The speed at which the magnetic interaction propagates along a chain of classic dipoles is discussed here. While in the quantum information counterpart for long-range interacting spins, where the speed of propagation of the information plays a paramount role, it is not strictly clear whether a light cone exists or not, here we provide numerical evidence that interacting dipoles do possess a linear light cone shortly after a perturbation takes place. Specifically, a power-law expansion occurs which is followed by a linear propagation of the associated interaction. As opposed to the quantum case, and in analogy with the so-called speed of gravity problem, we find that the speed of propagation of information can be arbitrarily large in the classic context. In order to agree with special relativity, we propose the derivation of a frame-independent Landau-Lifshitz equation.
dipole-dipole interaction. In magnetically ordered crystals, the ground state of the spin system is determined by the exchange interaction. There, the role of dipole-dipole force is, among other effects, to stimulate the formation of domains. Now, at this juncture, we have a “natural” interaction decreasing with $\alpha = 3$, whose leading term resembles the Heisenberg one. Therefore, the study of the dipole dynamics in an array of dipoles will become the main goal of the present work. Fortunately, there exists an equation, derived by Landau and Lifshitz [18], that can describe the time evolution of each dipole. However, as opposed to the quantum case, one needs to stress the fact that the spatial dimension, regardless of the arrangement of dipoles, is always $D = 3$. This will have consequences for no Lieb-Robinson bound could be applied in this case (conditions relating $\alpha$ and $D$ need not hold in our case).

**Statement of the problem.—** The Landau-Lifshitz equation that governs the dipole dynamics is given by

$$\frac{d s_l}{dt} = -s_l \times \vec{H}_l, \quad \vec{H}_l = C_M \sum_{l' \neq l} \left( 3 \frac{s_l^{(z)} \cdot r_{l,l'}^{(z)}}{||r_{l,l'}||^3} s_{l'}^{(z)} - s_l^{(z)} \right) \frac{1}{||r_{l,l'}||^4},$$

$$\frac{1}{C_M} \frac{d s_l}{dt} = A \cdot s_l,$$  \hspace{1cm} (1)

where the evolution for individual dipoles is explicitly shown (third equation), $C_M = \frac{\mu_0}{4\pi}$, and the corresponding matrix $A$ is given by

$$
\begin{pmatrix}
0 & \sum_{l' \neq l} \frac{1}{||r_{l,l'}||^3} s_{l'}^{(z)} & 2 \sum_{l' \neq l} \frac{1}{||r_{l,l'}||^3} s_{l'}^{(y)} \\
-\sum_{l' \neq l} \frac{1}{||r_{l,l'}||^3} s_{l'}^{(z)} & 0 & \sum_{l' \neq l} \frac{1}{||r_{l,l'}||^3} s_{l'}^{(z)} \\
-2 \sum_{l' \neq l} \frac{1}{||r_{l,l'}||^3} s_{l'}^{(y)} & \sum_{l' \neq l} \frac{1}{||r_{l,l'}||^3} s_{l'}^{(y)} & 0
\end{pmatrix}.
$$

(2)

In a one dimensional system parallel to the $x$ axis, $e_{l,l'} \parallel e_x$ and

$$H_l = \sum_{l' \neq l} s_{l'} - 3s_{x,l} e_x$$

(3)

One way to see that the magnitude of each dipole is preserved is via the nature of $A$. This matrix is a skew-symmetric one, which implies that all non-zero eigenvalues come in pairs or purely imaginary numbers and, in turn, guarantees that $||s_l(t)||$ remains constant $\forall l,t$. Since no damping term is considered in (1), the total dipole-dipole energy is preserved.

Let us, for the sake of numerical ease, normalize to unity $s_l(t) = (s_{l}^{(x)}(t),s_{l}^{(y)}(t),s_{l}^{(z)}(t))$ and take differences in terms of $a$, the interdipole distance. Then $C_M$ becomes $\frac{\mu_0 m}{4\pi a^3}$, where $m$ is the magnitude of the dipole and $a$ the interdipole distance. Incidentally, and without loss of generality, $m$ can be numerically adjusted such that $C_M$ is equal to one.

