Abstract: After developing in a $SU(2)_L \times U(1)$ gauge theory for $J = 0$ mesons, I show in the case of two generations that:
- the electroweak mass eigenstates differ, especially in the $K$ and $D$ sector, from what they are assumed to be on the basis of a classification by the $SU(4)$ group of flavour;
- a new mechanism for $CP$ violation springs out, different from that of the standard model for quarks which requires at least three generations, and I construct a $SU(2)_L \times U(1)$ gauge invariant Lagrangian which is not invariant by $CP$.

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1 Introduction.

The kaons have a long and rich story of a puzzling and cumbersome system. We shall not recall here the many steps that went from the $\tau - \theta$ puzzle [2] to the intricacies of $CP$-violation [3][4]. With the increase of the available energy came the discovery of the system of the $D$ mesons, similar in many respects to the latter. Presently, other thresholds have been crossed, but we shall stay here beyond the bottom threshold and mainly concentrate on $K$ and $D$ mesons; this corresponds, in the quark language [5], to two generations. We do not, however, study these particles from the quark point of view, but follow [1] that brings forward an electroweak gauge theory for the $J = 0$ mesons themselves; those are thus both the fields and the particles of the model which is, besides, compatible with the electroweak standard model for quarks [6] (see section 2).

In [1], I exhibited in particular all their electroweak representations with a given $CP$ quantum number, in the form of $N^2/2$ quadruplets ($N/2$ is the number of generations), containing a neutral singlet and a triplet of the custodial $SU(2)_V$ symmetry; this symmetry, different from what it is generally assumed to be [7], was shown there to be linked to the quantization of the electric charge (the extension to the leptonic sector was studied in [8]). I particularize here the study and show that:

- the electroweak mass eigenstates are different from their usual quark content attributed from a classification by the $SU(4)$ group of flavour, symmetry of strong interactions; in particular, the alignment of strong and electroweak $K$ and $D$ mesons is impossible, even at the limit of vanishing mixing (Cabibbo) angle;
- unlike what happens in the electroweak standard model for quarks, [3][9], one can now construct already with two generations a $SU(2)_L \times U(1)$ invariant gauge Lagrangian for $J = 0$ mesons which is not $CP$ invariant, and the electroweak mass eigenstates of which are not eigenstates of $CP$.

2 Electroweak representations of $J = 0$ mesons.

I adopt the point of view that there is no loophole in the experimental determination of the parity of $J = 0$ mesons. We know that this has been questioned [2], and will see that it might still be, but it allows us to work with electroweak mesonic representations the entries of which have a definite parity, and we put aside in this paper the possibility of an admixture of scalars and pseudoscalars.

We know then from [1] that one can find quadruplet representations of the electroweak $SU(2)_L \times U(1)$ gauge group which are of two types: the first contains a scalar singlet and a pseudoscalar triplet of the custodial $SU(2)_V$:

\[(M^0, \vec{M}) = (S^0, \vec{P}), \]

and the second a pseudoscalar singlet and a scalar triplet:

\[ (M^0, \vec{M}) = (P^0, \vec{S}). \]

The $SU(2)_L$ group acts on both by:

\[ T^i_L M^j = -\frac{i}{2} (\epsilon_{ijk} M^k + \delta_{ij} M^0), \]

\[ T^i_L M^0 = \frac{i}{2} \epsilon_{ij} M^j, \]
and mixes, as expected from its “left-handed” nature, scalars and pseudoscalars. The action of the \( U(1) \) group results from the Gell-Mann-Nishijima relation as explained in [1].

All entries are \( N \times N \) matrices, where \( N \) is the number of “flavours”, and the quadruplets can be written

\[
\Phi(\mathcal{D}) = (M^\nu, M^\mu, M^+, M^-)(\mathcal{D})
\]

\[
= \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{D} & 0 \\ 0 & K^\dagger DK \end{pmatrix}, i \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{D} & 0 \\ 0 & -K^\dagger DK \end{pmatrix}, i \begin{pmatrix} 0 & \mathcal{D}K \\ 0 & 0 \end{pmatrix}, i \begin{pmatrix} 0 & 0 \\ K^\dagger & 0 \end{pmatrix} \right];
\]

(4)

\( K \) is a \( N/2 \times N/2 \) unitary matrix that reduces, for \( N = 4 \), to the Cabibbo matrix [11]; \( \mathcal{D} \) is also a real \( N/2 \times N/2 \) matrix. That the entries \( M^+ \) and \( M^- \) are, up to a sign, hermitian conjugate (i.e. charge conjugate), requires that the \( \mathcal{D} \)'s are restricted to symmetric or antisymmetric matrices.

