RANDOM POLYTOPES AND AFFINE SURFACE AREA

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ABSTRACT. Let $K$ be a convex body in $\mathbb{R}^d$. A random polytope is the convex hull $[x_1, ..., x_n]$ of finitely many points chosen at random in $K$. $\mathbb{E}(K, n)$ is the expectation of the volume of a random polytope of $n$ randomly chosen points. I. Bárány showed that we have for convex bodies with $C^3$ boundary and everywhere positive curvature

$$c(d) \lim_{n \to \infty} \frac{\text{vol}_d(K) - \mathbb{E}(K, n)}{(\text{vol}_d(K) n)^{-\frac{2}{d+1}}} = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} d\mu(x)$$

where $\kappa(x)$ denotes the Gauß-Kronecker curvature. We show that the same formula holds for all convex bodies if $\kappa(x)$ denotes the generalized Gauß-Kronecker curvature.
1. Introduction

Let $K$ be a convex body in $\mathbb{R}^d$. A random polytope in $K$ is the convex hull of finitely many points in $K$ that are chosen at random with respect to a probability measure on $K$. Here we consider the normalized Lebesgue measure on $K$. For a fixed number $n$ of points we are interested in the expectation of the volume of that part of $K$ that is not contained in the convex hull $[x_1, \ldots, x_n]$ of the chosen points. We denote

$$
\mathbb{E}(K, n) = \int_{K \times \cdots \times K} \text{vol}_d([x_1, \ldots, x_n]) dP(x_1, \ldots, x_n)
$$

where $P$ is the $n$-fold product of the normalized Lebesgue measure on $K$. We are interested in the asymptotic behavior of

$$
\text{vol}_d(K) - \mathbb{E}(K, n) = \int_{K \times \cdots \times K} \text{vol}_d(K \setminus [x_1, \ldots, x_n]) dP(x_1, \ldots, x_n)
$$

In [R-S1, R-S2] the asymptotic behavior of this expression has been determined for polygons and smooth convex bodies in $R^2$.

**Theorem 1.** Let $K$ be a convex body in $\mathbb{R}^d$. Then we have

$$
c(d) \lim_{n \to \infty} \frac{\text{vol}_d(K) - \mathbb{E}(K, n)}{\left(\frac{\text{vol}_d(K)}{n}\right)^{\frac{d+1}{2}}} = \int_{\partial K} \kappa(x) \frac{1}{d+1} d\mu(x)
$$

where $\kappa(x)$ is the generalized Gauß-Kronecker curvature and

$$
c(d) = 2\left(\frac{\text{vol}_{d-1}(B_2^{d-1})}{d+1}\right)^{\frac{d}{d+1}} \frac{(d+3)(d+1)!}{(d^2 + d + 2)(d^2 + 1)\Gamma\left(\frac{d^2+1}{d+1}\right)}
$$

This problem was posed by Schneider and Wieacker [Schn-W] and Schneider [Schn]. It has been solved by Báráný [B] for convex bodies with $C^3$ boundary and everywhere positive curvature. Our result holds for arbitrary convex bodies.

The main ingredients of the proof are taken from [B-L], [B], and [Schü-W 1].

We introduce the notion of generalized Gauß-Kronecker curvature. A convex function $f : X \to \mathbb{R}, X \subseteq \mathbb{R}^d$ is called twice differentiable at $x_0$ in a generalized sense if there are a linear map $d^2 f(x_0) \in L(\mathbb{R}^d)$ and a neighborhood $U(x_0)$ so that we have for all $x \in U(x_0)$ and all subdifferentials $df(x)$

$$
\| df(x) - df(x_0) - (d^2 f(x_0))(x-x_0) \| \leq \Theta(\|x-x_0\| \|x-x_0\|)
$$

where $\lim_{t \to 0} \Theta(t) = \Theta(0) = 0$ and where $\Theta$ is a montone function. $d^2 f(x_0)$ is symmetric and positive semidefinite. If $f(0)=0$ and $df(0)=0$ then the ellipsoid or elliptical cylinder.
is called the indicatrix of Dupin at 0. The general case is reduced to the case 
\( f(0)=0 \) and \( df(0)=0 \) by an affine transform. The eigenvalues of 
\( d^2 f(0) \) are called the principal curvatures and their product the 
Gauß-Kronecker curvature \( \kappa(0) \). Alek-
sandrov [A, Ba] proved that a convex surface is almost everywhere differentiable 
in the generalized sense. As surface measure on \( \partial K \) we take the restriction of the 
\((d-1)\)-dimensional Hausdorff measure to \( \partial K \). For \( x \in \partial K \) the normal at \( x \) to \( \partial K \) is 
denoted by \( N(x) \). \( N(x) \) is almost everywhere unique. We denote 
\( K_t = \{ x \in K | vol_d((−x + K) \cap (x − K)) \geq t \} \) 
for \( t \in [0, T] \) with 
\[ T = \max_{y \in K} vol_d((−y + K) \cap (y − K)) \]

\( K_t \) is a convex body and was studied and used in [St, F-R, K, Schm1] and was 
called convolution body in [K, Schm1]. It was shown in [St, F-R] that 
\( K_T \) consists of one point only. Therefore we may also interpret 
\( K_T \) - in abuse of notation - as a 
point. For a given \( x \in \partial K \) there is a unique point \( x_t \in \partial K_t \) that is the intersection 
of the interval \([K_T, x]\) and \( \partial K_t \).

\[ P_ξ \] denotes the orthogonal projection onto the hyperplane orthogonal to \( ξ \) and 
passing through the origin. \( B^d_2(x, r) \) is the Euclidean ball in \( R^d \) with center \( x \) and 
radius \( r \). \( B^d_2 \) is the ball with center 0 and radius 1. \( H(x, ξ) \) denotes the hyperplane 
through \( x \) and orthogonal to \( ξ \). For a given hyperplane \( H \) the closed halfspaces are 
denoted by \( H^+ \) and \( H^- \). Usually \( H^+ \) is the halfspace containing \( K_T \) if we consider 
a convex body \( K \).

2. Outline Of The Proof

We outline the proof of Theorem 1. We have that 
\[ vol_d(K) - E(K, n) = \int_K \mathbb{P}\{(x_1, ..., x_n) | x \notin [x_1, ..., x_n]\} dx \]
\[ = -\int_0^T \frac{d}{dt} \int_{K_t} \mathbb{P}\{(x_1, ..., x_n) | x \notin [x_1, ..., x_n]\} dx dt \]
The derivative can be computed and we get 
\[ \int_0^T \int_{\partial K_t} \mathbb{P}\{(x_1, ..., x_n) | x \notin [x_1, ..., x_n]\} \frac{d\mu_t(x)}{vol_{d-1}(P_{N(x)}((−x + K) \cap (x − K)) \cap (y - x))} dt \]
where \( \mu_t \) is the surface measure on \( \partial K_t \). We pass to an integral on \( \partial K \) instead 
of \( \partial K_t \).
where $x_t$ is the unique element on $\partial K_t$ that is on the line through $K_T$ and $x$. Since $\mathbb{P}\{(x_1, \ldots, x_n)| x_t \notin [x_1, \ldots, x_n]\}$ is concentrated for large $n$ near the boundary $\partial K$ we get

$$\lim_{n \to \infty} \frac{\text{vol}_d(K) - \mathbb{E}(K, n)}{(\frac{\text{vol}_d(K)}{n})^{\frac{2}{d+1}}} =$$

$$\lim_{n \to \infty} \left(\frac{n}{\text{vol}_d(K)}\right)^{\frac{2}{d+1}} \int_{\partial K} \int_0^{\frac{\log n}{n}} \mathbb{P}\{(x_1, \ldots, x_n)| x_t \notin [x_1, \ldots, x_n]\} \frac{\|x_t\|^d < x, N(x) >}{\text{vol}_{d-1}(P_N(x_t)((-x_t + K) \cap (x_t - K))))} \frac{\|x\|^d < x_t, N(x_t) >}{d \mu}

