Abstract

A general framework for the deformation of the single-mode oscillators is presented and all deformed single-mode oscillators are unified. The extensions of the Aric-Coon, genon, the para-Bose and the para-Fermi oscillators are proposed. The generalized harmonic oscillator considered by Brzezinski et al. is rederived in a simple way. Some remarks on deformation of $SU(1,1)$ and supersymmetry are made.

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Recently, different partial results for the deformed single-mode oscillator have been obtained\cite{1}-\cite{12}. Our motivation is to present a unified view of all known deformations and to discuss some extensions. In particular, we consider extensions of the Arik-Coon\cite{1}, para-Bose, para-Fermi oscillators\cite{13} and their deformations\cite{4} \cite{6} \cite{11}. We show that even cases which were previously discarded as unphysical, owing to the negative norms\cite{1} \cite{3}, can be acceptable deformations of single-mode oscillators. The ”genon” oscillators\cite{7} are also included. Finally, we discuss the most generally deformed $SU(1,1)$ algebra and comment on hidden supersymmetry. The simplicity of our approach is demonstrated on all known results \cite{1}-\cite{12}.

Let us consider a pair of operators $\bar{a}, a$ (not necessarily hermitian conjugate to each other) with the number operator $N$. The most general commutation relation linear in the $\bar{a}a$ and $a\bar{a}$ operators is

$$a\bar{a} - F(N)\bar{a}a = G(N), \quad (1)$$

where $F(N)$ and $G(N)$ are arbitrary complex functions. The number operator satisfies

$$[N, a] = -a,$$

$$[N, \bar{a}] = \bar{a}, \quad (2)$$

$$[N, \bar{a}a] = [N, a\bar{a}] = 0.$$

Hence, we can write

$$\bar{a}a = \varphi(N),$$

$$a\bar{a} = \varphi(N + 1), \quad (3)$$

where $\varphi(N)$ is, in general, a complex function satisfying the recurrence relation

$$\varphi(N + 1) - F(N)\varphi(N) = G(N). \quad (4)$$

If $\varphi(N)$ is the bijective mapping, then

$$N = \varphi^{-1}(\bar{a}a) = \varphi^{-1}(a\bar{a}) - 1. \quad (5)$$
Let us denote the hermitian conjugate of the operator \( a \) by \( a^\dagger \). Then it follows that

\[
[N, a^\dagger] = a^\dagger,
\]

\[
\bar{a} = c(N)a^\dagger,
\]

where \( c(N) \) is a complex function of \( N \). It is convenient to choose \( c(N) \) to be a "phase" operator, \(|c(N)| = 1\). Then we have

\[
a^\dagger a = |\varphi(N)|,
\]

\[
aa^\dagger = |\varphi(N + 1)|,
\]

\[
aa^\dagger - a^\dagger a = |\varphi(N + 1)| - |\varphi(N)| = G_1(N),
\]

\[
c(N) = e^{i \text{arg} \varphi(N)} = \frac{\varphi(N)}{|\varphi(N)|}.
\]

If \( \varphi(N) > 0 \), then \( \text{arg} \varphi(N) = 0 \) and \( c(N) = 1 \).

Let us further assume that \( |0> \) is a vacuum:

\[
a|0> = 0,
\]

\[
N|0> = 0, \quad \varphi(0) = 0,
\]

\[
<0|0> = 1.
\]

One can always normalize the operators \( a \) and \( \bar{a} \) such that \(|\varphi(1)| = 1\); then \(|G(0)| = 1\). The function \( \varphi(N) \) is determined by the recurrence relation (4) and is given by

\[
\varphi(n) = [F(n - 1)]! \sum_{j=0}^{n-1} \frac{G(j)}{[F(j)]!},
\]

where

\[
[F(j)]! = F(j)F(j-1)...F(1),
\]

\[
[F(0)]! = 1.
\]

