Transfer matrix approach for the real symmetric 1D random band matrices

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Abstract

This paper adapts the recently developed rigorous application of the supersymmetric transfer matrix approach for the 1d band matrices to the case of the orthogonal symmetry. We consider $N \times N$ block band matrices consisting of $W \times W$ random Gaussian blocks (parametrized by $j,k \in \Lambda = [1,n] \cap \mathbb{Z}$, $N = nW$) with a fixed entry's variance $J_{jk} = W^{-1}(\delta_{j,k} + \beta \Delta_{j,k})$ in each block. Considering the limit $W,n \to \infty$, we prove that the behavior of the second correlation function of characteristic polynomials of such matrices in the bulk of the spectrum exhibit a crossover near the threshold $W \sim \sqrt{N}$.

1 Introduction

Starting from the works of Erdős, Yau, Schlein with coauthors (see [14] and reference therein) and Tao and Vu (see, e.g., [32]), the significant progress in understanding the universal behavior of many random graph and random matrix models were achieved. However for the random matrices with spacial structure our understanding is much more limited.

One of the most important models with a spatial structure are random band matrices (RBM), which are interpolating model between mean-field type Wigner matrices (Hermitian or real symmetric matrices with i.i.d. random entries) and random Schrödinger operators, which have only a random diagonal potential in addition to the deterministic Laplacian on a box in $\mathbb{Z}^d$.

The main long standing problem in the field is to prove a fundamental physical conjecture formulated in late 80th (see [10], [15]). The conjecture states that the eigenvectors of $N \times N$ RBM are completely delocalized and the local spectral statistics governed by the Wigner-Dyson statistics for large bandwidth $W$ (i.e. the local behavior is the same as for Wigner matrices), and by Poisson statistics for a small $W$ (with exponentially localized eigenvectors). The transition is conjectured to be sharp and for RBM in one spatial dimension occurs around the critical value $W = \sqrt{N}$. This is the analogue of the celebrated Anderson metal-insulator transition for random Schrödinger operators.

The conjecture on the crossover in RBM with $W \sim \sqrt{N}$ is supported by physical derivation due to Fyodorov and Mirlin (see [15]) based on supersymmetric formalism, and also by the so-called Thouless scaling. On the mathematical level of rigour, localization of eigenvectors in the bulk of the spectrum was first shown for $W \ll N^{1/8}$ [25], and then the bound

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was improved to $N^{1/7}$ [24]. On the other side, by a development of the Erdős-Schlein-Yau approach to Wigner matrices (see [14]), there were obtained some results where the weaker form of delocalization was proved for $W \gg N^{6/7}$ in [12], $W \gg N^{4/5}$ in [13], $W \gg N^{7/9}$ in [16]. The combination of this approach with the new ideas based on quantum unique ergodicity gave first GUE/GOE gap distributions for RBM with $W \sim N$ [5], and then were developed in [6] – [7], [35] to obtain bulk universality and complete delocalization in the range $W \gg N^{3/4}$ (see review [4] for the details).

There is a completely different approach which allows to work with random operators with non-trivial spatial structures based on supersymmetry techniques (SUSY). It is widely used in the physics literature (see e.g. reviews [11], [21]) but its rigorous mathematical application is usually quite difficult and it requires to incorporate various analytic and statistical mechanics techniques. However for the Hermitian RBM of a certain type it was successfully done both for correlation functions of characteristic polynomials and for usual correlation functions. More precisely, combining SUSY with a delicate steepest descent method and transfer matrix techniques, we were able to perform a complete study of the local regime of characteristic polynomials for Hermitian Gaussian 1d RBM (see [27] for the regime $W \gg \sqrt{N}$, [28] for the regime $W \ll \sqrt{N}$, and [30] for the regime $W \sim \sqrt{N}$), and also to obtain the first rigorous universality result for the second order correlation function for the whole delocalized region $W \gg \sqrt{N}$ (see [29]).

There are much less rigorous application of SUSY techniques for the case of real symmetric matrices, since the SUSY integral representations are more complicated for the case of orthogonal symmetry. However, the techniques of [27] were successfully adapted in [31] to the study of characteristic polynomials for real symmetric Gaussian 1d RBM in the delocalized regime $W \gg \sqrt{N}$. In this paper we want to perform the complete study of characteristic polynomials for real symmetric Gaussian 1d RBM adapting the SUSY transfer matrix techniques of [28], [30] to the case of orthogonal symmetry. This is an important step towards the proof of the universality of the usual correlation functions for the case of real symmetric 1d RBM, as well as for the general development of rigour application of SUSY approach for the real symmetric case.

The model we are going to consider is different from the model of 1d RBM considered in [27] – [28], [30] and in [31], but coincides with the model considered in [29]. Namely, we consider real symmetric block band matrices, i.e. real symmetric matrices $H_N$, $N = nW$ with elements $H_{jk,\alpha\gamma}$, where $j, k \in 1, \ldots, n$ (they parametrize the lattice sites) and $\alpha, \beta = 1, \ldots, W$ (they parametrize the orbitals on each site). The entries $H_{jk,\alpha\gamma}$ are random Gaussian variables with mean zero such that

$$\langle H_{j_1k_1,\alpha_1\beta_1}H_{j_2k_2,\alpha_2\beta_2} \rangle = \delta_{j_1j_2}\delta_{k_1k_2}\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}J_{j_1k_1}. \quad (1.1)$$

Here $J_{jk} \geq 0$ are matrix elements of the positive-definite symmetric $n \times n$ matrix $J$, such that

$$\sum_{j=1}^n J_{jk} = 1/W.$$

The probability law of $H_N$ can be written in the form

$$P_N(dH_N) = \exp \left\{-\frac{1}{4} \sum_{j,k \in \Lambda} \sum_{\alpha,\gamma = 1}^W \frac{H_{jk,\alpha\gamma}^2}{J_{jk}} \right\} dH_N, \quad (1.2)$$
where
\[ dH_N = \prod_{j<k} \prod_{\alpha<\gamma} \frac{dH_{jk,\alpha\gamma}}{\sqrt{2\pi J_{jk}}} \cdot \prod_{j} \prod_{\alpha<\gamma} \frac{dH_{j\alpha,\gamma\alpha}}{\sqrt{2\pi J_{j\alpha}}} \cdot \prod_{j} \prod_{\alpha} \frac{dH_{jj,\alpha\alpha}}{\sqrt{4\pi J_{jj}}} \]

Such models were first introduced and studied by Wegner (see [26], [34]) (and sometimes also called Wegner’s orbital models).

We restrict ourselves to the case
\[ J = 1/W + \beta \Delta^{(0)}/W, \quad \beta < 1/4, \]
where \( W \gg 1 \) and \( \Delta^{(0)} \) is the discrete Laplacian on \([1,n] \cap \mathbb{Z}\) with Neumann boundary conditions. Clearly, this model is one of the possible realizations of the Gaussian random band matrices with the band width \( 2W + 1 \) (note that the model can be defined similarly in any dimensions \( d > 1 \) taking \( j, k \in [1,n] \cap \mathbb{Z}^d \) in (1.1)).

For 1D RBM it was shown in [3, 22] that \( N_N \) converges weakly, as \( N,W \to \infty \), to a non-random measure \( N \), which is called the limiting NCM of the ensemble. The measure \( N \) is absolutely continuous and its density \( \rho \) is given by the well-known Wigner semicircle law:
\[ \rho(E) = \frac{1}{2\pi} \sqrt{4 - E^2}, \quad E \in [-2,2]. \] (1.4)

In this paper we consider the correlation functions (or the mixed moments) of characteristic polynomials, which can be defined as
\[ F_{2k}(\Lambda) = \int \prod_{s=1}^{2k} \det(\lambda_s - H_N)P_n(dH_N), \] (1.5)
where \( P_n(dH_N) \) is defined in (1.2), and \( \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{2k}\} \) are real or complex parameters that may depend on \( N \). As in the Hermitian case, correlation functions of characteristic polynomials of real symmetric 1d RBM are expected to exhibit a crossover near the threshold \( W \sim \sqrt{N} \): it is expected that they the same local behavior as for GOE for \( W \gg \sqrt{N} \), and the different behavior for \( W \ll \sqrt{N} \).

The asymptotic local behavior in the bulk of the spectrum of the 2k-point mixed moment for GOE is well-known. It was proved for \( k = 1 \) by Brézin and Hikami [5] (based on SUSY approach), and for general \( k \) by Borodin and Strahov [9] (with a different techniques) that
\[ F_{2k}(\Lambda_0 + \xi/n\rho(E)) = C_{N,k} \text{Pf}\left\{ DS(\pi(\xi_i - \xi_j)) \right\}_{i,j=1}^{2k} \Delta(\xi_1, \ldots, \xi_{2k})^{(1 + o(1))}, \]
where \( C_{N,k} \) is some multiplicative constant depending on \( N, k \),
\[ DS(x) = -3 \cdot \frac{d}{dx} \frac{\sin x}{x} = 3 \left( \frac{\sin x}{x^3} - \frac{\cos x}{x^2} \right), \] (1.6)
\( \Delta(\xi_1, \ldots, \xi_{2k}) \) is the Vandermonde determinant of \( \xi_1, \ldots, \xi_k \), and
\[ \hat{\xi} = \text{diag}\{\xi_1, \ldots, \xi_{2k}\}, \quad \Lambda_0 = E \cdot I. \]
In particular, for \( k = 1 \) we have
\[ F_2(\Lambda_0 + \hat{\xi}/n\rho(E)) = C_N \left( \frac{\sin(\pi(\xi_1 - \xi_2))}{\pi^3(\xi_1 - \xi_2)^3} - \frac{\cos(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2} \right)(1 + o(1)). \]
The last formula was proved also for real symmetric Wigner and general sample covariance matrices (see [18]).

Set
\[ \lambda_1 = E + \frac{\xi}{2N\rho(E)}, \quad \lambda_2 = E - \frac{\xi}{2N\rho(E)}, \]
where \( E \in (-2, 2), \) \( \rho \) is defined in (1.4), and \( \xi \) is a real parameter varying in any compact set \( K \subset \mathbb{R} \), and define
\[ D_2 = F_2^{1/2}(E, E). \]

The main result of the paper is the following theorem:

**Theorem 1.1** For the real symmetric 1d block random band matrices \( H_N, N = nW \) of (1.1) – (1.3) we have
\[ \lim_{n \to \infty} \tilde{F}_2\left( E + \frac{\xi}{2N\rho(E)}, E - \frac{\xi}{2N\rho(E)} \right) = \begin{cases} DS(\pi\xi), & W \gg n \gg 1; \\ (e^{-C^*\Delta - i\xi\hat{\nu}} \cdot 1, 1), & n = C^*W \\ 1, & 1 \leq W \leq n/\log^2 n, \end{cases} \]
where \( DS(x) \) is defined in (1.6), \( C^* = C^*/(2\pi\rho(E))^2 \) with \( \rho(E) \) of (1.4), and \( \varepsilon \) is any sufficiently small positive number. In this formula \( \Delta \) is a Laplace-Bertrami operator on \( \hat{Sp}(2) = Sp(2)/Sp(1) \times Sp(1) \), and \((\cdot, \cdot)\) is an inner product on \( L^2[\hat{Sp}(2), d\mu] \), where \( d\mu \) is the Haar measure on \( \hat{Sp}(2) \). \( \hat{\nu} \) is an operator of multiplication by
\[ \nu(Q) = 1 - 2(|Q_{12}|^2 + |Q_{14}|^2) \]
on \( \hat{Sp}(2) \). Notice that the since \( N = nW \), the transition happens at \( W \sim \sqrt{N} \).

1.1 Notation

We denote by \( C, C_1 \), etc. various \( W \) and \( N \)-independent quantities below, which can be different in different formulas. Integrals without limits denote the integration (or the multiple integration) over the whole real axis, or over the Grassmann variables.

Moreover,
- \( W \) is a size of the block, and \( n \) is the number of blocks in a row, so \( N = nW \) is the size of the matrix \( H \) of (1.1);
- \( \mathbb{E}\{ \ldots \} \) is an expectation with respect to the measure (1.2);
- \( a_{\pm} = \frac{iE \pm \sqrt{4 - E^2}}{2} = e^{\pm i\alpha_0}; \)
- \( \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \)
- \( D_0 = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \hat{\xi} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \)
- \( D_{0,4} = \begin{pmatrix} D_0 & 0 \\ 0 & D_0 \end{pmatrix}, \quad \hat{\xi}_4 = \begin{pmatrix} \hat{\xi} \\ 0 \end{pmatrix}, \quad L_4 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}; \)
• \( \Lambda_0 = E \cdot I_2, \quad \Lambda_{0,4} = E \cdot I_4; \)
• \( U(n) \) is a group of \( n \times n \) unitary matrices; unitary symplectic group \( Sp(n) \) is a group of \( 2n \times 2n \) unitary matrices \( Q \) which admit the relation

\[
Q \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} Q^t = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

• \( \hat{U}(2) = U(2)/(U(1) \times U(1)), \quad \hat{Sp}(2) = Sp(2)/(Sp(1) \times Sp(1)); \)
• \( T = \{ z \in \mathbb{C} : |z| = 1 \}, \quad \omega_A = \{ z \in \mathbb{C} : |z| = 1 + A/n \}; \)
• \( d\mu \) is the Haar measure on \( \hat{U}(2), \ d\mu \) is the Haar measure on \( \hat{Sp}(2); \)
• \( c_\pm = 1 + a_\pm^{-2}; \quad t_\ast = (2\pi \rho(E))^2 \)
• We denote by \( \tilde{a} \) the vector \( (a_1, a_2); \)

2 Integral representation

The main aim of this section is to derive the following proposition

**Proposition 2.1** The second correlation function \( (1.5) \) of the characteristic polynomials for 1d real symmetric Gaussian block band matrices \( (1.1) - (1.3) \) can be represented as follows:

\[
F_2(\Lambda_0 + \frac{\hat{\xi}}{2N \rho(E)}) = C_{n,W} \int \exp \left\{ \frac{\beta W}{4} \sum_{j=2}^{n} \text{Tr} \left( F_j - F_{j-1} \right)^2 \right\} \times \exp \left\{ \frac{W}{4} \sum_{j=1}^{n} \left( \text{Tr} F_j^2 - 2i \text{Tr} F_j \left( \Lambda_{0,4} + \frac{\hat{\xi}_4}{2N \rho(E)} \right) \right) \right\} \prod_{j=1}^{n} (\det F_j)^{-W/2} \prod_{j=1}^{n} dF_j,
\]

where \( \Lambda_{0,4} \) and \( \hat{\xi}_4 \) are defined in Notation, \( N = nW, C_{n,W} \) is some constant depending on \( W \) and \( n \) but not on \( \xi, \) and \( F_j \in Sp(2) \) are unitary symplectic \( 4 \times 4 \) matrices.

