A Structure-Preserving Parametric Finite Element Method for Area-Conserved Generalized Curvature Flow

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Received: 26 December 2022 / Revised: 27 April 2023 / Accepted: 3 May 2023 / Published online: 19 May 2023
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Abstract
A structure-preserving numerical method is presented and analyzed to simulate the motion of closed curves governed by area-conserved generalized curvature flow. We first propose a variational formulation and rigorously prove that it preserves two geometric structures of enclosed area conservation and perimeter decrease. Then the parametric finite element method is adopted to develop a semi-discrete scheme. Both the area conservation and perimeter decrease structure properties are strictly proven for the semi-discrete scheme by utilizing the Cauchy–Schwarz inequality and power mean inequality. On this basis, a fully discrete scheme is established by further taking the backward Euler method in the temporal direction and discretizing the unit normal vector appropriately. We give a rigorous proof that the scheme is structure-preserving at the discretized level. Finally, numerical results verify the structure-preserving property and reveal that the proposed method has second-order accuracy and enjoys asymptotic equal mesh distribution during the evolution.

Keywords
Area-conserved generalized curvature flow · Structure-preserving · Parametric finite element method

Mathematics Subject Classification
65M60 · 65M12 · 35K55 · 53C44

1 Introduction
Assume that $\Gamma(t)$ is a closed curve in two dimensions (2D). It can be parameterized as $X := X(s, t) = (x(s, t), y(s, t))^T \in \mathbb{R}^2$, here $s$ represents the arc length. We consider the motion of $\Gamma(t)$ governed by area-conserved generalized curvature flow, which is described
as
\[
\begin{aligned}
\partial_t X &= (\alpha \kappa^\beta - \lambda) n, \\
\kappa &= -\partial_{ss} X \cdot n, \\
\end{aligned}
\]
(1.1)
where \( \kappa := \kappa(s, t) \) and \( n := -\partial_s X \perp \) are the curvature and the outer unit normal vector of \( \Gamma(t) \), respectively, the symbol \( \perp \) means clockwise rotation by \( \frac{\pi}{2} \), the real numbers \( \alpha \) and \( \beta \) satisfy \( \alpha \beta < 0 \), and the term \( \lambda \) is chosen as
\[
\lambda := \frac{\int_{\Gamma(t)} \alpha \kappa^\beta \, ds}{\int_{\Gamma(t)} 1 \, ds}.
\]
(1.2)

The choice of \( \lambda \) leads to \( \int_{\Gamma(t)} (\alpha \kappa^\beta - \lambda) \, ds = 0 \), which can ensure that the area enclosed by \( \Gamma(t) \) remains a constant. Moreover, we give the initial curve \( \Gamma(0) := \Gamma_0 \). It is parameterized by \( X(s, 0) = X_0(s) = (x_0(s), y_0(s))^T \).

Flows (1.1) without the term \( \lambda \) have different phenomena of evolution for various \( \alpha \) and \( \beta \). To be specific, for cases \( \alpha < 0 \) and \( \beta = 1 \), this flow is the classical curvature flow. The embedded curve governed by this flow exists in finite time and shrinks to a point ultimately [23, 26, 28]. For case \( \alpha < 0 \) and \( 0 < \beta \neq 1 \), we obtain the powers of curvature flow, which is a nonlinear contracting flow for convex curves (see [1, 41, 44–46]). For case \( \alpha > 0 \) and \( \beta = -1 \), this flow coincides with the inverse curvature flow, which leads to curves with positive curvature \( \kappa > 0 \) staying strictly convex and expanding for all time [1, 30, 31, 49]. When \( \alpha > 0 \) and \( -1 \neq \beta < 0 \), the flow can be viewed as the generalization of the inverse curvature flow (see [1, 25, 37, 43]).

The extra term \( \lambda \) as (1.2) appeared in (1.1) balances above contraction or expansion phenomena, and forces the area enclosed by curve to remain constant throughout the evolution of the curve. Moreover, flow (1.1)–(1.2) has another fundamental geometric property, i.e., the curve’s perimeter decreases with respect to time. Thus the curve governed by (1.1)–(1.2) eventually evolves into a circle [18, 22, 38, 48]. For more related works on this flow and its counterpart in high-dimension, one can refer to [2, 14, 15, 29, 35, 47]. In this case, (1.1)–(1.2) can be interpreted as a unified planar flow model, which enjoys the fixed area and perimeter decrease properties. The model has important applications in many research areas, such as statistical physics and image processing [3, 19, 42].

This paper focuses on numerical simulations of the area-conserved generalized curvature flow (1.1)–(1.2), paying special attention to the two essential geometric structure preserving aspects (area conservation and perimeter decrease). In the existing literature, there have been various numerical methods for the flow (1.1)–(1.2) with particular case \( \alpha = -1 \) and \( \beta = 1 \), i.e., classical area-conserved curvature flow. For example, [39] investigated the finite difference method, [40] proposed the level set method, [50] studied the crystalline algorithm, [34] considered the Merriman–Bence–Osher (MBO) method and [13] developed the parametric finite element method (PFEM). Among them, the PFEM is of great interest because it usually enjoys unconditional stability and preserves good mesh quality during time evolution. In fact, Dziuk first proposed a PFEM for the mean curvature flow based on the curvature identity, see Ref. [20]. Then Barrett, Garcke and Nürnberg (BGN) proposed a new formulation for the geometric curvature flows based on a formulation in mixed form, see Refs. [9, 10, 12]. Compared with other parametric methods in [4, 16, 17, 21, 33], the PFEM presented by BGN has an excellent performance in the mesh quality during time evolution, and has been extensively applied in the simulation of geometric flows [6, 7, 36, 51]. Numerical results in [13] also indicate that the PFEM presented based on the BGN approach for the classical area-conserved curvature flow has asymptotic equal mesh distribution property, that
is to say, mesh points tend to be equidistributed along the polygonal curve and eventually reach the equal distribution. This property prevents possible mesh distortion and avoids remesh during time evolution, which plays a vital role in numerical simulations. Moreover, the PFEM in [13] preserves unconditional perimeter decrease. However, the fully discrete scheme suffers from area loss at the discrete level, that is, the enclosed area of the discrete solution cannot be exactly conserved. Thus it is a natural requirement to develop a new numerical scheme for the area-conserved generalized curvature flow (1.1)–(1.2), which can inherit the good properties of the PFEM and further preserve the enclosed area.

Recently, for surface diffusion flow, which also has area conservation and perimeter decrease properties, we notice that a structure-preserving fully discrete scheme was introduced in [32] to maintain these two geometric properties at the discretized level. Nevertheless, the asymptotic equal mesh distribution property is no longer observed during time evolution. Subsequently, motivated by the original ideas of [32] and BGN [11], a structure-preserving PFEM (SP-PFEM) was constructed by Bao and Zhao in [8] for surface diffusion flow. The SP-PFEM attains area conservation exactly and perimeter decrease unconditionally, at the same time has asymptotic equal mesh distribution property. Very recently, by combining the skills from [8, 32] and axisymmetric variational discretizations, SP-PFEMs were developed in [5] for axisymmetric geometric evolution equations.

