We formulate axisymmetric general relativity in terms of real Ashtekar–Barbero variables. We study the constraints and equations of motion and show how the Kerr, Schwarzschild and Minkowski solutions arise. We also discuss boundary conditions. This opens the possibility of a midisuperspace quantization using loop quantum gravity techniques for spacetimes with axial symmetry and time dependence.

I. INTRODUCTION

Due to the complexities of the quantized version of the Einstein equations in loop quantum gravity, the study of mini and midisuperspaces has proved a valuable tool to gain insights into the physics of the theory. The study first started with homogeneous cosmologies, giving rise to loop quantum cosmology (see [1] and references therein). It was later expanded to include spherically symmetric spacetimes (see [2] and references therein), including charged black holes [3]. In both cases interesting physical insights, like the elimination of singularities due to quantum effects, were found. It is natural to try to extend these studies to situations with less symmetry, like the case of axisymmetric spacetimes, which include physically important situations, like the Kerr geometry. There is virtually no literature on the subject. An early study of spacetimes with one Killing vector field made some progress, partially addressing the situation of axial symmetry in complex connection variables [4]. Some progress was also made in planar space-times ([5] and references therein, [6]) and the case of two spatial Killing vector fields was also discussed for the Gowdy models [7], including the use of hybrid quantizations (see [8] for a review). Some of these studies were in terms of the early form of the Ashtekar variables which were complex.

Here we would like to discuss the case of axisymmetric space-time using real Ashtekar–Barbero variables. We introduce a suitable Killing vector field and coordinates adapted to it. We will also show how the Kerr, Schwarzschild and Minkowski solutions arise. Besides, we will make some remarks on boundary conditions. This completes a classical setup suitable to perform a loop quantization, which we will discuss in a subsequent paper. This is the first example of a system with only one Killing vector field to be formulated with the real Ashtekar–Barbero variables.

The organization of this paper is as follows. In section 2 we discuss a set of symmetry adapted variables and set up the kinematics of the problem. In section 3 we introduce the constraints of general relativity in terms of the reduced axisymmetric variables introduced. In section 4 we work out the equations of motion. In section 5 we check that some particular solutions of interest including the Kerr, Schwarzschild and Minkowski space-times solve the equations we present. In section 6 we discuss boundary conditions. We end with a discussion.
II. KINEMATICS: SYMMETRY ADAPTED VARIABLES

Here we will impose a symmetry reduction due to a spatial Killing field with orbits tangent to $S^1$. Let us consider a choice of fiducial coordinates \( \{x, y, \phi\} \), where $\phi \in S^1$ and $x, y \in \mathbb{R}$. The Killing field will be then

$$K^a = (\partial_\phi)^a.$$  \hspace{1cm} (2.1)

We will be following the typical reduction procedure adopted for connection variables. Namely, a connection $A = A^i_a \tau_i dx^a$ will be invariant under the Killing symmetries if it satisfies the condition

$$\mathcal{L}_{\tilde{K}} A^i_a = \epsilon_{ijk} \lambda^j A^k_a,$$  \hspace{1cm} (2.2)

where $\tilde{K} = \lambda_i K^i = \lambda_3 \partial_\phi$ and $\lambda_1 = 0 = \lambda_2$. The previous equation amounts to

$$\partial_\phi A^i_a = \epsilon_{i3k} A^k_a.$$  \hspace{1cm} (2.3)

Notice that we are imposing that the Lie derivative be proportional to a constant $O(2)$ gauge transformation [9].

The same equation is valid for the densitized triad $E^a_i$. The most general solution to these equations are

$$A = A^i_a \tau_i dx^a = \left((\cos(\phi)\tau_1 + \sin(\phi)\tau_2) a^1_i + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2) a^2_i + a^3_3 \tau_3 \right) dx^a \hspace{1cm} (2.4)$$

$$E = E^a_i \tau_i \partial_a = \left((\cos(\phi)\tau_1 + \sin(\phi)\tau_2) e^1_i + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2) e^2_i + e^3_3 \tau_3 \right) \partial_a \hspace{1cm} (2.5)$$

where the symmetry adapted variables $(a^i_a, e^b_j)$ do not depend on the angular coordinate $\phi$, i.e. only on $(x, y)$, and are canonically conjugate. In order to prove this, it is very easy to verify that

$$\Omega = \frac{1}{8\pi G\beta} \int dx dy d\phi \delta E^a_i \wedge \delta A^i_a = \frac{1}{4G\beta} \int dx dy \delta e^a_i \wedge \delta a^i_a,$$  \hspace{1cm} (2.6)

with $\beta$ the Immirzi parameter. In other words,

$$\{a^i_a(\vec{x}), e^b_j(\vec{x}')\} = 4G\beta \delta^i_j \delta^a_b \delta^{(2)}(\vec{x} - \vec{x}').$$  \hspace{1cm} (2.7)

