Spatiotemporal ETAS model with a renewal main-shock arrival process

Tom Stindl* and Feng Chen†
Department of Statistics, UNSW Sydney

Abstract

This article proposes a spatiotemporal point process model that enhances the classical Epidemic-Type Aftershock Sequence (ETAS) model by incorporating a renewal main-shock arrival process, which we term the renewal ETAS (RETAS) model. This modification is similar in spirit to the renewal Hawkes (RHawkes) process but the conditional intensity process supports a spatial domain. It empowers the main-shock intensity with the capability to reset upon the arrival of main-shocks and therefore allows for heavier clustering of earthquakes than the spatiotemporal ETAS model introduced by Ogata [1998]. We introduce a likelihood evaluation algorithm for parameter estimation and provide a novel procedure to evaluate the fitted model’s goodness-of-fit based on a sequential application of the Rosenblatt transformation. A simulation algorithm for the RETAS model is developed and applied to validate the numerical performance of the likelihood evaluation algorithm and goodness-of-fit test procedure. We illustrate the proposed model and procedures on various earthquake catalogs around the world each with distinctly different seismic activity. These catalogs will demonstrate that the RETAS model affords additional flexibility in comparison to the classical spatiotemporal ETAS model and has the potential for superior modeling and forecasting of seismicity.

Key words and phrases: goodness-of-fit, point process, RHawkes process, spatial analysis, statistical seismology

*This research includes computations using the Linux computational cluster Katana supported by the Faculty of Science, UNSW Sydney, and the National Computational Infrastructure (NCI) supported by the Australian Government.

†Chen was partly supported by a UNSW SFRGP grant.
1 Introduction

Point processes are a popular framework to describe patterns of points that cluster in time, space, or a combination thereof. Spatiotemporal point processes are frequently used to model the clustering phenomena in seismic activity. The Epidemic-Type Aftershock Sequence (ETAS) model has received considerable attention for analyzing and forecasting the arrival times and epicenters of earthquakes. Ogata (1988) developed the first ETAS model and then generalized it to a spatiotemporal model (Ogata 1998). The main-shock arrival process for the ETAS model is a spatiotemporal Poisson process that is temporally homogeneous but spatially inhomogeneous, and the aftershocks are triggered according to a parametric kernel function that depends on the time, location and magnitude of the triggering earthquake. The ETAS model has been the cornerstone for several related works in seismology such as Console and Murru (2001); Console et al. (2003); Zhuang et al. (2002, 2004, 2008); Ogata (2004); Zhuang et al. (2005); Console et al. (2006); Marzocchi and Lombardi (2009); Helmstetter et al. (2006); Werner et al. (2011).

The success of self-exciting point processes in the seismological literature is due primarily to the convenient branching process interpretation introduced by Hawkes and Oakes (1974), which interprets the points of the process as either immigrants or offspring events. In the ETAS model, the immigrant events are interpreted as main-shocks, which can induce subsequent offspring events or aftershocks, which may themselves induce additional aftershocks of their own. The originating main-shock and the sequence of aftershocks induced by it form a cluster. The stability conditions in the model formulation of the ETAS model guarantee that the cluster eventually ceases.

Generally, most of the modeling flexibility in the ETAS model is derived from the self-excited part of the model. It is from the self-exciting part of the ETAS model that credible forecasts of aftershocks can be obtained. However, an inadequate description of the background component can lead to incorrect classification of main-shocks and aftershocks since the classification significantly depends on the specified background rate (both temporally and spatially). This misspecification can also cause the Omori power-law decay parameter to be either over or under estimated, and therefore the aftershock sequence would appear shorter or more prolonged than it should be.

Since the main-shock arrival process significantly influences the clustering behaviour of an earthquake catalog, we suggest that a more flexible choice for the background rate will lead to a more appropriate distribution (and identification) for main-shocks and their associated aftershock sequences. For instance, in the New Zealand earthquake catalog to be analyzed in Section 6.1, the classical spatiotemporal ETAS model appears to be inadequate,
and a modification to the model is required to fit the catalog. One approach to modifying the spatiotemporal ETAS model of Ogata (1998) is to allow the background seismicity to vary both temporally and spatially. Allowing the background rate to vary temporally has been discussed previously in the works of Chen and Hall (2013) and Godoy et al. (2016). Specifically, in the context of seismicity modelling, the nonstationary ETAS model introduced by Kumazawa and Ogata (2014) accounts for temporal variation in the main-shock arrival rate. In their work, a time-dependent factor was introduced to the background rate in the form of a first-order spline function to account for systematic changes in seismicity at different times. However, this modification departs from the class of stationary point processes and therefore may not be suitable for modelling long term seismicity. As such, our proposed extension to the ETAS model can model the short-term transient variations in seismicity while maintaining long-term stationarity.

Although nearly all stationary models assume a constant background event rate in time, generally main-shocks do not occur uniformly in time. For instance, the accumulation and release of strain and stress may affect the short-term main-shock rate. Therefore, it is highly plausible that main-shocks do not merely reflect a spatiotemporal Poisson process that is homogeneous in time and spatially inhomogeneous. Alternatively, the main-shock arrival times might be better described by a general renewal process, which allows the short term main-shock rate to vary according to the lapsed time since the last main-shock, while still maintaining a constant main-shock rate in the long term.

Recently, in the work of Wheatley et al. (2016), the self-exciting Hawkes process was generalized to model the immigrants or background events using a general renewal process but focused purely on the temporal characteristics of the data. Chen and Stündl (2018) then applied the renewal Hawkes (RHawkes) process to an earthquake catalog from the Pacific Ring of Fire near the east coast of Japan (originally studied in Ogata (1988)). They reported that the improvement in fit based on the renewal main-shock arrival process was adequate to describe the temporal characteristics of the catalog. Kolev and Ross (2018) applied an analogous modification to the temporal ETAS model by assuming the waiting time distribution between main-shocks follows a gamma or Brownian passage time (BPT) distribution. Their model was illustrated on two earthquake catalogs from New Madrid and Northern California. The non-Poissonian main-shock arrival models studied thus far have focused mainly on the temporal aspects of the data and they do not account for the spatial dependencies that exist among earthquakes.

In this work we extend the classical spatiotemporal ETAS model of Ogata (1998) by modeling the main-shock arrival process as a renewal process.
which accommodates short term deviations of the background seismicity from
the long term mean level, while maintaining the overall stationarity of the
model. The process, termed the renewal ETAS model (RETAS), nests the
ETAS model and therefore allows for a determination of whether the con-
stant (temporal) background seismicity model is appropriate for particular
seismic sequences.

With the introduction of the renewal process for the background seis-
micity, the main-shock rate depends on the lapsed time since the previous
main-shock which is not observable. This makes likelihood evaluation chal-
lenging. However, using a recursive algorithm motivated by that of [Chen and
Stindl 2015] for the RHawkes process likelihood evaluation, we can directly
evaluate the likelihood of the RETAS model. We can then fit the RETAS
model by minimizing the negative log-likelihood and obtain the variance esti-
mate for the maximum likelihood estimator (MLE) by inverting the Hessian
matrix. For goodness-of-fit (GOF) assessment of the RETAS model, we pro-
pose a novel approach based on the Rosenblatt residuals [Rosenblatt 1952],
which can also be applied to the classical spatiotemporal ETAS model. The
approach avoids the simulations required by the thinning spatial residuals
based approach of [Schoenberg 2003] and can assess the GOF of the tempo-
ral and spatial components of the model either simultaneously or separately.

The remainder of this article is structured as follows. The general form
of the RETAS model will be described in Section 2. Following this, the like-
lihood of the model is discussed in detail in Section 3. The method to assess
the GOF of the temporal and spatial aspects of the fitted model are pre-
sented in Section 4. Section 5 discusses the simulation of the RETAS model
and reports the results of a simulation study to investigate the performance
of the MLE and GOF test procedure. In Section 6, we illustrate the RETAS
model by fitting various forms of the model to different earthquake catalogs
around the world.

