ON THE SPECTRUM OF DIOPHANTINE APPROXIMATION CONSTANTS

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Abstract. The approximation constant \( \lambda_k(\zeta) \) is defined as the supremum of real \( \eta \) such that \( \|\zeta^j x\| \leq x^{-\eta} \) for \( 1 \leq j \leq k \) has infinitely many integer solutions \( x \). Here \( \| \cdot \| \) denotes the distance to the closest integer. We establish a connection on the joint spectrum \( (\lambda_1(\zeta), \lambda_2(\zeta), \ldots) \) which will lead to various improvements of known results on the individual spectrum of the approximation constants \( \lambda_k(\zeta) \) as well. In particular, for given \( k \geq 1 \) and \( \lambda \geq 1 \), we construct \( \zeta \) in the Cantor set with \( \lambda_k(\zeta) = \lambda \).

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1. Introduction and main results

1.1. Definition of the constants. We begin with the definition of the quantities \( \lambda_k(\zeta) \) that we will predominately consider, and their uniform versions \( \hat{\lambda}_k(\zeta) \). We denote by \( \lambda_k(\zeta) \) resp. \( \hat{\lambda}_k(\zeta) \) the supremum of \( \eta \in \mathbb{R} \) such that the system

\[
| x | \leq X, \quad \max_{1 \leq j \leq k} | \zeta^j x - y_j | \leq X^{-\eta},
\]

has a solution \( (x, y_1, y_2, \ldots, y_k) \in \mathbb{Z}^{k+1} \) for arbitrarily large \( X \) resp. for all \( X \geq X_0 \). We will relate to the constants \( \lambda_k(\zeta), \hat{\lambda}_k(\zeta) \) at some places, similarly defined by replacing \( \zeta^j \) by \( \zeta_j \) for the more general case of \( \zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^k \), which we will assume to be \( \mathbb{Q} \)-linearly independent together with 1. Most results in the current Section 1.1 basically stay valid in this more general setting.

By Dirichlet’s box principle, we have

\[
\frac{1}{k} \leq \hat{\lambda}_k(\zeta) \leq \lambda_k(\zeta) \leq \infty
\]

for all \( \zeta \in \mathbb{R} \).

Although this paper is mainly dedicated to the quantities \( \lambda_k(\zeta) \), our results relate to the somehow dual constants \( w_k(\zeta) \) and the uniform constant \( \hat{w}_k(\zeta) \) as well, see Section 4.2. The quantity \( w_k(\zeta) \) resp. \( \hat{w}_k(\zeta) \) is defined as the supremum of real \( \nu \) such that

\[
\max_{0 \leq j \leq k} | x_j | \leq X, \quad | x_0 + \zeta x_1 + \cdots + \zeta^k x_k | \leq X^{-\nu},
\]
has a solution \((x_0, x_1, \ldots, x_k) \in \mathbb{Z}^{k+1}\) for arbitrarily large \(X\) resp. for all \(X \geq X_0\).

Dirichlet’s Theorem yields

\[
(2) \quad k \leq \hat{w}_k(\zeta) \leq w_k(\zeta) \leq \infty.
\]

It follows from the definitions that for any \(\zeta \in \mathbb{R}\) we have the chains of inequalities

\[
(3) \quad \cdots \leq \lambda_3(\zeta) \leq \lambda_2(\zeta) \leq \lambda_1(\zeta) = w_1(\zeta) \leq w_2(\zeta) \leq w_3(\zeta) \leq \cdots,
\]

\[
\cdots \leq \hat{\lambda}_3(\zeta) \leq \hat{\lambda}_2(\zeta) \leq \hat{\lambda}_1(\zeta) = \hat{w}_1(\zeta) \leq \hat{w}_2(\zeta) \leq \hat{w}_3(\zeta) \leq \cdots.
\]

Moreover, the approximation constants are invariant under rational transformation, i.e.

\[
(4) \quad \lambda_k(\zeta) = \lambda_k(\alpha \zeta + \beta), \quad w_k(\zeta) = w_k(\alpha \zeta + \beta), \quad \alpha, \beta \in \mathbb{Q},
\]

\[
\hat{\lambda}_k(\zeta) = \hat{\lambda}_k(\alpha \zeta + \beta), \quad \hat{w}_k(\zeta) = \hat{w}_k(\alpha \zeta + \beta), \quad \alpha, \beta \in \mathbb{Q}.
\]

Khintchine’s transference principle \([12]\) allows a connection between the constants \(\lambda_k\) and the constants \(w_k\). Indeed, the original version is shown to be equivalent to

\[
(5) \quad \frac{w_k(\zeta)}{(k-1)w_k(\zeta) + k} \leq \lambda_k(\zeta) \leq \frac{w_k(\zeta) - k + 1}{k}
\]

for all \(\zeta \in \mathbb{R}\) in \([3]\). The analogue for the uniform constants holds as well \([9]\), i.e.

\[
(6) \quad \frac{\hat{w}_k(\zeta)}{(k-1)\hat{w}_k(\zeta) + k} \leq \hat{\lambda}_k(\zeta) \leq \frac{\hat{w}_k(\zeta) - k + 1}{k}.
\]

Consequently, the relations

\[
(7) \quad \lambda_k(\zeta) = \frac{1}{k} \iff w_k(\zeta) = k, \quad \hat{\lambda}_k(\zeta) = \frac{1}{k} \iff \hat{w}_k(\zeta) = k
\]

hold for any positive integer \(k\) and any real \(\zeta\). It is not hard to construct \(\zeta\) such that the asymptotic constants \(\lambda_k(\zeta), w_k(\zeta)\) take the value \(\infty\) (which for given \(\zeta\) is true for all \(k\) simultaneously or for none by Corollary 2 in \([8]\)), which one may deduce from the following Theorem \([1,26]\). The uniform constants on the other hand can be effectively bounded. An elementary result by Khintchine \([12]\) implies that for any irrational \(\zeta\), the one-dimensional uniform approximation constants are given as

\[
(8) \quad \hat{\lambda}_1(\zeta) = 1 = \hat{w}_1(\zeta).
\]

For \(k = 2\) and \(\zeta\) neither rational nor quadratic irrational, we have

\[
\hat{\lambda}_2(\zeta) \leq \frac{\sqrt{5} - 1}{2}, \quad \hat{w}_2(\zeta) \leq \frac{3 + \sqrt{5}}{2},
\]

and these constants are optimal \([19]\). There is equality for so called extremal numbers that can be explicitly constructed, we will discuss them some more in Section \([4,1]\). For \(k \geq 3\) and \(\zeta\) not algebraic of degree \(\leq k\), the upper bounds

\[
(9) \quad \hat{\lambda}_k(\zeta) \leq \frac{1}{\left\lceil \frac{k}{2} \right\rceil}, \quad \hat{w}_k(\zeta) \leq 2k - 1
\]

are known, with a refinement for \(k = 3\) due to Roy \([20]\). The left estimate is a slight refinement due to \([14]\) of results from \([10]\), the right follows from Theorem 2b in \([10]\). These bounds are not supposed to be optimal. We will deal with this topic in Section \([4,1]\).

Finally, we quote Roth’s Theorem \([18]\), which we will apply in Section \([4,2]\).
Theorem 1.1 (Roth). Any algebraic irrational \( \zeta \) satisfies \( \lambda_1(\zeta) = w_1(\zeta) = 1 \).

1.2. Problems from \[8\] and partial results. We now introduce problems posed as Problem 1, Problem 2, Problem 3 in \[8\] and known partial results to this problems, which we will deal with.

Problem 1.2. Let \( k \) be a positive integer. Is the spectrum of the function \( \lambda_k \) equal to \([1/k, \infty]\)?

Problem 1.3. Let \( k \) be a positive integer and \( \lambda \geq 1/k \). Determine the Hausdorff dimensions of the sets

\[ \{ \zeta \in \mathbb{R} : \lambda_k(\zeta) = \lambda \}, \quad \{ \zeta \in \mathbb{R} : \lambda_k(\zeta) \geq \lambda \} \]

Problem 1.4. Given any sequence \( (\lambda_k)_{k \geq 1} \) of non-increasing real numbers satisfying

\[ \lambda_k \geq \frac{1}{k}, \quad k \geq 1, \]

and

\[ \lambda_{kn} \geq \frac{\lambda_n - k + 1}{k}, \quad n \geq 1, k \geq 1. \]

Does there exists a real number \( \zeta \) with \( \lambda_k(\zeta) = \lambda_k \) for all \( k \geq 1 \)?

Next we recall some notes and known partial results on the problems respectively, that can be found in \[8\] as well with the exception of Theorem 1.9.

A positive answer to Problem 1.2 has been established only for \( k \in \{1, 2\} \). In this cases it is a consequence of the subsequent Theorems 1.5, 1.6, 1.7. Up to now, for general \( k \) it is has been shown in Theorem 2 in \[8\] that the spectrum contains the interval \([1, \infty]\). We will not improve Problem 1.2 though.

We turn to the metrical results. For \( k = 1 \) the following result concerning Problem 1.3 is due to Jarnik \[11\].

Theorem 1.5 (Jarnik). Let \( \lambda \geq 1 \). Then the Hausdorff dimension of the set of real \( \zeta \) with \( \lambda_1(\zeta) = \lambda \) equals \( 2/(1 + \lambda) \).

For \( k \geq 2 \) the following is known \[9\].

Theorem 1.6 (Budarina, Dickinson, Levesley). Let \( k \geq 2 \) be an integer and \( \lambda \geq k - 1 \). Then the Hausdorff dimension of the set of real \( \zeta \) with \( \lambda_k(\zeta) = \lambda \) equals \( 2/[k(1+\lambda)] \).

Moreover, for \( k = 2 \) and \( 1/2 \leq \lambda \leq 1 \) the problem is solved as well in \[2\], \[28\].

