Deconstructing Supersymmetry

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Abstract

Two supersymmetric classical mechanical systems are discussed. Concrete realizations are obtained by supposing that the dynamical variables take values in a Grassmann algebra with two generators. The equations of motion are explicitly solved. A genuine Lie group, the supergroup, generated by supersymmetries and time translations, is found to act on the space of solutions. For each system, the solutions with zero energy need to be constructed separately. For these Bogomolny-type solutions, the orbit of the supergroup is smaller than in the generic case.

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I. Introduction

Supersymmetry is one of the most powerful ideas in theoretical physics, combining bosonic and fermionic fields into a unified framework. Most supersymmetric theories are defined by a Lagrangian, from which the classical field equations are derived. However the meaning of the fermionic fields in such equations is not always clear, because they need to be anticommuting. Moreover, there are usually sources for the bosonic fields which are bilinear in the fermionic fields, and such sources are not ordinary functions. So an interpretation of the bosonic fields as ordinary functions fails.

In fact, the formalism for making sense of classical supersymmetric theories is readily available, but perhaps not sufficiently appreciated by theoretical physicists. It is the substance of the book by de Witt [1], and is also repeatedly mentioned in the earlier chapters of Freund’s book [2]. Fields in a supersymmetric field theory must take their values in a Grassmann algebra $B$. $B$ is the direct sum of an even part $B_e$ and an odd part $B_o$. The bosonic fields are valued in $B_e$, and the fermionic fields in $B_o$. It is necessary to decide which algebra $B$ to work with. $B$ can have a finite number, $n$, of generators, or an infinite number, and the content of the theory will depend on the choice. With $n$ generators, a scalar bosonic field is represented by $2^{n-1}$ ordinary functions, and by an infinite number if $B$ is infinitely generated. This is rather daunting. However, we shall choose $n = 2$ in what follows, and the resulting equations are quite manageable. (The choice $n = 1$ leads to trivial equations.)

Mechanical models, with bosonic and fermionic dynamical variables taking values in a Grassmann algebra, and depending only on time, were investigated by Casalbuoni [3] and by Berezin and Marinov [4], although not solved except in very simple cases. Supersymmetry constrains the structure of such models. We analyse two supersymmetric mechanical models below. We present the Lagrangian and equations of motion, their symmetries and the associated conserved quantities, and proceed to find the explicit form of the general solution of the equations of motion. We believe that this has not been done before. The possibility of constructing general solutions of the nonlinear coupled ODE’s shows the power of the supersymmetry of these models. From the supersymmetry algebra we construct a genuine Lie algebra of infinitesimal symmetries, which generates a genuine Lie group of symmetries of the dynamics. This group, which depends on $n$, we call the supergroup. The solutions depend on a number of constants of integration, and we comment on the extent to which the supergroup relates solutions with different values of these constants.
For each of these models, the solutions with zero energy need to be constructed independently. Here, one of the bosonic equations of motion reduces to a first-order Bogomolny-type equation \[5\]. The solution space is still acted on by the supergroup, but the orbit is of lower dimension than in the generic case. This feature of Bogomolny equations is not unfamiliar, but the complete solution of the equations of motion, including the fermionic variables, is perhaps novel.

Section II discusses the \( N = 2 \) supersymmetric mechanics of a particle moving in one dimension, subject to a potential. The model is a variant of the one whose quantized version was analysed by Witten \[6\]. Section III is concerned with the zero energy, Bogomolny case. Section IV discusses the \( N = 1 \) supersymmetric mechanics of a particle moving in one dimension. Again the model is a variant of the standard one, as the Lagrangian depends on a constant odd parameter. We conclude in Section V with some comments on the analysis, and on potential generalizations of this work.

II. \( N = 2 \) Supersymmetric Mechanics

Consider the following \( N = 2 \) supersymmetric Lagrangian \[6\], \[1, \S 5.7\]

\[
L = \frac{1}{2} \dot{x}^2 + \frac{1}{2} U(x)^2 + \frac{1}{2} \dot{\psi}_1 \psi_1 - \frac{1}{2} \dot{\psi}_2 \psi_2 + U'(x) \psi_1 \psi_2 .
\] (2.1)

This describes the supersymmetric mechanics of a particle moving in one dimension in a potential \(-U^2\). \( x(t) \) is bosonic (i.e. commuting) and \( \psi_1(t) \) and \( \psi_2(t) \) are fermionic (i.e. anticommuting) variables. Thus \( x \) is valued in \( B_e \), whereas \( \psi_1 \) and \( \psi_2 \) are valued in \( B_o \). Any function of \( x \), e.g. \( U(x) \), commutes with \( x \). Such functions are defined as polynomials or power series with real coefficients. If \( U(x) = x^p \), with \( p \) a positive integer, then \( U'(x) = px^{p-1} \), with the obvious extension to polynomials and power series. An overdot denotes the derivative with respect to time \( t \). \( \dot{x} \) commutes with \( x \), and similarly, \( \dot{\psi}_1 \) and \( \dot{\psi}_2 \) anticommute with both \( \psi_1 \) and \( \psi_2 \); hence the dynamics is classical, rather than quantized. Note that the terms \( \dot{\psi}_1 \psi_1 \) and \( \dot{\psi}_2 \psi_2 \) are not total time derivatives.