**Proof.** Introduce the following Grassmann fields:

\[ \Psi_l = \{ \psi_{jl} \}_{j=1,...,n}, \quad \psi_{jl} = (\psi_{jl1}, \psi_{jl2}, ..., \psi_{jlW})^t, \quad l = 1, 2. \]

Using \( (8.7) \) (see Appendix A) we obtain

\[
F_2(\lambda_1, \lambda_2) = \mathbf{E} \left\{ \int \exp \left\{ -\Psi_1^+ (\lambda_1 - H_N) \Psi_1 - \Psi_2^+ (\lambda_2 - H_N) \Psi_2 \right\} d\Psi \right\} \]

\[
= \int d\Psi \exp \left\{ -\lambda_1 \Psi_1^+ \Psi_1 - \lambda_2 \Psi_2^+ \Psi_2 \right\} \times \mathbf{E} \left\{ \sum_{j<k} \sum_{\alpha, \gamma} H_{jk, \alpha \gamma} (\eta_{jk, \alpha \gamma} + \eta_{kj, \gamma \alpha}) + \sum_{j} \sum_{\alpha \leq \gamma} H_{jk, \alpha \gamma} (\eta_{jk, \alpha \gamma} + \eta_{kj, \gamma \alpha}) \right\},
\]

where

\[
d\Psi = \prod_{j=1}^{n} \prod_{\alpha = 1}^{2} \prod_{l=1}^{W} d\psi_{jl\alpha} d\psi_{j\alpha l},
\]

\[ \eta_{jk, \alpha \gamma} = \bar{\psi}_{j1\alpha} \psi_{k1\gamma} + \bar{\psi}_{j2\alpha} \psi_{k2\gamma}, \quad \text{if} \ j \neq k \text{ or} \ \alpha \neq \gamma; \]

\[ \eta_{jj, \alpha \alpha} = (\psi_{j1\alpha} \psi_{j1\alpha} + \psi_{j2\alpha} \psi_{j2\alpha})/2. \]
Averaging over (1.2), we get
\[ F_2(\lambda_1, \lambda_2) = \int d\Psi \exp\{-\lambda_1 \Psi_1^+ \Psi_1 - \lambda_2 \Psi_2^+ \Psi_2\} \]
\[ \times \exp\left\{ \frac{1}{2} \sum_{j<k,\alpha,\gamma} J_{jk} (\eta_{jk,\alpha\gamma} + \eta_{kj,\gamma\alpha})^2 + \frac{1}{2} \sum_{j,\alpha<\gamma} J_{jj} (\eta_{jj,\alpha\gamma} + \eta_{jj,\gamma\alpha})^2 + \sum_{j,\alpha} J_{jj} \eta_{jj,\alpha\alpha}^2 \right\}. \]

It is easy to see that
\[ \frac{1}{2} \sum_{\alpha,\gamma} (\eta_{jk,\alpha\gamma} + \eta_{kj,\gamma\alpha})^2 = - (\psi_{1j}^+ \psi_{j2})(\psi_{k1}^+ \psi_{k2}) - (\psi_{k1}^+ \psi_{k2})(\psi_{j1}^+ \psi_{j2}) - (\psi_{j1}^+ \psi_{j2})(\psi_{k1}^+ \psi_{k2}) - (\psi_{k1}^+ \psi_{k2})(\psi_{j1}^+ \psi_{j2}) \]
\[ = - \frac{1}{2} \text{Tr} \tilde{F}_j \tilde{F}_j, \]
\[ \sum_{l=1,2} \lambda_l \Psi_l^+ \Psi_l = \frac{1}{2} \sum_{j=1}^n \text{Tr} \tilde{F}_j \Lambda_4 \]

where
\[ \tilde{F}_j = \begin{pmatrix} \psi_{j1}^+ \psi_{j1} & \psi_{j2}^+ \psi_{j2} & 0 & \psi_{j1}^+ \psi_{j2} \\ \psi_{j1}^+ \psi_{j1} & \psi_{j2}^+ \psi_{j2} & \psi_{j1}^+ \psi_{j1} & 0 \\ 0 & \psi_{j2}^+ \psi_{j2} & \psi_{j1}^+ \psi_{j1} & \psi_{j2}^+ \psi_{j2} \\ \psi_{j2}^+ \psi_{j1} & 0 & \psi_{j2}^+ \psi_{j1} & \psi_{j2}^+ \psi_{j2} \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}. \]

Applying the superbosonization formula (see Proposition 8.1, Appendix A) we obtain
\[ F_2(\lambda_1, \lambda_2) = C_{nW} \int \exp \left\{ - \frac{1}{4} \sum_{j,k=1}^n J_{jk} \text{Tr} F_j F_k - \frac{1}{2} \sum_{j,k=1}^n \text{Tr} F_j \Lambda_4 \right\} \prod_{j=1}^n (\det F_j)^{-W/2} \prod_{j=1}^n dF_j, \]

where \( \{F_j\}_{j=1}^n \) are unitary symplectic \( 4 \times 4 \) matrices form \( Sp(2) \), and \( C_{nW} \) is some constant depending on \( W \) and \( n \) but not on \( \lambda_1, \lambda_2 \). Shifting \( F_j \to iWF_j \) and plugging in (1.7), we get Proposition 2.3.

\[ \square \]

### 3 Representation in the operator form

To study (2.1), we are going to apply the transfer matrix approach. Namely, introduce
\[ \mathcal{F}(X) = \exp \left\{ W \left( \frac{1}{8} \text{Tr} X^2 - \frac{iE}{4} \text{Tr} X - \frac{1}{4} \text{Tr} \log X - C_+ \right) \right\}, \quad (3.1) \]
\[ \mathcal{F}_\xi(X) = \mathcal{F}(X) \cdot \mathcal{F}_{n,\xi}(X), \quad \mathcal{F}_{n,\xi}(X) := \exp \left\{ - \frac{i}{8n \rho(E)} \text{Tr} X \xi_+ \right\} \]

where
\[ C_+ = \frac{a_+^2}{2} - iEa_+ - \log a_+ \]
is chosen in such a way that \(|\mathcal{F}(X)| = 1\) in the saddle-points (see (4.2) later).

Let also \(\mathcal{K}, \mathcal{K}_\xi : \text{Sp}(2) \to \text{Sp}(2)\) be the operators with the kernels

\[
\mathcal{K}(X, Y) = \frac{W^3}{2\pi^3} \mathcal{F}(X) \exp\left\{ \frac{\beta W}{4} \text{Tr} (X - Y)^2 \right\} \mathcal{F}(Y); \quad (3.2)
\]

\[
\mathcal{K}_\xi(X, Y) = \frac{W^3}{2\pi^3} \mathcal{F}_\xi(X) \exp\left\{ \frac{\beta W}{4} \text{Tr} (X - Y)^2 \right\} \mathcal{F}_\xi(Y). \quad (3.3)
\]

Then Proposition 2.1 can be reformulated as

\[
F_2\left( E + \frac{\xi}{2N\rho(E)}, E - \frac{\xi}{2N\rho(E)} \right) = \tilde{C}_{n,W}(\mathcal{K}^{n-1}_\xi, \mathcal{F}_\xi), \quad (3.4)
\]

where \((\cdot, \cdot)\) is a standard inner product in \(\text{Sp}(2)\) with respect to the Haar measure \(d\mu\), and \(\tilde{C}_{n,W}\) is some constant depending on \(W\) and \(n\) but not on \(\xi\).

For arbitrary compact operator \(M\) denote by \(\lambda_j(M)\) the \(j\)th (by its modulo) eigenvalue of \(M\), so that \(|\lambda_0(M)| \geq |\lambda_1(M)| \geq \ldots\).

Since \(\mathcal{K}_\xi\) is a compact operator, one can rewrite \((\mathcal{K}_\xi^{n-1} - \mathcal{F}_\xi, \bar{\mathcal{F}_\xi}) = \sum_{j=0}^{\infty} \lambda_j(\mathcal{K}_\xi)c_j, \quad \text{with} \quad c_j = (\mathcal{F}_\xi, \psi_j)(\bar{\mathcal{F}_\xi}, \bar{\psi}_j),\)

where \(\{\psi_j\}\) are eigenvectors corresponding to \(\{\lambda_j(\mathcal{K}_\xi)\}\), and \(\{\bar{\psi}_j\}\) are the eigenvectors of \(\mathcal{K}_\xi^*\). Similar equality is true if we replace \(\mathcal{K}_\xi\) and \(\mathcal{F}_\xi\) by \(\mathcal{K}\) and \(\mathcal{F}\). Hence, to study (2.1), it suffices to study the eigenvalues and eigenvectors of \(\mathcal{K}_\xi, \mathcal{K}\).

### 4 Sketch of the proof of Theorem 1.1

As was mentioned above, we are interested in the analysis of the spectral properties of \(\mathcal{K}_\xi\) of (3.3) (see (3.4)). It appears that it is simpler to work with the resolvent analog of (3.4)

\[
(K^{n-1}_\xi f, g) = -\frac{1}{2\pi i} \oint_{\mathcal{L}} z^{n-1}(G_\xi(z)f, g)dz, \quad G_\xi(z) = (K_\xi - z)^{-1}, \quad (4.1)
\]

where \(\mathcal{L}\) is any closed contour which enclosed all eigenvalues of \(\mathcal{K}_\xi\).

The idea of the proof is very close to [28] – [30]. To outline it, we start with the following definition

**Definition 4.1** We say that the operator \(A_{n,W}\) is equivalent to \(B_{n,W}\) \((A_{n,W} \sim B_{n,W})\) on some contour \(\mathcal{L}\) if

\[
\int_{\mathcal{L}} z^{n-1}((A_{n,W} - z)^{-1} f, \bar{g})dz = \int_{\mathcal{L}} z^{n-1}((B_{n,W} - z)^{-1} f, \bar{g})dz (1 + o(1)), \quad n, W \to \infty,
\]

with some particular functions \(f, g\) depending of the problem.

The aim is to find some operator equivalent to \(\mathcal{K}_\xi\) whose spectral analysis is more accessible. Now we are going to discuss how this was done on the ideological level. The specific choice of the contour \(\mathcal{L}\) and functions \(f, g\) for each step will be discussed in details in Section 6.
It is easy to check that the stationary points of the function $F$ of (3.1) are

$$X_+ = a_+ \cdot I_4, \quad X_- = a_- \cdot I_4;$$

(4.2)

$$X_\pm(Q) = QD_{O,4}Q^*, \quad Q \in \tilde{Sp}(2)$$

where $a_\pm, D_{O,4}$ are defined in Notation. Notice also that the value of $|F|$ at points (4.2) is 1.

The first step in the proof of Theorem 1.1 is to apply the saddle-point approximation. Roughly speaking, we show that if we introduce the projection $Pr_s$ onto the $W^{-1/2} \log W$-neighbourhoods of the saddle points $X_+, X_-$ and the saddle "surface" $X_\pm$, then in the sense of Definition 4.1

$$K_\xi \sim Pr_sK_\xi Pr_s =: K_{s,\xi}.$$  

Moreover, we can show that only the neighborhood of the saddle "surface" $X_\pm$ gives the main contribution to the integral. The proof is based on a study of a quadratic approximation of a function $F$ of (3.1). Let us also emphasize, that for the block band matrices (1.1) – (1.3) this step is much simpler than for the model considered in [28] – [30] due to the large coefficient $W$ in the exponent of $F$. This analysis will be performed in details in Section 5.

To study the operator $K_{s,\xi}$ near the saddle "surface" $X_\pm$ we use the "polar coordinates". Namely, the matrices from $Sp(2)$ have two eigenvalues $a_j, b_j \in T = \{z : |z| = 1\}$ of the multiplicity two and can be considered as quaternion $2 \times 2$ matrices. In this language $F_j$ are quaternion unitary matrices, and so they can be diagonalized by the quaternion unitary $2 \times 2$ matrices from $\tilde{Sp}(2)$ (see, e.g., [20], Chapter 2.4).

