Critical Independent Sets of a Graph

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Abstract

Let $G$ be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent, and by $\text{Ind}(G)$ we mean the family of all independent sets of $G$.

The number $d(X) = |X| - |N(X)|$ is the difference of $X \subseteq V(G)$, and a set $A \in \text{Ind}(G)$ is critical if $d(A) = \max\{d(I) : I \in \text{Ind}(G)\}$.

Let us recall the following definitions:

- $\text{core}(G) = \bigcap\{S : S \text{ is a maximum independent set}\}$.
- $\text{corona}(G) = \bigcup\{S : S \text{ is a maximum independent set}\}$.
- $\text{ker}(G) = \bigcap\{S : S \text{ is a critical independent set}\}$.
- $\text{diadem}(G) = \bigcup\{S : S \text{ is a critical independent set}\}$.

In this paper we present various structural properties of $\text{ker}(G)$, in relation with $\text{core}(G)$, $\text{corona}(G)$, and $\text{diadem}(G)$.

Keywords: independent set, critical set, ker, core, corona, diadem, matching

1 Introduction

Throughout this paper $G$ is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of $G$ induced by $X$. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set $W$, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$. If $A, B \subseteq V(G)$, then $(A, B)$ stands for the set $\{ab : a \in A, b \in B, ab \in E(G)\}$.

The neighborhood $N(v)$ of a vertex $v \in V(G)$ is the set $\{w : w \in V(G) \text{ and } vw \in E(G)\}$, while the closed neighborhood $N[v]$ of $v \in V(G)$ is the set $N(v) \cup \{v\}$; in order to avoid ambiguity, we use also $N_G(v)$ instead of $N(v)$. 

1
The neighborhood \( N(A) \) of \( A \subseteq V(G) \) is \( \{ v \in V(G) : N(v) \cap A \neq \emptyset \} \), and \( N[A] = N(A) \cup A \). We may also use \( N_G(A) \) and \( N_G[A] \), when referring to neighborhoods in a graph \( G \).

A set \( S \subseteq V(G) \) is independent if no two vertices from \( S \) are adjacent, and by \( \text{Ind}(G) \) we mean the family of all the independent sets of \( G \). An independent set of maximum size is a maximum independent set of \( G \), and the independence number \( \alpha(G) \) of \( G \) is \( \max\{|S| : S \in \text{Ind}(G)\} \). Let \( \Omega(G) \) denote the family of all maximum independent sets, and let

\[
\text{core}(G) = \bigcap\{S : S \in \Omega(G)\} \quad \text{[10]}, \quad \text{and} \quad \text{corona}(G) = \bigcup\{S : S \in \Omega(G)\} \quad \text{[2]},
\]

Clearly, \( N(\text{core}(G)) \subseteq V(G) - \text{corona}(G) \), and there are graphs with \( N(\text{core}(G)) \neq V(G) - \text{corona}(G) \) (for an example, see Figure 1). The problem of whether \( \text{core}(G) \neq \emptyset \) is \( \text{NP}-\text{hard} \) [2].

![Figure 1: core(G) = \{a, b\} and V(G) - corona(G) = N(core(G)) \cup \{d\} = \{c, d\}.](image1)

A matching is a set \( M \) of pairwise non-incident edges of \( G \). If \( A \subseteq V(G) \), then \( M(A) \) is the set of all the vertices matched by \( M \) with vertices belonging to \( A \). A matching of maximum cardinality, denoted \( \mu(G) \), is a maximum matching.

For \( X \subseteq V(G) \), the number \( |X| - |N(X)| \) is the difference of \( X \), denoted \( d(X) \). The critical difference \( d(G) \) is \( \max\{d(X) : X \subseteq V(G)\} \). The number \( \max\{d(I) : I \in \text{Ind}(G)\} \) is the critical independence difference of \( G \), denoted \( \text{id}(G) \). Clearly, \( d(G) \geq \text{id}(G) \). It was shown in [26] that \( d(G) = \text{id}(G) \) holds for every graph \( G \). If \( A \) is an independent set in \( G \) with \( d(X) = \text{id}(G) \), then \( A \) is a critical independent set [26]. All pendant vertices not belonging to \( K_2 \) components are included in every inclusion maximal critical independent set.

For example, let \( X = \{v_1, v_2, v_3, v_4\} \) and \( I = \{v_1, v_2, v_3, v_6, v_7\} \) in the graph \( G \) of Figure 2. Note that \( X \) is a critical set, since \( N(X) = \{v_3, v_4, v_5\} \) and \( d(X) = 1 = d(G) \), while \( I \) is a critical independent set, because \( d(I) = 1 = \text{id}(G) \). Other critical sets are \( \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4, v_6, v_7\} \).

