When the kernel of a complete hereditary cotorsion pair is the additive closure of a tilting module

Jian Wang\textsuperscript{a}, Yunxia Li\textsuperscript{a} and Jiangsheng Hu\textsuperscript{b}\textsuperscript{*}

\textsuperscript{a}Department of Basic Science, Jinling Institute of Technology, Nanjing 211169, China
\textsuperscript{b}School of Mathematics and Physics, Jiangsu University of Technology
Changzhou 213001, China
E-mail: wangjian@jit.edu.cn, liyunxia@jit.edu.cn and jiangshenghu@jsut.edu.cn

Abstract

In this paper, we study when the kernel of a complete hereditary cotorsion pair is the additive closure of a tilting module. Applications go in three directions. The first is to characterize when the little finitistic dimension is finite. The second is to obtain equivalent formulations for a Wakamatsu tilting module to be a tilting module. The third is to give some new characterizations of Gorenstein rings.

Key Words: cotorsion pair; additive closure; strongly Gorenstein projective module; tilting module.
2010 Mathematics Subject Classification: 18G10; 18G20; 16E65.

1. Introduction

Tilting theory started in the context of finitely generated modules over Artin algebras and was further generalized over arbitrary associative rings with unit and to infinitely generated modules (see [1, 2, 18, 19, 20, 28]). Recall that the tilting class $\mathcal{B}$ associated to a tilting module $T$ over a ring $R$ is the class of $R$-modules satisfying $\mathcal{B} = T^{\perp_\infty}$ [28]. Tilting modules and classes occur naturally in various areas of contemporary module theory. For example, finiteness of the left little finitistic dimension of a left Noetherian ring $R$ is equivalent to the existence of a particular tilting class (see [3, Theorem 2.6]).

Cotorsion pairs were invented by Salce [40] in the category of abelian groups and have been deeply studied in approximation theory of modules [28], especially in the proof of the Flat Cover Conjecture [9]. Let $\mathcal{K}_\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ be the kernel of the cotorsion pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$. The additive closure $\text{Add}M$ of a module $M$ over a ring $R$ is defined as the class of all modules that are isomorphic to direct summands of direct sums of copies $M$. An interesting and deep result in [1] is that an $n$-tilting class $\mathcal{B}$ can be characterized by the properties of cotorsion pairs: $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair, $\mathcal{A}$ consists of modules of projective dimension at most $n$ and $\mathcal{K}_\mathcal{C}$ is closed under arbitrary direct sums. We note that the proof of this result relies on the fact that for any complete hereditary cotorsion pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$, if $\mathcal{B}$ is an $n$-tilting class then $\mathcal{K}_\mathcal{C} = \text{Add}T$ for some $n$-tilting $R$-module $T$. However the converse is

\textsuperscript{*}Corresponding author.
not true in general (see Example 2.6). The following is our first main result which gives some criteria for the kernel of a complete hereditary cotorsion pair to be the additive closure of a tilting module.

**Theorem 1.1.** Let \( R \) be a ring and \( \mathcal{C} = (A, B) \) a complete hereditary cotorsion pair of \( R \)-modules. Then the following are equivalent for any nonnegative integer \( n \):

1. \( K_\mathcal{C} = \text{Add} T \), where \( T \) is an \( n \)-tilting \( R \)-module.
2. \( K_\mathcal{C} \subseteq \mathcal{P}_n \), \( A \subseteq \mathcal{G}\mathcal{P}_n \) and \( K_\mathcal{C} \) is closed under direct sums.
3. \( K_\mathcal{C} \subseteq \mathcal{P}_n \) and \( B = T^{\perp \infty} \cap \mathcal{X}^{\perp \infty} \), where \( T \) is an \( n \)-tilting \( R \)-module and \( \mathcal{X} \) is a class of strongly Gorenstein projective \( R \)-modules.

Moreover, if \( B = G^{\perp \infty} \) for an \( R \)-module \( G \), then the above conditions are equivalent to

4. \( B = T^{\perp \infty} \cap N^{\perp \infty} \), where \( T \) is an \( n \)-tilting \( R \)-module and \( N \) is a strongly Gorenstein projective \( R \)-module.

5. \( K_\mathcal{C} \) is closed under direct sums and there is a strongly Gorenstein projective \( R \)-module \( M \) in \( A \) such that \( \Omega^n G \) is a direct summand of \( M \).

6. \( K_\mathcal{C} \) is closed under direct sums and there is a strongly Gorenstein projective \( R \)-module \( N \) in \( A \) such that \( (\Omega^n G)^{\perp \infty} = N^{\perp \infty} \).

Let \( \mathcal{P}^{<\infty} (\mathcal{G}\mathcal{P}^{<\infty}) \) be the class of finitely generated modules with finite projective (Gorenstein projective) dimension. Recall that the left little finitistic dimension of a ring \( R \) is

\[
\text{findim}(R) = \sup\{pd_M \mid M \in \mathcal{P}^{<\infty}\}.
\]

As the first application of Theorem 1.1, the next result characterizes when the little finitistic dimension is finite. See 4.3 for the proof.

**Theorem 1.2.** Let \( R \) be a left Noetherian ring. Then the following are equivalent:

1. \( \text{findim}(R) < \infty \).
2. \( K_\mathcal{C} = \text{Add} T \), where \( \mathcal{C} = (\perp(\mathcal{P}^{<\infty})^{\perp \infty}, (\mathcal{P}^{<\infty})^{\perp \infty}) \) and \( T \) is a tilting \( R \)-module.
3. \( K_\mathcal{C} = \text{Add} T \), where \( \mathcal{C} = (\perp(\mathcal{G}\mathcal{P}^{<\infty})^{\perp \infty}, (\mathcal{G}\mathcal{P}^{<\infty})^{\perp \infty}) \) and \( T \) is a tilting \( R \)-module.

The famous Finitistic Dimension Conjecture states that the little finitistic dimension \( \text{findim}(R) \) is finite for every Artin algebra \( R \) (see [4, 6]). Theorem 1.2 above gives criteria for the validity of this conjecture.

Wakamatsu in [35] introduced and studied the so-called generalized tilting modules, which are usually called Wakamatsu tilting modules, see [12, 32]. Tilting modules are Wakamatsu tilting modules, but the converse is not true in general because a Wakamatsu tilting module can have infinite projective dimension. As the second application of Theorem 1.1, we have the next result which gives equivalent formulations for a Wakamatsu tilting module to be a tilting module. See Theorem 4.4.

**Theorem 1.3.** Let \( R \) be a ring and \( \omega \) a Wakamatsu tilting \( R \)-module. Fix an exact sequence

\[
0 \rightarrow R \rightarrow \omega_0 \xrightarrow{f_0} \cdots \rightarrow \omega_i \xrightarrow{f_i} \cdots \text{ with } \omega_i \in \text{add} \omega \text{ and } \omega \in \ker(f_i)^{\perp \infty} \text{ for } i \geq 0.
\]

If we set \( A = \bigoplus_{i \geq 0} \ker(f_i) \), then the following are equivalent for any nonnegative integer \( n \):

1. \( \omega \) is an \( n \)-tilting \( R \)-module.
(2) $\omega^{\perp \geq n} = A^{\perp \geq n}$ and $\mathcal{K}_\mathcal{C} = \text{Add}T$, where $\mathcal{C} = (\perp(A^{\perp \infty}), A^{\perp \infty})$ and $T$ is an $n$-tilting $R$-module.

(3) $\mathcal{K}_\mathcal{C} = \text{Add}T$, where $\mathcal{C} = (\perp(\omega \oplus A)^{\perp \infty}, (\omega \oplus A)^{\perp \infty})$ and $T$ is an $n$-tilting $R$-module.

As a consequence of Theorem 1.3, we characterize when a Wakamatsu tilting module of finite projective dimension is a tilting module, see Corollary 4.5.

Recall that a ring $R$ is called Gorenstein (or Iwanaga-Gorenstein) [30] if it is both left and right Noetherian and $R$ has finite self-injective dimension on either side. In the case of commutative rings, this definition of Gorenstein rings coincides with Gorenstein rings of finite Krull dimension originally defined by Bass in [7]. For more details about Gorenstein rings, see [5, Section 3] and [26, Chapter 9].

As the third application of Theorem 1.1, the following result gives some new characterizations of Gorenstein rings. See Propositions 4.8 and 4.9.

**Theorem 1.4.** Let $R$ be a commutative ring. Then the following are equivalent:

1. $R$ is a Gorenstein ring.
2. $\mathcal{K}_\mathcal{C} = \text{Add}T$, where $\mathcal{C} = (\perp GI, GI)$ and $T$ is a tilting $R$-module.
3. $\mathcal{K}_\mathcal{C} = \text{Add}T$, where $\mathcal{C} = (R\text{-Mod}, I)$ and $T$ is a tilting $R$-module.

Moreover, if $R$ is a commutative Noetherian ring of finite Krull dimension, then the above conditions are equivalent to

4. For any exact sequence $\cdots \rightarrow T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} \cdots$ of tilting $R$-modules, $\mathcal{K}_\mathcal{C} = \text{Add}T$, where $\mathcal{C} = (\perp((\oplus \ker(d_i))^{\perp \infty}), (\oplus \ker(d_i))^{\perp \infty})$ and $T$ is a tilting $R$-module.

5. For any exact sequence $\cdots \rightarrow T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} \cdots$ of tilting $R$-modules, each $\ker(d_i)$ has finite Gorenstein projective dimension.

