DECORATED MARKED SURFACES II: INTERSECTION NUMBERS AND DIMENSIONS OF HOMS

YU QIU AND YU ZHOU

Abstract. We study the 3-Calabi-Yau categories $D$ arising from quivers with potential associated to a decorated marked surface $S_\Delta$, in the sense of [5]. We prove two conjectures in [5], that under the bijection between certain objects in $D$ and certain arcs in $S_\Delta$, the dimensions of homomorphisms between these objects equal the intersection numbers between the corresponding arcs.

Key words: intersection numbers, spherical twist, Calabi-Yau categories

1. Introduction

In this paper, we study a class of finite-dimensional derived category $D$, associated to quivers with potential from triangulated marked surfaces $S$. These categories are 3-Calabi-Yau, originally arose in the study of homological mirror symmetry. For instance, in type $A$, such categories $D$ was first studied by Khovanov-Seidel [4]. They showed that there are faithful (classical) braid group actions on $D$, which plays an crucial role in understanding such categories. The key ingredients there are:

- there is a bijection between spherical objects in $D$ and (admissible) curves on a disk with some decorated points;
- the dimension of (double graded) Hom between these spherical objects equals the (double graded) intersection between the corresponding curves.

Such an idea was generalized by the first author in [5]. The main motivation comes from the work of Bridgeland-Smith, who studied the link between stability conditions and quadratic differentials for marked surfaces. He introduces the decorated marked surface $S_\Delta$ for an unpunctured marked surface $S$ and shows that

- there is a bijection $\tilde{X}$ between spherical objects in $D$ and simple (closed) arc on $S_\Delta$;
- the spherical twist group (up to isotopy) of $D$ can be identified with the braid twist group of $S_\Delta$.

A conjecture was made in [5], that under the bijection $\tilde{X}$, the dimensions of homomorphisms between these spherical objects equal the intersection numbers between the corresponding arcs. In this sequel, we will prove this conjecture (Theorem 4.1), together with another similar one (Theorem 4.4).

Moreover, in the next sequel [1], we will apply the result here to prove that there is an unique canonical way to identify the derived categories associated to any triangulation of $S_\Delta$. This will imply that the spherical twists acts faithfully on the principal component of the space of stability conditions.
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2. Preliminaries

2.1. Triangulated 3-Calabi-Yau categories and spherical twists. A triangulated category $\mathcal{T}$ is called 3-Calabi-Yau if for any objects $X, Y \in \mathcal{T}$, there is a functorial isomorphism

$$\text{Hom}_\mathcal{T}(X, Y) \cong D \text{Hom}_\mathcal{T}(Y, X[3])$$

where $D = \text{Hom}_k(-, k)$ is the canonical duality. An (indecomposable) object $S$ in a triangulated 3-Calabi-Yau category $\mathcal{T}$ is called (3-)spherical if $\text{Hom}_\mathcal{T}^\bullet(S, S) = \delta$ if $k = 0$ or $3$ and equals to zero otherwise. Recall, e.g. from [6], that the twist functor of a spherical object $S$ is defined by

$$\phi_S(X) = \text{Cone}(S \otimes \text{Hom}_\mathcal{T}^\bullet(S, X) \to X)$$

with inverse

$$\phi_S^{-1}(X) = \text{Cone}(X \to S \otimes \text{Hom}_\mathcal{T}^\bullet(X, S)).$$

2.2. Quivers with potentials, Ginzburg dg algebras and 3-Calabi-Yau categories. A quiver with potential is a pair $(Q, W)$, where $Q$ is a finite quiver and $W$ is a linear combination of cycles in $Q$. A differential graded quiver $\hat{Q}$ associated to $(Q, W)$ is defined as follows

- it has the same vertex set $Q_0$;
- its arrows consists of three parts:
  - the arrows of $Q$ with degree 0,
  - an arrow $a^* : j \to i$ with degree $-1$, for each arrow $a : i \to j$ of $Q$,
  - a loop $t_i : i \to i$ with degree $-2$, for each vertex $i$ of $Q$;
- its differential $d$ is defined by
  - $d(a) = 0$,
  - $d(a^*) = \partial_a W$, for $a$ an arrow of $Q$ and
  - $\sum_{a \in Q_0} d(t_i) = \sum_{a \in Q_1}[a, a^*]$.

Moreover, we have the following notation:

- $\Gamma = \Gamma(Q, W)$: the Ginzburg dg algebra $k\hat{Q}$;
- $\mathcal{P}(Q, W)$: the Jacobian algebra which is the zeroth homology of $\Gamma$;
- $\text{per} \Gamma$: the perfect derived category of $\Gamma$;
- $\mathcal{D}_{fd}(\Gamma)$: the finite derived category of $\Gamma$.

It is known that $\mathcal{D}_{fd}(\Gamma)$ is 3-Calabi-Yau [2]. Let $\mathcal{H}_\Gamma$ be the canonical heart of $\mathcal{D}_{fd}(\Gamma)$ and $\text{Sim} \mathcal{H}_\Gamma$ be the set of its simples. As in [5], we use the following notations:

- $\text{ST}(\Gamma)$: the spherical twist subgroup of $\text{Aut} \mathcal{D}_{fd}(\Gamma)$, generated by $\phi_S$ for $S \in \text{Sim} \mathcal{H}_\Gamma$;
- $\text{Sph}(\Gamma)$: the set of reachable spherical objects in $\mathcal{D}_{fd}(\Gamma)$, i.e. $\text{ST}(\Gamma) \cdot \text{Sim} \mathcal{H}_\Gamma$.
2.3. **Triangulations of marked surfaces.** An (unpunctured) marked surface $S$ is an orientated compact surface with a finite set $M$ of marked points lying in its non-empty boundary $\partial S$. Up to homeomorphism, a marked surface $S$ is determined by the following data:

- the genus $g$ of $S$;
- the number $b$ of boundary components of $S$; and
- the partition $(m_1, \ldots, m_b)$ of the number $m = |M|$ describing the number of marked points in each boundary component.

An arc $\gamma$ in $S$ is a curve in the surface satisfying

- the endpoints of $\gamma$ are in $M$;
- except for its endpoints, $\gamma$ is disjoint from $\partial S$;
- $\gamma$ has no self-intersections in $S - M$; and
- $\gamma$ is not isotopic to a point or a boundary segment.

The arcs are always considered up to isotopy.

A triangulation $T$ of $S$ is a maximal collection of arcs in $S$ which do not cross each other in the interior of $S$. Then we have the following formula

$$n := |T| = 6g + 3b + m - 6.$$

It follows that the number $\mathcal{N}$ of the triangles in a triangulation $T$ satisfies the following formula

$$\mathcal{N} = \frac{2n + m}{3}.$$  \hfill (2.1)

Labardini-Fragoso defined a quiver with potential $(Q_T, W_T)$, associated to each triangulation $T$ of $S$ as follows:

- the vertices of $Q_T$ are indexed by the arcs in $T$;
- each clockwise angle in a triangle of $T$ gives an arrow between the vertices indexed by the edges of the angle;
- each triangle in $T$ with three edges are initial arcs gives a 3-cycle (up to cyclic permutation) and the potential $W_T$ is the sum of the such 3-cycles.

2.4. **Decorated marked surfaces.** A decorated marked surface $S_\triangle$ is a marked surface with an extra set $\triangle$ of $\aleph$ decorated points in the interior of $S$.

A closed arc $\eta$ in $S_\triangle$ is a curve in $S$ such that

- its endpoints are two different decorated points;
- except for its endpoints, $\gamma$ is disjoint from $\triangle$ and from the boundary $\partial S$ of $S$; and
- it has no self-intersections.

A general closed arc in $S_\triangle$ is a curve in $S$ such that it satisfies the last two conditions above and its endpoints are decorated points (but not necessarily different).

An open arc $\gamma$ in $S_\triangle$ is a curve in $S$ such that

- its endpoints are in $M$;
- except for its endpoints, $\gamma$ is disjoint from $\triangle$ and from the boundary $\partial S$ of $S$;
- it has no self-intersections in $S - \triangle$; and
- it is not isotopic to a point or a boundary component.
Both of (general) closed arcs and open arcs in $S_\Delta$ are always considered up to isotopy. We denote by $\text{CA}(S_\Delta)$, $\overline{\text{CA}}(S_\Delta)$ and $\text{OA}(S_\Delta)$ the set of closed, general closed and open arcs in $S_\Delta$, respectively.

Recall from [5, § 3.1] that the intersection numbers between arcs in $S_\Delta$ are defined as follows.

I.: For an open arc $\gamma$ and an (open or closed) arc $\eta$, their intersection number is defined as the geometric intersection number in $S_\Delta - M$:

$$\text{Int}(\gamma, \eta) = \min\{|\gamma' \cap \eta' \cap (S_\Delta - M)| \mid \gamma' \simeq \gamma, \eta' \simeq \eta\}$$

II.: For two closed arcs $\eta_1, \eta_2$, their intersection number is defined to be a number in $\frac{1}{2}\mathbb{Z}$ as follows:

$$\text{Int}(\eta_1, \eta_2) = \frac{1}{2} \text{Int}_\Delta(\eta_1, \eta_2) + \text{Int}_{S-\Delta}(\eta_1, \eta_2)$$

where

$$\text{Int}_\gamma(\eta_1, \eta_2) = \min\{\eta_1 \cap \eta_2 \cap \gamma \mid \eta_1' \simeq \eta_1, \eta_2' \simeq \eta_2\}$$

for $\gamma = \Delta, S - \Delta$.

A triangulation $T$ is a maximal collection of open arcs in $S_\Delta$ such that

- for any $\gamma_1, \gamma_2 \in T$, $\text{Int}(\gamma_1, \gamma_2) = 0$; and
- each triangle of $T$ contains exactly one point in $\Delta$.

The forgetful map $F: S_\Delta \to S$ forgetting the decorated points induces a map from $\text{OA}(S_\Delta)$ to the set of arcs in $S$ which sends a triangulation $T$ of $S_\Delta$ to a triangulation $T = F(T)$ of $S$. The quiver with potential $(Q_T, W_T)$ associated to $T$ is defined to be $(Q_T, W_T)$.

Let $\gamma$ be an (open) arc in a triangulation $T$. The arc $\gamma^f = \gamma^f(T)$ (resp. $\gamma^b$) is the arc obtained from $\gamma$ by anticlockwise (resp. clockwise) moving its endpoints along the quadrilateral in $T$ whose diagonal is $\gamma$, to the next marked points. The forward (resp. backward) flip of a triangulation $T$ at $\gamma \in T$ is the triangulation $T_\gamma^f = T \cup \{\gamma^f\} - \{\gamma\}$ (resp. $T_\gamma^b = T \cup \{\gamma^b\} - \{\gamma\}$). See Figure 1 for example. The exchange graph $\text{EG}(S_\Delta)$ is the oriented graph whose vertices are triangulations of $S_\Delta$ and whose arrows correspond to forward and backward flips. From now on, fix a connected component $\text{EG}^o(S_\Delta)$ of $\text{EG}(S_\Delta)$. When we mention a triangulation $T$ of $S_\Delta$, we always mean that $T$ is in $\text{EG}^o(S_\Delta)$.
2.5. The braid twists. The mapping class group $\text{MCG}(S_\Delta)$ of $S_\Delta$ consists of the isotopy classes of the homeomorphisms of $S$ which fix $\partial S$ pointwise and fix the set $\Delta$.

For any closed arc $\eta \in \text{CA}(S_\Delta)$, the braid twist $B_\eta \in \text{MCG}(S_\Delta)$ along $\eta$ is defined as in Figure 2. The braid twist group $\text{BT}(S_\Delta)$ is defined as the subgroup of $\text{MCG}(S_\Delta)$ generated by $B_\eta$ for $\eta \in \text{CA}(S_\Delta)$.