We shall consider two important configurations for the dipoles. The first one consists of an finite chain of $N$ dipoles ($N$ sufficiently big enough) along the $x$ direction, as described in Fig. 1. Suppose that the total number of dipoles is odd $N = 2M + 1$. We do not choose a setting where a perturbation starts at the beginning of the chain in order to avoid any finite-size effects. Thus, to such an end, at $t = 0$ we will set all dipoles pointing upwards, and the one in the middle, that is, at position $l = M$, pointing perpendicularly to the rest. In this fashion, we will reproduce all results as if they occurred inside the bulk, and therefore avoiding the ends provided $N$ is large enough. Specifically, $s_l(t = 0) = (0, 1, 0) \forall l \neq M$ and $s_M(t = 0) = (1, 0, 0)$. Notice that no time-dependency would occur classically in (1) if there was no projection of dipole $l = M$ along the $y$ axis, as opposed to the quantum case. Needless to say, all results will be symmetric at both sides of $l = M$. The second instance considered is given in Fig. 2, with $s_l(t = 0) = (1, 0, 0) \forall l \neq M$ and $s_M(t = 0) = (0, 1, 0)$. At this point, one has to bare in mind that the speed of the dipole-dipole interaction is infinite classically, as seen in (1). However, when dipoles start to move, their interaction mediated among themselves is not instantaneous.

At this juncture, and previous to any numerical computation, we shall clarify how the detection of information propagation translates in practice. In quantum information, the fidelity quantity $\mathcal{F} = ||\langle \Phi | \Psi \rangle||$ measures...
how similar two quantum states are. Classically, and speaking of dipoles, in the first case of Fig. 1, we need to assess the first dipole \( t' \) such that at the time \( t = t' \), the classical fidelity [10] \( F \equiv \langle s(t') \lvert s(t) \rangle \) starts to differ from 1. Let us define \( F \equiv 1 - \delta \). Thus, the best that can be done is to compute the speed of propagation of information \( v_s \) along the chain of interacting dipoles from \( \Delta L = l' - M \) and \( t = t' \) as soon as \( s(t') \leq 1 - \delta \). This situation has profound consequences in defining an “exact” speed of propagation \( v_s \), for it depends on the value of \( \delta \). In point of fact, and for any numerical scheme, the resolution of Eq. (1) will essentially depend on the time step \( h t \) regardless of the desired accuracy. In other words, \( s(t') = 1 - \delta \) always depend on \( h t \) via \( \delta = f(h t) \) in an extremely non-trivial fashion. This means to directly “detect” the interaction heavily depends on the machine precision, and it is doomed to fail in order to provide a unique answer. However, although not shown here, this direct method qualitatively describes what is found later on. In order to obtain quantitative results, what shall be done in practice is to adjust on the position-time plane of the evolution of the dipoles a certain curve. The quantity over which we will consider constant values will be that of \( 1 - F \). Obviously, the lowest value provided by the numerical computations will be template over which we shall infer the propagation of the information.

Results.- The details of the numerical resolution of the time evolution of all dipoles in all settings is described in the Supplementary Material. What is relevant is that the numerical scheme guarantees, both the preservation of the total energy of the system and the magnitude of all time-evolved dipoles. Additionally, \( C_M \) is set to 1.

The first instance that we shall consider is the configuration of \( N = 213 \) dipoles as shown in Fig. 1. The middle dipole points towards the positive \( x \) axis, whereas all the rest point upwards. Although it is not maximal, this configuration is of high energy per dipole. The numerical solution of the coupled set of equations (1) returns that the position travelled from the central dipole by the interaction evolves as \( x(t) = A \cdot t^3 \cdot a \) (\( a \) is the inter-dipole distance) initially, which is followed by a linear expression of the type \( B + v_s t \), where \( v_s \) is the speed of propagation of the information. \( v_s \) is, generally, of \( O(10) \) in units of \( a \) per second. To be more precise, we obtained \( x(t) = 106 \pm 43 t^{1/3} \), followed by \( 121 \pm 56 t \).

The speed \( v_s(t) = \frac{dx(t)}{dt} = A \cdot t^3 \cdot a \) is initially infinite, and eventually becomes constant, usually beyond \( t = 0.1 \). The speed \( v_s \) at \( t = h t \) is \( A \cdot t \cdot h t^3 \cdot a \), which can be arbitrarily large as compared to the speed of light, which is in flagrant contradiction with the tenets of special relativity.

We have deliberately avoided finite-size effects in the system by not considering the interaction when it reaches the last dipoles at both ends. Considering larger chains only extends the linear regime. Therefore, our results can be regarded as effectively occurring in the bulk. Also, some further analysis shows that \( v_s \) in the linear light cone is indeed proportional to \( m \), the magnitude of the dipole. Obviously, \( m = 0 \) would imply considering no dipoles and, hence, no possible propagation. The bare analysis of the fidelity in Fig. 1 clearly shows the typical V shape of the linear light cone, as it extends towards the ends of the dipole lattice.

