Newtonian invariance amendment for \(n\)-dimensional position-dependent mass Lagrangians: nonlocal point transformations

Omar Mustafa

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10 - Turkey, Tel.: +90 392 6301378; fax: +90 3692 365 1604.

Abstract: We argue that, under multidimensional position-dependent mass (PDM) settings, the Euler-Lagrange textbook invariance falls short and turned out to be vividly incomplete and/or insecure for a set of PDM-Lagrangians. We show that the transition from Euler-Lagrange component presentation to Newtonian vector presentation is necessary and vital to guarantee invariance. The totality of the Newtonian vector equations of motion is shown to be more comprehensive and instructive than the Euler-Lagrange component equations of motion (they do not run into conflict with each other though). We have successfully used the Newtonian invariance amendment, along with some nonlocal space-time point transformation recipe, to extract exact solutions for a set of \(n\)-dimensional nonlinear PDM-oscillators. They are, Mathews-Lakshmanan type-I PDM-oscillators, power-law type-I PDM-oscillators, the Mathews-Lakshmanan type-II PDM-oscillators, the power-law type-II PDM-oscillators, and some nonlinear shifted Mathews-Lakshmanan type-I PDM-oscillators.

PACS numbers: 05.45.-a, 03.50.Kk, 03.65.-w

Keywords: \(n\)-dimensional position-dependent mass Lagrangians, nonlocal point transformation, Euler-Lagrange equations invariance, Newtonian invariance amendment.

I. INTRODUCTION

Classical and quantum mechanical particles endowed with position-dependent mass (PDM) have initiated a substantial amount of research interest over the last few decades \([1-44]\). The position-dependent mass concept is, basically, a consequential manifestation of either a position-dependent deformation in the standard constant mass setting, or a position-dependent deformation in the coordinates settings. Which, in turn, offers a mathematically challenging problem in both classical and quantum mechanics. In quantum mechanics for example, the ordering ambiguity associated with the non-unique representation of the PDM von Roos Hamiltonian \([1]\) possess a significant amount of arguments as to what are the most proper parametric ordering settings (e.g., \([2-17]\)). It has been only recently that a proper definition for the position-dependent mass momentum operator is introduced by Mustafa and Algadhi \([5]\), resolving, hereby, the ordering ambiguity conflict. In classical mechanics, nevertheless, exact solutions to multidimensional PDM Euler-Lagrange equations are hard to find (e.g., \([18-21, 30, 33, 38, 44]\) and references cited therein). One should, therefore seek some kind of nonlocal space-time point transformations that guarantees Euler-Lagrange invariance and facilitates exact solvability.

Based on the readily existing one-dimensional version \([38]\), Mustafa in \([44]\) has very recently embarked upon the \(n\)-dimensional extension of the PDM Lagrangians via a nonlocal space-time point transformation and sought invariance between the standard "constant" mass and PDM Euler-Lagrange equations. Two \(n\)-dimensional PDM Lagrangian models were used,

\[
L_I \left( \vec{x}, \vec{\dot{x}} ; t \right) = \frac{1}{2} m_0 \sum_{j=1}^{n} m_j \left( x_j \right) \dot{x}_j^2 - V_I \left( \vec{x} \right); \quad \dot{x}_j = \frac{dx_j}{dt}; \quad j = 1, 2, \cdots, n \in \mathbb{N},
\]

(1)

and

\[
L_{II} \left( \vec{x}, \vec{\dot{x}} ; t \right) = \frac{1}{2} m_0 \sum_{j=1}^{n} m \left( \vec{x} \right) \dot{x}_j^2 - V_{II} \left( \vec{x} \right).
\]

(2)

Where, \(m_0\) is the standard "constant" mass, \(m_j \left( x_j \right)\) in \(L_I \left( \vec{x}, \vec{\dot{x}} ; t \right)\) is a dimensionless scalar multiplier that deforms each coordinate \(x_j\) and/or velocity component \(\dot{x}_j\) in a specific functional form, and \(m \left( \vec{x} \right)\) in \(L_{II} \left( \vec{x}, \vec{\dot{x}} ; t \right)\) represents

*Electronic address: omar.mustafa@emu.edu.tr*
a common dimensionless scalar multiplier that deforms the coordinates $x_i$’s and/or velocity components $\dot{x}_i$’s. Hence, similar consequential position-dependent deformations in the potential force fields $V_i(\vec{x})$ and $\tilde{V}_i(\vec{x})$ are unavoidable in the process. Moreover, Mustafa [44] has considered a conventional constant-mass Lagrangian

$$L(\vec{q}, \vec{\dot{q}}; \tau) = \frac{1}{2} m_o \sum_{j=1}^{n} \vec{q}_j^2 - V(\vec{q}); \quad \vec{\dot{q}}_j = \frac{dq_j}{d\tau}; \quad j = 1, 2, \ldots, n,$$

(3)

where $\vec{q} = (q_1, q_2, \ldots, q_n)$ are some generalized coordinates and $\tau$ is a re-scaled time. The idea is simply a manifestation of Euler-Lagrange textbook invariance procedure. That is, the Euler-Lagrange equations for $L_i(\vec{q}, \vec{\dot{q}}; \tau)$ and $L_{ii}(\vec{q}, \vec{\dot{q}}; \tau)$ should be invariant with those for $L(\vec{q}, \vec{\dot{q}}; \tau)$, if the used nonlocal space-time point transformation is deemed useful. However, it turned out that whilst the Euler-Lagrange equations for $L_{ii}(\vec{q}, \vec{\dot{q}}; \tau)$ failed to satisfy invariance conditions for $n \geq 2$, the Euler-Lagrange equations for $L_i(\vec{q}, \vec{\dot{q}}; \tau)$ proved to satisfy the invariance conditions for $n \geq 1$. Such results would consequently render the $n$-dimensional extension of the used nonlocal point transformation [44] for the $L_i(\vec{q}, \vec{\dot{q}}; \tau)$ as a minor and/or trivial progress. The said approach [44] may very well copy and paste the very recent work for the one-dimensional PDM-Lagrangians of Mustafa [38] (along with all examples discussed and reported therein) for each degree of freedom $x_i$. For more details on this issue the readers may refer to Mustafa [44]. In the current methodical proposal, nevertheless, we propose a remedy to this invariance problem in the form of the what, hereinafter, should be called "Newtonian invariance amendment". This to be viewed as a comeback of the traditional textbook Euler-Lagrange and Newtonian dynamical correspondence under PDM-settings. To the best of our knowledge, this has never been reported elsewhere in the literature. The organization of our article is in order.

