CLS: New Problems and Completeness

John Fearnley\textsuperscript{1}, Spencer Gordon\textsuperscript{2}, Ruta Mehta\textsuperscript{2}, and Rahul Savani\textsuperscript{1}

\textsuperscript{1} University of Liverpool, Liverpool, UK
{john.fearnley, rahul.savani}@liverpool.ac.uk
\textsuperscript{2} University of Illinois at Urbana-Champaign, Urbana IL, USA
rutamehta@cs.illinois.edu, slgordo@illinois.edu

Abstract

The complexity class $\text{CLS}$ was introduced by Daskalakis and Papadimitriou in \cite{9} with the goal of capturing the complexity of some well-known problems in $\text{PPAD} \cap \text{PLS}$ that have resisted, in some cases for decades, attempts to put them in polynomial time. No complete problem was known for $\text{CLS}$, and in \cite{9}, the problems $\text{ContractionMap}$, i.e., the problem of finding an approximate fixpoint of a contraction map, and $\text{P-LCP}$, i.e., the problem of solving a P-matrix Linear Complementarity Problem, were identified as prime candidates.

First, we present a new $\text{CLS}$-complete problem $\text{MetametricContractionMap}$, which is closely related to the $\text{ContractionMap}$. Second, we introduce $\text{EndOfPotentialLine}$, which captures aspects of $\text{PPAD}$ and $\text{PLS}$ directly via a monotonic directed path, and show that $\text{EndOfPotentialLine}$ is in $\text{CLS}$ via a two-way reduction to $\text{EndOfMeteredLine}$. The latter was defined in \cite{16} to keep track of how far a vertex is on the $\text{PPAD}$ path via a restricted potential function. Third, we reduce $\text{P-LCP}$ to $\text{EndOfPotentialLine}$, thus making $\text{EndOfPotentialLine}$ and $\text{EndOfMeteredLine}$ at least as likely to be hard for $\text{CLS}$ as $\text{P-LCP}$. This last result leverages the monotonic structure of Lemke paths for $\text{P-LCP}$ problems, making $\text{EndOfPotentialLine}$ a likely candidate to capture the exact complexity of $\text{P-LCP}$; we note that the structure of Lemke-Howson paths for finding a Nash equilibrium in a two-player game very directly motivated the definition of the complexity class $\text{PPAD}$, which eventually ended up capturing this problem’s complexity exactly.

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1 Introduction

The complexity class $\text{TFNP}$, which stands for total function problems in $\text{NP}$, contains search problems that are guaranteed to have a solution, and whose solutions can be verified in polynomial time \cite{24}. While $\text{TFNP}$ is a semantically defined complexity class and is thus unlikely to contain complete problems, a number of syntactically defined subclasses of $\text{TFNP}$ have proven very successful at capturing the complexity of total search problems. For example, the complexity class $\text{PPAD}$, introduced in \cite{27} to capture the difficulty of search problems that are guaranteed total by a parity argument, attracted intense attention in the past decade culminating in a series of papers showing that the problem of computing a Nash-equilibrium in two-player games is $\text{PPAD}$-complete \cite{4,8}. There are no known polynomial-time algorithms for $\text{PPAD}$-complete problems, and recent work suggests that no such algorithms are likely to exist \cite{1,14}. The class of problems that can be solved by local search (in perhaps exponentially-many steps), $\text{PLS}$, has also attracted much interest since it was introduced in \cite{17}, and looks similarly unlikely to have polynomial-time algorithms. Examples of problems that are complete for $\text{PLS}$ include the problem of computing a pure Nash equilibrium in a congestion game \cite{10} and computing a locally optimal max cut \cite{29}.
If a problem lies in both PPAD and PLS then it is unlikely to be complete for either class, since this would imply a extremely surprising containment of one class in the other. Motivated by the existence of several total function problems in PPAD \cap PLS that have resisted researchers attempts to design polynomial-time algorithms, in their 2011 paper [9], Daskalakis and Papadimitriou introduced the class CLS, a syntactically defined subclass of PPAD \cap PLS. CLS is intended to capture the class of optimization problems over a continuous domain in which a continuous potential function is being minimized and the optimization algorithm has access to a continuous improvement function. Daskalakis and Papadimitriou showed that many classical problems of unknown complexity were shown to be in CLS including the problem of solving a simple stochastic game, the more general problem of solving a Linear Complementarity Problem with a P-matrix, and the problem of finding an approximate fixpoint to a contraction map. Moreover, CLS is the smallest known subclass of TFNP and hardness results for it imply hardness results for PPAD and PLS simultaneously.

Recent work by Hubáček and Yogev [16] proved lower bounds for CLS. They introduced a problem known as EndOfMeteredLine which they showed was in CLS, and for which they proved a query complexity lower bound of $\Omega(2^{n/2}/\sqrt{n})$ and hardness under the assumption that there were one-way permutations and indistinguishability obfuscators for problems in P/poly. Another recent result showed that the search version of the Colorful Carathéodory Theorem is in PPAD \cap PLS, and left open whether the problem is also in CLS [25].

Unfortunately CLS is not particularly well-understood, and a glaring deficiency is the current lack of any complete problem for the class. In their original paper, Daskalakis and Papadimitriou suggested two natural candidates for complete problems for CLS, ContractionMap and P-LCP, and this remains an open problem. Another motivation for studying these two problems is that the problems of solving Condon’s simple stochastic games can be reduced to each of them (separately) in polynomial time and, in turn, there is sequence of polynomial-time reductions from parity games to mean-payoff games to discounted games to simple stochastic games [15,19,28,31]. The complexity of solving these problems is unresolved and has received much attention over many years (see, for example, [2,6,11,12,18,31]). In a recent breakthrough, a quasi-polynomial time algorithm for parity games was presented [3]. For mean-payoff, discounted, and simple stochastic games, the best-known algorithms run in subexponential time [22]. The existence of a polynomial time algorithm for solving any of these games would be a major breakthrough. For ContractionMap and P-LCP no subexponential time algorithms are known, and providing such algorithms would be a major breakthrough. As the most general of these problems, and thus most likely to be CLS-hard, we study ContractionMap and P-LCP.

Our contribution. We make progress towards settling the complexity of both of these problems. We introduce a problem, MetametricContractionMap, which generalizes ContractionMap only slightly, and we show that MetametricContractionMap is CLS-complete, thus identifying the first natural CLS-complete problem.

Our second reduction is to show that P-LCP can be reduced to EndOfMeteredLine. The EndOfMeteredLine problem was introduced to capture problems that have a PPAD directed path structure while that also allow us to keep count of exactly how far the vertex is from the start of the path. In a sense, this may seem rather unnatural, as many common problems do not seem to have this property. In particular, while the P-LCP problem has a natural measure of progress towards a solution given by Lemke’s algorithm, this is given in the form of a potential function, rather than an exact measure of the number of steps from the beginning of the algorithm.

To address this, we introduce a new problem EndOfPotentialLine which captures
problems with a PPAD path structure that also allow have a potential function that decreases along this path. It is straightforward to show that EndOfPotentialLine is more general than EndOfMeteredLINE. However, despite its generality, we are also able to show that EndOfPotentialLine can be reduced to EndOfMeteredLINE in polynomial time, and so the two problems are equivalent under polynomial time reductions. We show that P-LCP can be reduced to EndOfPotentialLine, which provides an alternative proof that P-LCP is in CLS.

We believe that the EndOfPotentialLine problem is of independent interest, as it naturally unifies the circuit-based view of PPAD and of PLS, and is defined in the spirit of the canonical definitions of PPAD and PLS. There are two obvious lines for further research. Given the reduction we provide, EndOfPotentialLine and EndOfMeteredLINE, are more likely candidates for CLS-hardness than P-LCP. Alternatively, one could attempt to reduce EndOfPotentialLine to P-LCP, thereby showing that that P-LCP is complete for the complexity class defined by these two problems, and in doing so finally resolve the long-standing open problem of the complexity of P-LCP. We note that, in the case of finding a Nash equilibrium of a two-player game, which we now know is PPAD-complete \cite{8}, the definition of PPAD was inspired by the path structure of the Lemke-Howson algorithm, as our definition of EndOfPotentialLine is directly inspired by the path structure of Lemke paths for P-matrix LCPs.

2 Preliminaries

In this section, we define polynomial-time reductions between total search problems and the complexity class CLS.

Definition 1. For total functions problems, a (polynomial-time) reduction from problem $A$ to problem $B$ is a pair of polynomial-time functions $(f, g)$, such that $f$ maps an instance $x$ of $A$ to an instance $f(x)$ of $B$, and $g$ maps any solution $y$ of $f(x)$ to a solution $g(y)$ of $x$.

Following \cite{9}, we define the complexity class CLS as the class of problems that are reducible to the following problem ContinuousLocalOpt.

