Review on the determination of $\alpha_s$ from the QCD static energy

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We review the determination of the strong coupling $\alpha_s$ from the comparison of the perturbative expression for the Quantum Chromodynamics static energy with lattice data. We collect here all the perturbative expressions needed to evaluate the static energy at the currently known accuracy.

1. Introduction

There has been much progress, in the last few years, in the perturbative evaluation of the Quantum Chromodynamics (QCD) static energy $E_0(r)$, i.e. the energy between a static quark and a static anti-quark separated a distance $r$. Alongside, unquenched lattice computations of $E_0(r)$ at short distances have become available. These developments have made manifest the ability of perturbative calculations in QCD to reproduce the short-distance part of $E_0(r)$ calculated on the lattice, and have led to a determination of the strong coupling $\alpha_s$ from the comparison of the two, which is what we review here. As a preface, let us illustrate the aforementioned progress by showing: (i) a comparison of perturbative calculations for $E_0(r)$, at different orders of accuracy, with short distance lattice data with three light flavors [2], Fig. 1, and (ii) short-distance lattice data for $E_0(r)$, with zero [3] and three [2] light flavors, along with the corresponding perturbative predictions at the highest accuracy known at present, Fig. 2. We can see from the figures that one can perfectly describe the short-distance behavior of $E_0(r)$ obtained in the lattice, which can be considered as an important landmark in our understanding of QCD.

The rest of the paper is organized as follows: in Sec. [2] we present the currently known terms in the perturbative expansion of the static energy. Section [3] contains the comparison of lattice data with perturbation theory and the corresponding extraction of $\alpha_s$. In Sec. [4] we conclude and discuss the expected developments in the near future. The Appendix collects color factors and beta function coefficients.
2. Perturbative expression for the static energy

The present knowledge of $E_0(r)$ at short distances can be summarized as follows

$$E_0(r) = -\frac{C_F\alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} \tilde{a}_1 + \left( \frac{\alpha_s(1/r)}{4\pi} \right)^2 \tilde{a}_2 ight.$$  
$$+ \left( \frac{\alpha_s(1/r)}{4\pi} \right)^3 \left[ a_3^L \log C_A \alpha_s(1/r) + \tilde{a}_3 \right]$$  
$$+ \left( \frac{\alpha_s(1/r)}{4\pi} \right)^4 \left[ a_3^{L2} \log^2 C_A \alpha_s(1/r) + a_3^L \log C_A \alpha_s(1/r) + \tilde{a}_4 \right]$$  
$$+ \cdots \right\}. \quad (1)$$

Non-analytic terms in $\alpha_s$ appear in $E_0(r)$, starting at order $\alpha_s^4$, due to virtual emission of ultrasoft gluons \cite{2} (i.e., gluons with energy and momentum smaller than $1/r$ that can change the color state of the quark-antiquark pair from singlet to octet).

Terms up to next-to-next-to-leading order (N$^2$LO), i.e., $\tilde{a}_1$ and $\tilde{a}_2$ in Eq. (1),
Determination of $\alpha_s$ from the QCD static energy

Fig. 2. Comparison of the static energy with lattice data for $n_f = 0$ and $n_f = 2 + 1$ ($n_f$ is the number of light flavors). The theoretical curves include terms up to order $\alpha_s^{4+n} \ln^n \alpha_s$ (with $n \geq 0$), what is referred to as $N_3^{LL}$ accuracy, see Sec. 2 for more details and explicit expressions. The additive constant in the perturbative expression for the static energy is taken such that each curve coincides with the corresponding lattice data point at the shortest distance (see Eq. (28)). The bands are obtained by adding a term $\pm C_F \alpha_s^5 / r$ to the $N_3^{LL}$ curves, and give an idea of the perturbative uncertainty of the results. $r_0 \Lambda_{\text{MS}} = 0.637$ is used for the $n_f = 0$ curve, and $r_0 \Lambda_{\text{MS}} = 0.70$ for the $n_f = 3$ one ($\Lambda_{\text{MS}}$ is the QCD scale, in the MS scheme, and $r_0$ is the lattice reference scale, see Sec. 3).