Then we apply Lebesgue’s convergence theorem and obtain

$$\int_{\partial K} \lim_{n \to \infty} \left(\frac{n}{\text{vol}_d(K)}\right)^{\frac{2}{d+1}} \int_0^{\frac{\log n}{n}} \mathbb{P}\{(x_1, \ldots, x_n)| x_t \notin [x_1, \ldots, x_n]\} \frac{\|x_t\|^d < x, N(x) >}{\text{vol}_{d-1}(P_N(x_t)((-x_t + K) \cap (x_t - K))))} \frac{\|x\|^d < x_t, N(x_t) >}{d \mu}

The hypothesis of Lebesgue’s convergence theorem is fulfilled since the following function dominates the functions under the integral: For every $x \in \partial K$ let $r(x)$ be the largest radius so that

$$B_2^d(x - r(x)N(x), r(x)) \subseteq K$$

$r(x)$ may be 0, e.g. if $N(x)$ is not unique. The functions under the integral are uniformly smaller than a constant times

$$r(x)^{-\frac{d+1}{d}}$$

which is integrable on $\partial K$. Then we show that the expression under the integral converges to $r(x)^{1/(d+1)}$ times an appropriate constant.

3. Proof Of Theorem 1

**Lemma 2.** Let $K$ be a convex body in $R^d$, $f$ a continuous function on $K$, and $t \in (0, T)$. Then we have

$$\frac{d}{dt} \int_{K_t} f(x)dx = -\int_{\partial K_t} \frac{f(x)}{\text{vol}_{d-1}(P_N(x)((-x + K) \cap (x - K))))} d\mu_t(x)$$

where $\mu_t$ is the surface measure on $\partial K_t$.

For $f(x)$ identical to 1 this is an unpublished result of Schmuckenschläger [Schm 2]. A similar result for convex floating bodies instead of convolution bodies can be found in [Schü-W2].
Lemma 3. Let $K$ be a convex body in $\mathbb{R}^d$ and $K_t, t \in [0,T]$, the convolution bodies. Then we have for all $t_0 \in [0,T]$

$$\text{vol}_d(K) - \mathbb{E}(K,n) =$$

$$\int_{\partial K} \int_{0}^{t_0} \mathbb{P}\{(x_1,\ldots,x_n) \mid x_t \notin [x_1,\ldots,x_n]\} \left\| x_t \right\|^d < x, N(x) > dtd\mu(x)$$

$$+ \int_{t_0}^{T} \int_{\partial K_t} \mathbb{P}\{(x_1,\ldots,x_n) \mid x \notin [x_1,\ldots,x_n]\} d\mu_t(x)dt$$

where $\mu$ and $\mu_t$ are the surface measures on $\partial K$ and $\partial K_t$ respectively and $\{x_t\} = \partial K \cap [K_T, x]$.

Proof.

$$\text{vol}_d(K) - \mathbb{E}(K,n) = \int_{K_T \times \cdots \times K_T} \text{vol}_d(K \setminus [x_1,\ldots,x_n]) d\mathbb{P}$$

$$= \int_{K_T \times \cdots \times K_T} \chi_{K \setminus [x_1,\ldots,x_n]} dxd\mathbb{P}$$

$$= \int_{K} \int_{K_T \times \cdots \times K_T} \chi_{K \setminus [x_1,\ldots,x_n]} d\mathbb{P} dx$$

$$= \int_{K_T} \mathbb{P}\{(x_1,\ldots,x_n) \mid x \notin [x_1,\ldots,x_n]\} dx$$

Since

$$\int_{K_T} \mathbb{P}\{(x_1,\ldots,x_n) \mid x \notin [x_1,\ldots,x_n]\} dx$$

is a bounded, continuous, decreasing function on $[0,T]$ it is absolutely continuous. We get

$$\text{vol}_d(K) - \mathbb{E}(K,n) = -\int_{0}^{T} \frac{d}{dt} \int_{K_T} \mathbb{P}\{(x_1,\ldots,x_n) \mid x \notin [x_1,\ldots,x_n]\} dx dt$$

Since $\mathbb{P}\{(x_1,\ldots,x_n) \mid x \notin [x_1,\ldots,x_n]\}$ is a continuous function of $x$ we get by Lemma 2

$$\text{vol}_d(K) - \mathbb{E}(K,n) = \int_{0}^{T} \int_{\partial K_T} \text{vol}_{d-1}(P_{N(x)}((-x + K) \cap (x - K))) d\mu_t(x) dt$$

$\square$
Lemma 4. Let $\text{cap}(r, \Delta)$ be a cap of height $\Delta$ of a $d$-dimensional Euclidean sphere with radius $r$. Then we have

$$2\left(2 - \frac{\Delta}{r}\right)^{d-1} \frac{\text{vol}_{d-1}(B_2^{d-1})}{d+1} \frac{\Delta^{d+1}}{r^{d+1}} \leq \text{vol}_d(\text{cap}(r, \Delta)) \leq 2^{d+1} \frac{\text{vol}_{d-1}(B_2^{d-1})}{d+1} \frac{\Delta^{d+1}}{r^{d+1}}$$

Lemma 5. Let $K$ be a convex body in $\mathbb{R}^d$. Then there are constants $c, c' > 0$ so that we have for all $x \in \partial K$ and for all $r > 0$ with $B_2^d(x - rN(x), r) \subseteq K$

$$\text{vol}_{d-1}(P_{N(x_t)}((x_t + K) \cap (x_t - K))) \geq \begin{cases} c(tr)^{\frac{d-1}{d+1}} & \text{if } 0 \leq t \leq c'r^d \\ c't^{\frac{d-1}{d}} & \text{if } c'r^d \leq t \leq T \end{cases}$$

where $\{x_t\} = \partial K_t \cap [x, K_T]$.

Proof. We may assume that $K_T$ coincides with the origin. By convexity and the fact that $K_T$ is an interior point we get that there is $c_1 > 0$ so that we have for all $x \in \partial K$

$$< \frac{x}{\|x\|}, N(x) > \geq c_1$$

Now we choose

$$c = \frac{\text{vol}_{d-1}(B_2^{d-1})}{(d+1)\text{vol}_d(B_2^d)} \min\{(1 - \sqrt{1 - c_1^2})^{\frac{d+1}{d}}, (1 - \sqrt{\frac{3}{4}})^{\frac{d+1}{2}}\}$$

Then we have for all $t \in [0, cr^d\text{vol}_d(B_2^d)]$

$$\|x - x_t\| \leq r < \frac{x}{\|x\|}, N(x) >$$

$$\frac{1}{r} < | < N(x), N(x_t) > |$$
Geometrically (4) means the following: Let \( z \) be the midpoint of the interval 
\([K_T, x] \cap B_2^d(x - rN(x), r)\). Then \( x_t \in [z, x] \) (see figure 1).

