Calabi-Yau manifolds from NC Hermitian $U(1)$ instantons

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ABSTRACT

We show that Calabi-Yau manifolds are emergent from the commutative limit of six-dimensional noncommutative Hermitian $U(1)$ instantons. Therefore we argue that the noncommutative Hermitian $U(1)$ instantons correspond to quantized Calabi-Yau manifolds.

Keywords: Emergent gravity, Noncommutative field theory, Gauge-gravity duality

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1 Introduction to Emergent Gravity

The emergent gravity can be described by considering the deformation of a symplectic manifold \((M, B)\) where \(B\) is a nondegenerate, closed two-form on \(M\) \([1, 2, 3, 4]\). It may be emphasized that a symplectic manifold \((M, B)\) is necessarily an even-dimensional orientable manifold since \(\nu = \frac{1}{n!} B^n\) defines a nowhere vanishing volume form where \(\dim(M) = 2n\). This fact will be important to understand the mirror symmetry of Calabi-Yau (CY) manifolds emergent from the deformation complex on a symplectic manifold, as will be discussed in a separate paper \([5]\). Let us consider a line bundle \(L\) over a symplectic manifold \((M, B)\) whose connection one-form is denoted by \(A = A_\mu(x) dx^\mu\). The curvature \(F\) of a line bundle is a closed two-form, i.e., \(dF = 0\) and so locally given by \(F = dA\). Suppose that the line bundle \(L\) over \((M, B)\) admits a local gauge symmetry \(B_L\) which acts on the connection \(A\) as well as the symplectic structure \(B\) on the base manifold \(M\):

\[
B_L : (B, A) \mapsto (B - d\Lambda, A + \Lambda)
\]  

with \(\Lambda\) an arbitrary one-form on \(M\). The local gauge symmetry \(B_L\) is known as the \(\Lambda\)-symmetry or \(B\)-field transformation in string theory. This symmetry then dictates that the curvature \(F = dA\) of \(L\) appears only with the combination \(F \equiv B + F\) since the two-form \(F\) is a gauge invariant quantity under the \(\Lambda\)-symmetry. Since \(dF = 0\), the line bundle \(L\) over \((M, B)\) results in a “dynamical” symplectic manifold \((M, F)\) if \(\det(1 + F\theta) \neq 0\) where \(\theta \equiv B^{-1}\) \([4]\). Here we mean the “dynamical” for a fluctuating field around a background. Therefore the electromagnetic force \(F = dA\) manifests itself as the deformation of a symplectic manifold \((M, B)\).

Since \(B\) is a symplectic structure on \(M\), it defines a bundle isomorphism \(B : TM \to T^*M\) by \(X \mapsto \Lambda = -\iota_X B\) where \(X \in \Gamma(TM)\) is an arbitrary vector field. As a result, the \(B\)-field transformation (1.1) can be written as

\[
B_L : (B, A) \mapsto ((1 + \mathcal{L}_X)B, A - \iota_X B)
\]  

with \(\mathcal{L}_X = d\iota_X + \iota_X d\) is the Lie derivative with respect to the vector field \(X\). Note that the ordinary \(U(1)\) gauge symmetry, \(A \mapsto A + d\lambda\), is a particular case of the \(\Lambda\)-symmetry (1.1) for \(\Lambda = d\lambda = -\iota_X B\). In this case, the vector field \(X_\lambda = -\theta(d\lambda)\) is called a Hamiltonian vector field. Since a vector field is an infinitesimal generator of local coordinate transformations, in other words, a Lie algebra generator of \(\text{Diff}(M)\), the \(B\)-field transformation (1.2) can be identified with a local coordinate transformation generated by the vector field \(X \in \Gamma(TM)\). Consequently the
\(\Lambda\)-symmetry \([1, 1]\) can be considered on par with the (dynamical) diffeomorphism symmetry. This fact leads to a remarkable conclusion \([1, 2]\) that, in the presence of \(B\)-fields, the underlying local gauge symmetry is rather enhanced. Thus we fall into a situation similar to general relativity that \(\phi \in \text{Diff}(M)\) such that \(\phi^*(F) = B\). For example, \(\phi^* = (1 + \mathcal{L}_x)^{-1} \approx e^{-\mathcal{L}_x}\) if \(A = -\Lambda = i_X B\). In other words, it is always possible to find a local coordinate transformation eliminating dynamical \(U(1)\) gauge fields as far as spacetime admits a symplectic structure. This statement is known as the Darboux theorem or the Moser lemma in symplectic geometry \([6]\). It is arguably a novel form of the equivalence principle for the electromagnetic force \([4]\). It may be rewarding to revisit the footnote \([1]\) with this insight.

Let us introduce an anchor map \(\theta = B^{-1} : T^* M \to TM\) defined by \(\Lambda \mapsto X = -\theta(\Lambda)\). The bivector \(\theta \in \Gamma(\Lambda^2 TM)\) is called a Poisson structure on \(M\) \([6]\). It gives the vector space \(C^\infty(M)\) a Lie algebra structure, called a Poisson bracket, that is antisymmetric, bilinear map \(\{\cdot, \cdot\}_\theta : C^\infty(M) \times C^\infty(M) \to C^\infty(M)\) defined by \((f, g) \mapsto \theta(df, dg) \equiv \{f, g\}_\theta\). An important property is that the map \(f \mapsto X_f(g) = \{f, g\}_\theta\) is a derivation on \(C^\infty(M)\) for any fixed \(g \in C^\infty(M)\). In terms of local coordinates on a small patch \(U \subset M\), the Poisson structure \(\theta = B^{-1}\) is given by

\[
\theta = \frac{1}{2} \theta_{ab}(y) \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b}, \quad (1.3)
\]

Without loss of generality, \(\theta_{ab}\) can be chosen to be a constant skew-symmetric matrix of rank \(2n\), typically taking the form \([\theta_{ab}] = 1_n \otimes \sqrt{-1}\sigma^2\) with \(\theta^i := \theta^{2i-1, 2i}\), \(i = 1, \cdots, n\). According to the Darboux theorem stating that \(\phi^*(B + F) = B\), it is always possible to find a locally inertial frame, namely, Darboux coordinates, to eliminate the electromagnetic force \(F = dA\). Let us represent the local coordinate transformation \(\phi \in \text{Diff}(M)\) as

\[
\phi : y^a \mapsto x^a(y) = y^a + \theta^{ab} a_b(y). \quad (1.4)
\]

