EXTENSION IN GROUPOIDS

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Abstract. As groupoids generalize groups, motivated by group extension we propose a concept of groupoid extension for Lie groupoids, i.e.,

\[ 1 \to A \to G \to K \to 1 \]

where \( A, G \) and \( K \) are Lie groupoids. Similar to the theory of group extension, we show that the existence of extension is obstructed by a groupoid cohomology of \( H^1_{\text{aut}}(K, Z_A) \), and the extensions are classified by \( H^2_{\text{aut}}(K, Z_A) \) once exists. Here \( Z_A \) is the center of \( A \). This generalizes the theory of group extensions, of gerbes over manifolds/groupoids and etc.

1. Introduction

Group extension is an important concept in group theory. Let \( A \) and \( K \) be groups. By a group extension of \( K \) by \( A \), one means a group \( G \) satisfying a short exact sequence of groups

\[ 1 \to A \xrightarrow{f} G \xrightarrow{\phi} K \to 1. \]

Each extension is determined by a band \( \omega : K \to \text{Out}(A) \) and a cofactor \( \Omega : K \times K \to A \). Here \( \text{Out}(A) \) denotes the group of outer automorphism of \( A \). Conversely, given a morphism \( \omega : K \to \text{Out}(A) \), it defines an obstruction class \( c_\omega \in H^2_{\text{aut}}(K, Z_A) \), where \( Z_A \) is the center of \( A \). Extensions of \( K \) by \( A \) with \( \omega \) as its band exist if and only if \( c_\omega = 0 \). Moreover, once \( c_\omega = 0 \), the extensions with band \( \omega \) are classified by \( H^2_{\text{aut}}(K, Z_A) \) (see for example [2, 22]).

A group \( K \) is identified with a groupoid \( BK := [\bullet / K] = (K \rightrightarrows \bullet) \). In terms of groupoids, (1.1) can be read, at least, formally as

\[ 1 \to BA \xrightarrow{f} BG \xrightarrow{\phi} BK \to 1. \]

In this paper, we want to consider its generalizations

\[ 1 \to A \xrightarrow{\phi} G \xrightarrow{\psi} K \to 1, \]

where \( A, G, K \) are Lie groupoids and we call (1.2) an extension in Lie groupoids. Here by the exactness at \( A \) we mean that \( \phi \) is injective, and by the exactness at \( K \) we mean that \( \psi \) is surjective. The exactness at \( G \) will be defined in Section 3. In fact, there are many important examples can be thought as special cases of this concept: (1) when \( A \) is a manifold \( F \), \( G \) is a fiber bundle over \( K \) with fiber \( F \) (see for example [14, 19]); (2) when \( A \) is a manifold \( F \), \( G \) is a fiber bundle over \( K \) with fiber \( F \) (see for example [1]).

The main results of the paper show that the theory of group extensions may be similarly generalized to that of Lie groupoid extensions. In Section 4 we show that the existence of Lie groupoid extension (1.2) with given morphism

\[ \Lambda : K^1 \to \overline{\text{Aut}}(A) \]

is obstructed by a class \([\Xi_A]\) in the groupoid cohomology group \( H^1_{\text{aut}}(K, Z_A) \), where \( Z_A \) is the center of the Lie groupoid \( A \), and in Section 5, we show that when \([\Xi_A] = 0\), the isomorphic classes of such extensions with band \( \Lambda \) are bijective to \( H^2_{\text{aut}}(K, Z_A) \).

The study of extension in Lie groupoids is originally motivated by the study of Gromov–Witten theory of orbifolds ([10]) in terms of orbifold groupoids (i.e. proper étale Lie groupoids). This is due to an alternative view of (1.2): roughly speaking, \( G \) may be thought as a family of \( A \) parameterized by \( K \). In the Gromov–Witten theory of orbifolds, it is inevitable to consider families of curves parameterized by the Deligne–Mumford moduli spaces which are Lie groupoids (cf. [9]). This issue will be addressed in [5]. Note that, in this case, when \( K \) is a point, \( \Lambda \) is the groupoid \( \text{Mor}(\Sigma, M) \) of morphisms from an orbifold surface to a symplectic orbifold \( M \), which is partially studied in [6]. Apparently, similar issues appear once one studies moduli spaces which involve orbifolds.

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Another interesting case is to consider the symplectic extension. Suppose that we have a groupoid extension given by (1.2), one may ask when G can be a symplectic orbifold if A and K are symplectic orbifolds. This will be explained in Section 6. Furthermore, it is natural to ask the relations among the Gromov–Witten invariants of A, K and G. For instance, Tang and Tseng ([19]) studied some special cases of that G is an étale gerbe over a symplectic orbifold K. We will be studied the Gromov-Witten theory for general fibrations in the future.

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2. Preliminaries

2.1. Lie groupoids. We briefly review basic concepts of Lie groupoids. One may be referred to [1, 15, 16, 18] for more details.

A Lie groupoid, denoted by $G = (G^1 \rightrightarrows G^0)$, consists of two smooth manifolds $G^0$ and $G^1$, and structure maps:

1. the source and target maps by $s : G^1 \to G^0$ and $t : G^1 \to G^0$ respectively;
2. the composition/multiplication map $\mu : K^2 := K^1 \times_{t,s} K^1 \to K^1, m(g, h) = gh$ or $g \cdot h$;
3. the unit map by $u : G^0 \to G^1$, and $u(a)$ by $1_a$;
4. the inverse map $i : G^1 \to G^1$, and $i(g)$ by $g^{-1}$.

We denote an arrow in $G$ by $g : x \to y$. We denote the coarse space of $G$ by $\overline{G} = G^0/G^1$, which has the quotient topology, and the quotient map to the coarse space by $\pi : G^0 \to \overline{G}$.

$G$ is proper if $s \times t : G^1 \to G^0 \times G^0$ is proper; $G$ is étale if both $s$ and $t$ are local diffeomorphisms. A proper étale Lie groupoid is called an orbifold groupoid.

Let $G, H$ be two Lie groupoids. A strict Lie groupoid morphism (or strict morphism for short) $f : G \to H$ is a functor $f = (f^0, f^1)$ with both $f^0$ and $f^1$ are smooth maps. A strict Lie groupoid morphism $f : G \to H$ induces a continuous map on coarse space $f : \overline{G} \to \overline{H}$.

A natural transformation $\sigma : f \Rightarrow g : G \to H$ between two strict morphisms is a smooth map $\sigma : G^0 \to H^1$ such that for every arrow $g : x \to y$ in $G^1$ there is a commutative diagram in $H^1$:

$$
\begin{array}{ccc}
G^0(x) & \xrightarrow{\sigma(x)} & G^0(y) \\
\downarrow{f^0(g)} & & \downarrow{g^0(h)} \\
G^0(y) & \xrightarrow{\sigma(y)} & G^0(y)
\end{array}
$$

A strict Lie groupoid morphism $f : G \to H$ is called an equivalence if

1. $t \circ pr_2 : G^1 \times_{f^0, s} H^1 \to H^0$ is a surjective submersion, and
2. the square

$$
\begin{array}{ccc}
G^1 & \xrightarrow{f^1} & H^1 \\
\downarrow{s \times t} & & \downarrow{s \times t} \\
G^0 \times G^0 & \xrightarrow{f^0 \times f^0} & H^0
\end{array}
$$

is a fiber product.

Remark 2.1. Lie groupoids together with strict morphisms and natural transformations between strict morphisms form a 2-category $2Gpd$. We denote the vertical and horizontal compositions of natural transformations in $2Gpd$ by “$\circ$” and “$\circledast$” respectively. Therefore, for three natural transformations $\rho_1 : f \Rightarrow g : A \to B, \rho_2 : g \Rightarrow h : A \to B$ and $\rho_3 : k \Rightarrow j : B \to C$ we have $\rho_1 \circ \rho_2 : f \Rightarrow h : A \Rightarrow B$ with

$$
\rho_1 \circ \rho_2(a) = \rho_1(a) \cdot \rho_2(a), \quad \forall a \in A^0,
$$

and $\rho_3 \circ \rho_1 : k \circ f \Rightarrow j \circ g : A \Rightarrow C$ with

$$
\rho_3 \circ \rho_1(a) = (k^1 \circ \rho_1(a)) \cdot (\rho_3 \circ g^0)(a), \quad \forall a \in A^0.
$$

Moreover, the category of strict morphisms $SMor(A, B) = (SMor^1(A, B) \rightrightarrows SMor^0(A, B))$ is a groupoid with multiplication over $SMor^1(A, B)$ being the vertical composition of natural transformation.

\footnote{Note that this is different from the usual notation, in which $s(g) = t(h)$, that is the arrows goes from the right to the left. In this paper we use the convention that the composition of arrows goes from left to right.}
2.2. **Refinements of Lie groupoid via open covers.** Let \( G = (G^1 \rightrightarrows G^0) \) be a Lie groupoid. Let \( \mathcal{U} = \{ U_a \mid a \in \mathcal{A} \} \) be an open cover of \( G^0 \). Then we can form the **refinement groupoid** (or pullback groupoid) \( G[\mathcal{U}] \) as follows:

1. The object space is

\[
G[\mathcal{U}]^0 := \bigsqcup_{a \in \mathcal{A}} U_a,
\]

the disjoint union of open subsets in \( \mathcal{U} \); there is a natural inclusion map \( q^0_{\mathcal{U}} : G[\mathcal{U}]^0 \to G^0 \) that embeds each \( U_a \) into \( G^0 \);

2. The arrow space is

\[
G[\mathcal{U}]^1 = G[\mathcal{U}]^0 \times_{q^0_{\mathcal{U}}, s} G^1 \times_{t, q^0_{\mathcal{U}}} G[\mathcal{U}]^0 = \{(x, g, y) \in G[\mathcal{U}]^0 \times G^1 \times G[\mathcal{U}]^0 \mid g : q^0_{\mathcal{U}}(x) \to q^0_{\mathcal{U}}(y)\};
\]

there is also a natural map \( q^1_{\mathcal{U}} : G[\mathcal{U}]^1 \to G^1 \) that projects to the second factor.

With the obvious five structure maps we see that \( G[\mathcal{U}] := (G[\mathcal{U}]^1 \rightrightarrows G[\mathcal{U}]^0) \) is a Lie groupoid. Moreover

\[
q_{\mathcal{U}} := (q^0_{\mathcal{U}}, q^1_{\mathcal{U}}) : G[\mathcal{U}] \to G
\]

is an equivalence of Lie groupoids.

If there is another open cover \( \mathcal{W} = \{ W_b \mid b \in \mathcal{B} \} \) that refines \( \mathcal{U} \) via an refine map \( \iota_{\mathcal{W}, \mathcal{U}} : \mathcal{W} \to \mathcal{U} \), then it is direct to see that \( \iota_{\mathcal{W}, \mathcal{U}} \) induces a Lie groupoid equivalence \( \iota_{\mathcal{W}, \mathcal{U}} : G[\mathcal{W}] \to G[\mathcal{U}] \). Moreover we have the following commutative diagram of Lie groupoid equivalences

\[
\begin{array}{ccc}
G[\mathcal{W}] & \xrightarrow{\iota_{\mathcal{W}, \mathcal{U}}} & G[\mathcal{U}] \\
\downarrow{q_{\mathcal{W}}} & & \downarrow{q_{\mathcal{U}}} \\
K[\mathcal{W}] & \xrightarrow{q_{\mathcal{U}}} & K[\mathcal{U}]
\end{array}
\]

Suppose \( \phi : G \to K \) is a strict morphism and \( \mathcal{U} \) is an open cover of \( K^0 \). \( \phi \) pulls back \( \mathcal{U} \) to an open cover \( \phi^* \mathcal{U} \) of \( G^0 \). Then we have a pullback strict morphism \( q_{\mathcal{U}}^\phi : G[\phi^* \mathcal{U}] \to K[\mathcal{U}] \) and a commutative diagram of strict morphisms

\[
\begin{array}{ccc}
G[\phi^* \mathcal{U}] & \xrightarrow{q_{\phi^* \mathcal{U}}} & G \\
\downarrow{q_{\mathcal{U}}} & & \downarrow{\phi} \\
K[\mathcal{U}] & \xrightarrow{q_{\mathcal{U}}} & K
\end{array}
\]

2.3. **Center of groupoids.** We recall the center for a groupoid that is defined in \([6, \S 5.1]\). Let \( A \) be a given groupoid. For every \( x \in A^1 \), let \( \Gamma_x := \{ g \in A^1 \mid g : x \to x \} \) be the isotropy group of \( x \). Let \( Z^0 A := \{ g \in Z \Gamma_x \mid x \in A^0 \} \subseteq A^1 \) where \( Z \Gamma_x \) means the center of the group \( \Gamma_x \). There is an \( A \)-action\(^2\) on \( Z^0 A \), of which the anchor map is the natural projection

\[
p = s = t : Z^0 A \to A^0, \quad Z \Gamma_x \ni g \mapsto x
\]

and the action map is

\[
\mu : A^1 \times_{s,p} Z^0 A \to Z^0 A, \quad (h, g) \mapsto h^{-1} g h.
\]

The center groupoid \( ZA \) is the action groupoid

\[
ZA := A \rtimes Z^0 A = (A^1 \times_{s,p} Z^0 A \rightrightarrows Z^0 A),
\]

whose source map is projection to the second factor and target map is \( \mu \). Other structure maps are obvious. We have a natural projection \( p : ZA \to A \) with

\[
p^0 = p : Z^0 A \to A^0, \quad p^1 = pr_1 : A^1 \times_{s,p} Z^0 A \to A^1.
\]

**Definition 2.2.** The **center** of \( A \) is the set of global sections of \( p : ZA \to A \).

It is clear that \( ZA \) is an abelian group.

\(^2\)See for example \([8]\) for the definition of groupoid actions on spaces.
2.4. Automorphism groupoid of Lie groupoids. In [6] we studied the automorphism groupoid of a given groupoid. Here, we consider a simpler version which is used in this paper.

Let $A = (A^1 \rightarrow A^0)$ be a Lie groupoid. Denote the identity morphism of $A$ by $\text{id}_A$. A strict Lie groupoid morphism $f : A \rightarrow A$ is called a strict automorphism of $A$ if there is another Lie groupoid morphism $g : A \rightarrow A$ such that $f \circ g = \text{id}_A$ and $g \circ f = \text{id}_A$.

Let $\text{SAut}^0(A)$ be the set of automorphisms of $A$. With composition of functors as the multiplication, it becomes a group. Let $\text{SAut}^1(A)$ being the space of natural transformations between strict automorphisms of $A$. Then we have a groupoid of strict automorphisms

$$\text{SAut}(A) = (\text{SAut}^1(A) \rightrightarrows \text{SAut}^0(A)).$$

The structure maps are obviously, and the multiplication in $\text{SAut}^1(A)$ is the vertical composition "$\circ$" between natural transformations\(^{3}\). Let $s_{\text{saut}}, t_{\text{saut}} : \text{SAut}^1(A) \rightrightarrows \text{SAut}^0(A)$ denote the source and target maps. Set $s^{-1}_{\text{saut}}(\text{id}_A)$ to be $N_A$.

**Proposition 2.3.** We have the following facts.

1. $N_A$ is a group with respect to $\circ$.
2. $\text{SAut}(A)$ is a quotient groupoid $\text{SAut}^0(A) \rtimes N_A$.
3. The stabilization of $N_A$ action at $\text{id}_A$ is $Z_A$. So is the stabilization of other automorphisms in $\text{SAut}^0(A)$.