![Figure 2. The Braid twist](image)

2.6. Topological preparation. We start with a lemma. Let

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

be a disk with three punctures $P_0 = (0, \frac{1}{2})$, $P_1 = (0, -\frac{1}{2})$, $P_2 = (0, 0)$. See Figure 3. Set

$$D^{>0} = \{(x, y) \in D \mid y > 0\} \text{ and } D^{<0} = \{(x, y) \in D \mid y < 0\}.$$

With in this subsection, when we mention a curve, we always mean a continuous map from $[0, 1]$ to $D$ such that it is disjoint with the punctures except at its endpoints. A curve is called simple if it has no self-intersections except at its endpoints and is called closed if its endpoints coincide. For a curve $\eta : [0, 1] \to D$, its restriction $\eta|_{[t_1, t_2]}$ is a curve defined as $\eta|_{[t_1, t_2]}(t) = \eta((t_2 - t_1)t + t_1)$ for $t \in [0, 1]$ and its inverse $\eta^{-1}$ is a curve defined as $\eta^{-1}(t) = \eta(1 - t)$ for $t \in [0, 1]$. For any two curves $\eta_1, \eta_2$, if $\eta_1(1) = \eta_2(0)$, their composition $\eta_2 \eta_1$ is a curve defined as

$$\eta_2 \eta_1(t) = \begin{cases} \eta_1(2t) & 1 \leq t \leq \frac{1}{2}, \\ \eta_2(2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

For a closed curve $\eta$, denote by $D_\eta$ the disk (possibly with punctures) enclosed by $\eta$.

Let $\eta : [0, 1] \to D$ be an arc with $\eta(0) = P_0$ and $\eta(1) = P_1$. We assume that $\eta$ has minimal intersections with the line $y = 0$ in its homotopy class, i.e.

$$|\{0 < t < 1 \mid \eta(t) \in \{y = 0\}\}| \leq |\{0 < t < 1 \mid \eta'(t) \in \{y = 0\}\}|$$

for any $\eta' \simeq \eta$ relative to $\{0, 1\}$.

**Lemma 2.1.** Let $\eta$ as above, $\eta_0 : [0, 1] \to D^{>0}$ and $\eta_1 : [0, 1] \to D^{<0}$ be two arcs satisfying the following conditions:

i): $\eta_0(0) = P_0$, $\eta_1(1) = P_1$;
ii): there exist \( 0 < r_0 < s_0 < 1 \) such that \( \eta(r_0) \in \{ y = 0 \} \) and \( \eta_0(1) = \eta(s_0) \);

iii): there exist \( 0 < r_1 < s_1 < 1 \) such that \( \eta(r_1) \in \{ y = 0 \} \) and \( \eta_1(0) = \eta(s_1) \).

Then the arcs \( \eta_0, \eta_1 \) intersect \( \eta \) at \( \eta(s_0), \eta(s_1) \) respectively from different sides. Moreover, the arc \( \alpha_0 := \eta|_{[s_0,1]} \eta_0 \) is homotopic to \( \alpha_1 := \eta_1 \eta|_{[0,s_1]} \) relative to \( \{0,1\} \).

**Proof.** Let \( c_0 := \eta_0^{-1} \eta|_{[0,s_0]} \) and \( c_1 := \eta|_{[s_1,1]} \eta_1^{-1} \). Then both of them are simple closed curves. Thanks to that \( \eta \) has minimal intersections with the line \( y = 0 \), we have \( \eta_0 \neq \eta|_{[0,s_0]} \) because \( \eta(r_0) \in \{ y = 0 \} \) for some \( 0 < r_0 < s_0 \). Similarly, \( \eta_1 \neq \eta|_{[s_1,1]} \).

So both of the disks \( D_{c_0} \) and \( D_{c_1} \) contains at least one puncture. Now we claim that \( D_{c_0} \subseteq D_{c_1} \) or \( D_{c_1} \subseteq D_{c_0} \). Otherwise, they are disjoint since they do not intersect except for sharing a common boundary segment. Then they both have to contains the puncture \( P_2 \), which is a contradiction.

Without loss of generality, we assume that \( D_{c_0} \subseteq D_{c_1} \) and \( \eta_0 \) intersects with \( \eta \) at \( \eta(s_0) \) from the left side. Then up to homotopy, there are three cases in the Figure 4. In the first two cases, condition one holds and the disk \( D_{\alpha_1^{-1} \alpha_0} \) contains no punctures. Hence \( \alpha_0 \simeq \alpha_1 \) relative to \( \{0,1\} \). In case (c), the disk \( D_{\eta^{-1} \eta_1 \eta|_{[s_0,s_1]}} \eta_0 \) contains no punctures. So \( \eta \simeq \eta \eta|_{[s_0,s_1]} \eta_0 \). But \( \eta \eta|_{[s_0,s_1]} \eta_0 \) (\( \simeq \eta \)) has less intersections with \( y = 0 \) than \( \eta \), which is a contradiction. This completes the proof. \( \square \)

**Figure 3.** A sketch graph for Lemma 2.1.
Now we generalize [5, Lemma 3.14] to our case. Let $T$ be a triangulation of $S_\Delta$.

**Lemma 2.2.** Let $\eta$ be a closed arc in $\text{CA}(S_\Delta)$ which is not in $T^*$. Then there are two arcs $\alpha, \beta$ in $\text{CA}(S_\Delta)$ such that

\[
\text{Int}_{S-\Delta}(\alpha, \beta) = 0 \quad (s o \text{ Int}(\alpha, \beta) \leq 1),
\]

\[
\eta = B_\alpha(\beta) \quad \text{or} \quad \eta = B_\alpha^{-1}(\beta),
\]

\[
\text{Int}(\gamma_i, \alpha) < \text{Int}(\gamma_i, \eta), \quad \text{Int}(\gamma_i, \beta) < \text{Int}(\gamma_i, \eta),
\]

and

\[
|\text{Int}(\gamma_i, \eta) - 2 \text{Int}(\alpha, \beta) \text{Int}(\gamma_i, \alpha)| = \text{Int}(\gamma_i, \beta)
\]

for each $\gamma_i \in T$.

**Proof.** Assume that $\eta$ has minimal intersections with the arcs in $T$ without loss of generality. If $\eta$ intersects at least three triangles of $T$ or $\eta \in T^*$, then the assertion holds by the proof of [5, Lemma 3.14]. Thus we can suppose that $\eta$ intersects exactly two triangles of $T$ with the endpoints $P_0$ and $P_1$ and $\eta \notin T^*$.

Since the original marked surface $S$ is not once punctured torus, these two triangles that intersect with $\eta$ cannot share three edges. One the other hand, if they share only one edge, $\eta$, which is contained in these two triangles, can only intersect them once. Therefore, these two triangle share exactly two edges and they form a annulus $A$. As we only care the interior of the union of these two triangles, we are in the situation of Lemma 2.1:

1): the two boundaries of the $A$ correspond to the boundary of the disk and the puncture $P_2$ there;

2): the decorated points $P_0$ and $P_1$ correspond to the punctures $P_0$ and $P_1$ there;

3): the arcs of $T$ in $A$ topologically correspond to $\{y = 0\} \cap D$ there.

Now, since $\text{Int}(\eta, T)$ is at least 3, we can find the arcs $\eta_0$ and $\eta_1$ satisfying the conditions in Lemma 2.1. Denote by $\eta|_{s_0, s_1}$ the arc $\eta|_{s_0, s_1}$ for $s_0 < s_1$ or the arc $(\eta|_{s_1, s_0})^{-1}$ for $s_0 > s_1$. Let $\alpha = \alpha_1 = \alpha_2$ and $\beta = \eta_1\eta|_{s_0, s_1}\eta_0$. Then $\text{Int}_{S-\Delta}(\alpha, \beta) = 0$ and $\eta = B_\alpha(\beta)$ (for the case that $\eta_0$ intersects $\eta$ at $\eta(s_0)$ from the left side) or $\eta = B_\alpha^{-1}(\beta)$ (for the case that $\eta_0$ intersects $\eta$ at $\eta(s_0)$ from the right side). Note that $\eta_0$ and $\eta_1$ do not cross the arcs in $T$. Then

\[
\text{Int}(\gamma_i, \alpha) = \text{Int}(\gamma_i, \eta|_{0, s_1}) = \text{Int}(\gamma_i, \eta|_{s_0, 1}) < \text{Int}(\gamma_i, \eta)
\]
and
\[ \text{Int}(\gamma_i, \beta) = \text{Int}(\gamma_i, \eta|_{s_0, s_1}) < \text{Int}(\gamma_i, \eta) \]
(see Figure 4). Hence
\[ \text{Int}(\gamma_i, \eta) = \begin{cases} 
2 \text{Int}(\gamma_i, \alpha) + \text{Int}(\gamma_i, \beta), & \text{for the case } s_0 > s_1; \\
2 \text{Int}(\gamma_i, \alpha) - \text{Int}(\gamma_i, \beta), & \text{for the case } s_0 < s_1.
\end{cases} \]

Thus, we complete the proof. \(\square\)

As an immediate consequence, the proof in [5, Proposition 4.3 and Proposition 4.4] works for all cases (include two special cases there):

**Proposition 2.3.** [5, Proposition 4.3 and Proposition 4.4] For any triangulation \(T^*\) of \(S\), we have
\[ \text{BT}(S_\Delta) = \text{BT}(T) \quad \text{and} \quad \text{CA}(S_\Delta) = \text{BT}(S_\Delta) \cdot T^*. \]

3. **Intersection numbers and dimensions of Homs**

Recall that in [5], the author gives a bijection from the set of closed arcs in \(S_\Delta\) to the set of reachable spherical objects in \(D_{fd}(\Gamma_{T_0})\), for any triangulation \(T_0\) such that any two of its triangles share at most one edge. In this section, we generalize this bijection to arbitrary triangulation \(T\) (of any decorated marked surface).

3.1. **The string model.** Next we give a correspondence from closed arcs in \(\text{CA}(S_\Delta)\) to objects in \(\text{D}_{fd}(\Gamma_T)\) for any triangulation \(T\) of \(S_\Delta\), following [5, §5.3]. In Appendix A, we recall the construction there. A difference from [5] is that there may be double arrows in the quiver \(Q_T\) (cf. Figure 5 for example). Then the component of homomorphism space of a degree between simples may be not 1-dimensional. So to arc segments we don’t associate homomorphism spaces, instead, associate basis elements of homomorphism spaces which are induced by graded arrows in \(Q_T\) (see Figure 10). Hence we have the followings.

- For each \(\eta \in \text{CA}(S_\Delta)\), \(X_\eta := X_{w(\eta)}\) defined in Appendix A is a perfect dg \(\mathcal{E}\)-module.
- \(X_\eta\) is well-defined up to shifts (since \(X_\eta = X_{\eta^{-1}}[m_{l(w)}(w_\eta)]\)). Denote the map by
  \[ \hat{X}_T: \text{OA}(S_\Delta) \to \text{per} \mathcal{E}_T / [1], \]
  \[ \eta \mapsto \hat{X}_T(\eta). \]

We will use the convention that \(X_\eta\) will be a representative in the shift orbits \(\hat{X}_T(\eta)\) and the notation \(X[Z]\) means the shift orbit that contains \(X\).
- If \(X_\eta[Z] = S[Z]\) for some initial closed arc \(s_i \in T^*\), then \(\eta = s_i\).

As shown in Appendix A.3, we have the following key proposition.

**Proposition 3.1.** Let \(\sigma \in \text{CA}(S_\Delta)\) and \(\tau \in \text{CA}(S_\Delta)\) satisfying \(\text{Int}_{S_\Delta}(\sigma, \tau) = 0\) and \(\eta = B_\sigma(\tau)\). Then there exists representatives \(X_?\) in \(\hat{X}_T(?)\) for \(? = \sigma, \tau, \eta\) such that there is a non-split triangle
\[ X_\tau \to X_\eta \to \bigoplus_{i=1}^{2\text{Int}(\sigma, \tau)} X_\sigma[m_i] \xrightarrow{(f)} X_\tau[1] \]
for some $m_i \in \mathbb{Z}$, where $f_i$’s are linearly independent.

