Harmonic morphisms and shear-free ray congruences

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We describe the relationship between complex-valued harmonic morphisms from Minkowski 4-space and the shear-free ray congruences of mathematical physics. Then we show how a horizontally conformal submersion on a domain of \( \mathbb{R}^3 \) gives the boundary values at infinity of a complex-valued harmonic morphism on hyperbolic 4-space.

INTRODUCTION

A harmonic morphism is a map that preserves Laplace’s equation. More explicitly, a smooth map \( \phi : M \to N \) between Riemannian or semi-Riemannian manifolds is called a harmonic morphism if its composition \( f \circ \phi \) with any locally defined harmonic function on the codomain \( N \) is a harmonic function on the domain \( M \); it thus ‘pulls back’ germs of harmonic functions to germs of harmonic functions. Harmonic morphisms are characterized as harmonic maps which are ‘horizontally weakly conformal’, see below.

Our purpose here is to describe the relationship between complex-valued harmonic morphisms from Minkowski 4-space and the shear-free ray congruences of mathematical physics. This paper is a revised and expanded version of Baird and Wood (1998); there are no plans to publish it. It can be regarded as a supplement to the book (Baird and Wood 2003), see

http://www.amsta.leeds.ac.uk/Pure/staff/wood/BWBook/BWBook.html,

which the reader may consult for background on harmonic morphisms and related topics; references in boldface such as ‘Definition 2.5.7’, ‘(2.5.8)’ refer to that book. In particular, the reader may find it helpful to read Chapters 2–4 of the book which give a self-contained introduction to harmonic maps and morphisms between Riemannian manifolds, Chapter 7 which gives the background in twistor theory and its relationship to harmonic morphisms in the Riemannian setting, and Chapter 14 which discusses harmonic maps and morphisms between semi-Riemannian manifolds.

In Sections S.1–S.5 of the present paper we describe one-to-one correspondences between conformal foliations by curves of \( \mathbb{R}^3 \), Hermitian structures on \( \mathbb{R}^4 \), shear-free ray congruences on Minkowski 4-space and complex-analytic foliations by null planes on \( \mathbb{C}^4 \); then, in Sections S.9–S.11 we explain how to find these quantities by twistor theory, on the way discussing group actions on the twistor spaces. A CR interpretation is given in Section S.12. Although this material is essentially known in mathematical physics, we give a detailed self-contained treatment as it is not widely known outside that domain.

In Sections S.6–S.8, we show how a complex-valued harmonic morphisms from Minkowski 4-space determines a shear-free ray congruence and conversely, and give a version relating complex-harmonic morphisms and null planes.

In Sections S.13–S.15 we show how a horizontally conformal submersion on a domain of \( \mathbb{R}^3 \) gives the boundary values at infinity of a complex-valued harmonic morphism on an open subset of hyperbolic 4-space; then we give explicit constructions of such submersions, relating these to the quantities described above.
S.1 CONFORMAL FOLIATIONS BY CURVES ON $\mathbb{R}^3$

Let $\mathbb{R}^3$ denote three-dimensional Euclidean space, i.e., $\mathbb{R}^3 = \{(x_1,x_2,x_3) : x_i \in \mathbb{R}\}$ equipped with its standard metric $g = dx_1^2 + dx_2^2 + dx_3^2$ and orientation. Let $C$ be a smooth foliation by curves of an open subset $A^3$ of $\mathbb{R}^3$. Assume that $C$ is oriented, i.e., we can find a smooth unit vector field $U$ tangent to its leaves. The orthogonal complement $U^\perp$ is then oriented.

For such a foliation we can reformulate the definition of conformal foliation (Definition 2.5.7), as follows. Let $J^\perp$ denote rotation through $+\pi/2$ on $U^\perp$, and let $\{\}^\perp$ denote orthogonal projection onto $U^\perp$. Let $L_Y$ denote the Lie derivative with respect to a vector field $Y$.

**Proposition S.1.1** The foliation $C$ is conformal if and only if, at all points of $A^3$,

$$\{(L_UJ^\perp)(X)\}^\perp = 0 \quad (X \in U^\perp),$$

(S.1.1)

and this holds if and only if

$$\nabla_{J^\perp X} U = J^\perp \nabla_X U \quad (X \in U^\perp).$$

(S.1.2)

**Proof** That (S.1.1) is equivalent to conformality follows from Proposition 2.5.16(ii). To show the equivalence of (S.1.1) and (S.1.2), we have, for $X \in \Gamma(U^\perp)$,

$$(L_UJ^\perp)(X) = \{(L_U(J^\perp X))\}^\perp - J^\perp \{(L_U(X))\}^\perp$$

$$= \{\nabla_U(J^\perp X)\}^\perp - \nabla_{J^\perp X} U - J^\perp \{\nabla_U X\}^\perp + J^\perp \nabla_X U$$

(S.1.3)

noting that $\nabla_X U \in \Gamma(U^\perp)$ since $g(\nabla_X U, U) = \frac{1}{2}X(g(U, U)) = 0$. Further, since $U^\perp$ has rank 2, from Proposition 2.5.16(i), we have $\nabla^\text{End} U^\perp J^\perp = 0$ so that

$$\{\nabla_U(J^\perp X)\}^\perp - J^\perp \{\nabla_U X\}^\perp = (\nabla^\text{End} U^\perp J^\perp)(X) = 0,$$

hence (S.1.1) holds if and only if (S.1.2) holds. \hfill \square

**Remark S.1.2** In terms of the Bott partial connection $\nabla_U^\text{Bott}$ on $\text{End}(U^\perp)$ (see (2.5.8) and (2.16)), (S.1.1) can be written as $\nabla_U^\text{Bott} J^\perp = 0$.

Recall that a $C^1$ map $M^m \rightarrow N^n$ between Riemannian or semi-Riemannian manifolds is called horizontally (weakly) conformal if, at each point of $M^m$, the adjoint of its differential is zero or conformal, see Section 2.4 for equivalent and more geometrical ways of saying this. In particular, a smooth map $f : M^m \rightarrow \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$, is horizontally weakly conformal if the vector fields $\text{grad}f_i$ are all orthogonal and of the same square norm:

$$\langle \text{grad} f_i, \text{grad} f_j \rangle = \langle \text{grad} f_j, \text{grad} f_j \rangle \quad \text{and} \quad \langle \text{grad} f_i, \text{grad} f_j \rangle = 0 \quad (i, j \in \{1, \ldots, n\}, \quad i \neq j).$$

(Here, $\langle \ , \ \rangle$ denotes the inner product on $TM$ given by the metric on $M$.) For a smooth function $f : A^3 \rightarrow \mathbb{C}$ this condition can be written nicely as

$$\langle \text{grad} f, \text{grad} f \rangle \equiv \sum_{i=1}^{3} (\partial f/\partial x_i)^2 = 0 \quad ((x_1,x_2,x_3) \in A^3).$$

(S.1.4)

By Corollary 2.5.12, conformal foliations $C$ are given locally as the fibres of horizontally conformal submersions; in the case of interest this can be said explicitly as follows.

**Lemma S.1.3** Let $f : A^3 \rightarrow \mathbb{C}$ be a smooth (respectively, real-analytic) function on an open subset of $\mathbb{R}^3$ which satisfies (S.1.4) Then its fibres form a smooth (respectively, real-analytic) conformal foliation $C$ by curves. All such foliations are given this way locally. \hfill \square
Remark S.1.4 The foliation $\mathcal{C}$ is oriented with unit positive tangent
\[ U = \nabla f_1 \times \nabla f_2 / |\nabla f_1 \times \nabla f_2|. \] (S.1.5)

Example S.1.5 Set $f(x_1, x_2, x_3) = a_1 x_1 + a_2 x_2 + a_3 x_3$ where the $a_i$ are complex constants satisfying $a_1^2 + a_2^2 + a_3^2 = 0$; this gives a foliation by parallel lines.

Example S.1.6 For any constant $c \in \mathbb{R}$, define $f : \mathbb{R}^3 \setminus \{x_1\text{-axis}\} \to \mathbb{C}$ by
\[ f(x_1, x_2, x_3) = \left\{ (x_1 + c)^2 + x_2^2 + x_3^2 \right\}/\{x_2 + ix_3\}. \]

Then $f$ is horizontally conformal and submersive. On identifying the extended complex plane $\mathbb{C} \cup \{\infty\}$ with the 2-sphere by stereographic projection (1.2.11), $f$ can be extended to a horizontally conformal submersive map from $\mathbb{R}^3 \setminus \{(-c, 0, 0)\}$ to $\mathbb{C} \cup \{\infty\} \cong S^2$. It can easily be checked that its level sets are the circles through $(-c, 0, 0)$ tangent to the $x_1$-axis, together with the $x_1$-axis; these are thus the leaves of a conformal foliation $\mathcal{C}_c$ on $\mathbb{R}^3 \setminus \{(-c, 0, 0)\}$. For later reference, the unit tangent to these leaves is calculated to be
\[ U = \frac{1}{\bar{x}_1^2 + x_2^2 + x_3^2} (\bar{x}_1^2 - x_1^2 - 2\bar{x}_1 x_2, 2\bar{x}_1 x_2, 2\bar{x}_1 x_3); \] (S.1.6)
where $\bar{x}_1 = x_1 + c$.

For more examples, see Section S.3.

S.2 HERMITIAN STRUCTURES ON EUCLIDEAN 4-SPACE

Let $\mathbb{R}^4$ denote four-dimensional Euclidean space, i.e., $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) : x_i \in \mathbb{R}\}$ equipped with the standard Euclidean metric $g = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$ and standard orientation. Let $x \in \mathbb{R}^4$. Recall (see, e.g., Section 7.1) that an almost Hermitian structure at $x$ is an isometric linear transformation $J_x : T_x M \to T_x M$ which satisfies $J_x^2 = -I$ (where $I$ denotes the identity map).

We can always find an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $T_x \mathbb{R}^4$ such that $J_x e_0 = e_1, J_x e_2 = e_3$. Then $J_x$ is called positive (respectively, negative) according as $\{e_0, ..., e_3\}$ is positively (respectively, negatively) oriented. As explained in Section 7.1, the set of all positive Hermitian structures at a point can be identified with $S^2 \cong CP^1$.

Let $A^4$ be an open subset of $\mathbb{R}^4$. Recall (see, e.g., Section 7.1 again) that by an almost Hermitian structure $J$ on $A^4$ we mean a smooth choice of almost Hermitian structure at each point of $A^4$, i.e., a smooth map (which we shall denote by the same letter) $J : A^4 \to CP^1$. Recall, also, that $J$ is integrable if and only if its Nijenhuis tensor $N$ satisfies
\[ N \equiv 0; \] (S.2.1)
by Proposition 7.1.3(ii) this is equivalent to $(\nabla_{JX}J)(Y) = (\nabla_XJ)(Y)$ \((x \in A^4, X, Y \in T_x \mathbb{R}^4)\).

A simple calculation shows that $(\nabla_XJ)(Y) = J(\nabla_XJ)(Y)$, so that (S.2.1) is equivalent to
\[ \nabla_{JX}J = J\nabla_XJ \quad (x \in A^4, X \in T_x \mathbb{R}^4), \] (S.2.2)
explicitly,
\[ (\nabla_{JX}J)(Y) = J((\nabla_XJ)(Y)) \quad (x \in M, X, Y \in T_x M). \]

If $A^4$ is given the almost Hermitian structure $J$, and $CP^1$ is given its standard Kähler structure, then (S.2.2) is the condition that the map $J : A^4 \to CP^1$ be holomorphic. An integrable almost Hermitian structure is called a Hermitian structure.

S.3 SHEAR-FREE RAY CONGRUENCES

Let $\mathbb{M}^4$ denote four-dimensional Minkowski space $\mathbb{M}^4 = \{(t, x_1, x_2, x_3) : t, x_i \in \mathbb{R}\}$ equipped with the standard Minkowski metric $g^M = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$ and time- and space-orientations.

By an $\mathbb{R}^3$-slice we mean a hyperplane given by $t = \text{const}$. Let $x \in \mathbb{M}^4$. We denote the $\mathbb{R}^3$-slice through $x$ by $\mathbb{M}_x^3$. Recall (e.g., from Definition 14.1.7) that a non-zero vector $w \in T_x \mathbb{M}^4$ is called null if $g^M(w, w) = 0$. It spans a one-dimensional null subspace, we shall call such a subspace
a null direction. Any non-zero null vector \( w \in T_x \mathcal{M}^4 \) can be normalized so that it is of the form \( w = \partial/\partial t + U \) where \( U \) is a unit vector in \( T_x \mathbb{R}_x^3 \). Let \( \ell \) be a smooth foliation by null lines (often called a ray congruence) of an open subset \( A^3 \) of Minkowski space, let \( w = \partial/\partial t + U \) be its tangent vector field and write \( W = \text{span}\{w\} \). Then \( W \) is a field of null directions. The distribution \( W^\perp \) orthogonal to \( W \) (with respect to the Minkowski metric \( g^M \)) is three-dimensional and contains \( W \). Choose any complement \( \Sigma \) of \( W \) in \( W^\perp \), such a complement is called a screen space, then the restriction of the Minkowski metric \( g^M \) to \( \Sigma \) is positive definite. Note that any screen space is naturally isomorphic to the factor space \( W^\perp/W \), however, it is usually more convenient to work with a specific screen space. A special choice of screen space at \( x \) is \( U_x^\perp \cap \mathbb{R}_x^3 \); this has a canonical orientation from which \( W^\perp/W \) and so all screen spaces, acquire an orientation.

Note that, for all \( X \in \Gamma(\Sigma) \), we have \( \langle \nabla_X w, w \rangle = \frac{1}{2} X \langle w, w \rangle = 0 \); also since the integral curves of \( w \) are geodesics, we have \( \nabla_w w = 0 \). It follows that \( \langle \mathcal{L}_w X, w \rangle = \langle \nabla_w X, w \rangle - \langle \nabla_X w, w \rangle = -\langle X, \nabla_w w \rangle - 0 = 0 \), so that \( \nabla_J w \in \Gamma(W^\perp) \) and \( \mathcal{L}_w X \in \Gamma(W^\perp) \) \( (X \in \Gamma(\Sigma)) \). \( \text{(S.3.2)} \)

Let \( J^\perp \in \Gamma(\text{End} \Sigma) \) denote rotation through \(+\pi/2\). Then the Lie derivative of \( J^\perp \) along \( W \) is measured by

\[
\{(\mathcal{L}_w J^\perp)(X)\}^\Sigma = \{(\mathcal{L}_w J^\perp X)\}^\Sigma - \{J^\perp(\mathcal{L}_w X)\}^\Sigma \quad (X \in \Gamma(\Sigma)).
\]

where \( \{ \}^\Sigma \) indicates projection from \( W^\perp \) onto \( \Sigma \) along \( W \); this is well-defined by (S.3.2)(ii).

Then \( \ell \) (or \( W \)) is said to be a shear-free ray (SFR) congruence if Lie transport along \( W \) of vectors in \( \Sigma \) is conformal, i.e.,

\[
\{(\mathcal{L}_w J^\perp)(X)\}^\Sigma = 0 \quad (X \in \Gamma(\Sigma));
\]

it is easily checked that this condition is independent of the choice of screen space. Comparison of equations (S.3.5) and (S.1.2) shows that the restriction of \( U \) to any \( \mathbb{R}^3 \)-slice is a vector field whose integral curves form an oriented conformal foliation by curves of an open subset \( A^3 \) of the slice. We shall call this process projection onto the slice. Conversely, given an oriented conformal foliation of an open subset \( A^3 \) of an \( \mathbb{R}^3 \)-slice, set \( U \) equal to its unit positive tangent vector field, then the null lines of \( \mathcal{M}^4 \) tangent to \( \partial/\partial t + U \) at points of \( A^3 \) define a ray congruence \( \ell \) on an open neighbourhood of \( A^3 \) in \( \mathcal{M}^4 \) which is shear-free at points of \( A^3 \). The next result shows that \( \ell \) is a SFR congruence; we shall call it the extension of \( \mathcal{C} \).

![Fig. S.1. Projection onto a slice.](image)

**Lemma S.3.1** If a ray congruence \( \ell \) is shear-free at one point of a ray, it is shear-free at all points of that ray.
Shear-free ray congruences

**Proof** Let \( X \) be a section of \( \Sigma \) and set \( Z = X + iJ^X \). Then it is easily seen that, at any point, condition (S.3.5) is equivalent to
\[
\langle \nabla_Z w, Z \rangle = 0. \tag{S.3.6}
\]
As in (S.3.2), \( L_w Z \equiv [w, Z] \in \Gamma(W^\perp) \). Extending the terminology of Section 2.5, call \( Z \) basic if the component of \( L_w Z \) in \( \Sigma \) is zero, i.e.
\[
[w, Z] \in \Gamma(W). \tag{S.3.7}
\]
Then, using (S.3.1), (S.3.7) and the zero curvature of \( M^4 \) we obtain
\[
w(\langle \nabla_Z w, Z \rangle) = \langle \nabla_w \nabla_Z w, Z \rangle + \langle \nabla_Z w, \nabla_w Z \rangle
= \langle \nabla_Z \nabla_w w, Z \rangle + \langle \nabla_Z w, \nabla_w w \rangle.
\]
Using (S.3.1) and (S.3.6) we see that this is zero; hence if (S.3.6) holds at one point of a ray, it holds at all points of that ray. \( \square \)

With projection and extension defined as above, the precise statement of our construction is the following.

**Theorem S.3.2** Let \( p \in M^4 \), and let \( A^3 \) be an open subset of \( \mathbb{R}^3 \). Then projection onto the slice \( \mathbb{R}^2_p \) defines a bijective correspondence between germs at \( A^3 \) of \( C^\infty \) shear-free ray congruences \( \ell \) defined on an open neighbourhood of \( A^3 \) in \( M^4 \) and \( C^\infty \) conformal foliations \( \mathcal{C} \) by curves on \( A^3 \). The inverse of projection is extension. \( \square \)

**Remark S.3.3** The twist, rotation or vorticity tensor of a ray congruence is defined at a point \( x \) by
\[
T(X, Y) = \frac{1}{2} g(\nabla_X Y - \nabla_Y X, w) = \frac{1}{2} g([X, Y], w)
= \frac{1}{2} \{g(\nabla_Y w, X) - g(\nabla_X w, Y)\} \quad (X, Y \in \Sigma_x).
\]

It measures how much (infinitesimally) nearby null lines passing through the screen space \( \Sigma_x \) twist around the null line through \( x \). It is easily seen that a congruence of null lines \( \ell \) is twist-free if and only if its projection \( \mathcal{C} \) onto one \( \mathbb{R}^2 \)-slice has integrable horizontal distribution.

**Example S.3.4** Let \( f(x_1, x_2, x_3) = (x_2 + ix_3)/(x_1 + |x|) \). It is easily checked that \( f \) is a horizontally conformal map from \( \mathbb{R}^3 \setminus \{(0, 0, 0)\} \) to \( \mathbb{C} \setminus \{\infty\} \cong S^2 \). Its level curves are radii from the origin and are the leaves of a conformal foliation \( \mathcal{C} \) of \( \mathbb{R}^3 \setminus \{(0, 0, 0)\} \) whose unit tangent vector field is given by
\[
U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} (x_1, x_2, x_3). \tag{S.3.8}
\]

We remark that, when \( \mathbb{R}^3 \) is conformally compactified to \( S^3 \) (see Section S.9), this example becomes projection from the poles: \( S^3 \setminus \{(1, 0, 0, 0)\} \to S^2 \) given by the formula \( (x_0, x_1, x_2, x_3) \mapsto \pm (x_1, x_2, x_3)/\sqrt{x_1^2 + x_2^2 + x_3^2} \).

It is easily checked that the shear-free ray congruence \( \ell \) extending \( U \) is defined on \( M^4 \setminus \{(t, 0, 0, 0) : t \in \mathbb{R}\} \) and has tangent vector field \( w = \partial/\partial t + U \) where
\[
U(t, x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} (x_1, x_2, x_3). \tag{S.3.9}
\]

For each \( t \), \( \ell \) projects to a conformal foliation on \( \mathbb{R}^3 \) with tangent vector field given by (S.3.9); note that this conformal foliation is independent of \( t \). See also Example S.15.2.

**Example S.3.5** Let \( f(x_1, x_2, x_3) = -ix_1 \pm \sqrt{x_2^2 + x_3^2} \). This is a horizontally conformal submersion from \( A^3 = \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) : x_2 = x_3 = 0\} \) to \( \mathbb{C} \). Its level curves are circles in planes parallel to the \((x_2, x_3)\)-plane and centred on points of the \( x_1 \)-axis; these give a conformal foliation \( \mathcal{C} \) of \( A^3 \) with a unit tangent vector field given by
\[
U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_2^2 + x_3^2}} (0, -x_3, x_2). \tag{S.3.10}
\]
We compute the tangent vector field to the SFR congruence $\ell$ which extends $U$. The affine null geodesic of $\ell$ in $M^4$ through $(x_1, x_2, x_3)$ with direction $\partial/\partial t + U$ is given parametrically by

$$T \mapsto \left( T, x_1, x_2, x_3 \mp x_3 T/\sqrt{x_2^2 + x_3^2}, x_3 \pm x_2 T/\sqrt{x_2^2 + x_3^2} \right)$$

(S.3.11)

Conversely, given $x = (T, X_1, X_2, X_3) \in M^4$ the null geodesic of $\ell$ through $x$ hits the $\mathbb{R}^3$-slice $t = 0$ at $(x_1, x_2, x_3)$ where

$$x_1 = X_1, \quad x_2 \mp x_3 T/\sqrt{x_2^2 + x_3^2} = X_2, \quad x_3 \pm x_2 T/\sqrt{x_2^2 + x_3^2} = X_3.$$

This has solution

$$(x_1, x_2, x_3) = \left( X_1, \frac{R}{X_2^2 + X_3^2} (RX_2 \pm TX_3), \frac{R}{X_2^2 + X_3^2} (RX_3 \mp TX_2) \right)$$

(S.3.12)

where $R = \sqrt{X_2^2 + X_3^2 - T^2}$. Hence the tangent vector field to the null geodesic of $\ell$ through $(t, x_1, x_2, x_3) \in M^4$ is given by $w = \partial/\partial t + U$ with

$$U = U_t(x_1, x_2, x_3) = U(t, x_1, x_2, x_3) = \frac{r}{\sqrt{x_2^2 + x_3^2}} \left( 0, -x_3 \pm \frac{t}{r} x_2, x_2 \pm \frac{t}{r} x_3 \right)$$

(S.3.13)

where $r = \sqrt{x_2^2 + x_3^2 - t^2}$.

