Electroweak two-loop Sudakov logarithms
for on-shell fermions and bosons

W. Beenakker and A. Werthenbach

1) Theoretical Physics, Univ. of Nijmegen, P.O. Box 9010, NL-6500 GL Nijmegen, The Netherlands
2) Deutsches Elektronen-Synchrotron, DESY Zeuthen, Platanenallee 6, D-15738 Zeuthen, Germany

Abstract

We calculate the virtual electroweak Sudakov (double) logarithms at one- and two-loop level for arbitrary on-shell/on-resonance particles in the Standard Model. The associated Sudakov form factors apply in a universal way to arbitrary non-mass-suppressed electroweak processes at high energies, although this universality has to be interpreted with care. The actual calculation is performed in the temporal Coulomb gauge, where the relevant contributions from collinear-soft gauge-boson exchange are contained exclusively in the self-energies of the external on-shell/on-resonance particles. In view of the special status of the time-like components in this gauge, a careful analysis of the asymptotic states of the theory is required. From this analysis we derive an all-order version of the Goldstone-boson Equivalence Theorem without the need for finite compensation factors. By exploiting conditions obtained from non-renormalization requirements, which are a consequence of our choice of gauge, we show that the Sudakov corrections can be extracted through a combination of energy derivatives and projections by means of external sources. We observe that the Standard Model behaves dynamically like an unbroken theory in the Sudakov limit, in spite of the fact that the explicit particle masses are needed at the kinematical (phase-space) level while calculating the Sudakov form factors.

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†wimb@sci.kun.nl
‡Anja.Werthenbach@desy.de
1 Introduction

At the next generation of colliders center-of-mass energies will be reached that largely exceed the electroweak scale. For instance, the energy at a future linear $e^+e^-$ collider is expected to be in the TeV range \[1\]. At these energies one enters the realm of large perturbative corrections. Even the effects arising from weak corrections are expected to be of the order of 10\% or more \[2\]–\[8\], i.e. just as large as the well-known electromagnetic corrections. In order not to jeopardize any of the high-precision studies at these high-energy colliders, it is therefore indispensable to improve the theoretical understanding of the radiative corrections in the weak sector of the Standard Model (SM). In particular this will involve a careful analysis of effects beyond first order in the perturbative expansion in the (electromagnetic) coupling $\alpha = e^2/(4\pi)$.

The dominant source of radiative corrections at TeV-scale energies is given by logarithmically enhanced effects of the form $\alpha^n \log^m(M^2/s)$ for $m \leq 2n$, involving particle masses $M$ well below the collider energy $\sqrt{s}$. A natural way of controlling the theoretical uncertainties would therefore consist in a comprehensive study of these large logarithms, taking into account all possible sources (i.e. ultraviolet, soft, and collinear). The potentially most important electroweak corrections are the so-called Sudakov logarithms $\propto \alpha^n \log^{2n}(M^2/s)$, arising from collinear-soft singularities \[9\]. It should be noted, however, that for pure fermionic final states (numerical) cancellations can take place between leading and subleading logarithms \[10\]. For on-shell bosons in the final state, the Sudakov logarithms in general tend to be dominant \[3, 8\].

Over the last few years various QCD-motivated methods have been applied to predict the electroweak Sudakov logarithms to all orders in perturbation theory \[11, 12, 13\]. The methods vary in the way that the QCD-motivated factorization and exponentiation properties are translated to the electroweak theory. This is caused by the fact that the electroweak theory is a spontaneously broken theory with two mass scales in the gauge-boson sector, whereas QCD is basically a single-scale theory. The main debate therefore focusses on the question “to what extent does the SM behave like an unbroken theory at high energies?” In fact, we already know that the transition from QCD to electroweak theory does not come without surprises. In Ref. \[14\] it was shown that the Bloch–Nordsieck cancellation between virtual and real collinear-soft gauge-boson radiation \[15\] is violated in the SM as soon as initial- or final-state particles carry an explicit weak charge (isospin) and summation over the partners within an $SU(2)$ multiplet is not performed. At an electron–positron collider, for instance, the weak isospin of the initial-state particles is fixed by the
accelerator and the Bloch–Nordsieck theorem is in general violated for left-handed initial states, even for fully inclusive cross-sections. The resulting electroweak effects can be very large, exceeding the QCD corrections for energies in the TeV range. With this in mind, explicit calculations of Sudakov corrections at two-loop level are needed to resolve any ambiguity in the translation from QCD to SM. Up to know the explicit two-loop calculations have been performed for pure fermionic processes, like $e^+e^- \rightarrow f\bar{f}$ [16] and fermion-pair production by an $SU(2) \times U(1)$-singlet source [17, 18].

In this paper we complete our previous analysis [16] of virtual Sudakov logarithms for fermions by extending the explicit two-loop calculation to scalar particles as well as transverse and longitudinal gauge bosons. By means of this explicit calculation we try to establish to what extent the SM behaves like an unbroken theory at high energies. Like in Ref. [16], we perform the calculation in the (temporal) Coulomb gauge, exploiting the fact that the Sudakov form factors are exclusively contained in the self-energies of the particles. A detailed description of the Coulomb-gauge method for massive particles, including a discussion of the asymptotic states, is presented in Sects. 2 and 3.

For longitudinal gauge bosons we carefully study the effects of the broken nature of the SM on the asymptotic states. This will result in a special, particularly simple form of the Goldstone-boson Equivalence Theorem where the longitudinal degrees of freedom can be substituted by the corresponding Goldstone-boson degrees of freedom without the need for finite compensation factors. Also in the transverse neutral gauge-boson sector, with its mixing between the $Z$ boson and the photon, a careful reanalysis of the asymptotic states beyond lowest order in perturbation theory is needed. By exploiting conditions obtained from non-renormalization requirements, which are a consequence of our choice of gauge, we will show that the Sudakov corrections can be extracted through a combination of energy derivatives and projections by means of external sources. In Sects. 4 and 5 we explicitly perform the one- and two-loop calculations of the Sudakov form factors. These explicit calculations, combined with the special properties of the Coulomb gauge, enable us to identify the similarities and differences between the SM and unbroken (single-scale) theories like QED and QCD.

2 The Coulomb gauge

In order to facilitate the calculation of the one- and two-loop Sudakov logarithms, we work in the Coulomb gauge for both massless and massive gauge bosons [16]. In the Coulomb gauge the
gauge-fixing Lagrangian for the gauge-boson mass eigenstates $V = W^\pm, Z, \gamma$ is given by
\[ L_{GF} = -\frac{\lambda}{2} \sum_{V=W^\pm,Z,\gamma} \left( \left( \partial_\mu - \frac{n \cdot \partial}{n^2} n_\mu \right) V^\dagger,\mu \right) \left( \left( \partial_\nu - \frac{n \cdot \partial}{n^2} n_\nu \right) V^\nu \right), \] (1)
with the temporal gauge vector $n^\mu = (1, 0, 0, 0)$. As a result of the gauge choice, the massive gauge bosons ($W^\pm$ and $Z$) will mix at lowest-order level with the corresponding would-be Goldstone bosons ($\phi^\pm$ and $\chi$), defined through the SM Higgs doublet
\[ \Phi(x) = \left( \frac{1}{\sqrt{2}} [v + H(x) + i\chi(x)] \right) \] (2)
with vacuum expectation value $\langle \Phi \rangle = \left( \frac{v}{\sqrt{2}} \right)$ and hypercharge $Y_\phi = 1$. This has to be contrasted with the covariant $R_\xi$ gauges, which by construction do not exhibit such mixing at lowest order. Selecting the bilinear interactions in the $W - \phi$ and $Z - \chi$ sectors, we obtain the following relevant pieces of Lagrangian in the Coulomb gauge:
\[ L_{W-\phi}^{\text{bilinear}} = (\partial^\mu W^+_\mu)(\partial^\nu W^-_\nu) - (\partial_\mu W^+_\mu)(\partial^\nu W^-_\nu) + (\partial_\mu \phi^+)(\partial^\mu \phi^-) + M_w^2 W^+_\mu W^-_\mu \]
\[ + \ i M_w \left[ (\partial^\mu W^+_\mu) \phi^- - (\partial^\mu W^-_\mu) \phi^+ \right] - \lambda \left[ \left( \partial_\mu - \frac{n \cdot \partial}{n^2} n_\mu \right) W^+ \right] \left[ \left( \partial_\nu - \frac{n \cdot \partial}{n^2} n_\nu \right) W^- \right] \] (3)
\[ L_{Z-\chi}^{\text{bilinear}} = \frac{1}{2} (\partial_\mu Z_\mu Z_\mu) - \frac{1}{2} (\partial_\mu Z_\nu Z_\nu) + \frac{1}{2} (\partial_\mu \chi)(\partial^\mu \chi) + \frac{1}{2} M_z^2 Z_\mu Z^\mu \]
\[ - \ M_z (\partial_\mu Z_\mu \chi - \frac{\lambda}{2} \left[ \left( \partial_\mu - \frac{n \cdot \partial}{n^2} n_\mu \right) Z_\mu \right] \left[ \left( \partial_\nu - \frac{n \cdot \partial}{n^2} n_\nu \right) Z^\nu \right] \] (4)
Hence the lowest-order interaction matrix in the charged-boson sector can be written as
\[ \begin{pmatrix} -i \left( k^2 - M_w^2 \right) \delta^{\mu\nu} - k^\mu k^\nu + \lambda (k^\mu - k_0 n^\mu) (k^\nu - k_0 n^\nu) & \pm i M_w k^\mu \\
\pm i M_w k^\nu & i k^2 \end{pmatrix} \]
\[ = \begin{pmatrix} W^{\pm,\mu}, k & W^{\pm,\nu}, k & W^{\pm,\mu}, k & W^{\pm,\nu}, k & \phi^{\pm}, k & \phi^{\pm}, k \\
\phi^{\pm}, k & W^{\pm,\nu}, k & \phi^{\pm}, k & W^{\pm,\nu}, k \end{pmatrix}, \] (5)
where the ± occurring in the first matrix of Eq. (5) correspond to $W^\pm$.3
The propagators in the Coulomb gauge are now obtained by inverting the interaction matrix and taking the limit \( \lambda \to \infty \):

\[
\begin{pmatrix}
\mu & \nu & \mu & \nu \\
\mu & \nu & \nu & \mu
\end{pmatrix}
\begin{pmatrix}
P_{w^\pm w^\pm}^{\nu \mu} & P_{w^\pm \phi^\pm}^{\nu \mu} \\
P_{\phi^\pm w^\pm}^{\nu \mu} & P_{\phi^\pm \phi^\pm}^{\nu \mu}
\end{pmatrix}
= \begin{pmatrix}
-\delta^\mu_\rho & 0 \\
0 & -1
\end{pmatrix}.
\]

(6)

This leads to the explicit form of the lowest-order propagators

\[
P_{w^\pm w^\pm}^{\mu \nu} = P_{w^\pm w^\pm}(k, M_w) = \frac{-i}{k^2 - M_{w^\pm}^2 + i \epsilon} \left( g_{\mu \nu} + \frac{k_\mu k_\nu}{k^2} - k_0 \frac{k_\mu n_\nu + k_\nu n_\mu}{k^2} \right)
\]

\[
P_{w^\pm \phi^\pm}^{\mu \nu} = \mp M_\mu(k, M_w) = \frac{\mp i M_w}{k^2 - M_{w^\pm}^2 + i \epsilon} \frac{k_0}{k^2} n_\mu
\]

\[
P_{\phi^\pm w^\pm}^{\mu \nu} = \mp M_\nu(k, M_w) = \frac{\mp i M_w}{k^2 - M_{w^\pm}^2 + i \epsilon} \frac{k_0}{k^2} n_\nu
\]

\[
P_{\phi^\pm \phi^\pm}^{\mu \nu} = \frac{i}{k^2 - M_{w^\pm}^2 + i \epsilon} \left( 1 + \frac{M_{w^\pm}^2}{k^2} \right),
\]

(7)

In the neutral \(Z - \chi\) sector the propagators are given by

\[
P_{Z Z}^{\mu \nu} = P_{Z Z}(k, M_Z) = \frac{-i}{k^2 - M_Z^2 + i \epsilon} \left( g_{\mu \nu} + \frac{k_\mu k_\nu}{k^2} - k_0 \frac{k_\mu n_\nu + k_\nu n_\mu}{k^2} \right)
\]

\[
P_{Z \chi}^{\mu} = -i M_\mu(k, M_Z) = \frac{M_Z}{k^2 - M_Z^2 + i \epsilon} \frac{k_0}{k^2} n_\mu
\]

\[
P_{\chi Z}^{\nu} = i M_\nu(k, M_Z) = \frac{-M_Z}{k^2 - M_Z^2 + i \epsilon} \frac{k_0}{k^2} n_\nu
\]

\[
P_{\chi \chi}^{\nu} = \frac{i}{k^2 - M_Z^2 + i \epsilon} \left( 1 + \frac{M_Z^2}{k^2} \right),
\]

(8)

and for the photon we obtain

\[
P_{\gamma \gamma}^{\mu \nu} = P_{\gamma \gamma}(k, 0) = \frac{-i}{k^2 + i \epsilon} \left( g_{\mu \nu} + \frac{k_\mu k_\nu}{k^2} - k_0 \frac{k_\mu n_\nu + k_\nu n_\mu}{k^2} \right).
\]

(9)

The properties of the above-given propagators, like the relation between \(P_{Z Z}^{\mu \chi}\) and \(P_{\chi Z}^{\mu \nu}\) etc., follow from the hermiticity of the bilinear Lagrangians (3) and (4).
The power of choosing the Coulomb gauge lies in the fact that in the kinematical region of interest the gauge-boson propagators become effectively transverse:

\[ P_{\mu\nu}(k, M) = -i \frac{\vec{k}^2 \varepsilon_{\mu\nu} + k_\mu k_\nu - k_0 (k_\mu n_\nu + n_\mu k_\nu)}{\vec{k}^2 (k^2 - M^2 + i\epsilon)} = \frac{-i}{k^2 - M^2 + i\epsilon} \left[ Q_{\mu\nu}(k) - \frac{k^2}{k^2} n_\mu n_\nu \right]. \quad (10) \]

The tensor

\[ Q_{\mu\nu}(k) = -\sum_{\lambda=\pm} \epsilon_{\mu}(k, \lambda) \epsilon^{*}_{\nu}(k, \lambda) \quad (11) \]

is the polarization sum for the transverse helicity states, characterized by

\[ \epsilon(k, \pm) \cdot n = 0 \quad , \quad \bar{\epsilon}(k, \pm) \cdot \vec{k} = 0 \quad , \quad \bar{\epsilon}(k, \pm) \cdot \bar{\epsilon}(k, \mp) = 0 \quad \text{and} \quad \bar{\epsilon}(k, \pm) \cdot \bar{\epsilon}(k, \pm) = 1. \quad (12) \]

Therefore the gauge bosons are effectively transverse if \( k^2 \ll \vec{k}^2 \), which is the case for collinear gauge-boson emission at high energies \( (k^2 \propto M^2 \text{ and } \vec{k}^2 \approx k_0^2 \gg M^2) \). As a result of the effective transversality, the virtual Sudakov logarithms originating from vertex, box etc. corrections are suppressed as long as the two defining conditions for Sudakov corrections are met, i.e. all kinematical invariants of the process under investigation have to be of the same order as the initial-state centre-of-mass (CM) energy squared and the lowest-order matrix element should not be suppressed by powers of \( M/k_0 \) to start with.

Hence, all virtual Sudakov logarithms are contained exclusively in the self-energies of the external on-shell particles (external wave-function factors) \[14, 15\] or the self-energies of any intermediate particle that happens to be effectively on-shell.\[2\] The latter is, for instance, needed for the production of near-resonance unstable particles. In that case the leading contribution can be determined by employing the so-called pole scheme \[21\] in the leading-pole approximation, which restricts the calculation to the on-shell residue belonging to the unstable particle that is close to its mass-shell. For the explicit formulation of this approximation as well as its subtleties we refer to the literature \[22\]. The elegance of the Coulomb-gauge method lies in the fact that, once all self-energies to all on-shell/on-resonance SM particles have been calculated, the prediction of the Sudakov form factor for an arbitrary electroweak process becomes more or less straightforward. It should be noted, however, that for an electroweak process like \( e^+e^- \rightarrow 4f \) it is in general not correct to assume universality and merely calculate the Sudakov form factors.

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1. We will come back to that in sect. 4.1, once we have established all the necessary ingredients.
2. Note that similar simplifications can probably be obtained equally well by working in an axial gauge, see for instance Ref. 20 for massless particles.
for the external particles (i.e. the six fermions). Depending on the final state and the kinematical configuration, the process $e^+e^- \rightarrow 4f$ can have different near-resonance subprocesses \[23\] (like $e^+e^- \rightarrow W^+W^- \rightarrow 4f$ or $e^+e^- \rightarrow ZZ \rightarrow 4f$). In that case the Sudakov correction factor is given by the wave-function factors of the near-resonance intermediate particles rather than the four final-state fermions. The reason for this is that the invariant mass of those intermediate particles is close to being on-shell and therefore not of the same order as the CM energy squared. The subsequent decay of the intermediate particles into the final-state fermions does not involve a large invariant mass and will as such not give rise to Sudakov logarithms.

We finally note that the relevant self-energies for the calculation of the Sudakov logarithms involve the exchange of collinear-soft gauge bosons, including their potential mixing with the corresponding would-be Goldstone bosons. The collinear-soft exchange of fermions and ghosts leads to suppressed contributions, since the propagators of these particles do not have the required pole structure. The fermion propagators $\propto 1/[p - m + i\epsilon]$ are not affected by the gauge choice and therefore lack the additional $1/\vec{k}^2$ poles, whereas the ghost propagators $\propto 1/\vec{k}^2$ lack the $1/[k^2 - M^2 + i\epsilon]$ poles. As we will see later, both poles are required for obtaining the Sudakov logarithms.

3 The Coulomb gauge: asymptotic states and external wave-function factors in the Sudakov limit

The calculation of the external wave-function factors for fermions is non-trivial \[24\], but due to the absence of mixing between different fermions no major complications arise in the Coulomb gauge. For massive gauge bosons, however, the mixing with the corresponding component of the Higgs doublet introduces an additional complication \[25\].

3.1 The charged-boson sector

Let us start off by considering the $W$ boson and the would-be Goldstone boson $\phi$. For a proper description of the on-shell $W$ bosons we need the asymptotic $W^{as}$ field, which generates the asymptotic $W$-boson states. It will have to be defined in terms of the interacting $W$ and $\phi$ fields:

$$W^{\pm,as}_\mu(x) = Z_w^{-\frac{1}{2}} W^{\pm}_\mu(x) \pm i \delta Z_1 \frac{\partial_\mu \phi^{\pm}(x)}{M_W} + \delta Z_{w,n} n_\mu n \cdot W^{\pm}(x) + \delta Z_2 \frac{\partial_\mu \partial \cdot W^{\pm}(x)}{M_W^2},$$

in such a way that the free-field propagators are retrieved for $W^{as}$ in the on-shell limit. This fixes the wave-function factors $Z$ and $\delta Z$ in terms of the self-energies of the interacting fields. The full
expression in Eq. (13) is in fact only needed to guarantee that the asymptotic vector field satisfies the physical polarization condition

$$\partial^\mu W_{\mu}^{\pm, \text{as}}(x) = 0$$

in the weak limit. For all practical purposes, i.e. calculating $S$-matrix elements, the asymptotic state will be connected to a source term $\epsilon^\mu(k)$ and it is sufficient to consider

$$W_{\mu}^{\pm, \text{as}}(x) \rightarrow Z_w^{-\frac{1}{2}} W_{\mu}^{\pm}(x) + \delta Z_{W,n} n \cdot W^{\pm}(x),$$

since the two terms containing $\partial_{\mu}$ vanish owing to $\epsilon(k) \cdot k = 0$. In the remainder of this section we will denote those irrelevant terms proportional to $k_{\mu}$ by ‘…’. Note that for transverse $W$ bosons ($W_T$) the second term in Eq. (13) will also vanish, since $\epsilon_T(k) \cdot n = \epsilon_T^0(k) = 0$. For longitudinal $W$ bosons ($W_L$) the full expression (15) will be of relevance, since $\epsilon_L^\mu(k)$ lies in the plane spanned by $k_{\mu}$ and $n_{\mu}$:

$$\epsilon_L^\mu(k) \equiv \epsilon^\mu(k, 0) = \frac{k^0}{M_w} k_{\mu} - \frac{M_w}{|k|} n_{\mu}. \quad (16)$$

Consequently, the wave-function factors for transverse and longitudinal $W$ bosons are different in the Coulomb gauge due to the special status of the time-like components.

