Zero modes in a system of Aharonov–Bohm fluxes

V. A. Geyler

Department of Mathematics, Mordovian State University
Bolshevistskaya 68, Saransk 430000, Russia

P. Štovíček

Department of Mathematics, Faculty of Nuclear Science
Czech Technical University
Trojanova 13, 120 00 Prague, Czech Republic

Abstract

We study zero modes of two-dimensional Pauli operators with Aharonov–Bohm fluxes in the case when the solenoids are arranged in periodic structures like chains or lattices. We also consider perturbations to such periodic systems which may be infinite and irregular but they are always supposed to be sufficiently scarce.

1. Introduction

The appearance of zero modes (wave functions at zero energy which are ground states for a positive quantum Hamiltonian) belongs to the most interesting phenomena in systems with topologically non-trivial configuration spaces; see the discussion and an extensive bibliography in [1]. Zero modes of the Dirac and Pauli operators are of great importance in many places in quantum field theory and mathematical physics [2, 3, 4]. They are the ingredients for the computation of the index of these operators and play a key role in understanding anomalies. One of the best known examples for such operators is the Pauli Hamiltonian of a two-dimensional charged particle moving in a magnetic field perpendicular to a plane and penetrating the plane in a bounded domain. In this case the field defines a vector bundle with a non-trivial connection and zero modes appear at sufficiently high strength of the field [5]. More precisely, it is easy to prove that the dimension $d$ of the space of zero modes is $d = \lceil \Phi \rceil$ where $\Phi$ is the total flux of the magnetic field measured in magnetic flux quanta, and for a real $x$, $x \geq 0$, $\lfloor x \rfloor$ denotes the lower integer part of $x$ ($\lfloor 0 \rfloor = 0$, $\lfloor n \rfloor = n - 1$ for $n \geq 1$ integer, and otherwise $\lfloor x \rfloor = \lceil x \rceil$, the integer part of $x$). It is worthy to note that in the three-dimensional case the appearance and the degeneracy of zero modes is a more subtle fact (see e.g. [6, 7, 8, 9, 10, 11] and the discussion therein).
In the current paper we restrict our consideration to two-dimensional systems only. More precisely, we consider Pauli operators which are Hamiltonians of an electron confined to a plane and subjected to a perpendicular time-independent magnetic field which is the sum of a uniform field and an additional field contributed by a (finite or infinite) array of singular flux tubes or, in other words, by an array of solenoids of zero width. We focus on zero modes in such systems. In more detail, the aim of the paper is to find conditions for appearance of zero modes in systems placed in a magnetic field with an infinite array of Aharonov–Bohm vortices. It has been shown in [12] on the physical level of rigor that zero modes occur if Aharonov–Bohm vortices are arranged in a periodic plane lattice provided that not all magnetic fluxes involved have integer values. In this paper we present a rigorous proof and show that under the same condition imposed on the flux, the result is true for a chain of Aharonov–Bohm solenoids or, more generally, for a uniformly discrete union of such chains. Moreover, the zero modes are retained if one adds to such a periodic structure of Aharonov–Bohm solenoids a not necessarily regular array of solenoids having sufficiently low density. This stability of zero-modes for the Hamiltonian that we call $H_{\text{max}}$ (its definition is discussed in Section 2) shows that their origin differs from that for localized states in the so called Aharonov–Bohm cages [13, 14], the latter are destroyed by arbitrarily small period modulations [15].

The main results of the paper are obtained with the help of a version of the Aharonov–Casher ansatz [5]. This version was proposed by Dubrovin and Novikov in [16] who employed it for an explicit construction of ground states of periodic magnetic Schrödinger operators (see Novikov’s review paper [17]). In our case, this ansatz reduces the problem of finding zero-modes to some estimates for entire functions. The mechanism of appearance of zero modes in the considered cases is close to that for a two-dimensional system in a uniform magnetic field in the presence of an infinite array of point scatterers [38, 18, 19, 20, 21, 22].

An interesting physical consequence of our result is the occurrence of oscillations of the type “localization–delocalization” in periodic systems of Aharonov–Bohm solenoids placed in a varying uniform magnetic field (Theorem 8.16). Another interesting result described in Theorems 8.8 and 8.16 is related to the problem of absolute continuity of the spectrum of the Schrödinger operator with periodic vector potential $A$. This absolute continuity has been proved for a wide class of potentials $A$ [23, 24, 25, 26, 27, 28]. An example of a vector potential $A$ having eigenvalues in the spectrum of the corresponding Schrödinger operator was given in [29] but only for dimensions higher than 3. Our results give such an example in dimension 2.

The paper is organized as follows. In Section 2 we try to point out some aspects regarding the history and the background of the problem. In Section 3 we introduce several basic examples of models with Aharonov–Bohm fluxes some of them are the main subject of this paper and are studied in detail in the sequel. Section 4 is devoted to a rigorous definition of the Pauli operator with Aharonov–Bohm fluxes. In Section 5 we discuss the elimination of integer-valued Aharonov–Bohm fluxes. In Section 6 we recall the Aharonov–Casher ansatz which makes it possible to construct
ground states of the Pauli operator using the theory of analytic functions. The main results of the paper are contained in Sections 8 and 9. In Section 8 we study zero modes of the Pauli operator with an infinite periodic system of Aharonov–Bohm solenoids. In Section 9 we address the question of perturbations of such periodic structures caused by translations and additions of Aharonov–Bohm solenoids. The subsystem formed by solenoids affected by the perturbation may be infinite and irregular but we always suppose that it is sufficiently scarce. Here we also discuss some examples of irregular Aharonov–Bohm systems. For the reader’s convenience we have included three appendices. In the first appendix we collect some basic definitions and auxiliary results concerning lattices. In the second appendix we recall some basic notions and results from the theory of analytic functions related to the growth of entire functions. The third appendix is devoted to the Weierstrass $\sigma$-function.

2. Additional comments on the history and the background of the problem

There are many interesting and important physical problems related to systems involving Aharonov–Bohm fluxes. Since the publication of the original paper due to Aharonov and Bohm [30] the physics of a magnetic flux in an infinitely thin solenoid (called Aharonov–Bohm flux or Aharonov–Bohm vortex) has been investigated both from theoretical and experimental points of view [31, 32]. The physical origin of the Aharonov–Bohm effect is even a subject of theoretical investigations up to now [33]. On the other hand, the motion of a charged particle (an electron, a hole or a composite fermion) in a plane perpendicular to a uniform magnetic field has found an important application in physics of the quantum Hall effect [34, 35]. The most striking feature of the Hamiltonian of such a system is the Landau quantization of the spectrum which consists of highly degenerated equidistant energy levels; this makes quantum Hall phenomena possible. Moreover, it is of interest to know how the quantum Hall system is altered by various defects, in particular, by impurities or by inhomogeneities of the magnetic field. Additional Aharonov–Bohm fluxes appear to be a minimal modification of the uniform magnetic field, while general inhomogeneous magnetic fields are extremely difficult to handle [36, 37]. Similarly, a minimal perturbation of the quantum Hall system is given by a point perturbation of the Landau operator (i.e., the Schrödinger operator with a uniform magnetic field) [38]. As shown below, both modifications require the operator extension theory for a correct construction of the corresponding Hamiltonian [39].

The vector potential of a system of Aharonov–Bohm solenoids has a strong singularity at the points where the plane intersects the solenoids. Therefore the differential operator defining the Hamiltonian is not essentially self-adjoint on its natural domain. This is true both in the non-relativistic case (for the Schrödinger and Pauli operators) and the relativistic one (for the Dirac operator). The boundary conditions for Schrödinger operators with an Aharonov–Bohm vortex as well as the corresponding self-adjoint extensions (i.e., Hamiltonians describing a spinless non-relativistic quan-
tum particle) are considered in many papers, let us mention e.g. [40, 41, 42, 43]. The multi-solenoid case is more difficult because of the rotational symmetry violation. This case was treated by means of the Krein resolvent formula in [44], and for an infinite chain of solenoids in [45]; different approaches are presented in [46, 47, 48]. The problem of defining the boundary conditions at the presence of a uniform background field has been investigated in [49, 50]. In the relativistic case, the problem of defining the appropriate Dirac operator is discussed e.g. in [51, 52, 53], and at the presence of a uniform component – in the recent articles [54, 55, 56, 57]. In all the mentioned papers, the spectral or scattering properties of the derived Hamiltonians are studied as well.

On sufficiently smooth functions from \( L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 = L^2(\mathbb{R}^2; \mathbb{C}^2) \) the two-dimensional Pauli operator for a charged particle with the spin \( s \) and the gyromagnetic ratio \( g \) acts as a formal differential operator [58]

\[
\hat{H} \equiv \hat{H}(A) = \frac{1}{2m^*} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} A \right)^2 - \mu \mathbf{B}
\]

where \( e \) and \( m^* \) are the charge and the mass of the particle, respectively, \( A = (A_x, A_y) \) is the vector potential of a magnetic field \( \mathbf{B} = B \mathbf{e}_z, B = \partial_x A_y - \partial_y A_x \), \( \mu \) is the magnetic momentum operator,

\[
\hat{\mu} = gs \mu_B \hat{s}_z ,
\]

with \( \mu_B \) being the Bohr magneton, \( \mu_B = -|e|\hbar/(2m^*c) \), and

\[
\hat{s}_z = \frac{1}{2} \sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(we consider the motion of a particle in the plane \( \mathbb{R}^2 \) canonically embedded in the space \( \mathbb{R}^3 \)). In general, the non-relativistic limit of the Dirac equation leads to the value \( g = 2 \), and the main part of our work deals with this value of the gyromagnetic ratio. In the case of an Aharonov–Bohm solenoid \( \mathbf{B} \) is proportional to the Dirac delta function, \( \delta(\mathbf{r}) \), and therefore the operator \( \hat{H} \) takes the form of the Schrödinger operator perturbed at a point and with a finite coupling constant \( \alpha \) standing in front of the “\( \delta \)-potential”. On the other hand, it is well known that in the two-dimensional case under consideration the expression \( \hat{H} \) defines a non-trivial perturbation of the operator

\[
\hat{H}_0 \equiv \hat{H}_0(A) = \frac{1}{2m^*} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} A \right)^2
\]

only if \( \alpha \) is in some sense ”infinitesimal” [39] (we suppose that appropriate boundary conditions defining \( \hat{H}_0 \) are chosen). This problem has been analyzed in [39] in detail for an arbitrary positive value of \( g \). To get around it, a solenoid of finite radius \( R \) is considered with a rotationally symmetric magnetic flux inside the solenoid but otherwise having an arbitrary profile (including the magnetic flux supported on the surface of the infinite cylinder), and the limit \( R \to 0 \) is discussed. In addition to [39] let us also mention papers [60, 63, 62, 61, 64, 65, 66, 67, 68]. Of course, the same approach is useful when a uniform component of the field is present or in the case of the Dirac operator (see [69, 70, 71] and references therein).
In the most important case when the gyromagnetic ratio $g$ equals 2 the Pauli operator has remarkable supersymmetry properties which makes it possible to use the Aharonov–Casher decomposition $[5]$. As a result, we have a convenient definition of the Pauli operator with a singular potential by means of a quadratic form (see Section $[5]$). More precisely, in this case we have, as usual, two natural quadratic forms associated to the expression ($[1]$ – the minimal and the maximal one (with the definition of the magnetic Schrödinger operator taken from $[2]$)). These forms provide us with two natural types of self-adjoint operators denoted $H_{\text{max}}^{\pm}$ and $H_{\text{min}}^{\pm}$ and playing the role of Pauli operators with Aharonov–Bohm solenoids (the sign $\pm$ stands for spin up and spin down supersymmetric partners, respectively). There is an important distinction between the operators $H_{\text{min}}^{\pm}$ and $H_{\text{max}}^{\pm}$. As it follows from the definitions, both operators $H_{\text{min}}^{\pm}$ coincide with the Friedrichs extension of the symmetric operator defined by expression $[2]$ with the vector potential $A$ corresponding to a system of Aharonov–Bohm fluxes. Therefore this extension (denoted simply by $H_{\text{min}}^{\pm}$) may be interpreted as the Hamiltonian of a “spinless” particle moving in a system of Aharonov–Bohm fluxes (this corresponds to physical problems for an electron when the spin–orbit coupling can be neglected and spin splitting is taken into account with the help of the perturbation theory $[58]$). Such a Hamiltonian has been considered e.g., in $[41, 59]$. On the other hand, the operators $H_{\text{max}}^{\pm}$ do not coincide in general which indicates that they directly take into account the energy of the spin–orbit interaction and therefore they may be regarded as the Pauli operators of the system under consideration. In the present article we concentrate mainly on zero modes of $H_{\text{max}}^{\pm}$. Note that boundary conditions defining the Hamiltonian $H_{\text{max}}$ are given in $[42, 43]$ (in the case of a single solenoid) and in $[73]$ (in the two-solenoid case).

For a finite system of Aharonov–Bohm solenoids, the existence problem of zero-energy eigenfunctions was considered in $[74, 75, 76, 77, 78]$. In this case the number $d$ of linearly independent zero-modes depends on the fractional parts of fluxes in the individual solenoids, $\{x\} = x - [x]$, rather than only on the total flux $\Phi$ in the system. This phenomenon is a consequence of the gauge invariance properties for the Aharonov–Bohm fluxes (see e.g. papers $[79, 80, 81, 82]$). In the case when the considered magnetic field has a “regular” component in addition to the magnetic field of Aharonov–Bohm solenoids the appearance of zero modes has been analyzed in $[83, 84]$. The results of $[84]$ are applicable also to the case when an infinite number of Aharonov–Bohm solenoids is present in the system but the total magnetic flux is necessarily finite (moreover, after some gauge transformation the total variation of the flux must be finite). On the other hand, it is clear that the thermodynamic limit of a bounded system with a fixed density of Aharonov–Bohm fluxes is a system with an infinite number of Aharonov–Bohm solenoids and with an infinite total flux. An example for a system of such a kind is the quasi-two-dimensional system with columnar defects in a uniform magnetic field directed along the defect axis $[85, 86, 87]$ or the GaAs/AlGaAs heterostructure coated with a film of type-II superconductor $[88]$ (in the latter case the Aharonov–Bohm fluxes are arranged in a honeycomb lattice, the so-called Abrikosov lattice).

As for the spectral properties of the operator $H_{\text{min}}$, they have been investigated
recently in detail by Melgaard, Ouhabaz and Rozenblum [89]. In particular, these authors proved with the help of results from [90] and [91] that $H_{\text{min}}$ has no zero modes at least for periodic lattices of Aharonov–Bohm solenoids, and therefore it differs from $H^+_{\text{max}}$ and $H^-_{\text{max}}$ for generic values of magnetic fluxes (and even it is not unitarily equivalent to these operators). Let us note that it is possible to extend this result to a chain of Aharonov–Bohm solenoids.

3. The Pauli operator with a singular magnetic field

In what follows we consider the motion of an electron with the gyromagnetic ratio $g = 2$, therefore

$$\hat{H} = \frac{\hbar^2}{2m^*} \left[ \left( i\partial_x + \frac{e}{c\hbar} A_x \right)^2 + \left( i\partial_y + \frac{e}{c\hbar} A_y \right)^2 - \frac{e}{c\hbar} \sigma_z B \right].$$

(3)

Let us denote for simplicity

$$\frac{e}{c\hbar} A = a, \quad \frac{e}{c\hbar} B = b,$$

(4)

so that $\partial_x a_y - \partial_y a_x = b$. In order to employ the dimensionless units we shall consider the operator

$$H \equiv H(a) = \frac{2m^*}{\hbar^2} \hat{H}(A).$$

(5)

Introducing a quantum of the magnetic flux,

$$\Phi^0 = \frac{2\pi c\hbar}{e},$$

(6)

we also have

$$a = \frac{2\pi}{\Phi^0} A, \quad b = \frac{2\pi}{\Phi^0} B,$$

(7)

$$H \equiv H(a) = (i\partial_x + a_x)^2 + (i\partial_y + a_y)^2 - \sigma_z b.$$

(8)

The operator $H$ (and respectively the operator $\hat{H}$) decomposes in a sum of two scalar operators,

$$H^\pm \equiv H^\pm(a) = (i\partial_x + a_x)^2 + (i\partial_y + a_y)^2 \mp b,$$

(9)

(respectively $\hat{H}^\pm(A) \equiv \hat{H}^\pm$) acting in $L^2(\mathbb{R}^2)$. We admit the vector potential $a$ to have singular points, more precisely, we assume that

$$a_x, a_y \in L^1_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \Omega),$$

(10)

where $\Omega$ is a discrete subset (possibly finite or empty) in $\mathbb{R}^2$. Consequently, the magnetic field $b = \partial_x a_y - \partial_y a_x$ is, in general, a distribution in $\mathbb{R}^2$ whose singular support is contained in $\Omega$. Expressions (11) and (9) represent symmetric operators with the
domain \( C_0^\infty(\mathbb{R}^2 \setminus \Omega) \); these operators will be denoted \( \tilde{H}^\pm(\mathbf{A}, \Omega) \) and \( H^\pm(\mathbf{a}, \Omega) \), respectively. If the singular support of \( B \) coincides with \( \Omega \) (in this case \( \Omega \) is determined by the vector potential \( \mathbf{A} \)) we shall simply write \( \tilde{H}^\pm(\mathbf{A}) \) and \( H^\pm(\mathbf{a}) \).

It is important to note that also in the case when \( b \) is a distribution the operator \( H^\pm(\mathbf{a}) \) depends, up to unitary equivalence, only on \( b \). More precisely, we have the following proposition.

**Theorem 3.1 (gauge invariance of the operator \( H^\pm(\mathbf{a}) \)).** Let \( \mathbf{a} \) and \( \mathbf{a} \) be vector potentials with the same magnetic field \( b \) (\( i.e. \), \( \mathbf{a}, \mathbf{a} \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}) \cap C^\infty(\mathbb{R}^2 \setminus \Omega; \mathbb{R}) \) and \( \partial_x a_y - \partial_y a_x = \partial_x \tilde{a}_y - \partial_y \tilde{a}_x = b \) in the sense of distributions). Then the operators \( H^\pm(\mathbf{a}, \Omega) \) and \( H^\pm(\mathbf{a}, \Omega) \) are unitarily equivalent. In more detail, there exists a real-valued function \( f \) belonging to \( C^\infty(\mathbb{R}^2 \setminus \Omega) \) such that \( \tilde{a} = a + \text{grad } f \), and \( H^\pm(\tilde{a}, \Omega) = W^{-1} H^\pm(\mathbf{a}, \Omega) W \) where \( W \) is the unitary operator acting via multiplication by the function \( \exp(-if) \).

Of course, this theorem is well known in the case when the field \( b \) is a function (not a distribution). In the case when \( b \) is a distribution the theorem is a consequence of the following lemma whose elementary proof was communicated to us by K. V. Pankrashkin.

**Lemma 3.2.** Assume that \( \mathbf{a} \in L^1_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \Omega) \) and the equality \( \partial_x a_y - \partial_y a_x = 0 \) holds true in \( \mathbb{R}^2 \) in the sense of distributions. Let \( \omega \in \Omega \) and let \( Q \) be a rectangle containing \( \omega \) but no other points from \( \Omega \). Then

\[
\int_{\partial Q} a_x \, dx + a_y \, dy = 0 .
\]  

(11)

**Proof.** Let us choose functions \( \varphi, \psi \in C_0^\infty(\mathbb{R}^2) \) so that \( \omega \notin \text{supp } \varphi, \varphi(x, y) = 1 \) in some neighborhood of the boundary \( \partial Q \), \( \psi(x, y) = \varphi(x, y) \) on \( \mathbb{R}^2 \setminus Q \) and \( \psi(x, y) = 1 \) on \( Q \). Using the Green formula we obtain

\[
\int_{\partial Q} a_x \, dx + a_y \, dy = \int_{\partial Q} \varphi a_x \, dx + \varphi a_y \, dy = \iint_Q (\partial_x (\varphi a_y) - \partial_y (\varphi a_x)) \, dxdy
\]

\[
= \iint_{Q \cap \text{supp}(\varphi)} \varphi (\partial_x a_y - \partial_y a_x) \, dxdy + \iint_Q (a_y \partial_x \varphi - a_x \partial_y \varphi) \, dxdy
\]

\[
= \iint_{\mathbb{R}^2} (a_y \partial_x (\varphi - \psi) - a_x \partial_y (\varphi - \psi)) \, dxdy = 0 .
\]

Here we have used the fact that the expression \( \partial_x a_y - \partial_y a_x \) represents a smooth function on \( \mathbb{R}^2 \setminus \Omega \) which necessarily vanishes on this domain.