The dynamical local coordinates \(a_a(y)\) will be called symplectic gauge fields, which are introduced to compensate local deformations of an underlying symplectic structure by \(U(1)\) gauge fields. The dynamical coordinates \(x^a(y)\) are covariant under a symplectic gauge transformation, i.e., \(\delta x^a(y) = -X_\lambda(x^a(y)) = \{x^a, \lambda\}_\theta(y)\) and so play an important role in emergent gravity \([7]\). It is convenient to introduce “covariant momenta” defined by

\[
D_a(y) \equiv B_{ab} x^b(y) = p_a + a_a(y) \in C^\infty(M) \quad (1.5)
\]

where \(p_a = B_{ab} y^b\). Note that

\[
\{D_a, D_b\}_\theta = -B_{ab} + f_{ab}, \quad (1.6)
\]

where \(f_{ab} = \partial_a a_b - \partial_b a_a + \{a_a, a_b\}_\theta\) is the field strength of symplectic gauge fields. One can see that symplectic gauge fields \(a_a(y) \in C^\infty(M)\) deform the background Poisson structure specified by
\( \{p_a, p_b\}_\theta = -B_{ab} \). In the end, the dynamical symplectic manifold \((M, F)\) can be described by a gauge theory of symplectic gauge fields introduced via the local coordinate transformation (1.4).

Since the symplectic manifold \((M, F)\) is a dynamical system, one may quantize the system like as quantum mechanics [4]. The quantization is straightforward as the dynamical system equips with an intrinsic Poisson structure given by (1.3). An underlying math is essentially the same as quantum mechanics. It results in a quantized (or NC) line bundle \( \hat{L} \) over a NC space [8], denoted by \( \mathbb{R}_\theta^{2n} \), whose coordinate generators satisfy the commutation relation

\[
[y^a, y^b] = i\theta^{ab}.
\] (1.7)

The NC \( \star \)-algebra generated by the Moyal-Heisenberg algebra (1.7) will be denoted by \( \mathcal{A}_\theta \) [9]. The quantization \( Q \) also lifts the coordinate transformation (1.4) to a local automorphism of \( \mathcal{A}_\theta \) defined by \( Q: \phi \mapsto D_A \) which acts on the NC coordinates \( y^a \) as [10]

\[
D_A(y^a) \equiv \hat{X}^a(y) = y^a + \theta^{ab} \hat{A}_b(y) \in \mathcal{A}_\theta.
\] (1.8)

One can see [9] that NC \( U(1) \) gauge fields are obtained by quantizing symplectic gauge fields, i.e., \( \hat{A}_a = Q(a_a) \). Let us define dynamical momentum variables \( \hat{D}_a(y) \equiv B_{ab} \hat{X}^b(y) = p_a + \hat{A}_a(y) \). Upon quantization, the Poisson bracket is similarly lifted to a NC bracket in \( \mathcal{A}_\theta \). For example, the Poisson bracket relation (1.6) is now defined by the commutation relation

\[
- i[\hat{D}_a, \hat{D}_b] = -B_{ab} + \hat{F}_{ab}
\] (1.9)

where the field strength of NC \( U(1) \) gauge fields \( \hat{A}_a \) is given by

\[
\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i[\hat{A}_a, \hat{A}_b] \star.
\] (1.10)

Hence we observe that NC \( U(1) \) gauge fields describe a dynamical NC spacetime (1.9) which is a deformation of the background NC spacetime (1.7). To sum up, a dynamical NC spacetime is defined by the quantization of a line bundle \( L \) over a symplectic manifold \((M, B)\) and described by a NC \( U(1) \) gauge theory [4].

An important point [9] is that a NC space such as the Moyal-Heisenberg algebra (1.7) always admits a nontrivial inner automorphism \( \mathfrak{A} \) defined by \( \mathcal{O} \mapsto \mathcal{O}' = U \ast \mathcal{O} \ast U^{-1} \) where \( U \in \mathfrak{A} \) and \( \mathcal{O} \in \mathcal{A}_\theta \). Its infinitesimal generators consist of an inner derivation \( \mathfrak{D} \). Then there is a well-known Lie algebra homomorphism between the NC \( \star \)-algebra \( \mathcal{A}_\theta \) and the inner derivation \( \mathfrak{D} \), defined by the map [2, 3, 4, 7]

\[
\mathcal{A}_\theta \to \mathfrak{D}: \mathcal{O} \mapsto \text{ad}_\mathcal{O} = -i[\mathcal{O}, \cdot] \star
\] (1.11)

for any \( \mathcal{O} \in \mathcal{A}_\theta \). Using the Jacobi identity of the NC \( \star \)-algebra \( \mathcal{A}_\theta \), it is easy to verify the Lie algebra homomorphism:

\[
[\text{ad}_{\mathcal{O}_1}, \text{ad}_{\mathcal{O}_2}] = -i\text{ad}_{[\mathcal{O}_1, \mathcal{O}_2]} \star.
\] (1.12)
for any $O_1, O_2 \in A_\theta$. In particular, we define the set of NC vector fields given by

$$\{ \hat{V}_a \equiv \text{ad}_{\hat{D}_a} \in D | \hat{D}_a \in A_\theta, \ a = 1, \cdots, 2n \}$$  \hspace{1cm} (1.13)$$

where $\hat{D}_a = p_a + \hat{A}_a$ are previously introduced dynamical NC momenta. One can apply the Lie algebra homomorphism (1.12) to the commutation relation (1.9) to yield

$$\text{ad}_{\hat{F}_{ab}} = [\hat{V}_a, \hat{V}_b] \in D.$$  \hspace{1cm} (1.14)

A basic idea of emergent gravity is to realize the gauge/gravity duality using the Lie algebra homomorphism (1.11) \cite{2, 3, 4, 7}. The gauge theory side of the duality is defined by a NC $U(1)$ gauge theory based on an associative algebra $A_\theta$ and its gravity side is defined by associating the derivation $D$ of the algebra $A_\theta$ with a (quantized) frame bundle of an emergent spacetime manifold $\mathcal{M}$. But, in order to identify global quantities such as vielbeins in gravity with elements of $D$, it is necessary to glue the local data of the derivation $D$ which are derived from NC gauge fields defined on Darboux charts. A detailed exposition for the globalization was recently given in Ref. \cite{4}. See also \cite{11}. For our purpose, it is enough to consider the globalization for the set of vector fields defined by Eq. (1.13). In particular, we are interested in the commutative limit, i.e. $|\theta| \to 0$, of the derivation algebra $D$. In this limit, the NC vector fields in Eq. (1.13) reduce to ordinary vector fields $V_a = V_{\mu}^a(y)\partial_{\mu} \in \Gamma(T_{\mathcal{M}})$, i.e.,

$$\hat{V}_a = V_a + O(\theta^2).$$  \hspace{1cm} (1.15)

