\textit{C}P^2S \textit{ sigma models described through hypergeometric orthogonal polynomials}

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\textbf{Abstract}

The main objective of this paper is to establish a new connection between the Hermitian rank-1 projector solutions of the Euclidean \textit{C}P^2S \textit{ sigma model in two dimensions and the particular hypergeometric orthogonal polynomials called Krawtchouk polynomials. We show that any such projector solutions of the }\textit{C}P^2S \textit{ model, defined on the Riemann sphere and having a finite action, can be explicitly parametrised in terms of these polynomials. We apply these results to the analysis of surfaces associated with }\textit{C}P^2S \textit{ models defined using the generalised Weierstrass formula for immersion. We show that these surfaces are homeomorphic to spheres in the }\mathfrak{su}(2s + 1) \textit{ algebra, and express several other geometrical characteristics in terms of the Krawtchouk polynomials. Finally, a connection between the }\mathfrak{su}(2) \textit{ spin-s representation and the }\textit{C}P^2S \textit{ model is explored in detail.}
Introduction

Among the various sigma models, the one which has been the most studied is the completely integrable two-dimensional Euclidean \( \mathbb{C}P^{2s} \) sigma model defined on the extended complex plane \( S^2 \) having finite action. This subject was first analysed in the work of Din and Zakrzewski \([1, 2]\), next by Borchers and Garber \([3]\), Sasaki \([4]\) and later discussed by Eells and Wood \([5]\), Uhlenbeck \([6]\). It was shown by Mikhailov and Zakharov \([7, 8]\) that the Euler–Lagrange (EL) equations can be reformulated as a linear spectral problem which proves to be very useful for the construction and analysis of explicit multi-soliton solutions of the \( \mathbb{C}P^{2s} \) model. The main feature of this model is that all rank-1 Hermitian projector solutions of this model are obtained through successive applications of a creation operator. This rich yet restrictive character makes the \( \mathbb{C}P^{2s} \) sigma models a rather special and interesting object to study. A number of attempts to generalise the \( \mathbb{C}P^{2s} \) models and their various applications can be found in the recent literature of the subject (see e.g. \([9, 10, 11, 12, 13, 14, 15, 16, 17]\) and references therein).

In this paper, we solve the problem of determining all rank-1 Hermitian projector solutions of the \( \mathbb{C}P^{2s} \) sigma models by representing them in terms of the Krawtchouk orthogonal polynomials \([18]\). We also find new explicit analytical expressions for the sequence (called the Veronese sequence) of solutions of the \( \mathbb{C}P^{2s} \) sigma model. The explicit parametrisation of solutions of this model in terms of the Krawtchouk polynomials has not been previously found. These solutions may in turn be used to study the immersion functions of two-dimensional (2D)-soliton surfaces. This task has been accomplished by introducing a geometric setting for a given set of \( 2s + 1 \) rank-1 projector solutions of the \( \mathbb{C}P^{2s} \) sigma models written in terms of conservation laws. The latter enable us to construct the so-called generalised Weierstrass formula for the immersion of the 2D-surfaces in the
\(\mathfrak{su}(2s+1)\) algebra. Consequently, the analytical results obtained in this paper allow us to explore some geometrical properties of these surfaces, including the Gaussian and the mean curvature and some global characteristics such as the Willmore functional, the topological charge and the Euler–Poincaré characters. We show that for any Veronese subsequent solutions the topological charge of the 2D-surfaces are integers, while their Euler–Poincaré characters remain constant and equal to 2. It is shown that all Gaussian curvatures are positive for all 2D-surfaces associated with the \(\mathbb{C}P^{2S}\) sigma models and therefore these surfaces are homeomorphic to spheres immersed in the Euclidean space \(\mathbb{R}^{4s(s+1)}\).

This paper is organized as follows. Section 2 contains a brief account of projector formalism associated with the \(\mathbb{C}P^{2S}\) sigma model. Section 3 is devoted to the construction and investigation of Veronese sequence solutions of the \(\mathbb{C}P^{2S}\) model which are expressed in terms of the Krawtchouk orthogonal polynomials. These results are then used in Section 4 to construct the explicit form of the Clebsh-Gordan coefficients associated with this model. In Section 5, we discuss in detail the \(\mathfrak{su}(2)\) spin-s representation associated with the Veronese sequence solutions of the \(\mathbb{C}P^{2S}\) model. In Section 6, we present a geometric formulation for Veronese immersions of 2D-surfaces associated with this model. Section 7 contains possible suggestions concerning further developments

2 Projector formalism

2.1 Preliminaries on the \(\mathbb{C}P^{2S}\) sigma model

In the study of \(\mathbb{C}P^{2S}\) models on Euclidean space, we search for the maps

\[
\mathbb{S}^2 \ni \Omega \ni \xi_{\pm} = \xi^1 \pm i\xi^2 \mapsto z = (z_0, z_1, ..., z_{2s}) \in \mathbb{C}^{2s+1}\{\emptyset\},
\]

(2.1)

with \(z^\dagger z = 1\), which are stationary points of the action functional \[19\]

\[
A(z) = \sum_{\mu=1}^{2} \int_{\Omega} (D_\mu z)^\dagger D_\mu z \ d\xi^1 \ d\xi^2,
\]

(2.2)
defined on a simply connected open subset $\Omega$ of the extended complex plane $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. The corresponding EL equations are given by

$$\sum_{\mu=1}^{2} D_{\mu} D_{\mu} z + z \left( (D_{\mu} z)^\dagger D_{\mu} z \right) = 0,$$  \hspace{1cm} (2.3)$$

where $D_{\mu}$ are the covariant derivatives defined by

$$D_{\mu} z = \partial_{\mu} z - (z^\dagger \partial_{\mu} z) z, \quad \partial_{\mu} = \frac{\partial}{\partial \xi^\mu}, \quad \mu = 1, 2.$$  \hspace{1cm} (2.4)$$

The action functional (2.2) is invariant under the local $U(1)$ transformation induced by $k : \Omega \to \mathbb{C} \setminus \{0\}$ with $|k| = 1$, since $D_{\mu}(kz) = kD_{\mu}(z)$ holds. In what follows, it is convenient to use the homogeneous variables

$$\mathbb{S}^2 \supset \Omega \ni \xi_{\pm} = \xi^1 \pm i\xi^2 \mapsto f \in \mathbb{C}^{2s+1} \setminus \{0\},$$  \hspace{1cm} (2.5)$$

and expand the model so that the action (2.2) becomes

$$\mathcal{A}(f) = \sum_{\mu=1}^{2} \int_{\Omega} \int f \left( \frac{D_{\mu} f}{f^\dagger f} \right)^\dagger D_{\mu} f \frac{d\xi^1 d\xi^2}{f^\dagger f f},$$  \hspace{1cm} (2.6)$$

where

$$D_{\mu} f = \partial_{\mu} f - \frac{f^\dagger \partial_{\mu} f}{f^\dagger f} f.$$  \hspace{1cm} (2.7)$$

The action integrals (2.2) and (2.3) are consistent with the relation

$$z = \frac{f}{|f|}, \quad |f| = (f^\dagger f)^{1/2},$$  \hspace{1cm} (2.8)$$

which links the inhomogeneous coordinates $z$ with the functions $f$ as homogeneous coordinates of the model. Note that for any functions $f, g : \Omega \to \mathbb{C}^{2s+1} \setminus \{0\}$ such that $f = kg$ for some $k : \Omega \to \mathbb{C} \setminus \{0\}$, the action functions remain the same, i.e. $\mathcal{A}(f) = \mathcal{A}(kg) = \mathcal{A}(g)$.