Another geometrical quantity that will be useful and can be computed now is the determinant of the symmetry-reduced densitized triad

$$E = \det(E) = \frac{1}{3!} \varepsilon_{abc} \varepsilon^{ijk} E^a_i E^b_j E^c_k = \frac{1}{3!} \varepsilon_{abc} \varepsilon^{ijk} e^a_i e^b_j e^c_k = \det(e) = e.$$  \hspace{1cm} (2.8)

The inverse of the densitized triad, $E^i_a$, takes a similar form as $E^a_i$, but replacing $e^a_i$ by $e^i_a$. One can easily see that $e^i_a$ fulfills $e^i_a e^a_i = \delta^i_j$ and $e^a_i e^b_i = \delta^a_b$, i.e. it is the inverse of $e^a_i$ (and therefore it can be written in terms of $e^a_i$). Then, the symmetry-reduced spatial metric can be written as

$$q_{ab} = EE^i_a E^i_b = e e^i_a e^i_b,$$  \hspace{1cm} (2.9)

and it only depends on $e^a_i$.

Similarly, the same reduction process can be applied to the extrinsic curvature $K = \cdots$
\[ K^i_a \tau_i dx^a, \text{ in its triadic form, and the spin connection } \Gamma = \Gamma^i_a \tau_i dx^a, \text{ namely} \]

\[ K = (\cos(\phi)\tau_1 + \sin(\phi)\tau_2)k^1_a + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2)k^2_a + k^3_a \tau_3 \] dx^a, \quad (2.10) \]

\[ \Gamma = (\cos(\phi)\tau_1 + \sin(\phi)\tau_2)\gamma^i_a + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2)\gamma^2_a + \gamma^3_a \tau_3 \] dx^a, \quad (2.11) \]

Actually, the components of the symmetry-reduced spin connection can be written in terms of the components of the symmetry-reduced triads as

\[ \gamma^i_a = \frac{1}{2} \varepsilon_{ijk} e^b_j (e^k_a + e^c_b c_{c,a} + e^k_c e^b_a) - \delta^i_a \delta^a_b = \gamma^i_a - \delta^i_a \delta^a_b. \quad (2.12) \]

where \( \gamma^i_a \) is the spin connection compatible with \( e^a_i \). This means that \( \{e^a_i, \gamma^j_b\} = 0 \), and therefore \( \{e^a_i, \gamma^j_b\} = 0 \).

The relation between the components of the symmetry-reduced extrinsic curvature with the ones of the symmetry-reduced Ashtekar-Barbero and the spin connections is

\[ \beta k^i_a = a^i_a - \gamma^i_a = a^i_a - \gamma^i_a + \delta^i_a \delta^a_b. \quad (2.13) \]

Finally, we will conclude this section by introducing some identities that can be useful for the calculations in the next sections. The first identity is the Poisson bracket of the connection with the inverse triad,

\[ \{a^i_a(x), e^j_b(x')\} = -4G\beta\delta^{(2)}(x - x') e^j_b e^i_a. \quad (2.14) \]

and it is easy to prove. The second identity,

\[ \{a^i_a(x), e(x)\} = 4G\beta\delta^{(2)}(x - x') e e^i_a, \quad (2.15) \]

is based on \( \delta e = e e^i_a \delta e^i_b \), for any given variation \( \delta e \) (in particular it is therefore valid for \( \partial e \)). For the next identity, we should first notice that, given any phase space tensor of the form \( T^a_i \) with density weight one, we can define

\[ \gamma(T) = \int dx dy T^a_i \gamma^i_a. \quad (2.16) \]

at least formally (we are not taking into account boundary terms neither fall-off conditions). These types of expressions appear, for instance, in the Lorentzian part of the symmetry-reduced Hamiltonian constraint. After a lengthy but simple calculation, one can prove that

\[ \{a^i_a(x), \gamma(T)\} = P^{ij}_{ac} e^b D_b T^i_j \bigg|_{x} + \int dx' dy' \{a^i_a(x), T^i_j(x')\} \gamma^i_b(x'), \quad (2.17) \]

where

\[ P^{ij}_{ac} = \frac{1}{2} \varepsilon_{ijk} e^c_j (e^i_a e^b_k + e^b_i e^c_k) + \varepsilon_{ijk} e^b_k (e^c_i e^j_a - e^a_i e^j_c) \quad (2.18) \]

and \( e^b D_b \) is the covariant derivative compatible with \( e^a_i \), namely \( e^b D_b e^a_i = 0 \).