In order to actually determine the wave-function factors $Z_w^{-\frac{1}{2}}$ and $\delta Z_{W,n}$ we study the Fourier Transform (FT) of the asymptotic-field propagator

$$\text{FT} \langle 0 | T \left( W_{\mu}^{+, \text{as}}(x) W_{\nu}^{-, \text{as}}(y) \right) | 0 \rangle =$$

$$= \text{FT} \langle 0 | T \left( Z_w^{-1} W_{\mu}^{+(x)}(x) W_{\nu}^{-}(y) + \delta Z_{W,n} Z_w^{-\frac{1}{2}} n_{\mu} n \cdot W^{+(x)}(x) W_{\nu}^{-}(y) \right. \right.$$

$$+ Z_w^{-\frac{1}{2}} \delta Z_{W,n}^{\nu} W_{\mu}^{+(x)}(x) n_{\nu} n \cdot W^{-(y)} + \delta Z_{W,n}^2 n_{\mu} n \cdot W^{+(x)} n_{\nu} n \cdot W^{-(y)} \right) | 0 \rangle + \ldots. \quad (17)$$

To further specify the above we need to gain knowledge about the dressed propagators for the interacting $W$ fields. Using the conventions introduced in the previous section, the interaction matrix can be written to all orders in perturbation theory as

$$\left( -i \left[ g^{\mu\nu}(k^2 - M_w^2) - k^{\mu} k^{\nu} + \lambda (k^{\mu} - k^0 n^{\mu})(k^{\nu} - k^0 n^{\nu}) - \Sigma_{\mu\nu}^{\mu\nu_{W}} \right] \pm i k^{\mu} M_w + i \Sigma_{\mu\phi W}^{\mu} \right) \pm i k^{\nu} M_w + i \Sigma_{\nu\phi W}^{\nu} + i [k^2 + \Sigma_{\phi}] \right).$$
Here \( \Sigma_{WW}^{\mu\nu} = \Sigma_{W\pm W\pm}^{\mu\nu} \) is the \( W \)-boson self-energy, \( \Sigma_{W^\pm \phi^\pm}^{\mu} = \Sigma_{\phi^\pm W^\pm}^{\mu} \) is the mixed \( W \)-boson/would-be Goldstone boson self-energy, and \( \Sigma_{\phi^\phi} = \Sigma_{\phi^\pm \phi^\pm} \) is the would-be Goldstone boson self-energy. For simplicity we will suppress the arguments (like \( k \) and \( n \)) of these self-energy functions. The Dyson-resummed (dressed) propagator matrix is obtained by inverting this interaction matrix and taking the limit \( \lambda \to \infty \) (see previous section). In order to make the derivation of these dressed propagators as compact as possible we now use the transverse tensor \( Q^{\mu\nu} \) as defined in Eq. (11) and introduce the space-like momentum

\[
q^\mu \equiv k^\mu - k^0 n^\mu. \tag{18}
\]

These quantities have the following useful properties

\[
n_\mu Q^{\mu\nu} = q_\mu Q^{\mu\nu} = Q^{\mu\nu} Q_\nu = 0
\]

\[
Q^{\mu\nu} Q_{\nu\rho} = Q^{\mu\rho}, \quad n \cdot q = 0, \quad q^2 = -\vec{k}^2. \tag{19}
\]

Next we use Lorentz covariance and decompose the \( W \)-boson self-energy according to

\[
\Sigma_{WW}^{\mu\nu} = Q^{\mu\nu} \Sigma_{WW, g} + q_\mu q_\nu + (q^\mu n^\nu + q^\nu n^\mu) \Sigma_{WW, m} + n^\mu n^\nu \Sigma_{WW, n}, \tag{20}
\]

bearing in mind that we have two independent four-vectors, \( k \) and \( n \), at our disposal. Similarly the mixed \( W \)-boson/would-be Goldstone boson self-energy can be written as

\[
\Sigma_{W^\pm \phi^\pm}^{\mu} = \pm q^\mu \Sigma_{W^+ \phi^+, q} \pm n^\mu \Sigma_{W^+ \phi^+, n} \tag{21}
\]

by virtue of the hermiticity of the interaction Lagrangian. Analogously the dressed-propagator matrix can be written in the generic form

\[
\begin{pmatrix}
W^\pm_{\mu} & W^\pm_{\nu} \\
\phi^\pm_{\mu} & W^\pm_{\nu}
\end{pmatrix} =
\begin{pmatrix}
W^\pm_{\mu} & W^\pm_{\nu} \\
\phi^\pm_{\mu} & \phi^\pm_{\nu}
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\imath [A_{WW} q_{\mu} q_{\nu} + B_{WW} (q_{\mu} n_{\nu} + n_{\mu} q_{\nu}) + D_{WW} n_{\mu} n_{\nu}] & \imath [E_{W^\pm \phi^\pm} q_{\mu} + F_{W^\pm \phi^\pm} n_{\mu}] \\
\imath [E_{\phi^\pm W^\pm} q_{\nu} + F_{\phi^\pm W^\pm} n_{\nu}] & \imath G_{\phi^\phi}
\end{pmatrix}.
\]
Making use of the Ward identities in the Coulomb gauge, which state that Green’s functions with a single gauge-boson line contracted with the corresponding space-like vector $\sqrt{\lambda} q^\mu$ should vanish for $\lambda \to \infty$, we immediately obtain $B_{WW} = C_{WW} = E_{w^\pm w^\pm} = E_{\phi^\pm w^\pm} = 0$. For the other coefficients we have to solve separate equations for the transverse sector,

$$A_{WW} = \left[k^2 - M_w^2 - \Sigma_{WW,n}\right]^{-1},$$

as well as for the longitudinal/scalar sector,

$$\begin{pmatrix}
\frac{k_0^2}{\Sigma_{WW,n}} \pm [k_0 M_w + \Sigma_{w^\pm w^\pm, n}] & k^2 + \Sigma_{\phi\phi} \\
\frac{k_0 M_w + \Sigma_{w^\pm w^\pm, n}}{\Sigma_{WW,n}} & \frac{k^2}{\Sigma_{\phi\phi}}
\end{pmatrix}
\begin{pmatrix}
-D_{WW} & F_{w^\pm w^\pm} \\
F_{\phi^\pm w^\pm} & G_{\phi\phi}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ (23)

Up to now the discussion has been completely general. At this point the calculation can be simplified by exploiting the special properties of the self-energies in the Sudakov limit ($k^2, M_w^2 \ll k_0^2$). In this limit Eq. (23) can be approximated by

$$\begin{pmatrix}
\frac{k_0^2}{\Sigma_{WW,n}} \pm [k_0 M_w + \Sigma_{w^\pm w^\pm, n}] & k^2 + \Sigma_{\phi\phi} \\
\pm [k_0 M_w + \Sigma_{w^\pm w^\pm, n}] & k^2 + \Sigma_{\phi\phi}
\end{pmatrix}
\begin{pmatrix}
-D_{WW} & F_{w^\pm w^\pm} \\
F_{\phi^\pm w^\pm} & G_{\phi\phi}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ (24)

as a result of the fact that the self-energy $\Sigma_{WW,n} = O(k^2, M_w^2)$ is suppressed with respect to $k_0^2$. The other self-energies are of the same order as the corresponding lowest-order terms, i.e. $\Sigma_{w^\pm w^\pm, n} = O(k_0 M_w)$, $\Sigma_{\phi\phi} = O(k^2, M_w^2)$ and neither self-energy contains inverse powers of $k^2$ or $M_w^2$ (see Appendix [1]), as required by hermiticity and analyticity. The resulting solutions for the dressed propagator functions read

$$G_{\phi\phi} = \frac{k_0^2}{k_0^2 \left[k^2 + \Sigma_{\phi\phi}\right] - \left[k_0 M_w + \Sigma_{w^\pm w^\pm, n}\right]^2},$$

$$D_{WW} = \frac{G_{\phi\phi}}{k_0^2} \left[k^2 + \Sigma_{\phi\phi}\right],$$

$$F_{w^\pm w^\pm} = \mp \frac{G_{\phi\phi}}{k_0^2} \left[k_0 M_w + \Sigma_{w^\pm w^\pm, n}\right] = F_{\phi^\pm w^\pm}.$$ (25)

Note the explicit hierarchy in the Sudakov limit, $|D_{WW}| \ll |F_{w^\pm w^\pm}| \ll |G_{\phi\phi}|$. This will play an important role later in the derivation of the Goldstone-boson Equivalence Theorem.

With the help of the dressed propagator functions and Eq. (19), the asymptotic-field propagator in Eq. (17) can be expressed as

$$\text{FT} \left\{ 0 | T \left( W_{\mu^+}^{a\sigma}(x) W_{\nu^\sigma}^{a\tau}(y) \right) | 0 \right\} = -i Z_w^{-1} A_{WW} Q_{\mu\nu} - i \left[Z_w^{-2} + \delta Z_{WW}\right] D_{WW} n_\mu n_\nu + \ldots.$$ (26)
The wave-function factors are subsequently obtained from the free-field constraint

\[ i \left( k^2 - M_{W, \text{phys}}^2 \right) \text{FT} \left\langle 0 \left| T \left( W^+_{\mu, \text{as}}(x) W^-_{\nu, \text{as}}(y) \right) \right| 0 \right\rangle \bigg|_{k^2 = M_{W, \text{phys}}^2} \]

\[ \equiv - \sum_{\lambda = \pm, 0} \epsilon_{\mu}(k, \lambda) \epsilon_{\nu}^*(k, \lambda) = \left[ Q_{\mu\nu}(k) - \frac{k^2}{k^2} n_\mu n_\nu + \ldots \right] \bigg|_{k^2 = M_{W, \text{phys}}^2}, \quad (27) \]

where \( M_{W, \text{phys}} \) is the physical pole mass of the \( W \) boson. The \( W \)-boson mass does not receive any corrections in the Sudakov limit, i.e. \( M_{W, \text{phys}} = M_W \), since a non-zero Sudakov correction to the mass would imply that either the pole mass or the mass counterterm of the \( W \) boson would become energy dependent. This special property of the Coulomb gauge can be understood in another way by realizing that in covariant \( R_\xi \) gauges no Sudakov logarithms occur in the self-energies. Consequently, the masses of the particles will not be shifted by the Sudakov corrections. This "non-renormalization" condition has far-reaching consequences for the determination of the wave-function factors. Applied to the above-given dressed propagators it leads to two identities:

\[ \Sigma_{WW, g} \propto (k^2 - M_W^2), \quad \Sigma_{\phi\phi} - \frac{2 M_W}{k_0} \Sigma_{\phi^+, n} - \frac{1}{k_0^2} \Sigma_{\phi^+, n}^2 \propto (k^2 - M_W^2), \quad (28) \]

which hold to all orders. In Appendix B we have verified explicitly that these identities hold at the one-loop level. Making use of the "non-renormalization" conditions we finally obtain for the wave-function factors:

\[ Z_{W}^{-1} = \frac{k^2 - M_W^2 - \Sigma_{WW, g}}{k^2 - M_W^2} \bigg|_{k^2 = M_W^2} = 1 - \frac{\partial \Sigma_{WW, g}}{\partial k^2} \bigg|_{k^2 = M_W^2} \]

\[ = 1 - \frac{i}{2 k_0} \epsilon_T, \mu(k) \left\{ \frac{\partial}{\partial k_0} \left[ i \Sigma_{WW}^{\mu\nu} \right] \right\} \epsilon_T, \nu(k) \bigg|_{k^2 = M_W^2} \]

\[ Z_{\phi}^{-1} = \frac{k_0^2 (k^2 - M_W^2) + k_0^2 \Sigma_{\phi\phi} - 2 M_W k_0 \Sigma_{\phi^+, n} - \Sigma_{\phi^+, n}^2}{k_0^2 (k^2 - M_W^2)} \bigg|_{k^2 = M_W^2} = 1 + \frac{\partial \Sigma_{\phi\phi}}{\partial k^2} \bigg|_{k^2 = M_W^2} \]

\[ = 1 - \frac{i}{2 k_0} \left\{ \frac{\partial}{\partial k_0} \left[ i \Sigma_{\phi\phi} \right] \right\} \bigg|_{k^2 = M_W^2} \]

\[ Z_{W}^{-\frac{1}{2}} + \delta Z_{W, n} = Z_{\phi}^{-\frac{1}{2}} \frac{k_0 M_W}{k_0 M_W + \Sigma_{W^+ + \phi^+, n}} \bigg|_{k^2 = M_W^2}. \quad (29) \]

Here \( Z_{\phi}^{-\frac{1}{2}} \) is the wave-function factor that enters the definition of an asymptotic state for the would-be Goldstone bosons:

\[ \phi^{\pm, \text{as}}(x) = Z_{\phi}^{-\frac{1}{2}} \phi^{\pm}(x) \quad (30) \]
with

\[ -i (k^2 - M_W^2) \langle 0 | T (\phi^{+..as}(x) \phi^{-..as}(y)) | 0 \rangle \bigg|_{k^2=M_W^2, \text{phys.}} \equiv 1. \quad (31) \]

In the first expression of Eq. (29) we have used the fact that the contraction with the transverse polarization vectors (sources) \( \epsilon_\mu^T(k) \) and \( \epsilon^{\nu T}_\nu(k) \) projects on \(-g_{\mu \nu} \), since \( \epsilon_\mu^T(k) \cdot k = \epsilon_T(k) \cdot n = 0 \) and \( \epsilon_T(k) \cdot \epsilon^{\nu T}_\nu(k) = -1 \), whereas the derivative \( \frac{1}{2k_0} \frac{\partial}{\partial k_0} \) projects on the on-shell wave-function factor by virtue of the “non-renormalization” condition. The drastic simplification of the second expression is due to the fact that the leading \( k^2 \) dependence is contained exclusively in \( \Sigma_{\phi \phi} \), since \( \Sigma_{W^++\phi^+, n} = \mathcal{O}(k_0 M_W) \). In fact, if we would add the unit sources for scalar particles, this second expression would bear a close similarity to the first one. As we will see later in this section and in Sect. 1, such a projection by means of sources and energy derivatives occurs in all bosonic and fermionic sectors.

Now we have all the ingredients to consider the \( S \)-matrix elements in the charged-boson sector. First some conventions. In the following we will denote the amputated Green’s functions generically by open circles. Only one external line will be given explicitly, namely the one that belongs to the asymptotic state under investigation. These asymptotic states are represented by double lines and the corresponding dressed propagators by hatched circles. We start with the asymptotic \( W^- \)-boson fields, which give rise to two diagrams. The first diagram involves the pure \( W^- \)-boson propagator:

\[ i (k^2 - M_W^2) \epsilon^\nu(k) \bigg|_{k^2=M_W^2} = \]

\[ i (k^2 - M_W^2) \left[ Z_W^{\frac{1}{2}} \epsilon^\nu(k) + \delta Z_{W,n} \, n^\nu \epsilon_0(k) \right] \bigg|_{k^2=M_W^2}. \]

Here we have left the polarization state unspecified, bearing in mind that for transverse \( W \) bosons \( \epsilon_T(k) \cdot n = 0 \) and for longitudinal \( W \) bosons \( \epsilon_L(k) \cdot n \approx k_0/M_W \) in the high-energy limit. Upon
amputation of the external legs we find in the Sudakov limit

\[
W^\pm \begin{pmatrix}
\left( k^2 - M_W^2 \right) Q^\mu \nu \\
k^2 - M_W^2 - \Sigma_{WW,g}
\end{pmatrix} \left\{ \frac{\left( k^2 - M_W^2 \right) \left[ k^2 + \Sigma_{\phi} \right] n^\mu n_\nu}{k_0^2 \left[ k^2 + \Sigma_{\phi} \right] - \left[ k_0 M_W + \Sigma_{W+\phi^+,n} \right]^2} \right\} \left[ Z_W^{-\frac{1}{2}} \epsilon^\nu(k) + \delta Z_{W,n} n^\nu \epsilon_0(k) \right] \bigg|_{k^2 = M_W^2}.
\]

From this general result we deduce that for transverse \( W \) bosons, with \( \epsilon_T^\nu(k) Q^\mu \nu = \epsilon_T^\nu(k) \) and \( \epsilon_T(k) \cdot n = 0 \), the contribution of Sudakov corrections simply amounts to multiplying each external transverse \( W \)-boson line of the matrix element by the factor \( Z_W^{-\frac{1}{2}} \). For longitudinal \( W \) bosons, with \( \epsilon_L^\nu(k) Q^\mu \nu = 0 \) and \( \epsilon_L(k) \cdot n \approx k_0/M_W \), one obtains a mass-suppressed contribution \( \propto M_W/k_0 \).

The second diagram involves the mixed \( W \)-boson/would-be Goldstone boson propagator:

\[
\phi^\pm \begin{pmatrix}
W^\pm, as \\
i (k^2 - M_W^2) \epsilon^\nu(k)
\end{pmatrix} \bigg|_{k^2 = M_W^2}.
\]

Upon amputation of the external legs we find in the Sudakov limit a vanishing contribution for transversely polarized \( W \) bosons and

\[
\phi^\pm \begin{pmatrix}
\left\{ \pm \frac{\left( k^2 - M_W^2 \right) \left[ k_0 M_W + \Sigma_{W+\phi^+,n} \right]}{k_0^2 \left[ k^2 + \Sigma_{\phi} \right] - \left[ k_0 M_W + \Sigma_{W+\phi^+,n} \right]^2} \right\} n^\nu \left[ Z_W^{-\frac{1}{2}} \epsilon_L^\nu(k) + \delta Z_{W,n} n^\nu \frac{k_0}{M_W} \right] \bigg|_{k^2 = M_W^2}.
\]

for longitudinally polarized \( W \) bosons. In other words, we find for longitudinal \( W \) bosons that the dominant contribution to any physical process originates from the amputated Green’s function where the amputated leg is a would-be Goldstone boson \( \phi \), provided of course that the matrix element is not mass-suppressed to start with (which is anyhow one of the basic defining conditions for the Sudakov corrections). The other contribution, where the amputated leg is a \( W \) boson contracted with the temporal gauge vector, is mass-suppressed. This is in fact the Goldstone-boson Equivalence Theorem [20], which is hence obtained quite naturally in the Coulomb-gauge

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approach as a result of the explicit mixing between gauge bosons and would-be Goldstone bosons.

Owing to the same mixing between gauge bosons and would-be Goldstone bosons, we have to address one more question before being able to wrap up the discussion of the Equivalence Theorem. Namely, we have to show that the $S$-matrix element with the asymptotic state $\phi^{\pm,as}$ will not exhibit a leading contribution for the amputated Green’s function where the amputated leg is a $W$ boson. The dominant contribution again has to be the one where the amputated leg is a would-be Goldstone boson. For the mixed $S$-matrix element we find

$$ W^\pm \bullet = \cdot (-i) (k^2 - M^2_W) \bigg|_{k^2=M^2_W}^\pm = - \cdot (-i) (k^2 - M^2_W) Z^{-\frac{1}{2}}_\phi \bigg|_{k^2=M^2_W} $$

and amputating the legs leads in the Sudakov limit to

$$ W^\pm \mu \bigg\{ \mp \frac{(k^2 - M^2_W)}{k^2} \frac{k_0}{\Sigma_{\phi^+}} \bigg[ k_0 M_W + \Sigma_{\phi^+} \bigg\} = n^\mu \frac{Z^{-\frac{1}{2}}_\phi}{k^2=M^2_W} $$

This indeed leads to a mass-suppressed contribution. Similarly we obtain for the diagram involving the pure scalar propagator

$$ \phi^\pm = \cdot (-i) (k^2 - M^2_W) \bigg|_{k^2=M^2_W}^\pm = \cdot (-i) (k^2 - M^2_W) Z^{-\frac{1}{2}}_\phi \bigg|_{k^2=M^2_W} $$

and amputation yields in the Sudakov limit

$$ \bigg\{ \frac{k^2}{k^2 + \Sigma_{\phi^+}} \bigg[ - k_0 M_W + \Sigma_{\phi^+} \bigg\} Z^{-\frac{1}{2}}_\phi \bigg|_{k^2=M^2_W}^\pm = \bigg\{ Z^{-\frac{1}{2}}_\phi \bigg|_{k^2=M^2_W} $$

This is not only the leading contribution, but identical to the leading contribution that we obtained from the asymptotic $W^\pm,as$ field. Applying the high-energy Sudakov limit to the Coulomb gauge, we do not only find the Equivalence Theorem $W^\pm,as \rightarrow \pm C \phi^{\pm,as}$ to hold for massive particles, but we find a very special case of the Equivalence Theorem, i.e. $C = 1$ to all orders in perturbation

\[\text{In covariant } R_\xi \text{ gauges there is by construction no mixing between gauge bosons and would-be Goldstone bosons at lowest order}\]
theory. This implies the identity of the two particles rather than mere proportionality.

