ENERGY-PRESERVING MIXED FINITE ELEMENT METHODS FOR A FERROFLUID FLOW MODEL

YONGKE WU AND XIAOPING XIE

ABSTRACT. In this paper, we develop a class of mixed finite element methods for the ferrofluid flow model proposed by Shliomis [Soviet Physics JETP, 1972]. We show that the energy stability of the weak solutions to the model is preserved exactly for both the semi- and fully discrete finite element solutions. Furthermore, we prove the existence and uniqueness of the discrete solutions and derive optimal error estimates for both the the semi- and fully discrete schemes. Numerical experiments confirm the theoretical results.

1. INTRODUCTION

Ferrofluids are colloidal liquids consisting of nanoscale ferromagnetic or ferrimagnetic particles suspended in carrier fluids. They have been wildly used in many technical areas [41] such as instrumentation, vacuum technology, lubrication, vibration damping and acoustics, and are expected to apply to some biomedical fields [30] like magnetic separation, drugs or radioisotopes targeted by magnetic guidance, hyperthermia treatments, and magnetic resonance imaging contrast enhancement.

There are two widely accepted ferrohydrodynamics (FHD) models: one has been obtained by Rosensweig [33, 32], and the other one is due to Shliomis [34, 35]. These two models both treat ferrofluids as homogeneous monophase fluids. The main difference between them is that Rosensweig’s model considers the internal rotation of the nanoparticles, while Shliomis’ model deals with the rotation as a magnetic torque. The existence of solutions for these two models were discussed in [1, 2, 3, 4, 28]. For Rosensweig’s FHD model, Amirat, Hamdache and Murat [4] gave the existence of global-in-time weak solutions, Amirat and Hamdache [3] showed the local-in-time existence of the unique strong solution, and Nochetto et al. [28] established the global existence of weak solutions in the absence of additional diffusion in the magnetization equation. For Shliomis’ model, Amirat and Hamdache [1] obtained the existence of global-in-time weak solutions, and in [2] derived the existence of local-in-time strong solutions.

Since the FHD models are coupled nonlinear partial differential equations, it is usually difficult to obtain their analytical solutions, and the only way to solve them is to seek approximation solutions by using numerical methods. There are limited works in this research field. In [36, 23, 22, 40], several numerical methods were applied to solve the reduced FHD models where some nonlinear terms of the original models are dropped so
as to get decoupled systems. In [27] Nochetto et al. showed the formal energy stability of the system of Rosensweig’s model, devised an energy-stable numerical scheme using finite elements, and proved the existence and, under some further simplifying assumptions, the convergence of the discrete solutions. We also refer to [43] for an unconditionally energy stable fully discrete finite element numerical scheme for a two-phase ferrohydrodynamics model.

In this paper, we shall develop a class of natural energy-preserving mixed finite element methods for Shliomis’ FHD model, which describes the flow of an incompressible ferrofluid submitted to an external magnetic field. We first reformulate Shliomis’ model into an equivalent formulation. By choosing proper finite element spaces, the semi-discrete scheme preserves the structure of the continuous equation naturally, thus the energy-preserving property holds for the scheme. We prove that the semi-discrete scheme has a unique solution under reasonable assumptions, and derive optimal error estimates by excavating deeply the properties of the $H(curl)$, $H(div)$ and $L^2$ finite element spaces; see Lemmas 2.8 and 2.9. Second, we apply the implicit Euler method to discretize the temporal derivatives in the semi-discrete scheme so as to obtain a fully discrete scheme. We show that the full discretization scheme admits, similar as the semi-discretization, an energy-preserving property and has at least one solution. We also obtain optimal error estimates for the fully discrete scheme. We mention that Shliomis’ model involves a quadrilinear term that Rosensweig’s model does not have, and this will lead to more difficulties in the mathematical analysis.

The rest of this paper is organized as follows. In section 2, we introduce several Sobolev spaces, give the governing equations of Shliomis’ FHD model, and construct the weak formulations. In section 3, we recall the finite element spaces and derive some properties of these spaces. We will show that the semi-discrete scheme preserves an energy similar as the continuous one and give the optimal order error estimates for the semi-discrete scheme. Section 4 will give the full discrete scheme and give the energy estimates and the optimal order error estimates of the full discrete scheme.

2. Preliminary

2.1. Sobolev spaces. Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected convex domain with Lipschitz boundary $\partial \Omega$, and $T > 0$ be the final time. We set $\Omega_T := \Omega \times (0, T]$ and $\Gamma_T := \partial \Omega \times (0, T]$. Let $n$ be the unit outward normal vector on $\partial \Omega$.

For any nonnegative integer $m$, we denote by $H^m(\Omega)$ the usual $m$-th order Sobolev space with norm $\| \cdot \|_m$ and semi-norm $| \cdot |_m$. In particular, $H^0(\Omega) = L^2(\Omega)$ denotes the space of all square integrable functions on $\Omega$, with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$. For the vector spaces $(H^m(\Omega))^3$ and $(L^2(\Omega))^3$, we use the same notations of norm, semi-norm and inner product as those for the scalar cases. We also introduce the spaces

\[ H(curl) = \{ v \in (L^2(\Omega))^3 : \text{curl} \, v \in (L^2(\Omega))^3 \} \]

and

\[ H(div) = \{ v \in (L^2(\Omega))^3 : \text{div} \, v \in L^2(\Omega) \}, \]

with the norms

\[ \| v \|_{\text{curl}} = (\| v \|^2 + \| \text{curl} \, v \|^2)^{1/2}, \quad \text{and} \quad \| v \|_{\text{div}} = (\| v \|^2 + \| \text{div} \, v \|^2)^{1/2}. \]
We set
\[ S := (H^1_0(\Omega))^3 = \{ v \in H^1(\Omega)^3 : v = 0 \text{ on } \partial \Omega \}, \]
\[ U := H_0(\text{curl}) = \{ v \in H(\text{curl}) : v \times n = 0 \text{ on } \partial \Omega \}, \]
\[ V := H_0(\text{div}) = \{ v \in H(\text{div}) : v \cdot n = 0 \text{ on } \partial \Omega \}, \]
\[ W := L^2_0(\Omega) = \{ v \in L^2(\Omega) : \int_\Omega v \, d\mathbf{x} = 0 \}, \]
where \( \text{curl } v = (\partial_y v_3 - \partial_z v_2, \partial_z v_1 - \partial_x v_3, \partial_x v_2 - \partial_y v_1)^T \), \( \text{div } v = \partial_x v_1 + \partial_y v_2 + \partial_z v_3 \) for \( v = (v_1, v_2, v_3)^T \).

For any scalar- or vector-valued space \( X \), defined on \( \Omega \), with norm \( \| \cdot \|_X \), we set
\[ L^p([0, T]; X) := \{ v : [0, T] \to X ; \| v \|_{L^p(X)} < \infty \}, \]
where
\[ \| v \|_{L^p(X)} := \begin{cases} \left( \int_0^T \| v(\cdot, t) \|_X^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \text{ess sup}_{0 \leq t \leq T} \| v(\cdot, t) \|_X & \text{if } p = \infty. \end{cases} \]

For simplicity, we set \( L^p(X) := L^p(0, T; X) \). For any integer \( r \geq 0 \), the spaces \( H^r(\Omega) := H^r(0, T) \) and \( C^r(\Omega) := C^r([0, T]; X) \) can be defined similarly.

### 2.2. Governing equations of the ferrofluid flow.

Consider the flow of an incompressible and viscous Newtonian ferrofluid, filling \( \Omega \), under the action of a known external magnetic field \( \mathbf{H}_e \) satisfying
\[ \mathbf{H}_e \cdot n = 0 \quad \text{on } \Gamma_T. \]

The magnetic field \( \mathbf{H}_e \) induces a demagnetizing field \( \mathbf{H} \) and a magnetic induction \( \mathbf{B} \) satisfying the law
\[ \mathbf{B} = \mathbf{H} + \mathbf{m}, \]
with \( \mathbf{m} \) the magnetization inside \( \Omega \). The governing equations of Shliomis’ model [34, 35] for this FHD flow read as follows: the fluid velocity \( \mathbf{u} \), the fluid pressure \( p \), the interior magnetization \( \mathbf{m} \), and the demagnetizing field \( \mathbf{H} \) satisfy
\[ \begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \eta \Delta \mathbf{u} + \nabla p - \mu_0 (\mathbf{m} \cdot \nabla) \mathbf{H} - \frac{\mu_0}{2} \text{curl } (\mathbf{m} \times \mathbf{H}) = 0, \\ \text{div } \mathbf{u} = 0, \\ \partial_t \mathbf{m} + (\mathbf{u} \cdot \nabla) \mathbf{m} - \sigma \Delta \mathbf{m} - \frac{1}{2} \text{curl } \mathbf{u} \times \mathbf{m} + \frac{1}{\tau} (\mathbf{m} - \chi_0 \mathbf{H}) + \beta \mathbf{m} \times (\mathbf{m} \times \mathbf{H}) = 0, \\ \text{curl } \mathbf{H} = 0, \\ \text{div}(\mathbf{H} + \mathbf{m}) = - \text{div } \mathbf{H}_e \end{cases} \]
in \( \Omega_T \), equipped with the boundary conditions
\[ (2) \quad \mathbf{u} = 0, \quad \text{curl } \mathbf{m} \times \mathbf{n} = 0, \quad \mathbf{m} \cdot \mathbf{n} = 0, \quad \text{and } \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_T, \]
and the initial conditions
\[ (3) \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{m}(x, 0) = \mathbf{m}_0(x) \quad \forall x \in \Omega. \]

Here \( \mathbf{u}_0 \) and \( \mathbf{m}_0 \) are given functions, and the parameters \( \eta, \mu_0, \sigma, \tau, \chi_0 \) and \( \beta \) are positive constants and their physical meanings can be found in, for example, [16, 34, 35, 38].

Introduce
\[ (4) \quad \mathbf{z} := \mathbf{u} \times \mathbf{m}, \quad \mathbf{k} := \text{curl } \mathbf{m}, \quad \tilde{p} := p + \frac{1}{2} \|\mathbf{u}\|^2 + \frac{1}{2} \mathbf{m} \cdot \mathbf{H}, \]
and let \( \varphi \in C^0(H_0^1(\Omega)) \) be such that \( H = \nabla \varphi \) due to the fact \( \text{curl } H = 0 \). Thus, applying the identities

\[
\begin{aligned}
(u \cdot \nabla)u &= \frac{1}{2} \nabla (u \cdot u) - u \times \text{curl } u, \\
(u \cdot \nabla)m &= \frac{1}{2} u \text{div } m - \frac{1}{2} m \text{div } u + \frac{1}{2} \nabla (u \cdot m) - \frac{1}{2} m \times \text{curl } u \\
((m \cdot \nabla)H,v) &= -(H \cdot v, \text{div } m) - ((m \cdot \nabla)v, H) \\
&= -\frac{1}{2}(H \cdot v, \text{div } m) - \frac{1}{2}(m \cdot H, \text{div } v) + \frac{1}{2}(v \cdot m, \text{div } H) + \frac{1}{2}(v \times (\text{curl } m), \dot{H}) + \frac{1}{2}(m \times \text{curl } v, H)
\end{aligned}
\]

leads to the following weak problem: Find \( u \in C^1(S) \), \( \bar{p} \in C^0(W) \), \( m \in C^1(V) \), \( z \in C^0(U) \), \( k \in C^0(U) \), \( H \in C^0(V) \), and \( \varphi \in C^0(W) \) such that

\[
(\text{5})
\]

\[
(\partial_t u, v) - (u \times \text{curl } u, v) + \eta(\nabla u, \nabla v) - (\bar{p}, \text{div } v) - \mu_0 b(v; H, m) - \frac{\mu_0}{2} (v \times k, H) = 0 \quad \forall \, v \in S,
\]

\[
(\partial_t m, F) + b(u; m, F) + \frac{1}{2} (u \times k, F) - \frac{1}{2} (\text{curl } z, F) + \sigma(\text{curl } k, F) + \sigma(\text{div } m, \text{div } F) + \frac{1}{2} (m, F) = 0 \quad \forall \, F \in V,
\]

\[
-(\text{div } H + \text{div } m, r) + (\text{div } H_e, r) = 0 \quad \forall \, r \in W,
\]

with the initial data \( (3) \), where

\[
b(v; m, F) := \frac{1}{2} (v \cdot F, \text{div } m) - \frac{1}{2} (v \cdot m, \text{div } F).
\]

It is obvious that

\[
(\text{6})
\]

\[
b(v; F, F) = 0, \quad \forall \, v \in S, \quad F \in V.
\]

For any \( t \in [0, T] \), define the energy

\[
\mathcal{E}(t) := \|u(\cdot, t)\|^2 + \|m(\cdot, t)\|^2 + \mu_0 \|H(\cdot, t)\|^2.
\]

To derive an energy estimate, we introduce an inequality first.

**Lemma 2.1** ([21], Lemma 1). Suppose that a nonnegative real number \( x \) satisfies the quadratic inequality

\[
x^2 \leq \gamma^2 + \beta x
\]

for \( \beta, \gamma \geq 0 \). Then

\[
x \leq \beta + \gamma.
\]

We have the following energy estimate.

**Theorem 2.2.** Given \( H_0 \in H^1(\text{div}) \), let \( u \in C^1(S) \), \( \bar{p} \in C^0(W) \), \( m \in C^1(V) \), \( z \in C^0(U) \), \( k \in C^0(U) \), \( H \in C^0(V) \), and \( \varphi \in C^0(W) \) solve the weak problem \( (5) \). Then the energy inequality

\[
\mathcal{E}(t) + C_1 \int_0^t \mathcal{F}(s) \, ds \leq \mathcal{E}(0) + C_2 \int_0^t \left( \|H_e(\cdot, s)\|^2_{H^{(\text{div})}} + \|\partial_t H_e(\cdot, s)\|^2 \right) \, ds
\]

for any \( t \in [0, T] \).
holds for all \( t \in [0, T] \), where \( C_1 \) and \( C_2 \) are positive constants depending only on \( \eta, \mu_0, \chi, \tau, \beta, \sigma \) and \( \Omega \), and the dissipated energy \( \mathcal{F}(t) \) is given by

\[
\mathcal{F}(t) := 2 \left( \eta \| \nabla u \|^2 + \sigma (1 + \mu_0) \| \mathbf{m} \|^2 + \sigma \| \mathbf{k} \|^2 + \frac{1}{\tau} \| \mathbf{m} \|^2 \right.
\]

\[
+ \frac{1}{\tau} [\mu_0 (1 + \chi_0) + \chi_0] \| \mathbf{H} \|^2 + \mu_0 \beta \| \mathbf{m} \times \mathbf{H} \|^2 \right).
\]

Proof. Taking \( v = u \) in the first equation of (5), we get

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \eta \| \nabla u \|^2 = \mu_0 b(u; H, m) + \frac{\mu_0}{2} (u \times \mathbf{k}, H).
\]

Taking \( F = H \) in the third equation of (5) and \( G = \text{curl} \ z \) and \( G = \text{curl} \ k \) in the sixth equation, and using the fact that \( \text{div} \, \text{curl} = 0 \), we obtain

\[
\mu_0 b(u; m, H) + \frac{\mu_0}{2} (u \times \mathbf{k}, H) = -\mu_0 (\partial_t, m, H) - \mu_0 \sigma (\text{div} m, \text{div} H)
\]

\[
- \frac{\mu_0}{\tau} (m, H) + \frac{\mu_0 \chi_0}{\tau} \| \mathbf{H} \|^2 + \mu_0 \beta \| \mathbf{m} \times \mathbf{H} \|^2.
\]

Therefore,

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 + \eta \| \nabla u \|^2 = \mu_0 (\partial_t, m, H) + \mu_0 \sigma (\text{div} m, \text{div} H) + \frac{\mu_0}{2} (m, H)
\]

\[
- \frac{\mu_0 \chi_0}{\tau} \| \mathbf{H} \|^2 - \mu_0 \beta \| \mathbf{m} \times \mathbf{H} \|^2.
\]

Taking \( r = \varphi \) in the last equation of (5), we have

\[
(\text{div} \, \mathbf{H} + \text{div} \, m, \varphi) + (\text{div} \, \mathbf{H}_e, \varphi) = 0.
\]

Taking \( G = H, G = m \) and \( G = H_e \) in the sixth equation of (5), respectively, we obtain

\[
\| H \|^2 + (\varphi, \text{div} \, H) = 0, \quad (m, H) + (\varphi, \text{div} \, m) = 0, \quad (H, H_e) + (\varphi, \text{div} \, H_e) = 0.
\]

Thus,

\[
(m, H) + \| H \|^2 = -(H_e, H).
\]

The fact that \( \text{div} \, H_0(\text{div}) = L^2_0(\Omega) \) implies that

\[
\text{div} \, (H + m) = - \text{div} \, H_e.
\]

Differentiating (9) with respect to \( t \), multiplying the resultant equation by \( \varphi \), and using the fact that

\[
(H, \partial_t \mathbf{H}) + (\text{div} \, \partial_t \mathbf{H}, \varphi) = 0, \quad (\partial_t m, H) + (\varphi, \text{div} \, \partial_t \mathbf{m}) = 0,
\]

\[
(H, \partial_t \mathbf{H}_e) + (\varphi, \text{div} \, \partial_t \mathbf{H}_e) = 0,
\]

we arrive at

\[
(\partial_t, m, H) = - \frac{1}{2} \frac{d}{dt} \| H \|^2 - (\partial_t H_e, H),
\]

which, together with (7), yields

\[
\frac{1}{2} \frac{d}{dt} \left( \| u \|^2 + \mu_0 \| H \|^2 \right) + \eta \| \nabla u \|^2 + \mu_0 \sigma \| \text{div} \, m \|^2 + \frac{\mu_0}{\tau} (1 + \chi_0) \| H \|^2 + \mu_0 \beta \| \mathbf{m} \times \mathbf{H} \|^2
\]

\[
= - \frac{\mu_0}{\tau} (H_e, H) - \mu_0 (\partial_t H_e, H) - \mu_0 \sigma (\text{div} \, H_e, \text{div} \, m).
\]
Taking $\mathbf{F} = \mathbf{m}$ in the third equation of (5), $\zeta = k$ in the fourth equation, $\kappa = z$ and $\kappa = k$ in the fifth equation, combining the resultant equations with (8), and using (6), we get
\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{m} \|^2 + \sigma \| \mathbf{k} \|^2 + \sigma \| \text{div} \mathbf{m} \|^2 + \frac{1}{\tau} \| \mathbf{m} \|^2 + \frac{\chi_0}{\tau} \| \mathbf{H} \|^2 = -\frac{\chi_0}{\tau} \langle \mathbf{H}_e, \mathbf{H} \rangle.
\]
This relation plus (10) implies
\[
\frac{1}{2} \frac{d}{dt} (\| \mathbf{u} \|^2 + \mu_0 \| \mathbf{H} \|^2 + \| \mathbf{m} \|^2) + \eta \| \nabla \mathbf{u} \|^2 + \sigma (1 + \mu_0) \| \text{div} \mathbf{m} \|^2 + \frac{1}{\tau} \| \mathbf{m} \|^2
\]
\[
+ \frac{1}{\tau} [\mu_0 (1 + \chi_0) + \chi_0] \| \mathbf{H} \|^2 + \sigma \| \mathbf{k} \|^2 + \mu_0 \beta \| \mathbf{m} \times \mathbf{H} \|^2
\]
\[
= -\frac{\mu_0 + \chi_0}{\tau} (\langle \mathbf{H}_e, \mathbf{H} \rangle - \mu_0 (\partial_t \mathbf{H}_e, \mathbf{H}) - \mu_0 \sigma (\text{div} \mathbf{H}_e, \text{div} \mathbf{m})).
\]
Integrate this equation with respect to $t$ on the interval $(0, t)$ and use the Cauchy-Schwarz inequality and Lemma 2.1, we then obtain the desired result. \hfill \square

2.3. Finite element spaces and properties. We consider some $H^1$, $H^0(\text{div})$ and $H(\text{curl})$—conforming finite element spaces that will be used in the spatial discretization of the weak problem (5).

Let $T_h$ be a quasi-uniform shape regular tetrahedron triangulation of $\Omega$ with mesh size $h := \max_{K \in T_h} h_K$, where, for any $K \in T_h$, $h_K$ denotes its diameter. For an integer $l \geq 0$, let $P_l(K)$ be the set of polynomials, defined on $K$, of degree no more than $l$.

For convenience, throughout the paper we use $a \lesssim b$ ($a \gtrsim b$) to denote $a \leq Cb$ ($a \geq Cb$), where $C$ is a generic positive constant independent of the mesh size $h$ and may be different at its each occurrence.

We introduce the following finite dimensional spaces:

- $S_h = (S_h)^3 \subset S = (H^1_0(\Omega))^3$, where $S_h$ is a Lagrange-element space [14] for the velocity $\mathbf{u}$, with $S_h|_K \supset P_{l+1}(K)$ for any $K$;
- $\Sigma_h \subset H^1_0(\Omega)$ is a Lagrange-element space with $\Sigma_h|_K \supset P_l(K)$ for any $K$; $\Sigma_h$ may be as same as $S_h$;
- $U_h \subset U = H^0(\text{curl})$ is an edge-element space [25, 26] for the new variables $z$ and $k$, with $U_h|_K \supset P_l(K)^3$ for any $K$;
- $V_h \subset V = H^0(\text{div})$ is a face-element space [31, 25, 10, 26, 9, 11] for the interior magnetization $\mathbf{m}$ and the demagnetizing field $\mathbf{H}$, with $V_h|_K \supset P_l(K)^3$ for any $K$;
- $W_h \subset W = L^2_0(\Omega)$ is a piecewise polynomial space for the new variable $\varphi$, with $W_h|_K \supset P_l(K)$ for any $K$;
- $L_h \subset W$ is a piecewise polynomial space for the modified pressure variable $\tilde{p}$, with $L_h|_K \supset P_l(K)$ for any $K$; In some cases one may take $L_h = W_h$.

In addition, we make the following assumptions for the above spaces.