More recently, the three-loop coefficient $\tilde{a}_3$ was computed by two different
groups It reads
\[ \tilde{a}_3 =: a_3 + \left( 8 \gamma_E^2 + 2 \gamma_E \pi^2 + 16 \zeta(3) \right) \beta_0^3 + 2 \gamma_E \beta_2 \]
\[ + \left( 12 \gamma_E^2 + \pi^2 \right) \beta_0^2 + 4 \gamma_E \beta_1 \] \[ a_1 + \left[ 6 a_2 \gamma_E + \frac{5}{2} \left( 4 \gamma_E^2 + \frac{\pi^2}{3} \right) \right] \beta_0, \] (5)
\[ a_3 =: a_3^{(3)} n_f^3 + a_3^{(2)} n_f^2 + a_3^{(1)} n_f + a_3^{(0)}, \] (6)
\[ a_3^{(3)} = - \left( \frac{20}{9} \right)^3 T_F^3, \] (7)
\[ a_3^{(2)} = \left( \frac{125411}{243} + \frac{368}{3} \zeta(3) + \frac{64 \pi^4}{135} \right) C_A T_F^2 + \left( \frac{14002}{81} - \frac{416}{3} \zeta(3) \right) C_F T_F^2, \] (8)
\[ a_3^{(1)} = \left( -709.717 \right) C_A^2 T_F + \left( \frac{-71281}{162} + 264 \zeta(3) + 80 \zeta(5) \right) C_A C_F T_F \]
\[ + \left( \frac{286}{9} + \frac{296}{3} \zeta(3) - 160 \zeta(5) \right) C_F^2 T_F + \left( -56.83(1) \right) \frac{d_{abcd} d_{abcd}}{N_A}, \] (9)
\[ a_3^{(0)} = 502.24(1) C_A^3 - 136.39(12) \frac{d_{abcd} d_{abcd}}{N_A} \]
\[ + \frac{8}{3} \pi^2 C_A^3 \left( - \frac{5}{3} + 2 \gamma_E + 2 \log 2 \right). \] (10)

Note that Refs. use a slightly different notation, in particular Ref. uses to denote just the first line of Eq. above. The color factors and beta function coefficients, which appear throughout the paper, are collected in the Appendix, \( \gamma_E = 0.5772 \ldots \) is the Euler constant, \( \zeta(x) \) is the Riemann zeta function, and \( n_f \) is the number of light flavors. For convenience, we also give here the numerical values of the \( \tilde{a}_{1,2,3} \) coefficients for \( N_c = 3 \)
\[ \tilde{a}_1 = 23.032 - 1.8807 n_f, \] (11)
\[ \tilde{a}_2 = 1396.3 - 192.9 n_f + 4.9993 n_f^2, \] (12)
\[ \tilde{a}_3 = 108654. - 21905.2 n_f + 1284.69 n_f^2 - 20.6009 n_f^3. \] (13)

The coefficients of the logarithmic terms in Eq. can be conveniently calculated within the framework of the effective theory potential Non-Relativistic QCD (pNRQCD). They read
\[ a_3^L = \frac{16 \pi^2}{3} C_A, \] (14)
\[ a_4^L = - \frac{16 \pi^2}{3} C_A \beta_0, \] (15)
\[ a_4^L = 16 \pi^2 C_A^3 \left[ a_1 + 2 \gamma_E \beta_0 + T_F n_f \left( - \frac{40}{27} + \frac{8}{9} \log 2 \right) \right. \]
\[ \left. + C_A \left( \frac{149}{27} - \frac{22}{9} \log 2 + \frac{4}{9} \pi^2 \right) \right]. \] (16)
pNRQCD can also be used to perform the resummation of these logarithms. This was done at leading order in Ref. [19], and at sub-leading order in Ref. [21]. When we include resummation of the ultrasoft logarithms $E_0(r)$ reads