Using the standard notation for the complex derivatives $\partial$ and $\overline{\partial}$ with respect to $\xi_{+}$ and $\xi_{-}$, i.e.

$$\partial = \frac{1}{2} \left( \frac{\partial}{\partial \xi^1} - i \frac{\partial}{\partial \xi^2} \right), \quad \overline{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial \xi^1} + i \frac{\partial}{\partial \xi^2} \right),$$  \hspace{1cm} (2.9)$$

we obtain that the $\mathbb{C}P^{2s}$ model equations expressed in terms of the $f$’s satisfy an unconstrained form of the EL equations

$$\left( \mathbf{1}_{2s+1} - \frac{f \otimes f}{f^\dagger f} \right) \left[ \partial \overline{\partial} f - \frac{1}{f^\dagger f} \left( (f^\dagger \overline{\partial} f) \partial f + (f^\dagger \partial f) \overline{\partial} f \right) \right] = 0,$$  \hspace{1cm} (2.10)$$

where $\mathbf{1}_{2s+1}$ is the $(2s+1) \times (2s+1)$ identity matrix. The holomorphic solution $f_0$ of (2.10) can be written as the Veronese sequence

$$f_0 = \left( 1, \sqrt{\binom{2s}{1}} \xi_+, \ldots, \sqrt{\binom{2s}{r}} \xi_+^r, \ldots, \xi_+^{2s} \right) \in \mathbb{C}^{2s+1} \setminus \{0\}.$$  \hspace{1cm} (2.11)$$
It was shown [1, 2] that for finite action integrals (2.6), subsequent solutions \( f_k \) can be obtained by acting with the creation (annihilation) operators. In terms of the homogeneous coordinates \( f_k \), new solutions of the EL equations (2.10) are obtained by the recurrence relations

\[
\begin{align*}
    f_{k+1} &= P_+ (f_k) = \left( 1_{2s+1} - \frac{f_k \otimes f^\dagger_k}{f_k f_k^*} \right) \partial f_k, \\
    f_{k-1} &= P_- (f_k) = \left( 1_{2s+1} - \frac{f_k \otimes f^\dagger_k}{f_k f_k^*} \right) \overline{\partial} f_k,
\end{align*}
\]

where \( P_\pm \) are raising and lowering operators with the properties

\[
P^0_\pm = 1_{2s+1}, \quad P^{2s+1}_\pm (f_k) = 0, \quad 0 \leq k \leq 2s.
\]

This procedure allows us to construct three classes of solutions; namely holomorphic \( f_0 \), antiholomorphic \( f_{2s} \) and mixed solutions \( f_k \), \( 1 \leq k \leq 2s - 1 \). Let us remark that the homogeneous coordinates \( \{ f_k \} \) are orthogonal, i.e. \( f_k^\dagger f_l = 0 \) if \( k \neq l \).

### 2.2 Rank-1 Hermitian projectors

The most fruitful approach to the study of the general properties of the \( \mathbb{C}P^{2S} \) model has been formulated through descriptions of the model in terms of rank-1 Hermitian projectors [21]. A matrix \( P_k(\xi_+, \xi_-) \) is said to be a rank-1 Hermitian projector if

\[
P^2_k = P_k, \quad P^\dagger_k = P_k, \quad tr(P_k) = 1.
\]

The target space of the projector \( P_k \) is determined by a complex line in \( \mathbb{C}^{2s+1} \), i.e. by a one-dimensional vector function \( f_k(\xi_+, \xi_-) \), given by

\[
P_k = \frac{f_k \otimes f^\dagger_k}{f_k f_k^*},
\]

where \( f_k \) is the mapping given by (2.5). Equation (2.15) gives an isomorphism between the equivalence classes of the \( \mathbb{C}P^{2S} \) model and the set of rank-1 Hermitian projectors. In this formulation, the action functional takes a more compact form which is scaling-invariant

\[
A(P_k) = \int\int_{S^2} tr (\partial P_k \overline{\partial} P_k) \ d\xi_+ d\xi_-
\]

and its extremum is subject to the constraints (2.14). The EL equations corresponding to (2.10), with constraints (2.14), take the simple form [22]

\[
[\partial \overline{\partial} P_k, P_k] = 0,
\]

and its extremum is subject to the constraints (2.14). The EL equations corresponding to (2.10), with constraints (2.14), take the simple form [22]
or can be equivalently written as the conservation law

$$\partial[\overline{\partial} P_k, P_k] + \overline{\partial}[\partial P_k, P_k] = 0. \quad (2.18)$$

For the sake of simplicity, we use the same symbol 0 for the scalar, vector and zero matrix throughout this paper.

It was shown [1, 2] that under the assumption that the $\mathbb{C}P^2_2$ model (2.17) is defined on the whole Riemann sphere $\Omega = S^2$ and that its action functional (2.16) is finite, all rank-1 projectors $P_k(\xi_+, \xi_-)$ can be obtained by acting on the holomorphic (or antiholomorphic) solution $P_0$ with raising (or lowering) operators $\Pi_\pm$. In terms of the projectors $P_k$, these operators take the form [21]

$$P_{k \pm 1} = \Pi_\pm(P_k) = \frac{(\partial \pm P_k) P_k (\partial \pm P_k)}{\text{tr}[(\partial \pm P_k) P_k (\partial \pm P_k)]} \quad \text{for} \quad \text{tr}[(\partial \pm P_k) P_k (\partial \pm P_k)] \neq 0, \quad (2.19)$$

and are equal to zero when $\text{tr}[(\partial \pm P_k) P_k (\partial \pm P_k)] = 0$, where $\partial_+$ and $\partial_-$ stand for $\partial$ and $\overline{\partial}$, respectively. Thus, from (2.19), the sequence of solutions in the $\mathbb{C}P^2_2$ model consists of $(2s + 1)$ rank-1 projectors $\{P_0, P_1, \ldots, P_{2s}\}$. These projectors satisfy the orthogonality and completeness relations [23]

$$P_j P_k = \delta_{jk} P_j, \quad (\text{no summation}) \quad \sum_{j=0}^{2s} P_j = 1_{2s+1}. \quad (2.20)$$

In what follows in this paper, we will assume that the $\mathbb{C}P^2_2$ model is defined on the Riemann sphere $S^2$ and has a finite action functional.

### 3 Solutions of the $\mathbb{C}P^2_2$ sigma model expressed in terms of the Krawtchouk polynomials

In this section, we complete the theory by showing that any rank-1 projector solution $P_k$ of the EL equations (2.17) can be explicitly expressed in terms of the Krawtchouk polynomials, which is one of the hypergeometric orthogonal polynomials of the Askey scheme [24]. We describe the links between the analytical properties of different solutions obtained through successive applications of a creation and annihilation operator (2.12). Separate classes of holomorphic, antiholomorphic and mixed type solutions are obtained explicitly and expressed in terms of Krawtchouk polynomials. As a result, we find new analytical expressions for the general rank-1 projector solutions of the EL equations (2.17).
As explained previously, a holomorphic solution \( f_0 \) of the \( \mathbb{C}P^{2S} \) sigma model (2.10) can be written as a Veronese sequence of the form (2.11). We note that this Veronese sequence may be expressed in a more compact way in terms of the Krawtchouk orthogonal polynomial

\[
(f_0)_j = \sqrt{\binom{2s}{j}} \xi^j_+ K_j(k; p, 2s), \quad \text{for } k = 0, \ 0 \leq j \leq 2s, \quad (3.1)
\]

\[
0 < p = \frac{\xi_+ \xi_-}{1 + \xi_+ \xi_-} < 1,
\]

where \((f_0)_j\) is the \(j\)th component of the vector \( f_0 \in \mathbb{C}^{2s+1} \). Also, we use the convention

\[
K_j(0; p, 2s) = 1. \quad (3.3)
\]

We recall that the Krawtchouk polynomial \( K_j(k; p, 2s) \) is defined in terms of the hypergeometric function [24]

\[
K_j(k) = K_j(k; p, 2s) = _2F_1(-j, -k; -2s; 1/p). \quad (3.4)
\]

Throughout this paper, for the sake of simplicity, we abbreviate some expressions by omitting their arguments. For the Krawtchouk polynomials, we write \( K_j \) instead of \( K_j(k; p, 2s) \) and \( K_j(k \pm 1) \) instead of \( K_j(k \pm 1; p, 2s) \).

The subsequent solutions of the \( \mathbb{C}P^{2S} \) model (2.17) can be obtained by acting with the creation or annihilation operators \( P_\pm \) through the recurrence relations (2.12). A set of subsequent solutions of the \( \mathbb{C}P^{2S} \) model, which consists of \( 2s + 1 \) vectors \( f_k \), is called a Veronese sequence of solutions if the subsequent solutions are obtained by acting with the creation operators (2.12) on the holomorphic Veronese solution (3.1). Thus the set of \( (2s + 1) \) rank-1 projectors \( \{P_0, P_1, ..., P_{2s}\} \) obtained through this procedure satisfies the EL equations (2.17) and the orthogonality and completeness relations (2.20).

This analysis has opened a new way for constructing and studying this type of solution in terms of the Krawtchouk polynomials. The advantage of expressing them in terms of these polynomials lies in the fact that they allow us to find the general class of analytical rank-1 solutions, provided that the Euclidean two-dimensional \( \mathbb{C}P^{2S} \) model admits a finite action (2.16). We obtain the following result.