2 General form of RETAS model

For an earthquake catalog, we let \( \{ (\tau_i, x_i, y_i, m_i) \}_{i \geq 1} \) denote the occurrence
time \( \tau_i \), location \( (x_i, y_i) \) and magnitude \( m_i \) of each earthquake. Let \( N \) be
the point process for the earthquakes with \( N(A) \) counting the number of
earthquakes in the set \( A \subset [0, T] \times S \times M \) where \( T \in \mathbb{R} \), \( S \subset \mathbb{R}^2 \), and
\( M \subset \mathbb{R} \). Denote the internal history of \( N \) by \( \mathcal{H} = \{ \mathcal{H}_t \}_{t \in [0,T]} \), where \( \mathcal{H}_t = \sigma \{ (\tau_i, x_i, y_i, m_i); \tau_i \leq t \} \) is the sigma-field representing the knowledge of all
the times, locations and magnitudes of earthquakes up to and including time
\( t \). The intensity process of \( N \) relative to \( \mathcal{H} \), or the conditional event rate at
time $t \in [0, T]$, location $(x, y) \in S$, and magnitude $m \in \mathcal{M}$ given the internal history of the process prior to time $t$, $\mathcal{H}_{t-}$, is defined as

$$
\lambda(t, x, y, m | \mathcal{H}_{t-}) := \lim_{\Delta t, \Delta x, \Delta y, \Delta m \to 0} \frac{\mathbb{E}[N([t, t + \Delta t) \times [x, x + \Delta x) \times [y, y + \Delta y) \times [m, m + \Delta m)|\mathcal{H}_{t-}]}{\Delta t \Delta x \Delta y \Delta m}.
$$

(1)

It is often assumed that the magnitudes of the earthquakes do not depend on their occurrence times and locations, or the times, locations, and magnitudes of previous earthquakes (Zhuang et al., 2002), and are independent and identically distributed ($i.i.d.$) with common density function $J$. Therefore, if we let $\lambda_g$ denote the ground intensity process for the times and locations of the earthquakes, that is

$$
\lambda_g(t, x, y) = \lim_{\Delta t, \Delta x, \Delta y \to 0} \frac{\mathbb{E}[N([t, t + \Delta t) \times [x, x + \Delta x) \times [y, y + \Delta y) \times \mathcal{M}) | \mathcal{H}_{t-}]}{\Delta t \Delta x \Delta y},
$$

then the conditional intensity process of the marked spatiotemporal point process takes the form

$$
\lambda(t, x, y, m | \mathcal{H}_{t-}) = \lambda_g(t, x, y | \mathcal{H}_{t-}) J(m).
$$

(2)

The RETAS process intensity can be specified by introducing the (unobservable) main-shock indicator $B_i$ such that $B_i = 0$ if the $i$-th earthquake is a main-shock and $B_i = 1$ if it is an aftershock. Furthermore, the function $I(t) := \max \{i \mid \tau_i < t, B_i = 0\}$ indicates the index of the most recent main-shock prior to time $t$ with the convention that $I(t) = 0$ for $t \leq \tau_1$ and $\tau_0 = 0$. For simplicity, we assume that there are no events before time zero and the first event is a main-shock. We introduce the extended history $\bar{\mathcal{H}}_t = \sigma(\mathcal{H}_t \cup \{I(s) ; s \leq t\})$ to include the index of the most recent main-shock (which is unobservable from the earthquake catalog). The ground intensity of the RETAS model with respect to the extended history $\bar{\mathcal{H}}_t$ is then specified as

$$
\lambda_g(t, x, y | \bar{\mathcal{H}}_{t-}) = \mu(t - \tau_{I(t)}) \nu(x, y) + \sum_{i; \tau_i < t} \kappa(m_i) g(t - \tau_i) f(x - x_i, y - y_i)
$$

$$
= \mu(t - \tau_{I(t)}) \nu(x, y) + \phi(t, x, y).
$$

(3)

The function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ describes the temporal variation of the main-shock arrival rate that renews upon the arrival of a main-shock. The spatial variation of main-shocks over the region $S$ is described by the function $\nu : \mathbb{R}^2 \to \mathbb{R}$. 

5
For stability of the main-shock arrival process, it is assumed that \( \int_0^\infty \exp(-\int_0^t \mu(s) ds) dt < \infty \), which implies that the expected waiting time between consecutive main-shocks is finite. For identifiability, it is also assumed that \( \int_0^\infty \nu(x,y) dy dx = 1 \). The temporal response function \( g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) represents the temporal distribution of an aftershock relative to its triggering earthquake, and the spatial response function \( f: \mathcal{S} \to \mathbb{R}_{\geq 0} \) describes the spatial distribution of an aftershock relative to its triggering earthquake. The functions \( g \) and \( f \) are both assumed to integrate to unity.

The ground intensity of the RETAS process relative to the internal history \( \mathcal{H} \) can be expressed in the form

\[
\lambda_g(t, x, y|\mathcal{H}_{t-}) = \mathbb{E} [\mu(t-\tau_I(t))|\mathcal{H}_{t-}] \nu(x, y) + \sum_{i: \tau_i < t} \kappa(m_i) g(t-\tau_i) f(x-x_i, y-y_i)
\]

where \( \mathbb{E}[\cdot] \) denotes the expected value, \( \kappa(m) \) denotes the mean number of triggered aftershocks by an earthquake with magnitude of size \( m \).

The ground intensity relative to the internal history is not readily available in general, unless \( \mu \) is a constant, and this has implications for likelihood evaluation.

For the remainder of this article, the following specific form for the RETAS model will be used. The main-shock arrival process is either Weibull with hazard function \( \mu_W(t) = \frac{\alpha}{\beta} t^{\alpha-1} e^{-t/\beta} \), \( t \geq 0 \), or gamma with hazard function \( \mu_G(t) = \frac{1}{\beta \Gamma((\alpha+1)/\beta)} (\frac{t}{\beta})^{\alpha-1} e^{-t/\beta} \), \( t \geq 0 \), where \( \alpha > 0 \) and \( \beta > 0 \) are the shape and scale parameters respectively, and \( \Gamma(x, k) = \int_x^\infty s^{k-1} e^{-s} ds \) is the upper incomplete gamma function. The bivariate density function \( \nu(x, y) \) is learned from historical data non-parametrically using a kernel density estimator. The temporal response function \( g \) takes the form of the modified Omori’s law (Omori, 1894; Utsu, 1961), \( g(t) = \frac{1}{\beta} t^{\alpha-1} e^{-t/\beta} \), \( t \geq 0 \), where \( \alpha > 0 \) and \( \beta > 0 \) are the shape and scale parameters respectively. The spatial response function \( f \) takes the form of a bivariate normal density, \( f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp(-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}) \), where \( \sigma_1 > 0 \) and \( \sigma_2 > 0 \) control the dispersion of the aftershock distribution in the \( x \) and \( y \) directions respectively. The boost function takes the form, \( \kappa(m) = Ae^{\delta(m-m_0)} \), where
A > 0 and δ > 0 control the expected number of induced aftershocks, and \( m_0 \) is the minimum or threshold magnitude of earthquakes in the catalog. The distribution of the magnitudes are given by the Gutenberg-Richter law (Gutenberg and Richter 1944), which is a shifted exponential distribution with density, \( J(m) = \gamma e^{-\gamma(m-m_0)}, m \geq m_0 \).

### 3 Maximum Likelihood Estimation

The difficulties that emerge from the complex expression for the intensity process with respect to the internal history implies that estimation of model parameters for the RETAS model cannot be readily performed using the classical likelihood formula for point process models directly. However, in this section, we develop an algorithm that computes the likelihood recursively, therefore enabling likelihood-based inferences such as maximum likelihood estimation (MLE). For marked spatiotemporal point processes with conditional intensity process defined as in (2), that is, with respect to the internal history, the log-likelihood can be represented in terms of the ground intensity process and an additional term relating to the marks

\[
\ell(\theta) = \sum_{i=1}^{n} \log (\lambda_g(\tau_i, x_i, y_i)) - \int_{0}^{T} \int_{S} \lambda_g(t, x, y) dy dx dt + \sum_{i=1}^{n} \log (J(m_i)),
\]

where \( \theta = (\alpha, \beta, p, c, d, A, \delta, \gamma)^T \) denotes the vector of parameters in the RETAS model, and \( n = N([0, T] \times S \times M) \) denotes the number of earthquakes in the study region in the interval \([0, T]\).

