Team and Person-by-Person Optimality Conditions of Differential Decision Systems

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Abstract—In this paper, we derive team and person-by-person optimality conditions for distributed differential decision systems with different or decentralized information structures. The necessary conditions of optimality are given in terms of Hamiltonian system of equations consisting of a coupled backward and forward differential equations and a Hamiltonian projected onto the subspace generated by the decentralized information structures. Under certain global convexity conditions it is shown that the optimality conditions are also sufficient.

I. INTRODUCTION

When the system model consists of multiple decision makers, and the acquisition of information and its processing is decentralized or shared among several locations, the decision makers actions are based on different information. We call the information available for such decisions, “decentralized information structures or patterns”. When the system model is dynamic, consisting of an interconnection of at least two subsystems, and the decisions are based on decentralized information structures, we call the overall system a “distributed system with decentralized information structures”.

Over the years several specific forms of decentralized information structures are analyzed mostly in discrete-time (see, for example [1], [2], [3], [4], [5] for the most recent approaches). However, at this stage the systematic framework addressing optimality conditions for distributed systems with decentralized information structures is [6], [7], where necessary and sufficient team game optimality conditions are given for distributed stochastic dynamical systems with decentralized information structures.

In this paper, we draw the corresponding results for deterministic continuous- and discrete-time systems with decentralized information structures. More specifically, we consider a team game reward (e.g., [8], [9], [10]) and we apply concepts from the classical theory of optimization to derive necessary and sufficient optimality conditions for nonlinear distributed systems with decentralized information structures. The optimality conditions developed in this paper can be applied to many architectures of distributed systems (see, for example, Fig. 1). The specific contributions of this paper are the following.

(a) Derive team games necessary and sufficient conditions of optimality for distributed deterministic differential decision systems with decentralized information structures.
(b) Derive person-by-person optimality conditions and discuss their relation with team optimality conditions;
(c) Apply the optimality conditions to certain types of differential team games.

In Section II the notation used throughout the paper is provided, along with some background on team games and information structures that is needed for our subsequent development. In Section III we first introduce the formulation of the team and person-by-person decision problems of differential systems, and then we derive the optimality conditions. In Section IV we compute the optimal strategies for specific pay-off and differential structures and in Section V we provide the equivalent formulation for discrete-time dynamical systems.

II. NOTATION AND PRELIMINARIES

The sets of real, integer and natural numbers are denoted by \( \mathbb{R} \), \( \mathbb{Z} \) and \( \mathbb{N} \), respectively; \( \mathbb{Z}_N \triangleq \{1, 2, \ldots, N\} \) and \( \mathbb{Z}_N^0 \triangleq \{0, 1, 2, \ldots, N\} \). The Borel algebra on \([0, T]\) is denoted by \( B([0, T]) \) and the linear transformation mapping of a vector space \( X \) into a vector space \( \mathcal{Y} \) is denoted by \( \mathcal{L}(X, \mathcal{Y}) \). \( \langle a, b \rangle \) represents the inner product in \( \mathbb{R}^n \), \( \forall a, b \in \mathbb{R}^n \) for some positive integer \( n \), whereas \( |a|_{\mathbb{R}^n} \triangleq \sqrt{\langle a, a \rangle} \) is the norm on \( \mathbb{R}^n \), \( \forall a \in \mathbb{R}^n \) for some positive integer \( n \). \( \mathcal{H} = M \oplus M^\perp \) is a direct sum representation of a Hilbert space \( \mathcal{H} \), where \( M \) is a closed subspace of \( \mathcal{H} \) and \( M^\perp \) its orthogonal complement. \( \Pi_M(x) \) is the orthogonal projection of a Hilbert space element \( x \in \mathcal{H} \) onto the subspace \( M \subset \mathcal{H} \).

Our derivations will make use of the following spaces.

\[
\begin{align*}
C([0, T], \mathbb{R}^n) & \triangleq \left\{ \text{continuous functions } \phi : [0, T] \rightarrow \mathbb{R}^n : \sup_{t \in [0, T]} |\phi(t)|_{\mathbb{R}^n} < \infty \right\}; \\
B^\infty([0, T], \mathbb{R}^n) & \triangleq \left\{ \text{measurable functions } \phi : [0, T] \rightarrow \mathbb{R}^n : \|\phi\|^2 \triangleq \sup_{t \in [0, T]} |\phi(t)|^2_{\mathbb{R}^n} < \infty \right\}
\end{align*}
\]

For Lebesgue measurable func-
tions φ we have the following spaces: $L^2([0, T], \mathbb{R}^n) \triangleq \left\{ \phi : [0, T] \to \mathbb{R}^n : \int_{[0, T]} \| \phi(t) \|^2_{L^2} dt < \infty \right\}$, $L^2([0, T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \triangleq \left\{ \phi : [0, T] \to \mathbb{R}^{m \times n} : \int_{[0, T]} \| \phi(t) \|^2_{L^2(\mathbb{R}^m, \mathbb{R}^n)} dt < \infty \right\}$.

III. TEAM GAMES OF DIFFERENTIAL SYSTEMS

We first introduce the mathematical formulation of the team and person-by-person (PbP) decision problems of differential systems, and then we derive the optimality conditions. We invoke decision maker (DM) strategies which are deterministic measurable functions, also known as regular strategies.

A. Elements of Team Games

The basic elements of a team game are the state space, the observation space, the DMs actions space, and the pay-off. These are described below.

Unobserved State Space

The unobserved state space is assumed to be a linear complete separable metric space $(\mathbb{X}_{[0,T]}, d)$, where $\mathcal{B}(\mathbb{X}_{[0,T]})$ are the measurable subsets of the unobserved state space $\mathbb{X}_{[0,T]}$ generated by open sets (with respect to metric $d$).

Elements $x \in \mathbb{X}_{[0,T]}$ are the unobserved state trajectories. Since state trajectories are solutions of differential equations, an envisioned scenario is $\mathbb{X}_{[0,T]} = C(0, T, \mathbb{R}^n)$, where $\mathcal{B}(\mathbb{X}_{[0,T]}) = \mathcal{B}(C(0, T, \mathbb{R}^n))$ is the $\sigma$-field generated by cylinder sets in $C(0, T, \mathbb{R}^n)$, and a state trajectory is $x(t) : t \in [0, T] \to \mathbb{X}_{[0,T]}$. We also introduce $\sigma$-fields generated by truncations of $x \in C([0, T], \mathbb{R}^n)$. $B_i(C([0, T], \mathbb{R}^n))$ defined by

$$B_i(C([0, T], \mathbb{R}^n)) \triangleq \sigma \left\{ x \in C([0, T], \mathbb{R}^n) : x(s) \in A \right\} : 0 \leq s \leq t, \quad A \in \mathcal{B}(\mathbb{R}^n), \quad t \in [0, T].$$

Thus, $\{B_i(C([0, T], \mathbb{R}^n)) : t \in [0, T]\}$ is a family of $\sigma$-fields which is nondecreasing.