The Lagrangian \( L \) may be obtained by dimensional reduction of the 1 + 1 dimensional \( N = 1 \) supersymmetric field theory with Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_+ \Phi \partial_- \Phi - \frac{1}{2} U(\Phi)^2 + \frac{i}{2} \psi_1 \partial_- \psi_1 + \frac{i}{2} \psi_2 \partial_+ \psi_2 + i \frac{dU}{d\Phi} \psi_1 \psi_2 ,
\] (2.2)

where \( \partial_+ \) and \( \partial_- \) are the standard light cone derivatives. By assuming that all fields are independent of the spatial coordinate, then absorbing certain factors of \( \sqrt{i} \) etc. in the fields and potential, and finally writing \( \Phi \) as \( x \), we recover the expression (2.1). The density (2.2)
is real in a certain sense related to quantization, but for our purposes the manifestly real expression (2.1) is a more convenient Lagrangian to discuss.

To obtain the equations of motion we calculate the formal variation $\Delta L$ due to variations $\Delta x$, $\Delta \psi_1$ and $\Delta \psi_2$. We combine $\Delta \dot{x}$, $\Delta \dot{\psi}_1$ and $\Delta \dot{\psi}_2$ into total time derivative terms, which are ignored, then move $\Delta x$, $\Delta \psi_1$ and $\Delta \psi_2$ to the left in each term. The result is

$$\Delta L = \Delta x(-\ddot{x} + UU' + U''\psi_1\psi_2) + \Delta \psi_1(-\dot{\psi}_1 + U'\psi_2) + \Delta \psi_2(\dot{\psi}_2 - U'\psi_1),$$

(2.3)

so the equations of motion are

$$\ddot{x} = UU' + U''\psi_1\psi_2$$

(2.4a)

$$\dot{\psi}_1 = U'\psi_2$$

(2.4b)

$$\dot{\psi}_2 = U'\psi_1.$$  

(2.4c)

The Lagrangian has two supersymmetries. The first is defined by the variations

$$\delta x = \epsilon \psi_1, \quad \delta \psi_1 = \epsilon \dot{x}, \quad \delta \psi_2 = \epsilon U,$$

(2.5)

where $\epsilon$ is an arbitrary infinitesimal constant in $B_\epsilon$. It is easily shown that the variation of $L$ is a total time-derivative

$$\delta L = \epsilon \frac{d}{dt}(\frac{1}{2} \dot{x}\psi_1 + \frac{1}{2} U\psi_2)$$

(2.6)

using $\dot{U} = U'\dot{x}$. The usual Noether method gives the conserved quantity

$$Q = \dot{x}\psi_1 - U\psi_2.$$  

(2.7)

The conservation of $Q$ is easily verified using the equations of motion:

$$\dot{Q} = \ddot{x}\psi_1 + \dot{x}\dot{\psi}_1 - U'\dot{x}\psi_2 - U\dot{\psi}_2$$

$$= UU'\psi_1 + U''\psi_1\psi_2\psi_1 + \dot{x}U'\psi_2 - U'\dot{x}\psi_2 - UU'\psi_1$$

(2.8)

$$= 0$$

since $\psi_1\psi_2\psi_1 = -\psi_1\psi_1\psi_2 = 0$. The second supersymmetry is defined by the variations

$$\tilde{\delta} x = \epsilon \psi_2, \quad \tilde{\delta} \psi_2 = -\epsilon \dot{x}, \quad \tilde{\delta} \psi_1 = \epsilon U,$$

(2.9)

and leads to the conserved quantity

$$\tilde{Q} = \dot{x}\psi_2 - U\psi_1.$$  

(2.10)
The supersymmetries relate different solutions of the equations of motion. To see this, consider the linearized variations of the equations (2.4)

\[ (\ddot{\Delta} x) = (UU')' \Delta x + U''' \Delta x \psi_1 \psi_2 + U'' \Delta \psi_1 \psi_2 + U' \Delta \psi_2 \]  \hspace{2cm} (2.11a)

\[ (\dot{\Delta} \psi_1) = U'' \Delta x \psi_2 + U' \Delta \psi_2 \]  \hspace{2cm} (2.11b)

\[ (\Delta \psi_2) = U'' \Delta x \psi_1 + U' \Delta \psi_1 \]  \hspace{2cm} (2.11c)

and assume that \( x, \psi_1 \) and \( \psi_2 \) satisfy (2.4). The linear equations (2.11) are satisfied by setting \( \Delta = \delta \) or \( \Delta = \bar{\delta} \), and using the variations defined in (2.5) and (2.9). Later, we shall see more concretely, and not just in the linearized approximation, how supersymmetry relates different solutions.

Since the Lagrangian (2.1) does not depend explicitly on time, we expect a conserved energy, associated with time translation symmetry. The coefficient of the time translation is an arbitrary infinitesimal element of \( B_e \). The energy is

\[ H = \frac{1}{2} \dot{x}^2 - \frac{1}{2} U^2 - U' \psi_1 \psi_2 , \]  \hspace{2cm} (2.12)

and its conservation is easily checked using the equations of motion.

We now simplify matters, and make the model more concrete, by supposing that the Grassmann algebra \( B \) is generated by just two elements \( \alpha, \beta \) satisfying

\[ \alpha^2 = 0 \ , \ \beta^2 = 0 \ , \ \alpha \beta + \beta \alpha = 0 . \]  \hspace{2cm} (2.13)

A basis for the algebra is \( \{1, \alpha, \beta, \alpha \beta\} \), and it follows from (2.13) that \( (\alpha \beta)^2 = 0 \). There is a matrix realization of these relations, although we will not use it. Let \( \{\gamma^\mu : 1 \leq \mu \leq 4\} \) denote Dirac matrices in four Euclidean dimensions, and set \( \alpha = \gamma^1 + i \gamma^2, \beta = \gamma^3 + i \gamma^4 \).