**Theorem 3.1** Let the \( \mathbb{C}P^{2S} \) model be defined on the Riemann sphere \( S^2 \) and have a finite action functional. Then the Veronese subsequent analytic solutions \( f_k \) of the \( \mathbb{C}P^{2S} \) model
(2.17) take the form

\[(f_k)_j = \frac{(2s)!}{(2s-k)!} \left( \frac{-\xi_-}{1+\xi_+\xi_-} \right)^k \sqrt{\binom{2s}{j}} \binom{2s}{j} K_j(k;p,2s), \quad 0 \leq k, j \leq 2s \] (3.5)

where \(K_j(k;p,2s)\) are the Krawtchouk orthogonal polynomials (3.4) and where \(p\) is given by (3.2). The rank-1 Hermitian projectors \(P_k\) corresponding to the vectors \(f_k\) have the form

\[(P_k)_{ij} = \binom{2s}{k} \frac{\xi_+^{\xi_-}}{(1+\xi_+\xi_-)^{2s}} \binom{2s}{i} \binom{2s}{j} K_i(k;p,2s)K_j(k;p,2s). \] (3.6)

**Proof.** The proof for the existence of the Veronese sequence of solutions \(f_k\) is straightforward if we use the holomorphic solution (2.11) and the recurrence relation (2.12). If \(f_k\) is given by (3.5), the orthogonal projector \(P_k\) given by (2.15) can be written as

\[(P_k)_{ij} = \frac{(f_k \otimes f_k^\dagger)_{ij}}{f_k^\dagger f_k} = \frac{\xi_i^\prime \xi_j^\prime \sqrt{\binom{2s}{i} \binom{2s}{j}} K_iK_j}{\sum_{q=0}^{2s} (\xi_+\xi_-)^q \binom{2s}{q} K_i^2}. \] (3.7)

By using the orthogonality relation (A.7) we find that \((P_k)_{ij}\) is given by (3.6).

We now show that, for \(0 \leq k \leq 2s\), the components of the vector functions \((f_k)_j\) given by (3.5) satisfy the recurrence relation (2.12)

\[(f_{k+1})_j - \partial(f_k)_j + (P_k \partial f_k)_j = 0. \] (3.8)

Firstly, let us compute the first derivative of \((f_k)_j\) with respect to \(\xi_+\). By using relation (A.3), we obtain

\[
\partial(f_k)_j = \frac{(2s)!}{(2s-k)!} \left( \frac{-\xi_-}{1+\xi_+\xi_-} \right)^k \sqrt{\binom{2s}{j}} \xi_+^{j-1} \left[ (j-k) K_j + \frac{k}{1+\xi_+\xi_-} K_j(k-1) \right]. \] (3.9)

Secondly, by using the explicit form (3.6) of \(P_k\) and the orthogonality relations (A.7) and (A.8), we get

\[(P_k \partial f_k)_j = 2 \frac{(2s)!}{(2s-k)!} \left( \frac{-\xi_-}{1+\xi_+\xi_-} \right)^{k+1} \sqrt{\binom{2s}{j}} \xi_+^{j+1} (k-s). \] (3.10)

Finally, we can compute

\[(f_{k+1})_j - \partial(f_k)_j + (P_k \partial f_k)_j = \frac{(2s)!}{(2s-k)!} \left( \frac{-\xi_-}{1+\xi_+\xi_-} \right)^k \sqrt{\binom{2s}{j}} \xi_+^{j-1} \]

\[\times [-p(2s-k)K_j(k+1) + (k - j + 2p(s-k))K_j - k(1-p)K_j(k-1)]. \] (3.11)
The last term vanishes because of (A.17). We have proved that the R.H.S. of (3.5) satisfies the recurrence relation and, for \( k = 0 \), is the holomorphic Veronese sequence. Therefore, we conclude that \( f_k \) is the Veronese sequence which completes the proof. □

It was demonstrated in [25] that the EL equations (2.17) with the idempotency property \( P^2 = P \) admit a larger class of solutions than the rank-1 Hermitian projectors \( P_k \). That is, linear combinations of rank-1 projectors

\[
P = \sum_{l=0}^{2s} \lambda_l P_l, \quad \lambda_l = 0 \text{ or } 1 \quad \text{for all } l \in \{ 0, 1, ..., 2s \} \tag{3.12}
\]

also satisfy the EL equations (2.17). In this case, the projector \( P \) maps the \( \mathbb{C}^{2s+1} \) space onto \( \mathbb{C}^k \), where \( k = \sum_{l=0}^{2s} \lambda_l \). Hence, in view of (3.6), the higher-rank projector \( P \) can also be expressed in terms of the Krawtchouk polynomials

\[
(P)_{ij} = \sum_{l=0}^{2s} \lambda_l \left( \frac{2s}{l} \right) \left( \frac{2s}{i} \right) \frac{(\xi_+\xi_-)^l}{(1 + \xi_+\xi_-)^{2s+1}} \xi_i^j \xi_j^i \sqrt{\left( \frac{2s}{l} \right) \left( \frac{2s}{i} \right)} K_i(k)K_j(k) \tag{3.13}
\]

which satisfy both the EL equations (2.17) and the idempotency condition (2.14) (the coefficients \( \lambda_l \) are 0 or 1). It was shown [25] that the inverse statement is not true. That is, there exist decompositions of higher rank projectors which satisfy the EL equations (2.17) into rank-1 projectors which do not satisfy them.

From Theorem 3.1, we are also able to deduce the following useful expressions, which are an analogue of the first Frenet formulae [23] given in terms of the Krawtchouk polynomials.

**Proposition 3.1** The following relations hold

\[
(P_k \partial P_k)_{il} = \left( \frac{2s}{k} \right) \sqrt{\left( \frac{2s}{l} \right) \left( \frac{2s}{i} \right)} k\xi_+^{k+i-1}\xi_-^{k+l} K_i(k-1),
\]

\[
(\partial P_k P_k)_{il} = \left( \frac{2s}{k} \right) \sqrt{\left( \frac{2s}{l} \right) \left( \frac{2s}{i} \right)} k\xi_+^{k+i}\xi_-^{k+l-1} K_i(k-1) K_l,
\]

\[
(P_k \overline{\partial P_k})_{il} = \left( \frac{2s}{k} \right) \sqrt{\left( \frac{2s}{l} \right) \left( \frac{2s}{i} \right)} k\xi_+^{k+i}\xi_-^{k+l-1} K_i K_l((l-2s+k)\xi_+\xi_-+l-k+kK_i(k-1)),
\]

\[
(\partial P_k \overline{P_k})_{il} = \left( \frac{2s}{k} \right) \sqrt{\left( \frac{2s}{l} \right) \left( \frac{2s}{i} \right)} k\xi_+^{k+i-1}\xi_-^{k+l} K_i(K_i((i-2s+k)\xi_+\xi_-+i-k+kK_i(k-1)).
\]

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Proof. By differentiating with respect to $\xi_+$ the expression (3.6) of the projectors $P_k$, one gets

$$
(\partial P_k)_{ij} = \binom{2s}{k} \sqrt{\binom{2s}{i} \binom{2s}{j}} \xi_{i+k+1} \xi_{j+k+1} \left[ (i - 2s)\xi_+ + i - k(1 - \xi_+ \xi_-) \right] K_i K_j + k(K_i(k-1) K_j + K_i K_j(k-1))].
$$

(3.14)

We have used relation (A.3) to get the previous result. We can obtain similar relations for $(\partial P_k)_{ij}$. The first relation of the proposition is proven by computing $\sum_{q=0}^{2s}(P_k)_{iq}(\partial P_k)_{ql}$ and by again using the orthogonality relations of the Lemma A.2. The three other relations are proven similarly. □

We now show that the linear spectral problem (LSP) can be expressed in terms of the Krawtchouk orthogonal polynomials $K_k$. Indeed, it was shown [7, 8] that the $CP^{2s}$ models with finite action integral (2.17) are completely integrable. The LSP associated with (2.17) is given by

$$
\begin{align*}
\partial \phi_k &= U(\lambda) \phi_k = \frac{2}{1 + \lambda} \{\partial P_k, P_k\} \phi_k, \\
\overline{\partial} \phi_k &= V(\lambda) \phi_k = \frac{2}{1 - \lambda} \{\overline{\partial} P_k, P_k\} \phi_k,
\end{align*}
$$