Observation Space

The observation space is assumed to be a linear complete separable metric space $(\mathbb{Y}_{[0,T]}, d')$, where $\mathcal{B}(\mathbb{Y}_{[0,T]})$ are measurable subsets generated by open sets, for $i = 1, \ldots, N$. Thus, elements $y \in \mathbb{Y}_{[0,T]}$ represent the observable trajectories. For unobserved state space $C([0, T], \mathbb{R}^n)$, the observable trajectories are generated by the maps

$$h^i : [0, T] \times C([0, T], \mathbb{R}^n) \to \mathbb{R}^{k_i}, \quad y^i(t) \triangleq h^i(t, x), \quad i = 1, \ldots, N$$

such that $\{h^i(t, x) : (t, x) \in [0, T] \times C([0, T], \mathbb{R}^{k_i})\}$ have the following property: for all $t \in [0, T]$, the map $(s, x) \to h^i(s, x)$ is $\mathcal{B}(\mathbb{R}^m) \cap B_i(C([0, T], \mathbb{R}^n))/B(\mathbb{R}^{k_i})$—measurable for $i = 1, \ldots, N$. When this property holds we say, $\{y^i(t) : t \in [0, T]\}$ is progressively measurable with respect to the family $\{B_i(C([0, T], \mathbb{R}^{k_i})) : t \in [0, T]\}$.

Often we shall assume observation trajectories which are square integrable $y^i \in L^2([0, T], \mathbb{R}^{k_i})$, and progressively measurable with respect to $\{B_i(C([0, T], \mathbb{R}^{k_i})) : t \in [0, T]\}$. Given an underlying Hilbert space $\mathcal{H}^i$ we denote by $\mathcal{H}_{0,t}^i \triangleq \text{Span} \{y^i(s) : 0 \leq s \leq t\}$ which is an element of the Hilbert space (i.e., $\mathcal{H}_{0,t}^i \subset \mathcal{H}^i$), $t \in [0, T]$, for $i = 1, \ldots, N$.

Note that the above constructions also embeds as a special case observation trajectories which are independent of $x$, by setting $y^i(t) = h^i(t), h^i : [0, T] \to \mathbb{R}^{k_i}, \forall i = 1, \ldots, N$.

Team Members

The team is assumed to consist of $N$ Decision Makers (DM) or players whose actions $\{u^i_t : t \in [0, T]\}$, take values in a closed convex set $\mathcal{A}^i$ of linear separable metric space $(\mathcal{M}^i, d^i)$, $i = 1, \ldots, N$. Unlike the centralized decision making, each DMs i actions depends only on own observation space $(\mathbb{Y}_{[0,T]}^i, d^i)$.

$$\forall u^i \in \mathcal{M}^i, \quad t \in [0, T], \quad u^i \in \mathcal{A}^i \subset \mathcal{R}^{d^i},$$

$$t \in [0, T], \quad u^i = \mathcal{B}_i(\mathbb{Y}_{[0,T]}^i : t \in [0, T])$$

and is a progressively measurable function, $\forall i \in \mathbb{Z}_N$. (3)

Clearly, $\mathcal{A}^i_{\text{reg}}[0, T]$ is a closed convex subset of $L^2([0, T], \mathbb{R}^{d^i})$, for $i = 1, 2, \ldots, N$. The set of admissible team or person-by-person strategies is denoted by $U_{\text{reg}}^N[0, T] \triangleq \times_{i=1}^N U_{\text{reg}}^i[0, T]$.

The DM actions, $\{u^i : t \in [0, T]\}$ are called:

Open Loop (OL), if $u^i_t = \mu^i(t)$, for $t \in [0, T]$, where $\mu^i : [0, T] \to \mathcal{A}^i$ are deterministic measurable functions, $i = 1, \ldots, N$;

Closed Loop Feedback (CLF), if $u^i_t = \mu^i(t, y^i)$ are nonanticipative functionals of the observation trajectory $y^i(t)$, for $t \in [0, T], \{\mu^i : [0, T] \times \mathcal{Y}^i_{[0,T]} \to \mathcal{A}^i, \mu^i \}$ are deterministic measurable mappings, $i = 1, \ldots, N$;

Closed Loop Markov (CLM), if $u^i_t = \mu^i(t, y^i(t))$, for $t \in [0, T], \mu^i : [0, T] \times \mathcal{R}^{k_i} \to \mathcal{A}^i$, are deterministic measurable functions, $i = 1, \ldots, N$.

Distributed Differential System

A distributed differential system consists of an interconnec-
Given the pay-off \(PbP\): Person-by-Person Optimality.

Define the augmented vectors by

\[
x(t) = f(t, x(t), u_t), \quad x(0) = x_0, \quad t \in (0, T], \quad i \in \mathbb{Z}_N.
\]

Payoff Functional

Consider the distributed system \((5)\) with a given admissible set of DMs strategies. Given a \(u \in \mathcal{U}^i_{reg}[0, T]\), we define the reward or performance criterion by

\[
J(u^1, u^2, \ldots, u^N) \triangleq \int_0^T \ell(t, x(t), u_t) dt + \varphi(x(T)), \quad (6)
\]

where \(\ell : [0, T] \times \mathbb{R}^n \times \mathcal{U} \rightarrow (-\infty, \infty]\) and \(\varphi : \mathbb{R}^n \rightarrow (-\infty, \infty]\). Notice that the performance of the decentralized system is measured by a single pay-off functional. The underlying assumption concerning the single pay-off instead of multiple pay-offs (one for each decision maker) is that the team objective can be met.

Team and Person-by-Person Optimality

Given the basic elements of the team game introduce above, we now introduce the definitions of team and Person-by-Person (PbP) or (player-by-player) optimality.