$$ E_0(r) = -\frac{C_F\alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} \tilde{a}_1 + \left( \frac{\alpha_s(1/r)}{4\pi} \right)^2 \tilde{a}_2 + \left( \frac{\alpha_s(1/r)}{4\pi} \right)^3 \tilde{a}_3 \right\} + \frac{2}{3} C_F r^2 \left\{ \frac{C_A \alpha_s(1/r)}{2} \left[ 1 + (a_1 + 2\gamma_E \beta_0) \frac{\alpha_s(1/r)}{4\pi} \right] \right\}^3 \times \left( \frac{2}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(1/r)} + \eta_0 \left[ \alpha_s(\mu) - \alpha_s(1/r) \right] \right) - \frac{C_F C_A^2}{12\pi r} \frac{\alpha_s^4(1/r)}{2\bar{r} \mu} \ln \frac{C_A \alpha_s(1/r)}{2\mu}, \quad (17) $$

with

$$ \eta_0 = \frac{1}{\pi} \left( \frac{\beta_1}{2\beta_0^2} + \frac{12B}{\beta_0} \right) ; \quad B = \frac{-10T_F n_f + C_A (6\pi^2 + 47)}{108}. \quad (18) $$

In Eq. (17) we only display the terms that are needed for next-to-next-to-next-to-leading logarithmic ($N^3LL$) accuracy, where by $N^3LL$ accuracy we mean that we include terms up to order $\alpha_s^{4+n} \log^n \alpha_s \ (n \geq 0)$. $\mu$ is the ultrasoft factorization scale, which takes a natural value $\mu \sim (C_A \alpha_s) / (2\bar{r})$; $E_0(r)$ is a physical observable and therefore $\mu$ independent, i.e. the $\mu$ dependence in Eq. (17) cancels order by order.

Let us recall at this point that in order to properly define the static limit of QCD, or Heavy Quark Effective Theory, one needs to introduce a residual mass term $\Lambda_s^2$, whose typical size is associated with the QCD hadronic scale, $\Lambda_{QCD}$. This residual mass term is inherited by pNRQCD. In the short-distance weak-coupling regime we are considering here, it can be encoded in a matching coefficient, that we denote by $\Lambda_s$, which should be added to the expressions for $E_0(r)$ above. i.e. we have

$$ E_0(r) \rightarrow E_0(r) + \Lambda_s. \quad (19) $$

The coefficient $\Lambda_s$ obeys ultrasoft renormalization group (RG) equations in pNRQCD. The solution of the RG equations, at the order we will need it, reads

$$ \Lambda_s(\mu) = K_1 + K_2 \alpha_s^2(1/r) C_F C_A^2 \frac{1}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(1/r)}, \quad (20) $$

where $K_1$ and $K_2$ are dimension-one constants of order $\Lambda_{QCD}$. The term involving $K_2$ starts contributing at $N^3LL$ accuracy, since one counts $K_{1,2} \sim \Lambda_{QCD} \sim \alpha_s^2/r \ll E_0 \sim \alpha_s/r$. When comparing the static energy with lattice data, it is important to perform the comparison in a way that is not affected by the presence of the so-called renormalon singularities. We will achieve this by explicitly...
working with a renormalon-free scheme all the time. We choose the so-called renormalon subtracted (RS) scheme introduced in Ref. [27]. In practice this means that we need to include a subtraction term to the perturbative expression for the static energy. If we compute the static energy at \( m \)-loop order in perturbation theory, the subtraction term reads

\[
\text{RS}_{\text{subtr.}} = R_s \rho \sum_{n=1}^{m} \left( \frac{\beta_0}{2\pi} \right)^n \alpha_s(\rho)^{n+1} \sum_{k=0}^{2} d_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)},
\]

with

\[
d_0 = 1,
\]

\[
d_1 = \frac{\beta_0^2 - \beta_2 \beta_0}{4b\beta_0^4},
\]

\[
d_2 = \frac{-2\beta_0^4 \beta_3 + 4\beta_0^3 \beta_1 \beta_2 + \beta_0^2 (\beta_2^2 - 2\beta_1^3) - 2\beta_0 \beta_1 \beta_2 + \beta_1^2}{32(b-1)b\beta_0^8},
\]

\[
b = \frac{\beta_1}{2\beta_0^2}.
\]

\( R_s \) in Eq. (21) is the normalization of the first renormalon singularity, it can be computed approximately using the procedure of Ref.[28]; \( \rho \) is a dimensional scale with a natural value around the center of the range of distances we consider, the presence of a dimensional scale is inherent in all schemes that explicitly cancel renormalon singularities. When we consider N^{3}LL accuracy, a corresponding subtraction term is also needed for the term in curly braces in the third line of Eq. (17) (this term arises from a difference of the color octet and color singlet potentials, therefore the renormalon subtraction here also involves an octet normalization constant \( R_o \)).