(A1) The diagram
\[
\begin{array}{cccc}
H^1_0 & \xrightarrow{\text{grad}} & U & \xrightarrow{\text{curl}} & V & \xrightarrow{\text{div}} & W \\
\downarrow \sigma_h & & \downarrow \varphi & & \downarrow \varphi & & \downarrow Q_h \\
\Sigma_h & \xrightarrow{\text{grad}} & U_h & \xrightarrow{\text{curl}} & V_h & \xrightarrow{\text{div}} & W_h
\end{array}
\]

is a commutative exact sequence in the sense that
\[
\ker(\text{curl}) = \text{img}(\text{grad}), \quad \ker(\text{div}) = \text{img}(\text{curl}).
\]
Here $\pi_h : H^1_0(\Omega) \to \Sigma_h$, $\pi^\perp_h : U \to U_h$, and $\pi^d_h : V \to V_h$ are the classical interpolation operators, and $Q_h : W \to W_h$ is the $L^2$ orthogonal projection operator. Note that the diagram (11) also indicates that

$$\text{grad} \Sigma_h \subset U_h, \quad \text{curl} U_h \subset V_h, \quad \text{div} V_h = W_h.$$

(A2) There holds the inf-sup condition

$$\sup_{v_h \in S_h} \frac{(q_h, \text{div} v_h)}{\|v_h\|} \geq \|q_h\| \quad \forall q_h \in L_h.$$

Note that there are many combinations of finite element spaces satisfy (A1) and (A2) (cf. [11, 19]).

We define three discrete weak operators,

$$\text{div}_h : U_h \to \Sigma_h, \quad \text{curl}_h : V_h \to U_h, \quad \text{grad}_h : W_h \to V_h,$$

as the adjoint operators of $-\text{grad}$, $\text{curl}$ and $-\text{div}$, respectively, i.e., for any $u_h \in U_h$, $\text{div}_h u_h \in \Sigma_h$ satisfies

$$\langle \text{div}_h u_h, s_h \rangle = -\langle u_h, \text{grad} s_h \rangle \quad \forall s_h \in \Sigma_h;$$

for any $v_h \in V_h$, $\text{curl}_h v_h \in U_h$ satisfies

$$\langle \text{curl}_h v_h, u_h \rangle = \langle v_h, \text{curl} u_h \rangle \quad \forall u_h \in U_h;$$

and for any $w_h \in W_h$, $\text{grad}_h w_h \in V_h$ satisfies

$$\langle \text{grad}_h w_h, v_h \rangle = -\langle w_h, \text{div} v_h \rangle \quad \forall v_h \in V_h.$$

Thus, we have the following reversed ordering exact sequence:

$$0 \leftarrow \Sigma_h \xleftarrow{\text{div}_h} U_h \xleftarrow{\text{curl}_h} V_h \xleftarrow{\text{grad}_h} W_h \leftarrow 0$$

Introduce the null spaces of the differential operators

$$\mathcal{Z}^c_h := U_h \cap \ker(\text{curl}) \quad \text{and} \quad \mathcal{Z}^d_h := V_h \cap \ker(\text{div}),$$

and the null spaces of the weak differential operators

$$\mathcal{R}^c_h := U_h \cap \ker(\text{div}_h) \quad \text{and} \quad \mathcal{R}^d_h := V_h \cap \ker(\text{curl}_h).$$

Similarly, we use the notations $\mathcal{Z}^c$ and $\mathcal{Z}^d$ to denote the null spaces in the continuous level, i.e.

$$\mathcal{Z}^c := U \cap \ker(\text{curl}), \quad \mathcal{Z}^d := V \cap \ker(\text{div}).$$

We also define

$$\mathcal{R}^c := U \cap (\mathcal{Z}^c)^\perp \quad \text{and} \quad \mathcal{R}^d := V \cap (\mathcal{Z}^d)^\perp$$

Let $\oplus^\perp$ stand for the $L^2$ orthogonal decomposition, then we have the following Hodge decompositions [7, 8]:

$$\begin{align*}
U_h &= \mathcal{Z}^c_h \oplus^\perp \mathcal{R}^c_h = \text{grad} \Sigma_h \oplus^\perp \text{div}_h U_h, \\
V_h &= \mathcal{Z}^d_h \oplus^\perp \mathcal{R}^d_h = \text{curl} U_h \oplus^\perp \text{grad}_h W_h.
\end{align*}$$

And there hold the following discrete Poincaré inequalities [7, 8, 13]:

$$\begin{align*}
\|\text{curl}_h u_h\| &\geq \|u_h\| \quad \forall u_h \in \mathcal{R}^c_h, \\
\|\text{div} v_h\| &\geq \|v_h\| \quad \forall v_h \in \mathcal{R}^d_h, \\
\|\text{grad}_h w_h\| &\geq \|w_h\| \quad \forall w_h \in W_h, \\
\|\text{curl}_h v_h\| &\geq \|v_h\| \quad \forall v_h \in \mathcal{Z}^d_h.
\end{align*}$$
For any \( s \in H^1_0(\Omega), u \in U \) and \( v \in V \), we define \( P^h_0s \in \Sigma_h, P^h_0u \in \mathfrak{R}^h_4 \) and \( P^d_4v \in \mathfrak{R}^d_4 \) such that

\[
(\text{grad } P^h_0s, \text{grad } \sigma_h) = (\text{grad } s, \text{grad } \sigma_h) \quad \forall \sigma_h \in \Sigma_h, \tag{23}
\]

\[
(\text{curl } P^h_0u, \text{curl } \phi_h) = (\text{curl } u, \text{curl } \phi_h) \quad \forall \phi_h \in \mathfrak{R}^h_4 \tag{24}
\]

and

\[
(\text{div } P^d_4v, \text{div } \psi_h) = (\text{div } v, \text{div } \psi_h) \quad \forall \psi_h \in \mathfrak{R}^d_4. \tag{25}
\]

Equation (23) determine \( P^h_0s \in \Sigma_h \) uniquely due to the Poincaré inequality in \( H^1_0(\Omega) \), and (24) and (25) also determine \( P^h_0u \in \mathfrak{R}^h_4 \) and \( P^d_4v \in \mathfrak{R}^d_4 \) uniquely, since the Poincaré inequalities (19) and (20) imply that \((\text{curl } \cdot), (\text{curl } \cdot)\) and \((\text{div } \cdot), \text{div } \cdot\) are inner products on the subspaces \( \mathfrak{R}^h_4 \) and \( \mathfrak{R}^d_4 \), respectively.

The Hodge decomposition on the continuous level implies that for any \( u \in U \) and \( v \in V \), there exist \( u_1 \in H^1_0(\Omega), u_2 \in \mathfrak{R}^4, v_1 \in \mathfrak{R}^4 \) and \( v_2 \in \mathfrak{R}^d \) such that

\[
u = \text{grad } u_1 \oplus v_2 \quad \text{and} \quad v = \text{curl } v_1 \oplus v_2.
\]

We introduce two projection-based quasi-interpolation operators, \( I^h_6 : U \to U_h \) and \( I^d_4 : V \to V_h \), defined by

\[
I^h_6u = \text{grad } P^h_0u_1 \oplus P^h_0u_2 \tag{26}
\]

and

\[
I^d_4v = \text{curl } P^d_4v_1 \oplus P^d_4v_2. \tag{27}
\]

The quasi-interpolation operators have the following properties:

**Lemma 2.3.** [39, Lemmas 3.3-3.5] The projection-based quasi-interpolation operators \( I^h_6 \) and \( I^d_4 \) have the following properties:

1. For any \( u \in U \) and \( v \in V \), there hold

\[
(I^h_6u, \text{grad } s_h) = (u, \text{grad } s_h) \quad \forall s_h \in \Sigma_h,
\]

\[
(I^d_4v, \text{curl } \phi_h) = (v, \text{curl } \phi_h) \quad \forall \phi_h \in U_h,
\]

\[
(\text{curl } I^h_6u, \text{curl } \psi_h) = (\text{curl } u, \text{curl } \psi_h) \quad \forall \psi_h \in U_h,
\]

\[
(\text{div } I^d_4v, \text{div } \phi_h) = (\text{div } v, \text{div } \phi_h) \quad \forall \phi_h \in V_h.
\]

2. For any \( u \in U \) and \( v \in V \), there hold

\[
\|I^h_6u\| \leq \|u\|, \quad \|I^d_4v\| \leq \|v\|.
\]

3. For any \( u \in H_0(\text{curl}) \cap H(\text{div}) \) and \( v \in H_0(\text{div}) \cap H(\text{curl}) \), there hold

\[
\text{div}_h I^h_6u = Q^h_6 \text{div } u, \quad \text{curl}_h I^d_4v = Q^d_4 \text{curl } v,
\]

where \( Q^h_6 : L^2(\Omega) \to \Sigma_h \) and \( Q^d_4 : L^2(\Omega) \to U_h \) are \( L^2 \) orthogonal projection operators. Therefore,

\[
\|\text{div}_h I^h_6u\| \leq \|\text{div } u\|, \quad \|\text{curl}_h I^d_4v\| \leq \|\text{curl } v\|.
\]

4. For any \( u \in U \cap H^{l+1}(\Omega) \) and \( v \in V \cap H^{l+1}(\Omega) \) with \( l \geq 0 \), there hold

\[
\|u - I^h_6u\| \lesssim h^r \|u\|_r \quad \text{for} \quad 1 \leq r \leq l + 1,
\]

\[
\|v - I^d_4v\| \lesssim h^r \|v\|_r \quad \text{for} \quad 1 \leq r \leq l + 1.
\]
Furthermore, if \( u \in H^{l+1}(\Omega) \) and \( v \in H^{l+1}(\Omega) \), there hold
\[
\| \text{curl} (I - I^h_u) u \| \lesssim h^r \| \text{curl} v \|, \quad \text{for } 1 \leq r \leq l + 1,
\]
\[
\| \text{div} (I - I^h_v) v \| \lesssim h^r \| \text{div} v \|, \quad \text{for } 1 \leq r \leq l + 1.
\]

We define \( Q^c_h : \mathbb{R}^c \to \mathbb{R}^c \) and \( Q^d_h : \mathbb{R}^d_h \to \mathbb{R}^d \) as the \( L^2 \) orthogonal projections, i.e., for any \( s_h \in \mathbb{R}^c_h \) and \( v_h \in \mathbb{R}^d_h \), \( Q^c_h s_h \in \mathbb{R}^c \) and \( Q^d_h v_h \in \mathbb{R}^d \) satisfy
\[
(Q^c_h s_h, \varphi) = (s_h, \varphi) \quad \forall \varphi \in \mathbb{R}^c,
\]
and
\[
(Q^d_h v_h, \phi) = (v_h, \phi) \quad \forall \phi \in \mathbb{R}^d,
\]
respectively. The definitions of \( Q^c_h s_h \) and \( Q^d_h v_h \) imply that
\[
Q^c_h s_h = s_h - \text{grad} \gamma, \quad Q^d_h v_h = v_h - \text{curl} \psi,
\]
where \( \gamma \in H^1_0(\Omega) \) and \( \psi \in \mathbb{R}^c \) are determined uniquely by
\[
(\text{grad} \gamma, \text{grad} r) = (s_h, \text{grad} r) \quad \forall r \in H^1_0(\Omega),
\]
and
\[
(\text{curl} \psi, \text{curl} \kappa) = (v_h, \text{curl} \kappa) \quad \forall \kappa \in \mathbb{R}^c,
\]
respectively. Therefore, we have
\[
\text{curl} Q^c_h s_h = \text{curl} s_h, \quad \text{div} Q^d_h v_h = \text{div} v_h.
\]
Furthermore, we have the following error estimates.

**Lemma 2.4.** For any \( s_h \in \mathbb{R}^c_h \) and \( v_h \in \mathbb{R}^d_h \), there hold
\[
\| s_h - Q^c_h s_h \| \lesssim h \| \text{curl} s_h \|,
\]
\[
\| v_h - Q^d_h v_h \| \lesssim h \| \text{div} v_h \|.
\]

**Proof.** We note that the proof of some special cases can be found in \([6, 19, 24]\). Set \( s = Q^c_h s_h \), and let \( \tau_h \in \mathbb{R}^c_h \) be the solution of the equation
\[
(\text{curl} \tau_h, \text{curl} \kappa_h) = (s_h - s, \kappa_h) \quad \forall \kappa_h \in \mathbb{R}^c_h.
\]
Since both \( \mathbb{R}^c_h \) and \( \mathbb{R}^c \) are \( L^2 \) orthogonal to \( \mathbb{R}_h = U_h \cap \ker(\text{curl}) \), the test function space of (28) can be enlarged to \( U_h \), i.e., the solution \( \tau_h \) of (28) satisfies
\[
(\text{curl} \tau_h, \text{curl} \kappa_h) = (s_h - s, \kappa_h) \quad \forall \kappa_h \in U_h.
\]
Therefore, by Lemma 2.3 we have
\[
\| s_h - s \|^2 = (s_h - s, s_h - I^c_h s + I^c_h s - s)
\]
\[
= (\text{curl} \tau_h, \text{curl} (s_h - I^c_h s)) + (s_h - s, I^c_h s - s)
\]
\[
= (s_h - s, I^c_h s - s)
\]
\[
\lesssim h \| s \|_1 \| s_h - s \|.
\]

Then the first desired estimate follows from the fact that \( \| s \|_1 \lesssim \| \text{curl} s \| = \| \text{curl} s_h \| \).

The second estimate follows similarly. In fact, let us set \( v = Q^d_h v_h \), and let \( w_h \in \mathbb{R}^d_h \) be the solution of
\[
(\text{div} w_h, \text{div} \phi_h) = (v_h - v, \phi_h) \quad \forall \phi_h \in \mathbb{R}^d_h.
\]
Since both \( \mathbb{R}^d \) and \( \mathbb{R}^d_h \) are \( L^2 \) orthogonal to \( \mathbb{R}_h = V_h \cap \ker(\text{div}) \), the test function space in (29) can be enlarged, i.e., the solution \( w_h \) of (29) satisfies
\[
(\text{div} w_h, \text{div} \phi_h) = (v_h - v, \phi_h) \quad \forall \phi_h \in V_h.
\]
Then the desired result follows from
\[ \|v_h - v\|^2 = (v_h - v, v_h - I_h^d v + I_h^d v - v) \]
\[ = (\text{div } w_h, \text{div}(v_h - I_h^d v)) + (v_h - v, I_h^d v - v) \]
\[ = (\text{div } w_h, \text{div}(v_h - v)) + (v_h - v, I_h^d v - v) \]
\[ = (v_h - v, I_h^d v - v) \]
\[ \lesssim h\|v\|_1 \|v_h - v\|. \]
and \( \|v\|_1 \lesssim \|\text{div } v\| = \|\text{div } v_h\| \).

\[\square\]

Let us define three “Hodge mappings”,
\[ Q_3^c : 3_h^c \to 3^c \cap H(\text{div}), \quad Q_3^d : 3_h^d \to 3^d \cap H(\text{curl}) \quad \text{and} \quad Q_{L_0^d} : W_h \to L_0^2(\Omega) \cap H^1(\Omega), \]
as follows:
\[ (\text{div}(Q_3^c s_h), \text{div } \tau) = (\text{div } s_h, \text{div } \tau) \quad \forall s_h \in 3_h^c, \ \tau \in 3^c \cap H(\text{div}), \]
\[ (\text{curl}(Q_3^d v_h), \text{curl } \phi) = (\text{curl } v_h, \text{curl } \phi) \quad \forall v_h \in 3_h^d, \ \phi \in 3^d \cap H(\text{curl}), \]
\[ (\text{grad } (Q_{L_0^d} w_h), \text{grad } q) = (\text{grad } w_h, \text{grad } q) \quad \forall w_h \in W_h, \ q \in L_0^2(\Omega) \cap H^1(\Omega). \]

**Lemma 2.5.** There holds
\[ \text{div}(3^c \cap H(\text{div})) = \text{div } H(\text{div}) = L^2(\Omega). \]

**Proof.** The Hodge decomposition
\[ L^2(\Omega)^3 = \text{curl } H(\text{curl}) \oplus ^\perp \text{grad } H_0^1(\Omega) = \text{curl } H(\text{curl}) \oplus ^\perp 3^c \]
implies
\[ H(\text{div}) = L^2(\Omega)^3 \cap H(\text{div}) = \text{curl } H(\text{curl}) \oplus ^\perp 3^c \cap H(\text{div}). \]
Then
\[ \text{div } H(\text{div}) = \text{div}(3^c \cap H(\text{div})), \]
and the conclusion follows. \[\square\]

We cite an estimate from [20, Lemma 2].

**Lemma 2.6.** If \( \Omega \in \mathbb{R}^3 \) is a bounded convex polyhedral domain, then there holds
\[ \|B_h - Q_3^d B\| \lesssim h\|\text{curl } B_h\| \quad \forall B_h \in 3_h^d. \]

By a similar proof as that of [20, Lemma 2], we can obtain the following results.

**Lemma 2.7.** If \( \Omega \in \mathbb{R}^3 \) is a bounded convex polyhedral domain in \( \mathbb{R}^3 \), there hold
\[ \|s_h - Q_3^c s_h\| \lesssim h\|\text{div } s_h\| \quad \forall s_h \in 3_h^c, \]
and
\[ \|q_h - Q_{L_0^d} q_h\| \lesssim h\|\text{grad } q_h\| \quad \forall q_h \in W_h. \]

**Proof.** For any \( s_h \in 3_h^c \), let \( s = Q_3^c s_h \), so \( \text{curl } (s_h - \pi_{\text{curl}}^h s) = 0 \), where \( \pi_{\text{curl}}^h \) is the bounded cochain projection to \( U_h \) [15], and there exists \( r_h \in \Sigma_h \) such that \( s_h - \pi_{\text{curl}}^h s = \text{grad } r_h \). The definition of \( Q_3^c \) and Lemma 2.5 imply
\[ (s_h, \text{grad } r_h) = -(\text{div } s_h, r_h) = -(\text{div } s, r_h) = (s, \text{grad } r_h). \]
Thus,
\[
\|s_h - s\|^2 = (s_h - s, s_h - \pi_h^{\text{curl}} s) + (s_h - s, \pi_h^{\text{curl}} s - s)
\]
\[
= (s_h - s, \text{grad } r_h) + (s_h - s, \pi_h^{\text{curl}} s - s)
\]
\[
= (s_h - s, \pi_h^{\text{curl}} s - s)
\]
\[
\leq h \|s\|_1 \|s_h - s\|
\]

Then the first desired conclusion follows from the fact that \(\text{curl } s = 0, 3^c \cap H(\text{div}) \hookrightarrow H^1(\Omega)\) and \(\|\text{div } s\| \leq \|\text{div}_h s_h\|\).

The thing left is to show the second conclusion of this lemma. For any \(q_h \in W_h\), let \(q = Q_{L^2} q_h\). Note that there exists a unique \(v_h \in \mathcal{R}_h^d\) such that \(q_h - Q_h q = \text{div } v_h\). Consider the following auxiliary problem: for any given \(f \in L^2(\Omega)^3\), find \(w \in L^2(\Omega) \cap H^1(\Omega)\) such that
\[
(\text{grad } w, \text{grad } r) = (f, \text{grad } r) \quad \forall r \in L^2(\Omega) \cap H^1(\Omega).
\]
The Poincaré inequality on the space indicates that this problem has a unique solution. This means
\[
L^2(\Omega)^3 \subset \text{grad } (L^2(\Omega) \cap H^1(\Omega)),
\]
which, together with the definition of \(Q_{L^2}\), yields
\[
(q_h, \text{div } v_h) = -(\text{grad}_h q_h, v_h) = -(\text{grad } q, v_h) = (q, \text{div } v_h).
\]

Therefore, there holds
\[
\|q_h - q\|^2 = (q_h - q, q_h - Q_h q) + (q_h - q, Q_h q - q)
\]
\[
= (q_h - q, \text{div } v_h) + (q_h - q, Q_h q - q)
\]
\[
\lesssim h \|\text{grad } q\| \|q_h - q\|,
\]
and the desired result follows from the fact that \(\|\text{grad } q\| \leq \|\text{grad}_h q_h\|\). \(\square\)

We have several \(L^p\) estimates (1 \(\leq p \leq 6\)) for the functions in the finite-dimensional spaces \(\mathcal{R}_h^c, \mathcal{R}_h^d\) and \(W_h\).

**Lemma 2.8.** If \(\Omega \subset \mathbb{R}^3\) is a bounded convex Lipschitz polyhedral domain, then for any integer 1 \(\leq p \leq 6\), there hold
\[
\|s_h\|_{L^p} \lesssim \|\text{div}_h s_h\| \quad \forall s_h \in \mathcal{R}_h^c,
\]
\[
\|B_h\|_{L^p} \lesssim \||\text{curl } B_h|| \quad \forall B_h \in \mathcal{R}_h^d,
\]
and
\[
\|q_h\|_{L^p} \lesssim \|\text{grad}_h q_h\| \quad \forall q_h \in W_h.
\]

**Proof.** For 1 \(\leq p \leq 2\) the desired results are just the discrete Poincaré inequalities [7, 8, 13]. For 2 \(\leq p \leq 6\), the triangular inequality implies
\[
\|s_h\|_{L^p} \leq \|s_h - \pi_h^{\text{curl}} Q_3^c s_h\|_{L^p} + \|\pi_h^{\text{curl}} Q_3^c s_h\|_{L^p}.
\]
The inverse inequality leads to
\[
\|s_h - \pi_h^{\text{curl}} Q_3^c s_h\|_{L^p} \lesssim h^{-3(\frac{3}{2} - \frac{1}{p})} \|s_h - \pi_h^{\text{curl}} Q_3^c s_h\|
\]
\[
\leq h^{-3(\frac{3}{2} - \frac{1}{p})} (\|s_h - Q_3^c s_h\| + \|Q_3^c s_h - \pi_h^{\text{curl}} Q_3^c s_h\|)
\]
\[
\lesssim \|\text{div}_h s_h\|.
\]
Using the stability of \(\pi_h^{\text{curl}}\) in the \(L^p\) norm, we have
\[
\|\pi_h^{\text{curl}} Q_3^c s_h\|_{L^p} \lesssim \|Q_3^c s_h\|_{L^p} \lesssim \|s_h\|_{L^p} \lesssim \|\text{div}_h s_h\|.
\]
As a result, the desired result \( \|s_h\|_{L^p} \lesssim \|\text{div}_h s_h\| \) follows. Similarly, we can prove the other two inequalities.

Furthermore, we have two estimates for functions in \( U_h \) and \( V_h \).

