FIXED POINT FORMULAS AND LOOP GROUP ACTIONS

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Dedicated to the memory of Professor Fredrick Almgren, Jr.

1. Introduction

In this paper we present a new fixed point formula associated with loop group actions on infinite dimensional manifolds. This formula provides information for certain infinite dimensional situations similarly as the well known Atiyah-Bott-Segal-Singer’s formula does in finite dimension. A generalization of the latter to orbifolds will be used as an intermediate step.

There exist extensive literature on loop groups, loop algebras and their representations. What set the present work and [C1] apart is the focus on representations of the central extensions of the loop groups, induced from Hamiltonian actions; getting explicit formulas which determine the multiplicities of the irreducible highest weight components, hence the structure of the induced representations.

The results in [C1] show that the highest weight vector occurring in an induced representation are carried by a compact variety, provided the setup meets certain general conditions. In particular geometric quantization is generalized to the setting of loop group actions. The results here link the representations with local data on fixed points. In that direction, we also obtain a new multiplicity formula.

The current paper can be read either as a sequel to [C1] or on its own.

The original motivation was to understand in geometric setting a conjecture by Verlinde [V]. From it the better known Verlinde formula was derived [V, MS]. As the project progressed, it became clear that Verlinde’s conjecture is the tip of an iceberg. Representations of the central extensions of loop groups, induced from Hamiltonian actions on infinite dimensional manifolds, can be related to local geometry at the fixed point sets.

Application in a forthcoming paper will include a direct proof of the aforementioned conjecture.

More interestingly the we construct a class of $G$-orbifolds called fusion product which are geometric dual to the product in Verlinde fusion algebra.

Much has been done about Verlinde formula, we refer the readers to [F,Be] for references.

1.1. Some notations. Let $G$ be a connected and simply connected compact simple Lie group, and $T$ be a maximal torus. Let $W$ be its Weyl group, $t_+$ be the positive Weyl chamber and $P, P_+$ the sets of weights and dominant weights respectively, after fixing a set of simple roots. Let $D = \prod_{\alpha > 0}(1 - e^{-\alpha})$ denote the Weyl denominator, $\theta$ be the highest root of $g$. On $g$, fix an invariant bilinear form $(\cdot | \cdot)$, so that $(\theta^\vee | \theta^\vee) = 2$ where $\theta^\vee$ is the coroot corresponding to $\theta$. The bilinear form induces a

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map $\nu : t \to t^*$ and a bilinear form on $t^*$. The lattice in $t$ generated by $\{W(\theta^v)\}$ is denoted by $M$, and
\[ M^* = \{ t \in t \mid (t|n) \in \mathbb{Z}, \forall n \in M \}. \]
Let $h^v$ be the dual Coxeter number, $\rho$ the half sum of positive roots. Define the affine alcove and the set of dominant weights in there respectively as
\[ C = t_+ \cap \{ a| (a|\theta) \leq 1 \}, \quad P^k_+ = \{ \lambda \in P_+ \mid (\lambda|\theta) \leq k \} = P_+ \cap kC. \]
The face $C^a$ in $C$ defined by $(\cdot|\theta) = 1$ is special, in particular
\[ \partial(\cup_{w \in W} wC) = \cup_{w \in W} wC^a. \]
The lattice $\frac{M^*}{k+h^v}$ induces a finite subgroup of $T$, $\exp(2\pi\nu^{-1}\frac{M^*}{k+h^v})$. The subset
\[ \{ e^{2\pi i \nu^{-1} \frac{\lambda + h^v}{\nu}} | \lambda \in P^k_+ \} \]
plays an important role in this paper. An useful observation is that each element $\tau$ in that subset is a regular element of $G$.

1.2. Assumptions on $X$. Let $X$ be a Banach manifold on which $LG$ acts, $\omega$ be an invariant symplectic form and $\mu$ be the associated moment map at level $k \in \mathbb{Z}_+$, i.e.,
\[ \mu : X \to \mathfrak{g}^0 \times \{ k \} \]
where elements of $\mathfrak{g}^0$ have one degree less of differentiability than those in $\mathfrak{g}$. Being a moment map, $\mu$ is equivariant with respect to the $LG$-action on $X$ and the co-adjoint action on $\mathfrak{g}^0 \times \{ k \}$. The co-adjoint action loses one degree of differentiability, hence the target of $\mu$ is $\mathfrak{g}^0 \times \{ k \}$. We assume in this paper the following unless stated otherwise:

- **H1**: $\mu$ proper;
- **H2**: $\mu(X)$ is transversal to $t \times \{ k \}$. More details are given in Section 2.

1.3. The highest weight modules of the induced representations. In [C1], we studied holomorphic actions by a loop group $LG$, when $X$ is complex. Let $N^+ \subset LG^C$ denote the subgroup whose Lie algebra is $\sum_{\alpha \geq 0}(\mathfrak{g})_{\mathbb{C}}^+$. It consists of non-constant boundary values of holomorphic maps from the unit disk to $G^C$, together with constant maps with values in the positive nilpotent group of $G^C$.

Suppose $L$ is a holomorphic $LG$-line bundle over $X$. We proved in [C1] that there is a compact complex $T$-orbifold $X_N$ and an orbifold $T$-line bundle $L_N$, so that $H^0(X_N, L_N)$ carries all the highest weight vectors of $H^0(X, L)$. In other words, as $T$-modules:
\[ H^0(X_N, L_N) \simeq H^0(X, L)^{N^+}. \]
The orbifold $X_N$ naturally can be viewed as the compactification of the quotient of $X/N^+$. Therefore this compact model of $X$ carries the same amount of information as $X$, in terms of understanding the induced representation.

To $X_N$ obviously one can apply the fixed point formula, a certain generalization of Atiyah-Bott-Segal-Singer results to orbifold, and get informations about the $T$-equivariant Riemann-Roch $RR(X_N, L_N)$, thus the structure of $H^0(X_N, L_N)$. But compactification involves adding certain locus, and additional fixed points sets on the locus. To understand those new fixed points is not an easy issue, even when dealing with finite dimensional groups, e.g. the compactification of symmetric spaces.
However here we will find a solution to resolve this problem here, utilizing the affine Weyl group $W^{\text{aff}}$. As it will be shown by examples, this solution is the best one can hope for.

1.4. **Description of the main result.** The main result in this paper does not require $X$ is complex, although that situation motivates the construction of $X_N$ and later consideration.

Let $Y = G \times T X_N$, the line bundle $L_N$ induces $L_Y$ on $Y$. If the original line bundle is of level $k \in \mathbb{Z}$ (or the moment map $\mu$ is of level $k$), which means the central part of $\tilde{L}G$, $S^1$, acts on $L$ with character $k$, the following function on $T$ will be uniquely determined by its restriction to \( \{ e^{2\pi i \nu - 1} \lambda + \rho_{k+h} \in P^k \} \):

$$\sum_{w \in W} w \frac{1}{D} \text{RR}(X_N, L_N) = \text{RR}(Y, L_G).$$

The function $\text{RR}(Y, L_G)$ can be defined directly from $(X, L)$. It is given by

$$\sum_{a \in P^k} \text{RR}(M_a, L_a) \chi_a$$

where $M_a$ is the reduced space of $X$ at $a$, $L_a$ is the line bundle induced from $L$, $\chi_a$ is the $G$-character function of the highest weight representation defined by $a$.

What geometric data are needed to determine this function? Before answering that question, let’s motivate the discussion by first detailing the holomorphic case. As mentioned earlier, the quotient $X/N^+$, after throwing away some bad orbits, can be compactified by $X_N$. The compactification locus has its image given by the boundary of the affine chamber $C$. The chamber is a simplex if $G$ is simple, and is a product of simplices if $G$ is semi-simple. For generic $X$, $X_N$ is an orbifold and strata in the compactification locus are in 1-1 correspondence with the sub-faces of $C$. Particularly interesting here are the strata $\{ X_Q \}$ whose images are on the affine wall $C^{\text{aff}}$. Each $X_Q$ has a corresponding subvariety in $Y = G \times T X_N$, $Y_Q$. The collection $\{ Y_Q \}$ is part of the compactifying strata in the $G$-space $Y$.

Each $\tau \in \{ e^{2\pi i \nu - 1} \lambda + \rho_{k+h} \lambda \in P^k \}$ is a regular element in $T$. Thus its fixed point sets $\{ V \}$ in $X$ has images under $\mu$ in $t$. Each component $V$ will induce a subvariety $V_\Delta$ in $Y_\Delta = wX_N \subset Y$. And it also induces a subvariety $V_Q = V_\Delta \cap Y_Q$. The collections $\{ V_\Delta \}, \{ V_Q \}$ will be used to determine $\text{RR}(Y)(\tau)$.

We emphasize that in general $\tau$ has lots more fixed point sets than $\{ V_\Delta \}, \{ V_Q \}$ on the compactification strata in $Y$. Not all of the fixed points in the compactification are in the closure of the interior ones. So the important feature of the main result is that only the closure of the interior $\tau$-fixed points $\{ V_\Delta \}$, and their intersection with strata in the compactification $\{ V_Q \}$, matter in determining $\text{RR}(Y)(\tau)$. This feature manifests the underlying affine Weyl group symmetries, and is not known to hold in finite dimension.
Main Theorem 1. At \( \tau \in \{ e^{2\pi i a^{-1} \frac{\lambda}{\nu}} | \lambda \in P^k_+ \} \), the following holds:

\[
RR(Y)(\tau) = \sum_{a \in P^k_+} RR(M_a, L_a) \chi_a(\tau)
\]

\[
= \sum_{V_\Delta} \left( \int_{V_\Delta} \frac{Td(V_\Delta) Ch(L_{V_\Delta})}{\det_{\text{nor}(V_\Delta, Y)}(1 - t^{-1}e^{-\Omega})} \right) + \sum_{V_Q \subset V_\Delta} \frac{1}{|W^\text{aff}_Q||I_{V_Q}|} \sum_{t \in I_{V_Q}} \int_{V_Q} \frac{Td(V_Q) Ch(L_{V_Q} \oplus \Lambda_{\text{nor}(V_Q, V_\Delta)}/V_Q)}{\det_{\text{nor}(V_Q, Y)}(1 - t^{-1}e^{-\Omega})}(\tau)
\]

where \( W^\text{aff}_Q \subset W^\text{aff} \) is the subgroup preserving \( Q \), \( I_{V_Q} \) is the isotropy group associated with \( V_Q \) and \( \tau I_{V_Q} \) is the set of liftings of \( \tau \). Furthermore, each integral above can be localized to the \( T^\text{fixed} \) point sets \( F \) in \( V_\Delta, V_Q \) respectively to yield:

\[
(1.2) \quad RR(Y)(\tau) = \sum_{\{F | \phi(F) \in W(C^\text{int})\}} FC(F)(\tau) + R(\tau)
\]

where

\[
FC_F(\tau) = \int_{F} \frac{Td(F) Ch(L_F)}{\det_{\text{nor}(F, Y)}(1 - t^{-1}e^{-\Omega})}(\tau);
\]

\[
R(\tau) = \sum_{\{F | \phi(F) \in W(C^\text{int})\}} \frac{1}{|W^\text{aff}_F||I_F|} \sum_{t \in I_F} \int_{F} \frac{Td(F) Ch(L_F \oplus \Lambda_{\text{nor}(F, Y)}/F)}{\det_{\text{nor}(F, Y)}(1 - t^{-1}e^{-\Omega})}(\tau).
\]

where \( C^\text{int} \) is the interior of \( C \), \( W^\text{aff}_F \) is the subgroup of \( W^\text{aff} \) preserving \( \phi(F) \), and \( I_F \) the isotropy group of \( F \).

Remark: 1). Similar to an earlier comment, the interesting feature in the second expression above is that only those \( F \) on the intersection \( V_\Delta \cap Y_Q \) matters. Other \( T^\text{fixed} \) points on the compactification locus \( Y_Q \) do exist and there are lots of them, but they do not contribute to \( RR(Y)(\tau) \) as it will be shown.

2). The presence of \( I_F, \tau F, I_{V_Q}, \tau I_{V_Q} \) in fixed point formula is a common feature in orbifold setting, this has been known for a while.

Consequences on \( LG \)-modules are given in Section 10.

1.5. Riemann-Roch of the reduced spaces. The previous result provides a way of computing the Riemann-Roch numbers of the reduced space \( M_a = \mu^{-1}(a)/(LG)_a \) via certain fixed point sets.

Corollary 1.1.

\[
RR(M_a, L_a) = \left( \frac{(-1)^l}{|M|} \sum_{\tau \in \{ e^{2\pi i a^{-1} \frac{\lambda}{\nu}} | \lambda \in P^k_+ \}} \chi_a(\tau) D^2(\tau) \left( \sum_{\{F | \mu(F) \in \{W(C^\text{int})\}} FC(F)(\tau) + R(\tau) \right) \right)
\]

where \( \tilde{a} = w_L(-a) \) with \( w_L \) being the longest element in \( W \) (or the highest weight in the contragredient representation to the one defined by \( a \)).
Remark: The above extends Verlinde’s formula for moduli space of flat connections over Riemann-surfaces.

If $X$ is holomorphic and $H^i(M_a, L_a) = 0$ for $i > 0$, then the above formula gives the multiplicities of the irreducible component with highest weight $(a, k)$ for the $\tilde{L}G$ representation on $H^0(X, L)$.

In finite dimension, the Riemann-Roch number of the reduced space can also be expressed in terms of the fixed points. The expression is obtained only recently as an application of Atiyah-Bott-Singer-Segal’s formula and the recent in [M].

1.6. What’s in the proof? The theorem is proved based on two main ingredients, both utilize the Weyl and affine Weyl group symmetries.

After applying directly fixed point formula for orbifolds to $Y$, one ends up with many terms of contributions from the fixed point sets on the compactifying locus. There are two kinds of them, those of the first kind have their images on $W(\partial C \setminus C^{\text{aff}})$, we prove using Weyl group symmetry that their contribution amounts to 0. The second kind are those with images on $W(C^{\text{aff}})$. Their contribution to the equivariant Riemann-Roch also amounts to 0, provided we restrict them as functions on $T$ to the subset \( \{ e^{2\pi i \nu - \frac{1}{k} \lambda} | \lambda \in P_k \} \), and each term has no pole on the subset. This is proved using affine Weyl group symmetries.

We shall refer to the above phenomenon as cancellation, it is based on the fundamental formula of Section 6 and several identities proved in Section 7. Also it requires detailed analysis of the fixed points on the compactification locus.

The second ingredient starts with the surgery formula in Section 12, it enables us to deal with the second kind of fixed points when poles are present. Their contribution is encoded in the function $R : \{ e^{2\pi i \nu - \frac{1}{k} \lambda} | \lambda \in P_k \} \to \mathbb{C}$. The exact expression of $R$ is determined using transformation rule of the affine Weyl group acting on the Riemann-Roch integrand, together with symplectic cuts. The calculation is rather elaborate.

The construction of the space $X_N$, was first done in 1993 prior the symplectic cuts. It shares certain similarity with the symplectic cuts, except the cuts are made along the degenerated parts of the two form $\omega | \mu^{-1}(t)$. The resulting space is only symplectic outside the inverse images of $\partial C$. The surgery formula for this kind of cuts is quite different from that using symplectic cuts, as shown by Prop. 12.1.

The cancellation mentioned earlier has a consequence called twin pair construction. It says that for a generic compact symplectic manifold with a Hamiltonian $G$-action, there is a different $G$-orbifold, with identical equivariant Riemann-Roch.

1.7. What is ahead? As mentioned earlier, applications will be given in a forthcoming paper.

In a separate paper, the result here will be improved so that all symplectic $LG$-manifolds, with compact quotient $X/LG$, will be covered. The present results assume the generic condition that $\mu(X) \subset lg \times \{ k \} \subset \tilde{lg}$ is transversal to $t$.

1.8. Organization of the paper. In section 2, we discuss the construction of $X_N$. The paper [C1] emphasizes on the holomorphic aspects of $X_N$ while here the construction builds around the symplectic structure. Several new results are presented here, including the existence of $LG$-invariant almost complex structures
on a class of $LG$-manifolds. The existence of $T$-invariant almost complex structure on $X_N$ is proved as well, which is not trivial considering that the symplectic form on $X_N$ is degenerate.

Section 3 contains the description of the fixed point sets, and their stratification which is a must since $X_N$ is an orbifold. And the fixed point formulas on orbifolds rely not only on the fixed points but their stratification as well. Section 4 includes the computations of weights of the induced $T$-action on the normal bundles to the fixed point sets and their lower strata, while Section 5 computes the curvatures of various components of the normal bundles.

The root of the cancellation is the fundamental formula presented in Section 6. I proved this formula in 1994. The original proof was based on comparison of two different compactifications of $G^{C}$. Both have the same Riemann-Roch but have different fixed points. Hence one yields an identity, then by induction on the rank of the group, one proves this formula. The result in Section 12 generalizes this ‘twin pair’ construction. The present proof is simplified by applying an identity which can be found in [M].

Another important component in proving the cancellation is described in Section 7. Section 8 addresses a complication which occurs when $G \neq SU(n)$. For those groups, the affine alcove may not be a simple simplex, with respect to the weight lattice. In constructing $X_N$, this fact introduces additional orbifold singularities. That section describes the extra components of the isotropy groups associated with the orbifold singularities.

Section 9 provides a way of computing the push-forward of certain cohomology classes on a fibration whose fibers are homogeneous spaces like $K/T$. Several integration formulas there can be viewed as localization formulas for families.

In section 10, we discuss the relations between $T$-spaces and $G$-spaces, characters of $T$-modules and $G$-modules. Several consequences of the main theorem are proved.

Section 11 proves the main cancellation which has a couple of consequences discussed in Section 12, including the ‘twin pair’. The proof of the main result is completed in the last section after we find an expression for the remainder term.

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2. Basic properties of the variety $X_N$

Many properties presented here were proved in [C1], with the exception of the existence of a $LG$-invariant almost complex structures on $X$, and a $T$-invariant one on $X_N$. In [C1], $X$ is assumed to be holomorphic, therefore it is not necessary. We list the important properties for $lg, X, X_N$ here.

2.1. Basics of affine Lie algebra. Let $g$ be the Lie algebra of $G$ and $T$ be a maximal torus with $t$ as its Lie algebra. The discussion works without much modification for semi-simple Lie groups. Let $LG$ be the loop group associated with $G$, and $lg = \text{Lie} LG$.

For functions on $S^1$, we use the $t \in [0, 1]$ to parameterize them. The Fourier series components are $\{e^{2\pi int}\}$. On $lg$, there is the following well defined form in terms of the invariant non-degenerate form $(\cdot | \cdot)$ on $g$:

$$(a|b) = \int_0^1 (a(t)|b(t))dt \in \mathbb{R}.$$  

It induces a symplectic form on $lg/g$:

$$B(a, b) = (a'|b) = \int_0^1 (a'(t)|b(t))dt$$

where $a' = da/dt, t \in [0, 1]$. The form is degenerate when restricted to the constant $g$. We choose the form on $g$ with the condition that $(\theta^\ast | \theta^\ast) = 2$ or equivalently $(\theta | \theta) = 2$, where $\theta^\ast$ is the coroot corresponding to the highest root $\theta$ of $g$. As pointed out in [PS, p. 46], the associated form $B$ defines the smallest integral class on $LG$.

The affine Lie algebra based on $g$, $g^{aff}$, is defined as

$$g^{aff} = lg \oplus \mathbb{R}d \oplus \mathbb{R}K,$$

where $K$ is the central element and $d = d/dt$ is the differentiation. The Lie bracket is

$$[\xi + \lambda d + cK, \eta + \chi d + c'K] = [\xi, \eta] + \lambda d\eta - \chi d\xi + B(\xi, \eta)K.$$

The central extension of $lg$, $\tilde{lg}$, is given by $g^{aff} = lg \oplus \mathbb{R}K$. It is the Lie algebra of $\tilde{LG}$ which is a circle bundle over $LG$, whose existence of $\tilde{LG}$ is proved in [PS].

The Lie algebra dual $g^{aff*}$ is given by

$$g^{aff*} = lg^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0.$$  

The bilinear form $(\cdot | \cdot)$ extends to $g^{aff}, g^{aff*}$, so that it is invariant. Its restriction to the 2-dim subspace $\mathbb{R}d \oplus \mathbb{R}K$, is of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. See [K, Ch. 6] for more details. Continue to use $\nu$ for the map $g^{aff} \rightarrow g^{aff*}$ defined by $(\cdot | \cdot)$. The bilinear form gives

$$\nu(lg \oplus \mathbb{R}d) = lg^* \oplus \mathbb{R}\Lambda_0.$$  

The simple roots of $g^{aff}$ consists of simple roots of $g$ together with $\alpha_0 := \delta - \theta$. Suppose that $\{E_i', F_i'\}$ are the Chevalley basis of $g$, let $E_0', F_0' \in g_0^{aff}$, so that

$$[E_0', F_0'] = -\theta^\ast,$$

where $-E_0$ is the Chevalley involution of $F_0'$, define

$$e_0 = z \otimes E_0', \quad f_0 = z^{-1} \otimes F_0'.$$
For $su(2)$, the pair is
\[
\begin{pmatrix}
0 & z \\
0 & 0 \\
z^{-1} & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix}.
\]
The pair $c_0, f_0$ together with $e_i = E_i', f_i = F_i'$, $1 \leq i \leq l$ generate $l g^C$.

The positive affine roots $\gamma \in \Delta_+(g^{aff})$ are
\[
\{n\delta + \alpha | n \geq 1, \alpha \in \Delta_+(g) \} \cup \Delta_+(g) \cup \{n\delta | n \geq 1\}
\]
and accordingly the basis of the root spaces are $E_\gamma = z^{n}E'_\pm, F'_\pm$ or $z^{n}h_\alpha$, where $\{E'_\alpha\}$ is the root space basis of $g$. The basis of the compact form are
\[
\{x_\gamma = e_\gamma - f_\gamma, y_\gamma = i[e_\gamma + f_\gamma]\}
\]
where $\gamma > 0$. The standard complex structure $J^\theta$ on $l g/t$ which inherits from the map $n^+ \to l g/t \simeq l g^C/(n^- + t^C)$ has this description:
\[
J^\theta(x_\gamma) = y_\gamma, \quad J^\theta(y_\gamma) = -x_\gamma.
\]
By direct computation, for $h \in t$, we obtain
\[
x'_\gamma = 2\pi ny_\gamma, \quad y'_\gamma = -2\pi nx_\gamma;
\]
\[
[h, x_\gamma] = (-1)^{\text{sign}(\alpha)}\alpha(h)/iy_\gamma = (-1)^{\text{sign}(\alpha)}i\alpha(h)y_\gamma;
\]
\[
[h, y_\gamma] = -(-1)^{\text{sign}(\alpha)}\alpha(h)/ix_\gamma = (-1)^{\text{sign}(\alpha)}i\alpha(h)x_\gamma.
\]
Hence for $\xi = \sum ax_\gamma + by_\gamma$,
\[
(J^\theta)^\prime [h, J^\theta] = -\sum (2\pi n - (-1)^{\text{sign}(\alpha)}i\alpha(h))(ax_\gamma + by_\gamma);
\]
\[
((J^\theta)^\prime [h, J^\theta])(\xi) = -\sum (2\pi n + (-1)^{\text{sign}(\alpha)}\alpha(h)/i)(|a|^2|x_\gamma|^2 + |b|^2|y_\gamma|^2).
\]
Now suppose that $\frac{1}{2\pi i}h$ is in the affine alcove $C$, then
\[
n + (-1)^{\text{sign}(\alpha)}\frac{1}{2\pi i}\alpha(h) = <n\delta + \alpha, \frac{d + h}{2\pi i}> \geq 0,
\]
because all the positive roots are positive on $C + \frac{d}{2\pi i}$, (our definition of $d$ differs from that in [K], hence the extra constant). Therefore $((J^\theta)^\prime [h, J^\theta])(\xi) \leq 0$. Let $\Omega(\xi, \eta) = (\xi' + [h, \xi]|\eta)$, earlier calculation shows that

**Lemma 2.1.** The form $\Omega(\cdot, J\cdot)$ is semi-positive definite.

### 2.2. Adjoint and co-adjoint action

Let $\xi, \eta \in l g$, then $[\xi + ad, \eta] = [\xi, \eta] + ad\eta$ which integrates to
\[
g^{-1}(\xi + ad)g = Ad_{g^{-1}} \xi + ag^{-1}dg + ad \in g^{aff}.
\]
The linear map $\nu : g^{aff} \to g^{aff}$ defined by $(-\cdot)$ has the effect that $\nu(\xi + ad) = \nu(\xi) + a\Lambda_0$, therefore the adjoint action by $g$ is given by

**Definition 2.1.** The adjoint action on $h + a\Lambda_0$ is
\[
ghg^{-1} + ag \frac{d}{dt}g^{-1} + ad.
\]

The level of $h + a\Lambda_0$ is $a$, thus the adjoint action preserves the level. In case of level 1, the projection to the $l g$ part is exactly the gauge transformation.
2.3. Topology on $LG$. Since the group $LG$ we are dealing with is the mapping space $\text{Map}(S^1, G)$, there is the question as to which norm is used for completion. The norms defined by $B(\cdot, J^\theta \cdot), \Omega(\cdot, J^\theta \cdot)$ are not strong enough. The weakest norm which is sensible geometrically, is the $H^1$-norm. We can use other Hilbert metrics or Banach metrics as long as they are stronger than $H^1$-norm.

**Convention:** The convention here is to let $l_g = H(S^1, g)$ be the space of maps completed under the norm of choice, and $LG$ be the corresponding group. Set $l_g^0$ to be the set with one degree less of derivative, i.e.,

$$l_g^0 = H'(S^1, g) = \{h' | h \in l_g\} \oplus g.$$  

It is another completion of $C^\infty(S^1, g)$, so that the map

$$h \in l_g \mapsto h' \in l_g^0$$

is Fredholm and bounded.

We will simply use $l_g$ for $l_g^0$, just keep on mind that the target of $\mu$ consists of elements with one degree less of differentiability.

2.4. Loop group actions and the assumptions $H1$, $H2$. Let $X$ be a Banach manifold with a differentiable action by $LG$,

$$\mu : X \rightarrow l_g \oplus \mathbb{R} \simeq l_g^* \oplus \mathbb{R}A_0$$

be a moment map associated with a symplectic 2-form $\omega$ on $X$. The isomorphism $\simeq$ is defined by the restriction of $\nu$. The moment map is equivariant with respect to the $LG$-action on $X$ and the adjoint action on $l_g^* \oplus \mathbb{R}A_0$.

**Remark:** The Banach norm on $TX$ does not have to be invariant; and the positive definite form $\omega(\cdot, J\cdot)$, in general, defines a topology weaker than the Banach norm, for any compatible almost complex structure on the tangent space.

**Definition 2.2.** $\mu$ is of level $k \in \mathbb{Z}_+$ if $\mu(X) \subset l_g \times \{k\}$.

**Remark:** The topology on $l_g \oplus \mathbb{R}$ as described makes the co-adjoint action a bounded map.

The following assumptions will be made:

**H1:** $\mu$ is proper with aforementioned topology.

**H2:** $\mu$ is transversal to $t \times \{k\}$ in $l_g \times \{k\}$.

The first one is essential and is equivalent to the compactness of $X/LG$, and the second is technical.

Assuming **H1** and **H2**, then $X_1 = \mu^{-1}(t \times \{k\})$ is a finite dimensional submanifold. It is not symplectic, $\omega|X_1$ has serious degeneracy. And it may not even be orientable. Whenever the stabilizer of $\mu(p)/lp$ in $l_g$, $(l_g)_{\mu(p)}$, has a semi-simple part, $\omega|_{T_pX_1}$ is null on $(l_g)_{\mu(p)}/T$.

2.5. Toric bundle $X$ over $LG/T$. It is a fundamental fact that the affine Weyl group $W^{aff}$ acts on $t \times \{k\}$, the quotient domain of the action is given by a simplex $k(C, 1)$ where $C$ is the affine alcove of $g$. One can also consider the action by $W^{aff}$ on the dual space $t^* \times \{k\}$, the quotient domain is $kC^*$ with $C^*$ given by

$$C^* = \{\lambda | (\alpha(\lambda) \geq 0, (\theta(\lambda) \leq 1) = \{\lambda | <\alpha^\vee, \lambda > \geq 0, <\theta^\vee, \lambda > \leq 1\}$$

where $\alpha^\vee, \theta^\vee$ are the coroots. The above descriptions of $C, C^*$ are the duals of each other, through the map $\nu : t \rightarrow t^*$.

When there is no confusion, we will not distinguish between $C, C^*$.
The simplex $C^*$ is not simple with respect to the weight lattice, the edges do not form a base of the weight lattice of $t$. In fact the edges are given by

$$\Lambda_i/a_i^*, \quad i = 1, ..., l$$

where $l$ is the rank of $g$, $\{\Lambda_i\}$ is the set of fundamental weights of $g$ and $\{a_i^*\}$ are the labels in the dual Dynkin diagram. More on this can be found in [C1].

From the theory of toric varieties, we know that there is an orbifold toric variety $X_g$ and an orbifold line bundle $L_N$ associated with $C$. The pair is the quotient of $\mathbb{C}P^l, H$ by a finite group, where $H$ is the hyperplane line bundle. The details are in [C1].

Given $X_g$, we can associate with it a toric bundle over $LG/T$:

$$X = LG \times_T X_g, \quad (gt, z) \simeq (g, tz).$$

The quotient is well defined since $T$ is compact and the action is free of fixed points. The construction of $X$ given here dated back to 1993.

There are many nice characteristics about $X$, we list a few needed later.

On $X$, there is an action by $LG$ and $T$ respectively, the two actions commute. The actions by $LG, T$ satisfy the Hamiltonian conditions with moment maps given by

$$\mu_X : X \to \mathfrak{g}^0 \times \{1\}, \quad \phi : X \to \mathfrak{t}.$$

Let $\tilde{\phi} = (\phi, 1) \in \mathfrak{t} \times \{1\}$. Then

$$\mu_X([g, z]) = \tilde{\text{Ad}}_g(\tilde{\phi}(z))$$

that the above is independent of the choice of $(g, z)$ in $[g, z]$ is evident. Let $T_{[t, z]} = \mathfrak{g}/t \oplus T_z X_g$. The 2-form on $X$ can be described as

$$\omega_{X_{[t, z]}((\xi, a), (\eta, b))} = (\xi + [\phi(z), \xi]|\eta) + \omega_{X_g}(a, b) \quad a, b \in T_z X_g, \quad \xi, \eta \in \mathfrak{g}/t.$$

The form is degenerate whenever $\phi(z)$ hits the boundary of the affine alcove $C$. The null space is generated by $(LG)_{\tilde{\phi}}/T$ where $(LG)_{\tilde{\phi}}$ is the stabilizer of $(\phi, 1)$. The complex structure $J$ is defined as: $J|_{\mathfrak{g}/t} = J|_{\mathfrak{g}}$ while on $T_z X_g$, it is given by the original one on $T_z X_g$. The choice of this complex structure is due to the following reasons:

1). The action by $t \in T$ on the left is

$$t(g, z) = (tg, z) \simeq (tg^{-1}, tz).$$

2). We want highest weight modules, rather than lowest weight ones.