For the renormalon to cancel order by order in \( \alpha_s \), one needs to expand \( \alpha_s(1/r) \) or vice versa, in order to have a single expansion parameter in the final expression for the static energy. Here we choose to expand \( \alpha_s(1/r) \) in terms of \( \alpha_s(\rho) \), note that in this case the explicit numerical value of the renormalon normalization constants is irrelevant for the lattice comparison in the next section.

The final expression for the static energy that we need to use is therefore given by

\[
E_0(r) = \left[ \text{Eq. (17)} \right] - \text{RS}_{\text{subtr.}} + \Lambda_s,
\]

where it is understood that each of the three terms in Eq. (20) is taken at the order needed to obtain the desired accuracy. Note that, in order to simplify the notation and to avoid a proliferation of symbols in the paper, earlier we denoted Eq. (17) (and Eq. (11)) also by \( E_0(r) \), i.e. we use \( E_0(r) \) as a generic denotation for the static energy, without specifying in the notation if ultrasoft resummation is performed or not, whether the perturbative expansion incorporates an explicit renormalon subtraction, or the presence of a residual mass term (which would only be absent in a purely perturbative result in \( \overline{\text{MS}} \)-like schemes).
3. Comparison with lattice data and extraction of $\alpha_s$

We can now compare the perturbative expressions for the static energy in the previous section with lattice data with three light flavors. This comparison allows us to extract the value of the QCD scale $\Lambda_{\text{MS}}$ (in the $\overline{\text{MS}}$ scheme), upon which the perturbative expressions depend. In order to obtain this extraction, we assume that perturbation theory, after implementing a cancellation of the leading renormalon singularity, is enough to describe lattice data in the range of distances we study.

We employ the $n_f = 2 + 1$ lattice data for the static energy obtained in Ref. 2. This lattice computation used a combination of tree-level improved gauge action and highly-improved staggered quark action. It employed the physical value for the strange-quark mass $m_s$ and light quark masses equal to $m_s/20$, which correspond to a pion mass of about 160 MeV in the continuum limit, very close to the physical value. The computation was performed for a wide range of gauge couplings, and was corrected for lattice artifacts. At each value of the gauge coupling one calculates the scale parameters $r_0$ and $r_1$ defined in terms of the static energy $E_0(r)$ as follows:

$$r_0^2 \frac{dE_0(r)}{dr} \big|_{r=r_0} = 1.65, \quad r_1^2 \frac{dE_0(r)}{dr} \big|_{r=r_1} = 1.$$  \hspace{1cm} (27)

The values of $r_0$ and $r_1$ were given in Ref. 2 for each gauge coupling. The static energy can be calculated in units of $r_0$ or $r_1$. For the present analysis, we only use lattice data for $r < 0.5r_0$, where perturbation theory should be reliable. Since we have lattice data points down to $r = 0.14r_0$, this means that we are studying the static energy in the $0.065 \text{ fm} \lesssim r \lesssim 0.234 \text{ fm}$ distance range, in physical units. The static energy has an additive ultraviolet renormalization (the self energy of the static sources) and one needs to normalize the results calculated at different lattice spacings to a common value at a certain distance (as an alternative to that one can also take a derivative and compute the force). The static energy in units of $r_0$ is fixed to 0.954 at $r = r_0$. For additional details about the lattice data see Refs. 2.