Inspired by [8, 32], we try to construct an SP-PFEM for the area-conserved generalized curvature flow (1.1)–(1.2) such that the two geometric structures and good mesh quality are well preserved. We have to deal with novel troubles and challenges brought by the nonlinear term $\kappa^\beta$ and the nonlocal term $\lambda$, and hope that the newly-established SP-PFEM is available for all $\alpha$ and $\beta$ cases. Firstly, we present a variational formulation and rigorously prove the properties of area conservation and perimeter decrease with the aid of the Cauchy–Schwarz inequality and power mean inequality. Then the variational formulation is approximated with PFEM in space to construct a semi-discrete scheme. By constructing appropriate functions, we provide a rigorous proof that the semi-discrete scheme preserves the two geometric structures. On this basis, by combining the backward Euler approximation in temporal direction and a suitable unit normal vector discretization, a structure-preserving fully discrete scheme is proposed successfully, which can maintain the two essential geometric structures simultaneously for the discrete solution. Numerical experiments reveal that the fully discrete scheme also enjoys asymptotic equal mesh distribution property. The main contribution of our work lies in: (I) we establish a unified structure-preserving PFEM for the area-conserved generalized curvature flow, which is accurate and efficient in practical simulations; (II) the structure-preserving properties are rigorously proved for the variational formulation, semi-discrete scheme and fully discrete scheme, respectively.

We organize this paper as follows. The variational formulation is introduced and investigated in Sect. 2. Section 3 focuses on the semi-discrete scheme and the rigorous proof of its structure-preserving property. On this basis, we propose the fully-discrete scheme in Sect. 4, rigorously prove the structure-preserving property, and further present iterative algorithms to solve the nonlinear system. After that, Sect. 5 provides numerical experiments to examine the accuracy, structure-preserving property and mesh quality during evolution. At last, Sect. 6 gives some conclusions.
2 A Variational Formulation

We will introduce and analyze a variational formulation for the area-conserved generalized curvature flow (1.1)–(1.2) in this section.

To begin with, we suppose that $\rho \in \mathbb{T}$ is a time independent variable over the periodic domain $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1]$, and further parameterize $\Gamma(t)$ by

$$X(\cdot, t) = (x(\cdot, t), y(\cdot, t))^T : \mathbb{T} \to \mathbb{R}^2, \quad (\rho, t) \mapsto X(\rho, t). \quad (2.1)$$

Then there holds

$$s(\rho, t) = \int_0^\rho |\partial_\rho X| d\rho,$$

which leads to $\partial\rho s = |\partial_\rho X|$. During the evolution, we need the following assumption: there exists a constant $C > 1$ to ensure $1/C \leq |\partial_\rho X| \leq C$, that is, $\Gamma(t)$ is regular for all $t$. Besides, the curvature $\kappa(\cdot, t)$ is supposed to satisfy the following requirement [38, 47, 48]:

$$\kappa(\cdot, t) \geq 0 \text{ for } 0 < \beta \neq 1 \quad \text{and} \quad \kappa(\cdot, t) > 0 \text{ for } \beta < 0, \quad \forall t \geq 0. \quad (2.2)$$

Next, we introduce a functional space as

$$L^2_p(\mathbb{T}) = \left\{ w : \mathbb{T} \to \mathbb{R} \bigg| \int_\mathbb{T} |w(\rho)|^2 d\rho < +\infty \right\},$$

and define the associated weighted $L^2$-inner product by

$$(w, \chi)_{\Gamma(t)} := \int_{\Gamma(t)} w(s)\chi(s)ds = \int_{\mathbb{T}} w(s(\rho, t))\chi(s(\rho, t))\partial_\rho s d\rho, \quad \forall w, \chi \in L^2_p(\mathbb{T}). \quad (2.4)$$

We note that the assumption $1/C \leq |\partial_\rho X| \leq C$ ensures the weighted inner products with respect to different $\Gamma(t)$s are equivalent, and thus the $L^2_p(\mathbb{T})$ can be interpreted as the usual $L^2(\mathbb{T})$ space. Then the Sobolev space $H^1_p(\mathbb{T})$ can be given as

$$H^1_p(\mathbb{T}) = \left\{ w : \mathbb{T} \to \mathbb{R} \bigg| w \in L^2_p(\mathbb{T}), \partial_\rho w \in L^2_p(\mathbb{T}) \right\}.$$ 

These notations can be easily extended to $\left[ L^2_p(\mathbb{T}) \right]^2$ and $\left[ H^1_p(\mathbb{T}) \right]^2$ for vector-valued functions.

In order to present the variational formulation, we adopt the identity $\kappa n = -\partial_\nu X$ given in [20] and reformulate (1.1) into

$$\partial_t X \cdot n = \alpha \kappa^\beta - \lambda, \quad (2.5a)$$

$$\kappa n = -\partial_\nu X. \quad (2.5b)$$

Then multiplying $\phi \in H^1_p(\mathbb{T})$ and $\psi = (\psi_1, \psi_2)^T \in \left[ H^1_p(\mathbb{T}) \right]^2$ to (2.5a) and (2.5b), respectively, and integrating over $\Gamma(t)$, we have

$$(\partial_t X \cdot n, \phi)_{\Gamma(t)} - (\alpha \kappa^\beta - \lambda, \phi)_{\Gamma(t)} = 0, \quad (2.6a)$$

$$(\kappa n, \psi)_{\Gamma(t)} - (\partial_\nu X, \partial_\nu \psi)_{\Gamma(t)} = 0. \quad (2.6b)$$
Therefore, a variational formulation for (1.1)–(1.2) is developed as follows: Suppose $\Gamma(0):=X(\cdot,0) \in \left[H_p^1(\mathbb{T})\right]^2$, find $(X(\cdot,t),\kappa(\cdot,t)) \in \left[H_p^1(\mathbb{T})\right]^2 \times H_p^1(\mathbb{T})$ for $t > 0$ satisfying
\[
\begin{align*}
\left(\partial_tX \cdot n, \phi\right)_{\Gamma(t)} - (\alpha\kappa^\beta - \lambda, \phi)_{\Gamma(t)} &= 0, \quad \phi \in H_p^1(\mathbb{T}), \\
(\kappa n, \psi)_{\Gamma(t)} - (\partial_sX, \partial_s\psi)_{\Gamma(t)} &= 0, \quad \psi \in [H_p^1(\mathbb{T})]^2.
\end{align*}
\tag{2.7}
\]
Let $\Gamma(t)$ be a simple closed curve, and denote its perimeter as $P(t)$, the region enclosed by $\Gamma(t)$ as $\Omega(t)$ with area $A(t)$, that is,
\[
P(t) := \int_{\Gamma(t)} 1 ds, \quad A(t) := \int_{\Omega(t)} 1 dx dy.
\tag{2.8}
\]
Then we have

**Theorem 1** (Area conservation and perimeter decrease) Assume that $(X(\cdot,t),\kappa(\cdot,t)) \in \left[H_p^1(\mathbb{T})\right]^2 \times H_p^1(\mathbb{T})$ is a solution of the variational formulation (2.7) with the assumption (2.2) holds on $\kappa(\cdot,t)$. Then for any $t \geq t_1 \geq 0$, there holds
\[
A(t) = A(t_1) \equiv A(0), \quad P(t) \leq P(t_1) \leq P(0),
\tag{2.9}
\]
i.e., area conservation and perimeter decrease.