In section 2, we recycle, in short, the Euler-Lagrange equations invariance for $L_i(\vec{q}, \vec{\dot{q}}; \tau)$ and $L_{ii}(\vec{q}, \vec{\dot{q}}; \tau)$ with those for $L(\vec{q}, \vec{\dot{q}}; \tau)$. This would make the current methodical proposal self-contained and vividly instructive. In the same section, we introduce our Newtonian invariance amendment. Hereby, we show that while the textbook Euler-Lagrange invariance (for $n \geq 2$) proved satisfactory only for $L_i(\vec{q}, \vec{\dot{q}}; \tau)$, the Newtonian invariance amendment is found to be satisfactory for both $L_i(\vec{q}, \vec{\dot{q}}; \tau)$ and $L_{ii}(\vec{q}, \vec{\dot{q}}; \tau)$, $n \geq 1$ (when compared with the conventional constant-mass Euler-Lagrange equations for $L(\vec{q}, \vec{\dot{q}}; \tau)$, of course). We may, therefore, label $L_i(\vec{q}, \vec{\dot{q}}; \tau)$ and $L_{ii}(\vec{q}, \vec{\dot{q}}; \tau)$ as "target-Lagrangians" and $L(\vec{q}, \vec{\dot{q}}; \tau)$ as "reference Lagrangian". We devote section 3 to illustrate our methodical proposal and use some conventional constant-mass $n$-dimensional nonlinear oscillators Lagrangian $L(\vec{q}, \vec{\dot{q}}; \tau)$ as a reference Lagrangian. Where, three different PDM-settings are successfully used within our nonlocal space-time point transformation recipe. They are, $\vec{q}(\vec{r}) = \sqrt{m(\vec{r})} \vec{r}, \vec{\dot{q}}(\vec{r}) = \sqrt{m(\vec{r})} \vec{\dot{r}}$ (where $\vec{\dot{r}}$ is a constant vector), and $\vec{\ddot{q}}(\vec{r}) = \sqrt{m(\vec{r})} (\vec{\dot{r}} + \vec{\ddot{r}})$. Such vector transformational recipes are mandatory substitutional settings for our Newtonian invariance amendment. In the same section, moreover, some $n$-dimensional illustrative examples are used. Amongst are, the Mathews-Lakshmanan type-I PDM-oscillators (36), the power-law type-I PDM-oscillators (40), the Mathews-Lakshmanan type-II PDM-oscillators (49), the power-law type-II PDM-oscillators (51), and some nonlinear shifted Mathews-Lakshmanan type-I PDM-oscillators (60). We conclude in section 4.

II. NEWTONIAN INvariance AMENDMENT TO PDM Euler-LAGRANGE EQUATIONS

We start with recollecting/recycling some vital parts of the $n$-dimensional extension of the PDM Lagrangians (via a nonlocal point transformation) work by Mustafa [44]. Therefore, we begin with the implementation of Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0; \quad i = 1, 2, \ldots, n \in \mathbb{N},$$

(4)

to obtain (with $m_o = 1$ throughout) $n$ PDM Euler-Lagrange equations (PDM EL-I)

$$\ddot{x}_i + \left( \frac{m_i(x_i)}{2m_i(x_i)} \right) \dot{x}_i + \left( \frac{1}{m_i(x_i)} \right) \partial x_i V_i(\vec{x}) = 0; \quad \ddot{x}_j = \frac{d^2 x_j}{dt^2},$$

(5)
for \( L_i \left( \vec{x}, \vec{\dot{x}}; t \right) \), and PDM EL-II

\[
\ddot{x}_i + \left( \frac{\dot{m}(\vec{x})}{m(\vec{x})} \right) \dot{x}_i - \frac{1}{2} \left( \frac{\partial_x m(\vec{x})}{m(\vec{x})} \right) \sum_{j=1}^n \ddot{x}_j + \left( \frac{1}{m(\vec{x})} \right) \partial_x V_{i\dot{t}}(\vec{x}) = 0, \tag{6}
\]

for \( L_{i\dot{t}} \left( \vec{x}, \vec{\dot{x}}; t \right) \), where \( \partial_{x_i} = \partial/\partial x_i \). Yet, the Euler-Lagrange equations for \( L\left( \vec{q}, \vec{\dot{q}}; \tau \right) \) (EL-G) yield

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \iff \frac{d}{d\tau} \dot{q}_i + \frac{\partial}{\partial q_i} V(\vec{q}) = 0 \tag{7}
\]

At this point, we shall seek some sort of feasible invariance for PDM EL-I of (5) and for PDM EL-II of (6) with EL-G of (7). In so doing, Mustafa [44] has suggested that we may extend/generalize the one-dimensional nonlocal point transformation [38] to fit into the \( n \)-dimensional settings and re-scale both time and space through

\[
d\tau = f(\vec{x}) \, dt, \quad dq_i = \delta_{ij} \sqrt{g(\vec{x})} \, dx_j \implies \frac{\partial q_i}{\partial x_j} = \delta_{ij} \sqrt{g(\vec{x})}. \tag{8}
\]

