New Foundations for Geometry
Two non-additive languages for arithmetical geometry

M.J. Shai Haran

August 20, 2015
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Chapter 0

Introduction

We spend our first years in the world of mathematics doing addition (using our fingers), and then we learn of multiplication (of natural numbers) as a kind of generalized addition. There is no wonder that the vast majority of structures in mathematics begin with addition, either explicitly (as an abelian group), or abstractly (as additive/abelian categories). The language of Grothendieck’s algebraic geometry is based on commutative rings having addition and multiplication. But when we compare arithmetic and geometry, we see that it is precisely the presence of addition in our language that causes all the problems. The ring of integers \( \mathbb{Z} \) is similar to the polynomial ring in one variable \( F[x] \) over a field \( F \). Taking for simplicity \( F \) algebraically closed, we have analogous diagrams of embeddings of rings in arithmetic and geometry:

\[
\begin{array}{ccc}
\text{Arithmetic} & \quad & \text{Geometry} \\
\mathbb{Z} & \xrightarrow{p} & \mathbb{Q}_p \\
\mathbb{Z} & \xleftarrow{p} & \mathbb{Q}_p \\
\mathbb{Q} & \xrightarrow{\eta} & \mathbb{Q}_\eta = \mathbb{R} \\
\mathbb{Z}_\eta = [-1, 1] & \xrightarrow{\eta} & \\
\mathbb{F}_p & \xrightarrow{\alpha} & \mathbb{F}_p[[x]] \\
\mathbb{F}_p & \xleftarrow{\alpha} & \mathbb{F}_p[[x]] \\
\mathbb{F}(x) & \xrightarrow{\eta} & \mathbb{F}((x - \alpha)) \\
\mathbb{F}(x) & \xleftarrow{\eta} & \mathbb{F}((x - \alpha)) \\
\mathbb{F}[x] & \xrightarrow{\eta} & \\
\mathbb{F}[x] & \xleftarrow{\eta} & \\
\end{array}
\]

Here the rational numbers \( \mathbb{Q} \) are analogous to the field of rational functions \( \mathbb{F}(x) \). The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is analogous to the field of Laurent series \( \mathbb{F}[[\frac{1}{x}]] \).
The $p$-adic numbers contain the (one dimensional local) ring of $p$-adic integers $\mathbb{Z}_p = \lim_{\longrightarrow} \mathbb{Z}/(p^n)$, which is analogous to the (one dimensional local) ring of power series $F[[x - \alpha]] = \lim_{\longrightarrow} F[x]/(x - \alpha)^n$.

The embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ is analogous to the embedding $F(x) \hookrightarrow F((x - \alpha))$, i.e. of expanding a rational function as a Laurent series around the point $\alpha$.

There are three basic problems where this analogy breaks down.

**The problem of the arithmetical plane:** In geometry when we have two objects we have their product. In particular, the product of the (affine) line with itself gives us the (affine) plane. This translates in the language of commutative rings into the fact that

$$F[X] \otimes_F F[X] = F[X_1, X_2]$$

the polynomial ring in two variables. When we try to find the analogous arithmetical plane, we find

$$\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$$

Having addition as part of the structures of a commutative ring forces the integers $\mathbb{Z}$ to be the initial object of the category of commutative rings, hence its categorical sum with itself reduces to $\mathbb{Z}$, and the arithmetical plane reduces to its diagonal.

**The problem of the absolute point:** The category of $F$-algebras has $F$ as an initial object, hence in geometry (over $F$) we have the point $\text{spec}(F)$ as a final object. Addition forces the integers $\mathbb{Z}$ to be the first object of commutative rings, hence $\text{spec}(\mathbb{Z})$ is the final object of Grothendieck’s algebraic geometry, and we are missing the absolute point $\text{spec}(\mathbb{F})$, where $\mathbb{F}$ is the field with one element - the non-existing common field of all finite fields $\mathbb{F}_p$, $p$ prime.

**The problem of the real prime:** In geometry over $F$ we realize that if we want to have global theorems we need to pass from affine to projective geometry. In particular, we have to add the point at infinity $\infty$ to the affine line to obtain the projective line. In our language of commutative rings this translates into the extra embedding $F(x) \hookrightarrow F((\frac{1}{x}))$, i.e. expanding a rational function as a Laurent series at infinity. There is nothing special about the point $\infty$, all the points of the projective line are the same, and the field $F((\frac{1}{x}))$ is isomorphic to each of the fields $F((x - \alpha))$, in particular: $F((\frac{1}{x}))$ contains the (one-dimensional local) ring of power series $F[[\frac{1}{x}]]$. The analog of the infinite point $\infty$ in arithmetic is the real prime, which we denote by $\eta$, and the analog of the extra embedding $F(x) \hookrightarrow F((\frac{1}{x}))$ is the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_\eta = \mathbb{R}$ of the rational numbers in the reals. The analog of the power-series subring $F[[\frac{1}{x}]] \subseteq F((\frac{1}{x}))$, is the ring of real integers $\mathbb{Z}_\eta = [-1, 1] \subseteq \mathbb{R}$. But the real interval $[-1, 1]$ is not closed under addition, and is not a commutative ring. The language of Grothendieck’s algebraic geometry cannot see the real prime, hence cannot
produce global theorems in arithmetic (this is the point of Arakelov’s geometry where the real prime \( \eta \) is added to \( \text{spec}(\mathbb{Z}) \) in an ad hoc way).

All this brings us to the inevitable conclusion, that if we want to see arithmetic as a true geometry, we have to change the language of geometry, and moreover, we have to give up addition as part of this language. Kurokawa, Ochiai, and Wakayama [KOW] were the first to suggest simply abandoning addition, and work instead with the language of multiplicative monoids. This approach of using (multiplicative) monoids was further developed by Deitmar [De], and indeed is the minimal concept included in all other approaches, as it will be in our approach, cf. §2.4. But this approach creates many new problems: the spectra of monoids always looks like the spectra of a local ring (the non-invertible elements of a monoid are the unique maximal ideal), and the primes of the (multiplicative) monoid \( \mathbb{Z} \) are arbitrary subsets of the (usual) primes. What we need is another operation that will replace addition.

The very same problem of the inadequacy of addition appears in physics in the theory of relativity: the interval of speeds \( (\pm c, c) \), \( c \) being the speed of light, is not closed under addition. Einstein’s solution, from which all of the theory of special relativity can be deduced, is to change addition into \( c \)-addition given by

\[
x +_c y = \frac{x + y}{1 - \frac{xy}{c^2}}
\]

Like addition, this operation is associative, commutative, 0 is the unit, and \(-x\) is the inverse:

\[
(x +_c y) +_c z = x +_c (y +_c z),
\]
\[
x +_c y = y +_c x,
\]
\[
x +_c 0 = x,
\]
\[
x +_c (-x) = 0
\]

There is also a kind of distributive law:

\[
z \cdot (x +_c y) = (z \cdot x) +_{|z|c} (z \cdot y)
\]

This approach of Einstein’s is not useful in arithmetic where we have also complex primes (e.g. the unique prime of \( \mathbb{Z}[i] \) over \( \eta \)). For complex primes, the real interval, \( \mathbb{Z}_\eta = [-1, 1] \), is replaced by the complex unit ball \( \mathbb{D}_\eta = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \), and the complex-\( c \)-addition, under which \( c \cdot \mathbb{D}_\eta = \{ z \in \mathbb{C} \mid |z| < c \} \) is closed, is given by (the non-associative non-commutative) operation:

\[
x +_c y = \frac{x + y}{1 + \frac{xy}{c^2}}, \ (y \text{ is the complex conjugate of } y).
\]

Perhaps also Nature was trying to tell us one of her secrets, when Heisenberg found matrix multiplication as the basic language of the microscopic world...

The hint comes from the mysteries of the real prime. We can change the underlying additive group we use to represent a ring, \( G_\alpha(B) = \text{Hom}(\mathbb{Z}[x], B) = B \).
to the general linear groups $GL_n(B)$. Then the analog for the real prime of
the (maximal compact) subgroup $GL_n(\mathbb{Z}_p) \subseteq GL_n(\mathbb{Q}_p)$ is the orthogonal group
- the (maximal compact) subgroup $^\wedge "GL_n(\mathbb{Z}_p)" = O_n \subseteq GL_n(\mathbb{Q}_p) = GL_n(\mathbb{R})$.
And for a complex prime of a number field it is the unitary group $U_n \subseteq GL_n(\mathbb{C})$.
Indeed, Macdonald [Mac] gives a q-analog interpolation between the zonal-
spherical-functions on $GL_n(\mathbb{Q}_p)/GL_n(\mathbb{Z}_p)$, and the zonal-spherical-functions on
$O_n/\mathbb{P}GL_n(\mathbb{R})$. Similarly, there is a q-analog interpolation between the zonal-spherical-function on the p-adic Grassmanian
$B_{n_1,n_2} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_n(\mathbb{Z}_p) \right\}$, and the real and complex Grassmanians
$O_n/\mathbb{P}O_n \times O_{n_2}$ and $U_n/\mathbb{P}U_{n_1} \times U_{n_2}$, $n_1 + n_2 = n$ (see [H08], [O]).

This point of view of the general-linear-group suggests also that for the
field with one element $\mathbb{F}$ we have $^\wedge "GL_n(\mathbb{F})" = S_n$, the symmetric group, which embeds as a common subgroup of all the finite group $GL_n(\mathbb{F}_p)$, $p$ prime (or the
"field" $\mathbb{F}\{\pm 1\}$, with $^\wedge "GL_n(\mathbb{F}\{\pm 1\})" = \{\pm 1\}^n \times S_n$).

But we need zero as a part of the language for geometry, and so we represented a
commutative ring $B$ by the collection of all $m$ by $n$ matrices over $B$, $Mat_{m,n}(B)$, for all $m,n$.
For the real integers $\mathbb{Z}_\eta$, $Mat_{m,n}(\mathbb{Z}_\eta)$ are the matrices in $Mat_{m,n}(\mathbb{R})$ that carry
the $n$ dimensional unit ball into the $m$ dimensional unit ball.

$$Z^n_\eta = \{ f \in \mathbb{R}^n, \sum_{i=1}^n |f(i)|^2 \leq 1 \}$$

$$(\mathbb{Z}_\eta)_{n,m} = Mat_{m,n}(\mathbb{Z}_\eta) = " \text{Hom}(\mathbb{Z}_\eta^m, \mathbb{Z}_\eta^n)" = \{ f \in Mat_{n,m}(\mathbb{R}), f(\mathbb{Z}_\eta^m) \subseteq \mathbb{Z}_\eta^n \}.$$ 

There is a natural involution:

$$(,)^t : (\mathbb{Z}_\eta)_{n,m} \to (\mathbb{Z}_\eta)_{m,n}$$

which only exists when we considered the $l_2$-norm.
The initial object $\mathbb{F}$ is represented by $" Mat_{m,n}(\mathbb{F})"$, the $m$ by $n$ matrices with
entries 0, 1 having at most one 1 in every row and column. As a category $\mathbb{F}$ is
equivalent to the category with objects the finite sets, and with morphisms the partial bijections.
We also keep as part of our language, the operations of matrix multiplication,
as well as the operations of direct-sum and transposition of matrices.
This language for geometry sees the real prime, there is a natural compactifi-
cation $\text{spec}(\mathbb{Z}) = \text{spec}(\mathbb{Z}) \cup \{\eta\}$ as a pro-object of the associated category of
schemes. The arithmetical plane does not reduce to its diagonal, and yet one
can do algebraic-geometry-Grothendieck-style over it.
Recently, there have been a few approaches to "geometry over $\mathbb{F}_1"$, such as Borger [Bo09], Connes Consani [CC09, CC14], Durov [Du], Lorscheid [Lo12], Soule [S], Takagi [Tak12], Töen- Vaquie [TV] and Haran [H07] and [H09].

For relations between these see [PL]. There are no inclusion relations between Haran’s and other approaches, except that Durov’s generalized rings are a subset of the $\mathbb{F}$-Rings of [H07]. But Durov’s use of monads forces him to use the $\ell_1$- metric at the real primes, so that the unit ball $Z^n_\eta$ is replaced by the polytop

$$ (Z^n_\eta)_{\ell_1} = \{ f \in \mathbb{R}^n, \sum_{i=1}^n |f(i)| \leq 1 \} $$

and $O_n = GL_n(Z_\eta)$ is replaced by the (finite) subgroup of $GL_n(\mathbb{R})$ preserving this polytop.

The present language of $\mathbb{F}$-Rings is the same as [H07], except that we omit the tensor-product of matrices from the structure: we use only matrix multiplication and direct-sum, and we add the involution to the structure.

An important observation of the present approach is that we do not need the tensor product to do geometry, and that the addition of an involution to the structure means that we have to work with the symmetric spectrum. We also analyze the notion of commutativity more carefully, and define the "commutative- $\mathbb{F}$-Rings" over which we can do geometry. The $\mathbb{F}$-Rings of [H07] are the subset of "totally-commutative- $\mathbb{F}$-Rings" of the present approach. We show the arithmetical surface (the categorical-sum of $\mathbb{Z}$ with itself in the category of commutative- $\mathbb{F}$-Rings) does not reduce to its diagonal, while in categories of totally-commutative $\mathbb{F}$-Rings [H07], or Durov’s [Du], as in ordinary commutative rings it does reduce to its diagonal: $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$.

We observe though that the geometric object $\text{Spec}(A)$, only depend on the operads $\{A_1, X\}$ and $\{AX_1, 1\}$, (and these can be identified in the presence of an involution). We therefore axiomatize the properties of the "self-adjoint operad" $\{A_1, X\}$, for an $\mathbb{F}$-Ring with involution $A$. This gives us the "Generalized-Rings" of [H10], the geometry of which was developed in [H09]. But in [H09] we assumed our generalized rings to be totally commutative, and self-adjoint, which are unnatural and limiting assumptions. Here we avoid these assumptions, and show that using the language of commutative-generalized-Rings, one can do "algebraic-geometry- Grothendieck-style". It includes "classical" algebraic geometry (fully-faithfully), and yet it solves the three basic problems of the analogy of arithmetic and geometry, as there are:

- A final object for Geometry, the "absolute point": $\text{Spec} \mathbb{F}$.
- natural compactifications $\text{Spec}(\mathbb{Z}) = \text{Spec}(\mathbb{Z}) \cup \{\eta\}$, and similarly for a number
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field \( K \),

\[ \text{Spec}(\mathcal{O}_K) = \text{Spec}(\mathcal{O}_K) \cup \{ \eta_i \}_{\eta_i : K \to \mathbb{C} \text{ mod conj}^n,} \]

as pro-object of the associated category of Grothendieck generalized schemes.

- The arithmetical plane does not reduce to the diagonal.

Unfortunately, it seems that to understand the geometric theory of generalized-Rings, one has first to go through the same theory using \( F \)-Rings with involution. Especially, the role of the symmetric spectrum \( \text{Spec}^t(A) \) (as opposed to the usual spectrum \( \text{Spec}(A) \)) as the basic building blocks for schemes of \( F \)-Rings with involution, and for schemes of generalized rings (non self-adjoint as in [H09]).

The contents of the chapters are as follows:

§1.1 We define "the field with one element \( F \)" (cf. [S], [H07]) to be the category of finite sets and partial bijections.

\[ \mathbb{F}(X, Y) = \{ f : D(f) \xrightarrow{\sim} I(f) \text{ bijection}, D(f) \subseteq X, I(f) \subseteq Y \} \]

and let \( \oplus \) be the disjoint union on this category.

\[ \oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \]

\[ X_0 \oplus X_1 = \{(i, x), i \in \{0, 1\}, x \in X_i \}, \]

\[(f_0 \oplus f_1)(i, x) = (i, f_i(x)), f_i \in \mathbb{F}_{Y_i, X_i}, \]

We have the associativity, commutativity, unit isomorphisms:

\[ a = a_{X_0, X_1, X_2} : (X_0 \oplus X_1) \oplus X_2 \xrightarrow{\sim} X_0 \oplus (X_1 \oplus X_2) \]

\[ c = c_{X_0, X_1} : X_0 \oplus X_1 \xrightarrow{\sim} X_1 \oplus X_0 \]

\[ u = u_X : X \oplus [0] \xrightarrow{\sim} X \]

We have an involution on \( \mathbb{F} \),

\[ (\ )^t : \mathbb{F} \xrightarrow{\sim} \mathbb{F}^{op} \]

\[ (f : D(f) \xrightarrow{\sim} I(f))^t = (f^{-1} : I(f) \xrightarrow{\sim} D(f)). \]

We shall assume the objects of \( \mathbb{F} \) form a (countable) set, containing \([n] = \{1, ..., n\}, n \geq 0.\)
§1.2 We define an $\mathbb{F}$-Ring (cf. [H07]) to be a symmetric monoidal category over $\mathbb{F}$: An $\mathbb{F}$-Ring $A$ is a category, together with a faithful functor $\epsilon : \mathbb{F} \to A$, which is the identity on objects, and a symmetric monoidal structure

$$\bigoplus : A \times A \to A$$

such that $\epsilon$ is strict monoidal; thus $A$ has objects the finite sets, for $X, Y \in A$, $X \bigoplus Y$ is the disjoint union, and $\bigoplus$ on arrows has associativity (resp. commutativity, unit) isomorphism given by $\epsilon(a)$ (resp. $\epsilon(c)$, $\epsilon(u)$). We also demand that $[0]$ is the initial and final object of $A$.

We write the arrows in $A$ from $X$ to $Y$ in "matrix" form $A(X,Y) := A_{Y,X}$.

An $\mathbb{F}$-Ring with involution is an $\mathbb{F}$-Ring $A$ with a functor

$$A \xrightarrow{(\cdot)^t} A^{op}$$

$$\mathbb{F} \xrightarrow{\sim} \mathbb{F}^{op}$$

such that $(a^t)^t = a$, and $(a_0 \bigoplus a_1)^t = a_0^t \bigoplus a_1^t$.

§1.3 We discuss the notion of commutativity for an $\mathbb{F}$-Ring. An $\mathbb{F}$-Ring is commutative if the following condition holds:

$$\forall a \in A_{Y,X}, b \in A_{1,J}, d \in A_{J,1}:$$

$$a \circ \bigoplus_X (b \circ d) = \bigoplus_Y (b \circ d) \circ a = \left(\bigoplus_J b\right) \circ \left(\bigoplus_X a\right) \circ \left(\bigoplus_X d\right) \in A_{Y,X}.$$  

It is totally commutative if

$$\forall a \in A_{Y,X}, \forall b \in A_{J,1}, \left(\bigoplus_X a\right) \circ \left(\bigoplus_J b\right) = \left(\bigoplus_Y b\right) \circ \left(\bigoplus_I a\right) \in A_{Y \bigoplus J, X \bigoplus I}.$$  

§1.4 For an $\mathbb{F}$-Ring $A$ we have a mapping

$$A(X,Y) = A_{Y,X} \to (A_{1,1})^{Y \times X}$$

$$a \mapsto a_{y,x} = j_y^* \circ a \circ j_x$$

where $(j_x : \{1\} \xrightarrow{\sim} \{x\}) \in F_{X,[1]}$. Although for most of the examples this mapping is an injection (we say $A$ is a "matrix - $\mathbb{F}$-Rings"), it need not be in general: for the "residue field" at the real prime it is not an injection! a replacement is the notion of tame $\mathbb{F}$-Ring.
§A.1 We show that the category of $F$-Rings has push-outs.

§A.2 We discuss equivalence - ideals and quotients.

§2.1 We show the category of rings is (fully) embedded in $F$-Rings ($R \mapsto F(R)$).

§2.2 We show the category of monoids is (fully) embedded in $F$-Rings ($M \mapsto F(M)$).

§2.3 The category of finite sets, $Set$; its opposite, $Set^{op}$; the category of finite sets and relations, $Rel(\equiv Set, Set^{op})$, are all examples of $F$-Rings.

§2.4 For every $p \geq 1$ the sub-$F$-Ring of real matrices $F(\mathbb{R})$ with operator $l_p$-norm $\leq 1$ is an $F$-Ring. For $p = 2$ (and only $p = 2$) it has involution.

§2.5 We discuss valuation-$F$-Rings, and prove Ostrowski theorem that for a number field $K$ the valuation $F$-Rings with involution correspond bijectively with the finite and infinite primes of $K$. (the proof itself is in the appendix B.1.)

§2.6 We show the finite directed graphs with no loops form an $F$-Ring.

§2.7 We construct the free-$F$-Ring on one generator of "degree" $(Y, X)$.

§2.8 We construct the $F$-Ring representing the functor $A \mapsto GL_n(A)$.

§2.9 We consider the arithmetical surface $F(\mathbb{N}) \otimes F(\mathbb{N})$. Its totally commutative quotient is reduced to the diagonal, but we prove that its commutative quotient is Not reduced to the diagonal!

§2.10 The $F$-Ring $F(\mathbb{N})$ (respectively, $F(R)$ for a ring $R$) is generated by the matrices $(1, 1)$ and $\begin{pmatrix} 1 \\ r \end{pmatrix}$, (resp. and $(r), r \in R$). We give the precises relations satisfied by these matrices.

§3 We show one can do algebraic geometry Grothendieck style (cf. \textbf{EGA}) over any commutative $F$-Ring.
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§4 The point of this chapter can be explained with ordinary commutative rings. If we want to develop algebraic geometry for commutative rings $A$ with (a possibly non trivial) involution, we can forget the involution and consider $\text{Spec } A$ - a topological space with an involution. But when we localize, or glue, such objects we lose the involution. The "right" way is to consider $\text{Spec } A^+$, where $A^+ = \{ a \in A \mid a^t = a \}$, over this space we have a sheaf of rings with involution.

§5 The point of this chapter can be explained with ordinary schemes (= the locally ringed spaces which are locally affine). While the locally - ringed - spaces (and the affine schemes) are closed under inverse limits, schemes are not. Given a point $x = \{ x_j \} \in \underset{\text{lim}}{\lim} X_j$, $X_j$ schemes, while each $x_j \in X_j$ has an affine neighborhood, these neighborhoods can shrink so that $x$ will not have an affine neighborhood. The real and complex primes of a number field are such points. We show that in the pro - category of schemes there exists the compactification of $\text{Spec } \mathbb{Z}$, and $\text{Spec } \mathcal{O}_K$, $K$ a number field. (this is the compactification of [H07], reproduced also in [Du]).

§6 We show that the process of real completion creating the continuum $\mathbb{R}^+ = GL_1(\mathbb{R})/(\pm 1)$ (and similarly $GL_n(\mathbb{R})/O_n$, $GL_n(\mathbb{C})/U_n$) can naturally be embedded in the language of pro- schemes. We define rank- $n$ vector bundle in such a way that for the compactified $\text{Spec } \mathcal{O}_K$, $K$ a number field, we obtain $GL_n(\mathbb{A}_K)/\prod_{p} GL_n(\mathcal{O}_{K,p}) \times O_n^\mathbb{R} \times U_n^\mathbb{C}$.

§7.1 For any $\mathbb{F}$-Ring $A$, we define $A$-mod as the full subcategory of the functor category $(Ab)^{A \times A^{op}}$ of $M$’s such that $M_{0,X} = M_{Y,0} = \{0\}$. The category of $A$-mod is complete and co- complete abelian category with enough projective and injectives. (and similarly the category $A$-mod$^t$, of $A$- modules with involution, when $A$ has an involution).

§7.2 We discuss the notion of commutativity for modules.

§7.3 We define the category of $\mathcal{O}_X$- mod$^{(t)}$ (possibly with involution) and its full subcategory $q.c. \mathcal{O}_X$- mod$^{(t)}$ of quasi- coherent $\mathcal{O}_X$- modules, so that for affine $X = \text{Spec } A$ localization gives an equivalence $A$-mod $\cong q.c. \mathcal{O}_X$- mod.

§7.4 For every homomorphism of $\mathbb{F}$-Rings$^{(t)}$, $\varphi : B \rightarrow A$, there is an adjunction (analogous with the commutative algebra’s extension and restriction of scalars):

$$A$-mod$^{(t)}(M^A, N) \cong B$-mod$^{(t)}(M, N_B),$$

$$M \in B$- mod$^{(t)}, \quad N \in A$-mod$^{(t)}.$$
§7.5 We define the infinitesimal extension of $A \in FR$ with an $M \in A\text{-mod}$,
$A \prod M$, which is an abelian group object in $FR/A$.

§7.6 We define the concept of derivations with values in $A$-modules and the
module of Kähler differentials representing them. For $\varphi \in FR^{(t)}(C, A)$, $B \in$
$C \setminus FR^{(t)} \diagdown A$ we have the adjunction,

\[
\left( C \setminus FR^{(t)} \diagdown A \right) \left( B, A\Pi M \right) \equiv \text{Der}^{(t)}_{\mathcal{E}}(B, M_B) \equiv A\text{-mod}^{(t)}(\Omega(B/C)^A, M).
\]

§7.7 We list the properties of differentials (such as the first and second exact
sequences).

§7.8 We give an explicit description of the modules $\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\})$ and
$\Omega(\mathbb{F}(\mathbb{N})/\mathbb{F})$. We show there is an exact sequence of $\mathbb{F}(\mathbb{Z})$-modules,

\[
\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\}) \xrightarrow{\partial} N \xrightarrow{\pi} \mathbb{F}(\mathbb{Z}) \to 0,
\]

where $N$ is also an $\mathbb{F}$-Ring and $\pi$ is a homomorphism of $\mathbb{F}$-Rings (with involution).

§7.9 We generalize the previous description to have an explicit description
for $\Omega(\mathbb{F}(R)/\mathbb{F}\{S\})$, $R$ a commutative $\mathcal{R}$Ring and $S \subseteq R$, a multiplicative set.

§7.10 Here we sketch the application of Quillen’s "non- additive homological
algebra", or "homotopical algebra", to our context. We give Quillen model
structures for modules and algebras and define the Quillen cotangent complex.

§8 Here we begin our "de-je-vu", with $\mathbb{F}$-Rings (with involution) replaced
by generalized-rings. We give more emphasis to the definitions and principal
examples, than to the proofs - which are very similar, and usually even simpler,
than the corresponding proofs for $\mathbb{F}$-Rings.

§8.1 is devoted to the definition of a generalized- ring, and in §8.2 we give
important remarks.

In §8.3 we associate with every ($\times$- commutative) $\mathbb{F}$-Ring with involution, a
(commutative) generalized ring. In particular, we describe the generalized rings:

- $\mathbb{F}$- the initial object of the category of generalized rings $\mathcal{G}R$, §8.3.1.
- $\mathcal{G}(B)$- the generalized ring associated with a commutative rig $B$, §8.3.2.
\[ O_{K,\eta} \] the generalized ring associated to an embedding \( \eta : K \rightarrow \mathbb{C} \), §8.3.3.

Valuation- generalized- rings of a generalized- field are given in §8.3.4, and we have Ostrowski’s theorem that for a number field \( K \), the valuations of \( \mathcal{G}(K) \) correspond to \( \mathcal{G}(O_{K,p}) \), \( p \) a finite prime, or to \( O_{K,\eta}, \eta : K \rightarrow \mathbb{C} \) (mod conjugation).

In §8.3.5 we give the generalized ring \( F_t \) associated to a commutative monoid \( M \).

In §8.3.6 we describe the free- commutative generalized- ring \( \Delta^W \), such that for any commutative generalized- ring \( A \) we have \( GR(\Delta^W, A) \equiv A_W \).

In §9 we give the various notions of ideals for a commutative generalized ring \( A \), and the relations between them.

In §9.1 we give the equivalence ideals \( eq(A) \); in §9.2 the functorial ideals, \( fun \cdot il(A) \); in §9.3 the operations on \( fun \cdot il(A) \); in §9.4 we give the homogeneous functorial ideals \( [1] \cdot il(A) \); and finally in §9.5 we give the useful notations of ideals \( il(A) \), and symmetric ideals \( il^s(A) \); generally, we have \( il^s(A) \subseteq [1] \cdot il(A) \subseteq il(A) \). For \( A = G(B) \), the generalized ring associated with a commutative ring \( B \) with an involution \( b \mapsto b^t \), \( il(A) \) corresponds bijectively with the ideals of \( B \), \( [1] - il(A) \) corresponds to the ideals \( b \) of \( B \) that are invariant under the involution: \( b = b^t \); and \( il^s(A) \) corresponds to the ideals of \( B \) that are generated as ideals by elements that are invariant under the involution \( b = (b_j)_{j \in J}, b_j^t = b_j \).

In §10 we give the geometry of commutative- generalized- rings (with no restrictions such as self- adjointness or total- commutativity). We discuss maximal (symmetric) ideals, and (symmetric) primes, and we have contravariant functors associating to a commutative- generalized- ring \( A \) its space of (symmetric) primes, \( \text{Spec}(A) \), (resp. \( \text{Spec}^t(A) \)), with its compact sober Zariski topology. There is a continuous map \( \text{Spec}(A) \rightarrow \text{Spec}^t(A) \).

In §11 we give the localizations of a commutative- generalized- ring \( A \), and the associated sheaf \( O_A \) of generalized rings over \( \text{Spec}^t(A) \).

In §12 we briefly describe the category \( \mathcal{LGRS} \) of locally-generalized- ringed- spaces, and its full subcategory of Grothendieck- generalized- schemes \( \mathcal{GGS} \). The category of generalized- schemes \( \mathcal{GS} \) is the pro- category of \( \mathcal{GGS} \): \( \mathcal{GS} = \text{pro-} \mathcal{GGS} \). It contains the compactified \( \overline{\text{Spec}} \mathcal{O}_K, K \) a number field.
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In §13.1 we briefly describe the tensor-products of commutative generalized-rings, and we write down the precise relations for the presentations

\[ \Phi : \Delta^2 \rightarrow G(N), \]
\[ \delta^2 \mapsto (1, 1), \]
\[ \Phi_Z : \mathbb{F} \{ \pm 1 \} \otimes \Delta^2 \rightarrow G(Z), \]
\[ \Phi_B : \mathbb{F} \{ B' \} \otimes \Delta^2 \rightarrow G(B), \]

\( B \) a commutative ring.

In §13.2 we give a description of \( G(N) \otimes G(N) \).

The tensor product gives the products in \( GGS, \) §13.3, and \( GS, \) §13.4, and we give the basic special case of the \textit{compactified arithmetical plane}

\[ \text{Spec} \mathbb{Z} \otimes_{\mathbb{F} \{ \pm 1 \}} \text{Spec} \mathbb{Z}. \]

To quote [CC14]:

"This note provides the algebraic geometric space underlying the non-commutative approach to RH. It gives a geometric framework reasonably suitable to transpose the conceptual understanding of the Weil proof in finite characteristic. This translation would require in particular an adequate version of the Riemann-Roch theorem in characteristic 1". see [H09] for more hints.

In §14.1 we briefly describe the theory of \( A \)-modules, for \( A \) a generalized ring, and its localization - the \( O_X \)-modules, \( X \in GGS. \)

In §14.2 we give the derivations and differentials (but only the even or odd ones, where the involution on \( M \) is the identity or minus the identity). We give explicitly the basic examples of the even derivations

\[ d = \frac{1}{2} dA_{[1]} : \mathbb{Z} \rightarrow \Omega^\mathbb{Z}_{[1]} \]

and \[ d = \frac{1}{2} dA_{[1]} : \mathbb{N} \rightarrow \Omega^\mathbb{N}_{[1]} \]

Here \( \Omega^\mathbb{N}_{[1]} \) is the free abelian group with generators \( \left\{ a \right\}, a, a' \in \mathbb{N}_{>0}, \) modulo the relations:
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Let \( \{a + a'\} \) be a cocycle \( \{ a' \} = \{ a' + a'' \} + \{ a' \} \) and \( a \) \( a \) \( \alpha \) \( a \) \( a \) \( \alpha \) \( a \) \( a \)

\[
\text{symmetric} \quad \begin{cases} a \in \{ a' \} \\ a' \end{cases} = \begin{cases} a' \end{cases} \quad a \]

\[
\text{linear} \quad \begin{cases} k \cdot a \end{cases} = \begin{cases} a' \end{cases} \quad k \cdot a \]

and \( d(n) = \left\{ \frac{1}{1} \right\} + \left\{ \frac{2}{1} \right\} + \cdots + \left\{ \frac{n-1}{1} \right\} \) satisfies

\[
d(n + m) = d(n) + d(m) + \left\{ \frac{n}{m} \right\}
\]

and Leibnitz:

\[
d(n \cdot m) = n \cdot d(m) + m \cdot d(n).
\]

Thus \( d(n) = \sum_p v_p(n) \cdot \frac{\alpha}{p} \cdot d(p) \) and \( \Omega_{[1]}^N \) is the free abelian group on \( d(p) \), \( \left\{ \frac{p-1}{1} \right\} \), \( p \) prime.

In a final appendix, we make contact with the analytic theory, and we give yet another explanation to the fact that the Gamma function gives the real analogue of the Euler factor \( (1 - p^{-s})^{-1} \) (or equivalently, via Mellin transform, that the Gaussian gives the real analogue of the characteristic function of \( \mathbb{Z}_p \).

For one explanation, which goes via the "quantum" \( q \)-analogue interpolation between the real and \( p \)-adic worlds, see [H01, H08]). We show that the Haar-Maak-\( O(N) \) or \( GL_N(\mathbb{Z}_p) \)-invariant probability measure \( \sigma_N \) on the real or \( p \)-adic sphere \( S^N_p \subseteq \mathbb{Q}_p^N \), behaves naturally with respect to the operations of multiplication and contraction, giving rise to the Beta function, and in particular, contracting \( \sigma_N \) with the vector \( \{1, \ldots, 1\} \in \mathbb{Q}_p^N \) we get in the limit \( N \to \infty \),

\[
\lim_{N \to \infty} \int_{S^N_p} |x_1 + \cdots + x_N|^{s-1} \sigma_N(dx) = \frac{\zeta_p(s)}{\zeta_p(1)}
\]

with \( \zeta_p(s) = (1 - p^{-s})^{-1} \) for the \( p \)-adic numbers, \( \zeta_\eta(s) = 2^\eta \Gamma(\frac{s}{\eta}) \) for the reals.

Alas, our idea and message are very simple: If one changes the operations of addition and multiplication to the more fundamental operations of multiplication and contraction of vectors (respectively, multiplication, direct-sum, and transposition of matrices), then one obtains a language in which arithmetic can be viewed as a geometry, and one proceeds exactly as in Grothendieck (with Quillen’s non-additive homotopical algebra).
This book would have not existed without the continuous efforts of Itai Cohen, who typed them into latex, and edited them.

It is dedicated to my father,
Prof. Menachem Haran,
and to my mother,
Dr. Raaya Haran-Twerski,
who taught us at an early stage that
"the trick is finding the simple explanation behind the complex phenomena"
and to the memory of
Daniel Quillen,
a teacher and a mentor, who taught us that mathematics is a language -
"when you have the right language to speak about a problem, you solved the problem".
Part I

F-Rings
Chapter 1

Definition of $\mathbb{F}$-Rings.

1.1 $\mathbb{F}$ the field with one element

For a category $C$ we write $X \in C$ for “$X$ is an object of $C$”, and we let $C(X,Y)$ denote the set of maps in $C$ from $X$ to $Y$. We denote by $\text{Set}_0$ the category with objects sets $X$ with a distinguished element $O_X \in X$, and with maps preserving the distinguished elements

$$\text{Set}_0(X,Y) = \left\{ f \in \text{Set}(X,Y) : f(O_X) = O_Y \right\}.$$

The category $\text{Set}_0$ has direct and inverse limits. The set $[o] = \{o\}$ is the initial and final object of $\text{Set}_0$. For $f \in \text{Set}_0(X,Y)$ we have

$$\ker f = f^{-1}(O_Y); \quad coker f = Y/f(X),$$

the set obtained from $Y$ by collapsing $f(X)$ to a point.

There is a canonical map

$$coker \ker f = X/f^{-1}(O_Y) \quad \longrightarrow \quad ker \ coker f = f(X).$$

Our first instinct is to take for $\mathbb{F}_0$, "the field with one element", or rather, the "finite dimensional $\mathbb{F}_0$-vector spaces", the full subcategory of $\text{Set}_0$ consisting of the finite sets. But note that the map that identifies two points to one (non-zero) point, has $\mathbb{R}$-linear extension the map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x_1, x_2) \mapsto x_1 + x_2$, and this map does not takes the unit $L_2$-disc into the unit interval $\left((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \mapsto \sqrt{2} > 1\right)$.

Thus we denote by $\mathbb{F}_0$ the subcategory of $\text{Set}_0$ with objects the finite pointed sets, and with maps

$$\mathbb{F}_0(X,Y) =$$
= \{ f \in \text{Set}_0(X,Y), \text{coker} \, \ker f \cong \ker \text{coker} f \text{ is an isomorphism} \} = \\
= \{ f \in \text{Set}_0(X,Y), f\rvert_{X\setminus f^{-1}(O_Y)} \text{ is an injection} \}

We let \text{Set}_\bullet denote the category with objects sets and with partially defined maps
\begin{equation}
\text{Set}_\bullet(X,Y) = \bigsqcup_{X' \subseteq X} \text{Set}(X',Y).
\end{equation}
Thus to \( f \in \text{Set}_\bullet(X,Y) \) there is associates its domain \( D(f) \subseteq X \), and \( f \in \text{Set}(D(f), Y) \).

We have an isomorphism of categories
\begin{equation}
\text{Set}_0 \cong \text{Set}_\bullet
\end{equation}
given by
\begin{align*}
X & \mapsto X_+ = X \setminus \{O_X\} \\
f & \mapsto f_+, \quad D(f_+) = X \setminus f^{-1}(O_Y);
\end{align*}
and inversly
\begin{align*}
x \in D(f) : & \quad \{O_X\} \coprod X = X_0 \leftrightarrow X \\
x \notin D(f) : & \quad \{ f(x) \} = f_0(x) \leftrightarrow f
\end{align*}

We let \( \mathbb{F} \) denote the subcategory of \( \text{Set}_\bullet \) corresponding to \( \text{Set}_0 \) under this isomorphism, it has objects the \textit{finite} sets, and maps are the partial bijections
\begin{equation}
\mathbb{F}(X,Y) = \{ f : D(f) \cong \text{I}(f) \text{ bijections}, D(f) \subseteq X, \text{I}(f) \subseteq Y \}.
\end{equation}

It is crucial that the objects of \( \mathbb{F} \) are finite sets, but we do not need \( \mathbb{F} \) to contain all finite sets.

To avoid problems with set theory we shall work with a countable set-model of \( \mathbb{F} \) that contains \( [0] = \emptyset \) (the empty set, the initial and final object), \( [1] = \{1\}, \ldots, [n] = \{1, \ldots, n\} \), \ldots and is closed under the operations of pull-back and push-out in \( \text{Set}_\bullet \).

The operation "\( \circ \)" will denote composition of partial - bijections, but note that if \( g \circ f \) is defined, then \( D(g \circ f) = f^{-1}(I(f) \cap D(g)) = D(f) \cap f^{-1}(D(g)) \).

Note that we can identify \( \mathbb{F}(X,Y) \) with \( Y \times X \) - matrices with value in \{0,1\}, having at most one 1 in every row or column, (and than \( \circ \) is matrix multiplication), and we will denote this set by \( \mathbb{F}_{Y,X} \).

Note that the category \( \mathbb{F} \) has \textit{no} sums or products, but we do have two symmetric monoidal structures on \( \mathbb{F} \), the disjoint union:
\begin{equation}
\bigoplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F}
\end{equation}
\begin{align*}
X_0 \oplus X_1 & = \{(i,x), i \in \{0,1\}, x \in X_i \}, \\
(f_0 \oplus f_1)(i,x) & = (i, f_i(x)), f_i \in \mathbb{F}_{Y_i,X_i}
\end{align*}
CHAPTER 1. DEFINITION OF F-RINGS.

(this is the categorical sum in Set•).

and the cartesian product:

\[ \otimes : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \]  
(1.1.9)

\[ X_0 \otimes X_1 = \{(x_0, x_1), x_i \in X_i\}. \]  
(1.1.10)

\[ (f_0 \otimes f_1)(x_0, x_1) = (f_0(x_0), f_1(x_1)), f_i \in \mathbb{F}_{X_i}. \]  
(1.1.11)

Thus in Set•,

\[ X \otimes Y = \text{coker}\{X \perp Y \to X \sqcup Y\} \]

or \( X \sqcup Y = X \perp Y \perp (X \otimes Y) \).

We have associativity, commutativity, and unit isomorphisms:

\[ a = a_{X_0, X_1, X_2} : (X_0 \oplus X_1) \oplus X_2 \xrightarrow{\sim} X_0 \oplus (X_1 \oplus X_2) \]  
(1.1.12)

\[ a(1, x_2) = (1, (1, x_2)) \]  
(1.1.13)

\[ a(0, (i, x_i)) = \begin{cases} (1, (0, x_1)), & i = 1 \\ (0, x_0), & i = 0 \end{cases} \]  
(1.1.14)

\[ c = c_{X_0, X_1} : X_0 \oplus X_1 \xrightarrow{\sim} X_1 \oplus X_0 \]  
(1.1.15)

\[ c(i, x) = (1 - i, x) \]  
(1.1.16)

\[ u = u_X : X \oplus [0] \xrightarrow{\sim} X \]  
(1.1.17)

\[ u(0, x) = x. \]  
(1.1.18)

where \([0]\) is the empty set, is the initial and final object of \(\mathbb{F}\). Similarly, there are associativity \(a^*\), commutativity \(c^*\), and unit \(u^*\) isomorphisms for the operation \(\otimes\), (the unit object for \(\otimes\) being \([1]\) = the one point set). Moreover, there is a distributivity isomorphism:

\[ d = d_{X, Y_0, Y_1} : X \otimes (Y_0 \oplus Y_1) \xrightarrow{\sim} (X \otimes Y_0) \oplus (X \otimes Y_1). \]  
(1.1.19)

Given a finite collection of finite sets \(\{Y_x\}_{x \in X}\), we can form the disjoint union:

\[ \bigoplus_X Y_x = \{(x, y), \ x \in X, \ y \in Y_x\} \]

and we have canonical isomorphisms:

\[ \bigoplus_X Y \xrightarrow{\sim} X \otimes Y \xrightarrow{\sim} \bigoplus_Y X. \]  
(1.1.20)

In order to keep our formulas simple, we shall abuse language and will not write these canonical isomorphisms.

Note that the category \(\mathbb{F}\) has involution:

\[ (\ )^t : \mathbb{F}^{\text{op}} \xrightarrow{\sim} \mathbb{F} \]  
(1.1.21)
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\[(f : D(f) \cong I(f))^t = (f^{-1} : I(f) \cong D(f)),\]  
\[(\text{or the transposed } \{0, 1\} - \text{matrix})\]

\[(f \circ g)^t = g^t \circ f^t, \quad (id_X)^t = id_X,\]  
\[(f^t)^t = f,\]

and this involution preserves the sum $\oplus$ (and the product $\otimes$):

\[(f_0 \oplus f_1)^t = f_0^t \oplus f_1^t.\]  

We usually let $X, Y, Z, W$ denote objects of $\mathbb{F}$, without explicitly saying so, and when we consider "$\text{Set}_*(X, Y)$" it is usually implicitly assumed that $X, Y \in \mathbb{F}$.

1.2 $\mathbb{F}$-Rings

In this section we define the category of $\mathbb{F}$-Rings, $\mathbb{FR}$, and the category of $\mathbb{F}$-Rings with involution, $\mathbb{FR}^t$. We show these categories are bi-complete.

Definition 1.2.1

An $\mathbb{F}$-Ring $A$ is a category, together with a faithful functor $\epsilon : \mathbb{F} \to A$, which is the identity on objects, and a symmetric monoidal structure

\[\oplus : A \times A \to A\]  

such that $\epsilon$ is strict monoidal; thus $A$ has objects the finite sets, for $X, Y \in A$, $X \oplus Y$ is the disjoint union, and $\oplus$ on arrows has associativity (resp. commutativity, unit) isomorphism given by $\epsilon(a)$ (resp. $\epsilon(c)$, $\epsilon(u)$, which we abuse language and omit from our formulas!). We also demand that $[0]$ is the initial and final object of $A$.

Thus an $\mathbb{F}$-Ring is a collection of pointed sets $A_{Y,X} = A(X, Y)$ for all finite sets $X, Y$, together with the operation of composition:

\[A_{Z,Y} \times A_{Y,X} \to A_{Z,X}\]  

\[g, f \mapsto g \circ f\]

which is associative:

\[(h \circ g) \circ f = h \circ (g \circ f),\]

unital:

\[f \circ id_X = f = id_Y \circ f \quad \text{for } f \in A_{Y,X},\]

and agree with composition on arrows of $\mathbb{F}$, where $\mathbb{F}_{Y,X} \subseteq A_{Y,X}$. 
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We also have the operation of direct sum:

$$A_{Y_0,X_0} \times A_{Y_1,X_1} \to A_{Y_0 \oplus Y_1, X_0 \oplus X_1} \tag{1.2.6}$$

$$f_0, f_1 \mapsto f_0 \oplus f_1 \tag{1.2.7}$$

such that we have:

naturality:

$$(g_0 \oplus g_1) \circ (f_0 \oplus f_1) = (g_0 \circ f_0) \oplus (g_1 \circ f_1) \tag{1.2.8}$$

$$id_{X_0} \oplus id_{X_1} = id_{X_0 \oplus X_1} \tag{1.2.9}$$

associativity:

$$(f_0 \oplus f_1) \oplus f_2 = f_0 \oplus (f_1 \oplus f_2) \tag{1.2.10}$$

(i.e. $a_{Y_0,Y_1,Y_2} \circ ((f_0 \oplus f_1) \oplus f_2) = (f_0 \oplus (f_1 \oplus f_2)) \circ a_{X_0,X_1,X_2}$, $f_i \in A_{Y_i,X_i}$) \tag{1.2.11}

commutativity:

$$f_0 \oplus f_1 = f_1 \oplus f_0 \tag{1.2.12}$$

(i.e. $c_{Y_0,Y_1} \circ (f_0 \oplus f_1) = (f_1 \oplus f_0) \circ c_{X_0,X_1}$, $f_i \in A_{Y_i,X_i}$) \tag{1.2.13}

unit:

$$f \oplus 0 = f \tag{1.2.14}$$

(i.e. $(f \oplus id_0) \circ u_X = u_Y \circ f$, $f \in A_{Y,X}$) \tag{1.2.15}

and $f_0 \oplus f_1$ agree with the sum in $\mathcal{F}$ for $f_i \in \mathcal{F}_{Y_i,X_i} \subseteq A_{Y_i,X_i}$.

**Definition 1.2.2**

For $\mathcal{F}$-Rings $A, A'$ we denote by $\mathcal{F}R(A, A')$ the collection of functors $\varphi : A \to A'$ over $\mathcal{F}$, and strict monoidal, i.e. $\varphi$ is a collection of set mappings $\varphi_{Y,X} : A_{Y,X} \to A'_{Y,X}$, with $\varphi(g \circ f) = \varphi(g) \circ \varphi(f)$; $\varphi(f_0 \oplus f_1) = \varphi(f_0) \oplus \varphi(f_1)$; $\varphi(\epsilon(f)) = \epsilon'(f)$ for $f \in \mathcal{F}_{Y,X}$. Thus we have a category of $\mathcal{F}$-Rings $\mathcal{F}R$.

We have for $X \in \mathcal{F}$ a functor $(\ )^X : \mathcal{F}R \to \mathcal{F}R$,

$$(A^X)_{Z,Y} := A_{X \otimes Z \otimes X \otimes Y} \tag{1.2.16}$$

We have a functor $(\ )^{op} : \mathcal{F}R \to \mathcal{F}R$,

$$(A^{op})_{Y,X} := A_{X \otimes Y} \tag{1.2.17}$$

We denote by $\mathcal{F}R^t$ the $\mathcal{F}$-Rings with involution, i.e. $A \in \mathcal{F}R$ and $(\ )^t : A^{op} \to A$ is an involution:

$$f^{tt} = f \tag{1.2.18}$$
CHAPTER 1. DEFINITION OF $\mathbb{F}$-RINGS.

\[(g \circ f)^t = f^t \circ g^t\]  
(1.2.19)

\[\epsilon(f^t) = \epsilon(f^t) \quad \text{for} \quad f \in \mathbb{F}_{Y,X}\]  
(1.2.20)

\[(f_0 \oplus f_1)^t = f_0^t \oplus f_1^t.\]  
(1.2.21)

For $A, A' \in \mathbb{F}^t$, we let

\[\mathbb{F}^t(A, A') = \{\varphi \in \mathbb{F}(A, A'), \quad \varphi(a^t) = \varphi(a)^t\}.\]  
(1.2.22)

We have a category of $\mathbb{F}$-Rings with involutions: $\mathbb{F}^t$.

For $A \in \mathbb{F}, X \in \mathbb{F}$, the set $A_{X,X}$ is an associative monoid, and we let $GL_X(A)$ denote its invertible elements:

\[GL_X(A) = \{a \in A_{X,X}, \text{ there exists } a^{-1} \in A_{X,X}, \quad a \circ a^{-1} = a^{-1} \circ a = id_X\}\]  
(1.2.23)

Clearly this constitute a functor form $\mathbb{F}$ to the category of groups,

\[GL_X : \mathbb{F} \to \text{Grps.}\]  
(1.2.24)

We have embeddings

\[GL_{X_0}(A) \times GL_{X_1}(A) \to GL_{X_0 \oplus X_1}(A)\]

\[(a_0, a_1) \mapsto a_0 \oplus a_1\]

In particular, we have the group $GL_X(A) = \lim_{n} GL_{[n]}(A)$, the direct limit with respect to $a \mapsto a \oplus 1$ (hence Quillen’s higher K-groups $K_i(A) = \pi_i(BGL_X(A)^+)$, cf. [Q73]).

**Theorem 1.1**

*The categories $\mathbb{F}$ and $\mathbb{F}^t$ are bi-complete.*

**Proof.** Inverse limits exists and can be calculated in sets:

\[(\lim_{j} A^{(j)})_{Y,X} = \lim_{j} (A^{(j)}_{Y,X})\]  
(1.2.25)

Also co-limits over a directed partially ordered set $J$, exists and can be calculated in Sets:

\[(\lim_{j} A^{(j)})_{Y,X} = \lim_{j} (A^{(j)}_{Y,X})\]  
(1.2.26)

The pushout $\bigsqcup_A B^0 \bigsqcup_A B^1$ is denoted (as usual) by $B^0 \otimes_A B^1$, it is constructed in the appendix A.1. Equalizers also exist, and in fact we can factorize an $\mathbb{F}$-$\text{Ring}$ by any equivalence - ideal, this is described in appendix A.2. This suffices for the existence of arbitrary co-limits. 

\[\square\]
CHAPTER 1. DEFINITION OF F-RINGS.

1.3 Commutativity

Note that we usually do not write canonical-isomorphisms of $\mathbb{F}$, especially, $\bigoplus X \cong X \otimes Y$.

Definition 1.3.1

Let $A \in \mathbb{F}$-Ring. We say $A$ is:

**Totally - commutative:**

$$\forall a \in A_{Y,X}, \forall b \in A_{J,I}, \ (\bigoplus_X a) \circ (\bigoplus_I b) = (\bigoplus_I b) \circ (\bigoplus_X a) \in A_{Y \otimes J, X \otimes I}. \quad (1.3.1)$$

**Left - commutative:**

$$\forall a \in A_{[1],X}, \forall b \in A_{[1],I}, \ a \circ (\bigoplus_I b) = b \circ (\bigoplus_X a) \in A_{[1],X \otimes I}. \quad (1.3.2)$$

**Right - commutative:**

$$\forall a \in A_{Y,[1]}, \forall b \in A_{J,[1]}, \ (\bigoplus_X a) \circ b = (\bigoplus_I b) \circ a \in A_{Y \otimes J,[1]}. \quad (1.3.3)$$

**1 - commutative:** If $A$ is both Left- and Right- commutative.

**\ast - commutative:**

$$\forall a \in A_{Y,[1]}, \forall b \in A_{[1],I}, \ a \circ (\bigoplus_I b) \circ (\bigoplus_X a) \in A_{Y,I}. \quad (1.3.4)$$

**Central:**

$$\forall a \in A_{Y,X}, \forall b \in A_{1,1}, \ a \circ (\bigoplus_X b) = (\bigoplus_I b) \circ a =: b \cdot a \in A_{Y,X}. \quad (1.3.5)$$

i.e. $A_{1,1}$ is a commutative monoid and it acts centrally on $A_{Y,X}$, and we shall denote this action by $b \cdot a$.

**Commutative:**

$$\forall a \in A_{Y,X}, \forall b \in A_{1,J}, \forall d \in A_{J,1} :$$

$$a \circ (\bigoplus_X b \circ d) = (\bigoplus_Y b \circ d) \circ a = (\bigoplus_Y b) \circ (\bigoplus_I a) \circ (\bigoplus_X d) \in A_{Y,X}. \quad (1.3.6)$$

We let $\mathbb{C}FR$ (resp. $\mathbb{F}R_{\text{tot-com}}, \mathbb{F}R_{\text{1-com}}, \mathbb{F}R_{\ast \text{-com}}, \mathbb{F}R_{\text{cent-l}}$) denote the full subcategories of $\mathbb{F}$-Rings consisting of the commutative (resp. totally-commutative, 1-commutative, $\ast$-commutative, central) $\mathbb{F}$-Rings. Noting that, $A$ commutative $\implies A$ central, we have embeddings of categories.
CHAPTER 1. DEFINITION OF $\mathbb{F}$-RINGS.

\[
\begin{array}{ccc}
\mathbb{C}\mathbb{F}\mathbb{R} & \xleftarrow{\mathbb{F}\mathbb{R}_{\text{cent}^1}} & \mathbb{F}\mathbb{R}_{\text{tot-com}} \\
\mathbb{F}\mathbb{R}_{\text{x-com.}} & \xrightarrow{\mathbb{F}\mathbb{R}_{\text{1-com.}}} & \mathbb{F}\mathbb{R}
\end{array}
\]  
(1.3.7)

1.4 Matrix coefficients and tameness.

For a set $X \in \mathbb{F}$, and an element $x \in X$, we denote by $j_x = j_x^X$, the morphisms of $\mathbb{F}$ given by

\[
j_x : [1] \to X, \quad j_x(1) = x \in X,
\]
(1.4.1)

and where

\[
j_x^t : X \to [1] \text{ is the partial bijection } \{x\} \to \{1\}.
\]
(1.4.2)

**Definition 1.4.1**

Define the matrix coefficients $J_{Y,X} : A_{Y,X} \to (A_{1,1})^{Y \times X}$ via

\[
a \mapsto \{j_y^t \circ a \circ j_x\},
\]
(1.4.3)

where $j_x \in \mathbb{F}_{X,1}$, $j_x(1) = x$, and $j_y^t \in \mathbb{F}_{1,Y}$.

**Definition 1.4.2**

We say $A \in \mathbb{F}\mathbb{R}$ is a "matrix $\mathbb{F}$-ring" if $J_{Y,X}$ is injective for any $X, Y \in \mathbb{F}$.

**Definition 1.4.3**

We say $A \in \mathbb{F}\mathbb{R}$ is tame: $\forall a, a' \in A_{Y,X}$,

\[
\forall b \in A_{1,Y}, \forall d \in A_{X,1} : b \circ a \circ d = b \circ a' \circ d \in A_{1,1} \implies a = a'.
\]
(1.4.4)

We have the implication (taking $b = j_y^t, d = j_x$):

\[
A \text{ matrix } \implies A \text{ tame.}
\]
(1.4.5)

We also have the implication:

\[
A \text{ commutative + tame } \implies A \times \text{-commutative.}
\]
(1.4.6)

(indeed, for $a \in A_{Y,1}, b \in A_{1,J}$, and any $d \in A_{J,1}, d' \in A_{1,Y}$, commutativity gives $d' \circ a \circ b \circ d = d' \circ \begin{array}{cc} 0_Y \circ b \circ a \circ d \end{array}$, hence by tameness $a \circ b = \begin{array}{cc} 0_Y \circ b \circ a \end{array}$).
Appendix A

A.1 Proof of existence of $B^0 \coprod_A B^1 = B^0 \otimes_A B^1$.

Theorem A.1

The category $\mathcal{K}$ has pushouts $\otimes_A = \coprod_A$.

\[
\begin{array}{c}
\xymatrix{
A \ar[rr]^f \ar[rd]_{\psi^0} & & B^0 \ar[dl]_{\phi^0} \\
& B^1 \ar[rr] \ar[dl]_{\psi^1} & & C \\
& A \ar[rr]^f & & B^0 \otimes_B B^1 \\
& & & \text{(A.1.1)}
}\end{array}
\]

Proof. Define the sets of chains of arrows,

\[
B_{Y,X} = \{(b_l, \ldots, b_0), l \geq 0, 1, b_j \in B_{X_{l+1}, X_j}^{j \mod 2}, Y = X_{l+1}, X = X_0\}
\]

(A.1.2)

Let $\sim$ be the equivalence relation on chains generated by:

1. $\ldots, b_{j+1} \circ \phi^{j+1}(a), b_j, \ldots \sim \ldots, b_{j+1}, \phi^j(a) \circ b_j, \ldots$,

2. $\ldots, b_{j+1}, f, b_j, \ldots \sim \ldots, b_{j+1} \circ f \circ b_j, \ldots, f \in \mathcal{K}$,

and the boundary cases:

\[
(f, b_j, \ldots) \sim (f \circ b_j, \ldots), \quad (\ldots, b_0, f) \sim (\ldots, b_0 \circ f)
\]

$f \in \mathcal{K}$ (or $f = \phi^j(a)$)

(A.1.3)

3. $\ldots, b_{j+2}, id_{X_{j+2}} \oplus \overline{b}_{j+1}, \overline{b}_j \oplus id_{X_j}, b_{j-1}, \ldots \sim$

\[
(\ldots, b_{j+2} \circ (\overline{b}_j \oplus id_{X_j}), (id_{X_j} \oplus \overline{b}_{j+1}) \circ b_{j-1}, \ldots)
\]
\[ b_{j+1} \in B_{j+1}^{\delta} \mod 2 \] or in diagram:

\[
\begin{array}{c}
(\ldots, b_{j+1}^\prime, b_j^\prime, \ldots) & \oplus & (\ldots, b_{j+1}, b_j, \ldots) \\
\oplus & & \oplus \\
(\ldots, b_{j+1}^\prime, b_j^\prime, \ldots) & \oplus & (\ldots, b_{j+1}, b_j, \ldots)
\end{array}
\]

The composition \( B_{Y,X} \times B_{Y,X} \rightarrow B_{Z,X} \) is given by,

\[
(b_l, \ldots, b_k) / \sim (b_l, \ldots, b_k) / \sim = \begin{cases} 
(b_l, \ldots, b_l^\prime, b_l, \ldots, b_k) / \sim \text{ if } \delta^l \neq 2 \text{ l} \\
(b_l, \ldots, b_l^\prime \circ b_l, \ldots, b_k) / \sim \text{ if } \delta^l = 2 \text{ l}
\end{cases}
\]

It is well defined, independent of representatives (since \( \circ \) is associative). Furthermore it is associative and unital,

\[
(\text{with identity: } (id_X^l) / \sim = (id_X^l) / \sim = id_X)
\]

and therefore \( B \) is a category. It has the natural maps \( \psi^l : B^l \rightarrow B \), and \( \psi^0 \varphi^0 = \psi \varphi^1 \) by (1).

Define the map,

\[
\oplus : B_{Y_0 \oplus X_0} \times B_{Y_1 \oplus X_1} \rightarrow B_{Y_0 \oplus Y_1 \oplus X_0 \oplus X_1}
\]

by:

\[
(b_l, \ldots, b_k) / \sim \oplus (b_l, \ldots, b_k) / \sim = (b_l \oplus b_l, \ldots, b_k \oplus b_k) / \sim
\]

Note that without loss of generality we can assume that \( l' = l, \delta' = \delta \), (otherwise add identities). The map \( \oplus \) is well defined and independent of representatives. Indeed, it is invariant by the 3 possible moves:

move (1):

\[
(\ldots, b_{j+1}^\prime, b_j^\prime, \ldots) \oplus (\ldots, b_{j+1} \circ \varphi^{j+1} (a), b_j, \ldots) =
\]

\[
(\ldots, b_{j+1} \circ \varphi^{j+1} (a), b_j^\prime \oplus b_j, \ldots) = (\ldots, b_{j+1} \circ b_{j+1} \circ (id_X^j \oplus \varphi^{j+1} (a)), b_j^\prime \oplus b_j, \ldots)
\]

\[
= (\ldots, b_{j+1} \circ b_{j+1}, b_j^\prime \oplus (\varphi^l (a) \circ b_j), \ldots)
\]

\[
\sim (\ldots, b_{j+1} \circ b_{j+1}, (id_X^j \oplus \varphi^l (a)) \circ (b_j^\prime \oplus b_j), \ldots)
\]

\[
\varphi \circ (id_X^j \oplus \varphi^l (a))
\]

\[
= (\ldots, b_{j+1}, \ldots) \oplus (\ldots, b_{j+1}, \varphi^l (a) \circ b_j, \ldots)
\]

move (2):

\[
(\ldots, b_{j+1}, b_j^\prime, b_j^\prime, \ldots) \oplus (\ldots, b_{j+1}, f, b_j, \ldots) =
\]
(\ldots, b'_{j+1} \oplus b_{j+1}, b'_j \oplus f, b'_{j-1} \oplus b_{j-1}, \ldots) = (\ldots, b'_{j+1} \oplus b_{j+1}, (b'_j \oplus id_{X_{j+1}}) \circ (id_{X'_j} \oplus f), b'_{j-1} \oplus b_{j-1}, \ldots)

\sim (\ldots, b'_{j+1} \oplus b_{j+1}, b'_j \oplus id_{X_{j+1}}, b'_{j-1} \oplus (f \circ b_{j-1}), \ldots)

= (\ldots, (b'_j \oplus id_{X_{j+2}}) \circ (id_{X'_{j+1}} \oplus b_{j+1}), (b'_j \oplus id_{X_{j+1}}) \circ id_{X'_{j+1}} \oplus b'_{j-1} \oplus (f \circ b_{j-1}), \ldots)

\sim (\ldots, b'_{j+1} \oplus id_{X_{j+2}}, b'_j \oplus id_{X_{j+1}}, b'_{j-1} \oplus (b_{j+1} \circ f \circ b_{j-1}), \ldots) =

= (\ldots, b'_{j+1}, b'_j, b'_{j-1}, \ldots) \oplus (\ldots, id_{X_{j+2}}, id_{X_{j+1}}, (b_{j+1} \circ f \circ b_{j-1}), \ldots) \quad (A.1.9)

move (3):

(\ldots, b'_{j+2}, b'_{j+1}, b'_j, b'_{j-1}, \ldots) \oplus (b_{j+2}, (id \oplus b_{j+1}), (b_j \oplus id), b_{j-1}) =

(\ldots, b'_{j+2} \oplus b_{j+2}, b'_{j+1} \oplus (id \oplus b_{j+1}), b'_j \oplus (b_j \oplus id), b'_{j-1} \oplus b_{j-1}, \ldots) =

= (\ldots, b'_{j+2} \oplus b_{j+2}, (b'_{j+1} \oplus id) \circ (id \oplus id), (id \oplus (b_j \oplus id)) \circ (b'_{j+1} \oplus id), (b'_{j-1} \oplus b_{j-1}) \circ (f \circ b_{j-1}), \ldots)

\sim (\ldots, b'_{j+2} \oplus b_{j+2}, b'_{j+1} \oplus id, id \oplus (b_j \oplus id), id \oplus (id \oplus b_{j+1}), b'_j \oplus (id \oplus b_j), b'_{j-1} \oplus (b_{j+1} \circ b_{j-1}) \circ (f \circ b_{j-1}), \ldots)

\sim (\ldots, b'_{j+2} \oplus (b_j \oplus id), (b'_{j+1} \oplus id), (b'_j \oplus id), b'_{j-1} \oplus ((id \oplus b_{j+1}) \circ b_{j-1}), \ldots)

= (\ldots, b'_{j+2}, b'_{j+1}, b'_j, b'_{j-1}, \ldots) \oplus (\ldots, (b_{j+2} \circ (b_j \oplus id)), id, id, (id \oplus b_{j+1}) \circ b_{j-1}) \quad (A.1.10)

It is then straightforward to check that B satisfy all the axioms of an \( F \)-RING.

For \( FR^i \) define the induced involution on B:

\[
(\_)^i : B_{Y,X} \rightarrow B_{X,Y}
\]

\[
((b_1, \ldots, b_i)/ \sim)^i = (b_i', \ldots, b_1')/\sim.
\]

This operation is well defined since all moves are self dual, and \( B \) is an \( F \)-RING with involution.

The map

\[
f^0 \otimes f^1 ((b_1, \ldots, b_8)/ \sim) = f^1(b_1) \circ \cdots \circ f^8(b_8), \quad f^j := f^{j \mod 2}
\]

is well defined since it remains invariant under each move (1), (2), (3), and it is (the unique) homomorphism of \( F \)-RING solving the universal property \( (\text{A.1.11}) \).
A.2 Equivalence ideals and quotients.

Definition A.2.1

For $A \in \mathbb{F}_R$, (resp. $A \in \mathbb{F}_R^{(t)}$) an equivalence ideal (resp. $t$-equivalence ideal) is a sub-$\mathbb{F}$-Ring (resp. with involution) $\mathcal{E} \subseteq A \prod A$ s.t.

$$\mathcal{E}_{Y,X} \subseteq A_{Y,X} \prod A_{Y,X} \text{ is an equivalence relation on } A_{Y,X},$$

(A.2.1)
equivalently, we have a collection of equivalence relations $\sim$ on the $A_{Y,X}$’s, compatible with the operations:

$$a \sim a' \implies b \circ a \circ d \sim b \circ a' \circ d$$

(A.2.2)

$$a \sim a' \implies a \oplus id \sim a' \oplus id$$

(A.2.3)

(and therefore $a_0 \sim a'_0, a_1 \sim a'_1 \implies a_0 \oplus a_1 \sim a'_0 \oplus a'_1$)

and in the presence of an involution: $a \sim a' \implies a^t \sim (a')^t$ (A.2.4)

Given an equivalence ideal $\mathcal{E}$ of $A$, let

$$A/\mathcal{E} = \prod_{Y,X \in \mathbb{F}} A_{Y,X}/\mathcal{E}_{Y,X},$$

(A.2.5)

and let $\pi : A \to A/\mathcal{E}$ denote the canonical map which associates with $a \in A_{Y,X}$ its equivalence class $\pi(a) \in A_{Y,X}/\mathcal{E}_{Y,X}$. It follows from (A.2.2-A.2.4) that we have well defined operations on $A/\mathcal{E}$,

$$\pi(f) \circ \pi(g) = \pi(f \circ g),$$

$$\pi(f) \oplus \pi(g) = \pi(f \oplus g)$$

resp. $\pi(f)^t = \pi(f^t)$ (A.2.6)

making $A/\mathcal{E}$ into an $\mathbb{F}$-Ring such that $\pi : A \to A/\mathcal{E}$ is a homomorphism of $\mathbb{F}$-Rings (resp. with involutions).

Given a homomorphism of $\mathbb{F}$-Rings $\varphi : A \to B$ denote by

$$\mathcal{K}\mathcal{E}\mathcal{R}(\varphi) = A \prod_B A = \prod_{Y,X \in \mathbb{F}} \mathcal{K}\mathcal{E}\mathcal{R}_{Y,X}(\varphi),$$

$$\mathcal{K}\mathcal{E}\mathcal{R}_{Y,X}(\varphi) = \{(a, a') \in A_{Y,X} \times A_{Y,X} | \varphi(a) = \varphi(a')\}. \quad (A.2.7)$$

It is clear that $\mathcal{K}\mathcal{E}\mathcal{R}(\varphi)$ is an equivalence-ideal of $A$, and that $\varphi$ induces an injection of $\mathbb{F}$-Rings $\overline{\varphi} : A/\mathcal{K}\mathcal{E}\mathcal{R}(\varphi) \to B$, such that $\varphi = \overline{\varphi} \circ \pi$, i.e.

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A/\mathcal{K}\mathcal{E}\mathcal{R}(\varphi) & \xrightarrow{\sim} & \varphi(A)
\end{array} \quad (A.2.8)
is a commutative diagram. Thus every map $\varphi$ of $\mathbb{F}$-Rings factors as epimorphism $(\pi)$ followed by an injection $(\varphi)$.

For a family $\{(a_i, a'_i)\} \in (A \times A)'$, $a_i, a'_i \in A_{Y_i}$, let $E$ be the equivalence ideal generated by $\{(a_i, a'_i)\}$. We have $(b, b') \in E$ iff $\exists$ path $b = b_0, b_1, ..., b_l = b'$, s.t. $\{b_{j-1}, b_j\}$ has the form

$$\{c_j \circ (a_{i(j)} \oplus id_{V_j}) \circ d_j, \quad c_j \circ (a'_{i(j)} \oplus id_{V_j}) \circ d_j\}$$

(A.2.9)

(or the form $\{c_j \circ (a'_{i(j)} \oplus id_{V_j}) \circ d_j, \quad c_j \circ (a''_{i(j)} \oplus id_{V_j}) \circ d_j\}$, in the presence of involution).

Indeed, if there is such a path $b = b_0, ..., b_l = b'$, than $(b_{j-1}, b_j) \in E$, and $E$ is an equivalence relation so $(b, b') \in E$. On the other hand, the set of $(b, b')$ s.t. there is a path $b = b_0, ..., b_l = b'$, with $\{b_{j-1}, b_j\}$ of the form (A.2.9), is an equivalence relation $b \sim b'$, satisfying

$$b \sim b' \implies c \circ b \circ d \sim c \circ b' \circ d, \quad b \oplus id \sim b' \oplus id, \quad (\text{and } b' \sim b''), \quad (A.2.10)$$

and hence it is precisely the equivalence ideal $E$ generated by $\{(a_i, a'_i)\}$.

For example, for any $\mathbb{F}$-Ring $A$, we have the equivalence ideals $E_? \subseteq A$ generated by the "?- commutativity" relation, and the associated quotient $A^? = A/E_?$, giving rise to the diagram of surjections,

\[
\begin{array}{ccc}
A^{cent'} &=& A/E_{cent'} \\
\downarrow & & \downarrow \\
A & & A^{com} = A/E_{com} \\
\downarrow & & \downarrow \\
A^{1-com} = A/E_{1-com} & & A^{tot-com} = A/E_{tot-com}.
\end{array}
\]

(A.2.11)

The functor $A \mapsto A^?$ is left adjoint to the embeddings $\{1.3.7\}$. $\mathbb{F}R \hookrightarrow \mathbb{F}R$

**Definition A.2.2**

For an $E \subseteq A \coprod A$ equivalence ideal, we say $E$ is tame if $A/E$ is tame. Thus a tame equivalence ideal $E$ is completely determined by the equivalence relation $E_{1,1}$ on $A_{1,1}$:

$$(a, a') \in E_{Y,X} \iff \forall b \in A_{1,Y \oplus Z}, \quad d \in A_{X \oplus Z, 1}, \quad (b \circ (a \oplus id_Z) \circ d, \quad b \circ (d' \oplus id_Z) \circ d) \in E_{1,1}.$$
We have the following bijection:

\[ \{ \text{tame equivalence ideals} \} \leftrightarrow \{ \mathcal{E} \subseteq A_{1,1} \times A_{1,1} \text{ equivalence relation such that} \}
\]

\[
\text{for } (a_i, a_i') \in \mathcal{E}^t, b \in A_{1,1}, d \in A_{1,1} \implies (b \circ (\oplus a_i) \circ d, b \circ (\oplus a_i') \circ d) \in \mathcal{E}
\]

For \( A \in \mathbb{R}^t \) take also \((a, a') \in \mathcal{E} \implies (a', (a')^t) \in \mathcal{E} \).

**Definition A.2.3**

\( a \subseteq A_{1,1} \) is an ideal if the following property hold:

\[ a_j \in a, b \in A_{1,1}, d \in A_{1,1} \implies b \circ (\oplus a_j) \circ d \in a \]

and is a t-ideal if also:

\[ a \in a \implies a' \in a. \]

**Proposition A.2.4**

There exists a Galois connection of,

\[ \{ a \subseteq A_{1,1} \text{ ideal} \} \overset{Z}{\leftrightarrow} \{ \mathcal{E} \subseteq A_1 \times A_1 \text{ is tame equivalence ideal} \} \]

\[ \{ a \subseteq A_{1,1} \text{ t-ideal} \} \overset{E}{\leftrightarrow} \{ \mathcal{E} \subseteq A_1 \times A_1 \text{ is tame t-equivalence ideal} \} \]

where

\[ Z(\mathcal{E}) = \{ a \in A_{1,1}, \ (a, 0) \in \mathcal{E} \} \]

\[ E(a) = \bigcap_{(a,0) \in \mathcal{E}} \mathcal{E} \]
Chapter 2

Examples of \( \mathbb{F} \)-Rings.

2.1 Rings

Definition 2.1.1

A "Rig" is a ring without negatives. Thus a Rig is a set with two associative operations \((+, \cdot)\), with units 0 and 1, addition \(+\) being commutative, and multiplication distributive over addition. A morphism between Rigs \(A \to B\) is a map \(\varphi : A \to B\) which preserves operations and units. Thus we have a category: \(\text{Rigs}\). We let \(\text{CRigs} \subseteq \text{Rigs}\) denote the full subcategory of commutative Rigs, i.e. where \(x \cdot y = y \cdot x\). We let \(\text{Rigs}^t\) denote the category of Rigs with involution, its objects are rigs \(R\) with involution:

\[
(\ )^t : R \to R
\]

\[
x^{tt} = x
\]

\[
(x \cdot y)^t = y^t \cdot x^t
\]

\[
(x + y)^t = x^t + y^t
\]

\[
1^t = 1
\]

\[
0^t = 0.
\]

and the morphisms are morphisms of \(\text{Rigs}\) that preserve the involution. Note that the identity is always an involution on a commutative Rig, and so we have a diagram of categories and functors

\[
\begin{array}{c}
\text{Rigs}^t \\
\text{\uparrow}
\end{array}
\]
Chapter 2. Examples of \( \mathbb{F} \)-Rings.

We shall write \( \mathbb{N} \) (resp. \( [0, \infty)^{(1)} \)) for the rigs of natural numbers (resp. non-negative reals) with the usual operation of multiplication \( \bullet \) and addition \( + \). We shall write \( \mathbb{N}^0 \) (resp. \( [0, \infty)^{(0)} \)) for the "frozen" rigs where \( x + y := \text{Max}\{x, y\} \). Moreover, for \( p \geq 1 \) (or \( 1/p \in (0, 1] \)), we write \( [0, \infty)^{(1/p)} \) for the rig of non-negative reals with \( x + y := (x^p + y^p)^{(1/p)} \), and with the usual multiplication \( \bullet \).

