TITS ALTERNATIVE IN HYPERKÄHLER MANIFOLDS

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Dedicated to Professor Yukihiko Namikawa on the occasion of his sixtieth birthday

ABSTRACT. We show an analogous result of the famous Tits alternative for a group $G$ of birational automorphisms of a projective hyperkähler manifold: Either $G$ contains a non-commutative free group or $G$ is an almost abelian group of finite rank. As an application, we show that the automorphism groups of the so-called singular K3 surfaces contain non-commutative free groups.

1. INTRODUCTION - BACKGROUND AND MAIN RESULTS

Our main results are Theorems (1.1)-(1.3) and (2.6). Though our actual proof is entirely algebraic, these results are much motivated by recent important works about automorphisms of K3 surfaces from the view of complex dynamics, notably, a work of McMullen [Mc] and works of Cantat [Ca1, 2].

1. Throughout this note, we work over the complex number field $\mathbb{C}$. By a hyperkähler manifold we mean a compact simply-connected complex Kähler manifold $M$ having everywhere non-degenerate holomorphic 2-form $\sigma_M$ s.t. $H^2(M, \Omega_M) = \mathbb{C} \sigma_M$. Such manifolds are even dimensional and form one of the three fundamental building blocks of compact Kähler manifolds of vanishing first Chern class. K3 surfaces are nothing but hyperkähler manifolds of dimension 2. We also note that the Hilbert scheme $\text{Hilb}^n(S)$, of 0-dimensional closed subschemes of length $n$ of a K3 surface $S$, is a hyperkähler manifold of dimension $2n$. We refer to the readers the excellent account [JHG, Part III] by Huybrechts about basics of hyperkähler manifolds. All of what we need here are also reviewed in [Og1, Section 2]. We denote the bimeromorphic (resp. biholomorphic) automorphism group of $M$ by $\text{Bir} \ M$ (resp $\text{Aut} \ (M)$). By an almost abelian group of finite rank, say $r$, we mean a group $G$ which containing a subgroup $A$ of finite index which fits in with the exact sequence

$$1 \rightarrow F \rightarrow A \rightarrow \mathbb{Z}^r \rightarrow 0.$$ 

(See eg. [Og1, Section 9].) The extreme counter part of an almost abelian group is a group which contains a non-commutative free group, or equivalently, a group having a subgroup isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. 

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2. The aim of this note is to study groups of birational automorphisms of a projective hyperkähler manifold. Note that any bimeromorphic automorphism is birational when a manifold is projective. Our main results are as follows:

**Theorem 1.1.** Let $M$ be a projective hyperkähler manifold and $G$ be a subgroup of $\text{Bir}(M)$. Then $G$ satisfies either:

1. $G$ is an almost abelian group of finite rank, or
2. $G$ contains a non-commutative free group.

In particular, so are $\text{Bir}(M)$ and $\text{Aut}(M)$.

**Theorem 1.2.** (Crescent Theorem) Let $M$ be a hyperkähler manifold having at least two complex torus fibrations, say $\varphi_i : M \to B_i$ ($i = 1, 2$). Then $M$ is projective. Assume furthermore that both $\varphi_i$ are Jacobian fibrations of positive Mordell-Weil rank. Then $\text{Bir}(M)$ contains a non-commutative free group.

Here by a complex torus fibration, we mean a surjective morphism $f : M \to B$ over a normal projective variety $B$, whose general fiber is a positive dimensional complex torus. (See [Ma] for fibered hyperkähler manifolds.) We call $f$ a Jacobian fibration if $M$ is projective and $f$ admits a rational section, say $O$. The rational sections of a Jacobian fibration $f$ naturally form an abelian subgroup of $\text{Bir}(M)$. We call this group the Mordell-Weil group of $f$ and denote it by $\text{MW}(f)$. By abuse of language, we call $\text{MW}(f)$ of positive rank if it contains an element of infinite order. See also [Ca1, 2], [CF], [Og2] for relevant results.

**Theorem 1.3.** Let $S$ be a singular K3 surface. Put $M := \text{Hilb}^n(S)$. Then, both $\text{Aut}(S)$ and $\text{Aut}(M)$, whence $\text{Bir}(M)$, contain a non-commutative free group.