Definition 2 (ContinuousLocalOpt \cite{9}). Given two arithmetic circuits computing functions $f : [0, 1]^3 \rightarrow [0, 1]^3$ and $p : [0, 1]^3 \rightarrow [0, 1]$ and parameters $\epsilon, \lambda > 0$, find either:

(C1) a point $x \in [0, 1]^3$ such that $p(x) \leq p(f(x)) - \epsilon$ or

(C2) a pair of points $x, y \in [0, 1]^3$ satisfying either

(C2a) $\|f(x) - f(y)\| > \lambda \|x - y\|$ or

(C2b) $\|p(x) - p(y)\| > \lambda \|x - y\|$.

In Definition 2, $p$ should be thought of as a potential function, and $f$ as a neighbourhood function that gives a candidate solution with better potential if one exists. Both of these functions are purported to be Lipschitz continuous. A solution to the problem is either an approximate potential minimizer or a witness for a violation of Lipschitz continuity.

Definition 3 (ContractionMap \cite{9}). We are given as input an arithmetic circuit computing $f : [0, 1]^3 \rightarrow [0, 1]^3$, a choice of norm $\|\cdot\|$, constants $\epsilon, c \in (0, 1)$, and $\delta > 0$, and we are promised that $f$ is $c$-contracting w.r.t. $\|\cdot\|$. The goal is to find

(CM1) a point $x \in [0, 1]^3$ such that $d(f(x), x) \leq \delta$,

(CM2) or two points $x, y \in [0, 1]^3$ such that $\|f(x) - f(y)\| / \|x - y\| > c$. 


In other words, the problem asks either for an approximate fixed point of \( f \) or a violation of contraction. As shown in [9], ContractionMap is easily seen to be in CLS by creating instances of ContinuousLocalOpt with \( p(x) = \| f(x) - x \| \), \( f \) remains as \( f \), Lipschitz constant \( \lambda = c + 1 \), and \( \epsilon = (1 - c)\delta \).

3 MetametricContractionMap is CLS-Complete

In this section, we define MetametricContractionMap and show that it is CLS-complete. In a metametric, all the requirements of a metric are satisfied except that the distance between identical points is not necessarily zero. The requirements for \( d \) to be a metametric are:

1. \( d(x, y) \geq 0 \);
2. \( d(x, y) = 0 \) implies \( x = y \) (but, unlike for a metric, the converse is not required);
3. \( d(x, y) = d(y, x) \);
4. \( d(x, z) \leq d(x, y) + d(y, z) \).

In the definition of the ContractionMap problem in [9], the assumed metric was one induced by a norm, where it was stated that the choice of norm did not matter. In the following definition of MetametricContractionMap, the contraction is with respect to a metametric, and this metametric is given as part of the input of the problem.

**Definition 4 (MetametricContractionMap).** We are given as input an arithmetic circuit computing \( f : [0, 1]^3 \to [0, 1]^3 \), an arithmetic circuit computing a metametric \( d : [0, 1]^3 \times [0, 1]^3 \to [0, 1] \), some \( p \)-norm \( \| \cdot \| \), and constants \( \epsilon, c \in (0, 1) \) and \( \delta > 0 \), and we are promised that \( f \) is \( c \)-contracting with respect to \( d \) and \( \lambda \)-continuous with respect to \( \| \cdot \| \), and that \( d \) is \( \gamma \)-continuous with respect to \( \| \cdot \| \). The goal is to find

(M1) a point \( x \in [0, 1]^3 \) such that \( d(f(x), x) \leq \epsilon \),
(M2) or two points \( x, y \in [0, 1]^3 \) such that

\( d(f(x), f(y))/d(x, y) > \epsilon \),
\( \| d(x, y) - d(x', y') \| / \|(x, y) - (x', y')\| > \delta \), or
\( \| f(x) - f(y) \| / |x - y| > \lambda \).

**Theorem 5.** MetametricContractionMap is CLS-hard.

**Proof.** Given an instance \( X = (f, p, \epsilon, \lambda) \) of ContinuousLocalOpt, we construct a metametric \( d(x, y) = p(x) + p(y) \). This satisfies the requirements to be a metametric. Furthermore, if \( p \) is \( \lambda \)-continuous with respect to the given \( p \)-norm \( \| \cdot \| \), then \( d \) is \( (2^{1/r-1}) \)-continuous with respect to \( \| \cdot \| \). For clarity, in the below proof we’ll omit the subscript \( r \) when writing the norm of an expression. To see this we observe that \( x, x', y, y' \in [0, 1]^n \), we have

\[
\frac{\| d(x, y) - d(x', y') \|}{\| (x, y) - (x', y') \|} = \frac{\| p(x) - p(x') \| + \| p(y) - p(y') \|}{\| (x, y) - (x', y') \|} \leq \frac{\lambda \| x - x' \| + \lambda \| y - y' \|}{\| (x, y) - (x', y') \|} \leq 2^{1/r-1} \lambda.
\]

We’ll output an instance \( Y = (f, d, \epsilon = \epsilon', c = 1 - \epsilon, \delta = \lambda, \lambda' = 2^{1/r-1} \lambda) \).

Now we consider solutions for the instance \( Y \) and show that they correspond to solutions for our input instance \( X \). If our solution is of type (M1), a point \( x \in [0, 1]^3 \) such that \( d(f(x), x) \leq \epsilon \), then we have \( p(f(x)) + p(x) \leq \epsilon \), and since the potential is non-negative, this implies \( p(x) \leq \epsilon \) and \( p(f(x)) \leq \epsilon \) and clearly \( p(f(x)) \geq 0 > p(x) - \epsilon \), so \( x \) is a solution to \( X \).
Now consider a solution that is a pair of points \( x, y \in [0, 1]^2 \) satisfying one of the conditions in \([M2]\). If the solution is of type \([M2a]\) we have \( d(f(x), f(y)) > cd(x, y) \), and by our choice of \( c \) this is exactly

\[
\frac{d(f(x), f(y))}{d(x, y)} > (1 - \epsilon) \quad \text{and} \quad p(f(x)) + p(f(y)) \geq (1 - \epsilon)(p(x) + p(y)),
\]

so either \( p(f(x)) > (1 - \epsilon)p(x) \) or \( p(f(y)) > (1 - \epsilon)p(y) \), and one of \( x \) or \( y \) must be a fixpoint solution to our input instance. Solutions of type \([M2b]\) or \([M2c]\) immediately give us violations of the \( \lambda \)-continuity of \( f \), and thus solutions to \( X \).

This completes the proof that \textsc{MetametricContractionMap} is \( \text{CLS} \)-hard.

\[\blacktriangleright\] Theorem 6. \textsc{MetametricContractionMap} is in \( \text{CLS} \).

Proof. Given an instance \( X = (f, d, \epsilon, c, \lambda, \delta) \) of \textsc{MetametricContractionMap}, we set \( p(x) \equiv d(f(x), x) \). Then our \textsc{ContinuousLocalOpt} instance is the following:

\[
Y = (f, p, \lambda', \delta, \epsilon' \equiv (1 - \epsilon)).
\]

Now consider any solution to \( Y \). If our solution is of type \([C1]\) a point \( x \) such that \( p(f(x)) > p(x) - \epsilon' \), then we have \( d(f(f(x)), f(x)) > d(f(x), x) - (1 - \epsilon)e \), and either \( d(f(x), x) \leq \epsilon \), in which case \( x \) is a solution for \( X \), or \( d(f(x), x) > \epsilon \). In the latter case, we can divide on both sides to get

\[
\frac{d(f(f(x)), f(x))}{d(f(x), x)} > 1 - \frac{(1 - \epsilon)e}{d(f(x), x)} \geq 1 - (1 - \epsilon) = e,
\]

giving us a violation of the claimed contraction factor of \( e \), and a solution of type \([M2a]\).

If our solution is a pair of points \( x, y \) of type \([C2a]\) satisfying \( \|f(x) - f(y)\| / \|x - y\| > \lambda' \geq \lambda \), then this gives a violation of the \( \lambda \)-continuity of \( f \). If instead \( x, y \) are of type \([C2b]\) so that \( \|p(x) - p(y)\| / \|x - y\| > \lambda' \), then we have

\[
|d(f(x), x) - d(f(y), y)| = |p(x) - p(y)| > (\lambda + 1)\delta \|x - y\|.
\]

We now observe that if

\[
|d(f(x), x) - d(f(y), y)| \leq \delta (\|f(x) - f(y)\| + \|x - y\|) \quad \text{and} \quad \|f(x) - f(y)\| / \|x - y\| \leq \lambda,
\]

then we would have

\[
|d(f(x), x) - d(f(y), y)| \leq \delta (\|f(x) - f(y)\| + \|x - y\|) \leq (\lambda + 1)\delta \|x - y\|,
\]

which contradicts the above inequality, so either the \( \delta \) continuity of \( d \) must be violated giving a solution to \( X \) of type \([M2b]\) or the \( \lambda \) continuity of \( f \) must be violated giving a solution of type \([M2c]\). Thus we have shown that \textsc{MetametricContractionMap} is in \( \text{CLS} \).

\[\blacktriangleright\]

4 \textbf{EndOfMeteredLine} to \textbf{EndOfPotentialLine} and Back

In this section, we define a new problem \textbf{EndOfPotentialLine}. Then, we design polynomial-time reductions from \textbf{EndOfMeteredLine} to \textbf{EndOfPotentialLine}, and from \textbf{EndOfPotentialLine} to \textbf{EndOfMeteredLine}, thereby showing that the two problems are polynomial-time equivalent. In Section 5 we reduce \textbf{P-LCP} to \textbf{EndOfPotentialLine}.