### 2.6. The variety $X_N$

Reverse the complex structure on $X$, so that $-\mu$ is the moment map, and $-\omega$ is compactible with the complex structure $-J$.

The variety $X_N$ which is important to our study can now be described as follows:

$$X_N = (\Psi^{-1}(0)/LG = \{(p, q)|\mu(p) = k\mu_X(q)\}/LG,$$

where $\Psi = -\mu + k\mu_X : X \times X \to \mathfrak{g}^0$ is the moment map associated with the diagonal action by $LG$ on the product space, with the 2-form $-\omega + \omega_X$.

The sign here is chosen so that no inversion of the complex structure on $X$ is necessary, this way we still get the highest vectors in the end.

Notice the level of $\Psi$ is 0. Set

$$Y_C = \mu^{-1}(kC \times \{k\}),$$
by H1,H2 it is a compact manifold with boundary.

The following shows that the toric variety can be used to close the gash which is the boundary of \( Y_C \).

**Proposition 2.1.** The space \( X_N \) is

\[
\{(p, q) \in Y_v \times X_\mathfrak{g} | \mu(p) = k\phi(q)\} / T.
\]

The proof is simple, since each pair \((p, q)\) with \( \Psi(p, q) = 0 \) can be conjugated to \((p', q') \in Y_v \times_{T} X_\mathfrak{g}\) with \( \mu(p') = k\mu_X(q') \in k(C, 1) \); the pair is unique up to \( T \).

From this description, considering the assumptions H1, H2, it is clear that \( X_N \) is a compact orbifold. The claim that \( X_N \) is holomorphic whenever \( X \) is complex is of more subtle nature, it is proved in [C1].

Because \( T, LG \) commute on \( X \), the action by \( T \) descends from the product \( X \times X \) to \( X_N \). So does the moment map \( \phi \). The form \( \omega - \omega_X \) when restricted to \( \Psi^{-1}(0) \) is invariant under the \( LG \)-action, thus it descends down to a form on \( X_N \). Denote it by \( \omega_N \). The pair \( \phi, \omega_N \) satisfy the conditions for Hamiltonian action, though \( \omega_N \) is degenerate.

When \( X \) is not complex, we will see there is an \( T \)-invariant almost complex structure \( J \) on \( X_N \), such that \( \omega_N(\cdot, J\cdot) \) is semi-positive definite. If \( \omega_N \) on \( X_N \) is symplectic, the existence of such a \( J \) is well known. For degenerate \( \omega_N \), the existence is not automatic.

2.7. **The existence of an LG-invariant \( J \) on \( X \).** The existence result only needs assuming H1.

**Step 1:** Existence of a positive bilinear \( LG \)-invariant form on \( X \).

Let \( \{v\} \) be the vertices of \( k(C, 1) \). There are \( l + 1 \) of them where \( l \) is the rank of \( \mathfrak{g} \). For each \( v \), let \( C^v \) be \( k(C, 1) \) after removing the face opposite to \( v \). Let \( W_v \) be the Weyl subgroup in \( W^{aff} \) generated by reflections with respect to walls passing \( v \).

It is well known that \( W_v \) is finite and is the Weyl group of \( (LG)_v \) which stabilizes \( v \) under the co-adjoint action. Now set

\[
O_v = \bigcup_{w \in W_v} wC^v,
\]

which is an open set. It is the star-shaped region with center \( v \) if we view the images of \( C \) under \( W^{aff} \) as a triangulation of \( t \times \{k\} \). E.g. when \( \mathfrak{g} = su(2), C = [0, 1] \) while \( O_0 = (-1, 1), O_1 = (0, 2) \).

Clearly \( \cup_v O_v \) is an open cover of \( k(C, 1) \), and there exists a partition of unity \( \{\psi_v\} \) subordinate to the covering, and \( \psi_v \) is invariant under \( W_v \). Define

\[
S_v = \mu^{-1}(\hat{\text{Ad}}_{(LG)_v}(O_v)); \quad S_a = \mu^{-1}(\hat{\text{Ad}}_{LG}O_a).
\]

Using the conditions on \( \mu \), we have:

\[
S_a = LG(S_a) = LG \times_{(LG)_a} (LG)_v(S_v);
\]

and \( \{S_a\} \) is a \( LG \)-invariant open cover of \( X \). Let \( g_a \) be an \( (LG)_a \)-invariant Riemannian metric on \( S_a \). It exists, since \( (LG)_a \) is compact and \( S_a \) is of finite dimension.

Next we define a positive bilinear \( LG \)-invariant form \( g_a^A \) on \( S_a \). Clearly

\[
T_pS_a = l\mathfrak{g} / (l\mathfrak{g})_a \oplus T_pS_a,
\]

on the second factor there is already a Riemannian metric \( g_a \). On the first one, the choice for \( J \) is \( J^\mathfrak{g} \). With this choice, we have the positivity of \( \Omega(\xi, J\xi) \) as shown earlier.
The form $\langle \cdot, \cdot \rangle$ is the restriction of the bilinear invariant form on $\mathfrak{g}^{aff} = \mathfrak{g} + \mathbb{R} \delta + \mathbb{R} d$, (see [K, Ch. 7]). And $[J \eta, x] = J[\eta, x]$ which can be verified using the property that if

$$[E_\alpha, E_\beta] = c_{\alpha, \beta} E_{\alpha + \beta},$$

then

$$[E_{-\alpha}, E_{-\beta}] = -c_{\alpha, \beta} E_{-\alpha - \beta}.$$

Thus $g_a^A$ is invariant under the conjugation on $\mathfrak{g}/(\mathfrak{g})_a$ by $(LG)_\mu$.

**Lemma 2.2.** *The decomposition*

$$TPS_a = \mathfrak{g}/(\mathfrak{g})_a \oplus TS_a$$

*is orthogonal with respect to $g_a^A, \omega_X$.*

*Pf:* It is orthogonal with respect to $g_a^A$ by construction. To check that for $\omega_X$, let $u \in TP_aS_a$ and $\xi \in \mathfrak{g}/(\mathfrak{g})_a$, by the definition of moment map,

$$\omega_X(u, \xi) = (D_a \mu(\xi)) = 0$$

where the last equality holds because $D_a \mu \in (\mathfrak{g})_a \perp \mathfrak{g}/(\mathfrak{g})_a$ under $\langle \cdot, \cdot \rangle$. QED

We can use the group action by $LG$ to extend $g_a^A$ to a $LG$-invariant form denoted by the same on $S_a$. This is made possible due to

1. the invariance of $g_a$ under $(LG)_a$ on $S_a$;
2. the complement of $TP_aS_a$ in $TPS_a$ is $\mathfrak{g}/(\mathfrak{g})_a$ whose positive bilinear form as in Eq. (??) is invariant under the conjugation.

Now for each $p$ in $S_a$, the image under $\mu$ of the orbit $LG(p)$ meets $O_a$ in a $W_a$-orbit on which $\psi_a$ is constant, since $\psi_a$ is $W_a$-invariant. Therefore, we can extend $\psi_a$ to $S_a$. The extension will still be denoted by $\psi_a$. Obviously, $\{\psi_a\}$ forms a partition of unity on $X$ subordinate to $\{S_a\}$.

Now clearly $g = \sum_i \psi_a g_a^A$ is an invariant positive bilinear form.

*Remark:* We do not call this form a Riemannian metric in order to avoid confusion, since the topology defined by $g$ is weaker than that on $X$.

*Step 2: Existence of an LG-invariant almost complex structure.*

On finite dimensional space $M$, given an invariant symplectic form $\omega$ and a positive definite form $g$, there is an uniquely well defined almost complex structure. The exact construction of $J$ in terms of $g, \omega$ is like this: the non-degenerate 2-form

$$\omega : TM \to T^*M,$$

the form $g$ defines a map from $T^*M \to TM$ which is the inverse of the map $\nu : TM \to T^*M$ defined by the metric.

Denote $\nu^{-1} \cdot \omega$ by $f : TM \to TM$. Then it is straightforward that $f^* = -f$, the dual is taken with respect to $g$. Therefore, $-ff = (f^*f)$ is positive definite and has an uniquely defined positive definite square root, $(-f^2)^{1/2}$ which is a linear map. Clearly $f, (-f^2)^{1/2}$ commute, so the following is the desired $J$,

$$J = f(-f^2)^{-1/2}.$$

This almost complex structure is clearly invariant.

*Two remarks:* 1). The positive form $\omega(\cdot, J \cdot)$ may not coincide with $g$; 2). If $g$ is given by $\omega(\cdot, J \cdot)$, then $J = J$.

Both of the above are easy to verify.
In the infinite dimensional situation, we refrain from defining $J$ on $TX$ by directly using the above because the definition of $f^* f$ may cause problem. In the particular situation we are facing, however, the map $f^* f$ will be shown to be $I$ except on a finite dimensional space. Let us examine the matter more closely.

Suppose $\mu = \mu(p) \in k(C, 1)$ is covered by $\{O_b\}$ but no other $O_a$. By definition of $O_a$, it is clear that the stabilizer of $\mu$ in $W^{aff}$, $W_\mu$, is

$$W_\mu = \cap_{\{b|O_b \ni \mu\}}W_b.$$  

When $\mu$ is in the interior, $W_\mu = \{I\}$ and all of $\{O_a\}$ cover it; when $\mu = v$ is a vertex, only $O_v$ covers it. Also $\mu$ is not in the support of $\psi_v$ if $\mu \notin O_a$. Therefore at $p$, the positive form $g = \sum_{\{b|O_b \ni \mu\}} \psi_b g_b^A$.  

On $T_p S_b = l g/(l g)_b \oplus T_p S_b$, $g_b^A$ agrees with the form

$$([\cdot, J \cdot]|\mu(p))$$  

(2.5) on the first factor, for each $b$ with $p \in O_b$. Therefore inside that tangent subspace, $g$ agrees with the expression in Eq. (2.5) on $\cap_{p \in O_b} l g/(l g)_b$.  

By the previous lemma, the finite dimensional subspace $E_p$ generated by $\{T_p S_b|\mu(p) \in O_b\}$ is orthogonal to $\cap_{\{b|O_b \ni \mu(p)\}} l g/(l g)_b$ with respect to $\omega, g$. Hence, $\omega$ and $g$ are in diagonal form in the decomposition

$$T_p X = \cap_{\{b|O_b \ni \mu(p)\}} l g/(l g)_b \oplus E_p.$$  

Thus the map $f = \nu^{-1} \omega$ is of diagonal form. On the first factor, again by the previous lemma and the definition of $g_b^A$,  

$$g = ([\cdot, J \cdot]|\mu),$$  

therefore over that subspace

$$f = J = J^g, \quad f(-f^2)^{-1/2} = f = J^g.$$  

On the finite dimensional subspace $E_p$, $J$ is uniquely defined by the restriction of $f$. Hence it is invariant by $(LG)\mu$. We can easily extend $J$ to the whole $X$ through invariance. Thus we have just proved the following

**Proposition 2.2.** There is a $LG$-invariant almost complex structure on $X$ such that $\omega(\cdot, J \cdot) = -(Df, \mu|\cdot\cdot)$ is positive definite.  

**Remark:** If $X$ is complex, the above equality $J = -J^g$ on $\cap_{\{b|O_b \ni \mu(p)\}} l g/(l g)_b$ is not true in general.

### 2.8. The almost complex structure $J$ on $X_N$.

We will prove the existence of an almost complex structure on $X_N$ by showing that there exists Hodge type decomposition for $T_{(p,q)} X \times \mathbb{X}$ where $\mu(p) = k \mu_X(q)$. Such a decomposition is used in the finite dimensional situation to show the existence of almost complex structure (or holomorphic structure if the original manifold is) on the reduced space.

Suppose $\mu(p) = k \mu_X(q) \in k(C, 1)$. It will be shown in the next section, as a consequence of $H2$, that $t_p \cap t_q = 0$ which implies $(l g)_p \cap (l g)_q = 0$, since $(l g)_q = t_q$.  

$$
\text{FIXED POINT FORMULAS AND LOOP GROUP ACTIONS 13}
$$
The following is easier to show than in the case when $X$ is holomorphic, for the almost complex structure $J^X$ constructed earlier has very special property.

From now on, let $X$ be equipped with the opposite of its usual complex structure. And continue to use the same one on $X$ so that $-\omega(\cdot, J\cdot) + \omega_X(\cdot, J\cdot)$ is non-negative definite.

**Proposition 2.3.** The action by $l_g$ on the product $T_{(p,q)}X \times X, \Psi(p,q) = -\mu(p) + k \mu_X(q) = 0$ has no kernel. There is a $L_g$-invariant (acting diagonally) decomposition of the tangent space:

$$T_{(p,q)}X \times X = V_{(p,q)} \oplus l_g(p,q) \oplus Jl_g(p,q),$$

where $l_g(p,q)$ denotes the induced tangent vectors on the product, $J$ is on the product and

$$V_{(p,q)} = \{(u,v) | D_{(u,v)}\Psi = D_{J(u,v)}\Psi = 0\}.$$

The decomposition is orthogonal with respect to

$$h(\cdot, \cdot) = -\omega(\cdot, J\cdot) + \omega_X(\cdot, J\cdot)$$
on the product space.

Remark: The above decomposition would have been obvious had $h$ defined a complete norm.

*Proof:* 1. Claim $D_{Jl_g}\Psi$ is onto $l_g^0$.

In the same notations as before, there is a decomposition of $T_pX$ as $\cap_b l_g/(l_g)_b \oplus E_p$ where $E_p$ is of finite dimension. By construction, $J^X = Jl_g$ and $J^X = -Jl_g$. Therefore, for $\xi \in \cap_b l_g/(l_g)_b$,

$$D_{J\xi}\Psi = -D_{J^X\xi}\mu(p) + D_{J\xi}\mu_X(q) = -2D_{J\xi}\mu(p) = 2\tilde{\text{ad}}_\mu(p)Jl_g\xi \in \cap_b l_g/(l_g)_b,$$

where one degree of differentiability is lost due to the adjoint action. Because $Jl_g$ preserves $l_g$ and $\tilde{\text{ad}}_\mu(p)(\cdot)$ with $\mu(p) \in k(C, 1)$ is an isomorphism on $\cap (O_b | \mu(p) \in O_b)/l_g/(l_g)_b$, the above implies that $D\Psi$ is onto $\cap_b l_g^0/(l_g)_b$. The orthogonal complement of $l_g^0/(l_g)_b$ is the subspace generated by $(l_g)_b$ such that $\mu \in O_b$. If $D\Psi$ is not onto that finite dimensional complement, there is a $\eta$, with

$$(D\Psi|\eta) = 0.$$

In particular,

$$0 = D_{J\eta}\Psi(\eta) = -\omega(J\eta, \eta) + \omega_X(J^X\eta, \eta),$$

which forces $\eta(p) = 0$ and $\eta(q)$ in the null direction of $\omega_X(\cdot, J^X\cdot)$. The analysis on the null space of $\omega_X(\cdot, J^X\cdot)$ shows that $\eta \in (l_g)_\mu^0$, where $\mu = \mu(p) = k\mu_X(q)$. But

**H2** is equivalent to

$$(l_g)_p \cap (l_g)_\mu = 0,$$
a contradiction.

2). Given a tangent vector in the product, $(u,v)$, from the claim there exists $\xi \in l_g$ such that $D_{J\xi}\Psi = D_{J(u,v)}\Psi$. And choose $\eta \in l_g$ to satisfy $D_{J\eta}\Psi = D_{(u,v)}\Psi$, then

$$(u', v') = (u,v) - J\eta(p,q) - \xi(p,q)$$

satisfies

$$D_{(u', v')}\Psi = 0; \quad D_{J(u', v')}\Psi = 0$$

because $D_{(u', v')} = -\tilde{\text{ad}}_\tau + \tilde{\text{ad}}_\tau^* \mu(q) = 0, \forall \tau \in l_g$. Therefore

$$(u', v') \in \ker D\Psi \cap \ker D\Psi = V_p.$$
3). The isomorphism between $V_p$ and the tangent space to the orbit $\ker D\Psi / l_g$ holds due to the claim in Step 1. Since $V_p$ is invariant under $J$, there exists $J$ on $TX_N$. QED
3. The Fixed Point Set on $X_N$

3.1. When a point is fixed by $T$. Let $u \in X_N$, then $u = [p, q]$ with $(p, q) \in X \times X$, the bracket denotes the equivalence class under the diagonal action by $LG$.

The point $q$ being in $X$ is itself an equivalent class, $q = [h, z]$ with $h \in LG$, $z \in X_g$. The equivalent class represented by $q$ is the orbit by the diagonal $T$ action on $LG \times X_g$. Here the $T$ action is from the right on $LG$.

Both equivalence relations can be kept tracked of by introducing the following:

**Definition 3.1.** For points in $X \times LG \times X_g$, define the equivalence relation:

\[(p, h, z) \simeq (gp, gh, sz)\]

for $g \in LG, s \in T$.

A triple defines a point in $X_N$ iff $\mu_X(p) = k\tilde{\text{Ad}}_g(\tilde{\phi}(z))$ where $\tilde{\phi} = (\phi, 1)$ and $\tilde{\text{Ad}}$ is the adjoint action.

On $X \times LG \times X_g$, the diagonal action by $LG$ on the first two factors and the diagonal $T$-action on the last two factors, commute with $T$ acting on the last one. And the $T$-action acting on $X_g$ alone preserves the moment maps $\mu - \Phi$, therefore it descends to one on $X_N$.

We make a semi-canonical choice for $[p, q]$, so that $q = [I, z]$. Because $\phi = \Psi(u) = \mu_X(p) - k\Phi(q)$, we have $\mu(p) \in k(t, 1)$. The ambiguity is due to the following:

\[(p, I, z) \simeq (s^{-1}p, I, sz)\]

as in Definition 3.1. Thus $(s^{-1}p, I, sz), s \in T$ represents the same point as $(p, I, z)$ in $X_N$.

**Lemma 3.1.** Let $t_z$ be the Lie algebra of the stabilizer $T_z \subset T$ of $z \in X_g$, $t_p$ be that for the point $p$ in $X$. Then a point $u = [p, I, z]$ is fixed by $T$, iff that $t = t_z + t_p$.

**Pf:** The condition is clearly sufficient. Suppose $u$ is fixed by $g$, i.e. $(p, I, gz) \simeq (p, I, z)$, that is for some $h \in T$,

\[(p, I, gz) = (h^{-1}p, I, hz).\]

Thus

\[hp = p, \quad h^{-1}gz = z\]

for $h \in T$. So $h \in T_p$ and $h^{-1}g \in T_z$, and $g = h \cdot h^{-1}g \in$. Therefore, any $g \in T$ can be decomposed into a product of elements in $T_p, T_z$, hence $t = t_z + t_p$. QED

Assume $(p, I, z)$ defines a point in $X_N$, then we have $\mu(p) = k\tilde{\phi}(z) = k(\phi(z), 1)$ by the definition of $X_N$.

**Lemma 3.2.** The following holds:

1. $t_z = (lg)^{\oplus 2} \cap t$.

2. Under the assumption that the image of $\mu_X$ is transversal to $(t, k)$ in $(lg, k)$, $t_z \cap t_p = 0$.

**Pf:** Let $C_\mu$ be the smallest wall of $\partial C$ containing $\phi(z)$, and let $V_\mu$ be the linear subspace in $t$ parallel to $C_\mu$. Then $V_\mu^\perp = t_z$, which is a basic fact in toric variety theory, or symplectic geometry.

The affine Lie algebra based on $g$, $g^{\text{aff}}$, is $lg + R\delta + R\Lambda_0$ as in [K]. The dual is $lg + R\delta + R\Lambda_0$. The simple roots are $\{\alpha_0, \alpha_1, ..., \alpha_l\}$ where $\{\alpha_1, ..., \alpha_l\}$ is the set of simple roots of $g$, and $\alpha_0 = \delta - \theta$ which acts on $(h, l) \in t + R\Lambda_0 \subset lg + R\delta + R\Lambda_0$ as $l - \alpha(h)$. The boundary of the alcove $C$ is defined by $\cap \alpha_i^{-1}(0)$. 

The stabilizer of $\tilde{\phi}$ is the same as that of $C_\mu$. And the stabilizer $(LG)_\mu$ of $C_\mu$ in $LG$ is generated by $t$ and $t_\mu(g)$, such that $\alpha_t(C_\mu) = 0$. Clearly $t_\mu(g) = \sum_{\alpha_t(C_\mu) = 0} t_{\alpha_t}$. We claim that

$$\langle \xi | \tilde{\phi} \rangle = 0, \quad \forall \xi \in (t_\mu)_{ss}.$$ 

This is clearly true for $x_\alpha, y_\alpha$ in $t_\mu(g)$, using the standard notation for $t_\mu(g) \simeq \text{sl}_2(\mathbb{C})$. As for the coroot $\alpha^\vee = [x_\alpha, y_\alpha]$, it holds as well because $\alpha(\mu) = 0$ and

$$(\alpha^\vee | \mu(p)) = (\alpha | [y_\alpha, \mu]) = \alpha(\tilde{\phi})(x_\alpha | x_\alpha) = 0.$$

Thus we have $t_\mu(g)_{ss} \cap t = \sum_{\alpha_t(C_\mu) = 0} \mathbb{R} \alpha^\vee$, the right hand side are in $t_z$ from the orthogonal condition just proved. We also know that the $V_\mu^+$ is exactly spanned by those $\alpha^\vee$, since $C_\mu$ as a subface is defined this way. Hence $t_z = t_\mu(g)_{ss} \cap t$.

2). The transversality condition is equivalent to

$$t_\mu(g)_{ss} \cap [t_\mu(g), t_\mu(g)] = \{0\},$$

but $t_\mu(g)_{ss} = [t_\mu(g), t_\mu(g)]$, and $t_\mu \subset t_\mu(g)$, therefore we have $t_\mu \cap [t_\mu(g), t_\mu(g)] = \{0\}$ which implies that $t_\mu \subset t_\mu(g)$, QED

Let $T_{p}^0$ be the connected component of $I$ in $T_p$. By the construction of the toric variety $X_p$, it is easy to see that $T_z$ is connected. If $t_p \oplus t_z = t$, naturally $(t_p, t_z) \in T_{p}^0 \times T_z \mapsto t_p t_z \in T$ is a covering map.

**Proposition 3.1.** Let $\tilde{F}_p$ be the connected component of $T_{p}^0$-fixed point set in $X$ containing $p$, and $M_z$ be the connected component of $T_z$-fixed point set in $X_g$ containing $z$.

Then the connected component containing $(p, I, z)$ of $T$-fixed point set in $X_N$ is given by $(\tilde{F}_p \times \{I\} \times M_z) \cap \Psi^{-1}(0)$.

**Pf:** The inclusion of the set in the connected component of the fixed point set by $T$ is clear. To see the other direction, suppose $(q, I, w)$ is in the connected component of $(p, I, z)$ which is fixed by $T$, and suppose $q$ is close to $p$, $w$ is close to $z$. The stabilizers in $T$ of $q, w$ must be subgroups of $T_{p}^0, T_z$, which is well known. On the other hand, we have shown $t_q + t_w = t, t_q \cap t_w = 0$. Therefore the inclusion of $t_q \subset t_p, t_w \subset t_z$ are equalities instead. Thus, $t_q, t_w$ vanish at $q, w$ respectively.

If $(q, I, w)$ is connected in the fixed point set to $(p, I, z)$ through a 1-parameter curve $(q_s, I, w_s)$, with $q_0 = p, w_0 = z$ and $q_1 = q, w_1 = z$, then for small $s$, by the above argument, $q_s, w_s$ are fixed by $t_p, t_w$. So they are in the connected components of $p, z$ fixed by $T_p, T_z$ respectively. The above argument shows also the set $\{s\}$ such that $(q_s, I, w_s)$ is in the desired product is open. Clearly the set is also closed. Therefore, $(q, I, w)$ is in the product. QED

3.2. More about the $T$-fixed point set on $X_N$. From earlier discussion, we have learned that $[p, I, z]$ is a fixed point of $T$ on $X_N$ iff $t_p \oplus t_z = t$. Let $T_{p}^0$ denote the connected component of $T_p$. To understand the structure of the fixed point set, we need the following:

**Definition 3.2.** Suppose $t_z \neq 0$, i.e. $\phi(z) \in \partial C$, let

$$K = \{g \in (LG)_\mu \mid \text{Ad}_g t = t, \forall t \in T_{p}^0\}, \quad N = \{g \in (LG)_\mu \mid \text{Ad}_g(T_{p}^0) = T_{p}^0\},$$

and Lie$K$, Lie$N$ be their Lie algebras.
Lemma 3.3. The groups \((LG)_\mu\) and \(K\) are compact and connected.

Suppose \((\mathfrak{g})^\mu_\mathfrak{g} \cap \mathfrak{t}_p = 0\) which is true under the assumption that the image of \(\mu\) is transversal to \(\mathfrak{t}\), then \(\text{Lie}N = \text{Lie}K\).

The weights of the \(t_p\) action on \((\mathfrak{g})_\mu/\text{Lie}K\) are non-trivial, therefore \(K\) is the largest connected group acting on \(\tilde{F}_p\).

\(\text{Pf}\): The assertion on \((LG)_\mu\) is well known. For \(K\), the argument is also standard but we include here anyway. Suppose \(K^0\) is the connected component passing \(I\). Let \(g \in K \setminus K^0\), then \(\text{Ad}_g K^0 = K^0\). The group \(T\) is a maximal torus in \((LG)_\mu\) and is contained in \(K^0\). Therefore \(\text{Ad}_g T\) is a maximal torus. By the uniqueness of maximal torus under the adjoint action, there is a \(h \in K^0\) such that \(\text{Ad}_h g T = T\). Obviously \(hg \in W(K^0)\), so \(hg\) is contained in the semi-simple part of \(K^0\). Thus \(g\) is connected to \(I\) and \(K = K^0\).

The adjoint action by \(T^0\) on \(\text{Lie}N \setminus \text{Lie}K\) has those roots of \(\text{Lie}N\) which are not roots of \(\text{Lie}K\) as eigenvalues. Let \(\alpha\) be one of them, then because \(t_p\) is normalized by \(N\), the reflection element in the Weyl group \(r_\alpha\) satisfies \(r_\alpha(t_p) = t_p\). Using the definition of the reflection, and the fact that \(\alpha(t_p) \neq 0\), one concludes the coroot \(\alpha^\vee \in t_p\). On the other hand \(\alpha^\vee \in t_z\), since \(t_z\) is the Cartan subalgebra of the \(\mathfrak{g}_p^{ss}\) which contains \(\text{Lie}N^{ss}\). But \(t_z \cap t_p = 0\), hence it is impossible that \(\alpha^\vee \in t_p\). So all the roots of \(\text{Lie}N\) are those of \(\text{Lie}K\). From this and the fact that \(t\) is the Cartan subalgebra of both groups, one concludes \(\text{Lie}N = \text{Lie}K\).

Now the assertion that the action by \(t_p\) on \(\mathfrak{q}_\mu/\text{Lie}K\) have non-trivial weights follows immediately. \(\text{QED}\)

3.3. \(Z\) and \(Z/T_z\) as fiber bundles.

Proposition 3.2. Let \(X\) be the connected component containing \(p\) in \(\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})\). Then the connected component of \(T\)-fixed point set on \(X_N\) passing \([p, I, z]\) is given by \(Z/T_z \times \{z\}\).

If \(\phi\) is in the interior of \(C\), \(T_z = I\).

In case \(\phi\) is on the boundary of \(C\), \(Z/T_z\) admits a projection to an orbifold \(E\) with fiber a finite quotient of \(K/T\).

\(\text{Pf}\): Suppose \([q, I, w]\) is in a same connected component as \([p, I, z]\) in \(X_N\) fixed by \(T\). It follows from Prop. 3.1 that \(q \in \tilde{F}_p\) and \(w \in M_z\). A basic property of moment map restricted to fixed point set dictates that

\[
\mu(\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})) \subset \mu(p) + \mathfrak{t}_p^1, \quad \phi(M_z) \subset \phi(z) + \mathfrak{t}_z^1,
\]

and \(\mathfrak{t}_p^1 \cap \mathfrak{t}_z^1 = 0\) since \(\mathfrak{t}_p \oplus \mathfrak{t}_z = \mathfrak{t}\). The condition \(\mu(q) = k\hat{\phi}(w) \in \bar{\mathfrak{t}}\) then forces

\[
\mu(q) = \mu(p) = k\hat{\phi}(z) = k\hat{\phi}(w).
\]

As points in the toric variety \(X_E\), the equality \(\mu_E(z) = \mu_E(w)\) implies that \(w = tz\) for some \(t \in T\). Therefore \([q, I, w]\) has a representative of the form \((t^{-1}q, I, z)\) which defines a point in the claimed set. Thus the association of \([q, I, w]\) with a point in the quotient space \((\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})/T_z) \times \{z\}\) is 1-1 and onto.

As for the last assertion, we already know that \(\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})\) admits the action of \(K\), since it commutes with \(T_p\) and preserves \(\mu^{-1}(\tilde{\phi})\). Let \(E = Z/K\). The fiber of the projection

\[
\pi : (\tilde{F}_p \cap \mu^{-1}(\tilde{\phi}))/T \to (\tilde{F}_p \cap \mu^{-1}(\tilde{\phi}))/KK = E
\]
can be explicitly described. If \([q] \in E\) with \(q \in \tilde{F}_p \cap \mu^{-1}(\tilde{\phi})\), then the fiber is \(\mathcal{K}(q) = \mathcal{K}/\mathcal{K}_q\). We know that \(\mathcal{K}_q\) has \(T^0_p\) as the maximal connected subgroup, since \(\text{Lie}\mathcal{K}_q\) has no semi-simple part from the condition \(\text{Lie}\mathcal{K}_q \cap [\text{Lie}\mathcal{K}_q, \text{Lie}\mathcal{K}_q] = 0\). Hence \(T^0_p\) is a normal subgroup and \(\mathcal{K}_q/T^0_p\) is a finite subgroup. Therefore the fiber is a finite quotient of the homogeneous space \(\mathcal{K}/T\) by \(\mathcal{K}_q/T^0_p\). QED

The following obviously holds:

\[
(3.2) \quad Z = \tilde{F}_p \cap \mu^{-1}(\tilde{\phi}) \to \tilde{F}_p \cap \mu^{-1}(\tilde{\phi})/T \to \tilde{F}_p \cap \mu^{-1}(\tilde{\phi})/K = E.
\]

Since \(T^0_p\) fixes points on \(Z\), the action by \(\mathcal{K}\) is not effective on \(Z\). In fact, we have

**Lemma 3.4.** Define \(t^0\) to be a complement of \(t_z \cap \text{Lie}\mathcal{K}_{ss}\) in \(t_z\), then

\[
\text{Lie}\mathcal{K} = \text{Lie}\mathcal{K}_{ss} \oplus t_0 \oplus t_p,
\]

where \(\text{Lie}\mathcal{K}_{ss}\) is the semi-simple part of \(\text{Lie}\mathcal{K}\).

