ON THE UNIRULED VOISIN DIVISOR ON THE LLSVS VARIETY

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Abstract. Let $Y$ be a smooth cubic fourfold, $F$ be its Fano variety of lines and $Z$ be its associated LLSvS variety, parametrizing families of twisted cubics and some of their degenerations. In this short note, we show that the divisor of singular cubic surfaces on $Z$ has two irreducible components, one of which coincides with the uniruled branch divisor of a resolution of the Voisin map $F \times F \rightarrow Z$.

Introduction

Given a smooth cubic fourfold $Y \subset \mathbb{P}^5$, its Fano variety of lines $F$ is a hyperkähler variety [1, Proposition 1]. Assuming further that $Y$ does not contain any plane, one constructs starting from the variety of twisted cubics another hyperkähler variety $Z$ of dimension 8 [9, Theorem B]. Their geometry is deeply related as Voisin showed by constructing a degree 6 rational map [12, Proposition 4.8]

$$\varphi : F \times F \rightarrow Z.$$ 

Twisted cubics have Hilbert polynomial $3t + 1$, their flat degenerations are called generalised twisted cubics. Generalised twisted cubics on a singular cubic surface of type $A_1$ fall in three different types, which we are going to call $\alpha, \beta$, and $\gamma$. They determine two divisors $D_\alpha, D_\beta$, on $Z$, see Definition 4.

Theorem 1. The divisor $D \subset Z$ of generalised twisted cubics lying on singular cubic surfaces has two irreducible components

$$D = D_\alpha \cup D_\beta.$$ 

The first component coincides with the uniruled branch divisor of a resolution of the Voisin map.

Acknowledgements. The content of this paper is part of my PhD thesis, I would like to express my gratitude to Christian Lehn for having introduced me to the problem and for his constant support. I was supported by DFG research grants Le 3093/2-1 and Le 3093/2-2.
Basic properties of the LLSvS variety

Construction of the LLSvS variety. We recollect basic properties of the variety $Z$ from [9]. Let $Y \subset \mathbb{P}(V) \simeq \mathbb{P}^5$ be a smooth cubic fourfold not containing any plane and let $M$ be the variety of generalised twisted cubics on $Y$, that is, the irreducible component of $\text{Hilb}^{3t+1}(Y)$ containing smooth twisted cubics. For any curve $C \in M$ its linear span $E := \langle C \rangle$ is a $\mathbb{P}^3$ that cuts $Y$ in the cubic surface $S_C := E \cap Y$. These associations give rise to the diagram

$$
\begin{array}{c}
M \\
\downarrow \sigma \\
\mathbb{P}(S^3W^*) \\
\downarrow s \\
\text{Gr}(V, 4)
\end{array}
$$

where $W$ is the universal quotient bundle over the Grassmannian $\text{Gr}(V, 4)$ and the section $s$ is given by cutting $Y$ with a $\mathbb{P}^3$. The morphism $\sigma$ factors through a smooth irreducible variety $Z'$ of dimension 8:

$$
\begin{array}{c}
M \\
\downarrow a \\
Z' \\
\downarrow b \\
\text{Gr}(V, 4)
\end{array}
$$

where $a$ is an étale locally trivial $\mathbb{P}^2$-fibration and $b$ is generically finite [9, Theorem B]. It is in effect finite over the open set $U_{ADE} \subset Z'$ of surfaces that are either smooth or with ordinary double points. For the complement we have the following estimate [9, Corollary 3.11, Proposition 4.2, Proposition 4.3]:

$$\dim(Z' \setminus U_{ADE}) \leq 6. \quad (1)$$

The irreducible holomorphically symplectic variety $Z$ is obtained by the contraction $\pi : Z' \to Z$ of the irreducible divisor $D_{nCM} \subset Z'$ of families of curves that are not arithmetically Cohen-Macaulay [9, Theorem 4.11].