First we recall the definition of \textbf{EndOfMeteredLine}, which was first defined in [16]. It is close in spirit to the problem \textbf{EndOfLine} that is used to define \textbf{PPAD} [27].
Definition 7 (EndOfMeteredLine [16]). Given circuits $S, P : \{0, 1\}^n \to \{0, 1\}^n$, and $V : \{0, 1\}^n \to \{0, \ldots, 2^n\}$ such that $P(0^n) = 0^n \neq S(0^n)$ and $V(0^n) = 1$, find a string $x \in \{0, 1\}^n$ satisfying one of the following:

- (T1) either $S(P(x)) \neq x \neq 0^n$ or $P(S(x)) \neq x$,
- (T2) $x \neq 0^n$, $V(x) = 1$,
- (T3) either $V(x) > 0$ and $V(S(x)) - V(x) \neq 1$, or $V(x) > 1$ and $V(x) - V(P(x)) \neq 1$.

Intuitively, an EndOfMeteredLine is an EndOfLine instance that is also equipped with an “odometer” function. The circuits $P$ and $S$ implicitly define an exponentially large graph in which each vertex has degree at most 2, just as in EndOfLine, and condition T1 says that the end of every line (other than $0^n$) is a solution. In particular, the string $0^n$ is guaranteed to be the end of a line, and so a solution can be found by following the line that starts at $0^n$.

The function $V$ is intended to help with this, by giving the number of steps that a given string is from the start of the line. We have that $V(0^n) = 1$, and that $V$ increases by exactly 1 for each step we make along the line. Conditions T2 and T3 enforce this by saying that any violation of the property is also a solution to the problem.

In EndOfMeteredLine, the requirement of incrementing $V$ by exactly one as we walk along the line is quite restrictive. We define a new problem, EndOfPotentialLine, which is similar in spirit to EndOfLine, but drops the requirement of always incrementing the potential by one as we move along the line.

Definition 8 (EndOfPotentialLine). Given Boolean circuits $S, P : \{0, 1\}^n \to \{0, 1\}^n$ such that $P(0^n) = 0^n \neq S(0^n)$ and a Boolean circuit $V : \{0, 1\}^n \to \{0, 1, \ldots, 2^n - 1\}$ such that $V(0^n) = 0$ find one of the following:

- (R1) A point $x \in \{0, 1\}^n$ such that $S(P(x)) \neq x \neq 0^n$ or $P(S(x)) \neq x$.
- (R2) Points $x, y \in \{0, 1\}^n$ such that $x \neq y$, $S(x) = y$, $P(y) = x$, and $V(y) - V(x) \leq 0$.

The key difference here is that the function $V$ is required to be strictly monotonically increasing as we walk along the line, but the amount that it increases in each step is not specified. At first glance, the definition of EndOfPotentialLine may seem more general and more likely to capture the whole class CLS. In fact, we will show that EndOfMeteredLine and EndOfPotentialLine are inter-reducible in polynomial-time.

Theorem 9. EndOfMeteredLine and EndOfPotentialLine are equivalent under polynomial-time reductions.

As expected, the reduction from EndOfMeteredLine to EndOfPotentialLine is relatively easy. It requires handling the difference in potential at $0^n$ and vertices with potential zero that are not discarded directly as possible solutions in EndOfPotentialLine. We make the latter self loops, but that creates extra starts and ends of lines which need to be handled. Full details of the reduction with proofs are in Appendix A.

The reduction from EndOfPotentialLine to EndOfMeteredLine is involved, and appears in detail in Appendix B. Here the basic idea is to insert missing single increments in between by introducing new vertices along the original edges. To allow this we need to encode potential itself in the vertex description. If there is an edge from $u$ to $u'$ in the EndOfPotentialLine instance whose respective potentials are $p$ and $p'$ such that $p < p'$ then we create edges $(u, p) \to (u, p + 1) \to \ldots \to (u, p' - 1) \to (u, p')$. However, this creates a lot of dummy vertices, namely those that never appear on any edge due to irrelevant potential values, i.e., in this example $(u, \pi)$ with $\pi < p$ or $\pi \geq p'$. We make them self loops (not an end-of-line) with zero potential, and since non-end-of-line solutions of
EndOfMeteredLine, namely $T2$ and $T3$, must have strictly positive potential, these will never create a solution of the EndOfMeteredLine instance.

In addition, a number of issues need to be handled with consistency: (a) a $T2$ type solution of EndOfMeteredLine may be neither at the end of any line nor be a potential violation in EndOfPotentialLine; we do extra (linear time) work to handle such solutions, (b) a $T3$ type potential violation may not be on a “valid” edge as required by EndOfPotentialLine. (c) “invalid” edges, (d) potential difference at the initial vertex $0^n$, etc.

5 Reduction from P-LCP to EndOfPotentialLine

In this section we present a polynomial-time reduction from the P-matrix Linear Complementarity Problem (P-LCP) to EndOfPotentialLine. A Linear Complementarity Problem (LCP) is defined as follows. Now on by $[n]$ we mean set $\{1,\ldots, n\}$.

▶ Definition 10 (LCP). Given a matrix $M \in \mathbb{R}^{d \times d}$ and a vector $q \in \mathbb{R}^{d \times 1}$, find a vector $y \in \mathbb{R}^{d \times 1}$ such that:

$$M y \leq q; \quad y \geq 0; \quad y_i (q - My)_i = 0, \forall i \in [n].$$  

(1)

In general, an LCP may have no solution, and deciding whether one does is NP-complete [5]. If the matrix $M$ is a P-matrix, as defined next, then the LCP $(M, q)$ has a unique solution for all $q \in \mathbb{R}^{d \times 1}$.

▶ Definition 11 (P-matrix). A matrix $M \in \mathbb{R}^{d \times d}$ is called a P-matrix if every principle minor of $M$ is positive, i.e., for every subset $S \subseteq [d]$, the sub-matrix $N = [M_{i,j}]_{i \in S, j \in S}$ has strictly positive determinant.

In order to define a problem that takes all matrices $M$ as input without a promise, Megiddo [23] defined P-LCP as the following problem (see also [24]).

▶ Definition 12 (P-LCP). Given a matrix $M \in \mathbb{R}^{d \times d}$ and a vector $q \in \mathbb{R}^{d \times 1}$, either:

(Q1) Find vector $y \in \mathbb{R}^{n \times 1}$ that satisfies (1)

(Q2) Produce a witness that $M$ is not a P-matrix, i.e., find $S \subseteq [d]$ such that for submatrix $N = [M_{i,j}]_{i \in S, j \in S}$ has strictly positive determinant.

Later, Papadimitriou showed that P-LCP is in PPAD [27], and then Daskalakis and Papadimitrou showed that it is in CLS [9] (based on the potential reduction method in [20]). Designing a polynomial-time solution for the P-LCP problem has been open for decades, at least since the 1978 paper of Murty [26] that provided exponential-time examples for complementary pivoting algorithms, such as Lemke’s algorithm [21], for P-matrix Linear Complementarity Problems. Murty’s family of P-matrices were based on the Klee-Minty’s cubes that had been used to give exponential-time examples for the simplex method, and which inspired the research that led to polynomial-time algorithms for Linear Programming. No similar polynomial-time algorithms are known for P-LCP though.