We close this discussion in the charged-boson sector with the following observations and conclusions. For transversely polarized external $W$ bosons the mixing with the $\phi$ field vanishes and the Sudakov correction factor amounts to multiplying each external transverse $W$-boson line of the matrix element by the factor $Z_W^\perp$. For longitudinally polarized external $W^\pm$ bosons the correction factor is not equal to $Z_W^\perp$, instead the dominant Sudakov correction factor amounts to multiplying each external longitudinal gauge-boson line of the matrix element by the factor $\pm Z_\phi^\perp$ (provided that the lowest-order matrix element is not mass-suppressed to start with). This statement is a special case of the Equivalence Theorem in the sense that the longitudinal $W$ bosons can be substituted by their would-be Goldstone boson counterparts, $\phi$, in the high-energy limit. Hence, we have effectively returned to the situation before spontaneous symmetry breaking where the Goldstone bosons represent the physical degrees of freedom. In this respect we could say that the SM behaves (dynamically) like an unbroken theory in the Sudakov limit, in spite of the fact that we cannot neglect the $W/\phi$ mass at the kinematical (phase-space) level while calculating the Sudakov correction factors.

3.2 The neutral-boson sector

In the neutral-boson sector of the SM we have to deal with four particles. The physical Higgs boson can be treated in a trivial way, since it does not mix with any of the other neutral particles. In the Sudakov limit the corresponding external wave-function factor is simply given by

$$Z_{H}^{-1} = 1 + \frac{\partial \Sigma_{HH}}{\partial k^2} \bigg|_{k^2=M_H^2} = 1 - \frac{i}{2k_0} \left\{ \frac{\partial}{\partial k_0} \left[ i \Sigma_{HH} \right] \right\} \bigg|_{k^2=M_H^2},$$

with $\Sigma_{HH}$ the Higgs-boson self-energy. The Sudakov correction factor then amounts to multiplying each external Higgs-boson line of the matrix element by the factor $Z_H^{-1}$. The remaining three particles are the photon ($\gamma$), the $Z$ boson, and the corresponding would-be Goldstone boson $\chi$.

At lowest order the situation is equivalent to the charged-boson sector, since in that case only the $Z$ boson and $\chi$ mix. However, beyond lowest order all three particles mix, which adds an extra level of complication. Before presenting the corresponding asymptotic states, we first address the

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4In fact, the $C = 1$ property is a general high-energy feature of the Coulomb gauge, even for non-double-logarithmic corrections. This has to be contrasted with covariant $R_\xi$ gauges where the Goldstone-boson fields occur in the gauge-fixing Lagrangian. This in general necessitates the introduction of a finite (renormalization-scheme- and $\xi$-dependent) factor $C_{\text{mod}} \neq 1$ in order to link the unphysical would-be Goldstone boson with the physical asymptotic Goldstone-boson state beyond lowest order [27, 31]. It requires a very special renormalization scheme to get $C = 1$ in that case [28, 30].
propagator functions in the Sudakov limit. They can be derived using the methods developed for
the charged-boson sector. Again only the $A$, $F$ and $G$ propagator functions survive, yielding in
the Sudakov limit:

\[
A_{\gamma\gamma} = \frac{k^2 - M_Z^2 - \Sigma_{zz,\gamma}^Z}{k^2 - M_Z^2 - \Sigma_{zz,\gamma}^Z} \quad \text{(33)}
\]

in the transverse sector, and

\[
A_{\gamma Z} = \frac{\Sigma_{\gamma Z,\gamma}}{k^2 - M_Z^2 - \Sigma_{zz,\gamma}^Z} = A_{Z\gamma}
\]

\[
A_{Z Z} = \frac{k^2 - \Sigma_{\gamma Z,\gamma}}{k^2 - M_Z^2 - \Sigma_{zz,\gamma}^Z} \quad \text{(33)}
\]

in the longitudinal/scalar sector. The functions occurring in these expressions have been obtained
by decomposing the various self-energies in the same way as prescribed for the charged-boson
sector. These self-energies have the following properties, by virtue of the hermiticity of the
interaction Lagrangian: $\Sigma_{\gamma Z} = \Sigma_{Z\gamma}$, $\Sigma_{\gamma \chi} = -\Sigma_{\gamma \chi}$, and $\Sigma_{Z \chi} = -\Sigma_{Z \chi}$.

The “non-renormalization” conditions for the photon and $Z$-boson masses give rise to the identities

\[
\left[ k^2 - M_Z^2 - \Sigma_{zz,\gamma}^Z \right] \left[ k^2 - \Sigma_{\gamma \gamma,\gamma}^Z - \Sigma_{\gamma Z,\gamma}^Z \right] \propto \Sigma_{\gamma Z}^Z \quad \text{(34)}
\]

These identities have been verified explicitly in Appendix \[B\] at the one-loop level. Note that the
photon-mass condition only applies to the transverse sector, since the on-shell photon only has
transverse degrees of freedom. At this point we note that there will be, in fact, one more “non-renormalization” condition, related to the electromagnetic charge. We will come back to this later.

According to the observations made in the charged-boson case, the wave-function factors in the transverse and longitudinal/scalar sectors are best treated separately. We start with the longitudinal sector. Bearing in mind that the asymptotic on-shell photon field \( A_{\mu}^{as} \) is transverse and decouples from the asymptotic \( Z_{\mu}^{as} \) and \( \chi^{as} \) fields, the asymptotic states can be defined as:

\[
A_{\mu}^{as}(x) = 0 \\
Z_{\mu}^{as}(x) \rightarrow Z^{-\frac{i}{2}} Z_{\mu}^{as}(x) + \delta Z_{\mu,n} n \cdot Z_L(x) \\
\chi^{as}(x) = Z^{\frac{i}{2}} \chi(x). \tag{36}
\]

The additional derivative terms needed for a proper asymptotic \( Z \)-boson state have again been left out, since these terms will not contribute to the physical \( S \)-matrix elements. By applying the “non-renormalization” conditions, the wave-function factors can be determined from the free-field constraints in the usual way:

\[
Z_{\chi}^{-1} = 1 + \frac{\partial \Sigma_{\chi\chi}}{\partial k^2} \bigg|_{k^2=M_Z^2} = 1 - \frac{i}{2k_0} \left\{ \frac{\partial}{\partial k_0} \left[ i \Sigma_{\chi\chi} \right] \right\} \bigg|_{k^2=M_Z^2} \\
Z^{-\frac{i}{2}} + \delta Z_{\mu,n} = Z^{-\frac{i}{2}} \frac{k_0 M_Z}{k_0 M_Z - i \Sigma_{\chi\chi,n}} \bigg|_{k^2=M_Z^2}. \tag{37}
\]

The relation between the \( S \)-matrix elements for outgoing longitudinal \( Z \) bosons and outgoing would-be Goldstone bosons \( \chi \) reads in the Sudakov limit (with \( N = \gamma, Z \))

\[
\begin{align*}
&= N \chi^{as} \bigg[ -i Z^{\frac{i}{2}} \bigg] \bigg|_{k^2=M_Z^2} + \gamma_{\mu} \left[ Z^{\frac{i}{2}} n^\mu \left( \frac{M_Z}{k_0} \right) \frac{i \Sigma_{\chi\chi,n}}{k_0 M_Z} \right] \bigg|_{k^2=M_Z^2} \\
&\quad - \chi^{as} \left[ (-i)^2 \left( k^2 - M_Z^2 \right) \right] \bigg|_{k^2=M_Z^2} + \chi^{as} \left[ (-i)^2 \left( k^2 - M_Z^2 \right) \right] \bigg|_{k^2=M_Z^2}.
\end{align*}
\]

\[\text{Eq. (36) is valid in double-logarithmic approximation. In more general situations one should replace } Z_{\mu} \text{ by } Z_{\mu} + A_{\mu} \Sigma_{\chi\chi,n} / [k_0^2 + \Sigma_{\chi\chi,n}] \text{ on the right-hand side of Eq. (36)}.
\]
Hence, for outgoing longitudinal $Z$ bosons the dominant Sudakov correction factor amounts to multiplying each outgoing longitudinal $Z$-boson line of the matrix element by the factor $-i Z_L^2$ (provided that the matrix element is not mass-suppressed to start with). For incoming longitudinal $Z$ bosons this Sudakov factor becomes $i Z_L^2$. So, just like in the charged-boson sector a special case of the Equivalence Theorem is obtained, $Z_{\gamma, \mu}^{as} \rightarrow -/+ i \chi^{as}$ for outgoing/incoming particles.

In the transverse sector the situation is quite different, since now the gauge bosons mix explicitly. The corresponding terms in the asymptotic states do not involve derivatives and therefore play an explicit role in the $S$-matrix elements. The easiest procedure to deal with this $Z-\gamma$ mixing is to first diagonalize the propagator matrix in the transverse sector according to

$$
\begin{pmatrix}
Z_{\gamma, \mu}^{diag} \\
A_{\gamma, \mu}^{diag}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta(k^2) & \sin \theta(k^2) \\
-\sin \theta(k^2) & \cos \theta(k^2)
\end{pmatrix}
\begin{pmatrix}
Z_{\gamma, \mu} \\
A_{\gamma, \mu}
\end{pmatrix},
$$

(38)

with

$$\theta(k^2) = \frac{1}{2} \arctan \left( \frac{2 \Sigma_{\gamma Z, g}}{M_Z^2 + \Sigma_{zz, g} - \Sigma_{\gamma \gamma, g}} \right).$$

(39)

Subsequently the asymptotic states are defined in terms of these diagonal interaction states through the relation

$$
\begin{pmatrix}
Z_{\gamma, \mu}^{as} \\
A_{\gamma, \mu}^{as}
\end{pmatrix}
= \begin{pmatrix}
C_{zz}^{-\frac{1}{2}} \cos \theta(M_Z^2) & 0 \\
0 & C_{\gamma \gamma}^{-\frac{1}{2}} \cos \theta(0)
\end{pmatrix}
\begin{pmatrix}
Z_{\gamma, \mu}^{diag} \\
A_{\gamma, \mu}^{diag}
\end{pmatrix},
$$

(40)

Using the “non-renormalization” conditions in Eq. (35) and the transverse part $\propto Q_{\mu \nu}$ of the free-field constraint, one obtains

$$\tan \theta(0) = \frac{\Sigma_{\gamma Z, g}}{M_Z^2 + \Sigma_{zz, g}} \bigg|_{k^2=0}, \quad C_{\gamma \gamma}^{-1} = 1 - \frac{\partial}{\partial k^2} \left[ \Sigma_{\gamma \gamma, g} + \frac{\Sigma_{\gamma Z, g}}{k^2 - M_Z^2 - \Sigma_{zz, g}} \right] \bigg|_{k^2=M_Z^2},$$

(41a)

$$\tan \theta(M_Z^2) = \frac{\Sigma_{\gamma Z, g}}{M_Z^2 - \Sigma_{\gamma \gamma, g}} \bigg|_{k^2=M_Z^2}, \quad C_{zz}^{-1} = 1 - \frac{\partial}{\partial k^2} \left[ \Sigma_{zz, g} + \frac{\Sigma_{\gamma Z, g}}{k^2 - \Sigma_{\gamma \gamma, g}} \right] \bigg|_{k^2=M_Z^2}. \quad (41b)$$

This leads to the following $S$-matrix elements for transverse neutral gauge bosons in the Sudakov limit (with $N = \gamma, Z$):

$$N \begin{array}{c}
\gamma^{as} \\
\nu
\end{array} i k^2 \epsilon_T^\nu(k) \bigg|_{k^2=0} = C_{\gamma \gamma}^{\frac{1}{2}} \epsilon_T^\nu(k) \left[ \begin{array}{c}
\gamma \\
\mu
\end{array} \bigg| - \tan \theta(0) \bigg|\begin{array}{c}
\gamma \\
\mu
\end{array} \right|_{k^2=0}$$
for the photon and

\[
N \frac{Z^{as}}{\nu} i(k^2 - M_Z^2) \epsilon^\nu_T(k) \bigg|_{k^2 = M_Z^2} = \frac{1}{2} C_{\gamma \gamma}^{Z \nu} \epsilon^\nu_T(k) \left[ Z \mu - \tan(\theta(M_Z^2)) \frac{\epsilon^\nu_T}{k^2 = M_Z^2} \right]
\]

for the Z boson. So \( C_{\gamma \gamma}^{Z \nu} \) and \( C_{\gamma \gamma}^{Z \nu} \) act as overall normalization factors, whereas \( \tan(\theta(0)) \) and \( \tan(\theta(M_Z^2)) \) account for the fact that the asymptotic neutral gauge-boson states have been obtained from a mixture of (interacting) photonic and Z-boson components. With the help of Eq. (35) we can bring the expressions for \( C_{\gamma \gamma}^{Z \nu} \) and \( C_{\gamma \gamma}^{Z \nu} \) in the familiar form of a projection by means of sources:

\[
C_{\gamma \gamma}^{Z \nu} = 1 + \tan^2(\theta(0)) - \frac{i}{\kappa_0} \epsilon_T,\mu(k) \left\{ i \frac{\partial}{\partial \kappa_0} \left[ \Sigma_{\gamma \gamma}^{\mu \nu} - 2 \tan(\theta(0)) \Sigma_{\gamma Z}^{\mu \nu} + \tan^2(\theta(0)) \Sigma_{ZZ}^{\mu \nu} \right] \right\} \epsilon^\nu_T(k) \bigg|_{k^2 = 0}
\]

\[
C_{\gamma \gamma}^{Z \nu} = 1 - \frac{i}{\kappa_0} \epsilon_T,\mu(k) \left\{ i \frac{\partial}{\partial \kappa_0} \left[ \Sigma_{ZZ}^{\mu \nu} + 2 \tan(\theta(M_Z^2)) \Sigma_{\gamma Z}^{\mu \nu} + \tan^2(\theta(M_Z^2)) \Sigma_{\gamma \gamma}^{\mu \nu} \right] \right\} \epsilon^\nu_T(k) \bigg|_{k^2 = M_Z^2}
\]

As promised we come back to the “non-renormalization” condition for the electromagnetic charge, which follows automatically from the requirement that the electromagnetic charge should not become energy-dependent. Combining the S-matrix element for the photon and this “non-renormalization” condition for the electromagnetic charge\(^6\), we obtain

\[
C_{\gamma \gamma}^{\frac{1}{2}} = 1 - \tan(\theta(0)) \frac{\sin \theta_w}{\cos \theta_w}, \tag{43}
\]

where \( \theta_w \) is the weak mixing angle. This condition will be crucial for limiting the calculation of the Sudakov correction factors to the calculation of derivatives of self-energies. An explicit check of Eq. (43) at the one-loop level can be found in Appendix [3].

### 4 Electroweak one-loop Sudakov logarithms

To establish the formalism that will be used in the following sections we are presenting here the one-loop calculation of the Sudakov logarithms in the Coulomb gauge [16]. For arbitrary on-shell/on-resonance SM particles our calculations are in agreement with the well known one-loop

\(^{\text{6}}\)This condition applies to the electromagnetic, non-isospin part of the coupling to the amputated Green’s functions, e.g. for couplings to fermions this consists of the complete \( f f \gamma \) coupling and the \( - Q_f \gamma \mu \sin \theta_w / \cos \theta_w \) part of the \( f f Z \) coupling.
contributions to the external wave-function factors $Z = 1 + \delta Z$. These one-loop contributions will be denoted by $\delta Z^{(1)}$.

4.1 The fermionic self-energy at one-loop level

As mentioned above, in order to determine the Sudakov logarithms in $s$-channel processes like $e^+e^- \rightarrow f\bar{f}$ ($f \neq e, \nu_e$), one has to calculate the external self-energies (i.e. the wave-function factors) of all four fermions involved in the process. Consider to this end the fermionic one-loop self-energy $\Sigma_f^{(1)}(p, n, M_1)$, originating from the emission of a gauge boson $V_1$ with loop-momentum $k_1$ and mass $M_1$ from an effectively massless fermion $f$ with momentum $p$:

$$- i \Sigma_f^{(1)}(p, n, M_1) = f(p) \rightarrow \gamma \rightarrow f_1(p-k_1) \rightarrow f(p)$$

Again $n$ is the unit vector in the time direction, which enters by virtue of using the Coulomb gauge. In the high-energy limit the fermion mass in the numerator of the fermion propagator can be neglected with respect to $p$ and similarly the contribution involving a mixed gauge-boson – Goldstone-boson propagator can be discarded. The self-energy $\Sigma_f^{(1)}$ then contains an odd number of $\gamma$-matrices, leading to the following natural decomposition in terms of the two possible structures $\gamma$ and $\not{p}$:

$$\Sigma_f^{(1)}(p, n, M_1) \approx \left[ \gamma \Sigma_p^{(1)}(p, n, M_1) + \not{p} \Sigma_n^{(1)}(p, n, M_1) \right] e^2 \Gamma_f^{V_{1V_1}} ,$$

with the proportionality factor of the second term being dictated by the “non-renormalization” condition for the fermion mass. The coupling factor $\Gamma_{f_{1V_1}}$ is defined according to

$$\Gamma_{f_{1V_1}} = V_{f_{1V_1}} - \gamma_5 A_{f_{1V_1}} ,$$

where $V_{f_{1V_1}}$ and $A_{f_{1V_1}}$ are the vector and axial-vector couplings of the fermion $f$ to the exchanged gauge boson $V_1$. In our convention these coupling factors read

$$\Gamma_{f_{1\gamma}} = -Q_f, \quad \Gamma_{f_{1Z}} = \frac{(1 - \gamma_5) I_f^3 - 2 Q_f \sin^2 \theta_w}{2 \cos \theta_w \sin \theta_w}, \quad \Gamma_{f_{1W}} = \frac{(1 - \gamma_5)}{2 \sqrt{2} \sin \theta_w} .$$

Whenever possible the fermion mass will be neglected. The massive case (e.g. top-quarks) can be treated in a similar way in view of the “non-renormalization” condition for the fermion mass (see the discussion in Sect. 3).
Here $I_f^3$ is the quantum number corresponding to the third component of the weak isospin, $e Q_f$ is the electromagnetic charge, and $\theta_w$ is the weak mixing angle. We have denoted the isospin partner of $f$ by $f'$.

