Optimizing Optimal Reduction: 
A Type Inference Algorithm for Elementary Affine Logic

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February 7, 2008

Introduction

The optimal reduction of \( \lambda \)-terms ([Lèv80]; see [AG98] for a comprehensive account and references) is a graph-based technique for normalization in which a redex is never duplicated. To achieve this goal, the syntax tree of the term is transformed into a graph, with an explicit node (fan) expressing the sharing of two common subterms (these subterms are always variables in the initial translation of a \( \lambda \)-term). Giving correct reduction rules for these sharing graphs is a surprisingly difficult problem, first solved in [Kat90, Lam90]. One of the main issues is to decide how to reduce two meeting fans, for which a complex machinery and new nodes have to be added (the oracle). There is large class of (typed) terms, however, for which this decision is very simple, namely those \( \lambda \)-terms whose sharing graph is a proof-net of Elementary Logic, both in the Linear [Gir98] (ELL) and the Affine [Asp98] (EAL) flavor. This fact was first observed in [Asp98] and then exploited in [ACM00] to obtain a certain complexity result on optimal reduction, where (following [Mai92]) we also showed that these \textit{EAL-typed} \( \lambda \)-terms are powerful enough to encode arbitrary computations of elementary time-bounded Turing machines. We did not know, however, of any systematic way to derive EAL-types for \( \lambda \)-terms, a crucial issue if we want to exploit in an optimal reducer the added benefits of this class of terms. This is what we present in this paper.

Main contribution of the paper is a type inference algorithm (Section 2), assigning EAL-types (formulas) to \textit{type-free} \( \lambda \)-terms (more precisely: to sharing graphs corresponding to type-free \( \lambda \)-terms). We will see in Section 4 that a typing inference for a \( \lambda \)-term \( M \) in EAL consists of a \textit{skeleton} – given by the assignment of a type to \( M \) in the simple type discipline – together with a \textit{box assignment}, essential because EAL allows contraction only on boxed terms. The
algorithm tries to introduce all possible boxes by collecting integer linear constraints during the exploration of the syntax tree of $M$. At the end, the integer solutions (if any) to the constraints give specific box assignments (i.e., EAL-derivations) for $M$. Correctness and completeness of the algorithm are proved with respect to a natural deduction system for EAL, introduced in Section 3.1 together with terms annotating the derivations.

The technique used in the paper, with minor modifications, can be used to obtain linear logic derivations as decorations of intuitionistic derivations, subsuming some of the results of [DJS95, Sch94]. In this way we may obtain linear derivations with a minimal number of boxes. We tackle this issue in Section 4.1.

A preliminary version of this work has already been published [CM01]. Besides giving more elaborated examples and technical details, several results are new. We prove that all EAL types can be obtained by applying the algorithm on the simple principal type schema; as a corollary, we may state the decidability of the type inference problem for EAL. We show how to use our technique to decorate full linear logic proofs. We show how the algorithm could be extended to allow arbitrary contractions.

In [CR03], the existence of a notion of principal type schema for EAL is investigated and established. Baillot [Bai02] gives a type-checking algorithm for Light Affine Logic, but it applies only to lambda terms in normal form. In [Bai03] the same author proves the decidability of LAL type inference problem for lambda-calculus following the approach proposed in [CR03].

1 Elementary Affine Logic

Elementary Affine Logic [Asp98] is a system with unrestricted weakening, where contraction is allowed only for modal formulas. There is only one exponential rule for the modality $!$ (of-course, or bang), which is introduced at once on both sides of the turnstile. The system is presented in Figure 1, where also $\lambda$-terms are added to the rules. We denote with $M[N/x]$ the usual notion of substitution of $N$ for the free occurrences of $x$ in $M$. In the contexts (or bases) $(\Gamma, \Delta, \text{etc.})$ a variable can occur only once (they are linear). Observe that, according to most literature on optimal reduction, we always write parenthesis around an application and we assume that the scope of a $\lambda$ is the minimal subterm following the dot; as a consequence, a term like $(\lambda x.M N)$ should be parsed as $((\lambda x.M)N)$. Cut-elimination may be proved for EAL in a standard way.

Given the sharing graph of a type-free $\lambda$-term, we are interested in finding a derivation of a type for it, according to Figure 1 (There is a subtle point in this notion, which is relevant for the completeness of our algorithm and which we will discuss at the end of this section. For the time being we may remain informal).

A simple inspection of the rules of EAL shows that any $\lambda$-term with an EAL type has also a simple type. Indeed, the simple type (and the corresponding
A ⊢ x : A

Γ ⊢ M : B

Γ, x : A ⊢ M : B

Γ, x : A ⊢ M : B

Γ ⊢ N : A, x : A, Δ ⊢ M : B

Γ, Δ ⊢ M/{N/x} : B

Γ, Δ ⊢ M/{N/x} : B

cut

weak

contr

Figure 1: (Implicational) Elementary Affine Logic

 derivation) is obtained by forgetting the exponentials, which must be present in an EAL derivation because of contraction. Therefore, in looking for an EAL-type for a λ-term M, we can start from a simple type derivation for M and try to decorate this derivation (i.e., add !-rules) to turn it into an EAL-derivation. Our algorithm implements this simple idea:

1. we find all “maximal decorations”;

2. these decorations correspond to well formed derivations only if certain linear constraints admit (integral) solutions.

We informally present the main point with an example on the term 

\[ \lambda xy. (x(x y)) \]

One simple type derivation for \( \text{two} \) (expressed as a sequent derivation) is:

\[
\begin{array}{ccc}
\text{w:0} & \vdash \text{w:0} \\
\text{x:a} & \vdash \text{y:a} \vdash \text{y:a} \\
\text{x:a} & \vdash \text{x:a} \vdash \text{y:a} \vdash \text{y:a} \\
\text{x:a} & \vdash \text{x:a} \vdash \text{y:a} \vdash \text{y:a} \\
\text{\vdash \lambda y.(x(x y));a:a} \\
\text{\vdash x:a} & \vdash \text{\lambda y.(x(x y));a:a} \\
\text{\vdash x:a} & \vdash \text{\lambda y.(x(x y));a:a} \\
\text{\vdash \lambda y.(x(x y));a:a} \\
\end{array}
\]

If we change every \( \rightarrow \) in \( \vdash \), the previous derivation can be viewed as the skeleton of an EAL derivation. To obtain a full EAL derivation (if any), we need to decorate this skeleton with exponentials, and to check that the contraction is performed only on exponential formulas.

We first produce a maximal decoration of the skeleton, interleaving \( n \) !-rules after each logical rule. For instance

\[
\begin{array}{ccc}
\text{w:0} & \vdash \text{w:0} \\
\text{x:a} & \vdash \text{y:a} \vdash \text{y:a} \\
\end{array}
\]
These constraints are equivalent to the last one from the fact that contraction is allowed only on exponential formulas. The second, third and fourth of these constraints come from unification; the last one becomes representing all EAL derivations with $n_1, n_2 \in \mathbb{N}$.

Continuing to decorate the skeleton of two (i.e., to interleave $!$-rules) we obtain

\[
\begin{array}{c}
\text{EC} \vdash w : \alpha \\
\text{EC} \vdash y : \alpha \\
\end{array}
\]

\[
\begin{array}{c}
\text{EC} \vdash x : \alpha \rightarrow \text{EC} \vdash y : \alpha \\
\text{EC} \vdash x : \alpha \\
\end{array}
\]

\[
\begin{array}{c}
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash x : \alpha \\
\end{array}
\]

\[
\begin{array}{c}
\text{EC} \vdash \lambda y. \alpha \rightarrow \text{EC} \vdash \lambda y. \alpha \\
\text{EC} \vdash \lambda y. \alpha \rightarrow \text{EC} \vdash \lambda y. \alpha \\
\text{EC} \vdash \lambda y. \alpha \\
\end{array}
\]

The last rule—contraction—is correct in EAL iff the types of $x$ are unifiable and banged. In other words iff the following constraints are satisfied:

\[
\begin{array}{c}
n_1, n_2, n_3, n_4, n_5, n_6 \in \mathbb{N} \\
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash \lambda y. \alpha \\
\end{array}
\]

\[
\begin{array}{c}
n_5 = n_3 + n_5 \\
n_1 + n_3 = n_2 \\
n_4 = n_1 \\
n_5 + n_6 \geq 1.
\end{array}
\]

The second, third and fourth of these constraints come from unification; the last one from the fact that contraction is allowed only on exponential formulas. These constraints are equivalent to

\[
\begin{array}{c}
n_1, n_5, n_6 \in \mathbb{N} \\
n_3 = 0 \\
n_1 = n_2 = n_4 \\
n_5 + n_6 \geq 1.
\end{array}
\]

Since clearly these constraints admit solutions, we conclude the decoration procedure obtaining

\[
\begin{array}{c}
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash x : \alpha \\
\text{EC} \vdash \lambda y. \alpha \\
\end{array}
\]

\[
\begin{array}{c}
\text{EC} \vdash \lambda y. \alpha \rightarrow \text{EC} \vdash \lambda y. \alpha \\
\text{EC} \vdash \lambda y. \alpha \rightarrow \text{EC} \vdash \lambda y. \alpha \\
\text{EC} \vdash \lambda y. \alpha \\
\end{array}
\]

Thus two has EAL types $\alpha \rightarrow \alpha$, $\alpha \rightarrow \alpha$, and $\alpha \rightarrow \alpha$, for any $n_1, n_5, n_6$ solutions of

\[
\begin{array}{c}
n_1, n_5, n_6 \in \mathbb{N} \\
n_5 + n_6 \geq 1.
\end{array}
\]

While simple and appealing, the technique of maximal decoration cannot be applied directly. The first problem is that sequent derivations are too constrained. There are many different (simple type) derivations for the same $\lambda$-term, depending on the position of $(\rightarrow L)$ rules, contractions, cuts, etc. Given a $\lambda$-term, we should therefore produce all possible derivations, and then decorate them. The problem stems from the fact that sequent derivations are not
driven by the syntax of the term. In fact, the standard simple type inference algorithm does not use a sequent-style presentation, but a natural deduction one, which is naturally syntax-driven. This is the solution we also follow in this paper — we decorate the λ-term. Unfortunately, it is well known (see Prawitz’s classical essay [Pra65]) that natural deduction for modal systems behave badly, since the obvious formulation for the modal rule (the one coinciding with rule ! of the sequent presentation) does not enjoy a substitution lemma. As a result, there are EAL type inferences which cannot be obtained directly as decoration of simple type derivations in natural deduction. Consider, for instance, the following simple type derivation (in the obvious natural deduction presentation of implicational logic) for \( M = \lambda x y k.(x y) : (A \to B) \to A \to (C \to B) \):

\[
\begin{align*}
x : A \to B & \vdash x : A \to B \\
y : A, k : C & \vdash y : A
\end{align*}
\]

\[
\begin{align*}
x : A & \vdash y : A, k : C \vdash (x y) : B
\end{align*}
\]

\[
\begin{align*}
x : A & \vdash (x y) : B \\
\vdash \lambda y k.(x y) : A \to (C \to B)
\end{align*}
\]

It is not difficult to see that in the system of Figure 1 there is a derivation establishing \( \vdash M : (A \to !B) \to A \to !((C \to B)) \). But no interleaving of ! rules into the derivation above can give this conclusion.

Indeed, to guarantee a substitution lemma, the modal rule for EAL in natural deduction must be formulated:

\[
\begin{array}{c}
\Delta_1 \vdash !A_1 \quad \ldots \quad \Delta_n \vdash !A_n \\
A_1, \ldots, A_n \vdash B
\end{array} \quad \Rightarrow \quad \Delta_1, \ldots, \Delta_n \vdash !B \quad \text{box}
\]

This rule, given a derivation of \( A_1, \ldots, A_n \vdash B \) (i.e., a λ-term \( M \) with the assignment of the type \( B \) from the basis \( A_1, \ldots, A_n \)): (i) “builds a box” around \( M \); (ii) allows the substitution of arbitrary terms for the free variables of \( M \).

Our algorithm will start from a simple type derivation in natural deduction for a term \( M \) (i.e., the syntax tree of the term decorated with simple types) and will try to insert (all possible) boxes around (suitable) subterms. We will sometimes use a graphical representation of this process. As an example, Figure 2 shows the decoration of the syntax tree of \( two \) we obtained in Section 1.

We are finally in the position to introduce formally the notion of EAL-typing for λ-terms. Recall that our main goal is to mechanically check whether a pure λ-term could be optimally reduced without the need of the oracle. While we lack a general characterization of this class of terms, we know that it contains any sharing graph coding the skeleton of a sequent proof in EAL. We already observed, however, that a single λ-term may correspond to more than one (sequent or natural deduction) proof. The position of the contraction is especially relevant in this context. Indeed, consider the term \( M = \lambda z x w.(x z) (x z) w \). Among the (infinite) EAL sequent derivations having \( M \) as a skeleton consider
the following two fragments:

\[
\begin{align*}
& \vdash z_1 : a, z_2 : a, x_1 : a \rightarrow (b \rightarrow b), x_2 : a \rightarrow (b \rightarrow b), w : b \vdash ((x_1 z_1) ((x_2 z_2) w)) : b \\
& \vdash a, z_1 : a, x_1 : !a \rightarrow (b \rightarrow b), x_2 : !a \rightarrow (b \rightarrow b), w : b \vdash ((x_1 z_1) ((x_2 z_2) w)) : !b \\
& \vdash z : a, x : !a \rightarrow (b \rightarrow b) \vdash \lambda w.((x) z ((x) z) w) : !(b \rightarrow !b) \rightarrow R
\end{align*}
\]

and

\[
\begin{align*}
& \vdash k_1, k_2 : b \rightarrow b \vdash \lambda w.(k_1 \ (k_2 w)) : b \rightarrow b \\
& \vdash k_1, k_2 : !b \rightarrow b \vdash \lambda w.(k_1 \ (k_2 w)) : !b \rightarrow b \\
& \vdash z : a \vdash z : a \\
& \vdash k : !b \rightarrow b \vdash \lambda w.(k \ (k w)) : !b \rightarrow b \\
& \vdash a, x : a \rightarrow (b \rightarrow b) \vdash \lambda w.((x) z ((x) z) w) : !b \rightarrow b \rightarrow L
\end{align*}
\]

If we display these derivations as annotated syntax tree with explicit fan nodes for contraction (that is, as sharing graphs), we obtain Figure 3 for the derivation (1), and Figure 4 for (2).