Since the distribution of the most recent main-shock index \( I(t) \) given the prior-\( t \) history \( \mathcal{H}_{t-} \) is not readily available, the loglikelihood expression in (5) is not amenable for direct evaluation. Therefore, we express the log-likelihood in an alternative form and propose an algorithm for its evaluation given in the subsequent theorem. For convenience, we use the following notations; \( s_i = (x_i, y_i) \), \( p_{ij} = P(I(\tau_i) = j|\mathcal{H}_{\tau_{i-}}) \), \( d_1 = \mu(\tau_1)\nu(s_1)e^{-\int_{0}^{\tau_2} \mu(t) dt} \), \( d_{ij} = (\mu(\tau_i - \tau_j)\nu(s_i) + \phi(\tau_i, s_i)) S_{ij} \), with \( S_{ij} = e^{-\int_{\tau_{i-1}}^{\tau_i} \mu(t-\tau_j) dt} \), for \( j = 1, \ldots, i-1, i = 2, \ldots, n \), \( S_{n+1,j} = e^{-\int_{\tau_{n}}^{\tau_i} \mu(t-\tau_j) dt} \), for \( j = 1, \ldots, n \), and, with \( \phi(t, x, y) \) as in [3],

\[
\Phi(T) = \int_{0}^{T} \int_{S} \phi(t, x, y) dx dy dt = \sum_{j=1}^{n} \kappa(m_j) \int_{\tau_j}^{T} g(t - \tau_j) dt \int_{S} f(x - x_j, y - y_j) dx dy.
\]
Theorem 3.1. The log-likelihood of the RETAS model with conditional intensity with respect to the extended history $\hat{H}$ in [3] is given by

$$\ell(\theta) = \log d_1 + \sum_{i=2}^{n} \log \left( \sum_{j=1}^{i-1} p_{ij} d_{ij} \right) + \log \left( \sum_{j=1}^{n} p_{n+1,j} S_{n+1,j} \right) + \Phi(T) + \sum_{j=1}^{n} \log J(m_j),$$

and the $p_{ij}$'s are recursively calculated using

$$p_{ij} = \begin{cases} \frac{p_{i-1,j} \phi(\tau_{i-1},s_i) S_{i-1,j}}{\sum_{j=1}^{i-2} p_{i-1,j} d_{i-1,j} + \sum_{k=1}^{i-2} p_{ik}}, & j = 1, \ldots, i-2, \quad i \geq 3, \\ 1 - \sum_{k=1}^{i-2} p_{ik}, & j = i-1 \end{cases}$$

from the initial value $p_{21} = 1.

Proof. For notational convenience, for a sequence $z_1, z_2, \ldots$, we use $z_{i:j}$ to denote $z_i, z_{i+1}, \ldots, z_j, i < j$. The likelihood function for the RETAS model is calculated by computing the sequential contribution to the likelihood between successive earthquakes as follows

$$L(\tau_{1:n}, s_{1:n}, m_{1:n}) = p(\tau_1, s_1, m_1) \left\{ \prod_{i=2}^{n} p(\tau_i, s_i, m_i | H_{\tau_i-}) \right\} \mathbb{P}(\tau_{n+1} > T | H_T).$$

Then by invoking the independent magnitude assumption, the likelihood can be separated into a spatiotemporal component for the ground intensity process and the marks as follows

$$L(\tau_{1:n}, s_{1:n}, m_{1:n}) = p(\tau_1, s_1) p(m_1 | \tau_1, s_1) \times
\left\{ \prod_{i=2}^{n} p(\tau_i, s_i | H_{\tau_i-}) p(m_i | \tau_i, s_i, H_{\tau_i-}) \right\} \mathbb{P}(\tau_{n+1} > T | H_T).$$

Therefore, the log-likelihood $\ell(\theta)$ is separable into a spatiotemporal component $\ell_s(\theta)$ and a likelihood contribution for the magnitudes $\ell_m(\theta)$, in which

$$\ell_s(\theta) = \log p(\tau_1, s_1) + \sum_{i=2}^{n} \log p(\tau_i, s_i | H_{\tau_i-}) + \log \mathbb{P}(\tau_{n+1} > T | H_T)$$

and

$$\ell_m(\theta) = \log p(m_1 | \tau_1, s_1) + \sum_{i=2}^{n} \log p(m_i | \tau_i, s_i, H_{\tau_i-}) = \sum_{i=1}^{n} \log J(m_i).$$
The form of $\ell_m$ is complete and the remainder of this proof deals with the evaluation of $\ell_s$. By conditioning on the index of the most recent main-shock, the expression in (10) has the equivalent form

$$\ell_s(\theta) = \log p(\tau_1, s_1) + \sum_{i=2}^n \log \left( \sum_{j=1}^{i-1} p(\tau_i, s_i | I(\tau_i) = j, \mathcal{H}_{\tau_i-})p_{i,j} \right)$$

$$+ \log \sum_{j=1}^n \mathbb{P} (I(\tau_{n+1}) = j | \mathcal{H}_T) p_{n+1,j},$$

where

$$p(\tau_i, s_i | I(\tau_i) = j, \mathcal{H}_{\tau_i-})$$

$$= (\mu(\tau_i - \tau_j) \nu(s_i) + \phi(\tau_i, s_i)) \exp \left( - \int_{\tau_{i-1}}^{\tau_i} \int_{\mathcal{S}} \mu(t - \tau_j) \nu(s) + \phi(t, s) \, ds \, dt \right)$$

$$= d_{ij} \exp \left( - \int_{\tau_{i-1}}^{\tau_i} \int_{\mathcal{S}} \phi(t, s) \, ds \, dt \right),$$

and

$$\mathbb{P} (I(\tau_{n+1}) = j | \mathcal{H}_T) = \exp \left( - \int_{\tau_n}^{T} \int_{\mathcal{S}} \mu(t - \tau_j) \nu(s) + \phi(t, s) \, ds \, dt \right)$$

$$= S_{n+1,j} \exp \left( - \int_{\tau_n}^{T} \int_{\mathcal{S}} \phi(t, s) \, ds \, dt \right).$$

Therefore, it follows that

$$\ell_s(\theta) = \log d_1 + \sum_{i=2}^n \log \left( \sum_{j=1}^{i-1} p_{i,j} d_{ij} \right) + \log \left( \sum_{j=1}^n p_{n+1,j} S_{n+1,j} \right) + \Phi(T). \quad (12)$$

To complete the proof, it remains to show the recursion (7) for the distribution of the most recent main-shock index. The proof is analogous to Eq. (S.6)-(S.9) in the file proofs.pdf contained in the online supplementary material accompanying Chen and Stindl (2018).

The MLE of the model parameter is obtained by directly maximizing the log-likelihood function in (6) using general purpose optimization routines, and inverting the Hessian matrix to estimate the variance of the MLE. For this purpose, fast and accurate computation of the double integral $\iint_S f(x - x_j, y - y_j) \, dx \, dy$ required in (6) is important. When the spatial domain $\mathcal{S}$ is large relative to the variance of the spatial response function $f$, the integral
can be approximated by 1. However, in general this approximation can be too crude and lead to substantial bias in the estimators. Numerical quadrature can be used to approximate the integral accurately, but 2-dimensional (2-d) quadrature can be intolerably slow. For some choices of the function $f(x,y)$, such as those that depend on the input $(x,y)$ only through its norm $\sqrt{x^2 + y^2}$, the 2-d integral can be reduced to 1-d integrals through a change of variables using polar coordinates and then evaluated using 1-d quadrature.

4 Goodness-of-fit Assessment

An assessment of the adequacy of the fit of the RETAS model to an earthquake catalog should be conducted once the fitted model has been obtained. There are two obvious aspects of the fitted model that should be assessed, that is, the temporal and the spatial aspects. These two aspects are typically assessed separately using different approaches, with the assessment of the temporal aspect based on the Papangelou time-change theorem (Theorem 7.4.1 [Daley and Vere-Jones 2003], and the assessment of the spatial aspect based on a thinning technique [Schoenberg, 2003]. However, the intensity process of the RETAS model relative to the internal history is cumbersome to work with, which makes it inconvenient to use these approaches. Instead, we propose a unified approach to evaluate the goodness-of-fit (GOF) of both aspects of the RETAS model based on the Rosenblatt (1952) transformation.