![Figure 2: core(G) = \{v_1, v_2, v_6, v_{10}\} is a critical set.](image2)
It is known that finding a maximum independent set is an NP-hard problem \[7\]. Zhang proved that a critical independent set can be find in polynomial time \[26\]. A sim-pler algorithm, reducing the critical independent set problem to computing a maximum independent set in a bipartite graph is given in \[1\].

**Theorem 1.1** \[3\] Each critical independent set can be enlarged to a maximum independent set.

Theorem 1.1 led to an efficient way of approximating \(\alpha(G)\) \[25\]. Moreover, it has been shown that a critical independent set of maximum cardinality can be computed in polynomial time \[8\]. Recently, a parallel algorithm computing the critical independence number was developed \[5\].

Recall that if \(\alpha(G) + \mu(G) = |V(G)|\), then \(G\) is a König-Egerváry graph \[6, 24\]. As a well-known example, each bipartite graph is a König-Egerváry graph as well.

**Theorem 1.2** \[11\] If \(G\) is a König-Egerváry graph, \(M\) is a maximum matching of \(G\), and \(S \in \Omega(G)\), then:

1. \(M\) matches \(V(G) - S\) into \(S\), and \(N(\text{core}(G))\) into \(\text{core}(G)\);
2. \(N(\text{core}(G)) = \cap \{V(G) - S : S \in \Omega(G)\}\), i.e., \(N(\text{core}(G)) = V(G) - \text{corona}(G)\).

The deficiency \(def(G)\) is the number of non-saturated vertices relative to a maximum matching, i.e., \(def(G) = |V(G)| - 2\mu(G)\) \[19\]. A proof of a conjecture of Graffiti.pc \[4\] yields a new characterization of König-Egerváry graphs: these are exactly the graphs, where there exists a critical maximum independent set \[9\]. In \[13\] it is proved the following.

**Theorem 1.3** \[13\] For a König-Egerváry graph \(G\) the following equalities hold
\[
d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = def(G).
\]

Using this finding, we have strengthened the characterization from \[9\].

**Theorem 1.4** \[13\] \(G\) is a König-Egerváry graph if and only if each of its maximum independent sets is critical.

For a graph \(G\), let denote
\[
\ker(G) = \bigcap \{S : S \text{ is a critical independent set}\} \[12\], and
\[
\text{diadem}(G) = \bigcup \{S : S \text{ is a critical independent set}\}.
\]

In this paper we present several properties of \(\ker(G)\), in relation with \(\text{core}(G)\), \(\text{corona}(G)\), and \(\text{diadem}(G)\).
2 Preliminaries

Let $G$ be the graph from Figure 2; the sets $X = \{v_1, v_2, v_3\}$, $Y = \{v_1, v_2, v_4\}$ are critical independent, and the sets $X \cap Y$, $X \cup Y$ are also critical, but only $X \cap Y$ is also independent. In addition, one can easily see that ker($G$) is a minimal critical independent set of $G$. These properties of critical sets and ker($G$) are true even in general.

**Theorem 2.1** [12] For a graph $G$, the following assertions are true:

(i) the function $d$ is supermodular, i.e., $d(A \cup B) + d(A \cap B) \geq d(A) + d(B)$ for every $A, B \subseteq V(G)$;

(ii) if $A$ and $B$ are critical in $G$, then $A \cup B$ and $A \cap B$ are critical as well;

(iii) $G$ has a unique minimal independent critical set, namely, ker($G$).

As a consequence, we have the following.

**Corollary 2.2** For every graph $G$, diadem($G$) is a critical set.

For instance, the graph $G$ from Figure 2 has diadem($G$) = \{v_1, v_2, v_3, v_4, v_6, v_7, v_10\}, which is critical, but not independent.

![Figure 3: Both $G_1$ and $G_2$ are not König-Egerváry graphs.](image)

The graph $G$ from Figure 1 has $d(G) = 1$ and $d(\text{corona}(G)) = 0$, which means that corona($G$) is not a critical set. Notice that $G$ is not a König-Egerváry graph. Combining Theorems 1.4 and 2.1(ii), we deduce the following.

**Corollary 2.3** If $G$ is a König-Egerváry graph, then both core($G$) and corona($G$) are critical sets.

Let consider the graphs $G_1$ and $G_2$ from Figure 3: core($G_1$) = \{a, b, c, d\} and it is a critical set, while core($G_2$) = \{x, y, z, w\} and it is not critical.

