Semi-simple groups that are quasi-split over a tamely-ramified extension
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To cite this version:
Philippe Gille. Semi-simple groups that are quasi-split over a tamely-ramified extension. Rendiconti del Seminario Matematico della Università di Padova, 2018, 140, pp.173-183. 10.4171/RSMUP/7. hal-01558507

HAL Id: hal-01558507
https://hal.archives-ouvertes.fr/hal-01558507
Submitted on 7 Jul 2017

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Let $K$ be a discretly valued henselian field with valuation ring $\mathcal{O}$ and residue field $k$. We denote by $K_{nr}$ the maximal unramified extension of $K$ and by $K_t$ its maximal tamely ramified extension. If $G/K$ is a semisimple simply connected group, Bruhat-Tits theory is available in the sense of [13, 14] and the Galois cohomology set $H^1(K_{nr}/K, G)$ can be computed in terms of the Galois cohomology of special fibers of Bruhat-Tits group schemes [6]. This permits to compute $H^1(K, G)$ when the residue field $k$ is perfect.

On the other hand, if $k$ is not perfect, “wild cohomology classes” occur, that is $H^1(K_t, G)$ is non-trivial. Such examples appear for example in the study of bad unipotent elements of semisimple algebraic groups [10]. Under some restrictions on $G$, we would like to show that $H^1(K_t/K_{nr}, G)$ vanishes (see Corollary 3.3). This is related to the following quasi-splitness result.

**Theorem 1.1.** Let $G$ be a semisimple simply connected $K$–group which is quasi-split over $K_t$.

1. If the residue field $k$ is separably closed, then $G$ is quasi-split.
2. $G \times_K K_{nr}$ is quasi-split.

*Date:* July 7, 2017.

The author is supported by the project ANR Geolie, ANR-15-CE40-0012, (The French National Research Agency).
This theorem answers a question raised by Gopal Prasad who found another proof by reduction to the inner case of type $A$ [14, th. 4.4]. Our first observation is that the result is quite simple to establish under the following additional hypothesis:

\textit{(⋆) If the variety of Borel subgroups of $G$ carries a 0-cycle of degree one, then it has a $K$-rational point.}

Property (⋆) holds away of $E_8$ (section 2). It is an open question if (⋆) holds for groups of type $E_8$. For the $E_8$ case (and actually for a strongly inner $K$–group $G$) of Theorem 1.1, our proof is a Galois cohomology argument using Bruhat-Tits buildings (section 3).

We can make at this stage some remarks about the statement. Since $K_{nr}$ is a discretely valued henselian field with residue field $k_s$, we observe that (1) implies (2). Also a weak approximation argument [7, prop. 3.5.2] reduces to the complete case. If the residue field $k$ is separably closed of characteristic zero, we have then $cd(K) = 1$, so that the result follows from Steinberg’s theorem [16, §4.2, cor. 1]. In other words, the main case to address is that of characteristic exponent $p > 1$.

\textbf{Acknowledgements.} We are grateful to G. Prasad for raising this interesting question and also for fruitful discussions.

2. THE VARIETY OF BOREL SUBGROUPS AND 0–CYCLE OF DEGREE ONE

Let $k$ be a field, let $k_s$ be a separable closure and let $\text{Gal}(k_s/k)$ be the absolute Galois group of $k$. Let $q$ be a nonsingular quadratic form. A celebrated result of Springer states that the Witt index of $q$ is insensitive to odd degree field extensions. In particular the property to have a maximal Witt index is insensitive to odd degree extensions and this can be rephrased by saying that the algebraic group $\text{SO}(q)$ is quasi-split iff it is quasi-split over an odd degree field extension of $k$. This fact generalizes for all semisimple groups without type $E_8$.

\textbf{Theorem 2.1.} Let $G$ be a semisimple algebraic $k$-group without quotient of type $E_8$. Let $k_1, \ldots, k_r$ be finite field extensions of $k$ with coprime degrees. Then $G$ is quasi-split if and only if $G_{k_i}$ is quasi-split for $i = 1, ..., r$.

