THE DYNAMICS OF THE NLS WITH THE COMBINED TERMS IN FIVE AND HIGHER DIMENSIONS

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Abstract. In this paper, we continue the study in [36] to show the scattering and blow-up result of the solution for the nonlinear Schrödinger equation with the energy below the threshold $m$ in the energy space $H^1(\mathbb{R}^d)$,

$$iu_t + \Delta u = -|u|^{4/(d-2)}u + |u|^{4/(d-1)}u, \quad d \geq 5.$$ (CNLS)

The threshold is given by the ground state $W$ for the energy-critical NLS: $iu_t + \Delta u = -|u|^{4/(d-2)}u$. Compared with the argument in [36], the new ingredient is that we use the double duhamel formula in [28, 45] to lower the regularity of the critical element in $L^\infty_t H^1_x$ to $L^\infty_t \dot{H}^{-\epsilon}_x$ for some $\epsilon > 0$ in five and higher dimensions and obtain the compactness of the critical element in $L^2_x$, which is used to control the spatial center function $x(t)$ of the critical element and furthermore used to defeat the critical element in the reductive argument.

1. Introduction

We consider the dynamics of the energy solutions for the nonlinear Schrödinger equation (NLS) with the combined nonlinearities in five and higher dimensions

$$\begin{cases} 
i u_t + \Delta u = f_1(u) + f_2(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0) = u_0(x) \in H^1(\mathbb{R}^d). \end{cases}$$ (1.1)

where $u : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{C}$, $d \geq 5$ and $f_1(u) = -|u|^{4/(d-2)}u$, $f_2(u) = |u|^{4/(d-1)}u$.

The equation has the following mass and Hamiltonian

$$M(u)(t) = \frac{1}{2} \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx; \quad E(u)(t) = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 \, dx + F_1(u(t)) + F_2(u(t))$$

where

$$F_1(u(t)) = -\frac{d-2}{2d} \int_{\mathbb{R}^d} |u(t, x)|^{2d} \, dx, \quad F_2(u(t)) = \frac{d-1}{2d+2} \int_{\mathbb{R}^d} |u(t, x)|^{2d+2} \, dx.$$ 

They are conserved for the sufficient smooth solutions of (1.1).

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In [44], Tao, Visan and Zhang made the comprehensive study of
\[ iu_t + \Delta u = |u|^{\frac{4}{d-2}} u + |u|^{\frac{4}{d-1}} u \]
in the energy space. They made use of the interaction Morawetz estimate established in [8] and the stability theory for the scattering solution. Their result is based on the scattering result of the defocusing, energy-critical NLS in the energy space, which is established by Bourgain [5, 6] for the radial case, I-team [9], Ryckman-Visan [41] and Visan [46] for the nonradial case. Since the classical interaction Morawetz estimate fails for (1.1), Tao, et al., leave the scattering and blow-up dichotomy of (1.1) below the threshold as an open problem in [44]. For other results, please refer to [1, 2, 17, 18, 37, 38, 39, 47, 48].

For the focusing, energy-critical NLS
\[ iu_t + \Delta u = -|u|^{\frac{4}{d-2}} u. \] (1.2)
Kenig and Merle first applied the concentration compactness in [4, 24, 25] into the scattering theory of the radial solution of (1.2) in [22] with the energy below that of the ground state of
\[-\Delta W = |W|^{\frac{4}{d-2}} W.\]
Subsequently, Killip and Visan made use of the double Duhamel argument in [26, 45] to removed the radial assumption in [27]. For the applications of the concentration compactness in the scattering theory and rigidity theory of the critical NLS, NLW, NLKG and Hartree equations, please see [10, 11, 12, 13, 14, 15, 20, 23, 26, 27, 29, 32, 35, 33, 34].

In [36], we made use of the concentration compactness argument and rigidity argument to show the dichotomy of the radial solution of (1.1) in $H^1(\mathbb{R}^3)$ with energy less than the threshold. In this paper, we continue this study in five and higher dimensions.

Now for $\varphi \in H^1$, we denote the scaling quantity $\varphi_{d,-2}^\lambda$ by
\[ \varphi_{d,-2}^\lambda(x) = e^{d\lambda} \varphi(e^{2\lambda} x). \]
We denote the scaling derivative of $E$ by $K(\varphi)$
\[ K(\varphi) = \mathcal{L}E(\varphi) := \left. \frac{d}{d\lambda} \right|_{\lambda=0} E(\varphi_{d,-2}^\lambda) = \int_{\mathbb{R}^d} 2|\nabla \varphi|^2 - 2|\varphi|^{\frac{4d}{d-2}} + \frac{2d}{d+1}|\varphi|^{\frac{4d+2}{d+1}} dx, \] (1.3)
which is connected with the Virial identity, and then plays the important role in the blow-up and scattering of the solution of (1.1).
The threshold is determined by the following constrained minimization of the energy $E(\varphi)$

$$m = \inf \{ E(\varphi) \mid 0 \neq \varphi \in H^1(\mathbb{R}^d), K(\varphi) = 0 \}. \quad (1.4)$$

Since we consider the $\dot{H}^1$-critical growth with the subcritical perturbation, we need the following modified energy

$$E^c(u) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla u(t,x)|^2 - \frac{d-2}{2d} |u(t,x)|^{\frac{2d}{d-2}} \right) \, dx.$$ 

As the nonlinearity $|u|^{\frac{4}{d-1}} u$ is the defocusing, $\dot{H}^1$-subcritical perturbation, one think that the focusing, $\dot{H}^1$-critical term plays the decisive role of the threshold of the scattering solution of (1.1) in the energy space. Just as the 3d case in [36], the first result is to characterize the threshold energy $m$ as following

**Proposition 1.1** ([36], Proposition 1.1). For $d \geq 5$, there is no minimizer for (1.4). But for the threshold energy $m$, we have

$$m = E^c(W),$$

where $W \in H^1(\mathbb{R}^d)$ is the ground state of the massless equation

$$-\Delta W = |W|^{\frac{4}{d-1}} W.$$

Main result in this paper is

**Theorem 1.2.** For $d \geq 5$. Let $u_0 \in H^1(\mathbb{R}^d)$ with

$$E(u_0) < m, \quad (1.5)$$

and $u$ be the solution of (1.1) and $I$ be its maximal interval of existence. Then

(a) If $K(u_0) \geq 0$, then $I = \mathbb{R}$, and $u$ scatters in both time directions as $t \to \pm \infty$ in $H^1$;

(b) If $K(u_0) < 0$ and $xu_0 \in L^2$ or $u_0$ is radial, then $u$ blows up both forward and backward at finite time in $H^1$.

By the above result, we conclude that the focusing, $\dot{H}^1$-critical term make the main contribution to the determination of the threshold of the scattering solution of (1.1). For the case $d = 3$, we verify the above result for the radial case in [36]. In this paper, we show the scattering result without the radial assumption in five and higher dimensions. Compared with the argument in [36], the new ingredient in five and higher dimensions is that we can use the double duhamel formula in [28] to lower the regularity of the critical element in $L^\infty H^1_x$ to $L^\infty \dot{H}^{-\epsilon}_x$ for some $\epsilon > 0$ and obtain the compactness of
the critical element in $L_x^2$, which is used to control the spatial center function $x(t)$ of the critical element and furthermore used to defeat the critical element in the reductive argument.

At last, from the assumption in Theorem 1.2 we know that the solution starts from the following subsets of the energy space,

$$
\mathcal{K}^+ = \left\{ \varphi \in H^1(\mathbb{R}^3) \left| \varphi \text{ is radial, } E(\varphi) < m, K(\varphi) \geq 0 \right. \right\}, \\
\mathcal{K}^- = \left\{ \varphi \in H^1(\mathbb{R}^3) \left| \varphi \text{ is radial, } E(\varphi) < m, K(\varphi) < 0 \right. \right\}.
$$

By the similar scaling argument to that in [36], we know that $\mathcal{K}^+ \neq \emptyset$. 

This paper is organized as follows. In Section 2 we give the basic well-known results, including the linear and nonlinear estimates, the local well-posedness, the perturbation theory and the monotonicity formula. In Section 3 we show the threshold by the variational method, which also give the proof of Proposition 1.1 and various variational estimates, which will be used in the proof of Theorem 1.2. In Section 4 we give the proof of the blow up in Theorem 1.2 in the radial case. In Section 5, we show the linear and nonlinear profile decompositions of the $H^1$-bound sequences of solution of (1.1). In Section 6, we make use of the stability theory and compactness argument to show the global wellposedness and scattering result in Theorem 1.2 in a reductive argument.

2. Preliminaries

In this section, we give some notation and some well-known results.

2.1. Littlewood-Paley decomposition and Besov space. Let $\Lambda_0(x) \in \mathcal{S}(\mathbb{R}^d)$ such that its Fourier transform $\tilde{\Lambda}_0(\xi) = 1$ for $|\xi| \leq 1$ and $\tilde{\Lambda}_0(\xi) = 0$ for $|\xi| \geq 2$. Then we define $\Lambda_k(x)$ for any $k \in \mathbb{Z}\setminus\{0\}$ and $\Lambda_{(0)}(x)$ by the Fourier transforms:

$$
\tilde{\Lambda}_k(\xi) = \tilde{\Lambda}_0(2^{-k}\xi) - \tilde{\Lambda}_0(2^{-(k+1)}\xi), \quad \tilde{\Lambda}_{(0)}(\xi) = \tilde{\Lambda}_0(\xi) - \tilde{\Lambda}_0(2\xi).
$$

Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The inhomogeneous Besov space $B_{p,q}^s$ is defined by

$$
B_{p,q}^s = \left\{ f \ | \ f \in \mathcal{S}'(\mathbb{R}^d), \left\| 2^{ks} \| \Lambda_k \ast f \|_{L^p_x} \right\|_{k \geq 0} < \infty \right\},
$$

where $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions. The homogeneous Besov space $\dot{B}_{p,q}^s$ can be defined by

$$
\dot{B}_{p,q}^s = \left\{ f \ | \ f \in \mathcal{Z}'(\mathbb{R}^d), \left( \sum_{k \in \mathbb{Z}\setminus\{0\}} 2^{ks} \| \Lambda_k \ast f \|_{L^p_x}^q + \| \Lambda_{(0)} \ast f \|_{L^p_x}^q \right)^{1/q} < \infty \right\}.
$$
\( \mathcal{Z}'(\mathbb{R}^d) \) denotes the dual space of \( \mathcal{Z}(\mathbb{R}^d) = \{ f \in \mathcal{S}(\mathbb{R}^d); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index} \} \) and can be identified by the quotient space of \( \mathcal{S}'/\mathcal{P} \) with the polynomials space \( \mathcal{P} \).