Another setting shall be given as follows: in a chain of \( N = 2M + 1 \) dipoles, \( s(t) = (0, 1, 0) \) and \( s_{M} = (0, 0, 1) \), as depicted in Fig. 2 for \( N = 257 \) dipoles. Notice that not only the functional form coincides with the case in Fig. 1, but the coefficient \( A \) for \( t^{1/3} \) and the speed \( v_s \) are almost the same in both cases, respectively. Also, if we assume that the propagation of the interaction occurs continuously, imposing continuity and derivability in both regimes we obtained that the term \( B \) in the linear regimes goes as \( B = \frac{2}{\sqrt{3}} \frac{1}{\sqrt{v_s}} A^{3/2} \). Thus, the only two quantities needed to quantitatively describe the propagation of the information along the chain, in both two regimes, are \( A \) and \( v_s \).

It is remarkable that the speed of the linear light cone \( v_s \) seems not to depend on the particular dipole configuration. In the setting of Fig. 1, the overall dipole-dipole energy interaction is greater than the one in Fig. 2, which is in fact the ground state in the thermodynamic limit. Thus, \( v_s \) seems to be a quantity that solely depends on the dipole-dipole interaction nature, as well as the \( A \) accompanying \( t^{1/3} \).

The evolution of the fidelity \( F \) or \( 1 - F \) for all dipoles, is shown in Fig. 2. Interestingly enough, this configuration leads to two solitons moving away from the central dipole at further times. When the interaction reaches the ends of the chain, the dipoles start to bounce back, precisely at the same speed at which the solitons are moving.

Admittedly, the previous method for finding the functional form for the time evolution may appear somehow artificial, although it is numerically correct. These result does not really come as a surprise: our initial configuration (in the second case) is with all dipoles parallel to \( x \) while the source, which for convenience we can place at the origin, is parallel to \( z \). Therefore the field felt by a dipole at location \( x \) writes, with all relevant constants set to one:

\[
H(x) = \frac{s(0)}{x^3} = \frac{s_z(0)}{x^3}\mathbf{e}_z.
\]

Now, this should be inserted into the Landau-Lifshitz equation (1). At the onset of the motion, we can consider \( s_z(0) = 1 \) to be constant as it evolves as a cosine : \( 1 - \gamma^2/2 \) where \( \gamma \) is the angle it makes with the \( z \) axis. Thus:

\[
\frac{ds(x, t)}{dt} = -\frac{s_z(0)}{x^3}\mathbf{e}_z \times s(x, t)
\]
and:

$$\frac{ds_y(x, t)}{dt} = -\frac{1}{x^3} s_x(x, t)$$

again, $s_x(x, t)$ evolves as the cosine of the angle with the $x$ axis, and can be considered constant $= 1$ for short time intervals $dt$, finally

$$x = \left( \frac{dt}{ds_y} \right)^{\frac{1}{3}}$$

That is, the distance $x$ from the source at which we observe a change $ds$ increases as $dt^{\frac{1}{3}}$.

The same reasoning with $H \propto x^{-\alpha}$ would yield

$$x = \left( \frac{dt}{ds_y} \right)^{\frac{1}{\alpha}} \quad (4)$$

and it so turns out that a simulation with $H \propto 1/r^4$ does indeed produce a $dt^{\frac{1}{4}}$ behavior. Equation (4) thus appears to be general for very short time intervals. This is certainly of paramount importance when compared to the quantum case: shortly after propagating, the information will travel along the classic chain of dipoles as $At^{\frac{1}{3}}$, for interactions decaying as $1/r^\alpha$.

The dependence of $v_s$ on the magnitude of the dipole is apparent from closer inspection of the Landau-Lifshitz equation (1). The transformation $t \rightarrow \frac{\mu}{4\pi m} t$ implies that $v_s \propto \frac{\mu}{4\pi m}$. Thus, the greater the magnitude of the dipoles, the faster the propagation of the interaction. Evidently, it also applies to changing the magnetic permeability and/or the interparticle distance. Remarkably, the value of $B = \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{\alpha}} A^{3/2}$ remains invariant under such rescaling.

We need to conciliate the finite speed of the magnetic interaction with the bound of information propagation defined by the speed of light $c$. This can be introduced ad hoc in the form of a “retarded” interaction, with $t \rightarrow t + \frac{1}{2} ||\vec{r} - \vec{r}_\text{source}||$ plus a covariant formulation for the Landau-Lifshitz equation.

