Energy invariant for shallow water waves and the Korteweg – de Vries equation.

Is energy always an invariant?

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It is well known that the KdV equation has an infinite set of conserved quantities. The first three are often considered to represent mass, momentum and energy. Here we try to answer the question of how this comes about, and also how these KdV quantities relate to those of the Euler shallow water equations. Here Luke’s Lagrangian is helpful. We also consider higher order extensions of KdV. Though in general not integrable, in some sense they are almost so.

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I. INTRODUCTION

There exists a vast number of papers dealing with the shallow water problem. Aspects of the propagation of weakly nonlinear, dispersive waves are still being studied. Last year we published two articles [1, 2] in which Korteveg–de Vries type equations were derived in weakly nonlinear, dispersive and long wavelength limit. The second order KdV type equation was derived. The second order KdV equation [3, 4], sometimes called "extended KdV equation", was obtained for the case with a flat bottom. In derivation of the new equation we adapted the method described in [4]. In [2], an analytic solution of this equation in the form of a particular soliton was found, as well.

It is well known, see, e.g. [5, 6], Ch. 5), that for the KdV equation there exists an infinite number of invariants, that is, integrals over space of functions of the wave profile and its derivatives, which are constants in time. Looking for analogous invariants for the second order KdV equation we met with some problems even for the standard KdV equation (which is first order in small parameters). This problem appears when energy conservation is considered.

In this paper we reconsider invariants of the KdV equation and formulas for the total energy in several different approaches and different frames of reference (fixed and moving ones). We find that the invariant \( I^{(3)} \), sometimes called the energy invariant, does not always have that interpretation.

We also give a proof that for the second order KdV equation, obtained in [1–4], \( \int_{-\infty}^{\infty} \eta^2 dx \) is not an invariant of motion.

The paper is organized as follows. In Section II several frequently used forms of KdV equations are recalled with particular attention to transformations between fixed and moving reference frames. In Section III the form of the three lowest invariants of KdV equations is derived for different forms of the equations. In Section IV we show that the energy calculated from the definition \( H = T + V \) has no invariant form. Section V describes the variational approach in a potential formulation which gives a proper KdV equation but fails in obtaining second order KdV equations. In the next section the proper invariants are obtained from Luke’s Lagrangian density. The closing section summarizes the conclusions.

II. THE EXTENDED KDV EQUATION

The geometry of shallow water waves is presented in Fig. 1.

In [1, 2] we derived an equation, second order in small parameters, in the fixed reference system and with scaled nondimensional variables containing terms for bottom fluctuations. They will not be considered here.

For a flat bottom that equation reduces to the second order KdV type equation, identical with [4, Eq. (21)] for \( \beta = \alpha \), that is,

\[
\eta_t + \eta_x + \alpha \frac{3}{2} \eta \eta_x + \beta \frac{1}{6} \eta_{3x} + \alpha^2 \left( -\frac{3}{8} \eta_x \eta_{3x} \right) + \alpha \beta \left( \frac{23}{24} \eta_x \eta_{2x} + \frac{5}{12} \eta_{3x} \right) + \beta^2 \frac{19}{360} \eta_{5x} = 0.
\]
Transformation to a moving frame. In both, in a fixed frame of reference and in a moving frame of reference for KdV equations in a fixed coordinate system extended KdV equation".

Equation (1) was earlier derived in [3] and called "the extended KdV equation".

Limitation to the first order in small parameters yields the KdV equation in a frame moving with the velocity of sound

$$h \eta_{xx} = \frac{a}{h}. \beta = \left( \frac{h}{l} \right)^2.$$  

FIG. 1. Schematic view of the geometry.

Subscripts denote partial differentiation. Small parameters $\alpha, \beta$ are defined by ratios of the wave amplitude $a$, the average water depth $h$ and mean wavelength $l$

$$\alpha = \frac{a}{h}, \quad \beta = \left( \frac{h}{l} \right)^2.$$  

Transformation to a moving frame in the form

$$\bar{x} = (x - t), \quad \bar{t} = t, \quad \bar{\eta} = \eta,$$  

allows us to remove the term $\eta_x$ in the KdV equation in a frame moving with the velocity of sound $\sqrt{gh}$

$$\bar{\eta}_t + \frac{3}{2} \bar{\eta} \bar{\eta}_x + \beta \frac{1}{6} \bar{\eta}_{xxx} = 0.$$  

The explicit form of the scaling leading to equations (1)–(4) is given by (29).

Problems with mass, momentum and energy conservation in the KdV equation were discussed in [8] recently.

In this paper the authors considered the KdV equations – (4) is given by (29).

What invariants can be attributed to equations (1) – (2) and (5) – (6)?

It is well known, see, e.g. [7, Ch. 5], that an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,$$  

where neither $T$ (an analog to density) nor $X$ (an analog to flux) contain partial derivatives with respect to $t$, corresponds to some conservation law. It can be applied, in particular, to KdV equations (where there exist an infinite number of such conservation laws) and to the equations of KdV type like (1). Functions $T$ and $X$ may depend on $x, t, \eta, \eta_x, \eta_{2x}, \ldots, h, h_x, \ldots$, but not $\eta_t$. If both functions $T$ and $X$ are integrable on $(-\infty, \infty)$ and $\lim_{x \to \pm \infty} X = \text{const}$ (soliton solutions), then integration of equation (7) yields

$$\int_{-\infty}^{\infty} T d\bar{x} = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} T d\bar{x} = \text{const},$$  

since

$$\int_{-\infty}^{\infty} X_x d\bar{x} = X(\infty, t) - X(-\infty, t) = 0.$$  

The same conclusion applies for periodic solutions (cnoidal waves), when in the integrals (6), (9) limits of integration $(-\infty, \infty)$ are replaced by $(a, b)$, where $b - a = \Lambda$ is the space period of the cnoidal wave (the wave length).

III. INVARIANTS OF KDV TYPE EQUATIONS

A. Invariants of the KdV equation

For the KdV equation (2) the two first invariants can be obtained easily. Writing (2) in the form

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left( \eta + \frac{3}{4} \eta^2 + \frac{1}{6} \beta \eta_{xx} \right) = 0.$$  

one immediately obtains the conservation of mass (volume) law

$$I^{(1)} = \int_{-\infty}^{\infty} \eta \, dx = \text{const}.$$  

Similarly, multiplication of (2) by $\eta$ gives
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \eta^2 \right) + \frac{\partial}{\partial x} \left( \frac{1}{2} \eta^2 + \frac{1}{2} \alpha \eta^3 - \frac{1}{12} \beta \eta_x^2 + \frac{1}{6} \beta \eta_{xx} \right) = 0,
\]
resulting in the invariant of the form
\[
I^{(2)} = \int_{-\infty}^{\infty} \eta^2 \, dx = \text{const.}
\] (13)

In the literature of the subject, see, e.g. [7, 8]. $I^{(2)}$ is attributed to momentum conservation.

Invariants $I^{(3)}$, $I^{(2)}$ have the same form for all KdV equations (2), (4), (A2), (5), (6).

