Matrix representations of multidimensional integral and ergodic operators

Anton A. Kutsenko

Jacobs University (International University Bremen), 28759 Bremen, Germany; email: akucenko@gmail.com
Saint-Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034, Russia

Abstract

We provide a representation of the $C^*$-algebra generated by multidimensional integral operators with piecewise constant kernels and discrete ergodic operators. This representation allows us to find the spectrum and to construct the explicit functional calculus on this algebra. In particular, it can be useful for various applications, since almost all discrete approximations of integral and differential operators belong to this algebra. Some examples are also presented: 1) we construct an explicit functional calculus for extended Fredholm integral operators with piecewise constant kernels, 2) we find a wave function and spectral estimates for 3D discrete Schrödinger equation with planar, guided, local potential defects, and point sources. Some problems of approximation of continuous multi-kernel integral operators by the operators with piecewise constant kernels are also discussed.

Keywords: operator algebras, integral equations, functional calculus, Schrödinger operator with defects

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be some domain. Consider the Hilbert space $L^2 = L^2(\Omega \to \mathbb{C}^M, \mu)$ of vector-valued functions acting on $\Omega$ ($\mu$ is the Lebesgue measure). The goal of the paper is the study of a discrete analogue of the following algebra of operators acting on $L^2$:

$$\mathcal{A}_c = \text{Alg}(\mathbf{A}, \int \cdot dx_1, ..., \int \cdot dx_N, \mathcal{T}).$$

In other words, $\mathcal{A}_c$ is generated by: 1) multiplication operators $\mathbf{A} \cdot$ (the dot $\cdot$ denotes the place of the operator argument $\mathbf{u} = (u_m)^M_{m=1} \in L^2$), where $\mathbf{A} = \mathbf{A}(x_1, ..., x_N)$ are bounded measurable $M \times M$ matrix-valued functions with complex entries defined on $\Omega$; 2) integral operators $\int \cdot dx_n = \int_{\Omega \cap I} \cdot dx_n$, where $I$ is the one-dimensional set (line) defined by

$$I = I(x_1, ..., x_{n-1}, x_{n+1}, ..., x_N) = \{y = (y_r)^N_{r=1} \in \mathbb{R}^N : y_r = x_r, \forall r \neq n\};$$

3) operators

$$\mathcal{T} \mathbf{u}(\mathbf{x}) = (u_m(T_m(\mathbf{x})))^M_{m=1}, \mathbf{x} \in \Omega, \text{ where } T_m : \Omega \to \Omega.$$
are measurable mappings. In particular, the algebra \( \mathcal{A} \) contains various integral operators of Fredholm type, ergodic operators (which are based on \( T_m \)), and their combinations. All such operators have different applications in mathematical physics, e.g., they describe a propagation of waves and other effects in complex structures with defects \([1, 2, 3, 4, 5, 6]\), diffusion \([7, 8]\), thermodynamic processes \([9, 10]\), random Schrödinger operators and various operators on discrete graphs \([11, 12, 13]\), electromagnetic scattering \([14]\). Some general aspects of the connection between integral and ergodic operators are discussed in \([15, 16, 17]\). Integral operators and some of their finite-dimensional approximations are discussed in \([18, 19, 20]\). The main problems for operators from \( \mathcal{A} \) are to find the spectrum, to find the inverse operators, square roots, or, more generally, to construct the functional calculus on this algebra. The difficulty is that \( \mathcal{A} \) is very complex. We try to find some discrete analogue \( \mathcal{A} \) for which the functional calculus can be constructed explicitly. One of the most important requirements to \( \mathcal{A} \) is to be finite dimensional. Because in this case \( \mathcal{A} \) can be expressed in terms of matrix algebras for which the functional calculus is well known. If \( \mathcal{A} \) is finite dimensional then, due to the Stone-Weierstrass theorem, all matrix-valued functions \( \mathcal{A} \) should be piecewise constant, otherwise the subalgebra generated by \( \mathcal{A} \) has an infinite dimension. This tells us how the operator algebra \( \mathcal{A} \) should be arranged. It is natural to suppose that \( \Omega \) is a union of a finite number of shifted copies of a cub \( H = [0, h)^N \) \( (h > 0) \)

\[
\Omega = \bigcup_{i=1}^{S} \Omega_i, \quad \Omega_i = a_i + H,
\]

(4)

where \( a_i \in \mathbb{R}^N \) are some vertices such that \( \Omega_i \) are disjoint. For any \( A \in \mathbb{C}^{M \times M} \), \( \alpha \subset \{1, ..., N\} \) and \( i, j \in \{1, ..., S\} \) introduce the following elementary operators \( \mathcal{E}^\alpha_{ij}[A] \)

\[
\mathcal{E}^\alpha_{ij}[A] u(x + a_i) = \begin{cases} 
  h^{-|\alpha|} A \int_{[0,h)^{\alpha}} u(x + a_j) dx_\alpha, & x \in H, \\
  0, & \text{otherwise},
\end{cases}
\]

(5)

where \( dx_\alpha = \prod_{n \in \alpha} dx_n \) and the number of elements in \( \alpha \) is denoted by \( |\alpha| \). Note that \( \alpha \) can be the empty set \( \emptyset \), in this case there is no \( \int \) in (5). Another useful remark is that \( \mathcal{E}^\alpha_{ij}[A] u(x + a_i) \) is constant along \( x_n \in [0, h) \) for \( n \in \alpha \). The examples of action of different operators \( \mathcal{E} \) is demonstrated in Fig. 1.

