A perturbative treatment of a generalized $\mathcal{PT}$-Symmetric Quartic Anharmonic Oscillator

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ABSTRACT

We examine a generalized $\mathcal{PT}$-symmetric quartic anharmonic oscillator model to determine the various physical variables perturbatively in powers of a small quantity $\varepsilon$. We make use of the Bender-Dunne operator basis elements and exploit the properties of the totally symmetric operator $T_{m,n}$.

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1. Introduction:

During recent years $\mathcal{PT}$-symmetric quantum mechanics has emerged as an area of high theoretical interest (e.g., [1-10] and references therein). For one thing, $\mathcal{PT}$-symmetry is a weaker condition compared to the usual Hermiticity but exhibits all the essential properties of a Hermitian quantum Hamiltonian. For another, $\mathcal{PT}$-symmetry opens up the window to the non-Hermitian world, thus enabling one to address a much broader class of Hamiltonians.

Although the current interest in $\mathcal{PT}$-symmetry stems from the 1998 seminal paper of Bender and Boettcher [1] where it was shown that for a certain class of $\mathcal{PT}$-symmetric Hamiltonians the spectrum remained entirely real, discrete and bounded below, the concept of $\mathcal{PT}$-symmetry had its roots in some earlier independent works as well. These include the ones of Caliceti et al [11] and Bessis and Zinn-Justin who studied a cubic anharmonic oscillator model with an imaginary coupling and that of Buslaev and Greechi [12] who analysed the spectra of certain non-Hermitian versions of the quartic anharmonic oscillator.

Recently Mostafazadeh [13] has revisited the question of observables for the $\mathcal{PT}$-symmetric cubic anharmonic oscillator problem and, in this regard, has performed a perturbative calculation of the physical observables including investigation of the classical limit. Motivated by Mostafazadeh’s work, we examine, in this note, the $\mathcal{PT}$-symmetric version of a generalized quartic anharmonic oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\mu^2}{2} x^2 + i \epsilon x^3 - m \hbar^2 \epsilon^2 x^4$$

with $(\mu, \nu \in \mathbb{R})$ that includes a cubic anharmonicity as well.

Noting that a $\mathcal{C}$-operator can be introduced [14] in the physical Hilbert space $\mathcal{H}_{phys}$ subject to a $\mathcal{CPT}$-inner product [15], and that it commutes with both $H$ and $\mathcal{PT}$, we show that for the above Hamiltonian an equivalent Hermitian Hamiltonian $h$ can be set up. The classical Hamiltonian $H_c$ is then obtained in the limit $\hbar \to 0$. The physical position and momentum operators $X$ and $P$ , which are actually $\eta_+$-pseudo-Hermitian for the metric operator $\eta_+$ and related to the conventional position ($x$) and momentum ($p$) operators by the same similarity transformation that links $H$ and $h$, clearly turns out to be $\mathcal{PT}$-symmetric, a result similar to the $\mathcal{PT}$-symmetric cubic oscillator. We also calculate the eigenvalues of $H$ based on the first-order Rayleigh-Schrödinger perturbation theory up to and including terms of order $\epsilon^3$. Further we determine the conserved probability density for a given state vector $\psi \in \mathcal{H}_{phys}$. It should be mentioned that our calculations are somewhat different from Mostafazadeh’s in that we have exploited the symmetrized objects $T_{m,n}$ [17] satisfying commutation (lowering type) and anti-commutation (raising type) relations to write down the perturbative expansion of the $\mathcal{C}$-operator.
2. Basic equations:

To ensure the reality of the spectrum of a diagonalizable operator it is necessary that the Hamiltonian \( H \) must be Hermitian with respect to a positive definite inner product \(<.,.>_{+}\). The latter can be expressed in terms of a positive definite metric operator \( \eta_{+} : \mathcal{H} \rightarrow \mathcal{H} \) of the reference Hilbert space \( \mathcal{H} \) in which \( H \) acts:

\[
<.,.>_{+}=<.,\eta_{+}.,> \tag{2}
\]

where \( \eta_{+} \) belongs to the set of all Hermitian invertible operators \( \eta : \mathcal{H} \rightarrow \mathcal{H} \) satisfying \( H^\dagger = \eta H \eta^{-1} \) \[18\] and can be expressed as

\[
\eta_{+} = e^{-Q} \tag{3}
\]

where \( Q \) is Hermitian. In terms of \( \eta_{+} \), \( C \) admits a representation

\[
C = \mathcal{P} \eta_{+} = \eta_{+}^{-1} \mathcal{P} \tag{4}
\]

The operator \( C \) commutes with both \( H \) and \( \mathcal{P} \mathcal{T} : [C, H] = 0, [C, \mathcal{P} \mathcal{T}] = 0 \) and mimicks the charge conjugation operator in particle theory.

