A Berry-Esseen theorem for Pitman’s $\alpha$-diversity

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Abstract

This paper is concerned with the study of the random variable $K_n$ denoting the number of distinct elements in a random sample $(X_1, \ldots, X_n)$ of exchangeable random variables driven by the two parameter Poisson-Dirichlet distribution, PD$(\alpha, \theta)$. For $\alpha \in (0,1)$, Theorem 3.8 in [23] shows that $K_n/n^\alpha \overset{a.s.}{\rightarrow} S_{\alpha,\theta}$ as $n \rightarrow +\infty$. Here, $S_{\alpha,\theta}$ is a random variable distributed according to the so-called scaled Mittag-Leffler distribution. Our main result states that

$$\sup_{x \geq 0} \left| P\left[\frac{K_n}{n^\alpha} \leq x\right] - P\left[S_{\alpha,\theta} \leq x\right]\right| \leq \frac{C(\alpha, \theta)}{n^\alpha}$$

holds with an explicit constant $C(\alpha, \theta)$. The key ingredients of the proof are a novel probabilistic representation of $K_n$ as compound distribution and new, refined versions of certain quantitative bounds for the Poisson approximation and the compound Poisson distribution.

Finally, we present the following application in the context of Bayesian nonparametric inference for species sampling problems: given an initial (observable) random sample $(X_1, \ldots, X_n)$ from the population, estimate of the number $K_{m,n}$ of hitherto unseen species that would be observed in $m$ additional (unobservable) samples. In the approach proposed in [8], $(X_1, \ldots, X_n)$ is a random sample from PD$(\alpha, \theta)$ featuring $K_n = \sum_{i=1}^{\alpha} S_{\alpha,\theta}$ as $n \rightarrow +\infty$, where $S_{\alpha,\theta}(n, j)$ is related to $S_{\phi,\theta}(n, j)$. Thus, we combine the previous main result with a new Berry-Esseen bound for de Finetti’s theorem recently obtained in [6], to obtain another Berry-Esseen theorem for the convergence of the distribution of $K_{m,n}^{(n)}$.

Keywords: Ewens-Pitman sampling model, Laplace method, Kolmogorov distance, Poisson approximation

1 Introduction

The two parameter Poisson-Dirichlet distribution was introduced by Perman et al. [20] as a generalization of the Poisson-Dirichlet distribution of Kingman [14]. For any $\alpha \in [0,1)$ and $\theta > \alpha$ let $(V_i)_{i \geq 1}$ be independent random variables such that $V_i$ is distributed according to a Beta distribution with parameter $(1-\alpha, \theta+\alpha)$. If $P_i := V_i$ and $P_i := V_i \prod_{1 \leq j \leq i-1} (1-V_j)$ for $i \geq 2$, then $\sum_{i \geq 1} P_i = 1$ almost surely, that is $(P_i)_{i \geq 1}$ is (almost surely) a random discrete distribution. The two parameter Poisson-Dirichlet distribution, denoted by PD$(\alpha, \theta)$, is defined as the distribution of the descending ordered statistics of $(P_i)_{i \geq 1}$; the Poisson-Dirichlet distribution arises as a special case by setting $\alpha = 0$. A random sample $(X_1, \ldots, X_n)$ from PD$(\alpha, \theta)$ induces an exchangeable random partition of $\{1, \ldots, n\}$ into $K_n \leq n$ blocks, with block frequency counts being the cardinalities of the equivalence classes arising from the (random) equivalence relation $i \sim j$ if and only if $X_i = X_j$ almost surely. In particular, let $M_n = (M_{1,n}, \ldots, M_{n,n})$ with
In view of the absolute continuity of $F$, pair of distribution functions $F_1$ and $F_2$ stand for the distribution functions of $K_n/n^\alpha$ and $S_{\alpha,\theta}$, respectively, i.e. $F_n(x) := P[K_n/n^\alpha \leq x]$ and $F_{\alpha,\theta}(x) := P[S_{\alpha,\theta} \leq x]$, for any $x > 0$. In order to measure the discrepancy between $F_n$ and $F_{\alpha,\theta}$, we consider the so-called Kolmogorov distance which, for any pair of distribution functions $F_1$ and $F_2$ supported in $[0, +\infty)$, is defined as
\[
d_K(F_1; F_2) := \sup_{x \geq 0} |F_1(x) - F_2(x)|.
\]
In view of the absolute continuity of $F_{\alpha,\theta}$, it is worth noticing that $d_K$ metrizes the weak convergence of $F_n$ towards $F_{\alpha,\theta}$. Our main result provides an upper bound for $d_K(F_n; F_{\alpha,\theta})$, showing how fast this discrepancy goes to zero as $n \to +\infty$.

**Theorem 1.** For any $\alpha \in (0, 1)$ and $\theta > 5$, there exists a positive constant $C_{\alpha,\theta}$, depending only on $\alpha$ and $\theta$, such that $d_K(F_n; F_{\alpha,\theta}) \leq n^{-\alpha}C_{\alpha,\theta}$ for every $n \in \mathbb{N}$. 
We present an application of Theorem 1 in the context of Bayesian nonparametric inference for species sampling problems. Consider a population \((X_i)_{i \geq 1}\) of individuals belonging to an infinite number of species \((S_j)_{j \geq 1}\) with unknown proportions \((p_j)_{j \geq 1}\). Given an initial (observable) random sample \((X_1, \ldots, X_n)\) from the population, a classical species sampling problem consists in the estimation of the number of hitherto unseen species that would be observed in \(m\) additional (unobservable) samples. See, e.g., Orłitsky et al. [18] and references therein. A Bayesian nonparametric approach to the estimation of the number of unseen species was proposed by Lijoi et al. [16], and further developed in Favaro et al. [8]. This approach relies on the PD\((\alpha, \theta)\) distribution as a prior distribution for the unknown species composition \((p_j)_{j \geq 1}\) of the population. Specifically, let \((X_1, \ldots, X_n)\) be a random sample from PD\((\alpha, \theta)\) featuring \(K_n = j \leq n\) species (blocks). Lijoi et al. [16] derived an expression of the posterior distribution, given \((X_1, \ldots, X_n)\), of the number \(K_m\) of new species in \(m\) additional sample. Then Favaro et al. [8] showed

\[
\frac{K_m(n)}{m^\alpha} \mid (X_1, \ldots, X_n) \xrightarrow{\text{a.s.}} S_{\alpha, \theta}(n, j)
\]

as \(m \to +\infty\), where \(S_{\alpha, \theta}(n, j) \overset{d}{=} B_j + \theta/n(\alpha-j) S_{\alpha, \theta+n}\), with \(S_{\alpha, \theta+n}\) being Pitman’s \(\alpha\) diversity and \(B_j + \theta/n(\alpha-j) S_{\alpha, \theta+n}\) is a random variable, independent of \(S_{\alpha, \theta+n}\), and distributed according to a Beta distribution with parameter \((j + \theta/n, \alpha-j)\). The random variable \(S_{\alpha, \theta}(n, j)\) is referred to as Pitman’s posterior \(\alpha\)-diversity. As extensively discussed in Favaro et al. [8], the importance of (4) is motivated by the fact that the computational burden for evaluating the posterior distribution of \(K_m\) becomes overwhelming for large \(m\). Then Pitman’s posterior \(\alpha\)-diversity has been extensively applied to obtain large \(m\) approximated posterior inferences for \(K_m\) via Monte Carlo sampling from \(S_{\alpha, \theta}(n, j)\). In this paper we formulate a Berry-Esseen theorem for Pitman’s posterior \(\alpha\)-diversity, thus quantifying the error of approximation in replacing the posterior distribution of \(K_m\) with Pitman’s posterior \(\alpha\)-diversity.

The paper is structured as follows. In Section 2 we prove Theorem 1: the proof relies on: i) a novel representation of the distribution of \(K_n\) in terms of the number of blocks in a random sample from a Poisson compound sampling model; ii) a quantitative version of the asymptotic expansion (in the sense of Poincaré) of a recurrent Laplace-type integral. In Section 3 we state and prove the Berry-Esseen theorem for Pitman’s posterior \(\alpha\)-diversity; the proof rely on: i) a novel representation of the posterior distribution of \(K_m\) in terms of a (compound) summation of independent Bernoulli random variables with random parameter; ii) a recent result in Dolera and Favaro [6] on the rate of convergence, in Kolmogorov distance, of the de Finetti’s law of large numbers for exchangeable Bernoulli sequences; iii) an application of Theorem 1.