This would necessarily mean that the unit vectors in the direction of \( q_i \) are obtained as

\[
\dot{q}_i = \frac{\sum_{k=1}^n \left( \frac{\partial x_k}{\partial q_i} \right) \ddot{x}_k}{\sqrt{\sum_{k=1}^n \left( \frac{\partial x_k}{\partial q_i} \right)^2}} \implies \dot{q}_i = \ddot{x}_i; \quad i = 1, 2, \ldots, n. \tag{9}
\]

Where the dimensionless functional structure of \( f(\vec{x}) \) and \( g(\vec{x}) \) shall be determined in the process below. Under such settings, one obtains

\[
\dot{q}_j = \frac{\sqrt{g(\vec{x})}}{f(\vec{x})} \dot{x}_j \quad \implies \quad \frac{d}{d\tau} \dot{q}_j = \frac{\sqrt{g(\vec{x})}}{f(\vec{x})^2} \left( \ddot{x}_j + \dot{x}_j \left[ \frac{\dot{g}(\vec{x})}{2g(\vec{x})} - \frac{f(\vec{x})}{g(\vec{x})} \right] \right), \tag{10}
\]

and hence EL-G of (7) would read

\[
\ddot{x}_j + \left( \frac{\dot{g}(\vec{x})}{2g(\vec{x})} - \frac{f(\vec{x})}{g(\vec{x})} \right) \dot{x}_j + \left( \frac{f(\vec{x})}{g(\vec{x})} \right) \partial_{x_j} V(q(\vec{x})) = 0; \quad j = 1, 2, \ldots, n. \tag{11}
\]

Obviously, the invariance between EL-I of (5) and the resulting EL-G of (11) is feasible and is simply summarized by the relations

\[
\frac{d}{d\tau_i} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \iff \begin{cases} 
\frac{\partial q_i}{\partial x_j} = \sqrt{g_i(\vec{x})}, \\
\frac{dr_i}{dt} = f_i(\vec{x}_i) \\
g_i(\vec{x}_i) = m_i(\vec{x}_i) f_i(\vec{x}_i)^2 \\
V(\vec{x}) = V(\vec{q}(\vec{x})) \\
\dot{q}_i(\vec{x}_i) = \frac{1}{r_i(\vec{x}_i)} \dot{q}_i(\vec{x}_i) = \ddot{x}_i \sqrt{m_i(\vec{x}_i)} 
\end{cases} \iff \frac{d}{d\tau} \left( \frac{\partial L_i}{\partial \dot{x}_i} \right) - \frac{\partial L_i}{\partial x_i} = 0. \tag{12}
\]

Nevertheless, it is clear that the dynamics of the \( n \)-dimensional PDM-system of \( L_i \left( \vec{x}, \vec{\dot{x}}; t \right) \) in (1) fully decouples and collapses into \( n \) one-dimensional dynamical systems for each degree of freedom \( x_i \) (i.e., \( n \)-dimensional PDM EL-I equations of motion). This would, in turn, render the \( n \)-dimensional extension proposal of Mustafa [44] as a minor and/or a trivial progress. For this approach [44] may very well copy and paste our very recent work for the one-dimensional PDM-Lagrangians in [38] along with all examples discussed and reported therein, for each degree of freedom \( x_i \). Whereas, the comparison between EL-G of (11) and PDM EL-II of (6) is only possible for the one-dimensional problems (i.e., for \( n = 1 \)). In this case, (6) collapses into (5) for \( i = 1 = n \). Nevertheless, for the multidimensional case \( n \geq 2 \), the third term in (6) has no counterpart in (11). This would, in effect, make the comparison incomplete/impossible and insecure. That is, for \( n \geq 2 \) the Euler-Lagrange equations (6) and (11) suggest...
that the invariance is, apparently, still far beyond reach. Hereby, "Newtonian invariance amendment" comes into action.

Apriori, one should be reminded that a transition from Euler-Lagrange into Newtonian dynamics is a simple textbook procedure and is in order. Let us recollect the PDM EL-II of (6) and phrase it to fit into Newtonian vector dynamics. We do so by associating with each degree of freedom a corresponding unit vector \( \hat{x}_i \) and sum up over \( i = 1, 2, \ldots, n \) to get

\[
m(\ddot{x}) \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \dot{m}(\ddot{x}) \sum_{i=1}^{n} \dot{x}_i \dot{x}_i - \frac{1}{2} \sum_{i=1}^{n} \partial_{x_i} m(\ddot{x}) \dot{x}_i \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) + \sum_{i=1}^{n} \dot{x}_i \partial_{x_i} V_{II}(\ddot{x}) = 0. \tag{13}
\]

This would allow us to preset the current equation in vector format settings, with \( m(\ddot{x}) = m(r) \) and \( m'(r) = dm(r)/dr \), as

\[
m(r) \ddot{a} + m'(r) \left[ \frac{\ddot{\upsilon} (\dddot{\upsilon} \cdot \dddot{\upsilon})}{r} \right] - \frac{1}{2} m'(r) \left[ \frac{\dddot{r} (\dddot{\upsilon} \cdot \dddot{\upsilon})}{r} \right] + \nabla V_{II}(\ddot{x}) = 0, \tag{14}
\]

where, we have used

\[
m(\ddot{x}) = \sum_{k=1}^{n} \partial_{x_k} m(r) \dot{x}_k = \frac{m'(r)}{r} \sum_{k=1}^{n} x_k \dot{x}_k = \frac{m'(r)}{r} (\dddot{r} \cdot \dddot{\upsilon}); \dddot{r} = \sum_{k=1}^{n} x_k \dddot{x}_k, \dddot{\upsilon} = \sum_{k=1}^{n} \dddot{x}_k \dot{x}_k. \tag{15}
\]

