STOCHASTIC LAGRANGIAN FLOWS FOR SDES WITH ROUGH COEFFICIENTS

GUOHUAN ZHAO

ABSTRACT. We prove the existence and uniqueness of Stochastic Lagrangian Flows and almost everywhere Stochastic Flows for nondegenerate SDEs with rough coefficients. As an application of our main result, we show that there exists a unique stochastic flow corresponding to each Leray-Hopf solution of 3D-Navier-Stokes equations.

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1. INTRODUCTION

Consider the following SDE in $\mathbb{R}^d$:

$$X_t = \xi + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s,$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are measurable functions and $W$ is a standard $m$–dimensional Brownian motion. In this work, we first study the well-posedness of (1.1) in the sense of DiPerna-Lions, when the coefficients only satisfy some very weak integrability conditions. Then under some additional mild regularity assumptions on the coefficients, we also investigate the well-posedness of above SDE in the stochastically strong sense. In particular, as an application, when $b$ is an arbitrary Leray-Hopf solution of 3D-Navier-Stokes (NS) equations and $\sigma = \sqrt{2} I_{3 \times 3}$ the stochastic flow corresponding to (1.1) is discussed.

The study of classic strong solution to SDEs in multidimensional spaces with singular drifts can at least date back to [20], where Veretennikov showed that (1.1) admits a unique strong solution, provided that $b$ is bounded measurable, $a := \frac{1}{2} \sigma \sigma'$ is uniformly elliptic and $\nabla_x \sigma \in L^2_{\text{loc}}$. Using Girsanov’s transformation and results from PDEs, Krylov and Röckner [8] obtained the existence and uniqueness of strong solutions to (1.1), when $\sigma = I_{d \times d}$ and $b$ satisfies the following Ladyzhenskaya-Prodi-Serrin’s type condition (abbreviated as LPS):

$$b \in L^q_{x} L^p_{t} \quad \text{with} \quad p, q \in [2, \infty), \quad \frac{d}{p} + \frac{2}{q} < 1.$$ 

See also [25, 27, 2, 21] and the references therein for further development. What we specifically point out is that Krylov made significant progress in his very recent work [9], where the strong well-posedness is proved when $b(t, x) = b(x) \in L^{d}_{x}$ and $\nabla_x \sigma \in L^{d}$.

Research of Guohuan is supported by the German Research Foundation (DFG) through the Collaborative Research Centre(CRC) 1283 Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.
On the other hand, classic martingale solutions to (1.1) were first considered by Stroock and Varadhan in [16] when the coefficients are continuous. Recently, in [29], Zhang and the author of this paper studied (1.1) far beyond the above LPS condition. Their main result shows that if

$$\sigma = I; \ b, (\text{div}b)^- \in L^p_t L^p_x \text{ with } p, q \in [2, \infty), \ \frac{d}{p} + \frac{2}{q} < 2,$$

then SDE (1.1) admits at least one weak (martingale) solution. However, in this paper, we will show that the weak uniqueness may fail in this case by providing a nontrivial counterexample (see Theorem 5.1 for more details). Therefore, the classic framework of martingale problem may be no longer suitable for studying SDEs with this kind of very singular drifts, and some other theoretical structures need to be considered.

In [6], Figalli proposed another important objects closely related to the martingale solution called Stochastic Lagrangian Flow (abbreviated as SLF and see Definition 2.3), which is a family of probability measures \(\{P_x\}_{x \in \mathbb{R}^d}\) such that (i) \(P_x\) is a martingale solution to (1.1) for \(\lambda_d\)-a.e. \(x \in \mathbb{R}^d\), where \(\lambda_d\) is the Lebesgue measure on \(\mathbb{R}^d\); (ii) for each \(t \in [0, T]\), the probability measure \(\int_{\mathbb{R}^d} \rho_t(x) dx\) on \(\mathbb{R}^d\) has a uniformly bounded density \(\rho_t\) (see Definition 2.3). In fact, the origin thought of SLF can be traced back to DiPerna and Lions’s celebrated work [5], where the authors studied the connection between the transport equation and the associated ODE

$$X_t(x) = x + \int_0^t b(s, X_s(x))ds.$$

Later, Ambrosio [1] developed the theory of Regular Lagrangian Flow (RLF), which relates existence and uniqueness for the continuity equation with well-posedness of the ODE. And the mentioned SLF can be regard as the stochastically analogy of RLF. Of course, the SLFs are related to the stochastically weak solutions to SDEs. After then, Zhang proposed the “strong” version of SLF in [24] and [26], which was named by almost everywhere Stochastic Flow (abbreviated as AESF, see Definition 2.6). In this framework, the filtered probability space \((\Omega, \mathcal{F}, F, \mathbb{P})\) and Brownian motion \(W\) are given, and the object is a random field \(X : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d\) such that (i) \(X(x)\) is a strong solution of (1.1) for a.e. \(x \in \mathbb{R}^d\); (2) \(\{\text{law}(X(x))\}_{x \in \mathbb{R}^d}\) is an AESF associated with (1.1) (see Definition 2.3). From their definitions, the well-posedness of (1.1) in both above senses should be understood “in average” respect to a.e. initial condition.

Due to the fact that (1.1) may be ill-posed in probabilistically weak sense under assumption (1.2), and inspired by [6], [19] and [29], we will from now concentrate on SLFs and AESFs for SDE (1.1), under nondegenerate condition on \(\sigma\), some integrability conditions on \(b\) and specially a couple of additional Sobolev regularity assumptions of \(b, \sigma\) for AESFs. As showed in [6] and [19], there are deep connections between well-posedness of Fokker-Planck equations associated with (1.1) in \(L^\infty\)-setting and well-posedness of associated SLFs, which provides efficient tools to study SDEs under low regularity assumptions. Following this approach, one need a deep understanding for the following Fokker-Planck equation:

$$\partial_t \mu_t - L^*_x \mu_t = \partial_i \mu_t - \partial_i (d^i \mu_t) + \partial_i (b^i \mu_t) = 0, \quad \mu_0 = \mu, \quad \text{FPE}_1,$$

where

$$d^i(t,x) := \frac{1}{2} \sigma^{ik}(t,x) \sigma^{jk}(t,x), \quad L^* f := \partial_i (d^i f) - \partial_i (b^i f).$$
If \( \mu_t \) is absolutely continuous with respect to the Lebesgue measure, and \( \mu_t(dx) = u(t,x)dx \), then the above equation (FPE1) can be rewritten as
\[
\partial_t u - \nabla \cdot (a \nabla u) + \nabla \cdot (Vu) = 0, \quad u(0) = \phi,
\]
where \( V^i := b^i - \partial_j a^{ij} \). To prove the well-posedness of (FPE2) in \( L^\infty \) space under weak integrability conditions, we first establish an energy inequality (3.6) for solutions to (FPE2) with bounded local energy. Then together with De-Giorgi’s iteration, we prove a global maximum principle and a stability result for solutions to (FPE2) in bounded function space with finite local energy, with the help of which, existence and uniqueness in the above space will be showed. Finally, we extend the previous uniqueness result to bounded function spaces by using a technique from [6]. Combining the above analytical conclusions and some probabilistic methods, we present the proof of our first main result, Theorem 2.4 in section 4. After that, for studying the corresponding AESL under additional condition (A4) below, we first prove a pathwise uniqueness result (see Lemma 4.7) for particular solutions to the original SDE (1.1), together with a Yamada-Watanabe’s type argument, we then show that there is a unique AESF corresponding to (1.1) (see Theorem 2.7). We should mention that under condition (1.2) the local maximum principle for homogenous Kolmogorov’s equation is proved by Nazarov and Uraltseva in [12] in the light of Moser’s iteration. When \( b \) is divergence-free and \( \|b\|_{L^p_t L^q_x} < \infty \) with \( 1 \leq \frac{d}{p} + \frac{2}{q} < 2 \), the Aronson’s type estimate for the heat kernel associated with operator \( \mathcal{L}^b_t = \Delta + b \cdot \nabla \) was established by Qian and Xi in [13].

Compared with the corresponding results in [6] and [19] for elliptic case, our Theorem 2.4 can allow \( b \) to only satisfy the integrability condition (1.2), instead of assuming it to be bounded in \( x \). We should emphasize that this is more than a technical promotion. In fact, a guiding motivation of singular drifts \( b \) in our work is given by the Leray-Hopf solutions \( u \) of the following 3D-NS equation:
\[
\begin{align*}
\partial_t u &= \Delta u + u \cdot \nabla u + \nabla p \\
\text{div} u &= 0 \\
u(0) &= \phi,
\end{align*}
\]
where \( u \) represents the velocity and \( p \) is the pressure. From the well-known fact that when the initial data \( \phi \) is square integrable, there exits at least one divergence free Leray-Hopf weak solution to (1.3) in
\[
V := \left\{ u : \|u\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} + \|\nabla u\|_{L^2([0,T];L^2(\mathbb{R}^3))} < \infty, \quad \forall T > 0 \right\},
\]
and by Sobolev’s embedding, \( u \) meets
\[
u \in L^p_t L^q_x \quad \text{with} \quad p, q \geq [2, \infty], \quad \frac{d}{p} + \frac{2}{q} = \frac{3}{2}.
\]
So according to our result, for each Leray-Hopf solution \( u \) of (1.3), the SDE
\[
dX_t = u(t,X_t)dt + \sqrt{2}dW_t, \quad X_0 = \xi
\]
admits a unique SLF (see Corollary 2.5). Moreover, Theorem 2.7 further suggests that (1.4) has a unique corresponding AESL (see Corollary 2.8). We point out that these results provide the possibility of studying the 3D-NS equation from the perspective of stochastic analysis. When \( u \)
is smooth in $x$, by Constantin and Iyer’s representation (see [4] and [23]), $u$ can be reconstructed from the unique strong solution $X_t$ to (1.4) as

$$u(t,x) = \mathcal{P}\mathcal{E}((\nabla^t X_t^{-1}(x) \cdot \varphi(X_t^{-1}(x)))), \quad (1.5)$$

where $\mathcal{P}$ is the Leray projection, $X_t(x)$ is the unique strong solution of (1.4) with $\xi \equiv x$, $X_t^{-1}(x)$ is the inverse of stochastic flow $x \mapsto X_t(x)$ and $\nabla^t$ stands for the transpose of the Jacobian matrix. However, so far the smoothness of $u$ in $x$ can only be proved in short time even if the initial datum is smooth and compactly supported. A natural question is that whether one can represent it and study their properties by investigating the corresponding stochastic Lagrangian representation for Leray-Hopf solutions to 3D-NS equations. However, we have to admit that the weak differentiability of the stochastic flow with respect to the starting point $x$ remains open—we plan to pursue this in the future.

This paper is organized as follows: In Section 2, we give some basic definitions of certain local Sobolev spaces and state our main results. In Section 3, we study the well-posedness of Fokker-Planck equation (FPPE) and state our main results. In Section 4, we present the proof of our main results, Theorem 2.4 and Theorem 2.7. The ill-posedness of (1.1) in probabilistically weak sense under condition (1.2) will be considered in section 5.

2. Definitions and Main Results

In this section, we first introduce some notations and definitions that will be used frequently in this paper and then present our main result.

Suppose $(E, \mathcal{E})$ is a measurable space, the collection of all $\sigma-$finite measures and probability measures on $E$ are denoted by $\mathcal{M}(E)$ and $\mathcal{P}(E)$, respectively. Given $T > 0$, let $C([0,T];\mathbb{R}^d)$ be the continuous function space equipped with the uniform topology, $\omega_t$ be the canonical process on it and $\mathcal{B}_t := \sigma\{\omega_s \in C([0,T];\mathbb{R}^d) : 0 \leq s \leq t\}$.

For $p, q \in [1, \infty]$, we define

$$\mathbb{L}^p_q(T) := L^p([0,T];L^p(\mathbb{R}^d)),$$

and $L^p(T) := \mathbb{L}^p_q(T)$. For $p, q \in (1, \infty), s \in \mathbb{R}$, we also define

$$\mathbb{H}^{s,p}(T) = L^p([0,T];H^{s,p}(\mathbb{R}^d)),$$

where $H^{s,p}$ is the Bessel potential space. The usual energy space is defined as the following way:

$$V(T) := \left\{ f \in \mathbb{L}^2_w(T) \cap L^2([0,T];H^1) : \|f\|_{V(T)} := \|f\|_{L^2} + \|\nabla_x f\|_{L^2(T)} < \infty \right\}.$$

Throughout this paper we fix a cutoff function

$$\chi \in C_0^\infty(\mathbb{R}^d;[0,1]) \text{ with } \chi|_{B_1} = 1 \text{ and } \chi|_{B_2^c} = 0,$$

and for $r > 0$ and $x \in \mathbb{R}^d$, define

$$\chi_r(x) := \chi(r^{-1}x), \quad \chi_r^*(x) := \chi_r(x-y), \quad x \in \mathbb{R}^d. \quad (2.1)$$

Next we introduction the localized Bessel potential spaces and energy space.
Definition 2.1. Let \( p, q \in [1, \infty] \), we define the Banach space: for fixed \( r > 0 \),

\[
\tilde{L}^p_q(T) := \left\{ f \in L^q([0,T];L^p_{\text{loc}}(\mathbb{R}^d)) : \| f \|_{\tilde{L}^p_q(T)} := \sup_{y \in \mathbb{R}^d} \| f \chi_y^r \|_{L^q(T)} < \infty \right\}
\]

and \( \tilde{L}^p(T) := \tilde{L}^p_\infty(T) \); For any \( p, q \in (1, \infty) \), \( s \in \mathbb{R} \), the localized Bessel potential space is defined by

\[
\tilde{H}^s_p(T) := \left\{ f \in L^q([0,T];H^s_{\text{loc}}) : \| f \|_{\tilde{H}^s_p(T)} := \sup_{y \in \mathbb{R}^d} \| f \chi_y^r \|_{H^s} < \infty \right\}.
\]

The localized energy space is defined by

\[
\tilde{V}(T) := \left\{ f \in \tilde{L}^2(\Omega) \cap \tilde{H}^{1,2}_p(T) : \| f \|_{\tilde{V}(T)} := \| f \|_{\tilde{L}^2(T)} + \| \nabla f \chi_y^r \|_{\tilde{L}^2(T)} < \infty \right\},
\]

\[
\tilde{V}^0(T) := \left\{ f \in \tilde{V}(T) : \text{for any } r > 0, y \in \mathbb{R}^d, t \mapsto f(t) \chi_y^r \right\}
\]

is strong continuous from \([0,T]\) to \( L^2(\mathbb{R}^d) \).

