Quantum Cosmological Correlations in an Inflating Universe: Can fermion and gauge fields loops give a scale free spectrum?

Kanokkuan Chaicherdsaku

Theory Group, Department of Physics, University of Texas
Austin, TX, 78712

Abstract

This paper extends the calculation of quantum corrections to the cosmological correlation $\langle \zeta \zeta \rangle$, which has been done by Weinberg for a loop of minimally coupled scalars, to other types of matter loops and a general and realistic potential. It is shown here that departures from scale invariance are never large even when Dirac, vector, and conformal scalar fields are present during inflation. No fine tuning is needed, in the sense that effective masses or coupling constants can have arbitrary values. Even when the mass is as large as $M_{Pl}$, the one-loop result is still naturally smaller than the classical one. Thus, the results show that the existence of these fields during inflation may not be ruled out and is consistent with natural reheating.

$^1$Electronic address: kanokkua@physics.utexas.edu
I. INTRODUCTION

If we would like to understand how electrons and photons arise during reheating, it is natural to look back and ask what happened at the time during inflation. The gaussian and nearly scale invariant spectrum predicted by the scalar field dominated universe theory agrees very well with current observations. It is therefore widely believed that the quantum fluctuations of a scalar field during inflation seeded the large scale structure of the universe we observe today. However, in order to understand where other matter such as fermions and photons observed today come from, the inflaton would have to couple with these fields during inflation. There is no reason why there must be only scalar fields and gravity but nothing else during inflation. Other matter would not have arisen during reheating if there were only a scalar field that coupled exclusively to itself and gravity. If other matter such as Dirac, vector, and conformal scalar fields during inflation do not give anything far larger than the observed values and do not break the scale invariant of the spectrum in the correlation functions, the existence of these fields during inflation may not be ruled out.

So far, the quantum theory of cosmological fluctuations is considered up to the quadratic term in the action [1]. Recently, non-gaussian terms in the scalar field(s) have been calculated classically [7,8]. The quantum effect to arbitrary order in cosmological fluctuations has been more recently formulated by Weinberg[2]. With a sample massless minimally coupled scalar loop calculation, Weinberg’s result shows that the momentum dependence goes as $q^{-3} \ln q$, with an additional suppression of $GH^2$ when compared to the classical result. Is this true for other kinds of matter such as Dirac, vector, and conformal scalar fields as well? In fact, the unbroken symmetry matter becomes non-negligible when we go beyond the quadratic term in the action in cosmological perturbation theories. It is therefore of great interest to investigate how the higher-order corrections to the bilinear correlation function $\langle \zeta \zeta \rangle_{\text{loops}}$ depend on momentum $q$ when gravitational fluctuations

\footnote{Although we do not actually know what the inflaton is and is not, here we treat any field that has unbroken symmetries, i.e $\langle \chi \rangle = \langle \psi \rangle = \langle A_\mu \rangle = 0$ as other matter. A scalar field $\varphi$ that has a non-zero expectation value is considered as an inflaton, as in conventional belief. Therefore in the quantum theory of cosmological fluctuation during inflation considered here, the inflaton $\varphi$, gravity $g_{\mu\nu}$, and other matter fields $\chi, \psi, A_\mu$ are expanded as $\varphi = \bar{\varphi} + \delta \varphi, g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \chi = 0 + \delta \chi, \psi = 0 + \delta \psi, A_\mu = 0 + \delta A_\mu$, respectively.}

\footnote{The classical correlation function is $\langle \zeta \zeta \rangle = \frac{8\pi G H^2}{2(2\pi)^3 q^3 |q|}$.
interact with general matter other than scalar fields. Will the result go approximately as $q^{-3}$ as in the scalar case?

If other fields, other than inflaton, acquire a VEV, it is not necessary that the result will not affect the scale invariance of the curvature perturbation. For example, it is considered in the literature that Dirac fields cannot give a scale invariant primordial spectrum of density perturbations because their momentum dependence $\langle \zeta \zeta \rangle$ is far from $q^{-3}$ and vector fields can generate a scale invariant spectrum only in a special kind of mass. However, the Dirac and vector fields in [6] and [13] respectively were considered only classically. In general, Dirac and vector fields in an expanding universe only exist as quantum fields with zero expectation value $\langle \psi \rangle = \langle A_\mu \rangle = 0$ and what we observe is the density correlation function related to $\langle \zeta \zeta \rangle$, not the product of the fields $\langle \delta \bar{\psi} \delta \psi \rangle$, $\langle \delta A_i \delta A_j \rangle$. Therefore, we have to learn how to quantize such fields at higher order in cosmological fluctuations. In this paper I calculate the quantum effect to the observable $\langle \zeta \zeta \rangle$ due to other matter loops. We use the in-in formalism, appropriate for calculating expectation values, rather than the S-Matrix in time dependent backgrounds. I mainly follow the calculation of Weinberg for a loop of minimally coupled scalars in [2] and extend it to the loops of (massive) Dirac, vector, and conformal scalar fields. We investigate how the $\zeta$ correlation function depends on its momentum and whether it can be large.

It is also important to investigate the momentum dependence when quantum corrections are applied to two-point correlation functions. If the momentum dependence of the loop spectrum goes as $q^{-n}$ such that $n$ is far greater than 3, this will produce a larger spectrum than the classical value outside the horizon when $q \to 0$. The existence of Dirac and vector fields during inflation can be easily ruled out if the spectrum is far from the scale invariant; therefore, those fields cannot be the candidate for the origin of structure. We have shown that this is not the case. We always obtain nearly scale invariant spectrums even when Dirac, vector, and conformal scalar fields are included.

In section II we explain why quantum effects could be large. In section III we present the aspects of the in-in formalism and the un-equal time (anti)commutators of all fluctuations that are needed for our present purposes. Section IV and V introduce a class of theories with a single inflaton field, plus an additional Dirac field with gravitational interactions. A (massive) fermion loop correlation function of $\langle \zeta \zeta \rangle$ is also calculated in these
sections. In sections VI, V II, and VIII we follow the same approach as in section IV and V except we replace the Dirac field with a vector field or conformal scalar field respectively. In section IX we summarize all the results of this paper and explain why the departure from scale invariance is still small even in a more general potential $V(\varphi) \to V(\varphi, \bar{\psi}\psi, A_\mu A^\mu)$. All results show that for all theories and matter with a general potential $V(\varphi, \bar{\psi}\psi, A_\mu A^\mu)$, the quantum correlation functions are never much larger than the classical (observed) value and are nearly scale invariant. In appendix we derive gravitational and general matter fluctuations used in the loop calculations of section IV-VIII to cubic order for the general reader.

II. PROBLEMS

There are some simple arguments that lead us to believe that quantum effects might contribute to the spectrum in the order of the observed values without getting suppressed by an additional factor of $G$. This could happen when matter couples with the inflaton and gives a vertex in the order of $\bar{\varphi}$. The fact that $\frac{\dot{\varphi}}{M_{Pl}}$ is not small raises the question whether loop effect could be large.

To clarify the problem, we take an example of the interaction of a fermion $\psi$, inflaton $\varphi$, and graviton $g_{\mu\nu}$ via

$$\mathcal{L}_{int} = \sqrt{-g} \bar{\varphi} \bar{\psi}\psi$$

In cosmological fluctuation, we normally expand the fields as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \varphi = \bar{\varphi} + \delta \varphi, \psi = 0 + \delta \psi$$

In general, fermions interact with fluctuations of both gravity $\delta g_{\mu\nu}$ and scalar field $\delta \varphi$ and thus affect the conserved quantity $\zeta$. However, we can choose a gauge in which the inflaton does not fluctuate ($\delta \varphi = 0$ gauge [7]) so that $\zeta$ is purely gravity in this gauge. Therefore, one of the interactions in eq. (1), for instance the trilinear interaction, has the form

$$H_{\zeta\bar{\psi}\psi}(t) = \int d^3 x a^3(t) \bar{\varphi}(t) \bar{\psi}(x, t) \psi(x, t) F[\zeta(x, t)]$$

where $F[\zeta]$ is some function of $\zeta$, depending on the details after the expansion of the metric. Let us choose $F[\zeta] = \zeta$ for simplicity. Now we calculate the
quantum contribution of $\langle \zeta \zeta \rangle$:

$$
\langle \zeta(x,t)\zeta(x',t) \rangle = -\int_{-\infty}^{t_2} dt_1 \int_{-\infty}^{t_2} dt_2 \left[ H_1, \left[ H_2, \zeta(x,t)\zeta(x',t) \right] \right]_0
$$

(4)

where $H_1 = H_I(t_1)$, $H_2 = H_I(t_2)$, and $H_I$ is the interaction part of the Hamiltonian (the part that is of third or higher order in fluctuations) in the interaction picture. By solving Dirac field equation in an inflating universe, the fermion pair $\bar{\psi}\psi$ goes as $a^{-3}$ at late time. Therefore, the factor $a^{-3}$ cancels with $\sqrt{-g}$ for each Hamiltonian. With zero factors of $a(t)$, the result of two time integrals grows as $(\ln a)^2$. But $(\ln a)^2$ is not as significant as $\bar{\phi}$ in producing an appreciable effect in the interaction eq. (3). Since there are a total of four $\zeta$s on the RHS of eq. (4), we get the factor $|\zeta_o|^4 \approx \left( \frac{8\pi G H^2}{e^2 \bar{\phi}} \right)^2$.

The two time integrals become

$$
\int dt_1 \int dt_2 = \frac{1}{H^2} \int \frac{d\tau_2}{\tau_2} \int \frac{d\tau_1}{\tau_1} \approx C \frac{H^2}{H^2}
$$

(5)

where $\tau$ is the conformal time $\tau \equiv \int_{t}^{\infty} \frac{dt'}{a(t')}$. Since $\bar{\phi}$ does not change very much during inflation, it can be approximated as $\bar{\phi}(t_1) \simeq \bar{\phi}(t_2) \simeq \bar{\phi}(t_q)$ at the time of horizon exit. Thus, $\bar{\phi}$ does not enter into the time integral. By collecting the factors of $H, \bar{\phi}$ and $8\pi G$, we can then approximate the correlation function as

$$
\langle \zeta \zeta \rangle_{\text{loop}} \rightarrow \frac{(8\pi G H^2(t_q))^2 \bar{\phi}^2(t_q) C_q}{H^2 e^2(t_q)}
$$

(6)

where $\epsilon \equiv -\frac{\dot{H}}{H^2}$ is a slow roll parameter and $C_q$ is the momentum dependence that depends on the results of time and momentum integrals and the details of the propagator to the momentum $p$ and $p'$. It is important to note that unlike the term $H$, the term $\bar{\phi}$, which could arise via Yukawa coupling, does not give a spectrum that is suppressed by a factor of $GH^2$. Therefore, if an unperturbed inflaton amplitude is as large as $M_{Pl}$, we have the correlation function

$$
\langle \zeta \zeta \rangle_{\text{loop}} \rightarrow \frac{(8\pi G) H^2 C_q}{e^2(t_q)}
$$

(7)

which is in the order of the classical result.

\[4\] See the full formula in the next section or in ref[2]
\[5\] This is shown explicitly in fermion sections
The large vertices in the order of the \( M_{Pl} \) raise the possibility that quantum effects are *not* suppressed by a factor of \( G \) as was previously believed. Therefore the large vertices could contribute to the loop spectrum in the order of the classical value. However, such realistic \( \bar{\phi}(t) \sim M_{Pl} \) coupling can only happen in massive, but not massless, matter fields at one-loop level. The reason is that the inflaton fluctuates around a non-zero background that always contributes to the non-derivative matter terms in the second order after field expansions, i.e., \( m\bar{\psi}\psi = \bar{\phi}\psi\psi \) or \( |\bar{\phi}|^2 A_i^2 = m^2 A_i^2 \). The massive case is more general because it allows the possibilities of interactions with other broken symmetry fields such as the inflaton and hence gives large vertices in the order of \( M_{Pl} \). Although the argument above is valid, we still need to find out what \( C_q \) is through actual calculations because \( C_q \) is also a function of mass that arises through the massive propagators. We have to investigate whether this \( C_q(m) \) gives other kind of suppression.

III. CORRELATION FUNCTION FORMULA

Since the Hamiltonian that governs fluctuations of fields is explicitly time dependent, we need an in-in formalism\[4\]. For the purpose of calculating the in-in correlation function to arbitrary order, it is convenient to use Weinberg’s formula\[2\]

\[
\langle Q(t) \rangle = \sum_{N=0}^{\infty} i^N \int_{-\infty}^t dt_N \int_{-\infty}^{t_N} dt_{N-1} \ldots \int_{-\infty}^{t_2} dt_1 \times \left\langle \left[ H_I(t_1), \left[ H_I(t_2), \ldots \left[ H_I(t_N), Q_I(t) \right] \ldots \right] \right] \right\rangle_0 \tag{8}
\]

The expectation value on the RHS of the equation above is annihilated in the free field vacuum by annihilation operators and on the LHS it is in interacting vacuum. We will use the equation above to calculate loop correlation functions.

We see from Weinberg’s formula above that quantization requires unequal time (anti) commutators. The quantity \( \zeta(x,t) \), the scalar field \( \chi(x,t) \), the Dirac field \( \psi(x,t) \), and the vector field \( A_i(x,t) \) can be Fourier decom-
posed as

\[
\zeta(x, t) = \int d^3q \left[ e^{iq \cdot x} \alpha(q) \zeta_q(t) + e^{-iq \cdot x} \alpha^*(q) \zeta_q^*(t) \right] \\
\chi(x, t) = \int d^3q \left[ e^{iq \cdot x} \chi_q(t) + e^{-iq \cdot x} \chi_q^*(t) \right] \\
\psi(x, t) = \int d^3q \sum_s \left[ e^{iq \cdot x} \alpha(q, s) U_{q,s}(t) + e^{-iq \cdot x} \beta^\dagger(q, s) V_{q,s}(t) \right] \\
A_i(x, t) = \int d^3q \sum_\lambda \left[ e^{iq \cdot x} e_i(q, \lambda) \alpha(q, \lambda) A_q(t) + e^{-iq \cdot x} e^*_i(q, \lambda) \alpha^*(q, \lambda) A_q^*(t) \right]
\]

where \( s = \pm \frac{1}{2} \) stands for the spin of fermion, \( \lambda = 1, 2 \) is a helicity index of a photon, \( \lambda = 1, 2, 3 \) is a helicity index of a massive vector field. \( e_i(q, \lambda) \) is a polarization vector. The creation and annihilation operators satisfy usual (anti) commutation relations as

\[
[a(q), a^*(q')] = \delta^3(q - q')
\]

\[
[a(q), a^*(q') \alpha(q), \alpha^*(q')] = \delta_{\lambda, \lambda'} \delta^3(q - q')
\]

\[
\{\alpha(q, s), \alpha(q', s')\} = \{\beta(q, s), \beta(q', s')\} = \delta_{ss'} \delta^3(q - q')
\]

\[
\{\alpha(q, s), \alpha(q', s')\} = \{\beta(q, s), \beta(q', s')\} = 0
\]

and \( \zeta_q(t), \chi_q(t), U_{q,s}(t), V_{q,s}(t) \) and \( A_q(t) \) satisfy interaction picture free field equations in an inflating universe. With the (anti) commutation relations above, the unequal time (anti) commutators of all matter are

\[
\left[ \zeta(x_1, t_1), \zeta(x_2, t_2) \right] = 2i \int d^3p e^{ip \cdot (x_1 - x_2)} Im \left( \zeta_p(t_1) \zeta^*_p(t_2) \right)
\]

\[
\left[ \chi(x_1, t_1), \chi(x_2, t_2) \right] = 2i \int d^3p e^{ip \cdot (x_1 - x_2)} Im \left( \chi_p(t_1) \chi^*_p(t_2) \right)
\]

\[
\{\psi(x_1, t_1), \psi(x_2, t_2)\} = \int d^3pe^{ip \cdot (x_1 - x_2)} \sum_s \left( U_{p,s}(t_1) \tilde{U}_{p,s}^+(t_2) + V_{-p,s}(t_1) \tilde{V}_{-p,s}^+(t_2) \right)
\]
and

$$\left[ A_i(x_1, t_1), A_j(x_2, t_2) \right] = 2i \int d^3p e^{i\mathbf{p} \cdot (x_1 - x_2)} \Pi_{ij} \Im \left( A_p(t_1) A^*_p(t_2) \right)$$  \hspace{1cm} (21)$$

where $\Pi_{ij}$ is the polarization factor in which

$$\Pi_{ij}(\hat{p}) = \sum_{\lambda=1}^{2} \hat{e}^*_i(\hat{p}, \lambda) \hat{e}_j(\hat{p}, \lambda) = \delta_{ij} - \frac{\hat{p}_i \hat{p}_j}{|\hat{p}|^2}$$  \hspace{1cm} (22)$$
is time independent for $m = 0$.