### 2.2. Linear estimates and nonlinear estimates.

We say that a pair of exponents \((q, r)\) is Schrödinger \( \dot{H}^s \)-admissible in \( d \geq 5 \) if

\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s
\]

and \( 2 \leq q, r \leq \infty \). If \( I \times \mathbb{R}^d \) is a space-time slab, we define the \( \dot{S}^0(I \times \mathbb{R}^d) \) Strichartz norm by

\[
\| u \|_{\dot{S}^0(I \times \mathbb{R}^d)} := \sup \| u \|_{L^2_t L^2_x(I \times \mathbb{R}^d)}
\]

where the sup is taken over all \( L^2 \)-admissible pairs \((q, r)\). We define the \( \dot{S}^s(I \times \mathbb{R}^d) \) and \( S^s(I \times \mathbb{R}^d) \) Strichartz norm to be

\[
\| u \|_{\dot{S}^s(I \times \mathbb{R}^d)} := \| D^s u \|_{\dot{S}^0(I \times \mathbb{R}^d)}, \quad \| u \|_{S^s(I \times \mathbb{R}^d)} := \| \langle \nabla \rangle^s u \|_{\dot{S}^0(I \times \mathbb{R}^d)}.
\]

We also use \( \dot{N}^0(I \times \mathbb{R}^d) \) to denote the dual space of \( \dot{S}^0(I \times \mathbb{R}^d) \) and

\[
\dot{N}^k(I \times \mathbb{R}^d) := \{ u; D^k u \in \dot{N}^0(I \times \mathbb{R}^d) \}.
\]

Before we introduce the linear estimate, we first give some exponents, which will be frequently used in the paper. For \( S(I) := L^\infty(I; L^2) \cap L^2(I; L^{2^*}) \), \( W_1(I) := L^{\frac{2(d+2)}{d+4}}(I; L^{\frac{2(d+4)}{d+2}}) \), \( V_1(I) := L^2(I; L^{\frac{4d+4}{d+2}}(I; L^{\frac{4d+4}{d+2}}(I; L^{\frac{2(d+2)}{d+4}}(I; L^{\frac{2(d+4)}{d+2}}(I; L^{\frac{4d+4}{d+2}}(I; L^{\frac{2d+4}{d+2}}(I; L^{2^*})))))) \),

\[
W_2(I) := L^{\frac{2(d+4)}{d+1}}(I; L^{\frac{2d+4}{d+1}}(I; L^{\frac{4d+4}{d+2}}(I; L^{\frac{2d+4}{d+1}}(I; L^{\frac{4d+4}{d+2}}(I; L^{\frac{2d+4}{d+1}}(I; L^{\frac{2d+4}{d+2}}(I; L^{2^*})))))),
\]

By definition and Sobolev’s inequality, we have

**Lemma 2.1.** For any \( \dot{S}^1 \) function \( u \) on \( I \times \mathbb{R}^d \), we have

\[
\| \nabla u \|_{S(I)} + \| \nabla u \|_{V_1(I)} + \| u \|_{W_1(I)} + \| \nabla u \|_{V_2(I)} + \| |\nabla|^{1/2} u \|_{W_2(I)} \lesssim \| u \|_{\dot{S}^1}.
\]

Now we state the standard Strichartz estimate.

**Lemma 2.2 ([7, 21, 43]).** Let \( I \) be a compact time interval, \( k \in [0,1] \), and let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be an \( \dot{S}^k \) solution to the forced Schrödinger equation

\[
iu_t + \Delta u = F
\]

for a function \( F \). Then we have

\[
\| u \|_{\dot{S}^k(I \times \mathbb{R}^d)} \lesssim \| u(t_0) \|_{\dot{H}^k(\mathbb{R}^d)} + \| F \|_{\dot{N}^k(I \times \mathbb{R}^d)},
\]
for any time \( t_0 \in I \).

For \( d \geq 5 \), let \( s_\alpha := \frac{d}{2} - \frac{2}{\alpha - 1} = 1 - \frac{2}{d} \), then \( \alpha = \frac{d^2 + 2d + 4}{d^2 - 2d + 4} \). Let \((\gamma, \rho)\) be the \( \dot{H}^{s_\alpha}\)-admissible pair such that

\[
\rho = \frac{\alpha + 1 + 2^*}{2}, \quad \frac{2}{\gamma} = d \left( \frac{1}{2} - \frac{1}{\rho} \right) - s_\alpha.
\]

**Lemma 2.3** ([16]). For \( d \geq 5 \), and any \( F \in L_t^2 \left( I; B_{q,2}^{\frac{2}{d^2 + 4}} \right) \), we have

\[
\left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L_t^\gamma \left( I; B_{q,2}^{\frac{2}{d^2 + 4}} \right)} \lesssim \| F \|_{L_t^2 \left( I; B_{q,2}^{\frac{2}{d^2 + 4}} \right)}.
\]

Let \((q_i, r_i)\), \( i = 1, \ldots, 6 \) be the exponentials such that

\[
\begin{align*}
\frac{1}{2} &= \frac{4/(d-2)}{q_1} + \frac{1}{\gamma} = \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{\gamma} = \frac{4/(d-1)}{q_4} + \frac{1}{\gamma} = \frac{1}{q_5} + \frac{1}{q_6} + \frac{1}{\gamma}, \\
\frac{d^2 + 4}{2d^2} &= \frac{4/(d-2)}{r_1} + \frac{1}{\rho} = \frac{1}{r_2} + \frac{1}{r_3} + \frac{d^2 - 2\rho}{d^2\rho} = \frac{4/(d-1)}{r_4} + \frac{1}{\rho} = \frac{1}{r_5} + \frac{1}{r_6} + \frac{d^2 - 2\rho}{d^2\rho},
\end{align*}
\]

where

\[
\begin{align*}
\frac{1}{q_2} &= \frac{1}{r_2} = \frac{d-2}{2(d+2)} \times \left( \frac{4}{d-2} - \frac{4}{d} \right), \\
\frac{1}{q_5} &= \frac{1}{r_5} = \frac{d-1}{2(d+2)} \times \left( \frac{4}{d-1} - \frac{4}{d} \right),
\end{align*}
\]

then

1. \( A_1 = (q_1, r_1) \) is \( \dot{H}^1\)-admissible pair; \( W_1 = \left( \left( \frac{4}{d-2} - \frac{4}{d} \right) q_2, \left( \frac{4}{d-2} - \frac{4}{d} \right) r_2 \right) \) is the diagonal \( \dot{H}^1\)-admissible pair; \( B_1 = (\frac{4}{d} q_3, \frac{4}{d} r_3) \) is \( \dot{H}^{1/2}\)-admissible pair.
2. \( A_2 = (q_4, r_4) \) is \( \dot{H}^{1/2}\)-admissible pair; \( W_2 = \left( \left( \frac{4}{d-1} - \frac{4}{d} \right) q_5, \left( \frac{4}{d-1} - \frac{4}{d} \right) r_5 \right) \) is the diagonal \( \dot{H}^{1/2}\)-admissible pair; \( B_2 = (\frac{4}{d} q_6, \frac{4}{d} r_6) \) is \( L^2\)-admissible pair.
3. \( ES = (\gamma, \rho) \), \( ES^* = \left( 2, \frac{d^2}{d^2 + 4} \right) \).

**Lemma 2.4** ([2] [46]). For \( d \geq 5 \), assume that \( h_1 \) and \( h_2 \) are Hölder continuous functions of order \( \frac{4}{d-2} \) and \( \frac{4}{d-1} \), respectively. Let \( I \) be an interval, then we have

\[
\begin{align*}
\| h_1(v) w \|_{L_t^2 \left( I; B_{q,2}^{\frac{d^2}{d^2 + 4}} \right)} &\lesssim \| v \|_{L_t^{\frac{4}{d-2}} \left( I; L_x^{r_1} \right)} \| w \|_{L_t^{\frac{4}{d-1}} \left( I; B_{q,2}^{\frac{d^2}{d^2 + 4}} \right)} \\
&\quad + \| v \|_{L_t^{\frac{4}{d-2}} \left( I; L_x^{\frac{4}{d-2}} q_2 L_x^{\frac{4}{d-2}} r_2 \right)} \| \nabla^{1/2} v \|_{L_t^{\frac{4}{d-2}} \left( I; L_x^{\frac{4}{d-2}} r_2 \right)} \| w \|_{L_t^{\frac{4}{d-1}} \left( I; B_{q,2}^{\frac{d^2}{d^2 + 4}} \right)};
\end{align*}
\]

\[
\| h_2(v) w \|_{L_t^2 \left( I; B_{q,2}^{\frac{d^2}{d^2 + 4}} \right)} \lesssim \| v \|_{L_t^{\frac{4}{d-1}} \left( I; L_x^{r_1} \right)} \| w \|_{L_t^{\frac{4}{d-1}} \left( I; B_{q,2}^{\frac{d^2}{d^2 + 4}} \right)}
\]
2.3. Local wellposedness and perturbation theory. Let us denote $ST(I)$ by $W_1(I) \cap W_2(I)$. By the analogue analysis as those in [2, 36], we have

**Theorem 2.5** (Local wellposedness, [2, 36, 44]). Let $u_0 \in H^1$, then for every $T > 0$, there exists $\eta = \eta(T)$ such that if

$$
\| (\nabla) e^{it\Delta} u_0 \|_{L^2([-T,T])} \leq \eta,
$$

then (1.1) admits a unique strong $H^1_x$-solution $u$ defined on $[-T,T]$. Let $(-T_{\min}, T_{\max})$ be the maximal time interval on which $u$ is well-defined. Then, $u \in S^1(I \times \mathbb{R}^d)$ for every compact time interval $I \subset (-T_{\min}, T_{\max})$ and the following properties hold:

1. If $T_{\max} < \infty$, then

$$
\| u \|_{ST((0,T_{\max}) \times \mathbb{R}^d)} = \infty.
$$

Similarly, if $T_{\min} < \infty$, then

$$
\| u \|_{ST((-T_{\min},0) \times \mathbb{R}^d)} = \infty.
$$

2. The solution $u$ depends continuously on the initial data $u_0$ in the following sense:

The functions $T_{\min}$ and $T_{\max}$ are lower semicontinuous from $H^1_x$ to $(0, +\infty]$. Moreover, if $u^{(m)}_0 \to u_0$ in $H^1_x$ and $u^{(m)}$ is the maximal solution to (1.1) with initial data $u^{(m)}_0$, then $u^{(m)} \to u$ in $ST(I \times \mathbb{R}^d)$ and every compact subinterval $I \subset (-T_{\min}, T_{\max})$. 

**Figure 1.** Admissible pairs: $A_i, B_i, V_i, W_i$, $i = 1, 2$, and $ES, ES^*$. 

\[ + \| v \|_{L^2(\mathbb{R}^d)} \| \nabla \|_{L^2(\mathbb{R}^d)} \| w \|_{L^\infty(\mathbb{R}^d)} \]
Proposition 2.6 ([2, 36]). Let $I$ be a compact time interval and let $w$ be an approximate solution to (1.1) on $I \times \mathbb{R}^d$ in the sense that

$$i \partial_t w + \Delta w = -|w|^{\frac{d}{4-2}} w + |w|^{\frac{d}{4}} w + e$$

for some suitable small function $e$. Assume that for some constants $L, E_0 > 0$, we have

$$\|w\|_{ST(I)} \leq L, \quad \|w(t_0)\|_{H^1_x(\mathbb{R}^d)} \leq E_0.$$ 

for some $t_0 \in I$. Let $u(t_0)$ close to $w(t_0)$ in the sense that for some $E' > 0$, we have

$$\|u(t_0) - w(t_0)\|_{H^1_x} \leq E'.$$

Assume also that for some $\varepsilon$, we have

$$\|\langle \nabla \rangle e^{i(t-t_0)} \Delta (u(t_0) - w(t_0))\|_{V_2(I)} \leq \varepsilon, \quad \|\langle \nabla \rangle e\|_{L^{\frac{d+2}{d+4}}(I)} \leq \varepsilon, \quad (2.1)$$

where $0 < \varepsilon \leq \varepsilon_0 = \varepsilon_0(E_0, E', L)$ is a small constant. Then there exists a solution $u$ to (1.1) on $I \times \mathbb{R}^d$ with initial data $u(t_0)$ at time $t = t_0$ satisfying

$$\|\langle \nabla \rangle (u - w)\|_{V_2(I)} \leq C(E_0, E', L) \varepsilon, \quad \text{and} \quad \|\langle \nabla \rangle u\|_{S(I)} \leq C(E_0, E', L).$$

2.4. Monotonicity formula.

Lemma 2.7 ([19]). Let $\phi \in C_0^\infty(\mathbb{R}^3)$, and $u$ be the solution of (1.1). Then we have

$$\partial_t \int_{\mathbb{R}^d} \phi(x)|u(t,x)|^2 \, dx = -2 \Delta \int_{\mathbb{R}^d} \nabla \phi \cdot \nabla \bar{u} \, u \, dx$$

$$\partial_t^2 \int_{\mathbb{R}^d} \phi(x)|u(t,x)|^2 \, dx = 4 \int_{\mathbb{R}^d} \phi_{ij}(x) u_i(t,x) \bar{u}_j(t,x) \, dx - \int_{\mathbb{R}^d} \Delta^2 \phi |u(t,x)|^2 \, dx$$

$$- \frac{4}{d} \int_{\mathbb{R}^d} \Delta \phi |u(t,x)|^{2^*} \, dx + \frac{4}{d+1} \int_{\mathbb{R}^d} \Delta \phi |u(t,x)|^{\frac{2d+2}{d+1}} \, dx.$$ 

3. Variational characterization

In this section, we show the threshold energy $m$ (Proposition 1.1) by the variational method, and various estimates for the solutions of (1.1) with the energy below the threshold. The argument is the analogue as the case $d = 3$ in [36].

