Classical Fuzzy Retrial Queue with Working Vacation using Hexagonal Fuzzy Numbers

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Abstract. Using fuzzy techniques “Classical Fuzzy Retrial Queue with Working Vacation (WV) using Hexagonal Fuzzy Numbers” is discussed in this paper. We acquire model in fuzzy environment as the average orbit length, Probability (Pr) that the server busy, and Pr (the server is in a WV period), the sojourn time of a customer in the queue. Finally numerical results are presented.

1. Introduction

Recently, the retrial queueing systems with working vacations have been investigated extensively Do [1] first studied an M/M/1 retrial queue with working vacations. Seyed Behrouz et al.[2] analyzed A fuzzy based threshold policy for a single server retrial queue with vacations. Following zadeh, many researchers and considered the problem of fuzzy queueing systems. On the basis of zadeh’s extension principle [4]. Dhurai.K, Karpagam.A[3] investigated A new membership function of hexagonal fuzzy numbers. In this paper the Part 2 describe fuzzy queue model. In Part 3 we discuss the average orbit length, P1 (the server is busy) and P1 (the server is in a WV period), the sojourn time of a customer in the queue are studied in fuzzy model. In Part 4 gives the numerical study of this model. Finally, conclusions are gained in part 5.

2. The fuzzy queue model

The model of this paper as M/M/1 retrial queue. we consider fuzzy arrival rate $\lambda$, then the fuzzy service rate is $\gamma$, fuzzy retrial rate $\beta$, fuzzy vacation time $\theta$, customer served fuzzy rate $\xi$ are assume to be fuzzy numbers respectively.

Now

$\lambda = \{(g, \mu_\lambda(g)), (g) \in s(\lambda)\}$

$\gamma = \{(h, \mu_\gamma(h)), (h) \in s(\gamma)\}$

$\beta = \{(i, \mu_\beta(i)), (i) \in s(\beta)\}$

$\theta = \{(j, \mu_\theta(j)), (j) \in s(\theta)\}$

$\xi = \{(l, \mu_\xi(l)), (l) \in s(\xi)\}$
Where $S(\bar{x}), S(\bar{y}), S(\bar{z}), S(\bar{\xi})$ are the universal sets of the arrival rate, service rate, retrial rate, vacation time, customer served rate respectively. It defines $f(g, h, i, j, l)$ as the system performance measure related to the above defined fuzzy queuing model, which depends on the fuzzy membership functions $S(\bar{x}), S(\bar{y}), S(\bar{z}), S(\bar{\xi})$. Applying Zadeh’s extension principle (1978), the membership functions of the measure of efficiency $f(\bar{x}, \bar{y}, \bar{z}, \bar{\xi})$ can be given as.

\[
\mu_{f}(\bar{x}, \bar{y}, \bar{z}, \bar{\xi})(H) = \sup_{g \in S(\bar{x})} \{ \min \{ \mu_{x}(g), \mu_{y}(h), \mu_{z}(i), \mu_{\xi}(l) / H = f(g,h,i,j,l) \} \} -----(1)
\]

If the $\alpha$-cuts of $f(\bar{x}, \bar{y}, \bar{z}, \bar{\xi})$ degenerate to some fixed value, then the system showing is a crisp number, otherwise it is a fuzzy number.

We acquire the membership function of some measure of efficiency, namely the average orbit length, $P_{t}$(the server is busy), $P_{r}$(the server is in a WV period), the sojourn time of a customer in the queue for the system in terms of these membership functions are, as follows

\[
\mu_{E(O_{L})}(L) = \sup_{g \in S(\bar{x})} \{ \min \{ \mu_{x}(g), \mu_{y}(h), \mu_{z}(i), \mu_{\xi}(l) / L = f(g,h,i,j,l) \} \} -----(2)
\]

Where

\[
L = \frac{g^{3}h + g^{2}[h(gh+j)+g(h+l)+i(h-g)^{2}] + [j+k(j+l)]g(j+l)}{(h-g)(j+l)[h(gh+j)+g(h-l)+gh] + g(h-g)(j+l)[h+l][h+g]}
\]

\[
\mu_{E(P_{b})}(M) = \sup_{g \in S(\bar{x})} \{ \min \{ \mu_{x}(g), \mu_{y}(h), \mu_{z}(i), \mu_{\xi}(l) / M = f(g,h,i,j,l) \} \} -----(3)
\]

Where

\[
M = \frac{g(h-g)-k(j+l)-\theta-g^{2}}{g(h-l)+h(1+\theta)(j+l)}
\]

\[
\mu_{E(W_{V})}(Q) = \sup_{g \in S(\bar{x})} \{ \min \{ \mu_{x}(g), \mu_{y}(h), \mu_{z}(i), \mu_{\xi}(l) / Q = f(g,h,i,j,l) \} \} -----(4)
\]