Similar identities also hold in the full theory, namely, for the original phase space variables \( A^i_a \) and \( E^j_b \), and the spin connection \( \Gamma^i_a \).
III. THE CONSTRAINTS

The total Hamiltonian of the full theory is a combination of 7 constraints: 3 Gauss
constraints, 3 vector constraints and the Hamiltonian constraint. Concretely,
\[ H_T = \frac{1}{16\pi G} \left[ G(\vec{\Lambda}) + D(\vec{N}) + C(N) \right], \tag{3.1} \]
where
\[ G(\vec{\Lambda}) = \frac{2}{\beta} \int d^3 x \Lambda^i \left( \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^a_k \right), \tag{3.2} \]
\[ D(\vec{N}) = \frac{2}{\beta} \int d^3 x N^a \left( E^b_i \partial_a A^i_b - \partial_b (E^b_i A^i_a) \right), \tag{3.3} \]
\[ C(N) = H_E(N) + H_L(N), \tag{3.4} \]
and where \( H_E(N) \) and \( H_L(N) \) are the Euclidean and Lorentzian parts of the Hamiltonian
constraint, given, respectively, by
\[ H_E(N) = -\int d^3 x N e^{-1} (A^i_{b,a} - A^i_{a,b} + \varepsilon_{ilm} A^i_a A^m_l) \varepsilon_{ijk} E^a_j E^b_k, \tag{3.5} \]
\[ H_L(N) = \int d^3 x (1 + \beta^2) e^{-1} \varepsilon_{ijk} \varepsilon_{ilm} E^a_j E^b_k K^i_m K^m_l. \tag{3.6} \]

Now, we replace the symmetry-reduced connection \( A^i_a \) and the densitized triad \( E^a_i \) in the
previous expressions. The symmetry-reduced Hamiltonian will be
\[ h_T = \frac{1}{8G} \left[ g(\vec{\Lambda}) + d(\vec{N}) + c(N) \right], \tag{3.7} \]
and the (smeared) constraints are,
\[ g(\vec{\Lambda}) = \frac{2}{\beta} \int dxdy \lambda^i \left( \partial_a e^a_i + \varepsilon_{ijk} a^j_a e^a_k + \varepsilon_{ijk} \delta^j_a \delta^k_a e^a_i \right), \tag{3.8} \]
\[ d(\vec{N}) = \frac{2}{\beta} \int dxdy N^a \left( e^b_j \partial_a a^i_b - \partial_b (e^b_j a^i_a) + \delta^j_a \varepsilon_{ijk} e^a_j e^b_k \right), \tag{3.9} \]
\[ h_E(N) = -\int dxdy \frac{N}{\sqrt{e}} \left[ (a^i_{b,a} - a^i_{a,b} + \varepsilon_{ilm} a^i_a a^m_l) \varepsilon_{ijk} e^a_j e^b_k + 2\delta^j_a \delta^k_a \varepsilon_{ijk} (a^i_a e^a_j e^b_k - a^i_a e^b_k) \right], \tag{3.10} \]
\[ h_L(N) = \int dxdy \frac{N}{\sqrt{e}} (1 + \beta^2) \varepsilon_{ijk} \varepsilon_{ilm} e^a_j e^b_k K^i_m K^m_l, \tag{3.11} \]
where, as before, we have written the scalar constraint as \( c(N) = h_E(N) + h_L(N) \).

IV. EQUATIONS OF MOTION

In this section we will provide the Poisson brackets of the components of the symmetry-
reduced connection and densitized triad with the constraints. We will perform a local
analysis, assuming suitable boundary terms have been chosen. We will discuss the issue of boundary terms for the asymptotically flat case in section 5, similar analyses can be carried out for other asymptotic behaviors. Let us start with the Gauss constraint. One can easily see that

\[
\{e^a_i(\vec{x}), g(\vec{\lambda})\} = \left(\epsilon_{ijk}\lambda^j e^a_k\right)_{|\vec{x}} 
\]

\[
\{a^a_i(\vec{x}), g(\vec{\lambda})\} = (-\lambda^i_a + \epsilon_{ijk}\lambda^j e^a_k + \epsilon_{ijk}\lambda^j a^k_a)_{|\vec{x}}. 
\]

We see that \(e^a_i\) transforms as a tensor and that \(a^a_i + \delta^a_i \delta\phi\) transforms as a connection.

The vector constraint yields

\[
\{e^a_i(\vec{x}), d(\vec{N})\} = \left((N^b e^a_i)_b - e^a_i N^b_b - \epsilon_{ilm}\epsilon_{mjk} a^i_l e^a_j e^b_k - 2N e^a_i e^b_j - e^a_i e^b_j - e^a_i e^b_j\right)_{|\vec{x}} 
\]

\[
\{a^a_i(\vec{x}), d(\vec{N})\} = \left(a^b_i N^a_b + N^b a^a_i + \epsilon_{ilm}\epsilon_{mjk} a^a_i e^b_j e^a_k\right)_{|\vec{x}}. 
\]

The Poisson brackets with the Euclidean and Lorentzian parts of the Hamiltonian constraint are,

\[
\{e^a_i(\vec{x}), h_E(N)\} = -\frac{\beta}{2} \left(\frac{N}{\sqrt{e}} \epsilon_{ijk} e^b_j e^a_k - 2\frac{N}{\sqrt{e}} \epsilon_{ilm}\epsilon_{mjk} a^i_l e^a_j e^b_k - 2\frac{N}{\sqrt{e}} \delta^a_i \delta\phi (e^a_i e^b_j - e^a_i e^b_j)\right)_{|\vec{x}} 
\]

\[
\{e^a_i(\vec{x}), h_L(N)\} = \frac{\beta}{2} \left(2(1 + \beta^2)\frac{N}{\sqrt{e}} \epsilon_{ilm}\epsilon_{mjk} a^i_l e^a_j e^b_k\right)_{|\vec{x}} 
\]