For each $t$, the integral curves of $U_t$ give a conformal (in fact, Riemannian) foliation $\mathcal{C}_t$ of the $\mathbb{R}^3$-slice $t = \text{const}$. it is easy to see that the leaves of $\mathcal{C}_t$ are the involutes of circles (see Figure S.2). See Examples S.13.8 and S.15.3 for further developments.

S.4 COMPLEX-ANALYTIC FOLIATIONS BY NULL PLANES ON $\mathbb{C}^4$

Definition S.4.1 (LeBrun 1983) A holomorphic metric on a complex manifold $M$ is a holomorphic section $g^C$ of $\otimes^2 T^*_{1,0}M$ which defines a (complex-symmetric bilinear) non-degenerate inner product on $T^*_{1,0}M$ for each $x \in M$.

Let $\mathbb{C}^4 = \{(x_0, x_1, x_2, x_3) : x_i \in \mathbb{C}^4\}$ equipped with the standard holomorphic metric $g^C = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. A vector $w$ is called (complex-)null or isotropic if $g^C(w, w) = 0$. Let $p \in \mathbb{C}^4$. A subspace in $T_p \mathbb{C}^4$ is called null if it consists of null vectors. For any $p = (p_0, p_1, p_2, p_3) \in \mathbb{C}^4$, by the $\mathbb{R}^4$-slice through $p$ we mean the real four-dimensional affine subspace

$$\mathbb{R}^4_p = \{(x_0, x_1, x_2, x_3) \in \mathbb{C}^4 : \text{Im } x_i = \text{Im } p_i \quad (i = 0, 1, 2, 3)\}$$

parametrized by

$$\mathbb{R}^4_p \ni (x_0, x_1, x_2, x_3) \mapsto (p_0 + x_0, p_1 + x_1, p_2 + x_2, p_3 + x_3) \in \mathbb{R}^4_p \subset \mathbb{C}^4$$

(S.4.1)

and with the orientation induced from the standard orientation of $\mathbb{R}^4$. If $p$ is the zero vector $0 = (0, 0, 0, 0)$ we write $\mathbb{R}^4$ for $\mathbb{R}^4_0$. Note that $T_p \mathbb{C}^4$ can be identified canonically with $T_p \mathbb{R}^4_p \otimes \mathbb{C}$; we shall frequently make this identification.

In a similar way, extending the notation of Section S.3, we define the $\mathbb{R}^3$-slice through $p$ by

$$\mathbb{R}^3_p = \{(x_0, x_1, x_2, x_3) \in \mathbb{C}^4 : x_0 = p_0, \text{Im } x_i = \text{Im } p_i \quad (i = 1, 2, 3)\}$$

parametrized by

$$\mathbb{R}^3_p \ni (x_1, x_2, x_3) \mapsto (p_0, p_1 + x_1, p_2 + x_2, p_3 + x_3) \in \mathbb{R}^3_p \subset \mathbb{C}^4$$

(S.4.2)

and with the orientation induced from the standard orientation of $\mathbb{R}^3$; we write $\mathbb{R}^3$ for $\mathbb{R}^3_0$.

Given any orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of the tangent space $T_p \mathbb{R}^4_p$, the plane $\Pi_p = \text{span}\{e_0 + ie_1, e_2 + ie_3\}$ is null; we call $\Pi_p$ an $\alpha$-plane (respectively, $\beta$-plane) according as the basis $\{e_0, e_1, e_2, e_3\}$ is positively (respectively, negatively) oriented.

By a complex-analytic distribution of $\alpha$-planes on an open set $A^C$ of $\mathbb{C}^4$ we mean a map $\Pi$ which assigns to each point $p$ of $A^C$ an $\alpha$-plane $\Pi_p \subset T_p \mathbb{C}^4$ in a complex-analytic fashion, i.e.,
Let \( \Pi_p = \text{span}\{w_1(p), w_2(p)\} \) where the maps \( w_i : A^C \to TC^4 \) are complex analytic. We can identify \( \Pi \) with its image, a complex-analytic subbundle of \( TC^4|_{A^C} \).

By Frobenius’ Theorem, a complex-analytic distribution \( \Pi \) of \( \alpha \)-planes on \( A^C \) is integrable if and only if

\[
[w_1, w_2] \in \Gamma(\Pi) \quad (w_1, w_2 \in \Gamma(\Pi)). \tag{S.4.3}
\]

**Lemma S.4.2** A complex-analytic distribution \( \Pi \) of \( \alpha \)-planes on \( A^C \) is integrable if and only if it is autoparallel, i.e.,

\[
\nabla w_1 w_2 \in \Gamma(\Pi) \quad (w_1, w_2 \in \Gamma(\Pi)). \tag{S.4.4}
\]

**Proof** Condition (S.4.3) holds if and only if

\[
g^C([w_1, w_2], w_1) = 0 \quad (w_1, w_2 \in \Gamma(\Pi)).
\]

Since \( g^C(\nabla w_2 w_1, w_1) = \frac{1}{2} w_2 (g^C(w_1, w_1)) = 0 \), this holds if and only if

\[
g^C(\nabla w_1 w_2, w_1) = 0 \quad (w_1, w_2 \in \Gamma(\Pi)). \tag{S.4.5}
\]

Now, \( g^C(\nabla w_1 w_2, w_2) = 0 \), hence (S.4.5) holds if and only if (S.4.4) holds. \( \square \)

Thus an integrable complex-analytic distribution of \( \alpha \)-planes has integral submanifolds which are (affine) \( \alpha \)-planes, i.e., planes of \( \mathbb{C}^4 \) whose tangent spaces are \( \alpha \)-planes; thus, an integrable complex-analytic distribution of \( \alpha \)-planes on \( A^C \) is equivalent to a complex-analytic foliation by affine \( \alpha \)-planes of \( A^C \). Similar considerations apply if we replace ‘\( \alpha \)-plane’ by ‘\( \beta \)-plane’.

**S.5 UNIFICATION**

The formulae (S.1.1), (S.2.1), (S.3.4) and (S.4.3) show that the four ‘distributions’ (or ‘fields’) studied in Sections S.1–S.4 are invariant if the standard metrics are replaced by conformally equivalent metrics. This suggests that there may be some relations between the distributions; we now explain those relations.

For any \( p = (p_0, p_1, p_2, p_3) \in \mathbb{C}^4 \), by analogy with the \( \mathbb{R}^4 \)-slice through \( p \) defined in Section S.4, we define the *Minkowski slice through \( p \)* to be the real four-dimensional affine subspace

\[
\mathbb{M}^4_p = \{ (x_0, x_1, x_2, x_3) : \text{Re} \, x_0 = \text{Re} \, p_0, \ \text{Im} \, x_i = \text{Im} \, p_i \ (i = 1, 2, 3) \}
\]

parametrized by

\[
\mathbb{M}^4 \ni (t, x_0, x_2, x_3) \mapsto (p_0 - it, p_1 + x_1, p_2 + x_2, p_3 + x_3) \in \mathbb{M}^4_p \subset \mathbb{C}^4.
\]

If \( p = 0 \), we write \( \mathbb{M}^4 \equiv \mathbb{M}^4_0 \). (Note that the minus sign before the term \( it \) is unimportant—it is simply to avoid minus signs elsewhere.)

Now we use the slice notation to show how all the four quantities discussed above are equivalent at a point.

**Proposition S.5.1** Let \( p \in \mathbb{C}^4 \). There are one-to-one correspondences between the quantities:

- (QP1) \( \alpha \)-planes \( \Pi_p \subset T_p \mathbb{C}^4 \),
- (QP2) positive almost Hermitian structures \( J_p : T_p \mathbb{R}^4_p \to T_p \mathbb{R}^4_p \),
- (QP3) unit vectors \( U_p \in T_p \mathbb{R}^3_p \),
- (QP4) null directions \( W_p \subset T_p \mathbb{M}^4_p \),

given by...
$$J_p = J_p(\Pi_p) = -i \text{ on } \Pi_p, \quad +i \text{ on } \Pi_p;$$

$$\Pi_p = \Pi_p(J_p) = (0, 1)\text{-tangent space of } J_p, \text{ i.e., } \Pi_p = \{X + iJ_pX : X \in T_p\mathbb{R}^4_p\};$$

$$U_p = U_p(J_p) = J_p(\partial/\partial x_0);$$

$$J_p = J_p(U_p) = \text{the unique positive almost Hermitian structure at } p$$

with $$J_p(\partial/\partial x_0) = U_p;$$

$$W_p = W_p(U_p) = \text{span}\{w_p\} \text{ where } w_p = \partial/\partial t + U_p;$$

$$U_p = U_p(W_p) = \text{normalized projection of } W_p \text{ onto } T_p\mathbb{R}^3_p, \text{ i.e. the unique}$$

$$U_p \in T_p\mathbb{R}^3 \text{ such that } W_p = \text{span}(\partial/\partial t + U_p);$$

$$W_p = W_p(\Pi_p) = \Pi_p \cap T_p\mathbb{M}^4_p;$$

$$\Pi_p = \Pi_p(W_p) = \text{the unique } \alpha\text{-plane containing } W_p.$$

\(\square\)

Note that we shall use an upright font: J, U, w, etc. to denote quantities which are related to each other as in this proposition.

Next we discuss correspondences between the various distributions.

Let $$p \in \mathbb{C}^4.$$ Suppose that $$J$$ is a real-analytic almost Hermitian structure on a neighbourhood $$A^4$$ of $$p$$ in $$\mathbb{R}^4_p.$$ Then we can extend $$J$$ to a neighbourhood $$A^C$$ of $$p$$ in $$\mathbb{C}^4$$ by asking that it be complex analytic. Note that this complex analyticity may be expressed by

$$\nabla_{iX}J = J\nabla_XJ \quad (X \in T_q\mathbb{C}^4, \ q \in A^C).$$

(S.5.2)

Via Proposition S.5.1, this defines a complex-analytic distribution of $$\alpha$$-planes.

Conversely, a complex-analytic distribution of $$\alpha$$-planes on a neighbourhood $$A^C$$ of $$p$$ defines a real-analytic almost Hermitian structure on $$A^4 = A^C \cap \mathbb{R}^4_p.$$ Similarly, a real-analytic distribution $$W$$ of null directions on a neighbourhood $$A^M$$ of $$p$$ in $$\mathbb{M}^4_p$$ gives rise to a complex-analytic distribution of $$\alpha$$-planes on a neighbourhood $$A^C$$ of $$p$$ in $$\mathbb{C}^4,$$ and conversely. Then we have the following.

**Proposition S.5.2** Let $$\Pi$$ be a complex-analytic distribution of $$\alpha$$-planes on an open subset $$A^C$$ of $$\mathbb{C}^4$$, let $$p \in A^C,$$ and let $$J$$ (respectively, $$W$$) be the corresponding almost Hermitian structure on $$A^4 = A^C \cap \mathbb{R}^4_p$$ (respectively, distribution of null directions on $$A^M = A^C \cap \mathbb{M}^4_p$$). Then the following conditions are equivalent:

(i) $$\Pi$$ is integrable on $$A^C,$$

(ii) $$J$$ is integrable on $$A^4,$$

(iii) $$W$$ defines a shear-free ray congruence on $$A^M.$$

**Proof** Firstly we show that $$\Pi$$ is integrable at points of $$A^4$$ if and only if $$J$$ is integrable on $$A^4.$$ To do this, let $$q \in A^4.$$ In view of Lemma S.4.2, the distribution $$\Pi$$ is integrable at $$q$$ if and only if

$$\nabla_{X+iJX}J = 0 \quad (X \in T_q\mathbb{R}^4_q).$$

(S.5.3)

By (S.5.2), this is equivalent to

$$\nabla_{iX}J = J\nabla_XJ \quad (X \in T_q\mathbb{R}^4_q)$$

which is the integrability condition (S.2.2) for $$J$$ at $$q.$$

Next we show that $$\Pi$$ is integrable at points of $$A^M$$ if and only if $$W$$ is a shear-free ray congruence on $$A^M.$$ To do this, take $$q \in A^M.$$ Then with $$U = J(\partial/\partial x_0),$$ since $$\partial/\partial x_0$$ is parallel, (S.5.3) is equivalent to

$$\nabla_{iX}U = J\nabla_XU \quad (X \in T_q\mathbb{R}^4_q),$$

(S.5.4)

and so, with $$w = \partial/\partial t + U,$$

$$\nabla_{X+iJX}w = 0 \quad (X \in T_q\mathbb{R}^4_q).$$

(S.5.5)

Choose $$X = \partial/\partial x_0 = i\partial/\partial t;$$ then this reads

$$\nabla_w w = 0;$$

(S.5.6)
this is the condition that the integral curves of \( w \) be geodesic at \( q \). With \( X \) chosen instead in the
screen space \( W_q \cap \mathbb{R}^3 = U_q \cap \mathbb{R}^3 \), we obtain the shear-free condition (S.3.5). Conversely, since the
two choices of \( X \) give vectors \( X + iX \) spanning \( \Pi_q \), if (S.5.6) and (S.3.5) hold so does (S.5.3).

To finish the proof, if any of the above conditions holds at all points of \( A^4 \) or \( A^M \), by analytic
continuation it holds throughout \( A^C \).

Combining this result with Theorem S.3.2 we obtain relations between the following four sorts of
distributions (for any \( p \in \mathbb{C}^4 \)):

- (Q1) holomorphic foliations \( \mathcal{F} \) by \( \alpha \)-planes \( \Pi \) of an open subset \( A^C \) of \( \mathbb{C}^4 \),
- (Q2) positive Hermitian structures \( J \) on an open subset \( A^4 \) of \( \mathbb{R}^4_p \),
- (Q3) real-analytic shear-free ray congruences \( \ell \) on an open subset \( A^M \) of \( \mathbb{M}^4_p \),
- (Q4) real-analytic conformal foliations \( \mathcal{C} \) by curves of an open subset \( A^3 \) of \( \mathbb{R}^3_p \).

Theorem S.5.3 Let \( A^C \) be an open subset of \( \mathbb{C}^4 \), and let \( p \in A^C \). Suppose that we are given
(i) a complex-analytic foliation by null planes on \( A^C \).

Then restriction to slices through \( p \) defines
(ii) a positive Hermitian structure on \( A^C \cap \mathbb{R}^4_p \),
(iii) a real-analytic shear-free ray congruence on \( A^C \cap \mathbb{M}^4_p \),
(iv) a real-analytic conformal foliation by curves on \( A^C \cap \mathbb{R}^3_p \).

Further, for a fixed open set \( A^3 \) of \( \mathbb{R}^3_p \), these maps define bijections between germs at \( A^3 \) of the
distributions (i), (ii), (iii), (iv).

Corollary S.5.4 Let \( p \in \mathbb{R}^4 \), and let \( A^3 \) be an open subset of \( \mathbb{R}^3_p \). Then projection \( J \mapsto U = J(\partial/\partial x_3) \) onto the slice \( \mathbb{R}^3_p \) defines a bijective correspondence between germs at \( A^3 \) of positive Hermitian structures \( J \) defined on an open neighbourhood of \( A^3 \) in \( \mathbb{R}^4 \) and real-analytic conformal foliations \( \mathcal{C} \) of \( A^3 \) by curves.

Remark S.5.5 Given a real-analytic conformal foliation \( \mathcal{F} \) by curves of an open subset \( A^3 \) of \( \mathbb{R}^3 \),
there corresponds a 5-parameter family of ‘associated’ real-analytic conformal foliations by curves of
open subsets of \( \mathbb{R}^3 \) obtained by extending \( \mathcal{F} \) to a holomorphic foliation by \( \alpha \)-planes and then
projecting this to all possible \( \mathbb{R}^3 \)-slices.

S.6 LOCAL COORDINATES AND HARMONIC MORPHISMS

A smooth map between Riemannian manifolds is a harmonic morphism if and only if it is both
harmonic and (ii) horizontally weakly conformal (Fuglede 1978, Ishihara 1979); this remains true for
semi-Riemannian manifolds (Fuglede 1996). For a complex-valued map \( \varphi : M \to \mathbb{C} \) from a (semi-)
Riemannian manifold these conditions read, respectively,

(i) \( \Delta^M \varphi = 0 \), and (ii) \( \langle \text{grad } \varphi, \text{grad } \varphi \rangle = 0 \),

where \( \Delta^M \) is the Laplace-Beltrami operator on \( M \), see Section 2.2 for more details. Note that
a complex-valued harmonic morphism from a real-analytic Riemannian manifolds is real analytic
(see Proposition 4.3.1); this is false, in general, for a semi-Riemannian domain (see Section 14.6).
The pair of equations (S.6.1) is conformally invariant in the sense that the composition \( f \circ \varphi \) of a
harmonic morphism \( \varphi : M \to \mathbb{C} \) and a weakly conformal (i.e., holomorphic or antiholomorphic) map
\( f \) to \( \mathbb{C} \) is a harmonic morphism. Hence, we can replace \( \mathbb{C} \) by a Riemann surface \( N \) and then (S.6.1)
for the equations is the harmonic morphism \( \varphi : M \to N \) in any complex (or isothermal) coordinates
(cf. Sections 1.1 and 4.1).

It is convenient to introduce standard null coordinates \( (q_1, \bar{q}_1, q_2, \bar{q}_2) \) on \( \mathbb{C}^4 \) by setting

\[
q_1 = x_0 + i x_1, \quad \bar{q}_1 = x_0 - i x_1, \quad q_2 = x_2 + i x_3, \quad \bar{q}_2 = x_2 - i x_3.
\]

Then \( \mathbb{R}^4 \) is given by \( \bar{q}_1 = \bar{q}_1, \bar{q}_2 = \bar{q}_2 \) and the holomorphic metric on \( \mathbb{C}^4 \) by
\( g^C = dq_1 d\bar{q}_1 + dq_2 d\bar{q}_2 \); this restricts to the Euclidean metric \( g = dq_1 d\bar{q}_1 + dq_2 d\bar{q}_2 \) on real slices \( \mathbb{R}^4_p \). (Here we write \( \bar{q}_i \) for
the complex conjugate \( \bar{q}_i \) of \( q_i \).)
For any $p \in \mathbb{C}^4$ and $[w_0, w_1] \in \mathbb{C}P^1$, set
\[
\Pi_p = \text{span}\left\{w_0 \frac{\partial}{\partial q_1} - w_1 \frac{\partial}{\partial q_2}, \ w_0 \frac{\partial}{\partial q_2} + w_1 \frac{\partial}{\partial q_1}\right\}.
\] (S.6.3)
It is easily checked that $\Pi_p$ is an $\alpha$-plane. Write $\mu = w_1/w_0 \in \mathbb{C} \cup \{\infty\}$; then if $\mu \neq \infty$, $\Pi_p$ has a basis
\[
\left\{\frac{\partial}{\partial q_1} - \mu \frac{\partial}{\partial q_2}, \ \frac{\partial}{\partial q_2} + \mu \frac{\partial}{\partial q_1}\right\};
\] (S.6.4)
if $\mu = \infty$, a basis for $\Pi_p$ is given by $\{\partial/\partial q_2, \ \partial/\partial q_1\}$.

The assignment (S.6.3) gives an explicit bijection
\[
\mathbb{C}P^1 \longleftrightarrow \{\alpha\text{-planes at } p\}.
\]

On applying Proposition S.5.1(ii), we obtain also an almost Hermitian structure $J_p = J_p(\Pi_p)$ on $T_p\mathbb{R}^4$. This has $(0, 1)$-tangent space with basis
\[
\left\{w_0 \frac{\partial}{\partial q_1} - w_1 \frac{\partial}{\partial q_2}, \ w_0 \frac{\partial}{\partial q_2} + w_1 \frac{\partial}{\partial q_1}\right\},
\] (S.6.5)
or, if $\mu = w_1/w_0 \neq \infty$,
\[
\left\{\frac{\partial}{\partial q_1} - \mu \frac{\partial}{\partial q_2}, \ \frac{\partial}{\partial q_2} + \mu \frac{\partial}{\partial q_1}\right\},
\] (S.6.6)
(cf. Section 7.4).

