Fractal dimensions of graph of Weierstrass-type function and local Hölder exponent spectra

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Abstract

We study several fractal properties of the Weierstrass-type function

\[ W(x) = \sum_{n=0}^{\infty} \lambda(x)\lambda(\tau x)\cdots\lambda(\tau^{n-1} x) g(\tau^n x), \]

where \( \tau : [0, 1) \rightarrow [0, 1) \) is a cookie cutter map with possibly fractal repeller, and \( \lambda \) and \( g \) are functions with proper regularity. In the first part, we determine the box dimension of the graph of \( W \) and Hausdorff dimension of its randomised version. In the second part, the Hausdorff spectrum of the local Hölder exponent is characterised in terms of thermodynamic formalism. Furthermore, in the randomised case, a novel formula for the lifted Hausdorff spectrum on the graph is provided.

Keywords: Weierstrass function, local Hölder exponent, Hausdorff dimension

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1. Introduction

Let \( \mathbb{T} := [0, 1) \) be the interval, topologically identified with the one-dimensional unit torus \( \mathbb{R}/\mathbb{Z} \), and \( (I_i)_{i \in \Sigma} \) be disjoint subintervals of \( \mathbb{T} \), where \( \Sigma_\ell = \{0, \ldots, \ell - 1\} \) for some \( \ell \in \mathbb{N} \). A map \( \tau : \mathbb{T} \rightarrow \mathbb{T} \) is called a cookie cutter if \( \tau(x) = 0 \) for \( x \in \mathbb{T} \setminus \bigcup_{i \in \Sigma} I_i \), and each restriction \( \tau_I^0 : I_i^0 \rightarrow (0, 1) \) is a \( C^{1+\alpha} \)-diffeomorphism. In this note, we study several fractal geometrical structures of the deterministic as well as the randomised Weierstrass-type function \( W_\Theta : \mathbb{T} \rightarrow \mathbb{T} \).
\[ W_\theta(x) = \sum_{n=0}^{\infty} \lambda(x)\lambda(\tau x) \cdots \lambda(\tau^{n-1} x) g(\tau^n x + \theta_n), \quad (1) \]

where \( \lambda: \mathbb{T} \to (0, 1) \) and \( g: \mathbb{T} \to \mathbb{R} \) are continuous maps which are piecewise \( C^{1+\alpha} \), i.e., their restrictions to each \( I_i^\circ \) are of class \( C^{1+\alpha} \). In addition, the variable \( \theta \in \mathbb{T}^N \) is only used as a random sequence. In the deterministic case, we study \( W := W_\theta \). Moreover, we always assume the following partial hyperbolic condition

\[ \inf_{\tau \in \Sigma_\alpha} \inf_{x \in (I_i^\circ)^\circ} |\tau'(x)| \lambda(x) > 1. \quad (2) \]

Note that the classical Weierstrass function is given by \( \tau(x) = \ell x \mod 1 \) and \( g(x) = \cos(2\pi x) \) for some \( \ell \in \mathbb{N} \) and \( \lambda \in (0, 1) \) with \( \ell \lambda > 1 \).

Since all summands of the absolutely convergent sum in (1) are continuous on \( \mathbb{T} \), so is \( W_\theta \). In fact, \( W_\theta \) is \( \alpha \)-Hölder continuous but generally, due to (2), no better regularity can be expected. This point will be deeply discussed in terms of a multifractal analysis of local Hölder exponent.

Before proceeding, we introduce a few notations. The intervals \((I_i)_{i \in \Sigma_\alpha}\) are called the monotonicity intervals of \( \tau \). Furthermore, the set of singular points and the repeller of \( \tau \) are respectively defined as

\[ \mathcal{N} = \bigcup_{n \in \mathbb{N}} \tau^{-n}\{0, 1\} \quad \text{and} \quad \mathcal{J} = \bigcap_{n \in \mathbb{N}} \overline{\tau^{-n}(0, 1)} . \]

Clearly, \( \tau(\mathcal{N}) = \tau^{-1}(\mathcal{N}) = \mathcal{N} \) and \( \tau(\mathcal{J}) \subseteq \mathcal{J} \). When we consider the restricted dynamics \((\mathcal{J}, \mathcal{J})\), we often write \( \tau' \) and \( \lambda \) etc instead of \((\tau|_\mathcal{J})'\) and \( \lambda|_\mathcal{J} \) etc. This restriction is essential from the point of view of the fractality, since \( W_\theta \) is ‘smooth’ outside of \( \mathcal{J} \) as shown below.

**Lemma 1.** If \( \mathcal{J} \neq \mathbb{T} \), then the restriction of \( W_\theta \) to \( \mathbb{T} \setminus (\mathcal{J} \cup \mathcal{N}) \) is of class \( C^{1+\alpha} \).

**Proof.** Let \( x \in \mathbb{T} \setminus (\mathcal{J} \cup \mathcal{N}) \). Then, we have \( x \in U := \mathbb{T} \setminus \overline{\tau^{-n}(0, 1)} \) for some \( n \in \mathbb{N} \). Observe that \( U \) is a neighbourhood of \( x \) such that

\[ W_\theta(u) = \sum_{j=0}^{n-1} \lambda(u)\lambda(\tau u) \cdots \lambda(\tau^{j-1} u) g(\tau^j u + \theta_j) + \lambda(u) \cdots \lambda(\tau^n u) \sum_{k=0}^{\infty} (\lambda(0))^k g(\theta_k + u) \]

for all \( u \in U \). Moreover, as \( x \notin \mathcal{N} \), there is a sub-neighbourhood \( x \in V \subseteq U \) such that the restrictions of \( \tau, \ldots, \tau^n \) to \( V \) are of class \( C^{1+\alpha} \). Thus \( W_\theta \) is continuously differentiable in \( x \). \( \Box \)

Hence we are only interested in the regularity of \( W_\theta \) over \( \mathcal{J} \). The following important dichotomy will be shown in a later section. For the classical Weierstrass function, the nowhere differentiability was proved by G. H. Hardy in [Har16].

**Lemma 2.** \( W \) is nowhere differentiable on \( \mathcal{J} \) or is of class \( C^{1+\alpha}(\mathbb{T}) \).

The above lemma was essentially shown in [Bed89], where the author referred to the latter case as the degenerate case. For some practical reasons, in this note, we use a slightly weaker definition: Given \( \theta \), \( W_\theta \) is degenerate if it is Lipschitz continuous.

Our study covers two main topics. One of them concerns the Hausdorff and box dimension of the graph \( G W_\theta = \{(x, W_\theta(x)) : x \in \mathbb{T}\} \). More generally, for any function \( \phi: D \to \mathbb{R} \), we define \( G\phi = \{(x, \phi(x)) : x \in D\} \). Theorem 1 gives the box dimension for \( W \), while theorem 2 does the Hausdorff dimension for the randomised function \( W_\theta \). In the case \( \mathcal{J} = \mathbb{T} \) and \( \tau' > 0 \), the box dimension of the graph of \( W \) was proved in [Bed89], while its Hausdorff dimension
in the classical case is proved in [She15, Bá15] and [Kel17]. In addition, the randomised function $W_\theta$ was studied in [Hun98] and [MW12]. Moreover, several similar objects were investigated e.g. in [Car13, Rom14] and [MU16]. The other topic is the local Hölder exponent, that is defined by

$$\text{hol}_{W_\theta}(x) = \sup \left\{ \alpha \in (0, 1) : \inf_{r > 0} \sup_{d(x,u) < r} \frac{|W_\theta(x) - W_\theta(u)|}{d(x,u)^\alpha} < \infty \right\},$$

(3)

where $d(x,u) := \max\{|x - u|, 1 - |x - u|\}$ is the torus metric. It turns out that these values are as a function on $\mathcal{J}$ highly complex unless it is constant, in the multifractal point of view. To see this, we study the level set

$$E_{\theta,\alpha} := \{ x \in \mathcal{J} : \text{hol}_{W_\theta}(x) = \alpha \}$$

in terms of the Hausdorff spectrum. For simplicity, let $E_\alpha := E_{\theta,\alpha}$ for $0 = (0, 0, \ldots)$. In particular, theorems 3 and 4 provide some formulas for the spectra

$$\alpha \mapsto \dim_H E_\alpha \quad \text{and} \quad \alpha \mapsto \dim_H (\mathcal{G}W_\theta \cap (E_{\theta,\alpha} \times \mathbb{R})).$$

Note that the first spectrum for a similar complex function was studied in [JS15].

2. Main results

Let $s_1, s_2 \in \mathbb{R}$ be the unique zeros of the Bowen equations

$$P_{\mathcal{J}}((1 - s_1) \log |r'| + \log \lambda) = 0 \quad \text{and} \quad P_{\mathcal{J}}(s_2 \log \lambda) = 0,$$

(4)

where $P_{\mathcal{J}}(\cdot)$ denotes the topological pressure on $(\mathcal{J}, \tau_{\mathcal{J}})$.

The next theorem generalises one of the major results in [Bed89].

**Theorem 1.** Suppose that $W$ is non-degenerate. Then, we have

$$\dim_B \mathcal{G}W_{1,\mathcal{J}} = s_1 \quad \text{and} \quad \dim_H \mathcal{G}W_{1,\mathcal{J}} \leq \min\{s_1, s_2\}.$$

In addition, if $\mathcal{J} \neq \mathbb{T}$, then $\dim_B \mathcal{G}W_{1,\mathcal{J} \setminus \mathcal{T}} = \dim_H \mathcal{G}W_{1,\mathcal{J} \setminus \mathcal{T}} = 1$.

As an immediate corollary, we can see that the Hausdorff dimension of $W_{1,\mathcal{J}}$ may be strictly smaller than its box dimension.

**Corollary 3.** Suppose that $W$ is non-degenerate. If $\dim_B \mathcal{G}W_{1,\mathcal{J}} < 1$, then $\dim_H \mathcal{G}W_{1,\mathcal{J}} < \dim_B \mathcal{G}W_{1,\mathcal{J}}$.

Now, a natural question is whether $\dim_H \mathcal{G}W_{1,\mathcal{J}} = \min\{s_1, s_2\}$. We will show this identity for a randomised function $W_\theta$ for a specific class of $g$.

Recall that $x$ is a simple zero of a function $f$ if $f(x) = 0$ but $f'(x) \neq 0$. Inspired by the notation in [MW12], we say that a function $g : \mathbb{T} \to \mathbb{R}$ satisfies the strong critical point hypothesis if $g \in C^1(\mathbb{T})$ and there is some number $m \in \mathbb{N}$ such that, for any $a \in (0, 1)$ and $c \in \mathbb{R}$, the function $g'(a + \cdot) - cg'$ has at most $m$ zeros and all of them are simple. For example, $g(x) = \cos(2\pi x)$ and $g(x) = \sin(2\pi x)$ belong to this class. However, in general, the verification of the condition is hard. The condition is only needed for lemma 22, which is originally stated in [MW12].

\[1\] In fact, our lemma 22 is implicitly claimed and applied in [MW12] under a far more general setting without detailed proofs. In this note, we give a proof for lemma 22 and do not access further generalisations.
Theorem 2. Suppose that $\vartheta \in T^{N_0}$ is an independent and identically uniformly distributed random sequence on $T$ and that $T_0 \cap J, T_1 \cap J, \ldots, T_{r-1} \cap J$ are pairwise disjoint as subsets of $T$. Furthermore, suppose that $g$ satisfies the strong critical point hypothesis. Then, almost surely with respect to $\vartheta$, we have
\[ \dim_H G^W_{\vartheta} = s_1 \quad \text{and} \quad \dim_H GW_{\vartheta} = \min\{s_1, s_2\}. \]

In order to describe the spectrum $\alpha \mapsto \dim_H E_{\alpha}$, we need to introduce further notation from the thermodynamic formalism. For each $q \in \mathbb{R}$, let $A_q$ be the number that is uniquely determined by
\[ P_J(-A_q \log |\tau'| + q \log \lambda) = 0. \] (5)

According to [Bar08], we can define the following quantities:
- Let $D$ be the Legendre transform of the map $q \mapsto A_q$, i.e. $D(\alpha) := \sup_{q \in \mathbb{R}} q \alpha + A_q$.
- Let $\alpha(q) := -A_q'$.
- Let $A := (\alpha_{\min}, \alpha_{\max})$, where $\alpha_{\min} := \inf_{q \in \mathbb{R}} \alpha(q)$ and $\alpha_{\max} := \sup_{q \in \mathbb{R}} \alpha(q)$.

Recall that, if $A \neq \emptyset$, the restriction of $D$ to the interval $A$ is a strictly concave non-negative analytic function, so $\alpha_c := \alpha(0) \in A$ is the unique maximum point (critical point). Observe that
\[ D(\alpha_c) = 1 \quad \text{and} \quad D'(\alpha_c) = 0. \]

In case $A = \emptyset$, we define $\alpha_c := \alpha_{\min} = \alpha_{\max}$.

The Hausdorff spectrum of the local Hölder exponent is characterised as follows.

Theorem 3. Suppose that $W$ is non-degenerate. In case $A = \emptyset$, we have
\[ E_{\alpha} = \begin{cases} J & \text{if } \alpha = \alpha_c \\ \emptyset & \text{otherwise} \end{cases}. \]

In case $A \neq \emptyset$, we have
\[ \dim_H E_{\alpha} = D(\alpha) \]
for all $\alpha \in A$. Moreover, for each $\alpha \in A$, there is a Gibbs measure $\nu_{\alpha}$ such that $\nu_{\alpha}(E_{\alpha}) = 1$.
\[ \dim_H \nu_{\alpha} = D(\alpha) = \frac{h_\tau(\nu_{\alpha})}{\int \log |\tau'| \, d\nu_{\alpha}} \quad \text{and} \quad \alpha = -\frac{\int \log \lambda \, d\nu_{\alpha}}{\int \log |\tau'| \, d\nu_{\alpha}}. \]

Remark 4. If $\lambda = |\tau'|^{-\theta}$ for some constant $\theta \in (0, 1)$, then $A = \emptyset$. Clearly, then $\alpha_c = \theta$. In particular, theorem 3 implies [Tod15, theorem 1], i.e. that the Lebesgue measure of $E_{\theta}$ is one.

We turn to the graph points over $E_{\alpha}$ for various $\alpha$. A natural upper bound can be immediately derived from theorem 3, applying the general formula presented in [Jin11, theorem 1].

Lemma 5. We have
\[ \dim_H (GW_{\vartheta} \cap (E_{\vartheta,\alpha} \times \mathbb{R})) \leq \min \left\{ \dim_H E_{\vartheta,\alpha} + 1 - \alpha, \frac{\dim_H E_{\vartheta,\alpha}}{\alpha} \right\} \]
for all $\alpha \in \mathbb{R}$ and $\vartheta \in T^{N_0}$.