Lemke’s algorithm introduces an extra variable, say $z$, to the LCP polytope, and follows a path on the 1-skeleton of the new polytope (like the simplex method for linear programming) based on complementary pivot rule (details below). A general LCP need not have a solution, and thus Lemke’s algorithm is not guaranteed to terminate with a solution. However, for P-matrix LCPs, Lemke’s algorithm terminates. Indeed, if Lemke’s algorithm does not terminate with a solution, it provides a witness that the matrix $M$ is not a P-matrix. The structure of the path traced by Lemke’s algorithm is crucial for our reduction, so let us first briefly describe the algorithm.
5.1 Lemke’s Algorithm

The explanation of Lemke’s algorithm in this section is taken from [13]. The problem is interesting only when $q \geq 0$, since otherwise $y = 0$ is a trivial solution. Let us introduce slack variables $s$ to obtain the following equivalent formulation:

$$My + s = q, \quad y \geq 0, \quad s \geq 0 \quad \text{and} \quad y_is_i = 0, \quad \forall i \in [d]. \quad (2)$$

Let $Q$ be the polyhedron in $2d$ dimensional space defined by the first three conditions; we will assume that $Q$ is non-degenerate (just for simplicity of exposition; this will not matter for our reduction). Under this condition, any solution to (2) will be a vertex of $Q$, since it must satisfy $2d$ equalities. Note that the set of solutions may be disconnected. An ingenious idea of Lemke was to introduce a new variable and consider the system:

$$My + s - z_1 = q, \quad y \geq 0, \quad s \geq 0, \quad z \geq 0 \quad \text{and} \quad y_is_i = 0, \quad \forall i \in [d]. \quad (3)$$

The next lemma follows by construction of (3).

▶ Lemma 13. Given $(M, q)$, $(y, s, z)$ satisfies (3) with $z = 0$ iff $y$ satisfies (1).

Let $P$ be the polyhedron in $2d+1$ dimensional space defined by the first four conditions of (3), i.e.,

$$P = \{(y, s, z) \mid My + s - z_1 = q, \quad y \geq 0, \quad s \geq 0, \quad z \geq 0\}; \quad (4)$$

we will assume that $P$ is non-degenerate.

Since any solution to (3) must still satisfy $2d$ equalities in $P$, the set of solutions, say $S$, will be a subset of the one-skeleton of $P$, i.e., it will consist of edges and vertices of $P$. Any solution to the original system (i.e., satisfying $z = 0$) will be a vertex of $P$.

Now $S$ turns out to have some nice properties. Any point of $S$ is fully labeled in the sense that for each $i$, $y_i = 0$ or $s_i = 0$. We will say that a point of $S$ has duplicate label $i$ if $y_i = 0$ and $s_i = 0$ are both satisfied at this point. Clearly, such a point will be a vertex of $P$ and it will have only one duplicate label. Since there are exactly two ways of relaxing this duplicate label, this vertex must have exactly two edges of $S$ incident at it. Clearly, a solution to the original system (i.e., satisfying $z = 0$) will be a vertex of $P$ that does not have a duplicate label. On relaxing $z = 0$, we get the unique edge of $S$ incident at this vertex.

As a result of these observations, we can conclude that $S$ consists of paths and cycles. Of these paths, Lemke’s algorithm explores a special one. An unbounded edge of $S$ such that the vertex of $P$ it is incident on has $z > 0$ is called a ray. Among the rays, one is special—the one on which $y = 0$. This is called the primary ray and the rest are called secondary rays. Now Lemke’s algorithm explores, via pivoting, the path starting with the primary ray. This path must end either in a vertex satisfying $z = 0$, i.e., a solution to the original system, or a secondary ray. In the latter case, the algorithm is unsuccessful in finding a solution to the original system; in particular, the original system may not have a solution. We give the full pseudo-code for Lemke’s algorithm in Appendix C.

5.2 Polynomial time reduction from P-LCP to EndOfPotentialLine

It is well known that if matrix $M$ is a P-matrix (P-LCP), then $z$ strictly decreases on the path traced by Lemke’s algorithm [7]. Furthermore, by a result of Todd [30, Section 5], paths
traced by complementary pivot rule can be locally oriented. Based on these two facts, next we derive a polynomial-time reduction from P-LCP to EndOfPotentialLine.

Let \( \mathcal{I} = (\mathcal{M}, \mathcal{q}) \) be a given P-LCP instance, and let \( \mathcal{L} \) be the length of the bit representation of \( \mathcal{M} \) and \( \mathcal{q} \). We will reduce \( \mathcal{I} \) to an EndOfPotentialLine instance \( \mathcal{E} \) in time \( \text{poly}(\mathcal{L}) \).

According to Definition 8, the instance \( \mathcal{E} \) is defined by its vertex set \( \text{vert} \), and procedures \( S \) (successor), \( P \) (predecessor) and \( V \) (potential). Next we define each of these.

As discussed in Section 5.1 the linear constraints of (3) on which Lemke’s algorithm operates forms a polyhedron \( \mathcal{P} \) given in (4). We assume that \( \mathcal{P} \) is non-degenerate. This is without loss of generality since, a typical way to ensure this is by perturbing \( \mathcal{q} \) so that configurations of solution vertices remain unchanged \[7\], and since \( \mathcal{M} \) is unchanged the LCP is still P-LCP.

Lemke’s algorithm traces a path on feasible points of (3) which is on 1-skeleton of \( \mathcal{P} \) starting at \( (y^0, s^0, z^0) \), where:

\[
y^0 = 0, \quad z^0 = \min_{i \in [d]} q_i, \quad s^0 = \mathcal{q} + z^1
\]  

(5)

We want to capture vertex solutions of (3) as vertices in EndOfPotentialLine instance \( \mathcal{E} \). To differentiate we will sometimes call the latter configurations. Vertex solutions of (3) are exactly the vertices of polyhedron \( \mathcal{P} \) with either \( y_i = 0 \) or \( s_i = 0 \) for each \( i \in [d] \). Vertices of (3) with \( z = 0 \) are our final solutions (Lemma 13). While each of its non-solution vertex has a duplicate label. Thus, a vertex of this path can be uniquely identified by which of \( y_i = 0 \) and \( s_i = 0 \) hold for each \( i \) and its duplicate label. This gives us a representation for vertices in the EndOfPotentialLine instance \( \mathcal{E} \).

EndOfPotentialLine Instance \( \mathcal{E} \).

- Vertex set \( \text{vert} = \{0, 1\}^n \) where \( n = 2d \).
- Procedures \( S \) and \( P \) as defined in Tables 1 and 3 respectively.
- Potential function \( V : \text{vert} \to \{0, 1, \ldots, 2^m - 1\} \) defined in Table 2 for \( m = \lceil \ln(2\Delta^3) \rceil \).

where

\[
\Delta = (n! \cdot I_{\text{max}}^{2d+1} + 1 \quad \text{and} \quad I_{\text{max}} = \max\{\max_{i,j \in [d]} M(i,j), \max_{i \in [d]} |q_i|\}.
\]

For any vertex \( u \in \text{vert} \), the first \( d \) bits of \( u \) represent which of the two inequalities, namely \( y_i \geq 0 \) and \( s_i \geq 0 \), are tight for each \( i \in [d] \). A valid setting of the second set of \( d \) bits will have at most one non-zero bit – if none is one then \( z = 0 \), otherwise the location of one bit indicates the duplicate label. Thus, there are many invalid configurations, namely those with more than one non-zero bit in the second set of \( d \) bits. These are dummies that we will handle separately, and we define a procedure IsValid to identify non-dummy vertices in Table 3 (in Appendix D.1). To go between “valid” vertices of \( \mathcal{E} \) and corresponding vertices of the Lemke polytope \( \mathcal{P} \) of LCP \( \mathcal{I} \), we define procedures EtoI and ItoE in Table 5 (in Appendix D.1). By construction of IsValid, EtoI and ItoE, next lemma follows. All the missing proofs of this section may be found in Appendix D.

\textbf{Lemma 14.} If \( \text{IsValid}(u) = 1 \) then \( u = \text{ItoE}(\text{EtoI}(u)) \), and the corresponding vertex \((y, s, z) \in \text{EtoI}(u) \) of \( \mathcal{P} \) is feasible in (3). If \((y, s, z) \) is a feasible vertex of (3) then \( u = \text{ItoE}(y, s, z) \) is a valid configuration, i.e., \( \text{IsValid}(u) = 1 \).

The main idea behind procedures \( S \) and \( P \), given in Tables 1 and 3 respectively, is the following (also see Figure 1): Make dummy configurations in \( \text{vert} \) to point to themselves with cycles of length one, so that they can never be solutions. The starting vertex \( 0^n \in \text{vert} \)
Let's consider Lemke's Path. Arrows on the edges represent increase or decrease in $z$ as we move along the edge.

**Figure 1** Construction of $S$ and $P$ for EndOfPotentialLine instance $E$ from the Lemke path. The first path is the Lemke path and the arrows on its edges indicate whether the value of $z$ increases or decreases along the edge. Note that the end or start of a path in $E$, which is an intermediate vertex in Lemke path that has either decreased and then increased, or increased and then decreased in the value of $z$, is a violation of $M$ being a $P$ matrix \[7\].