## 2 Proof of Theorem 1

The proof is split into four parts, developed in the next Subsections 2.1, 2.2, 2.3 and 2.4. In particular, the first contains a new probabilistic representation for \(K_n\) as compound distribution. The second is devoted to some technical results, culminating in a quantitative version of the asymptotic expansion (in the sense of Poincaré) of a recurrent Laplace-type integral, denoted by \(I_n(z)\). The third presents a new, refined version of a quantitative bound in the Poisson approximation, originally due to Hwang [13]. Finally, the fourth section makes use of the statements contained in the previous ones to carry out the proof of Theorem 1. As to notation, the present section will make frequent use of the concept of probability generating function of a random variable \(X\) which takes values in \(\mathbb{N}_0 := \{0, 1, 2, \ldots\}\), namely \(G_X(s) := \sum_{x \geq 0} P[X = x] s^x\).

### 2.1 A new probabilistic representation for \(K_n\)

The first result introduces a noteworthy probability distribution \(\rho(\cdot; \alpha, n, z)\) on \([1, \ldots, n]\), which involves the so-called generalized factorial coefficients, namely \(\mathcal{C}(k; n; \alpha) := \frac{1}{\alpha!} \sum_{i=1}^{k} (-1)^i \binom{k}{i} (\alpha)_{n} \).

[Note: The content above is a natural representation of the document text, formatted for readability and coherence. It includes necessary mathematical symbols and notation to convey the intended meaning accurately.]
With respect to the coefficients $C(n, k; \alpha)$ defined in Chapter 8 of Charalambides [3] (see, in particular, formula (8.48)), it is worth noticing that $C(n, k; \alpha) = (-1)^n k! C(n, k; \alpha)$.

**Lemma 1.** For $\alpha, q \in (0, 1)$, consider a sequence $\{Q_j(\alpha, q)\}_{j \geq 1}$ of i.i.d. random variables with zero-truncated extended negative binomial distribution, that is

$$P[Q_1(\alpha, q) = x] = \frac{1}{[1 - (1 - q)\alpha]} \left( \frac{\alpha}{x} \right) (-q)^x = \frac{\Gamma(\alpha + 1) \sin \pi \alpha \Gamma(x - \alpha)}{\pi [1 - (1 - q)\alpha]} x! q^x$$

for any $x \in \mathbb{N}$. Moreover, for $z > 0$, consider a Poisson random variable $N_\lambda$ with parameter $\lambda = z [1 - (1 - q)\alpha]$, independent of the sequence $\{Q_j(\alpha, q)\}_{j \geq 1}$. Then, putting $S(\alpha, q, z) := \sum_{j=0}^{N_\lambda} Q_j(\alpha, q)$, with the proviso that $P[Q_0(\alpha, q) = 0] = 1$, one gets

$$\rho(\{k\}; \alpha, n, z) := P[N_\lambda = k \mid S(\alpha, q, z) = n] = \frac{\mathcal{G}(n, k; \alpha) z^k}{\sum_{j=1}^{\infty} \mathcal{G}(n, j; \alpha) z^j}$$

for any $k \in \{1, \ldots, n\}$.

**Proof of Lemma 1.** For fixed $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, one has $P[S(\alpha, q, z) = n \mid N_\lambda = k] = P[\sum_{j=1}^{k} Q_j(\alpha, q) = n]$. To find the exact expression of this probability, consider the probability generating function $G(\cdot; k, \alpha, q)$ of $\sum_{j=1}^{k} Q_j(\alpha, q)$ to obtain, for $|s| < 1$,

$$G(s; k, \alpha, q) = \left( \mathcal{G}_Q(\alpha, q)(s) \right)^k = \left( \sum_{x=0}^{\infty} \frac{1}{[1 - (1 - q)\alpha]} \left( \frac{\alpha}{x} \right) (-sq)^x \right)^k = \left[ \frac{1 - (1 - sq)^\alpha}{1 - (1 - q)\alpha} \right]^k$$

by virtue of the binomial series. Since $[1 - (1 - u)^\alpha]^k = k! \sum_{n \geq k} \mathcal{G}(n, k; \alpha) \frac{u^n}{n!}$ holds whenever $|u| < 1$ (see Theorem 8.14 in Charalambides [3]), conclude that

$$P[S(\alpha, q, z) = n \mid N_\lambda = k] = \frac{k!}{[1 - (1 - q)\alpha]^k} \mathcal{G}(n, k; \alpha) \frac{q^n}{n!}$$

for $n \geq k$. Moreover, (6) holds also for $k > n$ since, in this case, $\mathcal{G}(n, k; \alpha) = 0$. Hence, using the explicit expression of $P[N_\lambda = k]$, deduce (5) directly from (6) by means of the Bayes formula. \(\square\)

The second result encapsulates the aforesaid probabilistic representation for $K_n$. For notational convenience, let $G_{\tau, \lambda}$ stand for a Gamma random variable with scale parameter $\lambda > 0$ and shape parameter $\tau > 0$, and let $\{R(\alpha, n, z)\}_{z > 0}$ denote a family of random variables with $P[R(\alpha, n, z) = k] = \rho(\{k\}; \alpha, n, z)$, for any $n \in \mathbb{N}$, $k \in \{1, \ldots, n\}$, $\alpha \in (0, 1)$ and $z > 0$.

**Proposition 1.** For fixed $n \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\theta > -\alpha$, there holds

$$K_n \overset{d}{=} R(\alpha, n, S_{\alpha, \theta} G_{\theta+n, 1}^{\alpha})$$

(7)

where $S_{\alpha, \theta}$, $G_{\theta+n, 1}$ and $\{R(\alpha, n, z)\}_{z > 0}$ are thought of as independent random elements, the identity being intended in distribution.

**Proof of Proposition 1.** Start from the well-known identity

$$P[K_n = k] = \frac{[\theta]_{(k, \alpha)} \mathcal{G}(n, k; \alpha)}{[\theta]_{(n, 1)}} \alpha^k$$

(8)
for any \( k \in \{1, \ldots, n\} \), whose proof is contained in Pitman [22] (see formula (99) therein). Then, thanks to the identity \( \int_0^{+\infty} x^{-\theta} f_\alpha(x) dx = \frac{\Gamma(\theta/\alpha)}{\alpha \Gamma(\theta)} \), it is easily checked that

\[
f(z; \alpha, \theta, n) = \frac{z^{\theta/\alpha+n/\alpha-1}}{\Gamma(\theta/\alpha)[\theta](n,1)} \left( \int_0^{+\infty} x^n e^{-xz^{1/\alpha}} f_\alpha(x) dx \right) \mathbb{I}\{z > 0\},
\]

is a probability density function. Thus, one has

\[
P[K_n = k] = \frac{\wp(n, k; \alpha)}{\Gamma(\theta/\alpha)[\theta](n,1)} \int_0^{+\infty} z^{k+\theta/\alpha-1} e^{-z} dz
\]

\[
= \frac{1}{\Gamma(\theta/\alpha)[\theta](n,1)} \int_0^{+\infty} z^{\theta/\alpha-1} e^{-z} \left( \sum_{j=1}^n \wp(n, j; \alpha)z^j \frac{\wp(n, k; \alpha)z^k}{\sum_{j=1}^n \wp(n, j; \alpha)z^j} dz
\]

\[
= \frac{1}{\Gamma(\theta/\alpha)[\theta](n,1)} \int_0^{+\infty} z^{\theta/\alpha+n/\alpha-1} \left( \int_0^{+\infty} x^n e^{-xz^{1/\alpha}} f_\alpha(x) dx \right) \frac{\wp(n, k; \alpha)z^k}{\sum_{j=1}^n \wp(n, j; \alpha)z^j} dz
\]

\[
= \int_0^{+\infty} \frac{\wp(n, k; \alpha)z^k}{\sum_{j=1}^n \wp(n, j; \alpha)z^j} f(z; \alpha, \theta, n) dz
\]

\[
= \int_0^{+\infty} P[R(\alpha, n, z) = k] f(z; \alpha, \theta, n) dz,
\]

where: the first identity follows from \( \alpha^{-k}[\theta](k, \alpha) = \Gamma(k + \theta/\alpha)/\Gamma(\theta/\alpha) \); the third exploits the relation

\[
\sum_{j=1}^n \wp(n, j; \alpha)z^j = e^z z^{n/\alpha} \int_0^{+\infty} x^n e^{-xz^{1/\alpha}} f_\alpha(x) dx \quad (9)
\]

displayed in Proposition 1 of Favaro et al. [9]; the fifth follows from (5). To conclude, it is enough to show that the probability distribution of \( S_{\alpha, \theta} G_{\theta+n,1}^{\alpha} \) possesses a density coinciding with \( f(\cdot; \alpha, \theta, n) \). In fact, one has

\[
P[S_{\alpha, \theta} G_{\theta+n,1}^{\alpha} \leq u] = \int_0^{+\infty} P[G_{\theta+n,1} \leq \left( \frac{u}{s} \right)^{1/\alpha}] \frac{\Gamma(\theta + 1)}{\alpha \Gamma(\theta/\alpha + 1)} \frac{e^{-s} f_\alpha(s^{-1/\alpha}) ds}{s^{\theta+1-1}}
\]

\[
= \int_0^{+\infty} \left( \int_0^{u/\alpha} t^{\theta+n-1-1} e^{-t} dt \right) \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} x^{-\theta} f_\alpha(x) dx
\]

\[
= \int_0^{+\infty} \left( \int_0^{u} \frac{1}{\Gamma(\theta + n)} e^{-\zeta^{1/\alpha}} z^{\frac{\theta+n-1}{\alpha} - 1} dz \right) \frac{\Gamma(\theta + 1)}{\alpha \Gamma(\theta/\alpha + 1)} x^n f_\alpha(x) dx
\]

where: the first identity follows from conditioning; the second and the third ensue from the changes of variable \( x = s^{-1/\alpha} \) and \( t = xz^{1/\alpha} \), respectively. \( \square \)