On applying Proposition S.5.1(iii) and (iv), we also obtain a unit vector $U_p$ and a null direction $W_p = \text{span}\{w_p\}$ where $w_p = \partial/\partial t + U_p$. We shall say that $\mu$ represents $\Pi_p$, $J_p$, $U_p$ and $W_p$. To get more explicit formulae for these quantities, we apply the unitary matrix $\frac{1}{\sqrt{|w_0|^2 + |w_1|^2}}\left(\begin{array}{cc}w_0 & w_1 \\ -w_1 & w_0\end{array}\right)$; this converts the basis of $\Pi_p$ to another basis \{e_0 + ie_1, e_2 + ie_3\} where \{e_0, e_1, e_2, e_3\} is a positive orthonormal basis for $\mathbb{R}^4$. Explicitly, write $u = iw_1/w_0 = i\mu$, and let $\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ denote stereographic projection from $(-1, 0, 0)$, then the basis \{e_0, e_1, e_2, e_3\} is given by
\[
e_0 = \partial/\partial x_0 \\
e_1 = \frac{1}{1 + |u|^2}(1 - |u|^2, 2\text{Re } u, 2\text{Im } u) = \sigma^{-1}(u) \\
e_2 + ie_3 = \frac{1}{1 + |u|^2}(-2u, 1 - u^2, i(1 + u^2)).
\] (S.6.7)

Thus $J_p$ is the unique positive almost Hermitian structure at $p$ which satisfies $J_p(\partial/\partial x_0) = \sigma^{-1}(i\mu)$.

Now let there be given a smooth distribution of any of the quantities $\Pi$, $J$, $W$, $U$ of (S.5.7) related as in Proposition S.5.1. Write $U = \sigma^{-1}(i\mu)$; then, after applying a Euclidean transformation, if necessary, to ensure that $\mu$ is finite, we see that (S.5.3) is equivalent to the equation
\[
Z(\mu) = 0 \quad (Z \in \Pi_q).
\] (S.6.8)
Recall that, in standard null coordinates $(q_1, \bar{q}_1, q_2, \bar{q}_2)$, a basis for $\Pi$ is given by (S.6.3) or (S.6.4). Thus (S.6.8) reads
\[
\left(\frac{\partial}{\partial q_1} - \mu \frac{\partial}{\partial q_2}\right)\mu = 0, \quad \left(\frac{\partial}{\partial q_2} + \mu \frac{\partial}{\partial q_1}\right)\mu = 0;
\] (S.6.9)
this is the condition that the distribution $\Pi$ of $\alpha$-planes represented by $\mu$ be integrable. These equations restrict on real slices to the equations
\[
\left(\frac{\partial}{\partial q_1} - \mu \frac{\partial}{\partial q_2}\right)\mu = 0, \quad \left(\frac{\partial}{\partial q_2} + \mu \frac{\partial}{\partial q_1}\right)\mu = 0;
\] (S.6.10)
this is the condition that the almost Hermitian structure \( J \) represented by \( \mu \) be integrable. On Minkowski slices, writing \( v = x_1 + t, w = x_1 - t \), these equations restrict to

\[
\left( \frac{\partial}{\partial v} + i\mu \frac{\partial}{\partial q_2} \right) \mu = 0, \quad \left( \frac{\partial}{\partial q_2} - i\mu \frac{\partial}{\partial w} \right) \mu = 0;
\]

this is the condition that the integral curves of the smooth distribution \( W \) of null directions represented by \( \mu \) form a shear-free ray congruence.

**Example S.6.1** Define \( \mu : \mathbb{C}^4 \setminus \{0\} \to \mathbb{C} \cup \{\infty\} \) by

\[
\mu = -q_2/\bar{q}_1.
\]

This satisfies equations (S.6.9), and so defines a holomorphic foliation \( \mathcal{F} \) by \( \alpha \)-planes. In fact, the leaves of this foliation form the holomorphic 2-parameter family of \( \alpha \)-planes:

\[
q_1 - \mu \bar{q}_2 = \nu, \quad q_2 + \mu \bar{q}_1 = 0 \quad (\mu \in \mathbb{C} \cup \{\infty\}, \nu \in \mathbb{C}).
\]

The map \( \mu \) restricts to the map \( \mathbb{R}^4 \setminus \{0\} \to \mathbb{C} \cup \{\infty\} \) given by \( \mu = q_2/\bar{q}_1 \); this satisfies (S.6.10) and so defines a positive Hermitian structure \( J \).

On \( \mathbb{M}^4 \) it restricts to

\[
\mu = -iq_2/(x_1 + t);
\]

this defines a shear-free ray congruence \( \ell \). A short calculation shows that the vector \( U = \sigma^{-1}(i\mu) \) is given by (S.1.6), so that the projection of \( \ell \) onto a slice \( t = c \) is the foliation \( \mathcal{C}_c \) described in Example S.1.6.

We now show how the quantities \( J \) and \( \ell \) define harmonic morphisms.

**Proposition S.6.2** Let \( \mu : A^4 \to \mathbb{C} \) be a smooth function from an open subset of \( \mathbb{R}^4 \) which satisfies (S.6.10), i.e., represents a Hermitian structure. Then \( \mu \) is a harmonic morphism.

**Proof** Simply note that, for any smooth function \( \mu : A^4 \to \mathbb{C} \),

\[
\frac{1}{4} \Delta \mu = \frac{\partial^2 \mu}{\partial q_1 \partial q_1} + \frac{\partial^2 \mu}{\partial q_2 \partial q_2} = \frac{\partial}{\partial q_1} \left( \frac{\partial \mu}{\partial q_1} - \mu \frac{\partial \mu}{\partial q_2} \right) + \frac{\partial}{\partial q_2} \left( \frac{\partial \mu}{\partial q_2} + \mu \frac{\partial \mu}{\partial q_1} \right),
\]

\[
\frac{1}{4} (\text{grad } \mu, \text{grad } \mu) = \frac{\partial \mu}{\partial q_1} \frac{\partial \mu}{\partial q_1} + \frac{\partial \mu}{\partial q_2} \frac{\partial \mu}{\partial q_2} = \frac{\partial \mu}{\partial q_1} \left( \frac{\partial \mu}{\partial q_1} - \mu \frac{\partial \mu}{\partial q_2} \right) + \frac{\partial \mu}{\partial q_2} \left( \frac{\partial \mu}{\partial q_2} + \mu \frac{\partial \mu}{\partial q_1} \right).
\]

By (S.6.10), the right-hand sides vanish, so that \( \mu \) is a harmonic morphism by the ‘if’ part of the Fuglede–Ishihara characterization (S.6.1) (see Lemma 4.2.1).

Since \( J = \sigma^{-1}(i\mu) \), by conformal invariance (see above or Section 4.1), it follows that \( J : A^4 \to S^2 \) is also a harmonic morphism.

**Remark S.6.3** That \( J \) is a harmonic morphism also follows from Proposition 7.9.1, as in Example 7.9.2.

We have an analogue for Minkowski space, as follows.

**Proposition S.6.4** Let \( \mu : A^M \to \mathbb{C} \) be a smooth function from an open subset of Minkowski space which represents a shear-free ray congruence, i.e., satisfies (S.6.11). Then \( \mu \) is a harmonic morphism.

**Example S.6.5** The direction field \( U \) of the SFR congruence of Example S.3.4 defines a harmonic morphism \( \mathbb{M}^4 \setminus \{(t,0,0,0) : t \in \mathbb{R}\} \to S^2 \). Note that this harmonic morphism is submersive and surjective.

A horizontally weakly conformal map \( \varphi : M \to N \) between semi-Riemannian manifolds is called degenerate at \( x \in M \) if \( \ker d\varphi_x \) is a degenerate subspace (Definition 14.5.6).
Example S.6.6  The direction field $U$ of the SFR congruence of Example S.3.5 defines a harmonic morphism from the cone given by
\[ A^M = \{(t, x_1, x_2, x_3) \in M^4 : x_2^2 + x_3^2 > t^2\} \]
to $S^2$ which is degenerate everywhere and has image given by the equator of $S^2$. Note that $U = \sigma(\mu)$ with
\[ \mu(t, x_1, x_2, x_3) = \frac{r}{x_2^2 + x_3^2} \left\{ \left(x_2 + \frac{t}{r} x_3\right) + i \left(x_3 - \frac{t}{r} x_2\right) \right\}. \]
The map $\mu : A^M \to \mathbb{C}$ defines a harmonic morphism $\mu : A^M \to \mathbb{C}$ equivalent to $U$; it is also degenerate, with image the unit circle. The fibre of $U$ (or $\mu$) through any point $p$ is the affine plane perpendicular to $U_p$; this is spanned by $U_p$, $\partial/\partial x_1$ and the vector in the $(x_2, x_3)$-plane perpendicular to $U_p$.

There is a complex version of the last two propositions which we explain in the next section.

S.7  COMPLEX-HARMONIC MORPHISMS

By a complex-harmonic function $\varphi : A^C \to \mathbb{C}$ on an open subset $A^C$ of $\mathbb{C}^4$ we mean a complex-analytic map satisfying the complexified Laplace’s equation:
\[ \sum_{i=0}^{3} \frac{\partial^2 \varphi}{\partial x_i^2} = 0 \quad ((x_0, x_1, x_2, x_3) \in A^C). \]  
(S.7.1)

By a complex-harmonic morphism $\varphi : A^C \to \mathbb{C}$ we mean a complex-analytic map satisfying (S.7.1) and a complexified version of the horizontal weak conformality condition:
\[ \sum_{i=0}^{3} \left( \frac{\partial \varphi}{\partial x_i} \right)^2 = 0 \quad ((x_0, x_1, x_2, x_3) \in A^C). \]  
(S.7.2)

The chain rule quickly shows that we can characterize complex-harmonic morphisms $A^C \to \mathbb{C}$ as those complex-analytic maps which pull back complex-analytic functions to complex-harmonic functions (cf. Theorem 4.2.2). Note also that, since these equations are invariant when $\varphi$ is composed with a holomorphic map, we can replace $\mathbb{C}$ by any Riemann surface.

Proposition S.7.1  (i) Let $\varphi : A^C \to \mathbb{C}$ be a complex-harmonic morphism from an open subset $A^C$ of $\mathbb{C}^4$ to a Riemann surface. Then, for any $p \in A^C$,
(a) $\varphi|_{A^C \cap \mathbb{R}^4}$ is a harmonic morphism (with respect to the standard Euclidean metric);
(b) $\varphi|_{A^C \cap \mathbb{M}^4}$ is a harmonic morphism (with respect to the standard Minkowski metric).

(ii) All harmonic morphisms from open subsets of $\mathbb{R}^4$ to Riemann surfaces and all real-analytic harmonic morphisms from open subsets of $\mathbb{M}^4$ to Riemann surfaces arise in this way.

Proof  (i) Immediate from the equations.

(ii) As noted above, any harmonic morphism from an open subset of $\mathbb{R}^4$ to a Riemann surface is real analytic. By analytic continuation, this is the restriction of a complex-analytic map on an open subset of $\mathbb{C}^4$, and this complex-analytic map is complex-harmonic. The $\mathbb{M}^4$ case is similar except that real analyticity is no longer automatic and must be assumed.

Then we have a complex-analytic version of Proposition S.6.2, as follows.

Proposition S.7.2  Let $\mu : A^C \to \mathbb{C}$ be a complex-analytic function from an open subset of $\mathbb{C}^4$. Suppose that $\mu$ satisfies (S.6.9), i.e., suppose that $\mu$ represents a complex-analytic foliation by null planes. Then $\mu$ is a complex-harmonic morphism.

S.8  HARMONIC MORPHISMS AND SHEAR-FREE RAY CONGRUENCES

We have just shown that a Hermitian structure, or any of the related distributions (S.5.7), defines a harmonic morphism. We now give some converses. We first of all reformulate Proposition 7.11.1. By ‘fibre component’ we shall mean ‘connected component of a fibre’.
Theorem S.8.1 Let \( \varphi : A^4 \to N^2 \) be a harmonic morphism from an open subset \( A^4 \) of \( \mathbb{R}^4 \) to a Riemann surface, with \( d\varphi \) nowhere zero. Then there exists a Hermitian structure \( J \) on \( A^4 \) which is parallel along each fibre component of \( \varphi \). Further, for any \( p \in A^4 \), there is a neighbourhood \( A^4_p \) of \( p \) in \( A^4 \) and a holomorphic map \( \rho : V \to \mathbb{C} \cup \{\infty\} \) from an open subset \( V \) of \( N^2 \) such that \( \mu = \rho \circ \varphi \) represents \( J \) on \( A^4_p \).

We have the following analogue for complex-harmonic morphisms:

Theorem S.8.2 Let \( \varphi : A^C \to N^2 \) be a complex-harmonic morphism, from an open subset \( A^C \) of \( \mathbb{C}^4 \) to a Riemann surface, with \( d\varphi \) nowhere zero. Then there exists a holomorphic foliation \( \mathcal{F} \) of \( A^C \) by \( \alpha \)-planes or by \( \beta \)-planes such that each fibre component of \( \varphi \) is the union of parallel null planes of \( \mathcal{F} \). Further, for any \( p \in A^C \), there is a neighbourhood \( A^C_p \) of \( p \) in \( A^C \), and a holomorphic map \( \rho : V \to \mathbb{C} \cup \{\infty\} \) from an open subset \( V \) of \( N^2 \), such that \( \mu = \rho \circ \varphi \) represents \( \mathcal{F} \) on \( A^C_p \).

Proof Let \( p \in A^C \). By Proposition S.7.1, \( \varphi \) restricts to a harmonic morphism on \( A^4 = A^C \cap \mathbb{R}^4_p \) which is easily seen to be submersive. By Theorem S.8.1, \( \varphi|_{A^4} \) is holomorphic with respect to some Hermitian structure \( J \) which is constant along each fibre component of \( \varphi \). By replacing \( q_2 \) with \( \tilde{q}_2 \), if necessary, we can assume that \( J \) is positively oriented. Represent \( J \) by the map \( \mu : A^4 \to \mathbb{C} \cup \{\infty\} \); then, since \( \varphi \) is holomorphic and \( J \) has \( (0,1) \)-tangent space with basis (S.6.6), we have at all points of \( A^4 \),

\[
\frac{\partial \varphi}{\partial q_1} - \mu \frac{\partial \varphi}{\partial q_2} = 0, \quad \frac{\partial \varphi}{\partial \bar{q}_1} + \mu \frac{\partial \varphi}{\partial \bar{q}_2} = 0. \tag{S.8.1}
\]

Since \( J \) is integrable, we have also equation (S.6.9):

\[
\frac{\partial \mu}{\partial q_1} - \mu \frac{\partial \mu}{\partial q_2} = 0, \quad \frac{\partial \mu}{\partial \bar{q}_1} + \mu \frac{\partial \mu}{\partial \bar{q}_2} = 0. \tag{S.8.2}
\]

Extend \( \mu \) to \( A^C \) by insisting that (S.8.1) holds; note that \( \mu \) is well defined by one of these equations—since the hypothesis that \( d\varphi \) is nowhere zero ensures that not all the partial derivatives of \( \varphi \) can vanish simultaneously—and is a complex-analytic function. By analytic continuation, (S.8.2) holds at all points of \( A^C \) so that \( \mu \) defines a holomorphic foliation \( \mathcal{F} \) by \( \alpha \)-planes. By (S.8.1), \( \varphi \) is constant along any \( \alpha \)-plane of \( \mathcal{F} \), so that each fibre of \( \varphi \) is the union of \( \alpha \)-planes of \( \mathcal{F} \). Further, \( J \) and so \( \mu|_{A^4} \) is constant along each fibre component of \( \varphi|_{A^4} \); by analytic continuation, \( \mu : A^C \to \mathbb{C} \cup \{\infty\} \) is constant along the fibre components of \( \varphi : A^C \to \mathbb{C} \cup \{\infty\} \), so that the \( \alpha \)-planes of \( \mathcal{F} \) making up a fibre component of \( \varphi \) are all parallel.

Lastly, since \( \mu \) is constant on the leaves of the foliation given by the fibres of \( \varphi \), it factors through local leaf spaces as \( \mu = \rho \circ \varphi \). Since \( \varphi \) and \( \mu \) are both holomorphic (with respect to \( i \) and \( J \) ), \( \rho \) must be holomorphic.

We have the following analogue for Minkowski space proved in a similar way.

Theorem S.8.3 Let \( \varphi : A^M \to N^2 \) be a real-analytic harmonic morphism from an open subset \( A^M \) of Minkowski 4-space \( \mathbb{M}^4 \) to a Riemann surface, with \( d\varphi \) nowhere zero. Then there is a shear-free ray congruence \( \ell \) on \( A^M \) such that each fibre component of \( \varphi \) is the union of parallel null lines of \( \ell \). Further, for any \( p \in A^M \), there is a neighbourhood \( A^M_p \) of \( p \) in \( A^M \), and a holomorphic map \( \rho : V \to \mathbb{C} \cup \{\infty\} \) from an open subset \( V \) of \( N^2 \), such that \( \mu = \rho \circ \varphi \) represents \( \ell \) on \( A^M_p \).

Note that in the real and complex cases the condition ‘\( d\varphi \) nowhere zero’ is equivalent to ‘submersive’. This is not so in the Minkowski case where \( \varphi \) may be degenerate (Definition 14.5.6). In this case we can be more precise as follows.

Corollary S.8.4 Let \( \varphi : A^M \to N^2 \) be a real-analytic harmonic morphism from an open subset \( A^M \) of \( \mathbb{M}^4 \) to a Riemann surface. Suppose that \( \varphi \) is degenerate at \( p \) (so that \( d\varphi_p \) has rank 1). Then there is a unique null direction \( W_p \in T_p \mathbb{M}^4 \) such that \( W_p \subset \ker d\varphi_p \). Furthermore, \( \ker d\varphi_p = W_p^\perp \). If, further, at each point \( q \) in the fibre component through \( p \), \( \varphi \) is degenerate, then that fibre component is an open subset of the affine null 3-space tangent to \( W_p^\perp \).
Harmonic morphisms and shear-free ray congruences

Proof Since \( d\varphi_p \) has rank 1, \( \ker d\varphi_p \) is three-dimensional. By Lemma 14.5.3, \( (\ker d\varphi_p)^\perp \subset \ker d\varphi_p \), so \( (\ker d\varphi_p)^\perp \) is one-dimensional and null. We set \( W_p = (\ker d\varphi_p)^\perp \), so \( \ker d\varphi_p = W_p^\perp \).

To prove uniqueness of \( W_p \), suppose that \( W_p \subset \ker d\varphi_p \) is another null direction. Then \( W_p \) and \( W_p^\perp \) together span a 2-dimensional subspace of null directions in \( W_p^\perp \), which is impossible.

This means that the distribution \( p \mapsto W_p \) must be tangent to the shear-free ray congruence of Theorem S.8.3, and so each \( W_p \) is parallel for all \( p \) in a fibre component. The last assertion follows from the fact that the connected component of the fibre is three-dimensional and has every tangent space parallel to \( W_p^\perp \).

We can also give the following version of Theorem S.8.3 for harmonic morphisms which are only smooth, though this needs the stronger hypothesis of submersivity and the conclusion is slightly weaker.

**Theorem S.8.5** Let \( \varphi : A^M \to N^2 \) be a smooth submersive harmonic map from an open subset \( A^M \) of \( \mathbb{M}^4 \) to a Riemann surface. Then there there is an open dense set of points \( A_p^N \) of \( A^M \) and a shear-free ray congruence \( \ell \) on \( A_p^N \) such that each fibre component of \( \varphi \) is the union of parallel null lines of \( \ell \). Further, for any \( p \in A_p^N \), there is a neighbourhood \( A^1_p \) of \( p \) in \( A_p^N \) and a holomorphic map \( \rho : V \to \mathbb{C} \cup \{\infty\} \) from an open subset \( V \) of \( N^2 \) such that \( \mu = \rho \circ \varphi \) represents \( \ell \).

Proof Let \( p \in U \). Let \( v, w, z, \tau \) be any null coordinates, i.e., coordinates in which the metric is \( dv \, dw + dz \, d\tau \). Then the harmonic morphism equations (S.6.1) read

\[
\begin{align*}
\varphi_v \varphi_w &+ \varphi_z \varphi_\tau = 0, \\
\varphi_{vw} &+ \varphi_z \varphi_\tau = 0.
\end{align*}
\]

Now choose the null coordinates such that \( \partial/\partial z, \partial/\partial \tau \) span the horizontal space at \( p \) and \( \partial/\partial v, \partial/\partial w \) span the vertical space, thus \( \varphi_v(p) = \varphi_w(p) = 0 \). Then from (S.8.3), \( \varphi_z(p) \varphi_\tau(p) = 0 \).

Without loss of generality we can assume that

\[
\varphi_z(p) \neq 0,
\]

so that \( \varphi_\tau(p) = 0 \). Now differentiation of (S.8.3) shows that all first derivatives of \( \varphi_\tau \) are zero at \( p \). On differentiating (S.8.3) with respect to \( z \) then \( \tau \), evaluating at \( p \) and using (S.8.5) we obtain \( \varphi_{z\tau}(p) = 0 \) then (S.8.4) gives \( \varphi_{zw}(p) = 0 \). On differentiating (S.8.3) with respect to \( v \) then \( w \) and evaluating at \( p \) we obtain \( \varphi_{vv}(p) \varphi_{ww}(p) = 0 \), so that

\[
\varphi_{vv}(p) = 0 \quad \text{or} \quad \varphi_{ww}(p) = 0.
\]

Suppose that \( \varphi_{vv}(p) = 0 \). Then, at \( p \), the function \( \mu = i\varphi_v/\varphi_z \) satisfies

\[
\begin{align*}
\frac{\partial \mu}{\partial v} + i\frac{\partial \mu}{\partial z} &= 0, \\
\frac{\partial \mu}{\partial \tau} - i\frac{\partial \mu}{\partial w} &= 0.
\end{align*}
\]

Comparing these with (S.6.11) shows that \( \mu \) represents a shear-free ray congruence \( W \) at \( p \). If, instead, \( \varphi_{ww}(p) = 0 \); interchange \( v \) and \( w \) in the above; again, we get a shear-free ray congruence \( W \) at \( p \).