We conclude this section by summarizing the contents of this paper. Section 2 contains some notations, definitions and lemmas for use throughout this paper. Section 3 is devoted to proving Theorem 1.1. Section 4 is some applications of Theorem 1.1, including the proofs of Theorems 1.2-1.4.

2. **Preliminaries**

Throughout this paper, $R$ is an associative ring with identity and $R\text{-Mod}$ is the category of left $R$-modules. Unless otherwise stated, all $R$-modules are left $R$-modules.

Next we recall some basic definitions and properties needed in the sequel. For more details the reader can consult [3, 26, 28].

**Notation.** Let $\mathcal{C}$ be a full subcategory of $R\text{-Mod}$ and $n$ a nonnegative integer. The classes $\mathcal{C}^{\perp}$, $\perp \mathcal{C}$, $\mathcal{C}^{\perp \geq n}$ and $\mathcal{C}^{\perp \infty}$ of $\mathcal{C}$ are defined as follows:

$$
\mathcal{C}^{\perp} = \{ M \in R\text{-Mod} \mid \text{Ext}_R^1(C, M) = 0 \text{ for all } C \in \mathcal{C} \}, \\
\perp \mathcal{C} = \{ M \in R\text{-Mod} \mid \text{Ext}_R^1(M, C) = 0 \text{ for all } C \in \mathcal{C} \}, \\
\mathcal{C}^{\perp \geq n} = \{ M \in R\text{-Mod} \mid \text{Ext}_R^n(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i \geq 1 \}, \\
\mathcal{C}^{\perp \infty} = \{ M \in R\text{-Mod} \mid \text{Ext}_R^i(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i \geq 1 \}.
$$
For $C = \{C\}$, we write for short $C^\perp$, $\perp C$, $C^{\perp \cdot n}$ and $C^{\perp \infty}$ in stead of $\{C\}^\perp$, $\perp \{C\}$, $\{C\}^{\perp \cdot n}$ and $\{C\}^{\perp \infty}$, respectively.

If $\cdots \rightarrow P_i \xrightarrow{f_i} \cdots \rightarrow P_0 \xrightarrow{f_0} M \rightarrow 0$ is a projective resolution of $M$, then $\text{im}(f_i)$ is called an $i$-th syzygy of $M$. We denote it by $\Omega^i M$ ($\Omega^0 M = M$).

Let $n$ be a nonnegative integer. For any $R$-module $M$, $\text{pd}_R M$ is the projective dimension of $M$. For convenience, we set $\mathcal{P}_n$ the class of $R$-modules $M$ with $\text{pd}_R M \leq n$.

Given an $R$-module $M$, we denote by $\text{Add}M$ (resp. $\text{add}M$) the class of all modules that are isomorphic to direct summands of direct sums (resp. finite direct sums) of copies $M$.

**Cotorsion pairs.** Let $\mathcal{A}$ and $\mathcal{B}$ be classes in $R\text{-Mod}$. Recall that a pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is called a cotorsion pair $[28, 40]$ if $\mathcal{A} = ^\perp \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$. The class $\mathcal{K}_{\mathcal{C}} = \mathcal{A} \cap \mathcal{B}$ is called the kernel of $\mathcal{C}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be hereditary $[28]$ if $\text{Ext}^i_R(A, B) = 0$ for all $i \geq 1$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, equivalently, if whenever $0 \rightarrow A_3 \rightarrow A_2 \rightarrow A_1 \rightarrow 0$ is exact with $A_2, A_1 \in \mathcal{A}$, then $A_3$ is also in $\mathcal{A}$, or equivalently, if whenever $0 \rightarrow B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow 0$ is exact with $B_1, B_2 \in \mathcal{B}$, then $B_3$ is also in $\mathcal{B}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete $[28]$ if one of the following two equivalent conditions holds:

- For each $R$-module $M$, there is an exact sequence $0 \rightarrow M \rightarrow B \rightarrow L \rightarrow 0$ with $B \in \mathcal{B}$ and $L \in \mathcal{A}$.
- For each $R$-module $M$, there is an exact sequence $0 \rightarrow D \rightarrow C \rightarrow M \rightarrow 0$ with $C \in \mathcal{A}$ and $D \in \mathcal{B}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is generated by a set $[28]$ provided that there is a set $\mathcal{S}$ of $R$-modules such that $\mathcal{S}^\perp = \mathcal{B}$ (i.e., $(\mathcal{A}, \mathcal{B}) = (\perp (\mathcal{S}^\perp), \mathcal{S}^\perp)$). In the literature, this is sometimes called the cotorsion pair cogenerated by $\mathcal{S}$. Here, however, we use the terminology from $[28]$. By $[28$, Theorem 3.2.1], each cotorsion pair generated by a set is complete. Moreover, we have the following lemma.

**Lemma 2.1.** Let $R$ be a ring and $M$ an $R$-module. Then $(\perp (M^{\perp \infty}), M^{\perp \infty})$ is a complete hereditary cotorsion pair.

**Proof.** Let $M$ be an $R$-module. It is easy to check that $M^{\perp \infty} = U^\perp$, where $U = \bigoplus_{i \geq 0} \Omega^i M$.

Thus $(\perp (M^{\perp \infty}), M^{\perp \infty})$ is a complete cotorsion pair by $[28$, Theorem 3.2.1]. One can check that $(\perp (M^{\perp \infty}), M^{\perp \infty})$ is also a hereditary cotorsion pair by the definition. \hfill \Box

In the following, for an $R$-module $M$, we set $\mathcal{K}_M = \perp (M^{\perp \infty}) \cap M^{\perp \infty}$. It follows from Lemma 2.1 that $\mathcal{K}_M = \mathcal{K}_{\mathcal{C}}$ whenever $\mathcal{C} = (\perp (M^{\perp \infty}), M^{\perp \infty})$. For more detailed information about cotorsion pairs, we refer the reader to $[26, 28]$.

**Gorenstein projective modules.** Following $[25, 29]$, an $R$-module $G$ is called Gorenstein projective if there is an exact sequence of projective $R$-modules

$$P : \cdots \rightarrow P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} \cdots$$
such that $G \cong \ker(f^0)$ and $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. In this case, the complex $P$ is also called a totally acyclic complex of projective $R$-modules. It is clear that each $\ker(f^i)$ is Gorenstein projective.

Let $n$ be a nonnegative integer. The Gorenstein projective dimension, $\text{Gpd}_R G$, of an $R$-module $G$ is defined by declaring that $\text{Gpd}_R G \leq n$ if, and only if there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to G \to 0$ with all $G_i$ Gorenstein projective (see [29, Definition 2.8]). By [29, Proposition 2.27], $\text{Gpd}_R M = \text{pd}_R M$ whenever $\text{pd}_R M < \infty$, and so any Gorenstein projective module with finite projective dimension is projective.

In the following, we use $\mathcal{GP}_n$ to denote the class of $R$-modules $M$ with $\text{Gpd}_R M \leq n$. By [29, Proposition 2.7], we get that an $R$-module $G$ belongs to $\mathcal{GP}_n$ if and only if any $n$-th syzygy $\Omega^n G$ of $G$ is Gorenstein projective.

**Lemma 2.2.** Let $G$ and $M$ be $R$-modules. Then the following are true for any nonnegative integer $n$:

1. If $\text{Ext}^i_R(G, M) = 0$ for all $i \geq n + 1$, then $\text{Ext}^i_R(W, M) = 0$ for all $W \in \perp(G^{\perp_\infty})$ and $i \geq n + 1$.
2. If $G \in \mathcal{P}_n$, then $\perp(G^{\perp_\infty}) \subseteq \mathcal{P}_n$.
3. If $G \in \mathcal{GP}_n$, then $\perp(G^{\perp_\infty}) \subseteq \mathcal{GP}_n$.

**Proof.** (1) Consider the exact sequence $0 \to M \to E_0 \to \cdots \to E_{n-1} \to L \to 0$ with each $E_i$ injective, we have $\text{Ext}^k_R(G, L) \cong \text{Ext}^{n+k}_R(G, M)$ for $k \geq 1$. By assumption, $\text{Ext}^k_R(G, L) = 0$ for $k \geq 1$. Thus $L \in G^{\perp_\infty}$. Note that $\perp(G^{\perp_\infty})$ is a hereditary cotorsion pair. Therefore $\text{Ext}^k_R(W, L) = 0$ for any $W \in \perp(G^{\perp_\infty})$ and $k \geq 1$. So $\text{Ext}^{n+k}_R(W, M) = 0$ for $k \geq 1$ by noting that $\text{Ext}^k_R(W, L) \cong \text{Ext}^{n+k}_R(W, M)$, as desired.

(2) Since $G \in \mathcal{P}_n$, we get that $\perp(G^{\perp_\infty}) \subseteq (\mathcal{P}_n)^{\perp_\infty})$. By [28, Theorem 4.1.12], $(\mathcal{P}_n, (\mathcal{P}_n)^{\perp_\infty})$ is a hereditary cotorsion pair. So $\perp((\mathcal{P}_n)^{\perp_\infty}) = \mathcal{P}_n$.

(3) Let $U = \bigoplus_{i=0}^{\infty} \Omega^i G$. It is clear that $G^{\perp_\infty} = U^{\perp}$. If $n = 0$ and $G$ is Gorenstein projective, then $U$ is also Gorenstein projective. So the result holds by [23, Theorem 3.2] and [28, Corollary 3.2.4].

