ON TRIPLE PRODUCT $L$-FUNCTIONS

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Abstract. Let $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ be a unitary cuspidal automorphic representation of $GL_3^3(\mathbb{A}_F)$ where $F$ is a number field. Assume that $\pi$ is everywhere tempered. Under suitable local hypotheses, for a sufficiently large finite set of places $S$ of $F$ we prove that the triple product $L$-function $L^S(s, \pi, \otimes^3)$ admits a meromorphic continuation to $\text{Re}(s) > \frac{3}{4}$. We also give some information about the possible poles.

Contents

1. Introduction 2
1.1. Summation formulae 3
1.2. Isolating a closed orbit 5
1.3. Remarks on the main theorem 6
1.4. Outline of the paper 6

Acknowledgements 7
2. Notation and preliminaries 7
2.1. Actions 7
2.2. Quasi-characters 7
2.3. Adelic quotients 7
2.4. Measures 8
2.5. Bounding sums by integrals 8
2.6. Schwartz spaces 9
3. Groups and orbits 10
4. A functional equation 12
5. The integrals $Z(\varphi, f, s)$ 17
6. Relation to Whittaker functions 20
7. The unramified computation 26
7.1. Computation of $J_i(\ell)$ 28
7.2. Return to $Z(W, f_0, s, z)$ 29
8. Nonvanishing of zeta integrals 33

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1. Introduction

Let $F$ be a number field with ring of adeles $\mathbb{A}_F$, let $m_1, m_2, m_3$ be a triple of positive integers and let $\pi := \otimes_{i=1}^3 \pi_i$ where the $\pi_i$ are all cuspidal automorphic representations of $GL_{m_i}(\mathbb{A}_F)$. Thus $\pi$ is a cuspidal automorphic representation of $\prod_{i=1}^3 GL_{m_i}(\mathbb{A}_F)$. We denote by

$$L(s, \pi, \otimes^3) = L(s, \pi_1 \times \pi_2 \times \pi_3)$$

the corresponding triple product $L$-function. It is the Langlands $L$-function defined by the tensor product representation

$$\otimes^3 : L(GL_{m_1} \times GL_{m_2} \times GL_{m_3}) \longrightarrow GL_{m_1 m_2 m_3}(\mathbb{C}).$$

The expected analytic properties of this $L$-function are unproven except in a handful of (important) cases [Ram00, KS02]. In some sense the case $m_1 = m_2 = m_3 = 3$ is the smallest case where their properties are unknown. We obtain some new results on the analytic properties of $L(s, \pi, \otimes^3)$ in this case.

Let $S$ be a sufficiently large set of places of $F$ (see §9) and let $v_3 \not\in S$. Moreover let $S' := S \cup \{v_3\}$. Our main theorem follows:

**Theorem 1.1.** Assume $\pi_{v_3}$ satisfies assumption (A) of §9. If $\pi$ is everywhere tempered the partial $L$-function

$$L^{S'}(s, \pi, \otimes^3)$$

admits a meromorphic continuation to $\text{Re}(s) > \frac{3}{4}$.

This is proved as Theorem 9.2 below. We also give some local description of the possible poles of $L^{S'}(s, \pi, \otimes^3)$ (see Theorem 9.2). Assumption (A) is the assertion that a certain local integral depending on $s$ is not identically zero as a function of $s$.

We now make some comments (no doubt well-understood to experts) that place this result in context. Let us call the collection of $L$-functions for which the Langlands-Shahidi method
is applicable the Langlands-Shahidi list of $L$-functions. The coarse analytic properties of this set of $L$-functions are due to Langlands [Lan71]. The work was refined and completed in the decisive work of Shahidi (see [Sha10] for a discussion). The list itself is now over 50 years old. In the prior literature on $L$-functions attached to automorphic representations of general linear groups every $L$-function that has been treated appears on the Langlands-Shahidi list. Certainly other constructions via Rankin-Selberg theory (see [Cog07]) or work of Godement-Jacquet [GJ72] are important, but at least for the general linear group they do not enlarge the set of $L$-functions that are understood. Theorem 1.1 establishes for the first times the analytic properties of an $L$-function attached to automorphic representations of the general linear group that is not on the Langlands-Shahidi list.

In the theory of automorphic $L$-functions there are many constructions that cannot be generalized to higher rank because they rely on dimension coincidences that hold only in small rank. Because of this it is important to point out that the techniques in this paper should generalize to treat triple product $L$-functions for arbitrary $m_1, m_2, m_3$. We plan to investigate this in the near future. In view of the converse theory of Cogdell and Piatetski-Shapiro [CPS99] we are cautiously optimistic that this could provide a path for proving functoriality for the Langlands transfer attached to the tensor product

$$\otimes^2 : L(GL_{m_1} \times GL_{m_2}) \rightarrow L(GL_{m_1m_2}).$$

Since every representation of an algebraic group can be obtained as a subrepresentation of an iterated tensor product of a given faithful representation and its dual representation, this case of the Langlands functoriality conjecture is of crucial importance.

1.1. Summation formulae. The technique we use in this paper has its origins in the fantastic conjectures of Braverman-Kazhdan [BK00], L. Lafforgue [Laf14], Ngô [Ngô14] and Sakellaridis [Sak12]. Very briefly, these works suggest in various settings that affine spherical varieties ought to admit Schwartz spaces, Fourier transforms, and summation formulae generalizing Poisson summation. The proof of Theorem 1.1 is the first time this circle of ideas has been used to prove properties of $L$-functions that were not already known by other methods.

Let us explain how summation formulae of this type are used in the present paper. Let $U_i = G_\alpha^3$ be the standard representation of $GL_3$ and $U_i^\vee$ its dual. Let $U := \oplus_{i=1}^3 U_i$ and $U^\vee := \oplus_{i=1}^3 U_i^\vee$; these are naturally representations of $GL_3$, as is

$$V := U^\vee \oplus U.$$  

For $F$-algebras $R$ let

$$Y(R) := \{(u_i^\vee, u_i) \in U^\vee(R) \times U(R) : u_i^\vee(u_1) = u_i^\vee(u_2) = u_i^\vee(u_3)\}.$$  

Following work of the author and Liu in [GL19b] we define a Schwartz space

$$\mathcal{S}(Y(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)).$$
Let $S(X(\mathbb{A}_F))$ be the Schwartz space of the Braverman-Kazhdan space $X$ (see (10.2.1)). It comes equipped with a Fourier transform $\mathcal{F}_X$. We have a diagram

$$
\begin{array}{ccc}
S(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) & \overset{\mathcal{F}_X}{\longrightarrow} & S(Y(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))
\end{array}
$$

where the left arrow is given by $f \mapsto I(f)$ and the right arrow is given by $f \mapsto I(\mathcal{F}_X(f))$.

Here $I(f)$ is defined as in (4.0.1) and $\mathcal{F}_X$ is the Fourier transform defined in (4.0.8). For suitable test functions $f \in S(Y(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))$ we also have a summation formula

$$
\sum_{\xi \in Y^{\text{ani}}(F) \times U^\vee(F)} I(f)(\xi) = \sum_{\xi \in Y^{\text{ani}}(F) \times U^\vee(F)} I(\mathcal{F}_X(f))(\xi)
$$

(see Theorem 11.9). Here $Y^{\text{ani}} \subset Y$ is the open complement of the vanishing locus of $(u_1^\vee, u_1) \mapsto u_1^\vee(u_1)$. This summation formula was proven by the author and Liu in [GL19b] under suitable assumptions on $f$. We require slightly different assumptions in this paper and the necessary modifications in the proof are explained in §11.

The space $V \times U^\vee = U^\vee \times U \times U^\vee$ is naturally a representation of $G_m \times GL_3^\vee$ with $GL_3^\vee$ acting diagonally and $G_m$ acting by scaling on $U$ (see §3). The closed subscheme

$$
Y \times U^\vee \subset U^\vee \times U \times U^\vee
$$

is preserved by this action.

We now follow the lead of Godement and Jacquet [GJ72]. Let $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ be a cuspidal automorphic representation of $GL_3^\vee(\mathbb{A}_F)$ and let $\varphi : GL_3^\vee(\mathbb{A}_F) \to \mathbb{C}$ be a smooth vector in the space of $\pi$. For $f \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))$ we then form the integral

$$
D(\varphi, f, s) := \int_{[G_m \times GL_3^\vee]} \varphi(g) \det g^{1/2} \sum_{\xi \in Y^{\text{ani}}(F) \times U^\vee(F)} I(f)((\lambda, g)^{-1}\xi) |\lambda|^{s+3} dg d^x \lambda.
$$

(1.1.2)

Under suitable assumptions on $f$ we prove that this admits a holomorphic continuation to the $s$ plane satisfying the functional equation

$$
D(\varphi, f, s) = D(\varphi^\vee, J \circ \mathcal{F}_X(f), 1 - s)
$$

(1.1.3)

where $J$ is the involution of $V$ that “switches $U$ and $U^\vee$ (see Theorem 4.1).

If we were following the usual paradigm, this would be the point where we would unfold the integral $D(\varphi, f, s)$ into an Eulerian integral and then interpret the Eulerian integral in terms
of \(L\)-functions. However \(D(\varphi, f, s)\) is not Eulerian. It is not even Eulerian if we integrate over the obvious action of \(\mathbb{G}_m^3\) on \(U^\vee\). Geometrically, the problem is that the scheme

\[ Y \times U^\vee \]

equipped with the action of \(\mathbb{G}_m \times \text{GL}_3^3 \times \mathbb{G}_m^3\) is not spherical. Suppose we drop the \(U^\vee\) factor. Then \(Y\) is spherical under the natural action of \(\mathbb{G}_m \times \text{GL}_3^3\), but the generic stabilizer of a point is isomorphic to \(\text{GL}_3^2\). Hence if we defined an analogue of \(D(\varphi, f, s)\) with a sum over \(\xi \in Y(F)\) instead of \(Y(F) \times U^\vee(F)\) it would vanish identically by the main result of [AGR93].

1.2. **Isolating a closed orbit.** For \(F\)-algebras \(R\) let

\[ W(R) := \{(u^\vee, u, w^\vee) \in Y(R) \times U^\vee(R) : w_i^\vee(u_i) = 0 \text{ for } 1 \leq i \leq 3\}. \]

It is a closed subscheme of \(Y \times U^\vee\) that is \(\mathbb{G}_m \times \text{GL}_3^3 \times \mathbb{G}_m^3\)-invariant and spherical. To deduce information about \(L\)-functions from the functional equation for \(D(\varphi, f, s)\) we choose an appropriate test function to isolate the contribution of this orbit. Let

\[ f = f_S f^S \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) \]

be chosen so that

\[ I(f_S) \in C^\infty_c(Y^{\text{ani}}(F_S) \times U^\vee(F_S)) \]

(this is possible by Lemma 11.3). Define

\[ Z(\varphi, f, s) = \int_{[G] \times [T]} \varphi(g) |\det g|^{1/2} \sum_{\xi \in W'(F)} I(f)((\lambda, g)^{-1} \xi h^{-t}) |\lambda|^{s+3} dh^x \lambda. \]

Here \(W' \subset W\) is the open \(\mathbb{G}_m \times \text{GL}_3^3 \times \mathbb{G}_m^3\)-orbit. (see (5.0.2)). In the proof of Theorem 5.1 we show that one can choose \(\tilde{f} \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))\) such that

\[ Z(\varphi, f, s) = D(\varphi, \tilde{f}, s). \]

Now \(D(\varphi, \tilde{f}, s)\) is well-behaved analytically, and \(Z(\varphi, f, s)\) can be expanded in terms of Whittaker functions (see §6). Using this expansion and assuming \(\pi\) is everywhere tempered we show in §9 that

\[ \frac{Z(\varphi, f, s)}{Z(\varphi, \tilde{f}, s)} \]

is holomorphic for \(\text{Re}(s) > -\frac{3}{4}\). After checking the usual local nonvanishing statements (see §8) this implies the main theorem.
1.3. Remarks on the main theorem. The main theorem and its hypotheses beg several questions which we now address. The first is if it is possible to remove the local assumption (A). The ideal method to remove this assumption would be to prove a more general version of the summation formula of Theorem 11.9 that removes assumption (3) of that theorem (it would be desirable to remove assumption (1) and (2) as well). We expect that this will introduce boundary terms into the summation formula that will explain the possible poles of $L(s, \pi^S, \otimes^3)$.

Second, it would be desirable to remove the assumption that $\pi$ is tempered. As of this writing we do not know whether it is possible to do this. The combinatorics involved in computing the local zeta functions is somewhat intricate and it seemed prudent to make assumptions allowing a direct approach to our main theorem.

Finally, we prove a functional equation for $D(\varphi, \tilde{f}, s)$ sending $s$ to $1 - s$ and use it to deduce the meromorphy of $Z(\varphi, f, s)$ and then $L(s + \frac{3}{2}, \pi^S, \otimes^S)$. However this does not imply a functional equation for $Z(\varphi, f, s)$. This is why the functional equation for $D(\varphi, \tilde{f}, s)$ does not match with the expected functional equation for $L(s + \frac{3}{2}, \pi^S, \otimes^3)$ (which would send $s$ to $-s - 2$).

1.4. Outline of the paper. We start in §2 by setting notation. Some simple results on the action of $G_m \times \text{GL}_3$ on $Y$ are explained in §3. In §4 we use the summation formula for $Y(F)$ proven in §11 to prove the analytic continuation and functional equation of $D(\varphi, f, s)$. In §5 we define $Z(\varphi, f, s)$ and prove its analytic continuation. The function $Z(\varphi, f, s)$ is related to Whittaker functions in §6. We use this to relate it to an Eulerian integral, that is, an integral that factors along the places of $F$. The local factors are computed in the nonarchimedean case in §7. In §8 we prove some necessary properties of local zeta integrals at the ramified places. In §9 we prove Theorem 1.1, restated more precisely as Theorem 9.2 below.

The work in this paper relies on the theory of the Schwartz spaces of Braverman-Kazhdan spaces in the sense of [GL19a] and on the theory of Schwartz spaces attached to triples of quadratic spaces developed in [GL19b]. These are recalled in §10 and §11, respectively. We have decided to place this discussion in the last sections of the paper because it is mostly a technical refinement of the author’s previous work with Liu. There is no circularity because the contents of §10 and §11 are independent of the results of the rest of the paper. Essentially we have to refine the definition of various Schwartz spaces in the archimedean case so that they are Fréchet spaces. This allows us to prove that $C_c^\infty(Y(F_\infty)) < S(Y(F_\infty))$ (see Lemma 11.3). We then check that the Poisson summation formula still holds for the refinement.

To aid the reader we have appended a list of notation. Most notation that is only used within a particular section is not included.
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2. Notation and preliminaries

2.1. Actions. If $G$ is a group acting on the left on a set $X$, $g \in G$, and $f : X \to \mathbb{C}$, we denote by

$$L(g)f : X \to \mathbb{C}$$

the map $x \mapsto f(g^{-1}x)$. Similarly, if $G$ acts on the right, then we denote by

$$R(g)f : X \to \mathbb{C}$$

the map $x \mapsto f(xg)$.

2.2. Quasi-characters. Let $F$ be a number field. If

$$\eta : [G_{\mathbb{G}m}] \to \mathbb{C}^\times$$

is a quasi-character we let $\text{Re}(\eta)$ the unique real number such that $\eta| \cdot |^{-\text{Re}(\eta)}$ is unitary. If $a \in (\mathbb{A}_{F}^\times)^3$ and $s \in \mathbb{C}^3$ we let

$$|a|^s := |a_1|^s_1|a_2|^s_2|a_3|^s_3$$

and if

$$\chi : (\mathbb{A}_{F}^\times)^3 \to \mathbb{C}^\times$$

is a quasi-character we let $\text{Re}(\chi) \in \mathbb{R}^3$ be the unique tuple of real numbers such that $\chi| \cdot |^{-\text{Re}(\chi)}$ is unitary. Moreover we let

$$[\chi] := \chi_1\chi_2\chi_3 : \mathbb{A}_{F}^\times \to \mathbb{C}^\times.$$  

We use the obvious analogues of this notation in the local setting, that is, when $F$ is replaced by $F_v$ for some place $v$ of $F$.

2.3. Adelic quotients. For an affine algebraic group $G$ over $F$ let

$$[G] := G(F) \backslash G(\mathbb{A}_F)$$

be the adelic quotient. Let $A_G$ be the neutral component of the real points of the maximal split torus of $\text{Res}_{F/\mathbb{Q}} Z_G$ and let

$$G(\mathbb{A}_F)^1 := \bigcap_{\chi \in \text{Hom}(G, G_{\mathbb{G}m})} \ker \left( | \cdot | \circ \chi : G(\mathbb{A}_F) \to \mathbb{C}^\times \right).$$
We set
\[(2.3.3) \quad [G]^1 := G(F) \setminus G(\mathbb{A}_F)^1.\]
If $G$ is connected and reductive then $G(\mathbb{A}_F)$ is the direct product of the subgroups $G(\mathbb{A}_F)^1$ and $A_G$.

As a first application of this notation, if $\eta : [G_m] \to \mathbb{C}^\times$ is a quasi-character then there is a unique $s_\eta \in \mathbb{C}$ and character $\eta^u : A_{G_m} \setminus [G_m] \to \mathbb{C}^\times$ such that
\[\eta = | \cdot |^{s_\eta} \eta^u.\]
Similarly, any quasi-character $\chi : [G_m]^3 \to \mathbb{C}^\times$ can be written as
\[\chi = | \cdot |^{s_\chi} \chi^u\]
for a unique $s_\chi \in \mathbb{C}^3$ and $\chi^u : A_{G_m^3} \setminus [G_m]^3 \to \mathbb{C}^\times$. We say that a function of $\eta$ is holomorphic, (resp. meromorphic) if it is holomorphic (resp. meromorphic) as a function of $s_\eta$ for each $\eta^u$, and similarly for $\chi$.

2.4. Measures. The ring of integers of $F$ is denoted by $\mathcal{O}$. If $v$ is a nonarchimedean place of $F$ then $\mathcal{O}_v$ is the ring of integers of $F_v$.

We fix, once and for all, a nontrivial character $\psi : \mathbb{A}_F \to \mathbb{C}^\times$. We let $dx = \otimes_v dx_v$ be the unique Haar measure on $\mathbb{A}_F$ such that the Haar measure $dx_v$ on $F_v$ is self-dual with respect to the Fourier transform defined by $\psi_v$ for all places $v$ of $F$. In particular, if $F_v$ is absolutely unramified and $\psi_v$ is unramified for some finite place $v$ then $dx_v(\mathcal{O}_v) = 1$. We let
\[d^v x_v := \zeta_v(1) \frac{dx_v}{|x|^v}.\]

2.5. Bounding sums by integrals. A standard method from calculus is to estimate the sum over $\mathbb{Z}$ of a smooth function by a corresponding integral. Finis and Lapid [FL11] have provided an elegant generalization of this to the adelic context that will prove useful below. We recall it for the convenience of the reader. As above let $G$ be an affine algebraic group over the number field $F$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g} := \text{Lie}G(F_\infty)$ and let $K \leq G(\mathbb{A}_F^\infty)$ be a compact open subgroup. Let $\mathcal{B}_G$ be a basis for the set of elements of $\mathcal{U}(\mathfrak{g})$ of degree less than or equal to $[F : \mathbb{Q}] \dim_F G$ with respect to the usual grading. Let
\[\mathcal{C}(G(\mathbb{A}_F), K) := \{ f : G(\mathbb{A}_F)/K \to \mathbb{C} : \| f * X \|_{L^1(G(\mathbb{A}_F)/K)} < \infty \text{ for all } X \in \mathcal{U}(\mathfrak{g}) \}\]
and for $f \in \mathcal{C}(G(\mathbb{A}_F), K)$ let
\[\mu_0(f) := \sum_{X \in \mathcal{B}_G} \| f * X \|_{L^1(G(\mathbb{A}_F)/K)}.\]
The following is proved when $F = \mathbb{Q}$ in [FL11, Lemma 3.3]. The general case stated below follows by restriction of scalars:
Lemma 2.1 (Finis and Lapid). There is a \( c \in \mathbb{R}_{>0} \) such that for all \( f \in \mathcal{C}(G(A_F), K) \) and \( g \in G(A_F) \) one has

\[
\sum_{\gamma \in G(F)} |f(g\gamma)| \leq c\mu_0(f).
\]

\( \square \)

2.6. Schwartz spaces. This work involves many Schwartz spaces. Assume for the moment that \( F \) is a local field and that \( X \) and \( Y \) are quasi-affine schemes of finite type over \( F \). Let \( X^{\text{sm}} \subset X \) and \( Y^{\text{sm}} \subset Y \) be the smooth loci. Any Schwartz space \( S(X(F)) \) we discuss will be a space of functions on \( X^{\text{sm}}(F) \). Functions in the Schwartz space need not be defined on all of \( X(F) \). We will not define Schwartz spaces of general quasi-affine schemes of finite type over \( F \). In fact obtaining a good definition for general spherical varieties is an important open problem \([\text{Sak12}]\). In this subsection we explain the definition for smooth quasi-affine schemes and how to form Schwartz spaces of products \( X(F) \times Y(F) \) given that the Schwartz spaces of each factor have been defined. Most of this is fairly obvious in the nonarchimedean case but decidedly less obvious in the archimedean case.

Assume for the moment that \( F \) is nonarchimedean. Then if \( X \) is smooth we set

\[ S(X(F)) = C^\infty_c(X(F)). \]

More generally, if we have already defined \( S(X(F)) \) and \( S(Y(F)) \) (whether or not they are smooth) we set

\[ S(X(F) \times Y(F)) := S(X(F)) \otimes S(Y(F)) \]

(the algebraic tensor product).

Now assume that \( F \) an archimedean local field and that \( X \) is smooth. In this case we define \( S(X(F)) \) as in \([\text{ES18}, \text{Remark 3.2}]\). Let us briefly recall the definition. Since \( X \) is quasi-affine we can choose an embedding

\[ \text{Res}_{F/\mathbb{R}} X(F) \to \mathbb{R}^n \]

in the category of real algebraic varieties such that the image is \( X'(\mathbb{R}) \) where \( X' \subset \mathbb{G}^n \) is a closed (affine) subscheme. We refer to \([\text{BCR98, Proposition 3.2.10}]\) for the proof. Incidentally this illustrates that not all morphisms in the real algebraic category can come from morphisms of schemes, since there are many examples of quasi-affine schemes that are not affine. For each \( D \in \mathbb{C}[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}] \), viewed as an (algebraic) differential operator on \( C^\infty(\mathbb{R}^n) \), let

\[
|f|_D := \inf \left\{ \sup_{x \in X(F)} |D\tilde{f}(x)| : \tilde{f} \in S(\mathbb{R}^n) \text{ and } \tilde{f}|_{X(F)} = f \right\}.
\]

(2.6.1)

Here \( S(\mathbb{R}^n) \) is the standard Schwartz space on \( \mathbb{R}^n \). We then let

\[
S(X(F)) := \left\{ f : X(F) \to \mathbb{C} : |f|_D < \infty \text{ for all } D \in \mathbb{C}[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}] \right\}.
\]

(2.6.2)
The seminorms \(|f|_D\) give \(S(X(F))\) the structure of a Fréchet space in a natural manner. The space \(S(X(F))\) and its topology do not depend on the choice of embedding [ES18, Lemma 3.6(i)]. Now suppose that we have defined Schwartz spaces \(S(X(F))\) and \(S(Y(F))\) that are additionally Fréchet spaces for some \(X\) and \(Y\) that are quasi-affine schemes of finite type over \(F\) which may not be smooth. We then define

\[
S(X(F) \times Y(F)) = S(X(F)) \hat{\otimes} S(Y(F))
\]

where the hat denotes the (complete) projective topological tensor product. Thus we obtain another Fréchet space. We warn the reader that we do not know whether or not the Schwartz spaces are nuclear, and hence we do not know whether the other various ways of defining topological tensor products coincide with this one. We also warn the reader that in [ES18] there is a definition of a Schwartz space for any quasi-affine scheme of finite type over the real numbers (not necessarily smooth). In the smooth case their definition coincides with ours. In the nonsmooth case it does not, essentially because functions in our Schwartz space need not extend to the singular set.