**Theorem 2.4** If core($G$) is a critical set, then

$$\text{core}(G) \subseteq \bigcap \{A : A \text{ is an inclusion maximal critical independent set}\}.$$  

**Proof.** Let $A$ be an arbitrary inclusion maximal critical independent set. According to Theorem 1.4 there is some $S \in \Omega(G)$, such that $A \subseteq S$. Since core($G$) \subseteq S, it follows that $A \cup \text{core}(G) \subseteq S$, and hence $A \cup \text{core}(G)$ is independent. By Theorem 2.1 we get that $A \cup \text{core}(G)$ is a critical independent set. Since $A \subseteq A \cup \text{core}(G)$ and $A$ is an inclusion maximal critical independent set, it follows that core($G$) \subseteq A, for every such set $A$, and this completes the proof.

**Remark 2.5** By Theorem 1.4 the following inclusion holds for every graph $G$.

$$\text{corona}(G) \supseteq \bigcup \{A : A \text{ is an inclusion maximal critical independent set}\}.$$
3 Structural properties of $\ker(G)$

Deleting a vertex from a graph may change its critical difference. For instance, $d(G - v_1) = d(G) - 1$, $d(G - v_13) = d(G)$, while $d(G - v_3) = d(G) + 1$, where $G$ is the graph of Figure 2.

**Proposition 3.1** [10] For a vertex $v$ in a graph $G$, the following assertions hold:

(i) $d(G - v) = d(G) - 1$ if and only if $v \in \ker(G)$;
(ii) if $v \in \ker(G)$, then $\ker(G - v) \subseteq \ker(G) - \{v\}$.

Note that $\ker(G - v)$ may differ from $\ker(G) - \{v\}$. For example, $\ker(K_{3,2})$ is equal to the partite set of size 3, but $\ker(K_{3,2} - v) = \emptyset$ whenever $v$ is in that set. Also, if $G = C_4$, then $\ker(G) - \{v\} = \emptyset - \{v\} = \emptyset$, while $\ker(G - v) = N_G(v)$ for every $v \in V(G)$.

**Theorem 3.2** [8] There is a matching from $N(S)$ into $S$ for every critical independent set $S$.

In the graph $G$ of Figure 2, let $S = \{v_1, v_2, v_3\}$. By Theorem 3.2 there is a matching from $N(S)$ into $S = \{v_1, v_2, v_3\}$, for instance, $M = \{v_2v_5, v_3v_4\}$, since $S$ is critical independent. On the other hand, there is no matching from $N(S)$ into $S - v_3$.

**Theorem 3.3** [10] For a critical independent set $A$ in a graph $G$, the following statements are equivalent:

(i) $A = \ker(G)$;
(ii) there is no set $B \subseteq N(A), B \neq \emptyset$ such that $|N(B) \cap A| = |B|$;
(iii) for each $v \in A$ there exists a matching from $N(A)$ into $A - v$.

The graphs $G_1$ and $G_2$ in Figure 4 satisfy $\ker(G_1) = \core(G_1)$, $\ker(G_2) = \{x, y, z\} \subset \core(G_2)$, and both $\core(G_1)$ and $\core(G_2)$ are critical sets of maximum size. The graph $G_3$ in Figure 4 has $\ker(G_3) = \{u, v\}$, the set $\{t, u, v\}$ as a critical independent set of maximum size, while $\core(G_3) = \{t, u, v, w\}$ is not a critical set.

![Graphs G1, G2, and G3](image)

Figure 4: $\core(G_1) = \{a, b\}$, $\core(G_2) = \{q, x, y, z\}$, $\core(G_3) = \{t, u, v, w\}$.

An independent set $S$ is *inclusion minimal* with $d(S) > 0$ if no proper subset of $S$ has positive difference. For example, in Figure 4 one can see that $\ker(G_1)$ is an inclusion minimal independent set with positive difference, while for the graph $G_2$ the sets $\{x, y\}, \{x, z\}, \{y, z\}$ are inclusion minimal independent with positive difference, and $\ker(G_2) = \{x, y\} \cup \{x, z\} \cup \{y, z\}$. 

5
Theorem 3.4 [16] If \( \ker(G) \neq \emptyset \), then

\[
\ker(G) = \bigcup \{ S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) = 1 \}
= \bigcup \{ S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0 \}.
\]

In a graph \( G \), the union of all minimum cardinality independent sets \( S \) with \( d(S) > 0 \) may be a proper subset of \( \ker(G) \). For example, consider the graph \( G \) in Figure 5, where \( \{x, y\} \subset \ker(G) = \{x, y, v, w\} \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Both \( S_1 = \{x, y\} \) and \( S_2 = \{u, v, w\} \) are inclusion minimal independent sets satisfying \( d(S) > 0 \).}
\end{figure}

Actually, all inclusion minimal independent sets \( S \) with \( d(S) > 0 \) are of the same difference.