Theorem 1.7 (Beresnevich, Dickinson, Vaughan, Velani). Let \( k = 2 \) and \( 1/2 \leq \lambda \leq 1 \). Then

\[ \dim(\{ \zeta \in \mathbb{R} : \lambda_k(\zeta) = \lambda \}) = \frac{2 - \lambda}{1 + \lambda} \]
It is worth noting that for any integer \( k \geq 2 \) and \( \lambda \geq 1/k \) we have
\[
\dim(\{ \zeta \in \mathbb{R} : \lambda_k(\zeta) \geq \lambda \}) \geq \frac{2}{k(1 + \lambda)}.
\]
For small values of \( \lambda \) and \( k \geq 3 \) the following is known [4].

**Theorem 1.8 (Beresnevich).** Let \( k \geq 3 \) be an integer and \( 1/k \leq \lambda \leq 3/(2k - 1) \). Then
\[
\dim(\{ \zeta \in \mathbb{R} : \lambda_k(\zeta) \geq \lambda \}) \geq \frac{k + 1}{1 + \lambda} - (k - 1).
\]

There is actually equality conjectured in Theorem 1.8. However, (11) and Theorem 1.8 do not even allow to answer Problem 1.2 for \( k \geq 3 \). As a last reference concerning Problem 1.3 we want to point out a result obtained a consequence of a famous result of Sprindžuk [26]. Sprindžuk showed that Lebesgue-almost all real numbers satisfy \( w_k(\zeta) = k \). With aid of (7) we infer the following.

**Theorem 1.9 (Sprindžuk).** For any positive integer \( k \), the set \( \{ \zeta : \lambda_k(\zeta) > 1/k \} \) has 1-dimensional Lebesgue-measure zero. In particular,
\[
\dim\left( \{ \zeta \in \mathbb{R} : \lambda_k(\zeta) = 1/k \} \right) = 1.
\]

Problem 1.4 bases on the fact that each of the assumptions are known to hold for any real \( \zeta \), in particular (10) was proved in [8], the other restrictions are due to (1), (3).

1.3. **Outline of new results on the problems.** Surprisingly, our methods yield a converse to (10), provided that the constants \( \lambda_k \) are strictly greater than 1. This is the key for a better understanding of the Problems 1.3, 1.4. We prove

**Theorem 1.10.** Let \( k \) be a positive integer and \( \zeta \) be a real number such that \( \lambda_k(\zeta) > 1 \). Then
\[
\lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k}.
\]
Moreover, if \( \zeta \) is irrational, for \( 1 \leq j \leq k \) we have \( \widehat{\lambda}_j(\zeta) = 1/j \).

First we give some remarks to this result.

**Remark 1.11.** Note that the restriction \( \lambda_k > 1 \) is necessary. Theorem 4.3 in [8] shows that there exist \( \zeta \) such that \( \lambda_k(\zeta) = 1 \) for all \( k \geq 1 \). Theorem 4.4 in [8] shows that for \( 1 \leq \lambda \leq 3 \), there exist \( \zeta \) with \( \lambda_1(\zeta) = \lambda \) and \( \lambda_2(\zeta) = 1 \). This shows that in general, \( \lambda_1 \) is not determined by \( \lambda_2 \) as in Theorem 1.10.

**Remark 1.12.** Note also, that in the case \( \lambda_k > 1 \), Theorem 1.10 implies the right inequality in Khintchine’s transference principle [5]. In view of (3) and Theorem 1.10 for \( \lambda_k(\zeta) > 1 \) indeed
\[
\lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k} = \frac{w_1(\zeta) - k + 1}{k} \leq \frac{w_k(\zeta) - k + 1}{k}.
\]
We should mention, though, that Khintchine’s results are formulated in the much more general context of real vectors \( \zeta \in \mathbb{R}^k \) linearly independent over \( \mathbb{Q} \) together with \( \{1\} \).
Remark 1.13. Theorem 1.10 can be easily extended to infer equality in (10) for \( n \geq 1 \) too, see Corollary 1.15.

From Theorem 1.10 we deduce an improvement of Theorem 1.6 concerning Problem 1.3.

Corollary 1.14. Let \( k \geq 2 \) be an integer and \( \lambda > 1 \). Then the Hausdorff dimension of the set of real \( \zeta \) with \( \lambda_k(\zeta) = \lambda \) equals \( 2/[k(1+\lambda)] \). For \( \lambda = 1 \), we have that the dimension of real \( \zeta \) with \( \lambda_k(\zeta) = 1 \) is at least \( 1/k \).

Proof. Theorem 1.10 implies for \( \lambda > 1 \) the set identity
\[
\{ \zeta \in \mathbb{R} : \lambda_k(\zeta) = \lambda \} = \{ \zeta \in \mathbb{R} : \lambda_1(\zeta) = k\lambda + k - 1 \}.
\]
In particular the Hausdorff dimensions coincide. The dimension of the right hand side can be determined with Theorem 1.5, which gives just \( 2/[k(1+\lambda)] \).

For \( \lambda = 1 \), note that in view of (10) the equality \( \lambda_1(\zeta) = 2k - 1 \) implies \( \lambda_k(\zeta) \geq 1 \). However, \( \lambda_k(\zeta) > 1 \) is impossible due to Theorem 1.10. Thus
\[
\{ \zeta \in \mathbb{R} : \lambda_k(\zeta) = 1 \} \supset \{ \zeta \in \mathbb{R} : \lambda_1(\zeta) = 2k - 1 \},
\]
and the assertion follows from Theorem 1.5 again. \( \square \)

Note that Theorem 1.8 suggests that we cannot expect Corollary 1.14 to hold for \( \lambda < 1 \). The proof of Corollary 1.14 shows that to determine the dimension \( s \) of the sets \( \{ \zeta \in \mathbb{R} : \lambda_k(\zeta) \geq \lambda \} \) for all \( \lambda > 1 \), it suffices to determine all the dimensions for \( k = 1 \).

Theorem 1.10 readily gives a negative answer to Problem 1.4 as well. The following corollary shows that all \( \lambda(\zeta) > 1 \) are determined by \( \lambda_1(\zeta) \), and unless \( \zeta \) is rational is a Liouville number, i.e. irrational and \( \lambda_1(\zeta) = \infty \), there can only be a finite number of those.

Corollary 1.15. Let \( \zeta \in \mathbb{R} \) and define \( k_0 := \lceil (\lambda_1(\zeta) + 1)/2 \rceil \). Then
\[
(13) \quad \lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k}, \quad 1 \leq k \leq k_0 - 1,
\]
\[
(14) \quad \max \left\{ \frac{\lambda_1(\zeta) - k + 1}{k}, \frac{1}{k} \right\} \leq \lambda_k(\zeta) \leq 1, \quad k \geq k_0.
\]
In particular, if \( \lambda_1(\zeta) < \infty \), there are only finitely many indices \( k \) with \( \lambda_k(\zeta) > 1 \). Moreover, the answer to Problem 1.4 is no.

Proof. Note that for all integers \( k < k_0 \) by construction \( (\lambda_1(\zeta) - k + 1)/k > 1 \). By (10) for \( n = 1 \), we have \( \lambda_k(\zeta) > 1 \) such that we can apply Theorem 1.10 to obtain the reverse inequalities in (13) for all \( k < k_0 \). The left inequalities in (14) are due to (10) and (1), the right inequality follows again from the definition of \( k_0 \) and Theorem 1.10.

For the second assertion, put \( \lambda_1 = \lambda_2 = 1.1 \) and \( \lambda_k = 1/k \) for \( k \geq 3 \). One readily checks the conditions of Problem 1.4 are satisfied, but it contradicts Theorem 1.10. \( \square \)
In view of Remark 1.11, the assertion on non-Liouville numbers becomes wrong if we relax the assumption to \( \lambda_k(\zeta) \geq 1 \). A rearrangement of Corollary 1.15 allows to determine \( \lambda_n(\zeta) \) from \( \lambda_m(\zeta) \) for arbitrary indices \( m, n \), provided both values are strictly larger than one. In particular, this implies equality in (10) for \( n > 1 \) as well. This is part of the following corollary.

**Corollary 1.16.** Let \( 1 \leq n \leq m \) be integers and \( \zeta \in \mathbb{R} \). If \( \lambda_m(\zeta) > 1 \), then

\[
\lambda_m(\zeta) = \frac{n\lambda_n(\zeta) + n - m}{m}.
\]

In particular, choosing \( m = kn \), we obtain equality in (10). If \( \lambda_n(\zeta) > 1 \), then we have the inequality

\[
\lambda_m(\zeta) \geq \frac{n\lambda_n(\zeta) + n - m}{m}.
\]

**Proof.** Assuming \( \lambda_m(\zeta) > 1 \), since \( n \leq m \) implies \( \lambda_n(\zeta) \geq \lambda_m(\zeta) > 1 \), we may apply Theorem 1.10 to both indices. This yields \( (\lambda_1(\zeta) - n + 1)/n = \lambda_n(\zeta) \) and \( (\lambda_1(\zeta) - m + 1)/m = \lambda_m(\zeta) \). It is now easy to check the identity.

Assuming \( \lambda_n(\zeta) > 1 \), Theorem 1.10 again implies \( (\lambda_1(\zeta) - n + 1)/n = \lambda_n(\zeta) \). On the other hand, (10) yields \( \lambda_m(\zeta) \geq (\lambda_1(\zeta) - m + 1)/m \). The assertion follows from basic rearrangements similar to the identity. \( \Box \)

The second assertion of Corollary 1.16 motivates to ask if the assumption \( \lambda_n(\zeta) > 1 \) is necessary, if not we would obtain an ultimate generalization of (10).

**Problem 1.17.** Let \( m \geq n \geq 1 \) be integers. Does the estimate

\[
\lambda_m(\zeta) \geq \frac{n\lambda_n(\zeta) + n - m}{m}
\]

hold for any \( \zeta \in \mathbb{R} \)?