**Proof of Theorem 3.1.** From Lemma 3.2 we derive in a standard manner that if \( \partial_x a_y - \partial_y a_x = 0 \) on \( \mathbb{R}^2 \) in the sense of distributions then there exists a real-valued function \( f \in C^\infty(\mathbb{R}^2 \setminus \Omega) \) such that \( \mathbf{a} = \text{grad } f \) on \( \mathbb{R}^2 \setminus \Omega \). Consequently, if \( \mathbf{a} \) and \( \tilde{a} \) obey
the assumptions of the theorem then for some function \( f \in C^\infty(\mathbb{R}^2 \setminus \Omega) \) we have \( \tilde{a} = a + \text{grad } f \). Let us denote by \( W \) the operator acting via multiplication by the function \( \exp(-if) \). Clearly, \( W \) is a well defined unitary operator in \( L^2(\mathbb{R}^2) \). Moreover, \( W \) leaves invariant the subspace \( C^\infty_0(\mathbb{R}^2 \setminus \Omega) \). A simple computation shows that \( W^{-1}H^\pm(a, \Omega)W = H^\pm(\tilde{a}, \Omega) \). Hence the operators \( H^\pm(a, \Omega) \) and \( H^\pm(\tilde{a}, \Omega) \) are unitarily equivalent. \( \square \)

**Remark 3.3.** Clearly, if \( a = \text{grad } f \) in the sense of distributions then \( \partial_x a_y - \partial_y a_x = 0 \) in the same sense.

**Remark 3.4.** A proposition analogous to that of Theorem 3.1 is also valid for the operator \( \hat{H}^\pm(A, \Omega) \). Namely, if \( \partial_x A_y - \partial_y A_x = \partial_y A_x - \partial_x A_y = B \) then \( \hat{A} = A + \text{grad } f \) and \( \hat{H}^\pm(A, \Omega) = W^{-1}H^\pm(\hat{A}, \Omega)W \) where \( W = \exp(-(ie/c)\hbar f) \).

Owing to the gauge invariance it is possible to require the vector potential \( A \) to have some additional properties. For example, the vector potential \( A \) can be frequently chosen so that it fulfills the Lorentz gauge condition

\[
\text{div } A = 0. \tag{12}
\]

**4. Basic examples**

In this section we recall several basic examples of magnetic fields fulfilling condition \( (10) \). At the same time, we introduce the necessary notation. The majority of results presented in the current paper concern Examples 5, 6 and 7. In what follows it will be convenient to identify the Euclidean plane \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \) and to work with the complex coordinates \( z = x + iy \) and \( \bar{z} = x - iy \).

**Example 1. The homogeneous field**

In this case \( B = \text{const} \) by definition and one can set

\[
A_x = -\frac{B}{2}y, \quad A_y = \frac{B}{2}x
\]

(the symmetric gauge). In the complex coordinates we have

\[
A_x = \frac{B}{2} \text{Im } \bar{z}, \quad A_y = \frac{B}{2} \text{Re } \bar{z}.
\]

In this example \( b = 2\pi \xi \) where \( \xi \) is the number of magnetic flux quanta through a unit area in \( \mathbb{R}^2 \) (the flux density). The Lorentz gauge condition \( (12) \) is obviously fulfilled.

**Example 2. The magnetic field of an Aharonov–Bohm solenoid**

Here \( B(\mathbf{r}) = \Phi \delta(\mathbf{r}) \) where \( \Phi \) is the magnetic flux through the solenoid. In this case one can set

\[
A_x = -\frac{\Phi}{2\pi} \frac{y}{r^2}, \quad A_y = \frac{\Phi}{2\pi} \frac{x}{r^2}.
\]
Equivalently, 

\[ a_x = \theta \, \text{Im} \, \frac{1}{z}, \quad a_y = \theta \, \text{Re} \, \frac{1}{z}, \]

where \( \theta = \Phi / \Phi^0 \) is the number of magnetic flux quanta through the Aharonov–Bohm solenoid. Actually, it is well known that

\[ \Delta \ln(|z|) = 2\pi \delta(z). \]

In the local coordinates we have

\[
B = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x = \frac{\Phi}{2\pi} \left( \frac{\partial}{\partial x} \text{Re} \, \frac{1}{z} - \frac{\partial}{\partial y} \text{Im} \, \frac{1}{z} \right) = \frac{\Phi}{2\pi} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \ln |z| = \Phi \delta(z).
\]

The vector potential \( \mathbf{a} \) can be also written as

\[ \mathbf{a} = \theta \, \text{grad} \ln |z|. \quad (13) \]

Here and everywhere in what follows \( \text{grad} \) stands for the symplectic gradient,

\[ \text{grad} f = \left( -\frac{\partial}{\partial y} f, \frac{\partial}{\partial x} f \right). \quad (14) \]

Hence \( b = 2\pi \theta \delta(z) \). The equality \( \text{div} \, \mathbf{a} = 0 \) trivially follows from (13).

**Example 3. An arbitrary system of Aharonov–Bohm solenoids**

Let now \( \Omega \) be a discrete subset of the plane \( \mathbb{R}^2 \) and let \( (\Phi_\omega)_{\omega \in \Omega} \) be an arbitrary family of real numbers with indices from \( \Omega \). We shall consider a system of Aharonov–Bohm fluxes intersecting the plane in the points from the set \( \Omega \) and perpendicular to the plane. The number \( \Phi_\omega \) equals the flux in the solenoid passing through the point \( \omega \in \Omega \). Then

\[
b = \frac{2\pi}{\Phi^0} B = 2\pi \sum_{\omega \in \Omega} \theta_\omega \delta(z - \omega)
\]

where, of course, \( \theta_\omega = \Phi_\omega / \Phi^0 \) is the number of magnetic flux quanta through the solenoid \( \omega \). For a vector potential \( \mathbf{a} \) fulfilling the Landau gauge condition one can choose a meromorphic function \( M(z) \) with the following properties:

1) \( M(z) \) has simple poles only,
2) the set of poles of \( M(z) \) coincides with \( \Omega \),
3) the residue of \( M(z) \) at the point \( \omega \) equals \( \theta_\omega \).

According to the Mittag-Leffler theorem such a function always exists. The computations carried out in Example 2 (jointly with the Cauchy–Riemann conditions) show that one can set

\[ a_x(z, \bar{z}) = \text{Im} \, M(z), \quad a_y(z, \bar{z}) = \text{Re} \, M(z). \]

The operator \( H^\pm(\mathbf{a}) \) will be also denoted by the symbol \( H^\pm(\Omega, \Theta) \) where \( \Theta = (\theta_\omega)_{\omega \in \Omega} \). The couple \( (\Omega, \Theta) \) determines the operator \( H^\pm(\Omega, \Theta) \) unambiguously up to unitary equivalence.
Example 4. An arbitrary system of Aharonov–Bohm solenoids with fluxes taking a finite number of values

Separately we consider the case when the number of mutually different fluxes in the family \((\Phi_\omega)_{\omega \in \Omega}\) is finite (equivalently, the family \((\theta_\omega)_{\omega \in \Omega}\) contains only a finite number of mutually different numbers \(\theta_\omega\)). We start from the case when all the involved solenoids carry the same flux: \(\theta_\omega = \theta, \forall \omega \in \Omega\). In this case we always set

\[
M(z) = \theta \frac{W'(z)}{W(z)}.
\]

Here the function \(W(z)\) differs from the Weierstrass canonical product \(W_\Omega(z)\) related to the set \(\Omega\) only by a multiplier \(\exp(g(z))\) where \(g(z)\) is an entire function. Obviously, the set of poles of the function \(W'(z)/W(z)\) coincides with \(\Omega\), all the poles are simple and all the residues are equal to 1. Thus one can set

\[
a = \theta \text{sgrad } \ln(|W(z)|) .
\] (15)

Actually, locally we have

\[
\frac{\partial}{\partial x} \ln(|W(z)|) = \frac{1}{2} \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) \ln (W(z) + \overline{W(\bar{z})}) = \text{Re} \frac{W'(z)}{W(z)},
\]

and analogously,

\[
\frac{\partial}{\partial y} \ln(|W(z)|) = -\text{Im} \frac{W'(z)}{W(z)}.
\]

In general, let \(\Omega_1, ..., \Omega_N\) be mutually disjoint discrete (possibly empty) sets, and let \(\theta_j, j = 1, ..., N\), be (not necessarily distinct) real numbers. The vector potential \(a\) is defined unambiguously, up to gauge equivalence, by the expression

\[
a = \sum_{j=1}^{N} \theta_j \text{sgrad } \ln(|W_j(z)|) = \text{sgrad } \ln \left( \prod_{j=1}^{n} |W_j(z)|^{|\theta_j|} \right)
\] (16)

where \(W_j\) is an entire function having simple zeros only and with the zero set being equal to \(\Omega_j\). The function \(W_j\) differs from the Weierstrass canonical product related to the set \(\Omega_j\) only by a multiplier of the form \(\exp(g_j(z))\) where \(g_j\) is an arbitrary entire function. An Aharonov–Bohm potential of the form \(16\) will be called a potential of finite type. The operator \(H^\pm(a)\) will be also denoted by the symbols \(H^\pm(\Omega_1, ..., \Omega_N; \theta_1, ..., \theta_N)\) or \(H^\pm(\Omega_j; (\theta_j))\).

The most important particular cases of potentials of finite type are those for which the Aharonov–Bohm field is invariant with respect to a discrete group \(\Lambda\) which is formed by motions of the Euclidean plane \(\mathbb{R}^2\) and whose action on \(\Omega\) is co-finite. First of all we shall be interested in the case when the group \(\Lambda\) is formed by parallel translations. Up to isomorphism, there exist just three groups of this type in the plane and they are characterized by their rank \(r (r = 0, 1, 2)\).

1. \(r = 0\). In this case \(\Lambda = \{0\}\) and the set \(\Omega\) is finite.
2. $r = 1$. In this case $\Lambda$ is isomorphic to $\mathbb{Z}$ and has the form $\Lambda = \{k\omega_0; k \in \mathbb{Z}\}$ where $\omega_0$ is a nonzero vector from $\mathbb{R}^2$. The set $\Omega$ has the form $\Omega = K + \Lambda$ where $K$ is a finite subset of the “elementary strip” $F = \{x \in \mathbb{R}^2; 0 \leq x \cdot \omega_0 < |\omega_0|^2\}$ (or, in the complex coordinates, $F = \{z \in \mathbb{C}; 0 \leq \text{Re } \bar{z} \omega_0 < |\omega_0|^2\}$). Since each $\omega \in \Omega$ is uniquely expressible in the form $\omega = \kappa + \lambda$, with $\kappa \in K$ and $\lambda \in \Lambda$, every $\Lambda$-invariant family $\Theta$ is unambiguously determined by its subfamily $\Theta_K = (\theta_\kappa)_{\kappa \in K}$.

3. $r = 2$. In this case $\Lambda$ is isomorphic to $\mathbb{Z}^2$ and has the form $\Lambda = \{k_1\omega_1 + k_2\omega_2; k_1, k_2 \in \mathbb{Z}\}$ where $\omega_1, \omega_2$ are linearly independent vectors from $\mathbb{R}^2$. The set $\Omega$ has the form $\Omega = K + \Lambda$ where $K$ is a finite subset of the elementary cell $F = \{t_1\omega_1 + t_2\omega_2; 0 \leq t_1, t_2 < 1\}$. We shall assume that the basis $\omega_1, \omega_2$ is positively oriented so that $\omega_1 \land \omega_2 = \text{Im } \bar{\omega}_1 \omega_2 > 0$. This expression is nothing but the area $S = S_\Lambda$ of the elementary cell $F$ of the lattice $\Lambda$.

We shall discuss each of these cases separately.

**Example 5. A finite number of Aharonov–Bohm solenoids**

Let $\Lambda = \{0\}$. In this case the set $\Omega$ is finite, $\Omega = \{\omega_1, \ldots, \omega_n\}$, and

$$b = 2\pi \sum_{j=1}^{n} \theta_j \delta(z - \omega_j).$$

As a rule, the vector potential in this case will be chosen in the form

$$a = \sum_{j=1}^{n} \theta_j \ \text{sgrad} \ln(|z - \omega_j|).$$

The operator $H^\pm(a)$ will be also denoted by $H^\pm(\omega_1, \ldots, \omega_n; \theta_1, \ldots, \theta_n)$.

**Example 6. A chain of Aharonov–Bohm solenoids**

Assume now that the rank of $\Lambda$ equals 1. Firstly we consider the case when $K$ contains only one element. Without loss of generality we assume that $K = \{0\}$. Then $\Omega = \Lambda$, $\theta_\omega = \theta$ for all $\omega$, and

$$W_\Omega(z) = z \prod_{k \in \mathbb{Z}, k \neq 0} \left(1 - \frac{z}{k\omega_0}\right) e^{z/k\omega_0} = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\omega_0^2}\right) = \frac{1}{\pi} \sin\left(\frac{\pi z}{\omega_0}\right).$$

Therefore one can set

$$W(z) = \sin\left(\frac{\pi z}{\omega_0}\right).$$

Consequently,

$$a = \theta \ \text{sgrad} \ln\left(|\sin\left(\frac{\pi z}{\omega_0}\right)|\right).$$
which means that
\[ a_x = \frac{\pi \theta}{\omega_0} \text{Im}\ ctg\left(\frac{\pi z}{\omega_0}\right), \quad a_y = \frac{\pi \theta}{\omega_0} \text{Re}\ ctg\left(\frac{\pi z}{\omega_0}\right). \]

Generally, \( \Omega = K + \Lambda \) with an arbitrary finite subset \( K \subset F \), and we have
\[ B = \sum_{\kappa \in K} \Phi_\kappa \sum_{\lambda \in \Lambda} \delta(z - \lambda - \kappa). \]

Then the vector potential reads
\[ a = \text{sgrad} \sum_{\kappa \in K} \theta_\kappa \ln \left( \left| \sin \left( \frac{\pi}{\omega_0} (z - \kappa) \right) \right| \right). \]

**Example 7. A lattice of Aharonov–Bohm solenoids**

Assume now that the rank of \( \Lambda \) equals 2 which means that \( \Lambda \) is a two-dimensional lattice. Again, we shall start from the case when \( K = \{0\} \), hence \( \Omega = \Lambda \). In this case \( W_\Omega(z) \) coincides with the Weierstrass \( \sigma \)-function of the lattice \( \Lambda \),
\[ \sigma(z; \omega_1, \omega_2) \equiv \sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left( 1 - \frac{z}{\omega} \right) \exp\left( \frac{z}{\omega} + \frac{z^2}{\omega^2} \right). \]

At the same time,
\[ \frac{\sigma'(z)}{\sigma(z)} = \zeta(z) = \zeta(z; \omega_1, \omega_2) \]
is the Weierstrass \( \zeta \)-function of the lattice \( \Lambda \). Thus
\[ a = \theta \ \text{sgrad} \ln(|\sigma(z)|) = \theta \ (\text{Im} \zeta(z), \text{Re} \zeta(z)). \]

In the general case \( \Omega = K + \Lambda \) with an arbitrary finite subset \( K \subset F \). Then the magnetic field takes the form
\[ B = \sum_{\kappa \in K} \Phi_\kappa \sum_{\lambda \in \Lambda} \delta(z - \lambda - \kappa). \]

One can set
\[ a = \text{sgrad} \sum_{\kappa \in K} \theta_\kappa \ln(|\sigma(z - \kappa)|). \]

**Remark 4.1.** In all the examples with an Aharonov–Bohm potential of finite type, and also in the case of a homogeneous magnetic field, there exists a function \( \varphi(x,y) \) such that \( a = \text{sgrad} \varphi \) or, equivalently,
\[ \Delta \varphi = b \] (17)

Namely, one can respectively set in Examples 1, 2, 3, 5, 6, 7:
\[ \varphi(x, y) = \frac{1}{2}\pi \xi(x^2 + y^2) = \frac{1}{2}\pi \xi|z|^2, \]

\[ \varphi(x, y) = \frac{1}{2}\theta \ln(x^2 + y^2) = \theta \ln(|z|), \]

\[ \varphi(x, y) = \sum_{j=1}^{n} \theta_j \ln|W_j(z)| = \ln \left( \prod_{j=1}^{n} |W_j(z)|^{\theta_j} \right), \]

\[ \varphi(x, y) = \sum_{j=1}^{n} \theta_j \ln(|z - \omega_j|), \]

\[ \varphi(x, y) = \sum_{\kappa \in K} \theta_\kappa \ln \left( \left| \sin \left( \frac{\pi(z - \kappa)}{\omega_0} \right) \right| \right) = \ln \left( \prod_{\kappa \in K} \left| \sin \left( \frac{\pi(z - \kappa)}{\omega_0} \right) \right|^{\theta_\kappa} \right), \]

\[ \varphi(x, y) = \sum_{\kappa \in K} \theta_\kappa \ln(|\sigma(z - \kappa; \omega_1, \omega_2)|) = \ln \left( \prod_{\kappa \in K} |\sigma(z - \kappa; \omega_1, \omega_2)|^{\theta_\kappa} \right). \]

Let us note that in the general case when \( B \) is a \( \Lambda \)-periodic continuous field the solution of the equation (17) is expressible in the form

\[ \varphi(z) = \frac{1}{2\pi} \int_{F} \ln(|\sigma(z - z'|)|) b(z') \ dx' dy', \]  

where \( F \) is an elementary cell of the lattice \( \Lambda \) [17]. Actually, we have already seen that

\[ \Delta \ln(|\sigma(z)|) = 2\pi \sum_{\lambda \in \Lambda} \delta(z - \lambda). \]

Therefore a formal computation yields

\[ \Delta \varphi(z) = \sum_{\lambda \in \Lambda} \int_{F} b(z') \delta(z - z' - \lambda) \ dx' dy'. \]  

For every \( z \in \mathbb{C} \) there exists a unique \( \lambda_0 \in \Lambda \) such that \( z \in F + \lambda_0 \), i.e., \( z - \lambda_0 \in F \). Then the summands in (19) with \( \lambda \neq \lambda_0 \) vanish and we have

\[ \Delta \varphi(z) = \int_{F} b(z') \delta(z' - (z - \lambda_0)) \ dx' dy = b(z - \lambda_0) = b(z). \]

In the case of a lattice formed by Aharonov–Bohm solenoids formula (18) makes still sense and it again yields

\[ \varphi(z) = \theta \ln(|\sigma(z)|). \]

Let us note that the Lorentz gauge condition (12) follows from the equality \( a = \text{sgrad} \varphi \).
In the sequel, the main results will be derived for Hamiltonians corresponding to three types of systems of Aharonov–Bohm solenoids. Namely, the set $\Omega$ formed by the intersection points of solenoids with the plane may be 1) a finite set, 2) a chain or a finite union of chains, 3) a lattice or a finite union of lattices. These systems will be called regular.

5. A rigorous definition of the Pauli operator as a self-adjoint operator

Let us return to the symmetric operators $H^\pm = H^\pm(a, \Omega)$ defined in \((9)\) while assuming that condition \((10)\) is satisfied. Let us introduce the momentum operators
\[
P_x \equiv P_x(a, \Omega) = -i\partial_x - a_x, \quad P_y \equiv P_y(a, \Omega) = -i\partial_y - a_y.
\]
In virtue of \((10)\) these operators can be considered as symmetric operators in $L^2(\mathbb{R}^2)$ with the domain $C^\infty_0(\mathbb{R}^2 \setminus \Omega)$. Following Aharonov–Casher \([5]\) we define the operators
\[
T_\pm \equiv T_\pm(a, \Omega) = P_x \pm iP_y,
\]
or $T_+ = -2i\partial_z - A(z, \bar{z})$, $T_- = -2i\partial_{\bar{z}} - \bar{A}(z, \bar{z})$ where $A = a_x + ia_y$. Then the following equalities hold true on $C^\infty_0(\mathbb{R}^2 \setminus \Omega)$:
\[
T_+T_- = H^-, \quad T_-T_+ = H^+.
\]
By a straightforward computation one can verify a simple but important lemma.

**Lemma 5.1.** The commutation relations
\[
[P_x, P_y] = ib, \quad [T_-, T_+] = -2b,
\]
are valid on $C^\infty_0(\mathbb{R}^2 \setminus \Omega)$. In particular, if $\text{supp } B \subset \Omega$ (including the case when $B$ corresponds to a system of Aharonov–Bohm solenoids) then the operators $P_x$ and $P_y$ (respectively $T_+$ and $T_-$) commute on the domain $C^\infty_0(\mathbb{R}^2 \setminus \Omega)$.

