New algorithms for computing the least trimmed squares estimator

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Abstract

Instead of minimizing the sum of all \( n \) squared residuals as the classical least squares (LS) does, Rousseeuw (1984) proposed to minimize the sum of \( h \ (n/2 \leq h < n) \) smallest squared residuals, the resulting estimator is called least trimmed squares (LTS). The idea of the LTS is simple but its computation is challenging since no LS-type analytical computation formula exists anymore.

Attempts had been made since its presence, the feasible solution algorithm (Hawkins (1994)), fastlts.f (Rousseeuw and Van Driessen (1999)), and FAST-LTS (Rousseeuw and Van Driessen (2006)), among others, are promising approximate algorithms. The latter two have been incorporated into R function ltsReg by Valentin TODOROV. These algorithms utilize combinatorial- or subsampling-approaches. With the great software accessibility and fast speed, the LTS, enjoying many desired properties, has become one of the most popular robust regression estimators across multiple disciplines.

This article proposes analytic approaches - employing first order derivative (gradient) and second order derivative (hessian matrix) of the objective function. Our approximate algorithms for the LTS are vetted in synthetic and real data examples. Compared with ltsReg - the benchmark in robust regression and well-known for its speed, our algorithms are comparable (and sometimes even favorable) with respect to both speed and accuracy criteria.

Other major contributions include (i) originating the uniqueness and the strong and Fisher consistency at empirical and population settings respectively; (ii) deriving the influence function in a general setting; (iii) re-establishing the asymptotic normality (consequently root-n consistency) of the estimator with a neat and general approach.

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Running title: Computing the least trimmed squares.
1 Introduction

In the classical linear regression analysis, it is assumed that there is a relationship for a given data set \( \{(x'_i, y_i)', i = 1, \cdots, n\} \):

\[
y_i = x'_i \beta_0 + e_i, \quad i = 1, \cdots, n,
\]

where \( y_i, e_i \) (error term) are in \( \mathbb{R}^1 \), \( ' \) stands for the transpose, \( \beta_0 = (\beta_{01}, \cdots, \beta_{0p})' \), the true unknown parameter, and \( x_i = (x_{i1}, \cdots, x_{ip})' \) are in \( \mathbb{R}^p \) \( (p \geq 1) \). Throughout, assume that \( x_{i1} = 1 \), that is, \( \beta_{01} \) is the intercept term of the model.

One wants to estimate the \( \beta_0 \) based on the given sample \( Z^{(n)} := \{(x'_i, y_i)', i = 1, \cdots, n\} \).

For a candidate coefficient vector \( \beta \), call the difference between \( y_i \) (observed) and \( x'_i \beta \) (predicted), the \( i \)-th residual \(- r_i \) (\( \beta \) is suppressed). That is,

\[
r_i = y_i - x'_i \beta.
\]

To estimate \( \beta_0 \), the classic least squares (LS) estimator is the minimizer of the sum of squares of residuals

\[
\hat{\beta}_{ls} = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} r_i^2.
\]

Alternatively, one can replace the square above by absolute value to obtain the least absolute deviations estimator (aka, \( L_1 \) estimator, in contrast to the \( L_2 \) (LS) estimator).

Due to its great computability and optimal properties when the error \( e_i \) follows a normal \( \mathcal{N}(\mu, \sigma^2) \) distribution, the least squares estimator is most popular in practice across multiple disciplines. It, however, can behave badly when the error distribution is slightly departed from the normal distribution, particularly when the errors are heavy-tailed or contain outliers. In fact, both \( L_1 \) and \( L_2 \) (LS) estimators have a pitiful 0% asymptotic breakdown point (see definition in section 2), in sharp contrast to the 50% of least trimmed squares estimator. The latter is one of robust alternatives to the least squares estimator.

Robust alternatives to the least squares regression estimator are abundant in the literature. The most popular ones are, among others, M-estimators (Huber(1964) \cite{12}), least median squares (LMS) and least trimmed squares (LTS) estimators (Rousseeuw (1984) \cite{20}), S-estimators (Rousseeuw and Yohai (1984) \cite{25}), MM-estimators (Yohai (1987) \cite{30}), \( \tau \) - estimators (Yohai and Zamar (1988) \cite{31}) and maximum depth estimators (Rousseeuw and Hubert (1999) \cite{21} and Zuo (2021a \cite{33}, 2021b \cite{34})).

In Rousseeuw (1984) \cite{20}, the famous least median squares (LMS) estimator was introduced, along with the least trimmed squares (LTS) estimator. The former was regarded as a \textit{breakthrough in Statistics} in the twentieth century (see Simpson(1987) \cite{18}). Recently Rousseeuw preferred LTS over LMS (see Rousseeuw and Van Driessen (2006) (RVD06) \cite{24}).
The LTS is defined as the minimizer of the sum of trimmed squares of residuals. Namely

\[ \hat{\beta}_{\text{lts}} := \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{h} (r^2)_{i:n}, \]

where \((r^2)_{1:n} \leq (r^2)_{2:n} \leq \cdots, (r^2)_{n:n}\) are the ordered squared residuals and \(\lfloor n/2 \rfloor \leq h < n\).

(a) Left panel: plot of seven artificial points and two candidate lines (\(L_1\) and \(L_2\)), which line would you pick? Intuitively points 1 and 2 are outliers. Note that LTS can resist up to \(\lfloor (n-p)/2 \rfloor = 2\) points in light of Theorem 6 on page 132 of Rousseeuw and Leroy (1987) (RL87) \[22\]. That is, it should pick a line similar to \(L_2\) as one expects. Unfortunately, this is not the case. If one employs \(\text{ltsReg}\) with the default number \(h = \lfloor (n+p+1)/2 \rfloor\) in RVD06 \[23\], then LTS-\(\text{ltsReg}\) prefers \(L_1\) to \(L_2\) (see right panel).

Right panel: the same seven points are fitted by LTS-\(\text{ltsReg}\), LTS-AA1, and the LS (benchmark). The solid black line is LTS with \(\text{ltsReg}\). The red dashed line is given by LTS with our new procedure AA1, and the green dotted line is given by LS (identical to LTS-\(\text{ltsReg}\) line in this case).

(b) Left panel: plot of seven highly correlated normal points (with mean being the zero vector and covariance matrix with diagonal entries being one and off-diagonal entries being 0.88) and three lines given by LTS-\(\text{ltsReg}\), LTS-AA1, and LS. The LS line is identical to LTS-\(\text{ltsReg}\) line again.

Right panel: LTS-\(\text{ltsReg}\) line (solid black) and LTS-AA1 line (dashed red), and LS (dotted green) for the same seven highly correlated normal points but with two points contaminated nevertheless. The LS line is again almost identical to LTS-\(\text{ltsReg}\) line. The phenomenon that sometimes LTS-\(\text{ltsReg}\) can behave like LS is predictable, it must be an LS line to some five points in this case due to the internal attribute of the algorithm in RVD06 \[24\], seeking to speed up the underlying computation.

Figure 1: (a) Difference between the two procedures: LTS based on \(\text{ltsReg}\) and LTS based on AA1 (see section 3). (b) Performance of LTS-\(\text{ltsReg}\) and LTS-AA1 when there are contaminated points (\(x\)-axis leverage points).

The idea of LTS is simple but to find out it is not that trivial since it involves searching all possible \(\beta\)s in \(\mathbb{R}^p\). Earlier attempts were made in Hawkin (1994) \[11\], Rousseeuw and Van Driessen (1999) \[23\], and Hawkin and Oliver (1999) \[9\], among others. The latest most
successful one was given in Rousseeuw and Van Driessen (2006) [24]. The latter idea and
that of their 1999 one have already been incorporated into the R function ltsReg. The ltsReg
has become a benchmark in robust regression and is well-known for its speed. One wonders if
there are any other procedures that could perform as good as ltsReg or even better (i.e. faster
and more accurate). This article proposes such procedures. Before going into details, let us
first appreciate the difference between new (LTS-AA1) and old (LTS-ltsReg) procedures.

Example 1.1 We borrow a small artificial data set (in $\mathbb{R}^2$) constructed in Zuo (2022) [35]
with $x = (5, 5.5, 4, 3.5, 3, 2.5, -2)$ and $y = (-5, -5, 6, 4, 2.4, 2, 5)$, they are plotted in the
left panel of the (a) of Figure 1. We also provide two candidate regression lines $L_1 (y = 0)$
and $L_2 (y = x)$. Which one would you pick to represent the overall pattern of the data set?

Intuitively points 1 and 2 are outliers and $L_2$ should be preferred in the left panel of part
(a) of Figure 1. But that is not the case if one employs the R function ltsReg which gives the
line in the right panel of (a) of Figure 1 along with it is the benchmark LS line (identical
to the LTS-ltsReg line) and the line by our new procedure LTS-AA1. Apparently, LTS-AA1
does the job that LTS should (and is expected to) do, it captures the major overall pattern
of the data set. One might immediately argue that the behavior of ltsReg for this artificial
data is not representative and current behavior is not reproducible for other data sets.

So in part (b) of Figure 1 we generated seven highly correlated normal points (with
correlation 0.88 between $x$ and $y$), when there is no contamination LTS-ltsReg (identical
to the LS again) and LTS-AA1 pick perfectly the linear pattern whereas if there are two
contaminated points (note that LTS allows $m := \lfloor (n - p)/2 \rfloor = 2$ contaminated points in this
case in light of Theorem 6 on page 132 of RL87 [22]), the line from LTS-ltsReg drastically
changes in this particular instance, which is identical to the LS line again. But LTS should
and could do much better than this, as illustrated by the line LTS-AA1.

The phenomenon that LTS-ltsReg sometimes behaves as an LS is purely predictable. This
is due to the internal attribute of the algorithm (FAST-LTS) in RVD06 [24], seeking to speed
up the computation and looking for the final solution among the least squares fits.

A similar example with an increased sample size ($n = 100$) is given in Section 3.