**Pf:** It is known that \(t \subset \text{Lie}\mathcal{K}\), so \(\text{Lie}\mathcal{K} = \text{Lie}\mathcal{K}_{ss} + \mathfrak{h}\) with \(\mathfrak{h} \subset t\). On the other hand, \([\text{Lie}\mathcal{K}_{ss}, t]\) = 0 by definition of \(\mathcal{K}\), hence \(\text{Lie}\mathcal{K}_{ss} \perp t_p\). But \(\text{Lie}\mathcal{K}_{ss} \subset (\text{Lie}\mathcal{K})_{ss}\) which has \(t_z\) as its Cartan subalgebra. Therefore \(\text{Lie}\mathcal{K}_{ss} \cap t \subset t_z\). We already knew that \(t = t_p \oplus t_z\), hence

\[
\text{Lie}\mathcal{K} = \text{Lie}\mathcal{K}_{ss} \oplus t_p \oplus t_0
\]

where \(t_0 \subset t_z\) and is perpendicular to \(t \cap \text{Lie}\mathcal{K}_{ss} \subset t_z\). QED

Let \(\mathcal{K}'\) be the group with \(\text{Lie}\mathcal{K} = \text{Lie}\mathcal{K}_{ss} \oplus t_0\) as its Lie algebra. This group acts effectively on \(Z\), and \(T_z\) is its Cartan subgroup.

So the fiber over \(E\) is a finite quotient of \(\mathcal{K}'/T_z\).

### 3.4. Connections of the orbifold fiberations

Let \(\text{Lie}\mathcal{K}' = t_z + n\) be the Cartan decomposition.

Fix a \(\mathcal{K}'\)-stable splitting of \(TZ = T^H Z \oplus T^\perp Z\), where \(T^\perp Z\) is tangent to the orbit by \(\mathcal{K}'\) on \(Z\). Such a splitting exists because the action by \(\mathcal{K}'\) is locally free, therefore there is automatically the bundle \(T^\perp Z\). The \(\mathcal{K}'\)-stable horizontal space can be obtained by an invariant metric on \(TZ\).

Let \(A : TZ \to T^\perp Z\) be the projection. There is a further decomposition:

\[
T^\perp Z = T'I Z + T''Z,
\]

where \(T'Z\) and \(T''Z\) are the vertical subbundle generated by vectors in \(t_z\) and \(n\) respectively.

**Lemma 3.5.** 1). If \((p, n) \in Z \times n \mapsto n(p) \in T''_p Z\), then the action by \(T_z\) on \(T''_p Z\) induced from the action on \(Z\), maps \((p, n)\) to \((pt, \text{Ad}_t n), \forall t \in T_z\).

2). \(T''(Z/T_z) = Z \times T_z n\) where the quotient is taken with the previous \(T_z\) action on \(T'' Z \times n\). And \(T(Z/T_z) = \pi^* T\mathcal{E} \oplus T''(Z/T_z)\), where \(\pi : Z/T \to E = Z/\mathcal{K}'\).

**Pf:** Let \(g_*\) denote the action on the tangent bundle. It is straightforward to verify that

\[
(3.3) \quad \xi(gp) = g_*(\text{Ad}_{g^{-1}} \xi)(p), \forall g \in \mathcal{K}', \xi \in \text{Lie}\mathcal{K}'.
\]

Hence \((gp, \xi) \mapsto (p, \text{Ad}_{g^{-1}} \xi)\). Or \((p, \xi) \mapsto (gp, \text{Ad}_g \xi)\), which implies the first assertion after applying it to \(\xi = n \in n\) and \(g = t \in T_z\).

2). This is simply the decomposition of \(T(Z/T_z)\) into horizontal and vertical parts. QED
Denote the map which identifies the vertical vectors with $\text{Lie}\mathcal{K}'$ by $P$. Combine with $A$, we obtain

$$PA : TZ \to \text{Lie}\mathcal{K}'.$$ 

Furthermore $P = P' + P''$ with $P'$, $P''$ take values in $t_z, n$ respectively. Obviously $PA$ is a connection on $Z$.

Let $u, v$ be vector fields in $T^HZ$ invariant under $\text{Lie}\mathcal{K}$, and let $\xi, \eta$ be vectors induced by two elements from $n$.

The decomposition of the curvature given below will be useful later.

**Lemma 3.6.** Suppose $T_z$ acts on $\mathbb{C}$ with character $\lambda$, then there is a connection $A_\lambda$ on the bundle $Z \times T_z \mathbb{C} \to Z/T_z$ such that its curvature is given by $\lambda(B) + \lambda(R)$; the 2-forms $B$ and $R$ are $t_z$-valued, and satisfy the following

$$B(u, v) = P' \cdot A([u, v]), \quad R(\xi, \eta) = P' \cdot A([\xi, \eta]).$$

**Pf:** On the trivial bundle $Z \times \mathbb{C}$, the action by Lie algebra of $t_z$ is given by $L_t = (t(q), \lambda(t))$, for $t \in t_z$. The connection defined by $\nabla = d + \lambda(P'A)$ satisfies

$$\nabla_{t(q)} = d_{t(q)} + \lambda(P'A)(t(p)) = d_{t(q)} + \lambda(t) = L_t$$

by the definitions of $P, A$. A connection, satisfying the above condition, descends to the quotient bundle $Z \times T_z \mathbb{C}$. The curvature is given by $d(\lambda(P'A))$. For vector fields $u, \xi$, by invariance of $u$, we know that

$$[u, \xi] = 0, \quad A(u) = 0, \quad PA(\xi(q)) \in n$$

hence $P'A(\xi(q)) = 0$ and the differential of the 1-form, $d(\lambda(P'A))(u, \xi) = 0$. Thus the curvature has no term mixing $T^HZ$ and $T''Z$, and consists only of the vertical and the horizontal part. The one coming from the horizontal vector fields $u, v$ is exactly

$$\lambda \cdot B(u, v) = \lambda \cdot P'A([u, v]) = \lambda \cdot PA([u, v]),$$

since $A(u) = A(v) = 0$ and $\lambda \cdot P'' = 0$. The one coming from the fiber, or the vertical directions, is $\lambda(PA([\xi(q), \eta(q)])) = \lambda(P'A([\xi(q), \eta(q)]))$. This follows from $\lambda(n) = 0$ and $P'A(\xi(q)) = P'A(\eta(q)) = 0$, hence in calculating $dP'A(\xi, \eta)$ only the said term remains. QED

Fix a base point $q$, one may ask how the forms $B, R$ transform along the fiber of $\pi$.

**Lemma 3.7.**

$$\lambda \cdot B(u, v)|_q = \lambda(\text{Ad}_q P A(u(q), v(q))),$$

$$\lambda \cdot R(\xi(q), \eta(q)) = \lambda(\text{Ad}_q[\xi(q), \eta(q)]).$$

**Pf:** Due to the invariance of $u, v$ and the fact that $g_*$ commutes with $[\cdot, \cdot]$, we have $[u(q), v(q)] = g_*(u(q), v(q))$. The map $g_*$ also commutes with $A$ by the invariance of $A$, therefore

$$A[u(q), v(q)] = g_*(A[u(q), v(q)]).$$
Let \( a = PA([u(q), v(q)]) \in \text{Lie}K^\circ \), then its induced vector \( a(q) = A([u(q), v(q)]) \) by the definition of \( P \). We know \( g_+(a(q)) = Ad_g(a(q)) \), thus

\[
PA[u(gq), v(gq)] = Pg_+(A([u(q), v(q)])) \\
= Pg_+(a(q)) \\
= PAd_g(a)(gq) \\
= Ad_g(a),
\]

hence \( \lambda(PA[u(gq), v(gq)]) = \lambda(Ad_g PA[u(q), v(q)]) \). For the same reason, we obtain

\[
\lambda \cdot R(\xi(gq), \eta(gq)) = \lambda \cdot Ad_g PA[\xi(q), \eta(q)].
\]

QED

This expression is crucial to a calculation in Section 9, because \( \lambda R \) behaves as the moment map on the fiber which is a coadjoint orbit.

3.5. Stratification of the fixed point set. Unlike the smooth case, the fixed point formula for orbifolds requires the contributions of lower strata of the fixed point set. Where do the strata come from? They are present due to the local isotropy groups on the fixed point set \( Z/T_z \).

To describe them locally, let \( p \in Z \), \( T_p \) may not be connected. Denote the connected component of \( I \) by \( T_p^0 \). Let \( I_p = T_p \cap T_z \), \( I_p^0 = T_p^0 \cap T_z \). They are finite groups since \( t_p \cap t_z = 0 \). When \( T_z \) acts on \( Z \), it has \( I_p \) as its stabilizer. All \( q \in Z \) are fixed by \( T_p^0 \), so the subgroup \( I_p^0 \) acts trivially on \( Z \). The effective isotropy group on \( Z \) is \( I_p/I_p^0 \), though \( I_p^0 \) may have non-trivial action on the normal bundle and can not be ignored. Obviously the discussion is unnecessary if for all \( p \in Z \), \( T_p = T_p^0 \).

For each \( h \in I_p/I_p^0 \), denote by \( Z_h \) its fixed points in \( Z \). The collection \( \{Z_h\} \) for \( h \in I_p/I_p^0, \forall p \in Z \) form stratification of \( Z \). And their quotient by \( T_z \) in \( F = Z/T_z \) contribute to the fixed point formula computations, which is different from the smooth case.

Let \( h \in I_p \setminus I_p^0 \), let \( Z_h = \{q \in Z | h(q) = q\} \). It is a submanifold. Clearly \( T_z \) acts on it, and the points there are fixed by \( h, T_p^0 \). Let \( K_h \) be the connected subgroup of \( K \) which commutes with \( h \), i.e., \( Ad_h h = h \), \( k \in K_h \). Then \( K_h \) acts on \( Z_h \), and it contains \( T \).

In the language above, \( Z \) itself can be thought of as \( Z_h, h \in I_p^0 \).

3.6. Lower stratum \( Z_h \) and \( Z_h/T_z \) as fiber bundles. Obviously one has

**Lemma 3.8.** The Lie algebra of \( K_h \), \( \text{Lie}K_h \), is the maximal subspace on which the action \( Ad_h | \text{Lie}K \) is \( I \).

This lemma implies that \( \text{Lie}K/\text{Lie}K_h \) induces a subspace normal to \( T_pZ_h \) in \( T_pZ \).

As in the case of \( Z/T_z \), we can realize \( Z_h/T_z \) as a fiber bundle. As in the case of \( \text{Lie}K \), \( \text{Lie}K_h \) splits into \( \text{Lie}K^\circ_h \) and \( t_p \). And the group \( K_h^\circ \simeq K_h/T_p^0 \) acts effectively on \( Z_h \). Associate with \( Z_h \) the space \( E_h = Z_h/K_h^\circ \). Similar to \( Z \), there is the following sequence:

\[
Z_h \rightarrow Z_h/T_z \rightarrow Z_h/K_h^\circ = E_h
\]

where the second projection yields a finite quotient of \( K_h^\circ/T_z \) as the fiber.
3.7. The action by a Weyl subgroup on the fixed points of $T$. Suppose $(p, I, z)$ defines a fixed point of $T$ in $X_N$, that is to say that $t_p, t_z$ generate $t$, and $\mu(p) = k\tilde{\phi}(z)$.

If $\phi(z)$ is in the interior of $C$, then $t_z = 0$, and $t_p = t$ by the above characterization of a fixed point.

Suppose $\phi(z) \in \partial C$, let $C_\phi$ be the smallest wall of $\partial C$ containing $\phi$. Then $l_{g_A} = (lg)_{\alpha}$ commutes with $\mu(p) = \phi(z)$, and it is generated by $l_{g_A}$ with $\alpha$ vanishing on the wall $C_\phi$.

Lemma 3.9. The subgroup $W^{aff} \cap (LG)^{ss}_\mu$ of $W^{aff}$ is the Weyl group of the finite dimensional semi-simple group $(LG)^{ss}_\mu$. It transforms one fixed point set to another in $\mu^{-1}(k\tilde{\phi})$ where $\tilde{\phi} = \mu(p)/k$.

For $w \in W(K)$, it preserves $Z$ but permutes among $\{Z_h\}$.

*Proof:* The group $(LG)_\mu$ is compact and connected, since it is the stabilizer of $\mu(p) = (\phi(z), k)$ under the adjoint action. The semi-simple part is generated by $(lg)_\alpha$ where $\alpha$ vanishes on $C_\phi$, therefore the Weyl group of $(LG)^{ss}_\mu$ is generated by the reflections with respect to those affine roots vanishing at $C_\phi$. In terms of the original affine Weyl group, it is exactly $W_\mu = W^{aff} \cap (LG)^{ss}_\mu$.

Suppose that $(p, I, z)$ defines a fixed point, then $\mu(gp) = Ad_g\mu(p) = \mu(p)$, for all $g \in (LG)^{ss}_\mu$. If $g$ is also in $W_\mu$, one has $g(t_z) = t_z$ because $t_z$ is the Cartan subalgebra of $(lg)_{\alpha}^{ss}$. Thus $g(t_p)$ and $g(t_z)$ generate $t$. To check that $(gp, I, z)$ is fixed by $T$, let $t \in t$,

$$t = t_1 + t_2, \quad t_1 \in g(t_p), t_2 \in g(t_z) = t_z.$$

Then

$$\exp(2\pi it)(gp, I, z) = (gp, I, \exp(2\pi it)z) \approx (\exp(2\pi it_1)gp, I, \exp(2\pi it_2)z) = (g \exp(2\pi i Ad_g^{-1} t_1)p, I, z) = (gp, I, z).$$

In the above, we have used the fact that $\exp(2\pi i Ad_g^{-1} t_1) \in T_p$ if $t_1 \in g(t_p)$. This shows the Weyl group of the semi-simple part of $\mu(p)$ acts on the fixed point set whose image is $\mu(p)$.

Fix a lifting of $w$ to $K$, it certainly preserves $Z$, since the whole $K$ does. If $Z_h$ is a stratum as described earlier, associated with $h \in I_p/I^0_p$, for some $h$ and $p$, then the point $w(p)$ has $Ad_w T_p$ as stabilizer, and its isotropy group is $Ad_w I_p$. Since $T_z$ is the Cartan subgroup of $(LG)^{ss}_\mu$, and $w$ is in its Weyl group, hence $w$ preserves $T_z$ and

$$Ad_w I_p = I_{w(p)}, \quad Ad_w I^0_p = I^0_{w(p)}.$$ 

Thus $w(Z_h) = Z_{w(h)}$. QED
4. Normal bundles to the fixed point sets in $X_N$ and weights

We will find out in this section the weights of the $T$-action on the normal space to the fixed point sets inside the compactification locus.

The last section describes the fixed point sets of $X_N$ in terms of data from $X$ and $\mathfrak{X}$. Suppose $(p, I, z)$ defines a fixed point in $X_N$, and $(p_0, h_0, z_0)$ is a curve with

$$(p_0, h_0, z_0) = (p, I, z), \quad (p_0', h_0', z_0') = (a, \xi, x) \in T_{(p, h, z)}(X \times LG \times X_\theta).$$

Let $t = t_p t_z$ so that $t_p \in T_p$ and $t_z \in T_z$. Then the action by $T$ on the tangent vectors is given by

$$(4.1) \quad t(a, \xi, x) = (a, \xi, t(x)) \simeq (t_p(a), \text{Ad}_{t_p} \xi, t_z(x))$$

which is obtained by the definitions of the action of $T$ on $X_N$, and the equivalence relation as in Def. 3.3.

By assumption, the equivalent class $[p, I, z]$ defines a fixed point in $X_N$. As before, $T^0_p$ is the connected component of $I$ in $T_p$. Then $T^0_p$ and $T_z$ generate $T$. Furthermore $T_p$ and $T_z$ act on the tangent spaces $T_p X$ and $T_z X$, respectively.

4.1. Tangent space. First let us describe the tangent space to $X_N$ at $[p, I, z]$.

Proposition 4.1. Let $\mu = \mu(p)$ and $V_p$ be defined as

$$V_p = \{ (a, b) \in T_p X | \exists b \in \mathfrak{g} \theta, D_{(a + i J a)} \mu(p) = D_{(b + i J b)} \mu_X \}$$

where $\mu_X : X = LG \times X_\theta \to \mathfrak{g} \times \{ k \}$. In the toric variety $X_\theta$, the point $z$ has $T_z$ as its stabilizer, let the subspace tangent to $T^C(z) \simeq (t_z^C)^C$ be denoted by $H_z$, then the tangent space to $X_N$ at $[p, I, z]$ admits the following decomposition:

$$T_u X_N \simeq \mathfrak{g} / \mathfrak{t} \oplus \mathfrak{h} / \mathfrak{t} \oplus H_z$$

where $(\mathfrak{g} / \mathfrak{t})$ is the Lie algebra of the stabilizer of the $\mu$ under the coadjoint action.

The tangent space $T_u X_N$ has a natural almost complex structure $J$ satisfying:

$$J|_{\mathfrak{g} / \mathfrak{t}} = J^X; \quad J|_{\mathfrak{g} / \mathfrak{t}} = J^\theta; \quad J|_{H_z} = -J^X\theta$$

where $J^X, J^\theta, J^X\theta$ are the almost complex structures on the space $X, \mathfrak{g} / \mathfrak{t}, X_\theta$ respectively.

Remark: The map $D_\mu$ after extended to the complexified tangent space, certainly is not holomorphic, thus the vectors in $V_p$ are very special.

Proof: The tangent space of $T_u X_N$ is isomorphic to the space

$$\{ (a, b) \in T_{(p, q)} X \times \mathbb{X} | D_{(a + i J a)} \mu(p) = D_{(b + i J b)} \mu_X(q) \},$$

as shown in Section 2 (or cf. [C1]). The subspace $H_z$ is contained there via $b \mapsto (0, 0, b)$, for $D_{(0, t + i J t)} \mu_X(I, z) = 0$. Let $q = [I, z] \in \mathbb{X}$. The subspace $\mathfrak{g}_\mu / \mathfrak{t}$ is contained there via the inclusion $\xi \in \mathfrak{g}_\mu / \mathfrak{t} \mapsto (0, \xi, 0)$, and it satisfies the equation

$$D_{(\xi + i J_\xi, 0)} \mu_X(I, z) = D_{(\xi - i J_\xi, 0)} \mu_X(I, z) = [\xi - i \xi J, \mu_X(I, z)] = 0,$$

since $[\xi, \mu_X(q)] = [\xi J, \mu_X(q)] = 0$. Also we have used the fact that $J^\theta \xi(q) = -J^\theta \xi(q) = -J^X \xi(q)$ with $J^\theta \xi = \xi J \in \mathfrak{g}_\mu / \mathfrak{t}$ being defined by the almost complex structure on $\mathfrak{g}_\mu / \mathfrak{t}$.

The subspace defined by $\mathfrak{g}_\mu / \mathfrak{t} \oplus H_z$ corresponds exactly to the subspace

$$U_0 = \{(a, b) \in T_{(p, q)} X | a = 0 \}.$$

The complement of $U_0$ in $T_u X_N$ is exactly $V_p$. 
There is a unique way to choose $b$ in the above definition of $V_p$: make it perpendicular to $H_z$, and choose it from

$$
\sum_{\alpha \not\in \Delta(l(g))} \lambda \cdot \mathfrak{g}_\alpha
$$

where the index means $\alpha$ is not a root of $l(g)$. This is possible since $D_{b} \mu_{X} = D_{J \cdot b} \mu_{X} = 0$ if $b \in H_z$ or $b$ is induced from $l(g)$.

Then the map $h : a \mapsto (a, b)$ satisfies $h \cdot J = J' \cdot h$. So the complex structure defined by $J'$ on $V_p$ is mapped to $J = (J^N, J')$ restricted to the image of $h$. Thus the assertion is verified. The other two identities involving $J'$ follow strictly from the description of $J'$ on $TX$. QED

4.2. Normal space to $Z$ and $Z_h$.

From the description of $V_p$ and $l(g)/t$, it is clear they inherit an almost complex structure from $T_{[p,I,z]}X_N$. In the following the weights refer to the weights by $t_p, h \in I_p \setminus I_p^0$ acting on the complex linear spaces.

Let $V_p^0$ and $N_p$ be the 0-weight and non-zero weight subspaces of $V_p$ under the linear $t_p$-action respectively. Then $V_p = N_p \oplus V_p^0$.

Similarly, denote by $\text{nor}^1(Z_h, Z) \subset \{V^0_p\}$ the maximal $h$-stable subbundle subspace on which $\det(I - h) \neq 0$.

**Proposition 4.2.** 1). The normal subspace to the fixed point set $Z/T_Z$ is

$$
N_p \oplus l(g)_\mu/LieK \oplus H_z.
$$

2). The normal bundle of $Z_h$ in $Z$ can be decomposed as

$$
\text{nor}^1(Z_h, Z) \oplus LieK/LieK_h.
$$

The bundle $\text{nor}^1(Z_h, Z)$ admits an action by $T_z$ which lifts the action on $Z_h$.

On the normal subspaces in both situation, there is the induced almost complex structure from $J$ on $T_aX_N$.

**Pf:** The first claim holds due to the fact that the normal space to the fixed point set is the non-zero weight space under the action by the $t_p$ on the tangent space. From the description of the tangent space and the definition of $N_p$, the assertion is evident.

Part 2 follows from essentially the same reasoning, with a slight variation since here we consider only the $h$-action. The the normal space is the maximal $h$-stable subspace in $T_pZ$, on which 1 is not an eigenvalue. There are two factors in the tangent to $T_pZ$, one is the Lie$K/t$, the other is $V_p^0$. Then from the definition of $\text{nor}^1(Z_h, Z)$ and Lemma 3.8, the decomposition holds.

The action by $T_z$ on $\text{nor}^1(Z_h, Z)$ exists since $T_z$ commutes with $h$, so the non-zero weight space under $h$ in $V_p$ at $p$ is isomorphic to that in $V_{t_p}$, $t \in T_z$.

The almost complex structure preserves those subspaces since $t_p$ and $h$-actions commute with $J$. QED

4.3. Description of weights.

From the last section, it is verified that $t = t_p \oplus t_z$ and $t_p \cap t_z = 0$. Let $t = t_p + t_z, t \in t$ denote this decomposition.

**Proposition 4.3.** Let the weights of the $T_p$-action on $N_p$ be denoted by $\{\gamma\}$, the weights of the $T_z$-action on $H_z$ denoted by $\{\lambda\}$, and the positive roots of $(l(g))_\mu/LieK$ be denoted by $\{\beta\}$ respectively.
Then the weights by the $T$-action on $T_{[p,t,z]}X_N$ are given by the corresponding three sets of weights: $\{\tilde{\gamma}\}$, $\{\tilde{\lambda}\}$, $\{\tilde{\beta}\}$ such that

$$
\tilde{\gamma}(t) = \gamma(t_p), \quad \tilde{\lambda}(t) = -\lambda(t_z), \quad \tilde{\beta}(t) = -\beta(t_p).
$$

**Remark:** The negative sign for $\tilde{\lambda}$ and $\tilde{\beta}$ reflects the complex structure on $X_N$ as described in Prop. 4.1.

**Pf:** The action by $t$ on the tangent space $T_{[p,t,z]}X_N$ is of the form:

$$
t_p \cdot t_z(a, \xi, x) = (t_p(a), [t_p, \xi], t_z(x))
$$

as shown by Eq. (4.1). Now apply this to the three types of normal vectors as in the last proposition, one has the assertion. The only thinking needed here is the observation that the embedding of $V_p$ into $T_uX_N$ is $T_p$ equivariant. The '-' sign reflects the choice of the complex structure on $X_N$ as described in Prop. 4.1.

4.4. **Weights on normal space to the lower strata $Z_h$ in the fixed point set.** As pointed earlier, the normal space to $Z_h$ consists of two parts: the normal space to $Z$ and the normal space of $Z_h$ in $Z$. The contribution of $Z_h$ to the fixed point formula, in the orbifold sense, depends on the weights of the action by $hT_p^0$ on the normal space. The calculation given earlier provides the answer for the action by $T$ on the normal space to $Z$, here we need only determine the weights on the second subspace which is $\text{nor}_p^{1}(Z_h, Z) \oplus (\text{Lie}K/\text{Lie}K_h)$. Recall that $h \in I_p/I_p^0 = (T_p \cap T_z)/(T_p^0 \cap T_z)$.

**Lemma 4.1.** Suppose the weights of the action by $h$ on $\text{nor}_p^{1}(Z_h, Z)$, $(\text{Lie}K/\text{Lie}K_h)$ are given by $\{\theta_i\}$, $\{\beta_i\}$ respectively. Let $(ht_p, h^{-1}t_z) \in hT_p^0 \times T_z$ be a lifting of $t \in T$ in $T_p \times T_z$, $\{\tilde{\theta}\}$, $\{\tilde{\beta}\}$ be the weights of the action by $(ht_p, h^{-1}t_z)$ on the two factors of the normal space. They are given by

$$
\tilde{\theta}(ht_p \cdot h^{-1}t_z) = \theta(h), \quad \tilde{\beta}(ht_p \cdot h^{-1}t_z) = \beta(h).
$$

**Pf:** Let $(p_s, k_s, z)$ be a curve, with $p_s, k_s$ tangent to $Z, K'$ but normal to $Z_h, K'_h$ and $p_0 \in Z_h, k_0 \in K'_h$; and $z$ is fixed by $T_z$. The action by $t \in T$ on that curve after choosing the lifting $ht_p \cdot h^{-1}t_z$, is simply

$$
(ht_{p_0}p_s, \text{Ad}_{ht_p} k_s, z),
$$

which is $(hp_s, \text{Ad}_h k_s, z)$ since $t_p$ acts trivially on $Z$, and $\text{Ad}_{t_p} k = k, \forall k \in K$ by the definition of $K$. Therefore, the weights are of the forms described. QED
4.5. **Three types of fixed points.** Suppose \((p, I, z)\) defines a fixed point in \(X_N\), we classify them according to the following:

1. \(\mu(p)/k = \tilde{\phi}(z)\) is in the interior of \((C, 1)\).
2. \(\mu(p)/k = \tilde{\phi}(z)\) is on \((\partial C, 1)\) but not a vertex of \(C\).
3. \(\mu(p)/k = \tilde{\phi}(z)\) is one of the vertices of the simplex \((C, 1)\).

Considering the decomposition of \(t\) into \(t_p, t_z\), the type 1) and 3) correspond to the cases \(t_z = 0\) and \(t_p = 0\) respectively.

4.6. **The \((LG)_{\mu}/T \) factor.** In case of type 3), \(t_z = t\), therefore \(t_p = 0\). Hence \(K = (LG)_{\mu}\), and \((LG)_{\mu}\) acts on the fixed point set, as shown for \(K\). What’s said earlier about the map \(\pi : F \to \mathcal{E}\) with fiber \(\mathcal{K}'/T_z\) now can be replaced by the factor \((LG)_{\mu}/T\).

4.7. **More on \(W^{\text{aff}}\).** Suppose the fixed point is of either type 2) or 3), then \(\mu(p)\) has stabilizer \((LG)_{\mu}\), under the co-adjoint action. The group has semi-simple part generated by \((q\lambda)_{\alpha}\), where \(\alpha\) is an affine root vanishing at \(\mu(p)/k\).

The following is a well known fact in affine algebra:

**Lemma 4.2.** The group generated by the reflections

\[ r_\alpha = \{ \alpha|\alpha(\mu) = 0 \} \]

is the Weyl group of \(T\) in \((LG)_{\mu}\), it acts on the type 2) fixed point set. Denote the Weyl group \(W_{\mu}\) which is the subgroup of \(W^{\text{aff}}\) fixing the smallest wall containing \(\mu\).

4.8. **Transformation weights by \(W_{\mu}\).** The discussion here is only meaningful for type 2) and 3) fixed points.

In the previous section, it was shown that the Weyl group \(W_{\mu}\) permutes among the collection of fixed point sets with the same value for \(\mu\).

Let \(\{\lambda'\}', \{\beta'\}', \{\gamma'\}'\) denote the three groups of weights at the point \((wp, I, z)\) as in Prop. 4.3.

Using the expressions in Eq. (4.1) and Prop. 4.3, we conclude that all the non-trivial weights are those \(\lambda'\)'s in case of type 3) fixed point, since \(t_p = 0\) and \(K = LG_{\mu}\), so there is no \(\gamma, \beta\). Hence, along the \((LG)_{\mu}\)-orbit of \((wp, I, z)\) which is fixed by \(T\) in \(X_N\), the weights are just those from \(H_z, \{\lambda\}'\).

In case of type 2) fixed point, we have the following relation between the stabilizers:

\[ T_{w(p)} = wT_p. \]

To derive the transformation rule for the weights, we need the following:

**Lemma 4.3.**

\[ V_{w(p)} = w_* (V_p), \]

where \(w_* : T_p X \to T_{w(p)}\) is the isomorphism induced from the diffeomorphism \(w\).

**Pf:** Let \(a \in V_p\), i.e. \(\exists \xi \in l\mathfrak{g}/t\) such that \(D_{a+iJa} \mu(p) = D_{\xi+iJ' \xi} k_{\tilde{\phi}}(z)\). Then using the invariance of \(J\) under \(w\), we have

\[
D_{w(a+iJa)} \mu(w(p)) = w(D_{a+iJa} \mu(p))
\]

\[
= w(D_{\xi+iJ' \xi} k_{\tilde{\phi}})
\]

\[
= D_{w(\xi)+iJ' w(\xi)}(\mu),
\]

since \(w\) preserves \(t\), we have \(w(\xi) \in l\mathfrak{g}/t\). So the assertion holds. QED
Write $t$ in two different way:

\[ t = t_p + t_z \in t_p \oplus t_z; \quad t = t_{wp} + t_{z}' \in t_{wp} \oplus t_z, \]

how are $t_p, t_{wp}$ related?

**Lemma 4.4.** \( wt_p = t_{wp} \in t_{wp}, \quad t_{z}' = t - wt + wt_z = t_p - wt_p + t_z \in t_z. \)

**Pf:** From the first equation, we have \( wt = wt_p + wt_z \). Also \( W_\mu = W(Lg_\mu, u) \) and \( t_z = Lg_\mu^s \cap t \), we claim \( wt - t \in t_z, \forall w \in W_\mu \). The claim holds easily for reflections, hence it holds for all $w$. Thus

\[ t = wt + (t - wt) = wt_p + (t - wt + wt_z) \in t_{wp} \oplus t_z. \]

Hence \( wt_p = t_{wp}, \quad t_{z}' = t - wt + wt_z \) by uniqueness of decomposition. Another expression for \( t_{z}' \) is

\[ t_{z}' = t - wt + wt_z = (t_p + t_z) - w(t_p + t_z) + wt_z = t_p - wt_p + t_z. \]

**QED**

Associated with \( \gamma' \) is a weight \( \tilde{\gamma}' \) which is a character of \( T_{wp}^0 \times T_z \), as shown in Prop. 4.3, defined by \( \tilde{\gamma}'(t) = \gamma'(t_{wp}) \). Hence

\[ \tilde{\gamma}'(t) = \gamma'(t_{wp}) = w(\gamma)(wt_p) = \gamma(t_p) = \tilde{\gamma}(t). \]

The first group of weights on \( T(p, l, z)X_N \), denoted by \( \{ \tilde{\gamma} \} \) as in Prop. 4.1, are invariant under \( W_\mu \), see Fig 4.1. The group \( W_\mu \) are generated by reflections w.r.t. planes containing \( \gamma \).