The operators \( \mathcal{E}^\alpha_{ij}[A] \) provide the interaction between the components of \( u \) in the various micro-domains \( \Omega_i \). Roughly speaking, if \( h > 0 \) is sufficiently small then \( \Omega \) can be approximately represented by (4) and the discrete analogue of \( \mathcal{A} \) can be chosen as

\[
\mathcal{A} = \text{Alg}(\mathcal{E}^\alpha_{ij}[A])
\]

(6)

In other words, \( \mathcal{A} \) is generated by \( \mathcal{E}^\alpha_{ij}[A] \) for all \( A \in \mathbb{C}^{M \times M} \), \( \alpha \subset \{1, ..., N\} \) and \( i, j \in \{1, ..., S\} \). Let us discuss why \( \mathcal{A} \) is a discrete analogue of \( \mathcal{A} \) defined by (1). The discrete analogues of the multiplication operators \( A(x) \cdot \) are the multiplication operators with piecewise constant functions \( A(x) \), i.e.

\[
A(x) = A_i = \text{const}, \quad x \in \Omega_i, \quad i = 1, ..., S.
\]

(7)
The corresponding operators are expressed in terms of $E$ as follows:

$$A(x) = \sum_{i=1}^{S} E_{ii}^{0}[A_i].$$  \hspace{1cm} (8)$$

The integrals are also expressed in terms of $E$:

$$\int \cdot dx_n = \sum_{i=1}^{S} \sum_{j \in \beta_i} h E_{ij}^{\{n\}}[I],$$  \hspace{1cm} (9)$$

where $I$ is the identity matrix,

$$\beta_i = \{ j : a_j(m) = a_i(m), \forall m \neq n \}$$  \hspace{1cm} (10)$$

and $a(m)$ is the $m$-th entry of the vector $a$. The discrete analogues of change-of-variables operators $T_m : \Omega \to \Omega$ (see (3)) should be based on discrete mappings $p_m : \{1, ..., S\} \to \{1, ..., S\}$ since $\Omega = \bigcup_{i=1}^{S} \Omega_i$. Taking arbitrary $p_m$ we construct

$$T_m x = x - a_i + a_{p_m(i)}, \quad x \in \Omega_i, \quad i = 1, ..., S.$$  \hspace{1cm} (11)$$

Then the discrete analogues of $T$ (3) are expressed in terms of $E$ as follows:

$$T = \sum_{m=1}^{M} \sum_{i=1}^{S} E_{i, p_m(i)}^{0}[I_m], \quad \text{where } I_m = (\delta_{im} \delta_{jm})_{i,j=1,1}^{M,M}$$  \hspace{1cm} (12)$$

and $\delta$ is the Kronecker delta.
The above arguments show that $\mathcal{A}$ can be considered as the discrete approximation of $\mathcal{A}_c$. As it is shown in Theorem 1.3 and Example 2 below, if $h \to 0$ then the approximation becomes better and better. Moreover, $\mathcal{A}$ is a closed $C^*$-subalgebra of $\mathcal{A}_c$ and, hence, deserves its own study. The properties of $\mathcal{A}$ are enough for most of practical applications.

Denoting by $^*$ the Hermitian conjugation, one can easily check the fundamental relations

$$E_{ij}^\alpha[A]^* = E_{ji}^\alpha[A^*], \quad E_{ij}^\alpha[A] + E_{ij}^\beta[B] = E_{ij}^\alpha[A + B], \quad E_{ij}^\alpha[A]E_{kl}^\beta[B] = \delta_{jk}E_{il}^\alpha[A'B].$$

(13)

Hence, $E_{ij}^\alpha[A]$ are basis elements and any operator $A \in \mathcal{A}$ has the form

$$A = \sum_{\alpha \subset \{1, \ldots, N\}} \sum_{i,j=1}^S E_{ij}^\alpha[A^\alpha_{ij}],$$

(14)

where $A^\alpha_{ij}$ are some $M \times M$ matrices. In practice, the form (14) is available after taking an approximation of the initial operator $A_c \in \mathcal{A}_c$. The question is how to find explicitly the spectrum of $A$, inverse $A^{-1}$, square root $\sqrt{A}$, etc. As mentioned above, if we provide a representation of $\mathcal{A}$ in terms of simple matrix algebras, then the answers on all these questions become explicit. We denote the simple matrix algebras as $\mathbb{C}^{n \times n}$, $n \geq 1$. Introduce the following matrices

$$A^\alpha = (A^\alpha_{ij})_{i,j=1}^S \in \mathbb{C}^{MS \times MS}, \quad B^\alpha = \sum_{\beta \subset \alpha} A^\beta$$

(15)

and the following mapping

$$\pi: \mathcal{A} \to (\mathbb{C}^{MS \times MS})^{2^N}, \quad \pi(A) = (B^\alpha)_{\alpha \subset \{1, \ldots, N\}}.$$  

(16)

The next theorem is our main result.

**Theorem 1.1.** The mapping $\pi$ is the *-algebra isomorphism between $C^*$-algebras $\mathcal{A}$ and $(\mathbb{C}^{MS \times MS})^{2^N}$. The inverse mapping has the form

$$\pi^{-1}(B^\alpha)_{\alpha \subset \{1, \ldots, N\}} = \sum_{\alpha \subset \{1, \ldots, N\}} \sum_{i,j=1}^S E_{ij}^\alpha[A^\alpha_{ij}], \quad \text{where}$$

(17)

$$A^\alpha = (A^\alpha_{ij})_{i,j=1}^S = \sum_{\beta \subset \alpha} (-1)^{\alpha \setminus \beta}B^\beta.$$  

(18)

Note that while the most of operators from $\mathcal{A}$ are infinite-dimensional and even non-compact, the algebra $\mathcal{A}$ has a finite dimension. We immediately obtain the following

