Variations in the magnetic torque acting on a wire

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Abstract
The relation $M = \mu \times B$ is presented in all elementary courses on electromagnetism, but it is usually given just for the simple case of a rectangular wire. We will present a completely general but elementary proof of this relation together with two more advanced proof methods. We will then provide some extensions: non-closed wires and non-uniform magnetic field.

(Some figures may appear in colour only in the online journal)

1. Introduction

The torque $M$ acting on a wire induced by a uniform magnetic field $B$ is given by the well-known formula

$$ M = \mu \times B, $$

where $\mu$ is the magnetic dipole moment of the wire. In all elementary textbooks on electromagnetism (see e.g. [1, 2]), this formula is introduced by studying the example of a rectangular wire, for which the magnetic dipole moment can be explicitly written as $A In$, where $A$ is the area of the wire, $I$ is the current passing through it and $n$ is the normal to the plane of the wire, with the orientation being consistent with that of the current.

Also in more advanced textbooks, the complete derivation of equation (1) is usually skipped and just the example of the rectangular wires is presented. Two notable exceptions to this rule are the eighth volume of the Landau theoretical physics course [3, p 128] and the book by Jackson [4, pp 188–90]: the (sketch of the) proof by Landau relies on a variation on the theme of the Stokes theorem:

$$ \oint d\ell \times X = \int d\sigma \times (\nabla \times X) + \int (d\sigma \cdot \nabla) X - \int d\sigma (\nabla \cdot X), $$

where $X$ is a generic vector field and the integrals on the left- and right-hand sides of the equation are line and surface integrals respectively. Jackson’s proof uses instead the identity

$$ \int (f Y \cdot \nabla g + g Y \cdot \nabla f + f g \nabla \cdot Y) d^3 r = 0, $$

where $f$ and $g$ are the scalar functions of the position and $Y$ is a vector field of compact support.
We will present a complete elementary proof of equation (1) together with the ones by Landau and Jackson reviewed here with some more details than in the original references. We will then show how the computation can be extended to the more general cases of non-closed wires and non-uniform magnetic field.

2. An elementary proof

The starting point is the Lorentz force acting on an element $d\ell$ of the wire, which in SI units is

$$dF = I d\ell \times B.$$  

(4)

where $I$ is the current. The torque acting on the wire is then

$$M = I \oint r \times (d\ell \times B).$$  

(5)

Let us now suppose the wire to be parametrized by the function $s(t)$, with $t \in [0, 1]$ being a real parameter. In this case, $d\ell = \dot{s} dt$ (we denote by the dot the derivative with respect to $t$) and equation (5) can be rewritten in the form

$$M = I \int_0^1 s \times (\dot{s} \times B) dt.$$  

(6)

By using the vectorial identity

$$X \times (Y \times Z) = Y(X \cdot Z) - Z(X \cdot Y),$$  

(7)

we can rewrite equation (6) in the form

$$M = I \int_0^1 [\dot{s}(s \cdot B) - B(\dot{s} \cdot s)] dt,$$  

(8)

and it is simple to show that the second term vanishes in a uniform field: the vector $B$ can be carried out of the integral, which finally reduces to

$$\int_0^1 \dot{s} \cdot s dt = \frac{1}{2} \int_0^1 \frac{d}{dt}|s|^2 dt = 0,$$  

(9)

since $s(0) = s(1)$ for a closed wire.

By expressing the first term of equation (8) in components, we have (summation on the repeated indices is always assumed when not otherwise stated)

$$\int_0^1 [\dot{s}(s \cdot B)]_i dt = B_j \int_0^1 \dot{s}_i s_j dt$$

$$= \frac{1}{2} B_j \int_0^1 (\dot{s}_i s_j + \dot{s}_j s_i) dt + \frac{1}{2} B_j \int_0^1 (\dot{s}_i s_j - \dot{s}_j s_i) dt$$

$$= \frac{1}{2} B_j \int_0^1 \frac{d}{dt}(s_i s_j) dt + \frac{1}{2} B_j \int_0^1 (\dot{s}_i s_j - \dot{s}_j s_i) dt$$

$$= \frac{1}{2} \int_0^1 [\dot{s}(s \cdot B) - s(B \cdot \dot{s})]_i dt,$$  

(10)

where the total derivative term vanishes for the same reason as the integral in equation (9) does. On the other hand, from the identity in equation (7) we have

$$(s \times \dot{s}) \times B = s(B \cdot \dot{s}) - s(B \cdot \dot{s}),$$  

(11)

