AUTOMORPHISMS AND IDEALS OF NONCOMMUTATIVE DEFORMATIONS
OF $\mathbb{C}^2/\mathbb{Z}_2$

XIAOJUN CHEN, ALIMJON ESHMATOV, FARKHOD ESHMATOV, AND VYACHESLAV FUTorny

Abstract. Let $O_\tau(\Gamma)$ be a family of algebras quantizing the coordinate ring of $\mathbb{C}^2/\Gamma$, where $\Gamma$ is a finite subgroup of $SL_2(\mathbb{C})$, and let $G_\Gamma$ be the automorphism group of $O_\tau$. We study the natural action of $G_\Gamma$ on the space of right ideals of $O_\tau$ (equivalently, finitely generated rank 1 projective $O_\tau$-modules). It is known that the later can be identified with disjoint union of algebraic (quiver) varieties, and this identification is $G_\Gamma$-equivariant. In the present paper, when $\Gamma \cong \mathbb{Z}_2$, we show that the $G_\Gamma$-action on each quiver variety is transitive. We also show that the natural embedding of $G_\Gamma$ into $Pic_{\mathbb{C}}(O_\tau)$, the Picard group of $O_\tau$, is an isomorphism. These results are used to prove that there are countably many non-isomorphic algebras Morita equivalent to $O_\tau$, and explicit presentation of these algebras are given. Since algebras $O_\tau(\mathbb{Z}_2)$ are isomorphic to primitive factors of $U(sl_2)$, we obtain a complete description of algebras Morita equivalent to primitive factors. A structure of the group $G_\Gamma$, where $\Gamma$ is an arbitrary cyclic group, is also investigated. Our results generalize earlier results obtained for the (first) Weyl algebra $A_1$ in [BW1] [BW2] and [S].

1. Introduction

Let $\mathbb{C}(x, y)$ be the free associative algebra on two generators. Then a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ acts naturally on $\mathbb{C}(x, y)$, and we can form the crossed product $R := \mathbb{C}(x, y) \ast \Gamma$. For each $\tau \in Z(\mathbb{C} \Gamma)$, the center of the group algebra $\mathbb{C} \Gamma$, let

$$S_\tau(\Gamma) := R/(xy - yx - \tau), \quad O_\tau(\Gamma) = eS_\tau e,$$

where $e$ is the symmetrizing idempotent $\frac{\sum_{g \in \Gamma} g}{|\Gamma|}$ in $\mathbb{C} \Gamma \subset S_\tau$. These algebras were introduced in [CBH]. One can easily verify that the associated graded algebra of $O_\tau$ with respect to its natural filtration is $\mathbb{C}[x, y]^\Gamma$. Hence algebras $O_\tau$ can be viewed as a quantization of the coordinate ring of the classical Kleinian singularity $\mathbb{C}^2/\Gamma$. When $\Gamma \cong \mathbb{Z}_2$ the algebra $O_\tau$ is isomorphic to a primitive factor of the enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$. On the other hand when $\tau = 1$ then $O_\tau$ is isomorphic to the algebra of $\Gamma$-invariant elements of $\mathbb{A}_1$.

Let $\mathbb{R}_\Gamma^+ \subset \mathbb{R}$ be the set of isomorphism classes of right ideals of $O_\tau$. Since, for generic values of $\tau$, $O_\tau$ is a simple hereditary domain, the set $\mathbb{R}_\Gamma^+$ can be identified with that of isomorphism classes of finitely generated rank one projective $O_\tau$-modules. In [BGK], a bijective map $\Omega$ was constructed from $\mathbb{R}_\Gamma^+$ to the disjoint union of quiver varieties $\mathcal{M}(\tau Q_\Gamma)$, where $Q_\Gamma$ is the quiver associated to $\Gamma$ under the McKay correspondence. Later in [E], the third author gave another more explicit construction of $\Omega$. Moreover, he showed that the group $G_\Gamma := \text{Aut}_{\mathbb{C}}(O_\tau)$ acts naturally on each $\mathcal{M}(\tau Q_\Gamma)$ and proved that $\Omega$ is a $G_\Gamma$-equivariant bijection.

In the case when $\Gamma \cong \mathbb{Z}_2$, the above $G_\Gamma$-equivariant bijection map is given by

$$\Omega : \mathcal{M}(\tau Q_\Gamma) = \bigcup_{\epsilon = 0, 1} \mathcal{M}(m, n; \epsilon) \longrightarrow \mathbb{R}_\Gamma^+,$$

where $L_\epsilon$ is a subset $(\mathbb{Z}_\geq 0)^2$ of dimension vectors of quiver representations $Q_\Gamma$ and $\epsilon = 0$ or 1 are indices corresponding to the trivial and the sign representations of $\mathbb{Z}_2$. Since the group $G_\Gamma$ acts on
each $\mathfrak{M}_r^G(m, n; \epsilon)$, we have the corresponding decomposition of $R_r^G$:

$$R_r^G = \bigcup_{\epsilon=0,1} \bigcup_{(m,n)\in I_\epsilon} R_r^G(m, n; \epsilon),$$

where $R_r^G(m, n; \epsilon) := \Omega(\mathfrak{M}_r^G(m, n; \epsilon))$.

The initial motivation for [BGK] and [E] came from a series of papers [BW1, BW2] by Y. Berest and G. Wilson. They showed that the space $\mathcal{R}$ of isomorphism classes of right ideals in $A_1$ are parametrized by the disjoint union of the Calogero-Moser algebraic varieties $\mathcal{C} = \bigsqcup_{n\geq 0} \mathcal{C}_n$:

$$\mathcal{C}_n := \{(X,Y) \in \text{Mat}_n(\mathbb{C}) \times \text{Mat}_n(\mathbb{C}) : \text{rk}([X,Y] + I_n) = 1) / \text{PGL}_n(\mathbb{C}),$$

where $\text{PGL}_n(\mathbb{C})$ acts on $(X,Y)$ by simultaneous conjugation. The space $\mathcal{C}_n$ is a basic example of quiver varieties: it is the quiver variety corresponding to $n$-dimensional representations of the quiver with one vertex and one loop. They defined an algebraic action of the group $G := \text{Aut}_C(A_1)$ on each $\mathcal{C}_n$ and showed that the bijective correspondence between $\mathcal{R}$ and $\mathcal{C}$ is $G$-equivariant. Moreover, they proved that this action is transitive, that is, each $\mathcal{C}_n$ is a $G$-orbit.

In [BL], motivated by the transitivity of the $G$-action on $\mathcal{C}_n$, R. Bocklandt and L. Le Bruyn proposed the following conjecture. For any quiver $Q$, they defined the algebra $A_Q := \mathbb{C}[N_Q] \otimes \mathbb{C}Q$, where $N_Q$ is the necklace Lie algebra of $Q$ and $\mathbb{C}Q$ is the path algebra of the double of $Q$. Then they claimed that the group $\text{Aut}_C(A_Q)$, the automorphism group of $A_Q$ preserving the symplectic element $\omega \in \mathbb{C}Q$, acts transitively on quiver varieties of $n$-dimensional representations of $Q$ for an appropriate choice of the dimension vector $\alpha$. Since $G_\Gamma$ is a subgroup of $\text{Aut}_C(A_Q)$, their first result is a proof the Bocklandt-Le Bruyn conjecture for $\mathbb{Z}_2$ cyclic quiver varieties:

**Theorem 1.1.** Let $\Gamma \cong \mathbb{Z}_2$. Then $G_\Gamma$ acts transitively on each $\mathfrak{M}_r^G(m, n; \epsilon)$.

The proof of transitivity in Theorem 1.1 has some advantages over that of Berest-Wilson. Most notably, it is purely geometric and does not use any fact from integrable systems. In fact, we will first give an alternative proof for transitivity of the $G$ action on $\mathcal{C}_n$.

Let us briefly outline our approach. We will show that $\mathfrak{M}_r^G(m, n; \epsilon)$ is $G_\Gamma$-flexible, i.e., the (co)tangent space at each point of this variety is spanned by the (co)tangent vectors to the orbits of $G_\Gamma$. To this end, we construct a family of functions $\{f_n\}_{n\geq 0}$ such that $\{df_n\}_{n\geq 0}$ span the cotangent space at each point. More precisely, we consider flows of two one-parameter subgroups of $G_\Gamma$ on $\mathfrak{M}_r^G(m, n; \epsilon)$ and, then, the action of $G_\Gamma$ is symplectic, we show that these are Hamiltonian flows of a family functions generating the algebra of functions $O(\mathfrak{M}_r^G(m, n; \epsilon))$, which implies the $G_\Gamma$-flexibility. Finally, using the fact that $\mathfrak{M}_r^G(m, n; \epsilon)$ is connected, we obtain that the later is a single orbit.

In general, we expect:

**Conjecture 1.1.** For $\Gamma = \mathbb{Z}_m(m \geq 3)$, $G_\Gamma$ acts transitively on each $\mathfrak{M}_r^G$.

The transitivity of the $G$-action on $\mathcal{C}_n$ in combination with Stafford’s theorem $G \cong \text{Pic}_C(A_1)$ (see [S] Corollary E) gives the following remarkable result. Let $D$ be a domain, which is Morita equivalent to $A_1$. Then all such algebras $D$ are classified, up to algebra isomorphism, by a single integer $n \geq 0$; the corresponding isomorphism classes are represented by the endomorphism rings $D_n := \text{End}_A(P_n)$ of the ideal $P_n := x^{n+1}A_1 + (xy + n)A_1$. Note here $A_1$ appears as the first member in the family $\{D_n\}_{n \geq 0}$.

Our next goal in this paper is to describe algebras Morita equivalent to $O_\Gamma(\mathbb{Z}_2)$. For that one needs to describe the orbits of $\text{Pic}_C(O_\Gamma)$ group on the space $R_r^G$. Since $G_\Gamma$ is only a subgroup of $\text{Pic}_C(O_\Gamma)$, the orbits of the later contain $G_\Gamma$-orbits. By Theorem 1.1, the $G_\Gamma$-orbits on $R_r^G$ are
For $\Gamma$ Theorem 1.4. We have $n,\lambda$ and $\Phi$ and $\text{Aut}_G$ coproduct structure (see [A]).

A $n,m$ counterexample for this. We will describe triples $(\tau)$ (Theorem 1.3. that by giving a description of all algebras equivalent to $O$ hence we have proved: the following generalization of Stafford’s result for presentation, we show the nonexistence of an isomorphism. 

Let $P_{m,n} \in R_{\Gamma}(m,n;\epsilon)$ be the image of this fixed point under $\Omega$. Then we shall prove:

**Theorem 1.2.** $End_{O_\tau}(P_{m,n}) \not\cong O_\tau$ unless $(m,n;\epsilon) = (0,0;0)$.

The proof of Theorem 1.2 is similar to that of Smith [Sm1], where he showed that $End_{A_1}(P_{1}) \not\cong A_1$. We will give an explicit presentation of the ideals $P_{m,n}$ in terms of generators of $O_\tau$ and an explicit description of $End_{O_\tau}(P_{m,n})$ as a subring of the field of fractions of $O_\tau$. Using this presentation, we show the nonexistence of an isomorphism.

It follows immediately from Theorem 1.2 that (1.2) holds if and only if $P,Q \in R_{\Gamma}(m,n;\epsilon)$, and hence we have proved:

**Corollary 1.1.** Let $D$ be a domain Morita equivalent to $O_\tau$. Then there exists a unique triple $(m,n;\epsilon) \in L_\epsilon \times \mathbb{Z}_2$ such that $D \cong End_{O_\tau}(P_{m,n})$.

Morita equivalence of primitive factors of $U(sl_2)$ was studied by Hodges in [H1]. Explicitly, he classified them up to Morita equivalence using the Hattori-Stallings trace. Our result generalizes that by giving a description of all algebras equivalent to $O_\tau$ not only among primitive factors.

By Corollary 1.1 the $G_\Gamma$-orbits and $\text{Pic}_C(O_\tau)$-orbits on $R_{\Gamma}$ coincide, and therefore, we can prove the following generalization of Stafford’s result for $A_1$ (see [S Corollary E]):

**Theorem 1.3.** The group monomorphism $G_\Gamma \rightarrow \text{Pic}_C(O_\tau)$ is an isomorphism.

The second part of [S Corollary E] states that if $D$ is Morita equivalent to $A_1$ but $D \not\cong A_1$ then $\text{Pic}_C(D)/\text{Aut}_\tau(D)$ is an infinite coset space. This means the automorphism group is an invariant distinguishing $A_1$ from $D$. In fact, Stafford suggested a conjecture that this might be the case for a large class of algebras similar to $A_1$ (see [S p.625]). However, the algebra $O_\tau$ provides a counterexample for this. We will describe triples $(n,m;\epsilon) \neq (0,0;0)$ such that $\text{Aut}_\tau(End(P_{m,n})) \cong \text{Pic}_C(End(P_{m,n}))$, which, by Morita equivalence, will imply $\text{Aut}_\tau(End(P_{m,n})) \cong \text{Aut}_\tau(O_\tau)$.

In the proof of Theorem 1.1 we have used generators of the group $G_\Gamma$ discovered by Bavula and Jordan in [BJJ]. For $\Gamma \cong \mathbb{Z}_m$, they proved that $G_\Gamma$ is generated by three abelian subgroups $\Theta_3$, $\Psi_\epsilon$, and $\Phi_{n,\lambda}$, where $n \in \mathbb{C}_*$ and $n \in \mathbb{Z}_{\geq 0}$ (see [2.4]–[3.10] for definition). For $\Gamma \cong \mathbb{Z}_2$, these generators were originally found in Fleury [F], who also showed that $G_\Gamma$ can be presented as the coproduct $SL_2(\mathbb{C}) \ast_U (\Theta_3,\Phi_{n,\lambda})$, where $U := SL_2(\mathbb{C}) \cap (\Theta_3,\Phi_{n,\lambda})$. We recall that $\text{Aut}_\tau(A_1)$ admits similar coproduct structure (see [A]).

So our goal here is twofold. First, we would like to extend a coproduct structure for $G_\Gamma$ when $\Gamma \cong \mathbb{Z}_n(m \geq 3)$. Second, we would like clarify the relationships between groups $G_\Gamma = \text{Aut}_\tau(O_\tau)$ and $\text{Aut}_\Gamma(S_\tau)$.

**Theorem 1.4.** We have
1. For $\Gamma \cong \mathbb{Z}_m (m \geq 3)$,
   \[ G_\Gamma \cong A \ast_U B, \]
   where $A := \langle \Theta_\lambda, \Psi_{n,\lambda} \rangle$, $B := \langle \Theta_\lambda, \Phi_{n,\lambda} \rangle$ and $U := A \cap B$.
2. For $\Gamma \cong \mathbb{Z}_m (m \geq 2)$, the automorphisms $\Theta_\lambda$, $\Psi_{n,\lambda}$ and $\Phi_{n,\lambda}$ of $O_\tau$ can be lifted to $\Gamma$-equivariant automorphisms of $S_\tau$ and
   \[ \text{Aut}_\Gamma(S_\tau) \cong H \rtimes G_\Gamma, \]
   where $H$ is an abelian subgroup defined in Lemma 2.4.

We would like to finish the introduction by mentioning one interesting consequence of Theorem 1.1 in the theory of integrable systems. In the seminal paper [W], G. Wilson has shown that rational solutions of KP hierarchy can be parametrized by the Calogero-Moser spaces $\mathcal{C} = \cup_{n \geq 0} \mathcal{C}_n$. Thus we have an action of the group $G$ on spaces of rational solutions, and the $G$-transitivity on each of $\mathcal{C}_n$ implies that choosing one solution from $\mathcal{C}_n$ one can obtain any other solution by applying an appropriate element of $G$. Recently, in [CS], a generalization of the KP hierarchy associated to the cyclic quiver has been introduced. By extending the result of [W], the authors have shown that rational solutions are parametrized by the corresponding quiver varieties. Hence Theorem 1.1 implies that any rational solution from $\mathcal{M}_2^{\mathbb{Z}_2} (m, n; \epsilon)$ can be obtained from any given solution by applying an appropriate element of $G_\Gamma$ for $\Gamma \cong \mathbb{Z}_2$.

The paper is organized as follows. In Section 2 we introduce the notations, review some basic facts about algebras $S_\tau$ and $O_\tau$ as well as their automorphism groups, and give a proof of Theorem 1.1. In Section 3 we explain the construction of the bijective map $\Omega$. The main result is Theorem 3.1 which shows how to construct an ideal for a given point of the quiver variety. The theorem also provides an important invariant of the corresponding ideal, $\kappa \in Q(O_\tau)$, where $Q(O_\tau)$ is the field of fractions of $O_\tau$. This invariant distinguishes ideals up to isomorphism. In Section 4 using $\kappa$, we provide a natural generating set for the coordinate ring of the quiver variety $\mathcal{M}_2^{\mathbb{Z}_2} (m, n; \epsilon)$. In Section 5 we give a proof of Theorem 1.1. In Section 6 we construct $\mathbb{C}^\ast$-fixed points of $\mathcal{M}_2^{\mathbb{Z}_2} (m, n; \epsilon)$. In Sections 7-8 we compute $\kappa$’s for $\mathbb{C}^\ast$-fixed points, their corresponding ideals and endomorphism rings. Finally, in Section 9 we give proofs of Theorems 1.2 and 1.3. It also discusses a counterexample to Stafford’s conjecture.

2. Automorphism groups of $S_\tau$ and $O_\tau$

2.1. Generalities. In this section we collect various facts about algebras $S_\tau(\Gamma)$ and $O_\tau(\Gamma)$ that we will need. Some of these facts will be used without comment in the body of the paper. All algebras will be considered over the field $\mathbb{C}$.

As mentioned above, we will be dealing with cyclic $\Gamma$’s. Thus let $m \geq 1$ and assume $\Gamma \cong \mathbb{Z}_m$. Fix an embedding $\Gamma$ into $\text{SL}_2(\mathbb{C})$ so that a generator $g$ of $\Gamma$ acts on $x$ and $y$ by $\epsilon$ and $\epsilon^{-1}$ respectively, where $\epsilon$ is a primitive $m$-th root of unity. Then $S_\tau$ is the quotient of $\mathbb{C}[x, y] \ast \Gamma$ subject to the following relations

\begin{align*}
(2.1) \quad & g^i \cdot x = \epsilon^i x \cdot g^i, \quad g^i \cdot y = \epsilon^{-i} y \cdot g^i \quad \text{for } i = 1, \cdots, m-1, \\
(2.2) \quad & x \cdot y - y \cdot x = \tau.
\end{align*}

One can replace (2.1) by another set of relations. Indeed, let $e_0, \cdots, e_{m-1}$ be the complete set of orthogonal primitive idempotents of $\mathbb{C} \Gamma$ given by

\begin{equation}
(2.3) \quad e_i := \frac{1}{m} \sum_{k=0}^{m-1} \epsilon^{-ik} g^k,
\end{equation}
where \( g \) is a generator of \( \Gamma \). Then relation (2.1) is equivalent to
\[
e_i \cdot x = x \cdot e_{i+1}, \quad e_i \cdot y = y \cdot e_{i-1} \quad \text{for } i \pmod{m}.
\] (2.4)

The homological and ring-theoretical properties of \( O_\tau \) depend on the values of the parameter \( \tau \). By the McKay correspondence we can associate to the group \( \Gamma \) an extended Dynkin diagram. The group algebra \( \mathbb{C}[\Gamma] \) is then identified with the dual of the space spanned by the simple roots of the corresponding root system. Following [CBH], we say that \( \tau \) is regular if it does not belong to any root hyperplane in \( \mathbb{C}[\Gamma] \). In the case when \( \tau \) is regular, the algebras \( S_\tau \) and \( O_\tau \) are Morita equivalent, the equivalence between the categories of right modules is given by (see [CBH] Theorem 0.4)
\[
F : \text{Mod}(S_\tau) \to \text{Mod}(O_\tau), \quad M \mapsto M \otimes_{S_\tau} \mathbb{C}[\Gamma].
\] (2.5)

Now let \( v \in \mathbb{C}[h] \). Then \( A(v) \) is the \( \mathbb{C} \)-algebra generated by \( a, b \) and \( h \) subject to the relations
\[
a \cdot h = (h - 1) \cdot a, \quad b \cdot h = (h + 1) \cdot b, \quad b \cdot a = v(h), \quad a \cdot b = v(h - 1).
\] (2.6)

These algebras have been studied in [B3a], [H2] and [Sm2]. One can easily establish the following relation between algebras \( O_\tau \) and \( A(v) \):

**Proposition 2.1.** Let \( \Gamma \cong \mathbb{Z}_m \) and let \( \tau := \tau_0 e_0 + \cdots + \tau_{m-1} e_{m-1} \in \mathbb{C}[\Gamma] \) be such that \( \tau_0 + \cdots + \tau_{m-1} \neq 0 \). Then the map \( \phi : O_\tau \to A(v) \) defined by
\[
ex^m \mapsto b, \quad ey^m \mapsto a, \quad eyx \mapsto (\tau_0 + \cdots + \tau_{m-1})h
\]
induces an algebra isomorphism, where
\[
v(h) := (\tau_0 + \tau_2 + \cdots + \tau_{m-1})^m \prod_{i=0}^{m-1} \left( h + \frac{\tau_0 + \cdots + \tau_i}{\tau_0 + \cdots + \tau_{m-1}} \right).
\]

**Corollary 2.1.** \( A^{\mathbb{Z}_m}_1 \cong A(v) \) for \( v(x) = \prod_{i=1}^{m} (x + \frac{1}{m}) \).

Now, if \( \deg(v) = 2 \) then \( A(v) \) is isomorphic to a primitive factor of the universal enveloping algebra of \( sl_2(\mathbb{C}) \). Indeed, let \( U = U(sl_2) \) be the universal enveloping algebra of \( sl_2(\mathbb{C}) \), and let \( F, E, H \) be its standard generators. Let \( \Omega = 4FE + H^2 + 2H \) be the Casimir element. Then primitive factors are of the form \( U_\alpha = U/\langle \Omega - \alpha \rangle \) (\( \alpha \in \mathbb{C} \)). If \( v(x) = \alpha/4 - x - x^2 \), then the map
\[
\phi : U_\alpha \to A(v), \quad E \mapsto a, \quad F \mapsto b, \quad H \mapsto 2h
\]
defines an isomorphism of algebras (see e.g. [H2] Example 4.7]).