A $2n$-dimensional manifold emergent from NC $U(1)$ gauge fields will be denoted by $\mathcal{M}$. The global vector fields $V_a = V_{\mu}^a(y)\partial_{\mu} \in \Gamma(T_{\mathcal{M}})$ are related to inverse vielbeins $E_a = E_{\mu}^a(y)\partial_{\mu} \in \Gamma(T_{\mathcal{M}})$ in general relativity as \cite{2, 3, 4, 7}

$$V_a = \lambda E_a$$  \hspace{1cm} (1.16)

where $\lambda$ is to be determined by a volume-preserving condition. We fix the conformal factor $\lambda$ by imposing the condition that the vector fields $V_a$ preserve a volume form

$$\nu = \lambda^p v^1 \wedge \cdots \wedge v^{2n}$$  \hspace{1cm} (1.17)

where $v^a = v_{\mu}^a(y)dy^\mu \in \Gamma(T^*_{\mathcal{M}})$ are coframes dual to $V_a$, i.e., $\langle v^a, V_b \rangle = \delta_b^a$. This means that the vector fields $V_a$ obey the condition

$$\mathcal{L}_{V_a} \nu = (\nabla \cdot V_a + (p - 2n)V_a \ln \lambda) \nu = 0.$$  \hspace{1cm} (1.18)

Note that any symplectic manifold always admits such volume-preserving vector fields. See the appendix B in Ref. \cite{4} for the discussion of modular vector fields on a symplectic manifold. In the end the dynamical metric emergent from NC gauge fields is given by

$$ds^2 = e^a \otimes e^a = \lambda^2 v_{\mu}^a(y)v^a_{\nu}(y)dy^\mu \otimes dy^\nu$$  \hspace{1cm} (1.19)
where $e^\alpha = e^\alpha_\mu dy^\mu = \lambda e^\alpha \in \Gamma(T^*M)$ are orthonormal one-forms on $M$ and

$$\lambda^p = \nu(V_1, \cdots, V_{2n}). \quad (1.20)$$

Since the gravitational metric (1.19) is completely determined by NC $U(1)$ gauge fields, a space-time geometry described by the metric (1.19) will be determined by the dynamical law of NC $U(1)$ gauge fields. In particular, Einstein gravity emerges from the commutative limit of NC $U(1)$ gauge fields [2, 3, 4, 7]. This emergent gravity picture may be strengthened by the fact that a theory of NC $U(1)$ gauge fields respects the diffeomorphism symmetry (1.2) and the Lie algebra homomorphism (1.11) realizes a duality between algebraic objects in $A_\theta$ and geometric objects in $D$. For instance, it was recently shown [12] that a holomorphic line bundle with a nondegenerate curvature two-form of rank $2n$ is equivalent to a $2n$-dimensional Kähler manifold and, in particular, CY $n$-folds for $n = 2$ and 3 are emergent from the commutative limit of NC $U(1)$ instantons in four and six dimensions, respectively. In this paper we will further elaborate the emergent gravity from NC $U(1)$ gauge fields and give a more elegant verification for emergent CY manifolds.

This paper is organized as follows. In Sect. 2, we derive the Hermitian Yang-Mills equations for NC $U(1)$ gauge fields [13]. In Sect. 3, we show that the commutative limit of NC Hermitian $U(1)$ instantons is isomorphically mapped to the Hermitian Yang-Mills equations for spin connections of an emergent CY manifold. In Sect. 4, we argue that the NC Hermitian $U(1)$ instantons should correspond to quantized CY manifolds which are describe by a matrix model or large $N$ gauge theory [4]. We also briefly discuss the mirror symmetry of CY manifolds from the emergent gravity picture that will be further illuminated in a separate paper [5]. In Appendix A, we present a calculational detail to verify that the self-duality equations for NC $U(1)$ instantons in four and six dimensions are transformed to geometrical equations for spin connections of an emergent CY manifold.

### 2 NC Hermitian $U(1)$ Instantons

Let $\mathcal{L} : \mathbb{R}_g^6 \rightarrow \mathbb{R}_g^6$ be a NC line bundle over a NC space $\mathbb{R}_g^6$ whose coordinates obey the commutation relation (1.7). Denote the connection of the NC line bundle $\mathcal{L}$ by $\tilde{A} = \tilde{A}_a(y)dy^a$ and its curvature $\tilde{F} = \frac{1}{2}\tilde{F}_{ab}dy^a \wedge dy^b$ is defined by [8]

$$\tilde{F} = d\tilde{A} - i\tilde{A} \wedge \tilde{A} = \frac{1}{2} \left( \partial_a \tilde{A}_b - \partial_b \tilde{A}_a - i[\tilde{A}_a, \tilde{A}_b]_\star \right) dy^a \wedge dy^b. \quad (2.1)$$

The structure group $U(1)_*$ of $\mathcal{L}$ acts on the connection as

$$\tilde{A} \mapsto \tilde{A}' = \tilde{g} \ast \tilde{A} \ast \tilde{g}^{-1} + i\tilde{g} \ast d\tilde{g}^{-1} \quad (2.2)$$

where $\tilde{g} \in U(1)_*$. In order to substantiate the idea that Riemannian manifolds emerge from NC $U(1)$ gauge fields obeying some equations, let us consider the six-dimensional NC $U(1)$ gauge theory
whose action is given by

\[ S = \frac{1}{4 G_{YM}^2} \int d^6 y \hat{F}_{ab} \hat{F}^{ab}. \] (2.3)

We will assume that the multiplication between NC fields is always the star product if it is not explicitly indicated for a notational simplicity. For example, \( \hat{f}(y) \hat{g}(y) := \hat{f}(y) \star \hat{g}(y) \) for \( \hat{f}, \hat{g} \in \mathcal{A}_\theta \). One can show \([14]\) that the action (2.3) can be written as the Bogomol’nyi form

\[ S = \frac{1}{8 G_{YM}^2} \int d^6 y \left[ \left( \hat{F}_{a_1 b_1} \pm \frac{1}{4} \varepsilon^{a_1 b_1 a_2 b_2 a_3 b_3} \hat{F}_{a_2 b_2} I_{a_3 b_3} \right)^2 - \frac{1}{2} (I_{ab} \hat{F}^{ab})^2 \right] \] (2.4)

where the constant symplectic matrix \( I_{ab} \) is given by

\[ I_{2i-1,2j} = \delta_{ij} = -I_{2j,2i-1}, \quad i, j = 1, 2, 3. \] (2.5)

