On the Intermittency and Chaos in High Energy Collisions

Fu Jinghua    Liu Lianshou    Wu Yuanfang

Institute of Particle Physics, Huazhong Normal University, Wuhan 430079 China

Tel: 027 87673313        FAX: 027 87662646        email: liuls@iopp.ccnu.edu.cn

Abstract

It is shown that an event sample from the Monte Carlo simulation of a random cascading $\alpha$ model with fixed dynamical fluctuation strength is intermittent but not chaotic, while the variance of dynamical fluctuation strength in different events will result in both the intermittency and the chaoticity behavior. This shows that fractality and chaoticity are two connected but different features of non-linear dynamics in high energy collisions.

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For classical system, the description of non-linear behavior is well established. It has been known by lattice calculation that the classical non-Abelian gauge theory generally exhibits deterministic chaos and that the Lyapunov exponent can be numerically determined [1]. But for quantum system, because of the ambiguity associated with quantum chaos in the realm of quantization, nonconservation of the number of degrees of freedom and lack of a meaningful definition of a trajectory, there is no corresponding description existed. The study of non-linear behavior in high energy physics has, therefore, to be started phenomenologically.

The first signal of such a behaviour came from the unexpectedly large local fluctuations in a single event of very high multiplicity recorded by the JACEE collaboration [2]. Such large fluctuations may not be simply due to statistical reason and was taken as a signal of the existence of non-linear dynamical fluctuations. It was soon realized that the idea can be applied to events of any multiplicity provided that a proper averaging of factorial moments is performed, as done in the pioneer work [3] of Bialas and Peschanski. These authors have been able to show that, if the statistical fluctuations are of Bernouli (fixed multiplicity case) or Poisson (variable multiplicity case) type, the averaged factorial moments $F_q$ is equal to the averaged dynamical probability moments $C_q$. The anomalous scaling of the latter has taken the name of intermittency (or fractal). This led to extensive experimental studies [4], and the expected anomalous scaling has been observed successfully in the experiments [5].

It should be realized, however, that the averaging procedure, apart from its clear advantages, brings also a danger of losing some important information on the spatial patterns from event to event. In particular, some interesting effects, if present only in a part of events produced in high-energy collisions, may be missed. A possible example of this kind is the quark-gluon plasma which is expected to be characterized by specific intermittency exponents [6]. It seems therefore important and urgent [7] to study the fluctuation of single-event moments $C_q^{(e)}$ inside an event sample. This fluctuation is related to the chaotic behavior of the system [7]. A quantity $\mu_q$ called entropy index can be introduced [8] as an adequate parameter in measuring the chaotic behavior. The positivity of entropy index $\mu_q > 0$ is proved to be a criterion for chaos [9].

Thus, two kind of non-linear phenomena — fractality (intermittency) and chaoticity have been proposed in high energy collisions. In this short note we will study the relation between them using Monte Carlo simulation of a random cascading $\alpha$ model. We will show that the anomalous scaling of the averaged probability moments (fractality or intermittency) and that of the event-space moments of single-event ones (chaoticity) are two connected but different features of non-linear dynamics. The system will exhibit both the fractal and the chaos behaviour only when the dynamical fluctuation strength is not fixed but is distributed over a certain range.

Let us first recall briefly the study of fractality (intermittency) in high energy collisions. This study is performed through the observation of anomalous scaling of averaged factorial moments $F_q$, which is equal to the averaged probabilty moments $C_q^e$

$$F_q(M) = C_q(M) \equiv \frac{1}{M} \sum_{i=1}^{M} \frac{\langle p_i^q \rangle}{\langle p_i \rangle^q} \propto M^{\phi q},$$

(1)

It has been shown [3] that the statistical fluctuations can be eliminated by using the factorial moments averaged over event sample. However, the extension of this method to single-event moments is highly non-trivial. It is easy to show that the elimination of statistical fluctuations in single-event factorial moments $F_q^{(e)}$ is incomplete. In order to avoid the complication caused by statistical fluctuations we will in this paper restrict ourself to the study of probability moments $C_q^{(e)}$ directly.
where a phase space region $\Delta$ is divided into $M$ sub-cells, $p_i$ is the probability for a particle to fall in the $i$th sub-cell.

For a flat inclusive distribution the moment $C_q^{(e)}$ for each event is defined as

$$C_q^{(e)} = M^{q-1} \sum_{i=1}^{M} \left( p_i^{(e)} \right)^q.$$  

(2)

We can now consider $C_q^{(e)}$ not only through its average — intended to get a better estimate of the hypothetical anomalous scaling of single-bin moments, cf. eqn.(1) — but also as a pattern-descriptor for particle fluctuations inside bins (just one among the many that could be devised).