Any Hermitian physical observable \( O \in \mathcal{H}_{phys} \) can be converted to a Hermitian operator \( o \in \mathcal{H} \) by the transformation

\[
O = \rho^{-1} o \rho \tag{5}
\]

where \( \rho = \sqrt{\eta_{+}} \) is a unitary operator and because of (3) may be given by

\[
\rho = e^{-Q/2} \tag{6}
\]

In view of (5) we can write

\[
H = \rho^{-1} h \rho \tag{7}
\]

where \( h \) is the corresponding Hermitian Hamiltonian. The classical Hamiltonian \( H_c(x_c, p_c) \) is obtained from \( h(x_c, p_c) \) by the relation

\[
H_c(x_c, p_c) = \lim_{\hbar \to 0} h(x_c, p_c) \tag{8}
\]

where the limit is assumed to exist.

For the sake of convenience let us introduce a set of new variables

\[
X := \hbar^{-1} x, \quad P := p, \quad \mathcal{M} := m^{1/2} \hbar \mu, \quad \varepsilon := m \hbar^3 \epsilon \tag{9}
\]

In terms of \( X \) and \( P \), \( H(x, p) \rightarrow H(X, P) \) with

\[
H(X, P) = H_0(X, P) + \varepsilon H_1(X, P) + \varepsilon^2 H_2(X, P) \tag{10}
\]
\[ H_0(X, P) = \frac{1}{2} P^2 + \frac{1}{2} \mathcal{M}^2 X^2 \] (11)
\[ H_1(X, P) = iX^3 \] (12)
\[ H_2(X, P) = -X^4 \] (13)
\[ H(X, P) = mH(x, p) \] (14)

3. Determining the Q and C-operators:

Using (3) in (4), we consider the general form of C as
\[ C = e^{Q(X, P)}P \] (15)

It has the following properties:
\[ [C, PT] = 0 \] (16)
\[ C^2 = 1 \] (17)
\[ [C, H(X, P)] = 0 \] (18)

but \([C, P] \neq 0\) and \([C, T] \neq 0\)

Substitution of C from (15) into (16) implies
\[ e^{Q(X, P)}PT = PT e^{Q(X, P)}P \] (19)

showing \(Q(X, P)\) to be an even function of X: \(e^{Q(X, P)} = e^{Q(-X, P)}\) . That \(Q(X, P)\) is an odd function of P follows from the consideration (17):
\[ e^{Q(X, P)}P e^{Q(X, P)}P = 1 \] (20)

which yields \(e^{Q(X, P)} = e^{-Q(-X, -P)}\) .

We now expand of \(Q(X, P)\) in a series of odd powers of \(\varepsilon\), namely
\[ Q(X, P) = \varepsilon Q_1(X, P) + \varepsilon^3 Q_3(X, P) + \varepsilon^5 Q_5(X, P) + \varepsilon^7 Q_7(X, P) + O(\varepsilon^8) \] (21)

Using (18), it follows that
\[ e^{Q(X, P)}H_0 - H_0 e^{Q(X, P)} = \varepsilon (e^{Q(X, P)}H_1 + H_1 e^{Q(X, P)}) - \varepsilon^2 (e^{Q(X, P)}H_2 - H_2 e^{Q(X, P)}) \] (22)