The group action [3] for the quadruplets is a particular case of a more general one which involves commuting and anticommuting \( N \times N \) matrices inside the \( U(N)_R \times U(N)_L \) chiral algebra; the generators of the \( SU(2)_L \) and \( U(1) \) subgroups are themselves represented by \( N \times N \) matrices [4],

\[
T^3_L = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^+ = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ K^\dagger & 0 \end{pmatrix},
\]

(5)

acting trivially on \( N \)-vectors of quarks if they are taken as the fundamental fields (the action of the gauge group on the mesons can be deduced from that on the quarks when the former are considered as \( \bar{q}_i q_j \) or \( \bar{q}_i \gamma_5 q_j \) composite states). Symmetric (\( S^0, \bar{P} \)) and antisymmetric (\( P^0, \bar{S} \)) representations are \( CP \)-even, while antisymmetric (\( S^0, \bar{P} \)) and symmetric (\( P^0, \bar{S} \)) representations are \( CP \)-odd. The \( C \), and hence the \( CP \) quantum number of a representation can be swapped by multiplying its entries by \( \pm i \).

Restricting to \( N = 4 \), we write below the four types of \( (M^0, M^3, M^+, M^-) \) quadruplets that appear, corresponding respectively to

\[
\mathcal{D}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{D}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{D}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};
\]

(6)

\[
\Phi(\mathcal{D}_1) = \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}, i \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}, i \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}, i \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \right] \]

(7)
\[
\Phi(D_2) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix},
\]

\[
\Phi(D_3) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -2c_\theta s_\theta & c_\theta & s_\theta \\
1 & c_\theta & -s_\theta & c_\theta \\
c_\theta & -s_\theta & c_\theta & s_\theta \\
c_\theta & s_\theta & -s_\theta & c_\theta
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & 2c_\theta s_\theta \\
1 & c_\theta \\
c_\theta & -s_\theta \\
c_\theta & s_\theta
\end{pmatrix},
\]

\[
\Phi(D_4) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -c_\theta \\
-c_\theta & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -s_\theta \\
-s_\theta & 1
\end{pmatrix},
\]

\[
\Phi(D) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -c_\theta \\
-c_\theta & 1
\end{pmatrix} \frac{i}{\sqrt{2}} \begin{pmatrix}
1 & -s_\theta \\
-s_\theta & 1
\end{pmatrix}.
\]

\[
c_\theta \text{ and } s_\theta \text{ stand respectively for the cosine and sine of the Cabibbo angle } \theta_c.\]

We shall also use in the following the notations

\[
(S^0, \overline{P})(D_1) = \Phi_1, \quad (S^0, \overline{P})(D_2) = \Phi_2, \quad (S^0, \overline{P})(D_3) = \Phi_3, \quad (S^0, \overline{P})(D_4) = \Phi_4. \quad (11)
\]

While there is of course no antisymmetric $D$ for $N = 2$ (one generation), $D_4$ is such a
matrix for \( N = 4 \) and is the only one in this case.

Though quarks never appear as fundamental fields, the reader can easily make the link between the mesons, represented above as \( 4 \times 4 \) matrices, and their quark “content”: it is simply achieved by sandwiching a given matrix belonging to a representation between fermionic 4-vectors \((\bar{u}, \bar{c}, \bar{d}, \bar{s})\) and \((u, c, d, s)\), and by remembering the parity of the corresponding particle. With the scaling that has to be introduced \([12, 11, 1]\), we have, for example

\[
\mathbb{P}^+(\mathbb{D}_1) = i f \left( g_0 (\pi^+ + D^+_s) + s_0 (K^+ - D^+) \right),
\]

where, according to the classification by flavour \( SU(4) \), we have translated, for pseudoscalars, \( \bar{u}d \) into \( \pi^+ \), \( \bar{u}s \) into \( K^+ \), \( \bar{c}d \) into \( D^+ \) and \( \bar{c}s \) into \( D^+_s \) etc.; \( f \) is the leptonic decay constant of the mesons (considered to be the same for all of them) and \( H \) is the Higgs boson.

We always refer to \( \pi, K, D, D_s \) as the eigenstates of strong interactions.

### 3 The mass and \( CP \) eigenstates for \( \theta_c = 0 \).

#### 3.1 Quadratic invariants.

For the sake of simplicity, we shall deal here with the case of vanishing Cabibbo angle. It teaches us the main features of the unavoidable misalignment between strong and electroweak eigenstates.