Denote the left hand side of (2) by KDV$(x,t)$ and take
\[
3\eta^2 \times \text{KDV}(x,t) = -2 \frac{\beta}{3 \alpha} \eta_x \times \frac{\partial}{\partial x} \text{KDV}(x,t).
\] (14)
The result, after simplifications is
\[
\frac{\partial}{\partial t} \left( \eta^3 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \right) + \frac{\partial}{\partial x} \left( \frac{9}{8} \alpha \eta^4 + \frac{1}{2} \beta \eta_x \eta^2 \right) - \beta \eta_x^2 \eta + \eta^3 + \frac{1}{18} \alpha \eta_x^2 - \frac{1}{18} \eta \eta_x \eta_{3x} - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 = 0.
\]
Then the next invariant for KdV in the fixed reference frame (2) is
\[
I^{(3)}_{\text{fixed frame}} = \int_{-\infty}^{\infty} \left( \eta^3 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \right) \, dx = \text{const.}
\] (16)

The same invariant is obtained for the KdV in the moving frame (4). The same construction like (14) but for equation (4) results in
\[
\frac{\partial}{\partial t} \left( \eta^3 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \right) + \frac{\partial}{\partial x} \left( \frac{9}{8} \alpha \eta^4 + \frac{1}{2} \beta \eta_x \eta^2 \right) - \beta \eta_x^2 \eta + \eta^3 + \frac{1}{18} \alpha \eta_x^2 - \frac{1}{18} \eta \eta_x \eta_{3x} - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 = 0.
\]
Then the next invariant for KdV equation in moving reference frame (2) is
\[
I^{(3)}_{\text{moving frame}} = \int_{-\infty}^{\infty} \left( \eta^3 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \right) \, dx = \text{const.}
\] (18)

The procedure similar to those described above leads to the same invariants for both equations (5) and (6) where KdV equations are written in dimensional variables. To see this, it is enough to take $3\eta^2 \times \text{kdv}(x,t) - \frac{1}{2} \frac{\beta}{\alpha} \frac{\partial}{\partial x} \text{kdv}(x,t) = 0$, where kdv$(x,t)$ is the lhs either of (5) or (6). For equation (5) the conservation law is
\[
\frac{\partial}{\partial t} \left( \eta^3 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \right) + \frac{\partial}{\partial x} \left( \frac{9c}{8h} \eta^4 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \eta_x \right)
\]
\[
- \frac{1}{2} \frac{\beta}{\alpha} \eta_x^2 \eta_{xx} + \frac{1}{18} \frac{\beta}{\alpha} \eta_x^2 \eta_{xxx} + \frac{1}{18} \frac{\beta}{\alpha} \eta_x^2 \eta_{xxx} = 0,
\]
whereas for equation (6) the flux term is different
\[
\frac{\partial}{\partial t} \left( \eta^3 - \frac{h^3}{3} \eta_x^2 \right) + \frac{\partial}{\partial x} \left( \frac{9c}{8h} \eta^4 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \eta_x \right)
\]
\[
+ \frac{1}{2} \frac{\beta}{\alpha} \eta_x^2 \eta_{xx} + \frac{1}{18} \beta \eta_x^2 \eta_{xxx} - \frac{1}{9} \frac{\beta}{\alpha} \eta_x^2 \eta_{xxx} = 0.
\] (20)

But in both cases the same $I^{(3)}$ invariant is obtained as
\[
I^{(3)}_{\text{dimensional}} = \int_{-\infty}^{\infty} \left( \eta^3 - \frac{h^3}{3} \eta_x^2 \right) \, dx = \text{const.}
\] (21)

**Conclusion** Invariants $I^{(3)}$ have the same form for fixed and moving frames of reference when the transformation from fixed to moving frame scales are the same way (e.g. $x' = x - t$ and $t' = t$). When scaling factors are different, like in (A11), then the form of $I^{(3)}$ in the moving frame differs from the form in the fixed frame, see Appendix (A).

For those solutions of KdV which preserve their shapes during the motion, that is, for cnoidal solutions and single soliton solutions, integrals of any power of $\eta(x,t)$ and any power of arbitrary derivative of the solution with respect to $x$ are invariants. That is,
\[
I^{(a,n)} = \int_{-\infty}^{\infty} (\eta^{a,n}) \, dx = \text{const.}
\] (22)

where $n = 0, 1, 2, \ldots$ and $a \in \mathbb{R}$ is an arbitrary real number. Then an arbitrary linear combination of $I^{(a,n)}$ is an invariant, as well.

**B. Invariants of the second order equations**

Can we construct invariants for KdV type equations of the second order? Let us try to take $T = \eta$ for equation (4). Then we find that all terms, except $\eta_t$, can be written as $X_t$, as
\[
\int \left[ \eta_x + \alpha \frac{3}{2} \eta \eta_x + \beta \frac{1}{6} \eta \eta_{3x} + \alpha^2 \left( -\frac{3}{8} \eta^2 \eta_x \right) + \alpha \beta \left( \frac{23}{24} \eta \eta_{2x} + \frac{5}{12} \eta \eta_{3x} \right) \right] \, dx = \eta + $$ \frac{3}{4} \alpha \eta^2 + \frac{1}{3} \beta \eta_{2x} - \frac{1}{6} \alpha^2 \eta^3 + $$ \alpha \beta \left( \frac{13}{48} \eta^2 + \frac{5}{12} \eta \eta_{2x} + \frac{19}{360} \beta^2 \eta_{4x} \right)
\] (23)

As (23) depends on $\eta$ and space derivatives and also since all those functions vanish when $x \to \pm \infty$, the conservation law for mass (volume)
\[
\int_{-\infty}^{\infty} \eta(x, t) \, dx = \text{const.,}
\] (24)
holds for the second order equation.

(Conservation law (24) holds for the equation with an uneven bottom, as well.)
Until now we did not find any other invariants for the second order equations. Moreover, we can show that the integral \(I^{(2)}\) is no longer an invariant of the second order KdV equation \(1\).

Upon multiplication of equation \(1\) by \(\eta\) one obtains
\[
0 = \frac{\partial}{\partial t} \left( \frac{1}{2} \eta^2 \right) + \frac{\partial}{\partial x} \left[ \frac{1}{2} \eta^2 + \frac{1}{2} \alpha \eta^3 + \frac{1}{6} \beta \left( - \frac{1}{2} \eta_x^2 + \eta_{xx} \right) \right] + \frac{3}{32} \alpha^2 \eta^4 + \frac{19}{360} \beta^2 \left( \frac{1}{2} \eta_x^2 - \eta_x \eta_{xx} + \eta_{4x} \right) \tag{25} + \frac{5}{12} \alpha \beta \eta^2 \eta_{xx}.
\]
The last term in \(25\) can not be expressed as \(\frac{\partial}{\partial x} X(\eta, \eta_x, \ldots)\). Therefore \(\int_{-\infty}^{x} \eta^2 \, dx\) is not a conserved quantity.

IV. ENERGY

The invariant \(I^{(3)}\) is usually referred to as the energy invariant. Is this really the case?