Now let \( A_1 = \{ p \in A^M : \varphi_{vv} = 0 \} \) and \( A_2 = \{ p \in A^M : \varphi_{ww} = 0 \} \). Then \( A_1 \cup A_2 = A^M \) and we obtain shear-free ray congruences on the open sets \( \text{int}(A_1) \) and \( \text{int}(A_2) \); by the Baire category theorem (Sims 1976, Section 6.4), their union is dense in \( A^M \).

**Remark S.8.6** On the open set \( \text{int}(A_1) \cap \text{int}(A_2) \) (which may be empty) both \( V \) and \( W \) are shear-free ray congruences; the fibres of \( \varphi \) are spanned by \( V \) and \( W \) and are totally geodesic.

**Example S.8.7** It can easily be checked that, for any holomorphic function \( f \), the composite map \( \varphi = f((x_2+i x_3)/(x_1+t)) \) is a complex-valued harmonic morphism on \( M^4 \setminus \{(t, x_1, x_2, x_3) : x_1+t = 0\} \). In fact, \( \varphi = f(i\mu) \) where \( \mu \) is given by (S.6.13). If \( df \) is nowhere zero, then \( \varphi \) is submersive, and the associated shear-free ray congruence \( \ell \) of Theorem S.8.3 (or S.8.5) is that in Example S.6.1.
Similar examples can be given for Theorems S.8.1 and S.8.2.

S.9 COMPACTIFICATIONS AND TWISTORIAL FORMULATION

Definition S.9.1 (LeBrun 1983) A holomorphic conformal structure on a complex manifold $M$ is a holomorphic line subbundle $L$ of the symmetric square $\otimes^2 T^*_{1,0} M$ such that, for every $p \in M$, any non-zero element of $L_p$ defines a (complex-symmetric bilinear) non-degenerate inner product on $T^*_p M$.

Any local nowhere-zero holomorphic section of $L$ is a holomorphic metric. If $g^C_1$, $g^C_2$ are any two such holomorphic sections defined on an open subset $A$ of $M$ then they are holomorphically conformally equivalent, i.e., $g^C_1 = \nu g^C_2$ for some holomorphic map $\nu : A \to \mathbb{C} \setminus \{0\}$. Conversely, a holomorphic conformal structure can be specified by giving a collection of holomorphic metrics whose domains cover $M$ such that any two are conformally related on the intersection of their domains.

Let $\mathbb{C}^4 = G_2(\mathbb{C}^4)$ denote the Grassmannian of all two-dimensional complex subspaces of $\mathbb{C}^4$ equipped with its standard complex structure. Then $\mathbb{C}^4$ can be embedded holomorphically in $\mathbb{C}^4$ by the mapping given in standard null coordinates by

$$c : x = (q_1, \bar{q}_1, q_2, \bar{q}_2) \mapsto \text{column space of } A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \bar{q}_1 & -q_2 \\ q_2 & \bar{q}_1 \end{pmatrix}. \quad (S.9.1)$$

We shall call $c$ the standard coordinate chart for $\mathbb{C}^4$; we identify $\mathbb{C}^4$ with its image under $c$. Points of $\mathbb{C}^4 \setminus \mathbb{C}^4$ will be called points at infinity.

Note that the matrix $A$ is of the form $A = \begin{pmatrix} I \\ Q \end{pmatrix}$ where $I$ denotes the $2 \times 2$-identity matrix and

$$Q = \begin{pmatrix} q_1 & -\bar{q}_2 \\ q_2 & \bar{q}_1 \end{pmatrix}. \quad (S.9.2)$$

Write $x = (x_0, x_1, x_2, x_3) = (q_1, q_2) \in \mathbb{R}^4 = \mathbb{C}^2$, then

$$Q = \begin{pmatrix} q_1 & -\bar{q}_2 \\ q_2 & \bar{q}_1 \end{pmatrix} = \begin{pmatrix} x_0 + ix_1 - x_3 + ix_4 \\ x_3 + ix_4 \\ x_0 - ix_3 \end{pmatrix} \quad (S.9.3)$$

which is a standard representation of the quaternion $q_1 + q_2 j = x_0 + ix_1 + jx_2 + kx_3$; we shall call such a matrix a quaternionic. Thus $Q$, and so $A$, represents a point of $\mathbb{R}^4$ if and only if $Q$ is quaternionic. If, on the other hand, $x = (t, x_1, x_2, x_3) \in \mathbb{M}^4$, then

$$Q = \begin{pmatrix} -it + ix_1 - x_3 + ix_4 \\ x_3 + ix_4 \\ x_0 - ix_3 \end{pmatrix} \quad (S.9.4)$$

so that $Q$, and so $A$, represents a point of $\mathbb{M}^4$ if and only if $Q$ is skew-Hermitian.

We give $\mathbb{C}^4$ a holomorphic conformal structure, as follows. Given any complex chart $c : U \to \mathbb{C}^4$, choose holomorphic maps $X, Y : U \to \mathbb{C}^4$ such that

$$c(q) = \text{span}\{X(q), Y(q)\} \quad (q \in U).$$

Choose a non-zero element $\omega^*$ of $\Lambda^4 \mathbb{C}^4$. For each $p \in U$ and $w \in T_p U$ define $g(w, w)$ to be the unique complex number such that

$$X \wedge Y \wedge dX(w) \wedge dY(w) = g(w, w) \omega^*. \quad (S.9.5)$$

On defining $g(v, w)$ for $v, w \in T_p U$ by polarization:

$$g(v, w) = \frac{1}{2} \{g(v + w, v + w) - g(v, v) - g(w, w)\}$$

we obtain a holomorphic metric on $U$, and so on $c(U)$. It is well defined up to multiplication by nowhere-zero holomorphic functions, and so defines a holomorphic conformal structure.
The equation of $q$ and the restriction of $T_p \mathbb{C}^4$ as well as holomorphic.

Now, at any point $p \in \mathbb{C}^4$, given any orthogonal basis $\{e_0, e_1, e_2, e_3\}$ of $T_p \mathbb{C}^4$ of vectors of the same length, the two-dimensional subspace

$$\Pi_p = \text{span}\{e_0 + ie_1, e_2 + ie_3\}$$

of $T_p \mathbb{C}^4$ is null; we call $\Pi_p$ an $\alpha$-plane (respectively, $\beta$-plane) according as the basis $\{e_0, e_1, e_2, e_3\}$ is positively (respectively, negatively) oriented. This agrees with our previous definition (Section S.4) if $p \in \mathbb{C}^4$. As before, any null two-dimensional subspace of $T_p \mathbb{C}^4$ is an $\alpha$-plane or a $\beta$-plane.

We now determine the null surfaces in $\tilde{\mathbb{C}}^4$, i.e. the surfaces whose tangent spaces are null. Fix $w \in \mathbb{CP}^3$, thus $w$ is a one-dimensional subspace of $\mathbb{C}^4$; then set

$$\tilde{w} = \{p \in G_2(\mathbb{C}^4) : \text{the plane } p \text{ contains } w\},$$

this defines a surface in $\tilde{\mathbb{C}}^4 \equiv G_2(\mathbb{C}^4)$. Now the formula (S.9.5) shows that all its tangent spaces are null; it is easily seen that they are all $\alpha$-planes. All such surfaces are given this way. If it is not at infinity, we easily see that $\tilde{w}$ is the image under $c$ of an affine $\alpha$-plane of $\mathbb{C}^4$ so we shall continue to call the surfaces affine $\alpha$-planes. (Affine $\beta$-planes have a similar definition and can be described as the sets of all two-dimensional subspaces of $\mathbb{C}^4$ contained in a given three-dimensional subspace.)

To understand all this better, embed $G_2(\mathbb{C}^4)$ in $\mathbb{CP}^5$ by the Plücker embedding $p$ which sends the plane spanned by the vectors $X$ and $Y$ to the point $[X \wedge Y] \in P(\Lambda^2 \mathbb{C}^4) \cong \mathbb{CP}^5$. Here $[ ] = \pi( )$ where $\pi : \Lambda^2 \mathbb{C}^4 \setminus \{0\} \to P(\Lambda^2 \mathbb{C}^4)$ is the standard projection. The image of $p$ is the complex quadric

$$Q^C = \{[\omega] : \omega \in \Lambda^2 \mathbb{C}^4 : \omega \wedge \omega = 0\},$$

the condition $\omega \wedge \omega = 0$ expressing the decomposibility of $\omega$. Identify $\Lambda^2 \mathbb{C}^4$ with $\mathbb{C}^6$ in the standard way; then the components of $pl$ are given by the $2 \times 2$ minors of the matrix $A$. Thus the composition $j = pl \circ c : \mathbb{C}^4 \to \mathbb{CP}^5$ is given by

$$j(q_1, q_2, q_3, q_4) = [z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}]$$

$$= [1, -\bar{q}_2, \bar{q}_1, -q_1, -q_2, q_1 \bar{q}_2 + q_2 \bar{q}_1]$$

(S.9.7)

and $Q^C$ has equation $z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23} = 0$. Change the coordinates linearly to

$$\xi_0 = z_{12} + z_{34}, \quad \xi_1 = z_{12} - z_{34}, \quad \xi_2 = z_{14} - z_{23}, \quad \xi_3 = i(z_{14} + z_{23}), \quad \xi_4 = -(z_{13} + z_{24}), \quad \xi_5 = -i(z_{13} - z_{24}).$$

(S.9.8)

In these coordinates, $Q^C$ has equation

$$\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 - \xi_5^2 = 0,$$

(S.9.9)

and the restriction of $j$ to $\mathbb{R}^4$ reads

$$q = (x_0, x_1, x_2, x_3) \mapsto [1 + |q|^2, 1 - |q|^2, 2x_0, 2x_1, 2x_2, 2x_3] = [1, \sigma^{-1}(q)]$$

where $\sigma^{-1} : \mathbb{R}^4 \to S^4 \subseteq \mathbb{R}^5$ is the inverse of stereographic projection. This shows that (i) the closure $\tilde{\mathbb{R}}^4$ of the image of $\mathbb{R}^4$ consists of the points of $Q^C$ with real coordinates $\xi$, this is the quadric $Q^R$ in $\mathbb{RP}^5$ with equation (S.9.9); (ii) $Q^R$ is conformally equivalent to $S^4$; (iii) the set of points at infinity of $\mathbb{R}^4$, i.e., the set $\tilde{\mathbb{R}}^4 \setminus \mathbb{R}^4$, consists of the single point $[1, -1, 0, 0, 0, 0]$.

In the coordinates

$$\tilde{\xi}_i = \xi_i \ (i \neq 2), \quad \tilde{\xi}_2 = i \xi_2$$

(S.9.10)

the equation of $Q^C$ becomes

$$g(\tilde{\xi}, \tilde{\xi}) \equiv \tilde{\xi}_0^2 - \tilde{\xi}_1^2 + \tilde{\xi}_2^2 - \tilde{\xi}_3^2 - \tilde{\xi}_4^2 - \tilde{\xi}_5^2 = 0$$

(S.9.11)

and the restriction of $j$ to $M^4$ reads

$$q = (t, x_1, x_2, x_3) \mapsto [1 + |q|^2, 1 - |q|^2, 2t, 2x_1, 2x_2, 2x_3].$$

(S.9.12)
Hence the closure $\overline{M}^4$ of the image of $M^4$ is the points of $Q^C$ with the coordinates $\xi_i$ real valued; this is the quadric $Q^M$ in $\mathbb{R}P^5$ with equation (S.9.11). The set of points at infinity of $M^4$, i.e., the set $\bar{M}^4 \setminus M^4$, consists of the null cone $\{1, −1, q : q \in M^4, |q|^2_1 = 0\}$.

**Remark S.9.2** The intersection of $\overline{M}^4$ with the chart $\tilde{\mathcal{C}}_0 = 1$ is the pseudosphere given by $S^4_1 = \{ (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \in \mathbb{R}^5 : \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 = 1 \}$. The map $j : M^4 \to Q^M$ restricts to the map $M^4 \setminus \{q \in M^4 : |q|^2_1 = -1\}$ given by

$$q \mapsto \frac{1}{1 + |q|^2_1} (1 - |q|^2_1, q).$$

This is a higher-dimensional version of the stereographic projection discussed in Example 14.1.22.

Note that the mapping from $S^1 \times S^3$ given by

$$((\xi_0, \xi_2), (\xi_1, \xi_3, \xi_4, \xi_5)) \mapsto \tilde{\xi}_0, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4, \tilde{\xi}_5$$

(S.9.13)

is a double covering; it is also conformal if we give $S^1 \times S^3$ the Lorentzian product metric $-g^{S^1} + g^{S^3}$. Think of $S^1 \times S^3$ as $\{ (z_1, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^2 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \}$, and define an equivalence relation $\sim$ on $S^1 \times S^3$ by $(z_1, z_2, z_3) \sim (\pm z_1, z_2, z_3)$. The map (S.9.13) factors to a conformal diffeomorphism

$$S^1 \times S^3/\sim \to Q^M.$$  

(S.9.14)

Note that the mapping

$$S^1 \times S^3 \ni (z_1, z_2, z_3) \mapsto (z_1^2, z_1z_2, z_1z_3) \in S^1 \times S^3$$

factors to a diffeomorphism of $S^1 \times S^3/\sim$ to $S^3 \times S^3$.

**Remark S.9.3** We give $S^1 \times S^3$ the time- and space-orientations induced from the standard orientations on $S^1$ and $S^3$. These factor to $S^1 \times S^3/\sim$. The map $j$ given by (S.9.12) sends $x \in \mathbb{R}^3$ to the equivalence class of the point $(1, 0), \sigma^{-1}(x)) \in S^1 \times S^3/\sim$. The differential of $j$ maps the vector $\partial/\partial t$ at $x$, which is normal to $\mathbb{R}^3$ in $M^4$, to $(u, 0) \in T_{(1, 0)}S^1 \times T_{\sigma^{-1}(x)}S^3$ where $u$ denotes the unit positive tangent vector. By a slight abuse of notation, we shall continue to denote $(u, 0)$ by $\partial/\partial t$ even at points at infinity. Then any null direction in $\overline{M}^4 \cong S^1 \times S^3/\sim$ is spanned by a vector $\partial/\partial t + U = (u, U) \in TS^1 \times TS^3$, as in $M^4$.

We give $Q^C$ a holomorphic conformal structure, as follows. Let $g^Q$ be a symmetric bilinear form on $\mathbb{C}^6$ such that $Q^C$ has equation $g^Q(\xi, \xi) = 0$, i.e., (S.9.11) in the coordinates $\xi_i$. Let $X : A \to T^1, 0Q^C$ be a holomorphic vector field defined on an open subset of $Q^C$, and let $\tilde{X} : A \to \mathbb{C}^6$ be a holomorphic lift of it, i.e., $d\pi(\tilde{X}) = X$ where $\pi : \mathbb{C}^6 \setminus \{0\} \to CP^5$ is the natural projection. Set $g(X, X) = g^Q(\tilde{X}, \tilde{X})$. Note that $\tilde{X}$ is defined up to a transformation $\tilde{X} = \lambda(p)\tilde{X} + \mu(p)p$ $(p \in A)$ where $\lambda : A \to \mathbb{C} \setminus \{0\}$ is holomorphic and $\pi(\hat{p}) = p$. Since $g(\hat{p}, \hat{p}) = g(\hat{p}, \hat{X}) = 0$, the first being the equation of $Q^C$ and the second the tangency condition, we have $g^Q(\hat{X}, \hat{X}) = \lambda^2 g^Q(\tilde{X}, \tilde{X})$. Hence $g$ is well defined up to multiplication by a nowhere-zero holomorphic function; it thus defines a holomorphic conformal structure on $Q^C$. Clearly this corresponds under $pl$ to the holomorphic conformal structure defined above on $G_2(\mathbb{C}^4)$. Thus $j$ is a holomorphic conformal inclusion.

More generally, for any $p \in \mathbb{C}^4$, let $\mathbb{R}^4_p$, $\mathbb{R}^3_p$, $\mathbb{M}^4_p$ denote the closures of the slices $j(\mathbb{R}^4_p)$, $j(\mathbb{R}^3_p)$, $j(\mathbb{M}^4_p)$ in $\mathbb{C}^4$, respectively; we write $\mathbb{R}^4, \mathbb{R}^3, \mathbb{M}^4$ if $p = 0$. Then the holomorphic conformal structure on $\mathbb{C}^4$ induces real conformal structures on these closures; clearly $c$ restricts to conformal inclusions $\mathbb{R}^4_{p} \to \mathbb{R}^4_{p'}, \mathbb{R}^3_{p} \to \mathbb{R}^3_{p'}$ and $\mathbb{M}^4_{p} \to \mathbb{M}^4_{p'}$.

Let $\bar{F}_{1, 2}$ be the complex flag manifold

$$\{(w, p) \in CP^3 \times G_2(\mathbb{C}^4) : \text{the line } w \text{ lies in the plane } p\};$$
this is called the correspondence space. The restrictions to \( F_{1,2} \) of the natural projections from \( \mathbb{C}P^3 \times G_2(\mathbb{C}^4) \) define a double holomorphic fibration:

\[
(w, p) \in \mathcal{F}_{1,2} \quad \mu \leftarrow \quad \nu
\]

\[
w \in \mathbb{C}P^3 \
p \in G_2(\mathbb{C}^4) \equiv \tilde{\mathbb{C}}^4
\]

For any \( w \in \mathbb{C}P^3 \), set

\[
\tilde{w} = \nu \circ \mu^{-1}(w);
\]

as in (S.9.6), \( \tilde{w} \) is an \( \alpha \)-plane in \( \tilde{\mathbb{C}}^4 \), which we call the \( \alpha \)-plane determined or represented by \( w \). Conversely, for any point \( p \in \tilde{\mathbb{C}}^4 \) we write

\[
\tilde{p} = \mu \circ \nu^{-1}(p);
\]

then \( \tilde{p} \) is a projective line (i.e., a one-dimensional projective subspace) of \( \mathbb{C}P^3 \) which represents the \( \mathbb{C}P^1 \)-worth of \( \alpha \)-planes through \( p \).

Explicitly, in the standard null coordinates \( c : \mathbb{C}^4 \hookrightarrow \tilde{\mathbb{C}}^4 \) given by (S.9.1), we have \( (w, p) \in \mathcal{F}_{1,2} \) if and only the following incidence relations are satisfied:

\[
w_0q_1 - w_1q_2 = w_2, \quad w_0q_2 + w_1q_1 = w_3;
\]

(S.9.18)

indeed, these express the condition that \( w \) be a linear combination of the columns of the matrix (S.9.1) and so lie in the plane represented by the point \( p \in \tilde{\mathbb{C}}^4 \). Now note that, for any \([w_0, w_1, w_2, w_3]\) with \([w_0, w_1] \neq [0, 0] \), (S.9.18) defines an affine \( \alpha \)-plane in \( \mathbb{C}^4 \); indeed, its tangent space at any point is the set of vectors annihilated by \([w_0dq_1 - w_1dq_2, w_0dq_2 - w_1dq_1] \) and so is given by (S.6.3). Points of \( \mathbb{C}P^3 \) on the projective line \( \mathbb{C}P^3_0 = \{[w_0, w_1, w_2, w_3] : [w_0, w_1] = [0, 0] \} \) correspond to affine \( \alpha \)-planes at infinity, i.e., in \( \tilde{\mathbb{C}}^4 \setminus \mathbb{C}^4 \). Thus \( \mathbb{C}P^3 \) parametrizes all \( \alpha \)-planes in \( \tilde{\mathbb{C}}^4 \) and (S.9.18) expresses the condition that the point in \( \mathbb{C}^4 \) with null coordinates \((q_1, q_2, q_3)\) lies on the \( \alpha \)-plane \( \tilde{w} \) determined by \( w = [w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 \setminus \mathbb{C}P^3_0 \), i.e., (S.9.18) is the equation of the \( \alpha \)-plane \( \tilde{w} \).

For \( w = [w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 \setminus \mathbb{C}P^3_0 \), the \( \alpha \)-plane \( \tilde{w} \) given by (S.9.18) intersects \( \mathbb{R}^4 = \mathbb{R}^4_0 \) at \((q_1, q_2)\) where

\[
\begin{align*}
w_0q_1 - w_1q_2 &= w_2, \\
w_0q_2 + w_1q_1 &= w_3.
\end{align*}
\]

(S.9.19)

This system has the unique solution

\[
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{|w_0|^2 + |w_3|^2} \begin{pmatrix} w_0w_2 + w_1w_3 \\ w_0w_3 - w_1w_2 \end{pmatrix};
\]

(S.9.20)

and so defines a map \( \mathbb{C}P^3 \setminus \mathbb{C}P^3_0 \to \mathbb{R}^4 \) which is the projection map of the twistor bundle of \( \mathbb{R}^4 \) given by (7.4.2). On mapping points of \( \mathbb{C}P^3_0 \) to the point at infinity \( \mathbb{R}^4 / \mathbb{R}^4 \), this map extends to a smooth map

\[
\pi = \pi_0 : \mathbb{C}P^3 \to \tilde{\mathbb{R}}^4 \cong S^4
\]

(S.9.21)

which is the projection map of the twistor bundle of \( S^4 \) as given in (7.3.3). Similarly, for any \( p \in \mathbb{C}^4 \) we have a smooth map \( \pi_p : \mathbb{C}P^3 \to \tilde{\mathbb{R}}^4_p \cong S^4 \) which sends \( w \in \mathbb{C}P^3 \) to the unique point of intersection of the \( \alpha \)-plane \( \tilde{w} \) with \( \tilde{\mathbb{R}}^4_p \); on writing \( p = (a_1, a_2, \tilde{a}_2) \) in standard null coordinates, a simple calculation shows that, in the chart (S.4.1) for \( \tilde{\mathbb{R}}^4_p \), this point of intersection is given by (S.9.20) with \( w_2, w_3 \) replaced by \( w'_2, w'_3 \) where

\[
w'_2 = w_2 - w_0a_1 + w_1\tilde{a}_2, \quad w'_3 = w_3 - w_0a_2 - w_1\tilde{a}_1
\]

(S.9.22)

(see Example S.10.2 for another explanation of these formulæ).