Both graphs are legal EAL sharing graphs, but only the first is a possible initial translation of \( M \) as a sharing graph, since in initial translations the fan nodes are used to share (contract) only variables, before abstracting them. Although our technique could be extended to cope with arbitrary contractions (see Section 4), we present it as a type inference algorithm for initial translations of type-free \( \lambda \)-terms, according to our original aim to use it as a tool in an optimal reducer. This is the motivation for the following notion.

**Definition 1.** A type-free \( \lambda \)-term \( M \) has EAL type \( A \) from the basis \( \Gamma \) (write: \( \Gamma \vdash _{EAL} M : A \)) iff there is a derivation of \( \Gamma \vdash M : A \) in the system of Fig-
Figure 3: One decoration of $\lambda z \ x \ w. ((x \ z) \ ((x \ z) \ w))$: the fan faces a lambda.

Figure 4: Another decoration of $\lambda z \ x \ w. ((x \ z) \ ((x \ z) \ w))$: the fan faces an application.
Remark 1. It is possible to formulate the previous definition directly in terms of sequent derivations, without any reference to the notion of sharing graph. It could be proved that \( \Gamma \vdash EAL M : A \) if and only if there is a sequent derivation of \( \Gamma \vdash M : A \) where all contractions either are immediately followed by \(-\circ R\), or are at the end of the derivation. However, the “only if” part is not trivial. In going from a sequent derivation to a sharing graph, in fact, we lose any information regarding the position of cuts and (to some extent) of \(-\circ L\). Therefore, given a term \( M \) for which \( \Gamma \vdash EAL M : A \) (that is, given a sharing graph that could be decorated with EAL-types and boxes) there are many sequent derivations corresponding to the skeleton coded by this sharing graph. Not all these derivations satisfy the constraint expressed by the “only if” part. It can be shown, however, that among these derivations there is one in which the constraint is satisfied. This could be obtained by using the notion of canonical form of an EAL derivation, introduced and exploited in [CR03].

Remark 2. There exist simply typeable terms without any EAL type. For instance the \( \lambda \)-term

\[
(\lambda n.(n \lambda y.(n \lambda z.z)) \lambda x.(x y))
\]

has a simple type, but no EAL decoration (see Appendix A for an analysis).

2 Type inference

The type inference algorithm is given as a set of inference rules, specifying several functions. The complete set of rules is given in Section 2.2; the properties of the algorithm will be stated and proved in Section 3. We start in the next section with the detailed discussion of an example, which will also introduce the various rules and the problems they have to face.

2.1 Example of type inference

A class of types for an EAL-typeable term can be seen as a decoration of a simple type with a suitable number of boxes.

Definition 2. A general EAL-type \( \Theta \) is generated by the following grammar:

\[
\Theta ::= n_1 \cdots + n_k \circ | \Theta \rightarrow \Theta
\]

where \( k \geq 0 \) and \( n_1, \ldots, n_k \) are variables ranging on \( \mathbb{N} \).

We shall illustrate our algorithm on the term \( (\lambda n.\lambda y.(n \lambda z.z) y) \lambda x.(x x \lambda w.w)) : \alpha \rightarrow \alpha \), whose simple type derivation in natural deduction is given in Figure 8 (\( I_\alpha \) stands for \( \alpha \rightarrow \alpha \)).
\[
\frac{
\begin{array}{l}
n : I_{\ell_1} \rightarrow I_{\ell_2} \vdash n : I_{\ell_2} \\
\vdash \lambda z. : I_{\ell_1} \\
\vdash z : I_{\ell_1} \\
\vdash z : I_{\ell_2}
\end{array}
}{
\begin{array}{l}
n : I_{\ell_1} \rightarrow I_{\ell_2} \vdash (n \mapsto z) : I_{\ell_2}
\end{array}
}\]

\[
\frac{
\begin{array}{l}
y : o \vdash y : o
\end{array}
}{
\begin{array}{l}
x : I_{\ell_1} \vdash x : I_{\ell_2}
\end{array}
}\]

\[
\frac{
\begin{array}{l}
\vdash \lambda n. \lambda y. ((n \mapsto z) y) : o
\end{array}
}{
\begin{array}{l}
\vdash \lambda n. \lambda y. ((n \mapsto z) y) : I_{\ell_2}
\end{array}
}\]

\[
\frac{
\begin{array}{l}
x : I_{\ell_1} \vdash x : I_{\ell_2}
\end{array}
}{
\begin{array}{l}
\vdash \lambda x. (x \mapsto (x \lambda w. w)) : I_{\ell_1} \rightarrow I_{\ell_2}
\end{array}
}\]

Figure 5: Simple type derivation of \((\lambda n. \lambda y. ((n \mapsto z) y) \lambda x. (x \mapsto (x \lambda w. w))) : o \rightarrow o\)

The algorithm searches for the leftmost innermost subterm for which there is no assignment of an EAL-type yet. In this case, it is the variable

\[n: ((o \rightarrow o) \rightarrow (o \rightarrow o)) \rightarrow (o \rightarrow o)\]

Its most general EAL-type is obtained from its simple type by adding \(p_i\) modalities wherever possible. This is the rôle of the function \(\mathcal{P}\):

\[\mathcal{P}(o) = \! p_0\] (3)

\[\mathcal{P}(\sigma) = \Theta \quad \mathcal{P}(\tau) = \Gamma\]

\[\mathcal{P}(\sigma \rightarrow \tau) = \! p_i(\Theta \rightarrow \Gamma)\] (4)

The main function of the algorithm—the type synthesis function \(\mathcal{S}\)—may now be applied. In the case of a variable \(x\) of simple type \(\sigma\) the rule is:

\[\mathcal{S}(x : \sigma) = (\Theta, \{x : \Theta\}, \emptyset, \emptyset)\] (5)

Observe that, given a term \(M\) of simple type \(\sigma\), \(\mathcal{S}(M : \sigma)\) returns a quadruple:

\(<\text{general EAL-type, base}^1 \{x_i : \Theta_i\}\text{; of pairs (variable:general EAL-type), set of line constraints, critical points}^2\>\).

In our example we obtain:

\[n : I_{p_1} (I_{p_2} (I_{p_3} (I_{p_4} o \rightarrow s p_5 o) \rightarrow s p_6 (I_{p_7} o \rightarrow s p_8 o)) \rightarrow s p_9 (I_{p_{10}} o \rightarrow s p_{11} o))\] (6)

for any \(p_i \in \mathbb{N}, 1 \leq i \leq 11\). In the following we will not explicit the “\(\in \mathbb{N}\)” for any variable we will introduce, being this constraint implicated by Definition.\(^2\)

**Notation 1.** We will write \((n \mapsto m)\) instead of \((!^n o \rightarrow \!^m o)\), for a better reading.

Analogously, \(z : (o \rightarrow o)\) is typed

\[z : p_{12}(p_{13} \circ p_{14})\] (7)

It is now the turn of the subterm \(\lambda z.\). The type synthesis rule for an abstraction \(\lambda x.\) takes the following steps:

\(^1\)A base here is a *multiset* where multiple copies of \(x : \Theta\) may be present.

\(^2\)We will discuss critical points in a moment.
1. infer the EAL-type for $M$;

2. add all possible boxes around $M$ (function $B$, which will be described later); the algorithm tries to build all possible decorations\(^3\) that in the case of an abstraction $\lambda x.M$ are the decorations of all subterms of $M$, already build by inductive hypothesis, plus all possible box-decorations of the whole $M$, performed at this stage of the inference by function $B$, plus all possible box decorations of $\lambda x.M$, eventually performed at the next step of the inference procedure;

3. contract all the types of abstracted variable $x$ (function $C$, which will be described later).

The rule is the following:

\[
C(\Theta_1, \ldots, \Theta_h) = A_3
\]

\[
B(M, B_1, \Gamma_1, cpts \cup \{\{x : \Theta_1\}, A_1\}) = \left(\bigcup \{x : \Theta_1\}, \Gamma, A_2\right)
\]

\[
S(M : \tau) = (\Gamma_1, B_1, A_1, cpts \cup \{sl_1(x), \ldots, sl_k(x)\})
\]

\[
S(\lambda x.M : \sigma \rightarrow \tau) = \Theta_1 \rightarrow \sigma, B, \{A_2, A_3, cpts\}
\]

In our example, there is only one occurrence of $z$ and therefore the contraction function $C$ is called with only one type and does not produce any constraint. Also the boxing function $B$ produce no result, being called on a variable, i.e., it acts as the identity returning a triple with the same base, type and (empty, in this case) set of constraints:

\[
B(z, \{\{z : p_{12}(p_{13} \circ p_{14})\}, p_{12}(p_{13} \circ p_{14}), \emptyset, \emptyset\} = \{\{z : p_{12}(p_{13} \circ p_{14})\}, p_{12}(p_{13} \circ p_{14}), \emptyset\}.
\]

The rôle of $cpts$ and $sl$ will be discussed in the context of the critical points, below. Coming back to our example, for $\lambda z.z : ((o \rightarrow o) \rightarrow (o \rightarrow o))$ we infer the EAL-type

\[
\lambda z.z : p_{12}(p_{13} \circ p_{14}) \rightarrow p_{12}(p_{13} \circ p_{14})
\]

When the algorithm infers the EAL-type for $(n \lambda z.z) : (o \rightarrow o)$, it:

1. adds all possible boxes around the argument $\lambda z.z$ with the boxing function, that in this case adds $b_1$ boxes around $\lambda z.z$ returning a triple with the same base, $b_1$ banged type and unmodified set of (again empty) constraints:

\[
B(\lambda z.z, \emptyset, p_{12}(p_{13} \circ p_{14}), \emptyset, \emptyset) = \emptyset, b_1(p_{12}(p_{13} \circ p_{14}), \emptyset, \emptyset)
\]

\(^3\)More precisely it builds all possible decorations without exponential cuts and with some other properties listed in Theorem 3. Decorations of these kinds are sufficient for the completeness of the algorithm.
2. imposes the EAL-type of \( n \) to be functional, i.e., the constraint
\[
p_1 = 0
\]  
(10)

3. unifies the EAL-type of the boxed \( \lambda z . z \) with the argument part of the EAL-type of \( n \):
\[
\Psi \left( \begin{array}{c}
b_1(p_{12}(p_{13} \to p_{14}) \to p_{12}(p_{13} \to p_{14})), \\
p_2(p_3(p_4 \to p_5) \to p_6(p_7 \to p_8))
\end{array} \right).
\]

Observe that the implicational structure of the types is already correct, since we start from a simple type derivation. Therefore, unification only produces a set of constraints on the variables used to indicate boxes. In our example, we get the constraints:
\[
\begin{align*}
& b_1 = p_2 \\
& p_{12} = p_3 \\
& p_{13} = p_4 \\
& p_{14} = p_5 \\
& p_{12} = p_6 \\
& p_{13} = p_7 \\
& p_{14} = p_8
\end{align*}
\]
\[
\begin{align*}
& b_1 = p_2 \\
& p_3 = p_6 = p_{12} \\
& p_4 = p_7 = p_{13} \\
& p_5 = p_8 = p_{14}.
\end{align*}
\]  
(11)

The type synthesis rule\(^4\) for an application, provided that \( M \) and \( N \) are not applications themselves, is:
\[
\Psi(\Theta_1, \Theta_3) = A_4 \\
\exists(N, B_2, \Theta_2, cpts_2, A_2) = (B_3, \Theta_3, A_3) \\
S(N : \sigma) = (\Theta_2, B_2, A_2, cpts_2) \\
S(M : \sigma \to \tau) = \langle !\sum_n (\Theta_1 \to \Gamma), B_1, A_1, cpts_1 \rangle
\]
\[
S((M \ N) : \tau) = \langle \Gamma, B_1 \cup B_3, \left\{ \begin{array}{c}
A_1 \\
A_3 \\
A_4 \\
cpts_1 \cup cpts_2 \\
\sum n_i = 0
\end{array} \right\} \rangle
\]  
(12)

\(^4\)We will explain \( \Psi \) later.