The proof is far to be uniform hence gathers several contributions [1, 8]. Note that the split version (in the absolutely almost simple case) is [9, th. C]. We remind the reader that a semisimple $k$-group $G$ is isomorphic to an inner twist of a quasi-split group $G^q$ and that such a $G^q$ is unique up to isomorphism. Denoting by $G^q_{ad}$ the adjoint quotient of $G^q$, this means that there exists a Galois cocycle $z : \text{Gal}(k_s/k) \to G^q_{ad}(k_s)$ such that $G$ is isomorphic to $zG^q$. We denote by $\pi : G^{sc,q} \to G^q_{ad}$ the simply connected cover of $G^q$. Then $zG^{sc,q}$ is the simply connected cover of $G^q$. Then $zG^{sc,q}$ is the simply connected cover of $zG^q \cong G$. 
Lemma 2.2. The following are equivalent:

(i) $G$ is quasi-split;
(ii) $[z] = 1 \in H^1(k, G^a_{ad})$;

If furthermore $[z] = \pi_*[z^{sc}]$ for a 1-cocycle $z^{sc}$ : Gal$(k_s/k) \to G^{sc,q}(k_s)$, (i) and (ii) also equivalent to

(iii) $[z^{sc}] = 1 \in H^1(k, G^{sc,q})$.

Proof. The isomorphism class of $G$ is encoded by the image of $[z]$ under the map $H^1(k, G^a_{ad}) \to H^1(k, \text{Aut}(G^q))$. The right handside map has trivial kernel since the exact sequence $1 \to G^q_{ad} \to \text{Aut}(G^q) \to \text{Out}(G^q) \to 1$ is split ([15, XXIV.3.10] or [12, 31.4]), whence the implication (ii) $\implies$ (i). The reverse inclusion (i) $\implies$ (ii) is obvious.

Now we assume that $z$ lifts to a 1-cocycle $z^{sc}$. The implication (iii) $\implies$ (ii) is then obvious. The point is that the map $H^1(k, G^{sc,q}) \to H^1(k, G^q)$ has trivial kernel [9, III.2.6] whence the implication (ii) $\implies$ (iii). \hfill \qedsymbol

We proceed to the proof of Theorem 2.1.

Proof. Let $X$ be the variety of Borel subgroups of $G$ [15, XXII.5.8.3], a projective $k$–variety. The $k$–group $G$ is quasi-split iff $X$ has a $k$-rational point. Thus we have to prove that if $X$ has a 0-cycle of degree one, then $X$ has a $k$-point.

Without loss of generality, we can assume that $G$ is simply connected. According to [15, XXIV.5] we have that $G \xrightarrow{\sim} \prod_{i=1,..,s} R_{l_i/k}(G_j)$ where $G_j$ is an absolutely almost simple simply connected group defined over a finite separable field extension $l_i$ of $k$ (the notation $R_{l_i/k}(G_j)$ stands as usual for the Weil restriction to $k$ to $l_i$).

The variety of Borel subgroup $X$ of $G$ is then isomorphic to $\prod_{j=1,..,s} R_{l_j/k}(X_j)$ where $X_j$ is the $l_j$-variety of Borel subgroups of $G_j$.

Reduction to the absolutely almost simple case. Our assumption is that $X(k_i) \neq \emptyset$ for $i = 1, .., r$ hence $X_j(k_i \otimes l_j) \neq \emptyset$ for $i = 1, .., r$ and $j = 1, .., s$. Since $l_j/k$ is separable, $k_i \otimes l_j$ is an étale $l_j$-algebra for $i = 1, .., r$ and it follows that $X_j$ carries a 0-cycle of degree one. If we know to prove the case of each $X_j$, we have $X_j(k_j) \neq \emptyset$ hence $X(k) \neq \emptyset$. From now on, we assume that $G$ is absolutely almost simple. We denote by $G_0$ the Chevalley group over $\mathbb{Z}$ such that $G$ is a twisted form of $G_0 \times_{\mathbb{Z}} k$.

