Theory of weak switching of arbitrary collector voltages in non-neutral plasma diodes

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Abstract. The response function concept is used to describe analytically the dynamics of electrons in a non-neutral diode in which the collector voltage is switched on at \( t = 0 \) from zero to a weak but otherwise arbitrary time-dependent voltage. It generalizes previous investigations of simple switching (Akimov et al 2003 J. Appl. Phys. 93 1246; Ender et al 2004 Phys. Plasmas 11 3212), where the final voltage is assumed to be constant. Use is made of the Laplace transformation technique and a remarkably simple expression is found for the Laplace-transformed emitter electric field from which analytic solutions for the time-dependent, highly transient response of the diode can be obtained. As an application, the switching to a final harmonic collector potential called cos-switching is investigated and the usefulness of an approximative response function approach is demonstrated by a comparison of the approximative with the exact results. For diodes with different charge non-neutrality parameter and branches with no electron reflection various scenarios for cos-switching are explored, revealing the exact time behaviour of the emitter electric field and of the net current density. The amplitude \( A_\infty(\omega) \) of the asymptotically driven oscillation as a function of the driven frequency \( \omega \) is also presented. Of upmost importance is a novel resonance phenomenon we found in that part of the second zone of generalized Pierce diodes for which the non-reflecting equilibria are linearly stable against aperiodic and oscillatory eigenmodes. In a narrow band region \( A_\infty(\omega) \) amplifies by a factor 50. This driven internal oscillation takes place on a high current level raising high promises e.g. for microwave generators and/or circuit current amplifiers as well as for diagnostic purposes.

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1. Introduction

The theoretical understanding of the internal dynamics of plasma and vacuum diodes, being often used as nonlinear elements in electronic circuits or as devices with medical or industrial applications, turns out to be crucial for a further progress and development of this field. Figure 1 shows schematically the underlying diode model. A beam of electrons of given density $n_{e0}$ and velocity $v_0$ enters the diode region at the emitter electrode at $x = 0$. The latter is kept at zero potential. If no reflection occurs, all electrons leave the diode at the collector, at $x = L$. A bias voltage $\Phi_1$ is applied across the diode region. Electrons which are reflected by the internal space charged potential return to the emitter electrode and are totally absorbed. It is furthermore assumed that the diode region is occupied uniformly by infinitely massive ions of constant density $n_i$. We allow any value of $n_i$, i.e. the neutralization parameter $\gamma := n_i/n_{e0}$ can vary from 0 to $\infty$. The ions are, hence, treated as immobile, a dynamical situation often found to be valid in lowest approximation as our focus is on the fast electron processes. (A practical example where ions do not participate in the dynamics and hence can be treated by a constant density is their perpendicular injection. If their velocity is sufficiently large they will leave the diode region essentially uncharged, i.e. without having experienced a change in their density distribution both in longitudinal and transversal directions. Hence, fresh ions with a homogeneous distribution will provide the necessary background [1].)

Of particular interest in conjunction with applications appears to be the switching of the collector voltage in order to control the electrical current flowing across the diode; e.g. to disrupt or to build up the current. Recent progress [2]–[4] revealed the highly transient nature of the electron beam dynamics which could be described analytically in case of a weak, instantaneous switching between two neighbouring equilibria of non-reflective type [1]. A further requirement for the validity of this theory was that the transition should not occur in the vicinity of a bifurcation point.

In the case of a pure electron diode, when $\gamma$ is zero, the bifurcation point is given along the non-reflective normal C-branch [5] by the point of maximum current better known as the space-charge-limited (SCL) current [6]–[8].
Figure 1. Schematic view of the electron dynamic in the diode. Some electrons, all of which enter the diode region at $x = 0$, are reflected by a virtual cathode. Ions contribute by their constant density $n_i$ only.

Furthermore, by admitting nonzero values of $\gamma$ through an immobile ion background, new branches of non-reflective equilibria could be found [1, 3], which allow to overcome the SCL and offer new scenarios for the current switch.

Both the Lagrangian fluid description treating the electron dynamics by the concept of trajectories of fluid elements [2], [9]–[14] as well as the Laplace transform technique applied to the Eulerian fluid description [3, 15] appeared to be successful tools to handle and solve the problem analytically.

On the other hand, concentrating on a pure electron diode, later referred to as Bursian diode [16], transitions close to an SCL-point and/or transitions of finite amplitude could be investigated numerically by means of a Vlasov code [4] resulting e.g. in a strong delay of the transition time provided that the transition took place close to an SCL-point. Also, the transition to a branch with partial electron reflection, denoted as the $B$-branch [3], could be studied numerically, showing the time-asymptotic approach to large amplitude oscillations of the dynamical quantities such as the collector current provided that an oscillatory unstable regime of the $B$-branch was selected [4]. In the language of nonlinear dynamics the approach to a limit cycle was seen, e.g. by plotting the minimum potential of the virtual cathode as a function of its position, a case which can be of strong interest for microwave generators such as vircators [17].

As mentioned already, analytically only the instantaneous switching of a weak, time-independent voltage has been treated so far, which, for ease of notation, is called SIMPLE SWITCHING further on.

In our present paper, we are getting rid of the limitation of a constant voltage and allow for an arbitrary time-dependence keeping the smallness of the amplitude perturbation.

In section 2, we develop the general theory of the so-extended switching process making use of both tools mentioned above. Firstly, the Bursian diode is solved within the Lagrangian
fluid description which is then generalized to a plasma diode with nonzero $\gamma$ by making use of the Laplace transform technique. As an application, section 3 is devoted to the special case of a harmonic time-dependence of the switched voltage perturbation. Of special interest are generalized Pierce diodes [18] because of the existence of new zones of equilibria. In the second zone, for instance, where $2\pi < \delta \gamma^{1/2} < 4\pi$ and $\delta$ is the normalized diode length, the oscillatory character of the leading eigenmodes of the linearized stability problem suggests a new diode quality and with it the emergence of an internal, wavy diode response. A resonance between the internally driven oscillation with the external driver can be expected which will also be investigated. Section 4 ends the paper with a summary and conclusions.