Figure 6: Decoration of \((n \lambda z . z)\).
Figure 6 shows the decoration obtained so far:

\[ n : b_1(p_3 \rightarrow p_5) \rightarrow p_3(p_4 \rightarrow p_5) \rightarrow p_9(p_{10} \rightarrow p_{11}) \vdash (n \ λz.z) : p_9(p_{10} \rightarrow p_{11}). \]

(13)

Next step is the inference of a general EAL-type \( p_{15} \) for \( y : o \). Then the algorithm starts to process \((n \ λz.z) y : o\). As before, the algorithm

1. applies \( \Box \) to the argument \( y \) (a void operation here, since the boxing function does nothing for variables);

2. imposes the EAL-type of \((n \ λz.z)\) to be functional:

\[ p_9 = 0. \]

(14)

3. unifies the EAL-types, to make type-correct the application:

\[ \varpi(p_{10}, p_{15}) = p_{10} = p_{15}. \]

(15)

However, the present case is more delicate than the application we treated before, since the function part is already an application. Two consecutive applications in \((n \ λz.z) y\) indicates that more than one decoration is possible. Indeed, there can be several derivations building the same term, that can be differently decorated. The issue is better appreciated if we look ahead for a moment and we consider the term \( \lambda y.(n \ λz.z) y\). There are two (simple) sequent derivations for this term, both starting with the term \((x y) : o\), for \( x : o \rightarrow o, y : o\). The first derivation, via a left \( \rightarrow \)-rule, obtains \((n \ λz.z) y : o\); then it binds \( y \), giving \( \lambda y.(n \ λz.z) y : o \rightarrow (o \rightarrow o)\). The second derivation permutes the rules: it starts by binding \( y \), obtaining \( \lambda y.(x y)\) and only at this point substitutes \((n \ λz.z)\) for \( x \), via the left \( \rightarrow \)-rule. When we add boxes to the two derivations, we see this is a critical situation. Indeed, in the first derivation we may box \((x y)\), then \((n \ λz.z) y\) and finally \( \lambda y.(n \ λz.z) y\). In the second, we box \((x y)\), then \( \lambda y.(x y)\) and finally the whole term. The two (incompatible) decorations are depicted in the two bottom trees of Figure 10. The critical edge—where the boxing radically differs—is the root of the subtree for \((n \ λz.z) y\), corresponding to the \( x \) that is substituted for in the left \( \rightarrow \)-rule. Let us then resume the discussion of the type inference for this term. At this stage we collect the critical point, marked with a star in Figure 7, indicating the presence of two possible derivations. When, in the future, it will be possible to add boxes, for example \( b_2 \) in Figure 7 during the type inference of \( \lambda y.(n \ λz.z) y\), the algorithm will consider the critical point as one of the closing points of such boxes, \( c_1 \) in Figure 7, eventually modifying the constraint in Equation (14) that impose type of \((n \ λz.z)\) to be functional and not exponential. Indeed, for completeness, the algorithm must take into account all possible derivations. When there will be more than one critical point, at every stage of the type inference, when it is possible to apply a \( ! \) rule, the algorithm will compute all possible
Figure 7: Critical point in the decoration of $\lambda y.((n \; \lambda z. z)y)$.

Figure 8: Combinations of two critical points.

combinations of the critical points (see Figure 8 showing a schematic example with two critical points) eventually modifying some constraints. We call slices\(^5\) such combinations of critical points; they are the data maintained by the algorithm and indicated in the rules as cpts. The task of combining the two lists of slices collected during the type inference of the function and argument part of an application is performed by $\sqcup$, whose rules are given in Section 2.2.4.

**Definition 3.** The list of free variable occurrences of a lambda term $M$ is defined in the following way:

(a) $FVO(x) = [x]$;
(b) $FVO(\lambda x. M) = FVO(M) - x$;
(c) $FVO((M_1 \; M_2)) = FVO(M_1) :: FVO(M_2)$ (the concatenation of lists).

\(^5\)We thank Philippe Dague for useful discussions and suggestions on the calculation of critical points.
**Definition 4.** A slice is a set of pairs (constraint, list of free variable occurrences) as in the following:

\[ sl = \{ (A^1_j, [y_1, \ldots, y_{1_h}]), \ldots, (A^k_j, [y_k, \ldots, y_{k_h}]) \} \]

A slice corresponds to a combination of critical points.

In our example the algorithm collects the slice \( (p_9 = 0, [n]) \). Notice that a slice partitions the set of free variable occurrences in a derivation: it marks the set of variable occurrences whose types should not be modified when the box is added. This is the intuitive meaning of the set of free variable occurrences in the data structure we use.

**Notation 2.**

- \( sl(x) \) indicates a slice having \( x \) as an element of every list of variables in it.
- \( x \in sl \) if and only if there exists one element of \( sl \) whose list of variables contains \( x \).
- \( A^j \in sl \) if and only if there exists one element of \( sl \) whose constraint is \( A^j \).
- Being \( A^j \) the constraint \( \sum_{i} n_i = 0 \), \( A^j - n \) corresponds to the constraint \( \sum_{i} n_i - n = 0 \).

The general type inference rule for the application we are considering now, i.e., \( ((M_1 M_2) N) \) when \( N \) is not an application, is the following:

\[
\begin{align*}
\text{cpts} & = (\text{cpts}_1 \cup \{ \sum n_i = 0, \text{FVO}(M_1 M_2) \}) \cup \text{cpts}_2 \\
\mathcal{B}(\Theta_1, \Theta_3) & = A_4 \\
\mathcal{B}(N, B_2, \Theta_2, \text{cpts}_2, A_2) & = \langle B_3, \Theta_3, A_3 \rangle \\
\mathcal{S}(N : \sigma) & = \langle \Theta_2, B_2, \mathcal{A}_2, \text{cpts}_2 \rangle \\
\mathcal{S}(\langle M_1, M_2 \rangle : \sigma \rightarrow \tau) & = \langle \sum n_i = 0, A_1, A_3 \rangle
\end{align*}
\]

In the example case we obtain:

\[
\begin{align*}
\{ n & : b_1(p_3(p_4 \rightarrow p_5) \rightarrow p_3(p_4 \rightarrow p_5)) \\
y & : p_{10} \}
\vdash ((n \lambda z. z) y) : p_{11}
\end{align*}
\]

and critical points \( \text{cpts} = \{ (p_9 = 0, [n]) \} \).

Typing \( \lambda y.((n \lambda z. z) y) : o \rightarrow o \) involves rule \( \Box \), the same we used for \( \lambda z. z \), but now the boxing procedure \( \mathcal{B} \) is called on a subterm that is not a single variable. The complete set of rules for \( \mathcal{B} \) is the following:

\[
\mathcal{B}(x, B, \Gamma, \text{cpts}, A) = \langle B, \Gamma, A \rangle
\]

\(^6A^j\) means the \( j \)-th row of the matrix \( A \), i.e., the \( j \)-th constraint.
Boxing of a variable produces no changes in the base, type and set of constraints.

\[ \mathcal{B}(B, \Gamma, cpts, A) = \langle B_1, \Gamma_1, A_1 \rangle \]
\[ \mathcal{B}(M, B, \Gamma, cpts, A) = \langle !^b B_1, !^b \Gamma_1, A_1 \rangle \] (19)

\[ \mathcal{B}(M, B, \Gamma, cpts, A) = \langle !^b B_1, !^b \Gamma_1, A_1 \rangle \]
\[ \mathcal{B}(B, \Gamma, \emptyset, A) = \langle B, \Gamma, A \rangle \] (20)

\[ \mathcal{B}(B, \Gamma, \emptyset, A) = \langle B, \Gamma, A \rangle \]
\[ \mathcal{B}(B_1, !^b \Gamma, cpts, A_2) = \langle B, \Delta, A_1 \rangle \]
\[ B_1 = \left\{ x_i : \begin{array}{ll} !^b \Theta_i & x_i \notin sl \\ \Theta_i & x_i \in sl \end{array} \right\} \]
\[ A_2 = \left\{ \begin{array}{ll} A^j & A^j \notin sl \\ A^j - c & A^j \in sl \end{array} \right\} \]
\[ \mathcal{B}(\{x_i : \Theta_i\}, \Gamma, \{sl\} \cup cpts, A) = \langle B, \Delta, A_1 \rangle \] (21)

Therefore, rule (8) gives in our case:

\[ S(\lambda y.((n \lambda z. z) y) : o \rightarrow o) = \]
\[ b_2 + c_1 + p_{10} \rightarrow b_2 + c_1 + p_{11}, \]
\[ \{ n : b_2 (b_1 (p_3 (p_4 \rightarrow p_5) \rightarrow p_3 (p_4 \rightarrow p_5) \rightarrow p_9 (p_{10} \rightarrow p_{11}))) \}, \]
\[ \left\{ \begin{array}{l} p_9 - c_1 = 0 \\ \vdots \end{array} \right\} \] (22)

where \( p_9 - c_1 = 0 \) is the unique constraint (Equation (14)) modified by \( \mathcal{B} \).

The decoration obtained is shown in Figure 7. Observe that, at this stage, the presence of incompatible derivations does not show up yet. It will be taken into account as soon as we will try to box a superterm of the one we just processed. If \( \lambda y.((n \lambda z. z) y) \) would be the whole term, on the contrary, an additional call to the function \( \mathcal{B} \) would be performed, see the rule (60) for function \( \mathcal{F} \).

When the algorithm processes \( \lambda n.\lambda y.((n \lambda z. z) y) : (((o \rightarrow o) \rightarrow (o \rightarrow o)) \rightarrow (o \rightarrow o)) \rightarrow (o \rightarrow o) \) it applies again rule (9). It adds \( c_2 \) boxes passing through
Figure 9: the critical point and $b_3$ boxes around the term, obtaining:

$$S(\lambda n.\lambda y.((n \lambda z.z) \ y) : (((o \to o) \to (o \to o)) \to (o \to o)) \to (o \to o)) =$$

$$b_3 + b_2(b_1(p_3(p_4 \to p_5) \to p_3(p_4 \to p_5)) \to p_9(p_{10} \to p_{11}))$$

$$\to b_3 + c_2(b_2 + c_1 + p_{10} \to b_2 + c_1 + p_{11})$$

$$\emptyset,$$

$$\begin{cases}
\vdots \\
p_9 - c_1 - c_2 = 0 \\
\vdots \\
\emptyset
\end{cases}$$

where $p_9 - c_1 - c_2 = 0$ is the unique constraints modified at this stage of the type synthesis.

The critical point $(p_9 - c_2 - c_2 = 0, [n])$ is removed. In fact, to bound $n$, the substitution of $n(\lambda z.z)$ for $x$ has to be already performed. It does not make sense to derive first $\lambda n.\lambda y.((x y)$, add boxes, and then substitute $n(\lambda z.z)$ for $x$, since this would be a free-variable catching substitution.

Figure 9 shows the decoration obtained. Notice that boxes $c_2$ and $b_2$ belong to the two incompatible EAL-derivations we already discussed before. The algorithm maintains at the same time these derivations guaranteeing (see Lemma 9) that if the final solution instantiates two incompatible derivations, we can always calculate an equivalent EAL-derivation (Figure 10 shows the two possible derivations for our example).

Going on with the type synthesis, the algorithm starts processing the leftmost occurrence of $x$ in $(x \ x \lambda w.w)$. We use superscripts (1) and (2) to discriminate the right and left occurrence, respectively. For the leftmost—$x^{(2)} : (o \to o) \to (o \to o)$—we infer the EAL-type

$$p_{16}(p_{17}(p_{18} \to p_{19}) \to p_{20}(p_{21} \to p_{22}));$$

(24)
analogously, for the rightmost $x^{(1)} : (o \to o) \to (o \to o)$ we get the EAL-type

$$p_{23}(p_{24}(p_{25} \to p_{26}) \to p_{27}(p_{28} \to p_{29})).$$ (25)

The EAL-type of $w : o$ is $p_{30}$ and then $\lambda w.w : o \to o$ is typeable in EAL with type $p_{30} \to p_{30}$.

The innermost application $(x^{(1)} \lambda w.w)$ is typed $p_{27}(p_{28} \to p_{29})$, once we have imposed

$$p_{23} = 0,$$ (26)

we have boxed $\lambda w.w$ with $b_4$ boxes, and we have unified the types

$$\mathcal{U}(p_{24}(p_{25} \to p_{26}), b_{4}(p_{30} \to p_{30})) = \begin{cases} b_4 = p_{24} \\ p_{25} = p_{30} = p_{26} \end{cases}.$$ (27)

When the algorithm processes $(x^{(2)} (x^{(1)} \lambda w.w))$, it adds $b_5$ boxes around the argument, imposes

$$p_{16} = 0$$ (28)

and unifies the types

$$\mathcal{U}(p_{17}(p_{18} \to p_{19}), b_{5} + p_{27}(p_{28} \to p_{29})) = \begin{cases} p_{17} = b_{5} + p_{27} \\ p_{18} = p_{28} \\ p_{19} = p_{29} \end{cases}.$$ (29)
Moreover, the presence of two consecutive applications makes the algorithm collect a new critical point \((p_{17} = b_5 + p_{27}, [x^{(1)}])\). The derivation obtained is:

\[
\begin{align*}
\{ & x^{(1)} : b_5(b_4(p_{25} \rightarrow o)(p_{25}) \rightarrow o p_{27}(p_{18} \rightarrow o p_{19})) , \\
& x^{(2)} : p_{17}(p_{18} \rightarrow p_{19}) \rightarrow o p_{20}(p_{21} \rightarrow o p_{22}) \} \vdash (x^{(2)} (x^{(1)} \lambda w.w)) : p_{20}(p_{21} \rightarrow o p_{22})
\end{align*}
\]

and its decoration is shown in Figure 11.

For the type inference of \(\lambda x.(x^{(2)} (x^{(1)} \lambda w.w)) : ((o \rightarrow o) \rightarrow (o \rightarrow o)) \rightarrow (o \rightarrow o)\), the algorithm applies the usual rule for abstractions seen above (8), but in this case there are two instances of the bound variable \(x\). Here comes to work the function \(C\), whose rules are the following:

\[
C(\Theta) = \emptyset
\]

\[
\begin{align*}
\forall (\ldots n_1 + \ldots + n_k \Theta_1, \ldots, \Theta_k) = A \\
C(\{n_1 + \ldots + n_k \Theta_1, \ldots, \Theta_k\}) = \left\{ \begin{array}{l}
\text{if } n_1 + \ldots + n_k \geq 1 \\
A
\end{array} \right.
\end{align*}
\]

Therefore the contraction of \(k\) general EAL-types is obtained by unification and the constraint that the contracted types have at least one “!” (since in EAL contraction is allowed only for exponential formulas).