The above action may be written in a more compact form as

\[ S = \frac{1}{8 G_{YM}^2} \int d^6 y \left[ (\hat{F}_{ab} \pm (\hat{F} \wedge \Omega)_{ab})^2 - \frac{1}{2} (I_{ab} \hat{F}^{ab})^2 \right] - \frac{1}{2} G_{YM}^2 \int \hat{F} \wedge \hat{F} \wedge \Omega \] (2.6)

where \( \Omega = \frac{1}{2} I_{ab} dy^a \wedge dy^b \) is the two-form of rank 6 and will be identified with the Kähler form of \( \mathbb{R}^6 \cong \mathbb{C}^3 \), i.e., \( d\Omega = 0 \). We assume that the wedge product between forms is defined under the star product, e.g., \( \hat{F} \wedge \hat{F} = \frac{1}{4} (\hat{F}_{ab} \star \hat{F}_{cd}) dy^a \wedge dy^b \wedge dy^c \wedge dy^d \). Then, using the Bianchi identity \( \hat{D}\hat{F} \equiv d\hat{F} - i(\hat{A} \wedge \hat{F} - \hat{F} \wedge \hat{A}) = 0 \), one can show that

\[ \hat{F} \wedge \hat{F} = d\hat{K} - \frac{i}{3} (\hat{C} \wedge \hat{A} + \hat{A} \wedge \hat{C}) \] (2.7)

where \( \hat{K} \) is a NC Chern-Simons term defined by

\[ \hat{K} \equiv \hat{A} \wedge \hat{F} + \frac{i}{3} \hat{A} \wedge \hat{A} \wedge \hat{A} \] (2.8)

and

\[ \hat{C} \equiv \hat{A} \wedge \hat{F} + \hat{F} \wedge \hat{A} + \frac{i}{2} \hat{A} \wedge \hat{A} \wedge \hat{A}. \] (2.9)

Note that the second term in Eq. (2.7) can be written as

\[ - \frac{i}{3} (\hat{C} \wedge \hat{A} + \hat{A} \wedge \hat{C}) = - \frac{i}{3 \cdot 3!} [\hat{C}_{\mu\nu\rho}, \hat{A}_\sigma] dy^\mu \wedge dy^\nu \wedge dy^\rho \wedge dy^\sigma \] (2.10)

and thus it vanishes under the integral thanks to the property \([9]\)

\[ \int d^6 y [\hat{f}, \hat{g}]_* = 0 \] (2.11)
for \( \hat{f}, \hat{g} \in A_\theta \). After all, the last term in Eq. (2.6) can be written as a boundary term on \( \partial \mathbb{R}^6 \cong S^5 \):

\[
\mp \frac{1}{2G_Y^2} \int_{S^5} \hat{K} \wedge \Omega. \tag{2.12}
\]

As was shown in Eq. (2.12), the last term in Eq. (2.6) is a topological term which depends only on the topological class of the vector bundle \( \hat{L} \) over \( \mathbb{R}^6 \). Then we can show that the minimum of the action (2.6) is achieved at the configuration obeying the equations

\[
\hat{F} = \mp * (\hat{F} \wedge \Omega) \tag{2.13}
\]

or, equivalently, using the fact \(*^2 \alpha = \alpha \) for any even form \( \alpha \),

\[
* \hat{F} = \mp \hat{F} \wedge \Omega. \tag{2.14}
\]

Note that the condition \( I_{ab} \hat{F}^{ab} = 0 \) needs not be imposed separately because it can be derived from Eq. (2.13) by using the identity \( \frac{1}{2} \varepsilon_{abcdef} I_{cd} I_{ef} = I_{ab} \). Therefore the term \( I_{ab} \hat{F}^{ab} \) in Eq. (2.6) identically vanishes as long as Eq. (2.13) is satisfied. It may be convenient to write the six-dimensional version of the self-duality equation (2.13) in terms of component notation:

\[
\hat{F}_{ab} = \pm \frac{1}{2} T_{cd} \hat{F}^{cd} \tag{2.15}
\]

where \( T_{cd} \equiv \frac{1}{2} \varepsilon_{abcdef} I_{ef} \). We will call Eq. (2.15) the Hermitian Yang-Mills (HYM) equations [13]. It is obvious that a solution of the HYM equations is automatically a solution of the equations of motion, \( \hat{D}^b \hat{F}_{ab} = 0 \), due to the Bianchi identity, \( \hat{D}_a \hat{F}_{bc} + \hat{D}_b \hat{F}_{ca} + \hat{D}_c \hat{F}_{ab} = 0 \).

In six dimensions, the four-form tensor \( T_{abcd} \) in Eq. (2.15) breaks the Lorentz symmetry \( SO(6) \cong SU(4)/\mathbb{Z}_2 \) to \( U(3) = SU(3) \times U(1) \). Thus it is useful to decompose the 15-dimensional vector space of two-forms \( \Lambda^2 T^* \mathcal{M} \) under the unbroken symmetry group \( U(3) \) into three subspaces [14]:

\[
\Lambda^2 T^* \mathcal{M} = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8 \tag{2.16}
\]

In general, the property (2.11) holds up to total derivative terms. Hence one may worry that the second term (2.10) may generate an additional boundary term on \( S^5 \). The asymptotic boundary condition for NC \( U(1) \) gauge fields is that \( \hat{F} \big|_{y \to \infty} \to 0 \) and so \( \hat{A} \big|_{y \to \infty} \to i \hat{g} \ast d \hat{g}^{-1} \) where \( \hat{g} \in U(1)_\ast \). Then the topological invariant (2.12) is coming from the second term in Eq. (2.8) which behaves like \( \sim \int_{S^5} (\hat{g} \ast d \hat{g}^{-1})^3 \wedge \Omega \). Note that the commutator term (2.10) contains more derivatives and thus more rapidly decays at the asymptotic boundary compared to the Chern-Simons term (2.8). As a result, total derivative terms in Eq. (2.10) do not contribute any nontrivial boundary term.