2. Analytical treatment of the switching process

This section is devoted to a generalization of the switching process as described in [2, 3] to which we refer for notation and basic set of equations, including boundary conditions. Firstly, we treat the Bursian diode in the Lagrangian fluid description allowing now for a generalization of the switched collector potential. We introduce the following decomposition of dynamical quantities in unperturbed and perturbed ones:

\begin{align}
V(t) &= V_0 + \tilde{V}(t) \cdot \Theta(t), \\
E_0(t) &= \varepsilon_0 + \tilde{\varepsilon}_0(t) \cdot \Theta(t), \\
T(t) &= T_0 + \tilde{T}(t) \cdot \Theta(t), \\
\chi(t_0, t) &= \chi(q) + \tilde{\chi}(t_0, t) \cdot \Theta(t),
\end{align}

where $V$ is the collector voltage, $E_0$ the emitter electric field, $T$ the transit time of a fluid element injected at $t_0$ into the diode, $\Theta(t)$ is the Heaviside faction, and $\chi(t_0, t)$ represents the actual trajectory of a fluid element at time $t$ governed by

\begin{equation}
\ddot{x}(t_0, t) - \left( t - t_0 \right) = -E_0(t)
\end{equation}

with the initial conditions

\begin{align}
\chi(t_0, t_0) &= 0, \\
\dot{x}(t_0, t_0) &= 1, \\
\chi'(t_0, t_0) &= -1.
\end{align}

Here ‘dot’ refers to $\partial_t$, whereas ‘prime’ represents $\partial_{t_0}$. Besides, two further conditions [2] have to be added:

\begin{equation}
\chi(t_0, t_0 + T(t)) = \delta, \quad \text{for } T,
\end{equation}

and

\begin{equation}
\int_{t_0}^{t_0 + T(t)} \mathrm{d}t_0 \chi'(t_0, t_0) \chi(t_0, t) = V(t), \quad \text{for } V.
\end{equation}

The equilibrium, which will be perturbed for $0 \leq t$ by a small, but otherwise arbitrary collector voltage $\tilde{V}(t)$, is represented by the first term on the right-hand side (rhs) of equation (1).
Its trajectory is given by

\[ \chi(q) = \frac{1}{6}q^3 - \frac{1}{2}\varepsilon_0 q^2 + q. \]  

(4)

where \( q = t - t_0 \). Furthermore, as found in [2] and the previous papers cited therein, the potential condition is given by \( \chi(T_0) = \sqrt{1 + 2V_0} \) and the transit time condition by \( \chi(T_0) = \delta \), whose \( \delta \) is the normalized diode length. This implies \( V_0 \geq -1/2 \) and means that reflection of electrons is not taken into account. This equation is referred to as the normal C-branch, for which the two mentioned constraints become by insertion of equation (4)

\[ \frac{1}{2}T_0^2 - \varepsilon_0 T_0 + 1 = \sqrt{1 + 2V_0}, \]  

(5a)

\[ \frac{1}{6}T_0^3 - \frac{1}{2}\varepsilon_0 T_0^2 + T_0 = \delta, \]  

(5b)

from which \( \varepsilon_0(\delta, V_0) \) and \( T_0(\delta, V_0) \) are found. Including now the small switching terms on the rhs of equation (1) we get by linearization the following set of equations

\[ \ddot{x}(t_0, t) = -\ddot{\varepsilon}_0(t), \]  

(6a)

\[ \dot{x}(t - T_0, t) + \dot{T}(t)\sqrt{1 + 2V_0} = 0, \]  

(6b)

\[ \delta\ddot{\varepsilon}_0(t) + \ddot{\bar{V}}(t) + \int_0^{T_0} dq \bar{x}(t - q, t) = 0, \]  

(6c)

representing the perturbed equation of motion, potential and transit time condition, valid for \( 0 \leq t \). They are subject to the boundary conditions

\[ \bar{x}(t_0, t_0) = \dot{x}(t_0, t_0) = \ddot{x}(t_0, t_0) = 0. \]  

(7)

For \( \delta = 1 \), \( V_0 = 0 \) and \( \bar{V}(t) = \Delta V \) the equations (6) coincide with equations (16), (18) and (24) of [2]. As in [2] \( \bar{x}(t_0, t) \) can be found in terms of \( \bar{\varepsilon}_0 \), by integrating equation (6a) which, inserted into equation (6c) yields an integral equation for \( \bar{\varepsilon}_0 \). We get for region 1, being defined by \( 0 \leq t < T_0 \),

\[ \int_0^t ds(T_0 - s) \bar{s}\bar{\varepsilon}_0(t - s) = \delta\bar{\varepsilon}_0(t) + \bar{V}(t), \]  

(8a)

and for regions 2, 3, \ldots, being defined by \( (n - 1)T_0 \leq t < nT_0, n = 2, 3, \ldots \), we find

\[ \int_0^{T_0} ds(T_0 - s) \bar{s}\bar{\varepsilon}_0(t - s) = \delta\bar{\varepsilon}_0(t) + \bar{V}(t). \]  

(8b)