Coming back to our example, the algorithm adds \(c_3\) boxes passing through the critical point and \(b_6\) boxes around the body of the abstraction. The \(B\) function modifies the first constraint in Equation (29):

\[
p_{17} = b_5 + p_{27} - c_3
\]
Then the algorithm contracts the types of $x$:

$$\mathcal{C} \left( b_6 + b_5(b_4(p_{25} \rightarrow p_{25}) \rightarrow p_{27}(p_{18} \rightarrow p_{19})), b_6 + c_3(p_{17}(p_{18} \rightarrow p_{19}) \rightarrow p_{20}(p_{21} \rightarrow p_{22})) \right) =$$

$$= \begin{cases} 
  b_6 + b + 5 \geq 1 \\
  b_5 = c_3 \\
  b_4 = p_{17} \\
  p_{18} = p_{19} = p_{21} = p_{22} = p_{25} \\
  p_{20} = p_{27} 
\end{cases} \quad (34)$$

Finally it removes the critical point $(p_{17} = b_5 + p_{27} - c_3, [x^{(1)}])$.

The derivation obtained, whose decoration is shown in Figure 12, is:

$$\vdash \lambda x.(x \ (x \ \lambda w.w)) : b_6 + b_5(b_4(p_{18} \rightarrow p_{18}) \rightarrow p_{20}(p_{18} \rightarrow p_{18}))$$

$$\rightarrow b_6 + b_5 + p_{20}(p_{18} \rightarrow p_{18}). \quad (35)$$

The algorithm process now the whole term $(\lambda n.\lambda y.(n \ (\lambda z.z) \ y) \ \lambda x.(x \ (x \ \lambda w.w))) : o \rightarrow o$. It adds $b_7$ boxes around the argument of the application and unifies the EAL-types for the correct application:

$$\mathcal{U} \left( b_7(b_6 + b_5(b_4(p_{18} \rightarrow p_{18}) \rightarrow p_{20}(p_{18} \rightarrow p_{18})) \rightarrow b_6 + b_5 + p_{20}(p_{18} \rightarrow p_{18})) \right) =$$

$$= \begin{cases} 
  b_7 = b_3 + b_2 \\
  b_1 = b_6 + b_5 \\
  b_4 = p_{20} \\
  p_{4} = p_{5} = p_{10} = p_{11} = p_{18} \\
  p_{9} = b_6 + b_5 + p_{20} 
\end{cases} \quad (36)$$

Since this is the complete term, the final step of the algorithm is a single call to the function $\mathcal{F}$, which in this case simply adds $b_8$ boxes around the term. Therefore, the simply typed lambda term

$$(\lambda n.\lambda y.(n \ (\lambda z.z) \ y) \ \lambda x.(x \ (x \ \lambda w.w)): o \rightarrow o \quad (37)$$
has EAL-type

\[ !b_8 + b_3 + c_2 \left( !b_2 + c_1 + p_4 + o \rightarrow !b_2 + c_1 + p_4 + o \right) \]

for any \( p_1, \ldots, p_{30}, b_1, \ldots, b_8, c_1, c_2, c_3 \in \mathbb{N} \) solutions of the set of constrains\(^7\) in equations (38):

\[
\begin{aligned}
  b_6 + b_5 &\geq 1 \\
  b_7 &= b_3 + b_2 \\
  b_1 &= p_2 = b_6 + b_5 \\
  b_5 &= c_3 \\
  p_1 &= p_{16} = p_{23} = 0 \\
  p_9 &= c_1 + c_2 = b_6 + b_5 + b_4 \\
  p_{17} &= b_5 + p_{27} - c_3 \\
  b_4 &= p_3 = p_6 = p_{12} = p_{17} = p_{20} = p_{24} = p_{27} \\
  p_4 &= p_5 = p_7 = p_8 = p_{10} = p_{11} = p_{13} = p_{14} = p_{15} = p_{18} \\
  p_4 &= p_{19} = p_{21} = p_{22} = p_{25} = p_{26} = p_{28} = p_{29} = p_{30}.
\end{aligned}
\]

The final decoration is shown in Figure 13. Considering the set of constraints in Equation (39) and the incompatibility of \( c_2 \) and \( b_2 \) stated above, the simply typed term

\[ (\lambda n. \lambda y. ((n \ \lambda z. z) \ y) \ \lambda x. (x \ (x \ \lambda w. w))) : o \rightarrow o \]

can be typed in EAL either:

1. for any \( n_1, \ldots, n_6 \in \mathbb{N}, n_1 \geq 1 \) with EAL-type \(!n_3 + n_5 \left( !n_1 + n_2 + n_4 + n_6 + o \rightarrow !n_1 + n_2 + n_4 + n_6 + o \right)\) and decoration shown in Figure 14 or

2. for any \( m_1, \ldots, m_7 \in \mathbb{N}, m_1 \geq 1 \ \land \ m_2 + m_3 = m_1 + m_5 \) with EAL-type \(!m_3 + m_4 + m_6 \left( !m_2 + m_7 + o \rightarrow !m_2 + m_7 + o \right)\) and decoration shown in Figure 15.

\(^7\)We have boxed the constraints which were not modified by \( \mathcal{B} \) until the end of the type
Figure 14: Final decoration.

Figure 15: Another possible final decoration.
2.2 The full algorithm

We define in this section the formal rules for the algorithm. An almost complete trace of its application to a simply typed term with no EAL type can be found in the Appendix.

Definition 5. (Type Synthesis Algorithm) Given a simply typeable lambda term \( M : \sigma \), the type synthesis algorithm \( S(M : \sigma) \) returns a triple \( (\Theta, B, A) \), where \( \Theta \) is a general EAL-type, \( B \) is a base (i.e., a multi-set of pairs variable, general EAL-type) and \( A \) is a set of linear constraints.

In the following \( n, n_1, n_2 \) are always fresh variables, \( o \) is the base type. Moreover, we consider \( !n_1 (!!n_2 \Theta) \) syntactically equivalent to \( !!n_1+n_2 \Theta \).

Notation 3. Given a set of linear constraints \( A \) and a solution \( X \) of \( A \), for any general EAL-type \( \Theta \) and for any base \( B = \{ x_1 : \Theta_1, \ldots, x_n : \Theta_n \} \), we denote with \( X(\Theta) \) the instantiation of \( \Theta \) with \( X \) and with \( X(B) \) the instantiation of \( B \) with \( X \), i.e., \( X(B) = \{ x_1 : X(\Theta_1), \ldots, x_n : X(\Theta_n) \} \).

2.2.1 Unification: \( U \)

Unification takes a set of \( h \geq 2 \) general EAL-types having the same underlying intuitionistic shape and returns a set of linear equations \( A \) such that for any solution \( X \) of \( A \), the instantiations of the \( h \) general EAL-types are syntactically identical.

\[
U(!!n_1 o, \ldots, !!n_h o) = \begin{cases} 
\sum n_{i_1} - \sum n_{i_2} = 0 \\
\vdots \\
\sum n_{i_{h-1}} - \sum n_{i_h} = 0 
\end{cases}
\]

(40)

\[
U(\Theta_1, \ldots, \Theta_{h-1}) = A_1 \\
U(\Theta_{h+1}, \ldots, \Theta_h) = A_2
\]

(41)

2.2.2 Contraction (\( C \)) and Type Processing (\( \mathcal{P} \))

Contraction in EAL is allowed only for exponential formulas. Thus, given \( k \) general EAL-types, \( C \) returns the same set of constraints of \( U \) with the additional inference process in the exposition above. They are now all collected in the set of constraints \( A \).
constraint that the number of external ! must be greater than zero.

\[
C(\Theta) = \emptyset
\]  

(42)

\[
\forall (n_1 + \cdots + n_h \Theta_1, \Theta_2, \ldots, \Theta_k) = A
\]

(43)

\[C(n_1 + \cdots + n_h \Theta_1, \ldots, \Theta_k) = \begin{cases} n_1 + \cdots + n_h \geq 1 \\ A \end{cases} \]

Given a simple type \( \tau \), \( P \) returns the most general EAL-type whose cancellation is \( \tau \), obtained by adding everywhere \( p \) exponentials (every \( p \) is a fresh variable).

\[
P(o) = \!p_o
\]  

(44)

\[
P(\sigma) = \Theta \quad P(\tau) = \Gamma
\]

(45)

2.2.3 Boxing: \( B \) and \( B \)

The boxing procedure \( B \) superimposes all boxes due to the presence of critical points. Recall the notion of slice (Definition 4) and Notation 2. \( B \) has no effect if there is no critical point:

\[
B(B, \Gamma, \emptyset, A) = \langle B, \Gamma, A \rangle
\]  

(46)

For any slice \( sl \), \( B \) adds \( c \) boxes around the subterm above the critical points belonging to \( sl \):

\[
B(B_1, \!^c \Gamma, cpts, A_2) = \langle B, \Delta, A_1 \rangle
\]

\[
B_1 = \left\{ x_i : \begin{cases} \!^c \Theta_i & x_i \notin sl \\ \Theta_i & x_i \in sl \end{cases} \right\},
\]

\[
A_2 = \left\{ \begin{array}{ll} A_j & A_j \notin sl \\ A_j - c & A_j \in sl \end{array} \right\}
\]

\[
B(\{x_i : \Theta_i\}_i, \Gamma, \{sl\} \cup cpts, A) = \langle B, \Delta, A_1 \rangle
\]  

(47)

Function \( B \) is the wrapper for \( B \). It calls \( B \) and then adds \( b \) external boxes:

\[
B(x, B, \Gamma, cpts, A) = \langle B, \Gamma, A \rangle
\]  

(48)

\[
B(B, \Gamma, cpts, A) = \langle B_1, \Gamma_1, A_1 \rangle
\]

(49)

**Proposition 1.** Let \( b, c_1, \ldots, c_k \) be the fresh variables introduced by \( B(M, B, \Gamma, cpts, A) = \langle \!^b B_1, \!^b \Gamma_1, A_1 \rangle \) and let \( X \) be a solution of \( A \), then
1. \( X_1 = (X, b = 0, c_1 = 0, \ldots, c_k = 0) \) is a solution of \( A_1 \);

2. \( X_1(\Gamma_1) = X(\Gamma) \);

3. \( X_1(B_1) = X(B) \).

**Proof.**

1. By Equation (47), for every variable \( c_i \) introduced by \( B \), there is a constraint \( \pm n_1 \pm \cdots \pm n_k = 0 \) that is changed in \( \pm n_1 \pm \cdots \pm n_k - c_i = 0 \), hence trivially, if the first one is solvable, then the second one is solvable too imposing \( c_i = 0 \). Moreover, by Equation (49), \( b \) is not added to the set of constraint, hence the thesis.

2. By Equation (47) \( \Gamma_1 = \{c_1 + \cdots + c_k \Gamma\} \).

3. By Equation (47) if \( B = \{x_i : \Theta_i\} \) then \( B_1 = \{x_i : \sum_{j \in J_i} \Theta_i\} \) where \( J_i \subseteq \{1, \ldots, k\} \).

2.2.4 **Product union:** \( \cup \)

Product union computes all possible combinations of critical points. It is the culprit for the exponential complexity of the algorithm.

\[
\emptyset \cup \text{cpts} = \text{cpts} \quad \emptyset = \text{cpts}
\]

\[
\begin{align*}
\{s_1, s_2, \ldots, s_n\} \cup \{s_1, s_2, \ldots, s_n\} &= \text{cpts} \\
\{s_1, \ldots, s_n\} \cup \{s_1, \ldots, s_n\} &= \{s_1, s_1 \cup s_2, \ldots, s_1 \cup s_n\} \cup \text{cpts}
\end{align*}
\]

2.2.5 **Type synthesis:** \( S \)

\( S \) is the main function of the algorithm. It is defined by cases on the structure of the \( \lambda \)-term. Its main cases have already been discussed in Section 2.1. Define \( \neg \text{app}(M) \) iff the term \( M \) is not an application.