**Proof.** (1), we could write down composition $\circ$ explicitly. First note that we have

$$N_A = \{ \sigma \in C^\infty(A^0, A^1) \mid s\sigma = id_A, \text{ and } \sigma \text{ is a diffeomorphism} \}.$$  \hspace{1cm} (2.5)

Let $\sigma, \gamma \in N_A$, we have (comparing with (2.3))

$$\gamma \circ \sigma(x) = \sigma(x) \cdot \gamma(t(\sigma(x))), \quad \text{i.e. } x \xrightarrow{\sigma(x)} y \xrightarrow{\gamma(y)} z.$$

The inverse of $\sigma$ is given by $\sigma^{-1}(x) = \sigma((t \circ \sigma)^{-1}(x))^{-1}$. That is, if $\sigma : \text{id}_A \Rightarrow f$, then $\sigma^{-1} : \text{id}_A \Rightarrow f^{-1}$, and $\sigma \circ \sigma^{-1} = \text{id}_A$.

(2) First of all, we write down the action of $N_A$ on $\text{SAut}^0(A)$. Take an $\alpha : \text{id}_A \Rightarrow f \in N_A$ and $g \in \text{SAut}^0(A)$. Then we set

$$\alpha(g) := t_{\text{saut}}(\alpha) \circ g = f \circ g.$$

This is an $N_A$-action since

$$(\beta \circ \alpha)(g) = t_{\text{saut}}(\beta \circ \alpha) \circ g = t_{\text{saut}}(\beta) \circ t_{\text{saut}}(\alpha) \circ g = \beta(\alpha(g)).$$

We next give an isomorphism $\phi : \text{SAut}(A) \rightarrow \text{SAut}^0(A) \rtimes N_A$. We set $\phi^0 = \text{id}_{\text{SAut}^0(A)}$, and

$$\phi^1(\alpha : f \Rightarrow g) = \alpha \rtimes 1_{f^{-1}} : \text{id}_A \Rightarrow g \circ f^{-1}.$$

The inverse of $\phi$ is

$$\psi^0 = \text{id}_{\text{SAut}^0(A)}, \quad \text{and } \psi^1(\alpha : \text{id}_A \Rightarrow f) = \alpha \rtimes 1_g.$$

We next show that $\psi$ is a morphism. We compute

$$\psi^1((f \circ g, \beta : \text{id}_A \Rightarrow h) \ast (g, \alpha : \text{id}_A \Rightarrow f)) = \psi^1(g, \beta \circ \alpha : \text{id}_A \Rightarrow h \circ f) = \beta \circ \alpha \rtimes 1_g$$

and

$$\psi^1(f \circ g, \beta : \text{id}_A \Rightarrow h) \circ \psi^1(g, \alpha : \text{id}_A \Rightarrow f) = (\beta \circ 1_{\text{fog}}) \circ (\alpha \rtimes 1_g) = (\beta \circ 1_{\text{fog}}) \circ (1_{\text{id}_A} \circ (\alpha \rtimes 1_g)) = (\beta \circ 1_{\text{id}_A}) \circ (\alpha \rtimes 1_g).$$

Therefore $\psi$ is a morphism, and $\text{SAut}(A) \cong \text{SAut}^0 \rtimes N_A$.

(3) Suppose that $\sigma \in N_A$ satisfies $\sigma : \text{id}_A \Rightarrow \text{id}_A$, then by the commutative diagram (2.1) we get $\sigma$ is a section of $p : Z_A \rightarrow A$, hence $\sigma \in Z_A$. It follows from the definition of $N_A$-action, the stabilization of every automorphism in $\text{SAut}^0(A)$ is the same as the one of $\text{id}_A$. This finishes the proof.

**Lemma 2.4.** The image $t_{\text{saut}}(N_A)$ is a normal subgroup of $\text{SAut}^0(A)$.  

\(^{3}\)In fact, $\text{SAut}(A)$ is a subgroupoid of $\text{SMor}(A, A)$. 

Proof. Suppose $\sigma : \text{id}_A \Rightarrow f$ and $h \in \text{SAut}^0(A)$. Let $1_h : h \Rightarrow h$ and $1_{h^{-1}} : h^{-1} \Rightarrow h^{-1}$ be the identity natural transformations over $h$ and $h^{-1}$ respectively. Then

$$1_h \circ \sigma \circ 1_{h^{-1}} : h \circ \text{id}_A \circ h^{-1} \Rightarrow h \circ f \circ h^{-1}.$$  

It follows from $h \circ \text{id}_A \circ h^{-1} = \text{id}_A$ that $h \circ f \circ h^{-1} = \rho(1_h \circ \sigma \circ 1_{h^{-1}})$. This finishes the proof. \hfill $\square$

To summarize, we have an exact sequence of groups

$$1 \longrightarrow Z_A \xrightarrow{\text{core}} N_A \xrightarrow{\text{taut}} \text{SAut}^0(A) \xrightarrow{\pi} \text{SAut}^0(A)/\text{Im} \text{taut} \longrightarrow 1. \quad (2.6)$$

**Corollary 2.5.** The coarse space $\overline{\text{SAut}(A)}$ is the group isomorphic to $\text{SAut}^0(A)/\text{Im} \text{taut}$.

Therefore the exact sequence (2.6) can also be written as

$$1 \longrightarrow Z_A \xrightarrow{\text{core}} N_A \xrightarrow{\text{taut}} \overline{\text{SAut}^0(A)} \xrightarrow{\pi} \overline{\text{SAut}(A)} \longrightarrow 1. \quad (2.7)$$

In the following we will use the classifying groupoid $(\overline{\text{SAut}(A)} \xrightarrow{\pi} \bullet)$ associated to the group $\overline{\text{SAut}(A)}$. For a Lie groupoid $K = (K^1 \xrightarrow{\rho} K^0)$, by a morphism $K^1 \rightarrow \overline{\text{SAut}(A)}$ we mean a strict Lie groupoid morphism $K \rightarrow (\overline{\text{SAut}(A)} \xrightarrow{\pi} \bullet)$.

### 2.5. Symplectic orbifold groupoid

We will also consider symplectic structure over orbifold groupoids (see for example [13]). Let $A = (A^1 \xrightarrow{\rho} A^0)$ be an orbifold groupoid. A **symplectic form** over $A$ is a symplectic form $\omega$ over $A^0$ such that

$$s^* \omega = t^* \omega$$

over $G^1$. A **symplectic orbifold groupoid** is an orbifold groupoid with a symplectic form.

Let $(A, \omega)$ be a symplectic orbifold groupoid. An automorphism $f \in \text{SAut}^0(A)$ is called a **symplectomorphism** or preserves the symplectic form, if

$$f^0, *\omega = \omega$$

over $A^0$.

### 3. Lie groupoid extensions

#### 3.1. Locally trivial fiber bundles

**Definition 3.1.** A topological product fiber bundle with fiber $A = (A^1 \xrightarrow{\rho} A^0)$ is a strict Lie groupoid morphism $\phi = (\phi^0, \phi^1) : G = (G^1 \xrightarrow{\rho} G^0) \rightarrow K = (K^1 \xrightarrow{\rho} K^0)$ such that

1. $G^1 = A^1 \times K^1$ and $\phi^1$ is the projection to the second factor for $i = 0, 1$;
2. the source map satisfies $s_G = s_A \times s_K$, and
3. the unit map satisfies $u_G = u_A \times u_K$, and
4. the target map satisfies the equation $t_G(\alpha, 1_x) = (t_A(\alpha), x)$.

$G \xrightarrow{\phi} K$ is called a **topologically trivial fiber bundle** if $G$ is isomorphic to a topological product fiber bundle.

$G \xrightarrow{\phi} K$ is called a **locally trivial fiber bundle** if there exists an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $K^0$ such that the pullback strict morphism

$$q_{U, \alpha}^* \phi : G|_{\phi^* \mathcal{U}} \rightarrow K|_{\mathcal{U}}$$

is a topologically trivial fiber bundle. $A$ is called the **fiber** of the bundle.

From this definition we see that the kernel$^4$ of a locally trivial fiber bundle $G \xrightarrow{\phi} K$,

$$\ker \phi := (\ker \phi^1 \xrightarrow{\rho} G^0) = \phi^{-1}(u(K^0) \xrightarrow{\rho} K^0), \quad (3.1)$$

is a locally trivial bundle of groupoids over $K^0$ with fiber isomorphic to $A$.

---

$^4$The kernel $\ker \phi$ is a subgroupoid of $G$. So we have the restriction morphism $\phi : \ker \phi \rightarrow K$ with images being $(u(K^0) \xrightarrow{\rho} K^0) = (K^0 \xrightarrow{\rho} K^0)$, the trivial groupoid that representing the manifold $K^0$. So we may view $\ker \phi$ as a groupoid fiber bundle over $K^0$ and write is as $\phi : \ker \phi \rightarrow K^0$. 
Definition 3.2. Two topologically trivial fiber bundles with fiber $A$, $G_i \overset{\phi_i}{\to} K_i$, $i = 1, 2$, are isomorphic if there is a groupoid isomorphism $G_1 \to G_2$ satisfying the following commutative diagram

$$
\begin{array}{ccc}
\ker \phi_1 & \overset{\subset}{\to} & G_1 \\
\downarrow & \simeq & \downarrow \\
\ker \phi_2 & \overset{\subset}{\to} & G_2 \\
\end{array}
$$

Let $\ker \phi_1 \hookrightarrow G_1 \overset{\phi_1}{\to} K_1$ and $\ker \phi_2 \hookrightarrow G_2 \overset{\phi_2}{\to} K_2$ be two fiber bundles with fiber $A$. We say that they are equivalent, if there are refinements $\mathcal{U}_i$, $i = 1, 2$, of $K_i$ such that the topologically trivial bundles $G_i[\mathcal{U}_i]$, $i = 1, 2$, are isomorphic.

It is direct to see that this is an equivalent relation between all locally trivial fiber bundles.

3.2. Lie groupoid extensions. Given a topological product fiber bundle $G = A \times K \overset{\phi}{\to} K$, for each arrow $x \overset{\xi}{\to} y$ in $K^1$, it induces a strict morphism from $(A, x)$ to $(A, y)$ via

$$
\begin{align*}
\Lambda_0^x : (a, x) & \overset{1_x \cdot \xi}{\to} (b, y); \quad b = t_\xi(1_x, \xi) \\
\Lambda_1^x : (a, 1_x) & \overset{}{\to} (\beta, 1_y); \quad \beta = (1_{1_y}(\alpha, \xi)^{-1}(1_x)(1_{t_\xi(\alpha, \xi)})
\end{align*}
$$

(3.2)

This induces a morphism from $A$ to $A$

$$
\Lambda : K^1 \to \text{SMor}^0(A, A); \quad \xi \mapsto \Lambda_\xi.
$$

(3.4)

From the definition of topological product fiber bundle one can easily see that

Lemma 3.3. (1) The map $\Lambda$ is smooth in the sense that the induced map $K^1 \times A \to A$ is smooth, (2) $\Lambda_{1_x} = \text{id}_A$ for every $1_x \in K^1$, (3) $\Lambda$ induces a morphism

$$
\bar{\Lambda} = \pi \circ \Lambda : K^1 \to \overline{\text{SMor}}(A, A).
$$

Definition 3.4. We call this $\Lambda$ the pre-action map of $G$ if the image of $\Lambda$ is in $\text{SAut}^0(A) \subseteq \text{SMor}^0(A, A)$. In this case we call $G = A \times A \overset{\phi}{\to} K$ a topological product Lie groupoid extension of $K$ by $A$, and the morphism

$$
\bar{\Lambda} : K^1 \to \overline{\text{SAut}}(A)
$$

the band of $G = A \times K \overset{\phi}{\to} K$.

Inspired by (3.1), we make the following definition.

Definition 3.5. A (locally trivial) fiber bundle $\phi : G \to K$ with fiber $A$ is called a Lie groupoid extension of $K$ by $A$ if there is a refinement $G[\phi^*\mathcal{U}] \to K[\mathcal{U}]$ that is isomorphic to a topological product Lie groupoid extension of $K[\mathcal{U}]$ by $A$. We write the extension as

$$
1 \to A \to G \overset{\phi}{\to} K \to 1.
$$

(3.5)

We call the band of $G[\phi^*\mathcal{U}] \to K[\mathcal{U}]$ the band of $G \to K$.

Definition 3.6. Two topological product Lie groupoid extensions with fiber $A$, $G_i \overset{\phi_i}{\to} K$, $i = 1, 2$ are isomorphic if the underlying topological product fiber bundle are isomorphic.

Two Lie groupoid extensions of $K$ by $A$, $G_i \overset{\phi_i}{\to} K$, $i = 1, 2$ are equivalent if the underlying fiber bundles are equivalent.
3.3. **Generalized cocycles.** In this subsection we assume that \( G = A \times K \xrightarrow{\phi} K \) is a topological product Lie groupoid extension, with pre-action map \( \Lambda : K^1 \to \text{SAut}^0(A) \) given by (3.4). For simplicity, for every arrow \( \xi \in K^1 \), we denote the corresponding strict automorphism by \( \Lambda_\xi = (\Lambda_\xi, \Lambda_\xi) \). Hence \( \Lambda_\xi = idA \) for all \( x \in K^0 \).

In general the pre-action map \( \Lambda : K^1 \to \text{SAut}^0(A) \) is not a morphism. The default of \( \Lambda \) being a homomorphism is measured by the following smooth map:

\[
\Omega : K^2 \times A^0 = (K^1 \times_{K^0, s} K^1) \times A^0 \to G^1, \quad \Omega(\xi, \eta, a) := (\xi, 1_a) \cdot (\eta, 1_{\Lambda_\xi(a)}) \cdot (\xi \eta, 1_{\Lambda_\xi \Lambda_\eta(a)})^{-1}. \tag{3.6}
\]

We call \( \Omega \) the **cofactor** associated to the pre-action map \( \Lambda \).

The images of \( \Omega \) lie in the kernel of \( \phi : G = A \times K \to K \), since

\[
\phi^1(\Omega(\xi, \eta, a)) = \xi \cdot \eta \cdot (\xi \eta)^{-1} = 1_{s(\xi)}.
\]

So in the rest of this subsection, we identify \( \Omega \) as the map

\[
\Omega : K^2 \times A^0 \to \ker \phi^1 \to A^1,
\]

and \( \Omega(\xi, \eta, a) \) will denote an element in \( A^1 \) and also an element \( (\Omega(\xi, \eta, a), 1_{s(\xi)}) \) in \( G^1 \).

We also have

\[
\Omega(\xi, 1_{t(\xi)}, a) = (1_{s(\xi)}, 1_a) = 1_{t(s(\xi)), a}, \quad \Omega(1_{s(\xi)}, \xi, a) = (1_{s(\xi)}, 1_a) = 1_{t(t(\xi)), a}. \tag{3.7}
\]

The associativity of multiplication in \( G^1 \) gives rise to some constraints on \( \Lambda \) and \( \Omega \), which we call the **generalized cocycle condition.** We next write down them explicitly.