A. Energy in a fixed frame as calculated from the definition

The hydrodynamic equations for an incompressible, inviscid fluid, in irrotational motion and under gravity in a fixed frame of reference, lead to a KdV equation of the form
\[
\ddot{\eta} + \dot{\eta} x + \frac{3}{4} \alpha \dot{\eta}^2 + \frac{1}{3} \beta \dot{\eta} \ddot{\eta} = 0. \tag{26}
\]
We will find the function
\[
\tilde{f}_x = \tilde{\eta} - \frac{1}{4} \alpha \dot{\eta}^2 + \frac{1}{3} \beta \dot{\eta} \ddot{\eta}, \tag{27}
\]
obtained as a byproduct in derivation of KdV, useful in what follows. For more details see Appendix \(13\) or \(12\), Chapter 5. Tildas denote scaled dimensionless quantities.

Now construct the total energy of the fluid from the definition.

The total energy is the sum of potential and kinetic energy. In our two-dimensional system the energy in original (dimensional coordinates) is
\[
E = T + V = \int_{-\infty}^{x} \left( \int_{0}^{\infty} \frac{\rho u^2}{2} \, dy \right) \, dx \tag{28}
+ \int_{-\infty}^{x} \left( \int_{0}^{\infty} \rho g y \, dy \right) \, dx.
\]
Equations \(26\) and \(27\) are obtained after scaling \(1, 2, 4\). We now have dimensionless variables, according to
\[
\tilde{\phi} = \frac{h}{la \sqrt{gh}} \phi, \quad \tilde{x} = \frac{x}{l}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{y} = \frac{y}{h}, \quad \tilde{t} = \frac{t}{l / \sqrt{gh}}, \tag{29}
\]
and
\[
V = \rho gh^2 l \int_{-\infty}^{x} \int_{0}^{\frac{1}{2} \alpha \eta} \rho \tilde{y} \, dy \, d\tilde{x}, \tag{30}
\]
\[
T = \frac{1}{2} \rho gh^2 l \int_{-\infty}^{x} \int_{0}^{\frac{1}{2} \alpha \eta} \left( \alpha^2 \tilde{\phi}^2 + \frac{\alpha^2 \beta^2}{2} \tilde{y}^2 \right) \, dy \, d\tilde{x}. \tag{31}
\]
Note, that the factor in front of the integrals has the dimension of energy.

In the following, we omit signs \(\sim\), having in mind that we are working in dimensionless variables. Integration in \(30\) with respect to \(y\) yields
\[
V = \frac{1}{2} gh^2 l \int_{-\infty}^{x} \left( \alpha^2 \eta^2 + 2 \alpha \eta + 1 \right) \, dx \tag{32}
= \frac{1}{2} gh^2 l \left[ \int_{-\infty}^{x} \left( \alpha^2 \eta^2 + 2 \alpha \eta \right) \, dx + \int_{-\infty}^{x} \, dx \right].
\]
After renormalization (subtraction of constant term \(\int_{-\infty}^{x} \, dx\)) one obtains
\[
V = \frac{1}{2} gh^2 l \rho \int_{-\infty}^{x} \left( \alpha^2 \eta^2 + 2 \alpha \eta \right) \, dx. \tag{33}
\]

In kinetic energy we use the velocity potential expressed in the lowest (first) order
\[
\phi_x = f_x - \frac{1}{2} \beta y f_{xxx} \quad \text{and} \quad \phi_y = - \beta y f_{xx}, \tag{34}
\]
where \(f_x\) was defined in \(27\). Now the bracket in the integral \(31\) is
\[
\left( \alpha^2 \phi_x^2 + \frac{\alpha^2}{\beta} \phi_y^2 \right) = \alpha^2 \left( f_x^2 + \beta y^2 (f_x f_{xxx} + f_{xx}^2) \right). \tag{35}
\]
Integration with respect to the vertical coordinate \(y\) gives, up to the same order,
\[
T = \frac{1}{2} \rho gh^2 l \int_{-\infty}^{x} \alpha^2 \left[ f_x^2 (1 + \alpha \eta) \right. \tag{36}
+ \beta (f_{xx} f_{xxx} + f_{xx}^2) \frac{1}{3} \left( 1 + \alpha \eta \right)^3 \left. \right] \, dx
= \frac{1}{2} \rho gh^2 l \int_{-\infty}^{x} \alpha^2 \left[ f_x^2 + \alpha f_x^2 \eta + \frac{1}{3} \beta \left( f_{xx} - f_{x} f_{xx} \right) \right] \, dx.
\]
In order to express energy through the elevation function \(\eta\) we use \(27\). We then substitute \(f_x = \eta\) in terms of the third order and \(f_x = \eta^2 - \frac{1}{2} \alpha \eta^3 + \frac{1}{3} \beta \eta x^2\) in terms of the second order
\[
T = \frac{1}{2} \rho gh^2 l \alpha^2 \left[ \int_{-\infty}^{x} \left( \eta^2 - \frac{1}{2} \alpha \eta^3 + \frac{1}{3} \beta \eta x^2 \right) \right. \tag{37}
+ \int_{-\infty}^{x} \frac{1}{3} \beta \left( \eta_x^2 + \eta_{xx} \right) \left. \right] \, dx.
\]
The last term vanishes as
\[ \int_{-\infty}^{+\infty} (\eta_x^2 + \eta_{xx}^2) \, dx = \int_{-\infty}^{+\infty} \eta_x^2 \, dx + \eta_{xx}^2|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \eta_x^2 \, dx = 0. \] (38)

Therefore the total energy in the fixed frame is given by
\[ E_{\text{tot}} = T + V = \rho gh^2 l \int_{-\infty}^{+\infty} \left( \alpha \eta + (\alpha)^2 \eta_x^2 \right) \, dx + \frac{1}{4} (\alpha \eta_x^2)^2 \, dx. \] (39)

The energy (39) in a fixed frame of reference does not contain the \( I^3 \) invariant.

The result (39) gives the energy in powers of \( \eta \) only. The same structure of powers in \( \eta \) was obtained by the authors of [8], who work in dimensional KdV equations [5] and [6]. On page 122 they present a non-dimensional energy density \( E \) in a frame moving with the velocity \( U \). Then, if \( U = 0 \) is set, the energy density in a fixed frame is proportional to \( \alpha \eta + \alpha^2 \eta_x^2 \) as the formula is obtained up to second order in \( \alpha \). However, the third order term is \( \frac{1}{4} \alpha^3 \eta_x^3 \), so the formula up to the third order in \( \alpha \) becomes
\[ E \sim \alpha \eta + \alpha^2 \eta_x^2 + \frac{1}{4} \alpha^3 \eta_x^3. \] (40)

This energy density contains the same terms like (39) and does not contain the term \( \eta_x^2 \), as well.

Energy calculated from the definition does not contain a proper invariant of motion.