The contribution to the external wave-function factor now amounts to multiplying the self-energy by $i/p$ on the side where it is attached to the rest of the scattering diagram and by the appropriate fermion source on the other side. Finally the square root should be taken of the external wave-function factor, i.e. the one-loop contribution should be multiplied by the usual factor $1/2$. For an initial-state fermion, for example, one obtains

$$\frac{1}{2} \frac{i}{p} \left[ -i \Sigma_f^{(1)}(p, n, M_1) \right] u_f(p) \approx \frac{e^2}{2} \Gamma_{f_f V_i}^2 \left[ \Sigma_f^{(1)}(n \cdot p, m_f^2, M_1) + 2 \Sigma_n^{(1)}(n \cdot p, m_f^2, M_1) \right] u_f(p)$$

$$\equiv \frac{1}{2} \delta Z_f^{(1)}(V_1) u_f(p),$$

(47)

where $m_f$ is the mass of the external fermion and $\sqrt{s} = 2p_0$ is the center-of-mass energy of the process $e^+e^- \rightarrow f\bar{f}$. This contribution to the external wave-function factor $Z_f = 1 + \delta Z_f$ can be extracted from the full fermionic self-energy by applying a projection by means of sources (see Sect. 3)

$$\delta Z_f^{(1)}(V_1) = \frac{i}{2p_0} \bar{u}_f(p) \left\{ \frac{\partial}{\partial p_0} \left[ -i \Sigma_f^{(1)}(p, n, M_1) \right] \right\} u_f(p)$$

$$= \frac{i}{2p_0} \bar{u}_f(p) \left\{ \frac{\partial}{\partial p_0} \int \frac{d^4k_1}{(2\pi)^4} (ie \gamma_\mu \Gamma_{f_f V_i}) \frac{i}{(p - k_1) - m_f} + i\epsilon (ie \gamma_\mu \Gamma_{f_f V_i}) P^{\mu\nu}(k_1, M_1) \right\} u_f(p)$$

$$\approx -e^2 \Gamma_{f_f V_i}^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{4p_\mu p_\nu}{[(p - k_1)^2 - m_f^2 + i\epsilon]^2} P^{\mu\nu}(k_1, M_1),$$

(48)

where we have made use of the Dirac equation for the spinor $u_f(p)$, its normalization condition $\bar{u}_f(p) \gamma^0 u_f(p) = 2p_0$, as well as

$$\frac{\partial}{\partial p_\mu} \frac{1}{p} = -\frac{1}{p} \gamma^\mu \frac{1}{p}.$$  

(49)

Note also that the loop-momentum $k_1$ has been neglected in the numerator of the fermion propagator, since only collinear-soft gauge-boson momenta will give rise to the Sudakov logarithms. Therefore it comes as no big surprise that we observe an eikonal factor in the integrand of the last integral in Eq. (48). The mass of the fermion inside the loop, $m_f$, is at best of the order of the

---

8For an outgoing fermion one obtains $\frac{1}{2} \bar{u}_f(p) \delta \tilde{Z}_f^{(1)}$, where $\delta \tilde{Z}_f^{(1)}$ can be derived from $\delta Z_f^{(1)}$ by reversing the sign in front of $\gamma_5$. 

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Z-boson mass (for the top-quark). At the leading-logarithmic level it therefore only enters as an independent mass scale if the exchanged gauge boson is a photon [i.e. $m_{f_1} = m_f$, as implemented in the last step of Eq. (48)], where the fermion mass is needed for the regularization of the collinear singularities. In the last step of Eq. (48) we have also exploited the fact that $\Delta Z_f^{(1)}$ will be multiplied on the right by $u_f(p)$, so writing $\Gamma_{f_1V_1}$ or its projection on left/right-handed chiral couplings $(V_{f_1V_1} \pm A_{f_1V_1})^2$ is effectively equivalent.

Making use of the explicit form of the gauge-boson propagator in the Coulomb gauge, given in Eq. (10), the numerator of the last integral in Eq. (48) can be simplified as follows

$$i \left[ k_1^2 - M_1^2 + i\epsilon \right] 4 p_\mu p_\nu P^{\mu\nu}(k_1, M_1) \approx \frac{4}{k_1^2} \left[ (p \cdot k_1)^2 - 2 k_{10} p_0 (p \cdot k_1) \right]$$

$$= \frac{1}{k_1^2} \left[ (p - k_1)^2 - p^2 - k_1^2 \right]^2 + \frac{4 k_{10} p_0}{k_1^2} \left[ (p - k_1)^2 - p^2 - k_1^2 \right]. \quad (50)$$

As we will see below, in order to obtain two logarithms both the fermion and the gauge-boson propagator are needed. Now $p^2 = m_f^2$ can be neglected and the terms $\propto k_1^2$ and $(p - k_1)^4$ will kill one of the types of denominators. Thus we are left with

$$i \left[ k_1^2 - M_1^2 + i\epsilon \right] 4 p_\mu p_\nu P^{\mu\nu}(k_1, M_1) \approx \frac{4 k_{10} p_0}{k_1^2} \left[ (p - k_1)^2 - m_f^2 \right]. \quad (51)$$

Therefore

$$\Delta Z_f^{(1)}(V_1) \approx -e^2 \Gamma_{f_1V_1}^{\mu\nu} \int \frac{d^4 k_1}{(2\pi)^4} \frac{4 k_{10} p_0}{k_1^2} \frac{-i}{[(p - k_1)^2 - m_f^2 + i\epsilon] [k_1^2 - M_1^2 + i\epsilon]}. \quad (52)$$

Having two canonical momenta at our disposal, i.e. $p$ and $n$, we define the following Sudakov parametrisation of the gauge-boson loop-momentum $k_1$:

$$k_1 = v_1 q + u_1 \bar{q} + k_{1\perp}, \quad (53)$$

with

$$p^\mu \equiv (E, \beta_f E, 0, 0), \quad \beta_f = \sqrt{1 - m_f^2/E^2}, \quad s = 4 E^2,$$

$$q^\mu = (E, E, 0, 0), \quad \bar{q}^\mu = (E, -E, 0, 0), \quad k_{1\perp}^\mu = (0, 0, \vec{k}_{1\perp}). \quad (54)$$

In terms of this parametrisation, the integration measure $d^4 k_1$, the invariants $(p \cdot k_1)$ and $k_1^2$,
and the gauge-boson energy $k_1^0$ read

$$d^4k_1 = \pi \frac{s}{2} dv_1 du_1 d\vec{k}_{1\perp}^2,$$

$$ (p \cdot k_1) = \frac{s}{4} \left[ v_1 (1 - \beta_f) + u_1 (1 + \beta_f) \right] \approx \frac{s}{2} \left( u_1 + \frac{m_f^2}{s} v_1 \right),$$

$$k_1^2 = s v_1 u_1 - \vec{k}_{1\perp}^2 \quad \text{and} \quad k_1^0 = \frac{\sqrt{s}}{2} (v_1 + u_1).$$

(55)

The term containing the fermion mass $m_f$ is needed for the exchange of photons only, regulating the collinear singularity at $u_1 = 0$. For the exchange of a massive gauge boson the mass $M_1$ will be the dominant collinear as well as infrared regulator.

The $v_1$-integration is restricted to the interval $0 \leq v_1 \leq 1$, as a result of the requirement of having poles in both hemispheres of the complex $u_1$-plane. The residue is then taken in the lower hemisphere in the pole of the gauge-boson propagator: $s v_1 u_1^{\text{res}} = \vec{k}_{1\perp}^2 + M_1^2 \equiv s v_1 y_1$. Finally, $\vec{k}_{1\perp}^2$ is substituted by $y_1$, with the condition $\vec{k}_{1\perp}^2 \geq 0$ translating into $v_1 y_1 \geq M_1^2/s$. The one-loop Sudakov contribution to $\delta Z_f$ now reads

$$\delta Z_f^{(1)}(V_1) \approx -\frac{\alpha}{\pi} \Gamma_{f f V_1} \int_0^\infty dy_1 \int_0^1 dv_1 \frac{\Theta(v_1 y_1 - M_1^2)}{(y_1 + \frac{m_f^2}{s} v_1)(v_1 + y_1)}$$

$$\approx -\frac{\alpha}{\pi} \Gamma_{f f V_1} \int_0^1 \frac{dy_1}{y_1} \int_{y_1}^1 \frac{dz_1}{z_1} K^{(1)}(s, m_f^2, M_1, y_1, z_1),$$

(56)

with the integration kernel $K^{(1)}$ given by

$$K^{(1)}(s, m_f^2, M_1, y_1, z_1) = \Theta(y_1 z_1 - M_1^2) \Theta(y_1 - \frac{m_f^2}{s} z_1).$$

(57)

Here we introduced the energy variable $z_1 = v_1 + y_1$ and made use of the fact that only collinear-soft gauge-boson momenta with $y_1 \ll z_1 \ll 1$ are responsible for the quadratic large-logarithmic effects. This can be read off directly from the first expression of Eq. (56), since for $v_1 \lesssim \mathcal{O}(y_1)$ the integrand of the $v_1$-integral does not exhibit a logarithmic $1/v_1$ type of evolution. Furthermore, one can use as rule of thumb that, in order to determine whether a certain term is negligible or not, the relevant kinematical region for quadratic large-logarithmic effects is given by “lower integration bound” $\ll$ integration variable $\ll$ “upper integration bound” (e.g. $M_1/\sqrt{s} \ll z_1 \ll 1$, $y_1 \ll z_1$, or $y_1 \gg M_1^2/s$). As a result, the gauge boson inside the loop is effectively on-shell and transversely polarized (see Eq. (10) with $k^2 \ll \vec{k}^2$ in the collinear regime). The same result can be obtained by means of the dispersion method. The dispersion method proceeds via the computation of the
absorptive part by applying the Cutkosky cutting rule, which effectively puts both the internal
gauge boson and fermion on-shell, whereas the external fermion becomes off-shell. Subsequently
the real part is obtained by using dispersion-integral (Cauchy-integral) techniques, turning the
internal fermion off-shell and allowing the external fermion to be on-shell.

The exchanged gauge boson can either be a massless photon (\(\gamma\)) or one of the massive weak
bosons (\(W^\pm\) or \(Z\)). The associated mass gap gives rise to distinctive differences in the two
types of contributions. Bearing in mind that the SM is not parity conserving and making use
of \((A.2)\) and \((A.3)\) we present the one-loop Sudakov correction factors for right- and left-handed
fermions/antifermions separately:

\[
\delta Z^{(1)}_{f_R}(\gamma) = \left[ \left( \frac{Y_R}{2} \right)^2 \right] L_\gamma(\lambda, m_f) = Q_f^2 L_\gamma(\lambda, m_f) ,
\]

\[
\delta Z^{(1)}_{f_L}(\gamma) = \left[ \left( \frac{Y_L}{2} \right)^2 + I_f^2 Y_f^L + \left( \frac{Y_f}{2} \right)^2 \right] L_\gamma(\lambda, m_f) = Q_f^2 L_\gamma(\lambda, m_f) ,
\]

\[
\delta Z^{(1)}_{f_R}(W) = 0 ,
\]

\[
\delta Z^{(1)}_{f_L}(W) = \frac{1}{2 \sin^2 \theta_w} L(M, M) ,
\]

\[
\delta Z^{(1)}_{f_R}(Z) = \frac{\sin^2 \theta_w}{\cos^2 \theta_w} \left( \frac{Y_R}{2} \right)^2 L(M, M) = \left[ \left( \frac{Y_f}{\cos \theta_w} \right)^2 - Q_f^2 \right] L(M, M) ,
\]

\[
\delta Z^{(1)}_{f_L}(Z) = \left[ \frac{\cos^2 \theta_w}{\sin^2 \theta_w} \left( I_f^3 \right)^2 - I_f^3 Y_f^L + \sin^2 \theta_w \cos^2 \theta_w \left( \frac{Y_L}{2} \right)^2 \right] L(M, M)
\]

\[
= \left[ \frac{1}{4 \sin^2 \theta_w} + \left( \frac{Y_f^L}{2 \cos \theta_w} \right)^2 - Q_f^2 \right] L(M, M) ,
\]

with

\[
L(M_1, M_2) = - \frac{\alpha}{4 \pi} \log \left( \frac{M_1^2}{s} \right) \log \left( \frac{M_2^2}{s} \right) ,
\]

\[
L_\gamma(\lambda, M_1) = - \frac{\alpha}{4 \pi} \left[ \log^2 \left( \frac{\lambda^2}{s} \right) - \log^2 \left( \frac{\lambda^2}{M_1^2} \right) \right] ,
\]

and \(\delta Z^{(1)}_{f_R} = \delta Z^{(1)}_{f_L}\) as well as \(\delta Z^{(1)}_{f_R} = \delta Z^{(1)}_{f_L}\) for all three gauge bosons. Note that these correction
factors are the same for incoming as well as outgoing particles. In Eq. \((58)\) \(Y_f^{R,L}\) denotes the right-
and left-handed hypercharge of the external fermion, which is connected to the third component
of the weak isospin \(I_f^3\) and the electromagnetic charge \(e Q_f\) through the Gell-Mann – Nishijima

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relation \( Q_f = I_f^3 + Y_f^{R,L} / 2 \). The parameter \( \lambda \) is the fictitious (infinitesimally small) mass of the photon needed for regularizing the infrared singularity at \( z_1 = 0 \). For the sake of calculating the leading Sudakov logarithms, the masses of the \( W \) and \( Z \) bosons can be represented by one generic mass scale \( M \).

In the process \( e^+ e^- \to f \bar{f} \) the one-loop correction factors presented in Eq. (58) contribute in the following way to the polarized matrix element, bearing in mind that at high energies the helicity eigenstates are equivalent to the chiral eigenstates:

\[
\mathcal{M}_{e_R^i e_L^j \to f_L^k f_R^l}^{1\text{-loop, sudakov}} = \frac{1}{2} \left[ \delta Z^{(1)}_{e_R^i} + \delta Z^{(1)}_{e_L^j} + \delta Z^{(1)}_{f_L^k} + \delta Z^{(1)}_{f_R^l} \right] \mathcal{M}_{e_R^i e_L^j \to f_L^k f_R^l}^{\text{born}},
\]

(61)

and similar expressions for the other possible helicity combinations.

As promised, we come back to two aspects of Sudakov logarithms in the Coulomb gauge that were anticipated in Sect. 2. First of all there was the question whether one could expect contributions to the Sudakov correction factor from self-energies with fermions or ghosts in the loop. We saw in this section that the \( 1/\vec{k}^2 \) part of the gauge-boson propagator in the Coulomb gauge is crucial for obtaining double logarithmic contributions. Obviously the fermion propagator does not possess such part. The ghost propagator does contain the required \( 1/\vec{k}^2 \) pole, but lacks the pole structure \( 1/(k^2 - M^2 + i\epsilon) \) and hence no contribution to the Sudakov correction factor can be obtained.

The second issue was the suppression of Sudakov logarithms originating from vertex corrections. To this end we consider the following vertex correction, where we assume for simplicity that the exchanged gauge boson as well as the incoming gauge boson are both photons and that the fermion is massless

\[
i \left[ k_2^2 + i\epsilon \right] p_1^\mu p_2^\nu P^{\mu\nu}(k_1, 0) = 4 E^2 \frac{x y}{(x + y)^2},
\]

(63)

where we have made use of the on-shell condition \( \vec{k}_{1\perp}^2 \approx 4 E^2 x y \). The remaining term will not lead to Sudakov logarithms since the numerator will kill both poles originating from the fermion
propagators. Hence we conclude that the piece of the gauge-boson propagator that would usually lead to Sudakov logarithms, \( i.e. k_0(k^\mu n^\nu + n^\mu k^\nu)/k^2 \), is effectively rendered inactive for vertex corrections by the \( g_{\mu\nu} \) part of the same gauge-boson propagator. The same argument holds for box or higher-point corrections.

4.2 The bosonic self-energies at one-loop level

As we have seen in Sect. 3, the transverse and longitudinal gauge bosons have to be treated separately. To all orders in perturbation theory the Sudakov correction factors for longitudinal \( W \) and \( Z \) bosons are given by \( Z_\phi^2 \) and \( Z_\chi^2 \), respectively (provided that the matrix element is not mass-suppressed to start with). These wave-function factors are obtained from the scalar \( \phi \) and \( \chi \) self-energies through the relations

\[
Z_{\phi}^{-1} = 1 + \frac{1}{2k_0} \frac{\partial \Sigma_{\phi\phi}}{\partial k_0} \big|_{k^2=M_W^2} \quad \text{and} \quad Z_{\chi}^{-1} = 1 + \frac{1}{2k_0} \frac{\partial \Sigma_{\chi\chi}}{\partial k_0} \big|_{k^2=M_Z^2}.
\]

The corresponding one-loop corrections can be calculated in a trivial way with the help of the (derivative) method described in Sect. 4.1.

Next we sketch the calculation of the transverse gauge-boson self-energies. We start with the charged sector (\( W^\pm \) bosons) and then move on to the neutral sector (\( Z \) bosons and photons). According to the discussion in Sect. 3, the Sudakov correction factor for transverse \( W \) bosons amounts to multiplying the corresponding external line of the matrix element by \( Z_{W}^{1/2} \). Recalling that

\[
Z_{W}^{-1} = 1 - \frac{i}{2k_0} \epsilon_{T,\mu}(k) \left\{ \frac{\partial}{\partial k_0} \left[ i \Sigma_{W,W}^{\mu\nu} \right] \right\} \epsilon_{T,\nu}^*(k) \big|_{k^2=M_W^2},
\]

the one-loop contribution to the external wave-function factor \( Z_W \equiv 1 + \delta Z_{W,T} \) can be extracted from the full one-loop \( W \)-boson self-energy by means of the projection

\[
\delta Z_{W,T}^{(1)} = \frac{i}{2k_0} \epsilon_{T,\mu}(k) \left\{ \frac{\partial}{\partial k_0} \left[ i \Sigma_{W,W}^{(1)}^{\mu\nu} \right] \right\} \epsilon_{T,\nu}^*(k) \big|_{k^2=M_W^2}.
\]

Note again that the transverse polarization vectors \( \epsilon_{T,\mu}(k) \) and \( \epsilon_{T,\nu}^*(k) \) project on \( -g_{\mu\nu} \). The vertex structures present in \( \Sigma_{W,W,g}^{(1)} \) will give rise to the usual eikonal factors, since we can neglect the loop-momentum with respect to the \( W \)-boson momentum \( k \). The rest of the calculation proceeds

\footnote{Recall that in the case of the self-energy \( p_\mu p_\nu g^{\mu\nu} = 0 \) and that therefore the terms relevant for Sudakov logarithms survive}

\footnote{The full (i.e. non-derivative) scalar self-energies can be found in Appendix B}
in the same way as worked out in Sect. 4.1 (for more details we refer to Appendix B), resulting in

\[ \delta Z_{W_T}^{(1)}(\gamma) = Q_w^2 L_\gamma(\lambda, M) \]  
\[ \delta Z_{W_T}^{(1)}(Z) = \cos^2 \theta_w \frac{L(M, M)}{\sin^2 \theta_w} = \left[ \frac{1}{\sin^2 \theta_w} - Q_w^2 \right] L(M, M) \]  
\[ \delta Z_{W_T}^{(1)}(W) = \frac{1}{\sin^2 \theta_w} L(M, M) \]

with \( L_\gamma(\lambda, M) \) and \( L(M, M) \) as defined in Eqs. (60) and (59), respectively.

We have applied these one-loop Sudakov corrections to the reactions \( e^+e^- \rightarrow W_T^+W_T^- \) and \( W_T^+W_L^- \) and found perfect agreement with the high-energy approximation in Ref. [3]. This indeed confirms the fundamental differences between transverse and longitudinal degrees of freedom.