Where

\[
Q = \frac{ghj+k(j+l)}{g(h-l)+h(1+\theta)(j+l)}
\]
\( \mu_{E[\delta]}(N) = \sup_{g \in ES(\delta)} \{ \min_{\lambda \in ES(\lambda)} [\mu_\delta(g), \mu_{\psi}(h), \mu_{\beta}(i), \mu_{\beta}(j), \mu_{\xi}(l) / N = f(g, h, i, j, l) ] \} \) ------ (5)

\[
g^3h + g^2[i(h-g) + (j(h-g) + h) + h[(h-g) + (j + l)] + (h-g)^2] + \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\lambda=1}^{N} g^3h + g^2[i(h-g) + (j(h-g) + h) + h[(h-g) + (j + l)] + (h-g)^2] + \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\lambda=1}^{N} = \]

Where \( N = \frac{g(h-g)(j+l)[(h-g)+g]}{g(h-g)(j+l)[(h-g)+g]} \)

Using the above principle is used to derive the membership values of \( \mu_{E[\delta]}, \mu_{E[\beta]}, \mu_{E[\psi]}, \mu_{E[\xi]} \) as a function of the parameter \( \alpha \). We are developing the membership functions of \( \mu_{E[\delta]}, \mu_{E[\beta]}, \mu_{E[\psi]}, \mu_{E[\xi]} \):

3. Performance of measure
We calculate the performance measures for this model in fuzzy model.

The average orbit length

Based on above principle \( \mu_{E[\delta]} \) is superimum of minimum over \( \{ \mu_\delta(g), \mu_{\psi}(h), \mu_{\beta}(i), \mu_{\beta}(j), \mu_{\xi}(l) \} \)

\[
L = \frac{g^3h + g^2[i(h-g) + (j(h-g) + h) + h[(h-g) + (j + l)] + (h-g)^2] + \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\lambda=1}^{N} g^3h + g^2[i(h-g) + (j(h-g) + h) + h[(h-g) + (j + l)] + (h-g)^2] + \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\lambda=1}^{N} = \]

to satisfying \( \mu_{E[\delta]}(L) = \alpha, \quad 0 \leq \alpha \leq 1 \)

We have the following five types:

Type (i) : \( \mu_\delta(g) = \alpha, \mu_{\psi}(h) \geq \alpha, \mu_{\beta}(i) \geq \alpha, \mu_{\beta}(j) \geq \alpha, \mu_{\xi}(l) \geq \alpha \)

Type (ii) : \( \mu_\delta(g) \geq \alpha, \mu_{\psi}(h) = \alpha, \mu_{\beta}(i) \geq \alpha, \mu_{\beta}(j) \geq \alpha, \mu_{\xi}(l) \geq \alpha \)

Type (iii) : \( \mu_\delta(g) \geq \alpha, \mu_{\psi}(h) \geq \alpha, \mu_{\beta}(i) = \alpha, \mu_{\beta}(j) \geq \alpha, \mu_{\xi}(l) \geq \alpha \)

Type (iv) : \( \mu_\delta(g) \geq \alpha, \mu_{\psi}(h) \geq \alpha, \mu_{\beta}(i) \geq \alpha, \mu_{\beta}(j) = \alpha, \mu_{\xi}(l) \geq \alpha \)

Type (v) : \( \mu_\delta(g) \geq \alpha, \mu_{\psi}(h) \geq \alpha, \mu_{\beta}(i) \geq \alpha, \mu_{\beta}(j) \geq \alpha, \mu_{\xi}(l) = \alpha \)

We can work out the lower and upper bounds of the \( \alpha \)-cuts of \( E[\delta L] \) for the all types past the corresponding parametric non-linear programme.

It can be written as

\[
E[\delta L]_a^{L} = \min_{\delta} \{ [L] \} \text{ and } E[\delta L]_a^{U} = \max_{\delta} \{ [L] \}
\]

Such that

\[
[g]_a^{L} \leq g \leq [g]_a^{U}, [h]_a^{L} \leq h \leq [h]_a^{U}, [i]_a^{L} \leq i \leq [i]_a^{U}, [j]_a^{L} \leq j \leq [j]_a^{U}, [l]_a^{L} \leq l \leq [l]_a^{U}
\]

Now,
\[ E[OL]_{\bar{a}}^{L} \text{ and } E[OL]_{\bar{a}}^{U} \text{ are inverse with respect to } \alpha, \text{ the left and right shape function,} \]
\[ L(L) = [E[OL]_{\bar{a}}^{L}]^{-1} \text{ and } R(L) = [E[OL]_{\bar{a}}^{U}]^{-1} \text{ can be derived from which the membership function} \]
\[ \mu_{E[OL]}(L) \text{ can be make as,} \]
\[ \mu_{E[OL]}(L) = \begin{cases} 
L(L), & E[OL]_{\bar{a}}^{L} \leq L \leq E[OL]_{\bar{a}}^{U} \\
1, & E[OL]_{\bar{a}}^{L} \leq L \leq E[OL]_{\bar{a}}^{U} \\
R(L), & E[OL]_{\bar{a}}^{L} \leq L \leq E[OL]_{\bar{a}}^{U} 
\end{cases} \quad (6) \]

Using the above method we get the upcoming results.