Those weights \( \{ \tilde{\beta}' \} \) are defined by \( \tilde{\beta}'(t) = -\beta(t_{wp}) \). But \( t_{wp} = w(t_p) \), therefore

\[ \tilde{\beta}'(t) = -\beta(t_{wp}) = -w(\beta)(t_p) = w(\tilde{\beta})(t). \]

The weights \( \{ \tilde{\lambda}' \} \), are defined as

\[ \tilde{\lambda}'(t) = -\lambda(t_{z}'). \]

The following transformation law for the weights is essential for future calculations.

**Proposition 4.4.** Let \( \Pi : t \to t_z \) be the orthogonal projection, then

\[ \tilde{\lambda}(t) = \lambda(\Pi t) = \lambda(\Pi t_p), \quad \tilde{\beta}(t) = \beta(\Pi t_p). \]

At the point \( (wp, l, z) \) with \( w \in W_\mu \), the weights are given by

\[ \tilde{\gamma}' = \tilde{\gamma}, \quad \tilde{\lambda}'(t) = \lambda(\Pi t) - \lambda(w\Pi t_p), \quad \tilde{\beta}'(t) = \beta(w\Pi t_p). \]

For \( v \in W_\mu \),

\[ \tilde{\gamma}'(vt) = \tilde{\gamma}'(t) = \tilde{\gamma}(t), \]

\[ \tilde{\lambda}'(vt) = v\lambda(\Pi t) - w\lambda(\Pi t_p), \]

\[ \tilde{\beta}'(vt) = w\beta(\Pi t_p). \]

**Remark:** 1) The projection \( \Pi \) can be removed since \( \beta, \lambda \) are linear functions on \( t_z \) and their extension to \( t \) by convention are compositions with \( \Pi \). 2) If \( w \in W(\text{Lie} K') \), then \( w t_p = t_p \), since \( [\text{Lie} K', t_p] = 0 \), and \( \tilde{\beta}' = \beta \).

**Pf:** The proof is divided into 3 steps for each group of equations in the above.
1). Since $t_z \in t_z$, and $t = t_z + t_p$, apply the map $\Pi$, one has $t_z = \Pi t - \Pi t_p$. Hence
\[ \tilde{\lambda}(t) = \lambda(t_z) = \lambda(\Pi t) - \lambda(\Pi t_p). \]
The root $\beta$ vanishes on the orthogonal complement of $t_z$, thus
\[ \tilde{\beta}(t) = \beta(t_p) = \beta(\Pi t_p). \]

2). An useful observation can be made here:
\[ \Pi(wt) = w\Pi(t) \] (4.10)
which is true for $w$ given by reflection $r_\beta$ with respect to a root $\beta$ of $(Lg)_\mu$, for
\[ \Pi(r_\beta t) = \Pi(t_\beta - \beta(t)\beta^\vee) = \Pi(t_\beta) - \beta(t)\beta^\vee \]
Since $\Pi(\beta^\vee) = \beta^\vee$, also $\beta(t) = \beta(\Pi t)$. Combine the two one gets
\[ \Pi(r_\beta t) = r_\beta(\Pi t). \]
Hence it holds for any $w \in W_\mu$.

It also has been shown that $\tilde{\gamma}' = \tilde{\gamma}$. As for $\tilde{\lambda}'$ and $\tilde{\beta}'$, using $t_z' = t - t_{wp} = t - wt_p$, we obtain
\[ \tilde{\lambda}'(t) = \lambda(t_z') = \lambda(\Pi t) - \lambda(\Pi t_{wp}), \]
\[ \tilde{\beta}'(t) = \beta(t_{wp}) = w\beta(t_p) = w\beta(\Pi t_p). \]

3). Decompose $t = t_p + t_z = v^{-1}t_p + t_z + (t_p - v^{-1}t_p)$, we've shown that $t_p - v^{-1}t \in t_z$, since $v \in W_\mu$. Therefore $t_z' = t_z + (t_p - v^{-1}t) \in t_z$ and $v(t) = t_p + v(t_z')$ with $v(t_z') \in t_z$, thus $(vt)_p = t_p$ and
\[ \tilde{\gamma}'(vt) = \tilde{\gamma}(vt) = \gamma((vt)_p) = \gamma(t_p) = \tilde{\gamma}(t). \]

As shown in Step 2), by replacing $t$ there with $vt$,
\[ \tilde{\lambda}'(vt) = \lambda(\Pi vt) - \lambda(w\Pi(\Pi t)_p), \]
but $\Pi vt = v\Pi t$, and it has just been shown $(vt)_p = t_p$, so we have
\[ (4.11) \]
\[ \tilde{\lambda}'(vt) = \lambda(v\Pi t) - \lambda(w\Pi t_p) \]
\[ = v\lambda(\Pi t) - w\lambda(\Pi t_p). \]

For $\tilde{\beta}'$, following Step 2), one has
\[ \tilde{\beta}'(vt) = w\beta(\Pi(\Pi t)_p), \]
but $(vt)_p = t_p$, therefore
\[ \tilde{\beta}'(vt) = w\beta(\Pi t_p). \] QED

Thus we have found how the weighs are related along $W_\mu$-orbit, and how they change under $t \mapsto vt$. 


4.9. **Transformation of weights for** $Z_h$. For the $p \in Z_h$, the group $W(K)/W(K_h)$ acts on it, in addition to the transformation by $w$ in $W((LG)_p)/W(K)$. The first group preserve $Z$ but permutes among $\{Z_h\}$. Since both $(LG)_\mu$ and $K$ contain $T$, there is the obvious exact sequence

$$ W(K)/W(K_h) \to W((LG)_p)/W(K_h) \to W((LG)_p)/W(K). $$

The strata $Z_{vh}/T_z \times \{z\}$ contains $[vp, I, z]$, where $Z_{vh} = vZ_h$. The normal space to $Z_{vh}/T_z \times \{z\}$ acquires two more set of weights $\{\theta_v\}$ and $\{\beta_v\}$ as shown earlier.

Let $t \in T$, with a fixed decomposition $t_zt_p$, the action on the normal space at $[vp, I, z]$ is that of $vht_p \cdot (vh)^{-1}t_z$.

The assertions in the next lemma are already verified in Lemma 4.1 and Prop. 4.4.

**Lemma 4.5.** Let $v \in W(K)/W(K_h)$. The weights $\{\tilde{\theta}^v\}$ and $\{\tilde{\beta}^v\}$ satisfy the following:

$$ \tilde{\theta}^v(vht_p \cdot (vh)^{-1}t_z) = \theta(h), \quad \tilde{\beta}^v(vht_p \cdot (vh)^{-1}t_z) = \beta(vh). $$

If $v \in W((LG)_p)/W(K)$, $v$ does not preserves $T^0_p$. The lifting of $t$ is given by $(vh)t_{vp} \cdot (vh)^{-1}t_z'$, where $t_{vp} = vt_p$. With that lifting, the weights $\{\tilde{\theta}^v\}$, $\{\tilde{\beta}^v\}$ are given by the same formula.

In both cases, the weights of the action by $((vh)t_{vp}, (vh)^{-1}t_z')$ in the normal space of $vZ$ in $vF$ are simply given by the evaluations of the weights given in Prop. 4.4 on $((vh)t_{vp}, (vh)^{-1}t_z')$.

4.10. **A word about lifting of action by** $t$. For orbifolds, in order to evaluate the contribution by the fixed point set, it is necessary to consider all the lifting of the action at a fixed point to the finite smooth cover. Once there, one needs to find the fixed point set of each lifting and find its contribution. The strata $Z_h/T_z$ for $h \in I_p \setminus I^0_p$ is one of those fixed point sets. What if $h \in I^0_p$? For such a $h$, $(ht_p, h^{-1}t_z) \in T_p \times T_z$ is a lifting of $t \in T$ as well, but $ht_p \in T^0_p$, and hence the consideration for that lifting is already incorporated when we study the action by general $(t_p, t_z) \in T^0_p \times T_z$. 
5. Curvatures of Various Bundles

In order to understand the contribution from the fixed points coming from compactification, i.e., those with images on \(W(\partial C)\), we need to know the curvature of their normal bundles. And their transformation law by certain subgroups of \(W\).

5.1. A general fact. Suppose \(S\) acts on a manifold \(N\) and \(\mathbb{C}^n\), the action on \(N\) is locally free and the action on \(\mathbb{C}^n\) is linear defined by \(\lambda : \text{Lie}S \to \text{gl}(\mathbb{C}^n)\). Let \(A\) be a \(S\)-invariant connection on \(N\) with \(\text{Lie}S\) identified with the vertical subspace in \(TN\). Denote the curvature of the connection by \(F_A\) which is a \(\text{Lie}S\)-values horizontal two form.

**Proposition 5.1.** 1). Then \((N \times \mathbb{C}^n)/S\) as vector bundle over \(N/S\) has a connection whose curvature is given by \(\lambda(F_A)\).

2). Suppose \(V = \mathbb{C}^n/\Lambda\) where \(\Lambda\) is a finite group acting on the vector space linearly. If \(S\) acts on \(N \times V\), such that the action is locally free on \(N\), and is a linear action on the orbifold line bundle \(V\), i.e. there is an extension of \(S\) by the finite group \(\Lambda\), \(S'\) acting on \(\mathbb{C}^n\) linearly. Then the orbifold line bundle \((N \times V)/S\) over \(N/S\) has the same curvature form as in 1).

*Proof:* 1). On the trivial bundle \(N \times \mathbb{C}^n\), defines the following connection \(d + \lambda \cdot A\), notice that \(\lambda \cdot A : TN \to \text{gl}(\mathbb{C}^n)\) is indeed a 1-form. This connection has the feature that the curve \((g_t(p), g_t(v))\) is horizontal for any 1-parameter subgroup \(\{g_t\}\). Therefore, it descends to a connection on the quotient space \(N/S\). For the quotient connection constructed this way, the curvature is given by the descent of the curvature upstairs which is of the given form.

2). Let \(S'\) act on \(N\) through \(S\). Then apply the above argument to the extension group \(S'\). QED

5.2. Action by \(T_z\) on bundles. The group \(T_z\) has a local free action on either \(Z\) or \(Z_h\). The action is obtained by restricting its action on \(X\) to the given sets. Therefore it acts on the various subbundles of the normal bundles. Denote the action on normal bundle by \(dt_z, \forall t_z \in T_z\).

**Lemma 5.1.** 1). \(dt_z : V_p \to V_{t_z p}\) is an isomorphism preserving the decomposition into \(V^0 \oplus N\).

The map \(dt_z\) extends to an isomorphism

\[
(5.1)\quad dt_z : N_p \oplus (\mathfrak{g})_p / \text{Lie}K \oplus H_z \to N_{t_z p} \oplus (\mathfrak{g})_p / \text{Lie}K \oplus H_z.
\]

2). The quotient by \(T_z\) as described in Eq. (5.1) defines the normal orbifold line bundle to the fixed point set \(F\) in \(X_N\).

**Warning:** The action by \(t_z\) above should not be confused with the \(T\)-action on normal bundle to the fixed points studied in Section 3. The action here defines the equivalent class, while the action in Section 3 is on the set of equivalent classes.

*Proof:* Let \((p_s, h_s, z_s)\) be a curve with \((p_0, h_0, z_0) = (p, I, z)\). Let \(q = [I, z] \in X\). Using the defining equivalence relation, the diagonal action by \(T_z\) on the first two factors is

\[(p_s, h_s, z_s) \mapsto (t_z p_s, t_z h_s, t_z z_s) \simeq (t_z p_s, t_z h_s, t_z^{-1} z_s).\]

Differentiate the above at \(s = 0\) to obtain the diagonal action by \(t_z\) on the tangent vectors:

\[
(5.2)\quad dt_z(p', h', z') \simeq (t_z p', Ad_t h', t_z^{-1}(z')).
\]
If \((p', h', z') \in V_p\), we have \(z' = 0\) and
\[
D_{p' + iJp'} \mu(p) = D_{h' + iJh'} k \mu(z)(q).
\]
Let \(t_{z*}\) denote the induced action by \(t_z\) on the tangent space. Apply the action \(\text{Ad}_t\) on both sides, and combine that with the properties of \(d\mu\), one has
\[
D_{t_{z*}(p' + iJp')} \mu(t_z p) = \text{Ad}_t \left( D_{(p' + iJp')} \mu(p) \right) = \text{Ad}_t \left( D_{(h' + iJh')} \mu(z)(q) \right) = D_{t_{z*}(h' + iJh')} \mu(z)(q),
\]
which shows that \((t_{z*}(p'), t_{z*}(h'), 0) \in V_{t_z p}\).

The calculation above also shows for general \(g \in LG\), the following holds:
\[
g_*(p' + iJp', h' + iJh', 0) = (g_*(p' + iJp'), \text{Ad}_g(h' + iJh'), 0).
\]

Since \(T_p, T_z\) commute, one has \(T_{t_z p} = T_p\), and clearly \((p' + iJp', h' + iJh', 0)\) has a non-zero weight under \(T_p\) iff \((t_{z*}(p' + iJp'), t_{z*}(h' + iJh'), 0)\) has non-zero weight under \(T_{t_z p} = T_p\). Hence \(N_p\) is \(N_{t_z p}\) under \(t_{z*}\).

The map \(t_{z*}\) preserves \((\mathfrak{g})_{\mathfrak{g}}/\text{Lie}\mathfrak{K}\) since \(t_z\) is the Cartan subalgebra of the semi-simple part of \((\mathfrak{g})_{\mathfrak{g}}\), and \(T_z\) preserves \text{Lie}\mathfrak{K} under the adjoint action. The map \(t_{z*}\) preserves \(H_z\) because \(T_z\) fixes \(z\) and \(H_z\) is the subspace in \(T_z X_g\) with non-zero weights. So it is stable under the action by \(T_z\).

2). The normal subspace to the fixed point set consists of non-zero weight vectors in the tangent space \(V_p \oplus (\mathfrak{g})_{\mathfrak{g}}/\text{Lie}\mathfrak{K} \oplus H_z\). Therefore, it is given by \(N_p \oplus (\mathfrak{g})_{\mathfrak{g}}/\text{Lie}\mathfrak{K} \oplus H_z\) before mod out by the local free action of \(T_z\). QED

5.3. Curvatures of the normal bundles. Next we calculate the curvatures of the three subbundles given in the above. The transformation of the curvatures under the action by \(W_{\mu}\) will be given as well.

Following Eq. (5.2), for \((p', h', z')\) in \((\mathfrak{g})_{\mathfrak{g}}/\text{Lie}\mathfrak{K}\), i.e., \(p' = 0, z' = 0\), and \(h' \in (\mathfrak{g})_{\mathfrak{g}}/\text{Lie}\mathfrak{K}\), the action by \(T_z\) is the adjoint action. And the action by \(T_z\) on \(H_z\) is the action by isotropy group, since \(T_z\) fixes \(z\).

Based on that Eq. (5.2), one can write the curvature in terms of the weights and the connection \(A\) as follows:

**Proposition 5.2.** 1). The curvature of the bundles
\[
Z \times_{T_z} ((\mathfrak{g})_{\mathfrak{g}}/\text{Lie}\mathfrak{K}), \quad Z \times_{T_z} H_z
\]
are given by
\[
- \oplus_{\beta \in \Delta_\mu / \Delta(\text{Lie}\mathfrak{K})} \beta \cdot dA, \quad - \oplus_{\lambda} \lambda \cdot dA
\]
respectively.

2). Let \(\nabla\) be a \(T_z\)-invariant connection on \(\tilde{N} = \{N_p\}_{p \in Z}\) over \(\tilde{F}_p \cap \mu^{-1}(\phi)\), and \(a_q(t) = L_t - \nabla_t\), where \(L_t\) is the Lie derivative of \(t\) acting on the bundle \(\tilde{N}\), then the curvature of \(\tilde{N}\) is given by
\[
(\nabla)^2 + a_q(dA).
\]

**Remark:** A simple but potentially important observation is that \(t_z\) acts in Eq. (5.2) on the last component \(z_3\) as \(t_z^{-1}\). This introduces the \(-\) sign in the next proposition. Hence the weights are \(\{-\lambda\}\) while the curvatures are given by \(\{\lambda dA\}\). For the bundle defined by \(l_{\mathfrak{g}_\mu}/t\), the weights are \(\{-\beta\} = -\Delta(l_{\mathfrak{g}_\mu})_+\) while the curvatures are \(-\{\beta dA\}\).
5.4. **Transformations of curvatures under** $W_\mu$. Recall that the fixed point sets $F_p \cap \mu^{-1}(\phi)$ transform under the action by the subgroup in $W_{\text{aff}}$, $W_\mu$, which is the Weyl group of $(LG)_\mu$. We want to know first how the transformations act on the normal bundles and how the curvatures change.

If the following, for $w \in W_\mu$, we fix a lifting to $(LG)_\mu$ and it will be denoted the same.

It follows directly from Eq. (5.3), at $(wp, I, z)$, the bundle $V_{wp} = w^*V_p$. Also, $T_{wp} = wT_z$, thus $v \in V_p$ is of non-zero weight with respect to $T_p$ iff $w^*(v)$ is of the same weight with respect to the action by $wT_p = T_{wp}$. Therefore, one has $N_{wp} = w^*N_p$. On the other hand, $T_z$ is the Cartan subgroup of $(LG)^{\mu}\vert_{\text{ss}}$ which is the semi-simple part of $(LG)_\mu$, whence $W_\mu$ is the normalizer of $T_z$. And conveniently we have $\{N_{wp}\}/T_{wp} = w(\{N_p\}/T_p)$. Hence we have proved the first part of the following:

**Proposition 5.3.** 1). The pull-back to $\{N_p\}$ by $w$ of curvature form $\nabla^2 + a_\emptyset(dA)$ of the bundle $\{N_{wp}\}$ at $wp$ is the same as that of $\{N_p\}$ at $p$ under the map $w^*$.

2). The pull-back under $w$ of the bundles $wZ \times_{T_z} (\mathfrak{g}_{\mu}/\text{Lie}K)$, $wZ \times_{T_z} H_z$ have curvatures given respectively by

$$- \oplus_\beta \Delta_{\mu/\Delta(\text{Lie}K)} \mu^{-1}(\beta) \cdot dA, \quad \oplus_\lambda \mu^{-1}(\lambda) \cdot dA.$$

**Pf:** The two expressions in Part 2 can be verified the same way. Let’s take the first one. For each subalgebra $I_{\beta}$, one has the isomorphism:

$$w^* : F_p \cap \mu^{-1}(\phi) \times \text{Ad}_{w^{-1}(\mathfrak{g}_{\beta})} \rightarrow F_{wp} \cap \mu^{-1}(\phi) \times \mathfrak{g}_{\beta}.$$

The action by $T_z$ on $\text{Ad}_{w^{-1}(\mathfrak{g}_{\beta})}$ has weight $-w^{-1}(\beta)$. Therefore the curvature is given by $-w^{-1}(\beta) \cdot dA$. QED

5.5. **More on the curvature of the bundle** $N$. Recall $\tilde{N} = \{N_p\}_{p \in Z}$, and $N$ is the quotient $\tilde{N}/T_z$. The group $K'$ acts on $\tilde{N}$ in a obvious manner, since $K'$ commutes with $T_p$. Let $\nabla$ be a $K'$-invariant connection on the bundle $\tilde{N}$, and the associated moment map $\epsilon$ be defined as

$$<\epsilon(q), (\xi)> := \mathcal{L}_\xi - \nabla_\epsilon \in \text{hom}(N_q).$$

Since we have $PA : TZ \rightarrow \text{Lie}K'$, we can change $\nabla$ to a new connection by adding a 1-form

$$\nabla = \nabla + <\epsilon(q), PA(\cdot) >.$$

The invariant sections of $\tilde{N}$ now are horizontal with respect to $\nabla$.

Using invariant sections on $\tilde{N}$, one proves easily the following:

**Proposition 5.4.** The connection $\nabla$ descends respectively to connections on $N \rightarrow F$ and $\tilde{N}/K' \rightarrow Z/K' = E$. The curvature 2-form on $F$ is the pull-back of that over $E$. In particular the curvature is trivial on the fiber $\pi : F \rightarrow E$.

5.6. **The curvature form of bundles over** $Z_h/T_z$. Along the strata $Z_h/T_z$, there is the normal bundle in $Z$: $\text{nor}^1(Z_h, Z) \oplus \text{Lie}K'/\text{Lie}K_h$ in addition to the restriction to $Z_h/T_z$ of the normal bundle of $Z/T_z \times \{z\}$ in $TX_N|_{Z_h/T_z \times \{z\}}$.

The curvatures and their transformations under $W_\mu$ of these two bundles can be written down similar to what we have just done for $Z$. Recall that $Z_h$ admits a locally free action by $K'_h$, and $Z_h/T_z$ is a fiber bundle with fiber given by a finite quotient of $K'_h/T_z$. On $Z_h$ fix a $K'_h$-invariant connection $A_h$, so that the tangent space is decomposed as $T^H Z_h \oplus T^\perp Z_h$, and $P_h : T^\perp Z_h \rightarrow Z_h \times \text{Lie}K'_h$ is the
trivialization of the vertical bundle. The map $P_h$ performs the same function as that of $P$ for $Z$. Fix a connection for the bundle $T^\perp Z$ which satisfies $\nabla_\xi = L_\xi, \forall \xi \in \text{Lie} K'$. Its existence follows the discussion on $Z$ with respect to the group $K'$.

For convenience, we group the two components in the normal space to $Z$, $(\text{lg})_\mu/\text{Lie} K$ and $\text{Lie} K/\text{Lie} K_h$ together as $(\text{lg})_\mu/\text{Lie} K_h$. They received separate treatment earlier since the first one is normal to $Z$ and the second one is tangent to $Z$ but normal to $Z_h$.

The proof of the following is identical to that of Prop. 5.3, just replace $Z_h$ by $Z$.

**Proposition 5.5.** The curvature of $\nabla$ is the pull-back of a form on $Z_h/K_h$.

The curvature of the bundle $Z_h \times_{T_z} ((\text{lg})_\mu/\text{Lie} K_h)$ on $Z_h/T_z$ is given by $- \sum \epsilon \cdot dP_hA_h$ where $\epsilon$ are the simple roots of $(\text{lg})_\mu$ not in $\text{Lie} K_h$. Furthermore, the curvature form $R_h = dP_hA_h$ can be decomposed into $dP_hA_h = B_h + R_h$

where $B_h(u, v)(pg)$ and $R_h(\xi, \eta)$ satisfy the same property as that of $B, R$ in Lemma 3.4.

For $w \in W_\mu$, the normal bundle at $wp \in Z_{wh}$ is given by

$$wV_p^{01} \oplus wN_p \oplus (\text{lg})_\mu/\text{Lie} K_{wh} \oplus H_z,$$

where $V_p^{01}$ is the maximal $h$-stable subspace of $V_p^0$ on which $\det(I-h) \neq 0$. The curvature of the first two components are the same as those at $p \in Z_h$. The curvature of the last two components are given respectively by

$$- \oplus w^{-1}(\beta) \cdot dP_hA_h, \quad \oplus w^{-1}(\lambda) \cdot dP_hA_h.$$
6. A FUNDAMENTAL FORMULA

In order to prove the desired fixed point formula, we need to understand the contribution from the fixed points in $X_N$ of type 2) and 3). Those do not come as fixed points of $X$ itself under the $T$-action, rather from certain compactification, $X_N$, of the quotient $X$ by the nilpotent subgroup $LG^+$. For in the space $X_N$, there is an open dense set which corresponds to $X/LG^+$.

We will prove in this section an important formula which enables us to calculate the contribution from the type 2) and 3) fixed point set in the next section.

Suppose $\mathfrak{t}$ is the Lie algebra of the semi-simple compact Lie group $K$, with $\mathfrak{t}$ as its Cartan subalgebra, given a choice of Weyl chamber in $\mathfrak{t}$ of $\mathfrak{k}$, assume that $\{\lambda\}$ is the set of fundamental weights in $\mathfrak{t}^*$, $\{\alpha\}$ is the set positive roots, and $W$ is the Weyl group. Let $k$ denote the rank of $\mathfrak{k}$.

For a character of $\mathfrak{t}$, $\mathfrak{l}$, define the symbolic notation

\[
(z')^j = \exp(2\pi i l(x)), \quad z_1^j = \exp(2\pi i l(y))
\]

where $x, y$ are $\mathfrak{t}$-valued forms of even order, e.g., $x = x' + x''$ with $x' \in \mathfrak{t}$ and $x''$ a $\mathfrak{t}$-valued curvature two form. Denote the number of simple roots by $m$, and the set of positive roots by $\Delta^+(K)$, the set of fundamental weights by $\{\lambda\}$.

**Proposition 6.1 (Fundamental Formula).**

\[
\frac{1}{z^m z_1^m \prod_{\alpha \in \Delta^+(K)} (1 - z^{-\alpha})(1 - z_1^{-\alpha})} \sum_{w,v \in W} \frac{(-1)^{m+\sigma(w)+\sigma(v)}}{\prod_{\lambda} (1 - z^{-w\lambda} z_1^{-v\lambda})}
\]

\[
= \sum_{w,v \in W} \frac{1}{\prod_{\alpha} (1 - z^{-w\alpha}) \prod_{\lambda} (1 - z^{-w\lambda} z_1^{v\lambda}) \prod_{\alpha} (1 - z_1^{-w\alpha})}
\]

\[
= 0.
\]

**Pf:** Step 1): Apply the well known formula that

\[
\sum_{w(\alpha) < 0} w(\alpha) = w(\rho) - \rho
\]

where $\rho$ is the half sum of the positive roots, to obtain

\[
\frac{1}{\prod_{\alpha} (1 - z^{-w\alpha}) \prod_{\lambda} (1 - z^{-w\lambda} z_1^{v\lambda}) \prod_{\alpha} (1 - z_1^{-w\alpha})}
\]

\[
= \frac{(-1)^{\sigma(w)+\sigma(v)}}{z^{-w(\rho)+\rho} z_1^{-z(w(\rho)+\rho)} \prod_{\alpha} (1 - z^{-\alpha}) \prod_{\lambda} (1 - z^{-w\lambda} z_1^{v\lambda}) \prod_{\alpha} (1 - z_1^{-\alpha})}.
\]

Next observe that $\sum_{\lambda} \lambda = \rho$, or $\sum w(\lambda) = w(\rho)$. Likewise for $v$, thus

\[
\prod_{\lambda} (1 - z^{-w\lambda} z_1^{v\lambda}) = (-1)^m z^{-w\rho} z_1^{v\rho} \prod_{\lambda} (1 - z^{-w\lambda} z_1^{v\lambda}).
\]

Denote the left side of the equation (6.2) by $L.H.$, after substituting the last expression into (6.3) to get

\[
L.H. = \frac{1}{z^m z_1^m \prod_{\alpha} (1 - z^{-\alpha}) \prod_{\alpha} (1 - z_1^{-\alpha})} \sum_{w,v \in W} \frac{(-1)^{m+\sigma(w)+\sigma(v)}}{\prod_{\lambda} (1 - z^{-w\lambda} z_1^{-v\lambda})}.
\]
First we rename the index $v$ to $vw$, this is legitimate for obvious reason, then

$$
\sum_{w, v \in W} (-1)^{\sigma(w)+\sigma(v)} \frac{1}{\prod_{\lambda} (1 - z^{w\lambda} z^{1_{\lambda}})}
$$

(6.6)

$$
= \sum_{w, vw \in W} (-1)^{\sigma(w)+\sigma(vw)} \frac{1}{\prod_{\lambda} (1 - z^{w\lambda} z^{vw\lambda})}
$$

$$
= \sum_{w, vw \in W} (-1)^{\sigma(v)} \frac{1}{\prod_{\lambda} (1 - z^{w\lambda} z^{vw\lambda})}.
$$

Step 2): Next we shall verify that the last sum vanishes.

It is well known that $\{ \lambda \}$ spans the positive Weyl cone $t^+$, and the set of cones of the form $w(t^+), \forall w \in W$ spans $t$, and there is no overlapping in the interiors of those cones.

Let $S$ denote the set of all possible subsets of $\{ \lambda \}$, and $\forall S \in S$, let $|S| = \#S$, it satisfies $1 \leq |S| \leq m$. The maximum number is $m$.

From [M], one obtains the following relation:

$$
\sum_{S \in S, w \in W} (-1)^{|S|} \prod_{\lambda \in S} \left( 1 - \exp <w\lambda, 2\pi ix > \right) = 1.
$$

(6.7)

From this we deduce that

$$
(-1)^m \sum_{w \in W} \frac{1}{\prod_{\lambda} \left( 1 - \exp <w\lambda, 2\pi ix > \right)} = 1 - \sum_{|S| < k, \forall w \in W} (-1)^{|S|} \prod_{\lambda \in S} \left( 1 - \exp <w\lambda, 2\pi ix > \right).
$$

(6.8)

Let $x = t + v(s)$ with $z = e^{2\pi it}, z_1 = e^{2\pi is}$, we have

$$
\exp <w\lambda, 2\pi ix > = z^{w\lambda} z_1^{vw\lambda}.
$$

Summing over $v \in W$ with sign $(-1)^{\sigma(v)}$, we have

$$
\sum_{w, v \in W} (-1)^{k+\sigma(v)} \prod_{\lambda} \left( 1 - z^{w\lambda} z_1^{vw\lambda} \right) = \sum_{\sigma(v)} (-1)^{\sigma(v)} - \sum_{w, v \in W, |S| < k} (-1)^{|S|+\sigma(v)} \prod_{\lambda \in S} \left( 1 - z^{w\lambda} z_1^{vw\lambda} \right)
$$

the first term in the last line is 0. When $|S| < k$, those weights in $S$ span a face of the Weyl chamber $t^+$. The face, or $S$, is fixed by a non-trivial subgroup of $W$. The subgroup is denoted by $W_S$, which is the Weyl group of a subgroup of $K$. And the subgroup $\text{Ad}_w W_S$ fixes the set of weights $w(S)$. 


Let $W$ be divided into cosets by $\text{Ad}_w W_S$ and let $[v_0]$ denote the coset, then for a fixed $S$ with $|S| < k$, one has

$$\sum_{w,v \in W} \frac{(-1)^{\sigma(v)}}{\prod_{\lambda \in S} \left( 1 - z^w \lambda z^v \lambda \right)}$$

(6.10)

$$= \sum_{[v_0] \in W/\text{Ad}_w W_S, w \in W} \sum_{v \in \text{Ad}_w W_S} \frac{(-1)^{\sigma(v)}}{\prod_{\lambda \in S} \left( 1 - z^w \lambda z^v \lambda \right)}$$

$$= \sum_{[v_0] \in W/\text{Ad}_w W_S, w \in W} \sum_{v \in \text{Ad}_w W_S} \frac{(-1)^{\sigma(v)}}{\prod_{\lambda \in S} \left( 1 - z^w \lambda z^v \lambda \right)}$$

$$= 0$$

because $\sum_{v \in \text{Ad}_w W_S} (-1)^{\sigma(v)} = 0$. Thus we have the desired claim. QED
7. Affine Weyl subgroups and the dual Coxeter numbers

In order to apply the formula proved in the last section, we need to know more about those fixed points of type 2) and 3) occurring along an affine wall of $C$.

Some familiarity with the affine Kac-Moody algebra is required here, see [K] for reference.

Let $(\rho, I, z)$ be a point which defines a fixed point in $X_N$. Assume that $\mu(p)/k = \hat{\phi}(z) \in (\partial C, 1)$, i.e. that fixed point is of type 2) and 3). Let $\mu$ denote $\mu(p)$.