In this section, we calculate a loop power spectrum of general matter valid for scalar, fermion and gauge fields. To see this in more detail, let us consider a general complex bosonic or fermionic field $\Psi$ with the interaction

$$H_I(t) = \int d^3x V(t) \Psi^*(x, t) \Psi(x, t) \zeta(x, t)$$  \hspace{1cm} (23)$$

where we omit the spinor and vector indices in this section, and $V(t)$ is any time dependent vertex function as a consequence of an expanding universe. The loop correction to the correlation function when $N = 2$ is

$$\langle Q(t) \rangle = -\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \left\langle \left[ H_1, \left[ H_2, Q \right] \right] \right\rangle_0$$  \hspace{1cm} (24)$$

If $Q$ is the product of the gravitational field $\zeta$ treated as the external legs, the vacuum expectation values of matter, either bosons or fermions, that circulate inside the loop on the RHS of eq. (24) can be evaluated independently from $Q$ because they have different types of creation and annihilation operators. We can write a real $\zeta$ field and a complex $\Psi$ field in the interaction picture in terms of creation and annihilation operators as

$$\zeta(x, t) = \int d^3p \left( e^{i\mathbf{p} \cdot \mathbf{x}} \alpha_{p} \zeta_p(t) + e^{-i\mathbf{p} \cdot \mathbf{x}} \alpha^*_p \zeta^*_p(t) \right)$$  \hspace{1cm} (25)$$

$$\Psi(x, t) = \int d^3p \sum_{\lambda} \left( e^{i\mathbf{p} \cdot \mathbf{x}} \alpha_{p, \lambda} X_{p, \lambda}(t) + e^{-i\mathbf{p} \cdot \mathbf{x}} \beta^\dagger_{p, \lambda} W_{p, \lambda}(t) \right)$$  \hspace{1cm} (26)$$

$^6$We can easily generalize to realistic interactions (such as the terms containing field derivative) later in the next section, once we obtain a general one-loop formula valid for any matter in this section.
\[ \Psi^*(x,t) = \int d^3p \sum_\lambda \left( e^{-iP \cdot x} \alpha_{p,\lambda}^\dagger X_{p,\lambda}^* (t) + e^{iP \cdot x} \beta_{p,\lambda} W_{p,\lambda}^* (t) \right) \quad (27) \]

where \( \alpha_p \) satisfies the commutation relations since \( \zeta \) is a boson but \( \alpha_{p,\lambda} \) and \( \beta_{p,\lambda} \) satisfy the (anti) commutation relations for (fermionic) bosonic matter loops. \( \lambda \) is either the spin index for the fermion or the helicity for the gauge field. \( X_{p,\lambda}(t) \) and \( W_{p,\lambda}(t) \) satisfy the matter free field equation in an inflating universe.

Let us now evaluate the RHS of eq. (24)

\[ [H_2, Q] = \int d^3x_2 V_2 [\Psi_2^* \Psi_2, Q] = \int d^3x_2 V_2 [\Psi_2^* \Psi_2 (\zeta Q - Q \zeta)] \quad (28) \]

\[ \langle [H_1, [H_2, Q]] \rangle_0 = \int \int d^3x_1 d^3x_2 V_1 V_2 \langle \left[ [\Psi_1^* \Psi_1, \Psi_2^* \Psi_2 Q], [\Psi_1^* \Psi_1, Q \Psi_2^* \Psi_2] \right] \rangle_0 - \int \int d^3x_1 d^3x_2 V_1 V_2 \langle (\Psi_1^* \Psi_1 \Psi_2^* \Psi_2 (\zeta_1 Q_2 - Q_1 \zeta_2)) \rangle_0 + \int \int d^3x_1 d^3x_2 V_1 V_2 \langle (\Psi_1^* \Psi_1 (Q \zeta_2 \zeta_1 - \zeta_2 Q \zeta_1)) \rangle_0 \quad (29) \]

Since \( \Psi \) is independent of \( \zeta \) and \( Q \), the vacuum expectation value can be evaluated independently.

For the \( \zeta \) part,

\[ \langle \zeta_1 \zeta_2 Q \rangle_0 = 2 \int d^3k d^3k' e^{i(k \cdot (x_1 - x) + k' \cdot (x_2 - x'))} \zeta_k (t_1) \zeta_{k'}^* (t_2) \zeta_{k^*} (t_1) \zeta_{k'^*}^* (t_2) \quad (30) \]

Hence,

\[ \langle \zeta_1 \zeta_2 Q \rangle_0 = \langle Q \zeta_2 \zeta_1 \rangle_0^* \quad (31) \]

Similarly,

\[ \langle \zeta_1 Q \zeta_2 \rangle_0 = 2 \int d^3k d^3k' e^{i(k \cdot (x_1 - x) + k' \cdot (x_2 - x'))} \zeta_k (t_1) \zeta_{-k'} (t_2) \zeta_{k^*}^* (t_1) \zeta_{-k'^*}^* (t_2) \quad (32) \]

Hence,

\[ \langle \zeta_1 Q \zeta_2 \rangle_0 = \langle \zeta_2 Q \zeta_1 \rangle_0^* \quad (33) \]
For the matter field, only $\Psi_1$ could pair with $\Psi_2$ at different times. To see this in more detail, we write the field operator $\Psi$ in terms of creation and annihilation operators similar to that in eq. (26). Hence, these products of the fields can be written in momentum space as

$$\langle \Psi_1^* \Psi_1 \Psi_2^* \Psi_2 \rangle_0 = \int d^3 p d^3 p' e^{i(p+p'-(x_1-x_2))} \times \sum_{\lambda,\lambda'} X_{p,\lambda}(t_1) X_{p,\lambda}^*(t_2) W_{p',\lambda'}(t_2) W_{p',\lambda'}^*(t_1)$$

(34)

Since

$$\langle \Psi_1^* \Psi_1 \Psi_2^* \Psi_2 \rangle_0 = \langle \Psi_2^* \Psi_2 \Psi_1^* \Psi_1 \rangle_0^*$$

(35)

Hence, the correlation function is

$$\int d^3 x e^{i q \cdot (x-x')} \langle Q(t) \rangle = - \int_{-\infty}^{t_2} dt_2 \int_{-\infty}^{t_1} dt_1 \int d^3 x \int d^3 x_2 \int d^3 x_1$$

$$\times 2 Re \left( \langle \Psi_1^* \Psi_1 \Psi_2^* \Psi_2 \rangle_0 (\langle \zeta_2^* Q \rangle_0 - \langle \zeta_1^* Q \zeta_2 \rangle_0) \right)$$

(36)

Note that many integrals over the momenta $k$ and $k'$ can be eliminated via the space integrations that produce delta functions, i.e.,

$$\int d^3 x \rightarrow (2\pi)^3 \delta^3(q - k)$$

(37)

$$\int d^3 x_1 \rightarrow (2\pi)^3 \delta^3(q + p + p')$$

(38)

$$\int d^3 x_2 \rightarrow (2\pi)^3 \delta^3(k' - p - p')$$

(39)

We therefore have the formula

$$\int d^3 x e^{i q \cdot (x-x')} \langle \zeta(x,t) \zeta(x',t) \rangle_{\text{loop}} = -4(2\pi)^9 \int d^3 p d^3 p' \delta^3(q + p + p')$$

$$\times \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_1 V_2 V_1 Re \left( \mathcal{Z} \mathcal{M} \right)$$

(40)

where

$$\mathcal{Z} = \zeta_q(t_1) \zeta_q^*(t) \left( \zeta_q(t_2) \zeta_q^*(t) - \zeta_q(t) \zeta_q^*(t_2) \right)$$

(41)

and

$$\mathcal{M} = \sum_{\lambda,\lambda'} X_{p,\lambda}(t_1) X_{p,\lambda}^*(t_2) W_{p',\lambda'}(t_2) W_{p',\lambda'}^*(t_1)$$

(42)
Eq. (42) is a formula for a general matter field loop.

For a real scalar field $\chi(x, t) = \chi^*(x, t)$, we have $X_p = \chi_p$ and $W_p = \chi_p^*$. Hence, eq. (42) becomes

$$M_{\chi} = 2\chi_p(t_1)\chi_p^*(t_2)\chi_{p'}(t_2)\chi_{p'}(t_1) \quad (43)$$

Note that we have an additional factor of 2 for any real field.

For the charged scalar field $\chi(x, t) \neq \chi^*(x, t)$, we still have $X_p = \chi_p$ and $W_p = \chi_p^*$. Hence, eq. (42) becomes

$$M_{\chi} = \chi_p(t_1)\chi_p^*(t_2)\chi_{p'}(t_2)\chi_{p'}(t_1) \quad (44)$$

which only differs from the real scalar field by a factor of 2.

For a fermionic field which is always complex, we have $X_{p, \lambda} = U_{p, s}$ and $W_{p, \lambda} = V_{p, s}$. Hence, eq. (42) becomes

$$M_{\psi} = \sum_{r, s=1, 2} U_{p, r}(t_1)U_{p, r}^*(t_2)V_{p', r, s}(t_2)V_{p', r, s}^*(t_1) \quad (45)$$

For a real vector field $A_i(x, t) = A_i^*(x, t)$, we have $X_{p, \lambda} = A_{p, \lambda}$ and $W_{p, \lambda} = A_{p, \lambda}^*$. Hence, eq. (42) becomes

$$M_{A} = 2\sum_{\lambda, \lambda'} A_{p, \lambda}(t_1)A_{p, \lambda}^*(t_2)A_{p', \lambda'}(t_2)A_{p', \lambda'}^*(t_1) \quad (46)$$

Note that the fermion and gauge fields require the summation over spin and helicity at different times, especially in the massive theories. In the subsequent sections we will use the formula for the general matter field loop shown above to calculate the power spectrums for more realistic interactions between various kinds of matter and gravity.

**IV. FERMION LOOP, INFLATON, AND GRAVITY**

In the known theory of cosmological fluctuation, only an inflaton and gravity are considered. Here we consider additional fermion as

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_\varphi + \mathcal{L}_f$$

$$= -\frac{1}{2} \sqrt{-g} \left[ \frac{1}{8\pi G} R + g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 2V(\varphi) + \bar{\psi} \gamma^\alpha (D_\alpha \psi) - (D_\alpha \bar{\psi}) \gamma^\alpha \psi + 2m\bar{\psi}\psi \right] \quad (47)$$
where $\mu, \nu, \ldots$ are the space time indices, and $\alpha, \beta, \ldots$ are the Lorentz indices raised and lowered by the vierbein $V^\alpha_\mu$. To deal with fermion, we need the tetrad formalism[10]. The metric in any general non-inertial coordinate system is related to the vierbein by

$$ g_{\mu\nu}(x) = V^\alpha_\mu(x)V_\nu^\beta(x)\eta_{\alpha\beta} \quad (48) $$

The covariant derivative to fermionic field due to the gravity interaction is

$$ D_\alpha \equiv V^\mu_\alpha \partial_\mu + \frac{1}{2} \sigma^{\beta\gamma} V^\nu_\beta V^\mu_\gamma V_{\nu+\mu} \quad (49) $$

where $\sigma^{\beta\gamma} \equiv \frac{1}{4}[\gamma^\beta, \gamma^\gamma]$. To find out what the time dependent propagators are, it is necessary to solve interaction free field equations in an inflating universe. For a fermion, it is

$$ a^{-\frac{3}{2}} \frac{d}{dt} \left( a^{\frac{3}{2}} \gamma^0 \psi \right) + \frac{\gamma^i}{a} \partial_i \psi + m\psi = 0 \quad (50) $$

We can re-scale the field $\psi \equiv a^{-\frac{3}{2}} S$ and work with the conformal time. We therefore have a simpler Dirac equation as

$$ \gamma^0 S' + \gamma^i \partial_i S + maS = 0 \quad (51) $$

where $S'$ denotes the conformal time derivative. Since the background is spatially translation invariant, the solution can be written in mode function as

$$ \psi(x,t) \equiv a^{-\frac{3}{2}}(t)S(x,t) = \frac{1}{a^{\frac{3}{2}}(t)} \int d^3 q \sum_s \left[ e^{i q \cdot x} \alpha(q,s) u_{q,s}(t) + e^{-i q \cdot x} \beta^\dagger(q,s) v_{q,s}(t) \right] \quad (52) $$

where $u_{q,s}(t)$ and $v_{q,s}(t)$ satisfy

$$ \gamma^0 u'_{q,s} + i\gamma^i q_i u_{q,s} + ima u_{q,s} = 0 \quad (53) $$

and

$$ \gamma^0 v'_{q,s} - i\gamma^i q_i v_{q,s} + ima v_{q,s} = 0 \quad (54) $$

For $\zeta_q(t)$, it satisfies Mukhanov’s equation [1] as

$$ \ddot{\zeta}_q + \frac{d}{dt} \left( \ln a^3(t) \right) \dot{\zeta}_q + \frac{q^2}{a^2} \zeta_q = 0 \quad (55) $$

We work with interaction picture free field equations here to obtain time dependent propagators via their solutions. When loop effect is included, Mukhanov’s equation is modified by varying the loop quantum effective action with respect to $\zeta$. 

11
We can expand matter and gravity fluctuations in the actions of (47) to arbitrary order in cosmological fluctuation. It is also convenient to write down the gravitational and all matters actions in ADM form [5] and solve for $N$ and $N_i$ in the constraint equations. The cubic term and higher order terms are time dependent vertices that are needed to calculate loop diagrams. The direct expansion of matter and gravitational fluctuations to higher order are complicated. However, as shown in the appendix, many terms are not necessary since they are cancelled via field equations and are removed by the field redefinition of $\zeta$. Therefore, the important terms of the trilinear interaction of any general matter are

$$H_{\zeta_{M, M}}(t) = - \int d^3 x \epsilon H a^5 (T^{00} + a^2 T^{ii}) \nabla^{-2} \zeta$$ (56)

The fermion energy momentum tensor in the presence of gravity is

$$T_{f}^{\mu \nu} = - \frac{1}{2} \left( \bar{\psi} \gamma^{\mu} \gamma^{\nu} \psi + \bar{\psi} \gamma^{\mu} \gamma^{\nu} \psi - \bar{\psi} \gamma^{\mu} \gamma^{\nu} \psi + \bar{\psi} \gamma^{\mu} \gamma^{\nu} \psi \right)$$ (57)

Therefore, we have the time component of the energy momentum tensor to quadratic order as

$$T_{f}^{00} = \bar{\psi} \gamma^{0} \psi - \bar{\psi} \gamma^{0} \psi$$

$$= (-i) \left( \bar{\psi}^\dagger \psi - \psi^\dagger \bar{\psi} \right)$$ (58)

where $\bar{\psi} \equiv \bar{\psi}^\beta \beta \equiv i \gamma^0, (\gamma^0)^2 = -1$, and $\partial^0 = -\partial_0 = -\frac{\partial}{\partial t}$. Similarly, the spatial component of the energy momentum tensor is

$$a^2 T_{f}^{ii} = \frac{1}{a} \left( \bar{\psi} \gamma^{i} (\partial_i \psi) - (\partial_i \bar{\psi}) \gamma^{i} \psi \right)$$ (59)

where $\partial^i = \frac{\partial}{\partial x}$ and $\gamma^\mu = V_\alpha^\mu \gamma^\alpha$. Therefore, $\gamma^{0} = V_0^0 \gamma^{0} = \gamma^{0}$ and $\gamma^{i} = V_i^m \gamma^m = \frac{\delta_i^m \gamma^m}{a}$. These give us the Hamiltonian interaction of fermion and gravity to the cubic order as

$$H_{\zeta \bar{\psi} \psi}(t) = - \int d^3 x \epsilon H a^5 \left[ \bar{\psi} \gamma^{0} \psi - \bar{\psi} \gamma^{0} \psi \right]$$

$$+ \frac{1}{a} \left( \bar{\psi} \gamma^{i} (\partial_i \psi) - (\partial_i \bar{\psi}) \gamma^{i} \psi \right) \nabla^{-2} \zeta$$ (60)

Note that the interaction above is real and also valid for massive fermions. The cubic order in the interaction above can be further simplified with

$$\bar{\psi} \gamma^{0} \psi - \bar{\psi} \gamma^{0} \psi + \frac{\partial_i \bar{\psi} \gamma^{i} \psi - \bar{\psi} \gamma^{i} \partial_i \psi}{a} - 2m \bar{\psi} \psi = 0$$ (61)
Hence, the trilinear interaction Hamiltonian is

$$H_{\zeta \bar{\psi} \psi}(t) = 2 \int d^3 x \epsilon H a^4 \left( \bar{\psi} \gamma^i (\partial_i \psi) - (\partial_i \bar{\psi}) \gamma^i \psi + m \bar{\psi} \psi \right) \nabla^{-2} \zeta$$

$$= -2 \int d^3 x \epsilon H a^5 \left( \bar{\psi} \gamma^0 \dot{\psi} - \dot{\bar{\psi}} \gamma^0 \psi + m \bar{\psi} \psi \right) \nabla^{-2} \zeta$$

(62)

We see that Dirac’s equations in an expanding universe (53) and (54) at $m = 0$ are the same as those in Minkowski space except that the physical time $t$ is replaced with the conformal time $\tau$. So we can expect the plane wave solutions for $u_{q,s}(t)$ and $v_{q,s}(t)$ to be

$$u_{q,s}(t) = u_{q,s}^0 e^{-i q \tau}, \quad v_{q,s}(t) = v_{q,s}^0 e^{i q \tau}$$