In the neutral gauge-boson sector we have to follow a step-wise procedure in order to express everything in terms of derivatives. First of all we exploit the fact that for \( N_{1,2} = \gamma, Z \) the self-energies \( \Sigma_{N_{1,2},g} = O(k^2, M_Z^2) \) do not contain inverse powers of \( k^2 \) or \( M_Z^2 \), as required by analyticity. The higher-order terms in \( k^2 \) will therefore be suppressed in the Sudakov limit, leading to the decomposition

\[ \Sigma_{N_{1,2},g} = k^2 \Sigma'_{N_{1,2},g} + M_Z^2 C_{N_{1,2},g}, \]

with both \( \Sigma'_{N_{1,2},g} \) and \( C_{N_{1,2},g} \) being independent of \( k^2 \). Next we expand Eqs. (41a) and (41b) to one-loop:

\[ C_{\gamma\gamma}^{-1} \xrightarrow{1-\text{loop}} 1 - \frac{1}{2k_0} \frac{\partial \Sigma_{\gamma\gamma}^{(1)}}{\partial k_0} k^2 = 0 \equiv 1 - \Sigma_{\gamma\gamma}^{(1)} \equiv 1 - \delta C_{\gamma\gamma} \]  
\[ C_{ZZ}^{-1} \xrightarrow{1-\text{loop}} 1 - \frac{1}{2k_0} \frac{\partial \Sigma_{ZZ}^{(1)}}{\partial k_0} k^2 = M_Z^2 \equiv 1 - \Sigma_{ZZ}^{(1)} \equiv 1 - \delta C_{ZZ}. \]

Both self-energies can be calculated by means of the derivative method explained in Sect. 4.1, resulting in

\[ \Sigma_{ZZ}^{(1)} = -\frac{\cos \theta_w}{\sin \theta_w} \Sigma_{\gamma\gamma}^{(1)} = \frac{\cos^2 \theta_w}{\sin^2 \theta_w} \Sigma_{\gamma\gamma}^{(1)} = \cos^2 \theta_w \left[ \frac{2}{\sin^2 \theta_w} L(M, M) \right] \equiv \cos^2 \theta_w \Sigma_{33}^{(1)}. \]

Only collinear-soft gauge-boson exchange contributes to the Sudakov correction (see Sect. 4.1), hence only the non-abelian \( W^3 \) components of the external neutral gauge bosons participate. To calculate the Sudakov correction factors occurring in the \( S \)-matrix elements for neutral gauge
bosons, we need one more ingredient according to Sect. 3: the tangent of the running \( \gamma - Z \) mixing angle, \( \tan \theta(k^2) \), at \( k^2 = 0 \) and \( k^2 = M_Z^2 \). From the “non-renormalization” condition (II3) for the electromagnetic charge and Eq. (II3) we derive

\[
\tan \theta(0)_{1\text{-loop}} = C^{(1)}_{\gamma Z, g} = \frac{\cos \theta_w}{2 \sin \theta_w} \Sigma^{(1)}_{\gamma \gamma, g} = \frac{\cos \theta_w}{\sin \theta_w} L(M, M). \tag{70}
\]

By means of Eqs. (66) and (II3), \( \tan \theta(M_Z^2) \) can be written at one-loop as

\[
\tan \theta(M_Z^2)_{1\text{-loop}} = \Sigma^{(1)}_{\gamma Z, g} + C^{(1)}_{\gamma Z, g} = -\frac{\cos \theta_w}{\sin \theta_w} L(M, M). \tag{71}
\]

Thus, due to the non-renormalization condition Eq. (II3) we do not have to explicitly calculate the full \( \gamma - Z \) self energy. Instead it is sufficient to know the derivatives \( \Sigma'_{N_1 N_2, g} \) up to the relevant order.\(^{11}\)

Now we have all the necessary ingredients for calculating the Sudakov correction factors that enter the \( S \)-matrix elements for transverse neutral gauge bosons (see Sect. 3). To this end we replace the \( Z \)-boson and photon fields in the amputated Green’s functions by the unbroken gauge fields \( B \) [belonging to \( U(1)_Y \)] and \( W^3 \) [belonging to \( SU(2)_L \)]:

\[
A_\mu = \cos \theta_W B_\mu - \sin \theta_W W^3_\mu,
\]

\[
Z_\mu = \sin \theta_W B_\mu + \cos \theta_W W^3_\mu. \tag{72}
\]

In this way we obtain different multiplicative Sudakov correction factors \( Z^{\frac{1}{2}}_{N_T, B} \) and \( Z^{\frac{1}{2}}_{N_T, W^3} \) for the \( B \) and \( W^3 \) components of an (asymptotic) transverse neutral gauge boson \( N_T \). Writing as usual \( Z = 1 + \delta Z \), the corresponding one-loop corrections read

\[
\delta Z^{(1)}_{\gamma, B}(W) = \delta C^{(1)}_{\gamma \gamma} - 2 \frac{\sin \theta_w}{\cos \theta_w} \tan \theta(0)_{1\text{-loop}} = 0
\]

\[
\delta Z^{(1)}_{\gamma, Z}(W) = \delta C^{(1)}_{\gamma Z} + 2 \frac{\cos \theta_w}{\sin \theta_w} \tan \theta(M_Z^2)_{1\text{-loop}} = 0
\]

\[
\delta Z^{(1)}_{\gamma, W^3}(W) = \delta C^{(1)}_{\gamma \gamma} + 2 \frac{\cos \theta_w}{\sin \theta_w} \tan \theta(0)_{1\text{-loop}} = \frac{2}{\sin^2 \theta_w} L(M, M)
\]

\[
\delta Z^{(1)}_{\gamma, W^3}(W) = \delta C^{(1)}_{\gamma Z} - 2 \frac{\sin \theta_w}{\cos \theta_w} \tan \theta(M_Z^2)_{1\text{-loop}} = \frac{2}{\sin^2 \theta_w} L(M, M). \tag{73}
\]

\(^{11}\)For completeness we give in Appendix B the full (non-derivative) one-loop \( \gamma - Z \) self-energy, which is found to be in agreement with the results presented above.
This can be represented generically by
\[ \delta Z_{\text{N},B}(W) = 0 \]
\[ \delta Z_{\text{N},W^3}(W) = \frac{2}{\sin^2 \theta_w} L(M, M), \]
irrespective of the particular on-shell limit. In this respect the SM behaves (dynamically) like an unbroken theory in the Sudakov limit, with the unbroken gauge fields \( B \) and \( W^3 \) being the relevant physical degrees of freedom.

### 4.3 General one-loop Sudakov logarithms

Gathering the knowledge from the previous subsections we can now make general statements. Upon summation over the allowed gauge-boson exchanges, one obtains the following expression for the full one-loop Sudakov correction to the external wave-function factor for an arbitrary on-shell/on-resonance particle with mass \( m \), charge \( Q \) and hypercharge \( Y \):
\[ \delta Z^{(1)} = \left[ \frac{C_2(R)}{\sin^2 \theta_w} + \left(\frac{Y}{2 \cos \theta_w}\right)^2 \right] L(M, M) + Q^2 \left[ L_\gamma(\lambda, m) - L(M, M) \right]. \] (75)

Here \( C_2(R) \) is the \( SU(2) \) Casimir operator of the particle. So, \( C_2(R) = C_{SU(2)}^F = 3/4 \) for particles in the fundamental representation: the left-handed fermions (\( f_L/\bar{f}_R \)), the physical Higgs boson (\( H \)) and the longitudinal gauge bosons (\( W^\pm_L \) and \( Z_L \), being equivalent to the Goldstone bosons \( \phi^\pm \) and \( \chi \)). For the particles in the adjoint representation of \( SU(2) \), i.e. the transverse \( W \) bosons (\( W^\pm_T \)) and the \( W^3 \) components of both the photon (at \( k^2 = 0 \)) and the transverse \( Z \) boson (at \( k^2 = M_Z^2 \)), one obtains \( C_2(R) = C_{SU(2)}^A = 2 \). For the \( SU(2) \) singlets, i.e. the right-handed fermions (\( f_R/\bar{f}_L \)) and the \( B \) components of both the photon (at \( k^2 = 0 \)) and the transverse \( Z \) boson (at \( k^2 = M_Z^2 \)), the \( SU(2) \) Casimir operator vanishes, \( C_2(R) = 0 \). Note that the terms proportional to \( Q^2 \) in Eq. (75) are the result of the mass gap between the photon and the weak bosons.

### 5 Electroweak two-loop Sudakov logarithms

Having established the method to calculate the Sudakov logarithms in the Coulomb gauge, we now perform the explicit two-loop calculation. Again the calculation is very similar for the various types of external particles. We use the fermion case as the major example to illustrate all the subtleties and then briefly give the results for the bosons.
5.1 The fermionic self-energy at two-loop level

At two-loop accuracy one has to take the following five generic sets of diagrams into account:

The fermions $f_i$ are fixed by the exchanged gauge bosons $V_i$. Various cancellations are going to take place between all these diagrams. In unbroken theories like QED and QCD merely the so-called ‘rainbow’ diagrams of set (a) survive. The same holds if all gauge bosons of the theory would have a similar mass. The unique feature of the SM is that it is only partially broken, with the electromagnetic gauge group $U(1)_{em} \neq U(1)_Y$ remaining unbroken. As such three of the four gauge bosons will acquire a mass, whereas the photon remains massless and will interact with the charged massive gauge bosons ($W^\pm$). As a consequence, merely calculating the ‘rainbow’ diagrams will not lead to the correct result. To illustrate the above we study each of the generic five topologies separately, indicating the corresponding two-loop contributions to the fermionic external wave-function factor by $\delta Z_f^{(2)}(a) - \delta Z_f^{(2)}(e)$, respectively.

First the ‘rainbow’ diagrams of set (a). Let the outer loop-momentum in the ‘rainbow’ be denoted by $k_1$ and the inner loop-momentum by $k_2$. For simplicity we use the generic mass $m_f$ for every fermion and do not distinguish between different fermion species. At one-loop level we learned that the fermion mass is only needed as a cut-off parameter to regularize the collinear singularity if the soft exchanged gauge boson is a photon. If this photon is attached to $f'$ rather then $f$, then it has to be preceded by the emission of a $W$ boson and, as we will see in the following, the heavy gauge-boson mass scale $M$ will replace $m_f'$ as dominant collinear cut-off. So, for all practical
purposes we can forget about $m_{f'}$. The ‘rainbow’ contribution $\delta Z_f^{(2)}(a)$ can then be written as
\[
\delta Z_f^{(2)}(a) \approx -(ie)^4 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \Gamma_{f_{1f}V_1}^2 \Gamma_{f_{1f}A_2}^2 \frac{4p_{\mu}p_{\nu}P^{\mu\nu}(k_1, M_1) 4p_{\mu'}p_{\nu'}P^{\mu'\nu'}(k_2, M_2)}{[(p-k_1)^2 - m_f^2 + i\epsilon][(p-k_1-k_2)^2 - m_f^2 + i\epsilon]}
\]
\[
\times \left( \frac{2}{[(p-k_1)^2 - m_f^2 + i\epsilon]} + \frac{1}{[(p-(k_1+k_2))^2 - m_f^2 + i\epsilon]} \right). 
\]
(76)

For the gauge-boson momentum $k_2$ we choose a Sudakov parametrisation equivalent to the one used for $k_1$, i.e.
\[
k_2 = v_2q + u_2\bar{q} + k_{2\perp},
\]
(77)
with $q$ and $\bar{q}$ defined in Eq. (54). The calculation simplifies if we perform the $u_2$ integration first, taking the residue in the lower hemisphere in the pole of the corresponding gauge-boson propagator. The rest of the calculation follows the steps of the one-loop calculation. Making use of Eq. (51) as well as the related identity
\[
i\left[ k_2^2 - M_2^2 + i\epsilon \right] \frac{4p_{\mu}p_{\nu}P^{\mu\nu}(k_2, M_2)}{k_2^2} \approx \frac{4k_2p_0}{k_2^2}((p-k_2)^2 - m_f^2)
\]
\[
\approx \frac{4k_2p_0}{k_2^2} \left[ (p-(k_1+k_2))^2 - m_f^2 \right] - \left[ (p-k_1)^2 - m_f^2 \right],
\]
(78)
one obtains in leading logarithmic approximation
\[
\delta Z_f^{(2)}(a) \approx \left( -\frac{\alpha}{\pi} \right) \Gamma_{f_{1f}V_1}^2 \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \Theta(y_1z_1 - \frac{M_f^2}{s}) \Theta(y_1 - \frac{m_f^2}{s}z_1)
\]
\[
\times \left( -\frac{\alpha}{\pi} \right) \Gamma_{f_{1f}A_2}^2 \int_0^1 \frac{dy_2}{y_2} \int_0^1 \frac{dz_2}{z_2} \Theta(y_2z_2 - \frac{M_f^2}{s}) \Theta(y_2 - \frac{m_f^2}{s}z_2)
\]
\[
\approx \left( -\frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \int_0^1 \frac{dy_2}{y_2} \int_0^1 \frac{dz_2}{z_2} \Gamma_{f_{1f}V_1}^2 \Gamma_{f_{1f}A_2}^2 \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1)
\]
\[
\times \Gamma_{f_{1f}A_2}^2 \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \Theta(y_2 - y_1)
\]
(79)
with $\mathcal{K}^{(1)}$ being defined in Eq. (57). As already hinted at above, the collinear region of the inner integral is restricted by the collinear region of the outer integral ($y_2 \gg y_1$). So, the ‘rainbow’ diagrams exhibit an explicit angular ordering.
Similarly the ‘crossed rainbow’ diagrams of set (b) yield

\[
\delta Z_f^{(2)}(b) = - (ie)^4 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \Gamma_{f_1v_1} \Gamma_{f_1f_2v_2} \Gamma_{f_2f_3v_1} \Gamma_{ff_3v_2} \\
\quad \times \frac{4p_\mu p_\nu P^{\mu\nu}(k_1, M_1) 4p'_\mu p'_\nu P'^{\mu'\nu'}(k_2, M_2)}{(p-k_1)^2 - m_f^2 + ie} \frac{1}{[(p-k_1-k_2)^2 - m_f^2 + ie]} \frac{1}{[(p-k_2)^2 - m_f^2 + ie]}
\]

\[
= - \Gamma_{f_1v_1} \Gamma_{f_1f_2v_2} \Gamma_{f_2f_3v_1} \Gamma_{ff_3v_2} \left( - \frac{\alpha}{\pi} \right) \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1)
\quad \times \left( - \frac{\alpha}{\pi} \right) \int_0^1 \frac{dy_2}{y_2} \int_0^1 \frac{dz_2}{z_2} \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2).
\]

(80)

Obviously the ‘reducible’ contribution from set (c) can only be the product of the two corresponding one-loop contributions

\[
\delta Z_f^{(2)}(c) = \left( - \frac{\alpha}{\pi} \right) \Gamma_{f_1v_1}^2 \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1)
\quad \times \left( - \frac{\alpha}{\pi} \right) \Gamma_{ff_3v_2}^2 \int_0^1 \frac{dy_2}{y_2} \int_0^1 \frac{dz_2}{z_2} \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2).
\]

(81)

These ‘reducible’ contributions follow from the fact that the irreducible fermionic self-energy enters \( Z_f \) in the denominator.

The two remaining sets of diagrams turn out to be more difficult to calculate. The main complication being that more than two gauge-boson propagators are involved and hence a variety of possible on-shell combinations enlarges the actual number of integrals to be performed. Defining \( k_3 = k_1 - k_2 \), the triple-gauge-boson (‘TGB’) diagrams of set (d) can be written in the following way

\[
\delta Z_f^{(2)}(d) = i (ie)^4 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \Gamma_{f_1v_1} \Gamma_{f_1f_2v_2} \Gamma_{f_2f_3v_1} \Gamma_{f_3v_2} \\
\quad \times \frac{2p_\mu 2p_\nu 2p'_\mu 2p'_\nu P^{\mu\nu}(k_1, M_1) P^{\mu\nu}(k_3, M_3) P'^{\mu'\nu'}(k_2, M_2) V_{\mu'\nu'\nu'}}{[(p-k_1)^2 - m_f^2 + ie][[(p-k_1-k_2)^2 - m_f^2 + ie][[(p-k_2)^2 - m_f^2 + ie]}
\]

\[
\equiv (i e)^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \Gamma_{f_1v_1} \Gamma_{f_1f_2v_2} \Gamma_{f_2f_3v_1} \Gamma_{f_3v_2} G_{132} \mathcal{K}(d) I K(d),
\]

(82)
with the integration kernel

\[
\text{IK}(d) = -i \frac{2 p_{\mu} 2 p_{\nu} 2 p_{\rho} P^{\mu\nu\rho}(k_1, M_1) P^{\rho\mu\nu}(k_3, M_3) P^{\nu\rho\mu}(k_2, M_2) V_{\mu'\rho'\nu'}}{[(p - k_1)^2 - m_f^2 + i\epsilon][(p - k_2)^2 - m_f^2 + i\epsilon]}
\]

\[
\times \left( \frac{1}{[(p - k_1)^2 - m_f^2 + i\epsilon]} + \frac{1}{[(p - k_2)^2 - m_f^2 + i\epsilon]} \right)
\]

(83)

and

\[
V_{\mu'\rho'\nu'} = (2k_2 - k_1)_{\mu'} g_{\nu'\rho'} + (-k_2 - k_1)_{\rho'} g_{\mu'\nu'} + (2k_1 - k_2)_{\nu'} g_{\mu'\rho'}. \tag{84}
\]

The totally antisymmetric coupling \( e G_{ijl} \) is the triple gauge-boson coupling with all three gauge-boson lines \((i, j, l)\) defined to be incoming at the interaction vertex. In our convention this coupling is fixed according to \( G_{\gamma W^+ W^-} = 1 \) and \( G_{ZW^+ W^-} = -\cos \theta_w / \sin \theta_w. \)

The integration kernel can be simplified by making use of the fact that the following generic contributions will not lead to \((\log)^4\) corrections:

- terms with only one gauge-boson propagator
- terms with no fermion propagator
- terms with one fermion propagator and only two gauge-boson propagators
- terms with two fermion propagators and two gauge-boson propagators but only one \(1/\vec{k}^2\)
- terms \(\propto (1/k_i)^l\) with \(l < 8\) in the soft \(k_i\) limit; four of those powers will be compensated by the loop integrals, hence four more are required to obtain four logarithms.