The potential function $V$, which if formally defined in Table 2, gives value zero to dummy vertices and the starting vertex $0^n$. To all other vertices, essentially it is $((z^0 - z) + \Delta^2) + 1$. Since value of $z$ starts at $z^0$ and keeps decreasing on the Lemke path this value will keep increasing starting from zero at the starting vertex $0^n$. Multiplication by $\Delta^2$ will ensure that if $z_1 > z_2$ then the corresponding potential values will differ by at least one. This is because, since $z_1$ and $z_2$ are coordinates of two vertices of polytope $P$, their maximum value is $\Delta$ and their denominator is also bounded above by $\Delta$. Hence $z_1 - z_2 \leq 1/\Delta^2$ (Lemma 10).

To show correctness of the reduction we need to show two things: (i) All the procedures are well-defined and polynomial time. (ii) We can construct a solution of $I$ from a solution of $E$ in polynomial time.
Functions $P$, $S$ and $V$ of instance $\mathcal{E}$ are well defined, making $\mathcal{E}$ a valid EndOfPotentialLine instance.

There are two possible types of solutions of an EndOfPotentialLine instance. One indicates the beginning or end of a line, and the other is a vertex with locally optimal potential (that does not point to itself). First we show that the latter case never arise. For this, we need the next lemma, which shows that potential differences in two adjacent configurations adheres to differences in the value of $z$ at corresponding vertices.

Lemma 15. Let $u \neq u'$ be two valid configurations, i.e., $\text{IsValid}(u) = \text{IsValid}(u') = 1$, and let $(y, s, z)$ and $(y', s', z')$ be the corresponding vertices in $\mathcal{P}$. Then the following holds:

(i) $V(u) = V(u') \iff z = z'$. (ii) $V(u) > V(u') \iff z < z'$.

Using the above lemma, we will next show that instance $\mathcal{E}$ has no local maximizer.

Lemma 16. Let $u, v \in \text{vert}$ s.t. $u \neq v$, $v = S(u)$, and $u = P(v)$. Then $V(u) < V(v)$.
Proof. Let \( x = (y, s, z) \) and \( x' = (y', s', z') \) be the vertices in polyhedron \( P \) corresponding to \( u \) and \( v \) respectively. From the construction of \( v = S(u) \) implies that \( z' < z \). Therefore, using Lemma 16 it follows that \( V(v) < V(u) \).

Due to Lemma 17, the only type of solutions available in \( E \) is where \( S(P(u)) \neq u \) and \( P(S(u)) \neq u \). Next two lemmas shows how to construct solutions of \( I \) from these.

**Lemma 18.** Let \( u \in \text{vert}, u \neq 0^o \). If \( P(S(u)) \neq u \) or \( S(P(u)) \neq u \), then \( \text{IsValid}(u) = 1 \), and for \( (y, s, z) = \text{EtoI}(u) \) if \( z = 0 \) then \( y \) is a Q1 type solution of \( P \)-LCP instance \( I = (M, q) \).

**Proof.** If \( u \) is a dummy configuration then clearly \( S(P(u)) = u \) and \( P(S(u)) = u \), therefore \( \text{IsValid}(u) = 1 \). Given this, from Lemma 14 we know that \( \text{IsValid}(u) = 1 \). Therefore, if \( z = 0 \) then using Lemma 13 we have a solution of the LCP (1), i.e., Q1 type of our \( P \)-LCP instance \( I = (M, q) \).

**Lemma 19.** Let \( u \in \text{vert}, u \neq 0^o \) such that \( P(S(u)) \neq u \) or \( S(P(u)) \neq u \), and let \( x = (y, s, z) = \text{EtoI}(u) \). If \( z \neq 0 \) then \( x \) has a duplicate label, say \( l \). And for directions \( \sigma_1 \) and \( \sigma_2 \) obtained by relaxing \( y_l = 0 \) and \( s_l = 0 \) respectively at \( x \), we have \( \sigma_1(z) \ast \sigma_2(z) \geq 0 \), where \( \sigma_i(z) \) is the coordinate corresponding to \( z \).

**Proof.** From Lemma 18 we know that \( \text{IsValid}(u) = 1 \), and therefore from Lemma 14 \( x \) is a feasible vertex in \( \mathcal{F} \). From the last line of Tables 1 and 3 observe that \( S(u) \) points to the configuration of vertex next to \( x \) on Lemke’s path only if it has lower \( z \) value otherwise it gives back \( u \), and similarly \( P(u) \) points to the previous only if value of \( z \) increases.

First consider the case when \( P(S(u)) \neq u \). Let \( v = S(u) \) and corresponding vertex in \( \mathcal{P} \) be \( (y', s', z') = \text{EtoI}(v) \). If \( v \neq u \), then from the above observation we know that \( z' > z \), and in that case again by construction of \( P \) we will have \( P(v) = u \), contradicting \( P(S(u)) \neq u \). Therefore, it must be the case that \( v = u \). Since \( z \neq 0 \) this happens only when the next vertex on Lemke path after \( x \) has higher value of \( z \) (by above observation). As a consequence of \( v = u \), we also have \( P(u) \neq u \). By construction of \( P \) this implies for \( (y'', s'', z'') = \text{EtoI}(P(u)) \), \( z'' > z \). Putting both together we get increase in \( z \) when we relax \( y_l = 0 \) as well as when we relax \( s_l = 0 \) at \( x \).

For the second case \( S(P(u)) \neq u \) similar argument gives that value of \( z \) decreases when we relax \( y_l = 0 \) as well as when we relax \( s_l = 0 \) at \( x \). The proof follows.

Finally, we are ready to prove our main result of this section using Lemmas 17, 18 and 19. Together with Lemma 19, we will use the fact that on Lemke path \( z \) monotonically decreases if \( M \) is a \( P \)-matrix or else we get a witness that \( M \) is not a \( P \)-matrix.

**Theorem 20.** \( P \)-LCP reduces to \text{EndOfPotentialLine} in polynomial-time.

**Proof.** Given an instance \( I = (M, q) \) of \( P \)-LCP, where \( M \in \mathbb{R}^{d \times d} \) and \( q \in \mathbb{R}^{d \times 1} \) reduce it to an instance \( E \) of \text{EndOfPotentialLine} as described above with vertex set \( \text{vert} = \{0, 1\}^{2d} \) and procedures \( S, P \) and \( V \) as given in Table 1, 2, and 3 respectively.

Among solutions of \text{EndOfPotentialLine} instance \( E \), there is no local potential maximizer, i.e., \( u \neq v \) such that \( v = S(u), u = P(v) \) and \( V(u) > V(v) \) due to Lemma 17. We get a solution \( u \neq v \) such that either \( S(P(u)) \neq u \) or \( P(S(u)) \neq u \), then by Lemma 13 it is valid configuration and has a corresponding vertex \( x = (y, s, z) \) in \( P \). Again by Lemma 18 if \( z = 0 \) then \( y \) is a Q1 type solution of our \( P \)-LCP instance \( I \). On the other hand, if \( z > 0 \) then from Lemma 19 we get that on both the two adjacent edges to \( x \) on Lemke path the value of \( z \) either increases or decreases. This gives us a minor of \( M \) which is non-positive, i.e., a Q2 type solution of the \( P \)-LCP instance \( I \).
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A EndOfMeteredLine to EndOfPotentialLine

Given an instance $I$ of EndOfMeteredLine defined by circuits $S, P$ and $V$ on vertex set $\{0, 1\}^n$ we are going to create an instance $I'$ of EndOfPotentialLine with circuits $S', P'$, and $V'$ on vertex set $\{0, 1\}^{(n+1)}$, i.e., we introduce one extra bit. This extra bit is essentially to take care of the difference in the value of potential at the starting point in EndOfMeteredLine and EndOfPotentialLine, namely 1 and 0 respectively.

Let $k = n + 1$, then we create a potential function $V' : \{0, 1\}^k \to \{0, \ldots, 2^k - 1\}$. The idea is to make $0^k$ the starting point with potential zero as required, and to make all other vertices with first bit 0 be dummy vertices with self loops. The real graph will be embedded in vertices with first bit 1, i.e., of type $(1, u)$. Here by $(b, u) \in \{0, 1\}^k$, where $b \in \{0, 1\}$ and $u \in \{0, 1\}^n$, we mean a $k$ length bit string with first bit set to $b$ and for each $i \in [2 : k]$ bit $i$ set to bit $u_i$.

Procedure $V'(b, u)$: If $b = 0$ then Return 0, otherwise Return $V(u)$.