Proposition 3.5 [16] If \( S_0 \) is an inclusion minimal independent set with \( d(S_0) > 0 \), then \( d(S_0) = 1 \). In other words,

\[
\{ S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0 \} = \\
\{ S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) = 1 \}.
\]

The converse of Proposition 3.5 is not true. For instance, \( S = \{x, y, u\} \) is independent in the graph \( G \) of Figure 5 and \( d(S) = 1 \), but \( S \) is not minimal with this property.

Proposition 3.6 [16] \( \min \{|S_0| : d(S_0) > 0, S_0 \in \text{Ind}(G)\} \leq |\ker(G)| - d(G) + 1 \) is true for every graph \( G \).

4 Relationships between \( \ker(G) \) and \( \text{core}(G) \)

Let us consider again the graph \( G_2 \) from Figure 8: \( \text{core}(G_2) = \{x, y, z, w\} \) and it is not critical, but \( \ker(G_2) = \{x, y, z\} \subseteq \text{core}(G_2) \). Clearly, the same inclusion holds for \( G_1 \), whose \( \text{core}(G_1) \) is a critical set.

Theorem 4.1 [12] For every graph \( G \), \( \ker(G) \subseteq \text{core}(G) \).

Let \( I_c \) be a maximum critical independent set of \( G \), and \( X = I_c \cup N(I_c) \). In [23] it is proved that \( \text{core}(G[X]) \subseteq \text{core}(G) \). Moreover, in [12], we showed that the chain of relationships \( \ker(G) = \ker(G[X]) \subseteq \text{core}(G[X]) \subseteq \text{core}(G) \) holds for every graph \( G \). Theorem 4.1 allows an alternative proof of the following inequality due to Lorentzen.

Corollary 4.2 [18, 22, 12] The inequality \( d(G) \geq \alpha(G) - \mu(G) \) holds for every graph.
Following Ore \cite{20,21}, the number $\delta(X) = d(X) = |X| - |N(X)|$ is the deficiency of $X$, where $X \subseteq A$ or $X \subseteq B$ and $G = (A, B, E)$ is a bipartite graph. Let

$$\delta_0(A) = \max \{\delta(X) : X \subseteq A\}, \quad \delta_0(B) = \max \{\delta(Y) : Y \subseteq B\}.$$ 

A subset $X \subseteq A$ having $\delta(X) = \delta_0(A)$ is $A$-critical, while $Y \subseteq B$ having $\delta(B) = \delta_0(B)$ is $B$-critical. For a bipartite graph $G = (A, B, E)$ let us denote $\ker_A(G) = \cap \{S : S \text{ is } A\text{-critical}\}$ and $\text{diadem}_A(G) = \cup \{S : S \text{ is } A\text{-critical}\}$. Similarly, $\ker_B(G) = \cap \{S : S \text{ is } B\text{-critical}\}$ and $\text{diadem}_B(G) = \cup \{S : S \text{ is } B\text{-critical}\}$.

It is convenient to define $d(\emptyset) = \delta(\emptyset) = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6}
\caption{G is a bipartite graph without perfect matchings.}
\end{figure}

For instance, the graph $G = (A, B, E)$ from Figure 6 has: $X = \{a_1, a_2, a_3, a_4\}$ as an $A$-critical set, $\ker_A(G) = \{a_1, a_2\}$, $\text{diadem}_A(G) = \{a_1 : i = 1, \ldots, 5\}$ and $\delta_0(A) = 1$, while $Y = \{b_i : i = 4, 5, 6, 7\}$ is a $B$-critical set, $\ker_B(G) = \{b_4, b_5, b_6\}$, $\text{diadem}_B(G) = \{b_i : i = 2, \ldots, 7\}$ and $\delta_0(B) = 2$.

As expected, there is a close relationship between critical independent sets and $A$-critical or $B$-critical sets.

**Theorem 4.3** \cite{13} Let $G = (A, B, E)$ be a bipartite graph. Then the following assertions are true:

(i) $d(G) = \delta_0(A) + \delta_0(B)$;

(ii) $\alpha(G) = |A| + \delta_0(A) = |B| + \delta_0(B) = \mu(G) + \delta_0(A) + \delta_0(B) = \mu(G) + d(G)$;

(iii) if $X$ is an $A$-critical set and $Y$ is a $B$-critical set, then $X \cup Y$ is a critical set;

(iv) if $Z$ is a critical independent set, then $Z \cap A$ is an $A$-critical set and $Z \cap B$ is a $B$-critical set;

(v) if $X$ is either an $A$-critical set or a $B$-critical set, then there is a matching from $N(X)$ into $X$.