Reduction to the characteristic zero case. If $k$ is of characteristic $p > 0$, let $O$ be a Cohen ring for the residue field $k$, that is a complete discrete valuation ring such that its fraction field $K$ is of characteristic zero and for which $p$ is an uniformizing parameter [3, IX.41]. The isomorphism class of $G$ is encoded by a Galois cohomology class in $H^1(k, \text{Aut}(G_0))$. Since $\text{Aut}(G_0)$ is a smooth affine $\mathbb{Z}$-group scheme [15, XXIV.1.3], we can use Hensel’s lemma $H^1_{\text{étale}}(O, \text{Aut}(G_0)) \xrightarrow{\sim} H^1(k, \text{Aut}(G_0))$ [15, XXIV.8.1] so that $G$ lifts in a semisimple simply connected group scheme $\mathfrak{G}$ over $O$. Let $X$ be the $O$–scheme of Borel subgroups of $\mathfrak{G}$ [15, XXII.5.8.3]. It is smooth and projective. For $i = 1, .., r$, let $K_i$ be an unramified field extension of $K$ of degree $[k_i : k]$ and of residue
field $k_i$. Denoting by $O_i$ its valuation ring, we consider the maps
\[ X(K_i) = X(O_i) \rightarrow X(k_i). \]
The left equality comes from the projectivity and the right surjectivity is Hensel’s lemma. It follows that $X(K_i) \neq \emptyset$ for $i = 1, \ldots, r$ so that $X_K$ has a 0-cycle of degree one. Assuming the result in the characteristic zero case, it follows that $X(K_i) \neq \emptyset$ for $i = 1, \ldots, r$, so that $X_K$ has a 0-cycle of degree one.

Assuming the result in the characteristic zero case, it follows that $X(K_i) = X(O_i) \neq \emptyset$ whence $X(k_i) \neq \emptyset$. We may assume from now that $k$ is of characteristic zero. We denote by $\mu$ the center of $G$ and by $t(G) \in H^2(k, \mu)$ the Tits class of $G$ [12, §31].

Since the Tits class of the quasi-split form $G^q$ of $G$ is zero, the classical restriction-corestriction argument yields that $t(G) = 0$. In other words $G$ is a strong inner form of its quasi-split form $G^q$. It means that there exists a Galois cocycle $z$ with value in $G^q(k)$ such that $G \cong z(G^q)$, that is the twist by inner conjugation of $G$ by $z$. Lemma 2.2 shows that our problem is rephrased in Serre’s question [17, §2.4] on the triviality of the kernel of the map
\[ H^1(k, G^q) \rightarrow \prod_{i=1, \ldots, r} H^1(k_i, G^q) \]
That kernel is indeed trivial in our case [2, Th. 0.4], whence the result.

We remind the reader that one can associate to a semisimple $k$-group $G$ its set $S(G)$ of torsion primes which depends only of its Cartan-Killing type [17, §2.2]. Since an algebraic group splits after an extension of degree whose primary factors belong to $S(G)$ [18], we get the following refinement.

**Corollary 2.3.** Let $G$ be a semisimple algebraic $k$-group without quotient of type $E_8$. Let $k_1, \ldots, k_r$ be finite field extensions of $k$ such that $\gcd([k_1 : k], \ldots, [k_r : k])$ is prime to $S(G)$. Then $G$ is quasi-split if and only if $G_{k_i}$ is quasi-split for $i = 1, \ldots, r$.

Lemma 2.2 together with the Corollary implies the following statement.