Now let us recall the definition of martingale solutions associated to the operator

\[ L := a^{ij} \partial_{ij} + b^i \partial_i. \]

Definition 2.2 (MP). A continuous process \( \{X_t\}_{t \in [0,T]} \) with value in \( \mathbb{R}^d \) defined on some filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) is a solution of the martingale problem (MP) associated to \((L, \mu_0)\) or martingale solution to (1.1), if it holds

\[ \mathbb{P} \circ X_0^{-1} = \mu_0 = \text{law}(\xi) \in \mathcal{P}(\mathbb{R}^d); \]

\[ \mathbb{E} \int_0^T |a(t,X_t)| + |b(t,X_t)|dt < \infty; \]

and for each \( f \in C^{1,2}_{t,x} \), the process

\[ t \mapsto M_t^f := f(t,X_t) - f(0,X_0) - \int_0^t [\partial_s + L_s]f(s,X_s)ds \]

is a \( \mathcal{F}_t \)-martingale. Or equivalently, a probability measure \( \mathbb{P} \) on \( C([0,T];\mathbb{R}^d) \) is a solution to MP associated to \((L, \mu_0)\) or martingale solution of (1.1), if the above relations hold for \((C([0,T];\mathbb{R}^d), B, B_t, \mathbb{P})\) and \( X = \omega \in C([0,T];\mathbb{R}^d) \).

The following definition of Stochastic Lagrangian Flow is taken from [6].

Definition 2.3 (SLF). Given a measure \( m_0 = \rho_0 \lambda_d \in \mathcal{M}(\mathbb{R}^d) \) with \( \rho_0 \in L^\infty \), we say that a measurable family of probability measures \( \{\mathbb{P}_x\}_{x \in \mathbb{R}^d} \) on \( C([0,T];\mathbb{R}^d) \) is an \( m_0 \)-Stochastic Lagrangian Flow (\( m_0 \)-SLF) associated with \( L \), if:

(i) for \( m_0 \)-a.e. \( x \), \( \mathbb{P}_x \) is a martingale solution of the SDE (1.1) starting from \( x \);
Theorem 2.4. Under \( \lambda \) and the \( u \in L^\infty \) such that

\[
\Lambda^{-1}\|\xi\|^2 \leq d^{ij}\xi_i\xi_j \leq \Lambda\|\xi\|^2; \\
\|b\|_{\tilde{L}^{p_1}(T)} + \|\partial_j a^{ij}\|_{\tilde{L}^{p_1}(T)} + \|(\nabla \cdot V)^-\|_{\tilde{L}^{p_2}(T)} \leq \kappa; \\
\partial_j a^{ij} \in L^\infty(T).
\]

Theorem 2.4. Under Assumption 1,

(1) for any \( m_0 = \rho_0 \lambda_d \in \mathcal{M}(\mathbb{R}^d) \) with \( \rho_0 \in L^\infty \), then there is a unique \( m_0 - \text{SLF} \) associated with \( L \);

(2) for any \( \mu_0 \in \mathcal{P}(\mathbb{R}^d) \) with bounded density with respect to \( \lambda_d \), there is a unique martingale solution \( \mathbb{P} \) associated to \( (L, \mu_0) \) such that \( \mu_t = \mathbb{P} \circ \omega_t^{-1} \ll \lambda_d \) and \( \mu_t = \rho_t \lambda_d \) with \( \rho_t \in L^\infty \) uniformly in \( t \).

If \( u \) is a Leray-Hopf solution to 3D-NS equation with initial condition \( u(0) \in L^2(\mathbb{R}^3) \), then \( u \in L^\infty([0,T];L^2) \cap L^2([0,T];H^1) \), by Sobolev embedding and interpolation theorem,

\[
u \in L^\alpha_q(T), \quad \frac{3}{p} + \frac{2}{q} = \frac{3}{2}, \quad p, q \in [2, \infty].
\]

Thus, Theorem 2.4 implies the following

Corollary 2.5. Suppose \( u \) is the Leray-Hopf weak solution to 3D-NS equation with \( L^2 \) initial datum, then

(1) for \( m_0 \in \mathcal{M}(\mathbb{R}^3) \) with a bounded density with respect to \( \lambda_3 \), there is a unique \( m_0 - \text{SLF} \) associated with \( (1.4) \);

(2) for any \( \mu_0 \in \mathcal{P}(\mathbb{R}^3) \) with bounded density with respect to \( \lambda_3 \), \( (1.4) \) admits a unique martingale solution \( \mathbb{P} \) such that \( \mu_t = \mathbb{P} \circ \omega_t^{-1} \ll \lambda_3 \) and \( \mu_t = \rho_t \lambda_3 \) with \( \rho_t \in L^\infty \) uniformly in \( t \).

From the probabilistic view, both results above are about the weak (martingale) solutions of SDE. Notice that a Leray-Hopf solution \( u \) of 3D-NS equation with \( L^2 \) initial datum is in \( \mathbb{H}^{1/2}_{1,2}(T) \). Our next main result show that the Sobolev regularity of \( u \) leads a sort of well-posedness of \( (1.4) \) in strong sense. Before presenting our statement of second theorem, let us give the definition of almost everywhere Stochastic Flow mentioned by Zhang in [26, Definition 2.1], which can be regard as the “strong” version of SLF.
Definition 2.6 (AESF). Suppose \((\Omega, \mathcal{F}, \mathbb{P})\) is a filtered probability space satisfying the common conditions and \(W\) is a standard \(d\)-dimensional Brownian motion on it. Given a measure \(m_0 = \rho_0 \lambda_\mathbb{R}^d \in \mathcal{M}(\mathbb{R}^d)\) with \(\rho_0 \in L^\infty\), we say a \(\mathbb{R}^d\)-valued measurable stochastic field on \([0, T] \times \Omega \times \mathbb{R}^d, X_t(\omega, x)\), is a \(m_0\)-almost everywhere Stochastic Flow (AESF) of (1.1) if

1. \(\{P_x\}_{x \in \mathbb{R}^d} := \{P \circ X^{-1}(x)\}_{x \in \mathbb{R}^d}\) is a \(m_0\)-SLF corresponding to \(L\);
2. for \(m_0\)-almost all \(x \in \mathbb{R}^d\), \(X_t(x)\) is a continuous \(\mathcal{F}_t\)-adapted process satisfying that

\[
X_t(x) = x + \int_0^t b(s, X_s(x)) ds + \int_0^t \sigma(s, X_s(x)) dW_s, \quad \forall t \in [0, T].
\]

In order to get the well-posedness of almost everywhere Stochastic Flow, we need a stronger assumption on the coefficients.

Assumption 2. The coefficients \(b\) and \(\sigma\) satisfy

\[
b \in L^1([0, T], W^{1,1}_\text{loc}(\mathbb{R}^d)), \quad \sigma \in L^2([0, T]; W^{1,2}_\text{loc}(\mathbb{R}^d)). \tag{A_4}
\]

Theorem 2.7. Under Assumption 1 and 2,

1. for any \(m_0 = \rho_0 \lambda_\mathbb{R}^d \in \mathcal{M}(\mathbb{R}^d)\) with \(\rho_0 \in L^\infty\), equation (1.1) admits a unique \(m_0\)-AESF;
2. if \(\xi \in \mathcal{F}_0\) is a random variable with bounded density, then equation (1.1) has a unique strong solution \(X_t\) such that the density of \(P \circ X^{-1}\) is uniformly bounded in \(t\).

We should point out that a similar result had been stated in [11] under the assumptions that \(\sigma = \mathbb{I}, \nabla \cdot b = 0, b \in \mathbb{H}_1^{1/2} \cap L^q\) with \(r > 1, \frac{d}{p} + \frac{2}{q} < 2\). Their argument essentially follows Zhang [24]. In this paper, we will give a different proof based on some techniques from [22] and [26]. Theorem 2.7 implies

Corollary 2.8. If \(d = 3\),

1. for any \(m_0 = \rho_0 \lambda_\mathbb{R}^d \in \mathcal{M}(\mathbb{R}^d)\) with \(\rho_0 \in L^\infty\), there is a unique \(m_0\)-AESF corresponding to each Leray-Hopf solution of 3D-NS equation;
2. for any random variable \(\xi \in \mathcal{F}_0\) with bounded density, equation (1.4) admits a unique strong solution \(X_t\) satisfying \(P \circ X^{-1} \in L^\infty(T)\).

As we mentioned before, the weak differentiability of the stochastic flow of (1.4) with respect to the starting point \(x\) is still open. To solve this problem, some profound properties about the NS equation may be needed.

3. KOLMOGOROV AND FOKKER-PLANCK EQUATION

In this section, we study the Fokker-Planck Equation associated to (1.1) and establish the well-posedness of (FPE) in \(L^\infty\) setting.

Here and in the sequel, we always assume \(d \geq 2, (p_i, q_i, e_i) \in (1, \infty)^2 \times (0, 1)\) and

\[
\frac{d}{p_i} + \frac{2}{q_i} = 2 - e_i. \tag{3.1}
\]

For any \((p_i, q_i)\) given above, we define \(p_i^*, q_i^* \in [2, \infty)\) by relations

\[
\frac{1}{p_i} + \frac{2}{p_i^*} = 1, \quad \frac{1}{q_i} + \frac{2}{q_i^*} = 1, \tag{3.2}
\]
which implies that
\[
\frac{d}{p_i} + \frac{2}{q_i} = 2 - e_i \iff \frac{d}{p_i^*} + \frac{2}{q_i^*} = \frac{d + e_i}{2}.
\] (3.3)

Let \( I \) be an open interval of \( \mathbb{R} \) and \( D \) be a domain in \( \mathbb{R}^d, Q := I \times D \). Consider the following PDE:
\[
\partial_t u - \nabla \cdot (a \nabla u) + \nabla \cdot (Vu) + cu = f \text{ in } Q.
\] (3.4)

**Definition 3.1.** We say \( u \in \bar{V}(Q) \) is a subsolution(supersolution) to (3.4) if for any almost every \( t \in I, \varphi \in C^\infty_c(Q) \) with \( \varphi \geq 0 \),
\[
\int_D u(t)\varphi(t) + \int_{D_t} \left[-u \partial_t\varphi + (a \nabla u) \cdot \nabla \varphi - uV \cdot \nabla \varphi + cu\varphi \right] \leq (\geq) \int_{D_t} f \varphi,
\] (3.5)
where \( D_t = (I \cap (-\infty, t]) \times D \).

**3.1. A maximum principle.** We first prove an energy inequality for the subsolution of (3.4), which is crucial for the De-Giorgi iteration technique.

We need the following assumption:
\[
\|V\|_{L^p_{\varphi_1}} + \|\left(\frac{1}{2} \nabla \cdot V + c\right)^-\|_{L^p_{\varphi_2}} + \|\left(\nabla \cdot V + c\right)^-\|_{L^p_{\varphi_2}} \leq \kappa'. \tag{A_2^*}
\]

**Lemma 3.2** (Energy inequality). Let \( 0 < \rho < R \leq 1, k > 0, I \subseteq \mathbb{R}, Q = I \times B_R \). Suppose \( u \in \bar{V}(Q) \) is a locally bounded weak subsolution to (3.4) and \( a, V, c \) satisfy (A_1), (A_2^*). \( \eta \) is a cut off function in \( x \), compactly supported in \( B_R, \eta(x) \equiv 1 \) in \( B_\rho \) and \( |\nabla \eta| \leq 2(R - \rho)^{-1} \). Then, for any \( u_k := (u - k)^+ \) and almost every \( s, t \in I \) with \( s < t \), we have
\[
\left( \int_D u_k^2 \eta^2 \right)(t) - \left( \int_D u_k^2 \eta^2 \right)(s) + \int_s^t \int_D |\nabla (u_k \eta)|^2 \leq \frac{C}{(R - \rho)^2} \left( \int D u_k^2 \right)^2 \left( \sum_{i=1}^3 \|u_k\|_{L^p_{\varphi_i}}^2 \right) + C \left( k^2 + \|f\|_{L^p_{\varphi_3}}^2 \right) \sum_{i=2}^3 \|I_{A_i^e}(k)\|_{L^p_{\varphi_i}}^2,
\] (3.6)
where \( A_i^e(k) = \{ u > k \} \cap [s, t] \times B_R \) and the constant \( C \) only depends on \( d, \Lambda, \kappa \) and \( (p_i, q_i) \).