Here a singular K3 surface means a K3 surface having maximum Picard number 20 ([Si]). Theorem (1.3) is a generalization of a famous result of Shioda and Inose [ibid], that *every singular K3 surface has an automorphism of infinite order*, in a non-commutative direction. Note that there is a projective K3 surface $S$ s.t. $\rho(S) = \rho$ and $|\text{Aut}(S)| < \infty$ for each integer $\rho$ with $1 \leq \rho \leq 19$ ([Ko, Ni]). So a statement similar to Theorem (1.3) is no more true for K3 surfaces of smaller Picard number $\rho \leq 19$. Theorem (1.3) also shows that the second alternative in Theorem (1.1) really occurs in each dimension.

3. It is also interesting to compare these results with the following existing result and interesting open Questions:

**Theorem 1.4.** [Og1] Let $M$ be a non-projective hyperkähler manifold. Then both $\text{Bir}(M)$ and $\text{Aut}(M)$ are almost abelian groups of finite rank. In particular, they are finitely generated.

**Question 1.5.** Let $M$ be a projective hyperkähler manifold. Are $\text{Bir}(M)$ and $\text{Aut}(M)$ finitely generated?

**Question 1.6.** (J.M. Hwang) Let $M$ be a projective hyperkähler manifold. Is $\text{Aut}(M)$ is of finite index in $\text{Bir}(M)$?

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1The second Theorem (1.2) is inspired by a discussion with Y. Kawamata about remarkable difference of the appearance of crescent in Japan and Korea; it looks vertical in Japan while horizontal in Korea. This reminded me of the situation in Theorem (1.2).
In dimension 2, the first question is affirmative by Sterk [St] and the second one is trivial. The second question is asked by J.M. Hwang after my talk on a relevant subject at KIAS on March 2005.

4. Our proof of Theorems (1.1)-(1.3) is based on two famous, deep results in linear algebraic groups; Lie-Kolchin Theorem and Tits Theorem ([Hm], [Ti]; see also [Ha] and Section 2 for the statements) and the notion of Salem polynomial (see eg. [Mc] and also Section 2). Unfortunately, our proof does not tell us much about algebro-geometrical reason why non-abelian free groups should be in the birational automorphism groups in Theorem (1.2). It would be interesting to find a more "geometrically visible" proof of Theorem (1.2), especially when $M$ is a K3 surface. For this, an observation of Cantat [Ca2] might give us some hint.

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2. Algebraic preparations

The goal of this section is Theorem (2.6), the technical heart in the proof of our main results. In 1, we recall Lie-Kolchin Theorem and Tits Theorem (Tits alternative). Both are very important for our proof. We recall basic notions about lattices in 2 and a few facts about Salem polynomials in 3. We show Theorem (2.6) in 4.

1. For simplicity, we shall work only over $\mathbb{C}$. Let $V \neq \{0\}$ be a finite dimensional vector space over $\mathbb{C}$. We regard the general linear group $GL(V)$ as an algebraic group defined over $\mathbb{C}$, with Zariski topology. We identify $GL(V)$ with the group $GL(V)(\mathbb{C})$ of $\mathbb{C}$-valued points in a usual way. A subgroup of $GL(V) = GL(V)(\mathbb{C})$ simply means a subgroup as an abstract group. If $G$ is a subgroup of $GL(V)$, then its Zariski closure $\overline{G}$, as well as the identity component of $\overline{G}$, is an algebraic subgroup of $GL(V)$. First, we notice the following well-known:

**Lemma 2.1.** Let $G$ be a solvable subgroup of $GL(V)$. Then:

(1) Any subgroup of $G$ and any quotient group of $G$ are solvable.

(2) $\overline{G}$ is also solvable.

**Proof.** We only show (2). It suffices to check that $[G,\overline{G}] = [G,G]$. Note that $[G,G]$ is closed in $\overline{G}$ (See for instance [Hm, 17.2]). Thus $[G,G] \subset [\overline{G},\overline{G}]$.