To every representation is associated a quadratic expression invariant by the electroweak gauge group

\[
\mathcal{I} = (M^0, \vec{M}) \otimes (M^0, \vec{M}) = M^0 \otimes M^0 + \vec{M} \otimes \vec{M};
\]

the “\( \otimes \)” product is a tensor product, not the usual multiplication of matrices and means the product of fields as functions of space-time; \( \vec{M} \otimes \vec{M} \) stands for \( \sum_{i=1,2,3} M^i \otimes M^i \).

The representations \( \Phi \) are such that the algebraic sum (to be specified below) of the corresponding invariants is diagonal both in the electroweak basis and in the basis of strong eigenstates; in particular, the (quadratic) part of the kinetic terms that involves ordinary derivatives

\[
\frac{1}{2} \sum_{i=1,2,3} \left( -\partial_\mu (S^0, \vec{S})(\mathbb{D}_i) \otimes \partial^\mu (S^0, \vec{S})(\mathbb{D}_i) + \partial_\mu (S^0, \vec{F})(\mathbb{D}_4) \otimes \partial^\mu (S^0, \vec{F})(\mathbb{D}_4) \right)
+ \partial_\mu (\mathbb{P}^0, \vec{S})(\mathbb{D}_i) \otimes \partial^\mu (\mathbb{P}^0, \vec{S})(\mathbb{D}_i) - \partial_\mu (\mathbb{P}^0, \vec{S})(\mathbb{D}_4) \otimes \partial^\mu (\mathbb{P}^0, \vec{S})(\mathbb{D}_4) \right)
\]

is also diagonal in the strong eigenstates, with the same normalization factor 1 for all of them; the relative signs that must be introduced for this purpose are due to the following:

- all pseudoscalars we define without an “\( i \)” (like \( \pi^+ = \bar{u}d_{\text{odd}} \)), such that the \( (\mathbb{P}^0, \vec{S}) \) quadruplets have to be multiplied by \( \pm i \);
- the skew-symmetry of \( \mathbb{D}_4 \) making the corresponding quadruplets have an opposite behaviour by charge conjugation as compared to the other six, introduces an extra minus sign in the corresponding quadratic invariants.

Another invariant is the “scalar product” of two representations transforming alike by the gauge group; for example such is

\[
\mathcal{I}_{12} = \Phi_1 \otimes \Phi_2 = S^0(\mathbb{D}_1) \otimes S^0(\mathbb{D}_2) + \vec{F}(\mathbb{D}_1) \otimes \vec{F}(\mathbb{D}_2).
\]
Because of the remark starting section 2, we \textit{a priori} exclude connecting mesons of different parities inside an invariant like \((S^0, P_\parallel)(D_1) \otimes (P^0, S_\parallel)(D_2) = S^0(D_1) \otimes P^0(D_2) + \bar{P}(D_1) \otimes S(D_2)\). Note that the \(SU(2)_L \times U(1)\) invariants do \textit{not} involve tensorial products of hermitian conjugate (charge conjugate) fields, but of the fields themselves; for example \(I_{12} = \Phi_1 \otimes \Phi_2\) is given by (15) and \(S^0 \otimes S^0 + P \otimes \bar{P}\). This underlies the results below.

### 3.2 A first possible attitude.

One can choose the electroweak mass eigenstates to match the quadruplets displayed above; one then does not introduce crossed mass terms, and we have \textit{a priori} eight independent mass scales. The mass eigenstates are also \(CP\) eigenstates, but it is obvious that electroweak and strong eigenstates differ.

Presumably, this choice is only reasonable in the case of three generations, because of the content of \(\Phi_1\): \(\Phi^\dagger_1\) is the Higgs boson and the three \(\Phi\) form the triplet of Goldstone bosons of the broken electroweak gauge group; once “eaten” by the three gauge fields that get massive by doing, they become their longitudinal degrees of freedom, the mass of which are consequently expected to match those of the gauge bosons; only in the case of three generations can we expect three (at least) pseudoscalar mesons to be as heavy as the \(W^\pm, Z\). We know now that some mesons interpreted to contain the “top” quark weight as much as 175 GeV \(\texttt{[13]}\), but the possibility is wide open that three among the eleven pseudoscalar mesons including the top quark are identical with the longitudinal \(W^\pm, Z\): indeed, in the present picture, the wave of the asymptotic states (mesons) is disconnected from that of their constituents fields, and different mass scales can be attributed, in a \(SU(2)_L \times U(1)\) invariant way, to different representations, independently of their “quark content”; since the eleven above mentioned “topped” pseudoscalar mesons will fit into several quadruplets, there is no reason why they should correspond to a single mass scale.