The representation (7) highlights the central role of the probability distribution \( \rho(\cdot; \alpha, n, z) \), which, unfortunately, seems not so easy to handle—even numerically—due to the computational complexity of the coefficients \( \wp(n, k; \alpha) \). For this reason, the next result provides the asymptotic behavior of this distribution for large values of \( n \), which will be used later as an approximation of \( \rho(\{k\}; \alpha, n, z) \).

**Lemma 2.** For fixed \( \alpha \in (0, 1) \) and \( z > 0 \), there holds

\[
\lim_{n \to +\infty} \rho(\{k\}; \alpha, n, z) = e^{-z} \frac{z^{k-1}}{(k-1)!}
\]

for any \( k \in \mathbb{N} \), which is tantamount to saying that \( R(\alpha, n, z) \xrightarrow{\mathcal{L}} 1 + N_z \) as \( n \to +\infty \).
Proof of Lemma 2. Letting $G(\cdot; \alpha, n, z)$ denote the probability generating function of $R(\alpha, n, z)$, observe that the thesis is equivalent to show that $G(s; \alpha, n, z) \to s \exp\{z(s - 1)\}$ as $n \to +\infty$, for any $s > 0$. Then, using the definition of $\mathcal{G}(n, k; \alpha)$, one has

$$G(s; \alpha, n, z) = \sum_{k=1}^{n} \mathcal{G}(n, k; \alpha)(sz)^k \sum_{m=1}^{k} \frac{1}{m!} z^m = \sum_{k=1}^{n} \frac{1}{k!} (sz)^k \sum_{m=1}^{k} (-1)^i \binom{k}{i} |i\alpha|_{(n, 1)}$$

observe that the thesis is equivalent to show that $G(s; \alpha, n, z) \to s \exp\{z(s - 1)\}$ as $n \to +\infty$, for any $s > 0$. Then, using the definition of $\mathcal{G}(n, k; \alpha)$, one has

$$G(s; \alpha, n, z) = \sum_{k=1}^{n} \frac{1}{k!} (sz)^k \sum_{m=1}^{k} (-1)^i \binom{k}{i} |i\alpha|_{(n, 1)}$$

for any $t > 0$. Upon noticing that $\frac{\Gamma(n,s)}{\Gamma(n)} \leq 1$ and that the relations

$$\left| \frac{\Gamma(n-s)}{\Gamma(n)} \right| \leq \left| \frac{\Gamma(n-i\alpha)}{i!} \right| \leq \frac{\Gamma(n-i\alpha)}{\Gamma(n) i!} \frac{\sin i\pi \alpha}{\pi}$$

hold for all $n \in \mathbb{N}$, $x > 0$ and $i \in \{1, \ldots, n\}$, one can write

$$\left| \sum_{i=2}^{n} (-1)^i \frac{|-i\alpha|_{(n, 1)}}{|-\alpha|_{(n, 1)}} \Gamma(n-i+1, t) t^i}{i!} \right| \leq \sum_{i=2}^{n} \frac{t^i}{i!} \left| \frac{|-i\alpha|_{(n, 1)}}{|-\alpha|_{(n, 1)}} \right| \Gamma(n-i+1, t) t^i \leq \Gamma(-\alpha) \sum_{i=2}^{n} \left| \frac{\Gamma(n-i\alpha)}{\Gamma(n-i)} \right| \frac{n^{i\alpha}}{i!} \Gamma(n-i\alpha) \frac{n^{i\alpha}}{i!}$$

for any $t > 0$. Upon noticing that $\frac{\Gamma(n,s)}{\Gamma(n)} \leq 1$ and that the relations

$$\left| \frac{\Gamma(n-s)}{\Gamma(n)} \right| \leq \left| \frac{\Gamma(n-i\alpha)}{i!} \right| \leq \frac{\Gamma(n-i\alpha)}{\Gamma(n) i!} \frac{\sin i\pi \alpha}{\pi}$$

hold for all $n \in \mathbb{N}$, $x > 0$ and $i \in \{1, \ldots, n\}$, one can write

$$\left| \sum_{i=2}^{n} (-1)^i \frac{|-i\alpha|_{(n, 1)}}{|-\alpha|_{(n, 1)}} \Gamma(n-i+1, t) t^i}{i!} \right| \leq \sum_{i=2}^{n} \frac{t^i}{i!} \left| \frac{|-i\alpha|_{(n, 1)}}{|-\alpha|_{(n, 1)}} \right| \Gamma(n-i+1, t) t^i \leq \Gamma(-\alpha) \sum_{i=2}^{n} \left| \frac{\Gamma(n-i\alpha)}{\Gamma(n-i)} \right| \frac{n^{i\alpha}}{i!} \Gamma(n-i\alpha) \frac{n^{i\alpha}}{i!}$$

At this stage, the monotonic increasing character of the function $(0, n) \ni x \mapsto n^x \Gamma(n-x)$, which follows from the inequality $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \leq \log(z)$ for any $z > 0$, entails that $\max_{i=2,\ldots,n} n^{i\alpha} \Gamma(n-i\alpha) = n^{n\alpha} \Gamma(n-n\alpha)$. Therefore, observe that $\frac{n^{n\alpha} \Gamma(n-n\alpha)}{\Gamma(n-n\alpha)} \sim n^{\alpha}$ to conclude that

$$\max_{i=2,\ldots,n} \frac{n^{i\alpha} \Gamma(n-i\alpha)}{\Gamma(n-n\alpha)} \sum_{i=2}^{n} \left( \frac{t}{n^{\alpha}} \right)^i \sim \left( \frac{t}{n^{\alpha}} \right)^{n-1} \frac{n^{\alpha}}{n^{\alpha}} \sim \frac{1}{n^{\alpha}}$$

as $n \to +\infty$, concluding the proof. □
2.2 A quantitative Laplace method for $\mathcal{I}_n(z)$

This subsection is focused on the analysis of the Laplace integral

$$\mathcal{I}_n(z) := \int_0^{+\infty} e^{-n\phi_z(y)} f_\alpha(y) dy$$

for $z > 0$, where $\phi_z(y) := zy - \log y$. This quantity is connected with (9) in view of the identity

$$d_n(x) := \sum_{j=1}^{n} \mathcal{G}(n,j;\alpha)(xn^\alpha) = e^{xn^\alpha} x^{n/\alpha} n^n \mathcal{I}_n(x^{1/\alpha})$$

valid for all $x > 0$. As first step, after noticing that $\overline{\mathcal{G}}(z) := 1/z$ is the only minimum point of $\phi_z(y)$, a direct application of the Laplace (see, e.g., Section 7 in Chapter 3 of [17]) methods shows that

$$\mathcal{I}_n(z) \sim \left( \frac{1}{z} \right)^{n+1} f_\alpha \left( \frac{1}{z} \right) e^{-n} \sqrt{\frac{2\pi}{n}} $$

as $n \to +\infty$. A more precise estimate is provided by the next

**Lemma 3.** For any $n \in \mathbb{N}$, there exists a continuous function $\delta_n : (0, +\infty) \to (0, +\infty)$ such that

$$\mathcal{I}_n(z) = \left( \frac{1}{z} \right)^{n+1} f_\alpha \left( \frac{1}{z} \right) e^{-n} \sqrt{\frac{2\pi}{n}} \left[ 1 + \delta_n(z) \right]$$

and $|\delta_n(z)| \leq \Delta(z)/n$ for any $z > 0$, where $\Delta : (0, +\infty) \to (0, +\infty)$ is a suitable continuous function which is independent of $n$. Moreover, $\Delta$ can be chosen in such a way that $\Delta(z) = O(z^{-4})$ as $z \to 0$, and $\Delta(z) \cdot f_\alpha \left( \frac{1}{z} \right) = O(z^{-\infty})$ as $z \to +\infty$.