(3.15)

where $\lambda \in \mathbb{C}$ is the spectral parameter and $P_k$ is the sequence of rank-1 orthogonal projectors (2.20) which map onto the one-dimensional direction of $f_k$. For all values of $\lambda$, the compatibility condition for equations (3.15) corresponds precisely to the EL equations (2.17). By using the results of Proposition 3.1, we find the following analytical expressions for $U$ and $V$

$$
\begin{align*}
(U(\lambda))_{il} &= 2 \frac{2s}{1 + \lambda} \binom{2s}{k} \sqrt{\binom{2s}{l} \binom{2s}{i} \xi_{i+k+1} \xi_{j+k+1}} \left[ (K_i K_j ((i - 2s + k)\xi_+ + i - k) - kK_i K_j(k-1) - kK_i K_j(k-1)) \right],
\end{align*}

(3.16)

$$
\begin{align*}
(V(\lambda))_{il} &= 2 \frac{2s}{1 - \lambda} \binom{2s}{k} \sqrt{\binom{2s}{l} \binom{2s}{i} \xi_{i+k+1} \xi_{j+k+1}} \left[ kK_i (k-1) K_j - K_i (K_i ((l - 2s + k)\xi_+ + l - k) + kK_i(k-1)) \right].
\end{align*}

(3.17)

An explicit soliton solution which vanishes at complex infinity was found for equations (2.17) in [7, 8]. Namely, in the asymptotic case, when $P_k$ tends to the identity matrix
1_{2s+1} as \( \lambda \) goes to \( \infty \), we have

\[
\phi_k = 1_{2s+1} + \frac{4\lambda}{(1-\lambda)^2} \sum_{j=0}^{k-1} P_j - \frac{2}{1-\lambda} P_k
\]

\[
\phi_k^{-1} = 1_{2s+1} - \frac{4\lambda}{(1+\lambda)^2} \sum_{j=0}^{k-1} P_j - \frac{2}{1+\lambda} P_k, \quad \lambda = it, \quad t \in \mathbb{R}.
\]  

(3.18)

Analytical results for (3.18) have explicitly been carried out for the wavefunction \( \phi_k \) but the expressions are rather involved, so we omit them here. In accordance with [26], we use the term soliton surfaces, which refers to surfaces associated with an integrable system.

4 The Clebsch-Gordan decomposition

Let us now explore certain properties of the \( \mathbb{C}P^{2s} \) model. From the Veronese sequence solutions \( f_k \) of this model given in Theorem 3.1, we obtain an explicit expression for the Lagrangian density \( L(P_k) = tr(\pi P_k \cdot \pi P_k) \) and its action integral \( A(P_k) = \int_{S^2} L \, d\xi_+ d\xi_- \) expressed in terms of the Krawtchouk polynomials.

**Proposition 4.1** Let the \( \mathbb{C}P^{2s} \) model be defined on the Riemann sphere \( S^2 \). We recall that the rank-1 Hermitian projectors \( P_k \), computed in Theorem 3.1, read

\[
(P_k)_{ij} = \binom{2s}{k} \binom{2s+k}{i-j} \frac{(\xi_+ + \xi_-)^k}{(1 + \xi_+ + \xi_-)^{2s}} \sqrt{\binom{2s}{i-j}} K_i(k) K_j(k).
\]

(4.1)

Then the Lagrange density is given by

\[
L(P_k) = \frac{2(s + 2sk - k^2)}{(1 + \xi_+ + \xi_-)^2},
\]

(4.2)

and the action integral (2.16) is finite and takes the form

\[
A(P_k) = \frac{2\pi(s + 2sk - k^2)}{1 + \xi_+ + \xi_-}, \quad 0 \leq k \leq 2s.
\]

(4.3)
Proof. The first derivative with respect to $\xi_+$ of $(P_k)_{ij}$ is given by (3.14). The complex conjugate provides the first derivative with respect to $\xi_-$. Therefore, one gets

$$L = tr((\overline{\partial} P_k \cdot \partial P_k) = \sum_{i,j=0}^{2s} (\overline{\partial} P_k)_{ij} (\partial P_k)_{ji}$$

(4.4)

$$= \sum_{i,j=0}^{2s} \left( \frac{2s}{k} \right)^2 \left( \frac{2s}{i} \right) \left( \frac{2s}{j} \right) \frac{1}{(1 + \xi_+ \xi_-)^{i+j+1}}$$

(4.5)

$$\cdot \left[ K_i K_j \left(j(1 + \xi_+ \xi_-) - 2s \xi_+ \xi_- - k(1 - \xi_+ \xi_-) + k(K_i(k - 1)K_j + K_iK_j(k - 1)) \right) \right]^2$$

By expanding the square in the last relation and by using the relations in Lemma A.2, after some algebraic manipulations, one proves relation (4.2). Thus the $\mathbb{C}P^{2s}$ model is defined from its action integral over the Riemann sphere

$$\mathcal{A}(P_k) = 2(s + 2sk - k^2) \int_{S^2} \frac{d\xi_+ d\xi_-}{(1 + \xi_+ \xi_-)^2} = \frac{2\pi(s + 2sk - k^2)}{1 + \xi_+ \xi_-},$$

(4.6)

which completes the proof. □

Next, suppose that we have constructed a set of rank-1 Hermitian projectors $P_k$ satisfying the EL equations (2.17) for which the action functional (2.16) is required to be finite. It was shown that each second mixed derivative of $P_k$ can be represented as a combination of at most three rank-1 neighbouring projectors [25]

$$\partial \overline{\partial} P_k = \hat{\alpha}_k P_{k-1} + (\hat{\alpha}_k + \check{\alpha}_k) P_k + \check{\alpha}_k P_{k+1},$$

(4.7)

where the Clebsch-Gordan coefficients $\hat{\alpha}_k$ and $\check{\alpha}_k$ are real-valued functions given by

$$\hat{\alpha}_k = tr(\overline{\partial} P_k \partial P_k) = \hat{\alpha}_k^\dagger, \quad \check{\alpha}_k = tr(\partial P_k \overline{\partial} P_k) = \check{\alpha}_k^\dagger.$$ (4.8)

The sum of these coefficients

$$\hat{\alpha}_k + \check{\alpha}_k = tr(\partial P_k \overline{\partial} P_k)$$

(4.9)

represents the Lagrangian density (4.2). Then we have the following

**Proposition 4.2** For the subsequent Veronese analytic solutions of the $\mathbb{C}P^{2s}$ model (2.17), the Clebsch-Gordan coefficients take the simple form

$$\hat{\alpha}_k = \frac{k(2s + 1 - k)}{(1 + \xi_+ \xi_-)^2} \quad \text{and} \quad \check{\alpha}_k = \frac{(k + 1)(2s - k)}{(1 + \xi_+ \xi_-)^2}.$$ (4.10)
**Proof.** From Proposition 3.1, we know the expression of \((P_k \partial P_k)_{il}\). Hence the Clebsch-Gordan coefficient \(\hat{\alpha}_k\) can be determined explicitly

\[
\hat{\alpha}_k = \text{tr}(\partial P_k P_k \partial P_k) = \sum_{i,l=0}^{2s} (\partial P_k)_{il} (P_k \partial P_k)_{il}
\]

\[
= \left(\frac{2s}{k}\right)^2 \frac{k(\xi_+ \xi_-)^{2k-1}}{(1 + \xi_+ \xi_-)^{4s+2}} \sum_{i,l=0}^{2s} \left(\frac{2s}{l}\right) \left(\frac{2s}{i}\right) \xi_+^{i+l} \xi_-^{i+l} K_i K_l (k - 1) \\
\cdot \left[i(1 + \xi_+ \xi_-) - 2s \xi_+ \xi_- - k(1 - \xi_+ \xi_-) \right] K_i K_l + k (K_l (k - 1) K_i + K_i K_l (k - 1) )
\]

\[
= \left(\frac{2s}{k}\right)^2 \frac{k^2(\xi_+ \xi_-)^{2k-1}}{(1 + \xi_+ \xi_-)^{4s+2}} \sum_{i,l=0}^{2s} \left(\frac{2s}{l}\right) \left(\frac{2s}{i}\right) \xi_+^{i+l} \xi_-^{l+i} K_i^2 (k - 1) K_l^2.
\]

By using the orthogonality relation of the Krawtchouk polynomials (A.7), we prove the first relation of (4.10). Bearing in mind that

\[
\bar{\alpha}_k = \text{tr}(\partial P_k P_k \partial P_k) = \text{tr}(\partial P_k \partial P_k) - \hat{\alpha}_k
\]

and making use of (4.2), we get the second relation of (4.10) which completes the proof. \(\square\)