### 3.3 The \(\pi - D_s\) mass splitting.

The second attitude is to attempt to align strong and weak eigenstates, at least some of them, and we first focus on the two representations \(\Phi_1\) and \(\Phi_2\) defined in \(\texttt{[14]}\).

Consider the mass term

\[
\mathcal{L}_m = \frac{1}{2}(m^2_1 \Phi_1 \otimes \Phi_1 + m^2_2 \Phi_2 \otimes \Phi_2 - 2m^2_{12} \Phi_1 \otimes \Phi_2).
\]

(16)

It is diagonalized, in the charged sector for example, by the states

\[
A^+ = \frac{1}{2} \left( \sqrt{\frac{m_1}{m_2}} (\pi^+ + D^+_s) + \sqrt{\frac{m_2}{m_1}} (\pi^+ - D^+_s) \right),
\]

\[
B^+ = \frac{1}{2} \left( \sqrt{\frac{m_1}{m_2}} (\pi^+ + D^+_s) - \sqrt{\frac{m_2}{m_1}} (\pi^+ - D^+_s) \right),
\]

(17)

and their charge conjugate. The same happens in the neutral sector.

For \(m_1 = m_2 = m\), the kinetic terms are also diagonal in \(\bar{A}\) and \(\bar{B}\), in which case \(\bar{A}\) and \(\pi\) are aligned, so are \(\bar{B}\) and \(\bar{D}_s\), with masses squared \(m^2 ± m^2_{12}\).
3.4 The $K - D$ system.

We shall now see that, because the matrices $D_3$ and $D_4$ have opposite symmetry properties, it is impossible to align strong and electroweak eigenstates in the $K - D$ system (the only exception is the trivial one, corresponding to degenerate $K$ and $D$ mesons).

Consider the two electroweak representations $U$ and $V$ obtained by combining $\Phi_3$ and $\Phi_4$ defined in (11)

$$\begin{aligned}
\Phi_3 &= \alpha U + \beta V, \\
\Phi_4 &= \delta U + \zeta V.
\end{aligned}$$

That the (CP invariant) kinetic term

$$L_{kin} = -\frac{1}{2}(\partial_\mu \Phi_3 \otimes \partial^\mu \Phi_3 - \partial_\mu \Phi_4 \otimes \partial^\mu \Phi_4)$$

stays diagonal in $U$ and $V$ requires

$$\alpha \beta - \delta \zeta = 0.$$  

Let us introduce the (CP invariant) mass terms

$$L_m = \frac{1}{2}(m_3^2 \Phi_3 \otimes \Phi_3 - m_4^2 \Phi_4 \otimes \Phi_4 - 2im_{34}^2 \Phi_3 \otimes \Phi_4).$$

We leave aside the case $m_3^2 = m_4^2$, $m_{34}^2 = 0$ which corresponds to degenerate $K$ and $D$.

If (21) can be diagonalized together with the kinetic terms, there should exist two mass scales $\mu_U^2$ and $\mu_V^2$ such that the quadratic Lagrangian invariant by $SU(2)_L \times U(1)$ is diagonal in $U$ and $V$: it then reads, using the condition (20)

$$L = \frac{1}{2}(\delta^2 - \alpha^2)(\partial_\mu U \otimes \partial^\mu U - \frac{\beta^2}{\delta^2 - \alpha^2} \partial_\mu V \otimes \partial^\mu V) - \frac{1}{2}(\mu_U^2 U \otimes U - \mu_V^2 V \otimes V).$$

The masses of $U$ and $V$ are

$$m_U^2 = \frac{1}{\delta^2 - \alpha^2} \mu_U^2, \quad m_V^2 = \frac{\delta^2}{\beta^2(\delta^2 - \alpha^2)} \mu_V^2.$$  

We shall take hereafter, without loss of generality

$$\delta^2 - \alpha^2 = 1$$

and look for real $\delta^2$ and $\alpha^2$. The condition of reality is not a restriction for what we look at since $U$ and $V$ can only be aligned with strong eigenstates for $\alpha$ and $\delta$ both real or both imaginary (see (24)); this is impossible as we show below.