Finally, we will provide the Poisson brackets of the components of the symmetry-reduced connection with the symmetry-reduced Hamiltonian constraint. The Euclidean part is simply given by

\[
\{a^a_i(\vec{x}), h_E(N)\} = -\frac{\beta}{2} \left[-\frac{N}{2} C_E e^i_a + 2\frac{N}{\sqrt{e}} \epsilon_{ijk} F^b_{ab} e^b_j + \frac{2N}{\sqrt{e}} (\delta^a_i \delta^j_\phi a^i_a e^a_j + \delta^a_i \delta^j_\phi a^i_b e^b_j - \delta^a_i \delta^j_\phi a^i_a e^a_j - \delta^a_i \delta^j_\phi a^i_a e^a_j)\right]_{|\vec{x}}, 
\]

where \(F^i_{ab} = a^i_{ab} - a^i_{a,b} + \epsilon_{ilm} a^i_la^m_b\). Here we have introduced

\[
C_E = \frac{1}{\sqrt{e}} \left[F^i_{ab}\epsilon_{ijk} e^a_j e^b_k + 2\delta^a_i \delta^j_\phi (a^i_a e^a_j - a^i_a e^a_j)\right], 
\]

that is nothing but the Euclidean part of the local Hamiltonian constraint \(C\). On the other hand, for the Lorentzian part we must notice that \(\{a^a_i, k^j_a\} = -\beta^{-1}\{a^a_i, \gamma^j_a\}\) and also remember the identity in (2.17) and the definition (2.18). After some manipulations, one


\[ \{a^i_a(x), h_L(N)\} = -\frac{\beta}{2} \left\{ -\frac{N}{2} C_L e^i_a - 2(1 + \beta^2) \frac{N}{\sqrt{e}} \varepsilon_{ijk} \varepsilon_{lmk} k^l_b k^m_e e^b_j \right. \\
\left. + \frac{1}{\beta(1 + \beta^2)} \left( \varepsilon_{ijk} e^m_a - \frac{1}{2} \varepsilon_{mjk} e^i_a \right) e^k_b e^k_c \left[ e^D_b \left( \frac{N k^m_c}{\sqrt{e}} \right) - e^D_c \left( \frac{N k^m_b}{\sqrt{e}} \right) \right] \right\} \bigg|_{\vec{x}}. \]  

(4.9)

Similarly,

\[ C_L = \frac{1}{\sqrt{e}} (1 + \beta^2) \varepsilon_{ijk} \varepsilon^l_{ilm} e^k_b k^m_e, \]  

(4.10)

is the Lorentzian part of the Hamiltonian constraint \( C \).

V. THE KERR, SCHWARZSCHILD AND MINKOWSKI SOLUTIONS

We have explicitly checked that the symmetry-reduced model admits the well-known solution of the full theory given by the Kerr metric.

In the usual spherical coordinates \((r, \theta, \phi)\), the densitized triad for Kerr in a diagonal gauge, identifying the internal directions with the coordinates, takes the following form:

\[ e^r_3 = \sin \theta \sqrt{(r^2 + a^2)(r^2 + a^2 \cos^2 \theta) + a^2 r r_s \sin^2 \theta}, \]

\[ e^\theta_1 = \frac{\sin \theta \sqrt{(r^2 + a^2)(r^2 + a^2 \cos^2 \theta) + a^2 r r_s \sin^2 \theta}}{\sqrt{a^2 + r^2 - r r_s}}, \]

\[ e^\phi_2 = \frac{r^2 + a^2 \cos^2 \theta}{a^2 + r^2 - r r_s}, \]  

(5.1)

where \( a = J/r_s \), while the rest of its components vanish. Together with the well-known choices of lapse and shift (e.g. \([10, 11]\))

\[ N = \sqrt{\frac{(a^2 + r(r - r_s))(a^2 + 2r^2 + a^2 \cos(2\theta))}{2(a^2 + r^2)(r^2 + a^2 \cos^2 \theta + 2a^2 r r_s \sin^2 \theta)}}, \]  

(5.2)

\[ N_\phi = -\frac{r r_s a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}, \quad N_r = 0, \quad N_\theta = 0, \]  

(5.3)

and one can easily verify that this solution corresponds to the Kerr metric in Boyer–Lindquist coordinates. Here, \( r_s \) is the Schwarzschild radius and \( a \) the angular momentum per unit mass. The spin connection can be computed out of the densitized triad. Finally, the connection components can be easily computed provided the extrinsic curvature in triadic form. Concretely,

\[ k^i_a = \delta^i_j \frac{e^b}{\sqrt{e}} K_{ab} \]  

(5.4)
where we obtain the extrinsic curvature from

\[ K_{ab} = \frac{1}{2N} \left( -\dot{h}_{ab} + \nabla_a N_b + \nabla_b N_a \right), \]  

keeping in mind that for our stationary solution \( \dot{h}_{ab} = 0 \). The expressions of the components of the connection are rather lengthy, as well as those of the Langrange multipliers \( \lambda^i \) of the Gauss constraint. We give them in the appendix.