For $n \geq 1$, we assume that $\text{Gpd}_R G \leq n$. Then $\Omega^n G$ is Gorenstein projective, and so every $R$-module in $\perp((\Omega^n G)^{\perp_\infty})$ is Gorenstein projective. Let $L$ be an $R$-module in $(\Omega^n G)^{\perp_\infty}$. Assume that $N$ is an $R$-module in $\perp(G^{\perp_\infty})$. Thus $\text{Ext}^{n+1}_R(N, L) = 0$ by (1), and hence $\text{Ext}^1_R(\Omega^n N, L) \cong \text{Ext}^{n+1}_R(N, L) = 0$. It follows that $\Omega^n N \in \perp((\Omega^n G)^{\perp_\infty})$ and then $\Omega^n N$ is Gorenstein projective. So $\text{Gpd}_R N \leq n$. This completes the proof.

**Strongly Gorenstein projective modules.** Recall that an $R$-module $M$ is called strongly Gorenstein projective [13] if there is an exact sequence of projective $R$-modules

$$P : \cdots \longrightarrow P \overset{f}{\longrightarrow} P \overset{f}{\longrightarrow} P \overset{f}{\longrightarrow} P \overset{f}{\longrightarrow} P \cdots$$

such that $M \cong \ker(f)$ and $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. 


All projective $R$-modules are strongly Gorenstein projective and the class of strongly Gorenstein projective modules is closed under direct sums. The principal role of the strongly Gorenstein projective modules is to give the following characterization of Gorenstein projective modules \cite[Theorem 2.7]{13}: an $R$-module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective $R$-module. Moreover, a careful reading of the proof of \cite[Theorem 2.7]{13} gives the following lemma.

**Lemma 2.3.** If $\cdots \to P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} \cdots$ is an exact sequence of projective $R$-modules with all $\ker(f^i)$ Gorenstein projective, then $\oplus \ker(f^i)$ is strongly Gorenstein projective.

Recall that a full subcategory $C$ of $R$-Mod is thick \cite{31} if $C$ is closed under direct summands and has the two out of three property: for every exact sequence of $R$-modules $0 \to A \to B \to C \to 0$ with two terms in $C$, then the third one is also in $C$.

Following \cite{11}, a complete hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be projective if $\mathcal{A} \cap \mathcal{B}$ is the class of projective $R$-modules.

**Lemma 2.4.** (\cite[Lemma 2.1]{38}) The following are true for any strongly Gorenstein projective $R$-module $N$:

1. $N^\perp$ is a thick subcategory of $R$-Mod.
2. $(\perp(N^\perp), N^\perp)$ is a projective cotorsion pair.

### Tilting modules

Let us recall some basic facts about (not necessarily finitely generated) tilting modules. An $R$-module $T$ is tilting \cite{1, 20} provided that the following hold:

(T1) $\pd_RT < \infty$.

(T2) $\Ext_R^i(T, T^{(\lambda)}) = 0$ for each $i \geq 1$ and for every cardinal $\lambda$.

(T3) There is a long exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ with $T_i \in \Add T$ for $0 \leq i \leq r$, where $r$ is the projective dimension of $T$.

The class $T^{\perp\infty}$ is called the tilting class induced by $T$. Further, $T$ and $T^{\perp\infty}$ are called $n$-tilting when $T \in \mathcal{P}_n$. An $n$-tilting class $\mathcal{B}$ can be characterized by the properties: $\mathcal{B}$ is closed under direct sums, $\perp(\mathcal{B})$ is a complete hereditary cotorsion pair and $\perp \mathcal{B} \subseteq \mathcal{P}_n$. By the proof of \cite[Theorem 4.1]{1}, we have the following lemma.

**Lemma 2.5.** Let $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair over a ring $R$. If $\mathcal{B}$ is an $n$-tilting class, then $K_\mathcal{C} = \Add T$ for a tilting $R$-module $T$.

We end this section with the following example which shows that the converse of Lemma 2.5 is not true in general.

**Example 2.6.** Let $R$ be a ring and $N$ a strongly Gorenstein projective $R$-module but not projective (see \cite[Example 2.5]{13}). Then $\mathcal{C} = (\perp(N^\perp), N^\perp)$ is a projective cotorsion pair by Lemma 2.4. Thus $K_\mathcal{C} = \Add R$. We claim that $N^\perp$ is not a tilting class. Indeed, if $N^\perp$ is a tilting class, then $\pd_R N < \infty$ by \cite[Theorem 4.1]{1}. Hence $N$ is projective. This is a contradiction.
3. Proof of Theorem 1.1

In the following part, we will prove our main theorem. For this purpose, we need some technical results.

One can check that an $R$-module $M$ is Gorenstein projective if and only if there is an exact sequence $0 \to M \to Q^0 \to Q^1 \to \cdots$ such that $Q^j$ is projective and $\text{Ext}^i_R(\ker(g^j), L) = 0$ for every projective $R$-module $L$, $j \geq 0$ and $i \geq 1$. Moreover, we have the following lemma.

**Lemma 3.1.** Let $G$ be an $R$-module and $n$ a nonnegative integer. The following are equivalent:

1. $\text{Gpd}_RG \leq n$.
2. There is an exact sequence of $R$-modules

$$0 \to G \to P^0 \to P^1 \to \cdots$$

with $\text{pd}_RP^j \leq n$ such that $\text{Ext}^i_R(\ker(f^j), P) = 0$ for every projective $R$-module $P$, $j \geq 0$ and $i \geq n + 1$.

**Proof.** (1) $\Rightarrow$ (2). By [17, Lemma 2.17], there is an exact sequence $0 \to G \to P^0 \to G^0 \to 0$ of $R$-modules with $G^0$ Gorenstein projective and $\text{pd}_RP^0 \leq n$. Thus there is an exact sequence of $R$-modules

$$0 \to G^0 \to P^1 \to P^2 \to \cdots$$

with each $P^j$ projective such that $\text{Ext}^i_R(\ker(f^j), P) = 0$ for every projective $R$-module $P$, $j \geq 1$ and $i \geq 1$. So we have the exact sequence of $R$-modules

$$0 \to G \to P^0 \to P^1 \to \cdots$$

with $f^0 = \beta \alpha$. By [29, Theorem 2.20], $\text{Ext}^i_R(G, P) = 0$ for every projective $R$-module $P$ and $i \geq n + 1$, as desired.

(2) $\Rightarrow$ (1). To prove that $\text{Gpd}_RG \leq n$, it is sufficient to prove that $\Omega^nG$ is Gorenstein projective. Consider the exact sequence $0 \to \ker(f^m) \to P^m \to \ker(f^{m+1}) \to 0$ for all $m \geq 0$,
we can construct the following commutative diagram as in [26, Lemma 8.2.1]

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker(f^m) & \rightarrow & P^m & \rightarrow & \ker(f^{m+1}) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & Q^{m,0} & \rightarrow & Q^{m,0} \oplus Q^{m+1,0} & \rightarrow & Q^{m+1,0} & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & \Omega^n \ker(f^m) & \rightarrow & \Omega^n P^m & \rightarrow & \Omega^n \ker(f^{m+1}) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0, & & 0, & & 0.
\end{array}
\]

where rows and columns are exact and each $Q^{t,k}$ is projective. Since $\text{pd}_R P^m \leq n$, $\Omega^n P^m$ is projective for all $m \geq 0$. Note that $\Omega^n G = \Omega^n \ker(f^0)$. Thus we have the following commutative diagram with exact row

\[
\begin{array}{cccccccccccc}
0 & \rightarrow & \Omega^n G & \rightarrow & \Omega^n P^0 & \rightarrow & \cdots & \rightarrow & \Omega^n P^{m-1} & \rightarrow & \Omega^n P^m & \rightarrow & \cdots \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & \Omega^n \ker(f^m) & & \cdots & & \Omega^n \ker(f^m) & & 0. \\
\end{array}
\]

By assumption, one can check that $\text{Ext}_{R}^{j}(\Omega^n \ker(f^m), P) \cong \text{Ext}_{R}^{n+j}(\ker(f^m), P) = 0$ for every projective $R$-module $P$, $j \geq 1$ and $m \geq 0$. Hence $\Omega^n G$ is Gorenstein projective, as desired. This completes the proof. \qed

**Lemma 3.2.** Let $R$ be a ring and $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ a complete hereditary cotorsion pair of $R$-modules. If $K_{\mathcal{C}} \subseteq \mathcal{P}_n$, $\mathcal{A} \subseteq \mathcal{G}\mathcal{P}_n$ and $G$ is an $R$-module in $\mathcal{A}$, then there is a strongly Gorenstein projective $R$-module $N$ in $\mathcal{A}$ such that $\Omega^n G$ is a direct summand of $N$.

**Proof.** Let $G$ be an $R$-module in $\mathcal{A}$. Note that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair. Then there is an exact sequence of $R$-modules

\[
0 \rightarrow G \rightarrow P^0 \xrightarrow{f^0} \cdots \rightarrow P^m \xrightarrow{f^m} \cdots
\]

with $P^j \in K_{\mathcal{C}}$ and $\ker(f^j) \in \mathcal{A}$ for $j \geq 0$. By assumption, $\mathcal{A} \subseteq \mathcal{G}\mathcal{P}_n$. Thus $\text{Gpd}_R \ker(f^j) \leq n$ for $j \geq 0$. It follows that $\text{Ext}_{R}^{n+i}(\ker(f^j), P) = 0$ for any projective $R$-module $P$, $j \geq 0$ and $i \geq 1$ by [29, Theorem 2.20]. Using a similar proof of (2) \Rightarrow (1) in Lemma 3.1, we get an exact sequence of $R$-modules
Proof. (1) There is a strongly Gorenstein projective $R$-module $N$ in $\perp(G^{\perp_\infty})$ such that $\Omega^n G$ is a direct summand of $N$. 
(2) $\perp(G^{\perp_\infty}) \subseteq GP_n$ and $K_G \subseteq P_n$. 