The next result on the randomised case is an application of lemma 13, which suggests the canonical representation for the lifted spectrum.
Theorem 4. Suppose that \( \vartheta \in \mathbb{T}^{\mathbb{N}} \) is an independent and identically uniformly distributed random sequence on \( \mathbb{T} \) and that \( \mathcal{I}_0 \cap \mathcal{J}, \mathcal{I}_1 \cap \mathcal{J}, \ldots, \mathcal{I}_{r-1} \cap \mathcal{J} \) are pairwise disjoint as subsets of \( \mathbb{T} \). Furthermore, suppose that \( g \) satisfies the strong critical point hypothesis. Then, almost surely with respect to \( \vartheta \), we have
\[
\text{dim}_H(\mathcal{GW}_\vartheta \cap (E_{\vartheta, \alpha} \times \mathbb{R})) = \min \left\{ \frac{D(\alpha)}{\alpha} + 1 - \alpha, \frac{D(\alpha)}{\alpha} \right\}
\]
for all \( \alpha \in A \).

Remark 6. If \( W \) is non-degenerate and \( A \neq \emptyset \), it is not hard\(^2\) to verify
\[
\text{dim}_H(\mathcal{GW} \times (\mathcal{J} \times \mathbb{R})) = \sup_{\alpha \in A} \min \left\{ \frac{D(\alpha)}{\alpha} + 1 - \alpha, \frac{D(\alpha)}{\alpha} \right\} = \sup_{\alpha \in A} \{ \text{dim}_H E_{\alpha} + 1 - \alpha \}.
\]

3. Preliminaries

Here is the basic notation. Most of the definitions are related to the basis dynamics \((\tau, \mathbb{T})\), while some can be only defined for the restriction \((\tau|_{\mathcal{J}}, \mathcal{J})\). Recall that \( \mathbb{T} := [0, 1) \) is endowed with the torus metric \( d(x, u) := \max\{|x - u|, 1 - |x - u|\} \). Given an interval \( I \), let \(|I|\) and \( \partial I \) denote its length and the set of its endpoints, respectively. That is, \(|I| := b - a \) and \( \partial I := \{a, b\} \), where \( I := [a, b] \).

- Let \([x]_n := (\kappa(x), \kappa(\tau x), \ldots, \kappa(\tau^{n-1} x))\) for \( x \in \mathcal{J} \) and \( n \in \mathbb{N} \), where \( \kappa(x) := i \) for \( x \in \mathcal{I}_i \).
- Let \( \rho_i : [0, 1) \to \mathbb{T} \) be the \( i \)th inverse branch for \( i \in \Sigma_\ell \), i.e. the \( C^\infty \)-extension of \((\tau|_{\mathcal{J}})^{-1}\).
- Let \( \rho_i := \rho_{i_1} \circ \cdots \circ \rho_{i_n} \) for \( i := (i_1, \ldots, i_n) \in (\Sigma_\ell)^n \) and \( n \in \mathbb{N} \).
- The \( m \)th monotonicity interval of \( x \in \mathcal{J} \) is the subinterval \( I_m(x) \subseteq \mathbb{T} \) defined by
  \[
  I_m(x) := I_{\kappa(x)} \cap \tau^{-1} I_{\kappa(\tau x)} \cap \cdots \cap \tau^{-(m-1)} I_{\kappa(\tau^{m-1} x)}.
  \]
- Given any function \( \phi : \mathcal{J} \to \mathbb{R} \) we simply write
  \[
  \phi_n(x) := \sum_{k=0}^{n-1} \phi(\tau^k x) \quad \text{and} \quad \phi(x) := \prod_{k=0}^{n-1} \phi(\tau^k x)
  \]
for \( x \in \mathcal{J} \) and \( n \in \mathbb{N}_0 \).
- Given Borel measurable subset \( A \subseteq \mathbb{R}^d \), let \( \mathcal{P}(A) \) denote the set of all Borel probability measures on \( A \). Moreover, let \( \mathcal{P}_\tau(\mathcal{J}) \) denote the set of all \( \tau \)-invariant Borel measures \( \nu \) on \( \mathcal{J} \), where the \( \tau \)-invariance means \( \nu \circ \tau^{-1} = \nu \).
- \( \nu \in \mathcal{P}_\tau(\mathcal{J}) \) is a Gibbs measure if there are constants \( C_\phi > 0 \) and \( P_\phi \) and a Hölder continuous function \( \phi : \mathcal{J} \to \mathbb{R} \) such that
  \[
  C_\phi^{-1} \leq \frac{\nu(I_m(x))}{\phi_n(x) - P_\phi} \leq C_\phi
  \]
for all \( x \in \mathcal{J} \) and \( n \in \mathbb{N} \).
- Let \( P(\phi) \) denote the topological pressure on \((\mathcal{J}, \tau)\) with respect to a Hölder continuous potential function \( \phi : \mathcal{J} \to \mathbb{R} \), i.e.

\(^2\)The second equation follows immediately from basic properties of \( D(\alpha) \). The first one is, however, not a trivial corollary of the theorem because Hausdorff dimension is only \( \sigma \)-stable. For verification, a slight modification of its proof seems to be necessary.
\[ P(\phi) = \sup_{\nu \in P^*(J)} h_\tau(\nu) + \int \phi \, d\nu, \]

where \( h_\tau(\nu) \) denotes the Kolmogorov–Sinai-entropy.

For more general definitions and details of Gibbs measures and topological pressure, see [Bar08] or [Pes97]. The next lemma summarizes several important relations between equilibrium state and Gibbs measure, more general statements are found in [Kel98].

**Lemma 7.** Let \( \phi : J \to \mathbb{R} \) be Hölder continuous. The equilibrium state for the potential \( \phi \) is a Gibbs measure for the same potential, and vice versa, where \( P_\phi = P(\phi) \) is always satisfied. In addition, any Gibbs measure is ergodic and atom-free.

### 3.1. Hausdorff dimension of measures

We use the standard definitions of several dimensions in [Bar08] or [Pes97], related to sets as well as measures, so we just recall a few things here. Let \( \mu \) be a Borel measure on a metric space \((E, d_E)\). The lower pointwise dimension of \( \mu \) is defined as

\[ d_\mu(u) := \liminf_{r \to 0} \frac{\log \nu(B_r(u))}{\log r} \]

for each \( u \in E \), where \( B_r(u) := \{ u' \in E : d_E(u, u') \leq r \} \) denote the closed balls. In addition, the Hausdorff dimension of the measure \( \mu \), denoted by \( \dim_H \mu \), is defined as

\[ \dim_H \mu := \inf \{ \dim_H Z : Z \subseteq E \text{ s.t. } \mu^*(E \setminus Z) = 0 \}, \]

where \( \mu^* \) is the outer measure extension of \( \mu \).

The next lemma provides an alternative definition of \( \dim_H \mu \).

**Lemma 8 ([Bar08, theorem 2.1.5 (3)])**. Let \( \mu \) be a Borel measure on an Euclidean space \( E \). Then \( \dim_H \mu \) is the essential supremum of \( d_\mu \) with respect to \( \mu \).

### 4. Dimensions of the graph

We give proofs of theorems 1 and 2 in this section. In addition, a short proof of lemma 2 can be also found here.

The following two basic lemmas are repeatedly used throughout the proof sections.

**Lemma 9.** There is a constant \( D > 0 \) such that

\[ \left( \frac{\|u\|}{\|v\|} \right) \in [D^{-1}, D], \quad \|\pi^n(x)\| \cdot |I_n(x)| \in [D^{-1}, D] \quad \text{and} \quad \frac{|\rho_{\pi^n}^t(v)|}{|I_n(x)|} \in [D^{-1}, D] \]

for all \( u \in I_n(x), x \in \mathcal{J}, v \in \mathcal{T} \) and \( n \in \mathbb{N} \).

In addition, there is a constant \( \delta_0 > 0 \) such that

\[ \delta_0 \leq \frac{|I_{n+1}(x)|}{|I_n(x)|} \leq \delta_0^{-1} \]

for all \( x \in \mathcal{J} \) and \( n \in \mathbb{N} \).
Proof. As $\tau'$ is piecewise $\alpha$-Hölder continuous on $\bigcup_{i \in \Sigma_\ell} I_i$, so is $\log |\tau'|$. Let $a := \sup_{i \in \Sigma_\ell} \sup_{u \in (I_i)^\circ} |1/\tau'(u)|$ so that, in view of the mean value theorem, $|I_k(x)| \leq a^k$ for all $x \in \mathcal{J}$ and $k \in \mathbb{N}$. The first estimate follows as

$$
\left| \log \left( \frac{\langle \tau^n \rangle'(x)}{(\tau^n)'(u)} \right) \right| \leq \sum_{j=0}^{n-1} \left| \log |\tau'|((\tau^j)'x) - \log |\tau'|((\tau^j)'u) \right| \leq C_0 \sum_{j=0}^{n-1} |I_{n-j}(\tau^j)'x|^{\alpha} \leq \frac{C_0}{1 - a^{\alpha}},
$$

where $C_0$ is the $\alpha$-Hölder constant of $\log |\tau'|$. The second one can be derived from the first one by the mean value theorem, the last one is a special case thereof with $u = \rho_{|I_1}(v)$. Finally, we can choose $\delta_0 = D^{-2}a$. The assertion is obtained by applying the second estimate twice. □

Lemma 10. The following statements are true.

- There is a constant $\overline{C} > 0$ such that
  $$
  \sup_{v \in \mathcal{J}(x)} |W_\theta(x) - W_\theta(v)| \leq \overline{C} \lambda^n(x)
  $$
  for all $x \in \mathcal{J}$, $n \in \mathbb{N}$ and $\theta \in \mathbb{T}^N$.

- If $W$ is non-degenerate, then there is a constant $c > 0$ such that
  $$
  c \lambda^n(x) \leq \sup_{v \in \mathcal{J} \cap \mathcal{J}(x)} |W(x) - W(v)|
  $$
  for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$.

- If $W_\theta$ is non-degenerate for almost all $\theta$, then there is a measurable function $c : \mathbb{T}^N \to [0, \infty)$ such that
  $$
  c(\sigma^n \theta) \lambda^n(x) \leq \sup_{v \in \mathcal{J} \cap \mathcal{J}(x)} |W_\theta(x) - W_\theta(v)|
  $$
  for all $x \in \mathcal{J}$, $n \in \mathbb{N}$ and $\theta \in \mathbb{T}^N$, where $\sigma : \mathbb{T}^N \to \mathbb{T}^N$ is the left shift operator. In addition, $\lim_{n \to \infty} \frac{\log c(\sigma^n \theta)}{n} = 0$ for almost all $\theta$.

Proof. The proof is provided in section 7. □

As a corollary of these lemmas, the nowhere-differentiability argument follows.

Proof of Lemma 2. Assume that $W$ is not of class $C^{1+\alpha}(\mathbb{T})$. In [Bed89, section 5], it is shown in case $\mathcal{J} = \mathbb{T}$ and $\tau' > 0$ that $W$ is not locally Lipschitz at each fixed point of $\tau$. In our slightly more general setting, the same statement can be similarly verified. In particular, we are in the non-degenerate situation. Let $x \in \mathcal{J}$ be arbitrary. By the second point of lemma 10, for each $n \in \mathbb{N}$ there is a $u_n \in \mathcal{J} \cap I_n(x)$ such that $c \lambda^n(x) \leq W(x) - W(u_n)$. On the other hand, by lemma 9, $|x - u_n| \leq |I_n(x)| \leq D/(\tau^n)'(x)$ for all $n \in \mathbb{N}$. Hence, with $\Delta$ denoting the value on the left hand side of the partial hyperbolicity condition (2), we have $\lim_{n \to \infty} u_n = x$ but

$$
\frac{|W(x) - W(u_n)|}{x - u_n} \geq \frac{c(\tau^n)'(x) \lambda^n(x)}{D} \geq (c/D) \Delta^n \underset{n \to \infty}{\longrightarrow} \infty.
$$

Thus the derivative of $W$ in $x$ does not exist. □
4.1. Proofs of theorems 1 and 2

The proofs consist of the three lemmas 11, 12 and 13. All upper bounds for the dimensions in the both theorems are provided in the first lemma, while the lower bounds are shown in several steps as follows. For the box dimension, the lower estimates follow from the second lemma, whose assumption is satisfied in view of the second and third point of lemma 10 in cases of theorems 1 and 2, respectively. In order to obtain the lower bound for the Hausdorff dimension stated in theorem 2, we need to apply the third lemma to the equilibrium states \( \nu_1 \) and \( \nu_2 \) for the pressures defined in (4), respectively. From that lemma, we can conclude the following for almost all \( \vartheta \): If \( s_1 \leq s_2 \), then the lift of \( \nu_1 \) on the graph of \( W_\vartheta \) has dimension \( s_1 \), whereas if \( s_1 > s_2 \), the lift of \( \nu_2 \) has dimension \( s_2 \). In view of lemma 8, it follows that 

\[
\min\{s_1, s_2\} \leq \dim_H(GW_\vartheta \cap (J \times \mathbb{R})) \text{ for almost all } \vartheta.
\]

**Lemma 11.** For any \( \vartheta \in \mathbb{N}^\mathbb{N} \), we have

\[
\overline{\dim_B}(GW_\vartheta \cap (J \times \mathbb{R})) \leq s_1 \quad \text{and} \quad \dim_H(GW_\vartheta \cap (J \times \mathbb{R})) \leq \min\{s_1, s_2\}.
\]

**Proof.** Let \( \vartheta \) be fixed. We first consider the box dimension. For \( r > 0 \), let \( N_r \) be the least number of closed squares with side length \( r \) that are needed to cover \( GW_\vartheta \cap (J \times \mathbb{R}) \). Recall that \( \overline{\dim_B}(GW_\vartheta \cap (J \times \mathbb{R})) = \limsup_{r \to 0} \frac{\log N_r}{\log r} \) by definition. To find the bound, we use Moran covers of \( J \), so let \( U_r := \{I_n(x) : x \in J\} \), where \( n_r(x) := \min\{n \in \mathbb{N} : |I_n(x)| < r\} \). Observe that \( J \subseteq \bigcup_{I \in \mathcal{I}} J \) and \( \delta_0 \leq |I| < r \) for all \( I \in \mathcal{I} \), where \( \delta_0 > 0 \) is the constant from lemma 9. In view of lemma 10, for each \( I_n(x) \in \mathcal{I} \), the part of the graph \( GW_\vartheta \cap (I_n(x) \times \mathbb{R}) \) can be covered by \( |C|^{\lambda^n(x)}/|I_n(x)| \) closed squares with side length \( r \). In view of the partial hyperbolicity (2) and lemma 9, this number does not exceed \( (\bar{C} + D) \lambda^n(x)/|I_n(x)| \). Now, let \( \nu_1 \in \mathcal{P}_+(J) \) be the equilibrium state for the topological pressure of the definition of \( s_1 \) in (4), which is a Gibbs measure according to lemma 7. Together with lemma 9, there is thus a constant \( C_1 > 0 \) such that

\[
C_1^{-1} \leq \frac{\nu_1(I_n(x))}{|I_n(x)|^{\lambda^n(x)}} \leq C_1
\]

for all \( x \in J \) and \( n \in \mathbb{N} \). Hence we can conclude

\[
N_r \leq \sum_{I_n(x) \in \mathcal{I}} (\bar{C} + D) \nu_1(I_n(x)|x|^{-s_1}) \leq (\bar{C} + D)C_1 \nu_1(I)|x|^{-s_1}
\]

\[
\leq (\bar{C} + D)C_1 \delta_0^{-s_1} \nu_1(I) \leq (\bar{C} + D)C_1 \delta_0^{-s_1} r^{-s_1}
\]

for all \( r > 0 \). This finishes the proof for the box dimension.