The adequate quantity to plot in order to compare with lattice data is:

$$E_0(r) - E_0(r_{\text{min}}) + E_0^{\text{latt}}(r_{\text{min}}) = E_0(r) + \text{const.},$$  \hspace{1cm} (28)

where $r_{\text{min}}$ is the shortest distance at which lattice data is available, and $E_0^{\text{latt}}(r_{\text{min}})$ is the value of the lattice data at that distance. Note that then, by construction, all perturbative curves coincide with the lattice point at the shortest distance available. Therefore, for instance, the $N^3\text{LL}$ curves that are shown in Fig. 2 are given by Eq. (28) with $E_0(r)$ from Eq. (20), taking each of the three terms in that last equation at $N^3\text{LL}$ accuracy (with $K_2$ fitted to the lattice data), recall also that we always express everything as an expansion in terms of $\alpha_s(\rho)$. Corresponding expressions hold for the rest of the curves. Let us also mention that in principle one can include finite strange-quark mass effects at one loop in Eq. (17) or Eq. (11), but they turn out to be negligible.
We can now search for the values of $\Lambda_{\overline{MS}}$ that are allowed by lattice data. The guiding principle we follow to achieve this is that the agreement with lattice should improve when the perturbative order of the calculation is increased. A procedure to perform the extraction following these guidelines was devised in Ref. [34], where it was applied to extract $r_0\Lambda_{\overline{MS}}$ for the $n_f = 0$ case. It consists of the steps described next.

First, to obtain the central value for $r_0\Lambda_{\overline{MS}}$, we:

1. Let $\rho$ vary by $\pm 25\%$ around its natural value at the center of the range where we have lattice data.

2. For each value of $\rho$, and at each order in the perturbative expansion of the static energy, we perform a fit to the lattice data ($r_0\Lambda_{\overline{MS}}$ is the parameter of each of the fits).

3. We select the $\rho$ values for which the reduced $\chi^2$ of the fits decreases when increasing the number of loops of the perturbative calculation.

Then we consider the set of $r_0\Lambda_{\overline{MS}}$ values in the $\rho$ range we have obtained and take their average, using the inverse reduced $\chi^2$ of each fit as weight. This gives the central value for $r_0\Lambda_{\overline{MS}}$. We can do that at different orders of accuracy, and obtain the results shown in Tab. 1. Note that the last row of the column is at three loop plus leading logarithmic resummation accuracy. If we perform the fits at $N^3\text{LL}$ accuracy, then an additional constant, $K_2$ in Eq. (20) above, enters in them and also needs to be fitted. If we try to do that we find that, with the present lattice data, the $\chi^2$ as a function of $r_0\Lambda_{\overline{MS}}$ is very flat; which means that at present we cannot improve our extraction of $r_0\Lambda_{\overline{MS}}$ by including the fits at $N^3\text{LL}$ accuracy in the analysis. Consequently, we take the numbers in the last row of Tab. 1 as our best result.

Then, to associate an error to this number we do the following. On the basis that the error associated to the result should reflect the uncertainties from unknown higher perturbative orders, we consider the weighted standard deviation in the range of $\rho$ obtained above, and the difference with the weighted average computed at the previous perturbative order. We take those two numbers as errors of our result, and add them linearly. We obtain $r_0\Lambda_{\overline{MS}} = 0.7024 \pm 0.0011 \pm 0.0665 = 0.70 \pm 0.07$.

Table 1. Values of $r_0\Lambda_{\overline{MS}}$ obtained at different levels of accuracy. “$N^2\text{LL}$” stands for next-to-next-to leading-logarithmic and “3 loop + us. res.” stands for three loop plus leading ultrasoft logarithmic resummation.

| Accuracy       | $r_0\Lambda_{\overline{MS}}$ |
|----------------|-------------------------------|
| tree level     | 0.395                         |
| 1 loop         | 0.848                         |
| 2 loop         | 0.636                         |
| $N^2\text{LL}$ | 0.756                         |
| 3 loop         | 0.690                         |
| 3 loop + us. res. | 0.702                     |
where the first error is due to the weighted standard deviation, and the second to the difference with the two-loop result. Note that assigning the difference with the result at the previous order as an error is a quite conservative estimate. To further assess possible systematic errors stemming from our procedure, we have redone the analysis using $p$-value weights and using constant weights. We find similar results, and in the final result quote an error that covers the whole range spanned by the three analyses. A partial additional cross-check of the result can be performed by redoing the whole analysis with the static energy normalized in units of the scale $r_1$, rather than $r_0$. Note that this is a cross-check, and not just a trivial re-scaling, because the systematics and errors entering the lattice analysis normalized in units of $r_0$ or $r_1$ are different. When we do that, we do find consistent results.