The divisor of singular cubic surfaces. The projection $\mathbb{P}(S^3W^*) \to \text{Gr}(V, 4)$ is equivariant for the natural action of $\text{PGL}(6)$. The locus $D_{\text{sing}} \subset \mathbb{P}(S^3W^*)$ of all singular cubic surfaces in $\mathbb{P}(V)$ surjects onto the Grassmannian, the fiber over a point $W_0$ is the divisor $D_{\text{sing}, W_0} \subset \mathbb{P}(S^3W^*_0)$ of singular cubic surfaces in $\mathbb{P}(W_0)$. Furthermore, $D_{\text{sing}, W_0}$ is irreducible [8, Theorem 2.2] and locally stratified depending on the singularity type of the parametrised surfaces, the $A_1$ locus forming an open set in $D_{\text{sing}, W_0}$. Since $D_{\text{sing}}$ coincides with the orbit of $D_{\text{sing}, W_0}$ under the action of $\text{PGL}(6)$ we conclude that $D_{\text{sing}}$ is an irreducible divisor and the $A_1$-locus is open in it. Pulling back this divisor along $s \circ b : Z' \to \mathbb{P}(S^3W^*)$ and taking advantage of both the finiteness of $s \circ b$ on the ADE-locus and the estimate (1) we get the following
Proposition 2. Let \( D' \subset Z' \) be the image under \( a \) of the locus of curves lying on singular cubic surfaces. Then \( D' \) is a divisor. Moreover, the preimage under \( s \circ b \) of singular cubic surfaces of type \( A_1 \) is a dense open set in \( D' \).

Analogously singular cubic surfaces determine a divisor \( D_{Gr} := s^{-1}(D_{sing}) \) in \( \text{Gr}(V, 4) \) and a divisor \( D := \pi(D') \) in \( Z \). In light of this proposition, when discussing the irreducible components of \( D' \) it will suffice to treat only the \( A_1 \) locus.

Twisted cubics on \( A_1 \)-singular cubic surfaces. Whereas twisted cubics on smooth cubic surfaces are a classical subject of study, twisted cubics on surfaces with ordinary double points are well explained in \([9, \S \, 2.1]\), whose content we briefly recall here and then make explicit in the specific case of \( A_1 \) singularities. For basic facts on the root lattice \( E_6 \) in connection with the geometry of the cubic surface we point to \([6, \S \, 9], [5]\), for lines on singular cubic surfaces \([2]\).

Given a singular cubic surface \( S \) with ordinary double points, its minimal resolution \( \tilde{S} \) is a weak del Pezzo surface and the orthogonal complement \( K_{\tilde{S}}^\perp \subset \text{Pic}(\tilde{S}) \) is a lattice of type \( E_6 \). The exceptional divisor of the resolution \( r : \tilde{S} \to S \) consists of \((-2)\)-curves, which form a subset of the root system \( R := \{ \alpha : \alpha^2 = 0 \} \subset K_{\tilde{S}}^\perp \) and generate a subroot system \( R_0 \subset R \). Let \( W(R_0) \) be the Weyl group generated by reflections of elements in \( R_0 \), then \([9, \text{Theorem 2.1}]\) gives a description of the Hilbert scheme with the reduced structure of generalised twisted cubics on \( S \):

\[
\text{Hilb}^{gtc}(S)_{\text{red}} \simeq R/W(R_0) \times \mathbb{P}^2.
\]

For any \( \alpha \in R \setminus R_0 \) and for any curve \( C \in |\alpha - K_{\tilde{S}}| \) the image \( r(C) \) is a generalised twisted cubic on \( S \). Conversely, the pullback of any aCM-curve on \( S \) lies in such a linear system \([9, \text{Proposition 2.2, Proposition 2.5, Proposition 2.6}]\). On the other hand, roots in \( R_0 \) correspond to families of nCM curves. We now want to make (2) explicit for \( A_1 \)-singular surfaces.