We turn to the Hausdorff dimension. As \( \overline{\dim_H}(GW_\vartheta \cap (J \times \mathbb{R})) \leq \dim_H(GW_\vartheta \cap (J \times \mathbb{R})) \leq s_1 \), it remains to show \( \dim_H(GW_\vartheta \cap (J \times \mathbb{R})) \leq s_2 \). To this end, we apply a general formula for local Hölder exponent that is stated as lemma 14 in a later section. Together with lemma 10, we have

\[
\overline{\lim}_{n \to \infty} \inf_{u \in \mathcal{U}(x)} \frac{\log |W_\vartheta(x) - W_\vartheta(u)|}{\log |I_n(x)|} \leq \overline{\lim}_{n \to \infty} \inf_{u \in \mathcal{U}(x)} \frac{\log \lambda^n(x)}{\log |I_n(x)|} =: \overline{h}(x)
\]

(6)
for all $x \in \mathcal{J} \setminus \mathcal{N}$. We consider the sub-level sets $E_\alpha^< := \{ x \in \mathcal{J} \setminus \mathcal{N} : \tilde{h}(x) < \alpha \}$ for $\alpha > 0$. The crucial fact is that
\[
\dim_H E_\alpha^< \leq s_2 \alpha
\] (7)
holds for all $\alpha > 0$. Postponing its proof, we first demonstrate how the rest of the main proof can be derived from this by means of the following very general inequality:
\[
\dim_H \{(x, W_\phi(x)) : x \in E_\alpha^< \text{ and } \text{hol}_{W_\phi}(x) \geq \beta \} \leq \frac{\dim_H E_\alpha^<}{\beta}
\]
for any $\alpha, \beta > 0$, see [Jin11, theorem 1]. Choose a sufficiently large interval, say, $[\alpha_{\min}, \alpha_{\max} + 1]$. Given $N \in \mathbb{N}$, we partition this interval equally into $N$ subintervals, so let $t_i := \alpha_{\min} + \frac{i}{N} (1 + \alpha_{\max} - \alpha_{\min})$ for $i = 0, \ldots, N$. Observe that for each $x \in \mathcal{J} \setminus \mathcal{N}$ there is some $i \in \{0, \ldots, N - 1\}$ such that $t_i \leq \tilde{h}(x) < t_{i+1}$, which together with (6) implies that $\text{hol}_{W_\phi}(x) \geq \tilde{h}(x) \geq t_i$. Thus we have $\mathcal{J} \setminus \mathcal{N} = \bigcup_{i=0}^{N-1} E_\alpha^< \cap \{ \text{hol}_{W_\phi} \geq t_i \}$. As the countable set $\mathcal{N}$ has Hausdorff dimension zero, it follows that
\[
\dim_H \mathcal{G}(W_\phi \cap (\mathcal{J} \times \mathbb{R})) = \max_{i=0, \ldots, N-1} \dim_H \left\{(x, W_\phi(x)) : x \in E_\alpha^< \text{ and } \text{hol}_{W_\phi}(x) \geq t_i \right\}
\leq \max_{i=0, \ldots, N-1} \frac{\dim_H E_\alpha^<}{t_i}
\leq \max_{i=0, \ldots, N-1} \frac{s_2 t_{i+1}}{t_i} \leq s_2 \left( 1 + \frac{1}{\alpha_{\min} N} \right).
\]
Hence, letting $N \to \infty$ yields the claimed upper bound $s_2$ for the Hausdorff dimension.

Finally, it remains to show the inequality (7). Given $r > 0$, we can define
\[
n_r(x) := \min \left\{ n \in \mathbb{N} : |I_n(x)| \leq r \text{ and } |I_n(x)|^\alpha \leq \sup_{u \in I_n(x)} \lambda^\alpha(u) \right\}
\]
for each $x \in E_\alpha^<$. Then, $\mathcal{U}_r := \{I_{n_r}(x) : x \in E_\alpha^<\}$ is a family of disjoint intervals such that $E_\alpha^< \subseteq \bigcup_{I \in \mathcal{U}_r} I$ and $|I| \leq r$ for all $I \in \mathcal{U}_r$. Let $\nu_2 \in \mathcal{P}_r(\mathcal{J})$ be the equilibrium state for the topological pressure of the definition of $s_3$ in (4). By lemma 7, there is a constant $C_2 > 0$ such that
\[
C^{-1}_2 \leq \frac{\nu_2(I_n(x))}{\lambda^\alpha(u)^\alpha} \leq C_2
\]
for all $u \in I_n(x)$, $x \in \mathcal{J}$ and $n \in \mathbb{N}$. In particular, for each $I_n(x) \in \mathcal{U}_r$ we have
\[
|I_n(x)|^{2\alpha} \leq \sup_{u \in I_n(x)} (\lambda^\alpha(u))^{\alpha} \leq C_2 \nu_2(I_n(x)).
\]
Thus, for any $d > s_2\alpha$, the $d$-Hausdorff measure of $E_\alpha^<$ is bounded as
\[
\mathcal{H}^d(E_\alpha^<) \leq \sum_{I \in \mathcal{U}_r} |I|^d = \sum_{I_{n}(x) \in \mathcal{U}_r} |I_n(x)|^{d-2\alpha} |I_n(x)|^{2\alpha}
\leq C_2 r^{d-s_2\alpha} \sum_{I \in \mathcal{U}_r} \nu_2(I) \leq C_2 r^{d-s_2\alpha}.
\]
Letting $r \to 0$, we obtain $\mathcal{H}^d(E_{n}^c) = 0$, so $\dim_{H} E_{n}^c \leq d$. As $d > s_2 \alpha$ was arbitrary, the claim is proved. \hfill \square

**Lemma 12.** Given $\vartheta$, assume that there is a sequence $(c_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $\lim_{n \to \infty} \frac{\log c_n}{n} = 0$ such that

$$c_n \lambda^n(x) \leq \sup_{v \in \mathcal{F} \cap \mathcal{L}(x)} |W_\vartheta(v) - W_\vartheta(x)|$$

for all $x \in \mathcal{F}$ and $n \in \mathbb{N}$. Then we have $\dim_{H}(G_{\vartheta} \cap (\mathcal{F} \times \mathbb{R})) \geq s_1$.

**Proof.** Let $\nu_1 \in \mathcal{P}_{\tau}(\mathcal{F})$ and $C_1 > 0$ be as in proof of lemma 11, so we have

$$C_1^{-1} \leq \frac{\nu_1(I_n(x))}{|I_n(x)|^{n-1} \lambda^n(x)} \leq C_1$$

for all $x \in \mathcal{F}$ and $n \in \mathbb{N}$. In addition, we again consider $N_r$, for $r > 0$, the least number of closed squares with side length $r$ that are needed to cover $\mathcal{H}_{n} \subset \mathbb{R}$. Recall that $\dim_{H}(G_{\vartheta} \cap (\mathcal{F} \times \mathbb{R})) = \lim_{r \to 0} \frac{\log N_r}{-\log r}$ by definition. Now, for an arbitrary $\varepsilon > 0$, let $D_N \subseteq \mathbb{T}$ denote the set of those $x \in \mathcal{F}$ which satisfy

$$\max \left\{ \log \lambda^n(x) \right\} - \int \log \lambda \, dv_1 \leq \frac{1}{n} \left( \log |v_1(I_n(x)) - v_1(v_1)| \right) \leq \frac{1}{n} \left( \log |I_n(x)| - \int \log |\tau'| \, dv_1 \right) < \varepsilon$$

for all $n \geq N$. Observe that $\lim_{N \to \infty} \nu_1(D_N) = 1$ in view of Birkhoff’s ergodic theorem and the Shannon–McMillan–Breiman theorem together with lemma 9. Hence we can choose a number $N_0 \in \mathbb{N}$ so that $\nu_1(D_{N_0}) > 0$. Let $\mathcal{U}_n := \{ I_n(x) : x \in D_{N_0} \}$ for $n \geq N_0$. Clearly, these are coverings of $D_{N_0}$. Since

$$C_1 \geq \frac{\nu_1(I_n(x))}{|I_n(x)|^{n-1} \lambda^n(x)} \geq \nu(I_n(x)) \lambda^n(x) e^{\frac{n(1-s_1)}{n} \int \log |\tau'| \, dv_1 - \log \lambda \, dv_1 - s_1 \varepsilon}$$

for each $I_n(x) \in \mathcal{U}_n$, we have

$$\# \mathcal{U}_n \geq \sum_{I \in \mathcal{U}_n} e^{\frac{n(1-s_1)}{n} \int \log |\tau'| \, dv_1 - \log \lambda \, dv_1 - s_1 \varepsilon} \nu_1(I) \geq \sum_{I \in \mathcal{U}_n} \nu_1(I) e^{\frac{n(1-s_1)}{n} \int \log |\tau'| \, dv_1 - \log \lambda \, dv_1 - s_1 \varepsilon}$$

for all $n \in \mathbb{N}$. On the other hand, in view of the choice of $(c_n)_n$, the height of $G_{\vartheta} \cap (I \times \mathbb{R})$ for $I \in \mathcal{U}_n$ is at least $e^{\frac{\int \log \lambda \, dv_1 - s_1 \varepsilon}{n}}$ so that $|I| \geq r_n$ for all $I \in \mathcal{U}_n$. Since $W_\vartheta$ is continuous, we have

$$N_{r_n} \geq \frac{\# \mathcal{U}_n}{2} e^{\frac{\int \log \lambda \, dv_1 - s_1 \varepsilon}{r_n} + \log c_n} \geq (2C_1)^{-1} \nu_1(D_{N_0}) e^{\frac{n(1-s_1)}{n} \int \log |\tau'| \, dv_1 - s_1 \varepsilon}.$$

Consequently,

$$\lim_{r \to 0} \frac{\log N_{r_n}}{-\log r} = \lim_{n \to \infty} \frac{\log N_{r_n}}{-\log r_n} \geq \frac{s_1 \int \log |\tau'| \, dv_1 - s_1 \varepsilon}{\int \log |\tau'| \, dv_1 + \varepsilon},$$

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where the above equality is due to monotonicity of the sequence \((N_t)_{t>0}\) and the fact \(\lim_{n \to \infty} \frac{\log r_n}{\log r_{n+1}} = 1\). Letting \(\varepsilon \to 0\) finishes the proof. \(\square\)

**Lemma 13.** Suppose that \(\vartheta \in T^{\mathbb{Z}}\) is an independent and identically uniformly distributed random sequence on \(\mathbb{T}\) and that \(T_0 \cap J, T_1 \cap J, \ldots, T_{-1} \cap J\) are pairwise disjoint as subsets of \(\mathbb{T}\). Furthermore, suppose that \(g\) satisfies the critical point hypothesis. Then, for any Gibbs measure \(\nu \in \mathcal{P}_r(J)\), we have

\[
\dim_H \mu_\vartheta = \min \left\{ \dim_H \nu + 1 + \frac{\int \log \lambda \, d\nu}{\int |\tau'| \, d\nu}, \frac{h_r(\nu)}{\int \log |\tau'| \, d\nu} \right\}
\]

for almost all \(\vartheta\).

**Proof.** The proof will be given in section 6. \(\square\)

### 5. Hölder exponent spectra

In this section, we study the spectra of the local Hölder exponent of \(W_\vartheta\), giving proofs of theorems 3 and 4.

Firstly, we introduce two useful formulas for the local Hölder exponent. The definition of the exponent for \(W_\vartheta\) was given in (3), which can be naturally applied for any continuous function \(\phi : \mathbb{T} \to \mathbb{R}\), too. Let \(B_r(x) := \{ u \in \mathbb{T} : d(x, u) \leq r \}\) be closed balls for \(x \in \mathbb{T}\) and \(r > 0\).

**Lemma 14.** For any continuous function \(\phi : \mathbb{T} \to \mathbb{R}\), we have

\[
\text{hol}_\vartheta(x) = \liminf_{r \to \vartheta} \inf_{u \in B_r(x)} \frac{\log |\phi(x) - \phi(u)|}{\log r}
\]

for all \(x \in \mathbb{T}\).

In addition, we have

\[
\text{hol}_\vartheta(x) = \liminf_{n \to \infty} \inf_{u \in B_n(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|}
\]

for all \(x \in J \setminus \mathcal{N}\).

**Proof.** We start with the first claim. Fix \(x \in \mathbb{T}\), and let \(h_x\) denote the value on the right hand side of the equation. Let \(\varepsilon > 0\) be arbitrary. Then we can choose a zero sequence \((r_k) \subset (0, 1)\) so that

\[
\inf_{u \in B_{r_k}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log r_k} < h_x + \varepsilon.
\]

In view of continuity, for each \(k\), there is a \(u_k \in B_{r_k}(x)\) such that \(|\phi(x) - \phi(u_k)| > r_k^{h_x + \varepsilon} \geq d(x, u_k)^{h_x + \varepsilon}\). Therefore,

\[
\sup_{u \in B_{r_k}(x)} \frac{|\phi(x) - \phi(u)|}{d(x, u)^{h_x + 2\varepsilon}} \geq \frac{|\phi(x) - \phi(u_k)|}{d(x, u_k)^{h_x + 2\varepsilon}} \geq d(x, u_k)^{-\varepsilon} \geq r_k^{-\varepsilon}.
\]

Thus, \(\inf_{r > 0} \sup_{u \in B_r(x)} \frac{|\phi(x) - \phi(u)|}{d(x, u)^{h_x + 2\varepsilon}} = \lim_{k \to \infty} \sup_{u \in B_{r_k}(x)} \frac{|\phi(x) - \phi(u)|}{d(x, u)^{h_x + 2\varepsilon}} = \infty\), so \(\text{hol}_\vartheta(x) \leq h_x + 2\varepsilon\). On the other hand, there is some \(r_0 \in (0, 1)\) such that \(\sup_{u \in B_{r_0}(x)} |\phi(x) - \phi(u)| < r_0^{h_x + \varepsilon}\) for all \(r \in (0, r_0)\). Hence we have \(\text{hol}_\vartheta(x) \geq h_x - \varepsilon\), since
\[ \inf_{r > 0} \sup_{u \in B_r(x)} \frac{|\phi(x) - \phi(u)|}{d(x,u)^{\nu_{-\varepsilon}}} \leq \sup_{u \in B_0(x)} \frac{\sup_{v \in B_{d(x,u)}(x)} |\phi(x) - \phi(v)|}{d(x,u)^{\nu_{-\varepsilon}}} \leq 1. \]

Letting \( \varepsilon \to 0 \) finishes the proof of the first claim.