Left multiplying both sides leads to,
Using Baker-Campbell-Hausdorff identity, i.e,
\[
e^{A}B e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \ldots
\] (23)
we can arrange the above expression as
\[\epsilon, \epsilon, 0 \] over all terms containing r-factor of P and s-factor of X. For example, we have
\[
-H_0, Q \left( - \frac{1}{3!}[[[H_0, Q], Q], Q], Q \right) - \frac{1}{3!}[[[[[H_0, Q], Q], Q], Q], Q]
\]
\[
-\frac{1}{5!}[[[[[[H_0, Q], Q], Q], Q], Q], Q]
\]
\[2\epsilon H_1 + \epsilon[H_1, Q] + \frac{\epsilon}{3!}[[H_1, Q], Q] + \frac{\epsilon}{5!}[[[H_1, Q], Q], Q] + \frac{\epsilon}{7!}[[[[H_1, Q], Q], Q], Q]
\]
\[
+ \frac{\epsilon^2}{3!}[[H_2, Q], Q] + \frac{\epsilon^2}{5!}[[[H_2, Q], Q], Q] + \frac{\epsilon^2}{7!}[[[[H_2, Q], Q], Q], Q]
\]
\[
+ \frac{\epsilon^3}{5!}[[[H_2, Q], Q], Q] + \frac{\epsilon^3}{7!}[[[[[H_2, Q], Q], Q], Q], Q]
\]
where we have taken the terms up to order \(\epsilon^7\)

Substituting (21) into (24) and equating terms of order \(\epsilon, \epsilon^3, \epsilon^5, \epsilon^7\) we get,
\[
[H_0, Q_1] = -2H_1
\] (25)
\[
[H_0, Q_3] = -\frac{1}{6}[Q_1, [Q_1, H_1]] + [Q_1, H_2]
\] (26)
\[
[H_0, Q_5] = -\frac{1}{6}([Q_3, [Q_1, H_1]] + [Q_1, [Q_3, H_1]]) + \frac{1}{360}[Q_1, [Q_1, [Q_1, H_1]]] + [Q_3, H_2]
\] (27)
\[
[H_0, Q_7] = -\frac{1}{6}([Q_5, [Q_1, H_1]] + [Q_3, [Q_3, H_1]] + [Q_1, [Q_5, H_1]])
\]
\[
+ \frac{1}{360}([Q_3, [Q_1, [Q_1, H_1]]] + [Q_1, [Q_3, [Q_1, H_1]]])
\]
\[
+ [Q_1, [Q_3, [Q_1, H_1]]] + [Q_1, [Q_1, [Q_3, H_1]]])
\]
\[
- \frac{1}{15120}([Q_1, [Q_1, [Q_1, [Q_1, H_1]]]] + [Q_5, H_2]
\] (28)

note that terms of order \(\epsilon^2, \epsilon^4, \epsilon^6\), i.e, even powers of \(\epsilon\) give no new results.

To solve for (25),(26),(27)and (28) we introduce, following Bender and Dunne [16] the totally symmetrized sum \(T_{r,s}\) over all terms containing r-factor of P and s-factor of X. For example, we have
\[T_{0,0} = 1\]
\[T_{1,0} = P\]
\[T_{1,2} = \frac{1}{3}(PX^2 + XPX + X^2P)\]
\[T_{0,3} = X^3\]
\[T_{3,1} = \frac{1}{4}(XP^3 + PX^2P + P^2XP + P^3X)\]
We thus get

\[ Q_1 = -\frac{4}{3} M^{-4} T_{3,0} - 2 M^{-2} T_{1,2} \quad (28a) \]

\[ Q_3 = \left( \frac{128}{15} M^{-10} - \frac{32}{5} M^{-8} \right) T_{5,0} + \left( \frac{40}{3} M^{-8} - 16 M^{-6} \right) T_{3,2} + \left( 8 M^{-6} - 8 M^{-4} \right) T_{1,4} - \left( 12 M^{-8} - 8 M^{-6} \right) T_{1,0} \quad (28b) \]

\[ Q_5 = \left( \frac{6398}{15} M^{-12} - 128 M^{-10} + 128 M^{-8} \right) T_{1,2} + \left( -64 M^{-10} + 32 M^{-8} - 32 M^{-6} \right) T_{1,6} \]

\[ + \left( \frac{41736}{45} M^{-14} - 256 M^{-12} + \frac{640}{3} M^{-10} \right) T_{3,0} + \left( -\frac{512}{3} M^{-12} + \frac{352}{3} M^{-10} - 128 M^{-8} \right) T_{3,4} \]

\[ + \left( -\frac{544}{3} M^{-14} + 128 M^{-12} - 128 M^{-10} \right) T_{5,2} + \left( -\frac{320}{3} M^{-16} + \frac{256}{7} M^{-14} - \frac{256}{7} M^{-12} \right) T_{7,0} \quad (28c) \]