The following result shows when \( A(v_1) \) and \( A(v_2) \) are isomorphic:

**Theorem 2.1.** [B3] Theorem 3.28.] Let \( v_1, v_2 \in \mathbb{C}[h] \). Then \( A(v_1) \cong A(v_2) \) if and only if \( v_2(h) = \eta v_1(h + \beta) \) for \( \beta \in \mathbb{C} \) and \( \eta \in \mathbb{C}^* \).

Now we get

**Corollary 2.2.** If \( \Gamma \cong \mathbb{Z}_2 \) and \( \tau = \tau_0 e_0 + \tau_1 e_1 \), then \( O_\tau \cong U_\alpha \) for \( \alpha = 1 - (\frac{\tau_0 + \tau_1}{2})^2 \).

**Proof.** By [B2a] \( U_\alpha \cong A(v_1) \), where \( v_1 = -(h + \frac{1 + \sqrt{-1}}{2})h + \frac{1 + \sqrt{-1}}{2} \). By Proposition 2.1 \( O_\tau \cong A(v_2) \), where \( v_2 = (\tau_0 + \tau_1)^2(h + 1) + \frac{\tau_0 + \tau_1}{2} \). Now, we can use Theorem 2.1 with \( \beta = \frac{1 + \sqrt{-1}}{2} \) and \( \eta = -(\tau_0 + \tau_1)^2 \) to prove our statement. \( \square \)
2.2. Proof of Theorem 2.2. Let $\text{Aut}_C(S_r)$ be the group of $C$-algebra automorphisms of $S_r$ and let $\text{Aut}_\Gamma(S_r)$ be the subgroup of $\Gamma$-invariant automorphisms. Let $\text{Aut}_\Gamma(\omega)(R)$ be the group of $\Gamma$-invariant automorphisms of $R$ preserving $\omega := xy - yx$. Then there is a natural group epimorphism

$$ \varphi : \text{Aut}_\Gamma(\omega)(R) \to \text{Aut}_\Gamma(S_r). $$

We also have the following group homomorphism

$$ \mu : \text{Aut}_\Gamma(S_r) \to \text{Aut}_C(O_r), \quad \mu(\sigma)(vbe) = e\sigma(b)ve. $$

Composing $\mu$ with the isomorphism in Proposition 2.1 we get a group homomorphism

$$ \rho : \text{Aut}_\Gamma(S_r) \to \text{Aut}_C(A(v)). $$

We now present generators of the group $\text{Aut}_C(A(v))$ discovered in [13]. Let $\sigma$ be a $C$-linear automorphism of $\mathbb{C}[h]$ such that $\sigma(h) = h - 1$. For $k \in \mathbb{N}$, let $\Delta_k$ be the linear map $\sigma^k - 1 : \mathbb{C}[h] \to \mathbb{C}[h]$. It is easy to check that $\Delta_k$ defines a $\sigma^n$-derivation, i.e.,

$$ \Delta_k(fg) = \Delta_k(f)g + \sigma^k(f)\Delta_k(g), $$

for all $f, g \in \mathbb{C}[h]$. Let $m := \text{deg}(v), n \geq 0$ be an integer and $\lambda \in \mathbb{C}$. Then the following families of automorphisms of $A(v)$:

$$ \Theta_\lambda : a \mapsto \lambda^m a, \quad h \mapsto h, \quad b \mapsto \lambda^{-m}b, $$

$$ \Psi_{k,\lambda} : a \mapsto a, \quad h \mapsto h - k\lambda a^k, \quad b \mapsto b + \sum_{i=1}^{m} \frac{t^i}{i!} \Delta^i_k(v) a^{i-1}, $$

$$ \Phi_{k,\lambda} : a \mapsto a + \sum_{i=1}^{m} \frac{(-\lambda)^i}{i!} \Delta^i_k(v) b^{i-1}, \quad h \mapsto h + k\lambda b^k, \quad b \mapsto b $$

were introduced in [13]. Moreover, they proved (see loc. cit. Theorem 3.29):

**Theorem 2.2.** $\text{Aut}_C(A(v))$ is generated by $\Theta_\lambda$, $\Psi_{k,\lambda}$ and $\Phi_{k,\lambda}$.

Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*/\mathbb{Z}_m$. Then the following maps define automorphisms both in $\text{Aut}_\Gamma(S_r)$ and in $\text{Aut}_\Gamma(\omega)(R)$:

$$ \theta_\lambda : x \mapsto \lambda^{-1}x, \quad y \mapsto \lambda y, \quad g \mapsto g; $$

$$ \psi_{k,\lambda} : x \mapsto x - k\lambda(\tau_0 + \cdots + \tau_m) y^{km-1}, \quad y \mapsto y, \quad g \mapsto g; $$

$$ \phi_{k,\lambda} : x \mapsto x, \quad y \mapsto y + n\lambda(\tau_0 + \cdots + \tau_m) x^{km-1}, \quad g \mapsto g. $$

Therefore, from now on, we will use the same notations for these automorphisms whether we consider them as elements of $\text{Aut}_\Gamma(S_r)$ or $\text{Aut}_\Gamma(\omega)(R)$. We can show

**Proposition 2.2.**  
(i) $\rho(\theta_\lambda) = \Theta_\lambda$, $\rho(\psi_{k,\lambda}) = \Psi_{k,\lambda}$, $\rho(\phi_{k,\lambda}) = \Phi_{k,\lambda}$ in $\text{Aut}_\Gamma(S_r)$.

(ii) $\rho \circ \varphi(\theta_\lambda) = \Theta_\lambda$, $\rho \circ \varphi(\psi_{k,\lambda}) = \Psi_{k,\lambda}$, $\rho \circ \varphi(\phi_{k,\lambda}) = \Phi_{k,\lambda}$.

**Proof.** (i) It is straightforward to see that $\rho(\theta_\lambda) = \Theta_\lambda$. We will only prove $\rho(\psi_{k,\lambda}) = \Psi_{k,\lambda}$ since the proof for $\rho(\phi_{k,\lambda}) = \Phi_{k,\lambda}$ is completely analogous.

By identification $\phi : O_r \to A(v)$, we obtain that $\psi_{k,\lambda}$ induces the automorphism

$$ (a, h, b) \mapsto (a, h - k\lambda a^k, e(x + \lambda(\tau_0 + \cdots + \tau_{m-1}) y^{mk-1})^m). $$

To prove $\rho(\psi_{k,\lambda}) = \Psi_{k,\lambda}$ it suffices to show

$$ e(x + \lambda(\tau_0 + \cdots + \tau_{m-1}) y^{mk-1})^m = b + \sum_{i=1}^{m} \frac{t^i}{i!} \Delta^i_k(v) a^{i-1}. $$

Set

$$ w_i := ey^i(x + \lambda(\tau_0 + \cdots + \tau_{m-1}) y^{mk-1})^i. $$
Note that $w_m$ is equal to the LHS of (2.17) multiplied by $ey^m = a$ from the left. Then using relation $xy - yx = \tau$, we obtain

\[
|n| = ey^{i-1}(xy + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
= ey^{i-1}(xy - \tau + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})y(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
= ey^{i-1}(xy + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})y(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
-ey^{i-1}(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
= ey^{i-1}(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})y(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
-\tau_{m-i+1}ey^{i-1}(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
= ey^{i-1}(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})y(x + \lambda(\tau_0 + \cdots + \tau_{m-1})y^{m^k})^{-1}
\]

\[
-\tau_{m-i+1}w_{i-1}.
\]

Repeating this process by moving further to the left $y$ term, we get

\[
w_i = w_{i-1}(w_{i-1} - \tau_{m-i+1} - \cdots - \tau_{m-1}).
\]

Hence

\[
w_m = w_1(w_1 - \tau_{m-1})(w_1 - \tau_{m-2} - \tau_{m-1})\cdots(w_1 - (\tau_1 + \cdots + \tau_{m-1}))
\]

or equivalently

\[
w_m = (\tau_0 + \cdots + \tau_{m-1})^m \prod_{i=0}^{m-1} \left( h + \lambda a^k \right) - \tau_0 + \cdots + \tau_{m-1} \right).
\]

Now if applying to $\Psi^{-1}_{k,\lambda}$, we have

\[
\Psi^{-1}_{k,\lambda}(w_m) = (\tau_0 + \cdots + \tau_{m-1})^m \prod_{i=0}^{m-1} \left( h - \frac{\tau_1 + \cdots + \tau_i}{\tau_0 + \cdots + \tau_{m-1}} \right),
\]

which is equal to $v(h - 1) = ab$, and therefore

\[
w_m = \Psi_{k,\lambda}(ab) = a\left(b + \sum_{i=1}^{n} \frac{\lambda_i}{i!} \Delta_i^m(v) a_{i-1} \right).
\]

Since $O_\tau$ is a domain, dividing by $a$ the last expression, we get (2.17). □

Now, let $G$ be the group generated by $\theta_\lambda, \psi_{m,\lambda}$ and $\phi_{m,\lambda}$. Again $G$ can be viewed both as a subgroup of Aut$_{F}(S_\tau)$ and of Aut$_{\tau}(R)$. Then:

**Corollary 2.3.** The homomorphisms $\rho$ and $\rho \circ \varphi$ map $G$ surjectively.

**Proposition 2.3.** Let $c_0, \cdots, c_{m-1} \in \mathbb{C}$. Then the assignment

\[
x \mapsto x_1 := (c_0 e_0 + \cdots + c_{m-1} e_{m-1}) x, \quad y \mapsto y_1 := (d_0 e_0 + \cdots + d_{m-1} e_{m-1}) y,
\]

defines an element in Aut$_{\tau}(R)$ and Aut$_{\tau}(S_\tau)$ if and only if $c_i \in \mathbb{C}^*$ and $d_i = 1/c_{i+1}$ for $i$ (mod $m$).

**Proof.** We will only prove that the above map defines an automorphism in Aut$_{\tau}(R)$, since for $S_\tau$ the arguments are similar. First, the above map is bijective if and only if $c_i \neq 0$, since the inverse is given by

\[
x_1 := (d_0 e_0 + \cdots + d_{m-1} e_{m-1}) x, \quad y_1 := (c_0 e_0 + \cdots + c_{m-1} e_{m-1}) y,
\]

\[
e_i \mapsto e_i.
\]
We need to show that \( e_i \cdot x_1 = x_1 e_{i+1} \) and \( e_i \cdot y_1 = y_1 e_{i-1} \) and \( x_1 \cdot y_1 - y_1 \cdot x_1 = x \cdot y - y \cdot x \). The first two relations are straightforward. The left hand side of the third relation is

\[
\begin{align*}
&(c_0 e_0 + \cdots + c_m e_{m-1})(d_0 e_1 + \cdots + d_{m-2} e_{m-1} + d_{m-1} e_0) \cdot x \cdot y \\
&\quad - (d_0 e_0 + \cdots + d_{m-1} e_{m-1})(c_0 e_1 + c_1 e_0 + \cdots + c_{m-1} e_{m-2}) \cdot y \cdot x \\
&\quad = (c_0 d_{m-1} e_0 + c_1 d_0 e_1 + \cdots + c_{m-1} d_{m-2} e_{m-1}) \cdot x \cdot y \\
&\quad - (c_1 d_0 e_0 + c_2 d_1 e_1 + \cdots + c_{m-1} d_{m-2} e_{m-2} + c_0 d_{m-1} e_{m-1}) \cdot y \cdot x \\
&\quad = xy -yx,
\end{align*}
\]

which proves our claim. \( \square \)

The group of automorphisms defined in Proposition 2.3 is isomorphic to \((C^*)^m\). Let \( H \) be the subgroup of \((C^*)^m\) consisting of \( m \)-tuples \((c_0, c_1, \ldots, c_{m-1}) \) such that \( c_0 c_1 \cdots c_{m-1} = 1 \).

**Lemma 2.1.** \( H \) is the kernel of \( \rho \) and \( \rho \circ \varphi \). In particular, \( H \) is a normal subgroup of \( \text{Aut}_{\Gamma, \omega}(R) \) and \( \text{Aut}_{\Gamma}(S_r) \).

**Proof.** Let \( \eta \in \text{Aut}_{\Gamma}(S_r) \) and let \( x_1 := \eta(x), \ y_1 := \eta(y) \). If \( \eta \in \text{Ker}(\rho) \) then

\[
ex_1^m = ex^m, \quad ey_1^m = ey^m, \quad ey_1 x_1 = eyx.
\]

Let \( x_1 = \sum_{i=0}^{m-1} c_i f_i(x, y) \). Then since \( c_i x_1 = x_1 c_{i+1} \), we have \( c_i f_i(x, y) c_{i+1} = c_{i+1} f_i(x, y) \) and hence \( ex_1^m = e f_0 f_1 \cdots f_{m-1} = ex^m \). The later implies \( c_i f_i = c_{i+1} f_i \) for some \( c_i \in C^* \). Thus \( x_1 = \sum_{i=0}^{m-1} c_i f_i x_1 \) and \( ex_1^m = c_0 c_1 \cdots c_{m-1} ex^m \). Similarly, we can show that \( y_1 = \sum_{i=1}^{m} d_i e_y y \) and \( d_0 \cdots d_{m-1} = 1 \). By Proposition 2.3, we can conclude that \( \eta \in H \). \( \square \)

Thus, combining Proposition 2.2 and Lemma 2.1 we have proved:

**Theorem 2.3.** Let \( G := \text{Aut}_{C}(O_r(Z_m)) \). Then

\[
\text{Aut}_{\Gamma}(S_r(Z_m)) \cong H \times G \cong \text{Aut}_{\Gamma, \omega}(R).
\]

Let \( G_1 := \langle \Theta_{\lambda}, \Psi_{k, \mu} \rangle, \ G_2 := \langle \Theta_{\lambda}, \Phi_{k, \mu} \rangle \) and \( G_3 := \langle \Theta_{\lambda} \rangle \). Then:

**Proposition 2.4.** We have \( G \cong G_1 *_{G_3} G_2 \).

**Proof.** By Theorem 2.3 the group \( G \) can be embedded in \( \text{Aut}_{\Gamma, \omega}(R) \) such that image of \( G_1 \) and \( G_2 \) are generated by \( \langle \theta_{\lambda}, \psi_{k, \mu} \rangle_{k \geq 1} \) and \( \langle \theta_{\lambda}, \phi_{k, \mu} \rangle_{k \geq 1} \) respectively. Recall \( \text{Aut}_{\Gamma, \omega}(R) \) can be presented as amalgamated product

\[
A *_{U} B,
\]

where \( A \) is the group of symplectic affine transformations, \( B \) is the group of triangular transformations and \( U = A \cap B \). Any element in \( g \in G \) can be written

\[
g = g_1 \cdots g_k,
\]

where \( g_i \) is either in \( G_1 \) or \( G_2 \) but not in \( G_3 \). Using the amalgamated product structure of \( \text{Aut}_{\Gamma, \omega}(R) \) we can rewrite it as

\[
\theta_{\lambda} h_1 \cdots h_k,
\]

where \( h_i \) is alternating in \( \psi_{q(y)} \) and \( \phi_{p(x)} \). This is a reduced word in \( \text{Aut}_{\Gamma, \omega}(R) \) hence \( g \) cannot be equal 1 unless \( \lambda = 1 \) and \( h_i = 1 \) for all \( i \). \( \square \)

3. **Ideal classes of \( O_r \) and quiver varieties**

In this section we recall some results proved by one of the authors [E] and by [BGK], related to a description of ideal classes of the algebra \( O_r \). More explicitly, we will show that there is a \( G \)-equivariant bijection between \( O_r \)-ideals and the disjoint union of certain quiver varieties \( \mathcal{M}_r \).
3.1. Nakajima's quiver varieties. Let $Q = (I, H)$ be a finite quiver (without loops) with the vertex set $I$ and the arrow set $H$, and let $\hat{Q} = (I, \hat{H})$ be its double quiver, obtained by adjoining a reverse arrow $a^*$ for each arrow $a \in H$. Let $V = (V_i)_{i \in I}$ and $W = (W_i)_{i \in I}$ be a pair of collections of vector spaces. We consider the vector space of linear maps

$$M(V, W) = E(V) \oplus L(W, V) \oplus L(V, W),$$

where

$$E(V) := \bigoplus_{a \in H} \text{Hom}(V_{t(a)}, V_{h(a)}), \quad L(W, V) := \bigoplus_{i \in I} \text{Hom}(W_i, V_i).$$

Note that the space $E(V)$ can be identified with $\text{Rep}(\hat{Q}, v)$, the space of $\hat{Q}$-representations of dimension vector $v = (\dim V_0, \cdots, \dim V_n) \in \mathbb{Z}^I$. There is a natural action of the group $G(V) := \prod \text{GL}(V_i)$ on $M(V, W)$ given by

$$(B_a, v_i, w_i) \mapsto (g_{h(a)} B_a g_{t(a)}^{-1}, g_i v_i, w_i g_i^{-1}).$$

For each $\tau = (\tau_i)_{i \in I} \in \mathbb{C}^I$, we define a subvariety $\mathcal{M}_\tau(V, W) \subseteq M(V, W)$ satisfying the following

- **Moment map equation:**

$$\sum_{a \in Q \atop h(a) = i} B_a B_a^* - \sum_{a \in Q \atop t(a) = i} B_a^* B_a + v_i w_i = \tau_i \text{Id}_{V_i}, \quad i \in I \quad (3.2)$$

- **Stability condition:**

$$B_a^*/V_{h(a)} \subseteq V_{t(a)}, \quad v_i(W_i) \subseteq V_i', \quad \text{then} \quad V' = V. \quad (3.3)$$

The action of $G(V)$ on $\mathcal{M}_\tau(V, W)$ is free, due to the stability condition. The Nakajima variety associated to the triple $(V, W, \tau)$ is defined as follows

$$\mathcal{M}_\tau^\infty(V, W) := \mathcal{M}_\tau(V, W) \sslash G(V) \quad (3.4)$$

where $\sslash$ is the GIT quotient.

**Remark.** The quiver variety $\mathcal{M}_\tau^\infty(V, W)$, associated to a quiver $Q$, can be identified with a representation variety of the deformed preprojective algebra $\Pi^\tau(Q)$ (see [CBH]).