**Proof** From the proposition 2.1 in [36], we obtain
\[
\frac{dA(t)}{dt} = (\partial_tX \cdot n, 1)_{\Gamma(t)},
\tag{2.10}
\]
which combining with (2.7) by taking $\phi \equiv 1$ yields
\[
\frac{dA(t)}{dt} = (\alpha\kappa^\beta - \lambda, 1)_{\Gamma(t)}
= \alpha \int_{\Gamma(t)} \kappa^\beta ds - \frac{\alpha}{P(t)} \int_{\Gamma(t)} \kappa^\beta ds \equiv 0.
\tag{2.11}
\]
Obviously, formula (2.11) indicates the area-preserving property in (2.9).

At the same time, similar to the argument of proposition 2.1 in [51] for the solid-state dewetting problem, we calculate the derivative of $P(t)$ as
\[
\frac{dP(t)}{dt} = (\partial_tX, \partial_s\partial_tX)_{\Gamma(t)}.
\tag{2.12}
\]
Setting $\psi = \partial_tX$ and $\phi = \kappa$ in the variational formulation (2.7), we get
\[
\frac{dP(t)}{dt} = (\kappa n, \partial_tX)_{\Gamma(t)} = (\alpha\kappa^\beta - \lambda, \kappa)_{\Gamma(t)}
= P(t) \alpha \left[ \frac{1}{P(t)} \int_{\Gamma(t)} \kappa^{\beta+1} ds - \frac{1}{P(t)^2} \left( \int_{\Gamma(t)} \kappa^\beta ds \right) \left( \int_{\Gamma(t)} \kappa ds \right) \right].
\tag{2.13}
\]
When $\beta = 1$, we know $\alpha < 0$. For any $\kappa \in \mathbb{R}$, by Cauchy–Schwarz inequality, we have
\[
\frac{dP(t)}{dt} \leq P(t) \alpha \left[ \frac{1}{P(t)} \int_{\Gamma(t)} \kappa^2 ds - \frac{1}{P(t)^2} \left( \int_{\Gamma(t)} \kappa ds \right) \left( \int_{\Gamma(t)} \kappa^2 ds \right) \right] = 0.
\tag{2.14}
\]
When $\beta = -1$, we know $\alpha > 0$ and $\kappa > 0$. Then by Cauchy–Schwarz inequality, we get

$$P(t)^2 \leq \int_{\Gamma(t)} \frac{1}{\kappa} ds \int_{\Gamma(t)} \kappa ds,$$

which combines with $\alpha > 0$ also implies

$$\frac{dP(t)}{dt} = P(t)\alpha \left[ \frac{1}{P(t)} \int_{\Gamma(t)} 1ds - \frac{1}{P(t)^2} \left( \int_{\Gamma(t)} \frac{1}{\kappa} ds \left( \int_{\Gamma(t)} \kappa ds \right) \right) \right] \leq 0. \quad (2.16)$$

Therefore the perimeter decrease property in (2.9) holds for $|\beta| = 1$. When $0 < |\beta| \neq 1$, to show $\frac{dP(t)}{dt} \leq 0$, we need the power mean inequality [27] [Page 144]:

Lemma 1 (Power mean inequality) For any finite interval $E = (c, d)$, and the non-zero parameters $p, q$ satisfy $-\infty < p < q < +\infty$. Suppose $f(z)$ is bounded and $f \geq 0$ in $E$ if $p > 0$, and $f > 0$ in $E$ if $p < 0$, we have

$$\left( \frac{\int_c^d f^p(z)dz}{|d-c|} \right)^\frac{1}{p} \leq \left( \frac{\int_c^d f^q(z)dz}{|d-c|} \right)^\frac{1}{q}. \quad (2.17)$$

For the case $0 < \beta \neq 1$, we know $\alpha < 0$ and $\kappa \geq 0$. By taking $E := \Gamma(t), (f, p, q) = (\kappa^\beta, 1, \frac{\beta+1}{\beta})$ and $(\kappa, 1, \beta + 1)$, respectively, we obtain

$$\frac{\int_{\Gamma(t)} \kappa^\beta ds}{P(t)} \leq \left( \frac{\int_{\Gamma(t)} \kappa^{\beta+1} ds}{P(t)} \right)^\frac{\beta}{\beta+1}, \quad \frac{\int_{\Gamma(t)} \kappa ds}{P(t)} \leq \left( \frac{\int_{\Gamma(t)} \kappa^{\beta+1} ds}{P(t)} \right)^\frac{1}{\beta+1}. \quad (2.18)$$

By substituting (2.18) into (2.13), we get

$$\frac{dP(t)}{dt} = P(t)\alpha \left[ \left( \frac{\int_{\Gamma(t)} \kappa^{\beta+1} ds}{P(t)} \right)^\frac{\beta}{\beta+1}, \left( \frac{\int_{\Gamma(t)} \kappa ds}{P(t)} \right)^\frac{1}{\beta+1} \right] \leq 0. \quad (2.19)$$

For the case $-1 \neq \beta < 0$, we know $\alpha > 0$ and $\kappa > 0$. By taking $E := \Gamma(t), (f, p, q) = (\kappa^\beta, \frac{\beta+1}{\beta}, 1)$ and $(\kappa, \beta + 1, 1)$, respectively, we obtain

$$\left( \frac{\int_{\Gamma(t)} \kappa^{\beta+1} ds}{P(t)} \right)^\frac{\beta}{\beta+1} \leq \frac{\int_{\Gamma(t)} \kappa^\beta ds}{P(t)} \leq \left( \frac{\int_{\Gamma(t)} \kappa^{\beta+1} ds}{P(t)} \right)^\frac{1}{\beta+1} \leq \frac{\int_{\Gamma(t)} \kappa ds}{P(t)}. \quad (2.20)$$

Similarly, by substituting (2.20) into (2.13), we obtain $\frac{dP(t)}{dt} \leq 0$. Thus, the perimeter decrease property in (2.9) is proven for all possible $\beta$ cases. The proof is completed. \qed

3 A PFEM Semi-discrete Scheme

By using PFEM to approximate (2.7) in space direction, we establish a semi-discrete scheme in this section and further rigorously prove that it is structure-preserving.