Let us write the dynamical variables in component form as

\[ x(t) = x_0(t) + x_1(t) \alpha \beta \]  \hspace{2cm} (2.14a)

\[ \psi_1(t) = a_1(t) \alpha + b_1(t) \beta \]  \hspace{2cm} (2.14b)

\[ \psi_2(t) = a_2(t) \alpha + b_2(t) \beta \]  \hspace{2cm} (2.14c)

where \( x_0, x_1, a_1, b_1, a_2, b_2 \) are ordinary functions of time. The “body”, \( x_0(t) \), can be regarded as classical.

Any positive power of \( x \) has the expansion

\[ x^n = x_0^n + n x_0^{n-1} x_1 \alpha \beta \]  \hspace{2cm} (2.15)
which extends to an arbitrary function of $x$ as

$$U(x) = U(x_0) + U'(x_0)x_1 \alpha \beta$$  \hfill (2.16)

where $U'(x_0)$ denotes the usual derivative of $U(x_0)$ with respect to $x_0$. Henceforth, if the argument of $U$ and its derivatives is not shown, it is $x_0$, with $x_0$ itself a function of $t$. The Lagrangian is the even function $L = L_0 + L_1 \alpha \beta$, where

$$L_0 = \frac{1}{2} \dot{x}_0^2 + \frac{1}{2} U^2$$ \hfill (2.17a)

$$L_1 = \dot{x}_0 \dot{x}_1 + UU' x_1 + \dot{a}_1 b_1 - \dot{a}_2 b_2 + U'(a_1 b_2 - a_2 b_1).$$ \hfill (2.17b)

Substituting (2.14) into (2.4), we obtain the equations of motion for the components

$$\ddot{x}_0 = UU'$$ \hfill (2.18a)

$$\ddot{x}_1 = (UU')' x_1 + U''(a_1 b_2 - a_2 b_1)$$ \hfill (2.18b)

$$\dot{a}_1 = U' a_2$$ \hfill (2.18c)

$$\dot{a}_2 = U' a_1$$ \hfill (2.18d)

$$\dot{b}_1 = U' b_2$$ \hfill (2.18e)

$$\dot{b}_2 = U' b_1.$$ \hfill (2.18f)

These equations can also be derived as the variational equations of $L_0$ and $L_1$. In fact, surprisingly, they can all be derived from $L_1$ alone, as the equation of motion for $x_0$, obtained from $L_0$, is the same as the equation obtained from $L_1$ by varying $x_1$.

There are a host of symmetries and conservation laws associated with the component form of the system. Some of these relate to supersymmetry. We may define two supersymmetry variations $\delta_\alpha$ and $\delta_\beta$, associated with $\delta$. $\delta_\alpha$ is defined, following (2.5), by

$$\delta_\alpha x = \epsilon \alpha \psi_1, \quad \delta_\alpha \psi_1 = \epsilon \alpha \dot{x}, \quad \delta_\alpha \psi_2 = \epsilon \alpha U(x),$$ \hfill (2.19)

where $\epsilon$ is now infinitesimal and real, and $\delta_\beta$ similarly by replacing $\alpha$ by $\beta$. In components, the first of these variations becomes

$$\delta_\alpha (x_0 + x_1 \alpha \beta) = \epsilon b_1 \alpha \beta$$ \hfill (2.20)

so $\delta_\alpha x_0 = 0$ and $\delta_\alpha x_1 = \epsilon b_1$. Similarly, by expanding out, we find the complete set of variations

$$\delta_\alpha x_1 = \epsilon b_1, \quad \delta_\alpha a_1 = \epsilon \dot{x}_0, \quad \delta_\alpha a_2 = \epsilon U$$ \hfill (2.21a)

$$\delta_\beta x_1 = -\epsilon a_1, \quad \delta_\beta b_1 = \epsilon \dot{x}_0, \quad \delta_\beta b_2 = \epsilon U.$$ \hfill (2.21b)
with all other variations, e.g. $\delta_{\beta}a_1$, vanishing. The supersymmetry $\tilde{\delta}$ leads similarly to the two independent sets of variations

\begin{align}
\tilde{\delta}_\alpha x_1 &= \epsilon b_2 , \quad \tilde{\delta}_\alpha a_1 = -\epsilon U , \quad \tilde{\delta}_\alpha a_2 = -\epsilon \dot{x}_0 \\
\tilde{\delta}_\beta x_1 &= -\epsilon a_2 , \quad \tilde{\delta}_\beta b_1 = -\epsilon U , \quad \tilde{\delta}_\beta b_2 = -\epsilon \dot{x}_0 .
\end{align} (2.22a)

$x_0$, and hence $L_0$ is unchanged by all these variations.