**Corollary 1.2.** i) The operator $A$ is invertible if and only if all matrices $B^\alpha$ are invertible. In this case, $A^{-1}$ can be computed explicitly

$$A^{-1} = \pi^{-1}(B^{-1})^{-1}_{\alpha \subset \{1, \ldots, N\}}.$$  

(19)
ii) Generalizing i) we can take rational functions $f$ and write

$$f(A) = \pi^{-1}((f(B_\alpha))_{\alpha \subseteq \{1, \ldots, N\}}).$$  \hspace{1cm} (20)

The extension to algebraic and transcendent functions $f$ is also obvious.

iii) The trace and the determinant of $A$ can be defined as

$$\det(A) = \prod \det B_\alpha, \quad \text{tr}(A) = \sum \text{tr}B_\alpha, \hspace{1cm} (21)$$

they satisfy the usual properties

$$\det(AB) = \det(A) \det(B), \quad \text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B), \hspace{1cm} (22)$$

$$\det(e^A) = e^{\text{tr}(A)}, \quad \text{sp}(A) = \{\lambda : \det(A - \lambda I) = 0\}, \hspace{1cm} (23)$$

where $I = \pi^{-1}((I)_{\alpha \subseteq \{1, \ldots, N\}})$ is the identity operator, $I$ is $MS \times MS$ identity matrix, and $a, b \in \mathbb{C}, A, B \in \mathcal{A}$.

iv) Since $\pi$ is $\ast$-isomorphism of $C^*$-algebras, the operator norm of $A \in \mathcal{A}$ can be computed explicitly

$$\|A\|_{L^2 \to L^2} = \max\{\lambda : \lambda \text{ are } s - \text{values of } B_\alpha, \text{ where } \pi(A) = (B_\alpha)_{\alpha \subseteq \{1, \ldots, N\}}\}. \hspace{1cm} (24)$$

Remark. The algebra

$$\mathcal{B} = \text{Alg} \left( A, \int_0^1 dx_1, \ldots, \int_0^1 dx_n \right)$$

with piecewise constant functions $A$ acting on $\Omega = [0, 1)^N$ is considered in [21]. The difference between $\mathcal{A}$ and $\mathcal{B}$ is the presence of more general class of sets $\Omega$ and the addition of change-of-variables operators $T$. In the case $\Omega = [0, 1)^N$ with the uniform partition of each of the segments $[0, 1)$ on $p$ intervals (i.e. $S = p^N$), $\mathcal{B}$ is a $C^*$-subalgebra of $\mathcal{A}$:

$$\mathcal{B} \simeq \prod_{n=0}^N \left( \mathbb{C}^{Mp^N \times Mp^N} \right)^{N-n} \subset \left( \mathbb{C}^{Mp^N \times Mp^N} \right)^{2N} \simeq \mathcal{A},$$

where $\binom{N}{n}$ are binomial coefficients. It is interesting to note that the generalization of the structure of $\mathcal{A}$ simplifies the proof of main results.

Let us discuss some approximation properties of continuous operators by discrete operators from $\mathcal{A}$. For simplicity, consider the case $\Omega = [0, 1)^N$ and $M = 1$. The generalization to $M > 1$ is similar. Consider a multi-kernel operator $\mathcal{A} : L^2(\Omega) \to L^2(\Omega)$ of the form, common in applications,

$$\mathcal{A}_\alpha u(k) = \sum_{a \subseteq \{1, \ldots, N\}} \int_{[0,1)^{|a|}} A_\alpha(k, x_a)u(k_\pi + x_a)dx_a, \quad k \in \Omega, \quad u \in L^2(\Omega). \hspace{1cm} (25)$$
Here, we also use the notation
\[ x_\alpha = (\tilde{x}_n), \quad \tilde{x}_n = \begin{cases} x_n, & n \in \alpha, \\ 0, & n \notin \alpha. \end{cases} \]

Note that \( x = x_\alpha + x_{\overline{\alpha}} \), where \( \overline{\alpha} = \{1, \ldots, N\} \setminus \alpha \) is the complement to the set \( \alpha \). Consider the uniform partition of \( \Omega \) onto \( p^N \) identical cubes
\[ \Omega = \bigcup_{i=1}^{p^N} \Omega_i, \quad \Omega_i = a_i + [0, 1/p)^N, \]
where
\[ a_i = \frac{1}{p^N} (b_j)_{j=1}^{N}, \quad i - 1 = \sum_{j=1}^{N} p^j b_j, \quad b_j \in \{0, \ldots, p-1\} \]
is the representation of \( i - 1 \) in the base \( p \) numeral system. Let us take the approximation of \( A_c \) in the form
\[ A = \sum_{\alpha \subset \{1, \ldots, N\}} p^N \sum_{i=1}^{p^N} \sum_{j=1}^{p^N} \frac{1}{p^{|\alpha|}} c_{ij}^\alpha [A_{ij}^\alpha], \]
where
\[ A_{ij}^\alpha = \begin{cases} A_\alpha(a_i + \varepsilon(a_j + \varepsilon)_\alpha), & (a_i)_\alpha = (a_j)_\overline{\alpha}, \text{ and } \varepsilon = (1/(2p))_{r=1}^N, \\ 0, & \text{otherwise}, \end{cases} \]
The next proposition shows us that \( A_c \to A \) in the operator norm.

**Theorem 1.3.** Suppose that \( A_\alpha(k, x_\alpha) \) in (25) are real functions with bounded first derivatives. Then
\[ \|A_c - A\|_{L^2 \to L^2} \leq \frac{N}{2p} \sum_{\alpha \subset \{1, \ldots, N\}} \|\nabla A_\alpha\|_{L^\infty} \to 0 \text{ for } p \to \infty. \]

Along with Theorem 1.1 and Corollary 1.2, we can use Theorem 1.3 to determine the spectrum and the inverse operator.