Note that the rigs \( [0, \infty)^{(\sigma)} \) interpolate continuously between the frozen \( (\sigma = 0) \) and the usual \( (\sigma = 1) \) reals, and that for \( \sigma \in (0, 1] \) they are all isomorphic via \( \Phi_{\sigma_1} : [0, \infty)^{(\sigma_1)} \rightarrow [0, \infty)^{(\sigma_2)} \), \( \Phi_{\sigma_2}(x) = x^{\sigma_2/\sigma_1} \). The multiplicative group of positive reals act as automorphisms of the frozen rig \( [0, \infty)^{(0)} \) via \( x \mapsto \Phi_{\sigma}(x) = x^\sigma \). For any \( q \in (0, 1) \), we have the sub-rigs \( M_q = \{0\} \cup q\mathbb{N} \subseteq [0, 1]^{(0)} \subseteq [0, \infty)^{(0)} \), and \( N_q = \{0\} \cup q\mathbb{Z} \subseteq [0, \infty)^{(0)} \), and the multiplicative monoid pf positive natural numbers act as endomorphisms of \( M_q \) and \( N_q \) via \( q^1 \mapsto \Phi_{n}(q^1) = q^{jn} \).

Definition 2.1.2

Let \( R \in \mathcal{R}\text{rig} \). Define \( R^{+/−} \) by

\[
R^{+/−} \equiv R \times R,
\]

\[
(m_+, m_-) + (m_+, m_-) = (m_+ + m_+, m_- + m_-),
\]

\[
(m_+, m_-) \cdot (m_+, m_-) = (m_+ \cdot m_+ + m_- \cdot m_-, m_+ \cdot m_- + m_- \cdot m_+).
\]

There is an equivalence relation \( \sim \) on \( R^{+/−} \):

\[
(m_+, m_-) \sim (m_+, m_-) \iff n_+ + m_- + r = m_+ + n_- + r, \text{ some } r \in R.
\]

such that we have,

\[
a \sim a', b \sim b' \implies a + b \sim a' + b',
\]

\[
a \cdot b \sim a' \cdot b'.
\]

We have the sequence of \( \mathcal{R}\text{rig} \) homomorphisms,

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & R^{+/−} / \sim \\
\downarrow & & \downarrow \\
K(R) & & K(R)
\end{array}
\]

\[
\begin{array}{ccc}
r \mapsto (r, 0) & & \\
\downarrow & & \\
(n_+, n_-) & \xrightarrow{\sim} & n_+ - n_- := (n_+, n_-)/ \sim .
\end{array}
\]

The rig \( K(R) \) is a ring, and for any ring \( A \),

\[
\mathcal{R}\text{ring}(K(R), A) = \mathcal{R}\text{rig}(R, A).
\]

If \( R \) has involution (resp. commutative), than \( R^{+/−} \) and \( K(R) \) have involution (resp. are commutative). We get adjoint functors,

\[
\begin{array}{cccc}
\mathcal{C}\text{ring} & \xrightarrow{K} & \mathcal{R}\text{rig} & \xrightarrow{\mathcal{C}} \mathcal{C}\text{ring} \\
\xrightarrow{\mathcal{C}\text{ring}} & & \xrightarrow{\mathcal{C}} & \\
\mathcal{R}\text{ring} & \xrightarrow{K} & \mathcal{R}\text{rig} & \xrightarrow{\mathcal{C}} \mathcal{R}\text{ring}
\end{array}
\]
CHAPTER 2. EXAMPLES OF $\mathbb{F}$-RINGS.

Definition 2.1.3

For any Rig $A$, let $A \cdot X = A^X$ be the free $A$-module with basis $X$, and define:

$$\mathbb{F}(A)_{Y,X} = \text{Hom}_A(A \cdot X, A \cdot Y) = Y \times X$$

- matrices with values in $A$.

The composition $\circ$ is matrix multiplication, and $\bigoplus$ is the direct sum of matrices. Clearly $\mathbb{F}(A)$ is an $\mathbb{F}$-Ring. Note that a morphism of Rigs $\varphi : A \to B$, induces a map of $\mathbb{F}$-Rings

$$F(\varphi) : \mathbb{F}(A) \to \mathbb{F}(B),$$

hence we have a functor

$$F( ) : \mathcal{Rigs} \to \mathcal{FR}.$$  

Note that if $A \in \mathcal{Rigs}^t$ has involution, than $\mathbb{F}(A) \in \mathcal{FR}^t$ also has involution: for $a = (a_{y,x}) \in \mathbb{F}(A)_{Y,X}$ we have $a^t \in \mathbb{F}(A)_{X,Y}$, $(a^t)_{x,y} = (a_{y,x})^t$, and it satisfies $(a \circ b)^t = b^t \circ a^t$. Thus we have a functor $F( ) : \mathcal{Rigs}^t \to \mathcal{FR}^t$. Note that if $A \in CRig$ is commutative, than $\mathbb{F}(A)$ is totally-commutative. Thus we have the diagram of categories and functors

$$\begin{array}{ccc}
\mathcal{Rigs}^t & \xrightarrow{F( )} & \mathcal{FR}^t \\
\downarrow & & \downarrow \\
\mathcal{Rigs} & \xrightarrow{F( )} & \mathcal{FR} \\
\downarrow & & \downarrow \\
\mathcal{CRigs} & \xrightarrow{F( )} & \mathcal{FR}_{\text{tot-com.}}
\end{array}$$ 

(2.1.3)

Note that $\mathbb{F}(A)$ is always a matrix- $\mathbb{F}$-Ring.

Moreover let $\varphi : \mathbb{F}(A) \to \mathbb{F}(B)$ be a map of $\mathbb{F}$-Rings. For $a \in \mathbb{F}(A)_{Y,X}$ write

$$a_{y,x} = j_y^t \circ a \circ j_x \in A = \mathbb{F}(A)_{[1],[1]},$$

for its matrix coefficients.

Since $\varphi$ is a functor over $\mathbb{F}$, and $j_y^t, j_x \in \mathbb{F}$, we have $\varphi(a)_{y,x} = \varphi(a_{y,x})$ and $\varphi$ is determined by $\varphi : A = \mathbb{F}(A)_{[1],[1]} \to B = \mathbb{F}(B)_{[1],[1]}$. This map is multiplicative: $\varphi(a_1 \cdot a_2) = \varphi(a_1) \cdot \varphi(a_2)$, $\varphi(1) = 1$, and moreover it is additive:

$$\varphi(a_1 + a_2) = \varphi \left[ (a_1, a_2) \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = (\varphi(a_1), \varphi(a_2)) \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \varphi(a_1) + \varphi(a_2)$$

(2.1.5)

Thus the functor $F( )$ is fully faithful.
Definition 2.1.4

Define,

\[(\mathbb{F}R^{(t)})^{Add} \equiv \text{commutative monoid objects in } \mathbb{F}R^{(t)},\]

That is, \(A \in (\mathbb{F}R^{(t)})^{Add}\) is the same as an addition map \(A_{Y,X} \times A_{Y,X} \rightarrow A_{Y,X}\) which is associative, commutative, has a unit element \(0_{Y,X}\), and satisfies,

\[a \circ (b + b') = a \circ b + a \circ b',\quad (a + a') \circ b = a \circ b + a' \circ b,\quad (\text{resp. } (a + b)^t = a^t + b^t).\]

The category of abelian group objects is the full subcategory \((\mathbb{F}R^{(t)})^{Ab} \subseteq (\mathbb{F}R^{(t)})^{Add}\).

We have \(A \in (\mathbb{F}R^{(t)})^{Ab} \iff A_{Y,X} \in Ab, \forall X, Y \in \mathbb{F}\).

For a rig (resp. ring) \(R\), the \(\mathbb{F}\)-Ring \(\mathbb{F}(R)\) is in \((\mathbb{F}R)^{Add}\), (resp. \((\mathbb{F}R)^{Ab}\)), and we have a similar diagram as before,

\[
\begin{array}{ccc}
C\text{Rigs} & \xrightarrow{\mathbb{F}(\_)} & (\mathbb{F}R^{(t)})^{Add} \\
\downarrow & & \downarrow \\
\text{Rigs} & \xrightarrow{\mathbb{F}(\_)} & (\mathbb{F}R)^{Add} \\
\downarrow & & \downarrow \\
C\text{Rings} & \xrightarrow{\mathbb{F}(\_)} & (\mathbb{F}R_{tot-com.})^{Add}
\end{array}
\quad \begin{array}{ccc}
\text{Rigs} & \xrightarrow{\mathbb{F}(\_)} & (\mathbb{F}R^{(t)})^{Add} \\
\downarrow & & \downarrow \\
\text{Rings} & \xrightarrow{\mathbb{F}(\_)} & (\mathbb{F}R)^{Add} \\
\downarrow & & \downarrow \\
C\text{Rings} & \xrightarrow{\mathbb{F}(\_)} & (\mathbb{F}R_{tot-com.})^{Ab}
\end{array}
\]

(2.1.6)

2.2 Monoids

Definition 2.2.1

Let \(M\) be a monoid with a unit \(1\) and a zero element \(0\). Thus we have an associative operation

\[M \times M \rightarrow M,\quad (a, b) \mapsto a \cdot b,
\]

\[a \cdot (b \cdot c) = (a \cdot b) \cdot c\] (2.2.1)

and \(1 \in M\) is the (unique) element such that

\[1 \cdot a = a = 1, \quad a \in M,\] (2.2.2)

and \(0 \in M\) is the (unique) element such that

\[0 \cdot a = a \cdot 0 = 0, \quad a \in M.\] (2.2.3)

Let \(\mathbb{F}\{M\}\) denote the \(\mathbb{F}\)-Ring with \(\mathbb{F}\{M\}_{Y,X}\) the \(Y \times X\) matrices with values in \(M\) with at most one non-zero entry in every row and column. Note that this is indeed an \(\mathbb{F}\)-Ring with the usual "multiplication" of matrices \(\circ\) (there is no addition involved - only multiplication in \(M\)) and direct sum \(\oplus\).
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Denoting by $\mathcal{M}on$ the category of monoids with unit and zero elements, and with maps respecting the operation and the elements 0, 1, the above construction yields a functor

$$\mathcal{M}on \to \mathbb{F}R, \ M \mapsto \mathbb{F}\{M\}. \quad (2.2.4)$$

This is the functor left-adjoint to the functor

$$\mathbb{F}R \to \mathcal{M}on, \ A \mapsto A_{[1],[1]} : \quad (2.2.5)$$

$$\mathbb{F}\{M\}, A = \mathcal{M}on(M, A_{[1],[1]}). \quad (2.2.6)$$

As a particular example, take $M = M_q$ to be the free monoid (with zero) generated by one element $q$,

$$M_q = q^n \cup \{0\}. \quad (2.2.7)$$

Then

$$\mathbb{F}\{M_q\}, A = A_{[1],[1]}. \quad (2.2.8)$$

Denote by $\mathcal{M}on^t$ the category of monoids with involution (i.e. the objects are monoids $M \in \mathcal{M}on$, with involution $(\ )^t : M \to M, \ x^t = x, \ (x \cdot y)^t = y^t \cdot x^t, \ 1^t = 1, \ 0^t = 0$, and the morphisms respect the involutions). We have an involution on $\mathbb{F}\{M\}$ for $M \in \mathcal{M}on^t$, so that we have a functor $\mathbb{F}\{} : \mathcal{M}on^t \to \mathbb{F}R^t$. Denote by $C\mathcal{M}on \subseteq \mathcal{M}on$ the full subcategory of commutative monoids (i.e. where $x \cdot y = y \cdot x$). For $M \in C\mathcal{M}on, \mathbb{F}\{M\}$ is totally commutative. Thus we have the diagram

$$\begin{array}{ccc}
\mathcal{M}on^t & \xrightarrow{\mathbb{F}\{\}} & \mathbb{F}R^t \\
\downarrow & & \downarrow \\
\mathcal{M}on & \xrightarrow{\mathbb{F}\{\}} & \mathbb{F}R \\
\uparrow & & \uparrow \\
C\mathcal{M}on & \xrightarrow{\mathbb{F}\{\}} & \mathbb{F}R_{\text{tot-com}}.
\end{array} \quad (2.2.9)
$$

Note that $\mathbb{F}\{M\}$ is always a matrix $\mathbb{F}$-Ring, and the functors $\mathbb{F}\{}$ are full and faithful.

2.3 $\textbf{Set}, \textbf{Set}^{op} \subseteq \textbf{Rel} \subseteq \mathbb{F}(\mathbb{N}^0)$.

Definition 2.3.1

Let $\textbf{Set}$ denote the $\mathbb{F}$-Ring of sets. The objects of $\textbf{Set}$ are the finite sets of $\mathbb{F}$, and we let $\textbf{Set}_{Y,X}$ be the partially defined maps of sets from $X$ to $Y$

$$\textbf{Set}_{Y,X} = \textbf{Set}_*(X,Y) = \textbf{Set}_0(X \cup \{0\}, Y \cup \{0\}). \quad (2.3.1)$$
CHAPTER 2. EXAMPLES OF $\mathbb{F}$-RINGS.

We can view the elements of $\text{Set}_{Y,X}$ as $Y \times X$-matrices with values in $\{0,1\}$, such that every column has at most one 1:

$$f \leftrightarrow (f)_{y,x} \quad \text{with} \quad (f)_{y,x} = \begin{cases} 1 & y = f(x) \\ 0 & \text{otherwise} \end{cases} \quad (2.3.2)$$

Then composition $\circ$ corresponds to matrix multiplication; The disjoint union $\oplus$ correspond to direct sum of matrices. These make $\text{Set}$ into an $\mathbb{F}$-Ring (with no involution), which is matrix and totally-commutative.

We have the opposite $\mathbb{F}$-Ring $\text{Set}^{\text{op}}$ with

$$\text{Set}^{\text{op}}_{Y,X} = \text{Set}_{X,Y}. \quad (2.3.3)$$

We have the $\mathbb{F}$-Ring of relations $\text{Rel}$ that contains both $\text{Set}$ and $\text{Set}^{\text{op}}$, with

$$\text{Rel}_{Y,X} = \{ F \subseteq Y \times X \text{ a subset} \} := \{0,1\}^{Y \times X}. \quad (2.3.4)$$

The composition of $F \in \text{Rel}_{Y,X}$ and $G \in \text{Rel}_{Z,Y}$ is given by

$$G \circ F = \{ (z,x) \in Z \times X | \exists y \in Y \text{ with } (z,y) \in G, (y,x) \in F \}, \quad (2.3.5)$$

and $G \circ F \in \text{Rel}_{Z,X}$.

When we view $G,F$ as $\{0,1\}$-matrices, this composition correspond to "matrix - multiplication" where we replace addition by $\text{Max}\{x,y\}$.

The sum $F_0 \oplus F_1 \in \text{Rel}_{Y_0 \oplus Y_1,X_0 \oplus X_1}$ of $F_i \in \text{Rel}_{Y_i,X_i}$ is given by the disjoint union of $F_0$ and $F_1$,

$$F_0 \oplus F_1 = \{(x,i), (y,i))|(x,y) \in F_i\} \text{ or by direct sum of matrices}. \quad (2.3.6)$$

Thus $\text{Rel} = \mathbb{F}(\{0,1\})$ is the $\mathbb{F}$-Ring with involution associated to the rig $\{0,1\}$ with usual multiplication $\bullet$, and $i + j = \text{Max}\{i,j\}$ as addition.

Thus $\text{Rel}$ is totally-commutative, matrix $\mathbb{F}$-Ring with involution. The embedding of rigs $\{0,1\} \hookrightarrow \mathbb{N}^0$ gives an embedding of $\mathbb{F}$-Rings $\text{Rel} = \mathbb{F}(\{0,1\}) \hookrightarrow \mathbb{F}(\mathbb{N}^0)$.

2.4 Real primes

Let $\eta : \mathbb{k} \hookrightarrow \mathbb{C}$ be an embedding of the rig $\mathbb{k}$ into the complex numbers. We have an injection of $\mathbb{F}$-Rings $\mathbb{F}(\mathbb{k}) \hookrightarrow \mathbb{F}(\mathbb{C})$. For $X \in \mathbb{F}$, let $\mathbb{k}^X = \mathbb{F}(\mathbb{k})_{X,[1]}$ denote the free $\mathbb{k}$-module over $X$. Thus for $a = (a_x) \in \mathbb{k}^X$, and $p \in [1, \infty]$, we have the vector $p$- norm:

$$|a|_p = \left( \sum_{x \in X} |\eta(a_x)|^p \right)^{1/p}, \quad p \in [1, \infty) \quad (2.4.1)$$
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$$|a|_{\infty} = \max_{x \in X} |\eta(a_x)|$$

and for $a = (a_{y,x}) \in k^{Y \times X}$ we have its operator $p$-norm:

$$\|a\|_p = \sup_{b \in k^{X}, |b|_p \leq 1} \left\{ |a \circ b|_p \right\}$$

(2.4.3)

e.g.,

$$\left\| (1, \ldots, 1) \right\|_p = \sup_{|\xi_1| + \cdots + |\xi_n| \leq 1} |\xi_1 + \cdots + \xi_n| = n^{1-\frac{1}{p}} \equiv n^{1/p'}.$$  

(2.4.4)

$$\left\| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|_p = \sup_{|\xi| \leq 1} \left( \sum_{i=1}^{n} |\xi|^p \right)^{1/p} = n^{1/p}.$$

(2.4.5)

$$\left\| (1, \ldots, 1) \right\|_p \cdot \left\| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|_p = n^{1/p'} \cdot n^{1/p} = n.$$

(2.4.6)

**Definition 2.4.1**

Define the sub-$\mathbb{F}$-Ring $O_{k,\eta}^{1/p} \subseteq \mathbb{F}(k)$ as follows:

$$(O_{k,\eta}^{1/p})_{Y,X} = \{ a \in \mathbb{F}(k)_{Y,X} = k^{Y \times X}, \|a\|_p \leq 1 \}. \quad (2.4.7)$$

As a sub-$\mathbb{F}$-Ring of $\mathbb{F}(k)$, the $\mathbb{F}$-Rings $O_{k,\eta}^\sigma$, $\sigma \in [0,1]$, are matrix and totally-commutative.

Note that in general, $O_{k,\eta}^\sigma$ has no involution, in fact we have(by Hölder’s inequality: the dual of the $p$-norm is the $p'$-norm, where $\frac{1}{p'} = 1 - \frac{1}{p}$):

$$(O_{k,\eta}^\sigma)^{op} \cong O_{k,\eta}^{1-\sigma} \quad \text{for} \quad 0 \leq \sigma \leq 1. \quad (2.4.8)$$

**Definition 2.4.2**

Let $O_{k,\eta} := O_{k,\eta}^{1/2} \subseteq \mathbb{F}(k)$. It is a sub $\mathbb{F}$-Ring of $\mathbb{F}(k)$, hence totally-commutative, and matrix, and at $\sigma = \frac{1}{2}$ (i.e. using the $L_2$-norm) it has involution!

**Definition 2.4.3**

Define $\mathbb{F}_{k,\eta} \in FR^t$,

$$(\mathbb{F}_{k,\eta})_{Y,X} = \{ f : D(f) \to I(f), D(f) \subseteq k^X, I(f) \subseteq k^Y \text{ are } k \text{-subspaces, and}, \quad f \text{ is } k\text{-linear and an isometry} : |f(v)|_2 = |v|_2 \} \quad (2.4.9)$$
CHAPTER 2. EXAMPLES OF \( \mathbb{F} \)-RINGS.

Note that when the composition \( g \circ f \) is defined we have:

\[
D(g \circ f) = f^{-1}(D(g) \cap I(f)), \quad I(g \circ f) = g(D(g) \cap I(f)). \tag{2.4.10}
\]

There is a surjective homomorphism of \( \mathbb{F} \)-Rings with involution, \( \phi : \mathcal{O}_{k,\eta} \to \mathbb{F}_{k,\eta} \).

For \( a \in (\mathcal{O}_{k,\eta})_{Y,X} \subset \mathbb{C}^{Y \times X} \), define \( \Delta_X = \pi' \circ a, \Delta_Y = a \circ \pi' \), and let \( V_X[\lambda] \) (resp. \( V_Y[\lambda] \)) denote the \( \lambda \)-eigenspace of \( \Delta_X \) (resp. \( \Delta_Y \)).

The operator \( \Delta_X \) (resp. \( \Delta_Y \)) is non-negative and we have the spectral decomposition \( \mathbb{C}^X = \bigoplus V_X[\lambda_i] \), \( \Delta_X = \bigoplus \lambda_i \cdot id_{V_X[\lambda_i]} \) (resp. \( \mathbb{C}^Y = \bigoplus V_Y[\lambda_i] \), \( \Delta_Y = \bigoplus \lambda_i \cdot id_{V_Y[\lambda_i]} \)), with eigenvalues \( \lambda_i \in [0,1] \), and the non-zero eigenvalues are the same for \( \Delta_X \) and \( \Delta_Y \) including multiplicities.

For \( \lambda > 0 \) we have isomorphisms:

\[
\xymatrix{ V_Y[\lambda] \ar@{~>}[r] & V_X[\lambda] }
\]

For \( \lambda = 1 \) we have an isometry:

\[
\phi(a) = \{ a : V_X[1] \xrightarrow{\sim} V_Y[1] \} \subset (\mathbb{F}_{k,\eta})_{Y,X}. \tag{2.4.11}
\]

This defines the homomorphism \( \phi : \mathcal{O}_{k,\eta} \to \mathbb{F}_{k,\eta} \). Note that \( \mathbb{F}_{k,\eta} \) is (as quotient of \( \mathcal{O}_{k,\eta} \)) totally - commutative. It is our first example of an \( \mathbb{F} \)-Ring which is not a matrix:

Indeed, any vector \( x = (x_1, \ldots, x_n) \in (\mathbb{F}_{k,\eta})_{1,n} \), with \( \sum_{i=1}^n |\eta(x_i)|^2 = 1 \), but with \( |\eta(x_i)| < 1 \) for \( i = 1, \ldots, n \), is non-zero, but all its matrix coefficients are zero. But note that \( \mathbb{F}_{k,\eta} \) is tame.

2.5 Valuation \( \mathbb{F} \)-Rings: Ostrowski theorem

Definition 2.5.1

A commutative \( \mathbb{F} \)-Ring with involution \( K \subset \mathbb{C}^{\mathbb{F}^1} \) is called an \( \mathbb{F} \)-field if every non-zero \( a \in K_{[1],[1]} \setminus \{0\} \) is invertible (have \( a^{-1} \in K_{[1],[1]} \) with \( a^{-1} a^{-1} = a^{-1} a = 1 = id_{[1]} \)).

A sub-\( \mathbb{F} \)-Ring with involution \( B \subset K \) is called full if

1. for every \( a \in K_{[X,Y]} \), \( X,Y \in \mathbb{F} \), there exists a non-zero element \( d \in B_{[1],[1]} \setminus \{0\} \) with \( d \cdot a = (\oplus Y) \circ a = a \circ (\oplus X) \in B_{Y,X} \).

(This means that \( K \) is the fraction-field of the domain \( B \), i.e. \( K = B_{[0]} = (B_{[1],[1]} \setminus \{0\})^{-1} \). The localization of \( B \) at the prime \( (0) \), cf. §3.3).

We will say that \( B \) is tame in \( K \) if for \( X,Y \in \mathbb{F} \) we have an equality:

2. \( B_{Y,X} = \{ a \in K_{Y,X} \mid \text{for all } b \in B_{1,Y} \text{, } d \in B_{X,1} \text{, } b \circ a \circ d \in B_{1,1} \} \).
A sub-$\mathcal{F}$-Ring with involution $B$, full and tame in $K$, will be called a valuation-$\mathcal{F}$-subring of $K$ if for every non-zero $a \in K_{1,1}\setminus\{0\}$,

3. either $a \in B_{1,1}$ or $a^{-1} \in B_{1,1}$.

Given $\mathcal{F}$-fields $k \subseteq K$, we denote by $\text{Val}(K/k)$ the set of all valuation-$\mathcal{F}$-subrings $B \subseteq K$, such that $B \supseteq k$.

Let $B$ be a valuation-$\mathcal{F}$-subring of an $\mathcal{F}$-field $K$. The group of units:

$$B^* = GL_1(B) = \{a \in B_{1,1} \mid \exists a^{-1} \in B_{1,1}, a \circ a^{-1} = 1\}$$

is a subgroup of

$$K^* = GL_1(K) = K_{1,1}\setminus\{0\}.$$

The quotient group $\Gamma = K^*/B^*$ is ordered: $|x| \leq |y| \iff x \cdot y^{-1} \in B_{1,1}$, where $|x| = x \cdot B^*$ is the quotient map $|| : K^* \rightarrow \Gamma$. We extend this quotient map by $|0| = 0$, to the map

$$| : K_{1,1} \rightarrow K_{1,1}/B^* = \Gamma \cup \{0\}$$

satisfying

(1)

$$|x| = 0 \iff x = 0$$

$$|x_1 \cdot x_2| = |x_1| \cdot |x_2|$$

$$|1| = 1 \text{ (= unit of )}.$$

We can embed $\Gamma$ in a complete ordered abelian group $\tilde{\Gamma}$, (e.g. $\tilde{\Gamma}$ = all dedekind subsets $D \subseteq \Gamma$), so that for every subset $Y \subseteq \tilde{\Gamma}$ which is bounded above (resp. below) there is a unique least upper bound $\sup Y \in \tilde{\Gamma}$ (resp. maximal lower bound $\inf Y \in \tilde{\Gamma}$). We can then define for $X, Y \in \mathcal{F}$ the two maps

$$|y|_{Y,X} : \Gamma \rightarrow \tilde{\Gamma} \cup \{0\}$$

(i) $|y|_{Y,X} = \sup\{|b \circ y \circ b'|, b \in B_{1,Y}, b' \in B_{X,1}\}$

(ii) $|y|_{Y,X}' = \inf\{|d^{-1}|, d \in K^*, d \cdot y \in B_{Y,X}\}$

We define:

$$|y|_{Y,X} = |y|_{Y,X}'.$$

This shows that the set in (i) (resp. (ii)) is bounded above (resp. below), and we have the inequality: $|y|_{Y,X} \leq |y|_{Y,X}'$.

Conversely, given $y \in K_{Y,X}$, if $d^{-1} \in K^*$ is such that $|d|_{1,1}^{-1} \geq |y|_{Y,X}$, that is: $|d|_{1,1}^{-1} \geq |b \circ y \circ b'|_{1,1}$ for all $b \in B_{1,Y}, b' \in B_{X,1}$, then $b \circ (d \cdot y) \circ b' \in B_{1,1}$ for all $b \in B_{1,Y}, b' \in B_{X,1}$, and (since $B$ is tame in $K$) this imply $d \cdot y \in B_{Y,X}$, hence $|d|_{1,1}^{-1} \geq |y|_{Y,X}'$, giving the reverse inequality: $|y|_{Y,X} \geq |y|_{Y,X}'$.\qed
(III) **Claim:**

(i) $|a \cdot a'| \leq |a| \cdot |a'|$

(ii) $|a_0 \oplus a_1| = \max\{|a_0|, |a_1|\}$

(iii) $|a'| = |a|$

**Proof.** (i): If $d, d' \in K^*$ are such that $|d|^{-1} \geq |a|, |d'|^{-1} \geq |a'|$, than $d \cdot a, d' \cdot a'$ are in $B$, so $(d \circ d') \cdot a \circ a' = (d \cdot a) \circ (d' \circ a')$ is in $B$, and $|d|^{-1}|d'|^{-1} \geq |a \circ a'|$. 

(ii): If $d_0, d_1 \in K^*$ are such that $d_0 \cdot a_0, d_1 \cdot a_1$ are in $B$, and if $|d_j| \leq |d_{j-1}|$ than $d_j \cdot a_0, d_j \cdot a_1$ are in $B$, so $d_j \cdot (a_0 \oplus a_1) = (d_j \cdot a_0) \oplus (d_j \cdot a_1)$ is in $B$, so $|a_0 \oplus a_1| \leq |d_j|^{-1} = \max\{|d_0|^{-1}, |d_1|^{-1}\}$ Taking the infimum over all such $d_0, d_1$ we get $|a_0 \oplus a_1| \leq \max\{|a_0|, |a_1|\} = |a_0|$, say. The inverse inequality follows from (i) since $a_{j_0} = f' \circ (a_0 \oplus a_1) \circ f$, with $f', f$ arrows of $F \subseteq B$, so $|a_{j_0}| \leq |f'| \cdot |a_0 \oplus a_1| \cdot |f| \leq |a_0 \oplus a_1|$. 

(iii) This follows since we are assuming $B \subseteq K$ to be stable under the involution, so $a \in B_{Y, X}$ if and only if $a^{t} \in B_{X, Y}$. $\square$

Let $\Gamma$ be a complete ordered abelian group, written multiplicatively, and form the ordered abelian monoid $\Gamma \cup \{0\}$, with $0 \cdot x = 0, 0 < x$ for all $x \in \Gamma$. Given a collection of mappings

$$|_{Y, X} : K_{Y, X} \to \Gamma \cup \{0\}$$

satisfying (III), with $|_{1, 1}$ satisfying (I), the subsets

$$B_{Y, X} = \{a \in K_{Y, X}, |a| \leq 1\}$$

form a sub-$F$-ring with involution $B \subseteq K$. If we have the equalities for $y \in K_{Y, X}$

\[ (II) \quad \begin{array}{l}
(i) \quad |y|_{Y, X} = \sup\{|b \circ y \circ b'|_{1, 1} : |b|_{1, Y}, |b'|_{X, 1} \leq 1\} \\
(ii) \quad = \inf\{|d|_{1, 1}^{-1} : d \in K^*, |d \cdot y|_{Y, X} \leq 1\}
\end{array} \]

then $B$ is full (by II ii), and tame in $K$ (by II i), and it follows that $B$ is a valuation-$F$-subring of $K$.

**Theorem Ostrowski I**

$$Val(F(Q)/F\{\pm 1\}) = \{F(Q), F(Z_{p}), p \text{ a finite prime, } O_{Q, \eta}\},$$

where $O_{Q, \eta}$ is the real prime.

**Proof.** cf. Appendix $\square$
Theorem Ostrowski II

For a number field $K$,

$$\text{Val}(\mathbb{F}(K)/\mathbb{F}\{\mu_K\}) = \{\mathbb{F}(K); \mathbb{F}(\mathcal{O}_{K,p}); \mathcal{O}_{K,\eta}\}$$

with $\mathcal{O}_{K,p}$ the localization of the ring of integers $\mathcal{O}_K$ at prime ideals $p \subseteq \mathcal{O}_K$, and with $\mathcal{O}_{K,\eta}$, the "real primes" of $K$, $\eta$ varies over the embeddings $\eta : K \hookrightarrow \mathbb{C}$ modulo conjugation.

Proof. cf. Appendix B. \hfill $\square$

Remark 2.5.1

Note that for any $\sigma \in [0, 1]$, the sub-$\mathbb{F}$-ring $\mathcal{O}_{K,\eta}^{(\sigma)} \subseteq \mathbb{F}(K)$ satisfies (1),(2),(3) of definition (2.5.1), i.e. it is full and tame valuation $\mathbb{F}$-subring. Alternatively, the operator $p = 1/\sigma$-norm satisfies (I),(II) and (IIIi),(IIIii). But only at $\sigma = \frac{1}{2}, p = 2$, we have an involution on $\mathcal{O}_{K,\eta}^{(\sigma)}$. Thus it is the presence of the involution that singles out the $L_2$-metric at the real primes.

2.6 Graphs

Definition 2.6.1

A graph $G$ is a pair of finite sets $(G_0, G_1)$ with two maps:

$$G = \left\{ G^1 \xrightarrow{\pi^0} G^0 \right\} \quad (2.6.1)$$

where $G^0$ - 'vertices', $G^1$ - 'edges'.

Given such a finite graph we get a category $\mathcal{C}G$: the objects of $\mathcal{C}G$ are the elements of $G_0$, and the arrows of $\mathcal{C}G$ are given by "paths":

$$\text{Ob}(\mathcal{C}G) \equiv G^0,$n

$$\mathcal{C}G(x, y) = \{e \equiv (e_1, \ldots, e_1) | e_i \in G^1, \pi^0(e_{j+1}) = \pi^1(e_j), \pi^1(e_i) = y, \pi^0(e_1) = x\}. \quad (2.6.2)$$

Definition 2.6.2

Given such a path $e = (e_1, \ldots, e_1) \in \mathcal{C}G(x, y)$ we shall say that $e$ "begins" at $x$, "ends" at $y$, and for a vertex $z \in G^0$, we say $e$ "goes through $z$" and we write:

$$z \in e \iff \exists k, z = \pi^1(e_k), i \in \{0, 1\}, \quad (2.6.3)$$

for an edge $e_0 \in G^1$, write:

$$e_0 \in e \iff \exists k, e_0 = e_k. \quad (2.6.4)$$
CHAPTER 2. EXAMPLES OF \( F \)-RINGS.

Note that for a vertex \( x \in G^0 \), we have
\[
id_x = \text{"empty path" at } x \in CG(x, x) \quad (2.6.5)
\]

Assume \( G \) has no loops: \( CG(x, x) = \{id_x\}, \forall x \in G^0 \).
\( \Rightarrow \) Every path can be extended to a maximal path.
\( \Rightarrow \) Every maximal path begins at \( In(G) = G^0\setminus\pi^1(G^1) \).
\( \Rightarrow \) Every maximal path ends at \( Out(G) = G^0\setminus\pi^0(G^1) \).

We denote by \( m(G) \) the set of all maximal paths.

We define an \( F \)-Ring with involution: \( Graph \in \mathbb{F}R^3 \).

\[
(Graph)_{Y,X} = \left\{ G = G^1 \xrightarrow{\pi^1} G^0, \text{no loops} \right\} /\text{isom.}
\]

(\( i.e. \) modulo isomorphisms of graphs that respects the embeddings \( i \) and \( o \).)

For \( G \in (Graph)_{Y,X} \) and \( X_0 \subseteq X \), define \( G[X_0] \):

\[
\begin{align*}
v & \in G^0, \exists e \in m(G), e \in e, \pi^0(e) \in X_0. \\
ev & \in G^1, \exists e \in m(G), e_0 \in e, \pi^0(e) \in X_0.
\end{align*}
\]

and for \( Y_0 \subseteq Y \), define \( [Y_0]G \):

\[
\begin{align*}
v & \in G^0, \exists e \in m(G), e \in e, \pi^1(e) \in Y_0. \\
ev & \in G^1, \exists e \in m(G), e_0 \in e, \pi^1(e) \in Y_0.
\end{align*}
\]

We also let,
\[
[Y_0][X_0] = [Y_0]G \cap G[X_0]. \quad (2.6.8)
\]

For \( G \in (Graph)_{Y,X} \), \( G' \in (Graph)_{Z,Y} \), let \( Y_0 = In(G) \cap Out(G') \subseteq Y \),

\[
(2.6.9)
\]
The operation of composition is defined by gluing \( G' \langle Y_0 \rangle \) and \([Y_0]G\) along \( Y_0\):
\[
G' \circ G = G' \langle Y_0 \rangle \sqcup \langle Y_0 \rangle G
\]
(2.6.10)

The sum in \( \text{Graph} \) is given by the disjoint union:
\[
G_i \in (\text{Graph})_{Y_i, X_i}, \quad G_0 \oplus G_1 = G_0 \sqcup G_1.
\]
(2.6.11)

The involution is given by reversing the directions of the edges of the graph:
\[
(G = G_1 \xrightarrow{\pi_0} G^0)^t = (G = G_1 \xrightarrow{\pi_1} G^0) : (\text{Graph})_{Y, X} \xrightarrow{\pi_1} (\text{Graph})_{X, Y}.
\]
(2.6.12)

E.g. the "discrete" graphs \( G \in (\text{Graph})_{Y, X} \), (i.e. \( G^1 = \emptyset \) and \( G \) is just a set with embeddings into \( X \) and into \( Y \)), give the elements of \( \mathbb{F} \):
\[
\mathbb{F}_{Y, X} = \{ G^0 \hookrightarrow X, \ G^0 \hookrightarrow Y\}/\text{isom} \subseteq (\text{Graph})_{Y, X}
\]
(2.6.13)

\( \text{Graph} \in \mathbb{F} \mathbb{R}^t \) is not even central, not matrix, but it is tame.

Note that we have a homomorphism of \( \mathbb{F} \)-Rings :
\[
\phi : \text{Graph} \rightarrow \mathbb{F}(N), \quad \phi(G)_{y, x} = \# \{ e \in m(G), \pi^0(e) = x, \pi^1(e) = y \}
\]
(2.6.15)

2.7 Free \( \mathbb{F} \)-Rings \( \mathbb{F}[\delta_{Y, X}] \)

We have an \( \mathbb{F} \)-Ring \( \mathbb{F}[\delta_{Y, X}] \in \mathbb{F} \mathbb{R} \), such that for any \( A \in \mathbb{F} \mathbb{R} \),
\[
\mathbb{F} \mathbb{R}(\mathbb{F}[\delta_{Y, X}], A) \equiv A_{Y, X}, \quad \varphi \mapsto \varphi(\delta_{Y, X}),
\]
(2.7.1)

and similarly, we have an \( \mathbb{F} \)-Rings with involution \( \mathbb{F}[\delta_{Y, X}, \delta'_{Y, X}] \in \mathbb{F} \mathbb{R}^t \), such that for any \( A \in \mathbb{F} \mathbb{R}^t \),
\[
\mathbb{F} \mathbb{R}^t(\mathbb{F}[\delta_{Y, X}, \delta'_Y], A) \equiv A_{Y, X}.
\]
(2.7.2)

The elements of \( \mathbb{F}[\delta_{Y, X}]_{W, Z} \) can be written as sequences of maps in \( \mathbb{F} \),
\[
(f_1, ..., f_j, ..., f_0), \text{ with } f_j \in \mathbb{F}(i_{j, j+1} \otimes X) \oplus V_{j+1} \oplus (i_{j, j+1} \otimes Y) \oplus V_j, \quad l > j > 0,
\]
\[
f_0 \in \mathbb{F}(i_{l, l+1} \otimes X) \oplus V_{l+1} \oplus (i_{l, l+1} \otimes Y) \oplus V_l,
\]
modulo certain identifications. Such a sequence represents the element
\[
f_l \circ \cdots \circ f_j \circ ((\oplus \delta_{Y, X}) \oplus id_{V_j}) \circ f_{j-1} \circ \cdots \circ f_1 \circ ((\oplus \delta_{Y, X}) \oplus id_{V_1}) \circ f_0.
\]
(2.7.3)
These elements can also be described as "\((Y, X)\)- marked graphs". The full \((Y, X)\)- graph is given by

\[
\delta_{Y,X} \equiv \left( Y \otimes X \mapright{\pi_0}^{\pi_1} Y \oplus X \right), \quad \begin{array}{l}
\pi_0(y, x) = x \\
\pi_1(y, x) = y.
\end{array}
\]  

(2.7.5)

\[
\delta_{Y,X} \in \text{Graph}_{Y,X}
\]

\[
\text{In}(\delta_{Y,X}) \equiv X
\]

\[
\text{Out}(\delta_{Y,X}) \equiv Y
\]

E.g.,

A \((Y, X)\)- marked graph from \(Z\) to \(W\) is given by a graph of the form

\[
G = (J \otimes Y \otimes X \oplus G^1 \mapright{\pi_0}^{\pi_1} (J \otimes Y) \oplus (J \otimes X) \oplus W_0 \oplus Z_0)
\]

(2.7.7)

with \(Z_0 \subseteq Z, W_0 \subseteq W, \pi^0(j, y, x) = (j, x), \pi^1(j, y, x) = (j, y)\), and \(\pi^0, \pi^1\) are injections on \(G^1:\)

\[
\pi^0 : G^1 \rightarrow \left( \bigoplus_{j \in J} Y \right) \oplus Z_0
\]

\[
\pi^1 : G^1 \rightarrow \left( \bigoplus_{j \in J} X \right) \oplus W_0
\]

(2.7.8)

we shall assume it has no loops and that for every \(j \in J\), there is \(y \in Y\), (resp. \(x \in X\)), and a path going from \((j, y)\) to \(W_0\), (resp. from \(Z_0\) to \((j, x)\)). Thus a \((Y, X)\)- marked graph is a graph that can be made out of a disjoint union of the full \((Y, X)\)- marked graphs \(\delta_{Y,X}\) (one copy for each \(j \in J\)), and some partial bijections. An isomorphism of such \((Y, X)\)- marked graph \(G = (J, G^1, \pi^0, \pi^1)\) and \(H = (I, H^1, \pi^0, \pi^1)\) is an isomorphism of graphs \(\phi : G \rightarrow H\), that is compatible with the maps into \(W\) and \(Z\), but is such that for some bijection

\[
\sigma : J \rightarrow I, \quad \begin{array}{l}
\phi(j, y, x) = (\sigma(j), y, x) \\
\phi(j, y) = (\sigma(j), y) \\
\phi(j, x) = (\sigma(j), x).
\end{array}
\]  

(2.7.9)
CHAPTER 2. EXAMPLES OF \( \mathcal{F} \)-RINGS.

The marked \((Y, X)\)-graph associated to a sequence \((f_1, \ldots, f_0)\) as above, \((2.7.3), (2.7.4)\), is obtained by taking \(J = \bigoplus_{j=1}^{l} I_j\), and adding an edge from \((i_1, y)\) to \((i_2, x)\) if and only if \((i_1, y) \in D(f_1), f_1(i_1, y) \in V_{i_1+1} \cap D(f_{i_1+1}), f_{i_1+1} \circ f_1(i_1, y) \in V_{i_1+2} \cap D(f_{i_1+2}), \ldots, f_{i_2-1} \circ \cdots \circ f_1(i_1, y) = (i_2, x)\) (and similarly for an element of \(Z\), resp. \(W\), instead of \((i_1, y)\), resp. \((i_2, x)\)). We can now describe \(\mathcal{F}[\delta_Y, X]_{W, Z}\) as \((Y, X)\) - marked - graphs from \(Z\) to \(W\) modulo isomorphism.

There is actually a "canonical form" (in fact two "dual" canonical forms) for \(G \in \mathcal{F}[\delta_Y, X]\). Thinking of \(G\) as a marked \((Y, X)\)-graph \(G = (J, G^1, \pi^0, \pi^1)\), let

\[
J_1 = \{ j \in J | \forall e \in G^1, \pi^0(e) = (j, y) \implies \pi^1(e) \in W \} \\
J_2 = \{ j \in J | \forall e \in G^1, \pi^0(e) = (j, y) \implies \pi^1(e) \in W \text{ or } \pi^1(e) = (j', x), j' \in J_1 \} \\
\vdots \\
J_k = \{ j \in J | \forall e \in G^1, \pi^0(e) = (j, y) \implies \pi^1(e) \in W \text{ or } \pi^1(e) = (j', x) \text{ with } j' \in \bigcup_{i=k}^{1} J_i \}
\]

Then the canonical form of \(G\) is given by the non-empty sets \(J_1, \ldots, J_k\), the finite sets \(\{u_{i,j}\}_{0 \leq i \leq j \leq k}\), and the embeddings

\[
(i) \quad \bigoplus_{j} u_{0,j} \hookrightarrow W, \quad \bigoplus_{j} u_{j,j} \hookrightarrow Z \\
(ii) \quad \bigoplus_{i \leq j} u_{i,j} \hookrightarrow J_{i_0} \otimes X \\
(iii) \quad \bigoplus_{i \leq j} u_{i,j_0} \hookrightarrow J_{j_0} \otimes X
\]

the embeddings in \((ii)\) and \((iii)\) are "dense" in the sense that for each \(j \in J_{i_0}\) (resp. \(j \in J_{j_0}\), there is an \(x \in X\) (resp. \(y \in Y\)), such that \((j, x)\) (resp. \((j, y)\)) is in the image.

**Remark 1**

There is a dual canonical form obtained by taking instead

\[
J_1 = \{ j \in J | \forall e \in G^1, \pi^1(e) = (j, x) \implies \pi^0(e) \in Z \} \\
J_2 = \{ j \in J | \forall e \in G^1, \pi^1(e) = (j, x) \implies \pi^0(e) \in Z \text{ or } \pi^0(e) = (j', y), j' \in J_1 \}
\]

etc.
Remark 2
Similarly, any element $G \in \mathbb{F}[\delta_{Y,X}, \delta^t_{Y,X}]_{W,Z}$ can be described as a graph made out of disjoint copies of $\delta_{Y,X}$ and of $\delta^t_{X,Y}$, and has canonical form

$$ (J_1, I_1, J_2, I_2, \ldots, J_k, I_k, u_{i,j}, 0 \leq i < j \leq k) $$ (2.7.13)

where now $J_i \oplus I_i \not= \emptyset$ for $i = 1, \ldots, k$, and the embeddings

$$ (i) \quad \bigoplus_j u_{0,j} \hookrightarrow W, \quad \bigoplus_j u_{j,j} \hookrightarrow Z $$

$$ (ii) \quad \bigoplus_{i \in \mathbb{J}} u_{i0,j} \hookrightarrow (J_{i0} \otimes X) \oplus (J_{i0} \otimes Y) $$

$$ (iii) \quad \bigoplus_{j \in \mathbb{J}} u_{i,j0} \hookrightarrow (J_{j0} \otimes Y) \oplus (J_{j0} \otimes X) $$ (2.7.14)

the embedding in (ii) and (iii) being dense (the image meets every copy of $X$ and $Y$).

Definition 2.7.1
When $G$ has such a canonical form we shall write $\deg G = k$.
Thus an element of degree 0 is just given by the set $u_{0,0}$ and the embeddings $u_{0,0} \hookrightarrow Z, u_{0,0} \hookrightarrow W$, i.e. it is just an element of $\mathbb{F}$:

$$ \mathbb{F}_{W,Z} = \{ G \in \mathbb{F}[\delta_{Y,X}]_{W,Z}, \ \deg G = 0 \} $$ (2.7.15)

When $\deg G = 1$, we have the embeddings

$$ u_{0,0} \oplus u_{0,1} \hookrightarrow W, \quad u_{0,0} \oplus u_{1,1} \hookrightarrow Z $$

$$ u_{0,1} \hookrightarrow \bigoplus_{J_1} Y, \quad u_{1,1} \hookrightarrow \bigoplus_{J_1} X $$ (2.7.16)

giving the elements

$$ f_{00} = (W \hookrightarrow u_{00} \hookrightarrow Z) \in \mathbb{F}_{W,Z} $$

$$ f_{01} = (W \hookrightarrow u_{01} \hookrightarrow \bigoplus_{J_1} Y) \in \mathbb{F}_{W,\bigoplus Y} $$

$$ f_{11} = (\bigoplus_{J_1} X \hookrightarrow u_{11} \hookrightarrow Z) \in \mathbb{F}_{\bigoplus X, Z} $$ (2.7.17)

and

$$ G = [f_{01} \circ (\bigoplus_{J_1} \delta_{Y,X}) \circ f_{11}] \oplus f_{00}. $$ (2.7.18)

In general we have:

$$ \deg(G_0 \oplus G_1) = \max\{\deg G_0, \deg G_1\} $$ (2.7.19)

$$ \deg(H \circ G) \leq \deg(H) + \deg(G) $$ (2.7.20)
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and for $\mathbb{F}[\delta_{Y,X}, \delta_{Y,X}^t]$:  
\[ \deg G^t = \deg G. \]  
(2.7.21)

More generally, given $C \in \mathbb{F}R$, and given $I \in \text{Set}/\mathbb{F} \times \mathbb{F}$, i.e. a set $I$ with a mapping $I \rightarrow \mathbb{F} \times \mathbb{F}$, $i \mapsto (Y_i, X_i)$, we have the $\mathbb{F}$-ring  
\[ C[\delta_I] \equiv C[\delta_{Y_i, X_i}; i \in I] := C \otimes \bigotimes_{i \in I} \mathbb{F}[\delta_{Y_i, X_i}], \]  
(2.7.22)

and we have the adjunction (with $U$ the forgetful functor):  
\[ C \downarrow \mathbb{F}R \quad \Downarrow \quad C \downarrow \mathbb{F}R \left(C[\delta_I], A\right) \equiv \text{Set}/\mathbb{F} \left(I, UA\right). \]  
(2.7.23)

Similarly, given $C \in \mathbb{F}R^t$, and given $I \in (\text{Set}/\mathbb{F} \times \mathbb{F})^t$, i.e. $I$ is a set with maps $d_i : I \rightarrow \mathbb{F}$, $i = 0, 1$, and an involution $I \rightarrow I$, $i \mapsto i^t$, $(i^t)^t = i$, $d_0 \circ (i^t) = d_1(i)$, $d_1(i^t) = d_0(i)$, we have the $\mathbb{F}$-Ring with involution  
\[ C[\delta_I, \delta_I^t] := C \otimes \bigotimes_{i \in I, i \sim i^t} C[\delta_{Y_i, X_i}, \delta_{Y_i, X_i}^t], \]  
(2.7.24)

and we have the adjunction  
\[ C \downarrow \mathbb{F}R^t \quad \Downarrow \quad C \downarrow \mathbb{F}R^t \left(C[\delta_I, \delta_I^t], A\right) \equiv (\text{Set}/\mathbb{F} \times \mathbb{F})^t \left(I, UA\right). \]  
(2.7.25)

2.8 \( \mathbb{F}[GL_X] \)

We have the functor $GL_X : \mathbb{F}R \rightarrow \text{Grps}$.  
\[ GL_X(A) = \{ a \in A_{X,X}, \exists a^{-1} \in A_{X,X} \ a \circ a^{-1} = a^{-1} \circ a = id_X \}. \]  
(2.8.1)

It is representable:  
\[ GL_X(A) = \mathbb{F}R(\mathbb{F}[GL_X], A) \]  
(2.8.2)

\[ \mathbb{F}[GL_X] = \mathbb{F}[\delta_{X,X}] \otimes \mathbb{F}[\delta_{X,X}^t] / \{ \delta_{X,X} \circ \delta_{X,X}^t \sim \delta_{X,X}^t \circ \delta_{X,X} \sim id_X \} \]  
(2.8.3)

The following structure exists on $\mathbb{F}[GL_X]$:  
\[ m^* : \mathbb{F}[GL_X] \rightarrow \mathbb{F}[GL_X] \otimes \mathbb{F}[GL_X] \quad \text{(co-multiplication)} \]  
\[ \delta_{X,X} \mapsto \delta_{X,X}^{(0)} \circ \delta_{X,X}^{(1)} \]  
\[ \delta_{X,X}^t \mapsto \delta_{X,X}^{(1)} \circ \delta_{X,X}^{(0)} \]  
(2.8.4)
2.9 The arithmetical surface: $\mathcal{L} = \mathbb{F}(\mathbb{N}) \otimes \mathbb{F}(\mathbb{N})$

Consider $\mathcal{L} = \mathbb{F}(\mathbb{N}) \otimes \mathbb{F}(\mathbb{N})$. As a particular example of (A.2.1), we have:

Note that the diagonal homomorphism $\text{diag} : \mathcal{L} = \mathbb{F}(\mathbb{N}) \otimes \mathbb{F}(\mathbb{N}) \rightarrow \mathbb{F}(\mathbb{N})$, factors through a surjection $d : \mathcal{L}^{\text{tot-com}} \rightarrow \mathbb{F}(\mathbb{N})$, since $\mathbb{F}(\mathbb{N})$ is totally commutative. We have the following

**Theorem 2.9.1**

1. The composition $d \circ f : \mathcal{L}^{1-\text{com}} \rightarrow \mathbb{F}(\mathbb{N})$ is an isomorphism (and hence both $d$ and $f$ are isomorphisms).
2. The composition $d \circ g : \mathcal{L}^{\text{com}} \rightarrow \mathbb{F}(\mathbb{N})$ is not an isomorphism.

**Proof.** (1) Note that the $\mathbb{F}$-Ring $\mathbb{F}(\mathbb{N})$ is generated by the elements $\sigma = (1, 1) \in \mathbb{F}(\mathbb{N})_{1, 1}$ and $\sigma^t = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \in \mathbb{F}(\mathbb{N})_{2, 1}$, with certain relations: $\mathbb{F}(\mathbb{N}) = \mathbb{F} < \sigma, \sigma^t >$. It follows that $\mathcal{L} = \mathbb{F} < \sigma, \sigma^t, \sigma', (\sigma')^t >$ is generated by $\sigma, \sigma^t$ and $\sigma', (\sigma')^t$ coming from the left and right factors of $\mathcal{L} = \mathbb{F}(\mathbb{N}) \otimes \mathbb{F}(\mathbb{N})$.

From $1 - \text{commutativity}$ we have:

$$\sigma \circ (\sigma^t \oplus \sigma') = \sigma' \circ (\sigma \oplus \sigma)$$
Thus 1 commutativity imply \( \sigma = \sigma' \), and similarly \( \sigma^t = (\sigma')^t \), and \( d \circ f : \mathcal{L}^{1-\text{com}} \to \mathbb{F}(\mathbb{N}) \) is an isomorphism.

In a pictorial way we see this as follows:

(2) Recall that elements of \( \mathcal{L}_{Y,X} = (\mathbb{F}(\mathbb{N}) \otimes_{\mathbb{R}} \mathbb{F}(\mathbb{N}))_{Y,X} \), are given by a sequence of sets \( X = X_0, X_1, \ldots, X_l = Y \), and \( X_{j+1} \times X_j \)-matrices over \( \mathbb{N} \), coming from the left "l", or right "r" copies of \( \mathbb{F}(\mathbb{N}) \) (depending on the parity of \( j \)). Thinking of an \( X_{j+1} \times X_j \)-matrix with values in \( \mathbb{N}, B_j \), as a set with maps

\[ Z_j \xrightarrow{\pi} X_{j+1}, \quad i = 0, 1. \]
(via \((B_j)_{x',x} = \#\pi^{-1}_1(x') \cap \pi_0^{-1}(x)\)) we obtain a graph with no loops

\[
G = \{ G^1 = \amalg \pi_j \to \pi_0 G^0 = \amalg_j X_j \}, \tag{2.9.4}
\]

with a mapping \(\mu : G^1 \to \{ l, r \}\).

Eliminating from our matrices rows \((i = 1)\), or columns \((i = 0)\), which are zero we may assume there are no such rows or columns.

**zero reduction:**

If \(\xi \in G^0 \setminus (Y \sqcup X)\) is such that \(\pi_i^{-1}(\xi) = \emptyset, \ i = 0 \text{ or } i = 1\)

\[\implies G \sim (G^1 \setminus \pi_i^{-1}(\xi), G^0 \setminus \{ \xi \}).\]

After a finite number of zero reductions we may assume with out loss of generality that \(G\) is zero-reduced; every path in \(CG\) extends to a maximal path beginning in \(In G \sim X\) and ending in \(Out G \sim Y\).

Set,

\[
A_{Y,X} = \left\{ G = \{ G^1 \xrightarrow{\pi_0} G^0 \}, \mu : G^1 \to \{ l, r \}, \mu(\pi_i^{-1}(x)) \equiv l \text{ or } \equiv r, \text{ no loops & } \begin{array}{l}
\text{zero-reduced,} \\
\text{In } G = G^0 \setminus \pi_1^{-1}(G^1) \leftrightarrow X, \text{ Out } G = G^0 \setminus \pi_0^{-1}(G^1) \leftrightarrow Y
\end{array} \right\} \sim
\]

(2.9.5)

If \((G, \mu) \in A_{Y,X}, x', x \in G^0, e \in G^1\) are such that \(\pi_0^{-1}(x') = \{ e \} = \pi_1^{-1}(x)\), i.e. \(e\) is the unique edge going out of \(x'\), and also the unique edge going into \(x\), than we can form the graph

\[
G' = \{ G^1 \setminus \{ e \}, G^0 / \{ x' \sim x \} \}. \tag{2.9.6}
\]

by throwing out the edge \(e\), and identifying the vertices \(x'\) and \(x\). We say \(G'\) is obtained from \(G\) by 1-reduction. If there are no such \(x, x', e\) in \(G\), we say \(G\) is one-reduced. After a finite number of 1-reductions we obtain a one reduced graph. If \((G, \mu)\) is both zero- & one-reduced, we say it is \(\mathbb{F}\)-reduced.

We can relax the condition on the matrices to alternate between "l" and "r", if we remember that consecutive matrices both "l" or both "r", are allowed to be multiplied, and in the description of \(N\)-valued matrices as sets, matrix multiplication corresponds to taking fiber products.

**Define \(l/r\)-reduction:** For a graph \(G\), and \(x \in G^0\) such that

\[
\mu(\pi_1^{-1}(x)) \equiv \mu(\pi_0^{-1}(x)) \quad (\equiv l \text{ or } \equiv r)
\]

\[\implies G \sim \left( \left[ G^1 \setminus (\pi_1^{-1}(x) \sqcup \pi_0^{-1}(x)) \right] \sqcup \left[ \pi_1^{-1}(x) \Pi \pi_0^{-1}(x) \right], \ G^0 \setminus \{ x \} \right) \tag{2.9.7}
\]

with: \((e, e') \in \pi_1^{-1}(x) \Pi \pi_0^{-1}(x), \ \pi_1(e, e') = \pi_1(e'), \ \pi_0(e, e') = \pi_0(e)\)
The inverse passage from $G$ to $G$ will be called $l/r$-inflation. A Graph $(G, \mu) \in A_{Y,X}$ is then $l/r$-reduced when for all $x \in G^0 \setminus (\text{In } G \cup \text{Out } G)$

$$\{\mu(\pi_1^{-1}(x)), \mu(\pi_0^{-1}(x))\} = \{l, r\}$$ (2.9.8)

We have a canonical form to any $(G, \mu) \in A_{Y,X}$ obtained after a finite number of $l/r$-reduction and 1-reduction, and it is characterized by the fact that $\#G^0$ is minimal, and so actually:

$$\mathcal{L} = \mathbb{F}(\mathbb{N}) \otimes \mathbb{F}(\mathbb{N}) \equiv \{(G, \mu), \mathbb{F} \text{-reduced and } l/r \text{-reduced}\}$$ (2.9.9)

Next let us look at the commutative quotient of $\mathcal{L}$,

$$\mathcal{L}^{\text{com}} = \mathbb{F}(\mathbb{N}) \otimes \mathbb{F}(\mathbb{N})/\approx \equiv \{(G, \mu), \mathbb{F} \text{-reduced}/\approx\},$$ (2.9.10)

where $\approx$ is the equivalence relation generated by $l/r$-reduction, $l/r$-inflation, and six commutative relations, passing from any vertex of the "commutativity triangle" to any other vertex, once one of the three patterns are recognized within our graph.

Commutativity triangle: for $a \in A_{Y,X}, b \in A_{1, J}, d \in A_{J, 1}$,

Here we have:

$$a = \quad d = \quad b = \quad \overline{X} = 2, \overline{Y} = 4, J = 3.$$

More specifically, $\approx$ is the equivalence relation such that $G \approx G'$ if and only if there is a path $G = G_0, \ldots, G_j, \ldots, G_l = G'$, where $\{G_{j-1}, G_j\}$ is one of the
following forms:

1. $G_j$ is obtained from $G_{j-1}$ via $l/r$-reduction.
2. $G_j$ is obtained from $G_{j-1}$ via $l/r$-inflation.
3. $G_j$ is obtained from $G_{j-1}$ via (one out of 6 possible) commutativity moves (see example above).

**Claim:** $\approx$ is an equivalence ideal.

Indeed, it is an equivalence relation and,

\[
\begin{align*}
(i) \quad G \approx G' & \implies G \oplus F \approx G' \oplus F \\
(ii) \quad G \approx G' & \implies H \circ G \circ F \approx H \circ G' \circ F
\end{align*}
\]

The first implication $(i)$ is trivial, while the latter implication $(ii)$ follows by the fact that (unlike total commutativity and 1-commutativity), the commutativity relation is $\mathbb{F}$-linear:

\[
G \approx G' \implies [Y_0]G[X_0] \approx [Y_0]G'[X_0].
\]

Indeed, $\mathbb{F}$-linearity implies $(ii)$ as (cf. (2.6.10)):

\[
H \circ G \circ F = H[Y_0][Y_0][Y_0][X_0][X_0][X_0].
\]

To show $\mathbb{F}$-linearity we have to show that if $G'$ is obtained from $G \in A_{Y,X}$ by any of the steps (1), (2), (3), then so is $[Y_0]G'[X_0]$ obtained from $[Y_0]G[X_0]$. This is clear for steps (1) and (2). For the 6 possible commutativity moves, we have to identify within $G$ one of the 3 possible patterns, and in particular subsets $\overline{X}, \overline{Y} \subseteq G_0^\circ$, and a subgraph $a \subseteq G$, $a$ from $\overline{X}$ to $\overline{Y}$. If $G'$ is obtained from $G$ via commutativity relation with pattern $(\overline{Y}, \overline{X}, J, a, b, d)$, then $[Y_0]G'[X_0]$ is obtained from $[Y_0]G[X_0]$ via the same commutativity relations with pattern $(\overline{Y_0}, \overline{X_0}, J, \overline{Y_0}a[\overline{X_0}], b, d)$, where

\[
\begin{align*}
\overline{X}_0 &= \{ x \in \overline{X} \mid \exists \text{ maximal path } e \in [Y_0]G[X_0] \text{ with } x \in e \} \\
\overline{Y}_0 &= \{ y \in \overline{Y} \mid \exists \text{ maximal path } e \in [Y_0]G[X_0] \text{ with } y \in e \}
\end{align*}
\]

This proves that $\approx$ is $\mathbb{F}$-linear, and hence an equivalence ideal.

Note that commutative relations does not preserve $l/r$-reduction, it is also needed to have $l/r$-inflation to get full commutative relations, e.g.,
Now using the presentation \((2.9.10)\) for \(L^{\text{com}}\) it follows that,

\[
d \circ g : L^{\text{com}} \rightarrow \mathbb{F}(\mathbb{N}) \text{ is an isomorphism}
\]

\[
\iff
\]

\[
\sigma = ((1, 1), l) \approx \sigma' = ((1, 1), r)
\]

\[
\iff
\]

\[
\exists \text{ path } (G_0, \mu_0), \ldots, (G_k, \mu_k) : \text{ with } \{G_{j-1}, G_j\} \text{ of the form (1), (2), or (3),}
\]

\[
\begin{align*}
\text{and } G_0 = & \sigma = 1 \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \\
& l, r \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} \\
\end{align*}
\]

\[
\cdots \quad G_k = \sigma' = 1 \quad \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \\
& \begin{array}{c}
\begin{array}{c}
l
\end{array}
\end{array}, r \begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} \\
\end{align*}
\]

\[
(2.9.14)
\]

where for any \(0 \leq j \leq k\), \(\text{Out } G_j \equiv \{1\}\), \(\text{In } G_j \equiv \{1, 2\}\), and \(\exists! \text{ path from } i \text{ to } 1, i = 1, 2\). (its image under the diagonal homomorphism is \((1, 1)\)), and \(G_j\) has no circuits and is zero -reduced. This implies that \(G_j\) must be of the following form:
2.10 Generators and relations for $\mathbb{F}$-rings

We have a surjection
\[ F[\delta_{1,2}, \delta_{1,2}] \twoheadrightarrow \mathbb{F}(N) \]
\[ \delta_{1,2} \mapsto (1,1) \]  
(2.10.1)

and a surjection
\[ \mathbb{F}\{\pm 1\}[\delta_{1,2}, \delta_{1,2}] \twoheadrightarrow \mathbb{F}(Z). \]  
(2.10.2)

Here $\mathbb{F}\{\pm 1\}[\delta_{1,2}, \delta_{1,2}] = \mathbb{F}\{\pm 1\} \otimes_{\mathbb{F}} \mathbb{F}[\delta_{1,2}, \delta_{1,2}]$.

Indeed, if $A \subseteq \mathbb{F}(N)$, (resp. $A \subseteq \mathbb{F}(Z)$) is a sub-$\mathbb{F}$-ring and contains $(1,1) \in A_{1,2}$, then $A_{Y,X}$ is closed under addition: $a, a' \in A_{Y,X}$:
\[
 a + a' = \left( \bigoplus_{Y} (1,1) \right) \circ (a \oplus a') \circ \left( \bigoplus_{X} \frac{1}{1} \right) \in A_{Y,X} \]  
(2.10.3)

And any matrix in $\mathbb{F}(N)_{Y,X}$ (resp. $\mathbb{F}(Z)_{Y,X}$) is a sum of matrices in $\mathbb{F}$ (resp. $\mathbb{F}\{\pm 1\}$).

Given $R \in \mathcal{R}(t)$, $A = \mathbb{F}(R) \in \mathcal{F}(t)$. We have a surjective homomorphism:
\[ \Phi : B = \mathbb{F}[\delta_{1,2}, \delta_{1,2} ; \delta_{1,1}(r), r \in R] \twoheadrightarrow A = \mathbb{F}(R), \]
\[ \Phi(\delta_{1,2}) = (1,1), \Phi(\delta_{1,2}^t) = \frac{1}{1}, \Phi(\delta_{1,1}(r)) = (r). \]  
(2.10.4)

Elements of $B_{Y,X}$ can be represented by graphs with no loops $G$, with $In G \leftarrow X, Out G \leftarrow Y$, with a map $\mu : G^1 \rightarrow R$ and the graph $G$ is made of the basic graphs
\[
 \delta \equiv \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}, \quad \delta^t \equiv \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}, \quad (r) \equiv \left\{ \begin{array}{c} r \end{array} \right\} \]  
(2.10.5)
The homomorphism \( \Phi \) takes \((G, \mu) \in B_Y, X\) into the \( Y \times X \)-matrix with values in \( R \),

\[
\Phi(G, \mu)_{y, x} = \sum_{(e_k, \ldots, e_1) \in m(G)} \mu(e_k) \ldots \mu(e_1). 
\]  

(2.10.6)

The equivalence ideal \( \text{KER} \Phi \) contains the following elements:

i. (1): (1) \( \equiv 1 \), (0) \( \equiv 0 \).

ii. \[
\begin{align*}
\text{Zero} : \delta \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \equiv (1). \\
\text{Zero}^t : (1, 0) \circ \delta^t & \equiv (1).
\end{align*}
\]

iii. \[
\begin{align*}
\text{Ass} : \delta \circ ((1) \oplus (1)) & \equiv \delta \circ ((1) \oplus \delta). \\
\text{Ass}^t : (\delta^t \oplus (1)) \circ \delta^t & \equiv ((1) \oplus \delta^t) \circ \delta^t.
\end{align*}
\]

iv. \[
\begin{align*}
\text{Comm} : \delta \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \equiv \delta. \\
\text{Comm}^t : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \delta^t & \equiv \delta^t.
\end{align*}
\]

v. Total - commutativity: \( \delta^t \circ \delta \equiv (\delta \oplus \delta) \circ (\delta^t \oplus \delta^t) \).

i.e. we have,

\[
\begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
1 \\
2
\end{array}
\quad = \quad 
\begin{array}{c}
(1, 1) \\
(2, 1)
\end{array}
\begin{array}{c}
1 \\
2
\end{array}
\begin{array}{c}
(1, 2) \\
(2, 2)
\end{array}
\]

(2.10.7)

the \( \Phi \)-image of this relation is

\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ (1, 1) \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]  

(2.10.8)

The indexing of the input of \( \delta \), and the output of \( \delta^t \), is important; the right hand side of (v) is Not equal to

\( (\delta \circ \delta^t) \oplus (\delta \circ \delta^t) \), whose \( \Phi \)-image is \( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \).
We have the relations,
\[
\begin{align*}
(r, \delta) : & \quad (r) \circ \delta \equiv \delta \circ ((r) \oplus (r)), \\
(\delta^t, r) : & \quad \delta^t \circ (r) \equiv ((r) \oplus (r)) \circ \delta^t, \\
(r_1 \cdot r_2) : & \quad (r_1) \circ (r_2) \equiv (r_1 \cdot r_2), \\
(r_1 + r_2) : & \quad \delta \circ ((r_1) \oplus (r_2)) \circ \delta^t \equiv (r_1 + r_2).
\end{align*}
\]

**Remark**

When working in the context of \( \mathcal{F}_R \), (with involution!), every relation is equivalent to its transpose, and we should add the relation,
\[(r)^t \equiv (r^t).
\]

**Theorem 2.10.1**

The equivalence ideal \( \text{KER} \Phi \) is generated by these relations.

**Proof.** Let \( G \in B \), we shall show that modulo these relations we can bring \( G \) to a canonical form which depends only on \( \Phi(G) \). By using (1), we can add to \( G \) identities \( \{\bullet \quad 1 \quad \bullet\} \), and assume without loss of generality that \( G \) has the form
\[
Y \supseteq X_t \biguplus X_{t-1} \biguplus \cdots \biguplus X_2 \biguplus X_1 \biguplus X_0 \subseteq X
\]
(2.10.9)

where each basic graph \( \{Z_j \supseteq X_j \biguplus X_{j-1}\} \) has the form
\[
\begin{align*}
X_j & \quad r_1 \quad X_{j-1} \\
\bullet & \quad r_2 \quad \bullet \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
\cdots & \quad \cdots \\
\bullet & \quad \bullet
\end{align*}
\]
(2.10.10)

i.e. is a direct sum of \( \delta, \delta^t, (r) \), with no composition. By further adding identities, we can assume each basic graph is either of the "left" form
-a direct sum of \( \delta, (r) \)'s, (no \( \delta^t \)), or of the "right" form
a direct sum of $\delta^t$, $(r)$'s, $(\delta^t, r)$ and total -comm. we can replace $Z_{j+1}$ of the "right" form, $Z_j$ of the "left" form, by a $Z'_{j+1}$ of the "left" form, and $Z'_j$ of the "right" form:

\[
\begin{align*}
\text{X}_1 + \cdots + \text{X}_{j+1} & \quad \text{X}_j \quad \text{X}_j - 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{X}_1 + \cdots + \text{X}_{j+1} & \quad \text{X}_j \quad \text{X}_j - 1 \\
\end{align*}
\]

Thus we can assume all the basic graphs of the "left" form appear to the left of all the basic graphs of the "right" form. Moreover, we can assume the basics graphs $Z_l, \ldots, Z_{j+1}$, are all direct sum of $\delta$, $(1)$'s, $Z_j, \ldots, Z_{j+1}$ are all direct sum of $(r)$'s, $r \in R$, and $Z_l, \ldots, Z_1$ are all direct sum of $\delta^t$, $(1)$'s. We can represent the graph $L = Z_l \circ \cdots \circ Z_{j+1}$ (resp $R = Z_j \circ \cdots \circ Z_l$) using $\text{Ass}$, $\text{Comm}$ (resp. $\text{Ass}^t$, $\text{Comm}^t$) as a direct sum of the graphs $\delta^{(n)}$ (resp. $(\delta^t)^{(n)}$), $\delta^{(1)} \equiv (1), \delta^{(2)} \equiv \delta$

and $\delta^{(n)} \equiv \delta \circ (\delta^{(n-1)} \oplus (1)) = \{ \begin{array}{c} 1 \\ \vdots \\ n \end{array} \}, (\delta^t)^{(n)} \equiv \{ \begin{array}{c} n \\ \vdots \\ 1 \end{array} \}$.

Using the $(r_1 \cdot r_2)$ relation, the graph $D = Z_j \circ \cdots \circ Z_{i+1}$ is equivalent to a "diagonal graph": a direct sum of $(r)$'s, with identifications $\text{Out } D \cong D^1 \to \text{In } D$. Thus, all in all, we can represent $G$ by the data of the set $D$, together with the maps $\pi_1 : D \to Y, \pi_0 : D \to X, \mu : D \to R$:
If \( d_1, d_2 \in D \), are such that \( \pi_1(d_1) = \pi_1(d_2), \pi_0(d_1) = \pi_0(d_2) \), we can use the \((r_1 + r_2)\)-relation to identify \( d_1 \) and \( d_2 \) to a point \( d \) with \( \pi_1(d) = \pi_1(d_i), \pi_0(d) = \pi_0(d_i), \mu(d) = \mu(d_1) + \mu(d_2) \):

\[
\begin{array}{c}
\vdots \\
r_1 \\
r_2 \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\vdots \\
r_1 + r_2 \\
\end{array}
\]

Thus we can assume \( D \) is a subset of \( Y \times X, \mu: D \to R \). Extending \( \mu \) by zeros we get \( \mu: Y \times X \to R \), i.e. a \( Y \times X \) matrix with values in \( R \), which is just \( \Phi(G) \).

The same proof shows that,

**Theorem 2.10.2**

For the surjective homomorphism \( \Phi: \mathbb{F}[\delta_{1,2}, \delta_{1,2}^t] \to \mathbb{F}(\mathbb{N}) \), the equivalence ideal \( KER(\Phi) \) is generated by relations (i)-(v).

**Theorem 2.10.3**

For the surjective homomorphism \( \Phi: \mathbb{F}[\pm 1][\delta_{1,2}, \delta_{1,2}^t] \to \mathbb{F}(\mathbb{Z}) \), the relations are (i)-(v), \((-1, \delta), (\delta^t, -1), ((-1) \cdot (-1))\), and cancellation

\[
\delta \circ ((1) \oplus (-1)) \circ \delta^t \equiv 0.
\]
Appendix B

Proof of Ostrowski’s theorem

Proof of Ostrowski I. We can describe the elements of $\text{Val}(\mathbb{F}(\mathbb{Q})/\mathbb{F}\{\pm 1\})$ as collection of mappings

$$\mid |_{Y,X} : \mathbb{F}(\mathbb{Q})_{Y,X} \to [0, \infty)$$

satisfying (I),(II),(III) and $|\pm 1|_{1,1} = 1$, where we identify the collection $\{ | |_{Y,X} \}$ with the collection $\{ |_{Y,X}^{\lambda} \}$, for any $\lambda > 0$. The "generic point" $\mathbb{F}(\mathbb{Q})$ corresponds to the trivial valuation $|/|_{Y,X} = \begin{cases} 1 & y \neq 0 \\ 0 & y = 0 \end{cases}$.

Let $\{ |_{Y,X} \}$ be a non-trivial valuation on $\mathbb{F}(\mathbb{Q})$, and let $B \subseteq \mathbb{F}(\mathbb{Q})$ be the associated valuation - $\mathbb{F}$ - subring, $B_{Y,X} = \{ b \in \mathbb{Q}^{Y \times X}, |b|_{Y,X} \leq 1 \}$.

For $q_1, q_2 \in \mathbb{Q}$ we have

$$|q_1 + q_2|_{1,1} = \left| (1,1) \circ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|_{1,1} \leq \max\{|q_1|_{1,1}, |q_2|_{1,1}\}.$$ 

Note that $|1,1|_{1,2} = \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|_{1,2}$ by III(iii).