We verify (4). Since \( B_2^d(x - rN(x), r) \subseteq K \) we have

\[
\text{vol}_d((-z + K) \cap (z - K)) \geq \text{vol}_d((-z + B_2^d(x - rN(x), r)) \cap (z - B_2^d(x - rN(x), r)))
\]

The last expression equals twice the volume of a cap whose height is greater than

\[
\Delta = r(1 - \sqrt{1 - c_1^2})
\]

By Lemma 4 and (3) we get

\[
\text{vol}_d((-z + K) \cap (z - K)) \geq 4(2 - \frac{\Delta}{r})^{\frac{d-1}{2}} \frac{\text{vol}_{d-1}(B_2^{d-1})}{d+1} \Delta^{\frac{d+1}{2}} r^{-\frac{d-1}{2}}
\]

\[
\geq 4 \text{cr}^d \text{vol}_d(B_2^d) \geq 4t \geq t
\]
Thus $z$ is an interior point of $K_t$ and $x_t \in [z, x]$. Moreover, $\|z - x\| = r < \frac{x}{\|x\|}, N(x) >$.

Now we check (5). We get by (4) and figure 1 that $x_t$ has to be in the shaded area of figure 2.

**Figure 2**

Assume that (5) is not true. Then it follows that the radius of the sphere $H(x_t, N(x_t)) \cap B_2^d(x - rN(x), r)$ is greater than $r$ and that $H(x_t, N(x_t))$ cuts off a cap of height greater than $(r(1 - \sqrt{3/4}))$. By Lemma 4 we get that the volume of this cap is greater than

$$2(2 - \frac{\Delta}{r})^{\frac{d-1}{d+1}} \frac{vol_{d-1}(B_2^{d-1})}{d+1} r^d(1 - \sqrt{3/4})^{\frac{d+1}{d+1}}$$

Therefore we have for the center $w$ of the sphere $B_2^d(x - rN(x), r) \cap H(x_t, N(x_t))$ that

$$vol_d((-w + K) \cap (w - K)) \geq 2vol_d(B_2^d(x - rN(x), r) \cap H^-(x_t, N(x_t)))$$

$$\geq 4 \frac{vol_{d-1}(B_2^{d-1})}{d+1} (1 - \sqrt{3/4})^{\frac{d+1}{d+1}} r^d \geq 4t > t$$

This means that $w$ is an interior point of $K_t$. On the other hand, $z$ is an element of the supporting hyperplane $H(x_t, N(x_t))$ to $K_t$. This gives a contradiction and we conclude that (5) is valid.
We denote $\Theta = \arccos(\frac{x}{\|x\|}, N(x))$.

From figure 3 it follows that the distance of $x_t$ to the boundary of $B_d^d(x - rN(x), r)$ equals

$$(\cos(\Theta) - \sin(\Theta) \cot(\frac{\pi}{2} - \frac{\alpha}{2})) \| x - x_t \|$$

By figure 1 we have $\alpha \leq \frac{\pi}{2} - \Theta$ so that the above expression is larger than

(6) $$(\cos(\Theta) - \sin(\Theta) \cot(\frac{\pi}{4} + \frac{\Theta}{2})) \| x - x_t \|$$

Please note that by (2) there is $\epsilon > 0$ so that we have for all $x \in \partial K$

(7) $$\cos(\Theta) - \sin(\Theta) \cot(\frac{\pi}{2} + \frac{\Theta}{2}) \geq \epsilon$$
We assume now that

\[(8) \quad \| x - x_t \| \geq \frac{1}{\epsilon} \frac{2^{d+1}}{\text{vol}_{d-1}(B^d_2)} \left( \frac{d + 1}{d - 1 + 1} \right) \]

Then

\((-x_t + K) \cap (x_t - K) \supseteq (-x_t + B^d_2(x - rN(x), r)) \cap (x_t - B^d_2(x - rN(x), r))

has a volume greater than twice the volume of a cap of a Euclidean ball of radius r and height (6). By Lemma 4, (6), and (7) we get as above

\[\text{vol}_d((-x_t + K) \cap (x_t - K)) \geq 4t\]

which cannot be true.

Therefore (8) does not hold. We deduce that the distance between the two parallel hyperplanes \(H(x, N(x))\) and \(H(2x_t - x, -N(x))\) is less than twice the right hand expression of (8). Moreover,

\[(-x_t + K) \cap (x_t - K) \subseteq H^+(x, N(x)) \cap H^+(2x_t - x, -N(x))\]

where both half spaces are chosen so that \(x_t\) is contained in them. Therefore there must be a hyperplane H parallel to \(H(x, N(x))\) so that

\[\text{vol}_{d-1}((-x_t + K) \cap (x_t - K) \cap H) \]

\[\geq \frac{\epsilon}{2} (rt) \frac{d+1}{d+1} \left( \frac{\text{vol}_{d-1}(B^d_2)}{d+1} \right)^{\frac{2}{d+1}}\]

By (5) we get that the same inequality holds for

\[P_{N(x_t)}((-x_t + K) \cap (x_t - K))\]

with another constant.

Now we consider the case

\[ct^d \text{vol}_d(B^d_2) \leq t \leq T\]

Since K is compact there are \(r_1, r_2 > 0\) so that

\[(9) \quad B^d_2(K_T, r_1) \subseteq K \subseteq B^d_2(K_T, r_2)\]

Suppose that

\[(10) \quad \text{vol}_{d-1}(P_{N(x_t)}((-x_t + K) \cap (x_t - K))) \leq ct \frac{d-1}{d}\]

with \(c \leq \frac{1}{r_1 \text{vol}_d(B^d_2)\frac{T}{t}}\). This implies that
and since \((-x_t + K) \cap (x_t - K)\) is symmetric with respect to \(x_t\) all other \(d-1\) dimensional sections of \((-x_t + K) \cap (x_t - K)\) that are parallel to \(H(x_t, N(x_t))\) have a smaller \(d-1\) dimensional volume. Therefore there must be a non-empty section whose distance to \(H(x_t, N(x_t))\) is at least \(\frac{1}{2c}t^\frac{d}{2}\). This means that there is \(z \in K\) with (see figure 4)

\[
(11) \quad d(z, H(x_t, N(x_t))) \geq \frac{1}{2c}t^\frac{d}{2}
\]

Let \(y\) be the unique point in the intersection

\([K_T, z] \cap H(x_t, N(x_t))\)

We show that \(y\) is an interior point of \(K_t\) which contradicts the fact that \(y \in H(x_t, N(x_t))\). The sphere

\([z, B^d_2(K_T, r_1)] \cap H(y, \frac{z - y}{\|z - y\|})\)

has a radius larger than

\[
\frac{1}{2c} \frac{r_1}{r_2} t^\frac{1}{2}
\]

This follows from (11). Thus we find that

\([z, B^d_2(K_T, r_1)] \supseteq B^d_2(y, \frac{1}{4c r_2} t^\frac{1}{2})\)

and as above that \(y\) is an interior point of \(K_t\).

\[\square\]

**Lemma 6.** [Schü-W1] Let \(K\) be a convex body in \(\mathbb{R}^d\), \(\alpha \in (0,1)\) and \(r(x)\) as defined by (1). Then we have

\[
\int_{\partial K} r(x)^{-\alpha} d\mu(x) < \infty
\]

where \(\mu\) is the surface measure on \(\partial K\).