Obviously, equation (14) reduces to \( m_o \ddot{a} = - \nabla V_{II}(\ddot{x}) \) for conventional constant mass settings (i.e., for \( m_o \neq 1 \) and \( m(r) = 1 \)). At this point, nevertheless, one may recall the cross product identity

\[
\ddot{a} \times (\dddot{\upsilon} \times \dddot{r}) = \dddot{b} (\dddot{a} \cdot \dddot{r}) - \dddot{c} (\dddot{a} \cdot \dddot{b}) \quad \Longrightarrow \quad \dddot{a} \times (\dddot{\upsilon} \times \dddot{r}) = \dddot{\upsilon} (\dddot{\upsilon} \cdot \dddot{r}) - \dddot{r} (\dddot{\upsilon} \cdot \dddot{\upsilon}). \tag{16}
\]

Which, obviously, suggests that for the case where \( \dddot{\upsilon} \) is parallel to \( \dddot{r} \) one obtains

\[
\dddot{\upsilon} (\dddot{\upsilon} \cdot \dddot{r}) = \dddot{r} (\dddot{\upsilon} \cdot \dddot{\upsilon}) \quad ; \quad \dddot{r} \parallel \dddot{\upsilon}, \tag{17}
\]

so that

\[
\dddot{a} + \frac{1}{2} \frac{m'(r)}{m(r)} \left[ \frac{\dddot{r} (\dddot{\upsilon} \cdot \dddot{\upsilon})}{r} \right] + \frac{1}{m(r)} \nabla V_{II}(\ddot{x}) = 0. \tag{18}
\]

This would be acceptable for the case of no rotational effects involved (i.e., the case we are considering here), otherwise the Lagrangian structure would include, in addition to the translational kinetic energy term of (2), a rotational kinetic energy term (c.f., e.g., the two-dimensional nonlinear oscillator kinetic energy term in \([29]\) and equation (3.1) in \([31]\)) and a different treatment would be required, therefore. Similarly, in a straightforward manner, one can show that EL-G of (11) can be rewritten (in the Newtonian vector form) as

\[
\dddot{a} + \left( \frac{g'(\ddot{x})}{2g(\ddot{x})} - \frac{f'(\ddot{x})}{f(\ddot{x})} \right) \left[ \frac{\dddot{r} (\dddot{\upsilon} \cdot \dddot{\upsilon})}{r} \right] + \left( \frac{f(\ddot{x})^2}{\sqrt{g(\ddot{x})}} \right) \nabla_q V(q(\ddot{x})) = 0; \quad \nabla_q = \sum_{i=1}^{n} \dddot{x}_i \partial_{x_i} ; \dddot{q}_i = \dddot{x}_i. \tag{19}
\]

We got now consistency and exact correspondence between (18) and (19). That is, the invariance between EL-G of (11) and PDM EL-II of (6) is now secured and mandates that

\[
\frac{1}{2} \frac{m'(r)}{m(r)} = \frac{g'(\ddot{x})}{2g(\ddot{x})} - \frac{f'(\ddot{x})}{f(\ddot{x})} \quad \iff \quad \frac{1}{m(r)} = \frac{f(\ddot{x})^2}{g(\ddot{x})} \quad \iff \quad g(r) = m(r) f(r)^2. \tag{20}
\]

Consequently, not only do we have consistency between (18) and (19) but also we have secured Newtonian invariance between (6) and (7). We may, therefore, safely rewrite (19) as

\[
\dddot{a} + \frac{1}{2} \frac{m'(r)}{m(r)} \left[ \frac{\dddot{r} (\dddot{\upsilon} \cdot \dddot{\upsilon})}{r} \right] + \frac{1}{m(r)} \nabla V(q(\ddot{x})) = 0, \tag{21}
\]

which immediately implies that \( V_{II}(\ddot{x}) = V(q(\ddot{x})) \). Wherein, we have used the relations

\[
\nabla V_{II}(\ddot{x}) = \sqrt{g(r)} \nabla_q V(q(\ddot{x})) \iff \sum_{i=1}^{n} \dddot{x}_i \partial_{x_i} V_{II}(\ddot{x}) = \sum_{i=1}^{n} \dddot{x}_i \partial_{x_i} V(q(\ddot{x})) \iff V_{II}(\ddot{x}) = V(q(\ddot{x})). \tag{22}
\]
This result is to be used to determine \( q_i (\tilde{x}) \)'s as well as the form of \( f(r) \) (consequently \( g(r) \)) for a given \( m(r) \). Moreover, in a straightforward manner, the same procedure can be followed to show that the PDM EL-I is also Newtonian invariant. Yet, we are now able to dismantle \((21)\) into \( n \) component equations

\[
\ddot{x}_i + \left( \frac{m(r)}{2m(r)} \right) \dot{x}_i + \left( \frac{1}{m(r)} \right) \partial_{x_i} V_{II} (\tilde{x}) = 0; \quad i = 1, 2, \ldots, n, \tag{23}
\]

or

\[
\ddot{x}_i + \frac{m'(r)}{2rm(r)} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + \frac{1}{m(r)} \partial_{x_i} V_{II} (\tilde{x}) = 0; \quad i = 1, 2, \ldots, n. \tag{24}
\]

Where both forms hold true under our current settings. As such, our nonlocal point transformation within our "Newtonian invariance amendment" is summarized by

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad \iff \quad \begin{cases}
\dot{q}_i = \dot{x}_i, \\
\frac{\partial q_i}{\partial x_i} = \sqrt{g(r)}, \\
\frac{d\tau}{dt} = f(r) \frac{dt}{d\tau}, \\
\frac{V(\tilde{x})}{\dot{q}_i} = \frac{V(\tilde{x})}{\dot{\tilde{x}}}, \quad \dot{\tilde{x}}_i = \frac{m(r)}{\sqrt{g(r)}} \dot{q}_i (\tilde{x}) = \dot{x}_i, \sqrt{m(r)}
\end{cases} \quad \iff \quad \frac{d}{d\tau} \left( \frac{\partial L_{II}}{\partial \dot{x}_i} \right) - \frac{\partial L_{II}}{\partial x_i} = 0; \quad i = 1, 2, \ldots, n \tag{25}
\]