Conventions:
In the following, we treat the case when $\mu$ is of level 1 to simplify the notation. The general case follows after replacing $\mu$ by $\mu/k$.

If $\mu \in \partial C$, there are two possibilities:
a). $\mu \in C \setminus C^{\text{aff}}$, i.e., $\mu$ is not on the wall defined by $\theta = 1$,
b). $\theta(\mu) = 1$, or $\mu \in C^{\text{aff}}$.

In the first case, $W_\mu$ is generated by reflections of a subset of simple roots, and $(LG)_\mu$ is generated by all the $g_\beta$ such that $\beta$ is a simple root vanishing at $\mu$. In particular, $(LG)_\mu$ is a subgroup of $G$ and $W_\mu$ is a subgroup of the regular Weyl group $W$.

In case of b), the affine root $\alpha_0 = \delta - \theta$ plays an important role. And the reflection with respect to $\alpha_0 = 0$ or $\theta = 1$ is the composition of the reflection defined by $\theta = 0$ and a translation element.

To understand this and the group $(LG)_\mu$ more thoroughly, we need a few things from the theory of affine Lie algebra.

7.1. A few facts on affine Lie algebra.

**Proposition 7.1.** Let $\Delta^+$ be the set of simple roots of $\mathfrak{g}$ with respect to the cone spanned by the alcove $C$, $\{\Lambda_i\}$ be the set of fundamental weights of $\mathfrak{g}$.

a). If $\mu$ is not on the affine wall, then

$$\Delta^+_\mu = \{\beta|\beta(\mu) = 0, \beta \in \Delta^+\}$$

is a set of simple roots for $(\mathfrak{l}g)_\mu$. The fundamental weights of $(\mathfrak{l}g)_\mu$ is the orthogonal projection with respect to the form $(\cdot|\cdot)$ of $\{\Lambda_\beta\}$ to the linear span of $\Delta^+_\mu$.

b). If $\mu$ is on the affine wall defined by $\theta = 1$, let

$$\Delta^0_\mu = \{\beta|\beta(\mu) = 0, \beta \in \Delta^+\},$$

$$\Delta^+_\mu = \{\delta - \theta\} \cup \Delta^0_\mu$$

is the set of simple roots. The fundamental weights are given by the orthogonal projection of

$$-\mu, \quad \Lambda_\beta - \alpha_i^\vee \mu, \beta \in \Delta^+_\mu$$

to the linear span of $\Delta^+_\mu$.

In both cases, the group $(LG)_\mu$ is connected and its Weyl group is the subgroup in $W^{\text{aff}}$ generated by reflections using elements in $\Delta^+_\mu$.

**Remark:** The fundamental weights of $(\mathfrak{l}g)_\mu$ may not be weights of $\mathfrak{g}$. Nevertheless, they are in the rational span of $\Delta^+$. 

**Pf:** For a), the Lie algebra $(\mathfrak{l}g)_\mu$ is a subalgebra of $\mathfrak{g}$, generated by $\mathfrak{g}_\beta, \beta \in \Delta^+_\mu$.

The assertions are well known facts about finite dimensional semi-simple algebras.
For part b), first we verify the assertion about the fundamental weights.

\[
2(\Lambda_\beta - a_\beta^\ast \mu | \alpha) / (\alpha | \alpha) = 2(\Lambda_\beta | \alpha) / (\alpha | \alpha) = \delta_\beta \alpha, \quad \alpha \in \Delta_0^0;
\]

\[
(-\mu | \alpha) = 0, \quad \alpha \in \Delta_0^0;
\]

\[
(\Lambda_\beta - a_\beta^\ast \mu | \theta) = a_\beta^\ast - a_\beta^\ast = 0,
\]

the last equality is due to the fact that \((\Lambda_\beta | \theta) = a_\beta^\ast\). Thus, the given set is dual to \(\Delta_\mu^+\), and therefore its orthogonal projection is the set of fundamental weights of \((l g)_\mu\).

The best way to see \(\Delta_\mu^+\) of part b) is a set of simple roots is to use the theory of affine Lie algebras, from which we learn that \(\Delta_\mu^0\) is a set of simple roots for the algebra \(g^{\text{aff}}\). Now it is known that any subset of simple roots is a set of simple roots for the subalgebra generated by the subset. Therefore,

\[
\Delta_\mu^+ = \{o_0 := \delta - \theta\} \cup \Delta_\mu^+
\]

is a set of simple roots. On the other hand,

\[
(\delta | \delta) = 0, \quad (\delta | \alpha) = 0, \quad \forall \alpha \in \Delta,
\]

see [K, Ch. 6]. Thus the inner product of a pair in \(\Delta^\prime\) is the same as that of the corresponding pair in \(\Delta_\mu^+\). In particular, the Dynkin diagrams formed by \(\Delta^\prime\) and \(\Delta_\mu^+\) are the same. Therefore, \(\Delta_\mu\) form a set of simple roots for the subalgebra \((l g)_\mu\).

From the characterization of the simple roots of \((L G)_\mu\), it is clear that its Weyl group is generated by reflections using \(\Delta_\mu^+\). In case \(\mu\) is on an affine wall, the reflections are with respect to \(\{\alpha = 0 | \alpha \in \Delta_0^0\}\) and \(\theta = 1\), as desired.

The connectedness is based on the well known argument in Lie theory. Decompose \((L G)_\mu = \cup K_i\) into connected component, and \(K_0\) contains \(I\). For \(g \in K_i, Ad_g K_0 = K_0\). Multiplying \(g\) with an element in \(K_0\) if necessary, we can assume that \(Ad_g T = T\) where \(T\) is the maximal torus. Therefore, \(g\) is in the Weyl group of \(K_0\), therefore \(g \in K_0\). QED

7.2. The half sum of positive roots of \((L G)_\mu\). Let \(\Delta_\mu\) be as before, and \(\{\lambda_i\}\) be the set of fundamental weights of \((l g)_\mu\). We have seen that \(\lambda_i\) is given by the orthogonal projection of \(\Lambda_\beta - a_\beta^\ast \mu\).

Let \(\rho_\mu\) be the half sum of positive roots of \((l g)_\mu\). For finite dimensional Lie algebra, it is well known that

\[
\rho_\mu = \sum_i \lambda_i.
\]

The following is as important as the fundamental formula:

**Proposition 7.2.** Let \(\rho\) be the half sum of positive roots of \(g\), \(\rho_{\text{aff}} = \rho + h^\ast \Lambda_0\), where \(h^\ast\) is the dual Coxeter number defined by \(h^\ast - 1 = \sum_{i=1,...,l} a_i^\ast\), then the following holds:

1. If \(\theta, \phi > 1\):

\[
w(\rho_\mu) - \rho_\mu = w(\rho) - \rho \mod \mathbb{Z} \delta, \quad \forall w \in W_\mu
\]
and
\[ 2\pi i \langle v \kappa \phi - \rho + v \rho \mu, t \rangle = e^{2\pi i \langle k \phi + \rho \mu, t \rangle} . \]

2). If \( \theta, \phi \geq 1 \) is on the affine wall of \( C \),
\[ r(\rho_\mu) - \rho_\mu = r(\rho) - \rho \mod \mathbb{Z} \delta \]
where \( r \) is a reflection defined by a simple root of \( \mathfrak{g} \). Let \( r_\theta \) be the reflection with respect to the affine wall \( \phi(\theta^\nu) = 1 \),
\[ r_\theta(\rho_\mu) - \rho_\mu = r_\theta(\rho) - \rho + h^\nu \theta, \]
where \( \nu : \mathfrak{g}^{\text{aff}} \to \mathfrak{g}^{\text{aff}^*} \) is the map induced by the bilinear form \( \langle \cdot | \cdot \rangle \) on \( \mathfrak{g}^{\text{aff}} \).

And when \( t \) is restricted to the lattice \( \mathcal{M}^*_{k+h^\nu} \), with \( M^* \) being the dual of the long root lattice,
\[ e^{2\pi i \langle v k \phi + v \rho + v \rho_\mu, t \rangle} = e^{2\pi i \langle k \phi + \rho_\mu, t \rangle}, \forall v \in W_\mu^0 \]
where \( W_\mu^0 \) is the subgroup of \( W_\mu \) generated by the reflections with respect to \( \alpha \in \Delta_\mu^0 \) and \( \theta \). It is isomorphic to \( W_\mu^0 \).

Pf: 1). It is enough to verify that for reflection \( r_\alpha, \alpha \in \Delta_\mu^+ \). In this case, \( \alpha \) is also a simple root of \( \mathfrak{g} \). Thus,
\[ r(\rho_\mu) - \rho_\mu = -\alpha = r(\rho) - \rho. \]
In this case, all the simple roots in \( \Delta_\mu^+ \) vanish at \( \tilde{\phi} \) by definition of \( (l_\mu) \). Therefore, \( v_\phi = \phi \), the eq. (7.3) holds as a consequence of the identity just proved.

2). If \( r \) is generated by a simple root of \( \mathfrak{g} \), which is the case if \( \alpha \in \Delta_\mu^0 \), then the previous argument works. If \( r = r_\theta \), \(-\theta \) is a simple root of \( (l_\mu) \) but not of \( \mathfrak{g} \), we obtain
\[ r_\theta(\rho_\mu) - \rho_\mu = \theta \mod \mathbb{Z} \delta. \]
On the other hand
\[ r_\theta(\rho) - \rho = -\langle \rho, \theta^\nu \rangle > \theta = -(h^\nu - 1)\theta, \]
where the last equation is from the definition of \( h^\nu \):
\[ h^\nu = 1 + \sum_i a_i^\nu = 1 + \langle \rho, \theta^\nu \rangle . \]
Now
\[ r_\theta k \phi = k \phi - k < \phi, \theta^\nu > = k \phi - k \theta, \]
thus we obtain
\[ e^{2\pi i \langle r_\theta k \phi - r_\theta \rho - r_\theta \rho_\mu, t \rangle} = e^{2\pi i \langle k \phi - (k + h^\nu)\theta - \rho_\mu, t \rangle} \]
which equals \( e^{2\pi i \langle k \phi - \rho_\mu, t \rangle} \) on the lattice since
\[ \langle (k + h^\nu)\theta, t \rangle \in \mathbb{Z}, \forall t \in \mathcal{M}^*_{k+h^\nu}. \]
The above holds now for the generators in \( W_\mu^0 \), therefore it holds on the lattice \( \mathcal{M}^*_{k+h^\nu} \) for all \( W_\mu^0 \).

The isomorphism between \( W_\mu \) and \( W_\mu^0 \), the only difference among the generators is the form has a reflection w.r.t. \( \theta = 1 \) while the latter has one w.r.t. \( \theta = 0 \). QED
8. Further orbifold complications

In order to apply fixed point principle to the space $X_N$, we need better understanding of the action by $T$ on the normal bundles to the fixed point set of all three types.

8.1. Weights on the toric variety $X_\phi$. We divide the discussion on the normal bundles according to the types of the fixed point sets.

We continue to use the convention from the last section.

8.2. When $\mu$ is on $(\partial C, 1)$. This is the more interesting case. Recall that $X_\phi$ is constructed as a global orbifold toric variety. First we investigate what is the stabilizer and the weights at a point on $X_\phi$.

From the last section, we have learned that the fundamental weights of $(LG)_\mu$ is given by orthogonal projections of the following vectors:

$$\Lambda_\beta - a_\beta^\gamma \mu, \quad \beta \in \Delta_\mu$$

together with $\mu$ if $\alpha_0 = \delta - \theta \in \Delta_\mu$. In the above, $\Lambda_\beta$ is a fundamental weight of $g$ as well since $\beta$ is a simple root of $g$. Because $X_\phi$ is an orbifold, the polytope $C$ is not a simple convex polytope (see [O] for definition) with respect to the weight lattice $M$ is not a simple convex polytope (see [O] for definition) with respect to the weight lattice $M$ generated by $\{\Lambda_\beta\}$, but rather it is a simple polytope in the larger lattice $M'$ generated by $\{\Lambda_\beta/a_\beta^\gamma\}$. The larger lattice defines a unique covering of $T$, $T'$ so that $\pi T' \to T$ has the quotient $M'/M$ as its kernel.

The dual lattice of $M$ is given by the coroot lattice

$$N = \sum \mathbb{Z} \alpha_\gamma^\ast.$$ 

The dual lattice of $M'$ is given by the sublattice

$$N' = \sum \mathbb{Z} (a_\gamma^\ast \alpha_\gamma^\ast).$$

So $T = \mathbb{R}^l/N$ and $T' = \mathbb{R}^l/N'$.

The polytope $C$ is integral with respect to $M'$, and it is actually a simple simplex. Thus it defines smooth toric variety $X'$ with respect to the group $(T')^C$, $X'$ is in fact the projective space $\mathbb{C}P^l$, and $X_\phi = X'/\ker \pi$, see [Od, p96].

**Proposition 8.1.** On $X' \simeq \mathbb{C}P^l$, assume $z' \in \phi^{-1}(\partial C)$. The stabilizer of $z'$, $T_z'$ and the weights of the action by $T_z'$ on the normal bundle is given by the following:

1. If $\mu = \phi(z')$ is not on the affine wall, the stabilizer is given by

$$\sum_{\beta \in \Delta_\mu} \mathbb{R} \beta^\gamma / (\sum_{\beta \in \Delta_\mu} a_\beta^\gamma \beta^\gamma).$$

The weights are given by the orthogonal projection to $\nu(x'_z)$ of

$$\Lambda_\beta/a_\beta^\gamma, \forall \beta \in \Delta_\mu.$$ 

2. If $\mu$ is on the affine wall, recall $\Delta_\mu = \{-\theta\} \cup \Delta^0_\mu$ and $\Delta^0_\mu = \{\beta \in \Delta_\mu | \beta(\mu) = 0\}$. The coroot $\theta^\gamma \in N'$ by definition is $\sum a_\gamma^\ast \alpha_\gamma^\ast$.

In particular $\sum a_\beta^\gamma \beta^\gamma$ is a sublattice of $N'$, where $a_\gamma^\ast = 1$. The stabilizer again is given by

$$\sum_{\beta \in \Delta_\mu} \mathbb{R} \beta^\gamma / (\sum_{\beta \in \Delta_\mu} a_\beta^\gamma \beta^\gamma) \subset T'.$$

The weights of the stabilizer are the orthogonal projection to $\nu(x'_z)$ of

$$\{-\mu\} \cup \{\Lambda_\beta/a_\beta^\gamma - \mu | \beta \in \Delta^0_\mu\}.$$
Therefore, the Lie algebra of the stabilizer \( t' \) is of the desired form. Clearly the lattice \( \sum_{\beta \in \Delta_\mu} \mathbb{Z}a_\beta \beta^\vee \) is in \( t'_\mu \), and in fact it is \( t'_\mu \cap N' \). Thus the stabilizer of \( z' \) in \( T'_\mu \) is \( \nu^{-1}(t'_\mu \cap N') \) whose explicit form is given by the proposition.

To understand the claim on the weights of the action by the stabilizer on the normal bundle, we recall first that each point on the toric variety \( X' \), the neighborhood is constructed as follows:

Let \( A \) be the tangent cone of the simplex \( C \) at the point \( \phi(z') \), \( A \) is a convex cone. Take the semi-group \( \sigma_\mu = N' \cap A \). Because \( C \) is a simple simplex, it is easy to see that

\[
\sigma_\mu = \sum_i \mathbb{Z}_{\geq 0} \eta_i + \sum_j \mathbb{Z} \xi_j,
\]

where \( \{\eta_i\} \cup \{\xi_j\} \) is a base of the lattice \( N' \). Since \( \sum_i \mathbb{Z} \xi_j \) is a sublattice, it is given by the lattice points in the maximal linear subspace contained in \( A \). Then the action of \( T' \) on a neighborhood of \( z' \) in the toric variety \( X' \) is given by

\[
t(z_1, \ldots, z_m, w_1, \ldots, w_n) = (t^m z_1, \ldots, t^m z_m, t^{\xi_1} w_1, \ldots, t^{\xi_n} w_n),
\]

the point \( z' \) correspond to a point with \( w_j = 0, z_i \neq 0 \) which also defines the fixed point set of the subgroup \( T'_\mu \). The above facts about toric varieties can be found in Ch. 1.2 and Ch. 2.4 in [Od]. From this it is easy to read off the weights by the action of \( T'_\mu \) near \( w_i = 0 \). They are given by the restriction of those weights \( \xi_j \) to \( T'_\mu \). Or the projections to \( \nu^{-1}t'_\mu \) of \( \{\xi_j\} \).

What are those weights \( \{\eta_i\} \cup \{\xi_j\} \)? First select a lattice point \( \Lambda_i/a_i^\vee \) on the affine subspace spanned by the smallest face \( \partial C_\mu \) passing \( \mu \), then

\[
\{\eta_k\} = \{\Lambda_k/a_k^\vee - \Lambda_i/a_i^\vee \in V_\mu\}, \quad \{\xi_j\} = \{\Lambda_j/a_j^\vee - \Lambda_i/a_i^\vee \notin V_\mu\}.
\]

On the other hand, if one replace \( \Lambda_i/a_i^\vee \) by a point on \( \partial C_\mu \), such as \( \mu \) itself, the projection to the orthogonal complement of \( V_\mu \) does not change. Therefore, the weights of the action by \( T'_\mu \) can also be given by the orthogonal projections of

\[
\{\Lambda_j/a_j^\vee - \mu | \Lambda_j/a_j^\vee - \mu \notin V_\mu\}.
\]

In the above construction, \( \{\Lambda_j/a_j^\vee = 0 \) is allowed to account for the weight which is the projection of \( -\mu \). QED

**Definition 8.1.** Let \( \epsilon' \) be an element of the coroot lattice so that it is given by

\[
(1/n) \sum_{\alpha \notin \Delta_\mu} a_\alpha \alpha_i \epsilon', \quad \text{where } n \geq 1 \text{ and no fraction of } \epsilon \text{ is in the coroot lattice.}
\]

Obviously, for \( su(l+1) \), there is only one choice \( n = 1 \).

Notice the definition of \( n \) depends on \( \Delta_\mu^0 \).
Corollary 8.1. 1). When \( \langle \phi, \theta \rangle < 1 \) where \( \phi = \phi(z) \in t^* \), the group \( T'_z/T_z \) is given by the finite group
\[
\exp(\sum_{\alpha_i \in \Delta_{\mu}} b_i \alpha_i^\vee), \quad 0 \leq b_i < a_i^\vee.
\]

2). When \( \langle \phi, \theta \rangle = 1 \) where \( \phi = \phi(z) \), the subgroup \( T'_z/T_z \) is given by
\[
\exp(b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee), \quad 0 \leq b < n, \quad 0 \leq b_i < a_i^\vee.
\]

\( Pf: \) The group in question satisfies \( T'_z/T_z \simeq N/N' \). From the expression of \( N, N' \), we can identify the elements in the kernel of the map \( T'_z \rightarrow T_z \) easily.

The second assertion is based on the simple observation that
\[
n e^\vee = \theta^\vee \mod \sum_{\beta \in \Delta_{\mu}^0} \mathbb{Z} \alpha_{\beta}^\vee \beta^\vee.
\]

Now we claim \( \{n e^\vee\} \cup \{\alpha_{\beta}^\vee \beta^\vee\}_{\beta \in \Delta_{\mu}^0} \) spans \( N' \) as well. To verify the claim, let \( m \) be in the sub-lattice \( N \cap (\sum_{\alpha_i \in \Delta_{\mu}^0} \mathbb{R} \alpha_i^\vee + \mathbb{R} \theta^\vee) \), then \( m = \sum_{\alpha_i \in \Delta_{\mu}^0} r_i \alpha_i^\vee + r \theta^\vee \), let \( \Lambda_i \) act on both sides, by assumption \( \Lambda_i(m) \in \mathbb{Z} \), therefore \( m_i = r_i + ra_i^\vee = \Lambda_t(m) \in \mathbb{Z} \).

So \( m = \sum_{\alpha_i \in \Delta_{\mu}^0} (m_i - ra_i^\vee) \alpha_i^\vee + r \theta^\vee \). Replace \( m \) by \( m' = -\sum_{\alpha_i \in \Delta_{\mu}} ra_i^\vee \alpha_i^\vee + r \theta^\vee = r \sum_{\alpha_i \notin \Delta_{\mu}^0} \alpha_i^\vee \alpha_j^\vee \), where in the last equation the definition of \( \theta^\vee \) is used. Clearly \( m' \) is in the same sub-lattice as \( m \), and \( m' \) is a integer multiple of \( e^\vee \) because \( ra_i^\vee \in \mathbb{Z} \).

This proves the claim.

From there, it is easy to identify what \( N/N' \simeq T'_z/T_z \) is. QED.

8.3. The transformation of \( T'_z/T_z \).

Lemma 8.1. Under the action by \( W_{\mu} \), the isotropy group \( T'_z/T_z \) transforms into itself.

For \( \phi \) with \( \langle \phi, \theta \rangle = 1 \) and \( \mu = (\phi, 1) \),
\[
e^{2\pi i \langle \phi, w(b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee) \rangle} = e^{2\pi i \langle \phi, w e^\vee \rangle} = e^{2\pi i \langle \phi, b e^\vee \rangle}.
\]

\( Pf: \) The first one is easy to verify using reflections defined by simple roots of \((LG)_\mu\), to be more explicit:
\[
r_\beta(b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee) = b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee - \langle b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee, \beta \rangle \beta^\vee
\]
where \( \beta \) is either \( \theta \) or in \( \Delta_{\mu}^0 \). In either case, because the coefficient of \( \beta^\vee \) above is in \( \mathbb{Z} \), after mod out the lattice defining \( T'_z \), it is clear that the element above is in \( T'_z/T_z \).

For the second part, first observe that \( e^{2\pi i \langle \phi, b e^\vee \rangle} = 1 \) if \( \beta \in \Delta_{\mu}^0 \) or \( \beta = \theta \). So the reflections do not change the value, hence it is invariant under \( W_{\mu} \). Or
\[
e^{2\pi i \langle \phi, w(b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee) \rangle} = e^{2\pi i \langle \phi, b e^\vee + \sum_{\alpha_i \in \Delta_{\mu}^0} b_i \alpha_i^\vee \rangle} = e^{2\pi i \langle \phi, b e^\vee \rangle}. \quad QED
\]

8.4. The relations among the groups \( T_z, T'_z \) and the maximal torus in \((LG)^{\mathbb{R}}_\mu\). We have just studied the relation between \( T_z, T'_z \). A third Abelian group is the maximal torus \( S \) of \((LG)^{\mathbb{R}}_\mu\), the three share the same Lie algebra. The difference is the defining lattice.
**Lemma 8.2.** The three groups are related as:

\[ T_z' \to S \to T_z \]

where each arrow is a covering. When \( \theta(\phi) \neq 1 \), \( S \simeq T_z \). When \( \theta(\phi) = 1 \), \( S/T_z \simeq \mathbb{Z}/n\mathbb{Z} \) where \( n \) is defined before.

*Pf:* If \( \theta(\phi) \neq 1 \), the coroot lattice of \( (L G)_{\mu}^{ss} \) is given by \( \{ \alpha^\vee \}_{\alpha \in \Delta^+_T} \), where each \( \alpha^\vee \) is also a coroot of \( g \). So the lattice defining \( T_z' \) is the same as that defines \( S \).

If \( \langle \theta, \phi \rangle = 1 \), the coroots are \( \{-\theta^\vee\} \cup \{\alpha^\vee\}_{\alpha \in \Delta^0} \) which form the lattice defining \( S \). On the other hand, the lattice defining \( T_z \) is generated by \( e^\vee \) and \( \{\alpha^\vee\}_{\alpha \in \Delta^0} \), therefore the claim is verified. QED
9. A couple of integration formulas

One of the key steps in proving the cancellation formula is the following evaluation of certain differential forms on a space which is a fiber bundle. To be more precise, let $Z ightarrow F ightarrow E$ be an sequence so that $F = Z/S$ and $E = Z/K$ where $S$ is a maximal torus of $K$ and $K$ admits a local free action on $Z$.

We shall use the notations introduced in Sect 1 on the connection $dPA$ defining the vertical and horizontal parts of $F$. Let $B, R$ be the vertical and horizontal part of $dPA$ respectively.

There is one exception here, $S$ is used instead of $T_z$ and $K$ is used instead of $K''$.

**Proposition 9.1.** 1. Let $\epsilon$ be a weight of $S$, then the bundle $Z \times_S \mathbb{C}$ with $(ps, v) \simeq (p, s'v)$ defines a line bundle on $Z/S = F$ with curvatures given by $\epsilon, dPA >$. The Chern class is given by $<\epsilon, B + R>$.

2. The following holds:

$$\text{Td}(TF) = Td(T^H F) \cdot Td(T'' F), \text{Td}(T^H F) = \pi^* \text{Td}(E)$$

where $\Delta_+$ is the set of positive roots of Lie$K$.

3. Localization for a family:

$$\int_{\pi^{-1}(p)} \text{Td}(TF) e^{i/2\pi <\epsilon, B + R>} = \text{Td}(E) \sum_{u \in W(K)} \prod_{\tau \in \Delta_+} \frac{e^{i/2\pi <u, B>}}{(1 - e^{-i/2\pi <\tau, uB>})}.$$  

(This equation and the next should be viewed as identities about differential forms.)

4. 

$$\int_{\pi^{-1}(p)} \frac{\text{Td}(TF)}{\prod_{\tau \in \Delta_+} (1 - e^{-i/2\pi <\tau, uB>})} = \text{Td}(E) \sum_{u \in W(K)} \prod_{\tau \in \Delta_+} \frac{1}{(1 - e^{-i/2\pi uB>})(1 - e^{2\pi i <\tau, uB>})}.$$  

(Pf: The first part repeats Prop. 5.1. Since $TF = T^H F \times T'' F$, the Todd class satisfies the equality

$$\text{Td}(F) = \text{Td}(TF) = \text{Td}(T^H F) \times \text{Td}(T'' F).$$

On the other hand, as discussion in Sect.1, $T'' F \simeq Z \times_S \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{s}^+ \subset \mathfrak{t}$. Hence

$$T'' F = \oplus_{\tau \in \Delta_+} Z \times_S \mathbb{C},$$

where $-\tau$ is the character of $S$ acting on $\mathbb{C}$, the sign reflects the choice of the complex structure on $X \times X$. According to Part 1), the Chern class of the line bundle is represented by $- <\tau, B + R>$ and the Todd class of $T'' F$ is the product of the Todd class of the line bundle corresponding to each positive root $\tau$. Thus the expression is verified.

To see 3), we employ the localization of equivariant cohomology class

$$\int_{K/S} \prod_{\tau \in \Delta_+} \frac{-2\pi i <\tau, 1/4\pi^2 + X > e^{2\pi i <\epsilon, X + 1/4\pi^2} R)}{(1 - e^{2\pi i <\tau, X + 1/4\pi^2} R)} = \sum_{u \in W(K)} \frac{e^{2\pi i <\epsilon, uX>}}{(1 - e^{2\pi i <\tau, uX>})}.$$
the above is an identity for analytic functions in $X, \forall X \in \mathfrak{s}$. If we plug the $\mathfrak{s}$-valued 2-form $B$, instead of $X$, we end up with an equality of forms. This is exactly the claim of Part 3).

To see the claim of Part 4, inserting $0 < r < 1$ and introducing the multi-index $\vec{n} \in \mathbb{Z}^m$ with $m = \#\{\epsilon\}$, and $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_m)$. Then

\[
\int_{\pi^{-1}(p)} \prod_{r}(1 - r e^{2\pi i \epsilon_t + 1/4\pi^2 R + 1/4\pi^2 B}) Td(TF) \cdot \prod_{\epsilon}(1 - r e^{2\pi i \epsilon_t + 1/4\pi^2 R + 1/4\pi^2 B}) = Td(T^H F) \sum_{\vec{n}} r |\vec{n}| e^{2\pi i \vec{n} \cdot \vec{\epsilon} t} \int_{\pi^{-1}(p)} Td(T^H F) e^{<\vec{n}, \vec{\epsilon}, i/2\pi(R+B)>}
\]

(9.3)

\[
\int_{\pi^{-1}(p)} \prod_{r}(1 - r e^{2\pi i \epsilon_t + 1/4\pi^2 R + 1/4\pi^2 B}) Td(TF) \cdot \prod_{\epsilon}(1 - r e^{2\pi i \epsilon_t + 1/4\pi^2 R + 1/4\pi^2 B}) = Td(T^H F) \sum_{\vec{n}} r |\vec{n}| e^{2\pi i \vec{n} \cdot \vec{\epsilon} t} \prod_{u \in W(K)} \frac{1}{\prod(1 - e^{i/2\pi <\tau, uB>})}
\]

once the formula is established for $0 < r < 1$, we can take limit $r \to 1$. QED
10. $T, G$-SPACES AND THE CONSEQUENCES OF THE MAIN RESULT

Before we tackle the technical difficulty, the main cancellation, let’s first see a couple of consequences of the main theorem.

There are two groups of results presented here, one is in the general case and the other is the holomorphic case.

10.1. Pass from $T$-modules to $G$-modules. If $V_T$ is a $T$-module with weight vectors in $t^*_+\subset t^*$, then one can apply the holomorphic induction to get a $G$-module from it. Namely, let

$$V_T = \oplus_{\lambda \in t^*_+} m_\lambda C_{v_\lambda}$$

where $v_\lambda$ is a weight vector with weight $\lambda$ and $m_\lambda \in \mathbb{Z}_+$ the multiplicity of the weight $\lambda$, then define

$$V_G = \oplus_{\lambda \in t^*_+} m_\lambda V_\lambda$$

where $V_\lambda$ is the unique highest weight $G$-module with $\lambda$ as the highest weight.

The above is just the usual holomorphic induction. Apply this construction to $H^0(X, L_N)$ to get a $G$-module which is denoted by $V_G(X_N, L_N)$.

The long passage we have taken that started from the $\widetilde{LG}$-module $H^0(X, L)$ to $H^0(X_N, L_N)$, and then $V_G(X_N, L_N)$ has some advantage. It is relatively easy to construct $X_N$ which should be viewed as the compactified quotient of $X$ by the maximal Borel $B^+ \subset LG$. We could have taken the quotient by $B^+_I \subset B^+$ which is $I$ at a fixed point on the circle. It would be more difficult to find its compactification, and will be done on another occasion. Although the space $X_N \times_T G$ which will be discussed more in the next section can be thought of as poor man’s version of $X/B^+_I$.

