On a non-vanishing Ext

LASZLO FUCHS and SAHARON SHELAH

Abstract. The existence of valuation domains admitting non-standard uniserial modules for which certain Ext’s do not vanish was proved in [1] under Jensen’s Diamond Principle. In this note, the same is verified using the ZFC axioms alone.

In the construction of large indecomposable divisible modules over certain valuation domains $R$, the first author used the property that $R$ satisfied $\text{Ext}^1_R(Q, U) \neq 0$, where $Q$ stands for the field of quotients of $R$ (viewed as an $R$-module) and $U$ denotes any uniserial divisible torsion $R$-module, for instance, the module $K = Q/R$; see [1]. However, both the existence of such a valuation domain $R$ and the non-vanishing of Ext were established only under Jensen’s Diamond Principle ♦ (which holds true, e.g., in Gödel’s Constructible Universe).

Our present goal is to get rid of the Diamond Principle, that is, to verify in ZFC the existence of valuation domains $R$ that admit divisible non-standard uniserial modules and also satisfy $\text{Ext}^1_R(Q, U) \neq 0$ for several uniserial divisible torsion $R$-modules $U$. (For the proof of Corollary 3, one requires only 6 such $U$.)

We start by recalling a few relevant definitions. By a valuation domain we mean a commutative domain $R$ with 1 in which the ideals form a chain under inclusion. A uniserial $R$-module $U$ is defined similarly as a module whose submodules form a chain under inclusion. $K = Q/R$ is a uniserial torsion $R$-module, it is divisible in the sense that $rK = K$ holds for all $0 \neq r \in R$. A divisible uniserial $R$-module is called standard if it is an epic image of the uniserial module $Q$; otherwise it is said to be non-standard. The existence of valuation domains admitting non-standard uniserials has been established in ZFC; see e.g. [3], [2], and the literature quoted there.

As the $R$-module $Q$ is uniserial, it can be represented as the union of a well-ordered ascending chain of cyclic submodules:

$$R = Rr_0 \subset Rr_1^{-1} \subset \ldots \subset Rr_\alpha^{-1} \subset \ldots \subset \bigcup_{\alpha < \kappa} Rr_\alpha^{-1} = Q \quad (\alpha < \kappa),$$

where $r_0 = 1$, $r_\alpha \in R$, and $\kappa$ denotes an infinite cardinal and also the initial ordinal of the same cardinality. As a consequence, $K = \bigcup_{\alpha < \kappa} (Rr_\alpha^{-1}/R)$ where $Rr_\alpha^{-1}/R \cong $
$R/R_\alpha$ are cyclically presented $R$-modules. We denote by $\nu^\beta_\alpha : Rr_{\alpha}^{-1}/R \to Rr_{\beta}^{-1}/R$ the inclusion map for $\alpha < \beta$, and may view $K$ as the direct limit of its submodules $Rr_{\alpha}^{-1}/R$ with the monomorphisms $\nu^\beta_\alpha$ as connecting maps.

A uniserial divisible torsion module $U$ is a ‘clone’ of $K$ in the sense of Fuchs-Salce [2], if there are units $e^\beta_\alpha \in R$ for all pairs $\alpha < \beta (< \kappa)$ such that

$$e^\beta_\alpha e^\kappa_\beta - e^\kappa_\alpha \in Rr_\alpha \quad \text{for all } \alpha < \beta < \gamma < \kappa,$$

and $U$ is the direct limit of the direct system of the modules $Rr_{\alpha}^{-1}/R$ with connecting maps $\nu^\beta_\alpha e^\beta_\alpha : Rr_{\alpha}^{-1}/R \to Rr_{\beta}^{-1}/R (\alpha < \beta)$; i.e. multiplication by $e^\beta_\alpha$ followed by the inclusion map. It might be helpful to point out that though $K$ and $U$ need not be isomorphic, they are ‘piecewise’ isomorphic in the sense that they are unions of isomorphic pieces.

Let $R$ denote the valuation domain constructed in the paper [1] (see also Fuchs-Salce [2]) that satisfies Ext$^1_R(Q,K) \neq 0$ in the constructible universe $L$. Moreover, there are clones $U_n$ of $K$, for any integer $n > 0$, that satisfy Ext$^1_R(Q,U_n) \neq 0$: for convenience, we let $K = U_0$.

This $R$ has the value group $\Gamma = \oplus_{\alpha < \omega_1} \mathbb{Z}$, ordered anti-lexicographically, and its quotient field $Q$ consists of all formal rational functions of $u^\gamma$ with coefficients in an arbitrarily chosen, but fixed field, where $u$ is an indeterminate and $\gamma \in \Gamma$. It is shown in [2] that such an $R$ admits divisible non-standard uniserials (i.e. clones of $K$ non-isomorphic to $K$), and under the additional hypothesis of $\diamondsuit_{\kappa_1}$, Ext$^1_R(Q,U_n) \neq 0$ holds; in other words, there is a non-splitting exact sequence

$$0 \to U_n \to H_n \xrightarrow{\phi} Q \to 0.$$

Using the elements $r_\alpha \in R$ introduced above, for each $n < \omega$ we define a tree $T_n$ of length $\kappa$ whose set of vertices at level $\alpha$ is given by

$$T_{n\alpha} = \{ x \in H_n \mid \phi(x) = r_\alpha^{-1} \}.$$

The partial order $<_T$ is defined in the following way: $x <_T y$ in $T_n$ if and only if, for some $\alpha < \beta$, we have $\phi x = r_\alpha^{-1}$ and $\phi y = r_\beta^{-1}$ such that

$$x = r_\alpha^{-1} r_\beta y \quad \text{in } H_n,$$

where evidently $r_\alpha^{-1} r_\beta \in R$.