3.2. A first formula. Fix an initial triangulation $T_0 = \{\gamma_1, \ldots, \gamma_n\}$. Write $\Gamma_0 = \Gamma_{T_0}$ and denote by $\mathcal{H}_0$ the canonical heart of $D_{fd}(\Gamma_{T_0})$. Let $T^*_0 = \{s_1, \ldots, s_n\}$ be the dual of $T_0$ and $S_i$ be the corresponding simples in $\mathcal{H}_0$.

**Assumption 3.2.** Suppose that $S$ admits a triangulation, such that any two triangles share at most one edge. This is equivalent to the condition that there is no double arrow in the corresponding quiver.

Without lose of generality, let the initial triangulation $T_0$ satisfies this assumption when it holds. Recall that $T^*_0$ is the dual of $T_0$ and $S_i$ be the corresponding simples in $D_{fd}(\Gamma_{T_0})$.

**Notations 3.3.** Let $\eta \in \mathcal{CA}(S_\Delta) - T_0$ intersect $T_0$ at $V_1, \ldots, V_m$ accordingly from an endpoint $Z_0$ to the other endpoint $Z_m$, where

$$m = l_0(\eta) := \text{Int}(\eta, T_0)$$

is the length of $\eta$, with respect to $T_0$. As $\eta \notin T_0$, we have $m \geq 2$. Denote by $\gamma_{j_s}$, $1 \leq s \leq m$, the open arcs in $T_0$ containing $V_s$. Denote by $Z_s$, $1 \leq s \leq m - 1$, the branching points contained in the same triangle as the segment $V_sV_{s+1}$. So the string $w(\eta)$ associated to $\eta$ is (cf. Definition A.3)

$$S_{j_1} \xrightarrow{\pi_{a_1}} S_{j_2} \xrightarrow{\pi_{a_2}} \cdots \xrightarrow{\pi_{a_{m-1}}} S_{j_m}.$$ 

By [3, Lemma 3.14], for each $Z_i$ with $Z_i \neq Z_0$ and $Z_i \neq Z_m$, there exists $1 \leq s \leq m - 1$ such that $Z_s = Z_i$ and there are closed arcs $\alpha$ and $\beta$ in $\mathcal{CA}(S_\Delta)$ such that their strings $w(\alpha)$ and $w(\beta)$ are

$$S_{j_1} \xrightarrow{\pi_{a_1}} S_{j_2} \xrightarrow{\pi_{a_2}} \cdots \xrightarrow{\pi_{a_{s-1}}} S_{j_s}, \quad \text{and}$$

$$S_{j_{s+1}} \xrightarrow{\pi_{a_{s+1}}} S_{j_{s+2}} \xrightarrow{\pi_{a_{s+2}}} \cdots \xrightarrow{\pi_{a_{m-1}}} S_{j_m},$$

respectively. Moreover, we have

$$l_0(\eta) = l_0(\alpha) + l_0(\beta). \quad (3.3)$$
Lemma 3.4. Keep the notations above. Let $s_i \in T^0_\alpha$. If in addition that
\begin{equation}
\text{Int}(s_i, \eta) = \text{Int}(s_i, \alpha) + \text{Int}(s_i, \beta),
\end{equation}
then we have
\begin{equation}
\dim \text{Hom}^*(S_i, X_\eta) = \dim \text{Hom}^*(S_i, X_\alpha) + \dim \text{Hom}^*(S_i, X_\beta)
\end{equation}
for some representatives $X_? \in \tilde{X}_0(?)$, where $? = \eta, \alpha$ and $\beta$.

Figure 6. The three subcases in the proof of Lemma 3.4
Proof. We assume that \( a_s \) is an arrow from \( V_s \) to \( V_{s+1} \) while the other case is similar. By Proposition 3.1, there are representatives \( X_\gamma \) with a nontrivial triangle

\[
X_\beta \rightarrow X_\eta \rightarrow X_\alpha \xrightarrow{\varphi} X_\beta[1].
\]

Note that (3.3) implies that the relative position of \( \alpha \) and \( \beta \) is one of the first three cases in Figure 12. Thus \( \varphi \) is in fact induced by a single map \( S_j \xrightarrow{\pi_{a_s}} S_{j+1} \) (cf. the proof of Lemma A.9). To prove (3.5), we only need to prove the morphism

\[
\text{Hom}(X_{s_i}, X_\alpha) \xrightarrow{\text{Hom}(X_{s_i}, \varphi)} \text{Hom}(X_{s_i}, X_\beta[1])
\]

is zero for any representative \( X_{s_i} = S_i[k] \), where \( k \in \mathbb{Z} \). There are two cases. Let \( d = \deg a_s \).

**Case I:** If \( Z_s \) is not an endpoint of \( s_i \), then \( \gamma_i \) is not an edge of the triangle containing \( Z_s \). So \( \gamma_i \), \( \gamma_j \), and \( \gamma_{j+1} \) do not form a triangle in \( T_0 \) since we require any two triangle in \( T_0 \) share at most one edge. Then by Lemma A.2,

\[
\text{Hom}^*(S_i, S_{j_s}) \otimes \text{Hom}(S_{j_s}, S_{j_{s+1}}[d]) = 0,
\]

which implies the morphism (3.6) is zero.

**Case II:** If \( Z_s \) is an endpoint of \( s_i \), there are the following three subcases.

(a) The other endpoint of \( s_i \) is \( Z_{s+1} \). Then (3.4) implies that \( V_s V_{s+1} \) intersects \( s_i \), i.e. picture (a) in Figure 6. So \( j_{s+1} = i \) and \( a_{s+1} \) is an arrow from \( V_{s+1} \) to \( V_{s+1} \). The only nonzero composition in \( \text{Hom}^*(S_i, S_{j_s}) \otimes \text{Hom}(S_{j_s}, S_{j_{s+1}}[d]) \) is

\[
\text{Hom}(S_i[d - 3], S_{j_s}) \otimes \text{Hom}(S_{j_s}, S_i[d]) \cong \text{Hom}(S_i[d - 3], S_i[d]).
\]

Then the corresponding (possible nonzero) morphism from \( X_{s_i} \) to \( X_\beta \) is as follows:

\[
\begin{align*}
X_{s_i} &: S_i[^?] \cong S_{j_{s+1}}[\deg a_{s+1} - 3] \\
X_{i_\beta} &: \cdots \xrightarrow{\pi_{a_s}} S_{j_s} \xrightarrow{\pi_{a_{j+1}}} \cdots \\
X_\beta &: \cdots \xrightarrow{\pi_{a_{j+1}}} S_{j_{s+1}}[\deg a_{s+1}] \xrightarrow{\pi_{a_{j+1}}} S_{j_{s+2}} \cdots
\end{align*}
\]

which is null-homotopic since \( \text{Hom}(S_i, S_i[^3]) \) factors through

\[
\pi_{a_{j+1}}: S_{j+2} \rightarrow S_{j_{s+1}}[\deg a_{j+1}] = S_i[\deg a_{j+1}].
\]

Thus, (3.6) is still zero.

(b) The other endpoint of \( s_i \) is \( Z_{s-1} \). Then (3.4) implies that \( V_{s-1} V_s V_{s+1} \) intersects \( s_i \), i.e. picture (b) in Figure 6. So \( j_s = i \) and \( a_{s-1} \) is an arrow from \( V_s \) to \( V_{s-1} \). If the composition \( \text{Hom}(S_i[z], S_{j_s}) \otimes \text{Hom}(S_{j_s}, S_{j_{s+1}}[d]) \) is nonzero, then \( z = 0 \) or \( z = 3 \). By Lemma A.2, such a composition is zero when \( z = 0 \). Moreover,
when \( z = 3 \), the corresponding morphism from \( X_s \) to \( X_\alpha \) is as follows:

\[
\begin{array}{c}
\xymatrix{ \cdots S_{j_{s-1}} \ar[r]^-{\pi_{a_{j-1}}} & S_j \ar[r]^-{\deg a_{j-1}} & \cdots }
\end{array}
\]

which is null-homotopic since \( \text{Hom}(S_i, S_j[3]) \) factors through

\[
\pi_{a_{j-1}} : S_{j_{s-1}} \to S_j[\deg a_{j-1}] = S_j[\deg a_{j-1}].
\]

Thus, (3.6) is still zero.

(c) The other endpoint of \( s_i \) is neither \( Z_{s+1} \) nor \( Z_{s-1} \). Then (3.4) implies that \( V_{s}V_{s+1} \) intersects \( s_i \), i.e., picture (c) in Figure 6. So \( i \neq j \) and \( i \neq j_{s+1} \). Thus \( d \geq 2 \) and \( \text{Hom}^*(S_i, S_j) \) is concentrated in degree \( 1 \leq d' \leq 2 \). Then the possible nonzero map in

\[
\text{Hom}^*(S_i, S_j, S_{j_{s+1}}[d])
\]

is of the form \( \text{Hom}(S_i[-d'], S_{j_{s+1}}[d]) \). But as \( d + d' \geq 3 \) and \( i \neq j_{s+1} \), Lemma A.2 implies such a map is zero. Thus, (3.6) is zero.

In all, we complete the proof. \( \square \)

**Proposition 3.5.** Let \( \eta \in \overline{\text{CA}}(S_\Delta) \). Under Assumption 3.2, we have

\[
\dim \text{Hom}^*(S_i, \overline{X}_0(\eta)) = 2 \text{Int}(s_i, \eta).
\]  

(3.7)

**Proof.** When \( \text{Int}(s_i, \eta) < 1 \) (i.e. is 0 or 1/2), the formula is proved in [5, Proposition 6.4]. Now we assume that \( \text{Int}(s_i, \eta) \geq 1 \). Use induction on \( l(\eta) \) starting with the trivial case \( m = 1 \) (i.e. \( \eta = s_i \)). Now suppose that the formula holds for \( l(\eta) \leq l \), with some \( l \geq 1 \).

Consider the case \( l(\eta) = l + 1 \). Then there are following cases:

A. \( \eta \in \text{CA}(S_\Delta) \) and \( \eta \) share both endpoints with \( s_i \).

Applying [5, Lemma 3.14] to decompose \( \eta \) into \( \alpha \) and \( \beta \), w.r.t. to some decorated point \( Z \), which is not an endpoint of \( \eta \) (or \( s_i \)). Then they satisfy the condition (3.4) in Lemma 3.4 (cf. Figure 7), and we also have (3.5). By the induction hypothesis, (3.7) holds for \( \alpha, \beta \) and hence for \( \eta \).

B. \( \eta \in \text{CA}(S_\Delta) \) and \( \eta \) share at most one endpoint with \( s_i \).

As \( \text{Int}(\eta, s_i) \geq 1 \), we have \( \text{Int}_{\Delta-\Delta}(\eta, s_i) \neq 0 \). Let \( Z \) be an endpoint of \( s_i \), such that the triangle containing \( Z \) contains intersections of \( \eta \) and \( s_i \). Choose the closest intersection \( Y \) between \( \eta \) and \( s_i \) from \( Z \). Applying [5, Lemma 3.14],
w.r.t. the decorated point $Z$ and the line segment $YZ(s)$ to decompose $\eta$ into $\alpha$ and $\beta$. Since (3.4) holds, we have (3.5) as above and hence (3.7) holds.

C. $\eta \in \overline{\text{CA}(S_\Delta)} - \text{CA}(S_\Delta)$ and $\text{Int}_\Delta(\eta, s_i) = 0$.

Same as Case B above, (3.7) holds.

D. $\eta \in \overline{\text{CA}(S_\Delta)} - \text{CA}(S_\Delta)$ and the endpoint of $\eta$ is an endpoint of $s_i$. There are two subcases.

First, $\eta$ is contained in one of the two triangles of $T_0$ that intersects $s_i$. Then we deduce that any $X_\eta$ in $\tilde{X}_0(\eta)$ is the extension between $S_i[d + 2]$ and $S_i[d]$. A direct calculation shows that (3.7) holds.