\[ Q_7 = \left( \frac{55398}{315} M^{-22} - \frac{124416}{315} M^{-20} + \frac{69612}{315} M^{-18} - \frac{2048}{9} M^{-16} \right) T_{9,0} \]

\[ + \left( \frac{97702}{35} M^{-20} - \frac{62208}{35} M^{-18} + \frac{34816}{35} M^{-16} - 1024 M^{-14} \right) T_{7,2} \]

\[ + \left( \frac{35734}{60} M^{-18} - \frac{35456}{15} M^{-16} + \frac{7424}{9} M^{-14} - 1536 M^{-12} \right) T_{5,4} \]

\[ + \left( \frac{721024}{315} M^{-16} - \frac{4096}{3} M^{-14} + \frac{2432}{3} M^{-12} - \frac{2560}{3} M^{-10} \right) T_{3,6} \]

\[ + \left( \frac{1792}{3} M^{-14} - 256 M^{-12} + 128 M^{-10} - 128 M^{-8} \right) T_{1,8} \]

\[ + \left( -\frac{2209024}{105} M^{-20} + \frac{619648}{75} M^{-18} - \frac{54272}{15} M^{-16} + 3584 M^{-14} \right) T_{5,0} \]

\[ + \left( -\frac{2875648}{105} M^{-18} + \frac{141824}{15} M^{-16} - \frac{15616}{3} M^{-14} + 5120 M^{-12} \right) T_{3,2} \]

\[ + \left( -\frac{390336}{35} M^{-16} + \frac{40832}{15} M^{-14} - 1216 M^{-12} + 1280 M^{-10} \right) T_{1,4} \]

\[ + \left( \frac{46974}{7} M^{-18} - \frac{40472}{15} M^{-16} + 1536 M^{-14} - 1280 M^{-12} \right) T_{1,0} \quad (28d) \]

Explicit forms of \( Q(X, P) \) and \( C(X, P) \) are then obtained by substituting the above expressions for \( Q_1, Q_3, Q_5, Q_7 \) into (21) and (15).

4. Determining the X and P operators:

Now we calculate the physical position and momentum operator \( X \) and \( P \) from the previously introduced variables \( X, P \) using the similarity transformation (5):

\[ X = \rho^{-1} X \rho = e^{\frac{Q}{2}} X e^{-\frac{Q}{2}} \quad (29) \]

\[ P = \rho^{-1} P \rho = e^{\frac{Q}{2}} P e^{-\frac{Q}{2}} \quad (30) \]
Using (23), we obtain for $X$ and $P$

\[
X = X + \varepsilon(2iM^{-4}P^2 + iM^{-2}X^2) + \varepsilon^2(2M^{-6}XP^2 - 2iM^{-6}P - M^{-4}X^3) \\
+ \varepsilon^3[(-\frac{172}{15}M^{-10} + 16M^{-8})iP^4 - (5M^{-6} - 4M^{-4})iX^4 \nonumber \\
- (\frac{128}{3}M^{-8} - 48M^{-6})XP + (\frac{64}{3}M^{-8} - 24M^{-6})iX^2P^2 \nonumber \\
+ (\frac{50}{3}M^{-8} - 16M^{-6})i] + O(\varepsilon^4) \tag{31}
\]

\[
P = P - \varepsilon(2iM^{-2}(XP - \frac{i}{2})) + \varepsilon^2(2M^{-6}P^3 - M^{-4}(X^2P - iX)) \nonumber \\
- i\varepsilon^3[(16M^{-8} - 16M^{-6})(XP^3 - \frac{3}{2}iP^2) \nonumber \\
+ (16M^{-6} - 16M^{-4})(X^3P - \frac{3}{2}iX^2)] + O(\varepsilon^4) \tag{32}
\]

where we retained terms of order of $\varepsilon^3$.

From (31) and (32), we easily see that

\[
\mathcal{P}X = -X \\
\mathcal{P}P = -P \\
\mathcal{T}X = X \\
\mathcal{T}P = -P
\]

but

\[
\mathcal{P}\mathcal{T}X = -X \\
\mathcal{P}\mathcal{T}P = P \tag{33}
\]

From (33) we conclude that the physical position and momentum operator, i.e., $X$ and $P \in \mathcal{H}_{\text{phys}}$ are $\mathcal{P}\mathcal{T}$-symmetric.