Moreover we can make use of effective identities like

\[
\frac{(p - k_1)^2}{[(p - k_2)^2 - m_f^2 + i\epsilon] \prod_{j=1}^3 [k_j^2 - M_j^2 + i\epsilon]} \rightarrow \frac{k_{10} k_{20}}{k_1^2} \frac{1}{[(p - k_2)^2 - m_f^2 + i\epsilon] \prod_{j=1}^3 [k_j^2 - M_j^2 + i\epsilon]},
\]

(85)

because the part of \((p - k_1)^2\) that is proportional to the \(\vec{k}_1\) component perpendicular to \(\vec{k}_2\) will not survive the \(\vec{k}_1\) integration.
The integration kernel $IK(d)$ of Eq. (82) can now be written as

\[
IK(d) \approx \frac{1}{k_1^2} \left[ \frac{1}{(p-k_1)^2 - m_f^2 + ie} \right] [k_1^2 - M_1^2 + ie] + \frac{1}{k_2^2} \left[ \frac{1}{(p-k_2)^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
+ \frac{1}{k_3^2} \left[ \frac{1}{(p-k_3)^2 - m_f^2 + ie} \right] [k_3^2 - M_3^2 + ie] \\
- \frac{1}{k_1^2} \left[ \frac{1}{(p-k_1)^2 - m_f^2 + ie} \right] [k_1^2 - M_1^2 + ie] \\
- \frac{1}{k_2^2} \left[ \frac{1}{(p-k_2)^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
- \frac{1}{k_3^2} \left[ \frac{1}{(p-k_3)^2 - m_f^2 + ie} \right] [k_3^2 - M_3^2 + ie] \\
+ \frac{1}{k_1^2} \left[ \frac{1}{(p-k_1)^2 - m_f^2 + ie} \right] [k_1^2 - M_1^2 + ie] \\
\times \frac{1}{k_3^2} \left[ \frac{1}{(p-(k_2+k_3))^2 - m_f^2 + ie} \right] [k_3^2 - M_3^2 + ie] \\
- \frac{1}{k_2^2} \left[ \frac{1}{(p-k_2)^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
\times \frac{1}{k_3^2} \left[ \frac{1}{(p-(k_1+k_3))^2 - m_f^2 + ie} \right] [k_3^2 - M_3^2 + ie] \\
- \frac{1}{k_1^2} \left[ \frac{1}{(p-k_1)^2 - m_f^2 + ie} \right] [k_1^2 - M_1^2 + ie] \\
\times \frac{1}{(k_1 - k_3)^2} \left[ \frac{1}{(p-(k_1+k_3))^2 - m_f^2 + ie} \right] [k_3^2 - M_3^2 + ie] \\
+ \frac{1}{k_2^2} \left[ \frac{1}{(p-k_2)^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
\times \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{(p-(k_1+k_2))^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
+ \frac{1}{k_1^2} \left[ \frac{1}{(p-k_1)^2 - m_f^2 + ie} \right] [k_1^2 - M_1^2 + ie] \\
\times \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{(p-(k_1+k_2))^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
+ \frac{1}{k_2^2} \left[ \frac{1}{(p-k_2)^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie] \\
\times \frac{1}{(k_1 - k_2)^2} \left[ \frac{1}{(p-(k_1+k_2))^2 - m_f^2 + ie} \right] [k_2^2 - M_2^2 + ie].
\]

Note here that the same result is obtained for the full gauge-boson propagator $P_{\mu\nu}$ as well as for the purely transverse part $\propto Q_{\mu\nu}$, as expected in the collinear regime. Apart from the coupling factor $\Gamma_{f_1V_1} \Gamma_{f_2V_3} \Gamma_{f_3V_2} G_{132}$ the integrals in Eq. (82) have been normalized in the usual way.
Therefore the first term $(86a)$ is easily identified as the product of two one-loop contributions $(52)$ with momenta $k_1$ and $k_2$, i.e.

$$(86a) \rightarrow \frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2).$$

In the second term $(86b)$ the momentum $k_1$ has to be expressed in terms of the momenta $k_2$ and $k_3$, i.e. $k_1 = k_2 + k_3$, in the fermion propagator as well as in the integration variable. This is convenient since those are the momenta appearing in the boson propagators of $(86b)$. (Remember that we have chosen to take the residue in the lower hemisphere in the pole of the gauge-boson propagators.) In doing this the ‘rainbow’-like structure can be immediately recognized and upon integrating first the $u_3$ variable belonging to the Sudakov parametrisation of $k_3$ we obtain instantly

$$(86b) \rightarrow \frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(y_3 - y_2).$$

Similarly, replacing $k_2 = k_1 - k_3$ and subsequently reversing the sign of the $k_3$ integration variable in $(86c)$ leads again to a ‘rainbow’-like structure and

$$(86c) \rightarrow \frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(y_3 - y_1).$$

The following four terms are unique in the sense that they only contain one fermion propagator and three gauge-boson propagators. As we will see later, those can be identified as so-called ‘frog’ contributions. Now having three propagators serving as potential poles we have to sum over all three possibilities of taking either two of them on-shell. Let us do this step by step for the example of $(86d)$. Starting by taking $k_1$ and $k_2$ as the integration variables, i.e. taking the corresponding propagators on-shell, the third gauge-boson propagator becomes

$$\frac{1}{k_3^2 - M_3^2 + i\epsilon} = \frac{1}{k_1^2 - 2 k_1 \cdot k_2 + k_2^2 - M_3^2 + i\epsilon} \approx \frac{1}{-2 k_1 \cdot k_2} \rightarrow \frac{1}{-s(z_1 y_2 + z_2 y_1)}.$$  

We need a $1/(y_1 z_2)$ contribution for a $(\log)^4$ correction, since from $k_{10}/k_1^2$ and from $1/(p-k_2)^2$ we obtained $1/(z_1 y_2)$ already. This leads to the $\Theta$-function $\Theta(z_2 y_1 - z_1 y_2)$. Furthermore, performing the $u_1$ integration first, the third gauge-boson propagator restricts the $v_1$ integration range to $0 \leq v_1 \leq v_2$. Hence we find for the first summand of kernel $(86d)$

$$\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \Theta(z_2 - z_1) \Theta(z_2 y_1 - z_1 y_2).$$

Taking $k_1$ and $-k_3$ as the next two integration variables and performing the $u_3$ integration first we obtain

$$-\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(y_3 - y_1) \Theta(z_3 - z_1) \Theta(z_3 y_1 - z_1 y_3).$$
Finally for $k_2$ and $k_3$ being the on-shell gauge-boson momenta

$$-\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(z_3 - z_2) \Theta(z_2 - z_3) \equiv 0,$$

since the two $\Theta$-functions cannot be fulfilled simultaneously. Note that the first $\Theta$-function originates from the $k_{10} = k_{30} + k_{20} \approx k_{30}$ constraint and the second $\Theta$-function arises due to the restricted $v_3$ integration range.

In order to combine Eqs. (91) and (92), we first relabel the integration variables of Eq. (92)

$$-\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_2 - z_1) \Theta(z_2 y_1 - z_1 y_2). \quad (93)$$

Adding to this the ‘one-way’ double ordered part of Eq. (91) leads to

$$\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \Theta(z_2 y_1 - z_1 y_2) \Theta(z_2 - z_1) \Theta(y_2 - y_1)$$

$$\times \left[ \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) - \mathcal{K}^{(1)}(s, m_f^2, M_3, y_2, z_2) \right]$$

$$= \frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \Theta(z_2 y_1 - z_1 y_2) \Theta(z_2 - z_1) \Theta(y_2 - y_1)$$

$$\times \Theta \left( y_2 - \frac{m_f^2}{s} z_2 \right) \left[ \Theta \left( y_2 z_2 - \frac{M_2^2}{s} \right) - \Theta \left( y_2 z_2 - \frac{M_3^2}{s} \right) \right], \quad (94)$$

which vanishes for all possible combinations of $M_i$ being the photon mass or the generic mass $M$. This is trivial for $M_2 = M_3$. For $M_2 = \lambda$ and $M_1 = M_3 = M$ the $\Theta(\cdot)$-functions $\Theta(z_2 - z_1) \Theta(y_2 - y_1)$ can be combined with $\Theta(y_1 z_1 - \frac{M_2^2}{s})$, restricting the $y_2, z_2$ integrations such that at least $y_2 z_2 \geq M^2/s$ and hence $\Theta \left( y_2 z_2 - \frac{M_2^2}{s} \right) - \Theta \left( y_2 z_2 - \frac{M_3^2}{s} \right)$ vanishes. The same holds for $M_3 = \lambda$ and $M_1 = M_2 = M$.

Hence we find for (86d)

$$\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \Theta(z_2 y_1 - z_1 y_2), \quad (95)$$

with $\Theta(z_2 y_1 - z_1 y_2)$ being obsolete for this combination of $\Theta$-functions. Analogously we find for (86d)

$$\frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \Theta(z_1 - z_2) \Theta(y_2 - y_1), \quad (96)$$

for (86d)

$$\frac{1}{2} \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(z_2 - z_3) \Theta(y_3 - y_2), \quad (97)$$
and eventually for (86g)
\[ \frac{1}{2} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(z_1 - z_3) \Theta(y_3 - y_1). \] (98)

Next (86h) can be identified as the following double ordered contribution
\[ (86h) \rightarrow \frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(y_3 - y_2) \Theta(z_2 - z_3). \] (99)

Similarly (86i) becomes
\[ (86i) \rightarrow \frac{1}{4} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_3, y_3, z_3) \Theta(y_3 - y_1) \Theta(z_1 - z_3). \] (100)

The remaining two contributions are
\[ (86j) \rightarrow \frac{1}{2} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \Theta(z_2 - z_1), \] (101)

and
\[ (86k) \rightarrow \frac{1}{2} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \Theta(z_1 - z_2), \] (102)

and can be combined to
\[ (86j) + (86k) \rightarrow \frac{1}{2} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2). \] (103)

After some relabeling, the two-loop correction factors originating from set (d) can be summarized as follows
\[
\delta Z_f^{(2)}(d) = \frac{1}{4} \Gamma_{ffV_1 \Gamma_{f_1f_2V_2}} \int \frac{dy_1}{y_1} \frac{dy_2}{y_2} \int \frac{dz_1}{z_1} \frac{dz_2}{z_2} \left\{ \left[ \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) + \mathcal{K}^{(1)}(s, m_f^2, M_2, y_1, z_1) \right] \times \mathcal{K}^{(1)}(s, m_f^2, M_3, y_2, z_2) \Theta(y_2 - y_1) \left[ 1 + 3 \Theta(z_1 - z_2) \right] + \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \times 
\left[ 3 + \Theta(y_1 - y_2) \Theta(z_2 - z_1) + \Theta(y_2 - y_1) \Theta(z_1 - z_2) \right] \right\},
\] (104)

Finally we calculate the ‘frog’ diagrams of set (e):
\[
\delta Z_f^{(2)}(e) = -e^2 \Gamma_{ffV_1} \Gamma_{ffV_2} \int \frac{d^4 k_1}{(2\pi)^4} \frac{4p_{\mu}\gamma_{\nu}P^{\mu\nu}(k_1, M_1) \left( i \Sigma_{V_1V_2,k}^{(1)} g_{\mu\nu} \right) P^{\mu\nu}(k_1, M_2)}{[(p - k_1)^2 - m_f^2 + i\epsilon]^2}
\]
where the expressions for the various $\Sigma^{(1)}_{V_1V_2g}$ can be found in Appendix B. Whenever the soft particle $V_3$ is a $W$ boson, the sum of the contributions from the gauge-boson propagator and the two mixed propagators is implicitly understood. With the purpose of making the bookkeeping as simple as possible for later summation over all possible combinations of particles in the various diagrams, we remove the explicit orientation in the inner loop and add the cases of both $V_3$ and $V_4$ being the soft gauge boson. After the usual simplifications we obtain with the help of the result from (86e)

$$\delta Z_f^{(2)}(e) = + \frac{1}{2} \Gamma_{fV_1} \Gamma_{fV_2} G_{134} G_{243} \left( - \frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \int_0^1 \frac{dy_2}{y_2} \int_0^1 \frac{dz_2}{z_2} \times \left[ \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) + \mathcal{K}^{(1)}(s, m_f^2, M_2, y_1, z_1) \right] \times \left[ \mathcal{K}^{(1)}(s, m_f^2, M_3, y_2, z_2) + \mathcal{K}^{(1)}(s, m_f^2, M_4, y_2, z_2) \right] \times \Theta(y_2 - y_1) \Theta(z_1 - z_2). \tag{105}$$

To summarize everything we write the generic two-loop contribution of Sudakov logarithms to $\delta Z_f^{(2)}$ as:

$$\delta Z_f^{(2)} \approx \left( - \frac{\alpha}{\pi} \right)^2 \Gamma_f^{(2)} \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \int_0^1 \frac{dy_2}{y_2} \int_0^1 \frac{dz_2}{z_2} \mathcal{K}^{(2)}(y_1, z_1, y_2, z_2). \tag{106}$$

For the five different topologies the various products $\Gamma_f^{(2)} \times \mathcal{K}^{(2)}$ of coupling factors and integration
kernels are given by

\[
\begin{align*}
\text{set (a):} & \quad \left[ \Gamma_{ffV_1}^2 K^{(1)}(s, m_f^2, M_1, y_1, z_1) \right] \left[ \Gamma_{ffV_2}^2 K^{(1)}(s, m_f^2, M_2, y_2, z_2) \right] \Theta(y_2 - y_1), \\
\text{set (b):} & \quad -\Gamma_{ffV_1} \Gamma_{f_1f_2V_2} \Gamma_{f_2f_3V_1} K^{(1)}(s, m_f^2, M_1, y_1, z_1) K^{(1)}(s, m_f^2, M_2, y_2, z_2), \\
\text{set (c):} & \quad \left[ \Gamma_{ffV_1}^2 K^{(1)}(s, m_f^2, M_1, y_1, z_1) \right] \left[ \Gamma_{ffV_2}^2 K^{(1)}(s, m_f^2, M_2, y_2, z_2) \right], \\
\text{set (d):} & \quad \frac{1}{4} \Gamma_{ffV_1} \Gamma_{f_1f_2V_2} \Gamma_{f_2f_3V_2} G_{132} \left\{ \left[ K^{(1)}(s, m_f^2, M_1, y_1, z_1) + K^{(1)}(s, m_f^2, M_2, y_1, z_1) \right] \times \right. \\
& \quad \times K^{(1)}(s, m_f^2, M_3, y_2, z_2) \Theta(y_2 - y_1) \left[ 1 + 3 \Theta(z_1 - z_2) \right] \\
& \quad + K^{(1)}(s, m_f^2, M_1, y_1, z_1) K^{(1)}(s, m_f^2, M_2, y_2, z_2) \times \\
& \quad \times \left[ 3 + \Theta(y_1 - y_2) \Theta(z_2 - z_1) + \Theta(y_2 - y_1) \Theta(z_1 - z_2) \right] \right\}, \\
\text{set (e):} & \quad \frac{1}{2} \Gamma_{ffV_1} \Gamma_{ffV_2} G_{134} G_{243} \left[ K^{(1)}(s, m_f^2, M_3, y_2, z_2) + K^{(1)}(s, m_f^2, M_4, y_2, z_2) \right] \times \\
& \quad \times \left[ K^{(1)}(s, m_f^2, M_1, y_1, z_1) + K^{(1)}(s, m_f^2, M_2, y_1, z_1) \right] \Theta(y_2 - y_1) \Theta(z_1 - z_2). \quad \text{(107)}
\end{align*}
\]

As a cross-check we also calculated the full two-loop result by means of the dispersion method. In that case the various \( \Theta \)-functions originate from the different two- and three-particle cuts that enter the calculation of the absorptive parts. The final result agrees with Eq. (107).

In Appendix A we have derived all relevant one- and two-loop integrals. Here we give the results, using the generic notation

\[
I^{(i)} = \left(-\frac{\alpha_s}{\pi}\right)^i \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \cdots \int_0^1 \frac{dy_i}{y_i} \int_0^1 \frac{dz_i}{z_i} K^{(i)}(y_1, z_1, \ldots, y_i, z_i). \quad \text{(108)}
\]

At one-loop level we found

\[
K^{(1)}(s, m_f^2, M, y_1, z_1) : I^{(1)} = L(M, M), \quad \text{(109)}
\]

\[
K^{(1)}(s, m_f^2, \lambda, y_1, z_1) : I^{(1)} = L_\gamma(\lambda, m_f). \quad \text{(110)}
\]

The functions \( L(M_1, M_2) \) and \( L_\gamma(\lambda, M_1) \) are the ones defined in Eqs. (103) and (104). At two-loop
level we found for the angular ordered integrals:

\[ K^{(1)}(s, m_f^2, M, y_1, z_1) K^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) : I^{(2)} = \frac{1}{2} L^2(M, M) , \]

\[ K^{(1)}(s, m_f^2, \lambda, y_1, z_1) K^{(1)}(s, m_f^2, \lambda, y_2, z_2) \Theta(y_2 - y_1) : I^{(2)} = \frac{1}{2} L^2(\lambda, m_f) , \]

\[ K^{(1)}(s, m_f^2, M, y_1, z_1) K^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) : I^{(2)} = \frac{7}{12} L^2(M, M) , \]

\[ K^{(1)}(s, m_f^2, \lambda, y_1, z_1) K^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) : I^{(2)} = L(M, M) L_\gamma(\lambda, m_f) \]

\[ - \frac{7}{12} L^2(M, M) . \]  \quad (111)

and for the double ordered integrals:

\[ K^{(1)}(s, m_f^2, M, y_1, z_1) K^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_1 - z_2) : I^{(2)} = \frac{1}{4} L^2(M, M) , \]

\[ K^{(1)}(s, m_f^2, M, y_1, z_1) K^{(1)}(s, m_f^2, \lambda, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_1 - z_2) : I^{(2)} = \frac{1}{3} L^2(M, M) , \]

\[ K^{(1)}(s, m_f^2, \lambda, y_1, z_1) K^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_1 - z_2) : I^{(2)} = \frac{2}{3} L(M, M) L(M, m_f) \]

\[ - \frac{1}{4} L^2(M, M) . \]  \quad (112)

Note that in the case of double ordering the collinear cut-off \( m_f^2 \) of the \( y_2 \) integral is in fact redundant.

Now the task at hand is to sum all possible contributions to obtain the full two-loop correction to the external wave-function factor. Using the abbreviations \( L \equiv L(M, M) \), \( L_{m_f} \equiv L(M, m_f) \) and
\[ L_\gamma \equiv L_\gamma(\lambda, m_f) \]

we find:

\[
\text{set}(a) = \begin{cases} 
\frac{1}{2} \left( \delta Z_{jL}^{(1)}(W) + \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 - \frac{1}{6} \frac{I_f^3 Y_f^L}{2 \sin^2 \theta_w} L^2 & \text{for } f_L, \bar{f}_R \\
\frac{1}{2} \left( \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 & \text{for } f_R, \bar{f}_L 
\end{cases}
\]

\[
\text{set}(b) = \begin{cases} 
- \left( \delta Z_{jL}^{(1)}(W) + \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 + 3 \delta Z_{jL}^{(1)}(W) \delta Z_{jL}^{(1)}(W) + \delta Z_{jL}^{(1)}(W) \left( 1 + 2 I_f^3 Y_f^L \right) [L_\gamma - L] & \text{for } f_L, \bar{f}_R \\
- \left( \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 & \text{for } f_R, \bar{f}_L 
\end{cases}
\]

\[
\text{set}(c) = \begin{cases} 
\left( \delta Z_{jL}^{(1)}(W) + \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 & \text{for } f_L, \bar{f}_R \\
\left( \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 & \text{for } f_R, \bar{f}_L 
\end{cases}
\]

and exclusively for left-handed fermions \((f_L, \bar{f}_R)\)

\[
\text{set}(d)_L = -2 \frac{I_f^3 Q_f}{\sin^2 \theta_w} \left[ L L_\gamma + \frac{2}{3} L L_m_f \right] - \frac{9}{8} \frac{I_f^3 Y_f^L}{\sin^2 \theta_w} L^2 \\
+ \left[ 3 - \frac{1}{6} \right] \frac{1}{4 \sin^2 \theta_w} L^2 + \left[ 3 + \frac{1}{6} \right] \frac{I_f^3 Y_f^L}{2 \sin^2 \theta_w} L^2
\]

\[
\text{set}(e)_L = \frac{7}{24} \frac{I_f^3 Q_f}{\sin^2 \theta_w} L^2 + \frac{1}{8} \frac{\cos^2 \theta_w}{\sin^2 \theta_w} L^2 + \frac{\cos^2 \theta_w}{4 \sin^2 \theta_w} L^2 + \frac{4}{3} \frac{I_f^3 Q_f}{\sin^2 \theta_w} L L_m_f - \frac{I_f^3 Q_f}{\sin^2 \theta_w} L^2.
\]

Here \(\delta Z_{jL/R}^{(1)}(V)\) are the one-loop corrections to the external wave-function factors given in Eq. (58). Hence the full two-loop fermionic Sudakov correction factor reads

\[
\delta Z_{j}^{(2)} = \begin{cases} 
\frac{1}{2} \left( \delta Z_{jL}^{(1)}(W) + \delta Z_{jL}^{(1)}(Z) + \delta Z_{jL}^{(1)}(\gamma) \right)^2 = \frac{1}{2} \left( \delta Z_{jL}^{(1)} \right)^2 & \text{for } f_L, \bar{f}_R \\
\frac{1}{2} \left( \delta Z_{jR}^{(1)}(Z) + \delta Z_{jR}^{(1)}(\gamma) \right)^2 = \frac{1}{2} \left( \delta Z_{jR}^{(1)} \right)^2 & \text{for } f_R, \bar{f}_L 
\end{cases}
\]

(113)

From Eq. (113) we deduce our main statement, namely that the virtual electroweak two-loop Sudakov correction factor is obtained by a mere exponentiation of the one-loop Sudakov correction factor. We also note that, in adding up all the contributions, we find that the ‘rainbow’ diagrams of set (a) yield the usual exponentiating terms plus an extra term for left-handed fermions. This
extra term originates from the charged-current interactions and is only non-vanishing as a result of the mass gap between the massless photon and the massive $Z$ boson. It cancels against a specific term originating from the triple gauge-boson diagrams of set (d). Similar (gauge) cancellations take place between the ‘crossed rainbow’ diagrams of set (b), the reducible diagrams of set (c), and another part of the triple gauge-boson diagrams of set (d). Finally, the left-over terms of set (d) get cancelled by the contributions from the gauge-boson self-energy (‘frog’) diagrams of set (e).