Problem 1: (Team and Person-by-Person Optimality)

**(T): Team Optimality.** Given the pay-off functional \((6)\), constraint \((5)\) the \(N\) tuple of strategies \(u^o = (u^1,o, u^2,o, \ldots, u^N,o) \in \mathcal{U}^o_{reg}[0, T]\) is called team optimal if it satisfies

\[
J(u^1,o, u^2,o, \ldots, u^N,o) \leq J(u^1, u^2, \ldots, u^N), \quad (7)
\]

for all \(u^1 \triangleq (u^1, u^2, \ldots, u^N) \in \mathcal{U}^i_{reg}[0, T]\). Any \(u^o \in \mathcal{U}^o_{reg}[0, T]\) satisfying \((7)\) is called an optimal regular decision strategy (or control) and the corresponding \(x^o(\cdot) \equiv x(\cdot; u^o(\cdot))\) (satisfying \((5)\)) the optimal state process.

**(PbP): Person-by-Person Optimality.** Given the pay-off functional \((6)\), constraint \((5)\) the \(N\) tuple of strategies \(u^o = (u^1,o, u^2,o, \ldots, u^N,o) \in \mathcal{U}^o_{reg}[0, T]\) is called person-by-person optimal if it satisfies

\[
\tilde{J}(u^{i,o}, u^{-i,o}) = J(u^o) \leq \tilde{J}(u^i, u^{-i,o}), \quad (8)
\]

for all \(u^i \in \mathcal{U}^i_{reg}[0, T], \; \forall i \in \mathbb{Z}_N,\) where

\[
\tilde{J}(v, u^{-i}) \triangleq J(u^1, u^2, \ldots, u^{i-1}, v, u^{i+1}, \ldots, u^N).
\]

Conditions \((8)\) are analogous to the Nash equilibrium strategies of team games consisting of a single pay-off and \(N\) DM. The rationale for the restriction to PbP optimal strategy is based on the fact that the actions of the \(N\) DM are not communicated to each other, and hence they cannot do better than restricting attention to this optimal strategy.

B. Existence of Solutions and Continuous Dependence

Herein, we study the question of existence of solutions to \((5)\) and its continuous dependence on the DM strategies based on the following assumptions.

**Assumptions 1:** The drift \(f\) associated with \((5)\) is a Borel measurable map defined by

\[
f : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}^n,
\]

and there exists a \(K \in L^{2,-}(\mathbb{R}^n, \mathbb{R})\) such that

\[
|f(t, x, u) - f(t, y, u)|_{\mathbb{R}^n} \leq K(t)|x - y|_{\mathbb{R}^n} \quad \text{ uniformly in } u \in A^{(N)};
\]

\[
|f(t, x, u) - f(t, x, v)|_{\mathbb{R}^n} \leq K(t)|u - v|_{\mathbb{R}^d} \quad \text{ uniformly in } x \in \mathbb{R}^n;
\]

\[
|f(t, x, u)|_{\mathbb{R}^n} \leq K(t)(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}).
\]

Assumptions \((1)\) are sufficient conditions for the existence of a unique \(C((0, T], \mathbb{R}^n)\) solution which is also an element of the space \(B^{\infty}((0, T], \mathbb{R}^n)\).

The following lemma establishes such results and continuous dependence of solutions on the DM strategies.

**Lemma 1:** Suppose Assumptions \((1)\) hold. Then for any \(u \in \mathcal{U}^i_{reg}[0, T]\), the following hold.

1. **System \((5)\)** has a unique solution \(x \in B^{\infty}((0, T], \mathbb{R}^n)\) which is continuous \(x \in C((0, T], \mathbb{R}^n)\).

2. **The solution of system \((5)\)** is continuously dependent on the DM strategies, in the sense that, as \(u^1,o \rightarrow u^i,o\) in \(\mathcal{U}^i_{reg}[0, T], \forall i \in \mathbb{Z}_N, \; x^o \rightarrow x^i\) in \(B^{\infty}((0, T], \mathbb{R}^n)\).

**Proof:** Similar to [6, Lemma 1].

C. Team and PbP Optimality Conditions

For the derivation of optimality conditions we shall require stronger regularity conditions for \(f\), as well as, for the running and terminal pay-offs functions \{\(\ell\), \(\varphi\)\}. These are given below.

**Assumptions 2:** The maps of \{\(f, \ell, \varphi\)\} satisfy the following conditions.

**(B1)** The map \(f : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow \mathbb{R}^n\) is continuous in \((t, x, u)\) and continuously differentiable with respect to \((x, u)\);

**(B2)** The first derivatives of \(f_x, f_u\) are bounded uniformly on \([0, T] \times \mathbb{R}^n \times A^{(N)}\).

**(B3)** The maps \(\ell : [0, T] \times \mathbb{R}^n \times A^{(N)} \rightarrow (-\infty, \infty]\) is Borel measurable, continuously differentiable with respect to \((x, u)\), the map \(\varphi : [0, T] \times \mathbb{R}^n \rightarrow (-\infty, \infty]\) is continuously differentiable with respect to \(x\), \(\ell(0, 0, t)\) is bounded, and there exist \(K_1, K_2 > 0\) such that

\[
|\ell_x(t, x, u)|_{\mathbb{R}^n} + |\ell_u(t, x, u)|_{\mathbb{R}^d} \leq K_1(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}),
\]

\[
|\varphi_x(x)|_{\mathbb{R}^n} \leq K_2(1 + |x|_{\mathbb{R}^n}).
\]
∀ \text{ convex equation and it is the unique solution of the variational differential to the state.} \\
Note that (B1), (B2) imply that |f(t, x, u)|_{\mathbb{R}^n} \leq K(1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d}), K > 0, and (B4) implies \( h_i \in \mathcal{B}^\infty([0, T], \mathbb{R}^n), i = 1, \ldots, N. \)

First, we derive necessary conditions for team and PhP optimality. For this derivation, we need the so-called variational equation. We note that for differential equations, the strategies can be either open-loop or feedback, and feedback strategies do not give smaller pay-off. Thus, the minimum pay-off attainable under open loop strategies is equal to the minimum pay-off attainable under feedback strategies. This is well known in deterministic optimal control theory. The point to be made is that when considering variations in the state trajectory the DM strategies do not react so we do not need to introduce derivatives of the u variable with respect to the state.