This decomposition can easily be understood by the Clifford isomorphism \( C \ell(d) = \bigoplus_{k=0}^d C \ell^k(d) \cong \Lambda^\ast \mathcal{M} = \bigoplus_{k=0}^d \Lambda^k T^* \mathcal{M} \), stating that there exists a vector space isomorphism between the Clifford algebra \( C \ell(d) \) in \( d \)-dimensions and the exterior algebra \( \Lambda^\ast \mathcal{M} \) of cotangent bundle \( T^* \mathcal{M} \) over \( \mathcal{M} \). In particular, the Clifford isomorphism implies that the Lorentz generators \( J^{ab} = \frac{i}{4} [\Gamma^a, \Gamma^b] \) in \( C \ell(d) \) are in one-to-one correspondence with two-forms in the vector space \( \Lambda^2 T^* \mathcal{M} \). Then the decomposition (2.16) is simply the branching of the vector space \( \Lambda^2 T^* \mathcal{M} \) under the symmetry reduction \( SO(6) \rightarrow U(3) = SU(3) \times U(1) \). It was argued in [14] that the Clifford isomorphism leads to an elegant picture for the mirror symmetry of CY manifolds.
where \( \Lambda_1^2 \), \( \Lambda_6^2 \), and \( \Lambda_8^2 \) are one-dimensional (singlet), six-dimensional and eight-dimensional vector spaces taking values in \( U(1) \subset U(3) \), \( SU(4)/U(3) = \mathbb{CP}^3 \), and \( SU(3) \subset U(3) \), respectively. One can show \(^{14}\) that, under the choice (2.5) for the Kähler form \( \Omega \) of \( \mathbb{R}^6 \), the NC \( U(1) \) gauge fields in \( \Lambda_6^2 \), i.e. \( \hat{F} = \frac{1}{2} \hat{F}_{ab} dy^a \wedge dy^b \in \Lambda_6^2 \), obey the (+)-equations

\[
\hat{F}_{ab} = \frac{1}{2} T_{ab}^{\ cd} \hat{F}_{cd}
\]

while, if \( \hat{F} \in \Lambda_8^2 \), they obey the (−)-equations

\[
\hat{F}_{ab} = -\frac{1}{2} T_{ab}^{\ cd} \hat{F}_{cd}.
\]

Explicitly, the (−)-equations (2.18), for example, are given by

\[
\hat{F}_{2i-1,2j-1} = \hat{F}_{2i,2j}, \quad \hat{F}_{2i-1,2j} = -\hat{F}_{2i,2j-1}, \quad i, j = 1, 2, 3, \quad (2.19)
\]

\[
\frac{1}{2} f^{ab} \hat{F}_{ab} = \hat{F}_{12} + \hat{F}_{34} + \hat{F}_{56} = 0. \quad (2.20)
\]

But note that, for the case of the (+)-equations (2.17), Eq. (2.19) has sign flips and Eq. (2.20) is replaced by \( \hat{F}_{12} = \hat{F}_{34} = \hat{F}_{56} = 0 \). Hence it consists of totally nine equations and so only six components in \( \Lambda_8^2 \) remain. For the reason to be explained later, NC Hermitian \( U(1) \) instantons or shortly NC instantons are given by the solutions of the (−)-equation (2.18) only. After all, the NC Hermitian \( U(1) \) instantons are constructed by projecting the vector space \( \Lambda^2 T^* \mathcal{M} \) into the eight-dimensional subspace \( \Lambda_8^2 \) which respects the \( SU(3) \) rotational symmetry. Hence it may be naturally expected that CY manifolds with \( SU(3) \) holonomy emerge from the NC Hermitian \( U(1) \) instantons obeying the (−)-equation (2.18) rather than the (+)-equation (2.17).

The symplectic structure (2.5) provides a natural pairing between coordinates which picks up a particular complex structure on \( \mathbb{R}^6 \). The complex coordinates specified by the symplectic matrix (2.5) are given by

\[
z^i = y^{2i-1} + \sqrt{-1} y^{2i}, \quad \bar{z}^i = y^{2i-1} - \sqrt{-1} y^{2i}, \quad i, \bar{i} = 1, 2, 3 \quad (2.21)
\]

and the corresponding NC \( U(1) \) gauge fields take the combination

\[
\hat{A}_i = \frac{1}{2} (\hat{A}_{2i-1} - \sqrt{-1} \hat{A}_{2i}), \quad \hat{A}_{\bar{i}} = \frac{1}{2} (\hat{A}_{2i-1} + \sqrt{-1} \hat{A}_{2i}). \quad (2.22)
\]

Then the field strengths of (2, 0) and (1, 1) parts are, respectively, given by

\[
\hat{F}_{ij} = \frac{1}{4} (\hat{F}_{2i-1,2j-1} - \hat{F}_{2i,2j}) - \frac{\sqrt{-1}}{4} (\hat{F}_{2i-1,2j} + \hat{F}_{2i,2j-1}), \quad (2.23)
\]

\[
\hat{F}_{ij} = \frac{1}{4} (\hat{F}_{2i-1,2j-1} + \hat{F}_{2i,2j}) + \frac{\sqrt{-1}}{4} (\hat{F}_{2i-1,2j} - \hat{F}_{2i,2j-1}). \quad (2.24)
\]

\(^4\)From now on, the imaginary unit \( i \) will be denoted by \( \sqrt{-1} \) to avoid a confusion with the frequently appearing holomorphic index \( i \).
Therefore Eq. (2.19) can be written as
\[ \hat{F}_{ij} = \hat{F}_{ij} = 0 \] (2.25)
which means that NC $U(1)$ gauge fields obeying the HYM equations (2.18) must be a connection of a NC holomorphic line bundle. The last equation (2.20) is equivalent to the condition
\[ \sum_{i=1}^{3} \hat{F}_{ii} = 0 \] (2.26)
which corresponds to the stability of the NC holomorphic line bundle [13].

For the (+)-equation (2.17), we get instead
\[ \hat{F}_{ij} = 0 \] (2.27)
and the totally six components from $\hat{F}_{ij}$ and $\hat{F}_{ij}$ survive. Therefore the NC $U(1)$ gauge fields obeying the (+)-equation (2.17) will not give rise to a NC Hermitian $U(1)$ instanton.

### 3 Emergent Calabi-Yau Manifolds

Using the Lie algebra homomorphism (1.14), one can translate the HYM equations (2.15) into some geometric equations for the vector fields determined by NC $U(1)$ gauge fields in Eq. (1.13) [2, 3, 4, 7].

