ON POISSON QUASI-NIJENHUIS LIE ALGEBROIDS

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Abstract. We propose a definition of Poisson quasi-Nijenhuis Lie algebroids as a natural generalization of Poisson quasi-Nijenhuis manifolds and show that any such Lie algebroid has an associated quasi-Lie bialgebroid. Therefore, also an associated Courant algebroid is obtained. We introduce the notion of a morphism of quasi-Lie bialgebroids and of the induced Courant algebroid morphism and provide some examples of Courant algebroid morphisms. Finally, we use paired operators to deform doubles of Lie and quasi-Lie bialgebroids and find an application to generalized complex geometry.

Introduction

The notion of Poisson quasi-Nijenhuis manifold was recently introduced by Stiénon and Xu [14]. It is a manifold $M$ together with a Poisson bivector field $\pi$, a $(1,1)$-tensor $N$ compatible with $\pi$ and a closed $3$-form $\phi$ such that $i_N \phi$ is also closed and the Nijenhuis torsion of $N$, which is nonzero, is expressed by means of $\phi$ and $\pi$. When $\phi = 0$ one obtains a Poisson-Nijenhuis manifold, a concept introduced by Magri and Morosi [11] to study integrable systems and which was extended to the Lie algebroid framework by Kosmann-Schwarzbach [6] and Grabowski and Urbański [5] who introduced the notion of a Poisson-Nijenhuis Lie algebroid. In this paper we propose a definition of Poisson quasi-Nijenhuis Lie algebroid, which is a straightforward generalization of a Poisson quasi-Nijenhuis manifold.

Quasi-Lie bialgebroids were introduced by Roytenberg [12] who showed that they are the natural framework to study twisted Poisson structures [13]. On the other hand, quasi-Lie bialgebroids are intimately related to Courant algebroids [9], because the double of a quasi-Lie bialgebroid carries a structure of Courant algebroid and conversely, a Courant algebroid $E$ that admits a Dirac subbundle $A$ and a transversal isotropic complement $B$, can be identified with the Whitney sum $A \oplus A^*$, where $A^*$ is identified with $B$ [12]. Generalizing a result of Kosmann-Schwarzbach [6] for Poisson-Nijenhuis manifolds and Lie bialgebroids, it is proved in [14] that a Poisson quasi-Nijenhuis structure on a manifold $M$ is equivalent to a quasi-Lie bialgebroid structure on $T^* M$. Extending the result of [14], we show that a Poisson quasi-Nijenhuis Lie algebroid has an associated quasi-Lie bialgebroid, so that it has also an associated Courant algebroid.

In an unpublished manuscript, Alekseev and Xu [1], gave the definition of a Courant algebroid morphism between $E_1$ and $E_2$ and, in the case where $E_1$ and $E_2$ are doubles of Lie bialgebroids $A, A^*$ and $B, B^*$, i.e $E_1 = A \oplus A^*$ and $E_2 = B \oplus B^*$, they established a relationship with a Lie bialgebroid morphism $A \rightarrow B$ [10]. Since doubles of quasi-Lie bialgebroids are Courant algebroids, it seems natural to obtain a relationship between Courant algebroid morphisms and quasi-Lie bialgebroid morphisms. This is the case when considering Courant algebroids associated with a Poisson quasi-Nijenhuis Lie algebroid of a certain type and with a twisted Poisson Lie algebroid, respectively. In a first step towards our result, we give the definition of a morphism of quasi-Lie bialgebroids which is, up to our
knowledge, a new concept that includes morphism of Lie bialgebroids as a particular case.

Another aspect of Poisson quasi-Nijenhuis manifolds that is exploited in [14] is the relation with generalized complex structures. We extend to Poisson quasi-Nijenhuis Lie algebroids some of the results obtained in [14] and also discuss the relation of Poisson quasi-Nijenhuis Lie algebroids with paired operators [3].

The paper is divided into three sections. In section 1 we introduce quasi-Lie bialgebroid morphisms and discuss their relationship with Courant algebroid morphisms. Section 2 is devoted to Poisson quasi-Nijenhuis Lie algebroids. We prove that each Poisson quasi-Nijenhuis Lie algebroid has an associated quasi-Lie bialgebroid and, in some particular cases, we construct a morphism of Courant algebroids. In the last section we use paired operators to deform doubles of Lie and quasi-Lie bialgebroids.

1. Quasi-Lie bialgebroids morphisms

1.1. Quasi-Lie bialgebroids. The main subject of this work are quasi-Lie bialgebroids. We begin by recalling the definition and give some examples.

**Definition 1.1.** [12] A quasi-Lie bialgebroid is a Lie algebroid \((A, [\cdot, \cdot], \rho)\) equipped with a degree-one derivation \(d_\ast\) of the Gerstenhaber algebra \((\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot])\) and a 3-section of \(A\), \(X_A \in \Gamma(\wedge^3 A)\) such that

\[d_\ast X_A = 0 \quad \text{and} \quad d_\ast^2 = [X_A, -]_A.\]

If \(X_A\) is the null section, then \(d_\ast\) defines a structure of Lie algebroid on \(A^\ast\) such that \(d_\ast\) is a derivation of \([\cdot, \cdot]_A\). In this case we say that \((A, A^\ast)\) is a Lie bialgebroid.

Examples of quasi-Lie bialgebroids arise from different well known geometric structures. We will illustrate some of them that will be needed in our work.

**Example 1.2.** Let \((A, [\cdot, \cdot], \rho)\) be a Lie algebroid and consider any closed 3-form \(\phi\). Equipping \(A^\ast\) with the null Lie algebroid structure, \((A^\ast, d_A, \phi)\) is canonically a quasi-Lie bialgebroid.

**Example 1.3.** [Lie algebroid with a twisted Poisson structure] Let \(\pi \in \Gamma(\wedge^2 A)\) be a bivector on the Lie algebroid \((A, [\cdot, \cdot], \rho)\) and denote by \(\pi^\#\) the usual bundle map

\[\pi^\# : A^\ast \to A, \quad \alpha \mapsto \pi^\#(\alpha) = i_\alpha \pi.\]

This map can be extended to a bundle map from \(\Gamma(\wedge^\bullet A^\ast)\) to \(\Gamma(\wedge^\bullet A)\), also denoted by \(\pi^\#\), as follows:

\[\pi^\#(f) = f \quad \text{and} \quad \langle \pi^\#(\mu), \alpha_1 \wedge \ldots \wedge \alpha_k \rangle = (-1)^k \mu(\pi^\#(\alpha_1), \ldots, \pi^\#(\alpha_k)),\]

for all \(f \in C^\infty(M)\) and \(\mu \in \Gamma(\wedge^k A^\ast)\) and \(\alpha_1, \ldots, \alpha_k \in \Gamma(A^\ast)\).

