Quantum thermodynamics aspects with a thermal reservoir based on $\mathcal{PT}$-symmetric Hamiltonians

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(Dated:)

We present results concerning aspects of quantum thermodynamics under the background of non-Hermitian quantum mechanics for the dynamics of a quantum harmonic oscillator. Since a better control over the parameters in quantum thermodynamics processes is desired, we use concepts from collisional model to introduce a simple prototype of thermal reservoir based on $\mathcal{PT}$-symmetric Hamiltonians and study its effects under the thermalization process of a single harmonic oscillator prepared in a displaced thermal state. We verify that controlling the $\mathcal{PT}$-symmetric features of the reservoir allows to reverse the heat flow between system and reservoir, as well as to preserve the coherence over a longer period of time and reduce the entropy production. Furthermore, we considered a modified quantum Otto cycle in which the standard hot thermal reservoir is replaced by the thermal reservoir based on $\mathcal{PT}$-symmetric Hamiltonians. By defining an effective temperature depending on the $\mathcal{PT}$-symmetric parameter, it is possible to interchange the quantum Otto cycle configuration from engine to refrigerator by varying the $\mathcal{PT}$-symmetric parameter. Our results indicate that $\mathcal{PT}$-symmetric effects could be useful to achieve an improvement in quantum thermodynamics protocols such as coherence protection and entropy production reduction.

I. INTRODUCTION

The ability to control and manipulate quantum information is the paramount key to develop the new generations of devices based on quantum properties. Any quantum system is under the action of its surround which induces the well-known decoherence phenomena on the system [1, 2]. Given that the coherence, for a specific basis, can be employed as a resource for many protocols of quantum information and quantum communication [3, 4], the protection of coherence along a quantum dynamics is an important task in modern devices, such as quantum computing [5] and quantum cryptography [6]. Furthermore, in many situations, as in quantum thermodynamics, the system is allowed to change heat with a thermal environment. For example, models of quantum heat engine and refrigerators employ two or more thermal reservoirs to absorb or release heat from/on the system functioning as working substance or refrigerator [7-9]. Thus, the performance as well as the functioning of quantum thermal machines depend crucially on how efficient is the quantum control involved in thermodynamic protocols.

Given a general protocol in quantum information or thermodynamics, one of the most important tasks is how to control the dynamics in order to optimize the results. For unitary processes (where entropy is kept unchanged), it is assumed a Hamiltonian representing an external agent that drives the system from an initial to a final state. This is particularly important in fluctuation theorems such as the Jarzynski [10] and Crooks [11] relations. On the other hand, in dissipative systems (where entropy changes), the control may be implemented through some parameter of the thermal reservoir. The entropy production during a thermalization, for instance, depends fundamentally on the quantum aspects of the initial state, given that some recent studies have shown that the entropy production is larger for states with initial coherence [12, 13].

In the last years, a new type of Hamiltonian control has arisen. Traditionally, the standard quantum mechanics assumes as a bona-fide operator all with the Hermiticity property, $O = O^\dagger$. This directly implies a set of real eigenvalues, with a complete set of eigenstates that are essential for defining a viable candidate for representing a physical observable in quantum theory [14]. However, since the paper of Bender and Boettcher [15], it has been shown that the set of Hamiltonians fulfilling the conditions of invariance by spatial reflection (parity $\mathcal{P}$) and time reversal $\mathcal{T}$ also has real eigenvalues and thus can represent physical systems. This new condition to guarantee real spectra is known as $\mathcal{PT}$-symmetry, and it gave rise to the non-Hermitian quantum mechanics [16], once $\mathcal{PT}$-symmetric Hamiltonians are not Hermitian in general. Indeed, the range of interest in effects by considering systems described by $\mathcal{PT}$-symmetric Hamiltonians is vast and includes quantum optics and photonics [17, 18], time-dependent Hamiltonians [19–21], applications to non-commutative geometries [22–24], as well as to fluctuation relations [25–27]. Since the conditions for an operator to be a viable candidate to represent a physical observable are that eigenvalues are real and that eigenvectors are complete, the condition of orthogonality for a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}$ can be relaxed and substituted by a biothogonality [28], and the connection with the biorthogonal system with the ortogonal is made by a nontrivial metric operator [29, 30]. Thus depending on the choice of the parameters associated to the metric the effects in employing the Hermitian Hamiltonian to some protocol can be considerably different [31, 32]. For example, Ref. [32] has shown that the decoherence dynamics can be modified changing the parameters of the metric. Another interesting set of results concerns the study of quantum thermodynamic with non-Hermitian Hamiltonian background. The pioneer paper in this direction is Ref. [27], in which the authors showed that the Jarzynski equality and the Carnot bound, both expressing the second law of thermodynamics, hold for non-Hermitian

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Hamiltonians.