1 See Note added in proof at the end of the paper.
so that from equations (8)–(10) we obtain

\[ M = \left( \frac{I}{2} \int_0^1 (s \times \dot{s}) \, dt \right) \times B, \]  

(12)

which is the desired equation (1) with the identification of the dipole magnetic moment

\[ \mu = \frac{I}{2} \int_0^1 s \times \dot{s} \, dt. \]  

(13)

Going back to the line integral form we finally obtain

\[ \mu = \frac{I}{2} \oint r \times d\ell, \]  

(14)

which for planar wires reduces to the simple expression \( ALn \), since the element of area is given by \( ndA = \frac{1}{2}r \times d\ell \).

3. Landau’s proof

In this section, we will present a proof of equation (1) by using the identity equation (2), whose proof is given in appendix A.

We first of all show how the form in equation (14) of the magnetic dipole moment can be simplified by using the extension of the Stokes theorem proven in the appendix. By using equation (2), we immediately obtain (since \( \nabla \times r = 0 \), \( a \cdot \nabla )r = a \) and \( \nabla \cdot r = 3 \))

\[ \oint d\ell \times r = \int d\sigma \times (\nabla \times r) + \int (d\sigma \cdot \nabla )r - \int d\sigma (\nabla \cdot r) \]

\[ = \int d\sigma - 3 \int d\sigma = -2 \int d\sigma, \]  

(15)

and thus equation (14) becomes

\[ \mu = I \int d\sigma, \]  

(16)

which is the simplest extension to non-planar wires of the expression \( \mu = ALn \) valid in the planar case. Clearly, the result of equation (16) does not depend on the choice of the surface of integration. If we denote by \( c \) a constant vector, then the difference between the (projection on \( c \) of the) results obtained with two different choices \( \Sigma_1 \) and \( \Sigma_2 \) is given by

\[ c \cdot (\mu_1 - \mu_2) = I \left( \int \frac{d\sigma}{\Sigma_1} (c \cdot e) - \int \frac{d\sigma}{\Sigma_2} (c \cdot e) \right) = I \int_{V_{12}} (\nabla \cdot c) \, d^3r = 0, \]  

(17)

where \( V_{12} \) is the volume bounded by the surfaces \( \Sigma_1 \) and \( \Sigma_2 \). Since this is true for every \( c \), we conclude that \( \mu_1 = \mu_2 \).

In order to apply equation (2) to the computation of the torque, it is convenient to use the following vectorial identity:

\[ X \times (Y \times Z) + Y \times (Z \times X) + Z \times (X \times Y) = 0 \]  

(18)

and rewrite equation (5) in the form

\[ M = -I \int d\ell \times (B \times r) + I \oint (r \times d\ell) \times B. \]  

(19)

By comparison with equation (14), the second term can be recast in the form \( 2\mu \times B \), while applying equation (2) to the first term we obtain

\[ I \oint d\ell \times (B \times r) = I \int d\sigma \times (\nabla \times (B \times r)) + I \int (d\sigma \cdot \nabla ) (B \times r) - I \int d\sigma (\nabla \cdot (B \times r)), \]  

(20)
By using the relations (which can be easily checked by direct computation)
\[ \nabla \times (B \times r) = 2B \]
\[ (\sigma \cdot \nabla)(B \times r) = B \times \sigma \]
\[ \nabla \cdot (B \times r) = 0 \]
and equation (16), we thus obtain
\[ I \oint d\ell \times (B \times r) = 2I \int d\sigma \times B + I \int B \times d\sigma = I \int d\sigma \times B = \mu \times B. \]
Using this result in equation (19), we finally obtain equation (1).

