Existence of solutions for a higher-order semilinear parabolic equation with singular initial data

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Abstract

We establish the existence of solutions of the Cauchy problem for a higher-order semilinear parabolic equation by introducing a new majorizing kernel. We also study necessary conditions on the initial data for the existence of local-in-time solutions and identify the strongest singularity of the initial data for the solvability of the Cauchy problem.

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1 Introduction

Consider the Cauchy problem for a higher-order nonlinear parabolic equation
\[
\begin{cases}
\partial_t u + (-\Delta)^m u = |u|^p, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = \mu(x) \geq 0, & x \in \mathbb{R}^N,
\end{cases}
\tag{1.1}
\]
where \(m = 2, 3, \ldots, p > 1\) and \(\mu\) is a nonnegative measurable function in \(\mathbb{R}^N\) or a nonnegative Radon measure in \(\mathbb{R}^N\). Problem \(1.1\) is one of the simplest evolution problems for higher-order nonlinear parabolic equations. In this paper we establish the existence of solutions of problem \(1.1\) by introducing a new majorizing kernel to the operator
\[
\partial_t u + (-\Delta)^m u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty).
\tag{1.2}
\]

We also study necessary conditions on the initial data for the existence of local-in-time solutions of \(1.1\) and we identify the strongest singularity of the initial data for the solvability of problem \(1.1\).

Before considering problem \(1.1\), we recall some results on the Cauchy problem for a semilinear parabolic equation
\[
\begin{cases}
\partial_t u - \Delta u = u^p, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = \mu(x) \geq 0, & x \in \mathbb{R}^N.
\end{cases}
\tag{1.3}
\]

In 1985 Baras and Pierre [3] studied necessary conditions for the existence of local-in-time solutions of \(1.3\) and proved the following (see also [14] and [23]).

**Theorem 1.1** Let \(u\) be a nonnegative solution of \(1.3\) in \(\mathbb{R}^N \times [0, T)\) for some \(T > 0\), where \(\mu\) is a nonnegative Radon measure in \(\mathbb{R}^N\). Then there exists \(c = c(N, p) > 0\) such that
\[
\sup_{x \in \mathbb{R}^N} \mu(B(x, \sigma)) \leq c \sigma^{N-\frac{2}{p-1}} \quad \text{for} \quad 0 < \sigma \leq T^\frac{1}{2}.
\tag{1.4}
\]

In particular, in the case of \(p = p_1 := 1 + 2/N\), there exists \(c' = c'(N) > 0\) such that
\[
\sup_{x \in \mathbb{R}^N} \mu(B(x, \sigma)) \leq c \left[ \log \left( e + \frac{T^\frac{1}{2}}{\sigma} \right) \right]^{-\frac{N}{2}} \quad \text{for} \quad 0 < \sigma \leq T^\frac{1}{2}.
\tag{1.5}
\]

We remark that, if \(1 < p < p_1\), then \(1.4\) is equivalent to
\[
\sup_{x \in \mathbb{R}^N} \mu(B(x, T^{1/2})) \leq c T^{\frac{N}{2} - \frac{1}{p-1}}.
\tag{1.5}
\]

By Theorem [1.1] we have:

(a) There exists \(c_1 = c_1(N, p) > 0\) such that, if \(\mu\) is a nonnegative measurable function in \(\mathbb{R}^N\) satisfying
\[
\mu(x) \geq c_1 |x|^{-N} \left[ \log \left( e + \frac{1}{|x|} \right) \right]^{-\frac{N}{2} - 1} \quad \text{if} \quad p = p_1,
\]
\[
\mu(x) \geq c_1 |x|^{-\frac{2}{p-1}} \quad \text{if} \quad p > p_1,
\]
Sufficient conditions for the existence of solutions of problem (1.3) have been studied in many papers since the pioneering work due to [25]. See e.g. [1, 2, 6, 11, 14, 17, 20, 21, 22, 23, 26] and references therein. Among others, by [14] and [22] we have:

(b) Let $1 < p < p_1$. Then there exists $c_2 = c_2(N, p) > 0$ such that, if

$$\sup_{x \in \mathbb{R}^N} \mu(B(x, T^\frac{1}{2})) \leq c_2 T^{\frac{N}{2} - \frac{1}{p-1}}$$

for some $T > 0$, then problem (1.3) possesses a solution in $\mathbb{R}^N \times [0, T]$.

(c) Let $p > p_1$. Then there exists $c_3 = c_3(N, p) > 0$ such that, if

$$0 \leq \mu(x) \leq c_3 |x|^{-N} \left[ \log \left( e + \frac{1}{|x|} \right) \right]^{-\frac{N}{2} - 1} + c_3$$

if $p = p_1$,

$$0 \leq \mu(x) \leq c_3 |x|^{-\frac{2}{p-1}} + c_3$$

if $p > p_1$,

then problem (1.3) possesses a local-in-time solution.

By assertions (a) and (c) we can identify the strongest singularity of the initial data for the existence of solutions of (1.3) with $p \geq p_1$. Assertions (b) and (c) are proved by the construction of suitable supersolutions of (1.3) and the order-preserving property and the semigroup property of the heat operator are crucial in the proofs.

The operator $\partial_t + (-\Delta)^m$ is not order-preserving and the study of the solvability of problem (1.1) is more delicate than that of problem (1.3). Indeed, the fundamental solution $G_m = G_m(x, t)$ of (1.2) changes its sign for $t > 0$. In the study of higher-order parabolic equations it is crucial to find a suitable majorizing kernel associated with $\partial_t + (-\Delta)^m$.

Galaktionov and Pohozaev [13] found a majorizing kernel of the form

$$\overline{G}_m(x, t) := D t^{-\frac{N}{2m}} \exp \left( -d \eta^{\frac{2m}{2m-1}} \right) \quad \text{with} \quad \eta = \eta(x, t) = t^{-\frac{1}{2m}} |x|, \quad (1.6)$$

where $D$ and $d$ are positive constants (see Section 2.1), and proved the existence of global-in-time solutions of (1.1) for any sufficiently small initial data in $L^1 \cap L^\infty$ in the case of $p > p_m := 1 + 2m/N$. They also proved nonexistence of global-in-time solutions of (1.1) provided that $1 < p \leq p_m$ and $\mu(x) \geq 0 \ (\not\equiv 0)$ in $\mathbb{R}^N$. Subsequently, the existence and the asymptotic behavior of global-in-time solutions with bounded initial data have been studied in several papers under suitable assumptions on the decay of the initial data at the space infinity. See e.g. [13, 17, 18]. (See also [9, 12].) On the other hand, it does not seem enough to study sufficient conditions for the existence of local-in-time solutions of problem (1.1) with singular initial data, although the results in [8] are available. As far as we know, there are no results related to the identification of the strongest singularity of the initial data for the existence of solutions of (1.1). One of the difficulties is that the integral operator associated with $\overline{G}_m$ does not have the semigroup property. Indeed, we can not apply the arguments in [14, 22, 25] with the majorizing kernel $\overline{G}_m$ to problem (1.1).
In this paper, by use of the fundamental solution of
\[ \partial_t u + (-\Delta)^{\theta/2} u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty), \] (1.7)
where \(0 < \theta < 2\), we introduce a new majorizing kernel \(K = K(x, t)\) satisfying
\[ |G_m(x, t)| \leq C_1 K(x, t), \]
\[ \int_{\mathbb{R}^N} K(x - y, t - s)K(y, s) \, dy \leq C_2 K(x, t), \] (1.8)
for \(x \in \mathbb{R}^N\) and \(0 < s < t\). Here \(C_1\) and \(C_2\) are positive constants. Applying the arguments in [14, 24] with an integral operator associated with \(K\), we establish the existence of solutions of problem (1.1). Furthermore, we modify the arguments in [7, 16] to study necessary conditions on the initial data for the existence of local-in-time solutions of (1.1). Then we can identify the strongest singularity of the initial data for the existence of local-in-time solutions of (1.1).

Now we are ready to state our main results of this paper. The first theorem concerns necessary conditions for the solvability of problem (1.1) and it corresponds to Theorem 1.1.

**Theorem 1.2** Let \(N \geq 1, m = 2, 3, \ldots\) and \(p > 1\). Let \(u\) be a solution of (1.1) in \(\mathbb{R}^N \times [0, T)\) for some \(T > 0\), where \(\mu\) is a nonnegative Radon measure in \(\mathbb{R}^N\). Then there exists \(\gamma = \gamma(N, m, p) > 0\) such that
\[ \sup_{x \in \mathbb{R}^N} \mu(B(x, \sigma)) \leq \gamma \sigma^{N - \frac{2m}{p-1}} \quad \text{for} \quad 0 < \sigma \leq T\frac{1}{2m}. \] (1.9)

In particular, if \(p = p_m := 1 + 2m/N\), then there exists \(\gamma' = \gamma'(N, m)\) such that
\[ \sup_{x \in \mathbb{R}^N} \mu(B(x, \sigma)) \leq \gamma' \left( \log \left( e + \frac{T^{\frac{1}{2m}}}{\sigma} \right) \right)^{-\frac{N}{2m}} \quad \text{for} \quad 0 < \sigma \leq T\frac{1}{2m}. \] (1.10)

Similarly to (1.5), if \(1 < p < p_m\), then (1.9) is equivalent to
\[ \sup_{x \in \mathbb{R}^N} \mu(B(x, T^{\frac{1}{2m}})) \leq \gamma T^{\frac{N}{2m} - \frac{1}{p-1}}. \]

As a corollary of Theorem 1.2 we have:

**Corollary 1.1** Let \(N \geq 1, m = 2, 3, \ldots\) and \(p \geq p_m\). Then there exists \(\gamma_1 = \gamma_1(N, m, p) > 0\) such that, if \(\mu\) is a nonnegative measurable function in \(\mathbb{R}^N\) satisfying
\[ \mu(x) \geq \gamma_1 |x|^{-N \left[ \log \left( e + \frac{1}{|x|} \right) \right]^{-\frac{N}{2m} - 1}} \quad \text{if} \quad p = p_m, \]
\[ \mu(x) \geq \gamma_1 |x|^{-\frac{2m}{p-1}} \quad \text{if} \quad p > p_m, \]
in a neighborhood of the origin, then problem (1.1) possesses no local-in-time solutions.
Corollary 1.1 corresponds to assertion (a). Next we state results on sufficient conditions for the existence of solutions of problem (1.1).