**Proof of Lemma 3.** The change of variables $s = zy - 1$ gives

$$\mathcal{I}_n(z) = \left( \frac{1}{z} \right)^{n+1} e^{-n} \int_{-1}^{+\infty} e^{-nh(s)} f_\alpha \left( \frac{s+1}{z} \right) ds$$

with $h(s) := s - \log(s + 1)$. Then, in order to exploit the analyticity of $h$ for $s \in (-1, 1)$, as in Example 1 in Chapter 2 of [24], fix $\sigma \in (0, 1)$ and split the above integral into the regions $s \in (\sigma, +\infty)$, $s \in (0, \sigma)$, $s \in (-\sigma, 0)$ and $s \in (-1, -\sigma)$.

First, write $h(s) \geq h'(\sigma)(s - \sigma) + h(\sigma)$ for every $s \in (\sigma, +\infty)$ by the convexity of $h$, yielding

$$\int_{\sigma}^{+\infty} e^{-nh(s)} f_\alpha \left( \frac{s+1}{z} \right) ds$$

\begin{align*}
\leq & \ (\sigma + 1)^n \exp \left\{ - \frac{n\sigma}{\sigma + 1} \right\} \int_{\sigma}^{+\infty} \exp \left\{ - \frac{nts}{\sigma + 1} \right\} f_\alpha \left( \frac{t+1}{z} \right) dt . \\
= & \ (\sigma + 1)^n \int_{\sigma+1}^{+\infty} \exp \left\{ - \frac{nts}{\sigma + 1} \right\} f_\alpha \left( \frac{t}{z} \right) dt
\end{align*}

The analysis of this term reduces to the study of

$$\sup_{n \in \mathbb{N}} n^{3/2}(\sigma + 1)^n \int_{\sigma+1}^{+\infty} \exp \left\{ - \frac{nts}{\sigma + 1} \right\} f_\alpha \left( \frac{t}{z} \right) dt .$$

Hence, deduce the boundedness of (15) for small values of $z$ by exploiting that $\|f_\alpha\|_{\infty} < +\infty$. For large values of $z$, put $\lambda := \frac{\sigma}{2(\sigma + 1)}$ and use the Cauchy-Schwartz inequality to obtain

$$\int_{\sigma+1}^{+\infty} \exp \left\{ - \frac{nts}{\sigma + 1} + \lambda t \right\} e^{-\lambda t} f_\alpha \left( \frac{t}{z} \right) dt$$
Therefore, the quantity to bound is now equal to

\[ \left( \int_{\sigma+1}^{+\infty} e^{-2nL} \exp \left\{ -\frac{2nt\sigma}{\sigma+1} + 2Lt \right\} dt \right)^{1/2} \left\| f_\alpha \right\|_\infty \int_0^{+\infty} e^{-2Lt} \left( \frac{t}{z} \right) dt \]

which leads to the complete control of (15).

Second, take into account the region \( s \in (-1, -\sigma) \). Writing \( h(s) \geq h'(-\sigma)(s + \sigma) + h(-\sigma) \) for every \( s \in (-1, -\sigma) \), again by the convexity of \( h \), leads to

\[ \int_{-1}^{-\sigma} e^{-nh(s)} f_\alpha \left( \frac{s + 1}{z} \right) ds \leq (1 - \sigma)^n \int_0^{1 - \sigma} \exp \left\{ \frac{nt\sigma}{1 - \sigma} \right\} f_\alpha \left( \frac{t}{z} \right) dt . \]

Therefore, the quantity to bound is now equal to

\[ \sup_{n \in \mathbb{N}} n^{3/2}(1 - \sigma)^n \int_0^{1 - \sigma} \exp \left\{ \frac{nt\sigma}{1 - \sigma} \right\} f_\alpha \left( \frac{t}{z} \right) dt . \tag{16} \]

The boundedness of (16) for small values of \( z \) follows once again from \( \left\| f_\alpha \right\|_\infty < +\infty \). For large values of \( z \), it is enough to observe that

\[ \int_0^{1 - \sigma} f_\alpha \left( \frac{t}{z} \right) dt \sim z^{1 - 3\alpha/2} \exp\{ -C(\alpha)z^{n/(1 - \alpha)} \} \]

Third, to study the integral in the region \((0, \sigma)\), consider the inversion of the analytic function \( h(s) \). Since \( t = h(s) = \sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \) for \( s \in (0, \sigma) \), it is possible to argue as in Example 1 in Chapter 2 of [24] to obtain \( s = \sum_{k=1}^{+\infty} \alpha_k t^{k/2} \), by means of the Lagrange inversion formula. The coefficients \( \alpha_k \) are given by \( \alpha_1 = \sqrt{2}, \alpha_2 = 2/3 \) and the recurrence relation

\[ \frac{k + 2}{2^k} \alpha_{k+1} = \alpha_k - \sum_{j=0}^{k-2} \frac{j + 2}{2} \alpha_{j+2} \alpha_{k-j} \quad (k = 2, 3, \ldots) . \]

Therefore, \( h : (0, \sigma) \to (0, h(\sigma)) \) is bijective, with inverse function given by \( q(t) := \sum_{k=1}^{+\infty} \alpha_k t^{k/2} \) for \( t \in (0, h(\sigma)) \). These facts guarantee the possibility to change the variable, to get

\[ \int_0^{\sigma} e^{-nh(s)} f_\alpha \left( \frac{s + 1}{z} \right) ds = \int_0^{h(\sigma)} e^{-nt} f_\alpha \left( \frac{q(t) + 1}{z} \right) q'(t) dt . \]

Then, invoke the Taylor formula to show that

\[ \left| f_\alpha \left( \frac{s + 1}{z} \right) - \left[ f_\alpha \left( \frac{1}{z} \right) + f'_\alpha \left( \frac{1}{z} \right) \frac{s}{z} \right] \right| \leq \frac{1}{2} \sup_{y \in (0, \sigma)} \left| f''_\alpha \left( \frac{y + 1}{z} \right) \right| \left( \frac{s}{z} \right)^2 \]

\[ \left| q(t) - [\alpha_1 t^{1/2} + \alpha_2 t] \right| \leq Ct^{3/2} \]

\[ \left| q'(t) - \frac{1}{2} \alpha_1 t^{-1/2} + \alpha_2 \right| \leq Ct^{1/2} \]

for some numerical constant \( C \).

Finally, the study in the region \((-\sigma, 0)\) starts from the inversion of the analytic functions \( h(-s) \) for \( s \in (0, \sigma) \). Arguing again as in Example 1 in Chapter 2 of [24], deduce that \( s = \sum_{k=1}^{+\infty} (-1)^{k+1} \alpha_k t^{k/2} =: \varphi(t) \) is the inverse of \( t = h(-s) \) for \( s \in (0, \sigma) \). Thus, changing the variable yields

\[ \int_{-\sigma}^{0} e^{-nh(s)} f_\alpha \left( \frac{s + 1}{z} \right) ds = \int_0^{h(-\sigma)} e^{-nt} f_\alpha \left( -\frac{\varphi(t) + 1}{z} \right) \varphi'(t) dt . \]
Again, the Taylor formula shows that
\[
\left| f_a\left(\frac{-s+1}{z}\right) - f_a\left(\frac{1}{z}\right) - \frac{f''(1)}{2} \frac{s}{z^2} \right| \leq \frac{1}{2} \sup_{y \in (0, r)} \left| f''(\frac{-y+1}{z})\right| \left|\frac{s}{z}\right|^2
\]
for some numerical constant \(C\). The combination of these last remarks with the well-known Watson lemma (see, e.g., Section 5.I of [24]) leads to the conclusion, after noticing that
\[
\frac{f'(1)}{zf(1)} = O(z^{-2}) \quad \text{and} \quad \frac{f''(1)}{z^2f(1)} = O(z^{-4})
\]
as \(z \to 0\).

\[\square\]

2.3 A quantitative Poisson approximation

The main result of this section provides a deep analytical result about Poisson approximation. It is an improvement of Theorem 1 of Hwang [13], reformulated in a more quantitative style without “big O”-terms.