They coincide with equations (28) and (29) of [2] in the considered limit. Another way of getting a solution is to derive and solve the corresponding retarded differential equation which is delegated to appendix A. The renewal integral equations (8a) and (8b) can also be solved by the Laplace transformation technique which we do next. Denoting the Laplace transform
of \( \tilde{\varepsilon}_0 \) by \( \varepsilon_{0p} \), we get by Laplace transformation of equations (8a) and (8b) as in appendix A of [2],

\[
\varepsilon_{0p} = \frac{-p^3}{\delta p^3 - [(pT_0 + 2)e^{-pT_0} + pT_0 - 2]} V_p.
\] (9)

It coincides with equation (A6) of [2] in the limit of simple switching noting that \( V_p \) becomes then

\[
V_p = \Delta V/p.
\] (10)

The simple switching of a non-neutral plasma diode was investigated in [3] by means of this technique and \( \tilde{\varepsilon}_0(t) \) was found by the inverse Laplace transformation. (In case of simple switching and of a Bursian diode the time-dependence of \( \tilde{\varepsilon}_0(t) \), as was found in [2], was obtained, too.) Note that \( V_p \) yields the boundary condition on the collector for the Laplace transform of the perturbed potential \( \eta_p(x) \), i.e. \( \eta_p(\delta) = V_p \). Using the boundary conditions at the emitter a general expression of \( \eta_p(x) \) of the form

\[
\eta_p(x) = F(p)h_p(x)
\] (11)

was obtained, where \( h_p(x) \) is given by equation (38) in [3]

\[
h_p(x) = - \left[ \frac{p^2 - \gamma}{\gamma^{1/2}} \sin(\gamma^{1/2}q) + 2p \cos(\gamma^{1/2}q) \right] e^{-pq} + (p^2 + \gamma)^2 x - (p^2 + \gamma)q + 2p,
\] (12)

in which the relationship between \( q \) and \( x \) is given by equation (41) of [1] or equation (4) of [3]

\[
x(q) = \frac{q}{\gamma} + \frac{\varepsilon_0}{\gamma} [\cos(\gamma^{1/2}q) - 1] + \frac{\gamma - 1}{\gamma^{3/2}} \sin(\gamma^{1/2}q).
\] (13)

The function \( F(p) \) in equation (11) was found by using the boundary condition (10) and the resulting solution for simple switching was given by

\[
\eta_p(x) = \frac{\Delta V}{p} \frac{h_p(x)}{h_p(\delta)}.
\] (14)

Hence, the corresponding expression for an arbitrary switching is obtained by

\[
\eta_p(x) = V_p \frac{h_p(x)}{h_p(\delta)}.
\] (15)

All other functions \( \varepsilon_p(x) \), \( n_p(x) \), etc can be obtained by spatial differentiation of equation (15). In particular, we get at the emitter, \( x = 0 \),

\[
\varepsilon_{0p} = \varepsilon_p(0) = -V_p \frac{(p^2 + \gamma)^2}{h_p(\delta)},
\] (16a)
generalizing equation (42) for simple switching of [3], which is given by

$$\varepsilon_{0p} = -\frac{\Delta V \left( p^2 + \gamma \right)^2}{p \ h_p(\delta)} \equiv \Delta V R_p,$$

(16b)

where $R_p$ is the Laplace transform of the response function $R(t)$ for weak and simple switching. In [3] we have outlined how to construct $R(t)$. Comparing equation (16a) with (16b) we see that for an arbitrary switching $\varepsilon_{0p}$ can be expressed by the Laplace transform of this response function

$$\varepsilon_{0p} = V_p p R_p.$$

(17)

Using the convolution theorem of the Laplace transformation we hence get

$$\tilde{\varepsilon}_0(t) = \int_0^t R(t - \tau) G(\tau) \, d\tau,$$

(18)

where $G(t)$ is the inverse Laplace transform of $G_p \equiv p V_p$.

It can be shown that for the other dynamical quantities a similar expression can be found involving a corresponding response function known already from simple switching. This not only holds for the normal $C$-branch (extended to $\gamma > 0$) but for all new branches as found in [3].

For the following we only consider the perturbed emitter electric field, drop for simplicity the index 0 in equation (18) and reformulate it for practical reasons to get

$$\tilde{\varepsilon}(t) = \int_0^t G(t - \tau) R(\tau) \, d\tau.$$

(19)

We emphasize that the linear response function concept offers the possibility of describing the time evolution in the diode for a given weak arbitrary switching of the collector voltage in a simple and general manner. Referring to equation (18) as a typical equation we see that the linear diode dynamics caused by a weak, instantaneous switching is given by a convolution of the intrinsic response function $R(t)$ with the applied collector voltage entering through $G(t)$.

Hence, the main task consists in finding the response function for different equilibria.

3. Evaluation of cos-switching

Then, let the collector to be supplied with a final harmonic potential, called cos-switching, i.e.

$$\tilde{V}(t) = \Delta V \cos(\omega t).$$

(20)

The Laplace transforms for the functions $\cos(\omega t)$ and $\sin(\omega t)$ take the form

$$\mathcal{L}\{\cos(\omega t)\} = \frac{p}{p^2 + \omega^2}, \quad \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{p^2 + \omega^2}.$$

(21)

Then, concerning the Laplace transform of a function $G$ we have

$$G_p \equiv p V_p = \Delta V \frac{p^2}{p^2 + \omega^2} = \Delta V \left( 1 - \omega \frac{\omega}{p^2 + \omega^2} \right),$$

(22)
and consequently, for $G_p R_p$ we obtain

$$G_p R_p = \Delta V \left( R_p - \omega \frac{\omega}{p^2 + \omega^2} R_p \right). \quad (23)$$

The first term in equation (23) refers to the response function $R(t)$ for weak and simple switching, and the second one is proportional to the Laplace transform of the sine function. Hence, we obtain for $\tilde{\varepsilon}$ from equation (23) using equation (19)

$$\tilde{\varepsilon}(t) = \Delta V \left\{ R(t) - \omega \int_0^t d\tau \sin [\omega(t - \tau)] \tilde{R}(\tau) \right\}. \quad (24)$$

Because of the smallness of the perturbation amplitude $\Delta V$, the result turns out to be proportional to it. Then, here and in further calculations, we omit the factor $\Delta V$.