Finally we discuss the adelic setting. Assume \(X\) and \(Y\) are quasi-affine schemes of finite type over the number field \(F\). Then \(X(\mathbb{A}_F^\infty)\) and \(Y(\mathbb{A}_F^\infty)\) are defined as topological spaces [Con12]. If \(X\) is smooth we define \(S(X(\mathbb{A}_F^\infty)) := C^\infty_c(X(\mathbb{A}_F^\infty))\) (i.e. locally constant functions of compact support). More generally if the Schwartz spaces \(S(X(\mathbb{A}_F^\infty))\) and \(S(Y(\mathbb{A}_F^\infty))\) have been defined then

\[
S(X(A_F^\infty) \times Y(A_F^\infty)) := S(X(A_F^\infty)) \otimes S(Y(A_F^\infty))
\]

(algebraic tensor product). If \(S(X(F_v))\) has already been defined for all \(v|\infty\) then

\[
S(X(F_\infty)) := \hat{\otimes}_{v|\infty} S(X(F_v))
\]

where the product is the (completed) projective topological tensor product, and

\[
S(X(\mathbb{A}_F)) := S(X(F_\infty)) \otimes S(X(\mathbb{A}_F^\infty))
\]

(algebraic tensor product).

3. Groups and orbits

In this section \(F\) is an arbitrary field of characteristic zero and constructions are understood to take place over \(F\). The symbol \(R\) denotes an \(F\)-algebra. Let

\[(3.0.1) \quad G := \mathbb{G}_m \times \text{GL}_3^3\]

and let \(\lambda : G \to \mathbb{G}_m\) be the projection to the first factor. Let \(U_i = \mathbb{G}_a^3\) and let \(U_i^\vee := \text{Hom}(\mathbb{G}_a^3, \mathbb{G}_a)\), both equipped with the natural action of \(\text{GL}_3\). Let \(U := \prod_{i=1}^3 U_i\) and \(U^\vee := \prod_{i=1}^3 U_i^\vee\). In computations we will identify the space of \(U_i^\vee\) with \(U_i\) via the pairing

\[
U_i(R) \times U_i(R) \rightarrow R
\]
\[(u, v) \mapsto u^t v.\]

For example we often write the action of \(GL_3(R)\) on \(U_i^\vee(R)\) as \(g^{-t}u^\vee\). Let \(V_i := U_i^\vee \times U_i\) and
\[ V := V_1 \times V_2 \times V_3. \]

Let
\[(3.0.2) \quad Y(R) := \{(u^\vee, u) \in U^\vee(R) \times U(R) : u^\vee_1(u) = u^\vee_2(u_2) = u^\vee_3(u_3)\}. \]

This scheme is preserved by the action of \(G\). Let \(Y^{\text{sm}} \subset Y\) be the smooth locus and let
\[(3.0.3) \quad Y^{\text{ani}} \subset Y \]
be the open complement of the vanishing locus of \((u^\vee, u) \mapsto u^\vee_i(u_i)\) (this is independent of \(i\)). Thus one has \(Y^{\text{ani}} \subset Y^{\text{sm}}\) and both subschemes are \(G\)-invariant.

For \(F\)-algebras \(R\) let
\[(3.0.4) \quad T(R) := \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \in (R^\times)^3 \right\}. \]

There is a left action of \(G \times T\) on \(U^\vee \times U \times U^\vee\):
\[(3.0.5) \quad G(R) \times T(R) \times U^\vee(R) \times U(R) \times U^\vee(R) \rightarrow U^\vee(R) \times U(R) \times U^\vee(R) \]
\[(\lambda, g, m) \cdot (u^\vee, u, w^\vee) \mapsto (g^{-t}u^\vee, \lambda^{-1}gu, g^{-t}w^\vee)m^{-t} \]

where \(m^{-t}\) acts via matrix multiplication. This action preserves \(Y \times U^\vee\).

Let us describe the orbits for this action. For \(\beta \in F\) let
\[(3.0.6) \quad \xi_\infty = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right), \quad \xi_\beta = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \]

all viewed as elements of \(V_i(F) \times U_i^\vee(F)\). For \(\beta \in (F \otimes \infty)^3\) let
\[(3.0.7) \quad \xi_\beta = (\xi_{\beta_1}, \xi_{\beta_2}, \xi_{\beta_3}) \in V(F) \times U^\vee(F) = \prod_{i=1}^3 V_i(F) \times U_i^\vee(F).\]

Let
\[ C_{\xi_\infty}(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, 1 : (a, b) \in \text{GL}_2(R) \right\}, \]
\[ C_{\xi_\beta}(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, c^{-1} : a, c \in R^\times, b \in R \right\} \]

and for \(\beta \in F^\times\) let
\[ C_{\xi_\beta}(R) = \left\{ \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}, 1 : a \in R^\times, b \in R \right\} \]

all viewed as subgroups of \(GL_3(R) \times R^\times\). For \(\beta \in (F \otimes \infty)^3\) we define \(C_{\xi_\beta}\) to be the product of the groups corresponding to the three factors of \(\xi_\beta\).

Let \(U_i^{\text{vo}} \subset U_i^\vee\) be the open complement of the origin and let \(U^{\text{vo}} := U_1^{\text{vo}} \times U_2^{\text{vo}} \times U_3^{\text{vo}}\).
Lemma 3.1. The set 
\[ \{ \xi_\beta : \beta \in (F \ll \infty)^2 \} \]
is a minimal complete set of representatives for the orbits of \( G(F) \times T(F) \) on \( Y^{ani}(F) \times U^{\psi_0}(F) \). The stabilizer of \( \xi_\beta \) is \( C_{\xi_\beta} \).

Proof. Every element in \( Y^{ani}(F) \) is in the \( G(F) \)-orbit of 
\[ \xi'_0 := ((e_2^\vee, e_2), (e_2^\vee, e_2), (e_2^\vee, e_2)) . \]
where \( e_2 = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) is the usual elementary column vector. The stabilizer of \( \xi'_0 \) is equal to
\[ 1 \times \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL_2^3(F) \right\} . \]
The proof is easy to deduce from these observations. \( \Box \)

For our later use we observe the following:

Lemma 3.2. The orbit \( O(\xi_0) \) of \( \xi_0 := \xi_{0,0,0} \) under \( G \times T \) is a closed subscheme of \( Y^{ani} \times U^{\psi_0} \).

Proof. The orbit is the intersection of the vanishing loci of the functions
\[ (u_i^\vee, u_i, w_i^\vee) \mapsto w_i^\vee(u_i) . \]
\( \Box \)

4. A functional equation

Let \( \pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \) be a cuspidal automorphic representation of \( GL_3^3(\mathbb{A}_F) \) with central character \( \omega = \omega_1 \otimes \omega_2 \otimes \omega_3 \). Let \( \varphi : [GL_3^3] \to \mathbb{C} \) be a smooth function in its space. Let \( S(X(\mathbb{A}_F)) \) be the Schwartz space of the compactification \( X \) of the Braverman-Kazhdan space \( X_F := [P, P] \backslash Sp_6 \) (see §10). For
\[ f_1 \otimes f_2 \otimes f_3 \in S(X(\mathbb{A}_F)) \otimes S(V(\mathbb{A}_F)) \otimes S(U^{\vee}(\mathbb{A}_F)) \]
and \( (\xi_1, \xi_2) \in Y^{sm}(\mathbb{A}_F) \times U^{\psi}(\mathbb{A}_F) \) we let
\[ I(f_1 \otimes f_2 \otimes f_3)(\xi_1, \xi_2) := \int_{N_0(\mathbb{A}_F) \backslash SL_2^3(\mathbb{A}_F)} f_1(\gamma_0 g) \rho(g) f_2(\xi_1) \rho(g) f_3(\xi_2) . \]
Here \( \gamma_0 \in X_F(F) \) is the representative for the open \( SL_2^3 \)-orbit given by (11.0.3) and
\[ N_0(R) := \{ ((1_{t_1}^{t_1}), (1_{t_2}^{t_2}), (1_{-t_1}^{-t_2})) : t_1, t_2 \in R \} \]
as in (11.0.4). For more details we refer to §11. This definition extends to yield a surjection
\[ I : S(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^{\vee}(\mathbb{A}_F)) \to S(Y(\mathbb{A}_F) \times U^{\vee}(\mathbb{A}_F)) . \]

Let \( f \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^{\psi}(\mathbb{A}_F)) \). Define
\[ D(\varphi, f, s) := \int_{[G]} \varphi(g) |\det g|^{s/2} \sum_{\xi \in Y^{ani}(F) \times U^{\psi_0}(F)} I(f)((\lambda, g)^{-1}\xi) |\lambda|^{s+3} dg \times \lambda . \]
that is, under suitable assumptions on \( f \) and \( \psi \) together with the fixed additive character \( \psi \) defined by (4.0.5) we will prove in this section that \( D(\varphi, f, s) \) admits a holomorphic continuation to \( (s_\omega, s) \in \mathbb{C}^4 \) that admits a functional equation.

For each \( i \) let \( O_{V_i} \) be the orthogonal group of \( V_i \) and let \( J \in \prod_{i=1}^3 O_{V_i}(\mathbb{Z}) \) be the element that is \( \left( I_3 \right) \) in all factors. We then have operators

\[
J : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))
\]

(4.0.5)

\[
f(x, u^\vee, u, w) \mapsto f(x, u, u^\vee, w)
\]

and

\[
J : \mathcal{S}(Y(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) \longrightarrow \mathcal{S}(Y(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))
\]

(4.0.6)

\[
f(u^\vee, u, w) \mapsto f(u, u^\vee, w)
\]

that satisfy the intertwining property

\[
I(J(f)) = J(I(f))
\]

since \( \rho \) is the restriction to \( \text{SL}_2(\mathbb{A}_F)^3 \) of the Weil representation of \( \prod_{i=1}^3 (\text{SL}_2(\mathbb{A}_F) \times O_{V_i}(\mathbb{A}_F)). \)

The pairing

\[
U^\vee_j(R) \times U^\vee_j(R) \longrightarrow R
\]

\[(u, v) \mapsto u^v v\]

together with the fixed additive character \( \psi \) induce a Fourier transform

\[
\mathcal{F}_{U^\vee_j} : \mathcal{S}(U^\vee_j(\mathbb{A}_F)) \longrightarrow \mathcal{S}(U^\vee_j(\mathbb{A}_F)).
\]

We let

\[
\mathcal{F}_{U^\vee} := \otimes_{i=1}^3 \mathcal{F}_{U^\vee_i}.
\]

We also denote by

\[
\mathcal{F}_{U^\vee} : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))
\]

the induced automorphism. We then have an automorphism

\[
\mathcal{F}_{X \times U^\vee} : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) \longrightarrow \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))
\]

(4.0.8)

\[
= \mathcal{F}_X \circ \mathcal{F}_{U^\vee} = \mathcal{F}_{U^\vee} \circ \mathcal{F}_X
\]

where \( \mathcal{F}_X \) acts via its action on \( \mathcal{S}(X(\mathbb{A}_F)) \).

We now prepare local assumptions so that we can state and prove a functional equation for \( D(\varphi, f, s) \). Assume that there are places \( v_1, v_2, v_3 \) of \( F \) (not necessarily distinct) with \( v_1, v_2 \) finite such that

\[
f = f_{v_1} f_{v_2} f_{v_3} f^{v_1 v_2 v_3}
\]

where \( f_{v_i} = f_{v_1} \otimes f_{v_2} \otimes f_{v_3} \) with

\[
(f_{v_1}, f_{v_2}, f_{v_3}) \in \mathcal{S}(X(F_{v_i})) \times \mathcal{S}(V(F_{v_i})) \times \mathcal{S}(U^\vee(F_{v_i})).
\]

Assume moreover that

(1) \( f_{v_1} \in C_c^\infty(X(F_{v_1})) \),
(2) $\mathcal{F}_0(x_{v_2}) \in C^\infty_c(X(F_{v_2}))$,
(3) for all $g \in \text{SL}_2(F_{v_3})$ one has $\text{supp}(\rho(g)f_2) \cap V_0(F_{v_3}) = \emptyset$

with $V_0$ defined as in (11.2.1). These are the same three assumptions as in Theorem 11.9.

**Theorem 4.1.** Under the assumptions (1-3) above the integral defining $D(\varphi, f, s)$ is absolutely convergent provided that $\text{Re}(s) \gg 1$ and $\text{Re}(s) \gg -\text{Re}(\omega_i)$. The function $D(\varphi, f, s)$ admits a holomorphic continuation to the $(s, s_\omega)$-plane satisfying

$$D(\varphi, f, s) = D(\varphi^\vee, J \circ \mathcal{F}_X \times U^\vee(f), 1 - s)$$

where $\varphi^\vee(g) := \varphi(g^{-t})$.

Our argument proving this theorem is roughly the same as that appearing in Tate’s thesis. We begin by proving convergence.

**Proposition 4.2.** For $f \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))$ the integral

$$\int_{[G]} |\varphi(g)| \det g^{|1/2} \sum_{\xi \in Y^\text{ani}(F) \times U^\vee(F)} |I(f)((\lambda, g)^{-1}\xi)||\lambda|^{\text{Re}(s)+3} dg \lambda$$

converges provided that $\text{Re}(s) \gg 1$ and $\text{Re}(s) \gg -\text{Re}(\omega_i)$ for all $i$.

**Proof.** The integral (4.0.9) can be written as

$$\int_{[G]} |\varphi(g)| \int_{[G_m] \times [G_m]} \sum_{\xi \in Y^\text{ani}(F) \times U^\vee(F)} |I(f)((\lambda, zg)^{-1}\xi)||\det zg|^{1/2} |\omega(z)||\lambda|^{\text{Re}(s)+3} dz \lambda dg$$

where the sum is over $\xi \in Y^\text{ani}(F) \times U^\vee(F)$. The inner integral here is

$$\int_{[G_m] \times [G_m]} |I(f)((\lambda, zg)^{-1}\xi)||\det zg|^{1/2} |\omega(z)||\lambda|^{\text{Re}(s)+3} dz \lambda.$$

It therefore suffices to show that (4.0.10) is of moderate growth on $GL_3^3(\mathbb{A}_F)$ (transforming under the center via $|\omega^{-1}|$) provided that $\text{Re}(s) \gg 1$ and $\text{Re}(s) \gg -\text{Re}(\omega_i)$ for all $i$.

By (11.1.3) there is an $f' \in \mathcal{S}(V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))$ such that

$$|I(f)(u^\vee, u, w^\vee)| \leq f'(u^\vee, u, w^\vee) \prod_{i=1}^3 \max\{|u_i^\vee|, |u_i|\}^{-4} \leq f'(u^\vee, u, w^\vee) \prod_{i=1}^3 |u_i^\vee|^{-4}$$

for $(u^\vee, u, w^\vee) \in Y^\text{ani}(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)$. To ease notation write

$$|u^\vee| := \prod_{i=1}^3 |u_i^\vee|.$$

Thus we are reduced to showing that

$$\left(4.0.11\right) \int_{[G_m] \times [G_m]} \sum_{\xi} f'((\lambda, zg)^{-1}(u^\vee, u, w^\vee)) \left|\det zg\right|^{1/2} |\omega(z)||\lambda|^{\text{Re}(s)+3} dz \lambda |u^\vee|^4$$
is of moderate growth as a function of \( g \) for any \( f' \) as above. Here the sum is over \( (u^\vee, u, w^\vee) \in Y^{\text{ani}}(F) \times U^{\vee}(F) \). Applying Lemma 2.1 to \( U^\vee \) we see that it suffices to show that

\[
\int_{[G_m] \times [G_m]} \sum_{(u^\vee, u) \in Y^{\text{ani}}(F)} \Psi \left((\lambda, zg)^{-1}(u^\vee, u)\right) \frac{|\omega(z)||\lambda_t^{\text{Re}(s)+1}d^\times zd^\times t|}{|zu^\vee v||\det zg|^{1/2}}
\]

is of moderate growth for all \( \Psi \in \mathcal{S}(V(A_F)) \).

Let \( V'_i \subset V_i \) be the open complement of the vanishing locus of the canonical quadratic form \( Q_i \). We will relate the sum above to a product of sums over the \( V_i \) in order to proceed. Let \( \Delta : \mathbb{G}_m \to \mathbb{G}_m^3 \) be the diagonal embedding. For any \( \Psi \) as above consider the function \( \Phi : (A_F^\times)^3/\Delta(A_F^\times) \times GL_3^\times(A_F) \to \mathbb{C} \)

given by

\[
\Phi(t, g) = \int_{[G_m] \times [G_m]} \sum_{(u^\vee, u) \in Y^{\text{ani}}(F)} \Psi(zg^t u^\vee, \lambda t z^{-1} g^{-1} u) \frac{|\lambda_t^{\text{Re}(s)+1}d^\times \lambda d^\times z|}{|zu^\vee v||\det zg|^{1/2}}.
\]

We are to show that \( \Phi(1, g) \) is of moderate growth for all such \( \Psi \).

By Lemma 2.1 applied to the group \( \mathbb{G}_m^3/\Delta(\mathbb{G}_m) \) it suffices to show for all \( \Psi \) as above

\[
\int_{(A_F^\times)^3/\Delta(A_F^\times)} \Phi(t, g) d^\times t
\]

\[
= \int_{(A_F^\times)^3/\Delta(A_F^\times)} \int_{[G_m] \times [G_m]} \sum_{(u^\vee, u) \in Y^{\text{ani}}(F)} \Psi(zg^t u^\vee, \lambda t z^{-1} g^{-1} u) \frac{|\omega(z)||\lambda_t^{\text{Re}(s)+1}d^\times \lambda d^\times t|}{|zu^\vee v||\det zg|^{1/2}}
\]

is of moderate growth as a function of \( GL_3^\times(A_F) \) (transforming under the center via \( |\omega|^{-1} \)). Assume that \( \Psi = \Psi_1 \otimes \Psi_2 \otimes \Psi_3 \). Then the above is the product over \( 1 \leq i \leq 3 \) of

\[
\int_{[G_m] \times [G_m]} \sum_{(u^\vee, u) \in V'_i(F)} \Psi_i(zg^t u^\vee, \lambda z^{-1} g^{-1} u) \frac{|\lambda z|^{\text{Re}(s)+1}d^\times \lambda d^\times z}{|zu^\vee v||\det zg|^{1/2}}
\]

so it suffices to show that this expression is of moderate growth as a function of \( GL_3(A_F) \) (transforming under the center via \( |\omega_i|^{-1} \)). Changing variables \( \lambda \mapsto \lambda z \) this is

\[
\int_{[G_m] \times [G_m]} \sum_{(u^\vee, u) \in V'_i(F)} \Psi_i(zg^t u^\vee, \lambda g^{-1} u) \frac{|\lambda z|^{\text{Re}(s)+1}d^\times \lambda d^\times z}{|zu^\vee v||\det zg|^{1/2}}.
\]

We can and do assume that \( \Psi_i \) is nonnegative. Then the above is dominated by the restriction to the subgroup

\[
\{(g^{-1}, g) \in GL_3(A_F)\}
\]

of the following function of \( (g_1, g_2) \in GL_3(A_F)^2 \):

\[
\int_{[G_m] \times [G_m]} \sum_{(u^\vee, u) \in (F^3 \setminus \{0\})^2} \Psi_i(zg_1^{-1} u^\vee, \lambda g_2^{-1} u) \frac{|\lambda z|^{\text{Re}(s)+1}d^\times \lambda d^\times z}{|zu^\vee v||\det zg_1^{-1}|^{1/2}}.
\]
This is a degenerate Eisenstein series on \( GL_3(\mathbb{A}_F)^2 \). It is easy to check that it is absolutely convergent provided that \( \text{Re}(s) \gg 1 \) and \( \text{Re}(s) \gg -\text{Re}(\omega_i) \) for all \( i \). Since Eisenstein series are of moderate growth we deduce the proposition. A reference for the moderate growth of these sorts of Eisenstein series (with sections normalized differently) is [GJ72, Lemmas 11.5-11.6].

\[ \text{Theorem 4.3.} \] Let \( (\lambda, g) \in G(\mathbb{A}_F) \). Under the assumptions of Theorem 4.1 on \( f \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U(\mathbb{A}_F)) \) one has

\[
\sum_{(\xi) \in Y^\text{ani}(F) \times U^\vee(F)} L(\lambda, g) I(f)(\xi) = \sum_{\xi \in Y^\text{ani}(F) \times U^\vee(F)} \frac{L(\lambda^{-1}, g^{-1} t) I(\mathcal{F}_{X \times U^\vee}(f))(\xi)}{|\det g||\lambda|^7}.
\]

The sums are absolutely convergent.

\[ \text{Proof.} \] By the Poisson summation formula on \( Y \) of Theorem 11.9 and [GL19b, Lemma 4.3] we have

\[
\sum_{(u^\vee, u, w^\vee) \in Y^\text{ani}(F) \times U^\vee(F)} I(f)(g^t u^\vee, \lambda g^{-1} u, g^t w^\vee)
= \sum_{(u^\vee, u, w^\vee) \in Y^\text{ani}(F) \times U^\vee(F)} \frac{I(\mathcal{F}_X(f))(\lambda^{-1} g^t u^\vee, g^{-1} u, g^t w^\vee)}{|\lambda|^7}.
\]

We now apply the Poisson summation formula on \( U^\vee \) with respect to the Fourier transform (4.0.7) to see that this is

\[
\sum_{(u^\vee, u, w^\vee) \in Y^\text{ani}(F) \times U^\vee(F)} \frac{I(\mathcal{F}_{X \times U^\vee}(f))(\lambda^{-1} g^t u^\vee, g^{-1} u, g^{-1} w^\vee)}{|\det g||\lambda|^7}.
\]

\[ \square \]

\[ \text{Proof of Theorem 4.1.} \] In the proof we identify

\[
\mathbb{R}_{\geq 0} \rightarrow A_{G_m}
\]

\[
t \mapsto t
\]

in such a way that the Haar measure on \( A_{G_m} \) corresponds to \( \frac{dt}{t} \) on \( \mathbb{R}_{>0} \) and \( |t| = t \).

We study

\[
D(\varphi, f, s)
\]

\[
= \int_0^1 \int_{F^\times \backslash (\mathbb{A}_F)^3 \times [GL_3]} \varphi(g)|\det g|^{1/2} \sum_{\xi \in Y^\text{ani}(F) \times U^\vee(F)} I(f)((t\lambda, g)^{-1}\xi)t^{s+3}dg d\lambda \frac{dt}{t}
+ \int_1^{\infty} \int_{F^\times \backslash (\mathbb{A}_F)^3 \times [GL_3]} \varphi(g)|\det g|^{1/2} \sum_{\xi \in Y^\text{ani}(U^\vee(F))} I(f)((t\lambda, g)^{-1}\xi)t^{s+3}dg d\lambda \frac{dt}{t}
\]
The second integral converges for all $s$ because it converges for $\text{Re}(s)$ sufficiently large by Proposition 4.2. Consider the first integral. We claim that it is

\begin{equation}
\int_0^1 \int_{F^\times \setminus (A_3^\infty)^1 \times [GL_3^2]} \varphi(g) \mid \det g \mid^{1/2} \sum_{\xi \in Y^\text{ani}(F) \times U^\vee(F)} I(f)((t\lambda, g)^{-1}\xi)t^{s+3}dg d\lambda \frac{dt}{t}.
\end{equation}

Indeed, the stabilizer in $GL_3^2$ of any $\xi \in Y^\text{ani}(F) \times U^\vee(F) - Y^\text{ani}(F) \times U^\text{vo}(F)$ contains an appropriately embedded copy of $GL_2$ (see the proof of Lemma 3.1). It follows from the main result of [AGR93] that these contributions vanish, proving the claim.