It is also very easy to check that in the limit \( a \to 0 \) we recover the Schwarzschild (static) solution. Concretely, the densitized triad reduces to

\begin{align*}
\mathbf{e}_3 &= r^2 \sin \theta, \\
\mathbf{e}_1^\theta &= \frac{r \sin \theta}{\sqrt{1 - \frac{r_s}{r}}}, \\
\mathbf{e}_2^\phi &= \frac{r}{\sqrt{1 - \frac{r_s}{r}}},
\end{align*}

the lapse now takes the familiar form

\[ N = \sqrt{1 - \frac{r_s}{r}}, \]  

while the shift vanishes, namely \( N_\phi = 0 \). Besides, we also have that \( K_{ab} = 0 \). Therefore, the connection is completely determined by the spin connection.

The Langrange multipliers \( \lambda^i \) of the Gauss constraint take the simple form:

\begin{align*}
\lambda^1 &= 0, \\
\lambda^2 &= 0, \\
\lambda^3 &= -\beta \frac{r_s}{r^2},
\end{align*}

Finally, the Minkowski solution can be recovered from the limit \( a \to 0 \) and \( r_s \to 0 \). The densitized triad reduces to

\begin{align*}
\mathbf{e}_3^r &= r^2 \sin \theta, \\
\mathbf{e}_1^\theta &= r \sin \theta, \\
\mathbf{e}_2^\phi &= r.
\end{align*}

The lapse function \( N = 1 \) becomes the usual one in flat spacetimes. As in the previous case, the spin connection completely determines the connection, and any other Lagrange multiplier (shift and \( \lambda^i \)) vanish.

**VI. BOUNDARY TERMS**

Up to now the analysis we made has been local. When one is in asymptotically flat spacetimes one needs to be mindful about falloff rates and integrations by parts. In particular, in addition to the constraints one has a true Hamiltonian associated to the generators of the
Lorentz group at infinity. In this section we will identify the boundary contributions needed to make the action differentiable in the asymptotically flat case for the Ashtekar–Barbero variables with axial symmetry. We will review individually each set of constraints to see if boundary terms are needed. We will follow closely [12, 13].

We start this section by writing the portion of the action that corresponds to the diffeomorphism constraints in ADM-variables:

\[ D[\vec{N}] = -2 \int_{\Sigma} d^3 x N^a \nabla_b P_{ab}, \]  

(6.1)

\( \nabla \) being the covariant derivative compatible with the metric \( g_{ab} \) and \( P_{ab} \) the ADM-momentum, given by:

\[ P_{ab} = -\frac{1}{16\pi G} \sqrt{q} \left( K_{ab} - q^{ab} K \right), \]  

(6.2)

where \( K_{ab} \) is the extrinsic curvature

\[ K_{ab} = \frac{1}{2N} (\dot{q}_{ab} + \nabla_b N_a + \nabla_a N_b). \]  

(6.3)

Taking variations of that term of the action with respect to the canonical variables yields the following boundary contribution,

\[ 2 \int_{\Sigma} d^3 x \nabla_b \left( N_a \delta P_{ab} \right), \]  

(6.4)

which must be canceled at infinity. We must thus add to the action the following surface term:

\[ \mathcal{P} = -2 \oint_{\delta \Sigma} d\mathcal{S}^b N^a P_{ab}. \]  

(6.5)

As we see, this boundary term not only depends on the phase space variables, but also on the Lagrange multipliers (the shift functions \( N^a \) in this case). In the following, we will assume that the latter will be prescribed functions at spatial infinity (determined by the asymptotic form of the Kerr metric given below in Eq. (6.6)). Therefore, we will not consider variations of these functions on the boundary.

The boundary term in Eq. (6.5) can be easily evaluated for the Kerr metric at spatial infinity. In spherical coordinates, its asymptotic form is given by

\[ dS^2 = - \left[ 1 - \frac{2m}{r} \right] dt^2 - \frac{2J \sin^2 \theta}{r} (dt d\phi + d\phi dt) + \left[ 1 + \frac{2m}{r} \right] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(6.6)

Moreover,

\[ dS^b = dSn^b = dS\delta^b_r, \]  