4. Jackson’s proof

This proof makes use of equation (3), which is easily proven. Since we assumed \( Y \) to be a function of compact support we have, by using the divergence theorem for a large enough volume \( V \) (bounded by the surface \( \Sigma \)),
\[ \int_V \nabla \cdot (fgY) \, d^3r = \int_\Sigma fg \sigma \cdot Y = 0, \]
(23)
since \( Y \) vanishes on \( \Sigma \). By using the identity
\[ \nabla \cdot (fgY) = fY \cdot \nabla g + gY \cdot \nabla f + fg \nabla \cdot Y, \]
we thus obtain equation (3).

The starting point is equation (5), which can be rewritten by noting that the current density \( j \) has a support on the wire and that \( I d\ell = j d^3r \), thus
\[ M = I \oint r \times (d\ell \times B) = \int r \times (j \times B) \, d^3r. \]
(25)
By using equation (7), this expression becomes
\[ M = \int j(r \cdot B) \, d^3r - \int B(r \cdot j) \, d^3r. \]
(26)
If we now use equation (3) with \( f = g = r_i \) (component \( i \) of \( r \)) and \( Y = j \), remembering that \( \nabla \cdot j = 0 \), we obtain (no summation over repeated indices)
\[ 0 = \int (fY \cdot \nabla g + gY \cdot \nabla f + fg \nabla \cdot Y) \, d^3r = 2 \int r_i j_i \, d^3r, \]
(27)
and by summing over \( i \) we obtain
\[ \int r \cdot j \, d^3r = 0, \]
(28)
so the second term of equation (26) vanishes in a uniform magnetic field.

By using instead \( f = r_i, g = r_k \) and \( Y = j \), we obtain (again no summation on indices)
\[ \int (r_i j_k + r_k j_i) \, d^3r = 0 \]
(29)
and by means of manipulations analogous to the ones in equation (10), we obtain
\[ \int j(r \cdot B) \, d^3r = \frac{1}{2} \int (j(j \cdot B) - r(B \cdot j)) \, d^3r = \frac{1}{2} \int (r \times j) \times B \, d^3r, \]
(30)
so that
\[ M = \frac{1}{2} \int (r \times j) \times B \, d^3r. \]
(31)
By using again the fact that \( I d\ell = j d^3x \), we see by comparison with equation (14) that
\[ \mu = \frac{1}{2} \int r \times j \, d^3r, \]
(32)
and equation (31) thus reduces to equation (1).
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5. Comments and extensions

We have seen in the previous sections that the torque generated by a uniform magnetic field \( B \) on a wire is given by (1), where the magnetic dipole moment \( \mu \) has the following equivalent definitions:

\[
\mu = \frac{I}{2} \int_0^1 s \times \dot{s} \, dt = \frac{I}{2} \oint r \times d\ell = I \int d\sigma = \frac{I}{2} \int r \times j \, d^3r.
\]  

(33)

It is instructive to go through the different proofs presented of the relation (1) to search for the key ingredients used. From the first proof it is clear that, for a formula like equation (1) to be valid, the wire must be closed, an aspect whose importance is not completely evident in the usual example of the rectangular wire. This same requirement is naturally fundamental also for the second proof, since the Stokes theorem could not be applied to an open wire, and, although in a less trivial way, it is fundamental also in the third proof. There the main ingredient was current conservation, but the current would not be conserved in an open wire (see also below). Also, in all the proofs, the uniformity of the magnetic field was crucial to carry \( B \) out of the integration and factorize \( \mu \).

The first form of equation (33) is usually the most direct one to use in computations involving non-planar wires which are not trivially decomposable into planar ones. A simple non-trivial example is the wire shown in figure 1, whose parametrization is

\[
s(t) = \begin{cases} 
\cos(4\pi t)\hat{x} + \sin(4\pi t)\hat{y} + \alpha t\hat{z} & t \in [0, 1/2] \\
\hat{x} + \alpha(1-t)\hat{z} & t \in (1/2, 1].
\end{cases}
\]  

(34)

It is simple to show that

\[
\int_0^1 s \times \dot{s} \, dt = \alpha \hat{y} + 2\pi \hat{z},
\]  

(35)

and thus the magnetic dipole moment of the wire parametrized by equation (34) is

\[
\mu = \frac{I\alpha}{2} \hat{y} + I\pi \hat{z}.
\]  

(36)

which clearly reduces to the planar result \( AI\hat{z} \), with \( A = \pi \), when \( \alpha = 0 \).