Assume that the interval $\mathbb{T}$ is divided into a uniform partition $\mathbb{T} = \bigcup_{i=1}^M T_i := \bigcup_{i=1}^M [\rho_{i-1}, \rho_i]$ with points $\{\rho_i := ih\}_{i=0}^M$, where $\rho_0 = \rho_M$ by periodic, the positive integer $M \geq 3$ and $h = 1/M$. Let $P_i(T_i)$ be a space composed by polynomials of degree no

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more than 1 defined on \( T_i \), then the conforming linear finite element space can be described by

\[
\mathbb{V}^h := \left\{ w^h \in C(\mathbb{T}) \big| w^h|_{T_i} \in \mathcal{P}_1(T_i), \ \forall i = 1, 2, \ldots, M \right\}.
\]

Let \( \Gamma^h(t) := X^h(\cdot, t) = (x^h(\cdot, t), y^h(\cdot, t))^T \in \mathbb{V}^h \) and \( \kappa^h(\cdot, t) \in \mathbb{V}^h \) be the finite element approximations of \( X(\cdot, t) \in \big[H^1_p(\mathbb{T})\big]^2 \) and \( \kappa(\cdot, t) \in H^1_p(\mathbb{T}) \), respectively. Then \( \Gamma^h(t) \) is a polygonal curve consists of ordered line segments \( \{h_i(t)\}_{i=1}^M := \{X^h(\rho_i, t) - X^h(\rho_{i-1}, t)\}_{i=1}^M \). Denote the length of \( h_i(t) \) by \( |h_i(t)| \), we always propose the following assumption

\[
\min_{1 \leq i \leq M} |h_i(t)| > 0, \quad \forall t > 0. \tag{3.1}
\]

In this situation, the two unit vectors \( n^h(t) \) and \( t^h(t) \), i.e., outward normal vector and tangential vector, are well defined and can be computed on each interval \( T_i \) by

\[
n^h|_{T_i} = -\frac{(h_i)^\perp}{|h_i|} := n_i^h, \quad t^h|_{T_i} = (n_i^h)^\perp = \frac{h_i}{|h_i|} := t_i^h, \quad \forall i = 1, 2, \ldots, M. \tag{3.2}
\]

Similar to the continuous equation, we further assume that

\[
\kappa^h(\rho_i, t) \geq 0 \text{ for } 0 < \beta \neq 1 \text{ and } \kappa^h(\rho_i, t) > 0 \text{ for } \beta < 0, \quad \forall i = 0, 1, \ldots, M, \ t \geq 0. \tag{3.3}
\]

Moreover, we employ the following inner product \( (\cdot, \cdot)^h_{\Gamma^h(t)} \) as

\[
(w, \chi)^h_{\Gamma^h(t)} := \frac{1}{2} \sum_{i=1}^M |h_i| \left[ (w \cdot \chi)(\rho_i^-) + (w \cdot \chi)(\rho_i^+) \right], \tag{3.4}
\]

where \( w \) and \( \chi \) represent piecewise continuous functions over \( \mathbb{T} \) and \( w(\rho_i^\pm) = \lim_{\rho \to \rho_i^\pm} w(\rho) \) for \( i = 0, 1, \ldots, M \).

Now we can set up a PFEM semi-discrete scheme according to the variational formulation \( 2.7 \) as follows: Let \( \Gamma^h(0) := \Gamma_0^h \) with \( X^h(\cdot, 0) \in \mathbb{V}^h \), find \( (X^h(\cdot, t), \kappa^h(\cdot, t)) \in \mathbb{V}^h \times \mathbb{V}^h \) for \( t > 0 \) satisfying

\[
\left\{ \begin{array}{l}
(\partial_t X^h \cdot n^h, \phi^h)^h_{\Gamma^h} - (\alpha(\kappa^h)^\beta - \lambda^h, \phi^h)^h_{\Gamma^h} = 0, \quad \phi^h \in \mathbb{V}^h, \\
(\kappa^h n^h, \psi^h)^h_{\Gamma^h} - (\partial_t X^h, \partial_n \psi^h)^h_{\Gamma^h} = 0, \quad \psi^h \in [\mathbb{V}^h]^2,
\end{array} \right. \tag{3.5}
\]

where \( \Gamma^h(0) \) is the linear interpolation function of \( \Gamma_0 \) meeting requirements \( X^h(\rho_i, 0) = X(\rho_i, 0), \forall i = 0, 1, 2, \ldots, M, \) and

\[
\lambda^h = \frac{(\alpha(\kappa^h)^\beta, 1)^h_{\Gamma^h}}{(1, 1)^h_{\Gamma^h}}. \tag{3.6}
\]

Denote \( (x_i^h(t), y_i^h(t))^T := X^h(\rho_i, t), i = 0, 1, \ldots, M \). Then for \( \Gamma^h(t) \), the enclosed area \( A^h(t) \) and perimeter \( P^h(t) \) can be computed by

\[
A^h(t) := \frac{1}{2} \sum_{i=1}^M [x_i^h(t) - x_{i-1}^h(t)] [y_i^h(t) + y_{i-1}^h(t)], \tag{3.7}
\]
and
\[ P_h(t) := \sum_{i=1}^{M} |h_i(t)| = (1,1)^h_{\Gamma_h}. \] (3.8)

Then we can rigorously prove that

**Theorem 2** (Area conservation and perimeter decrease) Suppose that \((X^h(\cdot,t), \kappa^h(\cdot,t)) \in \mathbb{V}^h \times \mathbb{V}^h\) satisfies (3.5) and the assumption (3.3) on \(\kappa^h(\cdot,t)\). Then for \(t \geq t_1 \geq 0\), there holds
\[
A^h(t) = A^h(t_1) \equiv A^h(0), \quad \text{(3.9)}
\]
\[
P^h(t) \leq P^h(t_1) \leq P^h(0), \quad \text{(3.10)}
\]
i.e., enclosed area conservation and perimeter decrease.

**Proof** By applying proposition 3.1 in [51], we obtain
\[
\frac{dA^h(t)}{dt} = (\partial_t X^h \cdot n^h, 1)^h_{\Gamma_h}. \quad \text{(3.11)}
\]

Taking \(\phi^h \equiv 1\) in (3.5), and employing (3.6) and (3.11), we are able to derive
\[
\frac{dA^h(t)}{dt} = \left(\alpha(\kappa^h)^\beta - \lambda^h, 1\right)^h_{\Gamma_h} = \frac{\alpha(\kappa^h)^\beta, 1}{(1,1)^h_{\Gamma_h}} (1,1)^h_{\Gamma_h} \equiv 0,
\]
which leads to (3.9) immediately. In other words, \(A^h(t)\) is conserved.

At the same time, computing the derivative of \(P^h(t)\), we have
\[
\frac{dP^h(t)}{dt} = (\partial_t \psi^h, \partial_t X^h)^h_{\Gamma_h}. \quad \text{(3.13)}
\]

In (3.5), we choose \(\phi^h = \kappa^h\) and \(\psi^h = \partial_t X^h\). Then using (3.5)–(3.6) and (3.13), we can deduce that
\[
\frac{dP^h(t)}{dt} = \left(\kappa^h n^h, \partial_t X^h\right)^h_{\Gamma_h} = \left(\alpha(\kappa^h)^\beta - \lambda^h, \kappa^h\right)^h_{\Gamma_h} \nonumber\]
\[
= \alpha \left[ (\kappa^h)^\beta, \kappa^h \right]^h_{\Gamma_h} - \left(\kappa^h, 1\right)^h_{\Gamma_h} \left(\kappa^h, 1\right)^h_{\Gamma_h}. \quad \text{(3.14)}
\]

When \(|\beta| = 1\), since \((\cdot,\cdot)^h_{\Gamma_h}\) is an inner product of the finite-dimensional space \(\mathbb{V}^h\), it follows from the Cauchy–Schwarz inequality that
\[
\left(\kappa^h, 1\right)^h_{\Gamma_h} \left(\kappa^h, 1\right)^h_{\Gamma_h} \leq \left(\kappa^h, \kappa^h\right)^h_{\Gamma_h} (1,1)^h_{\Gamma_h}, \quad \text{(3.15)}
\]
and
\[
(1,1)^h_{\Gamma_h} \left(\kappa^h, 1\right)^h_{\Gamma_h} \leq \left(\kappa^h, \kappa^h\right)^h_{\Gamma_h} (1,1)^h_{\Gamma_h}.
\]

\[ \leq \left(\kappa^h, \kappa^h\right)^h_{\Gamma_h} (1,1)^h_{\Gamma_h} \quad \text{(3.16)}
\]
\[
= \left(\kappa^h, 1\right)^h_{\Gamma_h}. \quad \text{(3.15)}
\]
Substituting (3.15)–(3.16) into (3.14) for $\beta = \pm 1$, respectively, and using $\alpha \beta < 0$, we deduce that $\frac{dp^h(t)}{dt} \leq 0$, which leads to the desired perimeter decrease result (3.10) for $|\beta| = 1$.