(6.7)

and \( dS = r^2 \sin \theta d\theta d\phi \). From the metric in Eq. (6.6), the only contributions to the integral
are
\[ \mathcal{P} = -2 \oint_{\delta \Sigma} dS N^\phi P_{\phi \tau} = -2 \oint_{\Sigma} d\theta d\phi r^2 \sin \theta N^\phi \left( \frac{\sqrt{q}}{16\pi G} (K_{\phi \tau} - q_{\phi \tau} K) \right), \tag{6.8} \]

where \( q_{ab} \) is the spatial metric and \( K^{ab} \) the extrinsic curvature. At the boundary \( \delta \Sigma \), we have
\[
q_{\phi \tau} = 0, \\
\sqrt{q} = r^{3/2} \sqrt{r + r_s} \sin \theta, \\
N_\phi = -\frac{2J \sin^2 \theta}{r} \Rightarrow N^\phi = -\frac{2J}{r^3}, \\
K_{\phi \tau} = \frac{3J \sin^2 \theta}{\sqrt{r^3(r - r_s) + 4J^2 \sin^2 \theta}}, \\
K = 0.
\]

Performing the integral, the leading term in the expansion in \( 1/r \) is
\[ \mathcal{P} = \lim_{r \to \infty} \frac{9\pi}{64G} \frac{J^2}{r} = 0. \tag{6.9} \]

Now, in Ashtekar-Barbero variables, the diffeomorphism constraint of the full theory takes the form
\[ \frac{1}{16\pi G} D(\tilde{N}) = \frac{1}{8\pi G \beta} \int_{\Sigma} N^a \left( E^b_i \partial_a A^i_b - \partial_b (E^b_i A^i_a) \right). \tag{6.10} \]

Its boundary term (for asymptotically flat spacetimes) takes the form
\[ \mathcal{P} = \frac{1}{8\pi G \beta} \oint_{\Sigma} N^a E^b_i A^i_a dS_b. \tag{6.11} \]

On the other hand, in our reduced theory, the reduced diffeomorphism constraint is given by
\[ \frac{1}{8\pi G} d(\tilde{N}) = \frac{1}{4\pi G \beta} \int_{\sigma} N^a \left( e^b_i \partial_a a^i_b - \partial_b (e^b_i a^i_a) + \delta^\phi_i \delta^i_j \epsilon_{ijk} a^j_k e^k_b \right). \tag{6.12} \]

where \( \sigma \) are the \( r, \theta \) 2D spatial sections of our reduced theory, with topology \( \mathbb{R}^2 \). The corresponding boundary term is
\[ p = \frac{1}{4G \beta} \oint_{\delta \sigma} N^a e^b_i a^i_a dS_b, \tag{6.13} \]

where
\[ ds^b = dsn^b = d\phi^b. \tag{6.14} \]
with \( ds = r^2 \sin \theta \, d\theta \). Since the only non-vanishing component of the shift is \( N^\phi \), the required boundary term is

\[
p = \frac{1}{4G\beta} \oint_{\delta\Sigma} N^\phi \, e_3^a \, a_3^\phi \, q_{rr} \, r^2 \sin \theta \, d\theta. \tag{6.16}
\]

Where we have chosen the triad as in (5.1). Again, we have at infinity:

\[
N^\phi = -\frac{2J}{r^3},
\]

\[
e^r_3 = r^2 \sin \theta,
\]

\[
q_{rr} = \frac{1}{1 + \frac{r_s}{r}},
\]

\[
a_3^\phi = \cos \theta + \gamma \frac{3Jr \sin^2 \theta}{\sqrt{r(r + r_s)(r^3(r - r_s) + 4J^2 \sin^2 \theta)}}.
\]

The term proportional to \( \cos \theta \) integrates to zero (since it contains the integral of \( \cos \theta \sin \theta \) between 0 and \( \pi \)) while the other term is easily seen to yield the same result as in (6.9).

### A. Hamiltonian constraint

Now we turn to the portion of the action involving the Hamiltonian constraint. In ADM variables:

\[
\frac{1}{16\pi G} C[N] = \frac{1}{16\pi G} \int_{\Sigma} N \left( q^{-1/2} \left( P_{ab}P^{ab} - \frac{1}{2} P \right) - q^{1/2} R \right). \tag{6.17}
\]

Where \( R \) is the Ricci scalar. In order for the variations with respect to the dynamical variables to be well defined, it is necessary to add to the action the surface term (see e.g. [12]):

\[
\mathcal{E} = \frac{1}{16\pi G} 2 \oint_{\delta\Sigma} dS_d \sqrt{q} \, q^{ac} \, q^{bd} \nabla_{[c} q_{d]} a. \tag{6.18}
\]

Where \( \nabla \) is the covariant derivative compatible with the order zero of expansion in \( 1/r \) of the spatial metric at infinity. This term corresponds to time translations at infinity. The surface term actually has another contribution coming from the fact that the Ricci tensor has second derivatives, requiring two integration by parts. That contribution corresponds to boosts at infinity, but due to our choice of adapted coordinates we do not allow such boosts.