If we have $|(1,1)|_{1,2} \leq 1$, than $|q_1 + q_2|_{1,1} \leq \max\{|q_1|_{1,1}, |q_2|_{1,1}\}$, and it follows that: $|n|_{1,1} \leq 1$ for all $n \in \mathbb{Z}$; $\{ n \in \mathbb{Z}, |n|_{1,1} < 1 \} = p \cdot \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$; $\mathbb{Z}_{(p)} \subseteq B_{1,1} \subseteq \mathbb{Q}$; and as $B_{1,1}$ is an (ordinary) subring of $\mathbb{Q}$: $\mathbb{Z}_{(p)} = B_{1,1}$.

For a matrix $b \in B_{Y,X} \subseteq \mathbb{Q}^{Y \times X}$, its coefficients $b_{y,x} = j_y^t \circ b \circ j_x \in B_{1,1} = \mathbb{Z}_{(p)}$, so $B \subseteq \mathbb{F}(\mathbb{Z}_{(p)})$. We have by II(i)

$$|(1,1,\ldots,1)|_{1,n} = \text{Sup} \left\{ \left| (1,1,\ldots,1) \circ \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right|_{1,1} \left|_{1,1} \mid_{1,1} \right) \in B_{n,1} \right\}$$

$$\leq \sup\{|b_1 + \cdots + b_n|_{1,1}, b_j \in \mathbb{Z}_{(p)}\} \leq 1,$$
and for a vector $b = (b_1, \ldots, b_n) \in \mathbb{F}(\mathbb{Z}_p)_{1,n}$, we have

$$|b|_{1,n} = (1, 1, \ldots, 1) \circ \left( \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right)_{1,n} \leq |(1, 1, \ldots, 1)|_{1,n} \max\{|b_j|_{1,1}\} \leq 1,$$

so $\mathbb{F}(\mathbb{Z}_p)_{1,n} = B_{1,n}$, and $\mathbb{F}(\mathbb{Z}_p)_{n,1} = B_{n,1}$.

For a matrix $b = (b_{y,x}) \in \mathbb{Z}_p^{Y \times X}$, we have by II(i)

$$|b|_{Y,X} = \sup \left\{ |d \circ b \circ d'|_{1,1}, d \in \mathbb{F}(\mathbb{Z}_p)_{1,X}, d' \in \mathbb{F}(\mathbb{Z}_p)_{Y,1} \right\} \leq 1$$

and so $\mathbb{F}(\mathbb{Z}_p)_{Y,X} = B_{Y,X}$.

Assume that we have $|(1, 1)|_{1,2} > 1$. Passing to an equivalent norm $\{ | \}_Y$, (with $\lambda \leq \frac{\log \sqrt{T}}{\log((1,1)_{1,2})}$), we can assume that $|(1, 1)|_{1,2} = \left| \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right|_{2,1}$, and

$$|q_1 + q_2|_{1,1} \leq 2 \cdot \max\{|q_1|_{1,1}, |q_2|_{1,1}\}, q_i \in \mathbb{Q}.$$ 

By induction we get

$$\left| \sum_{i=1}^{2^r} q_i \right|_{1,1} \leq 2^r \cdot \max\{|q_i|_{1,1}\}, q_i \in \mathbb{Q}$$

hence,

$$\left| \sum_{i=1}^{n} q_i \right|_{1,1} \leq 2 \cdot n \cdot \max\{|q_i|_{1,1}\}$$

hence $|n|_{1,1} \leq 2 \cdot |n|_\eta, |n|_\eta = \pm n$, the usual absolute value for $n \in \mathbb{Z}$.

We have for $q_1, q_2 \in \mathbb{Q},$

$$|q_1 + q_2|_{1,1} = |(q_1 + q_2)^n|_{1,1}^{1/n} = \sum_{k=0}^{n} \binom{n}{k} q_1^k \cdot q_2^{n-k} |_{1,1}^{1/n} \leq$$

$$(2(n + 1))^{1/n} \cdot \max\{|\binom{n}{k}|_{1,1} \cdot |q_1|_{1,1}^k \cdot |q_2|_{1,1}^{n-k}|\}^{1/n} \leq$$

$$(4(n + 1))^{1/n} \cdot \max\{|\binom{n}{k}|_{1,1} \cdot |q_1|_{1,1}^k \cdot |q_2|_{1,1}^{n-k}|\}^{1/n} \leq$$

$$(4(n + 1))^{1/n} \cdot (|q_1|_{1,1} + |q_2|_{1,1})^{n/2} = (4(n + 1))^{1/n} \cdot (|q_1|_{1,1} + |q_2|_{1,1}).$$

and taking the limit $n \to \infty$ we get the triangle inequality

$$|q_1 + q_2|_{1,1} \leq |q_1|_{1,1} + |q_2|_{1,1}.$$
APPENDIX B. PROOF OF OSTROWSKI’S THEOREM

We have by II(i),

\[ |(1,1,\ldots,1)|_{1,n} = \text{Sup}(|(1,1,\ldots,1) \circ q|_{1,1}, |q|_{n,1} \leq 1 \}
\]

\[ = \text{Sup}(|q_1 + \cdots + q_n|_{1,1} , |q|_{n,1} \leq 1 \}
\]

\[ \leq \text{Sup}(|q_1|_{1,1} + \cdots + |q_n|_{1,1} , |q|_{n,1} \leq 1 \} \leq n \]

Let \( a,b \in \mathbb{N} \), with \( a > 1 \), so we can expand \( b \) in the base \( a \):

\[ b = d_m \cdot a^m + \cdots + d_j \cdot a^j + \cdots + d_1 \cdot a + d_0 \]

\[ 0 \leq d_j < a , \quad m < \frac{\log b}{\log a} \]

We get

\[ |b|_{1,1} = |(1,1,\ldots,1) \circ \bigoplus_{i=0}^{m} a^i \circ \bigoplus_{j=0}^{m} d_j \circ (1,1,\ldots,1)|_{1,1} \leq \]

\[ |(1,1,\ldots,1)|_{1,m+1} \cdot \max \{|d|_{1,1}\} \cdot \max \{|a|^j|_{1,1}\} \leq \]

\[ (1 + m)^2 \cdot M_a \cdot \max \{|a|^j|_{1,1}\}, \]

with \( M_a = \max \{|d|_{1,1}\} \) a constant independent of \( b \). From this follows

\[ |b|_{1,1} = |b^n|_{1,1} \leq (1 + n \cdot \frac{\log b}{\log a})^{2/n} \cdot M_a^{1/n} \cdot \max \{|a|^j|_{1,1}\} \]

and letting \( n \to \infty \) we obtain

\[ |b|_{1,1} \leq \max \{|a|^j|_{1,1}\} \]

It follows that if \( |b|_{1,1} > 1 \) for some \( b \in \mathbb{Z} \), then \( |a|_{1,1} > 1 \) for all \( a \in \mathbb{Z} \\backslash \{\pm 1, 0\} \),

in which case \( |b|_{1,1}^{1/n} = |a|_{1,1}^{1/n} = e^\delta \) is a constant for \( a,b \in \mathbb{Z} \\backslash \{\pm 1, 0\} \), or

\[ |a|_{1,1} = |a|_{1,1}^{1/n} \]

for \( a \in \mathbb{Z} \), hence for \( a \in \mathbb{Q} \), \( |a|_{\eta} = \pm a \) the real absolute value.

Note that

\[ 2^\delta = |2|_{1,1} \leq |(1,1,1)|_{1,2} = \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right|_{2,1} \leq \sqrt{2} \sqrt{2} = 2, \quad \delta \leq 1, \]

and passing to an equivalent norms we may assume \( \delta = 1 \), and \( |q|_{1,1} = |q|_{\eta} \) is the usual real absolute value.

For a vector \( q = (q_1, \ldots, q_n) \in \mathbb{Q}^n \) we get

\[ \sum_{i=1}^{n} q_i^2 = |q \circ q^t|_{\eta} = |q \circ q^t|_{1,1} \leq |q|_{1,n} \cdot |q^t|_{n,1} = |q|_{1,n}^2 \]
or $|q|_\eta = \left( \sum_{i=1}^n q_i^2 \right)^{1/2} \leq |q|_{1,n}$.

On the other hand, from II(i) we get

$$|q|_{1,n} = \text{Sup} \left\{ |q \circ b|_{1,1}, \ |b|_{n,1} \leq 1 \right\}$$

$$\leq \text{Sup} \left\{ \sum_{i=1}^n q_i \circ b_i, \ \sum_{i=1}^n b_i^2 \leq 1 \right\} = \left( \sum_{i=1}^n q_i^2 \right)^{1/2} = |q|_\eta,$$

and so

$$|(q_1, \ldots, q_n)|_{1,n} = \left| \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \right|_{n,1} = \left( \sum_{i=1}^n q_i^2 \right)^{1/2}$$

and $B_{1,n} = (\mathcal{O}_{\mathbb{Q}_1})_{1,n}, \ B_{n,1} = (\mathcal{O}_{\mathbb{Q}_n})_{n,1}$.

Finally, for a matrix $a \in \mathbb{Q}^{Y \times X}$, from II(i) we get

$$|a|_{Y,X} = \text{Sup}\{|b \circ a \circ b'|, \ b = (b_y), \ \sum_y |b_y|^2 \leq 1, \ b' = (b'_x), \ \sum_x |b'_x|^2 \leq 1\}$$

is the usual $l_2$-operator norm, and $B_{Y,X} = (\mathcal{O}_{\mathbb{Q},o})_{Y,X}$

\[ \square \]

\textbf{Proof of Ostrowski II.} We can describe the elements of $\text{Val}(\mathbb{F}(K)/\mathbb{F}\{\mu_K\})$ as collection of mappings

$$\| \cdot \|_{Y,X} = \mathbb{F}(K)_{Y,X} = K^{Y \times X} \to [0, \infty).$$

satisfying (I),(II),(III), and $|\mu_K|_{1,1} = 1$, identifying $\{\| \cdot \|_{Y,X}\}$ with $\{\| \cdot \|_{X,Y}\}, \ \lambda > 0$. Let $B_{Y,X} = \{ b \in \mathbb{F}(K)_{Y,X}, \ \|b|_{Y,X} \leq 1 \}$ be the valuation $F$-subring of $\mathbb{F}(K)$, corresponding to a non-trivial valuation $\{\| \cdot \|_{Y,X}\}$. For $q_1, q_2 \in K$,

$$|q_1 + q_2|_{1,1} = \left| (1, 1) \circ \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|_{1,1,1} \leq \max\{|q_1|_{1,1}, |q_2|_{1,1}\},$$

Thus $\| \cdot \|_{1,1}$ is a valuation of $K$ (cf. [CE]), and passing to equivalent valuation we may assume $|q|_{1,1} = |q|_p, p \in \mathcal{O}_K$ a finite prime, or $|q|_{1,1} = |\eta q|$ with $\eta : K \to \mathbb{C}$ (modulo conjugation) a "real prime".

In the non-archimedean case, $|q|_{1,1} = |q|_p, B_{1,1} = \mathcal{O}_{K,p}$, and for any matrix $b = (b_{y,x}) \in B_{Y,X},$

$$|b_{y,x}|_{p} = |b_{y,x}|_{1,1} = |j_y^I \circ b \circ j_x|_{1,1} \leq |j_y^I|_{1,Y} \cdot |b|_{Y,X} \cdot |j_x|_{X,1} \leq |b|_{Y,X} \leq 1,$$
so $B_{Y,X} \subseteq \mathbb{F}(\mathcal{O}_{K_p})_{Y,X}$. Note that by II(i),

$$|(1, 1, \ldots, 1)|_{1,n} = \operatorname{Sup}\left\{ |b_1 + \cdots + b_n|_p : \left( \begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right)_{n,1} \leq 1 \right\}$$

$$\leq \operatorname{Sup}\{ |b_1 + \cdots + b_n|_p : |b_i|_p \leq 1 \} \leq 1.$$  

For a vector $b = (b_1, \ldots, b_n) \in \mathbb{F}(\mathcal{O}_{K_p})_{1,n},$

$$|b|_{1,n} = |(1, 1, \ldots, 1) \circ \left( \begin{array}{c} b_1 \\ 0 \\ \vdots \\ 0 \\ b_n \end{array} \right)_{n,1}| \leq |(1, 1, \ldots, 1)|_{1,n} \max\{|b_i|_p\} \leq 1,$$

so $B_{1,n} = \mathbb{F}(\mathcal{O}_{K_p})_{1,n}$, and $B_{n,1} = \mathbb{F}(\mathcal{O}_{K_p})_{n,1}$.

Finally, for a matrix $b = (b_{y,x}) \in \mathbb{F}(\mathcal{O}_{K_p})_{Y,X}$ we have by II(i),

$$|b|_{Y,X} = \operatorname{Sup}\{ |d \circ b \circ d'|_p : d \in \mathbb{F}(\mathcal{O}_{K_p})_{1,Y}, d' \in \mathbb{F}(\mathcal{O}_{K_p})_{X,1} \leq 1,$$

so $B_{Y,X} = \mathbb{F}(\mathcal{O}_{K_p})_{Y,X}$.

In the archimedean case, $|q|_{1,1} = |\eta q|$, $\eta : K \hookrightarrow \mathbb{C}$.

For a vector $q = (q_1, \ldots, q_n) \in K^n$, we get

$$\sum_{i=1}^{n} |\eta q_i|^2 = |q \circ \mathbf{1}|_{1,1} \leq |q|_{1,n}^2,$$

and $B_{1,n} = (\mathcal{O}_{K,q})_{1,n}$, $B_{n,1} = (\mathcal{O}_{K,q})_{n,1}$.

Conversely, from II(i) we get,

$$|q|_{1,n} = \operatorname{Sup}\{ |q \circ b|_{1,1} : |b|_{1,n} \leq 1 \}$$

$$\leq \operatorname{Sup}\{ |\eta(\sum_{i=1}^{n} q_i \cdot b_i)|, \sum_{i=1}^{n} |\eta b_i|^2 \leq 1 \} = (\sum_{i=1}^{n} |\eta q_i|^2)^{1/2}.$$

and $B_{1,n} = (\mathcal{O}_{K,q})_{1,n}$, and similarly $B_{n,1} = (\mathcal{O}_{K,q})_{n,1}$.

Finally, for a matrix $a \in K^{Y \times X}$, we get from II(i),

$$|a|_{Y,X} = \operatorname{Sup}\{ |b \circ a \circ b'|_{1,1} : b = (b_y), \sum_{y} |\eta b_y|^2 \leq 1, b' = (b'_x), \sum_{x} |\eta b'_x|^2 \leq 1 \}$$

is the usual $l_2$- operator norm, and $B_{Y,X} = (\mathcal{O}_{K,q})_{Y,X}$.
Chapter 3

Geometry

In this section $A \in \mathbb{C}^{2\mathbb{R}}$ is commutative.

3.1 Ideals, maximal ideals and primes

According to definition $[A.2.3]$, a subset $a \subseteq A_{1,1}$ is called an ideal if for

$$a_1, \ldots, a_n \in a, \ b \in A_{1,n}, \ b' \in A_{n,1} : \ b \circ (a_1 \oplus \cdots \oplus a_n) \circ b' \in a.$$  

(3.1.1)

Denote the set of ideals of $A$ by $I(A)$. Given an indexed set of ideals $a_i \subseteq A_{1,1}, \ i \in I$, their intersection $\cap_I a_i$ is again an ideal. Their sum $\Sigma_I a_i$ is an ideal generated by

$$\left\{ b \circ (\oplus a_j) \circ b' \bigg| a_j \in \cup a_j \right\}.$$  

(3.1.2)

The product $a \cdot a'$ of two ideals is an ideal generated by the product of elements of these ideals,

$$a \cdot a' = \left\{ b \circ (\oplus a_j \cdot a'_j) \circ b' \bigg| a_j \in a, a'_j \in a' \right\}.$$  

(3.1.3)

Let $\varphi : A \to B$ be a homomorphism of $\mathbb{F}$-Rings. If $b \in I(B)$ then $\varphi^*(b) = \varphi^{-1}(b) \in I(A)$, and we have a map

$$\varphi^* : I(B) \to I(A), \ b \mapsto \varphi^{-1}(b).$$  

(3.1.4)

If $a \in I(A)$, $\varphi(a)$ generates the ideal $\varphi_*(a)$,

$$\varphi_* : I(A) \to I(B), \ a \mapsto \varphi_*(a) = \{ b \circ (\oplus \varphi(a_i)) \circ b' \}.$$  

(3.1.5)
Proposition 3.1.1

We have the following:

1. \( a \subseteq \varphi_*\varphi_* a, \quad a \in \mathcal{I}(A) \).
2. \( b \supseteq \varphi_*\varphi_* b, \quad b \in \mathcal{I}(B) \).
3. \( \varphi_* b = \varphi_* \varphi_* \varphi_* b, \quad \varphi_* a = \varphi_* \varphi_* \varphi_* a \).
4. There is a bijection, via \( a \mapsto \varphi_* a \) (with inverse map \( b \mapsto \varphi_* b \)), from the set

\[
\{ a \in \mathcal{I}(A) \mid \varphi_* \varphi_* a = a \} = \{ \varphi_* b \mid b \in \mathcal{I}(B) \}
\]

(3.1.6) to the set

\[
\{ b \in \mathcal{I}(B) \mid \varphi_* \varphi_* b = b \} = \{ \varphi_* a \mid a \in \mathcal{I}(A) \}.
\]

(3.1.7)

Proof. The proofs of these are straightforward.

Given an ideal \( a \in \mathcal{I}(A) \), we write \( A/a \) for the quotient \( \mathbb{F}\text{-}\text{Ring} A/E(a) \), where \( E(a) \) is the equivalence ideal generated by \( a \).

Proposition 3.1.2

We have a one-to-one order-preserving correspondence

\[
\pi^* : \mathcal{I}(A)/a \rightarrow \{ b \in \mathcal{I}(A) \mid b \text{ satisfies } (*) \}
\]

(3.1.8)

where (*) means

\[
\text{for any } a \in \mathcal{I}: \quad b \circ (id_Z \oplus a) \circ b' \in b \iff b \circ (id_Z \oplus 0) \circ b' \in b.
\]

(3.1.9)

Proof. The proof is clear. (cf. (A.2.9) for the equivalence ideal \( E(a) \) generated by \( (a, 0), \ a \in \mathcal{I} \).)

Since the union of a chain of proper ideals is again a proper ideal, an application of Zorn’s lemma gives the following result.

Theorem 3.1.1 (Zorn)

There exists a maximal ideal \( m \not\subseteq A_{1,1} \).
Definition 3.1.1

An ideal \( p ⊆ A_{1,1} \) is called **prime**: 

\[ S_p = A_{1,1} \setminus p \text{ is multiplicatively closed } S_p \cdot S_p = S_p. \]  

(3.1.10)

We denote by \( \text{Spec} \ A \) the set of prime ideals.

For a homomorphism of \( \mathbb{F} \)-Rings \( ϕ : A → B \), the pullback \( ϕ^* = ϕ^{-1} \) induces a map 

\[ ϕ^* = \text{Spec}(ϕ) : \text{Spec} \ B → \text{Spec} \ A. \]  

(3.1.11)

Proposition 3.1.3

(1) If \( m \) is maximal then it is prime.

(2) More generally, if \( a ∈ \mathcal{I}(A) \), and given \( f ∈ A_{1,1} \) such that,

\[ ∀ n ∈ \mathbb{N} : f^n ≠ a. \]  

(3.1.12)

let \( m \) be a maximal element of the set

\[ \{ b ∈ \mathcal{I}(A) | b ⊇ a, b ≠ f^n ∀ n ∈ \mathbb{N} \} \]  

(3.1.13)

Then \( m \) is prime.

Proof. (1) If \( x, y ∈ A_{1,1} \setminus m \), the ideals \( (x) + m, (y) + m \) are the unit ideals. So we can write

\[ 1 = b \circ ( \bigoplus_j m_j ) \circ d, \quad \text{with } m_j = x \text{ or } m_j ∈ m \]  

(3.1.14)

\[ 1 = b' \circ ( \bigoplus_j m'_j ) \circ d', \quad \text{with } m'_j = y \text{ or } m'_j ∈ m \]  

(3.1.15)

It then follows that,

\[ 1 = 1 \cdot 1 = (b \circ ( \bigoplus_j m_j ) \circ d) \circ (b' \circ ( \bigoplus_j m'_j ) \circ d') \]

\[ = (b \circ \bigoplus_j m_j ) \circ (b' \circ ( \bigoplus_j m'_j ) \circ d') \circ d \]

\[ = b \circ \bigoplus_j (b' \circ ( \bigoplus_j (m_j \circ m'_j ) \circ d') \circ d) \]

\[ = (b \circ \bigoplus_j b') \circ ( \bigoplus_j (m_j \circ m'_j ) \circ (\bigoplus d' \circ d). \]  

(3.1.16)

but \( m_j \circ m'_j = x \circ y \) or \( m_j \circ m'_j ∈ m \), so 1 is in the ideal generated by \( m \) and \( x \circ y \), and since \( 1 ≠ m \) then \( x \circ y ≠ m \).

(2) Similarly, if \( x ≠ m \) then \( f^n \) is in the ideal generated by \( x \) and \( m \) so \( f^n = \)}
b \circ \bigoplus_{j} m_j \circ d$, with $m_j = x$ or $m_j \in m$. If $y \notin m$, $f^{n'} = b' \circ \bigoplus_{i} m_i' \circ d'$, with $m_i' = y$ or $m_i' \in m$. It then follows that

$$f^{n+n'} = b \circ \bigoplus_{j} m_j \circ d \circ b' \circ \bigoplus_{i} m_i' \circ d'$$

$$= b \circ ( \bigoplus_{j} m_j \circ b' \circ m_i') \circ ( \bigoplus_{j} d') \circ d$$  \hspace{1cm} (3.1.17)

but $m_j \circ m_i' = x \circ y$ or $m_j \circ m_i' \in m$, so $f^{n+n'}$ is in the ideal generated by $m$ and $x \circ y$, and since $f^{n+n'} \notin m$ then $x \circ y \notin m$. \hfill \square

**Definition 3.1.2**

For $a \in \mathcal{I}(A)$, the **radical** is

$$\sqrt{a} = \{ f \in A_{1,1} \mid f^n \in a \text{ for some } n \geq 1 \}$$ \hspace{1cm} (3.1.18)

It is easy to see that $\sqrt{a}$ is an ideal. This also follows from the following proposition.

**Proposition 3.1.4**

We have

$$\sqrt{a} = \bigcap_{a \subseteq p} p$$ \hspace{1cm} (3.1.19)

the intersection of prime ideals containing $a$.

**Proof.** If $f \in \sqrt{a}$, say $f^n \in a$, then for all primes $a \subseteq p$, $f^n \in p$ and so $f \in p$. If $f \notin \sqrt{a}$, let $m$ be a maximal element of the set (3.1.13), it exists by Zorn’s lemma, and it is prime by proposition 3.1.3(2), $a \subseteq m$ and $f \notin m$. \hfill \square

### 3.2 The spectrum: $\text{Spec } A$

**Definition 3.2.1**

For a set $\mathcal{U} \subseteq A_{1,1}$, we let

$$V_A(\mathcal{U}) = \{ p \in \text{Spec } A \mid \mathcal{U} \subseteq p \}.$$ \hspace{1cm} (3.2.1)

If $a$ is the ideal generated by $\mathcal{U}$, $V_A(\mathcal{U}) = V_A(a)$; we have

$$V_A(1) = \emptyset, \quad V_A(0) = \text{Spec } A,$$ \hspace{1cm} (3.2.2)

$$V_A(\Sigma a) = \cap_{i} V_A(a_i), \quad a_i \in \mathcal{I}(A),$$ \hspace{1cm} (3.2.3)

$$V_A(a \cdot a') = V_A(a) \cup V_A(a').$$ \hspace{1cm} (3.2.4)

Hence the sets $\{ V_A(a) \mid a \in \mathcal{I}(A) \}$ are the closed sets for the topology on $\text{Spec } A$, the **Zariski topology**.
Definition 3.2.2
For $f \in A_{1,1}$ we let
\[ D_A(f) = \text{Spec}(A) \setminus V_A(f) = \{ p \in \text{Spec } A | f \not\in p \}. \] (3.2.5)
We have
\[ D_A(f_1) \cap D_A(f_2) = D_A(f_1 \cdot f_2), \] (3.2.6)
\[ \text{Spec } A \setminus V_A(a) = \bigcup_{f \in a} D_A(f). \] (3.2.7)
Hence the sets $\{ D_A(f) | f \in A_{1,1} \}$ are the basis for the open sets in the Zariski topology. We have
\[ D_A(f) = \emptyset \iff f \in \bigcap_{p \in \text{Spec } A} p = \sqrt{0} \iff f^n = 0 \text{ for some } n \] (3.2.8)
and we say $f$ is a nilpotent. We have
\[ D_A(f) = \text{Spec } A \iff (f) = (1) \iff \exists f^{-1} \in A_{1,1} : f \cdot f^{-1} = 1 \] (3.2.9)
and we say $f$ is invertible. We denote by $GL_1(A)$ the (commutative) group of invertible elements.

Definition 3.2.3
For a subset $X \subseteq \text{Spec } A$, we have the associated ideal
\[ \mathcal{I}(X) = \bigcap_{p \in X} p. \] (3.2.10)

Proposition 3.2.1
We have
\[ \mathcal{I}V_A a = \sqrt{a}, \] (3.2.11)
\[ V_A \mathcal{I}(X) = \overline{X}, \text{ the closure of } X \text{ in the Zariski topology.} \] (3.2.12)

Proof. Equation (3.2.11) is just a restatement of proposition 3.1.4. For (3.2.12), $V_A \mathcal{I}(X)$ is clearly a closed set containing $X$, and if $C = V_A(a)$ is a closed set containing $X$, then $\sqrt{a} = IV_A(a) \subseteq \mathcal{I}(X)$, hence $C = V_A(\sqrt{a}) \supseteq V_A \mathcal{I}(X)$. \qed
CHAPTER 3. GEOMETRY

Corollary 3.2.1

We have a one-to-one order-reversing correspondence between closed sets $X \subseteq \text{Spec } A$, and radical ideals $a$, via $X \mapsto \mathcal{I}(X)$, $V_A(a) \leftrightarrow a$

\[
\{X \subseteq \text{Spec } A \mid \overline{X} = X\} \overset{1:1}{\longleftrightarrow} \{a \in \mathcal{I}(A) \mid \sqrt{a} = a\}. \tag{3.2.13}
\]

Under this correspondence the closed irreducible subsets correspond to the prime ideals. For $p_0, p_1 \in \text{Spec } A, p_0 \in \{p_1\} \iff p_0 \supseteq p_1$, we say that $p_0$ is a Zariski specialization of $p_1$, or that $p_1$ is a Zariski generalization of $p_0$. The space $\text{Spec } A$ is sober: every closed irreducible subset $C$ has the form $C = V_A(p) = \overline{\{p\}}$, and we call the (unique) prime $p$ the generic point of $C$.

Proposition 3.2.2

The sets $D_A(f)$, and in particular $D_A(1) = \text{Spec } A$, are compact (or 'quasi-compact': we do not include Hausdorff in compactness).

Proof. Note that $D_A(f)$ is contained in the union $\bigcup_i D_A(g_i)$ if and only if $V_A(f) \supseteq \bigcap_i V_A(g_i) = V_A(a)$, where $a$ is the ideal generated by $\{g_i\}$, if and only if $\sqrt{f} = \mathcal{I}V_A(f) \subseteq \mathcal{I}V_A(a) = \sqrt{a}$, if and only if $f^n \in a$ for some $n$, if and only if $f^n = b \circ (\bigoplus g_i) \circ b'$, and in such expression only a finite number of the $g_i$ are involved.

Let $\varphi : A \rightarrow B$ be a homomorphism of $\mathbb{F}$-Ring, $\varphi^* : \text{Spec } B \rightarrow \text{Spec } A$ the associated pullback map.

Proposition 3.2.3

We have

\[
\varphi^*^{-1}(D_A(f)) = D_B(\varphi(f)), \quad f \in A_{1,1}, \tag{3.2.14}
\]

\[
\varphi^*^{-1}(V_A(a)) = V_B(\varphi_*(a)), \quad a \in \mathcal{I}(A), \tag{3.2.15}
\]

\[
V_A(\varphi^{-1}b) = \varphi^*(V_B(b)), \quad b \in \mathcal{I}(B). \tag{3.2.16}
\]

Proof. The proofs of (3.2.14) and (3.2.15) are straightforward:

- For (3.2.14) we may assume $b = \sqrt{b}$ is a radical since $V_B(b) = V_B(\sqrt{b})$, $\varphi^{-1}(\sqrt{b}) = \sqrt{\varphi^{-1}(b)}$. Let $a = \mathcal{I}(\varphi^*(V_B(b)))$, so that $V_A(a) = \varphi^*(V_B(b))$ by (3.2.12). We have

\[
f \in a \iff f \in \mathfrak{p}, \forall \mathfrak{p} \in \varphi^*(V_B(b)) \iff f \in \varphi^{-1}(q), \quad \forall q \supseteq b
\]

\[
\iff f \in \varphi^{-1}(q), \quad \forall \mathfrak{p} \in \varphi^*(V_B(b))\]

\[
\iff f \in \varphi^{-1}(q), \quad \forall q \supseteq b
\]
\[ \varphi(f) \in \bigcap_{q \supseteq p} q = \sqrt{b} = b \iff f \in \varphi^{-1}(b). \quad (3.2.17) \]

It follows from (3.2.14), or from (3.2.15), that \( \varphi^* = \text{Spec}(\varphi) \) is continuous, hence \( A \mapsto \text{Spec} A \) is a contravariant functor from commutative \( F \)-Rings to compact, sober, topological spaces.

**Example 3.2.1**

Let \( A \) be a commutative ring, \( F(A) \) the associated \( F \)-Ring. An ideal \( a \subseteq A = F(a)_{1,1} \) is also an ideal in our sense, and conversely. Under this correspondence the primes of \( A \) correspond to the primes of \( F(A) \), and we have a homeomorphism with respect to the Zariski topologies:

\[ \text{Spec} A = \text{Spec} F(A). \quad (3.2.18) \]

**Example 3.2.2**

Let \( \eta : k \to \mathbb{C} \) be a real or complex prime of a number field, and let \( \mathcal{O}_{k,\eta} \) denote the \( F \)-Ring of real or complex 'integers'. Then

\[ m_\eta = \{ x \in k \mid |x|_\eta < 1 \} \quad (3.2.19) \]

is the (unique) maximal ideal of \( \mathcal{O}_{k,\eta} \), the closed point of Spec \( \mathcal{O}_{k,\eta} \).

### 3.3 Localization \( S^{-1}A \)

The theory of localization of an \( F \)-Ring \( A \), with respect to a multiplicative subset \( S \subseteq A_{1,1} \), goes exactly as in localization of commutative rings - since it is a multiplicative theory. We recall this theory next.

We assume \( S \subseteq A_{1,1} \) satisfies

\[ 1 \in S \quad (3.3.1) \]

\[ s_1, s_2 \in S \Rightarrow s_1 \cdot s_2 \in S \quad (3.3.2) \]

On the set

\[ A \times S = \coprod_{Y,X} A_{Y,X} \times S \]

we define for \( a_i \in A_{Y,X}, s_i \in S \)

\[ (a_1, s_1) \sim (a_2, s_2) \iff s \cdot s_2 \cdot a_1 = s \cdot s_1 \cdot a_2 \text{ for some } s \in S. \quad (3.3.3) \]

It follows that \( \sim \) is an equivalence relation, and we denote by \( a/s \) the equivalence class containing \( (a, s) \), and by \( S^{-1}A \) the collection of equivalence classes. On \( S^{-1}A \) we define the operations:

\[ a_1/s_1 \circ a_2/s_2 = (a_1 \circ a_2)/s_1s_2, \quad a_1 \in A_{Z,Y}, \quad a_2 \in A_{Y,X} \quad (3.3.4) \]

\[ a_1/s_1 \otimes a_2/s_2 = (s_2 \cdot a_1 \otimes s_1 \cdot a_2)/s_1s_2 \quad (3.3.5) \]
Proposition 3.3.1

The above operations are well defined, independent of the chosen representatives, and they satisfy the axioms of an $F$-Ring.

Proof. The usual proof works. For example, replacing $a_1/s_1$ in (3.3.5) by $a'_1/s'_1 \sim a_1/s_1$, say $s' \cdot s'_1 \cdot a_1 = s \cdot s_1 \cdot a_1'$, then

$$s \cdot s'_1 \cdot (s_2 \cdot s_1 a_1 \oplus s_1 a_2) = s \cdot s_1 \cdot (s_2 \cdot s'_1 \cdot a_2),$$

hence

$$(s_2 \cdot s_1 a_1 \oplus s_1 a_2)/s_1 s_2 = (s_2 a'_1 \oplus s'_1 a_2)/s'_1 s_2.$$ 

The $F$-Ring $S^{-1}A$ comes with a canonical homomorphism

$$\phi = \phi_S : A \to S^{-1}A, \ \phi(a) = a/1. \quad \text{(3.3.6)}$$

Proposition 3.3.2

We have the universal property of $\phi_S$:

$$\mathcal{HR}(S^{-1}A, B) = \{\varphi \in \mathcal{HR}(A, B) | \varphi(S) \subseteq GL_{1}(B)\}$$

$$\varphi \mapsto \varphi \circ \phi_S$$

$$\varphi(a/s) = \varphi(a) \cdot \varphi(s)^{-1} \quad \varphi$$

Proof. Clear. 

Note that $S^{-1}A$ is the zero $F$-Ring if and only if $0 \in S$.

The main examples of localizations are:

$$S_f = \{f^n\}_{n \geq 0}, \ f \in A_{1,1}. \ \text{We write } A_f \text{ for } S_f^{-1}A. \quad \text{(3.3.7)}$$

$$S_p = A_{1,1} \setminus p, \ p \in \text{Spec}(A). \ \text{We write } A_p \text{ for } S_p^{-1}A. \quad \text{(3.3.8)}$$

Consider the canonical homomorphism $\phi = \phi_S : A \to S^{-1}A, \ \phi(a) = a/1$. If $b \in \mathcal{I}(S^{-1}A)$, then $\varphi^{-1}(b) \in \mathcal{I}(A)$. If $a \in \mathcal{I}(A)$ is an ideal of $A$ then

$$S^{-1}a := \phi_S(a) = \{a/s \in (S^{-1}A)_{1,1} | a \in a, \ s \in S\} \quad \text{(3.3.9)}$$

is an ideal of $S^{-1}A$.

Proposition 3.3.3

If $b \in \mathcal{I}(S^{-1}A)$, then $S^{-1}(\varphi^{-1}\{b\}) = b$.

Proof. If $a/s \in b$, $a \in \varphi^{-1}\{b\}$, and $a/s \in S^{-1}(\varphi^{-1}\{b\})$; so $b \subseteq S^{-1}(\varphi^{-1}\{b\})$. The reverse inclusion is clear. 


Proposition 3.3.4
For \( a \in \mathcal{I}(A) \),
\[
\varphi^{-1}(S^{-1}a) = \{ a \in A \mid \exists \ s \in S \ : \ s \cdot a \in a \}.
\] (3.3.10)
In particular,
\[
S^{-1}a = (1) \iff a \cap S \neq \emptyset
\] (3.3.11)
Proof.
\[
a \in \varphi^{-1}(S^{-1}a) \iff a/1 = x/s, \ x \in a, \ s \in S \iff s \cdot a \in a, \ \text{for some} \ s \in S.
\]
\(\square\)

Proposition 3.3.5
The map \( \phi^*_S \) induces a bijection
\[
\phi^*_S : \text{Spec}(S^{-1}A) \xrightarrow{\sim} \{ p \in \text{Spec} A \mid p \cap S = \emptyset \},
\] (3.3.12)
which is a homeomorphism for the Zariski topology.
Proof. If \( q \in \text{Spec}(S^{-1}A) \), \( \phi^*_S(q) \) belongs to the right-hand-side. Conversely, if \( p \) belongs to the right-hand-side, \( S^{-1}p \) is a (proper) prime of \( S^{-1}A \). By propositions 3.3.3, 3.3.4, these operations are inverses of each other. \(\square\)

Corollary 3.3.6
We have homeomorphism for \( f \in A_{1,1} \),
\[
\phi^*_f : \text{Spec}(A_f) \xrightarrow{\sim} D_A(f).
\] (3.3.13)

Corollary 3.3.7
We have a homeomorphism for \( p \in \text{Spec}(A) \),
\[
\phi^*_p : \text{Spec}(A_p) \xrightarrow{\sim} \{ q \in \text{Spec} A \mid q \subseteq p \}.
\] (3.3.14)
In particular, \( A_p \) contains a unique maximal ideal \( \mathfrak{m}_p = S_p^{-1}p \); we say it is a local \( \mathbb{F} \text{-Ring} \).

Remark 3.3.8
For \( p \in \text{Spec}(A) \) we let \( \mathbb{F}_p = A_p/\mathfrak{m}_p \) denote the residue field at \( p \). Let \( \pi : A \rightarrow A/p \) be the canonical homomorphism, and \( S_p = \pi(S_p) \), we have also \( \mathbb{F}_p = S_p^{-1}(A/p) \). The commutative diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\phi_p} & A_p \\
\pi \downarrow & & \downarrow \\
A/p & \rightarrow & \mathbb{F}_p
\end{array}
\] (3.3.15)
is cartesian: $F_{p} = (A/p) \otimes_{A} A_{p}$

It is also functorial: given a homomorphism of $\mathbb{F}$-Rings $\varphi : A \to B$, $q \in \text{Spec } B$, $p = \varphi^*(q)$, we have a commutative cube

\[
\begin{array}{ccc}
A & \to & A_{p} \\
\downarrow & & \downarrow \\
B & \to & B_{q} \\
\downarrow & & \downarrow \\
A/p & \to & F_{p} \\
\downarrow & & \downarrow \\
B/q & \to & F_{q}
\end{array}
\]

(3.3.16)

3.4 Structure sheaf $O_{A}$

Next we define a sheaf $O_{A}$ of $\mathbb{F}$-Rings over $\text{Spec } A$.

Definition 3.4.1

For an open set $U \subseteq \text{Spec } (A)$, and for $Y, X \in \mathbb{F}$, we let $O_{A}(U)_{Y, X}$ denote the set of functions

\[
s : U \to \bigcup_{p \in U} (A_{p})_{Y, X},
\]

(3.4.1)
such that $s(p) \in (A_{p})_{Y, X}$, and $s$ is "locally a fraction":

\[
\forall \ p \in U, \exists \ a \text{ neighborhood } U_{p} \text{ of } p ; \exists \ a \in A_{Y, X} ; \exists f \in A_{1,1} \setminus \bigcup_{q \in U_{p}} q \quad (3.4.2)
\]
such that $s(q) = a/f \in A_{q}, \forall q \in U_{p}$

(★)

It is clear that

\[
O_{A}(U) = \bigcup_{Y, X} O_{A}(U)_{Y, X}
\]

(3.4.3)
is an $\mathbb{F}$-Ring. If $U' \subseteq U$, the natural restriction map $s \mapsto s|_{U'}$, is a homomorphism of $\mathbb{F}$-Rings $O_{A}(U) \to O_{A}(U')$, thus $O_{A}$ is a presheaf of $\mathbb{F}$-Rings. From the local nature of (★) we see that $O_{A}$ is in fact a sheaf of $\mathbb{F}$-Rings over $\text{Spec } A$, in the sense that for any $X, Y \in \mathbb{F}$, $U \mapsto O_{A}(U)_{Y, X}$ is a sheaf of (pointed) sets.
Proposition 3.4.2
For \( p \in \text{Spec}(A) \), the stalk
\[
O_{A, p} = \lim_{\to} O_A(U)
\]
(3.4.4)
of the sheaf \( O_A \) is isomorphic to \( A_p \).

Proof. The map taking a local section \( s \) in a neighborhood of \( p \) to \( s(p) \in A_p \), induces a homomorphism \( O_{A, p} \to A_p \), which is clearly surjective. It is also injective:
Let \( s_1, s_2 \in O_A(U)_{Y,X} \) have the same value at \( p, \ s_1(p) = s_2(p) \). Shrinking \( U \)
we may assume \( s_i = a_i/f_i \) on \( U, \ a_i \in A_{Y,X}, \ f_i \in A_{1,1} \), \( a_1/f_1 = a_2/f_2 \) in \( A_p \)
means \( h \cdot f_2 \cdot a_1 = h \cdot f_1 \cdot a_2, \ h \in A_{1,1} \setminus p, \) but then \( a_1/f_1 = a_2/f_2 \) in \( A_q \ \forall q \in U \cap D_A(h) \). \( \square \)

Proposition 3.4.3
For \( f \in A_{1,1} \), the \( \mathbb{F} \)-Ring \( O_A(D_A(f)) \) is isomorphic to \( A_f \).
In particular, the global sections \( \Gamma(\text{Spec}(A), O_A) \) def \( O_A(D_A(1)) \) \( \cong \) \( A \).

Proof. Define the homomorphism \( \psi : A_f \to O_A(D_A(f)) \) by sending \( a/f^n \) to the section whose value at \( p \) is the image of \( a/f^n \) in \( A_p \).
We shall show that \( \psi \) is injective:
If \( \psi(a_1/f^{n_1}) = \psi(a_2/f^{n_2}) \) then \( \forall p \in D_A(f) \) there is \( h_p \in A_{1,1} \setminus p \) with
\[
h_p f^{n_2} a_1 = h_p f^{n_1} a_2.
\]
(3.4.5)
Let \( a = ann_A(f^{n_2} a_1, f^{n_1} a_2) \), it is an ideal of \( A \), and \( \forall p \in D_A(f), \ p \notin V_A(a) \),
so \( D_A(f) \cap V_A(a) = \emptyset \), hence \( V_A(a) \subseteq V_A(f) \), hence \( f \in IV_A(a) = \sqrt{a} \), hence
\( f^n \in a \) for some \( n \geq 1 \), showing that \( a_1/f^{n_1} = a_2/f^{n_2} \) in \( A_f \).
We show next that \( \psi \) is surjective:
Let \( s \in O_A(D_A(f))_{Y,X} \). By proposition 3.2.2, \( D_A(f) \) is compact, so there exists
a finite open covering
\[
D_A(f) = \bigcup_{1 \leq i \leq N} D_A(h_i),
\]
(3.4.6)
such that for all \( p \in D_A(h_i) : \ s(p) = a_i/g_i \in A_p \), where \( a_i \in A_{Y,X} \) and \( g_i \in A_{[1],[1]} \) is such that \( D_A(g_i) \supseteq D_A(h_i) \) for \( 1 \leq i \leq N \).
We have \( V_A(g_i) \subseteq V_A(h_i) \), hence
\[
\sqrt{(g_i)} = IV_A(g_i) \supseteq IV_A(h_i) = \sqrt{(h_i)},
\]
(3.4.7)
hence \( h_i \in \sqrt{(g_i)} \) so that for some \( n_i \geq 1 \) we have \( h_i^{n_i} = c_i \cdot g_i \), hence \( s(p) = c_i a_i/h_i^{n_i} \). So we can replace \( h_i \) by \( g_i \). On the set
\[
D_A(g_i) \cap D_A(g_j) = D_A(g_i g_j)
\]
(3.4.8)
we have \( a_i/g_i = s(p) = a_j/g_j \), hence by the injectivity of \( \psi \)

\[
a_i/g_i = a_j/g_j \text{ in } A_{g_i/g_j}. \quad (3.4.9)
\]

This means \((g_i g_j)^n \cdot g_j a_i = (g_i g_j)^n \cdot g_j a_j\), and we can choose \( n \) big enough to work for all \( i, j \). We can replace \( g_i \) by \( g_i^{n+1} \) (since \( D_A(g_i) = D_A(g_i^{n+1}) \)), and replace \( a_i \) by \( g_i^n \cdot a_i \) (since \( s(p) = g_i^n a_i/g_i^{n+1} \)), and then have the simpler equation

\[
g_j \cdot a_i = g_i \cdot a_j \forall i, j.
\]

Since the sets \( D_A(g_i) \) cover \( D_A(f) \) we have, (cf., Proposition 3.2.2),

\[
f^m = b \circ (\oplus_i g_i) \circ b'. \quad (3.4.10)
\]

Set

\[
a = (\oplus b) \circ (\oplus a_i) \circ (\oplus b'). \quad (3.4.11)
\]

Then

\[
g_j \cdot a = (\oplus b) \circ (\oplus g_j a_i) \circ (\oplus b')
\]

\[
= (\oplus b) \circ (\oplus g_i a_j) \circ (\oplus b')
\]

\[
= \oplus (b \circ (\oplus g_i) \circ b') \circ a_j \text{ (by commutativity!)} \quad (3.4.12)
\]

Hence \( a_j/g_j = s(p) = a/f^m \) and \( s = \psi(a/f^m) \). \( \square \)

### 3.5 Grothendieck \( \mathbb{F} \)-Schemes and locally-\( \mathbb{F} \)-ringed spaces

We define the categories of \( \mathbb{F} \)-(locally)-ringed-spaces, and its full subcategory of (Grothendieck) \( \mathbb{F} \)-schemes.

**Definition 3.5.1**

An \( \mathbb{F} \)-ringed-space \((X, \mathcal{O}_X)\) is a topological space with a sheaf \( \mathcal{O}_X \) of \( \mathbb{F} \)-Rings: \( U \mapsto \mathcal{O}_X(U) \) is a pre-sheaf of \( \mathbb{F} \)-Rings such that for any \( W, Z \in \mathbb{F}, U \mapsto \mathcal{O}_X(U)_{W,Z} \) is a sheaf of (pointed) sets. That is, a collection \( \mathcal{O}_X(U) \) together with restriction maps \( \rho^U_V \in \mathbb{F} \mathcal{R}(\mathcal{O}_X(U), \mathcal{O}_X(V)) \) for every inclusion of open sets \( V \subseteq U \), such that:

\[
\begin{array}{ccc}
\mathcal{O}_X(U) & \xrightarrow{\rho^U_V} & \mathcal{O}_X(V) \\
\rho^U_W \downarrow & & \downarrow \rho^V_W \\
\mathcal{O}_X(W)
\end{array}
\]

is commutative for \( W \subseteq V \subseteq U \).
And for all \( Y, X \in \mathbb{F} \): \( U \mapsto \mathcal{O}_X(U)_{Y,X} \) is a sheaf, i.e. for any open covering \( U = \bigcup U_i \):

\[
0 \rightarrow \mathcal{O}_X(U)_{Y,X} \rightarrow \prod_i \mathcal{O}_X(U_i)_{Y,X} \cong \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)_{Y,X}
\]

is exact.

A map of \( \mathbb{F} \)-ringed-spaces \( f: X \rightarrow Y \) is a continuous map of the underlying topological spaces together with a map of sheaves of \( \mathbb{F} \)-Rings on \( Y \), \( f^\#: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X \), i.e. for \( U \subseteq Y \) open we have \( f^\#_U : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}U) \) a map of \( \mathbb{F} \)-Rings, such that for \( U' \subseteq U \):

\[
f^\#_U(s)|_{f^{-1}U'} = f^\#_{U'}(s|_{U'}). \tag{3.5.1}
\]

The \( \mathbb{F} \)-ringed-space \( X \) is \( \mathbb{F} \)-locally-ringed-space if for all \( p \in X \) the stalk \( \mathcal{O}_{X,p} \) is a local \( \mathbb{F} \)-Ring, i.e. contains a unique maximal ideal \( m_{X,p} \).

For a map of \( \mathbb{F} \)-ringed-spaces \( f : X \rightarrow Y \), and for \( p \in X \), we get an induced homomorphism of \( \mathbb{F} \)-Rings on the stalks

\[
f^\#_p : \mathcal{O}_{Y,f(p)} = \lim_{f(p) \in V} \mathcal{O}_Y(V) \rightarrow \lim_{p \in f^{-1}V} \mathcal{O}_X(f^{-1}V) \rightarrow \lim_{p \in U} \mathcal{O}_X(U) = \mathcal{O}_{X,p}
\]

\[
\tag{3.5.2}
\]

A map \( f : X \rightarrow Y \) of \( \mathbb{F} \)-locally-ringed-spaces is a map of \( \mathbb{F} \)-ringed-spaces such that \( f^\#_p \) is a local homomorphism for all \( p \in X \), i.e.

\[
f^\#_p (m_{Y,f(p)}) \subseteq m_{X,p} \text{ or equivalently } (f^\#_p)^{-1} m_{X,p} = m_{Y,f(p)}. \tag{3.5.3}
\]

We let \( \mathbb{F}RS_p \) (resp. \( \mathcal{L}\mathbb{F}RS_p \)) denote the category of \( \mathbb{F} \)-(resp. locally)-ringed-spaces.

For a homomorphism of commutative \( \mathbb{F} \)-Rings \( \varphi : A \rightarrow B \), for \( p \in \text{Spec}(B) \), we have a unique homomorphism \( \varphi_p : A_{\varphi^{-1}p} \rightarrow B_p \), such that we have a commutative diagram,

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
A_{\varphi^{-1}p} & \xrightarrow{\varphi_p} & B_p
\end{array}
\]

\[
\varphi_p(a/s) = \varphi(a)/\varphi(s), \text{ and } \varphi_p \text{ is a local homomorphism.}
\]

Thus \( A \mapsto \text{Spec}(A) \) is a contravariant functor from \( \mathbb{F} \)-Rings to \( \mathcal{L}\mathbb{F}RS_p \).

It is the adjoint of the functor \( \Gamma \) of taking global sections

\[
\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X), \quad \Gamma(f) = f^\#: \mathcal{O}_Y \rightarrow \mathcal{O}_X \tag{3.5.5}
\]
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Proposition 3.5.2

$$\mathcal{LFRSp}(X, \text{Spec}(A)) = \mathcal{FR}(A, \mathcal{O}_X(X)).$$ \hspace{1cm} (3.5.6)

Proof. For an \( \mathbb{F} \)-locally-ringed-space \( X \), and for a point \( x \in X \), the canonical homomorphism \( \phi_x : \mathcal{O}_X(X) \to \mathcal{O}_{X,x} \) gives a prime \( \mathcal{P}(x) = \phi_x^{-1}(\mathfrak{m}_{X,x}) \in \text{Spec} \mathcal{O}_X(X) \). The map \( \mathcal{P} : X \to \text{Spec} \mathcal{O}_X(X) \) is continuous:

$$\mathcal{P}^{-1}(D(f)) = \{ x \in X | \phi_x(f) \notin \mathfrak{m}_{X,x} \}.$$ \hspace{1cm} (3.5.7)

is open for \( f \in \mathcal{O}_X(X) \). We have an induced homomorphism

$$\mathcal{P}^\# : \mathcal{O}_X(X)_{\mathcal{P}(x)} \to \mathcal{O}_{X,x},$$ \hspace{1cm} (3.5.8)

making \( \mathcal{P} \) a map of \( \mathbb{F} \)-ringed-spaces, and taking the direct limit over \( f \) with \( \phi_x(f) \notin \mathfrak{m}_{X,x} \) we get

$$\mathcal{P}^\# : \mathcal{O}_X(X)_{\mathcal{P}(x)} \to \mathcal{O}_{X,x},$$ \hspace{1cm} (3.5.9)

showing \( \mathcal{P} \) is a map of \( \mathbb{F} \)-locally-ringed-spaces.

To a homomorphism of \( \mathbb{F} \)-Rings \( \varphi : A \to \mathcal{O}_X(X) \) we associate the map of \( \mathbb{F} \)-locally-ringed-spaces

$$X \xrightarrow{\mathcal{P}} \text{Spec} \mathcal{O}_X(X) \xrightarrow{\text{Spec}(\varphi)} \text{Spec} A.$$ \hspace{1cm} (3.5.10)

Conversely, to a map \( f : X \to \text{Spec} A \) of \( \mathbb{F} \)-locally-ringed-spaces (as in definition 3.5.1) we associate its action on global sections

$$\Gamma(f) = f^\#_{\text{Spec} A} : A = \mathcal{O}_A(\text{Spec} A) \to \mathcal{O}_X(X).$$ \hspace{1cm} (3.5.11)

Clearly, \( \Gamma(\text{Spec}(\varphi) \circ \mathcal{P}) = \varphi \).

Conversely, given a map \( f : X \to \text{Spec} A \) (as in definition 3.5.1), for \( x \in X \) we have a commutative diagram,

$$\begin{array}{ccc}
A & \xrightarrow{\Gamma(f)} & \mathcal{O}_X(X) \\
\downarrow{\phi_{f(x)}} & & \downarrow{\phi_x} \\
A_{f(x)} & \xrightarrow{f^\#_{f(x)}} & \mathcal{O}_{X,x}
\end{array}$$ \hspace{1cm} (3.5.12)

Since \( f^\#_{f(x)} \) is assumed to be local, \( (f^\#_{f(x)})^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{f(x)} \), and by the commutativity of the diagram we get \( \Gamma(f)^{-1}(\mathcal{P}(x)) = f(x) \), i.e. \( f = (\text{Spec} \Gamma(f)) \circ \mathcal{P} \) is the continuous map associated to the homomorphism \( \Gamma(f) \). Similarly, for \( g \in A \), the commutativity of the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\Gamma(g)} & \mathcal{O}_X(X) \\
\downarrow{\phi_{g(x)}} & & \downarrow{\phi_x} \\
A_g & \xrightarrow{f^\#_{f(g)}} & \mathcal{O}_{X,x}
\end{array}$$ \hspace{1cm} (3.5.13)

gives \( f^\#_{f(g)}(a/g^n) = \Gamma(f)(a)/(\Gamma(f)(g))^n \), hence \( f = (\text{Spec} \Gamma(f)) \circ \mathcal{P} \) as a map of \( \mathbb{F} \)-locally-ringed-spaces. \( \square \)
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Corollary 3.5.3
For \( \mathcal{F}\text{-Rings} \ A, B \):
\[
\mathcal{LRSp} \left( \text{Spec } B, \text{Spec } A \right) = \mathcal{F}(A, B).
\] (3.5.14)

Definition 3.5.4
A (Grothendieck) \( \mathcal{F}\)-scheme is an \( \mathcal{F}\)-locally-ringed-space \((X, \mathcal{O}_X)\), such that there is a covering by open sets \( X = \cup_i U_i \), and the canonical maps
\[
P : (U_i, \mathcal{O}_X|_{U_i}) \to \text{Spec } \mathcal{O}_X(U_i)
\] (3.5.15)
are isomorphisms of \( \mathcal{F}\)-locally-ringed-spaces. A morphism of \( \mathcal{F}\)-schemes is a map of \( \mathcal{F}\)-locally-ringed-spaces.

We denote the category of \( \mathcal{F}\)-schemes by \( \mathcal{gFS}\).

\( \mathcal{F}\)-schemes can be glued:

Proposition 3.5.5
Given a set of indices \( I \), and for \( i \in I \) given \( X_i \in \mathcal{gFS} \), and for \( i \neq j \), \( i, j \in I \), an isomorphism \( \varphi_{ij} : U_{ij} \cong U_{ji} \), with \( U_{ij} \subseteq X_i \) open (and hence \( U_{ij} \) are \( \mathcal{F}\)-schemes), such that
\[
\varphi_{ij} = \varphi_{ij}^{-1}
\] (3.5.16)
\[
\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}, \text{ and } \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \text{ on } U_{ij} \cap U_{ik}.
\] (3.5.17)
There exists \( X \in \mathcal{gFS} \), and maps \( \psi_i : X_i \to X \), such that
\( \psi_i \) is an isomorphism of \( X_i \) onto the open set \( \psi_i(X_i) \subseteq X \),
\[
X = \bigcup_i \psi_i(X_i),
\] (3.5.18)
\[
\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j),
\] (3.5.19)
\[
\psi_i = \psi_j \circ \varphi_{ij} \text{ on } U_{ij}.
\] (3.5.20)

Proof. Clear: glue the topological spaces and glue the sheaves of \( \mathcal{F}\)-Rings. For \( V \subseteq X \) open
\[
\mathcal{O}_X(V) = \ker \left\{ \prod_i \mathcal{O}_{X_i}(\psi_i^{-1}V) \Rightarrow \prod_{i,j} \mathcal{O}_{X_i}(\psi_i^{-1}V \cap U_{ij}) \right\}.
\] (3.5.21)
Chapter 4

Symmetric Geometry

In this section $A \in C\mathbb{F}R^i$ is commutative and has involution.

4.1 Symmetric ideals and symmetric primes

Definition 4.1.1

Consider the set $A^+ := \{ a \in A_{1,1} \mid a = a^t \}$. An ideal $a \subseteq A_{1,1}$ is called symmetric if it is generated by $a^+ := a \cap A^+$. (In particular $a = a^t$).

Denote by $I^+(A)$ the symmetric ideals of $A$.

Given an indexed set of symmetric ideals $a_i \subseteq A_i, i \in I$, their symmetric intersection $\bigcap_I a_i$ is the ideal generated by the mutual symmetric elements:

$$\bigcap_I^{+} a_i = \{ b \circ (\oplus a_j) \circ b' \mid a_j \in \bigcap_I^{+} a_i \}. \quad (4.1.1)$$

Their sum is again a symmetric ideal:

$$\sum_I a_i = \{ b \circ (\oplus a_j) \circ b' \mid a_j \in \bigcup_I^{+} a_i \}. \quad (4.1.2)$$

Their product $a \cdot a'$ is again the symmetric ideal:

$$a \cdot a' = \{ b \circ (\oplus a_j \cdot a'_j) \circ b' \mid a_j, a'_j \in a^+, a_j, a'_j \in a'^+ \}. \quad (4.1.3)$$

Let $\varphi : A \to B$ be a homomorphism of $\mathbb{F}$-Rings. If $b \in \mathcal{I}^+(B)$, we define:

$$\varphi^*(b) = \text{ideal generated by } \{ a = a' \in A^+, \varphi(a) \in b \} \equiv \langle \varphi^{-1}(b^+) \rangle \quad (4.1.4)$$

and if $a \in \mathcal{I}^+(A)$, we define,

$$\varphi_*(a) = \text{ideal generated by } \varphi(a^+) \subseteq B^+ \equiv \langle \varphi(a^+) \rangle. \quad (4.1.5)$$
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Proposition 4.1.1
For any \( a \in \mathcal{I}^+ (A), b \in \mathcal{I}^+ (B) \), we have the following:

1. \( a \subseteq \varphi^* \varphi_\ast a \).
2. \( b \supseteq \varphi^* \varphi_\ast b \).
3. \( \varphi^* b = \varphi^* \varphi_\ast \varphi^* b, \quad \varphi^* a = \varphi^* \varphi_\ast \varphi^* a \).
4. there is a bijection, via \( a \mapsto \varphi_\ast a \) (with inverse map \( b \mapsto \varphi^* b \)), from the set
   \[ \{ a \in \mathcal{I}^+ (A) | \varphi^* \varphi_\ast a = a \} = \{ \varphi^* b | b \in \mathcal{I}^+ (B) \} \]
   (4.1.6) to the set
   \[ \{ b \in \mathcal{I}^+ (B) | \varphi^* \varphi_\ast b = b \} = \{ \varphi_\ast a | a \in \mathcal{I}^+ (A) \}. \]
   (4.1.7)

Since the union of a chain of (proper) symmetric ideals is a (proper) symmetric ideal, we have

Theorem 4.1.1 (Zorn)
There exists a maximal symmetric ideal \( \mathfrak{m} \subseteq A_{1,1} \).

Definition 4.1.2
A symmetric ideal \( \mathfrak{p} \subseteq A_{1,1} \) is called symmetric prime:
   \[ S_\mathfrak{p}^+ = A^+ \setminus \mathfrak{p} \text{ is multiplicatively closed } \quad S_\mathfrak{p}^+ \cdot S_\mathfrak{p}^+ = S_\mathfrak{p}^+. \]
   (4.1.8)

We denote by \( \text{Spec}^+ A \) the set of symmetric prime ideals.

For a homomorphism of \( \mathbb{F}\text{-Rings} \phi : A \rightarrow B \), the pullback \( \varphi^* \) induce a map
   \[ \varphi^* = \text{Spec}(\varphi) : \text{Spec}^+ B \rightarrow \text{Spec}^+ A. \]
   (4.1.9)

Proposition 4.1.2
1. If \( \mathfrak{m} \in \mathcal{I}^+ (A) \) is a maximal symmetric ideal then it is symmetric prime.
2. More generally, if \( a \in \mathcal{I}^+ (A) \), and given \( f = f^t \in A^+ \) such that,
   \[ \forall n \in \mathbb{N} : \quad f^n \notin a. \]
   let \( \mathfrak{m} \) be a maximal element of the set
   \[ \{ b \in \mathcal{I}^+ (A) | b \supseteq a, f^n \notin b \ \forall n \in \mathbb{N} \} \]
   (4.1.10)
   Then \( \mathfrak{m} \) is symmetric prime.

Proof. (1) If \( x = x^t, y = y^t \in A_{1,1} \setminus \mathfrak{m} \), the ideals \( (x) + \mathfrak{m}, (y) + \mathfrak{m} \) are the unit ideals. So we can write
   \[ 1 = b \circ (\bigoplus_j m_j) \circ d, \quad \text{with } m_j = x \text{ or } m_j \in \mathfrak{m}. \]
   (4.1.11a)
\[ 1 = b' \circ (\bigoplus_j m'_j) \circ d', \quad \text{with } m'_j = y \text{ or } m'_j \in m. \]  

(4.1.11b) 

It then follows that, 
\[ 1 = 1 \cdot 1 = b \circ \bigoplus_j m_j \circ d \circ b' \circ \bigoplus_i m'_i \circ d' \]  

(4.1.12) 

but \( m_j \circ m'_i = x \circ y \) or \( m_j \circ m'_i \in m \), so 1 is in the ideal generated by m and \( x \circ y \), and since 1 \( \notin m \) then \( x \circ y \notin m \).

(2) Similarly, if \( x \notin m \) then \( f^n \) is in the ideal generated by \( x \) and \( m \) so \( f^n = b \circ \bigoplus_j m_j \circ d \), with \( m_j = x \) or \( m_j \in m \). If \( y \notin m \), \( f^n = b' \circ \bigoplus_i m'_i \circ d' \), with \( m_i = y \) or \( m_i \in m \). It then follows that 
\[ f^{n+n'} = b \circ \bigoplus_j m_j \circ d \circ b' \circ \bigoplus_i m'_i \circ d' \]  

(4.1.13) 

but \( m_j \circ m'_i = x \circ y \) or \( m_j \circ m'_i \in m \), so \( f^{n+n'} \) is in the ideal generated by \( m \) and \( x \circ y \), and since \( f^{n+n'} \notin m \) then \( x \circ y \notin m \).

Definition 4.1.3

For \( a \in \mathcal{I}^+(A) \), the symmetric radical is 
\[ \sqrt{a^+} = \text{ideal generated by } \{f = f^i \in A^+, f^n \in a \text{ for some } n \geq 1\} \]  

(4.1.14) 

\[ = \{b \circ \bigoplus_j f_j \circ d \mid b \in A_1, j, d \in A_{1,1}, f_j = f^i_j \in A^+, \text{ and } f^n_j \in a\}. \]

Note that \( \sqrt{a^+} \subseteq \sqrt{a} \), and for any \( a \in \sqrt{a}^+ \), we have \( a^n \in a \) for \( n >> 1 \).

Proposition 4.1.4

We have 
\[ \sqrt{a^+} = \bigcap_{a \in \mathfrak{p}} \mathfrak{p} = \text{i.e. the symmetric ideal generated by } \bigcap_{a \in \mathfrak{p}} \mathfrak{p}^+ \]  

(4.1.15) 

where \( \mathfrak{p} \) runs over symmetric primes containing \( a \).

Proof. If \( f = f^i \), \( f^n \in a \), then for all symmetric primes \( a \subseteq \mathfrak{p} \): \( f \in \mathfrak{p} \). If \( f = f^i \) and \( f^n \notin a, \forall n \), let \( m \) be a maximal element of the set (4.1.10), it exists by Zorn’s lemma, and it is symmetric prime by proposition (4.1.2b), \( a \subseteq m \) and \( f \notin m \).
4.2 The symmetric spectrum: \( \text{Spec}^+(A) \)

**Definition 4.2.1**

For a set \( \mathcal{U} \subseteq A^+ \), we let

\[
V_A^+(\mathcal{U}) = \{ \mathfrak{p} \in \text{Spec}^+(A) | \mathcal{U} \subseteq \mathfrak{p} \} \tag{4.2.1}
\]

If \( \mathfrak{a} \) is the ideal generated by \( \mathcal{U} \), \( V_A^+(\mathcal{U}) = V_A^+(\mathfrak{a}) \); we have

\[
V_A^+(1) = \emptyset, \quad V_A^+(0) = \text{Spec}^+(A), \tag{4.2.2}
\]

\[
V_A^+(\Sigma \mathfrak{a}) = \bigcap_{i} V_A^+(\mathfrak{a}_i), \quad \mathfrak{a}_i \in \mathcal{I}^+(A),
\]

\[
V_A^+(\mathfrak{a} \cdot \mathfrak{a}') = V_A^+(\mathfrak{a}) \cup V_A^+(\mathfrak{a}').
\]

Hence the sets \( \{ V_A^+(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}^+(A) \} \) are the closed sets for the topology on \( \text{Spec}^+(A) \), the Zariski topology.

**Definition 4.2.2**

For \( \mathfrak{f} = f^i \in A^+ \) we let

\[
D_A^+(\mathfrak{f}) = \text{Spec}^+(A) \setminus V_A^+(\mathfrak{f}) = \{ \mathfrak{p} \in \text{Spec}^+(A) | f \not\in \mathfrak{p} \}. \tag{4.2.3}
\]

We have

\[
D_A^+(\mathfrak{f}_1) \cap D_A^+(\mathfrak{f}_2) = D_A^+(\mathfrak{f}_1 \cdot \mathfrak{f}_2), \tag{4.2.4}
\]

\[
\text{Spec}^+ A \setminus V_A^+(\mathfrak{a}) = \bigcup_{\mathfrak{f} \in A^+} D_A^+(\mathfrak{f}).
\]

Hence the sets \( \{ D_A^+(\mathfrak{f}) | \mathfrak{f} \in A^+ \} \) are the basis for the open sets in the Zariski topology. We have

\[
D_A^+(\mathfrak{f}) = \emptyset \iff \mathfrak{f} \in \bigcap_{\mathfrak{p} \in \text{Spec}^+ A} \mathfrak{p} = \sqrt{0^+} \iff f^n = 0 \text{ for some } n \tag{4.2.5}
\]

and we say \( f \) is a nilpotent. We have

\[
D_A^+(\mathfrak{f}) = \text{Spec}^+ A \iff (\mathfrak{f}) = (1) \iff \exists f^{-1} \in A^+ : f \cdot f^{-1} = 1 \tag{4.2.6}
\]

and we say \( f \) is invertible. We denote by \( GL_1^+(A) \) the (commutative) group of symmetric invertible elements.

**Definition 4.2.3**

For a subset \( X \subseteq \text{Spec}^+(A) \), we have the associated ideal

\[
\mathcal{I}^+(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = \text{ideal generated by } \bigcap_{\mathfrak{p} \in X} \mathfrak{p}^+. \tag{4.2.7}
\]
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Proposition 4.2.1

We have
\[ \mathcal{I}^+ \mathcal{V}_A^+ a = \sqrt{a}^+, \quad (4.2.8a) \]
\[ \mathcal{V}_A^+ \mathcal{I}^+ (X) = \overline{X}, \text{ the closure of } X \text{ in the Zariski topology.} \quad (4.2.8b) \]

Proof. The first equation is just a restatement of proposition 4.1.4. Indeed,
\[ \mathcal{I}^+ \mathcal{V}_A^+ a = \text{ideal generated by } \bigcap_{a \in \mathfrak{p} \in \text{Spec}^+(A)} \mathfrak{p}^+ = \sqrt{a}^+ \quad (4.2.9) \]

For the second, \( \mathcal{V}_A^+ \mathcal{I}^+ (X) \) is clearly a closed set containing \( X \), and if \( C = \mathcal{V}_A^+ (a) \) is a closed set containing \( X \), then \( \sqrt{a}^+ = \mathcal{I}^+ \mathcal{V}_A^+ (a) \subseteq \mathcal{I}^+ (X) \), hence \( C = \mathcal{V}_A^+ (\sqrt{a}^+) \supseteq \mathcal{V}_A^+ \mathcal{I}^+ (X) \)

Corollary 4.2.1

We have a one-to-one order-reversing correspondence between closed sets \( X \subseteq \text{Spec}^+(A) \), and radical symmetric ideals \( a \), via \( X \mapsto \mathcal{I}^+ (X) \), \( \mathcal{V}_A^+ (a) \leftrightarrow a \)
\[ \{ X \subseteq \text{Spec}^+(A) | X = X \} \leftrightarrow \{ a \in \mathcal{I}^+ (A) | \sqrt{a}^+ = a \}. \quad (4.2.10) \]

Under this correspondence the closed irreducible subsets corresponds to symmetric prime ideals. For \( \mathfrak{p}_0, \mathfrak{p}_1 \in \text{Spec}^+(A), \mathfrak{p}_0 \in \{ \mathfrak{p}_1 \} \iff \mathfrak{p}_0 \supseteq \mathfrak{p}_1 \), we say that \( \mathfrak{p}_0 \) is a Zariski specialization of \( \mathfrak{p}_1 \), or that \( \mathfrak{p}_1 \) is the Zariski generalization of \( \mathfrak{p}_0 \). The space \( \text{Spec}^+(A) \) is sober: every closed irreducible subset \( C \) has the form \( C = \mathcal{V}_A^+ (\mathfrak{p}) = \{ \mathfrak{p} \} \), and we call the (unique) prime \( \mathfrak{p} \) the generic point of \( C \).

Proposition 4.2.2

The sets \( D_A^+(f) \), and in particular \( D_A^+(1) = \text{Spec}^+(A) \), are compact.

Proof. Note that \( D_A^+(f) \) is contained in the union \( \bigcup_i D_A^+(g_i) \) if and only if \( \mathcal{V}_A^+ (f) \supseteq \bigcap_i \mathcal{V}_A^+ (g_i) = \mathcal{V}_A^+ (a) \), where \( a \) is the ideal generated by \( \{ g_i \} \), if and only if \( \sqrt{a}^+ = \mathcal{I}^+ \mathcal{V}_A^+ (f) \subseteq \mathcal{I}^+ \mathcal{V}_A^+ (a) = \sqrt{a}^+, \) if and only if \( f^n \in a^+ \) for some \( n \), if and only if \( f^n = b \circ (\bigoplus g_i) \circ b^\prime \), and in such expression only a finite number of the \( g_i \) are involved.

Let \( \varphi : A \to B \) be a homomorphism of \( F \)-Ring with involution, \( \varphi^* : \text{Spec}^+(B) \to \text{Spec}^+(A) \) the associated pullback map.


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**Proposition 4.2.3**

We have

\[ \varphi^*(D_A^+(f)) = D_B^+(\varphi(f)), \quad f \in A^+, \quad (4.2.11) \]

\[ \varphi^*(V_A^+(a)) = V_B^+(\varphi^*(a)), \quad a \in \mathcal{I}^+(A), \quad (4.2.12) \]

\[ V_A^+(\varphi^*b) = \varphi^*(V_B^+(b)), \quad b \in \mathcal{I}^+(B). \quad (4.2.13) \]

**Proof.** The proofs of (4.2.11) and (4.2.12) are straightforward:

\[ q \in \varphi^{-1}(D_A^+(f)) \iff \varphi^*(q) \in D_A^+(f) \iff f \notin \varphi^{-1}(q) \iff \varphi(f) \notin q \iff q \in D_B^+(\varphi(f)), \]

\[ q \in \varphi^{-1}(V_A^+(a)) \iff \varphi^*(q) \in V_A^+(a) \iff a^+ \subseteq \varphi^{-1}(q) \iff \varphi^*(a) \subseteq q \iff q \in V_B^+(\varphi^*(a)). \]

For (4.2.13) we may assume \( b = \sqrt{b^+} \) is a radical since \( V_B^+(b) = V_B^+(\sqrt{b^+}), \varphi^*(\sqrt{b^+}) = \sqrt{\varphi^*(b)^+} \). Let \( a = \mathcal{I}^+(\varphi^*(V_B^+(b))) \), so that \( V_A^+(a) = \varphi^*(V_B^+(b)) \) by (4.2.8b).

We have

\[ f \in a \iff f \in p, \forall p \in \varphi^*(V_B^+(b)) \iff f \in \varphi^{-1}(q), \forall q \supseteq b \]

\[ \varphi(f) \in \bigcap_{q \supseteq b}^+ q = \sqrt{b^+} = b \iff f \in \varphi^*(b). \quad (4.2.14) \]

It follows from (4.2.11), or from (4.2.12), that \( \varphi^* = \text{Spec}^+(\varphi) \) is continuous, hence \( A \rightarrow \text{Spec}^+(A) \) is a contravariant functor from \( \text{CF-Rings}^t \) to compact, sober, topological spaces.

**Example 4.2.1**

Let \( A \) be a commutative ring with (nontrivial) involution \( a \mapsto a^t \), \( \mathbb{F}(A) \) the associated \( \mathbb{F} \)-Ring with involution \( (a^t)_{x,y} = (a_{y,x})^t, a = (a_{y,x}) \in A_{Y,X} \). An ideal \( a \subseteq A = \mathbb{F}(a)_{1,1} \) is also an ideal in our sense, and conversely. Under this correspondence the primes of \( A \) correspond to the primes of \( \mathbb{F}(A) \), and we have a homeomorphism with respect to the Zariski topologies: \( \text{Spec} \mathbb{F}(A) \cong \text{Spec} A \).

The symmetric primes correspond to primes of the subring

\[ A^+ = \{ a \in A \mid a^t = a \} \quad (4.2.15) \]

consisting of symmetric elements. (note that \( A/A^+ \) is always integral, \( \alpha \in A \) is a root of \( x^2 - (\alpha + \alpha^t)x + \alpha\alpha^t \in A^+[x] \)). We have as well a homeomorphism with respect to the topology defined in (4.2.1 – 2): \( \text{Spec}^+ \mathbb{F}(A) \cong \text{Spec} A^+ \).