From the obvious inclusions
\[
T_+^* \supset T_-
\]
we immediately deduce that the operators $T_\pm$ are closable and therefore the self-adjoint operators
\[
H^\pm_{\text{min}} \equiv H^\pm_{\text{min}}(a, \Omega) = T^*_\pm T_\pm
\]
are well defined (see, e.g., \([92, \text{Theorem X.25}]\)). The associated quadratic forms $h^\pm_{\text{min}}$ are closures of positive forms defined on $C^\infty_0(\mathbb{R}^2 \setminus \Omega)$ by the expressions
\[
\langle T_\pm \varphi | T_\pm \psi \rangle,
\]
respectively.
On the other side, let us consider a quadratic form defined on \( C_0^\infty(\mathbb{R}^2 \setminus \Omega) \) by the relation
\[
s^\pm(\varphi, \psi) = \langle P_x \varphi | P_x \psi \rangle + \langle P_y \varphi | P_y \psi \rangle \mp \langle b \varphi | \psi \rangle.
\] (27)

By a straightforward computation using relation (23) one can show the following lemma.

**Lemma 5.2.** The quadratic forms \( h^\pm_{\min} \) and \( s^\pm \) coincide on \( C_0^\infty(\mathbb{R}^2 \setminus \Omega) \).

In particular, if the support of \( B \) is contained in \( \Omega \) then the quadratic forms \( h^+_{\min} \) and \( h^-_{\min} \) coincide on \( C_0^\infty(\mathbb{R}^2 \setminus \Omega) \) and therefore they are necessarily equal.

**Corollary 5.3.** If \( B \) is a distribution with a support contained in \( \Omega \) then the operators \( H^+_{\min} \) and \( H^-_{\min} \) coincide. In particular, if the support of \( B \) is contained in \( \Omega \) then the operators \( H^+_{\min} \) and \( H^-_{\min} \) coincide on \( C_0^\infty(\mathbb{R}^2 \setminus \Omega) \) and therefore they are necessarily equal.

In view of Lemma 5.2 we shall sometimes simply write \( H_{\min} \) instead of \( H^\pm_{\min} \). The operator \( H_{\min} \) has been investigated in detail in [89].

Jointly with the operator \( H_{\min}(a, \Omega) \) let us consider the operators
\[
H^\pm_{\max} \equiv H^\pm_{\max}(a, \Omega) = T^\pm T^*_{\pm}
\] (29)
with the associated quadratic forms defined on \( \mathcal{D}(H^\pm_{\max}) \) by the expressions
\[
h^\pm_{\max}(\varphi, \psi) = \langle T^*_{\pm} \varphi | T^*_{\pm} \psi \rangle,
\] (30)
respectively.

The definitions of \( H^\pm_{\max}(a, \Omega) \) and \( H^\pm_{\min}(a, \Omega) \) in principle depend on the choice of the discrete set \( \Omega \). If \( \Omega \) coincides with the singular support of \( b \), however, we shall simply write, similarly as in Section 3, \( H^\pm_{\min}(a) \) and \( H^\pm_{\max}(a) \) since in that case the vector potential \( a \) determines \( \Omega \) unambiguously.

If the field \( B \) is sufficiently regular and \( \Omega = \emptyset \) then the operator \( H^\pm_{\min} \) coincides with the operator \( H^\pm_{\max} \). This is not true, however, for operators with Aharonov–Bohm fluxes (see [72], [89]). Since in this case \( H^\pm_{\min} \) is defined by expression (28) and is independent of spin, this operator is the Schrödinger operator of a spinless particle in the presence of the Aharonov–Bohm fluxes (or the Schrödinger operator of a particle with spin when interaction of the spin with the field can be neglected). On the other hand, \( H^\pm_{\min} \) are defined by expression (9), they depend on the spin and may be considered as the Pauli operators for an electron with the gyromagnetic ratio \( g = 2 \).

Below we are interested in the properties of ground states of the operator \( H^\pm_{\max} \).

For the analysis of operators \( H^\pm_{\max} \) the following description of the operators \( T^*_{\pm} \) will be useful. Namely, owing to condition (10) the differential operators \(-i\partial_x - ax\) and \(-i\partial_y - ay\) are well defined on the space of distributions \( \mathcal{D}'(\mathbb{R}^2 \setminus \Omega) \). Consequently, the operators \( T^*_{\pm} \) defined on \( C_0^\infty(\mathbb{R}^2 \setminus \Omega) \) can be naturally extended to linear mappings \( \tilde{T}^*_{\pm} \) defined on \( \mathcal{D}'(\mathbb{R}^2 \setminus \Omega) \). Using the fact that \( L^2(\mathbb{R}^2) \) is naturally embedded into \( \mathcal{D}'(\mathbb{R}^2 \setminus \Omega) \) we get the following lemma.
Lemma 5.4. The operator $T^*_\pm$ is a restriction of $\tilde{T}$ to the domain
\[ \{ f \in L^2(\mathbb{R}^2); \tilde{T} f \in L^2(\mathbb{R}^2) \} . \]

Using this observation we can prove the following lemma.

Lemma 5.5. Let $C$ be the operator of complex conjugation, $C f = \bar{f}$. Then $CH^\pm_{\text{max}}(a, \Omega) = H^\pm_{\text{max}}(-a, \Omega)C$ and $CH^\pm_{\text{min}}(a, \Omega) = H^\pm_{\text{min}}(-a, \Omega)C$.

Corollary 5.6. The operators $H^+_{\text{max}}(a, \Omega)$ and $H^-_{\text{max}}(-a, \Omega)$ have the same spectra. In particular, they have the same eigenvalues with equal multiplicities. An analogous proposition holds true for the couple of operators $H^+_{\text{min}}(a, \Omega)$ and $H^-_{\text{min}}(-a, \Omega)$.

6. Elimination of Aharonov–Bohm solenoids with integer fluxes

In this section we consider a vector potential $\tilde{a}$ of the form
\[ \tilde{a} = a + a_{AB} \]
where $a_{AB}$ is a vector potential corresponding to a system of Aharonov–Bohm solenoids intersecting the plane in the points of $\Omega$. We describe here briefly the ”gauge-periodicity” of the operators with the vector potential $\tilde{a}$; details can be found e.g. in [79, 80, 81, 82].

First we shall assume that the considered solenoids carry equal fluxes of the value $\theta_{AB}$. In this case we set $a_{AB} = \theta_{AB} \text{ sgrad } \ln(|W(z)|)$ (cf. Example 4 in Section 4). Let $\theta_{AB}$ be an integer. Then the function
\[ g(z, \bar{z}) = \exp\left( \theta_{AB} \ln\left( \frac{W(z)}{|W(z)|} \right) \right) = \exp\left( i \theta_{AB} \arg(W(z)) \right), \]
is well defined and continuous in the domain $\mathbb{C} \setminus \Omega$. Clearly, $|g(z, \bar{z})| = 1$, $\forall z \in \mathbb{C} \setminus \Omega$, and, moreover, $g \in C^\infty(\mathbb{R}^2 \setminus \Omega)$.

Lemma 6.1. If $\theta_{AB}$ is an integer then the following relations hold true
\[ g^{-1}P_x(\tilde{a}, \Omega)g = P_x(a, \Omega), \quad g^{-1}P_y(\tilde{a}, \Omega)g = P_y(a, \Omega). \]

Proof. It suffices to show that
\[ -i \text{ grad } g = g a_{AB}. \tag{31} \]
Actually, we have
\[
\frac{\partial}{\partial x} \ln\left( \frac{W(z)}{|W(z)|} \right) = \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) \left( \ln(W(z)) - \frac{1}{2} \left( \ln(W(z)) + \ln(\bar{W}(z)) \right) \right)
= i \text{ Im} \frac{W'(z)}{W(z)} = i \theta_{AB}^{-1} a_{AB,x}.
\]
Analogously,
\[
\frac{\partial}{\partial y} \ln \left( \frac{W(z)}{|W(z)|} \right) = i \Re \frac{W'(z)}{W(z)} = i \theta_{AB}^{-1} a_{AB,y}.
\]
Relation (31) obviously follows from these equalities. 

Assume now that \( a_{AB} \) is a vector potential corresponding to an Aharonov–Bohm field of finite type whose singular support coincides with \( \Omega = \Omega_1 \cup \ldots \cup \Omega_n \), and with an array of fluxes denoted by \( \Theta = (\theta_j)_{1 \leq j \leq n} \). Let \((m_j)_{1 \leq j \leq n}\) be an arbitrary array of integers and let \( \tilde{a}_{AB} \) be another Aharonov–Bohm potential of finite type defined by the same array of sets \((\Omega_j)\) but with the fluxes \( \tilde{\theta}_j = \theta_j + m_j \).

**Theorem 6.2.** Assume that \( a, a_{AB} \) and \( \tilde{a}_{AB} \) have the same meaning as described above. Then \( H_{\text{min}}^\pm (a + a_{AB}, \Omega) \) (respectively, \( H_{\text{max}}^\pm (a + a_{AB}, \Omega) \)) is unitarily equivalent to the operator \( H_{\text{min}}^\pm (a + \tilde{a}_{AB}, \Omega) \) (respectively, \( H_{\text{max}}^\pm (a + \tilde{a}_{AB}, \Omega) \)).

**Proof.** Let \( T_{1\pm} \), \( T_{2\pm} \) be the operators corresponding to the vector potentials \( a + a_{AB} \) and \( a + \tilde{a}_{AB} \), respectively, as described in Section 5. By construction,
\[
\mathcal{D}(T_{1\pm}) = \mathcal{D}(T_{2\pm}) = C_0^{\infty}(\mathbb{R}^2 \setminus \Omega).
\]
Applying repeatedly Lemma 6.1 one can show that there exists a unitary operator \( U \) such that
\[
U(C_0^{\infty}(\mathbb{R}^2 \setminus \Omega)) = C_0^{\infty}(\mathbb{R}^2 \setminus \Omega)
\]
and
\[
U^{-1}T_{2\pm}U = T_{1\pm}.
\]
From the unitarity of \( U \) it follows that
\[
U^{-1}T_{2\pm} = T_{1\pm} \quad \text{and} \quad U^{-1}T_{2 \pm}^*U = T_{1 \pm}^*.
\]
Consequently,
\[
U^{-1}H_{\text{min}}^\pm (a + a_{AB}, \Omega)U = U^{-1}T_{2 \pm}^*T_{2 \pm}U = T_{1 \pm}^*T_{1 \pm} = H_{\text{min}}^\pm (a + a_{AB}, \Omega)
\]
and
\[
U^{-1}H_{\text{max}}^\pm (a + a_{AB}, \Omega)U = U^{-1}T_{2 \mp}^*T_{2 \mp}U = T_{1 \mp}^*T_{1 \mp} = H_{\text{max}}^\pm (a + a_{AB}, \Omega).
\]
This shows the theorem. 

**Corollary 6.3.** If all fluxes \( \theta_j \) are integers then the operator \( H_{\text{min}}^\pm (a + a_{AB}, \Omega) \) (respectively, \( H_{\text{max}}^\pm (a + a_{AB}, \Omega) \)) is unitarily equivalent to the operator \( H_{\text{min}}^\pm (a, \Omega) \) (respectively, \( H_{\text{max}}^\pm (a, \Omega) \)).

Let us formulate separately two most important cases of this corollary. The first one is based on the fact that the both operators \( H_{\text{min}}^\pm (0, \Omega) \) and \( H_{\text{max}}^\pm (0, \Omega) \) do not depend on the choice of the discrete set \( \Omega \) and coincide with the Laplace operator \(-\Delta\).
Corollary 6.4. Let \( \Omega \) be a discrete set which is invariant with respect to a co-finite action of a lattice \( \Lambda \) of rank \( r \), \( 0 \leq r \leq 2 \). Assume that \( a = 0 \) and \( a_{AB} \) is a vector potential corresponding to a system of Aharonov–Bohm solenoids supported on the set \( \Omega \) and such that all fluxes are integers. Then each of the operators \( H_{\min}^\pm(a_{AB}, \Omega) \) and \( H_{\max}^\pm(a_{AB}, \Omega) \) is unitarily equivalent to the Laplace operator \(-\Delta\).

Proof. Since the action is co-finite we are again in the situation when \( \Omega \) splits into a finite union \( \Omega = \Omega_1 \cup \ldots \cup \Omega_n \). Hence one can apply Corollary 6.3. The unitary operator induced by multiplication with the function \( g \) acts locally in the form sense [93] and therefore each of the operators \( H_{\min}^\pm(0, \Omega) \) and \( H_{\max}^\pm(0, \Omega) \) is a point perturbation of \(-\Delta\) supported on the set \( \Omega \). The perturbed operator is clearly positive and local in the form sense [94]. On the other hand, every nontrivial point perturbation in the two-dimensional case is known to have a strictly negative infimum of the quadratic form over unit vectors [39]. \( \square \)

Since the minimum of spectrum in the case of a periodic point perturbation of the Landau operator is strictly smaller than the minimum of spectrum of the unperturbed operator [38], the following corollary is also true.

Corollary 6.5. Let \( a \) be a vector potential of a nonzero homogeneous magnetic field and assume again that the discrete set \( \Omega \) is invariant with respect to a co-finite action of a lattice \( \Lambda \) of rank \( r \), \( 0 \leq r \leq 2 \). Then for \( b > 0 \), each of the operators \( H_{\min}^+(a + a_{AB}, \Omega) \) and \( H_{\max}^+(a + a_{AB}, \Omega) \) is unitarily equivalent to the Landau operator \( H^+(a) \). For \( b < 0 \), an analogous statement is true for the operators \( H_{\min}^-(a + a_{AB}, \Omega) \) and \( H_{\max}^-(a + a_{AB}, \Omega) \).

To simplify the discussion to follow we shall assume once for all that an appropriate gauge transformation has been applied so that the values of all involved Aharonov–Bohm fluxes belong to the interval \([0, 1] \). If there are some zero values then \( \Omega \) is strictly larger then the singular support of \( b \). As shown by Corollaries [34] and [35] the zero values can be eliminated in some particular cases. We shall proceed in our simplifications even further. If not said otherwise, we assume everywhere in what follows that the values of Aharonov–Bohm fluxes belong to the interval \([0, 1]\) and, consequently, the singular support of \( b \) coincides with \( \Omega \).

7. The ground states (zero modes) of the Pauli operator

It follows immediately from the definition of the operators \( H_{\max}^\pm \) and \( H_{\min}^\pm \) that they are nonnegative. Consequently, if the equation

\[
H_{\min}^\pm \psi = 0
\]

(32)

or the equation

\[
H_{\max}^\pm \psi = 0
\]

(33)
has a solution in $L^2(\mathbb{R}^2)$ then this solution $\psi_\pm$ (called zero mode) is a ground state of the corresponding operator. Since the equality $H_{\text{min}}^\pm \psi = 0$ implies $\langle H_{\text{min}}^\pm | \psi \rangle = 0$, i.e., the equality $\| T_\pm \psi \|^2 = 0$, equation (32) is equivalent to the equality
\[ \mathcal{T}_\pm \psi = 0. \] (34)

Analogously, equation (33) is equivalent to the equality
\[ T_\pm^* \psi = 0, \] (35)
or, this is the same, to the condition
\[ \mathcal{T}_\pm \psi = 0, \quad \psi \in L^2(\mathbb{R}^2). \] (36)

Suppose that the vector potential $a$ was chosen to have the form $a = \text{sggrad} \, \varphi$ where $\varphi$ satisfies the equation $\Delta \varphi = b$ in the sense of distributions. We shall seek a solution of equation (36) in the form
\[ \psi_\pm(x, y) = \exp(\mp \varphi(x, y)) f(x, y) = \exp(\mp \varphi(z, \bar{z})) f(z, \bar{z}), \] (37)
where $f$ has to be chosen so that $\psi_\pm \in L^2(\mathbb{R}^2)$ (the Aharonov–Casher ansatz). In the space of distributions $\mathcal{D}'(\mathbb{R}^2 \setminus \Omega)$ we have
\[ T_\pm^* \psi_\pm = \mathcal{T}_\pm \psi_\pm = \exp(-\varphi) \left( (i \partial_x \varphi - a_y) f + (-\partial_y \varphi - a_x) f \right) - i (\partial_x f + i \partial_y f) \]
\[ = -2i \exp(-\varphi) \frac{\partial f}{\partial \bar{z}} \]
and
\[ T_\pm^* \psi_\pm = \mathcal{T}_\pm \psi_\pm = \exp(\varphi) \left( (i (-\partial_x \varphi + a_y) f + (-\partial_y \varphi - a_x) f \right) - i (\partial_x f - i \partial_y f) \]
\[ = -2i \exp(\varphi) \frac{\partial f}{\partial z}. \]

From here we deduce that the relation
\[ H_\text{max}^\pm \psi_\pm = 0, \quad \psi_\pm \in L^2(\mathbb{R}^2), \] (38)
is equivalent to the condition
\[ \frac{\partial f}{\partial \bar{z}} = 0 \quad (z \in \mathbb{C} \setminus \Omega), \quad \exp(-\varphi) f \in L^2(\mathbb{R}^2). \] (39)

Analogously, the relation
\[ H_\text{max}^\pm \psi_\pm = 0, \quad \psi_\pm \in L^2(\mathbb{R}^2), \] (40)
is equivalent to the condition
\[ \frac{\partial f}{\partial z} = 0 \quad (z \in \mathbb{C} \setminus \Omega), \quad \exp(\varphi) f \in L^2(\mathbb{R}^2). \] (41)

This shows the following theorem due to Aharonov and Casher [5].
Theorem 7.1. Assume that a vector potential \( a \) is expressed in the form \( a = \text{sgrad} \varphi \) where \( \varphi \) satisfies the equation \( \Delta \varphi = b \) in the sense of distributions. Then solutions of the equation \( H_{\text{max}}^+ \psi = 0 \) in \( L^2(\mathbb{R}^2) \) are exactly those functions from \( L^2(\mathbb{R}^2) \) which have the form \( \psi_+(z, \bar{z}) = \exp(-\varphi(z, \bar{z}))f(z) \) where \( f \) is a holomorphic function in the domain \( \mathbb{C} \setminus \Omega \).

Similarly, solutions of the equation \( H_{\text{max}}^- \psi = 0 \) in \( L^2(\mathbb{R}^2) \) are exactly those functions from \( L^2(\mathbb{R}^2) \) which have the form \( \psi_-(z, \bar{z}) = \exp(\varphi(z, \bar{z}))f(\bar{z}) \) where \( f \) is a holomorphic function in the domain \( \mathbb{C} \setminus \Omega \).

Let us point out an interesting consequence of the theorem.

Proposition 7.2. Assume that the both operators \( H_{\text{max}}^+ \) and \( H_{\text{max}}^- \) have zero modes. Then they are distinct. In particular, the set \( C_0^\infty(\mathbb{C} \setminus \Omega) \) is not a core for at least one of them.

Proof. Let \( \psi \) be a zero mode of \( H_{\text{max}}^+ \). Suppose that this operator coincides with \( H_{\text{max}}^- \). Then \( \psi \) is a zero mode for \( H_{\text{max}}^- \) as well. Using notation of Theorem 7.1 we have \( \psi = \exp(-\varphi)f = \exp(\varphi)g \) where \( f \) is holomorphic in the domain \( \mathbb{C} \setminus \Omega \) and \( g \) is antiholomorphic in the same domain. Since \( \varphi \) is real it holds true that \( |\psi|^2 = fg \).

Taking into account that \( g \) is holomorphic the last equality implies that \( fg \) is a constant function and hence the same is true for \( |\psi|^2 \). Since \( \psi \in L^2(\mathbb{R}^2) \) it follows that \( \psi = 0 \), a contradiction. \( \square \)

8. Zero modes of the operators \( H_{\text{max}}^\pm \) with Aharonov–Bohm potential of finite type

8.1. Formulation of the problem

In this section we shall study ground states of the operator \( H_{\text{max}}^\pm(a, \Omega) \) for an Aharonov–Bohm potential \( a \) of finite type determined by mutually disjoint discrete sets \( \Omega_1, \ldots, \Omega_n \) such that \( \Omega = \Omega_1 \cup \ldots \cup \Omega_n \), and by fluxes (not necessarily distinct) \( \theta_1, \ldots, \theta_n \) (cf. Example 4 from Section 4). Recall that we assume that \( 0 < \theta_j < 1 \), for all \( j \).

We can rephrase the formulation of the problem. Namely, according to Theorems 6.2 and 7.1 we have to study square integrability of a function \( \psi \) having the form

\[
\psi(z, \bar{z}) = f(z) \prod_{j=1}^n |W_j(z)|^{-\theta_j + m_j} \tag{42}
\]

where the numbers \( m_j \) are integers, \( f(z) \) is holomorphic or antiholomorphic in the domain \( \mathbb{C} \setminus \Omega \), and the functions \( W_j \) determine the potential \( a \) according to formula (16).