By Theorem 4.3 the integral (4.0.13) is equal to

\begin{equation}
\int_0^1 \int_{F^\times \setminus (A_3^\infty)^1 \times [GL_3^2]} \varphi(g) \mid \det g \mid^{-1/2} \sum_{\xi \in Y^\text{ani} \times U^\vee(F)} I(J \circ \mathcal{F}_{X \times U^\vee}(f))((t\lambda)^{-1}, g^{-t})^{-1}\xi)t^{s-4}dg d\lambda \frac{dt}{t}.
\end{equation}

Again the contribution of $\xi \in Y^\text{ani}(F) \times U^\vee(F) - Y^\text{ani}(F) \times U^\text{vo}(F)$ vanishes, so taking a change of variables $(\lambda, t, g) \rightarrow (\lambda^{-1}, t^{-1}, g^{-t})$ this is equal to

\begin{equation}
\int_1^\infty \int_{F^\times \setminus (A_3^\infty)^1 \times [GL_3^2]} \varphi^\vee(g) \mid \det g \mid^{1/2} \sum_{\xi \in Y^\text{ani} \times U^\vee(F)} I(J \circ \mathcal{F}_{X \times U^\vee}(f))((t\lambda, g)^{-1}\xi)t^{4-s}dg d\lambda \frac{dt}{t}.
\end{equation}

This again converges for all $s$ because it converges for $\text{Re}(s)$ sufficiently large by Proposition 4.2.

Thus we have shown that $D(\varphi, f, s)$ is equal to

\begin{equation}
\int_1^\infty \int_{F^\times \setminus (A_3^\infty)^1 \times [GL_3^2]} \varphi^\vee(g) \mid \det g \mid^{1/2} \sum_{\xi \in Y^\text{ani} \times U^\vee(F)} I(J \circ \mathcal{F}_{X \times U^\vee}(f))((t\lambda, g)^{-1}\xi)t^{4-s}dg d\lambda \frac{dt}{t} + \int_1^\infty \int_{F^\times \setminus (A_3^\infty)^1 \times [GL_3^2]} \varphi(g) \mid \det g \mid^{1/2} \sum_{\xi \in Y^\text{ani} \times U^\text{vo}(F)} I(f)((t\lambda, g)^{-1}\xi)t^{s+3}dg d\lambda \frac{dt}{t}.
\end{equation}

It is clear that (4.0.14) is holomorphic as a function of $s$. To prove the functional equation we observe that (4.0.14) is invariant under

\begin{equation}
(\varphi, f, s) \mapsto (\varphi^\vee, J \circ \mathcal{F}_{X \times U^\vee}(f), 1 - s).
\end{equation}

\[\square\]

5. The integrals $Z(\varphi, f, s)$

Let

\begin{equation}
W(R) := \{(u^\vee, u, w^\vee) \in Y(R) \times U^\vee(R) : w^\vee_i(u_i) = 0 \text{ for } i = 1, 2, 3\}
\end{equation}

and let

\begin{equation}
W^\text{ani} := W \cap Y^\text{ani} \times U^\vee,
\end{equation}

\begin{equation}
W^\prime := W \cap Y^\text{ani} \times U^\text{vo}.
\end{equation}
By the proof of Lemma 3.2 we have \( O(\xi_0) = W' \) and hence \( W \) is the closure of \( O(\xi_0) \) in \( Y \times U^\vee \).

For \( f \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F)) \) we form the zeta integral

\[
Z(\varphi, f, s) := \int_{[G]\times[T]} \varphi(g) |\det g|^{1/2} \sum_{\xi \in W'(F)} I(f)((\lambda, g)^{-1}\xi h)|\lambda|^{s+3}dh^\times \lambda dg
\]

where \( T \) is defined as in (3.0.4).

We will study this integral under appropriate assumptions on \( f \) which we now state. Let \( S \) be a finite set of places of \( F \) containing the set of infinite places such that \( \mathcal{O}^S \) has class number 1. Here \( \mathcal{O}^S \subset F \) is the ring of \( S \) integers. We assume that there are places \( v_1, v_2, v_3 \notin S \) such that \( f = f_{v_1} f_{v_2} f_{v_3} f^{Sl_3(v_1, v_2, v_3)} \) where \( f_{v_i} = f_{1v_i} \otimes f_{2v_i} \otimes f_{3v_i} \) with

\[
(f_{1v_i}, f_{2v_i}, f_{3v_i}) \in \mathcal{S}(X(F_{v_i})) \times \mathcal{S}(V(F_{v_i})) \times \mathcal{S}(U^\vee(F_{v_i}))
\]

and

\[
(1) I(f_{S}) \in C_{c}^{\infty}(Y_{\text{ani}}(F_{S}) \times U^{\text{vo}}(F_{S})),
\]

\[
(2) f_{1v_1} \in C_{c}^{\infty}(X_{P}(F_{v_1})),
\]

\[
(3) F_{X}(f_{1v_2}) \in C_{c}^{\infty}(X_{P}(F_{v_2})),
\]

\[
(4) \text{for all } g \in SL_3(F_{v_3}) \text{ one has supp}(\rho(g)f_2) \cap V_0(F_{v_3}) = \emptyset.
\]

Here \( V_0 \) is defined as in (11.2.1). We do not require the \( v_i \) to be distinct.

**Theorem 5.1.** Under the assumptions on \( f \) given above the integral \( Z(\varphi, f, s) \) converges absolutely for \( \Re(s) \) sufficiently large in a sense depending on \( \omega \). It admits a holomorphic continuation to \( (s, \omega), s \in \mathbb{C} \times \mathbb{C} \).

**Proof.** To prove this we will show \( Z(\varphi, f, s) = D(\varphi, \bar{f}, s) \) for a well-chosen test function \( \bar{f} \).

Since we have assumed that \( \mathcal{O}^S \) has class number 1 there is a compact measurable set \( \mathfrak{F}_S \subset T(F_3) \) such that \( \mathfrak{F}_S T(\hat{\mathcal{O}}^S) \) is a fundamental domain for \( T(F) \) acting on \( T(\mathbb{A}_F)^1 \). Thus

\[
Z(\varphi, f, s) = \int_{[G]} \varphi(g) |\det g|^{1/2} \sum_{\xi \in W'(F)} \int_{\mathfrak{F}_S T(\hat{\mathcal{O}}^S)} I(f)((\lambda, g)^{-1}\xi h) dh |\lambda|^{s+3}d^\times \lambda dg.
\]

By Lemma 11.5 there is an \( \alpha \in F^\times \) such that for all

\[
(u^\vee, u, u^\vee) \in \text{supp} \left( \int_{\mathcal{F}_S T(\hat{\mathcal{O}}^S)} R(h)I(f)dh \right) \cap Y_{\text{ani}}(F) \times U^\vee(F)
\]

one has

\[
u_i^\vee(u_i), w_i^\vee(u_i) \in \alpha \mathcal{O}^S
\]

for all \( i \) (in fact \( u_i^\vee(u_i) \) is independent of \( i \) by definition of \( Y \)). On the other hand since \( \int_{\mathcal{F}_S} R(h)I(f_S)dh \in C_{c}^{\infty}(Y_{\text{ani}}(F_{S}) \times U^{\text{vo}}(F_{S})) \) we have \( u_i^\vee(u_i) \neq 0 \) for all \( i \) and there is an \( A > 1 \) depending only on \( f \) and \( \mathfrak{F}_S \) such that \( A > |u_i^\vee(u_i)|_v \) for all \( v \in S \) and all \( i \). As \( \alpha \mathcal{O}^S \) is discrete in \( F_3 \) this implies that the \( u_i^\vee(u_i) \) lie in a finite subset of \( F^\times \) depending only on \( f \).
and $\mathfrak{F}_S$. Since $w^\vee(u_i) \in \alpha \mathcal{O}^S$ we conclude that there is a $\beta \in F^\times$ depending only on $f$ and $\mathfrak{F}_S$ such that
\[
\frac{w^\vee(u)}{u^\vee(u)} := \left(\frac{w^\vee_1(u_1)}{u^\vee_1(u_1)}, \frac{w^\vee_2(u_2)}{u^\vee_2(u_2)}, \frac{w^\vee_3(u_3)}{u^\vee_3(u_3)}\right) \in (\beta \mathcal{O}^S)^3.
\]
As the function
\[
Y^{\text{ani}}(\mathfrak{A}_F) \times U^\vee(\mathfrak{A}_F) \longrightarrow \mathfrak{A}_F^3
\]
(5.0.4)
\[
(u^\vee, u, w^\vee) \longmapsto \frac{w^\vee(u)}{u^\vee(u)}
\]
is invariant under the action of $G(\mathfrak{A}_F)$ we deduce that for all $(\lambda, g) \in G(\mathfrak{A}_F)$ and
\[
(u^\vee, u, w^\vee) \in \text{supp} \left( \int_{\mathcal{F}_s T(\mathfrak{O}^S)} L(\lambda, g) R(h) I(f) dh \right) \cap Y^{\text{ani}}(F) \times U^\vee(F)
\]
one has
\[
\frac{w^\vee(u)}{u^\vee(u)} \in (\beta \mathcal{O}^S)^3.
\]
Now choose $\mathcal{V} \in C_c(\mathfrak{A}_F^3)$ so that
(5.0.5)
\[
\mathcal{V}(0) = 1 \quad \text{and} \quad \text{supp}\mathcal{V} \cap (\beta \mathcal{O}^S)^3 = \{(0, 0, 0)\}.
\]
For $(u^\vee, u, w^\vee) \in Y^{\text{ani}}(\mathfrak{A}_F) \times U^\vee(\mathfrak{A}_F)$ let
\[
I_{\mathcal{V}}(f)(u^\vee, u, w^\vee) := \mathcal{V}\left(\frac{w^\vee(u)}{u^\vee(u)}\right) \int_{\mathcal{F}_s T(\mathfrak{O}^S)} R(h) I(f)(u^\vee, u, w^\vee) dh.
\]
Thus
\[
I_{\mathcal{V}}(f) \in C_c(\mathcal{F}_s(F) \times U^\vee(F)) \otimes \mathcal{S}\left(Y(\mathfrak{A}_F^S) \times U^\vee(\mathfrak{A}_F^S)\right) < \mathcal{S}(Y(\mathfrak{A}_F) \times U^\vee(\mathfrak{A}_F)).
\]
Since the function (5.0.4) is invariant under $G(\mathfrak{A}_F)$ we deduce that
\[
L(\lambda, g) I_{\mathcal{V}}(f)(\xi) = \begin{cases} L(\lambda, g) \int_{\mathcal{F}_s T(\mathfrak{O}^S)} R(h^{-1}) I(f)(\xi) dh & \text{if } \xi \in W'(F) \\ 0 & \text{if } \xi \in Y^{\text{ani}}(F) \times U^\vee(F) - W'(F) \end{cases}
\]
for all $(\lambda, g) \in G(\mathfrak{A}_F)$. Thus
\[
Z(\varphi, f, s) = \int_{[G]} \varphi(g) | \det g |^{1/2} \sum_{\xi \in Y^{\text{ani}}(F) \times U^\vee(F)} I_{\mathcal{V}}(f) ((\lambda, g)^{-1} \xi) |\lambda|^{s+3d^x} \lambda d^g.
\]
Here the sum is over $Y^{\text{ani}}(F) \times U^\vee(F)$, not just $W'(F)$.

Let
\[
\tilde{F}^S(u^\vee, u, w^\vee) := \int_{T(\mathfrak{O}^S)} f^S((u^\vee, u, w^\vee) h) dh.
\]
Using Lemma 11.3 choose a $\tilde{f}_S \in \mathcal{S}(X(F_S) \times V(F_S) \times U^\vee(F_S))$ such that (setting $\tilde{f} := \tilde{f}_S \tilde{F}^S$) one has $I(\tilde{f}) = I_{\mathcal{V}}(f)$. Then
\[
Z(\varphi, f, s) = \int_{[G]} \varphi(g) | \det g |^{1/2} \sum_{\xi \in Y^{\text{ani}}(F) \times U^\vee(F)} I(\tilde{f}) ((\lambda, g)^{-1} \xi) |\lambda|^{s+3d^x} \lambda d^g.
It is clear that the assumptions (1-4) placed on \( f \) before the proof remain valid for \( \tilde{f} \). Invoking Theorem 4.1 we deduce the theorem. \[\square\]

6. Relation to Whittaker functions

In this section we expand the integral \( Z(\varphi, f, s) \) defined as in (5.0.3) in terms of Whittaker functions. We take \( \text{Re}(s) \) sufficiently large in a sense depending on \( \omega \) (the central character of \( \pi \)) to justify our first manipulations using Proposition 4.2.

Unfolding and using Lemma 3.1 we obtain

\[
Z(\varphi, f, s) = \int_{[G] \times [T]^1} \varphi(g) \det g^{1/2} \sum_{\xi \in W'(F)} I(f)((\lambda, g)^{-1}\xi m)|\lambda|^{s+3}dmd^{\lambda} \lambda dg
\]

\[
= \int_{Q(F) \backslash G(\mathbb{A}_F) \times T(\mathbb{A}_F)^1} \varphi(g) \det g^{1/2} I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3}d^{\lambda} \lambda dg dm.
\]

Here

\[
(6.0.1) \quad \xi_0 := \xi_{0,0,0} \quad \text{and} \quad Q := C_{\xi_0}
\]

in the notation of (3.0.7). Let

\[
(6.0.2) \quad T_Q, N_Q \leq Q
\]

be the maximal torus of diagonal matrices and the unipotent radical of \( Q \), respectively. We place a measure on \( N_Q(\mathbb{A}_F) = \prod'_v N_Q(F_v) \) via the obvious isomorphism

\[
F_v^3 \xrightarrow{\sim} N_Q(F_v)
\]

\[
x \mapsto \begin{pmatrix} 1 & x \\ \bar{x} & 1 \end{pmatrix}.
\]

Our fixed nontrivial character \( \psi : F \backslash \mathbb{A}_F \to \mathbb{C}_\times \) induces a generic character of \( N_3^3(\mathbb{A}_F) \) via

\[
\psi : N_3^3(\mathbb{A}_F) \to \mathbb{C}_\times
\]

\[
\begin{pmatrix} 1 & \star \\ \bar{1} & y \end{pmatrix} \mapsto \prod_{i=1}^3 \psi(x_i + y_i).
\]

Let

\[
W^\varphi(g) := W^\varphi_\psi(g) := \int_{[N_3^3]} \varphi(n g) \overline{\psi}(n) dn
\]

be the associated Whittaker function. Here we choose the unique measure on \([N_3^3]\) such that

\[
\varphi(g) = \sum_{\gamma \in N_3^3(F) \backslash GL_3(F)} W^\varphi((\gamma \gamma_1)g).
\]
Proposition 6.1. The zeta integral \( Z(\varphi, f, s) \) is equal to the absolutely convergent integral
\[
\int_{N_Q(A_F) \backslash G(A_F) \times T(A_F)^1} W^\varphi(g) |\det g|^{1/2} I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3} d^\times \lambda dgdm
\]
provided that
\[
\begin{align*}
\operatorname{Re}(s) & \gg_\pi 1 \\
\operatorname{Re}(\omega_i) & \gg_\pi 1 \text{ for } 1 \leq i \leq 3.
\end{align*}
\]

The dependence on \( \pi \) in the bounds on \( \operatorname{Re}(s) \) and \( \operatorname{Re}(\omega_i) \) comes (unsurprisingly) from gauge estimates on \( W^\varphi \).

**Proof.** Replacing \( \varphi \) by its Whittaker expansion we have
\[
\int_{Q(F) \backslash G(A_F) \times T(A_F)^1} \varphi(g) |\det g|^{1/2} I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3} d^\times \lambda dgdm
\]
\[
= \int_{Q(F) \backslash G(A_F) \times T(A_F)^1} \sum_{\gamma \in N_Q^2(F) \backslash \GL_2(F)} |\det g|^{1/2} W^\varphi ((\gamma_1) g) I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3} d^\times \lambda dgdm
\]
\[
= \int_{T_Q(F) \backslash N_Q(A_F) \backslash G(A_F) \times T(A_F)^1} \left( \int_{[N_Q]} \sum_{\gamma \in N_Q^2(F) \backslash \GL_2(F)} |\det g|^{1/2} W^\varphi ((\gamma_1) ng) dn \right)
\]
\[
\times I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3} d^\times \lambda dgdm.
\]

By the Bruhat decomposition one has
\[
N_2(F) \backslash \GL_2(F) = T_2(F) \prod (1^1) B_2(F).
\]

Due to the integral over \([N_Q]\) the only \( \gamma \) that can contribute a nonzero summand are in \( T^3_2(F) \). Thus we can rewrite the integral above as
\[
\int_{T_Q(F) \backslash N_Q(A_F) \backslash G(A_F) \times T(A_F)^1} \sum_{\gamma \in T^3_2(F)} |\det g|^{1/2} W^\varphi ((\gamma_1) g) I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3} d^\times \lambda dgdm.
\]

Here we have used the fact that \( \text{meas}([N_Q]) = 1 \) because the measure on it is induced by a self-dual Haar measure on \( A_F^3 \). In the considerations above one can justify bringing the integral over \([N_Q]\) inside the sum over \( \gamma \) using the fact that the Whittaker expansion of a cusp form is absolutely convergent, uniformly on compact subsets of \([\GL_3^3]\). Since \( W^\varphi \) is left \( Z_{\GL_3}(F) \)-invariant we can rewrite this as
\[
\int_{N_Q(A_F) \backslash G(A_F) \times T(A_F)^1} |\det g|^{1/2} W^\varphi (g) I(f)((\lambda, g)^{-1}\xi_0 m)|\lambda|^{s+3} d^\times \lambda dgdm.
\]

provided this latter expression converges absolutely.
We now check this absolute convergence. Write \( K = \prod_{i=1}^{3} K_i \) where \( K_i = K_{\infty} \text{GL}_3(\hat{O}) \) with \( K_{\infty} \leq \text{GL}_3(F_{\infty}) \) a maximal compact subgroup. We decompose the Haar measure on \( \text{GL}_3(F) \) along the Iwasawa decomposition. To ease notation write
\[
d(a, c, \lambda, k, x, y) := d^x a_1 d^x a_2 d^x a_3 d^x c d^x \lambda d k d x d y.
\]

Then
\[
(6.0.6)
\]
\[
Z(\varphi, f, s)
= \int W^\varphi \left( \left( \begin{array}{c} x \\ a_2 \\ a_3 \end{array} \right) \right) k I(f) \left( \left( \begin{array}{c} \lambda \\ \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right) \end{array} \right) k, \right) -1 \xi_0 \left( \begin{array}{c} 1 \\ c \end{array} \right) |\lambda|^s \frac{|a_3|^{3/2} |\lambda|^3 d^x a}{|a_1|^{3/2} |a_2|^{1/2}} d(a, c, \lambda, k, x, y)
= \int W^\varphi \left( \left( \begin{array}{c} x \\ a_2 \\ a_3 \end{array} \right) \right) k I(f) \left( k^{-1} \left( \left( \begin{array}{c} 0 \\ a_2 \\ 0 \end{array} \right), \left( \begin{array}{c} -x \\ \lambda \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ a_3 c \end{array} \right) \right) \right) |\lambda|^s \frac{|a_3|^{3/2} |\lambda|^3 d^x a}{|a_1|^{3/2} |a_2|^{1/2}} d(a, c, \lambda, k, x, y).
\]

Here all integrals are over
\[
(a, c, \lambda, k, x, y) \in ((A_F^\times)^3) \times ((A_F^\times)^3) \times K \times A_F^3 \times A_F^3.
\]

We change variables
\[
x \mapsto \frac{a_1 a_2 x}{\lambda}
\]
to see that this is
\[
\int W^\varphi \left( \left( \begin{array}{c} a_1 a_2 x / \lambda \\ a_2 \\ a_3 \end{array} \right) \right) k I(f) \left( k^{-1} \left( \left( \begin{array}{c} 0 \\ a_2 \\ 0 \end{array} \right), \left( \begin{array}{c} -x \\ \lambda a_2^{-1} \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ a_3 c \end{array} \right) \right) \right) |\lambda|^s \frac{|a_3|^{3/2} |a_2|^{1/2} d(a, c, \lambda, k, x, y)}{|a_1|^{1/2} |a_3|^{3/2}}
\]

Taking a change of variables \((a_1, a_3) \mapsto (a_1 a_2, a_2 a_3^{-1})\) we arrive at
\[
(6.0.7)
\]
\[
Z(\varphi, f, s)
= \int W^\varphi \left( \left( \begin{array}{c} a_1 a_2 x / \lambda \\ a_2 \\ a_3 a_2^{-1} \end{array} \right) \right) k I(f) \left( k^{-1} \left( \left( \begin{array}{c} 0 \\ a_2 \\ 0 \end{array} \right), \left( \begin{array}{c} -x \\ \lambda a_2^{-1} \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ a_3 c \end{array} \right) \right) \right) |\lambda|^s \frac{|a_2|^{3/2} d(a, c, \lambda, k, x, y)}{|a_1|^{1/2} |a_3|^{3/2}}.
\]

To prove this converges we now replace \( I(f) \) and \( W \) by \(|I(f)|\) and \(|W|\). Using standard gauge estimates on Whittaker functions ([JPSS79a, Proposition 2.3.6 and 2.4.1] and [JPSS79b, Lemma 8.3.3]) we see that the integral is dominated by a finite sum of functions of the form
\[
\int f_0(a_1, a_3) |a_1 a_3|^{-1} |a_1 a_2^2 a_3^{-1}|^{\Re(\omega) / 3} 
\times h \left( \left( \begin{array}{c} 0 \\ a_2 \\ 0 \end{array} \right), \left( \begin{array}{c} -x \\ \lambda a_2^{-1} \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ a_3 c \end{array} \right) \right) |\lambda|^{\Re(s)} \frac{|a_2|^{3/2} d(a, c, \lambda, k, x, y)}{|a_1|^{1/2} |a_3|^{3/2}}
\]
where \( f_0 \in \mathcal{S} \left( ((A_F^\times)^3)^3 \right) \) is a nonnegative Schwartz function, \( t \in \mathbb{R}_>^3 \), and
\[
h(\xi) := \int_K |I(f)(k^{-1} \xi)| d k.
\]
The \( t_i \) do not depend on \( \omega_i \).
By well-known properties of Tate integrals provided that

\[(6.0.8) \quad \text{Re}(\omega_i)/3 > t_i + 3/2\]

the integral over \(a_1\) converges and hence the above is bounded by

\[
\int f_1 (a_3 |a_2|^\text{Re}(\omega)3/2 - f_1 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ \frac{-x}{a_2} \\ 0 \end{array} \right) \right) g_2 \left( \frac{a_2c}{a_3} \right) \left| \lambda \right|^{\text{Re}(s)} \frac{d\lambda^x a_2 d\lambda^x a_3 d\lambda^x d\lambda dx dy}
\]

for some \(f_1 \in \mathcal{S}(A_F^3)\). Here the integral is over

\[(a_2, a_3, c, \lambda, x, y) \in (A_F^3)^3 \times (A_F^\times)^3 \times (A_F^\times)^3 \times A_F^3 \times A_F^3 \times A_F^3\]

(the symbol \((A_F^\times)^1\) denotes the group of norm 1 ideles). By \((11.1.3)\) there is a nonnegative smooth function \(f_1 \in \mathcal{S}(V(A_F))\) and a nonnegative \(f_2 \in \mathcal{S}(U^\vee(A_F))\) such that the above is bounded by

\[
\int f_2 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ \frac{-x}{a_2} \\ 0 \end{array} \right) \right) f_2 \left( \frac{a_2c}{a_3} \right) \left| \lambda \right|^{\text{Re}(s)} \frac{d\lambda^x a_2 d\lambda^x a_3 d\lambda^x d\lambda dx dy}
\]

where the domain of the integral is as before.