**Corollary 2.4.** Let $G$ be a semisimple simply connected quasi-split algebraic $k$-group without factors of type $E_8$. Let $k_1, \ldots, k_r$ be finite field extensions of $k$ such that $\gcd([k_1 : k], \ldots, [k_r : k])$ is prime to $S(G)$. Then the maps
\[ H^1(k, G) \rightarrow \prod_{i=1, \ldots, r} H^1(k_i, G) \]
and
\[ H^1(k, G_{ad}) \rightarrow \prod_{i=1, \ldots, r} H^1(k_i, G_{ad}) \]
have trivial kernels.

We can proceed now on the proof of Theorem 1.1.(1) away of $E_8$ since Theorem 2.1 shows that the condition $(\ast)$ is fulfilled in that case.

**Proof of Theorem 1.1.(1) under assumption $(\ast)$.** Here $K$ is a discretely valued henselian field. We are given a semisimple $K$-group $G$ satisfying assumption $(\ast)$, and such that
$G$ becomes quasi-split after a finite tamely ramified extension $L/K$. Note that $[L : K]$ is prime to $p$. We denote by $X$ the $K$–variety of Borel subgroups of $G$. We want to show that $X(K) \neq \emptyset$. We are then reduced to the following cases:

(i) $K$ is perfect and the absolute Galois group $\text{Gal}(K_s/K)$ is a pro-$l$-group for a prime $l \neq p$.

(ii) $\text{Gal}(K_s/K)$ is a pro-$p$-group.

By weak approximation [7, prop. 3.5.2], we may assume that $K$ is complete. Note that this operation does not change the absolute Galois group (ibid, 3.5.1).

Case (i): We have that $\text{cd}_l(K) \leq \text{cd}_l(k) + 1 = 1$ [16, §II.4.3] so that $\text{cd}(K) \leq 1$. Since $K$ is perfect, Steinberg’s theorem [16, §4.2, cor. 1] yields that $G$ is quasi-split.

Case (ii): The extension $K$ has no proper tamely ramified extension hence our assumption implies that $G$ is quasi-split. □

Remarks 2.5. a) In case (i) of the proof, there is no need to assume that $K$ is perfect and $l$ can be any prime different from $p$. The point is that if $\text{Gal}(K_s/K)$ is a pro-$l$-group, then the separable cohomological dimension of $K$ is less than or equal to 1, and then any semi-simple $K$-group is quasi-split, see [13, §1.7]

b) It an open question whether a $k$–group of type $E_8$ is split if it is split after co-prime degree extensions $k_i/k$. A positive answer to this question would imply Serre’s vanishing conjecture II for groups of type $E_8$ [11, §9.2].

c) Serre’s injectivity question has a positive answer for an arbitrary classical group (simply connected or adjoint) and holds for certain exceptional cases [2].

3. Cohomology and buildings

The field $K$ is as in the introduction.

Proposition 3.1. Assume that $k$ is separably closed. Let $G$ be a split semisimple connected $K$-group. Then $H^1(K_s/K, G) = 1$.

Proof. We can reason at finite level and shall prove that $H^1(L/K, G) = 1$ for a given finite tamely ramified extension of $L/K$. We put $\Gamma = \text{Gal}(L/K)$, it is a cyclic group whose order $n$ is prime to the characteristic exponent $p$ of $k$.

Let $\mathcal{B}(G_L)$ be the Bruhat-Tits building of $G_L$. It comes equipped with an action of $G(L) \rtimes \Gamma$ [5, §4.2.12]. Let $(B, T)$ be a Killing couple for $G$. The split $K$–torus $T$ defines an apartment $\mathcal{A}(T_L)$ of $\mathcal{B}(G_L)$ which is preserved by the action of $N_G(T)(L) \rtimes \Gamma$.