The following lemma will be used further to give an alternative proof for the assertion that $\ker(G) = \text{core}(G)$ holds for every bipartite graph $G$.

**Lemma 4.4** If $G = (A, B, E)$ is a bipartite graph with a perfect matching, say $M$, $S \in \Omega(G)$, $X \in \text{Ind}(G)$, $X \subseteq V(G) - S$, and $G[X \cup M(X)]$ is connected, then

$$X^1 = X \cup M((N(X) \cap S) - M(X))$$

is an independent set, and $G[X^1 \cup M(X^1)]$ is connected.

**Proof.** Let us show that the set $M((N(X) \cap S) - M(X))$ is independent. Suppose, to the contrary, that there exist $v_1, v_2 \in M((N(X) \cap S) - M(X))$ such that $v_1v_2 \in E(G)$. Hence $M(v_1), M(v_2) \in (N(X) \cap S) - M(X)$. Hence $M(v_1), M(v_2) \in (N(X) \cap S) - M(X)$. Therefore, $G[X^1 \cup M(X^1)]$ is connected.
If \( M(v_1) \) and \( M(v_2) \) have a common neighbor \( w \in X \), then \( \{v_1, v_2, M(v_2), w, M(v_1)\} \) spans \( C_5 \), which is forbidden for bipartite graphs.

Otherwise, let \( w_1, w_2 \in X \) be neighbors of \( M(v_1) \) and \( M(v_2) \), respectively. Since \( G[X \cup M(X)] \) is connected, there is a path with even number of edges connecting \( w_1 \) and \( w_2 \). Together with \( \{w_1, M(v_1), v_1, v_2, M(v_2), w_2\} \) this path produces a cycle of odd length in contradiction with the hypothesis on \( G \) being a bipartite graph.

To complete the proof of independence of the set

\[
X^1 = X \cup M((N(X) \cap S) - M(X))
\]

it is enough to demonstrate that there are no edges connecting vertices of \( X \) and \( M((N(X) \cap S) - M(X)) \).

Assume, to the contrary, that there is \( vw \in E \), such that \( v \in M((N(X) \cap S) - M(X)) \) and \( w \in X \). Since \( M(v) \in (N(X) \cap S) - M(X) \) and \( G[X \cup M(X)] \) is connected, it follows that there exists a path with an odd number of edges connecting \( M(v) \) to \( w \). This path together with the edges \( vw \) and \( vM(v) \) produces cycle of odd length, in contradiction with the bipartiteness of \( G \).

Finally, since \( G[X \cup M(X)] \) is connected, \( G[X^1 \cup M(X^1)] \) is connected as well, by definitions of set functions \( N \) and \( M \).

Theorem 4.1 claims that \( \ker(G) \subseteq \text{core}(G) \) for every graph.

**Theorem 4.5** [13] If \( G \) is a bipartite graph, then \( \ker(G) = \text{core}(G) \).

**Alternative Proof.** The assertions are clearly true, whenever \( \ker(G) = \emptyset \), i.e., for \( G \) having a perfect matching. Assume that \( \ker(G) \neq \emptyset \).

Let \( S \in \Omega(G) \) and \( M \) be a maximum matching. By Theorem 1.2(i), \( M \) matches \( V(G) - S \) into \( S \), and \( N(\text{core}(G)) \) into \( \text{core}(G) \).

According to Theorem 3.3(ii), it is sufficient to show that there is no set \( Z \subseteq N(\text{core}(G)) \), \( Z \neq \emptyset \), such that \( |N(Z) \cap \text{core}(G)| = |Z| \).

Suppose, to the contrary, that there exists a non-empty set \( Z \subseteq N(\text{core}(G)) \) such that \( |N(Z) \cap \text{core}(G)| = |Z| \). Let \( Z_0 \) be a minimal non-empty subset of \( N(\text{core}(G)) \) enjoying this equality.

Clearly, \( H = G[Z_0 \cup M(Z_0)] \) is bipartite, because it is a subgraph of a bipartite graph. Moreover, the restriction of \( M \) on \( H \) is a perfect matching.
Claim 1. $Z_0$ is independent.