**Proof.** We claim that: for almost every \( s, t \in I \) with \( s < t \), it holds that
\[
\frac{1}{2} \left( \int_D u_k^2 \eta^2 \right)(t) - \frac{1}{2} \left( \int_D u_k^2 \eta^2 \right)(s) + \int_s^t \int_D \nabla u_k \cdot a \nabla (u_k \eta^2) \leq \int_s^t \int_D (u_k + k)V \cdot \nabla (u_k \eta^2) - \int_s^t \int_D c(u_k + k) u_k \eta^2 + \int_s^t \int_D f u_k \eta^2.
\] (3.7)
Indeed, if \([t, t + h] \subseteq I\), we define the Steklov’s mean of \( u \):
\[
u_k(t, x) := \frac{1}{h} \int_0^h u(t + s, x)ds = \frac{1}{h} \int_t^{t+h} u(s, x)ds,
\] (3.8)
and define $u_k^h := (u^h - k)^+$. Suppose $\varphi \in C^\infty_c(Q)$ with $\varphi \geq 0$, by (3.5) and choosing $h$ sufficiently small, we get

$$
\int_{I \times D} -u \partial_t \varphi^{-h} + (a \nabla u) \cdot \nabla \varphi^{-h} - (uV) \cdot \nabla \varphi^{-h} + (cu) \varphi^{-h} \leq \int_{I \times D} f \varphi^{-h}.
$$

Notice that for sufficiently small $h > 0$, $\partial_t u^h \in L^2_t(Q')$, by the above inequality, we obtain

$$
\int_{I \times D} \partial_t u^h \varphi + (a \nabla u)^h \cdot \nabla \varphi - (uV)^h \cdot \nabla \varphi + (cu)^h \varphi \leq \int_{I \times D} f^h \varphi. \quad (3.9)
$$

Now let $\varepsilon > 0$ sufficiently small such that $[s - \varepsilon, t + \varepsilon] \subseteq I$, define

$$
\xi_{s,t}^\varepsilon(r) = \begin{cases}
\varepsilon^{-1}(r + \varepsilon - s), & r \in [s - \varepsilon, s) \\
1, & r \in [s, t] \\
(1 - \varepsilon^{-1}(r - t)), & r \in (t, t + \varepsilon) \\
0, & I \setminus [s - \varepsilon, t + \varepsilon]
\end{cases}
$$

Let $\varphi = u_k^h \eta^2 \cdot \xi_{s,t}^\varepsilon$, integration by parts yields

$$
\int_{I \times D} \partial_t u^h \varphi = \frac{1}{2} \int_{I \times D} \partial_1 [(u_k^h)^2] \eta^2 \cdot \xi_{s,t}^\varepsilon = \frac{1}{2} \int_{I \times D} \partial_1 [(u_k^h \eta)^2] \cdot \xi_{s,t}^\varepsilon \xi_{s,t}^\varepsilon - \frac{1}{2} \int_{I \times D} (u_k^h \eta)^2 \xi_{s,t}^\varepsilon (\xi_{s,t}^\varepsilon)'
$$

$$
= \frac{1}{2\varepsilon} \int_{I} \int_{D} (u_k^h \eta)^2 \, - \frac{1}{2\varepsilon} \int_{s-\varepsilon}^{s} \int_{D} (u_k^h \eta)^2.
$$

By standard approximation argument one can see that (3.9) still holds for $\varphi = u_k^h \eta^2 \cdot \xi_{s,t}^\varepsilon$ (h is sufficiently small). Thus,

$$
\frac{1}{2\varepsilon} \int_{I} \int_{D} (u_k^h \eta)^2 \, - \frac{1}{2\varepsilon} \int_{s-\varepsilon}^{s} \int_{D} (u_k^h \eta)^2
$$

$$
+ \int_{I \times D} [(a \nabla u)^h \cdot \nabla (u_k^h \eta^2) \xi_{s,t}^\varepsilon] - (uV)^h \cdot \nabla (u_k^h \eta^2) \xi_{s,t}^\varepsilon + (cu)^h (u_k^h \eta^2) \xi_{s,t}^\varepsilon] \leq \int_{I \times D} f^h (u_k^h \eta^2) \xi_{s,t}^\varepsilon.
$$

Letting $h \to 0$ and then $\varepsilon \to 0$, by Lebesgue’s dominated convergence theorem and differentiation theorem, we obtain that for almost every $s, t \in I$,

$$
\frac{1}{2} \left( \int_{D} u_k^2 \eta^2 \right) (t) - \frac{1}{2} \left( \int_{D} u_k^2 \eta^2 \right) (s) + \int_{s}^{t} \int_{D} \nabla u_k \cdot a \nabla (u_k \eta^2)
$$

$$
\leq \int_{s}^{t} \int_{D} uV \cdot \nabla (u_k \eta^2) - \int_{s}^{t} \int_{D} cu u_k \eta^2 + \int_{s}^{t} \int_{D} f u_k \eta^2.
$$

Notice that $u \cdot 1_{\{u > k\}} = (u_k + k) 1_{\{u > k\}}$, we complete the proof for (3.7).
For almost every \( s,t \in I \), using integration by parts, we get

\[
\int_s^t \int_D (u_k + k)V \cdot \nabla (u_k \eta^2) = \frac{1}{2} \int_s^t \int_D \nabla^2 V \cdot \nabla (u_k^2) + 2 \int_s^t \int_D u_k^2 \eta V \cdot \nabla \eta \\
+ k \int_s^t \int_D \nabla^2 u_k + 2k \int_s^t \int_D u_k \eta V \cdot \nabla \eta \\
= - \int_s^t \int_D u_k^2 \eta V \cdot \nabla \eta - \frac{1}{2} \int_s^t \int_D \nabla \cdot \nabla u_k^2 + 2 \int_s^t \int_D u_k^2 \eta V \cdot \nabla \eta \\
- 2k \int_s^t \int_D u_k \eta V \cdot \nabla \eta - k \int_s^t \int_D \nabla \cdot \nabla u_k^2 + 2k \int_s^t \int_D u_k \eta V \cdot \nabla \eta \\
= \int_s^t \int_D u_k^2 \eta V \cdot \nabla \eta - \frac{1}{2} \int_s^t \int_D \nabla \cdot \nabla u_k^2 - k \int_s^t \int_D \nabla \cdot \nabla u_k \eta^2.
\]

(3.10)

Combining (3.7), (3.10), (A1) and using Hölder’s inequality, we obtain

\[
\frac{1}{2} \left( \int_D u_k^2 \eta^2 \right) (t) - \frac{1}{2} \left( \int_D u_k^2 \eta^2 \right) (s) + \frac{1}{\Lambda} \int_s^t \int_D |\eta \nabla u_k|^2 \\
\overset{(A1)}{\leq} \frac{1}{2} \left( \int_D u_k^2 \eta^2 \right) (t) - \frac{1}{2} \left( \int_D u_k^2 \eta^2 \right) (s) + \int_s^t \int_D \eta^2 \nabla u_k \cdot a \nabla \eta \\
\overset{(3.7),(3.10)}{\leq} -2 \int_s^t \int_D u_k \eta \nabla \nabla \cdot (a \nabla \eta) + \int_s^t \int_D u_k^2 \eta \nabla \eta - \int_s^t \int_D (\frac{1}{2} \nabla \cdot V + c) u_k^2 \eta^2 \\
- k \int_s^t \int_D (\nabla \cdot V + c) \eta^2 + \int_s^t \int_D f \eta^2 \\
\overset{(A1)}{\leq} 2\Lambda \int_s^t \int_D |\eta \nabla u_k| \cdot |u_k \nabla \eta| + \int_s^t \int_D u_k^2 |V| \cdot |\nabla \eta| + k^2 \int_s^t \int_D (\nabla \cdot V + c)^{-} \eta^2 \\
+ \int_s^t \int_D \left[ (\frac{1}{2} \nabla \cdot V + c)^{-} + (\nabla \cdot V + c)^{-} \right] u_k^2 \eta^2 + \int_s^t \int_D f u_k \eta^2.
\]

For any \( \delta > 0 \), by Hölder’s inequality, (3.2) and (A2*), we have

\[
2\Lambda \int_s^t \int_D |\eta \nabla u_k| \cdot |u_k \nabla \eta| \leq \delta \int_s^t \int_D |\nabla \eta|^2 + 4\Lambda^2 \delta^{-1} (R - \rho)^{-2} \left\| u_k \right\|_{L^2(A_1(k))}^2,
\]

where \( A_1'(k) = \{ u > k \} \cap [s,t] \times B_R \);

\[
\int_s^t \int_D u_k^2 |V| \cdot |\nabla \eta| \leq 2(R - \rho)^{-1} \kappa' \left\| u_k \right\|^2_{L^2((A_1'(k)))};
\]

\[
k^2 \int_s^t \int_D (\nabla \cdot V + c)^{-} \eta^2 \leq k^2 \kappa' \left\| \eta_{A_1'(k)} \right\|^2_{L^2(\eta^2)};
\]

\[
\int_s^t \int_D \left[ (\frac{1}{2} \nabla \cdot V + c)^{-} + (\nabla \cdot V + c)^{-} \right] u_k^2 \eta^2 \leq 2\kappa' \left\| u_k \right\|^2_{L^2(\eta^2)}.
\]
\[
\int_s^t \int_D f u_k \eta^2 \leq \|f\|_{L^q_{\rho_1}} \|u_k\|_{L^p_{\rho_1}} \|1_{A'_q(t)}\|_{L^p_{\rho_1}} \leq \|f\|_{L^q_{\rho_1}}^2 \|1_{A'_q(k)}\|_{L^p_{\rho_1}}^2 + \|u_k\|_{L^p_{\rho_1}}^2.
\]

Choosing \( \delta = (2\Lambda)^{-1} \) and combining the above inequalities, we get

\[
\left( \int_D u_k^2 \eta^2 \right) (t) - \left( \int_D u_k^2 \eta^2 \right) (s) + \int_s^t \int_D |\nabla(u_k \eta)|^2 \leq C(R - \rho)^{-2} \left( \|u_k\|_{L^p_{\rho_1}}^2 + \sum_{i=1}^3 \|u_k\|_{L^p_{\rho_1}}^2 \right) + C \left( k^2 + \|f\|_{L^q_{\rho_1}}^2 \right) \sum_{i=2}^3 \|1_{A'_q(k)}\|_{L^p_{\rho_1}}^2,
\]

where \( C \) only depends on \( d, \Lambda, \kappa' \) and \((p_i, q_i)\).

From now no, we assume \( Q = I \times D = (0, T) \times \mathbb{R}^d \). Using De Giorgi iteration, we will prove a global \( L^\infty \) estimate for the solutions to (3.4). A similar approach can be found in [10].

We need the following elementary lemma.

**Lemma 3.3.** Suppose \( \{y_j\}_{j \in \mathbb{N}} \) is a nonnegative nondecreasing real sequence,

\[
y_{j+1} \leq NC^j y_j^{1+\varepsilon}
\]

with \( \varepsilon > 0 \) and \( C > 1 \). Assume

\[
y_0 \leq N^{-1/\varepsilon} C^{-1/\varepsilon},
\]

then \( y_j \to 0 \) as \( j \to \infty \).

The following maximum principle is crucial.

**Theorem 3.4** (Global maximum principle). Assume \( u \in \tilde{V}^0(T) \) is a locally bounded weak subsolution to (3.4), \( u^+(0) \in L^\infty(\mathbb{R}^d) \) and \( V, c \) satisfy (A2'), then there is a constant \( C \) only depending on \( d, \Lambda, \kappa', T \) and \((p_i, q_i)\) such that for any \( f \in \tilde{L}^{p_3}_{\rho_1}(T) \),

\[
\|u\|_{\tilde{V}(T)} + \|u^+\|_{L^\infty} \leq C \left( \|u^+(0)\|_{L^\infty} + \|f\|_{\tilde{L}^{p_3}_{\rho_1}(T)} \right).
\]

**Proof.** Take \( R = 1, \rho = \frac{1}{3} \) in Lemma 3.2 and let \( \eta \) be the same function there. Define \( \eta_k(\cdot) := \eta(\cdot - x) \) and \( Q_{\tau, x} := (0, \tau] \times B_1(x) \).