We are given a Galois cocycle $z : \Gamma \to G(L)$; it defines a section $u_z : \Gamma \to G(L) \rtimes \Gamma, \sigma \mapsto z_\sigma \sigma$ of the projection map $G(L) \rtimes \Gamma \to \Gamma$. This provides an action of $\Gamma$ on $\mathcal{B}(G_L)$ called the twisted action with respect to the cocycle $z$. The Bruhat-Tits fixed point theorem [4, §3.2] provides a point $y \in \mathcal{B}(G_L)$ which is fixed by the twisted action. This point belongs to an apartment and since $G(L)$ acts transitively on the set of apartments of $\mathcal{B}(G_L)$ there exists a suitable $g \in G(L)$ such that $g^{-1} \cdot y = x \in \mathcal{A}(T_L)$. 
We observe that $\mathcal{A}(T_L)$ is fixed pointwise by $\Gamma$ (for the standard action), so that $x$ is fixed under $\Gamma$. We consider the equivalent cocycle $z'_\sigma = g^{-1} z_\sigma \sigma(g)$ and compute

$$z'_\sigma . x = z'_\sigma . \sigma(x) = (g^{-1} z_\sigma (g)) (\sigma(g^{-1}). \sigma(y)) = g^{-1}. ((z_\sigma \sigma). y) = g^{-1} \cdot y \quad [y \text{ is fixed under the twisted action}] = x.$$

Without loss of generality, we may assume that $z_\sigma. x = x$ for each $\sigma \in \Gamma$. We put $\mathcal{P}_x = \text{Stab}_{\mathcal{O}(L)}(x)$; since $x$ is fixed by $\Gamma$, the group $\mathcal{P}_x$ is preserved by the action of $\Gamma$. Let $\mathcal{P}_x$ the Bruhat-Tits $\mathcal{O}_L$-group scheme attached to $x$. We have $\mathcal{P}_x(\mathcal{O}_L) = \mathcal{P}_x$ and we know that its special fiber $\mathcal{P}_x \times_{\mathcal{O}_L} k$ is smooth connected, that its quotient $M_x = (\mathcal{P}_x \times_{\mathcal{O}_L} k)/U_x$ by its split unipotent radical $U_x$ is split reductive.

An important point is that the action of $\Gamma$ on $\mathcal{P}_x(\mathcal{O}_L)$ arises from a semilinear action of $\Gamma$ on the $\mathcal{O}_L$-scheme $\mathcal{P}_x$ as explained in the beginning of §2 of [14]. It induces then a $k$–action of the group $\Gamma$ on $\mathcal{P}_x \times_{\mathcal{O}_L} k$, on $U_x$ and on $M_x$. Since $x$ belongs to $A(T_L)$, $\mathcal{P}_x$ carries a natural maximal split $\mathcal{O}_L$–torus $\mathcal{T}_x$ and $T_x = T_x \times_{\mathcal{O}_L} k$ is a maximal $k$–split torus of $\mathcal{P}_x \times_{\mathcal{O}_L} k$ and its image in $M_x$ still denoted by $T_x$ is a maximal $k$-split torus of $M_x$. We observe that $\Gamma$ acts trivially on the $k$-torus $T_x$. But $T_x/C(M_x) = \text{Aut}(M_x, \text{id}_{T_x})$ [15, XXIV.2.11], it follows that $\Gamma$ acts on $M_x$ by means of a group homomorphism $\phi : \Gamma \rightarrow T_{x, \text{ad}}(k)$ where $T_{x, \text{ad}} = T_x/C(M_x) \subseteq M_x/C(M_x) = M_{x, \text{ad}}$. For each $m \in M_x(k)$, we have $\sigma(m) = \text{int}(\phi(\sigma)). m$.

Now we take a generator $\sigma$ of $\Gamma$ and denote by $a_\sigma$ the image in $M_x(k)$ of $z_\sigma \in \mathcal{P}_x$ and by $\overline{a}_\sigma$ its image in $(M_x/C(M_x))(k)$. The cocycle relation yields $\overline{a}_\sigma^2 = \overline{a}_\sigma \sigma(\overline{a}_\sigma) = \overline{a}_\sigma \phi(\sigma) \overline{a}_\sigma \phi(\sigma)^{-1}$ and more generally (observe that $\phi(\sigma)$ is fixed by $\Gamma$)