The rest of the article is organized as follows. Section 2 addresses the properties of LTS
estimator, its uniqueness at both sample and population settings, and its influence function
(in $p > 2$ cases), and its strong and Fisher consistency are established unprecedentedly. Its
asymptotic normality is re-established in a very different and neat approach (via stochastic
equicontinuity) than the previous ones in the literature. Section 3 proposes two new ap-
proximate algorithms for the computation of LTS. They are put into test in Section 4 where
examples of simulated and real data sets are presented. Compared with the benchmark LTS-
ltsReg, our new algorithms (LTS-AA1 and LTS-AA2) are comparable (and sometimes even
favorable) with respect to performance criteria accuracy and computation speed. Concluding
remarks in Section 5 end the article.
2 Properties of LTS

Definition Denote by \( F(x', y) \) the joint distribution of \( x' \) and \( y \) in model (1). Throughout \( F_Z \) stands for the distribution function of vector \( Z \). For a given \( \beta \in \mathbb{R}^p \) and an \( \alpha \in [1/2, c] \), \( c < 1 \), let \( q(\beta, \alpha) = F_{W}^{-1}(\alpha) \) be the \( \alpha \)th quantile of \( F_W \) with \( W := (y - x'\beta)^2 \), where constant \( c \) equal to one case is excluded to avoid unbounded \( q(\beta, \alpha) \) and the LS cases. Define

\[
O(F(x', y), \beta, \alpha) = \int (y - x'\beta)^2 \mathbb{1} \left( (y - x'\beta)^2 \leq q(\beta, \alpha) \right) dF(x', y),
\]

and

\[
\beta_{lts}(F(x', y), \alpha) = \arg \min_{\beta \in \mathbb{R}^p} O(F(x', y), \beta, \alpha), \tag{5}
\]

where \( \mathbb{1}(A) \) is the indicator of \( A \) (i.e., it is one if \( A \) holds and zero otherwise). Let \( F_n(x', y) \) be the sample version of \( F(x', y) \) based on a sample \( Z(n) := \{(x'_i, y_i), i = 1, 2, \cdots, n\} \). \( F_n(x', y) \) and \( Z(n) \) will be used interchangeably. Inserting the sample distribution function, one obtains

\[
nO(F_n(x', y), \beta, \alpha) = \sum_{i=1}^{\lfloor \alpha n \rfloor + 1} (r_i^2), \tag{6}
\]

and

\[
\hat{\beta}_{lts}^n := \beta_{lts}(F_n(x', y), \alpha) = \arg \min_{\beta \in \mathbb{R}^p} O(F_n(x', y), \beta, \alpha). \tag{7}
\]

It is readily seen that \( \hat{\beta}_{lts}^n \) above is identical to \( \hat{\beta}_{lts} \) in (3) with \( h = \lfloor \alpha n \rfloor + 1 \). From now on we prefer to treat \( \hat{\beta}_{lts}^n \) than \( \hat{\beta}_{lts} \).

The first natural question is the existence of the minimizer in the right-hand side (RHS) of (7), or the existence of \( \hat{\beta}_{lts}^n \). Does it always exist? If it exists, will it be unique?

2.1 Existence, uniqueness, and equivariance

Write \( O^n(\beta) \) for \( O(F_n(x', y), \beta, \alpha) \), it is readily seen that

\[
O^n(\beta) = \frac{1}{n} \sum_{i=1}^{n} r_i^2 \mathbb{1} \left( r_i^2 \leq (r_i^2)_{h,n} \right), \tag{8}
\]

where \( h = \lfloor \alpha n \rfloor + 1 \).

Existence and uniqueness

Theorem 2.1 We have

(i) \( \hat{\beta}_{lts}^n \) always exist.
To that end, it suffices to show that

\[
\sum_{i=1}^{n} (y_i - x_i' \beta)x_i \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right) = 0.
\]  

(ii) Furthermore it is the solution of the system of equations

\[
\sum_{i=1}^{n} (y_i - x_i' \beta)x_i \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right) = 0.
\]  

(iii) If \( M := M(Y_n, X_n, \beta, \alpha) = \sum_i x_i x_i' \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right) \) with \( Y_n = (y_1, \cdots, y_n)' \) and \( X_n = (x_1, \cdots, x_n)' \) is invertible when \( \beta = \beta_{lts} \), then

\[
\tilde{\beta}^n_{lts} = M(Y_n, X_n, \beta, \alpha)^{-1} \sum_i y_i x_i \mathbb{1} \left( r(\tilde{\beta}^n_{lts})_i^2 \leq r(\beta_{lts})_i^2 \right),
\]

where \( r(\beta)_i = y_i - x_i' \beta \).

(iv) If \( M \) is invertible, then \( \tilde{\beta}^n_{lts} \) is unique.

**Proof:**

(i) Taking derivative of \( O^n(\beta) \) with respect to (w.r.t.) \( \beta \), one obtains

\[
n \frac{\partial O^n(\beta)}{\partial \beta} = \sum_{i=1}^{n} 2r_i(\beta)x_i \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right) + \sum_{i=1}^{n} r_i^2 \frac{\partial}{\partial \beta} \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right).
\]

We now show that the second term on the RHS above equals to zero, or equivalently to show that

\[
\frac{\partial}{\partial \beta} \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right) = 0.
\]

To that end, it suffices to show that

\[
\frac{\partial}{\partial \beta} \mathbb{1} \left( r(\beta)_i^2 > (r(\beta)_{h:n})^2 \right) = 0,
\]

or it suffices to show that for any \( j \in \{1, 2, \cdots, p\} \)

\[
\frac{\partial}{\partial \beta_j} \mathbb{1} \left( r(\beta)^2_j > (r(\beta)_{h:n})^2 \right) = 0.
\]

On the other hand, one has

\[
\frac{\partial}{\partial \beta_j} \mathbb{1} \left( r(\beta)^2_j > (r(\beta)_{h:n})^2 \right) = \lim_{\delta \to 0} \frac{\mathbb{1} \left( r(\beta(\delta, j))^2 > (r(\beta(\delta, j))_{h:n})^2 \right) - \mathbb{1} \left( r(\beta)^2_j > (r(\beta)_{h:n})^2 \right)}{\delta}
\]

\[
= 0,
\]

where \( \beta(\delta, j) = (\beta_1, \beta_2, \cdots, \beta_{j-1}, \beta_j + \delta, \beta_{j+1}, \cdots, \beta_p)' \), and the last equality follows from the continuity of \( r(\beta)^2_j \) and \( (r(\beta)_{h:n})^2 \) in \( \beta \) so that the numerator above is zero.

If taking the second-order derivative of \( O^n(\beta) \) w.r.t. \( \beta \), one gets

\[
n \frac{\partial^2 O^n(\beta)}{\partial \beta^2} = \sum_{i=1}^{n} 2x_i x_i' \mathbb{1} \left( r_i^2 \leq (r)_{h:n}^2 \right).
\]
Call the matrix on the RHS above as $H$ (Hessian matrix up to a scale). It is readily seen that $H$ is positive semidefinite, which implies that the convexity of $Q^n(\beta)$ in $\beta$ and consequently further implies that the global minimizer of $O^n(\beta)$ exists, $\hat{\beta}_{ts}^n$.

(ii) It is seen from above that $O^n(\beta)$ is twice continuously differentiable, hence its first derivative evaluated at the global minimum must be zero. By (i) above, we have the equation $(9)$.

(iii) This part directly follows from (ii).

(iv) The given condition, in conjunction with (ii) above, implies that $O^n(\beta)$ is strictly convex in $\beta$. The uniqueness thus follows.

Remarks 2.1

(I) Existence of $\hat{\beta}_{ts}^n$ could also be established under the assumption that there are no $\lfloor (n + 1)/2 \rfloor$ sample points of $Z^{(n)}$ contained in any $p$-dimensional hyperplane, similarly to that of Theorem 1 (on page 113) of RL87 [22] for LMS or Theorem 2.2 for LST in Zuo (2022) [35]. It is established here with no assumption nevertheless.

(II) A sufficient condition for the invertibility of $M$ is that $x_1, \cdots, x_n$ are linear independent, or the rank of $X_n$ is $p$. This condition automatically holds true with probability one if $F(x',y)$ has a density.

(III) Víšek (2006a) [27] also addressed the existence of $\hat{\beta}_{ts}^n$ (Assertion 1), but the argument there is valid only when the sample space $\Omega$ in the probability triple $(\Omega, F, P)$ is finite, which is quite restrictive. Furthermore, $x_i$ there are non-random covariates (carriers).

Equivariance

A regression estimator $T$ is called regression, scale, and affine equivariant if, respectively (see page 116 of RL87)

\begin{align*}
T \left( \{ (x_i', y_i')', i \in N \} \right) &= T \left( \{ (x_i', y_i)' , i \in N \} \right) + b, \quad \forall b \in \mathbb{R}^p \\
T \left( \{ (x_i', s y_i)' , i \in N \} \right) &= s T \left( \{ (x_i', y_i)' , i \in N \} \right), \quad \forall s \in \mathbb{R}^1 \\
T \left( \{ (A' x_i)', y_i)' , i \in N \} \right) &= A^{-1} T \left( \{ (x_i', y_i)' , i \in N \} \right), \quad \forall \text{nonsingular } A \in \mathbb{R}^{p \times p}
\end{align*}

where $N = \{1, 2, \cdots, n\}$.

Lemma 2.2 [22] (page 132) $\hat{\beta}_{ts}^n$ defined in $(7)$ is regression, scale, and affine equivariant.

A regression estimation functional $T(\cdot)$ is said to be regression, scale, and affine equivari-
ant (see Zuo (2021b) [34]) if respectively
\[ T(F_{(y+x', b, x)}) = T(F_{(y, x)}) + b, \forall b \in \mathbb{R}^p; \]
\[ T(F_{(sy, x)}) = sT(F_{(y, x)}), \forall s \in \mathbb{R}; \]
\[ T(F_{(y, A'x)}) = A^{-1}T(F_{(y, x)}), \forall \text{nonsingular } A \in \mathbb{R}^{p \times p}; \]

**Theorem 2.2** \( \beta_{lts}(F_{(x', y)}, \alpha) \) is regression, scale, and affine equivariant.

**Proof** It is trivial and analogous to that of Lemma 2.2. \( \blacksquare \)

### 2.2 Robustness

Extending the original idea of the breakdown value (Hampel (1971) [6]) in a general and asymptotic setting, Donoho and Huber (1983) [4] advocated a finite-sample version of the breakdown value, in line with Hodges’s (1967) [10] study in the univariate framework. Roughly speaking, the finite sample breakdown value is the minimum fraction of ‘bad’ (or contaminated) data that an estimator can be affected to an arbitrarily large extent, which had been becoming the most prevailing quantitative measure of global robustness of any location or regression estimators in the finite sample practice.

**Finite sample breakdown point**

**Definition 2.1** [4] The finite sample replacement breakdown point (RBP) of a regression estimator \( T \) at the given sample \( Z^{(n)} = \{Z_1, Z_2, \ldots, Z_n\} \), where \( Z_i := (x_i', y_i)' \), is defined as
\[
\text{RBP}(T, Z^{(n)}) = \min_{1 \leq m \leq n} \left\{ \frac{m}{n} : \sup_{Z_m^{(n)}} \|T(Z_m^{(n)}) - T(Z^{(n)})\| = \infty \right\},
\]
where \( Z_m^{(n)} \) denotes an arbitrary contaminated sample by replacing \( m \) original sample points in \( Z^{(n)} \) with arbitrary points in \( \mathbb{R}^{p+1} \).