**Lemma 7.** [B-L] Let \(K\) be a convex body in \(\mathbb{R}^d\) and let \(x \in \partial K_t\). Then we have

\[
P\{(x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n] \} \leq 2 \sum_{i=0}^{d-1} \binom{n}{i} \left( \frac{t}{2 \text{vol}_d(K)} \right)^i \left( 1 - \frac{t}{2 \text{vol}_d(K)} \right)^{n-i}
\]
Lemma 8. Let $K$ be a convex body in $\mathbb{R}^d$, $t_0 \in (0, T]$ and $\mu_t$ the surface measure on $\partial K_t$. Then we have

$$\lim_{n \to \infty} n^{\frac{2}{d+1}} \int_{t_0}^{T} \int_{\partial K_t} \frac{\mathbb{P}\{(x_1, \ldots, x_n) | x \notin [x_1, \ldots, x_n]\}}{\text{vol}_{d-1}(P_{N(x)}((-x + K) \cap (x - K)))} d\mu_t(x) dt = 0$$

Proof. Since $t \in [t_0, T]$ and $t_0 > 0$ there is a constant $c > 0$ so that we have for all $x \in \partial K$.
\[ \text{vol}_{d-1}(P_{N(x)}((-x + K) \cap (x - K)) \geq c} \]

By Lemma 7 we get
\[
\frac{2}{c} n^{\frac{2}{d+1}} \int_{t_0}^{T} \int_{\partial K_t} \frac{\mathbb{P}\{x_1, \ldots, x_n \mid x \notin [x_1, \ldots, x_n]\}}{\text{vol}_{d-1}(P_{N(x)}((-x + K) \cap (x - K)))} d\mu_t(x) dt
\]
\[
\leq \frac{2}{c} n^{\frac{2}{d+1}} \sum_{i=0}^{d-1} \binom{n}{i} \int_{t_0}^{T} \int_{\partial K_t} \left( \frac{t}{2\text{vol}_d(K)} \right)^i (1 - \frac{t}{2\text{vol}_d(K)})^{n-i} d\mu_t(x) dt
\]

Since \( \text{vol}_{d-1}(\partial K_t) \leq \text{vol}_{d-1}(\partial K) \) we get that the last expression is smaller than
\[
\frac{4}{c} n^{\frac{2}{d+1}} \text{vol}_{d-1}(\partial K) \text{vol}(K) \sum_{i=0}^{d-1} \binom{n}{i} \frac{\Gamma(i+1)\Gamma(n-i+1)}{\Gamma(n+2)}
\]
\[
= \frac{4}{c} \text{vol}_{d-1}(\partial K) \text{vol}(K) n^{\frac{2}{d+1}} \frac{d}{n+1}
\]
\[
\leq \frac{4}{c} \text{vol}_{d-1}(\partial K) \text{vol}(K) d n^{-\frac{d-1}{d+1}}
\]

\[\blacksquare\]

**Lemma 9.** Let \( K \) be a convex body in \( \mathbb{R}^d \), let \( K_T \) be the origin, and \( 0 \leq t_1 \leq t_2 < T \).

Then there is a constant \( c > 0 \) so that we have for all \( n \in \mathbb{N} \)
\[
\int_{t_1}^{t_2} \frac{\mathbb{P}\{x_1, \ldots, x_n \mid x \notin [x_1, \ldots, x_n]\}}{\text{vol}_{d-1}(P_{N(x)}((-x + K) \cap (x - K)))} \frac{\|x_t\|^d}{\|x\|^d} < x, N(x) > dt
\]
\[\text{(12)}\]
\[
\leq cr(x) \frac{d-1}{d+1} \int_{t_1}^{t_2} \frac{\text{vol}_{d-1}(\partial K)}{\text{vol}(K)} \sum_{i=0}^{d-1} \binom{n}{i} s^i \frac{d-1}{d+1} (1 - s)^{n-i} ds
\]

where \( r(x) \) is defined by (1).

**Proof.** We have \( \|x_t\| \leq \|x\| \) and we have a constant \( c > 0 \) so that we have for all \( x \in \partial K < x_t, N(x_t) \geq c \) because \( t_2 < T \). Thus it is enough to estimate
\[
\int_{t_1}^{t_2} \frac{\mathbb{P}\{x_1, \ldots, x_n \mid x \notin [x_1, \ldots, x_n]\}}{\text{vol}_{d-1}(P_{N(x)}((-x + K) \cap (x - K)))} dt
\]

By Lemma 5 this is smaller than.
\[ \int_{t_1}^{c'r^d} c(r(x)t) - \frac{d+1}{d+2} \mathbb{P}\{(x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n]\}dt \]

\[ + \int_{c'r^d}^{t_2} ct^{-\frac{d-1}{d+1}} \mathbb{P}\{(x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n]\}dt \]

Since we have for \( t \in [c'r^d, t_2] \) that

\[ t^{-\frac{d-1}{d+1}} = t^{-\frac{d-1}{d+1}} t^{-\frac{d-1}{d+1}} \leq (c'r(x))^{-\frac{d-1}{d+1}} t^{-\frac{d-1}{d+1}} \]

we can estimate the above expression by

\[ cr(x)^{-\frac{d-1}{d+1}} \int_{t_1}^{t_2} t^{-\frac{d-1}{d+1}} \mathbb{P}\{(x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n]\}dt \]

where \( c \) is a new constant. Now it is left to apply Lemma 7. \( \square \)

**Lemma 10.** Let \( K \) be a convex body in \( \mathbb{R}^d \), let \( K_T \) be the origin, and let \( t_1 < T \). Then there is a constant \( c \) so that we have for all \( x \in \partial K \) and all \( n \in \mathbb{N} \)

\[ \int_0^{t_1} \frac{\mathbb{P}\{(x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n]\} \|x_t\|^d < x, N(x) >}{\text{vol}_{d-1}(P_{N(x)}((-x_t + K) \cap (x_t - K))) \|x\|^d < x_t, N(x_t) >} \] \[ \leq c r(x)^{-\frac{d-1}{d+1}} n^{-\frac{2}{d+1}} \sum_{i=0}^{d-1} \binom{n}{i} \frac{\Gamma(i + 1 -\frac{d-1}{d+1}) \Gamma(n + 1 - i)}{\Gamma(n + 2 -\frac{d-1}{d+1})} \]