Given \( A \in \text{CFR}^t \), we can forget the involution and consider \( \text{Spec} A \). There is a canonical map

\[ \pi^+_A : \text{Spec} A \rightarrow \text{Spec}^+ A \]
4.3 Symmetric localization

A set $S \subseteq A^{+}_{1,1}$ (consisting of symmetric elements!) is called "multiplicative" when

\begin{align*}
1 & \in S \\
S \cdot S & = S
\end{align*}

(4.3.1, 4.3.2)

For such $S$ the localized $\mathbb{F}$-Ring $S^{-1}A$ has involution:

\[(a/s)^t = a^t/s.\]

(4.3.3)

The following universal property holds:

\[\mathbb{F}R^t(S^{-1}A, B) = \{ \varphi \in \mathbb{F}R^t(A, B) \mid \varphi(S) \subseteq GL^+_1(B) \}. \]

(4.3.4)

The main examples of localizations are:

For $p \in \text{Spec}^+ A$, $S_p = A^+ \setminus p$, $S_p^{-1}A := A_p$

(4.3.5)

For $f = f^t \in A^+$, $S_f = \{ 1, f, f^2, \ldots, f^n, \ldots \}$, $S_f^{-1}A := A_f = A[\frac{1}{f}]$

(4.3.6)

4.4 Structure sheaf $\mathcal{O}_A^t/\text{Spec}^+ (A)$

**Definition 4.4.1**

A sheaf of $\mathbb{F}R^t$ (with involution) over a topological space $X$, $\mathcal{O} \in \mathbb{F}R^t/X$ is a pre-sheaf of $\mathbb{F}$-Rings with involutions $U \mapsto \mathcal{O}(U)$, such that for any $W, Z \in \mathbb{F}$, $U \mapsto \mathcal{O}(U)_{WZ}$ is a sheaf.

Next we define a sheaf $\mathcal{O}_A^t$ of $\mathbb{F}R^t$ over $\text{Spec}^+ A$.

**Definition 4.4.2**

For an open set $U \subseteq \text{Spec}^+ A$, $X, Y \in \mathbb{F}$:

\[\mathcal{O}_A(U)_{Y,X} := \{ s : U \rightarrow \bigsqcup_{p \in U} (A_p)_{Y,X}, s(p) \in (A_p)_{Y,X}, \text{ and satisfy } (*) \} \]

\[(*) \quad \forall p \in U, \exists f = f^t \notin p, p \in D^+(f) \subseteq U, \exists a \in A_{Y,X}, s(q) = a/f, \forall q \in D^+(f). \]
Theorem 4.4.1

(i) For \( p \in \text{Spec}^+ A \) we have
\[
\mathcal{O}_{A,p} = \lim_{\substack{\longrightarrow \\ \text{\scriptsize p} \in U}} \mathcal{O}_A(U) \xrightarrow{\sim} A_p.
\] (4.4.2)

(ii) For \( f = f^t \in A^+ \),
\[
\mathcal{O}_A(D^+(f)) \xrightarrow{\sim} A_f.
\] (4.4.3)

Proof. The proofs are exactly as in propositions 3.4.2 and 3.4.3.

\[\square\]

4.5 Schemes with involution \( \mathfrak{F} Sc^t \) and locally-\( \mathfrak{F} R^t \)-spaces \( \mathcal{L}\mathfrak{F} R^t S_p \)

Definition 4.5.1

An \( \mathfrak{F} R^t \)-space \((X, \mathcal{O}_X)\) is a topological space \( X \) with a sheaf \( \mathcal{O}_X \) of \( \mathfrak{F} \)-Rings with involutions. A map of \( \mathfrak{F} R^t \)-spaces \( f : X \rightarrow Y \) is a continuous map of the underlying topological spaces together with a map of sheaves of \( \mathfrak{F} R^t \) on \( Y \), \( f^\#: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X \), i.e. for \( U \subseteq Y \open \) we have \( f^\#: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}U) \) a map of \( \mathfrak{F} R^t \), such that for
\[
U' \subseteq U : \quad f^\# \left( \mathcal{O}_Y(U) \mid_{f^{-1}U'} \right) = f^\# \left( \mathcal{O}_X(U) \mid_{U'} \right).
\]

The \( \mathfrak{F} R^t \)-space \( X \) is a locally-\( \mathfrak{F} R^t \)-space if for all \( p \in X \) the stalk \( \mathcal{O}_{X,p} \) is a local \( \mathfrak{F} R^t \), i.e. contains a unique maximal symmetric ideal \( m_{X,p} \).

For a map of \( \mathfrak{F} R^t \)-spaces \( f : X \rightarrow Y \), and for \( p \in X \), we get an induced homomorphism of \( \mathfrak{F} R^t \) on the stalks
\[
f^\#: \mathcal{O}_{Y,f(p)} = \lim_{\substack{\longrightarrow \\ \text{\scriptsize f(p) \in V}}} \mathcal{O}_Y(V) \rightarrow \lim_{\substack{\longrightarrow \\ \text{\scriptsize p \in f^{-1}(V)}}} \mathcal{O}_X(f^{-1}V) \rightarrow \lim_{\substack{\longrightarrow \\ \text{\scriptsize p} \in U}} \mathcal{O}_X(U) = \mathcal{O}_{X,p}
\] (4.5.1)

A map \( f : X \rightarrow Y \) of locally-\( \mathfrak{F} R^t \)-spaces is a map of \( \mathfrak{F} R^t \)-spaces such that \( f^\# \) is a local homomorphism for all \( p \in X \), i.e.
\[
f^\# \left( m_{Y,f(p)} \right) \subseteq m_{X,p} \quad \text{or equivalently} \quad (f^\#)^* \left( m_{X,p} \right) = m_{Y,f(p)}.
\] (4.5.2)

We let \( \mathcal{L}\mathfrak{F} R^t S_p \) denote the category of \( \mathfrak{F} \)-locally-ringed-spaces with involution.

Definition 4.5.2

A pair \((X, \mathcal{O}_X)\) \( \in \mathcal{L}\mathfrak{F} R^t S_p \) is an (Grothendieck) \( \mathfrak{F} \)-Scheme with involution, if there exists an open covering \( \{U_i\}_{i \in I} \) of the topological space \( X \) s.t.
\[
(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\sim} \text{Spec}^+(\mathcal{O}_X(U_i)).
\] (4.5.3)
We denote the category of Grothendieck $\mathbb{F}$-Scheme with involution by $\mathfrak{g}\mathbb{F}Sc^t$. Note that we have the following full embedding of categories:

$$(\mathcal{F}R^t)^{op} \subseteq \mathfrak{g}\mathbb{F}Sc^t \subseteq \mathcal{L}\mathcal{F}R^t Sp.$$  

(4.5.4)
Chapter 5

Pro - limits

We can work with general filtered small category $J$ (cf. [AM]), or restrict attention to the case of $(J, \leq)$ a partial ordered set that is directed,

$$ \forall j_1, j_2 \in J, \exists j \in J, j \geq j_1, j \geq j_2. $$

and co-finite,

$$ \forall j \in J, \left| \{ i \in J | i \leq j \} \right| < \infty. $$

The following inverse limits over $J$-indexed inverse systems in $(\mathcal{FRI})^{op}$ and $\mathcal{L}\mathcal{FRI}Sp$ exist:

$$ \lim_{J} \mathcal{L}\mathcal{FRI}Sp^{J} \rightarrow \mathcal{L}\mathcal{FRI}Sp $$

Moreover in the affine case the following limit can be interchanged:

$$ \lim_{J} \text{Spec}^{+} (A_{j}) = \text{Spec}^{+} (\lim_{J} A_{j}). $$

But inverse limits over $(\mathcal{FSc})^{J}$ do not always exist! This leads us to define the "pro" category of $\mathcal{FSc}$ as our category of schemes.

5.1 Pro - Schemes

Definition 5.1

Define the pro- $\mathcal{FSc}$ (respectfully pro- $\mathcal{FSc}^{d}$) category to be the category with objects all inverse systems in $\mathcal{FSc}$ (respectfully $\mathcal{FSc}^{d}$) over arbitrary
directed co-finite partially ordered set. The morphisms between objects \( X = \{X_j\}_{j \in J}, Y = \{Y_i\}_{i \in I} \) are given by,

\[
\text{pro- } \mathcal{G} \mathcal{S} \mathcal{c} (X, Y) = \lim_{I} \lim_{J} \mathcal{G} \mathcal{S} \mathcal{c} (X_j, Y_i),
\]

(5.1.1)

(respectfully. \( \text{pro- } \mathcal{G} \mathcal{S} \mathcal{c}^t (X, Y) = \lim_{I} \lim_{J} \mathcal{G} \mathcal{S} \mathcal{c}^t (X_j, Y_i) \)).

For every \( i' \leq i, j \leq j' \) we have a commutative diagram in \( \text{Set} \):

\[
\begin{align*}
\mathcal{G} \mathcal{S} \mathcal{c} (X_j, Y_i) & \xrightarrow{\varphi_{j'}^i} \mathcal{G} \mathcal{S} \mathcal{c} (X_{j'}, Y_i) \\
\pi_{j'} \circ - & \downarrow \pi_i \circ - \\
\mathcal{G} \mathcal{S} \mathcal{c} (X_j, Y_{i'}) & \xrightarrow{\varphi_{j'}^i} \mathcal{G} \mathcal{S} \mathcal{c} (X_{j'}, Y_{i'})
\end{align*}
\]

(5.1.2)

(here \( \pi_{j'} : Y_i \rightarrow Y_{i'} \) and \( \pi_i : X_j \rightarrow X_{j'} \)).

We can describe a morphism between \( X \) and \( Y \) in the pro-category as collections of maps \( \varphi_{j'}^i : X_j \rightarrow Y_i \) defined for every \( i \) and for every \( j \geq \sigma(i) \) large enough (depending on \( i \)). The maps are such that the following conditions must hold for every \( i' \leq i \) and \( \sigma(i) \leq j \leq j' \),

\[
\varphi_{j'}^i \circ \pi_{j'}^{i'} = \varphi_{j'}^i
\]

\[
\pi_i \circ \varphi_{j'}^i = \varphi_{j'}^i
\]

(5.1.3)

The maps \( \{\varphi_{j'}^i\}, \{\tilde{\varphi}_{j'}^i\} \) are considered equivalent if for all \( i \in I \) and all \( j \in J \) large enough (depending on \( i \)):

\[
\varphi_{j'}^i = \tilde{\varphi}_{j'}^i
\]

We have a full and faithful embedding \( \mathcal{G} \mathcal{S} \mathcal{c}^t \rightarrow \text{pro- } \mathcal{G} \mathcal{S} \mathcal{c}^t \), (taking the indexing set to be a point). We have a functor \( \lim \) (which is generally not full, and not faithful, but is such on finitely presented objects), making a commutative diagram

\[
\begin{align*}
\text{pro- } \mathcal{G} \mathcal{S} \mathcal{c}^t & \xrightarrow{\lim} \mathcal{L} \mathcal{F} \mathcal{R} \mathcal{S} \\
\mathcal{G} \mathcal{S} \mathcal{c}^t & \xrightarrow{\lim} \mathcal{L} \mathcal{F} \mathcal{R} \mathcal{S} \\
\mathcal{G} \mathcal{S} \mathcal{c}^t & \xrightarrow{\lim} \mathcal{L} \mathcal{F} \mathcal{R} \mathcal{S}
\end{align*}
\]

(5.1.4)

5.2 The compactified \( \text{Spec } \mathcal{Z} \).

Let \( N \geq 2 \) be a square free integer. Let \( A_N \) be the \( \mathcal{F} \)-Ring defined by:

\[
A_N = \mathcal{F}(\mathbb{Z}[\frac{1}{N}]) \cap \mathcal{O}_{\mathbb{Q}, q},
\]

\[
(A_N)_{Y,X} = \left\{ a \in \mathcal{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right)_{Y,X} \mid |a|_q \leq 1 \right\},
\]

(5.2.1)
The map \( j : A_N \rightarrow \mathbb{F}(\mathbb{Z}[\frac{1}{N}]) \) is a localization, \((A_N)_{\frac{1}{N}} \cong \mathbb{F}(\mathbb{Z}[\frac{1}{N}])\), and so defines an injection:

\[
j^* : \text{Spec } \mathbb{Z}\left[\frac{1}{N}\right] \cong \text{Spec } \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) \cong \text{Spec } (A_N)_{\frac{1}{N}} \cong DA_N\left(\frac{1}{N}\right) \hookrightarrow \text{Spec } A_N.
\]

(5.2.2)

The space \( \text{Spec } A_N \) also contains the closed point,

\[
\eta = i^* (m_{\mathbb{Q}_p}) = \left\{ a \in (A_N)_{1,1} \middle| |a|_\eta < 1 \right\} = \mathbb{Z}\left[\frac{1}{N}\right] \cap (-1,1).
\]

(5.2.3)

which is the real prime given by the injection \( i : A_N \hookrightarrow \mathcal{O}_{\mathbb{Q}_p} \). The prime ideal \( \eta \) contains any other ideal of \( A_N \) thus it is the unique maximal ideal for \( A_N \), which is a local \( \mathbb{F} \)-ring (of Krull dimension 2).

Note that \( \text{Spec } A_N = \text{Spec } \mathbb{Z}[\frac{1}{N}] \cup \{\eta\} \) as sets. The point \( \eta \in \text{Spec } A_N \) is very closed in the sense that the only open set containing it is the whole space. For any non-trivial basic open set \( DA_N(f) \), say \( f = \frac{a}{N^k}, a \in \mathbb{Z}, |a|_\eta < N^k \), we have \((A_N)_f = \mathbb{F}(\mathbb{Z}[\frac{1}{N}])\), and so

\[
DA_N(f) = \text{Spec } (A_N)_f = \text{Spec } \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N^k}\right]\right) \cong \text{Spec } \mathbb{Z}\left[\frac{1}{N^k}\right].
\]

(5.2.4)

does not contain \( \eta \).

Let \( X_N \in \mathfrak{gFS}_S \) be the Grothendieck-\( \mathbb{F} \)-scheme defined by:

\[
X_N = \text{Spec } \mathbb{F}(\mathbb{Z}) \amalg_{\text{Spec } \mathbb{F}(\mathbb{Z}[\frac{1}{N}])} \text{Spec } A_N.
\]

(5.2.5)

i.e. gluing \( \text{Spec } \mathbb{F}(\mathbb{Z}) \) with \( \text{Spec } A_N \) along the common basic open set \( \text{Spec } \mathbb{F}(\mathbb{Z}[\frac{1}{N}]) \).

The open sets of \( X_N \) are open sets of \( \text{Spec } \mathbb{Z} \) and sets of the form \( U \cup \{\eta\} \) with \( \text{Spec } \mathbb{Z}[\frac{1}{N}] \subseteq U \subseteq \text{Spec } \mathbb{Z} \).

The points and specializations of \( X_N, N = p_1 \ldots p_k \),

\[
\begin{align*}
&: 
\end{align*}
\]

(5.2.7)
The structure sheaf $\mathcal{O}_{X_N}$ is defined as follows: for any open set $U \subseteq X_N$,

$$\mathcal{O}_{X_N}(U) = \left\{ <s_1, s_2> \big| s_1 \in \mathcal{O}_{F(Z)}(i_1^{-1}(U)) \text{ and } s_2 \in \mathcal{O}_{A_N}(i_2^{-1}(U)) \text{ and } \right.$$ \hspace{1cm} \left. s_1|_{i_1^{-1}(U) \cap \text{Spec } F(Z[1/\mathfrak{m}]^\#)} \simeq s_2|_{i_2^{-1}(U) \cap \text{Spec } F(Z[1/\mathfrak{m}]^\#)} \right\}$$  \hspace{1cm} (5.2.8)

For an open set $U = \text{Spec } Z[1/\mathfrak{m}] = D(1) \subseteq \text{Spec } Z$, we have,

$$s_1 \in \mathcal{O}_{F(Z)}(D(M)) \simeq F(Z[1/M]),$$  \hspace{1cm} (5.2.9)

$$s_2 \in \mathcal{O}_{A_N}(i_2^{-1}(U)), \quad i_2^{-1}(U) \subseteq \text{Spec } F(Z[1/N]).$$  \hspace{1cm} (5.2.10)

the isomorphism condition gives us in that case that

$$s_1|_{i_1^{-1}(U) \cap \text{Spec } F(Z[1/\mathfrak{m}]^\#)} = s_2,$$  \hspace{1cm} (5.2.11)

and therefore $\mathcal{O}_{X_N}(U) = F(Z[1/M])$.

For a set $U = \text{Spec } Z[1/\mathfrak{m}]$ with $M|N$, we have,

$$i_1^{-1}(U \cup \{\eta\}) = \text{Spec } F\left(Z[1/M]\right),$$  \hspace{1cm} (5.2.12)

$$i_2^{-1}(U \cup \{\eta\}) = \text{Spec } A_N.$$  \hspace{1cm} (5.2.13)

and so,

$$s_1 \in \mathcal{O}_{F(Z)}(D(M)) \simeq F(Z[1/M]),$$  \hspace{1cm} (5.2.14)

$$s_2 \in \mathcal{O}_{A_N}(D(1)) \simeq A_N,$$  \hspace{1cm} (5.2.15)

where $s_1|_{\text{Spec } F(Z[1/\mathfrak{m}])} = s_2|_{\text{Spec } F(Z[1/\mathfrak{m}])}$.

Thus $\mathcal{O}_{X_N}(U \cup \{\eta\})$ is the pullback of the diagram

$$\begin{array}{ccc}
A_N & \longrightarrow & F(Z[1/M]) \\
\downarrow & & \downarrow \\
\mathcal{O}_{F(Z[1/\mathfrak{m}]^\#)} & \longrightarrow & \mathcal{O}_{F(Z[1/\mathfrak{m}]^\#)}
\end{array}$$  \hspace{1cm} (5.2.16)

which is $A_M$, and so, $\mathcal{O}_{X_N}(U \cup \{\eta\}) = A_M$.

Alternatively, $X_N$ is "integral", and $\mathcal{O}_{X_N}(U)$, for $U \subseteq X_N$ open, are all $F$-sub-rings of the stalk at the generic point $\mathcal{O}_{X_N,\{0\}} = F(\mathbb{Q})$, given by

$$\mathcal{O}_{X_N}(U) = \begin{cases} 
\bigcap_{p \in U} F(Z(p)) & \eta \notin U, \\
\bigcap_{p \in U} F(Z(p)) \cap \mathcal{O}_{F(\mathbb{Q})} & \eta \in U
\end{cases}$$  \hspace{1cm} (5.2.17)
For $N$ dividing $M$ we have commutative diagrams

\[
\begin{array}{ccc}
A_N & \xrightarrow{\pi_N} & A_M \\
\mathbb{F}(\mathbb{Z}[\frac{1}{N}]) & \xrightarrow{\mathbb{F}(\mathbb{Z}[\frac{1}{M}])} & \mathbb{F}(\mathbb{Z}[\frac{1}{M}]) \\
\Spec A_N & \xleftarrow{\pi_N} & \Spec A_M \\
\Spec \mathbb{F}(\mathbb{Z}[\frac{1}{N}]) & \xleftarrow{\Spec \mathbb{F}(\mathbb{Z}[\frac{1}{M}])} & \Spec \mathbb{F}(\mathbb{Z}[\frac{1}{M}])
\end{array}
\]

(5.2.18)

and we obtain a map $\pi_N : X_M \to X_N$ which is a bijection on points and further $(\pi_M)_\ast \mathcal{O}_{X_M} = \mathcal{O}_{X_N}$, i.e. $(\pi_N)_\ast$ is the identity. But note that $X_M$ has more open neighborhoods of $\eta$ than $X_N$ such as $\Spec \mathbb{Z}[\frac{1}{N}] \cup \{\eta\} = \Spec A_M \subseteq \Spec A_N$. The "compactified $\Spec \mathbb{Z}$" is the inverse system $\{X_N\}$ with indices square free integers $N \geq 2$, maps $\pi_N^M$, and partial order given by divisibility. We denote it by $\Spec \mathbb{Z}$.

Note that the $\mathbb{F}$-locally-ringed-space $\mathcal{L}(\Spec \mathbb{Z}) = \lim_{N} X_N$ has for points $\Spec \mathbb{Z} \cup \{\eta\}$, with open sets of the form $U$ or $U \cup \{\eta\}$ with $U$ an arbitrary open set of $\Spec \mathbb{Z}$ (hence $\Spec \mathbb{Z}$ is of "Krull" dimension 1). Note that each $X_N$ is compact, and hence $\mathcal{L}(\Spec \mathbb{Z})$ is compact. Furthermore, the local $\mathbb{F}$-Ring $\mathcal{O}_{\Spec \mathbb{Z}, \eta}$ is just $\mathcal{O}_{\mathbb{Z}, \eta}$ (while the local $\mathbb{F}$-Ring $\mathcal{O}_{X_N, \eta}$ is only $A_N$).

For an open set $U = \Spec \mathbb{Z}[\frac{1}{N}]$ we have

\[
\mathcal{O}_{\Spec \mathbb{Z}}(U) = \mathbb{F}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right),
\]

and

\[
\mathcal{O}_{\Spec \mathbb{Z}}(U \cup \{\eta\}) = A_N.
\]

The global sections $\mathcal{O}_{\Spec \mathbb{Z}}(\Spec \mathbb{Z})$ is the $\mathbb{F}$-Ring $\mathbb{F}\{1\}$.

5.3 The compactified $\Spec \mathcal{O}_K$.

Similarly, for a number field $K$, with ring of integers $\mathcal{O}_K$, with real primes $\{\eta_i, i = 1, \ldots, r = r_h + r_c\}$, let $A_N,i = \mathbb{F}(\mathcal{O}_K[\frac{1}{N}]) \cap \mathcal{O}_{K, \eta_i}$, be the $\mathbb{F}$-Ring with

\[
(A_N,i)_{Y,X} = \left\{ a \in \mathbb{F}(\mathcal{O}_K[\frac{1}{N}]) : |a|_{\eta_i} \leq 1 \right\}
\]

(5.3.1)

the $Y \times X$ matrices with values in $\mathcal{O}_K[\frac{1}{N}]$ and with $\eta$-operator $L_\infty$- norm bounded by 1.

Let $X_N$ be the Grothendieck-$\mathbb{F}$-scheme obtain by gluing $\{\Spec A_N,i\}_{i=1}$, and $\{\Spec \mathbb{F}(\mathcal{O}_K)\}$, along the common open set $\Spec \mathbb{F}(\mathcal{O}_K[\frac{1}{N}])$. For $N$ dividing $M$ we obtain a map $\pi_N^M : X_M \to X_N$, with $\pi_N^M|_{\Spec A_M^i}$ induced by $A_N,i \subseteq A_M,i$. The inverse system $\{X_N\}$ is the pro- $\mathbb{F}$-scheme $\Spec \mathcal{O}_K$, the compactification of $\Spec \mathcal{O}_K$.

The space $\mathcal{L}(\Spec \mathcal{O}_K) = \lim_{N} X_N$ has for points $\Spec \mathcal{O}_K \cup \{\eta_i\}_{i \leq r}$, and open
sets are of the form \( U \cup \{ \eta_i \}_{i \in I} \) with \( U \) open in \( \text{Spec} \, \mathcal{O}_K \), and \( I \subseteq \{1, \ldots, r\} \) a subset (and hence it is of "Krull" dimension 1). The local \( \mathbb{F}\text{-}\mathbb{R}\text{ings} \) \( \mathcal{O}_{\text{Spec} \, \mathcal{O}_K, \eta_i} \) is the ring \( \mathcal{O}_{K, \eta_i} \). The global sections \( \mathcal{O}_{\text{Spec} \, \mathcal{O}_K} (\text{Spec} \, \mathcal{O}_K) \) is the \( \mathbb{F}\text{-}\mathbb{R}\text{ing} \) \( \mathbb{F}\{ \mu_K \} \), \( \mu_K \) the group of roots of unity in \( \mathcal{O}_K^\times \).
Chapter 6

Vector bundles

6.1 Meromorphic functions \( \mathcal{K}_{X_N} \).

Let \( X = \{X_N, \pi^M_N\}_{M \geq N \in \mathbb{N}} \) be a pro-object in the category \( \text{qFSh}^{(t)} \), let \( U \subseteq X_N \) be an open set and define a multiplicative set,

\[
S_N(U) = \left\{ s \in \mathcal{O}^\times_{X_N}(U)_{1,1} \mid \forall M \geq N, \forall V \subseteq (\pi^M_N)^{-1}(U) \right. \left. \forall a, a' \in \mathcal{O}_{X_M}(V)_{Y,X}, (\pi^M_N)^\#(s) \cdot a = (\pi^M_N)^\#(s) \cdot a' \implies a = a' \right\}.
\]

Define the sheaf of \( \mathbb{F} \)-Rings with involution, the "meromorphic functions" \( \mathcal{K}_N \), to be the sheaf associated to the pre-sheaf \( U \mapsto S_N(U)^{-1} \cdot \mathcal{O}_{X_N}(U)_{Y,X} \)

\[
\mathcal{K}_N = S_N^{-1} \mathcal{O}_{X_N} \in \mathbb{F} \mathcal{R}^t/X_N.
\]

For \( M \geq N \) we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_{X_N} & \xrightarrow{} & \mathcal{K}_N & \xrightarrow{} & S_N \\
\downarrow & & \downarrow & & \downarrow \pi^M_N \\
(\pi^M_N)_* \mathcal{O}_M & \xrightarrow{} & (\pi^M_N)_* \mathcal{K}_M & \xrightarrow{} & S_M
\end{array}
\]

6.2 Rank-d Vector Bundles at finite layer.

For any \( d \in \mathbb{F} \), we have an injection (of sheaves) of groups

\[
\left( GL_d(\mathcal{O}_{X_N}) \hookrightarrow GL_d(\mathcal{K}_N) \right) \in \mathcal{G} \text{rps}/X_N
\]

define,

\[
\mathcal{D}_d(X_N) = \Gamma(X_N, GL_d(\mathcal{K}_N)/GL_d(\mathcal{O}_{X_N})) \in \mathcal{S} \text{ets}_*.
\]
Elements of $\mathcal{D}_d(X_N)$ are represented as $\mathcal{D} := (u_\alpha, f_\alpha)$, where
\[
X_N = \bigcup_\alpha u_\alpha, \quad f_\alpha \in GL_d(K_N)(u_\alpha), \quad f_\alpha^{-1}|_{u_\alpha \cap u_\beta} \circ f_\beta|_{u_\alpha \cap u_\beta} \in GL_d(\mathcal{O}_{X_N}),
\]
and two such elements $D := (u_\alpha, f_\alpha), D' := (v_\beta, g_\beta)$ represent the same element of $\mathcal{D}_d(X_N)$:
\[
D = D' \iff \exists X_N = \bigcup_{\gamma} w_\gamma \text{ a common refinement of } \{u_\alpha\} \text{ and } \{v_\beta\},
\]
and $\exists u_\gamma \in GL_d(\mathcal{O}_{X_N})(w_\gamma)$ such that $f_\alpha|_{w_\gamma} = g_\beta|_{w_\gamma} \circ u_\gamma$, for $w_\gamma \subseteq u_\alpha \cap v_\beta$.

We obtain a category $\text{Vec}(X_N)$ of "vector bundles over $X_N$", with objects $\bigcup_{d \in \mathbb{N}} \mathcal{D}_d(X_N)$, and with arrows from $\mathcal{D} = (u_\alpha, f_\alpha) \in \mathcal{D}_d(X_N)$ to $\mathcal{D}' = (v_\beta, g_\beta) \in \mathcal{D}_d(X_N)$ given by
\[
\text{Vec}(X_N)(\mathcal{D}, \mathcal{D}') = \{h \in \Gamma(X_N, (\mathcal{K})_d', d), \quad g_\beta^{-1} \circ h|_{v_\beta \cap u_\alpha} \circ f_\alpha \in \mathcal{O}_{X_N}(v_\beta \cap u_\alpha)_d, d \subseteq \mathcal{K}_{N}(v_\beta \cap u_\alpha)_d, d\}
\]
(6.2.5)

For such $\mathcal{D} = (u_\alpha, f_\alpha), \mathcal{D}' = (v_\beta, g_\beta)$, we have the well defined element $\mathcal{D} \oplus \mathcal{D}' \in \mathcal{D}_{d+d'}(X_N)$,
\[
\mathcal{D} \oplus \mathcal{D}' := (u_\alpha \cap v_\beta, f_\alpha|_{u_\alpha \cap v_\beta} \oplus g_\beta|_{u_\alpha \cap v_\beta})
\]
(6.2.6)

We have associativity, commutativity, and unit isomorphisms, the unit object $(0) = (X_N, id_{[0]})$ (where: $\mathcal{D}_{[0]}(X_N) = \{(0)\}$) is the initial and final object of $\text{Vec}(X_N)$. For $h_1 \in \text{Vec}(\mathcal{D}_1, \mathcal{D}'_1)$, we have $h_0 \oplus h_1 \in \text{Vec}(\mathcal{D}_0 \oplus \mathcal{D}_1, \mathcal{D}'_0 \oplus \mathcal{D}'_1)$, thus $\text{Vec}(X_N)$ is a symmetric monoidal category.

For $d \geq 0$, we have the isomorphism classes of rank-$d$ vector bundles
\[
\text{Pic}_d(X_N) := \Gamma(X_N, GL_d(K_N)) \bigg/ \mathcal{D}_d(X_N)
\]
(6.2.7)

and we get from $\oplus$ an induced commutative monoid structure on $\text{Pic}_d(X_N) = \bigcup_{d \geq 0} \text{Pic}_d(X_N)$.

For $\mathcal{D}_1, \mathcal{D}'_1 \in \text{Pic}_d(X_N)$, we let $(\mathcal{D}_0, \mathcal{D}'_0) \sim (\mathcal{D}_1, \mathcal{D}'_1)$ iff $\mathcal{D}_0 \oplus \mathcal{D}'_0 \oplus \mathcal{D} \cong \mathcal{D}_1 \oplus \mathcal{D}'_1 \oplus \mathcal{D}$ for some $\mathcal{D} \in \text{Pic}_d(X_N)$. This defines an equivalence relation on pairs $(\mathcal{D}, \mathcal{D}')$. We let $\mathcal{D} - \mathcal{D}'$ denote the equivalence class of $(\mathcal{D}, \mathcal{D}')$, and we let $K(X_N) = \text{Pic}_d(X_N) \times \text{Pic}_d(X_N)/\sim$, the Grothendieck group of stable isomorphism classes of (virtual) vector bundles.

We have a homomorphism of monoids $\text{Pic}_d(X_N) \rightarrow K(X_N)$, which is universal.
For \( M \geq N \) we have pull-back of vector bundles

\[
(\pi^M_N)^\#: \mathcal{D}_d(X_N) \to \mathcal{D}_d(X_M) \quad \mathcal{D} = (u_\alpha, f_\alpha) \mapsto (\pi^M_N)^\# \mathcal{D} = ((\pi^M_N)^{-1}u_\alpha, (\pi^M_N)^\#(f_\alpha))
\]

(6.2.8)

It is (the object part of) a strict symmetric monoidal functor Vec\((X_N) \to \text{Vec}(X_M)\), and it induces a homomorphism of commutative monoids/abelian groups

\[
Pic_d(X_N) \to Pic_d(X_M) \\
K(X_N) \to K(X_M)
\]

(6.2.9)

**Remark 6.2.1**

We have a partial order on \( \mathcal{D}_d(X_N) \)

\[
\mathcal{D} = (u_\alpha, f_\alpha) \leq \mathcal{D}' = (v_\beta, g_\beta) \iff \text{id}_{[d]} \in \text{Vec}(X_N)(\mathcal{D}, \mathcal{D}')
\]

\[
\iff g_\beta^{-1} \circ f_\alpha \in \mathcal{O}_{X_N}(v_\beta \cap u_\alpha)_{d', d} \text{ for all } \beta, \alpha.
\]

(6.2.10)

The action of the group \( \Gamma(X_N, GL_d(K_N)) \) on \( \mathcal{D}_d(X_N) \) preserves this partial order. The maps \((\pi^M_N)^\#\) of (6.2.8) is order preserving, and is covariant with respect to \( \Gamma(X_N, GL_d(K_N)) \to \Gamma(X_M, GL_d(K_M)) \).

### 6.3 \( \mathcal{D}_d(X) \), Rank-\( d \) Vector Bundles in the limit.

Passing to the limit \( \lim_{N} X_{N} \), we have a symmetric monoidal category Vec\((X) = \lim_{N} \text{Vec}(X_N)\).

It has objects \( \bigcup_{d \in \mathbb{F}} \lim_{N} \mathcal{D}_d(X_N) \), and for \( \mathcal{D} \in \mathcal{D}_d(X_{N_0}) \), \( \mathcal{D}' \in \mathcal{D}_d(X'_{N_0}) \),

\[
\text{Vec}(X)(\mathcal{D}, \mathcal{D}') = \lim_{N \geq N_0, N_0'} \text{Vec}(X_N)((\pi^N_{N_0})^\# \mathcal{D}, (\pi^N_{N_0})^\# \mathcal{D}')
\]

(6.3.1)

The isomorphism classes of rank-\( d \) vector-bundles are given by

\[
Pic_d(X) = \lim_{N} Pic_d(X_N) = \lim_{N} \Gamma(X_N, GL_d(K_N)) \bigg/ \lim_{N} \mathcal{D}_d(X_N)
\]

The direct sum \( \oplus \) induces a commutative monoid structure on \( Pic_d(X) = \bigcup_{d \geq 0} Pic_d(X) \), and passing to stable isomorphism classes of virtual vector-bundles we obtain the Grothendieck group

\[
K(X) = \lim_{N} K(X_N) \in \text{Ab}.
\]

(6.3.2)
We describe next another passage to the limit (which is kind of dual to the one above, as we consider $\lim_N D_d(X_N)$). Set

$$\mathcal{B}_d(X) = \left\{ D = \{D_N\}, D_N \in D_d(X_N), \begin{array}{l} \forall M \geq N, (\pi_N^M)\# D_N \geq D_M, \\ \exists \delta_0 \in D_d(X_{N_0}), \forall N \geq N_0, D_N \geq (\pi_{N_0}^N)\# \delta_0 \end{array} \right\}$$

(6.3.3)
i.e. $\mathcal{B}_d(X)$ is the collection of bounded below (**), monotone decreasing (*), filters of vectors bundles.

Define a transitive reflexive relation on $\mathcal{B}_d(X)$,

$$\{D_N\} \geq \{D'_N\} \iff \left\{ \forall \delta_0 \in D_d(X_{N_0}) \text{ such that } \forall N \geq N_0 \ D'_N \geq (\pi_{N_0}^N)\# \delta_0 \right\}$$

(6.3.4)

and an equivalence relation,

$$\{D_N\} \sim \{D'_N\} \iff \forall N \geq N'_0(\geq N_0) \ D_N \geq (\pi_{N_0}^N)\# \delta_0$$

(6.3.5)

Finally, we define $\hat{\mathcal{B}}_d(X)$ to be the $\sim$ equivalence classes:

$$\hat{\mathcal{B}}_d(X) := \mathcal{B}_d(X)/\sim$$

(6.3.8)

It is a partially ordered set, with an (order preserving) action of the group $GL_d(\mathcal{K}(X)) := \lim_N GL_d(\mathcal{K}_N)(X_N)$.

**Example 6.3.1**

We have

$$\hat{\mathcal{B}}_d(\overline{\text{Spec } \mathbb{Z}}) \cong GL_d(\mathbb{A}_\mathbb{Q})/O_d \times \pi GL_d(Z_p)$$

(6.3.9)

where $\mathbb{A}_\mathbb{Q}$ is the ring of Adeles of $\mathbb{Q}$, and $O_d = GL_d(O_{\mathbb{R},n})$ the orthogonal group.

For a finite $p$, the symmetric space $\Xi_p = GL_d(\mathbb{Q}_p)/GL_d(Z_p)$ can be identified with the $GL_d(\mathbb{Q}_p)$-set

$$\mathcal{L}_p = \{L \subseteq \mathbb{Q}_p^{\oplus d}, L \text{ is a } \mathbb{Z}_p \text{-lattice} \}$$

(6.3.10)

via,

$$g \cdot GL_d(Z_d) \mapsto g(Z_p^{\oplus d})$$

(6.3.11)

since $GL_d(\mathbb{Q}_p)$ acts transitively on $\mathcal{L}_p$, and the stabilizer of $Z_p^{\oplus d}$ is precisely $GL_d(\mathbb{Z}_p)$. 

Similarly, for the real prime \( \eta \), the symmetric space \( \Xi_{\eta} = GL_d(\mathbb{R})/O_d \), can be identified with the \( GL_d(\mathbb{R}) \)-set \( \mathcal{L}_\eta = \{ Q \in Mat_{d,d}(\mathbb{R}), \text{symmetric and positive definite} \} \), of positive definite quadratic forms \( Q \), or alternatively by the associated ellipsoids \( \mathcal{L}_Q := \{ x \in \mathbb{R}^{\otimes d}, \ x \circ Q \circ x^t \leq 1 \} \), so that the correspondence is given (more like the case of finite \( p \)'s) as

\[
\Xi_{\eta} = GL_d(\mathbb{R})/O_d \overset{\sim}{\longrightarrow} \mathcal{L}_{\eta} = \{ \mathcal{L} \subseteq \mathbb{R}^{\otimes d} \text{ ellipsoid} \}
\]

(6.3.12)

\[ g \cdot O_d \mapsto g(\mathbb{Z}_\eta^{\otimes d}). \]  

(6.3.13)

with \( \mathbb{Z}_\eta^{\otimes d} = \{ x \in \mathbb{R}^{\otimes d}, \ x \circ x^t \leq 1 \} \) the \( d \)-dimensional unit ball.

The Adelic symmetric space \( \Xi_\mathbb{A} = GL_d(\mathbb{A})/O_d \times \prod_p GL_d(\mathbb{Z}_p) \) can be identified with the restricted-product \( \mathcal{L}_\mathbb{A} = \mathcal{L}_\eta \times \prod_p \mathcal{L}_p \), it is the subset of the product consisting of \( \mathcal{L} = (l_1, l_2, l_3, l_5, \ldots) \) such that \( l_p = \mathbb{Z}_p^{\otimes d} \) for all but finitely many \( p \)'s. Each such \( l \) is a compact subset of \( \mathbb{A}^{\otimes d} \).

In the space \( X_N = Spec \mathbb{F}(\mathbb{Z}) \prod_{\text{Spec} \mathbb{F}(\mathbb{Z}[\frac{1}{M}])} A_N, \ N = p_1 \ldots p_l \), we have the system \( \{ V^M_{\eta}, V^M_{p_1}, \ldots, V^M_{p_l} \} \) of cofinal coverings for \( M = q_1 \ldots q_k \) prime to \( N \), with

\[
V^M_{p_j} = Spec \mathbb{F}(\mathbb{Z}[\frac{p_j}{N \cdot M}]), \ j = 1, \ldots, l
\]

(6.3.14)

\[
V^M_{\eta} = Spec A_N \quad \text{(independent of \( M \!)}). \]  

(6.3.15)

(for every open cover \( \{ u_\alpha \} \) of \( X_N \), there exists \( M \) such that \( \{ V^M_{\eta}, V^M_{p_1} \} \) is a refinement of \( \{ u_\alpha \} \)).

The sheaf \( K_N \) of meromorphic functions is the constant sheaf \( \mathbb{F}(\mathbb{Q}) \), for all \( N \). Thus we can represent an element of \( D_d(X_N) \) by a sequence \( (g_\eta, g_{p_1}, \ldots, g_{p_l}) \in GL_d(\mathbb{Q})^{(l+1)} \), with \( g_\eta^{-1} \circ g_\eta \) and \( g_{p_j}^{-1} \circ g_{p_j} \) in \( \prod_{p | N \cdot M} GL_d(\mathbb{Z}_p) \), (a vacuous condition since \( M \) is arbitrary).

Two such sequences \( (g_\eta, g_{p_1}, \ldots, g_{p_l}) \), and \( (h_\eta, h_{p_1}, \ldots, h_{p_l}) \), represent the same element of \( D_d(X_N) \), if and only if \( h_\eta^{-1} \circ g_\eta \in GL_d(\mathbb{Z}_p) \), and \( h_{p_j}^{-1} \circ g_{p_j} \in O_d \cap GL_d(\mathbb{Z}[\frac{1}{M}]) \).

Thus we have the well defined map

\[
div_N : D_d(X_N) \to \mathcal{L}_\mathbb{A}
\]

(6.3.16)

\[
div_N(g_\eta, g_{p_1}, \ldots, g_{p_l}) = \{ \mathcal{L}_p \}, \ \mathcal{L}_p = \begin{cases} g_{p_j}(\mathbb{Z}_p^{\otimes d}) & \text{if } p = p_j | N \text{ or } p = \eta. \end{cases}
\]

(6.3.17)

For \( N = p_1 \cdots p_l \), and for \( N' = p'_1 \cdots p'_{l'} \) prime to \( N \), the map \( \pi_{N,N'}^*: X_{N,N'} \to X_N \), induce pullback \( (\pi_{N,N'}^*)^# : D_d(X_N) \to D_d(X_{N,N'}) \), and

\[
(\pi_{N,N'}^*)^#(g_\eta, g_{p_1}, \ldots, g_{p_l})/\sim = (g_\eta, g_{p_1}, \ldots, g_{p_l}, g_{p_{l'}}, \ldots, g_{p_{l'}})/\sim,
\]

(6.3.18)
i.e. $g_{\eta}$ placed in the $p'_i$ spots. Thus we have a commutative diagram:

$$
\begin{array}{c}
\mathcal{D}_d(X_N) \\
\downarrow \text{div}_N \\
\mathcal{D}_d(X_N) \\
\downarrow \text{div}_N' \\
\mathcal{L}_\mathbb{A} \\
\end{array}
$$

(6.3.19)

Note that \( \bigcup_N \text{div}_N(\mathcal{D}_d(X_N)) \) is the dense subset of \( \mathcal{L}_\mathbb{A} \) consisting of \( \{ \mathcal{L}_p \} \) with arbitrary \( \mathcal{L}_p \) at finite primes \( p \), and with \( \mathcal{L}_\eta \) defined over \( \mathbb{Q} \) (i.e. \( \mathcal{L}_\eta \in GL_d(\mathbb{Q}) \cdot (\mathbb{Z}_p^{\otimes d}) \subseteq \mathcal{L}_p \)) while \( GL_d(\mathbb{Q}) \cdot (\mathbb{Z}_p^{\otimes d}) \equiv \mathcal{L}_p \).

Given \( \mathcal{D} = \{ \mathcal{D}_N \} \in \mathcal{B}_d(X), X = \overline{\text{Spec } \mathbb{Z}} = \{ X_N \} \), monotone decreasing bounded below sequence, we let \( \text{div} \mathcal{D} := \bigcap_N \text{div}_N(\mathcal{D}_N) \), the intersection taken in \( \mathbb{A}^{\otimes d} \).

Thus \( \widehat{\text{div}}(\mathcal{D})_p = \text{div}_N(\mathcal{D}_N)_p \), for all finite \( p \), and all \( N \) divisible by (some fixed) \( N_0 \), and \( \text{div}(\mathcal{D})_\eta = \bigcap_N g_N(\mathbb{Z}_p^{\otimes d}) \) (where \( g_N \) is the \( \eta \)-component of \( \mathcal{D}_N \)) is an (arbitrary real) ellipsoid. We have \( \mathcal{D} = \{ \mathcal{D}_N \} \supseteq \mathcal{D}' = \{ \mathcal{D}'_N \} \), if and only if, \( \text{div}(\mathcal{D}) \supseteq \text{div}(\mathcal{D}') \), and so \( \widehat{\text{div}}(\cdot) \) induces a bijection

$$
\widehat{\mathcal{D}}_d(X) = \mathcal{B}_d(X)/\sim \xrightarrow{\text{div}_N} \mathcal{L}_\mathbb{A} \leftarrow X_\mathbb{A}.
$$

(6.3.20)

In exactly the same way, we obtain for any number field \( K \), a \( GL_d(K) \)-covariant identification

$$
\widehat{\mathcal{D}}_d(\overline{\text{Spec } \mathcal{O}_K}) \cong GL_d(\mathbb{A}_K)/\bigg/ \prod_\nu GL_d(\mathcal{O}_{K,\nu}),
$$

(6.3.21)

where \( \mathbb{A}_K \) is the ring of Adeles of \( K \), \( \mathcal{O}_{K,\nu} \) the local ring at \( \nu \) for finite \( \nu \)'s, and \( GL_d(\mathcal{O}_{K,\nu}) \cong O_d \) (resp. \( U_d \)) the orthogonal (resp. unitary) group for \( \nu \) real (resp. complex).

For \( d = 1 \), \( GL_1(\mathcal{K}_N)/GL_1(\mathcal{O}_{X_N}) \cong \mathbb{K}_N^*/\mathcal{O}_{X_N}^* \) is a sheaf of abelian groups, and we have the ordered abelian group \( \widehat{\mathcal{D}}_1(X) \), and its quotient by \( GL_1(\mathcal{K}(X)) = \lim_N GL_1(\mathcal{K}(X)_N) \), the completed- Picard group

$$
\widehat{\text{Pic}}_1(X) := GL_1(\mathcal{K}(X))/\widehat{\mathcal{D}}_1(X).
$$

(6.3.22)

For a number field \( K \),

$$
\widehat{\text{Pic}}_1(\overline{\text{Spec } \mathcal{O}_K}) = K^*/\mathbb{A}_K^*/\prod_\nu \mathcal{O}_{K,\nu}^*.
$$

(6.3.23)
Chapter 7

Modules

7.1 Definitions

Definition 7.1.1

Let $A \in \mathcal{F}$. We denote by $A\text{-mod}$ the full subcategory of the functor category $(\text{Ab})^{A \times A^{op}}$ given by

$$A\text{-mod} \equiv \left\{ M = \{M_{Y,X}\} \in (\text{Ab})^{A \times A^{op}}, M_{0,X} = \{0\} = M_{Y,0} \right\}.$$  \hspace{1cm} (7.1.1)

Thus an $A$-module $M$ is a collection of abelian groups $M_{Y,X}$, for $X, Y \in \mathcal{F}$, together with maps:

$$A_{Y',Y} \times M_{Y,X} \times A_{X,X'} \to M_{Y',X'}$$  \hspace{1cm} (7.1.2)

$$a, m, b \mapsto a \circ m \circ b$$  \hspace{1cm} (7.1.3)

such that,

$$a \circ (m + m') \circ b = a \circ m \circ b + a \circ m' \circ b,$$  \hspace{1cm} (homomorphism) \hspace{1cm} (7.1.4)

$$(a_1 \circ a_2) \circ m \circ (b_2 \circ b_1) = a_1 \circ (a_2 \circ m \circ b_2) \circ b_1,$$  \hspace{1cm} (associativity) \hspace{1cm} (7.1.5)

$$id_{Y} \circ m \circ id_{X} = m.$$  \hspace{1cm} (identity) \hspace{1cm} (7.1.6)

Notation: for $m \in M_{Y,X}$ we will write

$$a \circ m := a \circ m \circ id_{X},$$  \hspace{1cm} (7.1.7)

$$m \circ b := id_{Y} \circ m \circ b.$$  \hspace{1cm} (7.1.8)
Proposition 7.1.1

An $\mathcal{A}$-mod is complete and co-complete abelian category. It has enough projectives and injectives.

Proof. (Well known). All (co)limits can be taken pointwise $(\lim M)_{Y,X} = \text{lim}(M_{Y,X})$. The evaluation functor $i_{Y,X}^*: \mathcal{A}$-mod $\to$ Ab, $i_{Y,X}^* M = M_{Y,X}$, has a left (resp. right) adjoint $i_{Y,X}^!$ (resp. $i_{Y,X}^*$). Taking a surjection $p_{Y,X}^* \to M_{Y,X}$ (resp. injection $M_{Y,X} \hookrightarrow p_{Y,X}^*$) with $p_{Y,X}^*$ projective (resp. injective) in Ab, we obtain a surjection (resp. injection) $\bigoplus_{Y,X} i_{Y,X}^* p_{Y,X} \to M$ (resp. $M \hookrightarrow \prod_{Y,X} i_{Y,X}^* p_{Y,X}$) from a projective (resp. into an injective) $\mathcal{A}$-module.

We have the following injective and surjective maps:

\[ M_{Y_0,X_0} \oplus M_{Y_0,X_1} \oplus M_{Y_1,X_0} \oplus M_{Y_1,X_1} \xrightarrow{g} M_{Y_0Y_1,X_0X_1} \]

\[ M_{Y_0,Y_1} \oplus M_{X_0,X_1} \xrightarrow{f} M_{Y_0Y_1,X_0X_1} \]

given by,

\[ \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \mapsto \sum_{i,j=0,1} 1_{Y_j} \circ m_{i,j} \circ 1_{X_j}, \]

\[ (1_{Y_j} \circ m \circ 1_{X_j}) \longleftrightarrow m. \]

where,

\[ 1_{Y_j} \in \mathbb{F}_{Y_0 \oplus Y_1,Y_j}, 1_{X_j} \in \mathbb{F}_{X_j,X_0 \oplus X_1}. \]

\[ (1_{Y_j}^* \circ 1_{X_j} = id_{X_j}, 1_{X_j} \circ 1_{Y_j}^* = id_{X_j} \oplus 0_{X_1-j}, j = 0,1). \]

Note that the composition $g \circ f = id$, therefore it is a direct summand, but in general $f \circ g \neq id$, in fact,

\[ f \circ g = id \iff m = \sum_{i,j=0,1} 1_{Y_i} \circ 1_{Y_j}^* \circ m \circ 1_{X_j} \circ 1_{X_j}^*, \]

\[ \forall m \in M_{Y_0 \oplus Y_1,X_0 \oplus X_1}. \]

There is a similar correspondence:

\[ (M_{1,1})^{X \oplus Y} \xrightarrow{g} M_{Y,X} \]

\[ (M_{1,1})^{X \oplus Y} \xrightarrow{f} M_{Y,X} \]

where $(M_{1,1})^{X \oplus Y}$ is a direct summand,

\[ \left( 1_{y}^* \circ m \circ 1_{x} \right)_{(x,y) \in Y \oplus X} \longleftrightarrow m. \]
\[ m_{y,x} \mapsto \sum_{x \in X, y \in Y} 1_y \circ m_{y,x} \circ 1^t_x. \quad (7.1.14) \]

If these maps are isomorphisms for all \( X, Y \in \mathcal{F} \), we say \( M \) is a "matrix \( A \)-module".

In particular, the map \( f \) of (7.1.9), made up from the \( \mathcal{F} \)-action of \( 1_{Y_i} \) and \( 1^t_{X_j} \), gives the "direct- sum" for \( M \):

\[ M_{Y_0,X_0} \times M_{Y_1,X_1} \to M_{Y_0 \oplus Y_1, X_0 \oplus X_1} \quad (7.1.15) \]

\[ (m_0, m_1) \mapsto m_0 \oplus m_1 := 1_{Y_0} \circ m_0 \circ 1^t_{X_0} + 1_{Y_1} \circ m_1 \circ 1^t_{X_1}. \quad (7.1.16) \]

Note: associativity, commutativity, unit -isomorphisms are the canonical \( \mathcal{F} \)-isomorphisms. The map \( (m_0, m_1) \mapsto m_0 \oplus m_1 \) is strongly natural in the sense that

\[ a \circ (m_0 \oplus m_1) \circ a' = (a \circ 1_{Y_0}) \circ m_0 \circ (1^t_{X_0} \circ a') + (a \circ 1_{Y_1}) \circ m_1 \circ (1^t_{X_1} \circ a'), \quad (7.1.17) \]

In particular it is natural,

\[ (a_0 \oplus a_1) \circ (m_0 \oplus m_1) \circ (a'_0 \oplus a'_1) = (a_0 \circ m_0 \circ a'_0) \oplus (a_1 \circ m_1 \circ a'_1) \quad (7.1.18) \]

**Example 7.1.1**

Define \( M^{k,l} \in \mathcal{F} - \text{mod} \) by

\[ \left( M^{k,l} \right)_{Y,X} := \text{free abelian group on } (Y_0, X_0), \quad (7.1.19) \]

\[ Y_0 \subseteq Y \quad \text{,} \quad \#Y_0 := k \]

\[ X_0 \subseteq X \quad \text{,} \quad \#X_0 := l \quad (7.1.20) \]

and the \( \mathcal{F} \)-action of \( a \in \mathcal{F}_{Y', Y}, b \in \mathcal{F}_{X, X'} \), is defined on the generators \( (Y_0, X_0) \) of \( (M^{k,l})_{Y,X} \) by:

\[ a \circ (Y_0, X_0) \circ b = \begin{cases} (a(Y_0), b'(X_0)) & Y_0 \subseteq D(a), X_0 \subseteq I(b) \\ 0 & \text{otherwise} \end{cases} \quad (7.1.21) \]

The rank at every degree \( (Y, X) \) is easy to calculate and is given by,

\[ \text{Rank}_{\mathbb{Z}} \left( M^{k,l}_{Y,X} \right) = \binom{\#Y}{k} \cdot \binom{\#X}{l}. \quad (7.1.22) \]

where in case \( k > 1 \) or \( l > 1 \) and \( X = Y = 1 \) we have: \( \left( M^{k,l} \right)_{1,1} = \{0\} \).

It is an example of a non- matrix \( \mathcal{F} \)-module.
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Example 7.1.2

Define $M_{k,l} \in \mathbb{F}(\mathbb{Z})$-mod by,

$$
(M_{k,l})_{Y,X} = \text{Hom}_{\mathbb{Z}}\left(\bigwedge^l(Z \cdot X), \bigwedge^k(Z \cdot Y)\right)
$$

(7.1.23)

and the $\mathbb{F}(\mathbb{Z})$-action of

$$
a \in \mathbb{F}(\mathbb{Z})Y \cdot Y = \text{Hom}_{\mathbb{Z}}(Z \cdot Y, Z \cdot Y'), \quad b \in \mathbb{F}(\mathbb{Z})X \cdot X = \text{Hom}_{\mathbb{Z}}(Z \cdot X', Z \cdot X),
$$

(7.1.24)

is given by,

$$
a \circ m \circ b := \bigwedge^k(a) \circ m \circ \bigwedge^l(b).
$$

(7.1.25)

Note that it is non-matrix.

**Tensor product:**

For $M, N \in A$-mod, we have their tensor product $M \otimes A N \in A$-mod,

$$(M \otimes_A N)_{Y,X} := \bigoplus_{Z} M_{Y,Z} \otimes N_{Z,X} / \{(m \circ a) \otimes n - m \otimes (a \circ n), m \in M_{Y,Z'}, a \in A_{Z',Z}, a \in N_{Z,X}, n \in N_{Z,X}\},$$

and we have their left (resp. right) inner hom $\text{Hom}_{A}^{l/r}(M, N) \in A$-mod

$$
\text{Hom}_{A}^{l}(M, N)_{Y,X} := \{\varphi = \{\varphi_Z\}, \varphi_Z \in \text{Ab}(M_{Z,X}, N_{Z,Y}), \varphi(a \circ m) = a \circ \varphi(m)\}
$$

with $A$- action:

$$(a_1 \circ \varphi \circ a_2)(m) := \varphi(m \circ a_1) \circ a_2.\quad \text{resp.,}
$$

$$
\text{Hom}_{A}^{r}(M, N)_{Y,X} := \{\varphi = \{\varphi_Z\}, \varphi_Z \in \text{Ab}(M_{X,Z}, N_{Z,Y}), \varphi(m \circ a) = \varphi(m) \circ a\},
$$

with $A$- action:

$$(a_1 \circ \varphi \circ a_2)(m) := a_1 \circ \varphi(a_2 \circ m).$$

We have the adjunction:

$$
\text{A-mod}(M \otimes_A N, K) = \text{A-mod}(M, \text{Hom}_{A}^{l/r}(N, K)) = \text{A-mod}(N, \text{Hom}_{A}^{l}(M, K)).
$$

Note that $\otimes_A$ is associative, but Not commutative, and has unit $A$ only if $A$ is an $A$- module, i.e. $A \in (\mathbb{F}R)^{A\text{Ab}}$.

**Definition 7.1.2. $A$- modules with involution: $A$-mod$^\dagger$**

For any $A \in \mathbb{F}R$ define $A$-mod$^\dagger$ to be the category of $M = \{M_{Y,X}\} \in A$-mod with an involution,

$$(\cdot)^\dagger : M_{Y,X} \rightarrow M_{X,Y}
$$

(7.1.26)
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\[ m \mapsto m^t \] (7.1.27)

satisfying:

\[ (m_0 + m_1)^t = m_0^t + m_1^t \] (7.1.28)

\[ (m^t)^t = m \]

\[ (a \circ m \circ b)^t = b^t \circ m^t \circ a^t. \] (7.1.29)

The morphisms in \( \text{\textit{A}} \text{-mod}^t \) between two such modules \( M, N \), are given by the set:

\[ \text{\textit{A}} \text{-mod}^t(M, N) \equiv \{ \varphi \in \text{\textit{A}} \text{-mod}(M, N) \mid \varphi(m^t) = \varphi(m)^t \}. \] (7.1.30)

**Proposition 7.1.2**

The category \( \text{\textit{A}} \text{-mod}^t \) is complete and co-complete abelian category with enough injectives and projectives objects.

*Proof.* Same as proposition 7.1.1. \( \square \)

**Free \textit{A}-modules**

For \( X, Y \in \text{F} \) and an abelian group \( N \in \text{Ab} \), we have the adjunction:

\[ \text{\textit{A}} \text{-mod} \left( i^Y_X N, M \right) \equiv \text{Ab} \left( N, M_{Y,X} \right) \] (7.1.31)

where the \( i \) functor defined by,

\[ (i^Y_X N)_{W,Z} = \bigoplus_{A_{W,Y} \times A_{X,Z}} N. \] (7.1.32)

An element of \( (i^Y_X N)_{W,Z} \) has the form \( \sum_{i=1}^k n_i \cdot (a_i, b_i) \) with \( n_i \in N, a_i \in A_{W,Y}, b_i \in A_{X,Z} \).

\[ (\text{where } 0 \cdot (a, b) \equiv n \cdot (0, b) \equiv n \cdot (a, 0) \equiv 0, ) \] (7.1.33)

The \textit{A} action on such an element of \( a \in A_{W,Y}, b \in A_{X,Z} \) is given by:

\[ a \circ (\sum n_i(a_i, b_i)) \circ b = \sum n_i(a \circ a_i, b_i \circ b). \] (7.1.34)

In particular, we have the free \textit{A} module of degree \( (Y, X) \):

\[ A^{Y,X}_z := i^Y_X \mathbb{Z}, \] (7.1.35)

\[ \text{\textit{A}} \text{-mod}(A^{Y,X}, M) \equiv M_{Y,X}. \] (7.1.36)
For $A \in \mathbb{F}^{\mathbb{F}}$ (with involution!), $X, Y \in \mathbb{F}, N \in A b, M \in A\text{-mod}^t$, we have the adjunction:
\[
A\text{-mod}^t(i^{Y,X}_Y, M) \equiv Ab(N, M_{Y,X}) \tag{7.1.37}
\]
where now: $(i^{Y,X}_Y N)_{W,Z} = \left( \bigoplus_{A_{W,Y \times A_{X,Z}}} N \right) \oplus \left( \bigoplus_{A_{W,X \times A_{Y,Z}}} N \right) \tag{7.1.38}
\]
has $A$- action as above and has involution, interchanging the summands above:
\[
(\Sigma n_i(a_i, b_i))^t = \Sigma n_i(b_i, a_i). \tag{7.1.39}
\]
The free $\mathbb{A}$- module with involution of degree $(Y, X)$, $A^i_{Y,X}$, is obtained by taking $N = \mathbb{Z}$:
\[
A^i_{Y,X} := i^{Y,X}_{Y} \mathbb{Z} \quad \text{with involution,} \tag{7.1.40}
\]
\[
A\text{-mod}^t(A^i_{Y,X}, M) \equiv M_{Y,X} \tag{7.1.41}
\]
Similarly, for $I \in \text{Set/F} \times \mathbb{F}$, i.e. $I$ is a set together with a map $I \to \mathbb{F} \times \mathbb{F}, i \mapsto (Y_i, X_i)$, we have the free $A$- module, $A^I = A^i_{Y_i, X_i}$, giving the adjunction:
\[
A\text{-mod}^t \quad A\text{-mod}^t(I, M) \equiv \text{Set/F} \times \mathbb{F} (I, UM). \tag{7.1.42}
\]
If we let $(\text{Set/F} \times \mathbb{F})^t$ denote the category of sets over $\mathbb{F} \times \mathbb{F}, I \to \mathbb{F} \times \mathbb{F}$, together with involution i.e. a bijection $I \to I, i \mapsto i^t, i^{tt} = i, (I^{-1}(Y, X))^t = I^{-1}(X, Y)$, than we have the adjunction:
\[
A\text{-mod}^t \quad A\text{-mod}^t(A^I^{Y_X}, M) \equiv (\text{Set/F} \times \mathbb{F})^t(I, UM). \tag{7.1.43}
\]

### 7.2 Commutativity for Modules

Let $M \in A\text{-mod}^t$.

**Definition 7.2.1**

We say $M$ is **Total commutative** if:
\[
\forall m \in M_{Y,X}, \quad b \in A_{I,J}, \quad \left( \bigoplus_Y b \right) \circ \left( \bigoplus_J m \right) = \left( \bigoplus_J m \right) \circ \left( \bigoplus_X b \right), \tag{7.2.1}
\]
Commutative:
\[
\forall m \in M_{Y,X}, \quad b \in A_{1,1}, \quad d \in A_{J,1},
\]
\[
(\bigoplus_{Y} b) \circ (\bigoplus_{J} m) \circ (\bigoplus_{X} d) = (\bigoplus_{Y} b \circ d) \circ m = m \circ (\bigoplus_{X} d),
\] (7.2.2)

Central:
\[
\forall m \in M_{Y,X}, \quad b \in A_{1,1},
\]
\[
(\bigoplus_{Y} b) \circ m \circ (\bigoplus_{X} b) =: b \cdot m.
\] (7.2.3)
i.e. the monoid \( A_{1,1} \) acts centrally on \( M_{Y,X} \), and we denote this actions by \( b \cdot m \).

Note that,

\[
\text{total commutativity} \implies \text{commutativity} \implies \text{centrality}.
\]

We let \( CA\text{-}mod \) (resp. \( A\text{-}mod_{\text{tot-com}} \), \( A\text{-}mod_{\text{cent}} \)) denote the full subcategory of \( A\text{-}mod \) consisting of commutative (resp. totally commutative, central) \( A\text{-}modules \).

We have the following adjunctions between the categories,

\[
\begin{array}{cccc}
A\text{-}mod_{\text{tot-com}} & \cong & CA\text{-}mod & \cong \ A\text{-}mod_{\text{cent}} & \cong & A\text{-}mod
\end{array}
\]

the left adjoint given by the quotient maps:

\[
M_{\text{tot-com}} \leftarrow CM \leftarrow M_{\text{cent}} \leftarrow M
\] (7.2.4)

For \( M \in A\text{-}mod_{\text{cent}}^{(t)} \), and \( S \subseteq A_{1,1}^{(+)} \) multiplicative, we have the localization

\[
S^{-1}M \in S^{-1}A\text{-}mod^{(t)}.
\] (7.2.5)

We have

\[
(S^{-1}M)_{Y,X} := (M_{Y,X} \times S) / \sim,
\] (7.2.6)

\[
(m_0, s_0) \sim (m_1, s_1) \iff \exists s \in S \text{ such that } (s \cdot s_1) : m_0 = (s \cdot s_0) : m_1,
\]

and denoting by \( m/s \) the equivalence class of \( (m, s) \), we have

\[
(a'/s') \circ (m/s) \circ (a''/s'') := (a' \circ m \circ a'')/(s' \cdot s \cdot s'')
\] (7.2.7)

In particular we have localizations

\[
M_f := S_f^{-1}M, \quad S_f = \{1, f, f^2, \ldots, f^n, \ldots\}, \quad f \in A_{1,1}^{(+)}
\]

\[
M_p := S_p^{-1}M, \quad S_p = A_{1,1}^{(+)} \setminus p, \quad p \in \text{Spec}^{(+)}A.
\]
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Remark 0

If $A \in \mathbb{F}R$ is such that there exists $b \in A_{1,J}, d \in A_{J,1}, b \circ d = 1, b_j = b \circ 1_j = 0, (or$ $d_j = 1_j \circ d = 0)$ then $CA$-mod $= \{0\}$: If $M \in A$-mod is commutative, $m \in M_{Y,X}$,

$$m = (b \circ d) \cdot m = (\bigoplus_{Y} b) \circ (\sum_{j \in J} 1_{Y_j} \circ m \circ 1_{X_j}) \circ (\bigoplus_{X} d) = \sum_{j \in J} (\bigoplus_{Y} b_j) \circ m \circ (\bigoplus_{X} d_j) = \sum_{j} (b_j \circ d_j) \cdot m = 0.$$  
(7.2.8)

E.g., when $A = \mathbb{F}_\eta$.

Remark 1

If $M \in \mathbb{F} \cdot \text{mod}$, then $M$ is automatically commutative.

Remark 2

If $M \in \mathbb{F}\{S\} \cdot \text{mod}$, $M$ commutative $\iff$ $M$ central: $(\bigoplus_{Y} s) \circ m = m \circ (\bigoplus_{X} s), \forall s \in S, m \in M_{Y,X}$. For $S = \{\pm 1\}$, $(\bigoplus_{Y} (-1)) \circ m = m \circ (\bigoplus_{X} (-1))$; if this is $-m$ (= the inverse of $m$ in the abelian group $M_{Y,X}$), we shall say $M$ is "(-1)-true". For $S = \mathbb{N}$, $(\bigoplus_{Y} p) \circ m = m \circ (\bigoplus_{X} p)$ for all (prime) $p \in \mathbb{N}$; if this is equal to $p \cdot m$ (= $(m + m + \cdots + m)$, $p$ times), we shall say $M$ is "N-true". If it is both (-1)-true and N-true we shall say it is "Z-true".

Remark 3

For $R \in CRig, M \in \mathbb{F}(R) \cdot \text{mod}$, $m_0, m_1 \in M_{Y,X}$, the element $m_0 \ast m_1 \in M_{Y,X}$

$$m_0 \ast m_1 = (\bigoplus_{Y} (1,1)) \circ (m_0 \circ (\bigoplus_{X} (\bigoplus_{Y} (1,1))) = (\bigoplus_{Y} (1,1)) \circ (1_{Y_0} \circ m_0 \circ 1_{X_0} + 1_{Y_1} \circ m_1 \circ 1_{X_1} \circ (\bigoplus_{X} (1,1)).$$  
(7.2.9)

is equal to $m_0 + m_1$ (indeed, $(m_0 + m_0') \ast (m_1 + m_1') = (m_0 \ast m_1) + (m_0' \ast m_1')$).

Assume that $M$ is commutative as $\mathbb{F}(R) \cdot$-module. For $r \in R, m \in M_{Y,X}$, put

$$r \cdot m := (\bigoplus_{Y} (r)) \circ m = m \circ (\bigoplus_{X} (r)) \quad (M \text{ central}).$$  
(7.2.10)

We have $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m), 1 \cdot m = m, 0 \cdot m = 0, r \cdot (m + m') = r \cdot m + r \cdot m'$, (by definition!), and moreover,

$$r \cdot m + r' \cdot m = (\bigoplus_{Y} (1,1)) \circ (1_{Y_0} \circ r \cdot m \circ 1_{X_0} + 1_{Y_1} \circ r' \cdot m \circ 1_{X_1}) \circ (\bigoplus_{X} (1,1))$$

$$= (\bigoplus_{Y} (r, r')) \circ (m \circ (\bigoplus_{X} (1,1)))$$

$$= (\bigoplus_{Y} (r, r')) \circ (\bigoplus_{X} (1,1)) \circ m \quad (M \text{ commutative})$$

$$= (r + r') \cdot m.$$  
(7.2.11)
Thus \(M_{Y,X}\) is an \(R\) -module. For \(a \in \mathbb{F}(R)_{Y,Y'}, b \in \mathbb{F}(R)_{X,X'}\), the map \(m \mapsto a \circ m \circ b\) is an \(R\) -module homomorphism, so \(M \in (R\text{-mod})^{\mathbb{F}(R) \times \mathbb{F}(R)^{op}}\).

Note that \(M\) is \(\mathbb{N}\) - true, and if \(R \in \text{CRing}\), has negatives, \((-1) \cdot m = (\bigoplus (-1)) \circ m = m \circ (\bigoplus (-1))\) is equal to \(-m\), the inverse of \(m\), and \(M\) is \(\mathbb{Z}\) - true.

Conversely, if \(M \in (R\text{-mod})^{\mathbb{F}(R) \times \mathbb{F}(R)^{op}}\), is an \(\mathbb{F}(R)\)-module with values in \(R\) -modules such that for \(r \in R, m \in M_{Y,X}, (\bigoplus (r)) \circ m = m \circ (\bigoplus (r)) = r \cdot m\), (i.e. \(M\) is central, and the action of the monoid \(\mathbb{F}(R)_{1,1} = R\) is the given \(R\) -module structure on \(M_{Y,X}\)), then \(M\) is commutative as \(\mathbb{F}(R)\) -module:

\[
\left( \bigoplus_{i=1}^{n} m \right) \circ \left( \bigoplus_{i=1}^{n} m \right) = \left( \bigoplus_{i=1}^{n} m \right) \circ \left( \bigoplus_{i=1}^{n} m \right)
\]

\[
\left( \bigoplus_{i=1}^{n} r_{i} \cdot m \right) \circ \left( \bigoplus_{i=1}^{n} r_{i} \cdot m \right) = r_{1} \cdot r_{1}' \cdot m + \cdots + r_{n} \cdot r_{n}' \cdot m = [r_{1} r_{1}' + \cdots + r_{n} r_{n}'] \cdot m = (M_{Y,X} \in R\text{-mod})
\]

\[
\left[ (r_{1}, \ldots, r_{n}) \circ \left( \begin{array}{c} r_{1}' \\ \vdots \\ r_{n}' \end{array} \right) \right] \cdot m
\]

**7.3 Sheaves of \(O_{X}\)- modules**

**Definition 7.3.1**

Given \((X, O_{X}) \in \mathbb{F} \mathbb{R}(t)Sp\), define an \(O_{X}\)-mod \((t)\) \(M\) to be a functor

\[
U \mapsto M(U), \text{ for } U \subseteq X \text{ open},
\]

such that,

\[
M(U) \in CO_{X}(U)\text{-mod}^{(t)},
\]

for \(U \subseteq U' \subseteq X \text{ open}:

\[
b|_{U} \circ m|_{U} \circ d|_{U} = (b \circ m \circ d)|_{U},
\]

(resp. \((m|_{U})^{t} = m^{t}|_{U}\)),

and \(\forall X, Y \in \mathbb{F}\),

\[
U \mapsto M(U)_{Y,X} \text{ is a sheaf}.
\]
Definition 7.3.2

Let $A \in CFR^{(t)}$, $X = \text{Spec}(^+ A)$, $M \in CA$-$\text{mod}^{(t)}$. Define $\tilde{M} \in O_X$-$\text{mod}^{(t)}$,

$$\tilde{M}(U)_{Y,X} \equiv \{ s : U \to \prod_{p \in U} (M_p)_{Y,X} \mid s(p) \in (M_p)_{Y,X} \text{ such that locally } s(p) = m/f. \}$$  \hspace{1cm} (7.3.6)

We have,

**Proposition 7.3.1**

For $p \in \text{Spec}(^+ A)$,

$$\tilde{M}_p = M_p.$$  \hspace{1cm} (7.3.7)

**Proof.** see Proposition 3.4.2. 

**Theorem 7.3.2**

For $f \in A_{1,1}^{(+)x}$,

$$\tilde{M}(D^{(+)x}(f)) \equiv M_f.$$  \hspace{1cm} (7.3.8)

**Proof.** Replace $a \in A_{Y,X}$ by $m \in M_{Y,X}$ in Proposition 3.4.3. 

**Theorem 7.3.3**

Let $X \in gFR^{(t)}Sc$, $M \in O_X$-$\text{mod}^{(t)}$, the following conditions are equivalent,

1. $X = \bigcup \text{Spec}(^+ A_i)$, $\exists M_i \in A_i$-$\text{mod}^{(t)}$, $M|_{\text{Spec}(^+ A_i)} = \tilde{M}_i$,  \hspace{1cm} (7.3.9)

2. $\forall \text{Spec}(^+ A) \subseteq X$, $M(\text{Spec}(^+ A)) \tilde{\to} M|_{\text{Spec}(^+ A)}$,  \hspace{1cm} (7.3.10)

3. $\forall U \subseteq X$ open, $\forall g \in O_X(U)_{1,1}^{(+)x}$, letting $D(g) = \{ p \in U, g|_p \in GL_1(O_{X,p}) \} \subseteq U$, restriction induces isomorphism: $M(U)_g \tilde{\to} M(D(g)).$  \hspace{1cm} (7.3.11)

We say $M$ is "quasi-coherent" $O_X$-module, and we denote by $q.c.$ $O_X$-$\text{mod}^{(t)}$ the full subcategory of $O_X$-$\text{mod}^{(t)}$ consisting of the quasi coherent $O_X$-modules.

For an affine $X = \text{Spec}(^+ A)$, we have an equivalence

$$CA$-$\text{mod}^{(t)} \tilde{\to} q.c. \ O_X$-$\text{mod}^{(t)} \subseteq O_X$-$\text{mod}^{(t)},$$  \hspace{1cm} (7.3.12)

$$M \mapsto \tilde{M},$$

$$M(X) \mapsto M.$$  \hspace{1cm} (7.3.13)
7.4 Extension of scalars

For \( \varphi \in \mathcal{F}R^{(t)}(B, A) \), we have an induced pair of adjoint functors: We use geometric notations

\[
\begin{array}{c}
A \text{-mod}^{(t)} \xrightarrow{\varphi_*} B \text{-mod}^{(t)}
\end{array}
\]

The right adjoint is:

\[
\begin{array}{c}
N \xrightarrow{\varphi*} \varphi* N \equiv N_B := N
\end{array}
\] (7.4.1)

with \( B \)-action given via \( \varphi \):

\[
\begin{array}{c}
b_1 \circ n \circ b_2 := \varphi(b_1) \circ n \circ \varphi(b_2).
\end{array}
\] (7.4.2)

The left adjoint will be denoted by \( \varphi* M := M^A \). The abelian group \( M_{Y,X} \) is obtained from the free sum:

\[
\bigoplus_{W,Z \in \mathcal{F}A} M_{W,Z}
\] (7.4.3)

whose elements can be written as sums

\[
\sum_{i=1}^{k} a_i \circ [m_i] \circ a'_i, \quad a_i \in A_{Y,W_i}, \quad a'_i \in A_{Z_i,X_i}, \quad m_i \in M_{W_i,Z_i}.
\] (7.4.4)

with \( a \circ [m + m'] \circ a' = a \circ [m] \circ a' + a \circ [m'] \circ a' \), (7.4.5)

by dividing by the subgroup generated by all elements of the form

\[
a \circ [b \circ m \circ b'] \circ a' - (a \circ \varphi(b)) \circ [m] \circ (\varphi(b') \circ a').
\] (7.4.6)

If \( \varphi \in \mathcal{F}R^{t}(B, A) \), \( M \in B \text{-mod}^{t} \) then \( M^A \in A \text{-mod}^{t} \) has automatically an involution,

\[
\left( \sum a_i \circ [m_i] \circ c_i \right)^t = \sum c_i \circ [m_i'] \circ a'_i.
\] (7.4.7)

7.5 Infinitesimal extensions

Let \( A \in \mathcal{F}R, M \in A \text{-mod} \), We define the infinitesimal extension \( A \prod M \in (\mathcal{F}R/A)^{ab} \), an abelian group object of the category \( \mathcal{F}R/A \) of \( \mathcal{F}\text{-Rings} \) over \( A \):

\[
(A \prod M)_{Y,X} := A_{Y,X} \prod M_{Y,X}
\] (7.5.1)

\[
(a, m) \circ (b, n) := (a \circ b, a \circ n + m \circ b)
\] (7.5.2)

\[
(a_0, m_0) \oplus (a_1, m_1) := (a_0 \oplus a_1, m_0 \oplus m_1)
\] (7.5.3)

We have homomorphism

\[
\pi \in \mathcal{F}R(A \prod M, A), \quad \pi(a, m) = a,
\] (7.5.4)
Furthermore the map,

\[ \mu \in \mathcal{F}/A \left( (\text{AIM}) \prod_A (\text{AIM}), \text{AIM} \right), \quad \mu((a, m), (a, m')) = (a, m + m'). \]

(7.5.5)
satisfy associativity, commutativity, unit: \( \epsilon \in \mathcal{F}/A \left( A, \text{AIM} \right), \quad \epsilon(a) = (a, 0), \)
antipode: \( S(a, m) = (a, -m) \) and so makes \( A \prod M \) into an abelian group object in \( \mathcal{F}/A \).