In this section the following lemma will be useful.
Lemma 8.1. Assume that a function $f(z)$ is expressible in an annulus $r_1 < |z| < r_2$ as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$  

Then for $r_1 < r < r_2$ and arbitrary $n \in \mathbb{Z}$ it holds true that

$$\int_{|z|=r} |f(z)| \, dz \geq 2\pi |a_n| r^{n+1}. \quad (43)$$

In particular, if $r_1 = 0$ and $n > -2$ then

$$\iint_{|z|<r} |f(z)| \, dx \, dy \geq \frac{2\pi |a_n|}{n+2} r^{n+2}, \quad (44)$$

if $r_1 = 0$ and $a_n \neq 0$ for some $n \leq -2$ then

$$\iint_{|z|<r} |f(z)| \, dx \, dy = \infty, \quad (45)$$

if $r_2 = \infty$ and $a_n \neq 0$ for some $n \geq -2$ then

$$\iint_{|z|>r} |f(z)| \, dx \, dy = \infty. \quad (46)$$

Proof. The proof immediately follows from the simple estimate

$$\int_{|z|=r} |f(z)| \, |dz| = r \int_{0}^{2\pi} |f(re^{i\varphi})| \, d\varphi \geq r \int_{0}^{2\pi} e^{-in\varphi} f(re^{i\varphi}) \, d\varphi = 2\pi |a_n| r^{n+1}. \quad \square$$

An application of Lemma 8.1 yields the following auxiliary result.

Lemma 8.2. Assume that in (42) it holds $m_j = 0$, for all $j$, i.e.,

$$\psi(z, \bar{z}) = f(z) \prod_{j=1}^{n} |W_j(z)|^\beta_j$$

where $-1 < \beta_j < 0$, for each $j$, and $f(z)$ is holomorphic in the domain $\mathbb{C} \setminus \Omega$. If $\psi \in L^2(\mathbb{R}^2)$ then $f$ is an entire function.

Proof. Assume that $\omega \in \Omega_k$ and $r > 0$ is sufficiently small so that $D(\omega, r) \cap \Omega = \{\omega\}$ where $D(\omega, r)$ is the disc with radius $r$ centered at the point $\omega$. Since $W_j(\omega) \neq 0$, for $j \neq k$, and $\omega$ is a simple zero of $W_k(z)$ one can assume that $r$ is small enough so that it holds

$$\prod_{j=1}^{n} |W_j(z)|^\beta_j \geq c > 0,$$
for $0 < |z - \omega| < r$.

Now one can show that $\omega$ cannot be a pole nor an essential singularity of $f$. Otherwise $\omega$ would be a pole of even order or an essential singularity of $f^2$. In any case Lemma 8.1 implies that

$$\int \int_{D(\omega, r)} |\psi(z, \bar{z})|^2 dxdy \geq c^2 \int \int_{D(\omega, r)} |f(z)|^2 dxdy = \infty,$$

a contradiction. \qed

8.2. Finite number of Aharonov–Bohm fluxes

We start from the simplest case, i.e., from the Hamiltonian $H_{\text{max}}^\pm (\Omega, \Theta)$ corresponding to a finite number of Aharonov–Bohm fluxes (Example 5 from Section 4). Then $\Omega = \{\omega_1, \ldots, \omega_n\}$ is a finite set, $\Theta = (\theta_1, \ldots, \theta_n)$, $0 < \theta_j < 1$. In this case zero modes may occur under suitable assumptions on fluxes $\theta_j$. More precisely, the following theorem is true [74].

**Theorem 8.3.** A sufficient and necessary condition for the operators $H_{\text{max}}^+ (\Omega, \Theta)$ and $H_{\text{max}}^- (\Omega, \Theta)$ to have zero modes is

$$\sum_{j=1}^{n} \theta_j > 1, \quad (47)$$

in the former case, and

$$\sum_{j=1}^{n} \theta_j < n - 1 \quad (48)$$

in the latter case.

**Proof.** Let us start from the operator $H_{\text{max}}^+$. We have to find a nonzero function $f$ which is holomorphic in the domain $\mathbb{C} \setminus \Omega$ and such that the function

$$\psi(z, \bar{z}) = f(z) \prod_{j=1}^{n} |z - \omega_j|^{-\theta_j} \quad (49)$$

is square integrable. Suppose that condition (47) is satisfied. Taking for $f$ a constant function it is easy to verify that in that case we get a square integrable function $\psi$.

Conversely, assume that $\psi$ is square integrable but condition (47) is false. Then from (49) one easily deduces that $f$ cannot be a nonzero constant. Furthermore, from the equality

$$f(z) = \psi(z, \bar{z}) \prod_{\theta_j \in \Theta} |z - \omega_j|^{\theta_j},$$

we find that there exists a constant $c_1 > 0$ such that

$$|f(z)| \leq c_1 (1 + |z|) |\psi(z, \bar{z})| \quad \text{on } \mathbb{C}.$$
Consequently, if \( r > 1 \) then

\[
\int_{|z| < r} |f(z)|^2 \, dx \, dy \leq c_2 r^2
\]

where

\[
c_2 = 4 c_1^2 \int_{\mathbb{R}^2} |\psi(z, \bar{z})|^2 \, dx \, dy.
\]

From inequality (43) in Lemma 8.1 it follows that \( f(z) \) is a constant function. This contradiction proves the theorem in the case of the operator \( H^+_{\text{max}} \). To prove the theorem in the case of the operator \( H^-_{\text{max}} \) one can either modify the above argument or to apply Corollary 5.6. \( \square \)

8.3. A chain of Aharonov–Bohm fluxes

Here we show that the Hamiltonians \( H^\pm_{\text{max}} \) corresponding to a finite union of chains of Aharonov–Bohm fluxes have infinitely many zero modes. In this subsection we use the notation from Example 6 in Section 4.

The proof uses the following elementary estimate. Since for \( z = x + iy, x, y \in \mathbb{R} \), we have \( |\sin(z)|^2 = \cosh^2(y) - \cos^2(x) = \sinh^2(y) + \sin^2(x) \), it holds true that \( |\sin(y)| \leq |\sin(z)| \leq \cosh(y) \). Hence

\[
e^{|y|} - \frac{1}{2} \leq |\sin(z)| \leq e^{|y|}.
\]

(50)

**Theorem 8.4.** Let a uniformly discrete set \( \Omega \) be expressible as a disjoint union of a finite number of chains \( \Omega_1, \ldots, \Omega_n \), and let the chain \( \Omega_j = K_j + \Lambda_j \) carry Aharonov–Bohm fluxes \( (\theta_\kappa)_{\kappa \in K_j} \) \( (j = 1, \ldots, n, 0 < \theta_\kappa < 1) \). Then the Hamiltonians \( H^\pm_{\text{max}}(\Omega) \) have infinitely degenerate zero modes.

**Proof.** In the proof we shall consider only the operator \( H^+_{\text{max}} \). Using Lemma A.1 we may assume that each chain \( \Omega_j \) is contained in a line \( L_j \) and that it holds \( L_j \neq L_k \) for \( j \neq k \). Then the Bravais lattice \( \Lambda_j \) of the chain \( \Omega_j \) has the form \( \Lambda_j = \{ k\omega_j; k \in \mathbb{Z} \} \), \( \omega_j \in \mathbb{C}, \omega_j \neq 0 \). Without loss of generality we can suppose that \( \omega_1 > 0 \) and \( \kappa_1 = 0 \). Hence \( L_1 = \mathbb{R} \) and \( L_j = \omega_j \mathbb{R} + \kappa_j \) \( (j = 2, \ldots, n) \) where \( \kappa_j \) is a fixed element from \( K_j \).

For each line \( L_j \) we shall construct a strip \( P_j \) with border lines parallel to \( L_j \) and containing \( L_j \) in its interior. Furthermore, let \( Q \) be a sufficiently large disk centered at \( 0 \) such that outside \( Q \) the strips \( P_j \) do not intersect each other.

It suffices to show that there exists an infinite number of linearly independent entire functions \( f(z) \) for which the function

\[
\psi(z, \bar{z}) = f(z)g_1(z, \bar{z}) \cdot \ldots \cdot g_n(z, \bar{z}),
\]

with

\[
g_j(z, \bar{z}) = \prod_{\kappa \in K_j} \left| \sin \left( \frac{\pi(z - \kappa)}{\omega_j} \right) \right|^{-\theta_\kappa},
\]

(51)
is square integrable. We shall show that this condition is satisfied for any function

\[ f(z) = \frac{\sin(\alpha z)}{z} \]

where

\[ 0 < \alpha < \theta := \frac{\pi}{\omega_1} \sum_{\kappa \in K_1} \theta_\kappa. \]  

(52)

The verification follows from a series of claims.

(A) \( \psi \in L^2(Q) \),

(B) each function \( g_j \) is bounded outside the strip \( P_j \),

(C) the function \( g_1(z, \bar{z}) \sin(\alpha z) \) is bounded outside the strip \( P_1 \).

Claim (A) follows from the fact that \( f \) is bounded on \( Q \) and that the functions \( g_j \) have square integrable singularities. Moreover, only finitely many singularities are contained in \( Q \). Claims (B) and (C) are consequences of the inequalities in (50) and condition (52).

To complete the proof it remains to show that

(D) \( \psi \in L^2(P_j \setminus Q), \forall j = 1, \ldots, n \),

(E) \( \psi \in L^2(\mathbb{R}^2 \setminus (P_1 \cup \ldots \cup P_n)) \).

To show Claim (D) notice that (B) and (C) imply the estimate

\[ |\psi(z, \bar{z})| \leq c_j \frac{|g_j(z, \bar{z})|}{|z|}, \]

valid on \( P_j \setminus Q \) with some constant \( c_j > 0 \), and that the function \( g_j(z, \bar{z}) \) is periodic along the line \( L_j \). For \( j = 1 \) one uses also that \( \sin(\alpha z) \) is bounded on the strip \( P_1 \).

To show Claim (E) let us point out that the inequality

\[ |\psi(z, \bar{z})| \leq c' \frac{\sin(\alpha z)}{z} \left| g_1(z, \bar{z}) \right| \]

holds true on \( \mathbb{C} \setminus (P_2 \cup \ldots \cup P_n) \) with some constant \( c' > 0 \), as it follows from (B). From inequalities (50) one derives that

\[ |\psi(z, \bar{z})| \leq c'' \frac{\exp\left((\alpha - \theta)|y|\right)}{|z|}. \]

on \( \mathbb{C} \setminus (P_1 \cup P_2 \cup \ldots \cup P_n) \). Finally, condition (52) implies that \( \psi \in L^2(\mathbb{R}^2 \setminus (P_1 \cup \ldots \cup P_n)) \).

Under more restrictive conditions on the fluxes \( \theta_\kappa \) the assumption on the uniform discreteness can be dropped. Since every chain is a union of one-atom chains we can confine ourselves to such chains. Moreover, it is clear that a union of chains need not be a uniformly discrete set only in the case when among the chains in question there
are at least two contained in the same line. Consequently, it suffices to analyze the case when the chains are contained in a single line, say, in the real axis $\mathbb{R}$. Suppose that $\Omega_j = \kappa_j + \Lambda_j$ where $\Lambda_j = \{\omega_j k; k \in \mathbb{Z}\}$, $\kappa_j \in \mathbb{R}$, $\omega_j > 0$ ($j = 1, \ldots, n$), are mutually disjoint one-atom chains and $\theta_1, \ldots, \theta_n$ ($0 < \theta_j < 1$) are the corresponding Aharonov–Bohm fluxes.

**Theorem 8.5.** Assume that all chains $\Omega_j$, $j = 1, \ldots, n$, are contained in $\mathbb{R}$. Then the Hamiltonian $H^+_{\text{max}} = H^+_{\text{max}}(\Omega_1, \ldots, \Omega_n; \theta_1, \ldots, \theta_n)$ has an infinitely degenerate zero mode if one of the following conditions is satisfied:

1. $\theta_1 + \ldots + \theta_n < 1$,
2. $\frac{\theta_1}{\omega_1} + \ldots + \frac{\theta_n}{\omega_n} > \frac{1}{\omega_1} + \ldots + \frac{1}{\min \omega_j}$,
3. $n = 2$.

**Proof.** (i) In this case one can choose numbers $p_j > 1$ so that $p_j \theta_j < 1$, $\forall j = 1, \ldots, n$, and $\sum_{j=1}^n p_j = 1$ (e.g., $p_j = \theta_j/(\theta_1 + \ldots + \theta_n)$). Let us consider the functions

$$g_j(z, \bar{z}) = \frac{\sin(\alpha_j(z - \kappa_j))}{z - \kappa_j} \left| \sin \left( \frac{\pi}{\omega_j} \frac{z - \kappa_j}{\omega_j} \right) \right|^{-\theta_j},$$

where

$$0 < \alpha_j < \min \frac{\pi}{\omega_j} \theta_j.$$ 

Set $\psi = g_1 \cdots g_n$. It suffices to show that $\psi$ is square integrable. From the Jensen’s inequality it follows that

$$|\psi|^2 \leq \frac{|g_1|^{2p_1}}{p_1} + \ldots + \frac{|g_n|^{2p_n}}{p_n}.$$ 

(53)

Recalling that $p_j \theta_j < 1$, $p_j > 1$, and repeating the considerations from the proof of Theorem 8.4 one can show that each summand on the RHS of formula (53) is integrable.

Let us now discuss condition (ii). One can assume that $\min_j \omega_j = \omega_1$ and $\kappa_1 = 0$. We shall consider a function $\psi$ of the form

$$\psi(z, \bar{z}) = \frac{\sin(\alpha z)}{z} g_1(z, \bar{z}) \cdots g_n(z, \bar{z})$$

where

$$g_1(z, \bar{z}) = \left| \sin \left( \frac{\pi z}{\omega_1} \right) \right|^{-\theta_1},$$

(54)

$$g_j(z, \bar{z}) = \left| \sin \left( \frac{\pi(z - \kappa_j)}{\omega_j} \right) \right|^{1-\theta_j},$$

(55)
for \( j = 2, \ldots, n \), and \( \alpha \) obeys the condition

\[
0 < \alpha < \theta := \frac{\pi}{\omega_1} - \pi \sum_j \frac{1 - \theta_j}{\omega_j}.
\]  

(56)

Let \( T \) be a strip parallel to the real line \( \mathbb{R} \) and containing \( \mathbb{R} \) in its interior. One again concludes from (50) that outside the strip \( T \) it holds true that

\[
|\psi(z, \bar{z})| \leq c_1 \exp((\alpha - \theta)|y|).
\]

Furthermore, inside \( T \) we can use the estimate

\[
|\psi(z, \bar{z})| \leq c_2 \frac{\sin(\alpha z)}{z} |g_1(z, \bar{z})|.
\]

Therefore one can use a similar reasoning as in the proof of Theorem 8.4 to show that \( \psi \in L^2(\mathbb{R}^2) \).

Finally, let us discuss condition (iii). If \( \omega_1 = \omega_2 \) then one can refer to Theorem 8.4. In the opposite case we shall assume that \( \omega_1 < \omega_2 \). If \( \theta_1 + \theta_2 < 1 \) then we apply condition (i) from the theorem. If not then we have

\[
\frac{\theta_1}{\omega_1} + \frac{\theta_2}{\omega_2} > \frac{\theta_1 + \theta_2}{\omega_2} \geq \frac{1}{\omega_2} - \frac{1}{\min_j \omega_j},
\]

and we can apply condition (ii).

According to Lemma 5.5 we can reformulate the result for the operator \( H^{-}_{\text{max}} \) as follows.

**Theorem 8.6.** Assume that all chains \( \Omega_j, j = 1, \ldots, n \), are contained in \( \mathbb{R} \). Then the Hamiltonian \( H^{-}_{\text{max}} = H^{-}_{\text{max}}(\Omega_1, \ldots, \Omega_n; \theta_1, \ldots, \theta_n) \) has an infinitely degenerate zero mode if one of the following conditions is satisfied:

(i) \( \theta_1 + \ldots + \theta_n > n - 1 \),

(ii) \( \frac{\theta_1}{\omega_1} + \ldots + \frac{\theta_n}{\omega_n} < \frac{1}{\min_j \omega_j} \),

(iii) \( n = 2 \).

**8.4. A lattice of Aharonov–Bohm fluxes**

Let us now consider the Hamiltonian \( H^{\pm}_{\text{max}} \) for a lattice of Aharonov–Bohm solenoids \( \Omega = K + \Lambda \) where \( \Lambda \) is the Bravais lattice of the crystallographic lattice \( \Omega \) with a basis \( \{\omega_1, \omega_2\} \) (cf. Example 7 in Section 4). To analyze this case we shall use the Weierstrass \( \sigma \)-function \( \sigma(z) \equiv \sigma(z; \omega_1, \omega_2) \). Let us introduce, following [95, 96], the modified Weierstrass \( \sigma \)-function \( \tilde{\sigma}(z) \),

\[
\tilde{\sigma}(z) = e^{-\nu z^2} \sigma(z)
\]

(57)
where
\[ \nu = \frac{i}{4S}(\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1), \quad \eta_j = 2 \zeta\left(\frac{\omega_j}{2}\right), \quad S = \text{Im}(\bar{\omega}_1 \omega_2), \]
and \( \zeta(z) \) is the Weierstrass \( \zeta \)-function (cf. Appendix C). We shall need the following lemma. The number \( \mu \), occurring in the formulation of lemma and depending on the lattice \( \Lambda \), is defined by
\[
\mu = \frac{\pi}{2S}.
\]

**Lemma 8.7.** Let \( \alpha_j, j = 1, \ldots, n \), be real numbers such that \( 0 < \alpha_j < 1 \), let \( \beta \) be an arbitrary real number, and let \( a_j, j = 1, \ldots, n \), be an arbitrary array of complex numbers such that among them there is no couple congruent modulo \( \Lambda \). If the condition
\[
\beta < \mu \sum_{j=1}^n \alpha_j
\]
is satisfied then
\[
\exp(\beta|z|^2) \prod_{j=1}^n |\bar{\sigma}(z-a_j)|^{-\alpha_j} \in L^2(\mathbb{R}^2).
\]

*Proof.* We shall consider a shifted elementary cell of the lattice \( \Lambda \),
\[
L_\varepsilon = \{(t_1 + \varepsilon)\omega_1 + (t_2 + \varepsilon)\omega_2; \ 0 \leq t_1, t_2 < 1\},
\]
where \( \varepsilon > 0 \) is chosen so that the interior of \( L_\varepsilon \) contains exactly one zero for each of the functions \( \bar{\sigma}(z-a_j) \), and hence exactly one pole of the function \( 1/\bar{\sigma}(z-a_j) \). But in that case,
\[
\int_{L_\varepsilon} \left| \prod_{j=1}^n \bar{\sigma}(z-a_j) \right|^{-2\alpha_j} dxdy < \infty.
\]

Let \( \rho(z, \bar{z}) \) be a function defined by the formula
\[
|\sigma(z)|^2 = \exp(\nu z^2 + \bar{\nu} \bar{z}^2 + 2\mu z \bar{z}) \rho(z, \bar{z}),
\]
From formula (60) we deduce that
\[
I := \int\int_{L_\varepsilon} \prod_{j=1}^n |\rho(z-a_j, \bar{z}-\bar{a}_j)|^{-\alpha_j} dxdy < \infty.
\]
Since the function \( \rho(z, \bar{z}) \) is \( \Lambda \)-periodic and \( |\bar{\sigma}(z)|^2 = \exp(2\mu |z|^2) \rho(z, \bar{z}) \) (see Lemma C.1), it holds true that
\[
\int\int_{\mathbb{R}^2} \exp(2\beta|z|^2) \left| \prod_{j=1}^n \bar{\sigma}(z-a_j) \right|^{-2\alpha_j} dxdy
\]
\[
= \sum_{\lambda \in \Lambda} \int\int_{L_\varepsilon + \lambda} \exp\left(2(\beta - \mu \sum \alpha_j)|z|^2\right) \left| \prod_{j=1}^n \rho(z-a_j, \bar{z}-\bar{a}_j) \right|^{-\alpha_j} dxdy
\]
\[
\leq I \sum_{\lambda \in \Lambda} \sup \left\{ \exp\left(2(\beta - \mu \sum \alpha_j)|z|^2\right); \ z \in L_\varepsilon + \lambda \right\} < \infty.
\]
\]
**Theorem 8.8.** Let \( \Omega = K + \Lambda \) be a lattice of Aharonov–Bohm solenoids with an array of fluxes \( \Theta = (\theta_\kappa)_{\kappa \in K}, 0 < \theta_\kappa < 1 \). Then each of the operators \( H_{\max}^\pm(\Omega; \Theta) \) has an infinitely degenerate zero mode.