Let us denote the quadratic and nonlinear parts of $K$ by $K^Q$ and $K^N$, that is,

$$K(\varphi) = K^Q(\varphi) + K^N(\varphi),$$

where $K^Q(\varphi) = 2 \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dx$, and $K^N(\varphi) = \int_{\mathbb{R}^d} \left(-2|\varphi|^{\frac{2d}{d+2}} + \frac{2d}{d+1}|\varphi|^{\frac{2d+2}{d+1}}\right) \, dx$. 

Lemma 3.1. For any $\varphi \in H^1(\mathbb{R}^d)$, we have

$$\lim_{\lambda \to -\infty} K^Q(\varphi_{d,-2}^\lambda) = 0. \quad (3.1)$$

Proof. It is obvious by the definition of $K^Q$. \qed

Now we show the positivity of $K$ near 0 in the energy space.

Lemma 3.2. For any bounded sequence $\varphi_n \in H^1(\mathbb{R}^d) \setminus \{0\}$ with

$$\lim_{n \to +\infty} K^Q(\varphi_n) = 0,$$

then for large $n$, we have

$$K(\varphi_n) > 0.$$

Proof. By the fact that $K^Q(\varphi_n) \to 0$, we know that $\lim_{n \to +\infty} \|\nabla \varphi_n\|_{L^2}^2 = 0$. Then by the Sobolev and Gagliardo-Nirenberg inequalities, we have for large $n$

$$\|\varphi_n\|_{L^{2^*}}^{2^*} \lesssim \|\nabla \varphi_n\|_{L^2}^{2^*} = o(\|\nabla \varphi_n\|_{L^2}^2),$$

$$\|\varphi_n\|_{L^{2^*}}^{2d+2} \lesssim \|\varphi_n\|_{L^2}^{2^*} \|\nabla \varphi_n\|_{L^2}^{2d} = o(\|\nabla \varphi_n\|_{L^2}^2),$$

where we use the boundedness of $\|\varphi_n\|_{L^2}$. Hence for large $n$, we have

$$K(\varphi_n) = \int_{\mathbb{R}^d} \left( 2|\nabla \varphi_n|^2 - 2|\varphi_n|^{2^*} + \frac{2d}{d+1}|\varphi_n|^{\frac{2d+2}{d-1}} \right) dx \approx \int_{\mathbb{R}^d} |\nabla \varphi_n|^2 dx > 0.$$ 

This concludes the proof. \qed

By the definition of $K$, we denote two real numbers

$$\bar{\mu} = \max\{4, \frac{4d}{d-1}\} = \frac{4d}{d-1}, \quad \underline{\mu} = \min\{4, \frac{4d}{d-1}\} = 0.$$

Next, we show the behavior of the scaling derivative functional $K$ with respect to the scaling $\varphi_{d,-2}^\lambda$.

Lemma 3.3. For any $\varphi \in H^1$, we have

$$(\bar{\mu} - \mathcal{L}) E(\varphi) = \int_{\mathbb{R}^d} \left( \frac{2}{d-1}|\nabla \varphi|^2 + \frac{2}{d-1}|\varphi|^{2^*} \right) dx,$$

$$\mathcal{L} (\bar{\mu} - \mathcal{L}) E(\varphi) = \int_{\mathbb{R}^d} \left( \frac{8}{d-1}|\nabla \varphi|^2 + \frac{8d}{(d-1)(d-2)}|\varphi|^{2^*} \right) dx.$$

Proof. By the definition of $\mathcal{L}$, we have

$$\mathcal{L}\|\nabla \varphi\|_{L^2}^2 = 4\|\nabla \varphi\|_{L^2}^2, \quad \mathcal{L}\|\varphi\|_{L^{2^*}}^{2^*} = \frac{4d}{d-2}\|\varphi\|_{L^{2^*}}^{2^*}, \quad \mathcal{L}\|\varphi\|_{L^{\frac{2d+2}{d-1}}}^{\frac{2d+2}{d-1}} = \frac{4d}{d-1}\|\varphi\|_{L^{\frac{2d+2}{d-1}}}^{\frac{2d+2}{d-1}}.$$
which implies that
\[ (\bar{\mu} - L) E(\varphi) = \bar{\mu} E(\varphi) - K(\varphi) \]
\[ = \int_{\mathbb{R}^d} \left( \frac{2}{d-1} |\nabla \varphi|^2 + \frac{2}{d-1} |\varphi|^{2^*} \right) \, dx, \]
\[ L (\bar{\mu} - L) E(\varphi) = \frac{2}{d-1} \mathcal{L} \|\nabla \varphi\|_{L^2}^2 + \frac{2}{d-1} \mathcal{L} \|\varphi\|_{L^{2^*}}^{2^*} \]
\[ = \int_{\mathbb{R}^d} \left( \frac{8}{d-1} |\nabla \varphi|^2 + \frac{8d}{(d-1)(d-2)} |\varphi|^{2^*} \right) \, dx. \]

This completes the proof. \( \square \)

According to the above analysis, we will replace the functional \( E \) in (1.4) with a positive functional \( H \), while extending the minimizing region from ”the mountain ridge \( K(\varphi) = 0 \)” to “the mountain flank \( K(\varphi) \leq 0 \).” Let
\[ H(\varphi) := \left( 1 - \frac{\mathcal{L}}{\bar{\mu}} \right) E(\varphi) = \int_{\mathbb{R}^d} \left( \frac{1}{2d} |\nabla \varphi|^2 + \frac{1}{2d} |\varphi|^{2^*} \right) \, dx, \]
then for any \( \varphi \in H^1 \setminus \{0\} \), we have
\[ H(\varphi) > 0, \quad \mathcal{L} H(\varphi) \geq 0. \]

Now by the similar argument to that in [36], we can characterization the minimization problem (1.4) by making use of \( H \).

**Lemma 3.4** ([36], Lemma 2.9). For the minimization \( m \) in (1.4), we have
\[ m = \inf \{ H(\varphi) \mid \varphi \in H^1(\mathbb{R}^d), \varphi \neq 0, K(\varphi) \leq 0 \} \]
\[ = \inf \{ H(\varphi) \mid \varphi \in H^1(\mathbb{R}^d), \varphi \neq 0, K(\varphi) < 0 \}. \]  

Next we will use the \( H^1 \)-invariant scaling argument to remove the \( H^1 \)-subcritical growth term \( \int_{\mathbb{R}^d} |\varphi|^{\frac{2d+4}{d-2}} \, dx \) in \( K \), that is, to replace the constrained condition \( K(\varphi) < 0 \) with \( K^c(\varphi) < 0 \), where
\[ K^c(\varphi) := \int_{\mathbb{R}^d} \left( 2 |\nabla \varphi|^2 - 2 |\varphi|^{2^*} \right) \, dx. \]

In fact, we have

**Lemma 3.5** ([36], Lemma 2.10). For the minimization \( m \) in (1.4), we have
\[ m = \inf \{ H(\varphi) \mid \varphi \in H^1(\mathbb{R}^d), \varphi \neq 0, K^c(\varphi) < 0 \} \]
\[ = \inf \{ H(\varphi) \mid \varphi \in H^1(\mathbb{R}^d), \varphi \neq 0, K^c(\varphi) \leq 0 \}. \]
The above result holds for the defocusing perturbation, which implies that $K^c(\varphi) \leq K(\varphi)$. While the argument does not hold for the focusing perturbation from the proof in [36]. Please refer to [1, 2] for the related discussions. After these preparations, we can now make use of the sharp constant of the Sobolev inequality in [3, 42] to compute the minimization $m$ as following.

**Lemma 3.6** ([36], Lemma 2.11). For the minimization $m$ in (1.4), we have

$$m = E^c(W).$$

After the computation of the minimization $m$ in (1.4), we now give some useful variational estimates.

**Lemma 3.7** ([36], Lemma 5.4). Let $k \in \mathbb{N}$ and $\varphi_0, \ldots, \varphi_k \in H^1(\mathbb{R}^d)$. Assume that there exist some $\delta, \varepsilon > 0$ with $(3d - 1)\varepsilon < 2d\delta$ such that

$$\sum_{j=0}^{k} E(\varphi_j) - \varepsilon \leq E \left( \sum_{j=0}^{k} \varphi_j \right) < m - \delta, \text{ and } -\varepsilon \leq K \left( \sum_{j=0}^{k} \varphi_j \right) \leq \sum_{j=0}^{k} K(\varphi_j) + \varepsilon.$$

Then $\varphi_j \in K^+$ for all $j = 0, \ldots, k$.

**Lemma 3.8.** For $d \geq 5$ and any $\varphi \in H^1$ with $K(\varphi) \geq 0$, we have

$$\int_{\mathbb{R}^d} \left( \frac{1}{2d} |\nabla \varphi|^2 + \frac{1}{2d} |\varphi|^2 \right) dx \leq E(\varphi) \leq \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{d-1}{2d+2} |\varphi|^{\frac{2d+4}{d-2}} \right) dx. \quad (3.3)$$

*Proof.* On one hand, the second inequality in (3.3) is trivial. On the other hand, by the definition of $E$ and $K$, we have

$$E(\varphi) = \int_{\mathbb{R}^d} \left( \frac{1}{2d} |\nabla \varphi|^2 + \frac{1}{2d} |\varphi|^2 \right) dx + \frac{d-1}{4d} K(\varphi),$$

which implies the first inequality in (3.3). \qed

At the last of this part, we give the uniform bounds on the scaling derivative functional $K(\varphi)$ with the energy $E(\varphi)$ below the threshold $m$, which plays an important role for the blow-up and scattering analysis in Section 4 and Section 6.

**Lemma 3.9** ([36], Lemma 2.13). For any $\varphi \in H^1$ with $E(\varphi) < m$, then there exists a constant $\delta > 0$ such that

1. If $K(\varphi) < 0$, then

$$K(\varphi) \leq -\bar{\mu}(m - E(\varphi)). \quad (3.4)$$
(2) If $K(\varphi) \geq 0$, then

$$K(\varphi) \geq \min \left( \bar{\mu} \left( m - E(\varphi) \right), \frac{2}{2d - 3} \| \nabla \varphi \|_{L^2}^2 + \frac{2d}{(d + 1)(2d - 3)} \| \varphi \|_{{L^{2d+2}}}^{2d+2} \right).$$  \quad (3.5)

**Proof.** By Lemma 3.3 for any $\varphi \in H^1$, we have

$$\mathcal{L}^2 E(\varphi) = \bar{\mu} \mathcal{L} E(\varphi) - \frac{8}{d - 1} \| \nabla \varphi \|_{L^2}^2 - \frac{8d}{(d - 1)(d - 2)} \| \varphi \|_{L^{2^*}}^{2^*}. $$

Let $j(\lambda) = E(\varphi_{\lambda,-2}^{\lambda_0})$, then we have

$$j''(\lambda) = \bar{\mu} j'(\lambda) - \frac{8e^{4\lambda}}{d - 1} \| \nabla \varphi \|_{L^2}^2 - \frac{8de^{4d\lambda}}{(d - 1)(d - 2)} \| \varphi \|_{L^{2^*}}^{2^*}. $$ \quad (3.6)

**Case I:** If $K(\varphi) < 0$, then by (3.1), Lemma 3.2 and the continuity of $K$ in $\lambda$, there exists a negative number $\lambda_0 < 0$ such that $K(\varphi_{\lambda,-2}^{\lambda_0}) = 0$, and

$$K(\varphi_{\lambda,-2}^{\lambda_0}) < 0, \ \forall \ \lambda \in (\lambda_0, 0).$$

By (1.4), we obtain $E(\varphi_{\lambda,-2}^{\lambda_0}) \geq m$. Now by integrating (3.6) over $[\lambda_0, 0]$, we have

$$\int_{\lambda_0}^0 j''(\lambda) \, d\lambda \leq \bar{\mu} \int_{\lambda_0}^0 j'(\lambda) \, d\lambda,$$

which implies that

$$K(\varphi) = j'(0) - j'(\lambda_0) \leq \bar{\mu} (j(0) - j(\lambda_0)) \leq -\bar{\mu} (m - E(\varphi)),$$

which implies (3.4).