By the same argument used in the proof of Proposition 4.2 to prove that the above is convergent it suffices to prove that the following integral is convergent for all

\[(g_0, g_1, g_2) \in \mathcal{S}(A_F) \times \mathcal{S}(V_1(A_F)) \times \mathcal{S}(A_F),\]

and \(1 \leq i \leq 3:\)

\[
\int g_0 (a_3) |a_2|^{\text{Re}(\omega) - 5/2} - g_1 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ \frac{-x}{a_2} \\ 0 \end{array} \right) \right) g_2 \left( \frac{a_2c}{a_3} \right) \left| \lambda \right|^{\text{Re}(s)} \frac{d\lambda^x a_2 d\lambda^x a_3 d\lambda^x d\lambda dx dy}
\]

\[
= \int g_0 (a_3) |a_2|^{\text{Re}(\omega) - 5/2 + \text{Re}(s)} - g_1 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ \frac{-x}{a_2} \\ 0 \end{array} \right) \right) g_2 \left( \frac{a_2c}{a_3} \right) \left| \lambda \right|^{\text{Re}(s)} \frac{d\lambda^x a_2 d\lambda^x a_3 d\lambda^x d\lambda dx dy}
\]

Here the integral is over

\[(a_2, a_3, c, z, x, y) \in (A_F^\times)^2 \times (A_F^\times)^1 \times A_F^\times \times A_F^\times \times A_F^\times.
\]

Write the integral over \((A_F^\times)^1\) as an integral over the compact set \(F^\times \setminus (A_F^\times)^1\) and a sum over \(F^\times\). Applying Lemma 2.1 (in the case \(G = G_m\)), to prove that the integral above is convergent it suffices to show that for all \(g_0, g_1, g_2\) as above the integral

\[
\int g_0 (a_3) |a_2|^{2\text{Re}(\omega)3/2 - 11/2 + \text{Re}(s)} - t_1 |a_3|^{3/2} \frac{g_1 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ \frac{-x}{a_2} \\ 0 \end{array} \right) \right) \left| \lambda \right|^{\text{Re}(s)} \frac{d\lambda^x a_2 d\lambda^x a_3 d\lambda^x d\lambda dx dy}
\]

\[
= \int g_0 (a_3) |a_2|^{\text{Re}(\omega)3/2 - t_1 |a_3|^{3/2} g_1 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ \frac{-x}{a_2} \\ 0 \end{array} \right) \right) \left| \lambda \right|^{\text{Re}(s)} \frac{d\lambda^x a_2 d\lambda^x a_3 d\lambda^x d\lambda dx dy}
\]

converges absolutely. Here the integral is over

\[(a_2, a_3, c, z, x, y) \in (A_F^\times)^2 \times A_F^\times \times A_F^\times \times A_F^\times \times A_F^\times.
\]
The exponent of $3 + t_i + \text{Re}(\omega_i)/3$ in $|c|^{3+t_i+\text{Re}(\omega_i)/3}$ is somewhat arbitrary, any exponent larger than $3/2 + t_i + \text{Re}(\omega_i)/3$ would suffice. Since we have already assumed (6.0.8), it is not hard to see using known properties of Tate integrals that this integral converges provided that $\text{Re}(s) > 1$ and

$$2\text{Re}(\omega_i)/3 - 11/2 + \text{Re}(s) - t_i > 1.$$  

This implies the proposition. □

Define

$$Z(\varphi, f, s, z) := \int_{NQ(\mathbb{A}_F) \setminus G(\mathbb{A}_F) \times T(\mathbb{A}_F)} W^\varphi(g) \det g|^{1/2} I(f)((\lambda, g)^{-1}\xi_0 m)^{\det m|z|^{s+3d} d^x \lambda d\gamma dm}.$$  

Proposition 6.2. The integral $Z(\varphi, f, s, z)$ is absolutely convergent and

$$Z(\varphi, f, s) = \frac{1}{(2\pi i)^3} \int_{\text{Re}(z) = \sigma} Z(\varphi, f, s, z) dz$$

provided that

$$\text{Re}(s) \gg \pi 1,$$

$$\text{Re}(\omega_i) \gg \pi 1 \text{ for all } i,$$

$$\sigma_i \gg \pi \text{ Re}(\omega_i) \text{ for all } i,$$

$$\text{Re}(s) \gg \pi \sigma_i + \text{Re}(\omega_i) \text{ for all } i.$$  

Proof. Arguing as in the proof of Proposition 6.1, to show that $Z(\varphi, f, s, z)$ is absolutely convergent it suffices to show that for all $i$

$$\int g_0(a_3) \left| a_2 \right|^{\text{Re}(\omega_i) - 5/2 + s - z_i} g_1 \left( \left( \begin{array}{c} 0 \\ a_2 \\ y \end{array} \right), \left( \begin{array}{c} -x \\ 0 \end{array} \right) \right) g_2 \left( \left( \begin{array}{c} a_2 c \\ a_3 \\ z \end{array} \right), \left( \begin{array}{c} -z \\ 0 \end{array} \right) \right) |z|^s |c|^z d^x a_2^x d^x a_3 d^x d^x z dxdy$$

is absolutely convergent for all $g_0, g_2 \in S(\mathbb{A}_F), g_1 \in S(V_i(\mathbb{A}_F))$, where the integral is over

$$(a_2, a_3, c, z, x, y) \in (\mathbb{A}_F^\times)^2 \times \mathbb{A}_F^\times \times \mathbb{A}_F^\times \times \mathbb{A}_F \times \mathbb{A}_F.$$  

Using the standard range of absolutely convergence of Tate integrals we see that (6.0.13) is absolutely convergent provided that

$$\text{Re}(s) > 1$$

$$\text{Re}(z_i) > 1$$

$$-\text{Re}(\omega)/3 - t_i - 3/2 + z_i > 1$$

$$\text{Re}(\omega) - 5/2 + \text{Re}(s) - t_i > 1.$$
This proves the absolute convergence of the integral defining \( Z(\varphi, f, s, z) \) in the desired range.

Now consider the function
\[
A_T \rightarrow \mathbb{C}
\]
\[
x \mapsto \int_{N_Q(\mathbb{A}_F)\backslash G(\mathbb{A}_F) \times T(\mathbb{A}_F)^\dagger} W^\varphi(g) |\det g|^{1/2} I(f)((\lambda, g)^{-1}\xi_0m) \frac{|\det m|^2|\lambda|^{s+3}d\lambda d\sigma}{dn}.
\]
Choosing an isomorphism \( A_T \cong \mathbb{R}_{>0} \) we can consider the differential operator \( x \frac{d}{dx} \) applied to this function. Applying this differential operator has the effect of replacing \( f \) by another appropriate element in \( \mathcal{S}(Y(\mathbb{A}_F) \times U^\dagger(\mathbb{A}_F)) \). With this in mind the absolute convergence of \( Z(\varphi, f, s, z) \) (for all \( f \)) together with integration by parts (in the setting of \([\text{GL19a}, \text{Lemma} 5.8]\)) implies that the integral in (6.0.11) is absolutely convergent.

Given these absolute convergence statements the identity (6.0.11) follows from Mellin inversion.

The integral \( Z(\varphi, f, s, z) \) is Eulerian. In more detail, for every place \( v \) of \( F \) let \( \mathcal{W}(\pi_v, \psi_v) \) be the local Whittaker model of \( \pi_v \) with respect to \( \psi_v \) and assume that \( W^\varphi := \prod_v W_v \) with \( W_v \in \mathcal{W}(\pi_v, \psi_v) \). We assume that \( W_v(I) = 1 \) for almost all \( v \). Assume moreover that \( f = \otimes_v f_v \) is a pure tensor.

For \( z \in \mathbb{C} \) let
\[
(6.0.14) \quad Z(W_v, f_v, \eta_v, z) = \int_{N_Q(\mathbb{F}_v)\backslash G(\mathbb{F}_v)} W_v(g) I(f)((\lambda, g)^{-1}\xi_0m) \frac{|\det m|^2|\lambda|^{s+3}d\lambda d\sigma}{dn}
\]
whenever the integral is absolutely convergent. Then
\[
(6.0.15) \quad Z(\varphi, f, s, z) = \prod_v Z(W_v, f_v, \eta_v, z)
\]
provided that the bounds (6.0.12) on \( s, \omega, z \) are valid. We also have the following:

**Lemma 6.3.** Assuming the bounds on \( s, \omega, \) and \( z \) in (6.0.12) the integral \( Z(W_v, f_v, s, z) \) is equal to
\[
(6.0.16) \quad \int W_v \left( \left( \begin{array}{cc} a_1a_2 & \frac{a_1a_2^2\lambda}{a_2} \\ \frac{a_2}{a_2a_3} & 1 \end{array} \right) \frac{y}{a_2a_3} \right) k \right) I(f_v) \left( k^{-1} \left( \begin{array}{cc} 0 & a_2^x \\ -a_2^x & 0 \end{array} \right) , \left( \begin{array}{cc} 0 & \frac{a_2^x}{a_3} \\ \frac{a_2^x}{a_3} & 0 \end{array} \right) \right) \right) |c|^z|\lambda|^{s+3}d(a, c, \lambda, x, y) \frac{d(a, c, \lambda, x, y)dk}{|a_1|^{3/2}|a_2|^{3/2}|a_3|^{3/2}}
\]
where
\[
d(a, c, \lambda, x, y) = d^x a_1 d^x a_2 d^x a_3 d^x \lambda dxdy
\]
and the integral is over \( (a_1, a_2, a_3, c, \lambda, x, y) \in ((F_v^x)^3)^3 \times (F_v^x)^3 \times F_v^x \times (F_v^x)^3 \times K_v \).

**Proof.** The identity, assuming absolute convergence, is proven via the same computation used to prove (6.0.7). The absolute convergence statement is part of Proposition 6.2. \( \square \)
7. The unramified computation

Let $v$ be a nonarchimedean place of $F$ which we omit from notation, writing $F := F_v$. We assume $F$ is absolutely unramified and that $\pi$ is unramified, and that $W \in \mathcal{W}(\pi, \psi)^{GL_2(\mathcal{O})}$ is the unique unramified Whittaker functional satisfying $W(I) = 1$. We also assume that the residual characteristic is not 3 and that $\psi$ is unramified.

For $\alpha_0 \in \mathbb{Z}$ let

\begin{equation}
(7.0.1) \quad f_{\alpha_0} := 1_{\geq \alpha_0} \otimes 1_{V(\mathcal{O})} \otimes 1_{U^\vee(\mathcal{O})} \in S(X(F) \times V(F) \times U^\vee(F))
\end{equation}

where $1_{\geq \alpha_0}$ is defined as in (10.1.12). We compute

\[ Z(W, f_{\alpha_0}, s, z) \]

in this section. The manipulations below are all justified provided that the inequalities in (6.0.12) are valid by Lemma 6.3. We will always assume these inequalities hold in this section. Our basic expression for $Z(W, f_{\alpha_0}, s, z)$ (in the case $\alpha_0 = 0$) is Theorem 7.3 below. We explain in (7.0.2) below that the general case reduces to this case.

By lemmas 6.3 and 11.10,

\[
Z(W, f_{\alpha_0}, s, z) = q^{-\alpha_0} \int W \left( a_1 a_2, a_1 a_2 x/\lambda, y \right) \int 1_{\mathcal{O}} \left( \frac{\lambda}{\varpi^{2a_0} b_1 b_2 b_3} \right) 1_{b\varpi^{\alpha_0} V(\mathcal{O})} \left( \begin{pmatrix} 0 & -x \\ \lambda a_2 \end{pmatrix} \right) |b\varpi^{\alpha_0}|^{-2}
\times 1_{\mathcal{O}^3} \left( \frac{a_2 c}{a_3} \right) |c|^2 |\lambda|^s d^3 b |a_2|^{3/2} d(a, c, x, y) / |a_1|^{1/2} |a_3|^{3/2}
\]

where the integral over $b$ is over $b \in \mathcal{O}^3$ such that

\[ \max(|b_1^{-1} b_2 b_3|, |b_1 b_2 b_3^2|, |b_1 b_2 b_3^3|) \leq 1. \]

We now change variables $(a_2, \lambda, x, y) \mapsto (\varpi^{a_0} b a_2, \varpi^{a_0} \lambda, \varpi^{a_0} b x, \varpi^{a_0} b y)$ to see that the above is

\[
q^{-a_0(2s+1)} \int \int \omega(\varpi^{a_0} b) |b\varpi^{a_0}|^{3/2} W \left( \frac{a_1 a_2 a_1 a_2 x^2}{a_2 a_2 a_2 a_2} \right) 1_{\mathcal{O}} \left( \frac{\lambda}{b_1 b_2 b_3} \right)
\times 1_{\mathcal{O}^3} \left( \frac{a_2, y, x, \lambda b^{-2} a_2^{-1}}{a_3} \right) |c|^2 |\lambda|^s d^3 b |a_2|^{3/2} d(a, c, x, y) / |a_1|^{1/2} |a_3|^{3/2}
\]

\[
= q^{-a_0(2s+1)} \int \int \omega(\varpi^{a_0} b) |b\varpi^{a_0}|^{3/2} \psi \left( \frac{a_1 a_2 x^2}{\lambda} + \frac{a_3 y}{a_2} \right) W \left( \frac{a_1 a_2 a_2}{a_2 a_2 a_2} \right) 1_{\mathcal{O}} \left( \frac{\lambda}{b_1 b_2 b_3} \right)
\times 1_{\mathcal{O}^3} \left( \frac{a_2, y, x, \lambda b^{-2} a_2^{-1}}{a_3} \right) \frac{a_3 c}{\varpi^{a_0} b a_2} |c|^2 |\lambda|^s d^3 b |a_2|^{3/2} d(a, c, x, y) / |a_1|^{1/2} |a_3|^{3/2}.
\]

We have shown that

\begin{equation}
(7.0.2) \quad Z(W, f_{\alpha_0}, s, z) = q^{-11a_0/2} [\omega](\varpi^{a_0}) q^{a_0 a_2 - 2a_0 s} Z(W, f_0, s, z)
\end{equation}
where \( z_0 := z_1 + z_2 + z_3 \). Thus we can and do assume that \( a_0 = 0 \) for the remainder of the computation. We have shown

\[
Z(W, f_0, s, z) = \int \int \omega(b) |b|^{3/2} \psi \left( \frac{a_1 a_2 b^2}{\lambda} + \frac{a_3 y}{a_2} \right) W \left( \begin{smallmatrix} a_1 a_2 & a_2 & a_3 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{smallmatrix} \right) 1_\mathcal{O} \left( \frac{\lambda}{b_1 b_2 b_3} \right) 
\times 1_{(\mathfrak{O}^3)^6} \left( a_2, y, x, \lambda b^{-2} a_2^{-1}, c \right) \left| \frac{a_3 c}{ba_2} \right|^z |\lambda|^s d^x b |a_2|^{3/2} d(a, c, \lambda, x, y) / |a_1|^{1/2} |a_3|^{3/2} \]

Let

\[
(7.0.3) \quad \zeta(z) := \prod_{i=1}^3 \zeta(z_i).
\]

Executing the integrals over \( x, y, c \) we see that \( Z(W, f_0, s, z) \) is

\[
\zeta(z) \int \int \omega(b) |b|^{3/2} W \left( \begin{smallmatrix} a_1 a_2 & a_2 & a_3 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{smallmatrix} \right) 1_\mathcal{O} \left( \frac{\lambda}{b_1 b_2 b_3} \right) 
\times 1_{(\mathfrak{O}^3)^5} \left( a_2, a_3, a_1 a_2 b^2, \lambda b^{-2} a_2^{-1}, \lambda b^{-2} a_2^{-1} \right) \left| \frac{a_3 c}{ba_2} \right|^z |\lambda|^s d^x b |a_2|^{3/2} d(a, \lambda) / |a_1|^{1/2} |a_3|^{3/2} \]

where \( d(a, \lambda) = d^x a_1 d^x a_2 d^x a_3 d^x \lambda \) and the outer integral is over \( (a_1, a_2, a_3, \lambda) \in ((F^\times)^3)^3 \times F^\times \).

We now change variables \( (a_1, a_3) \mapsto (\lambda b^{-2} a_2^{-1} a_1, a_2 a_3) \) to arrive at

\[
Z(W, f_0, s, z) = \zeta(z) \int \int \omega(b) |b|^{3/2} W \left( \begin{smallmatrix} \lambda b^{-2} a_1 & a_2 & a_3 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{smallmatrix} \right) 1_\mathcal{O} \left( \frac{\lambda}{b_1 b_2 b_3} \right) 
\times 1_{(\mathfrak{O}^3)^4} \left( a_2, a_3, a_1, \lambda b^{-2} a_2^{-1} \right) \left| \frac{a_3 c}{b} \right|^z |\lambda|^s d^x b |a_2|^{1/2} d(a, \lambda) / \lambda b^{-2} a_1^{1/2} a_3^{3/2}.
\]

In analyzing this expression the following lemma is useful:

**Lemma 7.1.** If \( b_1, b_2, b_3 \in \mathcal{O}, b_1^{-1} b_2 b_3, b_2^{-1} b_3 b_1, b_3^{-1} b_1 b_2 \in \mathcal{O} \), and \( \ell \geq v(b_1) + v(b_2) + v(b_3) \) then \( \ell - 2v(b_i) \geq 0 \) for all \( i \).

**Proof.** By symmetry we may assume that \( v(b_3) \geq v(b_2) \geq v(b_1) \). Since \( v(b_1) + v(b_2) \geq v(b_3) \) and \( \ell \geq v(b_1) + v(b_2) + v(b_3) \) we have

\[
\ell \geq 2v(b_3) \geq 2v(b_2) \geq 2v(b_1).
\]

\[\square\]

At this point it is convenient to write the integral in terms of integrals that naturally factor in a fashion corresponding to the three copies of \( \text{GL}_3(F) \) in \( \text{GL}_3^3(F) \). Let \( W = \otimes_{i=1}^3 W_i \) where \( W_i \in W(\pi, \psi) \). For \( \ell \in \mathbb{Z}_{\geq 0} \) define

\[
(7.0.5) \quad J_i(\ell) := \int_{(F^\times)^3} W_i \left( \begin{smallmatrix} \omega' a_1 & a_2 & a_3 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{smallmatrix} \right) 1_{\mathfrak{O}^4} \left( a_1, a_2, \omega^\ell a_2^{-1}, a_3 \right) \frac{|a_3|^z |a_2|^{1/2} d^x a}{|a_2|^2 |a_1|^{1/2} |a_3|^{3/2}}
\]
where $d^*a = d^*a_1d^*a_2d^*a_3$.

Writing $k_i := v(b_i)$ and using Lemma 7.1 and (7.0.4) we see that

$$Z(W, f_0, s, z) = \zeta(z) \sum_{\ell=0}^{\infty} \sum_{k_0 \leq \ell} q^{-\ell s} \prod_{i=1}^{3} \omega_i(\varpi^{k_i}) q^{-k_i(3/2-z_i)} J_i(\ell - 2k_i)$$

where the sum is over $k \in \mathbb{Z}_{\geq 0}$ such that

$$k_1 \leq k_2 + k_3, \quad k_2 \leq k_1 + k_3, \quad k_3 \leq k_1 + k_2$$

and $k_0 := k_1 + k_2 + k_3$. In the following subsection we compute $J_i(\ell)$ for $1 \leq i \leq 3$.

### 7.1. Computation of $J_i(\ell)$

We assume that $W_i$ is normalized so that $W_i(I) = 1$. Let $\alpha_i$ be the Langlands class of $\pi_i$. We drop the subscript $i$ to ease notation.