(63)

where $u_{q,s}^0$ and $v_{q,s}^0$ stand for constant coefficients outside the horizon. These coefficients can be determined by matching the solutions at deep inside the horizon with the flat space solutions. Deep inside the horizon, the field does not feel the effect of the expansion of the universe. Therefore, the normalization factor can be chosen in the same way as in Minkowski space.

$$\sum_s u_{q,s}^0 \bar{u}_{q,s}^0 = \sum_s v_{q,s}^0 \bar{v}_{q,s}^0 = -i q^\mu q_\mu / 2 (2\pi)^3 q$$

(64)

where $q \equiv q^0 = \sqrt{q^2}$. We see that the momentum dependence $q$ of the expectation value of the fermion and anti-fermion pair $\langle \bar{\psi} \psi \rangle$ is far from the scale invariant spectrum $q^{-3}$. However, this does not rule out that fermions could not seed the large scale structure of the universe observed today. The reason is that we never observe the product of either scalar or fermionic fields but rather the product of temperature $\langle \delta T / T \rangle$ or density $\langle \delta \rho / \rho \rangle$ fluctuations, which are related to the conserved quantities $\langle \zeta \zeta \rangle$. Since fermions interact with the gravitational fluctuation, we therefore calculate how fermions affect $\zeta$ at the loop quantum level. We now calculate the one-loop graph with two vertices of the two point function. Owing to the interaction of gravity and fermionic fields, the quantum corrections to the $\zeta$ spectrum are

$$\left\langle \zeta(x, t) \zeta(x', t) \right\rangle_{\text{loop}} = - \int_t^t dt_2 \int_{-\infty}^{t_2} dt_1 \left\langle [H_1, [H_2, \zeta(x, t) \zeta(x', t)]] \right\rangle_0$$

(65)

For simplicity, we first start with massless fermions. We can use the formula (40) and (45) derived in the previous section. To match with the interaction

---

We emphasize that this formula is only valid for massless fermion. The situation is entirely different for massive fermion, as will be shown in the next section.
Hamiltonian in eq. (62), we replace the interaction in eq. (23) with

\[ \Psi^* \Psi \rightarrow \left( \bar{\psi} \gamma^i \frac{\partial_i}{a} \psi - \left( \frac{\partial_i}{a} \bar{\psi} \right) \gamma^i \psi \right) \]  

(66)

\[ \zeta_q(t_{1,2}) \rightarrow -\dot{\zeta}_q(t_{1,2})/q^2 \]  

(67)

\[ V(t) = 2\epsilon H a^5(t) \]  

(68)

Hence, \( Z \) in eq. (41) changes to

\[ Z \rightarrow \frac{1}{q^2} \left( \dot{\zeta}_q(t_1)\zeta_q^*(t_1)(\dot{\zeta}_q(t_2)\zeta_q^*(t_2) - \zeta_q(t)\dot{\zeta}_q^*(t_2)) \right) \]  

(69)

and \( M_{\psi} \) in eq. (15) changes to

\[ M_{\psi} \rightarrow -\frac{1}{a_1 a_2} (p_i - p'_i)(p_j - p'_j) \sum_{r,s} \gamma^i U_{p,s}(t_1) \bar{U}_{p,s}(t_2) \gamma^j V_{p',r}(t_2) \bar{V}_{p',r}(t_1) \]  

(70)

The equation above shows the need to sum over spins at different times. Fortunately, massless fermions are conformally flat so we can still use the spin sum formula from flat space. As seen from Dirac’s equation, the solutions of massless fermions are just plane waves with conformal time, \( u_{p,s}^o e^{-i\tau} \), after re-scaling the field such that \( U_{p,s} = a^{-3/2} u_{p,s}(t) \). With the spin sum equation (61), eq. (70) becomes

\[ M_{\psi} = (p_i - p'_i)(p_j - p'_j) \frac{p_0 p'_\beta}{4(2\pi)^6 p p' a_1^4 a_2^4} \text{tr}(\gamma^i \gamma^\alpha \gamma^j \gamma^\beta) e^{-i(p+p')(\tau_1 - \tau_2)} \]  

(71)

Since

\[ \text{tr}(\gamma^i \gamma^\alpha \gamma^j \gamma^\beta) = 4(\eta^i \eta^\beta - \eta^j \eta^\alpha + \eta^i \eta^\alpha) \]  

(72)

we have,

\[ M_{\psi} = \frac{1}{(2\pi)^6 a_1^4 a_2^4} (p - p')^2 (1 + \hat{p} \cdot \hat{p}') e^{-i(p+p')(\tau_1 - \tau_2)} \]  

(73)

Substituting \( M_{\psi} \) into eq. (40) with \( V(t) = 2\epsilon H a^5(t) \), we have

\[ \int d^3x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \left( \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t') \right)_{\text{loop}} = -16(2\pi)^3 \int d^3p d^3p' \delta^3(\mathbf{q} + \mathbf{p} + \mathbf{p}') (p - p')^2 (1 + \hat{p} \cdot \hat{p}') \int_{-\infty}^{t_2} dt_2 \epsilon_2 H_2 a_2 \int_{-\infty}^{t_2} dt_1 \epsilon_1 H_1 a_1 Re \left( Z e^{-i(p+p')(\tau_1 - \tau_2)} \right) \]  

(74)

where \( Z \) is the contribution of the \( \zeta \) part in eq. (69). To calculate \( Z \), we use the solution of the free field Mukhanov’s equation in the interaction picture

\[ \zeta_q(t) = \zeta_q^o e^{-iq\tau}(1 + iq\tau) \]  

(75)
\[ |\zeta_q|^2 = \frac{8\pi G H^2(t_q)}{2(2\pi)^3 \epsilon(t_q) q^3} \]  

(76)

Hence,

\[ Z = \frac{|\zeta_q|^4}{H_1 H_2 a_1^2 a_2^2} \left( e^{i q \tau - i q (\tau_1 + \tau_2)} - e^{-i q (\tau_2 - \tau_1)} \right) \]  

(77)

During slow roll inflation, we approximate \( \epsilon_1 \approx \epsilon_2 \approx \epsilon(t_q) \). Integrating over conformal time \( \tau_1 \), we get

\[
\int d^3 x e^{iq(x-x')}\left< \zeta(x,t)\zeta(x',t) \right>_{\text{loop}} = -16(2\pi)^3 |\zeta_q|^4 \epsilon^2(t_q) \int d^3 p d^3 p' \\
\times \delta^3(q+p+p')(p-p')^2(1+\hat{p} \cdot \hat{p}') \text{Re} \int_{-\infty}^{0} d\tau_2 \frac{i}{q+p+p'}(e^{-2i q \tau_2} - 1) \]  

(78)

where an upper limit \( t \to \infty \) or \( \tau \to 0 \) means a time still during inflation but sufficiently late so that \( a(t) \) is many e-foldings larger than its value when \( \frac{q}{a} \) falls below \( H \). Integrating over conformal time \( \tau_2 \) gives

\[
\text{Re} \int_{-\infty}^{0} d\tau_2 \frac{i}{q+p+p'}(e^{-2i q \tau_2} - 1) = -\frac{1}{2q(q+p+p')} 
\]  

(79)

Substituting eqs. (76) and (79) into eq. (78), we have

\[
\int d^3 x e^{iq(x-x')}\left< \zeta(x,t)\zeta(x',t) \right>_{\text{loop}} = \frac{2(8\pi G H^2(t_q))^2}{(2\pi)^3 q^7} \\
\times \int d^3 p d^3 p' \delta^3(q+p+p')(p-p')^2 \frac{1}{q+p+p'}(1+\hat{p} \cdot \hat{p}') \]  

(80)

Power counting shows that the results of the momentum integrals \( p, p' \) will go as \( q^4 \). If we use dimensional regulariization to remove UV divergences, the finite part of \( \langle \zeta^2 \rangle \) for the massless fermion loop will go as \( q^{-3} \ln q \). To determine the coefficient of the finite part of the momentum integral above, we will follow the calculation as done in [2] for the scalar case. Note that, in general,

\[
\int d^3 p \int d^3 p' \delta^3(p+p'+q)f(p,p',q) = \int d^3 p f(p,p') = |p'| = |p+q|,q \]  

(81)

and

\[
\int d^3 p = \int_{0}^{\infty} p^2 dp \int_{-1}^{1} d(cos \theta) \int_{0}^{2\pi} d\varphi \]  

(82)
The conservation of momentum $p' = |p + q|$ gives

$$p'^2 = p^2 + q^2 + 2pq \cos \theta_{pq}$$  \hspace{1cm} (83)$$

where $\theta_{pq}$ is an angle between the vectors $p$ and $q$. Since $q$ is a fixed external momentum, we can choose in the $z-$ direction. Hence, $\theta_{pq} = \theta$ and

$$p'dp' = pqd(\cos \theta)$$  \hspace{1cm} (84)$$

With eqs. (81), (82), and (84), we have

$$\int d^3p \int d^3p' \delta^3(p + p' + q)f(p, p', q) = 2\pi \int_0^\infty dp \int_{|p-q|}^{|p+q|} p'dp' f(p, p', q)$$  \hspace{1cm} (85)$$

Since $2p \cdot p' = q^2 - p^2 - p'^2$, eq. (80) can be written as

$$\int d^3x e^{i q \cdot (x - x')} \langle \zeta(x, t) \zeta(x', t) \rangle_{\text{loop}} = \frac{2(8\pi GH^2(t_q))^2}{(2\pi)^3 q^7} \left[ \frac{2\pi}{q} \mathcal{J}(q) \right]$$  \hspace{1cm} (86)$$

where

$$\mathcal{J}(q) \equiv \int_0^\infty dp \int_{|p-q|}^{|p+q|} p'dp' \left( \frac{p-p'}{q+p+p'} \right) \left( 1 + \frac{q^2 - p^2 - p'^2}{2pp'} \right)$$  \hspace{1cm} (87)$$

With dimensional regularization, the UV divergence of the integral above for $\delta = 0$ gives the pole term as

$$\frac{2\pi}{q} \mathcal{J}(q) \Rightarrow q^{4+\delta} F(\delta) = q^4 e^{\delta \ln q} F(\delta)$$  \hspace{1cm} (88)$$

where

$$F(\delta) \to \frac{F_0}{\delta} + F_1$$  \hspace{1cm} (89)$$

Therefore, in the limit $\delta \to 0$,

$$\frac{2\pi}{q} \mathcal{J}(q) = q^4 \left( \frac{F_0}{\delta} + F_0 \ln q + F_1 \right) = q^4 \left[ F_0 \ln q + L \right]$$  \hspace{1cm} (90)$$

where $L$ is a divergent constant. To calculate the coefficient $F_0$, it is necessary to evaluate the integral (87). The integral over $p'$ gives\textsuperscript{9}

$$\mathcal{J}(q) = \frac{1}{2} \int_0^\infty dp \left( p(15p^2 + 17pq + 4q^2)p' - \frac{pp'^2}{2}(11p + 6q) + \frac{p'^3}{3}(5p + q) - \frac{p'^4}{4} - 4p(p + q)(2p + q)^2 \log[p + q + p'] \right)_{p' = |p-q|}$$  \hspace{1cm} (91)$$

\textsuperscript{9}We use Mathematica for the integrals and six derivatives.
To eliminate the divergence in the momentum integral above, we differenti-
ate $J(q)$ in eq. (91) six times and then do the integration over $p$. This gives a finite result as

$$\frac{d(6)J(q)}{dq^6} = -\frac{8}{q}$$

(92)

where we take the limit $q \to 0$ after integrating over $p$. Hence,

$$J(q) = q^5 \left(-\frac{8}{5!} \ln q + L\right)$$

(93)

or

$$F_0 = -\frac{2\pi}{15}$$

(94)

Substituting $J(q)$ back into eq. (88), we have the finite part of correlation function as

$$\int d^3xe^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \left\langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t) \right\rangle_{\text{loop}} = -\frac{4\pi(8\pi GH^2(t_0))^2}{15(2\pi)^3 q^3} \left[ \ln q + C \right]$$

(95)

with $C$, an unknown constant. Notice that we have the same sign as that in the massless scalar loop because we do not have the time ordered product of fermion pairs in eq. (8). The opposite sign of fermion loops only arises in the in-out theory when we time order the product of fermion pairs in order to close the loop. Moreover, the result in eq. (95) is smaller than the classical result by a factor of $GH^2$.

V. MASSIVE FERMION

The calculation is much more difficult for massive fermions. This is because the mode solution of a massive fermion at arbitrary wavelength during inflation is not a simple plane wave like in the massless case or in flat space. We cannot rely on the trace technology normally used for spinors in flat space since the spin sum at different times cannot be written in the compact form of $\gamma$ matrices. We therefore need to solve the massive Dirac equation in an expanding universe during inflation and perform the spin sum at different times by multiplying the matrices.

We define $u_{s,\mathbf{p}}(\tau) \equiv [u_{+\mathbf{p}}(\tau)S, u_{-\mathbf{p}}(\tau)S]^T$ and $v_{s,\mathbf{p}}(\tau) \equiv [v_{+\mathbf{p}}(\tau)S, v_{-\mathbf{p}}(\tau)S]^T$ where $S$ are the two component eigenvectors of the helicity operators. We
use the Dirac representation of the gamma matrices.

\[ \gamma^0 = (-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  \hspace{1cm} (96)

\[ \gamma^i = (-i) \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \]  \hspace{1cm} (97)

Therefore, eq. (53) gives

\[ u'_{\pm,p} + i(\vec{\sigma} \cdot \vec{p})u_{\mp,p} \pm ima u_{\pm,p} = 0 \]  \hspace{1cm} (98)

We see that the equations above are first order coupled differential equations. To decouple them, we differentiate those equations one more time. With some algebra, we get two uncoupled second-order differential equations,

\[ u''_{\pm,p} + (\vec{p}^2 + (ma)^2 \pm i(am)')u_{\pm,p} = 0 \]  \hspace{1cm} (99)

where \((\vec{\sigma} \cdot \vec{p})^2 = (p_x^2 + p_y^2 + p_z^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) \(= \vec{p}^2 1_{2 \times 2} \equiv p^2\). Therefore, the equations above are solvable as

\[ u''_{\pm,p} + \left(\vec{p}^2 + \frac{1}{T^2}(r^2 \pm ir)\right)u_{\pm,p} = 0 \]  \hspace{1cm} (100)

The solutions are

\[ u_{FRW}(p, s, t) \equiv \begin{pmatrix} u_{\pm,p} \times S \\ u_{-\pm,p} \times S \end{pmatrix} = \begin{pmatrix} u_{\mu,p} \times S \\ (\vec{p} \cdot \vec{\sigma})u_{\pm,p} \times S \end{pmatrix} \]

\[ = \begin{pmatrix} e_{+p}\sqrt{-T}H_\mu^{(1)}(-p\tau) \times S \\ (\vec{p} \cdot \vec{\sigma})e_{-p}\sqrt{-T}H_\mu^{(1)}(-p\tau) \times S \end{pmatrix} \]  \hspace{1cm} (101)

where \(\mu \equiv \frac{1}{2} - ir\), \(\bar{\mu} \equiv \frac{1}{2} + ir\), and \(r \equiv \frac{m}{T}\).

We choose the initial conditions so that the positive frequency mode solutions match with the flat space-time solutions deep inside the horizon \[15\]

\[ u_{flat} = \left(\frac{E + m}{2(2\pi)^3 E}\right)^{\frac{1}{2}} \begin{pmatrix} 1 \times S \\ \frac{\vec{p} \cdot \vec{\sigma}/a}{E+m} \times S \end{pmatrix} e^{\int_0^\infty E(t')dt'} \]  \hspace{1cm} (102)

\[ ^{10}\text{Note that the asymptotic behavior of Hankel’s function for large } x \equiv -p\tau \text{ is } H^{(1)}_\nu(x) \to \sqrt{2\pi x} e^{ix - \nu\pi/2 - im/4}(1 + \frac{1}{2}\nu(\nu + \frac{1}{2})(\nu - \frac{1}{2}) + \ldots) \]
where \( E^2 \equiv m^2 + \vec{p}^2 \), \( S = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) for spin up and \( S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) for spin down. Note that \( S^\dagger S = 1 \). We can check that, in the massless limit, 
\[
    u_{\text{flat}} = \frac{1}{\sqrt{2(2\pi)^3}} e^{-ip\tau},
\]
as expected.