When $0 < |\beta| \neq 1$, similar to the proof of theorem 1, we will also use the power mean inequality in Lemma 1 to prove $\frac{dp^h(t)}{dt} \leq 0$.

Let $L_i = \sum_{j=1}^{i-1} |h_j|$ for $i = 2, 3, ..., M + 1$ and $L_1 = 0$. Then we define two functions by

$$f_1(z) = \begin{cases} (k^h)^\beta (\rho_{i-1}^+), & L_i \leq z < \frac{L_i + L_{i+1}}{2}, \\ (k^h)^\beta (\rho_i^-), & \frac{L_i + L_{i+1}}{2} \leq z < L_{i+1}, \end{cases} \quad (3.17)$$

and

$$f_2(z) = \begin{cases} (k^h)(\rho_{i-1}^+), & L_i \leq z < \frac{L_i + L_{i+1}}{2}, \\ (k^h)(\rho_i^-), & \frac{L_i + L_{i+1}}{2} \leq z < L_{i+1}. \end{cases} \quad (3.18)$$

Now for the case $0 < \beta \neq 1$, by taking $E := \Gamma^h$, $(f, p, q) = (f_1, 1, \frac{\beta + 1}{\beta})$ and $(f_2, 1, \beta + 1)$ in Lemma 1, respectively, we obtain

$$\frac{(k^h)^\beta, 1}{P^h(t)} = \frac{\sum_{i=1}^{M} \frac{1}{2} |h_i| ((k^h)^\beta (\rho_i^-) + (k^h)^\beta (\rho_{i-1}^+))}{P^h(t)}$$

$$= \int_{\Gamma^h} f_1(z) dz \leq \left( \int_{\Gamma^h} (f_1(z))^\frac{\beta + 1}{\beta} dz \right)^\frac{\beta}{\beta + 1}$$

$$= \left( \sum_{i=1}^{M} \frac{1}{2} |h_i| ((k^h)^{\beta + 1} (\rho_i^-) + (k^h)^{\beta + 1} (\rho_{i-1}^+)) \right)^\frac{\beta}{\beta + 1} \quad (3.19a)$$

and

$$\frac{(k^h, 1)}{P^h(t)} = \frac{\sum_{i=1}^{M} \frac{1}{2} |h_i| ((k^h)(\rho_i^-) + (k^h)(\rho_{i-1}^+))}{P^h(t)}$$

$$= \int_{\Gamma^h} f_2(z) dz \leq \left( \int_{\Gamma^h} (f_2(z))^\frac{\beta + 1}{\beta} dz \right)^\frac{1}{\beta + 1}$$

$$= \left( \sum_{i=1}^{M} \frac{1}{2} |h_i| ((k^h)^{\beta + 1} (\rho_i^-) + (k^h)^{\beta + 1} (\rho_{i-1}^+)) \right)^\frac{1}{\beta + 1} \quad (3.19b)$$
Then combining (3.19a) and (3.19b), we have
\[
\left(\frac{(k^h)_{\beta + 1}^h}{\rho^h(t)}\right)^{\beta} \geq \left(\frac{(k^h)_{\beta + 1}^h}{\rho^h(t)}\right)^{\beta} \left(\frac{(k^h)_{\beta + 1}^h}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}}
\]

Similarly, for the case $-1 \neq \beta < 0$, we set $E := \Gamma^h$, $(f, p, q) = (f_1^h, \beta + 1, 1)$ and $(f_2, \beta + 1, 1)$, respectively, then there holds
\[
\left(\frac{((k^h)_{\beta + 1}^h)}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}} \geq \left(\frac{((k^h)_{\beta + 1}^h)}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}} \left(\frac{((k^h)_{\beta + 1}^h)}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}}
\]

Using (3.21a) and (3.21b), by the same argument as (3.20), we have
\[
\left(\frac{(k^h)_{\beta + 1}^h}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}} \leq \left(\frac{(k^h)_{\beta + 1}^h}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}} \left(\frac{(k^h)_{\beta + 1}^h}{\rho^h(t)}\right)^{\frac{1}{\beta + 1}}
\]

Combining (3.14), (3.20), (3.22) and $0 \beta < 0$, we obtain $\frac{\rho^h(t)}{\rho^h(t)} \leq 0$, which means that the desired perimeter decrease result (3.10) is also valid for $0 < |\beta| \neq 1$. To sum up, the PFEM semi-discrete scheme (3.5) is perimeter decreasing for all $\beta$ cases. The proof is completed.

\[\square\]

4 A Structure-Preserving PFEM

4.1 A Fully-Discrete Scheme

In temporal direction, we take the size of time step as $\tau > 0$ and set $t_n = n\tau$ for integer $n \geq 0$ to represent the discrete time levels. Assume that $\Gamma^n := X^n(\cdot) = (x^n(\cdot), y^n(\cdot))^T \in \mathbb{R}^{h^h}$ and $k^n(\cdot) \in \mathbb{R}^{h^h}$ are the numerical approximations of $\Gamma^h(t_n) := X^h(\cdot, t_n)$ and $k^h(\cdot, t_n)$, respectively, here $(X^h(\cdot, t), k^h(\cdot, t)) \in \mathbb{R}^{h^h} \times \mathbb{R}^{h^h}$ satisfies the semi-discrete scheme (3.5).

Then $\Gamma^n$ is composed by segments \[\{h^n_i\}_{i=1}^M := (X^n(\rho_i) - X^n(\rho_{i-1}))_{i=1}^M\]. Similar to Sect. 3, for each $n \geq 0$, we also propose the following assumptions on $h^n_i$ and $k^n(\cdot)$ as
\[
\min_{1 \leq i \leq M} |h^n_i| > 0, \quad \forall i = 1, 2, \ldots, M,
\]

(4.1)
and
\[ \kappa^n(\rho_i) \geq 0, \text{ for } 0 < \beta \neq 1 \quad \text{and} \quad \kappa^n(\rho_i) > 0, \text{ for } \beta < 0, \quad \forall i = 0, 1, \ldots, M. \] (4.2)

After that, notations defined on \( \Gamma^n \), including the unit normal vector \( \mathbf{n}^n \), tangential vector \( \mathbf{t}^n \) and inner product \((\cdot, \cdot)^h_{\Gamma^n}\) can be defined similarly to that in (3.2) and (3.4) for \( \Gamma^h(I) \).