In this work, we investigate some quantum thermodynamics aspects in the context of non-Hermitian quantum mechanics. By introducing a particular $\mathcal{PT}$-symmetric Hamiltonian and a given metric, we obtain its Hermitian counterpart, from which we construct associated states. Using concepts of the collisional model theory [33-36], we write a Lindblad master equation where the system of interest is a single harmonic oscillator and the thermal reservoir contains $\mathcal{PT}$-symmetric features in the form of the parameter of the metric. We then consider a thermalization process in which the initial state of the harmonic oscillator is prepared in a displaced thermal state, such that it has an amount of coherence in the energy basis. We verify that depending on the $\mathcal{PT}$-symmetric features of the thermal reservoir, the heat flow between system and reservoir can be reversed, the coherence can be preserved along the dynamics, as well as the entropy production is considerably reduced. As a second part of our work, we consider a modified quantum Otto cycle where the standard hot thermal reservoir is replaced by a $\mathcal{PT}$-symmetric thermal reservoir prototype. By writing explicit expressions for the net work and for the heat exchanged with the cold and hot thermal reservoirs, we verify that depending on the $\mathcal{PT}$-symmetric features, the configuration of the Otto cycle can change from engine to refrigerator.

This work is organized as follows. In section II we review some concepts on $\mathcal{PT}$-symmetry and non-Hermitian quantum mechanics. Section III is dedicated to introduce the notion of $\mathcal{PT}$-symmetric thermal reservoir via collisional model concepts as well as to derive the thermal state with $\mathcal{PT}$-symmetric features. The thermalization process of a system composed of a single harmonic oscillator prepared in a displaced thermal state is analyzed in section IV, where the heat, coherence and entropy production are quantified in terms of $\mathcal{PT}$-symmetric feature of the thermal reservoir. In section V we study a modified version of the quantum Otto cycle in which the hot thermal reservoir is described by a $\mathcal{PT}$-symmetric thermal reservoir. Finally, our conclusion and final remarks are drawn in section VI.

II. REVIEW ON $\mathcal{PT}$-SYMMETRIC HAMILTONIAN

The standard quantum mechanics assumes that in order to give a given operator to be a viable candidate to represent a physical observable, it must have real spectra and a set of complete eigenstates. Hermitian operators fulfill these conditions, in order that many textbooks assume that any observable $O$ has to be Hermitian, $O = \mathcal{O}$ [37]. However, Hermitian operators are not the only operator class that has real spectra and a complete set of eigenvectors. As shown by Bender and Boettcher [15], operators that are invariant under parity $P$ and time reversal $T$ simultaneously also have real spectra and a complete set of eigenvectors, being a viable candidate to represent a physical observable. For example, for a general $N$-dimensional Hamiltonian $\mathcal{H}(q_\ell, p_\ell)$ with $\ell = 1, \ldots, N$, the reality of the spectrum is guaranteed by the unbroken $\mathcal{PT}$-symmetry [38], that can be translated into these two relations: $\{\mathcal{H}(q_\ell, p_\ell), \mathcal{PT}\} = 0$, and $\mathcal{PT}\{\Psi(t)\} = |\Psi(t)\rangle$, where $|\Psi(t)\rangle$ is the eigenstate of $\mathcal{H}(q_\ell, p_\ell)$. These conditions mean that the Hamiltonian $\mathcal{H}(q_\ell, p_\ell)$ must be invariant under the set of transformations $\mathcal{H}(q_\ell, p_\ell)$.

\[ \mathcal{PT} q_\ell (\mathcal{PT})^{-1} \to - q_\ell, \]
\[ \mathcal{PT} p_\ell (\mathcal{PT})^{-1} \to - p_\ell, \]
\[ \mathcal{PT} i (\mathcal{PT})^{-1} \to - i. \]