Second, suppose that $\eta$ is not contained in the two triangles of $T_0$ that intersect $s_i$. Then apply [5, Lemma A.2] to decompose $\eta$ into $\alpha$ and $\beta$ in Notations 3.3, w.r.t. some decorated point, which is not an endpoint of $s_i$. Then they satisfy the condition (3.4) in Lemma 3.4 (cf. Figure 7). and we also have (3.5). By the induction hypothesis, (3.7) holds for $\alpha, \beta$ and hence for $\eta$.

\[ \square \]

**Corollary 3.6.** Under Assumption 3.2,

\[ \dim \text{Hom}^*(\tilde{X}_0(\eta_1), \tilde{X}_0(\eta_2)) = 2 \text{Int}(\eta_1, \eta_2), \]

(3.8)

for any $\eta_i \in \text{CA}(S_\Delta)$.

**Proof.** By Proposition 2.3, there exists $s_i \in T_0^*$ and $b \in \text{BT}(T_0)$ such that $\eta_1 = b(s_i)$. Then by [5, Corollary 7.4], we have $\tilde{X}_0(b(s_i)) = \iota(b) \left( \tilde{X}_0(s_i) \right)$. Hence

\[ \dim \text{Hom}^*(\tilde{X}_0(\eta_1), \tilde{X}_0(\eta_2)) = \dim \text{Hom}^*(\tilde{X}_0(s_i), \tilde{X}_0(b^{-1}(\eta_2))) = 2 \text{Int}(s_i, b^{-1}(\eta_2)) = 2 \text{Int}(\eta_1, \eta_2). \]

\[ \square \]

3.3. **Independence.** As before, there is a fixed initial triangulation $T_0$ satisfying Assumption 3.2. Recall from [5], that two elements $\psi$ and $\psi'$ in $\text{Aut} D_{fd}(\Gamma_0)$ are isotopy, denote by $\psi \sim \psi'$, if $\psi^{-1} \circ \psi'$ acts trivially on $\text{Sph}(\Gamma_0)$. Let

\[ \text{Aut}^\sim D_{fd}(\Gamma_0) = \text{Aut} D_{fd}(\Gamma_0)/\sim. \]

By [5, (2.6)], $\psi \sim \psi'$ is equivalent to the condition that they acts trivially on $\text{Sim} H_0$. We will say an element $\varphi$ in $\text{Aut} D_{fd}(\Gamma_0)$ is null-homotopic if $\varphi \sim \text{id}$.

Now let $T$ be any other triangulation. Denote by $H_T$ the canonical heart in $D_{fd}(\Gamma_T)$. Recall that its simples can be indexed by closed arcs in $T^*$. Denote by $\text{Sph}(\Gamma_T)$ its set of reachable spherical objects.

**Definition 3.7.** We say two derived equivalences $\phi, \phi' : D_{fd}(\Gamma_0) \to D_{fd}(\Gamma_T)$ are isotopy if they only differ by null-homotopies, i.e. $\phi' = \varphi \circ \phi \circ \varphi_0$ for some $\varphi_0 \in \text{Aut} D_{fd}(\Gamma_0)$ and $\varphi \in \text{Aut} D_{fd}(\Gamma_T)$, which are null-homotopic.

**Lemma 3.8.** There is a canonical derived equivalence $\Phi_T : D_{fd}(\Gamma_0) \to D_{fd}(\Gamma_T)$, unique up to isotopy and shifts, such that the induced bijection

\[ \Phi_T : \text{Sph}(\Gamma_0)/[1] \to \text{Sph}(\Gamma_T)/[1] \]
satisfying the following condition:

- for any \( s \in \mathbb{T}^* \), the corresponding simple in \( \text{Sim} \mathcal{H}_T \) is in the shift orbit \( \tilde{X}_0(s) \).

\textbf{Proof.} First we show the uniqueness. Suppose that there are two such derived equivalences \( \Phi_T \) and \( \Phi_T' \). Then we have \( \Phi_T^{-1} \circ \Phi_T'(S_i) = S[m_i] \) for any simple \( S_i \) in the canonical heart \( \mathcal{H}_0 \). By calculating the Hom\(^{*} \), we deduce that all \( m_i \) should coincide, i.e. \( \Phi_T^{-1} \circ \Phi_T' \circ [-m] \) preserves \( \text{Sim} \mathcal{H}_0 \) and hence \( \text{Sph}(\Gamma_0) \). In other words, \( \Phi_T^{-1} \circ \Phi_T' \circ [-m] \) is null-homotopic in \( \text{Aut} \mathcal{D}_{fd}(\Gamma_0) \), as required.

Now we prove the lemma by induction, on the minimal number of flips from \( T_0 \) to \( T \), starting from the trivial case. Now suppose that we have a triangulation \( T \) of \( S_\Delta \) with dual graph \( T^* = \{s_i\}_{i=1}^n \) and a required derived equivalence \( \Phi_T \). Namely, let \( T_i \) be the simple in \( \text{Sim} \mathcal{H}_T \) that corresponds to \( s_i \) and then

\[ \Phi_T'(\tilde{X}_0(s_i)) = T_i[Z]. \]  

We only need to show that there exist a required derived equivalence \( \Phi_{T'} \) for any flip \( T' \) of \( T \) in \( S_\Delta \).

Without loss of generality, suppose that \( T' \) is the forward flip of \( T \) with respect to the arc \( \gamma \) and let \( s_j \) (for some \( j \)) be the dual arc of \( \gamma \) in \( T^* \). By Keller-Yang, there is a derived equivalence \( \Phi: \mathcal{D}(\Gamma_T) \to \mathcal{D}(\Gamma_{T'}) \) satisfying

\[ \Phi\left( (\mathcal{H}_T)^{p'}_{T_1} \right) = \mathcal{H}_{T'}. \]

Let \( (T')^* \) consists of closed arcs \( s'_i \) and and \( \text{Sim} \mathcal{H}_{T'} \) consists of the corresponding simples \( T'_i \). Note that the indexing of \( s'_i \) is induced by the indexing of \( s_i \) via the Whitehead move (e.g. \cite[Figure 10]{3}). By the tilting formula in \cite[Corollary 9.9]{5}, \( \Phi^{-1}(T'_i) \) equals one of \( T_i \), \( T_i[1] \) or \( \phi^{-1}_{T_j}(T_i) \). On the other hand, a straightforward calculation shows that

\[ s'_i = \begin{cases} s_i, & \text{if } \Phi^{-1}(S'_i) = S_i \text{ or } S_i[1] \\ B_{s_j}(s_i), & \text{if } \Phi^{-1}(S'_i) = \phi^{-1}_{s_j}(S_i). \end{cases} \]

Using formula \cite[(7.9)]{5}, that spherical twists are compatible with braid twist under \( \tilde{X}_0 \), we deduce that (3.9) implies

\[ \Phi^{-1}(T'_i[Z]) = \Phi_T(\tilde{X}_0(s'_i)), \]

or \( T'_i[Z] = \Phi \circ \Phi_T(\tilde{X}_0(s'_i)) \). Then \( \Phi_{T'} = \Phi_T \circ \Phi_T \) is the required derived equivalence. \( \square \)

As shown in § 3.1, there are bijections \( \tilde{X}_0: \text{CA}(S_\Delta) \to \text{Sph}(\Gamma_0)/[1] \) and \( \tilde{X}_T: \text{CA}(S_\Delta) \to \text{Sph}(\Gamma_T)/[1] \). We proceed to discuss the relationship between them.

\textbf{Proposition 3.9.} \textit{The following diagram commutes:}

\[ \begin{array}{ccc}
\text{CA}(S_\Delta) & \xrightarrow{\Phi_T} & \text{Sph}(\Gamma_T)/[1] \\
\tilde{X}_0 & \downarrow & \tilde{X}_T \\
\text{Sph}(\Gamma_0)/[1] & \xrightarrow{\Phi_T} & \text{Sph}(\Gamma_T)/[1],
\end{array} \]

where \( \Phi_T: \mathcal{D}_{fd}(\Gamma_0) \to \mathcal{D}_{fd}(\Gamma_T) \) is the equivalence in Lemma 3.8.
Proof. Firstly notice that $\Phi_T$ is only well-defined up to isotopy and shifts, which is compatible with the shifts orbits in the diagram above. Secondly, as $\Phi_T$ and $\tilde{X}_0$ are bijections, we only need to prove
\begin{equation}
\tilde{X}_T(\eta) = \Phi \circ \tilde{X}_0(\eta)
\end{equation}
for any $\eta \in CA(S_{\Delta})$.

Now use induction on $l(\eta) = \text{Int}(\eta, T)$. The starting step ($l(\eta) = 1$) is covered by Lemma 3.8. Now let us deal the inductive step for some $\eta$ with $l(\eta) > 1$ while assuming that (3.11) holds for any $\eta'$ with $l(\eta') \leq l(\eta)$. By Lemma 2.2, there are $\alpha$ and $\beta$ with the corresponding conditions. Without loss of generality, assume that $\eta = B_\alpha(\beta)$ and
\[\text{Int}(\gamma_i, \eta) - 2\text{Int}(\alpha, \beta)\text{Int}(\gamma_i, \alpha) = \text{Int}(\gamma_i, \beta)\]
By inductive assumption, we have
\[\tilde{X}_T(\alpha) = \Phi \circ \tilde{X}_0(\alpha) \quad \text{and} \quad \tilde{X}_T(\beta) = \Phi \circ \tilde{X}_0(\beta).
\]
Then on one hand, by Corollary 3.6, we have
\[\dim \text{Hom}^\bullet(\tilde{X}_0(\alpha), \tilde{X}_0(\beta)) = 2\text{Int}(\alpha, \beta).
\]
On the other hand, there is a non-trivial triangle
\[X_0[-1] \rightarrow \oplus_{i=1}^{2\text{Int}(\alpha, \beta)} X_\alpha[m_i] \rightarrow (f_i) \rightarrow X_\beta \rightarrow X_\eta,
\]
for some representatives $X_\gamma$ in $\tilde{X}_0(?)$, by Proposition 3.1. Then we deduce that
\[X_\eta = \phi_{X_\alpha}(X_\beta)
\]
or $\tilde{X}_0(\eta) = \phi_{\tilde{X}_0(\alpha)}(\tilde{X}_0(\beta))$. Noticing (3.12), we have $\tilde{X}_T(\eta) = \phi_{\tilde{X}_T(\alpha)}(\tilde{X}_T(\beta))$ for the same reason. Hence
\begin{align*}
\tilde{X}_T(\eta) &= \phi_{\tilde{X}_T(\alpha)}(\tilde{X}_T(\beta)) \\
&= \phi_{\tilde{X}_0(\alpha)}(\phi \circ \tilde{X}_0(\beta)) \\
&= \Phi \left( \phi_{\tilde{X}_0(\alpha)}(\tilde{X}_0(\beta)) \right) \\
&= \Phi \circ \tilde{X}_0(\eta)
\end{align*}
as required. \hfill \Box

Remark 3.10. By the proposition above, one can identify all sets $\text{Sph}(\Gamma_T)$ of reachable spherical objects, for any $T$ in $\text{EG}^\gamma(S_{\Delta})$, using the canonical derived equivalences in Lemma 3.8 between $D_{fd}(\Gamma_T)$. Hence, such equivalences also allow us to identify all the spherical twist group $\text{ST}(\Gamma_T)$ (cf. [5, Theorem 7.7]). Note that here we will consider that $\text{ST}(\Gamma_T)$ as a subgroup of $\text{Aut}_0 D_{fd}(\Gamma_T)$.

4. Main results

4.1. Intersection between closed arcs.

Theorem 4.1. [5, Conjecture 10.5] For any triangulation $T$ and $\eta_1 \in CA(S_{\Delta})$, we have
\begin{equation}
\dim \text{Hom}^\bullet(\tilde{X}_T(\eta_1), \tilde{X}_T(\eta_2)) = 2\text{Int}(\eta_1, \eta_2).
\end{equation}
Proof. If Assumption 3.2 holds, the theorem is equivalent to Corollary 3.6 since one can identify all bijections $\tilde{\chi}_T$ as in Remark 3.10.