Various extensions of the result in equations (1) and (33) can easily be performed. A particularly simple one is to consider a non-closed wire, i.e. \( \Delta = s(1) - s(0) \neq 0 \). Clearly, in

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**Figure 1.** The wire parametrically represented by equation (34) with \( \alpha = 1 \).
this case a net force is also acting on the wire; from equation (4) the expression for the total force
\[ F = I \Delta \times B \] (37)
can immediately be obtained, and collecting the previously vanishing terms in equations (9) and (10), we obtain for the torque:
\[ M = \mu \times B + \frac{I}{2} (s(1) \times (s(1) \times B) - \frac{l}{2} (s(0) \times s(0) \cdot B) - B |s(0)|^2) \]
\[ = \mu \times B + \frac{I}{2} s(1) \times (s(1) \times B) - \frac{l}{2} s(0) \times (s(0) \times B), \] (38)
where \( \mu \) is now defined by the first expression in equation (33). Since a net force is acting on the wire, the torque depends on the choice of the pole used (i.e. on the origin of the coordinates in our computation) and equation (38) cannot in general be written in terms of \( \Delta \) only. This happens only if we choose the pole in \( s(0) \), in which case equation (38) collapses to
\[ M = \mu \times B + \frac{I}{2} \Delta \times (s(1) \times B). \] (39)
This result can also be obtained using the methods in section 4 by noting that, for a non-closed wire, the current is not conserved and a source and a sink have to be present at the wire endings:
\[ \nabla \cdot j = I \delta(r - s(0)) - I \delta(r - s(1)). \] (40)
Another possible extension is the one to a non-uniform magnetic field (again for a closed wire). Let us consider for simplicity only the first linear correction to the uniform field case:
\[ B(r) = b + a r, \] (41)
where \( b \) is a constant vector and \( a \) is a linear operator, i.e. in components we have
\[ B_i(r) = b_i + a_{ij} r_j. \] (42)
The requirement \( \nabla \cdot B = 0 \) imposes the restriction \( \text{Tr} a = 0 \) and, if we further assume that the currents that generate \( B \) are far away from the wire (here ‘far away’ means that these currents do not contribute to the various line or surface integrals), from \( \nabla \times B = 0 \) the relation \( a_{ij} = a_{ji} \) follows, i.e. the matrix \( a \) is symmetric. Since equation (8) is linear in \( B \), we can calculate the corrective term to \( M \) simply by using \( b = 0 \). We then have (using \( s(0) = s(1) \))
\[ \int_0^1 [B(\dot{s} \cdot s)]_i \, dr = a_{ij} \int_0^1 s_j \dot{s}_k \, ds_k \, dt = \frac{1}{2} a_{ij} \int_0^1 s_j \frac{d}{dt}(s_k)^2 \, dt = -\frac{1}{2} \int_0^1 (s_k)^2 a_{ij} \dot{s}_j \, dt \] (43)
and thus
\[ \int_0^1 B(\dot{s} \cdot s) \, dr = -\frac{1}{2} \oint r^2 a \, dl. \] (44)
On the other hand,
\[ \int_0^1 \dot{s}(s \cdot B) \, dr = \oint r \, dl (r \cdot (ar)), \] (45)
and we thus obtain for the torque caused by the non-uniform magnetic field in equation (41) the expression
\[ M = \mu \times b + \frac{I}{2} \oint r^2 a \, dl - I \oint r \, dl (r \cdot (ar)). \] (46)
The second term can also be written as a surface integral; indeed, if we use equation (B.5) of appendix B, with \( \mathbf{B} \) being given by equation (41), we obtain
\[
M = \mu \times \mathbf{b} + \int d\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) + \int r \times (\mathbf{a} \cdot d\mathbf{a}).
\] (47)