For the reader’s convenience we restate the Weyl character formula in the case of interest for us (see, e.g. [FH91, (A.4)]). For integers $\lambda_1 \geq \lambda_2 \geq \lambda_3$ one has

$$\mathbb{S}_{\lambda_1, \lambda_2, \lambda_3}(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \text{sgn}(\sigma) x_{\sigma(1)}^{\lambda_1+2} x_{\sigma(2)}^{\lambda_2+1} x_{\sigma(3)}^{\lambda_3} \Delta(x)$$

where

$$\Delta(x) = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$$

and $S_3$ is the permutation group on $\{1, 2, 3\}$. We note that $\mathbb{S}_{\lambda_1, \lambda_2, \lambda_3}$ is a polynomial in $x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}$. By convention, $\mathbb{S}_{\lambda_1, \lambda_2, \lambda_3} = 0$ unless $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

**Lemma 7.2.** One has that

$$J(\ell) = L(\frac{1}{2}, x) L(z - \frac{1}{2}, \pi^*) \zeta(z)^{-1} \sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma) \alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 - q^{1/2})}{\Delta(\alpha)}.$$ 

**Proof.** We have

$$J(\ell) := \int_{(F \times)^3} W \begin{pmatrix} \omega^a_{a_1} & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \mathbb{1}_{\mathcal{O}^3} \left( a_1, a_2, \omega^a a_2^{-1}, a_3 \right) \frac{|a_3|^2 |a_2|^{1/2} d^* a}{|\omega^a|^{1/2} |a_1|^{1/2} |a_3|^{3/2}}.$$ 

Writing $k_i = v(a_i)$ and applying Shintani’s formula [Shi76] (see also [Cog07, §3.1.3]) we see that this is equal to

$$\sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \mathbb{S}_{\ell+k_1, k_2, -k_3}(\alpha) q^{-\ell/2+k_3(1/2-z)-k_1/2-k_2/2}$$

$$= \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma) \alpha_{\sigma(1)}^{\ell+k_1+2} \alpha_{\sigma(2)}^{k_2+1} \alpha_{\sigma(3)}^{-k_3} q^{-\ell/2+k_3(1/2-z)-k_1/2-k_2/2}}{\Delta(\alpha)}$$

$$= \sum_{\sigma \in S_3} \frac{\alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 - \frac{\alpha_{\sigma(1)}^{\ell+1}}{q^{1/2}})}{\Delta(\alpha)(1 - \frac{\alpha_{\sigma(1)}}{q^{1/2}})(1 - \frac{\alpha_{\sigma(2)}}{q^{1/2}})(1 - \frac{q^{1/2}}{q^{1/2}})}.$$
This is \( L\left(\frac{1}{2}, \pi\right)L(z - \frac{1}{2}, \pi^\vee) \) times
\[
\sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma)\alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 - \frac{q^{\ell+1}}{q^{(\ell+1)/2}})(1 - \frac{q^{1/2}}{q^{\sigma(3)}})(1 - \frac{q^{1/2}}{q^{\sigma(2)}})}{\Delta(\alpha)}.
\]

Taking a change of variables \( \sigma \to (12)\sigma \) and observing cancellation this is
\[
\sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma)\alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 - \frac{q^{\sigma(3)}}{q^{\ell/2}})(1 - \frac{q^{1/2}}{q^{\sigma(1)}})(1 - \frac{q^{1/2}}{q^{\sigma(2)}})}{\Delta(\alpha)} = \sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma)\alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 - \frac{q^{1/2}}{q^{\sigma(1)}})}{\Delta(\alpha)}.
\]

Using the transposition (23) instead of (12) and again observing cancellation this is
\[
\sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma)\alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 + \frac{q^{\sigma(3)}}{q^{\ell/2}})(1 - \frac{q^{1/2}}{q^{\sigma(1)}})(1 - \frac{q^{1/2}}{q^{\sigma(2)}})}{\Delta(\alpha)} = \sum_{\sigma \in S_3} \frac{\text{sgn}(\sigma)\alpha_{\sigma(1)}^{2+\ell} \alpha_{\sigma(2)} q^{-\ell/2}(1 - \frac{q^{1/2}}{q^{\sigma(1)}})}{\zeta(z)\Delta(\alpha)}.
\]

\( \square \)

7.2. **Return to** \( Z(W, f_0, s, z) \). We will denote the Langlands class of \( \pi_i \) by \( \alpha_i \) and let \( \alpha_{ij}, \) 
\( 1 \leq j \leq 3 \) be its eigenvalues. Let
\[
(7.2.1) \quad \alpha : = (\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}, \alpha_{33}).
\]

By (7.0.6) and Lemma 7.2 we have
\[
(7.2.2) \quad Z(W, f_0, s, z) = \prod_{i=1}^{3} L\left(\frac{1}{2}, \pi_i\right)L(z_i - \frac{1}{2}, \pi_i) = \sum_{k \leq \ell} \sum_{q \leq 0} q^{-\ell s} \sum_{i=1}^{3} \omega_i(\varpi_k)\frac{\text{sgn}(\sigma_i)\alpha_{i\sigma_i(1)}^{2+\ell-2k} \alpha_{i\sigma_i(2)} q^{-(\ell-2k)/2}(1 - \frac{q^{1/2}}{q^{\sigma_i(1)}})}{\Delta(\alpha_i)}.
\]

We change variables \( \ell \to k_0 + \ell \) and bring the sum over \( \sigma \) outside the product over \( 1 \leq i \leq 3 \) to obtain
\[
(7.2.3) \quad \sum_{\sigma \in S_3} \sum_{k \leq 0} \sum_{\ell=0}^{\infty} q^{-(\ell+k_0)s} \prod_{i=1}^{3} \frac{\omega_i(\varpi_k)\text{sgn}(\sigma_i)\alpha_{i\sigma_i(1)}^{2+\ell-2k_0} \alpha_{i\sigma_i(2)} q^{-(\ell-2k_0)/2}(1 - \frac{q^{1/2}}{q^{\sigma_i(1)}})}{\Delta(\alpha_i)} = \sum_{\sigma \in S_3} \left(1 - \frac{\alpha_{1\sigma_2(1)} \alpha_{2\sigma_2(1)} \alpha_{3\sigma_3(1)}}{q^{3/2+s}}\right)^{-1} \sum_{k_0} \prod_{i=1}^{3} \frac{\omega_i(\varpi_k)\text{sgn}(\sigma_i)\alpha_{i\sigma_i(1)}^{2+k_0} \alpha_{i\sigma_i(2)} (1 - \frac{q^{1/2}}{q^{\sigma_i(1)}})}{\Delta(\alpha_i)}.
\]

We now explain how to evaluate the sum over the \( k_i \). Let
\[
(b_{ij}) := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]
and let

\[ (7.2.4) \quad q(Y_1, Y_2, Y_3) = \sum_{\tau \in S_3} \frac{\prod_j Y_{\tau(1)}^{b_{1,j}} Y_{\tau(2)}^{b_{2,j}} Y_{\tau(3)}^{b_{3,j}}}{(1 - Y_{\tau(1)}^2 Y_{\tau(2)}^3 Y_{\tau(3)}) (1 - Y_{\tau(1)} Y_{\tau(2)} Y_{\tau(3)})}. \]

Here the product is over \( 1 \leq j \leq 2 \) such that \( \tau(j) > \tau(j + 1) \). This is a rational function in \( Y_1, Y_2, Y_3 \). Using a computer algebra system one computes that

\[
q(Y_1, Y_2, Y_3) = \frac{1 - (Y_1 Y_2 Y_3)^2}{(1 - Y_1 Y_2 Y_3)(1 - Y_1 Y_2)(1 - Y_1 Y_3)(1 - Y_2 Y_3)}.
\]

By [BGLS10, Theorem 1] one has an identity of formal power series

\[ (7.2.5) \quad \sum_{k} Y_1^{k_1} Y_2^{k_2} Y_3^{k_3} = q(Y_1, Y_2, Y_3) \]

where the sum is over \( k = (k_1, k_2, k_3) \in \mathbb{Z}_{\geq 0} \) such that \( k_1 \leq k_2 + k_3, k_2 \leq k_3 + k_1 \) and \( k_3 \leq k_1 + k_2 \). We note that in the published version of [BGLS10, Theorem 1] there is a typo: the indices on the variables \( z \) should be \( z_\tau(1) \) instead of \( z_1 \), etc.

We apply (7.2.5) with \( Y_i = \omega_i(\omega)_{\pi_1 \sigma_1}^{a_{1 \sigma_1}(1)} a_{2 \sigma_2(1)}^{a_{2 \sigma_2}(1)} a_{3 \sigma_3(1)}^{a_{3 \sigma_3}(1)} q^{\pi_1+\sigma_1} \)

\[
\sum_{\alpha \in S_3} \left( 1 + \frac{\omega_i(\omega)_{\pi_1 \sigma_1}^{a_{1 \sigma_1}(1)} a_{2 \sigma_2(1)}^{a_{2 \sigma_2}(1)} a_{3 \sigma_3(1)}^{a_{3 \sigma_3}(1)}}{q^{\pi_1+\sigma_1}} \right) \prod_{i=1}^{3} \frac{\text{sgn}(\sigma_i) a_{\pi_1 \sigma_1(2)}^{a_{\pi_1 \sigma_1(2)}} (1 - q^{1/2})_{q^{\pi_1+\sigma_1}} \Delta(\alpha_i)}{\omega_i(\omega) q^{4 - (z_0 - z_1 + 2s)}}.
\]

where

\[ z_0 := z_1 + z_2 + z_3. \]

This is

\[
L(s + \frac{3}{2}, \pi, \otimes) \prod_{i=1}^{3} L(4 - (z_0 - z_i) + 2s, \pi_i, \text{Sym}^2 \otimes [\omega]_{\omega_i}^{-1})
\]

\[
L(s + (z_0 - z_i) + 2s, \pi_i, \wedge^2 \otimes [\omega]_{\omega_i}^{-1})
\]

times

\[
\sum_{\alpha \in S_3} \left( 1 + \frac{\omega_i(\omega)_{\pi_1 \sigma_1}^{a_{1 \sigma_1}(1)} a_{2 \sigma_2(1)}^{a_{2 \sigma_2}(1)} a_{3 \sigma_3(1)}^{a_{3 \sigma_3}(1)}}{q^{\pi_1+\sigma_1}} \right) \prod_{i=1}^{3} \frac{\text{sgn}(\sigma_i) a_{\pi_1 \sigma_1(2)}^{a_{\pi_1 \sigma_1(2)}} (1 - q^{1/2})_{q^{\pi_1+\sigma_1}} \Delta(\alpha_i)}{\omega_i(\omega) q^{4 - (z_0 - z_1 + 2s)}}.
\]

We have proven the following theorem:

**Theorem 7.3.** One has

\[
Z(W, f_0, s, z) = L(s + \frac{3}{2}, \pi, \otimes) \left( \prod_{i=1}^{3} L(1/2, \pi_i) L(z_i - 1/2, \pi_i) \frac{L(4 - 2z_i + 2s, \pi_i, \text{Sym}^2 \otimes [\omega]_{\omega_i}^{-1})}{L(4 - 2z_i + 2s, \pi_i, \wedge^2 \otimes [\omega]_{\omega_i}^{-1})} \right)
\]
Proof. The implied constants are absolute.

As usual we say that \( \pi \) is essentially tempered if there is an abelian twist of it that is tempered.

**Corollary 7.4.** Assume that \( \pi \) is essentially tempered and that \( \text{Re}(\omega_i) \geq 0 \) for all \( i \). Let \( \epsilon > 0 \). If

\[
\text{Re}(s) \geq \max(-\frac{3}{2} + \text{Re}(z_i), -1) + \epsilon
\]

\[
\text{Re}(s) + \text{Re}(z_i) \geq \epsilon
\]

for each \( i \) then

\[
\frac{Z(W, f_0, s, z)}{L(s + \frac{3}{2}, \pi, \otimes) \prod_{i=1}^{3} L(\frac{1}{2}, \pi_i) L(z_i - \frac{1}{2}, \pi_i)} = 1 + O(q^{-1-\epsilon})
\]

where the implied constants are absolute.

**Proof.** We assume throughout the proof that \( \text{Re}(\omega_i) \geq 0 \) for each \( i \). The implied constants are all absolute. The expression in the corollary is equal to

\[
\prod_{i=1}^{3} \frac{L(4 - (z_0 - z_i) + 2s, \pi_i, \wedge^2 \otimes [\omega]_{\omega_i}^{-1})}{L(4 - (z_0 - z_i) + 2s, \pi_i, \text{Sym}^2 \otimes [\omega]_{\omega_i}^{-1})} = \prod_{i=1}^{3} \left( 1 + O(q^{-4-2s+2\text{Re}(z_0-z_i)}) \right) = 1 + O(q^{-1-\epsilon})
\]

times

\[
\sum_{g \in S^3} \left( \prod_{1 \leq i \leq 3} \left( 1 - \frac{\alpha_1 \alpha_2 \alpha_3}{q^{3/2+s}} \right) \right) \frac{\omega(\pi_{\alpha_1\alpha_2\alpha_3})}{q^{6-z_0+3s}}
\]

\[
\times \prod_{i=1}^{3} \text{sgn}(\sigma_i) \left( 1 - \frac{q^{1/2}}{\omega_i(\pi_{\omega_i}) q^{4-(z_0-z_i)+2s}} \right) \left( 1 - \frac{[\omega(\pi_{\omega_i}) \alpha_2^{\omega_i}(2)]}{\omega_i(\pi_{\omega_i}) q^{4-(z_0-z_i)+2s}} \right) \left( 1 - \frac{[\omega(\pi_{\omega_i}) \alpha_2^{\omega_i}(3)]}{\omega_i(\pi_{\omega_i}) q^{4-(z_0-z_i)+2s}} \right)
\]

\[
= \left( 1 + O(q^{-6+\text{Re}(z_0)-3s}) \right) \left( \prod_{i=1}^{3} \left( 1 + O(q^{-4+\text{Re}(z_0-z_i)-2s}) \right)^2 \right)
\]

\[
\times \sum_{g \in S^3} \left( \prod_{1 \leq i \leq 3} \left( 1 - \frac{\alpha_1 \alpha_2 \alpha_3}{q^{3/2+s}} \right) \right) \prod_{i=1}^{3} \text{sgn}(\sigma_i) \frac{\alpha_2^{\omega_i}(1) \alpha_2^{\omega_i}(2) \alpha_2^{\omega_i}(3)}{\Delta(\alpha_i)}
\]
To make it easier to observe these bounds we point out that the expressions we have bounded are decreasing as \( \Re(\omega_i) \to \infty \), so it suffices to bound them in the special case \( \Re(\omega_i) = 0 \). We have

\[
(1 + O(q^{-6+\Re(z_0)-3s})) \left( \prod_{i=1}^{3} (1 + O(q^{-4+\Re(z_0-z_i)-2s}))^2 \right) = 1 + O(q^{-1-\epsilon}).
\]

It therefore suffices to show that

\[(7.2.7) \quad \sum_{\sigma \in S_3^3} \left( \prod_{1 \leq i,j \leq 3} \left( 1 - \frac{\alpha_{1}\sigma_{1}(a_{1})\alpha_{2}\sigma_{2}(a_{2})\alpha_{3}\sigma_{3}(a_{3})}{q^{3/2+s}} \right) \right) \prod_{i=1}^{3} \frac{\text{sgn}(\sigma_{i})\alpha_{i}(\sigma_{i}(1))\alpha_{i}(\sigma_{i}(2))}{\Delta(\alpha_{i})} \left( 1 - \frac{q^{1/2}}{q^{\sigma(\sigma_{i}(1))}} \right)
\]

is \( 1 + O(q^{-1-\epsilon}) \). View \((7.2.7)\) as a polynomial in \( q^{-s} \). Using the Weyl character formula it is easy to check that the constant term is 1. Writing \( a \) for the coefficient of \( q^{-s} \) we see that \((7.2.7)\) is

\[1 + \frac{a}{q^s} + \sum_{j=2}^{28} O(q^{-j/2-j\epsilon}) = 1 + \frac{a}{q^s} + O(q^{-1-\epsilon}).\]

Thus to complete the proof it suffices to show that \( a = O(q^{-1-\min\{\Re(z_i)\}}) \). The expression \((7.2.7)\) is

\[(7.2.8) \quad L(s+\frac{3}{2}, \pi, \otimes^{3})^{-1} \sum_{\sigma \in S_3^3} \left( \prod_{1 \leq i,j \leq 3} \left( 1 - \frac{\alpha_{1}\sigma_{1}(a_{1})\alpha_{2}\sigma_{2}(a_{2})\alpha_{3}\sigma_{3}(a_{3})}{q^{3/2+s}} \right) \right) \prod_{i=1}^{3} \frac{\text{sgn}(\sigma_{i})\alpha_{i}(\sigma_{i}(1))\alpha_{i}(\sigma_{i}(2))}{\Delta(\alpha_{i})} \left( 1 - \frac{q^{1/2}}{q^{\sigma(\sigma_{i}(1))}} \right).
\]

The coefficient of \( q^{-s} \) in \( L(s+\frac{3}{2}, \pi, \otimes^{3}) \) is

\[
\sum_{\sigma \in S_3^3} \frac{\alpha_{1}\sigma_{1}(1)\alpha_{2}\sigma_{2}(1)\alpha_{3}\sigma_{3}(1)}{q^{3/2}}
\]

and the coefficient of \( q^{-s} \) in

\[
\sum_{\sigma \in S_3^3} \left( \prod_{1 \leq i,j \leq 3} \left( 1 - \frac{\alpha_{1}\sigma_{1}(a_{1})\alpha_{2}\sigma_{2}(a_{2})\alpha_{3}\sigma_{3}(a_{3})}{q^{3/2+s}} \right) \right) \prod_{i=1}^{3} \frac{\text{sgn}(\sigma_{i})\alpha_{i}(\sigma_{i}(1))\alpha_{i}(\sigma_{i}(2))}{\Delta(\alpha_{i})} \left( 1 - \frac{q^{1/2}}{q^{\sigma(\sigma_{i}(1))}} \right),
\]

is

\[(7.2.9) \quad \sum_{\sigma \in S_3^3} \frac{\alpha_{1}\sigma_{1}(1)\alpha_{2}\sigma_{2}(1)\alpha_{3}\sigma_{3}(1)}{q^{3/2}} + O(q^{-1-\min\{\Re(z_i)\}}).
\]

\[ \square \]

Outside of finitely many places our real interest is not in \( Z(W, f_0, s, z) \) but in \( Z(W, b_Y \times \mathbb{1}_{U^{\vee}(\mathcal{O})}, s, z) \). Now \( I(b_X \otimes \mathbb{1}_{V(\mathcal{O}) \times U^{\vee}(\mathcal{O})}) = \sum_{\alpha_0=0}^{\infty} q^{2\alpha_0} f_{2\alpha_0} \) by \((10.1.10)\) and

\[ Z(W, f_{\alpha_0}, s, z) = q^{-11\alpha_0/2} \omega(\omega^{\alpha_0}) q^{\alpha_0 z_0 - 2\alpha_0 s} Z(W, f_0, s, z)\]
by (7.0.2), so

\[
Z(W, b_X \otimes 1_{V(\mathcal{O}) \times U^\vee(\mathcal{O})}, s, z) = \sum_{\alpha_0=0}^{\infty} q^{2\alpha_0} q^{-11\alpha_0} [\omega](\varpi^{2\alpha_0}) q^{2\alpha_0 z_0 - 4\alpha_0 s} Z(W, f_0, s, z) \\
= L(9 + 4s - 2z_0, [\omega]^2) Z(W, f_0, s, z)
\]

where \(z_0 := z_1 + z_2 + z_3\). Similarly

\[
(7.2.11) \quad Z(W, f_0 - f_1, s, z) = (1 - q^{-11/2}[\omega](\varpi) q^{5s - 2s}) Z(W, f_0, s, z).
\]

Using Lemma 10.8 we similarly arrive at

\[
(7.2.12) \quad Z(W, F_X(1_0) \otimes 1_{V(\mathcal{O}) \times U^\vee(\mathcal{O})}), s, z)
\]

\[
= \sum_{j=0}^{\infty} q^{2j} \left( Z(W, f_{2j}, s, z) - q^{-6} Z(W, f_{2j-2} \otimes 1_{U^\vee(\mathcal{O})}, s, z) \right) \\
- q^{-4} Z(W, f_{2j-1} \otimes 1_{U^\vee(\mathcal{O})}, s, z) + q^{-10} Z(W, f_{2j-3} \otimes 1_{U^\vee(\mathcal{O})}, s, z)
\]

\[
= L(9 + 4s - 2z_0, [\omega^2]) \left( 1 - q^5[\omega](\varpi^{-2}) q^{4s-2z_0} - q^{3/2}[\omega](\varpi^{-1}) q^{2s-z_0} + q^{13/2}[\omega](\varpi^{-3}) q^{6s-3z_0} \right) \\
\times Z(W, f_0, s, z)
\]

\[
= L(9 + 4s - 2z_0, [\omega^2]) \left( 1 - q^{3/2}[\omega](\varpi^{-1}) q^{2s-z_0} \right) \left( 1 - q^5[\omega](\varpi^{-2}) q^{4s-2z_0} \right) Z(W, f_0, s, z).
\]

8. Nonvanishing of zeta integrals

Let \(v\) be a place of \(F\) which we omit from notation, writing \(F := F_v\), etc. Let

\[
Z_3 < T_3 < B_3 < P_3 < GL_3
\]

be the center, the maximal torus of diagonal matrices, the Borel subgroup of upper triangular matrices and the standard parabolic subgroup of type (2, 1). Let \(N_3\) be the unipotent radical of \(B_3\). Let \(\pi\) be a unitary generic representation of \(GL_3^3(F)\) with central character \(\omega\) and let \(\psi : N_3^3(F) \to \mathbb{C}^\times\) be a generic character. As usual, we say that \(\pi\) is essentially tempered if an abelian twist of \(\pi\) is tempered.

**Proposition 8.1.** Assume \(\pi\) is essentially tempered. One can choose \(f \in \mathcal{S}(X(F) \times U^\vee(F))\) so that

(a) \(I(f) \in C^\infty_c(Y_{sm}(F) \times U^\vee(F))\),

(b) \(Z(W, f, s, z)\) is absolutely convergent for all \(s\) provided that \(\text{Re}(\omega_i) \geq 0\) and \(\text{Re}(z_i) \geq \frac{1}{2} + \text{Re}(\omega_i)\) for all \(i\), and

(c) when \(F\) is archimedean, for fixed \(s\), \(Z(W, f, s, z)\) is rapidly decreasing in vertical strips as a function of \(z\).

Moreover, given \(s\), we can choose \(W\) and \(f\) so that (a), (b), (c) hold and \(Z(W, f, s, z)\) is not identically zero as a function of \(z\).
Proof. Consider the scheme-theoretic orbit \( O(\xi_0) \) of the point \( \xi_0 \in Y(F) \times U^\vee(F) \) under \( G \times T \). We claim that we can choose \( f \in \mathcal{S}(X(F) \times V(F) \times U^\vee(F)) \) so that \( I(f)|_{O(\xi_0)(F)} \) is any compactly supported smooth function on \( O(\xi_0)(F) \) that we wish. The set \( O(\xi_0)(F) \) is closed in \( Y^\text{sm}(F) \times U^\vee(F) \) by Lemma 3.2. In particular for any compactly supported smooth function \( \Phi \) on \( O(\xi_0)(F) \) we can choose a compactly supported smooth function on \( Y^\text{sm}(F) \times U^\vee(F) \) whose restriction to \( O(\xi_0)(F) \) is \( \Phi \). Thus the claim follows from Lemma 11.3.

In view of Lemma 3.1 the action map induces an isomorphism of quasi-affine \( F \)-schemes

\[
(8.0.1) \quad Q \setminus G \times T \to O(\xi_0).
\]

One has \( H^1(\text{Gal}(\bar{F}/F), Q) = 1 \) by [Ser02, §I.5.5 Proposition 38, §III.2.1 Proposition 6] and Hilbert’s theorem 90, so we deduce a homeomorphism (diffeomorphism in the archimedean setting)

\[
Q(F) \setminus G(F) \times T(F) = (Q \setminus (G \times T))(F) \to O(\xi_0)(F).
\]

Using the Iwasawa decomposition it follows that for any compactly supported smooth function \( \Phi \) on \( Q(F) \setminus G(F) \times T(F) \) we can choose \( f \in \mathcal{S}(X(F) \times V(F) \times U^\vee(F)) \) so that

\[
Z(W, f, s, z)
\]

\[
= \int_{N_Q(F) \setminus N_3^3(F) \times T_3^3(F) \times K \times F^\times \times M_Q(F)} W(tk)\psi(n)\Phi(ntk, \lambda, c)|\lambda|^s|c|^2 \frac{|\det t|^{1/2}d\times \lambda dntdkd^\times c}{\delta_{B_3^3}(t)dn_Q},
\]

where \( K = \text{GL}_3^3(\mathcal{O}) \) in the nonarchimedean case and \( K \) is a maximal compact subgroup of \( \text{GL}_3^3(F) \) in the archimedean case.

Now assume that we can show that for all \( \Phi' \in C^\infty_c(Q(F) \setminus G(F) \times T(F)) \) and \( W \in W(\pi, \psi) \) that

\[
(8.0.2) \quad \int_{N_Q(F) \setminus N_3^3(F) \times T_3^3(F) \times F^\times \times T(F)} W(t)\psi(n)\Phi'(nt, \lambda, c)|\lambda|^s|c|^2 \frac{|\det t|^{1/2}d\times \lambda dntdkd^\times c}{\delta_{B_3^3}(t)dn_Q}
\]

is absolutely convergent provided that \( \text{Re}(\omega_i) \geq 0 \) and \( \text{Re}(z_i) \geq \frac{1}{2} + \text{Re}(\omega_i) \) for all \( i \). It then follows that \( Z(W, f, s) \) is absolutely convergent, and in fact using integration by parts ones see that it is rapidly decreasing in vertical strips in the archimedean case.

Assume moreover that we can choose \( \Phi' \) and \( W \) so that (8.0.2) is nonzero as a function of \( z \) for a fixed \( x \). Then since \( Q(F) \setminus B_3^3(F) \times F^\times \times T(F) \) is closed in \( Q(F) \setminus G(F) \times T(F) \) we can choose \( f \) so that \( Z(W, f, s, z) \) is as close as desired to (8.0.2) (and in particular, is nonzero).

To summarize, it suffices to show that (8.0.2) is absolutely convergent for all \( \Phi' \) and nonzero as a function of \( z \) for some choice of \( \Phi' \) (given \( s \)).