Suppose \( u^* \triangleq (u^{1, o}, u^{2, o}, \ldots, u^{N, o}) \in \mathcal{U}_{rel}[N] [0, T] \) denotes the optimal decision and \( u \triangleq (u^1, u^2, \ldots, u^N) \in \mathcal{U}_{rel}[N] [0, T] \) any other decision. Since \( \mathcal{A}^i \) is convex then \( \mathcal{U}_{reg}[i, T] \) is convex \( \forall i \in \mathbb{N}, \) \( \forall \varepsilon \in [0, 1], \) \( u_{i}^{t, o} + \varepsilon(u_{i}^{t} - u_{i}^{t, o}) \in \mathcal{U}_{reg}[i, T], \) \( \forall i \in \mathbb{N}. \)

Let \( x^o(\cdot) \equiv x^o(\cdot; u^o(\cdot)) \) and \( x^o(\cdot) \equiv x^o(\cdot; u^o(\cdot)) \in \mathcal{B}^\infty([0, T], \mathbb{R}^n) \) denote the solutions of the system equation \( \mathcal{C} \) corresponding to \( u^o(\cdot) \) and \( u^o(\cdot), \) respectively. Consider the limit
\[
Z(t) \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ x^o(t) - x^o(t) \right\}, \quad t \in [0, T].
\]

We have the following result characterizing the variational equation.

**Lemma 2:** Suppose Assumptions 2 hold and consider strategies \( \mathcal{U}_{rel}[N] [0, T] \). The process \( \{Z(t) : t \in [0, T]\} \) defined by (9) is an element of the Banach space \( \mathcal{B}^\infty([0, T], \mathbb{R}^n) \) and it is the unique solution of the variational differential equation
\[
\dot{Z}(t) = f_x(t, z^o(t), u_{i}^{t}) Z(t) + \sum_{i=1}^{N} f_u(t, z^o(t), u_{i}^{t})(u_{i}^{t} - u_{i}^{t, o}), \quad Z(0) = 0.
\]

having trajectories \( Z \in C([0, T], \mathbb{R}^n). \)

**Proof:** Similar to [6, Lemma 2].

Before we show the optimality conditions we define the Hamiltonian system of equations, i.e.,
\[
\mathcal{H} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{A}^{(N)} \longrightarrow \mathbb{R}
\]
given by
\[
H(t, x, \psi, u) \triangleq \langle f(t, x, u), \psi \rangle + \ell(t, x, u), \quad t \in [0, T].
\]

For any \( u \in \mathcal{U}_{reg}[N] [0, T] \), the adjoint process \( \psi \in L^2([0, T], \mathbb{R}^n) \) satisfies the following backward differential equation
\[
\dot{\psi}(t) = - f^*_x(t, x(t), u_i) \psi(t) - \ell_x(t, x(t), u_i)
\]
\[
= - H_x(t, x(t), \psi(t), u_i), \quad t \in [0, T].
\]

(12a)

\[
\psi(T) = \varphi_\mathcal{X}(x(T)).
\]

(12b)

In terms of the Hamiltonian, the state process satisfies the differential equation
\[
\dot{x}(t) = f(t, x(t), u_i) = H(x(t), \psi(t), u_i), \quad t \in [0, T],
\]
\[
x(0) = x_0.
\]

Next, we state and prove the necessary conditions for team optimality. Specifically, given that \( u^o \in \mathcal{U}_{rel}[N] [0, T] \) is team optimal, we show that it leads naturally to the Hamiltonian system of equations (called necessary conditions).

**Theorem 1 (Necessary conditions for team optimality):** Consider Problem 1 under Assumptions 2 and assume \( \mathcal{A}^i \) are closed, bounded and convex subsets of \( \mathbb{R}^d, \) and \( \{y^i(s) : 0 \leq s \leq t\} \) generates \( \mathcal{H}^{y^i}_0 \) a closed subspace of a Hilbert space for \( i = 1, \ldots, N. \)

For an element \( u^* \in \mathcal{U}_{reg}[N] [0, T] \) with the corresponding solution \( x^* \in \mathcal{B}^\infty([0, T], \mathbb{R}^n) \) to be team optimal, it is necessary that the following conditions hold.

1) There exists a process \( \psi^o \in \mathcal{L}^2([0, T], \mathbb{R}^n), \)

2) The triple \( \{u^o, x^o, \psi^o\} \) satisfy the inequality:
\[
\sum_{i=1}^{N} \int_0^T \langle H_u^i(t, x^o(t), \psi^o(t), u_{i}^{t}), u_{i}^{t} - u_{i}^{t, o} \rangle dt \geq 0,
\]
\[
\forall u \in \mathcal{U}_{reg}[N] [0, T]. \quad (13)
\]

3) The process \( \psi^o \) is the unique \( C([0, T], \mathbb{R}^n) \) solution of the backward differential equation (12a), and \( u^o \in \mathcal{U}_{reg}[N] [0, T] \) satisfies the inequalities:
\[
\{ \Pi_{\mathcal{H}^{y^i}_0} \left( H_u^i(t, x^o(t), \psi^o(t), u_{i}^{t}) \right), u_{i}^{t} - u_{i}^{t, o} \} \geq 0,
\]
\[
\forall u \in \mathcal{A}^i, i \in [0, T], i = 1, 2, \ldots, N. \quad (14)
\]

**Proof:** For 1) and 2), this is similar to [6, Theorem 6]. For 3), consider \( \{H_u^i(t, x^o(t), \psi^o(t), u_{i}^{t}) : t \in [0, T]\} \) lying in the Hilbert space of square integrable functions \( \mathcal{H}^i, i = 1, \ldots, N, \) and the set of observables \( \{y^i(t) : t \in [0, T]\} \) generating a closed subspace \( \mathcal{H}^{y^i}_0 \triangleq \text{Span}\{y^i(s) : 0 \leq s \leq t\} \subset \mathcal{H}^i, i \in \mathbb{N}. \) Then for any \( H_u^i \) we have the decomposition
\[
H_u^i(t, x^o(t), \psi^o(t), u_{i}^{t}) = \Pi_{\mathcal{H}^{y^i}_0} \left( H_u^i(t, x^o(t), \psi^o(t), u_{i}^{t}) \right) + E(t),
\]
\[
E(t) \perp \mathcal{H}^{y^i}_0, \quad t \in [0, T], i \in \mathbb{N}.
\]