This computation gives an explicit form of the Clebsch-Gordan coefficients \(\hat{\alpha}_k\) and \(\bar{\alpha}_k\) in terms of the complex independent variables \((\xi_+, \xi_-) \in \mathbb{C}\). It follows from (4.7) and (4.10) that the second mixed derivative can be represented as a linear combination of three rank-1 neighbouring projectors, namely

\[
\partial \partial P_k = \frac{1}{(1 + \xi_+ \xi_-)^2} \left[k(2s - k - 1)P_{k-1} + 2(s + 2sk - k^2)P_k + (k + 1)(2s - k)P_{k+1}\right].
\]

From (2.20) and (4.12), we show that \(P_k\) also satisfies

\[
\text{tr}(P_k \partial \partial P_k) = \frac{2(s + 2sk - k^2)}{(1 + \xi_+ \xi_-)^2}.
\]

(4.13)

In [25], it was demonstrated that

\[
\bar{\partial} P_k \partial P_k = \hat{\alpha}_k P_{k-1} + \bar{\alpha}_k P_k \quad \text{and} \quad \partial P_k \bar{\partial} P_k = \hat{\alpha}_k P_k + \bar{\alpha}_k P_{k+1}.
\]

From the explicit form of the Clebsch-Gordan coefficients (4.10), these expressions become

\[
\bar{\partial} P_k \partial P_k = \frac{1}{(1 + \xi_+ \xi_-)^2} \left[k(2s - k + 1)P_{k-1} + (k + 1)(2s - k)P_k\right],
\]

(4.15)

\[
\partial P_k \bar{\partial} P_k = \frac{1}{(1 + \xi_+ \xi_-)^2} \left[(k + 1)(2s - k)P_{k+1} + k(2s - k + 1)P_k\right].
\]

(4.16)
5 The $\mathfrak{su}(2)$ spin-$s$ representation associated with the Veronese sequence of solutions of the $\mathbb{C}P^{2s}$ model

The objective of this section is to demonstrate a connection between the Euclidean sigma model \((2.17)\) in two dimensions and the spin-$s$ $\mathfrak{su}(2)$ representation. The spin matrix $S^z$ is defined as a linear combination of the \((2s+1)\) rank-1 Hermitian projectors $P_k$ by \([27, 28]\)

$$S^z = \sum_{k=0}^{2s} (k-s)P_k. \quad (5.1)$$

The eigenvalues of the generator $S^z$ are \([-s, -s+1, \ldots, s]\). They are either integer (for odd $2s+1$) or half-integer (for even $2s+1$) values. Under these assumptions, we have the following.

**Proposition 5.1** The spin matrix $S^z$ is given by the tridiagonal matrix

$$(S^z)_{ij} = \delta_{ij} \left( \frac{1-\xi_i \xi_j}{1+\xi_i \xi_j} \right) (i-s) - \delta_{i-1,j} \left( \frac{\xi_i}{1+\xi_i \xi_j} \right) \sqrt{i(2s+1-i)}$$

$$- \delta_{i,j-1} \left( \frac{\xi_i}{1+\xi_i \xi_j} \right) \sqrt{j(2s-j+1)}, \quad 0 \leq i, j \leq 2s. \quad (5.2)$$

**Proof.** By using the explicit form of the projector $P_k$ found in Theorem 3.1, one gets

$$(S^z)_{ij} = \frac{\xi_i \xi_j}{(1+\xi_i \xi_j)^{2s}} \sqrt{\binom{2s}{i}\binom{2s}{j}} \sum_{k=0}^{2s} (k-s) \binom{2s}{k} \binom{2s}{\xi_i \xi_j} K_i(k)K_j(k). \quad (5.3)$$

By using the orthogonality properties \((A.13)-(A.15)\), we obtain

$$(S^z)_{ij} = \sqrt{\binom{2s}{i}\binom{2s}{j}} \left[ \delta_{ij} \frac{2s \xi_i \xi_j + i-j \xi_i \xi_j}{\xi_i(1+\xi_i \xi_j)} \frac{j!(2s-j)!}{(2s)!} - \delta_{i-1,j} \frac{i \xi_i}{1+\xi_i \xi_j} \frac{(2s-i+1)!}{(2s)!} \right]$$

$$- \delta_{i,j-1} \frac{\xi_i}{1+\xi_i \xi_j} \frac{j!(2s-j+1)!}{(2s)!} - \delta_{ij} \frac{s!(2s-j)!}{(2s)!}. \quad (5.4)$$

After algebraic manipulations, the spin matrix $S^z$ takes the form \((5.2)\). \(\square\)

We recall that the generators $S^z$ and $S^\pm$ of the $\mathfrak{su}(2)$ Lie algebra satisfy the commutation relations

\[(i) \quad [S^z, S^\pm] = \pm S^\pm, \quad (ii) \quad [S^+, S^-] = 2S^z. \quad (5.6)\]

Due to the eigenvalues of the spin matrix $S^z$ \((5.1)\), we know that this matrix can be seen as the Cartan element of $\mathfrak{su}(2)$ in the spin-$s$ representation. We now compute the
corresponding \( S^+ \) and \( S^- \) in this representation. The usual spin-\( s \) representation of \( \mathfrak{su}(2) \) is identified with \cite{29,30}

\[
\begin{align*}
\sigma^z_{ij} &= (s - i)\delta_{ij}, \\
\sigma^+_{ij} &= \sqrt{(2s - j + 1)j}\delta_{ij}, \\
\sigma^-_{ij} &= \sqrt{(2s - i + 1)i}\delta_{ij}.
\end{align*}
\]

Due to Proposition 5.1, we note that we can decompose the spin matrix \( S^z \) as a linear combination of matrices \( \sigma^z \) and \( \sigma^\pm \), namely

\[
S^z = \frac{1}{1 + \xi_+\xi_-} \left( (\xi_+\xi_- - 1)\sigma^z - \xi_+\sigma^- - \xi_-\sigma^+ \right).
\]

Hence we have

**Proposition 5.2** Let us define the spin matrices by

\[
\begin{align*}
S^+ &= \frac{1}{1 + \xi_+\xi_-} \left( 2\xi_-\sigma^z - \sigma^- + \xi^2\sigma^+ \right), \\
S^- &= \frac{1}{1 + \xi_+\xi_-} \left( 2\xi_+\sigma^z + \xi^2\sigma^- - \sigma^+ \right).
\end{align*}
\]

Then the matrix \( S^z \) given by (5.10) and the two previous matrices \( S^+ \) and \( S^- \) satisfy the \( \mathfrak{su}(2) \) commutation relations (5.6).

Moreover, we have the following useful recurrence properties for \( S^zf_k \) and \( S^\pm f_k \) when \( f_k \) is an analytic vector given by (3.5)

\[
\begin{align*}
(i) \quad S^+ f_k &= \begin{cases} 
-(1 + \xi_+\xi_-)f_{k+1}, & \text{for } 0 \leq k \leq 2s - 1, \\
0, & \text{for } k = 2s,
\end{cases} \\
(ii) \quad S^- f_k &= \frac{1}{1 + \xi_+\xi_-} k(k - 1 - 2s)f_{k-1}, & \text{for } 0 \leq k \leq 2s, \\
(iii) \quad S^z f_k &= (k - s)f_k, & \text{for } 0 \leq k \leq 2s.
\end{align*}
\]

**Proof.** By using the fact that \( \sigma^z \) and \( \sigma^\pm \) satisfy the \( \mathfrak{su}(2) \) commutation relations, we prove that \( S^z \) and \( S^\pm \) also satisfy these relations. The property (5.13) follows directly from the definition (5.1) of \( S^z \) and of the definition (2.15) of \( P_k \). The action of the matrix \( S^+ \) on the vector \( f_k \) is given by

\[
(S^+ f_k)_j = \frac{1}{1 + \xi_+\xi_-} \left[ \xi^2\sqrt{(2s - j)(j + 1)}(f_k)_{j+1} + 2(s - j)\xi_-(f_k)_j - \sqrt{(2s - j + 1)}j(f_k)_{j-1} \right].
\]
By using the explicit form \((3.5)\) of \(f_k\) and identities on binomials we get

\[
(S^+ f_k)_j = -\frac{(2s)!((-\xi_+)^{j-1})}{(2s-k)!(1+\xi_+\xi_-)^{k+1}} \left[ \xi_-\xi_+(2s-j)K_{j+1}(k) + 2(s-j)K_j(k) - \frac{j}{\xi_+\xi_-} K_{j-1}(k) \right].
\] 

(5.17)

By using relation \((A.18)\), we obtain for \(0 \leq k < 2s\)

\[
(S^+ f_k)_j = -\frac{(2s)!((-\xi_+)^{j-1})}{(2s-k-1)!(1+\xi_+\xi_-)^{k}} K_j(k+1).
\] 

(5.18)

By using the explicit form for \(f_{k+1}\), we prove \((5.13)\). The case \(k = 2s\) is obtained by remarking that the right hand side of \((A.18)\) vanishes in this case. We perform similar computations to prove \((5.14)\) which concludes the proof. \(\Box\)

Let us emphasize that relation \((5.13)\) allows us to construct recursively the sequence \(f_k\) starting from \(f_0\). It provides another way to construct a set of solutions simpler than the ones given by the recurrence relation \((2.12)\).