Let us show the other inclusion. Take $g \in G$. Let us define the map $\alpha_g$ by

$$\alpha_g : \overline{G} \longrightarrow \overline{G} ; \ f \mapsto f^{-1}g^{-1}fg.$$ 

Clearly, $\alpha_g$ is continuous and satisfies $\alpha_g(G) \subset [G,G]$. Thus $\alpha_g(\overline{G}) \subset [\overline{G},\overline{G}]$. Hence $[G,G] \subset [\overline{G},\overline{G}]$. Let $f \in \overline{G}$. Let us define the map $\beta_f$ by

$$\beta_f : \overline{G} \longrightarrow \overline{G} ; \ g \mapsto f^{-1}g^{-1}fg.$$ 


Clearly, $\beta_f$ is continuous and satisfies $\beta_f(G) \subset [G, G]$. Since $[G, G] \subset [G, G]$, we have $\beta_f(G) \subset [G, G]$ as well. Thus $\beta_f(G) \subset [G, G]$ and hence $[G, G] \subset [G, G]$. □

Lie-Kolchin Theorem and Tits alternative are the following:

**Theorem 2.2.** (Lie-Kolchin Theorem, see eg. [Hm, Chap. VII, 17.6]) Let $G$ be a connected solvable subgroup of $\text{GL}(V)$. Then $G$ has a common eigenvector in $V$.

**Theorem 2.3.** (Tits alternative [Ti]) Let $G$ be a subgroup of $\text{GL}(V)$. Then $G$ is either virtually solvable or contains a non-commutative free group.

Here a group $G$ is called virtually solvable if $G$ contains a solvable subgroup of finite index. We also notice that any non-commutative free group contains $\mathbb{Z} \ast \mathbb{Z}$, i.e. the free group of rank 2.

2. By a lattice $L = (L, (\ast, \ast))$, we mean a pair consisting of a free abelian group $L \simeq \mathbb{Z}^r$ and its (possibly degenerate) integral-valued symmetric bilinear form $(\ast, \ast) : L \times L \to \mathbb{Z}$. By $L_K$, we denote the scalar extension $L \otimes \mathbb{Z}K$ of $L$ by a field $K$. The signature of $L$ is the pair of the numbers of positive-, zero- and negative-eigenvalues of a symmetric matrix associated to $(\ast, \ast)$. We call $L$ hyperbolic (resp. parabolic, elliptic) if the signature is $(1, 0, r - 1)$ (resp. $(0, 1, r - 1)$, $(0, 0, r)$).

We call an element $v \in L \setminus \{0\}$ (resp. a sublattice $M$) primitive if $L/\langle v \rangle$ (resp. $L/M$) is torsion-free. We denote by $O(L)$ the group of isometries of a lattice $L$. Note that $|O(L)| < \infty$ if $L$ is elliptic.

*From now on until the end of this section, we choose and fix a hyperbolic lattice $L$ of rank $r$.*

Then the set $\{v \in L_R | (v^2) > 0\}$ consists of two connected components (w.r.t. Euclidean topology of $L_R$). We choose and fix one of them and denote it by $P(L)$. We call $P(L)$ the positive cone of $L$. In general, there is no canonical choice of the positive cone. (However, when $L$ is the Néron-Severi group of a projective hyperkähler manifold, with Beauville-Bogomolov-Fujiki’s bilinear form, we always choose the positive cone so that it contains ample classes.)

Let $\overline{P}(L)$ (resp. $\partial P(L)$) be the closure (resp. the boundary) of the positive cone in $L_R$ (w.r.t. Euclidean topology). By the Schwartz inequality, we have $(x, y) \geq 0$ for $x, y \in \overline{P}(L) \setminus \{0\}$ and the equality holds iff $R_{>0}x = R_{>0}y \in \partial P(L)$.

Let $M$ be a primitive sublattice of $L$. Then $M$ is either hyperbolic, parabolic, or elliptic, and $M_L^\perp$ is elliptic, parabolic, hyperbolic respectively. Here and hereafter we denote by $M_L^\perp$, the primitive sublattice $\{v \in L | (v, w) = 0 \forall w \in M\}$. Note that $M \cap M_L^\perp = \{0\}$ and $M \oplus M_L^\perp$ is of finite index in $L$ when $M$ is hyperbolic or elliptic, while $M \cap M_L^\perp = \mathbb{Z}e$ with $(e^2) = 0$ and $M + M_L^\perp$ is corank 1 in $L$ when $M$ is parabolic.