**Theorem 1.3** Let \( N \geq 1, m = 2, 3, \ldots \) and \( 1 < p < p_m \). Let \( \mu \) be a nonnegative Radon measure in \( \mathbb{R}^N \). Then there exists \( \gamma_2 = \gamma_2(N, m, p) > 0 \) such that, if

\[
\sup_{x \in \mathbb{R}^N} \mu(B(x, T^{\frac{1}{p_m}})) \leq \gamma_2 T^{\frac{N}{p_m} - \frac{1}{p - 1}} \tag{1.11}
\]

for some \( T > 0 \), then problem (1.1) possesses a solution in \( \mathbb{R}^N \times [0, T) \).

**Theorem 1.4** Let \( N \geq 1, m = 2, 3, \ldots \) and \( p \geq p_m \). Then there exists \( \gamma_3 = \gamma_3(N, m, p) > 0 \) such that, if

\[
0 \leq \mu(x) \leq \gamma_3 |x|^\frac{2m}{p - 1} + \gamma_3 \quad \text{if} \quad p > p_m,
\]

then problem (1.1) possesses a local-in-time solution.

Theorems 1.3 and 1.4 correspond to assertions (b) and (c), respectively. Theorem 1.4 is a direct consequence of Theorems 5.2 and 5.3. (See also Remarks 5.1 and 5.2.) Furthermore, as a corollary of Theorems 1.2 and 1.3, we have:

**Corollary 1.2** Let \( \delta \) be the Dirac delta function in \( \mathbb{R}^N \). Then problem (1.1) possesses a local-in-time solution with \( \mu = D\delta \) for some \( D > 0 \) if and only if \( 1 < p < p_m \).

The rest of this paper is organized as follows. In Section 2 we collect preliminary results on the operators \( \partial_t + (\Delta)^m \) \((m = 2, 3, \ldots)\) and \( \partial_t + (\Delta)^{\theta/2} \) \((0 < \theta < 2)\) and their associated semigroups. We also formulate the definition of solutions of problem (1.1). Furthermore, we formulate the definition of solutions of an integral equation associated with problem (1.1) and prove some properties of the solutions. In Section 3 we modify the arguments in [7, 16] to prove Theorem 1.2. In Section 4 we introduce a majorizing kernel \( K = K(x, t) \) associated with \( \partial_t + (\Delta)^m \) and prove (1.8). In Section 5 we establish the existence of solutions of problem (1.1).

### 2  Preliminaries

This section is divided into three subsections. In Sections 2.1 and 2.2 we recall some preliminary results on the operators \( \partial_t + (\Delta)^m \) \((m = 2, 3, \ldots)\) and \( \partial_t + (\Delta)^{\theta/2} \) \((0 < \theta < 2)\), respectively. In Section 2.3 we formulate the definition of solutions of problem (1.1). Furthermore, we introduce an integral equation associated with problem (1.1) and prove some properties of the solutions.

We introduce some notation. For any \( 1 \leq r \leq \infty \), we denote by \( \| \cdot \|_r \) the usual norm of \( L^r := L^r(\mathbb{R}^N) \). For any \( x \in \mathbb{R}^N \) and \( R > 0 \), we set \( B(x, R) := \{ y \in \mathbb{R}^N : |x - y| < R \} \). For any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{M} := (\mathbb{N} \cup \{0\})^N \), we write

\[
|\alpha| := \sum_{i=1}^N \alpha_i, \quad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.
\]
By the letter $C$ we denote generic positive constants and they may have different values also within the same line.

### 2.1 Fundamental solutions to $\partial_t + (-\Delta)^m$ ($m = 2, 3, \ldots$)

Let $G_m = G_m(x, t)$ ($m = 2, 3, \ldots$) be the fundamental solution of (1.2). Then $G_m$ is represented by

$$G_m(x, t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi} e^{-t|\xi|^{2m}} d\xi, \quad x \in \mathbb{R}^N, t > 0. \quad (2.1)$$

The function $G_m$ changes its sign and the operator $\partial_t + (-\Delta)^m$ is not order-preserving.

Let $G_m$ be as in (1.6). Then, under a suitable choice of $D$ and $d$, it follows that

$$|G_m(x, t)| \leq \overline{G}_m(x, t), \quad x \in \mathbb{R}^N, t > 0. \quad (2.2)$$

(See [13].) Furthermore, $G_m$ satisfies

$$G_m(x, t) = t^{-\frac{N}{2m}} G_m(t^{-\frac{1}{2m}} x, 1), \quad (2.3)$$

$$G_m(0, t) > 0, \quad (2.4)$$

$$|\partial_\alpha^a G_m(x, t)| \leq C_{\alpha} t^{-\frac{N+|\alpha|}{2m}} \exp\left(-C_{\alpha}^{-1} \eta^{2m-2}\right) \quad \text{with} \quad \eta = t^{-\frac{1}{2m}} |x|, \quad (2.5)$$

for $x \in \mathbb{R}^N, t > 0$ and $\alpha \in \mathbb{M}$, where $C_{\alpha}$ is a positive constant. (2.3) and (2.4) immediately follow from (2.1). For (2.5), see e.g. [8, Section 3] and [9].

We define an integral operator associated with $G_m$. For any (signed) Radon measure $\mu$ in $\mathbb{R}^N$, we set

$$[S_m(t)\mu](x) := \int_{\mathbb{R}^N} G_m(x - y, t) d\mu(y), \quad x \in \mathbb{R}^N, t > 0. \quad (2.6)$$

Similarly, for any measurable function $\phi$ in $\mathbb{R}^N$, we set

$$[S_m(t)\phi](x) := \int_{\mathbb{R}^N} G_m(x - y, t) \phi(y) dy, \quad x \in \mathbb{R}^N, t > 0. \quad (2.7)$$

Let $j = 0, 1, 2, \ldots$. It follows from the Young inequality and (2.3) that

$$\|\partial_\alpha^a S_m(t)\phi\|_q \leq C_m t^{-\frac{N}{2m} \left(\frac{1}{p} - \frac{1}{q}\right)} \|\phi\|_p, \quad t > 0, \quad (2.8)$$

for $\phi \in L^p$ and $\alpha \in \mathbb{M}$ with $|\alpha| = j$, where $1 \leq p \leq q \leq \infty$ and $C_m$ is a positive constant independent of $p$ and $q$. (See also [8, Section 2].) Furthermore,

$$\lim_{t \to +0} \|S_m(t)\phi - \phi\|_\infty = 0 \quad (2.9)$$

for $\phi \in C_0(\mathbb{R}^N)$. The convergence rate depends on the modulus of continuity of $\phi$.  

2.2 Fundamental solutions to \( \partial_t + (-\Delta)^{\theta/2} \) \( (0 < \theta < 2) \)

Let \( 0 < \theta < 2 \). Let \( G_\theta = G_\theta(x,t) \) be the fundamental solution of \((1.1)\), that is,

\[
G_\theta(x,t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix\cdot \xi} e^{-t|\xi|^\theta} d\xi.
\]

Then \( G_\theta = G_\theta(x,t) \) is a positive, smooth and radially symmetric function in \( \mathbb{R}^N \times (0,\infty) \) and satisfies the following properties (see [4, 5]):

\[
\begin{align*}
G_\theta(x,t) &= t^{-\frac{N}{\theta}} G_\theta(t^{-\frac{1}{\theta}} x, 1), \\
|\partial^\alpha_x G_\theta(x,t)| &\leq C_\alpha t^{-\frac{N+|\alpha|}{\theta}} (1 + t^{-\frac{1}{\theta}} |x|)^{-N-\theta-|\alpha|}, \\
G_\theta(x,t) &\geq C t^{-\frac{N}{\theta}} (1 + t^{-\frac{1}{\theta}} |x|)^{-N-\theta},
\end{align*}
\]

for \( x \in \mathbb{R}^N, \ t > 0 \) and \( \alpha \in \mathbb{M} \), where \( C_\alpha \) is a positive constant. Furthermore, it follows that

\[
G_\theta(x,t) = \int_{\mathbb{R}^N} G_\theta(x-y,t-s) G_\theta(y,s) dy, \quad x \in \mathbb{R}^N, \ 0 < s < t.
\]

Similarly to (2.6) and (2.7), we set

\[
[S_\theta(t)\mu](x) := \int_{\mathbb{R}^N} G_\theta(x-y,t) \mu(y), \quad [S_\theta(t)\phi](x) := \int_{\mathbb{R}^N} G_\theta(x-y,t) \phi(y) dy,
\]

for (signed) Radon measure \( \mu \) in \( \mathbb{R}^N \) and measurable function \( \phi \) in \( \mathbb{R}^N \). Then, for any \( j = 0,1,2,\ldots, \), by the Young inequality and (2.11) we find \( C_j > 0 \) such that

\[
\| \partial^\alpha_x S_\theta(t) \phi \|_q \leq C_j t^{-\frac{N}{\theta} \left( \frac{1}{p} + \frac{1}{q} \right) - \frac{\theta}{\sigma} \| \phi \|_p, \quad t > 0,
\]

for \( \phi \in L^q, \ 1 \leq p \leq q \leq \infty \) and \( \alpha \in \mathbb{M} \) with \( |\alpha| = j \). See e.g. [17]. Furthermore, we recall the following lemma on the decay of \( \|S_\theta(t)\mu\|_\infty \). See \([12, \text{Lemma 2.1}]\).

**Lemma 2.1** Let \( \mu \) be a nonnegative Radon measure in \( \mathbb{R}^N \) and \( 0 < \theta < 2 \). Then there exists \( C = C(N,\theta) > 0 \) such that

\[
\|S_\theta(t)\mu\|_\infty \leq Ct^{-\frac{N}{\theta}} \sup_{x \in \mathbb{R}^N} \mu(B(x,t^{\frac{1}{\theta}})), \quad t > 0.
\]

2.3 Definition of solutions of \((1.1)\)

We formulate a definition of solutions of problem \((1.1)\).