Let \(\phi \in \Gamma(\wedge^3 A^\ast)\) be a closed 3-form on \(A\). We say that \((\pi, \phi)\) defines a twisted Poisson structure on \(A\) [13] if

\[[\pi, \pi]_A = 2 \pi^\#(\phi).\]

In this case, the bracket on the sections of \(A^\ast\) defined by

\[[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\#\alpha}\beta - \mathcal{L}_{\pi^\#\beta}\alpha - d (\pi(\alpha, \beta)) + \phi(\pi^\#\alpha, \pi^\#\beta, -), \quad \forall \alpha, \beta \in \Gamma(A^\ast),\]

is a Lie bracket and \(A^\ast_{\pi, \phi} = (A^\ast, [\cdot, \cdot], \rho \circ \pi^\#)\) is a Lie algebroid. The differential of this Lie algebroid is given by

\[d^\#_\pi X = [\pi, X]_A - \pi^\#(i_X \phi), \quad \forall X \in \Gamma(A)\].
The pair \((A, A^*)\) is not a Lie bialgebroid but when we consider the bracket on \(\Gamma(A)\) defined by:

\[
[X, Y]' = [X, Y]_A - \pi^2(\phi(X, Y, -)), \quad \forall X, Y \in \Gamma(A),
\]

the associated differential \(d'\), given by

\[
d'f = df \quad \text{and} \quad d'\alpha = d\alpha - i_{\pi^\alpha} \phi, \quad \forall f \in C^\infty(M), \alpha \in \Gamma(A^*),
\]
defines on \(A^*_{\phi, \phi}\) a structure of quasi-Lie bialgebroid \((A^*_{\phi, \phi}, d', \phi)\).

One should notice that when \(\phi = 0, \pi\) is a Poisson bivector. The Lie algebroid \(A^*_{\pi, 0}\) is simply denoted by \(A^*_\pi\), and together with the Lie algebroid \(A\) it defines a special kind of Lie bialgebroid called a triangular Lie bialgebroid.

Any bundle map \(\Phi : A \to B\) induces a map \(\Phi^* : \Gamma(B^*) \to \Gamma(A^*)\) which assigns to each section \(\alpha \in \Gamma(B^*)\) the section \(\Phi^* \alpha\) given by

\[
\Phi^* \alpha(X)(m) = \langle \alpha(\phi(m)), \Phi_m X(m) \rangle, \quad \forall m \in M, X \in \Gamma(A),
\]

where \(\phi : M \to N\) is the map induced by \(\Phi\) on the base manifolds. We denote by the same latter \(\Phi^*\) the extension of this map to the multisectitions of \(B^*\), where we set \(\Phi^* f = f \circ \phi\), for \(f \in C^\infty(N)\).

Let \(A \to M\) and \(B \to N\) be two Lie algebroids. Recall that a Lie algebroid morphism is a bundle map \(\Phi : A \to B\) such that \(\Phi^* : (\Gamma(\wedge^* B^*), d_B) \to (\Gamma(\wedge^* A^*), d_A)\) is a chain map.

Generalizing the notion of Lie bialgebroid morphism we propose the following definition of morphism between quasi-Lie bialgebroids:

**Definition 1.4.** Let \((A, d_A^*, X_A)\) and \((B, d_B^*, X_B)\) be quasi-Lie bialgebroids over \(M\) and \(N\), respectively. A bundle map \(\Phi : A \to B\) is a quasi-Lie bialgebroid morphism if

1) \(\Phi\) is a Lie algebroid morphism;

2) \(\Phi^*\) is compatible with the brackets on the sections of \(A^*\) and \(B^*\):

\[
[\Phi^* \alpha, \Phi^* \beta]_{A^*} = \Phi^* [\alpha, \beta]_{B^*} ;
\]

3) the vector fields \(\rho_{B^*}(\alpha)\) and \(\rho_{A^*}(\Phi^* \alpha)\) are \(\phi\)-related:

\[
T\phi \cdot \rho_{A^*}(\Phi^* \alpha) = \rho_{B^*}(\alpha) \circ \phi;
\]

4) \(\Phi X_A = X_B \circ \phi\),

where \(\alpha, \beta \in \Gamma(B^*)\) and \(\phi : M \to N\) is the smooth map induced by \(\Phi\) on the base.

**Example 1.5.** A Lie bialgebroid morphism \([10]\) is a Lie algebroid morphism which is also a Poisson map, when we consider the Lie-Poisson structures induced by their dual Lie algebroids. We can easily see that in case we are dealing with Lie bialgebroids, the definition of quasi-Lie bialgebroid morphism coincides with the one of Lie bialgebroid morphism.

**Example 1.6.** Consider \((A, d_A^*, X_A)\) and \((B, d_B^*, X_B)\) two quasi-Lie bialgebroids over the same base manifold \(M\). We can see that a base preserving quasi-Lie bialgebroid morphism (such that \(\phi = \text{id}\)) is a bundle map \(\Phi : A \to B\) such that \(\Phi^* \circ d_B = d_A \circ \Phi^*\), \(\Phi \circ d_A^* = d_B^* \circ \Phi\) and \(\Phi X_A = X_B\).

Other examples of quasi-Lie bialgebroid morphisms will appear in the next section associated with quasi-Nijenhuis structures.
1.2. Courant algebroids. A Courant algebroid $E \to M$ is a vector bundle over a manifold $M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a vector bundle map $\rho : E \to TM$ and a bilinear bracket $\circ$ on $\Gamma(E)$ satisfying:

C1) $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
C2) $e \circ e = \rho^* d \langle e, e \rangle$
C3) $\mathcal{L}_{\rho(e)} \langle e_1, e_2 \rangle = \langle e \circ e_1, e_2 \rangle + \langle e_1, e \circ e_2 \rangle$
C4) $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
C5) $e_1 \circ f e_2 = f(e_1 \circ e_2) + \mathcal{L}_{\rho(e_1)} fe_2$,

for all $e, e_1, e_2, e_3 \in \Gamma(E), f \in C^\infty(M)$.

Associated with the bracket $\circ$, we can define a skew-symmetric bracket on the sections of $E$ by:

$$[e_1, e_2] = \frac{1}{2} (e_1 \circ e_2 - e_2 \circ e_1)$$

and the properties C1)-C5) can be expressed in terms of this bracket.