**Proof.** We shall confine ourselves to the case of the operator \( H_{\max}^+ \). Let us consider the function

\[
\psi(z, \bar{z}) = f(z) \prod_{\kappa \in K} |\tilde{\sigma}(z - \kappa)|^{-\theta_\kappa}.
\]

According to Lemma 8.7, \( \psi \in L^2(\mathbb{R}^2) \) if \( f \) is an arbitrary polynomial. \( \square \)

**Remark 8.9.** Owing to Theorem 8.8, we can describe an interesting example related to the question of absolutely continuous spectrum for the Pauli operator \( H_{\max}^\pm(a) \) with a magnetic field \( b = \partial_x a_y - \partial_y a_x \) which is supposed to be periodic with respect to a lattice \( \Lambda = \{k_1 \omega_1 + k_2 \omega_2; k_1, k_2 \in \mathbb{Z}\} \), with \( S = \text{Im}(\bar{\omega}_1 \omega_2) > 0 \). If the vector potential \( a \) is “sufficiently regular” and the flux of the field \( b \) through the elementary cell equals zero then the spectrum of the operators \( H_{\max}^\pm(a) \) is purely absolutely continuous (see 23, 24, 25, 26, 27, 28 and others). The same result is true for Schrödinger operators with “sufficiently regular” periodic vector potentials in the space \( L^2(\mathbb{R}^d) \) for any \( d \geq 2 \). In the case \( d \geq 3 \), N. D. Filonov described in 29 an example showing that the assumptions on the vector potential stated in 28 and other papers cannot be essentially weakened. Theorem 8.8 shows that two-dimensional Pauli operators with a singular two-periodic magnetic field may have (infinitely degenerate) eigenvalues. In more detail, let us take, for example, a set \( K \) containing two elements, \( K = \{\kappa_1, \kappa_2\} \), and suppose that \( \theta \equiv \theta_{\kappa_1} = -\theta_{\kappa_2} \in [0, 1[ \). Then by Theorem 8.8 the both operators \( H_{\max}^\pm(a) \) have an eigenvalue, namely the number zero. According to Example 7 from Section 11 the corresponding vector potential \( a \) reads \( a_x = \theta \text{Im}(\zeta(z - \kappa_1) - \zeta(z - \kappa_2)), a_y = \theta \text{Re}(\zeta(z - \kappa_1) - \zeta(z - \kappa_2)) \). Owing to quasi-periodicity of the Weierstrass function \( \zeta(z), \zeta(z + \omega_j) = \zeta(z) + \eta_j \) where \( \eta_j = 2 \zeta(\omega_j/2) \), the vector potential \( a \) is \( \Lambda \)-periodic.

Now we state an analog of Theorem 8.4 for lattices of Aharonov–Bohm fluxes. In view of the example described in Remark A.2 we cannot here repeat the arguments from the proof of Theorem 8.4 but the properties of the modified Weierstrass \( \sigma \)-function simplify matters considerably.

**Theorem 8.10.** Let a uniformly discrete set \( \Omega \) be expressible as a disjoint union of a finite number of lattices \( \Omega_1, \ldots, \Omega_n \), and let the lattice \( \Omega_j = K_j + \Lambda_j \) carry Aharonov–Bohm fluxes \( (\theta_\kappa)_{\kappa \in K_j} \) \( (j = 1, \ldots, n, 0 < \theta_\kappa < 1) \). Then the Hamiltonians \( H_{\max}^\pm(\Omega) \) have infinitely degenerate zero modes.

**Proof.** In the proof we shall consider only the operator \( H_{\max}^+ \). Without loss of generality we suppose that each \( K_j \) is a singleton: \( K_j = \{\kappa_j\} \), and we shall write \( \theta_j \) instead of \( \theta_{\kappa_j} \). By the hypothesis, there is a sufficiently small disk \( D \) centered at 0 such that for \( \omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2 \), the sets \( D + \omega_1 \) and \( D + \omega_2 \) are disjoint. Denote \( L_j := D + \kappa_j + \Lambda_j \), then for every \( j \) there exists \( c_j > 0 \) such that \( |\tilde{\sigma}(z - \kappa_j)|^{-\theta_j} \leq c_j \) for \( z \notin L_j \). It is clear that

\[
\prod_{j=1}^n |\tilde{\sigma}(z - a_j)|^{-\theta_j} \leq \sum_{j=1}^n \left( \prod_{k \neq j} c_k \right) |\tilde{\sigma}(z - a_j)|^{-\theta_j}.
\]
Now we can refer to Lemma 8.7.

Analogs of Theorem 8.5 and Theorem 8.6 are also valid and can be proved by the same method. In more detail, let \( \Omega_j, j = 1, \ldots, n \), be mutually disjoint simple crystallographic lattices, \( \Omega_j = \kappa_j + \Lambda_j \) where \( \kappa_j \in \mathbb{C}, \Lambda_j = \{k_1 \omega_1^{(j)} + k_2 \omega_2^{(j)}; \ k_1, k_2 \in \mathbb{Z}\} \). Furthermore, \( S_j = \text{Im}(\omega_1^{(j)} \omega_2^{(j)}) \) designates the area of the elementary cell of the Bravais lattice \( \Lambda_j \).

**Theorem 8.11.** The Hamiltonian \( H_{\text{max}}^+ = H_{\text{max}}^+(\Omega_1, \ldots, \Omega_n; \theta_1, \ldots, \theta_n) \) has an infinitely degenerate zero mode if one of the following conditions is satisfied:

(i) \( \theta_1 + \ldots + \theta_n < 1 \),

(ii) \( \frac{\theta_1}{S_1} + \ldots + \frac{\theta_n}{S_n} > \frac{1}{S_1} + \ldots + \frac{1}{S_n} - \frac{1}{\min_j S_j} \),

(iii) \( n = 2 \) and \( S_1 \neq S_2 \).

**Theorem 8.12.** The Hamiltonian \( H_{\text{max}}^- = H_{\text{max}}^-(\Omega_1, \ldots, \Omega_n; \theta_1, \ldots, \theta_n) \) has an infinitely degenerate zero mode if one of the following conditions is satisfied:

(i) \( \theta_1 + \ldots + \theta_n > n - 1 \),

(ii) \( \frac{\theta_1}{S_1} + \ldots + \frac{\theta_n}{S_n} < \frac{1}{\min_j S_j} \),

(iii) \( n = 2 \) and \( S_1 \neq S_2 \).

### 8.5. Superposition of a homogeneous magnetic field with a field corresponding to Aharonov–Bohm solenoids

Here we consider a perturbation of a homogeneous magnetic field by the field corresponding to a system of Aharonov–Bohm solenoids, i.e., we consider a vector potential \( a \) of the form \( a = a_0 + a_{AB} \) where \( a_0 \) is the vector potential of a homogeneous magnetic field \( b_0 = 2\pi \xi_0 \) with a flux density \( \xi_0, a_0 = \pi \xi_0 (-y, x) \), and \( a_{AB} \) is the vector potential of a system of Aharonov–Bohm fluxes. We shall suppose that the potential \( a_{AB} \) is of finite type. In that case we have a finite family of mutually disjoint discrete subsets in the complex plane, \( \Omega_1, \ldots, \Omega_n \), and in each point of the set \( \Omega_j \ (j = 1, \ldots, n) \) there is a flux of magnitude \( \theta_j \ (0 < \theta_j < 1) \) intersecting the plane.

Suppose for definiteness that \( b_0 > 0 \). Then the operator \( H_{\text{max}}^+(a_0) \) has an infinitely degenerate zero mode (the lowest Landau level shifted by the value \(-b_0\)) while the ground state of the operator \( H_{\text{max}}^-(a_0) \) is strictly positive (this is the lowest Landau level shifted by the value \( b_0 \)). Thus the latter operator has no zero mode. Intuitively, the results proved below in Theorems 8.13 and 8.16 mean that if the set \( \Omega = \Omega_1 \cup \ldots \cup \Omega_n \) has a finite density then a superposition with the potential \( a_{AB} \) does not remove the zero mode from the spectrum of the operator \( H_{\text{max}}^+ \), and a zero mode cannot occur in the spectrum of the operator \( H_{\text{max}}^- \) provided \( b_0 \) is *sufficiently large*. Moreover, if this set has zero density then the same statement about zero modes of the operators \( H_{\text{max}}^+ \) and \( H_{\text{max}}^- \).
and $H^+_{\text{max}}$ is true for any $b_0 > 0$. In the case when $\Omega$ is a lattice, a superposition with the potential $a_{AB}$ does not remove the zero mode from the spectrum of $H^+_{\text{max}}$ for any $b_0 > 0$ but a zero mode may occur in the spectrum of $H^-_{\text{max}}$ for particular values of fluxes $\theta_j$. An attentive reader can effortlessly guess what happens for $b_0 < 0$.

Let $\alpha$ be an arbitrary positive number. For any $r > 0$ we denote

$$S(r) = \sum_{\omega \in \Omega, 0 < |\omega| \leq r} \omega^{-\alpha},$$

$$T(r) = \sum_{\omega \in \Omega, 0 < |\omega| \leq r} |\omega|^{-\alpha},$$

$$n(r) = \#\{\omega \in \Omega; |\omega| \leq r\}.$$  

(63) (64) (65)

**Theorem 8.13.** Suppose that $a = a_0 + a_{AB}$ and that the Aharonov–Bohm vector potential $a_{AB}$ is of finite type. Let the following conditions be satisfied: (a) for any $\alpha > 2$, the sums $T(r)$ are uniformly bounded, (b) $n(r) = O(r^2)$, (c) the sums $S(r)$ are uniformly bounded for $\alpha = 2$. Then, for sufficiently large $b_0 > 0$, the Hamiltonian $H^+_{\text{max}}(a)$ has an infinitely degenerate zero mode and $H^-_{\text{max}}(a)$ has no zero mode.

If for $\alpha = 2$ the sums $T(r)$ are uniformly bounded then, for any $b_0 > 0$, the Hamiltonian $H^+_{\text{max}}(a)$ has an infinitely degenerate zero mode and $H^-_{\text{max}}(a)$ has no zero mode.

In the case when $b_0 < 0$ the same claim remains true when interchanging the role of $H^+_{\text{max}}(a)$ and $H^-_{\text{max}}(a)$.

**Proof.** Let us consider the operator $H^+_{\text{max}}(a)$. In view of Theorem 7.1, we can assume that its zero mode, if any, has the form

$$\psi(z, \bar{z}) = f(z) \exp\left(-\frac{1}{2} \pi \xi_0 |z|^2\right) \prod_{j=1}^n |W_j|^{-\theta_j}$$

(66)

where $W_j(z) = W_{\Omega_j}(z)$ is the Weierstrass canonical product for the set $\Omega_j$ (see Appendix B) and $f(z)$ is a nonzero entire function (cf. Lemma 8.2).

Let assumptions (a), (b), (c) be satisfied. Then, according to the Borel theorem and to the Lindelöf theorem (cf. Theorems B.1 and B.4 in Appendix), every function $W_j(z)$ has order 2 and a finite type, i.e., $|W_j(z)| \leq a_j \exp(c_j |z|^2)$ with some constants $a_j, c_j > 0$. It follows that for

$$\frac{b_0}{4} > c_1(1 - \theta_1) + \ldots + c_n(1 - \theta_n)$$

(67)

the function (66) is square integrable if we set $f(z) = p(z) \prod_{j=1}^n W_j(z)$ where $p(z)$ is an arbitrary polynomial.

If the functions $T(r)$ are uniformly bounded for $\alpha = 2$ then $p_{\Omega} \leq 1$ (see (101)) and, according to Theorem B.5 from Appendix, the functions $W_j(z)$ are of minimal type and so the constants $c_j$ can be chosen arbitrarily small. Consequently, the restriction on the field $b_0 > 0$ is not necessary anymore.
In the case of the operator \( H_{\text{max}}^- (a) \) we have to discuss the function

\[
\psi(z, \bar{z}) = f(z) \exp \left( \frac{1}{2} \pi \xi_0 |z|^2 \right) \prod_{j=1}^{n} |W_j|^\theta_j^{-1} .
\]  

(68)

If assumptions (a), (b), (c) are satisfied then \( |W_j(z)|^\theta_j^{-1} \geq a_j \exp(c_j(\theta_j - 1)|z|^2) \) with some constants \( a_j, c_j > 0 \). Consequently, for \( b_0 \) obeying (67), \( R > 0 \) sufficiently large and for some \( c > 0 \) we have the inequality \( |\psi(z)|^2 \geq c|f(z)|^2 \) if \( |z| \geq R \). But then, as one deduces from Lemma 8.1, \( \psi \) is not square integrable, hence \( H_{\text{max}}^- (a) \) has no zero modes.

Obviously, changing the sign at \( b_0 \) means that \( H_{\text{max}}^+ (a) \) and \( H_{\text{max}}^- (a) \) interchange their roles in the above considerations.

Remark 8.14. If conditions (a), (b), (c) from Theorem 8.13 are satisfied then \( \Omega \) has finite density, i.e., \( \limsup_{r \to \infty} n(r)/r^2 < \infty \). If, in addition, the sums \( T(r) \) are uniformly bounded for \( \alpha = 2 \) then the density of the set \( \Omega \) is zero (see inequalities (106)).

Remark 8.15. All assumptions of Theorem 8.13 are fulfilled if every set \( \Omega_j \) is either finite or a union of chains.

Let now \( \Omega \) be a lattice. Using the above introduced notation we write \( \Omega_j = \kappa_j + \Lambda \) where \( \Lambda = \{k_1 \omega_1 + k_2 \omega_2; \ k_1, k_2 \in \mathbb{Z}\} \). Suppose that \( S = \text{Im}(\omega_1 \omega_2) > 0 \) (\( S \) is the area of an elementary cell in the lattice \( \Lambda \)). Let \( \eta_0 = \xi_0 S \) designate the flux of the homogeneous component of the field through the elementary cell of the lattice \( \Lambda \).

Theorem 8.16. Suppose that \( a = a_0 + a_{AB} \) and that \( \Omega \) is a lattice. Let \( b_0 > 0 \). Then the Hamiltonian \( H_{\text{max}}^+ (a) \) has an infinitely degenerate zero mode. The inequality

\[
\eta_0 + \sum_{j=1}^{n} \theta_j < n
\]

is a sufficient and necessary condition for \( H_{\text{max}}^- (a) \) to have a zero mode, and if it is fulfilled then the zero mode is infinitely degenerate.

Let \( b_0 < 0 \). Then the Hamiltonian \( H_{\text{max}}^- (a) \) has an infinitely degenerate zero mode. The inequality

\[
|\eta_0| < \sum_{j=1}^{n} \theta_j
\]

is a sufficient and necessary condition for \( H_{\text{max}}^+ (a) \) to have a zero mode, and if it is fulfilled then the zero mode is infinitely degenerate.

Proof. We shall start the proof from the operator \( H_{\text{max}}^- \). In analogy with the proof of Theorem 8.8 we consider the function

\[
\psi(z, \bar{z}) = f(z) \exp \left( -\frac{\pi \eta_0}{2S} |z|^2 \right) \prod_{j=1}^{n} |\tilde{\sigma}(z - \kappa_j)|^{-\theta_j}
\]  

(69)
(recall that \( \eta_0 = \xi_0 S \)). From Lemma 8.7 we immediately deduce that for an arbitrary polynomial \( f \) it holds true that \( \psi \in L^2(\mathbb{R}^2) \), hence \( \psi \) is an infinitely degenerated zero mode.

Let us now turn to the operator \( H_{\max}^- \). According to Theorem 6.2 and Lemma 8.2, the ground state of the operator \( H_{\max}^- \), if any, has the form

\[
\psi(z, \bar{z}) = f(\bar{z}) \exp\left(\frac{\pi \eta_0}{2S} |z|^2\right) \prod_{j=1}^{n} |\tilde{\sigma}(z - \kappa_j)|^{\theta_j - 1} \tag{70}
\]

where \( f(z) \) is an entire function. Using formula (126) from Appendix C (where \( \mu = \pi / 2S \)) we get

\[
|\psi(z, \bar{z})|^2 = |f(\bar{z})|^2 \exp\left(\frac{\pi \eta_0}{S} |z|^2\right) \prod_{j=1}^{n} \exp\left(\frac{\pi}{S} (\theta_j - 1) |z|^2\right) \\
\times \prod_{j=1}^{n} |\rho(z - \kappa_j, \bar{z} - \bar{\kappa}_j)|^{2(\theta_j - 1)} \tag{71}
\]

\[
= |f(\bar{z})|^2 \exp(c |z|^2) \prod_{j=1}^{n} |\rho(z - \kappa_j, \bar{z} - \bar{\kappa}_j)|^{2(\theta_j - 1)}
\]

where

\[
c = \frac{\pi}{S} \left( \eta_0 + \sum_{j=1}^{n} (\theta_j - 1) \right). \]

The condition \( \eta_0 + \sum_{j=1}^{n} \theta_j < n \) is equivalent to the condition \( c < 0 \). But in that case the membership \( \psi \in L^2(\mathbb{R}^2) \) can be proved as in Lemma 8.7.

Conversely, assume that \( c \geq 0 \) and that there exists a non-zero entire function \( f(z) \) such that \( \psi \in L^2(\mathbb{R}^2) \). Then from (71) we derive that \( |f(\bar{z})|^2 \leq c_1 |\psi(z, \bar{z})|^2 \) with some constant \( c_1 \). Consequently, \( f \in L^2(\mathbb{R}^2) \) which contradicts Lemma 8.1.

In the case \( b_0 < 0 \) one can either repeat the above reasoning or apply Corollary 5.6 while noticing that \( \{-x\} = 1 - \{x\} \) for any \( x \in \mathbb{R} \) which is not an integer.

**Remark 8.17.** Similarly to the case \( b_0 = 0 \) (cf. Remark 8.9), both operators \( H_{\max}^\pm(a) \) may have localized states also when the total flux through the elementary cell is zero. Suppose, for example, that \( b_0 > 0, 0 < \theta_1 < 1, 0 < \eta_0 + \theta_1 < 1 \) and \( \theta_2 = 1 - \eta_0 - \theta_1 \). Then \( \eta_0 + \theta_1 + \theta_2 = 1 < 2 \) and the assumption of the theorem is satisfied.

**Remark 8.18.** Theorem 8.16 shows that, for \( b_0 > 0 \), an oscillation of the type “localization–delocalization” occurs after adding an Aharonov–Bohm flux to a system with the Hamiltonian \( H_{\max}^- \).
9. Conservation of zero modes under translations and additions of Aharonov–Bohm solenoids. Irregular Aharonov–Bohm systems

9.1. Translation and addition of finitely many Aharonov–Bohm solenoids

Up to now we have investigated zero modes of regular Aharonov–Bohm systems (in the sense of the definition given at the end of Section 4), with Theorem 8.13 representing the only exception. The proof of this theorem suggests that one should expect zero modes also in the case when the homogeneous component of the magnetic field is absent provided the perturbation corresponds to a (in general, irregular) “sufficiently scarce” system. Further we shall consider such scarce perturbations applied to systems of chains or lattices of Aharonov–Bohm solenoids. Before addressing this question we shall prove that the zero mode of the Hamiltonian corresponding to a system of solenoids of finite type does not disappear if a finite number of solenoids are moved or if one joins a finite number of solenoids to the system.

In the followings theorems, \( a \) designates a potential of finite type corresponding to a system of Aharonov–Bohm solenoids which is determined by an array of mutually disjoint discrete sets, \( \Omega_1, \ldots, \Omega_n \), and by an array of fluxes, \( \theta_1, \ldots, \theta_n \) (0 < \( \theta_j \) < 1). \( \Omega \) designates the union \( \Omega = \Omega_1 \cup \cdots \cup \Omega_n \).

**Theorem 9.1.** In addition to the above introduced notation let \( K_j = \{\omega_{1,j}, \ldots, \omega_{n,j}\} \) be a finite subset of \( \Omega_j \) and let \( K'_j = \{\omega'_{1,j}, \ldots, \omega'_{n,j}\} \) be a finite subset of \( \mathbb{C} \) such that the sets \( \Omega'_j = (\Omega_j \setminus K_j) \cup K'_j \), \( j = 1, \ldots, n \), are mutually disjoint. If the operator \( H_{\pm \text{max}}^+(a) \) has a zero mode then the operator \( H_{\pm \text{max}}^+(a') \) determined by the array \( (\Omega'_1, \ldots, \Omega'_n; \theta_1, \ldots, \theta_n) \) has also a zero mode with the same multiplicity as that of the zero mode for the operator \( H_{\pm \text{max}}^+(a) \).