Change the variables to

$$F_j = Q_j^*A_{j,4}Q_j,$$

where $A_{j,4} = \text{diag}\{a_{1j}, a_{2j}, a_{1j}, a_{2j}\}$, $a_{1j}, a_{2j} \in T$, and $Q_j \in \tilde{Sp}(2)$. Then $dF_j$ of (2.1) becomes (see, e.g., [20])

$$\frac{\pi^2}{12}(a_{1j} - a_{2j})^4 d\vec{a}_j d\mu(Q_j),$$

where

$$d\vec{a}_j = \frac{d\bar{a}_{1j}}{2\pi i} \frac{d\bar{a}_{2j}}{2\pi i},$$

and $d\mu(Q_j)$ is the normalized to unity Haar measure on the symplectic group $\tilde{Sp}(2)$. Thus we get

$$\left(K_{\xi}^{-1}F_\xi, \tilde{F}_\xi\right) = \pi^{2n} \left(\frac{1}{2^n} \int (a_{11} - a_{21})^2 F_\xi(a_{11}, a_{21}, Q_1)(a_{1n} - a_{2n})^2 F_\xi(a_{1n}, a_{2n}, Q_n) \times \prod_{j=1}^{n-1} \left(\frac{1}{2}\right)^2 K_\xi(\tilde{F}_j, \tilde{F}_{j+1}) \right) \prod_{j=1}^{n} d\vec{a}_j d\mu(Q_j).$$

Introduce

$$t = (a_1 - a_2)(a_1' - a_2').$$

Then we obtain

$$F_2\left(E + \frac{\xi}{2N\rho(E)}, E - \frac{\xi}{2N\rho(E)}\right) = \tilde{C}_{n,\xi}(K_{\xi}^{-1}f, \tilde{f}),$$

(4.4)
where now $(\cdot, \cdot)$ is a standard inner product in $L_2[T^2] \times L_2[\hat{S}p(2), d\mu(Q)]$, and $\hat{C}'_{nW}$ is some constant depending on $W$ and $n$ but not on $\xi$. Here

$$f(a_1, a_2, Q) = (a_1 - a_2)^2 F_\xi(a_1, a_2, Q),$$

and $K_\xi = F_{n, \xi} K F_{n, \xi}$ is an integral operator in $L_2[T^2] \times L_2[\hat{S}p(2), d\mu(Q)]$ defined by the kernel

$$K_\xi(X, Y) = F_{n, \xi}(a_1, a_2, Q) K(a_1, a_2, Q; a'_1, a'_2, Q') F_{n, \xi}(a'_1, a'_2, Q'),$$

where

$$K(a_1, a_2, Q; a'_1, a'_2, Q') = A_a(\bar{a}, \bar{a}') K_\ast(t, Q_1, Q_2);$$

$$K_\ast(t, Q, Q') := \frac{\beta^2 W^2 l^2}{6} \cdot \exp\{-t \beta W S(Q(Q')^*)\}, \quad S(Q) = |Q_{12}|^2 + |Q_{14}|^2;$$

$$F_{n, \xi}(a, b, Q) = \exp\{-i \xi \pi \cdot \nu(a - b, Q) / n\};$$

$$\nu(p, Q) = \frac{p}{4\pi \rho(E)} \text{Tr} Q L_4 Q^* L_4 = \frac{p}{2 \pi \rho(E)} (1 - 2 S(Q))$$

with $t$ of $(4.3)$. $K_\ast$ here is a contribution of the symplectic group $\hat{S}p(2)$ into operator $K$, and $\exp\{-i \xi \pi \cdot \nu(x, Q) / n\}$ comes from the $1/n$-order perturbation $F_{n, \xi}$ of $F$ appearing in $F_\xi$ (see $(3.1)$). Operator $A_a$ is a contribution of eigenvalues $a_1, a_2$ and it has the form

$$A_a(\bar{a}; \bar{a}') = A(a_1, a'_1) A(a_2, a'_2),$$

$$A(a, a') = \left(\frac{W}{2\pi}\right)^{1/2} e^{-W \Lambda(a, a')};$$

$$\Lambda(x, y) = \frac{\beta}{2} (x - y)^2 - \frac{1}{2} \varphi_0(x) - \frac{1}{2} \varphi_0(y) + \Re \varphi_0(a_+);$$

$$\varphi_0(x) = x^2 / 2 - ix E - \log x.$$  

Observe that the operator $K_\ast(t, Q, Q')$ with some $t > 0$ is self-adjoint and its kernel depends only on $S(Q(Q')^*)$. Thus by the standard representation theory arguments (see e.g. [17, 33]), its eigenfunctions are the the same as for Laplace-Bertrami operator on $Sp(2)$. More precisely:

**Proposition 4.1** Consider any self-adjoint integral operator $M$ in $L_2[\hat{S}p(2), d\mu(Q)]$. If its kernel $M(Q, Q')$ depends only on $Q(Q')^*$, then its eigenvectors coincide with eigenvectors of Laplace-Bertrami operator on $\hat{S}p(2)$. Moreover, if the subspace

$$L_2[S, d\mu(Q)] \subset L_2[\hat{S}p(2), d\mu(Q)]$$

of the functions depending on $S(Q)$ (see $(4.7)$) only is invariant under $M$, then it can be diagonalized by the eigenfunctions

$$\phi_j(Q) = (-1)^j P_{2j}(\sqrt{S(Q)}),$$

where $P_{2j}(x)$ are orthogonal with respect to the weight $(1 - x^2)^3$ on $[0, 1]$ polynomials of degree $2j$, $\phi_0(x) = 1$ (polynomials $P_{2j}$ can be written as $P_{2j}(x) = c_j F_{hg}(-j, j + 3, 2; 1 - x^2)$, where $F_{hg}$ is a hypergeometric function, and $c_j$ is a normalization constant, see [17], Ch. 5). In addition, the following holds

$$(2x^2 - 1) P_{2j}(x) = \frac{j + 3}{2j + 3} P_{2j+2}(x) + \frac{j}{2j + 3} P_{2j-2}(x),$$
so the operator \( \hat{\nu} \) of multiplication on \( \nu(x, Q) \) of (4.8) is three diagonal in basis (4.11), and
\[
(\hat{\nu} \cdot \phi_0, \phi_0) = 0.
\]
(4.13)

If \( M(Q_1, Q_2) = K_\epsilon(t, Q_1, Q_2) \) of (4.7), then the corresponding eigenvalues \( \{\lambda_j(t)\}_{j=0}^\infty \) if \( t > d > 0 \), where \( d \) is some absolute positive constant, have the form
\[
\lambda_j(t) = 1 - \frac{(j + 1)(j + 2)}{Wt} + O((j^2/Wt)^2) + O(e^{-tW}).
\]
(4.14)

The proof of the proposition can be found in Appendix B.

Notice that since \( F(Q) \), \( F_\epsilon(Q) \) are the functions of \( S(Q) \) only, and hence according to Proposition [4.1] in what follows we can consider restrictions of \( K_\epsilon, K \), and \( K_\epsilon \) of (4.7) to \( L_2[S, d\mu(Q)] \) (to simplify notations we will denote these restrictions by the same letters).

In addition, it follows from Proposition [4.1] that if we introduce the following basis in \( L_2[\mathbb{R}^2] \times L_2[S, d\mu(Q)] \)
\[
\Psi_{k,j}(\tilde{a}, Q) = \Psi_k(\tilde{a})\phi_j(Q),
\]
\[
\Psi_k(\tilde{a}) = \psi_{k_1}(a_1)\psi_{k_2}(a_2),
\]
where \( \tilde{k} = (k_1, k_2) \), and \( \{\psi_k(x)\}_{k=0}^\infty \) is a certain basis in \( L_2[\mathbb{R}] \), then the matrix of \( K \) of (4.7) in this basis has a “block diagonal structure”, which means that
\[
(K\Psi_{k',j}, \Psi_{k,j}) = 0, \quad j \neq j_1
\]
(4.15)
\[
(K\Psi_{k',j}, \Psi_{k,j}) = (K_j\Psi_{k'}, \Psi_k)
\]
\[
= \int \lambda_j(t)A_\alpha(\tilde{a}, \tilde{a}')\psi_{k_1}(a_1)\psi_{k_2}(a_2)\psi_{k'_1}(a'_1)\psi_{k'_2}(a'_2)\frac{da_1da_2da'_1da'_2}{(2\pi)^4}.
\]

The next step in the proof of Theorem [4.1] is to show that we can restrict the number of \( \phi_j \) to
\[
l = \max\{1, \log n \cdot \sqrt{W/\sqrt{n}}\}.
\]
(4.16)
l is chosen in such a way that \( l^2n/W \gg \log n \). More precisely, we are going to show that in the sense of Definition [4.1]
\[
K_{s,\xi} \sim P_lK_{s,\xi}P_l =: K_{s,l,\xi},
\]
where \( P_l \) is the projection on the linear span of \( \{\Psi_{k,j}(\tilde{a}, Q)\}_{j=1}^{l-1} \).

For the further resolvent analysis we want to integrate out \( \tilde{a} \) to change \( t \) in the definition of \( K \) and \( a_1 - a_2, a_1' - a_2' \) in the definition of \( F_{n,\xi} \) (see (4.3), (4.6) - (4.7)) by their saddle-point values \( t_* = (a_+ - a_-)^2 = 4\pi^2\rho(E)^2 \) and \( a_+ - a_- = 2\pi\rho(E) \) correspondingly. We are going to show that only top eigenvalue of \( A \) gives the contribution. More precisely we want to show that in the sense of Definition [4.1]
\[
\lambda_0(K_{s,t})^{-1}K_{s,l,\xi} \sim \tilde{K}_{s,\xi,t}
\]
(4.17)
where
\[
K_{s,\xi,t} = (\lambda_0(K_{s,0}))^{-1}P_lK_{s,\xi}P_l,
\]
(4.18)
\[
K_{s,\xi}(Q_1, Q_2) = \frac{W^2t_*^2\beta^2}{6}e^{-\beta t_*WS(Q_1Q_2)}e^{-i\pi(\nu(2\pi\rho(E)Q_1)+\nu(2\pi\rho(E)Q_2))/n}
\]
10
and $P_l$ is the projection on $\{\phi_j(Q)\}_{j \leq l-1}$. Here $K_{s0}$ is $K_{s\xi}$ with $\xi = 0$.

Now (4.17), (4.1) and Definition 4.1 give

$$F_2 \left( E + \frac{\xi}{2N \rho(E)}, E - \frac{\xi}{2N \rho(E)} \right) = C_{n,W} \left( \hat{K}^{n-1}_s f_{\xi}, \hat{f}_{\xi} \right) (1 + o(1))$$

$$= C_{n,W} \lambda_0(K_{s,l})^{n-1} f_0^2(\hat{K}^{n-1}_{s,l}, 1)(1 + o(1)),$$

where $f_0 = (f, \Psi_0)$, and we used that $f_\xi$ asymptotically can be replaced by $f \otimes 1$, where $f$ does not depend on $\xi$ and $Q_j$. Similarly

$$D_2 = C_{n,W} \left( \hat{K}^{n-1}_{s,0,l} f_{\xi}, \hat{f}_{\xi} \right) (1 + o(1)) = C_{n,W} \lambda_0(K_{s,0,l})^n f_0^2(\hat{K}^{n-1}_{s,0,l}, 1)(1 + o(1)).$$

According to Proposition 4.1 $\phi_0(Q) = 1$ is an eigenvector of $\hat{K}_{s0}$ of (4.18) with $\xi = 0$ and the corresponding eigenvalue is 1, thus

$$(\hat{K}^{n-1}_{s,0,l}, 1) = 1.$$

Hence

$$\hat{F}_2 \left( E + \frac{\xi}{2N \rho(E)}, E - \frac{\xi}{2N \rho(E)} \right) = (\hat{K}^{n-1}_{s,l}, 1)(1 + o(1)). \quad (4.19)$$

Recall that according to Proposition 4.1 the eigenvectors of $\hat{K}_{s,l}$ are (4.11) and the corresponding eigenvalues are (see (4.14))

$$\lambda_j := \lambda_j(t_s) = 1 - j(j + 3)/t_s W + O((j(j + 3)/W)^2), \quad j = 0, 1, \ldots, l - 1. \quad (4.20)$$

Moreover, it follows from (4.6) – (4.7) that

$$\hat{K}_{s,l} = \hat{K}_{s,0,l} - n^{-1} \pi i \hat{\nu}_l + o(n^{-1}), \quad \hat{\nu}_l = P_l \hat{\nu} P_l, \quad (4.21)$$

where $\hat{\nu}$ is the operator of multiplication by (1.9), and $o(1/n)$ means some operator whose norm is $o(1/n)$. Thus the eigenvalues of $\hat{K}_{s,l}$ are in the $n^{-1}$-neighbourhood of $\lambda_j$.

In the localized regime $W^{-1} \gg n^{-1}$ we have $l = 1$, thus only $\lambda_0(\hat{K}_{s,l})$ gives the contribution to (4.19). Since (see Proposition 4.1)

$$\langle \hat{\nu}, 1, 1 \rangle = 0,$$

we get

$$\lambda_0(\hat{K}_{s,l}) = 1 + o(n^{-1}),$$

and so the limit of (4.19) is 1 (see the end of Section 6 for more details).

In the regime of delocalization all eigenvalues of $K_{s,l}$ give contribution to (4.19), but $K^{n-1}_{s,0,l} \to I$ (roughly speaking, this means that the second term in the r.h.s. of (4.20) does not give a contribution). Hence we have

$$\hat{K}_{s,l} \approx 1 - n^{-1} i \pi \hat{\nu}_l \Rightarrow (K^{n-1}_{s,l}, 1) \to (e^{-i \pi \hat{\nu}_l}, 1) = DS(\pi \hat{\nu}).$$

with $DS(\pi \hat{\nu})$ of (1.6) (see (9.2) and the end of Section 6 for more details).

In the critical regime $W^{-2} = C_n n^{-1}$ all eigenvalues of $\hat{K}_{s,l}$ give contribution, but now both second term in the r.h.s. of (4.20) and $1/n$-order term in the r.h.s. of (4.21) make an impact.
As it was mentioned above, the Laplace-Bertrami operator $\Delta$ on $L_2[S,d\mu]$ has eigenvectors (4.11) with corresponding eigenvalues $\lambda_j^* = j(j+3)$.

Thus $1 - n^{-1} C^* \Delta$ with $C^* = C_* / t_*$ has the same basis of eigenvectors with eigenvalues $1 - j(j+3) / t_* W$.