The affine \( \alpha \)-plane (S.9.18) intersects the Minkowski slice \( \mathbb{M}^4_{\mathbb{R}_p} \) if and only if it intersects \( \mathbb{R}^4_p \); the intersection is then an (affine) null line. For \( p = 0 \) this holds if and only if the point (S.9.20) lies in \( \mathbb{R}^3 \), i.e., \( \text{Re} q_1 = 0 \), and this holds if and only if \([w_0, w_1, w_2, w_3] \) lies on the real quadric

\[
N^5 = \pi^{-1}(\mathbb{R}^3) = \{[w_0, w_1, w_2, w_3] : w_0\overline{w}_2 + w_1\overline{w}_3 + w_1w_2 = 0 \} \subset \mathbb{C}P^3.
\]

(S.9.23)
Points of $N^5 \setminus \mathbb{C}P^3_0$ thus represent affine null lines of $\mathbb{M}^4$; points of $\mathbb{C}P^3_0$ lie in $N^5$ and represent the null lines at infinity. Note that the map (S.9.21) restricts to the projection map of a bundle

$$\pi : N^5 \to \mathbb{R}^3 \cong S^3$$  \hspace{1cm} (S.9.24)

which gives the point of intersection of the null line with $\mathbb{R}^3$. For general $p$ we replace $N^5$ by $N^5_p = \pi^{-1}_p(\mathbb{R}^3_p)$.

Note that, on interchanging $q_2$ and $\bar{q}_2$ in (S.9.18) we obtain the standard parametrization of $\beta$-planes by $\mathbb{CP}^3$.

The incidence relations (S.9.18) define the following fundamental map which gives the point of $\mathbb{CP}^3$ representing the $\alpha$-plane through $(q_1, \bar{q}_1, q_2, \bar{q}_2)$ with direction vector $\sigma^{-1}(iw_1/w_0)$:

$$\mathbb{C}^4 \times \mathbb{CP}^3 \longrightarrow \mathbb{CP}^3 \setminus \mathbb{C}P^3_0 \subset \mathbb{CP}^3,$$

$$\iota : \left( (q_1, \bar{q}_1, q_2, \bar{q}_2), [w_0, w_1] \right) \mapsto [w_0, w_1, w_0q_1 - w_1\bar{q}_2, w_0q_2 + w_1\bar{q}_1].$$  \hspace{1cm} (S.9.25)

The restriction of this to $\mathbb{R}^4 \times \mathbb{CP}^3$ gives a trivialisation of the twistor bundle (S.9.21) over $\mathbb{R}^4$, cf. Section 7.4.

In summary, (i) the map $w \mapsto \bar{w}$ defined by (S.9.16) determines a bijection from $\mathbb{CP}^3$ to the set of all affine $\alpha$-planes in $\mathbb{C}^4$; if $w \in \mathbb{CP}^3 \setminus \mathbb{C}P^3_0$, $\bar{w}$ is given on $\mathbb{C}^4$ by (S.9.18); (ii) a point $(w, p) \in F_{1,2}$ represents the $\alpha$-plane $\Pi_p$ at $p$ tangent to $\bar{w}$.

S.10 \hspace{1mm} GROUP ACTIONS

In order to decide what distributions are conformally equivalent, we need to discuss group actions on the diagram (S.9.15). The group $SL(4, \mathbb{C})$ acts in a canonical way on $\mathbb{C}^4$ and induces transitive actions on the three spaces in (S.9.15) such that $\mu$ and $\nu$ are equivariant. From our description of the holomorphic conformal structure on $\mathbb{C}^4 = G_2(\mathbb{C}^4)$ it is clear that $SL(4, \mathbb{C})$ acts on that space by holomorphic conformal transformations.

The subgroup $\{ B \in GL(6, \mathbb{C}) : \det B = 1, g^C(B(\xi), B(\xi)) = g^C(\xi, \xi) \}$ is isomorphic to the complex orthogonal group $SO(6, \mathbb{C})$. This acts on $\mathbb{CP}^5$ preserving the complex quadric $Q^C$ and restricts to an action on $Q^C$ by holomorphic conformal transformations.

As before, identify $\Lambda^2 \mathbb{C}^4$ with $\mathbb{C}^6$ in the standard way. Then, given $P \in SL(4, \mathbb{C})$, set $R(v \wedge w) = Pv \wedge Pw$; this defines a homomorphism

$$SL(4, \mathbb{C}) \to SO(6, \mathbb{C})$$  \hspace{1cm} (S.10.1)

which is easily seen to be a double covering.

Write

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$  \hspace{1cm} (S.10.2)

where $A, B, C, D$ are $2 \times 2$ complex matrices. We discuss which subgroups give transformations of $\mathbb{R}^4$ and $\mathbb{M}^4$.

For $\mathbb{R}^4$, let $SL(2, \mathbb{H})$ be the subgroup of matrices $P$ of $SL(4, \mathbb{C})$ which commute with $x \mapsto jx$; i.e., with $A, B, C, D$ quaternionic. Then any $P$ sends the fibres of the twistor map (S.9.21) into other fibres and so restricts to a transformation of $\mathbb{R}^4$. Set $F_{1,2}^R = \{(w, p) \in F_{1,2} : w \in \mathbb{CP}^3, p \in \mathbb{R}^4 \}$; then $\mu$ restricts to a diffeomorphism on $F_{1,2}^R$, and so the double fibration collapses to the twistor map (S.9.21).

For $\mathbb{M}^4$, let $h$ be the (pseudo-)Hermitian form on $\mathbb{C}^4$ given by

$$h(v, w) = v_0\bar{w}_2 + v_1\bar{w}_3 + v_2\bar{w}_0 + v_3\bar{w}_1$$

so that $N^5$ is given by $h(w, w) = 0$. Let

$$SU(4, h) = \{ P \in SL(4, \mathbb{C}) : h(Pv, Pw) = h(v, w) \}.$$
Then \( P \in SU(4, h) \) if and only if

\[
\begin{bmatrix}
A^* & C^* \\
B^* & D^*
\end{bmatrix}
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix},
\]

i.e.,

\[A^*C = -C^*A, \quad A^*D + C^*B = I, \quad B^*D = -D^*B. \tag{S.10.3}\]

An easy linear change of coordinates shows that \( h \) has signature \((2, 2)\) so that \( SU(4, h) \) is isomorphic to the group \( SU(2, 2) \). The group \( SU(4, h) \) partitions \( \mathbb{CP}^3 \) into three disjoint orbits, namely, \( \mathbb{CP}^3_+ = \{ z = [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : h(z, z) > 0 \} \), \( \mathbb{CP}^3_- = \{ z = [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : h(z, z) < 0 \} \) and \( \mathbb{N}^5 \). The diagram (S.9.15) restricts to

\[
(w, p) \in \mathbb{F}_{1,2} \quad \mu
\]

\[w \in \mathbb{N}^5 \quad \nu
\]

where \( \mathbb{F}_{1,2} = \{ (w, p) : w \in \mathbb{N}^5, p \in \tilde{\mathbb{M}}^4 \} \).

Since the quadric hypersurface \( Q^M \) is given by the quadratic form (S.9.11) of signature \((2, 4)\), the group \( \{ B \in GL(6, \mathbb{C}) : g(B\xi, B\xi) = g(\xi, \xi) \} \) is isomorphic to \( O(2, 4) \). It clearly acts by conformal transformations on \( Q^M \); in fact we have a double cover

\[O(2, 4) \to C(1, 3) \tag{S.10.5}\]

An easy linear change of coordinates shows that \( h \) has signature \((2, 2)\) so that \( SU(4, h) \) is isomorphic to the group \( SU(2, 2) \). The group \( SU(4, h) \) partitions \( \mathbb{CP}^3 \) into three disjoint orbits, namely, \( \mathbb{CP}^3_+ = \{ z = [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : h(z, z) > 0 \} \), \( \mathbb{CP}^3_- = \{ z = [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : h(z, z) < 0 \} \) and \( \mathbb{N}^5 \). The diagram (S.9.15) restricts to

\[
(w, p) \in \mathbb{F}_{1,2} \quad \mu
\]

\[w \in \mathbb{N}^5 \quad \nu
\]

where \( \mathbb{F}_{1,2} = \{ (w, p) : w \in \mathbb{N}^5, p \in \tilde{\mathbb{M}}^4 \} \).

Since the quadric hypersurface \( Q^M \) is given by the quadratic form (S.9.11) of signature \((2, 4)\), the group \( \{ B \in GL(6, \mathbb{C}) : g(B\xi, B\xi) = g(\xi, \xi) \} \) is isomorphic to \( O(2, 4) \). It clearly acts by conformal transformations on \( Q^M \); in fact we have a double cover

\[O(2, 4) \to C(1, 3) \tag{S.10.5}\]

of the full conformal group of \( Q^M \).

The homomorphism (S.10.1) restricts to a double cover

\[SU(2, 2) \to O_0(2, 4) \tag{S.10.6}\]

of the identity component of \( O(2, 4) \). The composition of this with (S.10.5) gives a 4 : 1 covering

\[SU(2, 2) \to C_0(1, 3) = C_+^1(1, 3) \tag{S.10.7}\]

of the identity component of the conformal group of \( Q^M \) which consists of those conformal transformations which preserve time- and space-orientations.

**Remark S.10.1**

(i) The double cover (S.10.1) exhibits \( SL(4, \mathbb{C}) \) as the universal covering of \( SO(6, \mathbb{C}) \), i.e., \( SL(4, \mathbb{C}) \cong Spin(6, \mathbb{C}) \).

(ii) Since \( \mathbb{M}^4 \) is conformally embedded in \( \tilde{\mathbb{M}}^4 \), any conformal transformation \( R \in C(1, 3) \) of \( \tilde{\mathbb{M}}^4 \) induces local conformal transformations of \( \mathbb{M}^4 \), or a global conformal transformation if \( R \) maps the points at infinity to points at infinity.

**Example S.10.2**

Given \( P \in SL(4, \mathbb{C}) \) as in (S.10.2), in the standard chart (S.9.1), the action on \( \mathbb{C}^4 \) is given by

\[Q \mapsto (C + DQ)(A + BQ)^{-1} \text{ where } Q = \begin{pmatrix} q_1 & \bar{q}_2 \\ q_2 & \bar{q}_1 \end{pmatrix}. \tag{S.10.8}\]

If \( Q \in SL(2, \mathbb{H}) \) then this restricts to a conformal transformation of the space \( \mathbb{R}^4 \); if \( Q \in SU(4, h) \) it restricts to a conformal transformation of \( \tilde{\mathbb{M}}^4 \). We now discuss some special cases.

(i) If \( B = C = 0 \), the transformation (S.10.8) is \( Q \mapsto DQA^{-1} \). If \( D \) and \( A \) are in \( SU(2) \cong Sp(1) \), i.e., represent unit quaternions, then \( P \in SL(2, \mathbb{H}) \) and (S.10.8) restricts to a rotation of \( \mathbb{R}^4 \). If, on the other hand, \( A = (D^*)^{-1} \) with \( D \in SL(2, \mathbb{C}) \), then \( P \) lies in \( SU(4, h) \cong SU(2, 2) \) and (S.10.8) restricts to the Lorentz transformation \( Q \mapsto DQD^* \) of \( \mathbb{M}^4 \); in fact this is a restricted’ Lorentz transformation, i.e., it lies in the identity component \( O_0(1, 3) = O_+^1(1, 3) \) of \( O(1, 3) \) consisting of those Lorentz transformations which preserve time- and space-orientations, and so we obtain a double covering of \( O_0(1, 3) \) by \( SL(2, \mathbb{C}) \), important in spinor theory.

(ii) If \( A = \lambda^{-1/2}I \), \( B = \lambda^{1/2}I \) are multiples of the identity with \( \lambda \) real and positive, then \( P \in SL(2, \mathbb{H}) \cap SU(2, 2) \), and on \( \mathbb{C}^4 \), \( \mathbb{R}^4 \) or \( \mathbb{M}^4 \), the transformation (S.10.8) is the dilation \( Q \mapsto \lambda Q \).
(iii) If $A = D = I$ and $B = 0$, then (S.10.8) is the translation $Q \mapsto Q + C$. If $C$ is of the form (S.9.3), then $P \in \text{SL}(2, \mathbb{H})$ and the translation restricts to a translation through $x$ in $\mathbb{R}^4$; if $C$ is of the form (S.9.4), i.e., is skew-Hermitian, then $P \in \text{SU}(4, h)$ and it restricts to a translation through $x$ in $\mathbb{M}^4$.

(iv) If $A = D = 0$ and $B = C = I$, then (S.10.8) is the inversion $Q \mapsto Q^{-1}$, i.e.

$$x = (x_0, x_1, x_2, x_3) \mapsto \frac{1}{g^C(x, x)}(x_0, -x_1, -x_2, -x_3).$$

Since $P \in \text{SL}(2, \mathbb{H}) \cap \text{SU}(2, 2)$, this restricts to inversions in $\mathbb{R}^4$ and $\mathbb{M}^4$, the latter given by

$$x = (t, x_1, x_2, x_3) \mapsto \frac{1}{|x|^2}(t, -x_1, -x_2, -x_3). \quad (S.10.9)$$

**Remark S.10.3** If (S.10.3) is replaced by

$$A^*C = -C^*A, \quad A^*D + C^*B = -I, \quad B^*D = -D^*B, \quad (S.10.10)$$

then $P$ sends $h$ to $-h$ and so interchanges $\mathbb{CP}_3^2$. For example, if $A = C = 0$, $B = -I$ and $C = I$, then $P$ satisfies (S.10.10). The induced action (S.10.8) on $\mathbb{C}^4$ is reflection through the origin: $x \mapsto -x$, this restricts to an orientation preserving conformal transformation on $\mathbb{M}^4$; however this is time- and space-orientation reversing.

Similarly, $A = C = 0$, $B = I$, $C = -I$ satisfies (S.10.10) and induces the inversion $Q \mapsto -Q^{-1}$ which is, again, time- and space-orientation reversing.

**Example S.10.4** Take the coordinates for $\mathbb{R}^6 = \Lambda^2 \mathbb{R}^4$ given by

$$(\eta, \tilde{\xi}) = (\eta_0, \eta_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4, \tilde{\xi}_5)$$

where $\eta_0 = \tilde{\xi}_0 + \tilde{\xi}_1, \eta_1 = \tilde{\xi}_0 - \tilde{\xi}_1$ and $(\tilde{\xi}_0, \ldots, \tilde{\xi}_5)$ are the coordinates given in (S.9.10). Thus $Q^C$ has equation $\eta_0 \eta_1 - \tilde{\xi}_0^2 = 0$ and $Q^M$ is the set of points of $Q^C$ with real coordinates. Using these coordinates, given $R \in \text{SO}(6, \mathbb{C})$, write it in block form as $R = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ where $E$ is $2 \times 2$. Since the embedding of $\mathbb{C}^4$ in $Q^C$ is given by

$$x \mapsto (\eta_0, \eta_1, x) = 2(1, g^C(x, x), x),$$

the action of $R$ on $\mathbb{C}^4 \subset Q^C$ is easily calculated to be

$$x \mapsto \frac{1}{e_{11} + e_{12} g^C(x, x) + f_{11}} \tilde{\xi} \left( g_1 + g^C(x, x) g_2 + Hx \right) \quad (S.10.11)$$

where $E = (e_{ij})$, $f_{1j}$ (respectively, $g_{ij}$) denotes the first row of $F$ (respectively, $j$'th column of $G$) and $f_{11} \tilde{\xi} = \sum_{j=1}^{4} f_{1j} \tilde{\xi}_{j+1}$.

If all the entries of $R$ are real so that $R \in \text{SO}(2, 4)$, then this action restricts to a conformal transformation of $\mathbb{M}^4$. We now discuss some particular cases.

(i) Set $E = I, F = G = 0$; then $R \in \text{O}(2, 4)$ if $H \in \text{O}(1, 3)$. In this case the action (S.10.11) is the Lorentz transformation $x \mapsto Hx$, this is a restricted Lorentz transformation if $H$ is in the identity component of $\text{O}(1, 3)$, in which case $R \in \text{O}_0(2, 4)$.

(ii) Set $E = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda$ is a non-zero real number, and $F = G = 0$, $H = I$. Then $K \in \text{O}_0(2, 4)$ and the action is the dilation $x \mapsto \lambda x$.

(iii) For $a \in \mathbb{C}^4$ regarded as a column vector, set $E = \begin{pmatrix} 1 & 0 \\ a \bar{a}^2 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 2a \end{pmatrix}$, $G = \begin{pmatrix} a & 0 \end{pmatrix}$ and $H = 0$. Then $R \in \text{O}_0(2, 4)$ and the action is the translation $x \mapsto x + 2a$.

(iv) Set $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $F = G = 0$, then $R \in \text{O}(2, 4)$ if and only if $H \in \text{O}(1, 3)$, and the action on $\mathbb{M}^4$ is given by $x \mapsto Hx/|x|^2$. For example, if $H = \text{diag} \{1, -1, -1, -1\}$, then $R$ lies in $\text{O}_0(2, 4)$ and the transformation is the inversion (S.10.9).
On the other hand if $H = -I$, then $R$ is in $O(2,4)$ but not in the identity component; the corresponding transformation $x \mapsto -x/|x|^2$ is in $C(1,3)$ but not in $C^1_1(1,3)$.

Note that all the examples with matrices $R \in O_0(2,4)$ correspond to matrices $P$ in $SU(2,2)$ under the double cover $(S.10.7)$, the form of such $P$ for (i)–(iv) above being as in the corresponding number in Example S.10.2.

If we use the coordinates $(S.9.8)$ instead, we can give similar examples of matrices $R \in SO(1,5)$ which give conformal transformations of $\mathbb{R}^4$.

S.11 A KERR THEOREM

Let $J$ be a Hermitian structure on an open subset $A^4$ of $\mathbb{R}^4$. Then, as in Section S.2, $J$ is represented by a map $J : A^4 \to CP^1$; combining this with the fundamental map $(S.9.25)$ gives a map

$$A^4 \xrightarrow{(I,J)} A^4 \times CP^1 \rightarrow CP^3 \setminus CP^1_0 \subset CP^3,$$

(S.11.1)

where $I$ denotes the identity map. Comparing definitions we see that this is just the section $\sigma_J$ of the twistor bundle $(S.9.21)$ defined by $J$, as in Section 7.1. As proved there, $\sigma_J$ is a holomorphic map $(A^4, J) \to (CP^3, \text{standard})$; this can also be seen by noting that the two maps in (S.11.1) are both holomorphic. Thus, as in Section 7.1, $\sigma_J(A^4)$ is a complex hypersurface of $CP^3$. In the case of $\mathbb{R}^4$, or its compactification $\mathbb{R}^4$, Proposition 7.1.3(iii) gives the following Kerr-type result.

**Proposition S.11.1** Given a complex hypersurface $S$ of $CP^3$, any smooth map $w : A^4 \to CP^3$ from an open subset of $\mathbb{R}^4$ with image in $S$ defines a Hermitian structure $J$ on $A^4$.

All Hermitian structures $J$ on open subsets $A^4$ of $\mathbb{R}^4$ are given this way by setting $S = \sigma_J(A^4)$.

□

On using Theorem S.5.3 we obtain similar results for the other three sorts of distribution in (S.9.17), but now we must use the mapping $\wedge$ defined by (S.9.15). For completeness, we include the last theorem with this notation, as follows.

**Corollary S.11.2** Given a complex hypersurface $S$ of $CP^3$, any smooth map $w : A \to CP^3$ from an open subset of (i) $C^4$ (respectively, (ii) $R^4$, (iii) $M^4$, (iv) $\mathbb{R}^3$) with

$$w(p) \in \mathbb{P} \quad (p \in A)$$

(S.11.2)

defines a (i) holomorphic foliation $\Pi$ by $\alpha$-planes (respectively, (ii) a positive Hermitian structure $J$, (iii) a real-analytic shear-free ray congruence $\ell$, (iv) a real-analytic conformal foliation $\mathcal{C}$ by curves) on $A$.

All such distributions are given this way.

We call $S$ the twistor surface of $\Pi, J, \ell$ or $\mathcal{C}$.

Note that in case (ii) (respectively, (iv)) the ‘incidence condition’ (S.11.2) reduces to the condition that $w$ be a section of the twistor bundle $(S.9.21)$ (respectively, its restriction $(S.9.24)$).

We can give a more explicit version for uncompactified spaces, as follows.