Let \( I_L \) and \( I_R \) be the leftmost and rightmost intervals defined as follows. \( I_L := I_0 \) if \( 0 \in \overline{I}_0 \), and otherwise \( I_L \) is the closed interval from 0 to the left endpoint of \( I_0 \). Similarly, \( I_R := I_{\ell-1} \) if \( 1 \in \overline{I}_{\ell-1} \), and otherwise from the right endpoint of \( I_{\ell-1} \) to 1. Recall that the distance between a set and a point is defined as \( \text{dist}(A,x) := \inf \{d(x,u) : u \in A\} \) for \( A \subseteq \mathbb{T} \) and \( x \in \mathbb{T} \). With the constant \( D > 0 \) from lemma 9, let \( \delta := D^{-1} \min\{|I_L|,|I_R|\} \), so that holds the implication:

\[ \text{dist}(\partial(L_0(x)), \{x\}) < \delta|I_0(x)| \implies \tau^n x \in I_L \cup I_R. \]

To see this, let \( u \in \partial L_0(x) \) satisfy \( d(u,x) = \delta|I_0(x)| \). If \( x \in \{0,1\} \), the claim is trivial. Otherwise, by the mean value theorem,

\[ d(\tau^n x, \tau^n u) \leq \sup_{w \in L_0(x)} |(\tau^n)'(w)| \cdot d(x,u) < \frac{D}{|I_0(x)|} \cdot \delta|I_0(x)| \leq \min\{|I_L|,|I_R|\}. \]

As \( \tau^n u \in \{0,1\} \), this means \( \tau^n x \in I_L \cup I_R \), so the claimed implication is shown.

Using the above constant \( \delta \), let

\[ J_0 := \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{ x \in J : \text{dist}(\partial(L_n(x)), \{x\}) \geq \delta|I_n(x)| \}. \]

We claim \( J \setminus J_0 \subseteq J_0 \). Observe that it is equivalent to show \( J \setminus J_0 \subseteq N \), so let \( x \in J \setminus J_0 \).

Then, there is some \( N \in \mathbb{N} \) such that

\[ \text{dist}(\partial(L_n(x)), \{x\}) < \delta|I_n(x)| \]

for all \( n \geq N \). As shown above, this means \( \tau^n x \in I_L \cup I_R \) for all \( n \geq N \), which is only possible when \( \tau^{N+1} x \in \{0,1\} \). Thus, \( x \in \mathbb{N} \).

We proceed the proof of the lemma. Let \( x \in J \setminus J_0 \). Since \( x \in J_0 \), the inequalities

\[ \inf_{u \in B_{d(x,u)}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} \leq \inf_{u \in L_n(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} \leq \inf_{u \in B_{d(x,u)}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} \]

hold for infinitely many \( n \in \mathbb{N} \). In addition, observe that the first and thirds expressions have the same inferior limits as \( n \to \infty \), so the value must coincide with the inferior limit of the middle one. In view of \( |L_n(x)| \searrow 0 \) and \( \lim_{n \to \infty} \frac{\log |I_n(x)|}{\log |I_n(x)|} = 1 \), the value can be calculated as

\[ \lim_{n \to \infty} \inf_{u \in B_{d(x,u)}(x)} \frac{\log |\phi(x) - \phi(u)|}{\log |I_n(x)|} = \lim_{r \to 0} \inf_{u \in B_0(x)} \frac{\log |\phi(x) - \phi(u)|}{\log r} = \text{hol}_\phi(x). \]

The following is an immediate corollary of the preceding lemma with the specific choice \( \phi = W_\phi \), in view of lemmas 9 and 10.
Lemma 15. If $W$ is non-degenerate, we have

$$\text{hol}_W(x) = \liminf_{n \to \infty} \frac{-\log \lambda^n(x)}{\log |\tau^n(x)|}$$

for all $x \in \mathcal{J} \setminus \mathcal{N}$.

In addition, if $W_\varnothing$ is non-degenerate for almost all $\varnothing$, then the same formula for $W_\varnothing$ is true for almost all $\varnothing$.

The next lemma is a collection of several fundamental facts from thermodynamic formalism. Let us consider the Hausdorff spectrum of the following (sub-, sup-) level sets:

$$S_\alpha := \left\{ x \in \mathcal{J} : \liminf_{n \to \infty} \frac{-\log \lambda^n(x)}{\log |\tau^n(x)|} = \alpha \right\},$$

$$S_\alpha^\leq := \left\{ x \in \mathcal{J} : \liminf_{n \to \infty} \frac{-\log \lambda^n(x)}{\log |\tau^n(x)|} \leq \alpha \right\} \text{ and } S_\alpha^\geq := \left\{ x \in \mathcal{J} : \liminf_{n \to \infty} \frac{-\log \lambda^n(x)}{\log |\tau^n(x)|} \geq \alpha \right\}.$$

For the details of the following lemma, consult [Pes97] or [Bar08]. Note that the function $D : A \to [0, 1]$ as well as the related constants were introduced immediately before theorem 3.

Lemma 16. If $A \neq 0$, we have

$$D(\alpha) = \dim_H S_\alpha = \begin{cases} \dim_H S_\alpha^\leq & \text{for } \alpha \in (\alpha_{\min}, \alpha_c) \\ \dim_H S_\alpha^\geq & \text{for } \alpha \in [\alpha_c, \alpha_{\max}] \end{cases}.$$

Moreover, there is a Gibbs measure $\nu_\alpha \in \mathcal{P}_\tau(\mathcal{J})$ such that $\nu_\alpha(S_\alpha) = 1$ and $\dim_H \nu_\alpha = \dim_H S_\alpha$.

It is now immediate to determine the Hausdorff spectrum $\alpha \mapsto \dim_H E_\alpha$.

Proof of theorem 3. By lemma 15 we have $E_\alpha \Delta S_\alpha \subseteq \mathcal{N}$, and thus $\dim_H E_\alpha = \dim_H S_\alpha$ for all $\alpha \in \mathbb{R}$. Therefore, in case $A \neq 0$, the assertion of the theorem follows from lemma 16. Moreover, in case $A = 0$, we obtain $\text{hol}_W(x) = \alpha_c$ for all $x \in \mathcal{J} \setminus \mathcal{N}$ from that lemma. Assuming $A = 0$, it remains to show that $\text{hol}_W(x) = \alpha_c$ for all $x \in \mathcal{J} \cap \mathcal{N}$. Observe that the assumption $A = 0$ is equivalent to that $(\alpha, \log |\tau^n + \log \lambda|)_{\mathcal{J}}$ is cohomologous to $0^1$, i.e. it is equal to $\phi \circ \tau - \phi$ for some bounded function $\phi : \mathcal{J} \to \mathbb{R}$.

In particular, in view of lemma 9, there is a constant $C_1 > 0$ such that

$$C_1^{-1} \leq |I_n(u)|^{-\alpha_c} \lambda^n(u) \leq C_1$$

for all $u \in \mathcal{J}$ and $n \in \mathbb{N}$. Hence, by lemma 10,

$$\sup_{u \in I_n(x)} |W(u) - W(x)| \geq c \lambda^n(x) \geq c C_1^{-1} |I_n(x)|^{\alpha_c}.$$

for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. On the other hand, by the first formula of lemma 14 together with the fact $\lim_{n \to \infty} \frac{\log |I_n(x)|}{\log |I_{n+1}(x)|} = 1$, we have

\[\text{Proof: Assume that } (\alpha, \log |\tau^n + \log \lambda|)_{\mathcal{J}} \text{ is cohomologous to } 0. \text{ Then, from (5) follows } P_\tau(-A_q + \alpha_q) \log |\tau^n| = 0 \text{ for all } q \in \mathbb{R}. \text{ Then, by Rohlin's formula, } -A_q = \alpha_q - 1 \text{ for all } q \in \mathbb{R}. \text{ Thus } -A_q \text{ is constant, so } A = 0. \text{ Conversely, assume } A = 0, \text{ which means that } -A_q \text{ is a constant function of } q, \text{ whose value is called } \alpha_q, \text{ by definition. As } A_0 = 1 \text{ by Rohlin's formula, we have } -A_0 = \alpha_0 - 1. \text{ Inserting it into (5) yields } P_\tau(q \alpha_q \log |\tau^n + \log \lambda| + \log |\tau^n| = 0 \text{ for all } q \in \mathbb{R}. \text{ This is only possible when } (\alpha, \log |\tau^n + \log \lambda|)_{\mathcal{J}} \text{ is cohomologous to } 0.\]
for all $x \in \mathbb{T}$, where $B_r(x) := \{u \in \mathbb{T} : |x - u| \leq r\}$ for $x \in \mathbb{T}$ and $r > 0$. Since $I_n(x) \subseteq B_{r}(x)$, it follows immediately that $\text{hol}_W(x) \leq \alpha_c$ for all $x \in \mathcal{F}$, and especially for all $x \in \mathcal{F} \cap \mathcal{N}$. To show the inverse inequality, we distinguish three cases.

Firstly, if $\{0, 1\} \cap \mathcal{F} = \emptyset$, then $\mathcal{F} \cap \mathcal{N} = \emptyset$, so there is nothing to prove.

Secondly, assume $\{0, 1\} \subseteq \mathcal{F}$. Given $x \in \mathcal{F} \cap \mathcal{N}$ and $n \in \mathbb{N}$, there are $m_n \in \mathbb{N}$ and $x_n \in \mathcal{F}$ so that

$$\overline{I_n(x)} \cap \overline{I_{m_n}(x_n)} = \{x\} \quad \text{and} \quad \delta_0 |I_n(x)| \leq |I_{m_n}(x_n)| \leq |I_n(x)|,$$

where $\delta_0 > 0$ is the constant from lemma 9. Thus we have

$$B_{|I_n(x)|}(x) \subseteq \overline{I_n(x)} \cup I_{m_n}(x_n).$$

In addition, by lemma 10,

$$\sup_{u \in I_n(x) \cup I_{m_n}(x_n)} |W(u) - W(x)| \leq \mathcal{C}(\lambda^n(x) + \lambda^{m_n}(x_n)) \leq \mathcal{C} C_1 (1 + \delta_0^{-\alpha_c}) |I_n(x)|^{\alpha_c}.$$ 

Hence we can choose a constant $C_2 > 0$ so that

$$\sup_{u \in B_{|I_n(x)|}(x)} |W(u) - W(x)| \leq C_2 |I_n(x)|^{\alpha_c}$$

for all $x \in \mathcal{F} \cap \mathcal{N}$ and $n \in \mathbb{N}$, which in terms means that $\text{hol}_W(x) \geq \alpha_c$ for all $x \in \mathcal{F} \cap \mathcal{N}$.

Finally, assume that only one of 0 and 1 belongs to $\mathcal{F}$. We define the right ball $B^+_r(x) \subseteq \mathbb{T} = \mathbb{R} / \mathbb{Z}$ as the canonically projected image of $[x, x + r] \subseteq \mathbb{R}$, and similarly, the left ball $B^-_r(x)$. Let $x \in \mathcal{F} \cap \mathcal{N}$. For sufficiently large $n$, by the assumption here, the interval $\overline{I_n(x)}$ must correspond with $B^-_{I_n(x)}(x)$ or $B^+_{I_n(x)}(x)$, say with $B^-_{I_n(x)}(x)$. Let $r^+ := \sup\{r > 0 : B^+_{I_n(x)}(x) \cap \mathcal{F} = \{x\}\}$ where $\sup \theta := 0$. If $r^+ = 0$, then there exist $(m_n)_n$ and $(x_n)_n$ as in the second case and the same arguments apply. Thus it remains the case $r^+ > 0$. As $W$ is $C^1$ on $(B^+_{r_+}(x))^\theta$ by lemma 1, there is a number $C(x) \geq C_2$ in dependence of $x$ such that

$$\sup_{u \in B_{r_+}^+(x)} |W(u) - W(x)| \leq C(x) r$$

for all $r \in (0, 1)$. From $B_{I_n(x)}(x) = \overline{I_n(x)} \cup B^+_{I_n(x)}(x)$ follows

$$\sup_{u \in B_{I_n(x)}(x)} |W(u) - W(x)| = \sup_{u \in B_{I_n(x)} \cup B^+_{I_n(x)}(x)} |W(u) - W(x)| \leq (\overline{C} + C(x)) |I_n(x)|^{\alpha_c}$$

for all $n \in \mathbb{N}$, which implies $\text{hol}_W(x) \geq \alpha_c$. \qed

We turn to the lifted spectrum $\alpha \mapsto \dim_{tH}(\mathcal{G}W_\theta \cap (E_{\theta, \alpha} \times \mathbb{R}))$ for almost all $\theta$.

**Proof of theorem 4.** As its upper bound is already provided in lemma 5, we only discuss the lower estimate. uniformly distributed random sequence on $\mathbb{T}$. Assume that $\theta \in \mathbb{T}_N$ is an independent and identically uniformly distributed random sequence on $\mathbb{T}$ and that
\( \mathcal{T}_0 \cap \mathcal{J}, \mathcal{T}_1 \cap \mathcal{J}, \ldots, \mathcal{T}_{-1} \cap \mathcal{J} \) are pairwise disjoint as subsets of \( \mathcal{T} \). Let \( \nu_\alpha \) be the Gibbs measure of lemma 16, and let \( \mu_{\theta, \alpha} \) be its lift on the graph of \( W_\theta \), i.e.

\[
\mu_{\theta, \alpha}(A) := \nu_\alpha \{ x \in \mathcal{J} : (x, W_\theta(x)) \in A \}
\]

for measurable set \( A \subseteq T \times \mathbb{R} \). Observe that \( \mu_{\theta, \alpha}(GW_\theta \cap (S_\alpha \times \mathbb{R})) = 1 \). Furthermore, by lemma 13 and theorem 3, for each \( \alpha \in A \), we have almost surely

\[
\dim_H(GW_\theta \cap (S_\alpha \times \mathbb{R})) \geq \dim_H(\mu_{\theta, \alpha}) = \min \left\{ \dim_H \nu_\alpha + 1 + \frac{\int \log \lambda \, d\nu_\alpha}{\int \log |\tau'| \, d\nu_\alpha}, \frac{h_\tau(\nu_\alpha)}{\alpha} \right\}
\]

Thus there is a set \( Z \subseteq \mathbb{T}^{\mathbb{N}_0} \) of full probability such that

\[
\dim_H(GW_\theta \cap (S_\alpha \times \mathbb{R})) \geq \min \left\{ D(\alpha) + 1 - \frac{\alpha}{\alpha}, \frac{D(\alpha)}{\alpha} \right\} =: \tilde{D}(\alpha)
\]

for all \( \alpha \in A \cap \mathbb{Q} \) and \( \theta \in Z \). In view of lemma 15, by shrinking \( Z \) if needed, we may also assume that \( S_\alpha \Delta E_{\theta, \alpha} \subseteq N \) for all \( \alpha \in A \cap \mathbb{Q} \) and \( \theta \in Z \). Then, by lemma 16, we obtain

\[
\dim_H(GW_\theta \cap (E_{\theta, \alpha} \times \mathbb{R})) = \dim_H(GW_\theta \cap (S_\alpha \times \mathbb{R}))
\]

for all \( \alpha' \in (\alpha_{\min}, \alpha] \cap \mathbb{Q}, \alpha \in (\alpha_{\min}, \alpha_c] \) and \( \theta \in Z \). As \( \tilde{D} \) is continuous, it follows that

\[
\dim_H(GW_\theta \cap (E_{\theta, \alpha} \times \mathbb{R})) \geq \tilde{D}(\alpha)
\]

for all \( \alpha \in (\alpha_{\min}, \alpha_c] \) and \( \theta \in Z \). A similar argument works for \( \alpha \in [\alpha_c, \alpha_{\max}] \).