Since

\[ \lim_{k \to \infty} \frac{\Gamma(k + \frac{2}{d+1})}{\Gamma(k)} k^{-\frac{2}{d+1}} = 1 \]

we can estimate the last expression by

\[ cr(x)^{-\frac{d-1}{d+1}} \]

where \( c \) is a new constant that does not depend on \( n \) and \( x \). \( \square \)
Lemma 11. We have
\[
\lim_{n \to \infty} n^{\frac{1}{d+1}} \int_0^1 \log \left( \frac{n}{s} \right) s^{\frac{d}{d+1}} (1-s)^{n-s} ds = 0
\]

Lemma 12. Let \( K \) be a convex body in \( \mathbb{R}^d \), let \( K_T \) be the origin, and let \( t_1 < T \). Then we have for all \( x \in \partial K \) with \( r(x) > 0 \)
\[
\lim_{n \to \infty} n^{\frac{1}{d+1}} \int_0^{t_1} \frac{\mathbb{P}\{ (x_1, ..., x_n) \mid x_t \notin [x_1, ..., x_n] \}}{\text{vol}_{d-1}(P_{N(x_t)}((-x_t + K) \cap (x_t - K)))} \frac{\|x_t\|^d < x, N(x) > \|x\|^d < x_t, N(x_t) >}{\text{vol}_{d-1}(\partial B_2^d(0, r))} dt = 0
\]

Proof. The result follows from Lemmata 9 and 11. \( \square \)

Lemma 13. [Wie]
\[
\lim_{n \to \infty} n^{\frac{1}{d+1}} (\text{vol}_d(B_2^d(0, r)) - \mathbb{E}(B_2^d(0, r), n)) = \frac{(d^2 + d + 2)(d^2 + 1)(d+1)}{2(d+3)(d+1)!} \frac{\text{vol}_d(B_2^d)}{\text{vol}_{d-1}(B_2^{d-1})} \frac{\sqrt{\pi}}{d+1} \Gamma \left( \frac{d^2 + 1}{d+1} \right) \text{vol}_{d-1}(\partial B_2^d(0, r))
\]

By an affine transform we can change the indicatrix of Dupin into a Euclidean sphere or a cylinder with a sphere as its base.

Lemma 14. Let \( K \) be a convex body in \( \mathbb{R}^d \) with \( 0 \in \partial K \) and \( N(0) = (0, ..., 0, -1) \). Suppose that \( \partial K \) is twice differentiable in the generalized sense at \( 0 \).
(i) If the indicatrix of Dupin at \( 0 \) is a \( d-2 \) dimensional sphere with radius \( \sqrt{\rho} \), then there is a \( t_0 > 0 \) and a monotone, increasing function \( \psi \) on \( \mathbb{R}^+ \) with \( \lim_{t \to 0} \psi(t) = \psi(0) = 1 \) so that we have for all \( t \in (0, t_0] \)
\[
\left\{ \left( \frac{x_1}{\psi(t)}, ..., \frac{x_{d-1}}{\psi(t)}, t \right) \mid x \in B_2^d((0, ..., 0, \rho), \rho) \text{ and } x_n = t \right\}
\]
\[
\subseteq K \cap H(-tN(0), N(0))
\]
\[
\subseteq \left\{ (\psi(t)x_1, ..., \psi(t)x_{d-1}, t) \mid x \in B_2^d((0, ..., 0, \rho), \rho) \text{ and } x_n = t \right\}
\]
(ii) If the indicatrix of Dupin at \( 0 \) is a \( d-2 \) dimensional cylinder with radius \( \sqrt{\rho} \),
Then there is a function $\Phi$ on $\mathbb{R}^+$ so that for every $\epsilon > 0$ there is a $t_0 > 0$ so that
\[
\lim_{t \to 0} \sqrt{t} \Phi(t) = 0 \quad \text{and} \quad \frac{\sqrt{t}}{\Phi(t)} \text{ is increasing on } \mathbb{R}^+ \quad \text{and so that we have for all } t \in (0, t_0]
\]
\[
\{(y, x, t) \mid (x, t) \in B_2^{d-k}(0, \rho - \epsilon), y \in [-\Phi(t), \Phi(t)]^{k-1}\}
\]
\[
\subseteq K \cap H(-tN(0), N(0))
\]

**Lemma 15.** Let $K$ be a convex body in $\mathbb{R}^d$, $c > 2$, and $x \in \partial K$ such that $\kappa(x) > 0$. Then there is $t_c > 0$ so that we have for all $t \in (0, t_c]$: We have for the hyperplane $H$ whose normal coincides with $N(x)$ and that satisfies $\text{vol}_d(K \cap H^-) = c^{d+1}t$ and for all $n \in \mathbb{N}$ with $n \geq \frac{\text{vol}_d(K)}{ct}$

\[
|\mathbb{P}\{(x_1, ..., x_n)|x_t \notin \{x_1, ..., x_n\} \cap H^-\} - \mathbb{P}\{(x_1, ..., x_n)|x_t \notin [x_1, ..., x_n]\}| < 2^{d-1}e^{-c_1c}
\]

where $c_1$ is a constant that depends on $d$ only.

**Proof.** We show that

\[
\mathbb{P}\{(x_1, ..., x_n)|x_t \notin \{x_1, ..., x_n\} \cap H^-\} \text{ and } x_t \in [x_1, ..., x_n] \leq 2^{d-1}e^{-c_1c}
\]

If we have

\[
x_t \notin \{x_1, ..., x_n\} \cap H^- \text{ and } x_t \in [x_1, ..., x_n]
\]

then there is $y \in H^+ \cap K$ so that

\[
(y, x_t) \cap \{x_1, ..., x_n\} \cap H^- = \emptyset
\]

For the following argument let us assume that $N(x) = e_1$ and $x_t = 0$. Moreover, since $\kappa(x) > 0$ we may assume that the indicatrix of Dupin at $x$ is a Euclidean sphere and by Lemma 14 $\partial K$ can be approximated arbitrary well at $x$ by a sphere if we choose the height of the cap or correspondingly $t_c$ sufficiently small. We assume for the following arguments that $K \cap H^-$ is a cap of a sphere. Later we shall see that we control the error by choosing $t_c$ sufficiently small. We consider the following sets (figure 5).
Figure 5

\[ \text{corn}_\Theta = K \cap H^- \cap H^+(x_t, N(x)) \cap \{ \bigcap_{i=2}^d H^-(x_t, e_1 + \Theta_i \lambda e_i) \} \]

where \( \Theta_2, \ldots, \Theta_d = \pm 1 \). We have \( 2^{d-1} \) sets and they are best described as corner sets. \( \lambda \) is chosen so that

\[ H^-(x_t, e_1 + \Theta_i \lambda e_i) \cap H^+ \]

consists of exactly one point.