Fix an integer $n > 0$, and define $T$ as the union of the trees $T_0, T_1, \ldots, T_n$ with a minimum element $z$ adjoined. It is straightforward to check that $(T,<_T)$ is indeed a tree with $\kappa$ levels, and the inequalities Ext$^1_R(Q,U_i) \neq 0 (i = 0, 1, \ldots, n)$ guarantee that $T$ has no branch of length $\kappa$.

We now define a model $\mathcal{M}$ as follows. Its universe is the union of the universes of $R, Q, U_i, H_i$ ($i = 0, \ldots, n$), and it has the following relations:

(i) unary relations $R, Q, U_i, H_i, T$, and $S = \{ r_\alpha \mid \alpha < \kappa \}$,

(ii) binary relation $<_T$, and $<_S$ (which is the natural well-ordering on $S$),
(iii) individual constants 0_{R}, 0_{Q}, 0_{U_{i}}, 1_{R}, and

(iv) functions: the operations in $R, Q, U_{i}, H_{i}$, where $R$ is a domain, $Q, U_{i}, H_{i}$ are $R$-modules, $\phi_{i}$ is an $R$-homomorphism from $H_{i}$ onto $Q$ ($i = 0, \ldots, n$, and $\psi : Q \rightarrow K$ with $\ker \psi = R$ is the canonical map.

We argue that even though our universe $V$ does not satisfy $V=L$, the class $L$ does satisfy it, and so in $L$ we can define the model $M$ as above. Let $T$ be the first order theory of $M$. So the first order (countable theory) $T$ has in $L$ a model in which

$$(\ast)_{M}$$
the tree $(T, <_{T})$ with set of levels $(S, <_{S})$ and with the function $\phi = \cup \phi_{i}$ giving the levels, as interpreted in $M$, has no full branch (this means that there is no function from $S$ to $T$ increasing in the natural sense and inverting $\phi_{i}$, or any $\phi_{i}$).

Hence we conclude as in [3] (by making use of Shelah [4]) that there is a model $M'$ with those properties in $V$ (in fact, one of cardinality $\aleph_{1}$).

Note that $(S, <_{S})$ as interpreted in $M'$ is not well ordered, but it is still a linear order of uncountable cofinality (in fact, of cofinality $\aleph_{1}$), the property $(\ast)_{M'}$ still holds, and it is a model of $T$. This shows that all relevant properties of $M$ in $L$ hold for $M'$ in $V$, just as indicated in [3].

It should be pointed out that, as an alternative, instead of using a smaller universe of set theory $L$, we could use a generic extension not adding new subsets of the natural numbers (hence essentially not adding new countable first order theories like $T$).

If we continue with the same argument as in [3], then using [4] we can claim that we have proved in ZFC the following theorem:

**Theorem 1.** There exist valuation domains $R$ admitting non-standard uniserial torsion divisible modules such that $\Ext_{R}^{1}(Q, U_{i}) \neq 0$ for various clones $U_{i}$ of $K$.

Hence we derive at once that the following two corollaries are true statements in ZFC; for their proofs we refer to [2].

**Corollary 2.** There exist valuation domains $R$ such that if $U, V$ are non-isomorphic clones of $K$, then $\Ext_{R}^{1}(U, V)$ satisfies:

(i) it is a divisible mixed $R$-module;

(ii) its torsion submodule is uniserial.

More relevant consequences are stated in the following corollaries; they solve Problem 27 stated in [2].

**Corollary 3.** There exist valuation domains admitting indecomposable divisible modules of cardinality larger than any prescribed cardinal.

**Corollary 4.** There exist valuation domains with superdecomposable divisible modules of countable Goldie dimension.
References

[1] L. Fuchs, Arbitrarily large indecomposable divisible torsion modules over certain valuation domains, Rend. Sem. Mat. Univ. Padova 76 (1986), 247-254.

[2] L. Fuchs and L. Salce, Modules over non-Noetherian Domains, Math. Surveys and Monographs, vol. 84, American Math. Society (Providence, R.I., 2001).

[3] L. Fuchs and S. Shelah, Kaplansky’s problem on valuation rings, Proc. Amer. Math. Soc. 105 (1989), 25-30.

[4] S. Shelah, Models with second order properties. II. Trees with no undefined branches, Ann. of Math. Logic 14 (1978), 73-87.

Department of Mathematics
Tulane University, New Orleans, Louisiana 70118, USA
e-mail: fuchs@tulane.edu

and

Department of Mathematics
Hebrew University, Jerusalem, Israel 91904
and Rutgers University, New Brunswick, New Jersey 08903, USA
e-mail: shelah@math.huji.ac.il