6. Dimension of the randomised lifted measure

This section is dedicated to the proof of lemma 13, which gives the value of Hausdorff dimension of the lifts of any Gibbs measures on the graph of the randomised function \( W_\theta \). The actual sharp upper bound will be straightforwardly determined in lemma 18, however, to verify the equality is, in spite of the randomisation, harder and of a major interest here. The latter will be completed in lemmas 23 and 24, as concluded at the end of this section.

We start with a fundamental observation of Gibbs measures. Recall that the distance between a set and a point is defined as \( \text{dist}(A, x) := \inf \{ d(x, v) : v \in A \} \) for \( A \subseteq \mathbb{T} \) and \( x \in \mathbb{T} \).

**Lemma 17.** Let \( \nu \in \mathcal{P}_r(\mathcal{J}) \) be a Gibbs measure. Then,

\[
\lim_{n \to \infty} \frac{\log \text{dist}(\partial I_n(x), x)}{\log |I_n(x)|} = 1
\]

for \( \nu\text{-a.a. } x \in \mathcal{J} \). In addition, \( \dim_H \nu = \frac{h_\tau(\nu)}{\log |\tau'| \, d\nu} \).
Proof. First, we show that \( \lim_{n \to \infty} \nu(\rho_n^i(T)) = 0 \) converges exponentially fast for each \( i \in \Sigma_t \). Since \( x_i := \lim_{n \to \infty} \rho_n^i(0) \) is a fixed point of \( \tau \), we have
\[
C_\phi^{-1} e^{\phi(x) + P_\phi} \leq \nu(\rho_n^i(T)) \leq C_\phi e^{\phi(x) + P_\phi}
\]
for all \( n \in \mathbb{N} \), where \( C_\phi, P_\phi \) and \( \phi \) are as in the definition of Gibbs measure. As \( \phi(x_i) + P_\phi < 0 \) follows from the left inequality, the right one implies the exponential decay.

Now, we turn to the first claim of the lemma. Given an arbitrary \( \alpha \in (0, 1) \), let \( M_\alpha := \lfloor \alpha n \rfloor \) and
\[
E_n := \tau^{-n} \left( \rho^M_0(T) \cup \rho^{M_n}_{-1}(T) \right)
\]
for \( n \in \mathbb{N} \). Since \( \tau^n x \in \mathcal{T} \cap \left( \rho^M_0(T) \cup \rho^{M_n}_{-1}(T) \right) \), we have
\[
d(0, \tau^n x) \geq \min \{|\rho^M_0(T)|, |\rho^{M_n}_{-1}(T)|\} \geq (\inf |1/\tau'|)^{M_n}.
\]
Furthermore, by the mean value theorem and lemma 9,
\[
dist(\partial I_n(x), x) \geq D^{-1} d(0, \tau^n x) \inf_{\omega \in \mathbb{T}} |\rho'_\alpha(\omega)| \geq D^{-2} \text{dist}(\mathcal{T}, 0)|I_n(x)| \geq D^{-2}(\inf |1/\tau'|)^{M_n}|I_n(x)|
\]
for all \( n \in \mathbb{N} \) and \( x \in \mathcal{T} \). Thus, by lemma 9, we obtain the bound
\[
\limsup_{n \to \infty} \frac{\log \text{dist}(\partial I_n(x), x)}{\log |I_n(x)|} \leq 1 + \lim_{n \to \infty} \frac{M_n}{\log |I_n(x)|} \log(\inf |1/\tau'|) = 1 + \frac{\alpha \log(\inf |1/\tau'|)}{\log(\sup_{\mathcal{T}} |1/\tau'|)}
\]
for all \( x \in \mathcal{T} \setminus \bigcup_{n \in \mathbb{N}} E_n \). On the other hand, since
\[
\nu(E_n) = (\rho^M_0(T) \cup \rho^{M_n}_{-1}(T)) \leq (\rho^M_0(T)) + \nu(\rho^{M_n}_{-1}(T))
\]
decays exponentially as \( n \to \infty \), \( \nu \left( \bigcup_{n \in \mathbb{N}} E_n \right) = 0 \) follows from Borel–Cantelli lemma. Thus the bound holds for \( \nu \)-a.a. \( x \in \mathcal{T} \), independently of the choice of \( \alpha \in (0, 1) \). Letting \( \alpha \downarrow 0 \) finishes the proof of this part.

Now, we turn to the second claim on \( \dim_H \nu \). Since \( \lim_{n \to \infty} \frac{\log |I_n(x)|}{\log |\tau'|} = 1 \) by lemma 9, and since \( B_{\text{dist}(\partial I_n(x), x)}(x) \subseteq I_n(x) \subseteq B_{\text{dist}(\partial I_n(x), x)}(x) \), it follows from the first claim that
\[
d_{\nu}(x) := \liminf_{r \to 0} \frac{\log \nu(B_r(x))}{\log r} = \liminf_{n \to \infty} \frac{\log \nu(B_{\text{dist}(\partial I_n(x), x)}(x))}{\log |I_n(x)|} = \liminf_{n \to \infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} = \frac{h_{\nu}(x)}{\log |\tau'|} \delta_{\nu}
\]
for \( \nu \)-a.a. \( x \in \mathcal{T} \), where the last equality is due to the Shannon–McMillan–Breiman theorem and Birkhoff’s ergodic theorem together with lemma 9.

Lemma 18 (Upper bound for \( \dim_H \mu^0 \)). Let \( \nu \in \mathcal{P}_t(\mathcal{T}) \) be a Gibbs measure. Then we have
\[
\dim_H \mu^0 \leq \min \left\{ \dim_H \nu + 1 + \frac{\int \log \lambda \, d\nu}{\int \log |\tau'| \, d\nu} \frac{h_{\nu}(x)}{\log |\tau'| \, d\nu} \right\}
\]
for all \( \theta \in T^{N_t} \).
Proof. Let $\vartheta \in \mathbb{T}^{\mathbb{N}_0}$ be fixed. By lemma 10, there is a constant $\bar{C} > 0$ so that
\[
\sup_{\nu \in \mathcal{J} \cap I_n(x)} |W_{\vartheta}(x) - W_{\vartheta}(\nu)| \leq \bar{C} \lambda^{n}(x)
\]
for all $x \in \mathcal{J}$ and $n \in \mathbb{N}$. Given an arbitrary $\varepsilon \in (0, e^{- \int \log |r^\prime| d\nu})$, let $D_N \subseteq \mathcal{J}$ denote the set of those $x \in \mathcal{J}$ which satisfies
\[
\max \left\{ \left| \frac{\log \lambda^n(x)}{n} - \int \log \lambda d\nu \right|, \left| \frac{- \log \nu(I_n(x))}{n} - h_{\vartheta}(\nu) \right|, \left| \frac{- \log |I_n(x)|}{n} - \int \log |r^\prime| d\nu \right| \right\} < \varepsilon
\]
for all $n \geq N$. Observe that $\nu \left( \bigcup_{n \geq N} D_N \right) = 1$ by Birkhoff’s ergodic theorem, Shannon–McMillan–Breiman theorem and lemma 9. Furthermore, let $G_N := \{(x, W_{\vartheta}(x)) : x \in D_N\}$ and $U_n^N := \{I_n(x) : x \in D_N\}$ for $n \geq N$. Since $e^{\rho(h_{\vartheta}(\nu)+\varepsilon)} \leq \nu(I)$ for each $I \in U_n^N$, $U_n^N$ contains at most $e^{\rho(h_{\vartheta}(\nu)+\varepsilon)}$ intervals. On the other hand, in view of (8), for each $I_n(x) \in U_n^N$, we can choose a covering of $G_N \cap (I_n(x) \times \mathbb{R})$ by $\left[ \tilde{C} \frac{\lambda^n(x)}{|I_n(x)|} \right]$ cubes with edge length $|I_n(x)|$. In view of the definition of $D_N$, this number is bounded as
\[
\left[ \tilde{C} \frac{\lambda^n(x)}{|I_n(x)|} \right] \leq \tilde{C} e^{\int \log |r^\prime| d\nu + \int \log \lambda d\nu + 2\varepsilon} + 1.
\]
Let $\tilde{U}_n^N$ be the set of all those cubes for all $I_n(x) \in U_n^N$, which then forms a covering of $G_N$. In addition, using the definition of $D_N$, we can see
\[
\sum_{Q \in \tilde{U}_n^N} \text{edge} - \text{length}(Q)^s
\]
\[
\leq \sum_{I \in U_n^N} \left( C e^{\int \log |r^\prime| d\nu + \int \log \lambda d\nu + 2\varepsilon} + 1 \right) |I|^s
\]
\[
= \sum_{I \in U_n^N} \left( C e^{\int \log |r^\prime| d\nu + \int \log \lambda d\nu + \frac{\int \log |r^\prime| d\nu + 2\varepsilon}{|I|^s}} + |I|^s \right)
\]
\[
\leq \sum_{I \in U_n^N} \left( C e^{\int \log |r^\prime| d\nu + \int \log \lambda d\nu + \rho(- \int \log |r^\prime| d\nu + 2\varepsilon)} + e^{\rho(- \int \log |r^\prime| d\nu + \varepsilon)} \right)
\]
\[
eq e^{\rho(h_{\vartheta}(\nu)+\varepsilon)} \cdot \tilde{C} e^{\int \log |r^\prime| d\nu + \int \log \lambda d\nu + \rho(- \int \log |r^\prime| d\nu + 2\varepsilon)} + e^{\rho(- \int \log |r^\prime| d\nu + \varepsilon)}
\]
\[
= e^{\rho(h_{\vartheta}(\nu)+1-s)} \int \log |r^\prime| d\nu + \int \log \lambda d\nu + \rho(- \int \log |r^\prime| d\nu + 2\varepsilon) + e^{\rho(h_{\vartheta}(\nu)-s \int \log |r^\prime| d\nu + (1+s)\varepsilon)} - \lim_{n \to \infty} 0
\]
for all $s > f(\varepsilon) := \max \left\{ \frac{h_{\vartheta}(\nu)+\int \log \lambda d\nu + \int \log |r^\prime| d\nu + 2\varepsilon}{\rho(- \int \log |r^\prime| d\nu - \varepsilon)}, \frac{h_{\vartheta}(\nu)+\varepsilon}{\rho(- \int \log |r^\prime| d\nu - \varepsilon)} \right\}$. Thus, $\dim_{H}(\mathcal{G}_N) \leq f(\varepsilon)$. Furthermore, as the Hausdorff dimension is $\sigma$-stable, we have
\[
\dim_{H} \left( \bigcup_{N} G_N \right) = \sup_{N} \dim_{H}(\mathcal{G}_N) \leq f(\varepsilon).
\]
On the other hand, $\mu_{\vartheta} \left( \bigcup_{N} G_N \right) = \nu \left( \bigcup_{N} D_N \right) = 1$, so we have $\dim_{H}(\mu_{\vartheta}) \leq \dim_{H} \left( \bigcup_{N} G_N \right) \leq f(\varepsilon)$. Letting $\varepsilon \to 0$, we obtain
\[ \dim_H(\mu_\theta) \leq \lim_{\varepsilon \to 0} f(\varepsilon) = \max \left\{ 1 + \frac{h_\tau(\nu) + \int \log \lambda \, d\nu}{\int \log |\tau'| \, d\nu}, \frac{h_\tau(\nu)}{\int \log |\tau'| \, d\nu} \right\} = 1 + \frac{h_\tau(\nu) + \int \log \lambda \, d\nu}{\int \log |\tau'| \, d\nu}, \]

where the last equality is due to (2). In view of the formula for \( \dim_H \nu \) provided in lemma 17, it is shown that that the left value of the claimed maximum is a correct upper bound.

In order to derive the other bound, we consider the following rough estimate. Observe that for any \( \beta \in (0, 1) \) and \( C > 0 \) we have
\[
d_{\mu_\theta}(x, W_\theta(x)) = \liminf_{n \to \infty} \frac{\log \nu \left\{ v \in \mathbb{T} : |x - v| \leq \beta^n, |W_\theta(x) - W_\theta(v)| \leq C \beta^n \right\}}{\log \beta^n}
\]
for \( \nu \)-a.a. \( x \), which is a slight modification of [Bar08, proposition 2.1.4]. Given an arbitrary \( \varepsilon \in (0, e^{\int \log \lambda \, d\nu}) \), we consider the same set \( D_N \subseteq \mathcal{J} \), \( N \in \mathbb{N} \) as above. Observe that
\[
I_n(x) \subseteq \left\{ v \in [0, 1] : |v - x| \leq (e^{-\lambda})^n, |W_\theta(v) - W_\theta(x)| \leq C (e^{-\lambda})^n \right\}
\]
for all \( x \in D_N \) and \( n \geq N \), where \( \lambda := e^{\int \log \lambda \, d\nu} \). By choosing \( \beta = e^{-\lambda} \) and \( C = C \) in the above formula for \( d_{\mu_\theta} \), we have
\[
d_{\mu_\theta}(x, W_\theta(x)) \geq \liminf_{n \to \infty} \frac{\log \nu \left\{ v \in [0, 1] : |v - x| \leq (e^{-\lambda})^n, |W_\theta(v) - W_\theta(x)| \leq C (e^{-\lambda})^n \right\}}{\log(e^{-\lambda})^n}
\]
for all \( x \in D_N \) and \( N \in \mathbb{N} \). In view of lemma 8, letting \( \varepsilon \to 0 \) finishes the proof. \( \square \)

We turn to the lower bound for \( \dim_H \mu_\theta \). From now on, we always assume that \( \nu \in \mathcal{P}_r(\mathcal{J}) \) is a Gibbs measure and that \( g \) satisfies the strong critical point hypothesis. Furthermore, suppose that \( I_0 \cap \mathcal{J}, I_1 \cap \mathcal{J}, \ldots, I_{n-1} \cap \mathcal{J} \) are pairwise disjoint as subsets of \( \mathbb{T} \). We can choose the expansivity constant \( \delta \in \left( 0, \frac{1}{\|\mathcal{J} \cap \mathcal{J} \|_{\infty}} \right) \) so small that for any \( x, v \in \mathcal{J} \)
\[
d(x, v) \leq \delta \| (\mathcal{J} \cap \mathcal{J} ) \|_{\infty} \quad \text{implies} \quad v \in I_1(x).
\]

Let \( d_n(x, v) := \max \{ d(\tau x, \tau v) : j = 0, \ldots, n - 1 \} \) denote the \( n \)-the Bowen metric for \( n \in \mathbb{N} \). In addition, let \( B_{\nu^r}(x) := \{ u \in \mathcal{J} : d_n(x, u) \leq r \} \) denote the ball with respect to \( d_n \) with centre \( x \in \mathcal{J} \) and radius \( r > 0 \). \( B_{\nu}(x) := B_{\nu^1}(x) \).