In the case where \( A \in \mathcal{F}/\mathcal{T}, M \in A\text{-mod}^\mathcal{T}, A \prod M \), has a natural involution \( (a, m)^t := (a^t, m^t), \)
and so \( A \prod M \in (\mathcal{F}/\mathcal{T}/A)^{ab}. \)

Note that we have the (strict) implications

\[ A \text{ totally-commutative} \implies \text{AIM commutative} \implies A \text{ commutative (as } \mathcal{F}\text{-Ring}) \implies M \text{ commutative (as } A\text{-module).} \]

### 7.6 Derivations and differentials

**Definition 7.6.1**

Let \( \varphi \in \mathcal{F}/\mathcal{T}(C, A), M \in A\text{-mod}^\mathcal{T}. \)

Define the \( C \)-derivations from \( A \) to \( M \) to be the set:

\[ \text{Der}_{\mathcal{T}}^C(A, M) := \{ \delta = \{ \delta_{Y,X} : A_{Y,X} \to M_{Y,X} \} \text{ such that} \]

\[ \begin{align*}
\text{(∗) Leibnitz: } & \delta(a \circ a') = \delta(a) \circ a' + a \circ \delta(a') \\
\text{(∗∗) C-linear: } & \delta(\varphi(c)) = 0. \\
\text{(∗∗∗) } & \delta(a_0 \oplus a_1) = \delta(a_0) \oplus \delta(a_1) \\
\text{(resp. } & \delta(a)^t = \delta(a^t))
\end{align*} \]

(7.6.1)

It is a functor \( \text{Der}_{\mathcal{T}}^C(A, -) : A\text{-mod}^\mathcal{T} \to \text{Ab}, \) representable by the \( A \)-module of Kähler differentials \( \Omega(A/C) \in A\text{-mod}^\mathcal{T}. \) The abelian group \( \Omega(A/C)_{Y,X} \) has elements that are sums of the form

\[ \sum_{i=1}^k m_i \cdot a_i' \circ d(a_i) \circ a_i'' \]

(7.6.2)

\[ m_i \in \mathbb{Z}, a_i' \in A_{Y,W_i}, a_i \in A_{W_i,Z_i}, a_i'' \in A_{Z_i,X}. \]

(7.6.3)
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with the relations,

(*) Leibnitz: \( a' \circ d(b \circ b') \circ a'' = (a' \circ b) \circ d(b') \circ a'' + a' \circ d(b') \circ (b' \circ a'') \)

(**) C-linear \( a' \circ d(\varphi(c)) \circ a'' \equiv 0 \). \( (7.6.4) \)

(or equivalently,

\[ a' \circ \varphi(c') \circ d(a) \circ \varphi(c'') \circ a'' = a' \circ d(\varphi(c') \circ a \circ \varphi(c'')) \circ a''. \]

(***) \( a' \circ \varphi(c') \circ d(a) \circ \varphi(c'') \circ a'' = a' \circ d(\varphi(c'') \circ a \circ \varphi(c'')) \circ a''. \)

If \( \varphi \in \mathcal{F} \mathcal{R}^t(C, A) \), i.e. has involution, then \( \Omega(A/C) \in A\text{-mod}^t \):

\[
\left( \Sigma m_i \cdot a_i' \circ d(a_i) \circ a_i'' \right)^t = \Sigma m_i \cdot (a_i''')^t \circ d(a_i') \circ (a_i')^t. \quad (7.6.5)
\]

E.g. for \( A = C[\delta_{Y,X}] = C \otimes \mathbb{F}[\delta_{Y,X}] \) we have,

\( \Omega(C[\delta_{Y,X}]/C) \equiv \text{free } C[\delta_{Y,X}]-\text{module of degree } (Y, X) \text{ with generator } d(\delta_{Y,X}) \). \( (7.6.6) \)

and similarly, for \( A = C[\delta_{Y,X}, \delta_{Y,X}'] \) the free - \( C \)-Ring with involution,

\( \Omega(C[\delta_{Y,X}, \delta_{Y,X}']/C) \equiv \text{free } C[\delta_{Y,X}, \delta_{Y,X}']-\text{mod with involution of degree } (Y, X) \)

generated by \( d(\delta_{Y,X}) \). \( (7.6.7) \)

For \( \varphi \in \mathcal{F} \mathcal{R}^{t(i)}(C, A) \), \( B \in C \setminus \mathcal{F} \mathcal{R}^{t(i)} / A \), i.e. we have \( \mathcal{F} \mathcal{R}^{t(i)}-\text{maps} \)

\[
\epsilon : C \to B,
\pi : B \to A,
\pi \circ \epsilon = \varphi, \quad (7.6.8)
\]

we have the following identifications, for \( M \in A\text{-mod}^{t(i)} \):

\[
\left( C \setminus \mathcal{F} \mathcal{R}^{t(i)} / A \right)(B, \text{ALM}) = \left\{ \psi : B \to A \prod M, \quad \psi(b) = (\pi(b), \delta(b)) \right\}
\left\{ \begin{array}{l}
\delta_{Y,X} : B_{Y,X} \to M_{Y,X}, \delta(b \circ b') = \delta(b) \circ \pi(b') + \pi(b) \circ \delta(b')
\delta(c(C)) = 0
\delta(b_0 \oplus b_1) = \delta(b_0) \oplus \delta(b_1)
\end{array} \right\}
\]

\[
\equiv \mathcal{D} \mathcal{E}^{t(i)}_{C}(B, M_B) \equiv B\text{-mod}^{t(i)}(\Omega(B/C), M_B) \equiv A\text{-mod}^{t(i)}(\Omega(B/C)^A, M). \quad (7.6.9)
\]
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Thus we have the adjunction:

\[
\begin{array}{ccc}
\Omega(B/C)^A & \cong & A\text{-mod}^{(t)} \\
B & \downarrow & \downarrow \\
C \setminus \mathfrak{R}^{(t)} / A & \longrightarrow & A\Pi M
\end{array}
\] (7.6.10)

Restricting to commutative \(A\)-modules \(CA\text{-mod}^{(t)}\), we have similar adjunction, with \(\Omega(B/C)^A\) replaced by its commutative quotient \(C\Omega(B/C)^A\).

7.7 Properties of differentials

Given a homomorphism \(k \to A\) of \(\mathfrak{R}^{(t)}\), and \(M \in A\text{-mod}^{(t)}\), we have the bijection:

\[
\text{Der}_{k}^{(t)}(A, M) \cong A\text{-mod}^{(t)}(\Omega(A/k), M).
\]

\[\varphi \circ d_{A/k} \leftrightarrow \varphi,\] (7.7.1)

where \(d_{A/k} : A \to \Omega(A/k)\) is the universal derivation.

Property 0

Given any commutative diagram in \(C\mathfrak{R}^{(t)}\), and \(M' \in A'\text{-mod}^{(t)}\),

\[
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
\hookrightarrow & & \hookrightarrow \\
k & \longrightarrow & k'
\end{array}
\] (7.7.2)

we have a sequence of homomorphisms:

\[
\text{Der}_{k}^{(t)}(A', M') \longrightarrow \text{Der}_{k}^{(t)}(A', M') \longrightarrow \text{Der}_{k}^{(t)}(A, M')
\] (7.7.3)

represented by the \(A'\text{-mod}\) homomorphisms (with commutative diagrams of derivations)

\[
\begin{array}{ccc}
\Omega(A/k)^{A'} & \longrightarrow & \Omega(A'/k) \longrightarrow \Omega(A'/k') \\
\downarrow (d_{A/k})^{A'} & & \downarrow d_{A'/k} \downarrow d_{A'/k'} \\
A & \longrightarrow & A'
\end{array}
\] (7.7.4)
Property 1 (First exact sequence)

Given a commutative diagram in $\mathcal{F}$

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & A' \\
& \searrow k & \\
& & k
\end{array} 
$$

we have an exact sequence:

$$
\Omega(A/k)^{A'} \longrightarrow \Omega(A'/k) \longrightarrow \Omega(A'/A) \longrightarrow 0 \tag{7.7.6}
$$

Proof. Applying $A'$-mod($\_\_\_ M'$) this is equivalent to

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Der}_A(A', M') \longrightarrow \text{Der}_k(A', M') \longrightarrow \text{Der}_k(A, M') \\
\end{array} \tag{7.7.7}
$$

is exact for every $M' \in A'$-mod, which is clear. \qed

The first exact sequence (7.7.6) will be exact on the left, if and only if, any derivation $D : A \rightarrow M'$ into an $A'$-mod has an extension to a derivation $D'$ of $A'$

$$
\begin{array}{ccc}
A & \xrightarrow{D} & M' \\
\searrow & & \searrow D' \\
& A' & & A'
\end{array} \tag{7.7.8}
$$

e.g. this holds if $\varphi : A \rightarrow A'$ is a retract: have $\psi : A' \rightarrow A$, $\psi \circ \varphi = id_A$, and can take $D' = D \circ \psi$.

Property 2 (Second exact sequence)

For a surjective $\varphi : A \rightarrow A'$,

$$
\mathcal{K}\mathcal{E}\mathcal{R}(\varphi) = A \prod_{A'} A \xrightarrow{\varphi} A \xrightarrow{k} A' \tag{7.7.9}
$$

we have an exact sequence

$$
\begin{array}{c}
\Omega(A \prod_{A'} A/k)^{A'} \xrightarrow{\delta} \Omega(A/k)^{A'} \rightarrow \Omega(A'/k) \rightarrow 0 \\
\end{array} \tag{7.7.10}
$$

d(a_0, a_1) \xrightarrow{\delta} d(a_0) - d(a_1).
Proof. Applying the functor $A'$-mod($\prod_i M'$) this is equivalent to the left exact sequence,

$$0 \to \text{Der}_k(A', M') \to \text{Der}_k(A, M') \to \text{Der}_k(A \prod_{A'} A, M')$$

$$D \mapsto D(a_0, a_1) = D(a_0) - D(a_1). \quad (7.7.11)$$

which is clear. \qed

Moreover, we can replace $\Omega^{(2)} = \Omega(A \prod_{A'} A/k)^{A'}$ on the left of (7.7.11) by its

odd quotient (w.r.t. the involution permuting the factors), $\Omega^{(-)} = \Omega^{(2)}/d(a_0, a_1) + d(a_1, a_0)$. Furthermore, we have a map $\delta : \Omega^{(3)} = \Omega(A \prod_{A'} A/k)^{A'} \to \Omega^{(2)}$,

$$d(a_0, a_1, a_2) \mapsto d(a_1, a_2) - d(a_0, a_2) + d(a_0, a_1),$$

and since $\delta \circ \delta = 0$, we can replace $\Omega^{(2)}$ on the left of (7.7.11) by $\Omega^{(-)}/\delta(\Omega^{(3)})$.

**Property 3**

$$\Omega(A_1 \otimes_k A_2/k) \cong \Omega(A_1/k)^{A_1 \otimes_k A_2} \oplus \Omega(A_2/k)^{A_1 \otimes_k A_2}. \quad (7.7.12)$$

**Property 4**

$$\Omega(A \otimes_k k'/k) \cong \Omega(A/k)^{A \otimes_k k'}. \quad (7.7.13)$$

**Property 5**

Differentials commute with direct limits: we have isomorphisms of $A = \lim_{\longrightarrow} A_i$-modules,

$$\Omega(\lim_{\longrightarrow} A_i/k_i) \cong \lim_{\longrightarrow} \Omega(A_i/k_i)), \quad (7.7.14)$$

**Proof.** Taking $A$-mod($\lim_{\longrightarrow} M$) this is equivalent to

$$A$-mod($\Omega(A \lim_{\longrightarrow} k_i). M) \equiv \text{Der}_{\lim_{\longrightarrow} k_i}(A, M) \equiv \lim_{\longrightarrow} \text{Der}_{k_i}(A, M) \equiv \lim_{\longrightarrow} A$-mod($\Omega(A_i/k_i), M)$

$$\equiv A$-mod($\lim_{\longrightarrow} \Omega(A_i/k_i), M)$. \quad (7.7.15)$$

Properties (0) – (5) all hold with $\Omega$ replaced by its commutative quotient $C\Omega$.

Given multiplicative sets $\sigma \subseteq k_{1,1}, S \subseteq A_{1,1}$, such that $\varphi : k \to A$, takes $\varphi_{1,1}(\sigma) \subseteq S$, we have

$$C\Omega(S^{-1}A/\sigma^{-1}k) \cong S^{-1}C\Omega(A/k). \quad (7.7.16)$$
Example 7.7.1

For a map of monoids \( \varphi \in \text{Mon}(M_0, M_1) \), and the associated map of \( \mathbb{F}\text{-Rings} \) \( \Phi = \mathbb{F}(\varphi) \in \mathbb{F} \mathbb{R}(\mathbb{F}(M_0), \mathbb{F}(M_1)) \), the \( \mathbb{F}(M_1) \) module \( \Omega = \Omega(\mathbb{F}(M_1)/\mathbb{F}(M_0)) \) is generated by \( d(m) \in \Omega_{1,1}, m \in M_1 \), and so is a matrix- module \( \Omega_{Y,X} = (\Omega_{1,1})^{Y \times X} \), and \( \Omega_{1,1} = \Omega(\mathbb{Z}[M_1]/\mathbb{Z}[M_0]) \) is the usual bi- module of differentials of the associated rings. For \( M_1 \) commutative, \( C \Omega_{Y,X} = (C \Omega_{1,1})^{Y \times X} \), and \( C \Omega_{1,1} \) the usual module of Kähler differentials of \( \mathbb{Z}[M_1]/\mathbb{Z}[M_0] \).

Example 7.7.2

For a map of rings \( \varphi \in \text{Mon}(R_0, R_1) \), and the associated map of \( \mathbb{F}\text{-Rings} \) \( \Phi = \mathbb{F}(\varphi) \in \mathbb{F} \mathbb{R}(\mathbb{F}(R_0), \mathbb{F}(R_1)) \), the \( \mathbb{F}(R_1) \) module \( \Omega = \Omega(\mathbb{F}(R_1)/\mathbb{F}(R_0)) \) is generated by \( d(r) \in \Omega_{1,1}, r \in R_1 \), and so is a matrix- module \( \Omega_{Y,X} = (\Omega_{1,1})^{Y \times X} \), and \( \Omega_{1,1} = \Omega(\mathbb{Z}[R_1]/\mathbb{Z}[R_0]) \) is the usual bi- module of differentials of the associated rings. For \( R_1 \) commutative, \( C \Omega_{Y,X} = (C \Omega_{1,1})^{Y \times X} \), and \( C \Omega_{1,1} \) the usual module of Kähler differentials of \( \mathbb{Z}[R_1]/\mathbb{Z}[R_0] \).

7.8 Differentials of \( \mathbb{F}(\mathbb{Z}) \) and \( \mathbb{F}(\mathbb{N}) \).

Theorem 7.8.1

The module \( \Omega = \Omega(\mathbb{F}(\mathbb{N})/\mathbb{F}) \), (respectively, \( \Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\})) \) is defined as:

\[
\Omega_{n,m} = \text{free abelian group on generators:}
\]

\[
[a \begin{array}{c} b \\ b' \end{array}] = a \circ d(1,1) \circ \begin{array}{c} b \\ b' \end{array} \quad \text{and} \quad [a,a'|b] = (a,a') \circ d \begin{array}{c} 1 \\ 1 \end{array} \circ b, \quad \forall b,b' \in \mathbb{N}^m, \forall a,a' \in \mathbb{N}^n;
\]

(7.8.1)

(respectfully, \( b,b' \in \mathbb{Z}^m, a,a' \in \mathbb{Z}^n \),
modulo relations:

\[(0) : \quad \begin{array}{c} 0 \\ b' \end{array} = \begin{array}{c} a \\ b \end{array} = \begin{array}{c} a \end{array} \begin{array}{c} b \\ 0 \end{array} = [a,a'|0] = [0,a'|b] = [a,0|b] = 0,
\]

\[(\text{Comm}) : \quad \begin{array}{c} a \\ b' \end{array} = \begin{array}{c} a \\ b \end{array} \begin{array}{c} b' \\ b \end{array}, \quad [a,a'|b] = [a',a|b],
\]

\[(\text{Ass}) : \quad \begin{array}{c} a \\ b_3 \\ b_2 \end{array} = \begin{array}{c} a \\ b_1 \\ b_2 \\ b_3 \end{array} = \begin{array}{c} a \\ b_1 \\ b_2 \\ b_3 \\ b_3 \end{array} \]

\[a + a_2,a_3|b] + [a_1,a_2|b] = [a_1,a_2 + a_3|b] + [a_2,a_3|b],
\]

\[(\text{Almost linear}) : \quad [a_1,a_2|b_1 + b_2] + [a_1 + a_2|b_1] = [a_1|b_1] + [a_2|b_2] + [a_1,a_2|b_1] + [a_1,a_2|b_2].
\]
respectively and the relations,

\[
\begin{align*}
\text{(Cancellation):} & \quad \begin{bmatrix} a & b \\ -a & -b \end{bmatrix} + [a, -a|b] = 0 \\
\text{(-1):} & \quad \begin{bmatrix} -a & -b \\ b' & a' \end{bmatrix} = \begin{bmatrix} -b & a \\ -b' & a' \end{bmatrix}, \quad [a, a'] - b = [-a, -a'|b],
\end{align*}
\]

(7.8.2)

**Proof.** By the description of \([2.10]\), \(F(\mathbb{N}) = F[\delta_{1,2}, \delta^1_{1,2}] / \text{relations}\), (respectively \(F(\mathbb{Z}) = F[\pm 1][\delta_{1,2}, \delta^2_{1,1}] / \text{relations}\)), the second exact sequence (7.7.11) gives that \(\Omega\) is the free \(F(\mathbb{N})\) (respectively \(F(\mathbb{Z})\)) module on \(d\delta_{1,2} = d(1,1)\) and \(d\delta^1_{1,2} = d \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), modulo the derived - relations.

The (0) relation follows, from the implication,

\[
(1, 1) \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1) \implies d(1, 1) \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.
\]

(7.8.3)

The (Comm) relation follows since,

\[
(1, 1) \circ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (1, 1) \implies d(1, 1) \circ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = d(1, 1).
\]

(7.8.4)

The (Ass) relation follows since,

\[
(1, 1) \circ ((1, 1) \oplus id_1) = (1, 1) \circ (id_1 \oplus (1, 1))
\]

\[
\downarrow
\]

\[
d(1, 1) \circ ((1) \oplus (1)) + (1, 1) \circ (d(1, 1) \oplus 0) = d(1, 1) \circ ((1) \oplus (1)) + (1, 1) \circ (0 \oplus d(1, 1))
\]

\[
\downarrow
\]

\[
[a|b_1 + b_2] + [a|b_2^1] = [a|b_2 + b_3] + [a|b_3^2].
\]

(7.8.5)

Thus \(\begin{pmatrix} a \\ b \end{pmatrix}, [b, -|b]\) are symmetric normalized 2-cycles \(\forall b, \forall a\).

The (Almost linear) relation follows since by total commutativity,

\[
\begin{align*}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ (1, 1) & = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = ((1, 1) \oplus (1, 1)) \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\downarrow
\end{align*}
\]

\[
d \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ (1, 1) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ d(1, 1) = (d(1, 1)_{1/1,2} + d(1, 1)_{2/3,4}) \circ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} +
\]

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} +
\]

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]
We have a surjective map

\[ + \left( \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 \\ \end{array} \right) \circ (d(1)_{1,3/1} + d(1)_{2,4/2}). \]  

(7.8.6)

Here the subindices indicate how the matrices act on the differentials (left/right), so that multiplying \(7.8.6\) by \((a_1, a_2)\) on the left, and by \((b_1, b_2)\) on the right, we obtain the almost linear form of the theorem.

Respectively for \(\mathbb{Z}\), the \((\text{Cancellation})\) relation follows since,

\[(1,1) \circ \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \\ \end{array} \right) = (0) \implies d(1,1) \circ \left( \begin{array}{c} 1 \\ -1 \\ \end{array} \right) + (1,-1) \circ d(1) = 0. \]  

(7.8.7)

\[\square\]

The commutative quotient \(C\Omega = C\Omega(\mathbb{F}(\mathbb{N})/\mathbb{F})\) (respectively, \(C\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}(\pm 1))\)) is obtained by adding the relations for all \(\lambda \in \mathbb{N}\) (respectively, \(\mathbb{Z}\)), (cf. Remark 3 of \[7.2\]):

\[\lambda \cdot \left[ \begin{array}{c} a \\ b' \end{array} \right] = \left[ \begin{array}{c} \lambda \cdot a \\ \lambda \cdot b' \end{array} \right] \]

\[\lambda \cdot [a, a', b] = [\lambda \cdot a, \lambda \cdot a', b] = [a, a'|\lambda \cdot b]. \]

(7.8.8)

Define \(\tilde{N}_{Y,X} := \) free abelian group on generators \([a|b], a \in \mathbb{Z}^Y = \mathbb{F}(\mathbb{Z})_{Y,1}, b \in \mathbb{Z}^X = \mathbb{F}(\mathbb{Z})_{1,X}, \) (i.e. we think of the \(a\)'s as column vectors, the \(b\)'s as row vectors), modulo the relations \([a|0] = [0|b] = 0, [-a|b] = [a|b] = -[a|b]. \)

\(\tilde{N} = \{\tilde{N}_{Y,X}\}\) is an \(\mathbb{F}\)-Ring with involution,

\[\tilde{N}_{Z,Y} \times \tilde{N}_{Y,X} \rightarrow \tilde{N}_{Z,X}, \]

(7.8.9)

\[\left( \sum_j n_j [c_j|a'_j]\right) \circ \left( \sum_i m_i [a_i|b_i]\right) = \sum_{i,j} n_j \cdot m_i \cdot a'_j \circ a_i \cdot [c_j|b_i], \]

(7.8.10)

\[\left( \sum_i m_i [a_i|b_i]\right)' = \sum_i m_i [b_i'|a_i']. \]

(7.8.11)

and \(\tilde{N}\) is also an \(\mathbb{F}(\mathbb{Z})\) -module, via

\[\mathbb{F}(\mathbb{Z})_{Y',Y} \times \tilde{N}_{Y,X} \times \mathbb{F}(\mathbb{Z})_{X,X'} \rightarrow \tilde{N}_{Y',X'}, \]

(7.8.12)

\[A \circ \left( \sum_i m_i [a_i|b_i]\right) \circ B := \sum_i m_i [Aa_i|b_iB] \]

(7.8.13)

(and the monoidal structure, \(\oplus : \tilde{N}_{Y_0, X_0} \times \tilde{N}_{Y_1, X_1} \rightarrow \tilde{N}_{Y_0 \oplus Y_1, X_0 \oplus X_1}, \) which is part of the \(\mathbb{F}\)-Ring structure, comes from the \(\mathbb{F}(\mathbb{Z})\) module structure).

We have a surjective map

\[\tilde{\pi} : \tilde{N} \rightarrow \mathbb{F}(\mathbb{Z}) \]

(7.8.14)

\[\tilde{\pi}_{Y,X} : \tilde{N}_{Y,X} \rightarrow \mathbb{F}(\mathbb{Z})_{Y,X} \]

(7.8.15)

\[\tilde{\pi}(\sum m_i [a_i|b_i]) = \sum m_i a_i \otimes b_i \in \mathbb{Z}^Y \otimes \mathbb{Z}^X \equiv \mathbb{Z}^Y \otimes \mathbb{Z}^X \rightarrow \mathbb{F}(\mathbb{Z})_{Y,X} \]

(7.8.16)
and \( \pi \) is both a homomorphism of \( \mathbb{F} \)-rings with involution, and a homomorphism of \( \mathbb{F}(\mathbb{Z}) \)-modules.

We have a surjective map \( \partial : \Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\}) \rightarrow \text{Ker}(\pi) \), define on the generators by

\[
\partial[a_1, a_2|b] = [a_1|b] + [a_2|b] - [a_1 + a_2|b] \\
\partial[a_1|b_2] = -[a_1|b_1] - [a|b_2] + [a|b_1 + b_2]
\]

(\(7.8.17\))

(Indeed it is easy to verify that \( \partial \)(relations) = 0).

Similarly, let \( N_{Y,X} := \text{free abelian group on generators } [a|b], a \in \mathbb{Z}^Y, b \in \mathbb{Z}^X \mod \)

\[
\lambda \cdot [a|b] = [\lambda \cdot a|b] = [a|\lambda \cdot b], \text{for all } \lambda \in \mathbb{Z}.
\]

(\(7.8.18\))

\( N = \{N_{Y,X}\} \) is an \( \mathbb{F} \)-ring with involution, and a commutative \( \mathbb{F}(\mathbb{Z}) \)-module, and again we have a surjection

\[
\pi : N \rightarrow \mathbb{F}(\mathbb{Z})
\]

(\(7.8.19\))

which is both a homomorphism of \( \mathbb{F} \)-rings with involution, and a homomorphism of \( \mathbb{F}(\mathbb{Z}) \)-modules. We have a surjective map \( \tilde{\partial} : C\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\}) \rightarrow \text{Ker}(\pi) \), defined as in (\(\ast\)).

Thus we have exact sequence (of \( \mathbb{F}(\mathbb{Z}) \)-modules)

\[
\begin{array}{cccccc}
\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\}) & \xrightarrow{\partial} & N & \xrightarrow{\pi} & \mathbb{F}(\mathbb{Z}) & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
C\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\}) & \xrightarrow{\tilde{\partial}} & N & \xrightarrow{\pi} & \mathbb{F}(\mathbb{Z}) & \rightarrow 0
\end{array}
\]

(\(7.8.20\))

The derivation \( d = \tilde{\partial} \circ d_{\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\}} : \mathbb{F}(\mathbb{Z}) \rightarrow N \), is given by:

\[
d_{1,1} \equiv 0,
\]

(\(7.8.21\))

\[
d_{1,2}(a, b) = [a|1, 0] + [b|0, 1] - [1|a, b]
\]

(\(7.8.22\))

\[
d_{2,1} \left( \begin{array}{c}
a \\
b \\
\end{array} \right) = \left[ \begin{array}{c}
a | 1, 0 \\
b | 0, 1 \\
\end{array} \right] - \left[ \begin{array}{c}
0 | a, b \\
1 | c, d \\
\end{array} \right]
\]

(\(7.8.23\))

\[
d_{2,2} \left( \begin{array}{c}
a \\
b \\
c \\
d \\
\end{array} \right) = \left[ \begin{array}{c}
a | 1, 0 \\
b | 0, 1 \\
c | 0, a, b \\
d | 0, c, d \\
\end{array} \right] - \left[ \begin{array}{c}
0 | a, b \\
1 | c, d \\
\end{array} \right]
\]

(\(7.8.24\))

For \( A \in \mathbb{F}(\mathbb{Z})_{n,m} \), with columns \( A^{(1)}, \ldots, A^{(m)} \in \mathbb{Z}^n \), rows \( A_1, \ldots, A_n \in \mathbb{Z}^m \),

letting \( f_1^n, \ldots, f_n^n \) be the standard column basis, \( e_1^m, \ldots, e_m^m \) the standard row basis,

\[
f_1^n = \begin{pmatrix}
0 \\
0 \\
1^{(i)} \\
0 \\
\vdots \\
0
\end{pmatrix}, e_j^m = (0, \ldots, 0, 1^{(j)}, 0, \ldots, 0)
\]

(\(7.8.25\))
we have:

\[ d_{n,m}(A) = \sum_{j=1}^{m} \left[ A^{(j)}|e_j^m \right] - \sum_{i=1}^{n} [f_i^n|A_i]. \quad (7.8.26) \]

### 7.9 Differentials for commutative rings

Let \( R \in C\text{Ring} \), \( S \subseteq R \), a multiplicative set.

We have the exact sequences of \( \mathbb{F}(R) \)-modules

\[
\Omega(\mathbb{F}(\mathbb{Z})/\mathbb{F}\{\pm 1\})^{\mathbb{F}(R)} \rightarrow \Omega(\mathbb{F}(\mathbb{R})/\mathbb{F}\{\pm 1\}) \rightarrow \Omega(\mathbb{F}(R)/\mathbb{F}(\mathbb{Z})) \rightarrow 0
\]

\[
\Omega(\mathbb{F}(S)/\mathbb{F}\{\pm 1\})^{\mathbb{F}(R)} \rightarrow \Omega(\mathbb{F}(\mathbb{R})/\mathbb{F}\{\pm 1\}) \rightarrow \Omega(\mathbb{F}(R)/\mathbb{F}(\mathbb{S})) \rightarrow 0
\]

(7.9.1)

**Theorem 7.9.1**

For the \( \mathbb{F}(R) \)-module \( \Omega = \Omega(\mathbb{F}(R)/\mathbb{F}) \), the abelian group \( \Omega_{n,m} \) can be described as the free \( \mathbb{Z} \)-module with generators

\[
[a|b'] = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \circ d(1,1) \circ \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}
\]

(7.9.2)

\[
[a,a'|b] = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a'_1 \\ \vdots \\ a'_m \end{pmatrix} \circ d(1,1) \circ \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}
\]

(7.9.3)

\[
[a|b]^{(r)} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \circ d(r) \circ (b_1, \ldots, b_m),
\]

(7.9.4)
modulo the relations: \[ a_i \in R^n, \ b_i \in R^m, \ r \in R \]

Zero : \[ [a_0, b_0] = [a_0, b_0] = 0 \]

Zero' : \[ [0, a|b] = [0, a|0|b] = 0 \]

(0) : \[ [a, b]^{(0)} = 0 \]

(1) : \[ [a, b]^{(1)} = 0 \]

Comm : \[ [a_0, b_1] = [a_2, b_0] \]

Comm' : \[ [a_1, a_2|b] = [a_2, a_1|b] \]

Ass : \[ [a, b_1 + b_2] + [a, b_1] = [a, b_1 + b_3] + [a, b_3] \]

Ass' : \[ [a, b_1 + b_2] + [a, b_1] = [a, b_1 + b_3] + [a, b_3] \]

tot-com : \[ [a, b_1 + b_2] - [a_1, a_2|b_1] - [a_1, a_2|b_2] \]

\[ + [a_1 + a_2|b_1, b_2] - [a_1|b_1, b_2] - [a_2|b_1, b_2] = 0 \]

\( r, \delta \) : \[ [a, b_1 + b_2]^{(r)} = [a, b_1]^{(r)} + [a, b_2]^{(r)} \]

\( \delta', r \) : \[ [a, b_1 + b_2]^{(r)} = [a, b_1]^{(r)} + [a, b_2]^{(r)} \]

\( r_1 \cdot r_2 \) : \[ [a, b]^{(r_1 + r_2)} = [a, b]^{(r_1)} + [a, b]^{(r_2)} + [a, r_1 \cdot b]^{(r_2)} + [a, r_2 \cdot b]^{(r_1)} \]

(7.9.5)

The \( \mathbb{F}(R) \)-module \( \Omega(\mathbb{F}(R) / \mathbb{F}(S)) \) is the quotient of \( \Omega(\mathbb{F}(R) / \mathbb{F}(S)) \) obtained by adding the relations \([a|b]^{(s)} = 0 \ \forall s \in S\).

\[ \square \]

**Proof.** Same as the proof of 7.8.1, by derivation of the relations 7.2.10 of \( \mathbb{F}(R) \).

**Remark (cf. Remark 3 7.2)**

For the commutative quotient \( C \Omega = C \Omega(\mathbb{F}(R) / \mathbb{F}(S)) \), \( C \Omega_{n,m} \) is obtained from the \( R \)-module \( \Omega_{n,m} \otimes \mathbb{Z} \) by adding the relations

\[ r \cdot [a, b_1] = [r \cdot a, b_1] = [a, r \cdot b_1] \]

\[ r \cdot [a_1, a_2|b] = [r \cdot a_1, r \cdot a_2|b] = [a_1, a_2|b \cdot r] \]

(7.9.6)

Let \( N_{Y,X} \) denote the \( R \)-module obtained from the free \( R \)-module with generators \([a|b], a \in R^Y, b \in R^X \), modulo the relations \( r \cdot [a|b] = [r \cdot a|b] = [a|r \cdot b] \).

We think of the \( b \)'s (resp. \( a \)'s) as row (resp. column) vector, i.e. \( b \in \mathbb{F}(R)_{1,X} \) (resp. \( a \in \mathbb{F}(R)_{Y,1} \)).

The collection \( N = \{ N_{Y,X} \} \) forms an \( \mathbb{F} \)-Ring with involution with respect to the operations

\[ \circ : N_{Z,Y} \times N_{Y,X} \rightarrow N_{Z,X} \]

(7.9.7)
(\sum_j [c_j | \pi_j]) \circ (\sum_i [a_i | b_i]) = \sum_j [\pi_j \circ a_i] \cdot [c_j | b_i] \quad (7.9.8)

(\sum_i [a_i | b_i])^t = \sum_i [b_i^t | a_i^t]. \quad (7.9.9)

It is also a commutative \( \mathbb{F}(R) \) -module with respects to the operation
\[
\mathbb{F}(R)_{Y', Y} \times \mathbb{N}_{Y, X} \times \mathbb{F}(R)_{X', X} \to \mathbb{N}_{Y', X'}.
\]

(7.9.10)

\[
A \circ (\sum_i [a_i | b_i]) \circ B = \sum_i [A \circ a_i | b_i \circ B].
\]

(7.9.11)

We have a surjection of \( \mathbb{F}\text{-}\text{Rings} \) with involution
\[
\pi : N \to \mathbb{F}(R),
\]

(7.9.12)

\[
N_{n, m} \to R^n \otimes_R R^m \equiv R^{n \times m} \equiv \mathbb{F}(R)_{n, m,},
\]

(7.9.13)

\[
\pi(\Sigma r_i [a_i; b_i]) = \Sigma r_i \cdot a_i \otimes b_i.
\]

(7.9.14)

We have a surjection \( \partial : C \Omega(\mathbb{F}(R)/\mathbb{F}) \to \ker(\pi) \), defined on the generators by
\[
\partial[a_1 | b_1] = [a_1 | b_1] + [a_2 | b_2] - [a_1 | b_1 + b_2]
\]

(7.9.15)

\[
\partial[a_1, a_2 | b] = -[a_1 | b] - [a_2 | b] + [a_1 + a_2 | b]
\]

(7.9.16)

(Indeed, it is easy to check that \( \partial \) takes our relations to zero. The minus signs comes from the need to have \( \partial(\text{tot-com}) = 0 \) and \( \partial((r_1 + r_2)) = 0 \). Thus we have,

**Theorem 7.9.2**

For \( R \in \text{CRing} \), we have an exact sequence of commutative \( \mathbb{F}(R) \)-modules
\[
C \Omega(\mathbb{F}(R)/\mathbb{F}) \to N(R) \xrightarrow{\pi} \mathbb{F}(R) \to 0.
\]

(7.9.17)

The map \( \pi \) is a homomorphism of \( \mathbb{F}\text{-}\text{Rings} \) with involution.

### 7.10 Quillen model structures

Define a Quillen model structure on the category of simplicial \( \mathbb{F}\text{-}\text{Rings} \) (with involution) under \( C \in \text{FR}^{(i)} \), \( \Delta(C \setminus \text{FR}^{(i)}) := (C \setminus \text{FR}^{(i)})^{\Delta^\text{op}} \):

**Fibrations:** \( \mathcal{F} = \left\{ \varphi : B \to B' \mid \forall X, Y \in \mathbb{F}, \{ \varphi_{Y, X} : B_{Y, X} \to B'_{Y, X} \} \in \mathcal{F}_{\text{Set}_0} \right\} \).

**Weak equivalences:** \( \mathcal{W} = \left\{ \varphi : B \to B' \mid \forall X, Y \in \mathbb{F}, \{ \varphi_{Y, X} : B_{Y, X} \to B'_{Y, X} \} \in \mathcal{W}_{\text{Set}_0} \right\} \).

**Cofibrations:** \( \mathcal{C} = \mathcal{L}(\mathcal{W} \cap \mathcal{F}) \).

(7.10.1)
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where \( F_{\text{Set}_0} \) (resp. \( W_{\text{Set}_0} \)) denote the fibrations (resp. weak equivalences) of simplicial pointed sets, and where \( L(W \cap F) \) denotes maps satisfying the left lifting property with respects to all the acyclic fibrations \( W \cap F \).

Theorem 7.10.1

This is a closed model structure.

Proof. ([Q67], theorem 4,II,§4)

For \( C \in \mathcal{F}_{\mathcal{R}(t)}, \mathcal{I} \in (\text{Set}/F \times F) \) define,

\[
C[\mathcal{I}] = C[\delta_{Y,X}] = C \bigotimes_{F, i \in I} F[\delta_{Y_i,X_i}].
\]

We have the adjunction,

\[
\Delta(\text{Set}/F \times F)^{(t)} \xrightarrow{F = C[1]} \Delta(C \setminus F)\mathcal{R}^{(t)}
\]

The model structure on \( \Delta(C \setminus F)^{(t)} \) is cofibrantly generated. The set of generating cofibrations is:

\[
\mathcal{I} = \left\{ C[(\partial \Delta(n) \cap \Delta(n))^X,Y] \right\} \equiv \left\{ C[(\partial \Delta(n))_{Y,X}] \to C[(\Delta(n))_{Y,X}], n \geq 1, Y, X \in F \right\}
\]

The set of generating acyclic fibrations is:

\[
\mathcal{J} = \left\{ C[(\Lambda(n,k) \cap \Delta(n))^X,Y] \right\} \equiv \left\{ C[\Lambda(n,k)]_{Y,X} \to C[\Delta(n)]_{Y,X}, 0 \leq k \leq n, Y, X \in F \right\}
\]

A model structure on simplicial \( A \)-modules (with involution),

\[
\Delta(A\text{-mod}^{(t)}) := (A\text{-mod}^{(t)})^{\Delta^{op}}
\]

is given similarly by,

Fibrations: \( F = \left\{ \varphi : M \to M' \mid \forall X, Y \in F, \{ \varphi_{Y,X} : M_{Y,X} \to M'_{Y,X} \} \in F_{\text{Set}_0} \right\} \),

Weak equivalences: \( W = \left\{ \varphi : M \to M' \mid \forall X, Y \in F, \{ \varphi_{Y,X} : M_{Y,X} \to M'_{Y,X} \} \in W_{\text{Set}_0} \right\} \),

Cofibrations: \( \mathcal{C} = L(W \cap F) \).

For \( \mathcal{I} \in \text{Set}/F \times F \), define the functor \( A^{\mathcal{I}}_{(t)} : \Delta(\text{Set}/F \times F) \to \Delta(A\text{-mod}^{(t)}) \),

\[
A^{\mathcal{I}}_{(t)} = \bigoplus_{i \in I} A_{(t)}^{Y_i,X_i}.
\]
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$A_{(i)}^{Y_i, X_i}$: the free $A$-module (with involution) on a generator of degree $(Y_i, X_i)$.

This model structure on $\Delta(A\text{-mod}^{(t)})$ is cofibrantly generated with,

$$\mathcal{I} = \left\{ A^{(n)}_{(t)}(Y, X) \to A^{(n-1)}_{(t)}(Y, X), \ n \geq 1, \ Y, X \in \mathbb{F} \right\}.$$  \hspace{1cm} (7.10.9)

$$\mathcal{J} = \left\{ A^{(n, k)}_{(t)}(Y, X) \to A^{(n)}_{(t)}(Y, X), \ 0 \leq k \leq n \geq 1, \ Y, X \in \mathbb{F} \right\}.$$  \hspace{1cm} (7.10.10)

Under the Dold-Puppe equivalence $\Delta(A\text{-mod}^{(t)}) \xrightarrow{\sim} \mathcal{C}(A\text{-mod}^{(t)})$, these model structures correspond to the projective model structure on chain complexes, for free $\mathcal{C}\mathcal{F}R^{(t)}$, $B = C[\delta_{Y_i}, X_i]_{i \in \mathcal{I}}$ (resp. $B = C[\delta_{Y_i}, X_i, \delta_{Y_i}', X_i']_{i \in \mathcal{I}/\sim}$), there is an identification,

$$\Omega(\mathcal{B}/C) \equiv B^{(Y_i, X_i)}_{(t)}.$$  \hspace{1cm} (7.10.12)

Thus the left adjoint functor of (7.6.10), $B \mapsto \Omega(\mathcal{B}/C)^A$, takes (acyclic) cofibrations to (acyclic) cofibrations, and is a left Quillen functor.

For $\varphi \in \mathcal{F}R^{(t)}(C, A)$, Quillen’s cotangent bundle is the element of the derived category of $A\text{-mod}^{(t)}$, $\mathcal{D}(A\text{-mod}^{(t)}) = HO(\Delta(A\text{-mod}^{(t)}))$, given by,

$$\mathcal{L}\Omega(A/C) := \Omega(P_C(A)/C)^A \in \mathcal{D}(A\text{-mod}^{(t)}).$$  \hspace{1cm} (7.10.13)

where $P_C(A) \to A$ is a cofibrant replacement of $A$.

$\exists$ standard resolution associative to the pair of adjoint functors,

$$C \xrightarrow{\mathcal{F}R^{(t)}} \xrightarrow{\mathcal{U}} (\mathcal{S}et/\mathbb{F} \times \mathbb{F})^{(t)}$$

$$P_C(A) \equiv \ldots \xrightarrow{FU} \xrightarrow{FU} \xrightarrow{FU} \xrightarrow{FU} \xrightarrow{FU} \xrightarrow{FU} \xrightarrow{FU} (A) = C[a; a \in \prod_{Y, X} A_{Y, X}] \xrightarrow{a} A$$  \hspace{1cm} (7.10.15)
given by the unit and counit of the adjunction.

We have,

$$L \Omega(\lim_i B_i/\lim_i C_i) = \lim_i L \Omega(B_i/C_i). \quad (7.10.16)$$

Similarly, for a map of simplicial objects $C_\bullet \to A_\bullet$ in $\Delta(\mathcal{F} \mathcal{R}^{(i)})$, applying $P_{C_n}(A_n)$ at each dimension $n$, we obtain a bi-simplicial object $P_{C_n}(A_n)_m = (F_{C_n}U)^m(A_n)$, and taking the diagonal object ($n = m$) we obtain the resolution $P^n_C(A_\bullet) \to A_\bullet$, cf. [I].

Given $\mathcal{F} \mathcal{R}^{(i)}$-homomorphisms

$$
\begin{array}{ccc}
C & \to & B \\
\downarrow & & \downarrow \\
C & \to & Q = P_C(B) \\
\end{array}
\quad , \quad (7.10.17)
$$

we have compatible resolutions, hence an exact (also on the left!) sequence of $A$-modules,

$$0 \to \Omega(Q/C)^A \to \Omega(P^\vee_C(A)/C)^A \to \Omega(P^\vee_Q(A)/Q)^A \to 0. \quad (7.10.18)$$

This can be interpreted (see [I]) as the exact triangle in $\mathbb{D}(A\text{-mod})$,

$$L \Omega(B/C)^A \to L \Omega(A/C) \to L \Omega(A/B), \quad (7.10.19)$$

or as the long exact sequence of $A$-modules

$$
\begin{array}{cccccc}
& & \Omega(B/C)^A & \to & \Omega(A/C) & \to & \Omega(A/B) & \to & 0 \\
& & \downarrow & \searrow & & \downarrow & \swarrow & \\
& L_1 \Omega(B/C)^A & \to & L_1 \Omega(A/C) & \to & L_1 \Omega(A/B) & \to & \\
& \downarrow & & \cdots & & \downarrow & & \\
& L_2 \Omega(B/C)^A & \to & L_2 \Omega(A/C) & \to & L_2 \Omega(A/B) & \to & \\
\end{array}
\quad (7.10.20)
$$

Let $X$ be a finite (Krull) dimensional, noetherian, topological space.

Define a model structure on $\Delta(\mathcal{O}_X\text{-mod}^{(i)})$, $(X, \mathcal{O}_X) \in \mathcal{F} \mathcal{R}^{(i)} \mathcal{S}_p$,

Fibrations: $\mathcal{F} = \left\{ \varphi : m \to m', \quad \forall X, Y \in \mathcal{F}, \forall V \subseteq U \subseteq X, \right\}
\left\{ m(U)_{Y,X} \to m'(U)_{Y,X} \prod_{m(V)_{Y,X}} m(V)_{Y,X} \in \mathcal{F}_{\text{Set}_0} \right\}$

Weak equivalences: $W = \left\{ \varphi : m \to m', \quad \forall X, Y \in \mathcal{F}, \forall p \in X, (m_p)_{Y,X} \to (m'_p)_{Y,X} \in \mathcal{W}_{\text{Set}_0} \right\}$

Cofibrations: $C = L(W \cap \mathcal{F})$.

$\quad (7.10.21)$
Define a model structure on $\Delta(\mathcal{O}_X \backslash (\mathbb{F}R^{(t)}/X))$, the category of simplicial sheaves of $\mathbb{F}R^{(t)}$ over $X$, together with a map $\mathcal{O}_X \rightarrow B$, with arrows $B \rightarrow B'$ are homomorphisms of simplicial $\mathbb{F}R^{(t)}$-sheaves over $\mathcal{O}_X$.

Fibrations:
$$F = \left\{ \varphi : B \rightarrow B', \text{ such that } \forall X, Y \in \mathbb{F}, \forall V \subseteq U \subseteq X, \left\{ \begin{array}{l} \{B(U)_{Y,X} \rightarrow B'(U)_{Y,X} \} \in F_{\text{seto}} \end{array} \right\}, \right.$$ $$\text{Weak equivalences:} \quad W = \{ \varphi : B \rightarrow B', \forall X, Y \in \mathbb{F}, \forall p \in X, \{(B_p)_{Y,X} \rightarrow (B'_p)_{Y,X} \} \in W_{\text{seto}} \} \quad \text{(7.10.22)}$$

Cofibrations:
$$C = L(W \cap F). \quad \text{(7.10.23)}$$

That these constitute a Quillen model structure on $\Delta(\mathcal{O}_X \backslash (\mathbb{F}R^{(t)}/X))$, and on $\Delta(\mathcal{O}_X-\text{mod}^{(t)})$, follows as in [Q67] (theorem 4.II, § 4), with the aid of the Brown-Gerstein lemma [BG]: For $X$ finite dimensional, noetherian,

$$F \cap W \subseteq \left\{ \varphi : B \rightarrow B', \text{ such that } \forall X, Y \in \mathbb{F}, \forall V \subseteq U \subseteq X, \left\{ \begin{array}{l} \{B(U)_{Y,X} \rightarrow B'(U)_{Y,X} \} \in W_{\text{seto}} \end{array} \right\} \right\} \quad \text{the global weak equivalences.} \quad \text{(7.10.24)}$$

For a map $f \in \mathfrak{gFS}^{(t)}(X, Y)$, the cotangent bundle $\mathbb{L}\Omega(X/Y)$ is the element of the derived category of $\mathcal{O}_X$-modules, $\mathbb{D}(\mathcal{O}_X-\text{mod}) = HO(\Delta(\mathcal{O}_X-\text{mod}))$,

$$\mathbb{L}\Omega(X/Y) := \Omega(P_f*\mathcal{O}_Y)/(f^*\mathcal{O}_Y). \quad \text{(7.10.24)}$$

Given another map $g \in \mathfrak{gFS}^{(t)}(Y, Z)$, we have the exact triangle,

$$\mathbb{L}\Omega(Y/Z)^X \rightarrow \mathbb{L}\Omega(X/Z) \rightarrow \mathbb{L}\Omega(X/Y). \quad \text{(7.10.25)}$$
Part II

Generalized Rings
Chapter 8

Generalized Rings

8.1 Definitions

For $Y \in \mathbb{F}$ and $X \subseteq Y$, define the operation of contracting $X$ to a point $*_X$:

$$Y/X := (Y \setminus X) \cup \{*_X\}$$

(8.1.1)

equipped with a map,

$$\pi : Y \to Y/X, \quad \pi(X) = \{*_X\}.$$  

(8.1.2)

We have an "inverse" operation: for $X, Z \in \mathbb{F}$ and $z_0 \in Z$, set

$$(Z \triangleleft X) := (Z \setminus \{z_0\}) \cup X,$$

(8.1.3)

equipped with a map,

$$\pi : Z \triangleleft X \to Z, \quad \pi(X) = z_0.$$  

(8.1.4)

We have the following two trivial identifications:

$$Y/X \triangleleft X \equiv Y \quad \text{and} \quad (Z \triangleleft X)/X \equiv Z.$$  

(8.1.5)

Definition 8.1.1

A generalized ring is a functor $A \in (\mathit{Set}_0)^{\mathbb{F}}$, such that $A[0] = \{0\}$ with the two operations:

- **multiplication:** for $z_0 \in Z \in \mathbb{F}$,
- **contraction:** for $X \subseteq Y \in \mathbb{F}$,

$$A_Z \times A_X \to A_{Z \triangleleft X} \quad \text{and} \quad A_Y \times A_X \to A_{Y/X}$$

$$c, a \quad \text{and} \quad b, a \quad \to \quad (c \triangleleft a) \quad \text{and} \quad (b \parallel a)$$
such that the following axioms hold:

O. Functoriality of operations.

For \( \varphi \in \mathbb{F}(Z,Z') \), \( \varphi(z_0) = z'_0 \) and \( \psi \in \mathbb{F}(X,X') \), we have a map defined in the obvious way \( \varphi \circ \psi \in \mathbb{F}(Z \triangleleft X , Z' \triangleleft X') \) and we require the following diagram to commute:

\[
\begin{array}{ccc}
A_Z \times A_X & \longrightarrow & A_{Z \triangleleft X} \\
\varphi \times \psi & \downarrow & \varphi \triangleleft \psi \\
A_{Z'} \times A_{X'} & \longrightarrow & A_{Z' \triangleleft X'}
\end{array}
\]

(8.1.6)

that is,

\[
\varphi(a_Z) \triangleleft \psi(a_X) = \varphi \triangleleft \psi(a_Z \triangleleft a_X).
\]

(8.1.7)

Secondly, for \( \varphi \in \mathbb{F}(Y/X, Y'/X') \), \( \varphi(*_X) = *_{X'} \) and \( \psi \in \mathbb{F}(X,X') \), we have a diagram:

\[
\begin{array}{ccc}
(A_Y \times A_X) & \longrightarrow & A_{Y/X} \\
\varphi \circ \psi & \downarrow & \varphi \\
(A_{Y'} \times A_{X'}) & \longrightarrow & A_{Y'/X'}
\end{array}
\]

(8.1.8)

and we require,

\[
\varphi \circ \psi(a_Y) \parallel a_{X'} = \varphi(a_Y \parallel \psi^t(a_{X'}))
\]

(8.1.9)

e.g. for the contraction,

\[
\begin{array}{ccc}
A_X \times A_X & \longrightarrow & A_{[1]} \\
(a_1, a_2) & \longrightarrow & a_1 \parallel a_2
\end{array}
\]

(8.1.10)

and for \( \varphi \in \mathbb{F}(X,X') \) we have,

\[
\varphi(a_X) \parallel a_{X'} = a_X \parallel \varphi^t(a_{X'})
\]

(8.1.11)

(up-to the identifications, \( A_{*X} = A_{[1]} = A_{*X'} \)).

And we have the four zero-axioms:

(i) \( 0_X \triangleleft a_X = 0_{X \triangleleft X} \),

(ii) \( 0_Y \parallel a_X = 0_Y \parallel X \),
(iii) \( a_Z \triangleleft Z \) with \( \varphi : Z[Z_0] \to Z[Z_0] \) \( \in \mathbb{F}_Z \triangleleft Z, Z \). Also for \( a_Z \in A_Z \triangleleft Z_0 \subseteq A_Z \), viewed as an element of \( A_Z \), we have for any \( a_X \in A_X \),

\[
\begin{align*}
a_Z \triangleleft a_X &= \varphi(a_Z),
\end{align*}
\]

for \( \varphi \in \mathbb{F}(Z[Z_0], Z \triangleleft X) \) defined as above.

(iv) \( a_Y \parallel_0 X = \varphi(a_Y) \) with \( (\varphi : \overline{Y \setminus X} \to \overline{Y \setminus X}) \in \mathbb{F}_{Y \setminus X}. \)

I. Disjointness Axiom.

Suppose \( X_0 \cup X_1 \subseteq Y \in \mathbb{F} \) is a disjoint union of subsets. Then for any \( b \in A_Y, a_i \in A_X, \)

\[
(b \parallel a_1) \parallel a_0 \equiv (b \parallel a_0) \parallel a_1. \tag{8.1.12}
\]

Schematically we have,

\[
\begin{align*}
&b \\
\downarrow & X_0 \quad \downarrow \quad X_1 \\
& a_0 \quad a_1
\end{align*}
\]

and we can contract from \( b, a_0 \) first and then \( a_1 \) or the other way round.

\[
\begin{align*}
Y & \rightarrow \overline{(Y/\setminus X_0)/X_1} \\
\downarrow & \parallel \\
& (Y/\setminus X_1)/X_0 \\
\downarrow & \parallel \\
& (Y/\setminus (\{X_0 \cup X_1\}) \cup \{*X_0, *X_1\})
\end{align*}
\]

Generally, we denote such multiple contractions as,

\[
b \parallel (a_i) \in A_Y/\setminus X_i, \quad b \in A_Y, a_i \in A_X, \quad \cup X_i \subseteq Y. \tag{8.1.15}
\]
Definition 8.1.2

For a map of sets \( f \in \text{Set}_*(Y, Z) \), \((Y, Z \in \mathbb{F})\), we put

\[
A_f := \prod_{z \in Z} A_{f^{-1}(z)}.
\]  

We have the (multiple) contraction

\[
A_Y \times A_f \longrightarrow A_Z
\]

\[
a_Y, a^f \longmapsto a_Y \parallel a^f.
\]

for \( a_Y \in A_Y \), \( a^f = (a^f_z), a^f_z \in A_{f^{-1}(z)}, \ z \in Z \).

II. Disjointness Axiom.

Suppose \( z_0 \neq z_1 \in Z \) and \( X_0, X_1 \in \mathbb{F} \). We have for \( c \in A_Z \) and \( a_i \in A_{X_i} \),

\[
(c \triangleleft a_0) \triangleleft a_1 = (c \triangleleft a_1) \triangleleft a_0,
\]

\[ (8.1.18) \]

Schematically we have,

and a corresponding diagram of sets:

\[
\]

\[ (8.1.19) \]

Generally we denote such a multiple multiplications, indexed by a subset \( Z_0 \subseteq Z \) with \( \{X_z\}_{z \in Z_0} \) by,

\[
c \triangleleft \left( (a_z) \right)_{z \in Z_0}.
\]

\[ (8.1.20) \]

for \( c \in A_Z \) and \( \{a_z \in A_{X_z}\}_{z \in Z_0} \).
**Definition 8.1.3**

For a map of sets \( f \in \text{Set}_*(Y, Z) \), \((Y, Z) \in \mathbb{F}\), we get (multiple) multiplication

\[
A_Z \times A_f \longrightarrow A_Y
\]

\[
a_z, a^f \longrightarrow a_Z \triangleleft a^f
\]

**III. Disjointness Axiom.**

For \( b \in A_Z, a_i \in A_{X_i} \), and \( X_0 \subseteq Z, z_1 \in Z \setminus X_0 \), we have,

\[
(b \parallel a_0) \triangleleft a_1 \equiv (b \triangleleft a_1) \parallel a_0.
\]

Schematically,

(8.1.23)

**IV. Associativity Axiom.**

For \( b \in A_Z, a_i \in A_{X_i}, z_0 \in Z, x_0 \in X_0 \),

\[
(b \triangleleft a_0) \triangleleft a_1 = b \triangleleft (a_0 \triangleleft a_1).
\]

Schematically,

(8.1.25)

**V. Left adjunction Axiom.**

For \( X_0 \subseteq X_1 \subseteq Y, b \in A_Y, a_0 \in A_{X_0}, a_1 \in A_{X_1/X_0} \), we have,

\[
(b \parallel a_0) \parallel a_1 \equiv b \parallel (a_1 \triangleleft a_0).
\]

(8.1.27)
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Schematically seen,

$$b$$

$$a_0$$

$$a_1$$

with a corresponding identity of sets:

$$(Y/X_0)/(X_1/X_0) = Y/X_1.$$  

(8.1.29)

VI. **Right adjunction Axiom.**

For $$X \subseteq Y_0 \subseteq Y_1, b \in A_{Y_0}, a \in A_X, c \in A_{Y_1/X},$$ we have,

$$c \parallel (b \parallel a) \equiv (c \prec a) \parallel b$$  

(8.1.30)

Schematically seen,

$$c$$

$$*_{X_0}$$

$$a$$

$$b$$

with a corresponding identity of sets:

$$(Y_1/X)/(Y_0/X) \equiv Y_1/Y_0.$$  

(8.1.32)

VII. **Left linear Axiom.**

For $$a \in A_X, b \in A_Y, c \in A_Z$$ where $$X \subset Y, z_0 \in Z,$$ we have,

$$c \prec (b \parallel a) \equiv (c \prec b) \parallel a$$  

(8.1.33)

Schematically seen,

$$c$$

$$z_0$$

$$b$$

$$a$$

(8.1.34)
VIII. Unit Axiom.

We have an element \( 1 \in A_1 \). We obtain for any \( x \in X \in \mathbb{F} \),

\[
1_x \equiv 1_x^X \in A_X, 1_x^X = j_x^X(1) \tag{8.1.35}
\]

for \( j_x^X \in F_{X, 1}, j_x^X(1) = x \).

We require that the following identities holds: for any \( a \in A_X \), and any \( x \in X \),

\[
\begin{align*}
1 \triangleleft a & \equiv a, \\
1 \triangleleft 1 & \equiv a, \\
a /sslash 1 & \equiv a. \tag{8.1.36}
\end{align*}
\]

with a corresponding identities of sets:

\[
\begin{align*}
[1] \triangleleft X & \equiv X, \\
x [1] & \equiv X, \tag{8.1.37}
\end{align*}
\]

\[
X /\{x\} \equiv X.
\]

Schematically it can be seen as follows:

\[
\begin{align*}
\begin{tikzpicture}
  \node (a) at (0,0) {$1$};
  \node (b) at (1,0) {$x$};
  \node (c) at (2,0) {$1$};
  \node (d) at (1,-1) {$a$};
  \draw [->] (a) -- (b);
  \draw [->] (b) -- (c);
  \draw [->] (b) -- (d);
  \end{tikzpicture}
\end{align*}
\]

IX. Commutativity.

Definition 8.1.4

We say a generalized ring \( A \) is commutative if it satisfies for any \( b \in A_Y, X_0 \subseteq Y, a_i \in A_{X_i} \),

\[
(b / a_0) \triangleleft a_1 \equiv (b \triangleleft \bigwedge_{x_0 \in X_0} (a_1)) /\bigwedge_{x_1 \in X_1} \in A_{(X_0 \times X_1) \cup (Y \setminus X_0)} \tag{8.1.39}
\]

(where the \( x_1 \)’ the copy of \( a_0 \) is attached through the indices \( X_0 \times \{x_1\} \)).

We also call this identity "Right- linear”.

Schematically seen,
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We are mainly interested in the commutative generalized rings. We are not interested in the vertical-commutative generalized rings of the following:

**Definition 8.1.5**

We say a generalized ring \( A \) is vertical-commutative if it satisfies, for any \( a \in A_X, b \in A_Y \),

\[
\begin{align*}
&\quad a \mathrel{\triangleleft} (b) \equiv b \mathrel{\triangleleft} (a) \\
&\text{in } A_X \triangleleft Y \equiv A_X \times Y \equiv A_Y \triangleleft X
\end{align*}
\]  

(8.1.41)

Schematically seen,

\[
\begin{array}{c}
\text{b} \\
\text{b}
\end{array}
\]

(8.1.42)

We say that \( A \) is "totally-commutative" if it is both commutative and vertical-commutative.

**Definition 8.1.6**

A homomorphism of generalized rings \( \varphi : A \to A' \) is a natural transformation of functors (so for \( X \in \mathbb{F} \), we have \( \varphi_X \in \text{Set}_0(A_X, A'_X) \), and for \( f \in \mathbb{F}(X,Y) \), we have \( \varphi_Y \circ f_A = f_{A'} \circ \varphi_X \)), such that \( \varphi \) preserves multiplication, contraction, and the unit:

\[
\begin{align*}
\varphi(a \triangleleft b) &= \varphi(a) \triangleleft \varphi(b), \\
\varphi((a \parallel b)) &= (\varphi(a) \parallel \varphi(b)), \\
\varphi(1_A) &= 1_{A'}
\end{align*}
\]  

(8.1.43-8.1.45)

We remark that for generalized rings \( A, A' \), a collection of maps \( \varphi = \{ \varphi_X \in \text{Set}_0(A_X, A'_X) \} \) satisfying (8.1.43-8.1.44) is a homomorphism; i.e. it is automatically a natural transformation of functors by functoriality (O) and unit axiom (VIII).

Thus we have a category of generalized rings and homomorphisms which we denote by \( GR \).

We denote by \( GR_C \) the full subcategory of \( GR \) consisting of the commutative generalized rings.
There are three equivalent ways to describe the operations of a generalized ring $A$:

1. The elementary operations of multiplications and contraction as above:

   \[
   \begin{align*}
   &a \cdot b : A_Z \times A_X \to A_{Z \triangleleft X}, \quad z_0 \in Z \\
   &a \triangledown b : A_Y \times A_X \to A_{Y \triangledown X}, \quad X \subseteq Y
   \end{align*}
   \tag{8.1.46}
   \]

2. Using the disjointness axioms we can view these operations as maps:

   For $f \in \text{Set}_\bullet(Y, Z)$ a map of sets, $(Y, Z \in \mathbb{F})$, and $A_f = \prod_{z \in Z} A_{f^{-1}(z)}$,

   \[
   \begin{align*}
   &a \cdot b : A_Z \times A_f \to A_Y \\
   &((a_Z, (b_f))) \mapsto a_Z \cdot (b_f) \\
   &a \triangledown b : A_Y \times A_f \to A_Z \\
   &((a_Y, (b_f))) \mapsto a_Y \triangledown (b_f)
   \end{align*}
   \tag{8.1.47}
   \]

3. We can further extend the operations "fiber by fiber", so that for map of sets $Y \xrightarrow{f} Z \xrightarrow{g} W$ we obtain:

   \[
   \begin{align*}
   &a \cdot b : A_g \times A_f \to A_{g \circ f} \\
   &[[(a_g^0, (a_f^0))]w_0 := a_g^0 \triangleleft (a_f^0), \quad z \in g^{-1}(w_0) \\
   &a \triangledown b : A_{g \circ f} \times A_f \to A_g \\
   &[[(a_g^0 \circ f, (a_f^0))]w_0 := a_g^0 \triangledown (a_f^0), \quad z \in g^{-1}(w_0)
   \end{align*}
   \tag{8.1.48}
   \]

The axioms in the fiber-extended form.

We can now write the axioms again in the fiberwise extended form:

**Associativity:** For $W \xrightarrow{h} Z \xrightarrow{g} Y \xleftarrow{f} X$ in $\text{Set}_\bullet$, and for $d \in A_h$, $c \in A_g$, $b \in A_f$, we have in $A_{h \circ g \circ f}$:

\[
(d \triangleleft (c \triangleleft b)) = (d \triangleleft c) \triangleleft b
\tag{8.1.49}
\]

**Left-Adjunction:** For $W \xrightarrow{h} Z \xrightarrow{g} Y \xleftarrow{f} X$ in $\text{Set}_\bullet$, and for $d \in A_{h \circ g \circ f}$, $a \in A_g$, $c \in A_f$, we have in $A_h$:

\[
(d \triangledown (a \triangledown c)) = ((d \triangledown c) \triangledown a)
\tag{8.1.50}
\]
The axiom of right-linearity is the delicate one, as it captures the commutativity we need, and it will require the following notations for generalizing the indices adjustment we had in \((8.1.39)\).

For \(Z \xrightarrow{g} Y \xleftarrow{f} X \) in \(\text{Set}_\bullet\), we form the cartesian square,
\[
\begin{array}{c}
X \\
\downarrow^d \\
Y \\
\downarrow^a \\
W \\
\end{array}
\begin{array}{c}
X \\
\downarrow^c \\
Y \\
\downarrow^a \\
Z \\
\end{array}
\begin{array}{c}
X \\
\downarrow^a \\
Y \\
\downarrow^d \\
W \\
\end{array}
\begin{array}{c}
X \\
\downarrow^e \\
Y \\
\downarrow^a \\
Z \\
\end{array}
\]
\[
\begin{equation}
Z \prod_Y X = \{ (z,x) \in D(g) \times D(f), g(z) = f(x) \} \tag{8.1.53}
\end{equation}
\]
with $\tilde{f}$, $\tilde{g}$ the natural projections. Note that we have an identification of fibers

$$\tilde{f}^{-1}(z) \cong f^{-1}(g(z)) \text{ for } z \in Z$$

and we get a map

$$A_f \to A_f, \quad c \mapsto g^*c \in A_f \quad \text{with} \quad g^*c(z) = c(g(z)). \quad (8.1.54)$$

Similarly, $\tilde{g}^{-1}(x) \cong g^{-1}(f(x))$ for $x \in X$, and we get a map

$$A_g \to A_g, \quad a \mapsto f^*a \in A_g \quad \text{with} \quad f^*a(x) = a(f(x))$$

**Right-Linear:** For $f, g, \tilde{f}, \tilde{g}$ as above, and $W \xrightarrow{h} Y$ in $\text{Set}_\bullet$, and for $d \in A_{h \circ f}$, $a \in A_g$, $c \in A_f$, we have in $A_{h \circ g}$:

$$(d \parallel c) \triangleleft a = (d \triangleleft f^*a) \parallel g^*c \quad (8.1.55)$$

**Unit axioms:** We have a distinguished element $1 = 1_A \in A[1]$.

Hence for each singleton $\{x\} \in \mathcal{F}$, using the unique isomorphism $[1] \xrightarrow{\sim} \{x\}$ we get $1_{\{x\}} \in A_{\{x\}}$; and for $f \in \mathcal{F}(X, Y)$ we have $1_f = (1_{f(y)})_{y \in f(X)} \in A_f$.

We have for all $h \in \text{Set}_\bullet(X, W)$, $d \in A_h$:

$$d \triangleleft 1_{id_X} = d, \quad 1_{id_W} \triangleleft d = d, \quad (d \parallel 1_{id_X}) = d \quad (8.1.56)$$
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8.2 Remarks

Remark 8.2.1. Functoriality

In $A_{[1]}$ we have $0 \lhd 1 = 1 \lhd 0 = 0 \lhd 0 = 0$, and $1 \lhd 1 = 1$, hence for $f \in \mathcal{F}(X,Y)$, $g \in \mathcal{F}(Y,Z)$, we have

$$1_g \lhd 1_f = 1_{g \circ f} \tag{8.2.1}$$

Also we have $0 \parallel 1 = 1 \parallel 0 = 0 \parallel 0 = 0$, and $1 \parallel 1 = 1$, hence

$$1_{id_Y} \parallel 1_f = 1_f \tag{8.2.2}$$

We obtain for all $a \in A_X$, $f \in \mathcal{F}(X,Y)$,

$$a \lhd 1_f = (a \lhd 1_f) \parallel 1_{id_Y} = (a \parallel (1_{id_Y} \parallel 1_f)) = a \parallel 1_f. \tag{8.2.3}$$

This gives a structure of a functor $\mathcal{F} \to \text{Set}_0$ on $X \to A_X$, which is the given structure by the functoriality and the unit axiom: for all $a \in A_X$, $f \in \mathcal{F}(X,Y)$,

$$a \lhd 1_f = (a \parallel 1_f) = f_A(a). \tag{8.2.4}$$

Remark 8.2.2. The involution

Note that $(1 \parallel a)$ makes sense only for $a \in A_{[1]}$, we define

$$a^t = 1 \parallel a, \ a \in A_{[1]} \tag{8.2.5}$$

It is an involution of $A_{[1]}$:

$$(a^t)^t = 1 \parallel (1 \parallel a) = (1 \lhd a) \parallel 1 = a \parallel 1 = a \tag{8.2.6}$$

It preserves the operation of multiplication, and the special elements $0$, $1$:

$$(a \lhd b)^t = 1 \parallel (a \lhd b) = (1 \parallel b) \parallel a = \left(1 \lhd (1 \parallel b)\right) \parallel a = \left(1 \parallel a\right) \lhd (1 \parallel b) = a^t \lhd b^t \tag{8.2.7}$$

$$0^t = (1 \parallel 0) = 0 \quad , \quad 1^t = (1 \parallel 1) = 1$$

Definition 8.2.3

We shall say that $A$ is self-adjoint if

$$a^t = a \text{ for all } a \in A_{[1]}.$$

In general, we let

$$A_{[1]}^+ = \{a \in A_{[1]}, a^t = a\} \tag{8.2.8}$$

denote the subset of symmetric elements.
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Remark 8.2.4. The structure of $A_{[1]}$

It follows from the associativity and unit axioms that $A_{[1]}$ is an associative monoid with unit 1. It also follows from the commutativity axiom IX, that it is commutative:

$$a \circ b = (a \circ b) \parallel 1 = (1 \circ a) \parallel (1 \parallel b) \equiv (1 \parallel (1 \parallel b)) \circ a = b \circ a \quad (8.2.9)$$

For $X \in \mathbb{F}$, the commutative monoid $A_{id_X} = (A_{[1]})^X$ acts on the right on $A_X$.

$$A_X \times (A_{(1)})^X \longrightarrow A_X$$

$$b \circ (a^{(x)}) \longmapsto b \circ a^{(x)} \quad (8.2.10)$$

Note that for $a = (a^{(x)}) \in A_{id_X}$, and $b \in A_X$,

$$b \circ a = (b \circ a) \parallel 1_{id_X} = (b \parallel (1_{id_X} \parallel a)) = (b \parallel a^t) \quad (8.2.11)$$

The monoid $A_{[1]}$ also acts on the left on $A_X$, via

$$A_{[1]} \times A_X \longrightarrow A_X$$

$$a \cdot b \longmapsto a \circ b \quad (8.2.12)$$

and this is the diagonal right action: For $a \in A_{[1]}$, $b \in A_X$, putting $\tilde{a} \in A_{id_X}$, $\tilde{a}^{(x)} \equiv a$ for all $x \in X$,

$$a \circ b = (1 \parallel a^t) \circ b \equiv (1 \parallel b) \parallel \tilde{a}^t = b \parallel \tilde{a}^t = b \circ \tilde{a}.$$ \hspace{1cm} (8.2.13)

More generally, for $f \in Set_{\ast}(X,Y)$, $b \in A_f$, $a = (a^{(y)}) \in A_{id_Y} = (A_{[1]})^Y$, we get

$$f^*a \in A_{id_X} = (A_{[1]})^X, \quad f^*a^{(x)} = a^{(f(x))} \text{ for } x \in X,$$

and we have in $A_f$,

$$a \circ b = b \circ f^*a.$$ \hspace{1cm} (8.2.14)

Indeed checking (8.2.14) at a given fiber $A_{f^{-1}(y)}$, $y \in Y$, reduces to (8.2.13).