**Lemma 2.9.** If \( \Omega \subset \mathbb{R}^3 \) is a bounded convex Lipschitz polyhedral domain, then for any \( 1 \leq p \leq 6 \), there hold
\[
\|s_h\|_{L^p} \lesssim \|\text{div}_h s_h\| + \|\text{curl}_h s_h\| \quad \forall s_h \in U_h,
\]
and
\[
\|v_h\|_{L^p} \lesssim \|\text{div} v_h\| + \|\text{curl}_h v_h\| \quad \forall v_h \in V_h.
\]

**Proof.** For any \( s_h \in U_h \), the Hodge decomposition implies that there exist \( s_{h,1} \in \mathcal{Z}_h \) and \( s_{h,2} \in \mathcal{R}_h \) such that
\[
s_h = s_{h,1} \perp s_{h,2}.
\]

By Lemma 2.8, we have
\[
\|s_{h,1}\|_{L^p} \lesssim \|\text{div}_h s_{h,1}\| = \|\text{div}_h s_h\|.
\]

Denote \( s_2 = Q_h^{s_2} s_{h,2} \), and let \( Q_h^s : L^2(\Omega) \to U_h \) be the \( L^2 \) projection operator. Using the embedding result \( H(\text{div}) \cap H_0(\text{curl}) \to H^1(\Omega) \to L^p(\Omega) \), we obtain
\[
\|s_{h,2}\|_{L^p} \leq \|Q_h^s (s_{h,2} - s_2)\|_{L^p} + \|Q_h^s s_2\|_{L^p} \lesssim h^{-3(\frac{2}{p} - \frac{3}{2})}\|s_{h,2} - s_2\|_{L^p} + \|s_2\|_{L^p} \lesssim \|\text{curl}_h s_{h,2}\| = \|\text{curl}_h s_h\|.
\]

Then the first desired result follows from (30) and the triangular inequality. Similarly, we can prove the second inequality.

Applying Lemma 2.9, we get the following properties of the projection-based quasi-interpolation operators \( I_h^c \) and \( I_h^d \).

**Lemma 2.10.** For any \( u \in L^\infty(\Omega) \cap U \) and \( v \in L^\infty(\Omega) \cap V \), there hold
\[
\|I_h^c u\|_{L^\infty} \lesssim \|u\|_{L^\infty}
\]
and
\[
\|I_h^d v\|_{L^\infty} \lesssim \|v\|_{L^\infty}.
\]

**Proof.** We only give the proof of the first inequality, since the second one follows similarly.

The hodge decomposition implies
\[
U = \mathcal{Z}^c \perp \mathcal{R}^c,
\]
so
\[
L^\infty(\Omega) \cap U = (\mathcal{Z}^c \cap L^\infty(\Omega)) \perp (\mathcal{R}^c \cap L^\infty(\Omega)).
\]
This means there exist \( u_1 \in \mathcal{Z}^c \cap L^\infty(\Omega) \) and \( u_2 \in \mathcal{R}^c \cap L^\infty(\Omega) \) such that
\[
u = u_1 \perp u_2.
\]

The definition of \( I_h^c \) and the first part of Lemma 2.3 imply that
\[
I_h^c u = Q_h^{3_h^c} u_1 \perp p_h^c u_2,
\]
where \( Q_h^{3_h^c} : L^2(\Omega) \to 3_h^c \) is the \( L^2 \) orthogonal projection operator. Therefore, we have
\[
\|I_h^c u\|_{L^\infty} \leq \|Q_h^{3_h^c} u_1\|_{L^\infty} + \|p_h^c u_2\|_{L^\infty} \lesssim \|u_1\|_{L^\infty} + \|p_h^c u_2\|_{L^\infty}.
\]
For the term $\|P_h^c u_2\|_{L^\infty}$, there holds
\begin{equation}
\|P_h^c u_2\|_{L^\infty} = \|P_h^c (I - \pi_h^c) u_2 + P_h^c \pi_h^c u_2\|_{L^\infty}.
\end{equation}
For any $\kappa_h \in \mathcal{K}_h$, the Poincaré inequality (19) implies that there exists a unique $\zeta_h \in \mathcal{K}_h$ such that $\kappa_h = \text{curl}_h \text{curl} \zeta_h$. Thus, we obtain
\begin{align*}
(P_h^c \pi_h^c u_2, \kappa_h) &= (P_h^c \pi_h^c u_2, \text{curl}_h \text{curl} \zeta_h) = (\text{curl} P_h^c \pi_h^c u_2, \text{curl} \zeta_h) \\
&= (\text{curl} \pi_h^c u_2, \text{curl} \zeta_h) = (\pi_h^c u_2, \kappa_h).
\end{align*}
This implies that
\begin{equation}
P_h^c \pi_h^c u_2 = Q_{\mathcal{K}_h} \pi_h^c u_2,
\end{equation}
where $Q_{\mathcal{K}_h} : L^2(\Omega) \rightarrow \mathcal{K}_h$ is the $L^2$ orthogonal projection operator. Thus,
\begin{equation}
\|P_h^c \pi_h^c u_2\|_{L^\infty} \lesssim \|u_2\|_{L^\infty}.
\end{equation}
For the term $P_h^c (I - \pi_h^c) u_2$, there holds
\begin{align*}
\|P_h^c (I - \pi_h^c) u_2\|_{L^\infty} &\lesssim h^{-1/2} \|P_h^c (I - \pi_h^c) u_2\|_{L^a} \\
&\lesssim h^{-1/2} \|\text{curl} P_h^c (I - \pi_h^c) u_2\| \\
&\lesssim h^{-1/2} \|\text{curl} (I - \pi_h^c) u_2\| \\
&\lesssim \|\text{curl} u_2\|_{H^{1/2}} \lesssim \|u_2\|_{L^\infty},
\end{align*}
where in the last inequality we have used the fact that $L^\infty \hookrightarrow H^{3/2}$. This estimate, together with (31)-(33), yields the desired conclusion.

Using Lemma 2.9, we also have the following Lemma.

**Lemma 2.11.** For any $\psi_h \in \mathcal{K}_h$ and $\phi_h \in V_h$, there holds
\begin{equation*}
\|\psi_h \times \phi_h\| \lesssim \|\text{div} \psi_h\| (\|\text{div} \phi_h\| + \|\text{curl}_h \phi_h\|)
\end{equation*}
**Proof.** Since $\psi_h \in \mathcal{K}_h$, we have $\text{curl}_h \psi_h = 0$. Let $\psi = Q_{\mathcal{K}_h} \psi_h$, then
\begin{equation*}
\|\psi_h \times \phi_h\| \leq \|\phi_h \times (\psi_h - \psi)\| + \|\phi_h \times \psi\|.
\end{equation*}
The first term can be bounded as
\begin{equation*}
\|\phi_h \times (\psi_h - \psi)\| \leq \|\phi_h\|_{L^\infty} \|\psi_h - \psi\| \lesssim h \|\phi_h\|_{L^\infty} \|\psi_h\| \lesssim \|\phi_h\|_{L^3} \|\psi_h\|.
\end{equation*}
For the second term, we have
\begin{equation*}
\|\phi_h \times \psi\| \leq \|\psi\|_{L^\infty} \|\phi_h\|_{L^3} \lesssim \|\psi\|_1 \|\phi_h\|_{L^3} \lesssim \|\psi_h\| \|\phi_h\|_{L^3}.
\end{equation*}
Then from Lemma 2.9 the desired result follows.

**3. Semi-discrete finite element scheme**

In this section, we shall give the semi-discrete scheme of (1), prove the existence of semi-discrete solutions, and derive error estimates.
3.1. Semi-discretization. The semi-discrete formulation of the ferrofluid flow model (1) reads as: Find $u_h \in C^1(S_h), \tilde{p}_h \in C^0(L_h), m_h \in C^1(V_h), z_h \in C^0(U_h), k_h \in C^0(U_h), H_h \in C^0(V_h),$ and $\varphi_h \in C^0(W_h)$ such that

$$
\begin{align*}
(\partial_t u_h, v_h) - (u_h \times \text{curl} u_h, v_h) + \nu(\nabla u_h, \nabla v_h)
- (\tilde{p}_h, \text{div} v_h) - \mu_0 b(v_h; H_h, m_h) - \frac{\mu_0}{2}(v_h \times k_h, H_h) &= 0 \quad \forall v_h \in S_h, \\
(\text{div} u_h, q_h) &= 0 \quad \forall q_h \in L_h,
\end{align*}
$$

(34)

$$
\begin{align*}
(\partial_t m_h, F_h) + b(u_h; m_h, F_h) + \frac{1}{2}(u_h \times F_h, k_h)
- \frac{1}{2}(\text{curl} z_h, F_h) + \sigma(\text{curl} k_h, F_h) + \sigma(\text{div} m_h, \text{div} F_h)
+ \frac{1}{2}(m_h, F_h) - \frac{\nu}{2}(H_h, F_h) + \beta(m_h \times (m_h \times H_h), F_h)
\left(\begin{array}{l}
(z_h, \zeta_h) - (u_h \times m_h, \zeta_h) \\
(k_h, \kappa_h) - (m_h, \text{curl} \kappa_h)
\end{array}\right) &= 0 \\
(\text{div} H_h + \text{div} m_h, r_h) + (\text{div} H_e, r_h) &= 0 \quad \forall r_h \in W_h,
\end{align*}
$$

with the initial data

$$
\begin{align*}
u_h(\cdot, 0) = \pi_h u_0(\cdot), \\
m_h(\cdot, 0) = I_h^d m_0(\cdot).
\end{align*}
$$

(35)

In the following of this section, we will prove that (34) has solutions. First, the choice of finite element spaces $V_h$ and $W_h$ implies the inf-sup condition

$$
\sup_{\psi_h \in V_h} \frac{\langle \text{div} \psi_h, r_h \rangle}{\|\psi_h\|_{\text{div}}} \gtrsim \|r_h\|, \quad \forall r_h \in W_h.
$$

Thus, for a given $m_h \in V_h,$ there exists a unique pair $(H_h(m_h), \varphi_h(m_h)) \in V_h \times W_h$ satisfying the last two equations of (34), with

$$
H_h(m_h) = \text{grad}_h \varphi_h(m_h) \in \mathcal{R}_h^d \subset V_h
$$

and

$$
\|H_h(m_h)\|_{\text{div}} + \|\varphi_h(m_h)\| \lesssim \|m_h\|_{\text{div}} + \|H_e\|_{\text{div}}.
$$

Second, we define

$$
\mathcal{R}_h^d := \{v_h \in S_h : \langle \text{div} v_h, q_h \rangle = 0, \quad \forall q_h \in L_h\}
$$

and introduce an auxiliary problem: Find $u_h \in C^1(\mathcal{R}_h^d)$ and $m_h \in C^1(V_h)$ such that

$$
\begin{align*}
(\partial_t u_h, v_h) - (u_h \times \text{curl} u_h, v_h) + \nu(\nabla u_h, \nabla v_h)
- \mu_0 b(v_h; H_h(m_h), m_h) - \frac{\mu_0}{2}(v_h \times \text{curl} k_h, H_h(m_h)) &= 0 \quad \forall v_h \in \mathcal{R}_h^d, \\
(\partial_t m_h, F_h) + b(u_h; m_h, F_h) + \frac{1}{2}(u_h \times F_h, \text{curl} k_h)
+ \frac{1}{2}(m_h, F_h) - \frac{\mu_0}{2}(u_h \times m_h, \text{curl} F_h)
\left(\begin{array}{l}
\frac{\nu}{2}(H_h(m_h), F_h) + \beta(m_h \times (m_h \times H_h(m_h)), F_h)
\end{array}\right) &= 0 \quad \forall F_h \in V_h,
\end{align*}
$$

(36)

with the initial data (35).

**Lemma 3.1.** Given $H_e \in H^1(\text{div})$, the auxiliary problem (36) has at least one solution $(u_h, m_h)$ satisfying $u_h \in L^\infty(L^2(\Omega)) \cap L^2(H^1(\Omega) \cap \mathcal{R}_h^d)$ and $m_h \in L^\infty(L^2(\Omega)) \cap L^2(V_h)$. 

Proof. Let \( \{ \Phi_i \}_{i=1}^{s} \) and \( \{ \Psi_j \}_{j=1}^{s+v} \) be the sets of bases respectively for \( \mathbb{R}_h^d \) and \( V_h \), satisfying
\[
(\Phi_i, \Phi_j) = \delta_{ij}, \ i, j = 1, 2, \cdots, s
\]
and
\[
(\Psi_i, \Psi_j) = \delta_{ij}, \ i, j = s+1, s+2, \cdots, s+v.
\]
Let
\[
u_h = \sum_{i=1}^{s} x_i(t) \Phi_i, \quad m_h = \sum_{i=s+1}^{s+v} x_i(t) \Psi_i, \quad u_h(\cdot, 0) = \sum_{i=1}^{s} x_i^0 \Phi_i, \quad m_h(\cdot, 0) = \sum_{i=s+1}^{s+v} x_i^0 \Psi_i.
\]
Then the auxiliary problem (36) can be written as: Find \( x_i(t) (i = 1, 2, \cdots, s + v) \) such that
\[
x_i(t) - \sum_{j, d=1}^{s+v} (\Phi_j \times \text{curl} \Phi_d, \Phi_t)x_j(t)x_d(t) + \eta \sum_{j=1}^{s} (\nabla \Phi_j, \nabla \Phi_t)x_j(t) - \mu_0 \sum_{j, d=s+1}^{s+v} (\Phi_j \times \text{curl} \Psi_d, H_h(\Psi_d))x_j(t)x_d(t) = 0
\]
for \( i = 1, 2, \cdots, s \), and
\[
x_i(t) - \frac{1}{2} \sum_{j, d=s+1}^{s+v} (\Phi_j \times \text{curl} \Phi_d, \Psi_t)x_j(t)x_d(t) + \frac{1}{2} \sum_{j, d=s+1}^{s+v} (\Phi_j \times \text{curl} \Psi_d, \Phi_t)x_j(t)x_d(t) + \sigma \sum_{j=s+1}^{s+v} \left( \text{div} \Psi_j, \text{div} \Psi_t \right) x_j(t) - \mu_0 \sum_{j=s+1}^{s+v} (H_h(\Psi_j), \Psi_t)x_j(t)
\]
\[
+ \beta \sum_{j, d=s+1}^{s+v} (\Psi_j \times (\text{curl} \Psi_d, H_h(\Psi_d)), \Psi_t)x_j(t)x_d(t) = 0
\]
for \( i = s + 1, \cdots, s + v \), with the initial conditions
\[
x_i(0) = x_i^0 \quad i = 1, 2, \cdots, s + v.
\]
By Carathéodory’s theorem [42, Page 1044], the system (37) - (38) admits a local maximal solution on \([0, t]\) for some \( t \in T \).

Taking \( v_h = u_h \in \mathbb{R}_h^d \) and \( F_h = H_h(m_h) \in \mathbb{R}_h^d \subset V_h \) in (36), we have
\[
\frac{1}{2} \frac{d}{dt} \| u_h \|^2 + \eta \| \nabla u_h \|^2 - \mu_0 b(u_h; H_h, m_h) - \frac{\mu_0}{2} (u_h \times \text{curl} m_h, H_h) = 0
\]
and
\[
(\partial_t m_h, H_h) + b(u_h; m_h, H_h) + \frac{1}{2} (u_h \times H_h, \text{curl} m_h) + \frac{1}{\tau} (m_h, H_h)
\]
\[
+ \sigma (\text{div} m_h, \text{div} H_h) - \frac{\chi_0}{\tau} \| H_h \|^2 - \beta \| m_h \times H_h \|^2 = 0.
\]
Multiplying \(-\mu_0\) to the second equation and adding the resultant equation with the first equation, we get
\[
\frac{1}{2} \frac{d}{dt} \| u_h \|^2 + \eta \| \nabla u_h \|^2 + \frac{\chi_0 \mu_0}{\tau} \| H_h \|^2 + \beta \mu_0 \| m_h \times H_h \|^2
\]
\[
= \mu_0 (\partial_t m_h, H_h) + \frac{\mu_0}{\tau} (m_h, H_h) + \sigma \mu_0 (\text{div} m_h, \text{div} H_h).
\]
Note that \( H_h = \text{grad}_h \varphi_h \) and
\[
\text{div} H_h = -\text{div} m_h - Q_h \text{div} H_e.
\]
Testing the above equation with $\varphi_h$ and applying integration by part, we obtain

\begin{equation}
\|H_h\|^2 = -(m_h, H_h) - (H_e, H_h).
\end{equation}

Differentiating (39) with respect to $t$, testing the resultant equation with $\varphi_h$, and using integration by part, we have

\begin{equation}
(\partial_t m_h, H_h) = -\frac{1}{2} \frac{d}{dt} \|H\|^2 - (\partial_t H_e, H_h).
\end{equation}

Thus, there holds

\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|u_h\|^2 + \mu_0 \|H_h\|^2) + \eta \|\nabla u_h\|^2 + \frac{(\chi_0 + 1)\mu_0}{\tau} \|H_h\|^2 \\
+ \beta\mu_0 \|m_h \times H_h\|^2 + \sigma \mu_0 \|\text{div} m_h\|^2 \\
= -\mu_0 (\partial_t H_e, H_h) - \frac{\mu_0}{\tau} (H_e, H_h) - \sigma \mu_0 (\text{div} H_e, \text{div} H_h).
\end{align*}

Taking $F_h = m_h \in V_h$ in (36), and using (40), we get

\begin{equation}
\frac{d}{dt} \|m_h\|^2 + \frac{1}{\tau} \|m_h\|^2 + \sigma \|\text{curl} m_h\|^2 + \sigma \|\text{div} m_h\|^2 + \frac{\chi_0}{\tau} \|H_h\|^2 = -\frac{\chi_0}{\tau} (H_e, H_h).
\end{equation}

Integrating the above equation on the interval $(0, t)$ for any $t \in (0, T]$, and using the Hölder inequality, we obtain

\begin{align*}
\|u_h\|^2 + \|m_h\|^2 + \mu_0 \|H_h\|^2 + 2\eta \int_0^t \|\nabla u_h\|^2 \, ds \\
+ \frac{2(\chi_0 + 1)\mu_0 + 2\chi_0}{\tau} \int_0^t \|H_h\|^2 \, ds + 2\beta\mu_0 \int_0^t \|m_h \times H_h\|^2 \, ds + \frac{2}{\tau} \int_0^t \|m_h\|^2 \, ds \\
+ 2\sigma \int_0^t \|\text{curl} m_h\|^2 \, ds + 2\sigma (1 + \mu_0) \int_0^t \|\text{div} m_h\|^2 \, ds \\
\leq \|u_h(\cdot, 0)\|^2 + \|m_h(\cdot, 0)\|^2 + \|H_h(\cdot, 0)\|^2 + 2\mu_0 \int_0^t \|\partial_t H_e\| \|H_h\| \, ds \\
+ \frac{2\mu_0 + \chi_0}{\tau} \int_0^t \|H_e\| \|H_h\| \, ds + \sigma \mu_0 \int_0^t \|\text{div} H_e\| \|\text{div} H_h\| \, ds,
\end{align*}

which implies that $\|u_h\|_{L^\infty(I^2)}$, $\|m_h\|_{L^\infty(I^2)}$, $\|u_h\|_{L^2(I^2)}$, $\|H_h\|_{L^2(I^2)}$, and $\|m_h\|_{L^2(I^2)}$ are bounded. Therefore, $(u_h, m_h)$ is the global solution of the auxiliary problem (36) on $[0, T]$.

Furthermore, we need the following assumption to show the auxiliary problem (36) admits a unique solution.

**Assumption 3.2.** There exist constants $h_0 > 0$ and $M_0 > 0$, such that for any $0 < h \leq h_0$ and any solution $(u_h, m_h)$ of (36), there holds

\begin{equation}
\max\{\|u_h\|_{L^\infty}, \|m_h\|_{L^\infty}, \|\text{curl} m_h\|_{L^\infty}, \|\text{div} m_h\|_{L^\infty}\} \leq M_0.
\end{equation}
Lemma 3.3. Under Assumption 3.2, the auxiliary problem (36) has a unique solution.