Procedure $S'(b, u)$:
1. If $(b, u) = 0^k$ then Return $(1, 0^n)$
2. If $b = 0$ and $u \neq 0^n$ then Return $(b, u)$ (creating self loop for dummy vertices)
3. If $b = 1$ and $V(u) = 0$ then Return $(b, u)$ (vertices with zero potentials have self loops).
4. If $b = 1$ and $V(u) > 0$ then Return $(b, S(u))$ (the rest follows $S$)

Procedure $P'(b, u)$:
1. If $(b, u) = 0^k$ then Return $(b, u)$ (initial vertex points to itself in $P'$).
2. If $b = 0$ and $u \neq 0^n$ then Return $(b, u)$ (creating self loop for dummy vertices)
3. If $b = 1$ and $u = 0^n$ then Return $0^k$ (to make $(0, 0^n) \to (1, 0^n)$ edge consistent).
4. If $b = 1$ and $V(u) = 0$ then Return $(b, u)$ (vertices with zero potentials have self loops).
5. If $b = 1$ and $V(u) > 0$ and $u \neq 0^n$ then Return $(b, P(u))$ (the rest follows $P$)

Valid solutions of EndOfMeteredLine of type T2 and T3 requires the potential to be strictly greater than zero, while solutions of EndOfPotentialLine may have zero potential. However, a solution of EndOfPotentialLine can not be a self loop, therefore note that we made all such vertices to self loop. By construction, the next lemma follows:

Lemma 21. $S', P', V'$ are well defined and polynomial in the sizes of $S, P, V$ respectively.

Our main theorem in this section is a consequence of the following three lemmas.

Lemma 22. For an $x = (b, u) \in \{0, 1\}^k$, $P'(x) = S'(x) = x$ (self loop) iff $x \neq 0^k$, and $b = 0$ or $V(u) = 0$.

Proof. Follows by the construction of $V'$, second condition in $S'$ and $P'$, and third and fourth conditions in $S'$ and $P'$ respectively.

Lemma 23. Let $x = (b, u) \in \{0, 1\}^k$ be such that $S'(P'(x)) \neq x \neq 0^k$ or $P'(S'(x)) \neq x$ (R1 type solution of EndOfPotentialLine instance $I'$), then $u$ is a solution of EndOfMeteredLine instance $I$.

Proof. The proof does a careful case analysis. By first conditions of the construction of $S', P'$ and $V'$, we have $x \neq 0^k$. Further, since $x$ is not a self loop, Lemma 22 implies $b = 1$ and $V'(1, u) = V(u) > 0.$
Case I. If \( S'(P'(x)) \neq x \neq 0^k \) then we will show that either \( u \) is a genuine start of a line other than 0" giving T1 type solution of \( \text{EndOfMeteredLine} \) instance \( I \), or there is some issue with the potential at \( u \) giving either T2 or T3 type solution of \( I \). Since \( S'(P'(1,0^0)) = (1,0^0) \), \( u \neq 0^0 \). Thus if \( S(P(u)) \neq u \) then we get T1 type solution of \( I \) and proof follows. If \( V(u) = 1 \) then we get T2 solution of \( I \) and proof follows.

Otherwise, we have \( S(P(u)) = u \) and \( V(u) > 1 \). Now since also \( b = 1 \), \( (1,u) \) is not a self loop (Lemma 22). Then it must be the case that \( P'(1,u) = (1,P(u)) \). However, \( S'(1,P(u)) \neq (1,u) \) even though \( P(S(u)) = u \). This happens only when \( P(u) \) is a self loop because of \( V(P(u)) = 0 \) (third condition of \( P' \)). Therefore, we have \( V(u) - V(P(u)) > 1 \) implying T3 type solution of \( I \).

Case II. Similarly, if \( P'(S'(x)) \neq x \), then either \( u \) is genuine end of a line of \( I \), or there is some issue with the potential at \( u \). If \( P(S(u)) \neq u \) then we get T1 solution of \( I \). Otherwise, \( P(S(u)) = u \) and \( V(u) > 0 \). Now as \( b = 1 \), \( (1,u) \) is not a self loop and \( V(u) > 0 \), it must be the case that \( S'(b,u) = (1,S(u)) \). However, \( P'(1,S(u)) \neq (b,u) \) even though \( P(S(u)) = u \). This happens only when \( S(u) \) is a self loop because of \( V(S(u)) = 0 \). Therefore, we get \( V(S(u)) - V(u) < 0 \), i.e., \( u \) is type T3 solution of \( I \).

\[ \text{Lemma 24.} \quad \text{Let } x = (b,u) \in \{0,1\}^k \text{ be an R2 type solution of the constructed } \text{EndOfPotentialLine} \text{ instance } I', \text{ then } u \text{ is a type T3 solution of } \text{EndOfPotentialLine} \text{ instance } I. \]

\[ \text{Proof.} \quad \text{Clearly, } x \neq 0^k. \text{ Also } x \text{ is not a self loop, because there is } y = (b',u') = S'(x) \text{ such that } y \neq x \text{ and } x = P'(y). \text{ This also implies that } y \text{ is not a self loop, and hence } b = b' = 1 \text{ and } V(u) > 0 \text{ (Lemma 22). Further, } y = S'(1,u) = (1,S(u)), \text{ hence } u' = S(u). \text{ Also, } V'(x) = V'(1,u) = V(u) \text{ and } V'(y) = V'(1,u') = V(u'). \]

Since, \( V'(y) - V'(x) < 0 \) we get \( V(u') - V(u) \leq 0 \Rightarrow V(S(u)) - V(u) \leq 0 \Rightarrow V(S(u)) - V(u) \neq 1 \). Given that \( V(u) > 0 \), \( u \) gives type T3 solution of \( \text{EndOfMeteredLine} \). \[ \square \]

\[ \text{Theorem 25.} \quad \text{An instance of } \text{EndOfMeteredLine} \text{ can be reduced to an instance of } \text{EndOfPotentialLine} \text{ in linear time such that a solution of the former can be constructed in a linear time from the solution of the latter.} \]

\section{B \text{EndOfPotentialLine to EndOfMeteredLine}}

Like in the previous section we will give a linear time reduction from instance \( I \) of \( \text{EndOfPotentialLine} \) to an instance \( I' \) of \( \text{EndOfMeteredLine} \). Let the given \( \text{EndOfPotentialLine} \) instance \( I \) be defined on vertex set \( \{0,1\}^n \) and with procedures \( S, P \) and \( V \), where \( V : \{0,1\}^n \rightarrow \{0,\ldots,2^m-1\} \).

\text{Valid Edge.} \quad \text{We call an edge } u \rightarrow v \text{ valid if } v = S(u) \text{ and } u = P(v). \]

We construct an \( \text{EndOfMeteredLine} \) instance \( I' \) on \( \{0,1\}^k \) vertices where \( k = n + m \). Let \( S', P' \) and \( V' \) denotes the procedures for \( I' \) instance. The idea is to capture value \( V(x) \) of the potential in the \( m \) least significant bits of vertex description itself, so that it can be gradually increased or decreased on valid edges. For vertices with irrelevant values of these least \( m \) significant bits we will create self loops. Invalid edges will also become self loops, e.g., if \( y = S(x) \) but \( P(y) \neq x \) then set \( S'(x,.,) = (x,.). \) We will see how these can not introduce new solutions.

In order to ensure \( V'(0^k) = 1 \), the \( V(S(0^n)) = 1 \) case needs to be discarded. For this, we first do some initial checks to see if the given instance \( I \) is not trivial. If the input
We have vertex set is $\{0^n \mid n \in \mathbb{N}\}$. However, this can not kill all the solutions since there is a path starting at $0^n$ and $\pi$. Let us assume now on that $0^n$ and $S(0^n)$ are not solutions of the $\text{EndOfPotentialLine}$ instance $I$, and then by Lemma 26, we have $0^n \to S(0^n) \to S(S(0^n))$ are valid edges, and $V(S(S(0^n))) \geq 2$. We can avoid the need to check whether $V(S(0))$ is one all together, by making $0^n$ point directly to $S(S(0^n))$ and make $S(0^n)$ a dummy vertex.

We first construct $S'$ and $P'$, and then construct $V'$ which will give value zero to all self loops, and use the least significant $m$ bits to give a value to all other vertices. Before describing $S'$ and $P'$ formally, we first describe the underlying principles. Recall that in $I$ vertex set is $\{0,1\}^n$ and possible potential values are $\{0, \ldots, 2^m - 1\}$, while in $I'$ vertex set is $\{0,1\}^k$ where $k = m + n$. We will denote a vertex of $I'$ by a tuple $(u, \pi)$, where $u \in \{0,1\}^n$ and $\pi \in \{0, \ldots, 2^m - 1\}$. Here when we say that we introduce an edge $x \to y$ we mean that we introduce a valid edge from $x$ to $y$, i.e., $y = S'(x)$ and $x = P'(y)$.