The action of $(A_{[1]})^X$ on $A_X$ is self-adjoint with respect to the pairing (8.1.10): for $a \in (A_{[1]})^X$, $b, d \in A_X$, we have

$$(b \circ a) \parallel d = b \parallel (d \parallel a) = b \parallel (d \circ a^t) \quad (8.2.15)$$

We denote the group of invertible elements of $A_{[1]}$ by $A^*$:

$$A^* = \{ a \in A_{[1]}, \exists a^{-1} \in A_{[1]}, a \circ a^{-1} = a^{-1} \circ a = 1 \} \quad (8.2.16)$$
Remark 8.2.5. "one contraction suffices"

The axioms of a *commutative* generalized ring allow to transform any formula in the operations of multiplication and contraction, to an equivalent formula with only *one* contraction. Thus expressions of the form

\[(a \sslash b) = (a_1 \triangleright a_2 \triangleleft \cdots \triangleleft a_n) \sslash (b_1 \triangleright \cdots \triangleright b_m)\]  

(8.2.17)

are closed under multiplication and contraction. We have the following lemma:

**Lemma 8.2.6**

Suppose we have a commutative diagram in \(\text{Set}^\bullet\) and elements of \(A\) over it \((b_i \in A_{f_i}, b \in A_h)\):

\[
\begin{array}{ccc}
X_2 & \xrightarrow{\hat{h}} & \tilde{X}_2 \\
\downarrow{f_2} & & \downarrow{\tilde{f}_2} \\
X_1 & \xrightarrow{\hat{h}} & \tilde{X}_1 \\
\downarrow{f_1} & & \downarrow{\tilde{f}_1} \\
X_0 & \xrightarrow{h} & \tilde{X}_0 \\
\end{array}
\quad \begin{array}{ccc}
X_2 & \xrightarrow{h^*(b_2)} & \tilde{X}_2 \\
\downarrow{f^*_2(f^*_1(b))} & & \downarrow{\tilde{f}^*_2(f^*_1(b))} \\
X_1 & \xrightarrow{h^*(b_1)} & \tilde{X}_1 \\
\downarrow{f^*_1(b)} & & \downarrow{\tilde{f}^*_1(b)} \\
X_0 & \xrightarrow{b} & \tilde{X}_0 \\
\end{array}
\]

where the inner squares are fibre products. Then we have the following identities:

\[h^*(b_1 \triangleright b_2) = h^*(b_1) \triangleright \tilde{h}^*(b_2),\]  

(8.2.19)

\[f^*_1(f^*_2(b)) = (f_2 \circ f_1)^*(b).\]  

(8.2.20)

**Proof.** For any \(z \in \tilde{X}_0\) we get:

\[h^*(b_1 \triangleright b_2)(z) = (b_1 \triangleright b_2)(h(z)) = b_1^{(h(z))} \triangleright b_2|_{f_1|_{h(z)}} = b_1^{(h(z))} \triangleright (b_2 |_{z \in f^{-1}(h(z))}),\]

and

\[(h^*(b_1) \triangleright \tilde{h}^*(b_2))(z) = h^*(b_1)(z) \triangleright \tilde{h}^*(b_2)|_z =
\]

\[= b_1^{(h(z))} \triangleright (\tilde{h}^*(b_2)) |_{z \in \tilde{f}_1^{-1}(z)} = b_1^{(h(z))} \triangleright (b_2^{(h(z))}).\]

and we get the equality by the identification of fibers for any \(z \in \tilde{X}_0\) of \(f_1^{-1}(h(z))\) and \(\tilde{f}_1^{-1}(z)\) given by

\[(x, z) \in \tilde{f}_1^{-1}(z) \iff x \in f_1^{-1}(h(z)).\]
In the general case where there are $n$ consecutive fiber products and an element $b_1 \lhd b_2 \lhd \ldots \lhd b_n$, we have formulas:

$$h^*(b_1 \lhd b_2 \lhd \ldots \lhd b_n) = h^*(b_1) \lhd \tilde{h}^*(b_2) \lhd \tilde{h}^*(b_3) \lhd \ldots$$

$$(f_1 \circ \ldots f_n)^*(b_1 \lhd \ldots \lhd b_n) = f_n^*(f_{n-1}^*(\ldots \circ f_1^*(b)\ldots)).$$

We have the formulas:

**Multiplication 8.2.7:** $(a \ll c) \ll (d \ll b) = (a \ll g^*c) \ll (d \ll f^*b)$, for $b \in A_g$, $c \in A_f$

$$((a \ll c) \ll d)^{\text{comm.}} = ((a \ll g^*c) \ll (d \ll f^*b)) \ll d = (a \ll f^*c) \ll (d \ll g^*b).$$

**Contraction 8.2.8:** $(a \ll b) \ll (c \ll d) = (a \ll g^*d) \ll (c \ll f^*b)$, for $b \in A_g$, $d \in A_f$

We can also see this in a more explicit way: for elements $d_1 \lhd d_2 \lhd \ldots \lhd d_m$ over $Y_m \to \ldots \to Y_0$ and $b_1 \lhd b_2 \lhd \ldots \lhd b_n$ over $X_m \to \ldots X_0$, with an isomorphism $Y_m \cong X_n$, and an element $b$ over $h : \hat{X}_0 \to X_0$ we form the diagram:
By commutativity we have now,

\[(d_1 \lhd d_2 \lhd \cdots \lhd d_m) \parallel (b_1 \lhd \cdots \lhd b_n) \quad (8.2.27)\]

\[(d_1 \lhd \cdots \lhd d_m) \parallel (b_1 \lhd \cdots \lhd b_n) = (d_1 \lhd \cdots \lhd (f_n^* \circ \cdots \circ f_1^*)(b)) \parallel (h^*(b_1) \lhd \cdots \lhd h^*(b_n)).\]

(8.2.28)

**Remark 8.2.9. Matrixness and Tameness**

We say that \( A \) is a "matrix" generalized ring if for \( X \in \mathbb{F} \) we have an injection:

\[A_X \hookrightarrow (A_{[1]})^X, \quad a \mapsto (a \parallel 1_x)_{x \in X}.\]  

(8.2.29)

We say \( A \) is "tame" if for \( X \in \mathbb{F} \), and any \( a, a' \in A_X \), we have

\[a \parallel d = a' \parallel d \quad \text{for all} \quad d \in A_X \implies a = a'.\]  

(8.2.30)

Note that we have the implication

\[ A \text{ matrix} \implies A \text{ tame}. \]

### 8.3 Examples of generalized Rings

**8.3.0 Generalized Rings arising from \( \mathbb{F} \)-Rings.**

If \( A = \{A_{Y,X}\} \) is an \( \mathbb{F} \)-Ring with involution, we get a generalized ring \( \mathcal{G}(A) \), with

\[\mathcal{G}(A)_X := A_{1,X}.\]

and operations,

**multiplication:**

\[z_0 \in Z : \quad \mathcal{G}(A)_Z \times \mathcal{G}(A)_X = A_{1,Z} \times A_{1,X} \to \mathcal{G}(A)_{z_0} = A_{1,Z \uplus} \quad (8.3.1)\]

\[a_z \lhd a_X := a_Z \circ (a_X \uplus \bigoplus_{z \in Z \setminus \{z_0\}} 1).\]
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contraction:
\[ G(A)_Y \times G(A)_X = A_{1,Y} \times A_{1,X} \to \frac{G(A)_{Y/X}}{A_{1,Y/X}} = A_{1,Y/X} \]
\[ a_Y \parallel a_X := a_Y \circ (a'_X \oplus \bigoplus_{y \in Y \setminus X} 1). \]  \hspace{1cm} (8.3.2)

Equivalently, for a map of sets \( f : Y \to Z \), we have

multiplication:
\[ A_{1,Z} \times \prod_{z \in Z} A_{1,f^{-1}(z)} \to A_{1,Y} \]
\[ a_Z \triangleleft (a'_z) := a_Z \circ (\bigoplus_{z \in Z} a'_z) \] \hspace{1cm} (8.3.3)

contraction:
\[ A_{1,Y} \times \prod_{z \in Z} A_{1,f^{-1}(z)} \to A_{1,Z} \]
\[ a_Y \parallel (a'_z)^t := a_Y \circ (\bigoplus_{z \in Z} a'_z)^t \] \hspace{1cm} (8.3.4)

When \( A \) is \( \times \)-commutative (resp. totally-commutative) \( F \)-Ring with involution, the generalized ring \( G(A) \) is commutative (resp. totally-commutative).

We have the following diagram:

In particular we have the totally-commutative matrix generalized rings:

I. \( F \) the initial object of \( GR : F_X = X \cup \{0\} \)
II. \( G(R) \) \( R \) a commutative rig
III. \( O_{K,\eta} \) \( \eta : K \to \mathbb{C}, \ (\ell_2)! \)
IV. \( F\{M\} \) \( M \) commutative monoid.

We specify now each of these examples explicitly.

8.3.1 The field with one element \( F \)

We write \( F \) for the functor \( F \simeq F_0 \subseteq Set_0 \). Thus
\[ F_X = X_0 = X \bigsqcup \{0_X\}, \text{ and } F_f = \prod_y (f^{-1}(y))_0 \text{ for } f \in Set_\bullet(X,Y). \] \hspace{1cm} (8.3.6)
The multiplication is given by
\[
F_Y \times F_f \to F_X
\]
\[
y_0, (x_f^{(y)}) \mapsto y_0 \circ x_f = x_f^{(y_0)} \in (f^{-1}(y_0))_0 \subseteq X_0
\]
(8.3.7)
The contraction is given by
\[
F_X \times F_f \to F_Y
\]
\[
x_0, (x_f^{(y)}) \mapsto (x_0 \sslash x_f) = \begin{cases} 
y & x_0 = x_f^{(y)} \\
0 & \text{otherwise}
\end{cases}
\]
(8.3.8)
It is easy to check that \( F \) is a totally-commutative, self-adjoint matrix generalized ring.
For \( A \in GR \), and for \( x \in X \in F \), put
\[
\varphi_X(x) = (1_x \sslash 1_{j_x}) = 1_x \lhd 1_{j_x} \in A_X.
\]
(8.3.9)
with \( j_x \in \mathbb{F}(\{x\}, X) \) the inclusion.
We have \( \varphi_X \in \text{Set}_0(F_X, A_X) \), and the collection of \( \varphi_X \) define a homomorphism \( \varphi \in GR(F, A) \). It is easy to check this is the only possible homomorphism, and \( F \) is the initial object of \( GR \).

### 8.3.2 Commutative Rigs

For a rig \( A \), let \( G(A)_X = A \cdot X = A^X \) be the free \( A \)-module with basis \( X \). It forms a functor \( G(A) : \mathbb{F} \to A \cdot \text{mod} \subseteq \text{Set}_0 \). We define the multiplication for \( f \in \text{Set}_0(X, Y) \) by
\[
G(A)_Y \times G(A)_f = A^Y \times \prod_{y \in Y} A^{f^{-1}(y)} \to A^X = G(A)_X
\]
(8.3.10)
\[
a = (a_y), b = (b_x^{(y)})_{x \in f^{-1}(y)} \mapsto (a \triangleleft b)_x = a_{f(x)} \cdot b_x^{(f(x)}
\]
We define the contraction by
\[
G(A)_X \times G(A)_f = A^X \times \prod_{y \in Y} A^{f^{-1}(y)} \to A^Y = G(A)_Y
\]
(8.3.11)
\[
a = (a_x), b = (b_x^{(y)})_{x \in f^{-1}(y)} \mapsto (a \sslash b)_y = \sum_{x \in f^{-1}(y)} a_x \cdot b_x^{(y)}
\]
It is straightforward to check that \( G(A) \) is a self-adjoint, Matrix generalized ring.
When \( A \) is a commutative rig, \( G(A) \) is a totally-commutative generalized ring.
A homomorphism of (commutative) rigs \( \varphi \in \text{Rig}(A, B) \) gives a homomorphism \( G(\varphi) \in GR(G(A), G(B)) \), thus we have a functor
\[
G : \text{Rig} \to GR
\]
(8.3.12)
It is fully-faithful: if \( \varphi \in GR(G(A), G(B)) \), and \( a = (a_x) \in G(A)_X \), then \( \varphi_X(a)_x = (\varphi_{X}(a_x)) \) by functoriality over \( \mathbb{F} \), so \( \varphi \) is determined by
\( \varphi_{[1]} : A \rightarrow B \): the map \( \varphi_{[1]} \) is multiplicative (and preserves 1), but it is also additive

\[
\varphi_{[1]}(a_1 + a_2) = \varphi_{[1]}((a_1, a_2) \parallel (1, 1)) = (\varphi_{[1]}(a_1), \varphi_{[1]}(a_2)) \parallel (1, 1) = \varphi_{[1]}(a_1) + \varphi_{[1]}(a_2)
\]

(8.3.13)

Thus \( \varphi_{[1]} \in \text{Rig}(A, B) \), and \( \varphi = \mathcal{G}(\varphi_{[1]}) \); and we have

\[
\text{Rig}(A, B) = \mathcal{G}(\mathcal{G}(A), \mathcal{G}(B))
\]

(8.3.14)

Note that for every \( X \in \mathbb{F} \), we have a distinguished element

\[
1_X \in \mathcal{G}(A)_X, \quad (1_X)_x = 1 \text{ for all } x \in X
\]

(8.3.15)

hence for \( f \in \text{Set}_+(X, Y) \), the element \( 1_f = (1_{f^{-1}(y)}) \in \mathcal{G}(A)_f \). These elements satisfy

\[
1_g \triangleleft 1_f = 1_{g \circ f}, \quad 1_{[1]} = 1;
\]

(8.3.16)

which imply naturality

\[
1_X \triangleleft 1_{f^*} = 1_{f_1(f)} \in \mathcal{G}(A)_{f_1(f)} \subseteq \mathcal{G}(A)_Y \text{ for } f \in \mathcal{F}(X, Y)
\]

(8.3.17)

Note that any element \( a = (a_x) \in \mathcal{G}(A)_X \), gives an element of the monoid

\[
\langle a \rangle = (a_x) \in \mathcal{G}(A)_{id_X} = A^X
\]

(8.3.18)

and the vector \( 1_X \) is "cyclic" in the sense that

\[
a = 1_X \triangleleft \langle a \rangle
\]

(8.3.19)

We have the generalized rings \( \mathcal{G}(\mathbb{N}) \) and \( \mathcal{G}([0, \infty)) \), as well as the "tropical" examples \( \mathcal{G}(\mathbb{N}_0) \) and \( \mathcal{G}([0, \infty)_0) \) where we replace addition by the operation of taking the maximum \( \max \{x, y\} \). For \( \sigma = 1/p \in (0, 1] \) we have the generalized ring \( \mathcal{G}([0, \infty)_\sigma) \) where we replace addition by the operation \( x +_\sigma y := (x^p + y^p)^{\sigma} \).

### 8.3.3 Real primes

Let \( || : K \rightarrow [0, \infty) \) be a non-archimedean absolute value on a field \( K \). Let

\[
\mathcal{O}_X = \left\{ a = (a_x) \in \mathcal{G}(K)_X, \sum_{x \in X} |a_x|^2 \leq 1 \right\}
\]

(8.3.20)

Note that \( X \mapsto \mathcal{O}_X \) is a subfunctor of \( \mathcal{G}(K) : \mathbb{F} \rightarrow \text{Set}_0 \). But it is also a sub-generalized-ring, in the sense that it is closed under the operations of multiplication and contraction, and \( 1 \in \mathcal{O}_{[1]} \). The proof for multiplication is straightforward:

For \( a = (a_y) \in \mathcal{O}_Y, \quad b = (b_x^{(y)}) \in \mathcal{O}_f \), we have

\[
\sum_{x \in X} |(a \triangleleft b)_x|^2 = \sum_{x \in X} |a_{f(x)}|^2 \cdot |b_x^{(f(x))}|^2
\]

(8.3.21)
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and so \(a \triangleleft b \in \mathcal{O}_X\).

The proof for contraction is just the Cauchy-Schwartz inequality:
For \(a = (a_x) \in \mathcal{O}_X\), \(b = (b^{(y)}_x) \in \mathcal{O}_f\), we have

\[
\sum_{y \in Y} \| \langle a, b \rangle_y \|^2 = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} a_x \cdot b^{(y)}_x \leq \sum_{y \in Y} \left( \sum_{x \in f^{-1}(y)} |a_x|^2 \right) \cdot \left( \sum_{x \in f^{-1}(y)} |b^{(y)}_x|^2 \right) 
\]

and so \(a \parallel b \in \mathcal{O}_Y\).
Thus \(\mathcal{O}\) is a generalized ring (totally-commutative, matrix, self-adjoint).

Note that

\[m_X = \left\{ a = (a_x) \in \mathcal{O}_X, \sum_{x \in X} |a_x|^2 < 1 \right\}\]

forms a subfunctor of \(\mathcal{O}\), and it is (the unique maximal) functorial-ideal of \(\mathcal{O}\), in the sense that we have

\[\mathcal{O} \triangleleft m, m \triangleleft \mathcal{O}, (m \parallel \mathcal{O}), (\mathcal{O} \parallel m) \subseteq m\]

By collapsing \(m_X\) to zero we obtain the quotient (in \(\text{Set}_0\)):

\[k_X = \mathcal{O}_X / m_X = \left\{ a = (a_x) \in \mathcal{O}_X, \sum_{x \in X} |a_x|^2 = 1 \right\} \cup \{0_X\}\]

There is a canonical projection map \(\pi_X : \mathcal{O}_X \rightarrow k_X\), with \(\pi_X(m_X) = 0_X\). Now (8.3.24) imply that there is a (unique) structure of a generalized ring on \(k\), such that \(\pi\) is a homomorphism, \(\pi \in \mathcal{GR}(\mathcal{O}, k)\). It is given by

\[a \triangleleft b = \left\{ \begin{array}{ll} a \triangleleft b & \text{if } ||a \triangleleft b|| = 1 \\ 0 & \text{if } ||a \triangleleft b|| < 1 \end{array} \right. ; (a \parallel b) = \left\{ \begin{array}{ll} a \parallel b & \text{if } ||a \parallel b|| = 1 \\ 0 & \text{if } ||a \parallel b|| < 1 \end{array} \right.\]

with the \(l_2\)-norm

\[||(a_x)|| = \left( \sum_{x \in X} |a_x|^2 \right)^{\frac{1}{2}}\]

Note that the (totally-commutative, self-adjoint) generalized ring \(k\) is not matrix, but is tame.
8.3.4 Ostrowski’s theorem

Let $K \in \mathcal{GR}_C$. We say $K$ is a field when any element in $K_1 \setminus \{0\}$ is invertible:

$$K_1^* = K_1 \setminus \{0\}. \quad (8.3.26)$$

For $A \subseteq K$ a sub-generalized ring of $K$ we say $A$ is full when:

$$\forall y \in K_X, \exists d \in A_1 \setminus \{0\}, d \prec y \in A_X. \quad (8.3.27)$$

(i.e. $K = A_{(0)}$ the localization of $A$ with respects to the prime $\{0\}$).

We say $A$ is tame when: for any $X \in F$

$$A_X = \{y \in K_X, y \not\divides a \in A_1, \forall a \in A_X\}. \quad (8.3.28)$$

We call $A$ a valuation of $K$ when it is full and tame and

$$\forall y \in K_1^*, y \in A_1 \text{ or } y^{-1} \in A_1. \quad (8.3.29)$$

For $k \subseteq K$ we denote the set of $K$-valuations by

$$\text{Val}(K/k) := \{A \subseteq K \text{ a valuation }, k \subseteq A\}. \quad (8.3.30)$$

For $B \in \text{Val}(K)$, the ordered-abelian-group $K_1^*/B_1^*$ can be embedded in a complete ordered-abelian-group $\Gamma$, and the quotient map,

$$| | : K_1 \rightarrow \Gamma \cup \{0\}, \quad |x| := x \prec B_1^*, \quad (8.3.31)$$

satisfies:

(I)

$$|x| = 0 \iff x = 0$$
$$|x_1 \prec x_2| = |x_1| \cdot |x_2|$$
$$|1| = 1 = \text{(unit of } \Gamma). \quad (8.3.32)$$

For $y \in K_X$ we have the equality (cf. Claim(II) of §2.5)

(II)

$$|y|_X := \sup\{|y \divides b|, b \in B_X\} = \inf\{|d^{-1}|, d \in K_1^*, d \prec y \in B_X\} \quad (8.3.33)$$

For $f \in \text{Set}_\bullet(Y, Z), a = (a_z) \in K_f = \prod_{z \in Z} K_{f^{-1}(z)}$, we put:

$$|a|_f = \text{Max}\{|a_z|_{f^{-1}(z)}\} \quad (8.3.34)$$

and we have,

(III)

$$|z \prec a|_Y \leq |z|_Z \cdot |a|_f, \quad z \in K_Z, \quad (8.3.35)$$
$$|y \divides a|_Z \leq |y|_Y \cdot |a|_f, \quad y \in K_Y.$$
CHAPTER 8. GENERALIZED RINGS

Conversely, given the maps for $X \in \mathbb{F}$,

$$|x| : K_X \rightarrow \Gamma \cup \{0\} \quad (8.3.36)$$

satisfying (III), with $|\cdot| = |\cdot|_1$ satisfying (I), $B = \{B_X\}$, with $B_X = \{b \in K_X, |b|_X \leq 1\}$ is a sub-generalized-ring of $K$. If further (II) is satisfied, then $B$ is tame and full in $K$, and is a valuation of $K$. When $\Gamma$ is rank-1, we can replace it by $\mathbb{R}^+ = (0, \infty)$, and two valuations

$$|x|, |x| : K_X \rightarrow [0, \infty) \quad (8.3.37)$$

are equivalent ($|x|_X \leq 1 \iff |x|'_X \leq 1$) if and only if $|x|'_X = |x|_X^\sigma$ for some $\sigma > 0$. We have exactly as in Appendix [B], the Ostrowski theorems. The proofs are the same as in Appendix [B], only written in the language of generalized rings. Thus e.g. $(1,1) \circ \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ should be replaced by $((1,1) \triangleleft (q_i)) \parallel (1,1)$. The proof for generalized ring is even shorter: we only need to show $B_X = (\mathbb{Z}_p)_X$ or $(\mathcal{O}_{Q,\eta})_X$, the rest of the proof (that $B_{Y,X} = (\mathbb{Z}_p)^{Y \times X}$ or $(\mathcal{O}_{Q,\eta})_{Y,X}$) is irrelevant.

**Theorem Ostrowski I**

$$\text{Val}(\mathcal{G}(\mathbb{Q})) = \text{Val}(\mathcal{G}(\mathbb{Q})/\mathbb{F}(\pm 1)) = \begin{cases} \mathcal{G}(\mathbb{Q}) & \text{p is prime} \\ \mathcal{G}(\mathbb{Z}_p) & \text{"the real prime"} \end{cases} \quad (8.3.38)$$

**Theorem Ostrowski II**

Let $K$ be a number field, $\mathcal{O}_K$ its ring of integers,

$$\text{Val}(\mathcal{G}(K)) = \text{Val}(\mathcal{G}(K)/\mathbb{F}(\mu_K)) = \begin{cases} \mathcal{G}(K) & \text{p } \subseteq \mathcal{O}_K \text{ a prime ideal, } \mathcal{O}_{K,p} = S_p^{-1}\mathcal{O}_K \\ \mathcal{G}(\mathcal{O}_{K,p}) & \text{p } \subseteq \mathcal{O}_K \text{ a prime ideal, } \mathcal{O}_{K,p} = S_p^{-1}\mathcal{O}_K \\ \mathcal{O}_{Q,\eta} & \eta : K \hookrightarrow \mathbb{C} \text{ mod conjugation} \end{cases} \quad (8.3.39)$$

### 8.3.5 Commutative Monoids

For $A \in \mathcal{G}_R$, we have $A_{[1]} \in \text{Mon}^t$, cf. Remarks 8.2.2-8.2.4, giving rise to a functor

$$\mathcal{G}_R \rightarrow \text{Mon}^t, \quad A \rightarrow A_{[1]} \quad (8.3.40)$$

This functor has a left adjoint. For $M \in \text{Mon}^t$, let

$$\mathbb{F}\{M\}_X = \bigsqcup_{x \in X} M = (X \times (M\setminus\{0\})) \cup \{0\} = [X \times M]/[x,0] \sim 0 \quad (8.3.41)$$

It forms a functor $X \rightarrow \mathbb{F}\{M\}_X : \mathbb{F} \rightarrow \text{Set}_0$. We define the operation of multiplication by

$$\triangleleft : \mathbb{F}\{M\}_Y \times \mathbb{F}\{M\}_f \rightarrow \mathbb{F}\{M\}_X \quad (8.3.42)$$
We define the operation of contraction by
\[(x \otimes y) : \mathbb{F}\{M\}_X \times \mathbb{F}\{M\}_Y \to \mathbb{F}\{M\}_X \times \mathbb{F}\{M\}_Y\]

\[ [x_0, m_0], [x^{(y)}, m^{(y)}] \mapsto [y, m_0(m^{(y)})^y] \]

It is straightforward to check that with these operations \( F \) forms a generalized ring, and for \( A \in \mathcal{G}R \), we have adjunction

\[ \mathcal{G}R(\mathbb{F}\{M\}, A) = \text{Mon}^t(M, A_{[1]}) \]

\[ \tilde{\psi}_X([x, m]) = \psi(m) \otimes 1 \]

with \( j_x \in \mathbb{F}\{[1], X\}, j_x(1) = x \) (8.3.44)

We let \( C\text{Mon} \subseteq \text{Mon}^t \) denote the full-subcategory of commutative monoids (with trivial involution). For \( M \in C\text{Mon} \), the generalized ring \( \mathbb{F}\{M\} \) is totally-commutative, matrix and self-adjoint.

### 8.3.6 The free commutative generalized ring \( \Delta^W \)

In this section we give a description of the commutative generalized ring \( \Delta^W \), characterized by the universal property that homomorphisms of commutative generalized rings \( \varphi : \Delta^W \to A \) correspond bijectively with elements of \( A_{[W]} \).

The elements of \( (\Delta^W)_X \), of "degree" \( X \in \mathbb{F} \), are the formulas of degree \( X \) that can be written using multiplication, contraction, the elements of \( \mathbb{F} \), and one variable \( \delta_W \) of degree \( W \), modulo the identifications of formulas that are consequences of the axioms of a commutative generalized ring. By Remark 8.2.5 - "one contraction suffice", these formulas have a concrete combinatorial shape which we describe next.

\[ \mathcal{C} \text{Mon} \subseteq \text{Mon}^t \] denote the full-subcategory of commutative monoids (with trivial involution). For \( M \in \mathcal{C} \text{Mon} \), the generalized ring \( \mathbb{F}\{M\} \) is totally-commutative, matrix and self-adjoint.

By a tree \( F \) we shall mean a finite set with a distinguished element \( 0_F \in F \), the root, and a map \( S = S_F : F \setminus \{0_F\} \to F \), such that for all \( a \in F \) there exists \( n \) with \( S^n(a) = 0_F \); we write \( n = \text{ht}(a) \), and put \( \text{ht}(0_F) = 0 \). For \( a \in F \), we put \( \nu(a) = 4S^{-1}(a) \). The boundary of \( F \) is the set

\[ \partial F = \{ a \in F, \nu(a) = 0 \} \]

(8.3.45)

The unit tree is the tree with just a root, \( \{0\} \), and \( \partial \{0\} = \{0\} \). The zero tree is the empty set, \( \emptyset \), and \( \partial \emptyset = \emptyset \). Given a subset \( B \subseteq \partial F \), we have the reduced tree \( F|_B \), obtained from \( F \) by omitting all the elements of \( \partial F \setminus B \), and then omitting all elements \( a \in F \) such that all the elements of \( S^{-1}(a) \) have been omitted and so on; we have \( \partial(F|_B) = B \).
Given for each $b \in \partial F$, a tree $G_b$, let $B = \{ b \in \partial F : G_b \neq \emptyset \}$, and form the tree $F < G := F|_B \amalg \prod_{b \in B}(G_b \setminus \{0_{G_b}\})$ (8.3.47)

with $S(a) = b$ if $a \in G_b$ and $S_{G_b}(a) = 0_{G_b}$, and otherwise $S$ being the restriction of the given $S_F$ and $S_{G_b}$.

Fix $W \in \mathbb{F}$. We say that a tree $F$ is $W$-labelled, if we are given for all $b \in F$ an injection

$$\mu_b : S^{-1}(b) \hookrightarrow W$$

(8.3.48)

We view $\mu$ as a map $\mu : F \setminus \{0_F\} \rightarrow W$, $\mu(b) = \mu_{S(b)}(b)$, injective on fibers of $S_F$.

Let for $X \in \mathbb{F}$,

$$\tilde{\Delta}_X = \{ F = (F_1 ; \{ T_x \}; \sigma) \}$$

(8.3.49)

consist of the data of a $W$-labelled tree $F_1$, a $W$-labelled tree $T_x$ for each $x \in X$, and a bijection $\sigma : \partial F_1 \rightarrow \prod_{x \in X} \partial T_x$. We view such data $F$ as being the same as $F' = (F'_1 ; \{ T'_x \}; \sigma')$ if there are isomorphisms of $W$-labelled trees $\tau_1 : F'_1 \sim F_1$, $\tau_x : T'_x \sim T_x$, with $\sigma' \circ \tau_1(b) = \tau_x \circ \sigma(b)$ for $b \in \partial F_1$, $\sigma(b) \in \partial T_x$.

Note that for such data $F = (F_1 ; \{ T_x \}; \sigma)$, we have an associated map

$$\bar{\sigma} : \partial F_1 \rightarrow X$$

(8.3.50)

$$\bar{\sigma}(b) = x \text{ iff } \sigma(b) \in \partial T_x$$

For $f \in Set^*(X, Y)$, we have $\tilde{\Delta}_f = \prod_{y \in Y} \tilde{\Delta}_{f^{-1}(y)}$, its elements are isomorphism classes of data $F = \{ (F_x)_{y \in f(X)} ; \{ T_x \}_{x \in D(f)} ; \sigma_y \}$, with bijections $\sigma_y : \partial F_y \sim \prod_{x \in f^{-1}(y)} \partial T_x$, for all $y \in Y$. 
We define the operation of multiplication by cf. (8.2.22):

\[ < : \tilde{\Delta}_Y \times \tilde{\Delta}_f \rightarrow \tilde{\Delta}_X \quad (8.3.51) \]

\[ G \triangleleft F = (G_1; G_y; \tau) \triangleleft (F_y; F_x; \sigma_y) = (G_1 \triangleleft G_y; F_x \triangleleft G_f(x); \tau \circ \sigma) \]

with

\[ \tau \circ \sigma : \partial (G_1 \triangleleft G_y) = \]

\[ = \partial G_1 \times \partial F_{y(1)} \overset{\tau}{\rightarrow} \prod_{y \in Y} \partial G_y \times \partial F_y \overset{\Pi \sigma_y}{\sim} \prod_{y \in Y} \partial G_y \times \partial F_x = \]

\[ = \prod_{x \in X} \partial F_x \times \partial G_{f(x)} = \prod_{x \in X} \partial (F_x \triangleleft G_{f(x)}) \]

We define the operation of contraction by, cf. (8.2.25):

\[ (\parallel) : \tilde{\Delta}_X \times \tilde{\Delta}_f \rightarrow \tilde{\Delta}_Y \quad (8.3.53) \]

\[ (G \parallel F) = ((G_1; G_x; \tau) \parallel (F_y; F_x; \sigma_y)) = (G_1 \parallel G_x; F_y \parallel G_{f(x)}; (\tau, \sigma)) \]

with

\[ (\tau, \sigma) : \partial (G_1 \parallel G_x) = \partial G_1 \times \partial F_{x(1)} \overset{\tau}{\rightarrow} \prod_{x \in X} \partial G_x \times \partial F_x = \]

\[ = \prod_{y \in Y} \prod_{x \in f^{-1}(y)} \partial F_x \times \partial G_x \overset{\Pi \sigma_y^{-1}}{\sim} \prod_{y \in Y} \partial F_y \times \partial G_{f(x)} = \prod_{y \in Y} \partial (F_y \parallel G_{f(x)}) \]

It is straightforward to check that \( \tilde{\Delta} \) with these operations is a (non-commutative!) generalized-ring, in the sense that all the axioms of a generalized ring except the axioms of right-linearity (8.1.39) are satisfied.

We let \( \Delta_X = \tilde{\Delta}_X / \varepsilon \) denote the quotient commutative generalized ring, where \( \varepsilon \) is the equivalence ideal generated by right-linearity: \((F, F') \in \varepsilon_X\) if and only if there exists a path \( F = F^0, \ldots, F^l = F' \), with \( \{F^k, F^{k+1}\} \) of the form

\[ \{((A_j \parallel B_j) \triangleleft C_j) \parallel D_j, ((A_j \triangleleft f_j^* C_j) \parallel g_j^* B_j) \parallel D_j\} \quad (8.3.55) \]

where the diagram is

\[ \begin{array}{c}
W_j \mid \prod Y_j \\
\downarrow f_j^* C_j \\
W_j \\
\downarrow A_j \\
[1] \leftarrow Z_j \\
\downarrow B_j \\
\downarrow f_j \\
Y_j \\
\downarrow C_j \\
\downarrow g_j \\
D_j \\
\downarrow g_j \\
X 
\end{array} \quad (8.3.56) \]
or of the form

\[ \{ D_j \parallel ((A_j \parallel B_j) \triangleleft C_j), D_j \parallel ((A_j \triangleleft f_j^* C_j) \parallel g_j^* B_j) \} \quad (8.3.57) \]

where the diagram is

\[ \begin{tikzcd}
   & W_j 
   & Y_j \\
   W_j 
   & Z_j 
   & f_j^* C_j \\
   X 
   & A_j 
   & B_j \\
   D_j 
   & C_j 
   & g_j^* B_j \\
   & Y_j \\
 \end{tikzcd} \quad (8.3.58) \]

Note that if \( \{ F, F' \} \in \varepsilon \), then \( \{ E \triangleleft F, E \triangleleft F' \} \), \( \{ F \triangleleft E, F' \triangleleft E \} \), \( \{ E \parallel F \}, E \parallel F' \} \), \( \{ F \parallel E, F' \parallel E \} \) are in \( \varepsilon \), so that \( \varepsilon \) is indeed an equivalence ideal, and the operations of multiplication and contraction descend to well defined operations on the \( \varepsilon \)-equivalence classes \( \Delta = \overline{\Delta} / \varepsilon \).

We get in this way a commutative generalized ring \( \Delta^W \),

\[ \Delta^W_X = \left\{ \{F_1; \{ F_x \}_{x \in X}; \sigma \}/\sim, \quad F_1, F_x \text{-labeled trees}, \quad \sigma : \partial F_1 \to \prod \partial F_x \text{ bijection} \right\} \quad (8.3.59) \]

The element \( \delta^W = \{ \{0\} \uplus W; \{0_w\}_{w \in W}; \sigma \} \in \Delta^W_X \) generates \( \Delta^W \).

For any commutative generalized ring \( A \), we have a bijection

\[ \mathcal{GR}_C(\Delta^W, A) = A_W \quad (8.3.60) \]

\[ \varphi \mapsto \varphi(\delta^W) \]

\[ \varphi(\alpha) \leftrightarrow a \]

Given an injection \( j : W \to W' \), every \( W \)-labelled tree \( F \) is naturally \( W' \)-labelled, and we have an injective homomorphism

\[ \Delta^j \in \mathcal{GR}(\Delta^W, \Delta^{W'}), \Delta^j_X(\{F_1\}_{W}; \{ F_x \}_{W}; \sigma) = (\{F_1\}_{W'}; \{ \overline{F_x} \}_{W'}; \sigma) \quad (8.3.61) \]

It is dual via \( (8.3.60) \) to the map \( A_{W'} \to A_W, a \mapsto a < 1_j \).

Conversely, if \( F \) is a \( W' \)-labelled tree, and

\[ B_F = \{ b \in \partial F, \mu(S^n(b)) \in W \text{ for } n = 0, \ldots, ht(b) - 1 \} \quad (8.3.62) \]

than the reduced tree \( F|_{B_F} \), c.f. \( (8.3.40) \), is \( W \)-labelled. We have a surjective homomorphism

\[ \Delta^j \in \mathcal{GR}(\Delta^{W'}, \Delta^W), \quad (8.3.63) \]
generalized ring \((\Delta^f)\) (i.e. \(\Delta^f\) generate commutative generalized rings). It is dual to the map \(A_W \leftrightarrow A_{W^*}, a \mapsto a^* \in \mathcal{I}_j^*, \) and we have \(\Delta^f \circ \Delta^g = \text{id}_{\Delta^f}. \) Thus the map \(W \mapsto \Delta^W \) is a functor

\[ \Delta : \mathcal{F} \to \mathcal{GR}_C \quad (8.3.64) \]

Fixing a family \(\{W_i\}_{i \in I} W_i \in \mathcal{F}, \) we say that a tree \(F = \{W_i\}. \) labelled, if for all \(b \in F \cap \partial F \) we are given \(j(b) \in I, \) and an injection \(\mu_b : S_{\mathcal{F}_F}^{-1}(b) \leftrightarrow W_{j(b)}. \)

Repeating the above construction, with \(\{W_i\}\)-labelled trees, we get a generalized ring \(\Delta^{(W_i)} \). The elements \(\delta^{W_i} = \{\{0\} \cup W_i \}; \{0_w \}_{w \in W_i}; \sigma \} \in \Delta^{(W_i)}\) generate \(\Delta^{(W_i)} \), and we have for any commutative generalized ring \(A, \)

\[ \mathcal{GR}_C(\Delta^{(W_i)}, A) = \prod_{i \in I} A_{W_i} \quad (8.3.65) \]

\(\varphi \mapsto (\varphi(\delta^{(W_i)})) \)

(i.e. \(\Delta^{W_1 \cdots W_n} = \Delta^{W_1} \otimes \cdots \otimes \Delta^{W_n} \) cf. §13.1 for the tensor product \(\otimes \) of commutative generalized rings.)

Thus we have the adjunction:

\[ \begin{array}{ccc}
I & \to & \mathcal{F} \\
\delta & & \mathcal{GR}_C \\
\mathcal{Set}/\mathcal{F} & \to & \prod_{x \in \mathcal{F}} A_X \\
UA & \mapsto & U \end{array} \]

\[ \Delta^{(W_i)} \]

\[ \mathcal{GR}_C \]

\(\Delta \theta \)

\(A \)

\[ \varphi \to (\varphi(\delta^{(W_i)})) \]

\[ \prod_{i \in I} A_{W_i} \]

For an element \(F = (F_1; \{F_x\}; \sigma) \in \Delta^{(W_i)}\), we have its (well-defined) degree

\[ \deg F = \max_{b \in F_1} \{\text{ht}(b) + \text{ht}(\sigma(b))\} \in \mathbb{N} \quad (8.3.67) \]

\[ \deg F = 0 \iff F \in \mathcal{F}_X \subseteq \Delta^{(W_i)} \quad (8.3.68) \]

For \(f \in \mathcal{Set}(Y, Z), G = \{G(z)\} \in \Delta^{(W_i)}\), putting \(\deg G = \max_{z \in Z} \{\deg G(z)\}, \) we have:

\[ \deg(F_Z \triangleleft G) \leq \deg F_Z + \deg G, F_Z \in \Delta^{(W_i)} \]

\[ \deg(F_Y \parallel G) \leq \deg F_Y + \deg G, F_Y \in \Delta^{(W_i)} \quad (8.3.69) \]

Given a map \(f \in \mathcal{Set}(Z, \mathcal{W}), \) we have the element

\[ \delta^f = ([\{0\} \cup f(Z) \cup D(f)]; \{0_z\}_{z \in D(f)}; \sigma = \text{id}_{D(f)} \in \Delta^{W,f^{-1}(w)}_{X} = (\Delta^W \otimes \otimes_{w \in W} \Delta^{f^{-1}(w)}_{Z}) \quad (8.3.70) \]
where the tree $F = \{0\} \cup f(Z) \cup D(f)$, has $S_F|_{D(f)} = f$, $S_F(f(Z)) = 0$, and
is labelled by $\mu_0 : f(Z) \rightarrow W$, and $\mu_w = id_{f^{-1}(w)}$ for $w \in f(Z)$. This element
gives a homomorphism of generalized rings, co-multiplication,
$$\Delta_f^I \in \mathcal{GR}_C(\Delta^Z, \Delta^{W,f^{-1}(w)}_Z), \quad \delta^Z \rightarrow \delta^f,$$
which is dual to multiplication $A_W \times \prod_{w \in W} A_{f^{-1}(w)} \rightarrow A_Z$.

On the other hand we have the element
$$\varepsilon^f = \left(\{\{0\} \cup D(f)\}; \{\{0\} \cup f^{-1}(w)\}\right)_{w \in f(Z)}; \sigma = id_{D(f)} \in \Delta^Z_{f^{-1}(w)} = (\Delta^Z \otimes \otimes_{w \in W} \Delta^{f^{-1}(w)}_W)_{\mathcal{C}}$$
giving rise to a homomorphism of generalized rings, co-contraction,
$$\Delta_f^J \in \mathcal{GR}_C(\Delta^W, \Delta^{Z,f^{-1}(w)}_Z), \quad \delta^W \rightarrow \varepsilon^f$$
which is dual to contraction $A_Z \times \prod_{w \in W} A_{f^{-1}(w)} \rightarrow A_W$.

The functor $\Delta : F \rightarrow \mathcal{GR}_C$, with its structure (of co-multiplication, co-contraction, and co-unit) is thus a co-generalized-ring-object in the tensor category $(\mathcal{GR}_C, \mathcal{C})$, i.e. the dual of our axioms are satisfied. (Just as the polynomial ring $\mathbb{Z}[X]$, with co-multiplication $\Delta_+(X) = X \otimes X$, and co-addition
$\Delta_+(X) = X \otimes 1 + 1 \otimes X$, is a co-ring object in the tensor category of commutative rings and $\mathcal{C}$).

**Example 8.3.1**

Taking for $W = [1]$, the unit set, a $[1]$-labelled-tree is just a "ladder" $\{x_0, x_1, \ldots, x_n\}$,
$S(x_j) = x_{j-1}$, and is determined by its length $n$. Thus the element $F = ([F_1]; [\mathcal{T}_x]; \sigma) \in \Delta^{[1]}_C$, is determined by the length $n$ of $F_1$, the point $x \in X$ such
that $F_x \neq 0$, and the length $m$ of $\mathcal{T}_x$. We write $F = (x, z^n \cdot (z^j)^m)$, and we have an isomorphism
$$\Delta^{[1]}_x \sim \mathbb{F}\{z^n \cdot (z^j)^m\}$$
with the generalized ring associated with the free monoid-with-involution on one element, $M = z^n \cup \{0\}$.

The self-adjoint quotient $\Delta^{[1]}_x$ of $\Delta^{[1]}$ is isomorphic to the generalized ring associated with the free monoid on one element $M = z^n \cup \{0\},$
$$\Delta^{[1]}_+ \sim \mathbb{F}\{z^n\}$$

**8.3.7 Limits**

Given a partially ordered set $I$, a functor $A \in (\mathcal{GR})^{f}$ is given by objects $A^{(i)} \in \mathcal{GR}$ for $i \in I$, and homomorphism $\phi_{i,j} : A^{(j)} \rightarrow A^{(i)}$ for $i \leq j, i,j \in I$, such that
\[ \varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k} \text{ for } i \leq j \leq k, \text{ and } \varphi_{ii} = id_{A(i)}. \]
The inverse limit of \( A \) exists, and can be computed in \( Set_0 \). We put
\[
(\lim_I A^{(i)})_X = \{ a = (a^{(i)}) \in \prod_{i \in I} A^{(i)}_X, \varphi_{i,j}(a_j) = a_i \text{ for all } i \leq j \} \tag{8.3.76}
\]

With the operations of componentwise multiplication and contraction \( \lim_I A^{(i)} \)
is a generalized-ring (sub-ring of \( \prod_{i \in I} A^{(i)} \)). We have the universal property
\[
\mathcal{G}R(B, \lim_I A^{(i)}) = \lim_I \mathcal{G}R(B, A^{(i)}) = \tag{8.3.77}
\]
\[
= \{(\psi_i) \in \prod_I \mathcal{G}R(B, A^{(i)}), \varphi_{i,j} \circ \psi_j = \psi_i \text{ for } i \leq j \}
\]

If the set \( I \) is directed (for \( j_1, j_2 \in I \) there is \( i \in I \) with \( i \leq j_1, j_2 \)) the direct limit of \( A \) exists, and can be computed in \( Set_0 \). We have
\[
(\lim_I A^{(i)})_X = \lim_I A^{(i)}_X = (\prod_{i \in I} A^{(i)}_X)/(a \sim \varphi_{i,j}(a)) \tag{8.3.78}
\]

There are well defined operations of multiplication and contraction making \( \lim_I A^{(i)} \) into a generalized ring. We have the universal property
\[
\mathcal{G}R(\lim_I A^{(i)}, B) = \lim_I \mathcal{G}R(A^{(i)}, B) = \tag{8.3.79}
\]
\[
= \{(\psi_i) \in \prod_I \mathcal{G}R(A^{(i)}, B), \psi_i \circ \varphi_{i,j} = \psi_j \text{ for } i \leq j \}
\]

We shall see below that \( \mathcal{G}R_C \) has sums and push-outs §13.1, and quotients by equivalence-ideals §9.1, hence arbitrary co-limits, and we have:

**Theorem 8.3.8**

\( \mathcal{G}R_C \) is complete and co-complete.
Chapter 9

Ideals

9.1 Equivalence ideals

Definition 9.1.1

For $A \in \mathcal{GR}$ an equivalence ideal $\varepsilon$ is a sub-generalized-ring $\varepsilon \subseteq A \times A$, such that $\varepsilon_X$ is an equivalence relation on $A_X$, (we write $a \sim a'$ for $(a, a') \in \varepsilon_X$; and for $a = (a_y), a' = (a'_y) \in A_f$, we write $a \sim a'$ for $(a_y, a'_y) \in \varepsilon_{f^{-1}(y)}$ for all $y \in Y$). Thus, we have an equivalence relation $\sim$ that respects the operations: if $a \sim a'$ and $b \sim b'$ then

\[ a \triangleleft b \sim a' \triangleleft b', \]
\[ a \parallel b \sim a' \parallel b'. \]

(whenever these operations are defined.)

We let $eq(A)$ denote the set of equivalence ideals of $A$.

For $\varepsilon \in eq(A)$ we can form the quotient $A/\varepsilon$, with

\[ (A/\varepsilon)_X = A_X/\varepsilon_X = \varepsilon_X\text{-equivalence classes in } A_X \]

There is a natural surjection $\pi_X : A_X \to A_X/\varepsilon_X$, and there is a unique structure of a generalized ring on $A/\varepsilon$ such that $\pi \in \mathcal{GR}(A, A/\varepsilon)$.

Definition 9.1.2

For $\varphi \in \mathcal{GR}(A, A')$, we let $KER(\varphi) = \bigcap_{A'} A$, $\varepsilon_X \equiv \varepsilon_X\text{-equivalence classes in } A_X$

\[ KER(\varphi)_X = \{(a_1, a_2) \in A_X \times A_X, \varphi(a_1) = \varphi_X(a_2)\} \]

$KER(\varphi)$ is an equivalence ideal.

We have the universal property of the quotient

\[ \mathcal{GR}(A/\varepsilon, A') = \{ \varphi \in \mathcal{GR}(A, A'), KER(\varphi) \supseteq \varepsilon \} \]
Every homomorphism \( \varphi \in \text{GR}(A, A') \) has a canonical factorization

\[
\varphi = j \circ \tilde{\varphi} \circ \pi
\]

with \( \pi \) surjection, \( j \) injection, and \( \tilde{\varphi} \) an isomorphism.

## 9.2 Functorial Ideals

**Definition 9.2.1**

For \( A \in \text{GR}_C \), a **functorial ideal** \( a \) is a collection of subsets \( a_X \subseteq A_X \), with \( 0_X \in a_X \), and with \( A \triangleleft a \), \( a \triangleleft A \), \( A \parallel a \), \( a \parallel A \subseteq a \).

We let \( \text{fun} \cdot \text{il}(A) \) denote the set of functorial ideals of \( A \).

For \( \varepsilon \in \text{eq}(A) \), we have the associated functorial ideal \( Z(\varepsilon) \):

\[
Z(\varepsilon)_X = \{ a \in A_X, (a, 0_X) \in \varepsilon_X \}
\]

(9.2.1)

For \( a \in \text{fun} \cdot \text{il}(A) \), we have the equivalence ideal \( E(a) \) generated by \( a \), it is the intersection of all equivalence ideals containing \( (a, 0) \) for all \( a \in a \).

These give a Galois correspondence:

\[
\text{fun} \cdot \text{il}(A) \xrightarrow{E} \text{eq}(A)
\]

(9.2.2)

It is monotone,

\[
a_1 \subseteq a_2 \Rightarrow E(a_1) \subseteq E(a_2)
\]

\[
\varepsilon_1 \subseteq \varepsilon_2 \Rightarrow Z(\varepsilon_1) \subseteq Z(\varepsilon_2)
\]

(9.2.3)

and we have

\[
a \subseteq ZE(a) , \quad EZ(\varepsilon) \subseteq \varepsilon
\]

(9.2.4)

It follows that we have

\[
ZEZ(\varepsilon) = Z(\varepsilon) , \quad E(a) = EZE(a)
\]

(9.2.5)

and the maps \( E, Z \) induce inverse bijections

\[
E \cdot \text{il}(A) \xleftarrow{\sim} Z \cdot \text{eq}(A)
\]

(9.2.6)

with

\[
E \cdot \text{il}(A) = \{ a \in \text{fun} \cdot \text{il}(A), a = ZE(a) \} = \{ Z(\varepsilon), \varepsilon \in \text{eq}(A) \}
\]

(9.2.7)
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the stable functorial ideals, and

\[
Z eq(A) = \{ \varepsilon \in eq(A), \varepsilon = EZ(\varepsilon) \} = \{ E(a), a \in il(A) \} 
\] (9.2.8)

Let \( a \in \text{fun} \cdot il(A) \), and let \( \varepsilon = (\varepsilon_X) \) be defined by

\[
\varepsilon_X = \left\{ (a, a') \in A_X \times A_X, \text{there exists a path } a = a_1, \ldots, a_n = a', \\
\text{with } \{a_j, a_{j+1}\} \text{ of the form either } \{(d_j < c_j) \parallel b_j, (d_j < c_j) \parallel b_j\}, \\
\text{or } \{d_j \parallel (b_j < c_j), d_j \parallel (b_j < c_j)\} \text{ with } d_j \in A_{Y_j}, b_j \in A_{Y_j}, \\
f_j \in \text{Set}_\ast(Z_j, X), c_j, c_j' \in A_{Y_j}, g_j \in \text{Set}_\ast(Z_j, Y_j) \text{ or resp.} \\
g_j \in \text{Set}_\ast(Y_j, X) \text{ and with } c_j^{(z)} = t_j^{(z)}, \text{ or } c_j^{(z)}, c_j'^{(z)} \in a, \\
\text{for all } z \in Y_j (\text{resp. } z \in Z_j) \right\} 
\] (9.2.9)

Claim 9.2.2

\[ E(a) = \varepsilon \]

Proof. It is clear that \( \varepsilon_X \) is an equivalence relation on \( A_X \), and that \( \varepsilon_X \subseteq E(a)_X \), and we need to show that \( \varepsilon \) respects the operations \( \{1, \ldots, n\} \). If \( (a, a') \in \varepsilon \), so there is a path \( a = a_1, \ldots, a_n = a' \) as in \( \{1, \ldots, n\} \), then \( h \triangleleft a_j \text{ (resp. } a_j < h, \\
(h \parallel a_j), (a_j \parallel h)) \text{ is a path, showing } (h \triangleleft a, h \triangleleft a') \text{ (resp. } (a \triangleleft h, a' \triangleleft h), \\
(h \parallel a, h \parallel a'), (a \parallel h, a' \parallel h)) \text{ is in } \varepsilon \). This follows from the

Basic identities 9.2.3

\[
h \triangleleft (d \triangleleft c \parallel b) = ((h \triangleleft d) \triangleleft c) \parallel b, \quad h \triangleleft (d \parallel (b \triangleleft c)) = (h \triangleleft d) \parallel (b \triangleleft c) \]

resp.

\[
((d \triangleleft c) \parallel b) \triangleleft h = (d \triangleleft (c \triangleleft h)) \parallel b, \quad (d \parallel (b \triangleleft c)) \triangleleft h = (d \triangleleft h) \parallel (b \triangleleft c) \]

\[
h \parallel ((d \triangleleft c) \parallel b) = (h \parallel (d \triangleleft c)) \parallel (h \triangleleft b), \quad h \parallel (d \parallel (b \triangleleft c)) = ((h \triangleleft b) \triangleleft c) \parallel d \]

\[
((d \triangleleft c) \parallel b) \parallel h = (h \triangleleft c) \parallel (h \triangleleft b), \quad (d \parallel (b \triangleleft c)) \parallel h = d \parallel ((h \triangleleft b) \triangleleft c) 
\]

It follows that a functorial ideal \( a \) is stable, \( a = ZE(a) \), if and only if for all \( b, d, c, \varpi \) as in \( \{1, \ldots, n\} \),

\[
(d \triangleleft c) \parallel b \in a \iff (d \triangleleft \varpi) \parallel b \in a 
\]

and \( d \parallel (b \triangleleft c) \in a \iff d \parallel (b \triangleleft \varpi) \in a \) (9.2.10)

9.3 Operations on functorial ideals

The intersection of functorial ideals is a functorial ideal,

\[
a_i \in \text{fun} \cdot il(A) \Rightarrow \bigcap_i a_i \in \text{fun} \cdot il(A) 
\]
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Given a collection \( B = \{B_X \subseteq A_X\} \), the functorial ideal generated by \( B \) will be denoted by \( \{B\}_A \), it is the intersection of all functorial ideals containing \( B \).

If \( B \) satisfies \( B \triangleleft A \subseteq B \), it can be described explicitly as

\[
\{B\}_A = \{a \in A_X, a = (d \triangleleft c) \parallel b \textit{ or } a = d \parallel (b \triangleleft c) \textit{ for some } d \in A_Y, b \in A_f, \]

\[
f \in \text{Set}_\bullet(Z, X), c \in A_g, g \in \text{Set}_\bullet(Z, Y),
\]

or resp. \( g \in \text{Set}_\bullet(Y, Z) \)

\[
\text{and with } c = (c(z)), c(z) \in B_{g^{-1}(z)} \text{ for all } z \in Y \text{ (resp. } z \in Z) \} \quad (9.3.1)
\]

It is clear that the set described in (9.3.1), contains \( B_X \), and is contained in \( \{B\}_A \), and we only have to check that it is a functorial ideal - this follows from the basic identities 9.2.3.

In particular, for \( a_i \in \text{fun} \cdot \text{il}(A) \), we can take \( B = \bigcup_i a_i \), and we obtain the smallest functorial ideal containing all the \( a_i \)'s.

\[
\sum_i a_i = \left\{ \bigcup_i a_i \right\}_A \quad (9.3.2)
\]

Thus \( \text{fun} \cdot \text{il}(A) \) is a complete lattice, with minimal element the zero ideal \( 0 = \{0_X\} \), and maximal element the unit ideal \( \{1\}_A = \{A_X\} \).

Note that for arbitrary \( B \) we can similarly describe the functorial ideal it generates as

\[
\{B\}_A = \{a \in A_X, a = (d \triangledown c \triangledown e) \parallel b \textit{ or } a = d \parallel (b \triangledown c \triangledown e), \textit{with } c(z) \in B \textit{ for all } z \} \quad (9.3.3)
\]

Note that for a subset \( B \subseteq A_{[1]} \), we have

\[
\{B\}_A = \{a \in A_X, a = d \parallel (b \triangledown c), d \in A_Y, b \in A_f, \]

\[
f \in \text{Set}_\bullet(Y, X), c = (c(y)) \in A_{id_Y} = (A_{[1]})^Y, c(z) \in B \cup B^1 \} \quad (9.3.4)
\]

We do not need the \( e \)'s in (9.3.3) because we have commutativity \( c \triangleleft e = e \triangleleft \tilde{c} \), c.f. (8.2.14), and we do not need the two shapes of (9.3.3) because the \( (A_{[1]})^Y \)-action is self-adjoint \( d \parallel (b \triangleleft c) = (d \triangleleft c') \parallel b \), c.f. (8.2.15).
9.4 Homogeneous ideals

**Definition 9.4.1**

A functorial ideal \( a \in \text{fun} \cdot \text{il}(A) \) is called *homogeneous* if it is generated by \( a_{[1]} \).

The subset \( a_{[1]} \subseteq A_{[1]} \) satisfies for all \( X \in \mathbb{F} \),

\[
A_X \parallel (A_X \triangleleft (a_{[1]})^X) \subseteq a_{[1]}
\]

(9.4.1)

Conversely, if a subset \( a_{[1]} \subseteq A_{[1]} \) satisfies (9.4.1), then \( a^t_{[1]} = a_{[1]} \) (because for \( a \in a_{[1]} \), \( a^t = 1 \parallel (1 \triangleleft a) \in a_{[1]} \)), and \( a_{[1]} \triangleleft A_{[1]} = a_{[1]} \) (because for \( a \in a_{[1]} \), \( b \in A_{[1]} \), \( a \triangleleft b = b \parallel (1 \triangleleft a^t) \in a_{[1]} \)). Moreover, the functorial ideal \( b \) generated by \( a_{[1]} \), has

\[
b_X = \bigcup_{f \in \text{Set}_{s}(Y,X)} A_Y \parallel (A_f \triangleleft (a_{[1]})^Y)
\]

(9.4.2)

and in particular \( b_{[1]} = a_{[1]} \). Thus we identify the set of homogeneous functorial ideals with the collection of subsets \( a_{[1]} \subseteq A_{[1]} \) satisfying (9.4.1), and we denote this set by \([1]\)-il(\( A \)),

\[
[1]\text{-il}(A) = \{ a \subseteq A_{[1]}, A_X \parallel (A_X \triangleleft (a)^X) \subseteq a \}
\]

(9.4.3)

The set \([1]\text{-il}(A)\) is a complete lattice, with minimal element \( \{0\} \), maximal element \( \{1\} A = A_{[1]} \). For \( a_i \in [1]\text{-il}(A) \), we have \( \bigcap_i a_i \in [1]\text{-il}(A) \), and \( \sum_i a_i \in [1]\text{-il}(A) \), where

\[
\sum_i a_i = \left\{ a \in A_{[1]}, a = b \parallel (d \triangleleft c), b, d \in A_X, c = (c(x)) \in \left( \bigcup_i a_i \right)^X \subseteq A_{id_X}, X \in \mathbb{F} \right\}
\]

(9.4.4)

Note that the homogeneous functorial ideal generated by elements \( a_i \in A_{[1]} \), \( i \in I \), can be described as

\[
\{ a_i \}_A = \left\{ a \in A_{[1]}, a = b \parallel (d \triangleleft c), b, d \in A_X, c = (c(x)) \in A_{id_X}, c(x) = a_i(x) \right. \\
\text{or } c(x) = a^t_{i(x)} \text{ for } x \in X \}
\]

(9.4.5)

We have also the operation of multiplication of homogeneous ideals. For \( a_1, a_2 \in [1]\text{-il}(A) \), we let \( a_1 \cdot a_2 \) denote the homogeneous ideal generated by the product \( a_1 \triangleleft a_2 = \{ a_1 \triangleleft a_2, a_i \in a_i \} \).
Thus
\[ a_1 \cdot a_2 = \ \{ a \in A_{[1]}, a = b \{ (d < c), b, d \in A_X, c = (c^{(x)}) \in (a_1 < a_2)^X \leq A_{id_X} \} \] (9.4.6)

This operation is associative, we have for \( a_1, a_2, a_3 \in [1]\cdot il(A) \),
\[ (a_1 \cdot a_2) \cdot a_3 = a_1 \cdot a_2 \cdot a_3 = a_1 \cdot (a_2 \cdot a_3) \] (9.4.7)

with
\[ a_1 \cdot a_2 \cdot a_3 = \ \{ a \in A_{[1]}, a = b \{ (d < c), b, d \in A_X, c = (c^{(x)}) \in (a_1 < a_2 < a_3)^X \leq A_{id_X} \} \]
(use \( (b \{ (d < a_1 < a_2) \{ a_3 = b \{ (d < a_1 < a_2 < a_3) \) and \( a_1 \{ (b \{ (d < a_2 < a_3) = b \{ (d \circ a_2 \circ a_3 \circ a_1)) \).

The multiplication of homogeneous ideals is clearly commutative, \( a_1 < a_2 = a_2 < a_1 \), has unit element \( \{1\}_A = A_{[1]}, a \cdot \{1\} = a \), and has zero element \( \{0\}, a \cdot \{0\} = \{0\} \). Thus \([1]\cdot il(A) \in Mon\).

For a homomorphism \( \varphi \in GR_C(A, B) \), and for \( b \in fun \cdot il(B) \), (resp. \( b \in [1]\cdot il(B) \)), its inverse image \( \varphi^*(b)_X = \varphi^{-1}_X(b)_X \), (resp. \( \varphi^*[1](b) = \varphi^{-1}[1](b) \subseteq A_{[1]} \)) is clearly a (resp. homogeneous) functorial ideal of \( A \). For \( a \in fun \cdot il(A) \), (resp. \([1]\cdot il(A) \)) let \( \varphi(a) \subseteq B \) denote the (homogeneous) functorial ideal generated by the image \( \varphi(a) \) (resp. \( \varphi[1](a) \)). We have Galois correspondences
\[ [1]\cdot il(A) \xrightarrow{\varphi^*[1]} [1]\cdot il(B) \quad \text{and} \quad fun \cdot il(A) \xrightarrow{\varphi^*} fun \cdot il(B) \] (9.4.8)

The maps \( \varphi^* \), \( \varphi_* \) are monotone, and satisfy
\[ a \leq \varphi^* \varphi_*(a), \quad \varphi_* \varphi^*(b) \leq b \] (9.4.9)

It follows that we have
\[ \varphi_*(a) = \varphi_* \varphi^* \varphi_*(a), \quad \varphi^*(b) = \varphi^* \varphi_* \varphi^*(b) \] (9.4.10)
and \( \varphi^* \), \( \varphi_* \) induce inverse bijections,
\[ \{ a \in [1]\cdot il(A), a = \varphi^* \varphi_*(a) \} = \{ \varphi^*(b), b \in [1]\cdot il(B) \} \sim \{ \varphi_* (a), a \in [1]\cdot il(A) \} = \{ b \in [1]\cdot il(B), b = \varphi_* \varphi^*(b) \} \] (9.4.11)

and similarly with \( fun \cdot il(A) \) and \( fun \cdot il(B) \).
Definition 9.4.2

For an equivalence ideal \( \varepsilon \in eq(A) \), and for a functorial ideal or a homogeneous ideal \( a \), we say \( a \) is \( \varepsilon \)-stable if for all \( (a, a') \in \varepsilon : \)

\[
a \in a \iff a' \in a.
\]

We denote by \( \text{fun} \cdot il(A)^\varepsilon \) (resp. \( [1] \cdot il(A)^\varepsilon \)) the set of \( \varepsilon \)-stable (homogeneous) ideals.

Letting \( \pi_\varepsilon : A \to A/\varepsilon \) denote the canonical homomorphism, we have bijections

\[
\text{fun} \cdot il(A)^\varepsilon \leftrightarrow \text{fun} \cdot il(A/\varepsilon) \quad \text{and} \quad [1] \cdot il(A)^\varepsilon \leftrightarrow [1] \cdot il(A/\varepsilon)
\]

\[
a \mapsto \pi_\varepsilon(a) \quad \pi_\varepsilon^{-1}(b) \leftarrow b
\]

(9.4.12)

Definition 9.4.3

For a (resp. homogeneous) functorial ideal \( a \), we say \( a \) is stable if it is \( E(a) \)-stable. We denote by \( E \cdot \text{fun} \cdot il(A) \) (resp. \( E[1] \cdot il(A) \)), the set of stable (homogeneous) functorial ideals. Note that by the explicit description of \( E(a) \) in Claim 9.2.2, a subset \( a \subseteq A_{[1]} \) is a stable homogeneous ideal if and only if

\[
\text{for } X, Y \in F, b, d \in A_X \otimes Y, c, \overline{c} \in (A_{[1]})^{X \otimes Y},
\]

with \( c^{(x)} = \overline{c}^{(x)} \) for \( x \in X \) and \( c^{(y)} = \overline{c}^{(y)} \in a \) for \( y \in Y \),

\[\text{have: } b \parallel (d \triangleleft c) \in a \iff b \parallel (d \triangleleft \overline{c}) \in a \]

(9.4.13)

(taking \( X = [0] \), \( \overline{c}^{(y)} \equiv 0 \), we see that this condition includes \( a \) being a homogeneous ideal).

9.5 Ideals and symmetric ideals

Definition 9.5.1

For \( A \in \mathcal{GR}_C \), a subset \( a \subseteq A_{[1]} \) will be called ideal if for all \( X \in F, b, d \in A_X, c = (c^{(x)}) \in (a)^X \subseteq A_{id_X} \), we have

\[
(b \triangleleft c) \parallel d \in a
\]

(9.5.1)

We denote by \( il(A) \) the set of ideals of \( A \).

Comparing with the definition of homogeneous ideals \( [1] \cdot il(A) \), we see that the homogeneous ideals are precisely the self-adjoint ideals

\[
[1] \cdot il(A) = \{a \in il(A), a = a^t\}
\]

(9.5.2)
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The set \( il(A) \) is a complete lattice, with minimal element \((0)\), maximal element \((1) = A_{[1]}\). For \( a_i \in il(A) \), we have \( \bigcap_i a_i \in il(A) \), and \( \sum_i a_i \in il(A) \), where (c.f. (9.4.3))

\[
\sum_i a_i = \left\{ a \in A_{[1]}, a = (b \triangleleft c) \triangledown d, b, d \in A_X, c = (c^{(x)}) \in \left( \bigcup_i a_i \right)^X \subseteq A_{idX} \right\}
\]

(9.5.3)

Note that the ideal generated by elements \( a_i \in A_{[1]}, i \in I \), can be described as (c.f. (9.4.5))

\[
(a_i)_A = \left\{ a \in A_{[1]}, a = (b \triangleleft c) \triangledown d, b, d \in A_X, c = (c^{(x)}) \in \{a_i\}^X \subseteq A_{idX} \right\}
\]

(9.5.4)

In particular, for \( a \in A_{[1]} \) the principal ideal generated by \( a \) is just

\[
(a)_A = a \triangleleft A_{[1]}
\]

(9.5.5)

Indeed, if \( c^{(x)} = a \triangleleft e_x, e_x \in A_{[1]} \) for \( x \in X \), then for \( b, d \in A_X \),

\[
(b \triangleleft c) \triangledown d = a \triangleleft ((b \triangleleft c) \triangledown d) \in a \triangleleft A_{[1]}
\]

(9.5.6)

(while the homogeneous ideal generated by \( a \) is the ideal generated by \( a \) and \( a^t \), c.f. (9.4.3) for \( \{a\}_A \)).

**Definition 9.5.2**

An ideal \( a \in il(A) \) will be called symmetric if it is generated by its subset of self-adjoint elements

\[
a^+ = a \cap A^+_{[1]} = \{ a \in a, a = a^t \}
\]

(9.5.7)

Such an ideal is clearly homogeneous. We write \( il^0(A) \) for the collection of symmetric ideals. It is again a complete lattice. We have multiplication of ideals, for \( a_1, a_2 \in il(A) \),

\[
a_1 \cdot a_2 = \left\{ a \in A_{[1]}, a = (b \triangleleft c) \triangledown d, b, d \in A_X, c = (c^{(x)}) \in (a_1 \triangleleft a_2)^X \subseteq A_{idX} \right\}
\]

(9.5.8)

It is associative,

\[
(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_3)
\]

(9.5.9)

with \( a_1 \cdot a_2 \cdot a_3 = \left\{ a \in A_{[1]}, a = (b \triangleleft c) \triangledown d, b, d \in A_X, c = (c^{(x)}) \in (a_1 \circ a_2 \circ a_3)^X \subseteq A_{idX} \right\} \)

(9.5.10)

(Use now: \( (b \triangleleft a_1 \triangleleft a_2) \triangledown d ) \triangleleft a_3 = (b \triangleleft a_1 \triangleleft a_2 \triangleleft a_3) \triangledown d \), and \( a_1 \triangleleft ((b \triangleleft a_2 \triangleleft a_3) \triangledown d) = (b \triangleleft a_1 \triangleleft a_2 \triangleleft a_3) \triangledown d \)).
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It is clearly commutative, $a_1 \cdot a_2 = a_2 \cdot a_1$; has unit (1), $a \cdot (1) = a$; has zero ($0$), $a \cdot (0) = (0)$.

Thus, $il(A)$ is a monoid with involution, $[1]-il(A)$ is its submonoid of element fixed by the involution, and $il'(A)$ is a submonoid of $[1]-il(A)$.

We can also divide ideals. For $a_0, a_1 \in il(A)$, we let

$$\langle a_0 : a_1 \rangle = \{ c \in A_{[1]}, c < a_1 \leq a_0 \} \quad (9.5.10)$$

This is an ideal, $(a_0 : a_1) \in il(A)$. Indeed, for $c(x) \in (a_0 : a_1)$, and for any $b, d \in A_X$, and any $a_1 \in a_1$, we have

$$((b < (c(x)) \mid d) < a_1 = (b < (c(x) < a_1)) \mid d \in a_0$$

and so $(b < (c(x)) \mid d \in (a_0 : a_1)$.

For elements $m_1, m_2 \in A_X$, we have their annihilator

$$ann_A(m_1, m_2) = \{ a \in A_{[1]}, a < m_1 = a < m_2 \} \quad (9.5.11)$$

This is an ideal, $ann_A(m_1, m_2) \in il(A)$. Indeed for $c(y) \in ann_A(m_1, m_2)$, and for any $b, d \in A_Y$ we have

$$((b < (c(y)) \mid d) < m_1 = (b < (c(y) < m_1)) \mid d = ((b < (c(y)) \mid d) < m_2$$

and so $(b < (c(y)) \mid d \in ann_A(m_1, m_2)$.

The ideal generated by the symmetric elements $ann_A(m_1, m_2) \cap A^+_{[1]}$ is a subideal:

$$ann_A^+(m_1, m_2) \subseteq ann_A(m_1, m_2), \quad (9.5.12)$$

it is clearly a symmetric ideal.

Let $\varphi \in GR_C(A, B)$. For an ideal $b \in il(B)$, its inverse image $\varphi^*(b) = \varphi^{-1}_{[1]}(b) \subseteq A_{[1]}$ is clearly an ideal of $A$. For $b \in il'(B)$, its inverse image $\varphi^*(b)$ is the ideal of $A$ generated by

$$\{ a = a^t \in A^+_{[1]} : \varphi_{[1]}(a) \in b \} \quad (9.5.13)$$

For an ideal $a \in il(A)$ (resp. a symmetric ideal $a \in il'(A)$), we let $\varphi_*(a) \subseteq B_{[1]}$ denote the ideal generated by the image $\varphi_{[1]}(a)$, $\varphi_*(a) = \lim_{X} (B_X \lhd (\varphi_{[1]}(a))^X) \parallel B_X$, (resp. by $\varphi_{[1]}(a^+)$).

We have a Galois correspondence

$$il(A) \xrightarrow{\varphi_*} il(B) \text{ and } il'(A) \xrightarrow{\varphi_*} il'(B) \quad (9.5.14)$$

The maps $\varphi^*, \varphi_*$ are monotone, and satisfy

$$a \subseteq \varphi^* \varphi_*(a), \quad \varphi_* \varphi^*(b) \subseteq b \quad (9.5.15)$$
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It follows that we have

\[ \varphi_*(a) = \varphi_* \varphi^* \varphi_*(a), \quad \varphi^*(b) = \varphi^* \varphi_* \varphi^*(b) \] (9.5.16)

and \( \varphi^*, \varphi_* \) induce inverse bijections,

\[ \{ a \in \text{il}(A), a = \varphi^* \varphi_*(a) \} = \{ \varphi^*(b), b \in \text{il}(B) \} \overset{\sim}{\longleftarrow} \{ \varphi_*(a), a \in \text{il}(A) \} = \{ b \in \text{il}(B), b = \varphi_\ast \varphi^* \varphi_\ast(b) \} \] (9.5.17)

and similarly with symmetric ideals \( \text{il}^t(A) \) and \( \text{il}^t(B) \).

In summary, for a general \( A \in \mathcal{GR}_C \), we have defined the sets

\[
\begin{align*}
\text{il}(A) & \quad \overset{\sim}{\longleftarrow} \quad [1]\text{-il}(A) & \quad \overset{\sim}{\longrightarrow} \quad \text{fun} \cdot \text{il}(A) \\
\text{il}^t(A) & \quad \overset{\sim}{\longleftarrow} \quad E[1]\text{-il}(A) & \quad \overset{\sim}{\longrightarrow} \quad E\text{-fun} \cdot \text{il}(A) \\
E\text{-il}^t(A) & \quad \overset{\sim}{\longleftarrow} \quad E[1]\text{-il}^t(A) & \quad \overset{\sim}{\longrightarrow} \quad E\text{-fun} \cdot \text{il}^t(A)
\end{align*}
\] (9.5.18)

For a self adjoint \( A \) we have equality \( \text{il}(A) = [1]\text{-il}(A) = \text{il}^t(A) \). It is easy to check that for \( A = \mathcal{G}(B) \), \( B \) a commutative ring, all the inclusions in (9.5.18) are equalities, and are identified with the set of ideals of \( B \). For \( A = \mathcal{G}(B) \), \( B \) a commutative ring with involution \( (\cdot)^t : B \to B \), \( \text{il}(A) \) is just the set of ideals of \( B \), (on which we have involution \( b \mapsto b^t \)); the set

\[
\begin{align*}
[1]\text{-il}(A) & = \text{fun} \cdot \text{il}(A) = \text{eq}(A) \\
E[1]\text{-il}(A) & = E\text{-fun} \cdot \text{il}(A) = Z \cdot \text{eq}(A)
\end{align*}
\] (9.5.19)

is the set of ideals \( b \) of \( B \) fixed by the involution \( b = b^t \) (so that \( B/b \) has involution); and finally, \( E\text{-il}^t(A) = \text{il}^t(A) \) are the ideals \( b \subseteq B \) that are generated by their symmetric elements \( b^+ = b \cap B^+ \) (with \( B^+ = \{ b \in B, b^t = b \} \subseteq B \) the subring of symmetric elements), which in turn correspond bijectively with the ideals of \( B^+ \).
Chapter 10

Primes and Spectra

10.1 Maximal ideals and primes

We say that an equivalence ideal $\varepsilon \in eq(A)$ is proper if $(1, 0) \notin \varepsilon$, or equivalently $\varepsilon_X \subseteq A_X \times A_X$ for some/all $X \in \mathbb{F}$, or equivalently $A/\varepsilon \neq 0$. We say that a functorial ideal, or an ideal, $a$ is proper if $1 \notin a$, or equivalently $a_{l[1]} \subseteq A_{l[1]}$. Since a union of a chain of proper ideals is again a proper ideal, an application of Zorn’s lemma gives

**Proposition 10.1.1**

For $A \in \mathcal{GR}_C$, there exists maximal proper ideal.

We let $Max(A) \subseteq il(A)$ denote the set of maximal ideals.

**Definition 10.1.2**

A (proper) ideal $p \in il(A)$ is called prime if $\mathcal{S}_p = A_{l[1]} \backslash p$ is closed with respect to multiplication, i.e. if for all $a, b \in A_{l[1]}$,

$$a \prec b \in p \quad \text{implies} \quad a \in p \quad \text{or} \quad b \in p \quad (10.1.1)$$

We let $spec(A) \subseteq il(A)$ denote the set of primes of $A$.

**Proposition 10.1.3**

$$Max(A) \subseteq spec(A).$$

**Proof.** Let $p \in Max(A)$, and take any elements $a, a' \in A_{l[1]} \backslash p$. Since $p$ is maximal, the ideals $(p, a)_A$ and $(p, a')_A$ are the unit ideal. We have therefore $1 = (b \prec c) \parallel d$, and $1 = (b' \prec c') \parallel d'$, with $b, d \in A_X$, $b', d' \in A_{X'}$, $c \in (p \cup \{a\})^X$, $c' \in (p \cup \{a'\})^{X'}$. 

