Potentials for (p,0) and (1,1) supersymmetric sigma models with torsion

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ABSTRACT

Using (1,0) superfield methods, we determine the general scalar potential consistent with off-shell (p,0) supersymmetry and (1,1) supersymmetry in two-dimensional non-linear sigma models with torsion. We also present an extended superfield formulation of the (p,0) models and show how the (1,1) models can be obtained from the (1,1)-superspace formulation of the gauged, but massless, (1,1) sigma model.
1. Introduction

The bosonic two-dimensional non-linear sigma-model with target space $\mathcal{M}$ is a field theory with fields $\phi$ that are maps from the two-dimensional Minkowski spacetime to a Riemannian manifold $\mathcal{M}$. The general action with second-order field equations is determined by three tensors on $\mathcal{M}$. These are (i) the metric tensor $g$, (ii) a closed three-form $H$, which is interpreted as a torsion tensor, and (iii) a scalar $V$. In light-cone coordinates $x^+, x^-$ for the two-dimensional spacetime (where the number of plus or minus signs indicates the $SO(1,1)$ charge) the action is

$$S = \int d^2x \left\{ \partial_+ \phi_i \partial_- \phi_j (g_{ij} + b_{ij}) - V(\phi) \right\}$$

(1.1)

where $b_{ij}$ is a locally defined potential for $H$: $H_{ijk} = (3/2) \partial_i b_{jk}$ ($i,j,k = 1 \cdots D, D = \dim \mathcal{M}$). When $V = 0$ the model is classically conformal invariant. Of particular interest, e.g. for applications to superstring theory, are the $(p,q)$-supersymmetric extensions of (1.1), which have $p$ spinor charges of one chirality and $q$ spinor charges of the other chirality.

Two-dimensional supersymmetric sigma models with $V \neq 0$ were first investigated about ten years ago [1] for $p = q$ and zero torsion as a means of introducing an infrared regulator into the massless models. In recent years $(2,2)$ models with $V \neq 0$ have been investigated in connection with Landau-Ginsberg formulations and integrable deformations of N=2 supersymmetric conformal field theory [2]. Of particular interest is the fact that the deformed models have solitons stabilized by a complex topological charge which appears in the supersymmetry algebra as a central charge. There also exist $(4,4)$-supersymmetric models with $V \neq 0$ with solitons carrying a quaternionic charge [3].

All two-dimensional supersymmetric sigma models, with or without potential, are special cases of the general $(1,0)$ model, which can be written in terms of $(1,0)$ scalar superfields $\{\phi^i(x, \theta^+); i = 1, \ldots, D\}$ and spinor superfields $\{\psi^a(x, \theta^+); a = 1, \ldots, n\}$. The superfields $\phi^i$ define a map $\phi$ from $(1,0)$ superspace, $\Sigma^{(1,0)}$, into $\mathcal{M}$. The superfield $\psi$ is a section of the vector bundle $S_- \otimes \phi^* \xi$ where $\xi$ is a vector bundle over $\mathcal{M}$ of rank $n$ and $S_-$ is the spin bundle over $\Sigma^{(1,0)}$. The $(1,0)$-superspace action without a potential has been given in previous work [4]. The generalization to include a potential is†

$$S = \int d^2x d\theta^+ \left\{ D_+ \phi^i \partial_- \phi^j (g_{ij} + b_{ij}) + i \psi^a \nabla_+ \psi^b_{-} h_{ab} + ims_a \psi^a \right\} ,$$

(1.2)

where $D_+$ is the spinor derivative satisfying $D_+^2 = i \partial_+$, $h_{ab}$ is a fibre metric of $\xi$ and

$$\nabla_+ \psi^b_{-} \equiv (D_+ \psi^b_{-} + D_+ \phi^i \Omega^b_{-c} \psi^c_{-} ) ,$$

(1.3)

where $\Omega(\phi)$ is a connection for $\xi$. The parameter $m$ is a constant with dimensions of mass and $s_a$ is a section of $\xi^*$. Without loss of generality one may choose a connection such that the fibre metric is covariantly constant, i.e. $\nabla_i h_{ab} = 0$.