Since \( u_{i}^{t} - u_{i}^{t, o} \in \mathcal{H}^{y^i}_0 \) by substituting the above decomposition in (13) we obtain
\[
\sum_{i=1}^{N} \int_0^T \langle \Pi_{\mathcal{H}^{y^i}_0} \left( H(t, x^o(t), \psi^o(t), u_{i}^{t}) \right), u_{i}^{t} - u_{i}^{t, o} \rangle dt \geq 0, \quad \forall u \in \mathcal{U}_{reg}[N] [0, T]. \quad (15)
\]
Let $t \in (0, T)$, and $\varepsilon > 0$, and consider the set $I_\varepsilon \equiv [t, t + \varepsilon] \subset [0, T]$ such that $|I_\varepsilon| \to 0$ as $\varepsilon \to 0$, for $i = 1, 2, \ldots, N$. For any $H_{0,t}^N$-progressively measurable $v^i_t \in \mathbb{A}^i$, construct

$$
    u^i_t = \begin{cases} 
        v^i_t & \text{for } t \in I_\varepsilon \\
        u^i_{t,0} & \text{otherwise} \quad i = 1, 2, \ldots, N. 
    \end{cases} \quad (16)
$$

Clearly, it follows from the above construction that $u^i_t \in U_{reg}^N[0, T]$. Substituting $16$ in $15$ we obtain the following inequality

$$
    \sum_{i=1}^N \int_{I_\varepsilon} \langle \Pi_{H_{0,t}^N} \left( H_{u^i_t}(t, x^o(t), \psi^o(t), u^o_{t,0}) \right), v^i_t - u^i_{t,0} \rangle \, dt \\
    \geq 0, \quad \forall v^i_t \in \mathbb{A}^i, \quad t \in [0, T], \quad i = 1, 2, \ldots, N. \quad (17)
$$

Letting $|I_\varepsilon|$ denote the Lebesgue measure of the set $I_\varepsilon$ and dividing the above expression by $|I_\varepsilon|$ and letting $\varepsilon \to 0$ we arrive at the following inequality.

$$
    \sum_{i=1}^N \int_{I_\varepsilon} \langle \Pi_{H_{0,t}^N} \left( H_{u^i_t}(t, x^o(t), \psi^o(t), u^o_{t,0}) \right), v^i_t - u^i_{t,0} \rangle \, dt \\
    \geq 0, \quad \forall v^i_t \in \mathbb{A}^i, \quad t \in [0, T], \quad i = 1, 2, \ldots, N. \quad (18)
$$

To complete the proof of 3) for a given $v^i_t \in \mathbb{A}^i$ (deterministic) define

$$
    g^i(t) \triangleq \langle \Pi_{H_{0,t}^N} \left( H_{u^i_t}(t, x^o(t), \psi^o(t), u^o_{t,0}) \right), v^i_t - u^i_{t,0} \rangle, \quad t \in [0, T], \quad i = 1, 2, \ldots, N. \quad (19)
$$

Then $g^i(t) \in H_{0,t}^N$. We shall show that

$$
    g^i(t) \geq 0, \quad \forall v^i_t \in \mathbb{A}^i, \quad t \in [0, T], \forall i \in \mathbb{Z}_N. \quad (20)
$$

Suppose for some $i \in \mathbb{Z}_N$, $20$ does not hold, and let $A^i \triangleq \{ t : g^i(t) < 0 \}$. Since $g^i(t) \in H_{0,t}^N, \forall t \in [0, T]$ we can choose $v^i_t$ in $18$ as

$$
    v^i_t \triangleq \begin{cases} 
        v & \text{on } A^i \\
        u^i_{t,0} & \text{outside } A^i 
    \end{cases}
$$

Together with $v^i_t = u^i_{t,0}, j \neq i, j \in \mathbb{Z}_N$. Substituting this in $15$ (with $u^i_{t,0} = v^i_t$) we arrive at $\int_{A^i} g^i(t) \, dt \geq 0$, which contradicts the definition of $A^i$, unless $A^i$ has Lebesgue measure zero. Hence, $20$ holds which is precisely $14$. This completes the derivation.

Next, we show that the necessary conditions of optimality $14$ are also sufficient under certain convexity conditions.

**Theorem 2 (Sufficient conditions for team optimality):** Consider Problem $[I]$ under the conditions of Theorem $[I]$ and let $(u^o(\cdot), x^o(\cdot))$ denote any control-state pair (decision-state) and let $\psi^o(\cdot)$ the corresponding adjoint processes. Suppose the following conditions hold:

1. $H(t, x, u) \in [0, T]$ is convex in $(x, u) \in \mathbb{R}^n \times \mathbb{A}^{(N)}$;
2. $\varphi(\cdot)$ is convex in $x \in \mathbb{R}^n$.

Then $(u^o(\cdot), x^o(\cdot))$ is team optimal if it satisfies $14$. In other words, necessary conditions are also sufficient.

**Proof:** Let $u^o \in U_{reg}^N[0, T]$ denote a candidate for the optimal team decision and $u \in U_{reg}^N[0, T]$ any other decision. Then,

$$
    J(u^o) - J(u) = \int_0^T \left( \ell(t, x^o(t), u^o_{t,0}) - \ell(t, x(t), u_t) \right) \, dt \\
    + \left( \varphi(x^o(T)) - \varphi(x(T)) \right). \quad (21)
$$

By the convexity of $\varphi(\cdot)$, we have

$$
    \varphi(x(T)) - \varphi(x^o(T)) \geq \langle \varphi_x(x^o(T)), x(T) - x^o(T) \rangle. \quad (22)
$$

Substituting $22$ into $21$ yields

$$
    J(u^o) - J(u) \leq \langle \varphi_x(x^o(T)), x(T) - x^o(T) \rangle \\
    + \int_0^T \left( \ell(t, x^o(t), u^o_{t,0}) - \ell(t, x(t), u_t) \right) \, dt. \quad (23)
$$

Applying the differential rule to $\langle \psi^o(x^o(t)), x(t) - x^o(t) \rangle$ on the interval $[0, T]$ we obtain the following equation.