**Step 1:** choose \( k \geq K_0 := \|u^+(0)\|_{L^\infty} + \|f\|_{\tilde{L}^{p_3}_{\rho_1}(T)} \), by (3.6) and letting \( s \downarrow 0 \), we have

\[
\sup_{t \in [0, \tau]} \left( \int_{B_1(x)} u_k^2 \eta_k^2 \right) (t) + \int_0^\tau \int_{B_1(x)} |\nabla(u_k \eta_k)|^2 \leq C \left( \|u_k\|_2^2_{L^2(Q_{\tau, x})} + \sum_{i=1}^3 \|u_k\|_{L^p_{\rho_1}}^2 \right) + CK^2 \sum_{i=2}^3 \|1_{A(x,k)}\|_{L^p_{\rho_1}}^2,
\]

where \( \tau \in (0, T) \) and \( A(x,k) := Q_{\tau,x} \cap \{u > k\} \).
Let $\tilde{\eta}_k(\cdot) = \eta\left(\frac{\cdot - x_k}{\delta}\right)$, $\ell_i = \frac{1}{p_i} - \frac{\alpha_i}{2d+4}$, $\ell_q = \frac{1}{q_i} - \frac{\alpha_i}{2d+4}$. By (3.3), we have $\frac{d}{\ell_i} + \frac{2}{\ell_q} = \frac{d}{2}$ and $L^\ell_i(\tau) \subseteq V(\tau)$, so H"{o}lder’s inequality yields,

$$\|u_k\|_{L^\ell_i(Q_{\tau, x})} \leq C \|u_k \tilde{\eta}_k\|_{L^\ell_i(\tau)} \|Q_{\tau, x}\|^{\frac{\rho_i}{2d+4}} \tag{3.13}$$

$$\leq C \|u_k \tilde{\eta}_k\|_{V(\tau)} \tau^{\frac{\rho_i}{2d+4}} \leq C \tau \frac{\rho_i}{2d+4} \|u_k\|_{V(\tau)}.$$ 

Obviously,

$$\|u_k\|_{L^\ell_i(Q_{\tau, x})} \leq C \tau \frac{\rho_i}{2d+4} \|u_k\|_{V(\tau)}.$$ 

By above estimates and (3.12), we get

$$\|u_k\|_{V(V(\tau))}^2 \leq C_{\delta} \sup_{x \in \mathbb{R}^d} \|u_k \eta_{k}\|_{V(\tau)}^2$$

$$\leq C \tau \delta \|u_k\|_{V(\tau)}^2 + Ck^2 \sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \|1_{A(x, k)}\|_{L^\ell_i}^2,$$

where $\delta = \min_i \{\frac{\alpha_i}{d+2}\}$. By choosing $\tau = (2C)^{-\delta^{-1}}$, we get

$$\|u_k\|_{V(V(\tau))}^2 \leq Ck^2 \sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \|1_{A(x, k)}\|_{L^\ell_i}^2. \tag{3.14}$$

Now let $\tilde{p}_i = (d + e_i)p_i^* / d$, $\tilde{q}_i = (d + e_i)q_i^* / d$, then by (3.3), $\frac{d}{\tilde{p}_i} + \frac{2}{\tilde{q}_i} = \frac{d}{2}$, so $L^\tilde{p}_i(\tau) \subseteq V(\tau)$. For any $h > k$, since $A(x, h) \subseteq \{u_k > h - k\} \cap Q_{\tau, x}$, by Chebyshev’s inequality, H"{o}lder’s inequality and (3.14), we get

$$\|1_{A(x, h)}\|_{L^\tilde{p}_i} \leq (h - k)^{-1} \|u_k\|_{L^\tilde{p}_i(\tau)} \leq (h - k)^{-1} \|u_k\|_{L^\tilde{p}_i(\tau, x)} \|1_{A(x, k)}\|_{L^{(d + e_i)p_i^* / e_i}} \tag{3.15}$$

$$\leq (h - k)^{-1} \|u_k \tilde{\eta}_k\|_{L^\tilde{p}_i(\tau)} \|1_{A(x, k)}\|_{L^{(d + e_i)p_i^* / e_i}} \leq C(h - k)^{-1} \|u_k \tilde{\eta}_k\|_{V(\tau)} \|1_{A(x, k)}\|_{L^{(d + e_i)p_i^* / e_i}} \leq C(h - k)^{-1} \|u_k \tilde{\eta}_k\|_{V(\tau)} \|1_{A(x, k)}\|_{L^{(d + e_i)p_i^* / e_i}} \leq C \frac{k}{h - k} \left(\sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \|1_{A(x, k)}\|_{L^\ell_i}^2\right)^{1+\varepsilon} \tag{3.15}$$

$$\leq C_1 \frac{k}{h - k} \left(\sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \|1_{A(x, k)}\|_{L^\ell_i}^2\right)^{1+\varepsilon} \quad (\forall x \in \mathbb{R}^d),$$

where $\varepsilon = \min_i \{\frac{\alpha_i}{d+e_i}\}$ and $C_1$ only depends on $d, A, \kappa'$ and $(p_i, q_i)$. Let $N > 1$ be a number will be determined later, define $k_j := NK_0(2 - 2^{-j})$ $(j \in \mathbb{N})$ and

$$y_j := \sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \|1_{A(x, k_j)}\|_{L^\ell_i}^2.$$ 

By (3.15), we have

$$y_{j+1} \leq 8C_1 2^j y_j^{1+\varepsilon}.$$
Thus, by Lemma 3.3, if
\[
\sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \| I_{A(x, N \mathbb{K}_0)} \|_{L^p_{q_i}} = y_0 \leq (8C_1)^{-1/\epsilon} 2^{1/\epsilon^2},
\]  
(3.16)
then \( \lim_{j \to \infty} y_j = 0 \), i.e. \( u^+ \leq 2NK_0 \) almost everywhere. Indeed, by (3.15), for any \( x \in \mathbb{R}^d \),
\[
\| I_{A(x, N \mathbb{K}_0)} \|_{L^p_{q_i}} \leq \frac{C_1}{N-1} \left( \sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \| Q_{x,x} \|_{q_i} \right)^{1+\epsilon}
\]
\[
\leq \frac{C_1}{N-1} \left( \sum_{i=2}^{3} \sup_{x \in \mathbb{R}^d} \| Q_{x,x} \|_{q_i} \right)^{1+\epsilon} \leq 2^{1+\epsilon} C_1/(N-1),
\]
which implies \( y_0 \leq 2^{2+\epsilon} C_1/(N-1) \). Let \( N = 1 + 2^{100} (C_1)^{1+\frac{1}{\epsilon}} \), then we have (3.16). Thus, there is a constant \( C_2 \) depending only on \( d, \Lambda, \kappa \) and \( (p_i, q_i) \) such that \( u^+(t,x) \leq C_2 K_0 = C_2(\| u^+(0) \|_{L^\infty} + \| f \|_{L^q(T)}) \) for almost every \( (t,x) \in [0, \tau] \times \mathbb{R}^d \). Since \( C_2 \) does not depend on the initial value of \( u \), we obtain that \( \| u^+ \|_{L^\infty(T)} \leq C_2(\| T/\tau + 1 \| K_0) \).

**Step 2:** choose \( k = 0 \), by (3.6) and similar argument in Step 1, we can obtain that for any \( \tau \in [0, T] \),
\[
\| u^+ \|_{L^2_T}^2 + \sup_{x \in \mathbb{R}^d} \| \nabla(u^+ \eta) \|_{L^2_T}^2 \leq \| u^+(0) \|_{L^2}^2 + C\tau^\delta \| u^+ \|_{L^2_T}^2 + C\| f \|_{L^q(T)}^2,
\]
and the constant \( C \) only depends on \( d, \Lambda, \kappa \) and \( (p_i, q_i) \). This yields
\[
\| u^+ \|_{L^2(T)} \leq C \left( \| u^+(0) \|_{L^\infty} + \| f \|_{L^q(T)} \right).
\]
So we complete our proof. \( \square \)

Next we give the precise definition of weak solution to Cauchy problem.

**Definition 3.5.** \( u \in \bar{V}^0(T) \) is called a weak solution of equation
\[
\begin{cases}
\partial_t u - \nabla \cdot (a \nabla u) + \nabla \cdot (Vu) + cu = f \\
u(0) = \phi
\end{cases}
\]  
(3.17)
in \([0, T] \times \mathbb{R}^d \), if for any \( \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d) \) and almost every \( t \in [0, T] \), it holds that
\[
\int_{\mathbb{R}^d} u(t)\varphi(t) - \int_{\mathbb{R}^d} \varphi(0)
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \left[ -u \partial_t \varphi + (a \nabla u) \cdot \nabla \varphi - uV \cdot \nabla \varphi + cu \varphi \right] = \int_0^t \int_{\mathbb{R}^d} f \varphi.
\]

### 3.2. Existence, uniqueness and stability

In this section, we will use the apriori estimate (3.11) to prove the existence-uniqueness and stability of weak solutions for equation
\[
\begin{cases}
\partial_t u - \nabla \cdot (a \nabla u) + \nabla \cdot (Vu) = f \\
u(0) = \phi
\end{cases}
\]  
(3.19)
Theorem 3.6 (Existence-uniqueness). Under \((A_1)\) and \((A_2)\), for each \(f \in L^p_0\), \(\phi \in L^\infty\) there exists a unique weak solution to (3.19) in \(\tilde{V}^0(T) \cap L^\infty(T)\).

Proof. The proof is essentially the same as the one of [29, Theorem 2.3]. First of all, the uniqueness is a direct consequence of (3.11). We prove the existence by weak convergence method. Let

\[ \rho_n(x) := n^d \rho(nx), \]

where \(0 \leq \rho \in C^\infty_c(B_1)\) with \(\int \rho = 1\). \(a_n(t,x) := a(t,\cdot) * \rho_n(x), V_n(t,x) := V(t,\cdot) * \rho_n(x), f_n(t,x) := f(t,\cdot) * \rho_n(x)\) and \(\phi_n = \phi * \rho_n\). By Proposition 4.1 of [29], we have

\[ V_n \in L^q_1([0,T];C^\infty_c(\mathbb{R}^d)), \quad f_n \in L^q_1([0,T];C^\infty_c(\mathbb{R}^d)), \]

and

\[ \sup_n \left( \|V_n\|_{L^q_1} + \|\nabla V_n\|_{L^q_2} + \|f_n\|_{L^q_2} \right) < \infty. \] (3.20)

It is well known that the following PDE has a unique smooth solution \(u_n \in C([0,T];C^\infty_c(\mathbb{R}^d))\):

\[ \partial_t u_n = \nabla \cdot (a_n \nabla u_n) - \nabla \cdot (V_n u_n) + f_n, \quad u_n(0) = \phi_n \]

holds in the distributional sense. In particular, for any \(\varphi \in C^\infty_c([0,T] \times \mathbb{R}^d)\) and \(t \in [0,T]\),

\[ \int_{\mathbb{R}^d} u_n(t) \varphi(t) - \int_{\mathbb{R}^d} \phi_n(0) \varphi(0) = \int_0^t \int_{\mathbb{R}^d} u_n \partial_t \varphi \\
- \int_0^t \int_{\mathbb{R}^d} -a_n \nabla u_n \cdot \nabla \varphi + u_n V_n \cdot \nabla \varphi + f_n \varphi. \] (3.21)

Since

\[ \|\partial_t u_n\|_{H^{1/2}_2(T)} \leq \|\nabla \cdot (a_n \nabla u_n) - \nabla \cdot (V_n u_n) + f_n\|_{H^{1/2}_2(T)} \]

\[ \leq C \left( \|a_n \nabla u_n\|_{L^2_2(T)} + \|V_n u_n\|_{L^2_2(T)} + \|f_n\|_{L^2_2(T)} \right) \]

\[ \leq C \left( \|a_n\|_{L^\infty} \|u_n\|_{H^{1/2}_2(T)} + \|V_n\|_{L^2(T)} \|u_n\|_{L^\infty(T)} + \|f_n\|_{L^p_0} \right) \]

\[ \leq C \left( \|u_n\|_{H^{1/2}_2(T)} + \|u_n\|_{L^\infty(T)} + \|f_n\|_{L^p_0(T)} \right). \]

By Theorem 3.4, we get for any \(T > 0\),

\[ \sup_n \left( \|u_n\|_{L^\infty(T)} + \|u_n\|_{\tilde{V}(T)} + \|\partial_t u_n\|_{H^{1/2}_2(T)} \right) < \infty. \] (3.22)

Hence, by the fact that every bounded subset of \(\tilde{V}(T)\) is relatively weak compact, there is a subsequence (still denoted by \(n\)) and \(\tilde{u} \in \tilde{V}(T) \cap L^\infty(T)\) such that for any \(\varphi \in C^\infty_c([0,T] \times \mathbb{R}^d)\)

\[ \int_0^t \int_{\mathbb{R}^d} u_n \partial_t \varphi + \int_0^t \int_{\mathbb{R}^d} -a_n \nabla u_n \cdot \nabla \varphi + u_n V_n \cdot \nabla \varphi + f_n \varphi \\
- \int_0^t \int_{\mathbb{R}^d} \tilde{u} \partial_t \varphi + \int_0^t \int_{\mathbb{R}^d} -(a \nabla \tilde{u}) \cdot \nabla \varphi + \tilde{u} V \cdot \nabla \varphi + f \varphi \] (3.23)

and

\[ \|\tilde{u}\|_{L^\infty(T)} + \|\tilde{u}\|_{\tilde{V}(T)} + \|\partial_t \tilde{u}\|_{H^{1/2}_2(T)} < \infty. \]
By Lions-Magenes lemma(cf. [18, Lemma 1.2, Chapter 3]), we obtain that $\bar{u} \in C([0,T];\overline{L}^2(\mathbb{R}^d))$, hence $\bar{u} \in \overline{V}^0(T) \cap L^\infty(T)$. On the other hand, by (3.22) and Aubin-Lions lemma (cf. [14]), there is a subsequence of $n$ (still be denoted by $n$) such that (3.23) holds and

$$\lim_{n \to \infty} \| u_n - \bar{u} \|_{L^2([0,T] \times B_R)} = 0, \quad \forall R > 0.$$ 