Example 1.7. [Standard Courant algebroid] Let $(A, [\cdot, \cdot], \rho_A)$ be a Lie algebroid. The double $A \oplus A^*$ equipped with the skew-symmetric bracket

$$[X + \alpha, Y + \beta] = [X, Y] + \left( \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) \right),$$

the pairing $\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X)$ and the anchor $\rho(X + \alpha) = \rho_A(X)$ is a Courant algebroid.

A standard Courant algebroid is a simple example of a Courant algebroid which is the double of a Lie bialgebroid. The construction of Courant algebroids as doubles of Lie bialgebroids is implicit in the next example, where we explicit the construction of the double of a quasi-Lie bialgebroid.

Example 1.8. [Double of a quasi-Lie bialgebroid] Let $(A, d_A, X_A)$ be a quasi-Lie bialgebroid. Its double $E = A \oplus A^*$ is a Courant algebroid if it is equipped with the pairing $\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X)$, the anchor $\rho = \rho_A + \rho_A^*$ and the bracket

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X^* \beta - \mathcal{L}_Y^* \alpha - \frac{1}{2} d_A(\alpha(Y) - \beta(X)) + X_A(\alpha, \beta, -) + \left( [\alpha, \beta], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) \right),$$

Taking $X_A = 0$ we have the Courant algebroid structure of a double of a Lie bialgebroid.

Another particular case that worths to be mentioned is the double of the quasi-Lie bialgebroid $(A^*, d, \phi)$ illustrated in Example 1.2. In this case the anchor is simply $\rho_E = \rho_A$ and the skew-symmetric bracket is a twisted version of the standard Courant bracket given by:

(1) $[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)) + \phi(X, Y, -)$.

1.3. Dirac structures supported on a submanifold. Dirac structures play an important role in the theory of Courant algebroids. Let us recall them before proceed.

A Dirac structure on a Courant algebroid $E$ is a subbundle $A \subset E$, which is maximal isotropic with respect to the pairing $\langle \cdot, \cdot \rangle$ and it is integrable in the sense that the space of the sections of $A$ is closed under the bracket on $\Gamma(E)$. Restricting the skew-symmetric bracket of $E$ and the anchor to $A$, we endow the Dirac structure with a Lie algebroid structure $(A, [\cdot, \cdot], \rho_{E|A})$. A Courant algebroid together with a Dirac structure is called a Manin pair.
As a way to generalize Dirac structures we have the concept of generalized Dirac structures or Dirac structures supported on a submanifold of the base manifold.

**Definition 1.9.** [1] On a Courant algebroid $E \to M$, a Dirac structure supported on a submanifold $P$ of $M$ or a generalized Dirac structure is a subbundle $F$ of $E_{|P}$ such that:

- D1) for each $x \in P$, $F_x$ is maximal isotropic;
- D2) $F$ is compatible with the anchor, i.e. $\rho_{|P}(F) \subset TP$;
- D3) For each $e_1, e_2 \in \Gamma(E)$, such that $e_1_{|P}, e_2_{|P} \in \Gamma(F)$, we have $(e_1 \circ e_2)_{|P} \in \Gamma(F)$.

Obviously, a Dirac structure supported on the whole base manifold $M$ is an usual Dirac structure of the Courant algebroid.

Generalizing the Theorem 6.11 on [1] to quasi-Lie bialgebroids we have:

**Theorem 1.10.** Let $E = A \oplus A^*$ be the double of a quasi-Lie bialgebroid $(A, d_A, Q_A)$ over the manifold $M$, $L \to P$ a vector subbundle of $A$ over a submanifold $P$ of $M$ and $F = L \oplus L^\perp$. Then $F$ is a Dirac structure supported on $P$ if and only if the following conditions hold:

1. $L$ is a Lie subalgebroid of $A$;
2. $L^\perp$ is closed for the bracket on $A^*$ defined by $d_A$;
3. $L^\perp$ is compatible with the anchor, i.e., $\rho_{A^*_{|P}}(L^\perp) \subset TP$;
4. $Q_{A_{|L^\perp}} = 0$.

**Proof.** Since $F = L \oplus L^\perp$, this is a Lagrangian subbundle of $E$. Suppose $F$ is a Dirac structure supported on $P$. By definition, we immediately deduce that $L$ is a Lie subalgebroid of $A$ and, for $\alpha, \beta$ sections of $A^*$ such that $\alpha_{|P}, \beta_{|P} \in \Gamma(L^\perp)$, we have

$$(\alpha \circ \beta)_{|P} = -Q_A(\alpha, \beta, -)_{|P} + \left[\alpha, \beta\right]_{A^*_{|P}} \in \Gamma(L \oplus L^\perp),$$

and this means that $\left[\alpha, \beta\right]_{A^*_{|P}} \in L^\perp$ and $Q_A(\alpha, \beta, -)_{|P} \in L$, or equivalently, $L^\perp$ is closed with respect to the bracket of $E$ and $Q_{A_{|L^\perp}} = 0$.

Moreover, since $F$ is compatible with the anchor,

$$\rho_{A^*_{|P}}(\alpha_{|P}) = \rho_A^*(\alpha)_{|P} = \rho_E(\alpha)_{|P} \in TP,$$

so $L^\perp$ is compatible with $\rho_A^*$.

Conversely, suppose $L$ is a Lie subalgebroid of $A$, $L^\perp \subset A^*$ is closed for $[\cdot, \cdot]_{A^*}$, $\rho_{A^*_{|P}}(L^\perp) \subset TP$ and $Q_{A_{|L^\perp}} = 0$. Obviously $F$ is compatible with the anchor. We are left to prove that $F$ is closed with respect to the bracket on $E$. Let $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$ such that $X + \alpha$ and $Y + \beta$ restricted to $P$ are sections of $F$, then

$$(X + \alpha) \circ (Y + \beta)_E = [X, Y]_A + i_\alpha d_A Y - i_\beta d_A X + d_A (\alpha(Y)) - Q_A(\alpha, \beta, -) + [\alpha, \beta]_{A^*} + \mathcal{L}_X \beta - i_Y d_A .$$

By hypothesis, we immediately have that

$$[X, Y]_{A_{|P}} = [X_{|P}, Y_{|P}]_L \in \Gamma(L),$$

$$[\alpha, \beta]_{A^*_{|P}} = [\alpha_{|P}, \beta_{|P}]_{L^\perp} \in \Gamma(L^\perp)$$

and

$$Q_A(\alpha, \beta, -)_{|P} \in \Gamma(L).$$

Now, notice that $\alpha(Y)_{|P} = 0$, so $d_A(\alpha(Y))_{|P} \in \nu^*(P) = (TP)^0$. Since $\rho_{A^*_{|P}}(L^\perp) \subset TP$, we have that

$$d_A(\alpha(Y))_{|P} = \rho_A^*(\alpha(Y))_{|P} \in \Gamma(L^\perp).$$

Analogously, $\rho_A^*(d_A(\alpha(Y)))_{|P} \in \Gamma(L^\perp)$. 