**Corollary S.11.3** Let $\psi$ be a homogeneous complex-analytic function of four complex variables. Let $[w_0, w_1] = \mu(q_1, q_2, q_3)$ be a smooth solution to the equation

$$\psi(w_0, w_1, w_0q_1 - w_1q_2, w_0q_2 + w_1q_1) = 0.$$

(S.11.3)
on an open subset $A$ of (i) $C^4$, (respectively, (ii) $R^4$, (iii) $M^4$, (iv) $\mathbb{R}^3$). Then $\mu = w_1/w_0$ represents (i) a complex-analytic foliation by null planes (respectively, (ii) a positive Hermitian structure, (iii) a real-analytic shear-free ray congruence, (iv) a real-analytic conformal foliation by curves) on $A$.

Further, the twistor surface of the distribution (i)–(iv) is given by

$$\psi(w_0, w_1, w_2, w_3) = 0.$$

(S.11.4)

All such distributions are given this way locally. □
Example S.11.4 Set $\psi(w_0, w_1, w_2, w_3) = w_3$. Then (S.11.3) reads

$$q_2 + \mu \bar{q}_1 = 0;$$

this has solution $\mu : \mathbb{C}^4 \setminus \{0\} \to \mathbb{C} \cup \{\infty\}$ given by (S.6.12), this defines the distributions $\Pi, J, \ell, \mathcal{C}_0$ of Example S.6.1.

For further examples, see Section S.15.

S.12 CR INTERPRETATION

A CR structure of class $C^r$ (Jacobowicz 1990) on an odd-dimensional manifold $M = M^{2k+1}$ is a choice of $C^r$ rank $k$ complex subbundle $\mathcal{H}$ of the complexified tangent bundle $T^\mathbb{C} M = TM \otimes \mathbb{C}$ with $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$ and

$$[C^\infty(\mathcal{H}), C^\infty(\mathcal{H})] \subset C^\infty(\mathcal{H}).$$  \hfill (S.12.1)

Given $\mathcal{H}$ we define the Levi subbundle $H$ of $TM$ by $H \otimes \mathbb{C} = \mathcal{H} \oplus \overline{\mathcal{H}}$ and a complex-linear endomorphism $J : H \otimes \mathbb{C} \to H \otimes \mathbb{C}$ by multiplication by $-i$ (respectively $+i$) on $\mathcal{H}$ (respectively $\overline{\mathcal{H}}$). Then $J$ restricts to an endomorphism of $H$ with $J^2 = -I$, and (S.12.1) is equivalent to the condition:

$$(i) \text{ if } X, Y \text{ are sections of } H \text{ then so is } [JX, Y] + [X, JY];$$

$$(ii) J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y].$$ \hfill (S.12.2)

Conversely, such a pair $(H, J)$ determines $\mathcal{H}$ by $\mathcal{H} = \{X + iJX : X \in H\}$; thus a CR structure can be defined equivalently as a subbundle $H$ of $TM$ of rank $k$ together with an endomorphism $J : H \to H$ with $J^2 = -I$ which satisfies (S.12.2).

Say that a map $\varphi : M \to N$ is CR with respect to CR structures $(H, J)$ on $M$ and $(H', J')$ on $N$ if $\varphi_* J$ maps $H$ to $H'$ and $\varphi_* \circ J' = J' \circ \varphi_*$ on $H$, equivalently $\varphi_*/_H : T^\mathbb{C} M \to T^\mathbb{C} N$ maps $\mathcal{H}$ to $\mathcal{H}'$. If $\varphi$ is also a diffeomorphism, it is called a CR diffeomorphism, clearly its inverse is automatically CR.

Replacing $H$ by $TN$ and $\mathcal{H}$ by $T^{0,1}N$, we obtain the definition of a CR map from a CR manifold to a complex manifold.

Example S.12.1 Let $A^3$ be an open subset of $\mathbb{R}^3$, or any of Riemanian 3-manifold, and let $U$ be the unit positive tangent to an oriented foliation $\mathcal{C}$ by curves of $A^3$. Give $A^3$ the CR structure defined by $(H, J) = (U^+, J^\perp)$.

Suppose that the leaves of the foliation $\mathcal{C}$ are given by the level sets of a submersion $f = f_1 + i f_2 : A^3 \to \mathbb{C}$. By replacing $f$ by its conjugate $f_1 - i f_2$, if necessary, we can assume that $\{\text{grad } f_1, \text{grad } f_2, U\}$ is positively oriented. Then $f$ is horizontally conformal if and only if it is CR with respect to the structure $(H, J)$, and this holds if and only if $\mathcal{C}$ is a conformal foliation.

Any real hypersurface $M^{2k+1}$ of a complex manifold $(\tilde{M}, J)$ has a canonical CR structure called the hypersurface CR structure given at each point $p \in M$ by (i) $H_p = T^{0,1} p \tilde{M} \cap T^\mathbb{C} M$. Equivalently, (ii) $H_p = T_p M \cap JT_p M$. Note that this necessarily has dimension $2k$ at all points $p$.

More generally, for any submanifold $M^{2k+1}$ of a complex manifold $(\tilde{M}, J)$, we may define $H_p$ and $H_p$ by (i) and (ii); if the dimension of $H_p$ is $2k$ at all points $p \in M^{2k+1}$, it defines a CR structure on $M^{2k+1}$ and $M^{2k+1}$ is called a CR submanifold of $(\tilde{M}, J)$. The inclusion map is then a CR map.

We give the unit tangent bundle $T^1 \mathbb{R}^3 = \mathbb{R}^3 \times S^2$ a CR structure $(H, J)$, as follows. At each $(p, U) \in \mathbb{R}^3 \times S^2$, use the canonical isomorphism $\mathbb{R}^3 \cong T_p \mathbb{R}^3$ to regard $U^\perp$ as a subspace $U^\perp_p$ of $T_p \mathbb{R}^3$; then set $H_{(p, U)} = U^\perp_p \oplus T_0 U^\perp$ and define $J$ to be rotation through $+\pi/2$ on $U^\perp_p$ together with the standard complex structure $J S^2$ on $T_0 U^\perp$. Define a complex structure $\mathcal{J}$ on $\mathbb{R}^4 \times S^2$ by

$$\mathcal{J} : \mathbb{R}^4 \times S^2 \to \text{End } T(\mathbb{R}^4 \times S^2), \quad (p, U) \mapsto (J(U)_{p}, J S^2) \quad \text{(S.12.3)}$$

with $J(U)_{p}$ the unique positive almost Hermitian structure which is determined by $J(U)_{p} (\partial/\partial x_0) = U^\perp_p$, cf. (S.5.1). A calculation, for example using Proposition S.7.1.3(ii), shows that $\mathcal{J}$ is integrable. Then the CR structure $(H, J)$ on $T^1 \mathbb{R}^3$ is the hypersurface CR structure given by regarding $\mathbb{R}^3 \times S^2$ as a real hypersurface of the manifold $$(\mathbb{R}^4 \times S^2, \mathcal{J})$$. 

CR interpretation
More generally, for any oriented Riemannian 3-manifold $M^3$, we can give the unit tangent bundle $T^1M^3$ a CR structure $(H, J)$, as follows. At each point $(p, U) \in T^1M^3 (p \in M^3, U \in T^1_pM^3)$, the Levi-Civita connection on $M^3$ defines a splitting $T_{(p, U)}(T^1M^3) = T_pM^3 \oplus U(T^1_pM^3)$. Since $T^1_pM^3$ is canonically oriented and is isometric to a 2-sphere, it has a canonical Kähler structure $J$. We set $H(p, U) = U \perp T(U(T^1_pM^3))$ and define $J$ to be rotation through $\pm \pi/2$ on $U \perp T_pS^2$ together with $J$ on $T(U(T^1_pM^3))$. It can be checked that (S.12.2) is satisfied.

For $M^3 = S^3$ this can be described more explicitly. The differential of the canonical embedding $S^3 \to \mathbb{R}^4$ defines an embedding $i : T^1S^3 \to T^1\mathbb{R}^4 = \mathbb{R}^5 \times \mathbb{R}^4$. At a point $(p, U) \in T^1S^3 (p \in S^3, U \in T_pS^3)$, we have

$$
\text{di}(H_i)(T^1S^3) = \{ (X, u) \in p \perp U \perp : \langle X, U \rangle + \langle p, u \rangle = 0 \} \subset \mathbb{R}^4 \times \mathbb{R}^4 \cong T(p, U)(\mathbb{R}^4 \times \mathbb{R}^4);
$$

then we choose $H(p, U) = (U \oplus p) \perp \times (U \oplus p) \perp$ and $J = \text{rotation through } \pm \pi/2$ on each oriented plane $(U \oplus p) \perp$. That this satisfies (S.12.2) can be seen either by direct calculation or by noting that any stereographic projection $S^3 \setminus \{ \text{point} \} \to \mathbb{R}^3$ is conformal and induces a CR diffeomorphism between $T^1(S^3 \setminus \{ \text{point} \})$ and $T^1\mathbb{R}^3$.

Let $A^3$ be an open subset of $\mathbb{R}^3$ or $S^3$, and let $U$ be the unit positive tangent to an oriented foliation by curves of $A^3$. We can regard $U$ as a smooth map from $A^3$ to $T^1\mathbb{R}^3$ or $T^1S^3$. Give $A^3$ the CR structure defined by $(H, J) = (U \perp, J \perp)$. Then we can interpret (S.1.2) as a CR condition, as follows.

**Proposition S.12.2** An oriented foliation on $A^3$ is conformal if and only if its positive unit tangent vector field $U : A^3 \to T^1\mathbb{R}^3$ or $T^1S^3$ is CR.

Define a map $k : T^1\mathbb{R}^3 \to N^5$ by sending a unit tangent vector $U$ at a point $p$ of $\mathbb{R}^3 \cong S^3$ to the point of $CP^3$ representing the affine null geodesic through $p$ tangent to $\partial / \partial t + U$. (This is well defined by Remark S.9.3.) Since every null geodesic of $M^4$ meets $\mathbb{R}^3$ in precisely one point, $k$ is a diffeomorphism. Its restriction to $\mathbb{R}^3$ is given by the restriction

$$
T^1\mathbb{R}^3 \cong \mathbb{R}^3 \times S^2 \to N^5 \quad \text{(S.12.4)}
$$

of the fundamental map (S.9.25). Since the latter map is holomorphic with respect to the complex structure $J$ of (S.12.3), the map (S.12.4) is a CR diffeomorphism.

Now, given a congruence of null lines on an open subset $A^M$ we have a map $w : A^M \to N^5$ which sends $p \in A^M$ to the point of $N^5$ representing (see Section S.9) the null line of the congruence through $p$. From Theorem S.3.2 we obtain a characterisation of shear-free, as follows.

**Proposition S.12.3** Let $\ell$ be a $C^\infty$ congruence of null lines on an open subset $A^M$ of $M^4$. Then $\ell$ is shear-free if and only if the map $w : A^M \to N^5$ representing it is CR when restricted to $A^3$.

**Corollary S.12.4** Let $N^3$ be a smooth CR submanifold of $CP^3$ contained in $N^5$. Then any smooth section $w : A^3 \to N^5$ of (S.9.24) defined on an open set of $\mathbb{R}^3$ with image in $N^3$ defines a conformal foliation by curves $C$ of $A^3$.

**Proof** It is easy to check that such a smooth section $w$ of $N^5 \cong T^1A^3$ is CR if and only if its image is a CR submanifold of $CP^3$.

Thus, given a conformal foliation $C$ by curves, or the associated SFR congruence $\ell$, its image under $w$ is a CR submanifold $N^3$ of $N^5$. If $\ell$ is real analytic then $N^3$ is the intersection of a complex hypersurface $S$ with $N^5$; the hypersurface $S$ is the twistor surface of $C$ and $\ell$, as in Corollary S.11.2. If $C$ (or, equivalently, $\ell)$ is only $C^\infty$, then it cannot necessarily be extended to an open subset of $C^4$, but it can be extended to one side, as follows.

**Proposition S.12.5** Let $C$ be a $C^\infty$ conformal foliation by curves of an open subset $A^3$ of $\mathbb{R}^3$, with horizontal distribution everywhere non-integrable, and let $U$ denote its unit positive tangent. Then the vector field $U$ can be extended to a positive Hermitian structure $J$ on one side of $\mathbb{R}^3$; precisely,
set \( \mathbb{R}^4_+ = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 > 0\} \) and \( \mathbb{R}^4_- = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 < 0\} \), then there exists a positive \( C^\infty \) Hermitian structure \( J \) on an open subset \( A^4 \) of \( \mathbb{R}^4_+ \cup \mathbb{R}^3 \) or of \( \mathbb{R}^4_- \cup \mathbb{R}^3 \) with \( A^4 \cap \mathbb{R}^3 = A^3 \) and \( J(\partial/\partial x_0) = U \) on \( A^4 \).

**Proof** The Levi form of a CR manifold \((M, \mathcal{H})\) is the mapping \( \mathcal{H} \times \mathcal{H} \to T^C M/(\mathcal{H} \oplus \mathcal{H}) \) induced by \( (Z, W) \mapsto -\frac{1}{2} i [Z, W] \). If \( C \) has horizontal distribution everywhere non-integrable, then clearly the Levi form of the associated CR structure on \( A^3 \) is non-zero, as is that of the isomorphic CR manifold \( N^3 \). Then, by a theorem of Harvey and Lawson (1975, Theorem 10.2), \( N^3 \) is the boundary of a complex hypersurface \( S \). This means that \( J \) can be defined on one side of \( \mathbb{R}^3 \).

**Remark S.12.6** The non-integrability of the horizontal spaces means that the twist \( (g([e_1, e_2], w) = g([e_1, e_2], U) \) (see Remark S.3.3) is non-zero, and \( J \) can be extended to the side \( x_0 > 0 \) (respectively, \( x_0 < 0 \)) for a positive orthonormal frame \( \{e_1, e_2, U\} \).

### S.13 The Boundary of a Hyperbolic Harmonic Morphism

We shall now discuss how horizontally conformal submersions are the boundary values at infinity of harmonic morphisms from hyperbolic 4-space.

**Proposition S.13.1** Let \( \varphi : A^4 \to \mathbb{C} \) be a submersive map from an open subset of \( \mathbb{R}^4 \) which is holomorphic with respect to a positive Hermitian structure \( J \) on \( A^4 \). Let \( p \in A^4 \) and set \( U = J(\partial/\partial x_0) \) on the open subset \( A^3 = A^4 \cap \mathbb{R}^3_p \) of \( \mathbb{R}^3_p \). Let \( C \) denote the foliation of \( A^3 \) given by the integral curves of \( U \). Set \( f = \varphi|_{A^3} \), so that \( f \) is real analytic. Then

\[
(i) \quad \frac{\partial \varphi}{\partial x_0} = 0 \quad \text{on} \quad A^3
\]

if and only if (ii) \( f = \varphi|_{A^3} \) is constant on the leaves of \( C \).

Further, in this case, (iii) \( f \) is a horizontally conformal submersion.

**Proof** Let \( q \in A^3 \). By the holomorphicity of \( \varphi \), since \( U_q = J_q(\partial/\partial x_0) \), equation (S.13.1) is equivalent to the vanishing of the directional derivative \( U_q(f) \), which is equivalent to constancy of \( f \) on the leaves of \( C \). In this case, let \( \{e_2, e_3 = Je_2\} \) be a basis for \( U_q \cap \mathbb{R}^3_p \); then holomorphicity of \( \varphi \) implies that \( e_3(f) = ie_2(f) \) so that \( f \) is horizontally conformal. Submersivity of \( f \) easily follows from that of \( \varphi \).

**Remark S.13.2** (i) In particular, \( C \) is a conformal foliation; this also follows from Corollary S.5.4. (ii) Any two of the conditions (i), (ii), (iii) in the proposition imply the other.

We now show how to find such functions \( \varphi \) using harmonic morphisms. Equip \( \tilde{\mathbb{R}}^4 = \mathbb{R}^4 \setminus \mathbb{R}^3 \) with the hyperbolic metric \( g^H = \left( \sum_{i=0}^{3} dx_i^2 \right)/x_0^2 \) so that each component \( \mathbb{R}^4 \setminus \mathbb{R}^3 \) is isometric to hyperbolic 4-space \( H^4 \). Let \( \tilde{A}^4 \) be an open subset of \( \tilde{\mathbb{R}}^4 \), then we call a smooth map \( \varphi : \tilde{A}^4 \to \mathbb{C} \) a hyperbolic harmonic map if it is a harmonic map with respect to the hyperbolic metric \( g^H \). By calculating the tension field from (3.2.7) we see that \( \varphi \) is hyperbolic harmonic if and only if

\[
x_0 \sum_{i=0}^{3} \frac{\partial^2 \varphi}{\partial x_i^2} - 2 \frac{\partial \varphi}{\partial x_0} = 0
\]

at all points of \( \tilde{A}^4 \).

Similarly, \( \pi : \tilde{A}^4 \to \mathbb{C} \) will be called a hyperbolic harmonic morphism if it is a harmonic morphism with respect to the metric \( g^H \); by the fundamental result of Fuglede and Ishihara (see Theorem 4.2.2), such maps are characterized as satisfying (S.13.2) and the horizontal weak conformality condition:

\[
\sum_{i=0}^{3} \left( \frac{\partial \varphi}{\partial x_i} \right)^2 = 0.
\]

Now, from Theorem 7.9.5 we have the following.
Corollary S.13.3  (i) Any submersive hyperbolic harmonic morphism \( \varphi : \tilde{A}^4 \to \mathbb{C} \) is holomorphic with respect to some Hermitian structure \( J \) on \( \tilde{A}^4 \) and has superminimal fibres with respect to \( J \), i.e., \( \ker d\varphi \subseteq \ker \nabla^H J \) on \( \tilde{A}^4 \) where \( \nabla^H \) is the Levi-Civita connection of the hyperbolic metric on \( \mathbb{R}^4 \).

(ii) Conversely, let \( J \) be a Hermitian structure on an open subset \( \tilde{A}^4 \) of \( \mathbb{R}^4 \), and \( \varphi : \tilde{A}^4 \to \mathbb{C} \) a non-constant map which is holomorphic with respect to \( J \). Then \( \varphi \) is a hyperbolic harmonic map, and hence a hyperbolic harmonic morphism, if and only if, at points where \( d\varphi \) is non-zero, its fibres are superminimal with respect to \( J \).

To formulate this analytically, let \( \Theta \) be the holomorphic contact form on complex projective space \( \mathbb{P}^3 \) given in homogeneous coordinates by (i.e., its lift to \( \mathbb{C}^4 \setminus \{0\} \) is given by

\[
\theta = \varphi \circ \pi.
\]

Then \( \ker \Theta \) gives the horizontal distribution of the restriction \( \pi : \mathbb{P}^3 \setminus N^3 \to (\mathbb{R}^4, g^H) \) of the twistor map (S.9.21). Set \( \Phi = \varphi \circ \pi \). Then \( \varphi \) has superminimal fibres if and only if

\[
\ker d\varphi \subseteq \ker \Theta;
\]

this is thus the condition that \( \varphi \) be a hyperbolic harmonic morphism. Equivalently, let \( w : \tilde{A}^4 \to \mathbb{C}^3 \) be the section of \( \pi \) corresponding to \( J \), i.e., with image the twistor surface \( S \) of \( J \) (see Theorem S.11.1); then we may pull back \( \Theta \) to a 1-form \( \theta = w^*\Theta \) on \( \tilde{A}^4 \). Condition (S.13.5) now reads \( \ker d\varphi \subseteq \ker \theta \), this condition being equivalent to superminimality of the fibres of \( \varphi \).

Note that, if \( A^4 \) is an open subset of \( \mathbb{R}^4 \), \( \mathbb{R}^4 \cup \mathbb{R}^3 \) or \( \mathbb{R}^4 \cup \mathbb{R}^3 \), with \( A^4 \cup \mathbb{R}^3 \) non-empty, any \( (C^2, \ldots, C^3) \) map which is a hyperbolic harmonic map on \( A^4 = A^4 \setminus \mathbb{R}^3 \) satisfies (S.13.2), and so (S.13.1), at the boundary \( A^3 = A^4 \cup \mathbb{R}^3 \). A key property for us is the following converse.

Proposition S.13.4 Let \( A^4 \) be a connected open subset of \( \mathbb{R}^4 \), \( \mathbb{R}^4 \cup \mathbb{R}^3 \) or \( \mathbb{R}^4 \cup \mathbb{R}^3 \) such that \( A^3 = A^4 \cap \mathbb{R}^3 \) is non-empty, and let \( \varphi : A^4 \to \mathbb{C} \) be a non-constant map which is holomorphic with respect to a Hermitian structure \( J \) on \( \tilde{A}^4 = A^4 \setminus \mathbb{R}^3 \) and submersive at almost all points of \( A^3 \). Then \( \varphi \) satisfies (S.13.1) on \( A^3 \) if and only if \( \varphi \mid_{A^4} \) is a hyperbolic harmonic morphism.

Proof It suffices to work at points where \( \varphi \) is submersive. At such points, note that (S.13.1) holds if and only if \( \ker d\varphi = \text{span}\{\theta/\partial x_0, J\theta/\partial x_0\} \). Now let \( S \) be the twistor surface of \( J \), and let \( \Phi : S \to \mathbb{C} \) be defined by \( \Phi = \varphi \circ \pi \).