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Thus we have

\[ 1 = 1 \triangleleft 1 = (b \triangleleft c) / d \triangleleft (b' \triangleleft c') / d' = (b \triangleleft c) / d \triangleleft (b' \triangleleft c') / d' \triangleleft (d' \triangleleft d) = (b \triangleleft c) / d \triangleleft (b' \triangleleft c') / d' \triangleleft (d' \triangleleft d) \]

But \( \tilde{c} \triangleleft \tilde{c} \in ((p \cup \{a\}) \triangleleft (p \cup \{a'\})) \times \times \times \subseteq (p \cup \{a \triangleleft a'\}) \times \times \times \),
and so \( 1 \in (p, a \triangleleft a') \), hence \( a \triangleleft a' \notin p \).

Similarly, a union of a chain of proper symmetric ideals is a proper symmetric ideal, and the set of maximal symmetric ideals \( \text{Max}'(A) \subseteq \text{il}'(A) \) is non-empty.

**Definition 10.1.4**

A (proper) symmetric ideal \( p \in \text{il}'(A) \) is called **symmetric prime** if

\[ S^+ \cap A_{[1]} \]

is closed with respect to multiplication, i.e. if for all \( a = a', b = b' \in A_{[1]} \),

\[ a \triangleleft b \in p \implies a \in p \text{ or } b \in p \quad (10.1.3) \]

Now a maximal symmetric ideal \( p \in \text{Max}'(A) \) is a symmetric prime. We let \( \text{spec}'(A) \) denote the set of symmetric primes:

\[ \text{Max}'(A) \subseteq \text{spec}'(A) \subseteq \text{il}'(A) \]

(10.1.4)

Note that for a prime \( p \in \text{spec}(A) \), the ideal \( A \cdot p^+ \), generated by the symmetric elements of \( p \), is a symmetric prime, and we have a canonical map

\[ \text{spec}(A) \to \text{spec}'(A), \quad p \mapsto A \cdot p^+ \]

(10.1.5)

**10.2 The Zariski topology**

**Definition 10.2.1**

The closed sets in \( \text{spec}(A) \) are the set of the form

\[ V(a) = \{ p \in \text{spec}(A), p \supseteq a \} \]

(10.2.1)

with \( a \subseteq A_{[1]} \), which we may take to be an ideal \( a \in \text{il}(A) \).

We have

\[ (i) \quad V(\sum a_i) = \bigcap_i V(a_i), \]
\[ (ii) \quad V(a \cdot a') = V(a) \cup V(a'), \]
\[ (iii) \quad V(0) = \text{spec}(A), \quad V(1) = \emptyset \]

(10.2.2)

This shows the sets \( V(a) \) define a topology on \( \text{spec}(A) \), the **Zariski topology**.

The closed sets in \( \text{spec}'(A) \) are similarly given by

\[ V'(a) = \{ p \in \text{spec}'(A), p \supseteq a \} \]

(10.2.3)
with \( \mathfrak{a} \in \pm(A) \); these satisfy \( \text{[10.2.2]} \), and define the Zariski topology on \( \text{spec}^t(A) \).

For a subset \( \mathcal{C} \subseteq \text{spec}(A) \), we have the ideal,

\[
I(\mathcal{C}) = \bigcap_{p \in \mathcal{C}} p
\]

(10.2.4)

For a subset \( \mathcal{C} \subseteq \text{spec}^t(A) \), we have the symmetric ideal \( I^t(\mathcal{C}) \) generated by \( \bigcup_{p \in \mathcal{C}} p^+ \).

We have Galois correspondences,

\[
\begin{align*}
\text{il}(A) & \quad \xrightarrow{V} \quad \{ \mathcal{C} \subseteq \text{spec}(A) \} \quad \text{and} \quad \xrightarrow{I^t} \quad \{ \mathcal{C} \subseteq \text{spec}^t(A) \} \\
\end{align*}
\]

(10.2.5)

The maps \( V, I \) (resp. \( V^t, I^t \)), are monotone

\[
\begin{align*}
\mathfrak{a}_1 & \subseteq \mathfrak{a}_2 \Rightarrow V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \\
C_1 & \subseteq C_2 \Rightarrow I(C_1) \supseteq I(C_2)
\end{align*}
\]

(10.2.6)

and we have

\[
\mathfrak{a} \subseteq IV(\mathfrak{a}) , \quad C \subseteq VI(C)
\]

(10.2.7)

It follows that we have

\[
V(\mathfrak{a}) = VIV(\mathfrak{a}) \quad \text{and} \quad I(C) = IVI(C)
\]

(10.2.8)

and the maps \( V, I \) induce inverse bijections between radical ideals and closed subsets of \( \text{spec}(A) \).

\[
\begin{align*}
\{ \mathfrak{a} \in \text{il}(A), \mathfrak{a} = IV(\mathfrak{a}) \} = \\
= \{ I(C), C \subseteq \text{spec}(A) \} \xrightarrow{\sim} \{ C \subseteq \text{spec}(A), C = VI(C) \} = \\
= \{ V(\mathfrak{a}), \mathfrak{a} \in il(A) \}
\end{align*}
\]

(10.2.9)

Similarly, we have a bijection between the radical symmetric ideals \( \mathfrak{a} = I^tV^t(\mathfrak{a}) \) and the closed subset of \( \text{spec}^t(A) \).

**Lemma 10.2.2**

For \( \mathfrak{a} \in il(A) \), we have

\[
IV(\mathfrak{a}) = \{ a \in A_{[1]}, a^n \in \mathfrak{a} \text{ for some } n > 0 \} \overset{\text{def}}{=} \sqrt[\text{def}]{\mathfrak{a}}
\]

(10.2.10)
Proof. If \( a \in \sqrt{\mathfrak{a}} \), say \( a^n \in \mathfrak{a} \), then for all \( p \supseteq \mathfrak{a}, a \in p \), and so \( \sqrt{\mathfrak{a}} \subseteq \bigcap_{a \in p} p = IV(\mathfrak{a}) \).

Assume \( a \notin \sqrt{\mathfrak{a}} \), so \( a^n \notin \mathfrak{a} \) for all \( n \). An application of Zorn’s lemma gives that there exists a maximal element \( p \) in the set
\[
\{ b \in i^l(A), b \supseteq \mathfrak{a}, a^n \notin b \text{ for all } n \} \tag{10.2.11}
\]
We claim \( p \) is prime. If \( x, x' \in A_{[1]} \setminus p \), then the ideals \((p, x)_A, (p, x')_A \) properly contain \( p \), and by maximality of \( p \) in the set \([10.2.11]\), we must have \( a^n \in (p, x)_A, a^n' \in (p, x')_A \), for some \( n, n' \). We get
\[
a^{n+n'} = a^n \cup a^n' = ((b \cup c) \setminus d) \cup ((b' \cup c') \setminus d') = (b \cup \tilde{b} \cup \tilde{c} \cup \tilde{c}') \setminus (d' \cup \tilde{d})
\]
with \( b, d \in A_X, b', d' \in A_{X'}, c \in (p \cup \{ x \})_X, c' \in (p \cup \{ x' \})_{X'} \).

But \( \tilde{c} \cup \tilde{c}' \in ((p \cup \{ x \}) \cup (p \cup \{ x' \}))_{X \cup X'} \subseteq (p \cup \{ x \cup x' \})_{X \cup X} \), and since \( a^{n+n'} \notin p \), we must have \( x \cup x' \notin p \); and \( p \) is indeed prime. Now \( p \supseteq \mathfrak{a} \), and \( a \notin p \), so \( a \notin \bigcap_{\mathfrak{a} \subseteq p} p = IV(\mathfrak{a}) \).

Similarly, for a symmetric ideal \( \mathfrak{a} \in i^l(A) \), we have \( I^+ V^+(\mathfrak{a}) = \sqrt{\mathfrak{a}^+} \), where \( \sqrt{\mathfrak{a}^+} \) is the ideal of \( A \) generated by the set \( \{ a = a^n \in A_{[1]} \mid a^n \in \mathfrak{a} \text{ for some } n > 0 \} \).

Lemma 10.2.3

For a subset \( C \subseteq \text{spec}(A) \), \( VI(C) = \overline{C} \) the closure of \( C \).

Proof. We have \( C \subseteq VI(C) \), and \( VI(C) \) is closed. If \( C \subseteq V(\mathfrak{a}) \), where we may assume \( \mathfrak{a} = \sqrt{\mathfrak{a}} \), then \( VI(C) \subseteq V IV(\mathfrak{a}) = V(\mathfrak{a}) \), and so
\[
VI(C) = \bigcap_{C \subseteq V(\mathfrak{a})} V(\mathfrak{a}) = \overline{C} \tag{10.2.12}
\]

Similarly, for \( C \subseteq \text{spec}^l(A) \), \( V^+ I^+(C) = \overline{C} \) the closure of \( C \). We can redefine \([10.2.9]\),
\[
\{ a \in i^l(A), a = \sqrt{\mathfrak{a}^+} \} \overset{\sim}{\longleftrightarrow} \{ C \subseteq \text{spec}^l(A), C = \overline{C} \} \tag{10.2.13}
\]

10.3 Basic open sets

A basis for the open sets of \( \text{spec}(A) \) is given by the basic open sets, these are defined for \( a \in A_{[1]} \) by
\[
D_a = \text{spec}(A) \setminus V(a) = \{ p \in \text{spec}(A), a \notin p \} \tag{10.3.1}
\]
A basis for the open sets of $\text{spec}^c(A)$ is given by

$$D_a^+ = \text{spec}^c(A) \setminus V^+(a)$$  \hspace{1cm} (10.3.2)

with symmetric $a = a^t \in A^+_1$.

We have,

$$D_{a_1} \cap D_{a_2} = D_{a_1 \cdot a_2}^+ D_{a_1}^+ \cap D_{a_2}^+ = D_{a_1, a_2}^+$$

$$D_1 = \text{spec}(A), \quad D_0 = \emptyset \quad D_1^+ = \text{spec}^c(A), \quad D_0^+ = \emptyset$$  \hspace{1cm} (10.3.3)

That every open set is the union of basic open sets, is shown by

$$\text{spec}(A) \setminus V(a) = \bigcup_{a \in A} D_a$$  \hspace{1cm} (10.3.4)

and

$$\text{spec}^c(A) \setminus V^+(a) = \bigcup_{a \in A^+} D_a^+ \text{, the union over } a^+ = a \cap A^+_1.$$  \hspace{1cm} (10.3.5)

Note that we have,

$$D_a = \text{spec}(A) \iff a \lhd A_1 = \{a\}_A = 1$$  \hspace{1cm} (10.3.6)

$$\iff \text{there exists } a \text{ (unique) } a^{-1} \in A_1, a \lhd a^{-1} = 1$$

We say that such $a$ is invertible, and we let $A^*$ denote the set of invertible elements. Note that $A^*$ is an abelian group (with a non-trivial involution for $A$ non self-adjoint), and $A \leftrightarrow A^*$ is a functor $\mathcal{GR}_C \to \mathcal{Ab}$ (= abelian groups).

Similarly, for $a = a^t$ symmetric, $D_a^+ = \text{spec}^c(A)$ if and only if $a$ is invertible.

Note that we have,

$$D_a = \emptyset \iff a \in \bigcap_{p \in \text{spec}(A)} p = \sqrt{0}$$  \hspace{1cm} (10.3.7)

$$\iff \text{there exists } n > 0 \text{ with } a^n = 0$$

We say that such $a$ is nilpotent. Similarly, for $a = a^t$, $D_a^+ = \emptyset$ if and only if $a$ is nilpotent.

**Lemma 10.3.1**

Let $a = \sqrt{a} \in \text{il}(A)$ be a radical ideal. Then

$$V(a) \text{ is irreducible } \iff a \text{ is prime}$$  \hspace{1cm} (10.3.8)
Proof. \((\Leftarrow\): If \(a\) is prime, \(V(a) = VI \{a\} = \overline{\{a\}}\) is the closure of a point, hence irreducible.

\((\Rightarrow\): For any \(a \in A_1\), we have
\[
V(a) \cap D_a \neq \emptyset \iff \exists p \in spec(A), p \supseteq a, p \neq a
\]
\[
\iff a \notin \bigcap_{p \subseteq a} p = \sqrt{a} = a
\]
Hence for any basic open sets \(D_a, D_b, a, b \in A_1\), we have
\[
V(a) \cap D_a \neq \emptyset \quad \text{and} \quad V(a) \cap D_b \neq \emptyset \iff a \neq a \quad \text{and} \quad b \neq a
\]
If \(V(a)\) is irreducible this implies
\[
\emptyset \neq V(a) \cap D_a \cap D_b = V(a) \cap D_{a \cap b} \iff a < b \neq a
\]

Similarly, for a radical symmetric ideal \(a = \sqrt{a^+}\), the set \(V^+(a)\) is irreducible if and only if \(a\) is a symmetric prime.

Thus the bijection \((10.2.13)\) induces a bijection
\[
spec(A) \overset{\sim}{\longleftarrow} \{ C \subseteq spec(A), C = \overline{C} \text{ closed and irreducible} \}
\]
\[
p \mapsto V(p) = \{ p \} \quad (10.3.9)
\]
and
\[
spec^f(A) \overset{\sim}{\longleftarrow} \{ C \subseteq spec^f(A), C = \overline{C} \text{ closed and irreducible} \} \quad (10.3.10)
\]
and the spaces \(spec(A)\) and \(spec^f(A)\) are Sober spaces (or Zariski spaces): every closed irreducible subset has a unique generic point.

**Proposition 10.3.2**

For \(a \in A_1\), (Respectively, \(a = a' \in A^+_1\)) the basic open set \(D_a\) (resp. \(D^+_a\)) is compact.

In particular, \(D_1 = spec(A)\) and \(D^+_1 = spec^f(A)\) are compact.

**Proof.** We have to show that in every covering of \(D_a\) by basic open sets \(D_{g_i}\), there is always a finite subcovering. We have
\[
D_a \subseteq \bigcup_i D_{g_i} \iff V(a) \supseteq \bigcap_i V(g_i) = V(\sum_i g_i \triangleleft A_1) \quad (10.3.9)
\]
\[
\iff \sqrt{\{a\}A} = IV(a) \subseteq IV(\sum_i g_i \triangleleft A_1) = \sqrt{\sum_i g_i \triangleleft A_1}
\]
\[
\iff \text{for some } n, a^n \in \sum_i g_i \triangleleft A_1
\]
\[
\iff \text{for some } n, X \in \mathbb{F}, b, d \in A_X, a^n = (b \triangleleft c) \parallel d,
\]
with \(c = (c^{(x)}) \in (\{g_i\})^X\) \quad (10.3.10)
Thus $c(x) = g_{i(x)}$, and going backwards in the above equivalences we get $D_a \subseteq \bigcup_{x \in X} D_{g(x)}$, a finite subcovering.

The canonical map $\pi_A : \text{spec}(A) \to \text{spec}^t(A)$, $\pi_A(p) = A \cdot p^+$, is continuous:

$$\pi_A^{-1}(V^+(a)) = V(a), \; \pi_A^{-1}(D_a^+) = D_a \quad (10.3.11)$$

### 10.4 Functoriality

For a homomorphism $\varphi \in \mathcal{GR}_{C}(A, B)$, the pull-back of a (symmetric) prime is a (symmetric) prime, and we have maps

$$\varphi^* = \text{spec}(\varphi) : \text{spec}(B) \to \text{spec}(A)$$

$$q \mapsto \varphi^*(q) = \varphi_{[1]}^{-1}(q) \quad (10.4.1)$$

and

$$\varphi^* = \text{spec}^t(\varphi) : \text{spec}^t(B) \to \text{spec}^t(A)$$

$$q \mapsto \varphi^*(q) = A \cdot (\varphi^{-1}(q) \cap A^+) \quad (10.4.2)$$

The inverse image under $\varphi^*$ of a closed set is closed, we have

$$\varphi_*^{-1}(V_A(a)) = \{ q \in \text{spec}(B), \varphi_{[1]}^{-1}(q) \supseteq a \} =$$

$$= \{ q \in \text{spec}(B), q \supseteq \varphi_{[1]}(a) \} = V_B(\varphi_{[1]}(a)) \quad (10.4.3)$$

and similarly, $\varphi_*^{-1}(V_A^+(a)) = V_B^+(\varphi_*(a))$ with $\varphi_*(a) = B \cdot \varphi(a^+)$. Also the inverse image under $\varphi^*$ of a basic open set is a basic open set, we have

$$\varphi_*^{-1}(D_a) = \{ q \in \text{spec}(B), \varphi_{[1]}^{-1}(q) \nsubseteq a \} =$$

$$= \{ q \in \text{spec}(B), \varphi_{[1]}(a) \nsubseteq q \} = D_B(\varphi_{[1]}(a)) \quad (10.4.4)$$

and similarly, $\varphi_*^{-1}(D_a^+) = D_B^+(\varphi_{[1]}(a))$ for $a = a^t$.

Thus the maps $\varphi^* = \text{spec}(\varphi)$ and $\text{spec}^t(\varphi)$ are continuous, and we see that $\text{spec}^t$ are contravariant functors from $\mathcal{GR}_{C}$ to the category $\text{Top}$, whose objects are (compact, sober) topological spaces, and continuous maps,

$$\text{spec}, \; \text{spec}^t : (\mathcal{GR}_{C})^{op} \to \text{Top} \quad (10.4.5)$$

**Lemma 10.4.1**

For $\varphi \in \mathcal{GR}_{C}(A, B)$, and for $b \in \text{il}(B)$, we have

$$V_A(\varphi_{[1]}^{-1}(b)) = \overline{\varphi^*(V_B(b))} \quad (10.4.6)$$
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Proof. We may assume without loss of generality that $b = \sqrt{b}$ is radical (noting that $\sqrt[\varphi_{[1]}^{-1}(b)]{\varphi_{[1]}^{-1}(\sqrt{b})}$).

Put \(a = I_{\varphi^*}(V(b))\), so that \(V(a) = \varphi^*(V(b))\) by lemma 10.2.3.

We have for any \(a \in A_{[1]}\),

\[
\begin{align*}
a \in a & \iff a \in p, \text{ for every prime } p \in \varphi^*(V(b)) \\
& \iff a \in \varphi^*(q) = \varphi_{[1]}^{-1}(q), \text{ for every prime } q \in V(b) \\
& \iff \varphi_{[1]}(a) \in \bigcap_{q \in V(b)} q = \sqrt{b} = b \quad (10.4.7) \\
& \iff a \in \varphi_{[1]}^{-1}(b)
\end{align*}
\]

Thus \(a = \varphi_{[1]}^{-1}(b)\), and the lemma is proved. \(\square\)
Chapter 11

Localization and sheaves

11.1 Localization

For $A \in \mathcal{GR}_C$, and $s \in A_{[1]}$, in the generalized ring $A[1/s]$ obtained from $A$ by adding an inverse to $s$, we have also an inverse to $s^t$, $(s^t)^{-1} = (s^{-1})^t$, and therefore also an inverse to the symmetric element $s \triangleleft s^t$, so $A[1/s] = A[1/s \triangleleft s^t]$.

We shall therefore localize only with respect to symmetric elements!

For $A \in \mathcal{GR}_C$, a subset $S \subseteq A_{[1]}$ is called multiplicative if

$$1 \in S, \text{ and } s \triangleleft S \subseteq S. \quad (11.1.1)$$

For such $S \subseteq A_{[1]}^+$, and for $X \in \mathcal{F}$, we let $(S^{-1}A)_X = (A_X \times S)\approx$ denote the equivalence classes of $A_X \times S$ with respect to the equivalence relation defined by

$$(a_1, s_1) \approx (a_2, s_2) \text{ if and only if } s_1 \triangleleft s_2 \triangleleft a_1 = s_1 \triangleleft s_2 \triangleleft a_2 \text{ for some } s \in S \quad (11.1.2)$$

We write $a/s$ for the equivalence class $(a, s)/\approx$. Note that by taking "common denominator" we can write any element $a = (a^{(y)}/s_y) \in (S^{-1}A)_f = \prod_{y \in Y} (S^{-1}A)_{f^{-1}(y)}$, in the form

$$a = (\overline{a^{(y)}/s}), \quad \left( \text{take } s = \prod s_y, \overline{a^{(y)}} = \left( \prod_{y' \neq y} s_{y'} \right) \triangleleft a^{(y)} \right) \quad (11.1.3)$$

For $f \in \mathcal{Set}_*(X, Y)$, $g \in \mathcal{Set}_*(Y, Z)$, we have well-defined operations of multiplication and contraction, independent of the choice of representatives,

$$\triangleleft : (S^{-1}A)_g \times (S^{-1}A)_f \longrightarrow (S^{-1}A)_{g \circ f}, \quad (a/s_1) \triangleleft (b/s_2) = (a \triangleleft b)/(s_1 \triangleleft s_2) \quad (11.1.4)$$

$$\parallel : (S^{-1}A)_{g \circ f} \times (S^{-1}A)_f \longrightarrow (S^{-1}A)_g, \quad (a/s_1) \parallel (b/s_2) = (a \parallel b)/(s_1 \triangleleft s_2) \quad (11.1.5)$$
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It is straightforward to check that these operations satisfy the axioms of a commutative generalized ring. The canonical homomorphism

\[ \phi : A \rightarrow S^{-1}A, \quad \phi(a) = a/1 \]  

(11.1.6)
satisfies the universal property

\[ \mathcal{GR}_C(S^{-1}A, B) = \{ \varphi \in \mathcal{GR}_C(A, B), \varphi(S) \subseteq B^* \} \]

(11.1.7)

with \( \varphi(s)^{-1} \varphi(a) = \tilde{\varphi}(a/s) \leftrightarrow \varphi \).

Example 11.1.1

For \( s \in A^+_{[1]} \), take \( S = \{ s^n, n \geq 0 \} \). We write \( A_s \) for \( S^{-1}A \), and \( \phi_s = \mathcal{GR}_C(A_s, S) \) satisfy

\[ \mathcal{GR}_C(A_s, B) = \{ \varphi \in \mathcal{GR}_C(A, B), \varphi(s) \in B^* \} \]  

(11.1.8)

Example 11.1.2

For \( p \in \text{spec}^d(A) \), take \( S_p = A^+_{[1]} \setminus p \). We write \( A_p \) for \( S^{-1}A \), and \( \phi_p = \mathcal{GR}_C(A_p, A_p) \) satisfy

\[ \mathcal{GR}_C(A_p, B) = \{ \varphi \in \mathcal{GR}_C(A, B), \varphi(A^+_{[1]}) \setminus p \subseteq B^* \} \]  

(11.1.9)

11.2 Localization and ideals

For an ideal \( \mathfrak{a} \in il(A) \), we let

\[ S^{-1}\mathfrak{a} = \{ a/s \in (S^{-1}A)_{[1]}, s \in S, a \in \mathfrak{a} \} \]  

(11.2.1)

By using common denominator, we see that \( S^{-1}\mathfrak{a} \) is an ideal of \( S^{-1}A \), \( S^{-1}\mathfrak{a} \in il(S^{-1}A) \):

For \( b/s_1, d/s_2 \in (S^{-1}A)_X \), and for \( a_x/s_x \in S^{-1}\mathfrak{a}, x \in X \), we have

\[ (b/s_1 \vartriangleleft (a_x/s_x)) \parallel (d/s_2) = ((b \vartriangleleft (s'_x \triangledown a_x)) \parallel d)/(s_1 \triangledown s'_x \cap \prod_{x \in X} s_x) \]

with \( s'_x = \prod_{x' \neq x} s_{x'} \), and this is in \( S^{-1}\mathfrak{a} \) since \( \mathfrak{a} \) is an ideal.

If \( \mathfrak{a} = A \cdot \mathfrak{a}^+ \) is symmetric, \( S^{-1}\mathfrak{a} \) is symmetric.

We have, therefore, the Galois correspondence

\[ il^{(t)}(A) \xrightarrow{S^{-1}} il^{(t)}(S^{-1}A) \xleftarrow{\phi^*} \]  

(11.2.2)
For \( b \in il(S^{-1}A) \), we have
\[
S^{-1}\phi^*b = b
\] (11.2.3)
Indeed, for an element \( a/s \in b \), we have \( a/1 \in b \), or \( a \in \phi^*b \), and \( a/s \in S^{-1}\phi^*b \); hence \( b \subseteq S^{-1}\phi^*b \), and the reverse inclusion is clear.
We have immediately from the definitions, for \( a \in il(A) \),
\[
\phi^*S^{-1}a = \{ a \in A_{[1]} \text{, there exists } s \in S \text{ with } s \triangleleft a \} = \bigcup_{s \in S}(a : s)
\] (11.2.4)
In particular,
\[
S^{-1}a = (1) \Leftrightarrow a \cap S \neq \emptyset
\] (11.2.5)
We say that \( a \in il^{(t)}(A) \) is \( S \)-saturated if \( \phi^*S^{-1}a = a \), that is if
\[
\text{for all } s \in S, a \in A_{[1]}^+ : s \triangleleft a \Rightarrow a \in a
\] (11.2.6)
We get that \( S^{-1} \) and \( \phi^* \) induce inverse bijections,
\[
\{ a \in il^{(t)}(A), \quad a \text{ is } S \text{-saturated} \} \overset{\sim}{\longrightarrow} il^{(t)}(S^{-1}A)
\] (11.2.7)
For an \( S \)-saturated (symmetric) ideal \( a \in il^{(t)}(A) \), let \( \pi_a : A \twoheadrightarrow A/a = A/E(a) \) be the canonical homomorphism, and let \( S = \pi_a(S) \subseteq (A/a)_{[1]}^+ \), then we have canonical isomorphism
\[
S^{-1}(A/a) \cong S^{-1}A/S^{-1}a
\] (11.2.8)
Note that for a (symmetric) prime \( p \in spec^{(t)}(A) \), \( p \) is \( S \)-saturated if and only if \( p \cap S = \emptyset \), and in this case \( S^{-1}p \) is a (symmetric) prime, \( S^{-1}p \in spec^{(t)}(S^{-1}A) \).
Note that for a (symmetric) prime \( q \in spec^{(t)}(S^{-1}A) \), \( \phi^*(q) \) is always an \( S \)-saturated (symmetric) prime. We get the bijections:
\[
\{ p \in spec^{(t)}(A), p \cap S = \emptyset \} \overset{\sim}{\longrightarrow} spec^{(t)}(S^{-1}A)
\] (11.2.9)
These are homeomorphisms for the Zariski topologies.

For \( s \in A_{[1]}^+ \), the homeomorphism \( \phi^*_s : spec^t(A_s) \overset{\sim}{\longrightarrow} D^t_s \subseteq spec^t(A) \) (11.2.10)

For a symmetric prime \( p \in spec^t(A) \), the homeomorphism \( \phi^*_p : spec^t(A_p) \overset{\sim}{\longrightarrow} \{ q \in spec^t(A), q \subseteq p \} \) (11.2.11)
The generalized ring \( A_p \) is a local-generalized-ring in the sense that it has a unique maximal symmetric ideal \( m_p = S^{-1}_p p \). The residue field at \( p \) is defined by
\[
\mathbb{F}_p = A_p/m_p = A_p/E(m_p).
\] (11.2.12)
We have the canonical homomorphism $\pi_p : A \to A/p$, and putting $\mathcal{S}_p = \pi_p(S_p)$, we have (11.2.8)

$$F_p = \mathcal{S}_p^{-1}(A/p) \quad (11.2.13)$$

The square diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi_p} & A_p \\
\downarrow{\pi_p} & & \downarrow{\pi_{m_p}} \\
A/p & \xrightarrow{\phi_0} & F_p
\end{array}
$$

is cartesian,

$$F_p = (A/p) \widehat{\otimes} A_p \quad (11.2.15)$$

$$\mathcal{G}RC(F_p, B) = \left\{ \varphi \in \mathcal{G}RC(A, B), \varphi(p) \equiv 0, \varphi(A^+_p \setminus p) \subseteq B^* \right\}$$

For a homomorphism of generalized rings $\varphi \in \mathcal{G}RC(A, B)$, and for $q \in spec^c(B)$ with $p = \varphi^*(q) \in spec^c(A)$, the square diagram (11.2.14) is functorial, and we have a commutative cube diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi_p} & A_p \\
\downarrow{\varphi} & & \downarrow{\varphi_p} \\
B & \xrightarrow{\phi_q} & B_q \\
\downarrow{A/p} & & \downarrow{F_p} \\
B/q & \xrightarrow{\phi_0} & F_q
\end{array}
$$

(11.2.16)

Note that the homomorphism $\varphi_p \in \mathcal{G}RC(A_p, B_q)$ is a local-homomorphism in the sense that

$$m_p = \varphi_p^*(m_q), \text{ or equivalently } \varphi_p(m_p) \subseteq m_q \quad (11.2.17)$$

**Definition 11.2.1**

We let $LGR$ denote the subcategory of $GRC$ with objects the local generalized rings, and with maps

$$LGR(A, B) = \{ \varphi \in \mathcal{G}RC(A, B), \varphi^*(m_B) = m_A \} \quad (11.2.18)$$
11.3 The structure sheaf $\mathcal{O}_A$

Definition 11.3.1

For $A \in \mathcal{GR}_C$, $U \subseteq \text{spec}^r(A)$ open, $X \in \mathcal{F}$, we denote by $\mathcal{O}_A(U)_X$ the set of sections

$$f : U \to \bigcap_{p \in U} (A_p)_X, \quad f(p) \in (A_p)_X$$

such that $f$ is locally a fraction:

for all $p \in U$, there exists open $U_p \subseteq U$, $p \in U_p$,

and there exist $a \in A_X$, $s \in A_{[1]}^{+} \setminus \bigcup_{q \in U_p} q$,

such that for all $q \in U_p$:

$$f(q) \equiv a/s \in (A_q)_X. \quad (11.3.1)$$

Note that $\mathcal{O}_A(U)$ is a commutative generalized ring, and for $U' \subseteq U$ restriction gives a homomorphism of generalized rings

$$\mathcal{O}_A(U) \to \mathcal{O}_A(U'), \quad f \mapsto f|_{U'}$$

Thus $\mathcal{O}_A$ is a pre-sheaf of generalized rings over $\text{spec}^r(A)$, and by the local nature of the condition (11.3.1) it is clear that it is a sheaf of generalized rings, i.e. for $X \in \mathcal{F}$, $U \mapsto \mathcal{O}_A(U)_X$ is a sheaf of sets. It is also clear that the stalks are given by

$$\mathcal{O}_{A,p} = \lim_{p \in U} \mathcal{O}_A(U) \xrightarrow{\sim} A_p \quad (11.3.2)$$

$$f(\sim) \mapsto f(p)$$

Theorem 11.3.2

For $s \in A_{[1]}^{+}$, we have a canonical isomorphism

$$\Psi : A_s \xrightarrow{\sim} \mathcal{O}_A(D^+_s), \quad \Psi(a/s^n) = \{f(p) \equiv a/s^n\} \quad (11.3.3)$$

In particular for $s = 1$,

$$A \xrightarrow{\sim} \mathcal{O}_A(\text{spec}^r(A))$$

Proof. The map $\Psi$ which takes $a/s^n \in A_s$ to the constant section $f$ with $f(p) \equiv a/s^n$ for all $p \in D^+_s$, is clearly well-defined, and is a homomorphism of generalized rings.

$\Psi$ is injective: Assume $\Psi(a_1/s^{n_1}) = \Psi(a_2/s^{n_2})$, and let

$$a = \text{ann}_A^r(a^{n_2} \lhd a_1, s^{n_1} \lhd a_2) \in \text{il}^r(A), \text{ cf. } (9.5.11, 9.5.12).$$

We have,

$$a_1/s^{n_1} = a_2/s^{n_2} \text{ in } A_p \text{ for all } p \in D^+_s$$
\[ s_{p} \preceq s_{n_{2}} \preceq a_{1} = s_{p} \preceq s_{n_{1}} \preceq a_{2} \text{ with } s_{p} \in A_{[1]} \setminus p \text{ for } p \in D_{s}^{+}\]
\[ \Rightarrow a \not\subset p \text{ for } p \in D_{s}^{+}\]
\[ \Rightarrow V^{+}(a) \cap D_{s}^{+} = \emptyset\]
\[ \Rightarrow V^{+}(a) \subseteq V^{+}(s)\]
\[ \Rightarrow s \in I^{+}V^{+}(a) = \sqrt{a}^{+}\]
\[ \Rightarrow s^{n} \in a \text{ for some } n\]
\[ \Rightarrow s^{n + n_{2}} \preceq a_{1} = s^{n + n_{1}} \preceq a_{2}\]
\[ \Rightarrow a_{1}/s^{n_{1}} = a_{2}/s^{n_{2}} \text{ in } A_{s}\]

\[ \Psi \text{ is surjective: Fix } f \in \mathcal{O}_{A}(D_{A}^{+})_{X}. \text{ Since } D_{A}^{+} \text{ is compact (Proposition 10.3.2), we can cover } D_{s}^{+} \text{ by a finite collection of basic open sets, } D_{s}^{+} = D_{g_{1}}^{+} \cup \ldots \cup D_{g_{N}}^{+}, g_{i}^{n} = g_{i}, \text{ such that on } D_{g_{i}}, \text{ the section } f \text{ is constant,}\]
\[ f(p) = a_{i}/s_{i} \text{ for } p \in D_{g_{i}},\]

We have \[ V^{+}(s_{i}) \subseteq V^{+}(g_{i}), \text{ hence } g_{i} \in I^{+}V^{+}(s_{i}) = \sqrt{s_{i}}^{+}, \text{ hence for some } n_{i}, \text{ and some } c_{i} \in A_{[1]}, g_{i}^{n_{i}} = c_{i} < s_{i}. \text{ Thus our section } f \text{ is given on } D_{g_{i}}^{+} \text{ by } a_{i}/s_{i} = c_{i} \preceq a_{i}/g_{i}^{n_{i}}. \text{ Noting that } D_{g_{i}}^{+} = D_{g_{i}}^{+}, \text{ we may replace } g_{i}^{n_{i}} \text{ by } g_{i}, \text{ and replace } c_{i} \preceq a_{i} \text{ by } a_{i}, \text{ and we have}\]
\[ f(p) = a_{i}/g_{i} \text{ for } p \in D_{g_{i}},\]

On the set \[ D_{g_{i}}^{+} = D_{g_{i}}^{+} \cap D_{g_{j}}, i \neq j, \text{ our section } f \text{ is given by both } a_{i}/g_{i} \text{ and } a_{j}/g_{j}. \text{ By the injectivity of } \Psi, \text{ we have}\]
\[ a_{i}/g_{i} = a_{j}/g_{j} \text{ in } A_{g_{i} < g_{j}}\]

Thus for some \( n \) we have
\[ (g_{i} \preceq g_{j})^{n} \preceq g_{j} < a_{i} = (g_{i} \preceq g_{j})^{n} \preceq g_{i} \preceq a_{j}\]

By finiteness we may assume one \( n \) works for all \( i, j \leq N \).

Replacing \( g_{i}^{n} \circ a_{i} \) by \( a_{i} \), and replacing \( g_{i}^{n+1} \) by \( g_{i} \), we may assume \( f \equiv a_{i}/g_{i} \) on \( D_{g_{i}}^{+}, \) and
\[ g_{j} \preceq a_{i} = g_{i} \preceq a_{j} \text{ for all } i, j\]

We have \[ D_{s}^{+} \subseteq \bigcup D_{g_{i}}^{+}, \] hence by (10.3.11) we have
\[ s^{M} = (b \preceq c) \parallel d\]

with \( b, d \in A_{Y}, c = (c^{(y)}) \in (A_{[1]})^{Y} \) with \( c^{(y)} = g_{i(y)} \),
\[ i(y) : Y \rightarrow \{1, \ldots, N\}\]

. Define \( a \in A_{X} \) by
\[ a = (b \preceq c) \parallel d, \text{ with } \hat{d} \in A_{\pi_{X}} = (A_{Y})^{X}, \hat{d}^{(x)} \equiv d,\]
\[ e \in A_{\pi_{X}} = (A_{X})^{Y}, e^{(y)} = a_{i(y)}\]
We have for $j = 1, \ldots, N$

$$g_j \triangleleft a = g_j \triangleleft ((b \triangleleft c) \parallel \tilde{d}) = (b \triangleleft g_j \triangleleft c) \parallel \tilde{d} = (b \triangleleft (g_j \triangleleft a_{i(y)})) \parallel \tilde{d}$$

$$= (b \triangleleft (g_i(y) \triangleleft a_j)) \parallel \tilde{d} = (b \triangleleft c \triangleleft \tilde{a}_j) \parallel \tilde{d} = ((b \triangleleft c) \parallel d) \triangleleft a_j = s^M \triangleleft a_j$$

(11.3.8)

Thus we have in $A_s$, $a_j/g_j = a/s^M$ for all $j$, and our section $f$ is constant $f = \Psi(a/s^M)$, and $\Psi$ is surjective. \qed
Chapter 12

Schemes

12.1 Locally generalized ringed spaces

Definition 12.1.1

For a topological space $\mathcal{X}$, we let $\mathcal{G}R_{C}/\mathcal{X}$ denote the category of sheaves of generalized rings over $\mathcal{X}$. Its objects are pre-sheaves $\mathcal{O}$ of commutative generalized rings, i.e. functors $U \rightarrow \mathcal{O}(U) : \mathcal{C}_{\mathcal{X}} \rightarrow \mathcal{G}R_{C}$, (with $\mathcal{C}_{\mathcal{X}}$ the category of open sets of $\mathcal{X}$, with $\mathcal{C}_{\mathcal{X}}(U, U') = \{ j_{U, U'}^{\mathcal{X}} \}$ for $U' \subseteq U$, otherwise $\mathcal{C}_{\mathcal{X}}(U, U') = \emptyset$), such that for all $X \in \mathcal{F}$, $U \rightarrow \mathcal{O}(U)$ is a sheaf. The maps $\mathcal{G}R_{C}/\mathcal{X}(\mathcal{O}, \mathcal{O}')$ are natural transformations of functors $\varphi = \{ \varphi(U) \}, \varphi(U) \in \mathcal{G}R_{C}(\mathcal{O}(U), \mathcal{O}'(U))$.

Definition 12.1.2

We denote by $\mathcal{G}R\mathcal{S}$ the category of (commutative) generalized ringed spaces. Its objects are pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, with $\mathcal{X} \in Top$, and $\mathcal{O}_{\mathcal{X}} \in \mathcal{G}R_{C}/\mathcal{X}$. The maps $f \in \mathcal{G}R\mathcal{S}(\mathcal{X}, \mathcal{Y})$ are pairs of a continuous function $f \in \text{Top}(\mathcal{X}, \mathcal{Y})$, and a map of sheaves of generalized rings over $\mathcal{Y}$, $f^{\sharp} \in \mathcal{G}R_{C}/\mathcal{Y}(\mathcal{O}_{\mathcal{Y}}, f_{\ast}\mathcal{O}_{\mathcal{X}})$; explicitly, for all open subsets $U \subseteq \mathcal{Y}$, we have a homomorphism of generalized rings

$$f^{\sharp}_{U} = \{ f^{\sharp}_{U, X} \} \in \mathcal{G}R_{C}(\mathcal{O}_{\mathcal{Y}}(U), \mathcal{O}_{\mathcal{X}}(f^{-1}U))$$

(12.1.1)

and these homomorphisms are compatible with restrictions: for $U' \subseteq U \subseteq \mathcal{Y}$ open, and for $a \in \mathcal{O}_{\mathcal{Y}}(U)_{X}$, we have $f^{\sharp}_{U, X}(a)|_{f^{-1}(U')} = f^{\sharp}_{U', X}(a|_{U'})$ in $\mathcal{O}_{\mathcal{X}}(f^{-1}U')_{X}$.

Remark 12.1.3

For a continuous map $f \in \text{Top}(\mathcal{X}, \mathcal{Y})$, we have a pair of adjoint functors

$$\mathcal{G}R_{C}/\mathcal{X} \xrightarrow{f^{\ast}} \mathcal{G}R_{C}/\mathcal{Y} \xleftarrow{f_{\ast}}$$

(12.1.2)
For sheaves of generalized rings $\mathcal{O}_X \in \mathcal{G}\mathcal{R}\mathcal{C}/\mathcal{X}$, $\mathcal{O}_Y \in \mathcal{G}\mathcal{R}\mathcal{C}/\mathcal{Y}$, we have
\[
f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}U), \quad U \subseteq \mathcal{Y} \text{ open}; \tag{12.1.3}
\]
\[
f^*\mathcal{O}_Y(U)_X = \text{ sheaf associated to the pre-sheaf}
U \mapsto \lim_{V \subseteq \mathcal{Y} \text{ open}} \mathcal{O}_Y(V)_X; \tag{12.1.4}
\]
and we have adjunction,
\[
\mathcal{G}\mathcal{R}\mathcal{C}/\mathcal{Y}(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathcal{G}\mathcal{R}\mathcal{C}/\mathcal{X}(f^*\mathcal{O}_Y, \mathcal{O}_X) \tag{12.1.5}
\]

**Remark 12.1.4**

For a map of generalized ringed spaces $f \in \mathcal{G}\mathcal{R}\mathcal{S}(\mathcal{X}, \mathcal{Y})$, and for a point $x \in \mathcal{X}$, we get the induced homomorphism on stalks $f^*_x \in \mathcal{G}\mathcal{R}(\mathcal{O}_{Y,f(x)}, \mathcal{O}_{X,x})$, via
\[
f^*_x : \mathcal{O}_{Y,f(x)} = \lim_{V \subseteq \mathcal{Y} \text{ open}} \mathcal{O}_Y(V) \xrightarrow{\quad \text{lim} f^*_x \quad} \lim_{x \in f^{-1}V} \mathcal{O}_X(f^{-1}V) \rightarrow \lim_{U \subseteq \mathcal{X} \text{ open}} \mathcal{O}_X(U) = \mathcal{O}_{X,x} \tag{12.1.6}
\]

**Definition 12.1.5**

We let $\mathcal{L}\mathcal{G}\mathcal{R}\mathcal{S} \subseteq \mathcal{G}\mathcal{R}\mathcal{S}$ denote the subcategory of $\mathcal{G}\mathcal{R}\mathcal{S}$ of locally generalized ringed spaces. Its objects are the objects $(\mathcal{X}, \mathcal{O}_X) \in \mathcal{G}\mathcal{R}\mathcal{S}$ such that for all points $x \in \mathcal{X}$ the stalk $\mathcal{O}_{X,x} \in \mathcal{L}\mathcal{G}\mathcal{R}$ is a local generalized ring, i.e. has a unique maximal symmetric ideal $m_{X,x}$. The maps $f \in \mathcal{L}\mathcal{G}\mathcal{R}\mathcal{S}(\mathcal{X}, \mathcal{Y})$ are the maps $(f, f^*_x) \in \mathcal{G}\mathcal{R}\mathcal{S}(\mathcal{X}, \mathcal{Y})$, such that for all points $x \in \mathcal{X}$, the induced homomorphism on stalks $f^*_x$ is a local homomorphism, $f^*_x \in \mathcal{L}\mathcal{G}\mathcal{R}(\mathcal{O}_{Y,f(x)}, \mathcal{O}_{X,x})$, $f^*_x(m_{Y,f(x)}) \subseteq m_{X,x}$.

**Theorem 12.1.6**

The functor of global sections
\[
\Gamma : \mathcal{L}\mathcal{G}\mathcal{R}\mathcal{S} \rightarrow (\mathcal{G}\mathcal{R}\mathcal{C})^{op}, \quad \Gamma(\mathcal{X}, \mathcal{O}_X) = \mathcal{O}_X(\mathcal{X}),
\]
\[
\Gamma(f, f^*_x) = f^*_x \text{ for } f \in \mathcal{L}\mathcal{G}\mathcal{R}\mathcal{S}(\mathcal{X}, \mathcal{Y}) \tag{12.1.7}
\]
and the spectra functor

$$\text{spec}^t : (\mathcal{GR}_C)^{op} \to \mathcal{LGRS}, \quad \text{spec}^t(A) = (\text{spec}^t(A), \mathcal{O}_A),$$

$$\text{spec}^t(\varphi) = \varphi^* \text{ for } \varphi \in \mathcal{GR}(A, B) \quad (12.1.8)$$

are an adjoint pair:

$$\mathcal{LGRS}(\mathcal{X}, \text{spec}^t(A)) = \mathcal{GR}_C(A, \mathcal{O}_X(\mathcal{X}))$$

functorially in $\mathcal{X} \in \mathcal{LGRS}$, $A \in \mathcal{GR}_C$ \hfill (12.1.9)

\textbf{Proof.} For a point $x \in \mathcal{X}$ we have the canonical homomorphism of taking the stalk at $x$ of a global section, $\phi_x \in \mathcal{GR}(\mathcal{O}_X(\mathcal{X}), \mathcal{O}_{X,x})$. Since $O_{X,x}$ is local with a unique maximal symmetric ideal $m_{X,x}$, we get by pullback a symmetric prime $p_x = \phi_x^*(m_{X,x}) \in \text{spec}^t(\mathcal{O}_X(\mathcal{X}))$. Thus we have a canonical map

$$p : \mathcal{X} \to \text{spec}^t(\mathcal{O}_X(\mathcal{X})), \quad x \mapsto p_x \quad (12.1.10)$$

The map $p$ is continuous: For a global section $g = g^t \in \mathcal{O}_X(\mathcal{X})_{[1]}^+\mathcal{O}_X$, we have the basic open set $D_g^+ \subseteq \text{spec}^t(\mathcal{O}_X(\mathcal{X}))$, and

$$p^{-1}(D_g^+) = \{ x \in \mathcal{X}, p_x \in D_g^+ \} = \{ x \in \mathcal{X}, \phi_x(g) \notin m_{X,x} \} \quad (12.1.11)$$

This set is open in $\mathcal{X}$, because if $\phi_x(g) \notin m_{X,x}$ we have in $\mathcal{O}_{X,x}$ some $v_x$ with $v_x < \phi_x(g) = 1$, hence there is an open set $U \subseteq \mathcal{X}$, with $x \in U$, and an element $v \in \mathcal{O}_X(U)_{[1]}^+$ with $v < g|_U = 1$, and for all $x' \in U$, $v_{x'} \circ \phi_{x'}(g) = 1$, and $\phi_x(g) \neq m_{X,x'}$. This shows $p$ is continuous. The uniqueness of the inverse $v_x = \phi_x(g)^{-1}$ for $x \in p^{-1}(D_g^+)$ shows we have a well defined inverse $v = (g|_{p^{-1}(D_g^+)})^{-1} \in \mathcal{O}_X(p^{-1}(D_g^+))_{[1]}^+\mathcal{O}_X$. Thus we have a homomorphism of generalized rings

$$p^t_{D_g^+} : \mathcal{O}_X(\mathcal{X})_g \to \mathcal{O}_X(p^{-1}(D_g^+)), \quad a/g^n \mapsto v^n \prec \left( a|_{p^{-1}(D_g^+)} \right) \quad (12.1.12)$$

The collection of homomorphisms $\{ p^t_{D_g^+}, g \in \mathcal{O}_X(\mathcal{X})_{[1]}^+\mathcal{O}_X \}$, are compatible with restrictions, and the sheaf property gives homomorphisms

$$p^t_U \in \mathcal{GR} \left( \mathcal{O}_{\text{spec}\mathcal{O}_X(\mathcal{X})}(U), \mathcal{O}_X(p^{-1}(U)) \right). \quad (12.1.13)$$

Thus we have a map of generalized ringed spaces

$$p = (p, p^t) \in \mathcal{LGRS}(\mathcal{X}, \text{spec}(\mathcal{O}_X(\mathcal{X}))). \quad (12.1.14)$$

For a point $x \in \mathcal{X}$, we can take the direct limit of $p^t_{D_g^+}$, over all global sections $g \in \mathcal{O}_X(\mathcal{X})_{[1]}^+\mathcal{O}_X$ with $\phi_x(g) \notin m_{X,x}$, and we get a local homomorphism $p_x^t \in \mathcal{LGR}(\mathcal{O}_X(\mathcal{X})_{p_x}, \mathcal{O}_{X,x})$. This shows $p$ is a map of \textit{locally}-ringed spaces, $p \in \mathcal{LGRS}(\mathcal{X}, \text{spec}^t(\mathcal{O}_X(\mathcal{X}))$.}
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Given a homomorphism of generalized rings \( \varphi \in \mathcal{GR}_C(A, \mathcal{O}_X(\mathcal{X})) \), we get the map in \( \mathcal{LGRS} \)

\[
(specl \varphi) \circ p : \mathcal{X} \to specl(\mathcal{O}_X(\mathcal{X})) \to specl(A)
\]

(12.1.15)

Given a map of locally ringed spaces \( f = (f, f^\sharp) \in \mathcal{LGRS}(\mathcal{X}, specl(A)) \), we get a homomorphism in \( \mathcal{GR}_C \),

\[
\Gamma(f) = f^\sharp_{specl(A)} : A = \mathcal{O}_A(specl(A)) \to \mathcal{O}_X(\mathcal{X})
\]

(12.1.16)

These correspondences give the functorial bijection of (12.1.9), we need only show they are inverses of each other. First for \( \varphi \in \mathcal{GR}_C(A, \mathcal{O}_X(\mathcal{X})) \), we have

\[
\Gamma(p) \circ \Gamma(specl(\varphi)) = id_{\mathcal{O}_X(\mathcal{X})} \circ \varphi = \varphi
\]

(12.1.17)

Fix a map \( f = (f, f^\sharp) \in \mathcal{LGRS}(\mathcal{X}, specl(A)) \). For a point \( x \in \mathcal{X} \), we have a commutative square in \( \mathcal{GR}_C \)

\[
\begin{array}{ccc}
A = \mathcal{O}_A(specl(A)) & \xrightarrow{\Gamma(f)} & \mathcal{O}_X(\mathcal{X}) \\
\phi_{f(x)} & \downarrow & \phi_x \\
A_{f(x)} = \mathcal{O}_{A, f(x)} & \xrightarrow{f^\sharp_x} & \mathcal{O}_{\mathcal{X}, x}
\end{array}
\]

(12.1.18)

Since the homomorphism \( f^\sharp_x \) is assumed to be local, we get

\[
\Gamma(f^*) (p_x) = \Gamma(f^*) (\phi_x^*(m_{X,x})) = \phi_{f(x)}^*(f^\sharp_x^*(m_{X,x})) = \phi_{f(x)}^*(m_{A_{f(x)}}) = f(x)
\]

(12.1.19)

This shows that \( (specl \Gamma(f)) \circ p = f \) as continuous maps.

For a symmetric element \( s = s^t \in A^{[1]} \), we have the commutative square in \( \mathcal{GR}_C \),

\[
\begin{array}{ccc}
A = \mathcal{O}_A(specl(A)) & \xrightarrow{\Gamma(f)} & \mathcal{O}_X(\mathcal{X}) \\
\downarrow & \downarrow & \\
A_s = \mathcal{O}_A(D_s^+) & \xrightarrow{f^\sharp_D} & \mathcal{O}_X(f^{-1}(D_s^+)) = \mathcal{O}_X(D_{f^1(s)})
\end{array}
\]

(12.1.20)

Thus for \( a/s^n \in A_s \), we must have

\[
f^\sharp_{D_s^+} (a/s^n) = \left( (\Gamma(f))(s^n) \right)_{f^{-1}(D_s^+)}^{-1} \circ (\Gamma(f)(a))_{f^{-1}(D_s^+)} = p_{D_{f^1(s)}}^\sharp \circ \Gamma(f)(a/s^n)
\]

(12.1.21)

This shows that \( f = (specl \Gamma(f)) \circ p \) also as maps of generalized-tringed spaces.
12.2 Schemes

We define the *Grothendieck-generalized-schemes* to be the objects of the full subcategory $\mathcal{GGS} \subseteq \mathcal{LGRS}$, consisting of the $(\mathcal{X}, \mathcal{O}_\mathcal{X})$’s which are locally affine. Later we shall define the category of *generalized-schemes* $\mathcal{GS}$ to be the pro-category of $\mathcal{GGS}$.

**Definition 12.2.1**

An object $\mathcal{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \in \mathcal{LGRS}$ will be called a *Grothendieck-generalized-scheme* if it is locally isomorphic to $\text{spec}(A)$’s: there exists a covering of $\mathcal{X}$ by open sets $U_i$, $\mathcal{X} = \bigcup_i U_i$, such that the canonical maps are isomorphisms

$$p : (U_i, \mathcal{O}_\mathcal{X}|_{U_i}) \xrightarrow{\sim} \text{spec}^i(\mathcal{O}_\mathcal{X}(U_i)) \quad (12.2.1)$$

We let $\mathcal{GGS}$ denote the full sub-category of $\mathcal{LGRS}$, with objects the Grothendieck-generalized-schemes.

**Open subschemes 12.2.2** Note that for $\mathcal{X} \in \mathcal{GGS}$, and for an open set $U \subseteq \mathcal{X}$, we have the *open subscheme* of $\mathcal{X}$ given by $(U, \mathcal{O}_\mathcal{X}|_{U})$. That this is again a scheme, $(U, \mathcal{O}_\mathcal{X}|_{U}) \in \mathcal{GGS}$, follows from the existence of affine basis for the Zariski topology on $\text{spec}^i(A)$, $A \in \mathcal{GRC}$, namely $(D^+_s, \mathcal{O}_A|_{D^+_s}) \cong \text{spec}^i(A_s)$ for $s \in A^+_1$.

**Gluing schemes 12.2.3** The local nature of the definition of Grothendieck-generalized-scheme implies that $\mathcal{GGS}$ admits gluing:

Given $\mathcal{X}_i \in \mathcal{GGS}$, and open subsets $U_{ij} \subseteq \mathcal{X}_i$, and maps $\varphi_{ij} \in \mathcal{GGS}(U_{ij}, U_{ij})$, satisfying the consistency conditions

1. $U_{ii} = \mathcal{X}_i$, and $\varphi_{ii} = id_{\mathcal{X}_i}$,
2. $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$, \hspace{1cm} (12.2.2)

there exists $\mathcal{X} \in \mathcal{GGS}$, and maps $\varphi_i \in \mathcal{GGS}(\mathcal{X}_i, \mathcal{X})$ such that

1. $\varphi_i$ is an isomorphism of $\mathcal{X}_i$ onto an open subset $\varphi_i(\mathcal{X}_i) \subseteq \mathcal{X}$
2. $\mathcal{X} = \bigcup_i \varphi_i(\mathcal{X}_i)$ \hspace{1cm} (12.2.3)
3. $\varphi_i(\mathcal{X}_i) \cap \varphi_j(\mathcal{X}_j) = \varphi_i(U_{ij})$, and $\varphi_j \circ \varphi_{ij} = \varphi_i$ on $U_{ij}$.

**Ordinary Schemes 12.2.4** For an ordinary scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, with $\mathcal{O}_\mathcal{X}$ a sheaf of commutative rings, there is a covering by open sets $\mathcal{X} = \bigcup_i U_i$, with $(U_i, \mathcal{O}_\mathcal{X}|_{U_i}) \cong \text{spec}(A_i)$, the ordinary spectrum of the commutative ring $A_i = \mathcal{O}_\mathcal{X}(U_i)$. We then have Grothendieck-generalized schemes $\mathcal{X}_i = \text{spec}^i(\mathcal{G}(A_i)) = (U_i, \mathcal{G}(\mathcal{O}_\mathcal{X})|_{U_i})$. These can be glued along $U_{ij} = U_i \cap U_j$, to a Grothendieck-generalized scheme denoted by $\mathcal{G}(\mathcal{X}) = (\mathcal{X}, \mathcal{O}_{\mathcal{G}(\mathcal{X})} = \mathcal{G}(\mathcal{O}_\mathcal{X}))$. It is just the
underlying topological space $X$ with the sheaf of generalized rings $G(O_X)$ associated to the sheaf of commutative rings $O_X$ via the functor $G: \text{Ring} \rightarrow \mathcal{G} \mathcal{R}$, \text{(8.3.12)}. Denoting by $\mathcal{RS}$ the category of (ordinary, commutative) ringed spaces, the functor $G$ applied to a sheaf of commutative rings $O_X$ gives a sheaf of commutative generalized rings $G(O_X)$, and we have a functor $G : \mathcal{RS} \rightarrow \mathcal{G} \mathcal{R} \mathcal{S}$.

Denoting by $\mathcal{LRS}$ (resp. by $\mathcal{S}$) the category of locally-(commutative)-ringed spaces (resp. the full subcategory of ordinary schemes), the fact that $G$ is fully-faithful implies that we have full-embeddings of categories:

$$
\begin{array}{c|c}
\text{ordinary} & \text{generalized} \\
\hline
\mathcal{LRS} & \mathcal{LGRS} \\
\uparrow & \uparrow \\
\mathcal{S} & \mathcal{GGS} \\
\text{spec} & \text{spec}^t \\
\mathcal{Ring}^{op} & (\mathcal{G} \mathcal{R}_C)^{op} \\
\end{array}
$$

$$
(12.2.4)
$$

The generalized Grothendieck scheme $G(X)$, associated with an ordinary scheme $X \in S$, has always a unique map: $G(X) \rightarrow \text{spec}^t G(\mathbb{Z})$.

Some Examples of Schemes over $\mathbb{F}$

\textbf{Example 12.2.5}

\textit{The (symmetric) affine line} over $\mathbb{F}$ is given by, cf. \text{(8.3.75)},

$$
\mathbb{A}^1_+ = \text{spec} \Delta^{[1]}_+ = \text{spec} \mathbb{F}\{z^N\}
$$

(12.2.5)

We have $F\{z^N\}_{[1]} = z^N \cup \{0\}$; (0) is a prime, the generic point of $\mathbb{A}^1_+$; and (z) is a prime, the closed point of $\mathbb{A}^1_+ = \{(0), (z)\}$.

Similarly, we have

$$
\mathbb{A}^1 = \text{spec}^t \Delta^{[1]} = \text{spec}^t \mathbb{F}\{z^n \cdot (z^t)^n\}.
$$

(12.2.6)

The symmetric prime (0) is the generic point of $\mathbb{A}^1$, and the symmetric prime $(z \cdot z^t)$ is the closed point: $\mathbb{A}^1 = \{(0), (z \cdot z^t)\}$. The homomorphism $F\{z^n \cdot (z^t)^n\} \rightarrow F\{z^n\}$, $z, z^t \mapsto z$, gives the immersion

$$
\mathbb{A}^1_+ \hookrightarrow \mathbb{A}^1
$$

(12.2.7)
Example 12.2.6

The (symmetric) multiplicative group over \( \mathbb{F} \) is given by

\[
\mathbb{G}_m^+ = \text{spec} \, \mathbb{F}\{z^2\} = \{(0)\} \subseteq \mathbb{A}_+^1
\]  

(12.2.8)

Note that for a commutative generalized ring \( A \),

\[
\mathcal{GR}_C(\mathbb{F}\{z^2\}, A) = A^* \cap A_{[1]}^+= \{a \in A_{[1]}^+, \text{ there is } a^{-1} \in A_{[1]}^+, a \circ a^{-1} = 1\}
\]

(12.2.9)

Similarly,

\[
\mathbb{G}_m = \text{spec}^t \, \mathbb{F}[z^2 \cdot (z')^2] = \{(0)\} \subseteq \mathbb{A}^1,
\]

(12.2.10)

and

\[
\mathcal{GR}_C(\mathbb{F}[z^2 \cdot (z')^2], A) = A^*. \text{ We have an immersion}
\]

\[
\mathbb{G}_m^+ \hookrightarrow \mathbb{G}_m
\]

(12.2.11)

Example 12.2.7

The (symmetric) projective line over \( \mathbb{F} \) is obtained by gluing two (symmetric) affine lines along \( \mathbb{G}_m^{(1)} \)

\[
\mathbb{P}_+^1 = \text{spec} \, \mathbb{F}\{z^N\} \coprod_{\text{spec} \, \mathbb{F}\{z^2\}} \text{spec} \, \mathbb{F}\{(z^{-1})^N\} = \{m_1, m_0, m_{\infty}\}
\]

(12.2.12)

Respectfully, the full

\[
\mathbb{P}^1 = \text{spec}^t \, \mathbb{F}\{z^N \cdot (z')^N\} \coprod_{\text{spec}^t \, \mathbb{F}\{z^2 \cdot (z')^2\}} \text{spec}^t \, \mathbb{F}\{(z^{-1})^N \cdot ((z')^{-1})^N\}
\]

(12.2.13)

It has a generic point \( m_1 = (0) \), and two closed points \( m_0 = (z) \), (resp. \( (z \cdot z') \)) \( m_{\infty} = (z^{-1}) \), (resp. \( (z^{-1} \cdot (z')^{-1}) \)). We have the immersion

\[
\mathbb{P}_+^1 \hookrightarrow \mathbb{P}^1
\]

(12.2.14)

Interchanging \( z \) and \( z^{-1} \) we get an involutive automorphism

\[
I : \mathbb{P}_+^1 \xrightarrow{\sim} \mathbb{P}_+^1, \quad I \circ I = \text{id}_{\mathbb{P}_+^1}
\]

(12.2.15)

interchanging \( m_0 \) and \( m_{\infty} \).

Every rational number \( f \in \mathbb{Q}^* \), defines a geometric map

\[
f_Z \in \mathcal{GG}(\text{spec} \, \mathbb{G}(\mathbb{Z}), \mathbb{P}_+^1)
\]

(12.2.16)

If \( f = \pm 1 \) this is given by the constant map

\[
\mathbb{F}\{z^2\} \xrightarrow{f} \mathbb{F}\{\pm 1\} \subseteq \mathbb{G}(\mathbb{Z}), z \mapsto f = \pm 1
\]

(12.2.17)
If $f \neq \pm 1$, let $N_0 = \prod_{\nu_p(j) > 0} p$, $N_\infty = \prod_{\nu_p(j) < 0} p$, then

\[ \text{spec } \mathbb{G}(\mathbb{Z}) = \text{spec } \mathbb{G}(\mathbb{Z}[\frac{1}{N_0}]) \bigg/ \text{spec } \mathbb{G}(\mathbb{Z}[\frac{1}{N_\infty}]) \] (12.2.18)

and the geometric map $f_\mathbb{Z}$ is given by the spec-maps associated to the homomorphisms:

\[ \begin{align*}
\mathbb{F}\{z^N\} & \rightarrow \mathbb{G}(\mathbb{Z}[\frac{1}{N_0}]) \\
\bigcap_{j \in J} & \rightarrow \bigcap_{j \in J} \mathbb{G}(\mathbb{Z}[\frac{1}{N_0}]) \\
\mathbb{F}\{(z^{-1})^N\} & \rightarrow \mathbb{G}(\mathbb{Z}[\frac{1}{N_\infty}]) \\
z & \mapsto f \\
z^{-1} & \mapsto f^{-1}
\end{align*} \] (12.2.19)

### 12.3 Projective limits

The category of locally generalized ringed spaces $\mathcal{LGRS}$ admits directed inverse limits. For a partially ordered set $J$, which is directed (for $j_1, j_2 \in J$, have $j \in J$ with $j \geq j_1$, $j \geq j_2$) and for a functor $\mathcal{X} : J \rightarrow \mathcal{LGRS}$, $j \mapsto \mathcal{X}_j$, $j_1 \geq j_2 \mapsto \pi_{j_2}^{j_1} \in \mathcal{LGRS}(\mathcal{X}_{j_1}, \mathcal{X}_{j_2})$, we have the inverse limit $\lim_J \mathcal{X} \in \mathcal{LGRS}$.

The underlying topological space of $\lim_J \mathcal{X}$ is the inverse limit of the sets $\mathcal{X}_j$, with basis for the topology given by the sets $\pi_{j_2}^{j_1}(U_j)$, with $U_j \subseteq \mathcal{X}_j$ open, and where $\pi_j : \lim_J \mathcal{X} \rightarrow \mathcal{X}_j$ denote the projection. The sheaf of generalized rings $\mathcal{O}_{\lim \mathcal{X}}$ over $\lim_J \mathcal{X}_j$, is the sheaf associated to the pre-sheaf $U \mapsto \lim_J \mathcal{O}_{\mathcal{X}_j}(U)$.

For a point $x = (x_j) \in \lim_J \mathcal{X}_j$, the stalk $\mathcal{O}_{\lim \mathcal{X}, x}$ is the direct limit of the local-generalized-rings $\mathcal{O}_{\mathcal{X}_j, x_j}$, and hence is local, and $(\lim_J \mathcal{X}_j, \mathcal{O}_{\lim \mathcal{X}}) \in \mathcal{LGRS}$.

An alternative explicit description of the sections $s \in \mathcal{O}_{\lim \mathcal{X}}(U)$, for $U \subseteq \lim_J \mathcal{X}_j$ open, are as maps

\[ s : U \rightarrow \prod_{x \in U} \mathcal{O}_{\lim \mathcal{X}, x}, \text{ with } s(x) \in \mathcal{O}_{\lim \mathcal{X}, x} \] (12.3.1)

such that for all $x \in U$, there exists $j \in J$, and open subset $U_j \subseteq \mathcal{X}_j$, with $x \in \pi_{j_2}^{j_1}(U_j) \subseteq U$ and there exists a section $s_j \in \mathcal{O}_{\mathcal{X}_j}(U_j)$, such that for all $y \in \pi_{j_2}^{j_1}(U_j)$, we have $s(y) = \pi_{j}^{j_1}(s_j)|_y$.

We have the universal property

\[ \mathcal{LGRS}(\mathbb{Z}, \lim_J \mathcal{X}_j) = \lim_J \mathcal{LGRS}(\mathbb{Z}, \mathcal{X}_j) \] (12.3.2)
Note that if \( X_j = spec^t(A_j) \) are affine generalized schemes, then the inverse limit

\[
\lim_{\rightarrow J}(spec^t(A_j)) = spec^t(\lim_{\rightarrow J} A_j)
\]

is the affine generalized scheme associated to \( \lim_{\rightarrow J} A_j \) in \( GR \). (Hence in \( Set_0 \), cf. [8.3.78].)

Note on the other hand that the category of Grothendieck- generalized schemes \( GGR \) is not closed under directed inverse limits (just as in the "classical" counterparts, the category \( LRS \) of locally ringed spaces (resp. \( Ring \)) is closed under directed inverse (resp. direct) limits, while the category \( S \) of schemes is not closed under directed inverse limits). The point is: for a point \( x = (x_j) \in \lim_{\rightarrow J} X_j \), in the inverse limit of the Grothendieck (generalized) schemes \( X_j \), while each \( x_j \in X_j \) has an open affine neighborhood, there may not be an open affine neighborhood of \( x \) in \( \lim_{\rightarrow J} X_j \).

**Definition 12.3.1**

The category of generalized schemes \( GS \) is the category of pro-objects of the category of Grothendieck-generalized schemes,

\[
GS = pro-GGS.
\]

Thus the objects of \( GS \) are inverse systems \( \mathcal{X} = (\{X_j\}_{j \in J}, \{\pi_{j_2}^{j_1}\}_{j_2 \geq j_1}) \), where \( J \) is a directed partially ordered set, \( X_j \in GGS \) for \( j \in J \), and \( \pi_{j_2}^{j_1} \in GGS(X_{j_1}, X_{j_2}) \) for \( j_1 \geq j_2, j_1, j_2 \in J \), with \( \pi_j^j = id_{X_j} \), and \( \pi_{j_2}^{j_1} \circ \pi_{j_3}^{j_2} = \pi_{j_3}^{j_1} \) for \( j_1 \geq j_2 \geq j_3 \). The maps from such an object to another object \( \mathcal{Y} = (\{Y_i\}_{i \in I}, \{\pi_{i_2}^{i_1}\}_{i_1 \geq i_2}) \) are given by

\[
GS(\mathcal{X}, \mathcal{Y}) = \lim_{\rightarrow I} \lim_{\rightarrow J} GGS(X_j, Y_i)
\]

i.e. the maps \( \varphi \in GS(\mathcal{X}, \mathcal{Y}) \) are a collection of maps \( \varphi_i^j \in GGS(X_j, Y_i) \) defined for all \( i \in I \), and for \( j \geq \tau(i) \) sufficiently large (depending on \( i \)), these maps satisfy:

for all \( i \in I \), and for \( j_1 \geq j_2 \) sufficiently large in \( J \):

\[
\varphi_i^{j_2} = \varphi_i^{j_1} \circ \pi_{j_2}^{j_1}
\]

for all \( i_1 \geq i_2 \) in \( I \), and for \( j \in J \) sufficiently large:

\[
\pi_{i_1}^{i_2} \circ \varphi_i^j = \varphi_{i_2}^j
\]

The maps \( \varphi = \{\varphi_i^j\}_{j \geq \tau(i)} \), and \( \tilde{\varphi} = \{\tilde{\varphi}_i^j\}_{i \geq \tau(i)} \), are considered equivalent if

for all \( i \in I \), and for \( j \in J \) sufficiently large:

\[
\varphi_i^j = \tilde{\varphi}_i^j
\]
The composition of $\varphi = \{\varphi^i_j\}_{j \geq \tau(i)} \in \mathcal{G}S(\mathcal{X}, \mathcal{Y})$, with $\psi = \{\psi^i_j\}_{k \geq \sigma(k)} \in \mathcal{G}S(\mathcal{Y}, \mathcal{Z})$, is given by $\psi \circ \varphi = \{\psi^i_j \circ \varphi^i_j\}_{j \geq \tau(\sigma(j))} \in \mathcal{G}S(\mathcal{X}, \mathcal{Z})$.

There is a canonical map (which in general is not injective or surjective, but is so for "finitely-presented" $\{X_j\}$ and $\{Y_i\}$, see [EGA]),

$$\lim_I^{\mathcal{LGRS}}(X_j, Y_i) \rightarrow \mathcal{LGRS}(\lim_{j} X_j, Y_i)$$

(12.3.9)

By the universal property (12.3.2) we have bijection

$$\lim_I^{\mathcal{LGRS}}(\lim_{j} X_j, Y_i) \rightarrow \mathcal{LGRS}(\lim_{j} X_j, \lim_{I} Y_i)$$

(12.3.10)

Composing (12.3.9) and (12.3.10) we obtain a map

$$\mathcal{L} : \lim_I^{\mathcal{LGRS}}(X_j, Y_i) \rightarrow \mathcal{LGRS}(\lim_{j} X_j, \lim_{I} Y_i)$$

(12.3.11)

Thus we have a functor

$$\mathcal{L} : \mathcal{G}S \rightarrow \mathcal{LGRS}, \quad \mathcal{L}(\{X_j\}_{j \in J}) = \lim_{J} X_j$$

(12.3.12)

We view the category $\mathcal{G}G\mathcal{S}$ as a full subcategory of $\mathcal{G}S$ (consisting of the objects $\mathcal{X} = \{X_j\}_{j \in J}$, with indexing set $J$ reduced to a singleton).

### 12.4 The compactified $\text{spec}\mathbb{Z}$

We denote by $\eta$ the real prime of $\mathbb{Q}$, so $|\eta| : \mathbb{Q} \rightarrow [0, \infty)$ is the usual (non-archimedean) absolute value, and we let $\mathcal{O}_\eta$ denote the associated generalized ring [8.3.20], $\mathcal{O}_\eta \subseteq \mathcal{G}(\mathbb{Q})$. For a square-free integer $N \geq 2$, we have the subgeneralized-ring

$$A_N = \mathcal{G}(\mathbb{Z}[\frac{1}{N}]) \cap \mathcal{O}_\eta \subseteq \mathcal{G}(\mathbb{Q})$$

(12.4.1)

The localization of $A_N$ with respect to $\frac{1}{N} \in A_N[1]$ gives $(A_N)_{\frac{1}{N}} = \mathcal{G}(\mathbb{Z}[\frac{1}{N}])$, so the inclusion $j_N : A_N \hookrightarrow \mathcal{G}(\mathbb{Z}[\frac{1}{N}])$ gives the basic open set

$$j_N^* : \text{spec}(\mathbb{Z}[\frac{1}{N}]) = \text{spec}^c \mathcal{G}(\mathbb{Z}[\frac{1}{N}]) \rightarrow D^+_\mathbb{Z} \subseteq \text{spec}^c(A_N)$$

(12.4.2)

The inclusion $i_N : A_N \hookrightarrow \mathcal{O}_\eta$, gives the real prime $\eta_N \in \text{spec}^c(A_N)$,

$$\eta_N = i_N^*(m_\eta), \quad (\eta_N)_X = \{a = (a_x) \in (\mathbb{Z}[\frac{1}{N}])^X, |a|^2 = \sum_{x \in X} |a_x|^2 < 1\}$$

(12.4.3)

Note that $\eta_N$ is the unique maximal ideal of $A_N$, and $A_N$ is a local generalized ring. Let $\mathcal{X}_N$ denote the Grothendieck generalized scheme obtained by gluing $\text{spec}^c(A_N)$ with $\text{spec}^c \mathcal{G}(\mathbb{Z})$ along the common (basic) open set $\text{spec}^c(\mathcal{G}(\mathbb{Z}[\frac{1}{N}]))$. 

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The open sets of $\mathcal{X}_N$ are the open sets $U_M = \text{spec}(\mathbb{Z}[\frac{1}{M}]) \subseteq \text{spec}(\mathbb{Z})$, (and $\mathcal{O}_{\mathcal{X}_N}(U_M) = \mathcal{G}(\mathbb{Z}[\frac{1}{M}])$), as well as the sets $\{\eta_N\} \cup U_M$, with $M$ dividing $N$ (and $\mathcal{O}_{\mathcal{X}_N}(\{\eta_N\} \cup U_M) = A_M, M|N$).

For $N_2$ dividing $N_1$, we have a map $\pi_{N_1\to N_2} \in \mathcal{G}G\mathcal{S}(\mathcal{X}_{N_1}, \mathcal{X}_{N_2})$ induced by the inclusions $A_{N_2} \hookrightarrow A_{N_1}$, and $\mathcal{G}(\mathbb{Z}[\frac{1}{N_2}]) \hookrightarrow \mathcal{G}(\mathbb{Z}[\frac{1}{N_1}])$.

Note that the associated locally-generalized-ring space

$$ \mathcal{O}_X = \text{spec}(\mathbb{Z}) = \lim_{\mathbb{N}} \mathcal{X}_N \in \mathcal{L}\mathcal{G}\mathcal{R}\mathcal{S} $$

(12.4.6)

has underlying topological space $X = \{\eta\} \bigsqcup \text{spec}(\mathbb{Z})$, with open sets $U_M = \text{spec}(\mathbb{Z}[\frac{1}{M}])$ (and $\mathcal{O}_X(U_M) = \mathcal{G}(\mathbb{Z}[\frac{1}{M}])$), as well as the sets $\{\eta\} \bigsqcup U_M$, with no restrictions on $M$, and $\mathcal{O}_X(\{\eta\} \bigsqcup U_M) = A_M$ for $M \geq 2$, while the global sections are $\mathcal{O}_X(X) = \mathbb{F}\{\pm 1\}$.