Also,
\[ d_Y (\alpha, \beta)|_P = (\rho_A^* (\alpha) \cdot \beta(Y) - \rho_B^* (\beta) \cdot \alpha(Y) - [\alpha, \beta]_{A^*} (Y))|_P = 0, \]
so \( i_Y d \beta \in \Gamma(L). \) Analogously, \( i_X d \beta \in \Gamma(L^\perp). \)

All these conditions allow us to say that \((X + \alpha) \circ (Y + \beta) \in \Gamma(L \oplus L^\perp)\) and, consequently, \(F\) is a Dirac structure supported on \(P.\)

**Corollary 1.11.** [1] Let \(E = A \oplus A^*\) be the double of a Lie bialgebroid then \(F = L \oplus L^\perp\) is a Dirac structure supported on \(P\) if and only if \(L\) and \(L^\perp\) are Lie subalgebroids of \(A\) and \(A^*.\)

Notice that when \(P = M\) we obtain Proposition 7.1 of [9].

**Corollary 1.12.** [1] Let \(E = TM \oplus T^* M\) be the standard Courant algebroid twisted by the \(\beta\)-form \(\phi \in \Omega^1(M)\) (see equation \((1)\) in Example 1.8). For any submanifold \(P \subset M\), \(F = TP \oplus \nu^* P\) is a Dirac structure supported on \(M\) if and only if \(i^* \phi = 0\), where \(i : P \rightarrow M\) is the inclusion map.

Like Lie bialgebroid morphisms, quasi-Lie bialgebroid morphisms give rise to Courant algebroid morphisms. Let us recall what is a Courant algebroid morphism.

**Definition 1.13.** [1] A Courant algebroid morphism between two Courant algebroids \(E \rightarrow M\) and \(E' \rightarrow M'\) is a Dirac structure in \(E \times E'\) supported on graph \(\phi\), where \(\phi : M \rightarrow M'\) is a smooth map and \(\overline{E'}\) denotes the Courant algebroid obtained from \(E'\) by changing the sign of the bilinear form.

**Theorem 1.14.** Let \(E_1 = A \oplus A^*\) and \(E_2 = B \oplus B^*\) be doubles of quasi-Lie bialgebroids \((A, d_A, Q_A)\) and \((B, d_B, Q_B)\) and \((\Phi, \phi) : A \rightarrow B\) a quasi-Lie bialgebroid morphism, then
\[ F = \{(a + \Phi^* b^*, \Phi a + b^*)| a \in A \text{ and } b^* \in B^* \text{ over compatible fibers}\} \subset E_1 \times \overline{E_2} \]
then \(F\) is a Dirac structure supported on graph \(\phi\), i.e. \(F\) is a Courant algebroid morphism.

**Proof.** The idea of the proof is analogous to the idea of the proof of Theorem 6.10 in [1] for Lie bialgebroid morphisms.

Consider \(M\) and \(N\) the base manifolds of \(A\) and \(B\), respectively. Consider the following subbundles over graph \(\phi\)
\[ L = \text{graph} \Phi = \{(a, \Phi a)| a \in A\} \subset A \times B \]
and
\[ L^\perp = \{(\Phi^* b^*, -b^*)| b^* \in B^*\} \subset A^* \times B^*. \]

Since \(\Phi\) is a Lie algebroid morphism, \(L\) is clearly a Lie subalgebroid of \(A \times B\). Analogously, we can also conclude that \(L^\perp\) is closed for the bracket on \(A^* \times B^*\) (where \(\overline{B^*}\) denotes the bundle \(B^*\) with bracket \([\cdot, \cdot]|_{\overline{B^*}} = -[\cdot, \cdot]_{B^*}\) and it is compatible with the anchor \(\rho_{A^* \times B^*} = (\rho_{A^*}, -\rho_{B^*})\). Also, since \(\Phi^* Q_A = Q_B \circ \phi\), we have that \((Q_A, Q_B)|_{L^\perp} = 0.\) So, Theorem 1.10 guarantees that \(L \oplus L^\perp\) is a Dirac structure supported on graph \(\phi\) of the double \(A \times B \oplus A^* \times B^*\) which is the Courant algebroid \(A \oplus A^* \times B \oplus \overline{B^*}\). Finally, observe that the bundle morphism \(b + b^* \mapsto b - b^*\) induces a canonical isomorphism between \(F\) and \(L \oplus L^\perp\) and the result follows. \(\square\)
2. Poisson Quasi-Nijenhuis Lie algebroids

Let \((A,[\cdot,\cdot],\rho)\) be a Lie algebroid over a manifold \(M\). The torsion of a bundle map \(N:A \to A\) (over the identity) is defined by

\[
T_N(X,Y) := [NX,NY] - N[X,Y], \quad X,Y \in \Gamma(A),
\]

where \([\cdot,\cdot]_N\) is given by:

\[
[X,Y]_N := [NX,Y] + [X,NY] - N[X,Y], \quad X,Y \in \Gamma(A).
\]

When \(T_N = 0\), the bundle map \(N\) is called a Nijenhuis operator, the triple \(A_N = (A,[\cdot,\cdot]_N,\rho_N = \rho \circ N)\) is a new Lie algebroid and \(N:A_N \to A\) is a Lie algebroid morphism.

Definition 2.1. On a Lie algebroid \(A\) with a Poisson structure \(\pi \in \Gamma(\wedge^2 A)\), we say that a bundle map \(N:A \to A\) is compatible with \(\pi\) if \(N\pi^\# = \pi^\# N^\ast\) and the Magri-Morosi concomitant vanishes:

\[
C(\pi,N)(\alpha,\beta) = [\alpha,\beta]_\pi N^\ast - [\alpha,\beta]_\pi N^\ast = 0,
\]

where \([\cdot,\cdot]_\pi\) is the bracket defined by the bivector field \(N\pi \in \Gamma(\wedge^2 A)\), and \([\cdot,\cdot]_\pi^\ast\) is the Lie bracket obtained from the Lie bracket \([\cdot,\cdot]_\pi\) by deformation along the tensor \(N^\ast\).