† The zero of the energy can be shifted by an arbitrary constant by the inclusion in the superspace Lagrangian of a constant proportional to $\theta^+$, but this would introduce a Lorentz non-covariant central charge into the supersymmetry algebra so we omit it.
The component action corresponding to (1.2) can be obtained by standard methods. After elimination of auxiliary fields one finds that the potential is given in terms of $s$ by

$$V(\phi) = \frac{1}{4} m^2 h^{ab} s_a s_b . \quad (1.4)$$

We shall restrict $h_{ab}$ to be positive definite, in which case the potential is positive semi-definite and the structure group of the bundle $\xi$ is a subgroup of $O(n)$. The potential vanishes at the zeros of $s_a (\phi)$ which are therefore the classical 'vacua' of the model. For many models of interest $s_a$ will have isolated zeros.\textsuperscript{†} If there is more than one isolated zero then there will be soliton solutions interpolating between them, as happens for the specific (2,2) and (4,4) models discussed in [3].

In the $m = 0$ case a (1,0)-supersymmetric sigma-model will be invariant under (p,q) supersymmetry provided that the target space satisfies certain conditions that have a natural interpretation in the language of complex geometry [5]. Similarly for $m \neq 0$, the requirement of (p,q) supersymmetry will place restrictions on the couplings $g_{ij}$, $b_{ij}$, $h_{ab}$ and $s_a$. The purpose of this paper is to find and analyse these conditions for (p,0) and (1,1) supersymmetry. Results on some of the (2,0) models that we obtain have already appeared [6] during the preparation of this paper.

The (p,0) case can be analysed either in (1,0) superspace or by a straightforward extension of the extended superfield methods of ref.[7], because the (p,0) algebra cannot develop a central charge; we shall discuss both methods. The inclusion of a scalar potential in models with supersymmetry charges of both chiralities, of which the (1,1) models are the simplest examples, is complicated by the fact that to obtain the general scalar potential one must include the possibility of central charges. The (1,1) model will be analysed in both (1,0) superspace and (1,1) superspace. Standard (1,1) superfield techniques exclude the possibility of central charges and therefore fail to give the general scalar potential, so a new approach is presented here that makes use of the (1,1) superfield methods of [8] and the geometry of supersymmetric gauged sigma models developed in [9].

2. (p,0) Supersymmetry

The variation of (1.2) with respect to the arbitrary variations $\delta \phi^i$ and $\delta \psi^a_-$ of $\phi^i$ and $\psi^a_-$ is (up to a surface term)

$$\delta S = \int d^2 x d\theta^+ \left\{ \delta \phi^i S_{-i} + \Delta \psi^a_- S_a \right\} , \quad (2.1)$$

where

$$\Delta \psi^a_- \equiv \delta \psi^a_- + \delta \phi^i \psi^b_- \Omega^a_{i b} \quad (2.2)$$

is the covariantization of $\delta \psi^a_-$, and

$$S_{-i} \equiv -2 g_{ij} \nabla_{-} \phi^j - i \psi^a_- \psi^b_- D_+ \phi^j F_{ijab} + im \nabla_i s_a \psi^a_-$$

$$S^a \equiv 2i \nabla_+ \psi^a_- + ims^a . \quad (2.3)$$

\textsuperscript{†} Note that generic sections of vector bundles over compact manifolds have isolated zeros.
The covariant derivatives involve the connections with torsion given by \( \Gamma_{jk}^i = \Gamma_{jk}^i \pm H_{jk}^i \) where \( \Gamma \) is the standard Levi-Civita connection. Using (2.3) it is readily verified that the action (1.2) is invariant under the transformations

\[
\delta_\epsilon \phi^i = -\frac{i}{2} D_+ \epsilon = D_+ \phi^i + \epsilon = \partial_\mp \phi^i
\]

\[
\Delta_\epsilon \psi^a_- = -\frac{i}{2} D_+ \epsilon = \nabla_+ \psi^a_- + \epsilon = \nabla_\mp \psi^a_- \tag{2.4}
\]

for \( x \)-independent (but \( \theta \)-dependent) superfield parameter \( \epsilon \). These transformations are those of (1,0) supersymmetry together with translations in the \( x^\mp \) direction.