Variable case, see equation (5):

\[
\mathcal{P}(\sigma) = \Theta \\
S(x : \sigma) = \langle \Theta, \{x : \Theta\}, \emptyset, \emptyset \rangle
\]
First abstraction case: in $\lambda x.M$, $x \in \text{FV}(M)$, see equation (5):

\[
\begin{align*}
h &\geq 1 \quad x \in \text{FV}(M) \\
C(\Theta_1, \ldots, \Theta_h) &= A_3 \\
\mathcal{B} &\left( M, B_1, \Gamma_1, \text{cpts} \cup \left\{ \frac{\text{sl}_1(x)}{\text{sl}_k(x)} \right\}, A_1 \right) = \left\langle B \cup \left\{ \frac{x : \Theta_1}{x : \Theta_h} \right\}, \Gamma, A_2 \right\rangle \\
S(M : \tau) &= \left\langle \Gamma_1, B_1, A_1, \text{cpts} \cup \left\{ \text{sl}_1(x) \right\} \right\rangle
\end{align*}
\]

\[S(\lambda x.M : \sigma \rightarrow \tau) = \left\langle \Theta_1 \rightarrow \Gamma, \left\{ \frac{A_2}{A_3}, \text{cpts} \right\} \right\rangle\] (53)

Second abstraction case: in $\lambda x.M$, $x \not\in \text{FV}(M)$ and $M$ is an application:

\[
\begin{align*}
x &\not\in \text{FV}((M_1 \ M_2)) \\
c\text{pts} &= c\text{pts}_1 \cup \left\{ (\sum n_i - n = 0, \text{FV}(M_1 \ M_2)) \right\} \\
\mathcal{S}(\sigma) &= \Theta \\
\mathcal{B} &((M_1 \ M_2), B_1, \Gamma_1, \text{cpts}_1, A_1) = \left\langle B, \sum n \cdot \Gamma, A \right\rangle \\
S((M_1 \ M_2) : \tau) &= \left\langle \Gamma_1, B_1, A_1, \text{cpts}_1 \right\rangle \\
\end{align*}
\]

\[S(\lambda x.(M_1 \ M_2) : \sigma \rightarrow \tau) = \left\langle \Theta \rightarrow !n \Gamma, B, \left\{ \frac{A}{\sum n_i - n = 0}, \text{cpts} \right\} \right\rangle\] (54)

Third abstraction case: in $\lambda x.M$, $x \not\in \text{FV}(M)$ and $M$ is not an application:

\[
\begin{align*}
&\neg\text{app}(M) \\
x &\not\in \text{FV}(M) \\
\mathcal{S}(\sigma) &= \Theta \\
\mathcal{B} &((M_1 \ M_2), B_1, \Gamma_1, \text{cpts}_1, A_1) = \left\langle B, \Gamma, A \right\rangle \\
S(M : \tau) &= \left\langle \Gamma_1, B_1, A_1, \text{cpts} \right\rangle \\
\end{align*}
\]

\[S(\lambda x.M : \sigma \rightarrow \tau) = \left\langle \Theta \rightarrow !n \Gamma, B, A, \text{cpts} \right\rangle\] (55)

First application case: in $M \ N$, neither $M$ nor $N$ are applications, see equation (12):

\[
\begin{align*}
&\neg\text{app}(M) \land \neg\text{app}(N) \\
\mathcal{S}(\Theta_1, \Theta_3) &= A_4 \\
\mathcal{B} &((N, B_2, \Theta_2, \text{cpts}_2, A_2) = \left\langle B_3, \Theta_3, A_3 \right\rangle \\
S(N : \sigma) &= \left\langle \Theta_2, B_2, A_2, \text{cpts}_2 \right\rangle \\
S(M : \sigma \rightarrow \tau) &= \left\langle !\sum n_i (\Theta_1 \rightarrow \Gamma), B_1, A_1, \text{cpts}_1 \right\rangle \\
\end{align*}
\]

\[S((M \ N) : \tau) = \left\langle \Gamma, B_1 \cup B_3, \left\{ \frac{A_1}{A_3}, A_4, \text{cpts}_1 \cup \text{cpts}_2 \right\}, \sum n_i = 0 \right\rangle\] (56)
Second application case: in \((M \ N)\), \(M\) is not an application:

\[
\begin{align*}
\neg \text{app}(M) \\
cpts &= cpts_1 \cup \{cpts_2 \cup \{(A_1^1, \text{FVO}((N_1 \ N_2)))\}\} \\
\forall (\Theta_3, \Theta_1) &= A_4 \\
\exists((N_1 N_2), B_2, \Theta_2, cpts_2, A_2) &= (B_3, \Theta_3, A_3) \\
\mathcal{S}((N_1 N_2) : \sigma) &= (\Theta_2, B_2, A_2, cpts_2) \\
\mathcal{S}(M : \sigma \rightarrow \tau) &= (\sum_i n_i (\Theta_1 \rightarrow \Gamma), B_1, A_1, cpts_1)
\end{align*}
\]

(57)

Notice that \(A_1^1\) indicates the equality constraints between the outermost number of \(!\) in the type of \((N_1 \ N_2)\) and in the function part of the type of \(M\).

Third application case: in \((M \ N)\), \(N\) is not an application, see equation (56):

\[
\begin{align*}
\neg \text{app}(N) \\
cpts &= \{cpts_1 \cup \{(\sum n_i = 0 \ FVO((M_1 \ M_2)))\}\} \cup cpts_2 \\
\forall (\Theta_1, \Theta_3) &= A_4 \\
\exists(N, B_2, \Theta_2, cpts_2, A_2) &= (B_3, \Theta_3, A_3) \\
\mathcal{S}(N : \sigma) &= (\Theta_2, B_2, A_2, cpts_2) \\
\mathcal{S}((M_1 M_2) : \sigma \rightarrow \tau) &= (\sum_i n_i (\Theta_1 \rightarrow \Gamma), B_1, A_1, cpts_1)
\end{align*}
\]

(58)

Fourth application case: in \((M \ N)\), both \(M\) and \(N\) are applications:

\[
\begin{align*}
cpts_4 &= cpts_2 \cup \{(A_1^1, \text{FVO}((N_1 \ N_2)))\} \\
cpts_3 &= cpts_1 \cup \{(\sum n_i = 0 \ FVO((M_1 \ M_2)))\} \\
\forall (\Theta_3, \Theta_1) &= A_4 \\
\exists((N_1 N_2), B_2, \Theta_2, cpts_2, A_2) &= (B_3, \Theta_3, A_3) \\
\mathcal{S}((N_1 N_2) : \sigma) &= (\Theta_2, B_2, A_2, cpts_2) \\
\mathcal{S}((M_1 M_2) : \sigma \rightarrow \tau) &= (\sum_i n_i (\Theta_1 \rightarrow \Gamma), B_1, A_1, cpts_1)
\end{align*}
\]

(59)

\[
\begin{align*}
\mathcal{S}((M_1 M_2) \ (N_1 N_2) : \tau) &= (\Gamma, B_1 \cup B_3, \left\{ \begin{array}{ll} A_1 \\ A_3 \\ A_4 \end{array} \right., cpts_3 \cup cpts_4) \\
\sum n_i &= 0
\end{align*}
\]

(60)

2.2.6 Type synthesis algorithm: \(\mathcal{S}\)

\(\mathcal{S}\) is the top level call for the algorithm. It passes the call to \(\mathcal{S}\), takes its result, boxes the term, forgets the critical points and eventually contracts the common
variables in the base.

\[ C(\Theta_{1}, \ldots, \Theta_{k_1}) = A_1 \quad \ldots \quad C(\Theta_{1h}, \ldots, \Theta_{kh}) = A_h \]

\[ B(M, B_1, \Theta_1, cpts, A') = \left\{ \begin{array}{l}
\{ x_1 : \Theta_{11}, \ldots, x_1 : \Theta_{k_1} \}, \\
\vdots \\
\{ x_h : \Theta_{1h}, \ldots, x_h : \Theta_{kh} \}
\end{array} \right\}, \Theta, A \]

\[ S(M : \sigma) = (\Theta_1, B_1, A', cpts) \]

3 Properties of the type inference algorithm

We will prove in this section that our algorithm \( S \) is complete with respect to the notion of \( \Gamma \vdash_{EAL} M : A \) introduced in Definition 1. Correctness and completeness of \( S \) are much simpler if, instead of EAL, we formulate proofs and results with reference to an equivalent natural deduction formulation, discussed in the following subsection. Before, we state the obvious fact that our algorithm does not loop, since any rule \( S \) decreases the structural size of the \( \lambda \)-term \( M \), any rule \( U \) decreases the size of the type \( \Theta \) and any rule \( B \) and \( \bowtie \) decreases the size of the set of critical points \( cpts \).

Proposition 2 (Termination). Let \( M \) be a simply typed term and let \( \sigma \) its simple type. \( S(M : \sigma) \) always terminates with a triple \( (\Theta, B, A) \).

The algorithm is exponential in the size of the \( \lambda \)-term, because to investigate all possible derivations we need to (try to) box all possible combinations of critical points (see the clauses for the product union, \( \bowtie \), in Section 2.2.4), that are roughly bounded by the size of the term.

3.1 NEAL

The natural deduction calculus (NEAL) for EAL in given in Figure 16, after [Asp98, BBdPH93, Rov98].

Lemma 1 (Weakening). If \( \Gamma \vdash_{NEAL} A \) then \( B, \Gamma \vdash_{NEAL} A \).

To annotate NEAL derivations, we use terms generated by the following grammar (elementary affine terms \( \Lambda^{EA} \)):

\[ M ::= x \mid \lambda x. M \mid (M \ M) \mid ! (M) \ [^M/x, \ldots, ^M/x] \ | \ [M]_{M=x,x} \]

Observe that in \( ! (M) \ [^M/x, \ldots, ^M/x] \), the \( [^M/x] \) is a kind of explicit substitution. To define ordinary substitution, define first the set of free variables of a term \( M \), \( \text{fv}(M) \), inductively as follows:
\[
\begin{align*}
\Gamma, A \vdash_{\text{NEAL}} A & \quad \frac{\Gamma \vdash_{\text{NEAL}} ! A \Delta, ! A \vdash_{\text{NEAL}} B}{\Gamma, \Delta \vdash_{\text{NEAL}} B} \quad \text{\textit{contr}} \\
\Gamma, A \vdash_{\text{NEAL}} B & \quad \frac{\Gamma \vdash_{\text{NEAL}} A \rightarrow B}{\Gamma \vdash_{\text{NEAL}} A \rightarrow B} \quad \text{\textit{\text{-o} I}} \\
\Gamma \vdash_{\text{NEAL}} A & \quad \frac{\Gamma \vdash_{\text{NEAL}} A \rightarrow B \Delta \vdash_{\text{NEAL}} A}{\Gamma, \Delta \vdash_{\text{NEAL}} B} \quad \text{\textit{\text{-o} E}} \\
\Delta_1 \vdash_{\text{NEAL}} ! A_1 \cdots \Delta_n \vdash_{\text{NEAL}} ! A_n & \quad \frac{A_1, \ldots, A_n \vdash_{\text{NEAL}} B}{\Gamma, \Delta_1, \ldots, \Delta_n \vdash_{\text{NEAL}} ! B} 
\end{align*}
\]

Figure 16: Natural Elementary Affine Logic in sequent style notation

- \(FV(x) = \{x\}\)
- \(FV(\lambda x. M) = FV(M) \setminus \{x\}\)
- \(FV(M_1 \ M_2) = FV(M_1) \cup FV(M_2)\)
- \(FV(! (M [M_1/x_1, \ldots, M_n/x_n]) = \bigcup_{i=1}^n FV(M_i) \cup FV(M) \setminus \{x_1, \ldots, x_n\}\)
- \(FV([M]_{N=x_1..x_2}) = (FV(M) \setminus \{x_1, x_2\}) \cup FV(N)\)

Ordinary substitution \(N\{M/x\}\) of a term \(M\) for the free occurrences of \(x\) in \(N\), is defined in the obvious way:

1. \(x\{M/x\} = M\);
2. \(y\{M/x\} = y\) if \(y \neq x\);
3. \(\lambda x. N\{M/x\} = \lambda x. N\);
4. \(\lambda y. N\{M/x\} = \lambda z. (N\{z/y\}\{M/x\})\) where \(z\) is a fresh variable;
5. \((N \ P)\{M/x\} = (N\{M/x\} \ P\{M/x\})\);
6. \(! (N) [P_1/x_1, \ldots, P_n/x_n]\{M/x\} = \bigcup_{i=1}^n FV(M_i) \cup FV(M) \setminus \{x_1, \ldots, x_n\}, where y_1, \ldots, y_n are all fresh variables;\)
7. \(! (N) [P_1/x_1, \ldots, P_n/x_n]\{M/x\} = ! (N) [P_1(M/x)/x_1, \ldots, P_n(M/x)/x_n]\) if \(\exists i\ s.t. \ x_i = x;\)
8. \([N]_{P=y,z}\{M/x\} = [N\{y'/y\}\{z'/z\}\{M/x\}]_{P(M/z)=y',z'}\) if \(x \notin \{y, z\}\), where \(y', z'\) are fresh variables;
9. \([N]_{P=y,z}\{M/x\} = [N]_{P(M/z)=y,z}\) if \(x \in \{y, z\}\).

Elementary terms may be mapped to \(\lambda\)-terms, by forgetting the exponential structure:

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\[
\begin{array}{c}
\Gamma, x : A \vdash_{\text{seal}} x : A \\
\Gamma \vdash_{\text{seal}} M : A \quad \Delta, x : A, y : A \vdash_{\text{seal}} N : B \\
\Gamma, \Delta \vdash_{\text{seal}} [N]_{M=x,y} : B
\end{array}
\]

\[
\begin{array}{c}
\Gamma, x : A \vdash_{\text{seal}} M : B \\
\Gamma \vdash_{\text{seal}} \lambda x. M : A \to B \\
\Gamma, \Delta \vdash_{\text{seal}} (M \ N) : B
\end{array}
\]

\[
\begin{array}{c}
\Delta_1 \vdash_{\text{seal}} M_1 : ! A_1, \ldots, \Delta_n \vdash_{\text{seal}} M_n : ! A_n \\
\Gamma, \Delta_1, \ldots, \Delta_n \vdash_{\text{seal}} ! (M_1/x_1) \ldots (M_n/x_n) : ! B
\end{array}
\]

\[
\begin{array}{c}
\Gamma, x : A \vdash_{\text{seal}} \exists x : A \\
\Gamma \vdash_{\text{seal}} \exists x : A \to B \\
\Gamma, \Delta \vdash_{\text{seal}} \exists (M \ N) : B
\end{array}
\]

\[
\begin{array}{c}
\Delta_1 \vdash_{\text{seal}} M_1 : ! A_1, \ldots, \Delta_n \vdash_{\text{seal}} M_n : ! A_n \\
\Gamma, \Delta_1, \ldots, \Delta_n \vdash_{\text{seal}} ! (M_1/x_1) \ldots (M_n/x_n) : ! B
\end{array}
\]

Figure 17: Term Assignment System for Natural Elementary Affine Logic

- \(x^* = x\)
- \((\lambda x. M)^* = \lambda x. M^*\)
- \((M_1 M_2)^* = (M_1^* M_2^*)\)
- \(! (M) [M_1/x_1, \ldots, M_n/x_n]^* = M^* \{M_1^*/x_1, \ldots, M_n^*/x_n\}\)
- \([M]_{N=x_1,x_2}^* = M^* \{N^*/x_1, N^*/x_2\}\)

**Definition 6.** (Legal elementary terms) The elementary terms are legal under the following conditions:

1. \(x\) is legal;
2. \(\lambda x. M\) is legal iff \(M\) is legal;
3. \((M_1 M_2)\) is legal iff \(M_1\) and \(M_2\) are both legal and \(\text{FV}(M_1) \cap \text{FV}(M_2) = \emptyset\);
4. \(! (M) [M_1/x_1, \ldots, M_n/x_n]\) is legal iff \(M\) and \(M_i\) are legal for any \(i \leq n\) and \(\text{FV}(M) = \{x_1, \ldots, x_n\}\) and \(i \neq j \Rightarrow \text{FV}(M_i) \cap \text{FV}(M_j) = \emptyset\);
5. \([M]_{N=x_1,x_2}\) is legal iff \(M\) and \(N\) are both legal and \(\text{FV}(M) \cap \text{FV}(N) = \emptyset\).