Conclusions.– We have shed new light on the existence of light cones in classic linear chains of interacting dipoles. By considering two different dipole configurations, we obtain a non-linear light cone that goes as $\propto t^{1/3}$ for very short initial times, which gives rise afterwards to a linear one, usually for $t \geq 0.1$. The conciliation of having a speed of information propagation less than the speed of light is possible by introducing a retarded action à la Liénard-Wiechert in the Landau-Lifshitz equation. Regarding the connection between long ranged interactions and quantum physics, the basic principle behind is that symmetry rules in quantum physics mean effective long ranged interactions which are evidenced in superconductivity, superfluidity, and so on such as the second sound in helium, albeit there is not a direct Hamiltonian for it. Finally, we conjecture that $A$ and $v_s$ in $At^{1/3}$, $B + v_s t$ solely depends on universal constants, as well as the exponent 3 in $1/r^3$. This conclusion easily extends to similar interactions of the type $1/r^\alpha$.

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SUPPLEMENTARY MATERIAL

I. INTEGRATION METHOD

Let us consider the second kind of dipole configuration described in the present work. Initial conditions were set as such: a total number of 1024 dipoles are set on the $x$ axis, all of them parallel to $\mathbf{e}_x$, except for $n = 512$ which is set parallel to $\mathbf{e}_z$. The propagation of this perturbation can be monitored by observing the normal component $S_N = \sqrt{s^2_{x_n}(t) + s^2_{y_n}(t)}$ of the dipoles.

This is shown in Fig. 3. A precursor is also revealed by the logarithmic color-scale. The white lines are, at this stage, guides for the eye, with a $t^{3/4}$ time-dependence.

The instantaneous motion generated by the Landau-Lifshitz equation is a precession motion around the field $\mathbf{H}_n$: standard integration methods, such as Euler or Runge-Kutta, do not usually conserve $|s_n|$, so one adequate method is to use the Rodrigues rotation matrix $R_n(t)$, so that:

$$s_n(t + ht) = R_n(t) s_n(t)$$

where $ht$ being the integration time-step, with

$$R_n(t) = \begin{pmatrix}
    h_x^2w + v & h_xh_yw - h_zu & h_xh_zw + h_yu \\
    h_yh_xw + h_zu & h_y^2w + v & h_yh_zw - h_xu \\
    h_zh_xw - h_yu & h_zh_yw + h_xu & h_z^2w + v
\end{pmatrix}$$

where $u = \sin \omega$, $v = \cos \omega$, $w = 1 - v$, while the precession angle is $\omega = |\mathbf{H}_n|ht$. Coordinates $h_{x,y,z}$ are those of the unit vector $h_n = \mathbf{H}_n/|\mathbf{H}_n|$. The so-called “improved Euler” or Heun method can then be used to integrate the Landau-Lifshitz equation.

The dipole-dipole interaction being long-ranged, the sum on $\ell$ for the field in the Landau-Lifshitz equation is bound to be costly: writing the interaction as a convolution and using fast Fourier transforms and zero-padding considerably accelerates computations.

The simulations were done without damping nor temperature regulation; the time-step $ht = 2.5 \times 10^{-3}$ was chosen to ensure energy conservation. The use of smaller $ht$ is important for a greater time-resolution at the very beginning of the time evolution provided by the Landau-Lifshitz equation.

II. PRECURSOR ANALYSIS METHOD

To analyze the precursor, one cannot simply rely upon observing one given contour line. To ensure that all dipoles with measurable motion have the same behavior, a master plot (Fig. 4) was done by rescaling $S_N$ to obtain superposition. This actually occurs within the interval $[0, 0.1]$ approximately. The master line is straight, proportional to $t$. The scaling coefficient turns out to be a constant $a$. A given contour line is characterized by $S_N = C$ a constant. Thus

$$x = \left(\frac{at}{C}\right)^{1/3}$$

Therefore, the distance covered by a contour line is proportional to $t^{3/4}$ as could be guessed from Fig. 3.

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FIG. 3. Normal components $S_N$ of $s_n(t)$ (logarithmic color-scale) for the central dipoles at the onset of the motion. White lines indicate $t^{3/5}$ behavior.

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FIG. 4. Master plot of $S_N$ for neighbors of the central perturbated dipole. Both scales are logarithmic. All curves are scaled to fit on the same line.