**Proposition 3.3.** Let $G$ be an $R$-module and $n$ a nonnegative integer. Then the following are equivalent:

1. There is a strongly Gorenstein projective $R$-module $N$ in $\perp(G^{\perp_\infty})$ such that $\Omega^n G$ is a direct summand of $N$.
2. $\perp(G^{\perp_\infty}) \subseteq GP_n$ and $K_G \subseteq P_n$.

**Proof.** (1) $\Rightarrow$ (2). By (1), there is a strongly Gorenstein projective $R$-module $N$ in $\perp(G^{\perp_\infty})$ such that $\Omega^n G$ is a direct summand of $N$. Then $\Omega^n G$ is Gorenstein projective. Thus $\operatorname{Gpd}_R G \leq n$, and so $\perp(G^{\perp_\infty}) \subseteq GP_n$ by Lemma 2.2(3).

For any $H \in K_G$, to prove that $H \in P_n$, by Lemma 2.4(2), we only need to show that $\Omega^n H \in \perp(N^\perp) \cap N^\perp$. It is clear that $H \in N^\perp$ since $N \in \perp(G^{\perp_\infty})$ and $H \in G^{\perp_\infty}$. Note that $N^\perp$ is thick by Lemma 2.4(1) and every projective $R$-module is in $N^\perp$. Thus $\Omega^n H \in N^\perp$. Next we will prove that $\Omega^n H$ is in $\perp(N^\perp)$. Let $K$ be an $R$-module in $N^\perp$. Since $N$ is strongly Gorenstein projective, there is an exact sequence $0 \to N \to P \to N \to 0$ with $P$ projective. It follows that $\Omega^n P$ is a direct summand of $N$, and hence $\Omega^n H$ is in $\perp(N^\perp)$. So $\Omega^n H \in \perp(N^\perp) \cap N^\perp$, as desired.
(2) ⇒ (1). Note that \((G^{\perp_{\infty}}, G^{\perp_{\infty}})\) is a complete hereditary cotorsion pair by Lemma 2.1. The proof follows from Lemma 3.2.

Remark 3.4. Assume that \(M\) is an \(R\)-module of finite projective dimension at most \(n\) and \(N\) is a strongly Gorenstein projective \(R\)-module. Note that \(\Omega^n(M \oplus N)\) can be taken to be \(\Omega^n M \oplus N\). Since \(\Omega^n M\) is projective, it is clear that \(\Omega^n M \oplus N\) is a strongly Gorenstein projective \(R\)-module in \(\frac{1}{2}((M \oplus N)^{\perp_{\infty}})\). Hence \(M \oplus N\) satisfies the condition (1) of Proposition 3.3.

The following well-known lemma will play a useful role in our investigations.

Lemma 3.5. (Dimension shifting) Let \(R\) be a ring, \(n\) a positive integer, and \(M\) an \(R\)-module. Fix an exact sequence of \(R\)-modules \(0 \to K \to T_n \to \cdots \to T_1 \to L \to 0\). The following are true:

1. If \(\text{Ext}_R^j(M, T_s) = 0\) for \(1 \leq s \leq n\) and \(i \geq 1\), then \(\text{Ext}_R^{n+j}(M, K) \cong \text{Ext}_R^j(M, L)\) for all \(j \geq 1\).

2. If \(\text{Ext}_R^j(T_s, M) = 0\) for \(1 \leq s \leq n\) and \(i \geq 1\), then \(\text{Ext}_R^{n+j}(L, M) \cong \text{Ext}_R^j(K, M)\) for all \(j \geq 1\).

Now we can give the proof of Theorem 1.1.

3.6. Proof of Theorem 1.1. (1) ⇒ (2). It is only need to prove that \(\mathcal{A} \subseteq \mathcal{GP}_n\). Let \(M\) be an \(R\)-module in \(\mathcal{A}\). Since \((\mathcal{A}, \mathcal{B})\) is a complete cotorsion pair, there is an exact sequence of \(R\)-modules

\[
0 \to M \xrightarrow{f^0} P^0 \xrightarrow{f^1} \cdots \xrightarrow{f^m} P^m \to \cdots
\]

such that each \(P^j \in \mathcal{K}_\mathcal{E}\) and each \(\ker(f^j) \in \mathcal{A}\). Note that there is an exact sequence \(0 \to R \to T_0 \to \cdots \to T_n \to 0\) of \(R\)-modules with each \(T_i \in \text{Add}T\). Thus, for every free \(R\)-module \(R^{(l)}\), we have the following exact sequence of \(R\)-modules

\[
0 \to R^{(l)} \to T_0^{(l)} \to \cdots \to T_n^{(l)} \to 0,
\]

where each \(T_i^{(l)} \in \mathcal{B}\) since \(\text{Add}T = \mathcal{K}_\mathcal{E} \subseteq \mathcal{B}\). It follows from Lemma 3.5(1) that \(\text{Ext}_R^{n+i}(A, R^{(l)}) \cong \text{Ext}_R^j(A, T_n^{(l)}) = 0\) for all \(A \in \mathcal{A}\) and \(i \geq 1\). Hence \(\text{Ext}_R^{n+i}(A, P) = 0\) for every projective \(R\)-module \(P\) and \(i \geq 1\). Therefore \(\text{Ext}_R^{n+i}(\ker(f^j), P) = 0\) for every projective \(R\)-module \(P\), \(j \geq 0\) and \(i \geq 1\). It is clear that \(\text{pd}_R P^j \leq n\) since \(P^j \in \mathcal{K}_\mathcal{E} = \text{Add}T\). Thus \(\text{Gpd}_R M \leq n\) by Lemma 3.1.

(2) ⇒ (3). Since \((\mathcal{A}, \mathcal{B})\) is a complete cotorsion pair, there is an exact sequence \(0 \to R \to T_0 \to \cdots \to T_{n-1} \to A \to 0\) of \(R\)-modules with \(A \in \mathcal{A}\) and \(T_j \in \mathcal{K}_\mathcal{E}\) for \(0 \leq j \leq n - 1\). Let \(T = A \oplus \oplus_{0}^{n-1} T_i\). We will check that \(T\) is an \(n\)-tilting \(R\)-module. Note that each \(T_j \in \mathcal{B}\) and \(\mathcal{A} \subseteq \mathcal{GP}_n\). \(\text{Ext}_R^j(M, A) \cong \text{Ext}_R^{n+i}(M, R) = 0\) for all \(M \in \mathcal{A}\) and \(i \geq 1\) by Lemma 3.5(1). It follows that \(A \in \mathcal{B}\), and then \(A \in \mathcal{K}_\mathcal{E}\). Therefore \(T \in \mathcal{K}_\mathcal{E}\), and \(T \in \mathcal{P}_n\) since \(\mathcal{K}_\mathcal{E} \subseteq \mathcal{P}_n\) by assumption. It is easy to check that \(\text{Ext}_R^j(T, T^{(l)}) = 0\) for all \(i \geq 1\) and each cardinal \(\lambda\) by noting that \(\mathcal{K}_\mathcal{E}\) is closed under direct sums. So \(T\) is an \(n\)-tilting \(R\)-module and \(\text{Add}T \subseteq \mathcal{K}_\mathcal{E}\).

Applying Lemma 3.2, for each \(L\) in \(\mathcal{A}\), we obtain a strongly Gorenstein projective \(R\)-module \(N_L\) in \(\mathcal{A}\) such that \(\Omega^n(L)\) is a direct summand of \(N_L\). Let \(\mathcal{X}\) be the class \(\{N_L \mid L \in \mathcal{A}\}\). Then
Next we check that \( B = T^{\perp_{\infty}} \cap X^{\perp_{\infty}} \). It is clear that \( B \subseteq T^{\perp_{\infty}} \cap X^{\perp_{\infty}} \). For the reverse containment, assume that \( L \) is an \( R \)-module in \( A \) and \( H \) is an \( R \)-module in \( T^{\perp_{\infty}} \cap N_{L}^{\perp_{\infty}} \). By [8, Theorem 3.11], there is an exact sequence of \( R \)-modules

\[
0 \rightarrow K_{n-1} \rightarrow T'_{n-1} \rightarrow \cdots \rightarrow T'_{1} \rightarrow T'_{0} \rightarrow H \rightarrow 0
\]

with \( T'_{j} \in \text{Add} T \). It is clear that \( \text{pd}_RT_{j} \leq n \) for \( 0 \leq j \leq n-1 \). Since \( N_{L} \) is strongly Gorenstein projective, \( T'_{j} \in N_{L}^{\perp_{\infty}} \) for \( 0 \leq j \leq n-1 \). Note that \( H \in N_{L}^{\perp_{\infty}} \) and \( N_{L}^{\perp_{\infty}} = N_{L}^{\perp_{\infty}} \). Applying Lemma 2.4(1), \( K_{n-1} \in N_{L}^{\perp_{\infty}} \). Since \( \Omega^{n}L \) is a direct summand of \( N_{L} \), \( \text{Ext}^{i}_{R}(\Omega^{n}L, K_{n-1}) = 0 \), and so \( \text{Ext}^{n+i}_{R}(L, K_{n-1}) = 0 \) for \( i \geq 1 \). Since \( T'_{j} \in K_{\infty} \) for \( 0 \leq j \leq n-1 \), \( \text{Ext}^{i}_{R}(L, H) \cong \text{Ext}^{n+i}_{R}(L, K_{n-1}) \) for \( i \geq 1 \) by Lemma 3.5(1). Then \( \text{Ext}^{i}_{R}(L, H) = 0 \) for \( i \geq 1 \), and so \( L \in \perp(T^{\perp_{\infty}} \cap N_{L}^{\perp_{\infty}}) \subseteq \perp(T^{\perp_{\infty}} \cap X^{\perp_{\infty}}) \). It follows that \( \mathcal{A} \subseteq \perp(T^{\perp_{\infty}} \cap X^{\perp_{\infty}}) \), and then \( T^{\perp_{\infty}} \cap X^{\perp_{\infty}} \subseteq B \). So \( B = T^{\perp_{\infty}} \cap X^{\perp_{\infty}} \).