For \( \varepsilon \in (0, 1) \) and \( N \in \mathbb{N} \), let \( C_{N, \varepsilon} \subseteq \mathcal{J} \) be the set of those \( x \) for which hold
\[
\begin{align*}
&\bullet \ B_{\varepsilon^{n-1}(\log \lambda d\nu - \varepsilon)}(x) \subseteq B_{\varepsilon^n}(x) \subseteq B_{\varepsilon^{n-1}(\log \lambda d\nu + \varepsilon)}(x), \\
&\bullet \ \varepsilon(\int \log \lambda \, d\nu - \varepsilon) \leq \lambda^n(x) \leq \varepsilon(\int \log \lambda \, d\nu + \varepsilon), \\
&\bullet \ \varepsilon^{\dim_H(\nu) + \varepsilon} \leq \nu(B_{\nu}(x)) \leq \varepsilon^{\dim_H(\nu) - \varepsilon}
\end{align*}
\]
for all \( n \geq N \) and \( r > 0 \).

**Lemma 19.** \( \lim_{N \to \infty} \nu(C_{N, \varepsilon}) = 1 \) for all \( \varepsilon \in (0, 1) \).

**Proof.** It suffices to show that three conditions are satisfied for all \( n \geq N \) for \( \nu \)-a.a. \( x \in \mathcal{J} \), where \( x \mapsto N \) is some function. Moreover, the three conditions can be separately considered, that is, if we know such a \( N \) for each condition, then their maximum is the one we need. The
existence of such $N_e$ for the second and third conditions is immediate since, by Birkhoff’s ergodic theorem and lemma 17,
\[
\lim_{n \to \infty} \frac{\log \lambda_n(x)}{n} = \int \log \lambda \, d\nu \text{ and } \lim_{r \to 0} \frac{\log \nu(B_r(x))}{\log r} = \dim_H \nu
\]
for $\nu$-a.a. $x$. Thus we focus on the first condition. W.l.o.g. assume $N \geq 2$ so large that $d(v, w) = |v - w|$ for all $w \in I_{\nu}(v)$ and $v \in \mathcal{J}$. Let $x \in \mathcal{J}$ and $n \geq N$.

For $v \in B_{\delta_n}(x)$, we have $v \in I_n(x)$, since $\tau_j v \in I_j(\tau x)$ for $j = 0, \ldots, n - 1$ due to the choice of $\delta_n$. In particular, $d(x, v) \leq |I_{n-1}(x)| \leq |I_n(x)|$. Thus we have $B_{\delta_n}(x) \subseteq B_{|I_n(x)|}(x)$ for all $x \in \mathcal{J}$ and $n \geq N$. Next, let $v \in \mathcal{J} \setminus B_{\delta_n}(x)$, so there is some $0 \leq j \leq n - 1$ such that $d(\tau_j x, \tau_j v) > \delta_n$. We consider the following two cases, separately. If $v \not\in I_n(x)$, then we have $d(x, v) \geq \text{dist}(\partial I_n(x), x)$. If $v \in I_n(x)$, by the mean value theorem and lemma 9,
\[
D[I_n(x)]^{-1} d(x, v) \geq D[I_n(x)]^{-1} d(x, v) \geq |(\tau^{-1})'(\tilde{v})| d(x, v) = d(\tau^{-1} x, \tau^{-1} v) > \delta_n
\]
for some $\tilde{v} \in I_n(x)$. Hence, together with the above observation, we have
\[
B_{r_j}(x) \subseteq B_{\delta_n}(x) \subseteq B_{|I_n(x)|}(x)
\]
for all $x \in \mathcal{J}$ and $n \geq N$, where $r_n(x) := \min \{ \text{dist}(\partial I_n(x), x), D^{-1} \delta_n |I_n(x)| \}$. This finishes the proof together with the fact that
\[
\lim_{n \to \infty} \frac{\log r_n(x)}{n} = \lim_{n \to \infty} \frac{\log |I_n(x)|}{n} = - \int \log |(\tau^{-1})'| d\nu
\]
for $\nu$-a.a. $x$, which follows from Birkhoff’s ergodic theorem and lemmas 9, 17.

Let $C_{c_e} := C_{N_c}$, for some fixed $N_c$ so that it has a positive $\nu$ measure. We consider then the restricted measure $\nu_c := \nu(C_{c_e} \cap \cdot)$ and its lift
\[
\mu_{\phi_c} := \nu \{ x \in C_{c_e} : (x, W_{\phi_c}(x)) \in \cdot \}.
\]
Recall that $s$-energy of a Borel measure $\mu$ on a metric space $(\mathcal{E}, d_{\mathcal{E}})$ is defined by
\[
I_s(\mu) = \int \int \frac{d\mu(x) d\mu(v)}{d_{\mathcal{E}}(x, v)^s}.
\]
Furthermore, $I_s(\mu) < \infty$ implies that $d_{\mu} \geq s$ holds $\mu$-almost surely (see e.g. [Fal05, section 4.3]).

Observe that
\[
\int \int I_s(\mu_{\phi_c}) d\phi = \int \int \int \frac{d\nu_{\phi_c}(x) d\nu_{\phi_c}(v) d\phi}{(d(x, v)^2 + (W_{\phi}(x) - W_{\phi}(v))^2)^{s/2}} \leq \frac{1}{\delta_{c_e}} + \int \int \int \{ (x, v) : d(x, v) \leq \delta_{c_e} \} \frac{d\nu_{\phi_c}(x) d\nu_{\phi_c}(v) d\phi}{(d(x, v)^2 + (W_{\phi}(x) - W_{\phi}(v))^2)^{s/2}} =: \frac{1}{\delta_{c_e}} + E_{s, c_{\phi}}(\phi).
\]
where $\Delta_{c_e} := \{ (x, v) \in \mathcal{J}^2 : d(x, v) \leq \delta_{c_e} \}$ is the $\delta_{c_e}$-diagonal set.

**Lemma 20.** If $E_{s, c_{\phi}}(\phi) < \infty$ for some $s > 0$ and $\phi \in (0, 1)$, then $\dim_H \mu_{\phi} \geq s$ for a.a. $\phi$. 


Proof. In view of lemma 8, it suffices to show for a.a. \( \vartheta \) that
\[
\nu \{ x \in C_c : d_{\vartheta,x} (x, W_\vartheta (x)) < s \} = 0.
\]

As \( E_{c,x} < \infty \), we have \( I_r (\mu_{\vartheta,x}) < \infty \) by (9) for a.a. \( \vartheta \). In the following, let \( \vartheta \) be such a parameter. Observe that the boundedness of \( s \)-energy implies that \( d_{\vartheta,x} (x, y) \geq s \) for \( \mu_{\vartheta,x} \)-almost all \((x, y) \in J \times R \), i.e., \( d_{\vartheta,x} (x, W_\vartheta (x)) \geq s \) for \( \nu \)-almost all \( x \in C_c \). On the other hand, by Borel density theorem we have
\[
\lim_{r \to 0} \frac{\mu_{\vartheta,x}(B_r(x, y))}{\mu_{\vartheta,x}(B_r(x, y))} = \frac{\mu_{\vartheta}(B_r(x, y) \cap (C_c \times R))}{\mu_{\vartheta}(B_r(x, y))} = 1
\]
for \( \mu_{\vartheta} \)-almost all \((x, y) \in C_c \times R \), so \( d_{\vartheta,x} (x, W_\vartheta (x)) = d_{\vartheta,x} (x, W_\vartheta (x)) \geq s \) for \( \nu \)-a.a. \( x \in C_c \).

For any \((x, v) \in J^2 \), let \( h_{x,v} \) denote the density of the random variable \( \vartheta \mapsto W_\vartheta (x) - W_\vartheta (v) \) with respect to the Lebesgue measure on \( R \).

The next key lemma is a variation of [MW12, lemma 4.2]. Our proof is based on its original proof.

**Lemma 21 (Key lemma).** There is a constant \( C_h > 0 \) such that
\[
\| h_{x,v} \|_\infty \leq \frac{C_h}{\lambda^p(x)}
\]
for all \( v \notin B_{n,\delta}(x) \setminus B_{n+1,\delta}(x) \), \( n \in \mathbb{N} \) and \( x \in J \).

In order to prove the preceding lemma, the following fact is essential. The proof is found in the last section.

**Lemma 22.** Assume that \( g \in C^1 (\mathbb{T}) \) satisfies the strong critical point hypothesis and that
then we have
\[
\sup_{(a,b) \in Q} \int_R |H_{a,c}(x)|^p dx < \infty
\]
for any compact subset \( Q \subset (0,1) \times R \) and \( p \in (1,2) \), where \( H_{a,c} \) denote the density of the measure \( m_T \{ x \in \mathbb{T} : x \mapsto c g(a + x) - c g(x) \in \cdot \} \) with respect to the Lebesgue measure on \( R \), where \( m_T \) denotes the Lebesgue measure on \( \mathbb{T} \).

**Proof of lemma 21.** Let \( v \in B_{n,\delta}(x) \setminus B_{n+1,\delta}(x) \) and \( n \in \mathbb{N} \) be fixed. Let \( h_{x,v,n} \) be the density of the random variable \( \vartheta_n \mapsto \lambda^v(x) g(\tau^x x + \vartheta_n) - \Lambda^v(x) g(\tau^x v + \vartheta_n) \) with respect to the Lebesgue measure on \( R \). As \( \vartheta_0, \vartheta_1, \ldots \) are independent, \( h_{x,v} \) can be considered as an infinite convolution of \( h_{x,v,n} \) for \( n \in \mathbb{N} \). In particular, by Hölder’s inequality and Young’s inequality,
\[
\| h_{x,v,n} \|_\infty \leq \| h_{x,v,n} * h_{x,v,n+1} * h_{x,v,n+2} \|_\infty \leq \| h_{x,v,n} \|_2 \| h_{x,v,n+1} \|_3 \| h_{x,v,n+2} \|_3/2,
\]
where \( \| \cdot \|_p \) denotes \( L^p \)-norm with respect to the Lebesgue measure on \( R \).

Let \( h_{x,v,n} \) be the density of
\[
\vartheta_n \mapsto g(\tau^x x + \vartheta_n) - \Lambda^v(x) g(\tau^x v + \vartheta_n).
\]
Observe that $h_{k,n}(t) = \frac{1}{X(t)} \tilde{h}_{k,n}(\frac{t}{X(t)})$, and thus $\|h_{k,n}\|_{3/2} = \frac{1}{(X(t))^{1/2}} \|\tilde{h}_{k,n}\|_{3/2}$ for any $k \in \mathbb{N}$. Therefore, we need to show that $\max_{j=0,1,2} \|h_{k,n,n+j}\|_{3/2}$ is bounded by a constant which is independent of the choice of $n, x, v$. To this end, we consider the random variable

$$\theta_k \mapsto g(\tau^x - \tau^v + \theta_k) - \frac{\lambda}{X(t)} g(\theta_k),$$

whose density to the Lebesgue measure on $\mathbb{R}$ is also represented by $\tilde{h}_{k,n}$. Firstly, $d_{x-1}(x,v) \leq \delta_k$ implies $v \in I_{x-1}(x)$. Similarly to lemma 9, in view of the Hölder continuity of $\log \lambda$, there is a constant $C > 0$ such that $C^{-1} \leq \frac{X(t)}{X(t)^2} \leq C$ for all $v \in I_k(x)$ and $k$. Thus, choosing $C := \left( \sup_{X(t)^2} \frac{X(t)}{X(t)^2} \right)^{1/2}$, we have $C^{-1} \leq \frac{X(t)}{X(t)^2} \leq C$ for $j \in \{0, 1, 2\}$. Secondly, for $j \in \{0, 1, 2\}$, since $|\tau^x - \tau^v| \leq |\tau^x - \tau^v| + |\tau^v - \tau^v| \leq \delta_k$ and $|\tau^v - \tau^v| \leq \delta_k$, we have $\tau^x - \tau^v \geq 0$ by the choice of $\delta_k$. Therefore, $|\tau^x - \tau^v| \geq |\tau^x - \tau^v| \geq \delta_k$. Now, we have shown that $\tau^x - \tau^v \in [\kappa, 1 - \kappa] + \mathbb{Z}$, where $\kappa := \min\{1 - \delta_k\} \in [0, 1]$. Observe that $\kappa$ and $C$ do not depend of $n, x, v$, and applying lemma 22 for $p = 2/3$ and $Q := [\kappa, 1 - \kappa] \times [C^{-1}, C]$, we obtain the bounding constant.

**Lemma 23.** Suppose that $-\int \log \lambda \, d\nu < h(x, \nu)$. Then we have for a.a. $\theta$ that

$$\dim_H \mu_\theta \geq \dim_H \nu + 1 + \int \frac{\log \lambda \, d\nu}{\log |\tau^x|} \nu.$$

**Proof.** We are going to show the assumption of lemma 20. By lemma 21 and the substitution formula of integral, we have

$$\int \frac{d\theta}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} = \int \frac{h(x,v)(z) \, dz}{(d(x,v)^2 + z^2)^{3/2}}$$

$$= \int_{R} \frac{d(x,v)^{1-\tau}}{(1 + r^2)^{3/2}} \left| \frac{d(x,v)}{d(x,v)^2 + z^2} \right|$$

$$\leq d(x,v)^{1-\tau} \int_{R} \frac{dr}{(1 + r^2)^{3/2}} ||h_{k,n}||_{\infty}$$

$$= d(x,v)^{1-\tau} K_\tau ||h_{k,n}||_{\infty} \leq \frac{K_\tau C_\kappa}{X^2(x)} d(x,v)^{1-\tau},$$

where $K_\tau := \int_{R} \frac{d\nu}{(1 + r^2)^{3/2}}$ is well-defined for $s > 1$. Let $\varepsilon > 0$ be small enough and let $x \in C_\varepsilon$. By Fubini’s lemma we have

$$\int \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$= \sum_{n=0}^{\infty} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta$$

$$\leq K_\tau C_\kappa \sum_{n=0}^{\infty} \frac{1}{X^2(x)} \int_{r_{x,i}} \frac{d\nu_{r,v}(\nu)}{(d(x,v)^2 + (W(x) - W(v))^2)^{3/2}} \, d\theta.$$
for all $s \in (1, S(\varepsilon))$, where

$$S(\varepsilon) := 1 + \left( \frac{\int \log |\tau'| d\nu - \varepsilon)(\dim_H \nu - \varepsilon) + \int \log \lambda d\nu - \varepsilon}{\int \log |\tau'| d\nu + \varepsilon} \right).$$

Observe that $\lim_{\varepsilon \to 0} S(\varepsilon) = \dim_H \nu + 1 + \frac{\int \log \lambda d\nu}{\int \log |\tau'| d\nu} > 1$ by the assumption, so the $(1, S(\varepsilon))$ is not empty. Finally, integrating the above inequalities related to $x$ by $\nu_{\theta, \varepsilon}$, we obtain $E_{\varepsilon, \theta} < \infty$ for all $s \in (1, S(\varepsilon))$, and thus $\dim_H \nu_{\theta} \geq S(\varepsilon)$ by lemma 20. Letting $\varepsilon \to 0$ finishes the proof.