Proof. Assume that \((u_h, m_h)\) and \((u_h^*, m_h^*)\) be any two solution of (36). We denote \(d_u := u_h - u_h^*\) and \(d_m := m_h - m_h^*\), and have

\[ (\partial_t d_u, v_h) + d_1(v_h) + \eta(\nabla d_u, \nabla v_h) - d_2(v_h) - d_3(v_h) = 0 \quad \forall \ v_h \in S_h \]

and

\[ (\partial_t d_m, F_h) + d_4(F_h) + d_5(F_h) + \frac{1}{\tau} (d_m, F_h) - d_6(F_h) \]

\[ + \sigma(\text{curl}_h d_m, \text{curl}_h F_h) + \sigma(\text{div} d_m, \text{div} F_h) \quad \forall \ F_h \in V_h, \]

with

\[
\begin{align*}
  d_1(v_h) &= (u_h \times \nabla u_h, v_h) - (u_h^* \times \nabla u_h^*, v_h), \\
  d_2(v_h) &= \mu_0 b(v_h; H_h(m_h), m_h) - \mu_0 b(v_h; H_h(m_h^*), m_h^*), \\
  d_3(v_h) &= \frac{\mu_0}{2} (v_h \times \nabla m_h, m_h) - \frac{\mu_0}{2} (v_h \times \nabla m_h^*, m_h^*), \\
  d_4(F_h) &= b(u_h; m_h, F_h) - b(u_h^*; m_h^*, F_h), \\
  d_5(F_h) &= \frac{1}{2} (u_h \times F_h, \text{curl}_h m_h) - \frac{1}{2} (u_h^* \times F_h, \text{curl}_h m_h^*), \\
  d_6(F_h) &= \frac{1}{2} (u_h \times m_h, \text{curl}_h F_h) - \frac{1}{2} (u_h^* \times m_h^*, \text{curl}_h F_h), \\
  d_7(F_h) &= \beta(m_h \times (m_h \times H_h(m_h)), F_h) - \beta(m_h^* \times (m_h^* \times H_h(m_h^*)), F_h).
\end{align*}
\]

Taking \(v_h = d_u\) and \(F_h = d_m\) in (42) and (43), respectively, and adding the resultant equations together, we obtain

\[
\frac{1}{2} \frac{d}{dt}(\|d_u\|^2 + \|d_m\|^2) + \eta \|\nabla d_u\|^2 + \frac{1}{\tau} \|d_m\|^2 + \sigma(\|\text{curl}_h d_m\|^2 + \|\text{div} d_m\|^2)
\]

\[
= \frac{\chi_0}{\tau} (H_h(d_m), d_m) - d_1(d_u) + d_2(d_u) + d_3(d_u) - d_4(d_m) - d_5(d_m) + d_6(d_m) - d_7(d_m).
\]

Taking \(F_h = H_h(d_m)\) in (43), we get

\[
(\partial_t d_m, d_H) + d_4(d_H) + d_5(d_H) + \frac{1}{\tau} (d_m, d_H) + \sigma(\text{div} d_m, \text{div} d_H) - \frac{\chi_0}{\tau} \|d_H\|^2 + d_7(d_H) = 0.
\]

Since \(H_h\) satisfying the sixth and seventh equations of (34), we can show that \(d_H\) satisfying the following linear saddle point problem: Find \(d_H \in V_h\) and \(d_\varphi \in W_h\) such that

\[ \left\{ \begin{array}{l}
  (d_H, \psi_h) + (d_\varphi, \text{div} \psi_h) = 0 \quad \forall \ \psi_h \in V_h, \\
  (\text{div} d_H, r_h) = -(d_m, r_h) \quad \forall \ r_h \in W_h.
\end{array} \right. \]

Since \(\text{div} V_h \subset W_h\), this system implies that

\[ d_H = \text{grad}_h d_\varphi \quad \text{and} \quad \text{div} d_H = -\text{div} d_m. \]

So

\[
(\text{div} d_m, \text{div} d_H) = -\|\text{div} d_H\|^2, \quad \|d_H\|^2 = -(d_m, d_H), \quad \frac{1}{2} \frac{d}{dt} \|d_H\|^2 = -(\partial_t d_m, d_H).
\]
Therefore, we have

\[
\frac{1}{2} \frac{d}{dt} (\|d_u\|^2 + \|d_m\|^2 + \mu_0 \|d_H\|^2) + \eta \| \nabla d_u \|^2 + \frac{1}{\tau} \| d_m \|^2 + \sigma (\| \text{curl} d_m \|^2 + \| \text{div} d_m \|^2) \\
+ \frac{\chi_0 (1 + \mu_0) + \mu_0}{\tau} \| d_H \|^2 + \sigma \mu_0 \| \text{div} d_H \|^2
\]

\[
= -d_1 (d_u) + \mu_0 d_4 (d_H) + d_2 (d_u) + \mu_0 d_5 (d_H) + d_3 (d_u) - d_4 (d_m) - d_5 (d_m) \\
+ \mu_0 d_6 (d_m) - d_7 (d_m) + \mu_0 d_7 (d_H).
\]

Denote

\[
D_1 := \| d_u \|^2 + \| d_m \|^2 + \mu_0 \| d_H \|^2,
\]

\[
\| F_h \|_{V_h}^2 := \| \text{div} F_h \|^2 + \| \text{curl} F_h \|^2 \quad \forall F_h \in V_h,
\]

\[
D_2 := \eta \| \nabla d_u \|^2 + \frac{1}{\tau} \| d_m \|^2 + \sigma \| d_m \|_{V_h}^2 + \frac{\chi_0 (1 + \mu_0) + \mu_0}{\tau} \| d_H \|^2 \\
+ \sigma \mu_0 \| \text{div} d_H \|^2 + \beta \mu_0 \| m_h^* \times d_H \|^2.
\]

Using the fact that

\[
\mu_0 d_4 (d_H) + d_2 (d_u) = \mu_0 b(u_h; d_m, d_H) + \mu_0 b(u_H; H_h, d_m),
\]

\[
\mu_0 d_5 (d_H) + d_3 (d_u) = \frac{\mu_0}{2} \left( (u_h \times d_H, \text{curl}_h d_m) + (d_u \times \text{curl}_h d_m, H_h) \right),
\]

\[
d_6 (d_m) - d_5 (d_m) = \frac{1}{2} \left( (d_u \times m_h, \text{curl}_h d_m) - (d_u \times d_m, \text{curl}_h m_h) \right),
\]

\[
d_7 (d_m) = \beta (d_m \times m_h, d_m \times H_h) - \beta (d_m \times m_h, d_m \times d_H) + \beta (d_m \times m_h, m_h \times d_H),
\]

\[
d_7 (d_H) = -\beta (m_h^* \times d_H)^2 + \beta (d_H \times m_h, d_m \times H_h) + \beta (d_m \times d_H, d_m \times H_h) \\
+ \beta (d_H \times d_m, m_h \times H_h),
\]
we obtain

\[ \frac{1}{2} \frac{d}{dt} D_1 + D_2 \leq \| u_h \|_{L^\infty} \| \nabla d_u \| \| d_u \| + \frac{\mu_0}{2} \| u_h \|_{L^\infty} (\| \text{div} d_m \| \| d_H \| + \| d_m \| \| \text{div} d_H \|) + \frac{\mu_0}{2} \| H_h \|_{L^\infty} \| d_u \| \| d_m \| + \| H_h \|_{L^\infty} \| d_u \| \| \text{curl}_h d_m \| + \| H_h \|_{L^\infty} \| d_u \| \| \text{curl}_h d_m \|
\]

\[ + \frac{1}{2} \| m_h \|_{L^\infty} \| d_u \| \| d_m \| + \frac{1}{2} \| m_h \|_{L^\infty} \| \text{curl}_h d_m \| + \| m_h \|_{L^\infty} \| d_u \| \| d_m \|
\]

\[ \leq M_0 \| d_u \| \| \nabla d_u \| + (\mu_0 + 2) \| d_m \|_{1,h} + M_0 \frac{\mu_0}{2} \| d_m \| \| \text{div} d_H \|
\]

\[ + \mu_0 M_0 \| d_H \| \| d_m \|_{1,h} + \beta M_0^2 \| d_m \| \| d_m \|_{1,h} + \beta (1 + 2 \mu_0) M_0^4 \| d_m \| \| d_H \| + \beta M_0 (1 + \mu_0) \| d_m \| \| d_H \| \| d_m \|
\]

\[ \leq (\frac{M_0^2}{2 \epsilon_1} + \frac{M_0^2 (\mu_0 + 1)^2}{2 \epsilon_2}) \| d_u \|^2 + \left( \frac{\mu_0^2 M_0^2}{2 \epsilon_4} + \frac{\beta^2 (1 + 2 \mu_0)^2 M_0^4}{2 \epsilon_6} \right) \| d_H \|^2
\]

\[ + \left( \frac{M_0^2 \mu_0^2}{8 \epsilon_3} + \frac{\beta M_0^4 (1 + \mu_0)^2}{2 \epsilon_5} + \frac{2 \beta M_0^4 (1 + \mu_0)^2}{\epsilon_7} \right) \| d_m \|^2
\]

\[ + \frac{\epsilon_1}{2} \| \nabla d_u \|^2 + \epsilon_2 + \epsilon_4 + \epsilon_5 \| d_m \|^2_{1,h} + \frac{\epsilon_3 + \epsilon_7}{2} \| \text{div} d_H \|^2 + \frac{\epsilon_6}{2} \| d_m \|^2.
\]

Take \( \epsilon_i > 0 \ (i = 1, 2, \cdots, 7) \) with

\[ \epsilon_1 \leq \eta, \ \epsilon_6 \leq \frac{2}{\gamma}, \ \epsilon_2 + \epsilon_4 + \epsilon_5 \leq 2 \sigma, \ \epsilon_3 + \epsilon_7 \leq 2 \sigma \mu_0,
\]

then by the assumption (41) we get

\[ \frac{dD_1}{dt} \leq CD_1,
\]

which, together with the Grönwall’s inequality, implies for any \( t \in (0, T) \)

\[ D_1(t) \leq D_1(0)e^{Ct}.
\]

Note that \( D_1(0) = 0 \), therefore, we have

\[ u_h = u^*_h, \ \ m_h = m^*_h.
\]

The desired result follows. \( \square \)

We are ready to show the existence and uniqueness of the solution for the semi-discrete scheme (34).

**Theorem 3.4.** Given \( H_e \in H^1(\text{div}) \), the semi-discrete scheme (34) has at least one solution. Furthermore, under Assumption I (34) admits a unique solution.

**Proof.** In view of Lemmas 3.1 and 3.3, we only need to prove that (34) and the auxiliary problem (36) are equivalent.

Assume that \( u_h \in \mathcal{R}_h \) and \( m_h \in V_h \) solve (36). Then there exists a unique \( (H_h, \varphi_h) \in V_h \times W_h \) satisfying the sixth and seventh equations of (34). Let \( z_h = Q_h^c (u_h \times m_h) \) and
\( k_h = \text{curl}_h m_h \), it is easy to check that \((z_h, k_h)\) is the unique solution of the fourth and fifth equations of (34). The definition of \( \mathcal{R}_h^s \) implies \( u_h \) satisfying the second equation of (34). The inf-sup condition (12) implies that there exists a unique \( \tilde{p}_h \in L_h \) such that \( \tilde{p}_h, u_h, m_h, H_h, \) and \( k_h \) satisfy the first equation of (34). From the second equation of (36) it is easy to see that the third equation of (34) holds.

It is obvious that the solution of (34) solves the auxiliary problem (36).

Now we turn to the energy estimation of the semi-discrete scheme (34). Define the energy \( \mathcal{E}_h(t) \) of the semi-discrete scheme (34) at time \( t \in [0, T] \) as

\[
\mathcal{E}_h(t) = \|u_h(\cdot, t)\|^2 + \|m_h(\cdot, t)\|^2 + \mu_0\|H_h(\cdot, t)\|^2.
\]

Since the semi-discrete scheme (34) inherits the structure of the weak formulation (5), by a similar proof as that of Theorem 2.2 we obtain the following energy estimate.

**Theorem 3.5.** Given \( H_e \in H^1(\text{div}), \) let \( u_h \in \mathcal{C}^1(S_h), \) \( m_h \in \mathcal{C}^1(V_h), \) \( H_h \in \mathcal{C}^0(V_h), \) \( \tilde{p}_h \in \mathcal{C}^0(L_h), \) \( z_h \in \mathcal{C}^0(U_h), \) and \( k_h \in \mathcal{C}^0(U_h) \) solve the semi-discrete scheme (34). Then the energy inequality

\[
\mathcal{E}_h(t) + C_1 \int_0^t \mathcal{F}_h(s) \, ds \leq \mathcal{E}_h(0) + C_2 \int_0^t \left( \|H_e(\cdot, s)\|^2_{H(\text{div})} + \|\partial_t H_e(\cdot, s)\|^2 \right) \, ds
\]

holds for all \( t \in [0, T] \), where \( C_1 \) and \( C_2 \) are positive constants depending only on the data \( \eta, \mu_0, \chi_0, \tau, \beta, \sigma \), and \( \Omega \), and the dissipated energy \( \mathcal{F}_h(t) \) is given by

\[
\mathcal{F}_h(t) := 2 \left( \eta \|\nabla u_h\|^2 + \sigma(1 + \mu_0) \|\text{div} m_h\|^2 + \sigma \|k_h\|^2 + \frac{1}{\tau} \|m_h\|^2 
+ \frac{2}{\tau} [\mu_0(1 + \chi_0) + \chi_0] \|H_h\|^2 + \mu_0 \beta \|m_h \times H_h\|^2 \right).
\]

### 3.2. Error analysis

We first make the following regularity assumptions for the solution \((u, \tilde{p}, m, z, k, H, \varphi)\) of the weak problem (5):

**Assumption 3.6.** Assume that \( u \in \mathcal{C}^1(S), \) \( \tilde{p} \in \mathcal{C}^0(W), \) \( m \in \mathcal{C}^1(V), \) \( z \in \mathcal{C}^0(U), \) \( k \in \mathcal{C}^0(U), \) \( H \in \mathcal{C}^0(V), \) and \( \varphi \in \mathcal{C}^0(W) \) solve (5) and satisfy

\[
\begin{align*}
\{ u & \in H^{l+2}(\Omega) \cap W^{1, \infty}(\Omega), \quad \partial_t u \in S \cap H^{l+2}(\Omega), \quad \tilde{p} \in H^{l+1}(\Omega), \\
m, H & \in H^{l+1}(\Omega) \cap W^{1, \infty}(\Omega), \quad \partial_t m \in V \cap H^{l+1}(\Omega), \quad \text{div} m, \text{div} H \in H^{l+1}(\Omega), \quad \text{curl} k, \text{curl} z \in H^{l+1}(\Omega).
\end{align*}
\]

To derive error estimates for the semi-discrete formulation (34), we introduce some notations:

\[
\begin{align*}
\xi_u & := \pi_h u - u_h, \quad \xi_m := I_h^d m - m_h, \quad \xi_z := I_h^d z - z_h, \\
\xi_k & := I_h^d k - k_h, \quad \xi_H := I_h^d H - H_h, \quad \xi_p := Q_h \tilde{p} - \tilde{p}_h, \\
\theta_u & := \pi_h u - u, \quad \theta_m := I_h^d m - m, \quad \theta_z := I_h^d z - z, \\
\theta_k & := I_h^d k - k, \quad \theta_H := I_h^d H - H, \quad \theta_p := Q_h \tilde{p} - \tilde{p}.
\end{align*}
\]

Here we recall that \( \pi_h : H^1_0(\Omega) \to \Sigma_h \) is the standard Lagrange interpolation operator, \( Q_h : W \to W_h \) is the \( L^2 \) orthogonal projection operator, and \( I_h^c : U \to U_h \) and \( I_h^d : V \to V_h \) are the quasi-interpolation operators defined in (26)-(27). Thus, under
Lemma 3.7. The estimates may depend on the regularity terms in view of the above estimates, we note that in the sequel the hidden constant factor \(C\) in all the estimates may depend on the regularity terms \(\|u\|_{l+2}, \|\hat{p}\|_{l+1}, \|m\|_{l+1}, \|\text{div} m\|_{l+1}, \|H\|_{l+1}, \|z\|_{l+1}, \|\text{curl} z\|_{l+1}, \|k\|_{l+1}, \|\text{curl} k\|_{l+1}\).

In view of the above estimates, we note that in the sequel the hidden constant factor \(C\) in all the estimates may depend on the regularity terms \(\|u\|_{l+2}, \|\hat{p}\|_{l+1}, \|m\|_{l+1}, \|\text{div} m\|_{l+1}, \|H\|_{l+1}, \|z\|_{l+1}, \|\text{curl} z\|_{l+1}, \|k\|_{l+1}, \|\text{curl} k\|_{l+1}\).

Taking the test functions of the first and third equations of (5) in the finite element spaces \(S_h\) and \(V_h\), respectively, and subtracting the first and third equations in (34) from the resultant equations, respectively, we get the following equations of the qualities \(\xi\) and \(\theta\):

(46) \[
(\partial_t \xi, \eta_h) + \eta(\nabla \xi, \nabla \eta_h) - (\xi, \text{div} \eta_h) = f_1(\eta_h) + f_2(\eta_h) + f_3(\eta_h) + (\partial_t \eta, \eta), \quad \forall \eta_h \in S_h,
\]

(47) \[
\left\{ \begin{aligned}
(\partial_t \xi, \eta) &+ \frac{1}{2}(\text{curl} \xi, F_h) + \sigma(\text{curl} \xi, F_h) + \frac{1}{l}(\xi, F_h) + \sigma(\text{div} \xi, \text{div} F_h) - \frac{\lambda_0}{\tau}(\xi, F_h) \\
&= \frac{1}{l}(\eta, F_h) - \frac{\lambda_0}{\tau}(\theta, F_h) + (\partial_t \theta, F_h) - \frac{1}{2}(\text{curl} \theta, F_h) + \sigma(\text{curl} \theta, F_h) - f_4(F_h) - f_5(F_h) - f_6(F_h), \quad \forall F_h \in V_h,
\end{aligned} \right.
\]

Define

(48) \[
\left\{ \begin{aligned}
J_1 &= f_1(\xi) + f_2(\xi) - f_3(\xi) + \mu_0 f_4(\xi) + \mu_0 f_5(\xi) + \mu_0 f_6(\xi) \\
J_2 &= -\mu(\theta, \xi) + \frac{\mu_0 \lambda_0}{\tau}(H, \xi) - \mu_0 \partial_t \theta (m, \xi) + \frac{1}{2} \left( \text{curl} \theta, \xi \right) \\
J_3 &= \frac{1}{l}(\theta, \xi) - \frac{\mu_0}{2}(H, \xi) - (\partial_t \theta, \xi) - \frac{1}{2}(\text{curl} \theta, \xi) + \sigma(\text{curl} \theta, \xi), \\
J_4 &= \frac{1}{2}(\text{curl} \xi, \xi) - f_4(\xi) - f_5(\xi) - f_6(\xi).
\end{aligned} \right.
\]

We have the following identity of \(\xi\) and \(\theta\):

Lemma 3.7. We have

\[
\frac{1}{2} \frac{d}{dt}(\|\xi\|^2 + \mu_0 \|\xi\|^2 + \|m\|^2 + \|\xi\|^2) + \eta \|\nabla \xi\|^2 + \frac{1}{l}(\mu_0 (1 + \chi_0) + \chi_0) \|\xi\|^2 \\
+ \sigma \|\xi\|^2 + \frac{1}{l} \|z\|^2 + \sigma \mu_0 \|\text{div} \xi\|^2 + \sigma \|\text{div} \xi\|^2
\]

\[
= J_1 + J_2 + J_3 + J_4.
\]
Proof. Taking \(v_h = \xi_u \in S_h\) in (46) and \(F_h = \xi_H \in V_h\) in (47), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\xi_u\|^2 + \eta \|\nabla \xi_u\|^2 = f_1(\xi_u) + f_2(\xi_u) - f_3(\xi_u) + (\partial_t \theta_u, \xi_u) + \eta (\nabla \theta_u, \nabla \xi_u) + (\tilde{p} - \tilde{p}_h, \div \xi_u)
\]
and
\[
(\partial_t \xi_m, \xi_H) + \sigma (\div \xi_m, \div \xi_H) + \frac{1}{\tau} (\xi_m, \xi_H) - \frac{\lambda_0}{\tau} \|\xi_H\|^2
\]
(50)
\[
= \frac{1}{\tau} (\theta_m, \xi_H) - \frac{\lambda_0}{\tau} (\theta_H, \xi_H) + (\partial_t \theta_m, \xi_H) - \frac{1}{2} (\curl \theta_m, \xi_H) + \sigma (\curl \theta_k, \xi_H)
\]
\[-f_4(\xi_H) - f_5(\xi_H) - f_6(\xi_H).
\]
Multiplying \(-\mu_0\) to the two sides of (50) and adding the resultant equation to (49), we have
\[
\frac{1}{2} \frac{d}{dt} \|\xi_u\|^2 + \eta \|\nabla \xi_u\|^2 + \frac{\mu_0 \lambda_0}{\tau} \|\xi_H\|^2 - \mu_0 (\partial_t \xi_m, \xi_H)
\]
(51)
\[-\sigma \mu_0 (\div \xi_m, \div \xi_H) - \frac{\mu_0}{\tau} (\xi_m, \xi_H) = J_1 + J_2.
\]
From the last equations of (1) and (34), we get
\[
\div H + \div m = -\div H_e,
\]
\[
\div H_h + \div m_h = -Q_h \div H_e.
\]
Therefore, we have
\[
\div \xi_H + \div \xi_m = -(I - Q_h) \div H_e + \div \theta_H + \div \theta_m.
\]
The sixth equations of (5) and (34) and the first item of Lemma 2.3 imply \(\xi_H \in \mathbb{R}^d\). Thus, there exists \(\phi_h \in W_h\) such that \(\xi_H = \grad_h \phi_h\). Test (52) with \(\phi_h\), use integration by part and the fact that there exists \(\psi_h \in V_h\) such that \(\phi_h = \div \psi_h\), then we obtain
\[
||\xi_H||^2 + (\xi_m, \xi_H) = 0.
\]
Differentiating (52) with respect to \(t\), testing the resultant equation by \(\phi_h\) and using integration by part, we have
\[
(\partial_t \xi_m, \xi_H) = -\frac{1}{2} \frac{d}{dt} \|\xi_H\|^2.
\]
Finally, take \(F_h = \xi_m \in V_h\) in (47), use (53) and the fact that \((\curl \xi_k, \xi_m) = ||\xi_k||^2 - (\theta_k, \xi_k)\), then we get
\[
\frac{1}{2} \frac{d}{dt} \|\xi_m\|^2 + \sigma ||\xi_k||^2 + \sigma \|\xi_k\|^2 + \frac{1}{\tau} \|\xi_m\|^2 + \frac{\lambda_0}{\tau} \|\xi_H\|^2 = J_3 + J_4.
\]
Adding (51) to (55) yields the desired result. \(\Box\)

Lemmas 3.8-3.11 are devoted to the estimation of \(J_i\) \((i = 1, 2, 3, 4)\).