For a $n$-dimensional random vector $Z = (Z_1, \ldots, Z_n)$, the Rosenblatt transformation $T(Z) = (T_1(Z), \ldots, T_n(Z))$ is defined by $T_1(z) = \mathbb{P}(Z_1 \leq z_1) = F_1(z_1)$, and $T_i(z) = \mathbb{P}(Z_i \leq z_i | Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}) = F_i(z_i | z_1, \ldots, z_{i-1})$ for $i = 2, \ldots, n$, where the $F_i$'s are the conditional distribution functions. It is known that the distribution of $T(Z)$ is uniform on the hypercube $[0,1]^n$. When the conditional distribution functions are replaced with their estimators $\hat{F}_i$'s, then the Rosenblatt residuals $\hat{F}_1(Z_1), \ldots, \hat{F}_n(Z_n | Z_1, \ldots, Z_{n-1})$ are approximately i.i.d. uniform on $[0,1]$ when the model is correctly specified. Uniformity can be checked using formal statistical tests such as the Kolomogorov-Smirnoff (K-S) test and independence by the Ljung-Box (L-B) test or visually assessed using graphical tools such as the uniform Q-Q plot and the ACF plot.

4.1 Temporal model assessment

Specific to the RETAS model, the Rosenblatt residuals to assess the temporal component of the model are given by $U_i = \hat{F}_i(\tau_i | H_{\tau_{i-1}})$ where $\hat{F}_i$ is the fitted
conditional distribution function of \( \tau_i \) given \( \mathcal{H}_{\tau_{i-1}} = \sigma \{ \tau_{1:i-1}, x_{1:i-1}, y_{1:i-1}, m_{1:i-1} \} \).

Let \( \hat{\mu}(\cdot) \) be the plugin estimate of \( \mu(\cdot) \) with the unknown parameters replaced with their estimated values, and \( \hat{\kappa}(\cdot), \hat{g}(\cdot), \hat{f}(\cdot, \cdot) \), and \( \hat{p}_{ij} \) be analogously defined. Then \( U_1 = \hat{F}_1(\tau_1) = 1 - \exp \left(-\int_0^{\tau_1} \hat{\mu}(s) \, ds \right) \) and for \( i = 2, \ldots, n, \)

\[
U_i = \hat{F}_i(\tau_i | \mathcal{H}_{\tau_{i-1}}) = 1 - \sum_{j=1}^{i-1} \hat{p}_{ij} \hat{S}_{ij},
\]

where

\[
\hat{S}_{ij} = \exp \left(-\int_{\tau_{i-1}}^{\tau_i} \left[ \hat{\mu}(s-\tau_j) + \sum_{k=1}^{N(t_i)} \hat{\kappa}(m_i) \hat{g}(s-\tau_k) \right] \int_S \hat{f}(x-x_k, y-y_k) \, dy \, dx \, ds \right).
\]

### 4.2 Spatial model assessment

For spatial model assessment, two residuals \( V_i \) and \( W_i \) are calculated for the longitude \( x_i \) and latitude \( y_i \) of the earthquakes respectively. First, we calculate the longitudinal residual \( V_i \) conditional on the time \( \tau_i \) of the quake and the history \( \mathcal{H}_{\tau_{i-1}} \) of the process prior to \( \tau_i \), and then we calculate the latitudinal residual \( W_i \) with the longitude \( x_i \) of the quake also included in the conditioning information. That is, we define the longitudinal residual \( V_i = G_i(x_i | \tau_i, \mathcal{H}_{\tau_{i-1}}) \), where \( G_i \) denotes the fitted conditional distribution of \( x_i \) given \( \tau_i \) and \( \mathcal{H}_{\tau_{i-1}} \), and the latitudinal residual \( W_i = H_i(x_i | \tau_i, x_i, \mathcal{H}_{\tau_{i-1}}) \), where \( H_i \) denotes the fitted conditional distribution of \( y_i \) given \( \tau_i, x_i, \) and \( \mathcal{H}_{\tau_{i-1}} \). In the following, when presenting the specific expressions of the residuals, we suppress the hat notation (‘\( \hat{\cdot} \)’) from various estimated parameters for convenience.

The longitudinal residual is given, for \( i = 1 \), by

\[
V_i = \int_{S \cap \{ (x, y); x \leq x_i \}} \nu(x, y) \, dy \, dx,
\]

and for \( i = 2, \ldots, n, \) by

\[
V_i = \sum_{j=1}^{i-1} \left\{ \frac{\mu(\tau_i - \tau_j) \int_{S \cap \{ (x, y); x \leq x_i \}} \nu(x, y) \, dy \, dx + \int_{S \cap \{ (x, y); x \leq x_i \}} \phi(\tau_i, x, y) \, dy \, dx}{\mu(\tau_i - \tau_j) + \int_S \phi(\tau_i, x, y) \, dy \, dx} \right\} \hat{p}_{ij}^{\gamma},
\]

where

\[
\int_S \phi(\tau_i, x, y) \, dy \, dx = \sum_{k=1}^{i-1} \kappa(m_k) g(t - \tau_k) \int_S f(x - x_k, y - y_k) \, dy \, dx,
\]

and \( \hat{p}_{ij}^{\gamma} = \mathbb{P}(I(\tau_i) = j | \tau_i, \mathcal{H}_{\tau_{i-1}}) \) are the most recent main-shock probabilities that are updated by including \( \tau_i \) in the condition, which differ slightly from
the $p_{ij}$ in (14). By updating the most recent main-shock probabilities $p_{ij}$ computed in the likelihood evaluation, we see that

$$p^\tau_{ij} = \frac{p(\tau_i | I(\tau_i) = j, H_{\tau_i-})}{p(\tau_i | H_{\tau_i-})} p_{ij},$$

(14)

where

$$p(\tau_i | I(\tau_i) = j, H_{\tau_i-}) = \lambda_i^j(\tau_i) \exp\left(-\int_{t_{i-1}}^{\tau_i} \lambda_i^j(t) dt\right),$$

and

$$p(\tau_i | H_{\tau_i-}) = \sum_{j=1}^{i-1} \lambda_i^j(\tau_i) \exp\left(-\int_{t_{i-1}}^{\tau_i} \lambda_i^j(t) dt\right) p_{ij},$$

with

$$\lambda_i^j(t) = \mu(t - \tau_j) + \sum_{k=1}^{N(t-)} \kappa(m_k) g(t - \tau_k) \int_S f(x - x_k, y - y_k) dy dx.$$

The latitudinal residual is defined similarly using the conditional distribution of $y_i$ given $H_{\tau_i-}$, $\tau_i$ and $x_i$. For $i = 1$, it is given by

$$W_1 = \frac{\int_{\{y \mid (x_1, y) \in S, y \leq y_1\}} \nu(x_1, y) dy}{\int_{\{y \mid (x_1, y) \in S\}} \nu(x_1, y) dy},$$

and for $i = 2, \ldots, n$, by

$$W_i = \sum_{j=1}^{i-1} \left\{ \frac{\mu(\tau_i - \tau_j)}{\mu(\tau_i - \tau_j)} \int_{\{y \mid (x_i, y) \in S, y \leq y_i\}} \nu(x_i, y) dy + \int_{\{y \mid (x_i, y) \in S, y \leq y_i\}} \phi(\tau_i, x_i, y) dy \right\} - p^\tau_{ij},$$

(15)

where $p^\tau_{ij} = P(I(\tau_i) = j | \tau_i, x_i, H_{\tau_i-})$ are the most recent main-shock probabilities, but now include both $\tau_i$ and $x_i$ in the condition. The previously computed $p^\tau_{ij}$ in (14) are updated to compute $p^\tau_{ij}$ as follows

$$p^\tau_{ij} = \frac{p(x_i | \tau_i, I(\tau_i) = j, H_{\tau_i-})}{p(x_i | \tau_i, H_{\tau_i-})} p_{ij},$$

where the densities are given by

$$p(x_i | \tau_i, I(\tau_i) = j, H_{\tau_i-}) = \mu(\tau_i - \tau_j) \int_{\{y \mid (x_i, y) \in S\}} \nu(x_i, y) dy + \int_{\{y \mid (x_i, y) \in S\}} \phi(\tau_i, x_i, y) dy,$$

and

$$p(x_i | \tau_i, H_{\tau_i-}) = \sum_{j=1}^{i-1} p(x_i | \tau_i, I(\tau_i) = j, H_{\tau_i-}) p^\tau_{ij}.$$
5 Simulations

This section reports the results of a simulation study for the RETAS model and confirms that the finite sample performances of the MLE are as expected. We first outline a procedure to simulate the RETAS model.