For instance, the (−)-equations (2.18) are equivalently stated as
\[ [\hat{V}_a, \hat{V}_b] = -\frac{1}{2} T^{cd}_{ab} [\hat{V}_c, \hat{V}_d]. \] (3.1)

In the commutative limit (1.15), NC vector fields $\hat{V}_a$ reduce to ordinary vector fields and Eq. (3.1) in this limit is defined by the Lie bracket. Hence we introduce the Lie bracket of the vector fields $V_a \in \Gamma(TM)$ defined by
\[ [V_a, V_b] = -g_{ab} c^c V_c. \] (3.2)

The Lie algebra (3.2) means that the dual covectors $v^a \in \Gamma(T^*M)$ obey the structure equations $dv^a = \frac{1}{2} g_{bc} a^b v^c$. Using Eq. (3.2), the commutative limit of Eq. (3.1) can be written as
\[ g^{ab} e^e = -\frac{1}{2} T^{cd}_{ab} g_{cd} e. \] (3.3)

In Appendix A, we show that, for a particular choice of volume form (1.17), Eq. (3.3) can be transformed into simple equations for spin connections given by
\[ \omega_{ab} = -\frac{1}{2} T^{cd}_{ab} \omega_{cd}. \] (3.4)

Now we will show that a six-dimensional manifold obeying Eq. (3.4) must be a Ricci-flat, Kähler manifold which is called a CY manifold [15, 16]. Therefore we come to a conclusion that CY manifolds are emergent from the commutative limit of six-dimensional NC Hermitian $U(1)$ instantons.
Suppose that $\mathcal{M}$ is a six-dimensional complex manifold whose metric $ds^2 = g_{\mu\nu} dy^\mu \otimes dy^\nu$ is given by Eq. (1.19). Let us choose local complex coordinates $y^\mu = (z^\alpha, \bar{z}^\beta)$, $\alpha = 1, 2, 3$ in which an almost complex structure takes the form $J^\alpha_{\beta} = \sqrt{-1} \delta^\alpha_{\beta}$, $J^\bar{\alpha}_{\bar{\beta}} = -\sqrt{-1} \delta^{\bar{\alpha}}_{\bar{\beta}}$. This complex structure is basically inherited from the symplectic structure $J = I_3 \otimes \sqrt{-1} \sigma^2$ which we already introduced in Eq. (2.5). We have split a curved space index $\mu = 1, \cdots, 6 = (\alpha, \bar{\alpha})$ into a holomorphic index $\alpha = 1, 2, 3$ and an anti-holomorphic one $\bar{\alpha} = 1, 2, 3$ and similarly, a tangent space index $a = 1, \cdots, 6 = (i, \bar{i})$ into $i = 1, 2, 3$ and $\bar{i} = 1, 2, 3$. We impose the Hermitian condition on the complex manifold $\mathcal{M}$ defined by $g(X, Y) = g(JX, JY)$ for any $X, Y \in \Gamma(T\mathcal{M})$ [16]. This means that the metric $g$ on the complex manifold $\mathcal{M}$ is a Hermitian metric, i.e., $g_{\alpha\bar{\beta}} = g_{\bar{\alpha}\beta} = 0$, $g_{\alpha\bar{\beta}} = g_{\bar{\alpha}\beta}$. The Hermitian condition can be solved by taking the vielbeins as

$$e^i_\alpha = e^\bar{i}_{\bar{\alpha}} = 0 \quad \text{and} \quad E^i_\alpha = E^{\bar{i}}_{\bar{\alpha}} = 0. \quad (3.5)$$

Then the two-form defined by $\Omega = \frac{1}{2} I_{ab} e^a \wedge e^b$ is a Kähler form, i.e., $\Omega(X, Y) = g(X, JY)$ and it is given by

$$\Omega = -\sqrt{-1} e^i \wedge e^{\bar{i}} = -\sqrt{-1} e^i_\alpha e^{\bar{i}}_{\bar{\alpha}} dz^\alpha \wedge d\bar{z}^{\bar{\alpha}} = -\sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}. \quad (3.6)$$

Using the torsion free condition, $de^a + \omega^a_{\ b} \wedge e^b = 0$, it is easy to show [15] that the Kähler condition, $d\Omega = 0$, is equivalent to the one that the spin connection $\omega_{ab}$ on a Hermitian manifold $(\mathcal{M}, g)$ is $U(3)$-valued, i.e.,

$$\omega_{ij} = \omega_{\bar{i}\bar{j}} = 0. \quad (3.7)$$

In this case, the Kähler metric is solely determined by a Kähler potential $K(z, \bar{z})$ as

$$g_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_{\bar{\beta}} K(z, \bar{z}). \quad (3.8)$$

It is well-known [15] [16] that the Ricci tensor of a Kähler manifold is the field strength of the $U(1)$ part of $U(3) = SU(3) \times U(1)$ spin connections and the $U(1)$ gauge field is given by the trace part of $U(3)$ spin connections, i.e.,

$$A^{(0)} \equiv \sqrt{-1} \sum_{i=1}^{3} \omega_{ii}. \quad (3.9)$$

Therefore a Kähler manifold $(\mathcal{M}, g)$ is Ricci-flat if $F^{(0)} = dA^{(0)} = 0$ or $A^{(0)} = d\lambda$. Note that the first cohomology for a simply connected manifold $\mathcal{M}$ identically vanishes, i.e., $H^1(\mathcal{M}) = 0$. Hence it is possible to choose a gauge, $A^{(0)} = 0$, for a simply connected Ricci-flat manifold which means that

$$\sum_{i=1}^{3} \omega_{ii} = 0. \quad (3.10)$$

A Ricci-flat and Kähler manifold is known as a CY manifold which plays an important role in string theory compactification [15]. Consequently a CY manifold with $SU(3)$ holonomy is characterized, up to a gauge choice, by Eqs. (3.7) and (3.10). In terms of real coordinates, they are succinctly
summarized by Eq. (3.4). In Appendix A, we show that the (generalized) self-duality equation (3.4) for spin connections is equivalent to the commutative limit of NC Hermitian $U(1)$ instantons defined by the Hermitian Yang-Mills equation (2.18) in six dimensions. Therefore we see that the commutative limit of six-dimensional NC Hermitian $U(1)$ instantons can be rephrased into the Ricci-flat and Kähler condition for Calabi-Yau manifolds.