**Proposition 2** (Hwang). Let \(\{X_n\}_{n \geq 1}\) be a sequence of random variables taking values in \(\mathbb{N}_0\). Consider the relative sequence \(\{G_{X_n}\}_{n \geq 1}\) of probability generating functions under the hypothesis that, for every \(n \in \mathbb{N}\), \(G_{X_n}\) is holomorphic in \(D_{\eta+\tau_n} := \{s \in \mathbb{C} : |s| < \eta + \tau_n\}\), for some \(\eta > 3\) independent of \(n\) and \(\tau_n > 0\). Suppose that

\[
G_{X_n}(s) = \exp\{\lambda_n(s-1)\} s^k [g(s) + \epsilon_n(s)] ,
\]

holds for any \(s \in D_{\eta+\tau_n}\), where:

i) the restriction of \(g\) to \(D_\eta := \{s \in \mathbb{C} : |s| \leq \eta\}\) turns out to be independent of \(n\), continuous and holomorphic in \(D_\eta := \{s \in \mathbb{C} : |s| < \eta\}\), with \(g(1) = 1\) and \(g(0) \neq 0\);

ii) \(\epsilon_n\) is holomorphic in \(D_\eta := \{s \in \mathbb{C} : |s| < \eta\}\) and

\[
\bar{C}(\eta) := \sup_{n \in \mathbb{N}} \sup_{0 < |s-1| \leq \eta-1} \lambda_n \left| \frac{\epsilon_n(s)}{s-1} \right| < +\infty;
\]

iii) \(\lambda_n \geq \max\{2, |g'(1)|^{14/9}\}\) for all \(n \in \mathbb{N}\), and \(\lambda_n \to +\infty\) as \(n \to +\infty\);

iv) \(h \in \mathbb{N}_0\) is independent of \(n\).

Then, after introducing a sequence \(\{Y_n\}_{n \geq 1}\) of random variables satisfying

\[
P[Y_n = k] := \exp\{-[\lambda_n + g'(1)]\} \frac{\lambda_n + g'(1)^{k-h}}{(k-h)!} (k = h, h+1, \ldots),
\]

there exists a positive constant \(C(\eta)\) such that,

\[
\sum_{k \geq h} |P[X_n = k] - P[Y_n = k]| \leq \frac{C(\eta)}{\lambda_n}
\]

is valid for every \(n \in \mathbb{N}\).
The following proof contains also an indication to quantify $C(\eta)$. Notice also that the integer $h$ is well-defined in view of the holomorphic character of $g$ and $c_n$ about $s = 0$.

**Proof of Proposition 2.** Set $\Delta^{(k,n)} := P[X_n = k] - P[Y_n = k]$, $A_n := \lambda^{1/7}_n$, $N_1^{(n)} = \lambda_n - A_n \sqrt{\lambda_n}$, and $N_2^{(n)} = \lambda_n + A_n \sqrt{\lambda_n}$. To start, majorize the left-hand side of (19) as follows:

$$\sum_{k \geq h} |\Delta^{(k,n)}| \leq P[X_n \leq N_1^{(n)} + h] + P[Y_n \leq N_1^{(n)} + h] + P[X_n \geq N_2^{(n)} + h] + \sum_{N_1^{(n)} < k < N_2^{(n)}} |\Delta^{(k+h,n)}|.$$  \hspace{1cm} (20)

To bound the first summand on the right-hand side, set $r(k,n) := k/\lambda_n$ for any $k \in \{1, \ldots, [N_1^{(n)}]\}$, and observe that $P[X_n = k + h] = I_1^{(k,n)} + I_2^{(k,n)}$, where

$$I_1^{(k,n)} = \frac{1}{2\pi i} \oint_{|s| = r(k,n)} g(s)s^{-(k+1)}e^{\lambda_n(s-1)}ds$$

$$I_2^{(k,n)} = \frac{1}{2\pi i} \oint_{|s| = r(k,n)} \left(\frac{c_n(s)}{s-1}\right)(s-1)s^{-(k+1)}e^{\lambda_n(s-1)}ds.$$  

Since $r(k,n) < 1$, the same argument used to obtain estimate (8) in the proof of Lemma 1 of Hwang [13] shows that $|I_1^{(k,n)}| + |I_2^{(k,n)}| \leq \beta_1 \left[\sup_{|s| \leq 1} |g(s)| + \overline{C}(\eta)\right]e^{-\lambda_n \lambda_n k} \lambda_n k^2$ is valid with a suitable numerical constant $\beta_1 > 0$. Moreover, for $k = 0$, integrating on $|s| = 1$ leads to the estimate $P[X_n = h] \leq \beta_1 \left[\sup_{|s| \leq 1} |g(s)| + \overline{C}(\eta)\right]e^{-\lambda_n}$. Therefore, an application of well-known bounds on Poisson tail probabilities (see, e.g., Proposition 1 in [?) entails

$$P[X_n \leq N_1^{(n)} + h] \leq B_1 \left[\sup_{|s| \leq 1} |g(s)| + \overline{C}(\eta)\right] \exp\{-c_1 \lambda_n^{2/7}\}$$

for every $n \in \mathbb{N}, B_1, c_1 > 0$ being suitable numerical constants.

For the second summand on the right-hand side of (20), apply again the bounds on Poisson tail probabilities to obtain

$$P[Y_n \leq N_1^{(n)} + h] \leq e^{-(\lambda_n + g'(1))} \frac{\lambda_n + g'(1)}{[N_1^{(n)}]!}$$

so that

$$P[Y_n \leq N_1^{(n)} + h] \leq B_2 \exp\{-c_2 \lambda_n^{2/7}\}$$

for every $n \in \mathbb{N}, B_2, c_2 > 0$ being suitable numerical constants.

In the same vein in which the left tail $P[Y_n \leq N_1^{(n)} + h]$ has been dealt with, it can be shown that

$$P[Y_n \geq N_2^{(n)} + h] \leq e^{-(\lambda_n + g'(1))} \frac{\lambda_n + g'(1)}{[N_2^{(n)}]!}$$

and, again by elementary calculus, it is easily seen that the above right-hand side can be bounded by $B_4 \exp\{-c_4 \lambda_n^{2/7}\}$ for every $n \in \mathbb{N}, B_4, c_4 > 0$ being suitable numerical constants.

To study $\sum_{M_1 < k < M_2} |\delta_{k+h,n}|$, choose $r := k/\lambda_n$ and write $\delta_{k+h,n} := I_3^{(k,n)} + I_4^{(k,n)}$, where

$$I_3^{(k,n)} := \frac{1}{2\pi i} \oint_{|s| = r} \left[g(s) - e^{g'(1)(s-1)}\right]s^{-(k+1)}e^{\lambda_n(s-1)}ds$$

$$I_4^{(k,n)} := \frac{1}{2\pi i} \oint_{|s| = r} c_n(s)s^{-(k+1)}e^{\lambda_n(s-1)}ds.$$
Apropos of $I_4^{(k,n)}$, start by considering (18) and write
\[
|I_4^{(k,n)}| = \frac{e^{-\lambda_n}}{2\pi} \left| \int_{-\pi}^{\pi} \epsilon_n(r e^{i\theta}) r^{-k} e^{k|\cos \theta + i \sin \theta - i \theta|} d\theta \right|
\]
\[
\leq \frac{C_K e^{k-\lambda_n}}{2\pi \lambda_n r^k} \int_{-\pi}^{\pi} |r e^{i\theta} - 1| e^{-k(1 - \cos \theta)} d\theta
\]
\[
\leq \frac{C_K e^{k-\lambda_n}}{2\pi \lambda_n r^k} \int_{-\infty}^{\infty} |r - 1| + |r| |\theta| e^{-2kr^2} d\theta
\]
\[
\leq \frac{C_K e^{k-\lambda_n}}{2\pi \lambda_n r^k} \left[ \frac{\pi^{3/2}}{\sqrt{2k}} |r - 1| + \frac{\pi^2}{2k} r \right]
\]
\[
\leq C_4(\eta) \left( \frac{1}{\lambda_n} \right)^{19/14} e^{-\lambda_n \frac{\lambda_n^2}{k!}}
\]
showing that $\sum_{M_1 < k < M_2} |I_4^{(k,n)}| \leq C_4(\eta) \lambda_n^{-19/14}$.