Thus, for a weak cos-switching voltage, the perturbed emitter electric field differs from the field perturbation in the case of simple switching by a term proportional to the convolution of the latter with a sine function.

Furtheron, we can calculate the total current flowing through the circuit. For a prescribed time-dependent collector voltage it will be time-dependent too but spatially independent in one-dimension. We, therefore, can determine its deviation from the equilibrium current by the displacement current at the emitter, noting that the convective current is constant there in the case of no electron reflection. In fact, to have a positive convective current, as used in our previous papers, we take the negative sign and define this as our net current density. Differentiating equation (24) and integrating by parts, we obtain for the net current density

$$\tilde{j}_{\text{net}}(t) := -\frac{d}{dt} \tilde{\varepsilon}(t) := \tilde{j}_{\text{net}}^0(t) + \omega \Delta V \tilde{R}(+0) \sin(\omega t) - \omega \int_0^t d\tau \sin [\omega(t - \tau)] \tilde{j}_{\text{net}}^0(\tau). \quad (25)$$

Here, $\tilde{j}_{\text{net}}^0$ is the net current density for simple switching.

### 3.1. Model response function

In [3], we obtained an analytical expression of $\tilde{R}(t)$ for a normal $C$-branch at arbitrary values of the neutralization ratio $\gamma$. For the normal $C$-branch, the switching process proceeds aperiodically and includes three sections (see figure 4 in [3]): an initial jump for a switching-on value of $-\Delta V/\delta$, a transition process during a time-gap $T_1$ being of the order of several time-of-flights $T_0$ of the electrons across the interelectrode gap (far from SCL-point $T_1 \sim (2-3)T_0$), and an asymptote with a constant value $-\Delta V \cdot S$, the value of which is known for any point on the normal $C$-branch (see equation (50b) in [3]).

For an understanding of the general peculiarities of a process with cos-switching, we represent a response function as a straight line within a region $0 < t < T_1$ and a constant at $t > T_1$

$$\tilde{R}(t) = \begin{cases} -B - Kt, & 0 \leq t < T_1, \\ -S, & t > T_1. \end{cases} \quad (26)$$

Here $B = -\tilde{R}(+0) = 1/\delta$ and $K = (S - B)/T_1$. The function (26) is described by three parameters: $B$, $S$ and $T_1$. An example of $\tilde{R}(t)$ and the approximating function $\tilde{R}(t)$ for the
Figure 2. The exact response function $R(t)$ (——–) and the model response function $\bar{R}(t)$ (24) (-----) as functions of time.

Bursian diode ($\gamma = 0$) with $\delta = 1$ and $T_1 = 1.4$ is shown in figure 2 by a solid and dashed line, respectively. As a result, the problem is strongly simplified. Calculating the simplified integral in equation (24) for $\tilde{\varepsilon}$ we obtain

$$\tilde{\varepsilon}(t) = \begin{cases} -B \cos(\omega t) - (K/\omega) \sin(\omega t), & t < T_1, \\ -B \cos(\omega t) - (K/\omega)(\sin(\omega t) - \sin[(\omega(t - T_1))]), & t > T_1. \end{cases}$$

Equation (27) can be rewritten as follows

$$\tilde{\varepsilon}(t) = \begin{cases} \omega^{-1}A_1 \cos(\omega t + \pi - \varphi_1), & t < T_1, \\ \omega^{-1}A_2 \cos(\omega t + \pi - \varphi_2), & t > T_1. \end{cases}$$

Here

$$A_1 = (\omega^2 B^2 + K^2)^{1/2}, \quad \varphi_1 = \arctan\left(\frac{K}{\omega B}\right),$$

$$A_2 = [K^2 + A_1^2 + 2K A_1 \sin(\omega T_1 - \varphi_1)]^{1/2}, \quad \varphi_2 = \arctan\left(\frac{K[1 - \cos(\omega T_1)]}{\omega B + K \sin(\omega T_1)}\right).$$

An expression for the net current density can be immediately obtained from equation (28)

$$\tilde{j}_{\text{net}}(t) = -\frac{d}{dt}\tilde{\varepsilon}(t) = \begin{cases} A_1 \cos(\omega t + \pi/2 - \varphi_1), & t < T_1, \\ A_2 \cos(\omega t + \pi/2 - \varphi_2), & t > T_1. \end{cases}$$

Figures 3(a) and (b) show the temporal functions $\tilde{\varepsilon}$ and $\tilde{j}_{\text{net}}$ for frequency values $\omega = 1$ and 0.5 for the Bursian diode under consideration. Note that a jump in the current at the point
Figure 3. The temporal dependence of the perturbed emitter electric field (a) and of the net current density (b), obtained by the model response function. Curve 1 (2) corresponds to the driving frequency $\omega = 1.0 (0.5)$.

Figure 4. $A_2(\omega)/\omega$ as a function $\omega$ obtained by the model response function.

$t = T_1$ occurs due to response function’s approximation. It is clearly seen that a transient process proceeds during a certain time of order of $T_1$, and that after the termination of this process, both functions result in harmonic oscillations of constant amplitude and phase.