Still making use of the condition (20), and of the relation (24), eqs. (18) invert to

$$\begin{aligned}
U &= \delta \Phi_4 - \alpha \Phi_3, \\
V &= \frac{\delta}{\beta}(\delta \Phi_3 - \alpha \Phi_4).
\end{aligned}$$

Replacing in (22), one gets, by matching it with (21), the system
\[
\begin{align*}
\begin{cases}
m_3^2 &= -\alpha^2 \mu_U^2 + \delta^2 \frac{\mu_U^2}{\mu_V^2} \mu_V^2 = \mu_U^2 - \delta^2 (\mu_U^2 - \frac{\mu_U^2}{\mu_V^2} \mu_V^2), \\
m_4^2 &= \delta^2 \mu_U^2 - \alpha^2 \frac{\mu_U^2}{\mu_V^2} \mu_V^2 = \mu_U^2 + \alpha^2 (\mu_U^2 - \frac{\mu_U^2}{\mu_V^2} \mu_V^2), \\
m_{34}^2 &= i \alpha \delta (\mu_U^2 - \frac{\mu_U^2}{\mu_V^2} \mu_V^2),
\end{cases}
\end{align*}
\]

One can extract from the first two equations of (26):
\[
m_U^2 = \frac{m_4^2 + \alpha^2 (m_2^2 + m_3^2)}{1 + 2 \alpha^2}, \quad m_V^2 = \frac{m_3^2 + \alpha^2 (m_2^3 + m_3^2)}{1 + 2 \alpha^2},
\]
and, from the third equation of (26) together with (24), one finds that \(\alpha^2\) must satisfy
\[
\alpha^4 + \alpha^2 + \frac{\xi^2}{4(1 + \xi^2)} = 0, \quad \xi = \frac{2 m_{34}^2}{m_4^2 - m_3^2}.
\]
It only has real solutions in \(\alpha^2\) for \(\alpha^2 < 1\); we then go to \(\rho = -i \alpha\), and the solutions of (28) are
\[
\rho^2 = \frac{1 \pm \sqrt{1 + 4 \xi^2}}{2}. \tag{29}
\]
\(\rho^2\) is always smaller than 1, such that \(\delta^2 = 1 - \rho^2\) is always positive and \(\delta\) real. \(U\) has \(\text{CP} = -1\), and \(V\) has \(\text{CP} = +1\) if \(\beta\) is chosen to be real.

Due in particular to \(\alpha = i \rho\) in eqs. (25), both \(U\) and \(V\) are different from “strong” eigenstates. The case \(\rho = 0\) corresponds to a vanishing crossed mass term \(m_{34}^2 = 0\) and to the alignment of \(U\) with \(\Phi_3\), and of \(V\) with \(\Phi_4\) (see subsection 3.2).

The apparently singular case \(\alpha^2 = -1/2\) in (27) is better treated directly from eq. (21) since it also corresponds to \(m_3^2 = m_4^2 = m^2\): the eigenvectors are \(\Phi_3 \pm i \Phi_4\) and the corresponding masses \(m^2 \pm m_{34}^2\).

To fix the ideas, let us take a very simple example: \(\xi^2 = 3, \rho = 1/2, \delta = \sqrt{3}/2\); one has then
\[
m_U^2 = \frac{3 m_4^2 - m_3^2}{2}, \quad m_V^2 = \frac{3 m_3^2 - m_4^2}{2}; \tag{30}
\]
\(U^\pm\) and \(V^\pm\) write
\[
\begin{align*}
U^+ &= \frac{1}{2} \left( (1 + i \sqrt{3}) K^+ + (1 - i \sqrt{3}) D^+ \right), \\
U^- &= \frac{1}{2} \left( (1 - i \sqrt{3}) K^- + (1 + i \sqrt{3}) D^- \right) = U^+ = -\text{CP} \quad U^+, \\
V^+ &= \frac{\sqrt{3}}{4 \beta} \left( (1 + i \sqrt{3}) K^+ - (1 - i \sqrt{3}) D^+ \right), \\
V^- &= -\frac{\sqrt{3}}{4 \beta} \left( (1 - i \sqrt{3}) K^- - (1 + i \sqrt{3}) D^- \right) = -V^+ = +\text{CP} \quad V^+.
\end{align*}
\]
Since \(i K^+\) and \(K^+\) have opposite \(\text{CP}\) transformation, and likewise \(i D^+\) and \(D^+\), the charged electroweak mass eigenstates are expected to decay into two as well as three pions. This provides a natural explanation for the \(\tau - \theta\) puzzle in the charged sector \(2\).

