Can a Borel group be generated by a Hurewicz subspace?

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Abstract

In this paper we formulate three problems concerning topological properties of sets generating Borel non-$\sigma$-compact groups. In case of the concrete $F_{\sigma\delta}$-subgroup of $\{0,1\}^{\omega\times\omega}$ this gives an equivalent reformulation of the Scheepers diagram problem.

Introduction

The Hurewicz property was introduced in [5] as a cover counterpart of the $\sigma$-compactness: a topological space $X$ is said to have this property, if for every sequence $(u_n)_{n\in\omega}$ of open covers of $X$ there exists a sequence $(v_n)_{n\in\omega}$, where each $v_n$ is a finite subset of $u_n$, such that each element $x \in X$ belongs to $\bigcup v_n$ for all but finitely many $n \in \omega$. It is easy to see that each $\sigma$-compact space is Hurewicz (= has the Hurewicz property). The converse statement is known to fail in ZFC, see [6]. By a Borel space we mean a separable metrizable space which is a Borel subset of its completion. This paper is devoted to problems close to the subsequent one.

Problem 1. Can a Borel non-$\sigma$-compact group be generated by its Hurewicz subspace?

This problem is especially interesting for the concrete subgroup $G$ of $\{0,1\}^{\omega\times\omega}$ (standardly endowed with the coordinatewise addition modulo 2) being equivalent to the “Hurewicz” part of the Scheepers diagram problem (see [6 Problems 1,2], [14 Problems 4.1,4.2], [12 Problem 1], and [13 Problem 3.2]), where

$$G = \{ x \in \{0,1\}^{\omega^2} : \text{for every } j \in \omega \text{ and for all but finitely many } i \in \omega \ (x_{i,j} = 0) \}. $$

In order to formulate the Scheepers diagram problem we have to recall some definitions. M. Scheepers in his work [10] introduced a long list of new properties looking similar to the Hurewicz one, and thus gave rise to the branch of set- theoretic topology known as Selection Principles. Selection principles may be thought as some combinatorial conditions on the family of open covers of a topological space. Let $\mathcal{A}$ and $\mathcal{B}$ be a families of covers of a topological space $X$. Following [10] we say that $X$ has the property

- $\bigcup_{\text{fin}}(\mathcal{A},\mathcal{B})$, if for every sequence $(u_n)_{n\in\omega}\mathcal{A}^{\omega}$ there exists a sequence $(v_n)_{n\in\omega}$, where each $v_n$ is a finite subset of $u_n$, such that $\{\bigcup v_n : n \in \omega\} \in \mathcal{B}$;

- $S_{\text{fin}}(\mathcal{A},\mathcal{B})$, if for every sequence $(u_n)_{n\in\omega} \in \mathcal{A}^{\omega}$ there exists a sequence $(v_n)_{n\in\omega} \in \mathcal{A}^{\omega}$, where each $v_n$ is a finite subset of $u_n$, such that $\bigcup\{v_n : n \in \omega\} \in \mathcal{B}$.
Throughout the paper $\mathcal{A}$ and $\mathcal{B}$ run over the families $\mathcal{O}$, $\Omega$, and $\Gamma$ of all open ($\omega$, $\gamma$-) covers of $X$. Given a family $u = \{U_i : i \in I\}$ of subsets of a set $X$, we define the map $\mu_u : X \rightarrow \mathcal{P}(I)$ letting $\mu_u(x) = \{i \in I : x \in U_i\}$ ($\mu_u$ is nothing else but the Marczewski “dictionary” map introduced in [9]). In what follows $I \in \{\omega, \omega^2\}$. Depending on the properties of $\mu_u(X)$ a family $u = \{U_n : n \in \omega\}$ is defined to be

- an $\omega$-cover $\mathbb{[\mathbb{I}]}$, if the family $\mu_u(X)$ is centered, i.e. for every finite subset $K$ of $X$ the intersection $\bigcap_{x \in K} \mu_u(x)$ is infinite;
- a $\gamma$-cover of $X$ $\mathbb{[\mathbb{I}]}$, if for every $x \in X$ the set $\mu_u(x)$ is cofinite in $\omega$, i.e. $\omega \setminus \mu_u(x)$ is finite.