Let \( S \subset \mathbb{P}^3 \) be a singular cubic surface of type \( A_1 \) with singular point \( P \), then its minimal resolution \( \tilde{S} \) is the blow-up with center \( P \), whose exceptional divisor is a \((-2)\)-curve \( \Gamma \). The linear system \( |-\Gamma - K_{\tilde{S}}| \) realises \( \tilde{S} \) as a blow-up of \( \mathbb{P}^2 \) in 6 points \( P_1, \ldots, P_6 \) lying on a quadric \( Q \), which is exactly the image of \( \Gamma \). In a picture:

\[
\begin{array}{c}
\pi \\
\downarrow \\
\mathbb{P}^2
\end{array} \quad \begin{array}{c}
\longrightarrow \quad r \\
\downarrow \\
\tilde{S} \\
\longrightarrow \quad \rho \\
\downarrow \\
S \subset \mathbb{P}^3.
\end{array}
\]

Here \( \rho \) is the rational map given by the linear system of cubics passing through the 6 points.

The Picard group of \( \tilde{S} \) is generated by the pullback \( H \) of the hyperplane class of \( \mathbb{P}^2 \) and the 6 exceptional divisors \( E_1, \ldots, E_6 \). The surface \( S \) contains 21
Proposition 3. Let $\rho$ be mapped via the singular point; moreover, the line trough any two distinct points $P_i, P_j$ is mapped via $\rho$ to the unique line $R_{ij}$ on $S$ intersecting both $E_i$ and $E_j$ not in $P$.

The canonical bundle $K_S$ has class $-3H + E_1 + E_2 + E_3 + E_4 + E_5 + E_6$, its orthogonal complement $K_S^\perp \subset \text{Pic}(\tilde{S})$ is a root lattice of type $E_6$. It has 72 roots:

$$\alpha_{ij} = E_i - E_j, \text{ for } i \neq j$$
$$\pm \beta_{ijk} = \pm(H - E_i - E_j - E_k), \text{ for } i, j, k \text{ pairwise distinct; }$$
$$\pm \gamma = \pm(2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6).$$

We are going to call the roots $\alpha_{i,j}$ of type $\alpha$, the roots $\pm \beta_{i,j,k}$ of type $\beta$, and the roots $\pm \gamma$ of type $\gamma$. The unique effective root is $\gamma$, the reflection with center $\gamma$ fixes the roots of type $\alpha$, whereas its action on $\beta$-roots is:

$$\pm \beta_{ijk} \mapsto \mp \beta_{lmn}$$

with $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$. Hence according to [9, Theorem 2.1] the Hilbert scheme with the reduced structure is the disjoint union of 51 copies of $\mathbb{P}^2$

$$\text{Hilb}^{\text{gic}}(S)_{\text{red}} \simeq \bigcup_{i,j,i\neq j} |\alpha_{ij} - K_\tilde{S}| \cup \bigcup_{i,j,k \text{ pairwise distinct}} |\beta_{ijk} - K_\tilde{S}| \cup | - \gamma - K_\tilde{S}|.$$

The curves parametrised by $| - \gamma - K_\tilde{S}|$ are not arithmetically Cohen Macaulay and will play no role in the following. For any curve $C$ in $|\alpha_{ij} - K_\tilde{S}|$ or in $|\beta_{ijk} - K_\tilde{S}|$ the schematic image $r(C)$ is a generalised twisted cubic on $S$.

An ordered pair $(E_i, E_j)$ of distinct lines through the singular point determines the family of generalised twisted cubic on $S$ corresponding to the linear system $|E_i - E_j - K_\tilde{S}|$. In contrast any triple $(E_i, E_j, E_k)$ of pairwise distinct lines through the singular point determines the family of generalised twisted cubic on $S$ corresponding to the linear system $|H - E_i - E_j - E_k - K_\tilde{S}|$.

We recapitulate what so far discussed in the following

**Proposition 3.** Let $S$ be a singular cubic surface of type $A_1$, let $P$ be the singular point.

(i) The surface $S$ has 6 distinguished lines $E_1, ..., E_6$, they are the only ones through the singular point. For each pair $(E_i, E_j)$ there exists a unique line $R_{ij}$ meeting both of them not in $P$.

(ii) Each ordered pair $(E_i, E_j)$ determines the family of twisted cubics of type $\alpha$ corresponding to the linear system $|E_i - E_j - K_\tilde{S}|$.