We may now conclude that, whilst the conventional textbook Euler-Lagrange invariance could only address the PDM EL-I settings (documented, in short, above and in a sufficiently comprehensive details in [44]) it could not address the current PDM EL-II settings. Whereas, the what should be called, hereinafter, "Newtonian invariance amendment" works to perfection for both PDM EL-I and PDM EL-II settings. One should also be reminded that "Newtonian invariance amendment" offered a vivid invariance to what is seemed to be incomplete and/or insecure transition into the Newtonian vector presentation of the equation of motion. However, the Newtonian invariance amendment offered a vivid invariance to what is seemed to be incomplete and/or insecure invariance of the PDM Euler-Lagrange equations. In the forthcoming illustrative examples, we clarify our methodical proposal reported above.

### III. NONLINEAR \( n \)-DIMENSIONAL PDM-OSSCILLATORS

The nonlinear \( n \)-dimensional PDM-oscillators, in the generalized coordinates, are generated from the force field

\[
V (\tilde{q}) = \frac{1}{2} m_\omega \omega^2 \sum_{j=1}^{n} q_j^2 = \frac{1}{2} m_\omega \omega^2 (\tilde{q} \cdot \tilde{q}). \tag{26}
\]

For which, one may use the EL-G equations of motion to yield (with \( m_\omega = 1 \)) \( n \) EL-G equations of motion

\[
\frac{d}{d\tau} \tilde{q}_i + \omega^2 q_i = 0; \quad i = 1, 2, \ldots, n, \tag{27}
\]

that admit exact solutions in the form of

\[
q_i = B \cos (\omega_\tau + \varphi_i). \tag{28}
\]

This is going to be our reference case for the forthcoming target PDM Lagrangians, using different functional settings for \( \tilde{q}(\tilde{x}) \).
A. Nonlinear $n$-dimensional PDM-oscillators: $\vec{q}(\vec{r}) = \sqrt{m(\vec{r})} \vec{r}$

The substitutions of

$$\vec{q}(\vec{r}) = \sqrt{m(\vec{r})} \vec{r} ; \quad r = \sqrt{\sum_{j=1}^{n} x_j^2},$$  \hspace{1cm} (29)

in (26) would imply the $n$-dimensional PDM-oscillators potential in the form of

$$V(\vec{q}(\vec{r})) = V(\vec{r}) = \frac{1}{2} m(\vec{r}) \omega_0^2 \sum_{j=1}^{n} x_j^2$$  \hspace{1cm} (30)

Before we proceed any further, we need first to put the substitution (29) to the test and see whether it satisfies our nonlocal point transformation condition $\dot{q}_i (\vec{x}) = \sqrt{m(\vec{r})} f(\vec{r}) \dot{x}_i$, of (25), or not. This is done in order.

$$\dot{q}(\vec{r}) = \sum_{i=1}^{n} \dot{x}_i q_i = \sqrt{m(\vec{r})} \sum_{i=1}^{n} \dot{x}_i x_i \iff \frac{d}{dt} \dot{q}(\vec{r}) = \sum_{i=1}^{n} \dot{x}_i \dot{q}_i = \sqrt{m(\vec{r})} \left[ \vec{v} + \frac{m'(\vec{r})}{2rm(\vec{r})} (\vec{r} \cdot \vec{v}) \right].$$  \hspace{1cm} (31)

We may now swap $\vec{r}$ and $\vec{v}$ in (17) and rewrite (31) as

$$\sum_{i=1}^{n} \dot{x}_i \dot{q}_i = \sqrt{m(\vec{r})} \left[ \vec{v} + \frac{m'(\vec{r})}{2rm(\vec{r})} (\vec{r} \cdot \vec{v}) \right] = \sqrt{m(\vec{r})} \left[ 1 + \frac{r m'(\vec{r})}{2m(\vec{r})} \right] \vec{v} \iff \dot{q}_i = \sqrt{m(\vec{r})} \left[ 1 + \frac{r m'(\vec{r})}{2m(\vec{r})} \right] \dot{x}_i.$$  \hspace{1cm} (32)

Comparing this result with $\dot{q}_i (\vec{r}) = \sqrt{m(\vec{r})} f(\vec{r}) \dot{x}_i$, of (25), we obtain

$$f(\vec{r}) = 1 + \frac{r m'(\vec{r})}{2m(\vec{r})}.$$  \hspace{1cm} (33)

We, therefore conclude that the substitution (29) satisfies our Newtonian invariance amendment provided that $f(\vec{r})$ is given by (33). As such, equations (24) read

$$\ddot{x}_i + \frac{m'(\vec{r})}{2rm(\vec{r})} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + f(\vec{r}) \omega_0^2 x_i = 0; \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (34)

which are our target PDM Euler-Lagrange equations to be solved for different PDM settings.

1. Mathews-Lakshmanan type-I $n$-dimensional PDM-oscillators: $m(\vec{r}) = 1/(1 \pm \lambda r^2)$

With a PDM function $m(\vec{r}) = 1/(1 \pm \lambda r^2)$ one can show that

$$f(\vec{r}) = 1 + \frac{r m'(\vec{r})}{2m(\vec{r})} = \frac{1}{1 \pm \lambda r^2}.$$  \hspace{1cm} (35)

Under such settings, the PDM Euler-Lagrange equations of (34) imply the Mathews-Lakshmanan type-I $n$-dimensional PDM-oscillators equations of motion

$$\ddot{x}_i + \left( \frac{\lambda x_i}{1 \pm \lambda r^2} \right) \sum_{j=1}^{n} \dot{x}_j^2 + \left( \frac{1}{1 \pm \lambda r^2} \right) \omega_0^2 x_i = 0; \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (36)