B. Energy in a moving frame

Now consider the total energy according to (28) calculated in a frame moving with the velocity of sound \( c = \sqrt{\frac{gh}{\rho}} \). Using the same scaling (29) to dimensionless variables we note that in these variables \( c = 1 \). As pointed out by Ali and Kalisch [8, Sect. 3] working in such system one has to replace \( \phi \) by the horizontal velocity in a moving frame, that is by \( \tilde{\phi} = \frac{\phi}{\alpha} = \alpha \eta - \frac{1}{4} \alpha^2 \eta_x^2 + \beta \left( \frac{3}{2} - \frac{\alpha}{\beta} \right) \eta_{xx} - \frac{1}{\alpha} \eta_{xxx} \).

Then repeating the same steps as in the previous subsection yields the energy expressed by invariants
\[ E_{\text{tot}} = \rho gh^2 l \int_{-\infty}^{+\infty} \left[ \frac{1}{2} \alpha \eta + \frac{1}{4} (\alpha \eta_x^2 + \frac{1}{2} \alpha^3 \left( \eta^3 - \frac{1}{3} \beta \eta_x^2 \right) \right] \, d\tilde{x}. \]
\[ = \rho gh^2 l \left[ \frac{1}{2} \alpha I^{(1)} + \frac{1}{4} \alpha^2 I^{(2)} + \frac{1}{2} \alpha^3 I^{(3)} \right]. \] (41)

The crucial term \( \frac{1}{6} \alpha^3 \beta \eta_x^2 \) in (41) appears due to integration over vertical variable of the term \( \frac{\beta}{\alpha} \eta_{xx} \) supplied by \( (\tilde{\phi} - \frac{1}{4} \eta_x^2)^2 \).

V. VARIATIONAL APPROACH

A. Lagrangian approach, potential formulation

Some attempts at the variational approach to shallow water problems are summarized in G.B. Whitham’s book [3, Sect 16.14].

For KdV as it stands, we can not write a variational principle directly. It is necessary to introduce a velocity potential. The simplest choice is to take \( \eta = \varphi_x \). Then equation (21) in the fixed frame takes the form
\[ \varphi_{xt} + \varphi_{xx} + \frac{3}{2} \alpha \varphi_x \varphi_{xx} + \frac{1}{6} \beta \varphi_{xxxx} = 0. \] (42)

The appropriate Lagrangian density is
\[ L_{\text{fixed frame}} := -\frac{1}{2} \varphi_x \varphi_{xx} - \frac{1}{2} \varphi_x^2 - \frac{\alpha}{4} \varphi_x^3 + \frac{\beta}{12} \varphi_{xx}^2. \] (43)

Indeed, the Euler–Lagrange equation obtained from Lagrangian (43) is just (12).

For our moving reference frame the substitution \( \eta = \varphi_x \) into (41) gives
\[ \varphi_{xt} + \frac{3}{2} \alpha \varphi_x \varphi_{xx} + \frac{1}{6} \beta \varphi_{xxxx} = 0. \] (44)

So, the appropriate Lagrangian density is
\[ L_{\text{moving frame}} := -\frac{1}{2} \varphi_x \varphi_{xx} - \frac{\alpha}{4} \varphi_x^3 + \frac{\beta}{12} \varphi_{xx}^2. \] (45)

Again, the Euler–Lagrange equation obtained from Lagrangian (45) is just (14).

B. Hamiltonians for KdV equations in the potential formulation

The Hamiltonian for the KdV equation in a fixed frame (2) can be obtained in the following way. Defining generalized momentum \( \pi = \frac{\partial L}{\partial \varphi_t} \), where \( L \) is given by (13), one obtains
\[ H = \int_{-\infty}^{+\infty} \left[ \pi \varphi - L \right] \, dx = \int_{-\infty}^{+\infty} \left[ \frac{1}{12} \varphi_x^2 + \frac{\alpha}{4} \varphi_x^3 - \frac{\beta}{12} \varphi_{xx}^2 \right] \, dx \]
\[ = \int_{-\infty}^{+\infty} \left[ \frac{1}{2} \eta_x^2 + \frac{1}{4} \alpha \left( \eta^3 - \frac{\beta}{3 \alpha} \eta_x^2 \right) \right] \, dx. \] (46)

The energy is expressed by invariants \( I^{(2)}, I^{(3)} \) so it is a constant of motion.

The same procedure for KdV in a moving frame (41) gives
\[ H = \int_{-\infty}^{+\infty} \left[ \pi \varphi - L \right] \, dx = \int_{-\infty}^{+\infty} \left[ \frac{\alpha}{4} \varphi_x^3 - \frac{\beta}{12} \varphi_{xx}^2 \right] \, dx \]
\[ = \frac{1}{4} \alpha \int_{-\infty}^{+\infty} \left( \eta^3 - \frac{\beta}{3 \alpha} \eta_x^2 \right) \, dx. \] (47)

The Hamiltonian (47) only consists \( I^{(3)} \).

The constant of motion in a moving frame is
\[ E = \frac{1}{4} I^{(3)} = \text{const.} \] (48)

The potential formulation of the Lagrangian, described above, is successful for deriving KdV equations both for
fixed and moving reference frames. It fails, however, for the second order KdV equation (11). We proved that there exists a nonlinear expression of $L(\phi_1, \phi_2, \phi_{xx}, \ldots)$, such that the resulting Euler–Lagrange equation differs very little from equation (11). The difference lies only in the value of one of the coefficients in the second order term $\alpha \beta (\phi_3, \phi_{xx})$. Particular values of coefficients in this term also cause the lack of the $I(2)$ invariant for second order KdV equation, (see (25)).

VI. LUKE’S LAGRANGIAN AND KDV ENERGY

The full set of Euler equations for the shallow water problem, as well as KdV equations (2), (A2), and second order KdV equation (11) can be derived from Luke’s Lagrangian (11), see, e.g. [5]. Luke points out, however, that his Lagrangian is not equal to the difference of kinetic and potential energy. Euler–Lagrange equations obtained from $L = T - V$ do not have the proper form at the boundary. Instead, Luke’s Lagrangian is the sum of kinetic and potential energy supplemented by the $\phi_1$ term which is necessary in dynamical boundary condition.

A. Derivation of KdV energy from the original Euler equations according to [12]

In Chapter 5.2 of the Infeld and Rowlands book the authors present a derivation of the KdV equation from the Euler (hydrodynamic) equations using a single small parameter $\varepsilon$. Moreover, they show that the same method allows us to derive the Kadomtsev–Petviashvili (KP) equation (17) for water waves (19, 21, 22) and also nonlinear equations for ion acoustic waves in a plasma [20, 23]. The authors first derive equations of motion, then construct a Lagrangian and look for constants of motion. For the purpose of this paper and for comparison to results obtained in the next section it is convenient to present their results starting from Luke’s Lagrangian density. That density can be written as (here $g = 1$)

$$L = \int_0^{1+\eta} \left( \phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + \varepsilon \right) dz. \quad (49)$$

In Chapter 5.2.1 of [12] the authors introduce scaled variables in a moving frame ($\varepsilon$ plays a role of small parameter and if $\varepsilon = \alpha = \beta$, then KdV equation is obtained). Then (for details, see Appendix B or [12] Chapter 5.2)]

$$\phi_x = -\varepsilon^2 f_{\xi\xi}, \quad \phi_z = \varepsilon f_t - \varepsilon^2 \frac{3}{2} f_{\xi\xi\xi}.$$