As a straightforward generalization of the definition of quasi-Poisson Nijenhuis manifolds presented in [14], we have:

Definition 2.2. A Poisson quasi-Nijenhuis Lie algebroid \((A,\pi,N,\phi)\) is a Lie algebroid \(A\) equipped with a Poisson structure \(\pi \in \Gamma(\wedge^2 A)\), a bundle map \(N:A \to A\) compatible with \(\pi\) and a closed 3-form \(\phi \in \Gamma(\wedge^3 A^\ast)\) such that

\[
T_N(X,Y) = -\pi^\# (i_X A_Y \phi) \quad \text{and} \quad d i_N \phi = 0.
\]

Theorem 2.3. If \((A,\pi,N,\phi)\) is a Poisson quasi-Nijenhuis Lie algebroid then \((A^\ast, d_N, \phi)\) is a quasi-Lie bialgebroid.

Proof. First notice that \(d\phi = 0\) and \(d i_N \phi = 0\) imply that

\[
d_N \phi = [i_N,d] \phi = i_N d\phi - d i_N \phi = 0.
\]

Secondly, we notice that since the bundle morphism \(N\) and the Poisson structure \(\pi\) are compatible, then \(d_N\) is a derivation of the Lie bracket \([\cdot,\cdot]_\pi\). In fact, first one directly sees that \(d\) is a derivation of \([\cdot,\cdot]_\pi\) and, since \(C(\pi,N)\) vanishes and

\[
d(\pi,\pi)(\alpha,\beta) = d_N [\alpha,\beta]_\pi - d_N [d_N\alpha,\beta]_\pi - [\alpha,d_N\beta]_\pi
- d [\alpha,\beta]_\pi N^\ast + [d\alpha,\beta]_\pi N^\ast + [\alpha,d\beta]_\pi N^\ast,
\]

we immediately conclude that \(d_N\) is a derivation of \([\cdot,\cdot]_\pi\) (the particular case where \(A = TM\) can be found in [6]).

It remains to prove that \(d_N^2 = [\phi,-]_\pi\). Using the definition of \(d_N\), we have:

\[
d_N^2 \alpha(X,Y,Z) = T_N(X,Y) \langle \alpha, Z \rangle - \langle \alpha, [T_N(X,Y), Z] \rangle + T_N([X,Y], Z) + c.p.
\]
The fact that \( T_N(X,Y) = -\pi^2 i_{X,Y} \phi \) yields:
\[
d^2_{N^*} \alpha(X,Y,Z) = -\phi(X,Y,\pi^4 d\langle \alpha, Z \rangle) - \langle \alpha, \mathcal{L}_Z (\pi^4 i_{X,Y} \phi) \rangle - \pi^2 i_{[X,Y] \wedge Z} \phi + \text{c.p.}
\]
\[
= -\phi(X,Y,\pi^4 d\langle \alpha, Z \rangle) - \langle \alpha, \mathcal{L}_Z \pi^4 i_{X,Y} \phi \rangle + \pi^2 \langle \mathcal{L}_Z i_{X,Y} \phi \rangle + \phi([X,Y],Z,\pi^4 \alpha) + \text{c.p.}
\]
\[
= -\phi(X,Y,\pi^4 d\langle \alpha, Z \rangle) - \langle \alpha, \mathcal{L}_Z \pi^4 i_{X,Y} \phi \rangle + \pi^2 (i_{X,Y} \mathcal{L}_Z \phi) - \pi^2 i_{[Z,X,Y]} \phi + \phi([X,Y],Z,\pi^4 \alpha) + \text{c.p.}
\]
\[
= -\phi(X,Y,\pi^4 d\langle \alpha, Z \rangle) - \phi(X,Y,(\mathcal{L}_Z \pi^4) \alpha) - \mathcal{L}_Z \phi(X,Y,\pi^4 \alpha) -\phi(X,[Z,Y] \pi^4 \alpha) + \text{c.p.}
\]
Since
\[
[\phi,\alpha]_\pi(X,Y,Z) = -\mathcal{L}_{\pi^4 (\alpha)} \phi(X,Y,Z)
\]
\[
- \{ \phi(X,Y,\pi^4 d\langle \alpha, Z \rangle) - \phi(X,Y,\mathcal{L}_Z \pi^4 (\alpha)) \} + \text{c.p.}
\]
and by hypothesis, \( \phi \) is closed, we finally have that
\[
(d^2_{N^*} \alpha - [\phi,\alpha]_\pi)(X,Y,Z) = -d\phi(X,Y,Z,\pi^4 \alpha) = 0.
\]
\[\square\]

Suppose \((A,\pi,N,\phi)\) is a Poisson quasi-Nijenhuis Lie algebroid. The double of the quasi-Lie bialgebroid \((A^*_\pi,\mathcal{d}_N,\phi)\) is a Courant algebroid (see Example 1.8) that we denote by \( E^\psi_\pi \).

An interesting case is when the 3-form \( \phi \) is the image by \( N^* \) of another closed 3-form \( \psi \):
\[
\phi = N^* \psi \quad \text{and} \quad d\psi = 0.
\]
In this case \((A,N^* \pi,\psi)\) is a twisted Poisson Lie algebroid because
\[
[N^* \pi, N^* \pi] = 2\pi^2 (\phi) = 2\pi^2 (N^* \psi) = 2N^* \pi^2 (\psi)
\]
and \( A^* \) has a structure of Lie algebroid: \( A^*_{\pi^2} = (A^*, [\cdot, \cdot]_{N^*}^\pi, N^* \pi^2) \) (see Example 1.3).

Equipping \( A \) with the differential \( d' \) given by
\[
d' f = df, \quad \text{and} \quad d' \alpha = d\alpha - i_{N^* \pi^4 \alpha} \psi,
\]
for \( f \in C^\infty(M) \) and \( \alpha \in \Gamma(A^*) \), we obtain a quasi-Lie bialgebroid: \((A^*_{\pi^2},d',\psi)\). Its double is a Courant algebroid and we denote it by \( E^\psi_{N^* \pi} \).

**Theorem 2.4.** Let \((A,\pi,N,\phi)\) be a Poisson quasi-Nijenhuis Lie algebroid and suppose that \( \phi = N^* \psi \), for some closed 3-form \( \psi \), then
\[
F = \{(a + N^* \alpha, Na + \alpha) | a \in A \text{ and } \alpha \in A^* \} \subset E^\psi_{N^* \pi} \times \overline{E^\phi_\pi}
\]
defines a Courant algebroid morphism between \( E^\psi_{N^* \pi} \) and \( E^\phi_\pi \).