Any additional supersymmetries of (1.2) of the same chirality must have Noether charges that anticommute with the first one. This implies that the additional supersymmetry transformations can be expressed in terms of (1,0) superfields and a set of constant, anticommuting, parameter(s) \( \{ \eta_-^r; r = 1, \ldots, p - 1 \} \). The form of these transformations for \( m = 0 \) is fixed by dimensional analysis; when \( m \neq 0 \) we must allow for an additional fermion variation proportional to \( m \). We are thus led to consider

\[
\delta_\eta \phi^i = i \eta_-'^r \Gamma_{rj}^i (\phi) D_+ \phi^j
\]

\[
\Delta_\eta \psi^a_- = \frac{1}{2} \eta_-^r \Gamma_{r'b}^a (\phi) \mathbf{S}^b + \frac{im}{2} \eta_-^r G_r^a (\phi) \tag{2.5}
\]

where \( \Gamma_r \) are tensors on \( M \), and \( G_r \) and \( \hat{\Gamma}_r \) are sections of \( \xi^* \) (the dual of \( \xi \)) and \( \xi \otimes \xi^* \), respectively. The field equation term in the above transformations is not necessary for a determination of the conditions arising from on-shell closure of the (p,0) algebra (see, for example, the discussion of the massless (2,0) models in ref. [4]). However, this term is required for off-shell closure [7], and hence for an extended superspace formulation to be possible.

The conditions required for off-shell closure [7] on \( \phi \) are

\[
\Gamma_r \Gamma_s = -\delta_{rs} + f_{rs}^t \Gamma_t \tag{2.6}
\]

(in matrix notation) and

\[
N(\Gamma_r, \Gamma_s)^i_{jk} = 0 \tag{2.7}
\]

where \( f_{rs}^t \) is zero for \( p=2 \) and equal to the quaternion structure constants \( \epsilon_{rst} \) for \( p=4 \), and \( N \) is the generalised Nijenhuis tensor defined by

\[
N(\Gamma_r, \Gamma_s)^i_{jk} \equiv 2 [\partial_i \Gamma_r^i [k \Gamma_s^l_j] - \Gamma_r^i [k \partial_j \Gamma_s^l_j] + (r \leftrightarrow s)] . \tag{2.8}
\]

Off-shell closure on \( \psi^a_- \) requires

\[
F_{ij}^a \Gamma_r^i[k \Gamma_s^l_j] = F_{kl}^a \delta_{rs} \tag{2.9}
\]

and

\[
\hat{\Gamma}_i^a \psi^a_- \equiv -\delta_{rs} \delta^a_b + f_{rs}^t \hat{\Gamma}_t^a \tag{2.10}
\]
and
\[ I^j_r \nabla_j I^a_s_{\ b} - \hat{I}^a_r \nabla_i I^c_s_{\ b} + (r \leftrightarrow s) = 0 \, . \] (2.11)

The above (off-shell) conditions are sufficient when \( m = 0 \) [7]. When \( m \neq 0 \) we find the additional condition
\[ [I^i_r \nabla_i G^a_s + (r \leftrightarrow s)] + 2\delta_{rs} \nabla_i s^a = 0 \, . \] (2.12)

When \( m = 0 \) the conditions for invariance of the action are
\[ I^k_r (ig)_{jk} = 0 \quad \nabla^{(+)}_i I^j_r = 0 \] (2.13)

and
\[ \hat{I}_{r(ab)} \equiv h^c_{(a} \hat{I}^c_{r(b)} = 0 \, . \] (2.14)

When \( m \neq 0 \) we require additionally that
\[ (G^a_r s_a) = \text{constant} \] (2.15)

and
\[ \nabla_i G^a_r = I^j_r \nabla_j s^a \quad (r = 1 \ldots p - 1) \, . \] (2.16)

The integrability condition for (2.16) is (2.9). The condition (2.15) was stated previously for the (2,0) model but with the constant set equal to zero [6].

We now turn to the analysis of these conditions. We shall first recall the (previously established) results for \( m=0 \). For \( p=2 \) \( \mathcal{M} \) is a complex manifold with complex structure \( I \); \( g \) is an hermitian metric with respect to \( I \), and the holonomy of the connection \( \Gamma^{(+)} \) is a subgroup of \( U(D/2) \). Furthermore, as implied by (2.9)-(2.11) and (2.14), the vector bundle \( \xi \) is holomorphic and \( \xi \) is hermitian. For \( p = 4 \) \( \mathcal{M} \) admits a quaternionic structure, i.e. the three (integrable) complex structures obey the algebra of imaginary unit quaternions, the metric \( g \) is tri-Hermitian and the holonomy of the connection \( \Gamma^{(+)} \) is a subgroup of \( Sp(D/4) \). Furthermore, the bundle \( \xi \) is tri-holomorphic and tri-hermitian, i.e. holomorphic and hermitian with respect to all three complex structures.