We will evaluate the previous boundary term at spatial infinity in order to show that it is finite. For convenience, we will introduce an asymptotically Cartesian coordinate system with coordinates \( \{ x^a \} \). We then expand our metric asymptotically as \( g_{ab} = \eta_{ab} + h_{ab} \), with \( \eta_{ab} \) the flat space metric and \( h_{ab} \) a small perturbation around \( \eta_{ab} \). We also expand the lapse as \( N = 1 + \mathcal{O}(1/r) \). Then, in the limit \( r \to \infty \), the leading contribution to the boundary
term takes the form
\[ \mathcal{E} = \frac{1}{16\pi G} \int_{\partial \Sigma} \left( \frac{\partial h_a^b}{\partial x^b} - \frac{\partial h_b^b}{\partial x^a} \right) dS^a = \frac{r_s}{2G}. \] (6.19)

Now, a direct calculation shows that, in Ashtekar variables, the equivalent boundary term to (6.18) takes the following form:
\[ \mathcal{E} = -\frac{1}{8\pi G} \int_{\partial \Sigma} dS_a \frac{N}{\sqrt{E}} \left( E^a_i \bar{D}_b E^b_i + E^b_i \bar{D}_b E^a_i \right), \] (6.20)
where, similar to the derivative \( \bar{\nabla} \) defined earlier, \( \bar{D} \) is the covariant derivative compatible with the order zero component of the triad at infinity.

In our reduced theory, the boundary term is given by
\[ \mathcal{E} = -\frac{1}{4G\beta} \int_{\partial \sigma} dS_a \frac{N}{\sqrt{e}} \left( E^a_i \bar{D}_b E^b_i + E^b_i \bar{D}_b E^a_i \right). \] (6.21)
Its evaluation in Cartesian coordinates agrees with the result given in (6.19).

\[ \text{B. Gauss constraint} \]

The contribution to the action of the Gauss constraint in the full theory is given by
\[ \frac{1}{16\pi G} G(\tilde{\lambda}) = \frac{1}{8\pi G\beta} \int_{\Sigma} \lambda^i \left( \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^a_k \right). \] (6.22)
The variation of this contribution also requires another boundary term in order to make the full variational problem well defined. It is given by
\[ Q = \frac{1}{8\pi G\beta} \int_{\partial \Sigma} dS_a (E^a_i - \bar{E}_a^i) \lambda^i, \] (6.23)
where \( \bar{E}_a^i \) is the densitized triad at spatial infinity. Without this term, inserting the asymptotic form of the triad, the result is divergent. Since at spatial infinity the metric is flat, \( \bar{E}_a^i \) is independent of \( M \) and \( J \). Therefore, any variational derivative of this term will be zero. Similar boundary terms have been suggested in previous treatments [12, 13].

In our symmetry reduced theory, the reduced Gauss constraint is given by:
\[ \frac{1}{8G} g(\tilde{\lambda}) = \frac{1}{4G\beta} \int_{\sigma} \lambda^i \left( \partial_a e^a_i + \varepsilon_{ijk} a^j_a e^a_k + \varepsilon_{ijk} \delta_e^3 \delta^a_c e^a_k \right). \] (6.24)
The boundary term takes the same form in terms of the reduced densitized triad, namely,
\[ q = \frac{1}{4G\beta} \int_{\partial \sigma} dS_a (e^a_i - \bar{e}^a_i) \lambda^i, \] (6.25)
where \( \mathbf{e}_i^a \) is the reduced triad at spatial infinity. After evaluation at spatial infinity, one gets

\[
q = \frac{5\pi^2 J^2}{64G r_S}.
\]

(6.26)

We note that it only depends on \( M \) and \( J \). Therefore, no new observables appear. This is due to our choice of diagonal triads, which ties spatial rotations generated by \( J \) to internal rotations.

## VII. CONCLUSIONS

We have developed the Ashtekar–Barbero framework for axisymmetric spacetimes. We found triads and connections adapted to the symmetry and wrote the Gauss law, vector and Hamiltonian constraints. We showed that the Kerr solution indeed solves the constraints and the evolution equations. We also discussed the boundary terms needed to make the action differentiable in a canonical treatment. This lays out a framework to attempt a loop quantization of axially symmetric spacetimes, which we will pursue in a further publication. These represent the most complex midisuperspaces considered up to date.

### Acknowledgments

We wish to thank Miguel Campiglia for discussions. This work was supported in part by Grant No. NSF-PHY-1603630, funds of the Hearne Institute for Theoretical Physics, CCT-LSU, Pedeciba, Grant No. PHY-1505411, the Eberly research funds of Penn State, Project. No. MINECO FIS2014-54800-C2-2-P from Spain and its continuation Project. No. MINECO FIS2017-86497-C2-2-P.