In order to prove the theorem, we need to remark the following property.

**Lemma 2.5.** Let \((A,\pi,N,\phi)\) be a Poisson quasi-Nijenhuis Lie algebroid, then
\[
\langle T_{N^*} (\alpha, \beta), X \rangle = \phi(\pi^4 \alpha, \pi^4 \beta, X),
\]
for all \( X \in \Gamma(A) \) and \( \alpha, \beta \in \Gamma(A^*) \).

**Proof.** The compatibility between \( N \) and \( \pi \) implies that (see [7])
\[
\langle T_{N^*} (\alpha, \beta), X \rangle = \langle \alpha, T_N (X, \pi^4 \beta) \rangle,
\]
so
\[
\langle T_{N^*} (\alpha, \beta), X \rangle = \langle \alpha, -\pi^4 (i_{X^\pi \pi \pi \pi} \phi) \rangle = -\phi(X, \pi^4 \beta, \pi^4 \alpha) = \phi(\pi^4 \alpha, \pi^4 \beta, X).
\]
\[\square\]
Proof of the Theorem. First notice that \( N^* : A^*_{N^*} \to A^* \) is a Lie algebroid morphism because it is obviously compatible with the anchors and
\[
N^* [\alpha, \beta]_{N^*} = N^* [\alpha, \beta]_{N} + N^* \psi(\pi^2 \alpha, \pi^2 \beta, -) = [N^* \alpha, N^* \beta]_{N}.
\]

Let \([, ]'\) be the bracket on the sections of \( A \) induced by the differential \( d' \). Notice that
\[
[X, f]' = \langle d'f, X \rangle = \langle df, X \rangle,
\]
so \( N^* d' f = d_N f \), for all \( f \in C^\infty(M) \) and \( X \in \Gamma(A) \).

And since
\[
[X, Y]' = [X, Y] - (N \pi)'(\psi(X, Y, -)),
\]
we have:
\[
N [X, Y]_{N} = [NX, NY] - T_N(X, Y) = [NX, NY] + \pi^2(i_{X,Y} N^* \psi) = [NX, NY] + \psi(NX, NY, N \pi t -) = [NX, NY]',
\]
for all \( X, Y \in \Gamma(A) \).

This way we conclude that \( N^* : A^*_{N^*} \to A^* \) is a quasi-Lie bialgebroid morphism (see definition 1.4) and the result follows from Theorem 1.14.

\( \square \)

Remark 2.6. As a trivial particular case, consider \( A \) a Lie algebroid and \( \psi \) a closed 3-form. We have that \( (A^*, d, \psi) \) is quasi-Lie bialgebroid (see Example 1.2). If \( N : A \to A \) is a Nijenhuis operator, then \( A_N = (A, [\cdot, \cdot]_N, \rho \circ N) \) is a Lie algebroid and \( N : A_N \to A \) is a Lie algebroid morphism. So,
\[
d_N N^* \psi = N^* d \psi = 0
\]
and \( (A_N, \pi = 0, N, N^* \psi) \) is a quasi-Nijenhuis Lie algebroid if \( dN^* \psi = 0 \). Then \( (A^*, d_N, \phi = N^* \psi) \) is a quasi-Lie bialgebroid and \( N^* : (A^*, d, \psi) \to (A^*, d_N, N^* \psi) \) is obviously a quasi-Lie bialgebroid morphism.

In fact, we can directly and immediately see that \( (A^*, d_N, N^* \psi) \) is a quasi-Lie bialgebroid (without thinking about quasi-Nijenhuis structures). This way we avoid the condition \( dN^* \psi \) needed to prove that \( (A_N, \pi = 0, N, N^* \psi) \) is a quasi-Nijenhuis algebroid and \( N^* : (A^*, d, \psi) \to (A^*, d_N, N^* \psi) \) is obviously still a quasi-Lie bialgebroid morphism.

3. Paired operators

Let \( (A, d_A, X_A) \) be a quasi-Lie bialgebroid over \( M \) and consider a bundle map over the identity, \( \mathcal{N} : A \oplus A^* \to A \oplus A^* \). This bundle map can be written in the matrix form \( \mathcal{N} = \begin{pmatrix} N & \pi \\ \sigma & N_{A^*} \end{pmatrix} \) with \( N : A \to A, N_{A^*} : A^* \to A^*, \pi : A^* \to A \) and \( \sigma : A \to A^* \).

Definition 3.1. The operator \( \mathcal{N} \) is called paired if
\[
\langle X + \alpha, \mathcal{N}(Y + \beta) \rangle + \langle \mathcal{N}(X + \alpha), Y + \beta \rangle = 0,
\]
for all \( X + \alpha, Y + \beta \in A \oplus A^* \), where \( \langle \cdot, \cdot \rangle \) is the usual pairing on the double \( A \oplus A^* \).

As it is observed in [3], \( \mathcal{N} \) is paired if and only if \( \pi \in \Gamma(\wedge^2 A), \sigma \in \Gamma(\wedge^2 A^*) \) and \( N_{A^*} = -N^* \).
3.1. Paired operators on the double of Lie bialgebroids. Let us now take the Lie bialgebroid \((A, A^*)\), where \(A^*\) has the null Lie algebroid structure. In this case, the double \(A \oplus A^*\) is the standard Courant algebroid of Example 1.7.

Now we consider, on the sections of \(A \oplus A^*\), the bracket deformed by \(\mathcal{N}\):
\[
[X + \alpha, Y + \beta]_\mathcal{N} = [\mathcal{N}(X + \alpha), Y + \beta] + [X + \alpha, \mathcal{N}(Y + \beta)] - \mathcal{N}[X + \alpha, Y + \beta]
\]
and the Courant-Nijenhuis torsion of \(\mathcal{N}\):
\[
\mathcal{T}_\mathcal{N}(X + \alpha, Y + \beta) := \mathcal{N}(X + \alpha), \mathcal{N}(Y + \beta)] - \mathcal{N}[X + \alpha, Y + \beta]_{\mathcal{N}}.
\]

A simple computation shows that for all \(\alpha, \beta \in \Gamma(A^*)\),
\[
\llbracket \alpha, \beta \rrbracket_\mathcal{N} = [\alpha, \beta]_\pi.
\]

**Proposition 3.2.** Let \(\mathcal{N}\) be a paired operator on \(A \oplus A^*\). If \(\mathcal{T}_\mathcal{N}|_{A^*} = 0\), then the vector bundle \(A^*\) is equipped with the Lie algebroid structure \(A^*_\pi\).