In the neutral sector, one gets:
\[ U^3 = \frac{1}{2\sqrt{2}} \left( (1 + i\sqrt{3})(D^0 - K^0) + (1 - i\sqrt{3})(D^0 - K^0) \right) = +U^3 = -CP \ U^3, \]
\[ V^3 = \frac{\sqrt{3}}{4\sqrt{2}\beta} \left( (1 + i\sqrt{3})(D^0 - K^0) - (1 - i\sqrt{3})(D^0 - K^0) \right) = -V^3 = +CP \ V^3. \]

(32)

\( U^3 \) and \( \pm iU^3 \) have opposite \( C \), thus opposite \( CP \), and are degenerate in mass; so are \( V^3 \) and \( \pm iV^3 \); in fact, because of the \( C \) quantum number, we do not deal, for \( N = 4 \), with sixteen pseudoscalar mesons, but with twice as many; they are pairwise degenerate when the mass eigenstates are also \( CP \) eigenstates. The same occurs in the scalar sector.

That the lightest pair, for example \( U^3, \pm iU^3 \) could be identified as the short-lived and long-lived neutral electroweak kaons is left as an open possibility.

4 \( CP \) violation with two generations (case \( \theta_c = 0 \)).

We have all the necessary ingredients to construct an \( SU(2)_L \times U(1) \) invariant Lagrangian which is not \( CP \) invariant: hence, the corresponding mass eigenstates do not have a definite \( CP \); it only requires the existence of at least one antisymmetric \( \mathbb{D} \) matrix, that is two generations. The principle is to find eigenstates which diagonalize the entire quadratic Lagrangian, but which are linear combinations of \( \Phi_i, i = 1, 2, 3 \) and \( \Phi_4 \) with real coefficients: the two types of quadruplets having different \( CP \) properties, the eigenstates will not have a definite transformation by this operation.

We shall work again, for example, in the mesonic subspace spanned by the two representations \( \Phi_3 \) and \( \Phi_4 \), but it must be clear that the phenomenon is more general and can occur with any set of quadruplets corresponding to two \( \mathbb{D} \) matrices \( \mathbb{D}_i, i = 1, 2, 3 \) and \( \mathbb{D}_4 \).

With the same notations as in the previous section, we introduce the \( SU(2)_L \times U(1) \) invariant, but now \( CP \) non-invariant (without “\( i \)” in the crossed mass term) quadratic mass Lagrangian

\[ \mathcal{L}_m = \frac{1}{2}(m_3^2 \Phi_3 \otimes \Phi_3 - m_4^2 \Phi_4 \otimes \Phi_4 + 2m_{34}^2 \Phi_3 \otimes \Phi_4). \]

(33)

The \( CP \) invariant kinetic terms are kept unaltered, though expressed in terms of \( U \) and \( V \); that they stay diagonal as before requires again that the condition (20) be satisfied.

The argument goes exactly along the same lines as in the previous section, except that eq. (28) is now replaced by

\[ \alpha^4 + \alpha^2 - \frac{\xi^2}{4(1 - \xi^2)} = 0, \quad \xi = \frac{2m_{34}^2}{m_4^2 - m_3^2}. \]

(34)

and has for solution

\[ \alpha^2 = \frac{-1 \pm \sqrt{1 - \xi^2}}{2}. \]

(35)

The existence of a positive real solution for \( \alpha^2 \) requires \( \xi^2 < 1 \), that is \( 2m_{34}^2 \leq |m_4^2 - m_3^2| \).

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\(^1\)In \cite{1}, I proposed to look for \( K_L \) and \( K_S \) respectively as the neutral \( P^3 \) of a \((\bar{S}^0, \bar{F})\) representation and the \( P^0 \) of a \((\bar{p}^0, \bar{S})\). That one is then at a loss to explain the near mass degeneracy between the two motivates the alternate proposition made above.
To fix the ideas as in the previous section, let us take the simple example $\xi^2 = 3/4, \alpha = \sqrt{1/2}, \delta = \sqrt{3/2}$. One has

$$
U^+ = -\frac{i}{\sqrt{2}} \left( (1 - \sqrt{3})K^+ + (1 + \sqrt{3})D^+ \right),
U^- = -\frac{i}{\sqrt{2}} \left( (1 + \sqrt{3})K^- + (1 - \sqrt{3})D^- \right) \pm CP \ U^+,
V^+ = -\frac{i\sqrt{3}}{2\beta} \left( (1 - \sqrt{3})K^+ - (1 + \sqrt{3})D^+ \right),
V^- = -\frac{i\sqrt{3}}{2\beta} \left( (1 + \sqrt{3})K^- - (1 - \sqrt{3})D^- \right) \pm CP \ V^+,
$$