To find the normalization coefficients \( c_{\pm}(p) \), we match the solutions of eqs. (101) with (102) by using the asymptotic property of Hankel’s functions. Hence,
\[
    c_{\pm,p} = i \sqrt{\frac{\pi}{2}} e^{\pm \frac{\pi r}{2}} \tag{103}
\]

To calculate \( \mathcal{M}_\psi \), we use the formula (40) and (45) derived in the previous section. To take account of the more realistic interaction Hamiltonian that arises after the expansion in eq. (62), we replace the interaction in eq. (23) with
\[
    \Psi^* \Psi \rightarrow \left( \bar{\psi} \gamma^0 \dot{\psi} - \dot{\bar{\psi}} \gamma^0 \psi + m \bar{\psi} \psi \right) \tag{104}
\]
\[
    \dot{\zeta}_q(t_{1,2}) \rightarrow -\dot{\zeta}_q(t_{1,2})/q^2 \tag{105}
\]
\[
    V(t) = -2eH\alpha^5(t) \tag{106}
\]

\( \mathcal{Z} \) is still the same as that in eq. (69). However, \( \mathcal{M}_\psi \) for massive fermion in eq. (45) becomes
\[
    \mathcal{M}_\psi = \frac{2}{a_3^2 a_2^3} \left( \hat{p} \sigma_2^* + \hat{p}' \sigma_1^* \right) \cdot \left( \hat{p} \sigma_1 + \hat{p}' \sigma_1 \right)
\]
\[
= \frac{2}{a_3^2 a_2^3} \vec{\pi}_2^* \cdot \vec{\pi}_1 \tag{107}
\]

where the \( \sigma \)s are
\[
\sigma = u_{\mu,p}^* \dot{u}_{\mu,p} - \dot{u}_{\mu,p}^* u_{\mu,p} + imu_{\mu,p} u_{\mu,p} \tag{108}
\]
\[
\dot{\bar{\sigma}} = \sigma(m \rightarrow -m) \tag{109}
\]
and \( u_{\mu,\bar{\mu}} \) are the massive mode solutions of Dirac equations
\[
    u_{\mu}(x) = \frac{i\sqrt{\pi x}}{2(2\pi)^{3/2}} e^{\frac{\pi x}{2}} H_{\mu}^{(1)}(x) \tag{110}
\]
\[
    u_{\bar{\mu}}(x) = \frac{i\sqrt{\pi x}}{2(2\pi)^{3/2}} e^{-\frac{\pi x}{2}} H_{\bar{\mu}}^{(1)}(x) \tag{111}
\]
for \( x = -p\tau \). The factor \( e^{\pm \pi r^2} \) arises due to the fixing of the coefficients \( c_{\pm,p} \) of mode solutions deep inside the horizon with the flat space-time solutions. We can check that in the massless limit \( m = 0 \), \( u_\mu = u_\bar{\mu} = u_\eta = \frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}} e^{-ip\tau} \) and \( \sigma(t) = \bar{\sigma}(t) = \frac{i}{2(2\pi)^{\frac{3}{2}}} \), hence,

\[
\mathcal{M}|_{m=0} = \frac{4}{a_1^3 a_2^3} \sigma_2^*(1 + \hat{p} \cdot \hat{p}') \quad \text{and} \quad \mathcal{M}_\psi = \text{Re} \int_{-\infty}^{\infty} dt_1 a_1^3 e^{i\eta \tau_1} \mathcal{M}_\psi
\]

which agrees with the result in massless fermion section in eq. (73). Substituting eqs. (76) and (77) back into the power spectrum formula eq. (40) with \( V(t) = -2\epsilon H a^5(t) \), we have \( \langle \zeta \zeta \rangle \) due to massive fermion loop as

\[
\int d^3 x e^{i q \cdot (x - x')} \langle \zeta(x, t) \zeta(x', t) \rangle_{\text{loop}} = -\frac{4(2\pi)^3 (8\pi GH^2)^2}{q^6} \times \int d^3 p d^3 p' \delta^3(q + p + p') I
\]

where

\[
I = \text{Re} \int_{-\infty}^{\infty} dt_2 a_2^3 (e^{-i\eta \tau_2} - e^{i\eta \tau_2}) \int_{-\infty}^{t_2} dt_1 a_1^3 e^{-i\eta \tau_1} \mathcal{M}_\psi
\]

So far, the result in eq. (107) is exact. There is no approximation involved in eq. (107). However, the exact result involves the integrand of Hankel’s functions and their time derivatives and this makes the integration quite challenging. Nevertheless, we get some idea that the time integrals will converge and go at most as \((\log a)^2\). However, in order to determine the momentum dependence of the power spectrum, we need to integrate over unequal times \( t_1, t_2 \) and momentums \( p, p' \) associated with fermion fluctuations. We will first integrate over times and then over momentums. The direct calculation is complicated because it involves integrating over products of Hankel’s functions with complex order. Since \( p \) is the running momentum from 0 to \( \infty \) whereas \( q \) is the fixed external momentum associated with a conserved quantity \( \zeta_q \), it is helpful to divide the integrals over momentum \( p \) in eq. (113) as an integral when \( \Lambda q \leq p \leq \infty \) and \( 0 \leq p \leq \Lambda q \). The first integral when \( p \to \infty \) can be approximated as if the fermion is massless due
to its high momentum. So, the result at high momentum after dimensional regularization will be close to that in the massless case in eq. (95). The second integral, when \( p \leq \Lambda q \), indicates that the mass effect may become important in the result. Therefore, additional calculations are needed to determine the momentum dependence.

The momentum \( p \) corresponds to the fermion fluctuation \( \psi_p \) and the momentum \( q \) corresponds to the conserved quantity \( \zeta_q \) fluctuation, this implies that a massive fermion \( \psi_p \) exits the horizon before \( \zeta_q \) exits the horizon if \( \Lambda \) is in the order of 1. In other words, the fermion fluctuation \( a^3 \bar{\psi}_p \psi_p \) is frozen by the time the fluctuation \( \zeta_q \) crosses the horizon. Outside the horizon of \( \zeta_q \), \( p, p' \) are sufficiently small and Hankel’s function can be approximated as

\[
H_{\beta}^{(1)}(x) \approx -\frac{i\Gamma(\beta)}{\pi} \left( \frac{x}{2} \right)^{-\beta}
\]  
which is valid for \( x \ll 1 \) and \( \beta > 0 \). Hence, the mode solutions in eq. (110) become

\[
u^{\mu}_{\rho}(x) = \frac{\Gamma(\mu)}{(2\pi)^2} e^{\frac{x}{2}x} \nu^\mu_{\rho}(x)
\]

\[
u^\mu_{\rho}(x) = \frac{\Gamma(\mu)}{(2\pi)^2} e^{\frac{x}{2}x} \nu^\mu_{\rho}(x)
\]

With this approximation, their (conformal) time derivatives are

\[
u^{\mu}_{\rho} = i \frac{r}{\tau} \nu^\mu_{\rho}, \nu^\mu_{\rho} = -i \frac{r}{\tau} \nu^\mu_{\rho}
\]

Hence,

\[
\sigma^0 = -i m \nu^0_{\rho,\rho,\rho} \nu^0_{\mu,\rho}
\]

\[
u^0_{\mu,\rho,\rho} \nu^0_{\mu,\rho} = \frac{|\Gamma(\mu)|^2}{(2\pi)^4} p^i p^j p^{i-j}
\]

\[
\sigma_2^0 \sigma_1^0 = \sigma_2^0 \sigma_1^0 = \frac{m^2 |\Gamma(\mu)|^4}{(2\pi)^8} p^2 p^2 p^2
\]

\[
\sigma_2^0 \sigma_1^0 = (\sigma_2^0 \sigma_1^0)^* = \frac{m^2 |\Gamma(\mu)|^4}{(2\pi)^8} p^2 p^2 + p^2 i
\]

Notice that \( \sigma(p, p') \) and \( \bar{\sigma}(p, p') \) are both time independent. The exponential factors \( e^{\pm i\pi r} \) arise when we fix the coefficients to match the solutions with
those inside the horizon. Hence, from eq. (107)

$$\mathcal{M}^\circ_{\psi} = \frac{2m^2|\Gamma(\mu)|^4}{(2\pi)^8a_1^2a_2^2} \left( 2 - (\hat{p} \cdot \hat{p}')(p'^2r p^2r + c.c.) \right)$$

$$= a_1^{-3}a_2^{-3} F^o_{p,p'}$$ (123)

where the factors $a_1^{-3}$, $a_2^{-3}$ from the fermionic part will cancel with the factor $\sqrt{-g}$ in eq. (114). Therefore,

$$I^o = F^o_{p,p'} \text{Re} \int_{-\infty}^{\infty} dt_2 (e^{-i\tau_2} - e^{i\tau_2}) \int_{-\infty}^{t_2} dt_1 e^{-i\tau_1}$$

$$= \frac{F^o_{p,p'}}{H^2(t_q)} \text{Re} \int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} (e^{-i\tau_2} - e^{i\tau_2}) \int_{-\infty}^{\tau_2} \frac{d\tau_1}{\tau_1} e^{-i\tau_1}$$ (124)

Although we see from eq. (124) that the time integral is of the order of $(\log a)^2$, we need to evaluate this integral if we want to see the momentum dependence $q$ for the correlation function of $\zeta$. From eq. (124), we have

$$I^o = \frac{F^o_{p,p'}}{H^2(t_q)} \text{Re} \int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} (e^{-i\tau_2} - e^{i\tau_2}) Ei(-i\tau_2)$$ (125)

Using $Ei(-ix) = ci(x) - isi(x)$, we have

$$I^o = -\frac{2F^o_{p,p'}}{H^2(t_q)} \int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} \sin(q\tau_2)\sin(q\tau_2)$$ (126)

With Mathematica,

$$\int_{-\infty}^{0} \frac{d\tau_2}{\tau_2} \sin(q\tau_2)\sin(q\tau_2) = -\frac{\pi^2}{8}$$ (127)

Therefore,

$$I^o = \frac{\pi^2 F^o_{p,p'}}{AH^2}$$ (128)

Notice that the result of the time integral above is $q$-independent. Hence,

$$\int_{|p-q|}^{p+q} p' dp' I^o_{p,p'} = G^o(p + q) - G^o(|p - q|) = 2q \frac{\partial G^o}{\partial p} + O(q^3)$$ (129)
Note that we only keep the leading order in $q$ in the last equation. Hence,

$$
\frac{2\pi}{q} \int_{0}^{q} dp \int_{|p-q|}^{|p+q|} dp' p' \mathcal{I}^{0} = \frac{7\pi^{3} |\Gamma(\mu)|^{4} q^{3} m^{2}}{3(2\pi)^{8} H^{2}} \tag{130}
$$

Substituting eqs. (130) into (113), we have the ζ correlation function due to a massive fermion loop as

$$
\int d^{3}x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t) \rangle_{\text{loop}} = \frac{-28\pi^{3} |\Gamma(\mu)|^{4} (8\pi G)^{2} m^{2} H^{2}}{3(2\pi)^{5} q^{3}} \tag{131}
$$

Using $|\Gamma(\mu)|^{2} = |\Gamma(\frac{\sqrt{2}}{2} \pm ir)|^{2} = \frac{\pi}{\cosh \pi r}$, we have

$$
\int d^{3}x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t) \rangle_{\text{loop}} = \frac{-7(8\pi G)^{2} m^{2} H^{2}}{24 q^{3} \cosh^{2} \frac{\pi m}{H}} \tag{132}
$$

which goes as $q^{-3}$ at low momentum.

It should be noted that the power spectrum will not be zero when we take $m = 0$. Eq. (132) for massive fermions is the result when the fermion pairs $\bar{\psi}_{p} \psi_{p}$ exit the horizon before or at the same time as $\zeta_{q}$ exits the horizon ($p \leq q$). We keep only the most dominant mode solutions for massive fermion after horizon exit. For the massless fermion case, the solution is simple enough so that we can do the integration exactly without any approximation. The integrand contributed by the massive fermion $\mathcal{F}^{0} \equiv a_{1}^{3} a_{2}^{3} M_{\psi}$ becomes frozen after horizon exit. The negative power of $(-\tau)$ that arises in the time integrals $\int \int dt_{2} dt_{1}$ of the massive fermion, but not the massless fermion, is more important than the exponential function when $\tau \to 0$. Therefore, massive fermion loops can contribute the $(\log a)^{2}$ factor because the interaction goes as $a^{0}$. In comparison, log $a$ does not arise in the massless fermion case because the interaction goes as $a^{-1}$ rather than $a^{0}$.

As mentioned in section II, a fermion mass could arise from the non-zero vacuum expectation value of an inflaton field in a flat potential. Therefore, the effective fermion mass during inflation could be as large as $M_{Pl}$. We know that in order to generate all matter observed today, the inflaton $\phi$ must couple to matter such as fermions sometime during inflation. Since we work in the gauge where an inflaton does not fluctuate $\delta \phi = 0$, the Yukawa coupling that can arise for the general inflaton potential does not change the result in eq. (132) but only shifts the fermion mass to be

$$
m \to m + \bar{\phi}(t_{q}) \tag{133}
$$
However, during inflation, the fermion mass could be large because the non-zero expectation value of the scalar field $\bar{\phi}(t)$ could be large. If the unperturbed inflaton amplitude at the time of horizon exit is as large as $M_{Pl}$, we have $m \simeq M_{Pl} \equiv \sqrt{8\pi G}$. Therefore, the power spectrum due to massive fermion loops is

$$\int d^3x e^{iq(x-x')} \langle \zeta(x,t)\zeta(x',t) \rangle_{\text{loop, } m=M_{Pl}} = -\frac{7(8\pi G)H^2}{24q^3 \cosh \frac{\pi m}{H}}$$

We see that even when we include the large $\bar{\phi} \sim M_{Pl}$ coupling that seems to give the quantum effect that does not get suppressed by the factor of $G$, the result is suppressed by the factor $\cosh^2 \frac{\pi m}{H}$ instead. This happens because the inflaton fluctuates around a non-zero background, implying that the massive fermion propagators are of the order of $M_{Pl}$. The factor $\cosh \frac{\pi m}{H}$ arises when we fix the mass dependent coefficients of the mode solution at late time.

It should be mentioned here that the large mass term does not get suppressed to the quantity like $a^3 \langle \bar{\psi}\psi \rangle \rightarrow \tanh \frac{\pi m}{H}$. Terms like $\bar{u}u$ or $\bar{v}v$ approach constants with the mass dependent constant coefficient going as $\tanh \frac{\pi m}{H}$, which does not have large mass suppression. However, to close the fermion loop, the tree $\langle \bar{\psi}\psi \rangle$ is not the only quantity we need to calculate. We also must calculate $\langle \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 \rangle$, which is the trace over the multiplied matrices such as $\sum_{r,s} \bar{v}_{p',r}(t_1)u_{p,s}(t_1)\bar{u}_{p,s}(t_2)v_{p',r}(t_2)$. Therefore, the bilinear $\bar{v}_1 u_1$ gives a constant mass dependent coefficient of $\frac{1}{\cosh \frac{\pi m}{H}}$, resulting in a small one-loop result even when the fermion couples to the inflaton in the order of $M_{Pl}$.

Apart from ultraviolet divergences, no infrared divergence can arise due to the late time behavior. The reason for infrared safety comes from the fact that the function $F_0$ approaches a constant at low momentum. This is similar to viewing $\sigma$ and $\tilde{\sigma}$ in eq. (107) as scalars which approach constants after horizon exit. Provided that the integral over time is infrared safe, the integral over time only comes from the $\zeta$ correlator, whereas the fermionic part only contributes an $a^{-3}$ factor that always cancels with the factor $\sqrt{-g}$ in each interaction Hamiltonian. After all integrations, the power spectrum gives a $q^{-3}$ momentum dependence. The result of the massive fermion case is valid at low momentum modes only where we need to cut off the momentum integral $p$ to some value i.e., $\Lambda q$, so that the approximation of small $p, p'$ in the Hankel function (eq. (115)) is still valid. This case means that,
if $\Lambda$ is of order 1, the fermion momentum modes $p, p'$ exit the horizon before the $\zeta$ momentum mode $q$ crosses the horizon ($p, p' \leq q = a(t_q)H(t_q)$). For higher momentum modes $p$, the fermion behaves like it is massless and is always suppressed by the factor $G$ and a negative power of the Robertson Walker $a$ as shown in massless fermion section.

To investigate whether the quantum effect is truly small, more careful consideration is needed for the mass effect. The reason is that various mass dependent coefficients can arise when matching the general solution with that inside the horizon for the general graphs. However, a fermion has two components that are needed to form a pair with its conjugate. Bilinears like $\bar{u}u$ or $\bar{v}v$, but not $\bar{v}u$, contribute a constant factors like $|\Gamma(\mu)|^2(e^{\frac{\pi m}{H}} - e^{-\frac{\pi m}{H}})$ which are $\tanh\frac{\pi m}{H} \to 1$ in the large mass limit. Bilinears like $u^\dagger u$ or $v^\dagger v$ contribute a constant factor like $\frac{\cosh \frac{m\pi}{H}}{\cosh \frac{m\pi}{H}} = 1$, which is mass independent. Therefore, by considering this alone, a fermion has no exponential suppression and seems to give a large quantum effect if a vertex is as large as $M_{Pl}$. However, as shown in the detailed calculation here, this is not possible for the loop graph that has two external legs with two trilinear vertices because it requires a bilinear like $\bar{v}u$ instead. Bilinear terms like $\bar{\psi}\gamma^i\psi$ get suppressed at late time because $\gamma^i$ can only have a contraction with $\frac{\dot{a}}{a}$. Therefore, the result is suppressed by an additional negative power of $a$ and its low momentum outside the horizon. Interaction terms like $\bar{\psi}\gamma^0\dot{\psi}$ contribute both $\frac{\dot{a}}{a}\bar{\psi}\gamma^i\psi$ and $m\bar{\psi}\psi$ factors via Dirac equation and its conjugate. Therefore, the maximum result of $\bar{\psi}\gamma^0\dot{\psi}$ cannot exceed the result of $\bar{\psi}\dot{\psi}$. The other powers of bilinear terms like $(\bar{\psi}\psi)^n$ for $n > 1$ cannot couple to the mass dimension in the order of $M_{Pl}$ and can only give higher fermion loops (by dimension counting in the action). Hence we expect the $(\bar{\psi}\psi)^{n>1}$ interaction type to be suppressed by a negative power of $a(t)$.