It is easy to verify that all four sets of variations $\delta_\alpha, \delta_\beta, \tilde{\delta}_\alpha, \tilde{\delta}_\beta$ are Noether symmetries of the Lagrangian $L_1$, giving total time derivatives. For example

$$
\delta_\alpha L_1 = \epsilon (\dot{x}_0 b_1 + UU'b_1 + \ddot{x}_0 b_1 - U\dot{x}_0 b_2 + U'(\dot{x}_0 b_2 - U b_1))
$$

$$
= \epsilon \frac{d}{dt}(\dot{x}_0 b_1) .
$$ (2.23)

In the usual way, we obtain the conserved Noether charges

\begin{align}
Q_\alpha &= \dot{x}_0 b_1 - U b_2 \\
Q_\beta &= -\dot{x}_0 a_1 + U a_2 \\
\tilde{Q}_\alpha &= \dot{x}_0 b_2 - U b_1 \\
\tilde{Q}_\beta &= -\dot{x}_0 a_2 + U a_1 ,
\end{align} (2.24a-d)

and may verify their conservation using the equations of motion (2.18). Of course, these charges are just the components of the supersymmetry charges we found earlier, although with labels switched, namely

\begin{align}
Q &= -\tilde{Q}_\beta \alpha + Q_\alpha \beta \\
\tilde{Q} &= -\tilde{\tilde{Q}}_\beta \alpha + \tilde{Q}_\alpha \beta .
\end{align} (2.25a-b)

Both $L_0$ and $L_1$ are invariant under time translations, leading to the conservation of two energies

\begin{align}
H_0 &= \frac{1}{2} \dot{x}_0^2 - \frac{1}{2} U^2 \\
H_1 &= \dot{x}_0 \dot{x}_1 - UU' x_1 - U'(a_1 b_2 - a_2 b_1) .
\end{align} (2.26a-b)

The conserved energy we found earlier is $H = H_0 + H_1 \alpha \beta$.

There is a further symmetry, a mini-time-translation symmetry, arising from an infinitesimal time translation with coefficient proportional to $\alpha \beta$

$$
\Delta x = \epsilon \alpha \beta \dot{x} , \quad \Delta \psi_1 = \epsilon \alpha \beta \dot{\psi}_1 , \quad \Delta \psi_2 = \epsilon \alpha \beta \dot{\psi}_2 .
$$ (2.27)
Expanding out in components, we find a single nonzero variation

$$\Delta x_1 = \epsilon \dot{x}_0 .$$

The associated variation of $L_1$ is

$$\Delta L_1 = \epsilon (\dot{x}_0 \ddot{x}_0 + UU' \dot{x}_0)$$

$$= \epsilon \frac{d}{dt} \left( \frac{1}{2} \dot{x}_0^2 + \frac{1}{2} U^2 \right),$$

and the conserved quantity is

$$\frac{1}{2} \dot{x}_0^2 - \frac{1}{2} U^2 ,$$

which is $H_0$. So we see that the equations of motion and both conserved energies, and all four components of the supersymmetry charges, can be derived from $L_1$.

There are yet more symmetries which mix the functions $a_1, a_2, b_1, b_2$. The combined variations

$$\Delta a_1 = \epsilon b_1 , \quad \Delta a_2 = \epsilon b_2$$

leave $L_1$ invariant, as do the variations

$$\Delta b_1 = \epsilon a_1 , \quad \Delta b_2 = \epsilon a_2 .$$

Finally, $L_1$ is invariant under

$$\Delta a_1 = \epsilon a_2 , \quad \Delta a_2 = \epsilon a_1 , \Delta b_1 = \epsilon b_2 , \Delta b_2 = \epsilon b_1 .$$

These symmetries imply that

$$R_a = \frac{1}{2} (b_1^2 - b_2^2)$$

$$R_b = \frac{1}{2} (a_1^2 - a_2^2)$$

$$R = a_1 b_2 - a_2 b_1$$

are all conserved.

The conservation of $R$ can also be understood from the symmetry of the original Lagrangian $L$ under the infinitesimal variations

$$\Delta \psi_1 = \epsilon \psi_2 , \quad \Delta \psi_2 = \epsilon \psi_1$$

(2.35)
with $\epsilon$ real, which implies the conservation of $\psi_1\psi_2$.

We turn now to the solution of the coupled equations (2.18). We start with the equation for $x_0$. This is the classical equation of the model without fermionic variables. It has the first integral

$$\dot{x}_0^2 - U^2 = 2E ,$$  \hspace{1cm} (2.36)

where $H_0 = E$ is the conserved energy, hence

$$\dot{x}_0 = (2E + U^2)^{\frac{1}{2}} .$$  \hspace{1cm} (2.37)

The solution in integral form is

$$\int_{X_0}^{x_0} \frac{dx'_0}{(2E + U(x'_0)^2)^{\frac{1}{2}}} = t ,$$  \hspace{1cm} (2.38)

where $x_0 = X_0$ at $t = 0$.

Given $x_0(t)$, and hence $U'(x_0(t))$, we can solve the linear equations for $a_1, a_2, b_1, b_2$. Of course, one solution is that these four functions all vanish. The supersymmetry variations (2.21) and (2.22) suggest the solution

$$a_1 = \lambda \dot{x}_0 + \mu U ,$$  \hspace{1cm} (2.39a)

$$a_2 = \lambda U + \mu \dot{x}_0 ,$$  \hspace{1cm} (2.39b)

$$b_1 = \sigma \dot{x}_0 + \tau U ,$$  \hspace{1cm} (2.39c)

$$b_2 = \sigma U + \tau \dot{x}_0 ,$$  \hspace{1cm} (2.39d)

where $\lambda, \mu, \sigma, \tau$ are arbitrary constants, and $U$ denotes $U(x_0(t))$. These functions do satisfy the equations of motion, e.g.