Which admit exact solutions of the form

$$x_i = B_i \cos (\Omega t + \varphi) ; \quad \Omega^2 = \frac{\omega_0^2}{1 \pm \lambda \sum_{i=1}^{n} B_i^2},$$  \hspace{1cm} (37)

and a total energy

$$E = \frac{1}{2} \left( \frac{\omega_0^2}{1 \pm \lambda \sum_{i=1}^{n} B_i^2} \right) \sum_{i=1}^{n} B_i^2 = \frac{1}{2} \Omega^2 \sum_{i=1}^{n} B_i^2.$$  \hspace{1cm} (38)
2. $n$-dimensional PDM power–law type-I oscillators: $m ( r ) = kr^{2v}$

A power-law type PDM function $m ( r ) = kr^{2v}$ would imply that

$$f ( r ) = 1 + \frac{r m' ( r )}{2m ( r )} = 1 + v. \quad (39)$$

Hence, the PDM Euler-Lagrange equations of (34) yield the $n$-dimensional PDM-oscillators equations of motion

$$\ddot{x}_i + n x_i \left( \sum_{j=1}^{n} \dot{x}_j^2 + (1 + v) \omega^2 x_i \right) = 0, \quad (40)$$

which admit exact solutions in the form

$$x_i = C_i \left[ \cos \left( \Omega t + \varphi_i \right) \right]^{1/(v+1)} ; \quad \Omega = (1 + v) \omega_v, \quad (41)$$

where $v \neq -1, 0$, otherwise trivial solutions are manifested, and a total energy

$$E = \frac{1}{2} \omega_v^2 k \left( \sum_{i=1}^{n} C_i^2 \right)^{v+1} = \frac{1}{2 (1 + v)^2} \Omega^2 k \left( \sum_{i=1}^{n} C_i^2 \right)^{v+1} \quad (42)$$

B. Nonlinear $n$-dimensional PDM-oscillators: $\ddot{q} (r) = \sqrt{m ( r )} \zeta$

Let us now use the assumption that

$$\ddot{q} (r) = \sqrt{m ( r )} \zeta ; \quad \zeta = \sum_{j=1}^{n} \zeta_j \dot{x}_j , \quad \zeta = \sqrt{\sum_{j=1}^{n} \zeta_j^2}, \quad (43)$$

in (26), would imply the $n$-dimensional PDM-oscillators potential

$$V (\ddot{q} (r)) = V (\ddot{r}) = \frac{1}{2} m ( r ) \omega_v^2 \sum_{j=1}^{n} \zeta_j^2 = \frac{1}{2} m ( r ) \omega_v^2 \zeta^2. \quad (44)$$

Where $\zeta$ is a constant vector and is parallel to $\ddot{r}$ and $\ddot{v}$ (i.e., $\ddot{r} \parallel \ddot{v} \parallel \zeta$) and satisfies the vector identity in (17). Under such settings, one would obtain

$$\frac{d}{dt} \ddot{q} (r) = \sum_{j=1}^{n} \dot{x}_j \dot{q}_j = \left( \frac{m ( r )}{2 \sqrt{m ( r )}} \right) \ddot{\zeta} = \frac{m' ( r )}{2 r \sqrt{m ( r )}} (\ddot{r} \cdot \ddot{v}) \zeta = \frac{m' ( r )}{2 r \sqrt{m ( r )}} (\ddot{r} \cdot \ddot{v}) \zeta = \sqrt{m ( r )} \left[ \frac{m' ( r )}{2 m ( r )} \right] \ddot{v}, \quad (45)$$

which immediately implies that

$$\sum_{j=1}^{n} \dot{x}_j \dot{q}_j = \sqrt{m ( r )} \left[ \frac{m' ( r )}{2 m ( r )} \right] \sum_{j=1}^{n} \dot{x}_j \dot{x}_j \iff \dot{q}_j = \sqrt{m ( r )} \left[ \frac{m' ( r )}{2 m ( r )} \right] \dot{x}_j. \quad (46)$$

This result is consistent with $\ddot{q} (r) = \sqrt{m ( r )} f ( r ) \dot{x}_i$ of (25) provided that

$$f ( r ) = \frac{m' ( r )}{2 m ( r )} \zeta, \quad (47)$$

Therefore, the PDM Euler-Lagrange equations of (24) read

$$\ddot{x}_i + \frac{m' ( r )}{2 r m ( r )} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + \frac{f ( r )}{r} \omega_v^2 \zeta x_i = 0. \quad (48)$$

This result represent now our new target PDM Euler-Lagrange equation to be solved for different PDM settings.
1. Mathews-Lakshmanan type-II $n$-dimensional PDM-oscillators: $m(r) = 1/(1 \pm \lambda r^2)$

With the substitution

$$m(r) = \frac{1}{1 \pm \lambda r^2} \iff f(r) = \frac{m'(r)}{2m(r)} \zeta = \pm \frac{\lambda r}{1 \pm \lambda r^2} \zeta,$$

the PDM Euler-Lagrange equations of (48) now read

$$\ddot{x}_i \mp \left( \frac{\lambda x_i}{1 \pm \lambda r^2} \right) \sum_{j=1}^{n} \dot{x}_j^2 + \left( \mp \lambda \zeta^2 \right) \omega_o^2 x_i = 0; \quad i = 1, 2, \ldots, n. \quad (49)$$

Which is exactly the same as (36) provided that $\zeta^2 = \mp 1/\lambda$ (hence the notion "Mathews-Lakshmanan type-II $n$-dimensional PDM-oscillators" is deemed appropriate). As such, it inherits the exact solutions of (37) and (38).