In the case of a non-Hermitian Hamiltonian $\mathcal{H}(q_\ell, p_\ell)$ fulfilling the condition in Eq. (1), it is known as $\mathcal{PT}$-symmetric Hamiltonian and the relation $\mathcal{H}(q_\ell, p_\ell) = \mathcal{H}(q_\ell, p_\ell)$ is achieved. The connection between the standard quantum mechanics and the non-Hermitian quantum mechanics is performed by employing the similarity transformation

\[ H(q_\ell, p_\ell) = \eta \mathcal{H}(q_\ell, p_\ell) \eta^{-1}, \]

in which $\eta = \eta(q_\ell, p_\ell)$ is the Dyson map such that $\eta\eta^{-1} = \mathbb{I}$, with $\mathbb{I}$ the identity operator [29, 30, 40]. Furthermore, using Eq. (2) and the Hermiticity relation of the Hamiltonian $\mathcal{H}(q_\ell, p_\ell)$, it is easy to show that the non-Hermitian Hamiltonian, $\mathcal{H}(q_\ell, p_\ell)$, satisfies the well-known quasi-Hermiticity relation $\Theta \mathcal{H}(q_\ell, p_\ell) \Theta = \mathcal{H}(q_\ell, p_\ell) \Theta$, where $\Theta = \eta \eta^{-1}$ is the metric operator to ensure probability conservation [30, 40]. Like the Hamiltonian, non-Hermitian observables $\mathcal{O}$ are connected to Hermitian observables through the similarity transformation, $O = \eta \mathcal{O} \eta^{-1}$. This implies that the expected value for these observables is the same, $\langle \phi(t) \rangle \mathcal{O} \phi(t) = \langle \Psi(t) \rangle \mathcal{O} \Psi(t) \rangle$, where $\langle \phi(t) \rangle = \eta^{-1} \langle \Psi(t) \rangle$ [40]. This allows us to obtain observables, choosing a similarity transformation (or Dyson map operator) transforming our non-Hermitian Hamiltonian into its Hermitian isospectral partner and working in the standard quantum mechanics with this partner.

III. MODELING A THERMAL RESERVOIR THROUGH $\mathcal{PT}$-SYMMETRIC HAMILTONIANS

In this section we use collisional model techniques to obtain the master equation in which the thermal reservoir carries $\mathcal{PT}$-symmetric features. In order to review shortly the collisional model concepts, one assumes a system $S$ in contact with a bath. In turn, the bath is assumed to be a sufficient large collection of auxiliary systems (ancilla), each of them prepared in the same thermal state. Besides, the ancillas are considered to be non-interacting, a fact that is associated to a Markovian dynamics of the system $S$ [34]. Writing the initial state of the system and each ancilla as $\rho_0$ and $\rho_{\text{ancilla}} = \zeta_\ell(n)$, respectively, where $\bar{n} = \text{Tr} (a a^\dagger)$ is the average number of photons in a mode, with $a$ (a $a^\dagger$) the annihilation (creation) operator, the joint state of the system plus bath is given by $\sigma = \rho_0 \otimes (\otimes_{j=1}^N \zeta_\ell(n))$. In the collisional model, the system $S$ is assumed to interact with just one ancilla at a time during a time interval $\delta t$. After the collision with the ancilla $\zeta_\ell(n)$, it is discarded and a subsequent new ancilla $\zeta_{\ell+1}(n)$ is brought to interact with the system. This process is repeated many times. An illustration of the collisional model is depicted in Fig. 1. Following Refs. [33, 34], by assuming that the interaction time between the system and each ancilla is sufficiently short, $\delta t \to 0$, the collisional model provides a Lindblad master equation [33], which can be given by

\[ \frac{d\rho}{dt} = -i [H_S, \rho] + \gamma (N + 1) D[a] \rho + \gamma N D[a^\dagger] \rho, \]
with $H_S = \hbar \omega_S (a^\dagger a + 1/2)$ the Hamiltonian of the system, $D[O] = \mathcal{O}\mathcal{P}_\omega - \frac{1}{2} (\mathcal{O}\mathcal{P}_\omega + \mathcal{O}_\omega^\dagger \mathcal{P}_\omega)$ are the Lindblad operators, $N = (e^{\beta \hbar \omega} - 1)^{-1}$ is the average number of photons associated to the bath, $\beta = 1/k_B T$ is the inverse temperature, and $\gamma$ is the decay rate. For $N = 0$, Eq. (3) represents pure loss.