$C_q^{(e)}$ may fluctuate greatly from event to event. In a sample consisting of a large number $N$ of events, we get a distribution of $C_q^{(e)}$, denoted by $P\left( C_q^{(e)} \right)$, which is normalized to unity. The conventionally defined factorial moments, cf eqn.(1), give only an estimate of the mean of $P\left( C_q^{(e)} \right)$. By taking the normalized moments of $P\left( C_q^{(e)} \right)$ in event-space defined as

$$C_{p,q} = \frac{\langle C_q^{(e)p} \rangle}{\langle C_q^{(e)} \rangle^p},$$  

(3)

we have a quantification of the fluctuation of the spatial patterns, i.e. we can investigate the full shape of the distribution and, especially, the way it changes with the resolution $\delta = \Delta/M$. The value of $p$ can be any positive real number. If $C_{p,q}(M)$ has a power law behaviour in $M$, i.e.

$$C_{p,q}(M) \propto M^{\psi_q(p)},$$  

(4)

then a new entropy index can be defined as,

$$\mu_q = \frac{d}{dp} \psi_q(p) \bigg|_{p=1}.$$  

(5)

It is easy to see that finite, nonvanishing positive values of $\mu_q$ corresponds to wide $P\left( C_q^{(e)} \right)$, which in turn means unpredictable spatial pattern from event to event. By applying the measure to known classical chaotic system, it has been shown [9] that $\mu_q$ can be used as a measure of chaos in problems where only the spatial patterns can be observed and the positivity of $\mu_q$ is a criterion for chaos.

An alternative way of calculating $\mu_q$ [9] is to express $C_{p,q}$ as

$$C_{p,q} = \langle \Phi_q^{(e)p} \rangle,$$  

(6)

in which,

$$\Phi_q^{(e)} = C_q^{(e)} / \langle C_q^{(e)} \rangle.$$  

(7)

With the definition

$$\Sigma_q = \langle \Phi_q^{(e)} \ln \Phi_q^{(e)} \rangle,$$  

(8)

we can obtain

$$\mu_q = \frac{\partial \Sigma_q}{\partial \ln M}.$$  

(9)
in the scaling region, i.e. where $\Sigma_q$ exhibits a linear dependence on $\ln M$. We will use this formula in calculating the entropy indices $\mu_q$.

Let us turn now to the consideration of the relation between the fractality and chaoticity in high energy collisions. Since the random cascading $\alpha$-model \[3\] is often used to study the dynamical fluctuations in these collisions, we will use this simple model as a tool for our investigation.

In the random cascading $\alpha$-model, the $M$ divisions of a phase space region $\Delta$ are made in steps. At the first step, it is divided into two equal parts; at the second step, each part in the first step is further divided into two equal parts, and so on. The steps are repeated until $M = \Delta Y/\delta y = 2^\nu$. How particles are distributed from step-to-step between the two parts of a given phase space cell is defined by independent random variable $\omega_{\nu, j, \nu}$, where $j, \nu$ is the position of the window ($1 \leq j, \nu \leq 2^{\nu-1}$) and $\nu$ is the number of steps. It is given by \[10\]:

$$
\omega_{\nu, 2j - 1} = \frac{1}{2}(1 + \alpha r) \quad ; \quad \omega_{\nu, 2j} = \frac{1}{2}(1 - \alpha r), \quad j = 1, \ldots, 2^{\nu-1}
$$

where, $r$ is a random number distributed uniformly in the interval $[-1, 1]$. $\alpha$ is a positive number less than unity, which determines the region of the random variable $\omega$ and describes the strength of dynamical fluctuations in the model. After $\nu$ steps, the probability in the $m$th window ($m = 1, \ldots, M$) is $p_m = \omega_{\nu, j_1, \nu} \omega_{\nu, j_2, \nu} \ldots \omega_{\nu, j_{\nu}}$. Then according to eqn(1), probability moment $C_q^{(c)}$ in each event of different division steps are calculated, and the moment $C_{p,q}$ and entropy index $\mu_q$ of the sample are obtained using eqn.(3) and eqn.(9).

Our research is done in the following two steps.

(A) Fix the model parameter to a definite value, say $\alpha = 0.34$, the experimental results being around this value. The results of $\ln C_q$, $\ln C_{p,q}$ and $\Sigma_q$ vs. $\ln M$ from 6000 MC simulation events are shown in Fig.1(a), (b), (c) respectively.

In Fig.1(a) we see a straight line in bi-logarithm plot, which is an indication of intermittency (or fractality). However, the behavior of $\ln C_{p,q}$ vs. $\ln M$ in the model, cf. Fig.1(b), is much different from the expected result for chaos \[7\]. It does not show any scaling behavior or upward bending when $M$ goes larger as the chaotic behaviour requires \[7\]. The first going up of $C_{p,q}$ is due to an intrinsic uncertainty of the intermittency parameters \[11\]. The cascade responsible for intermittent behaviour has different realizations in different events, and the intermittency exponents determined from different realizations of the same random cascade are scattered around the average, i.e. the method has a finite resolution with respect to the parameters of the random cascade. The $C_{p,q}$ saturates when $M$ goes large means that there isn’t any essential fluctuation of spatial pattern from event to event. Therefore, this kind of $\alpha$ model cannot reflect the feature of chaoticity. From the result showing in Fig.1(c), using eqn.(9), we can get for this case $\mu_q \sim 0$. If we donot consider the finite resolution of model, there isn’t any chaotic behavior.