**Lemma 3.8.** Under Assumption 3.6, we have
\[
J_2 \lesssim h^{l+1} (||\xi_H|| + h ||\xi_u|| + ||\nabla \xi_u|| + ||\xi_p||),
\]
\[
J_3 \lesssim h^{l+1} (||\xi_m|| + ||\xi_k||).
\]

**Proof.** Note that
\[
(\tilde{p} - \tilde{p}_h, \div \xi_u) = - (\theta_p, \div \xi_u) + (\xi_p, \div \xi_u).
\]
Using the fact that \((\xi_p, \div u_h) = (\xi_p, \div u) = 0\), we have
\[
(\tilde{p} - \tilde{p}_h, \div \xi_u) = - (\theta_p, \div \xi_u) + (\xi_p, \div \theta_u).
\]
Lemma 3.9. Under Assumption 3.6, we have

\[ \|\xi \times \xi_m\| \lesssim \|\text{div} \xi_H\| (h^{l+1} + \|\xi_k\| + \|\text{div} \xi_m\|), \]
\[ \|\xi_m \times \theta_m\| \lesssim h^{l+1/2} (h^{l+1} + \|\xi_k\| + \|\text{div} \xi_m\|), \]
\[ \|\xi_m \times \theta_H\| \lesssim h^{l+1/2} (h^{l+1} + \|\xi_k\| + \|\text{div} \xi_m\|). \]

Proof. The sixth equation of (34) implies that
\[ \text{curl}_h \mathbf{H}_h = 0, \]
which, together with the fact that \(\text{curl} \mathbf{H} = 0\), gives
\[ \text{curl}_h \xi_H = \text{curl}_h (I_h \xi_H) - \text{curl}_h \mathbf{H}_h = Q_h^c \text{curl} \mathbf{H} = 0. \]
This means \(\xi_H \in \mathcal{R}_h^d\). The Hölder inequality and Lemma 2.11 indicate
\[ \|\xi_H \times \xi_m\| \lesssim \|\text{div} \xi_H\| (\|\text{curl}_h \xi_m\| + \|\text{div} \xi_m\|). \]
The inverse inequality and Lemma 2.9 imply
\[ \|\xi_m \times \theta_m\| \leq \|\xi_m\|_{L^\infty} \|\theta_m\| \lesssim h^{-1/2} \|\xi_m\|_{L^\infty} \|\theta_m\| \lesssim h^{l+1/2} \|\mathbf{m}\|_{t+1} (\|\text{curl}_h \xi_m\| + \|\text{div} \xi_m\|) \]
and
\[ \|\xi_m \times \theta_H\| \leq \|\xi_m\|_{L^\infty} \|\theta_H\| \lesssim h^{-1/2} \|\xi_m\|_{L^\infty} \|\theta_H\| \lesssim h^{l+1/2} \|\mathbf{H}\|_{t+1} (\|\text{curl}_h \xi_m\| + \|\text{div} \xi_m\|). \]

The fifth equation of (34) yields
\[ \mathbf{k}_h = \text{curl}_h \mathbf{m}_h, \]
which, together with the fact that \(\mathbf{k} = \text{curl} \mathbf{m}\), leads to
\[ \|\text{curl}_h \xi_m\| = \|Q_h^c \mathbf{k} - \mathbf{k}_h\| \leq \|Q_h^c (I - I_h^c) \mathbf{k}\| + \|\xi_k\| \lesssim h^{l+1} \|\mathbf{k}\|_{t+1} + \|\xi_k\|. \]
Combining this inequality with the estimates (56)-(58) gives the desired results.

Lemma 3.10. Under Assumption 3.6, we have
\[ J_1 + \frac{\beta}{4} \|\mathbf{m}_h \times \xi_H\|^2 \lesssim h^{l+1} \|\xi_u\| + h^{l+1} (\|\xi_H\| + \|\text{div} \xi_H\|) \]
\[ + h^{l+1/2} \|\xi_k\| \|\nabla \xi_u\| + h^{l+1/2} \|\xi_H\| \|\nabla \xi_u\| + h^{2(l+1)} \]
\[ + h^{2l+1} (h^{l+1} + \|\xi_k\| + \|\text{div} \xi_m\|)^2 + ||\xi_u|| ||\xi_m||_{\text{div}} \]
\[ + ||\xi_m||_{\text{div}} ||\xi_H|| + ||\xi_u|| ||\xi_k|| + ||\xi_H|| ||\xi_k|| \]
\[ + ||\nabla \xi_u|| ||\xi_u|| + ||\xi_H|| ||\xi_m|| + ||\xi_m||^2. \]
Proof. For the terms $f_1(\xi_u)$ and $f_4(\xi_H)$, we get
\[
|f_1(\xi_u) + \mu_0 f_4(\xi_H)| \\
\leq \mu_0 |b(\xi_u; \theta_H, m)| + \mu_0 |b(\xi_u; \mathcal{I}_h^d H, \theta_m)| + \mu_0 |b(\xi_u; \mathcal{I}_h^d \xi_H, \xi_m)| \\
+ \mu_0 |b(\theta_u; m, \xi_H)| + \mu_0 |b(\pi_h u; \theta_m, \xi_H)| + \mu_0 |b(\pi_h u; \xi_m, \xi_H)| \\
\leq \| \text{div } m \|_{1, \infty} \| \xi_u \| \| \theta_H \|_{\text{div}} + \| \xi_u \| \| H \|_{1, \infty} \| \theta_m \|_{\text{div}} + \| H \|_{1, \infty} \| \xi_u \| \| \xi_m \|_{\text{div}} \\
+ \| \xi_u \|_{1, \infty} \| \xi_H \|_{\text{div}} \| \theta_u \| + \| \xi_u \|_{L^\infty} \| \xi_H \|_{\text{div}} \| \theta_m \|_{\text{div}} + \| \xi_u \|_{L^\infty} \| \xi_m \|_{\text{div}} \| \xi_H \|_{\text{div}} \\
\leq (\| \text{div } m \|_{L^\infty} \| H \|_{t+1} + \| \xi_u \|_{L^\infty} \| \xi_m \|_{\text{div}} h^{l+1} \| \xi_H \| + \| \xi_u \|_{1, \infty} \| \xi_H \|_{\text{div}} \| \xi_m \|_{\text{div}}). \\
\]

where in the last inequality we have used the fact that $\xi_H \in \mathcal{R}_h^d$ and $\| \xi_H \|_{\text{div}} \lesssim \| \text{div } \xi_H \|$. For the terms $f_2(\xi_u)$ and $f_3(\xi_H)$, we have
\[
f_2(\xi_u) + \mu_0 f_3(\xi_H) \\
= -\frac{\mu_0}{2} (\xi_u \times \theta_k, H) + \frac{\mu_0}{2} (\xi_u \times \xi_k, \theta_H) + \frac{\mu_0}{2} (\xi_u \times \xi_k, \xi_H) \\
- \frac{\mu_0}{2} (\xi_u \times k, \theta_H) - \frac{\mu_0}{2} (\theta_u \times \xi_H, k) + \frac{\mu_0}{2} (\xi_u \times \xi_H, \theta_k) \\
- \frac{\mu_0}{2} (\pi_h u \times \xi_H, \theta_k) + \frac{\mu_0}{2} (\pi_h u \times \xi_H, \xi_k) \\
\leq \| H \|_{L^\infty} \| \xi_u \| \| \theta_k \| + \| H \|_{L^\infty} \| \xi_u \| \| \xi_k \| + \| \xi_u \|_{L^\infty} \| \xi_k \|_{L^\infty} \| \theta_H \| \\
+ \| k \|_{L^\infty} \| \xi_u \| \| \theta_H \| + \| k \|_{L^\infty} \| \xi_H \| \| \theta_u \| + \| \xi_u \|_{L^\infty} \| \xi_H \|_{L^\infty} \| \theta_k \| \\
+ \| u \|_{L^\infty} \| \xi_H \| \| \theta_k \| + \| u \|_{L^\infty} \| \xi_H \| \| \xi_k \| \\
\leq (\| k \|_{L^\infty} \| H \|_{t+1} + \| H \|_{L^\infty} \| k \|_{t+1}) h^{l+1} \| \xi_u \| + (h \| k \|_{L^\infty} \| u \|_{t+2} + \| u \|_{L^\infty} \| k \|_{t+1}) h^{l+1} \| \xi_H \| \\
+ \| H \|_{t+1} h^{l+1/2} \| \xi_u \| \| \nabla \xi_u \| + \| k \|_{t+1} h^{l+1/2} \| \xi_H \| \| \nabla \xi_u \| + \| H \|_{L^\infty} \| \xi_u \| \| \xi_k \| + \| u \|_{L^\infty} \| \xi_H \| \| \xi_k \|, \\
\]

where in the last inequality, we have used the fact $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the inverse inequality $\| \xi_k \|_{L^3} \lesssim h^{-1/2} \| \xi_k \|$. For the term $f_3(\xi_u)$, it holds
\[
f_3(\xi_u) = - (\theta_u \times \text{curl } u, \xi_u) - (\pi_h u \times \text{curl } \theta_u, \xi_u) + (\pi_h u \times \text{curl } \xi_u, \xi_u) \\
\lesssim \| u \|_{1, \infty} \| \theta_u \|_{L^\infty} \| \xi_u \| + \| u \|_{L^\infty} \| \theta_u \|_{1} \| \xi_u \| + \| u \|_{\infty} \| \nabla \xi_u \| \| \xi_u \| \\
\lesssim (h \| u \|_{1, \infty} \| u \|_{t+2} + \| u \|_{L^\infty} \| u \|_{t+2}) h^{l+1} \| \xi_u \| + \| u \|_{L^\infty} \| \nabla \xi_u \| \| \xi_u \|. \\
\]
For the term $f_6(\xi_H)$, it holds
\[
\begin{align*}
f_6(\xi_H) &= \beta \left((\theta_m \times \xi_H, m \times H) + (\xi_H \times \xi_m, m \times H) - (\xi_H \times m_h, \theta_m \times H)\right) \\
&\quad + (\xi_H \times m_h, \xi_m \times H) + (\xi_H \times m_h, m_h \times \theta_H) - \|m_h \times \xi_H\|^2 \right) \\
&\leq \beta \left\{ \|m \times H\|_{L^\infty} \|\theta_m\| \|\xi_H\| + \|m \times H\|_{L^\infty} \|\xi_H\| \|\xi_m\| + \|\xi_H \times m_h\| \|\theta_m \times H\| \\
&\quad + \|\xi_H \times m_h\| \|\xi_m \times H\| + \|\xi_H \times m_h\| \|m_h \times \theta_H\| - \|m_h \times \xi_H\|^2 \right\} \\
&\leq \beta \left\{ \|m \times H\|_{L^\infty} \|\theta_m\| \|\xi_H\| + \|m \times H\|_{L^\infty} \|\xi_H\| \|\xi_m\| + \|\theta_m \times H\|^2 \\
&\quad + \|\xi_m \times H\|^2 + 2 \|\xi_m \times \theta_H\|^2 + 2 \|\xi_m \times \theta_H\|^2 - \frac{1}{4} \|m_h \times \xi_H\|^2 \right\},
\end{align*}
\]
which, together with Lemma 3.9, gives
\[
f_6(\xi_H) + \frac{\beta}{4} \|m_h \times \xi_H\|^2 \leq \|m \times H\|_{L^\infty} \|m\|_{L^{l+1}} \|\xi_H\| + \|m \times H\|_{L^\infty} \|\xi_H\| \|\xi_m\| \\
+ (\|H\|_{L^\infty} \|m\|_{L^{l+1}} + \|m\|_{L^\infty} \|H\|_{L^{l+1}}) \|\xi_m\|^2 + \|H\|_{L^\infty} \|\xi_m\| \|\theta_m \times H\|^2 \\
+ \|H\|_{L^{l+1}} \|\xi_m\| \|\theta_m \times H\| + \|\xi_m\| \|\theta_m \times H\|^2.
\]
Finally, the desired result follows from the above obtained estimates and the definition of $J_1$. \hfill \Box

For the term $J_4$, we have the following conclusion.

**Lemma 3.11.** Under Assumption 3.6, we have
\[
J_4 - \frac{\beta}{4} \|m_h \times \xi_H\|^2 \leq h^{l+1} \|\xi_k\| + h^{l+1} \|\xi_m\| \|\xi_m\| + h^{l+1} \|\xi_m\| \\
+ h^{l+1/2} \|\nabla u_s\| \|\xi_k\| + \|\xi_m\| \|\div\xi_m\| \\
+ h^{2l+1} (h^{l+1} \|\xi_k\| + \|\xi_m\| \|\div\xi_m\|) \|^2 \\
+ h^{2l+3/2} (h^{l+1} \|\xi_k\| + \|\div\xi_m\|) \\
+ h^{l+1/2} \|\xi_m\| (h^{l+1} \|\xi_k\| + \|\div\xi_m\|) \\
+ \|\xi_u\| \|\div\xi_m\| \|\xi_m\| + \|\div\xi_m\| \|^2.
\]

**Proof.** The fifth equations of (5) and (34) imply that
\[
(\xi_k, \kappa) = (\xi_m, \curl \kappa) + (\theta_k, \kappa) \quad \forall \kappa \in U_h.
\]
Taking $\kappa = \xi_z \in U_h$ in this equation, we have
\[
(\curl \xi_z, \xi_m) = (\xi_k, \xi_z) - (\theta_k, \xi_z).
\]
The fourth equations of (5) and (34) indicate that
\[
(\xi_z, \zeta) = (u \times m - u_h \times m_h, \zeta) + (\theta_z, \zeta) \quad \forall \zeta \in U_h.
\]
Taking $\zeta = \xi_k \in U_h$ in this relation, we get

$$\text{curl} \xi_k, \xi_m) = (u \times m - u_h \times m_h, \xi_k) + (\theta, \xi_k) + (\theta, \xi_k)$$

$$= -\left(\frac{1}{2}(\theta_u \times m, \xi_k) + (\xi_u \times m, \xi_k) + (\xi_u \times m, \xi_k) - (\pi_h u \times \theta_m, \xi_k)
- (\xi_u \times \xi_m, \xi_k) + (\pi_h u \times \xi_m, \xi_k) + (\theta, \xi_k) - (\theta, \xi_k),
$$

which, together with the fact that

$$f_5(\xi_m) = -\frac{1}{2}(\theta_u \times m, \xi_k) + \frac{1}{2}(\xi_u \times m, \xi_k) + \frac{1}{2}(\xi_u \times \xi_m, \xi_k) - \frac{1}{2}(\pi_h u \times \xi_m, \eta_k)$$

$$- \frac{1}{2}(\xi_u \times \xi_m, \xi_k) + \frac{1}{2}(\pi_h u \times \xi_m, \xi_k),$$

yields

$$\frac{1}{2}(\text{curl} \xi_k, \xi_m) - f_5(\xi_m)$$

$$= -\frac{1}{2}(\theta_u \times m, \xi_k) + \frac{1}{2}(\xi_u \times m, \xi_k) + \frac{1}{2}(\xi_u \times \xi_m, \xi_k)$$

$$- \frac{1}{2}(\pi_h u \times \theta_m, \xi_k) - \frac{1}{2}(\theta_u \times \xi_m, \xi_k) + \frac{1}{2}(\xi_u \times \xi_m, \xi_k)$$

$$+ \frac{1}{2}(\xi_u \times \xi_m, \theta_k) - \frac{1}{2}(\pi_h u \times \xi_m, \theta_k) + \frac{1}{2}(\theta, \xi_k) - \frac{1}{2}(\theta, \xi_k)$$

$$\lesssim \|m\|_{L^\infty} \|\theta_u\|_{L^\infty} \|\xi_k\| + \|m\|_{L^\infty} \|\xi_u\|_{L^\infty} \|\xi_k\| + \|\xi_u\|_{L^2} \|\xi_k\|_{L^2} \|\theta_m\|$$

$$+ \|u\|_{L^\infty} \|\theta_m\|_{L^\infty} \|\xi_k\| + \|k\|_{L^\infty} \|\theta_u\|_{L^1} \|\xi_m\| + \|k\|_{L^\infty} \|\xi_u\|_{L^1} \|\xi_m\|$$

$$+ \|\xi_u\|_{L^2} \|\xi_m\|_{L^2} \|\theta_k\| + \|u\|_{L^\infty} \|\xi_m\| \|\theta_k\| + \|\theta\|_{L^\infty} \|\xi_k\| + \|\theta\|_{L^\infty} \|\xi_k\|$$

$$\lesssim h^{l+1}(\|\xi_k\| + \|\xi_m\| + \|\xi_z\| + h^{l+1/2} \|\nabla u\|_{L^2} (\|\xi_k\| + \|\xi_m\|) + \|\xi_u\| (\|\xi_k\| + \|\xi_m\|).$$

For the term $f_4(\xi_m)$, we have

$$f_4(\xi_m) = -b(\theta_u; m, \xi_m) + b(\xi_u; m, \xi_m) + b(\xi_u; \theta_m, \xi_m) - b(\pi_h u; \theta_m, m, \xi_m)$$

$$\lesssim \|m\|_{1,\infty} \|\theta_u\|_{L^1} \|\xi_m\|_{L^1} + \|m\|_{1,\infty} \|\xi_u\|_{L^1} \|\xi_m\|_{L^1}$$

$$+ \|\xi_u\|_{L^\infty} \|\theta_m\|_{L^1} \|\xi_m\|_{L^1} + \|u\|_{L^\infty} \|\theta_m\|_{L^1} \|\xi_m\|_{L^1}$$

$$\lesssim h^{l+1} \|\xi_m\|_{L^1} + h^{l+1/2} \|\nabla u\|_{L^2} \|\xi_m\|_{L^1} + \|\xi_u\| \|\xi_m\|_{L^1},$$

where in the last inequality, we have used the inverse inequality $\|\xi_u\|_{L^\infty} \lesssim h^{-1/2} \|\xi_u\|_{L^6}$ and the inclusion result $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

For the term $f_6(\xi_m)$, we have

$$f_6(\xi_m) = \beta \left\{ \left( \xi_m \times \xi_m, m \times H \right) + \left( I_h^4 m \times \xi_m, \theta_m \times H \right) + \left( \xi_m \times I_h^4 m, \xi_m \times H \right) 
+ \left( \xi_m \times \xi_m, \xi_m \times H \right) \right\}$$

$$\leq \beta \left\{ \|m \times H\|_{L^\infty} \|\theta_m\|_{L^1} \|\xi_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\theta_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\xi_m\|_{L^1} \right\}$$

$$\leq \beta \left\{ \|m \times H\|_{L^\infty} \|\theta_m\|_{L^1} \|\xi_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\theta_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\xi_m\|_{L^1} \right\}$$

$$\leq \beta \left\{ \|m \times H\|_{L^\infty} \|\theta_m\|_{L^1} \|\xi_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\theta_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\xi_m\|_{L^1} \right\}$$

$$+ \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\theta_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\theta_m\| + \|m \times H\|_{L^1} \|\xi_m\|_{L^1} \|\xi_m\|_{L^1} \right\}.$$
Proof. The fourth equations of (5) and (34) imply that

\[ f_0(\xi_m) - \frac{\beta}{4} \| \mathbf{m}_h \times \xi_H \|^2 \leq h^{l+1} \| \xi_m \|^2 + h^{2l+1}(h^{l+1} + \| \xi_k \| + \| \text{div} \xi_m \|)^2 + h^{2l+3/2}(h^{l+1} + \| \xi_k \| + \| \text{div} \xi_m \|) \]
\[ + h^{l+1/2} \| \xi_m \|(h^{l+1} + \| \xi_k \| + \| \text{div} \xi_m \|) + h^{2l+1}(h^{l+1} + \| \xi_k \| + \| \text{div} \xi_m \|)^2. \]

Combining all the above estimates gives the desired result. □

Lemma 3.12. Under Assumption 3.6, we have

\[ \| \xi_z \| \lesssim h^{l+1} + h^{l+1/2} \| \nabla \xi_u \| + \| \xi_u \|. \]

Proof. The fourth equations of (5) and (34) imply that

\[ (\xi_z, \zeta_h) = (\mathbf{u} \times \mathbf{m} - \mathbf{u}_h \times \mathbf{m}_h, \zeta_h) + (\theta_z, \zeta_h) \quad \forall \zeta_h \in U_h. \]

Taking \( \zeta_h = \xi_z \in U_h \) in the above equation, we have

\[ \| \xi_z \|^2 = (\mathbf{u} \times \mathbf{m} - \mathbf{u}_h \times \mathbf{m}_h, \xi_z) + (\theta_z, \xi_z) \]
\[ = - (\theta_u \times \mathbf{m}, \xi_z) + (\xi_u \times \mathbf{m}, \xi_z) + (\xi_u \times \theta_m, \xi_z) - (\pi_h \mathbf{u} \times \theta_m, \xi_z) + (\theta_z, \xi_z) \]
\[ \lesssim \| \mathbf{m} \|_{L^\infty} \| \theta_u \| \| \xi_z \| + \| \mathbf{m} \|_{L^\infty} \| \xi_u \| \| \xi_z \| + \| \xi_u \|_{L^\infty} \| \theta_m \| \| \xi_z \|
\]
\[ + \| \pi_h \mathbf{u} \|_{L^\infty} \| \xi_z \| + \| \theta_z \| \| \xi_z \| \]
\[ \lesssim h^{l+1} \| \xi_z \| + h^{l+1/2} \| \nabla \xi_u \| \| \xi_z \| + \| \xi_u \| \| \xi_z \|, \]

where in the last inequality, we have used Lemma 2.9 and the estimate of \( \| \text{curl}_h \xi_m \| \) in the proof of Lemma 3.9. Then the desired result follows by cancelling \( \| \xi_z \| \) from the above inequality. □

Lemma 3.13. Under Assumption 3.6, for any \( t \in (0, T] \) we have

\[ \int_0^t \| \xi_p \| dt \lesssim \| \xi_u \| + hT^{l+2} + h^{l+1} + h^{l+1} \int_0^t \| \nabla \xi_u \| dt + h^{l+1/2} \int_0^t \| \xi_k \| dt \]
\[ + \int_0^t \| \text{div} \xi_H \| dt + \| \text{div} \mathbf{H}_h \|_{L^\infty(L^2)} \int_0^t \| \xi_m \| dt + \int_0^t \| \xi_k \| dt \]
\[ + \int_0^t \| \nabla \xi_u \| dt + \int_0^t (\| \nabla \xi_u \|^2 + \| \xi_k \| \| \text{div} \xi_H \|) dt. \]

Proof. The inf-sup condition (12) and equation (46) imply that for any \( 0 \leq t \leq T \),

\[ \int_0^t \| \xi_p \| dt \lesssim \sup_{\mathbf{v}_h \in \mathbf{S}_h} \int_0^t \frac{(\xi_p, \nabla \cdot \mathbf{v}_h)}{\| \mathbf{v}_h \|_1} dt \]
\[ \lesssim \| \xi_u (\cdot, t) \| + \int_0^t \| \nabla \xi_u \| dt + h^{l+2} \int_0^t \| \partial_t \mathbf{u} \|_{L^2} dt \]
\[ + h^{l+1} \int_0^t \| \mathbf{u} \|_{H^1} dt + h^{l+1} \int_0^t \| \tilde{p} \|_{L^2} dt + T_1 \]

with

\[ T_1 := \sup_{\mathbf{v}_h \in \mathbf{S}_h} \int_0^t \frac{|f_1(\mathbf{v}_h) + f_2(\mathbf{v}_h) - f_3(\mathbf{v}_h)|}{\| \mathbf{v}_h \|_1} dt, \]

where \( f_1(\mathbf{v}_h), f_2(\mathbf{v}_h) \) and \( f_3(\mathbf{v}_h) \) are as same as those in (46).
For any $v_h \in S_h$, we have

$$\begin{align*}
|f_1(v_h)| &= \mu_0 |b(v_h; \xi_H, m) - b(v_h; \theta_H, m) - b(v_h; H_h, \theta_m) + b(v_h; H_h, \xi_m)| \\
&\lesssim \|m\|_{1,\infty} \|v_h\| \|\xi_H\|_{\text{div}} + \|m\|_{1,\infty} \|v_h\| \|\theta_H\|_{\text{div}} \\
&\quad + \|v_h\|_{L^\infty} \|\text{div} H_h\| \|\theta_m\|_{\text{div}} + \|v_h\|_{L^\infty} \|\text{div} H_h\| \|\xi_m\|_{\text{div}} \\
&\lesssim \|v_h\| \|\xi_H\|_{\text{div}} + h^{l+1} \|v_h\| + \|\nabla v_h\| \|\text{div} H_h\|_{L^3} (h^{l+1} + \|\xi_m\|),
\end{align*}$$

and

$$\begin{align*}
|f_2(v_h)| &= \frac{\mu_0}{2} |(v_h \times \xi_k, H) - (v_h \times \theta, H) + (v_h \times \xi_k, \theta_H) \\
&\quad - (v_h \times I^H_k, \theta_H) - (v_h \times \xi_H, \theta_H) + (v_h \times I^H_k, \xi_H)| \\
&\lesssim \|H\|_{L^\infty} \|v_h\| \||\xi_k\|_{\text{div}} + \|H\|_{L^\infty} \|v_h\| \|\theta_k\|_{\text{div}} + \|v_h\|_{L^\infty} \|\xi_k\|_{\text{div}} \|\theta_H\|_{\text{div}} \\
&\quad + \|I^H_k\|_{L^\infty} \|v_h\| \|\theta_H\|_{\text{div}} + \|v_h\|_{L^\infty} \|\xi_H\|_{L^3} \|\xi_k\| + \|v_h\| \|I^H_k\|_{L^\infty} \|\xi_H\|_{\text{div}} \\
&\lesssim \|v_h\| \|\xi_k\| + h^{l+1} \|v_h\| + \|\xi_k\| \|\nabla v_h\| (h^{l+1/2} + \|\text{div} \xi_H\|) + \|v_h\| \|\xi_H\|,
\end{align*}$$

and

$$\begin{align*}
|f_3(v_h)| &= \|\xi_u \times \text{curl} u, v_h\| - \|\theta_u \times \text{curl} u, v_h\| + \|\xi_u \times \text{curl} \theta_u, v_h\| \\
&\lesssim \|\xi\|_{1,\infty} \|\xi_u\| \|v_h\| + \|\theta\|_{1,\infty} \|\theta_u\| \|v_h\| + \|\nabla \xi_u\| \|\theta_u\| \|\nabla v_h\| \\
&\quad + \|\xi\|_{1,\infty} \|\xi_u\| \|\theta_u\| + \|\nabla \xi_u\| \|\xi_u\| \|\nabla v_h\| \\
&\lesssim \|\xi\| \|\xi_u\| + h^{l+2} \|v_h\| + h^{l+1/2} \|\nabla \xi_u\| \|v_h\| \\
&\quad + h^{l+1} \|v_h\| + \|\nabla v_h\| \|\nabla \xi_u\| + \|\nabla \xi_u\| \|v_h\|.
\end{align*}$$

As a result, the desired result follows.