**Case II:** $K(\varphi) \geq 0$. We divide it into two subcases:

When $2\bar{\mu}K(\varphi) \geq \frac{8d}{(d - 1)(d - 2)} \| \varphi \|_{L^{2^*}}^{2^*}$. Since

$$\frac{8d}{(d - 1)(d - 2)} \| \varphi \|_{L^{2^*}}^{2^*} = -\frac{4d}{(d - 1)(d - 2)} K(\varphi) + \int_{\mathbb{R}^d} \left( \frac{8d}{(d - 1)(d - 2)} |\nabla \varphi|^2 + \frac{8d^2}{(d + 1)(d - 1)(d - 2)} |\varphi|^{\frac{2d+2}{d-1}} \right) \, dx,$$

then we have

$$2\bar{\mu}K(\varphi) \geq -\frac{4d}{(d - 1)(d - 2)} K(\varphi) + \int_{\mathbb{R}^d} \left( \frac{8d}{(d - 1)(d - 2)} |\nabla \varphi|^2 + \frac{8d^2}{(d + 1)(d - 1)(d - 2)} |\varphi|^{\frac{2d+2}{d-1}} \right) \, dx,$$

which implies that

$$K(\varphi) \geq \frac{2}{2d - 3} \| \nabla \varphi \|_{L^2}^2 + \frac{2d}{(d + 1)(2d - 3)} \| \varphi \|_{{L^{2d+2}}}^{2d+2}. $$
When $2\tilde{\mu} K(\varphi) \leq \frac{8d}{(d-1)(d-2)} \|\varphi\|_{L^{2^*}}^{2^*}$. By (3.6), we have for $\lambda = 0$

$$0 < 2\tilde{\mu} j'(\lambda) < \frac{8d e^{\frac{4d}{d-2} \lambda}}{(d-1)(d-2)} \|\varphi\|_{L^{2^*}}^{2^*},$$

$$j''(\lambda) = \tilde{\mu} j'(\lambda) - \frac{8d e^{\frac{4d}{d-2} \lambda}}{d-1} \|\nabla \varphi\|_{L^2}^2 - \frac{8d e^{\frac{4d}{d-2} \lambda}}{(d-1)(d-2)} \|\varphi\|_{L^{2^*}}^{2^*} \geq -\tilde{\mu} j'(\lambda). \quad (3.7)$$

By the continuity of $j'$ and $j''$ in $\lambda$, we know that $j'$ is an accelerated decreasing function as $\lambda$ increases until $j'(\lambda_0) = 0$ for some finite number $\lambda_0 > 0$ and (3.7) holds on $[0, \lambda_0]$.

By $K(\varphi_{d,-2}^{\lambda_0}) = j'(\lambda_0) = 0$, we know that

$$E(\varphi_{d,-2}^{\lambda_0}) \geq m.$$

Now integrating (3.7) over $[0, \lambda_0]$, we obtain that

$$-K(\varphi) = j'(\lambda_0) - j'(0) \leq -\tilde{\mu} \left( j(\lambda_0) - j(0) \right) \leq -\tilde{\mu} (m - E(\varphi)).$$

This completes the proof. \qed

4. Part I: Blow up for $\mathcal{K}^-$

In this section, we prove the blow-up result of Theorem 1.2 in the case that $u_0$ is radial. The case $xu_0 \in L^2$ is trivial. We can also refer to [36] for the similar discussions to the case $d = 3$. Now let $\phi$ be a smooth, radial function satisfying $\partial_t^2 \phi(r) \leq 2$, $\phi(r) = r^2$ for $r \leq 1$, and $\phi(r)$ is constant for $r \geq 3$. For some $R$, we define

$$V_R(t) := \int_{\mathbb{R}^d} \phi_R(x)|u(t,x)|^2 \, dx, \quad \phi_R(x) = R^2 \phi \left( \frac{|x|}{R} \right).$$

By Lemma 2.7, $\Delta \phi_R(r) = 2d$ for $r \leq R$, and $\Delta^2 \phi_R(r) = 0$ for $r \leq R$, we have

$$\partial_t^2 V_R(t) = 4 \int_{\mathbb{R}^d} \partial_{ij} (\phi_R) u_i(t,x) \bar{u}_j(t,x) \, dx - \int_{\mathbb{R}^d} \Delta^2 \phi_R |u(t,x)|^2 \, dx$$

$$- \frac{4}{d} \int_{\mathbb{R}^d} \Delta \phi_R |u(t,x)|^2 \, dx + \frac{4}{d+1} \int_{\mathbb{R}^3} \Delta \phi_R |u(t,x)|^{\frac{2d+2}{d+1}} \, dx$$

$$\leq 4 \int_{\mathbb{R}^d} \left( 2|\nabla u(t)|^2 - 2|u(t)|^2 \right) \frac{2d}{d+1} |u(t)|^{\frac{2d+2}{d+1}} \, dx$$

$$+ \frac{c}{R^2} \int_{R \leq |x| \leq 3R} |u(t)|^2 \, dx + c \int_{R \leq |x| \leq 3R} \left( |u(t)|^{\frac{2d+2}{d+1}} + |u(t)|^{2^*} \right) \, dx.$$

By the radial Sobolev inequality, we have

$$\|f\|_{L^\infty(|x| \geq R)} \leq \frac{c}{R^{(d-1)/2}} \|f\|_{L^2(|x| \geq R)}^{1/2} \|\nabla f\|_{L^2(|x| \geq R)}^{1/2}.$$
Therefore, by the mass conservation and Young’s inequality, we know that for any \( \epsilon > 0 \) there exist sufficiently large \( R \) such that
\[
\partial^2_t V_R(t) \\
\leq 4K(u(t)) + \epsilon \| \nabla u(t) \|^2_{L^2} + \epsilon^2.
\]
\[
= \frac{16d}{d-2} E(u) - \left( \frac{16}{d-2} - \epsilon \right) \| \nabla u(t) \|^2_{L^2} - \frac{8d}{(d+1)(d-2)} \| u(t) \|_{L^{\frac{2d+2}{d-2}}}^2 + \epsilon^2.
\] (4.1)

By \( K(u) < 0 \), the mass and energy conservations, Lemma 3.9 and the continuity argument, we know that for any \( t \in I \), we have
\[
K(u(t)) \leq -\bar{\mu} (m - E(u(t))) < 0.
\]

By Lemma 3.4, we have
\[
m \leq H(u(t)) < \frac{1}{d} \| u(t) \|_{L^{2^*}}^{2^*}.
\]
where we have used the fact that \( K(u(t)) < 0 \) in the second inequality. By the fact \( m = \frac{1}{d} (C_d^*)^{-d} \) and the sharp Sobolev inequality, we have
\[
\| \nabla u(t) \|^2_{L^2} \geq (C_d^*)^{-2^*} \| u(t) \|_{L^{2^*}}^{2^*} > (dm)^\frac{d}{d-2},
\]
which implies that \( \| \nabla u(t) \|^2_{L^2} > dm \).

In addition, by \( E(u_0) < m \) and energy conservation, there exists \( \delta_1 > 0 \) such that \( E(u(t)) \leq (1 - \delta_1)m \). Thus, if we choose \( \epsilon \) sufficiently small, we have
\[
\partial^2_t V_R(t) \leq \frac{16d}{d-2} (1 - \delta_1)m - d\left( \frac{16}{d-2} - \epsilon \right) m + \epsilon^2 \leq -\frac{8d}{d-2} \delta_1 m,
\]
which implies that \( u \) must blow up at finite time. \( \square \)

5. Profile decomposition

In this part, we will use the method in [4, 20, 24, 36] to show the linear and non-linear profile decompositions of the \( H^1 \)-bounded sequences of solutions of (1.1) in five and higher dimensions, which will be used to construct the critical element (minimal energy non-scattering solution) and show its properties, especially the compactness and regularity. In order to do it, we cite the similar notation to those in [20, 36]. Now we introduce the complex-valued function \( \overrightarrow{v}(t, x) \) by
\[
\overrightarrow{v}(t, x) = \langle \nabla \rangle v(t, x), \quad v(t, x) = \langle \nabla \rangle^{-1} \overrightarrow{v}(t, x).
\]
Given \((t_n^l, x_n^l, h_n^j) \in \mathbb{R} \times \mathbb{R}^d \times (0, 1]\), let \(\tau_n^j, T_n^j\) denote the scaled time drift, the unitary operator in \(L^2(\mathbb{R}^d)\), defined by
\[
\tau_n^j = - \frac{t_n^j}{(h_n^j)^2}, \quad T_n^j \varphi(x) = \frac{1}{(h_n^j)^{d/2}} \varphi\left(\frac{x - x_n^j}{h_n^j}\right).
\]

### 5.1. Linear profile decomposition.

By the similar arguments to that in \([36]\), we can establish that

**Proposition 5.1** \([36], \text{Proposition 5.1 and Lemma 5.3}\). Let
\[
\overrightarrow{v}_n(t, x) = e^{it\Delta} \overrightarrow{v}_n(0)
\]
be a sequence of the free Schrödinger solutions with bounded \(L^2\) norm. Then up to a subsequence, there exist \(K \in \{0, 1, 2, \ldots, \infty\}\), \(\{\varphi_j\}_{j \in [0, K]} \subset L^2(\mathbb{R}^d)\) and \(\{t_n^l, x_n^l, h_n^j\}_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{R}^d \times (0, 1]\) satisfying
\[
\overrightarrow{v}_n(t, x) = \sum_{j=0}^{k-1} \overrightarrow{v}_n^j(t, x) + \overrightarrow{w}_n^k(t, x), \tag{5.1}
\]
where \(\overrightarrow{v}_n^j(t, x) = e^{i(t-t_n^j)\Delta} T_n^j \varphi_j\), and
\[
\lim_{k \to K} \lim_{n \to +\infty} \|\overrightarrow{w}_n^k\|_{L^\infty(\mathbb{R}; B^{-d/2}_{\infty}(\mathbb{R}^d))} = 0, \tag{5.2}
\]
and for any \(l < j < k \leq K\),
\[
\lim_{n \to +\infty} \left(\log \frac{h_n^j}{h_n^l} \right) + \frac{t_n^j - t_n^l}{(h_n^j)^2} + \left|\frac{x_n^j - x_n^l}{h_n^j}\right| = \infty, \tag{5.3}
\]
\[
\lim_{k \to K} \lim_{n \to +\infty} \left| M(v_n(0)) - \sum_{j=0}^{k-1} M(v_n^j(0)) - M(w_n^k(0)) \right| = 0, \tag{5.4}
\]
\[
\lim_{k \to K} \lim_{n \to +\infty} \left| E(v_n(0)) - \sum_{j=0}^{k-1} E(v_n^j(0)) - E(w_n^k(0)) \right| = 0, \tag{5.5}
\]
\[
\lim_{k \to K} \lim_{n \to +\infty} \left| K(v_n(0)) - \sum_{j=0}^{k-1} K(v_n^j(0)) - K(w_n^k(0)) \right| = 0. \tag{5.6}
\]
Moreover, each sequence \(\{h_n^j\}_{n \in \mathbb{N}}\) is either going to 0 or identically 1 for all \(n\).