In order to reproduce both intermittency (fractal) and chaos we take the second step.

(B) Instead of giving $\alpha$ a fixed value we let it be a random variable having a Gaussian distribution. The mean value and variance of the Gaussian are both chosen as 0.22. Calculating from 6306 events, the result of $\ln C_q$ vs. $\ln M$ are shown in Fig.2. It can be seen from the figure that there is very good power-law or scaling behavior, which means that though we have changed the method of setting model parameter, the anomalous scaling of the mean value of $C_q^{(c)}$ (intermittency phenomenon) survives.
With the method developed in Ref. [12] we can get the effective fluctuation strength in this case as $\alpha_{\text{eff}} = 0.337$. This value of $\alpha_{\text{eff}}$ is within the limited range available in actual experiments. However, the behavior of $\ln C_{p,q}$ vs. $\ln M$ in the present case is much different from the case (A) and shows a typical behaviour of chaoticity, cf. Fig.3.

From this result we see that a distribution of $\alpha$ will cause a distribution of single event probability moment $C_{q}^{(e)}$, i.e. for an event sample we have a wide $P(C_{q}^{(e)})$. For increasing $M$, along with the expected increase of the average, $P(C_{q}^{(e)})$ will show a rapid broadening, i.e. a more violent fluctuation of $C_{q}^{(e)}(M)$ for different events, which will result in a even more unpredictable spatial pattern from event to event.

Using eqn.(8) and eqn.(9), we can calculate entropy indices for this case. The result of $\Sigma_{q}$ vs. $\ln M$ are shown in Fig.4(a). By performing a linear fit of $\Sigma_{q}$ vs. $\ln M$ in the range $M = 8$ to $M = 64$ (i.e. omitting the first three points), $\mu_{q}$ is obtained and plotted in Fig.4(b).

By this two steps of MC simulation we can see that the procedure of doing simulation with random cascading $\alpha$ model of fixed strength parameter $\alpha$, as has been widely used before, captured only one aspect of the non-linear property (intermittency) but cannot reproduce the fluctuation of spatial patterns from event to event. It will cause the losing of information on the spatial patterns in different events and some interesting effects if they are present only in a part of events, may be lost. If we want to give a more complete description of the non-linear properties using $\alpha$ model, the model parameter cannot be fixed.

In conclusion, the nonvanishing positive values of $\mu_{q}$, which is an indication of chaos, correspond to wide $P(C_{q}^{(e)})$, which in turn means unpredictable spatial pattern from event to event. Such an unpredictability of wide $P(C_{q}^{(e)})$ is caused by different dynamical fluctuation strength in different events, i.e. by a distribution of dynamical fluctuation strength in an event sample. Events in one sample are all beginning with similar initial condition. During the collision process each event will evolve with a different strength of dynamical fluctuation and result in a fluctuation of spatial pattern in final state event space.

Dynamical fluctuation strength is directly related to the dynamical mechanism in a particular collision. We take the distribution of dynamical fluctuation strength, i.e. the distribution of model parameter $\alpha$, to be a Gaussian only because it is the most common distribution of random variables in nature. We have also tried a uniform distribution of $\alpha$ and non-zero positive entropy indices $\mu_{q}$ can be obtained too. (The result is not shown here). Revealing the distribution of dynamical fluctuation strength in different collisions will be a very constructive work and it will certainly help us a lot in studying the mechanism of strong interactions. How the different distributions of dynamical fluctuation strength together with it’s mean value and width will influence the entropy index of an event sample is also a problem worthwhile further investigation.

As has been stressed in the introduction, this study is restricted to the probability moments and the problem of eliminating statistical fluctuations in experimental data analysis has been postponed. To develop an effective method for eliminating the statistical fluctuations for single-event moments is a challenge for future investigation.
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Figure Captions

**Fig.1** (a) Averaged $C_2$, (b) $C_{p,2}$, (c) $\Sigma_2$ for fixed $\alpha$. Full lines are for guiding the eye.

**Fig.2** Averaged $C_2$ for Gaussian-distributed $\alpha$.

**Fig.3** The $\ln C_{p,q}$ vs. $\ln M$ for the random cascading model with Gaussian-distributed $\alpha$. Full lines are for guiding the eye.

**Fig.4** $\Sigma_q$ and $\mu_q$ for Gaussian-distributed $\alpha$. Full lines are for guiding the eye.
Fig. 1 (a) Averaged $C_2$, (b) $C_{p,2}$, (c) $\Sigma_2$ for fixed $\alpha$. Full lines are for guiding the eye.
Fig. 2 Averaged $C_2$ for Gaussian-distributed $\alpha$. 
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Fig. 4 $\Sigma_q$ and $\mu_q$ for Gaussian-distributed $\alpha$. Full lines are for guiding the eye.