ON QUANTUM FLAG ALGEBRAS

ALEXANDER BRAVERMAN

Abstract. Let $G$ be a semisimple simply connected algebraic group over an algebraically closed field of characteristic 0. Let $V$ be a simple finite-dimensional $G$-module and let $y$ be its highest weight vector. It is a classical result of B. Kostant that the algebra of functions on the closure of $G \cdot y$ is quadratic. In this paper we generalize this result to the case of the quantum group $U_q(g)$. The proof uses information about $R$-matrix due to Drinfeld and Reshetikhin.

0. Introduction

0.1. Notations. Let $g$ be a semisimple Lie algebra over an algebraically closed field $k$ of characteristic 0. Let us choose a Borel subalgebra $b \subset g$ and a Cartan subalgebra $h \subset b$. Let $\Pi \subset h^*$ be the set of simple roots of $h$ with respect to these choises. We shall denote by $P(\Pi)$ the corresponding weight lattice and by $P(\Pi)^+$ the set of dominant weights in $P(\Pi)$. We shall denote by $\rho \in P(\Pi)^+$ the half sum of all positive roots. For any $\lambda, \mu \in h^*$ the symbol $\lambda > \mu$ will mean that $\lambda - \mu$ is a sum of positive roots. Choose an invariant bilinear form on $g$. We shall denote by $(\cdot, \cdot)$ its restriction to $h$ (and to $h^*$ by transport of structure) and by $\| \cdot \|$ the corresponding norm on $h^*$.

0.2. Let $K = k(q)$ where $q$ is transcendental over $k$. Let $U_q(g)$ denote the quantized universal enveloping algebra of $g$ (cf. for example [D1], [L]) constructed using the invariant form of 0.1 (as in [D1]). For any $\lambda \in P(\Pi)^+$ we shall denote by $V(\lambda)$ the simple $U_q(g)$-module with highest weight $\lambda$ (cf. [L]). We shall also denote by $\mathcal{M}_q(g)$ the tensor category of locally finite integrable (cf. [L]) $U_q(g)$-modules.

0.3. Quadratic algebras. For any $K$-vector space $V$ we shall denote by $T(V)$ the tensor algebra of $V$. Let $A = \oplus_{n=0}^\infty A_n$ be a graded $K$-algebra.

Definition. We say that $A$ is quadratic if

1. $A_0 = K$ and $A$ is generated by $A_1$

2. the ideal of relations of $A$ is quadratic, i.e. if we let $i$ denote the natural map $T(A_1) \to A$ then

$$\ker i = \sum_{i,j \in \mathbb{Z}^+} V^\otimes i \otimes W \otimes V^\otimes j$$

where $W = \ker i \cap V^\otimes 2$.
0.4. The result. Let $\lambda \in P(\Pi)^+$ and let $V = V(\lambda)$. Let us define a graded algebra $A(V) = \oplus_{n=0}^{\infty} A_n(V)$ by

$$A_n(V) := V^{\otimes n} / \sum_{V(\mu) \subset V^{\otimes n}, \mu < n\lambda} V(\mu)$$

and multiplication coming from the tensor algebra of $V$. In the classical ($q=1$) case the algebra $A(V)$ is isomorphic to the algebra of functions on the closure of $G \cdot y$ where $G$ is the simply connected algebraic group which corresponds to $\mathfrak{g}$ and $y$ is a highest weight vector in the $\mathfrak{g}$-module $V^*$, dual to $V$ and hence can be identified with the coordinate ring of the flag variety of $\mathfrak{g}$ with respect to the invertible sheaf defined by $V$. This is why in our situation the algebra $A(V)$ deserves the name of a quantum flag algebra.

**Theorem.** The algebra $A(V)$ is quadratic.

In the classical case this the analogous theorem was proven by B. Kostant (cf. for example [FH]).

0.5. Remarks. 1) In fact one can easily see that the algebra $A(V)$ is defined as an algebra in the category $\mathcal{M}_q(\mathfrak{g})$ and theorem 0.4 is equivalent to the statement that $A(V)$ is quadratic as an $\mathcal{M}_q(\mathfrak{g})$-algebra (cf [HS] for the relevant definitions).

2) In the case $\mathfrak{g} = sl_n$ theorem 0.4 can be deduced from [TT].

1. Proof of the theorem

1.1. The braiding. Let $(\mathcal{C}, \otimes)$ be a monoidal category (cf. [SS]). A **braiding** on $\mathcal{C}$ is an isomorphism $B$ of two functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (namely, $(V,W) \to V \otimes W$ and $(V,W) \to W \otimes V$ for any $(V,W) \in Ob \mathcal{C} \times \mathcal{C}$), which satisfies the hexagon identities (cf. [SS]). Drinfeld in [D1] has shown that $\mathcal{M}_q(\mathfrak{g})$ has a braiding and it was shown by D. Gaitsgory ([G]) that this braiding was essentially unique. For any $V,W \in Ob \mathcal{M}_q(\mathfrak{g})$ we shall denote by $B_{V,W}$ the corresponding morphism $B_{V,W} : V \otimes W \to W \otimes V$. Let also $S_{V,W} = B_{W,V} \circ B_{V,W} \in \text{End} V \otimes W$.