Dropping the assumption \( m \geq n \), counterexamples are provided by taking \( m = 1, n \geq 2 \) and any \( \zeta \) with \( \lambda_1(\zeta) < 2 \), noting \( n\lambda_n(\zeta) \geq 1 \) by (11). Note also that (11) is trivial in case of \( \lambda_n \leq 1 \), whereas (15) is in general not for \( m \geq 4, n \geq 3 \). Indeed, the smallest pair \( (m, n) \) that leads to a non-trivial case of (15) can be determined as \( m = 4, n = 3 \). If \( \lambda_3(\zeta) \in (2/3, 1] \), then (15) would lead to some bound \( \lambda_4(\zeta) > \eta > 1/4 \), where 1/4 is the trivial lower bound from (11).

Similar to the proof of Lemma 2.4, which is the main tool for the proof of Theorem 1.10, we will derive the following result in Section 3.

**Theorem 1.18.** Let \( k \) be a positive integer and \( \zeta \in \mathbb{R} \setminus \mathbb{Q} \). Then

\[
\hat{\lambda}_k(\zeta) \leq \max \left\{ \frac{1}{k}, \frac{1}{\lambda_1(\zeta)} \right\}.
\]

Again, we want to make some remarks.
Remark 1.19. Consider $\zeta$ with $\lambda_{k,1}(\zeta)$ bounded above by some $C > 1/k$. The assertion of Theorem 1.18 on bounds for $\hat{\lambda}_{k,1}(\zeta)$ is then trivial in case of $C \leq 1$, since we have $\hat{\lambda}_{k,1}(\zeta) \leq \hat{\lambda}_{1,1}(\zeta) = 1$ for irrational real $\zeta$, see Lemma 6 on page 8 and page 9 in [29]. More general, in case of $C \leq \lceil k/2 \rceil$ the assertion is implied by (9), such that Theorem 1.18 is interesting only for $C > \lceil k/2 \rceil$. Moreover, in the case $k = 1$, we have equality in Theorem 1.18.

Remark 1.20. Theorem 1.18 is wrong for rational $\zeta$, where we have $\lambda_{1,1}(\zeta) = \lambda_{k,1}(\zeta) = \hat{\lambda}_{k,1}(\zeta) = \infty$ for all positive integers $k$. Indeed, this is an obvious consequence of the fact that if $\zeta = a/b$ then $b^k \zeta^j \in \mathbb{Z}$ for any $1 \leq j \leq k$. Note that Theorem 1.10 does not require $\zeta$ to be irrational.

Remark 1.21. Theorem 1.18 improves the assertion on $\hat{\lambda}_j, 1 \leq j \leq k$ from Theorem 1.10, since $\lambda_k(\zeta) > 1$ is a stronger assumption than $\lambda_k(\zeta) \geq k$ by Theorem 1.10. See also Theorem 4.4 in Section 4.2.

Remark 1.22. As Theorem 1.10, Theorem 1.18 heavily depends on the fact that we are dealing with successive powers. In the more general context mentioned in Section 1.1 we can for instance construct cases where $\lambda_1(\zeta) = \lambda_k(\zeta) = \infty$ and $\hat{\lambda}_k(\zeta) = 1 > 1/k$ simultaneously, see Theorem 4 in [21]. Note $\hat{\lambda}_k(\zeta) \leq 1$ always holds.

1.4. Restriction to the Cantor set. The Cantor set is defined as the real numbers $a$ in $[0, 1]$ that can be written in the form

\[ a = c_1 3^{-1} + c_2 3^{-2} + \cdots, \quad c_i \in \{0, 2\}. \]

The spectrum of the quantity $\lambda_k(\zeta)$ with the restriction that $\zeta$ belongs to the Cantor set has been studied. For $k = 1$, the question is solved by the following constructive result which is Theorem 2 in [7]. We use a slightly different notation than the one in [7] for correlation with our upcoming results.

Theorem 1.23 (Bugeaud). Let $\hat{\rho} \in [1, \infty)$ and $\alpha > 0$. Any number

\[ \zeta = 2 \sum_{n \geq 1} 3^{-[\alpha(1+\hat{\rho})^n]} \]

belongs to the Cantor set and satisfies $\lambda_1(\zeta) = \hat{\rho}$. In particular, the spectrum of $\lambda_1$ restricted to the Cantor set equals $[1, \infty]$.

Indeed, the case $\lambda_1(\zeta) = \infty$ not explicitly mentioned in Theorem 2 in [7] is obtained similarly by a sequence with hyper-exponential growth in the exponent, such that indeed we can include the value $\infty$ in Theorem 1.23. We point out that concerning the approximation constants $w_k(\zeta)$ for $\zeta$ as in (16), it is shown in Theorem 7.7 in [5] that in case of $\hat{\rho} > (2k - 1 + \sqrt{4k^2 + 1})/2$ we have

\[ w_1(\zeta) = w_2(\zeta) = \cdots = w_k(\zeta) = \hat{\rho}. \]

Theorem 1.10 Theorem 1.23 and (17) yield

\[ \lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k} = \frac{\hat{\rho} - k + 1}{k} = \frac{w_k(\zeta) - k + 1}{k}. \]
In other words, \( \zeta \) as in (16) with \( \hat{\rho} > (2k - 1 + \sqrt{4k^2 + 1})/2 \) are explicit examples of real number with equality in the right hand inequality of the transference principle (5). This result was already established in the proof of Theorem 2 in [8], since (10) and (17) are sufficient. Recall that (18) is known solely for \( \hat{\rho} > (2k - 1 + \sqrt{4k^2 + 1})/2 \).

The best current result concerning the spectrum of \( \lambda_k \) within the Cantor set for \( k \geq 2 \), Theorem BL in [8], originates in \( \zeta \) as in (16), incorporating the above results.

**Theorem 1.24** (Bugeaud, Laurent). Let \( k \geq 2 \) be an integer. The spectrum of \( \lambda_k \) with \( \zeta \) in the Cantor set contains the interval \([1 + \sqrt{4k^2 + 1})/(2k), \infty]\).

Again, as conjectured for real \( \zeta \) in Problem 1.2, there is reason to believe that actually the spectrum equals \([1/k, \infty]\). The following immediate consequence of Theorem 1.10 yields an improvement of Theorem 1.24.

**Theorem 1.25.** Let \( k \geq 2 \) be an integer and \( \alpha > 0, \rho > 0 \). Then \( \zeta = 2 \sum_{n \geq 1} 3^{-\lceil \alpha[n(1+p)] \rceil} \) belongs to the Cantor set. If \( \rho \in (0,1] \), then

\[
\max \left\{ \rho, \frac{1}{k} \right\} \leq \lambda_k(\zeta) \leq 1.
\]

If \( \rho \in (1,\infty] \), we have equality

\[
\lambda_k(\zeta) = \rho.
\]

**Proof.** By Theorem 1.23 with \( \hat{\rho} := k\rho + k - 1 > 1 \), we have \( \lambda_1(\zeta) = k\rho + k - 1 \). If \( \rho > 1 \), then \( k < k_0 := \lceil (\lambda_1(\zeta) + 1)/2 \rceil \) such that with Corollary 1.15 we obtain (20). If \( \rho \in (0,1] \), then \( k \geq k_0 \) and the assertion (19) again follows from Corollary 1.15. \( \square \)

Theorem 1.25 obviously yields the improvement of Theorem 1.24 that the spectrum of \( \lambda_k \) contains \([1, \infty]\). However, we want to prove the more general statement Theorem 1.26. It extends Theorem 1.25 to expansions in an arbitrary base, and besides allows to include the value 1 in the spectrum.

**Theorem 1.26.** Let \( k \geq 2, b \geq 2 \) be integers and \( \rho \in (0, \infty) \). Let \( (a_n)_{n \geq 1} \) be a strictly increasing sequence of positive integers with the property

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = k(\rho + 1).
\]

Let

\[
\zeta = \sum_{n \geq 1} b^{-a_n}.
\]

If \( \rho \in (0,1] \), then

\[
\max \left\{ \frac{1}{k}, \rho \right\} \leq \lambda_k(\zeta) \leq 1.
\]

If \( \rho \in [1, \infty] \), we have equality

\[
\lambda_k(\zeta) = \rho.
\]

Moreover, if \( \rho \geq 1/k \), then \( \hat{\lambda}_k(\zeta) = 1/k \).
Note the similarity to the constructions of Theorem 1.23, where the analogue result was established for \( k = 1 \) and the sequence \( a_n = \lceil \alpha [k(1 + \rho)]^n \rceil \) for \( \alpha > 0 \). Roughly speaking, the additional factor \( k \) in the quotient \( a_{n+1}/a_n \) allows to generalize the one-dimensional result. However, the methods of the proofs of Theorem 1.23 and Theorem 1.26 are much different. The approach in this paper is rather connected to the one in [15], where a slightly weaker result than Theorem 1.23 was established.

We encourage the reader to compare the following corollary to Theorem 1.26 with Theorem 1 in [7], which we will not state, where a more general result in the special case \( k = 1 \) was established.

**Corollary 1.27.** Let \( k \geq 2, b \geq 2 \) be integers and \( \mathcal{A} \subset \{0, 1, \ldots, b - 1\} \) of cardinality \( |\mathcal{A}| \geq 2 \). The spectrum of the approximation constant \( \lambda_k(\zeta) \), restricted to \( \zeta \in (0, 1) \) whose expansion in base \( b \) have all digits in \( \mathcal{A} \), contains \([1, \infty)\).

In particular, for any \( \epsilon > 0 \) there exists a set \( B \) of Hausdorff dimension less than \( \epsilon \) such that the spectrum of \( \lambda_k \) within \( B \) contains \([1, \infty)\).

**Proof.** First additionally assume \( 0 \in \mathcal{A} \). Since \( |\mathcal{A}| \geq 2 \), there is another element in \( \mathcal{A} \) besides \( 0 \), say \( a \). We may assume \( a = 1 \), for else we may consider \( \zeta/a \) instead of \( \zeta \) without affecting the quantity \( \lambda_k \) by (1). The construction from Theorem 1.26 yields the assertion.