The stalks of $\mathcal{O}_X$ are given by

$$ \mathcal{O}_{X, p} = \mathcal{G}(\mathbb{Z}_{(p)}), \quad p \in \text{spec}(\mathbb{Z}), \quad \mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q}, p \nmid n\}, $$

(12.4.7)

Similarly for a number field $K$, with ring of integers $\mathcal{O}_K$, and with real and complex primes $\eta_i, i = 1, \ldots, \gamma = \gamma_R + \gamma_C$, we have the sub-generalized-ring of $\mathcal{G}(K)$ given by

$$ A_{\mathcal{X}_N, \eta} = \mathcal{G}(\mathcal{O}_K[\frac{1}{N}]) \cap \mathcal{O}_{K, \eta} \subseteq \mathcal{G}(K) $$

(12.4.8)

Let $\mathcal{X}_N$ be the Grothendieck generalized scheme obtained by gluing $\{\text{spec}(A_{\mathcal{X}_N, \eta})\}_{i \in \gamma}$ and $\{\text{spec}(\mathcal{G}(\mathcal{O}_K))\}$ along the common (basic) open set $\text{spec}(\mathcal{G}(\mathcal{O}_K[\frac{1}{N}]))$.

For $N_2|N_1$, we have $\pi_{N_1\to N_2} \in \mathcal{G}G\mathcal{R}(\mathcal{X}_{N_1}, \mathcal{X}_{N_2})$ induced by the inclusions $A_{N_2, \eta} \hookrightarrow A_{N_1, \eta}$. We get the compactified $\text{spec}(\mathcal{G}\mathcal{O}_K)$, it is the object of $\mathcal{G}\mathcal{S}$ given by the $\mathcal{X}_N$’s and $\pi_{N_1\to N_2}$’s.

The space

$$ \mathcal{X}_K = \mathcal{L}\left(\text{spec}(\mathcal{G}\mathcal{O}_K)\right) = \lim_{\mathbb{N}} \mathcal{X}_N \in \mathcal{L}\mathcal{G}\mathcal{R}\mathcal{S} $$

(12.4.9)

has for points the set $\text{spec}(\mathcal{G}\mathcal{O}_K) \bigsqcup \{\eta_i\}_{i \in \gamma}$, and for open subsets the sets $U \bigsqcup \{\eta_i\}_{i \in I}$, $U \subseteq \text{spec}(\mathcal{G}\mathcal{O}_K)$ open, $I \subseteq \{1, \ldots, \gamma\}$, where

$$ \mathcal{O}_{\mathcal{X}_K}(U \bigsqcup \{\eta_i\}_{i \in I}) = \bigcap_{p \in U} \mathcal{G}(\mathcal{O}_{K, p}) \cap \bigcap_{i \in I} \mathcal{O}_{K, \eta_i} $$

(12.4.10)
In particular, the global sections are
\[ \mathcal{O}_{X_K}(X_K) = \bigcap_{p \in \text{spec} \mathcal{O}_K} \mathcal{G}(\mathcal{O}_{K,p}) \cap \bigcap_{i \in \gamma} \mathcal{O}_{K,\eta_i} = \mathbb{F}\{\mu_K\} \tag{12.4.11} \]
with \( \mu_K \subseteq \mathcal{O}_K^* \) the group of roots of unity in \( \mathcal{O}_K^* \).

Returning for simplicity to the rational \( X = \text{spec} \mathbb{Z} \) case of (12.4.5), every rational number \( f \in \mathbb{Q}^* \), defines a geometric map
\[ f \in \mathcal{G}\mathcal{S}(X, \mathbb{P}_+^1) \tag{12.4.12} \]
i.e. a collection of maps \( f_N \in \mathcal{G}\mathcal{S}(X_N, \mathbb{P}_+^1) \), for \( N \) divisible by \( N_0 \cdot N_\infty \),
\[ N_0 = \prod_{\nu_p(f) > 0} p, \quad N_\infty = \prod_{\nu_p(f) < 0} p, \]
with \( f_N \circ \pi^M_{N_0} = \pi^M_{N_0} \).

For \( f = \pm 1 \) it is the constant map given by
\[ \mathbb{F}\{z^N\} \rightarrow \mathbb{F}\{\pm 1\} = \mathcal{G}(\mathbb{Z}) \cap A_N, \text{ for any } N \]
\[ z \mapsto f = \pm 1 \tag{12.4.13} \]
For \( f \neq \pm 1 \), we may assume \( |f|_\eta < 1 \), by the commutativity of the diagram (with \( I \) the inversion \(12.2.15\)),
\[ \begin{tikzpicture}
  \node (A) at (0,0) {$\mathcal{X}$};
  \node (B) at (0,-1) {$\mathbb{P}_+^1$};
  \node (C) at (-1,0) {}; \node (D) at (1,0) {};
  \draw[->] (A) -- (B) node at (-0.5,0) {$f$};
  \draw[->] (A) -- (D) node at (0.5,-0.5) {$f^{-1}$};
  \draw[->] (B) -- (C) node at (-1.5,-0.5) {};\node at (-0.5,0) {\scriptsize{$\eta$}};
  \draw[->] (B) -- (D) node at (0.5,-0.5) {};\node at (0.5,0) {\scriptsize{$\eta$}};
\end{tikzpicture} \tag{12.4.14} \]

Thus for \( N \) divisible by \( N_0 \cdot N_\infty \) we have \( f \in A_N \), and the map \( f_N \) is given by \( f_N = f_{\eta,N} \bigcup f_{\eta,N}^2 \), with \( f_{\eta,N} = \text{spec} f_{\eta,N}, f_{\eta,N}^2 = \mathcal{G}\mathcal{R}(\mathbb{F}\{z^N\}, A_N), \)
the unique homomorphism with \( f_{\eta,N}^2(z) = f \), and \( f_{\eta,N} \) is as in \(12.2.19\).
\[ X_N = \text{spec} \mathcal{G}(\mathbb{Z}) \bigcup_{\text{spec} \mathcal{G}(\mathbb{Z}[\frac{1}{N}])} \text{spec} A_N \quad \mathcal{G}(\mathbb{Z}[\frac{1}{N}]) \subset A_N \ni f \]
\[ \mathbb{P}_+^1 = \text{spec} \mathbb{F}\{z^{-1}\} \bigcup_{\text{spec} \mathbb{F}\{z\}} \text{spec} \mathbb{F}\{z\} \ni f \tag{12.4.15} \]

Similarly for a number field \( K \), every element \( f \in K^* \) defines a geometric map
\[ f \in \mathcal{G}\mathcal{S}(\text{spec} \mathcal{O}_K, \mathbb{P}_+^1) \tag{12.4.16} \]
Chapter 13

Products

13.1 Tensor product

The category $\mathcal{GR}_C$ of commutative generalized rings has tensor-products, i.e.

fibred sums: Given homomorphisms $\phi^j \in \mathcal{GR}_C(A,B^j)$, $j = 0,1$, there exists $B^0 \otimes B^1 \in \mathcal{GR}_C$, and homomorphisms $\psi^j \in \mathcal{GR}_C(B^j, B^0 \otimes A)$, $j = 0,1$, such that

$\psi^0 \circ \phi^0 = \psi^1 \circ \phi^1$, and for any $C \in \mathcal{GR}_C$,

$\mathcal{GR}_C(B^0 \otimes B^1, C) = \mathcal{GR}_C(B^0, C) \prod_{\mathcal{GR}_C(A,C)} \mathcal{GR}_C(B^1, C)$. 

So given homomorphisms $f^j \in \mathcal{GR}_C(B^j, C)$ with $f^0 \circ \phi^0 = f^1 \circ \phi^1$, there exists a

unique homomorphism $f^0 \otimes f^1 \in \mathcal{GR}_C(B^0 \otimes B^1, C)$, such that $(f^0 \otimes f^1) \circ \psi^j = f^j$.

The construction of $B^0 \otimes B^1$ goes as follows. First for a finite set

$\{b_0^0, \ldots, b_1^0, b_1^1, \ldots, b_m^1\}$, where $b_j^j \in B_j^j$, we have the free commutative general-

ized ring on the sets $\{X_0^0, X_0^1, \ldots, X_m^1\} \subseteq \mathcal{F}$, and we write $b_0^0, \ldots, b_1^0, b_1^1, \ldots, b_m^1$ for its canonical generators. Taking the direct limit over such finite subsets, cf. $\S8.3.7$, we have the free generalized ring $\Delta$ with generators $b$ with $b \in B_0^1$ or $b \in B_1^1$, and any $X \in F$. We divide $\Delta$ by the equivalence ideal $\mathcal{E}_A$ generated by

$\frac{b}{\mathcal{E}_A} \sim \frac{b}{\mathcal{E}_A}$, $b, b' \in B^j$, $j = 0,1$;

$\frac{b}{\mathcal{E}_A} \sim \frac{b}{\mathcal{E}_A}$, $b, b' \in B^1$, $j = 0,1$;

$1^j \sim 1^j$, where $1^j \in B^j_{[1]}$ is the unit;

$\varphi^0(a) \sim \varphi^1(a)$, for $a \in A$. 

The quotient generalized ring $\Delta/\mathcal{E}_A$ is the tensor product $B^0 \otimes A$, the homomorphism $\psi^j$ is given by $\psi^j(b) = \frac{b}{\mathcal{E}_A}$, $b \in B^j$. 

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Note that every element of $(B^0 \otimes B^1)_X$ can be expressed (non-uniquely) as

$$(a, b) = (a_1 \triangleleft a_2 \triangleleft \cdots \triangleleft a_m) \parallel (b_1 \triangleleft \cdots \triangleleft b_n) \mod e_A$$

with $a_i \in B_i^{(\text{mod}2)}$, $b_j \in B_j^{(\text{mod}2)}$, and $f_1 \circ \cdots \circ f_n = c_X \circ g_1 \circ \cdots \circ g_m$ (where $c_X \in \mathcal{S} \mathcal{E} \mathcal{T}_\bullet(X, [1])$ is the canonical map, $c_X(x) = 1$ for all $x \in X$). These elements are multiplied and contracted by the formulas of multiplication 8.2.7 and contraction 8.2.8.

Example 13.1.1

For monoids $M_0$, $M_1$, $N$, and homomorphisms $\psi^i \in \text{Mon}(N, M_i)$, $i = 0, 1$, we have (by adjunction 8.3.44),

$$\mathbb{F}\{M_0\} \otimes \mathbb{F}\{M_1\} = \mathbb{F}\{M_0 \otimes M_1\}$$

where $M_0 \otimes M_1$ is the fibered sum in the category $\text{Mon}$. The monoid $M_0 \otimes M_1$ is given by elements $m_0 \otimes m_1$, $m_i \in M_i$, with relations

$$m_0 \otimes 0 = 0 \otimes 0 = 0 \otimes m_1 \quad m_i \in M_i$$

and

$$m_0 \cdot \psi^0(n) \otimes m_1 = m_0 \otimes \psi^1(n) \cdot m_1 \quad n \in N$$

Example 13.1.2

For a commutative ring $B$, let $B^1$ denote the underlying multiplicative monoid of $B$ (i.e. forget addition), and let $\mathbb{F}\{B^1\}$ denote the associated generalized ring, cf. 8.3.5. From the identity map $B^1 \rightarrow \mathcal{G}(B)[1]$, we obtain by adjunction 8.3.44 the canonical injective homomorphism $J_B \in \mathcal{G}(\mathbb{F}\{B^1\}, \mathcal{G}(B))$. The unique homomorphism of rings $\mathbb{N} \rightarrow B$, gives the unique homomorphism of generalized rings $I_B \in \mathcal{G}(\mathbb{N}, \mathcal{G}(B))$. We get a canonical homomorphism of generalized rings,

$$\Psi_B = I_B \otimes J_B \in \mathcal{G}(\mathbb{N}) \otimes \mathcal{F}\{B^1\}, \mathcal{G}(B)$$

The homomorphism $\Psi_B$ is always surjective (as follows from 8.3.19). For any monoid $B$, the elements of the generalized ring $\mathcal{N}^B = \mathcal{G}(\mathbb{N}) \otimes \mathbb{F}\{B\}$, can be described as in (13.1.2), but we can move the elements of $\mathbb{F}\{B\}$ to the right (using 8.2.14), and we can take the elements of $\mathcal{G}(\mathbb{N})$ to be the generators $1_\mathbb{N}$; thus we can write every element of $\mathcal{N}^B_X$ as $\left(1_\mathbb{N} \circ \mu, 1_x\right)$, with $\pi \in \mathcal{S} \mathcal{E} \mathcal{T}_\bullet(X, X)$, and $\mu \in (B)\widetilde{X}$,

$$\mathcal{N}^B_X = \left\{\left(\pi : \widetilde{X} \rightarrow X, \mu : \widetilde{X} \rightarrow B\right)\right\} / \approx$$

(13.1.7)
The elements of $\mathcal{N}_X^B$ are (isomorphism classes of) sets over $X \prod B$, where the equivalence relation $\approx$ is invariant by isomorphisms, i.e.

$$(\pi : \tilde{X} \to X, \mu : \tilde{X} \to B) \approx (\pi' : \tilde{X}' \to X, \mu' : \tilde{X}' \to B)$$

if there is a bijection $\sigma : \tilde{X} \to \tilde{X}'$, $\pi = \pi' \circ \sigma$, $\mu = \mu' \circ \sigma$, and by zero, i.e.

$$(\tilde{X}, \pi, \mu) \approx (\tilde{X} \setminus \{x\}, \pi|_{\tilde{X} \setminus \{x\}}, \mu|_{\tilde{X} \setminus \{x\}}) \text{ if } \mu(x) = 0.$$ (13.1.8)

For $f \in \text{Set}_*(X, Y)$, and for $(\tilde{X}, \mu) \in \mathcal{N}_X^B$, $(Z, \lambda) \in \mathcal{N}_f^B$, we have the contraction, cf. §8.2.8,

$$((\tilde{X}, \mu) \parallel (Z, \lambda)) = \left(\tilde{X} \prod_X Z, (\mu \parallel \lambda)\right)$$ (13.1.9)

$$\mu \parallel \lambda)(x, z) = \mu(x) \cdot \lambda(z).$$

For $(\tilde{Y}, \mu) \in \mathcal{N}_f^B$ we have the multiplication, cf. §8.2.7.

$$(\tilde{Y}, \mu) \cdot (Z, \lambda) = (\tilde{Y} \prod_Y Z, \mu \cdot \lambda)$$ (13.1.10)

$$\mu \cdot \lambda(y, z) = \mu(y) \cdot \lambda(z).$$

For a commutative rig $B$, the canonical homomorphism $\Psi_B \in \mathcal{GR}(\mathcal{N}^{B!}, \mathcal{G}(B))$ is given in this description as

$$\left(\Psi_B(\tilde{X}, \mu)\right)_z = \sum_{\tilde{x} \in \tilde{X}} \mu(\tilde{x})$$ (13.1.11)

To get such a surjective homomorphism we can use any multiplicative submonoid $B_0 \subseteq B'$ such that $\mathbb{N}\{B_0\} = B$. For example, for $B = \mathbb{Z}$ the integers, we can take $B_0 = \{0, \pm 1\}$, and we get a surjective homomorphism

$$\Psi \in \mathcal{GR}(\mathcal{G}([\mathbb{N}]) \otimes \mathbb{F}\{\pm 1\}, \mathcal{G}(\mathbb{Z}))$$ (13.1.12)

### 13.1.3 generators and relations for $\mathcal{G}(B)$, $B$ commutative ring.

We have a surjective homomorphism

$$\Phi : \Delta^{[2]} \to \mathcal{G}(\mathbb{N})$$

$$\delta = \delta^{[2]} \mapsto (1, 1) \in \mathcal{G}(\mathbb{N})^{[2]}$$

$$\Phi_X(F_1, \{\overline{F}_x\}, \sigma) / \approx = (\# \sigma \overline{F}_x)_{x \in X}$$ (13.1.13)
Theorem 13.1.4
The equivalence-ideal $\mathcal{E}\mathcal{R}(\Phi) = \Delta^2 \prod \Delta^2$ is generated by $\mathcal{G}(\mathbb{N})$

**Zero:** $\delta \triangleright 1_1 = 1, \quad 1_1 \in F_{[2],[1]}, 1_1(1) = 1 \in [2]$

**Ass:** $\delta \triangleright \delta = \delta \triangleright \delta$

**Comm:** $\delta \triangleright 1_{(i \quad \delta)} = \delta$

We get a surjective homomorphism

$$\Phi_\pm : F\{\pm 1\} \otimes \Delta^2 \to \mathcal{G}(\mathbb{Z})$$

(13.1.15)

Theorem 13.1.5
The equivalence ideal $\mathcal{E}\mathcal{R}(\Phi_\mathbb{Z})$ is generated by the relations (13.1.14), and the relation

**Cancelation:** $(\delta \triangleright (-1)_{i=1,2}) / \delta = 0$

(13.1.16)

For a commutative ring $B$ we get a surjective homomorphism

$$\Phi_B : F\{B\} \otimes \Delta^2 \to \mathcal{G}(B)$$

(13.1.17)

Theorem 13.1.6
The equivalence-ideal $\mathcal{E}\mathcal{R}(\Psi_B)$ is generated by the relations (13.1.14), and the relations for $b_1, b_2 \in B$

$$(b_1, b_2): \quad (\delta \triangleright (b_i)_{i=1,2}) / \delta = (b_1 + b_2) \in F\{B\}[1] \equiv B.$$  

(13.1.18)

The proofs of theorems 13.1.4-6 are the same as the proof given in Theorem 2.10.1.

Every element $G \in (F\{B\} \otimes \Delta^2)_X$ can be represented (after moving the elements of $F\{B\}$ to the boundary using (13.2.14)) as $G = (G_1; \{G_x\}_{x \in X}; \sigma; \mu)$ $G_1, G_x$ are $\{1, 2\} = [2]$-labelled binary trees

$$\sigma : \partial G_1 \to \partial G_x \quad x \in X$$

(13.1.19)

and

$$\Phi_B(G) = \left( \sum_{z \in G_x} \mu(\sigma^{-1}(z)) \right)_{x \in X}$$

(13.1.20)
Note that the associated graph $\tilde{G} \in \text{Graph}^{[1],X}$ from $X$ to $[1]$, obtained by going from $X$ up the trees $\overline{G}_x$, and then via $\sigma^{-1}$, down the tree $G_1$,

$$\tilde{G}_0 \equiv G_1 \prod_{x \in X} (\prod_{x \in X} G_x) \equiv G_1 \circ (\prod_{x \in X} \overline{G}_x) \quad (13.1.21)$$

is already in the ("left")\(\rightarrow ("right")\) form of the proof of theorem 2.10.1.

### 13.2 The arithmetical plane $\mathcal{G}(\mathbb{N}) \otimes_f \mathcal{G}(\mathbb{N})$

We next give a description of the arithmetical plane $\mathcal{G}(\mathbb{N}) \otimes \mathcal{G}(\mathbb{N})$.

An **oriented-tree** is a (rooted) tree $F$ together with a map

$$\varepsilon_F : F \setminus \hat{F} \to \{0, 1\} \quad (13.2.1)$$

It is **1-reduced** if $\nu(a) \neq 1$ for all $a \in F$.

If for some $a \in F$, $S_F^{-1}(a) = \{a'\}$, we obtain by 1-reduction the tree

$$1_a(F) = F \setminus \{a\} \quad (13.2.2)$$

with $S_F(a') = S_F(a)$.

For every oriented tree $F$ there is a unique 1-reduced tree $F_{1-req}$; it is obtained from $F$ by a finite sequence of 1-reductions.

The oriented tree $F$ is **<r reduction** if for all $a \in F \setminus (\hat{F} \prod \{0_F\})$, $\varepsilon(a) \neq \varepsilon(S(a))$.

If for some $a \in F \setminus (\hat{F} \prod \{0_F\})$, $\varepsilon(a) = \varepsilon(S(a))$, we obtain by **<r reduction** the tree

$$O_a(F) = F \setminus \{a\} \quad (13.2.3)$$

with $S_{O_a(F)}(a') = S_F(a)$ if $S_F(a') = a$.

For every oriented tree $F$ there is a unique <r-reduced tree $F_{<r-req}$; it is obtained from $F$ by a finite sequence of <r-reductions. For a <r-reduced oriented tree $F$, the orientation $\varepsilon_F$ is completely determined by its value at the root $\varepsilon_F(0_F)$, since $\varepsilon_F(x) = \varepsilon_F(0_F) + \text{ht}(x) (\text{mod} 2)$. Thus we view <r-reduced oriented trees $F$ as ordinary trees together with an orientation of the root $\varepsilon_F = \varepsilon_F(0_F) \in \{0, 1\}$.

Note that the operations of 1-reduction and <r-reduction do not alter the boundary of a tree.

We let $\approx$ denote the equivalence relation on oriented trees generated by 1-reductions and <r-reductions. We let $[F]$ denote the equivalence class of the oriented tree $F$. Thus $[F] = [F']$ if and only if there exist $F = F_0, F_1, \ldots, F_l = F'$, such that for $j = 1, \ldots, l$, the pair $\{F_j, F_{j-1}\}$ is related by 1-reduction, or <r-reduction; it follows that there is a canonical identification of the boundaries: $\partial F = \partial F'$.

For a finite set $X \in F$, let $\Upsilon_X$ denote the collection of isomorphism classes of data

$$\Upsilon_X = \{F = ([F_1]; [\overline{F}_x], x \in X; \sigma_F)) / \approx \quad (13.2.4)$$
where \( F_1, F_x \) are oriented trees taken modulo \( \approx \)-equivalence, and \( \sigma_F \) is a bijection \( \sigma_F : \partial F_1 \rightarrow \prod_{x \in X} \partial F_x \) and the data is taken up to isomorphism and consistent-commutativity. Thus explicitly, the data \( F \) is equivalent to the data \( F' \), if and only if there exists \( F = F^0, F^1, \ldots, F^l = F' \) such that for \( j = 1, \ldots, l \) the pair \( \{ F^j, F^{j-1} \} = \{ G, G' \} \) is related by either:

**Isomorphism:** have isomorphism \( \tau_1 : G_1 \rightarrow G_1', \tau_x : G_x \rightarrow G'_x, x \in X \) such that \( \sigma_{G'} \circ \tau_1 (b) = \tau_x \circ \sigma_G (b) \) for \( b \in \partial G_1, \sigma_G (b) \in \partial G_x \).

**1-reduction:** have \( G' = 1_a G \), for some \( a \in G_1 \prod_{x \in X} \partial G_x \) with \( \nu (a) = 1 \), cf. (13.2.2).

**\( \langle - \rangle \)-reduction:** have \( G' = O_a G \), for some \( a \in (G_1 \setminus (\partial G_1 \prod \{0\})) \prod_{x \in X} \partial G_x \setminus (\partial G_x \prod \{0\}) \) with \( \varepsilon (a) = \varepsilon (S(a)) \), cf. (13.2.3).

**Consistent-commutativity:** \( \{ G, G' \} \) of the form (8.3.55) or (8.3.57). The operations of multiplication (8.3.51), and of contraction (8.3.53), induce well defined operations on equivalent classes of data, and make \( \Upsilon \) into a commutative generalized ring. It is straightforward to check that

\[
F \triangleleft G \equiv F \triangleleft (1_a G) \equiv F \triangleleft (O_a G) \\
\equiv (1_a F) \triangleleft G \equiv (O_a F) \triangleleft G \\
F \parallel G \equiv (F \parallel 1_a G) \equiv (F \parallel O_a G) \\
\equiv (1_a F \parallel G) \equiv (O_a F \parallel G) \tag{13.2.5}
\]

whenever the operations \( 1_a, O_a \) are relevant, and that \( \Upsilon \) satisfies the axioms of a commutative generalized ring.

Note that for \( \varepsilon = 0, 1 \), we have the elements

\[
\delta_\varepsilon^X = \left( [X \prod \{0\}]; [0_x], x \in X; \sigma \right) \in \Upsilon_X \tag{13.2.6}
\]

where \( X \prod \{0\} \) is the oriented tree with \( \varepsilon (0) = \varepsilon \), \( S(x) = 0 \) for \( x \in X \), and \( \sigma : X \rightarrow \prod_{x \in X} \{0_x\} \) is the natural bijection \( \sigma (x) = 0_x \).

For \( f \in \mathbf{Set}_\bullet (X, Y) \), and \( (\delta_f^\varepsilon)(y) = \delta_{f^{-1}(y)}^\varepsilon \), \( y \in Y \), we have via \( \langle - \rangle \)-reduction

\[
\delta_\varepsilon^Y \triangleleft \delta_f^\varepsilon \cong \delta_{D(f)}^\varepsilon \tag{13.2.7}
\]

we also have by 1-reduction

\[
\delta_{[1]} = ([\{0\} \prod \{1\}]; [0_1]; \sigma) \cong ([0_1]; [0_1]; id) = 1 \in \Upsilon_{[1]} \tag{13.2.8}
\]

Thus we get homomorphisms, \( \Psi^\varepsilon \in \mathcal{G} \mathcal{R} (\mathcal{G} (\mathbb{N}), \Upsilon) \) with \( \Psi^\varepsilon (\mathbb{I}_X) = \delta_\varepsilon^X \). It is clear that \( \Upsilon \) is generated by the \( \delta_\varepsilon^X \), and the only relations they satisfy are (13.2.7),
and consistent commutativity. It follows that $\Upsilon$ is the sum of $G(N)$ with itself in the category of commutative generalized rings: for any $A \in GR_C$,

$$GR_C(G(N), A) \times GR_C(G(N), A) \overset{\sim}{\to} GR_C(\Upsilon, A)$$

\((\varphi \circ \psi_0, \varphi \circ \psi_1) \mapsto \varphi$$

$$\varphi_0, \varphi_1 \mapsto \varphi_0 \otimes \varphi_1(\delta_X^X) := \varphi^*(1_X) \quad (13.2.9)$$

The diagonal homomorphism

$$\nabla \in GR_C(\Upsilon, G(N)) \quad (13.2.10)$$

is determined by $\nabla_X(\delta_X^X) = 1_X$, and is given explicitly by

$$\nabla_X([F_1]; [\hat{F}_x]_{x \in X}; \sigma) = (x\hat{F}_x)_{x \in X} \quad (13.2.11)$$

The homomorphism $\nabla$ is surjective, but it is not injective.

For a monoid $B$, the tensor product $\Upsilon \otimes F\{B\}$ can be described as isomorphism classes of data

$$(\Upsilon \otimes F\{B\})_X := \{F = ([F_1]; ([\hat{F}_x]_{x \in X}; \sigma; \mu_F))/ \cong \quad (13.2.12)$$

Here the data $([F_1]; ([\hat{F}_x]_{x \in X}; \sigma; \mu_F)$ is the data for $\Upsilon_X$, and $\mu_F$ is a map $\mu_F: \partial F_1 \to B$, and isomorphisms are required to preserve the $B$-valued maps, and the zero law holds in the form:

$$\mu_F(b) = 0, \sigma_F(b) = x_0 \Rightarrow F \cong ([F_1\{b\}; ([\hat{F}_x]_{x \neq x_0}; \{\hat{F}_{x_0}\}\{\sigma_F(b)\}] \quad (13.2.13)$$

The operations of multiplication and contraction are the given ones on the $\Upsilon$-part of the data (i.e. given by $(8.3.51)$ and $(8.3.53)$), and are given on the $B$-valued maps by (using the notations of $(8.3.52)$ and $(8.3.54)$):

$$\mu_{G \otimes F}(b, a) = \mu_G(b) \cdot \mu_{G_{\partial F}}(a), \quad b \in \partial G_1, a \in \partial F(b)$$

$$\mu_{G \otimes F}(b, a) = \mu_G(b) \cdot \mu_{F_{\partial F}(\sigma^{-1} a)}, \quad b \in \partial G_1, a \in \partial F_{\partial F}(b) \quad (13.2.14)$$

For commutative rings $B_0, B_1$, taking $B = B_0 \otimes B_1^1$ (the sum in Mon, cf. $(13.1.4)$), we get the generalized ring

$$\Upsilon \otimes F\{B_0 \otimes B_1^1\} = G(N) \otimes F\{B_0\} \otimes G(N) \otimes F\{B_1^1\} \quad (13.2.15)$$

which maps surjectively onto $G(B_0) \otimes G(B_1^1)$.

For the integers $\mathbb{Z}$, taking $B = \{0, \pm 1\}$, we get the generalized ring

$$\Upsilon \otimes F\{\pm 1\} = (G(N) \otimes F\{\pm 1\}) \otimes (G(N) \otimes F\{\pm 1\}) \quad (13.2.16)$$
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with a surjective homomorphism

$$\pi: \mathcal{Y} \otimes_{\mathbb{F}} \mathbb{F}\{\pm 1\} \to \mathcal{G}(\mathbb{Z}) \otimes_{\mathbb{F}\{\pm 1\}} \mathcal{G}(\mathbb{Z}).$$ \tag{13.2.17}

Note that $\ker \pi = \mathcal{E}$ is the equivalence ideal generated by the homogeneous ideal $\mathcal{a}$ generated by the elements giving the "cancellations" on the left and right $\mathcal{G}(\mathbb{Z})$: \[x = (F_\epsilon; \sigma = id; \mu = id) \in (\mathcal{Y} \otimes \mathbb{F}\{\pm 1\})_{[1]}, \quad \epsilon = 0, 1 \tag{13.2.18}\]

with the reduced oriented tree $F_\epsilon = \{0\} \cup \{\pm 1\}, \mathcal{S}(\pm 1) = 0, \mathcal{S}(0) = \epsilon$.

13.3 Products of Grothendieck-Generalized-schemes

The category $\mathcal{GGS}$ has fibred products:

Given maps $f^j \in \mathcal{GGS}(X^j, Y)$, there exists $X^0 \Pi Y^1 \in \mathcal{GGS}$, and maps $\pi_j \in \mathcal{GGS}(X^0 \Pi Y^1, X^j)$, with $f^0 \circ \pi_0 = f^1 \circ \pi_1$, and for any $g^j \in \mathcal{GGS}(Z, X^j)$, with $f^0 \circ g^0 = f^1 \circ g^1$, there exists a unique map $g^0 \circ \pi_0 \in \mathcal{GGS}(Z, X^0 \Pi Y^1)$, such that $\pi_j \circ (g^0 \circ \pi_0) = g^j$, $j = 0, 1$.

Writing $Y = \bigcup \text{spec}^0(A_i)$, $(f^j)^{-1}(\text{spec}^0(A_i)) = \bigcup B^j_{i, k}$, the fibred product $X^0 \Pi Y^1$ is obtained by gluing $\text{spec}^0(B^j_{i, k} \otimes B^1_{i, k})$. See the construction of fibred product of ordinary schemes, e.g. [Hart, Theorem 3.3, p. 87].

13.4 Products of Generalized-schemes

The category $\mathcal{GS}$ has fibred products. This is an immediate corollary of 13.3.

Given maps $\varphi = \{\varphi_j^j\}_{j \geq \sigma(i)} \in \mathcal{GS}(\{X_j\}_{j \in J}, \{Y_i\}_{i \in I})$, and $\varphi' = \{\varphi'_j^j\}_{j' \geq \sigma'(i)} \in \mathcal{GS}(\{X'_j\}_{j' \in J'}, \{Y'_i\}_{i' \in I'}) \tag{13.4.1}$

the fibred product of $\varphi$ and $\varphi'$ in $\mathcal{GS}$ is given by the inverse system $\{X_j \Pi Y_i, X'_j\}$, the indexing set is

$$\{(j, j', i) \in J \times J' \times I \mid j \geq \sigma(i), j' \geq \sigma'(i)\} \tag{13.4.2}$$

13.5 The Arithmetical plane: $X = \text{spec} \mathbb{Z} \prod_{\mathbb{F}\{\pm 1\}} \text{spec} \mathbb{Z}$

This is a special case of 13.4. The (compactified) arithmetical plane $X$ is given by the inverse system $\{X_N \prod_{\text{spec} \mathbb{F}\{\pm 1\}} X_M\}$, with indexing set.
\{(N, M) \in \mathbb{N} \times \mathbb{M} | N, M \text{ square-free} \} \text{ and with}

\[ X_N = \text{spec} \mathcal{G}(\mathbb{Z}) \prod_{\text{spec} \mathcal{G}(\mathbb{Z}[\frac{1}{N}])} \text{spec}(\mathcal{G}(\mathbb{Z}[\frac{1}{N}]) \cap \mathcal{O}_\eta) \quad (13.5.1) \]

as in (12.4.5). This generalized scheme \( X \) contains the affine open dense subset,

\[ \text{spec} \mathcal{G}(\mathbb{Z}) \prod_{\mathfrak{p}(\pm 1)} \text{spec} \mathcal{G}(\mathbb{Z}) = \text{spec}^t(\mathcal{G}(\mathbb{Z}) \otimes \mathcal{G}(\mathbb{Z})) \quad (13.5.2) \]

e.g. basis for neighborhoods of \((p, \eta)\) is given by

\[ \text{spec}^t \left[ \mathcal{G}(\mathbb{Z}[\frac{1}{N}]) \otimes_{\mathfrak{p}(\pm 1)} (\mathcal{G}(\mathbb{Z}[\frac{1}{M}]) \cap \mathcal{O}_\eta) \right] \quad (13.5.3) \]

where \( p \) does not divide \( N \), and \( M \) is arbitrary.

Similarly, for any number field \( K \) we have the compactified surface

\[ \text{spec} \mathcal{O}_K \prod_{\text{spec} \mathcal{O}_K \mathfrak{p}(\mu_K)} \text{spec} \mathcal{O}_K \quad (13.5.4) \]

It contains the affine open dense subset \( \text{spec}^t(\mathcal{G}(\mathcal{O}_K) \otimes_{\mathfrak{p}(\mu_K)} \mathcal{G}(\mathcal{O}_K)) \).
Chapter 14

Modules and differentials

14.1 A-module

Definition 14.1.1
Let $A \in \cal{G} \cal{R}$. An A-module is a functor $M \in (Ab)^{\cal{G}}$, with $M_{[0]} = \{0\}$, and operations:

**Multiplication**: for $z_0 \in Z \in \cal{F}$,
$$\preceq_{z_0} : M_Z \times A_X \rightarrow M_{Z \prec X}$$ (14.1.1)

**Contraction**: for $X \subseteq Y \in \cal{F}$,
$$\lnot / \lnot : M_Y \times A_X \rightarrow M_{Y / X}$$ (14.1.2)

These are assumed to satisfy the Disjointness Axioms I, II, III, and so for $f \in \text{Set}_\bullet(Y, Z)$, (with $A_f = \prod_{z \in Z} A_{f^{-1}(z)}$), we have "multiple"

**Multiplication**:
$$\preceq : M_Z \times A_f \rightarrow M_Y$$ (14.1.3)

**Contraction**:
$$\lnot / \lnot : M_Y \times A_f \rightarrow M_Z$$ (14.1.4)

We further assume these operations satisfy the following axioms:

**Unit and Functoriality**:
$$\text{for } f \in \cal{F}_{Z,Y}, m \in M_Y, m \lnot / = 1_f \preceq m = f_M(m)$$ (14.1.5)

**Homomorphism**:
$$\begin{align*}
(m_1 + m_2) \triangleleft a &= (m_1 \triangleleft a) + (m_2 \triangleleft a) \\
(m_1 + m_2) \lnot / a &= (m_1 \lnot / a) + (m_2 \lnot / a)
\end{align*}$$ (14.1.6)
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Associativity:
\[(m \trianglelefteq a_0) \trianglelefteq a_1 = m \trianglelefteq (a_0 \trianglelefteq a_1)\] (14.1.7)

Left Adjunction:
\[(m \trianglerightleftlefteq a_0) \trianglerightleftlefteq a_1 = m \trianglerightleftlefteq (a_1 \trianglerightleftlefteq a_0)\] (14.1.8)

Right Adjunction:
\[m \trianglerightleftlefteq (b \trianglerightleftlefteq a) = (m \trianglerightleftlefteq a) \trianglerightleftlefteq b\] (14.1.9)

Left Linearity:
\[m \trianglelefteq (b \trianglerightleftlefteq a) = (m \trianglerightleftlefteq b) \trianglerightleftlefteq a\] (14.1.10)

We are only interested in the commutative-\(A\)-modules that further satisfy

Right Linearity:
\[(m \trianglerightleftlefteq a) \trianglelefteq b = (m \trianglelefteq f^*b) \trianglerightleftlefteq g^*a \quad \text{for} \quad a \in A_f, b \in A_g.\] (14.1.11)

Note that for \(M\) an \(\mathbb{F}\)-module, i.e. a functor \(M \in (A\mathbb{b})^{\mathbb{F}}\) with \(M_{[0]} = \{0\}\); and for \(X \subseteq Y \subseteq \mathbb{F}, M_X\) is a subgroup of \(M_Y\), a direct summand, and the projection \(M_Y \to M_X\) is denoted by \(m \mapsto m|_X\).

In particular we have the "matrix-coefficient" map
\[J_X : M_X \to (M_{[1]})^X\]
\[J_X(m) = (m|_{(x)})_{x \in X} \quad \text{(identifying} \quad M_{[1]} = M_{(x)} \subseteq M_X).\] (14.1.12)

If these maps are injective for all \(X \in \mathbb{F}\) we say \(M\) is a "matrix"-\(A\)-module.

**Definition 14.1.2**

A homomorphism of \(A\)-modules \(\varphi : M \to M'\) is a natural transformation of functors that commutes with the \(A\)-action.

Thus we have an abelian category \(A\text{-mod}\). It is complete and co-complete: all (co-) limits can be taken pointwise. It has enough projectives and injectives: the evaluation functor at \(X \in \mathbb{F}, i^X : A\text{-mod} \to \mathbb{A}b, i^X M = M_X\), has a left (resp. right) adjoint \(i^X_*\) (resp. \(i^X^*\)).

In particular, we have the

**14.1.3 Free \(A\text{-mod}\) of degree \(X \in \mathbb{F}\):**

\[A^X = i^X_*\mathbb{Z},\quad \text{generated by} \quad \delta_X \in A^X_X, \quad \text{and} \]

\[A\text{-mod}(A^X, M) \cong M^X \text{ via } \varphi \mapsto \varphi_X(\delta_X).\] (14.1.13)

The elements of degree \(Z \in \mathbb{F}, m \in A^X_Z\), are linear combinations
\[m = \sum_{i=1}^k m_i \cdot (\delta_X \trianglelefteq a_i) \trianglerightleftlefteq b_i \quad \text{with} \quad m_i \in \mathbb{Z}, a_i \in A_{f_i : X_i \rightarrow X}, b_i \in A_{g_i : X_i \rightarrow Z}\] (14.1.14)
and for \( f \in \text{Set}_\bullet(Y, Z) \), (resp. \( f \in \text{Set}_\bullet(Z, Y) \)), and \( a \in A_f \), the action of \( a \) on such an element \( m \) is given by
\[
m \triangleleft a = \sum_{i=1}^{k} m_i(\delta_X \triangleleft (a_i \triangleleft f_i^* a_i)) \triangleright f_i^* b_i
\]  
(14.1.15)

resp. \( m \triangleright a = \sum_{i=1}^{k} m_i(\delta_X \triangleright a_i) \triangleright (a \triangleright b_i) \).

The elements \( (\delta_X \triangleright a) \triangleright b \) are subjected to the axioms of an \( A \)-mod, and we have
\[
(\delta_X \triangleright (a \triangleright c)) \triangleright b = (\delta_X \triangleright a) \triangleright (b \triangleright c)
\]  
for \( Y \xrightarrow{c} W \xrightarrow{b} Z \) \footnote{Diagram not shown.

\[
(\delta_X \triangleright (a \triangleright c)) \triangleright b = (\delta_X \triangleright a) \triangleright (b \triangleright c)
\]  
(14.1.16)

Example 14.1.4

For \( A = G(R) \), \( R \) a commutative rig (so every \( a = (a_x) \in A_X \), can be written as
\[a = 1_X \triangleleft a_x,
\]  
with \( a_x >: (A_{[1]}^X = A_{id_X}) \), we have
\[
(A^X)_Z := \mathbb{Z} \cdot A_{X \otimes Z} / \mathbb{Z} ,\]  
(14.1.17)

the free abelian group on \( X \) by \( Z \) matrices over \( R \) (modulo \( \mathbb{Z} \cdot 0_X \otimes Y \), because
\[A_{[0]}^X = \{0\} \). For \( a = (a_{x,z}) \in R^{X \otimes Z} \), we have the generators
\[
\delta_X \cdot (a) := (\delta_X \triangleleft (a_{x,z})) \triangleright (1_X \triangleright z) = (\delta_X \triangleright 1_z) \triangleright (a_{x,z})
\]  
(14.1.18)

and the \( G(R) \)-action on these generators is the diagonal action at each \( x \in X \).

Localization 14.1.5

For a multiplicative set \( S \subseteq A_{[1]}^+ \), and \( M \in A \)-mod, we have the localization
\[S^{-1}M \in S^{-1}A \text{-mod described as in . In particular, we have the localizations}
\]  
(14.1.19)

\( M_p = S_p^{-1}M \in A_p \text{-mod, } p \in \text{Spec}^\ell(A), \) and \( M_s \in A_s \text{-mod, } s = s^t \in A_{[1]}^+ \).
CHAPTER 14. MODULES AND DIFFERENTIALS

Definition 14.1.6
For \((X, \mathcal{O}_X) \in \mathcal{GRS}\), an \(\mathcal{O}_X\)-module is a functor (where \(C_X\) is the category of open subsets of \(X\) and inclusions),

\[
\begin{align*}
M : & C_X^{op} \times \mathbb{F} \to Ab \\
& U, Z \mapsto M(U)_Z
\end{align*}
\]

(14.1.19)

such that for \(U \subseteq X\) open, \(M(U) = \{M(U)_Z\} \in \mathcal{O}_X(U)-mod\); for \(U \subseteq U' \subseteq X\) open, the homomorphisms \(M(U')_Z \to M(U)_Z, m \mapsto m|_U\) is compatible with the operations: \((m \cdot a)|_U = m|_U \cdot a|_U, (m \cdot a)|_U = m|_U \cdot (a|_U); \) and for fixed \(Z \in \mathbb{F} : U \to M(U)_Z\) is a sheaf.

A homomorphism of \(\mathcal{O}_X\)-modules \(\varphi : M \to M'\), is a natural transformation of functors \(\varphi(U)_Z : M(U)_Z \to M'(U)_Z\), such that for \(U \subseteq X\) open, \(\{\varphi(U)_Z\}_{Z \in \mathbb{F} \in \mathcal{O}_X(U)-mod} \left( M(U), M'(U) \right)\), and for \(Z \in \mathbb{F}, \{\varphi(U)_Z\}_{U \subseteq X} \in (Ab)^{C_X^{op}}(M_Z, M'_Z)\).

Thus we have an abelian category: \(\mathcal{O}_X\)-mod.

For \(A \in \mathcal{GRC}, M \in A\text{-mod}\), we have \(\hat{M} \in \mathcal{O}_A\text{-mod}\), defined as in Definition 7.3.2. It has stalks at \(p \in \text{Spec}^t A\) given by \((\hat{M})_p = M_p\), cf. (14.1.7), and it has global sections over a basic open set \(D^+_s \subseteq \text{Spec}^t A\) given by \(M(D^+_s) = M_s, s = s^t \in A^+_t\) cf. Theorem 11.3.2 (The commutativity of \(M\) is essential!).

For \(X \in \mathcal{GGS}\), a Grothendieck-generalized-scheme, we have the full subcategory of "quasi-coherent" \(\mathcal{O}_X\)-modules, \(q.c.\mathcal{O}_X\text{-mod} \subseteq \mathcal{O}_X\text{-mod}\). Its objects are the \(\mathcal{O}_X\)-modules satisfying the equivalent conditions of theorem 7.3.3, and for \(X = \text{Spec}^t A\) affine, localization gives an equivalence

\[
A\text{-mod} \xRightarrow{\sim} q.c.\mathcal{O}_A\text{-mod} \subseteq \mathcal{O}_A\text{-mod}
\]

(14.1.20)

Restriction and extension of scalars 14.1.7

For \(\varphi \in \mathcal{GRC}(B, A)\), we have the adjoint functors (using geometric notations):

\[
A\text{-mod} \xRightarrow{\varphi_*} B\text{-mod}
\]

(14.1.21)

The right adjoint takes \(N \in A\text{-mod}\) to \(\varphi_* N = N\) with \(B\)-action given using \(\varphi : n \cdot b = n \cdot \varphi(b), n \parallel b = n \parallel \varphi(b)\).

The left adjoint takes \(M \in B\text{-mod}\) into the \(A\)-module \(\varphi^* M = M^A\), whose elements in degree \(Z \in \mathbb{F}, m \in (M^A)_Z\), can be written as sums

\[
m = \sum_{i=1}^{k} [(m_i)_i \cdot a_i] / a'_i \text{ with } m_i \in M_{X_i}, a_i \in A_{f_i : Y_i \to X_i}, a'_i \in A_{g_i : Y_i \to Z}\]

(14.1.22)
CHAPTER 14. MODULES AND DIFFERENTIALS

and \( a \in A_f, f \in \text{Set}_\bullet(Y, Z) \), (resp. \( f \in \text{Set}_\bullet(Z, Y) \)), acting via

\[
m \triangleleft a = \sum_{i=1}^{k} ([m_i] \triangleleft (a_i \triangleleft g_i^*a)) \parallel f^*a'_{i}
\]

resp. \( m \parallel a = \sum_{i=1}^{k} ([m_i] \parallel a_i) \parallel (a \parallel a'_{i})
\]

and we have the relations:

\[
([m + m'] \triangleleft a) \parallel a' = ([m] \triangleleft a) \parallel a' + ([m'] \triangleleft a) \parallel a'
\]

\[
([m \triangleleft b] \triangleleft a) \parallel a' = ([m] \triangleleft (\varphi(b) \triangleleft a)) \parallel a'
\]

\[
([m \parallel b] \triangleleft a) \parallel a' = ([m] \triangleleft h^*(a)) \parallel (a' \triangleleft f^* \varphi(b)), \quad b \in B_h, a \in A_f.
\]

14.2 Derivations and differentials

Definition 14.2.1

For \( A \in \mathcal{GR}_C, M \in A\text{-mod} \), we define the even and odd infinitesimal extensions \( A \prod^\pm M \), an abelian group object of \( \mathcal{GR}/A \), by

\[
\mathcal{F} \ni X \mapsto (A \prod^M)_{X} = A_X \prod M_X \quad (14.2.1)
\]

and for \( f \in \text{Set}_\bullet(Y, Z), a = (a(z)) \in A_f, m = (m(z)) \in M_f := \prod_{z \in Z} M_{f^{-1}(z)} \), we have

**multiplication:** for \( a_Z \in A_Z, m_Z \in M_Z \),

\[
(a_Z, m_Z) \triangleleft (a, m) := (a_Z \triangleleft a, m_Z \triangleleft a + \sum_{z \in Z} m^{(z)} \parallel (a_Z|_z)) \quad (14.2.2)
\]

**\pm-contraction:** for \( a_Y \in A_Y, m_Y \in M_Y \),

\[
(a_Y, m_Y) \parallel (a, m) := (a_Y \parallel a, m_Y \parallel a \pm \sum_{z \in Z} m^{(z)} \parallel (a_Y|_{f^{-1}(z)})) \quad (14.2.3)
\]

We have,

**projection:** \( \pi \in \mathcal{GR}(A \prod^M A), \quad \pi(a, m) = a, \)

**addition:** \( \mu \in \mathcal{GR}/A \left( (A \prod^M A)(A \prod^M M), A \prod^M M \right), \mu((a, m), (a, m')) = (a, m + m') \)

**unit:** \( \epsilon \in \mathcal{GR}/A(A, A \prod^M M), \epsilon(a) = (a, 0), \)

**antipode:** \( S \in \mathcal{GR}/A(A \prod^M M, A \prod^M M), S(a, m) = (a, -m). \)

(14.2.4)

(Note: When \( A \in \mathcal{GR}_C, M \) a (commutative) \( A \)-module, the generalized rings \( A \prod^\pm M \) need not be commutative.)
Definition 14.2.2

For $\varphi \in \mathcal{GR}(C, A), M \in A\text{-mod}$, an even/odd $C$-linear derivations from $A$ to $M$, is a collection of maps $\delta = \{\delta_X : A_X \to M_X\}_{X \in \mathbb{P}}$ satisfying:

\((\ast)\) Leibnitz: for $f \in \text{Set}_\bullet(Y, Z), a_Z \in A_Z, a_Y \in A_Y, a_f \in A_f,$

\[\delta(a_Z \lhd a_f) = \delta(a_Z) \lhd a_f + \sum_{y \in Y} \delta(a_f^y) \lhd (a_Z(y))\]

\[\delta(a_Y \rhd a_f) = \delta(a_Y) \rhd a_f + \sum_{y \in Y} \delta(a_f^y) \rhd (a_Y|_{f^{-1}(y)})\]  \hspace{1cm} (14.2.5)

\((\ast\ast)\) $C$-linear: for $c \in C$, $\delta \varphi(c) \equiv 0$.

Note that, since $\mathbb{P} \subseteq C$, we get $\delta(a \lhd 1_f) = \delta(a) \lhd 1_f$, and $\delta$ is always a natural transformation.

We denote by $\text{Der}_C^\pm(A, M)$ the collection of even/odd $C$-linear derivations $\delta : A \to M$. These are functors: $A\text{-mod} \to Ab, M \mapsto \text{Der}_C^\pm(A, M)$, represented by

The module of even/odd differentials $\Omega^\pm(A/C) \in A\text{-mod}$:

\[\text{Der}_C^\pm(A, M) \equiv A\text{-mod}(\Omega^\pm(A/C), M)\]

\[\varphi \circ d_{A/C}^\pm \leftarrow \varphi\]  \hspace{1cm} (14.2.6)

with the universal even/odd derivation $d_{A/C}^\pm : A \to \Omega^\pm(A/C)$.

For $Z \in \mathbb{P}$, the elements of $\Omega^\pm(A/C)_Z$ are sums of the form

\[\sum_{i=1}^k m_i \cdot (d^\pm(a_i) \lhd a'_i) \rhd a''_i\]

\[m_i \in \mathbb{Z}, a_i \in A_{W_i}, a'_i \in A_{f_i : Y_i \to W_i}, a''_i \in A_{\delta_{Y_i}}\]  \hspace{1cm} (14.2.7)

subjected to the $\pm$ Leibnitz and $C$-linearity relations.

14.2.4 Example

For $A = C[\delta_{W}] = C \otimes \Delta^W$, the free commutative generalized ring over $C$ generated by $\delta_W \in A_W$, we have

\[\Omega^\pm(C[\delta_{W}]/C) = \text{free } C[\delta_{W}]-\text{module generated by } d^\pm(\delta_{W})\text{ in degree }W,\]  \hspace{1cm} (14.2.8)

modulo "$\pm$ almost - linearity" - the derived commutativity relation (i.e. the $C[\delta_{W}]-$sub-module generated by the difference of $d^\pm$ applied to $(a \lhd \delta_{W}) \lhd \delta_{W}$ and to $(a \lhd \delta_{W}) \rhd \delta_{W}$, $a \in C[\delta_{W}]_Y, W \subseteq Y$).

For $\varphi \in \mathcal{GR}_C(C, A), B \in C\setminus \mathcal{GR}/A$, i.e. $\varphi = \pi \circ \epsilon \in \mathcal{GR}(C, B), \pi \in \mathcal{GR}(B, A)$,
and for $M \in A$-mod we have the natural identifications

$$C\setminus \mathcal{G}R / A(B, \mathcal{A}P^\pm M) \equiv \text{Der}_C(B, M) \equiv B$-mod($\Omega^\pm(B/C), M) \equiv A$-mod($\Omega^\pm(B/C)^A, M)$

$$\pi \prod \delta \leftrightarrow \delta, \quad \varphi \circ d_{B/C} \leftrightarrow \varphi$$

(14.2.9)

We obtain,

**The adjunctions 14.2.5**

$$\Omega^\pm(B/C)^A \quad A$-mod \quad M$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$B \quad C \setminus \mathcal{G}R / A \quad \mathcal{A}P^\pm M$$

(14.2.10)

Thus the even and odd differentials satisfy the ($\mathcal{G}R$-analogues of) all the properties (0 to 5) of §7.7.

**14.2.6 Example**

In particular we have the $\mathcal{G}(\mathbb{N})$, (resp. $\mathcal{G}(\mathbb{Z})$) -modules $\Omega^\mathbb{N}_\pm = \Omega^\pm(\mathcal{G}(\mathbb{N})/\mathbb{F})$, (resp. $\Omega^\mathbb{Z}_\pm = \Omega^\pm(\mathcal{G}(\mathbb{Z})/\mathbb{F}(\pm 1))$), with the universal even/odd derivation in degree $X \in \mathbb{F}$

$$d_X^\pm : \mathcal{G}(\mathbb{N})X = \mathbb{N}^X \to (\Omega^\mathbb{N}_\pm)_X$$

resp. $d_X^\pm : \mathcal{G}(\mathbb{Z})X = \mathbb{Z}^X \to (\Omega^\mathbb{Z}_\pm)_X$

(14.2.11)

The module $\Omega^\mathbb{N}_\pm(X)$ (resp. $\Omega^\mathbb{Z}_\pm(X)$) is obtained from the free $\mathcal{G}(\mathbb{N})$ (resp. $\mathcal{G}(\mathbb{Z})$) module of degree $[2]$, with generator $d^\pm(1,1)$, modulo the derived relations $\text{(14.2.12±) } \text{ and } \text{(14.2.13)}$. Thus $\Omega^\mathbb{N}_X$ (resp. $\Omega^\mathbb{Z}_X$) is the free abelian group with generators $\left\{ (a_z^\pm), (a_\pm^\pm) \in \mathbb{N}^X \right\}$ (resp. $\mathbb{Z}^X$), modulo the relations, $a, a', a'' \in \mathbb{N}^X$ (resp. $\mathbb{Z}^X$):

$$\pm \text{ almost linearity: } \left\{ \begin{array}{l}
\{ x \cdot a \} + \{ y \cdot a' \} + \{ x \cdot a' \} = \\
\{ x \cdot (a + a') \} + \{ y \cdot (a + a') \} = \\
\{ (x + y) \cdot a \} + \{ x \cdot (a + a') \} = \\
\{ (x + y) \cdot a' \} + \{ y \cdot (a + a') \}
\end{array} \right\}, \ (x,y) \in \mathbb{N} \text{ resp. } \mathbb{Z}.
$$

(14.2.12±)

(apply $d^\pm$ to the identity

$$((x,y) \parallel (1,1)) \parallel (1,1) = ((x,y) \parallel (1,1)) \parallel (1,1)$$

(apply $d^\pm$ to the identity

$$((x,y) \parallel (1,1)) \parallel (1,1) = ((x,y) \parallel (1,1)) \parallel (1,1)$$
or schematically

\(\begin{array}{ccc}
\text{cocycle:} & \{a + a'\} + \{a\} = \{a\} + \{a' + a''\}
\end{array}\)

\((1, 1) \triangleleft \{1, (1, 1)\} = (1, 1) \triangleleft \{(1, 1), 1\}\)

\(\begin{array}{ccc}
\text{normalized:} & \{a\} = 0 = \{a'\}
\end{array}\)

\((1, 1) \parallel (1, 0) = 1\)

\(\begin{array}{ccc}
\text{symmetric:} & \{a'\} = \{a\}
\end{array}\)

\((1, 1) \triangleleft \{1, (a, 1)\} = (1, 1)\)

\(\begin{array}{ccc}
\text{(resp. and cancellation:)} & 2 \cdot \{a\} = 0
\end{array}\)

\(\begin{array}{ccc}
\text{(apply } d^\pm \text{ to the identity)} & (1, 1) \triangleleft \{-a\} = (1, 1) = 0
\end{array}\)

The \(-1\)-almost linearity in \((\text{14.2.12})\) is a consequence of the other relations \((\text{14.2.13})\). (Applying the cocycle relation to the term in the square brackets) we have:

\[
\begin{align*}
\left\{ (x + y)a \right\} - \left\{ (x + y)a' \right\} - \left\{ x \cdot (a + a') \right\} - \left\{ y \cdot (a + a') \right\} &= - \left\{ xa \right\} + \left\{ \frac{xa}{ya} \right\} + \left\{ \frac{x(a + a')}{ya} \right\} + \left\{ \frac{(x + y)a'}{ya} \right\} - \left\{ \frac{x \cdot (a + a')}{ya} \right\} \\
&= - \left\{ xa \right\} - \left\{ y(a + a') \right\} + \left\{ xa \right\} + \left\{ y(a + a') \right\} - \left\{ x(a + a') \right\} \\
&= - \left\{ xa \right\} - \left\{ y(a + a') \right\} + \left\{ xa \right\} - \left\{ y(a + a') \right\} - \left\{ y(a + a') \right\} + \left\{ ya \right\} \\
&= - \left\{ xa \right\} + \left\{ xa \right\} + \left\{ y(a + a') \right\} - \left\{ ya \right\} + \left\{ ya \right\}.
\end{align*}
\]

\((\text{14.2.14})\)

Thus only the relations \((\text{14.2.13})\) are involved in \(\Omega^N_X, \Omega^Z_X\). The universal odd derivation \(d^-_X : N^X \to (\Omega^N_X)_X\) is trivial for \(X = [1]\), but is non-trivial for \(|X| > 1\). Letting \(N^N_X\) (resp. \(N^Z_X\)) denote the free abelian group with generators
[a], a ∈ ℤ[X] (resp. ℤ[X]), modulo the relation \([k ⋅ a] = k ⋅ [a], k ∈ ℤ\) (resp. ℤ), we have the exact sequence

\[
\begin{array}{ccccccccc}
\Omega^N_X & \xrightarrow{\partial} & N^X_X & \xrightarrow{\pi} & G(ℤ)_X & \rightarrow & 0 \\
\Omega^Z_X & \xrightarrow{\partial} & N^Z_X & \xrightarrow{\pi} & G(ℤ)_X & \rightarrow & 0 \\
\end{array}
\]

(14.2.15)

with \(\partial \left\{ \frac{a}{a'} \right\} = [a + a'] - [a] - [a'], \pi(\sum_i k_i [a_i]) = \sum_i k_i ⋅ a_i\).

The +1- almost-linearity, is thus equivalent modulo 2 torsion (i.e. after taking \(Ω^Z[X] \cong ℤ[\frac{1}{2}]\)), to left and right linearity: \(a, a' \in ℤ[X]\) (resp. ℤ[X]), \(x, y \in ℤ\) (resp. ℤ).

\[
\begin{align*}
\text{right: } \left\{ \begin{array}{l}
x(a + a') \\
y(a + a')
\end{array} \right\} &= \left\{ \begin{array}{l}
xa \\
ya'
\end{array} \right\} + \left\{ \begin{array}{l}
xa' \\
ya
\end{array} \right\} \\
\text{left: } \left\{ \begin{array}{l}
(x + y)a \\
(x + y)a'
\end{array} \right\} &= \left\{ \begin{array}{l}
xa \\
x a'
\end{array} \right\} + \left\{ \begin{array}{l}
ya \\
ya'
\end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l}
ka \\
ka'
\end{array} \right\} = k \left\{ \begin{array}{l}
a' \\
a
\end{array} \right\}
\end{align*}
\]

(14.2.16) (14.2.17)

Conversely, left and right linearity (14.2.16–14.2.17) imply + and − almost linearity (14.2.12–14.2.13). For \(X = [1]\), the left linearity conditions (14.2.16) and the right one (14.2.17) are one and the same!

Thus letting \(Ω^N_X\) (resp. \(Ω^Z_X\)) denote the free abelian group with generators \(\left\{ \frac{a}{a'} \right\}, a, a' \in ℤ[X]\) (resp. ℤ[X]), modulo the relations (14.2.13) and (14.2.16–14.2.17), we get the even and odd derivation

\[
\begin{align*}
d^+_{\{X\}} : & \mathbb{N}^X \rightarrow \Omega^N_{\{X\}} \\
d^+_{\{X\}} : & \mathbb{Z}^X \rightarrow \Omega^Z_{\{X\}}
\end{align*}
\]

(14.2.18)

In particular taking \(X = [1]\), we get the even differential

\[
\begin{align*}
d^+_{\{1\}} : & \mathbb{N} \rightarrow \Omega^N_{\{1\}} \\
& \text{resp. } d^+_{\{1\}} : \mathbb{Z} \rightarrow \Omega^Z_{\{1\}}.
\end{align*}
\]

(14.2.19)

It satisfies:

\[
d^+_{\{1\}}(0) = d^+_{\{1\}}(1) = 0
\]

(14.2.20)

For \(n \geq 2\),

\[
\begin{align*}
d^+_{\{1\}}(n) &= 2 \cdot \left( \begin{array}{l}
\frac{n - 1}{1} \\
\frac{n - 2}{1} \\
\vdots \\
\frac{1}{1}
\end{array} \right) + \left( \begin{array}{l}
\frac{n - 1}{-1} \\
\frac{2 - n}{-1} \\
\vdots \\
\frac{-1}{-1}
\end{array} \right) \\
\text{resp. } d^+_{\{1\}}(-n) &= 2 \cdot \left( \begin{array}{l}
\frac{1 - n}{1} \\
\frac{2 - n}{-1} \\
\vdots \\
\frac{-1}{-1}
\end{array} \right) + \left( \begin{array}{l}
\frac{1 - n}{1} \\
\frac{2 - n}{1} \\
\vdots \\
\frac{-1}{1}
\end{array} \right) = -d^+_{\{1\}}(n).
\end{align*}
\]

(14.2.21)
(use induction on \(n\), and apply \(d_{[1]}^+\) to the identity \((n+1) = ((1, 1) \triangleq (n, 1)) \triangleq (1, 1)\) for the induction step)

\[
Leibnitz \, d_{[1]}^+(n \cdot m) = m \cdot d_{[1]}^+(n) + n \cdot d_{[1]}^+(m)
\]

(14.2.22)

It follows that

\[
d_{[1]}^+(q^n) = n \cdot q^{n-1} \cdot d_{[1]}^+(q)
\]

(14.2.23)

and hence

\[
d_{[1]}^+(n) = \sum_p v_p(n) \frac{n}{p} d_{[1]}^+(p)
\]

(14.2.24)

After extending scalars to \(\mathbb{Q}\) this is equivalent to

\[
\frac{d_{[1]}^+(n)}{n} = \sum_p v_p(n) \frac{n}{p} d_{[1]}^+(p)
\]

(14.2.25)

This is the arithmetical analogue of the formula for \(f = f(z) \in \mathbb{C}(z)\),

\[
\frac{df}{f} = d\log f = \sum_{\alpha \in \mathbb{C}} v_\alpha(f) \frac{d(z - \alpha)}{z - \alpha} = \sum_{\alpha \in \mathbb{C}} v_\alpha(f) \frac{dz}{z - \alpha}
\]

The derivation \(d_{[1]}^+\) is not additive, but we do have the identity

\[
d_{[1]}^+(n_1 + n_2) = d_{[1]}^+(n_1) + d_{[1]}^+(n_2) + 2 \left\{ \frac{n_1}{n_2} \right\}
\]

(14.2.26)

(apply \(d_{[1]}^+\) to the identity \((n_1 + n_2) = ((1, 1) \triangleq (n_1)) \triangleq (1, 1)\) and use Leibnitz).

To see that \(\Omega_{[1]}^\mathbb{N}\) (resp. \(\Omega_{[1]}^\mathbb{Z}\)) is non-trivial, note that for each prime \(p\) we have a homomorphism \(\varphi_p\) from it onto \(\mathbb{Z}\), given on the generators \(\left\{ \frac{a}{a'} \right\}\) by

\[
\varphi_p\left( \frac{a}{a'} \right) = v_p(a + a') \cdot \frac{a + a'}{p} - v_p(a) \cdot \frac{a}{p} - v_p(a') \cdot \frac{a'}{p}
\]

(14.2.27)

\((\text{Indeed, } \Omega_{[1]}^\mathbb{N}\text{ is the free abelian group generated by } \left\{ \frac{1}{p - 1} \right\}, \text{ } p \text{ prime}).\)

The generalized ring \(H = G(\mathbb{Z}) \prod \Omega_{[1]}^\mathbb{Z}\) is an (absolute) abelian group via the maps

\[
H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to H_{\mathbb{Z}}
\]

\((a_1, m_1) + (a_2, m_2) := (a_1 + a_2, \left\{ \frac{a_1}{a_2} \right\} + m_1 + m_2)
\]

(14.2.28)

and we get an exact sequence

\[
0 \to \Omega(\mathbb{Z}) \to H \to G(\mathbb{Z}) \to 0.
\]

(14.2.29)
Appendix C

Beta integrals and the local factors of zeta

We shall concentrate on the case of the rational numbers \( \mathbb{Q} \). We denote by \( p \) the close points of \( \lim \text{Spec } \mathbb{Z} \), that is the finite primes (denoted by "\( p \neq \eta \)"), and the real prime (denoted by "\( p = \eta \)"); when we want to emphasise that a formula holds for all primes, finite or real, we write "\( p \geq \eta \)". For each \( p \geq \eta \) we have the completion \( \mathbb{Q}_p \), the \( p \)-adic numbers for \( p \neq \eta \), \( \mathbb{Q}_{\eta} = \mathbb{R} \) the reals, and we have the (maximal-compact) sub-generalized ring \( \mathcal{G}(\mathbb{Z}_p) \subseteq \mathcal{G}(\mathbb{Q}_p) \), with \( \mathcal{G}(\mathbb{Z}_\eta) = \mathbb{Z}_\eta \) the real prime (cf. §8.3.3). We let for \( p \geq \eta \),

\[
S^n_p = \{(x_1, \ldots, x_n) \in \mathcal{G}(\mathbb{Z}_p)[1], \ |x_1, \ldots, x_n|_p = 1 \} \tag{C.1}
\]

where,

\[
|x_1, \ldots, x_n|_p = \begin{cases} \text{Max}\{|x_1|_p, \ldots, |x_n|_p\} & p \neq \eta \\ (|x_1|^2 + \cdots + |x_n|^2)^{1/2} & p = \eta \end{cases} \tag{C.2}
\]

(cf. §2.5). Thus for \( p = \eta \) we have the \((n - 1)\) dimensional sphere (while for \( p \neq \eta \), we get an open subset of \( n \) dimensional space). Note that for all \( p \geq \eta \) (with \( \mathbb{Z}_\eta^* = \{ \pm 1 \} \)):

\[
S^n_p / \mathbb{Z}_p^* \cong \mathbb{P}^{n-1}(\mathbb{Z}_p) \cong \mathbb{P}^{n-1}(\mathbb{Q}_p) \tag{C.3}
\]

For \( p \neq \eta \),

\[
S^n_p = \lim_{k} \mathbb{P}^n(\mathbb{Z}/p^k), \quad \mathbb{P}^{n-1}(\mathbb{Z}_p) = \lim_{k} \mathbb{P}^{n-1}(\mathbb{Z}/p^k). \tag{C.4}
\]

For all \( p \geq \eta \), \( S^n_p \) is a homogenous space of the compact group \( GL_n(\mathbb{Z}_p) \) (where \( GL_n(\mathbb{Z}_\eta) = O(n) \) the orthogonal group).

For all \( p \geq \eta \), we denote by \( \sigma^n_p \) the unique \( GL_n(\mathbb{Z}_p) \)-invariant probability measure on \( S^n_p \). We write the local factors of the zeta function as

\[
\zeta_p(s) := \begin{cases} (1 - p^{-s})^{-1} & p \neq \eta \\ 2\pi \cdot \Gamma(\frac{s}{2}) & p = \eta \end{cases} \tag{C.5}
\]

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(so the global zeta, \( \zeta(s) = \prod_{p \geq \eta} \zeta_p(s) \), \( \Re(s) > 1 \), satisfies the functional equation \( \zeta(s) = (2\pi)^{s-\frac{1}{2}} \zeta(1-s) \)). We shall eventually recover these local factors \( \zeta_p(s), p \geq \eta \), in a uniform way. We denote for \( p \geq \eta \), the Beta function

\[
\beta_p(\alpha_1, \ldots, \alpha_n) = \frac{\zeta_p(\alpha_1) \cdots \zeta_p(\alpha_n)}{\zeta_p(\alpha_1 + \cdots + \alpha_n)} = \begin{cases} 
\frac{1-p^{-\alpha_1-\cdots-\alpha_n}}{(1-p^{-\alpha_1}) \cdots (1-p^{-\alpha_n})} & p \neq \eta \\
\frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\frac{\alpha_1 + \cdots + \alpha_n}{p})} & p = \eta 
\end{cases}
\] (C.6)

and the normalized Beta function

\[
B_p(\alpha_1, \ldots, \alpha_n) = \frac{\beta_p(\alpha_1, \ldots, \alpha_n)}{\beta_p(1, \ldots, 1)} = \frac{\zeta_p(n)}{\zeta_p(1)^n} \cdot \frac{\zeta_p(\alpha_1) \cdots \zeta_p(\alpha_n)}{\zeta_p(\alpha_1 + \cdots + \alpha_n)}. \] (C.7)

We have for all \( p \geq \eta \) the following Beta-integral:

\[
\text{Beta-} \int_{S_p^n} |x_1|^{\alpha_1-1} \cdots |x_n|^{\alpha_n-1} \sigma_1^p(dx) = B_p(\alpha_1, \ldots, \alpha_n), \ \Re(\alpha_i) > 0. \] (C.8)

here \( x = (x_1, \ldots, x_n) \in S_p^n \).

(This can be verified directly for \( p \neq \eta \), and for \( p = \eta \), or can be obtained as appropriate limits of a \( q \)-analogue. cf. [H08].)

Note that for real \( \alpha_i > 0 \), we get a probability measure, the "Beta-measure"

\[
\mu_p^{(\alpha_1, \ldots, \alpha_n)}(dx) = |x_1|^{\alpha_1-1} \cdots |x_n|^{\alpha_n-1} \frac{\sigma_p(dx)}{B_p(\alpha_1, \ldots, \alpha_n)} \] (C.9)

on \( S_p^n \), it is \( \mathbb{Z}_p^* \)-invariant, hence can be viewed as a probability measure on \( \mathbb{P}^{n-1}(\mathbb{Q}_p) \) (hence for \( p \neq \eta \), we get a Markov chain on \( \prod_k \mathbb{P}^{n-1}(\mathbb{Z}/p^k) \), cf. [H08] for the \( q \)-analogue, which also gives the real analogue).

On the other hand we have for all \( p \geq \eta \), and all \( y = (y_1, \ldots, y_n) \in \mathbb{Q}_p^n \),

\[
\text{Beta-} \int_{S_p^n} y_1 y_2 \cdots y_n |y_1|^{s-1} \sigma_p^*(y) dx = \int_{S_p^n} |x|^{s-1} \sigma_p^*(dx) = \frac{\zeta_p(n)}{\zeta_p(1)} \frac{\zeta_p(s)}{\zeta_p(n-1+s)} |y|^{s-1}. \] (C.10)

Note that the operations of multiplications and contraction are very natural for the Beta measures: we have for \( N = n_1 + \cdots + n_k, p \geq \eta \), a surjection

\[
\phi: S_p^k \times S_p^{n_1} \times \cdots \times S_p^{n_k} \to S_p^N \]
\[
(t, x^{(1)}, \ldots, x^{(k)}) \mapsto t \phi(x^{(i)}) \] (C.11)
and the measure $\sigma^N$ is obtained as the image of the measures $\sigma^{n_j}$ and the Beta measure $\mu_{p^n_1, \ldots, n_k}$.

**multiplication formula:**

\[
\int_{S^N_p} f(x) \sigma^N_p(dx) = \int_{S^k_p} \frac{|t_1|^{n_1-1} \cdots |t_k|^{n_k-1}}{B_p(n_1, \ldots, n_k)} \int_{S^1_p} \sigma^1_p(dx(1)) \cdots \int_{S^n_p} \sigma^n_p(dx(n)) \ f(t \circ (x(i)))
\]

(C.12)

Writing $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in Q^N_p$, with $x_j, y_j \in Q^N_{p_j}$, we have for $p \geq \eta, N = n_1 + \cdots + n_k$:

**Contraction**

\[
\int_{S^N_p} |x_1/\eta^1 \cdots |x_k/\eta^k|\alpha_N(dx) = \frac{\zeta_p(N)\zeta_p(\alpha_1) \cdots \zeta_p(\alpha_k)}{\zeta_p(1)^k \cdot \zeta_p(N-k + \alpha_1 + \cdots + \alpha_k)}|y_1/\eta^1 \cdots y_k/\eta^k|\alpha_N
\]

(C.13)

Note that (C.13) for $k = 1$ is (C.10) and (C.13) for $n_1 = \cdots = n_N = 1$, and $y_j \equiv 1$ is (C.8), and conversely, (C.13) follows form (C.8), (C.10) and the multiplication formula, (C.12).

Taking the vectors $y = 1_N = (1, 1, \ldots, 1) \in Q^N_p$, we get a probability measure $\varphi^N_p = \sigma^N_p / 1_N$ on $Q_p$ for all $p \geq \eta$:

\[
\int_{Q_p^N} f(x) \varphi^N_p(dx) = \int_{S^N_p} f(x_1 + \cdots + x_N) \sigma^N_p(dx)
\]

(C.14)

We have:

\[
\varphi^N_p(dx) = \left[\frac{1 - p^{-1-N}}{1 - \mu^N_p} \phi_{x_p}(x) + \frac{p^{1-N}}{1 - p^{-1-N}} \phi_{x_p}(x)\right] dx, \ p \neq \eta,
\]

(C.15)

where $dx$ is the additive Haar measure on $Z_p, dx(Z_p) = 1$, and $\phi_{x_p}$ (resp $\phi_{x_p^*}$) is the characteristic function of $Z_p$ (resp. $Z_p^*$);

\[
\varphi^N_\eta(dx) = \frac{\Gamma(\frac{N}{2})}{\sqrt{\pi}N^{N/2}} \cdot (1 - |x|^2/N)^{N-1} dx, \ x \in [-\sqrt{N}, \sqrt{N}],
\]

(C.16)

with the usual Haar measure $dx$ on $\mathbb{R}, dx([0,1]) = 1$.

When we take the limit $N \to \infty$ we obtain the measures

\[
\varphi^N_p(dx) = \phi_{x_p}(x)dx, \ p \neq \eta
\]

\[
\varphi^N_\eta(dx) = e^{-|x|^2/2} \frac{dx}{\sqrt{2\pi}}, \ p = \eta, \ dx([0,1]) = 1
\]

(C.17)

and for all $p \geq \eta$ we have from (C.10), with $y_i = 1$, in the limit $n \to \infty$:

\[
\frac{\zeta_p(s)}{\zeta_p(1)} = \lim_{N \to \infty} \int_{Q^N_p} |x|^s \varphi^N_p(dx) = \lim_{N \to \infty} \int_{S^N_p} |x_1 + \cdots + x_N|^s \sigma^N_p(dx).
\]

(C.18)
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