We shall consider here four selection principles: $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$, $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$, $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$ and $S_{\text{fin}}(\Gamma, \Omega)$. Let us note that $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ is nothing else but the Hurewicz property. Concerning $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$, it is the classical Menger covering property introduced in [8]. We are in a position now to formulate the

**Scheepers diagram problem.**

1. Does the property $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$ imply $S_{\text{fin}}(\Gamma, \Omega)$?
2. And if not, then does $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ imply $S_{\text{fin}}(\Gamma, \Omega)$?

One may ask the same question as in Problem $\mathbb{[\mathbb{I}]}$ for properties $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$ and $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$.

**Problem 2.** Can a Borel non-$\sigma$-compact group be generated by its subspace with the property $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$?

**Problem 3.** Can a Borel non-$\sigma$-compact group be generated by its subspace with the property $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$?

The subsequent theorem, which is the main result of this paper, is the reformulation of a Scheepers diagram problem in algebraic manner.

**Theorem 4.** The property $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ (resp. $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$, $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$) implies $S_{\text{fin}}(\Gamma, \Omega)$ if and only if the group $G$ is not generated by its subspace with the property $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ (resp. $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$, $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$).

In other words, the positive answer onto the Scheepers diagram problem (1) (resp. (2)) is equivalent to the negative answer onto Problem $\mathbb{[\mathbb{II}]}$ (resp. Problem $\mathbb{[\mathbb{III}]}$) in case of the group $G$.

The group $G$ is a rather simple object from the point of view of Descriptive Set Theory. For every $j \in \omega$ its projection onto $\{0, 1\}^{\omega \times \{j\}}$ is homeomorphic to $\mathbb{Q}$ being a countable metrizable space without isolated points. From the above it follows that $G$ is a countable intersection of $F_\sigma$ subsets of $\{0, 1\}^{\omega^2}$ (i.e. it is an $F_{\sigma\delta}$- or, equivalently, $\Pi^0_3$- subset) homeomorphic to $\mathbb{Q}^{\omega^2}$. Therefore it is a nowhere locally-compact, and it fails to have the property $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$. For more simple groups from the point of view of Borel hierarchy Problem $\mathbb{[\mathbb{II}]}$ can be answered in negative.

**Proposition 5.** No Borel non-$\sigma$-compact group $B$ can be generated by its subspace $X$ with the property $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ provided $B$ is an $F_\sigma$- or $G_{\delta}$- subspace of a complete metric space.
Recall that a map \( f \) from a topological space \( X \) to a topological space \( Y \) is \textit{Borel}, if for every Borel subset \( B \) of \( Y \) its preimage \( f^{-1}(B) \) is a Borel subset of \( X \). The subsequent statement answers Problem 3 in positive under the Continuum Hypothesis. On the other hand, it is known that the properties \( \bigcup_{f_{\text{in}}}(\Theta, \Omega) \) and \( \bigcup_{f_{\text{in}}}(\Theta, \Theta) \) coincide in some models of ZFC, see [17]. Therefore the negative answer onto Problem 2 would imply that the negative answer onto Problem 3 is consistent as well.

**Proposition 6.** Under the Continuum Hypothesis a metrizable separable group \( B \) can be generated by its subspace \( X \) with the property \( \bigcup_{f_{\text{in}}}(\Theta, \Theta) \) provided it is a Borel homomorphic image of a nonmeager metrizable separable group. In particular, \( G \) is generated by its subspace with the property \( \bigcup_{f_{\text{in}}}(\Theta, \Theta) \) under CH.