(36)

and, in the neutral sector,

$$
U^3 = -\frac{i}{2} \left( (1 - \sqrt{3})(D^0 - K^0) + (1 + \sqrt{3})(D^0 - K^0) \right),
V^3 = -\frac{i\sqrt{3}}{2\sqrt{2}\beta} \left( (1 - \sqrt{3})(D^0 - K^0) - (1 + \sqrt{3})(D^0 - K^0) \right).
$$

(37)

$\bar{U}^3 \neq \pm U^3, \bar{V}^3 \neq \pm V^3$: the mass eigenstates $U^3$ and $V^3$ are consequently not CP eigenstates.

The masses of $\bar{U}$ and $\bar{V}$ are

$$
m_{\bar{U}}^2 = \frac{3m_4^2 + m_3^2}{4}, \quad m_{\bar{V}}^2 = \frac{3m_4^2 + m_3^2}{4}.
$$

(38)

Treating in perturbation a Lagrangian like (22) requires being able to use Green functions of the form $(T\varphi(x)\bar{\varphi}(y))$. When $U$ and $V$ were $C$ (and $CP$) eigenstates as in the previous section, all their entries were related to their charge conjugate by a simple sign, but this is no longer the case. For this reason, one must now switch to the fields $A = (U + \bar{U})/2, B = (U - \bar{U})/2, C = (V + \bar{V})/2, D = (V - \bar{V})/2$, the entries of which have definite properties by charge conjugation; this transforms (22) into

$$
\mathcal{L} = \frac{1}{2}(\beta^2 - \alpha^2)((\partial_\mu A \otimes \partial^\mu \bar{A} - \partial_\mu B \otimes \partial^\mu \bar{B} + 2\partial_\mu A \otimes \partial^\mu B)
- \frac{\beta^2}{\delta^2}(\partial_\mu C \otimes \partial^\mu \bar{C} - \partial_\mu D \otimes \partial^\mu \bar{D} + 2\partial_\mu C \otimes \partial^\mu D))
- \frac{1}{2}\left(\mu_{\bar{U}}^2(A \otimes \bar{A} - B \otimes \bar{B} + 2A \otimes B) - \mu_{\bar{V}}^2(C \otimes \bar{C} - D \otimes \bar{D} + 2C \otimes D)\right);
$$

(39)

though the “masses” stay unchanged, unavoidable non-diagonal quadratic terms, including derivative ones, appear.

The whole kinetic Lagrangian, obtained from (14) by replacing normal derivatives with covariant derivatives with respect to $SU(2)_L \times U(1)$, is $CP$ invariant. But while, by the construction given above, the part that only includes normal derivatives can be diagonalized either in $\Phi_3$ and $\Phi_4$ or in $\bar{U}$ and $\bar{V}$, there is no reason why the same should happen for the remaining terms which couple to the electroweak gauge bosons. As a consequence, loop corrections with gauge fields are expected to induce transitions between the $U$ and $V$ mass eigenstates, and the independent symmetries $U \rightarrow \pm iU, V \rightarrow \pm iV$ mentioned at the end of subsection 3.4 are broken.
5 The case of non-vanishing Cabibbo angle.

The Cabibbo rotation makes any pseudoscalar eigenstate of strong interactions a linear combination of no longer two but, for the charged states, of the four \((S^0, \bar{P})(D_i), i = 1 \cdots 4\), and for the neutral states, of even more since \((P^0, S)\) representations have to be included too.

Since the coefficients of the linear combinations that determine the charged states are all real, the presence of both \(\Phi_4\) and \(\Phi_i, i = 1, 2, 3\) makes impossible the alignment of strong and electroweak eigenstates in such a way that, inside a given representation, the charged \(P^\pm\) are related by charge conjugation.

Let us indeed consider, for example, the two quadruplets

\[
\Xi_1 = \frac{1}{2\lambda}(c_\theta(\Phi_1 + \Phi_2) - s_\theta(\Phi_3 + \Phi_4))
\]

and

\[
\Xi_2 = \frac{1}{2\lambda}(c_\theta(\Phi_1 + \Phi_2) - s_\theta(\Phi_3 - \Phi_4)).
\]

Their charged states are

\[
\begin{align*}
\Xi_1^+ &= \pi^+, \\
\Xi_1^- &= c_\theta^2 \pi^- + c_\theta s_\theta (K^- - D^-) - s_\theta^2 D^+, \\
\Xi_2^+ &= c_\theta^2 \pi^+ + c_\theta s_\theta (K^+ - D^+) - s_\theta^2 D^-; \\
\Xi_2^- &= \pi^-;
\end{align*}
\]

so, even \(\pi^\pm\) cannot be now electroweak mass eigenstates.