We show that the pull-back \( \theta = w^*\Theta \) to \( A^4 \) satisfies

\[
\text{span}\{\theta/\partial x_0, J\theta/\partial x_0\} \subseteq \ker \theta
\]

at all points of \( A^3 \). To do this, since the (complexified) normal to \( \mathbb{R}^3 \) is given by the annihilator of \( \text{span}\{dq_1 - d\bar{q}_1, dq_2, d\bar{q}_2\} \), it suffices to show that, on \( \mathbb{R}^3 = \{q_1 + \bar{q}_1 = 0\} \), the form \( \theta \) is a linear combination of those three forms. Now, on taking differentials in (S.9.18) we obtain

\[
dw_2 = q_1 dw_0 + w_0 dq_1 - \bar{q}_2 dw_1 - w_1 d\bar{q}_2,
\]
\[
dw_3 = q_2 dw_0 + w_0 dq_2 + \bar{q}_1 dw_1 + w_1 d\bar{q}_1.
\]

Substituting these into (S.13.4) and rearranging gives the expression

\[
\theta = (w_3 - w_0 q_2 + w_1 q_1) dw_0 - (w_2 + w_0 \bar{q}_1 + w_1 \bar{q}_2) dw_1 + w_0 w_1 (d\bar{q}_1 - dq_1 - w_0^2 dq_2 - w_1^2 d\bar{q}_2).
\]

But, by (S.9.18), the coefficients of \( dw_0 \) and \( dw_1 \) vanish when \( q_1 + \bar{q}_1 = 0 \), and (S.13.6) follows.

Thus (S.13.1) is equivalent to the condition \( \ker d\varphi \subseteq \ker \theta \) of superminimality of the fibres of \( \varphi \) at points of \( A^3 \) or, equivalently \( \ker d\Phi \subseteq \ker \Theta \) on the real hypersurface \( N^3 = w(A^3) \) of \( S \). But this is a complex-analytic condition, so by analytic continuation, if \( \varphi \) has superminimal fibres at points of \( A^3 \) then it has superminimal fibres on the whole of \( A^4 \). By Corollary S.13.3, \( \varphi \) is a harmonic morphism.

Combining Propositions S.13.1 and S.13.4, we obtain our main result, as follows.
**Theorem S.13.5** Let $f : A^3 \to \mathbb{C}$ be a real-analytic horizontally conformal submersion on an open subset of $\mathbb{R}^3$. Then there is an open subset $A^4$ of $\mathbb{R}^4$ with $A^4 \cap \mathbb{R}^3 = A^3$, and a real-analytic submersion $\varphi : A^4 \to \mathbb{C}$ with $\varphi|_{A^3} = f$, such that $\varphi|_{A^4 \setminus \mathbb{R}^3}$ is a hyperbolic harmonic morphism. In fact the restriction $\varphi \mapsto f = \varphi|_{A^3}$ defines a bijective correspondence between germs at $A^3$ of real-analytic submersions $\varphi : A^4 \to \mathbb{C}$ on open neighbourhoods of $A^3$ in $\mathbb{R}^4$ which are hyperbolic harmonic on $A^4 \setminus \mathbb{R}^3$ and real-analytic horizontally conformal submersions $f : A^3 \to \mathbb{C}$.

**Proof** Let $C$ be the conformal foliation on $A^3$ given by the level sets of $f$, and let $U$ be the unit tangent vector field of $C$ such that $(U, \operatorname{grad} f_1, \operatorname{grad} f_2)$ is positively oriented. Let $J$ be the unique positive almost Hermitian structure on $A^3$ with $U = J(\partial/\partial x_0)$ and set $\varphi = f$ on $A^3$. Then, as in Theorem S.3.2, the null lines tangent to the vectors $\partial/\partial t + U$ define a shear-free ray congruence $\ell$ on some open neighbourhood $A^M$ of $A^3$ in $\mathbb{R}^4$; we extend $J$ and $\varphi$ to $A^M$ by making them constant along the leaves of $\ell$. Write $U = \sigma^{-1}(i\mu)$, then $\mu$ satisfies equations (S.6.11), and $\varphi$ satisfies a similar pair of equations. We extend these quantities to an open neighbourhood of $A^M$ in $\mathbb{C}^4$ by analytic continuation, i.e., by insisting that they be complex analytic, and finally we restrict to $\mathbb{R}^4$. We have thus extended $J$ and $\varphi$ to an open neighbourhood $A^4$ of $A^3$ in $\mathbb{R}^4$; then $\mu$ satisfies (S.6.10) and $\varphi$ satisfies similar equations, hence $J$ is a Hermitian structure on $A^4$ and $\varphi$ is holomorphic with respect to $J$. Since $U = J(\partial/\partial x_0)$, we have $\partial \varphi/\partial x_0 = -iU(\varphi) = 0$. It follows from Proposition S.13.4 that $\varphi$ is a hyperbolic harmonic morphism. □

**Remark S.13.6** If $f$ is only $C^\infty$, then, as in Proposition S.12.5, if the distribution given by $\operatorname{span}\{\operatorname{grad} f_1, \operatorname{grad} f_2\}$ is nowhere integrable, we can extend $f$ to one side of $\mathbb{R}^3$ precisely, there is an open subset $A^4$ of $\mathbb{R}^4 \setminus \mathbb{R}^3$ or $\mathbb{R}^4 \setminus \mathbb{R}^3$ with $A^4 \cap \mathbb{R}^3 = A^3$ and a $C^\infty$ map $\varphi : A^4 \to \mathbb{C}$ with $\varphi|_{A^3} = f$ such that $\varphi|_{A^4 \setminus \mathbb{R}^3}$ is a hyperbolic harmonic morphism.

**Corollary S.13.7** (i) Let $C$ be a real-analytic conformal foliation by curves of an open subset of $\mathbb{R}^3$. Then there is a real-analytic foliation of an open subset $A^4$ of $\mathbb{R}^4$ by surfaces which are minimal in $A^4 \setminus \mathbb{R}^3$ with respect to the hyperbolic metric and which intersect $\mathbb{R}^3$ in leaves of $C$.

(ii) Let $c$ be an embedded real-analytic curve in $\mathbb{R}^3$. Then there is an embedded real-analytic surface $S$ in an open subset $A^4$ of $\mathbb{R}^4$ which is minimal in $A^4 \setminus \mathbb{R}^3$ with respect to the hyperbolic metric and which intersects $\mathbb{R}^3$ in $c$.

**Proof** (i) Represent the leaves of $C$ as the level curves of a real-analytic horizontally conformal submersion $f : A^3 \to \mathbb{C}$ on an open subset of $\mathbb{R}^3$ and construct a hyperbolic harmonic morphism $\varphi$ as in the theorem; then its fibres give the desired foliation.

(ii) Embed $c$ in a real-analytic conformal foliation by curves of an open subset of $\mathbb{R}^3$, as follows. Construct the normal planes to $c$ and integrate the vector field given by the normals to these. This gives a foliation on an open neighbourhood of $c$ in $\mathbb{R}^3$ which has totally geodesic integrable horizontal distribution; by Proposition 2.5.8, it is Riemannian. (To get a conformal foliation which is not Riemannian, replace the planes by spheres, possibly of varying radii.) Now apply (i). □

**Example S.13.8** Let $f$ be the horizontally conformal submersion of Example S.3.5. Recall that its level sets are given by the leaves of the conformal foliation $C$ with tangent vector field $U$ given by (S.3.10). The extension of this to a shear-free ray congruence $\ell$ is described by (S.3.11). As in the proof of Theorem S.13.5, extend $f$ to a function $\varphi$ on an open subset of $\mathbb{M}^4$ by insisting that it be constant along the leaves of $\ell$. Using (S.3.12) we see that this function is

$$\varphi(t, x_1, x_2, x_3) = f\left(x_1, \frac{r}{x_2^2 + x_3^2}(rx_2 + tx_3), \frac{r}{x_2^2 + x_3^2}(rx_3 - tx_2)\right)$$

$$= -ix_1 + r \quad \text{where} \quad r = \sqrt{x_2^2 + x_3^2 - t^2};$$

this is smooth on the cone $x_2^2 + x_3^2 > t^2$. It extends by analytic continuation to the function

$$\varphi(x_0, x_1, x_2, x_3) = -ix_1 + \sqrt{x_2^2 + x_3^2 + x_0^2}$$

(S.13.7)
which is a complex-analytic function on suitable domains of $\mathbb{C}^4$. Its restriction to $\mathbb{R}^4$ is smooth on $\mathbb{R}^4 \setminus \{x_1\text{-axis}\}$ and defines the hyperbolic harmonic morphism $\varphi$ with boundary values at infinity given by $f$. See Example S.15.3 for further developments.

S.14 FINDING HORIZONTALLY CONFORMAL FUNCTIONS

We now show how to use Theorem S.13.5 to find explicitly the horizontally conformal submersion and conformal foliation by curves on an open subset of $\mathbb{R}^3$ which corresponds to a given complex hypersurface $S$ of $\mathbb{C}P^3$. In fact, by introducing a parameter $a \in \mathbb{C}^4$, we can obtain a 5-parameter family of horizontally conformal submersions whose level sets give the 5-parameter family of conformal foliations of open subsets of $\mathbb{R}^3$ discussed in Remark S.5.5. For this, we need to translate the hyperbolic metric to different slices. First, recall from Section S.9 the map $\pi_a : \mathbb{C}P^3 \to \mathbb{R}^4_a$ defined by $w \mapsto$ the intersection of the $\alpha$-plane $\bar{w}$ given by (S.9.18) with $\mathbb{R}^4_a$. Then, with the first component of $a$ denoted by $a_0$, we have the following.

**Lemma S.14.1** Let $a \in \mathbb{C}^4$. Equip $\mathbb{R}^4_a = \mathbb{R}^4 \setminus \mathbb{R}^3_a$ with the hyperbolic metric

$$g^H_a = (\sum_{i=0}^{3} dx_i)^2 / (x_0 - \text{Re} a_0)^2,$$

and let $N^a = \pi^{-1}_a(\mathbb{R}^3_a)$. Then the kernel of the holomorphic contact form

$$\Theta_a = -2a_0(w_1 dw_0 - w_0 dw_1) + w_1 dw_2 - w_2 dw_1 - w_0 dw_3 + w_3 dw_0$$

 restricted to the manifold $\mathbb{C}P^3 \setminus N^a$ gives the horizontal distribution of the map $\pi_a : \mathbb{C}P^3 \setminus N^a \to (\mathbb{R}^4_a, g^H_a)$.

**Proof** Write $a$ in null coordinates as $(a_1, a_2, a_3, a_4)$. From Example S.10.2 we see that the translation $T_a : x \mapsto x + a$ in $\mathbb{C}^4$ corresponds to the map $\tilde{T}_a : \mathbb{C}P^3 \to \mathbb{C}P^3$, $[w_0, w_1, w_2, w_3] \mapsto [w_0, w_1, w_2 + a_1 w_0 - \bar{a}_2 w_1, w_3 + a_2 w_0 + \bar{a}_1 w_1]$; i.e., $\pi_a \circ \tilde{T}_a = T_a \circ \pi$. In fact, this is a rephrasing of (S.9.22). Then $\Theta_a = (\tilde{T}_a^{-1})^* \Theta = \tilde{T}_a^* \Theta$; on calculating this, (S.14.1) follows.

Now let $S \subset \mathbb{C}P^3$ be a given complex hypersurface, and let $a \in \mathbb{C}^4$. Let $U$ be an open set in $S$ such that $\pi_a$ maps $U$ diffeomorphically onto an open set $A^4$ of $\mathbb{R}^4_a$. Then $U$ defines a Hermitian structure $J$ on $A^4$ represented by the section $w : A^4 \to U$ of $\pi_a$. If $S$ is given by

$$\psi(w_0, w_1, w_2, w_3) = 0 \quad (\psi \text{ homogeneous complex-analytic})$$

$w$ is given by solving (S.11.3). Given a submersive holomorphic map $\zeta : U \to \mathbb{C} \cup \{\infty\}$, set $\varphi_a = \zeta \circ w : A^4 \to \mathbb{C} \cup \{\infty\}$. Then $\varphi_a$ is holomorphic with respect to $J$ and so, by Proposition 7.9.1, it is a harmonic morphism with respect to the hyperbolic metric on $\mathbb{R}^4_a$ if and only if the level surfaces of $\zeta$ are horizontal, i.e. tangent to ker $\Theta_a$.

Set $\Sigma^S_a = \{w \in S : \ker \Theta_a = TS\}$. Then on $S \setminus \Sigma^S_a$, ker $\Theta_a \cap TS$ is a one-dimensional holomorphic distribution, so that its integral (complex) curves foliate $S \setminus \Sigma^S_a$. Note that $\pi_a(\Sigma^S_a) \cap A^4$ is the set of Kähler points of $J$, i.e., the set $\{p \in A^4 : \nabla^H_p J = 0 \forall w \in T_p A^4\}$; here $\nabla^H$ denotes the Levi-Civita connection of the hyperbolic metric on $\mathbb{R}^4_a$.

To find $\varphi_a$ we proceed as follows. Firstly, for an open subset $V$ of $U$, let $c : V \to \mathbb{C}^2$, $[w_0, w_1, w_2, w_3] \mapsto (\zeta, \eta)$ be complex coordinates for $S$. Then we can solve (S.9.18) locally to find the composite map $c \circ w$, $p \mapsto (\zeta(p), \eta(p))$.

Next let $\tilde{\zeta} = \zeta(\zeta, \eta)$ be a holomorphic function with $\partial \tilde{\zeta} / \partial \zeta \neq 0$ and set $\tilde{\eta} = \eta$. Then $(\zeta, \eta) \mapsto (\tilde{\zeta}, \tilde{\eta})$ is a locally complex analytic diffeomorphism. The map $\varphi_a(p) = \tilde{\zeta}(\zeta(p), \eta(p))$ restricts to a hyperbolic harmonic morphism on $\mathbb{R}^4_a$ if and only if the level sets of $\zeta$ are superminimal, the condition for this is

$$\Theta_a \left( \frac{\partial}{\partial \tilde{\eta}} \right) = 0.$$  

By the chain rule,

$$\frac{\partial}{\partial \tilde{\eta}} = \frac{\partial}{\partial \eta} - \frac{\partial \tilde{\zeta}}{\partial \eta} \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \eta} - \left( \frac{\partial \tilde{\zeta}}{\partial \eta} \frac{\partial}{\partial \zeta} \right) \frac{\partial}{\partial \tilde{\zeta}}.$$


so that (S.14.3) reads
\[
\Theta_a \left( \frac{\partial}{\partial \zeta} \right) \frac{\partial \tilde{\eta}}{\partial \eta} - \Theta_a \left( \frac{\partial}{\partial \eta} \right) \frac{\partial \tilde{\zeta}}{\partial \zeta} = 0 .
\]  
(S.14.4)

This equation can be solved to get a holomorphic function \( \tilde{\zeta} = \tilde{\zeta}_0(\zeta, \eta) \) such that \( \varphi_a : p \mapsto \tilde{\zeta}_0(\zeta(p), \eta(p)) \) restricts to a submersive hyperbolic harmonic morphism on \( (\mathbb{R}^4_a, g^H_a) \). Note that the solution is not unique; however, as the level sets of \( \tilde{\zeta}_0 \) are uniquely determined on \( V \), the map \( \varphi_a \) is unique up to postcomposition with a biholomorphic function. Restriction of \( \varphi_a \) to \( \mathbb{R}^3_a \) then gives a horizontally conformal submersion \( f \) on an open subset of \( \mathbb{R}^3_a \) and hence the conformal foliation by curves corresponding to \( S \) and \( a \). Note that the method actually finds a holomorphic function \( \varphi_a \) on an open subset of \( \mathbb{C}^4 \).

In summary, given a complex hypersurface \( S \) of \( \mathbb{C}P^3 \), there is defined locally a real Hermitian structure \( J \) and all the related distributions (S.5.7), in particular, \( U = J(\partial/\partial x_0) \). Then given any \( a \in \mathbb{C}^4 \), the integral curves of \( U \) define a real-analytic conformal foliation \( C_a \) on an open subset of \( \mathbb{R}^3_a \) and all such foliations are given this way. Then we have shown above how to find a holomorphic function \( \varphi_a : A^C \to \mathbb{C} \) on an open subset of \( \mathbb{C}^4 \) such that the level curves of \( \varphi_a |_{\mathbb{R}^3_a} \) give the leaves of the conformal foliation \( C_a \).

S.15 EXAMPLES

We start by considering twistor surfaces \( S \) in \( \mathbb{C}P^3 \) which are linear. The first example may be compared with Example 7.11.3.

Example S.15.1 Let \( S \) be the \( \mathbb{C}P^2 \) given by the zero set of the (homogeneous) linear function
\[
\psi(w_0, w_1, w_2, w_3) = b_0 w_0 + b_1 w_1 + b_2 w_2 + b_3 w_3
\]  
(S.15.1)
where \( b = (b_0, b_1, b_2, b_3) \neq (0, 0, 0, 0) \). We consider the case
\[
[b_0, b_1, b_2, b_3] = [0, s, 0, 1]
\]
with \( s \) a real parameter so that (S.15.1) reads
\[
sw_1 + w_3 = 0 .
\]  
(S.15.2)
Note that \( s = 0 \) if and only if \([b_0, b_1, b_2, b_3] \in N^5\); this will be a special case. Equation (S.11.3) reads
\[
s\mu + (q_2 + \mu \tilde{q}_1) = 0 ,
\]  
so that the direction field \( U \) of the distributions (S.5.7) with twistor surface \( S \) is given by \( U = \sigma^{-1}(i\mu) \) where
\[
\mu = -q_2 / (\tilde{q}_1 + s) .
\]
Parametrize \( S \setminus \{w_0 = 0\} \) by
\[
(\zeta, \eta) \mapsto [1, w_1, w_2, w_3] = [1, \zeta, \eta, -s\zeta] .
\]
In terms of these parameters, we have from equation (S.14.1),
\[
\theta_a = 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3
\]
\[
= 2a_0 d\zeta + \zeta d\eta - \eta d\zeta + s d\zeta
\]
so that equation (S.14.4) reads
\[
(-2a_0 + \eta - s) \frac{\partial \tilde{\zeta}}{\partial \eta} + \zeta \frac{\partial \tilde{\zeta}}{\partial \zeta} = 0 ;
\]  
this has a solution
\[
\tilde{\zeta} = -(-2a_0 + \eta - s)/\zeta .
\]
Harmonic morphisms and shear-free ray congruences

The incidence relations (S.9.18) read
\[
\begin{align*}
q_1 - \zeta \bar{q}_2 &= \eta \\
q_2 + \zeta \bar{q}_1 &= -s \zeta
\end{align*}
\]  
(S.15.3)
with solution
\[
\zeta = -\frac{q_2}{q_1 + s}, \quad \eta = \frac{q_1 \bar{q}_1 + q_2 \bar{q}_2 + q_1 s}{q_1 + s}
\]
so that
\[
\varphi_a = \frac{q_1 \bar{q}_1 + q_2 \bar{q}_2 - 2a_0 (\bar{q}_1 + s) + (q_1 - \bar{q}_1)s - s^2}{q_2}.
\]  
(S.15.4)
Set \(a_0 = -ic\); this gives the horizontally conformal function on the slice \(\mathbb{R}^3_+\):
\[
\varphi_c = \frac{(x_1 + c)^2 + x_2^2 + x_3^2 - s^2 + 2i(x_1 + c)s}{x_2 + ix_3}.
\]  
(S.15.5)
Write \(\rho = |s|\); if \(s > 0\), the map \(\varphi_c\) is the composition
\[
\mathbb{R}^3 \xrightarrow{T^1} \mathbb{R}^3 \xrightarrow{D_{1/\rho}} \mathbb{R}^3 \xrightarrow{\sigma^{-1}} \text{S}^3 \xrightarrow{\overline{H}} \text{S}^2 \xrightarrow{\sigma} \mathbb{C} \cup \{\infty\}
\]
where \(T^1\) is the translation \((x_1, x_2, x_3) \mapsto (x_1 + c, x_2, x_3)\), \(D_{1/\rho}\) is the dilation \(x \mapsto x/\rho\), \(\sigma^{-1}\) is the inverse of stereographic projection from \((-1, 0, 0, 0)\), and \(\overline{H}\) is the ‘conjugate’ Hopf fibration given by \(\sigma \circ \overline{H}(q_1, q_2) = \overline{q}_1/q_2\). If \(s < 0\) we replace \(\overline{H}\) by the Hopf fibration, given by \(\sigma \circ H(q_1, q_2) = q_1/q_2\). In either case the fibres of \(\varphi\) give the classic conformal foliation \(\mathcal{C}\) of \(\mathbb{R}^3\) by the circles of Villarceau, see Fig. 2.2. The shear-free ray congruence \(\ell\) whose projection is \(\mathcal{C}\) (see Theorem S.3.2) is called a Robinson congruence. For \(s = 0\), the foliation degenerates to the foliation by the bunch of circles tangent to the \(x_1\)-axis at \((-c, 0, 0)\) described in Example S.1.6, and \(\ell\) is called a special Robinson congruence. The hyperbolic harmonic morphism given by (S.15.4) with \(a_0 = -ic\) and with boundary values given by (S.15.5) (with \(s = 0\) in both cases) has fibres consisting of Euclidean hemispheres based on these circles. Now note that \(A \in SU(4, h)\) transforms the coefficients of the linear function (S.15.1) by
\[
(b_0, b_1, b_2, b_3) \mapsto (b_0, b_1, b_2, b_3) A.
\]
Since \(SU(4, h)\) acts transitively on the three subsets \(\mathbb{C}P^3_+\), \(\mathbb{C}P^3_-\) and \(N^5\) of \(\mathbb{C}P^3\), if \([b_0, b_1, b_2, b_3] \in \mathbb{C}P^3_+\) (respectively, \(\mathbb{C}P^3_-\)), the linear function is \(SU(4, h)\)-equivalent to (S.15.2) with \(s > 0\) (respectively, \(s < 0\)); if \([b_0, b_1, b_2, b_3] \in N^5\), it is equivalent to (S.15.2) with \(s = 0\).