Namely, the RBP of an estimator is the minimum replacement fraction that could drive the estimator beyond any bound. It turns out that both \( L_1 \) (least absolute deviations) and \( L_2 \) (least squares) estimators have RBP \( 1/n \) (or 0%, asymptotically), the lowest possible value whereas LTS can have \( (\lfloor (n-p)/2 \rfloor + 1)/n \) (or 50%) (with an optimal \( h \)) (see below), the highest possible value for any regression equivariant estimators (see pages 124-125 of RL87).

**Lemma 2.3** [22] (page 132) The RBP of the LTS method defined in [7] with \( h := \lceil an \rceil + 1 = \lfloor n/2 \rfloor + \lfloor (p+1)/2 \rfloor \) equals
\[
\text{RBP}(\hat{\beta}_{lts}^{n}, Z^{(n)}) = (\lfloor (n-p)/2 \rfloor + 1)/n.
\]

In terms of RBP, one can not ask more from LTS. RBP depicts the global robust perspectives of the underlying estimator in finite sample practice. Next we will focus on the local
robustness of $\beta_{ltls}(F(x', y), \alpha)$ defined in (5) at population setting via its influence function (Hampel, Ronchetti, Rousseeuw and Stahel (1986) (HRRS86) [7]).

**Influence function**

For a given distribution $F$ defined on $\mathbb{R}^{p+1}$ and an $\varepsilon \in (0, 1/2)$, the version of $F$ contaminated by an $\varepsilon$ amount of an arbitrary distribution $G$ on $\mathbb{R}^{p+1}$ is denoted by $F(\varepsilon, G) = (1 - \varepsilon)F + \varepsilon G$ (an $\varepsilon$ amount deviation from the assumed $F$).

**Definition 2.2** [7] The influence function (IF) of a functional $T$ at a given point $x \in \mathbb{R}^{p+1}$ for a given $F$ is defined as

$$IF(x; T, F) = \lim_{\varepsilon \to 0^+} \frac{T(F(\varepsilon, \delta_x)) - T(F)}{\varepsilon}, \quad (13)$$

where $\delta_x$ is the point-mass probability measure at $x \in \mathbb{R}^{p+1}$.

The function $IF(x; T, F)$ describes the relative effect (influence) on $T$ of an infinitesimal point-mass contamination at $x$ and measures the local robustness of $T$.

To establish the IF for the functional $\beta_{ltls}(F(x', y), \alpha)$, we need to first show its existence and uniqueness with or without point-mass contamination. To that end, write

$$F_{\varepsilon}(z) := F(\varepsilon, \delta_z) = (1 - \varepsilon)F(x', y) + \varepsilon\delta_z,$$

with $w = (s', t) \in \mathbb{R}^{p+1}$, $s \in \mathbb{R}^p$, $t \in \mathbb{R}^1$ as the corresponding random vector (i.e. $F_w = F_{\varepsilon}(z) = F(\varepsilon, \delta_z)$). The versions of (4) and (5) at the contaminated $F(\varepsilon, \delta_z)$ are respectively

$$O(F_{\varepsilon}(z), \beta, \alpha) = \int (t - s'\beta)^2 \mathbb{1}((t - s'\beta)^2 \leq q_{\varepsilon}(z, \beta, \alpha)) dF_w(s, t), \quad (14)$$

with $q_{\varepsilon}(z, \beta, \alpha)$ being the $\alpha$th quantile of the distribution function of $(t - s'\beta)^2$, and

$$\beta_{ltls}(F_{\varepsilon}(z), \alpha) = \arg \min_{\beta \in \mathbb{R}^p} O(F_{\varepsilon}(z), \beta, \alpha). \quad (15)$$

For $\beta_{ltls}(F(x', y), \alpha)$ defined in (5) and $\beta_{ltls}(F_{\varepsilon}(z), \alpha)$ above, we have an analogous result to Theorem 2.1. (Assume that the counterpart of model (5) is $y = x'\beta_0 + \varepsilon$).

**Theorem 2.3** We have

(i) $\beta_{ltls}(F(x', y), \alpha)$ and $\beta_{ltls}(F_{\varepsilon}(z), \alpha)$ always exist.

(ii) Furthermore, they are the solution of system of equations, respectively

$$\int (y - x'\beta)x \mathbb{1}((y - x'\beta)^2 \leq q(\beta, \alpha)) dF(x', y)(x, y) = 0 \quad (16)$$

$$\int (t - s'\beta)s \mathbb{1}((t - s'\beta)^2 \leq q_{\varepsilon}(z, \beta, \alpha)) dF_w(s, t) = 0 \quad (17)$$
(iii) $\beta_{lt}(F(x', y), \alpha)$ and $\beta_{lt}(F_{\varepsilon}(z), \alpha)$ are unique provided that
\[\int x x' \mathbb{1} \left((y - x' \beta)^2 \leq q(\beta, \alpha)\right) dF_{(x', y)}(x, y), \]
\[\int s s' \mathbb{1} \left((t - s' \beta)^2 \leq q_\varepsilon(z, \beta, \alpha)\right) dF_{w}(s, t) \]
are respectively invertible.

**Proof:** In light of Lebesgue Dominated Convergence Theorem, the proof is analogous to that of Theorem 2.1.

**Fisher consistency**

**Theorem 2.4** $\beta_{lt}(F(x', y), \alpha) = \beta_0$ provided that
\[(i) \ E_{(x', y)}\left(x x' \mathbb{1} (r(\beta)^2 \leq F^{-1}_{r(\beta)^2}(\alpha)) \right) \text{ is invertible}, \]
\[(ii) \ E_{(x', y)}(e x \mathbb{1} (e^2 \leq F^{-1}_{e^2}(\alpha)) = 0, \]
where $r(\beta) = y - x' \beta$.

**Proof** By theorem 2.3, (i) guarantees the existence and the uniqueness of $\beta_{lt}(F(x', y), \alpha)$ which is the unique solution of the system of the equations
\[\int (y - x' \beta) x \mathbb{1} ((y - x' \beta)^2 \leq q(\beta, \alpha)) dF_{(x', y)}(x, y) = 0. \]
Notice that $y - x' \beta = x' (\beta - \beta_0) + e$, insert this into the above equation we have
\[\int (x' (\beta - \beta_0) + e) x \mathbb{1} ((x' (\beta - \beta_0) + e)^2 \leq F^{-1}_{(x' (\beta - \beta_0) + e)^2}(\alpha)) dF_{(x', y)}(x, y) = 0. \]
By (ii) it is readily seen that $\beta = \beta_0$ is a solution of the above system of equations. Uniqueness leads to the desired result.

**Theorem 2.5** Put $\beta_{lt} := \beta_{lt}(F(x', y), \alpha)$, then for any $z_0 := (s_0', t_0) \in \mathbb{R}^{d+1}$, we have that
\[\text{IF}(z_0; \beta_{lt}, F_{(x', y)}) = \begin{cases} 0, & \text{if } (t_0 - s_0', \beta_{lt})^2 > q(\beta_{lt}, \alpha), \\ M^{-1}(t_0 - s_0', \beta_{lt})s_0, & \text{otherwise}. \end{cases} \]
provided that $M = E_{(x', y)} \left(x x' \mathbb{1} ((y - x' \beta_{lt})^2 \leq q(\beta_{lt}, \alpha)) \right)$ is invertible.

**Proof:** Write $\beta_{lt}(z_0)$ for $\beta_{lt}(F_{(z_0)}, \alpha)$ and insert it for $\beta$ into (17) and take derivative with respect to $\varepsilon$ in both sides of (17) and let $\varepsilon \to 0$, we obtain (in light of dominated theorem)
\[\left(\int \frac{\partial}{\partial \beta_{lt}(z_0)} r(\beta_{lt}(z_0)) ss' \mathbb{1} (r(\beta_{lt}(z_0))^2 \leq q_\varepsilon(z, \beta_{lt}(z_0), \alpha)) \right) \text{IF}(z_0, \beta_{lt}, F_{(x', y)}) \]
\[+ \int (y - x' \beta_{lt}(F_{(x', y)}, \alpha)) x \mathbb{1} ((y - x' \beta_{lt}(F_{(x', y)}, \alpha))^2 \leq q(\beta_{lt}(F_{(x', y)}, \alpha), \alpha)) d(\delta z_0 - F_{(x', y)}) \]
\[= 0, \tag{20} \]
where \( r(\beta) = y - x'\beta \). Call the two terms on the LHS as \( T_1 \) and \( T_2 \) respectively and call the integrand in \( T_1 \) as \( T_0 \), then it is seen that (see the proof (i) of Theorem 2.1)

\[
T_0 = \frac{\partial}{\partial \beta'_{lts}(z_0)}(y - x'\beta'_{lts}(z_0))(y - x'\beta'_{lts}(z_0))^2 \leq q(\beta, \alpha) \]

\[
= -xx' \left( (y - x'\beta_{lts})^2 \leq q(\beta_{lts}, \alpha) \right).
\]

Focus on the \( T_2 \), it is readily seen that

\[
T_2 = \int (y - x'\beta_{lts}(F'(x', y), \alpha))x1 (y - x'\beta_{lts}(F'(x', y), \alpha))^2 \leq q(\beta_{lts}(F'(x', y), \alpha), \alpha)) d\delta_0
\]

\[
- \int (y - x'\beta_{lts}(F'(x', y), \alpha))x1 (y - x'\beta_{lts}(F'(x', y), \alpha))^2 \leq q(\beta_{lts}(F'(x', y), \alpha), \alpha)) dF'(x', y)
\]

\[
= \int (y - x'\beta_{lts}(F'(x', y), \alpha))x1 (y - x'\beta_{lts}(F'(x', y), \alpha))^2 \leq q(\beta_{lts}(F'(x', y), \alpha), \alpha)) d\delta_0
\]

\[
= \begin{cases} 
0, & \text{if } (t_0 - s_0', \alpha)^2 > q(\beta_{lts}, \alpha), \\
(t_0 - s_0', \alpha), & \text{otherwise}.
\end{cases}
\]

where the second equality follows from (16). This, \( T_0 \), and display (20) lead to the desired result.

**Remarks 2.2**

(I) When \( p = 1 \), the problem in our model (1) becomes a location problem (see page 158 of RL87 [22]) and the IF of LTS estimation functional has been given on page 191 of RL87 [22]. In the location setting, Tableman (1994) [26] also studied the IF of LTS. When \( p = 2 \), namely in the simple regression case, Óllerer, et al (2015) [16] studied IF of the spare-LTS functional under the assumption that \( x \) and \( e \) are independent and normally distributed. The above result is much more general and valid for any \( p \geq 1 \), \( x' \), and \( e \).