(3) \( \Rightarrow \) (1) It is clear that \( \text{Add} T \subseteq \mathcal{A} \). Since \( T \) is a tilting \( R \)-module and each object in \( \mathcal{X} \) is strongly Gorenstein projective, it follows that \( \text{Add} T \subseteq \mathcal{X}^{\perp_{\infty}} \), and so \( \text{Add} T \subseteq \mathcal{K}_{\infty} \) since \( \mathcal{B} = T^{\perp_{\infty}} \cap \mathcal{X}^{\perp_{\infty}} \). For the reverse containment, we assume that \( K \in \mathcal{K}_{\infty} \). Let \( M \) be an \( R \)-module in \( T^{\perp_{\infty}} \). Then there exists an exact sequence \( 0 \rightarrow L \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_{0} \rightarrow M \rightarrow 0 \) of \( R \)-modules with \( T_{j} \in \text{Add} T \) for \( 0 \leq j \leq n-1 \) by [8, Theorem 3.11]. Since \( \text{Add} T \subseteq \mathcal{K}_{\infty} \) by the proof above, \( T_{j} \in \mathcal{K}_{\infty} \) for \( 0 \leq j \leq n-1 \). Then each \( T_{j} \) is in \( K^{\perp_{\infty}} \), and so \( \text{Ext}^{i}_{R}(K, M) \cong \text{Ext}^{n+i}_{R}(K, L) \) for \( i \geq 1 \) by Lemma 3.5(1). By assumption, \( \mathcal{K}_{\infty} \subseteq \mathcal{P}_{n} \). Then \( K \in \mathcal{P}_{n} \). Thus \( \text{Ext}^{i}_{R}(K, M) \cong \text{Ext}^{n+i}_{R}(K, L) = 0 \) for \( i \geq 1 \). Therefore \( K \in \perp(T^{\perp_{\infty}}) \). It is clear that \( K \in T^{\perp_{\infty}} \). Then \( K \in \text{Add} T \) by noting that \( \mathcal{K}_{T} = \text{Add} T \). So \( \mathcal{K}_{T} \subseteq \text{Add} T \), as desired.

(3) \( \Rightarrow \) (4). By assumption, \( \mathcal{B} = G^{\perp_{\infty}} \) for an \( R \)-module \( G \). Using a similar proof of (2) \( \Rightarrow \) (3), one can check that \( \mathcal{B} = T^{\perp_{\infty}} \cap N_{G}^{\perp_{\infty}} \), where \( N_{G} \) is a strongly Gorenstein \( R \)-module such that \( \Omega^{n}(G) \) is a direct summand of \( N \).

(4) \( \Rightarrow \) (3). Note that \( T^{\perp_{\infty}} \cap N^{\perp_{\infty}} = (T \oplus N)^{\perp_{\infty}} \). By Proposition 3.3 and Remark 3.4, we obtain the desired result.

(2) \( \Rightarrow \) (5) follows from Lemma 3.2.

(5) \( \Rightarrow \) (2) holds by Proposition 3.3 by noting that \( \mathcal{K}_{G} = \mathcal{K}_{\infty} \) and \( \perp(G^{\perp_{\infty}}) = \mathcal{A} \).

(5) \( \Rightarrow \) (6). By assumption, we only need to prove that \( (\Omega^{n}G)^{\perp_{\infty}} = M^{\perp_{\infty}} \). It is clear that \( M^{\perp_{\infty}} \subseteq (\Omega^{n}G)^{\perp_{\infty}} \) since \( \Omega^{n}G \) is a direct summand of \( M \). Let \( L \) be an \( R \)-module in \( (\Omega^{n}G)^{\perp_{\infty}} \). Then \( \text{Ext}^{n+i}_{R}(G, L) = 0 \) for \( i \geq 1 \). By Lemma 2.2, \( \text{Ext}^{n+i}_{R}(M, L) = 0 \) for \( i \geq 1 \). Since \( M \) is strongly Gorenstein projective, there is an exact sequence \( 0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0 \) with \( P \) projective. It follows that \( \text{Ext}^{n}_{R}(M, L) = 0 \) for \( j \geq 1 \). Then we have \( (\Omega^{n}G)^{\perp_{\infty}} = M^{\perp_{\infty}} \).

(6) \( \Rightarrow \) (5). By assumption, \( N \) is strongly Gorenstein projective and \( (\Omega^{n}G)^{\perp_{\infty}} = N^{\perp_{\infty}} \). Note that \( N^{\perp_{\infty}} = N^{\perp_{\infty}} \). It follows that \( \Omega^{n}G \in \perp(N^{\perp_{\infty}}) \). Applying Lemma 2.4(2), \( (\perp(N^{\perp_{\infty}}), N^{\perp_{\infty}}) \) is a projective cotorsion pair. By Lemma 3.2, there is a strongly Gorenstein projective \( R \)-module \( M \in \perp(N^{\perp_{\infty}}) \) such that \( \Omega^{0}(\Omega^{n}G) \) is direct summand of \( M \). It is clear that \( \Omega^{0}(\Omega^{n}G) = \Omega^{n}G \), and so \( \Omega^{n}G \) is a direct summand of \( M \). By assumption, \( N \in \perp(G^{\perp_{\infty}}) \). It follows that \( M \in \perp(N^{\perp_{\infty}}) = \perp(N^{\perp_{\infty}}) \subseteq \perp(G^{\perp_{\infty}}) \). Hence \( M \) is in \( \perp(G^{\perp_{\infty}}) \). This completes the proof.

**Corollary 3.7.** Let \( T \) be an \( n \)-tilting \( R \)-module and \( N \) a strongly Gorenstein projective \( R \)-module. Then \( \mathcal{K}_{T \oplus N} = \text{Add} T \).
Proof. This can be checked by the proof of (3) ⇒ (1) in Theorem 1.1. □

Given an infinite cardinal number $\lambda$, an $R$-module $M$ is said to be $\lambda^\prec$-generated if it is generated by less than $\lambda$ elements. A class of $R$-modules $\mathcal{C}$ is said to be a strong Kaplansky class [24] if there exists an infinite cardinal number $\lambda$ such that each $R$-module in $\mathcal{C}$ is a direct sum of $\lambda^\prec$-generated $R$-modules. Furthermore, we have the following corollary.

Corollary 3.8. Suppose that the class of Gorenstein projective $R$-modules is a strong Kaplansky class. Let $\mathcal{E} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair of $R$-modules such that $\mathcal{K}_\mathcal{E} = \text{Add}T$ for a tilting $R$-module $T$. Then there is an $R$-module $G$ such that $\mathcal{B} = G^{\perp_\infty}$.

Proof. Note that $\mathcal{K}_\mathcal{E} = \text{Add}T$ for a tilting $R$-module by hypothesis. Using Theorem 1.1, $\mathcal{B} = T^{\perp_\infty} \cap \mathcal{X}^{\perp_\infty}$, where $\mathcal{X}$ is a class of strongly Gorenstein projective $R$-modules. By assumption, each Gorenstein projective $R$-modules is a direct sum of $\lambda^\prec$-generated modules for an infinite cardinal number $\lambda$. Thus each module in $\mathcal{X}$ is a direct sum of such modules. It is easy to see that there is an $R$-module $M$ such that $\mathcal{X}^{\perp_\infty} = M^{\perp_\infty}$. Let $G = T \oplus M$. It is clear that $B = G^{\perp_\infty}$. □

4. Applications

Following [10], we denote by $\mathcal{FP}_\infty$ the class of $R$-modules possessing a projective resolution consisting of finitely generated modules. The objects of $\mathcal{FP}_\infty$ are sometimes referred to as the finitely $\infty$-presented modules (see [15]). If $R$ is left Noetherian (left coherent), then $\mathcal{FP}_\infty$ coincides with the class of finitely generated (finitely presented) $R$-modules.

Lemma 4.1. Let $R$ be a ring and $M$ be a Gorenstein projective $R$-module. If $M \in \mathcal{FP}_\infty$, then there is a strongly Gorenstein projective $R$-module $N$ such that $M$ is a direct summand of $N$, where $N$ is a direct sum of finitely generated Gorenstein projective $R$-modules.