As we will note throughout the work, all the states considered are Gaussian [41–44]. In this situation, a suitable method to treat the thermalization dynamics is employing the covariance matrix, defined as $\sigma_{ij} = \langle \mathbf{R}_i \mathbf{R}_j \rangle - 2 \langle \mathbf{R}_i \rangle \langle \mathbf{R}_j \rangle$, with the vector of operators $\mathbf{R} = (q, p)$, with $q(p)$ the position (momentum) operator. For a general Gaussian state $\rho = \rho(\vec{d}, \sigma)$ with first moments $\vec{d} = \langle q \rangle, \langle p \rangle$ and covariance matrix $\sigma$, the thermalization dynamics with a Markovian ansatz is enough to elucidate all relevant results in the following we use the expression "$\mathcal{PT}$-symmetric Hamiltonians".

where $\sigma_{\text{aspt}}$ is the asymptotic covariance matrix reached when $t \to \infty$ (complete thermalization). This formalism also provides an expression to obtain the internal energy of the system using the covariance matrix [45], given by $\mathcal{U}_t = \hbar \omega \text{Tr} | \sigma(t) | / 4$.

We now pass to consider a $\mathcal{PT}$-symmetric Hamiltonian following the concepts presented in Sec. II to construct thermal states for the ancillas of the bath. Let us assume the following $\mathcal{PT}$-symmetric Hamiltonian given by [32]

$$\mathcal{H}^{\mathcal{PT}} = \frac{\mu^2}{2m} + \frac{1}{2} m \omega^2 q^2 + 2 i \omega \epsilon p, \tag{6}$$

which has a real spectra. A similar structure for the $\mathcal{PT}$-symmetric Hamiltonian was employed in Ref. [32] to claim that $\mathcal{PT}$-symmetry can be useful to control the decoherence of a two-level system. For simplicity we assume the following expression for the Dyson operator, $\mathcal{G} = e^{\frac{\pi i}{2} \mathcal{P}_o^2}$. As we will see, this metric is enough to elucidate all relevant results in the present work. In performing the Hermitization procedure as in Sec. II we obtain the Hermitian Hamiltonian

$$H = \eta \mathcal{H}^{\mathcal{PT}} \eta^{-1} = \frac{\mu^2}{2m} p^2 + \frac{1}{2} m \omega^2 q^2 + \hbar \omega \epsilon, \tag{7}$$

with $\mu^2 = (1 + 4 \epsilon^2)$. Hamiltonian in Eq. (7) is clearly that of a one-mode harmonic oscillator with a shift of energy. Thus, it is possible to construct a thermal state which will be the ancillary states of the bath. A thermal state can be written in general as $\rho_{\text{th}} = e^{-\beta H}/Z$, where $Z = \sum_n e^{-\beta H}$ is the partition function. By using Eq. (7), the associated thermal state reads

$$\rho_{\text{th}} = e^{-\beta \hbar \omega (n + 1/2)} / \sum_n e^{-\beta \hbar \omega (n + 1/2)}, \tag{8}$$

Writing the thermal state in Eq. (8) in the Fock basis one has

$$\rho_{\text{th}}(\bar{m}) = \sum_n \frac{\bar{m}^n}{(\bar{m} + 1)^{n+1}} |n \rangle \langle n|, \tag{9}$$

with $\bar{m} = (e^{\beta \hbar \omega} - 1)^{-1}$. Throughout the rest of the work we assume $N = \bar{m}$, meaning that the thermal states of the ancilla in the bath are given by Eq. (9). This implies that the master equation in Eq. (3) shall also includes the effects of the $\mathcal{PT}$-symmetric Hamiltonian which physically is included by the factor $\mu$. In order to elucidate the physical meaning of $\mathcal{PT}$-symmetric features in the thermal reservoir prototype, we observe from the average number of photons $\bar{m}$ that it is possible to define an effective temperature as $\beta_{\text{eff}} = (1 + 4 \epsilon^2) \beta$, where obviously for $\epsilon = 0$ we have $\beta_{\text{eff}} = \beta$. In order to be clear, in the following we use the expression "$\mathcal{PT}$-symmetric thermal reservoir" as being that modeled with ancillas prepared in thermal states using $\mathcal{PT}$-symmetric Hamiltonians.