(II) The influence function of \( \beta_{lts} \) remains bounded if the contaminating point \((s_0', t_0)\) does not follow the model (i.e. its residual is extremely large), in particular for bad leverage points and vertical outliers. This shows the good robust properties of LTS.

The influence function of \( \beta_{lts} \), unfortunately, might be unbounded (in \( p > 1 \) case), sharing the drawback of that of spare-LTS (in the \( p = 2 \) case). The latter was shown in Óllerer, et al (2015) [16]. Trimming sheely based on the residuals (or squared residuals) will have this type of drawback since the term \( x'\beta \) can be bounded, but \( ||x|| \) might not.

### 2.3 Asymptotic properties

Víšek (2006 a [27], b [28], and c [29]) have addressed the consistency, root-n consistency, and normality of LST under a restrictive setting (\( x \)s are non-random covariates) and many assumptions on \( x \)s and on the distribution of \( e \) in three lengthy series papers. Here we will
address asymptotic properties of \( \hat{\beta}_{lts}^n \) in a much more concise approach and in a broader and different setting. Strong consistency has not been addressed by Víšek or in the literature.

**Strong consistency**

Following the notations of Pollard (1984) (P84) [17], write

\[
O(\beta, P) := O(F(x', y), \beta, \alpha) = P[(y - x'\beta)^2 \mathbb{1}(r(\beta)^2 \leq F_{r(\beta)}^{-1}(\alpha))] = Pf,
\]

\[
O(\beta, P_n) := O(F_n(x', y), \beta, \alpha) = O(Z(\alpha), \beta, \alpha) = \frac{1}{n} \sum_{i=1}^{n} r_i^2 \mathbb{1}(r_i^2 \leq (r_{h:n})^2) = Pnf,
\]

where \( f := f(x, y, \beta, \alpha) = (y - x'\beta)^2 \mathbb{1}(r(\beta)^2 \leq F_{r(\beta)}^{-1}(\alpha)), \) \( h = \lfloor an \rfloor + 1. \)

Under corresponding assumptions in Theorems 2.1 and 2.3, \( \hat{\beta}_{lts}^n \) and \( \beta_{lts} \) are unique minimizers of \( O(\beta, P_n) \) and \( O(\beta, P) \) over \( \beta \in \mathbb{R}^p \), respectively.

To show that \( \hat{\beta}_{lts}^n \) converges to \( \beta_{lts} \) almost surely, one can take the approach given in Section 4.2 of Zuo (2022) [35]. But here we take a different direct approach.

To show that \( \hat{\beta}_{lts}^n \) converges to \( \beta_{lts} \) almost surely, it will be suffice to prove that \( O(\hat{\beta}_{lts}^n, P) \to O(\beta_{lts}, P) \) almost surely, because \( O(\beta, P) \) is bounded away from \( O(\beta_{lts}, P) \) outside each neighborhood of \( \beta_{lts} \) in light of continuity and compactness (also see Lemma 4.3 of Zuo (2022) [35]).

By theorems 2.1 and 2.3, assume, without loss of generality (w.l.o.g.), that \( \hat{\beta}_{lts}^n \) and \( \beta_{lts} \) belong to a ball centered at \( \beta_{lts} \) with large enough radius \( r_0, B(\beta_{lts}, r_0) \). Assume, w.l.o.g., that \( \Theta = B(\beta_{lts}, r_0) \) is our parameter space of \( \beta \) hereafter. Define a class of functions

\[
\mathcal{F}(\beta, \alpha) = \left\{ f(x, y, \beta, \alpha) = (y - x'\beta)^2 \mathbb{1}(r(\beta)^2 \leq F_{r(\beta)}^{-1}(\alpha)) : \beta \in \Theta, \alpha \in [1/2, c] \right\}
\]

If we prove uniform almost sure convergence of \( P_n \) to \( P \) over \( \mathcal{F} \) (see Lemma 2.4 below), then we can deduce almost surely that \( O(\hat{\beta}_{lts}^n, P) \to O(\beta_{lts}, P) \) from

\[
O(\hat{\beta}_{lts}^n, P_n) - O(\hat{\beta}_{lts}^n, P) \to 0 \quad \text{(in light of Lemma 2.4), and}
\]

\[
O((\hat{\beta}_{lts}^n, P_n) \leq O(\beta_{lts}, P_n) \to O(\beta_{lts}, P) \leq O(\hat{\beta}_{lts}^n, P).
\]

Above discussions and arguments lead to

**Theorem 2.6.** Under corresponding assumptions in Theorems 2.1 and 2.3 for uniqueness of \( \hat{\beta}_{lts}^n \) and \( \beta_{lts} \) respectively, we have \( \hat{\beta}_{lts}^n \) converges almost surely to \( \beta_{lts} \) (i.e. \( \hat{\beta}_{lts}^n - \beta_{lts} = o(1), \) a.s.).

**Lemma 2.4.** \( \sup_{f \in \mathcal{F}} |P_n f - Pf| \to 0 \) almost surely.

**Proof:** We invoke Theorem 24 of II.5 of Pollard (1984) [17]. The first requirement of the theorem is the existence of an envelope of \( \mathcal{F} \). The latter is \( \sup_{\beta \in \Theta} F_{r(\beta)}^{-1}(c) \), which is bounded
in light of boundedness of \( \Theta \) and continuity of \( F_{\delta(r,\beta)^{2}}^{-1}(c) \) in \( \beta \). To complete the proof, we only need to verify the second requirement of the theorem.

For the second requirement, that is, to bound the covering numbers, it suffices to show that the graphs of functions in \( \mathcal{F}(\beta, \alpha) \) have only polynomial discrimination (see Theorem 25 of II.5 of P84 [17]).

The graph of a real-valued function \( f \) on a set \( S \) is defined as the subset (see page 27 of P84 [17])
\[
G_{f} = \{(s, t) : 0 \leq t \leq f(s) \text{ or } f(s) \leq t \leq 0, s \in S \}.
\]

The graph of a function in \( \mathcal{F}(\beta, \alpha) \) contains a point \((x(\omega), y(\omega), t)\) if and only if \(0 \leq t \leq f(x, y, \beta, \alpha)\) or \(f(x, y, \beta, \alpha) \leq t \leq 0\). The latter case could be excluded since the function is always nonnegative (and equals 0 case covered by the former case). The former case happens if and only if \(0 \leq \sqrt{t} \leq y - x'\beta\).

Given a collection of \( n \) points \((x_{i}, y_{i}, t_{i})\) \((t_{i} \geq 0)\), the graph of a function in \( \mathcal{F}(\beta, \alpha) \) picks out only points that belong to \(\{\sqrt{t_{i}} \geq 0\} \cap \{y_{i} - \beta'x_{i} - \sqrt{t} \geq 0\}\). Introduce \(n\) new points \((x_{i}, y_{i}, z_{i}) := (x_{i}, y_{i}, \sqrt{t_{i}})\) in \(\mathbb{R}^{p+2}\). On \(\mathbb{R}^{p+2}\) define a vector space \(\mathcal{G}\) of functions
\[
g_{a,b,c}(x, y, z) = a'x + by + cz,
\]
where \(a \in \mathbb{R}^{p}, b \in \mathbb{R}^{1},\) and \(c \in \mathbb{R}^{1}\) and \(\mathcal{G} := \{g_{a,b,c}(x, y, z) = a'x + by + cz, a \in \mathbb{R}^{p}, b \in \mathbb{R}^{1},\) and \(c \in \mathbb{R}^{1}\}\) which is a \(\mathbb{R}^{p+2}\)-dimensional vector space.

It is clear now that the graph of a function in \( \mathcal{F}(\beta, \alpha) \) picks out only points that belong to the sets of \(\{g \geq 0\}\) for \(g \in \mathcal{G}\). By Lemma 18 of II.4 of P84 (page 20) [17], the graphs of functions in \( \mathcal{F}(\beta, \alpha) \) pick only polynomial numbers of subsets of \(\{w_{i} := (x_{i}, y_{i}, z_{i}), i = 1, \cdots, n\}\); those sets corresponding to \(g \in \mathcal{G}\) with \(a \in \{0, -\beta\}, b \in \{0, 1\}\), and \(c \in \{-1, 1\}\) pick up even few subsets from \(\{w_{i}, i = 1, \cdots, n\}\). This in conjunction with Lemma 15 of II.4 of P84 (page 18) [17], yields that the graphs of functions in \( \mathcal{F}(\beta, \alpha) \) have only polynomial discrimination.

By Theorem 24 of II.5 of P84 [17] we have completed the proof.  

Root-n consistency and asymptotic normality

Instead of treating the root-n consistency separately, we will establish asymptotic normality of \(\hat{\beta}_{\text{tls}}^{n}\) directly via stochastic equicontinuity (see page 139 of P84 [17], or the supplementary of Zuo (2020) [32]).

**Stochastic equicontinuity** refers to a sequence of stochastic processes \(\{Z_{n}(t) : t \in T\}\) whose shared index set \(T\) comes equipped with a semi metric \(d(\cdot, \cdot)\). (a semi metric has all the properties of a metric except that \(d(s, t) = 0\) need not imply that \(s\) equals \(t\).)
Definition 2.3 [IIV. 1, Def. 2 of P84] [17]. Call $Z_n$ stochastically equicontinuous at $t_0$ if for each $\eta > 0$ and $\epsilon > 0$ there exists a neighborhood $U$ of $t_0$ for which

$$
\limsup_P \left( \sup_U |Z_n(t) - Z_n(t_0)| > \eta \right) < \epsilon.
$$

(21)

Because stochastic equicontinuity bounds $Z_n$ uniformly over the neighborhood $U$, it also applies to any randomly chosen point in the neighborhood. If $\tau_n$ is a sequence of random elements of $T$ that converges in probability to $t_0$, then

$$
Z_n(\tau_n) - Z_n(t_0) \to 0 \text{ in probability},
$$

(22)
because, with probability tending to one, $\tau_n$ will belong to each $U$. The form above will be easier to apply, especially when behavior of a particular $\tau_n$ sequence is under investigation.

Again following the notations of P84 [17]. Suppose $\mathcal{F} = \{f(\cdot, t) : t \in T\}$, with $T$ a subset of $\mathbb{R}^k$, is a collection of real, $P$-integrable functions on the set $S$ where $P$ (probability measure) lives. Denote by $P_n$ the empirical measure formed from $n$ independent observations on $P$, and define the empirical process $E_n$ as the signed measure $n^{1/2}(P_n - P)$. Define

$$
F(t) = P f(\cdot, t),
$$

$$
F_n(t) = P_n f(\cdot, t).
$$

Suppose $f(\cdot, t)$ has a linear approximation near the $t_0$ at which $F(\cdot)$ takes on its minimum value:

$$
f(\cdot, t) = f(\cdot, t_0) + (t - t_0)'\nabla(\cdot) + |t - t_0|r(\cdot, t).
$$

(23)

For completeness set $r(\cdot, t_0) = 0$, where $\nabla$ (differential operator) is a vector of $k$ real functions on $S$. We cite theorem 5 of IIV.1 of P84 [17] (page 141) for the asymptotic normality of $\tau_n$.