From equations (29)–(30) one can see that the amplitude $A_2$ and phase $\pi/2 - \varphi_2$ of the steady-state oscillations of the net current depend on the frequency of the external voltage. Since at high frequencies the amplitude $A_1$ tends to $\omega B = \omega / \delta$, and $A_2 = A_1$, the $A_2/\omega$-dependence is better suited than $A_2(\omega)$. The $A_2/\omega$ function is just shown in figure 4. It is seen that it has an oscillating feature of decreasing amplitude. The phase exhibits, also, damped oscillations with a change in frequency. When the parameter $T_1$ is varied, the features of both dependencies do not experience any qualitative change. Only the frequency of oscillations does vary.

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Consider the case of high frequencies. Here the amplitude \( A_2 \) tends to \( \omega/\delta \) and the phase to a \( \pi/2 \) threshold. Consider the ratio of the amplitude of current \( \tilde{j} \) to that of voltage \( \tilde{u} \), i.e. dimensionless conductance, at \( \omega \gg 1 \). We have \( y = \tilde{j}/\tilde{u} = \omega/(\delta \tilde{u}) \). If we now go over to the dimension units, taking \( e_n v_0 \) as a current density unit, \( m v_0^2/\epsilon \) as a voltage unit, \( \lambda_D = [m v_0^2/(4\pi e^2 n_0)]^{1/2} \) as a length unit, and \( \Omega_0 = [4\pi e^2 n_0/m]^{1/2} \) as a frequency unit where \( n_0 \) and \( v_0 \) are an electron density and velocity at the emitter, we then obtain for the dimensional conductance \( Y = \Omega/(4\pi d) = (\Omega C) \), \( Y = J/U \), where \( C \) is the diode capacity per unit area, \( J, U \) and \( \Omega \) are dimensional current density, voltage and frequency. Thus, at high frequencies the diode conductance turns out to be a pure capacitor one. It owes to the fact that the inertia of the electrons is practically negligible if \( \omega \gg 1 \).

Concerning the peculiarities of the amplitude, especially of the jump in the current density at \( t = 1.4 \) (see figure 3(b)), certain doubts arise about their correctness because they may be due to the approximation used. We expect that they originate from the jump in the derivative of the model response function at \( t = T_1 \) and that they disappear or are at least weakened if the smooth exact response function is used instead. To verify this, we undertook a calculation of the dependences \( \tilde{e}(t) \) and \( \tilde{j}_{\text{net}}(t) \) via the exact formulas in the following section.

3.2. Exact response function

The time-dependence of the emitter electric field \( \tilde{e}(t) \) in the case of simple switching of a non-neutral diode was presented in [3], by equation (60). It corresponds to the inverse Laplace transform of (16b) if \( R(t) \) has the following form:

\[
R(t) = -\sum_{n=1}^{N} f^{(n)}(t_n) \Theta(t_n) \tag{31a}
\]

with

\[
f^{(n)}(t_n) = \frac{4}{\pi} \sum_{l=1}^{n} \sum_{k=1}^{4} \frac{\Phi_{nkl}(\alpha_k)}{(n-l)!l!(l-1)!} t_n^{n-l} e^{\alpha_k t_n} + C_n. \tag{31b}
\]

In equations (31a) and (31b) \( N = [t/T_0] + 1 \) with \([ \ldots ]\) being the integer part of the argument, and \( t_n = t - (n - 1) T_0 \), where \( T_0 \) is the unperturbed transit time of the electrons across the diode. Furthermore, \( \alpha_k \) is a root of the polynomial

\[
R(p) = \delta p^4 + (2\gamma \delta - T_0) p^2 + 2p + \gamma (\gamma \delta - T_0), \tag{32}
\]

and the coefficients \( \Phi_{nkl} \) and \( C_n \) are given by

\[
\Phi_{nkl}(p) = \frac{d^{l-1}}{dp^{l-1}} \left[ \frac{Q_n(p)}{P_{nk}(p)} \right], \quad C_n = \gamma^{2-n} \left[ -\gamma^{1/2} \sin(\gamma^{1/2} T_0) \right]^{n-1} (\gamma \delta - T_0)^{-n}. \tag{33}
\]

where \( P_{nk}(p) = P_n/(p - \alpha_k)^n \), \( P_n(p) = p[R(p)]^n \), \( Q_n(p) = (p^2 + \gamma^2)[S(p)]^{n-1} \) and \( S(p) = \gamma^{-1/2} \sin(\gamma^{1/2} T_0) p^2 + 2 \cos(\gamma^{1/2} T_0) p - \gamma^{1/2} \sin(\gamma^{1/2} T_0) \).

From equation (31a) we can see that each time when \( t \) enters a new zone \( (n \to n+1) \) a new term \( f^{(n+1)}(t_n) \) adds. Continuity of \( R(t) \) is guaranteed by \( f^{(n)}(t_n)|_{t_n=0} = 0 \) as it was already.
shown in [3]. For \( n = 1 \) the sum over \( l \) in (31b) consists only of one member and hence the
time-dependence of \( f^{(n)}(t_n) \) is simple. For large \( n \), respectively time, however, when the sum
over \( l \) is large many terms of the type \( r_n^{n-l} \exp(\alpha_k t_n) \) contribute rendering the time evolution more
complex. Nevertheless, we emphasize that we know the analytic dependence of each contributing
term completely.

An investigation of equation (30) for arbitrary \( \gamma \) shows, that this polynomial has either two
real and two complex conjugate roots, \( \Gamma \pm i\Omega \), or two pairs of complex conjugate roots. In no
situation four real solutions have been found yet. For the pure electron diode (Bursian diode, \( \gamma = 0 \)) on the
\( C \)-branch only the first situation is met in which one of the real root is always zero. In any case, one can replace the sum of complex terms by a one in which only real terms appear
according to a formula like

\[
\Phi_{n1}(\alpha_1)e^{(\Gamma_1+i\Omega_1)t_n} + \Phi_{n2}(\alpha_2)e^{(\Gamma_1-i\Omega_1)t_n} = e^{\Gamma_1 t_n}[\Phi_{nl}^c \cos(\Omega_1 t_n) + \Phi_{nl}^s \sin(\Omega_1 t_n)],
\]

where \( \Phi_{nl}^c = 2\text{Re}[\Phi_{n1}(\alpha_1)] \) and \( \Phi_{nl}^s = -2\text{Im}[\Phi_{n1}(\alpha_1)] \), and \( \alpha_{1,2} = \Gamma \pm i\Omega \).