\section{5. The equivalent Hermitian Hamiltonian:}

For the operator $\rho$, the corresponding Hermitian Hamiltonian $\mathbf{h}(X, P)$ is given according to (7) with previously introduced variables $X, P$ as

\[
\mathbf{h}(X, P) = e^{-Q/2}H(X, P)e^{Q/2} \tag{34}
\]

Introducing the perturbative expansion for $\mathbf{h}(X, P)$ as

\[
\mathbf{h} = \sum_{i=0}^{\infty} h^{(i)}\varepsilon^i \tag{35}
\]
and using (10),(21),(25)-(28) and (34) along with the Baker-Campbell-Hausdorff identity (23), we obtain for the various coefficients \( h^{(i)} \), \( i=0,1,2,3,\ldots \), the results

\[
\begin{align*}
    h^{(0)} &= H_0 \\
    h^{(2)} &= H_2 + \frac{1}{3}[H_1, Q_1] \\
    h^{(4)} &= \frac{1}{4}[H_1, Q_3] - \frac{1}{192}[[[H_1, Q_1], Q_1], Q_1] \\
    h^{(6)} &= \frac{1}{4}[H_1, Q_5] - \frac{1}{192}([[H_1, Q_1], Q_1], Q_3) + \frac{1}{7680}([[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1) 
\end{align*}
\]

(36)

with the odd ones vanishing: \( h^{(i)} = 0, i=1,3,5,7,\ldots \).

Keeping terms up to the order \( \varepsilon^5 \); \( h(X, P) \) can thus be expressed as

\[
\begin{align*}
    h(X, P) &= \frac{1}{2}(T_{2,0} + \mathcal{M}^2 T_{0,2}) + \varepsilon^2[-\frac{1}{2}\mathcal{M}^{-4} + (\frac{3}{2}\mathcal{M}^{-2} - 1) T_{0,4} + 3\mathcal{M}^{-4} T_{2,2}] \\
    &\quad+ \varepsilon^4[-(36\mathcal{M}^{-10} - 24\mathcal{M}^{-8}) T_{4,2} + (27\mathcal{M}^{-10} - 24\mathcal{M}^{-8}) T_{2,0} - (\frac{54}{2} \mathcal{M}^{-8} - 36\mathcal{M}^{-6}) T_{2,4}] \\
    &\quad+ \varepsilon^6[-(12\mathcal{M}^{-6} - 6\mathcal{M}^{-4}) T_{0,2} - (\frac{7}{2} \mathcal{M}^{-6} - 6\mathcal{M}^{-4}) T_{0,6} + 2\mathcal{M}^{-12} T_{6,0}] + O(\varepsilon^6) 
\end{align*}
\]

(37)

If now we consider the normalized eigen vector \( |n > \) of the conventional Harmonic oscillator \( H_0 \), then we can easily calculate \( E_n \) for \( H \) by the first order Rayleigh-Schrödinger perturbation theory. We obtain up to the terms of order \( \varepsilon^3 \)

\[
\begin{align*}
    E_n &= \mathcal{M}(n + \frac{1}{2}) + \varepsilon^2 < n| h^{(2)} | n > + O(\varepsilon^4) \\
    &= \mathcal{M}(n + \frac{1}{2}) + \varepsilon^2 \left[ \frac{1}{4} \frac{1}{2\mathcal{M}^4} (30n^2 + 30n + 11) - (6n^2 + 6n + 3) \right] + O(\varepsilon^4) 
\end{align*}
\]

(38)

6. Classical Hamiltonian:

Employing (9), we have,

\[
T_{r,s} = \hbar^{-s} S_{r,s} 
\]

(39)

where \( S_{r,s} \) be the totally symmetrized sum of all terms containing r-factor of p and s-factor of x.

Specifically

\[
\begin{align*}
    S_{0,0} &= 1 \\
    S_{0,1} &= x \\
    S_{3,0} &= p^3 \\
    S_{1,3} &= \frac{1}{4} (x^3 p + xp x^2 + x^2 px + px^3)
\end{align*}
\]

and so on.