(iii) Each triple $(E_i, E_j, E_k)$ determines the family of twisted cubics of type $\beta$ corresponding to the linear system $|H - E_i - E_j - E_k - K_\tilde{S}|$.

\[1\text{Even though } H \text{ seems to depend on the choice of the resolution it does not because } H = -\Gamma - K_\tilde{S} \text{ where } \Gamma \text{ is the } (-2)-\text{curve.}\]
The general element of the divisor $D'$ is a family of generalised twisted cubic lying on a singular surface of type $A_1$ and can be of type $\alpha$, $\beta$ or $\gamma$.

**Definition 4.** We define $D'_\alpha$, respectively $D'_\beta$ and $D'_\gamma$, as the closure of the sets in $Z'$ of families of generalised twisted cubics lying on $A_1$-singular cubic surfaces of type $\alpha$, respectively $\beta$ and $\gamma$. We call $D_\alpha$, respectively $D_\beta$, the image $\pi(D'_\alpha)$ in $Z$, respectively $\pi(D'_\beta)$.

**Proposition 5.** The closed set $D'_\gamma$ is an irreducible divisor in $Z'$.

*Proof.* The set $D'_\gamma$ coincides with the divisor of nCM generalised twisted cubics, which is irreducible [9, Proposition 4.5].

In the next sections we show the irreducibility of $D'_\alpha$, $D_\alpha$ (Corollary 8) and of $D'_\beta$, $D_\beta$ (Corollary 10).

**The irreducible components of $D$**

**The irreducible component** $D_\alpha$. Let $Y$ be a smooth cubic fourfold not containing a plane and let $F$ be its Fano variety of lines, which is an irreducible holomorphically symplectic variety of dimension 4. Voisin [12, Proposition 4.8] constructed a degree 6 rational map

$$\varphi: F \times F \rightarrow Z.$$ 

The construction goes as follows. Let $(l, l')$ be a general point in $F \times F$. It corresponds to a pair of non-coplanar lines $L, L'$. After having chosen a point $x$ on $L$, one takes the residual conic $Q_x$ to $L'$ of the intersection $Y \cap \langle x, L' \rangle$. The union of $Q_x$ and $L$ determines then the class of a generalised twisted cubic on the cubic surface $S := \langle L, L' \rangle \cap Y$. In other words, the lines $L$ and $L'$ determine the family $|O_S(-K_S + L - L')|$ of generalised twisted cubics on $S$.

The indeterminacy locus of $\varphi$ coincides with the variety $I$ of incident lines, which is irreducible of dimension 6, [10, Theorem 1.2, Lemma 2.3]. The branch divisor of a resolution of $\varphi$ is a uniruled divisor, as remarked in [12, Remark 4.10], for details see [10, Lemma 4.4]. We consider the rational maps

$$\varphi' := \pi^{-1} \circ \varphi: F \times F \rightarrow Z'$$

$$\psi := b \circ \pi^{-1} \circ \varphi: F \times F \rightarrow Gr$$

where $\pi : Z' \rightarrow Z$ is the morphism discussed above and $\pi^{-1}$ is its rational inverse map. Here $Gr := Gr(V, 4)$ parametrises 3-dimensional projective spaces in $\mathbb{P}^5 = \mathbb{P}(V)$. As natural resolution of the indeterminacy locus of $\psi$ we consider the closure $\Gamma$ of its graph with projections $p: \Gamma \rightarrow F \times F$, $q: \Gamma \rightarrow Gr$. We study its points by taking flat limits along curves $\lambda$ on $F \times F$ through points where the rational map is not defined.
Lemma 6. About the fiber of the graph over $I$ we have:

$$q(p^{-1}(I)) = D^{Gr}.$$ 

Moreover, for the general point $(l, l')$ in $I$ we have:

$$p^{-1}(l, l') = \{ E \in Gr : \langle L, L' \rangle \subset E \subset T_y Y \} \simeq \mathbb{P}^1$$

where $y$ is the unique intersection point of $L$ and $L'$.