- Vertices $(S(0^n), \pi), \forall \pi \in \{0,1\}^m$ and $(0^n, 1)$ are dummies and hence have self loops.
- If $V(S(S(0^n))) = 2$ then we introduce an edge $(0^n, 0) \to (V(S(S(0^n))), 2)$, otherwise
  - for $p = V(S(S(0^n)))$, we introduce the edges $(0^n, 0) \to (0^n, 2) \to (0^n, 3) \ldots (0^n, p - 1) \to (V(S(S(0^n))), p)$.
- If $u \to u'$ valid edge in $I$ then let $p = V(u)$ and $p' = V(u')$
  - If $p = p'$ then we introduce the edge $(u, p) \to (u', p')$.
  - If $p < p'$ then we introduce the edges $(u, p) \to (u, p + 1) \to \ldots \to (u, p' - 1) \to (u', p')$.
  - If $q > p'$ then we introduce the edges $(u, p) \to (u, p - 1) \to \ldots \to (u, p' + 1) \to (u', p')$.
- If $u \neq 0^n$ is the start of a path, i.e., $S(P(u)) \neq u$, then make $(u, V(u))$ start of a path by ensuring $P'(u, V(u)) = (u, V(u))$.
- If $u$ is the end of a path, i.e., $P(S(u)) \neq u$, then make $(u, V(u))$ end of a path by ensuring $S'(u, V(u)) = (u, V(u))$.

Last two bullets above remove singleton solutions from the system by making them self loops. However, this can not kill all the solutions since there is a path starting at $0^n$, which has to end somewhere. Further, note that this entire process ensures that no new start or end of a paths are introduced.

**Procedure** $S'(u, \pi)$.

1. If $(u = 0^n$ and $\pi = 1)$ or $u = S(0^n)$ then Return $(u, \pi)$.
2. If $(u, \pi) = 0^n$, then let $u' = S(S(0^n))$ and $p' = V(u')$.
   - a. If $p' = 2$ then Return $(u', 2)$ else Return $(0^n, 2)$.
3. If $u = 0^n$ then
   - a. If $2 \leq \pi < p' - 1$ then Return $(0^n, \pi + 1)$.
   - b. If $\pi = p' - 1$ then Return $(S(S(0^n)), p')$.
   - c. If $\pi \geq p'$ then Return $(u, \pi)$.
4. Let \( u' = S(u), p' = V(u'), \) and \( p = V(u) \).
5. If \( P(u') \neq u \) or \( u' = u \) then Return \( (u, \pi) \).
6. If \( \pi = p = p' \) or \( \pi = p \) and \( p' = p + 1 \) or \( \pi = p' = p - 1 \) then Return \( (u', p') \).
7. If \( \pi < p < p' \) or \( p < p' \leq \pi \) or \( \pi > p > p' \) or \( p > p' \geq \pi \) then Return \( (u, \pi) \).
8. If \( p > p' \), then if \( p \leq \pi < p' - 1 \) then Return \( (u, \pi + 1) \). If \( \pi = p' - 1 \) then Return \( (u', p') \).
9. If \( p = p' \), then if \( p \geq \pi > p' + 1 \) then Return \( (u, \pi - 1) \). If \( \pi = p' + 1 \) then Return \( (u', p') \).

Procedure \( P'(u, \pi) \).

1. If \( (u = 0^n \) and \( \pi = 1 \) or \( u = S(0^n) \) then Return \( (u, \pi) \).
2. If \( u = 0^n \), then
   a. If \( \pi = 0 \) then Return \( 0^k \).
   b. If \( \pi < V(S(0^n)) \) and \( \pi \notin \{1, 2\} \) then Return \( (0^n, \pi - 1) \).
   c. If \( \pi < V(S(0^n)) \) and \( \pi = 2 \) then Return \( 0^k \).
3. If \( u = S(0^n) \) and \( \pi = V(S(0^n)) \) then
   a. If \( \pi = 2 \) then Return \( (0^n, 0) \), else Return \( (0^n, \pi - 1) \).
4. If \( \pi = V(u) \) then
   a. Let \( u' = P(u), p' = V(u'), \) and \( p = V(u) \).
   b. If \( S(u') \neq u \) or \( u' = u \) then Return \( (u, \pi) \).
   c. If \( p = p' \) then Return \( (u', p') \).
   d. If \( p' < p \) then Return \( (u', p - 1) \) else Return \( (u', p + 1) \).
5. Else \% when \( \pi \neq V(u) \)
   a. Let \( u' = S(u), p' = V(u'), \) and \( p = V(u) \).
   b. If \( P(u') \neq u \) or \( u' = u \) then Return \( (u, \pi) \).
   c. If \( p' = p \) or \( p < p < p' \) or \( p < p' \leq \pi \) or \( \pi > p > p' \) or \( p > p' \geq \pi \) then Return \( (u, \pi) \).
   d. If \( p < p' \), then If \( p < \pi < p' - 1 \) then Return \( (u, \pi - 1) \).
   e. If \( p > p' \), then if \( p > \pi \geq p' + 1 \) then Return \( (u, \pi + 1) \).

As mentioned before, thumb-rule for potential function procedure \( V' \) is to return zero for self loops, return 1 for \( 0^k \), and return the number formed by lowest \( m \) bits for the rest.

Procedure \( V'(u, \pi) \). Let \( x = (u, \pi) \) for notational convenience.

1. If \( x = 0^k \), then Return 1.
2. If \( S'(x) = x \) and \( P'(x) = x \) then Return 0.
3. If \( S'(x) \neq x \) or \( P'(x) \neq x \) then Return \( \pi \).

The fact that procedures \( S', P' \) and \( V' \) give a valid \texttt{EndOfMeteredLine} instance follows from construction.

\textbf{Lemma 27}. Procedures \( S', P' \) and \( V' \) gives a valid \texttt{EndOfMeteredLine} instance on vertex set \( \{0, 1\}^k \), where \( k = m + n \) and \( V': \{0, 1\}^k \to \{0, \ldots, 2^k - 1\} \).

Next three lemmas shows how to construct a solution of \texttt{EndOfPotentialLine} instance \( T \) from a type \( T1, T2, \) or \( T3 \) solution of constructed \texttt{EndOfMeteredLine} instance \( T' \). The main idea for next lemma, which handles type \( T1 \) solutions, is that we never create spurious end or start of a path.

\textbf{Lemma 28}. Let \( x = (u, \pi) \) be a type \( T1 \) solution of constructed \texttt{EndOfMeteredLine} instance \( T' \). Then \( u \) is type \( R1 \) solution of the given \texttt{EndOfPotentialLine} instance \( T \).
Proof. Let $\Delta = 2^n - 1$. In $I'$, clearly $(0^n, \pi)$ for any $\pi \in 1, \ldots, \Delta$ is not a start or end of a path, and $(0^n, 0)$ is not an end of a path. Therefore, $u \neq 0^n$. Since $(S(0^n), \pi), \forall \pi \in \{0, \ldots, \Delta\}$ are self loops, $u \neq S(0^n)$.

If to the contrary, $S(P(u)) = u$ and $P(S(u)) = u$. If $S(u) = u = P(u)$ then $(u, \pi), \forall \pi \in \{0, \ldots, \Delta\}$ are self loops, a contradiction.

For the remaining cases, let $P'(S'(x)) \neq x$, and let $u' = S(u)$. There is a valid edge from $u$ to $u'$ in $I$. Then we will create valid edges from $(u, V(u))$ to $(S(u), V(S(u))$ with appropriately changing second coordinates. The rest of $(u, \pi)$ are self loops, a contradiction.

Similar argument follows for the case when $S'(P'(x)) \neq x$.

The basic idea behind the next lemma is that T2 type solution in $I'$ has potential 1. Therefore, it is surely not a self loop. Then it is either an end of a path or near an end of a path, or else near a potential violation.

Lemma 29. Let $x = (u, \pi)$ be a type T2 solution of $I'$. Either $u \neq 0^n$ is start of a path in $I$ (type R1 solution), or $P(u)$ is an R1 or R2 type solution in $I$, or $P(P(u))$ is an R2 type solution in $I$.

Proof. Clearly $u \neq 0^n$, and $x$ is not a self loop, i.e., it is not a dummy vertex with irrelevant value of $\pi$. Further, $\pi = 1$. If $u$ is a start or end of a path in $I$ then done.

Otherwise, if $V(P(u)) > \pi$ then we have $V(u) \leq \pi$ and hence $V(u) - V(P(u)) \leq 0$ giving $P(u)$ as an R2 type solution of $I$. If $V(P(u)) < \pi = 1$ then $V(P(u)) = 0$. Since potential can not go below zero, either $P(u)$ is an end of a path, or for $u'' = P(P(u))$ and $u' = P(u)$ we have $u' = S(u'')$ and $V(u') - V(u'') \leq 0$, giving $u''$ as a type R2 solution of $I$.

At type T3 solution of $I'$ potential is strictly positive, hence they are not self loops. If they correspond to potential violation in $I$ then we get type R2 solution. But this may not be the case, if we made $S'$ or $P'$ self pointing due to end or start of a path respectively. In that case, we get type R1 solution. Next lemma formalizes this intuition.