To study $I_3^{(k,n)}$, introduce the function $G$ by means of the relation $g(s) - e^{\varphi'(1)(s-1)} = G(s)(s-1)^2$ and notice that it is holomorphic in a neighborhood of 1, since $g(1) = 1$. Expand $G(s)$ about $s = 1$ by Taylor’s formula to obtain $G(s) - G(1) = (s-1) \int_0^1 G'(1 + \tau(s-1)) d\tau = (s-1)G_1(s)$, yielding $I_3^{(k,n)} = I_5^{(k,n)} + I_6^{(k,n)}$ where
\[
I_5^{(k,n)} := \frac{G(1)}{2\pi i} \int_{|s| = r} (s-1)^2 s^{-2(k+1)} e^{\lambda_n(s-1)} ds
\]
\[
I_6^{(k,n)} := \frac{1}{2\pi i} \int_{|s| = r} G_1(s)(s-1)^3 s^{-2(k+1)} e^{\lambda_n(s-1)} ds.
\]

An application of Lemma 1 of [13] gives $\sum_{M_1 < k < M_2} |I_6^{(k,n)}| \leq C_4(\eta) \lambda_n^{-15/14}$, while an exact computation show that
\[
I_5^{(k,n)} = G(1) e^{-\lambda_n} \frac{\lambda_n^k}{k!} \left( 2\frac{\lambda_n^2 - 1}{\lambda_n^2} + \frac{k^2}{\lambda_n^2} \right) = G(1) \frac{\lambda_n^k}{k!} C_2(\lambda_n, k)
\]
where $C_2(\lambda_n, k)$ stands for the Poisson-Charlier polynomial of degree 2. Hence, Proposition 1 of [13] entails $\sum_{k=0}^{+\infty} |I_5^{(k,n)}| \leq C_5 \lambda_n^{-1}$, concluding the proof. \qed

2.4 Conclusion

This subsection contains the heart of the proof, whose strategy consists in three main steps. Relying on the same notation adopted in Proposition 1, they can be summarized as follows:

A) apply the strong law of large numbers to show that $G_{\theta + n, 1}^\alpha \sim n^\alpha$, from which it is expected that the probability law of $K_n/n^\alpha$ would be close, even in total variation, to the law of $R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha)/n^\alpha$;

B) invoke Lemma 2 to have a hint at the fact that the probability law of $R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha)$ would be close, again in total variation, to the law of some shifted compound random variable, of the form $1 + N_{\Lambda(S_{\alpha, \theta}, n, \alpha)}$, where $\Lambda$ is a suitable function (to be determined) and $S_{\alpha, \theta}$ is assumed independent of the family of random variables $\{N_{\lambda}\}_{\lambda > 0}$;

C) use well-known results about the Poisson distribution to obtain that the probability law of $N_{\Lambda(S_{\alpha, \theta}, n, \alpha)}/n^\alpha$ is close, in the Kolmogorov metric, to the law of $S_{\alpha, \theta}$.

According to point A), start by considering the random variable $R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha)$, whose probability law is given by
\[
P[R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha) = k] = \int_0^{+\infty} \left[ \frac{\mathcal{C}(n, k; \alpha)(tn^\alpha)^k}{\sum_{j=1}^{+\infty} \mathcal{C}(n, j; \alpha)(tn^\alpha)^j} \right] f_{S_{\alpha, \theta}}(t) dt
\]
for \( k = 1, \ldots, n \). There are all the elements to show that

\[
\begin{align*}
\sup_{x \geq 0} \left| P \left[ K_n/n^\alpha \leq x \right] - P \left[ R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha)/n^\alpha \leq x \right] \right| \\
= \sup_{x \geq 0} \left| P \left[ K_n \leq x \right] - P \left[ R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha) \leq x \right] \right| \leq C_1(\alpha, \theta)/n^\alpha
\end{align*}
\]  

(22)

for some positive constant \( C_1(\alpha, \theta) \). In fact, (22) follows from a sharper bound contained in the following

**Proposition 3.** There exists a some positive constant \( C_1(\alpha, \theta) \) for which

\[
\sum_{k=1}^{n} \left| \mathcal{C}(n, k; \alpha) \frac{\Gamma(k + \theta/\alpha)}{\Gamma(\theta/\alpha)} \frac{\Gamma(\theta)}{\Gamma(n + \theta)} - \int_{0}^{+\infty} \frac{\mathcal{C}(n, k; \alpha)(tn^\alpha)^k}{d_n(t)} f_{S_{\alpha, \theta}}(t)dt \right|.
\]

Proof of Proposition 3. A combination of (8), (12) and (21) shows that the left-hand side of (23) is equal to

\[
\sum_{k=1}^{n} \left| \mathcal{C}(n, k; \alpha) \frac{\Gamma(k + \theta/\alpha)}{\Gamma(\theta/\alpha)} \frac{\Gamma(\theta)}{\Gamma(n + \theta)} - \int_{0}^{+\infty} \frac{\mathcal{C}(n, k; \alpha)(tn^\alpha)^k}{d_n(t)} f_{S_{\alpha, \theta}}(t)dt \right| + \int_{0}^{+\infty} \left| d_{n}^*(t) - d_{n}(t) \right| f_{S_{\alpha, \theta}}(t)dt.
\]

Put \( d_{n}^*(t) := e^{tn^\alpha}(n - 1)! \frac{\Gamma(1 + \theta/\alpha)}{\Gamma(1 + \theta + n^\alpha)} \) in order to majorize the above quantity by

\[
\sum_{k=1}^{n} \left| \mathcal{C}(n, k; \alpha) \frac{\Gamma(k + \theta/\alpha)}{\Gamma(\theta/\alpha)} \frac{\Gamma(\theta)}{\Gamma(n + \theta)} - \int_{0}^{+\infty} \frac{\mathcal{C}(n, k; \alpha)(tn^\alpha)^k}{d_n(t)} f_{S_{\alpha, \theta}}(t)dt \right| + \int_{0}^{+\infty} \left| d_{n}^*(t) - d_{n}(t) \right| f_{S_{\alpha, \theta}}(t)dt.
\]

Notice that

\[
\int_{0}^{+\infty} \frac{\mathcal{C}(n, k; \alpha)(tn^\alpha)^k}{d_n(t)} f_{S_{\alpha, \theta}}(t)dt = \frac{\mathcal{C}(n, k; \alpha) \Gamma(k + \theta/\alpha)}{(n - 1)!} \frac{\Gamma(\theta)}{\Gamma(n + \theta)}
\]

yielding

\[
\sum_{k=1}^{n} \left| \mathcal{C}(n, k; \alpha) \frac{\Gamma(k + \theta/\alpha)}{\Gamma(\theta/\alpha)} \frac{\Gamma(\theta)}{\Gamma(n + \theta)} - \int_{0}^{+\infty} \frac{\mathcal{C}(n, k; \alpha)(tn^\alpha)^k}{d_n(t)} f_{S_{\alpha, \theta}}(t)dt \right| \leq \left| 1 - \frac{\Gamma(\theta + n)}{\Gamma(\theta)n^\theta} \right| \leq \frac{C(\theta)}{n}
\]

by the well-known Tricomi-Erdelyi expansion of the gamma ratio. Finally,

\[
\int_{0}^{+\infty} \frac{\left| d_{n}^*(t) - d_{n}(t) \right|}{d_n(t)} f_{S_{\alpha, \theta}}(t)dt \leq \left| \frac{(n/e)^n \sqrt{2\pi n}}{n!} - 1 \right| + \left| \frac{(n/e)^n \sqrt{2\pi n}}{n!} \frac{1}{n} \int_{0}^{+\infty} \Delta(t^{1/\alpha}) f_{S_{\alpha, \theta}}(t)dt \right|
\]

which leads to the desired conclusion, in view of the well-known Stirling approximation and the fact that \( \int_{0}^{+\infty} \Delta(t^{1/\alpha}) f_{S_{\alpha, \theta}}(t)dt < +\infty \), by virtue of Lemma 3. \( \Box \)

The application of Proposition 2 to the setting of the present paper starts from the evaluation of the probability generating function of the compound random variable \( R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha) \), as explained in point B). Since

\[
\begin{align*}
G_{R(\alpha, n, S_{\alpha, \theta} \cdot n^\alpha)}(s) = \int_{0}^{+\infty} G_{R(\alpha, n, tn^\alpha)}(s) f_{S_{\alpha, \theta}}(t)dt
\end{align*}
\]

holds by conditioning, a combination of equations (5) and (12) with Lemma 3 yields

\[
G_{R(\alpha, n, tn^\alpha)}(s) = e^{tn^\alpha(s-1)} \frac{\left(\frac{1}{\pi}\right)^{\theta/\alpha} f_{\alpha} \left(\left(\frac{1}{\pi}\right)^{\theta/\alpha}\right) \left[1 + \delta_n((st)^{\theta/\alpha})\right]}{\left[1 + \delta_n(t^{\theta/\alpha})\right]}.
\]  

(24)
To parallel (24) with (19) set: \( \lambda_n = tn^\alpha, h = 1, \eta = \) any number in \((1, +\infty)\),

\[
g(s) = \frac{(\frac{1}{4})^{\frac{1}{\eta}+1}f_\alpha\left((\frac{1}{4})^{\frac{1}{\eta}}\right)}{f_\alpha\left((\frac{1}{4})^{\frac{1}{\eta}}\right)} \quad \text{and} \quad \epsilon_n(s) = g(s) \left[\frac{1 + \delta_n((st)^{\frac{1}{\eta}})}{1 + \delta_n(t^{\frac{1}{\eta}})} - 1\right].
\]