**IV. REVERSING THE HEAT FLOW AND PROTECTING COHERENCE USING A $\mathcal{PT}$-SYMMETRIC THERMAL RESERVOIR**

In order to investigate how a thermal reservoir carrying $\mathcal{PT}$-symmetric features could influence the thermalization dynamics of a single-mode Gaussian state, we consider an example of initial state with coherence in the Fock basis. The motivation for this is that we are able to study not only the heat exchanged between system and reservoir but also the coherence dynamics introduced by the $\mathcal{PT}$-symmetric features of the bath.

The system under consideration is a single harmonic oscillator. The most paradigmatic Gaussian state associated to the Hamiltonian of the system is a thermal state $\zeta^{\text{th}}(\bar{n})$. To introduce coherence on the state $\zeta^{\text{th}}(\bar{n})$, we use the mechanism consisting in applying the displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$, such that the state $\rho = D(\alpha) \zeta^{\text{th}}(\bar{n}) D(\alpha)^\dagger$ possesses an amount of coherence.

To quantify the coherence of a Gaussian state, we use an entropic quantifier based on the relative entropy $S(\sigma_1 || \sigma_2) = \sigma_1 \ln(\sigma_1) - \sigma_1 \ln(\sigma_2)$ [46]. In Ref. [47] the same measure is particularized for Gaussian states, such that one way of quantifying

![Figure 1. Illustration of the collisional model. The system (blue circle) interacts with a particle of the thermal bath, which is composed of a large number of non-interacting identical particles, all prepared in the same thermal state. Once the system interacts with the particle $k$, a new particle, $k+1$, is put to interact with the system.](image)
\[ \rho_0 = D(\alpha) \zeta^{th}(\bar{n}) D(\alpha)^\dagger \]

\[ \rho_{t \to \infty} = \zeta^{th}(N) \]

Figure 2. An illustration of the thermalization process of a single harmonic oscillator prepared in a displaced thermal state with average number of photons \( \bar{n} \) interacting with the \( \mathcal{P}\mathcal{T} \)-symmetric thermal reservoir with effective temperature \( \beta_{\text{eff}} = (1 + 4 \epsilon^2) \beta \).

coherence of a general \( N \)-mode Gaussian state \( \rho = \rho(\vec{d}, \sigma) \) is given by

\[ C(\rho) = S(\zeta^{th}) - S(\rho), \tag{10} \]

where \( S(\cdot) \) is the von Neumann entropy,

\[ S(\rho) = - \sum_{j=1}^N \left[ \frac{\nu_j - 1}{2} \ln \left( \frac{\nu_j - 1}{2} \right) - \frac{\nu_j + 1}{2} \ln \left( \frac{\nu_j + 1}{2} \right) \right], \tag{11} \]

with \( \{\nu_j\}_{j=1}^N \) the symplectic eigenvalues of \( \sigma \), and \( \zeta^{th} \) is a \( N \)-mode reference thermal state with average number of photons \( \{\bar{k}_j\}_{j=1}^N \) written in terms of the first moments and the covariance matrix of \( \rho \) and given by [47]

\[ \bar{k}_j = \frac{1}{4} \left[ \sigma_{11}^j + \sigma_{22}^j + (d_j^\dagger)^2 + (d_j^\dagger)^2 - 2 \right]. \tag{12} \]

The coherence quantifier in Eq. (10) satisfies the following properties: \( C(\rho) \geq 0 \), \( C(\rho) = 0 \) if and only if \( \rho \) is a tensor product of thermal states, and \( C(\rho) \geq C(\Phi_{\text{IGC}}) \), where \( \Phi_{\text{IGC}} \) is an incoherent Gaussian channel [47]. An illustration of the thermalization protocol is depicted in Fig. 2.