Let $\Gamma \subset \text{SL}_2(\mathbb{C})$ be a finite subgroup. Then recall that one can associate to $\Gamma$ a quiver $Q_\Gamma = (I_\Gamma, H_\Gamma)$, whose underlying graph is an extended Dynkin diagram (see [McK]). Now let $\{U_i\}_{i \in I_\Gamma}$ be a complete set of irreducible representations of $\Gamma$. Then for a pair of $\Gamma$-modules $(V, W)$, we define collections of vector spaces $V := (V_i)$ and $W := (W_i)$, where $V \cong \oplus_i V_i \otimes U_i$ and $W \cong \oplus_i W_i \otimes U_i$ are decompositions into irreducible modules. Finally, let $\tau \in Z(\mathbb{C} \Gamma)$. Then $\tau = (\tau_i)_{i \in I_\Gamma}$ via the identification $Z(\mathbb{C} \Gamma) \cong \mathbb{C}^{I_\Gamma}$. Thus, given such a triple $(V, W, \tau)$, we can associate to it the quiver variety

$$\mathcal{M}_\tau^\infty(V, W) := \mathcal{M}_\tau^\infty(Q, V, W).$$

Since we will be concerned with the case when $\Gamma$ is cyclic and $W$ is one-dimensional $\Gamma$-module, we can write quiver varieties for this case more explicitly. Indeed, let $\Gamma \cong \mathbb{Z}_m$ and $W \cong U_k$ for some $0 \leq k \leq m - 1$. Then $Q_\Gamma$ has type $\tilde{A}_{m-1}$; it consists of $m$ vertices $\{0, 1, \cdots, m - 1\}$ and $m$ arrows.
as has been defined in [BW1, BW2]. A classical result by Dixmier [D] states that

\[ \Phi_{a,n} := (x + a y^n, y) \]

where

\[ \Phi \]

\[ a, b \in \mathbb{C} \] and \( n, m \in \mathbb{Z}_{\geq 0} \). Then for \( \sigma \in G \) and \( (X, Y, v, w) \in C_n \), let

\[ \sigma(X, Y, v, w) := (\sigma^{-1}(X), \sigma^{-1}(Y), v, w) \]
where $\sigma^{-1}$ is the inverse of $\sigma$ in $G$. For example, if $\sigma = \Phi_{a,n}$ then

$$\Phi_{a,n}(X, Y, v, w) = (X - aY^n, Y, v, w).$$

To define the action of $G\Phi$ on $M^\Gamma_\text{F}$ for $\Gamma \cong \mathbb{Z}_m$, we give a presentation of this quiver variety similar to that of $C_n$ as in (3.1). For a point

$$(X_0, X_1, \ldots, X_{m-1}; Y_0, Y_1, \ldots, Y_{m-1}, v_k, w_k) \in M^\Sigma_{Z_3}(n_0, \ldots, n_{m-1}; k),$$

set

$$X := \begin{pmatrix} 0 & X_0 & 0 & \ldots & 0 \\ 0 & 0 & X_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ X_{m-1} & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 & \ldots & Y_{m-1} \\ Y_0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & Y_{m-2} & 0 \end{pmatrix}.$$

These are $m \times m$ block matrices of size $n \times n$, where $n := n_0 + \cdots + n_{m-1}$, such that $X$ (resp. $Y$) is a block matrix whose only nonzero entries are on the $1$st and $(-m+1)$-st (resp. $(-1)$-st and $(m-1)$-st) diagonals. Next, let

$$w := (0, \ldots, 0, w_k, 0, \ldots, 0), \quad v := (0, \ldots, 0, v_k, 0, \ldots, 0)^T,$$

where $t$ is taking the transpose of the matrix. So $w$ is an $n$-dimensional row vector and $v$ is an $n$-dimensional column vector. Finally, let

$$T := \text{Diag} \left[ \tau_0 \text{Id}_{n_0}, \ldots, \tau_{m-1} \text{Id}_{n_{m-1}} \right]$$

be an $n \times n$ block diagonal matrix. Then it is easy to see that relations in (3.14) are equivalent to

$$XY - YX + T = vw.$$ 

Thus we can refer to points of $M^\Sigma_{Z_3}(n_0, \ldots, n_{m-1}; k)$ as quadruples $(X, Y, v, w)$ satisfying condition (3.13). This identification allows us to define the action of $G_{Z_3} := \text{Aut}_C(O_r(Z_m))$ on each $M^\Sigma_{Z_3}$ as follows. By Theorem 2.13 the group $G_{Z_3}$ is generated by automorphisms defined by (2.4) - (2.10). So it suffices to define an action on these generators. Let $\sigma \in \text{Aut}_C(S_\Gamma)$ be one of (2.4) - (2.10). Then for a $(X, Y, v, w) \in M^\Sigma_{Z_3}(n_0, \ldots, n_{m-1}; k)$, define exactly the same action as for $C_n$:

$$\sigma(X, Y, v, w) := (\sigma^{-1}(X), \sigma^{-1}(Y), v, w).$$

This is a well-defined action, since the RHS of (3.14) satisfies (3.13) and $\sigma^{-1}(X)$ (resp. $\sigma^{-1}(Y)$) has the same non-zero diagonals as $X$ (resp. $Y$). For instance, if $\sigma = \psi_{k,\lambda}$ then

$$\psi_{k,\lambda}(X, Y, v, w) = (X + k\lambda(\tau_0 + \tau_1 + \cdots + \tau_{m-1})Y^{km-1}, Y, v, w).$$

One can easily see that the action $\psi_{k,\lambda}$ on $(X_0, \ldots, X_{m-1}; Y_0, \ldots, Y_{m-1}; v_k, w_k)$ corresponding via identification (3.13) to $(X, Y, v, w)$ is given by

$$X_0 \mapsto X_0 + k\lambda(\tau_0 + \cdots + \tau_{m-1})Y_{m-1}Y_{m-2} \cdots Y_1,$$

$$X_i \mapsto X_i + k\lambda(\tau_0 + \cdots + \tau_{m-1})Y_{i-1}Y_{i-2} \cdots Y_0Y_{m-1}Y_{m-2} \cdots Y_{i+1}, \quad i = 1, \ldots, m-1,$$

$$Y_i \mapsto Y_i \quad i = 0, 1, \ldots, m-1, \quad v_k \mapsto v_k, \quad w_k \mapsto w_k.$$

The next statement follows directly from the definition of quiver varieties:

**Corollary 3.1.** The map in Lemma 3.3 is $G_{Z_3}$-equivariant.
3.3. $G_\tau$-equivariant bijective correspondence. Let $\mathcal{R}_\tau$ be the set of isomorphism classes $S_\tau$-submodules of $e S_\tau$ and let $\mathcal{R}'_\tau$ be the set of isomorphism classes of $O_\tau$-ideals. Then the functor $F$ (see (2.3)) gives a natural bijection between $\mathcal{R}'_\tau$ and $\mathcal{R}_\tau$. Let $K_0(\Gamma)$, $K_0(S_\tau)$ and $K_0(O_\tau)$ be the Grothendieck groups of the algebras $C\Gamma$, $S_\tau$ and $O_\tau$ respectively. Using Quillen’s theorem and Morita equivalence between $S_\tau$ and $O_\tau$, we can identify all three groups. Next, we recall that $\hat{\Gamma}$ is a set of irreducible $\Gamma$-modules. Then there is a map $\gamma : \mathcal{R}_\tau \to K_0(\Gamma) \times \hat{\Gamma}$ which sends a submodule $M$ to a pair $(V,W)$ so that $V$ does not contain $C\Gamma$. One can show (see [E, Theorem 7]) that $\gamma(M_1) = \gamma(M_2)$ if and only if $[M_1] = [M_2]$ in $K_0(S_\tau)$. Hence, the $K$-theoretic description produces a decomposition of $\mathcal{R}_\tau$ (and of $\mathcal{R}'_\tau$):

$$\mathcal{R}_\tau = \bigsqcup_{V,W} \mathcal{R}_\tau(V,W),$$

where $V$ runs over all finite-dimensional $\Gamma$-modules and $W \cong \mathcal{U}_k$ for some $k = 0, \ldots, m - 1$.

The action of $G_{Z_m}$ on $\mathcal{R}_\tau$ is pointwise, i.e., if $\sigma \in G_{Z_m}$ and $M \subset O_\tau$, then $\sigma M := \{ \sigma(m) \mid m \in M \}$. Moreover, this action respects the decomposition (3.15). The following result is proved in [E]:

**Theorem 3.1.** Let $V$ be a finite-dimensional $\Gamma$-module. Then for any $0 \leq k \leq m - 1$, there is a natural $G_{Z_m}$-equivariant bijection $\Omega : \mathcal{M}_{Z_m}^\mathbb{C}(V, \mathcal{U}_k) \to \mathcal{R}_\tau(V, \mathcal{U}_k)$ sending a point $(X,Y,v,w)$ in $\mathcal{M}_{Z_m}^\mathbb{C}(V, \mathcal{U}_k)$ to the class of the fractional ideal of $S_\tau$:

$$M = e_k \det(Y - y \Id) S_\tau + e_k \kappa \det(X - x \Id) S_\tau,$$

where $\Id$ is the identity matrix on $V$ and $\kappa$ is the following element

$$\kappa = 1 - w(Y - y \Id)^{-1}(X - x \Id)^{-1} v$$

in $Q(S_\tau)$, the classical ring of quotients of $S_\tau$.

**Remark.** 1. $\kappa$ is, indeed, an element of $Q(S_\tau)$, since $(Y - y \Id)^{-1}(X - x \Id)^{-1}$ is an $n \times n$ matrix with entries from $Q(S_\tau)$ and multiplying from the left by $w$ $(1 \times n$ matrix) and from the right by $v(n \times 1$ matrix) produces an element in $Q(S_\tau)$.

2. The bijective correspondence part (without $G$-equivariance and a presentation for an ideal) was proved earlier by Baranovsky, Ginzburg and Kuznetsov [BGK].

3. The case when $\Gamma = \{1\}$, that is, when $O_\tau \cong A_1$ and the corresponding quiver variety is $C_n$, was proved by Berest and Wilson [BW1, BW2], and they also showed the transitivity of the $G$-action.

Since our main results are concerning quiver varieties for $\Gamma \cong Z_2$, we will restate the above theorem more explicitly in this case. First, let us identify all triples $(n_0, n_1; \epsilon)$ for which the corresponding quiver variety $\mathcal{M}_{Z_m}^\mathbb{C}(n_0, n_1; \epsilon)$ is non-empty. If we assume that $n_0 \leq n_1$ and take $n_0 = n - k$ and $n_1 = n$, then by (3.6), we have

$$\dim_{\mathbb{C}} \mathcal{M}_{Z_m}^\mathbb{C}(n-k, n; \epsilon) = \begin{cases} 2(n-k-k^2), & \text{for } \epsilon = 0, \\ 2(n-k^2), & \text{for } \epsilon = 1. \end{cases}$$

So we need to assume $n \geq k^2 + k$ for $\epsilon = 0$ and $n \geq k^2$ for $\epsilon = 1$. Now assuming $n_1 \geq n_0$, we obtain similar constraints on $n$ and $k$. Thus $\mathcal{M}_{Z_m}^\mathbb{C}(n_0, n_1; \epsilon) \neq \emptyset$ if and only if $(n_0, n_1) \in L_c \subset (\mathbb{Z}_{\geq 0})^2$, where

$$L_0 := \{(n-k,n) \mid k \geq 0, n \geq k^2 + k\} \cup \{(n,n-k) \mid k \geq 0, n \geq k^2\},$$

$$L_1 := \{(n-k,n) \mid k \geq 0, n \geq k^2\} \cup \{(n,n-k) \mid k \geq 0, n \geq k^2 + k\}.$$ 

Thus, by Theorem 3.1 for $\Gamma \cong Z_2$ there is a decomposition

$$\mathcal{R}_\tau = \bigsqcup_{c=0,1} \bigsqcup_{(n_0, n_1) \in L_c} \mathcal{R}_\tau(n_0, n_1; \epsilon),$$

where

$$\mathcal{R}_\tau(n_0, n_1; \epsilon) \cong \mathcal{M}_{Z_m}^\mathbb{C}(n-k, n; \epsilon).$$
and a $G_{zm}$-equivariant bijection defined for any $\epsilon = 0, 1$ and $(n_0, n_1) \in L_z$:

\begin{equation}
\Omega : \mathfrak{M}_{zm}^x(n_0, n_1; \epsilon) \simeq \mathcal{R}_e(n_0, n_1; \epsilon), \quad (X, Y, v, w) \mapsto M,
\end{equation}

where $M$ is defined in (3.10)-(3.17).

4. Generating set for $\mathcal{O}(\mathfrak{M}_{zm}^x)$

Our goal for this section is to produce a set of generators for $\mathcal{O}(\mathfrak{M}_{zm}^x(V, U_k))$, which will be used to show the transitivity of the $G$-action.

Let $(X, Y, v, w)$ be a point in $\mathfrak{M}_{zm}^x(V, U_k)$. Then by Theorem 3.1, the ideal corresponding to this point is uniquely determined by $\kappa$. Expand $\kappa$ into the formal power series:

\begin{equation}
\kappa = 1 + \sum_{l, q \geq 0} (wY^l X^q v) y^{-l-1} x^{-q-1},
\end{equation}

where $wY^l X^q v \in \mathbb{C}$. This defines an embedding

\[ \mathfrak{M}_{zm}^x(V, U_k) \rightarrow \mathbb{C}^\infty, \quad (X, Y, v, w) \mapsto (wY^l X^q v)_{l, q \geq 0}. \]

Let $p := \dim(V)$. Then by the Cayley-Hamilton identity any power of $X$ or $Y$ can be expressed in terms of powers of $X$ or $Y$ less than or equal to $p$. So we have an embedding

\[ \mathfrak{M}_{zm}^x(V, U_k) \rightarrow \mathbb{C}^N, \]

where $N = p^2$. Dually, we have $\mathbb{C}[x_1, \ldots, x_N] \rightarrow \mathcal{O}(\mathfrak{M}_x(V, U_k))$.

Lemma 4.1. (a) $\mathcal{O}(\mathfrak{M}_{zm}^x(V, U_k))$ is generated by $(wY^q X^r v)_q, r \leq p$.

(b) $wY^q X^r v = 0$ unless $q = r \pmod{m}$.

Proof. (a) It follows from the above arguments.

(b) This is a special case of the next Proposition. \qed

Proposition 4.1. If $k_1, \ldots, k_l, l_1, \ldots, l_t$ are non-negative integers and $\epsilon = 0, 1$, then

\begin{equation}
wY^{k_1} X^{l_1} \cdots Y^{k_l} X^{l_t} \cdots X^r v = cw Y^{k_1} X^{l_1} \cdots Y^{k_l} X^{l_t} v,
\end{equation}

for some constant $c$. Moreover, both sides of (4.2) are zero unless $K = L \pmod{m}$, where $K = k_1 + \cdots + k_t$ and $L = l_1 + \cdots + l_t$.

Proof. The first part of the statement for $\epsilon = 0$ is trivial. Proof of the second part for $\epsilon = 0$ and $\epsilon = 1$ are the same, so we may assume $\epsilon = 1$. Recall (see (3.10)) that $X$ (resp. $Y$) is an $m \times m$ block matrix with only non-zero entries on the $(m-1)$-st and on the $(-1)$-st (resp. 1-st and $-(m-1)$-st) diagonals. Those non-zero entries are the matrices $X_0, \ldots, X_{m-1}$ and $Y_0, \ldots, Y_{m-1}$ respectively. Set for $0 \leq i, j \leq m - 1$,

\[ D_{[i, j]} := \begin{cases} Y_i Y_{i-1} \cdots Y_j, & \text{if } i \geq j, \\ Y_i Y_{i-1} \cdots Y_0 Y_{m-1} \cdots Y_j, & \text{if } i < j, \end{cases} \]

and

\[ C_{[i, j]} := \begin{cases} X_i X_{i+1} \cdots X_m X_0 \cdots X_j, & \text{if } i > j, \\ X_i X_{i+1} \cdots X_j, & \text{if } i \leq j. \end{cases} \]

Let $k_i = k_i' m + q_i$ and $l_i = l_i' m + r_i$, for $i = 1, \ldots, t$, where $k_i', l_i' \geq 0$ and $0 \leq r_i, q_i \leq m - 1$. Then $Y^{k_i}$ is an $m \times m$ block matrix whose only nonzero entries are on the $(m - q_i)$-th and the $(-q_i)$-th diagonals. The entries on the $(m - q_i)$-th diagonal are

\[ (D_{[j-1, i]} Y^{k_i} D_{[j-1, m-q_i, r_i]}), \quad 0 \leq j \leq m - 1 - q_i - 1, \]

where $D_{[i-1, i]} := D_{[m-1, i]}$, while the entries on the $(-q_i)$-th diagonal are

\[ (D_{[j-1, i]} Y^{k_i} D_{[j-1, m, j-q_i]}), \quad q_i \leq j \leq m - 1. \]
Similarly, $X^{li}$ is an $m \times m$ block matrix whose two nonzero diagonals are the $(r_i)$-th and $(r_i - m)$-th ones. The entries on the $(r_i)$-th one are

$$C_{[j,j+r_i-1]}(C_{[j+r_i,j+r_i-1]})^l_{ij}$$

for $0 \leq j \leq m - r_i - 1$, where $C_{[j,\cdot]} := C_{[j,m-1]}$ and the entries on the $(r_i - m)$-th diagonal are

$$C_{[j,j+m-r_i-1]}(C_{[j+m-r_i,j+m-r_i-1]})^l_{ij}$$

for $m - r_i \leq j \leq m - 1$.

Without loss of generality, we may assume $r_i \leq q_i$. Then non-zero diagonals of $Y^{ki}X^{li}$ are the $(m - q_i + r_i)$-th and the $(r_i - q_i)$-th ones and their entries are

$$(D_{[j-1,j]})^{ki}_1(D_{[j-1,m-q_i+j]}C_{[m-q_i+j,m-q_i+j+r_i-1]}(C_{[m-q_i+j,m-q_i+j-1]})^l_{ij},$$

for $0 \leq j \leq q_i - r_i - 1$ and

$$(D_{[j-1,j]})^{ki}_1(D_{[j-1,j-q_i]}C_{[j-q_i,j-q_i-r_i-1]}(C_{[j-q_i-r_i,q_i-r_i-1]})^l_{ij},$$

for $q_i - r_i \leq j \leq m - 1$. Thus both matrices

$$Y^{k_1}X^{l_1} \cdots Y^{k_l} \mathcal{T}^r X^{l_l} \cdots Y^{k_i}X^{l_i}$$

are $m \times m$ block matrices whose only non-zero entries are on the $m - \sum_i (q_i - r_i)$-th and on the $- \sum_i (q_i - r_i)$-th diagonals. Hence these matrices are block diagonal matrices if and only if $\sum_i q_i = \sum_i r_i$. The later is equivalent to $K = L \mod m$. Now if $A = (A_{ij})$ is an $m \times m$ block matrix of size $n \times n$ then $wAv = w_k(A_{kk})v_k$ (see (4.1)). This proves the second part of the statement, i.e., both sides of (4.2) are zero unless $K = L \mod m$.

We can now assume the later condition and hence matrices in (4.3) are block diagonal. Moreover, the diagonal entries of the first matrix are multiples of the diagonal entries of the second one, where the multiples $c_1, \ldots, c_m$ are certain permutations of $\tau_0, \tau_1, \ldots, \tau_{m-1}$. Once again using $wAv = w_k(A_{kk})v_k$, we get that the LHS of (4.2) is a multiple of the RHS.

**Lemma 4.2.** For $(X, Y, v, w) \in \mathcal{M}_\mathbb{Z}_m(V, U_k)$ we have

$$(YX)^l = Y^l(YX) - \sum_{i=1}^{l-1} Y^{l-i} \mathcal{T} Y^{l-i-1} + \sum_{i=1}^{l-1} Y^{l-i} v w Y^{l-i-1}.$$ 

**Proof.** Since $XY = YX = \mathcal{T} + vw$, we have

$$(YX)^l = Y(YX - \mathcal{T} + vw) = Y(YX) - Y\mathcal{T} + Yvw.$$ 

Now the statement can be easily proved by induction. \hfill \blacksquare

**Proposition 4.2.** Let $(X, Y, v, w) \in \mathcal{M}_\mathbb{Z}_m(V, U_k)$. Then

$$wY^{k_1}X^{l_1} \cdots Y^{k_l}X^{l_l}v = w(Y^K X^L)v + f(wY^iX^jv)v_{K' < K, j < L},$$

where $K = k_1 + \cdots + k_l$, $L = l_1 + \cdots + l_l$ and $f$ is some polynomial function.