**Remark.** None of the known methods of construction of spaces with the property \( \bigcup_{f_{\text{in}}}(\Theta, \Gamma) \) can give a subspace of a Borel non-\( \sigma \)-compact group generating it. All finite powers of spaces with the property \( \bigcup_{f_{\text{in}}}(\Theta, \Gamma) \) constructed in [6] Theorem 5.1, [15] Theorem 5.1, and [2] Theorem 10(1)] have the property \( \bigcup_{f_{\text{in}}}(\Theta, \Theta) \) or even \( \bigcup_{f_{\text{in}}}(\Theta, \Gamma) \), and hence so is any group they generate. But every Borel (even analytic) space with the property \( \bigcup_{f_{\text{in}}}(\Theta, \Theta) \) is \( \sigma \)-compact, see [1]. While the Sierpinski sets \( S \) considered in [6] and [11] have the subsequent property: for every Borel subset \( B \) containing \( S \) there exists a \( \sigma \)-compact \( L \) such that \( S \subset L \subset B \), see [3].

Concerning the property \( \bigcup_{f_{\text{in}}}(\Theta, \Omega) \), all known examples (besides Sierpinski sets) have the property \( \bigcup_{f_{\text{in}}}(\Theta, \Theta) \) in all finite powers, and hence can not generate non-\( \sigma \)-compact Borel group.

**Proofs**

In what follows \( A \subset^{*} B \) standardly means that \( A \setminus B \) is finite. In our proofs we shall exploit set-valued maps. By a \textit{set-valued map} \( \Phi \) from a set \( X \) into a set \( Y \) we understand a map from \( X \) into \( \mathcal{P}(Y) \) and write \( \Phi : X \rightarrow Y \) (here \( \mathcal{P}(Y) \) denotes the set of all subsets of \( Y \)). For a subset \( A \) of \( X \) we put \( \Phi(A) = \bigcup_{x \in A} \Phi(x) \subset Y \). The set-valued map \( \Phi \) between topological spaces \( X \) and \( Y \) is said to be

- \textit{compact-valued}, if \( \Phi(x) \) is compact for every \( x \in X \);
- \textit{upper semicontinuous}, if for every open subset \( V \) of \( Y \) the set \( \Phi^{-1}(V) = \{ x \in X : \Phi(x) \subset V \} \) is open in \( X \).

For a set \( X \) we can identify \( \mathcal{P}(X) \) with the compact space \( \{0, 1\}^{X} \) via the map \( \chi_{A} \rightarrow \chi_{A} \in \{0, 1\}^{X} \) assigning to a subset of \( X \) its characteristic function. A family \( \mathcal{A} \) of subsets of a set \( X \) is called \textit{upward closed}, for every \( A \in \mathcal{A} \) and \( B \supseteq A \) we have \( B \in \mathcal{A} \). For a set \( A \subset X \) we make the subsequent notation: \( \uparrow A = \{ B \subset X : A \subset B \} \). The following lemma is a more convenient reformulation of Theorem 4.

**Lemma 7.** Let \( \mathcal{P} \) be a topological property preserved by images under upper semicontinuous compact-valued maps. Then the following conditions are equivalent:

1. The property \( \mathcal{P} \) implies \( S_{f_{\text{in}}}(\Gamma, \Omega) \);
2. for every (upward-closed) \( \mathcal{F} \subset \mathcal{P}(\omega^{2}) \) with the property \( \mathcal{P} \) such that \( \omega \times \{ j \} \subset^{*} F \) for every \( F \in \mathcal{F} \) and \( j \in \omega \), there exists a sequence \((K_{j})_{j \in \omega} \) of finite subsets of \( \omega \) such that each element of the smallest filter containing \( \mathcal{F} \) meets \( \bigcup_{n \in \omega} K_{j} \times \{ j \} \).
Proof. (1) ⇒ (2). It simply follows from definition of the property $S_{\text{fin}}(\Gamma, \Omega)$ and the observation that $\{\{F \in \mathcal{F} : F \ni (i, j)\} : i \in \omega\}$ is an open $\gamma$-cover of $\mathcal{F}$ for every $j \in \omega$.