From the above discussion, our final result for $r_0 \Lambda_{\text{MS}}$ reads

$$r_0 \Lambda_{\text{MS}} = 0.70 \pm 0.07,$$

which using $r_0 = 0.468 \pm 0.004$ fm\cite{2} corresponds to

$$\alpha_s(\rho = 1.5 \text{GeV}, n_f = 3) = 0.326 \pm 0.019,$$

the uncertainty in $r_0$ is negligible in the final error above. $\rho \sim 1.5$ GeV, which corresponds to the center of the range where we have lattice data, is the natural scale of our $\alpha_s$ determination. When we evolve this value to the scale of the Z mass, $M_Z$, we obtain

$$\alpha_s(M_Z, n_f = 5) = 0.1156^{+0.0021}_{-0.0022},$$

where we have used the Mathematica package RunDec\cite{35} to obtain the above number (4 loop running, with the charm-quark mass equal to 1.6 GeV and the bottom-quark mass equal to 4.7 GeV).

Before concluding, let us mention that some studies for the $n_f = 2$ case have been presented in Refs.\cite{36,37}; and that previous analyses for $n_f = 0$, using perturbative expressions for the static energy at the two-loop level, include Refs.\cite{38,39}.

4. Conclusions and outlook

We have reviewed the determination of $\alpha_s$ from the comparison of lattice data with perturbative expressions for the QCD static energy. This determination was possible due to the recent advances in both the perturbative computation and the lattice evaluation of the static energy. It can be viewed as a nice example where a three loop computation is needed, and leads to an improved determination of a Standard Model parameter, as was expected to happen (see for instance Ref.\cite{40}). We have collected here (in Sec.\cite{2}) all the perturbative expressions needed to evaluate the static energy at the currently known accuracy, which were scattered over different papers. The final result of the current analysis is

$$\alpha_s(M_Z, n_f = 5) = 0.1156^{+0.0021}_{-0.0022}.$$


which uses lattice data in the 0.8-2.9 GeV energy range. The result is mostly compatible with other recent lattice determinations of \( \alpha_s \), although the central value is a bit lower, see Fig. 3 for a graphical comparison with recent lattice results. In Fig. 4 we illustrate where the result in Eq. (32) lays with respect to a few other recent non-lattice determinations. New lattice data for the static energy, also at shorter distances, will be available in the near future. Therefore, an updated result for \( \alpha_s \), with, in principle, reduced errors, can be expected to appear in the next few months.

**Appendix A. Color factors and beta function coefficients**

The color factors that appear in the paper read

\[
\begin{align*}
C_F &= T_F \frac{N_c^2 - 1}{N_c}; & C_A &= N_c; & T_F &= \frac{1}{2}; \\
\frac{d_{\text{abcd}} d_{\text{abcd}}}{N_A} &= \frac{N_c^3 + 6N_c}{48}; & \frac{d_{\text{abcd}} d_{\text{abcd}}}{N_A} &= \frac{18 - 6N_c^2 + N_c^4}{96N_c^2}; \\
\frac{d_{\text{abcd}} d_{\text{abcd}}}{N_A} &= \frac{N_c^4 + 36N_c^2}{24},
\end{align*}
\]  

(A.1)

where \( N_c \) is the number of colors.
We define the beta function as
\[ \alpha_s \beta(\alpha_s) = \frac{d \alpha_s}{d \ln \nu} = -\frac{\alpha_s^2}{2\pi} \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^n \beta_n = -2\alpha_s^2 \left[ \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \cdots \right], \]
(A.2)