If \( 0 \notin \mathcal{A} \), write \( \mathcal{A} = \{a_1, a_2, \ldots, a_g\} \) with \( a_1 < a_2 \cdots < a_g \). Let \( \mathcal{A}' := \{0, a_2 - a_1, \ldots, a_g - a_1\} \). For \( \mathcal{A}' \) the assertion holds. So for arbitrary \( \lambda \geq 1 \) we have \( \lambda_1(\zeta') = \lambda \) for some \( \zeta' = \zeta'(\lambda) \) with all base \( b \) digits in \( \mathcal{A}' \). Then

\[
\zeta := \zeta' + a_1(b^{-1} + b^{-2} + \cdots) = \zeta' + \frac{a_1}{b-1}
\]

has base \( b \) digits in \( \mathcal{A} \), and as \( \zeta \) is a rational transformation of \( \zeta' \), with (1) we infer \( \lambda_k(\zeta) = \lambda_k(\zeta') = \lambda \). Since \( \lambda \geq 1 \) was arbitrary, the assertion follows.

Eventually, choosing \( |\mathcal{A}| = 2 \) and \( b > e^{\log 2/\epsilon} = 2^{1/\epsilon} \) gives the last assertion because the set \( B \) of reals with all digits in base \( b \) in \( \mathcal{A} \) then has dimension \( \log 2/\log b \), which can be verified similarly to the case of the Cantor set \( b = 3, \mathcal{A} = \{0, 2\} \), see [27].

In Section 3 we will prove Theorems 1.10, 1.26 using methods we will establish in Section 2. We want to mention that Theorem 1.18 in Section 2, a consequence of Lemma 2.4, is an interesting result on its own. Section 4 is devoted to some more applications of (mostly) Theorem 1.10.

### 2. Preparatory results

First we need the basically well-known fact Lemma 2.2 from one-dimensional Diophantine approximation. It can be inferred easier by facts from continued fraction expansion, however we want to infer it from a more general result which might be helpful for generalizations. We quote Minkowski’s second lattice point Theorem as an auxiliary result.
Theorem 2.1 (Minkowski). In \( \mathbb{R}^{k+1} \), let \( K \) be a convex body with volume \( \text{vol}(K) \) and \( \Lambda \) be a full lattice with determinant \( \det \Lambda > 0 \). Let \( \mu_1, \mu_2, \ldots, \mu_{k+1} \) be the successive minima of \( K \) with respect to \( \Lambda \), i.e. \( \mu_j \) is the infimum of real \( \nu > 0 \) such that \( \nu K = \{ \nu k : k \in K \} \) contains at least \( j \) lattice points. Then

\[
\frac{2^{k+1} \det(\Lambda)}{(k+1)! \text{vol}(K)} \leq \mu_1 \mu_2 \cdots \mu_{k+1} \leq 2^{k+1} \frac{\det(\Lambda)}{\text{vol}(K)}.
\]

We infer a fairly easy consequence of Theorem 2.1 in the case \( k = 1 \), in order to further simplify the structure of the proofs in Section 3.

Lemma 2.2. Let \( \zeta \in \mathbb{R} \). Suppose we have the estimate

(25) \[ \| \zeta x \| < \frac{1}{2} x^{-1}. \]

Then there exist uniquely determined positive integers \( x_0, y_0, M_0 \) such that \( x = M_0 x_0 \), \( (x_0, y_0) = 1 \) and

\[ |\zeta x_0 - y_0| = \| \zeta x_0 \| = \min_{1 \leq v \leq x} \| \zeta v \|. \]

Moreover, we have the identity

(26) \[ \| \zeta x \| = M_0 \| \zeta x_0 \|. \]

Furthermore, \( y_0/x_0 \) is a convergent of the continued fraction expansion of \( \zeta \).

Proof. By (25), we have \( |\zeta x - y| < (1/2)x^{-1} \) for some integer \( y \). Assume there is an integer pair \( (x_1, y_1) \) linearly independent from \( (x, y) \) with \( 1 \leq x_1 \leq x \) and \( |\zeta x_1 - y_1| < (1/2)x^{-1} \). Then for \( Q = x \) and some \( \delta > 0 \), the system

(27) \[ |m| \leq Q, \quad |\zeta m - n| \leq \left( \frac{1}{2} - \delta \right) Q^{-1} \]

would have two linearly independent solutions \( (m, n) \in \mathbb{Z}^2 \). This implies that for the lattice \( \Lambda \) and any of the rectangles \( K_{Q, \delta} \) defined by

\( \Lambda := \{(m, m \zeta + n) \in \mathbb{R}^2 : m, n \in \mathbb{Z} \}, \quad K_{Q, \delta} := \{(a, b) \in \mathbb{R}^2 : |a| \leq Q, |b| \leq (1/2-\delta)Q^{-1} \} \)

for fixed \( \delta > 0 \), both successive minima are at most \( 1 \). Noting \( \det \Lambda = 1 \) and \( \text{vol}(K_{Q, \delta}) < 2 \), this contradicts Minkowski’s Theorem 2.1 for \( k = 1 \).

Hence for large \( x \), any pair \( (x_1, y_1) \) with \( 1 \leq x_1 \leq x \) and \( |\zeta x_1 - y_1| < (1/2)x^{-1} \) is linearly dependent to \( (x, y) \). Say \( d = \gcd(x, y) \) and \( x_0 := x/d, y_0 := y/d \). Any pair linearly dependent to \( (x, y) \) is an integral multiple of \( (x_0, y_0) \), and if \( (x_1, y_1) = N(x, y) \), then

\[ |\zeta x_1 - y_1| = N|\zeta x_0 - y_0| \geq |x_0 - y_0|. \]

This holds in particular for \( (x_1, y_1) = (x, y) \). Thus, \( x_0 \) is as given in the Lemma and \( M_0 = x/x_0 \). The uniqueness and the fact that \( x_0, y_0 \) are coprime are obvious by the construction.

The last assertion on continued fractions is Satz 11 in [17]. □
Definition 2.3. We call a positive integer \( x \) a best approximation for \( \zeta \) if it satisfies
\[
\| \zeta x \| = \min_{1 \leq v \leq x} \| \zeta v \|.
\]

The numbers \( x_0 \) from Lemma 2.2, that are denominators of convergents of \( \zeta \), are obviously best approximations. By law of best approximation, Satz 11 in [17], any best approximation is the denominator of a convergent of the continued fraction expansion of \( \zeta \), and vice versa since the convergents converge monotonically to \( \zeta \). Moreover, it is shown in Satz 14 in [17], that at least one of two consecutive convergents satisfies (25). It is well-known (25) is valid for any convergent if the factor \( 1/2 \) is replaced by 1, Satz 10 in [17].

The most technical ingredient in the proofs of the Theorems 1.10, 1.26, is the following

Lemma 2.4. Let \( k \) be a positive integer and \( \zeta \) be a real number.

Then there exists a constant \( C = C(k, \zeta) > 0 \) such that for any integer \( x > 0 \) the estimate
\[
\max_{1 \leq j \leq k} \| \zeta^j x \| < C \cdot x^{-1},
\]
implies \( y/x = y_0/x_0 \) for integers \( (x_0, y_0) = 1 \) and \( x_0^k \) divides \( x \), where \( y \) denotes the closest integer to \( \zeta x \). A suitable choice for \( C \) is given by \( C = C_0 := (1/2) \cdot k^{-1}(1 + |\zeta|)^{1-k} \).

Moreover, \( y_0^k/x_0^j \) is a convergent of the continued fraction expansion of \( \zeta^j \) for \( 1 \leq j \leq k \). Furthermore, provided (28) holds for some pair \( (x, C) \), then it holds for any pair \( (x', C') \) with \( x' \) a positive integral multiple of \( x_0^k \) not larger than \( x \), and the best possible value \( C \) in (28) is obtained for \( x' = x_0^k \).

Proof. Suppose (28) holds for some \( x \) and \( C = C_0 \). Denote \( y \) the closest integer to \( \zeta x \) and let \( y_0/x_0 \) be the fraction \( y/x \) in lowest terms.

Assumption (28) for \( j = 1 \) leads to
\[
\left| \frac{y_0}{x_0} - \zeta \right| = \frac{y}{x} - \zeta < C_0 x^{-2}.
\]
Since \( C_0 < 1/2 < 1 \), we have \( |y_0/x_0 - \zeta| \leq 1 \) and thus \( |y_0/x_0| \leq 1 + |\zeta| \). Combination of these facts yields for \( 1 \leq j \leq k \) the estimate
\[
\left| \frac{y_0}{x_0} - \zeta^j \right| = \left| \frac{y_0}{x_0} - \zeta \right| \cdot \left( \frac{y_0}{x_0} \right)^{j-1} + \cdots + \zeta^{j-1} < C_0 x^{-2} \cdot k (1 + |\zeta|)^{k-1} = \frac{1}{2} x^{-2}.
\]
Suppose \( x_0^k \nmid x \). Then, since \( x_0 \mid x \), the integer \( x \) has a representation in base \( x_0 \) as
\[
x = b_1 x_0 + b_2 x_0^2 + \cdots + b_{k-1} x_0^{k-1} + b_k x_0^k + \cdots + b_j x_0^j,
\]
where at least one of \( \{b_1, b_2, \ldots, b_{k-1}\} \) is not zero. Put \( u = i + 1 \in \{2, 3, \ldots, k\} \) with \( i \) the smallest index such that \( b_i \neq 0 \). By construction, for all \( j \neq i \) we have \( x_0^j (y_0^n / x_0^m) \in \mathbb{Z} \).
Hence, using that \((x_0, y_0) = 1\) and \(b_i \neq 0\), we have the estimate

\[
\| \frac{y_0^j}{x_0^j} \| = \| b_i x_0^j \frac{y_0^j}{x_0^j} \| = \| \frac{b_i y_0^j}{x_0} \| \geq x_0^{-1}.
\]

On the other hand, the estimate \((29)\) for \(j = u\) implies

\[
\| x (\zeta^u - \frac{y_0^j}{x_0^j}) \| \leq \frac{1}{2} x^{-1} \leq \frac{1}{2} x_0^{-1}.
\]

Combination of \((31), (32)\) and triangular inequality imply

\[
\max_{1 \leq j \leq k} \| \zeta^j x \| \geq \| \zeta^u x \| > \frac{1}{2} x_0^{-1} \geq \frac{1}{2} x^{-1},
\]

contradicting \((28)\) since \(C_0 < 1/2\). Hence indeed \(x_0^j|x|\).