In order to study the transient processes, which are involved in the simple switching of
non-neutral diodes, we developed a special numerical code. In it, for given parameters \( \gamma \), \( V_0 \)
and \( \delta \), firstly, the equilibrium branch is selected, as indicated in figure 2 of [3]; which means
that the starting state is defined. Then, all complex roots of the polynomial (32) are found. The
next step is to determine numerically the complex derivatives of the functions \( Q_n(p)/P_{nk}(p) \),
according to relation (33), to get the coefficients \( \Phi_{nk}(\alpha_k) \) in (31b). (We mention in parentheses
that \( \Phi_{nk}(\alpha_k) \) is non-singular, because \( P_n(p) \sim (p - \alpha_k)^n \) and hence denominator in \( P_{nk}(p) \)
drops out as \( p \to \alpha_k \)). As a result, the response function \( R(t) \) is determined. A first application of
this recipe was already given in [3].

To get the emitter electric field for cos-switching, we substitute this \( R(t) \) in the last term of
(24) and find

\[
\tilde{\varepsilon}(t) = \Delta V \left\{ R(t) + \omega \sum_{n=1}^{N} \int_{(n-1)T_0}^{t} \sin[\omega(t_n - \tau_n)] f^{(n)}(\tau_n) \right\},
\]

where \( \tau_n = \tau - (n-1)T_0 \) and use is made of (31a). By a change of the intergation variable
(\( \tau \to \tau_n \to x \)), we can rewrite the integral under the sum in (34), which is denoted by \( I_n(t) \), as

\[
I_n(t) = \int_{0}^{t_n} \sin[\omega(t_n - x)] f^{(n)}(x).
\]

Then by application of the addition theorem for the trigonometric functions twice we obtain for \( I_n(t) \)

\[
I_n(t) = [\cos[\omega(n-1)T_0]I^c_n - \sin[\omega(n-1)T_0]I^s_n] \sin(\omega t)
- [\sin[\omega(n-1)T_0]I^c_n + \cos[\omega(n-1)T_0]I^s_n] \cos(\omega t),
\]

where

\[
I^c_n = \int_{0}^{t_n} \cos(\omega x) / \sin(\omega x) f^{(n)}(x).
\]
3.3. Normal C-branch of Bursian diode

In figure 5, we present for the normal C-branch of the Bursian diode the time-dependence of the emitter electric field (figure 5(a)) and of the net current density (figure 5(b)), in analogy to figures 3(a) and (b). In both figures the curves labelled 1, 2 and 3 correspond to $\omega = 1.0$, 0.5 and 0.25, respectively. Several properties are worth to be mentioned. First of all, we see that the asymptotic form of an exact harmonic behaviour with $\omega$ given by the driving frequency is achieved very fast, more or less already after one and two transit times. The switching is hence accomplished very effectively. Obviously, in this frequency domain there is no $\omega$-influence on
the amplitude of the oscillation in $\tilde{\epsilon}$ in contrast to that of $\tilde{j}_{\text{net}}$. The latter is a consequence of the former, since $\tilde{j}_{\text{net}}$ is obtained by a time differentiation of $\tilde{\epsilon}$ which brings in a factor $\omega$.

Asymptotically the differences to the model response function results of figure 3 are negligibly small amounting up to 2–3 per cent. The model response function is hence well suited to describe this forced diode oscillation given by the formulas (28)–(30). In the transition region, however, a strong influence of $\omega$ is seen in both series of curves. Firstly, we recognize that the first minimum in $\tilde{\epsilon}$ is dropping with $\omega$ which is in accordance with equation (24), approaching (except for the constant $\Delta V$) the response function in the limit of $\omega \to 0$. Secondly, at $t = T_0 = 1.12$ a strong dependence of $\tilde{j}_{\text{net}}$ is seen, similarly to that of the model response function case, in figure 3(b), but now the transition is less singular which has, of course, to do with the more regular $R(t)$. Physically, it is the passage of the old electrons which brings in a new diode quality by the presence of fresh electrons [2].

In figure 6 the time-asymptotic amplitude $A_\infty(\omega)$ divided by $\omega$ is plotted as a function of $\omega$ for three values of the diode length $\delta = 0.8, 1.0$ and 1.25, respectively. For large $\omega$ all curves settle to a constant given by $\delta^{-1}$ which expresses the pure capacity behaviour of the diode: $A_{\infty}^{\text{cap}} = \omega/\delta$. In this regime electrons can no longer respond due to their finite inertia. Except for the first period, the oscillations turn out to be less pronounced (seen for $\delta = 1$) than in the model problem (figure 4). Further new features are that the maxima seem to be tangent to the straight line $A_\infty(\omega)/\omega = \delta^{-1}$ and that the period of oscillations in $\omega$ is stretched.

As a resume we state that the switching in this diode regime is accomplished very rapidly and that the gross diode dynamics is already well represented by the model response function.