We have that (8.0.2) is equal to

\[
\int_{T_3^3(F) \times T(F) \times F^3} \omega(t_2)W \begin{pmatrix} t_1 & 1 \\ t_3^{-1} & 1 \end{pmatrix}
\]
\[
\times \psi(x + y) \Phi_1 \left( \left( \begin{array}{ccc}
t_1 t_2 & t_2 x & x \\
t_2 & t_2 t_3^{-1} & t_3 \\
t_2 t_3^{-1} & t_3 & 1 \\
\end{array} \right), c \right) \frac{|t_1 t_2 t_3^{-1}|^{1/2} |c|^z d^x t_1 d^x t_2 d^x t_3 d^x c dx dy}{|t_1 t_3|^2}
\]

\[
\omega(t_2) W \left( \frac{t_1 t_3^{-1}}{t_3^{-1}} \right)
\times \psi(x + y) \Phi_1 \left( \left( \begin{array}{ccc}
t_1^{-1} x & x \\
t_2 & t_2 t_3^{-1} & t_3 \\
1 & t_1 & 1 \\
\end{array} \right), t_2 t_3^{-1} c \right) \frac{|c|^z d^x t_1 d^x t_2 d^x t_3 d^x c dx dy}{|t_1|^{3/2} |t_3|^{5/2}}.
\]

Taking a change of variables \((x, y, c) \mapsto (t_1 x, t_3 t_2^{-1} y, t_3 t_2^{-1} c)\) we arrive at

\[(8.0.3)
\int_{A_3 \times T(F) \times (F^2)^3} \omega(t_2) W \left( \frac{t_1 t_3^{-1}}{t_3^{-1}} \right) \psi(t_1 x + \frac{t_2 y}{t_2}) \Phi_1 \left( \left( \begin{array}{ccc}
1 & x \\
t_2 & 1 \\
1 & 1 \\
\end{array} \right), c \right) \frac{|c|^z d^x t_1 d^x t_2 d^x t_3 d^x c dx dy}{|t_1|^{1/2} |t_2|^{-1/2 + z} |t_3|^{3/2 - z}}.
\]

This is absolutely convergent for all \(W\) since we assumed that \(\text{Re}(\omega_1) \geq 0\) and \(\text{Re}(z_i) > 1/2 + \text{Re}(\omega_i)\) for all \(i\) and that \(\pi\) is tempered by the estimates on Whittaker functions given in [Jac09, Proposition 3.5] in the archimedean case and [JS83, Proposition 2.5] in the nonarchimedean case.

We can choose \(\Phi_1\) so that this is equal to

\[
\int_{A_3 \times T(F)} \omega(t_2) W \left( \frac{t_1 t_3^{-1}}{t_3^{-1}} \right) \Phi_2 \left( \frac{t_1, t_3}{t_2} \right) \Phi_3(t_2, c) \frac{|c|^z d^x t_1 d^x t_2 d^x t_3 d^x c dx dy}{|t_1|^{1/2} |t_2|^{-1/2 + z} |t_3|^{3/2 - z}}
\]

for any \(\Phi_2 \in S((F^2)^3)\) that is the Fourier transform of an element of \(C^\infty_c((F^2)^3)\) and any \(\Phi_3 \in C^\infty_c(((F^x)^2)^3)\). Pick \(t_1', t_3' \in (F^x)^3\) and \(W\) such that

\(W \left( \frac{t_1'}{t_3'} \right) \neq 0\).

Despite the fact that in the archimedean case \(\Phi_2\) cannot be compactly supported (unless it is zero), we can choose \(\Phi_2\) to approximate the \(\delta\)-distribution supported at the point \((t_1', t_3')\) and \(\Phi_3\) to approximate the \(\delta\)-distribution supported at \((1, 1)\) and thereby ensure that \(8.0.3\) is not identically zero as a function of \(z\).

9. Proof of the main theorem

Assume that \(\pi_1, \pi_2, \pi_3\) are unitary cuspidal automorphic representations of \(GL_3(\mathbb{A}_F)\) that are everywhere tempered. Let \(S\) be a set of places of \(F\) containing the primes dividing 3 and the infinite places that is large enough that \(\mathcal{O}^S\) has class number 1 and \(F_v\) is absolutely unramified for \(v \not\in S\). We also assume that every place \(v\) of \(F\) not in \(S\) has the property that the order of the residue field at \(v\) is sufficiently large in the sense that the quotient \((7.2.6)\) is nonzero for \(\text{Re}(z_i) > 1/2\) and \(\text{Re}(s) > -1\). This technical condition is automatically satisfied provided that the order of the residue field is greater than an absolute constant.
Let \( v_3 \not\in S \). We assume that the \( \pi_v \) are unramified for \( v \not\in S \cup \{v_3\} \). Consider the following local assumptions on \( \pi_{v_3} \) and

\[
 f_{v_3} = f_{1v_3} \otimes f_{2v_3} \otimes f_{3v_3} \in \mathcal{S}(X(F_{v_3})) \otimes \mathcal{S}(V(F_{v_3})) \otimes \mathcal{S}(U^\vee(F_{v_3})): 
\]

(1') One has \( I(f_{v_3})|_{O(\xi_0)(F_{v_3})} \in C_c^\infty(O(\xi_0)(F_{v_3})) \).

(2') For all \( g \in \text{SL}_2^3(F_{v_3}) \) one has \( \text{supp}(\rho(g)f_{2v_3}) \cap V_0(F_{v_3}) = \emptyset \).

Here \( O(\xi_0) \) is the orbit of the point \( \xi_0 \) under \( G \times T \). Under these assumptions the argument proving Proposition 8.1 implies that the integral defining \( Z(W, f_{v_3}, s, z) \) is absolutely convergent for all \( s \) and \( z \) with \( \text{Re}(z_i) > \frac{1}{2} \). We require the following assumption:

(A) There is a \( W \in \mathcal{W}(\pi_{v_3}, \psi_{v_3}) \) such that \( Z(W, f_{v_3}, s, z) \) is not identically zero as a function of \( s \).

We say that \( s \in \mathbb{C} \) lies in the **exceptional set** for \( \pi_{v_3} \) if \( Z(W, f_{v_3}, s, z) = 0 \) for all \( W \) and \( f_{v_3} \) satisfying assumptions (1') and (2').

Unfortunately we do not have a good understanding of which representations \( \pi_{v_3} \) satisfy (A). However, we can at least produce many functions satisfying (1') and (2'):

**Lemma 9.1.** One can always choose \( f_{v_3} \) so that (1') and (2') are valid.

**Proof.** We drop the subscript \( v_3 \) in the proof for notational simplicity. We first show that we can choose a nonzero \( f_2 \in \mathcal{S}(V(F)) \) such that (2') is valid. Let \( \sigma \) be a supercuspidal representation of \( \text{SL}_2^3(F) \) and let \( m \in C_c^\infty(\text{SL}_2^3(F)) \) be a matrix coefficient of \( \sigma^\vee \). There is a maximal quotient of \( \mathcal{S}(V(F)) \) on which \( \text{SL}_2^3(F) \) acts via \( \sigma \), and it is nonzero since we are in the stable range. It follows that

\[
 f_2 := \int_{\text{SL}_2^3(F)} m(g) \rho(g) f \, dg \neq 0
\]

for some \( f \in \mathcal{S}(V(F)) \). Let \( N_2 \leq \text{GL}_2 \) be the unipotent radical of the Borel subgroup of upper triangular matrices. For \( \xi \in V_0(\xi) \) and \( n \in N_2(F)^3 \) one has

\[
 \rho(g)f_2(\xi) = \rho(ng)f_2(\xi) = \int_{\text{SL}_2^3(F)} m(g') \rho(ng'g) f(\xi) \, dg' = \int_{\text{SL}_2^3(F)} m(g^{-1}n^{-1}g') \rho(g')f_2(\xi) \, dg'.
\]

But \( m \) is a supercuspidal function, hence

\[
 \int_{N_2(F)^3} m(g^{-1}n^{-1}g') \, dn = 0.
\]

It follows that \( \rho(g)f_2(\xi) = 0 \), so \( f_2 \) satisfies (2'). In particular its support is contained in \( Y_{\text{ani}}(F) \). The orbit \( O(\xi_0) \subset Y_{\text{ani}} \times U^\vee \) is closed by Lemma 3.2. Thus for any \( f_3 \in \mathcal{S}(U^\vee(F)) \) the restriction of \( f_2 \otimes f_3 \) to \( O(\xi_0) \) will be compactly supported and smooth, and we can choose \( f_3 \) so that it is nonzero. We now choose \( f_1 \in C_c^\infty(X(F)) \) so that \( I(f_1 \otimes f_2 \otimes f_3)|_{Y_{\text{ani}}(F)} = f_2 \otimes f_3 \) using the argument of Lemma 11.3. Thus (1') is satisfied. \( \square \)
Theorem 9.2. Assume $\pi_{v_3}$ satisfies (A). Then the partial $L$-function

$$L^{S \cup \{v_3\}}(s, \pi, \otimes)$$

admits a meromorphic continuation to $\text{Re}(s) > \frac{3}{4}$. It is holomorphic away from the exceptional set of $\pi_{v_3}$.

Lemma 9.3. Let $\sigma_1 < \sigma < \sigma_2$ be real numbers and let $f(z)$ be a nonzero function that is holomorphic and rapidly decreasing in the vertical strip

$$(9.0.1) \{ z \in \mathbb{C} : \sigma_1 \leq \text{Re}(z) \leq \sigma_2 \}.$$ 

Then for any $r \in \mathbb{R}_{>0}$ the integral

$$(9.0.2) \frac{1}{2\pi i} \int_{\text{Re}(z) = \sigma} r^z f(z)dz$$

is absolutely convergent. If $f$ is not identically zero then $(9.0.2)$ is nonzero for some $r \in \mathbb{R}_{>0}$.

Proof. The absolute convergence statement is clear. We have

$$\frac{1}{2\pi i} \int_{\text{Re}(z) = \sigma} r^z f(z)dz = \frac{1}{2\pi i} \int_{\text{Re}(z) = \sigma} e^{z \log r} f(z)dz.$$ 

Up to a nonzero constant this is the Fourier transform of the function

$$(9.0.3) \quad t \mapsto f(\sigma + it)$$

evaluated at $-\frac{\log r}{2\pi}$ (with one common normalization of the Fourier transform). The function $(9.0.3)$ is nonzero and is in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Its Fourier transform is therefore nonzero. □

Proof of Theorem 9.2. Let $v_1, v_2, v_3 \not\in S$ and let $S' = S \cup \{v_3\}$. Let $\varphi$ be a smooth function in the space of $\pi$ that is a pure tensor and is that is unramified outside of $S'$. Let $f \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F) \times U^\vee(\mathbb{A}_F))$ be a pure tensor. Fix a place $v_0|\infty$, let $\iota : \mathbb{R}^3_{>0} \to (F_{v_0})^3$ be the natural injection, and for $r \in \mathbb{R}^3_{>0}$ let

$$f_r(u^\vee, u, w^\vee) := f(u^\vee, u, w^\vee \iota(r^{-1})).$$

So that we can use analytic continuation in the $\omega_i$ we drop the assumption that $\pi$ is unitary until further notice. In view of Proposition 6.2 one has

$$(9.0.4) \quad \frac{Z(\varphi, f_r, s)}{L^{S'}(s + \frac{3}{2}, \pi, \otimes^3)} = \frac{1}{(2\pi i)^3} \int_{\text{Re}(z) = \sigma} \frac{Z(\varphi, f_r, s, z)}{L^{S'}(s + \frac{3}{2}, \pi, \otimes^3)}dz \quad = \quad \frac{1}{(2\pi i)^3} \int_{\text{Re}(z) = \sigma} r^z \frac{Z(\varphi, f, s, z)}{L^{S'}(s + \frac{3}{2}, \pi, \otimes^3)}dz.$$
provided that
\[ \Re(s) \gg_\pi 1, \]
\[ \Re(\omega_i) \gg_\pi 1 \text{ for all } i, \]
(9.0.5)
\[ \sigma_i \gg \Re(\omega_i) \text{ for all } i, \]
\[ \Re(s) \gg_\pi \sigma_i + \Re(\omega_i) \text{ for all } i. \]

Choose \( W_v \in \mathcal{W}(\pi_v, \psi) \) for all \( v \) such that \( \prod_v W_v = W^\ast \) and \( W_v(1) = 1 \) for all \( v \not\in S \). Still assuming (9.0.5) in view of (6.0.15) we have
\[ \frac{Z(\varphi, f_r, s)}{L_s'(s + \frac{3}{2}, \pi, \otimes^3)} = \frac{1}{(2\pi i)^3} \int_{\Re(z) = \sigma} r^z Z(W_{s'}, f_{s'}, s, z) \frac{Z(W_{s''}, f_{s''}, s, z)}{L_{s''}(s + \frac{3}{2}, \pi, \otimes^3)} dz. \]

We now explain more explicitly how to choose \( f \). We can and do choose \( f_S \) so that \( f_S \) satisfies assumption (1) of Theorem 5.1 (that is \( I(f_S) \in C^\infty_c(Y^\text{ani}(F_S) \times U^{\text{\check{v}}^\infty}(F_S)) \)) and \( Z(W_v, f_v, s, z) \) is holomorphic as a function of \( s \) and nonzero for any particular choice of \( s \) by Proposition 8.1. We take
\[ f_{v_1} = 1_0 \otimes 1_{V(\mathcal{O}) \times U^\vee(\mathcal{O})}, \quad f_{v_2} = F_X(1_0) \otimes 1_{V(\mathcal{O}) \times U^\vee(\mathcal{O})} \]
in the notation of (10.1.9) and \( f_{v_3} \) as in (1') and (2'). Finally for \( v \not\in \mathcal{S}' \cup \{v_1, v_2\} \) we let \( f_v = b_{X,v} \otimes 1_{U(\mathcal{O}) \times U^\vee(\mathcal{O})}. \) Then assumptions (1-4) of Theorem 5.1 are satisfied as we required at the beginning of the proof and we conclude that \( Z(\varphi, f, s) \) is holomorphic as a function of \( (s_\omega, s) \in \mathbb{C}^3 \times \mathbb{C} \).

By assumption (A) and Proposition 8.1 we can choose \( W_{s'} \in \mathcal{W}(\pi_{s'}, \psi_{s'}) \) so that \( Z(W_{s'}, f_{s'}, s, z) \) is analytic and nonzero as a function of \( s \) and \( z \) provided that \( \Re(\omega_i) \geq 0 \) and \( \Re(z_i) \geq \frac{1}{2} + \Re(\omega_i) \) for each \( i \). Under this assumption we claim that the identity (9.0.6) is valid when \( \pi \) is unitary, \( \sigma_1 = \sigma_2 = \sigma_3 = \frac{3}{4}, \) and \( \Re(s) > -\frac{3}{4}. \)

We claim first that (9.0.6) remains valid when the \( \pi_i \) are unitary, the \( \sigma_i \) are sufficiently large, and \( \Re(s) \) is sufficiently large (in a sense depending on the \( \sigma_i \)). We argue by analytic continuation. The contribution of \( Z(W_{s'}, f_{s'}, s, z) \) poses no problems as it is analytic as a function of \( s \) and \( z \) (for \( \Re(z) > \frac{1}{2} \)) and rapidly decreasing as a function of \( z \) in fixed vertical strips by Proposition 8.1.

On the other hand by (7.2.10), (7.2.11) and (7.2.12) we have
\[ Z(W_{s''}, f_{s''}, s, z) = (1 - q^{-11/2}[\omega](\omega)q^{z_0-2s}) \left( 1 - \frac{q^{3/2+2s-3z_0}}{[\omega](\omega_{v_2})} \right) \left( 1 - \frac{q^{5+4s-6z_0}}{[\omega](\omega_{v_2}^2)} \right) \]
\[ \times L(9 + 4s - 2z_0, [\omega^{S\cup\{v_1,v_3\}}]^2) Z(W_{s''}, f_{\hat{0}''}, s, z) \]
where
\[ f_{\hat{0}''} = \otimes_{v \not\in S'} 1_{\geq 0} \otimes 1_{V(\mathcal{O})} \otimes 1_{U^\vee(\mathcal{O})}. \]
In view of Corollary 7.4 for \( z_i \) with \( \Re(z_i) = \sigma \) and \( s \) sufficiently large the quotient

\[
(9.0.7) \quad \frac{Z(W^S, f^S, s, z)}{L(s + \frac{3}{2}, \pi^S, \otimes^3)} \prod_{i=1}^{3} L(\frac{1}{2}, \pi^S \sigma_i) L(z_i - \frac{1}{2}, \pi^S \sigma_i) =: D(s, z)
\]
is given by an absolutely convergent Dirichlet series for \( \Re(\omega_i) \geq 0 \). Thus for \( \Re(s) \) sufficiently large the identity (9.0.6) is valid for unitary \( \pi \) by analytic continuation. Still taking \( \Re(s) \) sufficiently large we can shift the contour to \( \Re(z_i) = \frac{3}{4} \) since

\[
L(z_i - \frac{1}{2}, \pi^S \sigma_i)
\]
is of at most polynomial growth in vertical strips [Bru06, §1] and \( Z(W^S, f^S, s, z) \) is rapidly decreasing as a function of the \( \Im(z_i) \) in a fixed vertical strip. Thus (9.0.6) is valid for \( \pi \) unitary and \( \sigma_i = \frac{3}{4} \) for all \( i \).

We henceforth assume that \( \pi \) is unitary and that \( \sigma_i = \frac{3}{4} \) for each \( i \). We have proven the identity

\[
(9.0.8) \quad \frac{Z(\varphi, f^S, s)}{L^S(s + \frac{3}{2}, \pi, \otimes^3)} = \prod_{i=1}^{3} L(\frac{1}{2}, \pi^S \sigma_i) \int_{\Re(z)=\left(\frac{3}{4}, \frac{3}{4}\right)} \frac{Z(W^S, f^S, s, z) r^z}{(2\pi i)^3} \prod_{i=1}^{3} L(z_i - \frac{1}{2}, \pi^S \sigma_i) \\
\times (1 - \Re[\omega])^{-1/2 + z_0 - z_0} (1 - \frac{q_{v_2}^{3/2 + 2s - z_0}}{[\omega](\varphi_{v_2}^{2})} (1 - \frac{q_{v_2}^{5 + 4s - 6z_0}}{[\omega](\varphi_{v_2}^{2})}) \\
\times L(9 + 4s - 2z_0, [\omega^{S(v_1, v_2)}])^2 D(s, z) dz
\]

and we wish to show that the left hand side of (9.0.8) is holomorphic and not identically zero for \( \Re(s) > -\frac{3}{4} \). By replacing \( \pi \) by a unitary twist we can and do assume \( \prod_{i=1}^{3} L(\frac{1}{2}, \pi^S \sigma_i) \neq 0 \), so this factor can be ignored. The factor

\[
\prod_{i=1}^{3} L(z_i - \frac{1}{2}, \pi^S \sigma_i)
\]
is independent of \( s \), not identically zero as a function of \( z_i \), and bounded in vertical strips as a function of \( z_i \). By Corollary 7.4 and our assumptions on the places in \( S \) given at the beginning of this section \( D(s, z) \) is given by a Dirichlet series in \( s \) that is absolutely convergent and nonzero for \( \Re(s) > -\frac{3}{4} \). Moreover

\[
(1 - q_{v_2}^{3/2}[\omega](\varphi_{v_2}^{-1})q_{v_2}^{2s-z_0})(1 - q_{v_2}^{5}[\omega](\varphi_{v_2}^{-2})q_{v_2}^{4s-2z_0}) L(9 + 4s - 2z_0, [\omega^{S(v_1, v_2)}])^2
\]
is absolutely convergent for \( \Re(s) > -1 \) and \( \Re(z_i) > \frac{1}{2} \). Thus altogether we have

\[
(9.0.9) \quad \frac{Z(\varphi, f^S, s)}{L^S(s + \frac{3}{2}, \pi, \otimes^3)} = \frac{1}{(2\pi i)^3} \int_{\Re(z)=\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)} r^z Z(W^S, f^S, s, z) g(s, z) dz
\]
where \( g(s, z) \) is holomorphic for \( \text{Re}(s) > -\frac{3}{4} \), bounded in vertical strips, and not identically zero as a function of \( z \). Invoking assumption (A), Proposition 4.2, and Lemma 9.3 we deduce the theorem.

\[ \square \]

10. Braverman-Kazhdan spaces

In [GL19a], following work of Braverman and Kazhdan, we define Schwartz spaces for certain Braverman-Kazhdan spaces \( X \) to be defined below and proved a Poisson summation formula for them. In this section we refine the definition of the Schwartz space in the archimedean case and endow it with the structure of a Fréchet space. This technical refinement is necessary for Lemma 11.3 below which is used in the proof of Theorem 5.1 above.

Let \( \text{Sp}_{2n} \) denote the symplectic group on a \( 2n \)-dimensional vector space and let \( P \leq \text{Sp}_{2n} \), \( M \leq P \) denote the Siegel parabolic and Levi subgroup. Specifically, for \( \mathbb{Z} \)-algebras \( R \) set

\[
\text{Sp}_{2n}(R) := \{ g \in \text{GL}_{2n}(R) : g^t \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} g := \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} \},
\]

\[
M(R) := \{ \begin{pmatrix} A & -I_n \\ A^{-t} & I_n \end{pmatrix} : A \in \text{GL}_n(R) \},
\]

\[
N(R) := \{ \begin{pmatrix} I_n & Z \\ I_n & -Z \end{pmatrix} : Z \in \text{gl}_n(R), Z^t = -Z \}
\]

and \( P = MN \). Apart from this section we will only use the \( n = 3 \) case, but since it is no more difficult to treat the general case we include it. We define a character

\[
\omega : M(R) \longrightarrow \mathbb{R}^* \\
(\begin{pmatrix} m & -I_n \\ m^{-t} & I_n \end{pmatrix}) \longmapsto \det m.
\]

Let

\[
X_P := [P, P]\backslash \text{Sp}_{2n}.
\]

We note that there is a left action

\[
M^{ab}(R) \times \text{Sp}_{2n}(R) \times X_P(R) \longrightarrow X_P(R) \\
(m, g, x) \longmapsto mxg^{-1}.
\]

One has the Plücker embedding

\[
\text{Pl} : X_P \longrightarrow \wedge^n \mathbb{G}_a^{2n}
\]

given by taking the wedge product of the last \( n \) rows of a representative \( g \in \text{Sp}_{2n}(R) \) for \([P, P](R)g \). We denote by \( X \) the closure of \( \text{Pl}(X_P) \):

\[
X := \overline{\text{Pl}(X_P)}.
\]

It is an affine variety (in fact a spherical variety, for many more details see [Li18, §7.2]). As explained in loc. cit., \( X \) is the affine closure of \( X_P : \overline{X_P}^\text{aff} = X \).
10.1. The local Schwartz spaces. Let $F$ be a local field of characteristic zero. When $F$ is archimedean we let $K \leq \text{Sp}_{2n}(F)$ denote a maximal compact subgroup and when $F$ is nonarchimedean let $K$ be a conjugate of $\text{Sp}_{2n}(O)$. For $f \in C^\infty(X_P(F))$ and $g \in \text{Sp}_{2n}(F)$ let

$$f_{\chi_s}(g) := \int_{M^\text{ab}(F)} \delta_P(m)^{1/2}\chi_s(\omega(m))f(m^{-1}g)dm$$

be its Mellin-transform. Here $\chi_s := \chi \cdot \epsilon^s$. It is a section of

$$I(\chi_s) := \text{Ind}_{P}^{\text{Sp}_{2n}}(\chi_s \circ \omega).$$

Here the induction is normalized. We regard $I(\chi_s)$ as a representation in the category of smooth representations (in other words we require sections to be smooth).