The excited states with unit norms are

\[
|n> = \frac{(a^\dagger)^n}{\sqrt{|\varphi(n)|!}}|0> = \frac{(e^{-\frac{i}{2} \bar{a}})^n}{\sqrt{|\varphi(n)|!}}|0>.
\]
\[ < n|m > = \delta_{mn}, \quad n, m = 0, 1, 2..., \]
\[ < n-1|a|n > = < n|a|^0|n-1 > = \sqrt{|\varphi(n)|}. \]  
\[ (11) \]

If \( \varphi(n) \neq 0 \) for \( \forall n \in \mathbb{N} \), then there is an infinite set ("tower") of states. However, if \( \varphi(n_0) = 0 \) for some \( n_0 \), then the state \( (a^\dagger)^{n_0}|0 > \) has zero norm and, consistently, we can put \( |n_0 > \equiv 0 \). The corresponding representation is finite-dimensional and the representation matrices are of the \( n_0 \times n_0 \) type ((11)).

The operators \( a \) and \( a^\dagger \) can be related to the Bose operators \( b \) and \( b^\dagger \) by mapping
\[ a = b\sqrt{\frac{|\varphi(N)|}{N}}, \quad a^\dagger = \sqrt{\frac{|\varphi(N)|}{N}}b^\dagger, \]
\[ (12) \]
where
\[ bb^\dagger - b^\dagger b = 1, \quad N = b^\dagger b. \]
\[ (13) \]

Note that this transformation preserves the number operator, i.e. \( N^{(a)} = N^{(b)} \).

If \( \varphi(n) \neq 0 \) for \( \forall n \in \mathbb{N} \), the Fock space of the deformed algebra is identical to the Fock space of the Bose oscillator. However, if \( \varphi(n) = 0 \) for some \( n_0 \in \mathbb{N} \), the Fock space of the Bose oscillator reduces to finite dimensional subspace. Then \( (a^\dagger)^n|0 > = 0 \) for some \( n \geq n_0 \).

Note that we can define a new vacuum for every \( n_0 \in \mathbb{N} \), for which \( \varphi(n_0) = 0 \):
\[ |n_0 > = \frac{b^{n_0}}{\sqrt{(n_0)!}}|0 >, \]
\[ a|0 > = a|n_0 > = b\sqrt{\frac{|\varphi(N)|}{N}}\frac{(b^\dagger)^{n_0}}{\sqrt{(n_0)!}}|0 > = 0, \]
\[ a^\dagger|0 > = a^\dagger|n_0 > = \sqrt{|\varphi(n_0 + 1)|}|n_0 + 1 > \neq 0. \]
\[ (14) \]

The vacuum \( |0 > \) is identical to the \( n_0 \)-particle state and \( |1 > \) corresponds to the new one-particle state. Hence, we can define as many vacua as there are solutions of
the equation $\varphi(n_0) = 0, n_0 \in \mathbb{N}$. The Fock space of the Bose oscillator is split into subspaces corresponding to different vacua. The new number operator $\aleph$ is defined as $\aleph|0\rangle = \aleph|n_0\rangle > = 0, \aleph = N - n_0$.

The new excited states are $|n\rangle = |n + n_0\rangle = \frac{(a^\dagger)^n}{\sqrt{|\varphi(n + n_0)|...|\varphi(n_0 + 1)|}}|0\rangle$, (15) $\aleph = N - n_0$.

Different vacua are not necessarily degenerate. Their relative positions depend on the Hamiltonian expressed in terms of $a$ and $a^\dagger$. (For example, if $H = a^\dagger a$, then all the vacua are degenerate). Note that $(a^\dagger)^n \neq 0$ for $\forall n \in \mathbb{N}$ in the case of different vacua.

The function $|\varphi(n)|$ uniquely determines the type of deformed oscillator algebra and vice versa. If $\varphi_1 \neq \varphi_2$ but $|\varphi_1| = |\varphi_2|$, the corresponding algebras are isomorphic.