3.4. Non-reflective branches of generalized Pierce diode

3.4.1. First zone. Our second case refers to a diode with a fixed ion background, called a generalized Pierce diode. As it was shown in figure 2 of [3], a nonzero $\gamma$ introduces new
Figure 7. Emitter electric field $\varepsilon_0$ as function the diode length $\delta$ for $\gamma = 0.9$ drawn for $E$, $C$- and $\tilde{C}$-branches. Crossed solid lines separate regions, containing these branches. Below the dashed curve lie stable solutions. The bold points correspond to $\delta = 2, 3, 4, 4.5$ and 5, respectively.

branches of non-reflective equilibria besides the $C$-branch, namely the $\tilde{C}$ and the $E$ and $\tilde{E}$ branches. Whereas the $\tilde{E}$ branch is always linear unstable, the other branches appear to be stable below the stability line in the $(\varepsilon_0, \delta)$-parameter space. The branches are distinguished by different external dc-voltages on which the weak cos-switching is applied for $t \geq 0$. In figure 7 we repeated this diagram for $\gamma = 0.9$ and two selected voltages $V = 0.2$ and 0.3 valid for the first zone, $0 < \delta \gamma^{3/2} < 2\pi$. Below the stability line (dashed curve) the normal $C$-branch, the $E$-branch and the $\tilde{C}$-branch are stable. The dots in the diagram mark the values of $\varepsilon_0$ and $\delta$ for which a further look is taken. In this first zone, the exact response function $R(t)$ behaves qualitatively similar independent of the chosen branch and of the selected $\delta$. As shown exemplary for $\delta = 2$ for $V = 0.2$ ($C$-branch) and $V = 0.3$ ($E$-branch) (figure 8(a)), it relaxes monotonically to the asymptotic value, given by equation (50b) in [3], indicating the correctness of the above-described analytical procedure. Only near the SCL-points of both $C$- and $\tilde{C}$-branches a slowing down of the relaxation to the asymptotic value is observed, as expected from physical grounds (see [4]). Near the critical SCL point of the $\tilde{C}$-branch (called previously the $C_2$ critical point) given by $\delta = 4.2983$ for $V = 0.2$ this difference is seen in figure 8(b) by comparison of $R(t)$ for $\delta = 4.5$ on the $\tilde{C}$-branch and on the $E$-branch respectively, the former relaxing by a factor 2 slower than the latter.

We also emphasize that the initial $R(0)$ and the asymptotic $R(\infty)$ correctly reflects the analytic formulas, (50a) and (50b), respectively. In any case, no big qualitative changes in $R(t)$ are seen with respect to that of the Bursian diode, as long as we stay in the first zone.

Also the asymptotic amplitude $A_\infty(\omega)$, shown in figure 9 for $\delta = 2, 3, 4$ and 5, respectively, behaves rather similar to that of the Bursian diode, which was shown in figure 6.

3.4.2. Second zone-resonance. For the second zone, $2\pi < \delta \gamma^{3/2} < 4\pi$, one can expect qualitatively different and novel diode properties because the leading mode in the spectrum of eigenmodes of the linearized equations for the perturbations is, as a rule, oscillatory in the second zone, rather than purely aperiodic as in the first zone (see [1], figures 29 and 30 for $\gamma = 1$, and references cited therein). This is indeed reflected in the response function $R(t)$ which
we will show in figure 11 for $\gamma = 1$, $V = -0.01$ and several $\delta$’s. Before that we repeat a plot of the nonreflective equilibrium branches in the second zone shown in figure 10, being part of figure 33(a) of [1]. It shows in the $(\varepsilon_0, \delta)$-plane the $E$- and $\bar{C}$-branches in this zone for $\varepsilon_0 < 0$ together with the region of their oscillatory and aperiodic stability, exhibited by the shaded double-triangle shaped area. The equilibrium points, we select for figure 11, on a $\bar{C}$-branch lie in this stable region (far enough away from the SCL-point, for which the relaxation would take much longer). (We note in parentheses that the stability gap for $V = 0$, given by the $\varepsilon_0 \equiv 0$ line,
Figure 10. Emitter electric field $\varepsilon_0$ as function the diode length $\delta$ for $\gamma = 1$ and different $\delta$-values drawn for $E$- and $\bar{C}$-branches. The solid line in between separates regions containing these branches. Boundaries of the shaded area correspond to aperiodic (· · · ·) and oscillatory (- - - -) stable solutions.

has been found earlier by Godfrey [20], who detected interesting nonlinear phenomena in this region.)

Indeed, $R(t)$ in figure 11 reflects this oscillatory behaviour which is superimposed on the monotonically decaying relaxation. Its mutual relationship depends on the distance of the chosen point to a stability line. Whereas the monotonically decaying part becomes stronger the more distant $\delta$ is from $\delta_{SCL}$ for $\bar{C}$-overlap (i.e. in the aperiodic stable region which has as a border the dotted line in figure 10), the oscillations become more intense the closer the point is to the border of oscillatory instability represented by a dashed line, the latter being similar also for the $E$-branch.

The amplitude of the forced oscillation in the time-asymptotic region $A_\infty(\omega)$ divided by $\omega$ is shown as a function of $\omega$ in figure 12. As an insert the evolution is plotted for larger times which show an approach to $A_\infty(\omega)/\omega \sim 1/\delta$ for $\omega \to \infty$, as we had before.

What is really new is the strong and relatively narrow resonance, in which the externally applied frequency meets the internal frequency, the response function being characterized with. In comparison with the capacitor limit ($\omega \to \infty$) an amplitude enhancement by a factor of 50 is obtained. The tiny oscillations superimposed on the gross $\omega$-behaviour of $A_\infty(\omega)$ are due to the replacement of the oscillatory $R(t)$ behaviour beyond $t = 120$ by the straight asymptotic line, as mentioned above, and disappears if this time limit is shifted to still higher values. They are hence a relic of the procedure and are not of physical origin.