Recall that we are interested in $j \leq l - 1 \sim \log W$ (since $P_1$ is the projection on $\{\phi_j\}_{j \leq l-1}$). Hence, according to (4.20) – (4.21), in the regime $W^{-1} = C_* n^{-1}$ we can write

$$\hat{K}_{s \xi,t} = P_t (1 - n^{-1} (C^* \Delta + i \xi \pi \nu)) P_t + o(n^{-1}),$$

which implies

$$(\hat{K}^n_{s \xi,t} 1, 1) \to (e^{-C^* \Delta - i \xi \pi \nu} 1, 1), \quad (4.22)$$

and finishes the proof of Theorem 1.1 The detailed proof of (4.22) is given in Section 6 (see Lemma 6.5).

5 Saddle-point analysis

Recall that the stationary points of the function $F$ of (3.1) are defined in (4.2).

We start the proof from the restriction of the integration with respect to $\tilde{a}_i, \tilde{a}'_i$ by the neighbourhood of $a_\pm$. Set

$$\Omega_+ = \{x : |x - a_+| \leq \log W/W^{1/2}\}, \quad \Omega_- = \{x : |x - a_-| \leq \log W/W^{1/2}\},$$

$$\tilde{\Omega}_+ = \{a_1, a'_1 \in \Omega_+, a_2, a'_2 \in \Omega_+\},$$

$$\tilde{\Omega}_- = \{a_1, a'_1, a_2, a'_2 \in \Omega_-\}$$

and let $1_{\tilde{\Omega}_+}, 1_{\tilde{\Omega}_+}, 1_{\tilde{\Omega}_-}$ be indicator functions of the above domains.

Lemma 5.1 Given $A(a, a')$ of (4.7), we have

$$\int_{T \setminus (\Omega_+ \cup \Omega_-)} |A(a, a')||da'| \leq Ce^{-c \log^2 W}. \quad (5.2)$$

Proof. Recall that

$$a_\pm = e^{\pm i \alpha_0},$$

and write for the parametrization $a = e^{i \varphi}, a' = e^{i \varphi'}$

$$-\Re(e^{i \varphi}, e^{i \varphi'}) = -\beta (\cos \varphi - \cos \varphi')^2 / 2 + \beta (\sin \varphi - \sin \varphi')^2 / 2 - \frac{\sin^2 \varphi + \sin^2 \varphi'}{2}
+ \frac{E(\sin \varphi + \sin \varphi')}{2} + \sin^2 \alpha_0 - E \sin \alpha_0
= -\beta (\cos \varphi - \cos \varphi')^2 / 2 + \beta (\sin \varphi - \sin \varphi')^2 / 2
- (\sin \varphi - \sin \alpha_0)^2 / 2 - (\sin \varphi' - \sin \alpha_0)^2 / 2
\leq \beta (\sin \varphi - \sin \varphi')^2 / 2 - (\sin \varphi - \sin \alpha_0)^2 / 2 - (\sin \varphi' - \sin \alpha_0)^2 / 2
\leq -(1 - 2\beta)(\sin \varphi - \sin \alpha_0)^2 / 2 - (1 - 2\beta)(\sin \varphi' - \sin \alpha_0)^2 / 2.$$
Here we have used \( \sin \alpha_0 = E/2 \). We have also for \( d' \in \mathbb{T} \setminus (\Omega_+ \cup \Omega_-) \)

\[
|\sin \varphi' - \sin \alpha_0| \geq C \log W/\sqrt{W}.
\]

Since \( \beta < 1/4 \), this implies (5.2)

\[\square\]

Lemma 5.1 yields that

\[
\int dQ'd\delta'(1 - 1_{\tilde{\omega}_z} - 1_{\tilde{\omega}_+} - 1_{\tilde{\omega}_-})\|K\| \leq e^{-c \log^2 W} \quad (5.3)
\]

Let us prove the following simple proposition

**Proposition 5.1** Let the matrix \( H(z) \) has the block form

\[
H(z) = \begin{pmatrix} H_{11}(z) & H_{12}(z) \\ H_{21}(z) & H_{22}(z) \end{pmatrix}.
\]

Then

\[
G(z) := H^{-1}(z) = \begin{pmatrix} G_{11} & -G_{11}H_{12}H_{22}^{-1} \\ -H_{22}^{-1}H_{21}G_{11} & H_{22}^{-1} + H_{22}^{-1}H_{21}G_{11}H_{12}H_{22}^{-1} \end{pmatrix} \quad (5.4)
\]

\[
G_{11} = (H_{11} - H_{12}H_{22}^{-1}H_{21})^{-1},
\]

If \( H_{22}^{-1} \) is an analytic function for \( |z| > 1 - \delta \), and \( \|H_{22}^{-1}\| \leq C \), then

\[
\oint_{\omega_A} z^{n-1}(G(z)f, g)dz = \oint_{\omega_A} z^{n-1}(G_{11}f^{(1)}(z), g^{(1)}(z))dz + O(e^{-nc}) \quad (5.5)
\]

where \( \omega_A = \{z : |z| = 1 + A/n\} \), \( f = (f_0, f_1) \), \( g = (g_0, g_1) \) where \( f_0 \) and \( g_0 \) are the projection of \( f \) and \( g \) on the subspace corresponding to \( H_{11} \), while \( f_1 \) and \( g_1 \) are the projection of \( f \) and \( g \) on the subspace corresponding to \( H_{22} \).

**Proof.** Formula (5.4) is the well-known block matrix inversion formula. Now apply the formula (5.4) and write

\[
\oint_{\omega_A} z^{n-1}(G(z)f, g)dz = \oint_{\omega_A} z^{n-1}(G_{11}f^{(1)}(z), g^{(1)}(z))dz + \oint_{\omega_A} z^{n-1}(H_{22}^{-1}f_1, g_1)dz.
\]

For the second integral change the integration contour from \( \omega_A \) to \( |z| = 1 - \delta \). Then the inequality

\[
|z|^{n-1} \leq (1 - \delta)^{n-1} \leq Ce^{-nc}
\]

yields (5.5).

\[\square\]

Notice that since \( \|K\| \leq 1 \) and \( |F_{n,\xi}| \leq 1 + C/n \), we can find such \( A \) that all eigenvalues of \( K_\xi \) lies inside \( \omega_A = \{z : |z| = 1 + A/n\} \).

Set

\[
H_{11}(z) = H_{11} - z = (1_{\tilde{\omega}_+} K_\xi 1_{\tilde{\omega}_+}) \oplus (1_{\tilde{\omega}_+} K_\xi 1_{\tilde{\omega}_+}) \oplus (1_{\tilde{\omega}_-} K_\xi 1_{\tilde{\omega}_-}) - z = K_{\xi,+} \oplus K_{\xi,+} \oplus K_{\xi,-} - z.
\]
Then (5.3) yields
\[ \|H_{22}\| + \|H_{12}\| + \|H_{21}\| \leq Ce^{-c \log^2 W}. \]
Therefore for any \(|z| > \frac{1}{2}\)
\[ \|H_{12}(H_{22} - z)^{-1}H_{21}\| \leq Ce^{-c \log^2 W}. \]
Moreover, it will be proven below that
\[ \|(H_{11} - z)^{-1}\| \leq Cn, \quad z \in \omega_A, \]
and so for \(G_{11}\) of (5.4) we have
\[ \|G_{11} - (H_{11} - z)^{-1}\| \leq e^{-c \log^2 W/2}. \]
Here we have used \(W \geq n^\varepsilon\). Thus we obtain by Proposition 5.1
\[ \oint_{\omega_A} z^{-1}(G_\xi(z)f,g)dz = \oint_{\omega_A} z^{-1}((H_{11} - z)^{-1}f,g)dz + O(e^{-c \log^2 W/2}) + O(e^{-nc_1}), \quad (5.6) \]
where \(G_\xi(z)\) is a resolvent of \(K_\xi\) (see (4.1)). In view of the block structure of \(H_{11}\), its resolvent also has a block structure, hence
\[ \oint_{\omega_A} z^{-1}(G_\xi(z)f,g)dz = \oint_{\omega_A} z^{-1}(G_{\xi,\pm}(z)f_\pm,g_\pm)dz + \oint_{\omega_A} z^{-1}(G_{\xi,\mp}(z)f_-,g_-)dz = I_{\xi,\pm} + I_{\xi,\mp} + I_{\xi,-}, \quad (5.7) \]
where
\[ G_{\xi,\pm} = (K_{\xi,\pm} - z)^{-1}, \quad G_{\xi,\mp}(z) = (K_{\xi,\mp} - z)^{-1}, \quad G_{\xi,-}(z) = (K_{\xi,-} - z)^{-1} \]
and \(f_\pm, f_-, f_+, g_\pm, g_+, g_-\) are projections of \(f\) and \(g\) onto the subspaces corresponding to \(K_{\xi,\pm}, K_{\xi,\mp}, K_{\xi,-}\). One can perform similar analysis for \(K\) instead of \(K_\xi\) and define \(I_\pm, I_+, I_-\), and \(I_{\xi,\mp}\).

In the next sections we are going to study each integral \(I_{\xi,\pm}, I_{\xi,\mp}, I_{\xi,-}\) separately. It will be shown below (see Section 7) that that \(I_{\xi,\mp}\) and \(I_{\xi,-}\) are exponentially small comparable to \(I_{\xi,\pm}\), so the main task is to study \(I_{\xi,\pm}\).

6 Analysis of \(I_{\xi,\pm}\)

As was mentioned in Section 4, to analyze \(K_\pm\) and \(K_{\xi,\pm}\) we are going to use the polar decomposition (4.6) – (4.10).

We start with the analysis of operator \(A_a\) of (4.9) in the domain \(\tilde{\Omega}_\pm\) of (5.1).

To this end, we are going to consider quadratic approximation of \(A(a, a')\) defined in (4.9). Make a change of variables
\[ a_{1i} = a_+(1 + i\theta_+\bar{a}_{1i}/\sqrt{W}), \quad a_{2i} = a_-(1 + i\theta_-\bar{a}_{2i}/\sqrt{W}), \quad (6.1) \]
where \( \theta_{\pm} \) are some complex constants with \(|\theta_{\pm}| = 1\) which will be determined later (see (6.5)). Notice that the Jacobian of (6.1) is a constant depending on \(n, W\) but not on \(\xi\), thus it does not give contribution to \(\bar{C}_{n, W}^{\prime}\) (see (4.4)). Define

\[
A^+(\tilde{a}, \tilde{a}') = 1_{\Omega_+} A(a_+(1 + i\theta_+\tilde{a}/\sqrt{W}), a_+(1 + i\theta_+\tilde{a}'/\sqrt{W})) 1_{\Omega_+},
\]

\[
A^-(\tilde{a}, \tilde{a}') = 1_{\Omega_-} A(a_-(1 + i\theta_-\tilde{a}/\sqrt{W}), a_-(1 + i\theta_-\tilde{a}'/\sqrt{W})) 1_{\Omega_-}.
\]

Then

\[
K_{\xi, \pm}(a_1, a_2, Q; a'_1, a'_2, Q') = A^+(\tilde{a}_1, \tilde{a}_1') A^-(\tilde{a}_2, \tilde{a}_2') K_\ast(t, Q, Q') e^{-\frac{i\epsilon_\ast}{\pi} \left( \nu(a_1-a_2, Q) + \nu(a'_1-a'_2, Q') \right)}.
\]

Since \( \varphi_{\nu_0}^\ast(a_+) = c_+ \) (see (4.10) and (1.1)), it is easy to see that the kernel \( A^+_\ast \) of (6.2) takes the form

\[
A^+(\tilde{a}, \tilde{a}') = A^+_\ast(\tilde{a}, \tilde{a}') (1 + W^{-1/2} \hat{p}_+(\tilde{a}))(1 + W^{-1/2} \hat{p}_+(\tilde{a}')) + O(e^{-c\log^2 W}),
\]

\[
A^+_\ast(\tilde{a}, \tilde{a}') = \frac{a_+ \theta_+}{\sqrt{2\pi}} \exp \left\{ (a_+ \theta_+)^2 [\beta(\tilde{a} - \tilde{a}')^2/2 - c_+ \tilde{a}^2/4 - c_+(\tilde{a}')^2/4] \right\}
\]

\[
\hat{p}_+(\tilde{a}) = ic_{3+} \tilde{a}^3 - c_{4+} \tilde{a}^4 W^{-1/2} - ic_{5+} \tilde{a}^5 W^{-1} + \ldots
\]

where the coefficients \( c_{3+}, c_{4+}, \ldots \) are expressed in terms of the derivatives of \( \varphi_0 \) at \( a_+ \). Similarly \( A^- \) of (6.2) can be approximated via \( A^-_\ast \) defined similarly to \( A^+_\ast \) in (6.4).