2. $n$-dimensional PDM power–law type-II oscillators: $m(r) = \lambda r^{2\nu}$

A power-law type PDM function $m(r) = \lambda r^{2\nu}$ would imply that

$$f(r) = \frac{m'(r)}{2m(r)} \zeta \iff f(r) = \frac{\nu \xi}{r^{2\nu}}, \quad (50)$$

and consequently the PDM Euler-Lagrange equations of (48) read

$$\ddot{x}_i + \frac{\nu}{r^{2\nu}} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) x_i + \frac{\nu \xi^2}{r^{2\nu}} \omega_o^2 x_i = 0; \quad i = 1, 2, \ldots, n. \quad (51)$$

Which admit exact solutions in the form

$$x_i = B_i \cos(\Omega t + \varphi); \quad \Omega^2 = \frac{\omega_o^2}{\lambda \left( \sum_{j=1}^{n} B_j^2 \right)} \quad (52)$$

provided that $\nu = -1$, and $\lambda = -1/\xi^2$. Hence, the total energy reads

$$E = \frac{1}{2} \omega_o^2 \xi^2 \left( \frac{\lambda}{\sum_{j=1}^{n} B_j^2} \right) = -\frac{1}{2} \Omega^2 \lambda = \frac{1}{2} \Omega^2 \xi^2. \quad (53)$$

C. Nonlinear $n$-dimensional shifted PDM-oscillators: $\vec{q}(\vec{y}) = \sqrt{m(\vec{y})} \vec{y}$, $\vec{y} = (\vec{r} + \vec{\zeta})$

A mixture of the two cases above, $\vec{q}(\vec{r}) = \sqrt{m(\vec{r})} \vec{r}$ and $\vec{q}(\vec{\zeta}) = \sqrt{m(\vec{\zeta})} \vec{\zeta}$ would imply that

$$\vec{q}(\vec{y}) = \sqrt{m(\vec{y})} \vec{y}, \quad \vec{y} = (\vec{r} + \vec{\zeta}) \iff f(y) = 1 + \frac{y m'(y)}{2m(y)}. \quad (54)$$

In this case, the oscillator potential of (26) yields an $n$-dimensional shifted PDM oscillator potential

$$V(\vec{q}(\vec{y})) = V(\vec{y}) = \frac{1}{2} m(y) \omega_o^2 \sum_{j=1}^{n} y_j^2 = \frac{1}{2} m(y) \omega_o^2 \sum_{j=1}^{n} \left(x_j + \xi_j\right)^2. \quad (55)$$
Under such shifted PDM settings, one may rewrite the transformation recipe (25) as

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\tilde{q}}_i} \right) - \frac{\partial L}{\partial \dot{\tilde{q}}_i} = 0 \iff \begin{cases} 
\dot{\tilde{q}}_i = \dot{x}_i = \dot{\tilde{y}}_i = \sum_{k=1}^{n} (x_k + \xi_k) \dot{x}_k \\
\frac{\partial q_i}{\partial y_i} = \sqrt{g(y)}, \\
g(y) = m(y) f(y)^2 \\
V(\tilde{y}) = V(\tilde{q}(\tilde{y})) \\
\dot{\tilde{q}}_i(\tilde{y}) = \frac{1}{f(y)} \dot{\tilde{q}}_i(\tilde{y}) = \dot{\tilde{y}}_i \sqrt{m(y)} 
\end{cases} \iff \frac{d}{dt} \left( \frac{\partial L_{ij}}{\partial \dot{y}_j} \right) - \frac{\partial L_{ij}}{\partial y_j} = 0; \; i = 1, 2, \cdots, n. \quad (56)
\]

Which, consequently, suggests that the \( n \)-dimensional PDM Euler-Lagrange equations are

\[
\ddot{x}_i + \frac{m'(y)}{2 y m(y)} \left( \sum_{j=1}^{n} \dot{x}_j^2 \right) (x_i + \xi_i) + f(y) \omega_0^2 (x_i + \xi_i) = 0, \quad (57)
\]

where

\[
y = \sqrt{\sum_{k=1}^{n} (x_k + \xi_k)^2}. \quad (58)
\]

It is obvious that for a PDM function of the form

\[
m(y) = \frac{1}{1 \pm \lambda y^2} = \frac{1}{1 \pm \lambda \sum_{k=1}^{n} (x_k + \xi_k)^2} \iff f(y) = m(y) = \frac{1}{1 \pm \lambda \sum_{k=1}^{n} (x_k + \xi_k)^2}, \quad (59)
\]

equations (57) would read the what may, very well, be called the \textit{shifted Mathews-Lakshmanan type-I} \( n \)-dimensional PDM-oscillators

\[
\dddot{x}_i \mp \left[ \frac{\lambda (x_i + \xi_i)}{1 \pm \lambda \sum_{k=1}^{n} (x_k + \xi_k)^2} \right] \left( \sum_{j=1}^{n} x_j^2 \right) + \left( \frac{1}{1 \pm \lambda \sum_{k=1}^{n} (x_k + \xi_k)^2} \right) \omega_0^2 (x_i + \xi_i) = 0; \; i = 1, 2, \cdots, n, \quad (60)
\]

that admit exact solutions in the form of

\[
x_i = A_i \cos (\Omega t + \varphi) - \xi_i; \quad \Omega^2 = \frac{\omega_0^2}{1 \pm \lambda \sum_{k=1}^{n} A_k^2}, \quad (61)
\]

The a total energy is then

\[
E = \frac{1}{2} \left( \frac{\omega_0^2}{1 \pm \lambda \sum_{k=1}^{n} A_k^2} \right) \sum_{i=1}^{n} A_i^2 = \frac{1}{2} \Omega^2 \sum_{i=1}^{n} A_i^2, \quad (62)
\]

which is in an obvious resemblance as that in (38).

\[\text{IV. CONCLUDING REMARKS}\]