Substitution of the above formulas into the expression $[\cdot]$ under the integral in (19) gives

$$[\cdot] = z - \varepsilon f_t + \varepsilon^2 \left( f_t + \frac{3}{2} f_{\xi\xi} + \frac{3}{2} f_{\xi\xi\xi} \right) \quad (51)$$

and its derivatives by $\eta$ and its derivatives. We use (24) replacing $\alpha$ and $\beta$ by $\varepsilon$, that is,

$$f_t = \eta - \frac{1}{4} \varepsilon^2 \eta^2 + \frac{1}{3} \varepsilon \eta \xi. \quad (57)$$

Remark The full Lagrangian is obtained by integration of the Lagrangian density (19) with respect to $x$. Integration limits are $(\eta, \infty)$ for a soliton solutions, or $[a, b]$ where $b - a = X$-wave length (space period) for cnoidal solutions. Integration by parts and properties of the solutions at the limits, see (19), allow us to use the equivalence

$$\int_{-\infty}^{\infty} \left( f_{\xi\xi} - f_t f_{\xi\xi\xi} \right) d\xi = \int_{-\infty}^{\infty} 2 f_{\xi\xi}^2 d\xi.$$

Thereupon

$$[\cdot] = z - \varepsilon f_t + \varepsilon^2 \left( f_t + \frac{3}{2} f_{\xi\xi} + \frac{3}{2} f_{\xi\xi\xi} \right) \quad (51)$$

$$\quad + \varepsilon^3 \frac{z^2}{2} \left[ f_{\xi\xi\xi \xi} - \frac{f_{\xi\xi\xi\xi}}{2} + f_{\xi\xi\xi} f_{\xi\xi} \right] + O(\varepsilon^4).$$

Integration over $y$ (note that $1 + \eta \implies 1 + \varepsilon \eta$)

$$L = \frac{1}{2} \left( 1 + \varepsilon \eta \right)^2 \left( 1 + \varepsilon \eta \right) - \varepsilon f_t + \frac{3}{2} \left( 1 + \varepsilon \eta \right) \left( f_{\xi\xi} + \frac{3}{2} f_{\xi\xi\xi} \right) + \frac{1}{3} \left( 1 + \varepsilon \eta \right)^3 \left( 1 + \varepsilon \eta \right) \left( f_{\xi\xi\xi} - \frac{3}{2} f_{\xi\xi\xi\xi} \right) + O(\varepsilon^4).$$

Write (53) up to third order in $\varepsilon$

$$L = L^{(0)} + \varepsilon L^{(1)} + \varepsilon^2 L^{(2)} + \varepsilon^3 L^{(3)} + O(\varepsilon^4).$$

It is easy to show, that

$$L^{(0)} = \frac{1}{2}, \quad L^{(1)} = \eta - f_t,$$

$$L^{(2)} = f_t + \frac{1}{2} \eta^2 - \eta f_t + \frac{1}{2} f_{\xi\xi} + \frac{1}{6} f_{\xi\xi\xi\xi}, \quad (54)$$

$$L^{(3)} = \eta f_t + \frac{1}{2} \eta f_{\xi\xi} + \frac{1}{2} \eta f_{\xi\xi\xi} - \frac{1}{6} f_{\xi\xi\xi} + \frac{1}{3} \eta f_{\xi\xi\xi\xi}. \quad (55)$$

The Hamiltonian density reads as

$$H = f_t \frac{\partial L}{\ partial f_t} + f_{\xi\xi\xi} \frac{\partial L}{\ partial f_{\xi\xi\xi}} - L \quad (55)$$

$$= \left[ 1 + \varepsilon \eta \left( \eta - f_t \right) + \varepsilon^2 \left( \frac{1}{2} \eta^2 - \eta f_t + \frac{1}{2} f_{\xi\xi} + \frac{1}{6} f_{\xi\xi\xi} \right) + \varepsilon^3 \left( \frac{1}{2} \eta f_{\xi\xi} + \frac{1}{2} \eta f_{\xi\xi\xi} + \frac{1}{3} \eta f_{\xi\xi\xi\xi} \right) \right].$$

Dropping the constant term one obtains the total energy as

$$E = \int_{-\infty}^{\infty} \left[ \varepsilon \left( \eta - f_t \right) + \varepsilon^2 \left( \frac{1}{2} \eta^2 - \eta f_t + \frac{1}{2} f_{\xi\xi} + \frac{1}{6} f_{\xi\xi\xi} \right) + \varepsilon^3 \left( \frac{1}{2} \eta f_{\xi\xi} + \frac{1}{2} \eta f_{\xi\xi\xi} + \frac{1}{3} \eta f_{\xi\xi\xi\xi} \right) \right] d\xi. \quad (56)$$

Now, we need to express $f_t$ and its derivatives by $\eta$ and its derivatives. We use (24) replacing $\alpha$ and $\beta$ by $\varepsilon$, that is,
Then the total energy in a moving frame is expressed in terms of the second and the third invariants

$$\mathcal{E} = -\left[\frac{1}{4} \int_{-\infty}^{\infty} \eta^2 \, dx + \frac{3}{2} \int_{-\infty}^{\infty} \left( \eta^3 - \frac{1}{3} \eta_0^2 \right) \, dx \right]. \quad (58)$$

Note that the term $\frac{1}{2} \eta_0^2$ occuring in the third order invariant originates from three terms appearing in $\phi_z^2$, $\phi_x^2$ and $\phi_t$ (see terms $f_{\xi \xi}$ and $f_{\xi \xi \xi}$ in (59)).

### B. Luke’s Lagrangian

The original Lagrangian density in Luke’s paper [11] is

$$L = \int_0^{h(x)} \rho \left[ \phi_t + \frac{1}{2} (\phi_x^2 + \phi_y^2) + gy \right] \, dy. \quad (59)$$

After scaling as in [1, 2, 4]

$$\tilde{\phi} = \frac{h}{a \sqrt{gh}} \phi, \quad \tilde{x} = \frac{x}{l}, \quad \tilde{y} = \frac{y}{h}, \quad \tilde{t} = \frac{t}{l / \sqrt{gh}}, \quad (60)$$

we obtain

$$\phi_t = gh \phi_t, \quad \phi_x^2 = gh \phi_x^2, \quad \phi_y^2 = gh \phi_y^2. \quad (61)$$

The Lagrangian density in scaled variables becomes ($dy = dh \tilde{y}$)

$$L = \rho gh \left[ \tilde{\phi}_t + \frac{1}{2} \left( \tilde{\phi}_x^2 + \frac{\alpha^2}{\beta} \tilde{\phi}_y^2 \right) \right] \tilde{y} \, d\tilde{y} + \frac{1}{2} \rho gh^2 (1 + \alpha \eta^2). \quad (62)$$

So, in dimensionless quantities

$$\frac{L}{\rho gh \alpha} = \int_0^{1+\alpha \eta} \left[ \tilde{\phi}_t + \frac{1}{2} \left( \alpha \tilde{\phi}_x^2 + \frac{\alpha^2}{\beta} \tilde{\phi}_y^2 \right) \right] \tilde{y} \, d\tilde{y} + \frac{1}{2} \alpha \eta^2, \quad (63)$$

where the constant term and the term proportional to $\eta$ in the expansion of $(1 + \alpha \eta)^2$ are omitted. The form [63] is identical with Eq. (2.9) in Marchant & Smyth [8].