**Proof.** We prove this by induction. Suppose the statement holds for $K' < K$ and $L' < L$. Then

$$wY^{k_1}X^{l_1-1}(XY)^lY^{k_2-1}X^{l_2} \cdots Y^{k_l}X^{l_l}v$$

$$= wY^{k_1}X^{l_1-1}(YX - \mathcal{T} + vw)Y^{k_2-1}X^{l_2} \cdots Y^{k_l}X^{l_l}v$$

$$= wY^{k_1}X^{l_1-1}(YX)Y^{k_2-1}X^{l_2} \cdots Y^{k_l}X^{l_l}v - wY^{k_1}X^{l_1-1} \mathcal{T} Y^{k_2-1}X^{l_2} \cdots Y^{k_l}X^{l_l}v$$

$$+ wY^{k_1}X^{l_1-1} v w Y^{k_2-1}X^{l_2} \cdots Y^{k_l}X^{l_l}v.$$
If we use Lemma 4.2 to the first term, Proposition 4.1 to the second one and the induction assumption to the third term in the last expression, then

\[
w Y^{k_1} X^{l_1-1} \left[ Y^{k_2-1} (YX) - \sum_{i=1}^{k_2-1} Y^i TY^{k_2-i-1} + \sum_{i=1}^{k_2-1} Y^{i+v} w Y^{k_2-i-1} \right] X l_2 \ldots Y^{k_1} X^{l_1-v} - cw Y^{k_1} X^{l_1-1} Y^{k_2-1} X l_2 \ldots Y^{k_1} X^{l_1-v} + g(wY^i X^j v)_{i < K}, j < L \]

\[
= w Y^{k_1} X^{l_1-1} Y^{k_2} X^{l_2+1} \ldots Y^{k_1} X^{l_1-v} - \sum_{i=1}^{k_2-1} w Y^{k_1} X^{l_1-1} Y^i TY^{k_2-i-1} X l_2 \ldots Y^{k_1} X^{l_1-v} + g(wY^i X^j v)_{i < K}, j < L, \]

where in the last equality we again used Proposition 4.1 and the induction assumption. Now repeating this procedure we can move \( Y^{k_2} \) further to the left and obtain

\[
w Y^{k_1+k_2} X^{l_1+l_2} Y^{k_3} \ldots Y^{k_1} X^{l_1-v} + h(wY^i X^j v) \]

for some polynomial \( h \). Similarly moving other powers of \( Y \) to the left we get

\[
w \left( Y^{k_1+\ldots+k_i} X^{l_1+\ldots+l_i} \right) v + f(wY^i X^j v)_{i < K}, j < L, \]

for some polynomial \( f \).

\[\Box\]

Recall that for \( m = 1 \) we have \( \mathfrak{m}_{v,m}^{\mathbb{Z}_m} = \{ \mathfrak{m}_{v,m}^{\mathbb{Z}_m} \} \cong \mathcal{C}_n \) (see Lemma 4.1), the \( n \)-th Calogero-Moser space. Then one has:

**Theorem 4.1.** The algebra \( \mathcal{O}(\mathcal{C}_n) \) is generated by the set \( W_1 := \{ w(Y + cX)^n v \}_{c \in \mathcal{C}, n \in \mathbb{N}}. \)

**Proof.** Similar to the case \( m = 2 \) below.

\[\Box\]

Next we prove our main result in this section.

**Theorem 4.2.** The set \( W_2 := \{ w(Y + cX)^{2n} v \}_{c \in \mathcal{C}, n \in \mathbb{N}} \) generates \( \mathcal{O}(\mathfrak{m}_{v,m}^{\mathbb{Z}_m}(V,U_k)) \).

**Proof.** We prove this statement by expressing generators \( wY^i X^j v \) of \( \mathcal{O}(\mathfrak{m}_{v,m}) \) (see Lemma 4.1) as polynomials of elements of \( W_2 \). The proof is by induction on the total degree \( i + j \) of \( wY^i X^j v \). First recall that, by Lemma 4.1(b), \( wY^i X^j v \) are zero for odd values of \( i + j \), so we will only consider \( i + j = 2n \). For \( n = 1 \), expanding \( w(Y + cX)^{2n} v \), one gets

\[
w Y^2 v + cw(YX + XY) v + c^2 wX^2 v, \]

and since

\[
w(YX + XY)v = 2wYXv - wTv + (wv)^2 = 2wYXv + d, \]

for some \( d \in \mathcal{C} \), we have

\[
w(Y + cX)^2 = wY^2 v + 2c wYX v + c^2 wX^2 v + cd. \tag{4.4} \]

By choosing distinct constants \( c_1, c_2 \) and \( c_3 \) for \( c \) in (4.4), we can show that \( wY^2 v, wYX v, wX^2 v \) can be expressed by elements of \( W_2 \). Now assume that all elements \( wY^i X^j v \) for \( i + j < 2n \) can be generated by elements of \( W_2 \). Then the expansion of \( w(Y + cX)^{2n} v \) by powers of \( c \) gives

\[
wY^{2n} v + cw(Y^{2n-1} X + \ldots) v + c^2 w(Y^{2n-2} X^2 + \ldots) v + \ldots + c^{2n} wX^{2n} v. \]
By Lemma 4.2, the coefficient of $c^j$ can be written as
\[ w(Y^iX^j + \cdots)v = swY^iX^jv + f(wY^pX^qv)_{p+q<2n}. \]
Then, by the induction assumption, $f(wY^pX^qv)$ can be generated by $W_2$ and hence each $wY^iX^jv$ for $i + j = 2n$ can be expressed in terms of $W_2$. \( \square \)

5. Transitivity of the $G_\Gamma$-action

Let $X$ be a smooth algebraic variety. A subgroup $G$ of $\text{Aut}_C(X)$ is called *algebraically generated* if it is generated as an abstract group by a family $\mathfrak{G}$ of connected algebraic subgroups of $\text{Aut}_C(X)$. Suppose $\mathfrak{G}$ is closed under conjugation by elements of $G$.

**Definition 5.1.** We say that a point $x \in X$ is $G$-flexible if the tangent space $T_xX$ is spanned by the tangent vectors to the orbits $H \cdot x$ of subgroups $H \in \mathfrak{G}$. The variety $X$ is called $G$-flexible if every point $x \in X$ is $G$-flexible.

Let us give some comments on this definition. First, since $X$ is smooth, $G$-flexibility of $X$ can be defined in terms of cotangent spaces instead of tangent spaces. Second, one can easily show that $X$ is $G$-flexible if one point of $X$ is $G$-flexible and $G$ acts transitively on $X$.

We have the following characterization of flexible points (see [AFKKZ, Corollary 1.11]):

**Proposition 5.1.** A point $x \in X$ is $G$-flexible if and only if the orbit $G \cdot x$ is open in $X$. An open $G$-orbit (if it exists) is unique and consists of all $G$-flexible points in $X$.

We need the following result proved in [CB, Theorem 1.3]:

**Theorem 5.1.** $\mathcal{M}_\Gamma(T, U)$ is a reduced and irreducible scheme. In particular, it is a smooth, connected affine algebraic variety.

Recall that $\mathcal{M}_\Gamma(T, U)$ has a natural symplectic structure with the symplectic form $\omega = \text{Tr}(dX \wedge dY)$ (see e.g. [CB]). This form gives an isomorphism between 1-forms and vector fields:
\[ \text{Tr}(f dX + g dY) \mapsto g \frac{\partial}{\partial X} - f \frac{\partial}{\partial Y} = (g, -f). \]
Since algebraic vector fields are in one-to-one correspondence with derivations, for any Hamiltonian $H \in \mathcal{O}(\mathcal{M}_\Gamma)$, the one-form $dH$ defines a derivation of $\mathcal{O}(\mathcal{M}_\Gamma)$, given in terms of the Poisson bracket $\{H, -\}$.

One can easily verify:

**Lemma 5.1.** The action of $G_\Gamma$ on $\mathcal{M}_\Gamma$ is symplectic.

Therefore any one parameter subgroup of $G_\Gamma$ defines Hamiltonian flow on $\mathcal{M}_\Gamma$ and hence corresponds to some Hamiltonian in $\mathcal{O}(\mathcal{M}_\Gamma)$.

**Example 5.1.** We show that
\[ \phi_t = (X + aY^2, Y) (X, Y + tX^2) \] is a Hamiltonian flow with Hamiltonian $\text{Tr}((X + aY^2)^3)$. In fact, we have
\[ \frac{d\phi_t}{dt}_{|t=0} = (-aY(X + aY^2)^2 + (X + aY^2)^2Y, (X + aY^2)^2). \]
On the other hand,
\[ d\text{Tr}((X + aY^2)^3) = \text{Tr}((X + aY^2)^2 (dX + aY dY + a dY Y)) = \text{Tr}((X + aY^2)^2 dX) + \text{Tr}(a[(X + aY^2)^2Y + Y(X + aY^2)^2] dY). \]
Using the symplectic form \( \text{Tr}(dX \wedge dY) \) which gives the isomorphism between 1-forms and vector fields:

\[
\text{Tr}(f \, dX + g \, dY) \mapsto g \frac{\partial}{\partial X} - f \frac{\partial}{\partial Y} = (g, -f),
\]

one obtains the corresponding vector field

\[
(-a(Y(X + aY^2)^2 + (X + aY^2)^2Y), (X + aY^2)^2).
\]

In fact there is a well-known and more general result.

**Lemma 5.2.** Let \( \phi_t \) be a Hamiltonian flow with Hamiltonian function \( H \) and \( \psi \) is any symplectic automorphism, then \( \psi \phi_t \psi^{-1} \) is a Hamiltonian flow with Hamiltonian \( \psi^* H = H \circ \psi \).

**Proof.** Let \( X_H \) be a Hamiltonian vector field for \( H \), namely

\[
X_H \big| \omega = dH.
\]

A flow \( \phi_t \) being the Hamiltonian flow of \( X_H \) implies

\[
\frac{d\phi_t}{dt}(x) = X_H(x).
\]

Equivalently for any function \( f \) we have

\[
\lim_{t \to 0} \frac{\phi_t(f) - f}{t} = X_H(f).
\]

For the flow \( \psi \phi_t \psi^{-1} \) we have

\[
\lim_{t \to 0} \frac{(\psi \phi_t \psi^{-1})^*(f) - f}{t} = X_H(\psi^*(f)) = \psi_*(X_H)(f).
\]

Thus \( \psi_* (X_H) \) is the vector field for the flow \( \psi \phi_t \psi^{-1} \). Therefore

\[
\psi_*(X_H) \big| \omega = \psi^*(dH) = d(\psi^* H). \quad \square
\]

From the above lemma, it follows immediately that:

**Corollary 5.1.** For \( m \in \mathbb{N} \) and \( n_1, n_2 \in \mathbb{Z}_{\geq 0} \), we have

\( a \) \( w \left( X + a Y^{m_1 n_1 - 1} \right)^{m_2} v \) is Hamiltonian with the flow

\( \phi_t = (x + a y^{m_1 n_1 - 1}, y)(x, y + t x^{m_2 n_2 - 1})(x - a y^{m_1 n_1 - 1}, y). \)

\( b \) \( w \left( Y + b X^{m_1 n_1 - 1} \right)^{m_2} v \) is Hamiltonian with the flow

\( \phi_t = (x, y + b x^{m_1 n_1 - 1})(x + t y^{m_2 n_2 - 1}, y)(x, y - b x^{m_1 n_1 - 1}). \)

**Lemma 5.3.** Let \( X \) be a smooth affine algebraic variety. If \( \{f_i\}_{i=1}^n \) generates \( \mathcal{O}(X) \) as an algebra, then \( \{df_i\}_{i=1}^n \) generates the cotangent space at any point of \( X \).

Let \( m = 1 \). Then \( \mathfrak{M}_{\Gamma = \{1\}}(\mathbb{C}^n, \mathbb{C}) = C_n \) is the \( n \)-th Calogero-Moser space and \( G_{\Gamma = \{1\}} \) is isomorphic to the subgroup of \( \text{Aut}_C(\mathbb{C}[x, y]) \) preserving the canonical symplectic form \( dx \wedge dy \). The later group can be identified with \( \text{Aut}_C(A_1) \), the automorphism group of the first Weyl algebra \([\text{XIII}]. \) We recall that \( G_{\Gamma = \{1\}} \) is generated by two families of automorphisms

\( (x + a y^n, y) \) and \( (x, y + b x^m) \),

where \( a, b \in \mathbb{C} \) and \( n, m \in \mathbb{Z}_{\geq 0} \). Then we recover the following result of Berest and Wilson \([\text{BW1}], \) Theorem 1.3):

**Theorem 5.2.** \( G_{\Gamma = \{1\}} \) acts on \( C_n \) transitively for each \( n \in \mathbb{Z}_{\geq 0} \).
Proof. By Theorem 4.1, $W_1$ generates $\mathcal{O}(\mathcal{C}_n)$ as an algebra. On the other hand, by Corollary 5.1, the family $W_1$ is Hamiltonian for the family of one-parameter subgroups $\phi_t = (x + ay, y)(x, y + tx^{k-1})(x - ay, y)$. Therefore, by Lemma 5.3, differentials of elements in $W_1$ span the cotangent space at any point. Hence $\mathcal{C}_n$ is a $G$-flexible. Since $\mathcal{C}_n$ is connected, by Proposition 5.1, $\text{Aut}_G(\mathbb{C}^2)$ acts transitively on $\mathcal{C}_n$.

Similarly for $\Gamma \cong \mathbb{Z}_2$ we have:

**Theorem 5.3.** $G_{\Gamma}$ acts transitively on $\mathcal{M}^*_{\mathbb{Z}_2}(n_0, n_1; \varepsilon)$ for each $(n_0, n_1) \in \mathbb{L}_c$.

**Proof.** Recall that $G_{\mathbb{Z}_2} = (\Theta_{\lambda}, (x + \lambda y^{2^k-1}, y), (x, y + \mu x^{2^k-1}))_{k \geq 1}$. By Theorem 4.2, elements of $W_2$ generate $\mathcal{O}(\mathcal{M}^*_{\mathbb{Z}_2})$. On the other hand, by Corollary 5.1, elements of $W_2$ are Hamiltonian with Hamiltonian flows $\phi_t = (x + ay^{2^k-1}, y)(x, y + tx^{k-1})(x - ay^{2^k-1}, y)$. Hence, by Lemma 5.3, differentials of the functions from $W_2$ span the cotangent space at any point. Hence $\mathcal{M}^*_{\mathbb{Z}_2}$ is a $G_{\mathbb{Z}_2}$-flexible variety. Since $\mathcal{M}^*_{\mathbb{Z}_2}$ is connected, by Proposition 5.1, $G_{\mathbb{Z}_2}$ acts transitively on $\mathcal{M}^*_{\mathbb{Z}_2}$. □

6. $C^*$-fixed points of $\mathcal{M}^*_{\mathbb{Z}_2}$

In this section we present for each quiver variety $\mathcal{M}^*_{\mathbb{Z}_2}(n_0, n_1; \varepsilon)$ a distinct point for which the computation of the corresponding $\kappa$ is relatively simple. More precisely, we will find a point of $\mathcal{M}^*_{\mathbb{Z}_2}(n_0, n_1; \varepsilon)$ represented by a quadruple $(X, Y, v, w)$, or equivalently by $(X_0, X_1, Y_0, Y_1, v_e, w_e)$, such that it is fixed by the $C^*$-action, where $C^*$ is a subgroup of $G_{\mathbb{Z}_2} = \text{Aut}_G(\mathbb{C})$. The later implies that the corresponding matrices $X$ and $Y$ are nilpotent and therefore the expansion (3.1.1) of $\kappa$ is a non-commutative Laurent polynomial in $x^{-1}$ and $y^{-1}$.

We now briefly outline how to construct these points. Recall that $\mathcal{M}^*_{\mathbb{Z}_2}(n_0, n_1; \varepsilon)$ is nonempty if and only if $(n_0, n_1) \in \mathbb{L}_c \subset (\mathbb{Z}_{\geq 0})^2$, where $\mathbb{L}_c$ is defined in (3.1.8). We may assume that $n_0 \leq n_1$ since, by Lemma 3.1, $\mathcal{M}^*_{\mathbb{Z}_2}(n_0, n_1; 0) \cong \mathcal{M}^*_{\mathbb{Z}_2}(n_1, n_0; 1)$. First we construct points for special values of $n_0$ and $n_1$, namely for $\mathcal{M}^*_{\mathbb{Z}_2}(k^2, k^2 + k; 0)$ and $\mathcal{M}^*_{\mathbb{Z}_2}(k^2 - k, k^2; 1)$, where $k \geq 1$. Note that, by (3.1.1), the later varieties are zero-dimensional and by connectedness each of them consists of a single point. Since $G_{\mathbb{Z}_2}$ acts on each of these singleton varieties, they are fixed by $G_{\mathbb{Z}_2}$ and, in particular, by its subgroup $C^*$. Let $(X, Y, v, w)$ be the point of $\mathcal{M}^*_{\mathbb{Z}_2}(k^2 - k, k^2; 1)$ such that (see (3.1.1))

$$X = \begin{pmatrix} 0 & X_0 \\ X_1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & Y_1 \\ Y_0 & 0 \end{pmatrix}.$$ 

Then we will describe a procedure how to add $n - k^2$ rows and $n - k^2$ columns to matrices $X_0, X_1, Y_0$ and $Y_1$ to get a nilpotent point of $\mathcal{M}^*_{\mathbb{Z}_2}(n - k, n; 1)$. Similarly, one can add $n - k - k^2$ rows and $n - k - k^2$ columns to $X_0, X_1, Y_0$ and $Y_1$ to obtain a point of $\mathcal{M}^*_{\mathbb{Z}_2}(n - k, n; 0)$.

6.1. Preliminaries. In this section we give decomposition of sets $\{1, 2, \ldots, k^2\}$ and $\{1, 2, \ldots, k^2 - k\}$ into disjoint union of subsets, which will be used as index sets for defining column vectors of matrices $X_i$ and $Y_i$ ($i = 0, 1$). Let

$$S_1 : = \{ i(i+1) - j + 1 \mid 1 \leq j \leq \left[ \frac{k}{2} \right], \quad j \leq i \leq k - j \},$$

$$S_2 : = \{ (i+1)^2 - j \mid 0 \leq j \leq \left[ \frac{k-1}{2} \right], \quad j \leq i \leq k - j - 1 \},$$

$$S_3 : = \{ i(i+1) - j + 1 \mid \left[ \frac{k}{2} \right] + 1 \leq i \leq k - 1, \quad k - i + 1 \leq j \leq i \},$$

$$S_4 : = \{ (i+1)^2 - j \mid \left[ \frac{k+1}{2} \right] \leq i \leq k - 1, \quad k - i \leq j \leq i \}.$$ 

We have:
Lemma 6.1. \( (i) \) For any two distinct pairs \((i, j)\) the defining relation in \(S_l\) \((l = 1, \cdots, 4)\) yields distinct values.

\((ii) \) \(S_i \cap S_j = \emptyset\) for \(i \neq j\).

\((iii) \) \(\sum_{l=1}^{4} |S_l| = k^2\), where \(|S_l|\) is the cardinality of \(S_l\).

In particular,
\[
\bigcup_{l=1}^{4} S_l = \{1, \cdots, k^2\}.
\]

Now we introduce another sets of integers:

\[
T_1 : = \left\{ i(i+1) - j + 1 \mid 1 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor, j \leq i \leq k - j - 1 \right\},
\]

\[
T_2 : = \left\{ (i+1)^2 - j \mid 0 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor, j \leq i \leq k - j - 2 \right\},
\]

\[
T_3 : = \left\{ i(i+1) - j + 1 \mid \left\lfloor \frac{k+1}{2} \right\rfloor \leq i \leq k - 1, k - i \leq j \leq i \right\},
\]

\[
T_4 : = \left\{ (i+1)^2 - j \mid \left\lceil \frac{k}{2} \right\rceil \leq i \leq k - 2, k - i - 1 \leq j \leq i \right\},
\]

then one can prove a statement similar to that of Lemma 6.1:

Lemma 6.2. \( (i) \) For any two distinct pairs \((i, j)\) the defining relation in \(T_l\) \((l = 1, \cdots, 4)\) yields distinct values.

\((ii) \) \(T_i \cap T_j = \emptyset\) for \(i \neq j\).

\((iii) \) \(\sum_{l=1}^{4} |T_l| = k(k-1)\).