**Definition 2.1** Let \( N \geq 1, \ m = 2,3,\ldots, \ p > 1 \ and \ T > 0 \). Let \( u \) be a locally integrable function in \( \mathbb{R}^N \times [0,T) \). Then we say that \( u \) is a solution of \((1.1)\) in \( \mathbb{R}^N \times [0,T) \) if \( u \) satisfies

\[
- \int_{\mathbb{R}^N} \varphi(x,0) \mu(dx) + \int_0^T \int_{\mathbb{R}^N} \left[ -u \partial_t \varphi + u(-\Delta)^m \varphi \right] dx \, dt = \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi \, dx \, dt
\]

for all \( \varphi \in C^\infty(\mathbb{R}^N \times [0,T)) \) with \( \text{supp} \varphi \subset B(0,R) \times [0,T-\epsilon] \) for some \( R > 0 \) and \( 0 < \epsilon < T \).
We also formulate a definition of solutions of the integral equation
\[ u(x,t) = \int_{\mathbb{R}^N} G_m(x - y, t) \, d\mu(y) + \int_0^t \int_{\mathbb{R}^N} G_m(x - y, t - s) |u(y,s)|^p \, dy \, ds. \quad (I) \]

**Definition 2.2** Let \( N \geq 1, \, m = 2, 3, \ldots, \, p > 1 \) and \( \mu \) be a nonnegative Radon measure in \( \mathbb{R}^N \). Let \( u \) be a continuous function in \( \mathbb{R}^N \times (0,T) \) for some \( T > 0 \) and set
\[
\begin{align*}
\mathfrak{m}_1(x,t) &:= \int_{\mathbb{R}^N} |G_m(x - y, t)| \, d\mu(y), \\
\mathfrak{m}_2(x,t) &:= \int_0^t \int_{\mathbb{R}^N} |G_m(x - y, t - s)| |u(y,s)|^p \, dy \, ds.
\end{align*}
\]

We say that \( u \) is a solution of integral equation (I) in \( \mathbb{R}^N \times [0,T) \) if
\[
\sup_{\tau \leq t < T} \|\mathfrak{m}_1(t)\|_\infty + \sup_{\tau \leq t < T} \|\mathfrak{m}_2(t)\|_\infty < \infty \quad \text{for} \quad \tau \in (0,T) \tag{2.15}
\]
and \( u \) satisfies integral equation (I) for \( (x,t) \in \mathbb{R}^N \times (0,T) \).

In the rest of this section we show that the solution of integral equation (I) is a solution of (1.1).

**Proposition 2.1** Let \( u \) be a solution of integral solution (I) in \( \mathbb{R}^N \times [0,T) \) for some \( T > 0 \).

(a) For any \( \tau \in (0,T) \), \( u_\tau \) defined by \( u_\tau(x,t) := u(x, t + \tau) \) is a solution of problem (1.1) in \( \mathbb{R}^N \times [0,T - \tau) \) with the initial data \( u(\tau) \).

(b) Let \( \alpha \in \mathcal{M} \) and \( i \in \{0,1\} \) be such that \( |\alpha| + 4i \leq 2m \). Then \( \partial_t^\alpha \partial_x^i u \in BC(\mathbb{R}^N \times [\tau,T)) \) for \( \tau \in (0,T) \).

(c) \( u \) satisfies
\[
\partial_t u + (-\Delta)^m u = |u|^p, \quad (x,t) \in \mathbb{R}^N \times (0,T), \tag{2.16}
\]
in the classical sense.

Furthermore, \( u \) is a solution of (1.1) in \( \mathbb{R}^N \times [0,T) \).

**Proof of assertions (a), (b) and (c).** Let \( u \) be a solution of integral equation (I) in \( \mathbb{R}^N \times [0,T) \) for some \( T > 0 \). By (1.6), (2.2) and (2.15) we see that
\[
\begin{align*}
\int_{\mathbb{R}^N} |G_m(x - y, t - \tau)| \left[ \int_{\mathbb{R}^N} |G_m(y - z, \tau)| \, d\mu(z) \right] \, dy < \infty, \\
\int_{\mathbb{R}^N} |G_m(x - y, t - \tau)| \left[ \int_0^\tau \int_{\mathbb{R}^N} |G_m(y - z, \tau - s)| u(z,s) |^p \, dz \, ds \right] \, dy < \infty,
\end{align*}
\]
for $x \in \mathbb{R}^N$ and $0 < \tau < t$. It follows from the Fubini theorem that
\[
\begin{align*}
\int_{\mathbb{R}^N} G_m(x - y, t - \tau) u(y, \tau) \, dy \\
= \int_{\mathbb{R}^N} G_m(x - y, t - \tau) \\
\times \left[ \int_{\mathbb{R}^N} G_m(y - z, \tau) \, d\mu(z) + \int_0^\tau \int_{\mathbb{R}^N} G_m(y - z, \tau - s) |u(z, s)|^p \, dz \, ds \right] \, dy \\
= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} G_m(x - y, t - \tau) G_m(y - z, \tau) \, dy \right) d\mu(z) \\
+ \int_0^\tau \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} G_m(x - y, t - \tau) G_m(y - z, \tau - s) \, dy \right) |u(z, s)|^p \, dz \, ds \\
= \int_{\mathbb{R}^N} G_m(x - z, t) \, d\mu(z) + \int_0^\tau \int_{\mathbb{R}^N} G_m(x - z, t - s) |u(z, s)|^p \, dz \, ds
\end{align*}
\]
for $x \in \mathbb{R}^N$ and $0 < \tau < t$. This together with Definition 2.2 implies that
\[
\sup_{\tau \leq t < T} \|u(t)\|_\infty < \infty, \quad 0 < \tau < T, \quad (2.17)
\]
\[
u(x, t) = \int_{\mathbb{R}^N} G_m(x - y, t - \tau) u(y, \tau) \, dy \\
+ \int_\tau^t \int_{\mathbb{R}^N} G_m(x - y, t - s) |u(y, s)|^p \, dy \, ds, \quad x \in \mathbb{R}^N, \quad 0 < \tau < T,
\]
and assertion (a) holds. By (2.17) we apply similar arguments in regularity theorems for second order parabolic equations (see e.g. [10, Chapter 1]) to integral equation (I) and obtain assertions (b) and (c). $\square$

It remains to prove that $u$ is a solution of problem (1.1). For this aim, we modify the arguments in [14] to prepare the following two lemmas.

**Lemma 2.2** Let $u$ be a solution of integral equation (I) in $\mathbb{R}^N \times [0, T)$ for some $T > 0$. Then
\[
\lim_{t \to +0} \int_{\mathbb{R}^N \setminus B(0, R)} G_m(\lambda x, t) \, d\mu(x) = 0, \quad (2.18)
\]
\[
\lim_{t \to +0} \int_0^t \int_{\mathbb{R}^N \setminus B(0, R)} G_m(\lambda x, t - s) |u(x, s)|^p \, dx \, ds = 0, \quad (2.19)
\]
for $R > 0$ and $\lambda > 0$.

**Proof.** By (2.4) we find $R_* > 0$ and $c_* > 0$ such that
\[
\inf_{x \in B(0, R_*)} G_m(x, 1) \geq c_* > 0.
\]
Then it follows from (2.3) that
\[
G_m(x - y, t) \geq t^{- \frac{N}{2m}} c_* \quad \text{for} \quad x - y \in B(0, R_* t^{\frac{1}{2m}}).
\]
This together with (2.14) and (2.15) implies that
\[ \infty > \|\overline{u}_1(T_{\epsilon})\|_\infty \geq \overline{u}_1(x, T_{\epsilon}) \geq \int_{B(x, T^{\frac{1}{2m}})} |G_m(x - y, T_{\epsilon})| \, d\mu(y) \]
\[ \geq c_* T^{-\frac{N}{2m}} \mu(B(x, R, T^{\frac{1}{2m}})), \]
\[ \infty > \|\overline{u}_2(T_{\epsilon})\|_\infty \geq \overline{u}_2(x, T_{\epsilon}) \geq \int_0^{T_{2\epsilon}} \int_{B(x, R, T_{\epsilon} - s)^{\frac{1}{2m}}} |G_m(x - y, T_{\epsilon} - s)| u(y, s)|^p \, dy \, ds \]
\[ \geq c_* \int_0^{T_{2\epsilon}} (T_{\epsilon} - s)^{-\frac{N}{2m}} \int_{B(x, R, T_{\epsilon} - s)^{\frac{1}{2m}}} |u(y, s)|^p \, dy \, ds \]
\[ \geq c_* \epsilon^{-\frac{N}{2m}} \int_0^{T_{2\epsilon}} |u(y, s)|^p \, dy \, ds, \]
for \( x \in \mathbb{R}^N \), where \( T_{\epsilon} := T - \epsilon \), \( T_{2\epsilon} = T - 2\epsilon \) and \( 0 < 2\epsilon < T \). Since \( x \in \mathbb{R}^N \) is arbitrary, we deduce that
\[ \sup_{x \in \mathbb{R}^N} \mu(B(x, R)) < \infty, \]
\[ \sup_{x \in \mathbb{R}^N} \int_0^{T_{-\epsilon}} \int_{B(x, R)} |u(y, s)|^p \, dy \, ds < \infty, \]
for \( R > 0 \) and \( 0 < \epsilon < T/2 \). (See [19] Lemma 2.1.)

Let \( 0 < R < \infty \) and set \( R' := \min\{R/2, 1/2\} \). By the Besicovitch covering lemma we can find an integer \( n_* \) depending only on \( N \) and a set \( \{x_{k,i}\}_{k=1,...,n_*,i \in \mathbb{N}} \subset \mathbb{R}^N \setminus B(0, R) \) such that
\[ \overline{B}(x_{k,i}, R') \cap \overline{B}(x_{k,j}, R') \quad \text{if} \quad i \neq j, \]
\[ \mathbb{R}^N \setminus B(0, R) \subset \bigcup_{k=1}^{n_*} \bigcup_{i=1}^{\infty} \overline{B}(x_{k,i}, R') \subset \mathbb{R}^N \setminus B(0, R/2). \]
\[ (2.21) \]

Then we have
\[ \int_{\mathbb{R}^N \setminus B(0, R)} \overline{G}_m(\lambda x, t) \, d\mu(x) \leq \sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \int_{B(x_{k,i}, R')} \overline{G}_m(\lambda x, t) \, d\mu(x) \]
\[ \leq \sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \mu(B(x_{k,i}, R')) \sup_{x \in B(x_{k,i}, R')} \overline{G}_m(\lambda x, t) \]
\[ \leq \sup_{x \in \mathbb{R}^N} \mu(B(x, 1)) \sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \sup_{x \in B(x_{k,i}, R')} \overline{G}_m(\lambda x, t). \]
\[ (2.22) \]