The additional conditions that arise for \( m \neq 0 \) are just (2.12), (2.15) and (2.16), but (2.12) is implied by (2.6) and (2.16) which leaves (2.15) and (2.16). The analysis of these two conditions is facilitated by the definitions
\[
L^a_r = G^a_r + \hat{I}^a_r b^b_s, \\
M^a_r = G^a_r - \hat{I}^a_r b^b_s
\] (2.17)

in terms of which we have
\[
G^a_r = \frac{1}{2} (M^b_r + L^b_r), \\
s^a = \frac{1}{2} \hat{I}^a_r b^b (M^b_r - L^b_r)
\] (2.18)

where there is no summation over \( r \) in the second of these equations (i.e. for \( p=4 \) we have three different expressions for \( s^a \)). The conditions (2.16) and (2.15) can now be rewritten in terms of \( L_r \) and \( M_r \) as follows:
\[
I^j_r \nabla_j L^a_r - \hat{I}^a_r b^b_s \nabla_i L^b_r = 0 \\
I^j_r \nabla_j M^a_r + \nabla_i (\hat{I}^a_r b^b M^b_r) = - (\nabla_i \hat{I}^a_r b^b) L^b_r
\] (2.19)
and
\[(\hat{I}_r)_{ab}L^a_rM^b_r = \text{constant} \quad (r = 1 \ldots p - 1) \] (2.20)

where there is again no summation over \( r \). Similarly, the potential \( V \) can be expressed in terms of \( L_r \) and \( M_r \) as
\[ V = \frac{1}{16}m^2h_{ab}(M^a_r - L^a_r)(M^b_r - L^b_r) . \] (2.21)

It remains to understand the conditions (2.19) and (2.20) on \( L_r \) and \( M_r \). Let us assume that \( \nabla \hat{I}_r = 0 \).† To solve (2.19) we choose complex coordinates adapted to the pair \((I_r, \hat{I}_r)\) of complex structures. Then
\[ \nabla_\mu L^a_r = 0 \quad \nabla_\mu M^a_r = 0 \] (2.22)

where \( \mu = 1, \ldots, (D/2) \) and \( \alpha = 1, \ldots, (n/2) \). Thus \( L_r \) is a holomorphic and \( M_r \) an anti-holomorphic section of \( \xi \) with respect to \((I_r, \hat{I}_r)\). The potential is expressed in terms of these sections as in (2.21) (and for \( p=4 \) this can be done in three different ways).

An interesting special case of this result is found by setting \( M_r = 0 \). In this case (2.22) implies that \( s \) is a holomorphic section of \( \xi \) for \( p=2 \) and a triholomorphic section of \( \xi \) (i.e. holomorphic with respect to all three pairs of complex structures) for \( p=4 \). The zeros of the potential (the ‘classical vacua’) are then given by the zeros of a (tri)holomorphic section of \( \xi \).

Finally, we remark that a more general class of \((p,0)\) models is obtained when the condition of off-shell closure of the supersymmetry algebra is relaxed to on-shell closure. Note that the first term in \( \Delta_{\eta}\psi^a \) of (2.5) is trivially a symmetry by itself provided (2.14) is satisfied; it was included as part of the definition of the supersymmetry transformation in order to ensure off-shell closure. For on-shell closure it can be dropped, in which case the conditions (2.10) and (2.11) clearly do not arise and can also be dropped. Then \( \hat{I} \) is restricted only by (2.14) which indeed allows the choice \( \hat{I} = 0 \) made in [4]. The implications of the remaining conditions in this case will be discussed elsewhere.

### 3. \((p,0)\) superfields

We shall now give an off-shell \((p,0)\) superspace formulation of the \((p,0)\) models just described, following the treatment of the massless case in [7]. The \((p,0)\) superspace \( \Sigma^{(p,0)} \) has coordinates \( \{x^+, x^-, \theta^+_l; \ l = 0, r; \ r = 1, \ldots, p - 1\} \) and the supercovariant derivatives \( D_{l+} \) satisfy
\[ \{D_{l+}, D_{m+}\} = 2i\partial_{l\pm} \delta_{lm} . \] (3.1)