In particular,
\[
\bigcup_{l=1}^{4} T_l = \{1, \cdots, k(k-1)\}.
\]

Next we establish some important relations between sets \(\{S_l\}\) and \(\{T_l\}\). Let \(S_4 = S_1^I \cup S_4^I\), where

\[
S_4^I = \{k(k-1) + i \mid 1 \leq i \leq k - 1 \}
\]

and \(S_4^I = S_4 \setminus S_4^I\). Hence using Lemmas 6.1 and 6.2 one obtains

\[
S_1 \sqcup S_2 \setminus \{k^2\} \sqcup S_3 \sqcup S_4^I = \bigcup_{l=1}^{4} T_l.
\]

The following proposition is easy to prove:

Proposition 6.1.

\[
T_3 \cap S_1 = S_1 \setminus T_1 = \left\{ (k - j)(k - j + 1) - j + 1 \mid 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor \right\},
\]

\[
T_4 \cap S_2 = S_2 \setminus (T_2 \cup \{k^2\}) = \left\{ (k - j)^2 - j \mid 1 \leq j \leq \left\lfloor \frac{k - 1}{2} \right\rfloor \right\}.
\]

In particular, we have

\[
T_3 = S_3 \cup (T_3 \cap S_1), T_4 = S_4^I \cup (T_4 \cap S_2).
\]
6.2. \( C^*\)-fixed point of \( \mathfrak{U}_{22}(k^2-k, k^2; 1) \). Recall that this point is represented by \( (X_0, X_1, Y_0, Y_1; v_1, w_1) \), where \( X_0, Y_1 \in \text{Mat}_{n_0 \times n_1}(\mathbb{C}), \ X_1, Y_0 \in \text{Mat}_{n_1 \times n_0}(\mathbb{C}), \ v_1 \in \mathbb{C}^{n_1} \) and \( w_1 \in (\mathbb{C}^{n_1})^* \) such that

\[
\begin{align*}
(6.1) & \quad X_0 Y_0 - Y_1 X_1 + \tau_0 \text{Id}_{n_0} = 0, \\
(6.2) & \quad X_1 Y_1 - Y_0 X_0 + \tau_1 \text{Id}_{n_1} = v_1 w_1.
\end{align*}
\]

**Notation.** Let us introduce two more notations which will be in this and later sections: \( a := \tau_1, b := \tau_0 + \tau_1 \).

6.2.1. **Matrix** \( Y_1 \). Let \( e_1, \ldots, e_{k(k-1)} \) be the standard basis for \( \mathbb{C}^{k(k-1)} \). Then define vectors \( v_1, \ldots, v_{k^2} \) as follows.

For \( S_1 \) indices:

\[
(6.3) \quad v_{i(i+1)-j+1} := (2j - 1) b e_{i+1-j}
\]

\[
(6.4) \quad + (a + 2(k - i + j - 1)b) \left( \sum_{l=1}^{[i+1]} e_{i-l+1} x_{-i-j+1} + \sum_{l=1}^{[i]} e_{i-l} (i-l+1) - i-j+1 \right),
\]

where \([a]\) is the integer part of \( a \).

For \( S_2 \):

\[
(6.5) \quad v_{i+1} - j := 2j b e_{i+1-j+1}
\]

\[
(6.6) \quad -(a + 2(k - i + j - 1)b) \left( \sum_{l=0}^{k-i-1} e_{i+l+1} x_{i-j+1} + \sum_{l=0}^{k-i-2} e_{i+l+1} (i+l+2) - i-j+1 \right).
\]

For \( S_3 \):

\[
(6.6) \quad v_{i+1} - j := -(a + 2(k - i + j - 1)b) e_{i+1-j+1}
\]

\[
(6.6) \quad -(a + 2(k - i + j - 1)b) \left( \sum_{l=0}^{k-i-2} e_{i+l+1} (i+l+2) - i-j \right) + \sum_{l=0}^{k-i-2} e_{i+l+1} (i+l+2) - i-j \right) + \sum_{l=0}^{k-i-2} e_{i+l+1} (i+l+2) - i-j \right).
\]

Define \( Y_1 \) as an \( k(k-1) \times k^2 \) matrix with columns consisting of \( v_i \)'s

\[
Y_1 := [v_1 v_2 \cdots v_{k^2}]\]

6.2.2. **Matrix** \( Y_0 \). Let \( f_1, \ldots, f_{k^2} \) be the standard basis for \( \mathbb{C}^{k^2} \). Then we introduce the following vectors. For \( T_1 \):

\[
(6.7) \quad u_{i(i+1)-j+1} := (a + 2j - 1)b f_{i+1-j+1}
\]

\[
(6.7) \quad + (a + 2(k - i + j - 1)b) \left( \sum_{l=1}^{[i+1]} f_{i-l+1} x_{i-l+1} + \sum_{l=1}^{[i]} f_{i-l+1} (i-l+2) - i-j \right),
\]
For $T_2$:
\[
\mathbf{u}_{(i+1)^2-j} = (a + 2j b) \mathbf{f}_{(i+1)^2-j} + (a + 2(k - i - j - 1) b) \left( \sum_{l=1}^{k-i-1} f_{(i+l+2)^2-i-j-1} + \sum_{l=1}^{k-i-2} f_{(i+l+2)^2-i-j-1} \right).
\]

(6.8)

For $T_3$:
\[
\mathbf{u}_{(i+1)^2-j} := -(2k - 2i - 1) b \mathbf{f}_{(i+1)^2-j+1}
\]

(6.9)

For $T_4$:
\[
\mathbf{u}_{(i+1)^2-j} := -2(k - i - 1) b \mathbf{f}_{(i+1)^2-j}
\]

(6.10)

Thus, we can define $Y_0$ as a $k^2 \times k(k-1)$ matrix with columns consisting of $\mathbf{u}_i$'s
\[
Y_0 := [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_{k(k-1)}].
\]

6.2.3. Matrices $X_0$ and $X_1$. We define
\[
X_0 := [\mathbf{e}_1 \cdots \mathbf{e}_{k(k-1)} 0 \cdots 0],
\]
where the number of 0 column vectors is exactly $k$. For
\[
X_1 := [\mathbf{w}_1 \cdots \mathbf{w}_{k(k-1)}],
\]
where $\mathbf{w}_i$'s are defined as follows. For $1 \leq i \leq k - 1$ and $1 \leq j \leq 2i$,
\[
\mathbf{w}_{i(j-1)+j} := \mathbf{f}_{i+j}.
\]

6.2.4. The vector $v_1$ and the covector $w_1$. To compute $v_1$ and $w_1$, first we need to see effects of multiplications by matrices $X_0$ and $X_1$. Indeed, one can easily check that the multiplication by $X_0$ from the left to the $k^2 \times k(k-1)$ matrix $A$ produces a $k(k-1) \times k(k-1)$ matrix obtained from $A$ by removing the last $k$ rows. If we multiply by $X_0$ from the right to $A$ then this gives the $k^2 \times k^2$ matrix obtained by adding $k$ zero columns to the right end of $A$.

Next, if we multiply by $X_1$ from the left to a $k(k-1) \times k^2$ matrix $B$, then one obtains the $k^2 \times k^2$ matrix whose $i^2$-th ($i = 1, \ldots, k$) rows are zero rows and if we remove those rows, we get exactly $B$. While multiplication by $X_1$ from the right to $B$ produces the $k(k-1) \times k(k-1)$ matrix obtained from $B$ by removing columns on the $i^2$-th positions where $i = 1, \ldots, k$.

Thus, one has
\[
X_0 Y_0 = [\tilde{\mathbf{u}}_1 \tilde{\mathbf{u}}_2 \cdots \tilde{\mathbf{u}}_{k(k-1)}], \quad Y_0 X_0 = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_{k(k-1)} 0 \cdots 0],
\]

(6.11)

\[
X_1 Y_1 = [\tilde{\mathbf{v}}_1 \tilde{\mathbf{v}}_2 \cdots \tilde{\mathbf{v}}_{k^2}], \quad Y_1 X_1 = [\mathbf{v}'_1 \mathbf{v}'_2 \cdots \mathbf{v}'_{k^2}],
\]

(6.12)

where $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$ and $\mathbf{v}'$ are defined as follows. First, for $i \in T_1 \sqcup T_2$, the vector $\tilde{\mathbf{u}}_i$, can be obtained from the corresponding $\mathbf{u}_i$ by replacing the standard vectors $\mathbf{f} \in \mathbb{C}^{k^2}$ in (6.7) or (6.8) by the standard vectors $\mathbf{e} \in \mathbb{C}^{k(k-1)}$. Next, the vector $\tilde{\mathbf{u}}_i$, for $i \in T_3$, can be presented as in (6.9), where $\mathbf{f}$ is replaced by $\mathbf{e}$ and $k - i - 1$, the upper limit in the first summation, is replaced by $k - i - 2$. Then, the vector $\tilde{\mathbf{u}}_i$, for $i \in T_4$, can be presented as in (6.10), where $\mathbf{f}$ is replaced by $\mathbf{e}$ and $k - i - 2$, the upper limit in the second summation, is replaced by $k - i - 3$. 
Next, we define $\tilde{v}_i$. For $S_1$ indices:

$$\tilde{v}_{i(i+1)-j+1} := (2j - 1) b f_{i(i+1)-j+1}$$

(6.13) $$+(a + 2(k - i + j - 1) b) \left( \sum_{l=1}^{[i-j] \over 2} f_{i-l+1(i-l+2)-i-j} + \sum_{l=1}^{[i-k] \over 2} f_{i-l+1(i-l+2)-i-j} \right).$$

For $S_2$ indices:

$$\tilde{v}_{(i+1)^2-j} := 2j b f_{(i+1)^2-j}$$

(6.14) $$+(a + 2(k - i + j - 1) b) \left( \sum_{l=0}^{k-i-1} f_{i+l+1(i+l+2)-i-j} + \sum_{l=0}^{k-i-2} f_{i+l+2(i+l+3)-i-j-1} \right).$$

For $S_3$ indices:

$$\tilde{v}_{i(i+1)-j+1} := -(a + (2k - 2i - 1) b) f_{i(i+1)-j+1}$$

(6.15) $$-(a + 2(k - i + j - 1) b) \left( \sum_{l=0}^{k-i-2} f_{i+l+1(i+l+2)-i-j} + \sum_{l=0}^{k-i-2} f_{i+l+2(i+l+3)-i-j-1} \right).$$

For $S_4$ indices:

$$\tilde{v}_{(i+1)^2-j} := -(a + 2(k - i - 1) b) f_{(i+1)^2-j}$$

(6.16) $$-(a + 2(k - i + j - 1) b) \left( \sum_{l=0}^{k-i-2} f_{i+l+2(i+l+3)-i-j} + \sum_{l=0}^{k-i-2} f_{i+l+2(i+l+3)-i-j-1} \right).$$

Finally, define $v'_i$ as follows

(6.17) $$T_1 \cup T_3 : \quad v'_{i(i+1)-j+1} = \tilde{v}_{i(i+1)-j+1},$$

(6.18) $$T_2 \cup T_4 : \quad v'_{(i+1)^2-j} = \tilde{v}_{(i+1)^2-j}.$$  

Thus, by using this and the above description of $\tilde{v}_i$’s, we obtain

$$X_0 Y_0 - Y_1 X_1 = [\tilde{u}_1 - v'_1 : \tilde{u}_{k(k-1)} - v'_{k(k-1)}] = (a - b) \text{Id}_{k(k-1)},$$

where $\text{Id}_{k(k-1)}$ is the $k(k-1) \times k(k-1)$ identity matrix.

Next by (6.11) and (6.12), one can write

$$X_1 Y_1 - Y_0 X_0 + a \text{Id}_{k^2}$$

as

$$[\tilde{v}_1 - u_1 - a f_1 : \tilde{v}_{k(k-1)} - u_{k(k-1)} + a f_{k(k-1)} : \tilde{v}_{k(k-1)+1} + a f_{k(k-1)+1} : \tilde{v}_{k^2} + a f_{k^2}].$$

To compute this expression, we use (6.7)-(6.10), (6.13)-(6.16) and the following decomposition (see Proposition (5.1) and discussion before)

$$\{1, 2, \cdots, k^2\} = T_1 \cup T_2 \cup S_3 \cup S_4 \cup (T_3 \cap S_1) \cup (T_4 \cap S_2) \subseteq \{k^2\}.$$  

Explicitly, for $T_1$ indices, one computes

$$\tilde{v}_{i(i+1)-j+1} - u_{i(i+1)-j+1} + a f_{i(i+1)-j+1} = 0.$$  

The same vanishing result is true for $T_2, S_3$ and $S_4$ indices.
Now for $T_3 \cap S_1$, we have

$$
\hat{v}_{(k-j)(k-j+1)-j+1} - u_{(k-j)(k-j+1)-j+1} = (4j - 2) b \, f_{(k-j)(k-j+1)-j+1} \\
+ \left( a + (4j - 2) b \right) \left( \sum_{l=1}^{k+1-j} f_{(k-j-l+1)(k-j-l+2)-k} + \sum_{l=1}^{k-j} f_{(k-j-l+1)^2 - k} \right) \\
+ \left( a + (4j - 2) b \right) \left( \sum_{l=0}^{j-1} f_{(k-j+l+1)(k-j+l+2)-k} + \sum_{l=0}^{j-2} f_{(k-j+l+2)^2 - k} \right).
$$

(6.22)

By change of variables in the above summations, one gets

$$
\hat{v}_{(k-j)(k-j+1)-j+1} - u_{(k-j)(k-j+1)-j+1} = \left( a + (4j - 2) b \right) \left( \sum_{l=1}^{k+1-j} f_{(k-j-l+1)(k-j-l+2)-k} + \sum_{l=1}^{k-j} f_{(k-j-l+1)^2 - k} \right).
$$

(6.23)

Similarly, for $T_3 \cap S_2$, we get

$$
\hat{v}_{(k-j)^2 - j} - u_{(k-j)^2 - j} + a f_{(k-j)^2 - j} = \left( a + 4j b \right) \left( \sum_{l=1}^{k+1} f_{(k-l+1)(k-l+2)-k} + \sum_{l=1}^{k-1} f_{(k-l+1)^2 - k} \right)
$$

(6.24)

and finally for $k^2$, one has

$$
\hat{v}_{k^2} + a f_{x^2} = a \left( \sum_{l=1}^{k+1} f_{(k-l+1)(k-l+2)-k} + \sum_{l=1}^{k-1} f_{(k-l+1)^2 - k} \right).
$$

(6.25)

Thus, by (6.21)-(6.25) all the columns (6.20) except those from the set $(T_3 \cap S_1) \cup (T_4 \cap S_2) \cup \{k^2\}$ are zero. Moreover, we can state

**Proposition 6.2.** (6.20) is a rank one matrix and the $(i,j)$-th entry is non-zero if and only if $i, j \in (T_3 \cap S_1) \cup (T_4 \cap S_2) \cup \{k^2\}$.

**Proof.** First, (6.23) - (6.25) are linearly dependent vectors and hence (6.20) is a rank one matrix.

The second assertion is also easy to show. The vectors in (6.20) - (6.26) are indexed by the set $(T_3 \cap S_1) \cup (T_4 \cap S_2) \cup \{k^2\}$ and they are colinear to

$$
\sum_{l=1}^{k+1} f_{(k-l+1)(k-l+2)-k} + \sum_{l=1}^{k-1} f_{(k-l+1)^2 - k}.
$$

(6.26)

So one only needs to show

$$
(T_3 \cap S_1) \cup (T_4 \cap S_2) \cup \{k^2\} = \left\{ (k-l+1)(k-l+2)-k, 1 \leq l \leq \frac{k+1}{2} \right\} \\
\bigcup \left\{ (k-l+1)^2 - k, 1 \leq l \leq \frac{k-1}{2} \right\}.
$$

(6.27)

For $2 \leq l \leq \frac{k+1}{2}$,

$$
(k-l+1)(k-l+2)-k = (k-l+1)^2 - l + 1.
$$
which are exactly the elements of $T_3 \cap S_2$, while for $l = 1$, $(k - l + 1)(k - l + 2) - k = k^2$. On the other hand, for $1 \leq l \leq \frac{k-1}{2}$,

$$(k - l + 1)^2 - k = (k - l)(k - l + 1) - l + 1,$$

which are exactly the elements of $T_3 \cap S_1$. \hfill \Box

**Remark.** Note that the identity (6.27) can be rewritten as

$$(T_3 \cap S_1) \cup (T_4 \cap S_2) \cup \{k^2\} = \left\{l(l + 1) - k, \frac{k + 1}{2} \leq l \leq k\right\} \cup \left\{l^2 - k, \frac{k + 3}{2} \leq l \leq k\right\},$$

and (6.26) is equivalent to

$$\sum_{l=1}^{k} f_{l(l+1)-k} + \sum_{l=\frac{k+1}{2}}^{k} f_{l^2-k}.$$  

Thus, from Proposition 6.2 and the Remark after it, we conclude:

**Corollary 6.1.** Let $v_1$ and $w_1$ be defined by

$$v_1 := \sum_{l=1}^{k} f_{l(l+1)-k} + \sum_{l=\frac{k+1}{2}}^{k} f_{l^2-k},$$

$$w_1 := \sum_{l=1}^{k} (a + 4(k - l) b) f_{l(l+1)-k} + \sum_{l=\frac{k+1}{2}}^{k} (a + 4(k - 4l + 2) b) f_{l^2-k},$$

where $t$ is taking the transpose of $f$. Then $(X_0, X_1; Y_0, Y_1; v_1, w_1)$ is the point of $\mathcal{M}_{2k}^\tau(k^2 - k, k^2; 1)$.

6.3. $\mathbb{C}^*$-fixed point of $\mathcal{M}_{2k}^\tau(n - k, n; 1)$. Using the above description of the nilpotent point of $\mathcal{M}_{2k}^\tau(k^2 - k, k^2; 1)$, we construct a $\mathbb{C}^*$-fixed point of $\mathcal{M}_{2k}^\tau(n - k, n; 1)$.

