The Dirichlet series that generates the Möbius function is the inverse of the Riemann zeta function in the right half of the critical strip

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Abstract:
In this paper I introduce a criterion for the Riemann hypothesis, and then using that I prove $\sum_{k=1}^{\infty} \mu(k)/k^s$ converges for $\Re(s) > \frac{1}{2}$. I use a step function $\nu(x) = 2\{x/2\} - \{x\}$ for the Dirichlet eta function ($\{x\}$ is the fractional part of $x$), which was at the core of my investigations, and hence derive the stated result subsequently.
In 1859, Bernhard Riemann showed the existence of a deep relationship between two very different mathematical entities, viz. the zeros of an analytic function and prime numbers.

The Riemann Hypothesis is usually stated as, the non-trivial zeros of the Riemann zeta function lie on the line $\Re(s) = \frac{1}{2}$. Although, this is the standard formulation, one of the exciting features of this problem is, it can be formulated in many different and unrelated ways.

The approach I take in this paper is influenced by Beurling’s 1955 paper: A closure problem related to the Riemann zeta function and Báez-Duarte’s 2001 paper: New versions of the Nyman-Beurling criterion for the Riemann Hypothesis, although it takes a new approach. In this paper I would be studying a simple step function $\nu$ relating it to the Dirichlet eta function $\eta$. I will show how the step function $\nu$ convolves with the M"obius function $\mu(n)$ and gives a constant, which I think is a new result significant at attacking RH.

**Theorem 1:** For the Dirichlet eta function is defined as, for all $\Re(s) > 0$

$$\eta(s) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} - \frac{1}{(2k)^s}$$

We get an equivalent expression in the form for all $\Re(s) > 0$ where $\nu(x) = 2 \{x/2\} - \{x\}$, and the expression being given by

$$\eta(s) = s \int_{1}^{\infty} \nu(x)x^{-s-1}dx$$

**(1)**

**Proof:** A simplification of the integral shall prove this case. We have for all $x \in [1, \infty]$ $\nu(x) = 0$ or $1$. It is not hard to see that $\nu(x) = 0$ whenever $x \in [2k, 2k+1)$ and $1$ whenever $x \in [2k-1, 2k)$ for all positive integers $k$. Hence, we can write the integral as

$$s \int_{1}^{\infty} \nu(x)x^{-s-1}dx = \sum_{k=1}^{\infty} \int_{2k-1}^{2k} s x^{-s-1}dx$$

Giving us the sum

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} - \frac{1}{(2k)^s}$$

which is nothing but $\eta(s)$. Since we already know that this sum converges for $\Re(s) > 0$, we get our result.
Alternatively, we can write equation (1) as
\[ \eta(s) = \int_0^1 \nu \left( \frac{1}{x} \right) x^{s-1} dx \]  
(2)

**Discussion:** \( \nu(x) \) is a simple function that oscillates between 0 and 1 at every integer. As mentioned in the above theorem, \( \nu(x) = 0 \) whenever \( x \in [2k, 2k + 1) \) and 1 whenever \( x \in [2k - 1, 2k) \) for all positive integers \( k \). Note that \( \nu(x/k) = 0 \) if \( x < k \). The next theorem relates \( \nu \) and Möbius \( \mu \) in a very interesting way. I think this might be a new and interesting result, where relating \( \mu \) with the simple oscillating step function gives a constant value of \(-1\). I think this result is significant for the proof of RH.

**Theorem 2:**
\[ \sum_{k=1}^{\infty} \mu(k) \nu \left( \frac{x}{k} \right) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x \in [1, 2) \\ -1 & \text{if } x \in [2, \infty) \end{cases} \]

**Proof:**
We have for \( 0 < \theta \leq 1 \)
\[ \int_0^\theta \nu \left( \frac{\theta}{x} \right) x^{s-1} dx = \frac{\theta^s \eta(s)}{s} \]
Since \( \nu(\theta/x) = 0 \) where \( \theta < x \), or \( \theta = 1 \) otherwise, hence
\[ \int_0^1 \nu \left( \frac{\theta}{x} \right) x^{s-1} dx = 0 \]
We get the following important equation,
\[ \int_0^1 \nu \left( \frac{\theta}{x} \right) x^{s-1} dx = \frac{\theta^s \eta(s)}{s} \]  
(3)

If we set \( f_\mu(x) = \sum_{k=1}^{\infty} \mu(k) \nu(x/k) \) we get
\[ \int_0^1 f_\mu \left( \frac{1}{x} \right) x^{s-1} dx = \frac{\eta(s)}{s} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \]  
(4)