Note that the special Robinson congruence is conformally equivalent via the inversion (Example S.10.2(iv)) to (S.15.1) with \(b = (0, 1, 0, 0)\); this gives \(\mu = 0\) and corresponds to a shear-free ray congruence consisting of parallel rays. The corresponding Hermitian structure \(J\) is Kähler; in fact, any globally defined Hermitian structure on \(\mathbb{R}^4\) is Kähler, see Lemma 7.11.5.

It is easy to see that any linear hypersurface (S.15.1), whether \(b\) is in \(N^5\) or not, is \(SL(2, H)\)-equivalent to (S.15.1) with \(b = (0, 1, 0, 0)\); so that any two Hermitian structures on a subset of \(S^4\) given by a linear equation are conformally equivalent.

We now give three examples where the twistor surface is a quadric so that we can solve all equations explicitly.

**Example S.15.2** Let the twistor surface be the quadric
\[
S = \{[w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 : w_0 w_3 - w_1 w_2 = 0\}.
\]
Then the direction field \(U\) of the corresponding distributions (S.5.7) is given by \(U = \sigma^{-1}(i\mu)\) where
\[
q_2 + \mu \bar{q}_1 - \mu(q_1 - \mu \bar{q}_2) = 0;
\]
this has solutions
\[
\mu = \frac{q_1 - \bar{q}_1 + \sqrt{(q_1 - \bar{q}_1)^2 - 4q_2 \bar{q}_2}}{2q_2}
\]  
(S.15.6)
so that

\[ i\mu = \frac{-x_1 \pm |x|}{x_2 - ix_3} = \frac{x_2 + ix_3}{x_1 \pm |x|} \]

where \( |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \). Thus, on any slice \( \mathbb{R}^3 \), \( U \) is given by

\[ U(t, x_1, x_2, x_3) = \pm(x_1, x_2, x_3)/|x| \]

which gives the tangent vector field of the foliation by radial lines of Example S.3.4.

Carrying out the calculations of Section S.14 gives \( \varphi_a = \mu \) for all \( a \); we omit the details which are similar to the next example. Then, for each \( a \), the map \( \sigma^{-1} \circ \varphi_a \) restricted to \( \mathbb{R}^3_a \) gives the harmonic morphism \( H^4 = (\mathbb{R}^4_a \setminus \{x_0\text{-axis}\}, \rho_a^H) \to S^2 \) given by orthogonal projection to \( \mathbb{R}^3_a \) followed by radial projection.

**Example S.15.3** Let the twistor surface be the quadric

\[ S = \{[w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 : w_0 w_3 + w_1 w_2 = 0 \} . \]

Then the direction field \( U \) of the corresponding distributions is given by the vector field \( U = \sigma^{-1}(i\mu) \) where

\[ q_2 + \mu \bar{q}_1 + \mu(q_1 - \mu \bar{q}_2) = 0 ; \]

this has solutions

\[ \mu = \frac{q_1 + \bar{q}_1 \pm \sqrt{(q_1 + \bar{q}_1)^2 + 4q_2 \bar{q}_2}}{2\bar{q}_2} = \frac{x_0 \pm s}{x_2 - ix_3} \]

where we write \( s = \sqrt{x_0^2 + x_2^2 + x_3^2} \). Note that

\[ \mu|_{\mathbb{R}^3_a} = \pm \frac{\sqrt{x_2^2 + x_3^2}}{x_2 - ix_3} = \pm \frac{x_2 + ix_3}{\sqrt{x_2^2 + x_3^2}} \]

so that on \( \mathbb{R}^3_a \),

\[ U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_2^2 + x_3^2}} (0, -x_3, x_2) ; \]

this is the tangent vector field of the conformal foliation \( C \) by circles around the \( x_1 \)-axis of Example S.3.5. Further note that, writing \( r = \sqrt{x_2^2 + x_3^2 - t^2} \), the restriction \( \mu|_{\mathbb{M}^4} \) is given by

\[ \mu(t, x_1, x_2, x_3) = \frac{-it \pm r}{x_2 - ix_3} \]

\[ = \frac{r}{x_2^2 + x_3^2} \left\{ \pm x_2 + \frac{t}{r} x_3 \pm \left( \pm x_3 - \frac{t}{r} x_2 \right) \right\} , \]

and so, on the open set \( x_2^2 + x_3^2 > t^2 \), we have

\[ U(t, x_1, x_2, x_3) = \frac{r}{\sqrt{x_2^2 + x_3^2}} \left( 0, \mp x_3 + \frac{t}{r} x_2, \pm x_2 + \frac{t}{r} x_3 \right) ; \]

then \( w = \partial/\partial t + U \) gives the tangent field of the shear-free congruence \( \ell \) extending \( C \) discussed in Example S.3.5.

Parametrize \( S \) away from \( w_0 = 0 \) by \( (\zeta, \eta) \mapsto [1, w_1, w_2, w_3] = [1, \eta, -\zeta, \zeta \eta] \). Then the incidence relations (S.9.18) read

\[ \begin{cases} q_1 - \bar{q}_2 \bar{q}_2 = -\zeta \\ q_2 + q_1 \bar{q}_1 = \zeta \eta \end{cases} ; \]

these have solutions

\[ \zeta = -ix_1 \pm s \quad \text{and} \quad \eta = \frac{x_0 \pm s}{x_2 - ix_3} . \]

On \( \mathbb{M}^4 \),

\[ \zeta = -ix_1 \pm r \quad \text{and} \quad \eta = \frac{-it \pm r}{x_2 - ix_3} . \]
These solutions are real analytic except on the cone \( x_3^2 + x_3^2 = t^2 \). We obtain smooth solutions if we avoid this cone.

We remark that the 3-manifold \( S \cap N^5 \) has equation: \( (\zeta + \overline{\zeta}) (|\eta|^2 - 1) = 0 \) and so consists of the union of two submanifolds: \( S_1 : \text{Re} \zeta = 0 \) and \( S_2 : |\eta| = 1 \). Smooth branches of \((\zeta, \eta)\) will correspond to CR maps \( A \to \mathbb{C}P^3 \) with image lying in \( S_1 \setminus S_2 \) or \( S_2 \setminus S_1 \); \( S_1 \cap S_2 \) corresponds to the branching set (cf. Chapter 9) of \((\zeta, \eta)\) which is the subset of \( A \) on which the square root \( r \) vanishes. Explicitly, note from (S.15.7) that, if \( t > |q_2| \) then \( \text{Re} \zeta = 0 \), this corresponds to points of \( S_1 \); if \( t < |q_2| \) then \( |\eta| = 1 \), this corresponds to points of \( S_2 \).

We have

\[
\theta_a = 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3 \\
2a_0 d\eta - \eta d\zeta + \zeta d\eta - \zeta \eta d\zeta
\]

so that (S.14.4) reads

\[
\eta \frac{\partial \zeta}{\partial \eta} + a_0 \frac{\partial \zeta}{\partial \zeta} = 0;
\]

this has a solution

\[
\tilde{\zeta}_a = \zeta - a_0 \ln \eta.
\]  

Set \( \varphi_a = \tilde{\zeta}_a(\zeta(p), \eta(p)) \); then this is the complex-valued map on a dense subset of \( \mathbb{C}^4 \) given by

\[
\varphi_a(x_0, x_1, x_2, x_3) = -ix_1 \pm s - a_0 \ln \frac{x_0 \pm s}{x_2 - ix_3},
\]  

(S.15.9)

For any \( a \in \mathbb{C}^4 \), this restricts to a complex-valued hyperbolic harmonic morphism on a dense subset of \((\mathbb{R}^4, g^R)\). In particular, when \( a = 0 \), this simplifies to the hyperbolic harmonic morphism on \( \mathbb{R}^4 \setminus \{x_1\)-axis\} given by (S.13.7). As in Example S.13.8, this further restricts on \( \mathbb{R}^3 \) to

\[
\varphi(x_1, x_2, x_3) = -ix_1 \pm \sqrt{x_1^2 + x_3^2},
\]  

(S.15.10)

the level surfaces of which give the conformal foliation by circles round the \( x_1 \)-axis of Example S.3.5. The harmonic morphism (S.13.7) has fibres given by the Euclidean spheres having these circles as equators, these spheres are totally geodesic in \((\mathbb{R}^4, g^R)\).

For definiteness, take plus signs in the above. Set \( a_0 = -it \) in (S.15.9). Then the restriction of \( \varphi_a \) to the open set \( \{(x_1, x_2, x_3) : x_1^2 + x_2^2 > t^2 \} \subset \mathbb{R}^3 = \mathbb{R}^3_a \) is the horizontally conformal map

\[
\varphi_t = \varphi_a = -ix_1 + r + it \ln \frac{r - it}{x_2 - ix_3} \\
= -ix_1 + r - t \operatorname{arg} \frac{r - it}{x_2 - ix_3};
\]

the level curves of this are the leaves of the conformal (in fact, Riemannian) foliation \( C_t \) of Example S.3.5; they lie in the planes \( x_1 = \text{constant} \) and are the involutes of the unit circle, see Fig. S.2.

**Remark S.15.4** The matrix \( P = \text{diag}(1, i, i, 1) \)—or rather \( P = \text{diag}(\theta, i\theta, i\theta, \theta) \) where \( \theta^4 = -1 \) so that \( P \in \text{SL}(4, \mathbb{C}) \)—transforms Example S.15.2 into Example S.15.3. Thus they give conformally equivalent foliations by \( \alpha \)-planes on subsets of \( \mathbb{C}^4 \); however, \( P \) does not lie in \( \text{SU}(4, h) \) or \( \text{SL}(2, \mathbb{H}) \); in fact we can easily see that the other pairs of distributions associated to this pair of foliations as in (S.5.7) are not conformally equivalent.

**Example S.15.5** Let the twistor surface be the quadric

\[
S = \{[w_0, w_1, w_3, w_4] \in \mathbb{C}P^3 : w_0w_1 + w_2w_3 = 0 \}.
\]

Then \( \mu \) satisfies

\[
\mu + (q_1 - \mu q_2)(q_2 + \mu q_1) = 0;
\]
this has solutions
\[ \mu = \frac{(1 + q_1 \tilde{q}_1 - q_2 \tilde{q}_2) \pm \sqrt{(1 + q_1 \tilde{q}_1 - q_2 \tilde{q}_2)^2 + 4q_1 \tilde{q}_1 q_2 \tilde{q}_2}}{2q_1 q_2} \].

Parametrize \( S \) away from \( w_0 = 0 \) by
\[ (\zeta, \eta) \mapsto [1, w_1, w_2, w_3] = [1, \zeta \eta, -\eta, \zeta] \).

Then the incidence relations (S.9.18) read
\[ \begin{cases} q_1 - \zeta \eta \tilde{q}_2 = -\eta \\ q_2 + \zeta \eta \tilde{q}_1 = \zeta \end{cases} ; \]
solving these gives
\[ \zeta = \frac{1 + q_1 \tilde{q}_1 + q_2 \tilde{q}_2 \pm s}{2q_2} \] (S.15.11)

where \( s = \sqrt{(1 + q_1 \tilde{q}_1 + q_2 \tilde{q}_2)^2 - 4q_2 \tilde{q}_2} \). This has branching set when the square root \( s \) is zero; on \( \mathbb{R}^4 = \mathbb{R}_0^4 \) this occurs on \( C = \{(q_1, q_2) \in \mathbb{C}^2 = \mathbb{R}^4 : q_1 q_2 = 0, |q_1|^2 + |q_2|^2 = 1\}; this set is the disjoint union of two circles. On any open set \( A \subseteq \mathbb{R}_0^3 \setminus C \), on fixing the \( \pm \) sign in (S.15.11), we obtain a smooth solution. Next note that
\[ \theta_\alpha = 2a_0 \, dw_1 + w_1 \, dw_2 - w_2 \, dw_1 - dw_3 \\
= 2a_0(\eta \, d\zeta + \zeta \, d\eta) - \zeta \eta \, d\eta + \eta(\eta \, d\zeta + \zeta \, d\eta) - d\zeta \\
= (\eta^2 + 2a_0 \eta - 1) \, d\zeta + 2a_0 \zeta \, d\eta , \]
so that (S.14.4) reads
\[ (\eta^2 + 2a_0 \eta - 1) \frac{\partial \tilde{\zeta}}{\partial \eta} - 2a_0 \zeta \frac{\partial \tilde{\zeta}}{\partial \zeta} = 0 ; \]
this has a solution
\[ \tilde{\zeta} = \zeta \left\{ \frac{\eta + a_0 + \sqrt{a_0^2 + 1}}{\eta + a_0 - \sqrt{a_0^2 + 1}} \right\}^{-a_0/\sqrt{a_0^2 + 1}} . \]
When \( a_0 = 0 \) this gives \( \tilde{\zeta} = \zeta \), so that, for any smooth branch \( \zeta : A \to \mathbb{C} \) of (S.15.11), the level curves of \( \varphi = \zeta|_A \) give a conformal foliation. Explicitly,

\[
\varphi(x_1, x_2, x_3) = \frac{1 + x_1^2 + x_2^2 + x_3^2 \pm s}{2(x_2 + ix_3)}
\]

with \( s = \sqrt{(1 + x_1^2 + x_2^2 + x_3^2)^2 - 4(x_2^2 + x_3^2)} \); this defines a rotationally symmetric foliation \( C_0 \). In fact, on each plane \( \arg(x_2 + ix_3) = \text{const.} \) it restricts to a foliation by coaxal circles as shown in Fig. S.3 (the plus and minus signs give the same foliation).

Fig. S.3. Some leaves of the foliation \( C_0 \) in the plane \( \arg(x_2 + ix_3) = 0 \).

S.16 \ NOTÉES ET REMARQUES

Section S.3

The shear tensor of a foliation \( \ell \) of a Lorentzian manifold by null lines is defined by \( S(X, Y) = \text{trace-free part of} \ (X, Y) \mapsto \frac{1}{2} g(\nabla_X Y + \nabla_Y X, w) \quad (X, Y \in C^\infty(\Sigma)) \). (S.16.1)

Geometrically it measures how fast infinitesimal circles in the screen space distort into ellipses under Lie transport along the congruence. It is easy to see that \( \ell \) is shear-free if and only if its shear tensor vanishes.

The expansion tensor is defined as the trace of (S.16.1). Together with the twist tensor discussed in Remark S.3.3, these are the three fundamental tensors associated to a congruence of null lines. For more information see Benn (1994), Beem, Ehrlich and Easley (1996).

Section S.7

Consider the set of complex-analytic maps \( \mathbb{C}^m \supset A^C \to \mathbb{C}^n \) which satisfy

\[
\sum_{i=1}^m \frac{\partial^2 \varphi}{\partial x_i^2} = 0 \quad \text{and} \quad \sum_{i=1}^m \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_i} = \lambda \delta_{\alpha\beta} \quad ((x_1, \ldots, x_m) \in A^C, \ \alpha, \beta \in \{1, \ldots, n\}).
\]

By a complex-analytic version of Theorem 4.2.2, such maps can be characterized as those complex-analytic maps which pull back complex-harmonic functions to complex-harmonic functions, hence we shall call them complex-harmonic morphisms. If \( \lambda \equiv 0 \), they pull back arbitrary complex-analytic functions to complex-harmonic functions; we shall call them degenerate complex-harmonic morphisms. When \( m = 4, n = 1 \) this is the case studied in Section S.7; for arbitrary \( m \) and \( n = 1 \), they are the analogue of the degenerate harmonic morphisms studied in Sections 14.5 and 14.6. See Pambira (2002, 2003) for more on complex-harmonic morphisms, as well as harmonic morphisms between degenerate semi-Riemannian manifolds.
Notes and comments

Section S.8
The correspondence between Hermitian structures and harmonic morphisms from a 4-dimensional Einstein manifold was established by the second author (Wood 1992); see also Chapter 7.

Section S.9
1. An invariant description of the compactification of $\mathbb{R}^n_p$ for any dimension and signature is given by Cahen, Gutt and Trautman (1993); see Akivis and Goldberg (1996), Ward and Wells (1990), Fillmore (1977), Mason and Woodhouse (1996) for other approaches.

2. The double holomorphic fibration (S.9.15) is the basis of the well-known twistor correspondence of relativity theory, which has led to twistor theory as pioneered by Penrose and others, see Penrose (1967, 1975), Penrose and Rindler (1987, 1988) and Huggett, Mason, Tod, Tsou and Woodhouse (1998) for a more recent compendium of articles. The Twistor Newsletter has provided a forum for discussion on the subject and contributions have been collected into three volumes: Mason and Hughston (1990), Mason, Hughston, Kobak (1995), Mason, Hughston, Kobak and Pulverer (2001).

The space $\mathbb{C}P^3$ is thought of as (projectivized) twistor space and points of this space become the fundamental objects from a physical point of view. Points of the real quadric hypersurface $N^3 \subset \mathbb{C}P^3$ defined by (S.9.23) are called null twistors; these correspond to null geodesics in Minkowski space (see Section S.9). At one level, the correspondence is merely formal, but the distinction becomes fundamental when quantization takes place. As Penrose describes the situation ‘... in the usual view, one imagines a space-time in which the points remain intact but where (the metric) $g_{ab}$ becomes quantised ... according to the twistor view ... the concept of a twistor remains intact, while that of a space-time point becomes fuzzy.’

One motivation for this complex-analytic view of the world is the empirically observed discreteness in nature, e.g., charge, spin etc., which has its analogue in the discrete values arising from contour integration. This is the basis of the so-called twistor diagrams, which give a combinatorial description of particle interactions (Penrose 1975, Sparling 1975).

In general, defining twistors on a curved space-time presents formidable difficulties. Two possible extensions to this case are hypersurface twistors (Penrose and Rindler 1988, part of Section 7.4) and asymptotic twistors (Penrose and Rindler 1988, part of Section 9.8). See also Mason, Hughston, Kobak and Pulverer (2001), and the next Note.

3. The extension of the ideas in this paper to more general manifolds will be complicated by the presence of curvature which means that (i) there are shear-free ray congruences only if the curvature is sufficiently special (Goldberg–Sachs Theorem), (ii) Lemma S.3.1 no longer holds—instead we have the Sachs equations (Penrose and Rindler 1988, Chapter 7), (iii) the CR structure on the unit tangent bundle of a 3-manifold may not be realizable as a hypersurface CR structure (LeBrun 1984). For discussions of these and related matters, see also Trautman (1985), Robinson and Trautman (1986), Lewandowski, Nurowski and Tafel (1990), Nurowski (1996).

Section S.11
The original Kerr Theorem (Kerr 1963), expresses the correspondence between a complex-analytic surface in the twistor space $\mathbb{C}P^3$ and a shear-free null geodesic congruence in Minkowski space. This correspondence permits one to express solutions to the zero-rest-mass field equation in terms of contour integrals of holomorphic functions defined on domains of $\mathbb{C}P^3$ (twistor functions), see Penrose (1969). By analogy, Baird (1993) (see also Baird 2002) expresses harmonic morphisms defined on the three-dimensional space forms in terms of contour integrals of functions defined on the corresponding mini-twistor space (cf. Chapter 6).

Section S.12
1. For general information on CR manifolds and embeddability and extension problems, see Jacobowitz (1990) and Boggess (1991).
2. For descriptions of the circles of Villarceau, see Wilker (1986) and compare with a different treatment in Baird (1998).
3. For other descriptions of the CR structure on the unit tangent bundle of a Riemannian 3-manifold see Sato and Yamaguchi (1989, Proposition 5.1) and LeBrun (1984). Note that this is not the same as the CR structure on the unit tangent bundle of a Riemannian manifold of arbitrary dimension discussed, for example, in Blair (1976, Chapter 7) or Tanno (1992).

Section S.13
1. A description of harmonic morphisms on hyperbolic space $H^4$ in terms of holomorphic data—essentially the twistorial formulation described in Sections S.8 and S.9—is given by Baird (1992).
Harmonic morphisms and shear-free ray congruences

2. Thinking of $H^m$ as the unit disc with the Poincaré metric (see Example 2.1.6), Li and Tam (1991) show that, given any $C^3$ map from the boundary at infinity $S^{m-1}$ of $H^m$ to the boundary at infinity $S^{m-1}$ of $H^n$, whose energy density is nowhere zero, there exists a harmonic map from $H^m$ to $H^n$ which realizes the given boundary map; see Li and Tam (1993) for regularity and uniqueness, Li, Tam and Wang (1995) for an extension to Hadamard surfaces, and Donnelly (1994, 1999) for the case of maps between rank one symmetric spaces.

2. LeBrun (1984) shows that any real-analytic 3-manifold $M^3$ is the boundary at infinity of some anti-self-dual Einstein 4-manifold $P^4$. In view of the description of harmonic morphisms with values in a surface on such a 4-manifold, given in Sections 7.9 and 7.10 in terms of integrable Hermitian structures, it is tempting to conjecture that a horizontally weakly conformal map $\varphi : M^3 \to N^2$ onto a Riemann surface is the boundary values at infinity of a harmonic morphism $\Phi : P^4 \to N^2$.

S.17 REFERENCES

Articles marked with an asterisk (*) are, in part at least, about harmonic morphisms. This part of the list is based on the Bibliography of Harmonic Morphisms regularly updated by Sigmundur Gudmundsson, see http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html.

The numbers of the sections in which articles are referred to are shown at the end of each entry; those in italics refer to the ‘Notes and comments’ for that section.

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