$$
    \langle \psi^o(T), x(T) - x^o(T) \rangle \\
    = \langle \psi^o(0), x(0) - x^o(0) \rangle \\
    + \int_0^T \langle -f_x^o(x(t), u^o_{t,0}), \psi^o(t) \rangle \, dt \\
    - \int_0^T \ell_x(t, x^o(t), u^o_{t,0}) \, dt \\
    + \int_0^T \langle \psi^o(t), f(t, x(t), u_t) - f(t, x^o(t), u^o_{t,0}) \rangle \, dt \\
    = - \int_0^T \langle H_x(t, x^o(t), \psi^o(t), u^o_{t,0}), x(t) - x^o(t) \rangle \, dt \\
    + \int_0^T \langle \psi^o(t), f(t, x(t), u_t) - f(t, x^o(t), u^o_{t,0}) \rangle \, dt. \quad (24)
$$

Note that $\psi^o(T) = \varphi_x(x^o(T))$. Substituting $24$ into $23$ we obtain

$$
    J(u^o) - J(u) \leq \int_0^T \left( H(t, x^o(t), \psi^o(t), u^o_{t,0}) - H(t, x(t), \psi^o(t), u_t) \right) \, dt \\
    - \int_0^T \langle H_x(t, x^o(t), \psi^o(t), u^o_{t,0}), x(t) - x^o(t) \rangle \, dt. \quad (25)
$$

By hypothesis of convexity of $H$ in $(x, u) \in \mathbb{R}^n \times \mathbb{A}^{(N)}$, then $25$ reduces to

$$
    J(u^o) - J(u) \leq \sum_{i=1}^N \int_0^T \langle H_{u^i,0}(t, x^o(t), \psi^o(t), u^o_{t,0}), u^o_{i,0} - u^i_t \rangle \, dt \\
    \leq \sum_{i=1}^N \int_0^T \Pi_{H_{0,t}^N} \left( H_{u^i_t}(t, x^o(t), \psi^o(t), u^o_{t,0}) \right), u^o_{i,0} - u^i_t \rangle \, dt \\
    \leq 0, \quad \forall u \in U_{reg}^N[0, T]. \quad (26)
$$
where the last inequality follows from \( \text{I4} \). This proves that \( u^o \) optimal and hence the necessary conditions are also sufficient.

Under the conditions of Theorem \( \text{I} \) it can be shown that the necessary conditions for team optimality and PbP optimality are equivalent. Moreover, under the conditions of Theorem \( \text{I2} \) it can be shown that PbP optimality implies team optimality. We state the results as a corollary.

**Corollary 1: (Necessary and sufficient conditions for PbP optimality).** Consider the PbP optimality of Problem \( \text{I} \) under the conditions of Theorems \( \text{I1} \) and \( \text{I2} \). The necessary and sufficient conditions for PbP optimality of \((u^o(\cdot), x^o(\cdot))\) are those of team optimality given in Theorems \( \text{I1} \) and \( \text{I2} \) with the variational inequality \( \text{I3} \) replaced by

\[
\int_0^T \left\langle H_{u^o, t} \left( H_u(t, x^o(t), \psi^o(t), u^o_t) \right), u^o_t - u^o \right\rangle dt \geq 0, \\
\forall u^o \in U^{i,o}_{t, r, q}[0, T], \quad \forall i \in Z_N.
\] (27)

**Proof:** Similar to that of Theorems \( \text{I1} \) and \( \text{I2} \). 

We conclude this section by stating that the necessary optimality conditions, Pontryagin’s maximum principle are obtained following the classical theory of deterministic optimal control with centralized strategies. The only variation is the characterization of the optimal strategies described by the projection of the Hamiltonian onto the Hilbert space closed subspace generated by the observables (on which the different DM actions are based on). Consequently, we state the following observations.

(O1): By considering spike or needle variations, condition, the derivatives of \( f \) and \( \ell \) w.r.t. \( u \) can be removed and replaced by \( f, \ell, \varphi \) that are twice differentiable in \( x \in \mathbb{R}^n \), having first partial derivatives which are measurable in \( t \in [0, T] \) and continuous with respect to the rest of the arguments, and second partial derivatives which are uniformly bounded.

(O2): The team and PbP optimality conditions of Theorems \( \text{I1} \) and \( \text{I2} \) are based on the assumption that \( \mathcal{K}^i, i = 1, \ldots, N \) are convex. We can consider relaxed strategies, that is, controls which are conditional distributions, \( u^i_t(\delta t) (y^i(s) : 0 \leq s \leq t)), i = 1, \ldots, N, \) and remove the assumptions on the differentiability of \( f, \ell \) with respect to \( u \), and instead assume \( \mathcal{K}^i, i = 1, \ldots, N \) are compact subsets of finite-dimensional spaces. Based on this relaxed strategies formulation we can show existence of optimal strategies utilizing appropriate weak* topologies. Such relaxed strategies are important when the DM actions are based on a finite number of points, such as, \( \mathcal{K}^i = \{-1, +1\} \) which is not a convex set.

(O3): The team and PbP optimality conditions of Theorems \( \text{I1} \) and \( \text{I2} \) can be generalized to include pointwise and integral constraints, of \( x, u \) involving inequalities and equalities. Moreover, the terminal time can be free laying on a manifold, and hence subject to optimization rather than been fixed \( T \). Such problems are extensively investigated in the theory of optimal control. Some of these problems can be transformed into the team and PbP problems investigated earlier, by augmenting the Hamiltonian, and modifying the boundary conditions.

### IV. Examples

In this section, we give examples for two team games with special structures, namely, Generalized Normal Form (GNF) and Linear Quadratic Form (LQF).

#### A. Generalized Normal Form (GNF)

**Definition 1 (Generalized Normal Form):** The game is said to have “general normal form” if

\[
f(t, x, u) \overset{\Delta}{=} b(t, x) + g(t, x)u, \\
g(t, x)u \overset{\Delta}{=} \sum_{j=1}^N g^{(j)}(t, x)u^j, \\
\ell(t, x, u) \overset{\Delta}{=} \frac{1}{2} \langle u, R(t, x)u \rangle + \frac{1}{2} \lambda(t, x) + \langle u, \eta(t, x) \rangle,
\]

where

\[
\langle u, R(t, x)u \rangle \overset{\Delta}{=} \sum_{i=1}^N \sum_{j=1}^N u^i R_{ij}(t, x)u^j,
\]

and \( R(\cdot, \cdot) \) is symmetric uniformly positive definite, and \( \lambda(\cdot, \cdot) \) is uniformly positive semidefinite. GNF refers to the case when the drift coefficient \( f \) is linear with respect to (w.r.t.) the decision variable \( u \), and the pay-off function \( \ell \) is quadratic in \( u \), while \( f, \ell, \varphi \) are nonlinear with respect to \( x \).