$$\overline{a}_\sigma^j = \overline{a}_\sigma \phi(\sigma) \overline{a}_\sigma \phi(\sigma)^{-1} \cdots \phi(\sigma)^{-j} \overline{a}_\sigma \phi(\sigma)^{-1} \phi(\sigma)^j \overline{a}_\sigma \phi(\sigma)^{-j} = (\overline{a}_\sigma \phi(\sigma))^j \phi(\sigma)^{-j}$$

for $j = 2, \ldots, n$. Since $\phi(\sigma)^n = 1$, we get the relation

$$1 = (\overline{a}_\sigma \phi(\sigma))^n.$$

Then $\overline{a}_\sigma \phi(\sigma)$ is an element of order $n$ of $M_{x, \text{ad}}(k)$ so is semisimple. But $k$ is separably closed so that $\overline{a}_\sigma \phi(\sigma)$ belongs to a maximal $k$-split torus $mT_{x, \text{ad}}$ with $m \in M_x(k)$. It follows that $m^{-1} \overline{a}_\sigma \phi(\sigma) m \in T_{x, \text{ad}}(k)$. Since $\phi(\sigma)$ belongs to $T_{x, \text{ad}}(k)$, we have that $m^{-1} \overline{a}_\sigma \phi(\sigma) m \phi(\sigma)^{-1} \in T_{x, \text{ad}}(k)$ hence $m^{-1} a_\sigma \sigma(m) \in T_{x, \text{ad}}(k)$. It follows that $m^{-1} a_\sigma \sigma(m) \in T_x(k)$. Since the map $\mathcal{P}_x(\mathcal{O}_L) \rightarrow M_x(k)$ is surjective we can then assume that $a_\sigma \in T_x(k)$ without loss of generality so that the cocycle $\alpha$ takes value in $T_x(k)$. But $T_x(k)$ is a trivial $\Gamma$-module so that $\alpha$ is given by a homomorphism $f_\alpha : \Gamma \rightarrow T_x(k)$. This homomorphism lifts (uniquely) to a homomorphism $\tilde{f}_\alpha : \Gamma \rightarrow T_x(\mathcal{O}_L)^\Gamma$. The main technical step is

**Claim 3.2.** The fiber of $H^1(\Gamma, \mathcal{P}_x) \rightarrow H^1(\Gamma, M_x(k))$ at $[f_\alpha]$ is $\{[\tilde{f}_\alpha]\}$. 
Using the Claim, we have \([z] = [\overline{f}_a] \in H^1(\Gamma, P_x)\). Its image in \(H^1(\Gamma, G(L))\) belongs to the image of the map \(H^1(\Gamma, T_x(L)) \to H^1(\Gamma, G(L))\). But \(0 = H^1(\Gamma, T_x(L))\) (Hilbert 90 theorem) thus \([z] = 1 \in H^1(\Gamma, G(L))\) as desired.

It remains to establish the Claim. We put \(P_x^* = \ker(P_x \to M_x(k))\) and this group can be filtered by a \(\Gamma\)-stable decreasing filtration by normal subgroups \(U^{(i)}_{\geq 0}\) such that for each \(i \leq j\) there is a split unipotent \(k\)-group \(U^{(i,j)}\) equipped with an action of \(\Gamma\) such that \(U^{(i)}/U^{(j)} = U^{(i,j)}(k)\) \([14, \text{ page 6}]\). We denote by \(\overline{f}_a P_x^*\) the \(\Gamma\)-group \(P_x^*\) twisted by the cocycle \(\overline{f}_a\); there is a surjection \(H^1(\Gamma, \overline{f}_a P_x^*)\) on the fiber at \([f_a]\) of the map \(H^1(\Gamma, P_x) \to H^1(\Gamma, M_x(k))\) \([16, \text{I.5.5, cor. 2}]\). It is then enough to show that \(H^1(\Gamma, \overline{f}_a P_x^*) = 1\). It happens fortunately that the filtration is stable under the adjoint action of the image of \(\overline{f}_a\). By using the pro-unipotent \(k\)-group \(U = \varprojlim U^{(0,j)}\) and Lemma 4.1 in the next subsection, we have that \(H^1(\Gamma, \overline{f}_a P_x^*) = H^1(\Gamma, (\overline{f}_a U)(k)) = 1\). Since \(H^1(\Gamma, (\overline{f}_a U)(k))\) maps onto the kernel of fiber of \(H^1(\Gamma, P_x) \to H^1(\Gamma, M_x(k))\) at \([\overline{f}_a]\) \([16, \text{§I.5.5, cor. 2}]\), we conclude that the Claim is established.