Since $H$ is a bipartite graph with a perfect matching it has two maximum independent sets at least. Hence there exists $W \in \Omega (H)$ different from $M (Z_0)$. Thus $W \cap Z_0 \neq \emptyset$. Therefore, $N (W \cap Z_0) \cap \text{core}(G) = M (W \cap Z_0)$. Consequently,

$$|N (W \cap Z_0) \cap \text{core}(G)| = |M (W \cap Z_0)| = |W \cap Z_0|.$$  

Finally, $W \cap Z_0 = Z_0$, because $Z_0$ has been chosen as a minimal subset of $N (\text{core}(G))$ such that $|N (Z_0) \cap \text{core}(G)| = |Z_0|$. Since $|Z_0| = \alpha (H) = |W|$ we conclude with $W = Z_0$, which means, in particular, that $Z_0$ is independent.

Claim 2. $H$ is a connected graph.

Otherwise, for any connected component of $H$, say $\hat{H}$, the set $V (\hat{H}) \cap Z_0$ contradicts the minimality property of $Z_0$.

Claim 3. $Z_0 \cup (\text{core}(G) - M (Z_0))$ is independent.

By Claim 1 $Z_0$ is independent. The equality $|N (Z_0) \cap \text{core}(G)| = |Z_0|$ implies $N (Z_0) \cap \text{core}(G) = M (Z_0)$, which means that there are no edges connecting $Z_0$ and $\text{core}(G) - M (Z_0)$. Consequently, $Z_0 \cup (\text{core}(G) - M (Z_0))$ is independent.

Claim 4. $Z_0 \cup (\text{core}(G) - M (Z_0))$ is included in a maximum independent set.

Let $Z_i = M ((N (Z_{i-1}) \cap S) - M (Z_{i-1}))$, $1 \leq i < \infty$. By Lemma 4.4 all the sets $Z^i = \bigcup_{0 \leq j \leq i} Z_j$, $1 \leq i < \infty$ are independent. Define

$$Z^\infty = \bigcup_{0 \leq i \leq \infty} Z_i,$$

which is, actually, the largest set in the sequence $\{Z^i, 1 \leq i < \infty\}$.

Figure 8: $S \in \Omega (G)$, $Q = \text{core}(G) - M (Z_0)$, $Y_0 = M (Z_0)$, $Y_1 = (N (Z_0) - M (Z_0)) \cap S$, $Y_2 = \ldots$, and $Z_i = M (Y_i)$, $i = 1, 2, \ldots$.

The inclusion

$$Z_0 \cup (\text{core}(G) - M (Z_0)) \subseteq (S - M (Z^\infty)) \cup Z^\infty$$

is justified by the definition of $Z^\infty$. 

9
Since $|M(Z^\infty)| = |Z^\infty|$ we obtain $|(S - M(Z^\infty)) \cup Z^\infty| = |S|$. According to the definition of $Z^\infty$ the set

$$(N(Z^\infty) \cap S) - M(Z^\infty)$$

is empty. In other words, the set $(S - M(Z^\infty)) \cup Z^\infty$ is independent. Therefore, we arrive at

$$(S - M(Z^\infty)) \cup Z^\infty \in \Omega(G).$$

Consequently, $(S - M(Z^\infty)) \cup Z^\infty$ is a desired enlargement of $Z_0 \cup (\text{core}(G) - M(Z_0))$.

Claim 5. $\text{core}(G) \cap ((S - M(Z^\infty)) \cup Z^\infty) = \text{core}(G) - M(Z_0)$.

The only part of $(S - M(Z^\infty)) \cup Z^\infty$ that interacts with $\text{core}(G)$ is the subset

$$Z_0 \cup (\text{core}(G) - M(Z_0)).$$

Hence we obtain

$$\text{core}(G) \cap ((S - M(Z^\infty)) \cup Z^\infty) = \text{core}(G) \cap (Z_0 \cup (\text{core}(G) - M(Z_0))) = \text{core}(G) - M(Z_0).$$

Since $Z_0$ is non-empty, by Claim 5 we arrive at the following contradiction

$$\text{core}(G) \notin (S - M(Z^\infty)) \cup Z^\infty \in \Omega(G).$$

Finally, we conclude with the fact there is no set $Z \subseteq N(\text{core}(G)), Z \neq \emptyset$ such that $|N(Z) \cap \text{core}(G)| = |Z|$, which, by Theorem [3,3] means that $\text{core}(G)$ and $\text{ker}(G)$ coincide.

Notice that there are non-bipartite graphs enjoying the equality $\text{ker}(G) = \text{core}(G)$; e.g., the graphs from Figure 9 where only $G_1$ is a König-Egerváry graph.