Proof. Let $M$ be a Gorenstein projective $R$-module in $\mathcal{FP}_\infty$. By [37, Lemma 2.7], there is an exact sequence $P_+ : 0 \to M \to P^0 \xrightarrow{f^0} \cdots \to P^n \xrightarrow{f^n} \cdots$ with $P^i$ finitely generated projective and $\text{ker}(f^i)$ finitely generated Gorenstein projective for $i \geq 0$. Note that $M$ is in $\mathcal{FP}_\infty$. It follows from [29, Theorem 2.5] that there exists an exact sequence $P_- : \cdots \to P^{-n} \xrightarrow{f^{-n}} \cdots \to P^{-1} \xrightarrow{f^{-1}} M \to 0$, where $P^i$ is finitely generated projective and $\text{ker}(f^i)$ is finitely generated Gorenstein projective for $i \leq -1$. Gluing the two sequences $P_-$ and $P_+$ above, we obtain an exact sequence $\cdots \to P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} \cdots$ of finitely generated projective $R$-modules with each $\text{ker}(f^i)$ finitely generated Gorenstein projective. Let $N = \oplus \text{ker}(f^i)$. It follows that $N$ is strongly Gorenstein projective by Lemma 2.3 and $M$ is a direct summand of $N$. □

The following lemma is essentially taken from [42, Lemma 4.4], where a variation of it appears. The proof given there carries over to the present situation.

Lemma 4.2. Let $R$ be a left Noetherian ring. Then $\text{findim}R = \sup \{ \text{Gpd}_RM \mid M \text{ is finitely generated and } \text{Gpd}_RM < \infty \}$.

We are now in a position to prove Theorem 1.2.
4.3. Proof of Theorem 1.2. (1) \(\Rightarrow\) (2) follows from [3, Theorem 2.6] and Lemma 2.5.

(2) \(\Rightarrow\) (1) holds by Theorem 1.1 by noting that \(\text{Gpd}_RM = \text{pd}_RM\) whenever \(\text{pd}_RM < \infty\).

(1) \(\Rightarrow\) (3). It is clear that there is a set \(S \subseteq \mathcal{GP}^\infty\) such that each module in \(\mathcal{GP}^\infty\) is isomorphic to an element in \(S\). Let \(G = \oplus G_i\) be the direct sum of all elements \(G_i\) in \(S\). Then \((\mathcal{GP}^\infty)^{\perp_\infty} = S^{\perp_\infty} = G^{\perp_\infty}\), and so \(\mathcal{C} = (\perp((\mathcal{GP}^\infty)^{\perp_\infty}), \mathcal{GP}^{\perp_\infty})\) is a complete hereditary cotorsion pair.

Assume that \(\text{findim} R = n < \infty\). Then each \(G_i \in \mathcal{GP}_n\) by Lemma 4.2. It is clear that \(\Omega^nG_i\) can be taken to be finitely generated and Gorenstein projective. For each \(\Omega^nG_i\), by Lemma 4.1, there is a strongly Gorenstein projective \(R\)-module \(N_i\) such that \(N_i\) is a direct sum of finitely generated Gorenstein projective \(R\)-modules and \(\Omega^nG_i\) is a direct summand of \(N_i\). Let \(\Omega^nG = \oplus \Omega^nG_i\) and \(N = \oplus N_i\). It is clear that \(N\) is strongly Gorenstein projective and \(\Omega^nG\) is a direct summand of \(N\). By the construction, \(N\) is also a direct sum of finitely generated Gorenstein projective \(R\)-modules.

Note that each finitely generated Gorenstein projective \(R\)-module is in \(\mathcal{GP}^\infty\). It follows that \(N \in \perp((\mathcal{GP}^\infty)^{\perp_\infty})\) since \(\mathcal{GP}^\infty \subseteq \perp((\mathcal{GP}^\infty)^{\perp_\infty})\). Since \(R\) is left Noetherian and each \(G_i\) is finitely generated, it is easy to check that \(\mathcal{K}_A = \mathcal{K}_G\) is closed under direct sums. By Theorem 1.1, \(\mathcal{K}_\mathcal{K} = \text{Add}T\), where \(T\) is a tilting \(R\)-module.

(3) \(\Rightarrow\) (1). By hypothesis and Theorem 1.1, \(\perp((\mathcal{GP}^\infty)^{\perp_\infty}) \subseteq \mathcal{GP}_n\). It follows that \(\mathcal{GP}^\infty \subseteq \mathcal{GP}_n\). So \(\text{findim} R \leq n\) by Lemma 4.2.

Recall that an \(R\)-module \(\omega\) is said to be a Wakamatsu tilting module [32, 36] if it has the following properties:

(W1) there exists an exact sequence \(\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_0 \rightarrow \omega \rightarrow 0\) with \(P_i\) finitely generated and projective for \(i \geq 0\);

(W2) \(\text{Ext}_R^i(\omega, \omega) = 0\) for \(i \geq 1\);

(W3) there exists an exact sequence \(0 \rightarrow R \rightarrow \omega_0 \xrightarrow{f_0} \cdots \rightarrow \omega_i \xrightarrow{f_i} \cdots\) with \(\omega_i \in \text{add} \omega\) and \(\omega \in \ker(f_i)^{\perp_\infty}\) for \(i \geq 0\).

The following result contains Theorem 1.3 from the introduction.

**Theorem 4.4.** Let \(R\) be a ring and \(\omega\) a Wakamatsu tilting \(R\)-module. Fix an exact sequence \(0 \rightarrow R \rightarrow \omega_0 \xrightarrow{f_0} \cdots \rightarrow \omega_i \xrightarrow{f_i} \cdots\) with \(\omega_i \in \text{add} \omega\) and \(\omega \in \ker(f_i)^{\perp_\infty}\) for \(i \geq 0\). If we set \(A = \bigoplus_{i \geq 0} \ker(f_i)\), then the following are equivalent for any nonnegative integer \(n\):

1. \(\omega\) is an \(n\)-tilting \(R\)-module.
2. \(\omega^{\perp_\infty} = A^{\perp_\infty}\) and \(\mathcal{K}_A = \text{Add}T\), where \(T\) is an \(n\)-tilting \(R\)-module.
3. \(\omega^{\perp_\infty} \supseteq A^{\perp_\infty}\) and \(\mathcal{K}_A = \text{Add}T\), where \(T\) is an \(n\)-tilting \(R\)-module.
4. \(\mathcal{K}_{\omega^{\perp_\infty}} = \text{Add}T\), where \(T\) is an \(n\)-tilting \(R\)-module.

**Proof.** (1) \(\Rightarrow\) (2). Assume that \(\omega\) is an \(n\)-tilting \(R\)-module. To prove \(\omega^{\perp_\infty} = A^{\perp_\infty}\), it is sufficient to show that \(A \in \mathcal{P}_n\). Note that \(\ker(f_i) \in \mathcal{FP}_\infty\) for any \(i \geq 0\) by [15, Theorem 1.8]. It follows from [28, Lemma 3.1.6] that \(A^{\perp_\infty}\) is closed under direct sums. Hence \(\mathcal{K}_A\) is closed under direct sums. Let \(P\) be any projective \(R\)-module. Since \(\omega\) is an \(n\)-tilting \(R\)-module, there is an exact sequence \(0 \rightarrow P \rightarrow T_0 \rightarrow \cdots \rightarrow T_n \rightarrow 0\)
with $T_i \in \text{Add}\omega$ for $0 \leq i \leq n$. By the proof above, $A^{\perp_\infty}$ is closed under direct sums. Then $\text{Add}\omega \subseteq A^{\perp_\infty}$. It follows from Lemma 3.5(1) that $\text{Ext}^{n+j}_{R}(A, P) \cong \text{Ext}^{n}_{R}(A, T_n)$ for $j \geq 1$, and so $\text{Ext}^{n+j}_{R}(A, P) = 0$ for $j \geq 1$. Hence $\text{Ext}^{n+j}_{R}(\text{ker}(f_i), P) = 0$ for $i \geq 0$ and $j \geq 1$. On the other hand, since $\text{pd}_P\omega_i \leq n$ for any $\omega_i \in \text{add}\omega$, we have that $\text{pd}_P\text{ker}(f_i) < \infty$. This implies that each $\text{pd}_P\text{ker}(f_i) \leq n$ by [29, Theorem 2.20 and Proposition 2.27], and then $A \in \mathcal{P}_n$. By Lemma 2.2, $\perp(A^{\perp_\infty}) \subseteq \mathcal{P}_n$. Applying Theorem 1.1, we obtain that $\mathcal{K}_A = \text{Add}T$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). By Theorem 1.1, there is a strongly Gorenstein projective $R$-module $N$ in $\perp(A^{\perp_\infty})$ such that $(\Omega^n A)^{\perp_\infty} = N^{\perp_\infty}$. By assumption, $\omega^{\perp_\infty} \supseteq A^{\perp_\infty}$. It follows that $(\Omega^n\omega)^{\perp_\infty} \supseteq (\Omega^n A)^{\perp_\infty}$. Then $(\Omega^n(\omega \oplus A))^{\perp_\infty} = (\Omega^n\omega)^{\perp_\infty} \cap (\Omega^n A)^{\perp_\infty} = N^{\perp_\infty}$. It is clear that $N$ is in $\perp((\omega \oplus A)^{\perp_\infty})$. Using a similar proof of (1) $\Rightarrow$ (2), one can check that $\mathcal{K}_{\omega \oplus_A}$ is closed under direct sums. Then by Theorem 1.1, we obtain that $\mathcal{K}_{\omega \oplus_A} = \text{Add}T$, where $T$ is an $n$-tilting $R$-module.