**Proof.** We shall confine ourselves to the discussion of the operator \( H_{\text{max}}^+(a) \). The zero modes of this operator can be written in the form

\[
\psi(z, \bar{z}) = f(z) \prod_{j=1}^n |W_j(z)|^{-\theta_j}
\]

where \( f \) is an entire function and \( W_j(z) = W_{\Omega_j}(z) \) is the Weierstrass canonical product for \( \Omega_j \). Then the function

\[
\tilde{\psi}(z, \bar{z}) = f(z) \prod_{j=1}^n \left( |W_j(z)|^{-\theta_j} \prod_{k=1}^{n_j} \left| \frac{z - \omega_{kj}}{z - \omega'_{kj}} \right|^{\theta_j} \right)
\]

represents a zero mode of \( H_{\text{max}}^+(a') \). One has only to verify that \( \tilde{\psi} \in L^2(\mathbb{R}^2) \). It is actually so because the additional singularities at the points \( \omega'_{kj} \) are square integrable and outside a compact set \( \tilde{\psi} \) differs from \( \psi \) by a bounded factor.
This argument clearly shows that the multiplicity of the zero mode for the operator $H^\pm_{\text{max}}(\mathbf{a}')$ is not smaller than the multiplicity of the zero mode for the operator $H^\pm_{\text{max}}(\mathbf{a})$. Since the operators play an equivalent role in the assumptions the converse is also true.

**Theorem 9.2.** Assume that additionally to the considered system of solenoids there are given a finite set $\Omega' = \{\omega'_1, \ldots, \omega'_m\} \subset \mathbb{C}$ not intersecting $\Omega$ and a corresponding family of fluxes $\{\theta'_1, \ldots, \theta'_m\}$ $(0 < \theta'_j < 1)$. Let $\mathbf{a}'$ be the vector potential determined by the array of sets $\Omega'_1, \ldots, \Omega'_n, \Omega'$, and by the array of fluxes $\theta_1, \ldots, \theta_n, \theta'_1, \ldots, \theta'_m$. If the operator $H^\pm_{\text{max}}(\mathbf{a})$ has a zero mode then the operator $H^\pm_{\text{max}}(\mathbf{a}')$ also has a zero mode whose multiplicity is not smaller than the multiplicity of the zero mode for the operator $H^\pm_{\text{max}}(\mathbf{a})$.

**Proof.** We shall confine ourselves to the discussion of the operator $H^\pm_{\text{max}}(\mathbf{a})$. The zero modes of this operator can be written in the form

$$\psi(z, \bar{z}) = f(z) \prod_{j=1}^{n} |W_j(z)|^{-\theta_j},$$

where $f$ is an entire function and $W_j(z) = W_{\Omega_j}(z)$ is the Weierstrass canonical product for $\Omega_j$. It turns out that the function

$$\tilde{\psi}(z, \bar{z}) = f(z) \prod_{j=1}^{n} |W_j(z)|^{-\theta_j} \prod_{k=1}^{m} |z - \omega'_k|^{-\theta'_k}.$$

is a zero mode of $H^\pm_{\text{max}}(\mathbf{a}')$. One has only to verify that $\psi \in L^2(\mathbb{R}^2)$. This again follows from the fact that the singularities at the points $\omega'_k$ are square integrable and the function $\tilde{\psi}$ differs from $\psi$ by a bounded factor outside a compact set.

**9.2. Additional notation**

Up to the end of the current section, $\Omega_j$ ($j = 1, \ldots, n$) designates either an array of mutually disjoint chains or an array of mutually disjoint lattices, $\Omega_j = K_j + \Lambda_j$ where $\Lambda_j$ is a Bravais lattice of rank 1 or 2 and $K_j = \{k_{1j}, \ldots, k_{mj,j}\}$ is a finite set. To each set $\Omega_j$ we relate an array of Aharonov–Bohm fluxes $\Theta_j = (\theta_{kj})_{1 \leq k \leq m_j}$. By $\Omega$ we denote the union $\Omega = \Omega_1 \cup \ldots \cup \Omega_n$. In addition, we shall consider another array of discrete subsets in the plane, $\Omega'_1, \ldots, \Omega'_m$, whose members are mutually disjoint as well as disjoint with the sets $\Omega_1, \ldots, \Omega_n$, and we relate to these additional sets an array of fluxes $\Theta' = (\theta'_j)_{1 \leq j \leq m}$. Finally, we consider a discrete set $\Omega'_0 \subset \Omega$ whose points are supposed to be removed from $\Omega$. Set $\Omega' = \Omega'_0 \cup \Omega'_1 \cup \ldots \cup \Omega'_m$. Furthermore, $W_j(z)$ is the Weierstrass canonical product for the set $\Omega_j$, $W_{\Omega_j}(z)$ is the Weierstrass canonical product for the set $\Omega_j \supset \cup_{\Omega_j} (k_{kj} + \Lambda_j)$. By $\tau'$ we denote the convergence exponent of the set $\Omega'$ and by $p'$ its genus. The symbol $n'(r)$ designates the number of points of the set $\Omega'$ contained in the disc $|z| \leq r$. The symbol $\mathbf{a}'$ designates the vector potential for the perturbed system of solenoids determined by the discrete sets $\Omega_1 \setminus \Omega'_0, \ldots, \Omega_n \setminus \Omega'_0, \Omega'_1 \cup \ldots \cup \Omega'_m$. 


and by the array of fluxes obtained by concatenating the arrays $\Theta_1, \ldots, \Theta_n$ and $\Theta'$.

Thus the set $\Omega'$ representing the total perturbation of the original set $\Omega$ need not be finite nor regular. On the other hand, we shall always suppose that the set $\Omega'$ is sufficiently scarce by imposing restrictive assumptions on its genus $p'$.

9.3. Addition of solenoids to a union of Aharonov–Bohm chains

In Theorem 8.4 it has been shown, roughly, that if $\Omega$ is a finite union of chains then the Hamiltonians $H_{\text{max}}^\pm(\Omega)$ have infinitely many zero modes. Below we show that this property survives provided the perturbation $\Omega'$ is sufficiently scarce and at least two chains are not parallel.

**Theorem 9.3.** Let $\Omega_1, \ldots, \Omega_n$ be chains whose union is a uniformly discrete set. Suppose that among the chains there are at least two which are not parallel. Furthermore, suppose that the genus of $\Omega'$ fulfills $p' = 0$. Then the Hamiltonians $H_{\text{max}}^\pm(a')$ have infinitely degenerate zero modes.

**Proof.** In virtue of Lemma A.1 one can assume that every chain $\Omega_j = K_j + \Lambda_j$, with $\Lambda_j = \{k \omega_j; k \in \mathbb{Z}\}$, $\omega_j \neq 0$, is contained in a line $L_j$ and that different chains are contained in different lines. We shall suppose, without loss of generality, that $L_1$ and $L_2$ are not parallel and that $L_1$ coincides with the real line $\mathbb{R}$, with $0 \in \Omega_1$.

Let us consider a function $\psi$ of the form

$$\psi(z, \bar{z}) = f(z) \prod_{j=1}^{n} g_j(z, \bar{z}) \prod_{k=1}^{m} |W_k(z)|^{1-\theta'_k}$$

(72)

where

$$g_j(z, \bar{z}) = \prod_{k=1}^{m} \left| \sin \left( \frac{\pi(z - k \omega_j)}{\omega_j} \right) \right|^{-\theta_k} |V_k(z)|^{\theta_k}.$$  

(73)

To prove the theorem it suffices to find an infinite, linearly independent family of entire function $f(z)$ such that $\psi \in L^2(\mathbb{R}^2)$.

Let us show that the functions

$$f(z) = \frac{\sin(\alpha z)}{z}$$

(74)

with sufficiently small $\alpha > 0$ suit this condition. To this end we shall need the following lemma.

**Lemma 9.4.** Assume that $\alpha_1, \alpha_2, a_1, a_2 \in \mathbb{C}$ fulfill $\alpha_1 \alpha_2 \neq 0$, $\alpha_1/\alpha_2 \notin \mathbb{R}$. Then for every $\varepsilon > 0$ there exist constants $\tilde{c}, c_1, c_2, \gamma_1, \gamma_2 > 0$ such that the inequality

$$c_1 e^{\gamma_1 |z|} \leq |\sin(\alpha_1(z - a_1)) \sin(\alpha_2(z - a_2))| \leq c_2 e^{\gamma_2 |z|}$$

(75)

holds true whenever $|z| \geq \tilde{c}$ and the distance from $z$ to the lines $L_1 = \alpha_1^{-1}\mathbb{R} + a_1$ and $L_2 = \alpha_2^{-1}\mathbb{R} + a_2$ is greater than $\varepsilon$. 

Proof of Lemma 9.4. Let $L$ be a line written in the form $L = \alpha^{-1}R + a$ where $\alpha, a \in \mathbb{C}$, $\alpha \neq 0$. Then inequality (50) implies that
\[
\frac{e^{i|\alpha|d} - 1}{2} \leq |\sin(\alpha(z - a))| \leq e^{i|\alpha|d}
\] (76)
where $d$ is the distance from the point $z$ to the line $L$. Actually, set $\alpha = |\alpha|e^{i\varphi}$. Then $|\text{Im}(\alpha(z - a))| = |\alpha|d$ where $d$ is the distance from $e^{i\varphi}(z - a)$ to $\mathbb{R}$. But, at the same time, $d$ is the distance from $z$ to $e^{-i\varphi}\mathbb{R} + a = \alpha^{-1}\mathbb{R} + a$. From (76) we deduce that for every $\varepsilon > 0$ one can find a constant $c > 0$ such that
\[
|\sin(\alpha(z - a))| \geq ce^{i|\alpha|d}
\] (77)
whenever the distance $d$ from $z$ to $L$ is greater than $\varepsilon$. Actually, it suffices to choose $c = (1 - e^{-|\alpha|\varepsilon})/2$.

Let us now reconsider the lines $L_1$ and $L_2$ from the lemma. Since $\alpha_1/\alpha_2 \notin \mathbb{R}$ the lines intersect each other in a point $v \in \mathbb{C}$. Let us denote by $\varphi$ the angle between the lines $L_1$ and $L_2$ ($0 < \varphi < \pi$), and by $\theta$ the angle between the vector $z - v$ and the line $L_1$. Then the distances $d_1$ and $d_2$ from the point $z$ to the lines $L_1$ and $L_2$ are respectively equal
\[
d_1 = |z - v||\sin(\theta)|, \quad d_2 = |z - v||\sin(\theta - \varphi)|.
\]
From (76) we get
\[
|\sin(\alpha_1(z - a_1)) \sin(\alpha_2(z - a_2))| \leq e^{i|\alpha_1||d_1| + |\alpha_2||d_2|} \leq e^{\max(|\alpha_1|, |\alpha_2|)(d_1 + d_2)}.
\]
Notice that $d_1 + d_2 \leq 2|z - v| \leq 2|z| + 2|v|$, hence
\[
|\sin(\alpha_1(z - a_1)) \sin(\alpha_2(z - a_2))| \leq c_2e^{\gamma_2|z|}
\]
where
\[
\gamma_2 = 2\max(|\alpha_1|, |\alpha_2|), \quad c_2 = e^{||v||\gamma_2},
\]
(this is true for any $z \in \mathbb{C}$).

Using (77) we can relate to every $\varepsilon > 0$ a constant $c > 0$ such that if $d_1, d_2 > \varepsilon$ then
\[
|\sin(\alpha(z - a_1)) \sin(\alpha(z - a_2))| \geq c_2e^{\min(|\alpha_1|, |\alpha_2|)(d_1 + d_2)}.
\]
On the other hand,
\[
d_1 + d_2 \geq |z - v|(\sin^2 \theta + \sin^2(\theta - \varphi)) = |z - v|(1 - \cos \varphi \cos(\varphi - \theta)) \geq |z - v|(1 - |\cos \varphi|) \geq |z|(1 - |\cos \varphi|) - |v|(1 - |\cos \varphi|).
\]
From here we deduce that the inequality
\[
|\sin(\alpha(z - a_1)) \sin(\alpha(z - a_2))| \geq c_1e^{\gamma_1|z|}
\]
holds true for
\[
c_1 = c_2e^{-\min(|\alpha_1|, |\alpha_2|)|v||(1 - |\cos \varphi|)|}, \quad \gamma_1 = \min(|\alpha_1|, |\alpha_2|)(1 - |\cos \varphi|).
\]
This proves the lemma. \qed
Proof of Theorem 9.3 (continued). Let us return to the proof of Theorem 9.3. From the assumptions of the theorem \( \rho' = 0 \) it follows that the functions \( V_{kj}(z) \) and \( W_j(z) \) are of growth \((1,0)\) (see Appendix \( \text{B} \)). This implies that for every \( \varepsilon > 0 \) there exists a constant \( c_\varepsilon > 0 \) such that

\[
\prod_{l=1}^{m_j} |W_l(z)|^{1-\theta_l} \prod_{j=1}^{n} \prod_{k=1}^{m_j} |V_{kj}(z)|^{\theta_{kj}} \leq c_\varepsilon \exp(\varepsilon |z|). \tag{78}
\]

Let \( P_j \) be a strip with border lines parallel to \( L_j \) and containing \( L_j \) in its interior, and let \( Q \) be a sufficiently large disk centered at 0. In virtue of Lemma 9.4 the disk \( Q \) can be chosen so that, for \( z \notin Q \cup P_1 \cup P_2 \), the inequality

\[
|\psi(z, \bar{z})| \leq c_1 |f(z)||\tilde{g}_1^B(z, \bar{z})||\tilde{g}_2^B(z, \bar{z})| \prod_{j=3}^{n} |\tilde{g}_j^A(z, \bar{z})| \tag{79}
\]

holds true with some constant \( c_1 > 0 \). Here we have set

\[
\tilde{g}_j^A(z, \bar{z}) = \prod_{k=1}^{m_j} \sin \left( \frac{\pi(z - \kappa_{kj})}{\omega_j} \right)^{-\theta_{kj}}
\]

and

\[
\tilde{g}_j^B(z, \bar{z}) = \prod_{k=2}^{m_j} \sin \left( \frac{\pi(z - \kappa_{kj})}{\omega_j} \right)^{-\beta_{kj}} \prod_{k=1}^{m_j} \sin \left( \frac{\pi(z - \kappa_{kj})}{\omega_j} \right)^{-\theta_{kj}},
\]

with \( 0 < \beta_{j} < \theta_{1j} \).

On the other hand, one can choose \( Q \) so that, for \( z \in P_2 \setminus Q \), we have the inequality \( |z| \leq c'd_1 \) where \( d_1 \) is the distance from \( z \) to the line \( L_1 \), and \( c' > 0 \) does not depend on \( z \). Then from (77) we deduce that \( Q \) can be replaced by a larger disk such that the inequality

\[
|\psi(z, \bar{z})| \leq c'_1 |f(z)||\tilde{g}_1^B(z, \bar{z})| \prod_{j=2}^{n} |\tilde{g}_j^A(z, \bar{z})|, \tag{80}
\]

with a constant \( c'_1 > 0 \), holds true for \( z \in P_2 \setminus Q \). An analogous assertion is true when interchanging the strips \( P_1 \) and \( P_2 \).

Finally, for any choice of the disk \( Q \), the inequality

\[
|\psi(z, \bar{z})| \leq c''_1 |f(z)| \prod_{j=1}^{n} |\tilde{g}_j^A(z, \bar{z})| \tag{81}
\]

holds true in the interior of \( Q \). Formulas (77), (80) and (81) make it possible to complete the proof by arguing in the same way as in the proof of Theorem 8.4. \( \square \)

In the case when all chains are parallel we have a somewhat weaker result.

Theorem 9.5. Let \( \Omega_1, \ldots, \Omega_n \) be parallel chains whose union is a uniformly discrete set. Assume that the convergence exponent of \( \Omega' \) satisfies either \( \tau' < 1/2 \) or \( \tau' \leq 1/2 \) and \( n'(r) = o(r^{1/2}) \). Then the Hamiltonians \( H_{\text{max}}^\pm(a') \) have infinitely degenerate zero modes.
Remark 9.6. Under the assumptions of Theorem 9.5 it holds true that $p' = 0$ but this equality does not imply the assumptions of the theorem.

Proof. In virtue of Lemma A.1 one can assume that every chain $\Omega_j = K_j + \Lambda_j$, $\Lambda_j = \{k\omega_j; \ k \in \mathbb{Z}\}$, is contained in a line $L_j$, with different chains being contained in different lines, and that all lines are parallel to the real axis. Hence one can assume that $\omega_j > 0$ for all $j$ and that all lines $L_j$ are contained in a half-plane $\text{Im} \ z > a$ where $a > 0$.

Let us consider a function $\psi$ of the form

$$\psi(z, \bar{z}) = f(z) \prod_{j=1}^{n} g_j(z, \bar{z}) \prod_{k=1}^{m} |W_k(z)|^{-\theta_k}$$  \hspace{1cm} (82)

where

$$g_j(z, \bar{z}) = \prod_{k=1}^{m_j} \sin\left(\frac{\pi(z - \kappa_{kj})}{\omega_j}\right)^{-\theta_{kj}} |V_{kj}(z)|^{\theta_{kj}}.$$  \hspace{1cm} (83)

To prove the theorem it suffices to find an infinite, linearly independent family of entire functions $f(z)$ such that $\psi \in L^2(\mathbb{R}^2)$. Let us show that in this case the functions

$$f(z) = \frac{\sin(\alpha z)}{\sin(\sqrt{\pi\alpha z}) \sin(\sqrt{-\pi\alpha z})},$$  \hspace{1cm} (84)

with sufficiently small $\alpha > 0$, will do. Here the function $f(z)$ is well defined and analytic in the upper half-plane provided the usual branch of the square root has been chosen. If $\text{Im} \ z > 0$ then $\sqrt{z} \sqrt{-z} = -iz$. Since $\sin(\sqrt{z})/\sqrt{z} = 1 - z/3! + \ldots$ is in fact an entire function, also the function $f(z)$ extends as an entire function.

We start the verification from several preliminary observations. The first one follows from inequality (54).

(A) For every $\varepsilon > 0$ there exists $c > 0$ such that

$$|\sin(\sqrt{z})| \geq \frac{1}{3} \exp(|z|^{1/2}) \quad \text{for } |z| \geq c, \ \varepsilon \leq |\arg z| \leq \pi,$$

$$|\sin(\sqrt{-z})| \geq \frac{1}{3} \exp(|z|^{1/2}) \quad \text{for } |z| \geq c, \ 0 \leq |\arg z| \leq \pi - \varepsilon.$$

Suppose that $n \in \mathbb{N}$ and $0 < \delta < \pi/4$. Let us denote

$$B_n(\delta) = \{z \in \mathbb{C}; |\sqrt{\pi z} - \pi n| < \delta\}.$$  \hspace{1cm} (54)

(B) For $n \neq m$ the sets $B_n(\delta)$ and $B_m(\delta)$ are disjoint.

Actually, suppose that $n > m$. If $z \in B_n(\delta) \cap B_m(\delta)$ then there exist $u, v \in \mathbb{C}$, $|u|, |v| < \delta$, such that $\pi z = (\pi n + u)^2 = (\pi m + v)^2$. Hence $\pi^2(n - m)(n + m) = 2\pi(mv - nu) + v^2 - u^2$. At the same time it holds true that $\pi^2(n - m)(n + m) \geq \pi^2(n + m)$,

$$|2\pi(mv - nu) + v^2 - u^2| \leq 2\pi \delta(m + n) + 2\delta^2 < 4\pi \delta(m + n) < \pi^2(n + m).$$
Set
\[ Q(\varepsilon) = \{ z \in \mathbb{C}; |\sin(\sqrt{\pi z})| < \varepsilon \}. \]

Let us denote by \( Q_n(\varepsilon) \) the connected component of the set \( Q(\varepsilon) \) containing the point \( \pi n^2 \), and by \( U(\varepsilon) \) the connected component of the set \( \{ w \in \mathbb{C}; |\sin(w)| < \varepsilon \} \) containing 0. Observe that (B) implies the following claim.

(C) For sufficiently small \( \varepsilon > 0 \) the connected components \( Q_n(\varepsilon) \) are mutually disjoint.

To complete the proof we shall need the following two lemmas.

**Lemma 9.7.** For every \( \delta > 0 \) there exists a constant \( c > 0 \) such that
\[
\left| \frac{\sin(z)}{\sin(\sqrt{\pi z})} \right| \leq c \max(\sqrt{|\pi z|} + 1, |\sin(z)|)
\]
on the strip \( |\text{Im} z| \leq \delta \).

**Proof of Lemma 9.7.** The equality \( |\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \), with \( z = x + iy \), \( x, y \in \mathbb{R} \), implies that \( |\sin z| \leq (x^2 + c_\delta^2 y^2)^{1/2} \leq c_\delta |z| \) for \( |\text{Im} z| \leq \delta \) where we have set \( c_\delta = \sinh(\delta)/\delta \).