Let \( \epsilon > 0 \) be such that \( 2(1 - \epsilon) > 1 + \epsilon \). For \( k = 1, \ldots, n_* \) and \( i \in \mathbb{N} \), since \( x_{k,i} \not\in B(0, R) \) and \( R' \leq R/2 \), we have
\[ \frac{|x_{k,i}|}{R'} \geq \frac{R}{R'} \geq 2 > \frac{1 + \epsilon}{1 - \epsilon}, \]
which implies that \( |x_{k,i}| - R' \geq \epsilon (|x_{k,i}| + R') \). Then it holds that
\[ |y| \geq |x_{k,i}| - R' \geq \epsilon (|x_{k,i}| + R') \geq \epsilon |x| \]
for \( y, z \in \overline{B(x_k, R')} \), \( k = 1, \ldots, n_* \) and \( i \in \mathbb{N} \). Therefore we observe from (1.6) that

\[
\sup_{x \in B(x_k, R')} \overline{G}_m(\lambda x, t) \leq \inf_{x \in B(x_k, R')} \overline{G}_m(\lambda \epsilon x, t) \leq \frac{1}{|B(0, R')|} \int_{B(x_k, R')} \overline{G}_m(\lambda \epsilon z, t) \, dz
\]

for \( k = 1, \ldots, n_* \) and \( i \in \mathbb{N} \), where \(|B(0, R')|\) is the volume of \( B(0, R') \). This together with (2.21) implies that

\[
\sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \sup_{x \in B(x_k, R')} \overline{G}_m(\lambda x, t) \leq C n_* R'^{-N} \int_{\mathbb{R}^N \setminus B(0, R/2)} \overline{G}_m(\lambda \epsilon z, t) \, dz
\]

\[
= C n_* R'^{-N} \int_{\mathbb{R}^N \setminus t^{-1/\nu} B(0, R/2)} \overline{G}_m(\lambda \epsilon z, 1) \, dz \to 0
\]

as \( t \to +0 \). Combining (2.22) and (2.23), we obtain (2.18).

Since

\[
\overline{G}_m(\lambda x, t - s) \leq C (t - s)^{-N} \exp \left(-C^{-1} \eta(\lambda x, t - s)^{2m-2}\right)
\]

\[
\leq \exp \left(-(2C)^{-1} \eta(\lambda x, t - s)^{2m-2}\right)
\]

\[
\leq \exp \left(-(2C)^{-1} \eta(\lambda x, t)^{2m-2}\right) =: \hat{G}_m(\lambda x, t)
\]

for \( x \in \mathbb{R}^N \setminus B(0, R) \) and \( 0 < s < t \), we have

\[
\int_0^t \int_{\mathbb{R}^N \setminus B(0, R)} \overline{G}_m(\lambda x, t - s)|u(x, s)|^p \, dx \, ds
\]

\[
\leq \sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \int_0^t \int_{B(x_k, R')} \overline{G}_m(\lambda x, t - s)|u(x, s)|^p \, dx \, ds
\]

\[
\leq \sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \sup_{x \in B(x_k, R')} \hat{G}_m(\lambda x, t) \int_0^t \int_{B(x_k, R')} |u(x, s)|^p \, dx \, ds
\]

\[
\leq \sup_{x \in \mathbb{R}^N} \int_0^{T/2} \int_{B(x, 1)} |u(x, s)|^p \, dx \, ds \sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \sup_{x \in B(x_k, R')} \hat{G}_m(\lambda x, t)
\]

for \( 0 < t \leq T/2 \). Similarly to (2.23), we observe that

\[
\sum_{k=1}^{n_*} \sum_{i=1}^{\infty} \sup_{x \in B(x_k, R')} \hat{G}_m(\lambda x, t) \leq C n_* R'^{-N} \int_{\mathbb{R}^N \setminus B(0, R/2)} \hat{G}_m(\lambda \epsilon z, t) \, dz \to 0
\]

as \( t \to +0 \). Combining (2.24) and (2.25), we see that

\[
\lim_{t \to +0} \int_0^t \int_{\mathbb{R}^N \setminus B(0, R)} \overline{G}_m(\lambda x, t - s)|u(x, s)|^p \, dx \, ds = 0,
\]

which implies (2.19). Thus Lemma 2.2 follows. □
Lemma 2.3 Let $u$ be a solution of integral equation (I) in $\mathbb{R}^N \times [0, T)$ for some $T > 0$. Then
\[
\lim_{t \to +0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x, t) G_m(x - y, t) \, d\mu(y) \, dx = \int_{\mathbb{R}^N} \varphi(y, 0) \, d\mu(y), \quad (2.26)
\]
\[
\lim_{t \to +0} \int_{\mathbb{R}^N} \int_{0}^{t} \int_{\mathbb{R}^N} G_m(x - y, t - s) \varphi(x, t) |u(y, s)|^p \, dy \, ds \, dx = 0, \quad (2.27)
\]
for $\varphi \in C^\infty(\mathbb{R}^N \times [0, T))$ with $\text{supp} \varphi \subset B(0, R) \times [0, T - \epsilon]$ for some $R > 0$ and $\epsilon \in (0, T)$.

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N \times [0, T))$ be such that $\text{supp} \varphi \subset B(0, R) \times [0, T - \epsilon]$ for some $R > 0$ and $\epsilon \in (0, T)$. Set
\[
\Phi(x, t : \tau) := [S_m(t) \varphi(\tau)](x) = \int_{\mathbb{R}^N} G_m(x - y, t) \varphi(y, \tau) \, dy, \quad x \in \mathbb{R}^N, \ t > 0, \ \tau \in (0, T).
\]
By (2.8) we have
\[
\|\Phi(t : \tau)\| \leq C \|\varphi(\tau)\| \leq C \|\varphi\|_{L^\infty(\mathbb{R}^N \times (0, T))}, \quad t > 0, \ \tau \in (0, T). \quad (2.28)
\]

On the other hand, it follows from the Fubini theorem that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x, t) G_m(x - y, t) \, d\mu(y) \, dx
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x, t) G_m(x - y, t) \, dx \, d\mu(y) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x, t) G_m(y - x, t) \, dx \, d\mu(y) \quad (2.29)
\]
\[
= \int_{\mathbb{R}^N} \Phi(y, t; t) \, d\mu(y) = \int_{\mathbb{R}^N} \varphi(y, 0) \, d\mu(y) + \int_{\mathbb{R}^N} [\Phi(y, t : t) - \varphi(y, 0)] \, d\mu(y).
\]

Since $|x - y| \geq |x|/2$ for $x \in \mathbb{R}^N \setminus B(0, 2R)$ and $y \in B(0, R)$, by (2.2) we can find $\lambda > 0$ such that
\[
|\Phi(x, t; \tau)| \leq \|\varphi\|_{L^\infty(\mathbb{R}^N \times (0, T))} \int_{B(0, R)} |G_m(x - y, t)| \, dy \quad (2.30)
\]
\[
\leq C \|\varphi\|_{L^\infty(\mathbb{R}^N \times (0, T))} \overline{G_m}(\lambda x, t)
\]
for $x \in \mathbb{R}^N \setminus B(0, 2R), \ t > 0$ and $\tau \in (0, T)$. Furthermore, by the uniform continuity of $\varphi$ in $B(0, 2R) \times [0, T - \epsilon]$ and (2.9) we observe that
\[
\sup_{x \in B(0, 2R)} |\Phi(x, t : t) - \varphi(x, 0)|
\]
\[
\leq \sup_{x \in B(0, 2R)} |\Phi(x, t : t) - \varphi(x, t)| + \sup_{x \in B(0, 2R)} |\varphi(x, t) - \varphi(x, 0)| \to 0 \quad (2.31)
\]
as $t \to +0$. Therefore, by (2.18), (2.28) and (2.31) we have
\[
\int_{\mathbb{R}^N} [\Phi(y, t : t) - \varphi(y, 0)] \, d\mu(y)
\]
\[
\leq \int_{B(0, 2R)} |\Phi(y, t : t) - \varphi(y)| \, d\mu(y) + \int_{\mathbb{R}^N \setminus B(0, 2R)} |\Phi(y, t : t)| \, d\mu(y)
\]
\[
\leq \sup_{x \in B(0, 2R)} |\Phi(x, t : t) - \varphi(x, 0)| \mu(B(0, 2R))
\]
\[
+ C \|\varphi\|_{L^\infty(\mathbb{R}^N \times (0, T))} \int_{\mathbb{R}^N \setminus B(0, 2R)} \overline{G_m}(\lambda y, t) \, d\mu(y) \to 0
\]
as \( t \to +0 \). Combining (2.29) and (2.32), we have (2.26). Furthermore, by (2.19), (2.28) and (2.30) we have

\[
\left| \int \int \int G_m(x - y, t - s) \varphi(x, t) u(y, s) \, dy \, ds \, dx \right|
\]

\[
= \left| \int \int \Phi(y, t - s) u(y, s) \, dy \, ds \right|
\]

\[
\leq C \| \varphi \|_{L^\infty(\mathbb{R}^N \times (0, T))} \int_0^t \int_{B(0, 2R)} |u(y, s)| \, dy \, ds
\]

\[
+ C \| \varphi \|_{L^\infty(\mathbb{R}^N \times (0, T))} \int_0^t \int_{\mathbb{R}^N \setminus B(0, 2R)} G_m(\lambda y, t - s) |u(y, s)| \, dy \, ds \to 0
\]

as \( t \to +0 \). This implies (2.27). Thus Lemma 2.3 follows. ✷

Now we are ready to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. Let \( u \) be a solution of integral equation (I) in \( \mathbb{R}^N \times (0, T) \) for some \( T > 0 \). It suffices to prove that \( u \) is a solution of (1.1) in \( \mathbb{R}^N \times (0, T) \).

Let \( \varphi \in C^\infty(\mathbb{R}^N \times [0, T]) \) be such that \( \text{supp} \varphi \subset B(0, R) \times [0, T - \epsilon] \) for some \( R > 0 \) and \( \epsilon \in (0, T) \). Then it follows from Definition 2.2 and Lemma 2.3 that

\[
\int \int u(x,t) \varphi(x,t) \, dx
\]

\[
= \int \int \varphi(x,t) G_m(x - y, t) \, dy \, dx
\]

\[
+ \int_0^t \int G_m(x - y, t - s) |u(y, s)| \, dy \, ds \, dx 
\]

\[
\to \int \varphi(x,0) \, dx
\]

as \( t \to +0 \). On the other hand, by (2.16) we see that

\[
- \int \varphi(x, \tau) u(x, \tau) \, dx + \int_\tau^T \int [\varphi(x, t) - u\varphi + u(-\Delta)^m \varphi] \, dx \, dt = \int_\tau^T \int |u|^p \varphi \, dx \, dt.
\]

Letting \( \tau \to +0 \), by (2.20) and (2.33) we have

\[
- \int \varphi(x, 0) \, dx + \int_0^T \int [\varphi(x, t) - u\varphi + u(-\Delta)^m \varphi] \, dx \, dt = \int_0^T \int |u|^p \varphi \, dx \, dt.
\]

This means that \( u \) is a solution of (1.1) in \( \mathbb{R}^N \times [0, T) \). Thus Proposition 2.1 follows. ✷

3 Proof of Theorem 1.2

In this section we modify the arguments in [10] (see also [7]) to prove Theorem 1.2.