The full Lagrangian is obtained by integration over $x$. In dimensionless variables ($dx = l \, d\tilde{x}$) it gives

$$\mathcal{L} = E_0 \int_0^{1+\alpha \eta} \left[ \tilde{\phi}_t + \frac{1}{2} \left( \alpha \tilde{\phi}_x^2 + \frac{\alpha^2}{\beta} \tilde{\phi}_y^2 \right) \right] \tilde{y} \, d\tilde{y} + \frac{1}{2} \alpha \eta^2 \, d\tilde{x}. \quad (64)$$

The factor in front of the integral, $E_0 = \rho gh \alpha = \rho gh^2 l \alpha$, has the dimension of energy.

Next, the signs ($\sim$) will be omitted, but we have to remember that we are working in scaled dimensionless variables in a fixed reference frame.

### C. Energy in the fixed reference frame

Express the Lagrangian density by $\eta$ and $f = \phi^{(0)}$. Now, up to first order in small parameters

$$\phi = f - \frac{1}{2} \beta y^2 f_{xx}, \quad \phi_t = f_t - \frac{1}{2} \beta y^2 f_{xxt}, \quad \phi_x = f_x - \frac{1}{2} \beta y^2 f_{xxx}, \quad \phi_y = -\beta f_{xx}. \quad (65)$$

Then the expression under the integral in (63) becomes

$$\left[ = f_t - \frac{1}{2} \beta y^2 f_{xxt} + \frac{1}{2} \alpha f_x^2 + \frac{1}{2} \alpha \beta y^2 (f_{xxx} + f_{xx}). \quad (66)$$

From properties of solutions at the limits ($-f_x f_{xxx} + f_{xx}^2$) $\Rightarrow 2 f_{xx}^2$. Integration of (66) over $y$ yields

$$\frac{L}{\rho gh \alpha} = \left( f_t + \frac{1}{2} \alpha f_x^2 \right) \left( 1 + \alpha \eta \right) - \frac{1}{2} \beta f_{xxt} \frac{1}{3} (1 + \alpha \eta)^3$$

$$+ \alpha \beta f_{xx}^2 \frac{1}{3} (1 + \alpha \eta)^3 + \frac{1}{2} \alpha \eta^2. \quad (67)$$

The dimensionless Hamiltonian density is

$$H \left( \frac{\partial f_t}{\partial \tilde{x}}, \frac{\partial f_x}{\partial \tilde{x}}, \frac{\partial f_y}{\partial \tilde{x}} - L \right)$$

$$= -\alpha \left[ \frac{1}{2} \alpha f_x^2 (1 + \alpha \eta) + \alpha \beta f_{xx}^2 \frac{1}{3} (1 + \alpha \eta)^3 + \frac{1}{2} \alpha \eta^2 \right]. \quad (68)$$

Again, we need to express the Hamiltonian by $\eta$ and its derivatives, only. Inserting

$$f_x = \eta - \frac{1}{4} \alpha \eta^2 + \frac{1}{2} \beta \eta_{xx} \quad (69)$$

into (68) and leaving terms up to third order one obtains

$$\frac{H}{\rho gh^2 l} = -\alpha \left[ \alpha \eta^2 + \frac{1}{4} \alpha^2 \eta^3 + \frac{1}{3} \alpha \beta (\eta_x^2 + \eta_{xx}) \right]. \quad (70)$$

The energy is

$$\frac{E}{\rho gh^2 l} = -\alpha \int_0^{\infty} \left[ \alpha \eta^2 + \frac{1}{4} \alpha^2 \eta^3 + \frac{1}{3} \alpha \beta (\eta_x^2 + \eta_{xx}) \right] \, dx$$

$$= \left[ \alpha^2 \int_0^{\infty} \eta^2 \, dx + \frac{1}{4} \alpha^3 \int_\infty^\infty \eta^3 \, dx \right] \quad (71)$$

since the integral of the $\alpha \beta$ term vanishes. Here, in the same way as in calculations of energy directly from the definition [39], the energy is expressed by integrals of $\eta^2$ and $\eta^3$. The term proportional to $\alpha \eta$ is not present in (71), because it was dropped earlier [3].

### D. Energy in a moving frame

Transforming into the moving frame

$$\tilde{x} = x - t, \quad \tilde{t} = \alpha t, \quad \partial_x = \partial_{\tilde{x}}, \quad \partial_t = -\partial_{\tilde{t}} + \alpha \partial_{\tilde{t}}. \quad (72)$$
\[ \phi = f - \frac{1}{2} \beta y^2 f_x \phi_x = f_x - \frac{1}{2} \beta y^2 f_{xxx} \phi_y = -\beta y f_{xx}, \]

(73)

\[ \phi_t = -f_x + \frac{1}{2} \beta y^2 f_{xxx} + \alpha (f_x - \frac{1}{2} \beta y^2 f_{xx}). \]

(74)

Up to second order

\[ \frac{1}{2} \left( \alpha \phi_x^2 + \frac{\alpha}{\beta} \phi_y^2 \right) = \frac{1}{2} \left[ \alpha f_x^2 + \alpha \beta y^2 (-f_x f_{xxx} + f_x^2) \right] \]

\[ = \frac{1}{2} \alpha f_x^2 + \alpha \beta y^2 f_{xx}. \]

(75)

Therefore the expression under the integral in (63) is

\[ [ ] = -f_x + \frac{1}{2} \beta y^2 f_{xxx} + \alpha (f_x - \frac{1}{2} \beta y^2 f_{xx}) + \frac{1}{2} \alpha f_x^2 + \alpha \beta y^2 f_{xx}. \]

Integration yields

\[ \frac{L}{\rho gh} = \left( -f_x + \frac{1}{2} \alpha f_x^2 \right) (1 + \alpha \eta) \]

\[ + \frac{1}{3} (1 + \alpha \eta)^3 \left( \frac{1}{2} \beta (f_{xxx} - f_{xx}) + \alpha f_x^2 \right) + \frac{1}{2} \alpha \eta^2. \]

(77)

Like in (63) above, the Hamiltonian density is

\[ \frac{H}{\rho gh^2 l} = -\alpha \left[ \left( -f_x + \frac{1}{2} \alpha f_x^2 \right) (1 + \alpha \eta) \right] \]

\[ + \frac{1}{3} (1 + \alpha \eta)^3 \left( \frac{1}{2} \beta f_{xxx} + \alpha f_x^2 \right) + \frac{1}{2} \alpha \eta^2. \]

(78)