Next we deal with the two special case when Assumption 3.2 does not hold: $S$ is an unpunctured annulus with two marked points or an unpunctured torus with one marked point. In both of these cases, we can obtain new marked surfaces satisfying Assumption 3.2 by adding a marked point as in [5, § 8]. More precisely, consider a marked surface $S'$ satisfying Assumption 3.2, with a triangulation $T$ such that by cutting along one of its arcs, we obtain $S$ with a triangulation $T$ (and another triangle $T$). As $\mathcal{D}_{fd}(\Gamma)$ is in fact the smallest triangulated category in $\mathcal{D}(\Gamma)$, generated by the simples in the canonical heart $\mathcal{H}_{\Gamma}$, one can regard $\mathcal{D}_{fd}(\Gamma_{T'})$ as a sub-triangulated category of $\mathcal{D}_{fd}(\Gamma_{T})$. As the theorem holds for $\mathcal{D}_{fd}(\Gamma_{T})$, it is straightforward to see that it also holds for $\mathcal{D}_{fd}(\Gamma_{T'})$. □

4.2. Independence revisit. As we have the intersection formula (4.1), the discussion in § 3.3 holds for any triangulation $T_0$ of any $S_\Delta$ (without Assumption 3.2). More precisely, the only place that we use Assumption 3.2 in § 3.3 is for (3.13). At that time, we only prove the intersection formula (3.8) under Assumption 3.2. Thus, Proposition 3.9 works in general as follows.

Theorem 4.2. We can identify all $\mathcal{D}_{fd}(\Gamma_{T})$ up to isotopy and shifts, for any $T$ in $\text{EG}^0(S_\Delta)$, such that the bijections $\tilde{\chi}_T: \text{CA}(S_\Delta) \to \text{Sph}(\Gamma_{T})/[1]$ is compatible (i.e. commute as in (3.10)) with this identification.

4.3. Intersection between open and closed arcs. Let $\Gamma$ be the Ginzburg dg algebra of some quiver with potential $A$ silting set $P$ in a category $\mathcal{D}$ is an $\text{Ext}^{>0}$-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\text{Ext}^i(P, T) = 0$ for any $P, T \in P$ and integer $i > 0$. The silting object associated to $P$ is $\bigoplus_{T \in P} T$. By abuse of notation, we will not distinguish a silting set and its associated silting object. For example, $\Gamma$ is the canonical silting object/set in $\text{per} \, \Gamma$.

Moreover, one can forward/backward mutate a silting object to get new ones (cf. [5, § 9] for details). A silting set $P$ in $\text{per} \, \Gamma$ is reachable if it can be repeatedly mutated from $\Gamma$. Denote by $\text{SEG}^0(\Gamma)$ the set of reachable silting sets in $\text{per} \, \Gamma$ and $\text{RR}_{\text{per} \, \Gamma_0}$ the set of reachable rigid objects in $\text{per} \, \Gamma$. Recall a result from [5].

Lemma 4.3. [5, § 9] There is a canonical bijection

$$\rho_T: \text{OA}^0(S_\Delta) \to \text{RR}_{\text{per} \, \Gamma_0}.$$ 

We finish the paper by proving another conjecture in [5].

Theorem 4.4. [5, Conjecture 10.6] For any triangulation $T$ and $\eta \in \text{CA}(S_\Delta)$, we have

$$\dim \text{Hom}^\bullet(\rho_T(\gamma), \tilde{\chi}_T(\eta)) = \text{Int}(\gamma, \eta).$$

(4.2)

Proof. First, for any two triangulations $T$ and $T'$, we actually have a canonical identification $\Phi: \mathcal{D}(\Gamma_T) \to \mathcal{D}(\Gamma_{T'})$, as in Lemma 3.8. Note that there is a simple-projective duality between silting set in $\text{per} \, \Gamma$ and the set of simples of a heart in $\mathcal{D}_{fd}(\Gamma)$. Thus,
as $\Phi$ preserves reachable spherical objects, up to shift, $\Phi$ preserves reachable rigid objects, up to shift. Second, by [5, Lemma 5.13], the theorem holds for $\gamma \in T$ and any $\eta \in OA(S)$. Now, choose any $\gamma \in OA^\circ(S_\triangle)$. Let $T'$ be a triangulation in $EG^\circ(S_\triangle)$ that contains $\gamma$. Then we have
\[
\dim \text{Hom}^\bullet(p_T(\gamma), \tilde{X}_T(\eta)) = \dim \text{Hom}^\bullet(p_{T'}(\gamma), \tilde{X}_{T'}(\eta)) = \dim \text{Hom}^\bullet(p_T(\gamma), \tilde{X}_{T'}(\eta)) = \text{Int}(\gamma, \eta).
\]
\[
\square
\]

Appendix A. The string model

This appendix also serves for [5].

A.1. Homological preparation. Let $S_i$, $1 \leq i \leq n$, be the simples in mod $\Gamma$, for some Ginzburg dg algebra $\Gamma$ arising from quivers with potential associated to triangulations of (unpunctured) marked surfaces. Consider the following short exact sequence for each $i$:
\[
0 \to \ker \pi_i \to P_i \xrightarrow{\zeta_i} S_i \to 0
\]
in mod $\Gamma$, where $p_i$ is the canonical map from $P_i$ to $S_i$ and $\ker \zeta_i = \bigoplus_{a:i \to k} aP_k$, where the direct sum is over all such arrows in $Q_1$ (similarly for later). Then there is a triangle in $\text{per}(\Gamma)$:
\[
\ker \zeta_i \to P_i \xrightarrow{\zeta_i} S_i \to \ker \zeta_i[1]
\]
which implies that $S_i$ is isomorphic to
\[
P_i \oplus (\ker \zeta_i[1]) = P_i \oplus \bigoplus_{a:i \to k} aP_k[1]
\]
as a dg module, with the induced differential consists of
- differential of $P_i$,
- differential of each $P_k[1]$ and
- embedding from each $aP_k[1]$ to $P_i[1]$.
Moreover, $\text{Hom}^\tau(aP_i, bP_j)$ is span by the morphisms that sends $ax$ to $bpx$ ($x \in P_i$) induced by paths $p : j \to i$ of degree $\deg a - \deg b + r$. Therefore, $\text{Hom}_{\text{per}(\Gamma)}(S_i, S_j[r])$ has a basis
\[
\{ \pi_b : b : i \to j \in Q_1 \text{ of degree } -r + 1 \text{ for } i \neq j \} \cup \{ \pi_{e_i} : \text{if } i = j \text{ and } r = 0 \}
\]
where $e_i$ is the trivial path at $i$ and $\pi_b$, regarded as a morphism:
\[
P_i \oplus \bigoplus_{a:i \to k} aP_k[1] \to P_j \oplus \bigoplus_{c:j \to l} aP_l[1],
\]
its component from $bP_l(b)[1]$ to $P_j$ sends $bx$ to $x$ while its component from any other $aP_l(a)$ to $P_j$ is zero (since $\text{Hom}(P_l(a), S_j) = 0$).

Lemma A.1. If $\deg b = 0$, $\pi_b$ is induced by a homomorphism of degree 1 in (A.2) with $\pi_{b}(t_i) = b^*$. 
Proof. Since the degree $\pi_b$ is 1, we have that $\pi_b(t_i) = xb^* + yb'^*$ for some $x, y \in k$, where $b'$ is the other arrow starting at $j$ (note that there are at most two arrows start at $j$). On one hand, $d(t_i) = bb^* + b'b'^* + s$ for $s$ is a sum of items whose first arrow is not $b$ or $b'$. So the restriction of $\pi_b(d(t_i))$ in $P_j[-1]$ is $b^*$. On the other hand, the restriction of $d\pi_b(t_i)$ in $P_j[-1]$ is $xb^* + yb'^*$. Hence $x = 1$ and $y = 0$.

\[\square\]

**Lemma A.2.** Let $a_i, i = 1, 2, 3$, be three arrows of $Q$ induced from the same triangle in $T$. Then

\[\pi_{a_{i+1}}\pi_{a_i} = \pi_{a_{i+2}} \quad \text{and} \quad \pi_{a_i}\pi_{a_i^*} = \pi_{t_i}, \quad (A.3)\]

for $i = 1, 2, 3$, where $a_{i+3} = a_i$ for $i \in \mathbb{Z}$. Moreover, let $a$ be a arrow of $Q$ induced from another triangle, then

\[\pi_x\pi_y = 0 = \pi_y\pi_x, \quad (A.4)\]

where $x = a$ or $a^*$, and $y = a_i$ or $a_i^*$ for $i = 1, 2, 3$.

Proof. Denote by $i$ the starting point of $a_i, i = 1, 2, 3$. Since there are at most two arrows starting at a vertex and at most two stop, we denote by $b_i$ (resp. $c_i$) the other arrow starting (resp. stopping) at $i$ if they exist. (See the following figure and note that these edges $b_i$ and $c_i$ may coincide and so do the vertices $t(b_i)$ and $s(c_i)$).

![Diagram showing arrows and vertices](image)

We only prove $\pi_{a_2}\pi_{a_1} = \pi_{a_3^*}$ and the rests are similar. First, the map $\pi_{a_1}$ is of the form $f_{a_1} = (g_{jl})_{6 \times 6}$ of degree 1 from

\[\ker \zeta_1[1] \oplus P_1 = t_1P_1[1] \oplus a_1P_2[1] \oplus b_1P_{l(b_1)}[1] \oplus a_3^*P_3[1] \oplus c_1^*P_{s(c_1)}[1] \oplus P_1\]

with the induced differential to

\[\ker \zeta_2[1] \oplus P_2 = t_2P_2[1] \oplus a_2P_3[1] \oplus b_2P_{l(b_2)}[1] \oplus a_4^*P_4[1] \oplus c_2^*P_{s(c_2)} \oplus P_2\]

with the induced differential. We claim that $g_{k2}$ vanishes except for $k = 4$ that sends $a_3^*p$ to $xap$ for some $x \in k$. Suppose this is true. Since $\pi_{a_1}$ is induced by the canonical morphism $\rho_{a_2}$ from $a_2P_3$ to $S_3$, so is $\pi_{a_2}\pi_{a_1}$. Therefore, $\pi_{a_2}\pi_{a_1} = \pi_{a_3^*}$.

Now, since $\text{Hom}^i(aP_i, bP_j)$ is span by the morphisms that sends $ax$ to $bpx$ ($x \in P_i$) induced by paths $p : j \rightarrow i$ of degree $\deg a - \deg b + r$, we have $g_{k2} = 0$ for $k = 1, 2, 3, 6$. 


Further, we have
\[
g_{52}(c_1^* p) = \begin{cases} 
    gao_2 b_3 p & \text{if } t(b_3) = s(c_1), \\
    gao_2 p & \text{if } s(c_1) = 3, \\
    0 & \text{otherwise,}
\end{cases}
\]
for some \( y \in k \). By Lemma A.1, \( g_{1k} = 0 \) except for \( k = 4 \) that sends \( t_1p \) to \( a_1^* p \). Then
\[
df_{a_1}(t_1) = fa_1(dt_1) = fa_1(a_1 a_1^* + b_1 b_1^* + a_3^* a_3 + c_1^* c_1)
\]
implies that
\[
d(a_1^*) = a_2 a_3 = g_{42}(a_3^* a_3) + g_{52}(c_1^* c_1) = xa_2 a_3 + y?c_1,
\]
where \( ? = a_2 \) or \( a_2 b_3 \). Hence \( x = 1 \) and \( y = 0 \), or \( g_{52} = 0 \) as required. \( \square \)

A.2. The construction. Following [5, § 5], we recall the string model of decorated marked surfaces for perfect dg modules.