(4) $\Rightarrow$ (1). Since $\text{Ext}^{n}_{R}(\text{im}(f_n), N) = 0$ for any $N \in \text{add}\omega$ and $i \geq 1$ by hypothesis, $\text{Ext}^{n}_{R}(\text{im}(f_n), \text{im}(f_{n-1})) \cong \text{Ext}^{n+1}_{R}(\text{im}(f_n), R)$ by Lemma 3.5(1). It follows from Theorem 1.1 that $\text{im}(f_n) \in \mathcal{GP}_n$ and hence $\text{Ext}^{n}_{R}(\text{im}(f_n), \text{im}(f_{n-1})) = 0$. Consequently, we obtain the following exact sequence

$$0 \to R \to \omega_0 \xrightarrow{f_0} \omega_1 \xrightarrow{f_1} \cdots \to \omega_{n-1} \xrightarrow{f_{n-1}} \text{im}(f_{n-1}) \to 0$$

with $\text{im}(f_{n-1}) \in \text{add}\omega$ and $\omega_i \in \text{add}\omega$ for $0 \leq i \leq n - 1$. So $\omega$ is an $n$-tilting $R$-module. \hfill $\Box$

**Corollary 4.5.** Let $\omega$ be a Wakamatsu tilting $R$-module with finite projective dimension. Keep the notations as in Theorem 4.4. Then the following are equivalent:

1. $\omega$ is a tilting $R$-module.
2. $\mathcal{K}_A = \text{Add}T$, where $T$ is a tilting $R$-module.
3. $\mathcal{K}_{\omega \oplus_A} = \text{Add}T$, where $T$ is a tilting $R$-module.

**Remark 4.6.** It is still an open problem whether a Wakamatsu tilting $R$-module of finite projective dimension must be a tilting $R$-module whenever $R$ is an Artin algebra. This is known as Wakamatsu Tilting Conjecture (see [12, Chapter IV]). Mantese and Reiten [32] showed that the Wakamatsu Tilting Conjecture is a special case of the Finistic Dimension Conjecture. These conjectures are also related to many other homological conjectures and attract many algebraists, see for instance [3, 12, 16, 32, 39, 42, 43].

Let $R$ be a ring. A left $R$-module $M$ is called $\mathcal{FP}_\infty$-injective (or absolutely clean) [14] if $\text{Ext}^{1}_R(N, M) = 0$ for all $R$-modules $N \in \mathcal{FP}_\infty$. Let $\mathcal{FP}_\infty \text{-Inj}$ be the class of $\mathcal{FP}_\infty$-injective $R$-modules, then $\mathcal{FP}_\infty \text{-Inj} = (\mathcal{FP}_\infty)^{\perp_\infty}$ by [14, Proposition 2.7]. It is clear that there is an $R$-module $G$ such that $\mathcal{FP}_\infty \text{-Inj} = G^{\perp_\infty}$. Thus $(\mathcal{FP}_\infty \text{-Inj}, \mathcal{FP}_\infty \text{-Inj})$ is a complete hereditary cotorsion pair over a general ring.

Recall that a left $R$-module $M$ is $\mathcal{FP}$-injective (or absolutely pure) [33, 41] provided that $\text{Ext}^{1}_R(N, M) = 0$ for all finitely presented left $R$-modules $N$. If $R$ is left coherent, then $\mathcal{FP}_\infty$-injective $R$-modules coincide with the class of $\mathcal{FP}$-injective $R$-modules.

The $\mathcal{FP}$-injective dimension of $M$ is defined to be the least nonnegative integer $n$ such that $\text{Ext}^{n+1}_R(N, M) = 0$ for all finitely presented left $R$-modules $N$. 

14
A coherent ring $R$ is called $n$-$FC$ [21] if $R$ has left and right $FP$-injective dimension at most $n$. In the case of Noetherian rings, an $n$-$FC$ ring coincides with an $n$-Gorenstein ring originally defined by Iwanaga in [30].

**Proposition 4.7.** Let $R$ be a ring and $\mathcal{K}_\mathcal{E}$ the kernel of $\mathcal{E} = (\perp (FP_\infty \text{-}{\text{Inj}}), FP_\infty \text{-}{\text{Inj}})$. Then the following are true for any nonnegative integer $n$:

1. $FP_\infty \subseteq GP_n$ if and only if $\mathcal{K}_\mathcal{E} = \text{Add}T$, where $T$ is an $n$-tilting $R$-module.
2. If $R$ is a commutative coherent ring, then $R$ is $n$-$FC$ if and only if $\mathcal{K}_\mathcal{E} = \text{Add}T$, where $T$ is an $n$-tilting $R$-module.

**Proof.** (1) “$\Rightarrow$”. Assume that $FP_\infty \subseteq GP_n$. It is clear that $FP_\infty \text{-}{\text{Inj}} = G_{\perp \infty}$ for an $R$-module $G$. Using a similar proof of Theorem 1.2, there is a strongly Gorenstein projective $R$-module $N$ in $\perp (FP_\infty \text{-}{\text{Inj}})$ such that $\Omega^nG$ is a direct summand of $N$. It is clear that $\mathcal{K}_\mathcal{E}$ is closed under direct sums. By Theorem 1.1, $\mathcal{K}_\mathcal{E} = \text{Add}T$, where $T$ is an $n$-tilting $R$-module.

“$\Leftarrow$”. The proof follows from Theorem 1.1.

(2) Let $FP$ be the class of finitely presented $R$-modules. If $R$ is a commutative coherent ring, then $FP = FP_\infty$ and $FP_\infty$-injective $R$-modules coincides with the class of $FP$-injective $R$-modules. The proof follows from [21, Theorem 7] and (1).

Recall that an $R$-module $M$ is called Gorenstein injective [25] if there exists an exact sequence $I : \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ of injective $R$-modules with $M \cong \text{im}(I_0 \rightarrow I^1)$ such that $\text{Hom}_R(E, I)$ is exact for every injective $R$-module $E$.

Let $\mathcal{G}I$ be the class of Gorenstein injective $R$-modules. By [34, Theorem 4.6], the cotorsion pair $\mathcal{E} = (\perp \mathcal{G}I, \mathcal{G}I)$ is complete. It is easy to check that $\mathcal{E}$ is hereditary and $\mathcal{K}_\mathcal{E}$ coincides with the class of injective $R$-modules. The following result contains (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) of Theorem 1.4 from the introduction.

**Proposition 4.8.** Let $R$ be a ring and $\mathcal{I}$ the class of injective $R$-modules. The following are equivalent for a nonnegative integer $n$:

1. $R$ is left Noetherian and $R$-$\text{Mod} \subseteq \mathcal{G}P_n$.
2. $\mathcal{K}_\mathcal{E} = \text{Add}T$, where $\mathcal{E} = (\perp \mathcal{G}I, \mathcal{G}I)$ and $T$ is an $n$-tilting $R$-module.
3. $\mathcal{K}_\mathcal{E} = \text{Add}T$, where $\mathcal{E} = (R$-$\text{Mod}, \mathcal{I})$ and $T$ is an $n$-tilting $R$-module.

**Proof.** (1) $\Rightarrow$ (2). It is easy check that $\mathcal{K}_\mathcal{E} = \mathcal{I}$. Then $\mathcal{K}_\mathcal{E}$ is closed under direct sums since $R$ is left Noetherian. By [22, Theorem 4.1], $\mathcal{K}_\mathcal{E} \subseteq \mathcal{P}_n$. Applying Theorem 1.1, we obtain that $\mathcal{K}_\mathcal{E} = \text{Add}T$, where $T$ is an $n$-tilting $R$-module.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). By assumption, $\mathcal{I}$ is closed under direct sums. It follows that $R$ is left Noetherian. By Theorem 1.1, we have that $R$-$\text{Mod} \subseteq \mathcal{G}P_n$. This completes the proof.

Let $R$ be a commutative Noetherian ring of finite Krull dimension, in [27], it is proved that $R$ is a Gorenstein ring if and only if every acyclic complex of projective $R$-modules is totally acyclic. We end this paper with the following result which contains (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) of Theorem 1.4 from the introduction.
Proposition 4.9. Consider the following conditions for a ring $R$:

(1) $R$ is a Gorenstein ring.

(2) For any exact sequence $\cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} \cdots$ of tilting $R$-modules, $\mathcal{C} = \text{Add}T$, where $\mathcal{C} = (\mathcal{C}_\infty, (\mathcal{C}_k, \ker(d_i))_{i \geq n})$ and $T$ is a tilting $R$-module.

(3) For any exact sequence $\cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} \cdots$ of tilting $R$-modules, each $\ker(d_i)$ has finite Gorenstein projective dimension.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The converses hold if $R$ is a commutative Noetherian ring of finite Krull dimension.