Expressing \( f_x \) by (69) one obtains

\[ \frac{H}{\rho gh^2 l} = -\alpha \left[ -\frac{1}{4} \alpha \eta^2 + \frac{1}{3} \beta \eta_\infty - \frac{1}{2} \alpha^2 \eta^3 \right] \]

\[ + \alpha \beta \left( \beta \eta^2 - \frac{5}{12} \eta_\infty \right) - \frac{1}{18} \alpha^2 \eta_{xxx}. \]

(79)

Finally the energy is given by

\[ \frac{E}{\rho gh^2 l} = \alpha \left( \frac{1}{4} \int_{-\infty}^{\infty} \eta^2 dx + \frac{1}{3} \int_{-\infty}^{\infty} \left( \eta^3 - \frac{1}{3} \alpha \eta^2 \right) dx \right) \]

\[ + \frac{1}{4} \int_{-\infty}^{\infty} \eta^2 dx \]

(80)

since integrals from terms with \( \beta, \beta^2 \) vanish at integration limits, and \( -\frac{5}{12} \eta_\infty \) is not present in (81). If we include that term, the total energy is a linear combination of all three lowest invariants, \( I^{(1)}, I^{(3)}, I^{(3)}. \)

Comment An almost identical formula for the energy in a moving frame, for KdV expressed in dimensional variables, was obtained in [3]. That energy is expressed by all three lowest order invariants

\[ \mathcal{E} = -\frac{1}{2} \frac{c^2}{h} \int_{-\infty}^{\infty} \eta \ dx + \frac{1}{4} \int_{-\infty}^{\infty} \eta^2 \ dx \]

\[ + \frac{1}{2} \frac{c^2}{h^2} \int_{-\infty}^{\infty} \left( \eta^3 - \frac{k^3}{3} \eta^2 \right) \ dx, \]

(81)

as well. Translation of (81) to nondimensional variables yields

\[ \mathcal{E} = ggh^2 l \left( -\frac{1}{2} \alpha I^{(1)} + \frac{1}{4} \alpha^2 I^{(2)} + \frac{1}{2} \alpha^3 I^{(3)} \right). \]

E. How strongly is energy conservation violated?

The total energy in the fixed frame is given by equation (39). Taking into account its non-dimensional part we may write

\[ E_1(t) = \frac{T + V}{ggh^2 l} = \int_{-\infty}^{\infty} \left[ \alpha \eta + (\alpha \eta)^2 + \frac{1}{4} (\alpha \eta)^3 \right] dx \]

\[ = \alpha I^{(1)} + \alpha^2 I^{(2)} + \frac{1}{4} \int_{-\infty}^{\infty} (\alpha \eta)^3 dx \]

\[ (82) \]

In order to see how much the changes of \( E_1 \) violate energy conservation we will compare it to the same formula but expressed by invariants

\[ E_2(t) = \alpha I^{(1)} + \alpha^2 I^{(2)} + \frac{1}{4} \alpha^3 I^{(3)}. \]

(83)

The time dependence of \( E_1 \) and \( E_2 \) is presented in Fig. 2 for a 3-soliton solution of KdV [2]. Presented is time evolution in the interval \( t \in [-12, 0] \). The shape of the 3-soliton solution is presented only for three times \( t = -12, -6, 0 \) in order to show shapes changing during the collision.

For presentation the example of a 3-soliton solution with amplitudes equal 1.5, 1 and 0.5 was chosen. In Fig. 3 the positions of solutions at given times were artificially shifted to set them closer to each other. The plots in Figs. 2 and 3 for \( t > 0 \) are symmetric to those which are shown in the figures.
For this example the relative discrepancy of the energy $E_1$ from the constant value, is very small

$$\delta E = \frac{E_1(t = -12) - E_1(t = 0)}{E_1(t = -12)} \approx 0.000258.$$  (84)

However, the $E_2$ energy is conserved with numerical precision of thirteen decimal digits in this example. In a similar example with a 2-soliton solution (with amplitudes 1 and 0.5) the relative error (84) was even smaller, with the value $\delta E \approx 0.00014$. This suggests that the degree of nonconservation of energy increases with $n$, where $n$ is the number of solitons in the solution.

![Image of shape evolution of 3-soliton solution during collision](image)

**FIG. 3.** Shape evolution of 3-soliton solution during collision.

### VII. CONCLUSIONS

The main conclusions can be formulated as follows

- The invariants of KdV in fixed and moving frames have the same form. (Of course when we have the same scaling factor for $x$ and $t$ in the transformation between frames).

- We confirmed some known facts. Firstly, that the usual form of the energy $H = T + V$ is not always expressed by invariants only. The reason lies in the fact, as pointed out by Luke in [11], that the Euler–Lagrange equations obtained from the Lagrangian $L = T - V$ do not supply the right boundary conditions. Secondly, the variational approach based on Luke’s Lagrangian density provides the right Euler equations at the boundary and allows for a derivation higher order KdV equations.

- In the frame moving with the velocity of sound all energy components are expressed by invariants. Energy is conserved.

- Numerical calculations confirm that invariants $I^{(1)}, I^{(2)}, I^{(3)}$ in the forms [11], [13], [10], [15] are exact constants of motion for two- and three-soliton solutions, both for fixed and moving coordinate systems. In all performed tests the invariants were exact up to fourteen digits in double precision calculations.

- For the extended KdV equation (11) we have only found one invariant of motion $I^{(1)}$ [24].

- The total energy in the fixed coordinate system as calculated in (39) is not exactly conserved but only altered during collisions, even then by minute quantities (an order of magnitude smaller than expected). Details in figure caption of figure 2.

A summary of these conclusions can be found in Table 2.

### Appendix A

The simplest, mathematical form of the KdV equation is obtained from (2) by passing to the moving frame with additional scaling

$$\tilde{x} = \sqrt{\frac{3}{2}} (x - t), \quad \tilde{t} = \frac{1}{4} \sqrt{\frac{3}{2}} \alpha t, \quad u = \eta, \quad (A1)$$

which gives a standard, mathematical form of the KdV equation

$$u_t + 6u u_x + \frac{\beta}{\alpha} u_{xxx} = 0, \quad \text{or} \quad u_t + 6u u_x + 6 u_{xxx} = 0 \quad \text{for} \quad \beta = \alpha. \quad (A2)$$

Equations (A2), particularly with $\beta = \alpha$ are favored by mathematicians, see, e.g. [13]. This form of KdV is the most convenient for ISM (the Inverse Scattering Method, see, e.g. [14–16]).