6.3.1. Matrices $X_0$ and $X_1$. They are obtained from their $\mathcal{M}_{2k}^\tau(k^2 - k, k^2; 1)$ counterparts by forming two by two block diagonal matrices where on the top left corner we place the identity matrix of size $n - k^2$. More explicitly, let $e_1, e_2, \cdots, e_{n-k}$ and $f_1, f_2, \cdots, f_n$ be the standard basis of $\mathbb{C}^{n-k}$ and $\mathbb{C}^n$ respectively. Then

$$X_0 := [e_1 \quad e_2 \quad \cdots \quad e_{n-k} \quad 0 \quad \cdots \quad 0],$$

where the number of $0$ columns is exactly $k$, while $X_1$ is

$$X_1 := [w_1 \quad w_2 \quad \cdots \quad w_{n-k}],$$

where $w_i$’s are defined as follows. For $1 \leq s \leq n - k^2$, $w_s := f_{s+1}$, and for $1 \leq i \leq k - 1$ and $1 \leq j \leq 2i$,

$$w_{i(i-1)+j+n-k^2} := f_{i^2+j+n-k^2}.$$

6.3.2. Matrix $Y_0$. We introduce vectors $h_i \in \mathbb{C}^n (1 \leq i \leq n - k)$ so that

$$Y_0 := [h_1 \quad h_2 \quad \cdots \quad h_{n-k}].$$

First, we recall that $Y_0 := [u_1 \quad u_2 \cdots \quad u_{k(k-1)}] \in \mathcal{M}_{2k}^\tau(k^2 - k, k^2; 1)$, where $u_i$’s are defined as in (6.7) - (6.10). Let $h'_i (i = 1, \cdots, k^2 - k)$ be vectors in $\mathbb{C}^n$ defined as follows. If $1 \leq i \leq k - 1$, we set $h_i'$ to be defined by the same formulas as $u_i$. If $k \leq i \leq k^2$ then $h_i'$ is defined as $u_i$, except indices of $f$ shifted down by $n - k^2$, that is, a term $f_i$ should be replaced by $f_{i+n-k^2}$. Next, let $T_{3d}$ and $T_{4d}$ be subsets of $T_3$ and $T_4$ respectively, consisting elements for which $j = i$:

$$T_{3d} = \left\{i^2 + 1 \left| \left\lfloor \frac{k+i}{2} \right\rfloor \leq i \leq k - 1\right\}, \quad T_{4d} = \left\{(i+1)^2 - i \left| \left\lfloor \frac{k}{2} \right\rfloor \leq i \leq k - 2\right\}.\right.$$
Then define
\[
\mathbf{h}_s := \begin{cases} 
\mathbf{h}'_s, & \text{for } 1 \leq s \leq k - 1, \\
(a + (s - 1) b) \mathbf{e}_s, & \text{for } k \leq s \leq n - k^2 + k - 1, \\
h'_{s-n+k^2}, & \text{for } s \in \{n - k^2 + k - n \mid (n - k^2 + T_{3d} \cup T_{4d})\}.
\end{cases}
\]
It remains to define \( \mathbf{h}_s \) for \( s \in n - k^2 + T_{3d} \cup T_{4d} \). For \( s \in n - k^2 + T_{3d} \),
\[
\mathbf{h}_s := -2(k - 2i - 1) b f_{(i+1)2-i+n-k^2}
\]
and for \( s \in n - k^2 + T_{4d} \),
\[
\mathbf{h}_s := -2(k - i - 1) b f_{(i+1)2-i+n-k^2}
\]
so that \( \mathbf{h}_s \) is defined as \( \mathbf{u}_i \) except indices of \( \mathbf{e} \) are shifted down by \( n - k^2 \). Then we define
\[
\mathbf{g}_s := \begin{cases} 
g'_s, & \text{for } 1 \leq s \leq k - 1, \\
(s - 1) b \mathbf{e}_{s-1}, & \text{for } k \leq s \leq n - k^2 + k - 1, \\
g'_{s-n+k^2}, & \text{for } s = n - k^2 + k, \\
g'_{s-n+k^2+1} + (n - k^2) b \mathbf{e}_{n-k^2+1}, & \text{for } s = n - k^2 + k + 1, \\
g'_{s-n+k^2}, & \text{for } s \in \{n - k^2 + k + 2, n - k \} \text{ but } s \notin \{n - k^2 + S_{3d} \cup S_{4d}\},
\end{cases}
\]
where
\[
S_{3d} = \left\{ i^2 + 1 \left\lfloor \frac{k}{2} \right\rfloor + 1 \leq i \leq k - 1 \right\}, \quad S_{4d} = \left\{ (i + 1)^2 - i \left\lfloor \frac{k+1}{2} \right\rfloor \leq i \leq k - 1 \right\}.
\]
For \( s \in n - k^2 + S_{3d} \),
\[
\mathbf{g}_s := -(a + (2k - 2i - 1) b) \mathbf{e}_{i^2-i+1+n-k^2}
\]
and for \( s \in n - k^2 + S_{4d} \),
\[
\mathbf{g}_s := -(a + 2(k - i - 1) b) \mathbf{e}_{i^2+1+n-k}
\]
6.3.4. The vector $v_l$ and the covector $w_l$. We can carry out the same type of computations like in the subsection 6.2.4 to find $v_1$ and $w_1$ and prove the following proposition:

**Proposition 6.3.** Let $X_0, X_1, Y_0, Y_1$ be defined as above, and let

$$v_1 := \sum_{l=\lceil \frac{k}{2} \rceil +1}^{k} f_{(l+1)-k+n-k^2} + \sum_{l=\lceil \frac{k+1}{2} \rceil +1}^{k} f_{l-k+n-k^2},$$

$$w_1 := \sum_{l=\lceil \frac{k}{2} \rceil +1}^{k} \left( a + 4(k-l) b \right) f_{(l+1)-k+n-k^2} + \sum_{l=\lceil \frac{k+1}{2} \rceil +1}^{k} \left( a + 4(k-l) b \right) f_{l-k+n-k^2} + \sum_{l=\lceil \frac{k+1}{2} \rceil +1}^{k} \left( a + 4(k-l) b \right) f_{l-k+n-k^2}.$$ 

Then $(X_0, X_1; Y_0, Y_1; v_1, w_1)$ is a $C^*$-fixed point of $\mathcal{M}_{k_2}^\tau(n-k, n; 1)$.

6.4. $C^*$-fixed points of $\mathcal{M}_{k_2}^\tau(n-k, n; 0)$. $C^*$-fixed points of quiver varieties $\mathcal{M}_{k_2}^\tau(n-k, n; 0)$ can be defined in a similar way as has been done for $\mathcal{M}_{k_2}^\tau(n-k, n; 1)$. We leave the details to interested reader.

7. Computations of $\kappa$ for $C^*$-fixed points

Recall that for $(X, Y, v, w) \in \mathcal{M}_{k_2}^\tau$ the corresponding $\kappa$ can be computed as follows

$$\kappa = 1 - \sum_{l,q \geq 0} (wY^lX^q) y^{-l-1} x^{-q-1}.$$ 

If $(X, Y, v, w)$ is a $C^*$-fixed point then $wY^lX^q = 0$ unless $l = q$, since the $C^*$-action is given by $(X, Y, v, w) \mapsto (\lambda X, \lambda^{-1} Y, v, w)$, where $\lambda \in C^*$. Thus, we have

$$\kappa = 1 - \sum_{l=q \geq 0} (wY^lX^q) y^{-l-1} x^{-q-1}.$$ 

To compute coefficients $wY^lX^q$ we need:

**Proposition 7.1.** Let $(X, Y, v, w)$ be a point of $\mathcal{M}_{k_2}^\tau(V, U_0)$. Then for $l \in \mathbb{Z}_{\geq 0},$

$$wY^lX^q = w_1 \prod_{q=0}^{\lceil \frac{l}{2} \rceil} (Y_0X_0 + q(\tau_0 + \tau_1) \text{Id}) \prod_{q=1}^{\lceil \frac{l}{2} \rceil} (Y_0X_0 + q\tau_0 + (q-1)\tau_1) \text{Id} v_1.$$ 

**Proof.** We may assume that $l$ is odd and leave the proof of the even case to the interested reader.

Using (6.1) and (6.2) we have

$$wY^lX^q = w_1 (Y_0Y_1) \frac{i}{\tau} Y_0X_0(X_1X_0) \frac{i}{\tau} v_1$$

$$= w_1 (Y_0Y_1) \frac{i}{\tau} (X_1Y_1 + \tau_1 \text{Id}_{n_1} - v_1w_1)(X_1X_0) \frac{i}{\tau} v_1$$

$$= w_1 (Y_0Y_1) \frac{i}{\tau} \left( X_1Y_1 + \tau_1 \text{Id}_{n_1} \right)(X_1X_0) \frac{i}{\tau} v_1 - wY^lX^q \cdot wX^q$$

$$= w_1 (Y_0Y_1) \frac{i}{\tau} Y_0(Y_1X_1)(Y_1X_1 + \tau_1 \text{Id}_{n_0}) X_0(X_1X_0) \frac{i}{\tau} v_1$$

$$= w_1 (Y_0Y_1) \frac{i}{\tau} Y_0(X_0X_0 + \tau_0 \text{Id}_{n_0})(X_0X_0 + (\tau_0 + \tau_1) \text{Id}_{n_0}) X_0(X_1X_0) \frac{i}{\tau} v_1$$

Now if $l = 3$ then we are done, otherwise we repeatedly use (6.1) and (6.2) to obtain our formula. □
7.1. $\kappa$ for the $C^*$-fixed point of $\mathfrak{M}_{C_2}(k^2 - k, k; 1)$. In view of Proposition 7.1, we need to see the action of $Y_0X_0$ on $v_1$. Recall that in this case

$$v_1 = \sum_{l=\lfloor \frac{k}{2} \rfloor + 1}^{k} f_{l(l+1)} - k + \sum_{l=\lfloor \frac{k+1}{2} \rfloor + 1}^{k} f_{l^2 - k}.$$ 

Using (6.11), for $\left[ \frac{k}{2} \right] + 1 \leq l \leq k - 1$, we have

$$Y_0X_0 f_{l(l+1)-k} = u_{l(l+1)-k} = -2(k - l)b f_{l(l+1)-k}$$

(7.1)

and for $l = k$, $l(l+1) - k = k^2$, we have

$$Y_0X_0 f_{k^2} = 0.$$ 

(7.2)

For $\left[ \frac{k+1}{2} \right] + 1 \leq l \leq k$, we have

$$Y_0X_0 f_{l^2-k} = u_{l^2-k} = -(2k - 2l + 1)b f_{l^2-k}$$

(7.3)

Let $\{\tilde{f}_i\}_{i=1}^k$ be a set of vectors defined as follows:

$$\tilde{f}_{2l-\left(\left[ \frac{k+1}{2} \right] - \left[ \frac{k}{2} \right] \right)} := f_{(l + \left[ \frac{k}{2} \right])(l + \left[ \frac{k}{2} \right] + 1)-k} \quad \text{for} \quad 1 \leq l \leq \left[ \frac{k+1}{2} \right],$$

$$\tilde{f}_{2l+\left(\left[ \frac{k+1}{2} \right] - \left[ \frac{k}{2} \right] \right)} := f_{(l + \left[ \frac{k+1}{2} \right])^2-k} \quad \text{for} \quad 1 \leq l \leq \left[ \frac{k}{2} \right].$$

Then relations (7.1)-(7.3) are equivalent to

$$Y_0X_0 \tilde{f}_i = -(k - i) b \tilde{f}_i - (a + 2(k - i)b) \sum_{s=i+1}^{k} \tilde{f}_s, \quad 1 \leq i \leq k.$$ 

(7.4)

Thus $Y_0X_0$ acts as a linear transformation on the subspace of $C^k$ spanned by $\{\tilde{f}_i\}$, and we denote by $A$ the corresponding matrix of this transformation with respect to to the basis $\{\tilde{f}_i\}$ (note they are linearly independent, since $f_i$’s are). It is clear from (7.4) that $A$ is a lower triangular matrix. Now it is elementary linear algebra to show that $A$ can be diagonalized by choosing the basis $\{g_i\}_{i=1}^k$ defined by

$$g_i := \sum_{j=1}^{k} c_{ij} \tilde{f}_j,$$

where $c_{ij} = 0$ for $i > j$, $c_{ii} = 1$ and for $i < j$,

$$c_{ij} := \frac{(z + 2(k - i))_{j-i}}{(j-i)!},$$

here $z := a/b$ and $(z)_n = \prod_{r=0}^{n-1} (z - r)$ is the lower Pochhammer symbol. Explicitly, we have

$$D = CAC^{-1},$$

where $C = (c_{ij})$ and $D = \text{Diag}[-(k - 1)b, -(k - 2)b, \cdots, 0]$.

Next, we express $v_1$ and $w_1$ in the basis $\{g_i\}$. Recall that

$$v_1 = \sum_{i=1}^{k} \tilde{f}_i, \quad w_1 = \sum_{i=1}^{k} (a + 2(k - i)b) \tilde{f}_i.$$
It is easy to verify that $C^{-1} = (d_{ij})$, where $d_{ij} = 0$ for $i > j$, $d_{ii} = 1$ and for $i < j$,

\[(7.8)\]

\[d_{ij} = (-1)^{i-j} \frac{z + 2(k-i) + (z + 2k - i - j - 1)_{j-i-1}}{(j-i)!}.\]

Since $\tilde{f}_i = \sum_{j=1}^k d_{ij}g_j$ we have $v_1 = \sum_{i=1}^k \sum_{j=1}^k d_{ij}g_j$. Now we can compute $w^Y X^l v$. To make notations simpler, we compute $w^Y Z^l v$, since computations for odd ones will be essentially the same. By Proposition 7.1 we get

\[w^Y Z^l v = \sum_{q=0}^l \prod_{q=0}^{l-1} (Y_0 X_0 + q \tau_0 + \tau_1) \cdot \prod_{q=1}^l (Y_0 X_0 + (q \tau_0 + (q-1) \tau_1)) v_1\]

\[= \sum_{i=1}^k \sum_{j=1}^k d_{ij} \prod_{q=0}^{l-1} ( - (k-j) b + q \tau_0 + (q-1) \tau_1) \prod_{q=1}^l ( - (k-j) b + q \tau_0 + (q-1) \tau_1) w_1 \cdot g_j\]

\[= \sum_{m=1}^k \sum_{i=1}^m \sum_{j=1}^m (a + 2(k-m) b) d_{ij} \left[ \sum_{q=0}^{l-1} (q-k+j)b \prod_{q=1}^l ( - (a + (k-j) b) b) \sum_{m=j} c_{jm} w_1 \cdot \tilde{f}_m \right] \]

\[(7.9)\]

\[= b^{2l+1} \sum_{m=1}^k \sum_{i=1}^m \sum_{j=1}^m d_{ij} c_{jm} (z + 2(k-m)) (k-j) (z + k - j - 1),\]

where $w_1 \cdot \tilde{f}_m = a + 2(k-m) b$ is the usual dot product of vectors and we recall that $z = a/b$.

Next we need to find $d_{ij} c_{jm}$. For $i = j = m$ we have $d_{ij} c_{jm} = 1$. For $i < m$, using (7.6) and (7.8), we get

\[d_{ij} c_{jm} = (-1)^{i-j} \frac{z + 2(k-i) + (z + 2k - i - j - 1)_{j-i-1}}{(j-i)!} \frac{(z + 2k - i - j - 1)_{m-j}}{(m-j)!}\]

\[= (-1)^{i-j} \frac{z + 2(k-i)}{(j-i)! (m-j)!} (z + 2k - i - j - 1)_{m-j}.\]

Thus from (7.9) one obtains $w^Y Z^l v = b^{2l+1} F_{k,2l}(z)$, where

\[F_{k,2l}(z) := \sum_{m=2}^k \left[ \sum_{i=1}^{m-1} (z + 2(k-i)) (z + 2(k-i)) \sum_{j=1}^m (-1)^{j-i} (z + 2k - i - j - 1)_{m-j-1} \frac{m-j}{(j-i)! (m-j)!} \right] \times (k-j)_l (z + k - j - 1)_l + \sum_{m=1}^k (z + 2(k-m)) (k-m)_l (z + k - m - 1)_l.

We claim the following lemma, whose proof will be given in the Appendix:

**Lemma 7.1.** For all $k \geq 2$ and $0 \leq l \leq k-1$,

\[F_{k,2l}(z) = \binom{k+l}{2l+1} (z + k + l - 1)_{2l+1},\]

and, similarly, for all $k \geq 2$ and $0 \leq l \leq k-2$,

\[F_{k,2l+1}(z) = - \binom{k+l}{2l+2} (z + k + l)_{2l+2}.\]

For all other values of $k$ and $l$ the corresponding $F_{k,1}(z) = 0$.

It follows from Theorem 5.1 and Lemma 7.1 that:
Corollary 7.1. For \((X_0, X_1; Y_0, Y_1; v_1, w_1) \in \mathcal{M}_{2}^{k}(k^2 - k, k^2; 1)\), we have

\[
\kappa = 1 + \sum_{l=1}^{2k-1} A_l y^{-l}x^{-l},
\]

where

\[
A_{2l-1} = -\left(\frac{k + l - 1}{2l - 1}\right)^{2l-1} \prod_{r=1}^{2l-1} ((k + l - r - 1)\tau_0 + (k + l - r)\tau_1),
\]

\[
A_{2l} = \left(\frac{k + l - 1}{2l}\right)^{2l-1} \prod_{r=0}^{2l-1} ((k + l - r - 1)\tau_0 + (k + l - r)\tau_1).
\]

7.2. \(\kappa\) for \(\mathbb{C}^*\)-fixed points of \(\mathcal{M}_{2}^{n-k}(n-k, n; 1)\) and \(\mathcal{M}_{2}^{n-k}(n, n-k; 0)\). The computation of \(\kappa\) for the point \((X_0, Y_0, Y_0, Y_1; v_1, w_1)\) defined in Proposition 6.3 is essentially the same as in the previous section with some minor modifications. First, recall that \(C\) is the change of basis matrix from \(\{\tilde{f}_i\}\) to \(\{g_i\}\) for the point of \(\mathcal{M}_{2}^{k}(k^2 - k, k^2; 1)\) (see (6.30) - (6.30)). Let \(\tilde{C} = (\tilde{c}_{ij})\) be that one for the above point of \(\mathcal{M}_{2}^{n-k}(n-k, n; 1)\). Then \(\tilde{c}_{ij} = c_{ij}\) if \(i \neq 1\) and

\[
\tilde{c}_{1j} = c_{1j} + \frac{(n-k^2)(z+2k-3)_{j-2}}{(j-1)!},
\]

where \(c_{ij}\) are defined in (1.6). The vectors \(v_1\) and \(w_1\) in the basis \(\{\tilde{f}_i\}\) will be presented as (compare to (7.7))

\[
v_1 = \sum_{i=1}^{k} \tilde{f}_i, \quad w_1 = (n - k^2) b \tilde{f}_1 + \sum_{i=1}^{k} (a + 2(k - i) b) \tilde{f}_i.
\]

Next, it is easy to verify that \(\tilde{C}^{-1} = (\tilde{d}_{ij})\) is given by \(\tilde{d}_{ij} = d_{ij}\) if \(i \neq 1\) and

\[
\tilde{d}_{1j} := d_{1j} + (-1)^{j-1} \frac{n - k^2}{(j-1)!} (z+2k-j-2)_{j-2},
\]

where \(d_{ij}\) is defined in (1.8). If we let \(\tilde{F}_{k,l}(z) := b^{l+1}wY^lX^l v\), then arguing as in (7.9) we get

\[
\tilde{F}_{k,2l}(z) = F_{k,2l}(z) + (n - k^2) \left( (k-1)_l (z+k-2)_l + \sum_{m=2}^{k} (z+2(k-m)) \right) \\
\quad \times \sum_{j=1}^{m} (-1)^{j-1} \frac{(z+2k-j-2)_{m-2} (k-j)_l (z+k-j-1)_l}{(j-1)! (m-j)!}
\]

\[
= F_{k,2l}(z) + (n - k^2) \tilde{H}_{k-1,2l}(z),
\]

where \(F_{k,2l}(z)\) and \(\tilde{H}_{k-1,2l}(z)\) are computed in Lemma 7.1. Similarly, we can show

\[
\tilde{F}_{k,2l+1}(z) = F_{k,2l+1}(z) + \tilde{H}_{k-1,2l+1}(z).
\]

Thus we obtain:
Proposition 7.2. For \((X_0, X_1; Y_0, Y_1; v_1, w_1) \in \mathcal{M}_{\mathcal{L}_2}(n - k, n; 1)\) we have

\[
\kappa = 1 - \sum_{l=0}^{k-1} \left( \frac{k + l}{2l + 1} \right) \prod_{r=2}^{2l+1} ((k + l - r)\tau_0 + (k + l - r + 1)\tau_1) \times \left((k + l - 1)\tau_0 + (k + l)\tau_1 + \frac{(n - k^2)(2l + 1)}{k + l}(\tau_0 + \tau_1)\right) y^{-2l-1} x^{-2l-1} 
\]

\[
+ \sum_{l=0}^{k-2} \left( \frac{k + l}{2l + 2} \right) \prod_{r=1}^{2l+1} ((k + l - r)\tau_0 + (k + l - r + 1)\tau_1) \times \left((k + l - 1)\tau_0 + (k + l + 1)\tau_1 + \frac{(n - k^2)(2l + 2)}{k + l}(\tau_0 + \tau_1)\right) y^{-2l-2} x^{-2l-2}. 
\]

(7.12)

We now recall that (see Lemma 6.1) \(\mathcal{M}_{\mathcal{L}_2}(n, n - k; 0)\) can be identified with \(\mathcal{M}_{\mathcal{L}_2}^r(n, n - k; 1)\), where \(\tau' = (\tau_1, \tau_0)\). Since this identification is \(\text{Aut}_C(O_r)\)-equivariant, fixed points are mapped to fixed points under this identification. Hence, we have:

Corollary 7.2. For the fixed point of \(\mathcal{M}_{\mathcal{L}_2}(n, n - k; 0)\) corresponding to the fixed point of \(\mathcal{M}_{\mathcal{L}_2}^r(n - k, n; 1)\) under the bijection defined in Proposition 6.3, \(\kappa\) is given by (7.12) where \(\tau_0\) and \(\tau_1\) are reversed.

7.3. \(\kappa\) for \(\mathbb{C}^*\)-fixed points of \(\mathcal{M}_{\mathcal{L}_2}^r(n - k, n; 0)\) and \(\mathcal{M}_{\mathcal{L}_2}^r(n, n - k; 1)\).