5.2 The bosonic self-energies at two-loop level

In a similar way, simply by adjusting the relevant couplings, we find that the two-loop Sudakov correction factors in the charged-boson sector can be obtained from the one-loop results by means of ‘exponentiation’. The same holds in an equally straightforward way for the non-transverse neutral-boson sector. Also the gauge cancellations conspire in a way very similar to what we already saw in the fermionic case. For transverse $W$ bosons, for instance, we observe the following. Due to the mass gap between the massive $Z$ boson and the massless photon, the ‘rainbow’ contributions from set (a) exhibit an extra term, which in the transverse $W$-boson case is canceled in part by the contributions from the triple gauge-boson diagrams of set (d) and in part by contributions from the ‘frog’ diagrams of set (e). The extra terms in the ‘crossed rainbow’ contribution of set (b), arising due to forbidden combinations of one charged and one neutral particle, are in the case of the photon compensated by contributions from the triple gauge-boson diagrams and in the case of the $Z$ boson by contributions from both the triple gauge-boson diagrams and the ‘frog’ diagrams. Eventually we are left with the very simple result $\delta Z^{(2)}_{WT} = \frac{1}{2} \left( \delta Z^{(1)}_{WT} \right)^2$ for the two-loop contribution to the external wave-function factor.

In the transverse neutral gauge-boson sector there is again the extra complication of having to determine $\tan(\theta(k^2))$ at $k^2 = 0$ and $k^2 = M^2_Z$. In analogy to the one-loop case discussed in Sect. 4.2, we expand Eqs. (41a) and (41b) to two-loop order, which yields with the help of Eqs. (66) and (43)

\[
C^{-1}_{\gamma\gamma} \rightarrow \begin{array}{c}
2\text{-loop} \\
1 - \Sigma^{(1)}_{\gamma\gamma, g} - \Sigma^{(2)}_{\gamma\gamma, g} - \frac{3}{4} \left( \Sigma^{(1)}_{\gamma Z, g} \right)^2 \\
\end{array}
\]

(114)

\[
C^{-1}_{ZZ} \rightarrow \begin{array}{c}
2\text{-loop} \\
1 - \Sigma^{(1)}_{ZZ, g} - \Sigma^{(2)}_{ZZ, g} - \frac{3}{4} \left( \Sigma^{(1)}_{\gamma Z, g} \right)^2. \\
\end{array}
\]

(115)
The various self-energies can be calculated by means of the derivative method explained in Sect. 5.1, resulting in

\[ \Sigma_{Zg}^{(2)} = -\frac{\cos \theta_w}{\sin \theta_w} \Sigma_{\gamma g}^{(2)} = \frac{\cos^2 \theta_w}{\sin^2 \theta_w} \Sigma_{\gamma g}^{(2)} = \cos^2 \theta_w \left[ -\frac{2}{\sin^4 \theta_w} L^2(M, M) \right] \equiv \cos^2 \theta_w \Sigma_{33}^{(2)}, \]

Next we derive from the “non-renormalization” condition (13) for the electromagnetic charge and Eq. (11a)

\[ \tan(0)_{2\text{-loop}} = C_{\gamma g, g}^{(2)} - C_{\gamma g, g}^{(1)} C_{\gamma g, g}^{(1)} = \frac{\cos \theta_w}{2 \sin \theta_w} \left[ \Sigma_{\gamma g}^{(2)} + \frac{1}{4} \left( \Sigma_{\gamma g}^{(1)} \right)^2 + \frac{3}{4} \left( \Sigma_{\gamma g}^{(1)} \right)^2 \right] \]

\[ = \frac{\cos \theta_w}{2 \sin^3 \theta_w} \left( 2 \cos^2 \theta_w - 1 \right) L^2(M, M). \]

Since

\[ C_{\gamma g, g}^{(1)} = -\Sigma_{\gamma g, g}^{(1)}, \quad C_{\gamma g, g}^{(1)} = 0 \]

as a result of the “non-renormalization” condition (33) for the masses, we find

\[ C_{\gamma g, g}^{(2)} = -\frac{\cos \theta_w}{2 \sin^3 \theta_w} \left( 1 + 2 \cos^2 \theta_w \right) L^2(M, M). \]

Now all ingredients are known and \( \tan(M_Z^2)_{2\text{-loop}} \) can be determined trivially

\[ \tan(M_Z^2)_{2\text{-loop}} = C_{\gamma g, g}^{(2)} + \Sigma_{\gamma g, g}^{(2)} + \left( C_{\gamma g, g}^{(1)} + \Sigma_{\gamma g, g}^{(1)} \right) \left( C_{\gamma g, g}^{(1)} + \Sigma_{\gamma g, g}^{(1)} \right) \]

\[ = \frac{\cos \theta_w}{2 \sin^3 \theta_w} \left( 2 \cos^2 \theta_w - 1 \right) L^2(M, M). \]

Replacing again the \( Z \)-boson and photon fields in the amputated Green’s functions by the unbroken gauge fields \( B \) and \( W^3 \), the final result in the transverse neutral gauge-boson sector reads:

\[ \delta Z_{NT, B}^{(2)} = 0 \]

\[ \delta Z_{NT, W^3}^{(2)} = \frac{1}{2} \left( \delta Z_{NT, W^3}^{(1)} \right)^2. \]

In fact, this step-wise procedure can be performed to all orders in perturbation theory, yielding

\[ \tan(0) = \frac{\sin \theta_w \cos \theta_w \left[ 1 - \sqrt{1 - \Sigma_{33}^{(1)}} \right]}{\sin^2 \theta_w + \cos^2 \theta_w \sqrt{1 - \Sigma_{33}^{(1)}}}, \quad C_{\gamma \gamma}^{\perp} = \cos^2 \theta_w + \frac{\sin^2 \theta_w}{\sqrt{1 - \Sigma_{33}^{(1)}}}, \quad C_{\gamma \gamma}^{\parallel} = \cos^2 \theta_w + \frac{\sin^2 \theta_w}{\sqrt{1 - \Sigma_{33}^{(1)}}}. \]

\[ \tan(M_Z^2) = -\frac{\sin \theta_w \cos \theta_w \left[ 1 - \sqrt{1 - \Sigma_{33}^{(1)}} \right]}{\cos^2 \theta_w + \sin^2 \theta_w \sqrt{1 - \Sigma_{33}^{(1)}}}, \quad C_{ZZ}^{\perp} = \sin^2 \theta_w + \frac{\cos^2 \theta_w}{\sqrt{1 - \Sigma_{33}^{(1)}}}, \quad C_{ZZ}^{\parallel} = \sin^2 \theta_w + \frac{\cos^2 \theta_w}{\sqrt{1 - \Sigma_{33}^{(1)}}}. \]

This automatically leads to

\[ Z_{NT, B}^{-1} = 1, \quad Z_{NT, W^3}^{-1} = 1 - \Sigma_{33}^{(1)}. \]
5.3 General two-loop Sudakov logarithms

We conclude this section with our general result. The full two-loop Sudakov correction to the external wave-function factor for an arbitrary on-shell/on-resonance particle is given by

$$\delta Z^{(2)} = \frac{1}{2} \left( \delta Z^{(1)} \right)^2,$$

(124)

in terms of the one-loop correction given in Eq. (75). In summary we would like to point out that to calculate the two-loop Sudakov correction factor for an arbitrary species of initial- or final-state particles, i.e. fermions, gauge bosons or would-be Goldstone bosons, the knowledge of the corresponding one-loop correction factor is sufficient. This is a well-known fact in massless or one-mass-scale theories, such as QED, QCD or generally $SU(N)$, where in fact in covariant gauges the two-loop results are effectively obtained from so-called ladder diagrams, corresponding to our ‘rainbow’ diagrams. We would like to stress again that for the SM, as a broken theory with two mass scales, the result $\delta Z^{(2)} = \frac{1}{2} \left( \delta Z^{(1)} \right)^2$ is identical, but at all intermediate stages extra terms arise due to the mass gap. Therefore the calculation of only one topology, i.e. the ‘rainbow’ diagrams, does not lead (not even effectively) to the correct two-loop Sudakov correction factor.

6 Conclusions

We have calculated the virtual electroweak Sudakov (double) logarithms at one- and two-loop level for arbitrary on-shell/on-resonance particles in the Standard Model. The associated Sudakov form factors apply, in principle in a universal way, to arbitrary non-mass-suppressed electroweak processes at high energies. We would like to stress that the universality of the Sudakov form factors has to be interpreted with care. Depending on the final state and the kinematical configuration, the process may possess various near-resonance subprocesses, which all have their own Sudakov correction factor. These correction factors are given (in leading-pole approximation) by the Sudakov form factors of the external particles involved in the subprocess. In this way the Sudakov form factors for unstable particles, like the massive gauge bosons and the Higgs boson, participate in the high-energy behaviour of reactions with exclusively stable particles in the final state.

For the explicit calculation we adopted the temporal Coulomb gauge for both massless and massive particles. About the latter case basically nothing is known in the literature, so in this paper we have tried to give as much detail as possible about its salient details. In view of the special status of the time-like components in this gauge, a careful analysis was required for the derivation of
the asymptotic fields of the theory, which are needed for a proper description of the on-shell/on-resonance states. In particular the presence of lowest-order mixing between the massive gauge bosons and the corresponding would-be Goldstone bosons required special attention.

In the Sudakov limit, our calculation was significantly aided by a few special properties of the temporal Coulomb gauge. First of all, in this special gauge all the relevant contributions involve the exchange of effectively on-shell, transverse, collinear-soft gauge bosons. Moreover, these relevant contributions are contained exclusively in the self-energies of the external on-shell/on-resonance particles (wave-function factors). This has to be contrasted with covariant gauges, where all contributions are residing in vertex and higher-point diagrams. Second, the various self-energies are subject to explicit “non-renormalization” conditions to all orders in perturbation theory. This allows us to obtain the Sudakov wave-function factors through a combination of energy derivatives and projections by means of sources. As a result, we observe that the Standard Model behaves dynamically like an unbroken theory in the Sudakov limit, in spite of the fact that the explicit particle masses are needed at the kinematical (phase-space) level while calculating the Sudakov correction factors. For instance, we obtain automatically a special version of the Equivalence Theorem, which states that the longitudinal degrees of freedom of the massive gauge bosons can be substituted by the corresponding Goldstone-boson degrees of freedom. As a result, the Sudakov form factors for longitudinal gauge bosons exhibit features that are typical for particles in the fundamental representation of \( SU(2) \), whereas for the transverse gauge bosons the usual adjoint features are obtained. Moreover, in the transverse neutral gauge-boson sector the mass eigenstates decompose into the unbroken fields \( W^3 \) and \( B \), each multiplied by the corresponding Sudakov form factor. At the kinematical level, though, the large mass gap between the photon and the weak gauge bosons remains.

Our explicit one- and two-loop calculations of the Sudakov form factors in the Standard Model reveal the following. The one-loop results are in agreement with the available calculations in the literature, including the distinctive terms originating from the mass gap between the photon and the weak gauge bosons. At two-loop level our findings are in agreement with an exponentiation of the one-loop results. We therefore conclude that also as far as the balance between the one- and two-loop virtual Sudakov logarithms are concerned, the Standard Model behaves like an unbroken theory at high energies. We would like to stress, though, that for the Standard Model, as a broken theory with two mass scales, all two-loop diagram topologies are needed to arrive at the correct
result. In general it is not possible to get the correct result by singling out one particular topology, such as the ‘ladder’-like diagrams in unbroken theories.

All these conclusions can be extended to real-emission processes in a relatively straightforward way. After all, since the Sudakov logarithms originate from the exchange of soft, effectively on-shell transverse gauge bosons, many of the features derived for the virtual corrections will be intimately related to properties of the corresponding real-emission processes. In this context we note that the Bloch–Nordsieck cancellation between virtual and real collinear-soft gauge-boson radiation \cite{15} is violated in the Standard Model \cite{14} as soon as initial- or final-state particles carry an explicit weak charge (isospin) and summation over the partners within an $SU(2)$ multiplet is not performed. In the case of final-state particles the event-selection procedure might (kinematically) favour one of the partners within the $SU(2)$ multiplet, leading to a degree of ‘isospin-exclusiveness’. In the initial state the situation is more radical. In that case the weak isospin is fixed by the accelerator, in contrast to QCD where confinement forces average over initial colour at hadron colliders. At an electron–positron collider, for instance, the Bloch–Nordsieck theorem is in general violated for left-handed initial states, even for fully inclusive cross-sections. The resulting electroweak effects can be very large, exceeding the QCD corrections for energies in the TeV range. They are such that at infinite energy the weak charges will become unobservable as asymptotic states \cite{14}, which implies for instance an $SU(2)$ charge averaging of the initial-state beams.

As a matter of fact, for a complete understanding of the perturbative structure of large logarithmic correction factors, subleading logarithms originating from soft, collinear, or ultraviolet singularities cannot be ignored \cite{8, 10}. For pure fermionic final states (numerical) cancellations can take place between leading and subleading logarithms \cite{10}. For on-shell bosons in the final state, however, the Sudakov logarithms in general tend to be dominant \cite{3, 8}, being anyhow intrinsically larger than the Sudakov logarithms for fermions owing to the larger adjoint $SU(2)$ factors.

Appendix

A One- and two-loop integrals

In this appendix we give the relevant one- and two-loop integrals that occur in the Sudakov correction factors. At one-loop accuracy we have to distinguish between two different cases, i.e. the exchanged soft gauge boson being a photon with the fictitious mass $\lambda$ or a massive gauge boson ($W$ or $Z$) with the generic mass $M$. The exchanged gauge boson being massive, we extract
from Eq. (57) with \( M_1 = M \)

\[
J^{(1)}(M) = \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_1 - \frac{m_f^2}{s} z_1).
\] (A.1)

From the first \( \Theta \)-function we obtain the integration boundaries

\[
J^{(1)}(M) = \int_{\frac{M}{\sqrt{s}}}^1 \frac{dz_1}{z_1} \int_0^\frac{M_1}{s} \frac{dy_1}{y_1} \Theta(y_1 - \frac{m_f^2}{s} z_1),
\]

which makes the second \( \Theta \)-function redundant, since \( m_f \leq O(M) \). Therefore

\[
J^{(1)}(M) = \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dy_1}{y_1} = \frac{1}{4} \log^2 \left( \frac{M^2}{s} \right). \quad (A.2)
\]

Similarly we obtain for \( M_1 = \lambda \)

\[
J^{(1)}(\lambda) = \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_1}{z_1} \Theta(y_1 z_1 - \frac{\lambda^2}{s}) \Theta(y_1 - \frac{m_f^2}{s} z_1)
\]

\[
= \int_{\frac{\lambda}{\sqrt{s}}}^1 \frac{dz_1}{z_1} \int_0^{\frac{\lambda^2}{s}} \frac{dy_1}{y_1} + \int_{\frac{\lambda}{\sqrt{s}}}^1 \frac{dz_1}{z_1} \int_0^{\frac{\lambda^2}{s}} \frac{dy_1}{y_1} = \frac{1}{4} \log^2 \left( \frac{\lambda^2}{s} \right) - \frac{1}{4} \log^2 \left( \frac{\lambda^2}{m_f^2} \right). \quad (A.3)
\]

The two-loop integrals fall into two categories, namely the angular ordered integrals and the integrals that are (double) ordered in energy and angle simultaneously. For the angular ordered two-loop integrals, see Eq. (79), we find for \( M_1 = M_2 = M \)

\[
J^{(2)}_{\text{angular}}(M, M) = \int_0^1 \frac{dz_1}{z_1} \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^1 \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_2 z_2 - \frac{M^2}{s}) \Theta(y_2 - y_1).
\] (A.4)

By means of symmetry arguments, \( i.e. \Theta(y_2 - y_1) \rightarrow \frac{1}{2} [\Theta(y_2 - y_1) + \Theta(y_1 - y_2)] = \frac{1}{2} \), we find

\[
J^{(2)}_{\text{angular}}(M, M) = \frac{1}{2} \left[ \frac{1}{4} \log^2 \left( \frac{M^2}{s} \right) \right]^2,
\] (A.5)

and similarly

\[
J^{(2)}_{\text{angular}}(\lambda, \lambda) = \frac{1}{2} \left[ \frac{1}{4} \log^2 \left( \frac{\lambda^2}{s} \right) - \frac{1}{4} \log^2 \left( \frac{\lambda^2}{m_f^2} \right) \right]^2. \quad (A.6)
\]

Furthermore

\[
J^{(2)}_{\text{angular}}(M, \lambda) = \int_0^1 \frac{dz_1}{z_1} \int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^1 \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_2 z_2 - \frac{\lambda^2}{s})
\]

\[
\times \Theta(y_2 - \frac{m_f^2 z_2}{s}) \Theta(y_2 - y_1)
\]

\[
= \int_{\frac{M}{\sqrt{s}}}^1 \frac{dz_1}{z_1} \int_{\frac{M^2}{s}}^1 \frac{dy_1}{y_1} \int_{\frac{m_f^2}{s}}^1 \frac{dz_2}{z_2} \int_{\frac{m_f^2}{s}}^1 \frac{dy_2}{y_2} = \frac{7}{12} \log^4 \left( \frac{M}{\sqrt{s}} \right), \quad (A.7)
\]
and hence by means of $\Theta(y_2 - y_1) = 1 - \Theta(y_1 - y_2)$ we find for $M_1 = \lambda$ and $M_2 = M$

$$J^{(2)}_{\text{angular}}(\lambda, M) = \left[ \frac{1}{4} \log^2 \left( \frac{M^2}{s} \right) \right] \times \left[ \frac{1}{4} \log^2 \left( \frac{\lambda^2}{s} \right) - \frac{1}{4} \log^2 \left( \frac{\lambda^2}{m_f^2} \right) \right] - \frac{7}{12} \log^4 \left( \frac{M}{\sqrt{s}} \right). \quad (A.8)$$

For the double (energy and angular) ordered integrals, see Eq. (\ref{eq:double_ordered_integral}), we find for the $M_1 = M_2 = M$ case

$$J^{(2)}_{\text{double ordered}}(M, M) =$$

$$= \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_2 z_2 - \frac{M^2}{s}) \Theta(y_2 - y_1) \Theta(z_1 - z_2)$$

$$= \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} = \frac{1}{4} \left[ \frac{1}{4} \log^2 \left( \frac{M^2}{s} \right) \right]^2. \quad (A.9)$$

For $M_1 = M$ and $M_2 = \lambda$ we obtain

$$J^{(2)}_{\text{double ordered}}(M, \lambda) =$$

$$= \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_2 z_2 - \frac{\lambda^2}{s}) \Theta(y_2 - y_1) \Theta(z_1 - z_2)$$

$$= \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} = \frac{1}{3} \log^4 \left( \frac{M}{\sqrt{s}} \right). \quad (A.10)$$

Finally for $M_1 = \lambda$ and $M_2 = M$ (note here that in the ‘frog’ configurations no two photons can appear in the integration kernel) the double-ordered integral reads

$$J^{(2)}_{\text{double ordered}}(\lambda, M) =$$

$$= \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{\lambda^2}{s}) \Theta(y_1 - \frac{m_f^2}{s z_1}) \Theta(y_2 z_2 - \frac{M^2}{s}) \Theta(y_2 - y_1) \Theta(z_1 - z_2)$$

$$= \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_2 z_2 - \frac{\lambda^2}{s}) \Theta(y_2 - \frac{m_f^2}{s z_2}) \Theta(y_1 - y_2) \Theta(z_2 - z_1). \quad (A.11)$$

Now we can write

$$\Theta(y_1 - y_2) \Theta(z_2 - z_1) = [1 - \Theta(y_2 - y_1)] \Theta(z_2 - z_1)$$

$$= \Theta(z_2 - z_1) - \Theta(y_2 - y_1) + \Theta(y_2 - y_1) \Theta(z_1 - z_2) \quad (A.12)$$
and hence

\begin{align*}
J^{(2)}_{\text{double ordered}}(\Lambda, M) &= -\frac{7}{12} \log^4 \left( \frac{M}{\sqrt{s}} \right) + \frac{1}{3} \log^4 \left( \frac{M}{\sqrt{s}} \right) \\
&\quad + \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \Theta(y_1 z_1 - \frac{M^2}{s}) \Theta(y_2 z_2 - \frac{\lambda^2}{s}) \Theta(y_2 - \frac{m^2 f z_2}{s}) \Theta(z_2 - z_1) \\
&= -\frac{1}{4} \log^4 \left( \frac{M}{\sqrt{s}} \right) + \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dy_1}{y_1} \int_0^1 \frac{dz_2}{z_2} \int_0^{z_2} \frac{dy_2}{y_2} \\
&= -\frac{1}{4} \log^4 \left( \frac{M}{\sqrt{s}} \right) + \frac{2}{3} \log^3 \left( \frac{M}{\sqrt{s}} \right) \log \left( \frac{m f}{\sqrt{s}} \right) .
\end{align*}