We call \(\overrightarrow{v}_n^j\) and \(\overrightarrow{w}_n^k\) the free concentrating wave and the remainder, respectively. According to the above result and Lemma \([37]\) we conclude
Proposition 5.2 ([36], Proposition 5.5). Let \( \overrightarrow{v}_n(t, x) \) be a sequence of the free Schrödinger solutions satisfying \( v_n(0) \in \mathcal{K}^+ \) and \( E(v_n(0)) < m \). Let

\[
\overrightarrow{v}_n(t, x) = \sum_{j=0}^{k-1} \overrightarrow{v}_j^k(t, x) + \overrightarrow{w}_k^n(t, x),
\]

be the linear profile decomposition given by Proposition 5.1. Then for large \( n \) and all \( j < K \), we have

\[
v_j^n(0) \in \mathcal{K}^+, \quad w_K^n(0) \in \mathcal{K}^+,
\]

such that (5.4)–(5.6). Moreover for all \( j < K \), we have

\[
0 \leq \lim_{n \to +\infty} E(v_j^n(0)) \leq \lim_{n \to +\infty} E(v_k^n(0)) \leq \lim_{n \to +\infty} E(v_n(0)),
\]

where the last inequality becomes equality only if \( K = 1 \) and \( v_1^n \to 0 \) in \( L_t^\infty \dot{H}_x^1 \).

5.2. Nonlinear profile decomposition. After the linear profile decomposition of a sequence of initial data in the last subsection, we now give the nonlinear profile decomposition of a sequence of solutions of (1.1) with the same initial data in the energy space \( H^1(\mathbb{R}^d) \). The procedure is the same as the 3d case in [36].

Let \( v_n(t, x) \) be a sequence of solutions for the free Schrödinger equation with initial data in \( \mathcal{K}^+ \), that is, \( v_n \in H^1(\mathbb{R}^d) \) and

\[
(i\partial_t + \Delta) v_n = 0, \quad v_n(0) \in \mathcal{K}^+.
\]

Let

\[
\overrightarrow{v}_n(t, x) = \langle \nabla \rangle v_n(t, x),
\]

then by Proposition 5.1, we have a sequence of the free concentrating wave \( \overrightarrow{v}_n^j(t, x) \) with \( \overrightarrow{v}_n^j(t_n^j) = T_n^j \varphi^j, \quad v_n^j(0) \in \mathcal{K}^+ \) for \( j = 0, \ldots, K \), such that

\[
\overrightarrow{v}_n(t, x) = \sum_{j=0}^{k-1} \overrightarrow{v}_n^j(t, x) + \overrightarrow{w}_k^n(t, x) = \sum_{j=0}^{k-1} e^{i(t-t_n^j) \Delta} T_n^j \varphi^j + \overrightarrow{w}_k^n
\]
\[
= \sum_{j=0}^{k-1} T_n^j e^{i \left( \frac{t-t_n^j}{(h_n^j)^2} \right) \Delta} \varphi^j + \overrightarrow{w}_k^n.
\]

Now for any concentrating wave \( \overrightarrow{v}_n^j, \quad j = 0, \ldots, K \), we undo the group action, i.e., the scaling and translation transformation \( T_n^j \), to look for the linear profile \( V^j \). Let

\[
\overrightarrow{v}_n^j(t, x) = T_n^j \overrightarrow{V}^j \left( \frac{t-t_n^j}{(h_n^j)^2} \right),
\]
then we have
\[
(i\partial_t + \Delta) \nabla \varphi^j = 0, \quad \nabla \varphi^j(0) = \varphi^j.
\]

Now let \( u^j_n(t, x) \) be the nonlinear solution of (1.1) with initial data \( v^j_n(0) \), that is
\[
(i\partial_t + \Delta) \nabla u^j_n(t, x) = (\langle \nabla \rangle^j f_1(\langle \nabla \rangle^{-1} \nabla u^j_n) + \langle \nabla \rangle f_2(\langle \nabla \rangle^{-1} \nabla u^j_n),
\]
\[
\nabla u^j_n(0) = \nabla \varphi^j(0) = T_n^j \nabla \varphi^j(\tau^j_n), \quad u^j_n(0) \in K^+,
\]
where \( \tau^j_n = -t^j_n/(h^j_n)^2 \). In order to look for the nonlinear profile \( \nabla \varphi^j_n \) associated to the free concentrating wave \( (\nabla \varphi^j_n; h^j_n, t^j_n, x^j_n) \), we also need undo the group action. We denote
\[
\nabla \varphi^j_n(t, x) = T_n^j \nabla \varphi^j_n \left( \frac{t - t^j_n}{(h^j_n)^2} \right),
\]
then we have
\[
(i\partial_t + \Delta) \nabla \varphi^j_n = (\langle \nabla \rangle^j f_1 \left( (\langle \nabla \rangle^j)^{-1} \nabla \varphi^j_n \right) + (h^j_n)^2 \nabla \nabla \varphi^j_n \left( \langle \nabla \rangle^j f_2 \left( (\langle \nabla \rangle^j)^{-1} \nabla \varphi^j_n \right),
\]
\[
\nabla \varphi^j_n(\tau^j_n) = \nabla \varphi^j(\tau^j_n).
\]

Up to a subsequence, we may assume that there exist \( h^j_\infty \in \{0, 1\} \) and \( \tau^j_\infty \in [-\infty, \infty] \) for every \( j = \{0, \ldots, K\} \), such that
\[
h^j_n \to h^j_\infty, \quad \text{and} \quad \tau^j_n \to \tau^j_\infty.
\]
As \( n \to +\infty \), the limit equation of \( \nabla \varphi^j_n \) is given by
\[
(i\partial_t + \Delta) \nabla \varphi^j_\infty = (\langle \nabla \rangle^j f_1 \left( (\langle \nabla \rangle^j)^{-1} \nabla \varphi^j_\infty \right) + (h^j_\infty)^2 \nabla \nabla \varphi^j_\infty \left( \langle \nabla \rangle^j f_2 \left( (\langle \nabla \rangle^j)^{-1} \nabla \varphi^j_\infty \right),
\]
\[
\nabla \varphi^j_\infty(\tau^j_\infty) = \nabla \varphi^j(\tau^j_\infty) \in L^2(\mathbb{R}^d).
\]

Let
\[
\hat{\nabla} \varphi^j_\infty := \left( \langle \nabla \rangle^j \right)^{-1} \nabla \varphi^j_\infty,
\]
then
\[
(i\partial_t + \Delta) \hat{\nabla} \varphi^j_\infty = f_1 \left( \hat{\nabla} \varphi^j_\infty \right) + (h^j_\infty)^2 \nabla \nabla \varphi^j_\infty \left( f_2 \left( \hat{\nabla} \varphi^j_\infty \right),
\]
\[
\hat{\nabla} \varphi^j_\infty(\tau^j_\infty) = \left( \langle \nabla \rangle^j \right)^{-1} \nabla \varphi^j(\tau^j_\infty).
\]

The unique existence of a local solution \( \nabla \varphi^j_\infty \) around \( \tau^j_\infty \) is known in all cases, including \( h^j_\infty = 0 \) and \( \tau^j_\infty = \pm \infty \). \( \nabla \varphi^j_\infty \) on the maximal existence interval is called the nonlinear profile associated with the free concentrating wave \( (\nabla \varphi^j_n; h^j_n, t^j_n, x^j_n) \).
The nonlinear concentrating wave $u^j_{(n)}$ associated with $(\vec{\nu}^j_n; h^j_n, t^j_n, x^j_n)$ is defined by

$$\vec{u}^j_{(n)}(t, x) = T^j_n \vec{U}^j_\infty \left( \frac{t - t^j_n}{(h^j_n)^2} \right),$$

then we have

$$(i\partial_t + \Delta) \vec{u}^j_{(n)} = \langle \nabla \rangle^j \cdot f_1 \left( (\langle \nabla \rangle^j)^{-1} \vec{u}^j_{(n)} \right) + (h^j_n)^{\frac{1}{2}} \cdot \langle \nabla \rangle^j f_2 \left( (\langle \nabla \rangle^j)^{-1} \vec{u}^j_{(n)} \right),$$

which implies that

$$\| \vec{u}^j_{(n)}(0) - \vec{u}^j_{(n)}(0) \|_{L^2} = \| T^j_n \vec{U}^j_\infty (\tau^j_n) - T^j_n \vec{V}^j_\infty (\tau^j_n) \|_{L^2} = \| \vec{U}^j_\infty (\tau^j_n) - \vec{V}^j_\infty (\tau^j_n) \|_{L^2} \lesssim \| \vec{U}^j_\infty (\tau^j_n) - \vec{U}^j_\infty (\tau^j_\infty) \|_{L^2} + \| \vec{V}^j_\infty (\tau^j_n) - \vec{V}^j_\infty (\tau^j_\infty) \|_{L^2} \to 0.$$

We denote

$$\vec{u}^j_{(n)} = \langle \nabla \rangle u^j_{(n)}.$$

If $h^j_\infty = 1$, we have $h^j_n \equiv 1$, then $u^j_{(n)} \in H^1(\mathbb{R}^d)$ and satisfies

$$(i\partial_t + \Delta) u^j_{(n)} = f_1(u^j_{(n)}) + f_2(u^j_{(n)}).$$

If $h^j_\infty = 0$, then $u^j_{(n)} \in H^1(\mathbb{R}^d)$ satisfies

$$(i\partial_t + \Delta) u^j_{(n)} = \frac{|\nabla|}{\langle |\nabla| \rangle} f_1 \left( \frac{\langle |\nabla| \rangle}{|\nabla|} u^j_{(n)} \right).$$

Let $u_n$ be a sequence of (local) solutions of (1.1) with initial data in $K^+$ at $t = 0$, and let $v_n$ be the sequence of the free solutions with the same initial data. We consider the linear profile decomposition given by Proposition 5.1

$$\vec{u}^j_n(t, x) = \sum_{j=0}^{k-1} \vec{U}^j_n(t, x) + \vec{w}^j_n(t, x), \quad \vec{v}^j_n(t^j_n) = T^j_n \varphi^j_n, \quad v^j_n(0) \in K^+.$$

With each free concentrating wave $\{\vec{U}^j_n\}_{n \in \mathbb{N}}$, we associate the nonlinear concentrating wave $\{\vec{u}^j_{(n)}\}_{n \in \mathbb{N}}$. A nonlinear profile decomposition of $u_n$ is given by

$$\vec{u}^j_{(n)}(t, x) := \sum_{j=0}^{k-1} \vec{u}^j_{(n)}(t, x) = \sum_{j=0}^{k-1} T^j_n \vec{U}^j_\infty \left( \frac{t - t^j_n}{(h^j_n)^2} \right). \quad (5.7)$$

Since the smallness condition (5.2) and the orthogonality condition (5.3) ensure that every nonlinear concentrating wave and the remainder interacts weakly with the others,
we will show that $\overline{u}^{<k}_{(n)} + \overline{w}^{k}_{n}$ is a good approximation for $\overline{u}^{n}$ provided that each nonlinear profile has the finite global Strichartz norm.