Proof of Theorem 1.2. Let \( u \) be a solution of problem (1.1) in \( \mathbb{R}^N \times [0, T) \) for some \( T > 0 \). Set

\[
u_T(x, t) := T^{\frac{1}{p-1}} u(T^{\frac{1}{2m}} x, T t), \quad \mu_T(x) := T^{\frac{1}{p-1}} \mu(T^{\frac{1}{2m}} x).
\]

(3.1)
Then $u_T$ is a solution of problem (1.1) in $\mathbb{R}^N \times [0,1)$ with the initial data $\mu_T$. Due to similar transformation (3.1), it suffices to consider the case of $T = 1$ for the proof of Theorem 1.2.

Let
\[
f(s) := e^{-\frac{1}{s}} \quad \text{if} \quad s > 0, \quad f(s) = 0 \quad \text{if} \quad s \leq 0.
\]

Set
\[
\eta(s) := \frac{f(2-s)}{f(2-s) + f(s-1)}.
\]

Then $\eta \in C^\infty([0,\infty))$ and
\[
\eta'(s) = \frac{-f'(2-s)f(s-1) - f(2-s)f'(s-1)}{(f(2-s) + f(s-1))^2} \leq 0, \quad s \geq 0,
\]

\[
\eta(s) = 1 \quad \text{on} \quad [0,1], \quad \eta(s) = 0 \quad \text{on} \quad [2,\infty).
\]

Set
\[
\eta^*(s) = 0 \quad \text{for} \quad s \in [0,1), \quad \eta^*(s) = \eta(s) \quad \text{for} \quad s \geq 1.
\]

Since $p > 1$, for $k = 1,2,\ldots$, it follows that
\[
|\eta^{(k)}(s)| \leq C\eta^*(s)^{\frac{1}{p}} \quad \text{for} \quad s \geq 1. \quad (3.2)
\]

Let $u$ be a solution of problem (1.1) in $\mathbb{R}^N \times [0,1)$. Let $x_0 \in \mathbb{R}^N$ and $0 < r^* < 1$ be such that
\[
\mu\left(B\left(x_0, (r^*/3)^{\frac{1}{m}}\right)\right) > 0.
\]

For any $R \in (0,1]$, we set
\[
\psi_R(x,t) := \eta\left(3\frac{|x-x_0|^{2m} + t}{R}\right), \quad \psi_R^*(x,t) := \eta^*\left(3\frac{|x-x_0|^{2m} + t}{R}\right).
\]

By (3.2), for $k = 1,2,\ldots$, we have
\[
|\partial_t \psi_R(x,t)| \leq CR^{-1}\psi_R^*(x,t)^{\frac{1}{p}}, \quad |\nabla^k_x \psi_R(x,t)| \leq CR^{-\frac{k}{2m}}\psi_R^*(x,t)^{\frac{1}{p}}, \quad (3.3)
\]

for $x \in \mathbb{R}^N$ and $0 < t \leq 1$. It follows from (3.3) that
\[
\int_{\mathbb{R}^N} \psi_R(x,0) \, d\mu + \int_0^1 \int_{\mathbb{R}^N} |u(x,t)|^p \psi_R(x,t) \, dx \, dt
= \int_0^1 \int_{\mathbb{R}^N} u(x,t)(-\partial_t + (-\Delta)^m)\psi_R(x,t) \, dx \, dt
\]

\[
\leq CR^{-1} \int_0^1 \int_{\mathbb{R}^N} |u(x,t)|^{\psi_R^*(x,t)^{\frac{1}{p}}} \, dx \, dt
\]

\[
\leq CR^{-1}\left(\int_0^1 \int_{\mathbb{R}^N} \chi_{\{\psi_R(x,t) > 0\}} \, dx \, dt\right)^{1-\frac{1}{p}} \left(\int_0^1 \int_{\mathbb{R}^N} |u(x,t)|^p \psi_R^*(x,t) \, dx \, dt\right)^\frac{1}{p} \quad (3.4)
\]
for $0 < R \leq 1$. On the other hand, it follows that
\[
\int_0^R \int_{\mathbb{R}^N} \chi_{\{\psi_R(x,t) > 0\}} \, dx \, dt = R^{N+1} \int_0^1 \int_{\mathbb{R}^N} \chi_{\{\psi_1(t,x) > 0\}} \, dx \, dt.
\]
This together with (3.4) implies that
\[
m_R + \int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \psi_R(x,t) \, dx \, dt \leq CR^p \left( \frac{N(p-1)}{2m} - 1 \right) \left( \int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \psi_R^*(x,t) \, dx \, dt \right)^{\frac{1}{p}}
\] (3.5)
for $0 < R \leq 1$, where
\[
m_R := \mu \left( B \left( x_0, (R/3) \frac{1}{2m} \right) \right).
\]
Let $\epsilon$ be a sufficiently small positive constant. For any $0 < r \leq R \leq 1$, set
\[
z(r) := \int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \psi_r^*(x,t) \, dx \, dt, \quad Z(R) := \int_0^R z(r) \min\{r^{-1}, \epsilon^{-1}\} \, dr.
\] (3.6)
Since $\eta^*$ is decaying on $[1, \infty)$ and $\text{supp} \eta^* \subset [1, 2]$, for any $(x,t) \in \mathbb{R}^N \times (0,1)$ with $3(|x-x_0|^{2m} + t) \geq R$, we have
\[
\int_0^R \psi_r^*(x,t) \min\{r^{-1}, \epsilon^{-1}\} \, dr \leq \int_0^R \eta^* \left( \frac{3|x-x_0|^{2m} + t}{r} \right) r^{-1} \, dr 
\]
\[
\leq \int_{3(|x-x_0|^{2m} + t)/R}^{\infty} \eta^*(s)s^{-1} \, ds 
\]
\[
\leq \eta^* \left( \frac{3|x-x_0|^{2m} + t}{R} \right) \int_1^2 s^{-1} \, ds \leq C \psi_R^*(x,t).
\] (3.7)
Since $\psi_r^*(x,t) = 0$ if $3(|x-x_0|^{2m} + t) < R$, by (3.6) and (3.7) we obtain
\[
\int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \psi_R(x,t) \, dx \, dt \geq \int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \psi_r^*(x,t) \, dx \, dt 
\]
\[
\geq C^{-1} \int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \left( \int_0^R \psi_r^*(x,t) \min\{r^{-1}, \epsilon^{-1}\} \, dr \right) \, dx \, dt 
\]
\[
= C^{-1} \int_0^R \int_0^R \int_{\mathbb{R}^N} |u(x,t)|^p \psi_r^*(x,t) \min\{r^{-1}, \epsilon^{-1}\} \, dx \, dt \, dr = C^{-1} Z(R).
\] (3.8)
Therefore we deduce from (3.5), (3.6) and (3.8) that
\[
m_R + C^{-1} Z(R) \leq CR^p \left( \frac{N(p-1)}{2m} - 1 \right) (\max\{R, \epsilon\} Z'(R))^{\frac{1}{p}}.
\] (3.9)
Since $m_R \geq m_r \geq m_{r_*} > 0$ for $r \geq r_*$, it follows from (3.9) that
\[
[m_{r_*} + Z(R)]^{-p} Z'(R) \geq C^{-1} R^{-\left( \frac{N(p-1)}{2m} - 1 \right)} (\max\{R, \epsilon\})^{-1}
\]
for $0 < r_* \leq R \leq 1$. Therefore we have

$$\int_{Z(r)}^Z [m_{r*} + s]^{-p} ds \geq C^{-1} \int_R^1 R^{-\left(\frac{N(p-1)}{2m} - 1\right)} \left(\max\{R, \epsilon\}\right)^{-1} dR \quad (3.10)$$

for $0 < r_* \leq r < 1$. Since

$$\int_{Z(r)}^Z [m_{r*} + s]^{-p} ds \leq \frac{1}{p-1} (Z(r) + m_{r*})^{-p+1} \leq \frac{1}{p-1} m_{r*}^{-p+1},$$

by (3.10) we obtain

$$\frac{1}{p-1} m_{r*}^{-p+1} \geq C^{-1} \int_R^1 R^{-\left(\frac{N(p-1)}{2m} - 1\right)} \left(\max\{R, \epsilon\}\right)^{-1} dR$$

for $0 < r_* \leq r \leq 1$. Letting $\epsilon \to +0$, we see that

$$\frac{1}{p-1} m_{r*}^{-p+1} \geq C^{-1} \int_R^1 R^{-\frac{N(p-1)}{2m}} dR$$

for $0 < r_* \leq r < 1$. This implies that

$$\mu(\mathbb{B}(x_0, (r_*/3)^{\frac{1}{2m}})) = m_{r*} \leq C \left(\int_R^1 R^{-\frac{N(p-1)}{2m}} dR\right)^{-\frac{1}{p-1}} \leq C \left(\int_R^{3r} R^{-\frac{N(p-1)}{2m}} dR\right)^{-\frac{1}{p-1}} \leq C r^{\frac{N}{2m} - \frac{1}{p-1}} \quad (3.11)$$

for $0 < r_* \leq r < 3r < 1$. Set $\sigma = (r/3)^{2m} = (r_*/3)^{2m} \in (0, 9^{-2m})$. Since $x_0 \in \mathbb{R}^N$ is arbitrary, we deduce from (3.11) that

$$\sup_{x \in \mathbb{R}^N} \mu(\mathbb{B}(x, \sigma)) \leq C \sigma^{\frac{N}{2m} - \frac{1}{p-1}}, \quad 0 < \sigma < 9^{-2m}. \quad (3.12)$$

On the other hand, for any $k \geq 1$, we find $C_k > 0$ such that

$$\sup_{x \in \mathbb{R}^N} \mu(\mathbb{B}(x, k\eta)) \leq C_k \sup_{x \in \mathbb{R}^N} \mu(\mathbb{B}(x, \eta)) \quad (3.13)$$

for $\eta > 0$ (see e.g. [19, Lemma 2.1]). This together with (3.12) implies (1.9).