The heat exchanged between system and thermal reservoir can be written in terms of the covariance matrix as

\[ \langle Q \rangle = \frac{\hbar \omega}{4} (\text{Tr} \left[ \sigma(\bar{t}) \right] - \text{Tr} \left[ \sigma(0) \right]) \tag{13} \]

\[ = \hbar \omega (N - \bar{n}) (1 - e^{-\gamma t}), \tag{14} \]

with \( N = (e^{\beta \hbar \omega \mu} - 1)^{-1} \). Another important quantity during a thermalization is the entropy production, which can be written as

\[ \langle \Sigma \rangle = -\beta \Delta U_{\tau_2, \tau_1} + \Delta S_{\tau_2, \tau_1}, \tag{15} \]

where \( \Delta U_{\tau_2, \tau_1} = U_{\tau_2} - U_{\tau_1} \) is the variation of internal energy associated to the thermalization process and \( \Delta S_{\tau_2, \tau_1} = S(\rho_{\tau_2}) - S(\rho_{\tau_1}) \).

Figure 3 presents the heat exchanged, coherence, and the entropy production as a function of the thermalization time \( t \) with the \( \mathcal{P}\mathcal{T} \)-symmetric thermal reservoir. The initial state is assumed to be a displaced thermal state \( \rho = D(\alpha) \zeta^{th}(\bar{n}) D(\alpha)^\dagger \), with an average number of photons \( \bar{n} = 2 \), and \( \alpha \) such that the initial position of the state on the phase-space is \( (q_0, p_0) = (1, 1) \). For the thermal reservoir, it was considered an effective temperature \( \beta_{\text{eff}} = (1 + 4 \epsilon^2) \beta \), with \( \beta \hbar \omega = 0.2 \). To see how \( \mathcal{P}\mathcal{T} \)-symmetric features of the thermal bath influence the thermalization dynamics we have set the value of \( \epsilon \) to be 0 (solid black lines), 0.5 (dotted red lines), and 1.0 (dashed dotted blue lines). The same results would be verified in the case of a squeezed thermal state as initial one-mode. In considering these two class of initial states, we are encompassing all the mechanics to generate coherence in a single-mode Gaussian state.

For \( \epsilon = 0 \) we recover the standard thermal bath. Figure 3-a) shows the heat exchanged between system and thermal bath as a function of the thermalization time. It can be observed that, depending on the value of the \( \mathcal{P}\mathcal{T} \)-symmetric parameter \( \tau \), the heat flow can be reversed, passing to flow from the reservoir to the system. This is physically explained by looking at the effective temperature \( \beta_{\text{eff}} = (1 + 4 \epsilon^2) \beta \) which changes for different values of \( \epsilon \). In Fig. 3-b) we present the coherence as a function of the thermalization time. The results shows that increasing the value of \( \epsilon \), i.e., becoming the \( \mathcal{P}\mathcal{T} \)-symmetric feature more intensive, it is possible to effectively protect the coherence of the initial state from the decoherence effects due the thermalization. Finally, Figure 3-c) depicts the entropy production as a function of the thermalization time, where it can be noted that the entropy production is reduced when \( \epsilon \) is increased.

Some of the results presented in Fig. 3 have been discussed in previous papers. In Ref. [32] the authors showed that the decoherence dynamics is modified if the environment is assumed to carry non-Hermitian signatures. Furthermore, Ref. [31] exploits a \( \mathcal{P}\mathcal{T} \)-symmetric interaction Hamiltonian in the linear response theory scenario and argues that depending on the \( \mathcal{P}\mathcal{T} \)-symmetric properties of the interaction, there could be a change in the heat flow from the system to the reservoir as well as a coherence protection of the initial state. Our results indicate that the same effects could be obtained by considering a simple model of \( \mathcal{P}\mathcal{T} \)-symmetric thermal reservoir. Besides, the introduction of an effective temperature shows physically how \( \mathcal{P}\mathcal{T} \)-symmetry affects the dynamics of the system.