Lemma 2.5 [17]. Suppose $\{\tau_n\}$ is a sequence of random vectors converging in probability to the value $t_0$ at which $F(\cdot)$ has its minimum. Define $r(\cdot, t)$ and the vector of functions $\nabla(\cdot)$ by (23). If

(i) $t_0$ is an interior point of the parameter set $T$;

(ii) $F(\cdot)$ has a non-singular second derivative matrix $V$ at $t_0$;

(iii) $F_n(\tau_n) = o_p(n^{-1}) + \inf_t F_n(t)$;

(iv) the components of $\nabla(\cdot)$ all belong to $L^2(P)$;

(v) the sequence $\{E_n(\cdot, t)\}$ is stochastically equicontinuous at $t_0$;

then

$$
n^{1/2}(\tau_n - t_0) \overset{d}{\longrightarrow} \mathcal{N}(0, V^{-1}[P(\nabla\nabla') - (P\nabla)(P\nabla)']V^{-1}).
$$
In order to apply the Lemma, we first realize that in our case, \( \hat{\beta}_{n}^{*} \) and \( \beta_{n}^{*} \) correspond to \( \tau_{n} \) and \( t_{0} \) (assume, w.l.o.g. that \( \beta_{n}^{*} = 0 \) in light of regression equivariance); \( \beta \) and \( \Theta \) correspond to \( t \) and \( T \); \( f(\cdot, t) := f(x', y, \alpha, \beta) = (y - x'\beta)^{2}1(\tau(\beta)^{2} \leq F_{\tau(\beta)}^{-1}(\alpha)) \). In our case, 

\[
\nabla(x, y, \alpha, \beta) = \frac{\partial}{\partial \beta} f(x, y, \alpha, \beta) = 2(y - x'\beta)x1(\tau(\beta)^{2} \leq F_{\tau(\beta)}^{-1}(\alpha)).
\]

We will have to assume that \( P(\nabla_{i}^{2}) = P(4(y - x'\beta)^{2}x_{i}^{2}1(\tau(\beta)^{2} \leq F_{\tau(\beta)}^{-1}(\alpha))) \) exists to meet (iv) of the lemma, where \( i \in \{1, \cdots, p\} \) and \( x' = (x_{1}, \cdots, x_{p}) \). It is readily seen that a sufficient condition for this assumption to hold is the existence of \( P(x_{i}^{2}) \). In our case, \( V = P(xx'1(\tau(\beta)^{2} \leq F_{\tau(\beta)}^{-1}(\alpha))) \), we will have to assume that it is invertible when \( \beta \) is replaced by \( \beta_{n}^{*} \) (this is covered by (18)) to meet (ii) of the lemma. In our case, 

\[
r(\cdot, t) = \left( \frac{\beta^{'}(\|\beta\|)}{\beta^{'}(\|\beta\|)} \right) \|\beta\|.
\]

We will assume that \( \lambda_{\min} \) and \( \lambda_{\max} \) are the minimum and maximum eigenvalues of positive semidefinite matrix \( V \) overall \( \beta \in \Theta \) and \( \alpha \in [1/2, c] \).

**Theorem 2.7** Assume that

(i) the uniqueness assumptions for \( \hat{\beta}_{n}^{*} \) and \( \beta_{n}^{*} \) in theorems 2.1 and 2.3 hold respectively;

(ii) \( P(x_{i}^{2}) \) exists;

then 

\[
r^{1/2}(\hat{\beta}_{n}^{*} - \beta_{n}^{*}) \xrightarrow{d} N(O, V^{-1}[P(\nabla'\nabla') - (P\nabla)(P\nabla)']V^{-1}),
\]

where \( \beta \) in \( V \) and \( \nabla \) is replaced by \( \beta_{n}^{*} \) (which could be assumed to be zero).

**Proof:** To apply Lemma 2.5, we need to verify the five conditions, among them only (iii) and (v) need to be addressed, all others are satisfied trivially. For (iii), it holds automatically since our \( \tau_{n} = \hat{\beta}_{n}^{*} \) is defined to be the minimizer of \( F_{n}(t) \) over \( t \in T (= \Theta) \).

So the only condition that needs to be verified is the (v), the stochastic equicontinuity of \{\( E_{n}r(\cdot, t) \}\) at \( t_{0} \). For that, we will appeal to the Equicontinuity Lemma (VII.4 of P84 [17], page 150). To apply the Lemma, we will verify the condition for the random covering numbers satisfy the uniformity condition. To that end, we look at the class of functions 

\[
\mathcal{R}(\beta, \alpha) = \left\{ r(\cdot, \cdot, \alpha, \beta) = \left( \frac{\beta^{'}(\|\beta\|)}{V \beta^{'}(\|\beta\|)} \right) \|\beta\| : \beta \in \Theta, \alpha \in [1/2, c] \right\}.
\]

Obviously, \( \lambda_{\max}r_{0} \) is an envelope for the class \( \mathcal{R} \) in \( L^{2}(P) \), where \( r_{0} \) is the radius of the ball \( \Theta = B(\beta_{n}^{*}, r_{0}) \). We now show that the covering numbers of \( \mathcal{R} \) is uniformly bounded, which amply suffices for the Equicontinuity Lemma. For this, we will invoke Lemmas II.25 and II.36 of P84 [17]. To apply Lemma II.25, we need to show that the graphs of functions in \( \mathcal{R} \) have only polynomial discrimination. The graph of \( r(x, y, \alpha, \beta) \) contains a point \( (x, y, t), t \geq 0 \) if and only if \( \left( \frac{\beta^{'}(\|\beta\|)}{V \beta^{'}(\|\beta\|)} \right) \|\beta\| \geq t \) for all \( \beta \in \Theta \) and \( \alpha \in [1/2, c] \).
Equivalently, the graph of \( r(x, y, \alpha, \beta) \) contains a point \((x, y, t)\), \( t \geq 0 \) if and only if \( \lambda_{\min} \|\beta\| \geq t \). For a collection of \( n \) points \((x'_i, y_i, t_i)\) with \( t_i \geq 0 \), the graph picks out those points satisfying \( \lambda_{\min} \|\beta\| - t_i \geq 0 \). Construct from \((x_i, y_i, t_i)\) a point \( z_i = t_i \) in \( \mathbb{R} \). On \( \mathbb{R} \) define a vector space \( G \) of functions

\[
g_{a,b}(x) = ax + b, \quad a, b \in \mathbb{R}.
\]

By Lemma 18 of P84 [17], the sets \( \{ g \geq 0 \} \), for \( g \in G \), pick out only a polynomial number of subsets from \( \{ z_i \} \); those sets corresponding to functions in \( G \) with \( a = -1 \) and \( b = \lambda_{\min} \|\beta\| \) pick out even fewer subsets from \( \{ z_i \} \). Thus the graphs of functions in \( G \) have only polynomial discrimination. ■

Remarks 2.3

(I) In the case of \( p = 1 \), that is in the location case, the asymptotic normality of LTS has been given in RL87 (page 180) [22], Rousseeuw (1983) [19], Butler (1982) [3], and Bednarski and Clarke, (1993) [1].

(II) Hössjer (1994) [11], under the rank-based optimization framework and stringent assumptions on error term \( e_i \) (even density that is strictly decreasing for positive value, bounded absolute first moment) and on \( x_i \) (bounded fourth moment), covers the asymptotic normality of LTS. Viške (2006c) [29] also treated the general case \( p \geq 1 \) and obtained the asymptotic normality of LTS under many stringent conditions on the non-random covariates \( x_i \)'s and the distributions of \( e_i \) in a twenty-seven long article. Our approach here is different from former classical analyses and much more neat and concise. ■

3 Approximate Algorithms for LTS

Unlike the least sum of squares estimator, which has an analytical computation formula \( (\hat{\beta}_{ls} = (X_nX'_n)^{-1}X'_nY_n) \), for the least trimmed squares (LTS) estimator, we do not have such a formula. The formula given in [10] can not serve our purpose directly (due to the circular dependency: the RHS depends on the LHS). That is, at this moment, we are unable to compute LTS exactly and have to appeal to approximate algorithms (AAs).

Before proceeding to the approximate computation, let us first recall our minimization problem and the objective function/quantity that needs to be minimized. For a given data set \( Z^{(n)} = \{(x'_i, y_i)\} \) in \( \mathbb{R}^{p+1} \) and an \( \alpha \in [1/2, c] \), \( c < 1 \) and a \( \beta \in \mathbb{R}^p \), consider the objective

\[
O(Z^{(n)}, \beta, \alpha) := \sum_{i=1}^n r_i^2 \mathbb{1} \left( r(\beta)^2_i \leq (r(\beta))_{h:n}^2 \right),
\]

where \( r_i := r(\beta)_i = y_i - x'_i\beta \) and \( h = \lfloor \alpha n \rfloor + 1 \). We like to minimize \( O(Z^{(n)}, \beta, \alpha) \) over \( \beta \in \mathbb{R}^p \) (actually within a bounded ball is sufficient) to obtains the least sum of trimmed
squares (LTS) of residuals estimator (for a fixed $\alpha$),

$$\hat{\beta}_{lts}^n := \hat{\beta}_{lts}(Z^{(n)}, \alpha) = \arg \min_{\beta \in \mathbb{R}^p} O(Z^{(n)}, \beta, \alpha).$$

It is readily seen that $O(Z^{(n)}, \beta, \alpha)$ is twice continuously differentiable and convex in $\beta \in \mathbb{R}^p$.

To treat the above minimization problem, we like to address the problem in a more general setting, that is, an unconstrained minimization problem. Here we have the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^p} f(x), \quad (24)$$

where $f: \mathbb{R}^p \to \mathbb{R}$ is convex and twice continuously differentiable. It is readily seen that the problem is solvable, i.e., there exists an optimal point $x^*$. Since $f$ is differentiable and convex, a necessary and sufficient condition for a point $x^*$ to be optimal is

$$\nabla f(x^*) = 0, \quad (25)$$

where $\nabla$ is the vector differential operator.