**Proposition 3.** If \(M\) is a legal term, then every free variable \(x \in \text{FV}(M)\) is linear in \(M\).

**Proof.** By trivial induction on the structure of \(M\) using definitions of legal terms and \(\text{FV}\).

**Note 1.** From now on we will consider only legal terms.

**Notation 4.** Let \(\Gamma = \{x_1 : A_1, \ldots, x_n : A_n\}\) be a basis. \(\text{dom}(\Gamma) = \{x_1, \ldots, x_n\}\); \(\Gamma(x_i) = A_i\); \(\Gamma \vdash V = \{x : A | x \in V \land A = \Gamma(x)\}\).

The term assignment system is shown in Figure 17, where all bases in the premises of the contraction, \(\to\) elimination and !-rule, have domains with empty intersection.
Lemma 2.

1. If $\Gamma \vdash_{\text{NEAL}} M : A$ then $\text{FV}(M) \subseteq \text{dom}(\Gamma)$;

2. if $\Gamma \vdash_{\text{NEAL}} M : A$ then $\Gamma \vdash \text{FV}(M) \vdash_{\text{NEAL}} M : A$.

Lemma 3 (Substitution). If $\Gamma, x : A \vdash_{\text{NEAL}} M : B$ and $\Delta \vdash_{\text{NEAL}} N : A$ and $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$ then $\Gamma, \Delta \vdash_{\text{NEAL}} M\{N/x\} : B$.

Proof. Recalling that both $M$ and $N$ are legal terms, by easy induction on the structure of $M$. □

Theorem 1 (Equivalence). $\Gamma \vdash_{\text{EAL}} A$ if and only if $\Gamma \vdash_{\text{NEAL}} A$.

Proof. (if) By induction, using the cut rule. It is also possible to prove, by an easy inspection of the cut-elimination theorem for EAL, that it is possible to eliminate just the exponential cuts, leaving the logical ones.

(only if) The only interesting case is $\neg \circ L$. The proof is identical to the case of intuitionistic logic. □

Lemma 4 (Unique Derivation). For any legal term $M$ and formula $A$, if there is a valid derivation of the form $\Gamma \vdash_{\text{NEAL}} M : A$, then such derivation is unique (up to weakening).

A notion of reduction is needed to state and obtain completeness of the type inference algorithm. We define two logical reductions ($\to_\beta$ and $\to_{\text{dup}}$) corresponding to the elimination of principal cuts in EAL. The other five reductions are permutation rules, allowing contraction to be moved out of a term.

$$\lambda x. M\{N/x\} \to_\beta M\{N/x\}$$

$$[N][M^{M_1}_1/x_1, \ldots, M_n/x_n] = x_1, y \quad \to_{\text{dup}}$$

$$[[N][M^{i}(x_1, \ldots, x_n)]/x_1]([M^{i}/y_1, \ldots, y_n]/y)]_{M_1 = x_1', y_1'} \cdots [M_n = x_n', y_n']$$

$$!\{M/M_1/x_1, \ldots, !\{N/P_i/y_1, \ldots, P_m/y_m\}/x_1, \ldots, M_n/x_n\}] \to_{!\varpi}$$

$$!\{M/N/x_1\}[[M_1/x_1, \ldots, P_i/y_1, \ldots, P_m/y_m]/x_1, \ldots, M_n/x_n\}]$$

$$(\{M_{M_1 = x_1, x_2}\ N) \to_{\emptyset - \varepsilon} ([M/N_{x_1, x_2}]_{N_{1 = x_1, x_2}})$$

$$(M\ N)_{N_1 = x_1, x_2} \to_{\emptyset - \varepsilon} ([M\ N_{x_1, x_2}]_{N_{1 = x_1, x_2}})$$

$$!(M)\{M_1/x_1, \ldots, !\{M_{M_1 = y_1, \ldots, M_n/x_n}\}/x_1, \ldots, M_n/x_n\}] \to_{!\varepsilon}$$

$$![M/N_{y_1, y_2}]_{N = y_1, y_2} \to_{\varepsilon - \varepsilon} ([M/N_{y_1, y_2}]_{N = y_1, y_2})$$

$$\lambda x. [M]_{N = y, z} \to_{\lambda - \varepsilon} [\lambda x. M]_{N = y, z}$$

where $x \notin \text{FV}(N)$.
where $M'$ in the $\rightarrow_{\text{dup}}$-rule is obtained from $M$ replacing all its free variables with fresh ones ($x_i$ is replaced with $y_i$): $x'_1$ and $x'_2$ in the $\rightarrow_{@-c}$-rule, $y'$ and $z'$ in the $\rightarrow_{!-c}$-rule and $y'_1, y'_2$ in the $\rightarrow_{c-c}$-rule are fresh variables.

**Definition 7.** The reduction relation on legal terms $\rightsquigarrow$ is defined as the reflexive and transitive closure of the union of $\rightarrow_{@}, \rightarrow_{\text{dup}}, \rightarrow_{!-!}, \rightarrow_{@-c}, \rightarrow_{!-c}, \rightarrow_{c-c}, \rightarrow_{\lambda-c}$.

**Proposition 4.** Let $M \rightsquigarrow N$ and $M$ be a legal term, then $N$ is a legal term.

**Proposition 5.** Let $M \rightarrow r$, $N$ where $r$ is not $\rightarrow_{\beta}$, then $M^* = N^*$.

**Lemma 5.** Let $M$ be a well typed term in $\{\text{dup}, !-, @-c, !-c, c-c, \lambda-c\}$-normal form, then

1. if $R = [N]_{p=x,y}$ is a subterm of $M$, then either $P = (P_1 \ P_2)$ or $P$ is a variable;
2. if $R = ! (N) [P_1/x_1, \ldots, P_k/x_k]$ is a subterm of $M$, then for any $i \in \{1, \ldots, k\}$ either $P_i = (Q_i, S_i)$ or $P_i$ is a variable.

**Theorem 2 (Subject Reduction).** Let $\Gamma \vdash_\text{NEAL} M : A$ and $M \rightsquigarrow N$, then $\Gamma \vdash_\text{NEAL} N : A$.

### 3.2 Properties of the Type Inference Algorithm

The following Lemma states that any slice in the set of critical points bars the rest of the term.

**Lemma 6.** Let $S(M : \sigma) = (\Theta, B, A, \text{cpts})$. For any slice $sl$ in $\text{cpts}$, $sl = \{\text{cpt}_1, \ldots, \text{cpt}_k\}$, for every path from the root of the syntax tree of $M$ to any leaf, there exists at most one $\text{cpt}_i$ in the path.

**Proof.** By induction on $M$. The unique interesting case is $M = (M_1 \ M_2)$. The thesis holds by inductive hypothesis and by a simple inspection of rules for $S$ and for the product union. \[ \square \]

The following lemma illustrates the relation between the set of critical points calculated by the algorithm for a given term $M$ and a particular class of decompositions of $M$.

**Lemma 7.** Let $S(M : \sigma) = (\Theta, B, A, \text{cpts})$.

1. $\forall \{\text{cpt}_1, \ldots, \text{cpt}_k\} = sl \in \text{cpts}$ there exist $P, (N_1, N_2), \ldots, (N_{1k}, N_{2k})$ such that $P$ is not a variable, $x_1, \ldots, x_k \in \text{FV}(P)$ and $M = P\{(N_1, N_2)/x_1, \ldots, (N_{1k}, N_{2k})/x_k\}$;
2. $\forall P, (N_1, N_2), \ldots, (N_{1k}, N_{2k})$ such that $P$ is not a variable, $x_1, \ldots, x_k \in \text{FV}(P)$ and $M = P\{(N_1, N_2)/x_1, \ldots, (N_{1k}, N_{2k})/x_k\}$, there exists $\{\text{cpt}_1, \ldots, \text{cpt}_k\} = sl \in \text{cpts}$ such that $\text{cpt}_i$ is the critical point at the root of $(N_1, N_2)$.\[ \square \]

**Proof.** By structural induction on $M$. 

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1. If $M$ is a variable, the thesis trivially holds being $\text{cpts} = \emptyset$. If $M = \lambda x.M'$, either $\text{sl}$ consists of a single critical point corresponding to the root of $M'$, then $P = \lambda x.y$, or $\text{sl}$ is a slice of $M'$, then by inductive hypothesis there exists $P'$ s.t. the thesis holds for $M'$. We take $P = \lambda x.P'$. Finally if $M = (M_1 M_2)$, if in $\text{sl}$ there is a critical point $\text{cpt}_i$ corresponding to the root of $M_1$ then by Lemma 6 all the other critical points in $\text{sl}$ belong to $M_2$ or there is only one critical point corresponding to the root of $M_2$. In the first case by inductive hypothesis there exists $P_2$ s.t. the thesis holds for $M_2$ and $\text{sl}$ without $\text{cpt}_i$. Then we take $P = (y P_2)$. The other cases are analogous.

2. If $M$ is a variable then $\not\exists P$ and the thesis trivially holds. If $M = \lambda x.M'$ then $P = \lambda x.P'$. If $P'$ is a variable, then the slice to consider is the one containing only the critical point corresponding to the root of $M'$. Such a slice has been added to $\text{cpts}$ in the rule for $S(\lambda x.(M_1 M_2) : \sigma)$ where $x \notin \text{FV}((M_1 M_2))$. Otherwise the thesis holds by inductive hypothesis. Finally if $M = (M_1 M_2)$, then $P = (P_1 P_2)$. If both $P_1$ and $P_2$ are not a variable, then by inductive hypothesis there exists $\text{sl}_1$ and $\text{sl}_2$. Then the thesis holds by definition of product union. The other cases are analogous.

Consider the length $L(M)$ of an EAL-term $M$ defined inductively:

$$L(x) = 0$$
$$L(\lambda x.M) = 1 + L(M)$$
$$L((M N)) = 1 + L(M) + L(N)$$

$$L(! (M) [M_1/x_1, \ldots, M_n/x_n] = L(M) + \sum_{i=1}^{n} L(M_i)$$

$$L([M]_{N=x,y}) = L(M) + L(N).$$

**Definition 8.** An EAL-term $M$ is simple if and only if

1. $M$ has no subterm of the form $[M_1]_{M_2=x,y}$ where $(M_2)^* \text{ is not a variable},$$
2. $L(M) = L((M)^*)$

**Fact 1.** A simple EAL-term contracts at most variables.

**Definition 9.** The set of candidate EAL-terms is the set of all EAL-terms $P$ such that

1. $P$ is in $\{!-, @-c, !-c, c-c, \lambda-c, \text{dup}\}$-normal form;
2. $P$ is simple;
3. if $[R]_{Q=x,y}$ is a subterm of $P$, then $x, y \in \text{FV}(R)$;
4. if $!(R) [Q_1/x_1, \ldots, Q_k/x_k]$ is a subterm of $P$, then $R$ is not a variable.
Definition 10. Given a general EAL-type $\Theta$ we define its erasure $\overline{\Theta}$ as the simple type obtained by erasing all the exponentials “!” and changing $\topo$ into $\to$.

Lemma 8. For any $\Theta$ general EAL-type there exists $X$ s.t. $X(\mathcal{P}(\overline{\Theta})) = \Theta$.

Theorem 3 (Completeness). Let $\Gamma \vdash_{NEAL} P : \Psi$ and let $P$ be a candidate EAL-term. Let $\mathcal{P}(P^* : \overline{\Psi}) = (\Theta, B, A)$, then there exists $X$ integer solution of $A$ such that $X(B) \subseteq \Gamma$, $\Psi = X(\Theta)$ and $X(B) \vdash_{NEAL} P : X(\Theta)$.

Proof. By induction on $P$.

- If $\Gamma, x : \Psi \vdash_{NEAL} x : \Psi$ then $S(x : \overline{\Psi}) = \langle \mathcal{P}(\overline{\Psi}), \{x : \mathcal{P}(\overline{\Psi})\}, \emptyset \rangle$ and the thesis holds by Lemma being any $X$ solution of the empty set of constraints.