There is a family of functions $(F, G)$ leading to the same algebra, with identical functions $\varphi(N)$. Therefore we can fix the "gauge", for example:

(a) $F(N) = 1, \ a\bar{a} - \bar{a}a = G_1(N)$,

(b) $G(N) = 1, \ a\bar{a} - F_1(N)\bar{a}a = 1$, (16)

(c) $F(N) = q, \ a\bar{a} - q\bar{a}a = G_q(N)$.

The connection between cases (a), (b) and (c) is

$$G_1(N) = G_q(N) + (q - 1)\varphi(N) = 1 + [F_1(N) - 1]\varphi(N).$$ (17)

Let us examine case (c). We assume that $G_q(N)$ can be expanded in powers of $N$ around any $n \in \mathbb{N}$. Then

$$a\bar{a} - q\bar{a}a = G_q(N) = \sum_{k=0}^{\infty} c_k N^k, \ c_0 = 1,$$

$$\varphi(n) = q^{n-1} \sum_{j=0}^{n-1} \frac{G_q(j)}{q^j} = q^{n-1} \sum_{k=0}^{\infty} c_k S_k(n, q),$$ (18)
where
\[ S_k(n, q) = \sum_{j=0}^{n-1} \frac{j^k}{q^j} \equiv q^{1-n} \{ \sum_{j=0}^{n-1} j^k \} (k, q). \] (19)

The recurrence relations for \( S_k(n, q) \) are
\[
(-q)^k \frac{d^k}{dq^k} S_0(n, q) = \sum_{l=1}^{k} e_l^{(k)} S_l(n, q),
\]
\[ S_k(n, q) = (-q \frac{d}{dq}) S_{k-1}(n, q), \] (20)

where
\[
e_l^{(k)} = \frac{1}{l!} \left[ \frac{d^l}{dx^l} x(x+1)\ldots(x+l-1) \right]_{x=0}. \] (21)

From eq. (20) it follows that
\[ S_k(n, q) = (-q \frac{d}{dq})^k S_0(n, q) = \sum_{l=1}^{k} e_l^{(k)} S_l(n, q) \] (22)

and from eq. (18)
\[ \varphi(n) = q^{n-1} G_q(-q \frac{d}{dq}) S_0(n, q), \] (23)

where
\[ S_0(n, q) = q^{1-n} \frac{q^n-1}{q-1} = [n]_{q^{-1}}. \] (24)

Specially, for case (a) we obtain
\[ \varphi(n) = \sum_{j=0}^{n-1} G_1(j) = [G_1(-q \frac{d}{dq}) S_0(n, q)]_{q=1}. \] (25)

Using these formulas, all kinds of deformed oscillator can be unified and the notion of deformation can also be extended to cases in which states with negative norms appeared.

In the following we discuss some examples.
(I) Extension of the Arik-Coon algebra:

\[ \begin{align*}
    aa^\dagger - qa^\dagger a &= 1, \\
    \varphi(n) &= \frac{q^n - 1}{q - 1},
\end{align*} \]  

(26)

The algebra is defined for \( q \geq -1 \), since for all other \( q \), \( a^\dagger a = \varphi(n) \) is negative. However, we propose the following deformed algebra instead of eq. (26),

\[ \begin{align*}
    a\bar{a} - q\bar{a}a &= 1, \quad q \in \mathbb{C}, \\
    \bar{a}a &= \frac{q^n - 1}{q - 1}, \\
    a^\dagger a &= |q^n - 1| = \frac{|q|^{2n} - 2|q|^n \cos n\varphi + 1}{|q|^2 - 2|q| \cos \varphi + 1}.
\end{align*} \]  

(27)

This is a generalization of the genon oscillator \([7]\) with \( q = e^{i2\pi M} \) and \( M \in \mathbb{N} \).