It is easy to check that for \( \beta < 1/4 \) the real parts of the eigenvalues \( \alpha_{1, +}, \alpha_{2, +} \) of the quadratic form

\[
\begin{pmatrix}
  a_+^2 (\frac{c_+}{4} - \beta) & a_+^2 \beta \\
  a_+^2 \beta & a_+^2 (\frac{c_+}{4} - \beta)
\end{pmatrix}
\]

in the exponent of \( A^+_\ast \) of (6.4) are positive. Same is true for \( A^-_\ast \). Denote

\[
\theta_{\pm} = (|\kappa_{\pm}|/|\kappa_{\pm}|)^{1/2}, \quad \kappa_{\pm} = (\alpha_{1, \pm} \alpha_{2, \pm})^{1/2} = a_+^2 (c_\pm^2/4 - \beta c_\pm)^{1/2},
\]

with \( c_\pm \) of (1.1). Notice that \( \theta_{\pm} \) is defined in such a way that

\[
\Re(\theta_{\pm}^2 \alpha_{1, \pm}) > 0, \quad \Re(\theta_{\pm}^2 \alpha_{2, \pm}) > 0.
\]

Now introduce the orthonormal bases

\[
\psi_k^\pm(\tilde{a}) = |\kappa_{\pm}|^{1/4} H_k(|\kappa_{\pm}|^{1/2} \tilde{a}) e^{-|\kappa_{\pm}|\tilde{a}^2/2},
\]

where \( \{H_k(x)\} \) are Hermite polynomials which are orthonormal with the weight \( e^{-x^2} \):

\[
H_k(x) = (2^{k-1/2} k! \sqrt{2\pi})^{-1/2} e^{x^2} \left( \frac{d}{dx} \right)^k e^{-x^2}.
\]

Below we will need the following lemma
Lemma 6.1  (i) Let $\kappa_+, \kappa_-$ be defined as in (6.3). Then the matrices of the operators $A^\pm_+$ and $A^\pm_-$ are diagonal in the basis $\{\psi^\pm_k\}$ and $\{\bar{\psi}^\pm_k\}$ and the corresponding eigenvalues have the form

$$\lambda^\pm_k = \lambda_k(A^\pm_+) = \lambda^\pm_0 q^k, \quad k = 0, 1, 2, \ldots$$

(6.7)

with

$$\lambda^\pm_0 = (\kappa^\pm/\alpha_\pm^2 + c_\pm/2 - \beta)^{-1/2},$$

(6.8)

$$q^\pm = \frac{\beta}{\kappa^\pm/\alpha_\pm^2 + c_\pm/2 - \beta}, \quad |q^\pm| < 1.$$

Notice that $|q^\pm| < 1$ implies

$$|\lambda^\pm_0| \leq \beta^{-1/2}.$$  

(6.9)

The matrices of operators $A^+$ and $A^-$ of (6.2) have the form

$$(A^\pm)_{k,k} = \lambda^\pm_0 q^k + O(1/W),$$

(6.10)

$$(A^\pm)_{k,k'} = O(W^{-1/2})(\delta[q-k',1] + \delta[k-k',3])$$

$$+ O(W^{-1})\delta[k-k',2] + O(W^{-(k-k')^2}) \quad k \neq k'.$$

(ii) The eigenvalues of operator

$$A_\pm = 1_{\Omega_\pm}(\lambda_0(t)A_0)1_{\Omega_\pm}$$

(6.11)

are $\lambda^\pm_0 q^k_0 q^l_0 + O(1/W)$, $k, l = 0, 1, \ldots$ and they are solutions of the equation

$$(A^\pm)_{0,0} - z - (A^\pm)^{(12)}((A^\pm)^{(22)} - z)^{-1}(A^\pm)^{(21)} = 0,$$

(6.12)

where

$$A_\pm = \begin{pmatrix} A_{00} & A^{(12)}_0 \\ A^{(21)}_0 & A^{(22)}_0 \end{pmatrix}$$

according to the decomposition $\{\psi^+_k, \bar{\psi}^-_k\} = \{\psi_0^+ \psi_0^-\} \oplus \{\psi^+_k, \bar{\psi}^-_k\}_{k \neq 0}$ with $\bar{k} = (k_1, k_2)$.

Here $\lambda_0(t)$ is the top eigenvalue of $K_*(t, Q, Q')$ (see (4.14)).

The top eigenvalue of $K_\pm$ has the form

$$\lambda_0(K_\pm) = \lambda_0(A_\pm) = \lambda^+_0 \lambda^-_0 + O(1/W).$$

Proof. To simplify formulas, we consider the kernel (see (6.4) - (6.5))

$$M(x, y) = a_+(2\pi)^{-1/2}e^{-(Ax, x)/2}, \quad \bar{x} = (x, y), \quad A = \begin{pmatrix} \mu & \nu \\ \nu & \mu \end{pmatrix}, \quad \lambda_\pm = \mu \pm \nu, \quad \Re\lambda_\pm > 0.$$

Then, taking $\kappa = \sqrt{\mu^2 - \nu^2} = \sqrt{\lambda_+ \lambda_-}$, we obtain that

$$a_+(2\pi)^{-1/2} \int e^{-(Ax, x)/2 + \kappa y^2/2} \left(\frac{d}{dy}\right)^k e^{-\kappa y^2} \frac{d}{dy}$$

$$= q^k \cdot a_+(\mu + \kappa)^{-1/2} e^{\kappa x^2/2} \left(\frac{d}{dy}\right)^k e^{-\kappa x^2}, \quad q = \frac{\nu}{\mu + \kappa},$$
so \( e^{k^2/2(\frac{d}{dy})^k} e^{-k^2}, k = 0,1, \ldots \) are the eigenvectors of \( M \). Since \( M \) is compact, we have \( |q| < 1 \). Notice also that

\[
a_{\pm}(\mu + \kappa_{\pm})^{-1/2} = \lambda_{0,\pm}^{\pm}.
\]

Now if we change the variables

\[
x_1 = \theta x, \ y_1 = \theta y, \ \ \ \theta = e^{-i(\arg \lambda_{\pm} + \arg \lambda_{\mp})/4} = e^{-i\arg \kappa/2},
\]

then for the new matrix \( \tilde{A} = \theta^2 A \) has eigenvalues \( \theta^2 \lambda_{\pm}, \theta^2 \lambda_{\mp} \), whose real parts are still positive, \( \tilde{\kappa} = |\kappa| \), and \( \tilde{q} = q \). This finishes the proof of (6.7) – (6.8).

Formula (6.10) follows directly from (6.4) and the fact that the Gaussian integral of \( x^{2k+1} \) is zero, and it immediately gives the statement about eigenvalues of \( A_{\pm} \) (it is easy to see that \( \lambda_{0}(t) \) does not change anything since it has only \( \tilde{a}/W^{3/2} \) and \( \tilde{a}'/W^{3/2} \)).

Equation (6.12) can be obtained from the standard Schur inversion formula. The rest of part (ii) follows directly from (i) and Proposition 1.1.

□

Now we are going to normalize \( K_{\pm}, K_{\xi,\pm} \) by \( \lambda_{0}(K_{\pm}) \):

\[
\hat{K}_{\pm} = \lambda_{0}(K_{\pm})^{-1} K_{\pm}, \ \ \ \hat{K}_{\xi,\pm} = \lambda_{0}(K_{\pm})^{-1} K_{\xi,\pm} \tag{6.13}
\]

with \( K_{\pm,\xi} \) of (6.3). Notice that

\[
\hat{K}_{\pm} = \tilde{A}_{\pm} \cdot \hat{K}_s \tag{6.14}
\]

where

\[
\tilde{A}_{\pm} = (\lambda_{0}(A_{\pm}))^{-1} A_{\pm}, \ \ \ \hat{K}_s(t, Q, Q') = (\lambda_{0}(t))^{-1} K_s(t, Q, Q'),
\]

so both top eigenvalues of \( \tilde{A}_{\pm}, \hat{K}_s \) is 1, and

\[
\hat{\lambda}_j(\hat{K}_s) = 1 - \frac{j(j+3)}{tW} + O((j^2/tW)^2), \ \ j = 1, 2, \ldots . \tag{6.15}
\]

Therefore, it is easy to see that all eigenvalues of \( \hat{K}_{\pm}, \hat{K}_{\xi,\pm} \) lies inside \( \omega_A = \{ z : |z| = 1 + A/n \} \).

Thus we get

\[
I_{\pm,\xi} = -2\pi i (K_{\xi,\pm}^{n-1} f, g) = -2\pi i \cdot \lambda_{0}(K_{\pm})^{n-1} (\hat{K}_{\xi,\pm}^{n-1} f, g) = \lambda_{0}(K_{\pm})^{n-1} \int_{\omega_A} z^{n-1}(\hat{G}_\xi(z) f, g) dz,
\]

where

\[
\hat{G}_\xi(z) = (\hat{K}_{\xi,\pm} - z)^{-1}.
\]

Similarly we can rewrite \( I_{\pm} \).

Consider the matrix of \( \hat{K}_{\xi,\pm} \) in the basis

\[
\Psi_{\hat{k},\hat{j}}(\hat{a}, \hat{B}, Q) = \psi_{\hat{k}_1}^+(\hat{a}_1) \psi_{\hat{k}_2}^-(\hat{a}_2) \phi_j(Q), \ \ k_1, k_2, j \geq 0, \tag{6.16}
\]

with \( \psi_{\hat{k}}^\pm \) of (6.6), and \( \phi_j \) of (4.14). Let \( \mathcal{H}_1 = \{ \Psi_{\hat{k},\hat{j}} \}_{j \leq l-1} \) and

\[
L_2(\mathbb{R}^2) \times L_2(\hat{Sp}(2), d\mu(Q)) = \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{6.17}
\]

and write

\[
\hat{K}_{\pm} = \begin{pmatrix} \hat{K}^{(11)}_{\xi} & \hat{K}^{(12)}_{\xi} \\ \hat{K}^{(21)}_{\xi} & \hat{K}^{(22)}_{\xi} \end{pmatrix}, \ \ \ \hat{K}_{\xi,\pm} = \begin{pmatrix} \hat{K}^{(11)}_{\xi} & \hat{K}^{(12)}_{\xi} \\ \hat{K}^{(21)}_{\xi} & \hat{K}^{(22)}_{\xi} \end{pmatrix} \tag{6.18}
\]

according to this decomposition. We will need the following simple lemma
Lemma 6.2 Given decomposition (6.18), we have

\[ \hat{K}^{(12)} = \hat{K}^{(21)} = 0, \quad \|\hat{K}^{(12)}\| \leq \frac{C}{n}, \quad \|\hat{K}^{(21)}\| \leq \frac{C}{n}; \quad (6.19) \]

and for \(|z| \geq 1 + A/n\) with big enough \(A\) we have

\[ \|\hat{K}^{(11)}_\xi - z\|^{-1} \leq Cn, \quad (6.20) \]

\[ \|\hat{K}^{(22)}_\xi - z\|^{-1} \leq CW/l^2, \quad (6.21) \]

and same is valid for \(\hat{K}\).

Proof of Lemma 6.2. The bound (6.19) follows from the block-diagonal structure of \(\hat{K}_\pm\) with respect to the basis (6.16) (see (4.15)), and the fact that \(\hat{K}_{\pm,\xi}\) is \(1/n\)-order perturbation of \(\hat{K}_\pm\).

The bound (6.20) follows from

\[ \|\hat{K}^{(11)}_\xi\| \leq 1 + c/n, \]

since for big enough \(A\)

\[ \|\hat{K}^{(11)}_\xi - z\|^{-1} \leq |z|^{-1} \sum_{k=0}^{\infty} \left( \frac{\|K^{(11)}_\xi\|}{|z|} \right)^k \leq Cn. \]

Similarly, according to (4.14) – (4.15), we get

\[ \|\hat{K}^{(22)}_\xi\| \leq 1 - \frac{Cl(l + 3)}{W} \]

and, since \(l^2/W \sim \log^2 n/n\) for \(W \geq Cn\) and \(l^2/W \gg n^{-1}\) for \(W \ll n\),

\[ \|\hat{K}^{(22)}_\xi\| \leq 1 - \frac{Cl^2}{W}, \]

which implies (6.21). □

The next step is to prove that we can consider only the upper-left block \(K^{(11)}_\xi\) of \(K_\xi\) (see (6.18)). More precisely, we are going to prove

Lemma 6.3 We have

\[ \int_{\omega_A} z^{n-1}(\hat{G}_\xi(z)f, \bar{f})dz = \int_{\omega_A} z^{n-1}(\hat{G}_{1,\xi}(z)f_1, \bar{f}_1)dz + O\left(\frac{W \log n}{l^2n}\right), \]

where

\[ \hat{G}_{1,\xi}(z) = (\hat{K}^{(11)}_\xi - z)^{-1}, \]

and we decomposed \(f = (f_1, f_2)\) with respect to decomposition (6.17). Notice that

\[ \frac{W \log n}{l^2n} \leq \frac{1}{\log n}. \]
Proof. Using the well-known Schur inversion formula we get

\[
(\hat{K}_\xi - z)^{-1} = \begin{pmatrix}
\hat{G}^{(11)}_\xi & -\hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \\
-\hat{G}^{(21)}_\xi \hat{G}^{(11)}_\xi & \hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi + \hat{G}^{(21)}_\xi \hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \hat{G}^{(21)}_\xi
\end{pmatrix},
\]

where

\[
\hat{G}^{(11)}_\xi = (\hat{K}^{(11)}_\xi - z)^{-1},
\]

\[
\hat{G}^{(21)}_\xi = (\hat{K}^{(11)}_\xi - z - \hat{K}^{(12)}_\xi \hat{K}^{(12)}_\xi)^{-1} = (1 - \hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \hat{G}^{(21)}_\xi)^{-1} \hat{G}^{(11)}_\xi.
\]

Thus

\[
\int_{\omega_A} z^{n-1}((K_\xi - z)^{-1} f, f)dz = \int_{\omega_A} z^{n-1}(G^{(11)}_\xi f, f_1)dz - \int_{\omega_A} z^{n-1}(G^{(11)}_\xi K^{(12)}_\xi G^{(21)}_\xi f, f_1)dz
\]

\[
- \int_{\omega_A} z^{n-1}(G^{(21)}_\xi K^{(12)}_\xi G^{(11)}_\xi f, f_1)dz + \int_{\omega_A} z^{n-1}((G^{(11)}_\xi + G^{(21)}_\xi K^{(12)}_\xi G^{(11)}_\xi K^{(12)}_\xi) f, f_2)dz.
\]

Denoting

\[
R = (1 - \hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \hat{G}^{(21)}_\xi)^{-1},
\]

we get

\[
\hat{G}^{(11)}_\xi = R\hat{G}^{(11)}_\xi.
\]

According to (6.19) – (6.21), we obtain

\[
\|\hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \hat{G}^{(21)}_\xi \| \leq Cn \cdot \frac{1}{n^2} \cdot \frac{W}{l^2} = \frac{CW}{l^2 n}.
\]