Denote

$$\begin{align*}
\xi_h(t) &:= \|\xi_u(\cdot, t)\|^2 + \|\xi_H(\cdot, t)\|^2 + \|\xi_m(\cdot, t)\|^2 \\
\tilde{\xi}_h(t) &:= \|\nabla \xi_u(\cdot, t)\|^2 + \|\xi_k(\cdot, t)\|^2 + \|\xi_H(\cdot, t)\|^2_{\text{div}} + \|\xi_m(\cdot, t)\|^2_{\text{div}}.
\end{align*}$$

We have the following Lemma.

**Lemma 3.14.** Under Assumption 3.6 and the condition $\|\text{div} H_h\|_{L^\infty(L^3)} \leq C$, for any $t \in [0, T]$ we have

$$\int_0^t \left( \xi_h(r) + \int_0^r \tilde{\xi}_h(s) \, ds \right) \, dr \lesssim h^{2(l+1)}.$$

**Proof.** By Lemmas 3.7, 3.8, 3.10 and 3.11, we have

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|\xi_u\|^2 + \mu_0 \|\xi_H\|^2 + \|\xi_m\|^2) + \eta \|\nabla \xi_u\|^2 + \frac{1}{\tau} (\mu_0 (1 + \chi_0) + \chi_0) \|\xi_H\|^2 \\
+ \sigma \|\xi_k\|^2 + \frac{1}{\tau} \|\xi_m\|^2 + \sigma \mu_0 \|\text{div} \xi_H\|^2 + \sigma \|\text{div} \xi_m\|^2 \\
\lesssim h^{l+1} \xi_1 + h^{l+1/2} \xi_2 + \xi_3,
\end{align*}$$

with

$$\begin{align*}
\xi_1 &:= \|\nabla \xi_u\| + \|\xi_H\|_{\text{div}} + \|\xi_m\|_{\text{div}} + \|\xi_k\| + \|\xi_z\| + \|\xi_p\|, \\
\xi_2 &:= \|\nabla \xi_u\| (\|\xi_k\| + \|\xi_H\| + \|\xi_m\|_{\text{div}}) + \|\xi_m\| (\|\xi_k\| + \|\text{div} \xi_m\|), \\
\xi_3 &:= \|\xi_u\| \|\xi_k\| + \|\xi_m\|_{\text{div}} + \|\nabla \xi_u\| + \|\xi_m\|_{\text{div}} \|\text{div} \xi_H\| + \|\xi_H\| \|\xi_k\| \\
&\quad + \|\xi_m\|^2 + \|\xi_m\| \|\xi_H\|.
\end{align*}$$
For any $t \in [0, T]$, using Lemmas 3.12 and 3.13, we get
\[
\int_0^t \mathfrak{T}_1 \, dt \lesssim h^{l+1} + \|\xi_u\| + \int_0^t (\|\xi_H\| + \|\xi_m\| + \|\nabla \xi_u\| + \|\xi_k\|) \, dt \\
+ \int_0^t (\|\nabla \xi_u\|^2 + \|\xi_k\| \, \div \xi_H) \, dt.
\]
Then, integrate (59) on the interval $(0, t)$, and we obtain
\[
\mathcal{E}_h(t) + \int_0^t \mathcal{F}_h(r) \, dr = h^{l+1} \int_0^t \mathfrak{T}_1 \, dt + h^{l+1/2} \int_0^t \mathfrak{T}_2 \, dt + \int_0^t \mathfrak{T}_3 \, dt \\
\lesssim h^{2(l+1)} + h^{l+1} \mathcal{E}_h^{1/2}(t) + \int_0^t \mathcal{E}_h^{1/2}(r) \mathcal{F}_h^{1/2}(r) \, dr \\
\lesssim h^{2(l+1)} + h^{l+1} \left( \mathcal{E}_h(t) + \int_0^t \mathcal{F}_h(r) \, dr \right)^{1/2} \\
+ \left( \int_0^t \left( \mathcal{E}_h(r) + \int_0^r \mathcal{F}_h(s) \, ds \right) \right)^{1/2} \left( \mathcal{E}_h(t) + \int_0^t \mathcal{F}_h(r) \, dr \right)^{1/2},
\]
which shows
\[
\mathcal{E}_h(t) + \int_0^t \mathcal{F}_h(r) \, dr \lesssim h^{2(l+1)} + \int_0^t (\mathcal{E}_h(r) + \int_0^r \mathcal{F}_h(s) \, ds) \, dr.
\]
Integrating this inequality on the interval $(0, \bar{t})$ for any $\bar{t} \in [0, T]$ yields
\[
\int_0^\bar{t} \left( \mathcal{E}_h(t) + \int_0^t \mathcal{F}_h(r) \, dr \right) \, dt \lesssim \bar{t}^2 h^{2(l+1)} + \int_0^\bar{t} \left[ \int_0^t \left( \mathcal{E}_h(r) + \int_0^r \mathcal{F}_h(s) \, ds \right) \, dr \right] \, dt,
\]
which, together with the Gronwall’s inequality [29, Theorem 1.2.2], implies the desired result.

Define the errors
\[
\text{err}_1(t) := \|u(\cdot, t) - u_h(\cdot, t)\|^2 + \|m(\cdot, t) - m_h(\cdot, t)\|^2 + \|H(\cdot, t) - H_h(\cdot, t)\|^2
\]
and
\[
\text{err}_2(t) := \|\nabla (u - u_h)(\cdot, t)\|^2 + \|\text{div} (m - m_h)(\cdot, t)\|^2 + \|\text{div} (H - H_h)(\cdot, t)\|^2.
\]
We finish this section by giving the following error estimates for the semi-discrete scheme (34):

**Theorem 3.15.** Under Assumption 3.6 and the condition $\|\text{div} H_h\|_{L^\infty(L^3)} \leq C$, for any $t \in [0, T]$ we have
\[
\int_0^t \left( \text{err}_1(s) + \int_0^s \text{err}_2(r) \, dr \right) \, ds \lesssim h^{2(l+1)},
\]
\[
\int_0^t \|\tilde{p}(\cdot, s) - \bar{p}_h(\cdot, s)\|^2 \, ds \lesssim h^{2(l+1)}.
\]

**Proof.** For any $s \in [0, t]$, the triangular inequality implies
\[
\text{err}_1(s) \lesssim \|\theta_u(\cdot, s)\|^2 + \|\theta_m(\cdot, s)\|^2 + \|\theta_H(\cdot, s)\|^2 + \mathcal{E}(s),
\]
\[
\text{err}_2(s) \lesssim \|\nabla \theta_u(\cdot, s)\|^2 + \|\text{div} \theta_m(\cdot, s)\|^2 + \|\text{div} \theta_H(\cdot, s)\|^2 \\
+ \|\nabla \xi_u(\cdot, s)\|^2 + \|\text{div} \xi_m(\cdot, s)\|^2 + \|\text{div} \xi_H(\cdot, s)\|^2,
\]
and

\[ \|\tilde{p}(\cdot, s) - \tilde{p}_h(\cdot, s)\| \leq \|\theta_p(\cdot, s)\| + \|\xi_p(\cdot, s)\|. \]

Then Lemmas 3.13 and 3.14 yield the desired results.

\[ \square \]

4. Fully discrete finite element scheme

In this section, we give a fully discrete scheme for the ferrofluid model (1), based on the semi-discretization (34). We will prove that this full-discretization also preserves the energy exactly and has optimal convergence.

4.1. Full discretization. For any positive integer \( N \), let

\[ \mathcal{J}_{\Delta t} = \{ t_n : t_n = n \Delta t, \ 0 \leq n \leq N \} \]

be a uniform partition of the time interval \([0, T]\) with the time step size \( \Delta t = T/N \).

For any \( 1 \leq n \leq N \), we denote by \( I_n \) the interval \((t_{n-1}, t_n)\) and by \( v^n \) the numerical approximation of \( v(t_n) \) for any quantity \( v(t) \) (both scalar or vector). We also set

\[ \delta_t v^n := \frac{v^n - v^{n-1}}{\Delta t}. \]

Then the fully discrete scheme of the ferrofluid flow model (1) reads as: Given \( u_h^{n-1} \in S_h \) and \( m_h^{n-1} \in V_h \) for \( n = 1, 2, \ldots \), find \( u^n_h \in S_h \), \( \tilde{p}_h^n \in L_h \), \( m^n_h \in V_h \), \( \kappa^n_h \in U_h \), \( H^n_h \in V_h \), and \( \varphi^n_h \in W_h \) such that

\[ \delta_t u^n_h - (u^n_h \times \nabla u^n_h, \nabla v_h) + \eta (\nabla u^n_h, \nabla v_h) - (\tilde{p}_h^n, \nabla v_h) - \mu_0 b(v^n_h, H^n_h, m^n_h) - \frac{\mu_0}{2} (v^n_h \times k^n_h, H^n_h) = 0 \quad \forall \ v_h \in S_h, \]

\[ (\delta_t m^n_h, F_h) + b(u^n_h, m^n_h, F_h) + \frac{1}{2}(u^n_h \times F_h, k^n_h) + \frac{1}{2}(m^n_h, F_h) - \frac{1}{2} (\nabla \times (H^n_h, F_h) + \beta (m^n_h \times (m^n_h \times H^n_h), F_h) = 0 \quad \forall \ F_h \in V_h, \]

\[ (z^n_h, \chi_h) - (u^n_h \times m^n_h, \chi_h) = 0 \quad \forall \chi_h \in U_h, \]

\[ (k^n_h, \kappa_h) - (m^n_h, \nabla k^n_h) = 0 \quad \forall \kappa_h \in U_h, \]

\[ (H^n_h, G_h) + (\varphi^n_h, \nabla G_h) = 0 \quad \forall G_h \in V_h, \]

\[ (\nabla H^n_h + \nabla m^n_h, r_h) + (\nabla H^n_h, r_h) = 0 \quad \forall r_h \in W_h. \]

In the rest of this subsection, we will focus on the solvability of the scheme (60). Denote

\[ \mathcal{W}_h := \mathcal{H}_h \times V_h \times \mathcal{H}_h, \quad \mathcal{V}_h := \mathcal{H}_h \times V_h \times W_h, \]

and define a functional \( \mathcal{A}(:, :, :) : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R} \) by

\[
\mathcal{A}(U^n_h, V_h) := (u^n_h, v_h) + \mu_0 (m^n_h, F_h) - \Delta t (u^n_h \times \nabla u^n_h, v_h) + \Delta t \eta (\nabla u^n_h, \nabla v_h) - \Delta t \mu_0 b(v^n_h, H^n_h, m^n_h) - \frac{\mu_0 \Delta t}{2} (v^n_h \times \nabla m^n_h, H^n_h) + \Delta t \mu_0 b(u^n_h, m^n_h, F_h) + \frac{\mu_0 \Delta t}{2} (u^n_h \times F_h, \nabla m^n_h) - \frac{\mu_0 \Delta t}{2} (u^n_h \times m^n_h, \nabla F_h) + \mu_0 \Delta t (m^n_h, F_h) + \sigma \mu_0 \Delta t \nabla \cdot (\nabla m^n_h, F_h) + \mu_0 \Delta t \nabla \cdot (\nabla F_h, m^n_h)
\]

\[ - \frac{\mu_0 \Delta t}{\tau} (H^n_h, F_h) + \beta \mu_0 \Delta t (m^n_h \times (m^n_h \times H^n_h), F_h) + \mu_0 \Delta t (m^n_h, \phi_h), \]

with

\[ U^n_h = (u^n_h, m^n_h, H^n_h) \in \mathcal{W}_h, \quad V_h = (v_h, F_h, \phi_h) \in \mathcal{V}_h. \]
Define the norms $\| \cdot \|_{2G}$ and $\| \cdot \|_{2\mathcal{G}}$ on $\mathbb{G}_h$ and $\mathcal{M}_h$, respectively as follows:

$$
\|U_h\|_{2G}^2 := \|u_h\|^2 + \Delta t \eta \|\nabla u_h\|^2 + \frac{\mu_0 \Delta t(2 + \chi_0)}{\tau} \|H_h\|^2 + \mu_0 \sigma \Delta t \|\text{div} \, H_h\|^2 \\
+ \frac{\mu_0 \Delta t}{\tau} \|m_h\|^2 + \mu_0 \sigma \Delta t \|\text{curl}_h m_h\|^2 + \mu_0 \sigma \Delta t \|\text{div} \, m_h\|^2,
$$

$$
\|V_h\|_{2\mathcal{G}}^2 := \|v_h\|^2 + \Delta t \eta \|\nabla v_h\|^2 + \Delta t \|F_h\|^2 + \Delta t \|\text{curl}_h F_h\|^2 + \|\text{div} \, F_h\|,
$$

Introduce an auxiliary problem: Find $U_h^n = (u_h^n, m_h^n, H_h^n) \in S_h \times V_h \times \mathcal{M}_h$ and $\tilde{p}_h^n \in L_h$ such that

$$
\text{(61)} \quad 
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \tilde{p}_h^n + \text{div} \, v_h = l(V_h) \\
(\text{div} \, u_h^n, q_h) = 0
\end{array} \right. \quad \forall \, q_h \in L_h,
$$

with

$$
l(V_h) := (u_h^{n-1}, v_h) + \mu_0 (m_h^{n-1}, F_h) - \mu_0 \Delta t (Q_h \text{div} \, H_h^n, \phi_h).
$$

**Remark 4.1.** From the exact sequence (11) we know that $\text{div} \, \mathcal{R}_h^d = W_h$ and that for any $\phi_h \in W_h$, there exists a unique $v_h \in \mathcal{R}_h^d$ such that $\text{div} \, v_h = \phi_h$. Since $\text{div}$ is a linear operator, it holds $\text{dim} \, \mathcal{R}_h^d = \text{dim} \, W_h$. Therefore, the matrix representation of $\mathcal{A}$ in (61) is a square matrix.

We have the following lemma.

**Lemma 4.2.** The full-discrete form (60) and the auxiliary problem (61) are equivalent in the following sense:

- if $u_h^n \in S_h, \tilde{p}_h^n \in L_h, m_h^n \in V_h, z_h^n \in U_h, k_h^n \in U_h, H_h^n \in V_h$ and $\varphi_h^n \in W_h$ solve (60), then $\mathcal{L}_h^n = (u_h^n, m_h^n, H_h^n) \in \mathcal{M}_h$ and $\tilde{p}_h^n \in L_h$ solve (61);
- if $\mathcal{L}_h^n = (u_h^n, m_h^n, H_h^n) \in \mathcal{M}_h$ and $\tilde{p}_h^n \in L_h$ solve (61), then $u_h^n, \tilde{p}_h^n, m_h^n, z_h^n = Q_h^n (u_h^n \times m_h^n), k_h^n = \text{curl}_h m_h^n$, and a unique $\varphi_h^n \in W_h$ satisfying $H_h^n = \text{grad}_h \varphi_h^n$ solve (60).

**Proof.** We assume that $u_h^n \in S_h, \tilde{p}_h^n \in L_h, m_h^n \in V_h, z_h^n \in U_h, k_h^n \in U_h, H_h^n \in V_h$, and $\varphi_h^n \in W_h$ solve (60). The fourth, fifth and sixth equations of (60) imply that

$$
k_h^n = \text{curl}_h H_h^n, \quad z_h^n = Q_h^n (u_h^n \times m_h^n), \quad H_h^n = \text{grad}_h \varphi_h^n \in \mathcal{R}_h^d.
$$

Substitute them into the first and third equations of (60), then we get

$$
\left\{ \begin{array}{l}
(u_h^n, v_h) - \Delta t (u_h^n \times \text{curl} \, u_h^n, v_h) + \eta \Delta t (\nabla u_h^n, \nabla v_h) \\
- \Delta t (\tilde{p}_h^n, \text{div} \, v_h) - \mu_0 \Delta t (v_h, H_h^n, m_h^n) \\
- \frac{\mu_0 \Delta t}{2} (v_h \times \text{curl} \, m_h^n, H_h^n) - (u_h^{n-1}, v_h) = 0 \\
\forall \, v_h \in S_h,
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{l}
(m_h^n, F_h) + \Delta t (u_h^n, m_h^n, F_h) + \Delta t (u_h^n \times F_h, \text{curl}_h m_h^n) \\
- \frac{\mu_0 \Delta t}{2} (u_h^n \times m_h^n, \text{curl}_h F_h) + \mu_0 \Delta t (\text{curl}_h m_h^n, \text{curl}_h F_h) \\
+ \sigma \Delta t (\text{div} \, m_h^n, \text{div} \, F_h) + \Delta t (m_h^n, F_h) - \mu_0 \Delta t (H_h^n, F_h) \\
+ \beta \Delta t (m_h^n \times m_h^n, H_h^n) - (m_h^{n-1}, F_h) = 0 \\
\forall \, F_h \in V_h,
\end{array} \right.
$$

which, together with the last and second equations of (60), gives (61).

On the other hand, we assume that $\mathcal{L}_h^n = (u_h^n, m_h^n, H_h^n) \in \mathcal{M}_h$ and $\tilde{p}_h^n \in L_h$ solve (61), then it is easy to check that $z_h^n = Q_h^n (u_h^n \times m_h^n)$ and $k_h^n = \text{curl}_h m_h^n$ satisfy the fourth and fifth equations of (60). The definition of $\mathcal{R}_h^d$ implies that there exists a unique $\varphi_h^n \in W_h$ such that $H_h^n = \text{grad}_h \varphi_h^n$ satisfying the sixth equation of (60). Taking $V_h = (v_h, 0, 0) \in \mathcal{M}_h, q_h = 0 \in L_h$ and $V_h = (0, 0, 0) \in \mathcal{M}_h, q_h \in L_h$ in (61), respectively, we get the
first and the second equations of (60). Taking $V_h = (0, F_h, 0) \in M_h$, $q_h = 0 \in L_h$ in (61), we get the third equation of (60). For any $r_h \in W_h$, we take $V_h = (0, 0, r_h) \in M_h$ in (61), we get the last equation of (60). □

In the following, we will prove that the auxiliary problem (61) has solutions, and then Lemma 4.2 implies the full-discrete scheme (60) also has solutions.

We first show that the following inf-sup condition holds.