It remains to prove (1.10). Let $p = p_m$. By (3.11) we have

$$\mu(\mathbb{B}(x_0, (r_*/3)^{\frac{1}{2m}})) \leq C \left(\int_R^1 R^{-\frac{N(p-1)}{2m}} dR\right)^{-\frac{1}{p-1}} \leq C \left|\log r\right|^{-\frac{N}{2m}} \leq C \left|\log \frac{r}{3}\right|^{-\frac{N}{2m}}$$

for $0 < r_* \leq r < 3r < 1$. Then, similarly to (3.12), we have

$$\sup_{x \in \mathbb{R}^N} \mu(\mathbb{B}(x, \sigma)) \leq C \left|\log \frac{r}{3}\right|^{-\frac{N}{2m}} \leq C \left[\log \left(e + \frac{1}{\sigma}\right)\right]^{-\frac{N}{2m}}, \quad 0 < \sigma < 9^{-2m}.$$

This together with (3.13) implies (1.10). Thus Theorem 1.2 follows. \[\square\]
4 Majorizing kernel

Let \( G_m = G_m(x,t) \) \((m = 2,3,\ldots)\) and \( G_\theta = G_\theta(x,t) \) \((0 < \theta < 2)\) be the fundamental solutions to \( \partial_t + (-\Delta)^m \) and \( \partial_t + (-\Delta)^\theta \) in \( \mathbb{R}^N \times (0,\infty) \), respectively. Define

\[
K(x,t) := G_\theta \left(x,t^{\frac{\theta}{2m}}\right), \quad x \in \mathbb{R}^N, \ t > 0.
\]  

(4.1)

Similarly to (2.6) and (2.7), we define an integral operator \( S_K(t) \) by

\[
[S_K(t)\mu](x) := \int_{\mathbb{R}^N} K(x-y,t) \, d\mu(y), \quad [S_K(t)\phi](x) := \int_{\mathbb{R}^N} K(x-y,t) \phi(y) \, dy,
\]

for (signed) Radon measure \( \mu \) and measurable function \( \phi \) in \( \mathbb{R}^N \). The aim of this section is to prove the following theorem, which is one of the main ingredients of this paper.

**Theorem 4.1** Let \( N \geq 1, \ m = 2,3\ldots \) and \( \theta \in (0,2) \). Let \( K \) be as in (4.1). Then \( K = K(x,t) > 0 \) in \( \mathbb{R}^N \times (0,\infty) \) and the following properties hold.

(a) For any \( j = 0,1,2,\ldots \), there exists \( d_j > 0 \) and \( d_j' > 0 \) such that

\[
|\partial^\alpha \partial_t^j G_m(x,t)| \leq d_j t^{-\frac{N}{2m}} K(x,t) \leq d_j' t^{-\frac{N}{2m} - \frac{1}{2m}}
\]

for \( x \in \mathbb{R}^N \), \( t > 0 \) and \( \alpha \in \mathbb{M} \) with \( |\alpha| = j \).

(b) There exists \( d'' > 0 \) such that

\[
\|S_K(t)\mu\|_\infty \leq d'' t^{-\frac{N}{2m}} \sup_{x \in \mathbb{R}^N} \mu(B(x,t^{\frac{1}{2m}})), \quad t > 0,
\]

for nonnegative Radon measure \( \mu \) in \( \mathbb{R}^N \).

(c) There exists \( d_* > 0 \) such that

\[
\int_{\mathbb{R}^N} K(x-y,t-s) K(y,s) \, dy \leq d_* K(x,t)
\]

for \( x \in \mathbb{R}^N \) and \( t > s > 0 \).

**Proof.** The positivity of \( K \) follows from the positivity of \( G_\theta \) (see Section 2.2). Let \( j = 0,1,2,\ldots \) and \( \alpha \in \mathbb{M} \) with \( |\alpha| = j \). By (2.5) we find \( C_1 > 0 \) such that

\[
|\partial^\alpha \partial_t^j G_m(x,t)| \leq C_1 t^{-\frac{N}{2m}} \exp \left( -C_1^{-1} \eta^{\frac{2m}{2m-1}} \right) \quad \text{with} \quad \eta = t^{-\frac{1}{2m}} |x|
\]  

(4.2)

for \( (x,t) \in \mathbb{R}^N \times (0,\infty) \). On the other hand, it follows from (2.11) and (2.12) that

\[
C_2^{-1} (1 + |x|)^{-N-\theta} \leq G_\theta(x,1) \leq C_2 (1 + |x|)^{-N-\theta}, \quad x \in \mathbb{R}^N,
\]

for some \( C_2 > 0 \). Then we find \( C_3 > 0 \) such that

\[
\exp \left( -C_3^{-1} |x|^{\frac{2m}{2m-1}} \right) \leq C_3 G_\theta(x,1), \quad x \in \mathbb{R}^N.
\]  

(4.3)

17
Let $\tau := t^{\theta/2m}$. By (2.10), (4.2) and (4.3) we obtain

$$|\partial^{\alpha}_{x} G_{m}(x, t)| \leq C_{1} C_{3} t^{-\frac{N}{2m} - \frac{1}{2m}} G_{\theta} \left(t^{-\frac{1}{2m}} x, 1\right)$$

$$= C t^{-\frac{N}{2m} - \frac{1}{2m}} G_{\theta} \left(\tau^{-\frac{\theta}{\tau}} x, 1\right)$$

$$= C t^{-\frac{N}{2m} - \frac{1}{2m}} \tau^{-\frac{N}{2m}} G_{\theta}(x, \tau) = C t^{-\frac{1}{2m}} G_{\theta}(x, t^{\frac{\theta}{2m}})$$

$$= C t^{-\frac{1}{2m}} K(x, t)$$

for $(x, t) \in \mathbb{R}^{N} \times (0, \infty)$. This implies assertion (a). On the other hand, by Lemma 2.1 and (4.1) we have

$$\|S_{K}(t)\mu\|_{\infty} = \left\|S_{g}(t^{\frac{\theta}{2m}}) \mu\right\|_{\infty} \leq C(t^{\frac{\theta}{2m}})^{-\frac{N}{2m}} \sup_{x \in \mathbb{R}^{N}} \mu \left(B(x, \left(t^{\frac{\theta}{2m}}\right)^{\frac{1}{\theta}})\right)$$

$$= C t^{-\frac{N}{2m}} \sup_{x \in \mathbb{R}^{N}} \mu(B(x, t^{\frac{1}{2m}})), \quad t > 0,$$

for nonnegative Radon measure $\mu$ in $\mathbb{R}^{N}$. This implies assertion (b).

We prove assertion (c). For any $0 < s < t$, set

$$\omega_{t,s} := \left(t - s\right)^{\frac{\theta}{2m}} + s^{\frac{\theta}{2m}}.$$

It follows from $\theta/2m \in (0, 1)$ that

$$t^{\frac{\theta}{2m}} \leq \omega_{t,s} = \left(t - s\right)^{\frac{\theta}{2m}} + s^{\frac{\theta}{2m}} \leq 2t^{\frac{\theta}{2m}}. \quad (4.4)$$

Then, by (2.13) we have

$$\int_{\mathbb{R}^{N}} K(x - y, t - s) K(y, s) \, dy = \int_{\mathbb{R}^{N}} G_{\theta}(x - y, (t - s)^{\frac{\theta}{2m}}) G_{\theta}(y, s^{\frac{\theta}{2m}}) \, dy$$

$$= G_{\theta}(x, \omega_{t,s}) = \omega_{t,s}^{-\frac{N}{2m}} G_{\theta}\left(\omega_{t,s}^{-\frac{1}{\theta}} x, 1\right) \leq t^{-\frac{N}{2m}} G_{\theta}\left(\omega_{t,s}^{-\frac{1}{\theta}} x, 1\right)$$

$$\leq C(1 + t^{-\frac{1}{2m}} |x|)^{-N-\theta} \leq C G_{\theta}\left(t^{-\frac{1}{2m}} x, 1\right). \quad (4.5)$$

for $x \in \mathbb{R}^{N}$ and $0 < s < t$. Furthermore, we observe from (2.11), (2.12) and (4.4) that

$$G_{\theta}\left(\omega_{t,s}^{-\frac{1}{\theta}} x, 1\right) \leq C \left(1 + \omega_{t,s}^{-\frac{1}{\theta}} |x|\right)^{-N-\theta} \leq C \left(1 + 2^{-\frac{1}{\theta}} t^{-\frac{1}{2m}} |x|\right)^{-N-\theta}$$

$$\leq C \left(1 + t^{-\frac{1}{2m}} |x|\right)^{-N-\theta} \leq CG_{\theta}\left(t^{-\frac{1}{2m}} x, 1\right). \quad (4.6)$$

Combining (4.5) and (4.6), we obtain

$$\int_{\mathbb{R}^{N}} K(x - y, t - s) K(y, s) \, dy \leq C t^{-\frac{N}{2m}} G_{\theta}\left(t^{-\frac{1}{2m}} x, 1\right) = CG_{\theta}\left(x, t^{\frac{\theta}{2m}}\right) = CK(x, t)$$

for $x \in \mathbb{R}^{N}$ and $0 < s < t$. This implies assertion (c). Thus Theorem 4.1 follows. □
5 Sufficient conditions on the solvability

In this section, by use of the majorizing kernel $K$ we establish the existence of solutions of problem (1.1).

5.1 Existence of solutions of integral equation (I)

We modify the argument in [24] to obtain sufficient conditions on the existence of solutions of integral equation (I) (see Section 2.3). Let $T > 0$ and

$$X := \left\{ f \in C(\mathbb{R}^N \times (0, T)) : \sup_{\tau \leq t < T} \| f(t) \|_{\infty} < \infty \quad \text{for} \quad \tau \in (0, T) \right\}.$$  

Let $K$ be as in Theorem 4.1. Let $U \in X$ be such that

$$d_0 U(x, t) \geq \int_{\mathbb{R}^N} K(x - y, t - s) U(y, s) \, dy > 0, \quad x \in \mathbb{R}^N, \ 0 < s < t < T,$$

where $d_0$ is as in Theorem 4.1. Let $\Psi$ be a positive continuous function in $(0, \infty)$ and set $V = \Psi(U)$. Assume that

$$D_0 := \sup_{0 < t < T} \left\| \frac{U(t)}{\Psi(U(t))} \right\|_{\infty} \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_{\infty} \, ds < \infty.$$  

Define

$$X_V := \{ f \in X : \| f \| < \infty \} \quad \text{with} \quad \| f \| := \sup_{0 < t < T} \sup_{x \in \mathbb{R}^N} \left| \frac{f(x, t)}{V(x, t)} \right|.$$  

Then the set $X_V$ is a Banach norm equipped with the norm $\| \cdot \|$. We apply the fixed point theorem in $X_V$ to prove the existence of solutions of integral equation (I).