- If the type derivation ends with

$$\Gamma \vdash_{NEAL} x : \Phi \Delta, y : \Phi, z : \Phi \vdash_{NEAL} N : \Psi$$

then the thesis holds by inductive hypothesis on $\Delta, y : \Phi, z : \Phi \vdash_{NEAL} N : \Psi$.

- If $P$ is an abstraction then the type derivation is

$$\Gamma, x : \Psi \vdash_{NEAL} M : \Phi \quad \Gamma \vdash_{NEAL} \lambda x. M : \Psi \to \Phi$$

The thesis holds by inductive hypothesis. Notice that the solution $X$ instantiates all variables introduced by the $B$ call of the rule for $S$ to 0. It is easy to see looking at the rules for $B$ that if in the solution $X$ there is one variable introduced by $B$ that is not set to zero, then the type is exponential and $\Psi \to \Phi$ is not.

- If $P$ is an application

$$\Gamma \vdash_{NEAL} M : \Phi \quad \Delta \vdash_{NEAL} N : \Phi$$

By inductive hypothesis there are solutions $X_1$ for $M$ and $X_2$ for $N$. Now, by the same considerations of the previous point, $X_1$ sets all variables introduced by the last $B$ call to 0. Thus the constraint $\sum n_j = 0$ of the rule for $S$ is satisfied. Moreover $X_1, X_2$ satisfies the constraints for the unification of types, because they are identical by hypothesis. Hence the thesis holds.

- Finally, if the derivation is

$$\Delta_1 \vdash_{NEAL} M_1 : \Phi_1 \cdots \Delta_n \vdash_{NEAL} M_n : \Phi_n \quad x_1 : \Phi_1, \ldots, x_n : \Phi_n \vdash_{NEAL} N : \Psi$$

then $\Gamma, \Delta_1, \ldots, \Delta_n \vdash_{NEAL} (N \ [M_1/x_1, \ldots, M_n/x_n] : \Psi$
then by Lemma 5 either $M_i$ is a variable or an application. If all $M_i$ are variables, then the thesis holds getting the solution of the inductive hypothesis and increasing the variable $b$ introduced by $B$ by one.

If there is an $M_i$ that is an application, then by Lemma 7 there is a critical point collected by the algorithm at the root of $M_i$. Then we take as solution $X$ the union of the solutions obtained by inductive hypothesis with the variable introduced by $B$ for the critical point corresponding to $M_i$ increased by one.

In the statement of the previous theorem, the request on the \{!−!, @−c, c−c, λ−c, dup\}-normal form is not a loss of generality, for the subject reduction lemma and Proposition 5. By Lemma 5, the only restriction induced by the request of contracting at most variable is the exclusion of elementary terms with subterms of the form $[R](Q_1, Q_2)=x,y$ or $!(R)[P_1/x_1, \ldots, (Q, Q_2)/x, \ldots, P_n/x_n]$ with $[S]_{x=y,z}$ subterm of $R$. Recalling the discussion at the end of Section 1, we see that these terms, in a sense, “contract too much” — in the sharing graph of the corresponding $\lambda$-term $P^*$, there would be fan nodes corresponding to non-variable contractions. We also do not take into account elementary affine terms with “false contractions”. This is not a limitation by Lemma 1 and Theorem 2.

Finally we discard term such $!(x)[M/x]$. Again this is not a limitation, in fact $(!(x)[M/x])^* = M^*$ and $\Gamma \vdash_{\text{neal}} !(x)[M/x] : !\Psi$ if and only if $\Gamma \vdash_{\text{neal}} M : \Psi$.

Notation 5. We use

$$
\Gamma \vdash M : !^n A \quad x : A \vdash N : B
$$

as a shorthand for

$$
\begin{array}{c}
\ldots \\
x_2 : ! ! A \vdash x_2 : ! ! A \\
x_1 : ! A \vdash ! (N)[x_1/x] : ! B \\
\vdots \\
\Gamma \vdash M : ! \ldots ! A \\
x_{n-1} : ! \ldots ! A \vdash ! (\cdots (N)[x_1/x] \ldots )[x_{n-1}/x_{n-2}] : ! \cdots ! B \\
\Gamma \vdash ! (\cdots (N)[x_1/x] \ldots )[M/x_{n-1}] : ! \cdots ! B
\end{array}
$$

Lemma 9 (Superimposing of derivations). Let $\mathcal{S}(M : \sigma) = (\Theta, B, A)$ and let $A$ be solvable. If there is a solution $X_1$ of $A$ that instantiates two boxes belonging to two superimposed derivations that are not compatible, then there exists another solution $X_2$ where all the instantiated boxes belong to the same derivation.

Moreover $X_1(\Theta) = X_2(\Theta)$ and $X_1(B) = X_2(B)$.
Proof. The proof of the lemma can be easily understood if we follow the intuition explained below with an example.

We may think of boxes as levels; boxing a subterm can then be seen as raising that subterm, as in Figure 18, where also some types label the edges of the syntax tree of a simple term. In particular, the edge starting from the \( @ \)-node and ending in \( x_0 \) has label \( !^{n_2}(\alpha \rightarrow !^{n_1}(\beta \rightarrow \gamma)) \) at level 0 (nearest to \( x_0 \)) and has label \( (\alpha \rightarrow \gamma) \) at level \( n_2 \). This is the graphical counterpart of the \(!\)-rule

\[
\ldots, x_0 : T, \ldots \vdash \ldots
\]
\[
\ldots, x_0 : !^{n_2}T, \ldots \vdash \ldots
\]

The complete decoration of Figure 18 can be produced in NEAL in two ways: by the instantiation of

\[
!^{n_2} (((x_0 \ x_1)y)((x_4 \ x_5)w)) \ [x_2 \ x_3]/y, (x_6 \ x_7)/w]
\]

and\footnote{The correct legal terms should have all free variable inside the square brackets. We omit to write variables when they are just renamed, for readability reasons (compare the first elementary term above with the (fussy) correct one \(!^{n_2} (((x_0 \ x_1)y)((x_4 \ x_5)w)) [x'_0/x_0, x'_1/x_1, (x_2 \ x_3)/y, x'_4/x_4, x'_5/x_5, (x_6 \ x_7)/w].\)
}

\[
!^{n_1} (((z\ x_2 \ x_3))((x_4 \ x_5)w)) \ [x_0 \ x_1]/z, (x_6 \ x_7)/w],
\]

which are boxes belonging to two different derivations. Graphically such an instantiation can be represented as in the first row of Figure 19 where incompatibility is evident by the fact that the boxes are not well stacked, in particular the rectangular one covers a hole. To have a correct EAL-derivation it is necessary to find the equivalent, well stacked configuration (that corresponds to the subsequent application of boxes from the topmost to the bottommost).

The procedure by which we find the well stacked box configuration is visualized in Figure 19. The reader may imagine the boxes subject to gravity (the passage from the first to the second row of Figure 19) and able to fuse each other when they are at the same level (the little square in the third row fuse with the solid at its left in the passage from the third to the fourth row).

The “gravity operator” corresponds to finding the minimal common subterm of all the superimposed derivations and it is useful for finding the correct order of
Figure 19: Equivalences of boxes.

application of the ! rule. The “fusion operator” corresponds to the elimination of a cut between two exponential formulas. Moreover, the final configuration of Figure 19 corresponds to a particular solution of the set of constraints produced by the type synthesis algorithm, that instantiates the following boxes:

\[ !^{n_1} ( !^{n_2 - n_1} ( !^{n_1} (((z \, y)((x_4 \, x_5)w))) [(x_0 \, x_1)/z])) [(x_2 \, x_3)/y]) (x_6 \, x_7)/w] \]

Finally, notice that during the procedure all types labeling the boundary edges of the lambda-term never changes, i.e., the instantiations of the term type (the label of the topmost edge) and the base types (the labels of the edges at the bottom) remain unchanged.

Now let \( S(M : \sigma) = (\Theta, B, A) \) and let \( X \) be the solution that instantiates \( k \) overlapping—thus incompatible—boxes. Consider the boxed syntax tree of \( M \) and associate to any node its level, i.e., the number of boxes containing the node. Notice that if there is a wire connecting the nodes \( a \) of level \( \ell \) and \( b \) of level \( \ell + k \), then the type labeling the wire is \( !^{k}\Psi \) near \( a \) and \( \Psi \) near \( b \), i.e., the sum of level and number of exponentials for types labeling the syntax tree is an invariant. We break the boxes using the following procedure: starting from the root of the syntax tree of \( M \), we are at level \( i = 0 \); we proceed with a breath first visit and whenever encounter a node of level \( \ell \neq i \) we close \( i \) boxes, open \( \ell \) boxes and set \( i \) to \( \ell \).

At the end of the procedure described above there are no more overlapping boxes, but it could be happen that there is a variable \( x \) not in the same boxes of its binding lambda node. Such configuration of boxes is not correct. However the level of the variable and lambda node is the same because the procedure of breaking boxes does not change level of nodes. Moreover all nodes belonging to the path from the lambda node to the variable have level higher or equal to the level of the variable since they all were initially in the same box and some
of them were eventually also in some overlapping boxes that increase the level. Hence we can fuse boxes until variable and corresponding binder are in the same box. The fusion operation is shown in Figure 20 and described by the following equation:

\[
!^k(M^{(h-k)(h^Q/z)/x})[^P/y] \text{ if } k \leq h \\
!^k(M[^P/y, (h^Q/z)]/x) \text{ if } k > h
\]

After all fusions are performed, all variables are in the same boxes of their lambda binders and there are no more overlapping boxes, thus the decoration obtained corresponds to an EAL-derivation. By completeness exists \( X_2 \) solution corresponding to such decoration. Moreover types labeling the syntax tree are unchanged by the transformations applied, hence the thesis.

**Theorem 4 (Soundness).** Let \( \mathcal{S}(M : \sigma) = (\Theta, B, A) \). For every integer solution of \( A \), there exists \( P \) candidate EAL-term such that \( P^* = M \) and \( X(B) \vdash_{\text{EAL}} P : X(\Theta) \).

**Proof.** By induction on the structure of \( M \), using the superimposing lemma. We first need a definition:

**Definition 11.** A syntax tree \( T \) is correctly decorated if the edges of the graph are labeled according to Figure 21 (in the rightmost picture, \( \Theta \) is inside \( n \) boxes). Moreover all edges connecting a variable \( x \) occurring multiple, are labeled with the same type \( !^n \Gamma \). In the case the variable is abstracted, the type label of variable is syntactically identical to the argument part of the type label of the edge at the root of the abstraction.
Given a correctly decorated syntax tree, and an instantiation $X$ for the general EAL-types labeling its edges such that the number of exponentials for types of multiple variables is greater than 1, it is easy to build the corresponding NEAL derivation, using the Curry-Howard isomorphism and eventually applying a contraction before the $\to$ introduction for binded variables and at the end of the derivation for free variables.

Thus, in order to prove soundness of our algorithm, it is sufficient to prove by structural induction on $M$ that we can build a correctly decorated syntax tree. If the solution taken into account instantiates two overlapping boxes we use Lemma 9. Hence without loss of generality we can consider $X$ such that all boxes are compatible. The only interesting part of the proof is the checking

\[ n = \sum n_i - c \quad \sum n_i = \sum n_i - c \quad \sum n_i - c = 0 \]

Figure 22: Decorations given by $B$.

of rules for $B$. In Figure 22 it is shown how build a correctly decorated syntax tree when the solution $X$ instantiates a box passing through a critical point (all three cases of critical points are depicted).

Finally we need to prove that $P$ is a candidate EAL-term. Points 2 and 3 of Definition 9 hold by construction of the NEAL derivation from the correctly decorated syntax tree, which also guarantees that $P$ is in \{@$ - c, ! - c, c - c, \lambda - c, \text{dup}\}-normal form. Point 4 holds by definition of $B$, and $P$ is in !-!-normal form by the superimposing lemma.

**Theorem 5 (Main theorem).** Let $M$ be a simply typeable $\lambda$-term. For any basis $\Gamma$ and EAL formula $C$: 

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\[ \Gamma \vdash_{\text{EAL}} M : C \text{ iff } \mathcal{S}(M : \Gamma) = (\Theta, B, A) \text{ and } A \text{ admits an integral solution } X \text{ such that } X(B) \subseteq \Gamma \text{ and } C = X(\Theta). \]

Proof. (\(\Rightarrow\)) \(\Gamma \vdash_{\text{EAL}} M : C\) is established by a sharing graph where no fan node faces the root of a subgraph. It is ready to see that the corresponding EAL-term is a candidate EAL-term. Theorem 3 allows to conclude.

(\(\Leftarrow\)) By Theorem 4, there is an EAL-term \(P\) such that \(P^* = M\) and \(X(B) \vdash_{\text{NEAL}} P : X(\Theta)\). The NEAL-term \(P\) codes a sharing graph establishing \(X(B) \vdash_{\text{EAL}} P^* : X(\Theta)\).

Lemma 10. Let \(M\) be a simply typeable \(\lambda\)-term; let \(\sigma\) be its principal type schema, and let \(\tau\) be any other type for \(M\). If \(\mathcal{S}(M : \tau) = (\Theta', B', A')\) and \(A\) admits a solution \(X\), then \(\mathcal{S}(M : \sigma) = (\Theta', B', A')\) and there exists \(X'\) solution of \(A'\).