**Lemma 4.3.** We have

$$\sup_{V_h \in M_h} \frac{A(U_h, V_h)}{||V_h||_{\Omega}} \geq \alpha ||U_h||_{\Omega} \quad \forall U_h = (u_h, m_h, H_h) \in M_h.$$

**Proof.** For any $U_h = (u_h, m_h, H_h) \in M_h$, taking $V_{h,1} = (u_h, -H_h, 0) \in M_h$ and using the fact that $\text{curl}_h H_h = 0$, we have

$$A(U_h, V_{h,1}) = \|u_h\|^2 + \Delta t \|\nabla u_h\|^2 + \frac{\mu_0 \chi_0 \Delta t}{\tau} \|H_h\|^2 + \beta \mu_0 \Delta t \|m_h \times H_h\|^2$$

$$\quad - \mu_0 \sigma \Delta t (\text{div} m_h, \text{div} H_h) - \mu_0 (\frac{\Delta t}{\tau} + 1)(m_h, H_h).$$

Taking $V_{h,2} = (0, m_h, 0) \in M_h$, we get

$$A(U_h, V_{h,2}) = \mu_0 \sigma \Delta t \|\text{curl}_h m_h\|^2 + \mu_0 \sigma \Delta t \|\text{div} m_h\|^2 + \mu_0 (\frac{\Delta t}{\tau} + 1)\|m_h\|^2$$

$$\quad - \frac{\mu_0 \chi_0 \Delta t}{\tau} (m_h, H_h).$$

Note that $H_h \in \mathbb{R}^d \Rightarrow$ implies that there exists $\psi_h \in W_h$ such that $H_h = \text{grad}_h \psi_h$. Taking $V_{h,3} = (0, 0, -(\frac{1+\chi_0}{\tau} + 1/\Delta t) \varphi_h + \sigma \text{div} H_h) \in M_h$, we obtain

$$A(U_h, V_{h,3}) = \left(\frac{\mu_0 \Delta t (1+\chi_0)}{\tau} + \mu_0\right)\|H_h\|^2 + \mu_0 \sigma \Delta t (\text{div} H_h, \text{div} m_h)$$

$$\quad + \left(\frac{\mu_0 \Delta t (1+\chi_0)}{\tau} + \mu_0\right)(m_h, H_h) + \mu_0 \sigma \Delta t \|\text{div} H_h\|^2$$

Let $V_h = V_{h,1} + V_{h,2} + V_{h,3} \in M_h$, then we have

$$A(U_h, V_h) = \|u_h\|^2 + \Delta t \|\nabla u_h\|^2 + \left(\frac{\mu_0 \Delta t (1+2\chi_0)}{\tau} + \mu_0\right)\|H_h\|^2$$

$$\quad + \mu_0 \sigma \Delta t \|\text{div} H_h\|^2 + \left(\frac{\mu_0 \Delta t}{\tau} + 1\right)\|m_h\|^2 + \mu_0 \sigma \Delta t \|\text{curl}_h m_h\|^2$$

$$\quad + \mu_0 \sigma \Delta t \|\text{div} m_h\|^2 + \beta \mu_0 \Delta t \|m_h \times H_h\|^2$$

$$\geq \|U_h\|^2_{\Omega}.$$

Using the fact that $\|\varphi_h\| \lesssim \|\text{grad}_h \varphi_h\| = ||H_h||$, we obtain

$$||V_h||_\Omega^2 = \|u_h\|^2 + \Delta t \|\nabla u_h\|^2 + \Delta t \|m_h - H_h\|^2 + \Delta t \|\text{curl}_h (m_h - H_h)\|^2$$

$$\quad + \Delta t \|\text{div} (m_h - H_h)\|^2 + \Delta t \|((1+\chi_0)/\tau + 1/\Delta t) \varphi_h - \sigma \text{div} H_h\|^2$$

$$\lesssim \|U_h\|^2_{\Omega}.$$

Then the desired result follows. □

We define a linear operator $\Pi : M_h \rightarrow M_h$ by

$$\Pi U_h := (u_h, m_h - H_h, -\frac{1+\chi_0}{\tau} w_h + \sigma \text{div} H_h) \in M_h$$
for any \( U_h = (u_h, m_h, H_h) \in \mathfrak{W}_h \), with \( w_h \in W_h \) satisfying \( H_h = \nabla_h w_h \). We have the following properties of \( \Pi \).

**Lemma 4.4.** The linear operator \( \Pi \) is surjective, and the adjoint operator \( \Pi^\top \) of \( \Pi \) with respect to the \( L^2 \) inner product is injective.

**Proof.** For any \( \mathcal{V}_h = (v_h, F_h, q_h) \in \mathfrak{W}_h \), there exist a unique pair \( w_h \in W_h \) and \( H_h \in \mathfrak{V}_h \) such that
\[
H_h = \nabla_h w_h \quad \text{and} \quad q_h = -\frac{1 + \chi_0}{\tau} w_h + \sigma \, \nabla w_h,
\]
since \( H_h \) and \( w_h \) solve the saddle point problem
\[
\begin{aligned}
(H_h, G_h) + (w_h, \nabla G_h) &= 0 \quad \forall G_h \in V_h, \\
\sigma(\nabla H_h, r_h) - \frac{1 + \chi_0}{\tau} (w_h, r_h) &= (q_h, r_h) \quad \forall r_h \in W_h.
\end{aligned}
\]
Therefore, choosing \( U_h = (v_h, F_h + H_h, H_h) \in \mathfrak{W}_h \), we have \( \Pi U_h = \mathcal{V}_h \). The surjective of \( \Pi \) implies that the adjoint operator \( \Pi^\top \) of \( \Pi \) with respect to the \( L^2 \) inner product is injective.

We define \( A_h : \mathfrak{W}_h \to \mathfrak{W}_h \) as
\[
A_h(U_h, \mathcal{V}_h) = A(U_h, \mathcal{V}_h) \quad \forall U_h \in \mathfrak{W}_h, \quad \forall \mathcal{V}_h \in \mathfrak{W}_h.
\]
The proof of Lemma 4.3 also implies that
\[
\forall U_h \in \mathfrak{W}_h, \quad \forall \mathcal{V}_h \in \mathfrak{W}_h.
\]
We consider the following problem: Find \( U_h^n \in \mathfrak{W}_h \) such that
\[
A(U_h^n, \mathcal{V}_h) = I(\mathcal{V}_h) \quad \forall \mathcal{V}_h \in \mathfrak{W}_h.
\]
Here \( I(\mathcal{V}_h) \) is the same as in (61). The problem (63) is equivalent to the operator form
\[
A_h(U_h^n) = Q^2_h I,
\]
where \( Q^2_h \) is the \( L^2 \) projection operator to \( \mathfrak{W}_h \). We have the following Lemma.

**Lemma 4.5.** The problem (63) has at least one solution \( U_h^n \in \mathfrak{W}_h \).

**Proof.** Let \( \Phi : \mathfrak{W}_h \to \mathfrak{W}_h \) be defined as
\[
\Phi(U_h^n) := A_h(U_h^n) - Q^2_h I.
\]
Then, for any \( U_h^n \in \mathfrak{W}_h \), we have
\[
\Pi^\top \Phi(U_h^n, U_h^n) = (\Phi(U_h^n), \Pi U_h^n) = (A_h(U_h^n), \Pi U_h^n) - I(\Pi U_h^n) \geq (\alpha \|U_h^n\|_{\mathfrak{W}}^2 - \|U_h^n\|_{\mathfrak{W}}^2)\|U_h^n\|_{\mathfrak{W}}.
\]
Taking \( c = \|U_h^n\|_{\mathfrak{W}} / \alpha \), we get
\[
\Pi^\top \Phi(U_h^n, U_h^n) \geq 0 \quad \forall U_h^n \in \mathfrak{W}_h \quad \text{with} \quad \|U_h^n\|_{\mathfrak{W}} = c.
\]
By [18, Chapter IV, Corollary 1.1], there exists an element \( U_h^n \in \mathfrak{W}_h \) such that
\[
\Pi^\top \Phi(U_h^n) = 0 \quad \text{with} \quad \|U_h^n\|_{\mathfrak{W}} \leq c.
\]
The injective of the linear operator \( \Pi^\top \) implies that
\[
\Phi(U_h^n) = 0 \quad \text{with} \quad \|U_h^n\|_{\mathfrak{W}} \leq c.
\]
This completes the proof.

We are now at a position to state the following existence theorem.
Theorem 4.6. The full-discrete scheme (60) has at least one solution.

Proof. By Lemma 4.2, we only need to prove that (61) has at least one solution $U_h^n = (u_h^n, m_h^n, H_h^n) \in \mathcal{U}_h$ and $\tilde{p}_h^n \in L_h$. From Lemma 4.5, let $U_h^n$ be a solution of (63), and we determine $\tilde{p}_h^n \in L_h$ such that

$\Delta t(\tilde{p}_h^n, \text{div } v_h) = \mathcal{L}(v_h) - \mathcal{A}(U_h^n, v_h) \quad \forall v_h \in S_h \times V_h \times W_h.$

The existence of $\tilde{p}_h^n \in L_h$ is then guaranteed by the inf-sup condition (12).

For the fully discrete scheme (60), define the energy

$\mathcal{E}_h^n := \|u_h^n\|^2 + \|m_h^n\|^2 + \mu_0\|H_h^n\|^2.$

Then the following energy estimate holds.

Theorem 4.7. Assume that $u_h^n \in S_h, \tilde{p}_h^n \in L_h, m_h^n \in V_h, z_h^n \in U_h, k_h^n \in U_h, H_h^n \in V_h$, and $\varphi_h^n \in \mathcal{W}_h$ solve (60), then we have

$\mathcal{E}_h^n + 2 \Delta t \mathcal{F}_h^n \leq \mathcal{E}_h^{n-1} + C \Delta t \left( \frac{\mu_0 + \chi_0}{\tau} \|H_h^n\|^2 + \mu_0\|\delta_t H_h^n\|^2 + \mu_0\|\sigma \text{div } H_h^n\|^2 \right)$

with

$\mathcal{F}_h^n = \frac{1}{\Delta t} \|\nabla u_h^n\|^2 + \mu_0 b(u_h^n; H_h^n, m_h^n) - \frac{\mu_0}{2} (u_h^n \times k_h^n, H_h^n)$

and

$(\delta_t m_h^n, H_h^n) + b(u_h^n; m_h^n, H_h^n) + \frac{1}{2} (u_h^n \times H_h^n, k_h^n) + \sigma(\text{div } m_h^n, \text{div } H_h^n)$

$= - \frac{1}{\tau} (m_h^n, H_h^n) + \frac{\chi_0}{\tau} \|H_h^n\|^2 + \beta\|m_h^n \times H_h^n\|^2,$

which further imply

$\Delta t \|u_h^n\|^2 + \eta \|\nabla u_h^n\|^2 = \frac{1}{\Delta t} (u_h^{n-1}, u_h^n) + \mu_0 b(u_h^n; H_h^n, m_h^n) + \frac{\mu_0}{2} (u_h^n \times k_h^n, H_h^n)$

$+ \frac{\mu_0}{\tau} (m_h^n, H_h^n) - \frac{\chi_0 \mu_0}{\tau} \|H_h^n\|^2 - \mu_0\beta\|m_h^n \times H_h^n\|^2.$

The sixth equation of (60) implies that $H_h^n = \text{grad } \varphi_h^n$. Taking $r_h = \varphi_h^n$ in the seventh equation of (60) at time levels $t_n$ and $t_{n-1}$ and using integration by part, we get

$(m_h^n, H_h^n) + \|H_h^n\|^2 = -(\pi_h^n H_h^n, H_h^n)$

and

$(m_h^{n-1}, H_h^n) + (H_h^{n-1}, H_h^n) = -(\pi_h^n H_h^{n-1}, H_h^n).$

These two equations give

$(\delta_t m_h^n, H_h^n) = - \frac{1}{\Delta t} \|H_h^n\|^2 + \frac{1}{\Delta t} (H_h^{n-1}, H_h^n) - (\pi_h^n \delta_t H_h^{n-1}, H_h^n).$

Taking $r_h = \text{div } m_h^n$ in the seventh equation of (60), we get

$(\text{div } m_h^n, \text{div } H_h^n) = - \|\text{div } m_h^n\|^2 - (\text{div } H_e, \text{div } m_h^n).$
Using the equations (64)-(67), we have
\[
\frac{1}{\Delta t} (\|u_h^n\|^2 + \mu_0 \|H^n_h\|^2) + \eta \|\nabla u_h^n\|^2 + \mu_0 \sigma \|\text{div } m_h^n\|^2 \\
+ \frac{\mu_0}{\tau} (1 + \chi_0) \|H^n_h\|^2 + \mu_0 \beta \|m_h^n \times H^n_h\|^2
\]
\[
= - \frac{\mu_0}{\tau} (\pi_h^n H^n_h, H^n_h) - \mu_0 (\pi_h^n \delta_t H^n_h, H^n_h) - \mu_0 \sigma (\text{div } H^n_h, \text{div } m_h^n) \\
+ \frac{1}{\Delta t} \left[ (u_h^{n-1}, u_h^n) + \mu_0 (H_h^{n-1}, H^n_h) \right].
\]

Taking \( F_h = m_h^n \) in the third equation of (60), we have
\[
\frac{1}{\Delta t} \|m_h^n\|^2 + \sigma \|k_h^n\|^2 + \sigma \|\text{div } m_h^n\|^2 + \frac{1}{\tau} \|m_h^n\|^2 + \frac{\chi_0}{\tau} \|H^n_h\|^2
\]
\[
= - \frac{\chi_0}{\tau} (\pi_h^n H^n_h, H^n_h) + \frac{1}{\Delta t} (m_h^{n-1}, m_h^n),
\]

The above two relations show
\[
\frac{1}{\Delta t} (\|u_h^n\|^2 + \mu_0 \|H^n_h\|^2 + \|m_h^n\|^2) + \eta \|\nabla u_h^n\|^2 + \sigma (\mu_0 + 1) \|\text{div } m_h^n\|^2 + \sigma \|k_h^n\|^2 \\
+ \frac{1}{\tau} \|m_h^n\|^2 + \frac{1}{\tau} [\mu_0 (1 + \chi_0) + \chi_0] \|H^n_h\|^2 + \mu_0 \beta \|m_h^n \times H^n_h\|^2
\]
\[
= - \frac{\mu_0 + \chi_0}{\tau} (\pi_h^n H^n_h, H^n_h) - \mu_0 (\pi_h^n \delta_t H^n_h, H^n_h) - \mu_0 \sigma (\text{div } H^n_h, \text{div } m_h^n) \\
+ \frac{1}{\Delta t} \left[ (u_h^{n-1}, u_h^n) + \mu_0 (H_h^{n-1}, H^n_h) + (m_h^{n-1}, m_h^n) \right],
\]

which, together with the inequality
\[
(u_h^{n-1}, u_h^n) + \mu_0 (H_h^{n-1}, H^n_h) + (m_h^{n-1}, m_h^n)
\]
\[
\leq \frac{1}{2} (\|u_h^n\|^2 + \mu_0 \|H^n_h\|^2 + \|m_h^n\|^2) + \frac{1}{2} (\|u_h^{n-1}\|^2 + \mu_0 \|H_h^{n-1}\|^2 + \|m_h^{n-1}\|^2),
\]
leads to
\[
\mathcal{E}_h^n + 2 \Delta t \mathcal{F}_h^n
\]
\[
\leq \mathcal{E}_h^{n-1} + 2 \Delta t \left( - \frac{\mu_0 + \chi_0}{\tau} (\pi_h^n H^n_h, H^n_h) - \mu_0 (\pi_h^n \delta_t H^n_h, H^n_h) - \mu_0 \sigma (\text{div } H^n_h, \text{div } m_h^n) \right).
\]

Finally, the desired result follows from the Cauchy-Schwarz inequality and Lemma 2.1.

\[ \square \]

**Remark 4.8.** When \( H_e = 0 \), from Theorem 4.7 we easily have the following energy decaying result:
\[
\mathcal{E}_h^n \leq \mathcal{E}_h^{n-1}.
\]

In particular, if \( \mathcal{F}_h^n \neq 0 \), then
\[
\mathcal{E}_h^n < \mathcal{E}_h^{n-1}.
\]
4.2. Error analysis. To give the error analysis of the fully discrete scheme (60), we first introduce the following notations:

\[ \xi_u^n := \pi_h u(\cdot, t^n) - u_h^n, \quad \xi_m^n := I_h^d m(\cdot, t^n) - m_h^n, \quad \xi_z^n := I_h^e z(\cdot, t^n) - z_h^n, \]

\[ \xi_k^n := I_h^d k(\cdot, t^n) - k_h^n, \quad \xi_H^n := I_h^d H(\cdot, t^n) - H_h^n, \quad \xi_p^n := Q_h \tilde{p}(\cdot, t^n) - \tilde{p}_h^n, \]

\[ \theta_u^n := \pi_h u(\cdot, t^n) - u(\cdot, t^n), \quad \theta_m^n := I_h^d m(\cdot, t^n) - m(\cdot, t^n), \quad \theta_z^n := I_h^e z(\cdot, t^n) - z(\cdot, t^n), \]

\[ \theta_k^n := I_h^d k(\cdot, t^n) - k(\cdot, t^n), \quad \theta_H^n := I_h^d H(\cdot, t^n) - H(\cdot, t^n), \quad \theta_p^n := Q_h \tilde{p}(\cdot, t^n) - \tilde{p}(\cdot, t^n). \]

Next, we rewrite the first and third equations of (5) at the time level \( t_n \) as:

\[
\begin{align*}
&\left( \delta_t \pi_h u^n, v \right) - \left( u^n \times \text{curl} u^n, v \right) + \eta(\nabla \pi_h u^n, \nabla v) \\
&\quad - (Q_h \tilde{p}, \text{div} v) - \mu_0 b(v; H^n, m^n) - \frac{\mu_0}{2} (v \times k^n, H^n) \\
&\quad = \left( \delta_t \theta_u^n, v \right) + \left( \delta_t u^n - \delta_t u^n, v \right) + \eta(\nabla \theta_u^n, v) - \left( \theta_p^n, \text{div} v \right) \quad \forall v \in S_h,
\end{align*}
\]

and

\[
\begin{align*}
&\left( \delta_t I_h^d m^n, F_h \right) + b(u^n; m^n, F_h) + \frac{1}{2} (u^n \times F_h, k^n) - \frac{1}{2} (\text{curl} I_h^e z^n, F_h) \\
&\quad + \sigma(\text{curl} I_h^d k^n, F_h) + \sigma(\text{div} I_h^d m^n, \text{div} F_h) + \frac{1}{\tau} (I_h^d m^n, F_h) \\
&\quad - \frac{\chi_0}{\tau} (I_h^d H^n, F_h) + \beta(m^n \times (m^n \times H^n), F_h) \\
&\quad = \left( \delta_t \theta_m^n, F_h \right) + \left( \delta_t m^n - \delta_t u^n, F_h \right) - \frac{1}{2} (\text{curl} \theta_z^n, F_h) \\
&\quad + \sigma(\text{curl} \theta_k^n, F_h) + \frac{1}{\tau} (\theta_m^n, F_h) - \frac{\chi_0}{\tau} (\theta_H^n, F_h) \quad \forall F_h \in V_h.
\end{align*}
\]

Subtracting the first and third equations of (60) from the above two equations, respectively, we get

\[
\begin{align*}
&\left( \delta_t \xi_u^n, v_h \right) + \eta(\nabla \xi_u^n, \nabla v_h) - \left( \xi_p^n, \text{div} v_h \right) \\
&\quad = f_t^u(v_h) + f_2^u(v_h) - f_3^u(v_h) + (\delta_t \theta_u^n, v_h) \\
&\quad + (\delta_t u^n - \delta_t u^n, v_h) + \eta(\nabla \theta_u^n, \nabla v_h) - \left( \theta_p^n, \text{div} v_h \right) \quad \forall v_h \in S_h
\end{align*}
\]

and

\[
\begin{align*}
&\left( \delta_t \xi_m^n, F_h \right) - \frac{1}{2} (\text{curl} \xi_z^n, F_h) + \sigma(\text{curl} \xi_u^n, F_h) + \frac{1}{\tau} (\xi_m^n, F_h) \\
&\quad + \sigma(\text{div} \xi_m^n, \text{div} F_h) - \frac{\chi_0}{\tau} (\xi_H^n, F_h) \\
&\quad = \left( \delta_t \theta_m^n, F_h \right) + \left( \delta_t \theta_m^n, F_h \right) + (\delta_t m^n - \delta_t u^n, F_h) - \frac{1}{2} (\text{curl} \theta_z^n, F_h) \\
&\quad + \sigma(\text{curl} \theta_k^n, F_h) - \frac{\chi_0}{\tau} (\theta_H^n, F_h) - f_t^u(F_h) - f_2^u(F_h) - f_3^u(F_h) \quad \forall F_h \in V_h.
\end{align*}
\]
We define

\[ J^n_0 := \frac{\mu_0}{\tau} (\theta_m, \xi_H) + \frac{\mu_0 \chi_0}{\tau} (\theta_{H}, \xi_H) - \mu_0 (\delta_t \theta_m, \xi_H) + \frac{1}{2} \mu_0 (\nabla \theta, \xi_H) \\
- \sigma \mu_0 (\nabla \theta, \xi_H) + (\delta_t \theta, \xi_H) + \eta (\nabla \theta, \nabla \xi_H) + (\hat{p} - \hat{p}_h, \nabla \xi_H) \\
+ (\delta_t u^n - \delta_t \xi_H^n, \xi_H^n) - \mu_0 (\delta_t \xi_H^n - \partial_r m^n, \xi_H^n). \]

Similar to Lemma 3.7, the following result holds:

**Lemma 4.9.** We have

\[ (\delta_t \xi^n_m, \xi^n_H) + \mu_0 (\delta_t \xi^n_H, \xi^n_H) + (\delta_t \xi^n_m, \partial_r \xi^n_H) + \frac{1}{\tau} (n_1 + \chi_0 + \chi_0) \xi^n_H + \sigma \xi^n_H + \sigma \xi^n_H \\
= J^n_0 + J^n_2 + J^n_2 + J^n_4. \]

**Proof.** Taking \( v_h = \xi^n_m \) in (68) and \( F_h = \xi^n_H \) in (69), respectively, we obtain

\[ (\delta_t \xi^n_m, \xi^n_H) + \eta (\nabla \xi^n_m, \xi^n_H) + (\delta_t \xi^n_m, \xi^n_H) + \frac{1}{\tau} (n_1 + \chi_0 + \chi_0) \xi^n_H + \sigma \xi^n_H + \sigma \xi^n_H \]

and

\[ (\delta_t \xi^n_m, \xi^n_H) + \frac{1}{\tau} (\xi^n_m, \xi^n_H) + \sigma (\xi^n_m, \xi^n_H) - \frac{\chi_0}{\tau} \xi^n_H + \eta (\nabla \xi^n_m, \nabla \xi^n_H) + (\delta_t \xi^n_m, \xi^n_H) + \frac{1}{\tau} (\nabla \theta, \xi^n_H) \\
+ \sigma (\nabla \theta, \xi^n_H) - \frac{\chi_0}{\tau} (\theta, \xi^n_H) - f^n_0 (\xi^n_H) - f^n_0 (\xi^n_H) - f^n_0 (\xi^n_H). \]

These two equations indicate

\[ (\delta_t \xi^n_m, \xi^n_H) + \eta (\nabla \xi^n_m, \xi^n_H) + \frac{\mu_0 \chi_0}{\tau} (\xi^n_H, \xi^n_H) \]

\[ - \mu_0 (\delta_t \xi^n_m, \xi^n_H) - \sigma \mu_0 (\xi^n_m, \xi^n_H) - \frac{\mu_0}{\tau} (\xi^n_m, \xi^n_H) \]

\[ = J^n_0 + J^n_2. \]

The last equations of (1) and (60) imply that

\[ \text{div} \ H^n + \text{div} \ m^n = - \text{div} \ H^n_e, \]

\[ \text{div} \ H^n + \text{div} \ m^n = - Q_h \text{div} \ H^n_e. \]

Therefore,

\[ \text{div} \ \xi^n_H + \text{div} \ \xi^n_H = - (I - Q_h) \text{div} \ H^n_e + \text{div} \ \theta^n_H + \text{div} \ \theta^n_H. \]

Since \( \xi^n_H \in R^n H \), there exists a unique \( \phi_h \in W_h \) such that \( \xi^n_H = \text{grad} \phi_h \). Testing (71) with \( \phi_h \), we get

\[ \| \xi^n_H \|^2 + (\xi^n_H, \xi^n_H) = 0. \]

Subtracting (71) on the time level \( t_{n-1} \) from (71) on the time level \( t_n \), and testing the resultant equation with \( \varphi_h \), we obtain

\[ (\delta_t \xi^n_H, \xi^n_H) + (\delta_t \xi^n_m, \xi^n_H) = 0. \]
Taking $F_h = \xi^n_m$ in (69) and using the fact that
\[(\text{curl } \xi^n_k, \xi^n_m) = \|\xi^n_k\|^2 - (\theta^n_k, \xi^n_k),\]
we get
\[(73) \quad (\delta_t \xi^n_m, \xi^n_m) + \sigma \|\xi^n_m\|^2 + \sigma \|\text{div } \xi^n_m\|^2 + \frac{1}{\tau} \|\xi^n_{mH}\|^2 + \frac{\gamma_0}{\tau} \|\xi^n_{mH}\|^2 = \tilde{J}^2_n + J^1_n.\]
Applying (72) and adding (70) and (73) together, we finally obtain the desired result. □

By following a similar routine as in the proof of Lemma 3.13, for any $n$

Then the desired result follows.