$$\dot{a}_1 = \lambda \ddot{x}_0 + \mu U' \dot{x}_0$$

$$= \lambda UU' + \mu U' \dot{x}_0$$

$$= U' a_2 ,$$  \hspace{1cm} (2.40)

and the presence of four constants implies that (2.39) is the general solution. The value of the conserved supersymmetry charge $Q_\alpha$ is

$$Q_\alpha = \dot{x}_0 b_1 - Ub_2$$

$$= \sigma (\dot{x}_0^2 - U^2)$$

$$= 2E\sigma ,$$  \hspace{1cm} (2.41)

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and similarly \( Q_\beta = -2E\lambda, \tilde{Q}_\alpha = 2E\tau \) and \( \tilde{Q}_\beta = -2E\mu \). The \( R \) charges take the values \( R_a = E(\sigma^2 - \tau^2), R_b = E(\lambda^2 - \mu^2) \) and \( R = 2E(\lambda\tau - \mu\sigma) \). There is a problem, however, if \( E = 0 \), for then

\[
\dot{x}_0 = \pm U
\]  

(2.42)

and the expressions (2.39) depend on only two arbitrary constants. Eq.(2.42) is the Bogomolny equation for this system. We postpone discussion of the general solution in this case to the next Section.

The remaining equation for \( x_1 \) is the inhomogeneous linear equation

\[
\ddot{x}_1 = (UU')'x_1 + 2E(\lambda\tau - \mu\sigma)U'',
\]  

(2.43)

where we have substituted the conserved value of \( R = a_1b_2 - a_2b_1 \). The supersymmetry transformations and the mini-time-translation suggest that solutions can be constructed from \( U \) and \( \dot{x}_0 \). It may be verified, using (2.18a) and (2.36), that a particular integral of (2.43) is

\[
x_1 = (\lambda\tau - \mu\sigma)U.
\]  

(2.44)

A solution of the homogeneous equation \( \ddot{x}_1 = (UU')'x_1 \) is \( x_1 = C_1\dot{x}_0 \), with \( C_1 \) a constant, since \( \frac{d^2x_0}{dt^2} = \frac{d(UU')}{dt} = (UU')\dot{x}_0 \). A second solution must satisfy \( \dot{x}_1\dot{x}_0 - x_1\ddot{x}_0 = C_2 \) for some constant (Wronskian) \( C_2 \). Write \( x_1 = f(t)x_0 \). Then \( f \) must satisfy \( \dot{f} = C_2/\dot{x}_0^2 \), so

\[
\frac{df}{dx_0} = \frac{C_2}{\dot{x}_0^2} = \frac{C_2}{(2E + U^2)^{\frac{1}{2}}}.
\]  

(2.45)

and hence the second solution is

\[
x_1 = C_2(2E + U^2)^{\frac{1}{2}} \int_{X_0}^{x_0(t)} \frac{dx'_0}{(2E + U(x'_0)^2)^{\frac{1}{2}}}.
\]  

(2.46)

The complete solution of (2.43) is therefore

\[
x_1 = (\lambda\tau - \mu\sigma)U + C_1(2E + U^2)^{\frac{1}{2}} + C_2(2E + U^2)^{\frac{1}{2}} \int_{X_0}^{x_0(t)} \frac{dx'_0}{(2E + U(x'_0)^2)^{\frac{1}{2}}}.
\]  

(2.47)

The value of the energy constant \( H_1 \) is \( C_2 \).

We have therefore found the general solution of the equations of motion (2.18), in terms of eight constants of integration \( X_0, E, \lambda, \mu, \sigma, \tau, C_1, C_2 \). Our solution is incomplete, however, if \( E = 0 \).
We conclude this Section with a brief discussion of the supersymmetry algebra and how it is realized on the dynamical variables. In the model considered here there are two supersymmetry operators \( Q \) and \( \tilde{Q} \) (we use the same notation as for the associated conserved charges). Together with \( \frac{d}{dt} \) they are a basis for a super Lie algebra over the reals with nontrivial relations

\[
Q^2 = \frac{d}{dt}, \quad \tilde{Q}^2 = -\frac{d}{dt}, \quad Q\tilde{Q} + \tilde{Q}Q = 0.
\]

Formally, the algebra has a representation on the dynamical variables

\[
Qx = \psi_1, \quad Q\psi_1 = \dot{x}, \quad Q\psi_2 = U
\]

\[
\tilde{Q}x = \psi_2, \quad \tilde{Q}\psi_2 = -\dot{x}, \quad \tilde{Q}\psi_1 = -U
\]  

(2.49)

This “on shell” representation requires that the equations \( \dot{\psi}_1 = U'\psi_2 \) and \( \dot{\psi}_2 = U'\psi_1 \) are satisfied, so that, for example \( Q^2\psi_2 = QU = U'(Qx) = U'\psi_1 = \dot{\psi}_2 \). \( Q, \tilde{Q} \) and \( \frac{d}{dt} \) are all symmetries of the Lagrangian, provided \( Q \) and \( \tilde{Q} \) are treated as antiderivations (an extra minus sign in the Leibniz rule when \( Q \) or \( \tilde{Q} \) goes past a fermionic variable, e.g. \( Q(\psi_1\psi_2) = (Q\psi_1)\psi_2 - \psi_1Q\psi_2 = \dot{x}\psi_2 - \psi_1U \)). Now although the actions of \( Q \) and \( \tilde{Q} \) given by (2.49) make formal sense, they cannot be regarded as variations of the dynamical variables. A bosonic variable cannot be varied by a fermionic one. Moreover, the vague requirement that the coefficients of \( Q \) and \( \tilde{Q} \) should be anticommuting, common in the literature, is not sufficiently precise. However, requiring the coefficients to be elements of \( B_o \) is sufficiently precise, and leads to eqs.(2.5) and (2.9) as genuine variations.