Now we define the Strichartz norms. First let $ST(I)$ and $ST^*(I)$ be the function spaces on $I \times \mathbb{R}^d$ defined as Section 2.3

$$ST(I) := W_1(I) \cap W_2(I), \quad ST^*(I) := L^2_t \left( I; \mathcal{B}^{\frac{d}{2d+1} \frac{4}{2}} \right).$$

The Strichartz norm for the nonlinear profile $\hat{U}_j^\infty$ depends on the scaling $h_j^\infty$.

$$ST_j^\infty(I) := \begin{cases} W_1(I) \cap W_2(I), & \text{for } h_j^\infty = 1, \\ W_1(I), & \text{for } h_j^\infty = 0. \end{cases}$$

By the similar arguments to that in [20, 36], we have

**Lemma 5.3** ([36], Lemma 5.6). *In the nonlinear profile decomposition (5.7). Suppose that for each $j < K$, we have*

$$\| \hat{U}_j^\infty \|_{ST_j^\infty(\mathbb{R})} + \| \hat{U}_j^\infty \|_{L^\infty_t L^4_x(\mathbb{R}^d)} < \infty.$$  

*Then for any finite interval $I$, any $j < K$ and any $k \leq K$, we have*

$$\lim_{n \to +\infty} \| u_j^{(n)} \|_{ST(I)} \leq \| \hat{U}_j^\infty \|_{ST_j^\infty(\mathbb{R})}, \quad (5.8)$$

$$\lim_{n \to +\infty} \| u_j^{(n)} \|_{ST(I)} \leq \lim_{n \to +\infty} \sum_{j < k} \| u_j^{(n)} \|_{ST(I)}, \quad (5.9)$$

*where the implicit constants do not depend on $I, j$ or $k$. We also have*

$$\lim_{n \to +\infty} \left\| f_1 \left( u_j^{(n)} \right) - \sum_{j < k} \left( \hat{U}_j^\infty \right) f_1 \left( \hat{U}_j^\infty \right) \right\|_{ST^*(I)} = 0, \quad (5.10)$$

$$\lim_{n \to +\infty} \left\| f_2 \left( u_j^{(n)} \right) - \sum_{j < k} \left( h_j^\infty \right) f_2 \left( h_j^\infty \right) \right\|_{ST^*(I)} = 0. \quad (5.11)$$

*After this preliminaries, we now show that $\overline{u}^{<k}_{(n)} + \overline{w}^{k}_{n}$ is a good approximation for $\overline{u}^{n}$ provided that each nonlinear profile has finite global Strichartz norm.*

**Proposition 5.4** ([36], Proposition 5.7). *Let $u_n$ be a sequence of local solutions of (1.1) around $t = 0$ in $\mathcal{K}^+$ satisfying*

$$M (u_n) < \infty, \quad \lim_{n \to \infty} E(u_n) < m.$$
Suppose that in the nonlinear profile decomposition (5.7), every nonlinear profile $\hat{U}_\infty^j$ has finite global Strichartz and energy norms we have

$$\|\hat{U}_\infty^j\|_{ST^k_{\infty}(\mathbb{R})} + \|\hat{U}_\infty^j\|_{L^2_t L^2_x(\mathbb{R}^d)} < \infty.$$ 

Then $u_n$ is bounded for large $n$ in the Strichartz and the energy norms

$$\lim_{n \to \infty} \|u_n\|_{ST(\mathbb{R})} + \|\hat{u}_n\|_{L^2_t L^2_x(\mathbb{R})} < \infty.$$ 

**Proof.** We only need to verify the condition of Proposition 2.6. Note that $u_{(n)}^{<k} + w_n^k$ satisfies that

$$(i\partial_t + \Delta) (u_{(n)}^{<k} + w_n^k) = f_1 (u_{(n)}^{<k} + w_n^k) + f_2 (u_{(n)}^{<k} + w_n^k)$$

$$+ f_1 (u_{(n)}^{<k} - f_1 (u_{(n)}^{<k} + w_n^k) + f_2 (u_{(n)}^{<k} - f_2 (u_{(n)}^{<k} + w_n^k)$$

$$+ \sum_{j<k} \langle \nabla \rangle^j \nabla_i \left( u_{(n)}^{<k} \right) - f_1 (u_{(n)}^{<k}) + \sum_{j<k} (h_{(n)}^j) \langle \nabla \rangle^j \nabla_i \left( u_{(n)}^{<k} \right) - f_2 (u_{(n)}^{<k}).$$

First, by the construction of $\hat{u}_{(n)}^{<k}$, we know that

$$\left\| \left( \hat{u}_{(n)}^{<k}(0) + \hat{w}_n^k(0) \right) - \hat{u}_n(0) \right\|_{L^2_x} \leq \sum_{j<k} \left\| \hat{u}_n^j(0) - \hat{u}_n^j(0) \right\|_{L^2_x} \to 0,$$

as $n \to +\infty$, which also implies that for large $n$, we have

$$\left\| \hat{u}_{(n)}^{<k}(0) + \hat{w}_n^k(0) \right\|_{L^2_x} \leq E_0.$$

Next, by the linear profile decomposition in Proposition 5.1, we know that

$$\|u_n(0)\|_{L^2_x}^2 = \|v_n(0)\|_{L^2_x}^2 = \sum_{j<k} \|v_n^j(0)\|_{L^2_x}^2 + \|w_n^k(0)\|_{L^2_x}^2 + o_n(1)$$

$$\geq \sum_{j<k} \|v_n^j(0)\|_{L^2_x}^2 + o_n(1) = \sum_{j<k} \left\| u_{(n)}^j(0) \right\|_{L^2_x}^2 + o_n(1),$$

$$\|u_n(0)\|_{H^1_x}^2 = \|v_n(0)\|_{H^1_x}^2 = \sum_{j<k} \|v_n^j(0)\|_{H^1_x}^2 + \|w_n^k(0)\|_{H^1_x}^2 + o_n(1)$$

$$\geq \sum_{j<k} \|v_n^j(0)\|_{H^1_x}^2 + o_n(1) = \sum_{j<k} \left\| u_{(n)}^j(0) \right\|_{H^1_x}^2 + o_n(1),$$

which means except for a finite set $J \subset \mathbb{N}$, the energy of $u_{(n)}^j$ with $j \notin J$ is smaller than the iteration threshold, hence we have

$$\|u_{(n)}^j\|_{ST(\mathbb{R})} \lesssim \|\hat{u}_{(n)}^j(0)\|_{L^2_x}.$$
thus, for any finite interval $I$, by Lemma 5.3 we have

$$\sup_k \lim_{n \to +\infty} \|u^{<k}_{(n)}\|_{ST(I)}^2 \lesssim \sup_k \lim_{n \to +\infty} \sum_{j<k} \|u^j_{(n)}\|_{ST(I)}^2$$

$$= \sup_k \lim_{n \to +\infty} \left[ \sum_{j<k, j \in J} \|u^j_{(n)}\|_{ST(I)}^2 + \sum_{j<k, j \notin J} \|u^j_{(n)}\|_{ST(I)}^2 \right]$$

$$\lesssim \sum_{j<k, j \in J} \|\hat{U}^j_\infty\|_{ST^*_{\infty}(I)}^2 + \sup_k \lim_{n \to +\infty} \sum_{j<k, j \notin J} \|\hat{u}^j_{(n)}(0)\|_{L_x^2}^2$$

$$< \infty.$$  

This together with the Strichartz estimate for $w^k_n$ implies that

$$\sup_k \lim_{n \to +\infty} \|u^{<k}_{(n)} + w^k_n\|_{ST(I)}^2 < \infty.$$  

Last we need show the nonlinear perturbation is small in some sense. By Proposition 5.1 and Lemma 5.3 we have

$$\left\| f_1 \left( u^{<k}_{(n)} \right) - f_1 \left( u^{<k}_{(n)} + w^k_n \right) \right\|_{ST^*_{\infty}(I)} \to 0,$$

$$\left\| f_2 \left( u^{<k}_{(n)} \right) - f_2 \left( u^{<k}_{(n)} + w^k_n \right) \right\|_{ST^*_{\infty}(I)} \to 0,$$

and

$$\left\| \sum_{j<k} \langle \nabla \rangle^j \langle \nabla \rangle^j f_1 \left( \langle \nabla \rangle^j u^j_{(n)} \right) - f_1 \left( u^{<k}_{(n)} \right) \right\|_{ST^*_{\infty}(I)} \to 0,$$

$$\left\| \sum_{j<k} h^j_\infty \langle \nabla \rangle^j f_2 \left( \langle \nabla \rangle^j u^j_{(n)} \right) - f_2 \left( u^{<k}_{(n)} \right) \right\|_{ST^*_{\infty}(I)} \to 0,$$

as $n \to +\infty$. Therefore, by Proposition 2.6 we can obtain the desired result, which concludes the proof. \(\square\)

### 6. Part II: GWP and Scattering for $\mathcal{K}^+$

We now use the stability analysis of the scattering solution of \(\text{(1.1)}\) and the compactness analysis of a sequence of the energy solutions of \(\text{(1.1)}\) to show the scattering result of Theorem 1.2 by contradiction.

---

1Since we use the full derivative $\langle \nabla \rangle$, we need use the local smoothing effect about the free solution $w(t,x)$ as that in [2, 27] to verify the weak interaction between the nonlinear concentrating waves $u^j_{(n)}(t,x)$ and the remainder $w(t,x)$. 

---
For any finite positive number $C < \infty$, let $E_C^*$ be the threshold for the uniform Strichartz norm bound, i.e.,

$$
E_C^* := \sup\{A > 0, ST(A) < \infty\}
$$

where $ST(A)$ denotes the supremum of $\|u\|_{ST(I)}$ for any strong solution $u$ of (1.1) in $\mathcal{K}^+$ on any interval $I$ satisfying $E(u) \leq A$, $M(u) \leq C$.

The small solution scattering theory gives us $E_C^* > 0$. We are going to show that $E_C^* \geq m$ by contradiction. From now on, suppose that $E_C^* \geq m$ fails, that is, we assume that

$$
E_C^* < m.
$$

### 6.1. Existence of a critical element

This part is similar to section 6.1 in [36]. By the definition of $E_C^*$ and the fact that $E_C^* < m$, there exist a sequence of solutions $\{u_n\}_{n \in \mathbb{N}}$ of (1.1) in $\mathcal{K}^+$, which have the maximal existence interval $I_n$ and satisfy that for some finite number $C$

$$
M(u_n) \leq C, \quad E(u_n) \to E_C^* < m, \quad \|u_n\|_{ST(I_n)} \to +\infty, \quad \text{as } n \to +\infty,
$$

then we have $\|u_n\|_{H^1} < \infty$ by Lemma 3.8. By the compact argument (profile decomposition) and the stability theory, we can show that

**Theorem 6.1.** For $d \geq 5$. Let $u_n$ be a sequence of solutions of (1.1) in $\mathcal{K}^+$ on $I_n \subset \mathbb{R}$ satisfying

$$
M(u_n) \leq C, \quad E(u_n) \to E_C^* < m, \quad \|u_n\|_{ST(I_n)} \to +\infty, \quad \text{as } n \to +\infty.
$$

Then there exists a global solution $u_\infty$ of (1.1) in $\mathcal{K}^+$ satisfying

$$
E(u_\infty) = E_C^* < m, \quad K(u_\infty) > 0, \quad \|u_\infty\|_{ST(\mathbb{R})} = \infty.
$$

In addition, there are a sequence $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d$ and $\varphi \in L^2(\mathbb{R}^d)$ such that, up to a subsequence, we have as $n \to +\infty$,

$$
\left\| \frac{\nabla}{\langle \nabla \rangle} \left( \overrightarrow{u}_n(0, x) - e^{-it_n\Delta} \varphi(x - x_n) \right) \right\|_{L^2} \to 0. \quad (6.2)
$$

**Proof.** By the time translation symmetry of (1.1), we can translate $u_n$ in $t$ such that $0 \in I_n$ for all $n$. Then by the linear and nonlinear profile decomposition of $u_n$, we have

$$
e^{it \Delta} \overrightarrow{u}_n(0, x) = \sum_{j < k} \overrightarrow{v}^j_n(t, x) + \overrightarrow{w}^k_n(t, x), \quad \overrightarrow{v}^j_n(t, x) = e^{i(t-t_n^j)\Delta} T_n^j \varphi^j, \quad \overrightarrow{w}^k_n(t, x) = \sum_{j < k} \overrightarrow{v}^j_n(t, x), \quad \overrightarrow{u}^j_n(t, x) = T_n^j \overrightarrow{U}^j \left( \frac{x - x_n^j}{(h_n^j)^2} \right),
$$
By Proposition 5.2 and the following observations that

1. Every solution of (1.1) in $K^+$ with the energy less than $E^*_C$, the mass less than $C$ has global finite Strichartz norm by the definition of $E^*_C$.