It holds that for Lebesgue almost all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u_n(t,x) \to \bar{u}(t,x),$$

as $n \to \infty$ along an appropriate subsequence. Thus, for almost every $t \in [0,T]$,

$$\int_{\mathbb{R}^d} \tilde{u}(t)\phi(t) - \int_{\mathbb{R}^d} \phi_n\phi(0) \to \int_{\mathbb{R}^d} \bar{u}(t)\phi(t) - \int_{\mathbb{R}^d} \phi\phi(0).$$

(3.24)

Combing (3.21), (3.23) and (3.24), we obtain that for all $\varphi \in C^\infty_c([0,T] \times \mathbb{R}^d)$ and almost every $t \in [0,T]$

$$\int_{\mathbb{R}^d} \tilde{u}(t)\varphi(t) - \int_{\mathbb{R}^d} \phi\varphi(0) = \int_0^t \int_{\mathbb{R}^d} \tilde{u}\partial_t\varphi + \int_0^t \int_{\mathbb{R}^d} -(a\nabla\tilde{u}) \cdot \nabla\varphi + \tilde{u}\nabla \cdot \varphi + f \varphi, $$

i.e. $\tilde{u}$ solves (3.19).

\[\square\]

**Theorem 3.7.** (Stability) Let $(p_i, q_i) \in [2, \infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 2$, where $i = 1, 2, 3$, $T > 0$. For any $n \in \mathbb{N} \cup \{\infty\} =: \mathbb{N}_\infty$, let $b_n, f_n, \phi_n$ satisfy

$$\sup_{n \in \mathbb{N}_\infty} \left( \| V_n \|_{L^p_{q_1}} + \| (\nabla \cdot V_n)^- \|_{L^{p_2}_q} + \| f_n \|_{L^p_{q_3}(T)} + \| \phi_n \|_{L^\infty_c} \right) < \infty.$$ 

For $n \in \mathbb{N}_\infty$, let $u_n \in \overline{V}^0(T) \cap L^\infty(T)$ be the unique weak solutions of (3.19) associated with coefficients $(V_n, f_n, \phi_n)$ with initial value $u_n(0) = \phi_n$. Assume that for any $\varphi \in C_c(\mathbb{R}^d)$,

$$\lim_{n \to \infty} \left( \| (V_n - V_\infty)\varphi \|_{L^1_{q_1}(T)} + \| (f_n - f_\infty)\varphi \|_{L^p_{q_3}(T)} + \| \phi_n - \phi_\infty \|_{L^\infty_c} \right) = 0.$$ 

Then it holds that for Lebesgue almost all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\lim_{n \to \infty} u_n(t,x) = u_\infty(t,x).$$

The proof of above theorem is essentially same with Theorem 3.6, so we omit its proof here.

Let us also mention the following Kolmogorov’s equation

$$\begin{cases}
\partial_t u - Lu = \partial_i u - a^i_j \partial_j u - b^i_j \partial_i u = f \\
u(0) = \phi,
\end{cases}$$

which can be rewritten as

$$\begin{cases}
\partial_t u - \nabla \cdot (a \nabla u) - \nabla \cdot (Vu) + \nabla \cdot Vu = f, \\
u(0) = \phi.
\end{cases} \quad (KE)$$

If $V \in \overline{L}^{p_1}_{q_1}(T)$, $(\nabla \cdot V)^- \in \overline{L}^{p_2}_{q_2}(T)$, due to Theorem 3.4, any subsolution $u \in \overline{V}^0(T)$ satisfies (3.11). Using similar argument in Theorem 3.6 (see also [29]), we have
Proposition 3.8. Assume $a, b, V$ satisfy $(A_1)$ and $(A_2)$, then for each $f \in L^p_{\text{loc}}(T)$ and $\phi \in L^\infty$ equation (KE) admits a unique weak solution $u \in \tilde{V}^0(T) \cap L^\infty(T)$.

In order to apply the theory on SLF developed in [6] and [19], we first need to extend the uniqueness result in Theorem 3.6 to larger space $L^\infty(T)$.

We first give a standard lemma.

Lemma 3.9. Suppose $F \in \tilde{L}^2(T)$, then the following PDE:
\[
\begin{cases}
\partial_t u - \nabla \cdot (a \nabla u) = \nabla \cdot F & \text{in } (0, T) \times \mathbb{R}^d, \\
u(0) = \phi \in \tilde{L}^2.
\end{cases}
\] (3.25)

admits a unique weak solution $u \in \tilde{V}^0(T)$ and
\[
\|u\|_{\tilde{V}(T)} \leq \|u(0)\|_{\tilde{L}^2} + C\|F\|_{\tilde{L}^2(T)}.
\]

Proof. The proof is quite standard, here we prove the apriori estimate for reader's convenience. Take test function $\phi = u \eta^2$, where $\eta$ is the same cut off function in the proof of Theorem 3.4. By basic calculations and Hölder's inequality, we obtain that for almost every $s, t \in [0, T],
\[
(\int_{\mathbb{R}^d} u^2 \eta^2_k(t) \right) - (\int_{\mathbb{R}^d} u^2 \eta^2_k(s)) + \int_s^t \int_{\mathbb{R}^d} \nabla (u \eta_k)^2 
\leq C \int_s^t \int_{\mathbb{R}^d} u \nabla \eta_k^2 + C \int_s^t \int_{\mathbb{R}^d} \nabla^2 (|\eta_k|^2 + |\nabla \eta_k|^2).
\]

Thus,
\[
\|u\|^2_{\tilde{V}(T)} \leq \sup_{x \in \mathbb{R}^d} \left[ \sup_{t \in [0, T]} \left( \int_{\mathbb{R}^d} u^2 \eta^2_k(t) + \int_0^t \int_{\mathbb{R}^d} \nabla (u \eta_k)^2 \right) \right] 
\leq \|u(0)\|^2_{\tilde{L}^2} + C\|F\|^2_{\tilde{L}^2(T)} + C \int_0^\tau \|u\|^2_{\tilde{L}^2(t)} \, dt.
\]

Gronwall's inequality yields
\[
\|u\|_{\tilde{V}(T)} \leq \|u(0)\|_{\tilde{L}^2} + C\|F\|_{\tilde{L}^2(T)}.
\]

Now we extend the uniqueness result of Theorem 3.6 to larger space $L^\infty(T)$. Our proof mainly follows [6].

Theorem 3.10. Suppose $a, b$ satisfy $(A_1)$, $(A_2)$, for any $\phi \in L^\infty$, (FPE)$_2$ has a unique solution $u \in \tilde{V}^0(T) \cap L^\infty(T)$. If moreover, $a$ satisfies $(A_3)$, then uniqueness also holds in $L^\infty(T)$. In particular, any $L^\infty(T)$ distributional solution of (FPE)$_2$ with bounded initial value belongs to $\tilde{V}^0(T) \cap L^\infty(T)$.

Proof. Suppose $u \in L^\infty(T)$ is a distributional solution to (FPE)$_2$, then
\[
\partial_t u - \nabla \cdot (a \nabla u) = -\nabla (Vu), \quad u(0) \in L^\infty
\]
Notice that \( Vu \in \tilde{\mathbb{L}}^2(T) \), by Lemma 3.9, there exists \( \tilde{u} \in \tilde{V}^0(T) \) solves the above equation, with the same initial condition. Let us define \( g := \tilde{u} - u \), \( Ag := \nabla \cdot (a \nabla g) \). \( g \in \tilde{\mathbb{L}}^2(T) \) is a distributional solution to equation

\[
\partial_t g - Ag = \partial_t g - \nabla \cdot (a \nabla g) = 0, \quad g(0) = 0. \tag{3.26}
\]

Here \( \nabla \cdot (a \nabla g) \) should be read by \( \partial_i j (a^i j g) + \partial_j (a^j i g) \). Assume \( w \in \tilde{\mathbb{H}}^{1,2}_2(T) \) solves

\[
\lambda w - Aw = \lambda w - \nabla \cdot (a \nabla w) = g, \quad \lambda > 0. \tag{3.27}
\]

in \([0, T] \times \mathbb{R}^d\). Multiple the above equation by \( w \eta^2 \), integrate on \([0, t] \times \mathbb{R}^d\) obtaining

\[
\lambda \int_0^t \int_{\mathbb{R}^d} w^2 \eta^2_k + \frac{1}{\lambda} \int_0^t \int_{\mathbb{R}^d} |\nabla w \eta_k|^2 \leq C \int_0^t \int_{\mathbb{R}^d} (w \nabla \eta_k)(\nabla w \eta_k) + \int_0^t \int_{\mathbb{R}^d} (g \eta_k)(w \eta_k)
\]

\[
\leq C \|w\|_{\tilde{\mathbb{L}}_2(T)}^2 + \frac{1}{2\lambda} \|\nabla w \eta_k\|_{L^2(T)} + \|g\|_{\tilde{\mathbb{L}}_2(T)}^2,
\]

this yields that there is a constant \( \lambda_0 > 0 \) such that for any \( \lambda \geq \lambda_0 \),

\[
\lambda \|w\|_{\tilde{\mathbb{L}}_2(T)} + \|\nabla w\|_{\tilde{\mathbb{L}}_2(T)} \leq C \|g\|_{\tilde{\mathbb{L}}_2(T)}.
\]

This estimate implies that for any \( \lambda \geq \lambda_0 \), there is a unique solution \( w =: G_\lambda g \in \tilde{\mathbb{H}}^{1,2}_2(T) \), here \( G_\lambda \) is the solution map of (3.27). It is also easy to verify that \( G_\lambda \) is also bounded from \( \mathbb{L}^2(T) \) to \( \mathbb{H}^{1,2}_2(T) \) and

\[
\lambda \|G_\lambda g\|_{\mathbb{L}^2(T)} + \|\nabla G_\lambda g\|_{\mathbb{L}^2(T)} \leq C \|g\|_{\mathbb{L}^2(T)} \tag{3.28}
\]

By (3.26), we have

\[
0 = \partial_t G^{-1}_\lambda w - AG^{-1}_\lambda w = G^{-1}_\lambda (\partial_t w - Aw) + [\partial_i, G^{-1}_\lambda] w,
\]

thus formally

\[
\partial_t w - Aw = G_\lambda \{[G^{-1}_\lambda, \partial_i] w\} = G_\lambda [\nabla \cdot (\partial_i a \nabla w)] \tag{3.29}
\]
in the sense of distribution. One can find the rigorous proof for (3.29) in [6]. Like before, multiplying (3.29) by \(w\eta^2\), integrating on \([0, t] \times \mathbb{R}^d\), using Hölder’s inequality and (3.28), we obtain

\[
\frac{1}{2} \int_{\mathbb{R}^d} |w(t)\eta|^2 + \frac{1}{\Lambda} \int_{0}^{t} \int_{\mathbb{R}^d} |\nabla w \cdot \eta|^2 \\
\leq \int_{0}^{t} \int_{\mathbb{R}^d} \nabla \cdot (\partial_t a \nabla w) [G_\lambda (w \eta^2)] = - \int_{0}^{t} \int_{\mathbb{R}^d} \partial_t a \nabla w \cdot \nabla [G_\lambda (w \eta^2)] \\
\leq \|\partial_t a\|_{L^\infty} \sum_{z \in \mathbb{Z}^d / 2} \int_{B_{1/2}(z)} \nabla w \cdot \nabla [G_\lambda (w \eta^2)] \\
\leq C \sum_{z \in \mathbb{Z}^d / 2} \left( \int_{0}^{t} \int_{B_{1/2}(z)} |\nabla w \cdot \eta|^2 \right)^{1/2} \left( \int_{0}^{t} \int_{\mathbb{R}^d} |\nabla [G_\lambda (w \eta^2)]|^2 \right)^{1/2} \\
\leq C \left( \sup_{z \in \mathbb{Z}^d / 2} \int_{0}^{t} \int_{\mathbb{R}^d} |\nabla w \cdot \eta|^2 \right)^{1/2} \cdot \left( \int_{0}^{t} \int_{\mathbb{R}^d} |\nabla [G_\lambda (w \eta^2)]|^2 \right)^{1/2} \\
\leq C \left\| \nabla (w\eta) \right\|_{L^2_t(t)} \left\| w\eta^2 \right\|_{L^2_t(t)} \\
\leq \frac{1}{2\Lambda} \sup_{x \in \mathbb{R}^d} \left\| \nabla (w\eta) \right\|_{L^2_t(t)}^2 + C \left\| w \right\|_{L^2_t(t)}^2.
\]

In the first inequality, we use the fact that \(G_\lambda\) is a symmetric operator in \(L^2\) space. Taking supremum over \(x \in \mathbb{R}^d\) on the left side of above inequalities, we get

\[
\left\| w(t) \right\|_{L^2} \leq C \int_{0}^{t} \left\| w(s) \right\|_{L^2}^2 ds, \quad t \in [0, T].
\]

Gronwall’s inequality yields \(w \equiv 0\) and hence \(g \equiv 0\).

\[\square\]

4. PROOF OF MAIN RESULTS

In this section, we give the proofs for our main results. Before that, let us list some conclusions in [29] and [19] (see also [6]).