Proposition 7.3. For \((X_0, X_1; Y_0, Y_1; v_1, w_1) \in \mathcal{M}_{\mathcal{L}_2}(n - k, n; 0)\) we have

\[
\kappa = 1 + \sum_{l=1}^{2k} B_l y^{-l} x^{-l}, 
\]

where

\[
B_{2l-1} = - \left( \frac{k + l - 1}{2l - 1} \right) \prod_{r=2}^{2l-1} ((k + l - r)\tau_0 + (k + l - r + 1)\tau_1) \times \left((k + l - 1)\tau_0 + (k + l)\tau_1 + \frac{(n - k^2 - k)(2l - 1)}{k + l}(\tau_0 + \tau_1)\right), 
\]

\[
B_{2l} = \left( \frac{k + l}{2l} \right) \prod_{r=1}^{2l-1} ((k + l - r - 1)\tau_0 + (k + l - r)\tau_1) \times \left((k + l - 1)\tau_0 + (k + l)\tau_1 + \frac{(n - k^2 - k)(2l)}{k + l}(\tau_0 + \tau_1)\right). 
\]

(7.14)

Similarly to Corollary 7.2 we can conclude:

Corollary 7.3. For the fixed point of \(\mathcal{M}_{\mathcal{L}_2}^r(n, n - k; 1)\) corresponding to the fixed point of \(\mathcal{M}_{\mathcal{L}_2}^r(n - k, n; 0)\) under the bijection defined in Proposition 6.3, \(\kappa\) is given by (7.14) where \(\tau_0\) and \(\tau_1\) are reversed.

8. \(\mathbb{C}^*\)-fixed ideals and their endomorphism rings

Recall that for \(\Gamma \cong \mathbb{Z}_2\), by transitivitly of the \(G_1\)-action, we have the following decomposition of \(\mathcal{R}_G^r\) into \(G_1\)-orbits

\[
\mathcal{R}_G^r = \bigcup_{(m, n) \in E \times \mathbb{Z}_2} \mathcal{R}_G^r(m, n; \epsilon). 
\]

Let \(P^r_{(m, n)} \in \mathcal{R}_G^r(m, n; \epsilon)\) be the images, under the bijective map \(\Omega\), of \(\mathbb{C}^*\)-fixed points, defined in Section 6. In this section we give an explicit description of \(P^r_{(m, n)}\) and its endomorphism ring.
8.1. C*-fixed ideals. Recall that, by Proposition 3.1 there is an algebra isomorphism \( \phi : O_r(\mathbb{Z}_2) \to A(v) \), where \( v(h) = (\tau_0 + \tau_1)^2(h + 1)(h + \tau_0/(\tau_0 + \tau_1)) \). Using \( \phi \) we realize ideals of \( O_r \) in \( A(v) \).

**Lemma 8.1.** Let \( f_n(h) := \phi(e y^n x^n) \), where \( \phi \) is defined in Proposition 3.1. Then

\[
f_n(h) = \begin{cases} v(h - 1) \cdots v(h - k), & n = 2k, \\
(\tau_0 + \tau_1)(h - k + 1)v(h - 1) \cdots v(h - k + 1), & n = 2k - 1.
\end{cases}
\]

**Proof.** We prove by induction. For \( n = 1 \) it follows from the definition of \( \phi \). Suppose that the identity holds for all degrees less than \( n \). Then

\[
ey^x x^n = ey(y^{-1} y)_x x^{n-1}.
\]

Repeatedly using \( xy = xy - \tau \), we obtain

\[
ey^x x^n = ey(xy^{-1} y)_x x^{n-1} = (eyx)(ey^{n-1} x^{n-1}) - \sum_{i=0}^{n-2} ey^{i+1} y^{n-2-i} x^{n-1}.
\]

Suppose \( n = 2k \), then \( \sum_{i=0}^{2k-2} ey^{i+1} y^{2k-2-i} = (k(\tau_0 + \tau_1) - \tau_0)ey^{2k-1} \). Hence

\[
(eyx)(ey^{2k-1} x^{2k-1}) - (k(\tau_0 + \tau_1) - \tau_0)ey^{2k-1} x^{2k-1} = (eyx - k(\tau_0 + \tau_1))ey^{2k-1} x^{2k-1}.
\]

Since \( \phi(eyx) = (\tau_0 + \tau_1)h \), the image of the later expression is

\[
(\tau_0 + \tau_1)(h + \frac{\tau_0}{\tau_0 + \tau_1} - \tau) \phi(ey^{2k-1} x^{2k-1}).
\]

By induction assumption on \( ey^{2k-1} x^{2k-1} \), we obtain

\[
(\tau_0 + \tau_1)^2(h + \frac{\tau_0}{\tau_0 + \tau_1} - \tau)(h - k + 1)v(h - 1) \cdots v(h - k + 1) = v(h - k)v(h - 1) \cdots v(h - k + 1).
\]

Analogously one can prove the identity for the case \( n = 2k - 1 \). \( \square \)

Let

\[(8.1) \quad l(h) := \frac{v(h)}{h + 1} = h + \frac{\tau_0}{\tau_0 + \tau_1}, \]

and define

\[
s_{n,2k-\epsilon}(h) := \prod_{i=1}^{2k-1-\epsilon} l(h - i) \cdot l(h - n + (k - \epsilon)(k - 1)),
\]

\[
s'_{n,2k-\epsilon}(h) := \prod_{i=1}^{2k-1-\epsilon} (h - i) \cdot l(h - n + (k - \epsilon)(k - 1)).
\]

**Proposition 8.1.** By identification \( \phi : O_r(\mathbb{Z}_2) \to A(v) \), we have:

(a) For \( (n-k, n, \epsilon) \in L_\epsilon \times \mathbb{Z}_2 \),

\[
P_{(n-k, n)}^{(c)} \cong a^{n-(k-\epsilon)(k-1)} A(v) + s_{n,2k-\epsilon}(h) A(v).
\]

(b) For \( (n, n-k, \epsilon) \in L_\epsilon \times \mathbb{Z}_2 \),

\[
P_{(n,n-k)}^{(c)} \cong a^{n-(k+\epsilon-1)(k-1)+1} A(v) + s'_{n,2k+\epsilon-1}(h) A(v).
\]

**Proof.** We only prove part (a), since the proof of (b) is identical. Without loss of generality, we may assume \( n = k^2 + k \) for \( \epsilon = 0 \), and \( n = k^2 \) for \( \epsilon = 1 \). For simplicity, we also assume that \( k \) is even. Then by Theorem 3.1 we have

\[
P_{(k^2, k^2+k)}^{(0)} = ey^{2k^2+k} O_r + e_{k_0} x^{2k^2+k} O_r,
\]

\[
P_{(k^2-k, k^2)}^{(1)} = e_1 y^{2k^2-k} O_r + e_{1k_1} x^{2k^2-k} O_r.
\]
since the corresponding matrices $X$ and $Y$ are nilpotent. Here $\kappa_0$ is given by (7.10) for $n = k^2 + k$ and $\kappa_1$ is given by (7.10). If we multiply $P^{(0)}_{(k^2,k^2+k)}$ by $y^{2k}$ and $P^{(1)}_{(k^2-k^2,k^2)}$ by $y^{2k-1}$ then $P^{(0)}_{(k^2,k^2+k)} \cong (e^y y^{(k^2+3)}, e_2^{k-1} \kappa_{01}^2 x^{2k-1+k}$ and $P^{(1)}_{(k^2-k^2,k^2)} \cong (e^y y^{(k^2+1)-(2k-1)}, e_2^{k-1} \kappa_{11} x^{2k-1-k})$, where $(-,-)$ means generated as right $O_\ast$-modules.

First we show

$$\phi(e^y \kappa_{00} x^{2k}) = s_{k^2+k,2k}(h)$$ and $\phi(e^y \kappa_{01} x^{2k-1}) = s_{k^2,2k-1}(h)$.

We prove by induction. Both identities obviously hold for $k = 1$. Suppose that they hold for all degrees less than $2k$. Consider the finite difference operator $\Delta : \mathbb{C}[h] \to \mathbb{C}[h]$ given by $\Delta(f) = f(h+1) - f(h)$. Then it is easy to see that

$$\Delta(s_i(h)) = l(\tau_0 + \tau_1)s_{i-1}(h).$$

By (7.13) for $n = k^2 + k$, we have

$$e^y \kappa_{00} x^{2k} = e^y x^{2k} + B_1 e^y x^{2k-1} + B_2 e^y x^{2k-2} + \cdots + B_{2k}.$$ Applying $\Delta$ to the image of $\phi$, we get

$$\Delta(\phi(e^y \kappa_{00} x^{2k})) = \Delta(f_{2k}(h)) + B_1 \Delta(f_{2k-1}(h)) + B_2 \Delta(f_{2k-2}(h)) + \cdots + B_{2k-1} \Delta(f_1(h)).$$

Then, using Lemma 8.1 we obtain

$$\Delta(f_{2i}(h)) = (\tau_0 + \tau_1)^2 v(h-1) \cdots v(h-i+1) \left(2ih - i(i + 1) + \frac{i\tau_0}{\tau_0 + \tau_1}\right)$$

$$= 2i(\tau_0 + \tau_1)f_{2i-1} + (\tau_0 + \tau_1)^2 \left(i(i - 1) - \frac{(i - 1)\tau_0}{\tau_0 + \tau_1}\right)f_{2i-2},$$

and similarly

$$\Delta(f_{2i-1}(h)) = (2i - 1)(\tau_0 + \tau_1)f_{2i-2} + (\tau_0 + \tau_1)^2 \left(i(i - 1) - \frac{(i - 1)\tau_0}{\tau_0 + \tau_1}\right)f_{2i-3}.$$ It is easy to see that $f_{2i-2}$ appears in $\Delta(e^y \kappa_{00} x^{2k})$ only in $\Delta(f_{2i}(h))$ and $\Delta(f_{2i-1}(h))$. Therefore, since $\{f_i\}$ are linearly independent, the coefficient of $f_{2i-2}$ can be computed as

$$B_{2k-2i}(\tau_0 + \tau_1)^2 \left(i(i - 1) + \frac{i\tau_0}{\tau_0 + \tau_1}\right) + B_{2k-2i+1}(2i - 1)(\tau_0 + \tau_1).$$ Simplifying the later expression, we get that it is equal to $2k(\tau_0 + \tau_1)A_{2k-2i+1}$, where $A_i$ is defined as in Corollary 7.1. Similarly, we can show that the coefficient of $f_{2i-1}$ is $2k(\tau_0 + \tau_1)A_{2k-2i}$. Therefore we have

$$\Delta(\phi(e^y \kappa_{00} x^{2k})) = 2k(\tau_0 + \tau_1) \left(f_{2k-1} + A_1 f_{2k-2} + \cdots A_{2k-2} f_1 + A_{2k-1}\right),$$

and hence $\Delta(\phi(e^y \kappa_{00} x^{2k})) = \phi(e^y \kappa_{01} x^{2k-1})$. By (7.13) and by the induction assumption $e^y \kappa_{01} x^{2k-1} = 2k(\tau_0 + \tau_1)P_{2k-1}$, we have

$$\Delta(\phi(e^y \kappa_{00} x^{2k})) = \Delta(s_{2k}(h)).$$ Now comparing constant terms, we get $s_{2k}(0) = B_{2k}$ and hence $\phi(e^y \kappa_{00} x^{2k}) = s_{k^2+k,2k}(h)$. Using similar arguments, we also obtain $\phi(e^y \kappa_{01} x^{2k-1}) = s_{k^2,2k-1}(h)$.

Now, applying $\phi$ to the generators of $P^{(0)}_{(k^2,k^2+k)}$ and $P^{(1)}_{(k^2-k^2,k^2)}$, we obtain

$$P^{(0)}_{(k^2,k^2+k)} \cong a \frac{(k^2+3)}{2} A(v) + s_{k^2+k,2k}(h)b \frac{k^{2k-1}_2}{2} A(v)$$

and

$$P^{(1)}_{(k^2-k^2,k^2)} \cong a \frac{(k^2+3)(2k-1)}{2} A(v) + s_{k^2,2k-1}(h)b \frac{k^{2k-3}_2}{2} A(v).$$
Since
\[ a^{\frac{k(2k-3)}{2}} b^{\frac{k(2k-1)}{2}} = a^{2k} \cdot \prod_{i=1}^{\frac{k(2k-1)}{2}} v(h - i) \]
and
\[ s_{k^2+k,2k}(h) b^{\frac{k(2k-3)}{2}} a^{\frac{k(2k-3)}{2}} = a^{2k} s_{k^2+k,2k}(h + 2k) \cdot \prod_{i=0}^{\frac{k(2k-1)-1}{2}} v(h + i + 2k), \]
the greatest common divisor (GCD for short) of the left hand sides is \(a^{2k}\) and therefore \(a^{2k} \in P^{(0)}_{(k^2+k,k)}\). Then \(s_{k^2+k,2k}(h)\) divides \(a^{2k}b^{2k}\) and hence \(s_{k^2+k,2k}(h) \in P^{(0)}_{(k^2+k,k)}\). Thus
\[ P^{(0)}_{(k^2+k,k)} \cong a^{2k} A(v) + s_{k^2+k,2k}(h) A(v). \]
Similarly, we can show
\[ P^{(1)}_{(k^2+k,k)} \cong a^{2k-1} A(v) + s_{k^2+k-1}(h) A(v). \]

8.2. Endomorphism rings of \(P^{(c)}_{(n,n)}\). First we introduce a grading on the algebra \(A(v)\) as follows. For \(t \in \mathbb{Z}\), put \(D(t) = \{ c \in A(v) | h \cdot c - c \cdot h = tc \}\). Then, by \(2.6\), \(A(v) = \bigoplus_{k=-\infty}^{\infty} D(t)\), where
\[ D(t) = \begin{cases} a^t \mathbb{C}[h], & t \geq 0, \\ b^{-t} \mathbb{C}[h], & t < 0. \end{cases} \]

\(A(v)\) is a graded ring, since \(D(t_1)D(t_2) \subseteq D(t_1 + t_2)\).

Set
\[ (8.3) \quad D^{(c)}_{n,n} := \text{End}_A(v) (P^{(c)}_{(n,n)}). \]

Then we have:

**Proposition 8.2.** \(a\) For \((n-k,n) \in L\), \(D^{(c)}_{n-k,n} = \bigoplus_{-\infty}^{\infty} E(t)\), where \(E(t)\) is
\[ \begin{cases} a^t \mathbb{C}[h], & \text{for } t \geq n - k(k - \epsilon + 1) + 1, \\ a^t \cdot l(h - n + (k - \epsilon)(k - 1) + t) \mathbb{C}[h], & \text{for } 0 < t \leq n - k(k - \epsilon + 1), \\ \mathbb{C}[h], & \text{for } t = 0, \\ b^{-t} \cdot \frac{s_{n,2k}(h+t)}{s_{n,2k}(h)} l(h - n + (k - \epsilon)(k - 1)) \mathbb{C}[h], & \text{for } -n + k(k - \epsilon + 1) \leq t < 0, \\ b^{-t} \cdot \frac{s_{n,2k}(h+t)}{s_{n,2k}(h)} \mathbb{C}[h], & \text{for } t < -n + k(k - \epsilon + 1). \end{cases} \]

\(b\) For \((n-k,n) \in L\), \(D^{(c)}_{n,n-k} = \bigoplus_{-\infty}^{\infty} F(t)\), where \(F(t)\) is
\[ \begin{cases} a^t \mathbb{C}[h], & \text{for } t \geq n - k(k + \epsilon) + 2, \\ a^t \cdot l(h - n + (k + \epsilon - 1)(k - 1) + t) \mathbb{C}[h], & \text{for } 0 < t \leq n - k(k + \epsilon) + 1, \\ \mathbb{C}[h], & \text{for } t = 0, \\ b^{-t} \cdot \frac{s_{n,2k}(h+t)}{s_{n,2k}(h)} l(h - n + (k - \epsilon)(k - 1)) \mathbb{C}[h], & \text{for } -n + k(k + \epsilon) - 1 \leq t < 0, \\ b^{-t} \cdot \frac{s_{n,2k}(h+t)}{s_{n,2k}(h)} \mathbb{C}[h], & \text{for } t < -n + k(k + \epsilon) - 1. \end{cases} \]

**Proof.** We only prove part (a), since the proof of (b) is similar. By Proposition 8.1 \(P^{(0)}_{(n-k,n)}\) is generated by homogeneous elements and hence can be presented as \(P^{(0)}_{(n-k,n)} = \bigoplus_{-\infty}^{\infty} (P^{(0)}_{(n-k,n)} \cap D(t)). \)
Then $P_{(n-k,n)}^{(0)} \cap D(t)$ is

\[
\begin{align*}
&\begin{cases}
a^t \mathbb{C}[h], & t \geq n-k^2+k, \\
a^t \cdot l(h-n+k^2-k+t) \mathbb{C}[h], & 2k-1 \leq t < n-k^2+k, \\
a^t \cdot \prod_{i=1}^{2k-t-1} l(h-i) l(h-n+k^2-k+t) \mathbb{C}[h], & 1 \leq t \leq 2k-2, \\
b^{-t} \cdot s_{n,2k}(h+t) \mathbb{C}[h], & t \leq 0.
\end{cases}
\end{align*}
\]

Now let $c \in D_{n-k,n}^{(0)}$, then only if $c \cdot a^{-n-k^2+k}$ and $c \cdot s_{n,2k}(h) \in P_{(n-k,n)}^{(0)}$. Thus $D_{n-k,n}^{(0)} = P_{n-k,n}^{(0)} a^{-n-k^2+k} \cap P_{n-k,n}^{(0)} s_{n,2k}^{-1}(h)$, and since $a^{-n-k^2+k}$ and $s_{n,2k}(h)$ are homogeneous elements, $D_{m,n}^{(c)} = \oplus_{-\infty}^{\infty} E(t)$, where

\[E(t) = (P_{n-k,n}^{(0)} \cap D(t+n-k^2+k)) \cdot a^{-n+k^2-k} \cap (P_{n-k,n}^{(0)} \cap D(t)) \cdot s_{n,2k}^{-1}(h)\]

It follows from (2.6) that $a^{-1} = bv(h-1)^{-1}$ and $a^{-n+k^2-k} = b^{n-k^2+k}v(h-1)^{-1} \cdots v(h-n+k^2-k)^{-1}$. Then $P_{n-k,n}^{(0)} \cdot a^{-n+k^2-k}$ can be computed as

\[
\begin{align*}
&\begin{cases}
a^t \mathbb{C}[h], & t \geq 0, \\
b^{-t} \prod_{i=1}^{t} v(h-i)^{-1} \cdot l(h-n+k^2-k+t) \mathbb{C}[h], & -n+k^2+k-1 \leq t < 0, \\
b^{-t} \prod_{i=1}^{t} v(h-i)^{-1} \cdot l(h-n+k^2-k+t) \times \prod_{i=n-k^2+k+1}^{2k-t-1} l(h-i) \mathbb{C}[h], & -n+k^2-k+1 \leq t \leq -n+k^2+k-2, \\
b^{-t} \prod_{i=1}^{n-k^2+k} v(h-i)^{-1} s_{n,2k}(h+t) \mathbb{C}[h], & t \leq -n+k^2-k,
\end{cases}
\end{align*}
\]

while $P_{n-k,n}^{(0)} \cdot s_{n,2k}(h)^{-1}$ can be expressed as

\[
\begin{align*}
&\begin{cases}
a^t \cdot s_{n,2k}(h)^{-1} \mathbb{C}[h], & t \geq n-k^2+k, \\
a^t \cdot s_{n,2k}(h)^{-1} l(h-n+k^2-k+t) \mathbb{C}[h], & 2k-1 \leq t < n-k^2+k, \\
a^t \cdot s_{n,2k}(h)^{-1} \prod_{i=1}^{2k-t-1} l(h-i) l(h-n+k^2-k+t) \mathbb{C}[h], & 1 \leq t \leq 2k-2, \\
b^{-t} \cdot s_{n,2k}(h)^{-1} s_{n,2k}(h+t) \mathbb{C}[h], & t \leq 0.
\end{cases}
\end{align*}
\]

Now we can compute $E(t)$:

(i) Suppose $t \geq n-k^2-k+1$. For $t \geq n-k^2+k$, it is clear that $E(t) = a^t \mathbb{C}[h]$; for $n-k^2-k+1 \leq t < n-k^2+k$, we have

\[h - (2k-1) \leq h-n+k^2-k+t \leq h-1,
\]

which means $l(h-n+k^2-k+t)$ divides $s_{n,2k}(h)$, and hence $E(t) = a^t \mathbb{C}[h]$.

(ii) Suppose $0 < t \leq n-k^2-k$. In this case

\[h-n+k^2-k < h-n+k^2-k+t \leq h-2k,
\]

which implies $l(h-n+k^2-k+t)$ is relatively prime to $s_{n,2k}(h)$. Hence, $E(t) = a^t \cdot l(h-n+k^2-k+t) \mathbb{C}[h]$.