VI. VECTOR LOOP, INFLATON, AND GRAVITY

Ford considered a classical vector field driving inflation [12]. In this section, we consider a quantized vector field that affects the quantity $\zeta$ and its correlation function through the interaction with gravitational fluctuations. The
The action is

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_\phi + \mathcal{L}_V$$

$$= -\frac{\sqrt{-g}}{2} \left[ \frac{1}{8\pi G} R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2V(\phi) + \frac{1}{2} \hat{g}^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + m^2 g^{\mu\nu} A_\mu A_\nu \right]$$

To determine the vector field propagator, we need to solve the interaction free field equation in an inflating universe. It is

$$\partial_\mu \left( a^3(t) F^{\mu\nu} \right) - a^3(t) m^2 A^\nu = 0$$

(136)

For $\nu = 0$, this gives

$$A_0 = -\frac{i q_i \dot{A}_i}{q^2 + (ma)^2}$$

(137)

For $\nu = j$, this gives

$$\ddot{A}_j + H \dot{A}_j + \frac{q^2}{a^2} A_j + m^2 A_j = \frac{q_i q_j A_i}{a^2} + i q_j (\dot{A}_0 + HA_0)$$

(138)

To eliminate the auxiliary field $A_0$, we apply $\partial_\nu$ to eq. (136). We have

$$\partial_\nu (a^3 A^\nu) = 0$$

(139)

or

$$\dot{A}_0 + 3HA_0 - \frac{i q_j}{a^2} A_i = 0$$

(140)

Substituting (140) in (138), we have the dynamical field equation of $A_j$ in an expanding universe as

$$\ddot{A}_j + H \dot{A}_j + \left( 1 + \frac{2q_i q_j}{q^2 + (ma)^2} \right) \dot{A}_j + \left( \frac{q^2}{a^2} + m^2 \right) A_j = 0$$

(141)

For the transverse direction $q_i A_i = 0$, we have

$$\ddot{A}_j + H \dot{A}_j + \left( \frac{q^2}{a^2} + m^2 \right) A_j = 0$$

(142)

where this is valid for photons ($m = 0$) and massive vector bosons in the transverse direction ($\lambda = 1, 2$).
For the parallel direction \((\lambda = 3, m \neq 0\) only), we have

\[
\ddot{A}_j + \left(1 + \frac{2q^2}{q^2 + (ma)^2}\right)\dot{A}_j + \left(\frac{q^2}{a^2} + m^2\right)A_j = 0 \tag{143}
\]

It is impossible to solve equation (143) exactly \([13]\). However, at late time during inflation, \(a(t)\) grows more or less exponentially. Therefore, the second term proportional to \(\dot{A}_j\) in the equation above may be negligible. Hence, the vector field can be written as

\[
A_i(x, t) = \int d^3q \sum_\lambda \left[ e^{ix \cdot q}e_i(\hat{q}, \lambda)\alpha(q, \lambda)A_q(t) + e^{-ix \cdot q}e_i^*(\hat{q}, \lambda)\alpha^*(q, \lambda)A_q^*(t) \right] \tag{144}
\]

where

\[
\sum_{\lambda, \lambda' = 1}^2 e_i^*(\hat{q}, \lambda)e_j(\hat{q}, \lambda') = \delta_{ij} - \hat{q}_i\hat{q}_j \tag{145}
\]

for photons \(m = 0\) and

\[
\sum_{\lambda, \lambda' = 1}^3 e_i^*(\hat{q}, \lambda)e_j(\hat{q}, \lambda') \to \delta_{ij} \tag{146}
\]

for massive vector bosons during late time inflation. Therefore, \(A_q(t)\) is the solution that satisfies

\[
\frac{d}{dt}\left(a(t)\frac{d}{dt}A_q(t)\right) + \frac{q^2}{a(t)}A_q(t) + m^2 aA_q(t) = 0 \tag{147}
\]

To solve the equation above at a general momentum \(q\), we can work in the conformal time \(\tau\). Hence the massive gauge field equation in an inflating universe is

\[
\frac{d^2A_q}{d\tau^2} + \left(q^2 + \frac{r^2}{\tau^2}\right)A_q = 0 \tag{148}
\]

where \(r \equiv \frac{m}{H}\).

We see from eq. (148) that in the limit of \(m = 0\), the solution of a massless vector field is a plane wave. This solution is the same as those of the conformal scalar and massless fermion. The positive mode solution for a massless vector field at general wavelength is

\[
A_q(t) = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2q}}e^{-iq\tau}, m = 0 \tag{149}
\]
For a massive vector field \( m \neq 0 \), the field equation (148) has the Bessel’s equation type [11]

\[
u'' + \left( q^2 - \frac{4\nu^2}{4\nu^2} \right) u_\nu = 0 \quad (150)
\]

Therefore, the general solution of a massive vector field is

\[
A_\tau(\tau) = E_\nu \sqrt{-\tau} H^{(1)}_\nu(-q\tau) + F_\nu \sqrt{-\tau} H^{(2)}_\nu(-q\tau), \quad m \neq 0 \quad (151)
\]

where

\[
\nu = \sqrt{\frac{1}{4} - \tau^2} \quad (152)
\]

Since we want the solution to match the positive solution at deep inside the horizon \( e^{-i\omega\tau} \), only \( H^{(1)}_\nu(x) \) but not \( H^{(2)}_\nu(x) \) gives an \( e^{-i\omega\tau} \) factor in the large \(|x|\) limit. Hence, \( F_\nu = 0 \) and

\[
A_\nu(t) = E_\nu \sqrt{-\tau} H^{(1)}_\nu(-q\tau) \quad (153)
\]

A normalized constant \( E_\nu \) is chosen to match with the solution at deep inside the horizon. Inside the horizon, the positive frequency solution is the same as that in flat space, which is,

\[
A_\nu(t) \rightarrow \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_\nu}} \exp \left( -i \int_{-\infty}^{\tau} \omega_\nu(\tau')d\tau' \right) \quad (154)
\]

where \( \omega_\nu(\tau) \equiv \sqrt{q^2 + (ma)^2} \). With the property of Hankel’s function in the asymptotic limit, \(|x| \rightarrow \infty\)

\[
H^{(1)}_\nu(x) \rightarrow \frac{\sqrt{2}}{\pi x} \exp \left( i(x - \frac{\pi\nu}{2} - \frac{\pi}{4}) \right) \quad (155)
\]

Since we now allow the existence of a mass term which can be either large or small, the normalized constants \( E_\nu \) can be a function of mass and this may affect the result of the correlation function.

During inflation, the mass of the matter could be large due to the interaction of matter with the inflaton \( \dot{\phi} \). As mentioned earlier, the slow roll condition of some inflationary theories requires \( m = \dot{\phi} \simeq M_{Pl} \). This can make the mass term as large as \( M_{Pl} \) and may affect the final result of the correlation function. To determine the mass dependent coefficient \( E_\nu \), we match the solution with that inside the horizon. From eqs. (153), (151), and (155), we have the mass dependent coefficient \( E_\nu \) as

\[
E_\nu(m) = \frac{\sqrt{\tau}}{2(2\pi)^{\frac{3}{2}}} e^{\frac{i\pi}{4}(1+2\nu)} \quad (156)
\]
From eqs. (153) and (156), we therefore have the massive mode solution of the gauge field that we will use for the propagator as

\[ A_q(t) = \frac{\sqrt{\pi}}{2(2\pi)^{3/2}} \epsilon^{(1+2\nu)} \sqrt{-\tau} H^{(1)}_{\nu}(-q\tau) \]  

(157)

where \( \nu = \sqrt{\frac{1}{2} - r^2} \).

To calculate what the interaction vertices are, we need to expand beyond quadratic order in fluctuations. As derived in the appendix, the cubic order is

\[ H\zeta AA(t) = -\int d^3x \epsilon^5 \left( \frac{1}{a^2} \dot{A}_i^2 + \frac{1}{2a^4} (\partial_i A_j - \partial_j A_i)^2 \right) \nabla^2 \dot{\zeta} \]  

(158)

where we use the energy momentum tensor in [7] and choose gauge \( A_0 = 0 \) for \( m = 0 \). For \( m \neq 0 \), we can solve the constraint equation of \( A_0 \) and plug it back into the action. As seen from eq. (137), \( A_0 \) is decaying as \( \frac{a\dot{A}_i}{a_{\text{Harm}}} \) after horizon exit. Therefore, we can approximate the massive vertices as that of the massless case. It is only the propagators that will be different from massless case. To continue, we use the general formula in eq. (40) with the replacement

\[ \Psi^* \Psi \rightarrow \frac{1}{a^2} \dot{A}_i^2 + \frac{1}{a^4} \left( (\partial_i A_j)^2 - \partial_i A_j \partial_j A_i \right) \]  

(159)

\[ \zeta_q(t_1,2) \rightarrow -\zeta_q(t_1,2)/q^2 \]  

(160)

\[ V(t) = -\epsilon Ha^5(t) \]  

(161)

Since the purely electric term (through \( (\dot{A}_i)^2 \)), purely magnetic term (through \( (\partial_i A_j)^2 - (\partial_i A_j)(\partial_j A_i) \)), and two cross terms arise when we evaluate the commutator \([H_1, [H_2, Q]]\), eq. (40) becomes

\[ M_A = \frac{P_1}{a_1^2 a_2} \dot{A}_p(t_1) \dot{A}_{p'}(t_1) \dot{A}_p^*(t_2) \dot{A}_{p'}^*(t_2) + \frac{P_2}{a_1^4 a_2^2} A_p(t_1) A_{p'}(t_1) A_p^*(t_2) A_{p'}^*(t_2) + \frac{P_3}{a_1^2 a_2^4} A_p(t_1) A_{p'}(t_1) A_p^*(t_2) A_{p'}^*(t_2) + \frac{P_4}{a_1^2 a_2^3 a_4} \dot{A}_p(t_1) \dot{A}_{p'}(t_1) \dot{A}_p^*(t_2) \dot{A}_{p'}^*(t_2) \]  

(162)
and $\zeta$ remains the same as in eq. (69) because we are calculating the same correlation function $\langle \zeta \zeta \rangle$ with various kinds of matter loops.

We now need to calculate the polarization factor $P_i$ for $i = 1 \ldots 4$. The $P_1$ factor comes from the purely electric field term which is

$$
P_1 = 2 \sum_{\lambda, \lambda' = 1} (e_{i,p,\lambda} e_{j,p,\lambda})(e_{i,p',\lambda'} e_{j,p',\lambda'}) = 1 + (\hat{p} \cdot \hat{p}')^2
$$

(163)

The $P_2$ factor comes from the purely magnetic field term which is

$$
P_2 = 2 \sum_{\lambda, \lambda' = 1} |(p \cdot \hat{e}_{p,\lambda} \cdot \hat{e}_{p',\lambda'} - (p \cdot \hat{e}_{p',\lambda'})(p' \cdot \hat{e}_{p,\lambda})|^2 = p^2 p'^2 (1 + (\hat{p} \cdot \hat{p}')^2)
$$

(164)

The $P_{3,4}$ factors come from the cross terms which are

$$
P_3 = P_4 = -2p \cdot p'
$$

(165)

Substitute these into eq. (40), we get

$$
\int d^3x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t) \rangle_{\text{loop}} = -8(2\pi)^9 \int d^3p d^3p' \delta^3(q + p + p')
$$

$$
\times \int_{-\infty}^{t} dt_2 \epsilon_2 \mathcal{H}_2 a_2^5 \int_{-\infty}^{t_2} dt_1 \epsilon_1 \mathcal{H}_1 a_1^5 Re \left( Z \mathcal{M}_A \right)
$$

(166)

where

$$
\mathcal{M}_A = \left(1 + (\hat{p} \cdot \hat{p}')^2\right) a_1^{-2} a_2^{-2} \dot{\mathcal{A}}_p(t_1) \dot{\mathcal{A}}_{p'}(t_1) \dot{\mathcal{A}}_p(t_2) \dot{\mathcal{A}}_{p'}(t_2)
$$

$$
+ p^2 p'^2 \left(1 + (\hat{p} \cdot \hat{p}')^2\right) a_1^{-4} a_2^{-4} \mathcal{A}_p(t_1) \mathcal{A}_{p'}(t_1) \mathcal{A}_p(t_2) \mathcal{A}_{p'}(t_2)
$$

$$
-2pp' (\hat{p} \cdot \hat{p}') a_1^{-4} a_2^{-2} \mathcal{A}_p(t_1) \mathcal{A}_{p'}(t_1) \dot{\mathcal{A}}_p(t_2) \dot{\mathcal{A}}_{p'}(t_2)
$$

$$
-2pp' (\hat{p} \cdot \hat{p}') a_1^{-2} a_2^{-4} \dot{\mathcal{A}}_p(t_1) \dot{\mathcal{A}}_{p'}(t_1) \mathcal{A}_p(t_2) \mathcal{A}_{p'}(t_2)
$$

(167)

Since the solution of the massless vector field is just a plane wave,

$$
\dot{A}_q(t) = -\frac{iq}{a(t)} A_q(t)
$$

(168)
eq. (167) can be simplified further as

\[ M = \frac{2p^2p'^2}{a_1^2a_2^2} (1 + (\hat{p} \cdot \hat{p}'))^2 A_p(t_1)A_{p'}(t_1)A_p^*(t_2)A_{p'}^*(t_2) \]

\[ = \frac{p^2p'^2}{2(2\pi)^6a_1^2a_2^2pp'} (1 + (\hat{p} \cdot \hat{p}'))^2 e^{-i(p+p')(\tau_1 - \tau_2)} \]  

(169)

Substituting the \( Z \) part in eq. (77) and gauge field part in eq. (169) into eq. (166), we have the correlation function due to massless vector fields:

\[ \int d^3x e^{i q \cdot (x - x')} \langle \zeta(x, t)\zeta(x', t) \rangle_{loop} = -16(2\pi)^9 \epsilon^2 |\zeta_o|^4 \int d^3pd^3p' \]

\[ \times \delta^3(q + p + p') |A_p|^2 |A_{p'}|^2 p^2p'^2 (1 + (\hat{p} \cdot \hat{p}'))^2 T \]  

(170)

where

\[ T = Re \int_0^\infty \int_{-\infty}^{\tau_2} d\tau_2 e^{-i(q\tau_2 - q\tau_1)} e^{i(\hat{p} + \hat{p}')\tau_2} \int_{-\infty}^{\tau_2} d\tau_1 e^{-i(q + p + p')\tau_1} \]

\[ = -\frac{1}{2q(q + p + p')} \]  

(171)

and the constant coefficients after horizon exit are

\[ |A_o|^2 = \frac{1}{2(2\pi)^3q} \]  

(172)

Substituting eqs. (171) and (172) into (170), we have the loop power spectrum due to a massless vector field as

\[ \int d^3x e^{i q \cdot (x - x')} \langle \zeta(x, t)\zeta(x', t) \rangle_{loop} = \frac{(8\pi GH^2)^2}{2(2\pi)^3q^7} \int d^3pd^3p' \]

\[ \times \delta^3(q + p + p') \frac{pp'}{q + p + p'} (1 + (\hat{p} \cdot \hat{p}'))^2 \]  

(173)

Notice that the first term is the same as that of a massless minimal coupled scalar loop. The \( \hat{p} \cdot \hat{p}' \) terms come from summation over polarization vectors. We can follow the same method of dimensional regularization shown in the earlier section. We have the finite part of correlation function due to massless gauge field as

\[ \int d^3x e^{i q \cdot (x - x')} \langle \zeta(x, t)\zeta(x', t) \rangle_{loop} = -\frac{44\pi(8\pi GH^2(t_q))^2}{30(2\pi)^3q^3} \left( \ln q + C \right) \]  

(174)
Notice that the result is smaller than the classical result by a factor of $GH^2$ in order of magnitude. The numerical coefficient is slightly more than that of a scalar loop in [2] because there are additional polarization factors.