4 Discussion

Suppose that $\mathcal{F} = dA = B + F$ is the curvature of a holomorphic line bundle, i.e., $\mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i = 0$ and $\mathcal{F}_{\bar{i}\bar{j}} = \overline{\partial}_i A_{\bar{j}} - \overline{\partial}_j A_{\bar{i}} = 0$. It can be solved by $A_i = \overline{\partial}_i \phi(z, \bar{z})$ and $A_{\bar{i}} = -\overline{\partial}_i \phi(z, \bar{z})$ where $\phi(z, \bar{z})$ is a real smooth function on $\mathbb{C}^n$. Then the curvature of a holomorphic line bundle is given by

$$\mathcal{F} = -\overline{\partial}_i \overline{\partial}_j \phi(z, \bar{z}) dz^i \wedge d\bar{z}^j = -\overline{\partial}_i \overline{\partial}_j \phi(z, \bar{z}). \quad (4.1)$$

Note that the Kähler form (3.6) of a 2n-dimensional Kähler manifold is given by

$$\Omega = -\overline{\partial}_i \overline{\partial}_j K(z, \bar{z}). \quad (4.2)$$

It was shown in the Appendix of [12] that one can identify $\phi(z, \bar{z})$ and $K(z, \bar{z})$ if the curvature $\mathcal{F}$ of a holomorphic line bundle is a symplectic structure, i.e., a nondegenerate, closed two-form. This means that a holomorphic line bundle with a nondegenerate curvature two-form of rank $2n$ is equivalent to a 2n-dimensional Kähler manifold. A CY manifold is a Kähler manifold with a vanishing first Chern class [16]. In this paper we have verified a particular case for the equivalence between a Kähler manifold and a holomorphic line bundle.

Since we have considered a line bundle over $\mathbb{R}^6$ with a symplectic structure $B$, the emergent CY manifolds from NC $U(1)$ gauge fields are noncompact. But the result can be generalized to compact CY manifolds by considering a line bundle $L \to M$ over a compact Kähler manifold $M$ with a symplectic structure $B$ if the two-form $B$ is a Kähler form of the base manifold $M$, although an explicit construction may be more difficult. Actually the argument in [12] can be generalized to a holomorphic line bundle over a compact Kähler manifold $M$ whose Kähler structure is given by a background $B$-field. Then the result in [12] will be equally applied to the compact case.

We observed in Sect. 1 that a line bundle over a symplectic manifold $(M, B)$ results in a dynamical symplectic manifold $(M, \mathcal{F})$. The quantization of the dynamical symplectic manifold $(M, \mathcal{F})$ gives rise to a dynamical NC spacetime described by a NC $U(1)$ gauge theory [4]. Therefore NC $U(1)$ gauge fields correspond to a quantized dynamical spacetime. In this paper we showed that CY manifolds are emergent from a semi-classical limit of NC Hermitian $U(1)$ instantons. Note that the ordinary vector field $V_a \in \Gamma(TM)$ in Eq. (1.15) is just the leading part of the NC vector field $\hat{V}_a \in \mathcal{D}$ when the commutative limit is taken into account and a classical Riemannian manifold $M$ is constructed by the set of the usual vector fields $V_a$. Thus it is natural to think of NC Hermitian $U(1)$
instantons as a quantized geometry of CY manifolds. Since NC gauge fields can be represented by large $N$ matrices in $\text{End}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space representing the NC space $(1.7)$ (9, 17), the quantized CY manifold will be described by a matrix theory or large $N$ gauge theory. It is well-known (18, 19) that such a matrix model or large $N$ gauge theory describes a nonperturbative formulation of string/M theories. Therefore it is reasonable (4) that NC $U(1)$ gauge fields describe a quantum geometry.

We explained in Sect. 1 that emergent gravity is defined by considering the deformation of a symplectic manifold $(M, B)$ by a line bundle $L \to M$. The line bundle $L$ manifests itself only by introducing a new symplectic structure $\mathcal{F} = B + F$ where $F = dA$ is identified with the curvature of the line bundle (4). Then symplectic or NC $U(1)$ gauge fields are introduced via a local coordinate transformation $\phi \in \text{Diff}(M)$ eliminating dynamical $U(1)$ gauge fields, i.e., $\phi^*(\mathcal{F}) = B$. The underlying math for this argument is the well-known theorem in symplectic geometry known as the Darboux theorem (6). Note that a CY manifold $X$ always arises with a mirror pair $Y$ obeying the mirror relation (15)

$$h^{1,1}(X) = h^{2,1}(Y), \quad h^{2,1}(X) = h^{1,1}(Y)$$

where $h^{p,q}$ is a Hodge number of a CY manifold. Since we showed that six-dimensional CY manifolds are emergent from the commutative limit of NC Hermitian $U(1)$ instantons which are the connections in a stable holomorphic line bundle $L \to M$ (13), an interesting question arises when we conceive the emergent CY manifolds from the mirror symmetry perspective. What is the mirror symmetry from the emergent gravity picture? Emergent gravity seems to provide a very elegant picture for the mirror symmetry (14). Note that a symplectic manifold $(M, B)$ is necessarily an orientable manifold and so admits the Hodge dual operation $* : \Omega^k(M) \to \Omega^{6-k}(M)$ between vector spaces of $k$-forms and $(6 - k)$-forms. Suppose that $C$ is a nondegenerate four-form that is co-closed, i.e., $\delta C = 0$ where $\delta = -(1)^{6k} \ast d \ast : \Omega^k(M) \to \Omega^{k-1}(M)$ is the adjoint exterior differential operator. Define a two-form $\tilde{B} \equiv \ast C$. Then $\delta C = 0 \iff d \tilde{B} = 0$. Therefore $\tilde{B}$ defines another symplectic structure independent of $B$. Hence one can equally consider the deformation of the dual symplectic structure $\tilde{B}$ by considering a dual line bundle $\tilde{L} \to M$. The curvature $\tilde{F} = d \tilde{A}$ of the dual line bundle $\tilde{L}$ may be identified with the Hodge dual of a co-closed four-form $G$, i.e., $\tilde{F} = \ast G$. Then we have the property $\delta G = 0 \iff d \tilde{F} = 0$. Therefore the dual line bundle $\tilde{L}$ similarly results in a dynamical symplectic manifold $(M, \tilde{F})$ where $\tilde{F} = \tilde{B} + \tilde{F} = \ast (C + G)$. One can introduce dual NC $U(1)$ gauge fields by a local coordinate transformation $\tilde{\phi} \in \text{Diff}(M)$ such that $\tilde{\phi}^*(\tilde{F}) = \tilde{B}$. After all it should be possible to find dual NC Hermitian $U(1)$ instantons as a solution of Hermitian Yang-Mills equations defined by dual NC $U(1)$ gauge fields. It is obvious that a CY manifolds will also arise from the dual NC Hermitian $U(1)$ instanton which is independent of a CY manifold emergent from the line bundle $L$ over a symplectic manifold $(M, B)$. In other words, the variety of emergent CY manifolds is doubled thanks to the Hodge duality $* : \Omega^1(M) \to \Omega^2(M)$. Since two classes of emergent CY manifolds are independent of each other, it should be possible to arrange a pair $(X, Y)$ such that $\chi(X) = -\chi(Y)$ where $\chi(M)$ is an Euler characteristic of a CY manifold $M$. Since $\chi(M) = 2(h^{1,1}(M) - h^{2,1}(M))$. 