10.2. The character functions of $T$ and $G$ modules. The character functions of the modules $V_T, V_G$ are related. Let

$$\chi_G(g) = \text{tr}(t|_{V_G}), \quad \chi_T(t) = \text{tr}(t|_{V_T}).$$

It is known that $\chi_G(g)$ is a class function, so its values are determined by the restriction to $t \in T$. As an easy application of the famed Weyl character formula, one has the following:

$$\chi_G = \sum_{w \in W} w \cdot \frac{\chi_T}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})}.$$ (10.1)

10.3. Relating $T, G$-spaces. The description above has a generalization. Suppose that $P$ is a compact symplectic manifold (or orbifold) and $V$ is a complex line bundle on it, and the pair admits an action by $T$, so that the action by $T$ on $P$ is Hamiltonian. Assume the above data fit together in the sense of geometric quantization, c.f. [GS]. Then one can define the following $T$-equivariant Riemann-Roch:

$$\text{RR}^T_P(t) = \int_P \text{Td}(TP) \text{Ch}(V)(t),$$ (10.2)

the above is also the equivariant index of a $\text{spin}^c$-complex. Via the fixed point formulas of Atiyah-Bott-Segal-Singer, the above can be written as contribution as $\sum_F FC_P(t)$ where $\{F\}$ is set of connected components of fixed points, $FC_P(t)$ is an integral on $F$ involving Td, Ch and equivariant classes of the normal bundle of $F$ in $P$. The exact expression will be given later.
By assumption on \( P \), there is a moment map \( \phi : P \to t \). Suppose that \( \phi(P) \subset t^+ \) which is the positive Weyl chamber of \( g \), we can associate a \( G \)-space with \( P, P \times_T G \), a \( G \)-bundle with \( V \) over \( P \times_T G, \pi^*(V)/T \) where \( \pi : P \times G \to P \) is the projection, the action by \( t \in T \) on \( \pi^*(V) \) is given by the following:

\[
(p, g, v) \in \pi^*(V) \mapsto (tp, tg, t^{\phi(p)}v).
\]

The following proposition is an easy exercise, as an application of the Atiyah-Bott-Segal-Singer fixed point formula on both \( T \) and \( P \times_T G \):

**Proposition 10.1.**

\[
RR^G_{P \times_T G}(t) = \sum_{w \in W} RR^T_P(wt) \frac{RR^T_P(wt)}{\prod_{\alpha \in \Delta^+} (1 - (wt)^{-\alpha})}
\]

where the left side represents the \( G \)-equivariant Riemann-Roch number associated with \( (P \times_T G, \pi^*(V)/T) \).

**10.4. Contribution from fixed point sets.** Given a connected component \( F \) of \( T \)-fixed point set in \( X_t = \mu^{-1}(t \times \{k\}) \), define

\[
FC_F(t) = \int_F \frac{Td(F) \text{Ch}(L|_F)}{\text{det}(1-t^{-1}e^{-\Omega})\text{nor}^C(F,X_t)},
\]

where \( Td(F), \text{Ch}(L|_F, t) \) are Todd class and \( T \)-equivariant Chern class of \( TF, L|_F \) respectively; the denominator is the standard equivariant class in the finite dimensional fixed point formula of the complex normal bundle of \( F \) in \( X_t \). The existence of the complex structure on the normal bundle has been shown in Section 1.

**10.5. Coefficients of modular transformations.** Use the notations introduced in Section 1, and let \( \chi_a \) be the character function of the highest weight \( G \)-module defined by \( a \in P_+ \). The following is in [K, Ch. 13]:

**Proposition 10.2.**

\[
\frac{(-1)^l}{|M_0(Z, k + h^+)|} \sum_{\lambda \in P_+^k} (\chi_b \cdot \chi_{\bar{a}} \cdot D^2)(e^{2\pi i\nu^{-1}(\frac{\lambda + \rho}{\Delta})}) = \delta_{b,a},
\]

where \( \bar{a} \) is the weight whose highest weight module is contragredient to the one defined by \( a \). (or \( \bar{a} = w_L(-a) \) with \( w_L \) being the longest element in \( W \)).

**10.6. Consequence of the main theorem.** Assume the main theorem, how can we determine the function \( RR(Y) \) from its values on the subset \( \{ e^{2\pi i\nu^{-1}(\frac{\lambda + \rho}{\Delta})} | \lambda \in P_+^k \} \)? We will prove Cor. 1.1 here.

**Lemma 10.1.** 1).

\[
RR^T(X_N, L_N)(t) = \sum_{a \in P_+^k} m_a t^a
\]

where \( P_+^k \) is the set of weights in \( kC \).

2). \( m_a = RR(M_a, L_a) \) the Riemann-Roch number of the pair \( (M_a, L_a) \) where \( M_a = \phi^{-1}(a)/T \) and \( L_a \) is the induced orbifold line bundle on \( M_a \).

3). \( RR^G(Y, L_Y) = \sum_{a \in P_+^k} m_a \chi_a \) where \( \chi_a \) is the character function of the highest weight \( G \)-module defined by \( a \).
The first assertion follows from an important fact that \( m_a = 0 \) if \( a \) is outside the image of \( k\phi \). We already know that the image lies in \( kC \), so \( a \) must be in \( kC \) if \( m_a \neq 0 \). That \( a \) has to be a weight, because \( \text{RR}^T(X_N, L_N)(t) \) is a function on \( T \).

Part 2 follows the affirmative solution (the Abelian case) to a conjecture by Guillemin-Sternberg.

The last part uses part 1 together with the Weyl character formula:

\[
\text{RR}^G(Y, L_Y) = \sum_{w \in W} w \cdot \frac{\text{RR}^T(X_N, L_N)}{\prod_{a \in \Delta^+} (1 - e^{-a})} \\
(10.7)
\]

\[
= \sum_{w \in W} w \cdot \frac{\sum_{a \in P^+_\mathbb{R}} m_a t^a}{\prod_{a \in \Delta^+} (1 - e^{-a})} \\
= \sum_{a \in P^+_\mathbb{R}} m_a \chi_a. \quad \text{QED}
\]

**Pf. of Cor. 1.1:** Let \( \text{RR}(Y) = \sum_{a \in P^+} m_a \chi_a \), then from the discussion of the \( T \) and \( G \)-spaces, we know that

\[
\text{RR}(X_N)(t) = \sum_{a \in P^+} m_a t^a.
\]

Thus only those \( \{a\} \) inside \( kC \) will occur, since \( \phi(X_N) \subset kC \). So one can write \( \text{RR}(Y) \) as \( \sum_{a \in P^+} m_a \chi_a \). Multiplying \( D^2 \chi_\alpha \) on both sides and sum over

\[
\tau \in \{ e^{2\pi i \nu - \frac{\lambda + \nu}{\lambda + \nu}} | \lambda \in P^k_{+} \}
\]
to get

\[
m_a = \frac{(-1)^l}{M \prod_{(k+1)} M} \sum_{\tau} \left( \text{RR}(Y) \cdot D^2 \cdot \chi_\alpha \right)(\tau) \\
(10.8)
\]

\[
= \frac{(-1)^l}{M \prod_{(k+1)} M} \sum_{\tau} \chi_\alpha(\tau) \cdot D^2(\tau) \left( \sum_{\{F|\mu(F) \in kW(C^\infty)\}} \frac{FC_F(\tau) \cdot R(\tau)}{\mu(F)} \right).
\]

QED

10.7. **Application to the holomorphic case.** Assume \( X \) is holomorphic, and it satisfies the conditions that \( \mu \) is both transversal and proper.

The result in this subsection does not rely on the main theorem of [C1]. First we write down the character formula/fixed point formula of the module \( V_G(X_N, L_N) \) in terms of the fixed points on \( X \).

**Theorem 10.1.** Assume \( H^0(X_N, L_N) = \text{RR}(X_N, L_N) \), which holds if the higher cohomology groups vanish.

1). \( H^0(X_N, L_N) \cong \sum_{a \in P^+_\mathbb{R}} m_a C_a \) where \( C_a \) is the \( T \)-module on which \( T \) acts with weight \( a \).

2). Let \( \chi_G(e^{it}), t \in \mathbb{T} \) be the character of the \( G \)-module \( V_G(X_N, L_N) \), then

\[
\chi_G(\tau) = \left( \sum_F \text{FC}_F + R(\tau) \right)(\tau), \quad \tau \in \{ e^{2\pi i \nu - \frac{\lambda + \nu}{\lambda + \nu}} | \lambda \in P^k_{+} \}
\]

Furthermore, the multiplicity is given by an irreducible component with the highest weight \( a \) is given by the same expression as \( m_a \) in Eq. (10.8).
Pf:

\[ \chi_T(X_N, L_N)(t) := \text{tr}(t | H^0(X_N, L_N)) \]
\[ = \sum_i (-1)^i \text{tr}(t | H^i(X_N, L_N)) \]
\[ = \text{RR}(X_N, L_N) \]
\[ = \sum_{a \in P^+_k} m_a t^a. \]

(10.10)

The first part follows.

Using the earlier result relating \( \chi_T(X_N, L_N) \) and \( \chi_G \), we have

\[ \chi_G = \sum_{w \in W} w \chi_T(X_N, L_N) \]
\[ \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \]
\[ = \sum_{w \in W} w \text{RR}(X_N, L_N) \]
\[ \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \]
\[ = \text{RR}(Y), \]

(10.11)

where the last equality follows from Prop. [10.1]. From there, the assertion is shown to be true by the main theorem and Cor. 1.1. QED

10.8. The character function of the \( \tilde{L}G \)-modules. Now we can derive a character formula for the representations of \( \tilde{L}G \) on \( H^0(X, L) \), assuming the main theorem of [C1].

The derivation here requires certain formal manipulations. The formal aspect to the approach here can be traced back to the derivation of Weyl-Kac character formula.

Let \( \chi_{\tilde{L}G}(X, L)(t) \) be the trace of \( t \) acting on the part of \( H^0(X, L) \) generated by the highest weight modules of level \( k \). The qualifier here about the trace is included because we do not know at this point whether \( H^0(X, L) \) is generated by the highest weight modules. If \( H^0(X, L) \) is a representation of finite energy, then automatically it has the desired quality, see [PS].

**Theorem 10.2.** Define the regular and reduced Weyl-Kac denominators as follows

\[ D_{WK} = \prod_{\tilde{\alpha} > 0} (1 - e^{-\tilde{\alpha}}), \quad D_{0}^{WK} = \prod_{\tilde{\alpha} > 0, \tilde{\alpha} \not\in \Delta^+} (1 - e^{-\tilde{\alpha}}) \]

where \( \tilde{\alpha} \) in the first function runs through all the positive roots of \( \tilde{g} \), while the second does not contain the positive roots \( \tilde{\alpha}, \Delta^+ \). Under the same assumptions as in Thm 10.4, the character \( \chi_{\tilde{L}G}(X, L)(t) \) is given by

\[ \chi_{\tilde{L}G}(X, L)(t) \]
\[ = \sum_{F \in \mathcal{F}} \int_{F} \frac{Td(F)\text{Ch}(L|_{F}, t)}{\det(I - t^{-1}e^{-\Omega})|_{\text{nor}^c(F, X)}} + \sum_{w \in W_{\text{aff}} / W} w \frac{R(\tau)}{D_{0}^{WK}}. \]

(10.12)

where \( \text{nor}^c(F, X) \) is the complex normal bundle of \( F \) in \( X \).

**Remark:** It is known that Weyl-Kac formula has a formal flavor to it. The infinite sum and product in the denominator above reflects that.

**Pf:** It is clear that \( D^{WK} = D \cdot D_{0}^{WK} \) with \( D \) as in Section 1.1.
Given \( a \in P_+^k \), there is a unique highest weight \( \hat{L}_G \)-module at level \( k \), \( \hat{V}_a \). The character \( \chi(t|\hat{V}_a) \) is provided by the Weyl-Kac formula as

\[
\hat{\chi}_a(t) = \sum_{w \in W_{aff}} w \frac{e^a}{D_W^0(t)}.
\]

The resemblance to the characters of the \( G \)-modules is obvious.

From the representation theory of \( g_{aff} \), we learn that each \( T \)-module with weights in \( P_+^k \) induces such a \( \hat{L}_G \)-module, and these are all the irreducible highest weight module of \( \hat{L}_G \) at level \( k \).

Now the part of \( H^0(X,L) \) generated by the highest weight modules of level \( k \) is simply

\[
\oplus_{a \in P_+^k} m_a \hat{V}_a,
\]

from our identification of all the highest weight vectors in \( H^0(X,L) \). Thus we conclude for \( t \in \{e^{2\pi i \nu \lambda + \rho} | \lambda \in P_+^k \} \):

\[
\chi_{\hat{L}_G}(X,L)(t) = \sum_{w \in W_{aff}} w \frac{\text{RR}(X_{F_{|L}},L_N)}{D_W^0(t)}
= \sum_{w \in W_{aff} / W} w \frac{\text{RR}(Y)}{D_W^0(t)}
= \sum_{w \in W_{aff} / W} w \sum_{F} \text{FC}_F + R \frac{D_W^0(t)}{D_W^0(t)}.
\]

(10.13)

Next we treat the first term. Since each connected component of fixed point sets in \( X_t \) can be written as \( wF \) for some \( w \in W_{aff} \), and \( F \in F^1 \), it suffices to show

\[
w \sum_{F \in F^1} \text{FC}_F \frac{D_W^0(t)}{D_W^0(t)} = \int_{wF} \frac{\text{Td}(wF) \text{Ch}(L|wF,t)}{\det(1-t^{-1}e^{-\Omega})}_{\text{nor}(F,X)}
\]

(10.14)

The map \( w : X \to X \) preserves the complex structure, naturally \( w \) induces isomorphism between \( TF, TwF \). Also the symplectic form which is defines \( c_1(L|wF), c_1(L|F) \) up to a constant, and is invariant under \( L_G \). Hence the two Chern classes are equal under pull-back. The equivariant Chern classes are related by

\[
w \text{Ch}(L|F,t) = e^{2\pi i \mu(wt) + c_1(L|F)} = w_* e^{2\pi i \mu(wt) + c_1(L|wF)}
\]

where \( w_* \) pulls back forms. And \( \text{Td}(TF) = w_* \text{Td}(TwF) \). The only tricky part is the identification of the classes associated with the normal bundles.

Let \( D \) be in the denominator of \( FC_F \), i.e.,

\[
D(t) = \det(1-t^{-1}e^{-\Omega})
\]

which can be written in terms of Chern roots \( \{x_i\} \), the weights \( \{\theta_i\} \) and the roots of \( g \) as

\[
\prod_i (1-t^{-\theta_i}e^{-x_i}) \prod_{\alpha \in \Delta^+} (1-e^{-\alpha}),
\]

the index here is finite, since \( \text{nor}(F,X_t) \) is of finite dimension.
The manipulation of $wD^{WK}$ is similar to the compact case, and we obtain
\[
\det(1 - t^{-1}e^{-\Omega})w(D^{WK}_0) = w \prod_i (1 - t^{-\theta_i}e^{-x_i}) \prod_{\alpha \in \Delta^+(g_{aff})} (1 - t^{-\alpha}) \\
= \prod_i (1 - t^{-w\theta_i}e^{-x_i}) \prod_{\alpha \in \Delta^+(g_{aff})} (1 - t^{-w\alpha}) \\
= \det(1 - t^{-1}e^{-\Omega})|_{\text{nor}(F,X)},
\]
where we have used the fact that $\text{nor}_x(F,X) \simeq T_x\mu^{-1}(t) \oplus (\mathfrak{g}/t)$. We observe that 1). $w^*(x_i^\mu) = x_i$ where $x_i^\mu$ is the Chern root of the corresponding line bundle at $wF$. 2). Each $(\mathfrak{L}_i)_{\alpha}$ in $(\mathfrak{L}/t)$ induces a trivial bundle over $F$ by group action, hence it has no curvature. 3). The weights on $wF$ on $\text{nor}(wF,X)$ is given by $w\{\theta_i\} \cup w\Delta^+(g_{aff})$.

After observing the above, immediately we obtain
\[
R.H. = w_* \det(1 - t^{-1}e^{-\Omega})|_{\text{nor}(wF,X)}.
\]
Thus we complete the proof. QED
11. The proof of the main cancellation

Proposition 11.1. Let $F_h$ denote a connected component of fixed point sets (here $h$ may be 1). Suppose $\mu = \mu(F_h) \in k(\partial C, 1)$ and is preserved by $W^\text{aff}_\mu$. Let $w$ be a lifting in $W^\text{aff}_\mu$ of $[w] \in W^\text{aff}_\mu/W(\mathbb{K}_h)$, $wF_h = F_{w\phi}$ (or denoted by $F^\mu_{w\phi}$) is another component of fixed point sets with the same value under $\mu$.

1. If $\mu(F_h) = k(\phi, 1)$ with $\phi \in \partial C$, but $\phi \notin C^\text{aff} = \{\theta = 1\}$, then

$$\sum_{v \in W} \frac{\sum_{[w] \in W^\text{aff}_\mu/W(\mathbb{K}_h)} FC_{wF_h}(t) \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})(t) = 0. \tag{11.2}$$

2. If $\phi \in \partial C \cap \{\theta = 1\}$ and $vFC_{wF_h}$ has no pole on $\{e^{2\pi i w^{-1} \Delta \phi} \mid \lambda \in \mathbb{P}^\pm\}$, then

$$\sum_{v \in W} \frac{\sum_{[w] \in W^\text{aff}_\mu/W(\mathbb{K}_h)} FC_{wF_h} \prod_{\alpha \in \Delta^+} (1 - e^{-2\pi i \alpha}) = 0 \text{ on } \frac{M^*}{k + h^*}. \tag{11.3}$$

Pf:

Step 1: Lifting action by $T$ to a covering group $T_p \times T'_Z$.

A key ingredient in Atiyah-Bott-Segal-Singer’s fixed point formula is the contribution from the normal bundles which appear in the denominators of an integral on the fixed point set. Ignoring for a moment the complications from orbifolds, the fixed point formula requires evaluating:

$$FC_F(g) = \int_F Td(TF) Ch(L|_{F''), g) \det(1 - g^{-1}e^{-it}) \tag{11.1}$$

where $\Omega$ is $i/2\pi$ times the curvature operator of the complex normal bundle.

In what follows, we will evaluate the integral above for $wF, wF_h$ and for $g = t, vt$ where $w, v$ are elements in Weyl subgroups specified later. Furthermore, we need to include the consideration that $\{F\}$ are orbifolds, if $F$ is of type 2, 3, i.e., $\phi(F) \in \partial C$.

Suppose $v, w \in W^\mu$, then the decomposition of $t = (t_{wp}, t^\nu) \in t_{wp} \oplus t_z$ has the following property as shown in Section 4:

$$(vt)^w\text{wp} = wt_p, \quad (vt)^w_z = vt_z + (vt_p - t_p) \in t_z.$$

Use $t_z$ to denote $vt_z + (vt_p - t_p)$, there should be no confusion. And denote by $e^{2\pi i t'}, e^{2\pi i s'} \in T'_Z$ certain lifting of $e^{2\pi i t_s} \in T_z, e^{2\pi i s} \in T_z \cap T_p$ respectively.

1. If $\mu$ is on the affine wall, the lifting of $ve^{2\pi i t} \in T$ to $T'_{wp} \times T'_z$ are given by

$$(e^{2\pi i (wt_p + ws)}, e^{2\pi i (t'_s + \sum b_i \alpha^i + be^c)}) \in T'_{wp} \times T'_z \text{ with } e^{2\pi i (ws)} \in \mathbb{P}_p = t^0_p = t^0_{wp},$$

where $0 \leq b_i < a^i, 0 < b < n$ with $n, \epsilon$ the same as in Corollary 8.1.

2. If $\mu$ is not on the affine wall, then the lifting are given by

$$(e^{2\pi i (wt_p + ws)}, e^{2\pi i (t'_s + \sum b_i \alpha^i)}) \in T_p \times T_z \text{ with } e^{2\pi i (ws)} \in \mathbb{P}_p = t^0_p = t^0_{wp}. \tag{11.3}$$

Lemma 11.1. Let $g$ be a lifting as above, and $v \in W^\mu_0$ which is the subgroup of $W$ isomorphic to $W^\text{aff}_\mu$. For $\phi, \theta > 1$, $g^{\phi} = e^{2\pi i \phi} < v, t + b \epsilon > = e^{2\pi i < v, t + b \epsilon >}$;

for $\phi, \theta \neq 1$, $g^{\phi} = e^{2\pi i < v, t >}$.
Pf: First of all, \( \phi \) is a weight on \( T_p \), since \( p \) is fixed by \( T_p \), and \( \phi \) is the character of \( L \) at \( p \). Now use Lemma \( \text{[32]} \) Part 2 of Corollary \( \text{[34]} \) and the fact that \( \phi \) is a weight on \( S \) in the notation there, we have the assertion. QED

Step 2: Defining the denominators.

Let \( g \) denote a lifting just described. Recall \( F_h \) is a strata associated with \( F \) and is fixed by \( T \) and \( h \) in the isotropy group of \( F \). From now, we use \( F_h \) instead of \( F \) and treat \( F \) as the special case \( h = I \). We have treated \( F \) and \( F_h \) separately so far, this leads to repetitions on occasions. We will point out whenever the difference requires additional attention.

The following are various factors which will appear in the denominator as in Eq. (11.1) along \( wF_h = F_h^w \). The expressions are obtained using the weights and curvatures computations done in Sections 3-5.

Remark: The signs in front of the weights below are determined by the following:

a). \( X_N \) is the reduced space of the product \( X \times \mathbb{X} \), for the map \( \mu_X = \mu_X \). The weakly symplectic form is \( \omega = -\omega X \). Thus one chooses the original \( J \) on \( X \), and \( -J \) on \( TX \) to make the form semi-positive definite. Hence a negative sign for the weights \( \{ \beta \} \), \( \{ \lambda \} \) on the the normal subbundle \( \{ g_t \oplus H_z \} \) from \( TX \)

b). The expression in the denominator involves \( g^{-1}e^{-\Omega} \), therefore another negative sign.

c). The difference in the signs for the term \( dA_h \) in various determinants below was referred to in the Remark after Prop. 5.2.

Now we can write down the expressions for the denominators:

1). If \( \mu \) is on the affine wall, let \( \vec{b} = (b(\mu), b_1, \ldots, b_{\mu}) \) with \( \mu = \#\Delta^0 \mu \) the number of simple roots of \( g \) which vanish at \( \mu/k \).

\begin{equation}
\text{(11.4)}
\end{equation}

\[
D^w_\mu(\nu t) = \det(1 - g^{-1}e^{-\Omega})|_{N_w \oplus \text{nor}^+(wZ_h, wZ)} = \prod (1 - e^{-2\pi i \gamma (t + s + 1/4\pi^2 \nu^2)}),
\]

\[
D^w_\emptyset(\nu t, \vec{b}) = \det(1 - g^{-1}e^{-\Omega})|_{H_z},
\]

\[
= (1 - e^{-2\pi i \mu(t \cdot s + b^* + \sum b_i \alpha_i^* - 1/4\pi^2 dA_h)})
\]

\[
\times \prod (1 - e^{2\pi i \Lambda_{\mu} - \gamma^\mu(t \cdot s + b^* + \sum b_i \alpha_i^* - 1/4\pi^2 dA_h)}),
\]

\[
D^w(\nu t, \vec{b}) = (1 - e^{-2\pi i \mu(t \cdot s + b^* - 1/4\pi^2 dA_h)})
\]

\[
\times \prod (1 - e^{2\pi i \mu(\Lambda_{\mu} - a^\mu \mu)(t \cdot s + b^* - 1/4\pi^2 dA_h)}),
\]

\[
D^w(\nu t, \vec{b}) = \det(1 - g^{-1}e^{-\Omega})|_{\mathfrak{g}_\mu/\text{LieK}_{wh} = \prod_{\beta \in \Delta^+ \setminus \Delta^+(K_{wh})} (1 - e^{2\pi i \mu\beta(t \cdot s + 1/4\pi^2 dA_h)}),
\]

\[
D^w_a(\nu t) = \prod_{\alpha \in \Delta^+} (1 - e^{-2\pi i \alpha(\nu t)}).
\]

2). If \( \mu \) is off the affine wall, let \( \vec{b} = (b_1, \ldots, b_{\mu}) \), the one difference is:

\[
\text{(11.5)}
\]

\[
D^w(\nu t, \vec{b}) = \det(1 - g^{-1}e^{-\Omega})|_{H_z}
\]

\[
= \prod (1 - e^{2\pi i \mu(t \cdot s + \sum b_i \alpha_i^* - 1/4\pi^2 dA_h)}),
\]

\[
D^w(\nu t) = \prod (1 - e^{2\pi i \mu(\Lambda_{\mu} - a^\mu \mu)(t \cdot s + b^* - 1/4\pi^2 dA_h)}).
\]
Obviously in the above definition, when \( h = I \), \( \text{nor}(wZ, wZ_h) \) is trivial since \( Z = Z_h \).

**Step 3: Expressing \( \text{FC}_{F^w_h}(vt) \).**

Continue to let \( g \in T_p \times T'_z \) where the first component of \( g \) passes \( w(hT^0_p) \) in \( w(T_p) \), as a lifting of \( vt \in T \). This form of lifting appeared in Eq. (11.2), (11.3). In the notations just introduced, we have along the normal bundle of \( F^w_h \)

\[
\det(1 - g^{-1}e^{-\Omega}) = D^w_0(g)D^w_0(g)\tilde{D}^w(g)
\]

by definitions of the factors on the right and the structure of the normal bundle.

The line bundles \( L|_{F_h}, L|_{F^w_h} \) (orbifold bundles actually) are related through the map \( w : F_h \to wF_h = F_{w|_h} \) as

\[
w^*(L|_{F^w_h}) = L|_{F_h}, \quad w(\phi) = \phi.
\]

This relation holds for the pair \( w_{\text{aff}}, Z_h \) obviously, and it holds on the quotient \( T_z \) because \( w \in W_{\mu} \) normalizes \( T_z \). In terms of \( F_h \), following Lemma [11.1], one obtains

1. when \( \mu \) is on the affine wall and \( w \in W_{\mu}^{\text{aff}} \),

\[
\text{Ch}(L|_{F^w_h}, g) = g^{\omega \phi}e^{\omega} = g^\phi e^\omega = e^{2\pi i<k_{\psi^0}, t+b^+>}e^\omega,
\]

where \( \omega \) is the symplectic form on \( X_N \) restricted to \( F_h \).

2. when \( \mu \) is off the affine wall,

\[
\text{Ch}(L|_{F^w_h}, g) = g^{\omega \phi}e^{\omega} = g^\phi e^\omega = e^{2\pi i<k_{\psi^0}, t>}e^\omega.
\]

Depending on whether \( \mu \) is on or off the affine wall, one obtains now an expression of \( \text{FC}_{F^w_h} \) in terms of an integral on \( F_h \) as follows:

\[
\phi(\theta^*|_{T_z}) = 1:\]

\[
\text{FC}_{F^w_h}(vt) = \frac{1}{|F_p||T'_z/T_z|} \sum_g \int_{F_h} \frac{Td(TF_h)e^{2\pi i<k_{\psi^0}, t+b^+>}e^\omega}{D^w_0(g)D^w_0(g)\tilde{D}^w(g)};
\]

\[
\phi(\theta^*|_{T_z}) \neq 1:\]

\[
\text{FC}_{F^w_h}(vt) = \frac{1}{|F_p||T'_z/T_z|} \sum_g \int_{F_h} \frac{Td(TF_h)e^{2\pi i<k_{\psi^0}, t>}e^\omega}{D^w_0(g)D^w_0(g)\tilde{D}^w(g)}.
\]

In the above the summation is over all the possible lifting of \( vt \) in \( w(hT^0_p) \times T'_z \) which is the isotropy group of the strata \( F_{w|_h} \).

**Step 4: Summation over \( T'_z/T_z \).**

For an upcoming calculation, it is crucial to replace the fractional weights

\[
(\Lambda_i - a^\psi_\mu)/a^\gamma_\mu, \quad \alpha_i \in \Delta^0_\mu
\]

by the integral weights on \( T_z \),

\[
\Lambda_i - a^\psi_\mu, \quad \alpha_i \in \Delta^0_\mu.
\]

This amounts to replacing \( \tilde{D}^w \) by \( D^w \) and is an important step. The basic observation is for \( \mu \) on the wall,

\[
|T'_z/T_z| = n \prod_{\alpha_i \in \Delta^0_\mu} a^\gamma_i;
\]

for \( \mu \) off the wall,

\[
|T'_z/T_z| = \prod_{\alpha_i \in \Delta^0_\mu} a^\gamma_i.
\]
and more importantly:

\begin{equation}
\sum_{b_i} \frac{1}{D^w(g)} = \left( \prod_{\alpha_i \in \Delta^w\mu} a_i^{\alpha} \right) \frac{1}{D^w(g)}.
\end{equation}

The last equation is based on two observations:

1). Using geometric series expansion, and the fact that \(\sum_{b_i} e^{2\pi i < a, b_i + a_i^\vee>}\) is either 0 or \(\prod_{\alpha_i \in \Delta^w\mu} a_i^{\alpha}\), for all the possible weight \(a\), depending on \(a\) is a weight on \(T^\mu\) but not on \(T_z\), or \(a\) is a weight on \(T_z\).

2). Another observation used in the above is

\begin{equation}
e^{-2\pi i < w, \Lambda_i - a_i^\vee \mu, t' - s + b\epsilon - 1/4\pi^2 dA_k>}
\end{equation}

due to \(\phi(\alpha_i^\vee) = 0\); \(\Lambda_i(\epsilon'), \Lambda_i(a_i^\vee) \in \mathbb{Z}\); also \(-\mu, \Lambda_i - a_i^\vee \mu\) are fundamental weights of \(T_z\), their values at \(e^{2\pi i (t' - s)}\) are the same at the projections \(e^{2\pi i (t_z - s)} \in T_z\).

Thus one has:

\(\phi(\theta^\vee) = 1\):

\begin{equation}
\text{FC}_{F_h^w}(vt) = 1 \int_{F_h} \frac{w^*\text{TD}(TF_{wh})e^{2\pi i < kv, t + \epsilon>}}{\text{D}^w_0(vt)\text{D}^w_0(vt)\text{D}^w(vt, b)}
\end{equation}

\(\phi(\theta^\vee) \neq 1\):

\begin{equation}
\text{FC}_{F_h^w}(vt) = \frac{1}{|J^0_p|} \sum_{g} \int_{F_h} \frac{\text{TD}(TF_{wh})e^{2\pi i < kv, t>}}{\text{D}^w_0(vt)\text{D}^w_0(vt)\text{D}^w(vt)}.
\end{equation}

**Step 5: An identity of Todd classes.** Given \(w \in W_{\mu}\), there is the map \(w: F_h \rightarrow Z_h/T_z \rightarrow wZ_h/T_z = Z_{wh}/T_z\).

**Lemma 11.2.**

\begin{equation}
w^*(\text{TD}(T^HF_{wh})) = \text{TD}(T^HF_{wh}),
\end{equation}

\begin{equation}
w^*(\text{TD}(T''F_{wh})) = \prod_{\tau' \in \Delta^+(K_{wh})} \frac{-i/2\pi < w\tau', dA_k>}{1 - e^{i/2\pi < \tau', dA_k>}}.
\end{equation}

**Remark:** The sign convention reflects again the the complex structure chosen on \(k_g\).

**Pf:** If \(\pi: F_h \rightarrow E_h\) as in Section 3, then \(\text{TD}(T^HF_{wh}) = \pi^*\text{TD}(E)\). The same holds for \(F_{wh}, E_{wh}\). It is easy to see that \(w: Z_h \rightarrow Z_{wh}\) is an equivariant isomorphism with respect to the action by \(K_h, \text{Ad}_{w} K_h = K_{wh}\) on \(Z_h, Z_{wh}\) respectively. Therefore, \(\text{TD}(E_h) = w^*\text{TD}(E_{wh})\) whose pull-back to \(F_h, F_{wh}\) yield the first equation.