Now employing the backward Euler method and an appropriate discretization for unit normal vector, we construct a structure-preserving fully discrete scheme as: Let \( h \) and \( \tau \) satisfies
\[ h = \tau \] (Area conservation and perimeter decrease)

Then we can prove that the fully-discrete scheme (4.3) is structure-preserving.

\[ \begin{cases} 
\left( \frac{X^{n+1} - X^n}{\tau} \cdot \frac{\mathbf{n}^{n+\frac{1}{2}}}{\mathbf{t}^n} \right)^h_{\Gamma^n} - (\alpha(\kappa^{n+1})^\beta - \kappa^{n+1}, \phi^h_{\Gamma^n})^h_{\Gamma^n} = 0, \quad \phi^h_{\Gamma^n} \in \mathcal{V}^h, \\
(\kappa^{n+1} \mathbf{n}^{n+\frac{1}{2}}, \psi^h_{\Gamma^n})^h_{\Gamma^n} - (\partial_x X^{n+1}, \partial_x \psi^h_{\Gamma^n})^h_{\Gamma^n} = 0, \quad \psi^h_{\Gamma^n} \in \mathcal{V}^h, 
\end{cases} \] (4.3)

Here
\[ \lambda^{n+1} = \frac{(\alpha(\kappa^{n+1})^\beta, 1)^h_{\Gamma^n}}{(1, 1)^h_{\Gamma^n}}, \quad \mathbf{n}^{n+\frac{1}{2}} = - \left( \frac{\partial_x X^n(\rho) + \partial_x X^{n+1}(\rho)}{2|\partial_x X^n(\rho)|} \right)^\perp. \] (4.4)

### 4.2 Area Conservation and Perimeter Decrease

For \( \Gamma^n \), the enclosed area \( A^n \) and the perimeter \( P^n \) can be given by
\[ A^n = \frac{1}{2} \sum_{i=1}^{M} (x_i^n - x_{i-1}^n)(y_i^n - y_{i-1}^n), \quad P^n = \sum_{i=1}^{M} |h_{i}^n| = (1, 1)^h_{\Gamma^n}, \quad n \geq 0. \] (4.5)

Then we can prove that the fully-discrete scheme (4.3) is structure-preserving.

**Theorem 3** (Area conservation and perimeter decrease) *Suppose that \( (X^{n+1}(\cdot), \kappa^{n+1}(\cdot)) \in [\mathcal{V}^h]^2 \times \mathcal{V}^h \) satisfies (4.3) and the assumption (4.2). Then there holds*
\[ A^{n+1} = A^n \equiv A^0, \quad P^{n+1} \leq P^n \leq \ldots \leq P^0, \] (4.6)
i.e., the area is preserved and the perimeter is decreasing.

**Proof** By using \( X^{n+1} \) and \( X^n \), we introduce \( \Gamma^h(\vartheta) = X^h(\rho, \vartheta) \) as
\[ \Gamma^h(\vartheta) = (1 - \vartheta)X^n(\rho) + \vartheta X^{n+1}(\rho), \quad \rho \in \mathbb{T}, \quad \vartheta \in [0, 1], \] (4.7)
and use \( B(\vartheta) \) to represent the area enclosed by \( \Gamma^h(\vartheta) \). Then applying theorem 2.1 in [8] and taking \( \phi^h \equiv 1 \) in (4.3), we obtain
\[ B(1) - B(0) = \left( \frac{X^{n+1} - X^n}{\tau} \cdot \frac{\mathbf{n}^{n+\frac{1}{2}}}{\mathbf{t}^n} \right)^h_{\Gamma^n} = \tau \alpha \left[ \left( \frac{(\kappa^{n+1})^\beta, 1)^h_{\Gamma^n}}{(1, 1)^h_{\Gamma^n}} \right)_{\Gamma^n} - \frac{((\kappa^{n+1})^\beta, 1)^h_{\Gamma^n}}{(1, 1)^h_{\Gamma^n}} \right]_{\Gamma^n} \equiv 0, \] (4.8)
which implies \( B(1) = B(0) \), i.e., \( A^{n+1} = A^n \). Thus the first formula in (4.6) is true.
Now we aim to get the perimeter decrease property. Choosing $\phi^h = \kappa^{n+1}$ and $\psi^h = X^{n+1} - X^n$ in (4.3), we can arrive at
\[
\left( \partial_s X^{n+1}, \partial_s (X^{n+1} - X^n) \right)_{\Gamma_n}^h = \left( \kappa^{n+1} n^{n+\frac{1}{2}}, (X^{n+1} - X^n) \right)_{\Gamma_n}^h
\]
\[= \tau \left( \alpha(\kappa^{n+1})^\beta - \chi^{n+1}, \kappa^{n+1} \right)_{\Gamma_n}^h
\]
\[= \tau \alpha \left[ \left( (\kappa^{n+1})^\beta, \kappa^{n+1} \right)_{\Gamma_n}^h - \left( \frac{(\kappa^{n+1})^\beta}{1}, \frac{1}{h_{\Gamma_n}} \right)_{\Gamma_n}^h \right].
\]
(4.9)

Then taking a similar argument as (3.15)–(3.20) in theorem 2, we know that
\[
\left( \partial_s X^{n+1}, \partial_s (X^{n+1} - X^n) \right)_{\Gamma_n}^h \leq 0.
\]
(4.10)

On the other hand, following the theorem 2.2 in [8], there holds
\[
\left( \partial_s X^{n+1}, \partial_s (X^{n+1} - X^n) \right)_{\Gamma_n}^h \geq P^{n+1} - P^n.
\]
(4.11)

Then the desired second result in (4.6) follows from (4.10) and (4.11) immediately. Therefore, the fully discrete scheme (4.3) indeed preserves the two geometric structures. The proof is completed. \( \square \)

4.3 The Iterative Solver

The Newton’s iterative method can be adopted to derive the solution $(X^{n+1}(-), \kappa^{n+1}(-)) \in \left[ \mathcal{V}^h \right]^2 \times \mathcal{V}^h$ of the nonlinear equations (4.3):

(1) For each $n \geq 0$, we take the initial data as $X^{n+1,0} = X^n, \kappa^{n+1,0} = \kappa^n$.