The super Lie algebra over the reals becomes an ordinary Lie algebra if the coefficients lie in \( B \), with \( Q \) and \( \tilde{Q} \) having coefficients in \( B_o \) and \( \frac{d}{dt} \) having a coefficient in \( B_e \). With \( B \) generated by \( \alpha \) and \( \beta \), this real Lie algebra is six dimensional, with generators

\[
Q_\alpha, \quad Q_\beta, \quad \tilde{Q}_\alpha, \quad \tilde{Q}_\beta, \quad \frac{d}{dt}, \quad \tilde{d}.
\]  

(2.50)

where \( \frac{d}{dt} \) is the usual time derivative and \( \tilde{d} \) the mini-time-derivative. Almost all these generators commute, except that

\[
[Q_\alpha, \quad Q_\beta] = -2\tilde{d}, \quad [\tilde{Q}_\alpha, \quad \tilde{Q}_\beta] = 2\tilde{d}.
\]  

(2.51)

\( \frac{d}{dt} \) is a central element which acts in the obvious way. The action of the other generators is given by (2.21), (2.22) and (2.28). So rather than think of the super Lie algebra as
an extension of the one-dimensional Lie algebra with generator \( \frac{d}{dx} \), one may regard it as shorthand for a larger Lie algebra with a particular structure related to \( B \). There is an infinite family of ordinary Lie algebras, one for each choice of \( B \), all of which stem from the same super Lie algebra. This interpretation of a super Lie algebra as a family of ordinary Lie algebras is discussed by Freund [2].

The Lie group generated by the six elements (2.50) is the true symmetry group of our system, the supergroup. From the infinitesimal action on the constants of integration of the general solution, it is clear that the supergroup has six-dimensional orbits in the space of solutions. Only \( E \) and \( C_2 \) are invariant.

**III. Zero Energy Solutions**

When the energy \( E = 0 \), the method described above does not give the general solution of the equations of motion (2.18). For this value of \( E \)

\[
x^2_0 - U^2 = 0 ,
\]

so \( x_0 \) satisfies the first order Bogomolny equation

\[
\dot{x}_0 = \pm U .
\]

For either choice of sign, \( \dot{x}_0 \) and \( U \) are no longer independent functions of time, so the expressions (2.39) depend effectively on only two arbitrary constants, and are no longer the general solution.

For simplicity, let us choose the upper sign in (3.2). The lower sign choice is essentially the same, and corresponds to a time reversal. Then the solution of (3.2) is

\[
\int_{x_0}^{x_0} \frac{dx'_0}{U(x'_0)} = t .
\]

To find the general solution of the equations for \( a_1, a_2, b_1, b_2 \), it helps to consider the limit \( E \to 0 \) of the solution given earlier. Note that for small non-zero \( E \),

\[
\dot{x}_0 = (2E + U^2)^{\frac{1}{2}}
\]

\[
= U + \frac{E}{U} + O(E^2) .
\]

A suitable linear combination of \( \dot{x}_0 \) and \( U \) is proportional to \( \frac{1}{U} \) in the limit \( E \to 0 \). We therefore try

\[
a_1 = \lambda \frac{1}{U} + \mu U .
\]
Then
\[ \dot{a}_1 = -\frac{\lambda}{U^2} U' \dot{x}_0 + \mu U' \dot{x}_0 = U'\left(-\frac{\lambda}{U} + \mu U \right) \] (3.6)
if \( \dot{x}_0 = U \). Thus \( a_2 = -\frac{\lambda}{U} + \mu U \) gives a solution of (2.18c), and it is easily checked that (2.18d) is also satisfied. Similarly we can solve eqs.(2.18e) and (2.18f). So the general solution of eqs.(2.18c-f) is
\[ a_1 = \frac{\lambda}{U} + \mu U , \quad a_2 = -\frac{\lambda}{U} + \mu U \]
(3.7)
where \( \lambda, \mu, \sigma, \tau \) are arbitrary constants.

The constants of the motion take the following values
\[ Q_\alpha = -\bar{Q}_\alpha = 2\sigma , \quad -Q_\beta = \bar{Q}_\beta = 2\lambda , \]
\[ R_a = 2\sigma \tau , \quad R_b = 2\lambda \mu , \quad R = 2(\lambda \tau - \mu \sigma) . \] (3.8)
These values are generally nonzero because of the careful way the limit \( E \to 0 \) was taken, even though previously these quantities were proportional to \( E \).

The remaining equation for \( x_1 \) also needs special treatment. This equation is
\[ \ddot{x}_1 = (UU')' x_1 + RU''' \] (3.9)
where \( R \) is the constant given in (3.8). The previous solution had a particular integral proportional to \( U \), and one homogeneous solution proportional to \( \dot{x}_0 \). When \( E = 0 \), and \( \dot{x}_0 = U \), one homogeneous solution is still \( U \). But a new particular integral is required. Again the limiting procedure suggests that this should be proportional to \( \frac{1}{U} \), and this is correct. Finding a second homogeneous solution is as before, but with \( E = 0 \). The result is that the general solution of (3.9) is
\[ x_1 = -\frac{R}{2U} + C_1 U + C_2 U \int_{X_0}^{x_0(t)} \frac{dx_0'}{U(x_0')^3} , \] (3.10)
where \( C_1 \) and \( C_2 \) are arbitrary constants. \( H_1 = C_2 \), as before.