Therefore

\[
\|1 - R\| \leq \frac{CW}{l^2 n},
\]

which together with (6.20) imply

\[
\left| \int_{\omega_A} z^{n-1}\left((\hat{G}^{(11)}_\xi - \hat{G}^{(11)}_\xi) f_1, f_1\right) \right| = \left| \int_{\omega_A} z^{n-1}\left((1 - R)G^{(11)}_\xi f_1, f_1\right) \right| 
\]

\[
\leq C\|1 - R\| \cdot \|f_1\|^2 \cdot \int_{\omega_A} \frac{|dz|}{|z - 1|} \leq \frac{CW \log n}{l^2 n}.
\]

It is easy to see also that

\[
\|f_2\| \leq C/n,
\]

and because of the consideration above

\[
\|\hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \hat{G}^{(21)}_\xi \| \leq CW/l^2,
\]

\[
\|\hat{G}^{(11)}_\xi \hat{K}^{(12)}_\xi \hat{G}^{(21)}_\xi \| \leq CW/l^2,
\]

so other terms in (6.22) are also small.

\[
\square
\]

The next step is to show that we can consider only the projection of \(\hat{K}^{(11)}_\xi, \hat{K}^{(11)}_\xi\) on a linear span of \(\{\Psi_{0,j}\}_{j \leq t}\) (see (6.16)). We prove
Lemma 6.4 Let \( P_l \) be the projection on \( \{\phi_j\}_{j=0}^{t-1} \) of \( A^1 \), \( \Delta_l = P_l \Delta P_l \), and \( \hat{\nu}_l = P_l \hat{\nu} P_l \) with \( \hat{\nu} \) defined in (1.9). Then

\[
\int_{\omega_A} z^{n-1}(\hat{G}_{1,\xi}(z) f_1, \bar{f}_1)dz
= \int_{\omega_A} \zeta^{n-1}\left( (P_l - \frac{1}{i\pi W} \Delta_l - \frac{i\pi \xi}{n} \hat{\nu}_l - \zeta + O\left( \frac{(l-1)^4}{W^2} \right)^{-1} f_0, \bar{f}_0 \right)dz + O\left( \frac{(l-1)^2n}{W^{3/2}} \right) + O\left( \frac{1}{W^{1/2}} \right),
\]

where \( O(x) \) is an operator whose norm is bounded by \( Cx \) which does not depend on \( \zeta \), and \( f_0 = (f, \Psi_0) \).

Recall \( l = 1 \) for \( W \ll n \) and \( (l-1)^2n/W^{3/2} \sim \log^2 n/W^{1/2} \) for \( W \geq Cn \), and \( t_* = (2\pi p(E))^2 \). Similar formula is true for \( \hat{G}_1 \) (i.e. for \( \hat{K}^{(1)} \) instead of \( \hat{K}^{(11)} \)).

Proof. Now write \( \hat{K}^{(11)} - z, \hat{K}^{(11)} - z \) in the block form

\[
\hat{K}^{(11)} - z = \begin{pmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{pmatrix}, \quad \hat{K}^{(11)} - z = \begin{pmatrix} M_{1,\xi} & M_{12,\xi} \\ M_{21,\xi} & M_{2,\xi} \end{pmatrix}
\]

according to decomposition

\[ \mathcal{H}_1 = \mathcal{M}_1 \oplus \mathcal{M}_2, \]

where \( \mathcal{M}_1 \) is a linear span of \( \Psi^{(l)}_0 = \{\Psi_{0,j}\}_{j \leq l} \) (see (6.16)).

Set

\[
G_{1,\xi}^0(z) = (M_{11,\xi} - M_{12,\xi} M_{22,\xi}^{-1} M_{21,\xi})^{-1} \quad G_1^0(z) = (M_{11} - M_{12} M_{22}^{-1} M_{21})^{-1}
\]

Then, using Proposition 5.1 we get

\[
\oint_{\omega_A} z^{n-1}(G(z)f_1, \bar{f}_1)dz = \oint_{\omega_A} z^{n-1}(G_1^0 f^{(1)}(z), g^{(1)}(z))dz + O(e^{-nc}),
\]

\[
\oint_{\omega_A} z^{n-1}(G_\xi(z)f_1, \bar{f}_1)dz = \oint_{\omega_A} z^{n-1}(G_1^{0,\xi} f^{(1)}(z), g^{(1)}(z))dz + O(e^{-nc}),
\]

where \( f^{(1)}, g^{(1)} \) are defined as in (5.5). Recall that

\[ \hat{K}^{(11)} - z = (\hat{A} P_l \hat{K}_* P_l \Psi^{(l)}_0, \Psi^{(l)}_0) - z. \]

Set

\[ P_l \hat{K}_* P_l = P_l - \hat{K}_l. \]

Then \( \hat{K}_l \), according to (6.15), is a diagonal matrix with eigenvalues

\[ \tilde{\lambda}_j(t) = j(j+3)/tW + O((j^2/tW)^2), \quad j = 0, \ldots, l - 1. \]

Since (1.3) and (6.1) implies

\[
t = \left( a_+ - a_- + \frac{i\theta + a_+ \bar{a}_1 - i\theta_+ a_- \bar{a}_2}{\sqrt{W}} \right) \left( a_+ - a_- + \frac{i\theta + a_+ \bar{a}'_1 - i\theta_+ a_- \bar{a}'_2}{\sqrt{W}} \right),
\]

20
\( \tilde{K}_l \) can be rewritten as

\[
\tilde{K}_l = \Delta_l/t_* W + O_a\left(\frac{(l-1)^2}{W^{3/2}}\right) + O\left(\frac{(l-1)^4}{W^2}\right) \tag{6.23}
\]

with \( t_* = (a_+ - a_-)^2 = (2\pi \rho(E))^2 \). Here \( O_a(X) \) is an operator of the type \( O(X) \) whose eigenvalues are linear in \( a, a' \).

Now, according to Lemma 6.1, \( \hat{A}_{00} = 1 + O(1/W) \), substituting (6.23), we get

\[
(\hat{A}\tilde{K}_l\Psi^{(l)}_0, \Psi^{(l)}_0)_{jj} = (\hat{A}_j(t)\hat{A}_j^{1/2} \Psi_0^{(l)} \Psi_0^{(l)}, \Psi_0^{(l)} \Psi_0^{(l)}) = \frac{j(j+3)}{t_* W} \cdot \left(1 + O\left(\frac{j^2}{W}\right)\right)
\]

Therefore,

\[
\tilde{K}^{(11)} - z = \hat{A}_{00} P_l - z - \Delta_l/t_* W + O((l-1)^4/W^2).
\]

Similarly

\[
M_{12} = \hat{A}_{12} \otimes P_l + O\left(\frac{(l-1)^2}{W^{3/2}}\right),
\]

\[
M_{21} = \hat{A}_{21} \otimes P_l + O\left(\frac{(l-1)^2}{W^{3/2}}\right),
\]

\[
M_{22} = \hat{A}_{22} \otimes P_l - z + O\left(\frac{(l-1)^2}{W}\right).
\]

Notice also that because of Lemma 6.1

\[
\|\hat{A}_{12}\| \leq W^{-1/2}, \quad \|\hat{A}_{21}\| \leq W^{-1/2}, \quad \|\hat{A}_{22} - z\|^{-1} \leq C.
\]

Hence

\[
M_{11} - M_{12} M_{22}^{-1} M_{21} = A_{00} P_l - \Delta_l/t_* W - z - \hat{A}_{12}(\hat{A}_{22} - z)^{-1} \hat{A}_{21} P_l + O\left(\frac{(l-1)^2}{W^2}\right)
\]

\[
+ O\left(\frac{(l-1)^4}{W^2}\right) = P_l - \Delta_l/t_* W - z - W^{-1} g_1(z) P_l + O\left(\frac{(l-1)^4}{W^2}\right),
\]

where

\[
g_1(z) = \hat{A}_{00} - 1 - \hat{A}_{12} (\hat{A}_{22} - z)^{-1} \hat{A}_{21} + O\left((l-1)^2/W\right) - O_*((l-1)^2/W)
\]

is analytic and bounded in \( \{z : |z - 1| < \delta\} \) for small enough \( \delta \) (recall \( l^2/W \leq \log^2 n/n \) and \( \|\hat{A}^{22}\| \leq |q_\pm| < 1 \) with \( q_\pm \) of (6.8) according to Lemma 6.1). Here \( O_z(\cdot) \) is operator of type \( O(\cdot) \) which may depend on \( z \), and \( O_*(\cdot) \) is operator \( O_z(\cdot) \) with substitution \( z = 1 \). Lemma 6.1 implies also

\[
g_1(1) = 0. \tag{6.24}
\]

Now set

\[
\zeta(z) = z + W^{-1} g_1(z). \tag{6.25}
\]

Since \( g_1(z) \) has a bounded derivative in \( \{z : |z - 1| < \delta\} \), we get

\[
\zeta'(z) = 1 + O(1/W),
\]

\[21\]
the Implicit function theorem implies that there exists the inverse function \( z(\zeta) \) with a derivative of order \( 1 + O(1/W) \). In addition, by \((6.24)\), \( \zeta(1) = 1 \), so the image of \( \{ z : |z - 1| < \delta \} \) lies in \( \{ \zeta : \delta/2 < |\zeta - 1| < 2\delta \} \), and it is easy to show that

\[
 z(\zeta) = \zeta + W^{-1}\tilde{g}_1(\zeta),
\]

where \( \tilde{g}_1 \) is a bounded analytic in \( \{ \zeta : |\zeta - 1| < 2\delta \} \), and \( \tilde{g}_1(1) = 0 \).

Now we consider the contour \( \tilde{\omega}_A = \{ z : \text{dist}(z; [1 - C(l - 1)(l + 2)/W; 1]) \leq A/n \} \) and the contour \( L_2 = \{ |z| \leq \frac{|q_\pm| + 1}{2} < 1 \} \) with \( q_\pm \) of \((6.8)\). It is easy to see that \( \tilde{\omega}_A \cup L_2 \) encircle all the eigenvalues of \( \hat{K}^{(11)}_\zeta, \hat{K}^{(11)}_\overline{\zeta} \) (see \((4.14) - (4.15)\) and Lemma \(6.1\)). But for \( z \in L_2 \)

\[
 |z|^{n-1} \leq e^{-cn},
\]

so the contribution of the integral over \( L_2 \) is small, and we need to consider integral over \( \tilde{\omega}_A \) only. It follows from \((6.24) - (6.25)\) that and consideration above that \( \zeta(z), z \in \tilde{\omega}_A \) will be inside \( \tilde{\omega}_{2A} \), and so, since \( l = 1 \) for \( W \ll n \) and \( l^2/W \sim \log^2 n/n \) for \( W \geq Cn \), we get

\[
 z(\zeta) = \zeta + O((l - 1)^2/W^2) + O(1/nW),
\]

hence

\[
 z^{n-1} = \begin{cases} 
 z^{n-1} + O(\log^2 n/W), & W \geq Cn, \\
 z^{n-1} + O(1/W), & W \ll n.
\end{cases} \tag{6.26}
\]

Notice also that for \( z \in \tilde{\omega}_A \)

\[
 \| (P_l - \Delta_l/t_sW - \zeta(z))^{-1} \| \leq Cn,
\]

thus

\[
 \| G^0_l \| = \| (P_l - \Delta_l/t_sW - \zeta(z) + O((l - 1)^4/W^2))^{-1} \| \leq \| (P_l - \Delta_l/t_sW - \zeta(z))^{-1} \| \\
 \times \| (1 + O((l - 1)^4/W^2) \cdot (P_l - \Delta_l/t_sW - \zeta(z))^{-1} )^{-1} \| \leq Cn.
\]

Hence, recalling \( l = 1 \) and \( \| \tilde{\omega}_A \| = C/n \) for \( W \ll n \) and \( |\tilde{\omega}_A| \leq C(l - 1)^2/W \) for \( W \geq Cn \), and \( \| f^{(1)}(z) - f_0 \| \leq C/\sqrt{W} \), we obtain

\[
 \oint_{\tilde{\omega}_A} z^{n-1}(G^0_l(z)f^{(1)}(z), g^{(1)}(z))dz \\
 = \oint_{\tilde{\omega}_A} z^{n-1}(G^0_l(z)f_0, \bar{f}_0)dz + O((l - 1)^2n/W^{3/2}) + O(1/W^{1/2}),
\]

that according to \((6.26)\) can be further transformed as

\[
 \oint_{\tilde{\omega}_A} z^{n-1}(G^0_l(z)f_0, \bar{f}_0)dz \\
 = \oint_{(\zeta(\omega)_A)} z^{n-1}((P_l - \Delta_l/t_sW - \zeta + O((l - 1)^4/W^2))^{-1}f_0, \bar{f}_0)d\zeta + O\left(\frac{(l - 1)^4n^2}{W^3}\right) + O\left(\frac{1}{W}\right),
\]

22
and the contour now can be changed back to $\omega_A$ (notice $l^4 n^2 / W^3 = \log^4 n / W$, $l^4 / W^2 = \log^2 n / W n$ for $W \geq C n$).

In order to perform the same analysis for $\hat{K}_{\xi}^{(11)}$ notice that

$$
\|M_{12}\| \leq C / \sqrt{W}, \quad M_{12, \xi} = M_{12} + O\left(\frac{1}{n \sqrt{W}}\right);
$$

$$
\|M_{21}\| \leq C / \sqrt{W}, \quad M_{21, \xi} = M_{21} + O\left(\frac{1}{n \sqrt{W}}\right);
$$

$$
\|M^{-1}_{22}\| \leq C, \quad M_{22, \xi} = M_{22} + O\left(\frac{1}{n}\right),
$$

and

$$
M_{11, \xi} = (A P_l F_{n, \xi} K_{\ast} F_{n, \xi} P_l \Psi, \Psi_0) - z = M_{11} - A_{00} \cdot \frac{i \pi \xi}{n} P_l \hat{\nu} P_l + O\left(\frac{1}{n \sqrt{W}}\right).
$$

Thus, since $A_{00} = 1 + O(1 / W)$, we have

$$
M_{1, \xi} - M_{12, \xi} M^{-1}_{22, \xi} M_{21, \xi} = M_{11} - M_{12} M^{-1}_{22} M_{21} - \frac{i \pi \xi}{n} P_l \hat{\nu} P_l + O\left(\frac{1}{n \sqrt{W}}\right),
$$

and hence we can apply same consideration as above.