By the height \( h_1 \) of \( \text{corn}_\Theta \) we understand the minimal distance of \( H(x_t, N(x)) \) and a parallel hyperplane so that \( \text{corn}_\Theta \) lies between them. We get

\[ \text{vol}_d(\text{corn}_\Theta) \geq 2^{-d+1} \frac{h_1}{d} \text{vol}_{d-1}(K \cap H(x_t, N(x))) \]

Let \( h_2 \) denote the height of the cap \( K \cap H^-(x_t, N(x)) \) and \( h_3 \) the height of \( K \cap H^- \). By Lemma 4 we get that

\[ 2c^{d+1} = \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K \cap H^-(x_t, N(x)))} \leq 2^{\frac{d+1}{2}} \left( \frac{h_3}{h_2} \right)^{\frac{d+1}{2}} \]

or

\[ h_2 c^2 \leq 2h_3 \]

The height \( h_1 \) is of the order \( \frac{h_3}{d} \). Altogether we get

(14) \[ \text{vol}_d(\text{corn}_\Theta) \geq 2^{-d+1} c \cdot c^2 h_2 \cdot \text{vol}_{d-1}(K \cap H(x_t, N(x))) \geq c \cdot c^2 t \]
where \( c_2 \) is a constant depending only on \( d \). We have by (13)

\[
\{(x_1, \ldots, x_n) | x_t \notin \{x_1, \ldots, x_n \} \cap H^- \text{ and } x_t \in [x_1, \ldots, x_n] \} \subseteq \{(x_1, \ldots, x_n) | \exists H_{x_t} : x_t \in H_{x_t}, H_{x_t} \cap K \cap H^+ \neq \emptyset \text{ and } H_{x_t}^+ \supset \{x_1, \ldots, x_n \} \cap H^- \}
\]

Indeed, by the theorem of Hahn-Banach there is a hyperplane \( H_{x_t} \) separating the convex sets \([\{x_1, \ldots, x_n \} \cap H^- \]) and the ray \([x_t + \lambda(y - x_t) | \lambda \in \mathbb{R}] \). By (13) they are disjoint. We may assume that at least one point of the ray is an element of \( H_{x_t} \). So \( x_t \) is also an element of \( H_{x_t} \). Let \( H_{x_t}^- \) be the halfspace containing \( y \), then \( H_{x_t}^- \cap K \cap H^+ \) contains \( y \) and is not empty.

Such a halfspace always contains one of the corner sets \( \text{corn}_\Theta \). This follows since we have in \( \mathbb{R}^d \) for a hyperplane passing through the origin: The corresponding halfspaces contain at least one \( 2^d \)-tant. Therefore we get

\[
\{(x_1, \ldots, x_n) | \exists H_{x_t} : H_{x_t}^- \cap K \cap H^+ \neq \emptyset \text{ and } H_{x_t}^+ \supset \{x_1, \ldots, x_n \} \cap H^- \} \subseteq \bigcup_{\Theta} \{(x_1, \ldots, x_n) | \{x_1, \ldots, x_n \} \subseteq K \setminus \text{corn}_\Theta \}
\]

And consequently we get by (14)

\[
P\{(x_1, \ldots, x_n) | x_t \notin \{x_1, \ldots, x_n \} \cap H^- \text{ and } x_t \in [x_1, \ldots, x_n] \}
\leq 2^{d-1}P\{(x_1, \ldots, x_n) | \{x_1, \ldots, x_n \} \subseteq K \setminus \text{corn}_\Theta \}
\leq 2^{d-1}(1 - \frac{c_2c^2t}{\text{vol}_d(K)})^n \leq 2^{d-1}\exp(-n\frac{c_2c^2t}{\text{vol}_d(K)})
\]

By the assumption on \( n \) we get that the last expression is smaller than

\[
2^{d-1}\exp(-c_2c)
\]

This argument also works if the volumes of the considered sets differ by a small error. Therefore the proof also goes through if \( K \) is not a sphere but can be approximated arbitrarily well by a sphere at the point \( x \). \( \square \)

**Lemma 17.** Let \( K \) be a convex body in \( \mathbb{R}^d \) and \( B \) a Euclidean ball of the same volume. Let \( x \in \partial K \) and \( z \in \partial B \) and assume that \( \kappa(x) > 0 \). Then for every \( \epsilon > 0 \) there is \( t_\epsilon > 0 \) so that we have for all \( t \in (0, t_\epsilon] \) and all \( n \in \mathbb{N} \) with \( n \geq 2d \)

\[
\|\mathbb{P}\{(x_1, \ldots, x_n) | x_t \in [x_1, \ldots, x_n] \} - \mathbb{P}\{(z_1, \ldots, z_n) | z_t \in [z_1, \ldots, z_n] \}| \leq \epsilon.
\]
Proof. We show first that there is \( c > 1 \) so that we have (15) whenever \( n \leq \frac{\text{vol}_d(K)}{ct} \) or \( n \geq c \frac{\text{vol}_d(K)}{t} \). As \( c \) we can choose a number satisfying

\[
c \geq \max\{ \frac{1}{\epsilon}, d \}
\]

(16)

\[(d4c^{d+2})^{d+1}e^{-\frac{2}{c}} < \epsilon\]

\[2^{d-1}e^{-c_1c} < \epsilon\]

where \( c_1 \) is the constant introduced in Lemma 15.

We consider the case \( n \leq \frac{\text{vol}_d(K)}{ct} \). Since the curvature at \( x \) is strictly positive we may assume that the indicatrix of Dupin is a sphere. If we choose \( t_\epsilon \) small enough then by Lemma 14 there is a hyperplane \( H \) through \( x, 0 < t \leq t_\epsilon \), so that \( \text{vol}_d(K \cap H^-) \leq t \). Therefore we get

\[
1 \geq P_K \{ (x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n] \} \geq P_K \{ (x_1, \ldots, x_n) | \{x_1, \ldots, x_n\} \subseteq K \cap H^+ \}
\]

\[
\geq (1 - \frac{t}{\text{vol}_d(K)})^n \geq (1 - \frac{1}{cn})^n \geq 1 - \frac{1}{c}
\]

The same estimate holds for \( P_B \) and we get (15). Now we consider the case \( n \geq 2c \frac{\text{vol}_d(K)}{t} \). By Lemma 7 we get

\[
0 \leq P \{ (x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n] \}
\]

\[
\leq 2 \sum_{i=0}^{d-1} \binom{n}{i} \left( \frac{t}{2\text{vol}_d(K)} \right)^i (1 - \frac{t}{2\text{vol}_d(K)})^{n-i}
\]

The function \( s^i(1 - s)^{n-i} \) attains its maximum at \( \frac{1}{n} \). Since \( i < d, d \leq c, \) and \( 2d \leq n \) we get that the latter expression is less than

\[
\sum_{i=0}^{d-1} \binom{n}{i} \left( \frac{c}{n} \right)^i (1 - \frac{c}{n})^{n-i} \leq dc^d e^{-\frac{c}{2}} \leq \epsilon
\]

The same holds for \( P_B \) and we get (15) again.