Finally, we plot in figure 13 the initial $\tilde{j}_{net}(t)$ behaviour after switching near the resonance point for three values of $\omega$. At the resonance frequency $\omega_R = 0.2729$ we see an increase in $\tilde{j}_{net}(t)$ and a rather rapid approach to the stationary forced oscillations. For $\omega = 0.2 < \omega_R$, we not only see an oscillation of lower amplitude but also some non-harmonic features (which is a linear one and not due to nonlinearity, because of our linear approach). For $\omega = 0.3 > \omega_R$ a harmonically driven oscillation, similar to $\omega = \omega_R$, is seen with however a lower amplitude.
Figure 11. The exact response functions $R(t)$ as functions of time for $\gamma = 1$ and $V = -0.01$ drawn for $\bar{C}$-branches and different $\delta$-values. Curve 1 corresponds to $\delta = 9.8$, 2 to 10.2, 3 to 10.5 and 4 to 11.4, respectively.

Figure 12. $A(\omega)/\omega$ as a function $\omega$ obtained by the response function for the $\bar{C}$-overlap branch. $\gamma = 1$, $V = -0.01$ and $\delta = 9.8$. 
4. Summary and conclusions

In this paper, the small amplitude switching of the collector voltage of a plasma diode was treated analytically. A linear response function $R(t)$ was found which governs the highly transient transition from a dc-state without reflection to an arbitrary time–dependent state. As a result, the exact time–dependence of basic diode functions such as the emitter electric field or the net current density is now available. By means of the Laplace transform technique, $R(t)$ was shown (equation (29a)) to be composed of an increasing number of contributions each time the border $nT_0$ of the $n$th time interval, with $n = 1, 2, 3, \ldots$ and $T_0$ being the unperturbed transit time, is crossed. On this basis a numerical program was written which calculates the exact response for cos-switching. Also a simpler model response function $\bar{R}(t)$ for the switching of Bursian diodes was proposed through which the diode dynamics can already be well represented, as shown by a comparison with the exact result.

For generalized Pierce diodes the behaviour of the response function $R(t)$ changes from a monotonic decrease in the first zone (figure 8) to an essentially oscillatory one in the second zone (figure 11), exhibiting a characteristic periodicity. This has the consequence that in the double-triangle shaped region of linear stability in the second zone (figure 10) the external oscillation can resonantly beat the internal one.

The main achievement, we discovered, is a resonance phenomenon triggered by the cos-switching of the diode in the second zone of a generalized Pierce diode. This resonance, in which the diode electrons perform synchronized strong collective oscillations, promises interesting applications. For example, its position in $\omega$-space can be used for diagnostic purposes to learn about the prevailing electron beam and ion parameters. Moreover, such a strong diode response increases the effectiveness of the diode as a nonlinear element for the amplification of the circuit current. Another interesting perspective of this high current regime without electron reflection can be foreseen for microwave generation with the additional advantage that it can be controlled very sensitively by changing the equilibrium parameters.

Last but not least, nonlinear effect may participate in the diode dynamics, either at resonance when the applied voltage is no longer small enough to justify linearization or when it is from
the outset of finite amplitude. In both the cases only by numerical simulations an answer can be found to what extent they modify the small amplitude results, found analytically in this paper.

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Appendix A. Retarded differential equation

As in [2], we get by a three-fold differentiation of (8a) and (8b)

\[ \ddot{\tilde{\epsilon}}_0(t) - T_0 \dot{\tilde{\epsilon}}_0(t) + 2\tilde{\epsilon}_0(t) = [2\tilde{\epsilon}_0(t - T_0)\Theta(t - T_0) + T_0 \ddot{\tilde{\epsilon}}_0(t - T_0) \Theta(t - T_0) ] - \dddot{V}(t), \]  

(A.1)

which is a remarkably simple extension of the earlier switching result (equation (37) of [2]).

It can be solved by following the procedure described in [2]. This renewal (or retarded) differential equation is valid in all regions i.e. for \(0 \leq t \). It coincides with equation (37) of [2] in the limit of \(\delta = 1\), \(V_0 = 0\) and \(\ddot{V}(t) = V\). The initial values, to be satisfied by \(\tilde{\epsilon}_0(t)\), are

\[ \tilde{\epsilon}_0(0) = -\dddot{V}(0)/\delta, \quad \dot{\tilde{\epsilon}}_0(0) = -\ddot{V}(0)/\delta, \quad \ddot{\tilde{\epsilon}}_0(0) = -\dddot{V}(0)/\delta - \dot{T}_0 \ddot{V}(0)/\delta^2. \]  

(A.2)

Whereas \(\tilde{\epsilon}_0(t)\) and \(\dot{\tilde{\epsilon}}_0(t)\) are continuous everywhere, including \(t = T_0\), the second derivative \(\ddot{\tilde{\epsilon}}_0(t)\) experiences a jump at \(t = T_0\) (and only there)

\[ [\ddot{\tilde{\epsilon}}_0(T_0)] = \dddot{\tilde{\epsilon}}_0(T_0^+) - \dddot{\tilde{\epsilon}}_0(T_0^-) = -\dot{T}_0 \dddot{V}(0)/\delta^2 \]  

(A.3)

in agreement with equation (35) of [2]. It can be shown that the Laplace transform of equation (A.1) is identical with equation (9), where use is made of the initial conditions equation (A.2).

We finally note that with the solution \(\tilde{\epsilon}_0(t)\) we can get the perturbed trajectory \(\tilde{x}(t_0, t)\) by integrating equation (6a) twice and, by following section 4 of [2], we also can derive the density and current perturbations etc, in the Eulerian space. (Note the footnote, [15], of the paper [3] with respect to changes of some formulas in [2].)

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