(iii) Suppose $t = 0$. It can be easily seen that $E(0) = \mathbb{C}[h]$.

(iv) Suppose $-n+k^2+k \leq t \leq -1$. Then

\[
E(t) = b^{-t} \prod_{i=1}^{t} v(h-i)^{-1} \cdot l(h-n+k^2-k+t) \mathbb{C}[h] \cap b^{-t} \cdot s_{n,2k}(h)^{-1} s_{n,2k}(h+t) \mathbb{C}[h].
\]
Recall that we have

\begin{equation}
\frac{s_{n,2k}(h + t)}{s_{n,2k}(h)} \cdot l(h - n + k^2 - k) = \frac{\prod_{i=1}^{2k-1-t} l(h - i)}{\prod_{i=1}^{2k-1} l(h - i)} \cdot l(h - n + k^2 - k + t)
\end{equation}

and

\begin{equation}
\prod_{i=0}^{-t} v(h - i)^{-1} = \prod_{i=0}^{-t+1} \frac{1}{h - i} \prod_{i=1}^{-t} \frac{1}{l(h - i)}.
\end{equation}

Now we consider two sub-cases: first \( t \leq -2k + 1 \) and then \( t > -2k + 1 \). In the first case, GCD of denominators of (8.5) and (8.6) is the denominator of (8.5). In the second case, we can simplify the multiplier of LHS of (8.5) to

\[ \frac{\prod_{i=1}^{2k-1-t} l(h - i)}{\prod_{i=1}^{2k-1} l(h - i)} , \]

and again the GCD of denominators is the one of (8.5). Thus, we obtain

\[ E(t) = b^{-t} \cdot \frac{s_{n,2k}(h + t)}{s_{n,2k}(h)} \cdot l(h - n + k^2 - k) \mathbb{C}[h]. \]

(v) Suppose \( t < -n + k^2 + k \). Divide this case into three sub-cases: \( t = -n + k^2 + k - 1, -n + k^2 - k + 1 \leq t \leq -n + k^2 + k - 2 \) and \( t \leq -n + k^2 - k \).

(a) For \( t = -n + k^2 + k - 1 \), \( E(t) \) has the same presentation as (8.4) and

\begin{equation}
\frac{s_{n,2k}(h + t)}{s_{n,2k}(h)} = \frac{\prod_{i=1}^{2k-1-t} l(h - i)}{\prod_{i=1}^{2k-1} l(h - i)} \cdot \frac{l(h - n + k^2 - k + t)}{l(h - n + k^2 - k)} = \frac{\prod_{i=1}^{2k-1-t+2} l(h - i)}{\prod_{i=1}^{2k-1} l(h - i)} \cdot \frac{l(h - n + k^2 - k + t)}{l(h - n + k^2 - k)} ,
\end{equation}

where the last equality follows from \( 2k - t - 1 = n - k^2 + k \). So GCD of denominators of (8.6) and (8.7) is the one of (8.7). Hence, we get

\begin{equation}
E(t) = b^{-t} \cdot \frac{s_{n,2k}(h + t)}{s_{n,2k}(h)} \mathbb{C}[h].
\end{equation}

(b) Next, for \( -n + k^2 - k + 1 \leq t \leq -n + k^2 + k - 2 \),

\[ E(t) = b^{-t} \prod_{i=1}^{2k-1-t} v(h - i)^{-1} \cdot l(h - n + k^2 - k + t) \prod_{i=n-k^2+k+1}^{2k-1} l(h - i) \mathbb{C}[h] \]

\[ \cap b^{-t} : s_{n,2k}(h)^{-1}s_{n,2k}(h + t) \mathbb{C}[h] , \]

and (8.5) for this case follows using the same argument as the previous case, once we notice that \( -t + 1 \leq n - k^2 + k \).

(c) Finally, the proof of (8.8) for \( t \leq -n + k^2 - k \) is exactly the same as the proof of (iv).

This completes the proof.

One can easily verify:

**Corollary 8.1.** For \( k \geq 1 \),

\[ D_{k^2,k^2+k}^{(0)} \cong A(w_1), \quad D_{k^2-k,k^2}^{(1)} \cong A(w_2) , \]

where

\[ w_1(h) := h \cdot l(h - 2k - 1) , \quad w_2(h) := h \cdot l(h - 2k) . \]
9. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. In Proposition 5.2, we gave a description of the endomorphism rings $D^{(c)}_{m,n}$ of ideals of $O_{r} \cong A(v)$. Recall that Theorem 1.2 claims that $D^{(c)}_{m,n} \not\cong A(v)$ for any $(m, n; \epsilon) \in L_{c} \times \mathbb{Z}_{2}$ unless $(m, n; \epsilon) = (0, 0; 0)$. Suppose there is an isomorphism $f : A(v) \to D^{(c)}_{n-k,n}$. By Proposition 5.2, $h$ is strictly semisimple both in $A(v)$ and in $D^{(c)}_{n-k,n}$. It is clear that under isomorphism a strictly semisimple element is mapped to a semisimple one. Hence, by Proposition 3.26, there is an automorphism $\Phi$ of $A(v)$ such that $\Phi(f^{-1}(h)) = \gamma h + \alpha$ for some $\gamma, \alpha \in \mathbb{C}$. Therefore $\Phi \circ f$ is a graded isomorphism. In particular, the image of a $\mathbb{C}[h]$ must generate all positive degree elements in $D^{(c)}_{n-k,n}$. But this is impossible, because a $\mathbb{C}[h]$ is mapped to a $l(h + k^2 - 2k) \cdot \mathbb{C}[h]$ and hence all elements in the positive degrees in $A(v)$ must be multiple of $l(h + k^2 - 2k)$. Similarly, we can show $D^{(c)}_{m,n} \not\cong A(v)$ for all other cases. This finishes the proof of Theorem 1.2. \hfill \Box

Before proceeding to the proof of Theorem 1.3, let us recall some basic facts about Picard groups. Let $A$ be a $\mathbb{C}$-algebra. Then the Picard group of $A$, denoted by $\text{Pic}_{\mathbb{C}}(A)$, is the multiplicative group consisting of all bimodule isomorphism classes $(X)$ of invertible bimodules $X \overline{\otimes} A$. Multiplication is defined by the formula $(X) \cdot (X') := (X \otimes_{A} X')$ and the inverse of $X$ is given by $X^{-1} := \text{Hom}_{A}(X, A)$. By definition, $\text{Pic}_{\mathbb{C}}(A)$ is the group of autoequivalences of $\text{Mod}(A)$, the category of finitely generated right $A$-modules. In particular, $\text{Pic}_{\mathbb{C}}(A)$ acts on $P(A)$, the subcategory of finitely generated projective modules over $A$:

$$P(A) \times \text{Pic}_{\mathbb{C}}(A) \to P(A), \quad P \times (X) \mapsto P \otimes_{A} X.$$ 

The orbits consist of projective modules having isomorphic endomorphism rings. Indeed, by the dual basis lemma $P_{A}$ is a projective module if and only if $P \otimes_{A} P^{\ast} \cong \text{End}_{A}(P)$. Then, for $(X) \in \text{Pic}_{\mathbb{C}}(A)$, one has

$$\text{End}_{A}(P \otimes_{A} X) \cong (P \otimes_{A} X) \otimes_{A} (P \otimes_{A} X)^{\ast} \cong P \otimes_{A} (X \otimes_{A} X^{\ast}) \otimes_{A} P^{\ast} \cong P \otimes_{A} A \otimes_{A} P^{\ast} \cong P \otimes_{A} P^{\ast} \cong \text{End}_{A}(P).$$

Assuming that $A$ has no units except nonzero scalars in $\mathbb{C}$ then

\begin{equation}
\omega_{A} : \text{Aut}_{\mathbb{C}}(A) \to \text{Pic}_{\mathbb{C}}(A), \quad \sigma \mapsto 1(A)_{\sigma},
\end{equation}

where $1(A)_{\sigma}$ means that the right action twisted by $\sigma$, is a group monomorphism. The following is a well-known fact (see e.g. [K] Theorem 37.16):

**Lemma 9.1.** Let $(X), (Y) \in \text{Pic}_{\mathbb{C}}(A)$. Then $X_{A} \cong Y_{A}$ if and only if $(Y) \in (X) \cdot \text{Im}(\omega_{A})$, that is, if and only if $Y \cong 1(X_{A})_{\sigma}$ (as bimodules) for some $\sigma \in \text{Aut}_{\mathbb{C}}(A)$.

Using this result Stafford showed (see [L] Corollary E)] that the map $\omega_{A_{1}}$ is a bijection. Theorem 1.3 is about the bijection of $\omega_{O_{r}}$ and our proof is similar to that of Stafford’s.

**Proof of Theorem 1.3.** As we pointed out above, it is enough to show that $\omega_{O_{r}}$ is surjective. By Theorem 1.1 for $(m, n) \in L_{r}$ the ideals $P^{(c)}_{(m,n)}$ are representatives of $\text{Aut}_{\mathbb{C}}(A_{1})$-orbits on $P(O_{r})$. By Theorem 1.1, the corresponding endomorphism rings $D^{(c)}_{(m,n)} \not\cong O_{r}$, unless $(m, n; \epsilon) = (0, 0; 0)$ in which case $P^{(c)}_{(m,n)} \cong O_{r}$. If $(X) \in \text{Pic}_{\mathbb{C}}(O_{r})$, then as a right $O_{r}$-module $X$ is an ideal of $O_{r}$, and $X \otimes_{O_{r}} X^{\ast} \cong O_{r}$. But $X \otimes_{O_{r}} X^{\ast} \cong \text{End}_{O_{r}}(X)$ and hence, by Lemma 9.1, $X \in \text{Im}(\omega_{O_{r}})$, which shows that $\omega_{O_{r}}$ is surjective. \hfill \Box

We would like to finish this section with the following discussion. In the above mentioned statement [L] Corollary E], it was also shown that $\text{Aut}_{\mathbb{C}}(A_{1})$ is an invariant distinguishing $A_{1}$ from non-isomorphic algebras Morita equivalent to it. More explicitly, if $D$ is Morita equivalent to $A_{1}$ then $\omega_{D}$ is an isomorphism if and only if $A_{1} \cong D$. However, this is not the case for $O_{r}$. 

Theorem 9.1. For $k \in \mathbb{Z}_{\geq 0}$, let $D$ be either $D_{k^2,k^2+k}^{(0)}$ or $D_{k^2-k,k^2}^{(1)}$. Then $\omega_D$ is an isomorphism.

Proof. Let $D$ be a domain Morita equivalent to $O_r$. Then $D \cong \text{End}_{O_r}(P)$ for some right ideal $P \in \mathcal{R}_{\mathbb{Z}_2}$. Denote by $G_P$ the stabilizer subgroup of $P$ under the action of $\text{Aut}_C(O_r)$ on $\mathcal{R}_{\mathbb{Z}_2}$. Then we have the following group homomorphisms

$$\text{Aut}_C(D) \hookrightarrow \text{Aut}_C(O_r) \hookrightarrow G_P.$$ 

Here the right inclusion is the natural embedding of $G_P$, and the left inclusion is given by

$$\text{Aut}_C(D) \hookrightarrow \text{Pic}_C(D) \cong \text{Pic}_C(O_r) \cong \text{Aut}_C(O_r),$$

where the first map is $\omega_D$, the second one is induced from Morita equivalence and the third one is the inverse of $\omega_D$. Then it is easy to show (see e.g. [BEE2 Theorem 1]) that the images of $\text{Aut}_C(D)$ and $G_P$ in $\text{Aut}_C(O_r)$ coincide. Now for $k \in \mathbb{Z}_{\geq 0}$, the sets $\mathcal{R}_{\mathbb{Z}_2}^+(k^2,k^2+k;0)$ and $\mathcal{R}_{\mathbb{Z}_2}^+(k^2-k,k^2;1)$ are singletons consisting of the ideals $P_{k^2,k^2+k}^{(0)}$ and $P_{k^2-k,k^2}^{(1)}$ respectively. Hence the stabilizers coincide with $\text{Aut}_C(O_r)$ itself. Therefore both inclusions in (9.2) are bijections, which implies that $\omega_D$ is an isomorphism.

Appendix A. Proof of Lemma 7.3

First we prove the following statement:

**Proposition A.1.** For $k \geq 2$ and $0 \leq l \leq k$,

$$H_{k,2l}(z) := \sum_{j=0}^{k} (-1)^j \frac{(k-j)! (z+k-j-l)_{k+l}}{(k-j)! j!} = \frac{(k+l)!}{(2l)! (k-l)!} (z)_{2l}.$$ 

Proof. Recall that the finite difference operator $\Delta[f](z) := f(z+1) - f(z)$ acts on Pochhammer symbols as follows: $\Delta(x+a)_k = (x+a)_{k-1}$. Hence it suffices to show

$$\Delta^i H_{k,2l}(0) = 0 \text{ for } 0 \leq i < 2l, \quad \Delta^2 H_{k,2l}(z) = \frac{(k+l)!}{(k-l)!}, \quad \Delta^i H_{k,2l}(z) = 0 \text{ for } i > 2l.$$

Indeed, one can easily check that for any $i \geq 0$,

$$\Delta^i H_{k,2l}(z) = (k+l)_i \sum_{j=0}^{k} (-1)^j \frac{(k-j)! (z+k-j-l)_{k+l-i}}{(k-j)! j!}.$$ 

Then $\Delta^i H_{k,2l}(0) = 0$ for $0 \leq i < 2l$, since $(k-j-l)_{k+l-i} = 0$. Next, by taking derivatives from the expansion of $(z-1)^k$:

$$\frac{d^i}{dz^i} (z-1)^k = k! \sum_{j=0}^{k} (-1)^j \frac{(k-j)!}{(k-j)! j!} z^{k-j-i},$$

and then evaluating at $z = 1$, we obtain

$$\sum_{j=0}^{k} (-1)^j \frac{i^j}{(k-j)! j!} = 0 \text{ for } 0 \leq i < k \text{ and } \sum_{j=0}^{k} (-1)^j \frac{j^k}{(k-j)! j!} = 1.$$ 

From these identities one can easily derive $\Delta^2 H_{k,2l}(z) = \frac{(k+l)!}{(k-l)!}$ and $\Delta^i H_{k,2l}(z) = 0$ for $i > 2l$. □

**Proof of Lemma 7.3.** We proceed by induction on $k$. For $k = 2$, we have

$$F_{2,2l}(z) = (z+2)(1)(z)(z-1)(z-1) + (z+2)(1)(z)(z-1)(z+2)(1)(z) = 4(z+1)(z+2).$$

Then $F_{2,0}(z) = 2(z+1)$ and $F_{2,2l}(z) = 2(z+2)$, which is exactly our statement for $k = 1$. Assuming it is true for $k$, we show it for $k + 1$. To this end one notes

$$F_{k+1,2l}(z) - F_{k,2l}(z) = (z+2k)H_{k,2l}(z),$$
where \( \hat{H}_{k,2l}(z) \) is

\[
\sum_{m=1}^{k} \left( z + 2(k - m) \right) \sum_{j=0}^{m} (-1)^j \frac{(z + 2k - j - 1)m}{(m - j)! \cdot j!} (k - j)!(z + k - j - 1)_l + (k)_l(z + k - 1)_l.
\]

Since \( z + 2(k - m) = (z + 2k - j - m) - (m - j) \), we can express \( \hat{H}_{k,2l}(z) \) as follows

\[
\hat{H}_{k,2l}(z) = \sum_{m=1}^{k} \sum_{j=0}^{m} (-1)^j \frac{(z + 2k - j - 1)m}{(m - j)! \cdot j!} (k - j)!(z + k - j - 1)_l
\]

Now we see that \( \hat{H}_{k,2l}(z) = H_{k,2l}(z + k + l - 1) \) and hence by Proposition A.1,

\[
F_{k+1,2l}(z) = F_{k,2l}(z) + (z + 2k) \frac{(k + l)!}{(2l)! (k - l)!} (z + k + l - 1)_{2l},
\]

and we are done once we use the induction assumption for \( F_{k,2l}(z) \). \(\square\)

References

[A] J. Alev, *Un automorphisme non modéré de \( \mathfrak{u}(g_3) \)*, Comm. Algebra 14 (8) (1986), 1365–1378.

[AFKKZ] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch and M. Zaidenberg, Flexible varieties and automorphism groups, Duke Math. J. 162 (4) (2013), 767–823.

[BGK] V. Baranovsky, V. Ginzburg and A. Kuznetsov, Quiver varieties and noncommutative geometry, Math. Z. 268 (2011), 12–21.

[BEE1] Yu. Berest, A. Eshmatov and F. Eshmatov, *On subgroups of the Dizmier group and Calogero-Moser spaces*, Electronic Research Announcements, 18 (2011), 12–21.

[BEE2] Yu. Berest, A. Eshmatov and F. Eshmatov, *Dizmier groups and Borel subgroups*, Adv. Math., 286 (2016), 387–429.

[BW1] Y. Berest and G. Wilson, *Automorphisms and ideals of the Weyl algebra*, Math. Ann. 318 (2000), no. 1, 127–147.

[BW2] Y. Berest and G. Wilson, *Ideal classes of the Weyl algebra and noncommutative projective geometry*. With an appendix by Michel Van den Bergh. Int. Math. Res. Not. 2002, no. 26, 1347–1396.

[BL] R. Bocklandt and L. Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z. 240 (2002), 141–167.

[CS] O. Chalykh and A. Silantyev, *KP hierarchy for the cyclic quiver*, arXiv:math/1512.08551 [math.QA].

[CB] W. Crawley-Boevey, *Geometry of the Moment Map for Representations of Quivers*, Compos. Math., 126 (2001), 257–293.

[CBH] W. Crawley-Boevey and M. Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605–636.
AUTOMORPHISMS AND IDEALS OF NONCOMMUTATIVE DEFORMATIONS OF $C^2/\mathbb{Z}_2$

[D] J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France 96 (1968), 209–242.

[E] F. Eshmatov, *DG-models of projective modules and Nakajima quiver varieties*, Homology, Homotopy and Appl. 9 (2007), 177–208.

[F] O. Fleury, *Sur les sous-groupes fins de $\text{Aut}_C\mathfrak{U}(\mathfrak{sl}_2)$ et $\text{Aut}_C\mathfrak{U}(\mathfrak{n})$*, J. Algebra 200, 404–427, 1998.

[H1] T. J. Hodges, *Morita equivalence of primitive factors of $U(\mathfrak{sl}(2))$*, Contemp. Math., 139 (1992), 175–179.

[H2] T. J. Hodges, *Noncommutative deformations of type-A Kleinian singularities*, J. Algebra 161 (1993), 271–290.

[ML1] L. Makar-Limanov, *On automorphisms of Weyl algebra*, Bull. Soc. Math. France 112 (1984), no 3, 359–363.

[ML2] L. Makar-Limanov, *On groups of automorphisms of a class of surfaces*, Israel Journal of Mathematics 69 (2) (1990), 250–256.

[Mck] J. McKay, *Graph, singularities and finite groups*, In Proceedings of Symposia in Pure Mathematics, 37 (1980), 183–186.

[N] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Rhode Island, 1999.

[Re] I. Reiner, *Maximal orders*. Corrected reprint of the 1975 original. With a foreword by M. J. Taylor. London Mathematical Society Monographs. New Series, 28. The Clarendon Press, Oxford University Press, Oxford, 2003.

[Sc] J.-P. Serre, *Trees*, Springer-Verlag, Berlin New York, 1980.

[S] J. T. Stafford, *Endomorphisms of right ideals of the Weyl algebra*, Trans. Amer. Math. Soc. 299 (1987), 623–639.

[Smi1] P. Smith, *An example of a ring Morita equivalent to the Weyl algebra $\mathfrak{A}_1$*, J. Algebra, 73 (1981), 552–555.

[Smi2] P. Smith, *A class of algebras similar to the enveloping algebra of $\mathfrak{sl}_2$*, Trans. Amer. Math. Soc., 322 (1990), no. 1, 285–314.

[W] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian* (with an Appendix by I. G. Macdonald), Invent. Math. 133 (1998), 1–41.

X. Chen and F. Eshmatov: Department of Mathematics, Sichuan University, Chengdu 610064 P. R. China

E-mail address: xjchen@scu.edu.cn, olimjon55@hotmail.com

A. Eshmatov: Department of Mathematics, Cornell University, Ithaca, NY 14850, USA

E-mail address: aeshmat@math.cornell.edu

V. Futorny: Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970, Brasil

E-mail address: futorny@ime.usp.br