### Appendix

Gauss’ law Lagrange multipliers:

\[
\lambda^1 = 0,
\]

\[
\lambda^2 = \frac{\sqrt{2a} \sqrt{a^2 + r(r - r_s)} r_s(a^4 - 3a^2r^2 - 6r^4 + 4ar(a^2 + r^2) \beta \cos(\theta) + a^2(a^2 - r^2) \cos(2\theta)) \sin(\theta)}{((a^2 + 2r^2 + a^2 \cos(2\theta))(a^4 + 2r^4 + a^2r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta)))^{3/2}},
\]
\[ \lambda^3 = \left( \frac{a^2 + r(r - r_s)}{(a^2 + r^2)(r^2 + a^2 \cos(\theta)^2) + a^2 r (\sin(\theta))^2 + 8a^3 r(a^2 + r(r - r_s))^{1/2}} \right)^{1/2} \left\{ \frac{(2r - r_s) \sin(\theta)}{(a^2 + r(r - r_s))^{1/2}} + \frac{\beta (a^2 + 2r^2 + a^2 \cos(2\theta))(a^4 + 4r^4 + a^2 r(3r + r_s) + a^2(r(r - r_s))) \cos(2\theta)}{a^2 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta)} \right\} \]

Connection components:

\[ a_r^1 = \frac{-2a r_s \beta (r^2 + a^2 \cos(\theta)^2)(a^4 - 3a^2 r^2 - 6r^4 + a^2(a - r)(a + r) \cos(2\theta)) \sin(\theta)}{(a^2 + r(r - r_s))^{1/2}(a^2 + 2r^2 + a^2 \cos(2\theta))^2(a^4 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta))} \]

\[ a_r^3 = 0, \]

\[ a_r^0 = 0, \]

\[ a_\theta^1 = \frac{r(a^2 + r(r - r_s))^{1/2}}{a^2 + 2r^2 + a^2 \cos(2\theta)^2} \]

\[ a_\theta^2 = \frac{8a^3 r(a^2 + r(r - r_s))^{1/2} \beta \cos(\theta)(r^2 + a^2 \cos(\theta)^2)(\sin(\theta))^2}{(a^2 + 2r^2 + a^2 \cos(2\theta))^2(a^4 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta))} \]

\[ a_\theta^3 = 0, \]

\[ a_\phi^0 = 0, \]

\[ a_\phi^1 = \frac{8a^3 r \beta \cos(\theta)(\sin(\theta))^3((a^2 + r(r - r_s))(a^2 + r^2)(r^2 + a^2 \cos(\theta)^2)(a^2 + r(r - r_s))(\sin(\theta))^2}{(a^2 + 2r^2 + a^2 \cos(2\theta))^2(a^4 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta))^{1/2}} \]

\[ a_\phi^2 = \frac{2(a^2 + r(r - r_s))(5a^2 + 8r^2) \cos(3\theta) + a^2 \cos(5\theta)}{2\sqrt{2}(a^2 + 2r^2 + a^2 \cos(2\theta))^2(a^4 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta))^{1/2}} \]

\[ a_\phi^3 = \frac{2(5a^6 + 8r^4 + 4a^2 r^2)(3r + r_s) + a^4 r(17r + 3r_s) \cos(\theta)}{2\sqrt{2}(a^2 + 2r^2 + a^2 \cos(2\theta))^2(a^4 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta))^{1/2}} \]

\[ a_\phi^4 = \frac{\sqrt{2}a r_s \beta(-a^4 + 3a^2 r^2 + 6r^4 + a^2(-a^2 + r^2) \cos(2\theta))(\sin(\theta))^2}{(a^2 + 2r^2 + a^2 \cos(2\theta))^2(a^4 + 2r^4 + a^2 r(3r + r_s) + a^2(a^2 + r(r - r_s)) \cos(2\theta))^{1/2}}. \]
[1] A. Ashtekar and P. Singh, Class. Quant. Grav. 28, 213001 (2011) [arXiv:1108.0893 [gr-qc]].
[2] R. Gambini, J. Olmedo and J. Pullin, Class. Quant. Grav. 31, 095009 (2014) [arXiv:1310.5996 [gr-qc]].
[3] R. Gambini, E. M. Capurro and J. Pullin, Phys. Rev. D 91, no. 8, 084006 (2015) [arXiv:1412.6055 [gr-qc]].
[4] V. Husain and J. Pullin, Mod. Phys. Lett. A 5, 733 (1990).
[5] D. E. Neville, Phys. Rev. D 92, no. 4, 044006 (2015) [arXiv:1305.2599 [gr-qc]].
[6] F. Hinterleitner and S. Major, Class. Quant. Grav. 29, 065019 (2012) [arXiv:1106.1448 [gr-qc]].
[7] V. Husain and L. Smolin, Nucl. Phys. B 327, 205 (1989).
[8] B. Elizaga Navascus, M. Martín-Benito and G. A. Mena Marugán, Int. J. Mod. Phys. D 25, no. 08, 1642007 (2016) [arXiv:1608.05947 [gr-qc]].
[9] R. Benguria, P. Cordero, C. Teitelboim, Nucl. Phys. B122, 61 (1977).
[10] D. Wiltshire, M. Visser, S. Scott, “The Kerr spacetime: rotation black holes in general relativity”, Cambridge University Press, Cambridge (2009).
[11] S. Chandrasekhar, “The mathematical theory of black holes” Oxford University Press, Oxford (1998).
[12] T. Thiemann, Class. Quant. Grav. 12, 181198 (1995) [arXiv:9910008 [gr-qc]]
[13] M. Campiglia, Class. Quant. Grav. 32, 145011 (2015) [arXiv:1412.5531 [gr-qc]]