Let \( \Gamma_T = \Gamma(\mathcal{Q}_T, W_T) \) be the Ginzburg dg algebra from a triangulation \( T \). Recall that there is a canonical heart \( \mathcal{H}_f \) in \( D_{fd}(\Gamma_T) \) and let
\[
S_T = \bigoplus_{S \in \mathcal{H}_f} S
\]
to be the direct sum of the simples in \( \mathcal{H}_f \). Consider the (dg) endomorphism algebra
\[
\mathcal{E}_T = \text{RHom}(S_T, S_T) \tag{A.5}
\]
with
\[
D_{fd}(\Gamma_T) \xrightarrow{\text{RHom}_{\mathcal{E}_T}(S_T, ?)} \text{per } \mathcal{E}_T \tag{A.6}
\]
We will identify these two categories when there is no confusion. In particular, \( \{S\}_{S \in \mathcal{H}_f} \) in \( D_{fd}(\Gamma_T) \) becomes the (indecomposable) projectives in \( \text{per } \mathcal{E}_T \).

**Definition A.3.** A string of simples (string for short) in \( \mathcal{H}_f \) is a walk
\[
w : R_0 \xrightarrow{f_1} R_1 \xrightarrow{f_2} \cdots \xrightarrow{f_p} R_{l(w)}
\]
where \( l(w) \) is the length of the string \( w \), \( R_i \in \text{Sim } \mathcal{H}_f \) and \( f_i \) is a homomorphism of some degree in either direction such that the composition of neighboring homomorphisms is zero if they have the same direction. For technical reasons we use the notation \( \varepsilon_w(f_i) = + \) if \( f_i \) points to right and \( \varepsilon_w(f_i) = - \) if \( f_i \) points to left.

**Definition-Proposition A.4.** Each string \( w \) in \( \mathcal{H}_f \) with an integer \( m \) defines a perfect dg \( \mathcal{E}_T \)-module \( X_w[m] \), called string, as follows: the underling graded module of \( X_w \) is of the form
\[
\bigoplus_{i=0}^p R_i[m_i(w)]
\]
where \( m_i(w) = \sum_{j=1}^i \varepsilon_w(f_j)(\deg f_j - 1) \) and the differential \( d_X = d_0 + \sum_{i=1}^p f_i[\deg f_i - 1] \) where \( d_0 \) is the direct sum of the differential of the \( S_{ki} \).

It follows straightly that a string dg \( \mathcal{E}_T \)-module is a \( m \)-perfect dg \( \mathcal{E}_T \)-module in the sense of [5, Definition 5.4].
Construction A.5. Let $\eta$ be a closed arc in $S_\triangle$ such that it is in a minimal position with respect to $T_0$ (i.e. there is no digon shown as in Figure 8). We shall fix an orientation of $\eta$ so that we can talk about its starting point and ending point. Denote by $\overline{\eta}$ the same arc with reversed orientation. We associate a string $w(\eta)$ as follows.

- Suppose that $\eta$ intersects $T_0$ at $V_1, \ldots, V_p$ accordingly from one endpoint to the other, where $V_i$ is in the arc $\gamma_{k_i} \in T_0$ for $1 \leq i \leq p$ and some $1 \leq k_i \leq n$ (cf. Figure 9).

![Figure 9. The intersections between $\eta$ and $T_0$](image)

- Each arc segment $V_iV_{i+1}$ in $\eta$ corresponds to an arrow $a_i$ in $\tilde{Q}_T$ as in Figure 10. Then we obtain a string:

$$w(\eta) : S_{k_0} \pi_{a_1} S_{k_1} \pi_{a_2} \cdots \pi_{a_l(\eta)} S_{k_l(\eta)}$$

where $l(\eta) = l(w(\eta))$ is the length of $w(\eta)$. and $\pi_{a_i}$ is the canonical homomorphism between simples induced by $a_i$ (see § A.1 for the construction of $\pi$).

A.3. The generalized string model. Let $\sigma, \tau \in \text{CA}(S_\triangle)$ be two different arcs with $\text{Int}_{S_{\triangle}}(\sigma, \tau) = 0$. We will always suppose that $\sigma$ is in a minimal position relative to $\tau$. Fix a pair of orientations of $\sigma$ and $\tau$. Suppose that $\sigma$ and $\tau$ intersects $T$ at $V_0, \ldots, V_p$ and $W_0, \ldots, W_q$ respectively in order, where $V_i$ (resp. $W_i$) is in the arc $\gamma_{k_i} \in T_0$ (resp. $\gamma_{j_i} \in T_0$). For convenience, we denote by $V_{-1}, W_{-1}$ the starting points of $\sigma$ and $\tau$ and denote by $V_{p+1}, W_{q+1}$ their ending points. Denote by $\Lambda_i$ the triangle containing
the segment $V_{i-1}V_i$ for $0 \leq i \leq p+1$. By Construction A.5, we have the following corresponding strings:

$$w(\sigma) : S_{k_0} \xrightarrow{\pi_{a_1}} S_{k_1} \xrightarrow{\pi_{a_2}} \cdots \xrightarrow{\pi_{a_{l}}} S_{k_{l}}$$

and

$$w(\tau) : S_{j_0} \xrightarrow{\pi_{b_1}} S_{j_1} \xrightarrow{\pi_{b_2}} \cdots \xrightarrow{\pi_{b_{q}}} S_{j_{q}}$$

where for $1 \leq i \leq p$ (resp. $1 \leq i \leq q$), $a_i$ (resp. $b_i$) is the arrow in $\hat{Q}_{\mathcal{T}}$ induced by the segment $V_{i-1}V_i$ (resp. $W_{i-1}W_i$), see Figure 10 for the constructions of $a_i$ and $b_i$.

Now, suppose $\sigma$ and $\tau$ share the same starting point, i.e. $V_{-1} = W_{-1}$ and let $\Lambda_{-1}$ be the triangle that contains them. Define the (positive/negative) extension of $\tau$ by $\sigma$ w.r.t. $V_{-1}$, denoted by $L_{\tau}^\sigma(\tau) = L_{\tau}^\sigma |_{V_{-1}}$, as follows: it is the general closed arc obtained from $\tau$ by (positive/negative) braid twisting the start segment $W_{-1}W_0$ of $\tau$ along $\sigma$. Moreover, its orientation of is induced by $\tau$ (cf. Figure 11). For instance, we have

$$L_{\tau}^\sigma(\tau) |_{V_{-1}} = L_{\tau}^\sigma(\sigma) |_{W_{-1}}$$

as unoriented arcs.

**Construction A.6.** Suppose $V_{-1} = W_{-1}$ as above. Consider a class of maps $\varphi(\sigma, \tau)_{s,g}$ between components of $X_{\varphi(\sigma)}$ and $X_{\varphi(\tau)}[m]$ (when regarded as chain of complexes in per $\mathcal{E}$), indexed by an integer $0 \leq s \leq \min\{p, q\} + 1$ and some homomorphism $g \in \text{Home}(S_{k_i}, S_{j_i})$:

- the identities from $S_{k_i}$ to $S_{j_i}$, for $0 \leq i \leq s - 1$, and
- $g = 0$ or $g = \pi_c$ for some arrow $c$ of $\hat{Q}$ in the triangle $\Lambda_s$ (cf. Figure 10). We further require $\deg \pi_c < 3$ if $s > 0$.

Note that in the case when $s = \min\{p, q\} + 1$, $g$ is not well-defined and we will take $g = \emptyset$ for convention.
Lemma A.7. Let $\varphi_{s,g} = \varphi(\sigma, \tau)_{s,g}$ be constructed as above. Then it forms a homomorphism if and only if either $s > 0$ with the following conditions hold:

(C0): for each $0 \leq i \leq s - 1$, $\pi_{a_i} = \pi_{b_i}$ and they have the same direction,
(C1): if $\varepsilon(\pi_b_s) = +$, then $\varepsilon(\pi_{a_s}) = +$, $g = \pi_a$ and $\deg \pi_{b_s} = \deg \pi_{a_s} + \deg \pi_a$,
(C2): if $\varepsilon(\pi_{a_s}) = -$, then $\varepsilon(\pi_{b_s}) = -$, $g = \pi_a$ and $\deg \pi_{a_s} = \deg \pi_{b_s} + \deg \pi_a$,
(C3): if $\varepsilon(\pi_{a_s}) = +$ and $\varepsilon(\pi_{b_s}) = -$, then $g = 0$ or $g = \pi_a$ with $\deg \pi_a < 3$, $\deg \pi_a + \deg \pi_{a_s} > 3$ and $\deg \pi_a + \deg \pi_{b_s} > 3$,
(C4): if $s = \min\{p, q\} + 1$, then $\varepsilon(\pi_{a_s}) = +$ if $p > q$, and $\varepsilon(\pi_{b_s}) = -$ if $p < q$;

or $s = 0$. Recall that $\varepsilon(\pi_{a_i}) \in \{\pm\}$ indicates the direction of the map $S_{k_{i-1}} \xrightarrow{\pi_{a_i}} S_{k_i}$.

Similarly for $\varepsilon(\pi_{b_i})$.

Proof. When $s = 0$, $\varphi_{s,g}$ consists of a single map $g : S_{k_0} \to S_{j_0}$. Since $\pi_{a_0}$ and $\pi_{b_1}$ are from different triangles from $g$, we have $g\pi_{a_0} = 0 = g\pi_{b_1}$ by Lemma A.2 (when they make sense). So $\varphi_{s,g}$ is a homomorphism. Next, consider the case when $s > 0$. There are four cases:

- If $\varepsilon(\pi_{b_s}) = +$, we have the following diagram

$$
\begin{array}{cccccc}
S_{k_0} & \cdots & S_{k_{s-1}} & S_{k_s} \\
S_{j_0} & \cdots & S_{j_{s-1}} & S_{j_s} \\
\end{array}
\xrightarrow{\pi_{a_s}}
\begin{array}{c}
g \\
\end{array}
$$

Then $\varphi_{s,g}$ forms a homomorphism if and only if $g$ exists (by $\pi_{b_s} \neq 0$) and $\varepsilon(\pi_{a_s}) = 0$ with $g\pi_{a_s} = \pi_{b_s}$. Hence by Lemma A.2, $g = \pi_a$ with $\deg \pi_{b_s} = \deg \pi_{a_s} + \deg \pi_a$ as required in Condition (C1).

- If $\varepsilon(\pi_{a_s}) = -$, then similarly Condition (C2) is satisfied.

- If $\varepsilon(\pi_{a_s}) = +$ and $\varepsilon(\pi_{b_s}) = -$, we have the following diagram

$$
\begin{array}{cccccc}
S_{k_0} & \cdots & S_{k_{s-1}} & S_{k_s} \\
S_{j_0} & \cdots & S_{j_{s-1}} & S_{j_s} \\
\end{array}
\xrightarrow{\pi_{a_s}}
\begin{array}{c}
g \\
\end{array}
$$

- If $\varepsilon(\pi_{b_s}) = 0$, then similarly Condition (C3) is satisfied.
Then \( \varphi_{s,g} \) forms a homomorphism if and only if \( g \pi_{a_s} = 0 = \pi_{b_s} g \). That is exactly Condition (C3) by Lemma A.2.

- If \( s = \min\{p, q\} + 1 \) and assume without loss of generality that \( p > q \), we have the following diagram

\[
\begin{array}{c}
S_{k_0} \rightarrow \cdots \rightarrow S_{k_q}^\pi a_{q+1} \rightarrow S_{k_{q+1}} \\
S_{j_0} \rightarrow \cdots \rightarrow S_{j_q}^\pi b_{q+1}
\end{array}
\]

Since \( \pi_{a_{q+1}} \neq 0 \), \( \varphi_{s,g} \) forms a homomorphism if and only if \( \varepsilon(\pi_{a_q}) = 0 \) as in Condition (C4).

In all, the lemma follows.