**Proposition 4.1** (cf. [29]). Assume \(a, b\) satisfy \((A_1)\) and \((A_2)\), then for each \(\mu_0 \in \mathcal{P}(\mathbb{R}^d)\), there exists at least one martingale solution associated with \((L, \mu_0)\), say \(\mathbb{P}\), which satisfies the following Krylov’s type estimate: for any \(p, q \in [2, \infty)\) with \(\frac{d}{p} + \frac{2}{q} < 2\), there exist \(\theta = \theta(p, q) > 0\) and a constant \(C > 0\) such that for all \(0 \leq t_0 < t_1 \leq T\) and \(f \in C^0_c(\mathbb{R}^{d+1})\),

\[
\mathbb{E}^\mathbb{P} \left( \int_{t_0}^{t_1} f(t, \omega_t) dt \bigg| B_{t_0} \right) \leq C(t_1 - t_0)^\theta \| f \|_{\mathcal{E}^p_\theta(T)}.
\]

(4.1)
Define
\[ \mathcal{L}_+ := \left\{ \mu : [0, T] \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d) : \int_0^T \int_{\mathbb{R}^d} (|a(t,x)| + |b(t,x)|) \mu_t(dx)dt < \infty \right\}, \]

\[ \mu_t = \rho_t \lambda_d, \rho_t \in L^\infty \text{ uniformly for } t \in [0, T], \]

and for any \( \phi \in C_b(\mathbb{R}^d), t \mapsto \int_{\mathbb{R}^d} \phi d\mu_t \) is continuous.

The following two Propositions are consequences of [19, Theorem 2.5] and [19, Lemma 2.12] respectively.

**Proposition 4.2.** Suppose \( \{\mu_t\}_{t \in [0, T]} \in \mathcal{L}_+ \), then there exists \( \mathbb{P} \in \mathcal{P}(C([0,T];\mathbb{R}^d)) \) which is a solution to the MP associated to the diffusion operator \( L \) such that, for every \( t \in [0, T] \), it holds \( \mu_t = \mathbb{P} \circ \omega_t^{-1} \).

**Proposition 4.3.** Assume that forward uniqueness for the \( (\text{FPE}_1) \) hold in the class \( \mathcal{L}_+ \) for any initial time. Then, for any \( \mu_0 = \rho_0 \lambda_d \in \mathcal{P}(\mathbb{R}^d) \) with \( \rho_0 \in L^\infty \), the \( \mu_0 - \text{SLF} \) is uniquely determined \( \mu_0 - \text{a.e.} \).

**Lemma 4.4.** Under Assumption 1, assume that \( \mu_0 = \rho_0 \lambda_d \in \mathcal{P}(\mathbb{R}^d) \) with \( \rho_0 \in L^\infty \), then equation \( (\text{FPE}_1) \) admits a unique solution in \( \mu \in \mathcal{L}_+ \).

**Proof.** The uniqueness follows from Theorem 3.10, so we only need to show the existence. We prove this by using probabilistic method. Let \( a_n, V_n \) be the same functions in the proof of Theorem 3.6, then we can find a collection of probability measures \( \{\mathbb{P}^n\}_{n \in \mathbb{N}} \) on \( C([0,T];\mathbb{R}^d) \) such that \( \mathbb{P}^n \) is the unique martingale solution associated to \( L^n := a_n^{ij} \partial_{ij} + b_n^i \partial_i \) with initial data \( \mu_0 \). For any stopping time \( \tau, \delta > 0 \) with \( \tau + \delta \leq T \), thanks to (4.1), we have
\[
\sup_{n \in \mathbb{N}} \mathbb{E}^n \left| \int_\tau^{\tau+\delta} |b^n|(s, \omega_s)ds \right| \leq C \delta^\theta \|b\|_{\mathcal{L}_+^\infty(T)}.
\]

Using above estimate and BDG inequality, we get
\[
\mathbb{E}^n \left( \sup_{0 \leq s \leq \delta} |\omega_{\tau+s} - \omega_\tau| \right) \leq \mathbb{E}^n \left( \sup_{0 \leq s \leq \delta} \left| \int_{\tau}^{\tau+\delta} |b^n|(t, \omega_t)dt \right| + \mathbb{E}^n \left( \sup_{0 \leq s \leq \delta} \left| \int_{\tau}^{\tau+\delta} \sqrt{2a^n(t, \omega_t)}dW_t \right| \right) \leq C(\delta^\theta + \delta^{1/2}),
\]
where \( C \) is independent of \( n \). Thus by [28, Lemma 2.7], we obtain
\[
\sup_n \mathbb{E}^n \left( \sup_{|s-t| \leq \delta} |\omega_t - \omega_s|^{1/2} \right) \leq C(\delta^\theta + \delta^{1/2}).
\]

From this, by Chebyshev’s inequality, we derive that for any \( \varepsilon > 0 \),
\[
\lim_{\delta \to 0} \sup_n \mathbb{P}^n \left( \sup_{|s-t| \leq \delta} |\omega_t - \omega_s| > \varepsilon \right) = 0.
\]
Hence, \( \{\mathbb{P}^n\} \) is tight in \( \mathcal{P}(C([0,T];\mathbb{R}^d)) \). Suppose \( \mathbb{P} \) is an limit point of \( \{\mathbb{P}^n\} \), then for each \( t \in [0, T] \), \( \mu_t^n := \mathbb{P}^n \circ \omega_t^{-1} \Rightarrow \mathbb{P} \circ \omega_t^{-1} := \mu_t \) as \( n \to \infty \) along an appropriate subsequence. For each
Notice that the map $f \in B$ by Theorem 3.7, we obtain that $0 \leq \rho_t^n(x) \to \rho_t(x)$ for almost everywhere $(t, x) \in [0, T] \times \mathbb{R}^d$, where $\rho_t$ is the unique solution to (FPE$_2$) with $\phi = \rho_0$ in class $L^\infty(T)$ (or $\tilde{V}^0(T) \cap L^\infty(T)$). Moreover,

$$||\rho||_{\mathcal{L}^\infty(T)} \leq \sup_{n \in \mathbb{N}} ||\rho_n||_{\mathcal{L}^\infty(T)} < \infty.$$ 

Lebesgue’s dominated convergence theorem yields that for each $f \in C_b(\mathbb{R}^d)$ and almost every $t \in [0, T]$,

$$\int_{\mathbb{R}^d} f \rho_t = \lim_{n \to \infty} \int_{\mathbb{R}^d} f \rho_t^n = \lim_{n \to \infty} \int_{\mathbb{R}^d} f \mu_t^n = \int_{\mathbb{R}^d} f \mu_t.$$

Notice that the map $[0, T] \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)$ is continuous, so for any $t \in [0, T]$,

$$\sup_{\|f\|_{L^1}=1} \int_{\mathbb{R}^d} f \mu_t \leq \sup_{\|f\|_{L^1}=1} \text{esssup}_{t \in [0, T]} \int_{\mathbb{R}^d} f \rho_t \leq ||\rho||_{\mathcal{L}^\infty(T)} \leq C.$$

Thus, $\mu_t = \mathbb{P} \circ \omega_{\lambda_t}^{-1} \in \mathcal{L}_+.$

**Proof of Theorem 2.4.** (1). If $m_0$ is a probability measure, then the uniqueness of $m_0$–SLF is a consequence of Lemma 4.4 and Proposition 4.3. For arbitrary $m_0 \in \mathcal{M}(\mathbb{R}^d)$, one can find a probability measure $\mu_0$ such that $\mu_0(dx) = \rho'(x)m_0(dx)$ and $0 < \rho < C < \infty$, $m_0$–a.e.. Notice that each $m_0$–SLF is a $\mu_0$–SLF, by the uniqueness of $\mu_0$–SLF and the fact $m_0 \ll \mu_0$, we obtain the uniqueness of $m_0$–SLF.

For the existence, we only need to prove the case $m_0 = \lambda_d$. Let $\rho(x) = e^{-|x|^2/2}$. $m_0^k(dx) := \rho(x/k)dx \in \mathcal{M}(\mathbb{R}^d)$, $\mu_0^k(dx) := (2\pi k^2)^{-d/2} \rho(x/k)dx \in \mathcal{P}(\mathbb{R}^d)$. The existence of $\mu_0^k$–SLF (or $m_0^k$–SLF) is a consequence of Proposition 4.2 and Lemma 4.4. Suppose $\{\mathbb{P}_k^x\}_{x \in \mathbb{R}^d}$ is a $\mu_0^k$–SLF, notice that for any $k, k' \in \mathbb{N}$, $\lambda_d \ll m_0^k \ll m_0^{k'}$, by the uniqueness result proved above, we obtain that $\mathbb{P}_k^x = \mathbb{P}_k^x$ for all $k \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^d$. Thus, by the definition of $m_0^k$–SLF, for each $k$,

$$m_0^k := \int_{\mathbb{R}^d} \mathbb{P}_x \circ \omega_{\lambda_d}^{-1} m_0^k(dx)$$

has a bounded density with respect to $\lambda_d$, say $\rho^k$. $m_0^k(dx) = \rho^k(x)dx$ is the unique $\mathcal{L}_+$–solution to (FPE$_1$) with initial value $m_0^k(dx) = \rho(x/k)dx$. By Theorem 3.4 and Theorem 3.10,

$$\sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} \|\rho^k_t\|_{\mathcal{L}^\infty} \leq C \sup_{k \in \mathbb{N}} \|\rho(\cdot/k)\|_{\mathcal{L}^\infty} \leq C.$$

Hence, for any $A \in \mathcal{B}(\mathbb{R}^d)$, $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \mathbb{P}_x \circ \omega_{\lambda_d}^{-1}(A) \mathbb{P}_x = \lim_{k \to \infty} \int_{\mathbb{R}^d} \mathbb{P}_x \circ \omega_{\lambda_d}^{-1}(A) \rho(x/k)dx$$

$$= \lim_{k \to \infty} m_0^k(A) = \lim_{k \to \infty} \rho_t^k \leq C \lambda_d(A),$$

where $\lambda_d(A) := \text{meas} A$.
which implies \( \{P_x\}_{x \in \mathbb{R}^d} \) is also an SLF.

(2) Suppose \( \{P_x\}_{x \in \mathbb{R}^d} \) is the SLF associated with \( L \). Then for any \( \mu_0 \in \mathcal{P}(\mathbb{R}^d) \) with \( \mu_0 \ll \lambda_d \), \( \mathbb{P} := \int P_x \mu_0(\mathrm{d}x) \) is a martingale solution associated with \( (L, \mu_0) \). Now suppose \( \mathbb{P} \) is a martingale solution associated with \( (L, \mu_0) \) and \( \mu_t := \mathbb{P} \circ \omega_t^{-1} \mathbb{P} \) with \( \rho_t \in L^\infty \) uniformly in \( t \). Let \( \{Q_x\}_{x \in \mathbb{R}^d} \subseteq \mathcal{P}(\mathbb{R}^d) \) be the regular conditional distribution given by \( \omega_0 = x \). By [17, Theorem 6.1.2] for \( \mu_0 \)-a.e. \( x, Q_x \) is a martingale solution to corresponding to \( (L, \delta_x) \). Notice that

\[
\int_{\mathbb{R}^d} Q_x \circ \omega_t^{-1} \mu_0(\mathrm{d}x) = \left( \int_{\mathbb{R}^d} Q_x \mu_0(\mathrm{d}x) \right) \circ \omega_t^{-1} = \mathbb{P} \circ \omega_t^{-1} = \mu_t,
\]

we get \( \{Q_x\}_{x \in \mathbb{R}^d} \) is a \( \mu_0 \)-SLF. The uniqueness of \( \mathbb{P} \) follows by the uniqueness of \( \mu_0 \)-SLF. \( \square \)

Remark 4.5. If \( m_0(\mathrm{d}x) = \rho_0(x) \mathrm{d}x \in \mathcal{M}(\mathbb{R}^d) \) with \( 0 < \rho_0 \leq C < \infty \) and \( \rho_0 \in C(\mathbb{R}^d) \), by the proof of Theorem 2.4, one can see that under Assumption 1, any \( m_0 \)-SLF is an SLF and vice versa.

Next we state a lemma about the maximum functions. One can find its proof in [26, Lemma 3.6] and [15].

**Lemma 4.6.** (i) Let \( f \in W^{1,1}_{loc}(\mathbb{R}^d) \), \( \rho_n(x) := n^d \rho(x/n) \in C^\infty_c(\mathbb{R}^d) \) with \( \int \rho = 1 \). For almost every \( x, y \in \mathbb{R}^d \) with \( |x - y| \leq \sqrt{\varepsilon} \ll 1 \),

\[
\frac{|f(x) - f(y)|}{\sqrt{|x - y|^2 + \varepsilon^2}} \leq 2^d (F_{f,n}(x) + F_{f,n}(y)),
\]

where \( F_{f,n} \) is a function depends on \( f, \rho, \varepsilon, n \). And there is a constant \( C = C(\rho, d) \),

\[
\int_{B_r} F_{f,n}(x) \mathrm{d}x \leq Cn^d \|\nabla f\|_{L^1(B_{r+1})} + \log \varepsilon^{-1} \|\nabla (f_n - f)\|_{L^1(B_{r+1})}, \tag{4.2}
\]

(ii) For any \( p > 1, r, R > 0 \),

\[
\int_{B_r} (M_R f(x))^p \mathrm{d}x \leq C_{d, p} \int_{B_{R+r}} |f(x)|^p \mathrm{d}x \tag{4.3}
\]

Now we are on the point to prove Theorem 2.7. Instead of proving an stability result for the approximation solutions of (1.1), we first prove the pathwise uniqueness of (1.1) if \( \xi \) has a bounded density, then using an Yamada-Watanabe type argument (cf. [22]) we show the existence of AESF.