The result given in Proposition \(5.2\) can be interpreted as the matrix elements of the \(SU(2)\) irreducible representations, known as the Wigner D functions. It is well-known that these matrix elements can be expressed in terms of the Krawtchouk polynomials \([31, 32]\).

6 Geometrical aspects of surfaces associated with the \(\mathbb{CP}^{2s}\) model

Let us now study certain geometrical properties of 2-dimensional (2D) surfaces associated with the \(\mathbb{CP}^{2s}\) model. We show that these surfaces are immersed in the \(\mathfrak{su}(2s+1)\) algebra and may be expressed in terms of the Krawtchouk orthogonal polynomials. These geometrical properties include, among others, the Gaussian and mean curvatures, the topological charge, the Willmore functional and the Euler-Poincaré character \([33, 34]\).

Under the assumption that the \(\mathbb{CP}^{2s}\) model is defined on the Riemann sphere \(\mathbb{S}^2\) and that the associated action functional \((4.3)\) of this model is finite, we can show that these surfaces are conformally parametrised. The proof is similar to that given in \([21]\). The further advantage of using the \(\mathbb{CP}^{2s}\) model in this context lies in the fact that it allows us to provide an explicit parametrisation in terms of the Krawtchouk polynomials instead of the formalism of connected rank-1 projectors. This approach has opened a new way
for constructing and investigating 2D-surfaces immersed in multidimensional Euclidean spaces $\mathbb{R}^{4s(s+1)}$.

Let us now present certain geometrical aspects of 2D-surfaces immersed in the $\mathfrak{su}(2s + 1)$ algebras. For a given set of $(2s + 1)$ rank-1 projector solutions of the EL equations (2.17), the generalized Weierstrass formula for immersion (GWFI) of 2D-surfaces is defined by a contour integral \[ X_k(\xi_+, \xi_-) = i \int_{\gamma_k} \left[ -[\partial P_k, P_k]d\xi_+ + [\overline{\partial} P_k, P_k]d\xi_- \right] \in \mathfrak{su}(2s + 1) \simeq \mathbb{R}^{4s(s+1)}, \] which is independent of the path of integration $\gamma_k \subset \mathbb{C}$. The conservation laws (2.18) ensure that the contour integral is locally independent of the trajectory. For the surfaces corresponding to the rank-1 projectors $P_k$, the integration of (6.1) can be performed explicitly (since $d(dX_k) = 0$) with the result \[ X_k = i \left( P_k + 2 \sum_{j=0}^{k-1} P_j \right) \left( \frac{1 + 2k}{1 + 2s} \right) \mathbf{1}_{2s+1}, \quad 0 \leq k \leq 2s. \] Note that for each $k$, the projectors $P_j$ satisfy the eigenvalue equations \[ (X_k - i\lambda_k \mathbf{1}_{2s+1}) P_j = 0 \] with the eigenvalues \[ \lambda_k = \begin{cases} 
\frac{2(k - 2s) - 1}{1 + 2s} & \text{for } j < k \\
\frac{2(k - s)}{1 + 2s} & \text{for } j = k \\
\frac{1 + 2k}{1 + 2s} & \text{for } j > k 
\end{cases} \] The immersion functions $X_k$ span a Cartan subalgebra of $\mathfrak{su}(2s + 1)$ \[ [X_k, X_j] = 0, \quad 0 \leq k, j \leq 2s \] and satisfy the algebraic conditions given in [37]. For mixed solutions $P_k$, $1 \leq k \leq 2s$, we get a cubic matrix equation \[ \left[ X_k - i \frac{1 + 2k}{1 + 2s} \mathbf{1}_{2s+1} \right] \left[ X_k - i \frac{2(k - s)}{1 + 2s} \mathbf{1}_{2s+1} \right] \left[ X_k - i \frac{2(k - 2s) - 1}{1 + 2s} \mathbf{1}_{2s+1} \right] = 0 \] while for holomorphic ($k = 0$) and antiholomorphic ($k = 2s$) solutions of the $\mathbb{C}P^{2s}$ equation (2.17), the minimal polynomial for the real-valued matrix function $X_k$ is quadratic. Namely, we have \[ \left[ X_0 - i \frac{2is}{1 + 2s} \mathbf{1}_{2s+1} \right] \left[ X_0 + \frac{2is}{1 + 2s} \mathbf{1}_{2s+1} \right] = 0, \quad k = 0, \]
and
\[ [X_{2s} + \frac{i}{1+2s}1_{2s+1}] [X_{2s} + \frac{2i(k-s)}{1+2s}1_{2s+1}] = 0, \quad k = 2s. \] (6.8)

For the sake of uniformity, the inner product \( X_k \) is defined by \[38\]
\[ (A, B) = -\frac{1}{2} tr(A \cdot B), \quad \text{for any } A, B \in su(2s + 1), \] (6.9)
instead of the Killing form. In view of the analytical form \(6.2\) of the 2D-surfaces \( X_k \) given in terms of the projectors \( P_k \), which are expressed through the formula \(3.6\), we can determine that the 2D-surfaces associated with the \( CP^{2s} \) model are submanifolds of the compact sphere with radius
\[ (X_k, X_k) = -\frac{1}{2} tr(X_k)^2 = \frac{1}{2} \left( \frac{(1+2k)(2(2s-k)-1)}{1+2s} - 1 \right) \] (6.10)
immersed in \( \mathbb{R}^{4s(s+1)} \simeq su(2s + 1) \).

The projectors \( P_k \) fulfill the completeness relation \(2.20\) which implies in turn that the immersion functions \( X_k \) satisfy the linear relation \[37\]
\[ \sum_{k=0}^{2s} (-1)^k X_k = 0. \] (6.11)

Once we have determined the immersion functions \(6.2\) of the 2D-surfaces, we can describe their metric and curvatures properties. From \(6.1\) and Proposition \(3.1\) the complex tangent vectors are obviously
\[ (\partial X_k)_{rl} = -i ([\partial P_k, P_k])_{rl} = -i \binom{2s}{k} \sqrt{\binom{2s}{l}(\frac{2s}{l+1})} \xi^l \xi^r - 1 \xi^k \]
\[ \cdot [K_l (K_r ((r - 2s + k) \xi^l \xi^r + r - k) + k K_r (k - 1)) - k K_r K_l (k - 1)] , \]
\[ (\bar{\partial} X_k)_{rl} = i ([\bar{\partial} P_k, P_k])_{rl} = i \binom{2s}{k} \sqrt{\binom{2s}{l}(\frac{2s}{l+1})} \xi^l \xi^r \xi^k - 1 \xi^l \]
\[ \cdot [k K_r (k - 1) K_l - K_r (K_l ((l - 2s + k) \xi^l \xi^r + l - k) + k K_l (k - 1))]. \] (6.12)