Lemma 4.10. Under Assumption 3.6, for any $1 \leq L \leq N$ we have
\[
\Delta t \sum_{n=1}^{L} \|\xi^n_p\| \lesssim \|\xi^L_L\| + h^{l+1} + \Delta t + h^{l+1} \max_{1 \leq n \leq L} \|\text{div } H^n_n\| \lesssim L^2 + h^{l+1/2} \|\xi^n_k\|
\]
\[
+ \Delta t \sum_{n=1}^{L} \left( \|\nabla \xi^n_m\| + \|\xi^n_{mH}\| \|\xi^n_m\| + \|\xi^n_{mH}\| \|\xi^n_m\| + \max_{1 \leq n \leq L} \|\text{div } H^n_n\| \|\xi^n_m\| \right) + L \sum_{n=1}^{L} \|\xi^n_m\|
\]
\[
+ \Delta t \sum_{n=1}^{L} \left( \|\xi^n_k\| \|\xi^n_p\| + \|\xi^n_{pH}\| \|\xi^n_p\| \right)
\]

Proof. The inf-sup condition (12) and (68) imply that for any $1 \leq L \leq N$, we have
\[
\Delta t \sum_{n=1}^{L} \|\xi^n_p\| \lesssim \Delta t \sup_{v_h \in S_h} \sum_{n=1}^{L} \frac{(\xi^n_p, \nabla \cdot v_h)}{\|v_h\|_1}
\]
\[
\lesssim \|\xi^L_L\| + \Delta t \sum_{n=1}^{L} \|\nabla \xi^n_m\| + h^{l+2} \|u\|_{L^\infty(H^{l+2})} + \Delta t \|u\|_{L^\infty(W^{2,\infty}(L^2))}
\]
\[
+ h^{l+2} \|u_0\|_{H^{l+2}} + h^{l+1} \|\tilde{u}\|_{L^\infty(H^{l+2})} + h^{l+1} \|\tilde{p}\|_{L^\infty(H^{l+1})} + \tilde{T}_1^n,
\]
with
\[
\tilde{T}_1^n = \sup_{v_h \in S_h} \Delta t \sum_{n=1}^{L} \frac{|f^n_1(v_h) + f^n_2(v_h) - f^n_3(v_h)|}{\|v_h\|_1},
\]
where
\[
f^n_1(v_h) = \mu_0 b(v_h; H(\cdot, t_n), m(\cdot, t_n)) - \mu_0 b(v_h; H^n_h, m^n_h),
\]
\[
f^n_2(v_h) = \frac{\mu_0}{T}(v_h \times k(\cdot, t_n), H(\cdot, t_n)) - \frac{\mu_0}{T}(v_h \times k^n_h, H^n_h),
\]
\[
f^n_3(v_h) = (\bar{u}(\cdot, t_n) \times \text{curl } u(\cdot, t_n) - u^n_h \times \text{curl } u^n_h, v_h).
\]
By following a similar routine as in the proof of Lemma 3.13, for any $v_h \in S_h$ we can obtain
\[
|f^n_1(v_h)| \lesssim \|v_h\|_1 \left( h^{l+1} + \|\xi^n_{mH}\| \|\nabla v_h\| \|\text{div } H^n_n\| \right),
\]
\[
|f^n_2(v_h)| \lesssim \|v_h\|_1 \left( h^{l+1} + \|\xi^n_{mH}\| + \|\xi^n_{mH}\| + \left( h^{l+1/2} + \|\text{div } H^n_n\| \right) \|\xi^n_{mH}\| \|\nabla v_h\|, \right)
\]
\[
|f^n_3(v_h)| \lesssim \|v_h\|_1 \left( h^{l+1} + \|\xi^n_{mH}\| + \|\nabla v_h\| \|\nabla v_h\| \right).
\]
Then the desired result follows. □
Denote

\[ \mathcal{E}_h^n := ||e^n||^2 + ||\xi_H^n||^2 + ||\xi_m^n||^2, \]
\[ \mathcal{F}_h^n := ||\nabla e^n||^2 + ||\xi_k^n||^2 + ||\xi_H^n||^2 + ||\xi_m^n||^2, \]

Then we have the following estimate:

**Lemma 4.11.** Under Assumption 3.6 and the condition \( \max_{1 \leq n \leq N} ||\text{div} H_H^n||_{L^2} \leq C \), for any \( 1 \leq L \leq N \) we have

\[ \Delta t \sum_{n=1}^{L} \left( \mathcal{E}_h^n + \Delta t \sum_{\gamma=1}^{n} \mathcal{F}_h^n \right) \lesssim h^{2(l+1)} + \Delta t^2. \]

**Proof.** By Lemmas 4.9, 3.10 and 3.11 and the inequalities (74) and (74), we have

\[ \frac{1}{2} \Delta t \left( ||e^n||^2 + \mu_0 ||\xi_H^n||^2 + ||\xi_m^n||^2 - ||\xi_{n-1}^m||^2 - ||\xi_{n-1}^m||^2 \right) \]
\[ + \eta ||\nabla e^n||^2 + \frac{1}{\tau} (\mu_0 (1 + \chi_0) + \chi_0) ||\xi_H^n||^2 + \sigma ||\xi_k^n||^2 \]
\[ + \frac{1}{\tau} ||\xi_m^n||^2 + \sigma \mu_0 \text{div} \xi_H^n ||^2 + \sigma \text{div} \xi_m^n ||^2 \]
\[ \lesssim h^{l+1} \mathcal{T}_1 + h^{l+1/2} \mathcal{T}_2 + \mathcal{T}_3 + \Delta t \mathcal{T}_4, \]

with

\[ \mathcal{T}_1 := ||\nabla e^n|| + ||\xi_H^n|| + ||\xi_m^n|| + ||\xi_k^n||, \]
\[ \mathcal{T}_2 := ||\nabla e^n|| ||\xi_k^n|| + ||\xi_k^n|| ||\xi_H^n|| + ||\xi_k^n|| ||\xi_m^n||, \]
\[ \mathcal{T}_3 := ||\xi_k^n|| ||\xi_m^n|| + ||\xi_m^n|| ||\text{div} \xi_H^n|| + ||\xi_m^n|| ||\text{div} \xi_m^n||, \]
\[ \mathcal{T}_4 := ||\xi_k^n|| + ||\xi_m^n||. \]

For any \( 1 \leq L \leq N \), using Lemma 3.12 and 4.10, we have

\[ \Delta t \sum_{n=1}^{L} \mathcal{T}_1 \lesssim h^{l+1} + ||\xi_L^n|| + \Delta t \sum_{n=1}^{L} \left( ||\xi_H^n|| + ||\xi_m^n|| + ||\nabla \xi_m^n|| + ||\xi_k^n|| \right). \]

Summing up (76) with \( n \) from 1 to \( L \), we get

\[ \mathcal{E}_h^L + \Delta t \sum_{n=1}^{L} \mathcal{F}_h^n = h^{l+1} \Delta t \sum_{n=1}^{L} \mathcal{T}_1 + h^{l+1/2} \Delta t \sum_{n=1}^{L} \mathcal{T}_2 + \Delta t \sum_{n=1}^{L} \mathcal{T}_3 + \Delta t^2 \sum_{n=1}^{L} \mathcal{T}_4 \]
\[ \lesssim h^{2(l+1)} + h^{l+1} \frac{1}{2} \left( \mathcal{E}_h^L \right)^{1/2} + \Delta t \sum_{n=1}^{L} \left( \mathcal{E}_h^n \right)^{1/2} \left( \mathcal{F}_h^n \right)^{1/2} + \Delta t^2 \sum_{n=1}^{L} \mathcal{F}_h^n \]
\[ \lesssim h^{2(l+1)} + h^{l+1} \left( \mathcal{E}_h^L \right)^{1/2} + \left( \Delta t \sum_{n=1}^{L} \mathcal{E}_h^n \right)^{1/2} \left( \Delta t \sum_{n=1}^{L} \mathcal{F}_h^n \right)^{1/2} + \Delta t^2 \sum_{n=1}^{L} \mathcal{F}_h^n \]

Therefore, we obtain

\[ \left( \mathcal{E}_h^L + \Delta t \sum_{n=1}^{L} \mathcal{F}(r) \right)^{1/2} \lesssim h^{l+1} + \Delta t + \left( \Delta t \sum_{n=1}^{L} \left( \mathcal{E}_h^n + \Delta t \sum_{l=1}^{n} \mathcal{F}_h(l) \right) \right)^{1/2}, \]
and then
\[
\Delta t \sum_{L=1}^{M} \left( \varepsilon_h^L + \Delta t \sum_{n=1}^{L} \delta_h^n \right) \lesssim T h^{2(l+1)} + T \Delta t^2 + \Delta t \sum_{L=1}^{M} \left[ \Delta t \sum_{n=1}^{L} \left( \varepsilon_h^n + \Delta t \sum_{l=1}^{n} \delta_h^l \right) \right],
\]
which, together with the Gronwall’s inequality [29, Theorem 1.2.2], implies the desired result.

Using Lemma 4.11 and the triangular inequality, we have the following error estimates for the fully discrete scheme (60).

**Theorem 4.12.** Under Assumption 3.6 and the condition \( \max_{1 \leq n \leq N} \| \text{div} \ H_h^n \|_{L^3} \leq C \), for any \( 1 \leq L \leq N \) we have
\[
\Delta t \sum_{n=1}^{L} \left( \text{err}_1^n + \Delta t \sum_{l=1}^{n} \text{err}_2^l \right) \lesssim h^{2(l+1)} + \Delta t^2
\]
and
\[
\Delta t \sum_{n=1}^{L} \| \tilde{p}^n - p_h^n \| \lesssim h^{2(l+1)} + \Delta t^2.
\]

5. Numerical experiments

In this section, we provide three numerical examples to verify the performance of the fully discrete scheme (60). The numerical experiments are performed by using iFEM package [12], and the nonlinear system (60) is solved by the following quasi-Newton iteration with \( M = 2 \):

**Algorithm 5.1.** Given \( u_h^{n-1} \) and \( m_h^{n-1} \), to find \( u_h^n, \tilde{p}_h^n, m_h^n, z_h^n, k_h^n, H_h^n, \) and \( \varphi_h^n \) through three steps:

**Step 1.** Let \( u_h^{n-1} = u_h^{n-1} \) and \( m_h^{n-1} = m_h^{n-1} \).

**Step 2.** For \( \rho = 1, 2, \ldots, M \) do

(a) Solving the saddle point system: Find \( H_h \in V_h \) and \( \varphi_h \in W_h \) such that
\[
\begin{cases}
(H_h, G_h) + (\varphi_h, \text{div} G_h) = 0 & \forall G_h \in V_h, \\
(\text{div} H_h, r_h) = - (\text{div} H_h^n, r_h) - (\text{div} m_h^n, r_h) & \forall r_h \in W_h.
\end{cases}
\]

(b) Solving the magnetization equation: Find \( m_h \in V_h, z_h \in U_h \) and \( k_h \in U_h \) such that for all \( F_h \in V_h, \zeta_h \in U_h \) and \( \kappa_h \in U_h \), there hold
\[
\begin{align*}
&\left( 1 + \frac{\Delta t}{\tau} \right) (m_h, F_h) + \sigma \Delta t (\text{div} m_h, \text{div} F_h) + \Delta t b(u_h^{-}; m_h, F_h) \\
&+ \sigma \Delta t (\text{curl} k_h, F_h) + \frac{\Delta t}{2} (u_h^{-} \times F_h, k_h) + \beta \Delta t (m_h \times (m_h^{-} \times H_h), F_h) \\
&+ \beta \Delta t (m_h^{-} \times (m_h \times H_h), F_h) - \frac{\Delta t}{2} (\text{curl} z_h, F_h) \\
&= \frac{\lambda_0}{\tau} \Delta t (H_h, F_h) + \beta \Delta t (m_h^{-} \times (m_h^{-} \times H_h), F_h),
\end{align*}
\]
\[
(z_h, \zeta_h) - (u_h^{-} \times m_h, \zeta_h) = 0,
\]
\[
(k_h, \kappa_h) - (m_h, \text{curl} \kappa_h) = 0;
\]
(c) Solving the Navier-Stokes equation: Find $u_h \in S_h$ and $\tilde{p}_h \in L_h$ such that for all $v_h \in S_h$ and $q_h \in L_h$, there hold

\[
(u_h, v_h) + \Delta t \eta (\nabla u_h, \nabla v_h) - \Delta t (\tilde{p}_h, \text{div} \ v_h) = \Delta t \mu_0 b(v_h; H_h, m_h) + \mu_0 \Delta t (v_h \times k_h, H_h) + (u_{n-1}^h, v_h) + \Delta t (\nabla u_h \times \text{curl} \ u_h, v_h),
\]

\[
(\text{div} \ u_h, q_h) = 0.
\]

(d) Let $u^n_h = u_h$ and $m^n_h = m_h$.

Step 3. Let $u^n_h = u_h$, $\tilde{p}^n_h = \tilde{p}_h$, $m^n_h = m_h$, $H^n_h = H_h$, $z^n_h = z_h$ and $k^n_h = k_h$.

Remark 5.2. From the convergence theory of Newton-type methods [17, 37], we can see that the iterative solution of Algorithm 1 will converge to the exact solution, provided that the iteration number $M$ is big enough and the initial guess is nearby the exact solution. In fact, Step 1 in Algorithm 1 ensures the initial guess to be close to the exact solution, and in all the subsequent numerical examples we only need to choose $M = 2$ to attain the optimal convergence of the scheme.

In the numerical scheme (60), we use the mini-element pair $(P_1 \oplus \text{bubble})$-$P_1$ [5] to discretize the variables $u$ and $\tilde{p}$, the lowest order face element $RT_0$ [31] to discretize $m$ and $H$, $P_0$-element to discretize $\varphi$, and the lowest order edge element $NE_0$ [25, 26] to discretize the variable $z$ and $k$. Such a combination of finite element spaces corresponds to $l = 0$.

In Examples 5.3, 5.4 and 5.5, we take $\Omega = [0, 1]^3$ and use $N \times N \times N$ uniform tetrahedral meshes (cf. Figure 1) with $N = 4, 8, 16, 32$. In first two examples, we take the temporal step size as $\Delta t = h/\sqrt{3} = 1/N$. With these settings, we easily see from Theorem 4.12 that the theoretical accuracy of the scheme is $O(h + \Delta t)$.

**Figure 1.** The domain $\Omega = [0, 1]^3$: $2 \times 2 \times 2$ (left) and $4 \times 4 \times 4$ (right) meshes.
Example 5.3. The exact solution, \((u, m, H, \bar{p})\), of the FHD model (1) with \(T = 4\) is given by
\[
\begin{align*}
\mathbf{u}(t, x, y, z) &= \sin(t)(\sin(\pi y), \sin(\pi z), \sin(\pi x))^T, \\
\mathbf{m}(t, x, y, z) &= \sin(t)(\sin(\pi x) \sin(\pi y) \sin(\pi z), 0, 0)^T, \\
H(t, x, y, z) &= 200 \sin(t) \begin{pmatrix}
(2x - 1)(x^2 - x)(y^2 - y)(z^2 - z)^2 \\
(x^2 - x)^2(2y - 1)(y^2 - y)(z^2 - z)^2 \\
(x^2 - x)^2(2y - 1)(y^2 - y)(z^2 - z)^2
\end{pmatrix}, \\
\bar{p}(x, y, z) &= 120x^2yz - 40y^3z - 40yz^3.
\end{align*}
\]
The parameters \(\sigma, \mu_0, \eta, \chi_0, \beta\) and \(\tau\) are all chosen as 1.

Numerical results of the relative errors of the discrete solutions at the ending time \(T\) are listed in Table 1. In addition, the exact energy \(E(t) = ||u||^2 + \mu_0||H||^2 + ||m||^2\) and its numerical energy at different time levels are plotted in Figure 2.

Table 1. Errors and convergence orders at ending time \(T = 4\) for Example 5.3: \(\Delta t = 1/N\).

| \(N\) | \(\frac{||\mathbf{u}_h - \mathbf{u}_k||}{||\mathbf{u}_k||}\) | \(\frac{||\mathbf{m}_h - \mathbf{m}_k||}{||\mathbf{m}_k||}\) | \(\frac{||\mathbf{H}_h - \mathbf{H}_k||}{||\mathbf{H}_k||}\) | \(\frac{||\mathbf{H}_h - \mathbf{H}_k||}{||\mathbf{H}_k||}\) | \(\frac{||\mathbf{m}_h - \mathbf{m}_k||}{||\mathbf{m}_k||}\) | \(\frac{||\mathbf{m}_h - \mathbf{m}_k||}{||\mathbf{m}_k||}\) |
|---|---|---|---|---|---|---|
| 4 | 0.0557 | 0.2369 | 0.0629 | 0.3205 | 0.2849 | 0.2849 |
| 8 | 0.0140 | 0.1131 | 0.0146 | 0.1638 | 0.1446 | 0.1446 |
| 16 | 0.0045 | 0.0557 | 0.0037 | 0.0824 | 0.0740 | 0.0740 |
| 32 | 0.0009 | 0.0277 | 0.0010 | 0.0413 | 0.0388 | 0.0388 |
| order | 1.9959 | 1.0136 | 1.9506 | 0.9948 | 0.9486 | 0.9486 |

Example 5.4. The exact solution of the FHD model (1) with \(T = 1\) is given by
\[
\begin{align*}
\mathbf{u}(t, x, y, z) &= \sin(t)(\sin(\pi y) \sin(\pi z), \sin(\pi z) \sin(\pi x), \sin(\pi x) \sin(\pi y))^T, \\
\mathbf{m}(t, x, y, z) &= \sin(t)(0, (x^2 - x)(y^2 - y)(z^2 - z), 0)^T, \\
H(t, x, y, z) &= 200 \sin(t) \begin{pmatrix}
(2x - 1)(x^2 - x)(y^2 - y)(z^2 - z)^2 \\
(x^2 - x)^2(2y - 1)(y^2 - y)(z^2 - z)^2 \\
(x^2 - x)^2(2y - 1)(y^2 - y)(z^2 - z)^2
\end{pmatrix}, \\
\bar{p}(x, y, z) &= 120x^2yz - 40y^3z - 40yz^3.
\end{align*}
\]
The parameters \(\sigma, \mu_0, \eta, \chi_0, \beta\) and \(\tau\) are all chosen as 1. Numerical results of the relative errors of the discrete solutions at the ending time \(T\) are listed in Table 2.

Example 5.5 (Energy test). This example is to investigate the energy decaying phenomenon of the scheme (cf. Theorem 4.7 and Remark 4.8). We consider the FHD model (1) with the initial value functions
\[
\mathbf{u}_0(x, y, z) = \begin{pmatrix}
\sin(\pi y) \sin(\pi z) \\
\sin(\pi z) \sin(\pi x) \\
\sin(\pi x) \sin(\pi y)
\end{pmatrix}, \quad \mathbf{m}_0(x, y, z) = \begin{pmatrix}
\sin(\pi x) \sin(\pi y) \sin(\pi z) \\
0 \\
0
\end{pmatrix},
\]
and the external magnetic field $H_e = 0$. We show in Figure 3 the discrete energy curve at the spatial and temporal meshes with $N = 32$ and $\Delta t = 1/64$.

From Tables 1 and 2 and Figures 2 and 3, we have the following observations:

- The errors in $L^2$ norm for $u$ and $\tilde{p}$ have the second order of convergence rates, which is better than the theoretical prediction, since the finite element spaces for $u$ and $\tilde{p}$ contain the piecewise polynomials of degree up to 1.
- The errors in $H^1$ semi-norm for $u$, $L^2$ and $H(\text{div})$ norms for $H$ and $m$, and $L^2$ norm for $z$ and $k$, all have the first (optimal) order rates.
- The numerical energy curve in Figure 2 fits the exact one almost exactly, which means that our algorithm preserves the energy of the FHD model.
When there is no source term, i.e. the external magnetic field $H_e = 0$, the discrete energy $\tilde{E}_h$ decays with time, which is conformable to the theoretical prediction in Theorem 4.7; see also Remark 4.8.

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