Thus, solving the unconstrained minimization problem (24) is the same as finding a solution of (25), which is a set of $p$ equations in the $p$ variables $x_1, \cdots, x_p$. In a few special cases, we can find a solution to the problem (24) by analytically solving the optimality equation (25), but usually the problem must be solved by an iterative algorithm. By this we mean an algorithm that computes a sequence of points $x^{(0)}, x^{(1)}, \ldots \in \mathbb{R}^p$ with $f(x^{(k)}) \to f(x^*)$ as $k \to \infty$. The algorithm is terminated when $f(x^{(k)}) - f(x^*) \leq \varepsilon$, where $\varepsilon > 0$ is some specified tolerance.

### 3.1 An iterative weighted least squares procedure

For simplicity of description, we write $O^n(\beta)$ and $I(i, \beta, h)$ respectively for $O(Z^{(n)}, \beta, \alpha)$ and $I \left( r(\beta) \right) \leq (r(\beta))_{h \alpha}^2$. Recall equations (9) and (10), although the latter is not a direct solution formula for $\hat{\beta}_{lts}^n$, but if we provide an initial estimate $\hat{\beta}_{lts}^{(n,0)}$, then we have obtained an iterative formula for $\hat{\beta}_{lts}^{(n,k)}$, $k = 1, 2, \cdots$, the iterative process terminates when the stopping criterion is satisfied.

Here the stopping criterion could be (i) the derivative of $O^n(\beta)$ at $\beta^{(n,k)}$ or (ii) the difference between consecutive $\hat{\beta}_{lts}^{(n,k)}$ or $O^n(\hat{\beta}^{(n,k)})$ is below a threshold (e.g. $10^{-3}$), and the initial estimate $\hat{\beta}_{lts}^{(n,0)}$ could be any educated-guess based on inspection of data (say, e.g., $\hat{\beta}_{lts}^{(n,0)} = 0 \in \mathbb{R}^p$), or the LS estimate, even the LST (Zuo (2022) [35]) (for robustness consideration).

Likewise, from equation (9), one gets an iterative formula for $\hat{\beta}_{lts}^{(n,k)}$. Let us rewrite it as

$$X'_{n} W_{n} X_{n} \beta = X_{n} W_{n} Y_{n},$$

16
where \( X_n = (x_1, \cdots, x_n)' \), \( Y_n = (y_1, \cdots, y_n)' \), and \( W_n = \text{diag}(w_i) \) is a diagonal matrix with diagonal \( i \)th entries \( w_i = I(i, \beta, h) \), \( i = 1, \cdots, n \). Let \( M \) be the matrix defined in Theorem 2.1, then if it is invertible, we have reached the same iterative formula for \( \tilde{\beta}_{lst}^n \) just like the one given in (10).

Summarizing the above two procedures, we have

**AA1 Iterative weighted least squares (IWLS)**

Input a data set \( Z^{(n)} = \{(x_i', y_i)', i = 1, 2, \cdots, n\} \) and a fixed \( h \) (or equivalently \( \alpha \)) and an initial \( \beta^0 = \beta_{lst}^{(n,0)} \), where \( (1, x_i')' \) corresponds to \( x_i \) in model (1).

Repeat for \( k = 1, \cdots, \)

- compute \( w_i^k = I(i, \beta_{lst}^{(n,k-1)}, h), i = 1, 2, \cdots, n \),
- compute recursively
  \[
  \tilde{\beta}^{(n,k)} = (X_n' W_n^k X_n)^{-1} X_n' W_n^k Y_n,
  \]
  where \( W_n^k = \text{diag}(w_i^k) \) is a diagonal weight matrix.

Until stopping criterion is satisfied.

Output Final \( \tilde{\beta}^{(n,k)} \).

Remarks 3.1

(I) In the iterative process above, the matrix \( X_n' W_n^k X_n \) might not be invertible, so one might first check its determinant to see whether one needs a Moore-Penrose generalized inverse.

(II) The convergence of above procedure is guaranteed by Banach’s fixed-point theorem. That is, we treat equation (10) as \( x = \Phi(x) \), a fixed point problem, and \( \Phi \) is a contraction mapping. One might be unable to guarantee the objective value \( O^n(\tilde{\beta}^{(n,k)}) \) is reduced in the each step of the iterative process above. However, there are such procedures, which will be addressed next. The above procedure is a first-order derivative based one, next we consider a second-order derivative based procedure.
and \( t(k) > 0 \) (except when \( x(k) \) is optimal). \( \nabla x \) is a single entity, a vector in \( \mathbb{R}^p \) called the step or search direction (not necessarily have unit norm), and \( k = 0, 1, \cdots \) denotes the iteration number. The scalar \( t(k) \geq 0 \) is called the step size or step length at iteration \( k \). The terms ‘search step’ and ‘scale factor’ are more accurate, but ‘search direction’ and ‘step length’ are the ones widely used. Sometimes the superscripts are dropped and the lighter notation is adopted: \( x^+ = x + t \nabla x \), or \( x := x + t \nabla x \), in place of \( x^{(k+1)} = x^{(k)} + t^{(k)} \nabla x^{(k)} \).

A descent method means that
\[
 f(x^{(k+1)}) < f(x^{(k)}),
\]
except when \( x^{(k)} \) is optimal. From convexity we know that \( \nabla f(x^{(k)})' (y - x^{(k)}) \geq 0 \) implies \( f(y) \geq f(x^{(k)}) \), so the search direction in a descent method must satisfy
\[
(\nabla f(x^{(k)}))' \nabla x_{nt} < 0,
\tag{26}
\]
i.e., it must make an acute angle with the negative gradient. We call such a direction a descent direction (for \( f \), at \( x^{(k)} \)). In following we will focus on Newton’s method as a representative among the group of descent procedures.

**Newton step** The vector
\[
\nabla x_{nt} = - (\nabla^2 f(x))^{-1} \nabla f(x)
\]
is called the Newton step (for \( f \), at \( x \)). It is a descent direction (see (26)) when the Hessian \( \nabla^2 f(x) \) is positive definite, because if \( \nabla f(x) \neq 0 \):
\[
(\nabla f(x))' \nabla x_{nt} = - (\nabla f(x))' (\nabla^2 f(x))^{-1} \nabla f(x) < 0.
\]

There are several advantages of this search direction (step), see pages 484-485 of BV04 [2].

**Newton decrement** The quantity
\[
\lambda(x) = \left( (\nabla f(x))' (\nabla^2 f(x))^{-1} \nabla f(x) \right)^{1/2}
\]
is called the Newton decrement at \( x \). For explanations and justifications of this quantity, see pages 486-487 of BV04.

**AA2 Newton’s method**

**Input** a starting point \( x \) in the domain of \( f \), tolerance \( \varepsilon > 0 \).

**Repeat**

1. Compute the Newton step and decrement. \( \nabla x_{nt} := - (\nabla^2 f(x))^{-1} \nabla f(x); \lambda^2 := (\nabla f(x))' (\nabla^2 f(x))^{-1} \nabla f(x) \).

2. Stopping criterion. quit if \( \lambda^2 / 2 \leq \varepsilon \).
3. Line search. Choose step size $t$ by backtracking line search. (see page 464 of BV04)

4. Update. $x := x + t\nabla x_{nt}$.

**Output $x$.**

Note that in LTS case, or in $f(x) := O^n(x)$, $x := \beta$ case, the Hessian matrix $\nabla^2 f(x)$ is just positive semidefinite and might not be invertible at each step of search above. Denote by $H(x)$ for the Hessian matrix $\nabla^2 f(x)$. we address this issue next.

**Forcing the Hessian Matrix to Be Positive-Definite** (page 202 of Eetal01 [5]) Marquardt (1963) [14], Levenberg (1944) [13], and others have suggested that the Hessian matrix of $f(x)$ be modified on each stage of the search as needed to ensure that the modified $H(x), \tilde{H}(x)$, is positive-definite and well-conditioned. The procedure adds elements to the diagonal elements of $H(x)$

$$\tilde{H}(x) = H(x) + \gamma I$$

where $\gamma$ is a positive constant large enough to make $\tilde{H}(x)$ positive-definite when $H(x)$ is not. Note that with a $\gamma$ sufficiently large, $\gamma I$ can overwhelm $H(x)$ and the minimization approaches a steepest descent search (see page 475 of BV04 [2] or page 190 of Eetal01 [5]).

**A modified Marquardt-Levenberg method**

1. Pick $x^0$ the starting point. Let $\varepsilon$ be convergence criterion

2. Set $k = 0$ and $\gamma = 10^3$

3. Calculate $\nabla f(x^k)$

4. Is $\|\nabla f(x^k)\| < \varepsilon$? If yes, terminate. If no, continue

5. Solve $(H(x^k) + \gamma I)s^k = -\nabla f(x^k)$ for $s^k$

6. If $(\nabla f(x^k))'s^k < 0$, go to step 8.

7. Set $\gamma^k = 2\gamma^k$ and go to step 5.

8. Choose $a^k$ by a line search procedure (see page 464 of BV04) so that

$$f(x^k + a^k s^k) < f(x^k)$$

9. Reduce $\gamma$, set $x^k := x^k + a^k s^k$. Go to step 3 with $k$ replaced by $k + 1$.

Note that $s^k$ and $a^k$ above correspond to the $\nabla x_{nt}$ and $t$ in Newton’s method (AA2), respectively.

Next we shall illustrate the performance of procedures AA1 and AA2, compared with ItsReg, in data sets examples.
Standard bivariate normal points with 0% contamination, $\alpha = 1/2$

Table entries (a, b) are: a:=empirical mean squared error, b=total time consumed

| p  | n   | LTS-AA1          | LTS-AA2          | LTS-ltsReg       |
|----|-----|------------------|------------------|------------------|
| 50 | (0.0748, 1.0049) | (0.0739, 4.0203) | (0.0668, 9.1536) |
| 100 | (0.0525, 1.9697) | (0.0520, 5.2887) | (0.0302, 7.9755) |
| 2  | 200 | (0.0414, 5.7382) | (0.0411, 12.861) | (0.0154, 14.803) |
|    | 300 | (0.0311, 13.518) | (0.0305, 27.814) | (0.0096, 21.346) |
|    | 400 | (0.0261, 24.530) | (0.0252, 48.620) | (0.0073, 28.806) |

Table 1: Total computation time consumed for all 1000 samples (seconds) and empirical mean squared error (EMSE) of LTS-AA1, LTS-AA2, and LTS-ltsReg for various ns and $p = 2$.

4 Illustration examples and comparison

All R codes for simulation and examples as well as figures in this article (downloadable via https://github.com/zuo-github/comp-lts) were run on a desktop Intel(R)Core(TM) 21 i7-2600 CPU @ 3.40 GHz.

4.1 Simulated data

Example 4.1 Performance of LTS-AA1, LTS-AA2, and LTS-ltsReg (with Gaussian or contaminated Gaussian data).