First of all take three composable arrows \( \xi, \eta, \zeta \in K^1 \), and suppose

\[
\begin{align*}
  x & \xrightarrow{\xi} y \xrightarrow{\eta} z \xrightarrow{\zeta} w.
\end{align*}
\]

Take an \( a \in A^0 \). We have the following diagram of arrows in \( G^1 = K^1 \times A^1 \)

\[
\begin{array}{cccc}
(x, a) & \xrightarrow{(\xi, 1_a)} & (y, a_1) & \xrightarrow{(\eta, 1_{a_2})} & (z, a_2) & \xrightarrow{(\zeta, 1_{a_3})} & (w, a_3) \\
\xi & \xrightarrow{\phi^0} & \eta & \xrightarrow{\phi^0} & \zeta & \xrightarrow{\phi^0} & \omega.
\end{array}
\]

Therefore

\[
\begin{align*}
  \Lambda_\xi^{-1} \Lambda_\eta \Lambda_\zeta(a) & = a_4, \\
  \Lambda_\xi^{-1} \Lambda_\zeta \Lambda_\eta(a_1) & = a_6, \\
  \Lambda_\xi^{-1} \Lambda_\eta \Lambda_\xi(a) & = a_7.
\end{align*}
\]

Then

\[
\begin{align*}
  & \Omega(\xi, \eta, a) \cdot \Omega(\xi \eta, \zeta, a_4) \cdot (\xi \eta \zeta, 1_{a_7}) \\
  = & \ (\xi, 1_a) \cdot (\eta, 1_{a_1}) \cdot (\xi \eta, 1_{a_4})^{-1} \cdot (\xi \eta, 1_{a_4}) \cdot (\zeta, 1_{a_2}) \cdot (\xi \eta \zeta, 1_{a_7})^{-1} \cdot (\xi \eta \zeta, 1_{a_7}) \\
  = & \ (\xi, 1_a) \cdot (\eta, 1_{a_1}) \cdot (\zeta, 1_{a_2}) \\
  = & \ (\xi, 1_a) \cdot \Omega(\eta, \zeta, a_1) \cdot (\eta \zeta, 1_{a_6}) \\
  = & \ (\xi, 1_a) \cdot \Omega(\eta, \zeta, a_1) \cdot (\zeta, 1_{a_3})^{-1} \cdot (\xi, 1_{a_3}) \cdot (\eta \zeta, 1_{a_6}) \\
  = & \ (\xi, 1_a) \cdot \Omega(\eta, \zeta, a_1) \cdot (\zeta, 1_{a_3})^{-1} \cdot \Omega(\xi, \eta \zeta, a_5) \cdot (\xi \eta \zeta, 1_{a_7}).
\end{align*}
\]
which is
\[ \Omega(\xi, \eta, a) \cdot \Omega(\xi, \zeta, a_2) = \Lambda^{-1}_\xi(\Omega(\eta, \zeta, a_1)) \cdot \Omega(\xi, \eta, a_2). \] (3.8)

As a special case, let \( \eta = \xi^{-1} \) with \( a_1 = \Lambda_\xi(a) \) and \( a_2 = \Lambda^{-1}_\xi(a) \); in addition with (3.7) we get
\[ \Omega(\xi, \xi^{-1}, a) = \Lambda^{-1}_\xi(\Omega(\xi^{-1}, \xi, a_1)). \] (3.9)

We call (3.8) the \textit{generalized cocycle condition}, and \((\Lambda, \Omega)\) a \textit{generalized cocycle}.

The analysis above shows that for a topological product Lie groupoid extension \( G = A \times K \overset{\phi}{\to} K \), the pre-action map \( \Lambda : K^1 \to \text{SAut}^0(A) \) and the band \( \Lambda : K^1 \to \bar{\text{SAut}}(A) \) fit into the following diagram

\[
\begin{array}{ccc}
K^1 & \xrightarrow{\Lambda} & \bar{\text{SAut}}(A) \\
\downarrow & & \\
1 & \xrightarrow{\pi} & \text{SAut}(A) \\
\end{array}
\] (3.10)

The obstruction of \( \Lambda \) being a morphism is recorded by the smooth map \( \Omega : K^{[2]} \times A^0 \to G^1 \) (cf. (3.6)). The pair \((\Lambda, \Omega)\) satisfies the generalized cocycle condition (3.8).

4. \textsc{Existence and Obstruction of Lie Groupoid Extensions}

In this section we study the existence and obstruction of Lie groupoid extensions. Let \( K \) and \( A \) be two Lie groupoids. Suppose we have a morphism \( \bar{\Lambda} : K^1 \to \bar{\text{SAut}}(A) \) and a smooth lifting \( \Lambda \) of \( \bar{\Lambda} \) (cf. (3.10)) satisfying \( \Lambda_1 x := \Lambda(1_x) = \text{id}_A \) for all \( x \in K^0 \).

4.1. \textbf{Existence of Lie groupoid extensions}. In this subsection we study the question when will there is a Lie groupoid extension of \( K \) by \( A \) with band \( \Lambda \).

For two composable arrows \( \xi, \eta \in K^1 \) we have
\[ \pi(\Lambda^{-1}_\xi \circ \Lambda_\eta \circ \Lambda_\xi) = [\text{id}_A] \in \bar{\text{SAut}}(A). \]

Therefore there is a natural transformation \( \Phi_{\xi, \eta} : \text{id}_A \Rightarrow \Lambda^{-1}_\xi \circ \Lambda_\eta \circ \Lambda_\xi : A \to A \). We define
\[ \Omega : K^{[2]} \times A^0 \to A^1, \quad \Omega(\xi, \eta, a) := \Phi_{\xi, \eta}(a). \] (4.1)

In particular, when \( \xi = 1_x \) or \( \eta = 1_y \) we have
\[ \Omega(1_x, \eta, a) = 1_a, \quad \Omega(\xi, 1_y, a) = 1_a. \] (4.2)

For each \( \xi \in K^1 \), we omit the superscripts and write the corresponding automorphism \( \Lambda_\xi \) of \( A \) by
\[ \Lambda_\xi := (\Lambda_\xi, \Lambda_\xi) \]
for simplicity. Therefore we have the commutative diagram

\[
\begin{array}{ccc}
a & \xrightarrow{\Omega(\xi, \eta, a)} & \Lambda^{-1}_\eta \Lambda_\xi(a) \\
\downarrow & & \downarrow \Lambda^{-1}_\eta \Lambda_\xi(a) \\
b & \xrightarrow{\Omega(\xi, \eta, b)} & \Lambda^{-1}_\eta \Lambda_\xi(b), \\
\end{array}
\]
for every arrow \( \alpha \in A^1 \). So for every arrow \( \alpha \in A^1 \) we have
\[ \Lambda^{-1}_\eta \Lambda_\xi(\alpha) = \Omega(\xi, \eta, s(\alpha))^{-1} \cdot \alpha \cdot \Omega(\xi, \eta, t(\alpha)). \] (4.3)

\textbf{Theorem 4.1}. Let \( \Lambda : K^1 \to \text{SAut}^0(A) \) be a smooth lifting of the morphism \( \bar{\Lambda} : K^1 \to \bar{\text{SAut}}(A) \). Suppose that \( \Omega \), defined in (4.1), is smooth. If \((\Lambda, \Omega)\) satisfies the generalized cocycle condition (3.8), i.e.
\[ \Omega(\xi, \eta, a) \cdot \Omega(\xi, \zeta, a_2) = \Lambda^{-1}_\xi(\Omega(\eta, \zeta, a_1)) \cdot \Omega(\xi, \eta, a_2) \]
for every composable arrows \( \xi, \eta \in K^1 \) and objects \( a \in A^0 \), then there is a topological product Lie groupoid extension of \( K \) by \( A \), \( G \to K \), with band \( \Lambda \). Denote this Lie groupoid extension by \( A \times_{\Lambda, \Omega} K \).

\textbf{Proof}. We set \( G = A \times K = (A^1 \times K^1 \to A^0 \times K^0) \). We next define the structure maps:
(1) The source and target maps for $G$ are
\[ s_G(\alpha, \xi) := (s(\alpha), s(\xi)), \]
\[ t_G(\alpha, \xi) := (t(\Lambda_G(\alpha)), t(\xi)) = (\Lambda_G(t(\alpha)), t(\xi)), \]
where the $s$ and $t$ on the right hand side are the source and target maps for $K$ and $A$. We do not use subscripts to distinguish them since obviously there is no ambiguity.

(2) The multiplication over $A^1 \times K^1$ is
\[ (\alpha, \xi) \cdot (\beta, \eta) := \left( \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta), \xi \cdot \eta \right), \]
where the multiplications $\Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha)$ and $\xi \cdot \eta$ are taken in $A$ and $K$ respectively, and when there is no ambiguity we will omit the "·" in the following computation.

(3) The unit map is $u(x, a) := (1_x, 1_a)$.

(4) The inverse arrow of an arrow $(\alpha, \xi)$ is
\[ (\alpha, \xi)^{-1} := \left( \Lambda_\xi(\alpha^{-1}) \cdot \Lambda^{-1}_{\xi^{-1}}, [\Omega(\xi, \xi^{-1}, s(\alpha)^{-1}], \xi^{-1} \right). \]

All these maps are smooth. We next show that these structure maps actually give rise to a Lie groupoid structure over $G$. Once we prove this, it is obvious that the projections to the first factors give rise to a topological product Lie groupoid $A$-extension $G \to K$.

We first verify the associativity of the multiplication (4.6). Take three composable arrows $(\alpha, \xi), (\beta, \eta), (\gamma, \zeta) \in G^1$. We have
\[ ((\alpha, \xi) \cdot (\beta, \eta)) \cdot (\gamma, \zeta) \]
\[ = (\alpha, \xi) \cdot (\Omega(\eta, \zeta, s(\beta)) \cdot \Lambda^{-1}_{\eta \zeta} \Lambda_\zeta(\beta) \cdot \Lambda^{-1}_{\eta \zeta} \Lambda_\zeta(\gamma), \eta \cdot \zeta) \]
\[ = (\Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\gamma), \xi \cdot \eta \cdot \zeta) \]
and
\[ (\alpha, \xi) \cdot ((\beta, \eta) \cdot (\gamma, \zeta)) \]
\[ = (\alpha, \xi) \cdot (\Omega(\eta, \zeta, s(\beta)) \cdot \Lambda^{-1}_{\eta \zeta} \Lambda_\zeta(\beta) \cdot \Lambda^{-1}_{\eta \zeta} \Lambda_\zeta(\gamma), \eta \cdot \zeta) \]
\[ = (\Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\gamma), \xi \cdot \eta \cdot \zeta). \]

It is direct to see that the associativity follows from the following equality
\[ \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\gamma) \]
\[ = \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\beta) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta(\gamma). \]

To prove (4.7), we apply (4.3) to both sides. We have
\[ \text{LHS} = \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \]
\[ = \Omega(\xi, \eta, s(\alpha)) \cdot \left\{ \Omega(\xi, \eta, s(\alpha))^{-1} \cdot \left[ \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \right] \cdot \Omega(\xi, \eta, t(\Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha))) \right\} \]
\[ = \left[ \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \right] \cdot \Omega(\xi, \eta, t(\Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha))) \]
\[ = \alpha \cdot \Omega(\xi, \eta, t(\alpha)) \cdot \Omega(\xi, \eta, \zeta, t(\Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha))) \]
and
\[ \text{RHS} = \Omega(\xi, \eta, s(\alpha)) \cdot \Lambda^{-1}_{\xi \eta} \Lambda_\eta \Lambda_\xi(\alpha) \]
\[ = \Omega(\xi, \eta, s(\alpha)) \cdot \left\{ \Omega(\xi, \eta, s(\alpha))^{-1} \cdot \alpha \cdot \Lambda^{-1}_{\xi \eta} \Omega(\eta, \zeta, s(\beta)) \right\} \cdot \Omega(\xi, \eta, t(\Lambda^{-1}_{\xi \eta} \Omega(\eta, \zeta, s(\beta)))) \]
\[ = \alpha \cdot \Lambda^{-1}_{\xi \eta} \Omega(\eta, \zeta, s(\beta)) \cdot \Omega(\xi, \eta, t(\Lambda^{-1}_{\xi \eta} \Omega(\eta, \zeta, s(\beta)))). \]
Then since $\Omega$ and $S$ satisfies (3.8) we get LHS = RHS. Hence the multiplication is associative.

Now we show that the unit for this multiplication over $(a, x)$ is $u(a, x) = (1_a, 1_x)$. We have

\[(1_a, 1_x) \cdot (\alpha, \xi) = \left( \Omega(1_x, \xi, a) \cdot \Lambda^{-1}_a \Lambda_\xi (1_a) \cdot \Lambda^{-1}_x \Lambda_\xi (\alpha), 1_x \xi \right) = \left( \Omega(1_x, \xi, a) \cdot \alpha, \xi \right) = (\alpha, \xi) \]

where we have used $\Lambda_1 = id_A$ and $\Omega(1_x, \xi, a) = 1_a$. Similarly, since $\Omega(\xi, 1_y, a) = 1_a$ we get

\[(\alpha, \xi) \cdot (1_b, 1_y) = \left( \Omega(\xi, 1_y, a) \cdot \Lambda^{-1}_b \Lambda_\xi (\alpha) \cdot \Lambda^{-1}_x \Lambda_\xi (1_b), \xi 1_y \right) = \left( \Omega(\xi, 1_y, a) \cdot \alpha, \xi \right) = (\alpha, \xi). \]

We next show that the inverse map and multiplication are compatible. Note that for any $\xi \in K^1$ we have $\Lambda_{\xi^{-1}} = \Lambda_1 t(\xi) = id_A$ and $\Lambda_{\xi^{-1}} = \Lambda_1 s(\xi) = id_A$. Therefore

\[(\alpha, \xi) \cdot (\alpha, \xi)^{-1} = \left( \Omega(\xi, \xi^{-1}, s(a)) \cdot id_A \Lambda_{\xi^{-1}} (\alpha) \cdot id_A \Lambda_{\xi^{-1}} \left[ \Lambda_\alpha (\alpha^{-1}) \cdot \Lambda_{\xi^{-1}}^{-1} \Omega(\xi, \xi^{-1}, s(a))^{-1} \right], \xi^{-1} \right) = \left( \Omega(\xi, \xi^{-1}, s(a)) \cdot \Lambda_{\xi^{-1}} (\alpha) \cdot \Lambda_{\xi^{-1}}^{-1} \Omega(\xi, \xi^{-1}, s(a))^{-1}, 1_{\xi^{-1}} \right) = (1_{s(\alpha)}, 1_{s(\xi)}), \]

and

\[(\alpha, \xi)^{-1} \cdot (\alpha, \xi) = \left( \Omega(\xi, \xi^{-1}, t(\alpha \xi)) \cdot id_A \Lambda_{\xi^{-1}} (\alpha) \cdot id_A \Lambda_{\xi^{-1}} \left[ \Lambda_\alpha (\alpha^{-1}) \cdot \Lambda_{\xi^{-1}}^{-1} \Omega(\xi, \xi^{-1}, s(a))^{-1} \right], \xi^{-1} \right) = \left( \Omega(\xi, \xi^{-1}, t(\alpha \xi)) \cdot \Lambda_{\xi^{-1}} (\alpha) \cdot \Lambda_{\xi^{-1}}^{-1} \Omega(\xi, \xi^{-1}, s(a))^{-1}, \alpha^{-1} \right), \xi^{-1} \]

\[\overset{(0.3)}{=} \left( \Omega(\xi^{-1}, \xi, t(\alpha \xi)) \cdot \Lambda_{\xi^{-1}} (\alpha^{-1} \Omega(\xi, \xi^{-1}, t(\alpha \xi))), \xi^{-1} \right) \]

\[\overset{(0.9)}{=} (1_{t(\alpha \xi)}, 1_{t(\xi)}). \]

It is obvious from the definition of structure maps that the pre-action map is $\Lambda$, hence the band is $\bar{\Lambda}$. This finishes the proof. \hfill \Box

We have

**Corollary 4.2.** For a topological product Lie groupoid extension $G \to K$ of $K$ by $A$ with pre-action map $\Lambda$ given by (3.4) and the associated cofactor $\Omega$ given by (3.6), there is an isomorphism of topological product Lie groupoid extensions $\mathbb{G} \equiv A \rtimes \Lambda, \Omega K$.

