergeometric in the sense of Definition 1.1 and satisfies the assumptions in Theorem 3.8. □
Hypergeometric polynomials are optimal

In Fig. 6 we depict the amoeba of the optimal hypergeometric polynomial $F_1(-5, -4, -4, 3; x, y)$.

Observe however that not every polynomial instance of $F_1(a, b_1, b_2, c; x, y)$ is optimal. The optimal property is in general not possessed by the $F_1$ polynomials whose Newton polytopes do not have sides that are orthogonal to the gradients of the linear forms in the defining Ore–Sato coefficient. For instance, $F_1(-4, 5, -7, 9; x, y)$ is not an optimal polynomial, its Newton polygon being just a triangle.

We further remark that the zero locus of a rational instance of a classical hypergeometric function need not be an optimal hypersurface. For example, the numerator of the rational function $F_2(5; 3/2, 1; -1/2, 2; x, y)$ is not an optimal polynomial.

5 Examples

In this section we collect examples of multivariate hypergeometric polynomials together with their Newton polytopes and amoebas.

**Example 5.1** The hypergeometric Horn system defined by the Ore–Sato coefficient

$$\varphi(s, t) = \Gamma (s - 6) \Gamma (s + t - 10) \Gamma (t - 6) \Gamma (-s + t - 4) \Gamma (-s)$$

admits the following polynomial solution: $p_1(x, y) = 21x^2 + 64x^3 + 21x^4 + 126xy + 2016x^2y + 4704x^3y + 2016x^4y^2 + 2016x^5y + 21y^2 + 2016xy^2 + 22050x^2y^2 + 47040x^3y^2 + 22050x^4y^3 + 2016x^5y^3 + 21x^6y^2 + 64y^3 + 4704xy^3 + 47040x^2y^3 + 98000x^3y^3 + 47040x^4y^4 + 4704x^5y^3 + 64x^6y^3 + 21y^4 + 2016xy^4 + 22050x^2y^4 + 47040x^3y^4 + 22050x^4y^5 + 2016x^5y^4 + 21x^6y^4 + 126xy^5 + 2016x^2y^5 + 4704x^3y^5 + 2016x^4y^5 + 126x^5y^5 + 21x^6y^6 + 64x^3y^6 + 21x^4y^6$.

(The system itself is too cumbersome to display and we omit it.) This polynomial turns out to be optimal. The Newton polygon and the amoeba of $p_1(x, y)$ are shown in Fig. 7.
Example 5.2 The next example shows that the number of $\Gamma$-factors in the Ore–Sato coefficient of an optimal hypergeometric polynomial can be strictly smaller than the number of faces of its Newton polytope. The hypergeometric system defined by the Ore–Sato coefficient
\[
\varphi(s, t) = \Gamma(s + 2t - 5) \Gamma(-2s - t - 4) \Gamma(-s - 5t + 1)
\] (5.1)
has the following polynomial solution:
\[
p_2(x, y) = 2421619200x^5 + 172972800x^6 + 2882880x^7 + 14560x^8 + 20x^9 + 174356582400x^2y + 48432384000x^3y + 2421619200x^4y + 34594560x^5y + 160160x^6y + 208x^7y + 2421619200xy^2 + 691891200x^2y^2 + 21621600x^3y^2 + 160160x^4y^2 + 286x^5y^2 + 524160xy^3 + 14560x^2y^3 + 56x^3y^3 + 32y^4 + xy^4.
\]
The support of $p_2(x, y)$ (bounded by the singular divisors of the corresponding Ore–Sato coefficient) and its amoeba are depicted in Fig. 8.

Example 5.3 The bivariate hypergeometric polynomial supported in the quadrilateral with the vertices $(2, 0), (3, 2), (2, 3)$ and $(0, 1)$ is given by $p_3(x, y) = 240x^2 + 3y + 240xy + 1080x^2y + 30xy^2 + 180x^2y^2 + 36x^3y^2 + 2x^2y^3$. Figure 9a–c) show the affine amoeba $\mathcal{A}_{p_3}$.
Hypergeometric polynomials are optimal

The amoeba, the compactified amoeba, the weighted compactified amoeba of the 6th Hadamard power of \( p_3(x, y) \) and a vanishing connected component of the complement of a deformation of \( \mathcal{WCA}(p_3(x, y)) \)

Figure 9d shows the vanishing connected component with the order \((2, 2)\) in the complement of the weighted compactified amoeba of a deformed version of \( p_3(x, y) \). We remark that the small component vanishes exactly at the point with the coordinates \((2, 2)\), that is, at the order of this component.

**Example 5.4** The first maximal minor of the Toeplitz matrix

\[
\begin{pmatrix}
x & y & 1 & 0 & 0 & 0 \\
1 & x & y & 1 & 0 & 0 \\
0 & 1 & x & y & 1 & 0 \\
0 & 0 & 1 & x & y & 1 \\
0 & 0 & 0 & 1 & x & y \\
\end{pmatrix}
\]

is the degree 6 bivariate Chebyshev polynomial of the second kind [1]. It is optimal in the coordinates \( \xi = xy, \eta = y^2/x \).

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