Lemma 30. Let $x = (u, \pi)$ be a type T3 solution of $I'$. If $x$ is a start or end of a path in $I'$ then $u$ gives type R1 solution in $I$. Otherwise $u$ gives a type R2 solution of $I$.

Proof. Since $V'(x) > 0$, it is not a self loop and hence is not dummy, and $u \neq 0^n$. If $u$ is start or end of a path then $u$ is a type R1 solution of $I$. Otherwise, there are valid incoming and outgoing edges at $u$, therefore so at $x$.

If $V(S(x)) - V(x) \neq 1$, then since potential either remains same or increases or decreases exactly by one on edges of $I'$, it must be the case that $V(S(x)) - V(x) \leq 0$. This is possible only when $V(S(u)) \leq V(u)$. Since $u$ is not an end of a path we do have $S(u) \neq u$ and $P(S(u)) = u$. Thus, $u$ is a type T2 solution of $I$.

If $V(x) - V(P(x)) \neq 1$, then by the same argument we get that for $(u'', \pi'') = P(u)$, $u''$ is a type R2 solution of $I$.

Our main theorem follows using Lemmas 27, 28, 29, and 30.

Theorem 31. An instance of EndOfPotentialLine can be reduced to an instance of EndOfMeteredLine in polynomial time such that a solution of the former can be constructed in a linear time from the solution of the latter.
C Pseudo-code for Lemke’s algorithm

| If $q \geq 0$ then Return $y \leftarrow 0$
| $y \leftarrow 0, z \leftarrow \min_{i \in [d]} q_i, s = q + z 1$
| $i \leftarrow$ duplicate label at vertex $(y, s, z)$ in $P$. $flag \leftarrow 1$
| While $z > 0$ do
|   If $flag = 1$ then set $(y', s', z') \leftarrow$ vertex obtained by relaxing $y_i = 0$ at $(y, s, z)$ in $P$
|   Else set $(y', s', z') \leftarrow$ vertex obtained by relaxing $s_i = 0$ at $(y, s, z)$ in $P$
|   If $z > 0$ then
|     $i \leftarrow$ duplicate label at $(y', s', z')$
|     If $v_i > 0$ and $v'_i = 0$ then $flag \leftarrow 1$. Else $flag \leftarrow 0$
|     $(y, s, z) \leftarrow (y', s', z')$
| End While
| Return $y$
Missing Procedures and Proofs from Section 5

D.1 Procedures IsValid, ItoE, and EtoI

Table 4 Procedure IsValid(u)

| If u = 0\(^d\) then Return 1 |
|-------------------------------|
| Else let \(\tau = (u_{(d+1)} + \cdots + u_{2d})\) |
| If \(\tau > 1\) then Return 0 |
| Let \(S \leftarrow \emptyset\). %set of tight inequalities. |
| If \(\tau = 0\) then \(S = S \cup \{z = 0\}\). |
| Else |
| Set \(l \leftarrow\) index of the non-zero coordinate in vector \((u_{(d+1)}, \ldots, u_{2d})\). |
| Set \(S = \{y_l = 0, s_l = 0\}\). |
| For each \(i\) from 1 to \(d\) do |
| If \(u_i = 0\) then \(S = S \cup \{y_i = 0\}\), Else \(S = S \cup \{s_i = 0\}\) |
| Let \(A\) be a matrix formed by lhs of equalities \(My + s - 1z = q\) and that of set \(S\) |
| Let \(b\) be the corresponding rhs, namely \(b = [q; 0_{d \times 1}]\). |
| Let \((y', s', z') \leftarrow b * A^{-1}\) |
| If \((y', s', z') \in P\) then Return 1, Else Return 0 |

Table 5 Procedures ItoE(u) and EtoI(y,s,z)

| ItoE(y,s,z) |
|--------------|
| If \(\exists i \in [d]\) s.t. \(y_i * s_i \neq 0\) then Return \((0_{(2d-2) \times 1}; 1; 1)\) %In valid |
| Set \(u \leftarrow 0_{2d \times 1}\). Let \(DL = \{i \in [d] \mid y_i = 0 \text{ and } s_i = 0\}\). |
| If |\(|DL| > 1\) then Return \((0_{(2d-2) \times 1}; 1; 1)\) %In valid |
| If \(|DL| = 1\) then for \(i \in DL\), set \(u_i \leftarrow 1\) |
| For each \(i \in [d]\) If \(s_i = 0\) then set \(u_{d+i} \leftarrow 1\) |
| Return \(u\) |

| EtoI(u) |
|---------------|
| If \(u = 0^n\) then Return \((0_{d \times 1}; q + z^u + 1, z^u + 1)\) %This case will never happen |
| If IsValid(u)=0 then Return \(0_{(2d+1) \times 1}\) |
| Let \(\tau = (u_{(d+1)} + \cdots + u_{2d})\) |
| Let \(S \leftarrow \emptyset\). %set of tight inequalities. |
| If \(\tau = 0\) then \(S = S \cup \{z = 0\}\). |
| Else |
| Set \(l \leftarrow\) index of non-zero coordinate in vector \((u_{(d+1)}, \ldots, u_{2d})\). |
| Set \(S = \{y_l = 0, s_l = 0\}\). |
| For each \(i\) from 1 to \(d\) do |
| If \(u_i = 0\) then \(S = S \cup \{y_i = 0\}\), Else \(S = S \cup \{s_i = 0\}\) |
| Let \(A\) be a matrix formed by lhs of equalities \(My + s - 1z = q\) and that of set \(S\) |
| Let \(b\) be the corresponding rhs, namely \(b = [q; 0_{d \times 1}]\). |
| Return \(b * A^{-1}\) |
D.2 Proof of Lemma 14

Lemma 14 (restated): If $\text{IsValid}(u) = 1$ then $u = \text{ItoE}(\text{EtoI}(u))$, and the corresponding vertex $(y, s, z) \in \text{EtoI}(u)$ of $\mathcal{P}$ is feasible in $\mathcal{E}$. If $(y, s, z)$ is a feasible vertex of $\mathcal{E}$ then $u = \text{ItoE}(y, s, z)$ is a valid configuration, i.e., $\text{IsValid}(u) = 1$.

Proof. The only thing that can go wrong is that the matrix $A$ generated in IsValid and EtoI procedures are singular, or the set of double labels $DL$ generated in ItoE has more than one elements. Each of these are possible only when more than $2d + 1$ equalities of $\mathcal{P}$ hold at the corresponding point $(y, s, z)$, violating non-degeneracy assumption. ▷

D.3 Proof of Lemma 15

Lemma 15 (restated): Functions $P$, $S$ and $V$ of instance $\mathcal{E}$ are well defined, making $\mathcal{E}$ a valid EndOfPotentialLine instance.

Proof. Since all three procedures are polynomial-time in $\mathcal{L}$, they can be defined by $\text{poly}(\mathcal{L})$-sized Boolean circuits. Furthermore, for any $u \in \text{vert}$, we have that $S(u), P(u) \in \text{vert}$. For $V$, since the value of $z \in [0, \Delta - 1]$, we have $0 \leq \Delta^2(\Delta - z) \leq \Delta^3$. Therefore, $V(u)$ is an integer that is at most $2 \cdot \Delta^3$ and hence is in set $\{0, \ldots, 2^m - 1\}$. ▷

D.4 Proof of Lemma 16

Lemma 16 (restated): Let $u \neq u'$ be two valid configurations, i.e., $\text{IsValid}(u) = \text{IsValid}(u') = 1$, and let $(y, s, z)$ and $(y', s', z')$ be the corresponding vertices in $\mathcal{P}$. Then the following holds: (i) $V(u) = V(u')$ iff $z = z'$. (ii) $V(u) > V(u')$ iff $z < z'$.

Proof. Among the valid configurations all except $0$ has positive $V$ value. Therefore, wlog let $u, u' \neq 0$. For these we have $V(u) = \lfloor \Delta^2(\Delta - z) \rfloor$, and $V(u') = \lfloor \Delta^2(\Delta - z') \rfloor$.

Note that since both $z$ and $z'$ are coordinates of vertices of $\mathcal{P}$, whose description has highest coefficient of $\max\{\max_{i \in [d]} M(i, j), \max_{i \in [d]} |q_i|\}$, and therefore their numerator and denominator both are bounded above by $\Delta$. Therefore, if $z < z'$ then we have

$$z' - z \geq \frac{1}{\Delta^2} \Rightarrow (\Delta - z) - (\Delta - z') \ast \Delta^2 \geq 1 \Rightarrow V(u) - V(u') \geq 1.$$

For (i), if $z = z'$ then clearly $V(u) = V(u')$, and from the above argument it also follows that if $V(u) = V(u')$ then it can not be the case that $z \neq z'$. Similarly for (ii), if $V(u) > V(u')$ then clearly, $z' > z$, and from the above argument it follows that if $z' > z$ then it can not be the case that $V(u') \geq V(u)$. ▷