![Figure 9](image)

Figure 9: $\text{core}(G_1) = \text{ker}(G_1) = \{x, y\}$ and $\text{core}(G_2) = \text{ker}(G_2) = \{a, b\}$.

There is a non-bipartite König-Egerváry graph $G$, such that $\text{ker}(G) \neq \text{core}(G)$. For instance, the graph $G_1$ from Figure 10 has $\text{ker}(G_1) = \{x, y\}$, while $\text{core}(G_1) = \{x, y, u, v\}$. The graph $G_2$ from Figure 10 has $\text{ker}(G_2) = \emptyset$, while $\text{core}(G_2) = \{w\}$.

![Figure 10](image)

Figure 10: Both $G_1$ and $G_2$ are König-Egerváry graphs. Only $G_2$ has a perfect matching.
5 \( \ker(G) \) and \( \text{diadem}(G) \) in König-Egerváry graphs

There is a non-König-Egerváry graph \( G \) with \( V(G) = N(\text{core}(G)) \cup \text{corona}(G) \); e.g., the graph \( G \) from Figure 11.

Theorem 5.1 If \( G \) is a König-Egerváry graph, then

(i) \(|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)|; \)
(ii) \( \text{diadem}(G) = \text{corona}(G) \), while \( \text{diadem}(G) \subseteq \text{corona}(G) \) is true for every graph;
(iii) \(|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)|.

Proof. (i) Using Theorems 1.2 and 1.3, we infer that
\[
|\text{corona}(G)| + |\text{core}(G)| = |\text{corona}(G)| + |N(\text{core}(G))| + |\text{core}(G)| - |N(\text{core}(G))| = |V(G)| + d(G) = \alpha(G) + \mu(G) + d(G) = 2\alpha(G).
\]
as claimed.

(ii) Every \( S \in \Omega(G) \) is a critical set, by Theorem 1.4. Hence we deduce that \( \text{corona}(G) \subseteq \text{diadem}(G) \). On the other hand, for every graph each critical independent set is included in a maximum independent set, according to Theorem 1.1. Thus, we infer that \( \text{diadem}(G) \subseteq \text{corona}(G) \). Consequently, the equality \( \text{diadem}(G) = \text{corona}(G) \) holds.

(iii) It follows by combining parts (i),(ii) and Theorem 4.1.

Notice that the graph from Figure 11 has \( |\text{corona}(G)| + |\text{core}(G)| = 13 > 12 = 2\alpha(G) \).

For a König-Egerváry graph with \(|\ker(G)| + |\text{diadem}(G)| < 2\alpha(G)|\) see Figure 11.

Figure 11 shows that it is possible for a graph to have diadem(\( G \)) \( \not\subseteq \) corona(\( G \)) and ker(\( G \)) \( \not\subseteq \) core(\( G \)).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{\( G \) is not a König-Egerváry graph, and core(\( G \)) = \{x, y, z\}.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{\( G_1 \) is a non-bipartite König-Egerváry graph, such that \( \ker(G_1) = \text{core}(G_1) \) and \( \text{diadem}(G_1) = \text{corona}(G_1) \); \( G_2 \) is a non-König-Egerváry graph, such that \( \ker(G_2) = \text{core}(G_2) = \{x, y\}; \text{diadem}(G_2) \cup \{z, t, v, w\} = \text{corona}(G_2) \).}
\end{figure}

Proposition 5.2 Let \( G = (A, B, E) \) be a bipartite graph.

(i) \([21]\) If \( X = \ker_A(G) \) and \( Y \) is a \( B \)-critical set, then \( X \cap N(Y) = N(X) \cap Y = \emptyset \);\n(ii) \([20]\) \( \ker_A(G) \cap N(\ker_B(G)) = N(\ker_A(G)) \cap \ker_B(G) = \emptyset \).
Now we are ready to describe both \( \ker \) and \( \text{diadem} \) of a bipartite graph in terms of its bipartition.

**Theorem 5.3** Let \( G = (A, B, E) \) be a bipartite graph. Then the following assertions are true:

(i) \( \ker(A) \cup \ker(B) = \ker(G) \);
(ii) \( |\ker(G)| + |\text{diadem}(G)| = 2\alpha(G) \);
(iii) \( |\ker(A)| + |\text{diadem}_B(G)| = |\ker(B)| + |\text{diadem}_A(G)| = \alpha(G) \);
(iv) \( \text{diadem}_A(G) \cup \text{diadem}_B(G) = \text{diadem}(G) \).