From \(x_0^j|x\) we infer \(x_0^j \leq x\), and \((29)\) yields

\[
\left| \frac{y_0^j}{x_0^j} - \zeta^j \right| < \frac{1}{2} x^{-2} \leq \frac{1}{2} x_0^{-2k} = \frac{1}{2} x_0^{2j-2k} \cdot (x_0^j)^{-2} \leq \frac{1}{2} \cdot (x_0^j)^{-2}, \quad 1 \leq j \leq k.
\]

Since clearly \((x_0^j, y_0^j) = 1\) for any \(1 \leq j \leq k\), Lemma \(2.4\) implies \(y_0^j/x_0^j\) is indeed a convergent of \(\zeta^j\) for every \(1 \leq j \leq k\).

Finally, we show that if \((28)\) holds for some pair \((x, \tilde{C})\) with \(x = M x_0^k\), then for all values \(\{x_0^k, 2x_0^k, \ldots, M x_0^k\}\) as well. It follows form \((29)\) that \(y_j := M x_0^{k-j} y_0^j\) is the closest integer to \(\zeta^j x\) for \(1 \leq j \leq k\). For arbitrary \(1 \leq N \leq M\), putting \(x' := N x_0^k\) and \(y_j' := N x_0^{k-j} y_0^j\) for \(1 \leq j \leq k\), we obtain

\[
|\zeta^j x' - y_j'| = N x_0^{k-j} |\zeta^j x_0 - y_0^j| \leq M x_0^{k-j} |\zeta^j x_0 - y_0^j| = |\zeta^j x - y_j'| < \tilde{C} x^{-1} \leq \tilde{C}' x^{-1}, \quad 1 \leq j \leq k.
\]

Moreover, it is obvious by the above that \(N = 1\) is the optimal choice. \(\square\)

**Remark 2.5.** For \(k = 1\), the assertion is trivial. The constant \(C_0\) in Lemma \(2.4\) can be slightly improved by establishing better bounds in the inequality of \((29)\). A small variation of the proof shows that for any \(\epsilon > 0\), restricting to sufficiently large \(x \geq \hat{x}(\epsilon)\), Lemma \(2.4\) holds for \(C_0 = \min\{(1/2 - \epsilon) k^{-1} |\zeta|^{1-k}, 1/2\}\).

**Remark 2.6.** It is not hard to see Lemma \(2.4\) would be wrong with right hand side in \((28)\) replaced by \((1/2) \cdot x_0^{-1}\) for any \(\zeta\) with \(\lambda_k(\zeta) > 1/k\). In particular, for \(k = 2\) and extremal numbers \(\zeta\) mentioned in Section \(1.1\) The proof of the false stronger version fails since \((32)\) is no longer correct.

Finally, for the proof of Theorem \(1.26\) we need an estimate for the concrete numbers \(\zeta\) in \((22)\). The exponent in Lemma \(2.7\) is actually much better than sufficient for the concern of Theorem \(1.26\) see Remark \(2.9\).

**Lemma 2.7.** Let \(k \geq 2, b \geq 2\) be integers, \(\rho > 0\) and \(\zeta\) be as in \((22)\) for some sequence \((a_n)_{n \geq 1}\) as in \((21)\). Then for \((x, y) \in \mathbb{Z}^2\) with sufficiently large \(x\), the estimate

\[
|\zeta x - y| \leq x^{-\frac{b}{k}+1}
\]
implies \((x, y)\) is linearly dependent to some
\[(34) \quad \underline{x}_n := (x_n, y_n) := (b^{a_n}, \sum_{j \leq n} b^{a_n-a_j}).\]

Proof. Assume the opposite, there exist arbitrarily large \((x, y)\) linearly independent to all \(\underline{x}_n\) for which \((33)\) holds. Let \(\delta > 0\). Say \(n\) is the index with \(b^{a_n} \leq x < b^{a_{n+1}}\).

First suppose \(x \leq b^{a_{n+1}-(1+\delta)a_n}\). Put \(Q = x\). If \(n\) or equivalently \(x\) is sufficiently large, then \((21)\) implies
\[(35) \quad \|b^n \zeta\| = \sum_{j \geq n+1} b^{a_j} \leq 2^{b_1(a_n - 1 - \delta) a_n} \leq b^{a_n(1-k(\rho+1)+\frac{\delta}{2})},\]
and hence
\[-\frac{\log |\zeta x - y|}{\log Q} \geq \frac{a_n(k(\rho+1)-1-\delta)}{a_{n+1}-(1+\delta)a_n} \geq \frac{(1+\rho)k-1-\delta}{(1+\rho)k-1} > 1.\]
On the other hand, \((35)\) also implies
\[-\frac{\log |\zeta x_n - y_n|}{\log Q} = -\frac{\log |\zeta x - y|}{\log x} \geq \frac{a_{n+1} - a_n}{a_{n+1}-(1+\delta)a_n} > 1.\]
Hence, for some \(\epsilon > 0\) and arbitrarily large \(n\), the system
\[(37) \quad |m| \leq Q, \quad |m\zeta - n| \leq Q^{-1-\epsilon}\]
has two linearly independent integral solutions \((m, n) = (x, y), (m, n) = (x_n, y_n)\). For large \(Q\), this contradicts Theorem \(2.1\) similarly to the proof of Lemma \(2.2\).

In the remaining case \(b^{a_{n+1}-(1+\delta)a_n} \leq x < b^{a_{n+1}}\), put \(Q = b^{a_{n+1}}\). For sufficiently large \(n\), clearly \((35)\) for \(n\) replaced by \(n+1\) shows that \((x_{n+1}, y_{n+1})\) satisfies \((27)\) for some \(\epsilon > 0\) (actually any \(\epsilon \geq k(\rho+1)-1 \geq k-1 \geq 1\)). On the other hand, \((33)\) yields
\[(38) \quad -\frac{\log |\zeta x - y|}{\log Q} = -\frac{\log |\zeta x - y|}{\log x} \log Q \geq \left(\frac{k}{k-1}\right) \frac{a_{n+1} - (1+\delta)a_n}{a_{n+1}}.\]
For \(\rho > 0\), we have
\[\lim_{n \to \infty, \delta \to 0} \frac{a_{n+1} - (1+\delta)a_n}{a_{n+1}} = k(\rho+1)-1 > k-1.\]
Hence, the right hand side in \((38)\) is strictly greater than 1 and thus the left also, thus for some \(\epsilon > 0\) the system \((37)\) has linearly independent integral solutions \((x, y), (x_{n+1}, y_{n+1})\) again, contradiction. \(\blacksquare\)

Remark 2.8. The first part of the proof shows in fact, that for \(\delta > 0\) and \(n \geq n(\delta)\) large enough, there cannot be a best approximation \(x\) in the interval \((b^{a_n}, b^{a_{n+1}-(1+\delta)a_n})\).

Remark 2.9. The continued fraction expansion of numbers \(\zeta\) as in Theorem \(1.26\) can be explicitly established using some variant of the Folding Lemma, see \([7]\) or \([10]\). This allows to sharpen the exponent \(-k/(k-1)\), or maybe explicitly determine the optimal value, in dependence of \(\rho\) in Lemma \(2.7\). It is possible to show, though, that Lemma \(2.7\) is wrong with exponent \(-1-\epsilon\) for some \(\epsilon > 0\), which would simplify the proof of Theorem \(1.26\) if
it was true for any $\epsilon > 0$. However, improvements of this kind are not necessary for our purposes. In fact, we only need the much weaker bound $-2k + 1$ instead of $-k/(k-1)$ for the proof of Theorem 1.26.

3. Proof of Theorems 1.10, 1.18, 1.26

First we prove Theorem 1.18 with a method very similar to the proof of Lemma 2.4. It might be possible to deduce Theorem 1.18 directly from this lemma, however there are technical difficulties so we prefer to prove it directly.

**Proof of Theorem 1.18.** Consider $k, \zeta$ fixed. The assertion is trivial for $\lambda_1(\zeta) = 1$, so we may assume $\lambda_1(\zeta) > 1$.

Let $1 < T < \lambda_1(\zeta)$ be arbitrary. By definition of the quantity $\lambda_1(\zeta)$, and since $\zeta \not\in \mathbb{Q}$, there exist arbitrarily large coprime $x_0, y_0$ with the property that

$$|\zeta - \frac{y_0}{x_0}| \leq x_0^{-T-1}.$$ 

For sufficiently large $x_0$ and a constant $D_0 = D_0(k, \zeta)$, similarly as in (29) we deduce

$$\left|\zeta^j - \frac{y_0^j}{x_0^j}\right| < D_0x_0^{-T-1}, \quad 1 \leq j \leq k. \quad (39)$$

We distinguish the cases $\lambda_1(\zeta) > k$ and $\lambda_1(\zeta) \leq k$.