Note that in the zero energy, Bogomolny case, the orbits of the supergroup on the space of solutions are four-dimensional, rather than six-dimensional. Only the coefficients of \( U \) in (3.7) and (3.10) can be varied by the group action. This is consistent with the observation
that the supersymmetry generator $\delta + \tilde{\delta}$ produces no variation at all when $\dot{x}_0 = U$ and $a_1, a_2, b_1, b_2, x_1$ all vanish.

**IV. $N = 1$ Supersymmetric Mechanics**

Another example of a solvable supersymmetric mechanical model is that of a particle moving in one dimension with $N = 1$ supersymmetry (sometimes referred to as $N = \frac{1}{2}$ supersymmetry) [4,7]. The supersymmetry algebra is simply $Q^2 = \frac{d}{dt}$. The dynamical variables are a bosonic variable $x(t)$ and a single fermionic variable $\psi(t)$, taking values in $B_e$ and $B_o$ respectively. The Lagrangian is

$$L = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{\psi}\psi + \alpha U(x)\psi . \quad (4.1)$$

$\alpha$ is an odd constant, an element of $B_o$. It is necessary for $\alpha$ to be odd, and $L$ even, otherwise the equations of motion are contradictory. This model is a variant of the usual nontrivial $N = 1$ supersymmetric mechanical models. Normally, such a model has two or more fermionic variables [8]. Here, one of these is replaced by the odd constant $\alpha$.

Taking the variation of $L$, ignoring total time derivatives, and shifting the variations to the left, gives

$$\Delta L = -\Delta x(\ddot{x} - \alpha U'\psi) - \Delta \psi(\dot{\psi} + \alpha U) , \quad (4.2)$$

so the equations of motion are

$$\ddot{x} = \alpha U'\psi \quad (4.3a)$$

$$\dot{\psi} = -\alpha U . \quad (4.3b)$$

We see that both sides of eq.(4.3a) are in $B_e$, and both sides of (4.3b) in $B_o$.

The supersymmetry variations are

$$\delta x = \epsilon \psi , \quad \delta \psi = \epsilon \dot{x} , \quad (4.4)$$

where $\epsilon$ is an arbitrary infinitesimal odd constant. The corresponding variation of $L$ is

$$\delta L = \epsilon\left(\frac{1}{2} \dot{x}\dot{\psi} + \frac{1}{2} \ddot{x}\psi - \alpha U \dot{x}\right) . \quad (4.5)$$

Let us introduce $V(x)$, satisfying $V' = U$. Then we can write $\delta L$ as a total time derivative

$$\delta L = \epsilon \frac{d}{dt}\left(\frac{1}{2} \dot{x}\dot{\psi} - \alpha V\right) . \quad (4.6)$$
Hence $L$ is supersymmetric, and the conserved supersymmetry charge is

$$Q = \dot{x}\psi + \alpha V$$

Using standard arguments, we also obtain the energy

$$H = \frac{1}{2} \dot{x}^2 - \alpha U\psi.$$ (4.8)

Its conservation follows from the equations of motion, together with $\alpha^2 = 0$.

We may again obtain a concrete realization of this model by supposing that the Grassmann algebra $B$ has just two generators. Without loss of generality we may suppose that $\alpha$ is one of these generators, and that the other is $\beta$. The algebra is then identical to that in Section II. Note that if $B$ had only one generator, then $\alpha\psi$ would be zero, and the model would become trivial.

We write the component expansion of the dynamical variables as

$$x(t) = x_0(t) + x_1(t)\alpha\beta$$

$$\psi(t) = a(t)\alpha + b(t)\beta$$ (4.9)

where $x_0, x_1, a, b$ are ordinary functions. The Lagrangian has the expansion $L = L_0 + L_1\alpha\beta$, where

$$L_0 = \frac{1}{2} \dot{x}_0^2$$

$$L_1 = \dot{x}_0\dot{x}_1 + \frac{1}{2} \dot{a}b - \frac{1}{2} \dot{b}a + U(x_0)b.$$ (4.10)

The equations of motion become

$$\ddot{x}_0 = 0$$

$$\ddot{x}_1 = U'(x_0)b$$

$$\dot{a} = -U(x_0)$$

$$\dot{b} = 0.$$ (4.11)

These can be obtained as the components of eqs.(4.3). They are also the variational equations obtained from $L_1$ and $L_0$, and, as before, $L_0$ is redundant.
The equations (4.11) imply the conservation of
\[ Q_\alpha = \dot{x}_0 a + V(x_0) \]  
(4.12a)
\[ Q_\beta = \dot{x}_0 b \]  
(4.12b)
\[ H_0 = \frac{1}{2} \dot{x}^2_0 \]  
(4.12c)
\[ H_1 = \dot{x}_0 \dot{x}_1 - U(x_0)b \]  
(4.12d)

and these are the components of \( Q \) and \( H \).