Now we consider the case \( \frac{\text{vol}_d(K)}{ct} \leq n \leq c \frac{\text{vol}_d(K)}{t} \). By triangle inequality we get

\[
|P_K \{ (x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n] \} - P_B \{ (z_1, \ldots, z_n) | z_t \notin [z_1, \ldots, z_n] \}| \leq
\]

\[
|P \{ (x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n] \} - P \{ (x_1, \ldots, x_n) | x_t \notin [x_1, \ldots, x_n] \cap H^-] \}| \]

\[
+ |P \{ (z_1, \ldots, z_n) | z_t \notin [z_1, \ldots, z_n] \} - P \{ (z_1, \ldots, z_n) | z_t \notin [z_1, \ldots, z_n] \cap H^-] \}|
\]
respectively and \\
\[ |P_K \{(x_1, \ldots, x_n)|x_t \notin \{x_1, \ldots, x_n\} \cap H^-\}| - |P_B \{(z_1, \ldots, z_n)|z_t \notin \{z_1, \ldots, z_n\} \cap \tilde{H}^-\}| \]
\\
\[ + |P_B \{(z_1, \ldots, z_n)|z_t \notin \{z_1, \ldots, z_n\}\} - P_B \{(z_1, \ldots, z_n)|z_t \notin \{z_1, \ldots, z_n\} \cap \tilde{H}^-\}| \]
\\
where H and \( \tilde{H} \) are hyperplanes whose normals coincide with \( N(x) \) and \( N(z) \) respectively and \\
\( \text{vol}_d(K \cap H^-) = \text{vol}_d(B \cap \tilde{H}^-) = c^{d+1}t \). The first and third summands of the latter expression can be estimated by Lemma 15. We estimate \\
now the second summand. Again, we may assume that the indicatrix of Dupin \\
at \( x \in \partial K \) is a Euclidean sphere. Moreover, we may assume that the radius \\
of the indicatrix equals the radius of \( B \). This is done by a volume preserving, affine \\
transform. We have \\
\[ P_K \{(x_1, \ldots, x_n)|x_t \notin \{x_1, \ldots, x_n\} \cap H^-\} = \]
\\
\[ \sum_{k=0}^{n} \binom{n}{k} \left( \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K)} \right)^k (1 - \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K)})^{n-k} \text{P}_K \{x_l \notin \{x_1, \ldots, x_k\}\} \]
\\
and the same for \( P_B \). We get by Lemma 7 for \( k \geq 4dc^{d+2} \)
\\
\[ \text{P}_K \{x_l \notin \{x_1, \ldots, x_k\}\} \]
\\
\[ \leq 2 \sum_{i=0}^{d-1} \binom{k}{i} \left( \frac{1}{2c^{d+1}} \right)^i (1 - \frac{1}{2c^{d+1}})^{k-i} \leq 2dk^d e^{-\frac{1}{4}kc^{-d-1}} \]
\\
The function \( s^d e^{-as} \) attains its maximum at \( \frac{d}{a} \). Therefore, and because of \\
\( 4dc^{d+2} \leq k \) the last expression is smaller than \\
\[ 2d(d4c^{d+2})^d e^{-dc} < \epsilon \]
\\
We get the same for \( P_B \). Therefore we have \\
\[ |P_K \{(x_1, \ldots, x_n)|x_t \notin \{x_1, \ldots, x_n\} \cap H^-\}| - |P_B \{(z_1, \ldots, z_n)|z_t \notin \{z_1, \ldots, z_n\} \cap \tilde{H}^-\}| \]
\\
\[ \leq \sum_{0 \leq k \leq 4dc^{d+2}} \binom{n}{k} \left( \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K)} \right)^k (1 - \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K)})^{n-k} \]
\\
\[ |P_K \{(x_1, \ldots, x_n)|x_t \notin \{x_1, \ldots, x_n\}\} - P_B \{(z_1, \ldots, z_n)|z_t \notin \{z_1, \ldots, z_n\}\} | \leq 2\epsilon \]
On the other hand, if we choose \( t_\epsilon \) sufficiently small we have for all \( t \in (0, t_\epsilon) \) and all \( k, 0 \leq k \leq 4dc^{d+2} \)

\[
|P_{K \cap H^y\{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k]} - P_{B \cap H^y\{z_1, \ldots, z_k\} \in [z_1, \ldots, z_k]}| < \epsilon
\]

This finishes the proof. We establish now (18). For \( k = 0, \ldots, d \) the difference is trivially 0. Now we assume that \( x = z \) and \( N(x) = N(z) \). We have that

\[
P_{K \cap H} \{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k] \}
\]

\[
\sum_{m=0}^{k} \binom{k}{m} \left( \frac{\text{vol}_d(K \cap H^y \cap B \cap \tilde{H}^-)}{\text{vol}_d(K \cap H^-)} \right)^m (1 - \frac{\text{vol}_d(K \cap H^y \cap B \cap \tilde{H}^-)}{\text{vol}_d(K \cap H^-)})^{k-m}
\]

\[
P_{K \cap H^y \cap B \cap \tilde{H}^-} \{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k] \}
\]

If we choose \( t_\epsilon \) small enough we have by Lemma 14 for all \( t \in (0, t_\epsilon) \) that

\[
1 - \frac{\text{vol}_d(K \cap H^y \cap B \cap \tilde{H}^-)}{\text{vol}_d(K \cap H^-)}
\]

is so small that we get by (19) and \( k \leq 4dc^{d+2} \)

\[
|P_{K \cap H^y\{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k]} - P_{B \cap H^y\{z_1, \ldots, z_k\} \in [z_1, \ldots, z_k]}| \leq
\]

\[
P_{K \cap H^y \cap B \cap \tilde{H}^-} \{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k] \} + e^{-16dc}
\]

Moreover, we have

\[
P_{K \cap H^y \cap B \cap \tilde{H}^-} \{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k] \}
\]

\[
\geq P_{K \cap H^y \cap B \cap \tilde{H}^-} \{x_1, \ldots, x_k\} \mid \{x_1, \ldots, x_n\} \subset K \cap H^y(x_t, N(x)) \}
\]

\[
\geq (1 - e^{-d-1})4dc^{d+2} \geq e^{-8dc}
\]

The last inequality holds because we have \( 1 - \frac{1}{s} \geq e^{-2s} \) for \( s \geq 2 \). By (20) and (21) we get now

\[
P_{K \cap H^y\{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k]} - P_{B \cap H^y\{z_1, \ldots, z_k\} \in [z_1, \ldots, z_k]} | \leq (1 + \exp(-c))P_{K \cap H^y \cap B \cap \tilde{H}^-} \{x_1, \ldots, x_k\} \in [x_1, \ldots, x_k] \}
\]

Therefore we get now...
\[ \mathbb{P}_{K \cap H^-} \{ (x_1, \ldots, x_k) | x_t \notin [x_1, \ldots, x_k] \} \leq \]
\[ \mathbb{P}_{K \cap H^-} \{ (x_1, \ldots, x_k) | x_t \notin \{ x_1, \ldots, x_k \} \cap B \cap \tilde{H}^- \} \]
\[ \leq (1 + e^{-c}) \mathbb{P}_{K \cap H^- \cap B \cap \tilde{H}^-} \{ (x_1, \ldots, x_k) | x_t \notin [x_1, \ldots, x_k] \} \]
\[ \leq (1 + e^{-c}) \left( \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K \cap H^- \cap B \cap \tilde{H}^-)} \right)^k \]
\[ \mathbb{P}_{K \cap H^-} \{ (x_1, \ldots, x_k) | x_t \notin [x_1, \ldots, x_k] \} \text{ and } \{ x_1, \ldots, x_k \} \subset B \cap \tilde{H}^- \}
\[ \leq (1 + e^{-c}) \left( \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K \cap H^- \cap B \cap \tilde{H}^-)} \right)^k \mathbb{P}_{K \cap H^-} \{ (x_1, \ldots, x_k) | x_t \notin [x_1, \ldots, x_k] \} \]