For a general estimator $T$, if it is regression equivariant, then we can assume (w.l.o.g.) that the true parameter $\beta_0 = 0 \in \mathbb{R}^p$. We calculate $\text{EMSE} := \frac{1}{R} \sum_{i=1}^{R} \|T_i - \beta_0\|^2 / R$, the empirical mean squared error (EMSE) for $T$, where $R = 1000$, $\beta_0 = (0, \cdots, 0)\prime \in \mathbb{R}^p$, and $T_i$ is the realization of $T$ obtained from the ith sample with size $n$ and dimension $p$.

First, we consider the simple regression case (i.e. $p = 2$); since it is the most fundamental regression case and the most prevailing case in practice with the advantage of visual inspection and graphical illustration.

Simple regression We generate 1000 standard bivariate normal samples for different ns and calculate the EMSE and the total time consumed by LTS-AA1, LTS-AA2, and LTS-ltsReg. The EMSE and the total time consumed (in seconds) by different LTS procedures are listed in Table 1.

Inspecting the Table reveals that (i) LTS-ltsReg has an overwhelming advantage on EMSE (or accuracy, variance/efficiency) (smallest EMSE in all cases considered, this is mainly due to the inherited optimal property of the LS by FAST-LTS in RVD06 [24] and in ltsReg for perfect normal data sets) but not on the speed criterion (at least for $n < 300$); (ii) LTS-AA1
has the fastest speed for all \( n \) followed by LST-AA2 as the second for all \( n < 300 \) where as for \( n \geq 300 \) LTS-ltsReg becomes the second fastest; (iii) AA2 is superior over AA1 on EMSE for all cases considered; (iv) For larger \( n \) (e.g. \( n > 600 \)), the speed advantage of LTS-ltsReg, advocated in RVD06 \[24\], will demonstrate (results are not listed in the Table).

Perfect standard normal data are not realistic practically. Next we consider a 5% contamination of the bivariate standard normal data set with the contaminating points also from a bivariate normal distribution with mean vector \((7, -2)'\) and a covariant matrix with diagonal entries being 0.1 and off-diagonal entries being zero. With these data sets we redone the simulation and obtained the results listed in Table 2.

Bivariate normal data sets, each with 5% contamination. \( \alpha = 1/2 \)

Table entries \((a, b)\) are: \( a = \)empirical mean squared error, \( b = \)total time consumed

| \( p \) | \( n \) | LTS-AA1 | LTS-AA2 | LTS-ltsReg |
|------|------|--------|--------|------------|
| 50   | (0.0762, 1.0097) | (0.0743, 3.6453) | (0.0835, 9.0636) |
| 100  | (0.0571, 1.8714) | (0.0561, 4.7418) | (0.0638, 7.9453) |
| 2    | 200  | (0.0450, 5.3907) | (0.0442, 10.869) | (0.0552, 14.597) |
| 300  | (0.0383, 13.542) | (0.0377, 24.367) | (0.0527, 23.802) |
| 400  | (0.0337, 22.486) | (0.0327, 41.480) | (0.0519, 28.702) |

Table 2: Total computation time consumed for all 1000 samples (seconds) and empirical mean squared error (EMSE) of LTS-AA1, LTS-AA2, and LTS-ltsReg for various \( ns \) and \( p = 2 \).

Inspecting the Table immediately reveals that (i) the advantage of ltsReg on EMSE in the perfect bivariate standard norm case vanishes with just a typical/realistic contamination rate in practice and AA2 become the overall winner on EMSE; (ii) in terms of speed, AA1 is the overall winner, followed by AA2 when \( n \leq 200 \) and when \( n \geq 300 \), ltsReg becomes the second fastest; (iii) as expected, when \( n \) increases, EMSE decreases and time consumed increases for all three procedures.

When the contamination rate raises, the advantage of LTS-AA1 and LTS-AA2 over LTS-ltsReg remains and even more overwhelming. One of the performance instances of LTS-AA1, and LTS-ltsReg is graphically illustrated in Figure 2.

Due to the nature of FAST-LTS in RVD06 \[24\], the LTS-ltsReg chases the benchmark line-the LS line, paying a price for its high speed gain for large size \( (n > 600) \) data sets.

Note that LTS, asymptotically, allow up to 50\% contamination without breakdown. It certainly allows 30 contamination points in this case for \( n = 100 \) sample size. Indeed, according to Theorem 6 of RL87 (page 132), LTS actually allows \( [(n - p)/2] = 49 \) contaminating points in this case without breakdown. But this robust attribute can not be demonstrated by the approximate algorithm ltsReg, unless one invokes LTS-AA1 (or LTS-AA2).
Figure 2: 100 highly correlated (correlation coefficient between $x$ and $y$ is 0.88) normal points with 30% of them are contaminated by other normal points. Left panel: scatterplot of the uncontaminated data set which is fitted by LTS-AA1 (red dashed line), LTS-ltsReg (solid black line), and LS (green dotted line) (identical the LTS-ltsReg line). Right panel: LTS-AA1, LTS-ltsReg, and LS lines fit the 30% contaminated data set. Solid black is LTS-ltsReg line, dashed red is the LTS-AA1 induced line, dotted blue is the LS line which is parallel to the LTS-ltsReg line due the step “intercept adjustment” proposed in RVD06 [23].

Multiple regression. Above, we have restricted our attention to the case $p = 2$. In practice, there are important cases with $p > 2$. We now investigate the performance of LTS-AA1, LTS-AA2 versus LTS-ltsReg when $p > 2$. We continue the contaminated standard Gaussian data points scheme. Here we generate 1000 samples $Z_i = (x_i', y_i')'$ with various $n$s from the standard normal distribution $\mathcal{N}(\mu, \Sigma)$, where $\mu$ is a zero $p$-vector, and $\Sigma$ is a $p$ by $p$ identity matrix with diagonal entries being 1 and off-diagonal entries being 0. Then 30% of them are contaminated by normal points with $\mu$ being the $p$-vector with all elements being 7 except the last one being $-2$ and the covariance matrix being diagonal matrix with diagonal being 0.1. The results are listed in Table 3.
Standard normal data sets, each with 30% contamination, $h = \lfloor (n + p + 1)/2 \rfloor$
Table entries (a, b) are: a:=empirical mean squared error, b=total time consumed

| p | n     | LTS-AA1       | LTS-AA2       | LTS-ltsReg   |
|---|-------|---------------|---------------|--------------|
| 100 | (0.0980, 2.1303) | (0.0937, 5.3985) | (0.1302, 15.930) |
| 3  | 200   | (0.0789, 6.5610) | (0.0765, 12.212) | (0.0957, 29.530) |
|     | 300   | (0.0658, 15.020) | (0.0643, 26.755) | (0.0807, 46.401) |
| 100 | (0.1653, 3.4321) | (0.1462, 7.0947) | (0.2190, 26.080) |
| 6   | 200   | (0.1118, 11.502) | (0.1061, 17.078) | (0.1220, 46.475) |
|     | 300   | (0.0876, 27.389) | (0.0839, 36.070) | (0.0862, 70.385) |
| 100 | (0.2344, 3.8618) | (0.1961, 7.2528) | (0.3273, 40.421) |
| 9   | 200   | (0.1459, 13.541) | (0.1377, 19.695) | (0.1521, 79.052) |
|     | 300   | (0.1118, 33.770) | (0.1074, 41.667) | (0.1033, 117.75) |

Table 3: Total computation time consumed for all 1000 samples (seconds) and empirical mean squared error (EMSE) of LTS-AA1, LTS-AA2, and LTS-ltsReg for various ns and ps.

Examining the Table reveals that (i) in terms of computation speed, there is no hope with LTS-ltsReg when there are 30% contamination (this does not contradict the known fact that ltsReg is fast for perfect normal data sets). LTS-AA1 is the overall winner for all ns and ps; (ii) Among LTS-AA1 and LTS-AA2, the latter is the winner on EMSE whereas the former takes its turn on speed; (iii) the case $p = 9$ and $n = 300$ gives a bright ray of light in a dark tunnel for LTS-ltsReg, it has the smallest EMSE.

Remarks 4.1

(I) Parameters tuning Our default $\alpha$ value is $1/2$, when $p = 2$, $h = \lfloor \alpha n \rfloor + 1 = \lfloor n/2 \rfloor + 1$ is identical to the default value $h = \lfloor (n + p + 1)/2 \rfloor$ in ltsReg. When $p > 2$, it is no longer equal to the default value, so we adopt this $h$ in Table 3 for our LTS-AA1 and LTS-AA2.

Throughout, the parameter $c$ is set to be 0, i.e. the initial $\beta$ is set to be the zero vector; the parameter $N$, the total allowed iterative steps in LTS-AA1, is set to be 200; the parameter $\gamma$ in LTS-AA2 is set to be 100; the threshold $\epsilon$ (stopping criterion) in LTS-AA1 and LTS-AA2 is set to be $10^{-3}$.

(II) Asymptotically, LTS can resist up to 50% contamination without breakdown. In the finite sample case, in light of Theorem 6 of RL87 on page 132, LTS can resist $\lfloor (n - p)/2 \rfloor$ contaminating points. In Table 3 we have $30\%n < \lfloor (n - p)/2 \rfloor$ for all ns and ps. So 30% contamination is allowed. But it seems that LTS-ltsReg is greatly affected by this amount of contamination with the largest EMSE in all cases but one.
Example 4.2 Performance of the three procedures with respect to a given true $\beta_0$ So far we have assumed that the true $\beta_0$ is the zero vector based on the regression equivariance. One might be unconformable with this assumption.

Now we examine the performance of the three procedures for computing LTS in a slightly different setting. We generate 1000 samples $\{(x_i', y_i')' \in \mathbb{R}^p\}$ with a fixed sample size 100 from an assumed model: $y = \beta_0'x + e$, where $x = (1, x_1, \ldots, x_{p-1})'$ and $\beta_0 = (\beta_0, \ldots, \beta_{p-1})'$ are in $\mathbb{R}^p$ and $x_i$ and $e$ are all from $\mathcal{N}(\mu, \sigma^2)$ distribution.

We list the total time consumed (in seconds) and the EMSE (the same formula as before but the true $\beta_0$ is the given one no longer being the zero vector) for the three methods with respect to three different $\beta_0$’s in Table 4.