Let \( g_k \) be the metric tensor associated with the surface \( X_k \). Its components are indicated with indices outside of parentheses in order to distinguish them from the index of the surface \( X_k \). The diagonal elements of the metric tensor are zero (i.e. \( g_{k}^{11} = (g_k)_{22} = 0 \)). This property follows from the vanishing of \( tr(\partial P_k \cdot \partial P_k) \) or \( tr(\bar{\partial} P_k \cdot \bar{\partial} P_k) \), as proven in \[21\]. In view of relation \(4.10\), the non-zero off-diagonal elements are
\[ (g_k)_{12} = (g_k)_{21} = \frac{1}{2} tr(\partial X_k \cdot \bar{\partial} X_k) = \frac{1}{2} tr(\partial P_k \cdot \bar{\partial} P_k) = \frac{2s(k+1) - k^2}{(1+\xi^l \xi^r)^2}, \quad 0 \leq k \leq 2s. \] (6.13)
Thus the first fundamental forms reduce to
\[
I_k = \text{tr}(\partial P_k \cdot \overline{\partial P_k})d\xi_+d\xi_- = \frac{2s(k+1) - k^2}{(1 + \xi_+\xi_-)^2}d\xi_+d\xi_- \quad (6.14)
\]
The non-zero Christoffel symbols of the second type are
\[
(\Gamma_k)^1_{11} = \frac{-4s(k+1) + k^2}{1 + \xi_+\xi_-}\xi_-, \quad (\Gamma_k)^2_{22} = \frac{-4s(k+1) + k^2}{1 + \xi_+\xi_-}\xi_+.
\]
We observe that the 2D-surfaces \(X_k\) are torsion-free, i.e. \((T_k)_{abc} = (\Gamma_k)^a_{bc} - (\Gamma_k)^a_{cb} = 0\). Hence the second fundamental forms
\[
II_k = \left(\partial^2 X_k - (\Gamma_k)^1_{11}\partial X_k\right) d\xi_+^2 + 2\partial\overline{\partial} X_k d\xi_+d\xi_- + \left(\overline{\partial}^2 X_k - (\Gamma_k)^2_{22}\overline{\partial} X_k\right) d\xi_-^2 \quad (6.16)
\]
are easy to find but the expressions are rather involved, so we omit them here. The first and second fundamental forms allow us to formulate the following.

**Proposition 6.1** The Gaussian curvatures
\[
K_k = \frac{-2\partial\overline{\partial} \ln |\text{tr}(\partial P_k \cdot \overline{\partial P_k})|}{\text{tr}(\partial P_k \cdot \overline{\partial P_k})}.
\]
of 2D-surfaces associated with the Veronese subsequent analytic solutions \(f_k \quad (3.5)\) of the \(\mathbb{C}P^{2s}\) model \((2.17)\) have the following constant positive values
\[
K_k = \frac{2}{2sk + s - k^2}, \quad 0 \leq k \leq 2s. \quad (6.19)
\]

**Proof.** Using the complex tangent vectors \((6.12)\) and the relation \((4.16)\), we can compute explicit analytic expressions for the Gaussian curvatures. In fact, from the formula \((6.18)\), we get
\[
K_k = \frac{-2\partial\overline{\partial} \ln |2(2sk + s - k^2)(1 + \xi_+\xi_-)^{-2}|}{(2sk + s - k^2)(1 + \xi_+\xi_-)^{-2}}. \quad (6.20)
\]
which, after simplification, gives the expression \((6.19)\).

**Proposition 6.2** The mean curvature vectors written in the matrix form
\[
H_k = \frac{-4i[\partial P_k, \overline{\partial P_k}]}{\text{tr}(\partial P_k \cdot \overline{\partial P_k})}, \quad (6.21)
\]
of 2D-surfaces $X_k$ associated with the Veronese subsequent analytic solutions $f_k$ (3.5) of the $\mathbb{CP}^2$ model (2.17) are expressible in terms of the Krawtchouk polynomials only.

$$\begin{align*}
\langle H_k \rangle_{st} &= -2i \binom{2s}{k} \frac{\sqrt{2s}}{(1 + \xi_+ \xi_-)_{2s}} \frac{\xi_{k+l-1} \xi_{k+l-1}}{2sk + s - k^2}, \\
&\left\{K_i K_j \left[\alpha_2 (\xi_+ \xi_-)^2 + \alpha_1 \xi_+ \xi_- + \alpha_0\right] + k K_i K_j (k - 1) \left[(l - 2s + k) \xi_+ \xi_- + l - k\right] \\
+ k K_j K_i (k - 1) \left[(j - 2s + k) \xi_+ \xi_- + j - k\right]\right\},
\end{align*}
$$

(6.22)

where

$$\begin{align*}
\alpha_2 &= (j - 2s + k)(l - 2s + k), \\
\alpha_1 &= 2((j - s)(l - s) - (k - s)(k - s - 1)), \\
\alpha_0 &= (j - k)(l - k).
\end{align*}$$

The mean curvature vectors $H_k$ satisfy the following conditions

$$tr(H_k) = 0, \quad \langle H_k, \partial X_k \rangle = 0, \quad \langle H_k, \bar{\partial} X_k \rangle = 0. \quad (6.23)$$

**Proof.** The proof is straightforward if we use relations given by (4.15)-(4.16) and relations given in the Appendix.

At this point, we can explore certain global geometrical characteristics of soliton surfaces $X_k$. In particular, we assume that these surfaces are compact, oriented and connected. To compute them we perform the integration over the whole Riemann sphere $S^2$. Under these circumstances, the following holds.

**Proposition 6.3** The Willmore functionals

$$\mathcal{W}_k = \int \int \left[tr(\partial P_k, \bar{\partial} P_k)\right]^2 d\xi_+ d\xi_- \quad (6.24)$$

of 2D-surfaces associated with Veronese subsequent analytical solutions $f_k$ (3.5) of the $\mathbb{CP}^2$ model (2.17) have constant positive values

$$\mathcal{W}_k = \frac{2\pi}{3} \left[4s^2(k^2 + k + 1) - 2ks(2k^2 + k + 3) + k^2(k^2 + 3)\right]. \quad (6.25)$$

**Proof.** The proof is straightforward if we use (4.15)-(4.16) and relations of the Appendix.

**Proposition 6.4** For any value of the Veronese subsequent analytic solutions $f_k$ (3.5) of the $\mathbb{CP}^2$ model (2.17), the integral over the whole Riemann sphere $S^2$ of the topological
charge densities of the 2D-surfaces $X_k$

$$Q_k = \frac{1}{\pi} \int_{S^2} \partial \overline{\partial} \ln |f^\dagger_k f_k| d\xi_+ d\xi_-,$$  \hspace{1cm} (6.26)

are integers

$$Q_k = 2(s - k)$$  \hspace{1cm} (6.27)

**Proof.** The proof follows immediately from (3.5) since the relation

$$f^\dagger_k f_k = \frac{(2s)! k!}{(2s - k)!} (1 + \xi_+ \xi_-)^{2(s-k)}$$  \hspace{1cm} (6.28)

holds.

Note that the values of the topological charges are distinguished between a one-instanton state $Q_0 = 2s$ in the $\mathbb{C}P^{2s}$ model and an anti-instanton state $Q_{2s} = -2s$. This fact produces the same winding over the target sphere $S^2$ but in the opposite direction. This result coincides with the values of the topological charges obtained earlier for low-dimensional sigma models [21].

**Proposition 6.5** For any value of the Veronese subsequent analytic solutions $f_k$ (3.5) of the $\mathbb{C}P^{2s}$ model (2.17), the Euler-Poincaré characters of the 2D-surfaces $X_k$

$$\Delta_k = -\frac{1}{\pi} \int_{S^2} \partial \overline{\partial} \ln |\text{tr}(\partial P_k \cdot \overline{\partial} P_k)| d\xi_+ d\xi_-.$$  \hspace{1cm} (6.29)

are the integer

$$\Delta_k = 2.$$  \hspace{1cm} (6.30)

**Proof.** The proof follows directly from (4.16).

Note that all surfaces possess the same value of the Euler-Poincaré character equal to 2 and all Gaussian curvatures $K_k$ are positive. This means that all 2D-surfaces associated with the $\mathbb{C}P^{2s}$ model are homeomorphic to spheres with radius given by (6.10).

### 7 Concluding remarks

The approach elaborated in this paper based on the hypergeometric orthogonal polynomials for studying the $\mathbb{C}P^{2s}$ sigma model has proven to be a very useful and suitable
tool for investigating the main features of numerous problems that require the solving of this model. The task of obtaining an increasing number of solutions of this model is related to the construction of the consecutive projectors $P_k$ and the associated immersion functions $X_k$ expressed in terms of the Krawtchouk polynomials. Their main advantage appears when these polynomials make it possible to construct a regular algorithm for finding the Veronese subsequent analytical solutions of the $\mathbb{C}P^{2s}$ sigma model without referring to any additional considerations but proceeding only from the given model. A $\mathfrak{su}(2)$ spin-$s$ representation in connection with the Krawtchouk polynomials associated with this model has been explored in detail. Some geometrical aspects of soliton surfaces have also been described in terms of the Krawtchouk polynomials. This analysis has opened a new way for computing the associated metric, the first and second fundamental form, the Gaussian and mean curvatures, the Willmore functionals, the topological charge and the Euler-Poincaré characters.