**Corollary 1.4.** Under the assumptions of Theorem 1.3, we assume also that \( A_c \) is self-adjoint, i.e. \( A_\alpha(k, x_\alpha) = A_\alpha(k_\overline{\alpha} + x_\alpha, k_\alpha) \). Then \( A \) is self-adjoint and
\[ \text{sp}(A_c) \subset B_\delta(\text{sp}(A)), \quad \text{sp}(A) \subset B_\delta(\text{sp}(A_c)), \]
where \( B_\delta \) means \( \delta \)-neighbourhood and we can set \( \delta = \frac{N}{2p} \sum_{\alpha \subset \{1, \ldots, N\}} \|\nabla A_\alpha\|_{L^\infty} \). If \( 0 \notin \text{sp}(A_c) \) then \( A_c \) is invertible and there is \( p \) such that \( 0 \notin \text{sp}(A) \) and \( \|A^{-1}\|_{L^2 \to L^2} \leq \varepsilon^{-1} \), and the following estimate is true
\[ \|A_c^{-1} - A^{-1}\|_{L^2 \to L^2} \leq \frac{\delta \|A^{-1}\|_{L^2 \to L^2}}{1 - \delta \|A^{-1}\|_{L^2 \to L^2}}. \]
Moreover, RHS of (32) tends to 0 for \( p \to \infty \).
This paper is organized as follows. Section 2 contains two examples: 1) new formulas for the functions of 1D Fredholm integral operators with step kernels; 2) the application of the method for obtaining a solution (with arbitrary precision) and spectral estimates for 3D discrete Schrödinger equation with planar, guided, and local potential defects. A short proof of the main result based on the explicit representation of a semigroup algebra of subsets is given in Section 3. We conclude in Section 4.

2. Examples

**Example 1.** Consider the case \( N, M = 1 \), and the classical Fredholm integral operators (see [18])

\[
A : L^2([0, 1]) \to L^2([0, 1]), \quad Au(x) = \int_0^1 B(x, y)u(y)dy
\]

(33)

with \( S \)-step (piecewise constant) kernels

\[
B(x, y) = S \sum_{i,j=1}^S B_{ij} \chi_i(x)\chi_j(y), \quad B_{ij} \in \mathbb{C}, \quad \chi_i(x) = \begin{cases} 
1, & x \in \left[\frac{i-1}{S}, \frac{i}{S}\right), \\
0, & \text{otherwise}.
\end{cases}
\]

(34)

Such operators form an algebra isomorphic to \( \mathbb{C}^{S \times S} \) (see, e.g., [19]). But this algebra does not contain the identity operator (\( Iu = u \)). Let us supplement it by adding new operators

\[
Au(x) = A(x)u(x) + \int_0^1 B(x, y)u(y)dy,
\]

(35)

where \( A(x) = \sum_{i=1}^S A_i\chi_i(x) \).

(36)

In other words

\[
A(x) = \chi^\top(x)A\chi(x), \quad B(x, y) = S\chi^\top(x)B\chi(y), \quad \text{where}
\]

\[
\chi = (\chi_i)_{i=1}^S, \quad \mathbf{A} = \text{diag}(A_i), \quad \mathbf{B} = (B_{ij})_{i,j=1}^S.
\]

(37)

(38)

Taking \( a_i = (i - 1)/S \) and using the notations (4)-(5) we obtain

\[
\mathcal{A} = \sum_{i=1}^S \mathcal{E}_i^0[A_i] + \sum_{i,j=1}^S \mathcal{E}_{ij}^1[B_{ij}].
\]

(39)

Applying the results of Theorem 1.1 and Corollary 1.2 along with (35)-(38) we obtain that

\[
\mathcal{A}^{-1}u(x) = \chi^\top(x)A^{-1}\chi(x)u(x) + S \int_0^1 \chi^\top(x)((\mathbf{A} + \mathbf{B})^{-1} - \mathbf{A}^{-1})\chi(y)u(y)dy
\]

(40)
if and only if all matrices are invertible (otherwise \( \mathcal{A} \) is non-invertible). The spectrum of \( \mathcal{A} \) consists of all eigenvalues of \( \mathcal{A} \) and \( \mathcal{A} + \mathcal{B} \). In general,

\[
f(\mathcal{A})u(x) = \mathbf{x}^\top f(\mathcal{A})\mathbf{x}(x)u(x) + S \int_0^1 \mathbf{x}^\top (f(\mathcal{A} + \mathcal{B}) - f(\mathcal{A}))\mathbf{x}(y)u(y)dy
\]  

for various functions \( f \) (e.g., rational/algebraic/transcendent: \( \exp, \sqrt{\cdot}, \ldots \)). We have reduced explicitly the functional calculus on integral operators to the functional calculus on matrices. Actually, the method based on the expansion in \( \mathcal{E}_{nm}^\alpha \) works well for more complex mixed multidimensional integral operators.