2. Proposition 5.4 precludes that all the nonlinear profiles $\hat{U}_{j\infty}$ have finite global Strichartz norm.

we deduce that there is only one profile and

$$E(u_{(n)}^0(0)) \to E^*_C, \quad u_{(n)}^0(0) \in K^+, \quad \|\hat{U}_{\infty}^0\|_{ST_{\infty}^0(I)} = \infty, \quad \|w_n^1\|_{L^\infty H^1_x} \to 0.$$ 

If $h_{n}^0 \to 0$, then $\hat{U}_{\infty}^0 = (\nabla)^{-1}\hat{U}_{\infty}^{j\infty}$ solves the $H^1$-critical NLS

$$(i\partial_t + \Delta)\hat{U}_{\infty}^0 = f_1(\hat{U}_{\infty}^0)$$

and satisfies

$$E^c\left(\hat{U}_{\infty}^0(\tau_{\infty}^0)\right) = E^*_C < m, \quad K^c\left(\hat{U}_{\infty}^0(\tau_{\infty}^0)\right) \geq 0, \quad \|\hat{U}_{\infty}^0\|_{W_1(I)} = \infty.$$ 

However, it is in contradiction with Killip-Visan’s result in [27]. Hence $h_{n}^0 \equiv 1$, which implies (6.2).

Now we show that $\hat{U}_{\infty}^0 = (\nabla)^{-1}\hat{U}_{\infty}^{j\infty}$ is a global solution, which is the consequence of the compactness of (6.2). Suppose not, then we can choose a sequence $t_{n} \in \mathbb{R}$ which approaches the maximal existence time. Since $\hat{U}_{\infty}^0(t + t_n)$ satisfies the assumption of this theorem, then applying the above argument to it, we obtain that for some $\psi \in L^2$ and another sequence $(t'_n, x'_n) \in \mathbb{R} \times \mathbb{R}^d$, as $n \to +\infty$

$$\left\|\frac{\nabla}{\nabla} \left(\hat{U}_{\infty}^0(t_n) - e^{-it'_n\Delta}\psi(x - x'_n)\right)\right\|_{L^2} \to 0. \quad (6.3)$$

Let $\hat{v}(t) := e^{it\Delta}\psi$. For any $\varepsilon > 0$, there exist $\delta > 0$ with $I = [-\delta, \delta]$ such that

$$\|\hat{v}(t - t'_n)\|_{L^2(I)} \leq \varepsilon,$$

which together with (6.3) implies that for sufficiently large $n$

$$\|e^{it\Delta}\hat{U}_{\infty}^0(t_n)\|_{L^2(I)} \leq \varepsilon.$$

If $\varepsilon$ is small enough, this implies that the solution $\hat{U}_{\infty}^0$ exists on $[t_n - \delta, t_n + \delta]$ for large $n$ by the small data theory. This contradicts the choice of $t_n$. Hence $\hat{U}_{\infty}^0$ is a global solution and it is just the desired critical element $u_c$. By Proposition 1.1, we know that $K(u_c) > 0.$
6.2. Compactness of the critical element. Since (1.1) is symmetric in \( t \), we may assume that
\[
\| u_c \|_{ST(0, +\infty)} = \infty. \tag{6.4}
\]
We call it a forward critical element.

**Proposition 6.2.** For \( d \geq 5 \). Let \( u_c \) be a forward critical element. Then there exists \( x(t) : (0, \infty) \to \mathbb{R}^d \) such that the set
\[
\{ u_c(t, x - x(t)); 0 < t < \infty \}
\]
is precompact in \( \hat{H}^s \) for any \( s \in (0, 1] \).

**Proof.** By the conservation of the mass, it suffices to prove the precompactness of \( u_c(t_n) \) in \( \hat{H}^1 \) for any positive time \( t_1, t_2, \ldots \). If \( t_n \) converges, then it is trivial from the continuity in \( t \).

If \( t_n \to +\infty \). Applying Theorem 6.1 to the sequence of solutions \( \tilde{u}_c(t + t_n) \), we get another sequence \( (t'_n, x'_n) \in \mathbb{R} \times \mathbb{R}^d \) and \( \varphi \in L^2 \) such that
\[
\| \langle \nabla \rangle^{-1} e^{it'_n \Delta} \varphi(x - x'_n) \|_{ST(-t'_n, +\infty)} + o_n(1) \to 0.
\]
Hence \( u_c \) can solve (1.1) for \( t > t_n \) with large \( n \) globally by iteration with small Strichartz norms, which contradicts (6.4).

(1) If \( t'_n \to -\infty \), then we have
\[
\| \langle \nabla \rangle^{-1} e^{it \Delta} \tilde{u}_c(t_n) \|_{ST(0, +\infty)} = \| \langle \nabla \rangle^{-1} e^{it \Delta} \varphi \|_{ST(-t'_n, +\infty)} + o_n(1) \to 0.
\]
Hence \( u_c \) can solve (1.1) for \( t > t_n \) with large \( n \) with diminishing Strichartz norms, which implies \( u_c = 0 \) by taking the limit, which is a contradiction.

Thus \( t'_n \) is bounded, which implies that \( t'_n \) is precompact, so is \( u_c(t_n, x + x'_n) \) in \( \hat{H}^1 \). \( \square \)

As a consequence, the energy of \( u_c \) stays within a fixed radius for all positive time, modulo arbitrarily small rest (that is, the spatial scaling function of \( u_c \) is 1). More precisely, we define the exterior energy by
\[
E_{R,c}(u; t) = \int_{|x - c| \geq R} \left( |\nabla u(t, x)|^2 + |u(t, x)|^{2^*} + |u(t, x)|^{\frac{2d + 2}{d - 1}} \right) dx
\]
for any \( R > 0 \) and \( c \in \mathbb{R}^d \). Then we have
Corollary 6.3. For \( d \geq 5 \), let \( u_c \) be a forward critical element. Then for any \( \epsilon \), there exist \( R_0(\epsilon) > 0 \) and \( x(t) : (0, +\infty) \rightarrow \mathbb{R}^d \) such that at any time \( t > 0 \), we have

\[
E_{R_0, x(t)}(u_c, t) \leq \epsilon E(u_c).
\]

Corollary 6.4 ([45]). For \( d \geq 5 \), let \( u_c \) be the critical element as shown in Theorem 6.1. Then for all \( t \in \mathbb{R} \),

\[
\lim_{T \rightarrow +\infty} \int_t^T e^{i(t-s)\Delta} \left( f_1(u(s)) + f_2(u(s)) \right) ds = \lim_{T \rightarrow -\infty} \int_T^t e^{i(t-s)\Delta} \left( f_1(u(s)) + f_2(u(s)) \right) ds
\]
as weak limits in \( \dot{H}^s_x \) for any \( s \in (0, 1] \).

6.3. Zero momentum of the critical element. Next we show that the critical element cannot move with any positive speed in the sense of energy by its Galilean invariance.

Proposition 6.5. For \( d \geq 5 \), let \( u_c \) be the critical element as shown in Theorem 6.1, then its total momentum, which is a conserved quantity, vanishes:

\[
P(u_c) := 2 \Re \int_{\mathbb{R}^d} \overline{u_c(t, x)} \cdot \nabla u_c(t, x) \, dx \equiv 0.
\]

Proof. We drop the subscript \( c \) for simplicity. Note that the momentum \( P(u) \) and the mass \( M(u) \) are finite and conserved. Moreover, \( M(u) \neq 0 \), otherwise \( u \) would be identically zero and not a critical element.

Let \( \tilde{u} \) be the Galilean boost of \( u \) by \( \xi_0 \), which is determined later.

\[
\tilde{u}(t, x) := e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u(t, x - 2\xi_0 t),
\]
then we have

\[
\|\nabla \tilde{u}(t)\|_{L^2}^2 = \|\nabla u(t)\|_{L^2}^2 + |\xi_0|^2 M(u) + \xi_0 \cdot P(u).
\]

Equivalently, we have \( M(\tilde{u}) = M(u) \) and

\[
E(\tilde{u}) = E(u) + \frac{1}{2} |\xi_0|^2 M(u) + \frac{1}{2} \xi_0 \cdot P(u), \quad K(\tilde{u}) = K(u) + 2|\xi_0|^2 M(u) + 2\xi_0 \cdot P(u).
\]

If \( P(u) \neq 0 \), choosing \( \xi_0 \in \mathbb{R}^d \) such that

\[
-\frac{1}{2} K(u) \leq |\xi_0|^2 M(u) + \xi_0 \cdot P(u) < 0,
\]
then we can find another critical element \( \tilde{u} \) in \( K^+ \) with

\[
M(\tilde{u}) = M(u) \leq C, \quad E(\tilde{u}) < E(u) = E_C^*, \quad \|\tilde{u}\|_{ST(\mathbb{R})} = +\infty,
\]
which is in contradiction with the definition of $E^*_c$. Hence $P(u) \equiv 0$. \hfill \Box

### 6.4. Negative regularity

In this subsection, we show that

**Proposition 6.6.** For $d \geq 5$. Let $u_c$ be the critical element as shown in Theorem 6.1, then $u_c \in L^\infty \dot{H}^{-\epsilon}$ for some $\epsilon > 0$.

**Proof.** We drop the subscript $c$. Since we have $u \in L^\infty_t H^1$, then we have $u \in L^\infty_t L^p_x$, for any $p \in [2, \frac{2d}{d-2}]$.

$$
\| f_1(u) + f_2(u) \|_{L^\infty_t L^r_x} \lesssim \| u \|_{L^\infty_t L^2_x} \| u \|_{L^\infty_t L^{p_1}_x} + \| u \|_{L^\infty_t L^{p_2}_x} \| u \|_{L^\infty_t L^{p_2}_x}
$$

where $2 \leq p_1, p_2 \leq 2d/(d-2)$ and

$$
\frac{1}{r} = \frac{1}{2} + \frac{4}{d-2} \times \frac{1}{p_1} = \frac{1}{2} + \frac{4}{d-1} \times \frac{1}{p_2}.
$$

Therefore for any $r \in [\frac{2d-1}{d+1}, \frac{2d}{d+1}]$, we have

$$
f_1(u) + f_2(u) \in L^\infty_t L^r_x. \quad (6.5)
$$

Now from (6.5), we claim that

$$
u \in L^\infty_t \dot{B}^{-s_0}_{2,\infty}, \text{ for any } s_0 = \frac{d}{r} - \frac{d+4}{2} \in [0, \frac{2}{d-1}], \quad (6.6)
$$

which implies the negative regularity of $u$ by interpolation. Now we shows (6.6). By the time translation symmetry, it suffices to show that $u(0) \in \dot{B}^{-s_0}_{2,\infty}$. In fact, from Corollary 6.4 we have

$$
\| u_N(0) \|_{L^2}^2 = \left< i \int_0^\infty e^{-it\Delta} P_N \left( f_1(u(t)) + f_2(u(t)) \right) dt, -i \int_0^\infty e^{-it\Delta} P_N \left( f_1(u(\tau)) + f_2(u(\tau)) \right) d\tau \right>
\leq \left| \int_0^\infty \int_{-\infty}^0 \left< P_N \left( f_1(u(t)) + f_2(u(t)) \right), e^{i(t-\tau)\Delta} P_N \left( f_1(u(\tau)) + f_2(u(\tau)) \right) \right> \right| dt d\tau.
$$