**Proof.** A straightforward computation shows that
\[
\mathcal{T}_\mathcal{N}(\alpha, \beta) = 0 \Rightarrow [\pi^\# \alpha, \pi^\# \beta] = \pi^\# [\alpha, \beta]_\pi,
\]
for all sections \(\alpha\) and \(\beta\) of \(A^*\). This means that \(\pi\) is a Poisson bivector on \(A\) and the result follows. \(\square\)

Now we give sufficient conditions for a paired operator to define a Poisson quasi-Nijenhuis structure on a Lie algebroid.

**Theorem 3.3.** Let \(\mathcal{N} = \begin{pmatrix} \mathcal{N} & \pi^\# \\ \sigma & -\mathcal{N}^* \end{pmatrix} \) be a paired operator on \(A \oplus A^*\) such that
\[
\mathcal{N} \pi^\# = \pi^\# \mathcal{N}^* \quad \text{and} \quad i_{NX} \sigma = \mathcal{N}^*(i_X \sigma), \quad \forall X \in \Gamma(A).
\]
If \(\mathcal{T}_\mathcal{N}|_{A^*} = 0\) and \(\mathcal{T}_\mathcal{N}|_{A} = 0\), then \((A, \pi, N, d \sigma)\) is a Poisson quasi-Nijenhuis Lie algebroid.

**Proof.** First, notice that the condition \(i_{NX} \sigma = \mathcal{N}^*(i_X \sigma)\) means that
\[
\sigma(NX, Y) = \sigma(X, NY), \quad \forall X, Y \in \Gamma(A),
\]
and implies that \(N \sigma\) defined by \(N \sigma(X, Y) = \sigma(NX, Y)\) is a 2-form on \(M\). The condition \(\mathcal{N} \pi^\# = \pi^\# \mathcal{N}^*\) ensures that \(N \pi\) is a bivector field on \(A\).

For all \(\alpha, \beta \in \Gamma(A^*)\),
\[
\mathcal{T}_\mathcal{N}(\alpha, \beta) = 0 \iff \pi \text{ is a Poisson bivector and } [\alpha, \beta]_\mathcal{N}^\pi = [\alpha, \beta]_{\mathcal{N} \pi}.
\]
So we have that \(\pi\) and \(N\) are compatible. On the other hand, if \(X\) and \(Y\) are sections of \(A\), then
\[
\mathcal{T}_\mathcal{N}(X, Y) = 0 \iff \mathcal{T}_\mathcal{N}(X, Y) = \pi^\#(d \sigma(X, Y, -)) \quad \text{and} \quad d(N \sigma) = i_N d \sigma.
\]
According to Definition 2.2, \((A, \pi, N, d \sigma)\) is a Poisson quasi-Nijenhuis Lie algebroid. \(\square\)

From Theorem 2.3, we obtain:

**Corollary 3.4.** \((A^*_\pi, d_N, d \sigma)\) is a quasi-Lie bialgebroid.

**Remark 3.5.** We note that a paired operator \(\mathcal{N}\) that satisfies \(\mathcal{N}^2 = -\text{Id}_{A \oplus A^*}\), also satisfies
\[
\langle \mathcal{N}(X + \alpha), \mathcal{N}(Y + \beta) \rangle = \langle X + \alpha, Y + \beta \rangle, \quad \forall X + \alpha, Y + \beta \in \Gamma(A \oplus A^*).
\]
In this case \(\mathcal{N}\) defines a generalized complex structure on the Lie algebroid \(A\).

From \(\mathcal{N}^2 = -\text{Id}_{A \oplus A^*}\), we deduce that \(N \pi^\# = \pi^\# N^*, N^2 X + \pi^\#(i_X \sigma) = -X\) and \(i_{NX} \sigma = N^*(i_X \sigma)\), with \(X \in \Gamma(A)\).
Let us denote by \((A \oplus A^*)\) the vector bundle map equipped with the nondegenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle_N\) given by
\[
(X + \alpha, Y + \beta)_N = \langle N(X + \alpha), N(Y + \beta) \rangle,
\]
the bundle map \(\rho_N\) given by \(\rho_N(X + \alpha) = a(NX) + \pi^\#(\alpha)\) and the bracket \([, ]_N\) on its space of sections.

We can now establish a result that generalizes the one of \([14]\), for the case where the Lie algebroid \(A\) is \(TM\).

**Theorem 3.6.** Let \(N = \begin{pmatrix} N & \pi \\ \sigma & -N^* \end{pmatrix}\) be a paired operator on \(A \oplus A^*\) such that \(N^2 = -Id_{A \oplus A^*}\). If \(T_N|_{A^*} = 0\) and \(T_N|_A = 0\), then \((A \oplus A^*)_N\) is a Courant algebroid and it is identified with the double of the quasi-Lie bialgebroid \((A^*_N, d_N, \sigma)\).

**Proof.** From Corollary 3.4 and Remark 3.5 we have a Courant algebroid \(E_{\sigma}^d\) which is the double of the quasi-Lie bialgebroid \((A^*_N, d_N, \sigma)\). An easy computation shows that the bracket on \(\Gamma(E_{\sigma}^d)\) coincides with the bracket \([, ]_N\) on \(\Gamma((A \oplus A^*)_N)\), the anchor of \(E_{\sigma}^d\) is \(\rho_N\) and the nondegenerate bilinear form on \(E_{\sigma}^d\) is exactly \(\langle \cdot, \cdot \rangle_N\).

3.2. **Paired operators on the double of quasi-Lie bialgebroids.** Now we consider the quasi-Lie bialgebroid \((A^*, d_A, \phi)\) of Example 1.2 and the Courant algebroid structure on its double: the standard Courant bracket twisted by \(\phi, ([, ]^\phi)\), and the anchor \(\rho_A\). Let \(N = \begin{pmatrix} N & \pi \\ \sigma & -N^* \end{pmatrix}\) be a paired operator and consider the bracket on \(\Gamma(A \oplus A^*)\) deformed by \(N\):
\[
[X + \alpha, Y + \beta]^\phi_N = [N(X + \alpha), Y + \beta]^\phi + [X + \alpha, N(Y + \beta)]^\phi - N [X + \alpha, Y + \beta]^\phi.
\]

The Theorem 3.6 admits a direct extension for the case of quasi-Lie bialgebroids.