**Case 1:** $\lambda_1(\zeta) > k$. Then we may assume $T > k$ as well. Let $X := x_0^k/2$. Write $1 \leq x \leq X$ in base $x_0$ as in (30) where there might be an additional $b_0 \neq 0$, and $i, u, b$ defined as in Lemma 2.4. Since $x_0, y_0$ are coprime and $b_i \neq 0$, we have

$$\left\|\frac{x_0^u}{y_0^u}\right\| = \left\|\frac{b_i x_0^{-u} y_0^u}{x_0^u}\right\| = \left\|\frac{b_i y_0^u}{x_0^u}\right\| \geq \frac{1}{x_0}. \quad (40)$$

Moreover, (39) yields for $1 \leq x \leq X$ the upper bounds

$$\left|\zeta^u - \frac{y_0^u}{x_0^u}\right| \leq \frac{X}{2} \left|\zeta^u - \frac{y_0^u}{x_0^u}\right| \leq \frac{D_0}{2}x_0^{-k-1} \quad (41)$$

Since $T > k$, the right hand side is smaller than $(1/2)x_0^{-1}$ for large $x_0$, so combining (40), (41) with triangular inequality yields for $1 \leq x \leq X$ the estimate

$$M_x := \max_{1 \leq j \leq k} \left\|\zeta^j x\right\| \geq \left\|\zeta^u x\right\| \geq \frac{1}{2}x_0^{-1}.$$ 

Using the definition of $\hat{\lambda}_k$ we conclude

$$\hat{\lambda}_k(\zeta) \leq \liminf_{X \to \infty} \max_{1 \leq x \leq X} \frac{-\log M_x(\zeta)}{\log X} \leq \frac{1 + \log 2}{k - \log 2},$$

and with $X \to \infty$ or equivalently $x_0 \to \infty$ indeed $\hat{\lambda}_k(\zeta) \leq 1/k$. 


Case 2: \( \lambda_1(\zeta) \leq k \). Define \( X := (1/2)D_0^{-1}x_0^T \), and again write \( x \leq X \) in base \( x_0 \) and define \( i, u, b \) as in case 1. We have \( 0 \leq i \leq \lfloor T \rfloor \leq T \), such that by \( T < k \) we infer that \( 1 \leq u \leq k \). We have \((40)\) precisely as in case 1, such as

\[
|x \left( \zeta^u - y_0^u \right)| \leq X \left| \zeta^u - y_0^u \right| \leq \frac{1}{2D_0}x_0^T \cdot D_0x_0^{-T-1} = \frac{1}{2}x_0^{-1}. 
\]

Combining \((40), (42)\) and triangular inequality yield

\[
M_x(\zeta) := \max_{1 \leq j \leq k} \| \zeta^j x \| \geq \| \zeta^u x \| \geq \frac{1}{2}x_0^{-1}. 
\]

Again we conclude

\[
\hat{\lambda}_k(\zeta) \leq \liminf_{X \to \infty} \max_{1 \leq x \leq X} \frac{-\log M_x(\zeta)}{\log X} \leq \frac{1 + \log 2}{\log x_0} \frac{\log X}{\log D_0 + \log 2}. 
\]

As we may choose \( T \) arbitrarily close to \( \lambda_1(\zeta) \), indeed \( \hat{\lambda}_k(\zeta) \leq 1/\lambda_1(\zeta) \) follows again with \( X \to \infty \) or equivalently \( x_0 \to \infty \).

Next we prove Theorem 1.10 with aid of the Lemmas 2.4, 2.2.

**Proof of Theorem 1.10.** In view of \((10)\), for the assertion on \( \lambda_k(\zeta) \) we only have to show that provided that \( \lambda_k(\zeta) > 1 \) holds, we have

\[
\lambda_k(\zeta) \leq \frac{\lambda_1(\zeta) - k + 1}{k}. 
\]

The definition of the quantity \( \lambda_k(\zeta) \) implies that for any fixed \( 1 < T < \lambda_k(\zeta) \), the inequality

\[
\max_{1 \leq j \leq k} \| \zeta^j x \| \leq x^{-T} 
\]

has arbitrarily large integer solutions \( x \). One checks that for any \( \tau > 0 \) and sufficiently large \( x > \hat{x}(\tau,T) := \tau^{1/(1-T)} \) we have \( x^{-T} < \tau x^{-1} \). Choosing \( \tau \leq C_0 \) with \( C_0 < 1/2 \) from Lemma 2.4 (11) ensures we may apply both Lemma 2.4 and Lemma 2.2 for \( x \geq \hat{x} \), with coinciding pairs \( x_0, y_0 \) such that \( y_0/x_0 \) is the reduced fraction \( y/x \). Further let \( M_0 \) be as in Lemma 2.2. Writing \( M_0 = x_0^0 \), by Lemma 2.4 we infer \( \eta \geq k - 1 \). Moreover, define \( T_0 \) implicitly by \( x_0^{-T_0} = |\zeta x_0 - y_0| \), i.e.

\[
T_0 = -\frac{\log |\zeta x_0 - y_0|}{\log x_0}. 
\]

The derived properties yield

\[
T \leq -\frac{\log \| \zeta x \|}{\log x} = -\frac{\log(M_0 |\zeta x_0 - y_0|)}{\log(M_0 x_0)} \leq \frac{T_0 - \eta}{1 + \eta} \leq \frac{T_0 - (k - 1)}{1 + (k - 1)} = \frac{T_0 - k + 1}{k}. 
\]

Since this is true for arbitrarily large values of \( x \) (and thus \( x_0 \)) and we may choose \( T \) arbitrarily close to \( \lambda_k(\zeta) \), the definition of \( T_0 \) implies \((43)\).

Since \( \lambda_1(\zeta) = k\lambda_k(\zeta) + k - 1 > 2k - 1 \geq k \), the assertion on \( \hat{\lambda}_j(\zeta) \) follows from Theorem 1.18.
We actually proved something stronger than Theorem 1.10. We point out the more general results evolved from the proof as a corollary.

**Corollary 3.1.** Let $k \geq 2$ be an integer, $\zeta$ be a real number. For any fixed $T > 1$, there exists $\hat{x} = \hat{x}(T, \zeta)$, such that the estimate

$$\max_{1 \leq j \leq k} \| \zeta^j x \| \leq x^{-T}$$

for an integer $x \geq \hat{x}$ implies the existence of $x_0, y_0, M_0$ as in Lemma 2.2 with the properties

$$x \geq x_0^k, \quad M_0 \geq x_0^{k-1}, \quad |\zeta x_0 - y_0| \leq x_0^{-kT-k+1}.$$  

Similarly, if for $C_0 = C_0(k, \zeta)$ from Lemma 2.4 the inequality

$$\max_{1 \leq j \leq k} \| \zeta^j x \| < C_0 \cdot x^{-1}$$

has an integer solution $x > 0$, then (45) holds with $T = 1$.

For direct consequences of Corollary 3.1, see Section 4.

With aid of the Lemmas 2.2, 2.7 and Corollary 3.1, we can prove Theorem 1.26. Lower bounds for $\lambda_k(\zeta)$ in Theorem 1.26 will be rather straightforward to derive by looking at $x = b^a_n$ for large $n$, whereas the proof of more interesting upper bounds is slightly technical. We give an outline for the proof of upper bounds. We distinguish between integers $x$ with the property that $\| \zeta x_0 \| < x_0^{-2k+1}$ for $x_0$ the largest best approximation $\leq x$, and those for which this inequality is wrong. Lemma 2.2 and Lemma 2.7 allow an easy classification of the values $x$ belonging to the first class, to which Lemma 2.4 and Lemma 2.2 can be effectively applied to obtain upper bounds. For the remaining class of integers $x$, the negated assertion of Corollary 3.1 immediately yields the upper bound 1.

**Proof of Theorem 1.26.** We consider $k \geq 2$ and $\rho > 0$ fixed, and a corresponding sequence $(a_n)_{n \geq 1}$ and $\zeta$ as in (22) constructed via the sequence. Note first that

$$\| b^a_n \zeta \| = \sum_{j \geq n+1} b^{a_n-a_j} \leq 2 \cdot b^{a_n-a_{n+1}}.$$  

We first prove the assertion on $\hat{\lambda}_k$. Assuming $\rho \geq 1/k$, for any $\delta > 0$ and sufficiently large $n \geq \hat{n}(\delta)$ we have

$$a_{n+1} \geq (k + k\rho - \delta)a_n \geq (k + 1 - \delta)a_n.$$  

Choosing integers $x$ of the form $b^a_n$, the estimate (46) and $\delta \to 0$ imply

$$\lambda_1(\zeta) \geq \limsup_{n \geq 1} \frac{\log(2 \cdot b^{a_n-a_{n+1}})}{\log b^a_n} \geq \limsup_{n \geq 1} \frac{(k + 1)a_n - a_n}{a_n} = k.$$  

For any such $\zeta$, the assertion follows directly from Theorem 1.18.

To prove (23) and (24), we show

$$\max \left\{ \frac{1}{k}, \rho \right\} \leq \lambda_k(\zeta) \leq \max \{ 1, \rho \}.$$
We start with the left inequality. We only have to show \( \lambda_k(\zeta) \geq \rho \), the other inequality \( \lambda_k(\zeta) \geq 1/k \) is trivial by \((1)\). It suffices to consider integers \( x \) of the form \( x = b^{kan} \). Write \( \zeta = S_n + \epsilon_n \) with
\[
S_n = \sum_{j=1}^{n} b^{-a_j}, \quad \epsilon_n = \sum_{j=n+1}^{\infty} b^{-a_j}.
\]
Since \( S_n < 1, \epsilon_n < 1 \) and the binomial coefficients are bounded above by \( k! \), we have that
\[
\zeta^j = \sum_{i=0}^{j} \binom{j}{i} S_n^i \epsilon_n^{j-i} = S_n^j + O(\epsilon_n), \quad 1 \leq j \leq k,
\]
as \( n \to \infty \), with the implied constant depending on \( k \) only. The crucial point is now that \( xS_n^j \) is an integer for \( 1 \leq j \leq k \) by construction. Thus for some constant \( C_0 > 0 \) independent of \( n \) and \( C_1 = 2C_0 \), we have
\[
(48) \quad \| x\zeta^j \| \leq C_0 \sum_{j=n+1}^{\infty} b^{-a_j} x = C_0 \cdot b^{kan} \sum_{j=n+1}^{\infty} b^{-a_j} \leq C_1 \cdot b^{kan-a_n+1}, \quad 1 \leq j \leq k.
\]
The condition \((21)\) implies for any \( \nu > 0 \) and sufficiently large \( n \geq \hat{n}(\nu) \)
\[
kan-a_n+1 \leq (-k\rho+\nu)a_n.
\]
For sufficiently large \( n \) (or equivalently \( x \)), combination with \((48)\) yields
\[
\max_{1 \leq j \leq k} \| x\zeta^j \| \leq C_1 \cdot b^{(-k\rho+\nu)a_n} = C_1 \cdot x^{-\rho+\frac{\nu}{k}} \leq x^{-\rho+\frac{\nu}{k}},
\]
and with \( \nu \to 0 \) we indeed obtain \( \lambda_k(\zeta) \geq \rho \) for any fixed \( \rho > 0 \).