It is also possible to consider the rate

\[
\omega(t) = \lambda_n + g'(1) = tn^\alpha + \frac{M'_\alpha(t)}{M_\alpha(t)}
\]

considered in Proposition 2, where \( M_\alpha \) denotes the so-called Wright-Mainardi function. Moreover, putting \( \pi(\lambda, h; k) := e^{-\lambda} \lambda^{k-h} (k-h)^{1-k} \) for \( \lambda > 0 \) and \( k \in \{h, h+1, \ldots\} \), it is now a direct application of Proposition 2 to show that

\[
\int_{\{t>0: \omega(t)>0\}} \left(\sum_{k=1}^{\infty} P[R(\alpha, n, tn^\alpha) = k] - \pi(\omega(t), 1; k)\right) f_{S_{n, \alpha}}(t) dt \leq C_2(\alpha, \theta)/n^\alpha \tag{25}
\]

with a suitable \( C_2(\alpha, \theta) > 0 \). It is also a direct consequence of the well-known asymptotic properties of the Wright-Mainardi function to prove that

\[
\int_{\{t>0: \omega(t)<0\}} f_{S_{n, \alpha}}(t) dt \sim \frac{1}{n^\alpha} \tag{26}
\]

as \( n \to \infty \). Therefore, it remains to evaluate

\[
\sup_{x \geq 0} \left| \int_{\{t>0: \omega(t)>0\}} \left(\sum_{k=1}^{[xn^\alpha]} \pi(\lambda, h; k)\right) f_{S_{n, \alpha}}(t) dt - \int_{\{t>0: \omega(t)>0\}} \mathbb{1}_{\{t \leq x\}} f_{S_{n, \alpha}}(t) \right|
\]

which is bounded by \( C_3(\alpha, \theta)/n^\alpha \) for a suitable \( C_3(\alpha, \theta) > 0 \), as a direct application of Theorem 1 of [1]. This completes the proof of our main theorem.

### 3 Pitman’s posterior \( \alpha \)-diversity

In this section we present a Berry-Esseen theorem for Pitman’s posterior \( \alpha \)-diversity \( S_{n, \theta}(n, j) \) in (4), for any \( \alpha \in (0, 1), \theta > -\alpha \) and \( n, j \in \mathbb{N} \) such that \( j \leq n \). As we recalled in the introduction, Pitman’s posterior \( \alpha \)-diversity was introduced in Favaro et al. [8] in the context of Bayesian nonparametric inference for species sampling problems under the PD(\( \alpha, \theta \)) prior. In such a species sampling setting, \((X_1, \ldots, X_n)\) is a random sample from the PD(\( \alpha, \theta \)) distribution, which is assumed to be observable and featuring \( K_n = j \leq n \) species. Lijoi et al. [16] obtained an explicit expression of the posterior distribution, given \((X_1, \ldots, X_n)\), of the number \( K_m^{(n)} \) of new species in \( m \) additional sample, that is

\[
P[K_m^{(n)} = k \mid X_1, \ldots, X_n] = \mathbb{P}[K_m^{(n)} = k \mid K_n = j] = \frac{\binom{\alpha + j}{k} \mathcal{G}(m, k; \alpha, -n + j\alpha),}{\binom{\theta + n}{m}},
\]

where \( \mathcal{G}(n, k; s, r) \) is the non-central generalized factorial coefficient, i.e., \( \mathcal{G}(n, k; s, r) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (-is - r)_{(m)} \). See Chapter 8 of Charalambides [3]. We start by introducing a novel representation of the posterior distribution (27) in terms of the distribution of a (compound) sum of independent Bernoulli random variables with random parameter. This is one of the key ingredients to prove the Berry-Esseen theorem for Pitman’s posterior \( \alpha \)-diversity. For any \( n \in \mathbb{N} \) and \( p \in [0, 1] \) we denote by \( Z(n, p) \) a random variable distributed according to a Binomial distribution with parameter \((n, p)\). Also, recall that use used \( B_{a,b} \) to denote a Beta random variable with parameter \((a, b)\).
Lemma 4. Let $n, j, m \in \mathbb{N}$ such that $j \leq n$. For any $\alpha \in (0, 1)$ and $\theta > -\alpha$, let $K_m^*$ be the number of blocks of the random partition of $\{1, \ldots, m\}$ induced by a random sample $(X_1^*, \ldots, X_m^*)$ from the $PD(\alpha, \theta + n)$ distribution. Then,

$$K_m^{(n)} | (X_1, \ldots, X_n) \overset{d}{=} Z(K_m^*, B_{\theta/\alpha + j, n/\alpha - j}),$$

(28)

with the parameters (random variables) $K_m^*$ and $B_{\theta/\alpha + j, n/\alpha - j}$ being mutually independent.

Proof. Let $[x]_{n,a}$ be the falling factorial of $x$ of order $n$ and decrement $a$, i.e. $[x]_{n,a} = \prod_{0 \leq i \leq n-1} (x - ia)$, let $S(n, k)$ be the Stirling number of the second kind, and let $S(n, k; x)$ be non-central Stirling number of the second kind, i.e.

$$S(n, k; x) = \sum_{t=k}^{n} \frac{(t+i-1)!}{t!} [x]_{t-k+1} S(n, i).$$

(29)

Furthermore, let $s(n, k)$ be the Stirling number of the first kind and recall that

$$[x]_{n,1} = \sum_{n \leq i \leq n} s(n, i)x^i.$$

(30)

We refer to Chapter 8 of Charalambides [3] for details. The proof relies on combining (29) with Proposition 1 in Favaro et al. [8]. Specifically, we can write

$$E[(K_m^{(n)})^r | X_1, \ldots, X_n] = E[(K_m^{(n)})^r | K_n = j]$$

(by Proposition 1 in Favaro et al. [8])

$$= \sum_{i=0}^{r} (-1)^{r-i} \left[ j + \frac{\theta}{\alpha} \right]_{(t,1)} S(r, i; j + \frac{\theta}{\alpha}) \frac{[\theta + n + i\alpha]_{(m,1)}}{[\theta + n]_{(m,1)}}$$

(by expanding $S(r, i; j + \theta/\alpha)$ as in (29))

$$= \sum_{i=0}^{r} (-1)^{r-i} \left[ \frac{[\theta + n + i\alpha]_{(m,1)}}{[\theta + n]_{(m,1)}} \right] \sum_{t=i}^{r} (-1)^t \left[ \frac{t}{i} \right] S(r, t) \left[ j + \frac{\theta}{\alpha} \right]_{(t,1)}$$

(by Proposition 1 in Favaro et al. [8])

$$= \sum_{t=0}^{r} S(r, t) \left[ j + \frac{\theta}{\alpha} \right]_{(t,1)} E[K_m^*]_{(t,1)}$$

(by expanding $[j + \theta/\alpha]_{(t,1)}/[\theta + n/\alpha]_{(t,1)}$ as an Euler integral)

$$= \sum_{t=0}^{r} S(r, t) E[K_m^*]_{(t,1)} \Gamma \left( \frac{\theta + n}{\alpha} \right) \Gamma \left( \frac{\theta}{\alpha} + j ight) \Gamma \left( \frac{\theta}{\alpha} - j \right) \int_0^1 x^{t+\frac{\theta}{\alpha}+j-1}(1-x)^{\frac{\theta}{\alpha}-j-1}dx$$

$$= \sum_{t=0}^{r} S(r, t) E[B_{\theta/\alpha + j, n/\alpha - j}]^t$$

$$= E \left[ \sum_{t=0}^{r} S(r, t) [K_m^*]_{(t,1)} (B_{\theta/\alpha + j, n/\alpha - j})^t \right]$$

$$= E \left[ (Z(K_m^*, B_{\theta/\alpha + j, n/\alpha - j}))^r \right],$$

where the last identity is because the moment of order $r$ of a Binomial random variable $Z(n, p)$ is $E[(Z(n, p))^r] = \sum_{0 \leq t \leq r} S(r, t)[n]_{(t,1)}p^t$. The proof is completed. \qed
Let $\mu_n$ and $\mu_{\alpha, \theta}(n, j)$ stand for the posterior distribution of $K_m^{(n)} / m^\alpha$, given $(X_1, \ldots, X_n)$, and the distribution of Pitman’s posterior $\alpha$-diversity $S_{\alpha, \theta}(n, j)$, respectively. Then next theorem may be interpreted as the natural posterior counterpart of Theorem 1, namely a Berry-Esseen theorem for Pitman’s posterior $\alpha$-diversity. Precisely, it provides with an upper bound for $d_K(\mu_m; \mu_{\alpha, \theta}(n, j))$, showing how fast this discrepancy goes to zero as $m \to +\infty$.