where
\[ \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \quad \beta_1 = \frac{34}{3} C_A^2 - \frac{20}{3} C_A n_f T_F - 4C_F n_f T_F, \]
(A.3)
\[ \beta_2 = \frac{2857}{54} C_A^3 + \left( -\frac{1415}{27} C_A^2 - \frac{205}{9} C_A C_F + 2C_F^2 \right) n_f T_F \]
+ \left( \frac{158}{27} C_A + \frac{44}{9} C_F \right) n_f^2 T_F^2, \quad \beta_3 = \left( \frac{150653}{486} - \frac{44}{9} \zeta(3) \right) C_A^4 + \left( \frac{136}{3} \zeta(3) - \frac{39143}{81} \right) C_A n_f T_F \]
+ \left( \frac{7073}{243} - \frac{656}{9} \zeta(3) \right) C_A^2 C_F n_f T_F \]
+ \left( \frac{352}{9} \zeta(3) - \frac{4204}{27} \right) C_A C_F^2 n_f T_F \]
\[ + 46C_F^2 n_f T_F + \left( \frac{224}{9} \zeta(3) + \frac{7930}{81} \right) C_A^2 n_f^2 T_F^2 \]
+ \left( \frac{448}{9} \zeta(3) + \frac{17152}{243} \right) C_A C_F n_f^2 T_F^2 \]
+ \left( \frac{1352}{27} - \frac{704}{9} \zeta(3) \right) C_F^2 n_f^2 T_F^2 + \frac{424}{243} C_A n_f^3 T_F^3 \]
+ \frac{1232}{243} C_F n_f^3 T_F^3 \]
+ \frac{512}{9} \zeta(3) - \frac{1664}{3} \zeta(3) \right) n_f \frac{d^{abcd} d^{abcd}}{N_A} \]
+ \left( \frac{512}{3} \zeta(3) - \frac{704}{9} \right) n_f^2 \frac{d^{abcd} d^{abcd}}{N_A} + \left( \frac{704}{3} \zeta(3) - \frac{80}{9} \right) \frac{d^{abcd} d^{abcd}}{N_A}. \]
(A.6)

Acknowledgments

It is a pleasure to thank Alexei Bazavov, Nora Brambilla, Péter Petreczky, Joan Soto, and Antonio Vairo for collaboration on the work reported in this review. This work is supported by the Swiss National Science Foundation (SNF) under the Sinergia grant number CRSII2_141847_1.

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Fig. 4. Comparison of the result for $\alpha_s(M_Z)$ in Eq. (32) with a few other recent non-lattice $\alpha_s$ determinations. We include results from $\tau$ decays (Boito et al.\textsuperscript{45}; Abbas et al.\textsuperscript{46,47}; Caprini et al.\textsuperscript{48}; Pich\textsuperscript{49}), thrust (Abbate et al.\textsuperscript{50}; Gehrmann et al.\textsuperscript{51}), and parton distribution function (PDF) fits (ABM11\textsuperscript{52}, MSTW\textsuperscript{53}, NNPDF\textsuperscript{54}; note that in this case the error bars do not include effects from unknown higher-order perturbative corrections), along with the PDG average\textsuperscript{55}. 

Fig. 4 shows a comparison of the result for $\alpha_s(M_Z)$ in Eq. (32) with several recent non-lattice $\alpha_s$ determinations. The results include those from $\tau$ decays (Boito et al.\textsuperscript{45}; Abbas et al.\textsuperscript{46,47}; Caprini et al.\textsuperscript{48}; Pich\textsuperscript{49}), thrust (Abbate et al.\textsuperscript{50}; Gehrmann et al.\textsuperscript{51}), and parton distribution function (PDF) fits (ABM11\textsuperscript{52}, MSTW\textsuperscript{53}, NNPDF\textsuperscript{54}; note that in this case the error bars do not include effects from unknown higher-order perturbative corrections), along with the PDG average\textsuperscript{55}. 

- PDG'12 average
- NNPDF (Ball et al.'11)
- MSTW (Martin et al.'09)
- ABM11 (Alekhin et al.'12)
- thrust (Gehrmann et al.'12)
- thrust (Abbate et al.'10)
- $\tau$ decay FOPT (Pich'11)
- $\tau$ decay CIPT (Pich'11)
- $\tau$ decay BCIPT (Caprini et al.'11)
- $\tau$ decay BRGSPT (Abbas et al.'12)
- $\tau$ decay CIPT (Boito et al.'12)
- $\tau$ decay FOPT (Boito et al.'12)
- BBGPSV (this review)