3. In [Mc] and [Og1], the notion of Salem polynomial plays a very crucial role. In our proof, it also plays an important role. We recall the definition:
Definition 2.4. An irreducible monic polynomial \( \Phi(t) \in \mathbb{Z}[t] \) of degree \( n \) is called a Salem polynomial if the complex roots of \( \Phi(t) = 0 \) consists of two real roots \( a \) and \( 1/a \) s.t. \( a > 1 \) and \( n - 2 \) roots on the unit circle \( S^1 \).

In our argument, we need the following result. This is a formal generalization of [Mc, Proposition 3.3]:

**Proposition 2.5.** Let \( L \) be a hyperbolic lattice of rank \( r \) and \( g \) be an element of \( O(L) \), preserving a positive cone \( \mathcal{P}(L) \). Then the irreducible factors of the characteristic polynomial \( \Phi_g(t) := \det (tI - g) \) of \( g \) includes at most one Salem polynomial; and the remaining factors are cyclotomic. In particular, if \( c \) is an eigenvalue of \( g \), then so is \( 1/c \).

**Proof.** Since \( \Phi_g(t) \in \mathbb{Z}[t] \) is monic, all the eigenvalues of \( g \) are algebraic integers. Since \( L \) is hyperbolic and \( g \in O(L) \), it follows that \( g \) has at most one eigenvalue (counted with multiplicity) outside \( S^1 \). Therefore by \( |\det g| = 1 \), \( g \) satisfies either that all eigenvalues are on \( S^1 \), or that \( g \) has exactly one eigenvalue outside \( S^1 \), say \( a \), exactly one eigenvalue inside \( S^1 \), say \( b \), and \( r - 2 \) eigenvalues on \( S^1 \).

In the first case, the eigenvalues are all roots of unity by Kronecker’s Theorem. So, the irreducible factors of \( \Phi_g(t) \) are cyclotomic.

Let us consider the second case. Both \( a \) and \( b \) are real by the uniqueness. Then, one can choose an eigenvector \( v \) with eigenvalue \( a \) in \( L_{\mathbb{R}} \). Then, by \( (v^2) = (g(v)^2) = a^2(v^2) \), we have \( (v^2) = 0 \). Thus, \( v \in \mathcal{P}(L) \) (after replacing \( v \) by \( -v \) if necessary). The same holds for an eigenvector with eigenvalue \( b \). Thus \( a > 1 \) and \( ab = 1 \) by \( g(\mathcal{P}(L)) = \mathcal{P}(L) \) and \( |\det (g)| = 1 \). Thus \( \det g = 1 \) as well. Let \( f(t) \in \mathbb{Z}[t] \) be the minimal monic polynomial of \( a \). Then \( f(t)|\Phi_g(t) \). Since \( \Phi_g(0) = \pm 1 \), we have also \( f(b) = 0 \). Thus \( f(t) \) is a Salem polynomial. The zeros of the monic polynomial \( \Phi_g(t)/f(t) \in \mathbb{Z}[t] \) are now on \( S^1 \). Thus, again by Kronecker’s Theorem, \( \Phi_g(t)/f(t) \) includes only cyclotomic polynomials as its irreducible factors. The last statement is now clear. Indeed, if \( c \in S^1 \), then \( 1/c = \overline{c} \) is a zero of \( \Phi_g(t) \). If \( c \not\in S^1 \), then \( c = a \) or \( b \), and \( 1/c = b \) or \( a \) is also a zero of \( \Phi_g(t) \). \( \square \)

4. Let us now formulate our key result:

**Theorem 2.6.** Let \( L \) be a hyperbolic lattice of rank \( r \) and \( G \) be a subgroup of \( O(L) \). Assume that \( G \) is virtually soluble. Then \( G \) is almost abelian of finite rank.