If \( q < -1 \), we find that

\[ a^\dagger a = \frac{|q|^n + (-)^{n-1}}{|q| + 1} > 0, \quad \forall n \in \mathbb{N}, \]

\[ aa^\dagger - |q|^a^\dagger a = (-1)^N. \]  

(28)

When \( q = -1 \), it corresponds to the Fermi oscillator. Similarly, the algebra \( aa^\dagger + a^\dagger a = (-1)^N \) corresponds to the Bose oscillator.

Let us in the same way extend single para-Bose and para-Fermi oscillators \([13]\) as well as their deformations \([14]\). 

(II) Extension of the single para-Bose oscillator

\[ \begin{align*}
    a\bar{a} + \bar{a}a &= 1 + c_1N, \quad c_1 \in \mathbb{C}, \\
    \varphi(n) &= (-)^{n-1}\{(1 - c_1q \frac{d}{dq} \frac{q^{-n-1}}{q-1})\}_{q=-1} \\
    &= (-)^{n-1}\{S_0(n, -1) + c_1S_1(n, -1)\} = \frac{1}{2}c_1n + [n]_1(1 - \frac{1}{2}c_1).
\end{align*} \]  

(29)
For $c_1 = 0$, eq.(29) corresponds to the Fermi oscillator. When $c_1 = \frac{2}{p}$, $p \in \mathbb{N}$, eq.(29) corresponds to the para-Bose oscillator of the $p^{th}$ order [5][13][11], and the representation is infinite-dimensional. When $c_1 = -\frac{1}{p}$, $p \in \mathbb{N}$, the representations are finite dimensional but different from the para-Fermi oscillator. (If we admit a different vacuum, $|0\rangle = |p\rangle$, there is an infinite tower of states on it).

Different deformations of a single para-Bose oscillator are proposed in [4][6]. For example:

$$\varphi^{(a,b)}(n) = \left[ \frac{1}{2} c_1 n + [n]_{-1}(1 - \frac{1}{2} c_1) \right]_{q^2}^{(a,b)},$$

where

$$[x]_q^{(a)} = \frac{q^x - 1}{q - 1},$$

$$[x]_q^{(b)} = \frac{q^x - q^{-x}}{q - q^{-1}} = q^{1-x}[x]_q^{(a)}.$$ (31)

Note that the single para-Bose algebra (29) can also be written as

$$a\bar{a} - \bar{a}a = \frac{1}{2} c_1 + (1 - \frac{1}{2} c_1)(-)^N.$$ (32)

This is equivalent to the so-called ”modification” [8][9]

$$aa^\dagger - a^\dagger a = 1 + 2\nu(-)^N.$$ (33)

If we change the operators $a$ and $a^\dagger$ in eq.(32) into $a \rightarrow a\sqrt{1 + \nu}$ and put $\frac{1}{1+2\nu} = \frac{1}{2} c_1$, we obtain eq.(33).

Furthermore, the following deformation is proposed in ref.[9] :

$$aa^\dagger - qa^\dagger a = q^{-N}(1 + 2\nu(-)^N).$$ (34)

Using our equation (18) we obtain

$$\varphi(n) = q^{n-1}(S_0(n, q^2) + 2\nu S_0(n, -q^2))$$

$$= q^{1-n}\left\{ [n]_{q^2}^{(a)} + 2\nu(-)^{n-1}[n]_{-q^2}^{(a)} \right\}$$

$$= [n]_q^{(b)} + 2\nu(i)^{1-n}[n]_{iq}^{(b)}.$$ (35)
Extension of the single para-Fermi oscillator:

\[ a\bar{a} - \bar{a}a = 1 + c_1 N, \quad c_1 \in \mathbb{C} \]

\[ \varphi(n) = n(1 + \frac{1}{2}c_1(n - 1)). \] \hspace{1cm} (36)

For \( c_1 = 0 \), eq.(36) corresponds to the Bose oscillator. When \( c_1 = -\frac{2}{p}, p \in \mathbb{N} \) eq.(36) corresponds to the para-Fermi oscillator of the \( p^{th} \) order \([13][14]\), and corresponding representations are finite dimensional. (If we admit a different vacuum, \(|0\rangle = |p\rangle\), there is an infinite tower of states on it). When \( c_1 > 0 \), the representations are infinite dimensional but different from the para-Bose oscillator.