(Justification for the exchange of summation and integral: Let \( f_n(x) = \sum_{k=1}^{n} \mu(k) \nu(x/k) \). Now since for \( \Re(s) > 1 \), we have \( \sum_{k=1}^{n} \int_0^\infty |\mu(k)\nu(x/k)x^{s-1}| dx \leq \sum_{k=1}^{n} \frac{\mu(k)}{k^s} \int_0^\infty \nu(x)x^{s-1} dx \leq \frac{\zeta(s)\zeta(2s)}{\zeta(2^{s})} < \infty \). By Fubini-Tonelli’s theorem we can say, \( \int_0^\infty \sum_{k=1}^{\infty} \mu(k) \nu(x/k)x^{s-1} dx = \)
\[ \sum_{k=1}^{\infty} \mu(k) \nu(x/k)x^{-s-1}dx. \]

For \( \Re(s) > 1 \) we know that \( \sum_{k=1}^{\infty} \mu(k)/k^s = 1/\zeta(s) \). Hence, for \( \Re(s) > 1 \)
\[ \int_{0}^{1} f_{\mu} \left( \frac{1}{x} \right) x^{s-1}dx = \frac{1 - 2^{-1-s}}{s} \]

(5)

Now, since \( f_{\mu}(x) = 1 \) whenever \( x \in [1, 2) \) giving us for all \( \Re(s) > 1 \)
\[ \int_{0}^{\frac{1}{2}} \left( 1 + f_{\mu} \left( \frac{1}{x} \right) \right)x^{s-1}dx = 0 \]

(6)

Since, \( \sum \mu(k)\nu(x/k) \) is always constant in any given \([n, n+1)\). So due to equation (6) we have for all \( x \geq 2 \), \( f_{\mu}(x) = -1 \). (Properties of Dirichlet series).

**Discussion:** The derivation is not explained above. Notice, the integral in equation (6) above can be expanded as,
\[ -\frac{1 - f_{\mu}(2)}{2^s} + \frac{f_{\mu}(2) - f_{\mu}(3)}{3^s} + \frac{f_{\mu}(3) - f_{\mu}(4)}{4^s} + \ldots = 0 \]

The uniqueness property implies, \(-1 - f_{\mu}(2) = f_{\mu}(2) - f_{\mu}(3) = \ldots = 0 \). Since \( f_n(x) \) is a step function, we get the result for \( x \geq 2 \).

**Theorem 3:** If \( f_n = O(n^{s_0+\varepsilon}) \) where \( f_n = |\sup f_n(x)| \), then \( \sum_{k=1}^{\infty} \mu(k)/k^s \) converges for all \( \Re(s) > \sigma_0 \).

**Proof:** If we set \( f_n(x) = \sum_{k=1}^{n} \mu(k)\nu(x/k) \) we get for \( \Re(s) > 0 \)
\[ \int_{0}^{\frac{1}{2}} \left( 1 + f_n \left( \frac{1}{x} \right) \right)x^{s-1}dx = \frac{\eta(s)}{s} \sum_{k=1}^{n} \frac{\mu(k)}{k^s} - \frac{1 - 2^{-1-s}}{s} \]

(7)

Changing the integral we get
\[ \int_{2}^{\infty} \frac{1 + f_n(x)}{x^{s+1}}dx = \frac{\eta(s)}{s} \sum_{k=1}^{n} \frac{\mu(k)}{k^s} - \frac{1 - 2^{-1-s}}{s} \]

(8)

Since, \( 1 + f_n(x) = 0 \) for \( x \in [2, n) \), it follows from above,
\[ \left| \frac{\eta(s)}{s} \sum_{k=1}^{n} \frac{\mu(k)}{k^s} - \frac{1 - 2^{-1-s}}{s} \right| = \int_{2}^{\infty} \frac{|1 + f_n(x)|}{x^{s+1}}dx = \int_{n}^{\infty} \frac{|1 + f_n(x)|}{x^{s+1}}dx \leq \sup \frac{1 + f_n(x)}{\sigma n^\sigma} \]

(9)
If \( f_n = O(n^{\sigma_0 + \epsilon}) \) then LHS converges to 0 for \( \sigma > \sigma_0 \) as \( n \to \infty \), i.e.,

\[
\lim_{n \to \infty} \left| \frac{\eta(s)}{s} \sum_{k=1}^{n} \frac{\mu(k)}{k^s} - \frac{1}{s} - \frac{2^{1-s}}{s} \right| = 0
\]

which gives, for \( \Re(s) > \sigma_0 \)

\[
\frac{\eta(s)}{s} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} = \frac{1}{s} - \frac{2^{1-s}}{s}
\]

Now here \( \eta(s) \neq 0 \) for \( 1 > \Re(s) > \sigma_0 \). (This is because of, equation (9) with \( \eta(s) = 0 \) gives LHS = \( \frac{1}{s} \). Ignoring the line \( \Re(s) = 1 \), because it is known that Möbius converges on that line.)