**Proof.** We shall show Theorem (2.6) by induction on \( r \). The result is clear if \( r = 1 \). So, we may assume that \( r \geq 2 \). We shall proceed the proof by dividing into five steps.

**Step 1.** By the assumption, \( G \) has a solvable subgroup \( N \) s.t. \( [G : N] < \infty \). Note that \( G \) is almost abelian of finite rank iff so is \( N \) (cf. [Og1, Section 9]). Then by replacing \( G \) by \( N \), we may assume that \( G \) is solvable. We will do so from now on.

**Step 2.** Put \( V := L_C \). We have a natural embedding:

\[
G \subset O(L) \subset GL(V)
\]

Let \( \overline{G} \) be the Zariski closure of \( G \) in \( GL(V) \) and \( S \) be the identity component of \( \overline{G} \), i.e. the irreducible component containing the identity \( 1 \). Since \( G \) is solvable, so is \( G \) by Lemma (2.1)(2). Thus, by Lemma (2.1)(1), \( S \) is a connected solvable subgroup...
of $GL(V)$. Since $G$ is an algebraic subset of a noetherian $GL(V)$, it has only finitely many irreducible components. Thus, $[G : S] < \infty$. For the same reason as in Step 1, we may now assume that $G$ is a subgroup of a connected solvable group by replacing $G$ by its finite index subgroup $G \cap S$, and that the positive cone $\mathcal{P}(L)$ is $G$-stable by further replacing $G$ by its index two subgroup (if necessary). We will do so from now on.

Step 3. Recall that $G < GL(r, \mathbb{Z})$. Then, if all elements of $G$ is of finite order, then their orders are universally bounded by some constant, say $N$. Then $G$ is a finite group by Burnside’s Theorem. In particular, $G$ is almost abelian of rank $0$ and we are done. So, we may assume that there is $g_0 \in G$ s.t. $\text{ord} \, g_0 = \infty$. We will do so from now on.

Step 4. Since $G$ is in a connected solvable subgroup, say $S$, of $GL(V)$, by applying Lie-Kolchin Theorem (2.2) for $S$, we find a common eigenvector of $G$, say $v \in V = L_C$. Set $g(v) = \alpha(g)v$ for $g \in G$. Then $\alpha$ defines a group homomorphism

$$\alpha : G \longrightarrow C^\times ; \, g \mapsto \alpha(g) .$$

Let $M$ be the minimal primitive sublattice of $L$ s.t. $v \in M_C$. This $M$ is $G$-stable. Indeed, since $g \in G$ is defined over $\mathbb{Z}$ and $g(v) = \alpha(g)v$, one has $M \cap g(M) \subset M$, $(M \cap g(M))_C = M_C \cap g(M)_C$ and $v \in M_C \cap g(M)_C$. Thus $v \in (M \cap g(M))_C$, and therefore $g(M) = M$ by the minimality of $M$.

Step 5. Note that $M$ is either elliptic, parabolic, or hyperbolic. We completes the proof by dividing into these three cases.

The case where $M$ is elliptic. In this case $M^\perp_L$ is hyperbolic. Let $K := \text{Ker}(r_M : G \longrightarrow O(M))$. Then $K$ is of finite index in $G$ and $K \subset O(M^\perp_L)$. Since $\text{rank} M^\perp_L < \text{rank} L$, $K$ is almost abelian of finite rank by the induction hypothesis. Thus so is $G$.

The case where $M$ is parabolic. In this case there is a unique primitive element $u \in M$ s.t. $(u^2) = 0$ and $u \in \partial \mathcal{P}(L)$. By the uniqueness of $u$ and by $G(\mathcal{P}(L)) = \mathcal{P}(L)$, this $u$ is $G$-stable. Since $u \in \partial \mathcal{P}(L) \cap L \setminus \{0\}$, it follows from [Og2, Proposition 2.9] that $G$ is almost abelian of finite rank.

The case where $M$ is hyperbolic. In this case $M^\perp_L$ is elliptic (possibly $0$).