Thus we get that
\[ |\mathbb{P}_{K \cap H^-} \{ (x_1, \ldots, x_k) | x_t \notin [x_1, \ldots, x_k] \} - \mathbb{P}_{K \cap H^- \cap B \cap \tilde{H}^-} \{ (x_1, \ldots, x_k) | x_t \notin [x_1, \ldots, x_k] \}| < \epsilon \]
if we choose \( c \) sufficiently big. We have the same inequality for \( \mathbb{P}_{B \cap \tilde{H}^-} \). This implies (18). \( \square \)

**Lemma 18.** Let \( K \) be a convex body in \( \mathbb{R}^d \) and \( x \in \partial K \). Suppose that \( \partial K \) is twice differentiable at \( x \) in the generalized sense. Then we have

(i) \( \lim_{t \to 0} \frac{\text{vol}_d(K \cap H^-)}{\text{vol}_d(K \cap H^- \cap B \cap \tilde{H}^-)} = 1 \)

(ii)

\[ \lim_{t \to 0} \frac{\text{vol}_{d-1}(P_{N(x_t)}((-x_t + K) \cap (x_t - K)))}{\text{vol}_{d-1}(B_2^{d-1})} = \kappa(x) \frac{1}{d+1} \left( \frac{2}{d+1} \right)^{d+1} \text{vol}_{d-1}(B_2^{d-1})^{-\frac{2}{d+1}} \]

**Proof.** (i) The same arguments as in the proof of Lemma 5 are applied. We just sketch the argument. Suppose (i) is not true. Then we find a supporting hyperplane \( H(x_t, N(x_t)) \) so that \( x_t \) is very close to \( x \) but \( N(x_t) \) is not close to \( N(x) \). By the assumption we have that all the points in the set \( H(x_t, N(x_t)) \cap K \) do not belong to the interior of \( K_t \). On the other hand, the volume \( \text{vol}_d(K \cap H^-(x_t, N(x_t))) \) is so big that we can single out a point in \( H(x_t, N(x_t)) \cap K \) that is in the interior of \( K_t \).

(ii) We consider the case \( \kappa(x) > 0 \). The case \( \kappa(x) = 0 \) is treated in an analogous way. By Lemma 14 \( K \) can be approximated by an ellipsoid. We may assume it is
a Euclidean sphere. By (i) \(< x, N(x) >\) is as close to \(< x, N(x) >\) as we choose it to be for small t. Altogether we have that

\[
vol_{d-1}(P_{N(x_t}}((-x_t + K) \cap (x_t - K)))
\]
is up to some error equal to

\[
vol_{d-1}(H(x_t, N(x)) \cap K)
\]
or

\[
vol_{d-1}(H(x_t, N(x)) \cap B_{2^d}(x - N(x), \kappa(x) - \frac{1}{\pi^2} N(x))
\]

It is left to apply Lemma 4. \(\Box\)

**Proof of Theorem 1.** We may assume that \(K_T\) coincides with the origin. By Lemma 3 and 8 we have

\[
\lim_{n \to \infty} \frac{vol_d(K) - \mathbb{E}(K, n)}{(\frac{1}{n})^{\frac{d}{d+1}}}
\]

\[
\lim_{n \to \infty} n^{\frac{2}{d+1}} \int_{\partial K} \int_0^{\frac{t}{n}} \frac{\mathbb{P}\{(x_1, \ldots, x_n) \mid x_t \notin [x_1, \ldots, x_n]\} \|x_t\|^d < x, N(x) >}{vol_{d-1}(P_{N(x)}(\triangle x_t + K) \cap (x_t - K))} \|x\|^d < x_t, N(x_t) > dt d\mu
\]

provided the limit exists. We apply now Lebesgue’s convergence theorem in order to change limit and integration over \(\partial K\). The hypothesis of Lebesgue’s theorem is fulfilled because of Lemma 6 and 10. By Lemma 12 we get that the latter expression equals

\[
\int_{\partial K} \lim_{n \to \infty} n^{\frac{2}{d+1}} \int_0^{\frac{t}{n}} \frac{\mathbb{P}\{(x_1, \ldots, x_n) \mid x_t \notin [x_1, \ldots, x_n]\} \|x_t\|^d < x, N(x) >}{vol_{d-1}(P_{N(x)}(\triangle x_t + K) \cap (x_t - K))} \|x\|^d < x_t, N(x_t) > dt d\mu
\]

By Lemma 18 this expression equals

\[
\int_{\partial K} \lim_{n \to \infty} n^{\frac{2}{d+1}} \int_0^{\frac{t}{n}} \frac{\mathbb{P}\{(x_1, \ldots, x_n) \mid x_t \notin [x_1, \ldots, x_n]\}}{vol_{d-1}(P_{N(x)}(\triangle x_t + K) \cap (x_t - K))} dt d\mu
\]

By Lemma 18 (ii) we get

\[
\int_{\partial K} \kappa(x) \frac{1}{d+1} d\mu \lim_{n \to \infty} n^{\frac{2}{d+1}} \frac{(\frac{2}{d+1})^{\frac{d-1}{d+1}}}{vol_{d-1}(B_{2^d}(x - \frac{1}{\pi^2} N(x), \kappa(x) - \frac{1}{\pi^2} N(x))} \int_0^{\frac{t}{n}} \frac{\mathbb{P}\{(x_1, \ldots, x_n) \mid x_t \notin [x_1, \ldots, x_n]\}}{\kappa(x)^{\frac{1}{d+1}}} \|x\|^d < x, N(x) > dt
\]

By Lemma 17 we have for \(x \in \partial K\) with \(\kappa(x) > 0\)

\[
\lim_{n \to \infty} n^{\frac{2}{d+1}} \int_0^{\frac{t}{n}} \frac{\mathbb{P}\{(x_1, \ldots, x_n) \mid x_t \notin [x_1, \ldots, x_n]\}}{\kappa(x)^{\frac{1}{d+1}}} \|x\|^d < x, N(x) > dt
\]
\[
= \lim_{n \to \infty} n^{\frac{2}{d+1}} \int_0^{\log n} t^{-\frac{d-1}{d+1}} \mathbb{P}_B \{(z_1, \ldots, z_n) | z_t \notin [z_1, \ldots, z_n]\} \, dt
\]

where \(B\) is a Euclidean ball whose volume is the same as that of \(K\). The limit for \(B\) exists by Lemma 13. Thus we get

\[
\lim_{n \to \infty} \frac{\text{vol}_d(K) - \mathbb{E}(K, n)}{(\frac{1}{n})^{\frac{2}{d+1}}} = \int_{\partial K} \kappa(x) \frac{1}{d+1} \, d\mu \lim_{n \to \infty} n^{\frac{2}{d+1}} \frac{\left(\frac{2}{d+1}\right)^{\frac{d+1}{d+1}}}{\text{vol}_{d-1}(B_2^{d-1})} \int_0^{\log n} \mathbb{P}_B \{(z_1, \ldots, z_n) | z_t \notin [z_1, \ldots, z_n]\} \, dt
\]

Since this formula holds for all convex bodies it holds in particular for the Euclidean ball. By Lemma 13 we determine the coefficient. \(\square\)

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