**Case I** $\beta_0 = (-2, 0.1, 1)'$, all $x_i$ from $\mathcal{N}(0, 1)$ and $e$ from $\mathcal{N}(4, 2)$.

**Case II** $\beta_0 = (-2, 0.1, 1, 5)'$, all $x_i$ from $\mathcal{N}(0, 1)$ and $e$ from $\mathcal{N}(4, 1)$.

**Case III** $\beta_0 = (50, 0.1, -2, 15, 100)'$, all $x_i$ from $\mathcal{N}(0, 1)$ and $e$ from $\mathcal{N}(4, 0.1)$.

Replication 1000 times, $n = 100$

| Performance criteria | LTS-AA1 | LTS-AA2 | LTS-ltsReg |
|----------------------|---------|---------|------------|
| **Case I** $p = 3$   | EMSE    |         |            |
|                      | 12.69950| 12.67103| 16.14046   |
|                      | Total time consumed | 3.216512 | 7.506577 | 16.023746 |
| **Case II** $p = 4$  | EMSE    |         |            |
|                      | 15.35819| 15.28321| 16.05240   |
|                      | Total time consumed | 4.078485 | 9.978173 | 18.569919 |
| **Case III** $p = 5$ | EMSE    |         |            |
|                      | 15.99660| 15.86961| 15.99881   |
|                      | Total time consumed | 3.525999 | 12.125942 | 21.797242 |

Table 4: Performance of LST-AA1, LTS-AA2, versus LTS-ltsReg for three true $\beta_0$’s.

Inspecting the Table immediately reveals that (i) all three procedures have almost the same EMSE in all cases, however, they are yet distinguishable. LTS-AA2 always has the smallest EMSE, LTS-AA1 has the second smallest EMSE, LTS-ltsReg has the largest EMSE in all cases; (ii) in terms of computation speed, LTS-ltsReg distinguishes itself with an outstanding largest consumed time in all cases, LTS-AA1 is the overall winner in speed followed by LTS-AA2.

$\blacksquare$
Up to this point, we have dealt with synthetic data sets. Next we investigate the performance of three procedures with respect to real data sets in higher dimensions.

### 4.2 Real data

**Example 4.3 Performance of the three procedures for textbook size data sets**

We first look at real data sets with relatively small sample size $n$ and moderate dimension $p$. All data sets (taken from RL87), with names are listed below in the Table 5, explained in details in the textbook. Since all methods depend on randomness, we run the computation with replication number $R = 1000$ times to alleviate the randomness, we then calculate the total time consumed (in seconds) by different methods for all replications, and the EMSE (with true $\beta_0$ being replaced by the sample mean of 1000 $\hat{\beta}_{\text{lts}}^n$), which then is the sample variance of all $\hat{\beta}_{\text{lts}}^n$s up to a factor $R/(R - 1)$. The results are reported in Table 5.

Table entries (a, b) are: $a$: empirical variance of $\hat{\beta}_{\text{lts}}^n$, $b$: total time consumed

| data set | (n, p) | LTS-AA1 | LTS-AA2 | LTS-ltsReg |
|----------|-------|---------|---------|------------|
| Heart    | (12, 3)| (109.06, 2.4627) | (108.97, 35.631) | (5395.3, 3.5088) |
| Phosphor | (18, 3)| (2694.1, 3.4934) | (1926.1, 32.783) | (4838.9, 6.8473) |
| Delivery | (25, 3)| (37.628, 2.5511) | (9.1771, 24.883) | (13.986, 6.1227) |
| Salinity | (28, 4)| (363.94, 5.1979) | (1.8437, 37.423) | (2220.1, 8.2601) |
| Aircraft | (23, 5)| (120.96, 1.8730) | (16.800, 34.981) | (178.56, 8.3204) |
| Wood     | (20, 6)| (2.0774, 2.7545) | (0.8387, 44.820) | (2.0821, 9.2989) |
| Coleman  | (20, 6)| (2285.3, 5.3761) | (403.96, 51.202) | (1588.7, 9.3864) |

Table 5: Total time consumed (in seconds) and sample variance in 1000 replications by LTS-AA1, LTS-AA2, and LTS-ltsReg for various real data sets.

Inspecting the Table divulges that (i) in terms of sample variance (the EMSE), LTS-AA2 is the overall winner, and LTS-ltsReg is the loser (with exceptions for salinity and coleman where LTS-AA1 has the largest sample variance); (ii) in terms of speed, LTS-AA1 is the overall winner and LTS-AA2 is the loser, LTS-ltsReg is the second fastest runner.

The limitation of this example is that the data sets are still relatively small and not in very high dimensions. We examine a high dimension and large sample dataset next.

**Example 4.4 Performance of the three procedures for a large data set** (Plasma Retinol...
and Beta-Carotene Levels [15]). This data set contains 315 observations on 14 variables. The goal is to investigate the relationship between personal characteristics and dietary factors, and plasma concentrations of retinol, beta-carotene, and other carotenoids. For a more detailed description, see [http://lib.stat.cmu.edu/datasets/](http://lib.stat.cmu.edu/datasets/).

We fit the data with the three procedures. To evaluate the performance of the three procedures: (i) we sample \( m \) points (without replacement) \((m = 315, \text{ entire data set}, \text{ or } m = 100, 150, 200, 250)\), compute the \( \hat{\beta}_{ltS}^{n} \)s with different methods, we do this RepN= 10^4 times; (ii) we calculate the total time consumed (in seconds) for all replications by different methods, and the EMSE (with true \( \beta_0 \) being replaced by the sample mean of RepN \( \hat{\beta}_{ltS}^{n} \)s from (i)), which then is the sample variance of all \( \hat{\beta}_{ltS}^{n} \)s up to a constant factor. The results are reported in Table 6.

Table entries (a, b) are: a:=empirical variance of \( \hat{\beta}_{ltS}^{n} \)s, b=total time consumed

| \( p \) | \( m \) | \( \text{LTS-AA1} \) | \( \text{LTS-AA2} \) | \( \text{LTS-ltsReg} \) |
|---|---|---|---|---|
| 100 | 90.6298, 0.00698 | 1.89412, 0.04498 | 130.001, 0.03098 |
| 150 | 477.140, 0.18788 | 164.043, 0.56966 | 1103.01, 0.86400 |
| 200 | 71.8897, 0.08895 | 10.6509, 0.11293 | 87.6239, 0.10194 |
| 250 | 43.9857, 0.06796 | 20.0034, 0.17889 | 99.2728, 0.13192 |
| 315 | 65.7140, 0.22886 | 11.8136, 0.72457 | 194.587, 0.37577 |

Table 6: Total time consumed (in seconds) and sample variance in 10000 replications by LTS-AA1, LTS-AA2, and LTS-ltsReg for real data sets with various sample size \( m \)'s and \( p = 14 \).

Inspecting the Table reveals that (i) in terms of sample variance (or EMSE), LTS-AA2 is the overall winner with the smallest EMSE in all cases considered whereas LTS-ltsReg is the loser with the largest sample variance in all cases; (ii) in terms of speed, LTS-AA1 is the overall winner whereas LTS-ltsReg becomes the second fastest runner (except \( m = 150 \) case); (iii) the EMSEs in table are actually favorable to ltsReg since it often outputs a zero vector (initialized one) due to singular matrices frequent appearance in the routine.

So far we mainly focus on the evaluation of the performance of three algorithms for LTS. Next we will assess the quality (predictability) of estimates by the three procedures.

Our new evaluation scheme is: (i) we first sample \( m \) points from total \( n = 315 \) points, and get the estimates from different methods, (ii) then we calculate the prediction error by computing the sum of squares (SS) of the residuals of the \((n-m)\) left points with respect to the estimate just obtained based on the \( m \) points, (iii) we do this 10000 times with different \( m \)s, \( 100 \leq m \leq 300 \).
Due to the singular matrix frequent appearance in ltsReg, to have a fair comparison we compute the SS for all three procedures only when ltsReg does not have a such issue in a replication (that is, the total valid replication numbers might be much smaller than $10^4$). The results are listed in Table 7.

Table entries $(a, b)$ are: $a$: mean sum of squared prediction errors, $b$: total time consumed

| $p$ | $m$     | LTS-AA1           | LTS-AA2           | LTS-ltsReg         |
|-----|---------|-------------------|-------------------|-------------------|
| 100 | (3230.05, 0.00600) | (3383.77, 0.04897) | (3931.23, 0.05997) |
| 150 | (3404.29, 0.02499) | (3684.14, 0.08945) | (3981.58, 0.15791) |
| 200 | (1110.76, 0.01899) | (1258.55, 0.04797) | (1453.91, 0.05397) |
| 250 | (573.198, 0.02799)  | (742.520, 0.06296) | (852.563, 0.06496) |
| 300 | (461.078, 0.21439)  | (504.202, 0.52069) | (699.245, 0.46204) |

Table 7: Mean sum of squared errors of prediction based on $m$ sub-sample points and total time consumed (in seconds) in 10000 replications by LTS-ltsReg, LTS-AA1, and LTS-AA2 for Plasma data with various $m$’s and $p = 14$.

Inspecting the Table reveals that (i) LTS-AA1 is the overall winner both in the accuracy (sample variance) and speed criteria; (ii) in terms of accuracy, LTS-ltsReg is the loser, it is also the loser in terms of speed (except in the case $m = 300$ where LTS-AA2 is the slowest runner).

5 Concluding remarks

Combinatoric v.s. analytic approach Existing approximate algorithms in the literature, including, among others, the feasible solution algorithm (FSA) of Hawkins (1994), ltsReg or FAST-LTS, are either sub-sampling or combinatoric approach to compute LTS. LTS-AA1 and LTS-AA2 are employing an analytic approach to compute LTS. The advantage of the former approach is its speed of computation of LTS whereas the disadvantage is its large variability (or sample variance). This is confirmed in our examples. By taking the full advantage of differentiability of the objective function in regression parameter, LTS-AA1 and LTS-AA2 are not inferior in speed, compared with ltsReg - well-known for its speed, the fastest runner in the literature.

The phenomenon of ltsReg resembling LS ltsReg searches, among many LS estimators resulting from fitting $h$ points out of entire data and returns the one with the least sum of $h$ smallest (among $n$) squares of residuals. So the final solution is an LS estimate (up to an intercept difference) to a specific $h$-points subset. Therefore its resembling an LS behavior is predictable. It inherits both the advantage (easy for computation) and disadvantage (sensitive to outliers or contamination).
**Fairness of performance criteria**  
ItsReg calls the FORTRAN subroutine rftsltreg in its computation process whereas both AA1 and AA2 are pure R function based computation. The comparison among the speeds of LTS-AA1 and LTS-AA2, and LTS-ltsReg in Section 4 is based on the unfair ground. However, the results are not very favorable to LTS-ltsReg.

**Randomness of results in Tables**  
All entries in the Tables in Section 4 are not deterministic since all algorithms involve randomness. This is especially true for the Tables 6 and 7 where subsampling of $m$ point out of total $n$ points took place. That is, another run of the three algorithms will not yield identical results. That said, the overall pattern in the tables will remain consistent.

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