Proof. (1) $\Rightarrow$ (2). Suppose that $R$ is $n$-Gorenstein. Let $\cdots \to T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} \cdots$ be an exact sequence of tilting $R$-modules and $G = \oplus \ker(d_i)$. Note that $\ker(d_i) \in \mathcal{GP}_n$ and $T_i \in \mathcal{GP}_n$ for $i \in \mathbb{Z}$ by [26, Theorem 12.3.1]. Thus $T_i \in \mathcal{P}_n$ for $i \in \mathbb{Z}$ by [29, Proposition 2.27]. By the proof of (2) $\Rightarrow$ (1) in Lemma 3.1, there exists an exact sequence $\cdots \to P_2 \xrightarrow{\beta_2} P_1 \xrightarrow{\beta_1} P_0 \xrightarrow{\beta_0} \cdots$ of projective $R$-modules with each $\ker(\beta_i)$ Gorenstein projective such that $\Omega^n \ker(d_i) \cong \ker(\beta_i)$ for all $i \in \mathbb{Z}$ and $\Omega^n G \cong \oplus \ker(\beta_i)$. Hence $\Omega^n G$ is strongly Gorenstein projective by Lemma 2.3. It is clear that $\Omega^n G \in \mathcal{P}_n$ since $(\mathcal{P}_n, G_{1, \infty})$ is hereditary. Thus, to show that $\mathcal{C} = \text{Add}T$, it suffices to show $\mathcal{C}$ is closed under direct sums by Theorem 1.1.

Let $H = \oplus H_i$ with $H_i \in \mathcal{C}$. It is clear that $H \in \mathcal{P}_n$. To prove that $H \in \mathcal{C}$, we only need to show that $H \in \ker(d_i)$ for all $i \in \mathbb{Z}$. Note that each $H_i \in \ker(d_i)$ for all $i \in \mathbb{Z}$. Applying Hom$_R(-, H_j)$ to the short exact sequence $0 \to \ker(d_i) \to T_i \to \ker(d_i) \to 0$, one can check that $H_i \in T_i^\perp$ for each $i \in \mathbb{Z}$. Thus $H \in T_i^\perp$ for $i \in \mathbb{Z}$ since each $T_i$ is $n$-tilting. Consider the exact sequence $0 \to \ker(d_i) \to T_i \to T_0 \to \cdots \to T_{i-n+1} \to \ker(d_i) \to 0$, we have $\text{Ext}^k_R(\ker(d_i), H) \cong \text{Ext}^{k+n}_R(\ker(d_i), H)$ for $k \geq 0$ by Lemma 3.5(2). Note that $\mathcal{K} \subseteq \mathcal{P}_n$ by Proposition 3.3. It follows that $H_j \in \mathcal{P}_n$, and hence $H \in \mathcal{P}_n$. Since $\text{Gpd}_n \ker(d_i) \leq n$ for $i \in \mathbb{Z}$ by the proof above, $\text{Ext}^{k+n}_R(\ker(d_i), H) = 0$ for $k \geq 1$ by [29, Theorem 2.20]. Hence $\text{Ext}^{k}_R(\ker(d_i), H) = 0$ for $k \geq 1$ and so $H \in \ker(d_i)$ for all $i \in \mathbb{Z}$, as desired.

(2) $\Rightarrow$ (3) follows from Theorem 1.1.

(3) $\Rightarrow$ (1). Assume that $R$ is a commutative Noetherian ring of finite Krull dimension. Thus, to prove that $R$ is Gorenstein, it suffices to show that every acyclic complex of projective $R$-modules is totally acyclic by [27, Corollary 2]. Let $\mathcal{P}$ be an acyclic complex of projective $R$-modules, it must be a direct summand of an acyclic complex $\mathcal{F}$ of free $R$-modules. Let $N$ be the direct sum of all cycles of the complex $\mathcal{F}$. Using a similar proof in [13, Theorem 2.7], $N$ is a cycle of another acyclic complex $\mathcal{F}'$ of free $R$-modules. By assumption, there is a nonnegative integer $n$ such that $N \in \mathcal{GP}_n$. Thus each cycle of $\mathcal{F}$ has Gorenstein projective dimension at most $n$. It implies that each cycle of the complex $\mathcal{F}$ must be Gorenstein projective and so the complex $\mathcal{F}$ is totally acyclic. It follows that $\mathcal{P}$ is also totally acyclic. This completes the proof. \[\square\]
ACKNOWLEDGEMENTS

The results of this paper partially answer a question that was raised by Professor Changchang Xi during the third author’s visit at Capital Normal University in January 2016. He would like to thank Professor Changchang Xi for his support and suggestions. Results in this paper were presented at the conference “Homological and homotopical tools in category theory with applications in algebraic geometry, representation theory and module theory” in Hangzhou in May 2018. The authors thanks Serigo Estrada and Haiyan Zhu for their hospitality. The authors would like to thank Xiaowu Chen, Nanqing Ding, Xianhui Fu, Jan Šaroch and Jiaqun Wei for helpful discussions on parts of this article.

REFERENCES

[1] L. Angeleri-Hügel, F.U. Coelho, Infinitely generated tilting modules of finite projective dimension, Forum Math. 13 (2001) 239-250.
[2] L. Angeleri-Hügel, A. Tonolo, J. Trlifaj, Tilting preenvelopes and cotilting precovers, Algebr. Represent. Theory 4 (2001) 155-170.
[3] L. Angeleri-Hügel, J. Trlifaj, Tilting theory and the finitistic dimension conjecture, Trans. Amer. Math. Soc. 354 (2002) 4345-4358.
[4] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, Vol. 36, Cambridge University Press, Cambridge, 1995.
[5] L.L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. 85 (2002) 393-440.
[6] H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc. 95 (1960) 466-488.
[7] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963) 8-28.
[8] S. Bazzoni, A characterization of n-cotilting and n-tilting modules, J. Algebra 273 (2004) 359-372.
[9] L. Bican, R. El Bashir, E.E. Enochs, All modules have flat cores, Bull. London Math. Soc. 33 (2001) 385-390.
[10] R. Bieri, Homological Dimension of Discrete Groups, Queen Mary College Mathematics Notes, Mathematics Department, Queen Mary College, London, 1976.
[11] A. Beligiannis, Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, J. Algebra 288 (2005) 137-211.
[12] A. Beligiannis, I. Reiten, Homological and homotopical aspects of torsion theories, Mem. Amer. Math. Soc. vol. 188, 2007.
[13] D. Bennis, N. Mahdou, Strongly Gorenstein projective, injective, and flat modules, J. Pure Appl. Algebra 210 (2007) 437-445.
[14] D. Bravo, J. Gillespie and M. Hovey, The stable module category of a general ring, arXiv: 1405.5768.
[15] D. Bravo, M.A. Pérez, Finiteness conditions and cotorsion pairs, J. Pure Appl. Algebra, 221 (2017) 1249-1267.
[16] O. Celikbas, R. Takahashi, Auslander-Reiten conjecture and Auslander-Reiten duality, J. Algebra 382 (2013) 100-114.
[17] L.W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions-A functorial description with applications, J. Algebra 302 (2006) 231-279.
[18] R.R. Colby, K.R. Fuller, Tilting, cotilting, and serially tilted rings, Comm. Algebra 18 (1990) 1585-1615.
[19] R. Colpi, G. D’Este, A. Tonolo, Quasi-tilting modules and counter equivalences, J. Algebra 191 (1997) 461-494.
[20] R. Colpi, J. Trlifaj, Tilting modules and tilting torsion theories, J. Algebra 178 (1995) 614-634.
[21] N.Q. Ding and J.L. Chen, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra 24(9) (1996) 2963-2980.

[22] I. Emmanouil, *On the finiteness of Gorenstein homological dimensions*, J. Algebra 372 (2012) 376-396.

[23] E.E. Enochs, A. Iacob, O.M.G. Jenda, *Closure under transfinite extensions*, Illinois J. Math. 51 (2007) 561-569.

[24] E.E. Enochs, M. Cortés-Izurdiaga, B. Torrecillas, *Gorenstein conditions over triangular matrix rings*, J. Pure Appl. Algebra 218 (2014) 1544-1554.

[25] E.E. Enochs, O.M.G. Jenda, *Gorenstein injective and projective modules*, Math. Z. 220 (1995) 611-633.

[26] E.E. Enochs, O.M.G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin-New York, 2000.

[27] S. Estrada, X.H. Fu, A. Iacob, *Totally acyclic complexes*, J. Algebra 372 (2012) 376-396.

[28] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra 189 (2004) 167-193.

[29] Y. Iwanaga, *On rings with finite self-injective dimension*, Comm. Algebra 7 (1979) 393-414.

[30] H. Krause, Ø. Solberg, *Applications of cotorsion pairs*, J. London Math. Soc. 68 (2003) 631-650.

[31] F. Mantese, I. Reiten, *Wakamatsu tilting modules*, J. Algebra 278 (2004) 532-552.

[32] B.H. Maddox, *Absolutely pure modules*, Proc. Amer. Math. Soc. 18 (1967) 155-158.

[33] J. Šaroch, J. Štovíček, *Singular compactness and definability for Sigma-cotorsion and Gorenstein modules*, arXiv:1804.09080.

[34] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra 114 (1988) 106-114.

[35] T. Wakamatsu, *Tilting modules and Auslander’s Gorenstein property*, J. Algebra 275 (2004) 3-39.

[36] J. Wang, Y.X. Li, J.S. Hu, *Noncommutative G-semihereditary rings*, J. Algebra Appl. 16 (2018) 1850014, https://doi.org/10.1142/S0219498818500147.

[37] J. Wang, L. Liang, *A characterization of Gorenstein projective modules*, Comm. Algebra 44 (2016) 1420-1432.

[38] J.Q. Wei, *Auslander bounds and homological conjectures*, Rev. Mat. Iberoam. 27 (2011) 871-884.

[39] L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. 23 (1979) 11-32.

[40] B. Zimmermann-Huisgen, *The finitistic dimension conjectures-A tale of 3.5 decades*, in: Abelian Groups and Modules, Kluwer, Dordrecht, 1995, 501-517.