**Proof.** The object part of this groupoid isomorphism is identity. Take an arrow $(\alpha, \xi) \in G^1 = A^1 \times K^1$. Suppose $t(\alpha, \xi) = b$. Then the arrow part of this groupoid isomorphism is

\[(\alpha, \xi) \mapsto \left( pp_2 \left[ (\alpha, \xi) \cdot_G (1_{\Lambda_{\xi^{-1}}(b)^{-1}}), \xi \right], \xi \right). \]

The inverse on arrows is given by $(\alpha, \xi) \mapsto (\alpha, 1_{s(\xi)}) \cdot_G (1_{t(\alpha)}, \xi)$. It is direct to verify that this is an isomorphism of topological product Lie groupoid extensions of $K$ by $A$. \hfill \Box

### 4.2. Obstruction of Lie groupoid extensions.

Let $\Lambda$ be a smooth lifting of a morphism $\bar{\Lambda}: K^1 \to \mathcal{S}\text{Aut}(A)$ and $\bar{\Omega}: K^{[2]} \times A^0 \to A^1$ be a smooth family of natural transformations defined as (4.1). In this subsection we study the question when will $(\Lambda, \bar{\Omega})$ satisfy the generalized cocycle condition (3.8).
Take an \( a \in A^0 \) and three composable arrows \( \xi, \eta, \zeta \) in \( K^1 \), i.e., \( (\xi, \eta, \zeta) \in K^{[3]} = K^1 \times_{t,s} K^1 \times_{t,s} K^1 \). Consider the following diagram of automorphisms and natural transformations:

\[
\begin{array}{cccccc}
\xi & \Rightarrow & \eta & \Rightarrow & \zeta & \Rightarrow \\
\Omega(\xi, \eta, a_4) & \Rightarrow & \Lambda^{-1}_\xi(\Omega(\eta, \zeta, a_4)) & \Rightarrow & \Omega(\eta, \zeta, a_4) & \Rightarrow \\
\Delta_{\zeta} & \Rightarrow & \Lambda_{\xi} & \Rightarrow & \Lambda_{\eta} & \Rightarrow \\
\Delta_{\eta} & \Rightarrow & \Lambda_{\gamma} & \Rightarrow & \Lambda_{\eta} & \Rightarrow \\
\end{array}
\]

Take an arrow \( \alpha \in \Gamma_a \), the isotropy group of \( a \) in \( A \). Then by the definition of \( \Omega \) we have the following equalities in \( A^1 \)

\[
\Omega(\xi, \eta, a_4)^{-1} \cdot \Omega(\xi, \eta, a)_1 \cdot \Omega(\xi, \eta, a_4) = \Omega(\xi, \eta, a_4)^{-1} \cdot \Lambda_{\xi}^{-1} \Lambda_{\eta} \Lambda_{\xi}(\alpha) \cdot \Omega(\xi, \eta, a_4)
\]

and

\[
\Omega(\xi, \eta, a_5)^{-1} \cdot \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, a_4, a_1) \cdot \Lambda_{\eta} \Lambda_{\xi}(\alpha)
\]

It follows from these two equalities that the following arrow

\[
\Xi(\xi, \eta, \zeta, a) := \Omega(\xi, \eta, a) \cdot \Omega(\xi, \eta, \zeta, a_4) \cdot \Omega(\xi, \eta, a_5)^{-1} \cdot \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, a_4, a_1)
\]

belongs to the center of \( \Gamma_a \). Thus we get a smooth map

\[
\Xi : K^{[3]} \times A^0 = K^1 \times_{t,s} K^1 \times_{t,s} K^1 \times A^0 \to ZA^0.
\]

By (4.3) this map is invariant under the \( A \)-action on \( ZA^0 \). Hence it gives rise to a smooth map

\[
\Xi : K^{[3]} = K^1 \times_{t,s} K^1 \times A^0 \to ZA.
\]

It is obvious that

**Lemma 4.3.** When \( \Xi(\cdot, \cdot, a) \equiv 1_a \) for all \( a \in A^0 \), \((\Lambda, \Omega)\) satisfies the generalized cocycle condition (3.8), and we get an extension with band \( \tilde{\Lambda} \).

The morphism \( \tilde{\Lambda} : K^1 \to \overline{SAut(A)} \) induces a smooth \( K \)-action on \( ZA \) (see §A.2)

\[
\tilde{\Lambda} : K^1 \to Aut(ZA).
\]

Then we could define the groupoid cohomology \( H^*_A(K, ZA) \) of \( K \) with coefficients in \( ZA \) (see §A.1).

**Theorem 4.4.** \( \Xi \) is a 3-cocycle in the cochain complex \( C^*_A(K, ZA) \), hence represents a class \([\Xi]\) in \( H^3_A(K, ZA) \). Moreover, this class \([\Xi]\) only depends on \( \tilde{\Lambda} \), not on \((\Lambda, \Omega)\).

We will prove this theorem in the appendix, §A.3.

**Theorem 4.5.** There is a topological product Lie groupoid extension of \( K \) by \( A \) with band \( \tilde{\Lambda} : K^1 \to \overline{SAut(A)} \) if and only if \([\Xi]\) = 0 in \( H^3_A(K, ZA) \).
4.3. We get a topological trivial Lie groupoid extension $A \rtimes_{\Lambda, \Omega'} A$ with band $\Lambda$.

We call $[\Xi] \in H^3_A(K, Z_A)$ the obstruction class of Lie groupoid extensions of $K$ by $A$ with band $\Lambda$.

5. Classification of Lie groupoid extensions

Now suppose we are given a morphism $\bar{\Lambda} : K^1 \to \text{SAut}(A)$. In this section we classify all Lie groupoid extensions over $K$ by $A$ with band $\Lambda$ when the obstruction class $[\Xi] = 0 \in H^3(K, Z_A)$.

5.1. Topological product Lie groupoid extensions. We first classify topological product Lie groupoid extensions of $K$ by $A$ with band $\Lambda$. By Corollary 4.2, we only need to study topological product Lie groupoid extensions constructed out of generalized cocycles.

Since $[\Xi] = 0$, by Theorem 4.5 every smooth lifting $\Lambda$ of $\bar{\Lambda}$ and the corresponding smooth $\Omega$ determines a topological product Lie groupoid extension $\Lambda \rtimes_{\Lambda, \Omega} K$. However, there would exist different liftings $(\Lambda, \Omega)$ that yield isomorphic extensions.

First note that, every natural transformation $\rho : \Lambda_1 \Rightarrow \Lambda_2$ in $\text{SAut}^1(A)$ has an inverse $\rho^{\otimes, -1} : \Lambda_2^{-1} \Rightarrow \Lambda_1^{-1}$ under the horizontal composition, that is $\rho^{\otimes, -1} \circ \rho = \rho \circ \rho^{\otimes, -1} = 1_{\text{id}_A} : \text{id}_A \Rightarrow \text{id}_A : A \to A$, the identity natural transformation. It is given by

$$\rho^{\otimes, -1}(a) = \left(\Lambda_1^{-1} \circ \rho \circ \Lambda_2^{-1}(a)\right)^{-1}, \quad \forall a \in A^0. \quad (5.1)$$

It also has an inverse $\rho^{\otimes, -1} : \Lambda_2 \Rightarrow \Lambda_1$ with respect to the vertical composition. That is $\rho \circ \rho^{\otimes, -1} = 1_{\Lambda_1} : \Lambda_1 \Rightarrow \Lambda_1$ and $\rho^{\otimes, -1} \circ \rho = 1_{\Lambda_2} : \Lambda_2 \Rightarrow \Lambda_2$ are the identity natural transformations over $\Lambda_1$ and $\Lambda_2$ respectively. It is given by

$$\rho^{\otimes, -1}(a) = \rho(a)^{-1}, \quad \forall a \in A^0.$$

We say

**Definition 5.1.** Two generalized cocycles $(\Lambda, \Omega)$ and $(\Lambda', \Omega')$ are equivalent if there is a smooth family of natural transformations $\rho(\xi) : \Lambda_\xi \Rightarrow \Lambda'_\xi$ parameterized by $K^1$ such that

$$\Omega'(\xi, \eta) = \Omega(\xi, \eta) \circ [\rho(\xi)\rho^{\otimes, -1} \circ \rho(\eta) \circ \rho(\xi)]$$

and $\rho(1_A) = u$, the unit map $u : A^0 \to A^1$.

It is direct to see that

**Lemma 5.2.** Two extensions $A \rtimes_{\Lambda, \Omega} K$ and $A \rtimes_{\Lambda', \Omega'} K$ are isomorphic if and only if $(\Lambda, \Omega)$ is equivalent to $(\Lambda', \Omega')$.

Denote the set of isomorphic classes of topological product Lie groupoid extensions of $K$ by $A$ with band $\Lambda$ by $\text{Iso}(K, A, \Lambda)$. Denote the isomorphic class of $A \rtimes_{\Lambda, \Omega} K$ by $[A \rtimes_{\Lambda, \Omega} K]$.

**Theorem 5.3.** There is a 1-to-1 correspondence between $\text{Iso}(K, A, \Lambda)$ and $H^2_A(K, Z_A)$.

**Proof.** By the above analysis, the isomorphic classes of topological product Lie groupoid extensions of $K$ by $A$ with band $\Lambda$ correspond to the equivalent classes of generalized cocycles that lift $\Lambda$.

Let $A \rtimes_{\Lambda_0, \Omega_0} K$ be the groupoid extension corresponds to a fixed generalized cocycle $(\Lambda_0, \Omega_0)$ with $\pi \circ \Lambda_0 = \bar{\Lambda}$. The existence of $A \rtimes_{\Lambda_0, \Omega_0} K$ is guaranteed by the assumption. We next construct a map from $H^2_A(K, Z_A)$ to the set of equivalent classes of generalized cocycles by setting

$$[c] \mapsto [(\Lambda_0, c_0\Omega_0)], \quad (5.2)$$

where $c : K^{[2]} \to Z_A$ is a cocycle in $C^2_A(K, Z_A)$, representing $[c]$. First of all, $(\Lambda_0, c_0\Omega_0)$ is still a generalized cocycle. So there is a groupoid extension of $K$ by $A$ corresponds to $(\Lambda_0, c_0\Omega_0)$, hence with band $\bar{\Lambda}$.

This map is well-defined. If $c$ and $c'$ represent the same cohomology class, there is a 1-cocohian $\rho$ such that $d\rho = c' - c^{-1}$. Then $\rho$ gives us the equivalence between $(\Lambda_0, c\Omega_0)$ and $(\Lambda_0, c'\Omega_0)$. We next show that this map is bijective.
For the surjectivity, suppose \((\Lambda, \Omega)\) is a generalized cocycle with \(\Lambda\) being a smooth lifting of \(\bar{\Lambda}\). Then since both \(\Lambda\) and \(\Lambda_0\) are smooth liftings of \(\bar{\Lambda}\), there is a smooth family of natural transformations \(\rho(\xi) : \Lambda_\xi \Rightarrow \Lambda_0,\xi\) for all \(\xi \in K^1\), and \(\rho(1_x) = u\) for every \(x \in K^0\). Therefore for every pair of composible arrows \(\xi_1, \xi_2 \in K^1\), we have
\[
\rho(\xi_1 \circ \xi_2) \Rightarrow \rho(\xi_2) \Rightarrow \rho(\xi_1) : \Lambda_{\xi_1 \circ \xi_2} \Rightarrow \Lambda_{\xi_2} \Rightarrow \Lambda_{\xi_1},
\]
on the other hand, for \(\xi_1, \xi_2\) we have
\[
\Omega(\xi_1, \xi_2) : id_A \Rightarrow \Lambda_{\xi_1 \circ \xi_2} \Rightarrow \Lambda_{\xi_1}, \quad \Omega_0(\xi_1, \xi_2) : id_A \Rightarrow \Lambda_{\xi_2} \Rightarrow \Lambda_{\xi_1}.
\]
Therefore we obtain a natural transformation
\[
\Omega'(\xi_1, \xi_2) := \Omega(\xi_1, \xi_2) \circ (\rho(\xi_1) \circ \rho(\xi_2)) : id_A \Rightarrow \Lambda_{\xi_1 \circ \xi_2} \Rightarrow \Lambda_{\xi_1} \Rightarrow \Lambda_{\xi_2} \Rightarrow \Lambda_{\xi_1}.
\]
This means that \((\Lambda, \Omega)\) is equivalent to \((\Lambda_0, \Omega')\). The two natural transformations \(\Omega'(\xi_1, \xi_2)\) and \(\Omega_0(\xi_1, \xi_2)\) induce a natural transformation \(c(\xi_1, \xi_2) := \Omega'(\xi_1, \xi_2) \circ \Omega_0(\xi_1, \xi_2) \circ \rho(\xi_1) \circ \rho(\xi_2)\): \(id_A \Rightarrow \Lambda_{\xi_1 \circ \xi_2} \Rightarrow \Lambda_{\xi_1}\).

By item 3 of Proposition 2.3, \(c(\xi_1, \xi_2) \in Z_A\). Therefore \(c \in C^3_A(\Lambda, Z_A)\). The fact that both \((\Lambda_0, \Omega_0)\) and \((\Lambda_0, \Omega')\) are generalized cocycle implies that \(c\) is a cocycle. Moreover \(\Omega' = c\circ \Omega_0 = c\cdot \Omega_0\). Therefore the map \((5.2)\) is surjective.

Finally, we consider the injectivity. If \((\Lambda, \Omega_0)\) is equivalent to \((\Lambda_0, \Omega')\), then there is a smooth family of natural transformations \(\rho(\xi) : \Lambda_{0, \xi} \Rightarrow \Lambda_0,\xi\) such that for every composable pair of arrows \(\xi_1, \xi_2 \in K^1\),
\[
(c\Omega_0)(\xi_1, \xi_2) = (c\cdot \Omega_0)(\xi_1, \xi_2) \circ (\rho(\xi_1) \circ \rho(\xi_2)).
\]
This gives rise to a \(c' \in C^3_A(\Lambda, Z_A)\), since the isotropy group of \(\Lambda_0\) in \(SAut(A)\) is also \(Z_A\) by Proposition 2.3. Moreover \(c = c' \cdot d\rho\). Therefore the map \((5.2)\) is injective. This finishes the proof. \(\square\)

5.2. Classifying Lie groupoid extensions. The morphism \(\bar{\Lambda} : K^1 \rightarrow \overline{SAut(A)}\) induces a morphism
\[
\bar{\Lambda}_\mathcal{U} = \bar{\Lambda} \circ q_1^A : K[\mathcal{U}]^1 \rightarrow \overline{SAut(A)}. \tag{5.3}
\]
for every open cover \(\mathcal{U}\) of \(K^0\) via the refinement morphism \(q : K[\mathcal{U}] \rightarrow K\).