\( \square \)

**Lemma A.8.** Let \( \varphi_{s,g} \) be a homomorphism as in Lemma A.7 with \( s > 0 \). Then it is nonzero.

**Proof.** By the construction of \( \varphi_{s,g} \) (as a chain map), when \( s > 1 \), its component from \( S_{k_0} \) to \( S_{j_0} \) is identity, see the following diagram.

\[
\begin{array}{c}
S_{k_0}^\pi a_1 \rightarrow \cdots \\
S_{j_0}^\pi b_1 \rightarrow \cdots
\end{array}
\]

So if \( \varphi_{s,g} \) is null-homotopy, there exist \( f' \in \text{Hom}^*(S_{k_0}, S_{j_0}) \) and \( f'' \in \text{Hom}^*(S_{k_0}, S_{j_1}) \) such that \( f'\pi_{a_1} + \pi_{b_1} f'' \) is the identity from \( S_{k_0} \) to \( S_{j_0} \). Since \( \deg \pi_{a_1} > 1 \) (resp. \( \deg \pi_{b_1} > 1 \)) and \( f' \) (resp. \( f'' \)) lives in positive degrees, the composition \( f'\pi_{a_1} \) (resp. \( \pi_{b_1} f'' \)) lives in positive degrees (if non-zero), which is a contradiction. \( \square \)

Next, we give an explicit construction of a homomorphism in \( \text{Hom}^*(X_\sigma, X_\tau) \) induced by an intersection in \( \Delta \) between two closed arcs \( \sigma \) and \( \tau \).

**Lemma A.9.** Suppose in the situation of Construction A.6. There exists a unique nonzero homomorphism

\[\varphi(\sigma, \tau) = \varphi(\sigma, \tau)_{s,g} : X_{w(\sigma)} \rightarrow X_{w(\tau)}[m].\]

Moreover, if \( \sigma \in \text{CA}(S_\Delta) \), there exists a triangle

\[X_{w(\sigma)} \xrightarrow{\varphi(\sigma, \tau)} X_{w(\tau)}[m] \xrightarrow{\varphi(\pi, \tau)} X_{w(\varsigma)}[m'] \xrightarrow{\varphi(\varsigma, \tau)} X_{w(\sigma)}[1]\]

in per \( \mathfrak{C} \) for some integer \( m, m' \), where \( \varsigma = L^+_\tau(\tau) \).

**Proof.** Denote by \( \varphi_{s,g} = \varphi(\sigma, \tau)_{s,g} \), which is a homomorphism. By Lemma A.7, \( s = 0 \) or \( s = t \), for the unique integer \( 1 \leq t \leq \min\{p, q\} + 1 \) such that the segments \( V_{t-1}W_{t-1} \simeq W_{t-1}V_t \) and \( V_{t-1}W_t \neq W_{t-1}V_t \).

- If \( s = 0 \), there is a unique segment in the triangle \( A_0 \) such that its induced arrow \( \beta \) satisfies \( \pi_\beta \in \text{Hom}^\pi_{a_0}(S_{k_0}, S_{j_0}) \). Recall that \( \deg \pi_{a_0} = 1 - \deg \beta \) (cf. (A.1)).
• If \( 1 \leq s = t \leq \min\{p, q\} \), there is a unique segment in the triangle \( \Lambda_t \) such that its induced arrow \( \beta \) satisfies \( \pi_\beta \in \text{Hom}^{[\deg \pi_\beta]}(S_{k_t}, S_{j_t}) \).

• If \( s = t = \min\{p, q\} + 1 \), then \( g = \emptyset \).

Thus, \( g \) is uniquely determined by \( s \) so that \( \varphi_{s, g} \) is a homomorphism. There are four cases shown in Figure 12, according to the relative position of \( V_1V_0 \) and \( W_1W_0 \).

**The first two cases in Figure 12:** The segment \( AB \) in each case in the figure corresponds to the arrow \( \beta : k_0 \to j_0 \) in \( \overrightarrow{Q_T} \). Then

![Figure 12](image-url)

**Figure 12. The four cases for the starting segments of \( \sigma \) and \( \tau \)**

\[
\varphi_{0, \pi_\beta} : X_{w(\sigma)} \to X_{w(\tau)}[\deg \pi_\beta]
\]

is a homomorphism and we have \( \text{Cone}(\varphi_{0, \pi_\beta}) \cong X_{w(\varsigma)}[1 + m_{l(w)}(\sigma)] \), where \( \varsigma = L^+_p(\tau) \) with the corresponding string \( w(\varsigma) \) as

\[
S_{k_p} \xrightarrow{\pi_{a_p}} \cdots \xrightarrow{\pi_{a_1}} S_{k_0} \xrightarrow{\pi_\beta} S_{j_0} \xrightarrow{\pi_{b_1}} \cdots \xrightarrow{\pi_{b_3}} S_{j_q} .
\]

Hence there is a triangle

\[
X_{w(\sigma)} \xrightarrow{\varphi_{0, \pi_\beta}} X_{w(\tau)}[\deg \pi_\beta] \xrightarrow{\kappa} X_{w(\varsigma)}[1 + m_{l(w)}(\sigma)] \xrightarrow{\varphi \circ \vartheta} X_{w(\sigma)}[1] \quad (A.7)
\]

in per \( \mathfrak{C} \). More precisely, we have \( \kappa = \varphi(\tau, \varsigma)(p+1) \) and \( \vartheta = \varsigma, \tau \). Further, \( S_{k_0} \neq S_{j_0} \) forces \( s = 0 \) and \( \varphi_{0, \pi_\beta} \) is the unique homomorphism.

**The third case in Figure 12:** As above, we also have the homomorphism \( \phi_s \) and we shall proof the uniqueness of such a homomorphism. Note that \( \deg \pi_\beta = 3 \) and \( \varphi_{0, \pi_\beta} \) is as follows:

\[
\varphi_{0, \pi_\beta} : S_{k_0} \xrightarrow{\pi_{a_1}} S_{k_1} \xrightarrow{0} \cdots \xrightarrow{0} S_{k_{i-1}} \xrightarrow{0} S_{j_1} \xrightarrow{0} \cdots \xrightarrow{0} S_{j_{i-1}} \xrightarrow{0} \cdots
\]

We claim that this chain map \( \varphi_{0, \pi_\beta} \) is homotopic to

\[
\psi_{\beta_1} : S_{k_0} \xrightarrow{\pi_{a_1}} S_{k_1} \xrightarrow{0} \cdots \xrightarrow{0} S_{k_{i-1}} \xrightarrow{0} S_{j_1} \xrightarrow{0} \cdots \xrightarrow{0} S_{j_{i-1}} \xrightarrow{0} \cdots
\]

where \( \beta_1 \) is an arrow \( k_1 \to j_1 \) with degree 3. Without loss of generality, assume that \( \varepsilon(\pi_{a_1}) = \varepsilon(\pi_{b_1}) = + \). Then by Lemma A.2, there is a map \( \pi_c : S_{k_1} \to S_{j_0} \) with \( \deg \pi_c = \)
$3 - \deg \pi_{a_1}$ such that $\pi_{\beta} = \pi_{c}\pi_{a_1}$. Moreover, Lemma A.2 also implies $\pi_{b_1}\pi_{c} = \pi_{\beta_1}$, which assures the claim. By induction, the homomorphism $\varphi_{0,\pi_{\beta}}$ is homotopic to the homomorphism $\psi_{\beta_{t-1}}$:

$$
\psi_{\beta_{t-1}}: \begin{array}{cccccccc}
S_{k_0} & \pi_{a_1} & S_{k_1} & \cdots & S_{k_{t-1}} & S_{k_{t+1}} & \cdots \\
0 & 0 & & & \pi_{\beta_{t-1}} & 0 \\
S_{j_0} & \pi_{b_1} & S_{k_1} & \cdots & S_{j_{t-1}} & S_{j_{t+1}} & \cdots
\end{array}
$$

induced by the map $\beta_{t-1}$ with $\deg \pi_{\beta_{t-1}} = 3$. Now, consider the possible cases for $\pi_{a_i}$ and $\pi_{b_i}$ shown in Figure 13. We claim that $\varphi_{0,\pi_{\beta}}$ is not null-homotopic.

**Subcase (a):** We have $\varepsilon(\pi_{a_i}) = +, \varepsilon(\pi_{b_i}) = +$ and $\deg \pi_{a_i} > \deg \pi_{b_i}$. Then $\psi_{\beta_{t-1}}$ is as follows:

$$
\psi_{\beta_{t-1}}: \begin{array}{cccccccc}
S_{k_0} & \pi_{a_1} & S_{k_1} & \cdots & S_{k_{t-1}} & S_{k_{t+1}} & \cdots \\
0 & 0 & & & \pi_{\beta_{t-1}} & 0 \\
S_{j_0} & \pi_{b_1} & S_{k_1} & \cdots & S_{j_{t-1}} & S_{j_{t+1}} & \cdots
\end{array}
$$

If $\psi_{\beta_{t-1}}$ is null-homotopic, there is a homomorphism $h \in \text{Hom}^\bullet(S_{k_1}, S_{j_{t-1}})$ such that $\pi_{\beta_{t-1}} = \pi_{a_i}h$, with $\deg \pi_{a_i} + \deg h = 3$, and $h\pi_{b_i}$ factors through $\pi_{a_{t+1}}$ or $\pi_{b_{t+1}}$. See the diagram above. So $h = \pi_{a_i}$ for some arrow $a$ that corresponds to the same triangle as $a_t$ and $b_t$, but different triangle as $a_{t+1}$ or $b_{t+1}$. Since $\deg \pi_{b_i} < \deg \pi_{a_i}$, we deduce that $h\pi_{b_i} \neq 0$ by Lemma A.2. Again, by Lemma A.2, $h\pi_{b_i}$ doesn’t factor through $\pi_{a_{t+1}}$ or $\pi_{b_{t+1}}$ since $a_{t+1}$ and $b_{t+1}$ are induced from different triangles as $a_i$. Then there is a contradiction. So $\psi_{\beta_{t-1}}$ is not null-homotopic.

**Subcase (b):** We have that $\varepsilon(\pi_{b_i}) = -, \varepsilon(\pi_{a_i}) = -$ and $\deg \pi_{a_i} < \deg \pi_{b_i}$. Similar as above, we have that $\psi_{\beta_{t-1}}$ is not null-homotopic.

**Subcase (c), (d) or (e):** We have $\varepsilon(\pi_{a_i}) = -, \varepsilon(\pi_{b_i}) = +$, if $a_t$ exists, and $\varepsilon(\pi_{b_i}) = +$, if $b_t$ exists. Then it is straightforward to see that $\pi_{\beta_{t-1}}$ can’t factor through $\pi_{a_i}$ or $\pi_{b_i}$, which implies that $\psi_{\beta_{t-1}}$ is not null-homotopic.

In all, $\varphi_0 \sim \psi_{\beta_{t-1}} \neq 0$. Moreover, in these subcases, the condition (C1) or (C2) does not holds for $s = t$. Therefore $\varphi_{0,\pi_{\beta}}$ is the unique nonzero homomorphism, which induces a triangle as (A.7).

**The last case in Figure 12:** There are all subcases $(a) \to (e)$ for $\varepsilon(\pi_{a_i})$ and $\varepsilon(\pi_{b_i})$ as shown in Figure 13 (note that we need to switch $\sigma$ and $\tau$ in the figure here). We claim that $\varphi_{0,\pi_{\beta}} \sim \psi_{\beta_{t-1}} = 0$ and $\phi_i$ is the unique nonzero homomorphism. Let $p = l(\sigma)$ and $q = l(\tau)$ and we only discuss **Subcase (a)** in details for demonstration.

**Subcase (a):** We have $\varepsilon(\pi_{b_i}) = +, \varepsilon(\pi_{a_i}) = +$ and $\deg \pi_{a_i} < \deg \pi_{b_i}$.
In particular, \( \deg \pi_{\alpha t} < 3 \). Noticing that \( \deg \pi_{\beta_{t-1}} = 3 \), by Lemma A.2, there is an arrow \( c \) arising from the triangle \( \Lambda_t \) such that \( \pi_{e_{\alpha t}} \pi_{a_{t-1}} \). Moreover, \( \pi_{b_{t-1}} \pi_{c_{t-1}} = 0 \) since \( \deg \pi_{b_{t-1}} + \deg \pi_{c_{t-1}} > \deg \pi_{a_{t-1}} + \deg \pi_{c_{t-1}} = 3 \). Thus \( \psi_{\beta_{t-1}} \) is null-homotopic.