\( P_{r} \text{ (the server is busy)} \)
\[ \mu_{E[P_{\bar{a}}]}(M) = \begin{cases} 
L(M), & E[P_{\bar{a}}]^{L}_{\bar{a}} \leq M \leq E[P_{\bar{a}}]^{U}_{\bar{a}} \\
1, & E[P_{\bar{a}}]^{L}_{\bar{a}} \leq M \leq E[P_{\bar{a}}]^{U}_{\bar{a}} \\
R(M), & E[P_{\bar{a}}]^{L}_{\bar{a}} \leq M \leq E[P_{\bar{a}}]^{U}_{\bar{a}} 
\end{cases} \quad (7) \]

\( P_{r} \text{ (the server is in a WV period)} \)
\[ \mu_{E[WV]}(Q) = \begin{cases} 
L(Q), & E[WV]^{L}_{\bar{a}} \leq Q \leq E[WV]^{U}_{\bar{a}} \\
1, & E[WV]^{L}_{\bar{a}} \leq Q \leq E[WV]^{U}_{\bar{a}} \\
R(Q), & E[WV]^{L}_{\bar{a}} \leq Q \leq E[WV]^{U}_{\bar{a}} 
\end{cases} \quad (8) \]

The sojourn time of a customer in the queue
\[ \mu_{E[S]}(N) = \begin{cases} 
L(N), & E[S]^{L}_{\bar{a}} \leq N \leq E[S]^{U}_{\bar{a}} \\
1, & E[S]^{L}_{\bar{a}} \leq N \leq E[S]^{U}_{\bar{a}} \\
R(N), & E[S]^{L}_{\bar{a}} \leq N \leq E[S]^{U}_{\bar{a}} 
\end{cases} \quad (9) \]

4. Numerical Study

The average orbit length:

Opine the arrival rate \( \bar{\lambda} \), service rate \( \bar{\gamma} \), retrial rate \( \bar{\theta} \), vacation time \( \bar{\theta} \) and customer served rate \( \bar{\xi} \) are assumed to be hexagonal fuzzy numbers expressed by:

\( \bar{\lambda} = \{11,12,13,14,15,16\} \text{ and } \bar{\gamma} = \{71,72,73,74,75,76\} \text{ and } \bar{\theta} = \{21,22,23,24,25,26\} \text{ and } \bar{\theta} = \{51,52,53,54,55,56\} \text{ and } \bar{\xi} = \{91,92,93,94,95,96\} \) sequentially. Next

\[ \lambda(\alpha) = \min_{x \in \bar{s}(\bar{\lambda}), G(x) \geq \alpha} \max_{x \in \bar{s}(\bar{\lambda}), G(x) \geq \alpha} \]

Where
\[ G(x) = \begin{cases} 
\frac{1}{2} \left( \frac{x-a_{1}}{a_{2}-a_{1}} \right), & \text{for } a_{1} \leq x \leq a_{2} \\
\frac{1}{2} + \frac{1}{2} \left( \frac{x-a_{1}}{a_{3}-a_{1}} \right), & \text{for } a_{2} \leq x \leq a_{3} \\
1, & \text{for } a_{3} \leq x \leq a_{4} \\
1 - \frac{1}{2} \left( \frac{x-a_{4}}{a_{5}-a_{4}} \right), & \text{for } a_{4} \leq x \leq a_{5} \\
\frac{1}{2} \left( \frac{a_{6}-x}{a_{6}-a_{5}} \right), & \text{for } a_{5} \leq x \leq a_{6} \\
0, & \text{for otherwise} 
\end{cases} \quad (10) \]
(i.e.), λ (α) = [11 + α], [16 - α], γ(α) = [71 + α], [76 - α], β(α) = [21 + α], [26 - α], 
θ(α) = [51 + α], [56 - α], (α) = [91 + α], [96 - α]. It is clear that , when \( g = g_{α}^L \), \( h = h_{α}^L \), \( i = i_{α}^L \), \( j = j_{α}^L \), \( l = l_{α}^L \) L achieve its superlative value and when \( g = g_{α}^L \), \( h = h_{α}^L \), \( i = i_{α}^L \), \( j = j_{α}^L \), \( l = l_{α}^L \) L achieve its slightest value.