1.2. For any $\lambda \in \mathfrak{h}^*$ define $c(\lambda) = (\lambda, \lambda + 2\rho) = ||\lambda + \rho||^2 - ||\rho||^2$. Let $\lambda, \mu \in P(\Pi)^+$. For any $\gamma \in P(\Pi)^+$ denote by $P_{\lambda,\mu}^\gamma \in \text{End} (V(\lambda) \otimes V(\mu))$ the projector from $V(\lambda) \otimes V(\mu)$ to its $V(\gamma)$-isotypical component. The following statement is due to N. Reshetikhin ([R], cf. also [D2, § 5]).

**Lemma.**

$$S_{V(\lambda),V(\mu)} = \sum_{\gamma \in P(\Pi)^+} q^{2(c(\gamma) - c(\lambda) - c(\mu))} P_{\lambda,\mu}^\gamma$$

1.3. Let $V = V(\lambda)$ and $S = S_{V,V^{\otimes n}}$ (for some $n \in \mathbb{N}$).

**Corollary.**

$$\{x \in V^{\otimes n} | Sx = q^{2(c(n\lambda) - c(\lambda) - c((n-1)\lambda))}x\} = V(n\lambda) \subset V^{\otimes n}$$

i.e. $V(n\lambda)$ is a full eigenspace of $S$ with eigenvalue $q^{2(c(n\lambda) - c(\lambda) - c((n-1)\lambda))}$.

**Proof.** Let $P_{n-1} = \{\mu \in P(\Pi)^+ | V(\mu) \subset V^{\otimes n}\}$. We know from 1.2 that $V(n\lambda)$ is an eigenspace of $S$ with eigenvalue $q^{2(c(n\lambda) - c(\lambda) - c((n-1)\lambda))}$. On the other hand 1.2
Let \( V \) be the following algebra
\[
\gamma = A
\]
which finishes the proof. □

1.5. Now we are ready to prove theorem 0.4. Let \( \hat{\lambda} \) denote the highest weight of the module \( V^* \), dual to \( V \). We are going to prove the quadraticity of the algebra \( A(V^*) \). Let \( W = \sum_{V(\mu) \subset V^* \otimes V^*} V(\mu) \subset V^* \otimes V^* \). Then the quadraticity of the algebra \( A(V^*) \) is equivalent to
\[
\sum_{i=1}^{n-1} V^* \otimes V^* = 1 \otimes \sum_{V(\mu) \subset V^* \otimes V^*} V(\mu) \subset V^* \otimes V^* \quad (**)
\]
Let \( V^n = V \otimes V \otimes V \otimes \cdots \otimes V \). Then the orthogonal complement to the left hand side of (** in \( V^n \)) is equal to \( \bigcap_{i=1}^n V_i^n \) and the orthogonal complement to the right hand side of (** in \( V^n \)) is equal to \( V(n\lambda) \). Hence (** in \( V^n \)) is equivalent to the following

**Proposition.** \( \bigcap_{i=1}^n V_i^n = V(n\lambda) \)

**Proof.** After corollary 1.3 it is enough to check that the left hand side is an eigenspace of \( S \) (since we have obvious embedding \( V(n\lambda) \subset \bigcap_{i=1}^n V_i^n \)). For any \( 1 \leq i \leq n-1 \) denote by \( B^{i+1} \) the morphism \( id_{V \otimes V} \otimes B \). Then the hexagon identities imply that \( B_{V \otimes V} = B_{V \otimes V} \otimes id_{V \otimes V} \). Let \( \kappa \) be as in 1.4. Then
\[
S|_{\bigcap_{i=1}^n V_i^n} = B_{V \otimes V} \otimes id_{V \otimes V} \big|_{\bigcap_{i=1}^n V_i^n} = B^{1,2} \otimes \cdots \otimes B^{n-1,n} \otimes B^{n-1,n} \otimes \cdots \otimes B^{2,3} \otimes B^{1,2} \big|_{\bigcap_{i=1}^n V_i^n} = \kappa^{2(n-1)} \otimes id_{\bigcap_{i=1}^n V_i^n}
\]
which finishes the proof. □
1.6. Concluding remarks. In [Be] R. Bezrukavnikov extended Kostant’s result, showing that for $q = 1$ not only is the algebra $A(V)$ quadratic, but it is also a Koszul algebra (cf. for example [BG]). It would be interesting to extend this result to the quantum case. One idea in this direction could be the following. It is remarked in [BG] that if $V$ is a vector space and $B \in \text{End } V$ is a Yang-Baxter operator (cf. [D1] or [BG]) which is unitary (i.e. $B^2 = \text{id}$) then the subspace \( \{ x \in V \otimes V \mid Bx = x \} \) of $V \otimes V$ defines a Koszul algebra. In our situation one can show that in fact $V(2\lambda) \subseteq V \otimes V$ is an eigenspace of $B_{V,V}$. However $B^2_{V,V} \neq \text{id}$, but one may try to prove the statement of [BG] holds for a broader class of Yang-Baxter operators, which will include $B_{V,V}$. This will prove the Koszulity of the dual algebra of $A(V)$ and hence the Koszulity of $A(V)$ itself (we refer to [BG] for the definition of duality for quadratic algebras and its relation to Koszulity).

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School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Israel
E-mail address: braval@math.tau.ac.il