Let us choose \( \varepsilon > 0 \) small enough so that the sets \( Q_n(\varepsilon) \) are mutually disjoint, \( |\sqrt{\pi z}| > \pi n - 1 \) for \( z \in Q_n(\varepsilon) \) and \( |\sin(w)| \geq |w|/2 \) on \( U(\varepsilon) \). For \( z \in \mathbb{C} \setminus Q(\varepsilon) \), the desired inequality is valid with \( c = \varepsilon^{-1} \). If \( z \in Q_n(\varepsilon) \) then \( \sqrt{\pi z} - \pi n \in U(\varepsilon) \) and therefore
\[
|\sin(\sqrt{\pi z})| = |\sin(\sqrt{\pi z} - \pi n)| \geq |\sqrt{\pi z} - \pi n|/2.
\]
Furthermore, if \( |\text{Im} z| \leq \delta \) then
\[
|\sin(z)| = |\sin(z - \pi n^2)| \leq c_\delta |z - \pi n^2|.
\]

Consequently we have, for \( z \in Q_n(\varepsilon) \) and \( |\text{Im} z| \leq \delta \),
\[
\left| \frac{\sin(z)}{\sin(\sqrt{\pi z})} \right| \leq 2 c_\delta \frac{|z - \pi n^2|}{|\sqrt{\pi z} - \pi n|} \leq \frac{2c_\delta}{\pi} \left( |\sqrt{\pi z}| + \pi n \right) \leq \frac{4c_\delta}{\pi} (|\sqrt{\pi z}| + 1).
\]
This proves the lemma. \( \square \)

**Lemma 9.8.** For any \( b > 0 \) there exists \( \varepsilon > 0 \) such that
\[
\int\int_{Q(\varepsilon)} \left| \frac{\sin(z)}{\sin(\sqrt{\pi z})} \right|^2 \exp(-b|z|^{1/2}) \, dx \, dy < \infty
\]
where one chooses the principal branch of the square root on \( \mathbb{C} \setminus \mathbb{R}_- \).

**Proof of Lemma 9.8.** We choose \( \varepsilon \) small enough so that Claim (C) is true. In the integral
\[
I_n = \int\int_{Q_n(\varepsilon)} \left| \frac{\sin(z)}{\sin(\sqrt{\pi z})} \right|^2 \exp(-b|z|^{1/2}) \, dx \, dy
\]
where

\[ \phi \] is the property \( P \).

Furthermore, there clearly exists a constant \( c > \max L \in V \) such that \( \epsilon > 0 \), depending on \( \varepsilon \) but independent of \( n \), such that \( \sin(2nw + (w^2/\pi)) \leq c \exp(2\varepsilon) \) on \( U(\varepsilon) \). Thus we get

\[ I_n \leq \frac{16c^2}{\pi^2} \int_{U(\varepsilon)} \frac{\sin(2nw + (w^2/\pi))}{|w|^2} |w + \pi n|^2 \exp\left(-\frac{b}{\sqrt{\pi}} |w + \pi n|\right) dudv. \]

By modifying the constant in front of the integral we can simplify this inequality,

\[ I_n \leq c' n^2 \exp(-b\sqrt{\pi} n) \int_{U(\varepsilon)} \frac{\sin(2nw + (w^2/\pi))}{|w|^2} \exp\left(-\frac{b}{\sqrt{\pi}} |w + \pi n|\right) dudv. \]

Here again the constant \( c' > 0 \) does not depend on \( n \). If \( \sup\{|w|; w \in U(\varepsilon)\} < b\sqrt{\pi}/4 \) then

\[ \sum_{n=0}^{\infty} I_n \leq c' \int_{U(\varepsilon)} \sum_{n=0}^{\infty} n^2 \exp(-b\sqrt{\pi} n) \frac{\sin(2nw + (w^2/\pi))}{|w|^2} dudv < \infty. \]

This proves the lemma.

Proof of Theorem 9.5 (continued). Let us return to the proof of Theorem 9.5. We denote \( A_1 = \{ z \in \mathbb{C}; \text{Re} z \geq 0 \} \), \( A_2 = \{ z \in \mathbb{C}; \text{Re} z \leq 0 \} \), and we shall show that \( \psi \in L^2(A_j), j = 1, 2 \). More precisely, we shall prove only the membership \( \psi \in L^2(A_1) \), the property \( \psi \in L^2(A_2) \) can be shown analogously. We split the set \( A_1 \) into a union \( A_1 = P_1 \cup P_2 \) with \( P_1 = \{ z \in A_1; 0 \leq \text{Im} z \leq b \} \), \( P_2 = A_1 \setminus P_1 \). The bound \( b \) is chosen so that the strip \( P_1 \) contains the set \( \Omega \) in its interior.

From the assumptions it follows that the functions \( V_{kj}(z) \) and \( W_j(z) \) are of growth \((1/2, 0)\), i.e., for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that

\[ \max_{j,k} \{|V_{kj}(z)|, |W_k(z)|\} \leq C_\varepsilon \exp(\varepsilon |z|^{1/2}) \] (cf. Appendix 3). Let us denote

\[ G(z, \bar{z}) = \frac{1}{\sin(\sqrt{-\pi \alpha z})} \prod_{l=1}^{m} |W_l(z)|^{1-\theta_l} \prod_{j=1}^{n} \prod_{k=1}^{m_j} |V_{kj}(z)|^{\theta_{kj}}. \] (85)

Hence

\[ \phi(z, \bar{z}) = \frac{\sin(\alpha z)}{\sin(\sqrt{\pi \alpha z})} G(z, \bar{z}) \prod_{j=1}^{n} \tilde{g}_j(z, \bar{z}) \]

where

\[ \tilde{g}_j(z, \bar{z}) = \prod_{k=1}^{m_j} \left| \sin\left(\frac{\pi(z - \kappa_{kj})}{\omega_j}\right)\right|^{-\theta_{kj}}. \]
According to Claim (A) there exist constants $c_1, c_2 > 0$ such that

$$|G(z, \bar{z})| \leq c_1 \exp(-c_2|z|^{1/2}), \quad (86)$$

for all $z \in A_1$.

With the aid of Lemma 9.7 and formula (86) we can estimate the function $\psi$ on the strip $P_1$,

$$|\psi(z, \bar{z})| \leq c'(|\sqrt{\pi \alpha} z| + 1) \exp(-c_2|z|^{1/2}) \prod_{j=1}^{n} \tilde{g}_j(z, \bar{z}).$$

The singularities of $\psi$ in $P_1$ are square integrable and therefore, similarly as in the proof of Theorem 8.4, we obtain

1. $\psi \in L^2(P_1)$.

Since $\Omega \subset P_1$, on $P_2 \cap \alpha^{-1}Q(\varepsilon)$ we have the estimate

$$|\psi(z, \bar{z})| \leq c'' \frac{\sin(\alpha z)}{\sin(\sqrt{\pi \alpha} z)} \exp(-c_2|z|^{1/2})$$

If $\varepsilon$ is small enough then Lemma 9.8 implies that

2. $\psi \in L^2(P_2 \cap \alpha^{-1}Q(\varepsilon))$.

Finally, for $z \in P_2 \setminus \alpha^{-1}Q(\varepsilon)$ we have $|\sin(\sqrt{\pi \alpha} z)| \geq \varepsilon$ and hence the inequality

$$|\psi(z, \bar{z})| \leq c'' |\sin(\alpha z)| \prod_{j=1}^{n} \tilde{g}_j(z, \bar{z})$$

holds true. We conclude, similarly as in the proof of Theorem 8.4, that if

$$0 < \alpha < \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{\pi}{\omega_j} \theta_{kj}$$

then

3. $\psi \in L^2(P_2 \setminus Q_\alpha(\varepsilon))$.

This concludes the proof of the theorem.

9.4. Addition of solenoids to an Aharonov–Bohm lattice

The case when the Aharonov–Bohm fluxes are arranged in a lattice $\Omega$ has been discussed in Theorem 8.8. It turns out that for a scarce perturbation the result stated in the theorem is still true.

**Theorem 9.9.** Suppose that $\Omega$ is a lattice, i.e., $\Omega_j = \kappa_j + \Lambda$ where $\Lambda$ is a Bravais lattice of rank 2. Suppose further that the genus of the set $\Omega'$ fulfills $p' \leq 1$. Then the Hamiltonians $H_{\max}^\pm(\alpha')$ have infinitely degenerate zero modes.
Proof. Let \( \bar{W}(z) \) be the Weierstrass canonical product for the set \( \Omega' \), let \( g' \) be the growth order of \( \bar{W}(z) \), and let \( \tau' \) be the convergence exponent of \( \Omega' \). Then \( g' = \tau' \leq p' + 1 \leq 2 \) (see Appendix B). If \( g' < 2 \) then \( \bar{W}(z) \) is of growth \((2,0)\). If \( g' = 2 = p' + 1 \) then, by Theorem B.5 (b), \( \bar{W}(z) \) is of minimal type. Consequently, also in the latter case \( \bar{W}(z) \) is of growth \((2,0)\). This means that for any \( c > 0 \) there exist \( a > 0 \) and \( R > 0 \) such that for all \( z, |z| > R \), it holds true that

\[
|\bar{W}(z)| \leq a \exp(c|z|^2).
\] (87)

The same observation is clearly true for any subset of \( \Omega' \). In particular, the functions \( V_j(z) \) and \( W_\ell(z) \) are of growth \((2,0)\) and obey estimates similar to (87).

Zero modes of \( H_{\text{max}}^\pm (a') \) are gauge equivalent to functions of the form

\[
\psi(z, \bar{z}) = f(z) \prod_{j=1}^n |\bar{\sigma}(z - \kappa_j)|^{-\theta_j} \prod_{j=1}^n |V_j(z)|^{\theta_j} \prod_{\ell=1}^m |W_\ell(z)|^{1-\theta'_\ell}.
\]

From Lemma 8.7 and from the estimate (87) with sufficiently small \( c > 0 \) it follows that \( \psi \) is square integrable if \( f(z) \) is an arbitrary polynomial.

\[\square\]

9.5. Irregular systems of Aharonov–Bohm solenoids

All the preceding results were concerned with Aharonov–Bohm systems \( \Omega \) with bounded density, i.e., for which \( \limsup_{r \to \infty} n(r)/r^2 < \infty \) (cf. (65)). Here we show that zero modes may occur also in systems with infinite density, more precisely, in systems for which \( \liminf_{r \to \infty} n(r)/r^2 = \infty \). Moreover, we shall present examples of systems \( \Omega \) with arbitrarily large convergence exponent \( \tau_\Omega \). Let us fix a natural number \( N \geq 2 \) and set

\[
\Omega_N = \{ e^{\pi i k/N} m^{1/N} ; m \in \mathbb{N}, k = 0, 1, \ldots, 2N - 1 \}.
\]

Obviously, the convergence exponent \( \tau \) for the set \( \Omega_N \) equals \( N \). In particular, for \( N > 2 \) we have \( \lim_{r \to \infty} n(r)/r^2 = \infty \). Let \( \theta \) be an arbitrary number from the interval \([0,1]\). Then the vector potential \( a \) of the Aharonov–Bohm system determined by the couple \((\Omega_N, \theta)\) reads \( a(z, \bar{z}) = \theta \text{ sgrad } \ln(|W(z)|) \) where

\[
W(z) = \frac{\sin(\pi z_N)}{z^{N-1}}.
\]

Theorem 9.10. The Hamiltonians \( H_{\text{max}}^\pm (a) \) have infinitely degenerate zero modes.

Proof. It is sufficient to show that for \( 0 < \alpha < \pi \theta \) the function

\[
\psi(z, \bar{z}) = \frac{\sin(\alpha z_N)}{z^N} |W(z)|^{-\theta}
\]

is square integrable. Set

\[
S = \left\{ z \in \mathbb{C} ; -\frac{\pi}{2N} < \arg z < \frac{3\pi}{2N} \right\}.
\]
Then
\[ \iint_{\mathbb{R}^2} |\psi(z, \bar{z})|^2 \, dx \, dy = N \iint_{S} |\psi(z, \bar{z})|^2 \, dx \, dy \]
and therefore it suffices to verify that
\[ \iint_{S} |\psi(z, \bar{z})|^2 \, dx \, dy < \infty. \quad (88) \]

Let us make a substitution of the integration variable in (88), \( w = z^N \) where \( w = u + iv, \ u, v \in \mathbb{R} \). Since \( du \wedge dv = N^2 |z|^{2N-2} dx \wedge dy \) we can rewrite the integral as
\[ \iint_{S} |\psi(z, \bar{z})|^2 \, dx \, dy = \frac{1}{N^2} \iint_{\mathbb{R}^2} \frac{1}{|w|^\beta} |\sin(\alpha w)|^2 |\sin(\pi w)|^{-2\theta} \, du \, dv \quad (89) \]
where
\[ \beta = 4 - 2\theta - 2 \frac{1 - \theta}{N}. \]
Since \( 2 - 2\theta - \beta > -2 \) and \( \beta > 1 \) one can show that the integral (89) is finite using a reasoning as in the proof of Theorem 8.4.

\[ \square \]

**Appendix A. Lattices**

Here we collect basic definitions and some auxiliary results about lattices. Let \( E \) be a finite-dimensional real Euclidean space with dimension \( d \). A discrete subgroup \( \Lambda \) of the additive group \( E \) is called a Bravais lattice. For any Bravais lattice \( \Lambda \) there exist linearly independent vectors \( \omega_1, \ldots, \omega_r \in E \) such that
\[ \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \ldots + \mathbb{Z}\omega_r. \]
The array \( (\omega_j)_{1 \leq j \leq r} \) is called a basis of the Bravais lattice \( \Lambda \). The integer \( r \) does not depend on the choice of basis and is called the rank of the lattice \( \Lambda \). To every basis \( (\omega_j)_{1 \leq j \leq r} \) one relates the elementary cell \( F, \ F \subset E \), formed by all \( x \in E \) whose orthogonal projection \( x' \) onto the linear span \( L \) of the lattice \( \Lambda \) has a decomposition
\[ x' = t_1 \omega_1 + t_2 \omega_2 + \ldots + t_r \omega_r, \]
with \( 0 \leq t_j < 1 \) for all \( j \). If \( r = d \) (the dimension of the lattice \( \Lambda \) is maximal possible) then \( F \) is a convex parallelepiped. In the opposite case \( F \) is even not bounded.

A non-empty discrete subset \( \Omega \subset E \) is called a crystal with the Bravais lattice \( \Lambda \) if it is invariant with respect to the action of \( \Lambda \) on \( E \) and has a finite number of orbits. Obviously, every crystal \( \Omega \) can be written in the form \( \Omega = K + \Lambda \) where \( K \subset E \) is a finite set whose number of elements equals the number of orbits. Without loss of generality we may assume that \( K \subset F \). Conversely, every set of the form \( \Omega = K + \Lambda \) is a crystal. If \( |K| = 1 \) then the crystal is called mono-atomic or simple. In the general
case when $|K| = n$ the crystal $\Omega$ is called $n$-atomic. If $r = 1$ then $\Omega$ is called a chain (a simple chain if in addition $|K| = 1$). If $r = d$ then $\Omega$ is called a lattice (more precisely, a crystal lattice) in the space $E$. In other words, a crystal is such a discrete subset $\Omega \subset E$ whose group of parallel translations acts co-compactly on $E$.

Let us note that in our definition we do not exclude the case $r = 0$. If so then $\Lambda = \{0\}$ and a crystal with the Bravais lattice $\Lambda$ is simply a finite subset of $E$.

It is worth of noticing that a crystal $\Omega$ is always a uniformly discrete subset of $E$. This means that there exists a constant $c > 0$ such that $|\omega' - \omega''| \geq c$ whenever $\omega', \omega'' \in \Omega$, $\omega' \neq \omega''$.

We shall need the following lemma.

**Lemma A.1.** Assume that $\dim E = 1$ and that $\Omega_1, \ldots, \Omega_n$ are chains in $E$. The union $\Omega = \Omega_1 \cup \ldots \cup \Omega_n$ is a chain if and only if $\Omega$ is a uniformly discrete set.

**Proof.** We only need to prove that this condition is sufficient. Moreover, it suffices to consider the case $n = 2$. The general case then follows by mathematical induction. Let us write

$$\Omega_j = K_j + \Lambda_j \quad (j = 1, 2),$$

where $\Lambda_j$ is the Bravais lattice of $\Omega_j$. Let us identify $E$ with $\mathbb{R}$. Then $\Lambda_j = \mathbb{Z} \omega_j$, with $\omega_j > 0$.

We shall show that the number $p = \omega_1/\omega_2$ is rational. Actually, in the opposite case the set $\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2$ would be dense in $\mathbb{R}$. Let us choose $\kappa_1 \in K_1$ and $\kappa_2 \in K_2$. We can find a sequence $n_k \omega_2 - m_k \omega_1$ ($m_k, n_k \in \mathbb{Z}$) converging to $\kappa_1 - \kappa_2$ and such that $\kappa_1 - \kappa_2 \neq n_k \omega_2 - m_k \omega_1$ for all $k$. Obviously, this contradicts the assumption that the set $\Omega$ is uniformly discrete.

Hence $p = N/M$, with $N, M \in \mathbb{N}$. Then $M \omega_1 = N \omega_2$ and therefore $\Omega$ is invariant with respect to the lattice $\Lambda$ with the basis vector $M \omega_1$. It is easy to see that the number of orbits of the group $\Lambda$ in $\Omega$ is finite. But this means that $\Omega$ is a chain. \qed

**Remark A.2.** For $\dim E > 1$ an analog of Lemma A.1 is false. Actually, already for $\dim E = 2$ it can happen that a union of two simple lattices is uniformly discrete but not a lattice. This is demonstrated by the following example. Let $E = \mathbb{R}^2$. Let $\Lambda_1$ be a Bravais lattice with the basis $\omega_1 = e_1, \omega_2 = e_2$, and let $\Lambda_2$ be a Bravais lattice with the basis $\omega'_1 = \sqrt{2}e_1, \omega'_2 = e_2$. Consider the crystal lattices $\Omega_1 = \Lambda_1$ and $\Omega_2 = \kappa + \Lambda_2$ where $\kappa = (1/2)e_2$. Obviously, $\Omega = \Omega_1 \cup \Omega_2$ is a uniformly discrete set. Let $\Lambda$ be a group of parallel translations acting on $\Omega$. Suppose that the number of orbits of the group $\Lambda$ in $\Omega$ is finite. Then there exist $n, m \in \mathbb{Z}$, $n \neq m$, such that $n \omega_1$ and $m \omega_1$ belong to the same orbit. Hence $k \omega_1 \in \Lambda$ for some $k \in \mathbb{Z}$, $k \neq 0$. But $\kappa + k \omega_1 \notin \Omega$.

**Appendix B. Auxiliary results from the theory of analytic functions**

Here we recall some results from the theory of analytic functions that are necessary for our presentation, for the details see [101, 102, 106]. For an entire function $f$ we set
\[ M_f(r) \equiv M(r) = \max_{|z|=r} |f(z)| = \max_{|z| \leq r} |f(z)|. \] (90)

The order (more precisely, the growth order) of an entire function \( f \) is the number

\[ \varrho_f \equiv \varrho = \inf \{ \alpha; \exists R_\alpha > 0 \ \forall r > R_\alpha \ M(r) < \exp(r^\alpha) \}, \] (91)

or, equivalently,

\[ \varrho_f = \limsup_{r \to \infty} \frac{\ln \ln(M(r))}{\ln(r)}. \] (92)

Let us note that if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) then

\[ \varrho_f = \limsup_{n \to \infty} \frac{\ln(n)}{\ln(|a_n|^{1/n})}. \] (93)

If \( \varrho_f < \infty \) then one says that \( f \) is a function of finite order. For a function of finite order \( \varrho \) the number

\[ \varsigma_f \equiv \varsigma = \inf \{ K > 0; \ \exists R_K > 0 \ \forall r > R_K \ M(r) < \exp(K r^\varrho) \} \] (94)

is well defined and it is called the type of the function \( f \). The type can be equivalently defined by the formula

\[ \varsigma_f = \limsup_{r \to \infty} \frac{\ln(M(r))}{r^\varrho}. \] (95)

Moreover, \( \varsigma_f \) can be expressed in terms of the Taylor coefficients \( a_n \),

\[ (\varsigma_f e^{\varrho_f})^{1/\varrho} = \limsup_{n \to \infty} n^{1/\varrho} |a_n|^{1/n}. \] (96)

If \( \varrho_f < \infty \) and \( \varsigma_f = 0 \) then \( f \) is called a function of minimal type.