In the next stage of this research, it would be worthwhile to extend this investigation to the case of the Grassmannian sigma model which take values in the homogeneous spaces

$$G(m, n) = SU(2s + 1)/S(U(m) \times U(n)) .$$

These models are a generalisation of the model considered in this paper. Both models possess many common properties like an infinite number of conserved quantities and infinite dimensional dynamical symmetries which generate the affine Kac–Moody algebra. They admit Hamiltonian structures and complete integrability with well-formulated linear spectral problems. The investigation of the soliton surfaces for Grassmannian sigma model can lead to different classes and much more diverse types of surfaces. The question of the diversity and complexity of the associated surfaces still remains open for further research and should be answered in further work.

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A Properties of the Krawtchouk polynomials

In this appendix, we recall and prove useful properties of the Krawtchouk polynomials and of the projectors $P_k$ (3.6). According to R. Koekoek [24], the properties of the Krawtchouk polynomials follow directly from their definition

$$K_j(k; p, 2s) = \frac{\xi_+ \xi_-}{1 + \xi_+ \xi_-} \binom{-j, -k; -2s; 1/p}{1/p}, \quad 0 \leq j, k \leq 2s,$$ (A.1)

where we recall that $0 < p = \frac{\xi_+ \xi_-}{1 + \xi_+ \xi_-} < 1$. In view of the formula (1.10.6) from [24], the Krawtchouk polynomial forward shift operator is

$$K_j(k + 1; p, 2s) - K_j(k; p, 2s) = -\frac{j}{2sp} K_{j-1}(k; p, 2s - 1).$$ (A.2)

**Lemma A.1** The first derivative with respect to $\xi_\pm$ of the Krawtchouk polynomials are given by

$$\partial K_j(k; p, 2s) = \frac{-k}{\xi_+(1 + \xi_+ \xi_-)} (K_j(k; p, 2s) - K_j(k - 1; p, 2s)),$$ (A.3)

$$\overline{\partial} K_j(k; p, 2s) = \frac{-k}{\xi_- (1 + \xi_+ \xi_-)} (K_j(k; p, 2s) - K_j(k - 1; p, 2s)).$$ (A.4)

**Proof.** We recall the formula for the derivative of the hypergeometric function

$$\frac{\partial}{\partial x} \left(\binom{a, b; c}{x}\right) = \frac{ab}{c} \binom{a + 1, b + 1; c + 1; x}. \quad (A.5)$$

Hence, from the definition (A.1), we can evaluate the first derivatives with respect to $\xi_+$ of the Krawtchouk polynomial

$$\partial K_j \left( k; \frac{\xi_+ \xi_-}{(1 + \xi_+ \xi_-)}, 2s \right) = \partial \left( \binom{-j, -k; -2s; \frac{1 + \xi_+ \xi_-}{\xi_+ \xi_-}}{x} \right)$$

$$= -\frac{\xi_+ \xi_-}{(1 + \xi_+ \xi_-)} \binom{-j + 1, -k + 1; -2s + 1; \frac{1 + \xi_+ \xi_-}{\xi_+ \xi_-}}{x}$$

$$= \frac{1}{\xi_+ \xi_-} \frac{ik}{2s} K_{j-1}(k - 1; p, 2s - 1).$$ (A.6)

In view of (A.2), we obtain the expression (A.3). Similarly, we obtain the relation (A.4).

□
Lemma A.2 The following orthogonality relations hold

\[
\sum_{q=0}^{2s} \binom{2s}{q} (\xi_+\xi_-)^q K_q(k)K_q(\ell) = \frac{(1 + \xi_+\xi_-)^{2s}}{(\xi_+\xi_-)^{k}} \delta_{k,\ell}, \tag{A.7}
\]

\[
\sum_{q=0}^{2s} \binom{2s}{q} (\xi_+\xi_-)^q K_q = \frac{(1 + \xi_+\xi_-)^{2s-1}}{(\xi_+\xi_-)^{k}} (k + (2s - k)\xi_+\xi_-), \tag{A.8}
\]

\[
\sum_{q=0}^{2s} \binom{2s}{q} (\xi_+\xi_-)^q K_q K_q(k - 1) = -\frac{(1 + \xi_+\xi_-)^{2s-1}}{(\xi_+\xi_-)^{k}} (2s - k + 1), \tag{A.9}
\]

\[
\sum_{q=0}^{2s} \binom{2s}{q} (\xi_+\xi_-)^q K_q^2 = \frac{(1 + \xi_+\xi_-)^{2s-2}}{(\xi_+\xi_-)^{k}} ((\xi_+\xi_-)^2(k - 2s)^2 + 2\xi_+\xi_-(4sk - 2k^2 + s) + k^2). \tag{A.10}
\]

Proof. Relation (A.7) is directly obtained from [21]. Let us now differentiate the orthogonality relation (A.7) (for \(\ell = k\)) with respect to \(\xi_+\). By using (A.3), one gets

\[
\sum_{q=0}^{2s} \binom{2s}{q} q\xi_+^{q-1}\xi_-^q K_q^2 = \frac{2k}{\xi_+(1 + \xi_+\xi_-)} \sum_{q=0}^{2s} \binom{2s}{q} (\xi_+\xi_-)^q K_q(K_q - K_q(k - 1))
\]

\[
= \frac{(1 + \xi_+\xi_-)^{2s-1} [2s\xi_+\xi_- - k(1 + \xi_+\xi_-)]}{\xi_+(\xi_+\xi_-)^{k}} \binom{2s}{k}. \tag{A.11}
\]

Therefore, by again using (A.7), we get (A.8). Similarly, by differentiating the orthogonality relation (A.7) with respect to \(\xi_+\) (for \(\ell = k - 1\)), one gets (A.9). Finally, we differentiate relation (A.8) to get relation (A.10).

Let us recall that the Krawtchouk polynomials are self-dual i.e. they satisfy

\[
K_j(k) = K_k(j). \tag{A.12}
\]

Therefore, from Lemma A.2 one gets the other orthogonality properties

\[
\sum_{k=0}^{2s} \binom{2s}{k} (\xi_+\xi_-)^k K_j(k)K_\ell(k) = \frac{(1 + \xi_+\xi_-)^{2s}}{(\xi_+\xi_-)^{j}} \binom{2s}{j} \delta_{j,\ell}, \tag{A.13}
\]

\[
\sum_{k=0}^{2s} \binom{2s}{k} (\xi_+\xi_-)^k k K_j^2(k) = \frac{(1 + \xi_+\xi_-)^{2s-1}}{(\xi_+\xi_-)^{j}} (j + (2s - j)\xi_+\xi_-), \tag{A.14}
\]

\[
\sum_{k=0}^{2s} \binom{2s}{k} (\xi_+\xi_-)^k k K_j(k)K_{j-1}(k) = -\frac{(1 + \xi_+\xi_-)^{2s-1}}{(\xi_+\xi_-)^{j-1}} \binom{2s}{j} (2s - j + 1). \tag{A.15}
\]
Let us also remark that

\[ \sum_{k=0}^{2s} \binom{2s}{k} (\xi_+\xi_-)^k k K_j(k)K_\ell(k) = 0 \quad \text{if} \quad \ell \neq k, k \pm 1 . \]  

(A.16)

In our notation, the difference equation satisfied by the Krawtchouk polynomials (see (1.10.5) of [24]) read

\[-p(2s - k)K_j(k + 1) + (k - j + 2p(s - k))K_j - k(1 - p)K_j(k - 1) = 0 . \]  

(A.17)

Lemma A.3 The following relation between the Krawtchouk polynomials holds

\[ \frac{1}{1 + \xi_+\xi_-} \left[ 2(s - j)K_j + \xi_+\xi_- (2s - j)K_{j+1} - \frac{j}{\xi_+\xi_-}K_{j-1} \right] = (2s - k)K_j(k + 1) . \]  

(A.18)

Proof. By using the following properties of the hypergeometric functions [39]

\[ (b - c)_{2F1} (a, b - 1; c; x) + (c - a - b)_{2F1} (a, b; c; x) = a(x - 1)_{2F1} (a + 1, b; c; x) . \]  

(A.19)

one deduces that

\[ K_j(k + 1) = \frac{1}{2s - k} \left[ (2s - j - k)K_j - \frac{j}{\xi_+\xi_-}K_{j-1} \right] . \]  

(A.20)

Then, by replacing \( K_j(k + 1) \) in (A.18) in the above formula, we see that (A.18) is equivalent to the recurrence relation of the Krawtchouk polynomial (see (1.10.3) of [24]) which concludes the proof. \( \square \)