On one hand, by the dispersive estimate of $e^{it\Delta}$, we have

$$
\left| \left< P_N \left( f_1(u(t)) + f_2(u(t)) \right), e^{i(t-\tau)\Delta} P_N \left( f_1(u(\tau)) + f_2(u(\tau)) \right) \right> \right|
\lesssim \| P_N \left( f_1(u(t)) + f_2(u(t)) \right) \|_{L^\infty_t} \| P_N \left( f_1(u(\tau)) + f_2(u(\tau)) \right) \|_{L^\infty_t}
\lesssim \left| t - \tau \right|^{d(\frac{1}{2} - \frac{1}{2})} \| f_1(u(t)) + f_2(u(t)) \|_{L^\infty_x} \| f_1(u(\tau)) + f_2(u(\tau)) \|_{L^\infty_x}.
$$

On the other hand, by Bernstein’s inequality, we have

$$
\left| \left< P_N \left( f_1(u(t)) + f_2(u(t)) \right), e^{i(t-\tau)\Delta} P_N \left( f_1(u(\tau)) + f_2(u(\tau)) \right) \right> \right|
\lesssim \left| t - \tau \right|^{d(\frac{1}{2} - \frac{1}{2})} \| f_1(u(t)) + f_2(u(t)) \|_{L^\infty_x} \| f_1(u(\tau)) + f_2(u(\tau)) \|_{L^\infty_x}.
$$
\[ \|u_N(0)\|_{L^2}^2 \lesssim f_1(u) + f_2(u), \quad \int_0^\infty \int_{-\infty}^0 \min\left( |t-\tau|^{-1}, N^2 \right)^{d\left(\frac{1}{2} - \frac{1}{2}\right)} \, dt \, d\tau \lesssim N^{2s_0} \|f_1(u) + f_2(u)\|_{L^2_t L^\infty_x}, \]

Therefore, for \( d \geq 5 \), we have

\[ \|u_N(t)\|_{L^2}^2 \lesssim f_1(u) + f_2(u), \quad \int_{x-x(t) \geq C(\eta)} |u(t, x)|^2 \, dx \lesssim \eta. \]

**Proof.** The proof is the same as Lemma 8.2 in [27]. \( \square \)

### 6.5. Control of the spatial center function of the critical element

Now we will use the virial argument to control the spatial center function of the critical element by acknowledge of the zero momentum and the compactness in \( H^1 \) of the critical element.

**Proposition 6.8.** For \( d \geq 5 \). Let \( u_c \) be the critical element as shown in Theorem 6.1. Then

\[ |x(t)| = o(t), \quad \text{as} \quad t \to +\infty. \]

**Proof.** We argue by contradiction. Suppose that there exist \( \delta > 0 \) and a sequence \( t_n \to +\infty \) such that

\[ |x(t_n)| > \delta t_n \quad \text{for all} \quad n \geq 1. \]

Let \( \eta > 0 \) be a small constant to be chosen later. By compactness and Corollary 6.3 and Corollary 6.7, there exist \( x(t) \) and \( C(\eta) \) such that for any \( t \geq 0 \)

\[ \int_{|x-x(t)|>C(\eta)} (|\nabla u(t, x)|^2 + |u(t, x)|^2) \leq \eta. \]  

Define

\[ T_n := \inf \left\{ t \in [0, t_n] \mid |x(t)| = |x(t_n)| \right\} \leq t_n, \quad R_n := C(\eta) + \sup_{t\in[0,T_n]} |x(t)|. \]
Now let \( \phi \) be a smooth, radial function satisfying \( 0 \leq \phi \leq 1 \), \( \phi(x) = 1 \) for \( |x| \leq 1 \), and \( \phi(x) = 0 \) for \( |x| \geq 2 \). Define the truncation “position” as following

\[
X_R(t) = \int_{\mathbb{R}^d} x \phi \left( \frac{|x|}{R} \right) \cdot |u(t, x)|^2 \, dx,
\]

then we have

\[
|X_{R_n}(0)| \leq \left| \int_{|x| \leq C(\eta)} x \phi \left( \frac{|x|}{R_n} \right) |u(0, x)|^2 \, dx \right| + \left| \int_{|x| \geq C(\eta)} x \phi \left( \frac{|x|}{R_n} \right) |u(0, x)|^2 \, dx \right| \leq C(\eta) M(u) + 2\eta R_n,
\]

and

\[
|X_{R_n}(T_n)| \geq |x(T_n)| \cdot M(u) - |x(T_n)| \cdot \left| \int_{\mathbb{R}^d} \left( 1 - \phi \left( \frac{|x|}{R_n} \right) \right) |u(T_n, x)|^2 \, dx \right| - \left| \int_{|x - x(T_n)| \leq C(\eta)} (x - x(T_n)) \phi \left( \frac{|x|}{R_n} \right) |u(T_n, x)|^2 \, dx \right| - \left| \int_{|x - x(T_n)| \geq C(\eta)} (x - x(T_n)) \phi \left( \frac{|x|}{R_n} \right) |u(T_n, x)|^2 \, dx \right|
\]
\[
\geq |x(T_n)| \cdot (M(u) - \phi(T_n)) - C(\eta) M(u) - (R_n + |x(T_n)|) \cdot \phi(T_n)
\]
\[
\geq |x(T_n)| \cdot (M(u) - \eta) - C(\eta) \cdot (M(u) + \eta).
\]

Thus, taking \( \eta > 0 \) sufficiently small, we have

\[
|X_{R_n}(T_n) - X_{R_n}(0)| \geq \frac{M(u)}{2} |x(T_n)| - 2M(u) \cdot C(\eta).
\]

(6.8)

On the other hand, we have

\[
\partial_t X_{R_n}(t) = 23 \int_{\mathbb{R}^d} \phi \left( \frac{|x|}{R_n} \right) \nabla u(t, x) \cdot \overline{u(t, x)} \, dx
\]
\[
+ 23 \int_{\mathbb{R}^d} \frac{x}{|x|} \phi' \left( \frac{|x|}{R_n} \right) x \cdot \nabla u(t, x) \overline{u(t, x)} \, dx,
\]

which together with Proposition [6.3] and Corollary [6.3] implies that for any \( t \in [0, T_n] \)

\[
|\partial_t X_{R_n}(t)| \leq 23 \int_{\mathbb{R}^d} \left( 1 - \phi \left( \frac{|x|}{R_n} \right) \right) \nabla u(t, x) \cdot \overline{u(t, x)} \, dx
\]
\[
+ 23 \int_{\mathbb{R}^d} \frac{x}{|x|} \phi' \left( \frac{|x|}{R_n} \right) x \cdot \nabla u(t, x) \overline{u(t, x)} \, dx \leq C\eta.
\]

Hence, we have

\[
\frac{M(u)}{2} \cdot \delta T_n - 2M(u) \cdot C(\eta) \leq \frac{M(u)}{2} \cdot \delta t_n - 2M(u) \cdot C(\eta) < |x(T_n)| - C(\eta) \leq C\eta \cdot T_n.
\]
Taking $\eta$ sufficiently small such that $C\eta \leq \frac{M(u)}{4} \cdot \delta$, we obtain a contradiction with the fact $T_n \to +\infty$. \hfill \Box

6.6. **Death of the critical element.** We are in a position to preclude the soliton-like solution by a truncated Virial identity.

**Theorem 6.9.** For $d \geq 5$. The critical element $u_c$ of (1.1) cannot be a soliton in the sense of Theorem 6.1.

**Proof.** We still drop the subscript $c$. By Proposition 6.8 for any $\eta > 0$, there exists $T_0 = T_0(\eta) \in \mathbb{R}$ such that

$$|x(t)| \leq \eta t \text{ for all } t \geq T_0. \tag{6.9}$$

Now let $\phi$ be a smooth, radial function satisfying $0 \leq \phi \leq 1$, $\phi(x) = 1$ for $|x| \leq 1$, and $\phi(x) = 0$ for $|x| \geq 2$. For some $R$, we define

$$V_R(t) := \int_{\mathbb{R}^d} \phi_R(x)|u(t, x)|^2 \, dx, \quad \phi_R(x) = R^2 \phi \left( \frac{|x|^2}{R^2} \right).$$

On one hand, we have

$$\partial_t V_R(t) = 4 \Im \int_{\mathbb{R}^d} \phi_R(x) x \cdot \nabla u(t, x) \, u(t, x) \, dx.$$ 

Therefore, we have

$$|\partial_t V_R(t)| \lesssim R \tag{6.10}$$

for all $t \geq 0$ and $R > 0$.

On the other hand, by Hölder’s inequality, we have

$$\partial_t^2 V_R(t) = 4 \int_{\mathbb{R}^d} \partial_j (\phi_R) u_i(t, x) \bar{u}_i(t, x) \, dx - \int_{\mathbb{R}^d} (\Delta^2 \phi_R)(x)|u(t, x)|^2 \, dx$$

$$- \frac{4}{d} \int_{\mathbb{R}^d} (\Delta \phi_R)(x)|u(t, x)|^2^* \, dx + \frac{4}{d+1} \int_{\mathbb{R}^d} (\Delta \phi_R)(x)|u(t, x)|^{\frac{2d+2}{d-1}} \, dx$$

$$= 4 \int_{\mathbb{R}^d} \left( 2|\nabla u(t, x)|^2 - 2|u(t, x)|^2^* + \frac{2d}{d+1}|u(t, x)|^{\frac{2d+2}{d-1}} \right) \, dx$$

$$+ O \left( \int_{|x| \geq R} \left( |\nabla u(t, x)|^2 + |u(t, x)|^2^* + |u(t, x)|^{\frac{2d+2}{d-1}} \right) \, dx + \left( \int_{R \leq |x| \leq 2R} |u(t, x)|^2^* \, dx \right)^{(d-2)/d} \right)$$

$$= 4K(u(t)) + O \left( \int_{|x| \geq R} \left( |\nabla u(t, x)|^2 + |u(t, x)|^{\frac{2d+2}{d-1}} \right) \, dx + \left( \int_{R \leq |x| \leq 2R} |u(t, x)|^2^* \, dx \right)^{(d-2)/d} \right).$$
By Lemma 3.9, we have
\[
4K(u(t)) = 4 \int_{\mathbb{R}^d} \left( 2|\nabla u(t,x)|^2 - 2|u(t,x)|^{2^*} + \frac{2d}{d+1} |u(t,x)|^{\frac{2(d+2)}{d-1}} \right) \, dx
\]
\[
\geq \min \left( \bar{\mu}(m - E(u(t))), \frac{2}{2d-3} \|\nabla u(t)\|_{L^2}^2 + \frac{2d}{(d+1)(2d-3)} \|u(t)\|_{L^{\frac{2(d+2)}{d-1}}}^{\frac{2(d+2)}{d-1}} \right)
\]
\[
\geq E(u(t)),
\]
Thus, choosing \( \eta > 0 \) sufficiently small and \( R := C(\eta) + \sup_{t \in [T_0,T_1]} |x(t)| \) and by Corollary 6.3, we obtain
\[
\partial_t^2 V_R(t) \geq E(u(t)) = E(u_0),
\]
which implies that for all \( T_1 > T_0 \)
\[
(T_1 - T_0) E(u_0) \lesssim R = C(\eta) + \sup_{t \in [T_0,T_1]} |x(t)| \leq C(\eta) + \eta T_1.
\]
Taking \( \eta \) sufficiently small and \( T_1 \) sufficiently large, we obtain a contradiction unless \( u \equiv 0 \). But \( u \equiv 0 \) is not consistent with the fact that \( \|u\|_{ST(\mathbb{R})} = \infty \). \( \square \)

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