The (off-shell) \((p,0)\) superfields \( \phi \) and \( \psi \) are defined as follows: \( \phi \) is a map from \( \Sigma^{(p,0)} \) into the sigma model manifold \( \mathcal{M} \) and \( \psi \) is a section of the vector bundle \( \phi^*\xi \otimes S_- \) over \( \Sigma^{(p,0)} \), where \( \xi \) is a vector bundle over \( \mathcal{M} \). These superfields satisfy the chirality constraints
\[ D_{r+}\phi^i = I_{r+}^i D_{a+} \phi^j \]
\[ \nabla_{r+}\psi^a_- = \hat{I}_{r+}^a_+ \nabla_{a+}\psi_- + \frac{m}{2} I_r^a_- \] (3.2)

† In the \((2,0)\) case it is always possible to find a metric connection with this property [7].
where $I_r$, $\hat{I}_r$ and $L_r$ are the same tensors as those denoted previously by these symbols. The conditions found in the previous section for off-shell closure of the $(p,0)$-supersymmetry transformations can now be interpreted as the integrability conditions of the above constraints.

In terms of these $(p,0)$ superfields, the massive $(p,0)$-supersymmetric action is

$$S = \int d^2x d\theta^+_0 \left\{ D_{0+i} \phi^i \partial_\theta \phi^j (g_{ij} + b_{ij}) + i\psi^a \nabla_{0+i} \psi^b_- h_{ab} + im s_a \psi^-_a \right\}.$$  \hspace{1cm} (3.3)

This is similar to the action (1.2) for the $(1,0)$ supersymmetric sigma models, but note that for $p > 1$ the superspace measure appearing in (3.3) is not the full superspace one. Nevertheless, the action is invariant under the full $(p,0)$ supersymmetry transformations provided that the couplings satisfy all the conditions previously obtained.

4. $(1,1)$ Supersymmetry

Our task now is to find the conditions under which the $(1,0)$-supersymmetric action (1.2) has an additional supersymmetry of the opposite chirality. In this case we must also allow for a central charge which generates an isometry symmetry. On dimensional grounds the $(1,0)$ superfield form of these additional transformations must have the form

$$\delta \zeta \phi^i = D_+ \zeta e^i_a(\phi) \psi^a_- + m \zeta X^i(\phi)$$

$$\Delta \zeta \psi^a_- = -iD_+ \zeta e^a_i(\phi) \partial \phi^i + D_+ \zeta M^a_{bc}(\phi) \psi^b_--\psi^c_- + m \zeta U^a_b \psi^b_-$$ \hspace{1cm} (4.1)

where $e^a_i$, $e^a_i$, $M^a_{bc}$, $U^a_b$ and $X^i$ are globally defined tensors on $\mathcal{M}$ and/or the bundle $\xi$. The parameter $\zeta$ is an $x$-independent $(1,0)$ Grassman even scalar superfield with expansion $\zeta = \alpha + \theta^+ \epsilon_+$, so that it combines the left-handed supersymmetry parameter $\epsilon_+$ and the parameter $\alpha$ of central charge transformations.

We begin by checking the commutator of two $\zeta$-transformations. For $m = 0$ we find that the following conditions are required:

$$e^a_i e^a_j = \delta^a_j \quad e^a_i e^a_b = \delta^a_b$$ \hspace{1cm} (4.2)

and

$$M^a_{bc} = e^i_b e^a_c \left( \partial_i e^a_j + \Omega^a_{i b} e^b_{j} \right).$$ \hspace{1cm} (4.3)

When $m \neq 0$ we find additionally that

$$U^a_b = e^i_b \nabla_i X^a - 2X^c M^a_{bc},$$ \hspace{1cm} (4.4)

where $X^a \equiv X^i e^a_i$. The conditions (4.2) imply that the vector bundle $\xi$ of the fermionic sector is isomorphic to the tangent bundle of the target manifold $\mathcal{M}$, and $e^a_i$ can therefore be interpreted as a vielbein. Thus, the indices of $\xi$ and the tangent bundle of $\mathcal{M}$ become interchangeable.
A calculation of the commutator of a (1,0)-supersymmetry transformation with a \( \zeta \)-transformation yields the following result (using no geometric constraints or field equations):

\[
[\delta_\epsilon, \delta_\zeta] \phi^i = \left( -\frac{i}{2}D_+ \zeta D_+ \epsilon \right) mX^i
\]

\[
[\delta_\epsilon, \delta_\zeta] \psi^a = \left( -\frac{i}{2}D_+ \zeta D_+ \epsilon \right) mU^a_b \psi^b - (\cdots) \Omega_k^a b \psi^b.
\]

The right hand side of these commutators is just a \( \zeta \)-transformation with a \( \theta \)-independent parameter, and it is readily verified that such transformations commute with both (1,0) and (0,1) supersymmetries, i.e. the supersymmetry algebra has an off-shell central charge when both \( m \neq 0 \) and \( X^i \neq 0 \).