(2) ⇒ (1). Let $X$ be a topological space with the property $\mathcal{P}$ and $(u_j)_{j \in \omega}$ be a sequence of open $\gamma$-covers of $X$. Let us write $u_j$ in the form $u_j = \{U_{i,j} : i \in \omega\}$. Set $u = \{U_{i,j} : i, j \in \omega\}$. Consider the set-valued map $\Phi : X \to \mathcal{P}(\omega^2)$, $\Phi : x \mapsto \mu_x(x)$. Applying Lemma 2 of [17], we conclude that $\Phi$ is compact-valued and upper semicontinuous, and hence $\mathcal{F} := \Phi(X)$ has the property $\mathcal{P}$. The definition of $\Phi$ implies that $\mathcal{F}$ is upward closed. Since $u_j$ is a $\gamma$-cover of $X$ for every $j \in \omega$, $\omega \times \{j\} \subset^* F$ for each $F \in \mathcal{F}$. From the above it follows that there exists a sequence $(K_j)_{j \in \omega}$ of finite subsets of $\omega$ such that each element of the smallest filter $\mathcal{U}$ containing $\mathcal{F}$ meets some $K_j \times \{j\}$. Then the family $\{U_{i,j} : i \in K_j\}$ is easily seen to be an $\omega$-cover of $X$, which finishes our proof. \hfill \Box

The properties $\bigcup_{\text{fin}}(\mathcal{O}, \mathcal{O})$, $\bigcup_{\text{fin}}(\mathcal{O}, \Omega)$, and $\bigcup_{\text{fin}}(\mathcal{O}, \Gamma)$ satisfy the conditions of the above lemma by [17] Lemma 1.

**Proof of Theorem 4.** Let $\mathcal{P}$ be any of the properties $\bigcup_{\text{fin}}(\mathcal{O}, \mathcal{O})$, $\bigcup_{\text{fin}}(\mathcal{O}, \Omega)$, and $\bigcup_{\text{fin}}(\mathcal{O}, \Gamma)$. Assuming that $\mathcal{P}$ implies $S_{\text{fin}}(\Gamma, \Omega)$, fix a subspace $X$ of $\mathcal{G}$ with the property $\mathcal{P}$. Let us denote by $\varphi$ the map assigning to a subset $A$ of $\omega^2$ its characteristic function $\chi_A \in \{0, 1\}^{\omega^2}$. Then the space $\mathcal{F} = \{\omega^2 \setminus A : A \in \varphi^{-1}(X)\}$ has the property $\mathcal{P}$ being homeomorphic to $X$, and $\omega \times \{j\} \subset^* F$ for every $F \in \mathcal{F}$ by our choice of $\mathcal{G} \supset X$. Applying Lemma 4 we conclude that there exists a sequence $(K_j)_{j \in \omega}$ of finite subsets of $\omega$ such that $\bigcup_{j \in \omega} K_j \times \{j\}$ meets all elements of the smallest filter containing $\mathcal{F}$. Now, a direct verification shows that the characteristic function $\chi_{\bigcup_{j \in \omega} K_j \times \{j\}}$ can not be represented as a sum of elements of $X$, which means that $X$ does not generate $\mathcal{G}$.

Next, let us assume that $\mathcal{P}$ does not imply $S_{\text{fin}}(\Gamma, \Omega)$ and apply Lemma 4 to find an upward closed family $\mathcal{F}$ of subsets of $\omega^2$ such that for every sequence $(K_j)_{j \in \omega}$ of finite subsets of $\omega$ there exists a finite subset $\mathcal{A}$ of $\mathcal{F}$ such that

$$(\bigcup_{j \in \omega} K_j \times \{j\}) \cap \bigcap_{\mathcal{A}} \mathcal{A} = \emptyset.$$  