Clearly, also in the non-uniform field case, a non-vanishing net force is in general present, which, by using equation (2) (remembering that \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \times \mathbf{B} = 0 \)), can be written as
\[
F = \int (d\mathbf{a} \cdot \nabla) \mathbf{B} = a \mathbf{\mu} = \nabla (\mathbf{\mu} \cdot \mathbf{B}).
\] (48)

If we denote by \( b \) the modulus of \( \mathbf{b} \), by \( a \) a typical value of \( a \) and consider a wire of typical linear dimension \( L \), the contributions in equations (46)–(48) are of the order
\[
M \sim bL^2 + aL^3, \quad F \sim aL^2,
\] (49)
and the first of these equations can conveniently be rewritten as
\[
M \sim bL^2 \left( 1 + \frac{L}{\lambda} \right),
\] (50)

where \( \lambda = b/a \) is the typical length scale of variation of the magnetic field in equation (41). It is then clear that, as intuitively obvious, the non-uniformity of the magnetic field can be neglected as far as \( L \ll \lambda \). For dimensional reasons, the force in equation (49) has one power of \( L \) missing with respect to the torque, but dimensionality would also suggest the presence of a term \( bL \), which is absent, since in a uniform field no net force is acting on the wire. Because of the absence of this leading contribution, the non-uniformity of the magnetic field cannot be neglected even for small wires in the force computation.

For a generic non-uniform magnetic field, equation (41) is just the first term of a Taylor expansion, whose general form is
\[
B_i = a_i^{(0)} + a_{i_1}^{(1)} r_{j_1} + a_{i_2}^{(2)} r_{j_1} r_{j_2} + \cdots + a_{i_{j_1} \cdots j_n}^{(n)} r_{j_1} \cdots r_{j_n} + \cdots,
\] (51)
where \( a_{i_{j_1} \cdots j_n}^{(n)} \) is symmetric under permutations of \( j_1, \ldots, j_n \). The condition \( \nabla \cdot \mathbf{B} = 0 \) for the \( n \)th term becomes
\[
0 = \partial_i B_i = a_{i_{j_1} \cdots j_n}^{(n)} \partial_i (r_{j_1} \cdots r_{j_n}) = n a_{i_{j_1} \cdots j_n}^{(n)} r_{j_1} \cdots r_{j_n},
\] (52)

and the condition \( \nabla \times \mathbf{B} = 0 \) gives
\[
0 = \partial_\alpha B_\beta - \partial_\beta B_\alpha = n a_{\alpha_{j_1} \cdots j_n}^{(n)} - a_{\alpha j_1 \cdots j_n}^{(n)} r_{j_1} \cdots r_{j_n},
\] (53)
so that \( a_{i_{j_1} \cdots j_n}^{(n)} \) is again completely symmetric and traceless. It is then not difficult to generalize equations (46) and (47). We can introduce a characteristic length \( \lambda_{(n)} \) for every term in equation (51) by
\[
\lambda_{(n)} = \sqrt{\frac{a^{(0)}}{a^{(n)}}},
\] (54)
where \( a^{(0)} \) and \( a^{(n)} \) stand here for typical values, and equation (50) generalizes to
\[
M \sim a^{(0)} L^2 \left[ 1 + \frac{L}{\lambda_{(1)}} + \left( \frac{L}{\lambda_{(2)}} \right)^2 + \cdots + \left( \frac{L}{\lambda_{(n)}} \right)^n + \cdots \right].
\] (55)

For a typical magnetic field, we have
\[
\lambda_{(1)} \lesssim \lambda_{(2)} \lesssim \cdots \lesssim \lambda_{(n)} \lesssim \cdots,
\] (56)
and we thus see that the expansion (55) can be truncated to the \( n \)th term if the typical linear dimension of the wire satisfies
\[
\frac{L}{\lambda_{(n+1)}} \left( \frac{\lambda_{(n)}}{\lambda_{(n+1)}} \right)^n \ll 1.
\] (57)
6. Conclusions

We discussed three different methods to compute the torque acting on a generic wire in a uniform magnetic field, the first is completely elementary and the other two present a higher degree of mathematical sophistication. We have then shown how the computation can be generalized to the cases of non-closed wires and non-uniform magnetic field.