The obstruction class associated to \(\bar{\Lambda}_\mathcal{U}\) is the image of the obstruction class \([\Xi]\) associated to \(\bar{\Lambda}\) under the homomorphism
\[
q_1^A : H^3_k(K, Z_A) \rightarrow H^3_{\bar{\Lambda}_\mathcal{U}}(K[\mathcal{U}], Z_A),
\]
hence also vanishes by the assumption \([\Xi]\) = 0.

Therefore, for every open cover \(\mathcal{U}\) of \(K^0\), there exists topological product Lie groupoid extension of \(K[\mathcal{U}]\) by \(A\) with band \(\bar{\Lambda}_\mathcal{U}\).

In the subsection w.r.t equivalence of Lie groupoid extensions we classify all Lie groupoid extensions with band induced from \(\bar{\Lambda}\) via \((5.3)\). By definition of Lie groupoid extensions, every Lie groupoid extension \(G \xrightarrow{\phi} K\) has a refinement \(G[\phi^*\mathcal{U}] \rightarrow K[\mathcal{U}]\) being a topological product Lie groupoid extension; and by the Definition 3.6 of equivalence of Lie groupoid extensions \(G \xrightarrow{\phi} K\) is equivalent to \(G[\phi^*\mathcal{U}] \rightarrow K[\mathcal{U}]\). However, there would exist topological product Lie groupoid extension of refinement of \(K\) by \(A\) that is not a refinement of a Lie groupoid \(G \xrightarrow{\phi} K\) of \(K\) by \(A\). So we consider the following set
\[
\overline{\text{Ext}}(K, A, \bar{\Lambda}) := \left\{ A \times_{\bar{\Lambda}_\mathcal{U}}, \Omega_\mathcal{U} : K[\mathcal{U}] \mid \Omega_\mathcal{U}\text{ is an open cover of } K^0, \right\}.
\]
Denote the quotient of this set by the equivalence relation between Lie groupoid extensions given in Definition 3.6 by \(\text{Ext}(K, A, \bar{\Lambda})\).

First of all we have the following two simple lemmas.

Lemma 5.4. Every generalized cocycle \((\Lambda, \Omega)\) over \(K\) pulls back to a generalized cocycle \((\Lambda_\mathcal{U}, \Omega_\mathcal{U}) := q^\mathcal{U}_1(\Lambda, \Omega)\) over \(K[\mathcal{U}]\) with
\[
\Lambda_\mathcal{U} := \Lambda \circ q^\mathcal{U}_1 : K[\mathcal{U}]^1 \rightarrow SAut(A)
\]
and
\[
\Omega_\mathcal{U} := \Omega \circ (q^\mathcal{U}_1, q^\mathcal{U}_1, id_A) : K[\mathcal{U}]^1 \times_{\mathcal{U}} K[\mathcal{U}]^1 \times A^0 \rightarrow A^1.
\]
Moreover, there is a canonical isomorphism
\[
A \times_{\bar{\Lambda}_\mathcal{U}}, \Omega_\mathcal{U} : K[\mathcal{U}] \cong q^\mathcal{U}_1(A \times_{\Lambda, \Omega} K).
\]

Lemma 5.5. \(A \times_{\bar{\Lambda}_\mathcal{U}} K[\mathcal{U}]\) is equivalent to \(A \times_{\bar{\Lambda}'_\mathcal{U}}, \Omega'_\mathcal{U} : K[\mathcal{U}']\) if and only if there is a third open cover \(W\) of \(K^0\) refining both \(\mathcal{U}\) and \(\mathcal{U}'\), such that the pullback generalized cocycles \((\Lambda_\mathcal{W}, \Omega_\mathcal{W})\) and \((\Lambda'_\mathcal{W}, \Omega'_\mathcal{W})\) are equivalent.
For a fixed $\mathcal{U}$ we have the set of isomorphic classes of topological product Lie groupoid extensions $\text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U)$. If $\mathcal{V}$ is another open cover that refines $\mathcal{U}$ via $\iota_{\mathcal{W}, \mathcal{U}} : \mathcal{V} \to \mathcal{U}$, then we have the commutative diagram of morphisms

\[
\begin{array}{ccc}
K[\mathcal{V}] & \xrightarrow{\iota_{\mathcal{W}, \mathcal{U}}} & K[\mathcal{U}] \\
\downarrow{\pi_{\mathcal{W}}} & & \downarrow{\pi_{\mathcal{U}}} \\
K & & K
\end{array}
\] (5.4)

and

\[
\tilde{\Lambda}_U \circ \iota_{\mathcal{W}, \mathcal{U}} = \tilde{\Lambda}_\mathcal{V}.
\] (5.5)

Note that, the commutative diagram (5.4) and the equality (5.5) do not depend on the choice of the refine map $\iota_{\mathcal{W}, \mathcal{U}}$. So in the following, we do not specify the refinement map $\iota_{\mathcal{W}, \mathcal{U}}$.

The refinement morphism $\iota_{\mathcal{W}, \mathcal{U}}$ pulls back a generalized cocycle $(\Lambda_U, \Omega_U)$ over $K[\mathcal{U}]$ to a generalized cocycle $(\iota_{\mathcal{W}, \mathcal{U}}^* \Lambda_U, \iota_{\mathcal{W}, \mathcal{U}}^* \Omega_U)$ over $K[\mathcal{W}]$. Moreover, similar as Lemma 5.4 we have

\[
iota_{\mathcal{W}, \mathcal{U}}^* (A \ltimes_{\Lambda_U, \Omega_U} K[\mathcal{U}]) \cong A \ltimes_{\iota_{\mathcal{W}, \mathcal{U}}^* \Lambda_U, \iota_{\mathcal{W}, \mathcal{U}}^* \Omega_U} K[\mathcal{W}].
\]

Therefore we have a refinement map

\[
iota_{\mathcal{W}, \mathcal{U}}^* : \text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U) \to \text{Iso}(K[\mathcal{W}], A, \tilde{\Lambda}_\mathcal{W}), \quad [A \ltimes_{\Lambda_U, \Omega_U} K[\mathcal{U}]] \mapsto [A \ltimes_{\iota_{\mathcal{W}, \mathcal{U}}^* \Lambda_U, \iota_{\mathcal{W}, \mathcal{U}}^* \Omega_U} K[\mathcal{W}]].
\]

By taking all open covers of $K^0$, the collection of isomorphic classes $\text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U)$ and of refinement maps $\iota_{\mathcal{W}, \mathcal{U}}^* : \text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U) \to \text{Iso}(K[\mathcal{W}], A, \tilde{\Lambda}_\mathcal{W})$ forms a direct system. Thus we can form the direct limit

\[
\lim_{\mathcal{U}} \text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U).
\]

Then by definitions of equivalence of Lie groupoid $A$-extensions in Definition 3.6 and of direct limit we get the following result.

**Proposition 5.6.** There is a 1-to-1 map

\[
\text{Ext}(K, A, \tilde{\Lambda}) \xrightarrow{1-1} \lim_{\mathcal{U}} \text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U),
\]

induced by identity map on the set $\text{Ext}(K, A, \tilde{\Lambda})$.

We next consider the cohomology groups $H^*_{A_U}(K[\mathcal{U}], Z_A)$ for open covers of $K^0$. For a refinement $\iota_{\mathcal{W}, \mathcal{U}} : \mathcal{W} \to \mathcal{U}$ we also have a group homomorphism

\[
iota_{\mathcal{W}, \mathcal{U}}^* : H^*_{A_U}(K[\mathcal{U}], Z_A) \to H^*_{A_{\mathcal{W}}}(K[\mathcal{W}], Z_A).
\]

As the construction of Čech cohomology of manifolds in [21] and for simplicial spaces in [20], this homomorphism $\iota_{\mathcal{W}, \mathcal{U}}^*$ does not depend on the choice of the explicit refinement map $\iota_{\mathcal{W}, \mathcal{U}} : \mathcal{W} \to \mathcal{U}$. Then the collection of these cohomology groups and these group homomorphisms also forms a direct system. Then we also form the direct limit

\[
\lim_{\mathcal{U}} H^*_{A_U}(K[\mathcal{U}], Z_A).
\]

Now fix a reference lifting $\Lambda_0 : K^1 \to \text{SAut}(A)$ of $\tilde{\Lambda}$, and a $\Omega_0$ such that $(\Lambda_0, \Omega_0)$ is a generalized cocycle. The generalized cocycle $(\Lambda_0, \Omega_0)$ pulls back to generalized cocycle $\iota_{\mathcal{W}, \mathcal{U}}^*(\Lambda_0, \Omega_0) = (\iota_{\mathcal{W}, \mathcal{U}}^* \Lambda_0, \iota_{\mathcal{W}, \mathcal{U}}^* \Omega_0)$ over $K[\mathcal{U}]$ for every open cover $\mathcal{U}$ of $K^0$. By Theorem 5.3 we have the identification

\[
\Phi_{\mathcal{W}} : H^2_{A_{\mathcal{U}}}(K[\mathcal{U}], Z_A) \xrightarrow{1-1} \text{Iso}(K[\mathcal{U}], A, \tilde{\Lambda}_U), \quad [c] \mapsto [K \ltimes_{\Lambda_0, \Omega_0, \iota_{\mathcal{W}, \mathcal{U}}^*} A]
\]

for every open cover $\mathcal{U}$ of $K^0$.

**Theorem 5.7.** Two extensions $K[\mathcal{U}] \ltimes_{A_{\mathcal{U}}, c_{\mathcal{U}}} A$ and $K[\mathcal{V}] \ltimes_{A_{\mathcal{V}}, c_{\mathcal{V}}} A$ are equivalent, if there is a third cover $\mathcal{W}$ of $K^0$ that refines both $\mathcal{U}$ and $\mathcal{V}$ via $\iota_{\mathcal{W}, \mathcal{U}}$ and $\iota_{\mathcal{W}, \mathcal{V}}$ such that

\[
iota_{\mathcal{W}, \mathcal{U}}^*(c) = \iota_{\mathcal{W}, \mathcal{V}}^*(c')
\]

Moreover, we have a 1-to-1 map

\[
\text{Ext}(K, A, S) \xrightarrow{1-1} \lim_{\mathcal{U}} H^2_{A_{\mathcal{U}}}(K[\mathcal{U}], Z_A).
\]
Proof. The first assertion follows from the definition and is a restatement of Lemma 5.5. To prove the second assertion, we only have to show the following diagram\[ H^2_{\Lambda_0}(\mathcal{U}, Z_A) \xrightarrow{\phi_{\mathcal{U}}} \text{Iso}(\mathcal{U}, A, \tilde{\Lambda}_0) \]

\[ H^2_{\Lambda_0}(\mathcal{U}, Z_A) \xrightarrow{\phi_{\mathcal{U}}} \text{Iso}(\mathcal{U}, A, \tilde{\Lambda}_0) \]

is commutative. This follows from the fact that all reference generalized cocycles \((\Lambda_0, \Omega_0)\) are pull backs of the fixed generalized cocycle \((\Lambda_0, \Omega_0)\) via refinements. This finishes the proof. \qed

6. Symplectic structure over Lie groupoid extensions

Suppose that both \(K\) and \(A\) are orbifold groupoids. Then the Lie groupoid extension \(G = A \times_{\Lambda_0} K\) associated to a generalized cocycle \((\Lambda, \Omega)\) is also an orbifold groupoid. In this section we show that under some appropriate conditions, there is a natural symplectic structure over \(G = A \times_{\Lambda_0} K\) when both \(K\) and \(A\) are symplectic orbifold groupoids.

**Theorem 6.1.** Let \(\omega_K\) and \(\omega_A\) be the symplectic forms over the orbifold groupoids \(K\) and \(A\) respectively. If \(\Lambda : K^1 \to \text{SAut}^0(A)\) preserves the symplectic structure over \(A\) and is locally constant, then the symplectic form \(\omega_G := \omega_A \times \omega_K = pr_1^*\omega_A \wedge pr_2^*\omega_K\) is a symplectic form over \(G = A \times_{\Lambda_0} K\), where \(pr_i\) are the projections from \(G^0 = A^0 \times K^0\) to its two factors for \(i = 1, 2\).

**Proof.** By the definition of symplectic form over orbifold groupoid we need to show that for two arbitrary tangent vector fields \(X_G := (X_A, X_K), Y_G := (Y_A, Y_K) \in \Gamma(TG^1) = \Gamma(T(A^1 \times K^1))\) we have

\[ s_G^*\omega_G(X_G, Y_G) = t_G^*\omega_G(X_G, Y_G), \]

where \(s_G\) and \(t_G\) are the source and target maps of \(G\) defined in (4.4) and (4.5).

First of all, by the definition of source map in (4.4) we have

\[ s_G^*\omega_G(X_G, Y_G) = \omega_G(s_G, X_A, s_G, Y_G) \]

\[ = \omega_G((s_A, X_A, s_A, Y_A), (s_K, X_K, s_K, Y_K)) \]

\[ = \omega_A(s_A, X_A, s_A, Y_A) \cdot \omega_K(s_K, X_K, s_K, Y_K), \]

where \(s_K\) and \(s_A\) are the source maps of \(K\) and \(A\) respectively. Next we consider \(t_G^*\omega_G(X_G, Y_G)\). By the definition of pulling back of forms we have

\[ t_G^*\omega_G(X_G, Y_G) = \omega_G(t_G, X_G, t_G, Y_G) \]

However \(t_G, t_G\) is much more complicated by the definition in (4.5). We have

\[ t_G(\alpha, \xi) = (\Lambda_\alpha \circ t_A(\alpha), t_K(\xi)). \]

Write \(\Lambda_\alpha \circ t_A(\alpha)\) as \(f(\alpha, \xi)\). Then

\[ t_G(\alpha, \xi) = (\frac{\partial f}{\partial \alpha} X_A, t_K, X_K + \frac{\partial f}{\partial \xi} X_K). \]

However, by the assumption that \(\Lambda\) is locally constant we have \(\frac{\partial f}{\partial \xi} = 0\). Therefore \(t_G(\alpha, \xi) = (\frac{\partial f}{\partial \alpha} X_A, t_K, X_K).\)

Similarly \(t_G, Y_G = (\frac{\partial f}{\partial \alpha} Y_A, t_K, Y_K).\) So we have

\[ t_G^*\omega_G(X_G, Y_G) = \omega_G\left(\frac{\partial f}{\partial \alpha} X_A, t_K, X_K, \frac{\partial f}{\partial \alpha} Y_A, t_K, X_K\right) \]

\[ = \omega_A\left(\frac{\partial f}{\partial \alpha} X_A, \frac{\partial f}{\partial \alpha} Y_A\right) \cdot \omega_K(t_K, X_K, t_K, Y_K) \]

\[ = \omega_A\left(\frac{\partial f}{\partial \alpha} X_A, \frac{\partial f}{\partial \alpha} Y_A\right) \cdot \omega_K(s_K, X_K, s_K, Y_K), \]

Conclusion.
where for the last equality we have used the assumption that \((K, \omega_K)\) is a symplectic orbifold groupoid. On the other hand we have \(\frac{\partial f}{\partial \alpha} X_A = d(\Lambda \xi) \circ t_{A,*}(X_A)\), \(\frac{\partial f}{\partial \alpha} Y_A = d(\Lambda \xi) \circ t_{A,*}(Y_A)\). Therefore
\[
\omega_{\Lambda}(\frac{\partial f}{\partial \alpha} X_A, \frac{\partial f}{\partial \alpha} Y_A) = \omega_A(d(\Lambda \xi) \circ t_{A,*}(X_A), d(\Lambda \xi) \circ t_{A,*}(Y_A))
\]
\[
= \Lambda^* \omega_A(t_{A,*}X_A, t_{A,*}Y_A)
\]
\[
= \omega_A(t_{A,*}X_A, t_{A,*}Y_A)
\]
\[
= \omega_A(s_{A,*}X_A, s_{A,*}Y_A),
\]
where we have used the assumption that \(\Lambda\) preserves the symplectic form over \(A\) for the third equality and the assumption that \(\omega_A\) is a symplectic form over \(A\) for the last equality. Consequently,
\[
t_{G,G}^{*}\omega_G(X_G, Y_G) = s_{G,G}^{*}\omega_G(X_G, Y_G).
\]
Therefore \(\omega_G\) is a symplectic form over \(G = A \rtimes_{A,\Omega} K\).