A function \( f \) of finite order \( \varrho \) and of finite type \( \varsigma \) obeys the estimate

\[ |f(z)| \leq \exp((\varsigma + \varepsilon)|z|^\varrho) \] (97)

for arbitrary \( \varepsilon > 0 \) and \( |z| \) greater than a constant \( R_\varepsilon > 0 \). Conversely, if the estimate

\[ |f(z)| \leq c \exp(\varsigma_1 |z|^\varrho_1) \] (98)

is fulfilled with a constant \( c > 0 \) then the function \( f \) has both a finite order and a finite type, and it holds \( \varrho_f \leq \varrho_1 \) and \( \varsigma_f \leq \varsigma_1 \). A couple of numbers \( (\varrho_1, \varsigma_1) \), \( 0 \leq \varrho_1, \varsigma_1 \leq \infty \), determines the growth of a function \( f(z) \) if \( \varrho_f \leq \varrho_1 \) and \( \varrho_f = \varrho_1 \) implies \( \varsigma_f \leq \varsigma_1 \).

Functions of growth \((1, \varsigma_1)\), with \( \varsigma_1 < \infty \), are said to have exponential growth.

The order and the type of an entire function on one side and the distribution of its zeroes on the other side are deeply related. Let us arrange all nonzero elements of a discrete set \( \Omega \subset \mathbb{C} \) in a sequence \( \Omega_\ast = (\omega_k)_{k \geq 1} \) which is ascending in the absolute
value and ascending in the argument \(0 \leq \arg z < 2\pi\) in the case of equal absolute values. The convergence exponent of the set \(\Omega\) (or of the sequence \(\Omega^*_n\)) is the number

\[
\tau_\Omega \equiv \tau = \inf \left\{ \alpha > 0; \sum_{k=1}^{\infty} \frac{1}{|\omega_k|^\alpha} < \infty \right\},
\]

or, equivalently,

\[
\tau_\Omega = \limsup_{k \to \infty} \frac{\ln(k)}{\ln(|\omega_k|)}.
\]

If \(\tau_\Omega\) is finite then the number

\[
p_\Omega \equiv p = \begin{cases} \max \left\{ n \in \mathbb{N}; \sum_{k=1}^{\infty} \frac{1}{|\omega_k|^n} = \infty \right\} & \text{if } \Omega \text{ is infinite,} \\ -\infty & \text{if } \Omega \text{ is finite,} \end{cases}
\]

is well defined and it is called the genus of the set \(\Omega\) (or of the sequence \(\Omega^*_n\)). For \(\tau_\Omega = \infty\) we set \(p_\Omega = \infty\).

For \(r > 0\) we set

\[
n_\Omega(r) \equiv n(r) = \# \{ \omega \in \Omega; |\omega| \leq r \}.
\]

For a non-empty set \(\Omega\) the formula

\[
\tau_\Omega = \limsup_{r \to \infty} \frac{\ln(n(r))}{\ln(r)},
\]

shows that the convergence exponent of an discrete set characterizes its density \[106, \text{Theorem 2.5.8}\].

On the other hand, let \(\Omega_f \equiv \Omega\) be the zero set of an entire function \(f\). Then the following fundamental inequality is valid (the Hadamard theorem, see for example \[106, \text{Theorem 2.5.18}\]):

\[
\tau_\Omega \leq \varrho_f.
\]

If \(\varrho_f\) is not an integer then \(\tau_\Omega = \varrho_f\) \[106, \text{Theorem 2.9.1}\]. From \(104\) we deduce that

\[
n(r) = O(r^{\varrho_f + \epsilon})
\]

for any \(\epsilon > 0\). If \(\varrho_f > 0\) and \(\varsigma_f < \infty\) then a stronger estimate is valid \[106, \text{Theorem 2.5.13}\]:

\[
L \equiv \limsup_{r \to \infty} r^{-\varrho_f} n(r) \leq c \varrho_f \varsigma_f,
\]

\[
l \equiv \liminf_{r \to \infty} r^{-\varrho_f} n(r) \leq \varrho_f \varsigma_f.
\]

Moreover, if \(\tau_\Omega > 0\) then \(L e^{l/L} \leq \varrho_f \varsigma_f\), in particular, \(L + l \leq \varrho_f \varsigma_f\).

For \(\varrho_f < \infty\) it can never happen that \(n_\Omega(r) = o(r^{\varrho_f - \epsilon})\) \[106, \text{Theorem 2.9.3}\]. The following theorem is due to Lindelöf (see \[106, \text{Theorem 2.9.5 and Theorem 2.10.1}\]).
Theorem B.1. (1) Assume that \( q \equiv q_f < \infty \) is not an integer. An entire function \( f(z) \) is of finite type if and only if \( n_\Omega(r) = O(r^q) \), and it is of minimal type if and only if \( n_\Omega(r) = o(r^q) \).

(2) Assume that \( q_f \) is a positive integer. The function \( f(z) \) is of finite type if and only if \( n_\Omega(r) = O(r^q) \) and the sums

\[
S(r) = \sum_{0 < |\omega_k| \leq r} \omega_k^{-q}
\]

are bounded.

An entire function with simple zeroes is determined by its zero set up to an multiplier \( e^{g(z)} \) where \( g(z) \) is an entire function. Furthermore, for an arbitrary discrete set \( \Omega \subset \mathbb{C} \) there exists an entire function \( f(z) \) with simple zeroes whose zero set coincides with \( \Omega \). Denote by \( E(u, p) \) the Weierstrass canonical multiplier, with \( u \in \mathbb{C} \) and with \( p \in \mathbb{N} \),

\[
E(u, p) = (1 - u) \exp\left(u + \frac{u^2}{2} + \ldots + \frac{u^p}{p}\right)
\]

(by definition, \( E(u, 0) = 1 - u \)). Let \( \Omega_s = (\omega_k)_{k \geq 1} \) be, as above, the sequence formed by all nonzero elements of the set \( \Omega \) appropriately enumerated. Let us denote by \( \chi_\Omega \equiv \chi \) an integer which is equal 1 if 0 \( \in \Omega \) and 0 in the opposite case. The Weierstrass canonical product associated to \( \Omega \) is by definition an entire function \( W_\Omega(z) \) defined by the infinite product

\[
z^\chi \prod_{k=1}^\infty E(z/\omega_k, p_\Omega)
\]  

(107)

if the convergence exponent of \( \Omega \) is finite, and by the infinite product

\[
z^\chi \prod_{k=1}^\infty E(z/\omega_k, k)
\]  

(108)

in the opposite case.

Theorem B.2 (Weierstrass, Hadamard). The infinite product defining the function \( W_\Omega(z) \) converges absolutely and locally uniformly. Consequently, \( W_\Omega(z) \) is an entire function and its zero set coincides with \( \Omega \). Moreover, the zero set of an entire function \( f(z) \) with simple zeroes only equals \( \Omega \) if and only if the function \( f(z) \) is of the form

\[
f(z) = e^{g(z)} W_\Omega(z)
\]  

(109)

where \( g(z) \) is an entire function. The growth order of the function \( W_\Omega(z) \) equals the convergence exponent of the set \( \Omega \).

Moreover, the following theorem is true.

Theorem B.3 (Hadamard). If the function \( f \) from relation (109) has a finite order \( q_f \) then \( g(z) \) is a polynomial with a degree not exceeding \( [q_f] \).
Theorem B.4 (Borel Theorem). Conversely, if \(\tau < \infty\) and \(f(z)\) is a function written in the form \(g(z)\) where \(g(z)\) is a polynomial of degree \(n\) then \(f(z)\) has a finite order \(\varrho_f = \max(\tau, n)\). If either \(\tau < n\) or the series \(\sum_{k=1}^{\infty} |\omega_k|^{-\tau}\) is convergent then the function \(f(z)\) is of finite type.

The genus of a function \(f(z)\) having the form \(g(z)\), where \(g(z)\) is a polynomial of degree \(n\), is the integer \(\eta_f = \max(n, p_\Omega)\). The following theorem is a useful completion of Theorem B.1 due to Lindelöf [106, Theorem 2.10.3].

Theorem B.5. Under the assumptions of Theorem B.1 let \(\varrho_f\) be a positive integer. A function \(f(z)\) written in the form \(g(z)\), where \(g(z)\) is a polynomial, is of minimal type if and only if one of the following conditions is satisfied:

\[(a)\] \(n_\Omega(r) = o(r^\varrho_f)\), \(p_\Omega = \varrho_f\), and

\[
\sum_{k=1}^{\infty} \omega_k^{-\varrho_f} = -\varrho_f \alpha_0
\]

where \(\alpha_0\) is the coefficient (possibly vanishing) standing at \(z^{\varrho_f}\) in the polynomial \(g(z)\),

\[(b)\] \(p_\Omega = \varrho_f - 1\) and \(\alpha_0 = 0\).

In particular, if \(q_f < \varrho_f\) then \(f(z)\) is of minimal type.

We shall also need the following particular case of the Mittag-Leffler theorem (see [101, II.7.3.2] or [102]).

Theorem B.6. For an arbitrary discrete subset \(\Omega\) of the complex plane \(\mathbb{C}\) and for an arbitrary sequence of complex numbers \((\theta_\omega)_{\omega \in \Omega}\) there exists a meromorphic function \(M(z)\) obeying the following conditions:

\[(1)\] \(M(z)\) has only simple poles,

\[(2)\] the set of poles of the function \(M(z)\) coincides with \(\Omega\),

\[(3)\] the residuum of \(M(z)\) at the point \(\omega\) equals \(\theta_\omega\).

Appendix C. The Weierstrass \(\sigma\)-function and related functions

In our approach an important role is played by the order \(\varrho\) and by the type \(\varsigma\) of the Weierstrass \(\sigma\)-function \(\sigma(z)\),

\[
\sigma(z) \equiv \sigma(z; \omega_1, \omega_2) = z \prod_{\omega \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(z - z^2/2\omega^2\right).
\]

It is easy to see that the convergence exponent of any lattice in the plane equals 2. Actually, the series

\[
\sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{|n_1 \omega_1 + n_2 \omega_2|^\alpha}
\]
(the dash indicates, as usual, that the summand with indices \( n_1 = n_2 = 0 \) is omitted) converges if and only if \( \alpha > 2 \). Hence, by the Borel theorem, \( q = 2 \). Since for \( \alpha = 2 \) the series (110) diverges the Borel theorem does not say anything about the type of the function \( \sigma(z) \) (apart of the fact that it is finite). The type of this function has been found in the general case by A. M. Perelomov [95]. In order to make our presentation self-contained we reproduce below some details from his derivation.

Let us start from recalling the notation

\[
\eta_j = 2\zeta\left(\frac{\omega_j}{2}\right)
\]  

(111)

and the fact that the \( \sigma \)-function is quasi-periodic in the following sense:

\[
\sigma(z + \omega_j) = -\sigma(z) \exp\left(\eta_j\left(z + \frac{\omega_j}{2}\right)\right).
\]  

(112)

Recall also that \( S = \text{Im}(\bar{\omega}_1\omega_2) \) designates the area of the elementary cell.

**Lemma C.1** ([95]). The function \( |\sigma(z)|^2 \) can be expressed in the form

\[
|\sigma(z)|^2 = \exp(\nu z^2 + \bar{\nu} \bar{z}^2 + 2\mu z \bar{z})\rho(z, \bar{z})
\]  

(113)

where \( \rho \) is a \( \Lambda \)-periodic function,

\[
\nu = \frac{i}{4S}(\eta_1\bar{\omega}_2 - \eta_2\bar{\omega}_1), \quad \mu = \frac{\pi}{2S}.
\]  

(114)

**Proof.** From (112) we obtain

\[
|\sigma(z + \omega_j)|^2 = |\sigma(z)|^2 \exp\left(2\text{Re}\left(\eta_j\left(z + \frac{\omega_j}{2}\right)\right)\right).
\]  

(115)

On the other side, the function \( \rho \) defined by equality (113), with \( \nu \in \mathbb{C} \) and \( \mu \in \mathbb{R} \), is periodic if and only if it holds

\[
|\sigma(z + \omega_j)|^2 = \exp\left(2\nu z \omega_j + 2\bar{\nu} \bar{z} \omega_j + 2\mu z \bar{z} + \nu \omega_j^2 + \bar{\nu} \bar{\omega}_j^2 + 2\mu \omega_j \bar{\omega}_j\right) |\sigma(z)|^2.
\]  

(116)

Comparing (115) to (116) and taking into account the equality

\[
2\text{Re}\eta_j\left(z + \frac{\omega_j}{2}\right) = \eta_j z + \bar{\eta}_j \bar{z} + \frac{\eta_j \omega_j}{2} + \frac{\bar{\eta}_j \bar{\omega}_j}{2},
\]

we arrive at the system

\[
\nu \omega_j + \mu \bar{\omega}_j = \frac{1}{2} \eta_j, \quad j = 1, 2,
\]  

(117)

\[
\nu \omega_j^2 + \bar{\nu} \bar{\omega}_j^2 + 2\mu \omega_j \bar{\omega}_j = \frac{1}{2}(\eta_j \omega_j + \bar{\eta}_j \bar{\omega}_j), \quad j = 1, 2.
\]

The first couple of equations in (117) gives

\[
\nu = \frac{1}{2} \frac{\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1}{\omega_1 \bar{\omega}_2 - \bar{\omega}_1 \omega_2}, \quad \mu = \frac{1}{2} \frac{\omega_1 \eta_2 - \omega_2 \eta_1}{\omega_1 \bar{\omega}_2 - \bar{\omega}_1 \omega_2}.
\]  

(118)
Lemma C.1 implies the equality
\[ \eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i \]  \hspace{1cm} (119)
we find that relations (118) and (114) are equivalent. Using (118) and the fact that \( \mu \) is real one can check that the second couple of equations in (117) is satisfied identically.

**Lemma C.2 ([95]).** The type of the function \( \sigma(z; \omega_1, \omega_2) \) is given by the equality
\[ \varsigma = |\nu| + \mu = \frac{1}{4S} \left( |\eta_1 \omega_2 - \eta_2 \omega_1| + 2\pi \right). \]  \hspace{1cm} (120)

**Proof.** Let us rewrite (113) as follows,
\[ |\sigma(z)|^2 = \exp\left( (\nu + \bar{\nu})(x^2 - y^2) + 2i(\nu - \bar{\nu})xy + 2\mu(x^2 + y^2) \right) \rho(z, \bar{z}). \]  \hspace{1cm} (121)

The quadratic form occurring in the exponent, \( 2\Re(\nu)(x^2 - y^2) - 4\Im(\nu)xy \), can be diagonalized with the aid of a rotation of the coordinate system. Eigenvalues of the corresponding symmetric matrix are \( \lambda_1 = -\lambda_2 = 2|\nu| \). Set \( \varepsilon = e^{i\varphi} \) where \( \varphi \) is the angle of the rotation. Since the quadratic form \( x^2 + y^2 \) is rotationally invariant we have
\[ |\sigma(\varepsilon z)| = \exp\left( (\mu + |\nu|)x^2 + (\mu - |\nu|)y^2 \right) \rho^{1/2}(z, \bar{z}). \]  \hspace{1cm} (122)

Owing to the periodicity of the function \( \rho \) it holds true that
\[ \max_{|z|=r} |\sigma(z)| = \max_{|z|=r} |\sigma(\varepsilon z)| \leq c \exp\left( (\mu + |\nu|)r^2 \right). \]  \hspace{1cm} (123)

Consequently, \( \varsigma \leq |\nu| + \mu \).

To show the opposite inequality it suffices to construct a sequence \( z_k \) such that \( |z_k| \to \infty \) and
\[ |\sigma(\varepsilon z_k)| \geq c \exp\left( (\mu + |\nu| - \delta_k)|z_k|^2 \right), \]  \hspace{1cm} (124)
where \( \delta_k \downarrow 0 \) and \( c > 0 \) is a fixed constant. First we note that, by the uniqueness theorem for analytic functions in a real variable, there exists a point \( z_0 \) such that \( \rho(z_0, \bar{z}_0) \neq 0 \). Then there exists \( c > 0 \) such that \( |\rho(z, \bar{z})| > c \) on a neighborhood \( V \) of \( z_0 \). This gives the choice of \( c \). Further we consider the canonical mapping \( h : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda \).

Two cases are possible: either the image \( h(z_0 + \mathbb{R}) \) is a closed curve in the torus \( T = \mathbb{R}^2/\Lambda \) or this image is dense in \( T \). In the former case there exists a sequence \( \lambda_k \in \mathbb{R} \) such that \( \lambda_k \to \infty \) and
\[ h(z_0 + \lambda_k) = h(z_0), \]  \hspace{1cm} (125)
in the latter case condition (125) should be replaced by \( h(z_0 + \lambda_k) \to h(z_0) \). In the both cases condition (124) holds true with \( z_k = z_0 + \lambda_k \). \hfill \Box

Following ([95]) we introduce the function
\[ \tilde{\sigma}(z) = e^{-\nu z^2} \sigma(z). \]
Lemma C.1 implies the equality
\[ |\tilde{\sigma}(z)|^2 = \exp(2\mu|z|^2) \rho(z, \bar{z}). \]  \hspace{1cm} (126)
Lemma C.3 ([95]). Let \( f(z) \) be an entire function whose zero set coincides with \( \Lambda = Z_1 + Z_2 \), with all zeroes being simple. Then the order \( \sigma_f \) is at least 2, \( \sigma_f \geq 2 \), and if \( \sigma_f = 2 \) then the type \( \varsigma_f \) is at least \( \mu \), \( \varsigma_f \geq \mu = \pi/2S \). Moreover, in the case of the function \( \tilde{\sigma}(z) \) the minimal values are achieved both for the order \( \sigma \) and the type \( \varsigma \), i.e., \( \sigma_{\tilde{\sigma}} = 2 \) and \( \varsigma_{\tilde{\sigma}} = \mu = \pi/2S \).

Proof. Since the function \( \sigma(z) \) is expressed as a Weierstrass canonical product its order equals the convergence exponent \( \tau_\Lambda = 2 \). Let us consider the entire function \( f(z) = e^{-a z^2} \sigma(z) \), with \( a \in \mathbb{C} \). Then

\[
|f(z)|^2 = \exp\left(2 \text{Re}(\nu - \alpha)(x^2 - y^2) - 4 \text{Im}(\nu - \alpha)xy + 2\mu(x^2 + y^2)\right) \rho(z, \bar{z}). \tag{127}
\]

It is clear that the order of the function \( f(z) \) equals 2, and similarly as in the proof of Lemma C.2, the type of \( f(z) \) equals \( |\nu - \alpha| + \mu \). Obviously, the smallest type (namely, \( \mu \)) is achieved for \( \alpha = \nu \). In particular, the function \( \tilde{\sigma}(z) \) is of order 2 and its type equals \( \mu \).

Conversely, suppose that the zero set of an entire function \( f(z) \) coincides with \( \Lambda \) and that all zeroes of \( f(z) \) are simple. Since \( \tau_\Lambda = 2 \) the Hadamard theorem implies that \( \sigma_f \geq 2 \). Suppose that \( \sigma_f = 2 \). We can write \( f(z) \) in the form \( f(z) = e^{g(z)} \sigma(z) \). By the Hadamard theorem, \( g(z) = az^2 + bz + c \). If \( a = 0 \) then the type of \( f(z) \) equals the type of \( \sigma(z) \), if \( a \neq 0 \) then the type of \( f(z) \) equals the type of \( \exp(az^2) \sigma(z) \). In the both cases the type of \( f(z) \) is greater or equal \( \mu \). \( \square \)

Remark C.4. If \( \Lambda \) is a quadratic or hexagonal lattice then \( \nu = 0 \) and, consequently, \( \tilde{\sigma}(z) = \sigma(z) \). Actually, in the former case we can suppose that \( \omega_1 > 0, \omega_2 = i \omega_1 \). Then \( \eta_1 = \pi/\omega_1, \eta_2 = -\pi i/\omega_1 \) [104, 18.14.8 and 18.14.10], hence \( \nu = 0 \). In the latter case we can suppose that \( \omega_1 = k e^{-i \pi/3}, \omega_2 = k e^{i \pi/3} \), with \( k > 0 \). Then

\[
\eta_1 = \frac{2\pi e^{i \pi/3}}{\sqrt{3}(\omega_1 + \omega_2)}, \quad \eta_2 = \frac{2\pi e^{-i \pi/3}}{\sqrt{3}(\omega_1 + \omega_2)}
\]

[104] 18.13.16 and 18.13.19. In this case, too, \( \nu = 0 \).

Remark C.5. There exist lattices for which \( \nu \neq 0 \) and, consequently, \( \tilde{\sigma}(z) \neq \sigma(z) \). It suffices to consider a lattice with \( \eta_2 = 0 \) (such a lattice exists, see [104, 18.3.10]). Then, by the Lagrange formula, \( |\eta_1 \omega_2| = 2\pi \) and hence \( |\nu| = \pi/2S \). This means that the type of the \( \sigma \)-function for such a lattice equals \( \pi/S \). Since \( \nu \) depends on \( (\omega_1, \omega_2) \) continuously any value of \( |\nu| \) lying between 0 and \( \pi/2S \) is realized by a convenient lattice.

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