Now from (14),
Keeping terms up to the terms of order of $\varepsilon^3$ in (37) and using the relations (9),(39),(40) we finally obtain

\[ h(x,p) = \frac{p^2}{2m} + \frac{1}{2}\mu^2 x^2 + \varepsilon^2 m \left[ \frac{3}{2} m^{-1} \mu^{-2} - \hbar^2 \right] x^4 - 2 m^{-2} \hbar^2 \mu^{-4} - 6 i m^{-2} h \mu^{-4} x p - 3 m^{-2} \mu^{-4} x^2 p^2 + O(\varepsilon^4) \] (41)

The corresponding classical Hamiltonian can be read off from (8):

\[ H_c(x_c,p_c) = \lim_{\hbar \to 0} h(x_c, p_c) \]

\[ = \frac{p_c^2}{2M(x_c)} + \frac{1}{2}\mu^2 x_c^2 + \frac{3\varepsilon^2}{2\mu^2} x_c^4 + O(\varepsilon^4) \] (42)

where

\[ M(x_c) = \frac{m}{1 - 6\mu^{-4}\varepsilon^2 x_c^2} \] (43)

A position-dependent mass $M(x_c)$ is implied by (43) for the classical particle whose dynamics is dictated by the Hamiltonian $H_c(x_c, p_c)$.

### 7. Conserved probability density:

For a given state vector $\psi \in \mathcal{H}_{phys}$, the perturbation expansion for the corresponding physical wave function is

\[ \Psi(x) = <x | e^{-\frac{Q^2}{2}} | \psi > \]

\[ = <x | \sum_{k=0}^{\infty} \frac{(-1)^k Q^k}{2^k k!} | \psi > \]

\[ = \psi(x) + <x | -\frac{Q^2}{2} | \psi > \varepsilon + <x | \frac{Q^2}{8} | \psi > \varepsilon^2 + <x | (-\frac{Q^2}{2} - \frac{Q^3}{48}) | \psi > \varepsilon^3 + O(\varepsilon^4) \] (44)

Using (9),(28a) and (28b) we obtain from (44)

\[ \Psi(x) = (1 + \varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 L_3)\psi(x) + O(\varepsilon^4) \] (45)

where

\[ L_1 = -\frac{m}{2} h^3 \hat{Q}_1 \] (46)

\[ L_2 = \frac{m^2}{8} h^6 \hat{Q}_1^2 \] (47)

\[ L_3 = -\frac{m^3}{2} h^9 \hat{Q}_3 - \frac{m^3}{48} h^9 \hat{Q}_1^3 \] (48)

and,

\[ \hat{Q}_1 = -\frac{4}{3} m^{-2} \hbar^{-4} \mu^{-4} S_{3,0} - 2 m^{-1} \hbar^{-2} \mu^{-2} S_{1,2} \] (49)

\[ \hat{Q}_3 = (\frac{128}{15} m^{-5} \hbar^{-10} \mu^{-10} - \frac{32}{3} m^{-4} \hbar^{-8} \mu^{-8}) S_{5,0} + (\frac{40}{3} m^{-4} \hbar^{-10} \mu^{-8} - 16 m^{-3} \hbar^{-8} \mu^{-6}) S_{3,2} \]
Employing (46)-(50) into (45), we find the conserved probability density \( \rho \) associated with a given state vector \( \psi \in \mathcal{H}_{\text{phys}} \) as

\[
\rho(x) = N^{-1} \left| \Psi(x) \right|^2
\]

where

\[
N = \int_{-\infty}^{\infty} \left| \Psi(x) \right|^2 dx
\]

(51)

(52)

8. Conclusion:

We have carried out a perturbative treatment to study a \( \mathcal{PT} \)-symmetric quartic anharmonic oscillator model. We have shown possible to set up an equivalent Hermitian Hamiltonian by employing a similarity transformation. Such a Hamiltonian has a classical limit too. Physical position and momentum operators have been determined perturbatively and energy eigenvalues are obtained in the framework of first order Rayleigh-Schrödinger perturbation theory. In all these calculations we have kept terms up to and including those of order \( \varepsilon^3 \). The conserved probability density is also determined. Finally, let us mention that in the absence of the quartic term in (1) our results essentially reduce to those of Mostafazadeh's [13] for the physical observables \( X \) and \( P \), equivalent Hermitian Hamiltonian \( h(X,P) \) and the energy spectrum derived from the first order Rayleigh-Schrödinger perturbation theory.

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