Further, denote by \( \beta \) the arrow that segment \( AB \) corresponds to. By Lemma A.2, \( \pi_{b_{t-1}} = \pi_{\beta \pi_{\alpha t}} \). By Condition (C1) in Lemma A.7,

\[
\varphi_{t,g}: X_{w(\alpha)} \to X_{w(\tau)}
\]

is a homomorphism, for \( g = \pi_{\beta_{t-1}} \), and thus nonzero by Lemma A.8.

\[
\varphi_{t,g}:
\begin{align*}
S_{k_0} & \to S_{k_1} \to \cdots \to S_{k_{t-1}} \to S_{k_t} \to \cdots \\
\pi_{\alpha t} & \uparrow \pi_{\alpha t} \uparrow \cdots \uparrow \pi_{\alpha t} \\
S_{j_0} & \to S_{j_1} \to \cdots \to S_{j_{t-1}} \to S_{j_t} \to \cdots \\
\pi_{b_{t-1}} & \uparrow \pi_{b_{t-1}} \uparrow \cdots \uparrow \pi_{b_{t-1}} \\
\end{align*}
\]
Moreover, we have $\text{Cone} \varphi_{t,g} \cong X_{w(\varsigma)}[m']$, where $\varsigma = L_\sigma^+(\tau)$ with the corresponding string $w(\varsigma)$ as

$$
S_{k_p} \stackrel{\pi_{ap}}{\cdots} \stackrel{\pi_{aq+1}}{\longrightarrow} S_{k_t} \stackrel{\pi_\beta}{\longrightarrow} S_{j_t} \stackrel{\pi_{b_{t+1}}}{\cdots} \stackrel{\pi_{b_q}}{\longrightarrow} S_{j_q},
$$

and $m' = m_t(\tau) + m_{l(w)}(\sigma) - m_{l-1}(\sigma)$. Hence there is a triangle

$$
X_{w(\varsigma)} \xrightarrow{\varphi_{t,\beta}} X_{w(\tau)} \xrightarrow{\kappa} X_{w(\varsigma)}[m'] \xrightarrow{\theta} X_{w(\sigma)}[1]. \tag{A.8}
$$
in per $\mathcal{E}$. More precisely, we have $\kappa = \varphi(\tau, \varsigma)_{q-t, \pi_{aq}}$ and $\theta = \varphi(\varsigma, \tau)_{p-t, 0}$. Note that $p-t > 0$ and $q-t > 0$, so both of $\varphi(\tau, \varsigma)_{q-t, \pi_{aq}}$ and $\varphi(\varsigma, \tau)_{p-t, 0}$ are nonzero.

**Subcase (b):** We have $\varepsilon(\pi_{aq}) = \varepsilon(\pi_{bt}) = +$ and $\deg \pi_{aq} > \deg \pi_{bt}$. Then there is a triangle (A.8), where $\varsigma$ and $m'$ are the same as in **Subcase (a)**. More precisely, we have $\kappa = \varphi(\tau, \varsigma)_{q-t, 0}$ and $\theta = \varphi(\varsigma, \tau)_{p-t, \pi_{bq}}$ here.

**Subcase (c):** We have $\varepsilon(\pi_{aq}) = -$ and $\varepsilon(\pi_{bt}) = -$. Then $\text{Cone} \varphi_{t,0} \cong X_{w(\varsigma)}[m']$ where $\varsigma = L_\sigma^+(\tau)$ with the corresponding string $w(\varsigma)$ as

$$
S_{k_p} \stackrel{\pi_{ap}}{\cdots} \stackrel{\pi_{aq+1}}{\longrightarrow} S_{k_t} \stackrel{\pi_{aq}}{\longrightarrow} S_{j_t} \stackrel{\pi_{b_{t+1}}}{\cdots} \stackrel{\pi_{bq}}{\longrightarrow} S_{j_q},
$$

and $m' = 1 + m_{t-1}(\tau) + m_{l(w)}(\sigma) - m_{t-1}(\sigma)$. So there is an induced triangle

$$
X_{w(\varsigma)} \xrightarrow{\varphi_{t,0}} X_{w(\tau)} \xrightarrow{\kappa} X_{w(\varsigma)}[m'] \xrightarrow{\theta} X_{w(\sigma)}[1].
$$
in per $\mathcal{E}$. More precisely, we have $\kappa = \varphi(\tau, \varsigma)_{q-t, \pi_{aq}}$ and $\theta = \varphi(\varsigma, \tau)_{p-t, \pi_{bq}}$ here.

**Subcase (d):** We have $p > q, t = q + 1$ and $\varepsilon(\pi_{aq}) = +$. Then $\text{Cone} \varphi_{t,?} \cong X_{w(\varsigma)}[m']$, where $\varsigma = L_\sigma^+(\tau)$ with the corresponding string $w(\varsigma)$ as

$$
w(\varsigma) : S_{k_p} \stackrel{\pi_{ap}}{\cdots} \stackrel{\pi_{aq+2}}{\longrightarrow} S_{k_{q+1}}
$$

and $m' = 1 + m_{l(w)}(\sigma)$. So there is an induced triangle (A.8). More precisely, we have $\kappa = \varphi(\tau, \varsigma)_{0, \pi_{aq+1}}$ and $\theta = \varphi(\varsigma, \tau)_{p-q, 0}$ here.

**Subcase (e):** We have $p < q$ and $t = p + 1$. Then $\text{Cone} \varphi_{t,?} \cong X_{w(\varsigma)}[m']$, where $\varsigma = L_\sigma^+(\tau)$ with the corresponding string $w(\varsigma)$ as

$$
w(\varsigma) : S_{k_p} \stackrel{\pi_{ap+1}}{\cdots} \stackrel{\pi_{aq-1}}{\longrightarrow} S_{k_q}
$$

and $m' = 1 + m_{l(w)}(\sigma)$. So there is an induced triangle (A.8). More precisely, we have $\kappa = \varphi(\tau, \varsigma)_{q-p}$ and $\theta = \varphi(\varsigma, \tau)_{0, \pi_{bq+1}}$.

Therefore, $\varphi_{t,g}$ is the unique homomorphism in the last case of Figure 12, which completes the proof. □

**Remark A.10.** By the proof of the lemma above, $s = 0$ in the unique nonzero homomorphism $\varphi(\sigma, \tau) = \varphi(\sigma, \tau)_{s,g}$, for the first three cases in Figure 12, and $s > 0$, for the last case in Figure 12.

**Corollary A.11.** Let $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{CA}(S_\Delta)$ be three orientated curves with the same starting point and suppose that each of them is in a minimal position with respect to
the others. If their start segments are in clockwise order in the triangle which contains their starting point. Then

\[ \varphi(\sigma_2, \sigma_3) \varphi(\sigma_1, \sigma_2) = \varphi(\sigma_1, \sigma_3). \]

**Proof.** Consider the relative position of the first segments of \( \sigma_i \).

If \( \sigma_1 \) and \( \sigma_3 \) are as of \( \sigma \) and \( \tau \) in the last case in Figure 12, then \( \sigma_2 \) is between them by the clockwise order. By Remark A.10, \( \varphi(\sigma_i, \sigma_j) = \varphi(\sigma_i, \sigma_j) t_{ij}, g_{ij} \) with \( t_{ij} > 0 \), for \( (i, j) = (1, 2), (2, 3) \) or \( (1, 3) \). So the composition of homomorphisms \( \varphi(\sigma_2, \sigma_3) \varphi(\sigma_1, \sigma_2) \) is \( \varphi(\sigma_1, \sigma_3) \min \{ t_{12}, t_{23} \}, g \) for some \( g \). By the uniqueness of the homomorphism \( \varphi(\sigma_1, \sigma_3) \), we have \( \varphi(\sigma_2, \sigma_3) \varphi(\sigma_1, \sigma_2) = \varphi(\sigma_1, \sigma_3) \).

For any other possible relative position of the first segments of \( \sigma_i \), a similar deduction applies. See Figure 14 for all essential cases. \( \square \)

![Figure 14. The relative position of \( \sigma_i \)](image)

**Proposition A.12.** Let \( \sigma, \tau \) be two closed arcs in \( \text{CA}(S_\triangle) \) with \( \text{Int}_{S_\triangle}(\sigma, \tau) = 0 \) and \( \eta = B_\sigma(\tau) \). Then there exist representatives \( X_? \) in \( \tilde{X}(?) \) for \( ? = \sigma, \tau, \eta \) such that there is a non-split triangle

\[ X_\tau \to X_\eta \to \bigoplus_{i=1}^{2\text{Int}(\sigma, \tau)} X_\sigma[m_i] \xrightarrow{(f_i)} X_\tau[1] \]

(A.9)

for some \( m_i \in \mathbb{Z} \). In the case when \( \text{Int}_\triangle(\sigma, \tau) = 2 \), the maps \( f_i \) are \( \varphi(\sigma, \tau) \) and \( \varphi(\sigma, \tau) \) as in Lemma A.9 and they are linearly independent.

**Proof.** If \( \text{Int}_\triangle(\sigma, \tau) = 1 \), without loss of generality, we assume that their starting point coincide. Then \( B_\sigma(\tau) = L^+_\sigma(\tau) \). So the triangle in Lemma A.9 becomes (A.9).

Next suppose that \( \text{Int}_\triangle(\sigma, \tau) = 2 \). Fix orientations of \( \sigma \) and \( \tau \) so that they have the same starting point \( V_{-1} \) and the same ending point \( V_{\infty} \). Let \( \zeta = L^+_\sigma(\tau) |_{V_{-1}} \). Then we have

\[ \eta = B_\sigma(\tau) = L^+_\sigma(\tau) |_{V_{\infty}}, \]

where \( V_{\infty} \) is the common starting points of \( \sigma \) and \( \bar{\tau} \). Similarly, let \( \xi = L^+_\sigma(\bar{\tau}) |_{V_{\infty}} \) and then we have

\[ \eta = B_\sigma(\tau) = \left( L^+_\sigma(\zeta^{-1}) |_{V_{-1}} \right)^{-1}. \]
Note that the starting segments of $\bar{\sigma}$, $\bar{\tau}$ and $\bar{\varsigma}$ are in clockwise order in the triangle which contains their starting point, see Figure 15. By Corollary A.11, we have

$$\varphi(\bar{\tau}, \bar{\varsigma}) \varphi(\bar{\sigma}, \bar{\tau}) = \varphi(\bar{\sigma}, \bar{\varsigma}).$$

By Lemma A.9, each morphism above induces a triangle; then applying Octahedral Axiom to this composition gives the following commutative diagram of triangles

$$\begin{array}{ccc}
X_\eta & \longrightarrow & X_\sigma[l] \longrightarrow \longrightarrow X_\xi[m + l'] \\
\downarrow & \varphi(\sigma, \tau) & \downarrow \varphi(\sigma, \tau) \\
X_\sigma[m] & \longrightarrow & X_\xi[m + l] \longrightarrow \longrightarrow X_\xi[m + l'] \\
\downarrow \varphi(\bar{\tau}, \bar{\varsigma}) & \varphi(\bar{\tau}, \bar{\varsigma}) & \downarrow \varphi(\bar{\tau}, \bar{\varsigma}) \\
X_\xi[m'+l] & \longrightarrow & X_\xi[m'+l] \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
\end{array}$$

where $[?]$ are the proper shifts. Let $f_1 = \varphi(\sigma, \tau)$ and $f_2 = \varphi(\bar{\sigma}, \bar{\tau})$. Then we have the triangle (A.9). Since $\varphi(\bar{\tau}, \bar{\varsigma}) f_1 = 0$ and $\varphi(\bar{\tau}, \bar{\varsigma}) f_2 = \varphi(\bar{\sigma}, \bar{\varsigma}) \neq 0$, we deduce that $f_1, f_2$ are linearly independent. \hfill $\square$

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Yu Qiu: Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway  
email address: Yu.Qiu@bath.edu  

Yu Zhou: Fakultät für Mathematik, Universität Bielefeld, D-33501, Bielefeld, Germany  
Current address: Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway  
email address: yuzhoumath@gmail.com