Consider first the case where $M^\perp_L \neq \{0\}$, i.e. the case where $M \neq L$. Let $K := \text{Ker}(r_M : G \longrightarrow O(M^\perp_L))$. Then $K$ is of finite index in $G$ and $K \subset O(M)$. Since $\text{rank} M < \text{rank} L$ by the case assumption, $K$ is almost abelian of finite rank by the induction hypothesis. Thus so is $G$.

It remains to consider the case where $M = L$. In this case, the result follows from the next Lemma (2.7)(4).

Lemma 2.7. Assume that $M = L$. Then:

1. The homomorphism $\alpha : G \longrightarrow C^\times$ is injective. In particular, $G$ is an abelian group.

2. The characteristic polynomial $\Phi_{g_0}(t)$ is a Salem polynomial. Here $g_0$ is an element of $G$ with $\text{ord} \, g_0 = \infty$ in Step 3.
(3) Let \( g \in G \). Then there is \( \varphi(t) \in \mathbb{Q}[t] \) s.t. \( g = \varphi(g_0) \).
(4) \( G \) is an almost abelian group of finite rank.

Proof. Let us show (1). If \( \alpha(g) = 1 \), then \( v \in E(g, 1) \). Here \( E(g, 1) \subset L_C \) is the eigenspace of eigenvalue 1 of \( g \). Since \( g \) is defined over \( \mathbb{Z} \), there is a primitive sublattice \( E \subset L \) s.t. \( E_C = E(g, 1) \). Thus, by the minimality of \( L = M \), one has \( E = L \), i.e. \( g = 1 \). Thus \( \alpha \) is injective.

Let us show (2). If \( \alpha(g_0) \) is a root of unity, then there is a positive integer \( m \) s.t. \( \alpha(g_0^m) = 1 \). Then \( g_0^m = 1 \) by (1), a contradiction to \( \text{ord } g_0 = \infty \). Thus \( \alpha(g_0) \) is not a root of unity. Thus, by Proposition (2.5), \( \Phi_{g_0}(t) \) has a Salem polynomial, say \( f(t) \), as its irreducible factor. If \( \Phi_{g_0}(t) \neq f(t) \), then one can write \( \Phi_{g_0}(t) = f(t)h(t) \).

Here \( h(t) \) is a product of cyclotomic polynomials by Proposition (2.5). However, since \( (f(t), h(t)) = 1 \) and \( f(\alpha(g_0)) = 0 \), the decomposition \( \Phi_{g_0}(t) = f(t)h(t) \) would lead a non-trivial rational decomposition of \( L_Q \), say \( L_Q = L_1 \oplus L_2 \), s.t. \( v \in (L_2)_C \). However this contradicts the minimality of \( L = M \). Thus \( \Phi_{g_0}(t) = f(t) \) and the results follows.

Let us show (3). Since \( \Phi_{g_0}(t) \) is irreducible by (2), the eigenvalues of \( g_0 \) are mutually distinct. Thus the \( \mathbb{C} \)-linear space \( W' := \{ h \in \text{End}_C(L_C) | h g_0 = g_0 h \} \) is of dimension \( r \). Since \( g_0 \) is defined over \( \mathbb{Q} \), one has \( W' = W_C \), where \( W := \{ h \in \text{End}_Q(L_Q) | h g_0 = g_0 h \} \). Note that \( g_0^k (0 \leq k \leq r - 1) \) are linearly independent over \( \mathbb{Q} \) by the irreducibility of \( \Phi_{g_0}(t) \). Thus \( \langle g_0^k \rangle_{k=0}^{r-1} \) forms a basis of \( W \) over \( \mathbb{Q} \). Since \( G \subset W \) by (1), the result follows.

Let us show (4). Let \( F(\subset \mathbb{C}) \) be the minimal splitting field of \( \Phi_{g_0}(t) \) over \( \mathbb{Q} \). Let \( O_F \) be the ring of algebraic integers of \( F \) and \( U_F \) be the unit group of \( O_F \). Then \( \alpha(G) \subset F \) by (3). Both \( \alpha(g) \) and \( 1/\alpha(g) \) (\( g \in G \)) are zero of the characteristic polynomial \( \Phi_g(t) \in \mathbb{Z}[t] \) by Proposition (2.5). Thus both \( \alpha(g) \) and \( 1/\alpha(g) \) are algebraic integers in \( F \). Hence \( \alpha(G) \subset U_F \). By the Dirichlet unit Theorem, \( U_F \) is a finitely generated abelian group. Thus so is its subgroup \( \alpha(G) \). Since \( \alpha \) is injective, this implies the result.