5.1 Simulation algorithm for the RETAS model

Simulation of the RETAS model can be efficiently performed by utilizing the branching structure of the process. The algorithm operates as follows.

1. Simulate the occurrence times of main-shocks up to the censoring time $T$, as the cumulative sum of i.i.d. positive random variables with hazard rate function $\mu(\cdot)$, and denote these by $\tau_1^0, \ldots, \tau_{n_0}^0$.

2. For each main-shock $i = 1, \ldots, n_0$ simulate the location according to the bivariate probability density function $\nu(\cdot)$ and denote the locations as $s_1^0, \ldots, s_{n_0}^0$.

3. Next, simulate the main-shock magnitudes $m_i^0$ according to the density $J(\cdot)$ and store all the times, locations, and magnitudes of main-shocks as the generation 0 catalog $G(0) = \{(\tau_i^0, s_i^0, m_i^0), i = 1, \ldots, n_0\}$. Now set $l = 0$.

4. For each earthquake in the generation $l$ catalog $G(l)$, namely, $(\tau_k^l, s_k^l, m_k^l)$ for $k = 1, \ldots, |G(l)|$, simulate the potential number of aftershocks as a Poisson random variable with mean $\kappa(m_k^l)$. Then for each potential aftershock, simulate its occurrence time relative to $\tau_k^l$ according to the temporal response function $g(\cdot)$, and discard the aftershock if the simulated occurrence time is beyond $T$; if retained, simulate its location relative to $s_k^l$ according to the spatial response function $f(\cdot, \cdot)$, and discard the aftershock if the location is outside $S$; and if retained, simulate its magnitude according to the density function $J(\cdot)$. Record the times, locations and magnitudes of all retained aftershocks as the generation $l + 1$ catalog $G(l+1)$.

5. If the catalog $G(l+1)$ is non-empty, set $l \leftarrow l + 1$ and return to Step 4; otherwise, collect the simulated earthquakes of all generations and return them as the overall catalog $G = \bigcup_l G(l)$. 

13
5.2 Simulation Results

In this section, numerical evidence confirms that the simulation, likelihood evaluation, and goodness-of-fit test algorithms are performing correctly and establishes that the MLE has satisfactory finite sample performance. The RETAS model examined in this simulation study is the same as described in Section 2 but here we only investigate the Weibull hazard function \( \mu_W(t) \). The magnitudes \( m \) are simulated using a shifted exponential distribution with rate parameter \( \gamma = 5 \) and threshold magnitude \( m_0 = 6 \).

The simulations consist of 1000 realizations of the RETAS model up to the censoring time \( T = 200 \) over the whole plane, i.e. \( S = \mathbb{R}^2 \). The shape and scale parameters of the Weibull hazard function are chosen to be \((\alpha,\beta) = (0.5,0.5)\) or \((2,1)\) to generate both heavily (temporally) clustered main-shocks and under-dispersed main-shocks. The scale parameter \( \beta \) was chosen to have a mean waiting time between consecutive main-shocks close to unity. The main-shocks are spatially distributed according to an independent-marginal bivariate normal distribution with mean at the origin and standard deviations 0.25 and 0.5 in the \( x \) and \( y \) directions respectively. The parameters in the temporal response function \( g \) are fixed at \( p = 2 \) and \( c = 0.01 \). The spatial response function is a bivariate normal density function with independent marginals and variances \( \sigma_1 = 0.01 \) and \( \sigma_2 = 0.02 \) in the \( x \) and \( y \) directions respectively. The parameters for the boost function \( \kappa \) are \( A = 0.5 \) and \( \delta = 1 \), which implies that an earthquake will induce, on average, \( \int_{m_0}^{\infty} A e^{\delta (m - m_0) - \frac{1}{\gamma}(m - m_0)/\gamma} dm = 0.625 \) aftershocks. With such choice of the model parameters, the number of earthquakes in the first simulation model ranges from 718 to 1503 with mean 1063, and the number of earthquakes in the second simulation model ranges from 1008 to 1405 with mean 1189. The more volatile number of earthquakes in the first simulation model is to be expected, as the shape parameter \( \alpha = 0.5 \) for the Weibull distribution therein implies more volatile waiting times between main-shock arrivals.

The results of the simulation study are reported in Table 1, which contains the true value of each parameter (True), the mean of the 1000 parameter estimates by directly minimizing the negative log-likelihood function (Est), the empirical standard error of the 1000 parameter estimates (SE), the estimated standard error by computing the mean of the standard errors found by inverting the approximate Hessian matrix (\( \hat{SE} \)), and the empirical coverage probability (CP) of the 95% confidence interval obtained by assuming (asymptotic) normality of the estimator.

The MLE estimator demonstrates unbiasedness as the estimated parameters are, on average, close to their respective true values. The empirical standard errors and the average of the standard errors are very similar. Fur-
Table 1: Estimation results for the simulated data using MLE for the RETAS model over the region $S = \mathbb{R}^2$ with Weibull main-shock renewal process, an aftershock density following Omori’s law and bivariate normal aftershock spatial distribution.

|     | $\alpha$ | $\beta$ | $p$  | $c$  | $\sigma_1$ | $\sigma_2$ | $A$  | $\delta$ |
|-----|----------|----------|------|------|------------|------------|------|---------|
| True| 0.5      | 0.5      | 2    | 0.01 | 0.01       | 0.02       | 0.5  | 1       |
| Est | 0.5014   | 0.5197   | 2.0148 | 0.0104 | 0.0104   | 0.0218   | 0.5045 | 0.9756  |
| SE  | 0.0313   | 0.0899   | 0.1668 | 0.0026 | 0.0013   | 0.0042   | 0.0455 | 0.2737  |
| $\hat{SE}$ | 0.0307 | 0.0865 | 0.1783 | 0.0026 | 0.0011 | 0.0023 | 0.0474 | 0.2894 |
| CP  | 0.9539   | 0.9609   | 0.9529 | 0.9449 | 0.8898   | 0.8317   | 0.9539 | 0.9579  |

Table 2: The percentage of residual series that reject the uniformity hypothesis.

|     | Temporal | Longitudinal | Latitudinal |
|-----|----------|--------------|-------------|
| True| 2        | 1            | 2           |
| Est | 1.9913   | 0.9986       | 2.0205      |
| SE  | 0.1306   | 0.0408       | 0.1558      |
| $\hat{SE}$ | 0.1218 | 0.0407 | 0.1554 |
| CP  | 0.9280   | 0.9460       | 0.9460      |

Furthermore, the 95% coverage probabilities are all reasonably close to their nominal level. Therefore, we conclude that the MLE provides a satisfactory finite sample performance, and the simulation algorithm presented in Section 5.1 is simulating from the correct model specification.

The appropriateness of the sequential goodness-of-fit test procedure introduced in Section 4 will now be assessed. To this end, the three residual series $\{U\}$, $\{V\}$, and $\{W\}$ for each simulated catalog are computed under two scenarios using the estimated parameters and the true parameters. For both situations, the residuals series are examined for uniformity using the K-S test and independence using the L-B test. The results are presented in Table 2, which reports the percentage of $p$-values for the several tests that are less than the nominal significance level as indicated on the left-hand panel for both the 5% and 1% level, for the temporal, longitudinal and latitudinal components.