On \(T''Z_{wh}\), after decomposing it into a sum of line bundles according to the roots \(\{\tau'\}\), the curvature is given by

\[-\oplus < \tau', dA_{wh}> = -\oplus < w\tau', dA_h>\]

The second identity follows that. **QED**

**Step 6:** A sufficient condition for the validity of the cancellation.

First of all, recall

\begin{equation}
w^*\text{TD}(TF_{wh}) = \text{TD}(T^HF_{h})w^*\text{TD}(T''F_{wh})
\end{equation}
where $\text{Td}(T^H F_h) = \pi^* \text{Td}(E_h)$, with
\[
\pi : F_h = Z_h/T_z \to E_h = Z_h/K'_h.
\]
Also it was shown in Prop. 4.4 $D^w_T(vt) = D_0(t)$. We also know both forms below
\[
w^* \text{Td}(T^H F_{wh}) = \text{Td}(T^H F_h), \quad e^\omega
\]
are the pull-back of forms on $E_h$, because they are null in the fiber direction of the map $\pi$. Also clearly that $D^w_T(vt)$ is constant with respect to the integration variable. Therefore, in order to evaluate
\[
\sum_g \int_{F_h} w^* \text{Td}(TF_{wh}) e^{2\pi i \langle kv\phi, t + be^\gamma \rangle} e^\omega / D^w_T(vt) D_0^w(vt) D^w(vt, b)
\]
(for $< \phi, \theta > = 1$) or
\[
\sum_g \int_{F_h} w^* \text{Td}(TF_{wh}) e^{2\pi i \langle kv\phi, t \rangle} e^\omega / D^w_T(vt) D_0^w(vt) D^w(vt)
\]
(if $< \phi, \theta > \neq 1$), one can pull $D^w_T(vt), \text{Td}(T^H F_h)$ and $e^\omega$ out, when integrating along $\pi^{-1}([p])$; those factors except $D^w_T(vt)$ are independent of $v \in W^0_\mu, w \in W_\mu$. Thus to prove the proposition, which involving evaluating a sum over $W_\mu$ of the above integrals, it suffices to prove
\[
(11.14)
\]
\[
\phi(\theta^\nu) = 1:
\]
\[
\sum_{w \in W_\mu, \forall \pi^{-1}([p])} \int_{F_h} w^* \text{Td}(TF_{wh}) e^{2\pi i \langle t_\nu + s + b\gamma \rangle} / D^w_T(vt) D_0^w(vt) D^w(vt, b) = 0, \quad \text{on } M^* / k + h^\nu;
\]
\[
\phi(\theta^\nu) \neq 1:
\]
\[
\sum_{w \in W_\mu, \forall \pi^{-1}([p])} \int_{F_h} w^* \text{Td}(TF_{wh}) e^{2\pi i \langle t_\nu + s + b\gamma \rangle} / D^w_T(vt) D_0^w(vt) D^w(vt) = 0.
\]
In the above, a lifting of each $w \in W_\mu / W(K_h)$ to $W_\mu$ is fixed and denoted by the same.

**Step 7:** Turn the integrals along the fiber to a sum via equivariant cohomology.

Recall from Section 3 $dA_h = B_h + R_h$, where $B_h$ is the horizontal part of the curvature, $R_h$ is the vertical part tangent to $\pi^{-1}([p])$. Recall also that $\pi^{-1}([p])$ is a finite quotient of $K'_h/T_z$. Hence the integrals can be pulled to $K'_h/T_z$.

We will continue to use the same notations for the pull-back to $K'_h/T_z$ of various curvature forms on $\pi^{-1}([p])$. In Eq.\ref{(11.14)}, the term $D^w_T(vt)$ is a constant on $K'_h/T_z$. The integral of the rest were calculated using equivariant cohomology. As a straightforward application of Formula 4 in Prop. \ref{(9.1)}, the answers are
\[
\phi(\theta^\nu) = 1:
\]
\[
(11.15) \int_{K'_h/T_z} w^* \text{Td}(T^' F_{wh}) e^{2\pi i \langle t_\nu + s + b\gamma \rangle} / D^w_T(vt) D_0^w(vt) D^w(vt, b) = e^{2\pi i \langle t_\nu + s + b\gamma \rangle} \sum_{w \in W(K_h), \forall \pi^{-1}([p])} \frac{1}{\prod_{\pi' \in \Delta^+} \langle \tau_{\pi'}, 1/4\pi^2 wB \rangle} d_w^* d_w',
\]
where
\begin{equation}
(11.16) \quad d^w_b = \prod_{\beta} (1 - e^{2\pi i w \beta (t_p + s + 1/4\pi^2 u B_h)}),
\end{equation}
\begin{equation}
(11.17) \quad d^w = \prod_{i \in I_{\mu}} (1 - e^{2\pi i (u \Lambda_i) (t_z - s + b e^\tau - 1/4\pi^2 u B_h)}),
\end{equation}
in the above the \{ \beta \} are in \( \Delta^+_\mu \setminus \Delta^+(K_{wh}) \). (In applying Prop. \[ \text{[1]} \] above, we treat factors \( D^w_b(vt)D^w(vt, b) \) collectively as \( \prod_c (1 - e^{2\pi i c \tau v t + 1/4\pi^2 B_h + 1/4\pi^2 B_h}) \); and use \( w \tau' \) instead of \( \tau \) there.)

For the other case \( \mu(\theta^\nu) \neq 1 \), the only difference is in the definition of \( d^w_b \),
\begin{equation}
(11.18) \quad d^w = \prod_{i \in I_{\mu}} (1 - e^{2\pi i (u \Lambda_i) (t_z - s + b e^\tau - 1/4\pi^2 u B_h)}).
\end{equation}

**Step 8: The first lucky break.**

We will now simplify the expression just obtained to be in a position to apply the fundamental formula of Section 6.

The first break comes when we group the factor involving the positive roots \( \{ \tau' \} = \Delta^+(K_{wh}) \), with the one involving \( \{ \beta \} = \Delta^+_\mu \setminus \Delta^+(K_{wh}) \), we realize that the first factor can be made look just like the second one. To be more specific:
\begin{equation}
\prod_{\tau' \in \Delta^+(K_{wh})} (1 - e^{i/2\pi < w \tau', u B_h}>)
= \prod_{\tau' \in \Delta^+(K_{wh})} (1 - e^{2\pi i < w \tau', t_p + s + 1/4\pi^2 u B_h>}).
\end{equation}

There are two reasons for the above equation: 1). \( e^{2\pi i u (t_p + s)} = e^{2\pi i (t_p + s)} \in hT^0_p \) since \( u \in W(K_h) \) and \( K_h \) commutes with \( hT^0_p \) by its definition; 2). \( e^{2\pi i w \tau' (t_p + s)} = 1 \), the adjoint action by \( hT^0_p \) on \( \text{Lie}K_h \) is \( I \) since the two commute by the definition of \( \text{Lie}K_h \), and \( w \tau' \) is a root of \( K_h \).

In this form, the product is clearly in the same species as \( d^w_b \).

**Step 9: The second break and the finale.** The denominator needs to be written in a form so we can apply Prop. \([1] \). The denominator acquires an extra factor after integration as shown in the previous two steps. So the total is
\begin{equation}
D^w_u = D^w_a(vt)D^w_b(vt)D^w(vt, b) \prod_{\tau' \in \Delta^+(K_{wh})} (1 - e^{2\pi i < w \tau', t_p + s + 1/4\pi^2 u B_h>})
\end{equation}
if \( \phi(\theta^\nu) = 1 \). For \( \phi(\theta^\nu) \neq 1 \), \( D^w_v \) is defined the same way except \( D^w(vt, b) \) is replaced by \( D^w(vt) \) as defined in Step 2 above.

According to the previous step, \( D^w_b(vt) \) which has \( \beta \) running over \( \Delta^+_\mu \setminus \Delta^+(K_{wh}) \) can be written together with \( \prod_{\tau' \in \Delta^+(K_{wh})} (1 - e^{2\pi i < w \tau', t_p + s + 1/4\pi^2 u B_h>}) \) as
\begin{equation}
(11.19) \quad \prod_{\beta \in \Delta^+_\mu} (1 - e^{2\pi i < w \beta, t_p + s + 1/4\pi^2 u B_h>}),
\end{equation}
which can be further written as
\begin{equation}
(11.20) \quad \prod_{\beta \in \Delta^+_\mu} (1 - e^{2\pi i < w \beta, t_p + s + b e^\tau + 1/4\pi^2 u B_h>}),
\end{equation}
for \( \beta \) is a weight of \( T_z \), therefore is trivial on the \( T^I_z/T_z \) in which \( e^{2\pi i b e^\tau} \) lies.
Now the full expression of \( D^w_v \) is given by

\[
D^w_v = \prod_{\beta \in \Delta^+} (1 - e^{2\pi i <w\beta, t> + s + b e^\gamma + 1/4\pi^2 u B_h>) \prod_{\alpha \in \Delta^+} (1 - e^{2\pi i <v\alpha, t>})
\]

(11.21)

\[
x \prod_{\lambda} (1 - e^{2\pi i (\lambda, t) - <w\lambda, t> + s + b e^\gamma + 1/4\pi^2 u B_h>)
\]

where \( \lambda \) runs through the fundamental weights of \((\mathfrak{g})_\mu\). If \( \phi(\theta^\mu) \neq 1 \), just remove the term \( b e^\gamma \) in the above. Also

\[
e^{2\pi i <w\beta, t> + s + b e^\gamma>} = e^{2\pi i <w\beta, p + s + b e^\gamma>},
\]

\[
e^{-2\pi i <w\lambda, t> + s + b e^\gamma>} = e^{-2\pi i <w\lambda, u(t_p + s + b e^\gamma)>},
\]

(11.22)

for \( u \in W(K_h) \), this property can be deduced from Lemma 8.1 and the fact that \( u|_{\mathfrak{h}T^0_p} = I \), for \( u \in W(K_h) \) commutes with \( hT^0_p \) as \( K_h \) does.

As mentioned earlier, \( w \) is a lifting of \( W_\mu/W(K_h) \) in \( W_\mu \), and \( u \in W(K_h) \). The product \( wu \) on \( \mathfrak{g} \) runs through the whole \( W_\mu \). Or \( uw \) on \( \mathfrak{g}^* \) goes through \( W_\mu \).

Denote the combined by \( wu \in W_\mu \).

Let \( y = t_p + s + b e^\gamma + 1/4\pi^2 B_h \). Finally, for \( v \in W_\mu^0, w \in W_\mu \), and \( m \) being the rank of \( \text{Lie} K_h \) we have the following after pulling out some of the exponential terms:

\[
D^w_v = \prod_{\beta \in \Delta^+} (1 - e^{2\pi i <w\beta, y>}) \prod_{\alpha \in \Delta^+} (1 - e^{2\pi i <v\alpha, t>})
\]

(11.23)

\[
x \prod_{\lambda} (1 - e^{2\pi i (\lambda, t) - <w\lambda, y>)}
\]

\[
= (-1)^{m + \sigma(w) + \sigma(v)} e^{2\pi i (<w\rho_p - \rho_v - w \sum \lambda, y>) + <\rho - v p + v \sum \lambda, t>}
\]

\[
x \prod_{\beta \in \Delta^+} (1 - e^{-2\pi i <\beta, y>}) \prod_{\alpha \in \Delta^+} (1 - e^{-2\pi i <\alpha, t>}) \prod_{\lambda} (1 - e^{-2\pi i (\lambda, t) - <w\lambda, y>}),
\]

in the exponent of the first term above, there is the cancellation

\[
\rho_\mu - \sum \lambda = 0.
\]

The final calculation is

\[
\sum_{v \in W_\mu, w \in W_\mu} e^{2\pi i <v\phi, t + b e^\gamma>} \frac{D^w_v}{e^{2\pi i <\rho_\mu, y>}}
\]

(11.24)

\[
= \prod_{\alpha \in \Delta^+} (1 - e^{2\pi i <\rho_\mu, t>}) \prod_{\beta \in \Delta^+} (1 - e^{-2\pi i <\beta, y>})
\]

\[
x \sum_{v \in W_\mu^0, w \in W_\mu} \frac{(-1)^{m + \sigma(w) + \sigma(v)} e^{2\pi i (<v\phi, t + b e^\gamma> - <\rho - v p + v \rho_p, t>)}}{\prod_{\lambda} (1 - e^{-2\pi i (<\lambda, t> - <w\lambda, y>))}},
\]

after observing \( e^{2\pi i <v\phi, b e^\gamma>} = e^{2\pi i <v\phi, b e^\gamma>} \), hence it can be taken outside the summation. It now suffices to show the vanishing of

\[
\sum_{v \in W_\mu^0, w \in W_\mu} \frac{(-1)^{m + \sigma(w) + \sigma(v)} e^{2\pi i (<v\phi + \rho - v p + v \rho_p, t>)}}{\prod_{\lambda} (1 - e^{-2\pi i (<\lambda, t> - <w\lambda, y>))}}.
\]

(11.25)
The numerator, when \( \phi(\theta^v) = 1 \) and on the lattice \( \frac{M^*}{\mathbb{Z} h} \) as shown in Prop. 7.2 agrees with
\[ e^{2\pi i \langle k\phi + \rho_\mu, t \rangle} \]
which is independent of \( v, w \) and can be pulled outside the summation. When \( \phi(\theta^v) \neq 1 \), the numerator equals
\[ e^{2\pi i \langle k\phi + \rho_\mu, t \rangle} \]
everywhere. The lattice is invariant under \( W \). Therefore, over this lattice, the vanishing of the above sum is implied by
\[ \sum_{v \in W_\mu^0, w \in W_\mu} (-1)^{\sigma(w) + \sigma(v)} \prod_\lambda (1 - e^{-2\pi i (\langle v\lambda, t \rangle - \langle w\lambda, y \rangle)}) = 0, \]
whose validity is shown by Prop. 6.1. Thus we have completed the proof of Part 1, 2 of Prop. 11.1.
12. TWINS AND A NEW SURGERY FORMULA

12.1. A consequences from the proof: Twin pairs of compact $G$-manifolds.

Here we give an easy application for finite dimensional symplectic $G$-manifolds, or more generally symplectic $G$-orbifolds.

Let $M$ be a symplectic manifold (or orbifold) with a Hamiltonian $G$-action, $f$ be the moment map. Suppose that $M$ is compact and $f^{-1}(t)$ is smooth, i.e. the image of $f$ is transversal to $t$, then we can construct a $T$-orbifold $M_N$, just as we have done for the $LG$-space $X$. Specifically, $M_N$ is constructed as follows: Let $k \in \mathbb{Z}_+$ so that $f(M) \cap t_+ \subset kC$. Let $X_g$ be the same toric variety as in Section 2, with $\Phi$ as its moment map. Then

$$M_N = \{(p, q) \in f^{-1}(t) \times X_g | f(p) = k\Phi(q)/T.$$

Another way to see it is to start with $f^{-1}(t_+)$ which is a manifold with non-smooth boundary $f^{-1}(\partial t_+)$. Each face $Q$ on $\partial t_+$ has a semi-simple stabilizer $G_Q \subset G$ under the adjoint action. The normal subspace to $Q$ is the Lie algebra of a maximal torus of $G_Q$, $T_Q$. Then

$$M_N = \cup_Q f^{-1}(Q_{\text{int}})/T_Q.$$

The construction looks similar to symplectic cuts, but the two have quite different properties when one consider surgery formulas for equivariant Riemann-Roch.

The map $\phi(u) = f(p)$ with $u = [p, q]$, is well defined, since $f$ is invariant under $T$. The variety $M_N$ enjoys the same property as $X_N$ in Section 2.3. The $T$-fixed points on $M_N$ comes from either fixed points on $M$, or as a result of surgery along

$$(w, \phi^{-1}(\partial t_+)), \ w \in W.$$

The latter were studied in Section 3.

Similar to $X_N$, $M_N$ has degeneracy as a symplectic orbifold and has a $T$-invariant almost complex structure.

The space

$$M_G = G \times_T M_N$$

is a new $G$-space. In general, $M_G$ is different from $M$, when $f(M) \cap t$ intersects the boundary of the Weyl chambers. Though the restriction of the image of the moment maps of the two spaces coincide.

The figure 12.1 illustrates the cuts and the intersections of the images of the two spaces with $t$.

As already mentioned, the new fixed points all have their images on the boundary of the Weyl chambers. As shown in the proof in the last section, the sum of

![Figure 12.1. The extra cuts on $f(M_G)$](image-url)
contributions of fixed points with the same image vanishes. Therefore, we have the following interesting consequence for the pair:

**Corollary 12.1.** The two $G$-spaces $M$ and $M_G$ have the same $G$-equivariant Riemann-Roch numbers.

### 12.2. Another consequence: A new surgery formula

Let $M$ be a symplectic $G$-orbifold satisfying the transversality condition, i.e. $f(M)$ is transversal to $t \subset \mathfrak{g}$, and $V$ be a smooth symplectic subvariety of $M$. Suppose the $T$-action preserves $V$, with $V \subset f^{-1}(t)$.

**Definition 12.1.** 1). $(\tilde{U}, G_U)$ is an orbifold chart of $U$ if $G_U$ is a finite group acting with no non-trivial kernel on $\tilde{U}$ so that

$$\pi : \tilde{U} \to U = \tilde{U}/G_U, \quad \pi(\tilde{p}) = G_U(\tilde{p}) \in U.$$  

2). $I_V \subset T$ is the isotropy group of $V$ if for an open set $U$ with $U \cap V \neq \emptyset$, and $I_V \subset G_U$ is the stabilizer of $\pi^{-1}(V \cap U)$. For $\tau \in T$ which fixes $V$, let $\tau_V$ denotes all the local liftings of $\tau$.

From $[C]$, we know that up to isomorphism, the group $I_V$ is independent of the chart, and the open set $U$. So it is well defined over $V$.

**Remark:** The local liftings may not be extended over the entire $V$, since there could be global monodromy. On the other hand, the following characteristic class over $V$ can be defined, as an average over $\tau I_V$:

$$\frac{1}{|I_V|} \sum_{\tilde{p} \in \tau I_V} \int_V \frac{Td(V) \text{Ch}(L_{V})}{\text{det}_{\text{nor}(V,M)}(1 - t^{-1}e^{-\Omega})},$$

where $\Omega$ is the $i/2\pi$-curvature operator of the normal bundle of $V$ in $M$. We want to find another expression for it.

The expression is in terms of the subvarieties in $M_G$ and its lower strata.

For each face $Q$ in $\{w \in t_+ | w \in W\}$, $Q$ can be written as $wQ'$ with $Q'$ as a face of $t_+$. Thus each $Q$ has a stabilizer in $W$, denoted by $W_Q$. Let

$$\begin{align*}
M_Q &= \{(p, q) \in f^{-1}(Q) \times X_\mathfrak{g} | f(p) = kw\Phi(q)\}/T; \\
V_Q &= \{(p, q) \in V \times X_\mathfrak{g} | f(p) \in Q, \ f(p) = kw\Phi(q)\}/T.
\end{align*}
$$

Both $M_Q$ and $V_Q$ are orbifolds, as a consequence of the transversality of $f(M)$ to $t$, and both have $T$-invariant almost complex structures. Therefore one can define Todd class and makes sense of the equivariant normal bundles of $V_Q$ in $M_Q$.

The readers are warned that there is no natural almost complex structure on the normal bundle of $V$ in $M_I$.

**Lemma 12.1.** 1). Let $M_Q^G = G \times M_Q$. Then $M_Q, M_G^Q$ and $V_Q$ are subvarieties in $M_G = G \times_T M_N$.

2). If $F$ is a subvariety of $V_Q$, then the isotropy groups, $I(F, V_Q), I(F, M_G)$ of $F$ in $V_Q, M_G$ respectively satisfies the following exact short sequence:

$$1 \to I(V_Q, M_G) \to I(F, M_G) \to I(F, V_Q) \to 1$$

where the second term is the isotropy group of $V_Q$ in $M_G$. In particular:

$$|I(F, M_G)| = |I(V_Q, M_G)| \cdot |I(F, V_Q)|.$$
Weyl chambers. Assume $Q$ is a face of $F$ of $wC$. As a result of the cuts along the boundary of Weyl chambers. The first kind have their

There are two kinds of fixed point sets on $\{Q\}$. Without loss of generality, assume $Q$ is a face of $t_+$. Let $g$ be the stabilizer of $\phi(F) \in Q \subset g$. We claim

Proposition 12.1. Suppose $\tau \in T$ fixes $V$. Let $\Lambda_{\text{max}}(V_Q, V_{\Delta})$ be the determinant line bundle of $\text{nor}(V_Q, V_{\Delta})$, then

$$
\frac{1}{|I_V|} \sum_{s \in \tau I_V} \int_{V} \frac{\text{Td}(V) \text{Ch}(L_V)(t)}{\det_{\text{nor}(V,M)}(1 - s^{-1}e^{-t})} 
$$

where $W_Q, I_{V_Q}, I_V$ are stabilizer of $Q$ in $W$ and isotropy groups of $V_Q, V$ in $M_G, M$ respectively.

Proof: We prove the statement by first representing both sides in terms of the $T$-fixed points of $V$. Since $T$ acts on $V$ and its normal bundle, we can apply the localization of equivariant cohomology classes on $V$ to get

$$
\frac{1}{|I_V|} \int_{V} \frac{\text{Td}(V) \text{Ch}(L_V)}{\det_{\text{nor}(V,M)}(1 - t^{-1}e^{-t})} 
$$

where the denominator can be combined as $\det_{\text{nor}(F,V)}(1 - t^{-1}e^{-t})$ obviously.

As for the right hand side, for the same reason as above one can express it as

$$
\sum_{\Delta \subset \Delta} \sum_{Q \subset Q} \frac{1}{|I_Q|} \sum_{t \in T} \frac{\text{Td}(F) \text{Ch}(L_F)}{\det_{\text{nor}(F,V)}(1 - t^{-1}e^{-t})} 
$$

where $m = |W_Q||I_{V_Q}||I(F, V_Q)|$. There are two kinds of fixed point sets on $\{Q\}$: those already on $V$, and those as a result of the cuts along the boundary of Weyl chambers. The first kind have their isotropy groups come with the property that $I(F, M_G) = I(F, M)$ since the cut does not pass through $F$. The second kind have images under $\phi$ on the boundary of $wC$ for some $w$.

Also we may apply Lemma 12.12 to the orders of the isotropy groups, so that $|I_{V_Q}||I(F, V_Q)| = |I(F, M_G)|$ independent of $V_Q$, if $\phi(F)$ is on the boundary of the Weyl chambers.

Next we describe the weights on the various bundles. Without loss of generality, assume $Q$ is a face of $t_+$. Let $g_\phi$ be the stabilizer of $\phi(F) \in Q \subset g$. We claim
that the total contribution of all the terms involving $F$ vanishes, if $\phi(F)$ is on the boundary of the Weyl chambers.

Section 3 analyzed the formation of fixed point sets, here we continue to use the notations.

Let $g_\phi^s$ be the semi-simple part of $g_\phi$, $t_\phi = g_\phi^s \cap t$ is a Cartan subalgebra of $g_\phi^s$. Let $\Lambda_\phi$ be the set of fundamental weights of $g_\phi^s$, and

$$(12.4) \quad \Lambda_Q = \{ \lambda \in \Lambda_\phi | \lambda \in Q \}, \quad \Lambda_Q^\perp = \Lambda_\phi \setminus \Lambda_Q.$$  

Here $t_\phi$ is identical to $t_z$ in Section 3 and Section 4, which is true since both subalgebras are the orthogonal complement of the smallest face containing $\phi(F)$.

As shown in Prop. 4.3, $\Lambda_\phi$ induces the following weights along $F$,

$$s^\lambda = (s_\phi)^\lambda$$

for each lifting of $s$ to $(s_p, s_\phi) \in T_p \times T_\phi$. Or in terms of Lie algebra notation:

$$\exp 2\pi i < \lambda, t > = \exp 2\pi i < \hat{\lambda}, t_\phi > = \exp 2\pi i < \tilde{\lambda}, t - t_p > .$$

The set $\{ \lambda | \tilde{\lambda} \in \Lambda_Q^\perp \}$ are all the weights on $\text{nor}(V_\phi, V_\Delta)|_F$.

We remark that for each fixed point set $F$ with image on the boundary of $t^*_+$, there is a submanifold $Z \subset M$ with $Z/T_\phi$ where $T_\phi = T \cap G_\phi^s$, just as in Section 3. The action by $T_z$ is locally free, hence there is an associated connection $A$ on it. Thus the equivariant Chern class of $\Lambda^\text{maxnor}(V_\phi, V_\Delta)|_F$ is given by

$$\exp \sum_{\lambda \in \Lambda_Q^\perp} -2\pi i < \lambda, t - 1/4\pi^2 dA > = \exp \sum_{\lambda \in \Lambda_Q^\perp} -2\pi i < \tilde{\lambda}, t - t_p - 1/4\pi^2 dA >$$

where $\exp 2\pi i (t - t_p) = s$.

If $\Delta = wt_+$, then the above is replaced by

$$\exp \sum_{\lambda \in \Lambda_Q^\perp} -2\pi i < \lambda, wt - wt_p - 1/4\pi^2 ws dA > .$$

Let $\{ \gamma \}, \{ \beta \}, \{ \alpha \}$ be the same as in Prop. 4.3, where $X_N$ is replaced by $M_N$, then the weights to $F \subset V_\phi$ in $M_Q^G$ are given by the same expressions as in Prop. 4.4, except only those $\lambda$ with $\tilde{\lambda} \in \Lambda_Q$ contribute, because those in $\Lambda_Q^\perp$ are normal to $M_Q$ or to $M_Q^G$.

As for the $\det_{\text{nor}(F,M_Q^G)}(1 - s^{-1}e^{\Omega})$, similar to the expressions given in the last section, we can express it in terms of the weights $\{ \gamma \}, \{ \beta \}, \{ \alpha \}$:

$$\prod_{\lambda \in \Lambda_Q} D_0^w(ws)D_a^w(ws)D_b^w(ws)$$

where $D_0, D_a, D_b$ are the same as in Eq. (11.4) of the last section.

There might be a non-Abelian $\mathcal{K}$ commuting with $T_\phi$, hence it acts on $Z$. As was shown in Step 7 and Step 8 in the last section, the presence of a non-Abelian $\mathcal{K}$ after integrating the integrand along $\mathcal{K}/T_\phi$ leaves little trace behind, except replacing the two form $1/4\pi^2 dA$ by $1/4\pi^2 B$ which is a two form on $Z/\mathcal{K}$. Therefore we deal directly with the final expression of the denominator.
So the fixed points contribution of $F \subset M^G_Q$ is given by

\begin{equation}
\sum_{\Delta, Q \subset \Delta} \frac{1}{|W_Q|} \sum_{F \subset Q} \frac{1}{|I(F, M_G)|} \sum_{s \in I(F, M_G)} \left( \frac{Td(F) \text{Ch}(L_F \oplus A^\text{max nor}(V_Q, V_\Delta)|_F)}{\det_{\text{nor}(F, M^G_Q)}(1 - s^{-1}e^{-\Omega})} \right) d
\end{equation}

where $y = t_p + 1/4\pi^2 B$.

Recall that $D^w_0(ws) = D_0(s)$ as shown in Prop. 4.4, and use the expressions from Step 2 in the last section to obtain:

\begin{align}
D^w_b(ws) &= (-1)^{\sigma(w)} e^{2\pi i (\sum_{\omega < \alpha} 2\pi i < \omega, \beta, y)} \prod_{\beta \in \Delta_+(g_\beta)} (1 - e^{2\pi i < \beta, y}) \\
&= (-1)^{\sigma(w)} e^{2\pi i < \rho_\beta - \rho_\beta, y>} D_0(s);
\end{align}

\begin{align}
D^w_a(ws) &= (-1)^{\sigma(w)} e^{-\sum_{\omega < \alpha} 2\pi i < \omega, \alpha, t>} \prod_{\beta \in \Delta_+(g_\beta)} (1 - e^{-2\pi i < \alpha, t>}) \\
&= (-1)^{\sigma(w)} e^{2\pi i < \rho_\beta - \rho_\beta, t>} D_a(s).
\end{align}

This is the same procedure as in Eq. (11.23). From Lie theory, we have

\[ \sum_{\lambda \in \Lambda_Q} w\lambda + \sum_{\lambda \in \Lambda^\perp_Q} w\lambda = w\rho_\phi. \]

Now the denominator can be written as

\begin{equation}
d = e^{\sum_{\lambda \in \Lambda_Q} 2\pi i \lambda, wt - wy} \prod_{\lambda \in \Lambda_Q} (1 - e^{2\pi i < \lambda, wt - wy}) D_0(\tau) D^w_b(\tau) D^w_a(\tau) e^{2\pi i < \beta, t - y>} \prod_{\lambda \in \Lambda_Q} (e^{-2\pi i < \lambda, wt - wy} - 1).\end{equation}

The number of terms with a fixed $Q$, and varying $\Delta$ is exactly $|W_Q|$ which is the same number of $\Delta$ containing $Q$. Gather them together and get rid of the term $1/|W_Q|$. Now the summation over all the $Q \subset \Delta = wt$, containing $\phi(F)$ to yield the following:

\begin{align}
\sum_{Q \subset \Delta} \frac{1}{|W_Q|} \int_{F} \frac{Td(F) \text{Ch}(L_F \oplus A^\text{max nor}(V_Q, V_\Delta)|_F)}{\det_{\text{nor}(F, M^G_Q)}(1 - s^{-1}e^{-\Omega})} d \\
= \sum_{Q \subset \Delta} \int_{F} \frac{Td(F) \text{Ch}(L_F)}{D_0(\tau) D_b(\tau) D_a(\tau) e^{2\pi i < \rho_\beta, t - y>} \prod_{\lambda \in \Lambda_Q} (e^{-2\pi i < \omega, \lambda, t - y} - 1)} \sum_{Q} (\prod_{\lambda \in \Lambda_Q} (e^{-2\pi i < \lambda, wt - wy} - 1))
\end{align}

\begin{align}
= \sum_{Q \subset \Delta} \int_{F} \frac{Td(F) \text{Ch}(L_F)}{D_0(\tau) D_b(\tau) D_a(\tau) e^{2\pi i < \rho_\beta, t - y>} \prod_{\lambda \in \Lambda_Q} (1 - e^{-2\pi i < \lambda, wt - wy})).\end{align}
Summing over various $\Delta$ in the above notations is the same as going over $w \in W_\phi$, thus the total contributions of $F$ is 0 since

$$\sum_{w \in W_\phi} \sum_Q \frac{(-1)^{\# \Lambda_Q}}{\prod_{\lambda \in \Lambda_Q} (1 - e^{w\lambda})} = 0$$

by Eq. (6.7).

Thus only those $\{F\}$ on $V_Q$ with $\phi(F)$ not on the boundary contributes. They are exactly the $T$-fixed point components on $V$. Therefore, the right hand side is the same as the left one. QED
13. Expression for the remainder term \( R \)

If for each fixed point set component \( F \) of \( Y \), \( FC_F \) has no pole on
\[
\left\{ e^{2\pi i \nu - \frac{\lambda + m}{\nu + i}} | \lambda \in F^k \right\},
\]
then the cancellation of Section 11 shows all the \( T \)-fixed points on the boundary cancels on the subset. However, such poles may occur, thus the fixed points coming from compactification do not cancel readily. The question is how they contribute. Here we will find an explicit expression for the remainder term \( R \).