Obvious, the restriction of \(\omega_G\) over the fiber of \(A \rtimes_{A,\Omega} K \rightarrow K\), i.e. the fiber of the kernel, is isomorphic to \((A, \omega_A)\).

It is direct to see that the existence of symplectic forms is a Morita equivalence invariant (see for example [13, Proposition 7.3]), that is if \(f : G \rightarrow H\) is an equivalence between orbifold groupoids, then a symplectic form over \(G\) naturally induces a symplectic form over \(H\) and vice versa.

**Corollary 6.2.** Let \(G \overset{\phi}{\rightarrow} K\) be a Lie groupoid extension of \(K\) by \(A\) with \(G[\phi^*U] \rightarrow K[U]\) being a topological product Lie groupoid extension that refines \(G \rightarrow K\). Suppose that both \(A\) and \(K\) are symplectic orbifold groupoids. Then if the pre-action map of \(G[\phi^*U] \rightarrow K[U]\) satisfies the assumption in Theorem 6.1, then there is a natural symplectic form over \(G\).

**Example 6.3.** For example, for a finite group \(A\) consider an étale \(A\)-gerbe\n
\[
\begin{array}{ccc}
K^0 & \xrightarrow{i} & G^1 \\
\downarrow & & \downarrow \\
K^0 & \xrightarrow{j} & K^1
\end{array}
\]

over a symplectic orbifold groupoid \((K, \omega_K)\), where \(\ker j\) is a locally trivial \(A\)-bundle over \(K^0\). Let \(BA = (\bullet \times A \xrightarrow{\bullet} B)\) be the classifying groupoid of \(A\). Then \(G\) is a Lie groupoid \(BA\)-extension over \(K\), and \(G^0 = \bullet \times K^0 = K^0\). The symplectic form over \(BA\) is trivial. By the commutative diagram (6.1) we see that \(\omega_K\) on \(K^0\) is also compatible with the source and target maps of \(G\). Hence we get the induced symplectic structure over \(G\).

By taking an appropriate open cover \(U\) of \(K^0\) we get the refinement (see for example [14, 19])
\[
A \times K[U]^0 \xrightarrow{i} A \times K[U]^1 \xrightarrow{j} K[U]^1
\]
\[
\begin{array}{ccc}
K[U]^0 & \xrightarrow{\alpha} & K[U] \\
\downarrow & & \downarrow \\
K[U]^0 & \xrightarrow{\beta} & K[U]^0
\end{array}
\]

Then one see the pre-action map \(\Lambda_U\) for the refinement extension is locally constant. Since the symplectic form over \(BA\) is trivial, \(\Lambda_U\) preserves the symplectic form on \(BA\). Therefore by the previous corollary the \(A\)-gerbe \(G\) over the symplectic orbifold groupoid \(K\) also has a naturally induced symplectic form.

### 7. Induced Extension over Morphism Groupoid

Let \(G = A \rtimes_{A,\Omega} K \overset{\phi}{\rightarrow} K\) be a Lie groupoid extension corresponding to a generalized cocycle \((\Lambda, \Omega)\) over \(K\). \(\phi\) induces a map on coarse space \(\overline{\phi} : \overline{G} \rightarrow \overline{K}\). It is a fibration with fiber being the coarse space \(\overline{A}\) of \(A\). \(\overline{\phi}\) induces a group homomorphism over homology groups \(\overline{\phi}_* : H_*(\overline{G}, \mathbb{Z}) \rightarrow H_*(\overline{K}, \mathbb{Z})\). We call a homology classes \(\beta \in H_*(\overline{K}, \mathbb{Z})\) a fiber class if \(\overline{\phi}_*(\beta) = 0 \in H_*(\overline{K}, \mathbb{Z})\). A fiber class gives rise to a homology class in \(H_*(\overline{A}, \mathbb{Z})\), which is mapped to \(\beta\) via the induced homomorphism of the inclusion of \(\overline{A}\) into \(\overline{G}\) as a fiber. We denote this class in \(H_*(\overline{K}, \mathbb{Z})\) still by \(\beta\).

When \(A\) and \(K\) are both symplectic orbifold groupoid, under the assumption in Theorem 6.1, we get a symplectic structure over \(G\). Then we could study the orbifold Gromov–Witten theory of \(G\). The orbifold Gromov–Witten theory of \(G\) has a parameter being the homology classes in \(H_2(\overline{G}, \mathbb{Z})\). It is natural to expect that orbifold Gromov–Witten theory of \(G\) w.r.t. to a degree 2 fiber class is determined by the orbifold Gromov–Witten theory of \(A\) and the geometry.
of the Lie groupoid extension $G \to K$. The first step to study the orbifold Gromov–Witten theory of $G$ is to study the morphism groupoid of generalized morphisms from orbifold Riemannian surfaces to $G$. This was carried out for general groupoids in [6] by R. Wang and the first two authors. In this section we study the relation between the morphism groupoid of fiber class generalized morphisms from a Lie groupoid $H = (H^1 \rightrightarrows H^0)$ to $G$ and the morphism groupoid of generalized morphisms from $H$ to $A$.

### 7.1. Morphism groupoids.

**Definition 7.1 ([6]).** A generalized morphism $(U, \psi, u) : H \to A$ consists of a triple $(U, \psi, u)$, where $\psi : U \to H$ is an equivalence of Lie groupoids and $u : U \to A$ is a strict Lie groupoid morphism.

**Definition 7.2 ([6]).** Let $(U_1, \psi_1, u_1), (U_2, \psi_2, u_2) : H \to A$ be two generalized morphisms. An arrow $\alpha : (U_1, \psi_1, u_1) \to (U_2, \psi_2, u_2)$ is a natural transformation that fits into the following diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\psi_1} & U_1 \\
\downarrow{\pi_1} & & \downarrow{u_1} \\
U_1 \times_H U_2 & \xrightarrow{\pi_2} & A.
\end{array}
\]

where $U_1 \times_H U_2 = U_1 \times_{\psi_1,H,\psi_2} U_2$ is the fiber product (cf. [1, 6]) of $\psi_1 : U_1 \to H$ and $\psi_2 : U_2 \to H$.

Denote by $\text{Mor}^0(H, A)$ the space of generalized morphism from $H$ to $A$ and $\text{Mor}^1(H, A)$ the space of arrows between generalized morphisms in $\text{Mor}^0(H, A)$. Then there is a (vertical) composition $\circ$ between arrows that makes

\[\text{Mor}(H, A) := (\text{Mor}^1(H, A) \rightrightarrows \text{Mor}^0(H, A))\]

into a groupoid. In fact this composition is constructed from the vertical composition of natural transformations. See [6, Theorem 3.7] for the explicit construction.

We also have the morphism groupoid $\text{Mor}(H, G)$. This groupoid has several interesting sub-groupoids. Given a generalized morphism $(U, \psi, u) : H \to G$, there is an induced continuous map

\[
\overline{\text{H}} \xleftarrow{\psi} U \xrightarrow{\pi} \overline{G}.
\]

We say that this generalized morphism represents a fiber class if $\overline{\psi} \circ \overline{\pi} \circ \overline{\psi}^{-1} = \{pt\} \in \overline{K}$. Denote by $\text{Mor}^0(H, G)_F$ the space of all fiber class generalized morphisms in $\text{Mor}^0(H, G)$. Then we get a sub-groupoid

\[\text{Mor}(H, G)_F = \text{Mor}(H, G)\vert_{\text{Mor}^0(H, G)_F}\]

of $\text{Mor}(H, G)$.

We next assume that the coarse space $\overline{\text{H}}$ of $H$ has a fundamental class $[\overline{\text{H}}] \in H_*(\overline{\text{H}}, \mathbb{Z})$. For example when $\overline{\text{H}}$ is a compact oriented topological manifold or a compact oriented orbifold groupoid, this assumption is satisfied. Take a fiber class $\beta \in H_*(\overline{G}, \mathbb{Z})$, which determined a class $\beta \in H_*(\overline{A}, \mathbb{Z})$. Let $\text{Mor}^0(H, G)_\beta$ be the space of all fiber class generalized morphisms in $\text{Mor}^0(H, G)_F$ such that

\[\overline{\pi} \circ \overline{\psi}^{-1}(\overline{[\text{H}]} \circ \beta \circ \overline{\psi}^{-1}(\overline{[\text{H}]}) = \beta.\]

Then we have another sub-groupoid

\[\text{Mor}(H, G)_\beta = \text{Mor}(H, G)\vert_{\text{Mor}^0(H, G)_\beta}\]

of $\text{Mor}(H, G)$. Similarly, we have a sub-groupoid $\text{Mor}(H, A)_\beta$ of $\text{Mor}(H, A)$.

### 7.2. Induced groupoid extension.

In this subsection we show that the generalized cocycle $(\Lambda, \Omega)$ induces a generalized cocycle $(\Lambda, \Omega)$ with $\Lambda : K^1 \to S\text{Aut}^0(\text{Mor}(H, A))$ and an extension of $\text{K}$ by $\text{Mor}(H, A)$.

We first define $\tilde{\Lambda}$. For every $\xi \in K^1$, we have an automorphism $\Lambda_\xi : A \to A$. It induces a morphism

\[\tilde{\Lambda}_\xi : \text{Mor}(H, A) \to \text{Mor}(H, A)\]

by

\[(U, \psi, u) \mapsto (U, \psi, \Lambda_\xi \circ u), \quad \alpha \mapsto \Lambda_\xi \circ \alpha.\]

It is direct to see that this $\tilde{\Lambda}_\xi$ is an automorphism of $\text{Mor}(H, A)$. This gives us the $\tilde{\Lambda}$. 

Next we define $\tilde{\Omega}$. It should assign each pair $(\xi, \eta) \in \mathcal{K}^{[2]} = K^1 \times_{t,s} K^1$ a natural transformation $\tilde{\Omega}(\xi, \eta)$ from $\text{id}_{\text{Mor}(H, A)}$ to $\tilde{\Lambda}^{-1}_{\xi \eta} \circ \tilde{\Lambda}_{\xi \eta} \circ \tilde{\Lambda}_{\xi}$. We next write down the explicit expression of $\tilde{\Omega}$. First of all it is a map
\[
\tilde{\Omega} : K^1 \times_{t,s} K^1 \times \text{Mor}^0(H, A) \to \text{Mor}^1(H, A).
\]

Let
\[
(U_1, \psi_1, u_1) \xrightarrow{\alpha_{U_1}} (U_2, \psi_2, u_2)
\]
be an arrow in $\text{Mor}(H, A)^1$. For $i = 1, 2,$
\[
\tilde{\Lambda}^{-1}_{\xi \eta} \circ \tilde{\Lambda}_{\eta} \circ \tilde{\Lambda}_{\xi}(U_i, \psi_i, u_i) = (U_i, \psi_i, \Lambda_{\xi \eta}^{-1} \circ \Lambda_{\eta} \circ \Lambda_{\xi} \circ u_i).
\]
So $\tilde{\Omega}(\xi, \eta, (U_i, \psi_i, u_i))$ is an arrow from $(U_i, \psi_i, u_i)$ to $(U_1, \psi_1, \Lambda_{\xi \eta}^{-1} \circ \Lambda_{\eta} \circ \Lambda_{\xi} \circ u_i)$, i.e.

\[
\begin{array}{c}
\xymatrix{
U_i 
\ar[r]^{u_i} 
\ar[d]_{\pi_i} & A \\
U_i \times_H U_i 
\ar[u]^j 
\ar[r]_{\Lambda_{\xi \eta}^{-1} \circ \Lambda_{\eta} \circ \Lambda_{\xi} \circ u_i} & A.
}
\end{array}
\]

An object in the fiber product $U_i \times_H U_i$ is of the form
\[
a_1 \quad \xrightarrow{\alpha_{U_i}} \quad a_2
\]
with $a_i \in U_i^{0}$ and $\alpha_{U_i} : \psi_i^0(a_1) \to \psi_i^0(a_2)$ in $H$. Since $\psi_i$ is an equivalence, there is an arrow $\alpha_{U_i} : a_1 \to a_2$ in $U_i$ satisfying $\psi_i^1(\alpha_{U_i}) = \alpha_{U_i}$. Then the $\tilde{\Omega}(\xi, \eta, (U_i, \psi_i, u_i))$ we want is a map
\[
\tilde{\Omega}(\xi, \eta, (U_i, \psi_i, u_i)) : (U_i \times_H U_i)^0 \to A^1.
\]
We set
\[
\tilde{\Omega}(\xi, \eta, (U_i, \psi_i, u_i))(a_1 \xrightarrow{\alpha_{U_i}} a_2) := u_i(\alpha_{U_i}) \cdot \Omega(\xi, \eta, u_i(a_2))
= \Omega(\xi, \eta, u_i(a_1)) \cdot \Lambda_{\xi \eta}^{-1} \circ \Lambda_{\eta} \circ \Lambda_{\xi} \circ u_i(\alpha_{U_i}).
\]
The last equality follows from the fact that $\Omega(\xi, \eta) : \text{id}_A \Rightarrow \Lambda_{\xi \eta}^{-1} \circ \Lambda_{\eta} \circ \Lambda_{\xi}$ is a natural transformation. By direct computation we have

**Theorem 7.3.** The pair $(\tilde{\Lambda}, \tilde{\Omega})$ is a generalized cocycle. Therefore we get a groupoid extension of $K$ by $\text{Mor}(H, A)$, that is $\text{Mor}(H, A) \rtimes \tilde{\Lambda}, \tilde{\Omega} K$.

Moreover, when the fundamental class $[H]$ exists, $\tilde{\Lambda}$ preserves $\text{Mor}(H, A)_{\beta}$, therefore we could restrict $(\tilde{\Lambda}, \tilde{\Omega})$ to $\text{Mor}(H, A)_{\beta}$. Denote the restriction by $(\tilde{\Lambda}_{\beta}, \tilde{\Omega}_{\beta})$. Then we have a groupoid extension $\text{Mor}(H, A)_{\beta} \rtimes \tilde{\Lambda}_{\beta}, \tilde{\Omega}_{\beta} K$ of $K$ by $\text{Mor}(H, A)_{\beta}$.

### 7.3. Fiber class generalized morphisms of $G$.

Obviously, every object $(x, (U, \psi, u))$ in the groupoid $\text{Mor}(H, A) \rtimes \tilde{\Lambda}, \tilde{\Omega} K$ defines a generalized morphism
\[
H \xrightarrow{\psi} U \xrightarrow{u} A \xrightarrow{\tau_x} G
\]
via the inclusion $\tau_x$ of $A$ as the fiber over $x \in K^0$ (or $(1_x \Rightarrow x)$ of the kernel kernel $\phi$ of the extension $\phi : G \to K$).