The percentage of residual series that reject the uniformity or independence hypothesis is relatively close to the nominal levels indicating that the GOF test procedure is performing as expected at the given significance level. For the temporal component, the residuals series computed using the estimated parameters all pass the uniformity test even at the 5% level, which is to be expected since using the fitted parameters with a correctly specified model under the null hypothesis generally leads to inflated $p$-values in GOF tests. However, when the true parameters are used in place of the estimated parameters, the percentage of failed tests is much closer to the nominal lev-
Table 2: Percentage of K-S and L-B tests which lead to rejection of the null hypothesis for the two simulation models based on 1000 simulated datasets by computing residuals that are evaluated at both the estimated parameter values of the true simulated parameter values.

|          | U K-S | L-B | V K-S | L-B | W K-S | L-B |
|----------|-------|-----|-------|-----|-------|-----|
| Estimated parameter values |       |     |       |     |       |     |
| 5%       |       |     |       |     |       |     |
| Model 1  | 0.00% | 6.01% | 4.21% | 6.51% | 4.61% | 5.91% |
| Model 2  | 0.00% | 6.40% | 4.20% | 8.00% | 6.40% | 5.70% |
| 1%       |       |     |       |     |       |     |
| Model 1  | 0.00% | 2.10% | 0.80% | 2.20% | 0.60% | 1.20% |
| Model 2  | 0.00% | 1.70% | 0.70% | 2.20% | 1.60% | 1.40% |
| True parameter values |       |     |       |     |       |     |
| 5%       |       |     |       |     |       |     |
| Model 1  | 4.91% | 6.01% | 4.21% | 7.11% | 4.61% | 5.81% |
| Model 2  | 3.40% | 6.40% | 4.80% | 8.00% | 5.20% | 5.50% |
| 1%       |       |     |       |     |       |     |
| Model 1  | 1.10% | 2.10% | 0.90% | 2.40% | 0.30% | 1.60% |
| Model 2  | 0.80% | 1.70% | 0.80% | 2.20% | 1.10% | 1.20% |

In conclusion, the GOF tests based on these residuals appear to work well as expected.

6 Data Analysis

Many regions around the world have significantly distinct and unique seismic activity. The versatility of the RETAS model will be illustrated on various earthquake catalogs around the world, including New Zealand, Chile, Japan, and China. These catalogs will confirm that by introducing the renewal main-shock arrival process to the classical ETAS model improves the GOF. Since these two models are nested, the AIC is computed to establish that the RETAS model is the superior model of choice for these particular earthquake catalogs (when the self-exciting structure for aftershocks and the spatial variation for main-shocks have the same specification).

6.1 New Zealand earthquake catalog

In this subsection we investigate a large region that includes most of the seismically active regions of New Zealand. The earthquake catalog is compiled from the GeoNet Quake Search Database. The region under investigation is defined by the coordinates 164° – 182° N and 48° – 35° S which consists of the majority of New Zealand as seen in Figure 1. This region has previously
been studied in the work of Harte (2013, 2014). The catalog contains 463 earthquakes during the period 1997-01-01 until 2015-06-30 with threshold magnitude $m_0 = 5$. Figure 1 displays all earthquakes that have occurred since the 1800-01-01 until 2015-06-30, with the size of the circle indicating the relative size of the magnitude of the earthquake. Using historical and in-sample data we estimate the main-shock (background) spatial density by applying a two-dimensional Gaussian kernel density smoother to all the earthquakes that occurred in the region.

Figure 1: Earthquakes around New Zealand from 1800-01-01 until 2015-06-30 with the size of the circle indicating the relative size of the magnitude of the earthquake.

The renewal main-shock arrival process is fit with three different forms; exponential (classical ETAS), Weibull and gamma distributions. For each of the three main-shock renewal distributions, we report the estimation results in Table 3 which contains the estimated model parameters, the standard errors and the AIC estimate for the fitted model. General purpose optimization routines are used to obtain estimates of model parameter with the initial parameters set using a fit from a nested model. More specifically, we fit sequentially the following models: Poisson process, temporal Hawkes process, ETAS model, spatiotemporal ETAS model and RETAS model.
Table 3: Estimation results which includes parameter estimates and standard errors in parentheses for the New Zealand earthquake catalog with exponential (ETAS), Weibull and gamma RETAS models.

|       | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{p}$ | $\hat{c}$ | $\hat{\delta}_1$ | $\hat{\delta}_2$ | $\hat{A}$ | $\hat{\delta}$ | AIC   |
|-------|----------------|---------------|----------|---------|-----------------|-----------------|---------|-----------|-------|
| ETAS  | 1.095          | 0.065         | 0.017    | 0.217   | 1.719           |                 | 4599.21 |           |        |
|       | (1.66)         | (0.025)       | (0.00036)| (0.0088)| (0.0025)        | (0.044)         | (0.11)  |           |       |
| RETAS$_W$ | 0.864         | 23.00         | 1.128    | 0.0026  | 0.063           | 0.017           | 0.170   | 1.708     | 4590.99|
|       | (0.044)        | (1.79)        | (0.024)  | (0.0064)| (0.0085)        | (0.0024)        | (0.027) | (0.11)    |       |
| RETAS$_G$ | 0.752         | 33.425        | 1.092    | 0.0020  | 0.062           | 0.017           | 0.222   | 1.712     | 4588.95|
|       | (0.060)        | (3.68)        | (0.024)  | (0.0028)| (0.0084)        | (0.0024)        | (0.046) | (0.11)    |       |

The flexibility provided by the RETAS model is mainly derived from the parameters $\alpha$ and $\beta$ which describe the renewal main-shock arrival process and we examine these parameter first. The estimated shape parameters $\hat{\alpha} = 0.864$ and $\hat{\alpha} = 0.752$ for the Weibull and gamma RETAS models are both significantly less than one with 95% confidence intervals (0.778, 0.950) and (0.635, 0.869) respectively. This implies that the main-shock arrival process depart from homogenous temporal variation ($\alpha = 1$) and indicates that more substantial clustering is apparent than that of the classical ETAS model. The mean waiting time between consecutive main-shocks for the three models are all very close with 25.11, 24.78 and 25.14 days for the classical ETAS, Weibull, and gamma RETAS models respectively. However, the volatility of the main-shock waiting times are different. For instance, the standard deviation for these waiting times are 25.10 for the classical ETAS model but are higher for the Weibull and gamma RETAS model at 28.78 and 28.99, respectively.

The self-excitation component of each of the three fitted models have similar parameter estimates and infer comparable aftershock properties. The estimated parameters $\hat{p}$ and $\hat{c}$ are comparable among the models. Take for example the gamma RETAS model in which $\hat{p} = 1.092$ and $\hat{c} = 0.0020$. This implies that the aftershock occurs with probability 24.59% within the next hour, 43.47% within a day, 52.74% within a week and 67.15% within the next year after the triggering earthquake. The aftershocks generally occur shortly after the triggering earthquake but its effects can still last a long time thereafter. For instance, the median waiting time between an earthquake and its induced aftershock (if there is any) is 3 days and 19.13 hours for the gamma RETAS model, and this is significantly shorter than the median waiting time between main-shocks which is around 10.25 days. The estimated parameters $\hat{\delta}_1$ and $\hat{\delta}_2$ are also very close for all three models with $\hat{\delta}_1$ being more than three times larger than $\hat{\delta}_2$. The estimated boost function parameters $\hat{A}$ and $\hat{\delta}$ indicate the expected number of induced aftershocks from a single earthquake.
Table 4: GOF test results for the New Zealand earthquake catalog exponential (ETAS), Weibull and gamma RETAS models. This table contains the p-values for the K-S tests and L-B tests for the temporal, longitudinal, and latitudinal residuals, as well as the combined residual series.

| Model        | U   | V   | W   | Combined   |
|--------------|-----|-----|-----|------------|
|              | K-S | L-B | K-S | L-B | K-S | L-B | K-S | L-B |
| ETAS         | 0.023 | 0.145 | 0.224 | 0.754 | 0.937 | 0.119 | 0.072 | 0.251 |
| RETAS$_{Wei}$ | 0.159 | 0.058 | 0.120 | 0.693 | 0.902 | 0.104 | 0.135 | 0.117 |
| RETAS$_{Gam}$ | 0.360 | 0.035 | 0.188 | 0.715 | 0.907 | 0.113 | 0.255 | 0.107 |

of magnitude $m$. For example, an earthquake of magnitude 5, 5.5 and 6 are expected to induce 0.222, 0.522 and 1.229 aftershocks under the estimated gamma RETAS model. Under the Weibull RETAS model, these values are slightly smaller at 0.179, 0.421, and 0.988 respectively. Therefore, we can conclude that the choice of the renewal main-shock process is having a minimal impact on the temporal and spatial distribution of aftershocks around their triggering earthquake. However, even though the self-excited part of the models are similar, the identification of main-shocks and aftershocks will still be influenced by the main-shock arrival rate which changes depending on the time lapsed since the most recent main-shock.