For the moving reference frame, in which the KdV equation has a standard (mathematical) form (A2), the invariant $I^{(3)}$ looks different. To see this difference denote the lhs of (A2) by $\text{KDVm}(x, t)$ and construct

$$3\eta^2 \times \text{KDVm}(x, t) - \frac{\beta}{\alpha} \eta_x \times \frac{\partial}{\partial x} \text{KDVm}(x, t) = 0.$$ 

Then after simplifications one obtains

$$\frac{\partial}{\partial t} \left[ \eta^3 - \frac{1}{2} \eta_x^2 \right] + \frac{\partial}{\partial x} \left[ \frac{9}{2} \eta^4 - 6 \frac{\beta}{\alpha} \eta^2 \eta_x \right] + 3 \frac{\beta}{\alpha} \eta^2 \eta_{xx} - \frac{1}{2} \left( \frac{\beta}{\alpha} \eta_x^2 \right)^2 + \left( \frac{\beta}{\alpha} \right)^2 \eta_x \eta_{xxx} \right] = 0; \quad (A3)$$

which implies the invariant $I^{(3)}$ in the following form

$$I^{(3)}_{\text{moving frame}} = \int_{-\infty}^{\infty} \left( \eta^3 - \frac{1}{2} \frac{\beta}{\alpha} \eta_x^2 \right) \, dx = \text{const} \quad \text{for} \quad \beta = \alpha. \quad (A4)$$

$$I^{(3)}_{\text{moving frame}} = \int_{-\infty}^{\infty} \left( \eta^3 - \frac{1}{2} \eta_x^2 \right) \, dx = \text{const} \quad \text{for} \quad \beta = \alpha.$$
TABLE I. Comparison of different energy formulas. Here $\eta^{(3)} = \int_{-\infty}^{\infty} \eta^3 \, dx$. † Formulas in this column are written in $E/\rho g h^2 l$.

|                   | Euler equations | Luke’s Lagrangian | Integrals $T+V$ | Potential Lagrangian | KdV dimensional † |
|-------------------|-----------------|-------------------|-----------------|----------------------|------------------|
| Fixed frame       | $a_1 I_1^{(3)} + a_2 I_2^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $a_2 I_2^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $a_1 I_1^{(3)} + a_2 I_2^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $-\frac{1}{2} a_1 I_1^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $-\frac{1}{2} a_1 I_1^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ |
| Moving frame      | $\frac{1}{2} a_2 I_2^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $\frac{1}{2} a_2 I_2^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $-\frac{1}{2} a_1 I_1^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $-\frac{1}{2} a_1 I_1^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ | $-\frac{1}{2} a_1 I_1^{(3)} + \frac{1}{2} a_3 I_3^{(3)}$ |

We see, however, that the difference between (A4) and (13) is caused by additional scaling.

In the Lagrangian approach as described in Sect. V, the substitution $u = \varphi_x$ into (A2) gives

$$
\varphi_{xt} + 6 \varphi_x \varphi_{xx} + \varphi_{xxxx} = 0.
$$

(A5)

Then the appropriate Lagrangian density for equation (A2) with $(\alpha = \beta)$ is

$$
\mathcal{L}_{\text{standard KdV}} := -\frac{1}{2} \dot{\varphi}_t \varphi_x - \dot{\varphi}_x^3 + \frac{1}{2} \dot{\varphi}_{xx}^2.
$$

(A6)

Indeed, the Euler–Lagrange equation obtained from the Lagrangian (A6) is just (A4).

The Hamiltonian for KdV (A2) can be found e.g. in [10]. Defining generalized momentum $\pi = \partial \mathcal{L}/\partial \dot{\varphi}_t$, where $\mathcal{L}$ is given by (A6), one obtains

$$
H = \int_{-\infty}^{\infty} \left[ \pi \dot{\varphi} - \mathcal{L} \right] \, dx = \int_{-\infty}^{\infty} \left[ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_t} \dot{\varphi}_t - \mathcal{L} \right] \, dx.
$$

(A7)

This is the same invariant as $I^{(3)}_{\text{moving frame}}$ in (A4).

Appendix B

The set of Euler equations for irrotational motion of an incompressible and inviscid fluid can be written (neglecting surface tension) in dimensionless form:

$$
\nabla^2 \phi = 0 \quad \text{(B1)}
$$

$$
\phi_z = 0 \quad \text{on} \quad z = 0 \quad \text{(B2)}
$$

$$
\eta_t + \phi_x \eta_x - \phi_x = 0 \quad \text{on} \quad z = 1 + \eta \quad \text{(B3)}
$$

$$
\phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + \eta = 0 \quad \text{on} \quad z = 1 + \eta. \quad \text{(B4)}
$$

We look for solutions to the Laplace equation (B1) in the form

$$
\phi = \sum_{n=0}^{\infty} \varepsilon^n f^{(n)}(x, y, t) \quad \text{(B5)}
$$

yielding

$$
\sum_{n=0}^{\infty} \left[ n(n-1)z^{-2} f^{(n)} + z^n \nabla^2 f^{(n)} \right] = 0. \quad \text{(B6)}
$$

In two dimensions $(x, z)$ we obtain

$$
f^{(n+2)} = -\frac{1}{(n+1)(n+2)} \frac{\partial^2 f^{(n)}}{\partial x^2} \quad \text{(B7)}
$$

The boundary condition at the bottom, $\phi_z = 0$ at $z = 0$ implies $f^{(1)} = 0$ and then all odd $f^{(2k+1)} = 0$. Now

$$
\phi = \sum_{n=0}^{\infty} (-1)^m \varepsilon^{2m} \frac{\partial^m f}{(2m)! \partial x^{2m}}, \quad \text{(B8)}
$$

where $f := f^{(0)}$. In the stretched coordinates $\xi := \varepsilon \partial_x^2$ so

$$
\phi = \varepsilon^{\frac{3}{2}} \left( f + \sum_{n=0}^{\infty} (-1)^m \varepsilon^{2m} (\varepsilon \partial_x^2)^{2m} f \right). \quad \text{(B9)}
$$

Now both (B1) and (B2) are satisfied. We must also satisfy the boundary conditions on $z = 1 + \eta$.

In the derivation of KdV and Kadomtsev-Petivashvili [17] from the Euler equations (B1)–(B4) Infeld and Rowlands [12] applied scaling assuming the following relations

vavelength : depth : amplitude as $\varepsilon^{-1/2} : 1 : \varepsilon$.

They then applied a transformation to a frame moving with velocity of sound. The coordinates scales as

$$
\xi = \varepsilon^{\frac{3}{2}} (x - t), \quad \tau = \varepsilon^{\frac{3}{2}} t \quad \text{(B10)}
$$

$$
\partial_t = -\varepsilon^{\frac{3}{2}} \partial_\xi + \varepsilon^{\frac{3}{2}} \partial_\tau, \quad \partial_x = \varepsilon^{\frac{3}{2}} \partial_\xi. \quad \text{(B11)}
$$

For the wave amplitude and velocity potential the appropriate scaling was

$$
\eta = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \ldots, \quad \text{(B12)}
$$

and

$$
\phi = \varepsilon^{\frac{3}{2}} \phi^{(1)} + \varepsilon^{\frac{3}{2}} \phi^{(2)} + \ldots. \quad \text{(B13)}
$$
Then the lowest order expression for $\phi$ is

$$\phi \approx \varepsilon \frac{z}{2} f - \varepsilon \frac{z^2}{2} f \xi$$

Next, Infeld and Rowlands show that in order to simultaneously satisfy $\text{(B3)}$ and $\text{(B4)}$ the next order contributions to $\eta$ and $\phi$ cancel. It is enough to keep

$$1 + \eta = 1 + \varepsilon \eta^{(1)}$$

and $\phi = \varepsilon \frac{1}{2} \phi^{(1)}$

and drop upper index $^{(1)}$ in what follows.

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