We denoted this induced generalized morphism by $(U, \psi, u)_x$. It is obvious that for every object $(x, (U, \psi, u))$ in the groupoid $\text{Mor}(H, A) \rtimes \tilde{\Lambda}, \tilde{\Omega} K$, the induced generalized morphism $(U, \psi, u)_x$ into $G$ is a fiber class generalized morphism. Moreover this procedure gives rise to a groupoid morphism
\[
\tau := (\tau^0, \tau^1) : \text{Mor}(H, A) \rtimes \tilde{\Lambda}, \tilde{\Omega} K \to \text{Mor}(H, G)_F.
\]

Similarly, when the fundamental class $[H]$ exists, we have
\[
\tau_{\beta} : \text{Mor}(H, A)_{\beta} \rtimes \tilde{\Lambda}_{\beta}, \tilde{\Omega}_{\beta} K \to \text{Mor}(H, G)_{\beta}.
\]

Our main theorem in this section is

**Theorem 7.4.** Suppose that $[H]$ is compact. Then $\tau$ and $\tau_{\beta}$ are both equivalence between groupoids.
Proof. We prove this theorem for $\tau$. The proof for $\tau_{\beta}$ is similar. We only have to show that every fiber class generalized morphism $(U, \psi, u) : H \to G$ is connected to a generalized morphism in $\tau^0((\text{Mor}(H, A) \times K)^0)$ by an arrow in $\text{Mor}^1(H, G)_F$.

Let $(U, \psi, u) : H \to G$ be a fiber class generalized morphism. Suppose $\psi = (\psi^0, \psi^1), u = (u^0, u^1), U = (U^1 \rightrightarrows U^0)$ and

$$U^0 = \bigsqcup_{i \in I} U^0_i$$

be a decomposition of $U^0$ into open, connected components. Since $\overline{H}$ is compact, we could first modify $(U, \psi, u)$ into a new generalized morphism $\tilde{\psi} : H \to G$ such that $V^0$ has only finite connected components. This can be done as follows. The decomposition (7.1) of $U^0$ induces an open cover $\{\overline{U}_i^0\}_{i \in I}$ of $\overline{H}$. Since $\overline{H}$ is compact. The open cover $\{\overline{U}_i^0\}_{i \in I}$ has a finite cover, say $\{\overline{U}_i^0\}_{1 \leq k \leq n}$ for an $n \in \mathbb{Z}_\geq 1$. Let

$$V^0 := \bigsqcup_{1 \leq k \leq n} U^0_{i_k}$$

Then we take $V = U|_{V^0}$, and $\varphi = \psi|_V, \nu = u|_V$. Since $(U, \psi, u)$ is a fiber class generalized morphism, so is $(\psi, \varphi, \nu)$. On the other hand, the natural inclusion $i^0 : V^0 \to U^0$ induces an equivalence $i : V \to U$. Then it gives rise to an arrow in $\text{Mor}^1(H, G)_F$ which connects $(U, \psi, u)$ and $(\psi, \varphi, \nu)$. So in the following we always assume that for the original $(U, \psi, u), U^0$ has finite connected components and $I = \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_\geq 1$.

Since $(U, \psi, u)$ is a fiber class generalized morphism, $\varphi \circ \overline{u}(U) = \{pt\} \in \mathcal{K}$. Therefore each connected component $U^0_i$ is mapped into a fiber of $G^0 \to K^0$. Suppose $\psi^0(u^0_i(U^0_i)) = x_i \in K^0$. If all $x_i = x \in K^0$, then $(U, \psi, u) \in \text{Im} (\tau^0)$. So we next assume that these $x_i$ are not all the same.

Now we modify $u$ to a new morphism $w = (w^0, w^1) : U \to G$ such that $\phi^0(u^0_i(U^0_i)) = x_i$. We denote by $U^1[U^0_i, U^0_j]$ the space of arrows in $U^1$ that start from $U^0_i$ and end at $U^0_j$.

For every $x_i, 1 \leq i \leq n$, take an arrow $\xi_i : x_i \to x_1$ in $K^1$. In particular, for every $i$ with $x_i = x_1$ we require that $\xi_i = 1_{x_i} = 1_{x_1}$. Then we set

$$w^0(\cdot) := \Lambda_{\xi_i} \circ u^0(\cdot) = t_{\{w^0(\cdot), \xi_i\}} \text{ on } U^0_i,$$

and

$$w^1(\cdot) := \{1_{w^0(\xi_i)}, \xi_i\}^{-1} \cdot u^1(\cdot) \cdot \{1_{w^0(t(\xi_i))), \xi_j\} \text{ on } U^1[U^0_i, U^0_j].$$

Then one can see that the natural transformation from $u$ to $w$ is given by

$$\bigsqcup_{i \in I}(1_{u^0_i(\cdot), \xi_i}) : \bigsqcup_{i \in I} U^0_i \to G^1.$$

This finishes the proof. \qed

By using the axiom of choice we could removed the assumption on the compactness of $|H|$.

**Remark 7.5.** In [6], the authors also constructed another morphism groupoid $\text{FMor}(H, A)$ by using full-morphism (cf. [6, Def. 3.8]) and strict fiber products (cf. [6, Def. 2.10]). Moreover, there is a natural equivalence (cf. [6, Thm. 3.15])

$$i : \text{FMor}(H, A) \to \text{Mor}(H, A).$$

Similar as the constructions in Subsection 7.1, we have $\text{FMor}(H, A)_F$, and $\text{FMor}(H, A)_\beta$. Moreover, $i$ restricts to groupoid equivalences

$$i_F : \text{FMor}(H, A)_F \to \text{Mor}(H, A)_F, \text{ and } i_\beta : \text{FMor}(H, A)_\beta \to \text{Mor}(H, A)_\beta.$$

On the other hand, there are also induced groupoid extensions over $K$ with fibers being $\text{FMor}(H, A)_F$ and $\text{FMor}(H, A)_\beta$ respectively, i.e. similar results as Theorem 7.3 hold for $\text{FMor}(H, A)_F$ and $\text{FMor}(H, A)_\beta$. So we have

$$\text{FMor}(H, A)_F \times_{\Lambda, \overline{\Omega}} K, \text{ and } \text{FMor}(H, A)_\beta \times_{\Lambda, \overline{\Omega}} K.$$
A.1. Groupoid cohomology with coefficient in an abelian group. We fix some notation of groupoid cohomology. Let $G = (G^1 \rightrightarrows G^0)$ be a Lie groupoid. Let $E$ be an abelian (Lie) group. A $G$-action on $E$ means a (smooth) map $\Lambda : G^1 \to \text{Aut}(E)$ satisfying that for every composable arrows $g, h$ in $G^1$, i.e. $t(g) = s(h)$, it holds
$$\Lambda(h) \circ \Lambda(g) = \Lambda(gh).$$ \hfill (A.1)
Recall that the composition of two arrows $gh$ goes from left to right.
Consider the notation $G^{[n]}$ for the set of $n$-composable arrows\(^6\)
$$G^{[n]} := \{ \tilde{g} = (g_1, \ldots, g_n) \mid t(g_i) = s(g_{i+1}), 1 \leq i \leq n - 1 \}. \hfill (A.2)$$
When $n = 0$, $G^{[0]} = G^0$ is the space of objects. We could illustrate a point $\tilde{g} = (g_1, \ldots, g_n) \in G^{[n]}$ by
$$x = s(g_1) \xrightarrow{g_1} t(g_1) = s(g_2) \xrightarrow{g_2} \cdots \xrightarrow{g_n} t(g_n).$$
Now let $E$ be an abelian (Lie) group equipped with a $G$-action. For $n \in \mathbb{Z}_{\geq 0}$ an $n$-cochain on $G$ with values in $E$ is a (smooth) map $G^{[n]} \to E$. Denote the set of $n$-cochain on $G$ with values in $E$ by $C^n(G, E)$. The coboundary map
$$d : C^n(G, E) \to C^{n+1}(G, E)$$
that makes the the family $C^*(G, E)$ into a cochain complex is defined by
$$dc(g_1, \ldots, g_{n+1}) := \tilde{\Lambda}(g_1)^{-1}(c(g_2, \ldots, g_{n+1})) + \sum_{i=1}^n (-1)^i c(g_1, \ldots, g_ig_{i+1}, \ldots, g_n) + (-1)^{n+1} c(g_1, \ldots, g_n)$$
for $n \geq 1$, with
$$dc(g) = \tilde{\Lambda}(g)^{-1}(c(t(g))) - c(s(g)).$$
Here we write the operation in $E$ by “+”. Then one can checks that $dd = 0$ on $C^n(G, E)$ for all $n \in \mathbb{Z}_{\geq 0}$. The associated **groupoid cohomology group** is
$$H^n_{\Lambda}(G, E) := \frac{\ker\{ d : C^n(G, E) \to C^{n+1}(G, E) \}}{\text{Im}\{ d : C^n(G, E) \to C^{n+1}(G, E) \}}$$
for $n \geq 1$ and $H^0_{\Lambda}(G, E) = \ker\{ d : C^0(G, E) \to C^1(G, E) \}$.

A.2. Induced action of $K$ on $Z_A$ for a groupoid $A$-extension. Let $\tilde{\Lambda} : K^1 \to \text{SAut}(A)$ be a morphism with smooth liftings. Then there is an induced $K$-action on the group $Z_A$. We next describe this action. We first take a smooth lifting $\Lambda : K^1 \to \text{SAut}^0(A)$. So it assigns an arrow $\xi : x \to y$ in $K^1$ an isomorphism $\Lambda_\xi : A \to A$. For every $\sigma \in Z_A$, we define $\Lambda_\xi(\sigma)$ by
$$\Lambda_\xi(\sigma)(a) := \Lambda_\xi \circ \sigma \circ \Lambda_\xi^{-1}(a) \ \text{abbreviation} \quad = \quad \Lambda_\xi \sigma \Lambda_\xi^{-1}(a).$$ \hfill (A.3)

**Lemma A.1.** The formula (A.3) defines a $K$-action on $Z_A$.

**Proof.** As in (4.1) for every pair of composable arrows $\xi, \eta$ in $K^1$, we have the natural transformation $\Omega(\xi, \eta) : \text{id}_A \Rightarrow \Lambda_\eta^{-1} \Lambda_\xi \Lambda_\eta$. We will show that the $A$-invariance of $\sigma \in Z_A$ ensure that
$$\Lambda_\eta(\Lambda_\xi(\sigma)) = \Lambda_\xi(\sigma).$$
Take an element $a \in A^0$. Then
$$\Lambda_\eta(\Lambda_\xi(\sigma))(a) = \Lambda_\eta \Lambda_\xi(\sigma)(\Lambda_\eta^{-1}(a)) = \Lambda_\eta \Lambda_\xi \sigma \Lambda_\eta^{-1} \Lambda_\eta^{-1}(a), \quad \text{and} \quad \Lambda_\xi(\sigma)(a) = \Lambda_\xi \sigma \Lambda_\xi^{-1}(a).$$

\(^6\)As we have remark in the definition of multiplication of arrows of groupoids in the beginning of §2, the convention is different from the usual one, in which $s(g_i) = t(g_{i+1})$, that is the arrows goes from the right to the left. However, we emphasis that it goes from left to right. So in the following, the definition of natural maps from $G^{[n]}$ to $G^0$ will be different from the usual one (see for example [11, 3]), so is the definition of coboundary map.
Since $\Omega(\xi, \eta) : \text{id}_A \Rightarrow \Lambda_{\xi\eta}^{-1}A_\eta^1A_\xi$ is a natural transformation we have a commutative diagram

\[
\begin{array}{ccc}
\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a) & \xrightarrow{\Omega(\xi, \eta, \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a))} & \Lambda_{\xi\eta}^{-1}(a) \\
\sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)) & \xrightarrow{\Omega(\xi, \eta, \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a))} & \Lambda_{\xi\eta}^{-1}(a) \setminus \Lambda_{\xi\eta}^{-1}A_{\eta\xi}(\sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a))) \\
\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a) & \xrightarrow{\Omega(\xi, \eta, \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a))} & \Lambda_{\xi\eta}^{-1}(a).
\end{array}
\]

The $A$-invariance of $\sigma$ implies that for every arrow $h : \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a) \to \Lambda_{\xi\eta}^{-1}(a)$ in $A$, we have

\[h^{-1} \cdot \sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)) \cdot h = \sigma(\Lambda_{\xi\eta}^{-1}(a)).\]

Therefore for the arrow $\Omega(\xi, \eta, \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)) : \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a) \to \Lambda_{\xi\eta}^{-1}(a)$, the commutative diagram above implies

\[\Lambda_{\xi\eta}^{-1}A_{\eta}\Lambda_{\xi}(\sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a))) = \Omega(\xi, \eta, \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a))^{-1} \cdot \sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)) \cdot \Omega(\xi, \eta, \Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)) = \sigma(\Lambda_{\xi\eta}^{-1}(a)).\]

So we have

\[\Lambda_{\eta}\Lambda_{\xi}\sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)) = \Lambda_{\xi\eta}\sigma(\Lambda_{\xi}^{-1}A_{\eta}^{-1}(a)).\]

Consequently $\Lambda_{\eta}(\Lambda_{\xi}(\sigma)) = \Lambda_{\xi\eta}(\sigma)$. This shows that $\Lambda$ induces a $K$-action on $Z_A$. \qed

Moreover,

**Lemma A.2.** The action $\Lambda(\xi)$ does not depend on the smooth lifting $\Lambda$ of $\bar{A}$, but only on $\bar{A}$ itself.

**Proof.** Suppose both $\Lambda_1$ and $\Lambda_2$ are liftings of $\bar{A}$. Then there are natural transformations $\rho_\xi : \Lambda_1,\xi \Rightarrow \Lambda_2,\xi$ for all $\xi \in K^1$. Take an $a \in A^0$. Let $a_i = \Lambda_i^{-1}(a)$ for $i = 1, 2$. By the definition (A.3), we have

\[\Lambda_i,\xi(\sigma)(a) = \Lambda_i,\xi(\sigma)(a_i).\]

We next show that $\Lambda_1,\xi(\sigma)(a_1)) = \Lambda_2,\xi(\sigma)(a_2)$.

First of all, by the natural transformation $\rho_\xi : \Lambda_1,\xi \Rightarrow \Lambda_2,\xi$ we get an arrow

\[\rho_\xi(a_2) : \Lambda_1,\xi(a_2) \to \Lambda_2,\xi(a_2) = \Lambda_1,\xi(a_1) = a,
\]

which induces an arrow $\Lambda_1,\xi[\rho_\xi(a_2)] : a_2 \to a_1$. Therefore by the $A$-invariance of $\sigma$ we get

\[\sigma(a_1) = \Lambda_1,\xi[\rho_\xi(a_2)] \cdot \sigma(a_2) \cdot \Lambda_1,\xi[\rho_\xi(a_2)].\]

Apply $\Lambda_1,\xi$ to both sides we get

\[\Lambda_1,\xi(\sigma(a_1)) = \rho_\xi(a_2)^{-1} \cdot \Lambda_1,\xi(\sigma(a_2)) \cdot \rho_\xi(a_2).
\]

Again, since $\rho_\xi : \Lambda_1,\xi \Rightarrow \Lambda_2,\xi$, for the arrow $\sigma(a_2) : a_2 \to a_2$ we have a commutative diagram in $A^1$

\[
\begin{array}{ccc}
\Lambda_1,\xi(a_2) & \xrightarrow{\rho_\xi(a_2)} & \Lambda_2,\xi(a_2) \\
\Lambda_1,\xi(\sigma(a_2)) & \xrightarrow{\Lambda_2,\xi(\sigma(a_2))} & \Lambda_2,\xi(\sigma(a_2)).
\end{array}
\]

Therefore

\[\Lambda_1,\xi(\sigma(a_1)) = \rho_\xi(a_2)^{-1} \cdot \Lambda_1,\xi(\sigma(a_2)) \cdot \rho_\xi(a_2)^{-1} = \Lambda_2,\xi(\sigma(a_2)).\]

This finishes the proof. \qed

So we denote this induced action of $K$ on $Z_A$ by $\bar{A} : K^1 \to \text{Aut}(Z_A)$. By using this action we could define groupoid cohomology $H^*_A(K, Z_A)$ of $K$ with coefficients in $Z_A$. 