V. \( \mathcal{P}\mathcal{T} \)-SYMMETRIC THERMAL RESERVOIR IN A QUANTUM OTTO CYCLE

Motivated by the results in the last section, in particular the reversing of the heat flow between system and thermal reservoir, here we employ the model of \( \mathcal{P}\mathcal{T} \)-symmetric thermal reservoir in a modified quantum Otto cycle to observe its effects. We assume the simpler form for the quantum Otto cycle, i.e., the unitary strokes are performed quasistatically and the thermalization with the hot and cold reservoirs are complete. These restrictions are not relevant in our discussion, once the \( \mathcal{P}\mathcal{T} \)-symmetric parameter \( \epsilon \) introduces just an effective temperature \( \beta_{\text{eff}} \) on the thermal reservoir. Thus, taking in account a finite-time quantum Otto cycle will not bring any advantage in our discussion. The illustration of the cycle is shown in Fig. 4. The hot thermal reservoir, with inverse temperature \( \beta_{\text{hot}} \), is assumed to have \( \mathcal{P}\mathcal{T} \)-symmetric features in an exact way as in the Sec. IV, whereas the cold thermal reservoir is assumed to be purely thermal, i.e., it is a stan-
standard thermal bath with inverse temperature $\beta_{\text{cold}} > \beta_{\text{hot}}$. The quantum Otto cycle is fueled by a harmonic oscillator, prepared initially in thermal equilibrium with the cold thermal reservoir with state denoted by $\rho_0 = \zeta_{\text{cold}}^\text{eq}$. The quantum Otto cycle is described as follows.

First stroke. The working substance is submitted to a unitary process performed quasistatically such that the final state is given $\rho_{\text{f1}} = \zeta_{\text{cold}}^\text{eq}$ and the frequency of the Hamiltonian is changed from $\omega_i$ to $\omega_f$. The work performed in this stage is given by $W_i = (\hbar \omega_f \text{Tr}[\sigma_{\tau_1}] - \hbar \omega_i \text{Tr}[\sigma_0])/4$.

Second stroke. A complete thermalization between the working substance and the hot thermal reservoir is implemented, such that the final state is given by $\rho_{\tau_2} = \zeta_{\text{hot}}^\text{eq}$. Note that the inclusion of the parameter $\epsilon$ in the thermal state after the thermalization with the hot thermal reservoir means that the effective temperature of $\rho_{\tau_2}$ depends on $\epsilon$. The frequency of the Hamiltonian is kept unchanged during this process, with heat exchanged given by $Q_2^\text{eff} = \hbar \omega_f (\text{Tr}[\sigma_{\tau_2}] - \text{Tr}[\sigma_{\tau_1}])/4$.

Third stroke. The frequency of the Hamiltonian is quasistatically changed back to $\omega_i$, such that the working substance state at the final is $\rho_{\text{f3}} = \zeta_{\text{cold}}^\text{eq}$. Again, the frequency is kept unchanged. The heat exchanged with the cold thermal reservoir is $Q_4^\text{eff} = \hbar \omega_i (\text{Tr}[\sigma_{\tau_0}] - \text{Tr}[\sigma_{\tau_1}])/4$.

In a quantum Otto cycle, depending on the thermodynamic quantities, i.e., the net work $W_{\text{net}}^\epsilon = \langle W_i^\epsilon \rangle + \langle W_2^\text{eff} \rangle$, as well as the two heat exchanged $\langle Q_2^\text{eff} \rangle$ and $\langle Q_4^\text{eff} \rangle$, it works as engine or refrigerator. For the former, the conditions are $W_{\text{net}}^\epsilon < 0$, $\langle Q_2^\text{eff} \rangle > 0$, and $\langle Q_4^\text{eff} \rangle < 0$, whereas for the latter, $W_{\text{net}}^\epsilon > 0$, $\langle Q_2^\text{eff} \rangle < 0$, and $\langle Q_4^\text{eff} \rangle > 0$. The thermodynamic quantities can be written explicitly as

$$\langle Q_2^\text{eff} \rangle = \frac{\hbar \omega_f}{2} \left\{ \coth \left[ \frac{\hbar \omega_f \beta_{\text{hot}}^\text{eff}}{2} \right] - \coth \left[ \frac{\hbar \omega_i \beta_{\text{cold}}}{2} \right] \right\},$$

$$\langle Q_4^\text{eff} \rangle = \frac{\hbar \omega_i}{2} \left\{ \coth \left[ \frac{\hbar \omega_i \beta_{\text{cold}}}{2} \right] - \coth \left[ \frac{\hbar \omega_f \beta_{\text{hot}}^\text{eff}}{2} \right] \right\},$$

$$W_{\text{net}}^\epsilon = -\frac{\hbar (\omega_f - \omega_i)}{2} \left\{ \coth \left[ \frac{\hbar \omega_f \beta_{\text{hot}}^\text{eff}}{2} \right] - \coth \left[ \frac{\hbar \omega_i \beta_{\text{cold}}}{2} \right] \right\},$$

where $\beta_{\text{hot}}^\text{eff} = \sqrt{1 + \epsilon^2} \beta_{\text{hot}}$ and the conditions $\omega_f/\omega_i < \beta_{\text{cold}}/\beta_{\text{hot}}$ and $\omega_f/\omega_i > \beta_{\text{cold}}/\beta_{\text{hot}}$ must be fulfilled for the engine and refrigerator configuration, respectively.