**Example 2.** Let us consider a discrete normalized 3D time-dependent Schrödinger operator with three planar potentials \( V_1, V_2, V_3 \) with their intersections (see Fig. 2) and with attenuated harmonic source term \( e^{-(i\lambda + \epsilon)t} \) (\( \lambda \in \mathbb{R} \) is the frequency, \( \epsilon > 0 \) is the attenuation factor, and the amplitude is 1) located at the origin. Then the corresponding equation on the wave function \( \Psi \) is

\[
\frac{i}{\hbar} \frac{\partial \Psi_n}{\partial t} - \Delta_{\text{discr}} \Psi_n - V_n \Psi_n = \delta_{n0} e^{-(i\lambda + \epsilon)t}, \quad n = (n_i)_1^3 \in \mathbb{Z}^3, \tag{42}
\]
where

\[
V_n = \begin{cases} 
V, & n_i = 0, \prod_{j \neq i} n_j \neq 0, \\
V + V_j, & n_i = n_j = 0, n_k \neq 0 (k \notin \{i, j\}), \\
V_1 + V_2 + V_3, & n = 0, \\
0, & \text{otherwise}
\end{cases}
\]

and \( \bar{n} \) denotes points adjacent to \( n \). Our goal is to solve the Schrödinger equation and to find \( \Psi_n \). Assuming \( \Psi_n(t) = \hat{\Psi}_n e^{i(\varepsilon + \chi)t} \) with time-independent \( \hat{\Psi}_n \) and taking Fourier series \( \hat{\Psi}(k) = \sum_{n \in \mathbb{Z}^3} e^{2\pi in^\top k} \hat{\Psi}_n \), \( k = (k_i)_i \in [0, 1)^3 \) we can rewrite (12) in the integral form

\[
A(k)\hat{\Psi}(k)

- V_1 \int_0^1 \hat{\Psi}(x_1, k_2, k_3)dx_1 - V_2 \int_0^1 \hat{\Psi}(k_1, x_2, k_3)dx_2 - V_3 \int_0^1 \hat{\Psi}(k_1, k_2, x_3)dx_3 = 1,
\]

where \( A(k) = -i\varepsilon + \chi - 4\sum_{i=1}^3 \sin^2 \pi k_i \). This 3D system contains non-parallel potential defects and hence does not admit explicit procedure for finding \( \hat{\Psi} \) (see, e.g., [22] for 2D case). There are various methods to obtain the approximated solution of (5). We will use the method based on the expansion of the operator (5) in the basis \( \mathcal{E}_{nm}^\alpha \) to obtain the approximation of \( \hat{\Psi} \) with an arbitrary precision. Let \( p \in \mathbb{N}, h = 1/p \). Then, following (11) we have

\[
\Omega \equiv [0, 1)^3 = \bigcup_{n \in \mathcal{N}} \{a_n + [0, h]^3\}, \quad \mathcal{N} = \{0, \ldots, p - 1\}^3, \quad a_n = h n.
\]

It is convenient to use the indices \( n, m \in \mathcal{N} \) in (15) instead of numbers \( i, j \). Introducing \( A_n = A(a_n + 1h/2) \) (where \( 1 = (1, 1, 1) \)) we can write the approximation of the equation \( \mathcal{A}\hat{\Psi} = 1 \) (\( \mathcal{A} \) denotes the left-hand side of (14)) as \( \tilde{\mathcal{A}}\hat{\Psi} = 1 \), where

\[
\tilde{\mathcal{A}} = \sum_{n \in \mathcal{N}} \varepsilon^\theta_{nm}[A_n] - \sum_{n, m \in \mathcal{N}, n_1 = m_1, \ldots, n_3 = m_3} \varepsilon_{nm}^{[1]} [hV_1] - \sum_{n, m \in \mathcal{N}, n_1 = m_1, \ldots, n_3 = m_3} \varepsilon_{nm}^{[2]} [hV_2] - \sum_{n, m \in \mathcal{N}, n_1 = m_1, \ldots, n_3 = m_3} \varepsilon_{nm}^{[3]} [hV_3].
\]

Then the matrices \( A_\alpha \in \mathbb{C}^{p^3 \times p^3} \) (see (14), (15)) are defined by

\[
A_{\emptyset} = \text{diag}(A_n), \quad A_{\{1\}} = -hV_1(\delta_{n_2m_2}\delta_{n_3m_3})_{n, m \in \mathcal{N}}, \quad A_{\{2\}} = -hV_2(\delta_{n_1m_1}\delta_{n_3m_3})_{n, m \in \mathcal{N}}, \quad A_{\{3\}} = -hV_3(\delta_{n_1m_1}\delta_{n_2m_2})_{n, m \in \mathcal{N}}, \quad A_{\{1, 2\}} = A_{\{1, 3\}} = A_{\{2, 3\}} = A_{\{1, 2, 3\}} = 0.
\]