Lemma 24. Suppose that $-\int \log \lambda d\nu \geq h_s(\nu)$. Then we have $\nu_{\theta}$-a. that

$$d\nu_{\theta} \geq \frac{h_s(\nu)}{-\int \log \lambda d\nu}.$$

Proof. For $s < s' < 1$, by Jensen’s inequality and lemma 21, we have

$$\int \frac{d\theta}{(d(x, v)^2 + (W_\theta(x) - W_\theta(v))^2)^{s'/2}} \leq \left( \int \frac{d\theta}{(d(x, v)^2 + (W_\theta(x) - W_\theta(v))^2)^{s/2}} \right)^{s'/s} \leq C_{s', s}^r \frac{d(x, v)^{s'-s}||h_{x, \theta}||_{s'}}{(\lambda^s(x))^{s'}}.$$

where $K_s$ is the constant in proof of lemma 23 and well-defined for $t = s'/s > 1$. Note that we also used $d(x, v) \leq 1$. Let $\varepsilon > 0$ be small enough and let $x \in C_{\varepsilon}$. By Fubini’s lemma we have

$$\int \int_{B_{\delta}(x))} \frac{d\nu_{\varepsilon}(v)}{(d(x, v)^2 + (W_\theta(x) - W_\theta(v))^2)^{s'/2}} d\theta \leq \sum_{n=0}^{\infty} \int \int_{B_{\delta}(x) \setminus B_{\delta+1}(x))} \frac{d\nu_{\varepsilon}(v)}{(d(x, v)^2 + (W_\theta(x) - W_\theta(v))^2)^{s'/2}} d\theta \leq C_{s'}^r K_{s'/s} \sum_{n=0}^{\infty} \frac{\nu(B_{\delta+n}(x))}{(\lambda^s(x))^{s'}} \leq C_{s'}^r K_{s'/s} \sum_{n=0}^{\infty} \frac{\nu(B_{\delta+1}(-f \log |\tau'| d\nu + \varepsilon)(x))}{(\lambda^s(x))^{s'2}} \leq C_{s'}^r K_{s'/s} \sum_{n=0}^{\infty} e^{-f \log |\tau'| d\nu + \varepsilon} (\dim_H \nu - \varepsilon) e^{-s'(\int \log \lambda d\nu - \varepsilon)} < \infty.$$
for all $s \in (0, 1)$ and $s' \in (\max\{s, \tilde{S}(\varepsilon)\}, 1)$, where
\[
\tilde{S}(\varepsilon) := \frac{\left(\int \log|\tau'|\,d\nu - \varepsilon\right)(\dim_H \nu - \varepsilon)}{\int \log \lambda \,d\nu + \varepsilon}.
\]

Observe that $\tilde{S}(\varepsilon) < 1$ for $\varepsilon > 0$ so that the following approximation works, even in case $\tilde{S}(0) = 1$. Since $\lim_{\varepsilon \to 0} \tilde{S}(\varepsilon) = \frac{h_{\nu}(\nu)}{\int \log|\tau'|\,d\nu} < 1$, the claim follows by lemmas 19 and 20. \hfill \Box

**Conclusion of proof of lemma 13.** For almost all $\vartheta$, it is a combination of lemmas 23 and 24 that
\[
J_{\mu_{\vartheta}} \geq \min \left\{ \dim_H \nu + 1 + \frac{\int \log \lambda \,d\nu}{\int \log|\tau'|\,d\nu} \right\}
\]
holds $\mu_{\vartheta}$-almost surely. Hence, together with the upper bound obtained in lemma 18, the proof of lemma 13 is finished. \hfill \Box

### 7. Proof of lemma 10

We mainly follow the major arguments in [Bed89] except for the part of lemma 27, the novel lower bound for the oscillation of randomised graph. All things become a bit more tidy than in the mentioned work because of the additional variable $\vartheta$. Indeed, we consider the following dynamical system. For each $(\vartheta, i) \in \mathbb{T} \times \Sigma_{\vartheta}$, the contraction $F_{\vartheta, i} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is defined by
\[
F_{\vartheta, i}(x, y) = (\rho_i(x), \lambda(\rho_i(x)) y + g(\rho_i(x) + \vartheta)).
\]
Observe that
\[
F_{\vartheta, i}(x, W_{\vartheta}(x)) = (x, W_{\vartheta}(x))
\]
for all $x \in \mathbb{T}$ and $\vartheta \in \mathbb{T}_1$. In addition, the derivative matrix for $F_{\vartheta, i}$ can be calculated as
\[
DF_{\vartheta, i}(x, y) = \begin{bmatrix}
\rho'_i(x) & 0 \\
(y \cdot \lambda' + g'(+ \vartheta)(\rho_i(x)) \cdot \rho'_i(x) & \lambda(\rho_i(x))
\end{bmatrix}
= \begin{bmatrix}
a_i(x) & 0 \\
b_i(\vartheta, x, y) & c_i(x)
\end{bmatrix}
\]

for $(x, y) \in \mathbb{T} \times \mathbb{R}$. It is convenient for iterated maps to be denoted by
\[
F_{\vartheta, 1} := F_{\vartheta_{n+1}, i_{n+1}} \circ \cdots \circ F_{\vartheta, i_1} \circ F_{\vartheta_{i_0}, i_0}
\]
for $(\vartheta, i) \in \mathbb{T}^n \times \Sigma^n_{\vartheta}$. Observe that, applying (10) iteratively, we have
\[
F_{[\vartheta, n]}(x, W_{\vartheta}(x)) = (x, W_{\vartheta}(x))
\]
for all $x \in \mathbb{T}$, $\vartheta \in \mathbb{T}_1$ and $n \in \mathbb{N}$, where $[\vartheta, n] := ([\vartheta_{n-1}, \vartheta_{n-1}, \ldots, \vartheta_{n-1}])$.

In the following, a continuous map $C : I \to [0, 1] \times \mathbb{R}$ for any subinterval $I \subseteq \mathbb{T}$ will be called a curve if $C_1$ is strictly monotone, where $C(t) = (C_1(t), C_2(t))$. Given a curve $C : I \to \mathbb{T} \times \mathbb{R}$, its width, height and the height over $\mathcal{J}$ are, respectively, denoted by
\[
|C|_W := |C_1(I)|, \quad |C|_H := \sup_{t_1, t_2 \in I} |C_2(t_1) - C_2(t_2)| \quad \text{and} \quad |C|_{\mathcal{J}} := \sup_{t_1, t_2 \in I \cap C_1^{-1}(\mathcal{J})} |C_2(t_1) - C_2(t_2)|.
\]
Lemma 25. Let $M > 0$ and $I \subseteq \mathbb{T}$ be a subinterval. For any curve $C : I \to \mathbb{T} \times [-M, M]$ with $C_i$, we have
\[ \inf_{t \in I} c_i(C_i(t)) : |C_i|_{\mathcal{J} \mathcal{H}} - b \cdot |C_i|_W \leq |F_{\partial_i} \circ C|_{\mathcal{J} \mathcal{H}} \quad \text{and} \quad |F_{\partial_i} \circ C|_H \leq \sup_{t \in I} c_i(C_i(t)) : |C_i|_{\mathcal{J} \mathcal{H}} + b \cdot |C_i|_W, \]
where $b := \max \left\{ b_i(\partial, x, y) : i \in I, \partial \in \mathbb{T}, (x, y) \in \mathbb{T} \times [-M, M] \right\}$.

Proof. Due to the monotonicity assumption on $C_i$, we can reparametrise the curve so that $C_1(t) = t$ by implicit function theorem. Observe that after this step, it hold $|C_i|_W = |I|$ and $|C_i|_{\mathcal{J} \mathcal{H}} = \sup_{t \in I} |C_2(t_1) - C_2(t_2)|$. Furthermore, by the Stone–Weierstrass approximation theorem, we may assume $C_2 \in C^1(\mathbb{R})$. Now, we can write
\[ (F_{\partial_i} \circ C)'(t) = DF_{\partial_i}(C(t)) : \begin{bmatrix} C'_i(t) \\ C'_2(t) \end{bmatrix} = b_i(\partial, C(t)) + c_i(t) C'_2(t) \]
for all $t \in I$, so we have
\begin{align*}
|F_{\partial_i} \circ C|_{\mathcal{J} \mathcal{H}} &\geq \int_{t_1}^{t_2} b_i(\partial, C(t)) + c_i(t) C'_2(t) \, dt \\
&\geq \inf_{t \in [t_1, t_2]} c_i(t) \int_{t_1}^{t_2} C'_2(t) \, dt - b \cdot (t_2 - t_1) \\
&\geq \inf_{t \in I} c_i(t) \cdot (C_2(t_1) - C_2(t_2)) - b \cdot |C_i|_W
\end{align*}
for all $t_1, t_2 \in \mathcal{J} \cap I$. The first inequality in the claim follows from this, immediately. Moreover, there exist some $t_1, t_2 \in I$ such that
\[ |F_{\partial_i} \circ C|_H = F_{\partial_i} \circ C(t_2) - F_{\partial_i} \circ C(t_1) \]
\[ = \int_{t_1}^{t_2} b_i(\partial, C(t)) + c_i(t) C'_2(t) \, dt \leq b \cdot |C_i|_W + \int_{t_1}^{t_2} c_i(t) C'_2(t) \, dt. \]
Finally, since $c_i > 0$, by the mean value theorem, there is some $t_3 \in (t_1, t_2)$ such that
\[ \int_{t_1}^{t_2} c_i(t) C'_2(t) \, dt = c_i(t_3) \int_{t_1}^{t_2} C'_2(t) \, dt \leq \sup_{t \in I} c_i(t) |C_2|_{\mathcal{J} \mathcal{H}}. \]

Lemma 26. Given $M > 0$, there are constants $L', L'', L''' > 0$ such that
\[ (L' |C_i|_{\mathcal{J} \mathcal{H}} - L'' |C_i|_W) \lambda^n(\rho_l(x)) \leq |F_{\partial_j} \circ C|_{\mathcal{J} \mathcal{H}} \leq L''' \lambda^n(\rho_l(x)) \]
for any curve $C : I \to \mathbb{T} \times [-M, M]$ ($\partial, \mathbf{i}) \in \mathbb{T} \times \Sigma_2^n$, $x \in \mathbb{T}$ and $n \in \mathbb{N}$.

Proof. Let $C, (\partial, \mathbf{i}), x$ and $n$ be given as stated. By applying lemma 25, inductively, we obtain
\[ |F_{\partial_j} \circ C|_{\mathcal{J} \mathcal{H}} \geq \tilde{c}_0 \cdots \tilde{c}_{n-1} |C_i|_{\mathcal{J} \mathcal{H}} - b \sum_{j=1}^n |\rho|_{|\mathbf{i}|=j} \circ C_1(I) \prod_{k=n-j+1}^{n-1} \tilde{c}_k, \]
where $\tilde{c}_j := \max \left\{ c_{\partial_j}(\partial, x, y) : (x, y) \in \mathbb{T} \times [-M, M] \right\}$.
where \( \hat{c}_k := \inf_{t \in I} c_u (\rho_{\partial u} \circ C_1 (t)) \) and \( \prod_{k=\max}^{n-1} \hat{c}_k := 1 \). Let \( (x_k)_{k=0}^{n-1} \subset T \) be such that
\[
\hat{c}_k = \inf_{t \in I} \left( \rho_{[i]} \circ C_1 (t) \right) = \lambda \left( \rho_{[i]} (x_k) \right).
\]

For \( p = 0, \ldots, n-1 \), since
\[
\left| \log \prod_{k=p}^{n-1} \lambda \left( \rho_{[i]} (x_k) \right) \right| \leq \sum_{k=p}^{n-1} \left| (\log \lambda)' \right|_{\infty} \left| \rho_{[i]} (x_k) - \rho_{[i]} (x) \right|
\leq \frac{\left| (\log \lambda)' \right|_{\infty}}{1 - \max \left| (1/\tau') \circ \rho_{[i]} \right|_{\infty}} =: \log K_0,
\]
we have
\[
\prod_{k=p}^{n-1} \hat{c}_k = \lambda^{-p} (\rho_k (x)) \prod_{k=p}^{n-1} \lambda \left( \rho_{[i]} (x_k) \right) = \left[ K_0^{-1}, K_0 \right] \cdot \lambda^{n-p} (\rho_k (x)).
\]

Similarly, \( \left| \rho_{\partial u} \circ C_1 (t) \right| = \left| \rho_{\partial u} (u_p) \right| \left| C \right|_W = \left| (1/\tau')^p \left( \rho_{\partial u} (u_{n-j}) \right) \right| \left| C \right|_W \) for some \( u_p \in [0, 1] \), and there is a constant \( K_1 > 0 \) such that
\[
\left| (1/\tau')^p \left( \rho_{\partial u} (u_{n-j}) \right) \right| \leq \left| (1/\tau')^p \left( \rho_{\partial u} (x) \right) \right| \leq \left[ K_1^{-1}, K_1 \right] \cdot \left| (1/\tau')^p \left( \rho_{\partial u} (x) \right) \right|
\]
for \( p = 0, \ldots, n-1 \). Observe that since the derivative of \( \log \left| \tau' \right| \) on \( \bigcup_{t \in I} I_t \) may fail to exist, we need here to consider the \( \alpha \)-Hölder semi-norm \( \cdot \mid_{\alpha} \) in order to obtain the constant
\[
\log K_1 := \left| \log \left| \tau' \right| \right|_{\alpha} / (1 - \left| 1/\tau' \right|_{\infty}^{\alpha}).
\]
Consequently, the lower bound follows as
\[
\left| F_{\partial u} \circ C \right|_{\sigma, H} \geq K_0^{-1} \lambda^{n-1} (\rho_k (x)) \left| C \right|_{\sigma, H} - b K_0 K_1 \left| C \right|_W \sum_{j=1}^{n} \left| (1/\tau')^{-j} \left( \rho_{\partial u} (x) \right) \right| \lambda^{j-1} (\rho_k (x))
\]
\[
= \lambda^n (\rho_k (x)) \left( K_0^{-1} \left| C \right|_{\sigma, H} - b K_0 K_1 \lambda (\rho_{\partial u} (x)) \left| C \right|_W \sum_{j=1}^{n} \left| (1/\tau')^{-j} \left( \rho_{\partial u} (x) \right) \right| \lambda^{j-1} (\rho_k (x)) \right)
\]
\[
\geq \lambda^n (\rho_k (x)) \left( K_0^{-1} \inf \left( 1/\lambda \right) \left| C \right|_{\sigma, H} - b K_0 K_1 \frac{\lambda \left| C \right|_W}{1 - \max \left| (1/\tau') \circ \rho_{[i]} \right|_{\infty}} \right).
\]

Finally, the upper bound can be derived in a similar manner. Indeed, as done above for \( \hat{c}_k \), we can show
\[
\prod_{k=p}^{n-1} \hat{c}_k \in \left[ K_0^{-1}, K_0 \right] \cdot \lambda^{n-1} (\rho_k (x)),
\]
for \( p = 0, \ldots, n-1 \), where \( \hat{c}_k := \sup_{t \in I} c_u (\rho_{[i]} \circ C_1 (t)) \). From this follows that
\[
\left| F_{\partial u} \circ C \right|_{\sigma, H} \leq \lambda^n (\rho_k (x)) \left| C \right|_{\sigma, H} + b K_0 K_1 \left| C \right|_W \sum_{j=1}^{n} \left| (1/\tau')^{-j} \left( \rho_{\partial u} (x) \right) \right| \lambda^{j-1} (\rho_k (x))
\]
\[
\leq \lambda^n (\rho_k (x)) \left( \left| 1/\lambda \right|_{\infty} + b K_0 K_1 \frac{\lambda \left| C \right|_W}{1 - \max \left| (1/\tau') \circ \rho_{[i]} \right|_{\infty}} \right).
\]

\[\Box\]
Lemma 27. Assume that \( W_\vartheta \) is non-degenerate for almost all \( \vartheta \). Then, for any \( L > 0 \), we have

\[
\lim_{n \to \infty} \frac{\log a_L(\sigma^n \vartheta)}{n} = 0
\]

for a.a. \( \vartheta \), where \( a_L(\vartheta) := \sup_{u,v \in J} |W_\vartheta(u) - W_\vartheta(v)| - L|u - v| \).