Set $X = \{\chi_{\omega^2 \setminus F} : F \in \mathcal{F}\}$. Then $X$ has the property $\mathcal{P}$ being homeomorphic to $\mathcal{F}$. We claim that $X$ is a set of generators of $\mathcal{G}$. Indeed, let us fix any $g \in \mathcal{G}$ and set $K_j = \{i \in \omega : g_{i,j} = 1\}$. Then each $K_j$ is finite by the definition of $\mathcal{G}$. For the sequence $(K_j)_{j \in \omega}$ find a finite subset $\mathcal{A} = \{A_i : i \leq n\}$ of $\mathcal{F}$ as above. Using the upward closedness of $\mathcal{F}$, define inductively a finite subset $\mathcal{B} = \{B_i : i \leq n\}$ of $\mathcal{F}$ letting $B_0 = A_0$ and $B_k = A_k \cup \bigcup_{l < k}(\omega^2 \setminus B_l)$ for all $0 < k \leq n$. It is easy to prove by induction over $k \leq n$ that $(\omega^2 \setminus B_l) \cap (\omega^2 \setminus B_k) = \emptyset$ for all $l < k$ and $\bigcap_{l \leq k} B_k = \bigcap_{l \leq k} A_k$, consequently $\bigcap \mathcal{B} = \bigcap \mathcal{A} \subset (\omega^2 \setminus \bigcup_{j \in \omega} K_j \times \{j\})$. Let $C_k = B_k \cup (\omega^2 \setminus \bigcup_{j \in \omega} K_j \times \{j\})$, $k \leq n$. Then $C = \{C_k : k \leq n\}$ has the following properties

(i) $\bigcup C = \omega^2 \setminus \bigcup_{j \in \omega} K_j \times \{j\};$

(ii) $(\omega^2 \setminus C) \cap (\omega^2 \setminus D) = \emptyset$ for all $C, D \in \mathcal{C};$

(iii) $\mathcal{C} \subset \mathcal{F}.$

It suffices to note that $\{\chi_{\omega^2 \setminus C_k} : k \leq n\} \subset X$ by (iii) and $\chi_{\omega^2 \setminus C_0} + \cdots + \chi_{\omega^2 \setminus C_n} = \chi_{\bigcup_{j \in \omega} K_j \times \{j\}} = g$, which finishes our proof. \hfill \Box
Proof of Proposition 5. First assume that $B$ is a non-$\sigma$-compact $G_\delta$-subspace of a complete metric space and fix a subspace $X$ of $B$ with the property $\bigcup_{n \in \omega} (O, \Gamma)$. The same argument as in [6, Theorem 5.7] gives a $\sigma$-compact subset $L$ of $B$ such that $X \subset L$. Since $B$ is not $\sigma$-compact, it is not generated by $L$, and hence by $X$ as well.

Now consider a non-$\sigma$-compact Borel group $B$ which is an $F_\sigma$ subset of a complete metric space $Y$ and write $B$ in the form $\bigcup_{n \in \omega} B_n$, where each $B_n$ is closed in $Y$. Let $X$ be a subspace of $B$ with the property $\bigcup_{n \in \omega} (O, \Gamma)$. Since the property $\bigcup_{n \in \omega} (O, \Gamma)$ is preserved by closed subspaces, $X \cap B_n$ has the property $\bigcup_{n \in \omega} (O, \Gamma)$ for all $n \in \omega$. In addition, each $B_n$ is a $G_\delta$-subspace of $Y$ being closed. From the above it follows that there exists a $\sigma$-compact $L_n$ such that $X \cap B_n \subset L_n \subset B_n$, and consequently $X \subset \bigcup_{n \in \omega} L_n \subset B$. It suffices to apply the same argument as in the first part of the proof. 

Proof of Proposition 6. Let $C$ be a nonmeager metrizable separable topological group and $f : C \to B$ be a surjective Borel homomorphism. Almost literal repetition of the proof of Lemma 29 from [11] give us a subspace $Z$ of $C$ such that $Z$ generates $C$ and each Borel image of $Z$ has the property $\bigcup_{n \in \omega} (O, O)$, see [11, Corollary 30]. It suffices to note that $B$ is generated by $f(Z)$.

Next, let us show that under CH the group $G$ is generated by its subspace with the property $\bigcup_{n \in \omega} (O, O)$. Indeed, let us denote by $\tau$ the Tychonoff product topology on $\{0, 1\}^{\omega \times \omega} = \prod_{j \in \omega} \{0, 1\}^{\omega \times \{j\}}$, where $\{0, 1\}^{\omega \times \{j\}}$ is considered with the discrete topology for each $j \in \omega$. Then $\tau|G$ is stronger than the natural topology on $G$, and $(G, \tau|G)$ is a completely metrizable topological group being a countable product of countable discrete groups.

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