Proof. Let $i = (l, l')$ be a general point in $I$ and $\lambda \subset F \times F$ a smooth curve in an affine open intersecting $I$ exactly in $i$. By the properness of the Grassmannian we get a morphism $\lambda \rightarrow Gr$ and the curve parametrises a family $f : S \rightarrow \lambda$ of cubic surfaces contained in $Y$. Since singular cubic surfaces determine a divisor in the Grassmannian, we may assume that $S_t$ is smooth for every $i \neq t \in \lambda$. By [7, III, Theorem 10.2] the smoothness of the morphism $f$ is equivalent to the smoothness of the surface $S_i$.

We claim that the surface $S_i$ is not smooth. Suppose to the contrary that the family were smooth, then the Picard groups $\text{Pic}(S_t) = H^2(S_t, \mathbb{Z})$ would glue together in the local system $R^2f_*\mathbb{Z}$ on $\lambda$. Every point $t \in \gamma$ parametrises a pair of lines $(l_t, l'_t)$, which are disjoint for $t \neq i$. Taking their intersection product in $\text{Pic}(S_t)$ we then get

$$0 = L_t \cdot L'_t = L_i \cdot L'_i = 1.$$ 

Hence, we conclude that $S_i$ is singular.

This shows the factorisation $p^{-1}(I) \rightarrow D^{Gr} \subset Gr$. Choosing $\lambda$ accurately, one proves that the morphism is dominant onto $D^{Gr}$ and thus surjective.

Indeed, let $E \in D^{Gr}$ be a $\mathbb{P}^3$ that contains two distinct incident lines $L$ and $L'$ meeting in a point $y$, in which the cubic surface $E \cap Y$ is singular. We consider the following diagram involving the tangent space $T_y Y$ at $y$, the normal bundle $N_{L|Y}$ of $L$ in $Y$ and its stalk $N_{L|Y}(y)$ at the point $y$ with the natural maps:

$$\begin{array}{ccc}
T_y Y & \rightarrow & H^0(L, N_{L|Y}) \xrightarrow{ev} N_{L|Y}(y).
\end{array}$$

The general line $L$ is of type I [3, Definition 6.6], that is, $N_{L|Y} \cong \mathcal{O}_L^{\oplus 2} \oplus \mathcal{O}_L(1)$, thus we may assume that the evaluation map $ev$ is surjective. Let $e$ be a vector in $T_y E$ not contained in $T_y \langle L, L' \rangle$. The image of $e$ under $T_y E \subset T_y Y \rightarrow N_{L|Y}(y)$ lifts to a vector $\tilde{e} \in H^0(L, N_{L|Y})$, which corresponds to a deformation of $L$ and is represented by a curve $\lambda'$ in $F$ through $l$. If we set $\lambda = \lambda' \times \{l'\} \subset F \times F$, then the limit $\mathbb{P}^3$ computed along $\lambda$ coincides with $E$.

The first assertion is now proven.

The limit surface $\mathcal{S}_i$ is in effect singular in the intersection point $y$ of $L_i$ and $L'_i$. Indeed, we may assume $\mathcal{S}_i$ has one $A_1$-singularity in the point
In virtue of [11, §2], after restricting to an analytic neighbourhood of \( i \), there is a diagram

\[
\begin{array}{ccc}
T & \xrightarrow{r} & S \\
\downarrow & & \downarrow \\
(\lambda', i') & \xrightarrow{f} & (\lambda, i)
\end{array}
\]