By the definition of Hamiltonian \( \mathcal{H} \), its derivative is given by

\[
\mathcal{H}_u(t, x, \psi, Q, u) = g^s(t, x)\psi + R(t, x)u + \eta(t, x), \\
(t, x) \in [0, T] \times \mathbb{R}^n.
\]

By Theorem \( \text{I1} \) utilizing the fact that \( u^i_{t,o} \in \mathcal{H}_{u,t}^i \) for each \( i \in Z_N \), the explicit expression for \( u^i_{t,o} \) is given by

\[
u^i_{t,o} = - \left\{ \Pi_{\mathcal{H}^n_{u,t}} \left( R_{ij}(t, x^o(t), \psi^o(t)) \right) \right\}^{-1} \\
\left\{ \Pi_{\mathcal{H}^n_{u,t}} \left( \eta^i(t, x^o(t)) \right) \right\} \\
+ \sum_{j=1, j \neq i}^N \Pi_{\mathcal{H}^n_{u,t}} \left( R_{ij}(t, x^j(t))u^j_{t,o} \right) \\
+ \Pi_{\mathcal{H}^n_{u,t}} \left( g^{i,*}(t, x)\psi^j(t) \right), \quad i = 1, 2, \ldots, N
\]

#### B. Linear Quadratic Form (LQF)

**Definition 2 (Quadratic Form):** The game is said to have “linear quadratic form” if

\[
f(t, x, u) = A(t)x + B(t)u, \\
\ell(t, x) = \frac{1}{2} \langle u, R(t)u \rangle + \frac{1}{2} \langle x, H(t)x \rangle + \langle x, F(t) \rangle \\
+ \langle x, E(t)x \rangle + \langle u, m(t) \rangle,
\]

\[
\varphi(x) = \frac{1}{2} \langle x, M(T)x \rangle + \langle x, N(T) \rangle.
\]
and $R(\cdot)$ is symmetric uniformly positive definite, $H(\cdot)$ is symmetric uniformly positive semidefinite, and $M(T)$ is symmetric positive semidefinite.

From the optimal strategies under LQF, one obtains for $i = 1, 2, \ldots, N$:

$$u^\circ_{i,0} = - \left\{ R_{ii}(t) \right\}^{-1} \left\{ m_i(t) + \sum_{j=1}^{N} E_{ij}(t) \Pi_{H_{ii},t} \left( x^\circ_{j,0}(t) \right) \right\} + \sum_{j=1, j \neq i}^{N} R_{ij} \Pi_{H_{ji},t} \left( u^\circ_{j,0} \right) + B^{(t)} \psi^\circ(t) \Pi_{H_{ii},t} \left( \psi^\circ(t) \right).$$

Note that the previous equations can be put in the form of fixed point matrix equation.

1) Team games of Linear Quadratic Form - Explicit Expressions of Adjoint Processes: This is a necessary step before one proceeds with the computation of the explicit form of the optimal decentralized strategies, or the computation of them via fixed point methods. For a game of LQF, let $(x^\circ(\cdot), \psi^\circ(\cdot))$ denote the solutions of the Hamiltonian system, corresponding to the optimal control $u^\circ$, then

$$\frac{d}{dt} x^\circ(t) = A(t)x^\circ(t) + b(t) + B(t)u^\circ_t, \quad x^\circ(0) = x_0,$$

$$\frac{d}{dt} \psi^\circ(t) = -A^* (t) \psi^\circ(t) - H(t)x^\circ(t) - F(t) - E^* (t) u^\circ_t,$$

$$\psi^\circ(T) = M(T)x^\circ(T) + N(T).$$

Next, we find the form of the solution of the adjoint equation (30). Let $\{ \Phi(t, s) : 0 \leq s \leq t \leq T \}$ denote the transition operator of $A(\cdot)$ and $\Phi^*(\cdot, \cdot)$ that of the adjoint $A^* (\cdot)$ of $A(\cdot)$, then we have the identity $\frac{d}{ds} \Phi(t, s) = -A^* (s) \Phi^*(t, s), 0 \leq s \leq t \leq T$. One can verify by differentiation that the solution $\{ \psi^\circ(t) : t \in [0, T] \}$ of (29), is given by

$$\psi^\circ(t) = \Phi^*(T, t) M(T)x^\circ(T) + N(T) +$$

$$\int_{t}^{T} \Phi^*(s, t) \left( H(s)x^\circ(s) ds + F(s) ds + E^* (s) u^\circ_s \right) ds$$

(31)

Since for any control policy, $\{ x^\circ(s) : 0 \leq s \leq t \leq T \}$ is uniquely determined from (29) and its current value $x^\circ(t)$, then (31) can be expressed via

$$\psi^\circ(t) = \Sigma(t)x^\circ(t) + \beta^\circ(t), \quad t \in [0, T],$$

(32)

where $\Sigma(\cdot), \beta^\circ(\cdot)$ determine the operators to the one expressed via (31).

Next, we determine the operators $(\Sigma(\cdot), \beta^\circ(\cdot))$. Differentiating both sides of (32) and using (29), (30) yields

$$\dot{\Sigma}(t) + A^* (t) \Sigma(t) + \Sigma(t) A(t) + H(t) = 0,$$

$$\dot{\Sigma}(t) = M(T),$$

(33)

$$\dot{\beta^\circ}(t) + A^* (t) \beta^\circ(t) + \Sigma(t) b(t) + F(t) + \Sigma(t) B(t) u^\circ_t + E^* (t) u^\circ_t = 0,$$

$$\beta^\circ(T) = N(T).$$

(34)

Decentralized Information Structures

Here, we invoke the minimum principle to compute the optimal strategies for team games of LQF. Without loss of generality we assume the distributed dynamical decision systems consists of an interconnection of two subsystems, each governed by a linear differential equation with coupling. This can be generalized to an arbitrary number of interconnected subsystems.

Consider the distributed dynamics described below.