(2) For given $(X^{n+1,l}(-), \kappa^{n+1,l}(-)) \in \left[ \mathcal{V}^h \right]^2 \times \mathcal{V}^h$, we compute $X^{n+1,l+1}(-) = X^{n+1,l}(-) + X^\delta(-), \kappa^{n+1,l+1}(-) = \kappa^{n+1,l}(-) + \kappa^\delta(-)$ via the Newton direction $(X^\delta(-), \kappa^\delta(-)) \in \left[ \mathcal{V}^h \right]^2 \times \mathcal{V}^h$ obtained by
\[
\left( \frac{X^\delta}{\tau} \cdot n^{n+\frac{1}{2},l}, \phi^h \right)_{\Gamma_n}^h + \left( \frac{X^{n+1,l} - X^n}{\tau} \cdot \frac{(-\partial_{\rho} X^\delta)^\perp}{2|\partial_{\rho} X^n|}, \phi^h \right)_{\Gamma_n}^h
\]
\[= -\alpha \beta \left( (\kappa^{n+1,l})^\beta - 1, \kappa^\delta \right)_{\Gamma_n}^h + \alpha \beta \left( \frac{((\kappa^{n+1,l})^\beta - 1, \kappa^\delta)_{\Gamma_n}^h}{(1, 1)_{\Gamma_n}^h}, \phi^h \right)_{\Gamma_n}^h
\]
\[= -\left( \frac{X^{n+1,l} - X^n}{\tau} \cdot n^{n+\frac{1}{2},l}, \phi^h \right)_{\Gamma_n}^h + \alpha \left( (\kappa^{n+1,l})^\beta - \frac{((\kappa^{n+1,l})^\beta, 1)_{\Gamma_n}^h}{(1, 1)_{\Gamma_n}^h}, \phi^h \right)_{\Gamma_n}^h,
\]
(4.12)

and
\[
\left( \kappa^\delta n^{n+\frac{1}{2},l}, \psi^h \right)_{\Gamma_n}^h + \left( \kappa^{n+1,l} \frac{(-\partial_{\rho} X^\delta)^\perp}{2|\partial_{\rho} X^n|}, \psi^h \right)_{\Gamma_n}^h - \left( \partial_s X^\delta, \partial_s \psi^h \right)_{\Gamma_n}^h
\]
\[= -\left( \kappa^{n+1,l} n^{n+\frac{1}{2},l}, \psi^h \right)_{\Gamma_n}^h + \left( \partial_s X^{n+1,l}, \partial_s \psi^h \right)_{\Gamma_n}^h,
\]
(4.13)
where
\[ n^{n+\frac{1}{2},l} = \frac{1}{2|\partial\rho X^n|} \left( \partial_\rho X^n + \partial_\rho X^{n+1,l} \right) \downarrow. \]

(III) Set \( X^{n+1,l} = X^{n+1,l+1} \) and \( \kappa^{n+1,l} = \kappa^{n+1,l+1} \), then re-use step (II) to derive new \( X^{n+1,l+1} \) and \( \kappa^{n+1,l+1} \).

(IV) Introduce the tolerance \( tol \) and repeat the step (III) until
\[
\max_{0 \leq n \leq M} \left( |X^{n+1,l+1}(\rho_i) - X^{n+1,l}(\rho_i)| + |\kappa^{n+1,l+1}(\rho_i) - \kappa^{n+1,l}(\rho_i)| \right) \leq tol,
\]
then \( (X^{n+1,l+1}, \kappa^{n+1,l+1}) \) is the desired solution.

In addition, the Picard iteration method can be taken as another solver. We just need to replace step (II) with (II): For given \( \kappa^{n+1,l} \in \mathbb{R}^h \), seek \( (X^{n+1,l+1}, \kappa^{n+1,l+1}) \in [\mathbb{R}^h]^2 \times \mathbb{R}^h \) satisfying
\[
\begin{align*}
& \left( X^{n+1,l+1} \cdot n^{n+\frac{1}{2},l}, \phi^h \right)_{\Gamma^n} - a \left( (\kappa^{n+1,l})^{\beta-1} \kappa^{n+1,l+1}, \phi^h \right)_{\Gamma^n} \\
& \quad + a \left( (\kappa^{n+1,l})^{\beta-1} \kappa^{n+1,l+1}, 1 \right)_{\Gamma^n} \left( X^n \cdot n^{n+\frac{1}{2},l}, \phi^h \right)_{\Gamma^n} = 0.
\end{align*}
\]

5 Numerical Results

Numerical results of the SP-PFEM (4.3), including the convergence rate, area conservation, perimeter decrease and mesh quality will be reported in this section. Moreover, the morphological evolutions of closed curves are also simulated.

We compute the numerical error function \( e^h(t) \) to investigate the convergence rate, employ the relative area loss function \( \frac{\Delta A^h(t)}{A^n(0)} \) and normalized perimeter function \( \frac{W^h(t)}{W^n(0)} \) to examine the trends of area and perimeter, and introduce the mesh ratio function \( R^h(t) \) to estimate the quality of mesh, respectively. These functions are defined by
\[
e^h(t) \big|_{t=t^i} := M(\Gamma^n, \Gamma(t^i)) = |(\Omega_n \setminus \Omega_{t^i}) \cup (\Omega_{t^i} \setminus \Omega_n)| = |\Omega_n| - |\Omega_{t^i}| - 2|\Omega_n \cap \Omega_{t^i}|,
\]
\[
\frac{\Delta A^h(t)}{A^n(0)} \big|_{t=t^i} := \frac{A^n - A^0}{A^0}, \quad \frac{W^h(t)}{W^n(0)} \big|_{t=t^i} := \frac{P^n}{P^0}, \quad R^h(t) \big|_{t=t^i} := \max_{1 \leq i \leq M} \frac{|h_i^n|}{\min_{1 \leq i \leq M} |h_i^n|}.
\]

where \( M(\cdot, \cdot) \) is the manifold distance [51], \( \Omega_n \) and \( \Omega_{t^i} \) are the regions enclosed by \( \Gamma^n \) and \( \Gamma(t^i) \), respectively, \( |\Omega| \) represents the area of region \( \Omega \), \( A^n \) and \( P^n \) have been given in (4.5).

During the numerical experiments, the size parameters \( \tau \) and \( h \) are taken to meet \( \tau = O(h^2) \), e.g. \( \tau = h^2 \), \( \Gamma(t_{m}) \) is replaced with \( \Gamma^m \) computed with very small size parameters \( h = 2^{-8} \) and \( \tau = 2^{-16} \), and Newton’s iteration method with the tolerance \( 10^{-12} \) is used to solve the nonlinear equations (4.3).

We take the initial shape as an ellipse curve \( \frac{x^2}{\tau^2} + \frac{y^2}{\tau^2} = 1 \) and provide six choices of the parameters \( \alpha \) and \( \beta \) with \( |\alpha| = 1 \) satisfying \( \alpha \beta < 0 \), i.e., \( a) \ (\alpha, \beta) = (-1, 1) \), \( b) \ (\alpha, \beta) = \)
Fig. 1 Numerical errors \( e^h(t) \) with \( \tau = h^2 \) at different time points \( t \) for six types of parameters \( \alpha \) and \( \beta \)

\((-1, 2), (c) (\alpha, \beta) = (-1, 1/3), (d) (\alpha, \beta) = (1, -1), (e) (\alpha, \beta) = (1, -2), (f) (\alpha, \beta) = (1, -1/3)\). These choices are employed to verify the performance of SP-PFEM (4.3) in simulating the area-conserved generalized curvature flow (1.1)–(1.2).

To begin with, we plot numerical errors \( e^h(t) \) at different time points \( t \) for six types of parameters \( \alpha \) and \( \beta \) in Fig. 1. It is observed that the SP-PFEM (4.3) is convergent to order \( O(h^2) \) in the spatial direction.

We depict the relative area loss function \( \Delta A^h(t)/A^h(0) \) (blue line) for different parameters \( \alpha \) and \( \beta \) with \( h = 2^{-5} \), \( \tau = \frac{2}{25} h^2 \) and show the iteration numbers of Newton’s iterative (black line) in Fig. 2. From Fig. 2, we see that the relative area loss is of the order of magnitude about \( 10^{-15} \), which is close to the machine epsilon around \( 10^{-16} \). Thus the area conservation property of the SP-PFEM (4.3) is confirmed. In addition, at each time step, we observe that the number of Newton’s iteration is at most 3. This indicates the effectiveness of the algorithm.