**Lemma 4.7.** Suppose \( b, \sigma \) satisfy Assumption 2, \( \xi \in \mathcal{F}_0 \) is a random variable with bounded density. Assume \( X_t, Y_t \) are two strong solutions of (1.1) whose one dimensional distributions have uniformly bounded densities, then we have \( X = Y \) a.s.

**Proof.** For any \( \varepsilon > 0 \), let \( \phi_{\varepsilon} \) be a increasing smooth function on \([0, \infty)\),

\[
\phi_{\varepsilon}(s) = \begin{cases} s & s \in [0, \varepsilon/2] \\ \varepsilon & s \in [\varepsilon, \infty) \end{cases}
\]

and \( \phi_{\varepsilon}'(s) \leq C1_{[0,\varepsilon]}(s), \phi_{\varepsilon}''(s) \leq Ce^{-1}1_{[\varepsilon, \infty]}(s) \).

\[
\Phi_{\varepsilon}(z) := \log \left( 1 + \frac{\phi_{\varepsilon}(|z|^2)}{\varepsilon^2} \right), \quad Z_t := X_t - Y_t.
\]
Then,

\[ |\partial_i \Phi(z)| = \left| \frac{2\phi'_e(|z|^2)zi}{\varepsilon^2 + \phi_e(|z|^2)} \right| \leq \frac{C_1 \min\{|z|, \sqrt{\varepsilon}\}}{\varepsilon^2 + |z|^2}, \]

\[ |\partial_j \Phi(z)| = \left| \frac{2\phi'_e(|z|^2)\delta_{ij}}{\varepsilon^2 + \phi_e(|z|^2)} + \frac{4\phi''_e(|z|^2)zi^2j}{\varepsilon^2 + \phi_e(|z|^2)} - \frac{4\phi''_e(|z|^2)zi_j}{\varepsilon^2 + \phi_e(|z|^2)} \right| \leq \frac{C_1 \min\{|z|, \sqrt{\varepsilon}\}}{\varepsilon^2 + |z|^2}. \]

Denote \( \tau_R := \inf\{t > 0 : |X_t| > R, |Y_t| > R\} \). By Itô’s formula and Lemma 4.6,

\[
E \Phi_e(Z_{t \wedge \tau_R}) = \int_0^{t \wedge \tau_R} E \left[ \partial_i \Phi_e(Z_s) \cdot (b^i(s, X_s) - b^i(s, Y_s)) \right] ds \\
+ \frac{1}{2} \int_0^{t \wedge \tau_R} E \partial_{ij} \Phi_e(Z_s) \left[ (\sigma^j(s, X_s) - \sigma^j(s, Y_s)) \cdot (\sigma^k(s, X_s) - \sigma^k(s, Y_s)) \right] ds \\
\leq 2E \int_0^{t \wedge \tau_R} \frac{|b(s, X_s) - b(s, Y_s)|}{\varepsilon + |X_s - Y_s|^2} ds \\
+ CE \int_0^{t \wedge \tau_R} \frac{|\sigma(s, X_s) - \sigma(s, Y_s)|^2}{\varepsilon + |X_s - Y_s|^2} ds
\]

\[
\leq CE \int_0^{t \wedge \tau_R} [F_{e,n}(s, X_s) + F_{e,n}(s, Y_s)] ds + \\
+ CE \int_0^{t \wedge \tau_R} [M|\nabla \sigma|(s, X_s) + M|\nabla \sigma|(s, Y_s)] ds =: I_1(\varepsilon) + I_2,
\]

where \( F_{e,n}(s, x) = F_{e,n}^b(s)(x) \) in Lemma 4.6. Let \( \rho^X_t, \rho^Y_t \) be the density of \( X_t \) and \( Y_t \) respectively, then

\[
I_2 \leq C \int_0^{t \wedge \tau_R} E \left[ (M_R|\nabla \sigma|(X_s))^2 + (M_R|\nabla \sigma|(Y_s))^2 \right] ds \\
\leq C \int_0^t \int_{B_R} [M_R|\nabla \sigma|(s, x)]^2 (\rho^X_s(x) + \rho^Y_s(x)) dx ds \\
\leq C \int_0^t \int_{B_{2R}} |\nabla \sigma|^2(s, x) dx ds \leq C.
\]

For \( I_1(\varepsilon) \), by (4.2),

\[
I_1(\varepsilon) \leq C \int_0^t \int_{B_R} F_{e,n}(s, x)(\rho^X_s + \rho^Y_s) dx ds \\
\leq Cn^d \int_0^t \int_{B_{R+1}} |\nabla b(s, x)| dx ds + C|\log \varepsilon| \int_0^t \int_{B_{R+1}} |\nabla b(s, x) - \nabla b_n(s, x)| dx ds.
\]

Thus,

\[
E \Phi_e(Z_{t \wedge \tau_R}) \leq C(1 + n^d \|\nabla b\|_{L^1([0,t] \times B_{R+1})}) + C|\log \varepsilon| \|\nabla b - \nabla b_n\|_{L^1([0,t] \times B_{R+1})}.
\]

By Chebyshev’s inequality,

\[
P \left( |X_t - Y_t| > \sqrt{\varepsilon} ; t \leq \tau_R \right)
\]
Theorem 6.1.2, for a.e. $\Omega$ is defined in the same way. Denote

Let $\varepsilon(x)$, for $x \in X$, the map $\varepsilon(x)$ be uniformly bounded density. Thus, pathwise uniqueness yields $\hat{\psi} = \hat{\psi}'$.

Notice $X$, $Y$ are both continuous processes, we obtain

$$\mathbb{P}(\{X_t = Y_t \text{ a.s.}\}) = 0.$$ 

Proof of Theorem 2.7. Let $\mu_0(dx) = (2\pi)^{-d/2}e^{-|x|^2/2}dx$. By Remark 4.5, we only need to prove the existence and uniqueness of $\mu_0 - \text{AESF}$ associated to SDE (1.1). According to Proposition 4.1, there exists at least one weak solution (martingale solution), say $(\hat{X}, \hat{\omega})$ to (1.1) with law $\hat{\mu}_t = \mu_0$ and $\hat{p}_t := d\hat{P} \circ X_t^{-1}/d\lambda_t$ is uniformly bounded on $[0, T] \times \mathbb{R}^d$. Suppose $(\hat{X}', \hat{\omega}')$ is another weak solution to (1.1) and the one-dimensional distribution of $\hat{X}'$ is also uniformly bounded. Let $Q(x, w; d\omega)$ be the regular conditional distribution of $X$ given $(\hat{X}_0, \hat{\omega}) = (x, w)$ and $Q'(x, w; d\omega')$ is defined in the same way. Denote $\Omega := C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathbb{R}^m)$,

$$Q(d\omega, d\omega', dw) := \int_{\mathbb{R}^d} Q(x, w; d\omega) \times Q'(x, w; d\omega') \mu_0(dx) \eta(dw),$$

where $\eta$ is the Wiener on $C([0, T]; \mathbb{R}^m)$. Let $\mathcal{F}_T^0 = B(C([0, T]; \mathbb{R}^d)) \times B(C([0, T]; \mathbb{R}^d)) \times B(C([0, T]; \mathbb{R}^m))$, $\mathcal{N} = \bigcup_{t \geq 0} (\mathcal{F}_T^0 \lor \mathcal{N}_t)$. Suppose that $(\omega, \omega', w)$ is the canonical process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then $(\omega, w)$ and $(\omega', w)$ have the same distributions as $(X, W)$ and $(X', W')$, respectively. Moreover, $w$ is an $\mathcal{F}_t$-Brownian motion under $\mathbb{Q}$ (see [7, Lemma 1.2, Chapter IV]). Under the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, $(\omega, w)$ and $(\omega', w)$ are two solutions of (1.1). And $\mathbb{Q} \circ \omega_t^{-1}, \mathbb{Q} \circ \omega'_t^{-1}$ both enjoy uniformly bounded density.

Thus, pathwise uniqueness yields $\mathbb{Q}(\omega = \omega') = 1$, which implies

$$Q(x, w; \omega = \omega') = 1, \quad \mu_0 \times \eta - \text{a.s.} (x, w).$$

Hence, there exists a measurable map $\psi(x, w)$ such that for $\mu_0 \times \eta - \text{a.s.}$ $(x, w)$,

$$Q(x, w; \omega = \psi(x, w)) = Q'(x, w; \omega' = \psi(x, w)) = 1,$$

i.e.

$$Q(x, w; B) = 1_B(\psi(x, w)), \quad \forall B \in B(C([0, T]; \mathbb{R}^d)).$$

Moreover, for a.e. $x$, the map $w \mapsto \psi(x, w)$ is $B(C([0, T]; \mathbb{R}^m))/B(C([0, T]; \mathbb{R}^d))$-measurable (see [7, Lemma 1.1, Chapter IV]). Recalling that $Q(x, w; \cdot)$ is the regular conditional probability of $\omega$ given $(\omega_0 = x, w)$, so $\int Q(x, w; \cdot) \eta(dw)$ is the regular conditional probability of $\omega$ given $\omega_0 = x$. Notice that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a martingale solution to (1.1) with initial distribution $\mu_0$, by [17, Theorem 6.1.2], for a.e. $x$ the probability measure

$$\mathbb{B}(C([0, T]; \mathbb{R}^d)) \ni B \mapsto \int Q(x, w; B) \eta(dw) = \int 1_B(\psi(x, w)) \eta(dw) = \eta \circ \psi^{-1}(x, \cdot)(B)$$

is a martingale solution to (1.1) with initial data $\xi = x$. Thus, given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a standard Brownian motion $W$ on it, for a.e. $x \in \mathbb{R}^d$,

$$(X(x) := \psi(x, W), W)$$
is a strong solution to (1.1) with initial datum $\xi = x$. Moreover, for any $A \in B(\mathbb{R}^d)$

$$\mu_t(A) := \int_{\mathbb{R}^d} \mathbf{P} \circ X^{-1}_t(x)(A) \mu_0(dx)$$

$$= \int_{\mathbb{R}^d} \mu_0(dx) \int Q(x, w; \omega_t \in A) \eta(dw) \leq C\lambda_d(A).$$

Thus, $\{X(x)\}_{x \in \mathbb{R}^d}$ is a $\mu_0$-AESF. The proof for uniqueness of AESF is essentially the same with the one of Lemma 4.7, so it is left to the readers. □

5. ILL-POSEDNESS IN WEAK SENSE

In this section, assume $d \geq 3$ and $\sigma = \|$. For any $p \in (d/2, d)$, we construction a divergence free drift $b \in \widetilde L^p(\mathbb{R}^d)$ such that (1.1) have at least two weak solutions starting from the origin.

**Theorem 5.1.** Let $d \geq 3$ and $\sigma = \|$. For any $p \in (d/2, d)$ with $p \geq 2$, there is a divergence free vector field $b \in \widetilde L^p(\mathbb{R}^d)$ such that the weak uniqueness of (1.1) fails.

**Proof.** Step 1. Let $\alpha \in (1, d/p) \subseteq (1, 2)$, $g$ be a nonnegative smooth even function on $\mathbb{R}$ such that $g'(x) \geq 0$ for all $x > 0$, and $g(x) \equiv 0$ if $|x| < 1/2$ and $g \equiv 1$ if $|x| > 1$. Denote

$$r = \left(\sum_{i=1}^{d-1} x_i^2\right)^{1/2}, \quad H(x) = r^{d-1} x_d^{-\alpha} g(x_d/r).$$

For any $x_d > 0$, we define

$$b_i(x) := -N x_i r^{1-d} \partial_{x_i} H(x), \quad i = 1, \cdots, d-1,$$

$$b_n(x) := N r^{2-d} \partial_n H(x)$$

and $b(x_1, \cdots, x_{n-1}, -x_d) := -b(x_1, \cdots, x_{n-1}, x_d)$. Basic calculations yield

$$b_i(x) = N \alpha (x_i x_d^{-\alpha-1}) g(x_d/r) - N r^{-1} x_i x_d^{-\alpha} g'(x_d/r), \quad x_d > 0, 1 \leq i \leq d-1,$$

and

$$b_d = N(d-1) x_d^{-\alpha} g(x_d/r) - N r^{-1} x_d^{-\alpha+1} g'(x_d/r), \quad x_d > 0.\quad (5.2)$$

Noticing that

$$|b(x)| \leq C x_d^{-\alpha} \mathbf{1}_{\{x \leq 2x_d\}}, \quad \forall x_d > 0$$

and $p < d/\alpha$, we have

$$\|b\|_{L^p(Q_1)}^p \leq C \int_0^1 x_d^{-\alpha p} dx \int_{\{r \leq 2x_d\}} dx_1 \cdots dx_{d-1} \leq C \int_0^1 x_d^{d-1-\alpha p} dx_d < \infty,$$

where

$$Q_p := \{x \in \mathbb{R}^d : r < \rho, |x_d| < \rho\}.$$ 

Moreover, by the definition of $b$, we also have $\sup_{x \notin Q_1} |b(x)| < \infty$, hence $b \in \widetilde L^p$. On the other hand, by basic calculations,

$$\partial_d b_d(x) = N r^{2-d} \partial_{x_d} H(x), \quad x_d > 0,$$

$$\partial_i b_i(x) = -N r^{1-d} + (1-d) x_i^2 r^{d-1} \partial_{x_i} H(x) + N x_i^2 r^{d-1} \partial_{x_d} H(x), \quad x_d > 0, 1 \leq i \leq d-1,$$
so we have \( \text{div}b \equiv 0 \).