**Theorem 5.1** Let $T > 0$, $m = 2, 3, \ldots, p > 1$. Assume (5.1) and (5.2). Let $\delta > 0$ and $M > 0$ be such that

$$\delta + d_0 d_* D_* M^p \leq M, \quad 2pd_0 d_* D_* M^{p-1} < 1,$$

where $d_0$ and $d_*$ are as in Theorem 4.1. Assume that $u_0(t) = S_m(t) \mu \in X$ and $\| u_0 \| \leq \delta$. Then there exists a unique solution $u \in X_V$ with $\| u \| \leq M$ of integral equation (I) in $\mathbb{R}^N \times [0, T)$.

**Proof.** Set

$$B_M := \{ u \in X_V : \| u \| \leq M \}.$$  

Define

$$F u(t) := u_0(t) + N(t), \quad N(t) := \int_0^t S_m(t - s)|u(s)|^p \, ds,$$

for $u \in B_M$. Then

$$|F u(t)| \leq \delta V(t) + |N(t)|, \quad |N(t)| \leq d_0 M^p \int_0^t S_K(t - s)V(s)^p \, ds.$$  

(5.4)
By (5.6) and (5.7) we apply the Banach fixed point theorem to find 
for 
\( u \) \( F \) such that 
On the other hand, by (5.3) and (5.5) we find 
\( \nu \) \( x,t \) for 

\( 0 < t < T \)

Since

\[
V(x,t)^p = \frac{\Psi(U(x,t))^p}{U(x,t)} U(x,t) \leq \left\| \frac{\Psi(U(t))^p}{U(t)} \right\|_\infty U(x,t),
\]

\[
U(x,t) \leq \frac{U(x,t)}{\Psi(U(x,t))} \Psi(U(x,t)) \leq \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty V(x,t),
\]

for \( (x,t) \in \mathbb{R}^N \times (0, T) \), by (5.1) we have

\[
\int_0^t S_K(t-s)V(s)^p \, ds \leq \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty S_K(t-s)U(s) \, ds
\]

\[
\leq d_s U(t) \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty \, ds
\]

\[
\leq d_s \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty V(t) \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty \, ds \leq d_s D_s V(t)
\]

for \( 0 < t < T \). It follows from (5.3), (5.4) and (5.5) that

\[
|\| F u \| | \leq \delta + d_0 d_s D_s M^p \leq M \quad \text{for} \quad u \in B_M.
\]

(5.6)

On the other hand, by (5.3) and (5.5) we find \( \nu \in (0, 1) \) such that

\[
| F u_1(t) - F u_2(t) | \leq d_0 \int_0^t S_K(t-s)|| u_1 ||^p - || u_2 ||^p \, ds
\]

\[
\leq pd_0 \int_0^t S_K(t-s) \max\{|u_1(s)|^{p-1},|u_2(s)|^{p-1}\} V(s) \frac{|u_1(s) - u_2(s)|}{V(s)} \, ds
\]

\[
\leq 2pd_0 M^{p-1} || u_1 - u_2 || \int_0^t S_K(t-s) V(s)^p \, ds
\]

\[
\leq 2pd_0 d_s D_s M^{p-1} V(t) || u_1 - u_2 || \leq \nu V(t) || u_1 - u_2 ||
\]

for \( u_1, u_2 \in B_M \). This implies that

\[
|\| F u_1 - F u_2 || | \leq \nu || u_1 - u_2 || \quad \text{for} \quad u_1, u_2 \in B_M.
\]

(5.7)

By (5.6) and (5.7) we apply the Banach fixed point theorem to find \( u_* \in B_M \) uniquely such that \( F u_* = u_* \) in \( X_V \). This implies that \( u_* \in C(\mathbb{R}^N \times (0, T)) \) and \( u_* \) satisfies

\[
u_s(x,t) = u_0(x,t) + \int_0^t \int_{\mathbb{R}^N} G_m(x-y,t-s) |u_*(y,s)|^p \, dy \, ds
\]

for \( (x,t) \in \mathbb{R}^N \times (0, T) \). Furthermore, by (5.4) and (5.5) we have

\[
\sup_{\tau \leq t < T} || u_0(t) ||_\infty \leq \delta \sup_{\tau \leq t < T} V(t) < \infty,
\]

\[
\sup_{\tau \leq t < T} \left\| \int_0^t S_m(t-s) |u_*(s)|^p \, ds \right\|_\infty \leq d_0 d_s M^p D_s \sup_{\tau \leq t < T} V(t) < \infty,
\]

for \( \tau \in (0, T) \). Therefore we see that \( u_* \) is a solution of integral equation (I) in \( \mathbb{R}^N \times (0, T) \). Thus Theorem 5.1 follows. □
5.2 Sufficient conditions for solvability

We obtain sufficient conditions for the existence of solutions of problem (1.1) by combining Theorem 5.1 and the arguments in [14], [22] and [25]. (See also [15].) We prove Theorem 1.3.

Proof of Theorem 1.3. By similar transformation (3.1) and Proposition 2.1 it suffices to show the existence of solutions of integral equation (I) in $R^N \times [0,1)$.

We assume (1.11) with $T=1$ and show the existence of solution of integral equation (I) in $R^N \times [0,1)$. Let $K$ be as in Theorem 4.1, that is, $K(x,t) = G_\theta(x,t) = \frac{\theta}{m}(x,t) = \frac{\theta}{m}(x,t)$ with $0 < \theta < 2$.

Set $U(x,t) := 2d_0 [S_K(t)\mu](x)$ and $u_0(x,t) := [S_m(t)\mu](x)$. Then it follows from Theorem 4.1 that

$$\int_{R^N} K(x,y,t-s)U(y,s)dy = 2d_0 \int_{R^N} \int_{R^N} K(x,y,t-s)K(y,z,s)dyd\mu(z)$$

for $x \in R^N$ and $0 < s < t$, that is, $U$ satisfies (5.1). Furthermore, it follows from Theorem 4.1 that

$$|u_0(x,t)| \leq d_0 \int_{R^N} K(x,y,t)d\mu(y) = \frac{1}{2} U(x,t), \quad (x,t) \in R^N \times (0,1).$$

On the other hand, it follows from assertion (b) of Theorem 4.1 and (1.11) that

$$\|U(t)\|_\infty \leq Ct^{-\frac{N}{2m}} \sup_{x \in R^N} \mu(B(x,t^\frac{1}{2m})) \leq Ct^{-\frac{N}{2m}} \sup_{x \in R^N} \mu(B(x,1)) \leq C\gamma t^{-\frac{N}{2m}}$$

for $0 < t < 1$. Since $1 < p < 1 + 2m/N$, by (5.9) we have

$$\int_0^1 \|U(s)\|_\infty^{p-1} ds \leq (C\gamma)^{p-1} \int_0^1 s^{-\frac{N}{2m}(p-1)} ds \leq C\gamma^{p-1}.$$  

We apply Theorem 5.1 with

$$\Psi(s) = s, \quad V = U, \quad T = 1, \quad \delta = \frac{1}{2} \quad \text{and} \quad M = 1.$$  

Then, by (5.8) we have

$$|||u_0||| \leq \frac{1}{2}.$$  

Furthermore, by (5.2) and (5.10) we see that

$$D_\ast \equiv \sup_{0 < t \leq 1} \int_0^1 \|U(s)\|_\infty^{p-1} ds \leq C\gamma^{p-1}.$$  

21
Then, by (5.11) and (5.12), taking a sufficiently small $\gamma > 0$, we find a function $u \in B_M \subset X_V$ such that
\[ u(t) = S_M(t)\mu + \int_0^t S_M(t-s)|u(s)|^p \, ds, \quad 0 < t < 1. \]
Furthermore, we see that (2.13) also holds with $T = 1$. Therefore $u$ is a solution of integral equation (I). Thus Theorem 1.3 follows. □

**Remark 5.1** The argument in the proof of Theorem 1.3 is applicable to the case where $\mu$ is a signed Radon measure in $\mathbb{R}^N$. Indeed, the same conclusion as in Theorem 1.3 holds if $\mu$ is a signed Radon measure satisfying
\[ \sup_{x \in \mathbb{R}^N} |\mu|(B(x, T^{\frac{1}{2m}})) \leq \gamma_2 T^{\frac{N}{2m} - \frac{1}{p-1}} \]
for some $T > 0$, instead of (1.11). Here $|\mu|$ is the total variation of $\mu$.

Similarly to Remark 5.1 we consider problem (1.1) without the nonnegativity of the initial data and obtain sufficient conditions for the existence of solutions of problem (1.1).

**Theorem 5.2** Let $N \geq 1$, $m = 2, 3, \ldots$ and $1 < \alpha < p$. Then there exists $\gamma = \gamma(N, m, p, \alpha) > 0$ such that, if $\mu$ is a measurable function in $\mathbb{R}^N$ satisfying
\[ \sup_{x \in \mathbb{R}^N} \left[ \int_{B(x, \sigma)} |\mu(y)|^\alpha \, dy \right] \leq \gamma_\sigma^{-\frac{2m}{p-1}}, \quad 0 < \sigma \leq T^{\frac{1}{2m}}, \] (5.13)
for some $T > 0$, then problem (1.1) possesses a solution in $\mathbb{R}^N \times [0, T)$.