We now turn to the conditions obtained from \( \zeta \)-invariance of the action. It is very convenient to simplify them with the aid of the conditions already found above from closure of the algebra. Doing this one finds that some of them are not independent of those already derived above. For \( m = 0 \) the independent ones can most easily be summarised by the following equations

\[
h_{ab} = e_a^i e_b^j g_{ij}
\]

and

\[
\nabla_j^{(-)} e_i^a = 0.
\]

When \( m \neq 0 \) we find in addition that the Lie derivatives of both \( g_{ij} \) and \( H_{ijk} \) with respect to \( X^i \) must vanish, so that

\[
\nabla_{(i} X_{j)} = 0 ,
\]

and \( \iota_X H \) is a closed two-form which implies that there is some locally defined one-form \( u \) on \( M \) such that \( \iota_X H = du \), i.e.

\[
X^k H_{ijk} = \partial_{[i} u_{j]}. \tag{4.9}
\]

The one-form \( u \) is defined by this equation up to the derivative of a local function. Furthermore,

\[
X^i \nabla_i^{(-)} s_j + s_i \nabla_i^{(-)} s_j = 0 ,
\]

and

\[
s_i = u_i - X_i. \tag{4.11}
\]

The solution of the entire set of equations resulting from closure of the algebra and invariance of the action can be stated as follows. Firstly, for \( m = 0 \), the quantities \( e_i^a \) and \( e_a^i \) are vielbeins that relate the metric \( g \) of \( M \) with the fibre metric \( h \) of the bundle \( \xi \). The connection \( \Omega \) is the spin-connection of \( \Gamma^{(-)} \), and \( M_{ijk} = -H_{ijk} \). For \( m \neq 0 \) we also have that the vector \( X \) is a Killing vector field, that the torsion \( H_{ijk} \) is invariant with respect to the symmetry generated by this Killing vector field, that the section \( s_i \) is given by \( \nabla_i^{(-)} s_j + s_i \nabla_i^{(-)} s_j = 0 \), and that, after some computation,

\[
U_{ij} = \nabla_{[i}^{(+)} X_{j]} = \nabla_{[i} u_{j]} - \nabla_{[i} X_{j]} . \tag{4.12}
\]

In addition eq. (4.10) is equivalent (after more computation) to

\[
\partial_i (X^j u_j) = 0. \tag{4.13}
\]
Contracting (4.9) with the Killing vector field \(X\) and using (4.13) we learn that the one-form \(u\) is invariant with respect to the symmetry generated by \(X\), i.e.

\[
\mathcal{L}_X u = 0 .
\]  

(4.14)

From eq. (1.4) and using (4.11) we find that

\[
V(\phi) = \frac{1}{4} m^2 g^{ij}(u - X)_i(u - X)_j .
\]  

(4.15)

Using (4.13) we can rewrite the potential as

\[
V = \frac{1}{4} m^2 \left( g^{ij} u_i u_j + g_{ij} X^i X^j \right) + \text{const} .
\]  

(4.16)

where the constant is \(-(m^2/2)u \cdot X\). Note also that because of (4.14), the \(u^2\) part of the potential, and hence the entire potential, is required to be invariant under the symmetry generated by \(X\). In the special case for which the torsion vanishes, and hence for which \(u_i = \partial_i U\) for locally-defined scalar \(U\), we have

\[
V(\phi) = \frac{1}{4} m^2 \left( g^{ij} \partial_i U \partial_j U + g_{ij} X^i X^j \right) + \text{const} .
\]  

(4.17)

which agrees with [1].