**Theorem 3.7.** Let \(N\) be a paired operator on the double \(A \oplus A^*\) of the quasi-Lie bialgebroid \((A^*, d_A, \phi)\), such that \(N^2 = -Id_{A \oplus A^*}\). If \(T_{\mathcal{N}}|_{A^*} = 0\) and \(T_{\mathcal{N}}|_A = 0\), then \((A \oplus A^*)_N\) is a Courant algebroid and it is identified with the double of the quasi-Lie bialgebroid \((A^*_N, d', \sigma + i_N \phi)\), where \(d'\) the differential given by \(d'f = d_N f\) and \(d'\alpha = d_N \alpha - i_N \pi^\#(\alpha)\), for \(f \in C^\infty(M)\) and \(\alpha \in \Gamma(A^*)\).

**Proof.** Let \(\alpha, \beta \in \Gamma(A^*)\) and \(X, Y \in \Gamma(A)\). Then,
\[
T_{\mathcal{N}}(\alpha, \beta) = 0 \iff \begin{cases} \pi \text{ is a Poisson bivector} \\ [\alpha, \beta]_{\mathcal{N}^\pi} - [\alpha, \beta]_{\mathcal{N}^\pi} = \phi(\pi^\#(\alpha), \pi^\#(\alpha), -) \end{cases}
\]
and \(T_{\mathcal{N}}(X, Y) = 0\) iff
\[
\begin{cases} T_{\mathcal{N}}(X, Y) = \pi^\#(\pi^\#(d_N \sigma)(X, Y, -)) - N \pi^\#(\phi(X, Y, -)) \\ d_N \sigma(X, Y, -) + \phi(X, Y, -) = \phi(NX, NY, -) + \phi(NX, Y, N -) + \phi(X, NY, N -) + (i_N d\sigma)(X, Y, -). \end{cases}
\]

A straightforward generalization for Lie algebroids of the results presented in \([15]\) and in \([8]\) in the case of a manifold, establishes that the four equations corresponding to \(T_{\mathcal{N}}|_{A^*} = 0\) and \(T_{\mathcal{N}}|_A = 0\) are equivalent to the vanishing of the Courant-Nijenhuis torsion of \(\mathcal{N}\) with respect to the bracket \([, ]^\phi\) on \(\Gamma(A^*)\). Therefore, we have a
new Courant algebroid structure on the vector bundle $A \oplus A^*$, $(A \oplus A^*)_{N}^\phi = (A \oplus A^*, [\cdot, \cdot]_N^\phi; \rho_N^\phi, \langle \cdot, \cdot \rangle_N^\phi)$. The restriction of the bracket $[\cdot, \cdot]_N^\phi$ to the sections of $A^*$ is the bracket $[\cdot, \cdot]_\pi$ and since $T_N^A \pi = 0$, we have that $A^*_\pi$ is a Dirac structure of the Courant algebroid $(A \oplus A^*)_{N}^\phi$. On the other hand, the restriction of the bracket $[\cdot, \cdot]_N^\phi$ to the sections of $A$ gives

$$[X, Y]_N^\phi = [X, Y]_N - \pi^\#(\phi(X, Y, -)) + d\sigma(X, Y, -) + i_N \phi(X, Y, -)$$

and the anchor $\rho_N^\phi$ restricted to $\Gamma(A)$ is $\rho_A \circ N$. If we consider the bracket $[X, Y] = [X, Y]_N - \pi^\#(\phi(X, Y, -))$ on the sections of $A$ and the bundle map $\rho_A \circ N$, the differential corresponding to this structure on $A$ is $d'$ given by,

$$d'f = d_N f \quad \text{and} \quad d'\alpha = d_N \alpha - i_{\pi^\#(\alpha)} \phi,$$

with $f \in C^\infty(M)$ and $\alpha \in \Gamma(A^*)$.

The vector bundle $A$ is obviously a transversal isotropic complement of $A^*$, so that $(A^*_\pi, d', d\sigma + i_N \phi)$ is a quasi-Lie bialgebroid [12]. Finally, a simple computation shows that the double of this quasi-Lie bialgebroid is naturally identified with the Courant algebroid $(A \oplus A^*)_{N}^\phi$. \hfill \Box

REFERENCES

[1] A. Alekseev and P. Xu, Derived brackets and Courant algebroids. *Unpublished manuscript*.
[2] H. Bursztyn, D. Iglesias Ponte and P. Ševera, Courant morphisms and moment maps. Preprint arXiv:0801.1665.
[3] J. Cariñena, J. Grabowski and G. Marmo, Courant algebroid and Lie bialgebroid contractions. *J. Phys. A: Math. Gen.* 37 (2004), 5189–5202.
[4] M. Crainic, Generalized complex structures and Lie brackets. Preprint arXiv:math/0412097.
[5] J. Grabowski and P. Urbanski, Lie algebroids and Poisson-Nijenhuis structures. *Rep. Math. Phys.* 40 (1997) 195–208.
[6] Y. Kosmann-Schwarzbach, The Lie bialgebroid of a Poisson-Nijenhuis manifold. *Lett. Math. Phys.* 38 (1996) 421–428.
[7] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Nijenhuis structures. *Ann. Inst. Henri Poincaré* 53 (1990), 35–81.
[8] U. Lindström, R. Minasian, A. Tomasiello and M. Zabzine, Generalized complex manifolds and supersymmetry. *Comm. Math. Phys.* 257 (2005), 235–256.
[9] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids. *J. Diff. Geom.* 45 (1997), no. 3, 547–574.
[10] K. Mackenzie, General theory of Lie groupoids and Lie algebroids. *London Math. Soc. Lecture notes series* 213, Cambridge University Press, Cambridge 2005.
[11] F. Magri and C. Morosi, A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds. *Quaderno* S 19, 1984, University of Milan.
[12] D. Roytenberg, Quasi-Lie bialgebroids and twisted Poisson manifolds. *Lett. Math. Phys.* 61 (2002), 123–137.
[13] P. Ševera and A. Weinstein, Poisson geometry with a 3-form background. Noncommutative geometry and string theory. *Prog. Theor. Phys.*, Suppl. 144 (2001) 145–154.
[14] M. Stiennon and P. Xu, Poisson Quasi-Nijenhuis Manifolds. *Comm. Math. Phys.* 50 (2007), 709–725.
[15] I. Vaisman, Reduction and submanifolds of generalized complex manifolds. *Diff. Geom. Appl.* 25 (2007), 147–166.