For each model the temporal, longitudinal and latitudinal residual series are computed and examined for uniformity and independence using the K-S and L-B tests respectively. The p-values are reported in Table 4. The classical spatiotemporal ETAS model does not provide a satisfactory fit since the temporal residuals result in a rejection of the uniformity assumption as the 5% level with a p-value of only 0.023. Therefore, it is unable to describe the temporal pattern of the earthquake arrival times and additional flexibility in specifying the arrival time distribution between earthquakes is required.

The second observation is that the p-values from the K-S test for the temporal residual series of the RETAS model are notably higher and signify a meaningful improvement in fit for both the Weibull and gamma RETAS model. This shows that the RETAS model can sufficiently model the temporal variation of earthquakes for this New Zealand catalog. However, the p-value from the L-B test falls just short of the 5% significance level for the gamma RETAS model. For the spatial components, both the longitudinal and latitudinal residuals indicate that all three models are providing similar fits with p-values greater than 5% for both uniformity and independence. However, from the AIC values reported in Table 3, the two RETAS models have markedly smaller AIC values, with that of the gamma RETAS model slightly smaller than the Weibull RETAS model. Therefore, by combining the
GOF test results and the AIC values, we conclude that the gamma RETAS model is the favored model to describe the seismicity for this New Zealand earthquake catalog.

6.2 Earthquake catalogs around the world

In this subsection, we study earthquake catalogs from various regions around the world to confirm the utility of the RETAS model in modeling diverse range of seismically active regions. The section includes an analysis of the following earthquake catalogs:

- **Japan**: This catalog contains 577 earthquakes in the rectangular region defined by 141° – 145° E and 36° – 42° N during the period 1980-01-01 until 2015-06-30 with threshold magnitude $m_0 = 5.5$. This earthquake catalog was obtained from the Japan Meteorological Agency (JMA) database.

- **Chile**: This catalog contains 370 earthquakes in the rectangular region defined by 76° – 64° W and 40° – 18° S from 1997-01-01 until 2015-06-30 with threshold magnitude $m_0 = 5.5$. The data was sourced from the United States Geological Survey (USGS) database.

- **China**: This catalog contains 627 earthquakes in the region 97° – 107° E and 26° – 34° N, from 1997-01-01 to 2015-06-30 with a threshold magnitude of $m_0 = 4.5$. This data was also obtained from the USGS database.

For each of the three earthquake catalogs, we fit all three RETAS models and summarize the results in Table 5. For all three earthquake catalogs the AIC value is significantly smaller for the RETAS model than the classical ETAS model with the gamma RETAS model performing the best based on this criterion. The estimated shape parameter of the gamma renewals $\alpha$ is statistically less than one in all data sets, suggesting that the earthquakes in these catalogs are more heavily clustered than implied by the ETAS model. By examining the estimation results for the Japan earthquake catalog we observe that there is a noticeable difference in the estimated value $\hat{p}$ for each of the three models. For instance, in the classical ETAS model, $\hat{p} = 1.004$ which is the smallest estimated value, while for the gamma RETAS model $\hat{p} = 1.082$ is significantly higher. This implies that for the RETAS models, the induced shocks happen much quicker than in the classical ETAS model.

The estimated spatial response function which describes the distribution of aftershocks around their triggering earthquake is very similar in each the
Table 5: Estimation results including parameter estimates and standard errors in parentheses for the Japan, Chile and China earthquake catalog with exponential (ETAS), Weibull and gamma RETAS models.

|       | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{p}$ | $\hat{c}$ | $\hat{\sigma}^2_1$ | $\hat{\sigma}^2_2$ | $\hat{A}$ | $\hat{\delta}$ | AIC       |
|-------|----------------|---------------|-----------|------------|---------------------|---------------------|---------|--------------|-----------|
| Japan |                |               |           |            |                     |                     |         |              |           |
| ETAS  | 1              | 8.97          | 1.004     | 0.0024     | 0.084              | 0.024               | 9.864   | 0.893        | 3947.68   |
|       | (NA)           | (8.54)        | (0.00044) | (0.00041)  | (0.019)             | (0.0033)            | (1.451) | (0.100)      |           |
| RETAS  | 0.887          | 70.34         | 1.101     | 0.0059     | 0.084              | 0.024               | 0.664   | 0.814        | 3925.12   |
|       | (0.084)        | (8.55)        | (0.021)   | (0.016)    | (0.018)             | (0.0032)            | (0.097) | (0.112)      |           |
| RETAS  | 0.562          | 134.56        | 1.082     | 0.0052     | 0.075              | 0.023               | 0.727   | 0.829        | 3911.95   |
|       | (0.068)        | (22.75)       | (0.021)   | (0.014)    | (0.013)             | (0.0029)            | (0.128) | (0.108)      |           |
| Chile  |                |               |           |            |                     |                     |         |              |           |
| ETAS  | 1              | 39.63         | 1.043     | 0.0079     | 0.039              | 0.122               | 0.840   | 0.854        | 4320.26   |
|       | (NA)           | (3.61)        | (0.031)   | (0.0032)   | (0.0061)            | (0.027)             | (0.486) | (0.135)      |           |
| RETAS  | 0.708          | 32.37         | 1.018     | 0.0068     | 0.037              | 0.056               | 1.779   | 0.834        | 4303.60   |
|       | (0.052)        | (3.91)        | (0.015)   | (0.0025)   | (0.0051)            | (0.012)             | (1.406) | (0.127)      |           |
| RETAS  | 0.564          | 69.68         | 1.016     | 0.0068     | 0.036              | 0.053               | 1.951   | 0.853        | 4296.86   |
|       | (0.060)        | (10.98)       | (0.012)   | (0.0024)   | (0.0051)            | (0.012)             | (1.436) | (0.125)      |           |
| China  |                |               |           |            |                     |                     |         |              |           |
| ETAS  | 1              | 43.02         | 1.103     | 0.016      | 0.044              | 0.0094              | 0.688   | 1.152        | 3238.89   |
|       | (NA)           | (4.02)        | (0.024)   | (0.0045)   | (0.0036)            | (0.0012)            | (0.098) | (0.085)      |           |
| RETAS  | 0.618          | 36.90         | 1.118     | 0.013      | 0.044              | 0.0090              | 0.614   | 1.184        | 3232.49   |
|       | (0.059)        | (4.24)        | (0.022)   | (0.0048)   | (0.0036)            | (0.0013)            | (0.069) | (0.082)      |           |
| RETAS  | 0.694          | 60.20         | 1.100     | 0.016      | 0.044              | 0.0087              | 0.668   | 1.207        | 3229.45   |
|       | (0.081)        | (9.21)        | (0.024)   | (0.0046)   | (0.0036)            | (0.0012)            | (0.098) | (0.078)      |           |

Discussion

This article proposed an improved version of the classical ETAS model, in which the homogenous main-shock arrival process is replaced by inhomogeneous main-shock arrival process in the form of a renewal process. The three models and with each catalog, and this is consistent with the New Zealand catalog presented in the previous section except for the longitudinal dispersion parameter for the Chile catalog which has an estimated value that is more than double that of the RETAS models. However, unlike the fitted models for the New Zealand catalog, there are significant variations in the estimated values of the parameter $A$ and $\delta$ between the three models. For instance, for the Japan dataset, the estimated parameters for the classical ETAS model are $\hat{A} = 9.864$ and $\hat{\delta} = 0.893$ while for the gamma RETAS model they are both considerably smaller with $\hat{A} = 0.727$ and $\hat{\delta} = 0.829$. This implies that the classical ETAS model has a larger number of expected aftershocks induced from smaller magnitude earthquakes, and since $\delta$ is also bigger it implies that more aftershocks on average are induced from earthquakes of any magnitude.
RETAS model was applied to several earthquake catalogs from around the world including New Zealand, Japan, Chile, and China. These earthquake catalogs revealed that the RETAS model had a significant influence on the data fitting of main-shocks compared to the classical ETAS model. In fact, the classical ETAS model was shown to not be suitable for the New Zealand earthquake catalog due to its simple assumption about main-shock arrival times. The RETAS model overcame this shortfall and provided a superior quality of fit, as indicated by the newly proposed sequential GOF test procedure and the AIC value. After incorporating both Weibull and gamma renewal distributions, the estimated shape parameters $\hat{\alpha}$ of the main-shock renewal distributions (which is restricted to one in the classical ETAS model) become significantly smaller than one, around 0.5-0.8, indicating a much stronger clustering of main-shock earthquakes.