A.3. The cocycle \( \Xi \). Recall that given a smooth lifting \( \Lambda : K^{1} \rightarrow \text{SAut}^{0}(A) \) of \( \bar{\Lambda} \), and the corresponding smooth family of natural transformation \( \Omega : K^{[2]} \times A^{0} \rightarrow A^{1} \), we have the following elements of \( Z_{A} \) (cf. (4.8))

\[
\Xi(\xi, \eta, \zeta, a) = \Omega(\xi, \eta, a) \cdot \Omega(\xi, \eta, \zeta, \xi_{e}, \Lambda_{\xi}^{-1} \Lambda_{\eta} \Lambda_{\zeta} \Lambda_{\xi}(a) - 1) \cdot \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, \Lambda_{\xi}(a) - 1)
\]

for three composable arrows \( \xi, \eta, \zeta \) of \( K \). Therefore \( \Xi \) is a cochain in \( C_{A}^{3}(K, Z_{A}) \).

**Theorem A.3.** \( \Xi \) is a 3-cocycle in the cochain complex \( C_{A}^{3}(K, Z_{A}) \).

Before prove this theorem, we first analysis the properties of \( \Xi \).

**Lemma A.4.** We could cyclically permute the elements in the product of the definition of \( \Xi \) without change the values.

Take three composable arrows \( \xi, \eta, \zeta \) in \( K \). For any \( a \in A^{0} \), set

\[
\Xi_{1}(\xi, \eta, \zeta, a) = \Omega(\xi, \eta, \zeta) \cdot \Omega(\xi, \eta, \zeta, \xi_{e}, \Lambda_{\xi}^{-1} \Lambda_{\eta} \Lambda_{\zeta} \Lambda_{\xi}(a) - 1) \cdot \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, \Lambda_{\xi}(a) - 1) \\
\Xi_{2}(\xi, \eta, \zeta, a) = \Omega(\xi, \eta, \zeta, \xi_{e}, \Lambda_{\xi}^{-1} \Lambda_{\eta} \Lambda_{\zeta} \Lambda_{\xi}(a) - 1) \cdot \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, \Lambda_{\xi}(a) - 1) \\
\Xi_{3}(\xi, \eta, \zeta, a) = \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, \Lambda_{\xi}(a) - 1) \cdot \Omega(\xi, \eta, \zeta, \Lambda_{\xi}(a))
\]

Then \( \Xi = \Xi_{1} = \Xi_{2} = \Xi_{3} \).

**Proof.** We show that \( \Xi_{1}, \Xi_{2} \) and \( \Xi_{3} \) give rise to maps from \( K^{[3]} \) to \( Z_{A} \). Then by the \( A \)-invariance of elements of \( Z_{A} \). We get this lemma.

We give the explicit computation about \( \Xi_{1} \). We first show that it belongs to the center. Take an arrow \( a \in \Gamma_{a} \). By applying (4.3) repeatedly, we have

\[
\Omega(\xi, \eta, \zeta, a) \cdot \Omega(\xi, \eta, \zeta, \xi_{e}, \Lambda_{\xi}^{-1} \Lambda_{\eta} \Lambda_{\zeta} \Lambda_{\xi}(a) - 1) \cdot \Lambda_{\xi}^{-1} \Omega(\eta, \zeta, \Lambda_{\xi}(a) - 1) \\
= \Omega(\xi, \eta, \zeta, a) \cdot \Omega(\xi, \eta, \zeta, \Lambda_{\xi}^{-1} \Lambda_{\eta} \Lambda_{\zeta} \Lambda_{\xi}(a)) \cdot \Omega(\eta, \zeta, \Lambda_{\xi}(a))
\]

where the red part is the result of the previous part with an underline via applying (2.1). Therefore, \( \Xi_{1}(\xi, \eta, \zeta, a) \in ZA^{0} \). One can also see that, it is \( A \)-invariant, hence gives rise to a map \( K^{[3]} \rightarrow Z_{A} \). From the expression of \( \Xi \) and \( \Xi_{1} \) we have

\[
\Xi(\xi, \eta, \zeta, a) = \Omega(\xi, \eta, a) \cdot \Xi_{1}(\xi, \eta, \zeta, a) \cdot \Omega(\xi, \eta, \zeta, \Lambda_{\xi}(a)) \cdot \Omega(\xi, \eta, \zeta, \Lambda_{\xi}(a))^{-1} \\
= \Omega(\xi, \eta, a) \cdot \Xi_{1}(\xi, \eta, \zeta, a) \cdot \Omega(\xi, \eta, \zeta, a)^{-1}.
\]

Then since \( \Xi_{1} \) has images in \( Z_{A} \), we also have

\[
\Xi(\xi, \eta, \zeta, a) = \Omega(\xi, \eta, a) \cdot \Xi_{1}(\xi, \eta, \zeta, a) \cdot \Omega(\xi, \eta, a)^{-1}
\]
Consequently, $\Xi(\xi, \eta, \zeta, a) = \Xi_1(\xi, \eta, \zeta, a)$. By similar computations, we see that $\Xi_2$ and $\Xi_3$ also belong to $Z_A$, and $\Xi_2 = \Xi_3 = \Xi$. This finishes the proof.}

Now we proceed to prove Theorem A.3. We compute the differential of $\Xi$. Since the value of $\Xi$ lies in the center, we have

$$d\Xi(\xi_1, \xi_2, \xi_3, \xi_4) = \Lambda_{\xi_1}^{-1}(\Xi(\xi_2, \xi_3, \xi_4)) \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3) \cdot \Xi(\xi_1, \xi_2, \xi_3) \cdot \Xi(\xi_1, \xi_2, \xi_3)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3).$$

In the following we use some abbreviations of notations. For example we write

$$\Lambda_1 := \Lambda_{\xi_1}, \quad \xi_{12} := \xi_1 \xi_2.$$

We first compute $\Xi(\xi_1, \xi_2, \xi_3) = \Xi(\xi_1, \xi_2, \xi_3) \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a)$ and take an element $a \in A^\bullet$.

$$\Xi := \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a) \cdot \Xi(\xi_1, \xi_2, \xi_3)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a).$$

Denote the middle term by

$$A(\xi_1, \xi_2, \xi_3, \xi_4)(\Lambda_{\xi_1}^{-1} \Lambda_{\xi_2}^{-1} \Lambda_{\xi_3}^{-1} \Lambda_{\xi_4}^{-1} A_{\xi_234}(a)) := \Xi(\xi_1, \xi_2, \xi_3, \xi_4) \cdot \Xi(\xi_1, \xi_2, \xi_3)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3) \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a),$$

Then one finds that $A(\xi_1, \xi_2, \xi_3, \xi_4)(\Lambda_{\xi_1}^{-1} \Lambda_{\xi_2}^{-1} \Lambda_{\xi_3}^{-1} \Lambda_{\xi_4}^{-1} A_{\xi_234}(a))$ belongs to the center of $\Gamma_{\Lambda_{\xi_1}^{-1} \Lambda_{\xi_2}^{-1} \Lambda_{\xi_3}^{-1} \Lambda_{\xi_4}^{-1} A_{\xi_234}(a)}$. And by (4.3), one finds that $A(\xi_1, \xi_2, \xi_3, \xi_4)$ gives rise to an A-invariant section of $ZA \rightarrow A$, hence belongs to $ZA$. Therefore $\Xi$ equals to

$$\Xi := \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a) \cdot \Xi(\xi_1, \xi_2, \xi_3)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3) \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3)(a)^{-1} \cdot \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a).$$
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_3^{-1} A_{123}(a)) \cdot \Omega(\xi_{12}, \xi_3, A_1^{-1} A_3^{-1} A_{123}(a)) \]
\[ = \Omega(\xi_{12}, \xi_3, A_1^{-1} A_2^{-1} A_{123}(a))^{-1} \cdot \left\{ \Lambda_{12}^{-1} \left[ \Omega(\xi_3, \xi_4, A_3^{-1} A_{123}(a)) \right] \right\} \]
\[ = \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_3^{-1} A_{123}(a)) \cdot \Omega(\xi_{12}, \xi_3, A_1^{-1} A_3^{-1} A_{123}(a))^{-1} \]
\[ \cdot \Omega(\xi_{12}, \xi_3, A_1^{-1} A_2^{-1} A_{123}(a)) \cdot \Omega(\xi_{12}, \xi_3, A_1^{-1} A_3^{-1} A_{123}(a))^{-1} \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{123}(a)) \cdot \Omega(\xi_{12}, \xi_3, A_1^{-1} A_3^{-1} A_{123}(a))^{-1} \]

As above, denote the middle term by \( B(\xi_1, \xi_2, \xi_3, \xi_4)(A_1^{-1} A_3^{-1} A_{123}(a)) \). Then one also find that it gives rise to an \( A \)-invariant section \( B(\xi_1, \xi_2, \xi_3, \xi_4) \in Z_a \). So

\[ \Theta = B(\xi_1, \xi_2, \xi_3, \xi_4)(a) \]
\[ = \Lambda_{12}^{-1} \left[ \Omega(\xi_3, \xi_4, A_{12}(a)) \right] \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_3^{-1} A_4 A_{12}(a))^{-1} \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_4 A_{12}(a)) \right] \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_{12}(a)) \right]^{-1} \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_{12}(a)) \right] \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a)) \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_3^{-1} A_{12}(a)) \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a)) \]

We rewrite the RHS of this equality as

\[ \Theta = \Lambda_{12}^{-1} \left[ \Omega(\xi_3, \xi_4, A_{12}(a)) \right] \cdot \left\{ \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_3^{-1} A_4 A_{12}(a))^{-1} \right\} \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_4 A_{12}(a)) \right] \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_{12}(a)) \right]^{-1} \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_{12}(a)) \right] \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a)) \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_3^{-1} A_{12}(a)) \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a)) \]

As above, denote the middle term by \( C(\xi_1, \xi_2, \xi_3, \xi_4)(A_1^{-1} A_3^{-1} A_4 A_{12}(a)) \). Then one also see that it gives rise to an \( A \)-invariant section in \( Z_a \). Therefore

\[ \Theta = C(\xi_1, \xi_2, \xi_3, \xi_4)(a) \]
\[ = \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a))^{-1} \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_{12}(a)) \right] \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_4 A_{12}(a)) \right] \cdot \Lambda_1^{-1} \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_{12}(a)) \right]^{-1} \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a)) \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_3^{-1} A_{12}(a)) \]
\[ \cdot \Omega(\xi_1, \xi_2, A_1^{-1} A_2^{-1} A_{12}(a)) \]

\[ = \Lambda_{12}^{-1} A_2 \left[ \Omega(\xi_2, \xi_3, A_2^{-1} A_{12}(a)) \cdot \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_4 A_{12}(a)) \cdot \Omega(\xi_2, \xi_3, A_2^{-1} A_3^{-1} A_{12}(a)) \right]^{-1} \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \right] \]
\[ \cdot \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \]
\[ \cdot \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \]

\[ = \Lambda_{12}^{-1} A_2 \left[ \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \cdot \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \right]^{-1} \]
\[ \cdot \Lambda_1^{-1} \left[ \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \right] \]
\[ \cdot \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \]
\[ \cdot \Omega(\xi_3, \xi_4, A_3^{-1} A_4 A_{12}(a)) \]
where for the last equality we have used the fact that the induce action of $K$ on $Z_A$ is a morphism from $K^1$ to Aut$(Z_A)$. Therefore for every $a \in A^0$,
\[
d \Xi(\xi_1, \xi_2, \xi_3, \xi_4)(a) = \Lambda^{-1} \Xi(\xi_2, \xi_3, \xi_4)(a) \cdot \Theta = \Lambda^{-1} \Xi(\xi_2, \xi_3, \xi_4)(a) \cdot [S^{-1} \Xi(\xi_2, \xi_3, \xi_4)(a)]^{-1} = 1_a.
\]
This shows that $\Xi$ is a 3-cocycle, hence represents a class $[\Xi]$ in $H^3_3(K, Z_A)$. This finishes the proof of Theorem A.3.

Finally we show that

**Theorem A.5.** The cohomology class $[\Xi]$ depends only on $\Lambda$, not on $(\Lambda, \Omega)$.

**Proof.** First consider the case that, there is another $\Omega'(\cdot, \cdot)$ satisfying
\[
\Omega'(\xi, \eta, \cdot) : id_{\Lambda} \Rightarrow \Lambda^{-1} \Lambda_{\eta} \Lambda_{\xi}.
\]
Then we see that
\[
\Omega(\xi, \eta, \cdot) \circ \Omega'(\xi, \eta, \cdot) : id_{\Lambda} \Rightarrow id_{\Lambda}.
\]
Hence $\Omega(\xi, \eta, \cdot) \circ \Omega'(\xi, \eta, \cdot) \circ^{-1} : id_{\Lambda} \Rightarrow id_{\Lambda}$. Set
\[
\rho(\xi, \eta)(a) = \Omega(\xi, \eta, a) \cdot \Omega'(\xi, \eta, a)^{-1}.
\]
Then $\rho \in C^2_\Lambda(K, Z_A)$. Denote by $\Xi'$ the cocycle determined by $(\Lambda, \Omega')$ via (4.8). Then we find
\[
\Xi = d\rho \cdot \Xi'.
\]
So $[\Xi] = [\Xi']$.

We next consider the case that there is another lifting $\Lambda''$ of $\Lambda$ and a corresponding $\Omega''$. Then there are a smooth family of natural transformations
\[
\rho_\xi : \Lambda_\xi \Rightarrow \Lambda''_\xi.
\]
We define
\[
\Omega'(\xi, \eta, \cdot) := \Omega(\xi, \eta, \cdot) \circ (\rho_\xi^{-1} \circ \rho_\eta \circ \rho_\xi) : id_{\Lambda} \Rightarrow (\Lambda''_{\xi_{\eta}} )^{-1} \Lambda''_{\xi_{\eta}} \Lambda''_\xi.
\]
Denote by $\Xi$ the cocycle determined by $(\Lambda, \Omega)$, by $\Xi'$ the cocycle determined by $(\Lambda'', \Omega')$, and by $\Xi''$ the cocycle determined by $(\Lambda'', \Omega'')$. Then by the argument for first case we have
\[
[\Xi'] = [\Xi''].
\]
On the other hand, the $\rho$ in (A.4) gives rise to the conjugation transformation from $(\Lambda, \Omega)$ to $(\Lambda'', \Omega')$. Then since $\Xi$ and $\Xi'$ are both sections of $Z_A$, we have $\Xi' = \Xi$. Therefore $[\Xi] = [\Xi'']$. \qed

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