In [GL19a] the author and Liu defined a Schwartz space $\mathcal{S}(X(F), K)$ by a refinement of the method in [BK02]. It was also denoted by $\mathcal{S}(X(F), K)$ in loc cit., but we warn the reader that in the earlier paper $X$ denoted $X_P$, not its affine closure. In any case the Schwartz space comes equipped with a Fourier transform

$$\mathcal{F}_X : \mathcal{S}(X(F), K) \longrightarrow \mathcal{S}(X(F), K).$$

The Fourier transform depends on our choice of additive character $\psi$. Functions in $\mathcal{S}(X(F), K)$ are $K$-finite smooth functions on $X_P(F)$, and the space $\mathcal{S}(X(F), K)$ contains the space $C^\infty_c(X_P(F), K) \leq C^\infty_c(X_P(F))$ of compactly supported $K$-finite smooth test functions on $X_P(F)$ [GL19a, Proposition 4.7].

In the nonarchimedean case the space $\mathcal{S}(X(F), K)$ does not depend on $K$, but it does depend on $K$ in the archimedean case. This is aesthetically unpleasant. More seriously, in the current paper we must work with functions on $X_P(F)$ that are not $K$-finite. Thus we now explain how to define a space $\mathcal{S}(X(F))$ without reference to $K$ in the archimedean setting. It contains $C^\infty_c(X_P(F))$ and $\mathcal{S}(X(F), K)$ for all $K$. In particular it contains functions that are not $K$-finite.

We recall the meromorphic functions $a_w(s, \chi)$ of [Ike92, §1.2] indexed by $w \in \Omega_n$, the set of representatives for the Weyl group of $\text{Sp}_{2n}$ modulo the Weyl group of $M$ given by taking the element in each coset of smallest length. The function $a_w(s, \chi)$ is a certain product of $L$-functions evaluated at arguments depending on $s$, $\chi$ and $n$. A section $f(s)$ of $I(\chi_s)$ is good if it is meromorphic and if the section

$$g \mapsto \frac{M_wf(s)(g)}{a_w(\chi, s)}$$

is holomorphic for all $w \in \Omega_n$, where $M_w$ is the usual intertwining operator [GL19a, (3.0.2)].

In the nonarchimedean setting this is all one needs to define the Schwartz space. The following is [GL19a, Definition 4.1]:

**Definition 10.1.** Assume $F$ is nonarchimedean. The Schwartz space is defined to be the space of $f \in C^\infty(X_P(F))$ such that for each $g \in \text{Sp}_{2n}(F)$ and character $\chi$ of $F^\times$ the integral
(10.1.1) defining $f_{\chi_s}(g)$ is absolutely convergent for $\Re(s)$ large enough and defines a good section.

Assume until otherwise stated that $F$ is archimedean. In [GL19a] the author and Liu introduced a stronger notion of an excellent section. For real numbers $A \leq B$, $p(x) \in \mathbb{C}[x]$ and meromorphic functions $f: \mathbb{C} \to \mathbb{C}$ let

$$V_{A,B} := \{ s \in \mathbb{C} : A \leq \Re(s) \leq B \},$$

$$|f|_{A,B,p} := \sup_{s \in V_{A,B}} |p(s)f(s)|$$

(which may be $\infty$). Let $w_0 \in \Omega_n$ be the long Weyl element. A good section $f(s)$ is excellent if for all $g \in \text{Sp}_{2n}(F)$, real numbers $A < B$, $w \in \{ \text{Id}, w_0 \}$ and any polynomials $p_w := p_{w,\chi} \in \mathbb{C}[x]$ such that $p_w(s)a_w(s,\chi)$ has no poles for $V_{A,B}$ one has

$$|M_w f(s)(g)|_{A,B,p_w} < \infty.$$

This is the same as the definition in [GL19a], except we have removed the requirement of $K$-finiteness. The local properties of excellent sections proved in loc. cit. remain valid with identical proofs. As a warning, in [Ike92] Ikeda restricts to $K$-finite sections, but the local properties of intertwining operators used in loc. cit. do not depend on $K$-finiteness [Wal92, Chapter 10].

Consider the Lie algebra

$$\mathfrak{g} := \text{Lie}(M_{ab}(F) \times \text{Sp}_{2n}(F)).$$

It acts on $C^\infty(X_P(F))$ via the differential of the action (10.0.3) and hence we obtain an action of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Let $\hat{K}_{G_m}$ be a set of representatives for the (unitary) characters of $F^\times$ modulo equivalence, where $\chi$ is equivalent to $\chi'$ if and only if $\chi = | \cdot |^t \chi'$ for some $t \in \mathbb{R}$. It can be identified with the set of characters of the maximal compact subgroup $K_{G_m}$ of $F^\times$, which explains the notation.

**Definition 10.2.** The Schwartz space $\mathcal{S}(X(F))$ consists of the $f \in C^\infty(X_P(F))$ such that for all $D \in U(\mathfrak{g})$, $g \in \text{Sp}_{2n}(F)$ and each character $\chi$ of $F^\times$ the integral (10.1.1) defining $(D.f)_{\chi_s}(g)$ is absolutely convergent for $\Re(s)$ large enough and satisfies the following condition: For all $g \in \text{Sp}_{2n}(F)$, real numbers $A < B$, $w \in \{ \text{Id}, w_0 \}$ and any polynomials $p_{w,\eta} \in \mathbb{C}[s]$ such that $p_{w,\eta}(s)a_w(s,\eta)$ has no poles for $V_{A,B}$ one has

$$|f|_{A,B,p_{w,K,D}} := \sup_{\eta \in \hat{K}_{G_m}} \sum_{\eta \in \hat{K}_{G_m}} |M_{w,D.f_{\eta_s}}(k)|_{A,B,p_{w,\eta}} < \infty.$$

This definition is a refinement of the definition of the Schwartz space given in loc. cit. We have dropped the $K$-finiteness assumption. In its place we have added a condition on the behavior of the functions under the infinitesimal action of $\mathfrak{g}$ (to guarantee smoothness of sections as a function of $k$) and control over the sum over $\eta$ of the given sections. In the $K$-finite case the sum over $\eta$ is finite so asserting that the sum over $\eta$ is bounded is
superfluous. Though the definition of the seminorms \(|\Phi|_{A,B,p_w,K,D}\) depends on \(K\), since all maximal compact subgroups of \(\text{Sp}_n(F)\) are \(P(F)\)-conjugate it follows that the topology on \(\mathcal{S}(X(F))\) is independent of the choice of \(K\).

We still have a well-defined Fourier transform
\[
(10.1.3) \quad \mathcal{F}_X : \mathcal{S}(X(F)) \rightarrow \mathcal{S}(X(F))
\]
and we still have
\[
(10.1.4) \quad C_\infty^\infty(X_P(F)) < \mathcal{S}(X(F)).
\]
(see [GL19a, Proposition 4.7]).

To complete our discussion of \(\mathcal{S}(X(F))\) we must endow it with a topology. The expression (10.1.2) is a seminorm on \(\mathcal{S}(X(F))\) and the collection of these seminorms as \(A,B,p_w,D\) vary gives \(\mathcal{S}(X(F))\) the structure of a locally convex space. This structure, at least a priori, depends on \(K\), but as remarked above the underlying topology is independent of \(K\).

**Lemma 10.3.** The space \(\mathcal{S}(X(F))\) is a Fréchet space.

**Proof.** We first observe that we can replace the family of seminorms with a countable sub-family inducing the same topology. More specifically we can choose a countable basis of \(U(g)\), and restrict the \((A,B)\) to lie in the set \(\{(-N,N) : N \in \mathbb{Z}_{\geq 0}\}\). Since the poles of \(a_w(s,\eta)\) can only occur at points in \(\frac{1}{2}\mathbb{Z}\) (see [GL19a, (3.0.4)]) we can similarly restrict our attention to a countable family of \(p_w,\eta\).

Now suppose for each \(\eta \in \hat{K}_{\mathbb{G}_m}\) we are given a section \(f^{(s)} \in I(\eta_s)\) such that all of the seminorms described above are finite. Then by Mellin inversion \(f^{(s)} = f_{\eta_s}\) for a unique \(f \in \mathcal{S}(X(F))\) (see [GL19a, Lemma 4.3]). Thus the countable family of seminorms described above is separating. It follows that \(\mathcal{S}(X(F))\) is Hausdorff and metrizable. It is also clear that it is complete. \(\square\)

It is useful to explicitly state and prove a refinement of [GL19a, Lemmas 5.1 and 5.7]. The group \(\text{Sp}_{2n}\) acts on \(\wedge^n \mathbb{G}_a^{2n}\) via its action on \(\mathbb{G}_a^{2n}\). Choose a \(K\)-invariant bilinear form \((\cdot,\cdot)\) on \(\wedge^n F^{2n}\) and set \(|x| = (x,x)^{|F:R|/2}\). We then set
\[
(10.1.5) \quad |g| = |\text{Pl}(g)|
\]
where \(\text{Pl} : X_P \rightarrow \wedge^n \mathbb{G}_a^{2n}\) is the Plücker embedding from above.

**Lemma 10.4.** If \(F\) is nonarchimedean then any \(f \in \mathcal{S}(X(F))\) is compactly supported and satisfies
\[
|f(g)| \ll f \frac{|g|^{-\frac{n+1}{2}}}{2}.
\]
If \(F\) is archimedean then for each \(N \in \mathbb{Z}_{\geq 0}\) and \(D \in U(g)\) there is a continuous seminorm \(\nu_{D,N}\) on \(\mathcal{S}(X(F))\) such that for \(f \in \mathcal{S}(X(F))\) one has
\[
|f(g)| \leq \nu_{D,N}(f)|g|^{-\frac{n+1}{2}}.
\]
Proof. The first assertion is just [GL19a, Lemma 5.1]. For the second assertion using Mellin inversion [GL19a, Lemma 4.3] we write

\[(\omega(m))^{-N} D.f(mk) = \delta_P(m)^{1/2} \sum_{n \in \mathbb{Z}} \int_{i\mathbb{R}+\sigma} (D.f)_{\eta_{n+2N[\mathbb{R}]}^{-1}}(k) \eta_s(\omega(m)) \frac{ds}{4\pi|\mathbb{R}|^{1/2}}.\]

The factor \(a_{I_{2n}}(s, \chi)\) is holomorphic in the half plane \(\text{Re}(s) > -\frac{1}{2}\) (see [GL19a, (3.0.3)]), and

\[\frac{a_{I_{2n}}(s + \frac{2N}{[\mathbb{R}]}}, \chi)}{a_{I_{2n}}(s, \chi)}\]

is a polynomial in \(s\) by [GL19a, Lemma 5.6]. Thus using the definition of an excellent section we can shift the contour to \(\sigma = 0\) to see that the above is

\[\delta_P(m)^{1/2} \sum_{n \in \mathbb{Z}} \int_{i\mathbb{R}} (D.f)_{\eta_{n+2N[\mathbb{R}]}^{-1}}(k) \eta_s(\omega(m)) \frac{ds}{4\pi|\mathbb{R}|^{1/2}}\]

\[\leq \delta_P(m)^{1/2} \frac{\{2 |f|_{A,B,1,K,D} + |f|_{A,B,s^2,K,D}\}}{4\pi|\mathbb{R}|^{1/2}}\]

where \(A := -\frac{1}{4} + \frac{2N}{[\mathbb{R}]}\) and \(B := \frac{1}{4} + \frac{2N}{[\mathbb{R}]}\). Since \(|mk| = |\omega(m)|^{-1}\) and \(\delta_P(m) = |m|^{-n+1}\) we deduce the lemma. \(\square\)

We prove the following lemma for use in the proof of Theorem 11.9:

**Lemma 10.5.** Assume \(F\) is archimedean. The space \(\mathcal{S}(X(F))\) contains \(\mathcal{S}(X(F), K)\) as a dense subspace. The Fourier transform

\[\mathcal{F}_X : \mathcal{S}(X(F)) \rightarrow \mathcal{S}(X(F))\]

is continuous.

**Proof.** The first assertion follows from the usual argument (see [War72, §4.4.3.1]).

We recall from [GL19a, Theorem 4.4] that \(\mathcal{F}_X(f)\) is the unique function in \(\mathcal{S}(X(F))\) such that

\[(10.1.6) \quad \mathcal{F}_X(f)_{\chi_s} = M_{w_0}^*(f_{\chi_s})\]

where \(M_{w_0}^*\) is the normalized intertwining operator of [GL19a, (3.0.5)]. Thus

\[|D.M_w\mathcal{F}_X(f)_{\chi_s}(k)|_{A,B,pw} = |D.M_wM_{w_0}^*(f_{\chi_s})(k)|_{A,B,pw}.\]

Using the argument of [GL19a, Lemma 3.4], this is bounded by a constant depending on \(A\) and \(B\) and \(\chi\) times

\[(10.1.7) \quad |D.M_w'(f_{\chi_s})(k)|_{A,B,pw'}\]

where

\[w' = \begin{cases} w_0 & \text{if } w = \text{Id} \\ \text{Id} & \text{if } w = w_0. \end{cases}\]
The second assertion of the lemma follows. \( \Box \)

For the next lemma \( F \) can be archimedean or nonarchimedean:

**Lemma 10.6.** If \((m, g) \in M^{ab}(F) \times \text{Sp}_{2n}(F)\) and \( f \in \mathcal{S}(X(F))\) then \( L(m) R(g) f \in \mathcal{S}(X(F))\).

Moreover

\[
(10.1.8) \quad \mathcal{F}_X(L(m) R(g) f) = \delta_p(m)^{-1} L(m^{-1}) R(g) \mathcal{F}_X(f).
\]

**Proof.** One has

\[
(L(m) R(g) f)_{\chi_s} = \delta_p(m)^{-1/2} \chi_s(\omega(m)^{-1}) R(g) f_{\chi_s}.
\]

Thus the assertion that \( L(m) R(g) f \in \mathcal{S}(X(F))\) follows immediately from the definition of \( \mathcal{S}(X(F))\).

Now

\[
\mathcal{F}_X(L(m) R(g) f)_{\chi_s} = M_{\omega_0}(L(m) R(g) f)_{\chi_s} \text{ by } (10.1.6)
\]

\[
= \delta_p(m)^{-1/2 \chi_s(\omega(m)^{-1})} R(g) M_{\omega_0}(f)_{\chi_s}
\]

\[
= \delta_p(m)^{-1/2 \chi_s(\omega(m)^{-1})} R(g) \mathcal{F}_X(f)_{\chi_s} \text{ by } (10.1.6)
\]

\[
= \delta_p(m)^{-1} (L(m^{-1}) R(g) \mathcal{F}_X(f))_{\chi_s}.
\]

Thus by [GL19a, Theorem 4.4] one has \( \mathcal{F}_X(L(m) R(g) f) = \delta_p(m)^{-1} L(m^{-1}) R(g) \mathcal{F}_X(f) \). \( \Box \)

Assume now that \( F \) is nonarchimedean. By the Iwasawa decomposition, a \( \mathbb{C} \)-vector space basis for \( C_c^\infty(X_P(F))^{\text{Sp}_{2n}(\mathcal{O})} \) is given by the functions

\[
(10.1.9) \quad 1_k := 1_{[P,P](F)} \left( \begin{array}{cc} \omega^{-k} & I_{n-1} \\ I_n & \omega^k \end{array} \right)^{\text{Sp}_{2n}(\mathcal{O})}
\]

for \( k \in \mathbb{Z} \). The space \( \mathcal{S}(X(F))^{\text{Sp}_{2n}(\mathcal{O})} \) contains \( C_c^\infty(X_P(F))^{\text{Sp}_{2n}(\mathcal{O})} \) but it is larger. It contains, for example, the basic function

\[
(10.1.10) \quad b_X := \sum_{(j_1, \ldots, j_{[n/2]}, k) \in \mathbb{Z}_{\geq 0}^{[n/2]+1}} q^{2j_1+4j_2+\cdots+2[n/2]j_{[n/2]}} 1_{k+2j_1+\cdots+2j_{[n/2]}}.
\]

One has \( \mathcal{F}_X(b_X) = b_X \) [GL19a, Lemma 5.4] provided that \( \psi \) is unramified.

It will be convenient to isolate another family of functions in this space. Let

\[
(10.1.11) \quad m(\omega) := \left( \begin{array}{cc} \omega^{-1} & I_{n-1} \\ I_n & \omega \end{array} \right)
\]

and let

\[
(10.1.12) \quad 1_{\geq c} := \sum_{k \geq c} 1_k.
\]

**Lemma 10.7.** One has \( 1_{\geq c} \in \mathcal{S}(X(F))^{\text{Sp}_{2n}(\mathcal{O})} \).
Proof. One has \( L(m(\varpi)^c) 1_{\geq 0} = 1_{\geq c} \) so by Lemma 10.6 it suffices to show \( 1_{\geq 0} \in \mathcal{S}(X(F))^{Sp_{2n}(\mathbb{O})} \).

Since
\[
\left( \prod_{j=1}^{\lfloor n/2 \rfloor} (1 - q^{2j} L(m(\varpi)^2)) \right) b_X = 1_{\geq 0},
\]
we can apply Lemma 10.6 again to deduce the result. \( \square \)

The following computation was used in \( \S 7 \):

Lemma 10.8. When \( n = 3 \) one has
\[
\mathcal{F}_X(1_0) = \sum_{j=0}^{\infty} q^{2j} \left( 1_{\geq 2j} - q^{-6} 1_{\geq 2j-2} - q^{-4} 1_{\geq 2j-1} + q^{-10} 1_{\geq 2j-3} \right).
\]

Proof. One has \( 1_0 = 1_{\geq 0} - 1_{\geq 1} \). Thus by the computation in the proof of Lemma 10.7 one has
\[
\mathcal{F}_X(1_0) = \mathcal{F}_X(1_{\geq 0} - 1_{\geq 1}) = \mathcal{F}_X((1 - q^2 L(m(\varpi)^2))b_X - L(m(\varpi))(1 - q^2 L(m(\varpi)^2))b_X).
\]

Using Lemma 10.6 we see that this is
\[
\left( 1 - \frac{q^2}{\delta_P(m(\varpi)^2)} L(m(\varpi)^{-2}) \right) b_X - \delta_P(m(\varpi))^{-1} L(m(\varpi)^{-1}) \left( 1 - \frac{q^2}{\delta_P(m(\varpi)^2)} L(m(\varpi)^{-2}) \right) b_X
\]
\[
= (1 - q^{-6} L(m(\varpi)^{-2}) \sum_{j=0}^{\infty} q^{2j} 1_{\geq 2j} - q^{-4} L(m(\varpi)^{-1})(1 - q^{-6} L(m(\varpi)^{-2})) \sum_{j=0}^{\infty} q^{2j} 1_{\geq 2j}
\]
\[
= \sum_{j=0}^{\infty} q^{2j} \left( 1_{\geq 2j} - q^{-6} 1_{\geq 2j-2} - q^{-4} 1_{\geq 2j-1} + q^{-10} 1_{\geq 2j-3} \right).
\]
\( \square \)

10.2. The summation formula. We now revert to the global setting. We define
\[
\mathcal{S}(X(F_\infty)) := \hat{\otimes}_{v | \infty} \mathcal{S}(X(F_v))
\]
where the hat denotes the projective topological tensor product. We let
(10.2.1) \[
\mathcal{S}(X(\mathbb{A}_F)) := \mathcal{S}(X(F_\infty)) \otimes \hat{\otimes}_{v | \infty} \mathcal{S}(X(F_v)),
\]
where the restricted direct product is with respect to the basic functions \( b_{X,v} \). It comes equipped with a Fourier transform \( \mathcal{F}_X \) that is the tensor product of the local Fourier transforms.

There is one place in [GL19a] where dropping the assumption that \( f \in \mathcal{S}(X(\mathbb{A}_F)) \) is \( K_\infty \)-finite could potentially cause issues. This is that Eisenstein series formed from smooth sections (as opposed to \( K \)-finite sections) need not be of finite order [GL06, \S 2.3, Remark 1]. The fact that Eisenstein series are of finite order was used in [GL19a] to prove a Poisson
summation formula for $X$. We require an analogous formula for functions in $S(X(\mathbb{A}_F))$ so we must proceed slightly differently.

Recall that $C_c^\infty(X_P(F_v)) < S(X(F_v))$ by [GL19a, Proposition 4.7].

**Theorem 10.9.** Assume that some finite places $v_1, v_2$ (not necessarily distinct) one has $f = f_{v_1}f_{v_2}$ and $\mathcal{F}_X(f) = \mathcal{F}_X(f_{v_1})\mathcal{F}_X(f_{v_2})$ with

$$f_{v_1} \in C_c^\infty(X_P(F_{v_1})) \quad \text{and} \quad \mathcal{F}_X(f_{v_2}) \in C_c^\infty(X_P(F_{v_2})).$$

Then

$$\sum_{\gamma \in X(F)} f(\gamma) = \sum_{\gamma \in X(F)} \mathcal{F}_X(f)(\gamma). \quad (10.2.2)$$

**Proof.** Let $K_\infty \leq \text{Sp}_{2n}(F_\infty)$ be a maximal compact subgroup and let $S(X(F_\infty), K_\infty) \leq S(X(F_\infty))$ be the space of $K_\infty$-finite functions. We may assume $f = f_\infty f^\infty$ with $f_\infty \in S(X(F_\infty))$ and $f^\infty \in S(X(\mathbb{A}_F^\infty))$. Assume first that $f_\infty$ is $K_\infty$-finite. Then the stated identity follows from [GL19a, Theorem 1.1] and [GL19b, Theorem 10.1].

We now argue by continuity to deduce the identity in general. Using the estimates in Lemma 10.4 and the convergence argument in [GL19b, Proposition 9.2], it suffices to construct a sequence $f_{n, \infty} \in S(X(F_\infty))$ of $K_\infty$-finite functions such that $f_n \to f_\infty$ and $\mathcal{F}_X(f_{n, \infty}) \to \mathcal{F}(f_\infty)$ in the topology on $S(X(F_\infty))$. But $S(X(F_\infty), K_\infty)$ is dense in $S(X(F_\infty))$ and $\mathcal{F}_X$ is continuous by Lemma 10.5. \qed

We remark that Theorem 10.9 was already proved in [BK02], but with a different definition of the Schwartz space. At least at the nonarchimedean places, the two definitions should yield the same space of functions. At the archimedean places this is less clear. In any case, it is easier to just prove the theorem directly than to rigorously check the compatibility of the two definitions.

11. Triples of quadratic spaces

Let

$$V_i, \quad 1 \leq i \leq 3$$

be a triple of vector spaces of even dimension over the number field $F$ and for each $i$ let $Q_i$ be a nondegenerate quadratic form on $V_i$. A special case of this notation was used in §3. The quadratic form $Q_i$ together with our fixed additive character $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}^\times$ give rise to a Weil representation

$$\rho : \text{SL}_2(\mathbb{A}_F) \times O_{V_i}(\mathbb{A}_F) \times S(V_i(\mathbb{A}_F)) \longrightarrow S(V_i(\mathbb{A}_F))$$

where $O_{V_i}$ is the orthogonal group of $V_i$. 