This completes the proof of Theorem (2.6).

3. Proof of Theorems (1.1)-(1.3)

In this section, we shall prove Theorems (1.1)-(1.3). Let \( M \) be a projective hyperkähler manifold. Then the Néron-Severi group \( NS(M) \) is a hyperbolic lattice w.r.t. the Beauville-Bogomolov-Fujiki’s form. Let \( G \) be a subgroup of \( \text{Bir}(M) \). Then, there is a natural group homomorphism \( r_{NS} : G \rightarrow O(NS(M)) \). The kernel of \( r_{NS} \) is a finite group by [Og1, Corollary 2.7]. So, \( G \) contains a non-commutative free group if so does \( r_{NS}(G) \), and \( G \) is almost abelian of finite rank if so is \( r_{NS}(G) \).

Proof of Theorem (1.1). Assume that \( r_{NS}(G) \) does not contain a non-commutative free subgroup. Then, by the Tits alternative (2.3), \( r_{NS}(G) \) is virtually solvable. Thus \( r_{NS}(G) \) is almost abelian of finite rank by Theorem (2.6). Q.E.D.
Proof of Theorem (1.2). Since $B_i$ are assumed to be projective, one can write $\varphi_i = \Phi_{|\varphi_i^* H_i|}$. Here $H_i$ is a very ample divisor of $B_i$ and $\Phi_{|\varphi_i^* H_i|}$ is the morphism associated with the complete linear system $|\varphi_i^* H_i|$. Put $e_i := [\varphi_i^* H_i] \in NS(M)$. Then $(e_i^2) = 0$ (cf. [Ma]). Since $B_i$ is $Q$-factorial and $\rho(B_i) = 1$ by Matsushita [ibid], it follows that $R_{>0} e_i \neq R_{>0} e_2$. (Here we note that all the arguments in [ibid] are valid if we assume that the base space is normal and projective, even if $M$ is not assumed to be projective. Indeed, one can rewrite his argument [ibid] by using a Kähler class on $M$, instead of an ample class). Thus $(e_1 + e_2)^2 > 0$ and hence $NS(M)$ is hyperbolic. Thus $M$ is projective by the fundamental result of Huybrechts [Hu].

Choose $f_i \in MW(\varphi_i)$ s.t. ord $(f_i) = \infty$. We naturally regard both $f_i$ as elements of Bir $(M)$. Set $G := \langle f_1, f_2 \rangle$. Then $r_{NS}(G) = \langle r_{NS}(f_1), r_{NS}(f_2) \rangle$ and $r_{NS}(f_i)(e_i) = e_i$ for each $i = 1, 2$. By Theorem (1.1), it suffices to check that $r_{NS}(G)$ is not almost abelian of finite rank. However, the proof of this fact is the same as [Og2, Theorem 1.6(1)] (except a few obvious modifications). Q.E.D.

Remark 3.1. Our proof of Theorem (1.2) is based on Theorem (1.1) which, as we see above, involves a very deep result, the Tits alternative. By using the so-called table-tennis Lemma and some elementary results in hyperbolic geometry (see eg. [Ha, Sections 1, 3, 4]), one can also find a non-commutative free group in $\langle r_{NS}(f_1), r_{NS}(f_2) \rangle$ more directly. This is pointed out to us by S. Cantat.

Proof of Theorem (1.3). By [Og2, Theorem 1.6(1)], every singular K3 surface $S$ admits at least two Jacobina fibrations of positive Mordell-Weil rank. Thus $Aut(S)$ contains a non-commutative free group by Theorem (1.2). By the universality of the Hilbert scheme, $Aut(S)$ naturally acts on $M := Hilb^n(S)$. This action is also faithful. Thus $Aut(M)$, and hence Bir $(M)$, contains a non-commutative free group. Q.E.D.

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