For the quantum Otto cycle operating as an engine the efficiency is given by $\eta = -W_{\text{net}}^\epsilon/\langle Q_2^\text{eff} \rangle = 1 - \omega_f/\omega_i$, while operating as a refrigerator the coefficient of performance is $\text{COP} = \langle Q_4^\text{eff} \rangle/W_{\text{net}}^\epsilon = \omega_i/(\omega_f - \omega_i)$. Thus, the inclusion of $\mathcal{PT}$-symmetric features in the hot thermal reservoir does not affect the performance of the quantum Otto cycle. It must be stressed that these results would be obtained even considering a finite-time regime and partial thermalizations [8]. However, from the Sec. IV, depending on the value of $\epsilon$ it is possible to change the direction of the heat flow between the system and hot thermal reservoir. This fact implies that we can move from the engine to refrigerator configurations and vice-versa by controlling the $\mathcal{PT}$-symmetric parameter $\epsilon$. Figure 5 depicts exactly these results by showing the net work, and the hot and cold heat exchanged as a function of the parameter $\epsilon$. As it can be observed from Fig. 5, for a critical value of $\epsilon$, the quantum cycle configuration changes from engine to refrigerator. The exact value of $\epsilon_c$ is given by

$$\epsilon_c = \frac{\sqrt{(\omega_i \beta_{\text{cold}})^2 - (\omega_f \beta_{\text{hot}}^\text{eff})^2}}{2 \omega_f \beta_{\text{hot}}^\text{eff}}.$$

This result is the main concerning application of $\mathcal{PT}$-symmetric effects in quantum Otto cycle and in principle could be tested in quantum machines based on optical devices [48].
We set the parameters of the cycle to be a displaced thermal state. We verified that depending on the $\mathcal{PT}$-symmetric features of the thermal reservoir, the heat flow can be reversed between system and reservoir, the coherence is preserved over a longer period of time, as well as the entropy production is considerably reduced. As the second part of the work, we considered a modified quantum Otto cycle, where the standard hot thermal reservoir was replaced by a $\mathcal{PT}$-symmetric thermal reservoir. Although the performance of the Otto cycle does not depend on $\mathcal{PT}$-symmetric features, we showed that varying the $\mathcal{PT}$-symmetric parameter of the hot thermal reservoir implies in changing the configuration of the cycle from engine to refrigerator.

The results goes in the direction of recent contributions showing the relevance in considering non-Hermitian Hamiltonians in quantum protocols. We hope that this work can help to unveil the role played by $\mathcal{PT}$-symmetric Hamiltonians in quantum thermodynamics.

VI. CONCLUSION

The emergent interest in considering non-Hermitian Hamiltonians with real spectra, i.e., fulfilling $\mathcal{PT}$-symmetric conditions, in different branches of quantum physics have been evidenced by considerable theoretical and experimental developments. The powerful of modeling quantum systems with $\mathcal{PT}$-symmetric features could be useful for future quantum devices. In this work, we have investigated some quantum thermodynamics aspects in the scenario in which the thermal reservoir is modeled such that it carries $\mathcal{PT}$-symmetric effects. Employing concepts from the collisional model theory we write a Lindblad master equation which governs the thermalization dynamics of a single harmonic oscillator interacting with a $\mathcal{PT}$-symmetric thermal reservoir. It was possible to write an effective temperature for the thermal reservoir, $\beta_{\text{hot}}^{\text{eff}} = \sqrt{1 + \epsilon^2} \beta_{\text{hot}}$, which includes $\mathcal{PT}$-symmetric effects.

ACKNOWLEDGMENTS

Jonas F. G. Santos acknowledges São Paulo Research Grant No. 2019/04184-5 and Federal University of ABC for support. Fabricio S. Luiz kindly acknowledges National Council for Scientific and Technological Development (CNPq) Research Grant No. 151435/2020-0 and São Paulo State University for support.
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