**VII. MASSIVE VECTOR BOSON**

We consider the late time mode solution of the massive vector field because the mode solution for the propagator is no longer a simple plane wave as in the massless case. For $|x| = |pτ| → 0$, we have

$$J_\nu(x) \rightarrow \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \left(1 + \mathcal{O}(x^2)\right)$$

(175)

By definition of Hankel’s function,

$$H^{(1)}_\nu(x) = \frac{1}{i \sin \nu \pi} \left(J_{-\nu}(x) - e^{-\nu \pi i} J_\nu(x)\right)$$

(176)

Hence, the late time behavior of mode solution approaches

$$H^{(1)}_\nu(x) \rightarrow -\frac{i}{\sin \nu \pi} \left(x^{-\nu} \left(\frac{x^\nu}{2^\nu \Gamma(1 - \nu) - \frac{e^{-\nu \pi i} x^\nu}{2^\nu \Gamma(\nu + 1)}}\right) \left(1 + \mathcal{O}(x^2)\right)\right)$$

(177)

The exact solution in eq. (157) approaches

$$\mathcal{A}_q(t) = C_q a^{\lambda^+} + D_q a^{\lambda^-}$$

(178)

at late time where

$$\lambda_{\pm} = -\frac{1}{2} \pm \nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m^2}{H^2}}$$

(179)

and

$$C_q = -\frac{i \sqrt{\pi}}{2(2\pi)^{\frac{3}{2}} \sqrt{H} \sin \nu \pi} \frac{e^{\frac{i \pi}{2}(1 + 2\nu)}}{\Gamma(1 - \nu)} \left(\frac{2H}{q}\right)^\nu$$

(180)

$$D_q = \frac{i \sqrt{\pi}}{2(2\pi)^{\frac{3}{2}} \sqrt{H} \sin \nu \pi} \frac{e^{\frac{i \pi}{2}(1 - 2\nu)}}{\Gamma(1 + \nu)} \left(\frac{2H}{q}\right)^{-\nu}$$

(181)

The time derivative of the solution contributes the same power of $a$ as

$$\dot{\mathcal{A}}_q(t) = H \left(\lambda^+ C_q a^{\lambda^+} + \lambda^- D_q a^{\lambda^-}\right)$$

(182)
We see from the equation above that the time derivative of the propagator in massive theories contributes an additional factor

\[ \dot{A}_q \rightarrow \lambda_\pm f(t) \rightarrow \left( -\frac{1}{2} \pm \sqrt{\frac{1}{2} - \frac{m^2}{H^2}} \right) f(t) \]  

(183)

where \( f(t) \) has the same power of \( t \) as in \( A_q \). Therefore, there will be a term like

\[ \dot{A}_p(t_1) \dot{A}_{p'}(t_1) \dot{A}_p^*(t_2) \dot{A}_{p'}^*(t_2) \rightarrow \mathcal{O}(\lambda_\pm^4 F[t_1, t_2]) \]  

(184)

in the loop. Since at large mass limit,

\[ \lambda_\pm^4 \rightarrow \frac{M_{Pl}^4}{H^4} \]  

(185)

the time derivative propagators give an additional factor of \( (\lambda_\pm)^4 \) for four fields. Because of eq. (185), the loop spectrum does not seem to get suppressed by an additional factor of \( G \), but may get suppressed by the constant coefficient at large mass in eqs. (180) and (181) or the results of the loop integrals. We therefore investigate whether there is true suppression or not.

To analyze in more detail, we start from the interaction in eq. (158). Therefore, the massive vector field loop contributes

\[ \mathcal{M}_{A,m \neq 0} = \Pi^*(t_2) \Pi(t_1) \]  

(186)

where

\[ \Pi(t) = \frac{1}{a^2(t)} \dot{A}_{i,p,\lambda}(t) \dot{A}_{i,p',\lambda'}(t) + \frac{1}{a^4(t)} \left( p_p p_{p'} A_{j,p,\lambda}(t) A_{j,p',\lambda'}(t) ight. \\
- \left. p_p p_{p'} A_{j,p,\lambda}(t) A_{j,p',\lambda'}(t) \right) \]  

(187)

The exact solution of a massive vector field involves Hankel’s functions which are rather complicated to integrate over time and momentums. However, we can get some ideas about what the momentum dependence of the observable spectrum is by considering the long wavelength mode solutions. We see from eqs. (178) and (182) that \( \dot{A}_q \) gives the same power of \( a \) as \( A_q \) at late time. Therefore, we can keep the most leading order term as the universe rapidly expands

\[ \mathcal{M}_A \rightarrow \frac{3}{a^2 a_1^2} \dot{A}_p^*(t_2) \dot{A}_{p'}^*(t_2) \dot{A}_p(t_1) \dot{A}_{p'}(t_1) \]  

(188)
This means that the (massive) electric-like term is more dominating than the magnetic-like term after horizon exit. This result is different from the result of massless vector fields in which all electric and magnetic terms are equally important. Substituting eq. (188) into eq. (40), we have

$$\int d^3 x e^{i q \cdot (x - x')} \langle \zeta(x, t) \zeta(x', t) \rangle_{\text{loop}} \rightarrow -24(2\pi)^9 \int d^3 p d^3 p' \delta^3(q + p + p')$$

$$\int_{-\infty}^{t} dt_2 e^{2 H a_2^3} \int_{-\infty}^{t_2} dt_1 \epsilon_1 H_1 a_1^3 \text{Re} \left( \mathcal{Z} \hat{A}_p^* (t_2) \hat{A}_{p'}^* (t_2) \hat{A}_p (t_1) \hat{A}_{p'} (t_1) \right)$$

where $\mathcal{Z}$ is the contribution from the $\zeta$ part which is still the same as that in eq. (187). Hence,

$$\int d^3 x e^{i q \cdot (x - x')} \langle \zeta(x, t) \zeta(x', t) \rangle_{\text{loop}} \rightarrow -24(2\pi)^9 |\epsilon|^2 |\zeta_0^q|^4 \int d^3 p d^3 p' \delta^3(q + p + p')$$

where

$$T \equiv \text{Re} \int_{-\infty}^{t} dt_2 a_2 (e^{-i q \tau_2} - e^{i q \tau_2}) \hat{A}_p^* (t_2) \hat{A}_{p'}^* (t_2)$$

$$\times \int_{-\infty}^{t_2} dt_1 a_1 e^{-i q \tau_1} \hat{A}_p (t_1) \hat{A}_{p'} (t_1)$$

(191)

With the time derivative of the late time mode solution in eq. (182), we have

$$\hat{A}_p (t_1) \hat{A}_{p'} (t_1) = H^2 \left[ \lambda_+^2 C_p C_{p'} a_1^{2\lambda_+} + \lambda_+^2 D_p D_{p'} a_1^{2\lambda_-} + \lambda_+ \lambda_- (C_p D_{p'} + D_p C_{p'}) a_1^{\lambda_+ + \lambda_-} \right]$$

(192)

Therefore, the $t_1$ integral is

$$\int_{-\infty}^{t_2} dt_1 a_1 e^{-i q \tau_1} \hat{A}_p (t_1) \hat{A}_{p'} (t_1) = H^2 \int_{-\infty}^{t_2} dt_1 e^{-i q \tau_1} \left[ \lambda_+^2 C_p C_{p'} a_1^{2\nu} + \lambda_+^2 D_p D_{p'} a_1^{-2\nu} + \lambda_+ \lambda_- (C_p D_{p'} + D_p C_{p'}) \right]$$

$$\rightarrow H \left[ \frac{\lambda_+^2}{2\nu} c_+ a_2^{2\nu} + c_0 - \frac{\lambda_-}{2\nu} c_+ a_2^{-2\nu} - \lambda_+ \lambda_- c_0 E_i(-i q \tau_2) \right]$$

(193)

for $2\nu \neq 0$ and

$$c_0 = C_p D_{p'} + D_p C_{p'}$$

(194)

$$c_+ = C_p C_{p'}$$

(195)

$$c_- = D_p D_{p'}$$

(196)

34
and $\lambda_\pm = -\frac{1}{2} \pm (\nu = \sqrt{\frac{1}{4} - \frac{m^2}{H^2}})$ which can be either real or complex or zero, depending on its mass when compared to the expansion rate $H$.

**Small Mass: $m < \frac{H}{2}$**

For $m < \frac{H}{2}$, $\nu$ is real. Therefore,

$$-\frac{1}{2} < \lambda_+ < 0, -1 < \lambda_- < -\frac{1}{2}$$

and $|C_q|^2$ and $|D_q|^2$ are $q$-dependent, depending on its mass. Hence,

$$\mathcal{T} = H^3 \Re \int_{-\infty}^{t} dt_2 (-2i) \sin q\tau_2 \left[ \lambda_+^2 c_+^* a_2^{2\nu} + \lambda_-^2 c_- a_2^{-2\nu} + \lambda_+ \lambda_- c_0^* \right]$$

$$\times \left[ \frac{\lambda_+^2}{2\nu} c_+ a_2^{2\nu} - \frac{\lambda_-^2}{2\nu} c_- a_2^{-2\nu} - \lambda_+ \lambda_- c_0 Ei(-iq\tau_2) \right]$$

(198)

We can see that there is no contribution from the terms proportional to $|c_+|^2$ and $|c_-|^2$ because they are all real. With the factor $i$ in the integrand, the contribution is purely imaginary. Hence there is no contribution after taking the real part. Therefore, the terms that give non-zero result are

$$\mathcal{T} \to 2qH \lambda_- \lambda_0^3 \Im (c_+^* c_0) \left( \frac{1 - \eta \ln q\tau}{a^3 \eta^2} \right)$$

(200)

for $0 \ll 2\nu < 1$. Therefore, the correlation function due to a massive vector field loop is

$$\int d^3 x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t) \rangle \to -\frac{12(2\pi)^3 \lambda_- \lambda_0^3 (8\pi GH^2)^2 H}{q^5 a^3}$$

$$\times \frac{(1 - \eta \ln q\tau)}{\eta^2} \int d^3 p d^3 p' \delta^3(q + p + p') \Im (c_+^* c_0)$$

(201)
To calculate what \( \text{Im}(c^*_n c_0) \) is, we see from eqs. (180) and (181) that \( C_p^* \mathcal{D}_p \) is \( p \)-independent. Therefore,

\[
\text{Im}(c^*_n c_0) = \frac{(2H)^{2\nu} \Gamma^2(\nu)}{8\nu(2\pi)^7 H^2} \left[ \frac{1}{p^{2\nu}} + \frac{1}{p'^{2\nu}} \right] \tag{202}
\]

where we use \( \Gamma(1-x)\Gamma(x) = \frac{\pi \sin \pi x}{x} \) and

\[
\text{Im}(C_p^* \mathcal{D}_p) = \frac{1}{4(2\pi)^4 H \nu} \tag{203}
\]

\[
|C_p|^2 = \frac{\pi \Gamma^2(\nu)}{(2\pi)^5 H} \left( \frac{2H}{p} \right)^{2\nu} \tag{204}
\]

Hence,

\[
\int d^3x e^{iq \cdot (x-x')} \langle \zeta(x, t) \zeta(x', t) \rangle_{\text{loop, } m< \frac{q}{H}} =
\]

\[
- \frac{3(2)^{2\nu} \Gamma^2(\nu) \lambda_\nu \lambda_\nu^3 (8\pi GH^2)^2 (1 + (2\nu - 1) \ln q\tau)}{2(2\pi)^4 H^{1-2\nu} q^3 a^{1-2\nu} \nu(1-2\nu)^2} \times \int d^3p d^3p' \delta^3(q + p + p') \left[ \frac{1}{p^{2\nu}} + \frac{1}{p'^{2\nu}} \right] \tag{205}
\]

To determine the momentum dependence \( q \) of the spectrum, we integrate over internal momentum \( p, p' \) circulated inside the loop. Following the similar way as in the massive fermion section, we have the momentum dependent spectrum as

\[
\int d^3x e^{iq \cdot (x-x')} \langle \zeta(x, t) \zeta(x', t) \rangle =
\]

\[
- \frac{24\Gamma^2(\nu) \lambda_\nu \lambda_\nu^3 (8\pi GH^2)^2}{(2\pi)^3 (2H a(t))^\eta} \frac{(1 - \eta \ln q\tau)}{(1 - \eta)(2 + \eta)\eta^2 q^3 \eta^{-\eta}} \tag{206}
\]

where

\[
2\nu = \sqrt{1 - \frac{4m^2}{H^2}} \simeq 1 - \frac{2m^2}{H^2} \tag{207}
\]

or

\[
\eta \equiv 1 - 2\nu \simeq \frac{2m^2}{H^2} < \frac{1}{2} \tag{208}
\]

We see that the departure from scale invariance is still small.
Critical Mass: \( m = \frac{H}{2} \)

For \( m = \frac{H}{2} \), \( \nu \) is zero. Therefore,

\[
\lambda_+ = \lambda_- = -\frac{1}{2} \quad (209)
\]

and \( C_q \) and \( D_q \) are \( q \)-independent. Eq. (193) becomes

\[
\int_{-\infty}^{t_2} dt_1 a_1 e^{-i q \tau_1} \dot{A}(t_1) \dot{A}(t_1) = -\frac{H}{4} (C_p + D_p) (C_{\nu} + D_{\nu}) Ei(-i q \tau_2) \quad (210)
\]

Substituting equation above into eq. (191), we therefore have

\[
T = \frac{\pi^2 H^2}{64} |C_p + D_p|^2 |C_{\nu} + D_{\nu}|^2 \quad (211)
\]

We see from eqs. (180) and (181) that the coefficients are all momentum independent when \( \nu = 0 \). Since

\[
(C_q + D_q)|_{\nu=0,m=\frac{H}{2}} = 0 \quad (212)
\]

we need to use l’Hospital’s rule. From eqs. (180) and (181), we have

\[
\lim_{\nu \to 0} (C_q + D_q) = -i \sqrt{\pi} e^{\frac{i \pi}{4}} \lim_{\nu \to 0} \left[ \frac{e^{i \frac{\pi}{2} \left( \frac{2H}{q} \right) \nu}}{2(2\pi)^{\frac{3}{2}} \sqrt{H}} - \frac{e^{-i \frac{\pi}{2} \left( \frac{2H}{q} \right) \nu}}{2(2\pi)^{\frac{3}{2}} \sqrt{H}} \right] \quad (213)
\]

Note that

\[
\lim_{\nu \to 0} \frac{e^{i \frac{\pi}{2} \left( \frac{2H}{q} \right) \nu}}{\sin \nu \pi \Gamma(1 - \nu)} = \lim_{\nu \to 0} \pi \cos \nu \pi \Gamma(1 - \nu) - \sin \nu \pi \psi(1 - \nu) \Gamma(1 - \nu) = \frac{i}{2} \quad (214)
\]

Similarly,

\[
\lim_{\nu \to 0} \frac{e^{-i \frac{\pi}{2} \left( \frac{2H}{q} \right) \nu}}{\sin \nu \pi \Gamma(1 + \nu)} = -\frac{i}{2} \quad (215)
\]

Therefore,

\[
\lim_{\nu \to 0} (C_q + D_q) = \frac{\sqrt{\pi} e^{\frac{i \pi}{4}}}{2(2\pi)^{\frac{3}{2}} \sqrt{H}} \quad (216)
\]
We see that the coefficients are $q$-independent. Hence, eq. (211) becomes

$$T = \frac{\pi^4}{1024(2\pi)^6}$$

(217)

Therefore, the correlation function in eq. (190) becomes

$$\int d^3x e^{iq(x-x')} \langle \zeta(x,t)\zeta(x',t) \rangle = -\frac{\pi^2(8\pi GH^2)^2}{1024q^3}$$

(218)

**Large Mass: $m > \frac{H}{2}$**

For $m > \frac{H}{2}$, $\lambda_\pm$ are complex conjugates of each other as

$$\lambda_+ \equiv \lambda = -\frac{1}{2} + is, \lambda_- = \lambda^* = -\frac{1}{2} - is$$

(219)

where $\nu = is$ and $s \equiv \sqrt{\frac{m^2}{H^2} - \frac{1}{4}}$ is real. Notice that $|C_q|^2$ and $|D_q|^2$ are $q$-independent. Therefore, eq. (193) becomes

$$\int_{-\infty}^{t_2} dt_1 a_1 e^{-iq\tau_1} \mathcal{A}_p(t_1) \mathcal{A}'_{p'}(t_1) \to \frac{H^3}{2is} \left[ \lambda^2 c_+ a_2^{2is} - \lambda^* c_- a_2^{-2is} - 2is|\lambda|^2 c_0 Ei(-iq\tau_2) \right]$$

(220)

We see that the time components of the first two terms are complex conjugates of each other with different constant coefficients. Integrating over time $t_2$ gives

$$T = -\frac{H^3}{s} Re \int_{-\infty}^t dt_2 \sin q\tau_2 \left[ \lambda^2 c_+ a_2^{2is} - \lambda^* c_- a_2^{-2is} + |\lambda|^2 c_0^* \right]$$

$$\times \left[ \lambda^2 c_+ a_2^{2is} - \lambda^* c_- a_2^{-2is} - 2is|\lambda|^2 c_0 Ei(-iq\tau_2) \right]$$

$$= \frac{|\lambda|^4 H^2}{s} \int_{-\infty}^0 \frac{d\tau_2}{\tau_2} \sin q\tau_2 \left( |c_+|^2 - |c_-|^2 - 2s|c_0|^2 Siq\tau_2 \right)$$