Now let us analyze

$$
\int_{\omega_A} \zeta^{n-1} \left(\left(P_l - \frac{1}{t_{\ast} W} \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l - \zeta + O((l - 1)^4 / W^2)\right)^{-1} f_0, \bar{f}_0\right) d\zeta
$$

- **localized regime:** $W \ll n$. In this regime $l = 1$, so we need to study

$$
\int_{\omega_A} \zeta^{n-1} \left(\left(P_l - \frac{i \pi \xi}{n} \hat{\nu}_1 - \zeta\right)^{-1} f_0, \bar{f}_0\right) d\zeta
$$

$$
= -2 \pi i \cdot \|f_0\|^2 \left(\left(P_l - \frac{i \pi \xi}{n} \hat{\nu}_1\right)^{-1} 1, 1\right).
$$

But since $\phi_0 = 1$ and $\hat{\nu}_1 = P_l \hat{\nu} P_l$, $\hat{\nu} \cdot 1 = \phi_1$ (see (1.9) and Proposition 4.1), we obtain

$$
(P_l - \frac{i \pi \xi}{n} \hat{\nu}_1) 1 = 1 - \frac{i \pi \xi}{n} P_l \phi_1 = 1,
$$

which implies

$$
\left(\left(P_l - \frac{i \pi \xi}{n} \hat{\nu}_1\right)^{-\frac{1}{n-1}} 1, 1\right) = 1,
$$

thus Theorem 1.1 in the regime $W \ll n$.

- **critical regime:** $n = C_{\ast} W$. Again we need to study $\left(K_{0}^{-1} f_0, f_0\right)$ with

$$
K_0 = P_l - \frac{1}{t_{\ast} W} \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l + O(t^4 / W^2) = P_l - \frac{C_{\ast}}{n} \Delta_l - \frac{i \pi \xi}{n} \hat{\nu}_l + O(\log^4 n / n^2),
$$

where $C_{\ast} = C_{\ast} / t_{\ast}$. It is enough to prove
Lemma 6.5 Given (4.18), if \( n = C_s W, l = [\log W] \) we have
\[
(K_0^{n-1}, 1, 1) \rightarrow (e^{-C_s \Delta - i \xi \pi \hat{v}} 1, 1), \quad n, W \rightarrow \infty,
\]
with \( \hat{v}, \Delta \) as in Theorem 1.1.

Similar Lemma is proved in [30], but for the sake of completeness we repeat the proof here.

Proof of Lemma 6.5 Notice that
\[
K_0 = P_t - n^{-1} C_s \Delta_t - \frac{i \xi \pi}{n} \hat{v}_t + O(\log^4 n/n^2) = P_t e^{-n^{-1}(C_s \Delta_t + i \xi \pi \hat{v}_t) + O(\log^4 n/n^2)} P_t.
\]
Thus
\[
K_0^{n-1} = P_t e^{-C_s \Delta_t - i \xi \pi \hat{v}_t} P_t + O(\log^4 n/n),
\]
and so
\[
(K_0^{n-1}, 1, 1) = (e^{-C_s \Delta_t - i \xi \pi \hat{v}_t} 1, 1) + O(\log^4 n/n).
\]
Consider the basis \( \{ \phi_j \} \) of (4.11). In this basis Laplace operator \( \Delta \) is diagonal, and operator \( \hat{v} \) is three diagonal (since it corresponds to the multiplication by \( 2x^2 - 1 \), see (1.9) and (4.12)). To simplify notations, let \( F \) be an operator of multiplication by \( (i \pi \xi \nu) \) and \( \tilde{\Delta} = C_s \Delta \). Set
\[
D = \tilde{\Delta} + F,
\]
\[
D^{(l)} = \tilde{\Delta} + F^{(l)},
\]
where \( F^{(l)} \) be the matrix \( F \) where we put \( F_{l-1,l} = F_{l,l-1} = 0 \). It is evident that (recall \( \phi_0 = 1 \))
\[
(e^{-D^{(l)}} \phi_0, \phi_0) = (e^{-P_l D P_l} \phi_0, \phi_0) = (e^{-P_l (C_s \Delta + i \xi \pi \nu) P_l} 1, 1).
\]
Thus we are left to prove that
\[
\left( (e^{-D} - e^{-D^{(l)}}) \phi_0, \phi_0 \right) \rightarrow 0. \quad (6.28)
\]
Notice that both \( e^{-D} \), \( e^{-D^{(l)}} \) are bounded operators, and \( |F| \leq C, |F^{(l)}| \leq C \). We will use the well-known Duhamel formula
\[
e^{-tA_1} - e^{-tA_2} = \int_0^t e^{-(t-s)A_2} (A_1 - A_2) e^{-sA_1} ds. \quad (6.29)
\]
For \( A_1 = D, A_2 = D^{(l)} \) and \( t = 1 \) it gives
\[
\left| (e^{-D} - e^{-D^{(l)}}) \phi_0 \right| = \left| \int_0^1 e^{-(1-s)D^{(l)}} (F - F^{(l)}) e^{-sD} \phi_0 ds \right|
\]
\[
= \left| \int_0^1 e^{-(1-s)D^{(l)}} (F_{l-1,l} E_{l-1,l} + F_l \cdot E_{l,l-1}) e^{-sD} \phi_0 ds \right|
\]
\[
= \left| \int_0^1 e^{-(1-s)D^{(l)}} \left( F_l \phi_l (e^{-sD} \phi_0, \phi_{l-1}) + F_{l-1} \phi_{l-1} (e^{-sD} \phi_0, \phi_l) \right) ds \right|
\]
\[
\leq C \left( \left| (e^{-sD} \phi_0, \phi_{l-1}) \right| + \left| (e^{-sD} \phi_0, \phi_l) \right| \right).
\]
Here $E_{l-1,l}$ is an operator whose matrix in the basis $\{\phi_j\}$ has 1 at $(l-1,l)$ place and zeros everywhere else, and $E_{l,l-1}$ is defined in a similar way. $F_{l-1}, F_l$ are $(l-1,l)$ and $(l, l-1)$ elements of the matrix $F$ in the same basis.

Now let us bound $\left|\left(e^{-tD\phi_0, \phi_l}\right)\right|$. To this end apply Duhamel’s formula (6.29) $p = [l/2]$ times with $A_1 = D$ and $A_2 = \Delta$. We obtain

$$\left(e^{-tD\phi_0, \phi_l}\right) = \sum_{j=1}^{p} \int_{s_1 + \ldots + s_j \leq s} (e^{-s_1\Delta} F e^{-s_2\Delta} F \ldots e^{-s_j\Delta} \phi_0, \phi_l) ds_1 \ldots ds_j$$

$$+ \int_{s_1 + \ldots + s_p \leq s} (e^{-s_1\Delta} F e^{-s_2\Delta} F \ldots e^{-s_p\Delta} \phi_0, \phi_l) ds_1 \ldots ds_p.$$

Since $e^{-t\Delta}$ is diagonal in the basis $\{\phi_j\}$, and $F$ is only three diagonal, the expression $e^{-s_1\Delta} F e^{-s_2\Delta} F \ldots e^{-s_p\Delta} \phi_0$ is in the linear span of $\{\phi_k\}_{k=0}^j$, and thus the sum above is 0. Hence

$$\left|\left(e^{-tD\phi_0, \phi_l}\right)\right| \leq \left|\int_{s_1 + \ldots + s_p \leq s} (e^{-s_1\Delta} F e^{-s_2\Delta} F \ldots e^{-s_p\Delta} \phi_0, \phi_l) ds_1 \ldots ds_p\right|$$

$$\leq C\int_{s_1 + \ldots + s_p \leq s} ds_1 \ldots ds_p = \frac{C^l s^l}{l!} \leq C_1 e^{-l \log l} \to 0,$$

which finishes the proof of (6.28).

\[ \square \]

\textbf{delocalized regime:} $W \gg n$. Since in this regime $l^4/W^2 = C \log^4 n/n^2$, we get

$$\int_{\omega_A} \zeta^{-n-1} \left( \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \hat{\nu}_l - \zeta + O(l^4/W^2) \right)^{-1} f_0, \bar{f}_0 \right) d\zeta$$

$$= \int_{\omega_A} \zeta^{-n-1} \left( \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \hat{\nu}_l - \zeta \right)^{-1} f_0, \bar{f}_0 \right) d\zeta + O(\log^4 n/n),$$

Hence we need to study

$$|f_0|^2 \int_{\omega_A} \zeta^{-n-1} \left( \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \hat{\nu}_l - z \right)^{-1} 1, 1 \right) dz$$

Now let us define

$$m = \sqrt[3]{\frac{W}{n}}, \quad m \to \infty,$$

in order to get

$$\frac{m^2 n}{W} = \frac{1}{m} \to 0.$$

Set

$$G(z) = \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \hat{\nu}_l - z \right)^{-1}, \quad G^{(m)}(z) = \left( P_m - \frac{C_s}{W} \Delta_m - \frac{i\pi \xi}{n} \hat{\nu}_m - z \right)^{-1},$$

$$G^{(m,l)}(z) = \left( P_l - \frac{C_s}{W} \Delta_l - \frac{i\pi \xi}{n} \hat{\nu}_{l,m} - z \right)^{-1},$$

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where \( \hat{\nu}_{l,m} \) has the same matrix as a tridiagonal operator \( \hat{\nu}_l \) but with \((m, m + 1)\) and \((m + 1, m)\) elements equal to 0, and \( P_m \) is a projection on \( \{ \phi_j \}_{j \leq m} \), \( \Delta_m = P_m \Delta P_m \), \( \hat{\nu}_m = P_m \hat{\nu} P_m \). Notice that \( \hat{\nu}_{l,m} \) has a block diagonal structure with blocks \( m \times m \) and \((l - m) \times (l - m)\), thus

\[
(G^{(m,l)}(z)1, 1) = (G^{(m)}(z)1, 1). \tag{6.30}
\]

We are going to prove

\[
\int_{\omega_A} z^{n-1} (G(z) \cdot 1, 1)\,dz = \int_{\omega_A} z^{n-1} (G^{(m,l)}(z) \cdot 1, 1)\,dz + O(1/m). \tag{6.31}
\]

Then, if we define

\[
G^{(m)}_0(z) = (P_m - \frac{i\pi \xi n}{\hat{\nu}_m} - z)^{-1}, \tag{6.32}
\]

we can write using (6.30) and the standard resolvent identity

\[
(G^{(m,l)}(z) \cdot 1, 1) = (G^{(m)}(z)1, 1) = (G^{(m)}_0(z)1, 1) + (G^{(m)}(z)(\frac{C_*}{W} \Delta_m) G^{(m)}(z)1, 1).
\]

But

\[
\| \frac{C_*}{W} \Delta_m \| \leq \frac{C m^2}{W} \leq \frac{1}{mn},
\]

hence, since both resolvent can be bounded by \( |z - 1|^{-1} \), we get

\[
\left| \int_{\omega_A} z^{n-1} (G^{(m)}(z)(\frac{C_*}{W} \Delta_m) G^{(m)}_0(z)1, 1)\,dz \right| \leq \frac{C}{mn} \int_{\omega_A} \frac{|dz|}{|z - 1|^2} \leq \frac{C}{m}, \tag{6.33}
\]

where we have used

\[
\int_{\omega_A} \frac{|dz|}{|z - 1|^2} \leq C n. \tag{6.34}
\]

Now (6.30) – (6.33) imply

\[
| f_0 |^2 \int_{\omega_A} z^{n-1} (G(z)1, 1)\,dz = | f_0 |^2 \int_{\omega_A} z^{n-1} (G^{(m)}_0(z)1, 1)\,dz + O(1/m)
\]

\[
= -2\pi i \cdot | f_0 |^2 \cdot \left( (P_m - \frac{i\pi \xi n}{\hat{\nu}_m})^{n-1} 1, 1 \right) + O(1/m).
\]

Since \( \hat{\nu} \) is bounded, we can easily change \( \hat{\nu}_m \) to \( \hat{\nu} \) and use

\[
\left( 1 - \frac{i\pi \xi n}{\hat{\nu}} \right)^{n-1} = e^{-i\pi \xi \hat{\nu}} + O(1/n),
\]

which implies Theorem 1.1.

Therefore we are left to prove (6.31). First we will need a bound

**Lemma 6.6** For \( |z| \geq 1 + A/n \) we have

\[
|G_{ij}(z)| \leq \frac{C}{|z - 1|} e^{-\delta|i-j|},
\]

where \( C \) and \( \delta \) depends only on \( A \). Same is true for \( G^{(m,l)}(z) \).
Notice that since \( \hat{\nu} \) is bounded 3-diagonal matrix (see (4.12)), and \( P_l - \frac{C_s}{W} \Delta_l \) is diagonal, Lemma 6.6 follows from the standard Combes-Thomas arguments (see, e.g., [23], Ch 13, Proposition 13.13.1)).