The matrices \( B_\alpha \) (15) are

\[
B_{\emptyset} = A_{\emptyset}, \quad B_{\{i\}} = A_{\emptyset} + A_{\{i\}} (i = 1, 2, 3), \quad B_{\{1, 2\}} = A_{\emptyset} + A_{\{1\}} + A_{\{2\}}, \quad B_{\{1, 3\}} = A_{\emptyset} + A_{\{1\}} + A_{\{3\}}, \quad B_{\{2, 3\}} = A_{\emptyset} + A_{\{2\}} + A_{\{3\}}, \quad B_{\{1, 2, 3\}} = A_{\emptyset} + A_{\{1\}} + A_{\{2\}} + A_{\{3\}}.
\]
To compute $\tilde{A}^{-1}$ we follow (12) and (17), (18). Define the matrices $C_\alpha$ by
\[
C_\emptyset = B_\emptyset^{-1}, \quad C_{(i)} = B_{(i)}^{-1} - B_\emptyset^{-1} \quad (i = 1, 2, 3), \quad C_{(1,2)} = B_{(1,2)}^{-1} - B_{(1)}^{-1} - B_{(2)}^{-1} + B_\emptyset^{-1},
\]
\[
C_{(1,3)} = B_{(1,3)}^{-1} - B_{(1)}^{-1} - B_{(3)}^{-1} + B_\emptyset^{-1}, \quad C_{(2,3)} = B_{(2,3)}^{-1} - B_{(2)}^{-1} - B_{(3)}^{-1} + B_\emptyset^{-1},
\]
\[
C_{(1,2,3)} = B_{(1,2,3)}^{-1} - B_{(1,2)}^{-1} - B_{(1,3)}^{-1} - B_{(2,3)}^{-1} + B_{(1)}^{-1} + B_{(2)}^{-1} + B_{(3)}^{-1} - B_\emptyset^{-1}. \quad (49)
\]
All matrices $B_\alpha$ are invertible since $\tilde{A}^{-1}$ exists (because $\tilde{A} + i\varepsilon$ is self-adjoint). Then $\tilde{\Psi}$ has the form
\[
\tilde{\Psi} = \tilde{A}^{-1} = \chi^\top \left( C_\emptyset \chi + pC_{(1)} \int_0^1 \chi(x_1, k_2, k_3)dx_1 + pC_{(2)} \int_0^1 \chi(k_1, x_2, k_3)dx_2 +
\right.
\]
\[
\left. \quad + pC_{(3)} \int_0^1 \chi(k_1, k_2, x_3)dx_3 + p^2C_{(1,2)} \int_0^1 \int_0^1 \chi(x_1, x_2, k_3)dx_1dx_2 +
\right.
\]
\[
\left. \quad + p^2C_{(1,3)} \int_0^1 \int_0^1 \chi(x_1, k_2, x_3)dx_1dx_3 + p^2C_{(2,3)} \int_0^1 \int_0^1 \chi(k_1, x_2, x_3)dx_2dx_3 +
\right.
\]
\[
\left. \quad + p^3C_{(1,2,3)} \int_0^1 \int_0^1 \int_0^1 \chi(x_1, x_2, x_3)dx_1dx_2dx_3 \right), \quad (50)
\]
where the vector-valued function $\chi = (\prod_{i=1}^3 \chi_n(k_i))_{n\in\mathbb{N}}$ (see also (34) for $\chi_n$). Due to the simple form of $\chi$, all integrals in (50) can be computed explicitly: for example
\[
p^2 \int_0^1 \int_0^1 \chi(x_1, k_2, x_3)dx_1dx_3 = (\chi_{n_2}(k_2))_{n\in\mathbb{N}}.
\]
So, $\tilde{\Psi}$ in (50) is given explicitly. The norm of difference between $A$ (41) and $\tilde{A}$ (46) depends on how much $A_n$ approximates $A$, at least $\|A - \tilde{A}\|_{L^2\rightarrow L^2} \leq 24\pi h$, see also Theorem 1.3. The norm of the inverse operators $\|A^{-1}\|_{L^2\rightarrow L^2} \leq \varepsilon^{-1}$ and $\|\tilde{A}^{-1}\|_{L^2\rightarrow L^2} \leq \varepsilon^{-1}$ since $A + i\varepsilon$ and $\tilde{A} + i\varepsilon$ are self-adjoint. Hence $\|A^{-1} - \tilde{A}^{-1}\|_{L^2\rightarrow L^2} \leq 24\pi h/\varepsilon^2$. In particular,
\[
\|\tilde{\Psi} - \tilde{\Psi}\|_{L^2} \leq 24\pi h/\varepsilon^2, \quad \sum_{n\in\mathbb{Z}^3} |\tilde{\Psi}_n - \tilde{\Psi}_n|^2 \leq (24\pi h/\varepsilon^2)^2, \quad (51)
\]
where $\tilde{\Psi}_n$ are Fourier coefficients of $\tilde{\Psi}$. On the side, we obtain the following spectral estimates
\[
\text{dist}(\text{sp}(A), \bigcup_{\beta\in\{1,2,3\}} \text{sp}(B_\beta)) \leq 24\pi h. \quad (52)
\]
The estimates (51), (52) become better and better for $h \to 0$. Depending on the number $|\alpha|$, the components $\text{sp}(B_\alpha) \setminus (\cup_{|\beta|<|\alpha|} \text{sp}(B_\beta))$ approximate the volume, planar, guided, and local isolated spectral components of $A$ which corresponds to volume, planar, guided, and local wave functions (waves) $\Psi$. These waves propagate along the potential defects of the corresponding dimension $3 - |\alpha|$ and exponentially attenuate in the perpendicular directions to the defect, see, e.g., the discussion in [22].
3. Proof of Theorems 1.1 and 1.3

At first, let us consider the semigroup of subsets
\[ G = \{ e_\alpha : \alpha \subset \{1, \ldots, N\} \}, \quad e_\alpha e_\beta = e_{\alpha \cup \beta} \]
and the corresponding \( C^* \)-algebra
\[ \mathcal{M} = \left\{ \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha e_\alpha : A_\alpha \in \mathbb{C} \right\}. \]

The identity element in this algebra is \( 1 = e_\emptyset \), where \( \emptyset \) is the empty set. All basis elements \( e_\alpha^* = e_\alpha \) are self-adjoint. Define the mapping
\[ \pi_1 : \mathcal{M} \to \mathbb{C}^{2^N}, \]
\[ \pi_1 \left( \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha e_\alpha \right) = (B_\alpha)_{\alpha \subset \{1, \ldots, N\}}, \quad B_\alpha = \sum_{\beta \subset \alpha} A_\beta. \]

**Lemma 3.1.** The mapping \( \pi_1 \) is the *-algebra isomorphism. The inverse mapping is defined by
\[ \pi_1^{-1}((B_\alpha)_{\alpha \subset \{1, \ldots, N\}}) = \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha e_\alpha, \quad A_\alpha = \sum_{\beta \subset \alpha} (-1)^{|\alpha \setminus \beta|} B_\beta. \]