We are left to prove the right hand side of \((47)\), which we do indirectly. Suppose there exists \( \rho > 0 \) and \( \zeta \) as in Theorem \((1.26)\) such that \( \lambda_k(\zeta) > \max\{1, \rho\} \). Then for \( \epsilon = (\lambda_k + 1)/2 - 1 > 0 \), the inequality
\[
\max_{1 \leq i \leq k} \| \zeta^j x \| \leq x^{-1-\epsilon}
\]
has arbitrarily large solutions \( x \). Consequently Corollary \((5.1)\) applies. It yields that \( x = M_0 x_0 \) for some best approximation \( x_0 \) such that \( |\zeta x_0 - y_0| < x_0^{-2k+1} \) for some \( y_0 \) with \( (x_0, y_0) = 1 \), and \( M_0 \geq x_0^{k-1} \). Note that \( k/(k-1) < 2k - 1 \) for \( k \geq 2 \). Moreover, for any fixed \( n \) the entries \( x_n, y_n \) of the vectors \( \underline{x}_n = (x_n, y_n) \) defined in \((54)\) are coprime, since \( x_n \) consists of prime factors dividing \( b \) and \( y_n \equiv 1 \mod b \). Thus Corollary \((2.7)\) shows that for large \( x_0 \), the inequality \( |\zeta x_0 - y_0| < x_0^{-2k+1} \) can be satisfied only if \( (x_0, y_0) = \underline{x}_n \) for some \( n \). So say \( x_0 = b^{a_n} \), consequently \( M_0 \geq b^{(k-1)a_n} \) and \( x \geq b^{kan} \). Observing
\[
\| b^{a_n} \zeta \| = \sum_{j \geq n+1} b^{a_n-a_j} \geq b^{a_n-a_n+1},
\]
Lemma \((2.2)\) concretely \((26)\), yields
\[
(49) \quad \| \zeta x \| = M_0 |\zeta x_n - y_n| \geq b^{(k-1)a_n} \cdot b^{a_n-a_n+1} = b^{kan-a_n+1}.
\]
By the assumption \((21)\) on the sequence \((a_n)_{n \geq 1}\), we have for any \( \eta > 0 \) and sufficiently large \( n \geq \hat{n}(\eta) \) (or equivalently \( x \) large enough)
\[
kan-a_n+1 \geq (-k\rho-\eta)a_n.
\]
Together with \([19]\) we infer
\[
\max_{1 \leq j \leq k} \| \zeta x \| \geq \| \zeta x \| \geq b^{(k \rho - \eta) a_n} \geq x^{-\rho - \frac{\eta}{k}}.
\]

Thus, the approximation constant \(\lambda_k(\zeta)\) restricted to pairs \((x,y)\) linearly dependent to some \(x_n\) is bounded above by \(\rho + \eta/k\). As we may choose \(\eta\) arbitrarily small, this contradicts the assumption \(\lambda_k(\zeta) > \max\{1, \rho\}\) as well.

**Remark 3.2.** In fact we proved that for any \(\epsilon > 0\) and sufficiently large \(n \geq \hat{n}(\epsilon)\) for any integer \(x \in [b^{a_n}, b^{a_{n+1}}]\) not divisible \(b^{a_n}\) satisfies max_{1 \leq j \leq k} \| \zeta^j x \| \geq x^{-1-\epsilon}. Using the argument of case \(T = 1\) in Corollary [3.1] within the proof instead of the \(T > 1\) case, this can be sharpened to max_{1 \leq j \leq k} \| \zeta^j x \| \geq C_0 x^{-1} with \(C_0\) from Lemma 2.4.

Eventually, we state the two obvious conjectures concerning generalizations of Theorem [1.26]. Both would imply a positive answer to Problem 1.2.

**Conjecture 3.3 (Weak).** Let \(k \geq 2, b \geq 2\) be integers and \(\rho \geq 1/k\). Let \((a_n)_{n \geq 1}\) be a strictly increasing sequence of positive integers with the property (21) and \(\zeta\) as in (22). Then \(\lambda_k(\zeta) = \rho\).

**Conjecture 3.4 (Strong).** Let \(k \geq 2, b \geq 2\) be integers and \(\rho > 0\). Let \((a_n)_{n \geq 1}\) be a strictly increasing sequence of positive integers with the property (21) and \(\zeta\) as in (22). Then \(\lambda_k(\zeta) = \max\{\rho, 1/k\}\).

The crucial point why the methods in the proof of Theorem [1.26] do not allow to establish better upper bounds for \(\lambda_k(\zeta)\) in case of \(\rho < 1\), is that no extension of Lemma 2.4 to this case seems available.

### 4. Some More Consequences of Theorems 1.10, 1.18

#### 4.1. Maximizing \(\hat{\lambda}_k\)

For any positive integer \(k\), we consider the following quantity.

**Definition 4.1.** For an integer \(k \geq 2\), let \(W(k)\) be the supremum of \(\hat{\lambda}_k(\zeta)\) among all \(\zeta\) not algebraic of degree \(\leq k\).

It follows from (11), (9) that
\[
\frac{1}{k} \leq W(k) \leq \frac{1}{\left\lfloor \frac{k}{2} \right\rfloor}.
\]

Moreover \(W(k)\) is known for \(k \in \{1, 2\}\) and a better upper bound is known for \(k = 3\), see Section 1.1. Theorem 1.18 in combination with Theorem 1.10 yield information on the well-known problem of the determination of \(W(k)\).

**Corollary 4.2.** For an integer \(k \geq 2\), suppose \(W(k) > 1/k\) is true. Let \((\zeta_n)_{n \geq 1}\) be a sequence of reals such that \(\lim_{n \to \infty} \hat{\lambda}_k(\zeta_n) = W(k)\). Then \(\lambda_1(\zeta_n) < k\) and \(\lambda_k(\zeta_n) \leq 1\) for sufficiently large \(n \geq n_0\).
Proof. If for some $n$ we have $\lambda_k(\zeta_n) > 1$, then Theorem 1.10 yields $\hat{\lambda}_k(\zeta_n) = 1/k$, so the convergence condition gives a contradiction unless $n$ is small. Similarly, $\lambda_1(\zeta_n) \geq k$ is contradicted by Theorem 1.18 for large $n$ if $\zeta_n$ is irrational, and $\zeta_n \in \mathbb{Q}$ can be excluded for large $n$ as well, since then $\hat{\lambda}_k(\zeta_n) = \infty$. \hfill $\square$

Remark 4.3. For $k \geq 3$, no non-trivial lower bound for $W(k)$ seems to be known. Hence we made the very reasonable additional assumption $W(k) \neq 1/k$ (note $k \geq 2$), which means there exists $\zeta$ not algebraic of degree $\leq k$ with $\hat{\lambda}_k(\zeta) \neq 1/k$.

We want to discuss the connection of our result to $W(k)$ a little more. For $k = 2$, for the extremal numbers $\zeta$ from Section 1.1 we have \cite{19}

$$\lambda_2(\zeta) = 1, \quad \hat{\lambda}_2(\zeta) = W(2) = \gamma := (\sqrt{5} - 1)/2 \approx 0.6180.$$ 

This means the bound 1 in Corollary 4.2 is sharp for $k = 2$. Moreover, in the more general setting mentioned briefly in Section 1.1, $\gamma$ is the maximum value of $\hat{\lambda}_2(\zeta)$ among all $\zeta \in \mathbb{R}^2$ with $\lambda_2(\zeta) \leq 1$, a consequence of (1.21) in \cite{21}. The latter can be derived from the fact that extremal numbers induce what Summerer and Schmidt call a regular graph \cite{25}, for $k = 2$ and the parameter $\lambda_2(\zeta) = 1$. Numbers that induce a regular graph are conjectured to maximize $\hat{\lambda}_k(\zeta)$ for given $\lambda_k(\zeta)$ in the case $k \geq 3$ as well, which was proved for $k = 3$ in \cite{25}.

The question arises if for some $k \geq 3$, the value $W(k)$ is obtained for $\zeta$ with $\lambda_k(\zeta)$ with $\lambda_k(\zeta) = 1$ and/or that induce a regular graph. However, for $k \geq 3$ it is possible to show that not both can be true. Indeed, it was shown in \cite{21} that for the regular graph

$$\lambda_k(\zeta) = 1 + \hat{\lambda}_k(\zeta)$$

holds, which for $\lambda_k(\zeta) = 1$ implies $\hat{\lambda}_k(\zeta) > 1/2$. In case of $k \geq 3$, this contradicts \cite{8}. In fact, (50) and (9) show that the regular graph for $\zeta = (\zeta, \zeta^2, \ldots, \zeta^k)$ can be obtained only for small values of $\lambda_k(\zeta)$, more precisely $\lambda_k(\zeta) \leq (2/k) \cdot (1 + o(1))$ as $k \to \infty$.