Proof. Let \( L > 0 \) be arbitrary. Observe that \( a_L(\vartheta) > 0 \). Fix those parameters \( \vartheta \) for which \( W_\vartheta \) is non-degenerate. Let \( M := \sup_{\vartheta \in T \times T} |W_\vartheta(x)| \), and recall \( c_i := \lambda \circ \rho_i \). Given \( u,v \in T \) and \( i \in \Sigma_i \), the invariance relation of (10) yields

\[
W_\vartheta(p_i(u)) = c_i(u) W_\vartheta(u) + g(p_i(u) + \vartheta_0).
\]

On the other hand, there is some \( w \) between \( u,v \) such that \( |p_i(u) - p_i(v)| = |p_i'(w)| |u - v| \), so we have

\[
|c_i(u) W_\vartheta(u) - c_i(v) W_\vartheta(v)| = |c_i(w) (W_\vartheta(u) - W_\vartheta(v)) + W_\vartheta(u) (c_i(u) - c_i(w)) + W_\vartheta(v) (c_i(w) - c_i(v))| \\
\geq c_i(w) |W_\vartheta(u) - W_\vartheta(v)| - 2M |\lambda'| \|u - v|.
\]

Together with the facts

\[
-|g(p_i(u) + \vartheta_0) - g(p_i(v) + \vartheta_0)| \geq \|g'\| \|u - v|,
\]

and

\[
c_i(w) - |p_i'(w)| = c_i(w) \left( 1 - \frac{1}{\lambda |\vartheta'(\vartheta)|} \right) \geq (\inf_j (1/\vartheta'(\vartheta)) \rho_j) =: \delta_1 > 0,
\]

it follows that

\[
|W_\vartheta(p_i(u)) - W_\vartheta(p_i(v))| - L|p_i(u) - p_i(v)| \\
= |c_i(w) W_\vartheta(u) - c_i(v) W_\vartheta(v)| - L|p_i'(w)| |u - v| \\
\geq c_i(w) (|W_\vartheta(u) - W_\vartheta(v)| - L|u - v|) \\
+ (L c_i(w) - L |p_i'(w)| - 2M |\lambda'| \|u - v| - \|g'\| \|u - v|) \\
\geq c_i(w) (|W_\vartheta(u) - W_\vartheta(v)| - L|u - v|) + (L \delta_1 - 2M |\lambda'| |u - v| - \|g'\|) |u - v| \\
\geq c_i(w) (|W_\vartheta(u) - W_\vartheta(v)| - L|u - v|),
\]

for all \( L \geq L_0 := (2M |\lambda'| + \|g'\|) \delta_1^{-1} \). In particular, for those \( \vartheta \) such that \( a_L(\vartheta) > 0 \),

\[
a_L(\vartheta) \geq \sup_{u,v \in J \cap H} |W_\vartheta(u) - W_\vartheta(v)| - L|u - v| \geq (\inf \vartheta) a_L(\vartheta) \vartheta.
\]

Now, we prove the lemma for \( L \geq L_0 \). Fix any \( i \), say \( i = 0 \). We have shown that \( \log a_L(\vartheta) - \log a_L(\vartheta_0) \geq \log(\inf \vartheta) \) for a.a. \( \vartheta \), which implies \( \log a_L(\vartheta) - \log a_L(\vartheta_0) \in L_{\vartheta_0}^* \vartheta \) with \( \int a_L(\vartheta) - a_L(\vartheta_0) \vartheta d\vartheta = 0 \) in view of [Kel96, lemma 2]. Thus, by the telescoping sum and Birkhoff’s ergodic theorem, we have
for a.a. $\vartheta$. Finally, the claim holds also for $L \in (0, L_0)$ as $a_{L_0}(\vartheta) \leq a_L(\vartheta) \leq 2M + 1$ for all $\vartheta$.

**Proof of lemma 10.** Let $L', L'', L''' > 0$ be the constants of lemma 26 related to the given $M := \sup_{(\vartheta,x) \in \mathbb{T}^n \times \mathbb{T}} |W_\vartheta(x)|$. For the upper bound, let $x \in J$, $n \in \mathbb{N}$ and $\vartheta \in \mathbb{T}^n$ be arbitrary. Consider the curve $C : \mathbb{T} \rightarrow \mathbb{T} \times \mathbb{R}$ by $C(t) := (t, W_{\vartheta}^R(t))$. The invariance relation of (11) yields
\[ (t, W_\vartheta(t)) = F|_{\vartheta, [s]}(C(\tau^n t)) \]
for all $t \in I_n(x)$. Thus, by lemma 26 we have
\[
\sup_{t \in I_n(x)} |W_\vartheta(x) - W_{\vartheta}(\vartheta)| \leq \sup_{u \in I_n(x)} |W_\vartheta(u) - W_\vartheta(\vartheta)|
\]
\[
= |F|_{\vartheta, [s]} \circ C|_H \leq L'''' \lambda^n(\bar{\rho}_{[s]}(\tau^n x)) = L'''' \lambda^n(x).
\]
Therefore, the first claim is satisfied with $C := L'''$.

Next, we show the last claim. Suppose that $W_\vartheta$ is non-degenerate for almost all $\vartheta$. Let $a_L : \mathbb{T}^n \rightarrow [0, \infty)$ be defined as in lemma 27 for $L := L''/L'$. We claim that $c(\vartheta) := (L'/2) a_L(\vartheta)$ satisfies the claimed properties. By construction, it only remains to verify the claimed lower bound inequality. Let $x \in J$ and $n \in \mathbb{N}$.

Finally, the second claim follows also the above estimate. Indeed, if $W = W_0$ is non-degenerate, then we can choose $c := c(\sigma^n \vartheta) = (L'/2) a_L(\vartheta) > 0$.

**8. Proof of lemma 22**

**Proof.** Let $a, c$ be fixed. Let $u_1, \ldots, u_m$ be the zeros of $f_{a,c}$, where $f_{a,c}(x) := g(a + x) - cg(x)$. As $f_{a,c}$ has $m + 1$ invertible branches, we can compute that
Thus, since $m \leq \overline{m}$ by the strong critical point hypothesis, and since $Q$ is compact, it suffices to show that the integral on the right hand side is locally bounded, i.e. for each $(a, c) \in Q$ there is some $\delta > 0$ such that

$$\sup_{(\tilde{a}, \tilde{c}) \in U_{\delta}(a, c)} \int_0^1 \frac{dx}{|f'_{a,c}(x)|^{p-1}} < \infty,$$

where $U_{\delta}(a, c) := \{(\tilde{a}, \tilde{c}) \in Q : |a - \tilde{a}| + |c - \tilde{c}| < \delta\}$ denotes the $\delta$-neighbourhood of $(a, c)$. For each $a \in \mathbb{T}$, Taylor’s theorem says

$$f'_{a,c}(x) = f'_{a,c}(a) + (f''_{a,c}(u) + r_{a,c}(u, x)) (x - a),$$

where

$$r_{a,c}(u, x) := \frac{\int_0^1 f''_{a,c}(t) (x - t)^2 dt}{x - a}.$$

Taking $M := \sup_{(a,c) \in Q} \|f''_{a,c}\|_{\infty}$, we can roughly estimate $|r_{a,c}(u, x)| \leq M |x - a|^2$. In the following, we use the fact that

$$|f'^{(j)}_{a,c}(x) - f'^{(j)}_{a,c}(a)| \leq \left( \|g^{(j)}\|_{\infty} + \|g^{(j+1)}\|_{\infty} \right) (|a - \tilde{a}| + |c - \tilde{c}|)$$

for $j = 0, 1, 2$.

Observe that for each $u_i$, since $f'_{i,0}(u_i) = 0$, we have

$$\frac{1}{|f'_{a,c}(x)|} \leq \frac{1}{|f'_{a,c}(a)(x - u_i)|} \left| \frac{r_{a,c}(u_i, x)}{f'_{a,c}(a)} \right|. $$

Choosing $\varepsilon_i \in (0, \frac{\min_{1 \leq j \leq m} |u_j - u_i|}{2})$ so small that

$$\frac{M \varepsilon_i^2}{|f'_{a,c}(u_i)|} \leq \frac{1}{24} \quad \text{and} \quad \inf_{x \in B_{\varepsilon_i}(u_i)} |f''_{a,c}(x)| \geq |f''_{a,c}(u_i)| \geq \frac{|f''_{a,c}(u_i)|}{2},$$

we consider the decomposition

$$\int_0^1 \frac{dx}{|f'_{a,c}(x)|^{p-1}} = \sum_{i=1}^m \int_{B_{\varepsilon_i}(u_i)} \frac{dx}{|f'_{a,c}(x)|^{p-1}} + \int_{K_a} \frac{dx}{|f'_{a,c}(x)|^{p-1}},$$

where $K_a := \mathbb{T} \setminus \bigcup_{i=1}^m B_{\varepsilon_i}(u_i)$.

Now, we consider $(\tilde{a}, \tilde{c})$ close to $(a, c)$. Again we can consider the same decomposition

$$\int_0^1 \frac{dx}{|f'_{a,c}(x)|^{p-1}} = \sum_{i=1}^m \int_{B_{\varepsilon_i}(u_i)} \frac{dx}{|f'_{a,c}(x)|^{p-1}} + \int_{K_a} \frac{dx}{|f'_{a,c}(x)|^{p-1}}.$$
We first show that the last integral is locally bounded about \((a,c)\). In view of (13) and compactness of \(K_{a,c}\), there is a \(\delta_1 > 0\) such that

\[
\inf_{x \in K_{a,c}} |f'_{a,c}(x)| \geq \frac{\inf_{x \in K_{a,c}} |f_{a,c}(x)|}{2} > 0
\]

for all \((\tilde{a}, \tilde{c}) \in U_{\delta_1}(a,c)\). Thus,

\[
\sup_{(\tilde{a}, \tilde{c}) \in U_{\delta_1}(a,c)} \int_{K_{a,c}} |f'_{\tilde{a},\tilde{c}}(x)|^{p-1} \leq \frac{2^{p-1}}{\inf_{x \in K_{a,c}} |f'_{a,c}(x)|^{p-1}}.
\]

It remains to show that each \(\int_{B_{\delta_1}(u)} \frac{dx}{|f'_{a,c}(x)|^{p-1}}\) is locally bounded about \((a,c)\). In view of (13), we can choose \(\delta_1 > 0\) so that

\[
\inf_{x \in B_{\delta_1}(u)} |f'_{a,c}(x)| \geq \frac{|f''_{a,c}(u)|}{3}
\]

for all \((\tilde{a}, \tilde{c}) \in U_{\delta}(a,c)\). Let \(u \in B_{\delta}(u)\) such that \(|f'_{\tilde{a},\tilde{c}}(u)| = \min |f'_{\tilde{a},\tilde{c}}|\). Then, since

\[
\frac{|f_{\tilde{a},\tilde{c}}(u,x)|}{f''_{\tilde{a},\tilde{c}}(u)} \leq \frac{M(2\varepsilon)}{(1/3)|f''_{a,c}(u)|} \leq \frac{1}{2},
\]

we have

\[
\frac{1}{|f'_{a,c}(x)|} \leq \frac{1}{|f'_{\tilde{a},\tilde{c}}(x)-f'_{a,c}(u)|} = \frac{1}{|f''_{a,c}(u)(x-u)|} = \frac{2}{|f''_{a,c}(u)(x-u)|} \leq \frac{6}{|f''_{a,c}(u)|}.
\]

Thus,

\[
\int_0^1 \frac{dx}{|f'_{a,c}(x)|^{p-1}} \leq \left(\frac{6}{|f''_{a,c}(u)|}\right)^{p-1} \int_0^1 \frac{dx}{|x-u|^{p-1}} \leq \left(\frac{6}{|f''_{a,c}(u)|}\right)^{p-1} \frac{2}{2-p}.
\]

\(\square\)

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### References

[Bar08] Barreira L 2008 *Dimension and Recurrence in Hyperbolic Dynamics* (Berlin: Springer)
[Bed89] Bedford T 1989 The box dimension of self-affine graphs and repellers *Nonlinearity* 2 53–71
[Bá15] Bárány B 2015 On the Ledrappier–Young formula for self-affine measures Mathematical Proc. of the Cambridge Philosophical Society (Cambridge: Cambridge Philosophical Society)

[Car13] Carvalho A 2011 Hausdorff dimension of scale-sparse Weierstrass-type functions Fund. Math. 213 1–13

[Fal05] Falconer K 2005 Fractal Geometry: Mathematical Foundations and Applications 2nd edn (New York: Wiley)

[Har16] Hardy G H 1916 Trans. Am. Math. Soc. 17 301–25

[Hun98] Hunt B R 1998 The Hausdorff dimension of graphs of Weierstrass functions Proc. Am. Math. Soc. 126 791–800

[Jin11] Jin X 2011 The graph and range singularity spectra of b-adic independent cascade functions Adv. Math. 226 4087–5017

[JS15] Jaerisch J and Sumi H 2015 Multifractal formalism for expanding rational semigroups and random complex dynamical systems Nonlinearity 28 2913

[Kel96] Keller G 1996 A note on strange nonchaotic attractors Fund. Math 151 139–48

[Kel98] Keller G 1998 Equilibrium States in Ergodic Theory (London Mathematical Society Student Texts) (Cambridge: Cambridge University Press)

[Kel17] Keller G 2017 A simpler proof for the dimension of the graph of the classical weierstrass function Ann. Inst. Henri Poincaré 53 109–81

[MU16] Mihailescu E and Urbański M 2016 Random countable iterated function systems with overlaps and applications Adv. Math. 298 726–58

[MW12] Moss A and Walkden C P 2012 The Hausdorff dimension of some random invariant graphs Nonlinearity 25 743–60

[Pes97] Pesin Y B 1997 Dimension Theory in Dynamical Systems (Chicago, IL: University of Chicago Press)

[Rom14] Romanowska J 2014 Measure and Hausdorff dimension of randomized Weierstrass-type functions Nonlinearity 27 787

[She15] Shen W 2016 Hausdorff dimension of the graphs of the classical Weierstrass functions (arXiv:1505.03986)

[Tod15] Todorov D 2015 Hölder properties of Weierstrass-like solutions of θ-twisted cohomological equations (arXiv:1507.08938)