**Proof.** Consider the following basis in \( \mathcal{M} \)
\[ f_\alpha = e_\alpha \prod_{n \notin \alpha} (1 - e_{\{n\}}). \]

Direct calculations give us
\[ f_\alpha^* = f_\alpha, \quad f_\alpha f_\beta = \delta_{\alpha \beta} f_\alpha. \]

This means that \( \{f_\alpha\}_{\alpha \subset \{1, \ldots, N\}} \) is a basis in \( \mathbb{C}^{2^N} \) because the number of elements \( \# \{\alpha : \alpha \subset \{1, \ldots, N\}\} = 2^N \), and hence \( \mathcal{M} \) is isomorphic to \( \mathbb{C}^{2^N} \) with the isomorphism \( e_\alpha \leftrightarrow f_\alpha \). Using
\[ \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha e_\alpha = \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha e_\alpha \prod_{n \notin \alpha} (1 - e_{\{n\}} + e_{\{n\}}) = \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha \sum_{\beta \supset \alpha} e_\beta \prod_{n \notin \beta} (1 - e_{\{n\}}) \]
\[ = \sum_{\alpha \subset \{1, \ldots, N\}} A_\alpha \sum_{\beta \supset \alpha} f_\beta = \sum_{\alpha \subset \{1, \ldots, N\}} \left( \sum_{\beta \supset \alpha} A_\beta \right)f_\alpha = \sum_{\alpha \subset \{1, \ldots, N\}} B_\alpha f_\alpha \]
we obtain that the isomorphism \( e_\alpha \leftrightarrow f_\alpha \) coincides with \( \pi_1 \) (55)-(56). Similarly, identities
\[ \sum_{\alpha \subset \{1, \ldots, N\}} B_\alpha f_\alpha = \sum_{\alpha \subset \{1, \ldots, N\}} B_\alpha e_\alpha \prod_{n \notin \alpha} (1 - e_{\{n\}}) = \sum_{\alpha \subset \{1, \ldots, N\}} B_\alpha \sum_{\beta \supset \alpha} (-1)^{|\beta \setminus \alpha|} e_\beta \]
\[= \sum_{\alpha \in \{1, \ldots, N\}} \left( \sum_{\beta \subset \alpha} (-1)^{|\alpha \setminus \beta|} B_\beta \right) e_\alpha = \sum_{\alpha \in \{1, \ldots, N\}} A_\alpha e_\alpha \quad (63)\]

give us the form of the inverse mapping \(\pi^{-1}_1\) \[57\]. ■

By Lemma \[3.1\], the \(C^*\)-algebra \(\mathcal{M}_M = \mathcal{M}_{M \times M}\) of \(M \times M\) matrices with entries belonging to \(\mathcal{M}\) is isomorphic to \((C^{M \times M})^{2^N}\). The corresponding isomorphism is

\[\pi_M : \mathcal{M}_M \to (C^{M \times M})^{2^N}, \quad \pi_M(\sum_{\alpha \in \{1, \ldots, N\}} A_\alpha e_\alpha) = (B_\alpha)_{\alpha \in \{1, \ldots, N\}}, \quad B_\alpha = \sum_{\beta \subset \alpha} A_\beta, \quad (64)\]

\[\pi^{-1}_M((B_\alpha)_{\alpha \in \{1, \ldots, N\}}) = \sum_{\alpha \in \{1, \ldots, N\}} A_\alpha e_\alpha, \quad A_\alpha = \sum_{\beta \subset \alpha} (-1)^{|\alpha \setminus \beta|} B_\beta. \quad (65)\]

The same result holds for the \(C^*\)-algebra \(\mathcal{M}_{MS} \simeq \mathcal{M}^{S \times S}\) \(\simeq \mathcal{M}^{MS \times MS}\). For any \(A \in C^{M \times M}\) and \(\alpha \subset \{1, \ldots, N\}\) the basis elements \(F_{ij}^\alpha[A] = (\delta_{ij} A e_\alpha)_{i,j=1}^S \in \mathcal{M}_{MS}\) satisfy the same equations as the basis elements \(E_\alpha^\beta[[A] \in \mathcal{A}\) (see \[13\])

\[F_{ij}^\alpha[A] = F_{ji}^\alpha[A^*], \quad F_{ij}^\alpha[A] + F_{ij}^\alpha[B] = F_{ij}^\alpha[A + B], \quad F_{ij}^\alpha[A] F_{kl}^\beta[B] = \delta_{jk} F_{il}^\alpha [AB]. \quad (66)\]

This means that \(\mathcal{M} \simeq \mathcal{M}_{MS}\) with the natural isomorphism \(E_\alpha^\beta[A] \leftrightarrow F_{ij}^\alpha[A]\). Then, comparing \[13\]-\[16\] and \[64\]-\[65\] (with \(MS\) instead of \(M\)) we deduce that \(\pi\) in \[16\] is the isomorphism satisfying \[17\]-\[18\].