First let’s outline the steps in calculating the remainder term \( R \) as function on
\[
\left\{ e^{2\pi i \nu - \frac{\lambda + m}{\nu + i}} | \lambda \in F^k \right\}.
\]
Instead of using the fixed points, we will construct varieties which contain the fixed points on the compactifying locus. Those varieties are obtained using cuts which are transversal to \( \partial \mathfrak{c} \). Instead of transporting the fixed points by translating elements in \( W_{aff} \), which fails to work if the singular poles occur, we will move the varieties. The varieties have equivariant Riemann-Roch which are well defined functions on \( T \).

After transporting the varieties using the appropriate elements in \( W_{aff} \), one can cancel the contributions of fixed points lying on the compactifying locus, or on the boundary of certain Weyl chambers, just as was done in the last section. Then we will apply Prop. 12.1 and transport only \( \{ V_Q \} \) back to get a explicit formula of \( R \).

13.1. Partition of the affine alcove.

The union \( \bigcup_{w \in W_{aff}} w \mathcal{C} \) forms a tiling of \( \mathfrak{t} \).

Let \( x \) be in the interior of \( \mathcal{C} \), then \( W_{aff}(x) \) is the vertices of a dual \( W_{aff} \)-invariant decomposition of \( \mathfrak{t} \).

Each vertex \( a \) of \( \mathcal{C} \) is now the center of a convex polytope with \( W_{aff}(x) \) as vertices.

Let \( l \) be the rank of \( G \), a little experiment shows that under the \( W_{aff} \) translations there are \( l + 1 \) of different polytopes in the dual to \( \bigcup_{w \in W_{aff}} w \mathcal{C} \). Each one can be moved so that it contains exactly one of the vertices of \( \mathcal{C} \) as an interior point.

Denote those \( l + 1 \) polytopes by \( \{ R_a \} \). Let \( (LG)_a \) be the subgroup of \( LG \) preserving
\[
(a, 1) \in \mathfrak{t} \times \{ k \} = \mathfrak{t} \times \mathbb{R} \subset \mathfrak{t}_g,
\]
under the co-adjoint action of the central extension. By construction \( R_a \) is invariant under \( W_{aff}^a \) which is a subgroup of \( W_{aff} \) and is the Weyl group of \( (LG)_a \). The invariance comes from the fact that the edges of \( R_a \) are spanned by affine roots, hence invariant under the reflection by the corresponding root.

Let \( U_a = \mu^{-1}( (LG)_a(R_a) ) \), then \( U_a \) is a symplectic \( (LG)_a \)-manifold of finite dimension, since \( \mu \) is proper and \( (LG)_a(R_a) \) is compact in \( \mathfrak{t}_g \). The proof that it is symplectic is identical to the finite dimensional situation.

For \( H \subset T \), the \( H \)-fixed points on \( X_N \) have images under \( \phi \) as linear subsets in \( kC \), denote them by \( \{ P_d \} \), \( 0 \leq d \leq l - 1 \) is the dimension of the connected component. And each \( P_d \) is a relatively open set in \( kC \). The following is easy to verify:

**Lemma 13.1.** There is a point \( x \in C \), rational w.r.t. the weight lattice and in the interior or of \( C \), such that the corresponding decomposition of \( \mathfrak{t} \) as described above is transversal to all the linear subsets \( \{ P_d/k \} \).
13.2. Varieties corresponding to the partition.

For each $R_a$, $W_a^{\text{aff}}(R_a \cap C) = R_a$. So $R_a \cap C$ acts as the fundamental domain of $W_a^{\text{aff}}$ on $R_a$.

Let the faces of $R_a$ be denoted by $\{\Box\}$, then each $\Box$ is preserved by a subgroup in $W_a^{\text{aff}}$, $W_\Box$. Because each face of $R_a$ is spanned by roots, and the reflections defined by those roots preserve $\Box$. Another way to see $W_\Box$ is as follows: If $Q$ is the smallest face of $wC$ which meets $\Box$, then the intersection has to be a point, otherwise even a smaller face can be found to meet $\Box$. Let $p = \Box \cap Q$, then $\Box$ is perpendicular to $Q$, since it is defined by the roots vanishing on $Q$. Therefore the group $W_Q \subset W_a^{\text{aff}}$ which fixes $Q$, preserves $\Box$.

**Definition 13.1.** Let $(LG)_\Box$ denote the group which fixes the smallest face $Q$ intersecting $\Box$. $W_\Box \subset W_a^{\text{aff}}$ be the subgroup preserving $\Box$.

Next we shall use $\{\Box\}$ to define symplectic cuts on both $X, Y = G \times_T X_N$.

For each face $\Box$, the intersection $\Box \cap C$ is a convex polytope. The map $\phi : X_N \rightarrow kC$ defines for each $\Box$, $\phi^{-1}(k(C \cap \Box))$. For simplicity of notations, let $k = 1$.

For each sub-face $B$ of $\Box$ in the interior of $C$, let $t_B$ be the subalgebra of $t$ perpendicular to the linear set defined by the face $B$, $T_B$ be the group generated by $t_B$.

**Definition 13.2.** Let $X_{N,\Box}$ be the cut space associated with $\phi^{-1}(C \cap \Box)$, i.e.,

$$X_{N,\Box} = \cup_B \phi^{-1}(B) \simeq .$$

That $X_{N,\Box}$ is an $T$-orbifold with an invariant almost complex structure follows from the same argument for $X_N$ itself. Although $X_{N,\Box}$ has degenerate symplectic form, just as $X_N$ does. The orbifold line bundle $L_N$ also induces one on $X_{N,\Box}$, denoted by $L_{N,\Box}$ which can be defined using quotients by $T_B$ on $L_N|\phi^{-1}(C \cap \Box)$.

The proof of the following can be found in [M]:

**Lemma 13.2.** The $T$-equivariant Riemann-Roch satisfies the following

$$\text{RR}_T(X_N, L_N) = \sum_{\Box \cap C \neq \emptyset} (-1)^{\text{codim}\Box} \text{RR}_T(X_{N,\Box}, L_{N,\Box}).$$
Therefore the corresponding $G$-spaces $Y, Y_\square = G \times_T X_{N,\square}$ and the line bundles $L_Y, L_\square$ satisfy

$$RR_G(Y, L) = \sum_{\square \cap C \neq \emptyset} (-1)^{\text{codim} C} RR_G(Y_\square, L_\square).$$

**Remark:** Readers should compare this surgery formula with that in Prop. [2.1].

There is a map $\phi: Y_\square$ which serves as moment map, for the degenerate symplectic form, just as we have shown for $Y = G \times_T X_N$.

The image $\phi(Y_\square \cap t)$ is contained in $W(k(\square \cap C))$, since the image of $X_{N,\square}$ is in $(k(\square \cap C))$.

13.3. Another space $Z_\square$ associated with $\square$. Define $Z_\square$ to be the $(LG)_\square$-symplectic orbifold which is the cut space associated with $\mu^{-1}(LG_\square(\square))$. There are two ways of defining it. The first one is as in [C3], where a holomorphic $LG \times (LG)_\square$ symplectic orbifold $M_\square$ was constructed, then $Z_\square$ can be defined as the symplectic reduced space of the product $X \times M_\square$.

The other approach is given in [M].

The space $Z_\square$ is compact, has a moment map $\Phi$ whose image $\Phi(Z_\square \cap t) \subset \square$.

If $LG_\square \subset G$, i.e., $\square \cap C_{\text{aff}} = \emptyset$, the two space $G_\square \times_T X_{N,\square} = Z_\square$ are related as twins in the sense of the previous section, where $M, M_G$ would be $Z_\square$ and $G_\square \times_T X_{N,\square}$ respectively. In particular they share the same Riemann-Roch.

On the other hand if $\square \cap C_{\text{aff}} \neq \emptyset$, then we will see below how the Riemann-Roch are related.

13.4. Riemann-Roch of $X_{G,\square}$ and $Z_\square$. For $w \in W$, clearly $w(\square \cap C) \subset wC$. If $W_\square_{\text{aff}}$ preserves $\square$, then $Ad_w W_\square_{\text{aff}}$ preserves $w\square, w \in W$.

In the following $W_\square$ will be used in place of $W_\square_{\text{aff}}$.

For $W_\square \subset W_{\text{aff}}$, there is the isomorphic subgroup $W_\square^0 \subset W$. The proof of this simple fact is identical to that of the last statement in Prop. 7.2.

For each $w$ is in $W_\square^0$, let $w'$ denote the corresponding element in $W_\square$.

**Lemma 13.3.** For every pair $w, w'$, there is a translating element $v$ in the long root lattice so that $w = vv'$.

**Pf:** This is a fairly simple fact. Since I can not find a reference for that, the proof is included. The proof is based on induction of the length of $w$.

By applying an element in $W$, we may assume that $\square \cap C \neq \emptyset$ since $W$ preserves the long root lattice. For such a $\square$, all the simple roots of $(lg)_\square$ are all simple, or contain $\alpha_0$.

Suppose $w$ is $r_i$, then

$$r_i r'_i = r_i (r'_i)^{-1}$$
is either $I$, or $-\theta$ depending on whether $r_i$ is defined by a simple root of $g$ or by $\alpha_0$. Thus the assertion holds for elements of length 1.

Assume it’s proved for elements with length less than $n$. Suppose $w$ has length $n$, $w = rv_1 w_1$ with $v_1$ of length less than $n$, and $r$ is one of the generating reflections. By induction assumption $v_1 = v_1 w_1'$, and $v_1$ is a translation. One can write $w' = r' w_1'$, then

$$w = rv_1 w_1' = rv_1 r^{-1} r'r' w_1' = rv_1 r^{-1} r'r' w = vw',$$

where both $rv_1 r^{-1}$, $rr'$ are translations by elements in the long root lattice. So is $v$ as the composition of the two. QED.

The following is essential, it explains in geometric context the appearance of the lattice $\frac{M^*}{k+h^\vee}$. By the definition of $M^*$, we have $e^{(k+h)v} = 1$ on $\frac{M^*}{k+h^\vee}$. The first part generalizes Prop. 7.2.

**Proposition 13.1.** 1). Let $\rho_\Omega, \rho$ be the half sum of positive roots of $(lg)_C, g$ respectively. Let $D_\Omega, D$ be their Weyl denominators, and $w = vw'$ with $v, w', v$ as in the previous lemma.

Suppose $\lambda$ is of level $k$, then

$$(13.2) \quad w^\frac{\lambda}{D} = e^{(k+h)v} \frac{D_\Omega}{D} w' \frac{e^{\lambda}}{D_\Omega}.$$

In particular,

$$(13.3) \quad w^\frac{\lambda}{D} = \frac{D_\Omega}{D} w' \frac{e^{\lambda}}{D_\Omega} \text{ on } \frac{M^*}{k+h^\vee},$$

where $v$ is the translation in the previous lemma.

2). As function on $T$, the following holds

$$(13.4) \quad w \int_F \frac{Td(F) \text{Ch}(LF \oplus H)}{\text{det}(1-t^{-1}e^{-\Omega})} = e^{(k+h)v} \frac{D_\Omega}{D} w' \int_F \frac{Td(F) \text{Ch}(LF \oplus H)}{\text{det}(1-t^{-1}e^{-\Omega})}$$

where $H$ is a bundle of level 0 on which $W^{aff}$ acts. If there is no pole on $\frac{M^*}{k+h^\vee}$, then

$$(13.5) \quad w \int_F \frac{Td(F) \text{Ch}(LF \oplus H)}{\text{det}(1-t^{-1}e^{-\Omega})} = \frac{D_\Omega}{D} w' \int_F \frac{Td(F) \text{Ch}(LF \oplus H)}{\text{det}(1-t^{-1}e^{-\Omega})} \text{ on } \frac{M^*}{k+h^\vee}.$$
13.5. **Moving** $X_{N,\square}$ **and the consequence.** As we have mentioned earlier when $\text{FC}_F$ has poles on $\frac{M^*}{k+h^\nu}$, we can not replace $w \text{FC}_F$ by $\frac{D}{D'} w' \text{FC}_F$ on $\frac{M^*}{k+h^\nu}$. On the other hand, the function $RR_T(X_{N,\square}, L_{N,\square})$ is a polynomial, thus it can be evaluated everywhere. For that function, the following holds:

The above transformation rule yields the following

**Corollary 13.1.**

\begin{equation}
\frac{W}{D} \frac{RR_T(X_{N,\square}, L_{N,\square})}{D} = e^{(k+h')v} \frac{D}{D'} w' \frac{RR_T(X_{N,\square}, L_{N,\square})}{D'};
\end{equation}

Furthermore,

\begin{equation}
\frac{W}{D} \frac{RR_T(X_{N,\square}, L_{N,\square})}{D} = \frac{D_{\square}}{D} w \frac{RR_T(X_{N,\square}, L_{N,\square})}{D_{\square}} \quad \text{on} \quad \frac{M^*}{k+h^\nu};
\end{equation}

For $X_{G,\square} = G \times_T X_{N,\square}$, one has

\begin{equation}
RR(X_{G,\square}, L_{G,\square}) = \sum_{u \in W/W_{\square}} u \frac{D_{\square}}{D} u \frac{RR(Z_{\square}, L_{\square})}{D_{\square}} \quad \text{on} \quad \frac{M^*}{k+h^\nu}.
\end{equation}

**Pf:** The first one is an immediate consequence of Prop.13.1, after one writes both sides in terms of the $T$-fixed points contributions.

Since the function $RR_T(X_{N,\square}, L_{N,\square})$ is the equivariant index of a spin$\mathbb{C}$ complex defined by the pair $X_{N,\square}, L_{N,\square}$, it is a well defined function everywhere. Hence we can evaluate on the lattice $\frac{M^*}{k+h^\nu}$ to get the second formula.

To see the next identity, expand $RR(X_{G,\square}, L_{G,\square})$ in terms of $RR(X_{N,\square}, L_{N,\square})$, then apply the first and second identity to yield

\begin{equation}
RR(X_{G,\square}, L_{G,\square}) = \sum_{w \in W} w \frac{RR_T(X_{N,\square}, L_{N,\square})}{D} = \sum_{u \in W/W_{\square}} u w \frac{RR_T(X_{N,\square}, L_{N,\square})}{D} = \sum_{u \in W/W_{\square}} u \frac{D_{\square}}{D} \sum_{w' \in W_{\square}} w' \frac{RR_T(X_{N,\square}, L_{N,\square})}{D_{\square}} \quad \text{on} \quad \frac{M^*}{k+h^\nu},
\end{equation}

it is easy to recognize the sum $\sum_{w' \in W_{\square}} w' \frac{RR_T(X_{N,\square}, L_{N,\square})}{D_{\square}}$ is simply the Riemann-Roch of the space $(LG)_{\square} \times_T X_{N,\square}$. How is it related to $Z_{\square}$? They are twin-pairs as discussed in the last section, replacing $M, G$ there by $Z_{\square}, (LG)_{\square}$. Thus one has

\begin{equation}
RR(X_{G,\square}, L_{G,\square}) = \sum_{u \in W/W_{\square}} u \left( \frac{D_{\square}}{D} RR(Z_{\square}, L_{\square}) \right) \quad \text{on} \quad \frac{M^*}{k+h^\nu}. \quad \text{QED}
\end{equation}

So the above relates the Riemann-Roch of $G$-space $X_{G,\square}$ and the $LG_{u,\square}$-space $\{Z_{\square} := u Z_{\square}\}$.

The fig.13.5 illustrates the relations between the intersections with $t$ of the images of $X_{G,\square}$ and three $Z_{u,\square}$. The three separate regions, inside the middle hexagon, with dotted lines are associated with $X_{G,\square}$. In this case $W_{\square} \simeq W$, the image of $Z_{\square}$ meeting $t$ inside one of the regions filled with dashed lines. What are the other two identical regions? If one starts with $u \square$, in this case $u \neq I, R_3$, then $W_{u,\square} \simeq W$ holds as well. And one of the other two regions will contain the intersection of the image with $t$ of $u Z_{\square}$.
In the second figure, $W_\square \simeq \mathbb{Z}_2$, thus $W_\square^0$ is not the same as $W$. $W/W_\square^0$ has three elements. The union of the six short segments inside the hexagon contains $\mu(X_{G,\square}) \cap t$. The long segments contain the intersections with $t$ of the images of $\{uZ_{u,\square}\}$.

13.6. **The final step.** *Proof of the Main Theorem: Step 1:* By cutting and applying Cor. 13.1, we first cancel the contributions to $\text{RR}(Y)$, from the fixed point set $F$ with $\phi(F) \in w(\partial C \setminus C_{\text{aff}})$. The cancellation below is implicit when applying Cor. 13.1.

By the fundamental property of cutting, one has

$$\text{RR}(Y) = \sum_{w \in W} w \frac{\text{RR}(X_N)}{D}$$

$$= \sum_{w \in W} w \frac{1}{D} \sum_{\square \cap C \neq \emptyset} (-1)^{\text{codim} \square} \text{RR}_T(X_{N,\square})$$

$$= \sum_{\square \cap C \neq \emptyset} (-1)^{\text{codim} \square} \sum_{w \in W} w \frac{1}{D} \text{RR}_T(X_{N,\square})$$

$$= \sum_{\square \cap C \neq \emptyset} (-1)^{\text{codim} \square} \sum_{u \in W/W_\square^0} u \frac{D_\square}{D} \text{uRR}(Z_{u,\square}) \text{ on } M^* \frac{k+h^r}{k}$$

where the last step uses Cor. 13.1. See the figure for the translations of the images of $\phi(X_{N,\square})$, similar actions are taken place for the varieties with images lying on the lower dimensional $\square$ which are not illustrated.

**Step 2: Localize to $V$.** It should be clear that $u\text{RR}(Z_{u,\square}) = \text{RR}(Z_{u,\square})$. While the intersection of the image of $Z_{u,\square}$ with $t$ is in $W_{\square}^{\text{aff}}(\square)$, the image of $uZ_{\square}$ is in

$$uW_{\square}^{\text{aff}}(\square) = \text{Ad}_u(W_{\square}^{\text{aff}})(u\square) = W_{u,\square}^{\text{aff}}(u\square).$$

Each $\tau \in \{e^{2\pi i w^{-1}} \frac{\lambda + \rho}{\text{rank} \lambda} | \lambda \in P_k^+\}$ is a generic element in $T$, hence the connected components of $\tau$-fixed points in $Z_{u,\square}$, denoted by $\{V\}$, must have image under moment map in $t$. Each $V$ is an orbifold, and is symplectic, by general theory on fixed point sets and the fact that $Z_{u,\square}$ is a symplectic orbifold. We may apply Prop. [12.1], replacing $M, G, V$ there by $Z_{u,\square}, K, V$ where $K = (LG)_{u,\square}$. The collection $\{\Delta\}$
Figure 13.4. Comparing varieties with images differed by translation elements in $W^{\text{aff}}$

Figure 13.5. After translation and the twin pair comparison

there is now replaced by $\{uu'(\Box \cap C)|w' \in W^{\text{aff}}\}$ which is denoted by $\{\Delta'\}$ in the following. Denote by $Q_1$ a face of $u(\Box \cap \partial C)$, and

$$Q = \text{Ad}_u w(Q_1), \quad Q' = \text{Ad}_u w'(Q_1)$$

where $w' \in W^{\text{aff}}$ and $w$ is the corresponding element in $W^0$. We obtain

$$u(\text{RR}(Z_\Box))(\tau) = \text{RR}(Z_{u\Box})(\tau)$$

$$= \sum_{\Delta' \subset \Box} \sum_{Q' \subset \Delta'} |W_{Q'}||I_{V_{Q'}}| \sum_{t \in \tau I_{V_{Q'}}} \int_{V_{Q'}} \frac{\text{Td}(V_{Q'}) \text{Ch}(L_{V_{Q'}} \oplus \Lambda_{\text{max nor}}(V_{Q'}, V_{\Delta'}))}{\text{det}_{\text{nor}}(V_{Q'}, X_{Q'}) (1 - t^{-1}e^{-\Omega})}.$$  

The varieties $\{Z_{w\Box}\}$ have images as in the following figure:

Step 3: Transporting $V_{Q'}$. The transporting here are in the directions opposite to the arrows in the figure [13.6]. One does not simply get back the result in the
beginning, since the cancellation has already taken place. This is an important realization.

Two observations can be made here: Each integral \( \int_{Q'} \) in the above is finite, when evaluated at \( \tau \), because \( V_{\text{Ad}_u \cdot w(Q)} \) is a connected component of \( \tau \)-fixed points in \( Z_Q^K \), so the action by \( \tau \) on the normal bundle is non-trivial, thus the denominator is well defined at \( \tau \). Second observation is that one can apply Prop. [3.1.2] to this situation. To make this more explicit, let \( \text{Ad}_u \cdot w = s \text{Ad}_u \cdot w' \) where \( s \) is a translation defined by an element in the long root lattice, according to Lemma 13.3. For each \( \text{Ad}_u \cdot w'(\square \cap C) \), there is the corresponding \( \text{Ad}_u \cdot w(\square \cap C) \), obtained by shifting \(-s\). Likewise \( \text{Ad}_u \cdot w(Q_1) \) is obtained from \( \text{Ad}_u \cdot w'(Q_1) \) by shifting \(-s\).

The corresponding varieties \( V_Q \subset X_Q \) have a similar relation \( s^{-1}(V_Q) = V_{-s(Q)} \subset s^{-1}(X_Q) = X_{-s(Q)} \), and \( s^{-1}(V_Q) \) is a connected component of \( \tau \)-fixed points, since the translation commutes with \( T \)-action.

The denominator is given by

\[
\det_{\text{nor}(V_{Q'}, X_{Q'})} (1 - t^{-1}e^{-\Omega}) = uu'D_{Q'} \det_{\text{nor}(V_Q, X_Q)} (1 - t^{-1}e^{-\Omega}),
\]

thus we obtain the following relation as a consequence of Prop. [3.1.2]

\[
\frac{e^{k+h^*})s_a(D_{Q'})}{D_{Q'}} \int_{V_{Q'}} \frac{\text{Td}(V_{Q'}) \text{Ch}(LV_{Q'} \oplus \Lambda_{\text{max nor}}(V_{Q'}, V_{\Delta'}))}{\det_{\text{nor}(V_{Q'}, X_{Q'})}(1 - t^{-1}e^{-\Omega})} = \int_{V_Q} \frac{\text{Td}(V_Q) \text{Ch}(LV_Q \oplus \Lambda_{\text{max nor}}(V_Q, V_\Delta))}{\det_{\text{nor}(V_Q, X_Q)}(1 - t^{-1}e^{-\Omega})},
\]

where \( Q' = uu'(Q_1), Q = uu'(Q_1) \) with \( Q_1 \) a face of \( \square \cap C \).

Both sides have no poles at \( \tau \) since \( \tau \text{I}_{V_Q} \) acts on the normal bundles with no 0 eigenvalue. Evaluate them at \( \tau \in \text{exp}(\frac{X}{k+h^*}) \), i.e. \( e^{k+h^*})s_1 \text{ for } Q = sQ' \), \( \Delta = s\Delta', \) one has the following

\[
\text{RR}(Y)(\tau) = \sum_{\Delta \cap C \neq \emptyset} (-1)^{\text{codim} \Delta} \sum_{u \in W/Q} u \sum_{w \in \text{Ad}_u \cdot W_Q} \frac{1}{|W_Q||I_Q|} \sum_{t \in \text{I}_Q} |I_Q|,
\]

\[
(13.11)
\]

\[
\text{I}_{V_Q} = \int_{V_Q} \frac{\text{Td}(V_Q) \text{Ch}(LV_Q \oplus \Lambda_{\text{max nor}}(V_Q, V_\Delta))}{\det_{\text{nor}(V_Q, X_Q)}(1 - t^{-1}e^{-\Omega})}.
\]

**Step 4: Eliminate the extra things.** Next, we will write the above in a concise form. To see further cancellation, it is easiest to write the integrals above in terms of the \( T \)-fixed points contribution:

\[
\frac{1}{|I_Q|} \sum_{t \in \text{I}_Q} \int_{V_Q} \frac{\text{Td}(V_Q) \text{Ch}(LV_Q \oplus \Lambda_{\text{max nor}}(V_Q, V_\Delta))}{\det_{\text{nor}(V_Q, X_Q)}(1 - t^{-1}e^{-\Omega})} = \sum_{F \subset V_Q} \frac{1}{|I_F|} \sum_{t \in \text{I}_F} \int_{F} \frac{\text{Td}(F) \text{Ch}(LF \oplus \Lambda_{\text{max nor}}(V_F, V_{\Delta}|F))}{\det_{\text{nor}(F, X_Q)}(1 - t^{-1}e^{-\Omega})}.
\]

Substitute the above into Eq. (13.11), then we will treat those terms according to the type of \( F \):

a). \( \phi(F) \) is on a cut defined by \( \square \) with \( \dim \square < l = \dim t \).

b). \( \phi(F) \) is on some \( wQ, Q \subset \partial C \setminus C_{\text{aff}} \), but not on \( \square \) with \( \dim \square < l = \dim t \).

c). \( \phi(F) \) is on some \( wQ \) with \( Q \subset C_{\text{aff}} \); but not on \( \square \) with \( \dim \square < l = \dim t \).
d). \( \phi(F) \) is neither on any \( wQ, \forall w \in W, \forall Q \subset \partial C \) nor on \( \Box_1 \) with \( \dim \Box_1 < l = \dim t \).

Obviously, the above covers all possibilities for \( \phi(F) \). We claim the contributions of \( F \) of the first two kinds amount to 0 in Eq. (13.6).

In the first case, suppose \( u(\Box_1 \cap C), u(Q_1) \) are the smallest among \( u(\Box \cap C), u(Q) \) containing \( \phi(F) \). Fixing a \( u(Q) \supseteq u(Q_1) \), and \( u(\Box) \supseteq u(\Box_1) \), what is the integrand in the above integral?

For convenience of notations, assume below that \( u = I \).

Let us take a moment to discuss the weights on \( \text{nor}(F, V_{\Box}) \) of the group \( (LG)_Q \) which is the stabilizer in \( LG \) of \( Q \).

Each \( t \in T \) comes from a product of a triple

\[
t_P \cdot t_{\Box_1} \cdot t_Q \in T_P \cdot T_{\Box_1} \cdot T_Q,
\]

the finite ambiguity of the choice of the triple causes \( F \) to be an orbifold singularity. So the orbifold weights on the normal bundle of \( F \) in \( X_{\Box} \) can be described as follows:

\[
\begin{align*}
\lambda(t) &= \tilde{\lambda}(t_Q); \\
\gamma(t) &= \tilde{\gamma}(t_P); \\
a(t) &= \tilde{a}(t_{\Box_1}); \\
\beta(t) &= \tilde{\beta}(t_P)
\end{align*}
\]

(13.12)

the verification of the above is the same as Prop. 4.3.

The normal bundle \( \text{nor}(V_{\Box}, V_{\Box})|_F \) has weights given by \( \lambda(t) = \tilde{\lambda}(t_Q), \tilde{\lambda} \notin \Lambda_Q \) where \( \Lambda_Q \) is the subset of \( \{ \lambda \} \) not parallel to \( Q \).

One realizes in the above that for different \( \Box \supseteq \Box_1 \), the only difference in the integrand is the term \( \prod_{\lambda \in \Lambda_Q} (1 - e^{-2\pi i \frac{c_{\lambda} t_p + 1}{4s} dA}) \) appearing in the denominator, here \( \tilde{B} \) is the form representing the Chern class of the principle bundle corresponding to the orbit of \( T_{\Box_1} \).

Let \( D_\alpha(s), D_\beta(s) \) be the same as in Section 11, and \( D'_\beta \) be defined the same as \( D_\beta \) except only those \( \gamma \) tangent to \( \Box_1 \) will be involved. We have the following representation of the integrand:

\[
\begin{align*}
&\frac{(-1)^{\text{codim} \Box} \sum_{\Box \supseteq \Box_1} \int_F \frac{\text{Td}(F) \text{Ch}(L_F \oplus \Lambda_{\text{max}} \text{nor}(V_{\Box}, V_{\Box}))}{\det_{\text{nor}(F, X^{\Box})} (1 - s^{-1} e^{-\Omega})} Td(F) \text{Ch}(L_F)}{\sum_{\Box \supseteq \Box_1} \int_F e^{\sum_{\lambda \in \Lambda_Q} 2\pi i \frac{c_{\lambda} t_p + 1}{4s} dA} D_\alpha(s) D_\beta(s) D'_\beta(s) D_\alpha(s)} \\
&\quad \times \sum_{\Box} (-1)^{\text{codim} \Box} \frac{1}{\prod_{\lambda \in \Lambda_Q} (1 - e^{-c_{\lambda} t_p + 1/4s} dA)} \\
&= 0
\end{align*}
\]

(13.13)
since 
\[
\sum_{\mathbb{A}_n} (-1)^{\text{codim} \square} \frac{1}{\prod_{a \in \mathbb{A}_n} (1 - e^{-\angle_{a, \iota_F + 1/4\pi} dA})} = 0.
\]

As for $F$ of the second type, $\phi(F)$ is in the interior of $\cup_w w(C)$, but on a wall of Weyl chamber $w(Q)$. The sum of the integrals over various $Q'$ with $Q \subset Q'$ was already shown to be 0 in the proof of the Prop. 12.1. The argument there for those fixed points appearing on the boundary of the Weyl chamber shows the same cancellation here.

Therefore, only $F$ in c) or d) survives. We do recognize these two types, $F$ of type c) are those produced by the intersection of $\tau$ fixed point set $V$, with the compactification locus corresponding to $w(C^{\text{aff}})$, i.e. the affine walls. Those of type d) are the fixed point sets in $X$ with images in $W(C^{\text{int}})$. Thus we have the following form for $\text{RR}(Y)$ when evaluated at $\tau$:

\[
(13.14) \quad \text{RR}(Y) = \sum_{\{F|\phi(F)\in W(C^{\text{int}})\}} \text{FC}(F)
+ \sum_{\{F|\phi(F)\in W(C^{\text{aff}})\}} \frac{1}{|W_Q||I_F|} \sum_{t \in \iota_F} \int_F \frac{Td(F) \text{Ch}(L_F \oplus L^{\text{max}} \text{nor}(Y_Q, Y)|_F)}{\text{det}_{\text{nor}(F, Y_Q)}(1 - s^{-1}e^{-\Omega})}.
\]

The first sum is over the true fixed points on $X \cap \mu^{-1}(W(C))$, the absence of the isotropy group $I_F$ is due to the smoothness of $Y$ at the interior fixed points, since no cutting passes such $F$. The second sum is over those fixed points lying on the intersection of the compactification locus $Y_Q$ and a $\tau$-fixed point set component. The only explanation needed here is that

\[
\text{Ch}(\text{nor}(Y_Q, Y_{\Delta})|_F) = \text{Ch}(\text{nor}(V_Q \cap \square_1, V_{C \cap \square})|_F)
\]

which is obvious in terms of these weights $\lambda$ with $\tilde{\lambda} \in Q$.

Thus we obtain the second expression for $\text{RR}(Y)(\tau)$ in the main theorem. The first one can be obtained easily now by localize the integrals over $V_{\Delta}$, $V_Q$ to their fixed points. QED.

We emphasize that there are plenty of $T$-fixed point sets on the compactified locus, they do not contribute to the formula in Eq. (13.14) unless they are on the compactified $\tau$-fixed points.
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