From the generated for the given input values of \( \lambda \), \( \gamma \), \( \beta \), \( \beta \) and \( \xi \)

(i) If L decrease as g increases then for the fixed value h, i, j, l.
(ii) If L decrease as h increases then for the fixed value g, i, j, l.
(iii) If L decrease as i increases then for the fixed value g, h, j, l.
(iv) If L decrease as j increases then for the fixed value g, h, i, l.
(v) If L decrease as l increases then for the fixed value g, h, i, j.

i.e), their upper bound given by \( g = 11 + a \), \( h = 76 - a \), \( i = 26 - a \), \( j = 56 - a \), \( l = 96 - a \). And the superlative value of \( E[OL] \) occurs when \( g = 61 - a \), \( h = 71 + a \), \( i = 21 + a \), \( j = 51 + a \), \( l = 91 + a \). It can be build as:

\[
G(x) = \begin{cases} 
0.5(x - 1), & \text{for } a_1 \leq x \leq a_2 \\
0.5 + 0.5(x - 1), & \text{for } a_2 \leq x \leq a_3 \\
1, & \text{for } a_3 \leq x \leq a_4 \\
1 + 0.5(x - 4), & \text{for } a_4 \leq x \leq a_5 \\
0.5(6 - x), & \text{for } a_5 \leq x \leq a_6 \\
0, & \text{for } \text{otherwise}
\end{cases} \tag{11}
\]

The values of \( a_1, a_2, a_3, a_4, a_5, a_6 \) get from (6) are:

\[
\mu_{E[OL]}(L) = \begin{cases} 
0.5(x - 1), & \text{for } 1.057376 \leq x \leq 1.801349 \\
0.5 + 0.5(x - 1), & \text{for } 1.801349 \leq x \leq 2.017264 \\
1, & \text{for } 2.017264 \leq x \leq 2.017498 \\
1 + 0.5(x - 4), & \text{for } 2.017498 \leq x \leq 1.801290 \\
0.5(6 - x), & \text{for } 1.801290 \leq x \leq 1.075937 \\
0, & \text{for } \text{otherwise}
\end{cases} \tag{12}
\]

In likewise we arrived the successive outcomes.

\( P, \) (the server is busy):

\[
\mu_{E[P, \text{is busy}]}(M) = \begin{cases} 
0.5(x - 1), & \text{for } 0.00000 \leq x \leq 0.613480 \\
0.5 + 0.5(x - 1), & \text{for } 0.613480 \leq x \leq 0.617440 \\
1, & \text{for } 0.617440 \leq x \leq 0.618424 \\
1 + 0.5(x - 4), & \text{for } 0.618424 \leq x \leq 0.614848 \\
0.5(6 - x), & \text{for } 6.503594 \leq x \leq 0.00000 \\
0, & \text{for } \text{otherwise}
\end{cases} \tag{13}
\]
The server is in a WV period:

\[
\begin{align*}
\mu_{E[\bar{W}_V]} (Q) &= \begin{cases} 
0.5(x - 1), & \text{for } 0.5852 \leq x \leq 0.5928 \\
0.5 + 0.5(x - 1), & \text{for } 0.5928 \leq x \leq 0.6200 \\
1, & \text{for } 0.6200 \leq x \leq 0.6206 \\
1 + 0.5(x - 4), & \text{for } 0.6206 \leq x \leq 0.5936 \\
0.5(6 - x), & \text{for } 0.5936 \leq x \leq 0.5860 \end{cases} \\
& \text{otherwise}
\end{align*}
\]

The sojourn time of a customer in the queue:

\[
\begin{align*}
\mu_{E[\bar{S}]} (N) &= \begin{cases} 
0.5(x - 1), & \text{for } 0.102363 \leq x \leq 0.122603 \\
0.5 + 0.5(x - 1), & \text{for } 0.122603 \leq x \leq 0.127737 \\
1, & \text{for } 0.127737 \leq x \leq 0.127776 \\
1 + 0.5(x - 4), & \text{for } 0.127776 \leq x \leq 0.122727 \\
0.5(6 - x), & \text{for } 0.122727 \leq x \leq 0.102658 \\
0, & \text{otherwise}
\end{cases} 
\end{align*}
\]

The upcoming four graphs are constitute the measure of efficiency

![Graph showing Arrival rate, Service rate versus the average orbit length]
Fig. 2

Arrival rate, Service rate versus The probability that the server is busy

Fig. 3

Fig. 4
5. Conclusion

Here, we discuss Classical fuzzy retrial queue with WV using hexagonal fuzzy numbers. Also, we obtained the $P_r$ (the server is busy) and $P_s$ (server is in a WV period), the sojourn time of a customer in the queue. The numerical results show the effective of performance measure. Example for this fuzzy queue model is telephone psychological counselling and computer systems.

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