5. **Potentials from gauged (1,1) sigma models**

The (1,1) supersymmetry algebra with central charges is

\[
\{ \tilde{Q}_+, \tilde{Q}_+ \} = 2 P_+ \quad \{ \tilde{Q}_-, \tilde{Q}_- \} = 2 P_- \quad \{ \tilde{Q}_+, \tilde{Q}_- \} = Z ,
\]  

(5.1)

and these supercharges can be realised in (1,1)-superspace as

\[
\tilde{Q}_+ = Q_+ + \frac{1}{2} \theta^- Z \\
\tilde{Q}_- = Q_- + \frac{1}{2} \theta^+ Z
\]  

(5.2)

where

\[
Q_+ = \frac{\partial}{\partial \theta^+} + i \theta^+ \frac{\partial}{\partial x^+} \\
Q_- = \frac{\partial}{\partial \theta^-} + i \theta^+ \frac{\partial}{\partial x^-} .
\]  

(5.3)

The \(Q_\pm\) generate the usual (1,1) supersymmetry transformations without a central charge and \(Z\) is the central charge generator which acts by a diffeomorphism along a vector field \(X\). This vector field should leave the action invariant and this implies that \(X\) is Killing and satisfies (4.9).
It would clearly be desirable to have a (1,1) superspace formulation of the (1,1) models just described but the conventional (1,1) superspace formulation does not allow for central charges. This problem can be circumvented by using the superspace with central charges [8] and a relation between supersymmetric sigma-models with potentials and gauged (1,1)-supersymmetric sigma-models. Specifically, under certain conditions the central charge transformations can be thought of as certain gauge transformations.

The action of the gauged (1,1) supersymmetric sigma model with target manifold $\mathcal{M}$ and gauge group $G$ is given by [9]

\[
S = \int d^2 x \, d\theta^+ d\theta^- \left( g_{ij} \nabla_+ \phi^i \nabla_- \phi^j + b_{ij} D_+ \phi^i D_- \phi^j - \frac{1}{2} A_+ u_i D_- \phi^i - \frac{1}{2} A_- u_i D_+ \phi^i + m W(\phi) \right) \tag{5.4}
\]

where $\phi^i$ are now (1,1) superfields, the couplings $g$ and $b$ are defined as in the action (2.1), $u_i$ is defined as in (4.9), and $W$ is a scalar on $\mathcal{M}$. The superfields $A_\pm$ are the superspace components of the connection $A$ for $G$ which here we take to be one-dimensional abelian. The derivatives $\nabla_\pm$ are the associated covariant derivatives. In particular $\nabla_\pm \phi^i = D_\pm \phi^i + A_\pm X^i$, where $D_\pm$ are the spinor derivatives of (1,1) superspace ($D^2_+ = i \partial_+ , D^2_- = i \partial_- )$ and $X$ is the vector field on $\mathcal{M}$ generated by the action of the gauge group.

This action is invariant under the gauge transformations

\[
\delta \phi^i = \lambda X^i, \quad \delta A_\pm = -D_\pm \lambda \tag{5.5}
\]

for arbitrary superfield parameter $\lambda(x, \theta)$ provided that the following conditions hold:

\[
\nabla_{(i} X_{j)} = 0 \quad \mathcal{L}_X H_{ijk} = 0 \quad X^i u_i = 0 \quad X^i \partial_i W = 0 \tag{5.6}
\]

where $H$ is the torsion three form. Observe that this action is manifestly (1,1) supersymmetric.

We now choose particular values for the gauge fields as follows:

\[
A_\pm = \bar{A}_\pm \equiv \frac{m}{2} \theta_\pm \tag{5.7}
\]

so that the field strength $\bar{F}_{+-} = m$. Under combined (1,1) supersymmetry and gauge transformations the connections $\bar{A}_\pm$ transform as

\[
\delta \bar{A}_\pm = \frac{m}{2} \epsilon_\pm - D_\pm \lambda \tag{5.8}
\]

and so vanish if

\[
\lambda = \bar{\lambda} \equiv \frac{m}{2} (\theta^+ \epsilon^- + \theta^- \epsilon^+ ) . \tag{5.9}
\]

Thus the gauge fields (5.7) are invariant under supersymmetry transformations supplemented by compensating gauge transformations with parameters (5.9), and these modified supersymmetry transformations are precisely those generated by the modified supersymmetry operators (5.2). This means that a theory that is invariant under the action of these supersymmetry generators with central charge is given by setting the gauge fields to the values (5.7) in
the gauged sigma-model action. Moreover, this action is in fact invariant if the last two con-
ditions in (5.6) are relaxed to become respectively $u \cdot X = \text{constant}$ and $X^i \partial_i W = \text{constant}$, in which case the result agrees with that of the (1,0) superfield calculation.

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