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Note added in proof. During the processing of this paper it was pointed out to the author that a fourth proof method can be found in [6].

Appendix A. An extension of the Stokes theorem

The Stokes theorem relates the circulation of a field along a closed curve to the flux of its curl: in formulae (see e.g. [5])
\[ \oint d\ell \cdot X = \int d\sigma \cdot (\nabla \times X), \]
while we are interested in line integrals of the form
\[ I = \oint d\ell \times X. \]
In order to make use of the Stokes theorem in the computation of the rhs of equation (A.2), it is convenient to take the scalar product of \( I \) with a constant vector, which we will denote by \( c \):
\[ c \cdot I = \oint c \cdot (d\ell \times X) = \oint d\ell \cdot (X \times c) = \int d\sigma \cdot [\nabla \times (X \times c)], \]
where in the intermediate step we used the identity \( A \cdot (B \times C) = B \cdot (C \times A) \) and the last identity is just the usual Stokes theorem.

Passing in components and remembering that \( c \) is a constant vector, we obtain
\[ [\nabla \times (X \times c)]_i = \epsilon_{ijk} \partial_j \epsilon_{k\ell m} X_{\ell c_m} \]
\[ = -\epsilon_{kji} \epsilon_{k\ell m} \partial_j X_{\ell c_m} \]
\[ = -\delta_{ji} \delta_{\ell m} - \delta_{jm} \delta_{il} \partial_j X_{\ell c_m} \]
and thus
\[ \nabla \times (X \times c) = -c (\nabla \cdot X) + (c \cdot \nabla) X. \]
In a similar way, it can be shown that
\[ c \times (\nabla \times X) = -c (\nabla \cdot X) + (c \cdot \nabla) X, \]
and by summing these two equations we obtain
\[ \nabla \times (X \times c) = (\nabla \times X) \times c + \nabla (c \cdot X) - c (\nabla \cdot X). \]
By using the identity in equation (A.3), we obtain
\[ c \cdot I = \int d\sigma \cdot [(\nabla \times X) \times c] + \int d\sigma \cdot \nabla (c \cdot X) - \int c \cdot d\sigma (\nabla \cdot X) \]
\[ = \int c \cdot (d\sigma \times (\nabla \times X)) + \int d\sigma \cdot \nabla (c \cdot X) - \int c \cdot d\sigma (\nabla \cdot X), \]
and by replacing the constant vector \( c \) by the vectors of the coordinate axes we finally obtain the desired extension of the Stokes theorem:

\[
I = \int d\sigma \times (\nabla \times X) + \int (d\sigma \cdot \nabla)X - \int d\sigma (\nabla \cdot X). \tag{A.9}
\]

**Appendix B. Another variant of the Stokes theorem**

In this appendix, we will deduce another variant of the Stokes theorem, this time in reference to integrals of the form of the torque:

\[
J = \oint r \times (d\ell \times B). \tag{B.1}
\]

Multiplying \( J \) by the constant vector \( c \), we can use

\[
c \cdot (r \times (d\ell \times B)) = (d\ell \times B) \cdot (c \times r) = d\ell \cdot (B \times (c \times r)) \tag{B.2}
\]

to obtain, by the Stokes theorem,

\[
c \cdot J = \int d\sigma \cdot [\nabla \times (B \times (c \times r))]. \tag{B.3}
\]

By proceeding as in equation (A.4), it can be shown that

\[
\nabla \times (Y \times Z) = (Z \cdot \nabla)Y - (Y \cdot \nabla)Z + Y(\nabla \cdot Z) - Z(\nabla \cdot Y) \tag{B.4}
\]

and by using \( \nabla \cdot B = 0 \) and \( \nabla \cdot (c \times r) = 0 \), we obtain

\[
c \cdot J = \int d\sigma \cdot [(c \times r) \cdot \nabla]B + c \cdot \int d\sigma \times B. \tag{B.5}
\]

For a uniform magnetic field, only the last term survives and once again gives equation (1).

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