This permits to complete the proof of Theorem 1.1.

Proof of Theorem 1.1.(1). By the usual reductions, the question boils down to the semisimple simply connected case and even the absolutely almost \(K\)-simple semisimple simply connected case. Taking into account the cases established in section 2, it remains to deal with the case of type \(E_8\). Denote by \(G_0\) the split group of type \(E_8\), we have \(G_0 = \text{Aut}(G_0)\). It follows that \(G \cong G_0\) with \([z] \in H^1(K, G_0)\). Our assumption is that \(G_{K_1}\) is quasi-split so that \([z] \in H^1(K_1/K, G_0)\). Proposition 3.1 states that \(H^1(K_1/K, G_0) = 1\), whence \(G\) is split.

We record the following cohomological application.

**Corollary 3.3.** Let \(G\) be a semisimple algebraic \(K\)-group which is quasi-split over \(K_1\). We assume that \(G\) is simply connected or adjoint. Then \(H^1(K_1/K_{nr}, G) = 1\).

Proof. Theorem 1.1 permits to assume that \(G\) is quasi-split. We denote by \(\pi : G \to G_{ad}\) the adjoint quotient of \(G\). Since the map \(H^1(K, G) \to H^1(K, G_{ad})\) has trivial kernel \([9, \text{ lemme III.2.6}]\), we can assume that \(G\) is adjoint. Let \([z] \in H^1(K_1/K_{nr}, G)\). We consider the twisted \(K_{nr}\)-form \(G' = z G\) of \(G\). Since \(G'_{K_1}\) is isomorphic to \(G_{K_1}\), \(G'_{K_1}\) is quasi-split and Theorem 1.1 shows that \(G'\) is quasi-split hence isomorphic to \(G\). It means that \(z\) belongs to the kernel of the map \(\text{int}_* : H^1(K, G) \to H^1(K, \text{Aut}(G))\). But the exact sequence of \(K\)-groups \(1 \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1\) splits \([15, \text{XXIV.3.10}]\) so that the above kernel is trivial. Thus \([z] = 1 \in H^1(K_1/K_{nr}, G)\). \(\square\)

4. **Appendix: Galois cohomology of pro-unipotent groups**

Let \(k\) be a separably closed field. Let \(U\) be a pro-unipotent algebraic \(k\)-group equipped with an action of a finite group \(\Gamma\), that is \(U\) admits a decreasing filtration
\[ U = U_0 \supset U_1 \supset U_2 \supset \cdots \] by normal pro unipotent \( k \)-groups which are stabilized by \( \Gamma \) and such that \( U_i / U_{i+1} \) is an unipotent algebraic \( k \)-group for \( i = 1, \ldots, n \).

**Lemma 4.1.** We assume that \( \sharp \Gamma \) is invertible in \( k \) and that the \( U_i / U_{i+1} \)'s are smooth and connected. Then \( H^1(\Gamma, U(k)) = 1 \).

**Proof.** We start with the algebraic case, that is of a smooth connected unipotent \( k \)-group. According to [15, XVII.4.11], \( U \) admits a central characteristic filtration \( U = U_0 \supset U_1 \supset \cdots \supset U_n = 1 \) such that \( U_i / U_{i+1} \) is a twisted form of a \( k \)-group \( G_n \).