where \( f: (\lambda', i') \rightarrow (\lambda, i) \) is a finite Galois cover mapping \( i' \) to \( i \) and where \( T \rightarrow \lambda' \) is a family of smooth surfaces such that \( T_t \rightarrow \mathcal{J}_{f(t)} \) is an isomorphism for any \( i' \neq t \in \lambda' \) and \( T_{i'} \rightarrow \mathcal{J}_{i'} \) is the minimal resolution of \( \mathcal{J}_{i'} \). The surfaces \( T_t \) are isomorphic to blow-ups of \( \mathbb{P}^2 \) in 6 points, which are in general position for any \( t \neq i' \), hence the groups \( \text{Pic}(T_t) \) form a local system over the all \( \lambda' \). After further shrinking \( \lambda' \) to a contractible neighbourhood of \( i' \) the latter becomes trivial. The lines \( L_{f(t)} \subset \mathcal{J}_{f(t)} \approx T_t \) form a flat family over \( \lambda \setminus \{i'\} \). Taking its closure we find a curve \( X \) in \( T_{i'} \), which completes the family to a flat family over all \( \lambda' \) and corresponds to a section of the local system of Picard groups. The curve \( X \) can be either the strict transform \( \tilde{L} \) of the line \( L \) or the union of \( \tilde{L} \) and the \((-2)\)-curve \( \Gamma \), which arises as exceptional divisor of the resolution of \( \mathcal{J}_{i'} \). Analogously, for the lines \( L'_{i'} \) we find the curve \( X' \) in \( T_{i'} \), which can be either \( \tilde{L}' \) or the union of \( \tilde{L}' \) and the curve \( \Gamma \). Since the intersection product of \( X \) and \( X' \) in \( \text{Pic}(T_{i'}) \) must coincide with \( L_{f(t)} \cdot L'_{f(t)} = 0 \) for any \( t \neq i' \), the only possibility is that \( P \) lies on both the lines \( L \) and \( L' \).

Let \( \lambda \) be a curve as the one in the proof above, since \( Z' \) is proper the Voisin map extends to a well-defined morphism

\[ \varphi'_{\lambda}: \lambda \rightarrow Z'. \]

We are interested in the image of \( i \), which is represented by a family of generalised twisted cubic. By the previous lemma we know that any such curve lies on a singular surface.

**Lemma 7.** For the general point \( i \in I \) and the general curve \( \lambda \) the limit twisted cubic \( \varphi'_{\lambda}(i) \) is of type \( \alpha \).

**Proof.** We may assume that the limit family of twisted cubics lies on a singular surface \( S_t \) of type \( A_1 \) with one singularity at the point of intersection of \( L, L' \). The point \( \varphi'_{\lambda}(i) \) is represented by a family of generalised twisted cubics on \( S_t \), which in turn corresponds to a linear system \( A \) on the minimal resolution \( \tilde{S} \) of \( S_t \). In contrast, for any other point \( t \neq i \) in \( \lambda \) the image \( \varphi_{\lambda}(t) \) consists of the family of curves in \( |\mathcal{O}_{S_t}(-K_{S_t} + L_t - L'_t)| \). After passing to a Galois cover of an analytic neighbourhood of \( i \) in \( \lambda \) as before, we see that the linear system \( A \) is equal to \( |\mathcal{O}_{\tilde{S}}(-K_{\tilde{S}} + \tilde{L} - \tilde{L}')| \), where \( \tilde{L} \) and \( \tilde{L}' \) are the strict transforms of the two lines \( L \) and \( L' \). Thus the limit family \( \varphi'_{\lambda}(i) \) is of type \( \alpha \).

In terms of the geometry of \( Z \) we have thus proven the following.
Corollary 8. The closed set $D'_\alpha$ is an irreducible uniruled divisor in $Z'$. Its image $D_\alpha$ in $Z$ coincides with the branch locus of a resolution of the Voisin map.

The irreducible component $D_\beta$. We consider the variety of triples of lines with non-trivial common intersection:

\[ I_3 := \{(l_1, l_2, l_3) \in F \times F \times F : L_1 \cap L_2 \cap L_3 \neq \emptyset \}. \]

Lemma 9. The variety $I_3$ is irreducible of dimension 7.