Now, let us prove Theorem \[1.3\]. It is seen that

\[(A_c - A) u(k) = \sum_{\alpha \subset \{1, \ldots, N\}} \sum_{i=1}^{2^N} \int_{\Omega_i \cap \{|x| = 1\}} (A_\alpha(k, x_\alpha) - A_\alpha(\tilde{k}, (\tilde{x}_i)_\alpha)) u(k_\pi + x_\alpha) dx_\alpha,
\]

where \(\tilde{k}\) is the center of the cube \(\Omega_j\) for which \(k \in \Omega_j\) and \(\tilde{x}_i\) is the center of \(\Omega_i\). Then, the mean value estimates

\[\max_{x \in \Omega_i} |A_\alpha(k, x_\alpha) - A_\alpha(\tilde{k}, (\tilde{x}_i)_\alpha)| \leq N \frac{\|A\|_{L^\infty}}{2p} \|\nabla A_\alpha\|_{L^\infty}\]

finish the proof of \[30\]. Te estimates \[31\] are classical spectral estimates for \(A_c\) and its perturbation \(\mathcal{A} = A_c + A - A_c\) in terms of the norm of perturbation \(\delta = \|A - A_c\|_{L^2 \to L^2}\), see, e.g., \[23\]. If \(A_c\) is invertible then \(A\) is invertible for all sufficiently large \(p\) and \(\|A^{-1}\|_{L^2 \to L^2} \to \|A_c^{-1}\|_{L^2 \to L^2}\). Moreover,

\[\|A_c^{-1} - A^{-1}\|_{L^2 \to L^2} = \| \left( \sum_{n=1}^\infty (A_c^{-1} (A_c - A))^n \right) A^{-1}\|_{L^2 \to L^2} \leq \frac{\delta \|A_c^{-1}\|_{L^2 \to L^2}^2}{1 - \delta \|A_1^{-1}\|_{L^2 \to L^2}}.\]

The last term tends to 0 since \(\delta\) tends to 0 for \(p \to \infty\).
4. Conclusion

We have shown that the analysis of mixed multidimensional integral and some type of ergodic operators can be explicitly reduced to the analysis of special matrices. This allows us to compute functions of such operators and their spectra explicitly with an arbitrary precision.

Funding statement

This work was partially supported by the RSF project 18-11-00032 and DFG project TRR 181.

References

[1] A. B. Movchan, and L. I. Slepyan, Band gap green’s functions and localized oscillations, Proc. R. Soc. A 463 (2007) 2709–2727.
[2] D. J. Colquitt, M. J. Nieves, I. S. Jones, A. B. Movchan, and N. V. Movchan, Waves in lattices with imperfect junctions and localised defect modes, Proc. R. Soc. A 469 (2150) (2013) 20120579.
[3] M. Makwana, and R. V. Craster, Localised point defect states in asymptotic models of discrete lattices, Q. J. Mechanics Appl. Math. 66 (2013) 289–316.
[4] A. A. Kutsenko, Algebra of multidimensional periodic operators with defects, J. Math. Anal. Appl. 428 (2015) 221–230.
[5] A. A. Kutsenko, Recovery of defects from the information at detectors, Inverse Problems 32 (2016) 055005.
[6] V. Caudrelier, Multisymplectic approach to integrable defects in the sine-gordon model, J Phys A Math Gen 48 (19) (2015) 195203.
[7] M. F. Norman, Ergodicity of diffusion and temporal uniformity of diffusion approximation, J. Appl. Prob. 14 (1977) 399–404.
[8] J. Schenker, Diffusion in the mean for an ergodic Schrodinger equation perturbed by a fluctuating potential, Comm. Math. Phys. 339 (2015) 859–901.
[9] I. Prigogine, A. P. Grecos, and Cl. George , Kinetic theory and ergodic properties, Proc. Nati. Acad. Sci. USA 14 (1976) 1802–1805.
[10] E. Kreyszig, Advanced Engineering Mathematics, Wiley, 2011.
[11] V. Kirsh, and L. A. Pastur , Ergodic theory and discrete one-dimensional random Schrodinger operators: uniform existence of the Lyapunov exponent , Contemp. Math. 327 (2003) 1–23.
[12] D. Lenz, F. Schwarzengerber, and I. Veselic, A Banach space-valued ergodic theorem and the uniform approximation of the integrated density of states , Geom. Dedicata 150 (2011) 1–34.
[13] E. Korotyaev, N. Saburova, Effective masses for laplacians on periodic graphs, J. Math. Anal. Appl. 436 (2016) 104–130.
[14] F. Vico, M. Ferrando, L. Greengard, Z. Gimbutas, The decoupled potential integral equation for time-harmonic electromagnetic scattering, Comm. Pure Appl. Math. 69 (2016) 771–812.
[15] T. Eisner, B. Farkas, M. Haase, and R. Nagel, Operator theoretic aspects of ergodic theory, Springer, 2012.
[16] V. Kirsh, and L. A. Pastur , Analogues of Szego’s theorem for ergodic operators , Mat. Sb. 206 (2015) 93–119.
[17] C. Houdayer, and Y. Isono, Bi-exact groups, strongly ergodic actions and group measure space type III factors with no central sequence , Comm. Math. Phys. 348 (2016) 991–1015.
[18] E. I. Fredholm, Sur une classe d’équations fonctionnelles, Acta Math. 27 (1903) 365–390.
[19] M. S. Gockenbach, Finite-Dimensional Linear Algebra, Discrete mathematics and its applications., CRC Press, 2010.
[20] S. Albeverio, E. I. Gordon, A. Yu. Hrennikov, Finite-dimensional approximations of operators in the Hilbert spaces of functions on locally compact abelian groups, Acta Appl. Math. 64 (2000) 33–73.

[21] A. Kutsenko, Mixed multidimensional integral operators with piecewise constant kernels and their representations, Linear and Multilinear Algebra 0 (2017) 0. doi:10.1080/03081087.2017.1415294

[22] A. A. Kutsenko, Algebra of 2d periodic operators with local and perpendicular defects, J. Math. Anal. Appl. 442 (2016) 796–803.

[23] T. Kato, Perturbation theory for linear operators, Springer-Verlag Berlin Heidelberg, 1995.