(221)

where the other terms vanish in the limit of $t \to \infty$ because of the oscillating behavior of the integrand $e^{\pm iHt^2}$. We can integrate further with the use of Mathematica

$$\int_{-\infty}^0 \frac{d\tau_2}{\tau_2} \sin q\tau_2 Siq\tau_2 = -\frac{\pi^2}{8}$$

(222)

$$\int_{-\infty}^0 \frac{d\tau_2}{\tau_2} \sin q\tau_2 = \frac{\pi}{2}$$

(223)
Therefore, eq. (221) becomes

\[
T = \frac{\pi |\lambda|^4 H^2}{4s} \left[ 2(|c_+|^2 - |c_-|^2) + \pi s |c_0|^2 \right] \quad (224)
\]

We now need to calculate what \(|c_{\pm,0}|^2\) are. From eq. (194), we have

\[
|c_+|^2 = |C_p|^2 |C_p'|^2 \quad (225)
\]

\[
|c_-|^2 = |D_p|^2 |D_p'|^2 \quad (226)
\]

\[
|c_0|^2 = |C_p D_p' + D_p C_p'|^2 \quad (227)
\]

From eqs. (180) and (181), we have

\[
|C_q|^2 = \frac{\pi}{4(2\pi)^3 H^3 \sin \pi \nu} e^{-\pi s} \left| \Gamma(1 - \nu) \right|^2 \quad (228)
\]

and

\[
|D_q|^2 = \frac{\pi}{4(2\pi)^3 H^3 \sin \nu \pi} e^{\pi s} \left| \Gamma(1 + \nu) \right|^2 \quad (229)
\]

We can see that \(|C_q|^2\) and \(|D_q|^2\) are momentum independent because \(\nu \equiv is\) is purely imaginary in the large mass limit when \(m > \frac{H}{2}\). Therefore,

\[
|c_+|^2 - |c_-|^2 = \frac{\pi^2 (e^{-2\pi s} - e^{2\pi s})}{16(2\pi)^6 H^2 \sin \pi s} \left| \Gamma(1 + \nu) \right|^4 \quad (230)
\]

Using \(|\Gamma(1 + is)|^2 = |\Gamma(1 - is)|^2 = \frac{\pi s}{\sinh \pi s}\) and \(\sin is = i \sinh s\) for real \(s\), the equation above is simplified as

\[
|c_+|^2 - |c_-|^2 = -\frac{\coth \pi s}{4(2\pi)^6 s^2 H^2} \quad (231)
\]

where we use \(\sinh 2x = 2 \sinh x \cosh x\). Also,

\[
|c_0|^2 = \frac{\cos^2(s \ln \frac{p}{p'})}{4(2\pi)^6 s^2 H^2 \sinh^2 \pi s} \quad (232)
\]

Notice that the coefficients \(|c_+|^2 - |c_-|^2\) are completely momentum independent and \(|c_0|^2\) is nearly momentum independent (\(\ln \frac{|p|}{|p+q|} \to 0\) when \(q \to 0\)). Substituting eqs. (231), and (232) into eq. (224), we have

\[
T = \frac{\pi |\lambda|^4}{8s (2\pi)^6} \left[ -\frac{\coth \pi s}{s^2} + \frac{\pi \cos^2(s \ln \frac{p}{p'})}{2s \sinh^2 \pi s} \right] \quad (233)
\]
From eqs. (190) and (233), we have

\[
\int d^3x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t) \rangle_{\text{loop}} = -\frac{3\pi(8\pi G H^2)^2|\lambda|^4}{32s(2\pi)^3q^6} \int d^3p d^3p' \times \delta^3(\mathbf{q} + \mathbf{p} + \mathbf{p}') \left[ -\frac{\coth \pi s}{s^2} + \frac{\pi \cos^2(s \ln \frac{q}{p_0})}{2s \sinh^2 \pi s} \right]
\]

The result of the momentum integrals \( p, p' \) gives the momentum dependence as \( q^3 \), which cancels with the \( q^{-6} \) factor in eq. (234). We therefore have the approximated scale invariant spectrum after horizon exit as

\[
\int d^3x e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t) \rangle_{\text{loop,m}} = \frac{3\pi^2(8\pi G H^2)^2|\lambda|^4}{8s^3(2\pi)^3q^3} \coth \pi s (235)
\]

where \( \lambda = -\frac{1}{2} + is \) and \( s = \sqrt{|\frac{m^2}{2H^2} - \frac{1}{4}|} \).

**VIII. CONFORMAL SCALAR LOOP, INFLATON, AND GRAVITY**

We have learned that the spectrums of massless minimal coupled scalar, massless fermion, and massless vector fields loops all go as \((8\pi G H^2)^2 q^{-6} \ln q\). We would like to investigate whether this is also true for conformal scalar loop. The full action considered during inflation is

\[
\mathcal{L} = -\frac{1}{2} \sqrt{-g} \left[ \frac{1}{8\pi G} R + g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 2V(\varphi) + g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \xi R \chi^2 \right]
\]

where \( \varphi \) is an inflaton, \( \chi \) is additional conformal scalar matter in which \( \langle \chi \rangle = 0 \), and \( \xi = \frac{1}{6}, 0 \) for conformal and minimal couplings respectively. We consider this to see how the conformal scalar affects the spectrum \( \langle \zeta \zeta \rangle \) through the interaction with gravitational fluctuation. To arrive at the field equation of the conformal scalar field, we need the action up to the second order in the field fluctuations which is

\[
\mathcal{L}^{(2)}_\chi = \frac{a^3}{2} \left( \chi^2 - \frac{(\partial_\lambda \chi)^2}{a^2} - 12\xi H^2 \chi^2 \right)
\]

where \( \bar{R} = -12H^2 \). Varying the second order of the action with respect to \( \chi \), we have the field equation of the conformal scalar field as

\[
\ddot{\chi}_q + 3H \dot{\chi}_q + \left( \frac{q^2}{a^2} + 12\xi H^2 \right) \chi_q = 0
\]

(238)
Notice that if there is no extra term $12 \xi H^2 \chi$ (or $\xi = 0$), this is just a minimal coupled massless scalar, in which the dominant solution approaches a constant at late time. It is known that the massless minimal coupled scalar produces a scale free spectrum. We would like to investigate the momentum dependence of the power spectrum due to conformal scalar loops here when $\xi = \frac{1}{6}$. Eq. (235) can be solved exactly by re-scaling the field $\chi \equiv u/a$. Hence, for the conformal scalar,

$$u'' + \left(q^2 - \frac{a''}{a} + 2H^2 a^2\right)u_q = 0$$  \hspace{1cm} (239)

During inflation, $a \simeq -\frac{1}{H\tau}$, therefore, the last two terms of eq. (239) are cancelled. We arrive at a simple field equation of the conformal scalar field

$$u'' + q^2 u_q = 0$$  \hspace{1cm} (240)

Therefore, the solution to the equation above is just a simple plane wave valid to all wavelengths

$$\chi_q(t) = u_q(t)/a(t) = \frac{1}{(2\pi)^3 a(t) \sqrt{2q}} e^{-iq\tau}$$  \hspace{1cm} (241)

where we choose the constant coefficients to match with the positive mode solution inside the horizon. From eq. (241), the conformal scalar field correlation function to leading order is

$$\langle \chi(x, t) \chi(x', t) \rangle = \int d^3 q e^{iq(x-x')} |\chi_q(t)|^2$$

$$= \int \frac{d^3 q}{(2\pi)^3} e^{iq(x-x')} \frac{1}{2qa^2(t)}$$  \hspace{1cm} (242)

We see that its momentum dependence is far from scale invariant at the classical level. However, we never observe the product of the scalar fields in CMB anisotropy but rather the correlation function of the temperature or density fluctuations which is related to the conserved quantity $\zeta$. Therefore, we study how the conformal scalar field affects the observable power spectrum $\langle \zeta \zeta \rangle$ via the gravitational interactions at the quantum level. We can calculate the trilinear vertices due to the conformal scalar $\chi$ and gravity $\zeta$.

They are

$$L_{\zeta \chi \chi} = -\frac{a}{2} \zeta (\partial_i \chi)^2 + a \partial_i \left( \frac{\zeta}{H} - \epsilon H a^2 \nabla^2 \zeta \right) \partial_i \chi$$

$$- \frac{a}{2H} \zeta (\partial_i \chi)^2 - \frac{a^3}{2H} \zeta \dot{\chi}^2 + \frac{3a^3}{2} \zeta \dot{\chi}^2$$

$$- a^3 \left( 3H^2 \zeta + H \dot{\zeta} - \frac{\delta R}{12} \right) \chi^2$$  \hspace{1cm} (243)
where the last line above contains the additional terms from the massless minimal coupled scalar. Those terms in the last line arise from the conformal term which is $\frac{1}{12} R \chi^2$. We see that the interactions above are rather complicated. As derived in the appendix, the interaction vertices can be written in a more compact form as

$$H_{\zeta \chi \chi}(t) = - \int d^3 x \epsilon H a^5 (T^{00} + a^2 T^{ii}) \nabla^{-2} \dot{\zeta}$$

where $T^{\mu \nu}$ is the energy momentum tensor at the second order of arbitrary matter. The combination of time and space components of the energy momentum tensor is [14]

$$T^{00} + a^2 T^{ii} = 2(1 - 3 \xi) \dot{\chi}^2 + 12 \xi^2 H^2 \chi^2 + \frac{2 \epsilon}{a^2} \chi^2 - 2 \xi \chi \left( \chi_{;00} + \chi_{,ii} \right)$$

$$- \xi \chi^2 \left( \bar{R}_{00} + \frac{\bar{R}_{ii}}{a^2} - \bar{R} + 3 \xi \bar{R} \right)$$

We can check that $T^{00} + a^2 T^{ii} = 2 \dot{\chi}^2$ for minimal coupling $\xi = 0$. For conformal coupling $\xi = \frac{1}{6}$, we have

$$T^{00} + a^2 T^{ii} = \dot{\chi}^2 + \frac{1}{3} \left( \frac{(\partial_i \chi)^2}{a^2} - 2 \chi \ddot{\chi} - H^2 \chi^2 \right)$$

where the conformal field equation (238) is used and

$$\bar{R}_{00} + \frac{\bar{R}_{ii}}{a^2} - \bar{R} + 3 \xi \bar{R} = 0$$

due to $\bar{R}_{00} = 3 H^2$, $\bar{R}_{ii} = -9a^2 H^2$ and $\bar{R} = -12 H^2$ in de-Sitter phase inflation. Therefore, the trilinear interaction Hamiltonian of the conformal scalar $\chi$ and gravity $\zeta$ is

$$H_{\zeta \chi \chi}(t) = - \int d^3 x \epsilon H a^5 \left[ \dot{\chi}^2 + \frac{1}{3} \left( \frac{(\partial_i \chi)^2}{a^2} - 2 \chi \ddot{\chi} - H^2 \chi^2 \right) \right] \nabla^{-2} \dot{\zeta}$$

To calculate the loop spectrum in the commutator $[H_1, [H_2, Q]]$, we can use the general formula in eq.(40) with the replacement

$$\Psi^* \Psi \rightarrow \dot{\chi}^2 + \frac{1}{3} \left( \frac{(\partial_i \chi)^2}{a^2} - 2 \chi \ddot{\chi} - H^2 \chi^2 \right)$$

$$\zeta_q(t_{1,2}) \rightarrow - \frac{\zeta_q(t_{1,2})}{q^2}$$

$$V(t) = -cHa^5(t)$$
Hence,
\[
\mathcal{M}_\chi \equiv \pi^*(t_2)\pi(t_1) \tag{252}
\]
where
\[
\pi(t) = \dot{x}_p x_p - \frac{1}{3} x'_p \dot{x}_p - \frac{1}{3} \ddot{x}_p x_p - \frac{1}{3} \left( \frac{pp'}{a^2} + H^2 \right) x_p x_p \tag{253}
\]
With
\[
\dot{x}_q(t) = -(H + \frac{iq}{a}) x_q \tag{254}
\]
\[
\ddot{x}_q(t) = \left[ H^2 + 3iH \frac{q}{a} - \frac{q^2}{a^2} \right] x_q \tag{255}
\]
many terms are cancelled. Therefore,
\[
\pi(t) = \frac{1}{3a^2} \left[ p^2 + p'^2 - 4pp' \right] x_p(t)x_p'(t) \tag{256}
\]
Hence,
\[
\mathcal{M}_\chi = \frac{1}{9a_1^2a_2^2} \left( p^2 + p'^2 - 4pp' \right)^2 x_p^*(t_2)x_p^*(t_1)x_p(t_1)x_p(t_1) \tag{257}
\]
From eq. (40), we have
\[
\int d^3xe^{iq(x-x')}\left\langle \zeta(x,t)\zeta(x',t) \right\rangle_{\text{loop}} = -8(2\pi)^3 \int d^3pd^3p'\delta^3(q + p + p') \times \int_{-\infty}^{t_2} dt_2 \epsilon_2 H_2 a_2^5 \int_{-\infty}^{t_1} dt_1 \epsilon_1 H_1 a_1^5 Re \left( Z_{\pi_2^*\pi_1} \right) \tag{258}
\]
where \(\zeta\) remains the same as in eq. (77). Substituting the \(Z\) part eq. (77) and matter part eq. (256) into eq. (258), we have the correlation function due to the conformal scalar field loop
\[
\int d^3xe^{iq(x-x')}\left\langle \zeta(x,t)\zeta(x',t) \right\rangle_{\text{loop}} = -\frac{2(2\pi)^3(8\pi GH^2)^2}{9q^6} \int d^3pd^3p' \times \delta^3(q + p + p') \left[ p^2 + p'^2 - 4pp' \right]^2 Re \int_{-\infty}^{t_2} dt_2 a_2(e^{-iq\tau_2} - e^{iq\tau_2})x_p^*(t_2)x_p^*(t_2) \times \int_{-\infty}^{t_1} dt_1 a_1 e^{-iq\tau_1} x_p(t_1)x_p(t_1) \tag{259}
\]
With the mode solution in eq. (241) and the result of the time integrations, we have the loop correlation function as

\[
\langle \zeta(x,t)\zeta(x',t) \rangle_{\text{loop}} = \frac{(8\pi G H^2)^2}{36(2\pi)^3 q^3} \times \int d^3p d^3p' \delta^3(q + p + p') \frac{1}{pp'p + p + q} \frac{p^2 + p'^2 - 4pp'}{(p^2 + p'^2 - 4pp')} ^2
\]

(260)

With the dimensional regularization as done before, we have the conformal scalar loop correlation function as

\[
\langle \zeta(x,t)\zeta(x',t) \rangle_{\text{loop}} = \frac{-\pi}{90(2\pi)^3 q^3} \left[ \ln q + C \right]
\]

(261)

We see that it is nearly scale invariance and smaller than the classical result by a factor of \( \frac{\epsilon H^2 \ln q}{M_{Pl}} \).

**IX. SUMMARY OF ALL RESULTS**

We study quantum effects of cosmological correlations due to the interactions of gravitational and matter fluctuations. It is shown that departures from scale invariance are never large, regardless of what kind of theories, what kind of matter, or what kind of inflaton potential \( V(\varphi) \) are used. The results in this paper may be compared with the Weinberg’s result [2],

**Minimal Coupled Scalar Field Loops**

\[
\langle \zeta \zeta \rangle_{m=0} = -\frac{\pi (8\pi G)^2 H(t_q)^4}{15(2\pi)^3 q^3} \left[ \ln q + C \right]
\]

(262)

where \( H(t_q) \propto q^{-\epsilon} \) is the expansion rate at the time of horizon exit.

The results in this paper are

**Dirac Field Loops**

\[
\langle \zeta \zeta \rangle_{m_f=0} = -\frac{4\pi (8\pi G)^2 H(t_q)^4}{15(2\pi)^3 q^3} \left[ \ln q + C \right]
\]

\[
\langle \zeta \zeta \rangle_{m_f \neq 0} \rightarrow -\frac{7(8\pi G)^2 m_f^2 H(t_q)^2}{24q^3 \cosh^2 \frac{m_f}{H}}
\]

(264)
Gauge Field Loops

\[ \langle \zeta \zeta \rangle_{m_v=0} = -\frac{44\pi(8\pi G)^2 H(t_q)^4}{30(2\pi)^3q^3} \left[ \ln q + C \right] \]  

(265)

\[ \langle \zeta \zeta \rangle_{m_v<\frac{\mu}{2}} = -\frac{24\Gamma^2(\nu)\lambda_-^3(8\pi GH(t_q)^2)^2}{(2\pi)^3|2H(t_q)a(t)|\eta} \frac{(1-\eta \ln q\tau)}{(1-\eta)(2+\eta)\eta^2q^3-\eta} \]  

where \( \lambda_\pm = -\frac{1}{2} \pm \nu \), \( 0 < 2\nu = \sqrt{1 - \frac{4m_v^2}{H^2}} \) < 1 and \( \eta = 1 - 2\nu < 0.5 \).