From now on we take \( n = 3 \) in the constructions of \( \S 10 \), so \( X_P := [P, P] \backslash \mathrm{Sp}_6 \), etc. There is an injection

\[
\mathrm{SL}_3^2(R) \rightarrow \mathrm{Sp}_6(R)
\]

\[
\left( \begin{array}{cc}
(a_1, b_1) \\
(c_i, d_i)
\end{array} \right) \mapsto \left( \begin{array}{cccc}
a_1 & a_2 & b_1 & b_2 \\
0 & a_3 & 0 & b_3 \\
0 & 0 & d_3 & d_2 \\
c_1 & c_2 & c_3 & d_1
\end{array} \right).
\]

Identifying \( \mathrm{SL}_3^2 \) with its image it is well-known that \( X(F)/\mathrm{SL}_3^2(F) \) is finite. There is a unique open \( \mathrm{SL}_3^2 \)-orbit in \( X \) and as a representative we may take

\[
\gamma_0 := \left( \begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{array} \right).
\]

(11.0.3)

The stabilizer of \( \gamma_0 \) under \( \mathrm{SL}_3^2 \) is the group \( N_0 \) whose points in an \( F \)-algebra \( R \) are given by

\[
N_0(R) := \{((1 \ t_1), (1 \ t_2), (1 \ -t_1 - t_2)) : t_1, t_2 \in R \}.
\]

(11.0.4)

For proofs and or references for all of this we refer to [GL19b, \S 2].

Let \( V := \prod_{i=1}^3 V_i \). For \( F \)-algebras \( R \) let

\[
Y(R) := \{ (\xi_1, \xi_2, \xi_3) \in V(R) : Q_1(\xi_1) = Q_2(\xi_2) = Q_3(\xi_3) \}.
\]

(11.0.5)

In [GL19b] we defined Schwartz spaces for \( Y \) and proved a Poisson summation formula using them. For the current paper in the archimedean case we require a refinement of the notion of the Schwartz space from [GL19b] which is given below. In particular we give it the structure of a Fréchet space. The main motivation for the refinement is Lemma 11.3 below, which is used in Theorem 5.1 above. Unfortunately the proof of the Poisson summation formula in [GL19b] cannot be directly quoted in order to prove a Poisson summation formula for the refinement of the Schwartz space presented in this paper. Essentially one has to pass to a limit. Because of this we indicate the changes that must be made. As mentioned in the introduction, this has the advantage of making the current paper relatively self-contained.

For \( f_1 \in S(X(\mathbb{A}_F)) \), \( f_2 \in S(V(\mathbb{A}_F)) \) and \( \xi \in Y^\text{sm}(\mathbb{A}_F) \) let

\[
I(f_1 \otimes f_2)(\xi) := \int_{N_0(\mathbb{A}_F) \backslash \mathrm{SL}_3^2(\mathbb{A}_F)} f_1(\gamma_0 g) \rho(g) f_2(\xi) d\dot{g}.
\]

(11.0.6)

This is absolutely convergent by [GL19b, Propositions 6.3, 7.1, 8.2, 8.3]. We define \( I(f) \) for any \( f \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \) in the obvious manner and set

\[
S(Y(\mathbb{A}_F)) = \langle I(f) : f \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \rangle
\]

(11.0.7)

where the brackets denote the \( \mathbb{C} \)-span. We use the analogous local notation.
11.1. Comments on the local Schwartz spaces. We work locally in this section until the last paragraph. Thus let \( v \) be a place of \( F \) which we omit from notation, writing \( F := F_v \). Assume for the moment that \( F \) is archimedean.

**Lemma 11.1.** Let \( N_1, N_2, N_3 \in \mathbb{Z}_{\geq 0} \). There is a continuous seminorm \( \nu_{N_1, N_2, N_3} \) on \( \mathcal{S}(X(F) \times U^\vee(F)) \) such that

\[
|I(f)(v_1, v_2, v_3)| \leq \nu_{N_1, N_2, N_3}(f) \prod_{i=1}^{3} \max(|v_i|, 1)^{-N_i \min(|v_i|, 1)}^{-1 - \dim V_i/2}
\]

provided that no \( v_i = 0 \). If \( v_1 = 0 \) but \( v_2 \) and \( v_3 \) are nonzero then there is a continuous seminorm \( \nu_{N_2, N_3} \) on \( \mathcal{S}(X(F) \times U^\vee(F)) \) such that

\[
|I(f)(0, v_2, v_3)| \leq \nu_{N_2, N_3}(f) \prod_{i=2}^{3} \max(|v_i|, 1)^{-N_i \min(|v_i|, 1)}^{-\dim V_1/2 - \dim V_i/2}
\]

**Proof.** In view of Lemma 10.4 this follows from the proofs of [GL19b, Proposition 8.2 and 8.3].

**Lemma 11.2.** The kernel of the map

\( I: \mathcal{S}(X(F) \times V(F)) \longrightarrow C^\infty(Y^\text{sm}(F)) \)

is closed.

**Proof.** If \((v_1, v_2, v_3) \in Y^\text{sm}(F)\) then at most one \( v_i \) is equal to zero. Thus by Lemma 11.1 (and symmetry, in the case where \( v_2 = 0 \) or \( v_3 = 0 \)) for every \( \xi \in Y^\text{sm}(F) \) the linear form \( f \mapsto I(f)(\xi) \) is continuous. The kernel in the statement of the lemma is the intersection of the kernels of these linear forms.

We give \( \mathcal{S}(Y(F)) = \mathcal{S}(X(F) \times V(F))/\ker I \) the quotient topology (which is Fréchet). For \( F \) archimedean or nonarchimedean let

\( \mathcal{S} := \text{Im}(\mathcal{S}(V(F)) \rightarrow C^\infty(Y^\text{sm}(F))) \)

where the implicit map is restriction of functions. We observe that \( C^\infty_c(Y^\text{sm}(F)) < \mathcal{S} \).

Moreover, we have the following result:

**Lemma 11.3.** One has \( \mathcal{S} \leq \mathcal{S}(Y(F)) \).

**Proof.** Let \( f_1 \in C^\infty_c(X_P(F)) \), so \( f_1 \in \mathcal{S}(X(F)) \) (see (10.1.4)). We assume that

\[
g \mapsto f_1(\gamma_0 g)
\]

is supported in the open Bruhat cell \( (B(F)w_0B(F))^3 \subset \text{SL}_2^3(F) \) where \( B \leq \text{SL}_2 \) is the Borel subgroup of upper triangular matrices and \( w_0 = (\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}) \). For \( g \in Y^\text{sm}(F) \) and \( f \in \mathcal{S} \) consider

\[
I(f_1 \otimes f)(y) = \int_{N_0(F) \setminus \text{SL}_2^2(F)} f_1(\gamma_0 g) \rho(g) f(y) dg.
\]
Writing this as an integral over the open Bruhat cell we obtain
\[ \int_{F \times B(F)^3} f_1(\gamma_0(1 \frac{t}{1}) w_0 b) \rho(\gamma_0(1 \frac{t}{1}) w_0 b) f(y) dt \delta_{B^3}(b) dt b. \]

Here \((1 \frac{t}{1})\) is shorthand for \(((1 \frac{t}{1}), (1 \frac{t}{1}), (1 \frac{t}{1}))\).

By choosing \(f_1\) appropriately we can make this equal to
\[ \int_{F \times B(F)^3} h_1(t) h_2(b) \rho(\gamma_0(1 \frac{t}{1}) w_0 b) f(y) dt \delta_{B^3}(b) dt b \]
for any \(h_1 \in C_c^\infty(F)\) and \(h_2 \in C_c^\infty(B(F))\). In the nonarchimedean case it is clear that we can choose \(h_1\) and \(h_2\) so that this integral is equal to \(f'(y)\) for any \(f' \in \mathcal{S}(V(F))\).

The corresponding assertion in the archimedean case follows as well by taking appropriate limits. □

Let
\[ \nu_i : GO_{V_i} \longrightarrow \mathbb{G}_m \]
be the similitude norm. Temporarily let
\[ (11.1.1) \quad G(R) := \{ (g_1, g_2, g_3) \in \prod_{i=1}^{3} GO_{V_i}(R) : \nu_1(g_1) = \nu_2(g_2) = \nu_3(g_3) \}. \]

We let \(\prod_{i=1}^{3} GO_{V_i}\) act on \(V\) in the natural manner. The restriction of this action to \(G\) preserves the subscheme \(Y \subset V\). The following is an easy consequence of [GL19b, Lemma 4.3]:

**Lemma 11.4.** If \(f \in \mathcal{S}(Y(F))\) then \(L(g)f \in \mathcal{S}(Y(F))\). □

Finally we discuss bounds. For \(v = (v_1, \ldots, v_{\dim V_i}) \in V_i(F)\) let
\[ (11.1.2) \quad |v| = \max_i |v_i|. \]

**Lemma 11.5.** Assume that \(f \in \mathcal{S}(Y(F))\) and \(y = (y_1, y_2, y_3) \in Y(F)\) where no \(y_i\) is zero. There is a nonnegative \(\Phi \in \mathcal{S}(V(F))\) such that
\[ |f(y)| \leq \Phi(y) \prod_{i=1}^{3} |y_i|^{-1 - \dim V_i / 2}. \]

Let \(\varepsilon > 0\) be given. If \(F\) is nonarchimedean and absolutely unramified and the residual characteristic of \(F\) is large enough in terms of \(\varepsilon\) then
\[ I(b_X \otimes \mathbb{1}_{V(\mathcal{O})})(y) \leq \mathbb{1}_{V(\mathcal{O})}(y) \prod_{i=1}^{3} |y_i|^{1 - \dim V_i / 2 - \varepsilon}. \]

**Proof.** In the nonarchimedean case this is [GL19b, Lemma 6.4 and Proposition 7.1]. In the archimedean case this follows from Lemma 11.1. □
We now revert to global notation, so $F$ is a number field. Lemma 11.5 immediately implies the following adelic statement: for $f \in \mathcal{S}(Y(\mathbb{A}_F))$ there is a Schwartz function $\Phi \in \mathcal{S}(V(\mathbb{A}_F))$ such that
\[(11.1.3) \quad |f(y)| \leq \Phi(y) \prod_{i=1}^{3} |y_i|^{-1-\dim V_i/2}.
\]

11.2. A summation formula for triples of quadratic spaces. In [GL19b] a summation formula for $Y(F)$ was proved under some restrictions on the test function $f \in \mathcal{S}(Y(\mathbb{A}_F))$. It is convenient to work with slightly different test functions in the current setting and we explain how to derive the summation formula under these new assumptions. The main result is Theorem 11.9 below.

For $f_2 \in \mathcal{S}(V(\mathbb{A}_F))$ and $g \in \text{SL}_3^2(\mathbb{A}_F)$ let
\[\Theta_{f_2}(g) := \sum_{\xi \in V(F)} \rho(g)f_2(\xi).\]

Here $\rho$ is the Weil representation as in (11.0.2) above.

Let $V_0 \subset V$ be the closed subscheme whose points in an $F$-algebra $R$ are given by
\[(11.2.1) \quad V_0(R) := \{(\xi_1, \xi_2, \xi_3) \in V(R) : Q_i(\xi_i) = 0 \text{ for any } i\}.
\]

Consider the following assumption on $f_2 \in \mathcal{S}(V(F_v))$:
\[(11.2.2) \quad \text{For all } g \in \text{SL}_3^2(F_v) \text{ one has } \text{supp}(\rho(g)f_2) \cap V_0(F_v) = \emptyset.
\]

The import of this assumption is the following consequence:

**Lemma 11.6.** Under assumption (11.2.2), $\Theta_{f_2}(g)$ is a cuspidal function on $[\text{SL}_3^2]$.

**Proof.** Let $N_2 \leq \text{SL}_2$ be the unipotent radical of the Borel subgroup of upper triangular matrices and let $\iota_i : N_2 \hookrightarrow \text{SL}_3^2$ be the inclusion of $N_2$ into the $i$th factor. For all $g \in \text{SL}_3^2(\mathbb{A}_F)$ one has
\[
\int_{[N_2]} \Theta_{f_2}(\iota_i(n)g)dn = \sum_{\xi \in V(F)} \int_{[N_2]} \rho(\iota_i(n)g)f_2(\xi)dn
= \sum_{\xi \in V(F)} \rho(g)f_2(\xi).
\]

Let $f_1 \in \mathcal{S}(X(\mathbb{A}_F))$ (see §10 for notation). The following is an analogue of the well-known fact that Eisenstein series are of moderate growth:

**Lemma 11.7.** The function $g \mapsto \sum_{\gamma \in X(F)} |f_1(\gamma g)|$ is of moderate growth on $[\text{Sp}_6]$. 

Proof. There is an element \( r \in \mathcal{O} \cap F^\times \) such that for any \( N \geq 2 \)

\[
\sum_{\gamma \in X(F)} |f_1(\gamma g)| \ll_{N, f_1} \sum_{\xi \in \mathfrak{N}^{-1} \wedge \mathcal{O}^6 \setminus \{0\}} \left( \prod_{v \mid \infty} |\xi|_v^{-N} \right) \left( \prod_{v \in S_{-\infty}} |\xi|_v^{-2} \right) \prod_{v \not\in S} |\xi|_v^{-2-\varepsilon}
\]

by Lemma 10.4. We may as well assume that \( \Pi(g) \) lies in a given Siegel set for \( \text{GL}(\wedge^3 G^6) \) and then for any \( N' \geq 0 \) we can take \( N \) sufficiently large in the above to see that it is dominated by a constant depending on the Siegel set, \( \mathfrak{N} \), and \( N \) times

\[
(11.2.3) \sum_{\xi \in \mathfrak{N}^{-1} \wedge \mathcal{O}^6 \setminus \{0\}} \left( \prod_{v \mid \infty} |\xi|_v^{-N'} \right)
\]

This is bounded independently of \( \Pi(g) \) in a Siegel set (and hence for any \( g \)) by \([\text{Gar18}, \text{Claim 8.4.2}]\).

□

Lemma 11.8. Let \( f_1 \in \mathcal{S}(X(\mathbb{A}_F)) \) and \( f_2 \in \mathcal{S}(V(\mathbb{A}_F)) \). If \( \Theta f_2 \) is cuspidal one has

\[
(11.2.4) \int_{[\text{SL}_2^3]} \sum_{\gamma \in X(F)} f_1(\gamma g)\Theta f_2(g) dg = \sum_{\xi \in \mathfrak{Y}^{\text{type}}(F)} I(f_1 \otimes f_2)(\xi).
\]

The sum on the right is absolutely convergent.

Proof. We adopt the notation of the proof of \([\text{GL19b, Theorem 5.3}]\) in what follows. In particular for the proof we let \( G := \text{SL}_2^3 \). In loc. cit. the integral on the left was unfolded to a sum indexed by a particular (finite) minimal set of representatives \( \{ \gamma_a \} \) of \( X(F)/G(F) \):

\[
(11.2.5) \int_{[G]} \sum_{\gamma \in X(F)} f_1(\gamma g)\Theta f_2(g) dg = \sum_{\gamma_a} \int_{G_{\gamma_a}(F) \backslash G(\mathbb{A}_F)} f_1(\gamma_a g)\Theta f_2(g) dg.
\]

where \( G_{\gamma_a} \) is the stabilizer of \( \gamma_a \). By assumption \( \Theta f_2 \) is cuspidal and hence rapidly decreasing. Combining this with Lemma 11.7 we see that

\[
\int_{[G]} \sum_{\gamma \in X(F)} |f_1(\gamma g)\Theta f_2(g)| dg < \infty
\]

and hence each integral on the right of (11.2.5) is absolutely convergent.

In loc. cit. we placed an assumption on \( f_1 \) to ensure that only the open orbit, represented by \( \gamma_0 \), contributed. In the setting of the current lemma we can argue that the contribution of any class in \( X(F)/G(F) - [P, P](F)\gamma_0 G(F) \) vanishes as follows: If \( \gamma_a \neq \gamma_0 \) then by \([\text{GL19b, Lemma 2.3}]\) the stabilizer \( G_{\gamma_a} \) contains a subgroup forcing the contribution of \( \gamma_a \) to vanish since \( \Theta f_2(g) \) is cuspidal. Thus the sum above has just one nonzero term:

\[
\int_{G_{\gamma_0}(F) \backslash G(\mathbb{A}_F)} f_1(\gamma_0 g)\Theta f_2(g) dg.
\]

One then proceeds exactly as in the proof of \([\text{GL19b, Theorem 5.3}]\) to complete the proof. □
ON TRIPLE PRODUCT L-FUNCTIONS

The automorphism $F_X : S(X(F_\infty)) \to S(X(F_\infty))$ is continuous by Lemma 10.5 and hence

$$F_X := F_X \otimes 1 : S(X(\mathbb{A}_F)) \otimes S(V(\mathbb{A}_F)) \to S(X(\mathbb{A}_F)) \otimes S(V(\mathbb{A}_F))$$

extends by continuity to

$$F_X : S(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \to S(X(\mathbb{A}_F) \times V(\mathbb{A}_F)).$$

We now prepare local assumptions for our summation formula. Assume that there are places $v_1, v_2, v_3$ of $F$ (not necessarily distinct) with $v_1, v_2$ finite such that

$$f = f_{v_1} f_{v_2} f_{v_3}$$

where $f_{v_i} = f_{1v_i} \otimes f_{2v_i}$ with $f_{1v_i} \in S(X(F_{v_i}))$ and $f_{2v_i} \in S(V(F_{v_i}))$. Assume moreover that

1. $f_{1v_1} \in C_c^\infty(X_P(F_{v_1}))$,
2. $F_X(f_{1v_2}) \in C_c^\infty(X_P(F_{v_2}))$,
3. for all $g \in \text{SL}_2^3(F_{v_3})$ one has $\text{supp}(\rho(g)f_2) \cap V_0(F_{v_3}) = \emptyset$

with $V_0$ defined as in (11.2.1).

**Theorem 11.9.** Under assumptions (1-3) above one has

$$\sum_{\xi \in Y^{\text{an}}(F)} I(f)(\xi) = \sum_{\xi \in Y^{\text{an}}(F)} I(F_X(f))(\xi).$$

The sums over $\xi$ are absolutely convergent.

**Proof.** Assume first that $f = f_1 \otimes f_2$ where $(f_1, f_2) \in S(X(\mathbb{A}_F)) \times S(V(\mathbb{A}_F))$. The function $\Theta_{f_2}$ is cuspidal by Lemma 11.6 and the identity of Theorem 10.9 is valid. We take this expression and integrate it times $\Theta_{f_2}(g)$ over $[\text{SL}_2^3]$. This yields the identity by Lemma 11.8. For arbitrary $f \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$ the identity follows by a continuity argument using Lemma 11.1, Lemma 10.5, and the estimates in the proof of [GL19b, Proposition 9.2]. □

11.3. **Some unramified Schwartz functions.** Let $v$ be a nonarchimedean place of $F$ which we omit from notation, writing $F$ for $F_v$. Assume that $\psi$ is unramified. In this subsection we explicitly describe a family of unramified functions in $S(Y(F))$.

Let $m_i = \dim V_i/2$ and $1 - m := (1 - m_1, 1 - m_2, 1 - m_3)$ and let $b_X \in S(X(F))$ be the basic function. Let

$$\chi_Q : (F^\times)^3 \to \mathbb{C}^\times$$

be the quadratic character attached to the $Q_i$ as in [GL19b, (3.1.2)]. When the spaces $V_i$ are split quadratic spaces (that is, have totally isotropic subspace of dimension $m_i$) then $\chi_Q = 1$. By a minor modification of the proof of [GL19b, Proposition 6.3] one obtains the following useful lemma:
Lemma 11.10. For $\alpha_0 \in \mathbb{Z}$ and any function $f_2 \in S(V(F))$ such that $\rho(k)f_2 = f_2$ for $k \in \text{SL}_2(\mathcal{O})$ one has

\begin{equation}
I(\mathbb{1}_{\geq \alpha_0} \otimes f_2)(\xi) = q^{-\alpha_0} \int_{\mathcal{O}} \left( \sum_{i=1}^{3} Q_i(\xi) \right) f_2 \left( \frac{\xi}{b_1b_2b_3^2c} \right) \chi_{Q}(b)b^{\frac{1-m}{d}}d^x b
\end{equation}

where the integral is over $b_1, b_2, b_3 \in \mathcal{O}$ satisfying

\begin{equation}
\max(|b_1^{-1}b_2b_3|, |b_2^{-1}b_1b_3|, |b_3^{-1}b_1b_2|) \leq 1.
\end{equation}

Moreover

\begin{equation}
I(b_X \otimes f_2)(\xi) = \sum_{j=0}^{\infty} \int_{\mathcal{O}} \left( \sum_{i=1}^{3} Q_i(\xi) \right) f_2 \left( \frac{\xi}{b^{2j}} \right) \chi_{Q}(b)b^{2j}\frac{1-m}{d}d b
\end{equation}

where the integral is again over $b_1, b_2, b_3 \in \mathcal{O}$ satisfying (11.3.2).

This was used in a special case in §7 above. We have stated it in the more general setting here for future reference.
List of symbols

\( \mathbb{1}_k \) element of \( \mathcal{S}(X(F)) \) (10.1.9)
\( \mathbb{1}_{\geq c} := \sum_{k \geq c} \mathbb{1}_k \) element of \( \mathcal{S}(X(F)) \) (10.1.12)
\( A_G \) \S 2.3
\( |a|^s \) for \( s \in \mathbb{C} \) (2.2.3)
\( b_X \) basic function in \( \mathcal{S}(X(F)) \) (10.1.10)
\( [\chi] := \prod_{i=1}^{3} \chi_i \) product of characters (2.2.2)
\( f_{oo} \) (7.0.1)
\( \mathcal{F}_{U^\vee} \) Fourier transform (4.0.7)
\( \mathcal{F}_{X} \) generalized Fourier transform (10.1.3)
\( \mathcal{F}_{X \times U^\vee} \) generalized Fourier transform (4.0.8)
\( \gamma_0 \) element of \( \text{Sp}_6(\mathbb{Z}) \) (11.0.3)
\( [G] = G(F) \backslash G(\mathbb{A}_F) \) adelic quotient (2.3.1)
\( G(\mathbb{A}_F)^1 \) norm one subgroup of \( G(\mathbb{A}_F) \) (2.3.2)
\( [G]^1 := G(F) \backslash G(\mathbb{A}_F)^1 \) adelic quotient (2.3.3)
\( G \) \( \mathbb{G}_m \times \text{GL}_3^3 \) (3.0.1)
\( I \) map on Schwartz spaces (11.0.6), (4.0.1)
\( J \) (4.0.5)
\( N_0 \) unipotent subgroup of \( \text{SL}_3^3 \) (11.0.4)
\( Q \) (6.0.1)
\( \rho \) Weil representation (11.0.2)
\( \text{Re}(\chi) \) \S 2.3.1
\( s_X \) \S 2.3.1
\( \mathcal{S}(X(F)) \) local Schwartz space \S 10.1
\( \mathcal{S}(X(\mathbb{A}_F)) \) adelic Schwartz space (10.2.1)
\( \mathcal{S}(Y(F)), \mathcal{S}(Y(\mathbb{A}_F)) \) Schwartz spaces (11.0.7)
\( U_i = \mathbb{G}_a^3 \) \S 3
\( U = \prod_{i=1}^{3} U_i, \ U^\vee = \prod_{i=1}^{3} U_i^\vee \) \S 3
\( V = \prod_{i=1}^{3} V_i \) vector space \S 11 and 3
\( V_0 \) open subscheme of \( V \) (11.2.1)
\( |v| \) norm of a vector (11.1.2)
\( W \) (5.0.1)
\( X_P \) \( [P, P] \backslash \text{Sp}_6 \) (10.0.2)
\( X \) affine closure of \( X_P \) (10.0.5)
\( \xi_0 \) (6.0.1)
\( Y \) closed subscheme of \( V \) (11.0.5)
\( Y^{\text{ani}} \) open subscheme of \( Y \) (3.0.3)
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