In the neutral sector, the “strong” \(\pi^0\) can be written

\[
\pi^0 = \frac{(\bar{u}u - \bar{d}d)}{\sqrt{2}} |_{P \text{ odd}} = \frac{1}{2}(P^0 - iP^3)(D) = \frac{1}{2}(P^0 + iP^3)(\bar{D}) = \frac{1}{2}(\begin{pmatrix} c_\theta^2 & -c_\theta s_\theta \\ -c_\theta s_\theta & s_\theta^2 \end{pmatrix})
\]

\[
= \frac{1}{2}(-i(P^3(\bar{D}_1) + P^3(\bar{D}_2)) + c_\theta s_\theta (P^0 + iP^3)(\bar{D}_3) + s_\theta^2 (P^0 + iP^3)(\bar{D}_2));
\]

because of the presence of the \(\bar{P}^0\)’s it is now connected, by the action of the \(T^\pm\) generators of the gauge group \([3]\), not only to pseudoscalar charged particles but to charged scalars. Consequently, aligning the strong and electroweak neutral pions unavoidably leads to charged electroweak mass eigenstates that are mixtures of scalars and pseudoscalars.

Since we set aside this possibility from the beginning,\(^2\) we see that, as soon as the Cabibbo rotation is operating, there is no way either for the \(\pi^0\) to be an electroweak mass eigenstate.

Note that a mass term \(m^2 \Xi_1 \otimes \Xi_1\) is not \(CP\) invariant unless the one for \(\Xi_2\) has the same coefficient; a crossed mass term \(m^2 \Xi_1 \otimes \Xi_2\) is \(CP\) invariant.

The situation in the \(K\) and \(D\) sector is now even more intricate than in the case of a vanishing Cabibbo angle. The same construction as in the previous section can lead to electroweak mass eigenstates which are not \(CP\) eigenstates, but these states now involve \(\pi\) and \(D_s\) mesons too, which cannot be disentangled from \(K\) and \(D\): in general, \(CP\) violation cannot be restricted to the sole sector of \(D\) and \(K\) mesons.

\(^2\)We rejected it for the sake of simplicity, not because it is uninteresting: it is left here as an open question, to be investigated in further works.
6 Conclusion.

It it essentially made of open questions.
The electroweak standard model for quarks is simple and extremely seducing but:
- it could be that it only describe a limited set among all the features of mesons and thus be too restrictive;
- if one adds to it a gauge theory of quarks and gluons [14], the missing features could be thought to be hidden in the process of “confinement”; but we are unable to solve it, and the remarks of Feynman at the beginning of his 1981 paper [15] are still topical.

In general, going to smaller and smaller substructures aims at a greater simplicity; this may however not be optimal, in particular when the gap deepens between the notion of “fields” and “particles”, that is between what we compute and what we observe.

We have given in previous works [12, 11, 1] and here an example of a gauge theory for J = 0 mesons which is not only compatible with the SU(2)_L × U(1) standard model for quarks, but also richer without invoking quantum chromodynamics; it cannot pretend, of course, to describe mesons of higher angular momenta, for which compositeness is certainly appealing (though a Regge-like behaviour [16] has not yet been attached to quantum chromodynamics either). But famous examples remind us that different descriptions of the same reality should not be thought to exclude each other but to concur towards a better understanding of observed phenomena.

A very pressing question clearly concerns what is detected and measured in experiments, where all data are analyzed through the “filter” of a theoretical model; good compatibility with a given model does not exclude a better filter which would still improve the agreement. In this respect, I suggested in [1] that the custodial SU(2)_V described there as directly related to the quantization of the electric charge could be found to be an exact symmetry if the data were analyzed not with the “filter” of the standard model for quarks, but with an electroweak gauge theory for mesons, in which the internal lines in the perturbative diagrams are also the propagators of asymptotic states.

The suggestion made in the present work is that the nature of observed mesonic mass eigenstates may not yet be so well understood, specially as far as electroweak interactions are concerned: we have seen that, unlike what is expected, the latter strongly alter the SU(4) classification of eigenstates; the mixture of scalars and pseudoscalars is left, too, as an open question.

Our understanding of CP violation is also modified by the point of view developed here: that it can take place for two generations demonstrates how much our description of reality depends on the model that we use to interpret the experimental data.

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