(266)

\[ \langle \zeta \zeta \rangle_{m_v=\frac{\mu}{2}} = -\frac{\pi^2(8\pi G)^2 H(t_q)^4}{1024q^3} \]  

(267)

\[ \langle \zeta \zeta \rangle_{m_v>\frac{\mu}{2}} \rightarrow \frac{3\pi^2(8\pi G)^2 H(t_q)^4 |\lambda|^4}{8s^2(2\pi)^3q^3} \coth \pi s \]  

where \( \lambda = -\frac{1}{2} + is \) and \( s = \sqrt{|m_v^2 - \frac{1}{4}|} \).

(268)

\[ \langle \zeta \zeta \rangle_{m_v=M_{Pl}} \rightarrow \frac{3\pi^2(8\pi G)^2 M_{Pl} H(t_q)^3}{8(2\pi)^3q^3} \]  

(269)

Conformal Scalar Field Loops

\[ \langle \zeta \zeta \rangle = -\frac{\pi(8\pi G)^2 H(t_q)^4}{90(2\pi)^3q^3} \left[ \ln q + C \right] \]  

(270)

We see that even when the mass is as large as \( M_{Pl} \), the one-loop result is still naturally smaller than the classical one. Therefore, no fine tuning is needed. The result above is still valid in the realistic and the general potential \( V(\varphi, \bar{\psi}\psi, A_\mu A^\mu) \). The reason is that we choose the gauge in which the inflaton does not fluctuate \([7]\) \((\delta \varphi = 0)\).

Even when the additional interactions of inflaton and matters arise, the results in eqs. (264) and (268) do not change but rather the masses are shifted by

\[ V(\varphi) \rightarrow V(\varphi, \bar{\psi}\psi, A_\mu A^\mu) \]  

(271)

\[ m_f \rightarrow m_f + \frac{\partial^2 V}{\partial \psi \partial \bar{\psi}}|_{\psi=0} \]  

(272)

\[ m_v^2 \rightarrow m_v^2 + \frac{\partial^2 V}{\partial A_\mu \partial A^\mu}|_{A_\mu=0} \]  

(273)
The is because the mass shift \( \frac{\partial^2 V}{\partial \bar{\psi} \partial \psi} \bigg|_{\psi=0} + \frac{\partial^2 V}{\partial A_\mu \partial A_\mu} \bigg|_{A_\mu=0} \), which is a function of the unperturbed inflaton only, does not change much during inflation. We therefore can approximate the unperturbed inflaton at the time of horizon exit \( \bar{\phi}(t) \simeq \bar{\phi}(t_q) \). Hence there is no additional consequence to the momentum dependence of loop spectrums.

Therefore, the spectrums are nearly scale invariant even if we add interactions of arbitrary matter and the inflaton to the interactions of matter and gravity. These results imply that we and the things around us did not come from nothing or an unknown scalar field as in conventional beliefs. Rather it points to the fact that we originated from quantum fluctuations due to the interactions between gravity and various matters during the time of Big Bang inflation.

ACKNOWLEDGMENTS

For helpful conversations I am grateful to my supervisor S. Weinberg. I also thank the referee for remarks that to a clearer presentation and for pointing out some typos. I thank A. Fassi, C. Hong, and S. Young for correcting my English. This material is based upon work supported by the National Science Foundation under Grant No. PHY-0455649.

APPENDIX: HIGHER ORDER FLUCTUATIONS

This appendix is to clarify and derive interactions of matter and gravitational fluctuations used in the loop calculations. The method of expansion and quantization shown by Weinberg [2] has a more compact form than the direct expansion of matter and gravitational fluctuations. Following his method, we can extend the calculation to other matter such as fermion, gauge, and conformal scalar fields without many difficulties. We would like to show the calculation in detail for the general reader.

In cosmological fluctuations, we generally expand the gravity and an inflaton around a time dependent background such that

\[
g_{\mu\nu}(x, t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x, t) \\
\varphi(x, t) = \bar{\varphi}(t) + \delta \varphi(x, t)
\]
When we add any other kinds of matter which have unbroken symmetries, they can be expanded as

\[ M(x, t) = 0 + \delta M(x, t) \]  

(276)

where \( M \) represents any additional matter such as fermions, vector bosons, and conformal scalar fields. The perturbation to the metric around an FRW background can always be placed in the form of

\[ ds^2 = -(1 + E)dt^2 + 2a(t)F_i dt dx^i + a^2(t)((1 + A)\delta_{ij} + B_{ij})dx^i dx^j \]  

(277)

where we only consider the scalar mode which is the subject of interest here. The gauge invariant observable quantity is defined as

\[ \zeta_q \equiv \frac{A_q}{2} - \frac{H\delta\varphi_q}{\dot{\varphi}} \]  

(278)

to linear order. We see that \( \zeta \) is a gauge invariant quantity that relates to both matter and gravitational fluctuations. There is a need for us to learn how to quantize such theories with minimum complication.

Since the inflaton and gravity are related through Einstein’s equation, we have some choices in choosing a gauge. It is found to be more convenient to choose a gauge such that the inflaton does not fluctuate [7]. Therefore, we can write down all the components of the gravitational fluctuations \( \delta g_{\mu\nu} \) in terms of a single variable \( \zeta \) by solving Einstein’s equation in the Maldacena gauge \( \delta \varphi = B = 0 \). From the gravitational field equations and the energy conservation equations[9],

\[
0 = \dot{\varphi} - HE \\
0 = H\dot{E} + 2(3H^2 + \dot{H})E - a^{-2}\nabla^2 A - \dot{A} - 6H\dot{A} + 2a^{-1}H\nabla^2 F \\
0 = \frac{1}{2}\frac{d}{dt}(E\dot{H}) - 3H\dot{H}E - a^{-1}\dot{H}\nabla^2 F + \frac{3}{2}\dot{H}\dot{A}
\]

(280)

(281)

Solving the equations above, we therefore have

\[ A = 2\zeta, E = \frac{2\zeta}{H}, F = -\frac{\zeta}{aH} + \epsilon a\nabla^{-2}\dot{\zeta} \]  

(282)

where \( \epsilon \equiv -\frac{\dot{H}}{H^2} \). By eliminating \( E \) and \( F \) yields a differential equation for \( A \):

\[ \ddot{A} + \left(3H - \frac{2\dot{H}}{H} + \frac{\ddot{H}}{H} \right)\dot{A} - \frac{\nabla^2 A}{a^2} = 0 \]  

(283)
This is sometimes known as Mukhanov equation[1], in which $A = 2\zeta$. We can write the metric and its fluctuations in terms of $\zeta$ as

$$
g_{00} = -(1 + \frac{2\zeta}{H}) = N_i N^i - N^2 \quad (284)$$

$$
g_{0i} = \partial_i \left( -\frac{\zeta}{H} + \epsilon a^2 \nabla^{-2} \dot{\zeta} \right) = N_i \quad (285)$$

$$
g_{ij} = a^2 \delta_{ij} \left( 1 + 2\dot{\zeta} \right) = h_{ij} \quad (286)$$

The determinant of the metric is

$$
\sqrt{-g} = N \sqrt{h} = a^3 \left( 1 + \frac{\dot{\zeta}}{H} \right) e^{3\zeta} \quad (287)
$$

The gravitational, inflaton, and matter actions in $\delta \varphi = 0$ gauge are

$$
L = \frac{\sqrt{-g}}{2} \left[ \dot{\varphi}^2 + 2V(\varphi) + \frac{R}{8\pi G} \right] + L_M(\chi, \bar{\psi}\psi, A_\mu A^\mu) \quad (288)
$$

where $L_M(\chi, \bar{\psi}\psi, A_\mu A^\mu)$ are the additional types of matter such as the conformal scalar, fermion, and vector bosons that do not have the background. The first three terms give vertices of purely $\zeta$. We are presently interested in the interactions of matter and gravitational fluctuations in the last term ($L_M$ term) because, in general, the matter loops are larger than the $\zeta$ loops by a factor of $8\pi G$. Therefore, the time dependent tri-linear vertices of general matter are

$$
H_{\zeta MM}(t) = -\frac{1}{2} \int d^3x a^3 T^\mu \nu \delta g_{\mu \nu} \quad (289)
$$

where $T^\mu \nu$ is the energy momentum tensor of arbitrary matter evaluated at quadratic order in fluctuations. With the Bianchi Identity,

$$
T^\mu \nu = T^\mu \nu + \Gamma^\mu \mu \lambda T^\lambda \nu + \Gamma^\nu \nu \lambda T^{\mu \lambda} = 0 \quad (290)
$$

we have

$$
\frac{1}{a^3} \frac{d}{dt} (a^3 T^00) + a\dot{a} T^{i0} + \partial_i T^{00} = 0 \quad (291)
$$

where we use $\tilde{\Gamma}_i^{i0} = 3H, \Gamma_{ij}^0 = a\dot{a} \delta_{ij}, \Gamma^0_0 = \Gamma^0_{i0} = \tilde{\Gamma}_ij = 0$ for the unperturbed FRW metric. Integrating by parts in space and using the Bianchi Identity
eq. (291), eq. (289) becomes

\begin{align*}
H_{\tilde{\zeta}MM}(t) &= \int d^3 x a^3 \left( \frac{\dot{\zeta}}{H} T_{00} + \left( \frac{\zeta}{H a^3} - \frac{\epsilon}{a} \nabla^{-2} \dot{\zeta} \right) \frac{d}{dt} \left( a^3 T_{00} \right) - \epsilon H a^2 \nabla^{-2} \dot{\zeta} (a^2 T_{ii}) \right) \\
&\quad - \epsilon H a^2 \nabla^{-2} \dot{\zeta} (a^2 T_{ii}) \right) \right) \tag{292}
\end{align*}

where the term \( a^2 \zeta T_{ii} \) is cancelled. With the Mukhanov equation

\[ \ddot{\zeta} + \left( 3H + \frac{\dot{H}}{\epsilon} \right) \dot{\zeta} - \frac{\nabla^2}{a^2} \zeta = 0 \tag{293} \]

Eq. (292) is simplified as

\[ H_{\tilde{\zeta}MM}(t) = Z(t) + \dot{Y}(t) \tag{294} \]

where

\begin{align*}
Z(t) &= - \int d^3 x \epsilon H a^5 (T_{00} + a^2 T_{ii}) \nabla^{-2} \dot{\zeta} \\
Y(t) &= a^6 T_{00} \left( \frac{\zeta}{H a^3} - \frac{\epsilon}{a} \nabla^{-2} \dot{\zeta} \right) \tag{296}
\end{align*}

The term \( \dot{Y}(t) \) can be removed by the field redefinition of \( \tilde{\zeta} \equiv \exp(-iY)\zeta \exp(iY) \) as mentioned in [2]. To see more clearly, for any interaction Hamiltonian of the the form (294), Eq. (8) can be put in the form

\[ \langle Q(t) \rangle = \sum_{N=0}^{\infty} i^N \int_{-\infty}^{t} dt_N \int_{-\infty}^{t_{N-1}} dt_{N-1} \ldots \int_{-\infty}^{t_2} dt_1 \\
\times \langle [\tilde{H}_I(t_1), [\tilde{H}_I(t_2), \ldots [\tilde{H}_I(t_N), \tilde{Q}^I(t)]]] \rangle \tag{297} \]

where

\[ \tilde{H}_I(t) = e^{iY(t)} \left[ Z(t) + \dot{Y}(t) + i e^{-iY(t)} \left( \frac{d}{dt} e^{iY(t)} \right) \right] e^{-iY(t)} \]

\[ \tilde{H}_I(t) = Z(t) + i [Y(t), Z(t)] + \frac{i}{2} [Y(t), \dot{Y}(t)] + \ldots \tag{298} \]

and

\[ \dot{Q}^I(t) = e^{iY(t)} Q^I(t) e^{-iY(t)} \]

\[ \dot{Q}^I(t) = Q^I(t) + i [Y(t), Q^I(t)] - \frac{1}{2} [Y(t), [Y(t), Q^I(t)]] + \ldots \tag{299} \]
As mentioned in [2], the redefinition of the operators is necessary. It is only products of the redefined field operators whose expectation values may be expected to give results that converge at late times. The results contributed from $Y(t)$ part only give a sum of powers of $q$ with divergent coefficients, but with no logarithmic singularity in $q$. Therefore, $\zeta$ used in the sections IV-VIII is a new redefined variable, in which we can safely calculate the contribution from the $Z(t)$ part only. The calculation in this way is more simplified than the direct expansion of the fluctuations.

REFERENCES

1. V.S. Mukhanov, H.A. Feldman, and R.H. Brandenbeger, Physics Reports 215, 203 (1992) for a review of linearized classical and quantum theory of cosmological perturbation.

2. S. Weinberg, Quantum Contributions to Cosmological Correlations Phys. Rev. D72 (2005) 043514, (hep-th/0506236)

3. S. Weinberg, Quantum Contributions to Cosmological Correlations II, Phys. Rev. D74 (2006) 023508, (hep-th/0605244)

4. J. Schwinger, Proc. Nat. Acad. Sci. US 46, 1401 (1960); J. Math. Phys. (N.Y.) 2, 407 (1961). K.T. Mahanthappa, Phys. Rev. 126, 329 (1962); P.M. Bakshi and K.T. Mahanthappa, J. Math. Phys. (N.Y.) 4, 1 (1963); 4, 12 (1963); L.V. Keldysh, Sov. Phys. JETP 20, 1018 (1965); D. Boyanovsky, D. Cormier, H.J. de Vega, R. Holman, Phys. Rev. D 57 (1997) 3373-3388; D. Boyanovsky, D. Cormier, H.J. de Vega, R. Holman, S.P. Kumar, Phys. Rev. D 57 (1998) 2166-2185; B. DeWitt, The Global Approach to Quantum Field Theory (Clarendon Press, Oxford, 2003): Sec. 31 for in-in quantum effective action. In-In formalism has been applied to cosmology by E. Calzetta and B.L. Hu, Phys. Rev. D35, 495 (1987); N.C. Tsamis and R. Woodard, Ann. Phys. (N.Y.) 238, 1 (1995); 253, 1 (1997); Phys. Lett. B426, 21 (1998); V.K. Onemli and R.P. Woodard, Phy. Rev. D 70, 107301 (2004); D. Boyanovsky, H.J. de Vega, N.G. Sanchez, Nucl. Phys. B 747 (2006)25-54 (and earlier articles by Boyanovsky et al. referred to therein); T. Brunier, V.K. Onemli, and R.P. Woodard, Classical Quantum Gravity 22, 59 (2005); but not to the problem of calculating $\zeta$ correlation functions of the curvature perturbation, in presence of Dirac, vector, and conformal scalar fields during inflation.

50
5. R. S. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current Research, ed. L. Witten (Wiley, New York, 1962):227-gr-qc/0405109, C. Misner, K. Throne, J. Wheeler, Gravitation (W H Freeman and Company, 1970), and A. Ashtekar, Lectures on Non-Perturbative Canonical Gravity, Advanced Series in Astrophysics and Cosmology-Vol 6, ed. F. Zhi and R. Ruffini (World Scientific, 1991): Chapter 9 for ADM formalism of gravity and matter.

6. C. Armendariz-Picon, Patrick B. Greene, Spinor, Inflation, and Non-Singular Cyclic Cosmologies, Gen. Rel.Grav.35(2003)1637−1658(hep-th/0301129) for the density perturbation of classical spinor

7. J. Maldacena, JHEP 0305, 013 (2003) (astro-ph/0210603) for Non-Gaussian effect of Single field infaltion.

8. A. Gangui, F. Lucchin, S. Matarrese, and S. Mollerach, Astrophys. J. 430, 447 (1994) [astro-ph/9312033]; P. Creminelli, astro-ph /0306122; P. Creminelli and M. Zaldarriaga, astro-ph/0407059 G. I. Rigopoulos, E. S. Shellard, and B. J. W. van Tent, astro-ph/0410486; F. Bernardeau, T. Brunier, and J-P. Uzan, Phys. Rev. D 69, 063520 (2004). For a review, see N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, Phys. Rep. 402, 103, astro-ph/0406398

9. S. Weinberg, Cosmology Lecture Note, Lectures given to the cosmology classes during 2004 – 2005 academic years at The University of Texas at Austin, To be officially published in 2007.

10. S. Weinberg, Gravitaion and Cosmology , (John Wiley and Sons, 1972):

11. I. S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series, and Products, (Academic Press, 1965)

12. L.H. Ford, Inflation driven by a vector field, Phys. Rev. D 40, 967 (1989).

13. K. Dimopoulos, Can a vector field be responsible for the curvature perturbation in the Universe, Phys.Rev.D74(2006)083502 (hep-ph/0607229).

14. N.D. Birrell, P. C. W. Davies, Quantum fields in curved space Cambridge University Press 1982, eq. 3.190

15. D. H. Lyth, D. Roberts, Cosmological consequences of particle creation during inflation, Phys. Rev. D 57(1998)7120−7129(hep-ph/9609441).