At the end of this section, we want to relate to Corollary 3.1 for $k = 2$ to the extremal numbers such as the numbers constructed in Theorem 4.3 and in Theorem 4.4 in \cite{8} from Remark 1.11. Corollary 3.1 shows that for any real $\zeta$ with $\lambda_1(\zeta) < 3$, we have

$$\max\{\|\zeta x\|, \|\zeta^2 x\|\} \geq cx^{-1}$$

for some $c > 0$ and all integers $x > 0$. Extremal numbers, which we denote by $\mu$, are shown in Theorem 3.1 in \cite{19} to satisfy $\lambda_2(\mu) = 1$. From some general facts on extremal numbers one can show $\lambda_1(\mu) = 1$ as well, we will not go into detail (this is evident for the bad approximable extremal numbers with continued fraction given by a Fibonacci word, see Example 1 on page 13 in \cite{19}). Thus (51) must hold, which indeed is established in Theorem 3.1 in \cite{19} as well. The numbers constructed in Theorem 4.3 and in Theorem 4.4 with $\lambda < 3$ in \cite{8} must satisfy (51) by the same reason. Roughly speaking, Corollary 3.1 almost prohibits the existence of such numbers, but not quite. Note that Corollary 3.1 yields the explicit values $c = C_0(2, \zeta) = (1/4) \cdot (1 + |\zeta|)^{-1}$ in (51) in the mentioned cases.
4.2. Dual problem. We conclude with applications to the dual constants $w_k(\zeta), \hat{w}_k(\zeta)$. We say in advance that in the proofs in this section, we will assume $k \geq 2$, since for $k = 1$ the assertions follow from (1), (2) or (8) if they are not trivial at all. It will be convenient to use Roth’s Theorem 1.1 at some places to exclude the case that the numbers $\zeta$ involved are algebraic, in order to apply (9). Though, essentially due to Theorem 1.10, the implication $\lambda_1(\zeta) \leq m$ for $m$ the weaker Liouville Theorem, would mostly be sufficient (with one exception in Theorem 4.4). Note also that Corollary 3.1 yields the equivalence of $\lambda_k(\zeta) > 1$ and $w_1(\zeta) = \lambda_1(\zeta) > 2k - 1$.

First we want to point out a consequence of Theorem 1.18. It allows to improve the linear form bound in (9) provided that $\lambda_1(\zeta)$ is sufficiently large.

**Theorem 4.4.** Let $k$ be a positive integer and $\zeta$ an irrational real number. In case of $w_1(\zeta) \geq k$, which is in particular true if $\lambda_k(\zeta) > 1$, the equalities
\begin{equation}
\hat{w}_1(\zeta) = 1, \quad \hat{w}_2(\zeta) = 2, \quad \ldots, \quad \hat{w}_k(\zeta) = k
\end{equation}
hold. In case of $k - 1 < w_1(\zeta) < k$, we have the inequalities
\begin{equation}
k \leq \hat{w}_k(\zeta) \leq \min \left\{ \frac{k}{w_1(\zeta) - k + 1}, 2k - 1 \right\}.
\end{equation}

**Proof.** Note that $w_1(\zeta) = \lambda_1(\zeta)$ by (3). Thus Theorem 1.18 implies $\hat{\lambda}_j(\zeta) = 1/j$ for $1 \leq j \leq k$, and (52) follows from (7). Note that by assumption and Roth’s Theorem $\zeta$ is transcendental, such that the upper bound $2k - 1$ in the non-trivial right inequality in (53) comes from (9). The remaining upper bound follows from (8) and Theorem 1.18 via
\[
\frac{\hat{w}_k(\zeta)}{(k-1)\hat{w}_k(\zeta) + k} \leq \hat{\lambda}_k(\zeta) \leq \max \left\{ \frac{1}{k}, \frac{1}{\lambda_1(\zeta)} \right\} = \max \left\{ \frac{1}{k}, \frac{1}{w_1(\zeta)} \right\} = \frac{1}{w_1(\zeta)}
\]
by elementary rearrangements. \qed

One readily checks (53) is an improvement to (11) in case of $w_1(\zeta) > k - 1 + k/(2k - 1) = k - 1/2 + o(1)$.

Theorem 4.4 allows to determine the values of $\hat{w}_k(\zeta)$ for Liouville numbers $\zeta$. In addition to results that were established before, we finally can determine all the classical approximation constants introduced in Section 1.1 for Liouville numbers.

**Corollary 4.5.** Let $\zeta$ be a Liouville number, i.e. an irrational real number which satisfies $\lambda_1(\zeta) = \infty$. Then for any $k \geq 1$ we have
\[
\lambda_k(\zeta) = \infty, \quad \hat{\lambda}_k(\zeta) = \frac{1}{k}, \quad w_k(\zeta) = \infty, \quad \hat{w}_k(\zeta) = k.
\]

**Proof.** The assertion on $\hat{w}_k(\zeta)$ follows from (52). The assertion on $\lambda_k(\zeta)$ was established in Corollary 1 in [8]. The assertion on $\hat{\lambda}_k(\zeta)$ follows either from Theorem 1.18 (Corollary 3.5 in [22]) or (7), and the assertion on $w_k(\zeta)$ is due to (3). \qed
Similar as carried out in Section 1.4, Theorem 1.10 and (3) imply the following.

We want to emphasize a consequence of the case $T = 1$ in Corollary 3.1, which improves the assertion about Liouville numbers in Corollary 1.15 and gives a criterion for a number to be a Liouville number.

**Theorem 4.6.** An irrational real number $\zeta$ is a Liouville number if and only if for any positive integer $k$, the estimate

$$\max_{1 \leq j \leq k} \|\zeta^j x\| < C_0(k, \zeta) \cdot x^{-1}$$

with $C_0$ defined in Lemma 2.4, has an integer solution $x = x(k, \zeta) > 0$.

**Theorem 4.7.** Let $k$ be a positive integer and $\zeta$ a real number with $\lambda_k(\zeta) > 1$, or equivalently $w_1(\zeta) = \lambda_1(\zeta) > 2k - 1$. We have equality in the right inequality of (5) if and only if

$$w_1(\zeta) = w_2(\zeta) = \cdots = w_k(\zeta).$$

The only if statement is the contribution of Theorem 1.10, the if part can be inferred as in Section 1.4 by (10) without any restriction on $\lambda_k(\zeta)$.

Recall that (18) for $\zeta$ as in (16) and $\hat{\rho}$ large enough, gave explicit non-trivial equality cases for the left inequality in (5). We want to discuss equality cases in (5), (6) some more. Since (7) is inferred by (5), (6), we want to restrict to the case $w_k(\zeta) > k$ resp. $\hat{w}_k(\zeta) > k$ (or equivalently $\lambda_k(\zeta) > 1/k$ resp. $\hat{\lambda}_k(\zeta) > 1/k$). The left inequalities can clearly hold for values of $\lambda_k(\zeta)$ resp. $\hat{\lambda}_k(\zeta)$ in the range $[1/k, 1/(k - 1)]$ only.

In the more general context of $\zeta \in \mathbb{R}^k$, Schmidt and Summerer give a reformulation of the analogue transference principle (5) in terms of successive minima functions $\psi_j$ with derived values $\hat{\psi}_j$, $\overline{\psi}_j$ for $1 \leq j \leq k + 1$, where $k = n - 1$ in their notation. Basic properties of those functions are

$$\frac{1}{k} \geq \overline{\psi}_{k+1} \geq 0 \geq \psi_1 \geq \overline{\psi}_1 \geq -1.$$ 

Remark b on page 80 in [23] establishes that the left inequality in (5) is equivalent to

$$k\overline{\psi}_1 + \overline{\psi}_{k+1} \leq 0,$$

and equality in (55) is equivalent to equality in the left inequality in (5). Moreover, the estimate (1.18$_{i=1}$) in [24] can be written as

$$k\overline{\psi}_1 + \overline{\psi}_{k+1} \leq \psi_1 \left(\frac{k + 1}{k} - \psi_1 + \overline{\psi}_{k+1}\right).$$

By the properties (54), the parenthesis expression is positive and hence equality in (55) can hold only in case of $\overline{\psi}_1 = 0$, which is equivalent to $\hat{\lambda}_k = 1/k$ by Theorem 1.4 in [23]. Hence equality in left inequality in (5) implies $\hat{\lambda}_k(\zeta) = 1/k$. Similarly, with (1.18$_{j=n}$) in [24] and (55) one can show that equality in the right hand side of (5) always holds in case of $\lambda_k(\zeta) = \infty$, and in case of $\lambda_k(\zeta) < \infty$, equality implies $\hat{\lambda}_k(\zeta) = 1/k$. Note that if $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k)$, the condition $\lambda_k(\zeta) = \infty$ leads to $\hat{\lambda}_k(\zeta) = 1/k$ as well by virtue of Theorem 1.18. See also Proposition 1.15 in [24].
We turn to (6). Due to (9), the spectrum of $\hat{\lambda}_k(\zeta)$ for equality cases in both inequalities is small anyway. Also, $\lambda_k(\zeta) \leq 1$ is necessary due to Theorem 4.4 in the non-trivial case $\hat{\lambda}_k(\zeta) > 1/k$. Proceeding very similar to the proof mentioned Remark b, one can show that (6) is equivalent to $\psi_1 + k\psi_{k+1} \geq 0$ and $k\psi_1 + \psi_{k+1} \leq 0$, and again corresponding equality cases coincide similarly as above. A similar somehow dual result to (1.18) could give some information on equality cases in (9).

Finally, another application related to the connection between the simultaneous approximation and the dual linear form problem.

**Theorem 4.8.** Let $k$ be a positive integer and $\zeta$ be irrational real and suppose $\lambda_k(\zeta) > 1$, or equivalently $w_1(\zeta) = \lambda_1(\zeta) > 2k - 1$. Then $w_1(\zeta) > \hat{w}_k(\zeta)$.

**Proof.** First note that $\zeta$ is transcendental by the assumption and Roth’s Theorem. Hence, the bounds in (9) are valid, and combination with Theorem 1.10 yields

$$w_1(\zeta) = \lambda_1(\zeta) = k\lambda_k(\zeta) + k - 1 > 2k - 1 \geq \hat{w}_k(\zeta).$$

□

**Remark 4.9.** A similar problem to Problem 1.4 concerning the approximation constants $w_k(\zeta)$, is whether for any non-decreasing sequence $(w_k)_{k \geq 1}$ with $w_k \geq k$ there exists $\zeta \in \mathbb{R}$ such that $w_k(\zeta) = w_k$. Unfortunately, Khintchine’s principle and Theorem 1.10 yield no implications concerning this question. It is known that the individual spectrum of each $w_k$ equals $[k, \infty]$ thanks to Bernik [3].

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