Subsystem Dynamics 1:

$$\frac{d}{dt} x^1(t) = A_{11}(t)x^1(t) + B_{11}(t)u^1_t + A_{12}(t)x^2(t) + B_{12}(t)u^1_t, \quad x^1(0) = x^1_0, \quad t \in (0, T],$$

(35)

Subsystem Dynamics 2:

$$\frac{d}{dt} x^2(t) = A_{22}(t)x^2(t) + B_{22}(t)u^2_t + A_{21}(t)x^1(t) + B_{21}(u^2_t, x^2(t) = x^2_0, \quad t \in (0, T].$$

(36)

Pay-off Functional:

$$J(u^1, u^2) = \frac{1}{2} \left\{ \int_{0}^{T} \left( \left( x^1(t), x^2(t)^2 \right), H(t) \left( x^1(t), x^2(t) \right) \right) \right.$$  

$$+ \left( \left( u^1_t, u^1_t \right), R(t) \left( u^1_t, u^1_t \right) \right) \right\} dt + \left( \left( x^1(T), x^2(T)^2 \right), M(T) \left( x^1(T), x^2(T) \right) \right).$$

(37)

Define the augmented variables by

$$x \triangleq \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad u \triangleq \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad \psi \triangleq \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix},$$

(38)

and matrices by

$$A \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

$$B^{(1)} \triangleq \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \quad B^{(2)} \triangleq \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}.\$$

(39)

Let $(x^\circ(\cdot), \psi^\circ(\cdot))$ denote the solutions of the Hamiltonian system, corresponding to the optimal control $u^\circ$, then

$$\frac{d}{dt} x^\circ(t) = A(t)x^\circ(t) + B(t)u^\circ_t, \quad x^\circ(0) = x_0,$$

(39a)

$$\frac{d}{dt} \psi^\circ(t) = -A^* (t) \psi^\circ(t) - H(t)x^\circ(t),$$

$$\psi^\circ(T) = M(T)x^\circ(T),$$

(39b)

$$\psi^\circ(t) = \Sigma(t)x^\circ(t) + \beta^\circ(t),$$

(39c)

where $\Sigma(\cdot), \beta^\circ(\cdot)$ are given by (33), (34) with $b, F, E = 0$. The optimal decisions $\{ (u^1_{t_1}, u^2_{t_2}) : 0 \leq t_1, t_2 \leq T \}$ are given by

$$\Pi_{H^1, t} \left( \mathcal{H}_{u^1}(t, x^1, x^2, \psi^1, \psi^2, u^1_t, u^2_t) \right) = 0, \quad t \in [0, T].$$

(40a)

$$\Pi_{H^2, t} \left( \mathcal{H}_{u^2}(t, x^1, x^2, \psi^1, \psi^2, u^1_t, u^2_t) \right) = 0, \quad t \in [0, T].$$

(40b)
From (40a), (40b) the optimal decisions for \( t \in [0, T] \) are given by
\[
\begin{align*}
\psi_k^{1.o} &= -R_{11}^{-1}(t)B^{(1),o}(t)\Pi_{H_{0,1}^{0.t}}^{1.o}(\psi(t)) \\
&\quad - R_{11}^{-1}(t)R_{12}(t)\Pi_{H_{0,1}^{0.t}}^{2.o}(u_k^{2.o}), \quad (41a)
\end{align*}
\]
\[
\begin{align*}
\psi_k^{2.o} &= -R_{22}^{-1}(t)B^{(2),o}(t)\Pi_{H_{0,1}^{0.t}}^{2.o}(\psi(t)) \\
&\quad - R_{22}^{-1}(t)R_{21}(t)\Pi_{H_{0,1}^{0.t}}^{1.o}(u_k^{1.o}). \quad (41b)
\end{align*}
\]

One can proceed further to utilize the solution for \( \psi^o(\cdot) \) to express the projections in (41a), (41b) into projections of the state \( x^o(\cdot) \) onto the subspaces \( H_{0,1}^{0.t} \), \( i = 1, 2 \), and then find the equations governing these projections. This procedure is lengthy and hence it is omitted.

V. DISCRETE-TIME DYNAMICAL SYSTEMS

By either discretizing the continuous-time system (or considering the discrete-time analog), we have the Hamiltonian at each time step \( k \) given by

\[
\mathcal{H}(k, x, \psi, u) \triangleq \langle f(k, x, u), \psi(k+1) \rangle + \ell(k, x, u),
\]

where \( k \in \mathbb{Z}_{T-1}^0 \) and \( T \) is a positive integer. Note that the adjoint is one step ahead of the other terms. In terms of the Hamiltonian, the state process satisfies the differential equation
\[
\begin{align*}
x(k+1) &= f(k, x, u_k) \\
&= \mathcal{H}_{\psi}(k, x(k), \psi(k+1), u_k), \quad k \in \mathbb{Z}_{T-1}^0 \quad (42a)
\end{align*}
\]
\[
x(0) = x_0. \quad (42b)
\]

For any \( u \in \mathcal{U}_{reg}^{(N)}[0, T] \), the adjoint process is \( \psi \in \ell^2([0, T], \mathbb{R}^n) \) satisfies the following backward differential equation
\[
\psi(k) = -H_x(k, x(k), \psi(k+1), u_k), \quad k \in \mathbb{Z}_{T-1}^0, \quad (43a)
\]
\[
\psi(T) = \varphi(x(T)). \quad (43b)
\]

The process \( \psi^o \) is the unique solution of the backward difference equation (43a), (43b) and \( u^o \in \mathcal{U}_{reg}^{(N)}[0, T] \) satisfies the inequalities:
\[
\langle \Pi_{H_{0,k}^{0.t}}(H_u(k, x^o(k), \psi^o(k+1), u^o_{k+1})), u^i - u^i_{k+1} \rangle \geq 0,
\forall u^i \in \mathcal{A}^i, \quad k \in [0, T], i = 1, 2, \ldots, N. \quad (44)
\]

VI. CONCLUSIONS AND FUTURE WORK

In this paper we have considered team games for distributed decision systems, with decentralized information patterns for each DM. Necessary and sufficient optimality conditions with respect to team optimality and PbP optimality criteria are derived, based on Pontryagin’s maximum principle. The methodology is very general, and applicable to many areas. However, several additional issues remain to be investigated. Below, we provide a short list.

(F1) The derivation of optimality conditions can be used in other type of games such as Nash-equilibrium games with decentralized information structures for each DM, and minimax games of robust control.

(F2) The methodology can be extended to deal with exogenous inputs in the state dynamics and the measurements, by assuming these belong to \( L^2 \) or \( \ell^2 \) spaces. For distributed systems of control of linear quadratic form, with decentralized information structures, one may also invoke the minimax formulation found in [11] which invokes Krein spaces, instead of Hilbert spaces.

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