**Proof.** Similarly to the proof of Theorem 1.3 it suffices to show the existence of solution of integral equation (I) in $\mathbb{R}^N \times [0, 1)$. We apply Theorem 5.1 with
\[ T = 1, \quad u_0(x, t) := S_M(t)\mu, \quad U(x, t) := (2d_0)^\alpha S_K(t)\mu|^{\alpha}, \quad \Psi(s) := s^{\frac{1}{\alpha}}, \]
\[ V(x, t) := 2d_0 (S_K(t)\mu|^{\alpha})^{\frac{1}{\alpha}}. \] (5.14)
By (5.13) we have
\[ \sup_{x \in \mathbb{R}^N} \int_{B(x, \sigma)} |\mu(y)|^\alpha \, dy \leq C \gamma_\alpha \gamma^{-\frac{2m}{p-1}} \] for $0 < \sigma < 1$. This together with assertion (b) of Theorem 4.1 implies that
\[ \|U(t)\|_\infty \leq C t^{-\frac{N}{2m}} \sup_{x \in \mathbb{R}^N} \mu^\alpha(B(x, t^{\frac{1}{2m}})) \]
\[ \leq C \gamma_\alpha t^{-\frac{N}{2m}} \left( t^{\frac{1}{2m}} \right)^{N-\frac{2m}{p-1}} \leq C \gamma_\alpha t^{-\frac{N}{p-1}} \] (5.15)
for $0 < t < 1$. By (5.15) we obtain
\[ \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty \, ds = \int_0^t \|U(s)\|^{\frac{p\alpha}{\infty}} \, ds \leq C \gamma_\alpha^{p-\alpha} \int_0^t s^{-\frac{p\alpha}{p-1}} \, ds \leq C \gamma_\alpha^{p-\alpha} t^{-\frac{p\alpha}{p-1}}; \]
\[ \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty = \|U(t)\|^{\frac{\alpha}{\infty}} \leq C \gamma_\alpha^{-1} t^{-\frac{\alpha}{p-1}}, \]
for $0 < t < 1$. By (5.15) we obtain
\[ \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty \, ds = \int_0^t \|U(s)\|^{\frac{p\alpha}{\infty}} \, ds \leq C \gamma_\alpha^{p-\alpha} \int_0^t s^{-\frac{p\alpha}{p-1}} \, ds \leq C \gamma_\alpha^{p-\alpha} t^{-\frac{p\alpha}{p-1}}; \]
\[ \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty = \|U(t)\|^{\frac{\alpha}{\infty}} \leq C \gamma_\alpha^{-1} t^{-\frac{\alpha}{p-1}}, \]
for $0 < t < 1$. This implies that
\[
D_* \equiv \sup_{0 < t \leq 1} \left\| \frac{U(t)}{\Psi(U(t))} \right\|_{\infty} \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_{\infty} \, ds \leq C \gamma^{p-1}. \tag{5.16}
\]

On the other hand, it follows from Theorem 1.3 and the Jensen inequality that
\[
|S_m(t)\mu| \leq d_0 S_K(t)|\mu| \leq d_0 (S_K(t)|\mu|^{\alpha})^{\frac{1}{\alpha}} = \frac{1}{2} V(t) \tag{5.17}
\]
for $0 < t < 1$. Similarly to the proof of Theorem 1.3, taking a sufficiently small $\gamma > 0$ and applying Theorem 4.1 with $\delta = 1/2$ and $M = 1$, we see that integral equation (I) possesses a solution in $R^N \times [0,1)$. Thus Theorem 5.2 follows. \hfill \square

**Theorem 5.3** Let $N \geq 1$, $m = 2,3,\ldots, p = p_m$ and $\beta > 0$. For $s > 0$, set
\[
\Phi(s) := s|\log(e + s)|^\beta, \quad \rho(s) := s^{-N} \left[ \log \left( e + \frac{1}{s} \right) \right]^{-\frac{N^2}{2m}}.
\]

Then there exists $\gamma = \gamma(N,m,\beta) > 0$ such that, if $\mu$ is a nonnegative measurable function in $R^N$ satisfying
\[
\sup_{x \in R^N} \Phi^{-1} \left[ \int_{B(x,\sigma)} \Phi(T^{p-1}|\mu(y)|) \, dy \right] \leq \gamma \rho(\sigma T^{\frac{1}{2m}}), \quad 0 < \sigma \leq T^{\frac{1}{2}}, \tag{5.18}
\]
for some $T > 0$, then problem (1.1) possesses a solution in $R^N \times [0,T)$. \hfill \square

**Proof.** Similarly to the proof of Theorem 1.3, it suffices to show the existence of solutions of integral equation (I) in $R^N \times [0,1)$. Let $0 < \gamma < 1$ and assume (5.18). Let $L \geq e$ and set $\Phi_L(s) := s|\log(L + s)|^\beta$ for $s > 0$. Then, taking a sufficiently large $L \geq e$ if necessary, we have:

(a) $\Phi_L$ is convex in $(0,\infty)$;

(b) the function $(0,\infty) \ni s \mapsto s^{\frac{p-1}{2}}|\log(L + s)|^{-\beta p}$ is monotone increasing.

Define a positive function $\Psi_L = \Psi_L(s)$ in $(0,\infty)$ by $\Psi_L(s) := \Phi_L^{-1}(s)$. Then
\[
C^{-1} \Phi_L(s) \leq \Phi(s) \leq C \Phi_L(s),
\]
\[
C^{-1} s|\log(L + s)|^{-\beta} \leq \Psi_L(s) \leq C s|\log(L + s)|^{-\beta}, \tag{5.19}
\]
for $s > 0$. We apply Theorem 5.1 with
\[
T = 1, \quad u_0(x,t) := S_m(t)\mu, \quad U(x,t) := S_K(t)\Phi_L(|\mu|), \quad \Psi(s) := \Phi_L^{-1}(s),
\]
\[
V(x,t) := \Phi_L^{-1}(S_K(t)|\mu|).
\]

It follows from (5.18) and (5.19) that
\[
\sup_{x \in R^N} \Phi_L^{-1} \left[ \int_{B(x,\sigma)} \Phi_L(|\mu(y)|) \, dy \right] \leq C \gamma \rho(\sigma), \quad 0 < \sigma < 1. \tag{5.21}
\]
Applying assertion (b) of Theorem 4.1 with (5.21), we see that

\[ \|U(t)\|_\infty = \|S_K(t)\Phi_L(\|\mu\|)\|_\infty \leq Ct^{-\frac{N}{2m}} \sup_{x \in \mathbb{R}^N} \int_{B(x,t^{1/2m})} \Phi_L(\|\mu(y)\|) \, dy \]

\[ \leq Ct^{-\frac{N}{2m}} \left( t^{\frac{1}{2m}} \right)^N \Phi_L(C\gamma p(t^{\frac{1}{2m}})) \]

\[ \leq C\gamma p(t^{\frac{1}{2m}})[\log(L + C\gamma p(t^{\frac{1}{2m}}))]^\beta \]

\[ \leq C\gamma p(t^{\frac{1}{2m}})[\log(L + C\gamma p(t^{\frac{1}{2m}}))]^\beta \leq C\gamma t^{-\frac{N}{2m}} \left| \log \frac{t}{2} \right|^{-\frac{N}{2m} + \beta} =: \gamma \xi(t) \]

for \( 0 < t < 1 \). Since \( p = p_m = 1 + 2m/N \), it follows from property (b), (5.19) and (5.22) that

\[ \frac{\Psi_L(U(x,t))^p}{U(x,t)} \leq C(U(x,t))^{p-1}[\log(L + U(x,t))]^{-\beta p} \]

\[ = CU(x,t)^{\frac{p-1}{p}} U(x,t)^{\frac{p}{p-1}}[\log(L + U(x,t))]^{-\beta p} \]

\[ \leq C(\gamma \xi(t))^\beta (\gamma \xi(t))^{\frac{p}{p-1}}[\log(L + \gamma \xi(t))]^{-\beta p} \]

\[ \leq C\gamma^{\frac{p-1}{2}} \xi(t)^{\frac{p}{p-1} - 1}[\log(L + \xi(t))]^{-\beta p} \]

\[ \leq C\gamma^{\frac{p-1}{2}} t^{-\frac{N}{2m}(p-1)} \left| \log \frac{t}{2} \right|^{-\frac{N}{2m}(p-1) + \beta(p-1) - \beta p} \leq C\gamma^{\frac{p-1}{2}} t^{-1} \left| \log \frac{t}{2} \right|^{-1 - \beta} \]

for \((x,t) \in \mathbb{R}^N \times (0,1)\). Similarly, we have

\[ 0 \leq \frac{U(x,t)}{\Psi_L(U(x,t))} \leq C[\log(L + U(x,t))]^\beta \leq C[\log(L + \gamma \xi(t))]^\beta \]

\[ \leq C[\log(L + \xi(t))]^\beta \leq C \left| \log \frac{t}{2} \right|^{\beta} \]

for \((x,t) \in \mathbb{R}^N \times (0,1)\). By (5.23) and (5.24) we obtain

\[ D_s = \sup_{0 < t < 1} \left\{ \left| \frac{U(t)}{\Psi_L(U(t))} \right|_\infty \left\| \int_0^t \left\| \frac{\Psi_L(U(s))^p}{U(s)} \right\|_\infty ds \right\| \right\} \leq C\gamma^{\frac{p-1}{2}} \sup_{0 < t < 1} \left\{ \left| \log \frac{t}{2} \right|^{\beta} \int_0^t s^{-1} \left| \log \frac{s}{2} \right|^{-1 - \beta} ds \right\} \leq C\gamma^{\frac{p-1}{2}}. \]

On the other hand, it follows from Theorem 4.1 and the Jensen inequality that

\[ |u_0(t)| = |S(t)\mu| \leq d_0 S_K(t)|\mu| \leq d_0 \Phi_L^{-1}(S_K(t)\Phi_L(\|\mu\|)) = d_0 V(t) \]

for \( 0 < t < 1 \). Similarly to the proof of Theorem 1.3 by (5.25) and (5.26), taking a sufficiently small \( \gamma > 0 \) and applying Theorem 5.1 with (5.20), \( \delta = d_0 \) and \( M = 2d_0 \), we see that integral equation (I) possesses a solution in \( \mathbb{R}^N \times [0,1) \). Thus Theorem 5.3 follows. \( \Box \)

Theorem 1.4 easily follows from Theorems 5.2 and 5.3 (See also [14] and [15].)
Remark 5.2 Consider the Cauchy problem

\[
\begin{aligned}
\partial_t u + (-\Delta)^m u &= F(u), & x \in \mathbb{R}^N, & t > 0, \\
u(x, 0) &= \mu(x) \geq 0, & x \in \mathbb{R}^N,
\end{aligned}
\]

where \( m = 2, 3, \ldots \) and \( F \) is a continuous function in \( \mathbb{R} \). Assume that

\[
|F(u)| \leq |u|^p, \quad |F(u) - F(v)| \leq C_F(|u|^{p-1} + |v|^{p-1})|u - v|
\]

for \( u, v \in \mathbb{R} \). Then, applying the same arguments in Section 5, we can show that the same conclusions as in Theorems 1.3, 5.2 and 5.3 and the same statement as in Remark 5.1 hold for problem (P).

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26
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