(A.13)

\section{B Bosonic one-loop self-energies}

For completeness we give in this appendix the full (i.e. \textit{‘non-derivative’}) bosonic self-energies at one-loop level. As an example we present the explicit derivation of the $g_{\mu\nu}$ part of the $W$-boson self-energy, which is a basic building block in the calculation of all two-loop Sudakov correction factors (see Sect. 5). We first have to select the contributing diagrams. In principle there are 4×4 possible generic combinations of scalar, mixed and gauge-boson particle states in the upper and lower part of the loop, which are displayed in Fig. (1). As explained in Sect. 4.1, the fermion- and ghost-loop contributions do not have the right pole structure for producing Sudakov logarithms.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Diagrams that can give rise to Sudakov logarithms in the one-loop gauge-boson self-energy}
\end{figure}

The only way to obtain $g_{\mu\nu}$ contributions from the $W$-boson self-energy is by having a gauge boson as lower (energetic) particle in the loop. The mixed and scalar propagators do not contain a $g_{\alpha\beta}$ term and hence there is no way to contract the Lorenz index $\nu$ through to the other
side of the diagram along the lower line. In principle this can be circumvented by having a
gauge boson as upper (soft) particle in the loop. However, as we have seen in Sect. [11] the
presence of $1/|k_2^2|$ is crucial for obtaining Sudakov logarithms. This eliminates the $g_{\alpha\beta}$ term in
the propagator of the soft gauge boson as well as the $g_{\mu\nu}$ term from the tensor reduction of
$k_{2\mu}k_{2\nu}/k_2^2 \xrightarrow{\text{g}_{\mu\nu}} k_{2\perp\mu}k_{2\perp\nu}/k_2^2 \leq 1$. Hence we are left with the first four diagrams, out of which
diagram (2) does not contribute. In diagram (2) the scalar propagator exhibits the required pole
part of diagram (1):

$$i \Sigma_{WW,\mu\nu}^{(1)}(V_3) = \int \frac{d^4k}{(2\pi)^4} P^{\sigma\sigma'}(k_2, M_3) P^{\rho\rho'}(k_1 - k_2, M_4) (ie G_{134}) V_{\mu\sigma\rho} (ie G_{243}) V_{\nu\rho'\sigma'}.$$  

(B.1)

The totally antisymmetric coupling $e G_{ijl}$ is the triple gauge-boson coupling with all three gauge-

boson lines $(i,j,l)$ defined to be incoming at the interaction vertex, i.e. we are dealing with

$G_{134} G_{243} = G_{W^+ V_3 V_4} G_{W^+ V_4^* V_3^*} = M_W^2 V_3 V_4$ in the above expression. In our convention this coupling

is fixed according to $G_{\gamma W^+ W^-} = 1$ and $G_{ZW^+ W^-} = -\cos \theta_W / \sin \theta_W$. The tensor structures of the
two triple gauge-boson interactions read

$$V_{\mu\sigma\rho} = (k_1 + k_2)_\rho g_{\mu\sigma} + (-k_2 + k_1 - k_2)_\mu g_{\sigma\rho} + (-k_1 + k_2 - k_1)_\sigma g_{\mu\rho}$$

$$(B.2)$$

$$V_{\nu\rho'\sigma'} = -[(k_1 - k_2 + k_1)_{\sigma'} g_{\rho'\nu} + (k_2 - k_1 + k_2)_\nu g_{\sigma'\rho'} + (-k_1 - k_2)_{\rho'} g_{\sigma'\nu}]$$

$$(B.3)$$

where we have selected the part that will eventually lead to $g_{\mu\nu}$ once we consider the effective
replacement $P^{\rho\rho'}(k_1 - k_2, M_4) \rightarrow -ig^{\rho\rho'}/[(k_1 - k_2)^2 - M_W^2 + ie]$ (see discussion above). Let us
leave the soft particle $V_3$ unspecified for the time being, i.e. $V_3 = \gamma, Z, W$ are all possible. With

$$i \left[ k_2^2 - M_W^2 + ie \right] (2k_1 - k_2)_\sigma (2k_1 - k_2)_{\sigma'} P^{\sigma\sigma'}(k_2, M_3)$$

$$\approx - (2k_1 - k_2)_{\sigma} (2k_1 - k_2)_{\sigma'} \frac{k_{20}}{k_2^2} \left( k_{2\sigma} n_{\sigma'} + n_{\sigma} k_{2\sigma'} \right)$$

$$\approx - \frac{2k_{20}}{k_2^2} \left( 2k_1 \cdot k_2 - k_2^2 \right) \left( 2k_{10} - k_{20} \right) \approx \frac{4k_{10}k_{20}}{k_2^2} \left[ (k_1 - k_2)^2 - k_{10}^2 \right],$$

(B.4)
we obtain for the $g_{\mu\nu}$ part of the $W$-boson self-energy

$$i \Sigma^{(1),V_3}_{WW,g} (V_3) \approx e^2 G_{134} G_{243} \int \frac{d^4 k_2}{(2\pi)^4} \frac{(2 k_1 - k_2)_{\sigma}(2 k_1 - k_2)_{\sigma'} i P^{\sigma\sigma'}(k_2, M_3)}{[(k_1 - k_2)^2 - M_4^2 + i\epsilon]}$$

$$\approx e^2 G_{134} G_{243} \int \frac{d^4 k_2}{(2\pi)^4} \frac{4 k_{10} k_{20}}{k_2^2} \frac{(k_1 - k_2)^2 - M_4^2 + M_1^2 - k_1^2}{[(k_1 - k_2)^2 - M_4^2 + i\epsilon] [k_2^2 - M_3^2 + i\epsilon]}$$

$$\approx -e^2 G_{134} G_{243} \int \frac{d^4 k_2}{(2\pi)^4} \frac{4 k_{10} k_{20}}{k_2^2} \frac{k_1^2 - M_1^2}{[(k_1 - k_2)^2 - M_4^2 + i\epsilon] [k_2^2 - M_3^2 + i\epsilon]}, \quad (B.5)$$

in the Sudakov limit.

In the case of $V_3$ being a neutral gauge boson $(N)$ and hence $V_4$ being the $W$ boson $(M_4 = M_w)$ we are left with

$$i \Sigma^{(1)}_{WW,g} (\gamma) = [k_1^2 - M_w^2] \mathcal{F}(\lambda, M_w) \quad (B.6a)$$

$$i \Sigma^{(1)}_{WW,g} (Z) = \frac{\cos^2 \theta_w}{\sin^2 \theta_w} [k_1^2 - M_w^2] \mathcal{F}(M_Z, M_w), \quad (B.6b)$$

with

$$\mathcal{F}(M_3, M_4) = -e^2 \int \frac{d^4 k_2}{(2\pi)^4} \frac{4 k_{10} k_{20}}{k_2^2} \frac{1}{[(k_1 - k_2)^2 - M_4^2 + i\epsilon] [k_2^2 - M_3^2 + i\epsilon]}.$$ \quad (B.7)

Now from Eq. (23) we recall that

$$\delta Z^{(1)}_{WW,g} = \left. \frac{\Sigma^{(1)}_{WW,g}}{k_1^2 - M_w^2} \right|_{k_1^2=M_w^2}, \quad (B.8)$$

hence the Sudakov correction factor reads

$$\delta Z^{(1)}_{WW}(N) = (-i) G_{134} G_{243} \mathcal{F}(M_N, M_w). \quad (B.9)$$

Note here that diagrams (3) and (4) do not contribute in the above case of $V_3$ being a neutral particle. First of all, the photon does not have a would-be Goldstone boson partner. Secondly, the $Z\chi$ and $\chi Z$ mixing propagator is at both ends attached to two $W$ bosons, leading to a vanishing contribution since in the SM the $\chi$ does not couple to two $W$ bosons. Apart from the couplings, Eq. (B.3) is identical to Eq. (52). Hence the required steps to eventually obtain the Sudakov logarithms are identical to the ones given explicitly in the fermion sector. The one-loop Sudakov correction factors for transverse $W$ bosons and a soft neutral gauge boson in the loop are then given by

$$\delta Z^{(1)}_{WW}(\gamma) = Q_w^2 \mathcal{L}_\gamma(\lambda, M) \quad (B.10a)$$

$$\delta Z^{(1)}_{WW}(Z) = \frac{\cos^2 \theta_w}{\sin^2 \theta_w} \mathcal{L}(M, M) = \left[ \frac{1}{\sin^2 \theta_w} - Q_w^2 \right] \mathcal{L}(M, M), \quad (B.10b)$$

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with \( L_\gamma(\lambda, M) \) and \( L(M, M) \) being defined in Eqs. (51) and (59), respectively.

In the case of \( V_3 \) being the \( W \) boson and hence \( V_4 \) being either a photon or a \( Z \) boson we also have to calculate diagrams (3) and (4). Leaving the charge of the mixed propagator general, we obtain

\[
i \Sigma^{(1)}_{W', \nu \nu} (\phi W) = \frac{\phi(k_2)}{W^{\pm}(k_1, M_W) \nu(k_1 - k_2, M_N) W^{\pm}(k_1, M_W)}
\]

\[
= \int \frac{d^4k_2}{(2\pi)^4} (-i \epsilon \tilde{G}_N M_W) g_{\mu \rho} P^{\mu \rho'}(k_1 - k_2, M_N) \frac{\mp iM_w}{k_2^2 - M_w^2 + i\epsilon} \frac{k_{20}}{k_2^2} n^{\sigma'}(i\epsilon G_{W, NW} \mp) V_{\nu \rho \sigma'},
\]

where we have introduced the abbreviation \( \tilde{G}_\gamma = 1 \) and \( \tilde{G}_Z = \sin \theta_w / \cos \theta_w \). Again the triple gauge-boson vertex can be simplified according to Eq. (B.3): \( V_{\nu \rho \sigma'} \rightarrow -(2k_1 - k_2)_{\sigma'} g_{\rho \nu} \). Bearing in mind that \( k_{2, \sigma} n^{\sigma'} \) together with the already present factor \( k_{20} \) will kill the crucial factor \( 1/k_2^2 \), we can safely ignore the \( k_{2, \sigma} \) term. Selecting the \( g_{\nu \nu} \) part we obtain

\[
i \Sigma^{(1)}_{W', \nu \nu} (\phi W) = -e^2 \tilde{G}_N G_{W, NW} \int \frac{d^4k_2}{(2\pi)^4} 2 \frac{k_{20}}{k_2^2} \frac{2k_{20}k_{10}}{k_2^2} \frac{\mp M_w^2}{[(k_1 - k_2)^2 - M_w^2 + i\epsilon] [k_2^2 - M_w^2 + i\epsilon]},
\]

and for the contribution from diagram (4) we can immediately write

\[
i \Sigma^{(1)}_{W', \nu \nu} (W \phi) = -e^2 \tilde{G}_N G_{W, NW} \int \frac{d^4k_2}{(2\pi)^4} 2 \frac{k_{20}}{k_2^2} \frac{2k_{20}k_{10}}{k_2^2} \frac{\mp M_w^2}{[(k_1 - k_2)^2 - M_w^2 + i\epsilon] [k_2^2 - M_w^2 + i\epsilon]}.
\]

Note that Eqs. (B.12) and (B.13) are identical as a result of \( G_{W, W} = G_{W, NW} \). Hence upon adding the contributions corresponding to diagrams (1), (3) and (4), we find for the neutral particle being a photon \( i.e. G_{W, W} = G_{W, NW} = \pm 1, \tilde{G}_\gamma = 1 \]

\[
i \Sigma^{(1)}_{W, W} (w + \phi W \phi) = \left( [k_1^2 - \lambda^2] - M_w^2 \right) F(M_w, \lambda),
\]

where we can neglect the photon mass \( \lambda \) in the prefactor. For the neutral particle being the \( Z \) boson \( i.e. G_{W, W} = G_{W, ZW} = \mp \cos \theta_w / \sin \theta_w, \tilde{G}_Z = \sin \theta_w / \cos \theta_w \) we find

\[
i \Sigma^{(1)}_{W, W} (w + \phi W \phi) = \left( \frac{\cos^2 \theta_w}{\sin^2 \theta_w} [k_1^2 - M_z^2] + M_w^2 \right) F(M_w, M_z) = \frac{\cos^2 \theta_w}{\sin^2 \theta_w} [k_1^2 - M_w^2] F(M_w, M_z),
\]

making use of \( M_z^2 \cos^2 \theta_w = M_w^2 \). In all situations we find \( \Sigma^{(1)}_{W, \nu \nu} \propto (k_1^2 - M_w^2) \), in agreement with the “non-renormalization” condition (28) in Sect. 3. This proves that for the \( W \)-boson self-energy
the derivative method gives the correct results, which in fact is true to all orders in perturbation theory.

The $g_{\mu \nu}$ part of the full self-energy for neutral particles can be obtained from Eqs. (B.5), (B.12) and (B.13), bearing in mind that we have two identical contributions from the two soft limits:

$$i \Sigma^{(1)}_{\gamma_{\gamma}, g}(w + \phi w + \phi) = 2 \left( [k_1^2 - M_{w}^2] + M_{w}^2 \right) F(M_{w}, M_{\gamma})$$  \hspace{1cm} (B.16a)

$$i \Sigma^{(1)}_{\gamma_{Z}, g}(w + \phi w + \phi) = 2 \left( \frac{\cos^2 \theta_{w}}{\sin^2 \theta_{w}} [k_1^2 - M_{w}^2] - M_{w}^2 \right) F(M_{w}, M_{w})$$ \hspace{1cm} (B.16b)

$$i \Sigma^{(1)}_{\gamma_{\gamma}, g}(w + \phi w + \phi) = 2 \left( -\frac{\cos \theta_{w}}{\sin \theta_{w}} [k_1^2 - M_{w}^2] - \frac{1}{2} \left( \frac{\cos \theta_{w}}{\sin \theta_{w}} - \frac{\sin \theta_{w}}{\cos \theta_{w}} \right) M_{w}^2 \right) F(M_{w}, M_{w})$$ \hspace{1cm} (B.16c)

$$i \Sigma^{(1)}_{\gamma_{Z}, g}(w + \phi w + \phi) = 2 \left( -\frac{\cos \theta_{w}}{\sin \theta_{w}} [k_1^2 - M_{w}^2] + \frac{1}{2} \left( \frac{\sin \theta_{w}}{\cos \theta_{w}} - \frac{\cos \theta_{w}}{\sin \theta_{w}} \right) M_{w}^2 \right) F(M_{w}, M_{w})$$ \hspace{1cm} (B.16d)

Hence we find

$$i \Sigma^{(1)}_{\gamma_{\gamma}, g}(w + \phi w + \phi) = 2 \left( [k_1^2] \right) F(M_{w}, M_{w})$$  \hspace{1cm} (B.17a)

$$i \Sigma^{(1)}_{\gamma_{Z}, g}(w + \phi w + \phi) = 2 \left( \frac{\cos^2 \theta_{w}}{\sin^2 \theta_{w}} [k_1^2 - M_{w}^2] \right) F(M_{w}, M_{w})$$ \hspace{1cm} (B.17b)

$$i \Sigma^{(1)}_{\gamma_{\gamma}, g}(w + \phi w + \phi) = i \Sigma^{(1)}_{\gamma_{Z}, g}(w + \phi w + \phi) = -\frac{\cos \theta_{w}}{\sin \theta_{w}} \left( [k_1^2] + [k_1^2 - M_{w}^2] \right) F(M_{w}, M_{w})$$ \hspace{1cm} (B.17c)

or generically

$$i \Sigma^{(1)}_{\gamma_{N_1 N_2}, g}(w + \phi w + \phi) = G_{134} G_{243} \left( [k_1^2 - M_{N_1}^2] + [k_1^2 - M_{N_2}^2] \right) F(M_{w}, M_{w})$$ \hspace{1cm} (B.18)

with $G_{134} G_{243} = G_{N_1 W + W^+} G_{N_2 W^+ W^+}$. From this generic expression we see that the diagonal self-energies (with $N_1 = N_2 = N$) are proportional to the inverse pole ($k_1^2 - M_{N}^2$). This confirms the “non-renormalization” condition at the one-loop level. The explicit calculation of the full mixed $\gamma - Z$ self-energy confirms the prediction that we made in Sect. 4.2, which was based on “non-renormalization” of the masses as well as the electromagnetic charge. Since those “non-renormalization” conditions hold to all orders, we can make use of them to calculate the full two-loop mixed $\gamma - Z$ self-energy based on the knowledge of the derivatives of $\Sigma_{N_1 N_2, g}$.

We finish this appendix by giving the relevant set of bosonic one-loop (non-derivative) self-energies in the SM, using $F(M, M) = F(M, \lambda) = i L(M, M)$ and $F(\lambda, M) = i L_{\gamma}(\lambda, M)$. Denoting the momentum of the external particles by $k$, the list reads

\[52\]
• Charged-boson sector

\[ \Sigma_{WW, g}^{(1)} = [k^2 - M_W^2] \left[ L_\gamma(\lambda, M) + \left( \frac{2}{\sin^2 \theta_w} - 1 \right) L(M, M) \right] \] (B.19)

\[ \Sigma_{\phi\phi}^{(1)} = -[k^2 - M_W^2] \left[ L_\gamma(\lambda, M) + \frac{\cos \theta_w}{\sin \theta_w} L(M, M) \right] - k^2 \left[ \frac{1}{4 \cos^2 \theta_w} - \frac{1}{4 \sin^2 \theta_w} \right] L(M, M) \] (B.20)

\[ \Sigma_{W^+\phi^+, n}^{(1)} = -k_0 M_W \left[ \frac{1}{8 \cos^2 \theta_w} - \frac{1}{8 \sin^2 \theta_w} \right] L(M, M) \] (B.21)

• Neutral-boson sector

\[ \Sigma_{\gamma\gamma, g}^{(1)} = 2 \left[ k^2 \right] L(M, M) \] (B.22)

\[ \Sigma_{ZZ, g}^{(1)} = 2 \frac{\cos \theta_w}{\sin^2 \theta_w} \left[ k^2 - M_Z^2 \right] L(M, M) \] (B.23)

\[ \Sigma_{\gamma Z, g}^{(1)} = \Sigma_{\phi Z, g}^{(1)} = -\frac{\cos \theta_w}{\sin \theta_w} \left( k^2 + \left[ k^2 - M_Z^2 \right] \right) L(M, M) \] (B.24)

\[ \Sigma_{\chi, n}^{(1)} = -\Sigma_{\chi, n}^{(1)} = -i k_0 M_Z \frac{\cos \theta_w}{2 \sin \theta_w} L(M, M) \] (B.25)

\[ \Sigma_{\chi, n}^{(1)} = -\Sigma_{\chi, n}^{(1)} = \frac{i}{4} k_0 M_Z \left( \frac{\cos \theta_w}{\sin^2 \theta_w} - \frac{1}{2 \cos^2 \theta_w \sin^2 \theta_w} - 1 \right) L(M, M) \] (B.26)

\[ \Sigma_{\chi\chi}^{(1)} = -\frac{1}{4 \sin^2 \theta_w} \left( 2 k^2 + \frac{k^2}{\cos^2 \theta_w} - 4 M_Z^2 \cos^2 \theta_w \right) L(M, M) \] (B.27)

\[ \Sigma_{HH}^{(1)} = -[k^2 - M_H^2] \left( \frac{3}{4 \sin^2 \theta_w} + \frac{1}{4 \cos^2 \theta_w} \right) L(M, M) \] (B.28)

These expressions are in agreement with the various “non-renormalization” conditions listed in Sect. 3.

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