We draw the time evolutions of the normalized perimeter \( W^h(t)/W^h(0) \) with fixed \( h = 2^{-4} \) for different time step \( \tau \) in Fig. 3. It is easy to see that the normalized perimeter indeed decreases in time under different \( \tau \), which numerically substantiates the theoretical analysis in theorem 3. The mesh ratio function \( R^h(t) \) is depicted in Fig. 4 for different \( h \). In the beginning, an equal partition of the polar angle is employed to divide the initial curve, which results in the mesh ratio \( R^h(0) \) close to 3. As time evolves, we can see \( R^h(t) \) gradually decreases and eventually converges to 1, which indicates the asymptotic mesh equal distribution. This property can follow proposition 2.1 in [8].

The curve evolutions are depicted in Fig. 5. It can be observed that all curves evolve into a circle finally, which is in line with the theoretical results in the literature [38, 47, 48] on the equilibrium shape of the area-conserved generalized curvature flow (1.1)–(1.2). Moreover, we see that the generating curves remain convex during the simulation. This is also consistent with the convexity preserving property of the flow (1.1)–(1.2) for a simple smooth strictly convex closed initial curve [24, 47].

In addition, from the time points’ chosen in Fig. 5, one can discover that the larger \( |\beta| \), the faster the curve evolves into a circle. This fact can be revealed more clearly by comparing the perimeters under these six types of parameters \( \alpha \) and \( \beta \) (see Fig. 6). From Fig. 6, we observe
Fig. 2 Relative area loss function \( \frac{\Delta A^h(t)}{A^h(0)} \) (blue line) and iteration number (black line) of the SP-PFEM (4.3) for six types of parameters \( \alpha \) and \( \beta \), where we choose fixed \( h = 2^{-5} \) and \( \tau = \frac{2}{25} h^2 \) (Color figure online)

Fig. 3 Normalized perimeter \( \frac{W^h(t)}{W^h(0)} \) of the SP-PFEM (4.3) with fixed \( h = 2^{-4} \) for different \( \tau \) and six types of parameters \( \alpha, \beta \) that with increasing \( |\beta| \), the corresponding perimeter decreases to the stable value faster and faster. At the same time, we also make an investigation on how the parameter \( \alpha \) influences the curve evolution. The perimeter results for different \( \alpha \) choices are plotted in Fig. 7. It is easy to see that for fixed \( \beta \), the increase of \( |\alpha| \) will speed up the evolution of the curve.

At last, for the case \((\alpha, \beta) = (-1, 1)\), we further use the SP-PFEM (4.3) to simulate the motion of following three more complex initial curves

Curve I: \[
\begin{align*}
x &= (2 + \cos(6\theta)) \cos \theta \\
y &= (2 + \cos(6\theta)) \sin \theta
\end{align*}
\]

Curve II: \[
\begin{align*}
x &= \cos \theta \\
y &= 2 \sin \theta - 1.9 \sin^3 \theta
\end{align*}
\]
Fig. 4 Mesh ratio $R^h(t)$ of the SP-PFEM (4.3) with $\tau = h^2$ for different $h$ and six types of parameters $\alpha, \beta$

Fig. 5 Evolutions of the initial ellipse curve (red solid line) to the equilibrium shape (blue solid line) at different times (dashed lines) for six types of parameters $\alpha$ and $\beta$, where time points are chosen as follows: $t \in \{0.25, 0.75, 2\}$ in (a); $t \in \{0.25, 0.75, 2\}$ in (b); $t \in \{0.5, 2, 4\}$ in (c); $t \in \{0.05, 0.2, 0.5\}$ in (d); $t \in \{0.01, 0.1, 0.2\}$ in (e); $t \in \{0.5, 1.25, 2.5\}$ in (f). Here we choose $h = 2^{-6}$ and $\tau = h^2$ (Color figure online)

Curve III: \[
x = \cos \theta \\
y = \frac{1}{2} \sin \theta + \sin(\cos \theta) + (0.2 + \sin \theta \sin^2(3\theta)) \sin \theta
\]
with $\theta \in [0, 2\pi]$. The curve evolutions of these initial shapes are drawn in Figs. 8, 9 and 10, respectively. Figure 8 reveals that the six petals of Curve I gradually disappear during time evolution and a circle is finally formed. Analogous numerical results for Curve II and Curve III can be observed in Figs. 9 and 10, respectively. The reliability and applicability of the SP-PFEM (4.3) are further verified through these numerical examples.
Fig. 6 Normalized perimeter $\frac{W_h(t)}{W_h(0)}$ of the SP-PFEM (4.3) with fixed $h = 2^{-4}$ for six types of parameters $\alpha, \beta$.

Fig. 7 Normalized perimeter $\frac{W_h(t)}{W_h(0)}$ of the SP-PFEM (4.3) with fixed $h = 2^{-4}$ for different types of parameters $\alpha, \beta$. Here $\beta$ is fixed in every subplot.
Fig. 8  Evolution of Curve I with $h = 2^{-7}$ and $\tau = h^2$ at times: (i) $t = 0$, (j) $t = 0.05$, (k) $t = 0.15$, (l) $t = 0.25$, (m) $t = 0.4$, (n) $t = 1$

Fig. 9  Evolution of Curve II with $h = 2^{-7}$ and $\tau = h^2$ at times: (i) $t = 0$, (j) $t = 0.05$, (k) $t = 0.1$, (l) $t = 0.2$, (m) $t = 0.4$, (n) $t = 1$
Fig. 10 Evolution of Curve III with $h = 2^{-7}$ and $\tau = h^2$ at times: (i) $t = 0$, (j) $t = 0.05$, (k) $t = 0.1$, (l) $t = 0.2$, (m) $t = 0.5$, (n) $t = 1$

6 Conclusions

This work investigated numerical simulations of the area-conserved generalized curvature flow. Firstly, we give the variational formula. Then the semi-discrete scheme is constructed by using the PFEM method. Finally, the SP-PFEM fully discrete scheme is established by further combining the backward Euler method and a suitable discretization of the normal vector, which works for all possible cases of $\alpha$ and $\beta$. We rigorously prove that the two essential geometric structures are well preserved. Numerical results confirm the structure-preserving property and manifest that the SP-PFEM has second-order accuracy and enjoys asymptotic equal mesh distribution during the evolution. Therefore, this paper gives an accurate and efficient numerical method for the area-conserved generalized curvature flow in 2D. In the future, we will further explore the extension of the new SP-PFEM for volume-preserving generalized mean curvature flow in three dimensions.

Acknowledgements The authors wish to thank Prof. Weizhu Bao for his advice and encouragement while this work was undertaken.

Author Contributions LP: Paper writing, Theoretical analysis, Numerical tests. YL: Theoretical analysis, Writing-review and editing. All authors read and approved the final manuscript.

Funding This research was supported by National Natural Science Foundation of China (Grant No. [11701523]).

Data Availability The data generated or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.
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