Proof. We have to show that it is not the case that \(A\) admits a solution and \(A'\) is unsolvable. Constraints are added only by contraction \(\text{[13]}\) or unification \(\text{[41]}\). The former constraints depend only on the structure of the syntax tree of the term and hence they are not affected by the type change. As for the latter, changing \(\tau\) into \(\sigma\) makes some unification constraints disappear. In fact, it is possible to decompose \(\Theta\) in \(\Theta'\{x_1 \rightarrow \Sigma_1, \ldots, x_n \rightarrow \Sigma_n\}\). When the algorithm synthesizes \(M : \sigma\), all unification constraints in \(A\) regarding \(\Sigma_1 \ldots \Sigma_n\) disappear, and we obtain \(A'\) (up to renaming). In order to prove that \(A'\) is \(A\) minus the set of unification constraints produced by \(\Sigma_1 \ldots \Sigma_n\), it is sufficient to inspect the definitions of \(\mathcal{P}\) and \(\mathcal{W}\). As the solution space has increased, it is not possible that \(A'\) has no solution.

Corollary 1. Let \(M\) be a simply typeable \(\lambda\)-term and let \(\sigma\) be its principal type schema. For any basis \(\Gamma\) and EAL formula \(C\): \(\Gamma \vdash_{\text{EAL}} M : C\) iff \(\mathcal{S}(M : \sigma) = (\Theta, B, A)\), \(A\) admits an integral solution \(X\) and there exists a substitution \(S\) from type variables to EAL-types such that \(S(X(B)) \subseteq \Gamma\) and \(S(X(\Theta)) = C\).

The corollary gives a weak notion of principal type for EAL. Any EAL type of a term arises as an instance of a solution of the constraints obtained for its simple principal type schema. The result, however, does not say anything on the structure of these \(!\)-decorated instances. The study of a general notion of principal schema for EAL is the subject of \(\text{[CR03]}\). On the other hand, the corollary is enough to establish the decidability of type inference.

Theorem 6. It is decidable whether, given a type-free \(\lambda\)-term \(M\), there exist an EAL formula \(C\) and a basis \(\Gamma\) such that \(\Gamma \vdash_{\text{EAL}} M : C\).

4 Conclusions

We have presented an algorithm for assigning EAL types to type-free, pure \(\lambda\)-terms, obtained as the (technically non trivial) elaboration of the idea of “box decoration” of a simple type derivation. The algorithm is shown complete with
4.1 Linear decorations of intuitionistic derivations

The problem to obtain linear logic derivations from intuitionistic derivations has been thoroughly studied [DJS95, Sch94, Rov92]. Our linear constraints method can be used to obtain a variety of such decorations.

The implicational fragment of linear logic can be obtained from EAL by adding the rules:

\[ \Gamma, A \vdash B \]

\[ \Gamma, !A \vdash B \]

\[ \Gamma, !!A \vdash B \]

Introduce now the rule \((d+b)\)

\[ \frac{\Gamma, !A \vdash B}{\Gamma, !-(d+b)A \vdash B} \]

\[ (d+b) \]

which acts as a multiple \(\delta\) rule, except when \(d = 0\) and \(b = -1\). In this case it is the same of an \(\epsilon\) rule. It is easy to prove that \(\Gamma \vdash_{\text{LL}} B\) iff \(\Gamma \vdash_{\text{LL}} -\{\delta,\epsilon\} \cup (d+b) B\).

Extend now the maximal decoration method as follows. After each logical rule, interleave \(n\) \(!\)-rules, and then, for each formula \(A\) in the context, add one \((d_i+b_i)\) rule and \(e_i\) \(\epsilon\)-rules. For example

\[ A, B \vdash C \]

\[ A \vdash B \rightarrow C \]

becomes

\[ \frac{\text{\(n\)} A, \text{\(n\)} B \vdash \text{\(n\)} C}{\text{\(n\)} A, \text{\(n\)} B \vdash \text{\(n\)} C} \]

\[ (b+d) \]

\[ \frac{\text{\(n\)} - (d_1+b_1) + e_1 A, \text{\(n\)} - (d_2+b_2) B \vdash \text{\(n\)} C}{\text{\(n\)} - (d_1+b_1) + e_1 A, \text{\(n\)} - (d_2+b_2) B \vdash \text{\(n\)} C} \]

\[ (b+d) \]

\[ \frac{\text{\(n\)} - (d_1+b_1) + e_1 A, \text{\(n\)} - (d_2+b_2) + e_2 B \vdash \text{\(n\)} C}{\text{\(n\)} - (d_1+b_1) + e_1 A, \text{\(n\)} - (d_2+b_2) + e_2 B \vdash \text{\(n\)} C} \]

During the type inference, the set of constraints obtained from unification and contraction is augmented by the constraints of rules \((d_i+b_i)\). It is not difficult to see that any solution of the set of constraints collected by the algorithm gives a linear logic derivation having the original intuitionistic derivation as a skeleton.

Notice that the meta-derivations obtained by the above procedure represents a set of LL derivations complete for the provability of LL formulas. In fact, the
unique derivations of LL that are not direct instances of the previous meta-derivations are those where exponential rules are applied in a different order. However, it is easy to see that the rules under discussion may be freely permuted. For example, if \( \Gamma \vdash_{LL} B \) with an application of \(!\)-rule followed by an \( \epsilon\)-rule, then \( \Gamma \vdash_{LL} B \) with inverted order of exponential rules (the proof is similar for the other cases).

The use of linear constraints allows now the use of linear programming techniques to obtain decorations with specific properties. By minimizing the objective function \( \sum_i n_i + \sum_j (d_j + b_j) + \sum_k e_k \), we obtain decorations using a minimal number of boxes. Or, we may minimize only the use of \( \epsilon \) and \( \delta \) rules, if we minimize \( \sum_j (d_j + b_j) + \sum_k e_k \). In the language of optimal reduction, these are decorations introducing a minimal number of brackets and croissants, and are thus the natural candidates to be used as initial translations for those \( \lambda \)-terms which does not have an EAL type.

4.2 Arbitrary contractions

Instead of using Definition 1, we may would like an algorithm complete with respect to the notion given directly by Figure 1, that is, allowing arbitrary contractions (and not only variable contractions) in the sharing graphs. Proceed as follows. Given a generic sharing graph, first decompose it into several subgraphs with the property that no fan faces a subgraph; than readback them, obtaining a set of lambda-terms. For example, the graph of Figure 4 of Section 1 can be decomposed in \( \lambda z. \lambda x. \lambda w. (k k w) \) and \( (x z) \). After the decomposition, call the type synthesis algorithm separately on every subterm, calculate the suitable unification constraints with \( \mathcal{U} \), collect all the constraints in a single system, and solve it.

This procedure computes all possible decorations, except those boxes that surround more than one subterm. However, the proof of the superimposing lemma allows to conclude that there is a decoration with a box around more than one subterm if and only if there exists a decoration with boxes only around a single subterm, with the same type (see Figure 23 for a graphical intuition).
Acknowledgments

We are happy to thank Harry Mairson, for extended comments and criticism on previous versions of the paper; and Simona Ronchi della Rocca, for the many discussions, suggestions, and comments.

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A Appendix

We have already observed that the simply typed lambda term

\[(\lambda n. (n \, \lambda y. (n \, \lambda z. y)) \, \lambda x. (x \, (x \, y))) : o\]
is not typeable in EAL. If one knows optimal reduction [AG98], this can be seen in a simple way, writing the term as a sharing graph and reducing it in the abstract algorithm by matching fans by labels (see Figure 24 where the redexes fired at every step are indicated by a dashed oval). The sharing graph in normal form is a cycle, that is a sharing graph which does not correspond to any $\lambda$-term (least to say to $y$, which is the normal form of the given term). This means that the oracle is needed for the reduction of this term, and hence it cannot have a type in EAL.

Figure 24: Incorrect reduction of $$(\lambda n.(\lambda y.(n \lambda z.y)) \lambda x.(x \, x \, y)).$$
We can give a formal proof, by calling the type inference algorithm on such a term. The following is a trace of the execution, where each box delimits the call and return of a single type inference rule:

\( S(\lambda n.(n \lambda y.(n \lambda z.y)) \lambda x.(x (x y))) : o) \)

\[ S(\lambda n.(n \lambda y.(n \lambda z.y)) : (o \to o) \to o) \]

\[ S(n : (o \to o) \to o) \]

\[ \mathcal{P}(o \to o) = p_1(p_2(p_3 \to o) \to o) \]

\[ = (p_1(p_2(p_3 \to o) \to o), \{ n : p_1(p_2(p_3 \to o) \to o) \}, \emptyset, \emptyset) \]

\[ S(\lambda y.(n \lambda z.y) : o \to o) \]

\[ S(n : (o \to o) : o) \]

\[ \mathcal{P}(o \to o) = p_6(p_7(p_8 \to o) \to p_{10}) \]

\[ = (p_6(p_7(p_8 \to o) \to p_{10}), \{ n : p_6(p_7(p_8 \to o) \to p_{10}) \}, \emptyset, \emptyset) \]

\[ S(\lambda z.y : o \to o) \]

\[ S(y : o) \]

\[ \mathcal{P}(o) = p_{11} \]

\[ = (p_{11}, \{ y : p_{11} \}, \emptyset, \emptyset) \]

\[ \mathcal{B}(y, \{ y : p_{11} \}, p_{11}, \emptyset, \emptyset) = \{ \{ y : p_{11} \}, p_{11}, \emptyset \} \]

\[ \mathcal{P}(o) = p_{12} \]

\[ = (p_{12} \to p_{11}, \{ y : p_{11} \}, \emptyset, \emptyset) \]

\[ \mathcal{B}(\lambda z.y, \{ y : p_{11} \}, p_{12} \to p_{11}, \emptyset, \emptyset) \]

\[ \mathcal{B}(y, p_{11}, p_{12} \to p_{11}, \emptyset, \emptyset) = \{ \{ y : p_{11} \}, p_{12} \to p_{11}, \emptyset \} \]

\[ = (\{ y : b_1 + p_{11} \}, b_1(p_{12} \to p_{11}), \emptyset) \]

\[ \mathcal{W}(p_7(p_8 \to o), b_1(p_{12} \to p_{11})) = \left\{ \begin{array}{l} p_7 = b_1 \\ p_8 = p_{12} \\ p_9 = p_{11} \end{array} \right\} \]

\[ = \langle p_{10}, \{ n : p_6(p_7(p_8 \to o) \to p_{10}), y : b_1 + p_{11} \}, \left\{ \begin{array}{l} p_7 = b_1 \\ p_8 = p_{12} \\ p_9 = p_{11} \\ p_6 = 0 \end{array} \right\} \rangle \]

\[ \mathcal{B}(\lambda z.y, \{ n : b_1(p_8 \to o) \to p_{10}, y : b_1 + p_9, p_{10}, \emptyset, \emptyset) \}

\[ = (\{ n : b_1(p_8 \to o) \to p_{10}, y : b_1 + p_9, p_{10}, \emptyset, \emptyset) \}

\[ C(b_2 + b_1 + p_9) = \emptyset \]

\[ = (b_2 + b_1 + p_9 \to b_2 + p_{10}, \emptyset) \}

\[ \mathcal{B}(\lambda z.y, \{ n : b_2(p_8 \to o) \to p_{10}, b_2 + b_1 + p_9, \emptyset, \emptyset) \}

\[ = (\{ n : b_2(p_8 \to o) \to p_{10}, b_2 + b_1 + p_9, \emptyset, \emptyset) \}

\[ \mathcal{W}(p_2(p_3 \to o), b_3(b_2 + b_1 + p_9 \to b_2 + p_{10}) = \left\{ \begin{array}{l} p_2 = b_3 \\ p_3 = b_2 + b_1 + p_9 \\ p_4 = b_2 + p_{10} \end{array} \right\} \]
\[
\begin{align*}
&= \left\{ \begin{array}{l}
x : b_2(p_2 \rightarrow p_3), \\
y : b_1(p_6 \rightarrow p_7),
\end{array} \right\}, b_2 + p_3, \{ b_1 + p_7 - p_2 - b_2 = 0 \} \\
&= \left\{ \begin{array}{l}
x : b_3 + b_2(p_2 \rightarrow p_3), \\
y : b_3 + b_1 + p_6
\end{array} \right\}, b_3 + b_2 + p_3, \{ b_1 + p_7 - p_2 - b_2 = 0 \}
\end{align*}
\]

\[
C(b_3 + b_2(p_2 \rightarrow p_3), b_3 + b_1(p_6 \rightarrow p_7)) = \left\{ \begin{array}{ll}
b_3 + b_2 \geq 1 \\
b_3 + b_2 = b_3 + b_1 \\
p_2 = p_6 \\
p_3 = p_7
\end{array} \right\} = \left\{ \begin{array}{ll}
b_3 + b_2 \geq 1 \\
b_2 = b_1 \\
p_2 = p_6 \\
p_3 = p_7
\end{array} \right\}
\]

\[
\exists \lambda x. (x (x y)), \{ y : b_3 + b_1 + p_2, b_3 + b_1(p_2 \rightarrow p_3) \rightarrow b_3 + b_1 + p_2, \emptyset, \{ b_3 + b_1 \geq 1 \})
\]

\[
\forall (b_4((p_5 \rightarrow p_5) \rightarrow p_5, b_6(b_3 + b_1(p_2 \rightarrow p_3) \rightarrow b_3 + b_1 + p_2))
\]

\[
\begin{align*}
&= \left\{ \begin{array}{l}
b_4 = b_2 \\
p_5 = p_2 \\
p_5 = b_3 + b_1 + p_2
\end{array} \right\} = \left\{ \begin{array}{l}
b_4 = b_2 \\
p_5 = p_2 \\
b_3 + b_1 = 0
\end{array} \right\}
\end{align*}
\]

Notice that the last constraint \( b_3 + b_1 = 0 \) is incompatible with the previous \( b_3 + b_1 \geq 1 \) hence the set of solutions is empty.