Since in equation (11), \( \eta(s) \neq 0 \), therefore \( \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \) converges for \( \Re(s) > \sigma_0 \).

**Theorem 4:** \( \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \) converges for \( \Re(s) > \frac{1}{2} \)

**Proof:**

For any \( \sigma > 0 \), \( \sigma \neq 1 \) the following is justified geometrically.

\[
\sum_{k=1}^{n} \frac{|\mu(k)|}{k^\sigma} \leq \int_{1}^{n} \frac{dy}{y^\sigma} + 1 = \frac{n^{1-\sigma}}{1-\sigma} - \frac{1}{1-\sigma} + 1 = O(n^{1-\sigma})
\]

For \( \sigma = 1 \) we shall similarly have for any \( \epsilon > 0 \), \( \sum_{k=1}^{n} |\mu(k)|/k \leq \log(n) + 1 = O(n^\epsilon) \)

I. Assume \( 1 \geq \sigma_0 > 1/2 \) such that, \( \sum_{k=1}^{\infty} \frac{|\mu(k)|}{k^s} \) converges only for \( \Re(s) > \sigma_0 \).

II. Now consider for \( \sigma > 0 \),

\[
\left| \int_{1}^{\infty} \frac{f_n(x)}{x^{\sigma+1}} \, dx \right| \leq \int_{1}^{\infty} \left| \frac{f_n(x)}{x^{\sigma+1}} \right| \, dx = \int_{0}^{\infty} \left| \frac{f_n(x)}{x^{\sigma+1}} \right| \, dx \leq \int_{0}^{\infty} \sum_{k=1}^{n} |\mu(k)|/x^{\sigma+1} \, dx = \frac{\eta(\sigma)}{\sigma} \sum_{k=1}^{n} \frac{|\mu(k)|}{k^\sigma} \leq cn^{1-\sigma}
\]

Assume \( \sigma_0 > \sigma' > 1/2 \)

\[
\frac{1}{n^{\sigma'}} \left| \int_{1}^{\infty} \frac{f_n(x)}{x^{\sigma+1}} \, dx \right| \leq \frac{1}{n^{\sigma'}} \int_{1}^{\infty} \left| \frac{f_n(x)}{x^{\sigma+1}} \right| \, dx \leq cn^{1-\sigma' - \sigma}
\]
Now as \( n \to \infty \), equation (12) \( \to 0 \) for all \( \sigma > 1 - \sigma' \). If we consider

\[
g_n(x) = |f_n(x)|/n^{\sigma'}
\]

then the Mellin transform over \( g_n(x) \) can be expressed as a Dirichlet series for \( \sigma > 1 - \sigma' \), because \( g_n(x) \) is a step function (since \( f_n(x) \) is a step function) and the transform converges.

\[
D_n = \int_1^\infty g_n(x) \frac{dx}{x^{\sigma+1}} = \sum_{k=1}^{\infty} \frac{a_{n,k}}{k^{\sigma}}
\]

Since the Dirichlet series \( D_n \to 0 \) for all \( \sigma > 1 - \sigma' \) as \( n \to \infty \), hence \( a_{n,k} \to 0 \) as \( n \to \infty \) (by the uniqueness property of Dirichlet series, i.e., the series vanishes identically). Giving us \( g_n(1) \to 0, g_n(1) - g_n(2) \to 0, g_n(2) - g_n(3) \to 0 \), and so on... as \( n \to \infty \). Hence \( g_n(x) \to 0 \) as \( n \to \infty \), and therefore

\[
f_n = |\sup f_n(x)| = o(n^{\sigma'})
\]

III. Now considering theorem 3 and the result in II., we get \( \sum_{k=1}^{\infty} \mu(k)/k^s \) converges for all \( \Re(s) > \sigma' \).

IV. But this contradicts our assumption that \( \sum_{k=1}^{\infty} \mu(k)/k^s \) converges only for \( \Re(s) > \sigma_0 > 1/2 \), since \( \sigma_0 > \sigma' > 1/2 \). Therefore, we must have \( \sum_{k=1}^{\infty} \mu(k)/k^s \) converges for all \( \Re(s) > 1/2 \), thereby validating the Riemann hypothesis.

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