**Proof.** (i) By Theorem 4.3, \( \ker(A) \cup \ker(B) \) is critical in \( G \). Moreover, the set \( \ker(A) \cup \ker(B) \) is independent in accordance with Proposition 4.2(ii). Assume that \( \ker(A) \cup \ker(B) \) is not minimal. Hence the unique minimal \( d \)-critical set of \( G \), say \( Z \), is a proper subset of \( \ker(A) \cup \ker(B) \), by Theorem 2.1(iii). According to Theorem 4.3(iv), \( Z_A = Z \cap A \) is an \( A \)-critical set, which implies \( \ker(A) \subseteq Z_A \), and similarly, \( \ker(B) \subseteq Z_B \). Consequently, we get that \( \ker(A) \cup \ker(B) \subseteq Z \), in contradiction with the fact that \( \ker(A) \cup \ker(B) \neq Z \subseteq \ker(A) \cup \ker(B) \).

(ii), (iii), (iv) By Proposition 5.2(i), we have
\[
|\ker(A)| - \delta_0(A) + |\text{diadem}_B(G)| = |N(\ker(A))| + |\text{diadem}_B(G)| \leq |B|.
\]
Hence, according to Theorem 4.3(ii), it follows that
\[
|\ker(A)| + |\text{diadem}_B(G)| \leq |B| + \delta_0(G) = \alpha(G).
\]
Changing the roles of \( A \) and \( B \), we obtain
\[
|\ker(B)| + |\text{diadem}_A(G)| \leq \alpha(G).
\]

By Theorem 4.3(iv), \( \text{diadem}(G) \cap A \) is \( A \)-critical and \( \text{diadem}(G) \cap B \) is \( B \)-critical. Hence \( \text{diadem}(G) \cap A \subseteq \text{diadem}_A(G) \) and \( \text{diadem}(G) \cap B \subseteq \text{diadem}_B(G) \). It implies both the inclusion \( \text{diadem}(G) \subseteq \text{diadem}_A(G) \cup \text{diadem}_B(G) \), and the inequality
\[
|\text{diadem}(G)| \leq |\text{diadem}_A(G)| + |\text{diadem}_B(G)|.
\]

Combining Theorem 4.3 Theorem 6.1(i),(ii), and part (i) with the above inequalities, we deduce
\[
2\alpha(G) \geq |\ker(A)| + |\ker(B)| + |\text{diadem}_A(G)| + |\text{diadem}_B(G)| \geq |\ker(G)| + |\text{diadem}(G)| = |\text{core}(G)| + |\text{corona}(G)| = 2\alpha(G).
\]

Consequently, we infer that
\[
|\text{diadem}_A(G)| + |\text{diadem}_B(G)| = |\text{diadem}(G)|,
|\ker(G)| + |\text{diadem}(G)| = 2\alpha(G),
|\ker(A)| + |\text{diadem}_A(G)| = |\ker(B)| + |\text{diadem}_A(G)| = \alpha(G).
\]
Since \( \text{diadem}(G) \subseteq \text{diadem}_A(G) \cup \text{diadem}_B(G) \) and \( \text{diadem}_A(G) \cap \text{diadem}_B(G) = \emptyset \), we finally obtain that
\[
\text{diadem}_A(G) \cup \text{diadem}_B(G) = \text{diadem}(G),
\]
as claimed. ☐
6 Conclusions

In this paper we focus on interconnections between ker, core, diadem, and corona. In [15] we showed that $2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$ is true for every graph, while the equality holds whenever $G$ is a König-Egerváry graph, by Theorem 5.1(i).

According to Theorem 4.1, $\ker(G) \subseteq \text{core}(G)$ for every graph. On the other hand, Theorem 1.1 implies the inclusion $\text{diadem}(G) \subseteq \text{corona}(G)$. Hence

$$|\ker(G)| + |\text{diadem}(G)| \leq |\text{core}(G)| + |\text{corona}(G)|$$

for each graph $G$. These remarks together with Theorem 5.1(iii) motivate the following.

**Conjecture 6.1** $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$ is true for every graph $G$.

When it is proved one can conclude that the following inequalities:

$$|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$$

hold for every graph $G$.

By Corollary 2.3, $\text{core}(G)$ is critical for every König-Egerváry graph. It justifies the following.

**Problem 6.2** Characterize graphs such that $\text{core}(G)$ is a critical set.

Theorem 4.5 claims that the sets $\ker(G)$ and $\text{core}(G)$ coincide for bipartite graphs. On the other hand, there are examples showing that this equality holds even for some non-König-Egerváry graphs (see Figure 9). We propose the following.

**Problem 6.3** Characterize graphs with $\ker(G) = \text{core}(G)$.

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