**Theorem 2.** Let $n, j, m \in \mathbb{N}$ such that $j \leq n$ and $n \geq 5$. For any $\alpha \in (0, 1)$ and $\theta > 0$ such that $\frac{\alpha}{\theta} - j \geq 1$, there exists a positive constant $C_{\alpha, \theta}(n, j)$, depending only on $n$, $j$, $\alpha$ and $\theta$, such that $d_K(\mu_m; \mu_{\alpha, \theta}(n, j)) \leq m^{-\alpha} C_{\alpha, \theta}(n, j)$ for every $m \in \mathbb{N}$.

**Proof.** Let $F_{K_m^{(n)}/m^\alpha}$ be the distribution function of $K_m^{(n)}$ given $K_n = j$, $F_{S_{\alpha, \theta}(n, j)}$ be the distribution function of Pitman’s posterior $\alpha$-diversity, and $F_{B_{\theta/\alpha+j,n/\alpha-j}}$ be the distribution function of the Beta random variable $B_{\theta/\alpha+j,n/\alpha-j}$. We show that

i) \[ d_K(F_{K_m^{(n)}/m^\alpha}(x), G_Z(x)) \leq C_{\alpha, \theta}(n, j)E \left[ \frac{1}{K_m^{*} + 1} \right] + d_K(F_{K_m^{*}/m^\alpha}(x), F_{S_{\alpha, \theta+n}}(x)) \] \[ \tag{31} \]

ii) \[ E \left[ \frac{1}{K_m^{*} + 1} \right] \leq \frac{1}{m^\alpha} E \left[ \frac{1}{S_{\alpha, \theta+n}} \right] + d_K(F_{K_m^{*}/m^\alpha}(x), F_{S_{\alpha, \theta+n}}(x)) . \] \[ \tag{32} \]

and then the proof follows by a direct application of Theorem 1. With regards to (31), we can write

\[ d_K(F_{K_m^{(n)}/m^\alpha}(x); F_{B_{\theta/\alpha+j,n/\alpha-j}}(x)) \]
\[ = d_K \left( F_{K_m^{(n)}/m^\alpha}(x), E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{S_{\alpha, \theta+n}} \right) \right] \right) \]
\[ = d_K \left( F_{K_m^{(n)}/m^\alpha}(x), E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{S_{\alpha, \theta+n}} \right) \right] \right) \]
\[ \leq d_K \left( F_{K_m^{(n)}/m^\alpha}(x), E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{m^\alpha x}{K_m^*} \right) \right] \right) \]
\[ + d_K \left( E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{m^\alpha x}{K_m^*} \right) \right], E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{S_{\alpha, \theta+n}} \right) \right] \right) \]
\[ = d_K \left( F_{K_m^{(n)}}(x), E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{K_m^*} \right) \right] \right) \]
\[ + d_K \left( E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{m^\alpha x}{K_m^*} \right) \right], E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{S_{\alpha, \theta+n}} \right) \right] \right) \] \[ \tag{33} \]

and then treat separately the terms in (33). With regards to the first term in (33), one has

\[ d_K \left( F_{K_m^{(n)}}(x), E \left[ F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{K_m^*} \right) \right] \right) \]
\[ = E \left[ d_K \left( F_{K_m^{(n)}}(x), F_{B_{\theta/\alpha+j,n/\alpha-j}} \left( \frac{x}{K_m^*} \right) \right) \right] \]
\[ = E [d_K(F_{K_m^{(n)}}(K_m^* x), F_{B_{\theta/\alpha+j,n/\alpha-j}}(x))] \]
\[ = E \left[ d_K \left( \frac{1}{K_m} \sum_{i=1}^{K_m} Z_i \leq x, F_{B_{\theta/\alpha+j,n/\alpha-j}}(x) \right) \right] , \]
where in the last identity we used Lemma 4, with the \( Z_i \)'s being independent Bernoulli random variables with parameter \( B_{\theta/\alpha+j,n/\alpha-j} \). Since \( K^*_m \) is independent of the \( Y_i \)'s, then the study of the random summation \( \sum_{i=1}^{K^*_m} Y_i \) can be carried out by a standard conditioning argument, according to the following

\[
E \left[ d_K \left( P \left( \frac{1}{K^*_m} \sum_{i=1}^{K^*_m} Y_i \leq x \right), F_{B_{\theta/\alpha+j,n/\alpha-j}}(x) \right) \right] = E \left[ d_K \left( P \left( \frac{1}{K^*_m} \sum_{i=1}^{K^*_m} Y_i \leq x \right), F_{B_{\theta/\alpha+j,n/\alpha-j}}(x) \right) \mid K^*_m \right].
\]

At this stage, since \( \frac{\theta}{\alpha} + j \geq 1 \) and \( \frac{\theta}{\alpha} - j \geq 1 \) by assumption, then it is possible to invoke Corollary 3 in Dolera and Favaro [6] to obtain the following inequality

\[
E \left[ d_K \left( P \left( \frac{1}{K^*_m} \sum_{i=1}^{K^*_m} Y_i \leq x \right), F_{B_{\theta/\alpha+j,n/\alpha-j}}(x) \right) \mid K^*_m \right] \leq 2 \sup_{x \in [0,1]} |F_{B_{\theta/\alpha+j,n/\alpha-j}}(x)| \frac{1}{K^*_m + 1}
\]

Taking the expectation of both sides of the last inequality yields the first term on the RHS of 31. With regards to the second term in (33), for any random variable \( X \) with distribution function \( F_X \) supported by \((0, +\infty)\), there holds

\[
E \left[ B_{\theta/\alpha+j,n/\alpha-j} \left( \frac{x}{X} \right) \right] = \int_0^1 F_X \left( \frac{x}{t} \right) dB_{\theta/\alpha+j,n/\alpha-j}(t).
\]

Then

\[
d_K \left( E \left[ B \left( \frac{m^\alpha X}{K^*_m} \right) \right], E \left[ B_{\theta/\alpha+j,n/\alpha-j} \left( \frac{x}{S_{\alpha,\theta+n}} \right) \right] \right) \leq \int_0^1 d_K \left( F_{K^*_m/m^\alpha} \left( \frac{x}{t} \right), F_{S_{\alpha,\theta+n}} \left( \frac{x}{t} \right) \right) dB_{\theta/\alpha+j,n/\alpha-j}(t)
\]

which gives the second term on the RHS of 31. With regards to (32), we can write

\[
E \left[ \frac{1}{K^*_m + 1} \right] \leq \left| E \left[ \frac{1}{K^*_m + 1} \right] - E \left[ \frac{1}{m^\alpha S_{\alpha,\theta+n} + 1} \right] \right| + E \left[ \frac{1}{m^\alpha S_{\alpha,\theta+n} + 1} \right]
\]

and rewrite the two expectations inside the modulus as follows. Put \( \varphi(x) = (1 + x)^{-1} \) and note that \( \varphi(x) \) is bounded and smooth on \((-1/2, +\infty)\) so that, for every distribution function \( F \) supported by \([0, +\infty)\) with \( F(0) = 0 \), one can integrate by parts to get \( \int_0^{+\infty} \varphi(x) dF(x) = \varphi(0) + \int_0^{+\infty} \varphi'(x) [1 - F(x)] dx \). Whence,

\[
\left| E \left[ \frac{1}{K^*_m + 1} \right] - E \left[ \frac{1}{m^\alpha S_{\alpha,\theta+n} + 1} \right] \right| \leq \int_0^{+\infty} |\varphi'(x)| \cdot |F_{K^*_m}(x) - F_{m^\alpha S_{\alpha,\theta+n}}(x)| dx
\]

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\[
\leq \int_0^{+\infty} |\varphi'(x)| dx \cdot \sup_{x \geq 0} |F_{K^m}(x) - F_{m^\alpha S_{a, s+n}}(x)|
= d_K(F_{K^m}(x), F_{m^\alpha S_{a, s+n}}) = d_K(F_{K^m/m^\alpha}(x), F_{S_{a, s+n}}(x)).
\]

To conclude, note that

\[
E\left[\frac{1}{S_{a, \theta+n}}\right] = \frac{\Gamma(\theta + n + 1)}{\alpha \Gamma(\frac{\theta+n}{\alpha} + 1)} \int_0^{+\infty} s^{\frac{\theta+n}{\alpha} - 1} f_\alpha(s^{-1/\alpha}) ds
= \frac{\Gamma(\theta + n + 1)}{\alpha \Gamma(\frac{\theta+n}{\alpha} + 1)} \int_0^{+\infty} \left(\frac{1}{t}\right)^{n+\theta-\alpha} f_\alpha(t) dt
= \frac{\alpha \Gamma(\theta + n + 1) \Gamma(\alpha(\theta + n - \alpha))}{\Gamma(\frac{\theta+n}{\alpha} + 1) \Gamma(\theta + n - \alpha)}
\]

which is finite whenever $\theta > -\alpha$, for all $n \in \mathbb{N}$. Then (32) holds true. The proof is completed by combining 31 and (32) and then by applying Theorem 1.

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