It is straightforward to solve the equations (4.11), starting with
\[ x_0 = \lambda t + \mu , \ b = \nu \]  
(4.13)
where \( \lambda, \mu, \nu \) are arbitrary constants. The energy \( H_0 \) is \( \frac{1}{2} \lambda^2 \). We now regard \( Q_\alpha \) as a constant of integration, obtaining
\[ a = \frac{Q_\alpha}{\lambda} - \frac{1}{\lambda} V(\lambda t + \mu) \]  
(4.14)
as the solution of (4.11c). Finally, treating \( H_1 \) similarly, we have
\[ \dot{x}_1 = \frac{H_1}{\lambda} + \frac{\nu}{\lambda} U(\lambda t + \mu) \]  
(4.15)
so
\[ x_1 = \frac{H_1}{\lambda} t + X_1 + \frac{\nu}{\lambda^2} V(\lambda t + \mu) \]  
(4.16)
where \( X_1 \) is a constant. The general solution of the model involves six arbitrary constants \( \lambda, \mu, \nu, Q_\alpha, H_1, X_1 \).

We obtain a genuine Lie algebra of symmetries from the components of the supersymmetry transformation and time translation. Starting with \( \delta \) we obtain two independent supersymmetry variations
\[ \delta_\alpha x = \epsilon \alpha \psi , \ \delta_\alpha \psi = \epsilon \alpha \dot{x} \]  
(4.17a)
\[ \delta_\beta x = \epsilon \beta \psi , \ \delta_\beta \psi = \epsilon \beta \dot{x} \]  
(4.17b)
where \( \epsilon \) is now infinitesimal and real. Writing \( x \) and \( \psi \) in terms of components, we find
\[ \delta_\alpha x_1 = \epsilon b \ , \ \delta_\alpha a = \epsilon \dot{x}_0 \]  
(4.18a)
\[ \delta_\beta x_1 = -\epsilon a \ , \ \delta_\beta b = \epsilon \dot{x}_0 \]  
(4.18b)
with all other variations vanishing. These are symmetries of \( L_1 \), and trivially of \( L_0 \). In addition there is symmetry under an infinitesimal time translation of all the dynamical quantities. Finally, there is symmetry under the infinitesimal mini-time-translation

\[
\Delta x_1 = \epsilon \dot{x}_0 .
\]  

(4.19)

The supergroup of this system therefore has generators

\[
Q_\alpha , Q_\beta , \frac{d}{dt}, \tilde{d}
\]

(4.20)

where the action of \( Q_\alpha, Q_\beta \) is defined by (4.18), \( \frac{d}{dt} \) is the time derivative, and \( \tilde{d} \) the mini-time-derivative defined by (4.19). The only non-trivial bracket is

\[
[Q_\alpha , Q_\beta] = -2\tilde{d}.
\]  

(4.21)

Acting with the supergroup we may vary \( \mu, \nu, Q_\alpha, X_1 \), but not the constants defining the energy \( \lambda \) and \( H_1 \).

The solution as we have presented it doesn’t make sense if \( \lambda = 0 \). This is the zero energy, Bogomolny case. If \( H_0 = 0 \) then \( \dot{x}_0 = 0 \), so \( x_0 \) takes a constant value \( \mu \), hence \( U \) and \( U' \) take constant values \( U(\mu) \) and \( U'(\mu) \). The general solution is then easily found to be

\[
x_0 = \mu , b = \nu
\]

(4.22a)

\[
a = -U(\mu)(t - t_0)
\]

(4.22b)

\[
x_1 = \frac{1}{2} U'(\mu)\nu t^2 + rt + X_1
\]

(4.22c)

where \( \mu, \nu, t_0, r, X_1 \) are constants of integration. The second energy constant is \( H_1 = -U(\mu)\nu \). Supersymmetry transformations and time translations change the constants \( r, X_1 \) and \( t_0 \). However, unlike in the \( H_0 \neq 0 \) case, eq.(4.18b) implies that the value of \( b \) cannot be changed, and the orbits of the supergroup are three-dimensional rather than four-dimensional.

V. Conclusions

We have presented two supersymmetric classical mechanical models. By supposing that the dynamical variables take values in a Grassmann algebra \( B \) with two generators, we have deconstructed the models into component form and obtained equations of motion which
can be explicitly solved. These equations are the variational equations of a Lagrangian $L_1$ of non-standard form, and in each case, the “body” variable $x_0$ obeys a classical equation unaffected by the fermionic variables. A genuine Lie group, generated from the supersymmetry algebra, acts on the space of solutions.

One could ask how the solutions would look if the dynamical variables were reconstructed to be $B$-valued, or further combined into superspace dynamical variables. At first sight there is only a slight gain in elegance, but this needs more careful study. It is also of interest to know whether the equations remain solvable if $B$ is a larger algebra.

The model discussed in Section IV involved an odd constant $\alpha$. Possibly, Grassmann-valued constants are of use in other supersymmetric models. For example, it might be possible in certain “brane” models to have a non-real cosmological constant.

One of the motivations for this work was to better understand the solitons that occur in many supersymmetric field theories. These are solutions of the classical field equations, with the fermionic fields set equal to zero. They usually also satisfy first-order Bogomolny equations. It would be much more satisfactory if they could be regarded as special cases of solutions where the fermionic fields are nonzero. Our mechanical models suggest that the ”body” fields of the soliton will be unaffected by the fermionic fields. But the general solutions will involve nonzero fermionic fields coupled to the soliton, and in addition there will be nonzero bosonic fields with values in the even, non-real part of the Grassmann algebra.

The connection between the classical models discussed here and their quantized versions is also worth exploring. The Heisenberg equations of the quantized theory may be formally the same as the equations that we have solved, but $x, \dot{x}$ and $\psi, \dot{\psi}$ need to obey canonical commutation and anticommutation relations, respectively. It would be interesting to know whether the general classical solution describes a suitable limit of a quantum state.

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