Since \( U_{i+1} \) is smooth and \( k \) is separably closed, we have the following exact sequence of \( \Gamma \)-groups

\[ 1 \to U_{i+1}(k) \to U_i(k) \to (U_i / U_{i+1})(k) \to 1. \]

The multiplication by \( \sharp \Gamma \) on the abelian group \( (U_i / U_{i+1})(k) \) is an isomorphism so that \( H^1(\Gamma, (U_i / U_{i+1})(k)) = 0 \). The exact sequence above shows that the map \( H^1(\Gamma, U_{i+1}(k)) \to H^1(\Gamma, U_i(k)) \) is onto. By induction it follows that \( 1 = H^1(\Gamma, U_n(k)) \) maps onto \( H^1(\Gamma, U(k)) \) whence \( H^1(\Gamma, U(k)) = 1 \).

We consider now the pro-unipotent case. Since the \( U/U_i \)'s are smooth, we have that \( U(k) = \varprojlim(U/U_i)(k) \). Therefore by successive approximations the kernel of the map

\[ H^1(\Gamma, U(k)) \to \varprojlim H^1(\Gamma, (U/U_i)(k)) \]

is trivial. But according to the first case, the right hand side is trivial thus \( H^1(\Gamma, U(k)) = 1 \). \( \square \)

**References**

[1] E. Bayer-Fluckiger, H.W. Lenstra Jr., *Forms in odd degree extensions and self-dual normal bases*, Amer. J. Math. 112 (1990), 359–373.

[2] J. Black, *Zero cycles of degree one on principal homogeneous spaces*, J. Algebra 334 (2011), 232-246.

[3] N. Bourbaki, *Algèbre commutative* (Ch. 7–8), Springer–Verlag, Berlin, 2006.

[4] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. I*, Inst. Hautes Etudes Sci. Publ. Math. 41 (1972), 5–251.

[5] F. Bruhat, J. Tits, *Groupes algébriques sur un corps local II. Existence d’une donnée radicielle valuée*, Pub. Math. IHES 60 (1984), 5–184.

[6] F. Bruhat, J. Tits, *Groupes algébriques sur un corps local III. Compléments et application à la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo 34 (1987), 671–698.

[7] O. Gabber, P. Gille, L. Moret-Bailly, *Fibrés principaux sur les corps henséliens*, Algebraic Geometry 5 (2014), 573-612.

[8] S. Garibaldi, *The Rost invariant has trivial kernel for quasi-split groups of low rank*, Comment. Math. Helv. 76 (2001), 684–711.

[9] P. Gille, *La R-équivalence sur les groupes algébriques réductifs définis sur un corps global*, Publications Mathématiques de l'I.H.É.S. 86 (1997), 199-235.

[10] P. Gille, *Unipotent subgroups of reductive groups of characteristic \( p > 0 \)*, Duke Math. J. 114 (2002), 307-328.

[11] P. Gille, *Groupes algébriques semi-simples sur un corps de dimension cohomologique séparable \( \leq 2 \)*, monograph in preparation.
[12] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The book of involutions*, AMS Colloq. Publ. 44, Providence, 1998.

[13] G. Prasad, *A new approach to unramified descent in Bruhat-Tits theory*, preprint (2016), arXiv:1611.07430.

[14] G. Prasad, *Finite group actions on reductive groups and tamely-ramified descent in Bruhat-Tits theory*, preprint (2017), arXiv:1705.02906.

[15] Séminaire de Géométrie algébrique de l’I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math. 151-153. Springer (1970).

[16] J.-P. Serre, *Cohomologie galoisienne*, cinquième édition, Springer-Verlag, New York, 1997.

[17] J.-P. Serre, *Cohomologie galoisienne: Progrès et problèmes*, Séminaire Bourbaki, exposé 783 (1993-94), Astérisque 227 (1995).

[18] J. Tits, *Sur les degrés des extensions de corps déployant les groupes algébriques simples*, C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), 1131–1138.