Notice that the only values of \( c_1 \) for which eqs.(29) and (36) give the same oscillator are \( c_1 = (0, -2);(2, 0);(-1, -1) \), where the first (second) entry corresponds to the value of \( c_1 \) in eq.(29) (eq.(36)), respectively.

Simple deformations of the single para-Fermi oscillator \([4][5]\) are

\[ \varphi^{(a,b)}(n) = [n]_q^{(a,b)}[1 + \frac{1}{2}c_1(n - 1)]_q^{(a,b)}. \] \hspace{1cm} (37)

Similarly as in \([9]\), one can write

\[ a\bar{a} - q\bar{a}a = q^{-N}(1 + c_1 N), \]

\[ \varphi(n) = q^{n-1}(S_0(n, q^2) + c_1 S_1(n, q^2)). \] \hspace{1cm} (38)

Finally, we give a simple interpolation between the para-Bose, the para-Fermi, the genon oscillators and all other known deformed single-mode oscillators \([1]-[12]\) : 

\[ a\bar{a} - q\bar{a}a = p^{-N} \sum_{k=0}^{\infty} c_k N^k, \quad p, q, c_k \in \mathbb{C}, \]

\[ \varphi(n) = q^{n-1} \sum_{k=0}^{\infty} c_k S_k(n, pq) = \sum_{k=0}^{\infty} c_k \left\{ \sum_{j=0}^{n-1} j^k \right\}_{(k,pq)}. \] \hspace{1cm} (39)

Finite-dimensional representations are determined with solutions of the equation \( \varphi(n_0) = 0, n_0 \in \mathbb{N} \). The commutation relation, eq.(39), represents the most general
single-mode deformed oscillator. General multimode deformed oscillators are more complicated and some partial results are obtained in [14].

We conclude with two remarks.

Remark 1. We can define new operators $B_- = -\frac{1}{2}a^2$ and $B_+ = \frac{1}{2}\bar{a}^2$, satisfying the commutation relations

$$[N, B_{\pm}] = \pm 2B_{\pm},$$

$$[B_+, B_-] = \frac{1}{4}(\varphi(n + 2)\varphi(n + 1) - \varphi(n)\varphi(n - 1)),$$ (40)

and the Casimir operator

$$C = B_+B_- + \frac{1}{4}\varphi(n)\varphi(n - 1) + C_0$$

$$= B_-B_+ + \frac{1}{4}\varphi(n + 2)\varphi(n + 1) + C_0.$$ (41)

$C_0 = \text{const.}$

Using $\varphi(n)$ from eq.(35) we reproduce $U_q(SU(1, 1))$ as a spectrum-generating algebra for the algebra (34). In the limit $\nu \to 0$, $\varphi(n) = q^{n-1}S_0(n, q^2)$ and the standard deformation of the $SU_q(1, 1)$ is reproduced [15]. The undeformed -oscillator realization of the $SU(1, 1)$ algebra is recovered with $\nu \to 0$, $q \to 1$, i.e. with $\varphi(n) = n$.

Remark 2. In the same way as in [9] we can realize the supersymmetry algebra \{Q_i(p), Q_j(p)\} = 2\delta_{ij}H(p), i, j = 1, 2 ([15] [16]) with the Hermitian operators $Q_i(p)$, $p \in \mathbb{R}$, given by

$$Q_1(p) = a^\dagger p^{N+1}P_- + ap^NP_+$$

$$Q_2(p) = i(a^\dagger p^{N+1}P_- - ap^NP_+)$$ (42)

$$P_{\pm} = \frac{1}{2}(1 \pm (-)^N).$$

The hamiltonian $H(p)$ is given as

$$H(p) = |\varphi(N + 1)|p^{2(N+1)}P_- + |\varphi(N)|p^{2N}P_+.$$ (43)
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