We assume that the weak uniqueness of (1.1) holds for the above constructed \( b \) and \( P_x \in \mathcal{P}(C(\mathbb{R}_+; \mathbb{R}^d)) \) is the unique martingale solution to (1.1) with \( \xi \equiv x \).

Let \( f(x) = \text{sgn}(x_d)g(x_d) \) and
\[
F: C(\mathbb{R}_+; \mathbb{R}^d) \to \mathbb{R}; \quad \omega \mapsto \int_0^\infty e^{-t}f((\omega)_d)dt.
\]

By bounded dominated convergence theorem, \( F \) is a continuous map on \( C(\mathbb{R}_+; \mathbb{R}^d) \). Since \( b \) and \( f \) are anti-symmetric about \( x_d \), we have
\[
\mathbb{E}_xF(\omega) = 0, \quad \text{if } x_d = 0. \tag{5.3}
\]

Suppose that \( \mathbb{R}^d \ni x' \to 0 \), as shown in the proof of Lemma 4.4, \( \{P_x\}_x \) is tight in \( \mathcal{P}(C(\mathbb{R}_+; \mathbb{R}^d)) \) and its limit points are the martingale solutions to (1.1) with \( x = 0 \). Thanks to our uniqueness assumption, we have \( P_{x_n} \Rightarrow P_0 \), which implies
\[
\lim_{n \to \infty} \mathbb{E}_{x_n}F = \mathbb{E}_0F \tag{5.4}
\]
However, in our next step, we will find a sequence \( \mathbb{R}^d \ni x_n \to 0 \) such that
\[
\lim \inf_{n \to \infty} \mathbb{E}_{x_n}F \geq c_0 > 0 = \mathbb{E}_0F. \tag{5.5}
\]

And (5.4) and (5.5) indicate that our assumption is not true.

Step 2. Denote
\[
V_k = \{ x \in \mathbb{R}^d : x_d > kr, k > 0 \}, \quad \tau = \inf \{ t > 0 : \omega_t \not\in V_1 \}
\]
and
\[
T = \inf \{ t > 0 : (\omega_t)_d \leq 0 \}, \quad \sigma = \inf \{ t > 0 : (\omega_t)_d \geq 1 \}, \quad \sigma' = \inf \{ t > \sigma : (\omega_t)_d \leq 1 \}.
\]
Let \( \kappa \in (1, (d - 1)/\alpha) \), below we should prove that for some \( N \gg 1 \),
\[
\inf \{ P_x(\sigma < 1 \land \tau, \sigma' > 1 + \sigma) \} \geq p_N > 0. \tag{5.6}
\]

This is sufficient since \( \{P_x\}_{x \in \mathbb{R}^d} \) forms a strong Markov process (see [17]) and for any \( x \in V_k \cap Q_1 \),
\[
\mathbb{E}_xF(\omega) = \mathbb{E}_x \int_0^T e^{-t}f(\omega_t)dt + \mathbb{E}_x \left[ e^{-T} \left( \int_0^\infty e^{-t}f(\omega_t)dt \right) \circ \theta_T \right]
\geq \mathbb{E}_x \left[ \left\{ \sigma < 1 \land \tau, \sigma' > 1 + \sigma \right\} \int_\sigma^{\sigma'} e^{-t}dt \right]
\geq \mathbb{E}_x e^{-T} \mathbb{E}_{\omega_T} \int_0^\infty e^{-t}f(\omega_t)dt \overset{(5.3),(5.6)}{\geq} p_N(e^{-1} - e^{-2}) =: c_0 > 0.
\]

In order to show (5.6), we define
\[
\Omega_N := \left\{ \omega : |W_s(\omega) - W_t(\omega)| \leq N^{\frac{1}{1+\alpha}}|s - t|^{\frac{\alpha}{1+\alpha}}, \forall s, t \in [0, 1]\right\},
\]
where \( W_t := \omega_t - \omega_0 - \int_0^t b(\omega_s)ds \), which is a Brownian motion under \( P_x \). Noticing that \( 0 < 1/(1 + \alpha) < 1/2 \), we can choose \( N \gg 1 \) such that \( P_x(\Omega_N) \geq 1/2 \). Denote
\[
y_t = (\omega_t)_d - (W_t)_d, \quad \hat{\omega}_t = ((\omega)_1, \cdots, (\omega)_{d-1}), \quad \hat{x} = (x_1, \cdots, x_{d-1}).
\]
Notice that
\[ b_1(x) = N\alpha(x, x_d^{-\alpha - 1}), \quad b_n(x) = N(d - 1)x_d^{-\alpha}, \quad \forall x \in V_1, \]
we have
\[ y_t - x_d = N(d - 1) \int_0^t [y_s + (W_s)_d]^{-\alpha} ds \geq 0, \quad t \in [0, \tau], \]
\[ \hat{\omega}_t - \hat{x}_t = N\alpha \int_0^t \dot{\hat{x}}_s [y_s + (W_s)_d]^{-\alpha - 1} ds + \hat{W}_t, \quad t \in [0, \tau]. \tag{5.7} \]
For any \( x \in V_\kappa \cap Q_{\sqrt{N}} \), let
\[ t_x = \frac{1}{N-1} x_d^{1+\alpha} \in (0, 1). \tag{5.8} \]
For any \( t \in [0, t_x \wedge \tau] \) and \( \omega \in \Omega_N \), we have
\[ |W_t(\omega) - W_0(\omega)| \leq \varepsilon x_d \leq \varepsilon y_t, \tag{5.9} \]
where
\[ \varepsilon := N^{-\frac{1}{2(1+\alpha)}} \to 0 \quad (N \to \infty). \]
Thus,
\[ (1 + \varepsilon)^{-\alpha} N(d - 1)y_t^\alpha \leq \frac{dy_t}{dt} \leq (1 - \varepsilon)^{-\alpha} N(d - 1)y_t^\alpha, \quad \forall t \in [0, t_x \wedge \tau]. \]
Chaplygin’s Lemma yields
\[ \hat{y}_t := \left[ x_d^{1+\alpha} + (1 + \varepsilon)^{-\alpha} \right] \left\{ N(d - 1) \alpha + 1 \right\} \frac{1}{1+\alpha} \]
\[ \leq y_t \leq \left[ x_d^{1+\alpha} + (1 - \varepsilon)^{-\alpha} N(d - 1) \alpha + 1 \right] \frac{1}{1+\alpha} =: \tilde{y}_t, \quad \forall t \in [0, t_x \wedge \tau]. \tag{5.10} \]
This implies
\[ (\omega_t)_d = y_t - (W_t)_d \geq (1 - \varepsilon)y_t \geq (1 - \varepsilon)\tilde{y}_t. \tag{5.11} \]
Recalling that \( |\hat{x}| < x_d / \kappa \) and \( |\hat{\omega}| \leq (\omega_t)_d \) for all \( t \in [0, \tau] \), we have
\[ |\hat{\omega}_t| \leq |\hat{x}| + N\alpha \int_0^t \hat{\omega}_s [y_s + (W_s)_d]^{-\alpha - 1} ds + |W_t| \]
\[ \leq |\hat{x}| + \varepsilon y_t + N\alpha \int_0^t [y_s + (W_s)_d]^{-\alpha} ds \]
\[ \leq |\hat{x}| + \varepsilon y_t + \frac{\alpha}{d - 1} (y_t - x_d) \]
\[ \leq \left( \varepsilon + \frac{\alpha}{d - 1} \right) y_t + \left( \frac{1}{\kappa} - \frac{\alpha}{d - 1} \right) x_d \]
\[ \leq \left( \varepsilon + \frac{1}{\kappa} \right) y_t. \tag{5.12} \]
Hence, for any \( t \in [0, t_x \wedge \tau] \) and \( \omega \in \Omega_N \),
\[ \frac{(\omega_t)_d}{|\hat{\omega}|} \geq (1 - \varepsilon)(\varepsilon + 1 / \kappa)^{-1} > 1, \]
provided that $N$ is sufficiently large. So we get $\tau_{\omega_t} \leq \tau$ for all $\omega \in \Omega_N$. Moreover, for sufficiently large $N$, we also have $\tilde{y}_{t_x} \geq y_{t_x} \geq \tilde{y}_{t_x} \geq 2^{1/3}x_d$ and

\[
\frac{(X_{t_x})_{d}}{|X_{t_x}|} \geq \frac{(1 - \varepsilon)\tilde{y}_{t_x}}{(\varepsilon + \frac{\alpha}{d - 1})\tilde{y}_{t_x} + \left(1 - \frac{\alpha}{d - 1}\right)x_d} \geq \frac{(1 - \varepsilon)\tilde{y}_{t_x}}{\varepsilon + 2^{-1/3}k^{-1} + (1 - 2^{-1/3})(\frac{\alpha}{d - 1})} \tilde{y}_{t_x} \geq (5.10)^{\kappa},
\]

here we use the fact that $k < (d - 1)/\alpha$ and $\varepsilon \to 0$ as $N \to \infty$ in the last inequality. Thus,

$$\omega_t \in V_{\kappa} \text{ for each } \omega \in \Omega_N.$$ 

Now let $z^1 = x \in V_{\kappa} \cap Q_1$, $z^2 = \omega_{\sigma_1}$, $\cdots$, $z^n = \omega_{\sigma_{n-1}}$, $\cdots$.

Assume $\omega \in \Omega_N$ and $m(\omega)$ is an integer such that $\sum_{i=1}^{m(\omega)} t_{z_i(\omega)} < 1$ and $z_m \leq \sqrt[3]{N}$. Then, by the definition of $\Omega_N$ and the above discussion, we can see that for each $i = 1, \cdots, m(\omega)$, $z_i \in V_{\kappa} \cap Q_{\sqrt[3]{N}}$ and $\sum_{i=1}^{m(\omega)} t_{z_i(\omega)} < 1 \wedge \tau(\omega)$. From (5.9) and (5.10), we get

\[
c_1(z_d) := \left\{1 + (1 + \varepsilon)^{-\alpha}(d - 1)(\alpha + 1)\right\}^{\frac{1}{\alpha + 1}} - \varepsilon \right\}(z_d) \leq (z_d)_{d} \leq \left\{1 + (1 - \varepsilon)^{-\alpha}(d - 1)(\alpha + 1)\right\}^{\frac{1}{\alpha + 1}} + \varepsilon \right\}(z_d)_{d} =: c_2(z_d),
\]
for all $\omega \in \Omega_N$ and $i = 1, \ldots, m - 1$. Combing the above estimates and (5.8), we get
\[
\sum_{i=1}^{m} t_{d_i} = N^{-1} \sum_{i=1}^{m} (z_i^d)^{1+\alpha} \geq N^{-1} \sum_{i=1}^{m} (c_i^{d-1} x_d)^{1+\alpha}
\]
and
\[
\sum_{i=1}^{m} t_{d_i} = N^{-1} \sum_{i=1}^{m} (z_i^d)^{1+\alpha} \leq N^{-1} \sum_{i=1}^{m} (c_i^{d-1} x_d)^{1+\alpha}.
\]
Noticing that $c_2 \geq c_1 \rightarrow [1 + (d - 1)(\alpha + 1)]^{1/(\alpha+1)}$ and $c_1/c_2 \rightarrow 1$ as $N \rightarrow \infty$, one can choose sufficiently large $N$ and suitable $m$ such that
\[
\sum_{i=1}^{m} (c_i^{d-1} x_d)^{1+\alpha} < N (\text{which implies } \sum_{i=1}^{m} t_{d_i} < 1)
\]
and
\[
2 \leq c_1^{m-1} x_d \leq c_2^{m-1} x_d \leq \sqrt[3]{N}.
\]
Hence, for each $x \in V_\kappa \cap Q_1$ and $\omega \in \Omega_N$, we have $\sigma(\omega) < 1 \wedge \tau(\omega)$, which implies
\[
\mathbb{P}_x(\sigma < 1 \wedge \tau) \geq \mathbb{P}_x(\Omega_N) \geq 1/2.
\]
Since $b$ is uniformly bounded on the strip $S := \{x \in \mathbb{R}^d : 1 < x_d < 2\}$, we see
\[
\inf_{\{x \in \mathbb{R}^d : x_d = 2\}} \mathbb{P}_x(\sigma' > 1) \geq c_N > 0.
\]
Using the above two estimates and strong Markov property, we obtain that for all $x \in V_\kappa \cap Q_1$,
\[
\mathbb{P}_x(\sigma < 1 \wedge \tau, \sigma' > 1 + \sigma)
\]
\[
= \mathbb{P}_x(\sigma < 1 \wedge \tau, \sigma' \circ \theta_\sigma > 1)
\]
\[
\geq \mathbb{P}_x(\sigma < 1 \wedge \tau) \inf_{\{x \in \mathbb{R}^d : x_d = 2\}} \mathbb{P}_x(\sigma' > 1)
\]
\[
\geq c_N/2 =: p_N > 0.
\]
This proves (5.6). So we complete our proof. \qed

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GUOHUAN ZHAO: DEPARTMENT OF MATHEMATICS, BIELEFELD UNIVERSITY, GERMANY, EMAIL: zhaoguohuan@gmail.com