We also fit the RETAS model to two other seismically active regions around the world, in California and Italy for the period 1997-01-01 to 2015-06-30 with a threshold magnitude $m_0 = 4.5$ and $m_0 = 4$, respectively. For the California earthquake catalogue, the estimated shape parameter for the gamma main-shock renewal distribution was $\hat{\alpha} = 0.795$, which is again statistically significantly less than one. The AIC value was 2370.71, which is slightly smaller than the classical ETAS model at 2373.42. However, the parameter controlling the aftershock decay was estimated to be very close to one using the density function $g$. Therefore, to allow $p$ to have a values smaller than one, we use the temporal response function $g(t) = (1 + \frac{t}{c})^{-p}$ and found $\hat{p} = 0.964$. However, this means that $g$ can no longer be normalized to a properly density function since its integral on $(0, \infty)$ diverges. The approximate Hessian matrix was also not invertible and therefore estimates of the standard errors were easily obtainable.

For the Italian earthquake catalog, the estimated shape parameter of the gamma RETAS model was $\hat{\alpha} = 1.041$, which is slightly bigger than one. This indicates that the renewal main-shock arrival process is not too dissimilar to a homogeneous Poisson process, and the classical ETAS model would be appropriate model for this catalog. This is another advantage of the RETAS model since it can help in determining any departure or non-departure from a homogenous main-shock arrival process. This is further reinforced by the AIC values, which determines that the classical ETAS model is the superior model of choice when accounting for both the adequacy of model fit and model complexity (number of parameters).

A more meaningful analysis of an earthquake catalog using the RETAS model could be performed by utilizing a probability based stochastic declustering algorithm, in which each earthquake is assigned a probability to be either a main-shock or an aftershock induced by a previous earthquake. For
the classical ETAS model, this is straightforward to perform since the past points of the process are conditionally independent of future points conditioned on the history. However, this is not exactly true for the RETAS model, and therefore the probabilities required for the declustering algorithm must be conditioned on the complete earthquake catalog rather than only the past points. This provides an avenue for future research.

References

Chen, F. and Hall, P. (2013). Inference for a nonstationary self-exciting point process with an application in ultra-high frequency financial data modeling. *J. Appl. Probab.*, 50(4):1006–1024.

Chen, F. and Stindl, T. (2018). Direct likelihood evaluation for the renewal Hawkes process. *Journal of Computational and Graphical Statistics*, 27(1):119–131.

Console, R. and Murru, M. (2001). A simple and testable model for earthquake clustering. *Journal of Geophysical Research: Solid Earth*, 106(B5):8699–8711.

Console, R., Murru, M., and Lombardi, A. M. (2003). Refining earthquake clustering models. *Journal of Geophysical Research: Solid Earth*, 108(B10).

Console, R., Rhoades, D. A., Murru, M., Evison, F. F., Papadimitriou, E. E., and Karakostas, V. G. (2006). Comparative performance of time-invariant, long-range and short-range forecasting models on the earthquake catalogue of Greece. *Journal of Geophysical Research: Solid Earth*, 111(B9).

Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes Volume I: Elementary Theory and Methods*. Springer-Verlag, New York, 2nd edition.

Godoy, B. I., Solo, V., Min, J., and Pasha, S. A. (2016). Local likelihood estimation of time-variant Hawkes models. In 2016 *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 4199–4203.

Gutenberg, B. and Richter, C. F. (1944). Frequency of earthquakes in California. *Bulletin of the Seismological Society of America*, 34(4):185–188.
Harte, D. S. (2013). Bias in fitting the ETAS model: a case study based on New Zealand seismicity. *Geophysical Journal International*, 192(1):390–412.

Harte, D. S. (2014). An ETAS model with varying productivity rates. *Geophysical Journal International*, 198(1):270–284.

Hawkes, A. G. and Oakes, D. (1974). A cluster process representation of a self-exciting process. *Journal of Applied Probability*, 11(3):493–503.

Helmstetter, A., Kagan, Y. Y., and Jackson, D. D. (2006). Comparison of Short-Term and Time-Independent Earthquake Forecast Models for Southern California. *Bulletin of the Seismological Society of America*, 96(1):90–106.

Kolev, A. A. and Ross, G. J. (2018). Inference for ETAS models with non-poissonian mainshock arrival times. *Statistics and Computing*.

Kumazawa, T. and Ogata, Y. (2014). Nonstationary ETAS models for non-standard earthquakes. *Ann. Appl. Stat.*, 8(3):1825–1852.

Marzocchi, W. and Lombardi, A. M. (2009). Real-time forecasting following a damaging earthquake. *Geophysical Research Letters*, 36(21).

Ogata, Y. (1988). Statistical models for earthquake occurrences and residual analysis for point processes. *Journal of the American Statistical Association*, 83(401):9–27.

Ogata, Y. (1998). Space-time point-process models for earthquake occurrences. *Annals of the Institute of Statistical Mathematics*, 50(2):379–402.

Ogata, Y. (2004). Space-time model for regional seismicity and detection of crustal stress changes. *Journal of Geophysical Research: Solid Earth*, 109(B3).

Omori, F. (1894). On the aftershocks of earthquakes. *Journal of the College of Science, Imperial University of Tokyo*, 7:111–120.

Rosenblatt, M. (1952). Remarks on a multivariate transformation. *The Annals of Mathematical Statistics*, 23(3):470–472.

Schoenberg, F. P. (2003). Multidimensional residual analysis of point process models for earthquake occurrences. *Journal of the American Statistical Association*, 98(464):789–795.
Utsu, T. (1961). A statistical study of the occurrence of aftershocks. *Geophysical Magazine*, 30:521–605.

Werner, M. J., Helmstetter, A., Jackson, D. D., and Kagan, Y. Y. (2011). High-Resolution Long-Term and Short-Term Earthquake Forecasts for California. *Bulletin of the Seismological Society of America*, 101(4):1630–1648.

Wheatley, S., Filimonov, V., and Sornette, D. (2016). The Hawkes process with renewal immigration & its estimation with an EM algorithm. *Computational Statistics & Data Analysis*, 94:120 – 135.

Zhuang, J., Chang, C.-P., Ogata, Y., and Chen, Y.-I. (2005). A study on the background and clustering seismicity in the Taiwan region by using point process models. *Journal of Geophysical Research: Solid Earth*, 110(B5).

Zhuang, J., Christophersen, A., Savage, M. K., Vere-Jones, D., Ogata, Y., and Jackson, D. D. (2008). Differences between spontaneous and triggered earthquakes: Their influences on foreshock probabilities. *Journal of Geophysical Research: Solid Earth*, 113(B11).

Zhuang, J., Ogata, Y., and Vere-Jones, D. (2002). Stochastic declustering of space-time earthquake occurrences. *Journal of the American Statistical Association*, 97(458):369–380.

Zhuang, J., Ogata, Y., and Vere-Jones, D. (2004). Analyzing earthquake clustering features by using stochastic reconstruction. *Journal of Geophysical Research: Solid Earth*, 109(B5).