Using the resolvent identity we can write

\[
\int_{\omega_A} z^{n-1} \left( G(z) \cdot 1, 1 \right) dz = \int_{\omega_A} z^{n-1} \left( G^{(m,l)}(z) \cdot 1, 1 \right) dz \\
+ \int_{\omega_A} z^{n-1} \left( G(z) \left( \frac{\delta \hat{\nu}}{n} \right) G^{(m,l)}(z) \cdot 1, 1 \right) dz,
\]

where \( \delta \hat{\nu} = i \pi (\hat{\nu}_l - \hat{\nu}_m) \), i.e. the matrix with only two non-zero elements \((m, m + 1)\) and \((m + 1, m)\). Rewrite

\[
\left( G(z) \left( \frac{\delta \hat{\nu}}{n} \right) G^{(m,l)}(z) \right)_{1,1} = \frac{\delta \hat{\nu}}{n} \left( G^{(m,l)}_{m,1} G^n_{0,0}(z) + \frac{\delta \hat{\nu}}{n} G^{(m,l)}_{m,0} G^n_{0,1}(z) + \frac{\delta \hat{\nu}}{n} G^{(m,l)}_{m,1} G^n_{0,1}(z) \right).
\]

But according to Lemma 6.6

\[
|G^{(m,l)}_{m+1,0}(z)| \leq \frac{C}{|z - \bar{1}|} e^{-\delta m},
\]

and similar bounds hold for other resolvent elements in (6.36). Thus

\[
\left| \int_{\omega_A} z^{n-1} \left( G(z) \left( \frac{\delta \hat{\nu}}{n} \right) G^{(m,l)}(z) \cdot 1, 1 \right) dz \right| \leq \frac{Ce^{-2\delta m}}{n} \int_{\omega_A} \frac{|dz|}{|z - \bar{1}|^2} \leq Ce^{-2\delta m},
\]

where we have used (6.34). This and (6.35) yield (6.31).

7 Analysis of \( I_+ \) and \( I_- \)

Since the integrals \( I_+ \) and \( I_- \), we can consider \( I_+ \) only. In this case we will consider \( \{F_i\} \) of Proposition 2.2 like \( S_p(2) \) matrix which is in \( W^{-1/2} \)-neighbourhood of \( a_+ I_4 \). Then \( F_i \) can be parametrized as \( F_i = a_+ (I + i \theta_i X_i / \sqrt{W}) \), where \( X_i \) is a quaternion Hermitian matrix

\[
X_j = \left( \begin{array}{cccc}
\tilde{a}_{1j} & \tilde{w}_{1j} & 0 & \tilde{w}_{j2} \\
\tilde{w}_{1j} & \tilde{a}_{2j} & \tilde{w}_{j2} & 0 \\
0 & -\tilde{w}_{j2} & \tilde{a}_{1j} & \tilde{w}_{j1} \\
\tilde{w}_{j2} & 0 & \tilde{w}_{j1} & \tilde{a}_{2j}
\end{array} \right),
\]

where \( \tilde{w}_{j1} = (x_j + iy_j)/\sqrt{2}, \tilde{w}_{j2} = (p_j + iq_j)/\sqrt{2} \). This change transforms the measure \( dF_i \) to

\[
\frac{(ia_+ \theta_i)^6}{4} W^{-3} d\tilde{a}_1 d\tilde{a}_2 d\tilde{x}_1 d\tilde{y}_1 d\tilde{p}_1 d\tilde{q}_1.
\]

We need to keep the same \( \tilde{C}'_{n, W} \) as in [4,4], so in the parametrization above the operator \( K^+_n \) has the form

\[
K^+_n (X, X') = \beta^2 A' \left( \tilde{a}, x, y, p, q; \tilde{a}', x', y', p', q' \right) (1 + o(1)),
\]

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where

\[ A^+_n(\bar{a}, x, y, p, q; \bar{a}', x', y', p', q') = A^+(\bar{a}_1, \bar{a}_1')A^+(\bar{a}_2, \bar{a}_2')A^+(x, x')A^+(y, y')A^+(p, p')A^+(q, q'). \]

with \( A^+ \) of (6.4). Similarly to Lemma 6.1 one can get that the largest eigenvalue of \( A^+_n \) is \( \beta^2(\lambda_0^+)^n + O(W^{-1}) \) (see (6.8)), and the next eigenvalue is smaller then \( \beta^2(\lambda_0^+)^n(1 - \delta) \). Remember that we have normalization \( \lambda_0^-(K_\pm)^{-1} \), and \( \lambda_0^0(K_\pm) = \lambda_0^+ \lambda_0^- + O(1/W) \) (see Lemma 6.11). But according to (5.9) \( \beta|\lambda_0^-|^2 < 1 \), thus

\[ \|\lambda_0^-(K_\pm)^{-1}K_+\| < 1 - \delta, \]

and so

\[ I_+ = O(e^{-cn}). \]

8 Appendix A: SUSY techniques

Here we provide the basic formulas and definitions of SUSY approach used in Section 2.

Let us consider two sets of formal variables \( \{\psi_j \}_{j=1}^n; \{\bar{\psi}_j \}_{j=1}^n \), which satisfy the anticommutation conditions

\[ \psi_j \psi_k + \psi_k \psi_j = \bar{\psi}_j \bar{\psi}_k + \bar{\psi}_k \bar{\psi}_j = 0, \quad j, k = 1, \ldots, n. \quad (8.1) \]

Note that this definition implies \( \psi_j^2 = \bar{\psi}_j^2 = 0 \). These two sets of variables \( \{\psi_j \}_{j=1}^n \) and \( \{\bar{\psi}_j \}_{j=1}^n \) generate the Grassmann algebra \( \mathfrak{A} \). Taking into account that \( \psi_j^2 = 0 \), we have that all elements of \( \mathfrak{A} \) are polynomials of \( \{\psi_j \}_{j=1}^n \) and \( \{\bar{\psi}_j \}_{j=1}^n \) of degree at most one in each variable. We can also define functions of the Grassmann variables. Let \( \chi \) be an element of \( \mathfrak{A} \), i.e.

\[ \chi = a + \sum_{j=1}^n (a_j \psi_j + b_j \bar{\psi}_j) + \sum_{j \neq k} (a_{j,k} \psi_j \psi_k + b_{j,k} \psi_j \bar{\psi}_k + c_{j,k} \psi_j \bar{\psi}_k) + \ldots. \quad (8.2) \]

For any sufficiently smooth function \( f \) we define by \( f(\chi) \) the element of \( \mathfrak{A} \) obtained by substituting \( \chi - a \) in the Taylor series of \( f \) at the point \( a \). Since \( \chi \) is a polynomial of \( \{\psi_j \}_{j=1}^n; \{\bar{\psi}_j \}_{j=1}^n \) of the form (8.2), according to (8.1) there exists such \( l \) that \( (\chi - a)^l = 0 \), and hence the series terminates after a finite number of terms and so \( f(\chi) \in \mathfrak{A} \).

For example, we have

\[ \exp\{a \bar{\psi}_1 \psi_1 \} = 1 + a \bar{\psi}_1 \psi_1 + (a \bar{\psi}_1 \psi_1)^2/2 + \ldots = 1 + a \bar{\psi}_1 \psi_1, \]

\[ \exp\{a_{11} \bar{\psi}_1 \psi_1 + a_{12} \bar{\psi}_2 \psi_1 + a_{21} \bar{\psi}_1 \psi_2 + a_{22} \bar{\psi}_2 \psi_2 \} = 1 + a_{11} \bar{\psi}_1 \psi_1 \\
+ a_{12} \bar{\psi}_1 \psi_2 + a_{21} \bar{\psi}_2 \psi_1 + a_{22} \bar{\psi}_2 \psi_2 + (a_{11} \bar{\psi}_1 \psi_1 + a_{12} \bar{\psi}_2 \psi_2) \]

\[ + (a_{21} \bar{\psi}_1 \psi_2 + a_{22} \bar{\psi}_2 \psi_2)^2/2 + \ldots = 1 + a_{11} \bar{\psi}_1 \psi_1 + a_{12} \bar{\psi}_2 \psi_1 + a_{21} \bar{\psi}_2 \psi_1 \\
+ a_{22} \bar{\psi}_2 \psi_2 + (a_{11} a_{22} - a_{12} a_{21}) \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2. \quad (8.3) \]

Following Berezin [2], we define the operation of integration with respect to the anticommuting variables in a formal way:

\[ \int d\psi_j = \int d\bar{\psi}_j = 0, \quad \int \psi_j d\psi_j = \int \bar{\psi}_j d\bar{\psi}_j = 1, \quad (8.4) \]
and then extend the definition to the general element of $\mathfrak{A}$ by the linearity. A multiple integral is defined to be a repeated integral. Assume also that the “differentials” $d\psi_j$ and $d\bar{\psi}_k$ anticommute with each other and with the variables $\psi_j$ and $\bar{\psi}_k$. Thus, according to the definition, if

$$f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^{k} p_{j_1} \psi_{j_1} + \sum_{j_1 < j_2} p_{j_1,j_2} \psi_{j_1} \psi_{j_2} + \ldots + p_{1,2,\ldots,k} \psi_1 \ldots \psi_k,$$

then

$$\int f(\psi_1, \ldots, \psi_k) d\psi_k \ldots d\psi_1 = p_{1,2,\ldots,k}. \quad (8.5)$$

Let $A$ be an ordinary Hermitian matrix with positive real part. The following Gaussian integral is well-known

$$\int \exp \left\{ - \sum_{j,k=1}^{n} A_{jk} z_j \bar{z}_k \right\} \prod_{j=1}^{n} d\Re z_j d\Im z_j = \frac{1}{\det A}. \quad (8.6)$$

One of the important formulas of the Grassmann variables theory is the analog of this formula for the Grassmann algebra (see [2]):

$$\int \exp \left\{ - \sum_{j,k=1}^{n} A_{jk} \psi_j \bar{\psi}_k \right\} \prod_{j=1}^{n} d\psi_j d\bar{\psi}_j = \det A, \quad (8.7)$$

where $A$ now is any $n \times n$ matrix.

For $n = 1$ and 2 this formula follows immediately from (8.3) and (8.5).

We will also need the following proposition

**Proposition 8.1** (see [19] and references therein)

Let $\psi_j = (\psi_{j1}, \ldots, \psi_{jm})^t$, $j = 1, \ldots, p$ be the Grassman vectors, and let $F$ be some function that depends only on combinations

$$\psi^+ \psi := \left\{ \sum_{\alpha=1}^{m} \bar{\psi}_{j\alpha} \psi_{k\alpha} \right\}_{j,k=1}^{p}, \quad \psi \psi^t := \left\{ \sum_{\alpha=1}^{m} \psi_{j\alpha} \psi_{k\alpha} \right\}_{j,k=1}^{p}, \quad \psi^+ \bar{\psi} := \left\{ \sum_{\alpha=1}^{m} \bar{\psi}_{j\alpha} \bar{\psi}_{k\alpha} \right\}_{j,k=1}^{p}$$

and set

$$d\Psi = \prod_{j=1}^{p} \prod_{\alpha=1}^{m} d\bar{\psi}_{j\alpha} d\psi_{j\alpha}.$$

Assume also that $m \geq p$. Then

$$\int F \left( \begin{array}{c} \psi^+ \psi \\ \psi \psi^t \\ \psi^+ \bar{\psi} \end{array} \right) d\Phi d\Psi = C_{p,m} \int F(Q) \cdot \det^{-m/2} Q d\mu(Q),$$

where $C_{p,m}$ is some constant depending on $p$ and $m$, $Q \in \text{Sp}(p)$, and $d\mu(Q)$ is a Haar measure over $\text{Sp}(p)$. 

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Appendix B: Proof of Proposition 4.1

The first part of Proposition 4.1 follows from the standard representation theory arguments and can be found e.g. in [17], Ch.5. The recurrent relation (4.12) follows from the recurrent relation for hypergeometric functions, see e.g. [1].

Notice also that operator \( \hat{\nu} \) correspond to the multiplication on \( c(2x^2 - 1) \) with \( x = \sqrt{S(Q)} \) (see (4.8)). Thus (4.12) gives that \( \hat{\nu}\phi_0 \) is proportional to \( \phi_1 \), which implies (4.13).

To get the asymptotic expression (4.14) for the eigenvalues of \( K^* \) we need the Itzykson-Zuber formula of the integration over \( \mathcal{S}p(2) \) (for the proof see, e.g., [31])

Proposition 9.1 If \( p \neq 0 \), then

\[
\int_{\mathcal{S}p(2)} \exp\{-pS(Q(Q'))\} d\mu(Q') = \frac{6}{p^2} \left( 1 - \frac{2}{p} + e^{-p}(1 + 2/p) \right). \tag{9.1}
\]

Moreover,

\[
\int_{\mathcal{S}p(2)} \exp\{i\pi \xi - 2i\pi S(Q)\} d\mu(Q) = DS(\pi \xi). \tag{9.2}
\]

Given (4.12), it is easy to check that \( P_{2n}(0) = (-1)^n \), and the coefficient at \( x^2 \) of \( P_{2n} \) is \( (-1)^{n-1}n(n+3)/2 \). Therefore

\[
\lambda_j(t) = \frac{p^2}{6} \int_{\mathcal{S}p(2)} \exp\{-pS(Q)\}\left(1 - \frac{j(j + 3)}{2}S(Q) + \ldots\right) d\mu(Q)
\]

\[
= \frac{p^2}{6} \left(1 - \frac{2}{p} + \frac{p^2}{6} \left(1 - \frac{2}{p}\right) + \frac{6}{p^2} \cdot \frac{2}{p^2}\right) \cdot \frac{j(j + 3)}{2} + O((j^2/Wt)^2)
\]

\[
= 1 - \frac{(j + 1)(j + 2)}{Wt} + O((j^2/Wt)^2)
\]

with \( p = Wt \). Here we used \( j(j + 3) + 2 = (j + 1)(j + 2) \), (9.1), and

\[
\int_{\mathcal{S}p(2)} \exp\{-pS(Q)\}\{S(Q)\}^k d\mu(Q) = (-1)^k \left(\frac{d}{dp}\right)^k \int_{\mathcal{S}p(2)} \exp\{-pS(Q)\} d\mu(Q).
\]

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