Proof. Let $\mathbb{L} \subset F \times Y$ be the universal family of lines on $Y$ parametrised by $F$, its threefold product fits in the diagram

\[
\begin{array}{ccc}
\mathbb{L} \times \mathbb{L} \times \mathbb{L} & \xrightarrow{p} & Y \times Y \times Y \\
\downarrow q & & \\
F \times F \times F,
\end{array}
\]

where $p$ and $q$ denote the natural projections. The variety $I_3$ is the image via $q$ of $J := p^{-1}(\Delta)$, where $Y \cong \Delta \subset Y \times Y \times Y$ is the diagonal embedding. Since $J$ is locally cut out by 8 equations, all its irreducible components have dimension greater than or equal to 7. The restriction of $q$ to $J$ is birational and just contracts the large diagonal

\[ \{(l_1, l_2, l_3) \in I_3 : l_{i_1} = l_{i_2} \text{ for some } i_1 \neq i_2 \} \]

which has dimension 6. Thus all irreducible components of $I_3$ have dimension at least 7. Via the projection $F \times F \times F \to F \times F$ onto the first two factors the variety $I_3$ is fibred over the irreducible variety $I$ of dimension 6 [10, Lemma 2.3]:

\[ p_{12} : I_3 \to I. \]

We study its fibres.

- if $(l, l')$ lies on the diagonal of $F \times F$, that is $L = L'$, then its preimage is the variety $F_L$ of lines intersecting $L$.
- In contrast, if $(l, l') \in I$ is a point such that $L \cap L' = \{y\}$ then the fibre $p_{12}^{-1}(l, l')$ is the variety $C_y$ of lines through $y$. The variety $C_y$ of lines through a given point $y$ is a curve, which in general irreducible, except for finitely many points in $Y$ for which it is a surface [4, Proposition 2.4]. The variety $F_L$ admits a rational map to $L$ well-defined away from $l \in F_L$:

\[ F_L \dashrightarrow L, \; r \mapsto R \cap L. \]

The fibre over a point $y \in L$ is the variety $C_y$, thus $F_L$ is a surface. It follows that $I_3$ is irreducible.

A general triple $(l_1, l_2, l_3)$ in $I_3$ spans a $\mathbb{P}^3$ which intersects the cubic fourfold $Y$ in a singular cubic surface of type $A_1$: the singular point being the unique common intersection point of the three lines. According to our discussion in the previous section, this data determines the class of a
generalised twisted cubic of type $\beta$ (cf. Proposition 3). We have therefore constructed a rational map

$$\rho: I_3 \longrightarrow Z'.$$

which is dominant onto $D'_\beta$. As immediate consequence we get

**Corollary 10.** The closed set $D'_\beta$ in $Z'$ as well as its image $D_\beta$ in $Z$ is an irreducible divisor.

**References**

[1] Arnaud Beauville and Ron Donagi. La variété des droites d’une hypersurface cubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(14):703–706, 1985.

[2] J. W. Bruce and C. T. C. Wall. On the classification of cubic surfaces. *J. London Math. Soc. (2)*, 19(2):245–256, 1979.

[3] C. Herbert Clemens and Phillip A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[4] Izzet Coskun and Jason Starr. Rational curves on smooth cubic hypersurfaces. *Int. Math. Res. Not. IMRN*, (24):4626–4641, 2009.

[5] Michel Demazure, Henry Charles Pinkham, and Bernard Teissier, editors. *Séminaire sur les Singularités des Surfaces*, volume 777 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Held at the Centre de Mathématiques de l’École Polytechnique, Palaiseau, 1976–1977.

[6] Igor V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view.

[7] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[8] Daniel Huybrechts. The geometry of cubic hypersurfaces. available at [http://www.math.uni-bonn.de/people/huybrech/Notes.pdf](http://www.math.uni-bonn.de/people/huybrech/Notes.pdf).

[9] Christian Lehn, Manfred Lehn, Christoph Sorger, and Duco van Straten. Twisted cubics on cubic fourfolds. *J. Reine Angew. Math.*, 731:87–128, 2017.

[10] Giosuè Emanuele Muratore. The indeterminacy locus of the Voisin map. *Beitr. Algebra Geom.*, 61(1):73–88, 2020.

[11] Oswald Riemenschneider. Special surface singularities: a survey on the geometry and combinatorics of their deformations. Number 807, pages 93–118. 1992. Analytic varieties and singularities (Japanese) (Kyoto, 1992).

[12] Claire Voisin. Remarks on zero-cycles of self-products of varieties. In *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*, volume 179 of *Lecture Notes in Pure and Appl. Math.*, pages 265–285. Dekker, New York, 1996.

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