In this work, we have considered two-types of PDM-Lagrangians \( L_i \left( \vec{x}, \vec{x}^2; t \right) \) of (1) and \( L_{ij} \left( \vec{x}, \vec{x}^2; t \right) \) of (2). They represent two different PDM-Lagrangians structures. In \( L_i \left( \vec{x}, \vec{x}^2; t \right), m_i (x_j) \) is a dimensionless scalar deformation
in the coordinate $x_j$ and/or velocity component $\dot{x}_j$ in a specific functional form. Whereas, in $L_{II} \left( \vec{x}, \vec{\dot{x}}; t \right)$, $m \left( r \right)$ is a common dimensionless deformation for all coordinates $x_j$ and/or velocity components $\dot{x}_j$. The feasibility of their textbook Euler-Lagrange invariance with the conventional constant mass Lagrangians $L \left( \vec{q}, \vec{\dot{q}}; \tau \right)$ of (3) is studies via some $n$-dimensional nonlocal space-time point transformation recipe of (8), (9), and (10). We have shown that, while the PDM Euler-Lagrange equations for $L_i \left( \vec{x}, \vec{\dot{x}}; t \right)$ satisfy the invariance conditions with EL-G of $L \left( \vec{q}, \vec{\dot{q}}; \tau \right)$ for $n \geq 1$, the PDM Euler-Lagrange equations for $L_{II} \left( \vec{x}, \vec{\dot{x}}; t \right)$ failed to do so for $n \geq 2$. This issue has stimulated and/or inspired the current methodical proposal to introduce the new concept of “Newtonian invariance amendment”.

As long as the conventional constant mass setting are in point, both Euler-Lagrange invariance and Newtonian invariance coincide with each other. However, under the current PDM $n$-dimensional setting, it is deemed necessary and vital that the transition from the Euler-Lagrange component presentations, (6) and (11), to Newtonian vector presentations, (18) and (19), should be carried out in order to secure invariance. It was obvious that for $n \geq 2$ the invariance between Euler-Lagrange equations (6) and (11) is still far beyond reach. Whereas, in the Newtonian presentations the invariance between (18) and (19) is proved crystal clear. The totality of the Newtonian vector presentation of the dynamical equation of motion is shown to be more comprehensive/instructive than the Euler-Lagrange component presentations. Hence, the notion “Newtonian invariance amendment” is rendered unavoidable to what is seemed to be incomplete and/or insecure invariance of the PDM Euler-Lagrange equations. This is clarified and documented in our analytical discussions in section 2. To the best of our knowledge, this issue has never been reported elsewhere in the literature.

The $n$-dimensional nonlinear oscillators Lagrangian $L \left( \vec{q}, \vec{\dot{q}}; \tau \right)$ of (3) and (26)

\[
L \left( \vec{q}, \vec{\dot{q}}; \tau \right) = \frac{1}{2} m_0 \sum_{j=1}^{n} q_j^2 - \frac{1}{2} m_0 \omega_0^2 \sum_{j=1}^{n} q_j^2,
\]

is used as a reference Lagrangian in section 3. Therein, we have used the substitutions $\vec{q}(\vec{r}) = \sqrt{m(\vec{r})} \vec{r}$, $\vec{\dot{q}}(\vec{r}) = \sqrt{m(\vec{r})} \vec{\dot{r}}$, and $\vec{\ddot{q}}(\vec{y}) = \sqrt{m(\vec{y})} \vec{\ddot{y}}$ to find the corresponding $n$-dimensional nonlinear PDM-oscillators Lagrangians $L_{II} \left( \vec{x}, \vec{\dot{x}}; t \right)$. They are, respectively, the PDM target Lagrangians,

\[
L_{II} \left( \vec{x}, \vec{\dot{x}}; t \right) = \frac{1}{2} m_0 \sum_{j=1}^{n} m(\vec{r}) \dot{x}_j^2 - \frac{1}{2} m(\vec{r}) \omega_0^2 \sum_{j=1}^{n} x_j^2,
\]

(64)

\[
L_{II} \left( \vec{x}, \vec{\dot{x}}; t \right) = \frac{1}{2} m_0 \sum_{j=1}^{n} m(\vec{r}) \dot{x}_j^2 - \frac{1}{2} m(\vec{r}) \omega_0^2 \xi_j^2,
\]

(65)

and

\[
L_{II} \left( \vec{x}, \vec{\dot{x}}; t \right) = \frac{1}{2} m_0 \sum_{j=1}^{n} m(\vec{y}) \dot{x}_j^2 - \frac{1}{2} m(\vec{y}) \omega_0^2 \sum_{j=1}^{n} \left( x_j + \xi_j \right)^2.
\]

(66)

Their Euler-Lagrange equations (34), (48), and (57), respectively, satisfy the Newtonian invariance amendment and are solved for different PDM functional settings. Indulging within are some illustrative examples that include, the Mathews-Lakshmanan type-I PDM-oscillators (36), the power-law type-I PDM-oscillators (40), the Mathews-Lakshmanan type-II PDM-oscillators (49), the power-law type-II PDM-oscillators (51), and some nonlinear shifted Mathews-Lakshmanan type-I PDM-oscillators (60). Their exact solutions were successfully obtained.

Finally, we have not only introduced the new concept of the Newtonian invariance amendment into Euler-Lagrange invariance, but also we foresee that a new class of pseudo-superintegrable and/or pseudo-superseparable PDM-Lagrangians and consequently PDM-Hamiltonians is implicitly introduced in the current methodical proposal. The Lagrangian in (63), hence the corresponding Hamiltonian, represent a class of superintegrable Lagrangians/Hamiltonians in the Liouville-Arnold sense of integrability (c.f., e.g., [39–43] and related references cited therein). That is, they introduce more constants of motion (also called integrals of motion) than the degrees of freedom the system is moving within. As long as the superintegrable Lagrangian (63), and its corresponding Hamiltonian, are transformable (through the current nonlocal point transformation) into a set of PDM-Lagrangians/Hamiltonians that do not even admit separability, the descendent PDM-Lagrangians/Hamiltonians deserve to be labeled as pseudo-superintegrable and/or pseudo-superseparable, therefore.