12
and $h^{p,q}(M) \geq 0$, $\chi(X) = -\chi(Y)$ implies the mirror relation \[4.3\]. Consequently, the emergent gravity suggests a beautiful picture that the mirror symmetry of CY manifolds is simply the Hodge theory for the deformation of symplectic and dual symplectic structures. We will discuss the mirror symmetry in emergent gravity elsewhere \[5\].

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A Spin connections of NC $U(1)$ instantons

In this appendix, we will show that the (generalized) self-duality equations for NC $U(1)$ instantons in four and six dimensions can be transformed to geometrical ones for spin connections of an emergent Riemannian manifold using the Lie algebra homomorphism (1.14). In particular, it is shown that the resulting geometric equations are equivalent to the CY condition for a Ricci-flat, Kähler manifold.

Let us consider the generalized self-duality equations defined by Eq. (2.15) where

$$T_{ab}^{cd} = \begin{cases} \varepsilon_{ab}^{cd}, & d = 4; \\ \frac{1}{2} \varepsilon_{ab}^{cdef} I_{ef}, & d = 6. \end{cases}$$

(A.1)

We showed in Sect. 2 that, using the Lie algebra homomorphism (1.14), the self-duality equations (2.15) can be isomorphically mapped to those of vector fields in the commutative limit, viz., Eq. (1.15). To be specific, they are given by

$$g_{ab}^e = \pm \frac{1}{2} T_{ab}^{cd} g_{cd}^e$$

(A.2)

after using the Lie algebra relation (3.2). Let us introduce the Lie bracket for the frame basis $E_a = E_a^\mu(x) \frac{\partial}{\partial x^\mu} \in \Gamma(TM)$ defined by

$$[E_a, E_b] = - f_{ab}^c E_c.$$  

(A.3)

The Lie algebra (A.3) can be rephrased into the structure equations for the vielbeins $e^a = e^a_\mu(x) dx^\mu \in \Gamma(T^*M)$ given by

$$de^a = \frac{1}{2} f_{bc}^a e^b \wedge e^c.$$  

(A.4)

Combining Eq. (A.4) with the torsion free condition, $de^a + \omega^a_b \wedge e^b = 0$, leads to the relation

$$f_{abc} = \omega_{abc} - \omega_{bac}$$

(A.5)

where we used the symmetrization symbol $\omega_{[abc]} = \omega_{abc} + \omega_{bca} + \omega_{cab}$ for spin connections $\omega_{bc} = \omega_{abc} dx^\mu = \omega_{abc} e^a$. The structure equations in Eqs. (3.2) and (A.3) are related to each other by the relation (1.16) [2, 7]:

$$g_{ab}^c = \lambda (f_{ab}^c - E_a \ln \lambda \delta_b^c + E_b \ln \lambda \delta_a^c).$$  

(A.6)

Now we want to transform the self-duality equations (A.2) into some equations for spin connections using the relations (A.6) and (A.5) together with the volume-preserving condition (1.18) that is equal to

$$\omega_{bab} = f_{bab} = (p + 1 - d) E_a \ln \lambda$$

(A.7)

or equivalently

$$g_{ab} = pV_a \ln \lambda.$$  

(A.8)
First we will check on this approach with the well-established result in [2,20] for the four-dimensional case. Then we will apply it to the six-dimensional case.

In four dimensions, Eq. (A.2) can be written by using Eq. (A.6) as
\[ f_{ab}^e - \phi_{[a} \delta_{b]}^e = \pm \frac{1}{2} \varepsilon_{ab}^{cd} (f_{cd}^e - \phi_{[c} \delta_{d]}^e) \]  
(A.9)

where \( \phi_a := E_a \ln \lambda \). Contracting \( \varepsilon_{f abe} \) on both sides of Eq. (A.2) leads to the result
\[ g_{bab} = \mp \frac{1}{2} \varepsilon_{a bcd} g_{bcd}. \]  
(A.10)

Using Eqs. (A.6) and (A.8), the above equation can be written as
\[ p \phi_a = \mp \frac{1}{2} \varepsilon_{a bcd} f_{bcd} \]  
(A.11)

and it can be inverted as
\[ f_{[abc]} = \pm p \varepsilon_{abc d} \phi_d = 2 \omega_{[abc]}. \]  
(A.12)

Using this result together with the relation (A.5), the self-duality equation (A.9) takes the form
\[ \omega^e_{ab} + \phi_{[a} \delta_{b]}^e \mp \frac{p}{2} \varepsilon_{ab}^{ef} \phi_f = \pm \frac{1}{2} \varepsilon_{ab}^{cd} \left( \omega^e_{cd} + \phi_{[c} \delta_{d]}^e \mp \frac{p}{2} \varepsilon_{cd}^{ef} \phi_f \right). \]  
(A.13)

Note that the combination \( \phi_{[a} \delta_{b]}^e \mp \varepsilon_{ab}^{ef} \phi_f \) automatically obeys the same type of self-duality equations
\[ \phi_{[a} \delta_{b]}^e \mp \varepsilon_{ab}^{ef} \phi_f = \pm \frac{1}{2} \varepsilon_{ab}^{cd} \left( \phi_{[c} \delta_{d]}^e \mp \varepsilon_{cd}^{ef} \phi_f \right). \]  
(A.14)

Subtracting (A.14) from (A.13) gives us the result
\[ \omega^e_{ab} \mp \frac{p - 2}{2} \varepsilon_{ab}^{ef} \phi_f = \pm \frac{1}{2} \varepsilon_{ab}^{cd} \left( \omega^e_{cd} \mp \frac{p - 2}{2} \varepsilon_{cd}^{ef} \phi_f \right). \]  
(A.15)

Hence the choice \( p = 2 \) adopted in [2,20] leads to the self-duality equation for spin connections:
\[ \omega^e_{ab} = \pm \frac{1}{2} \varepsilon_{ab}^{cd} \omega^e_{cd}. \]  
(A.16)

If the spin connection \( \omega^a_{b} \) is (anti-)self-dual, the Riemann curvature tensor \( R^a_{b} = d\omega^a_{b} + \omega^a_{c} \wedge \omega^c_{b} \) is also (anti-)self-dual, i.e., \( R_{ab} = \pm \frac{1}{2} \varepsilon_{ab}^{cd} R_{cd} \). Conversely, if the curvature tensor is (anti-)self-dual, the spin connection also becomes (anti-)self-dual up to a gauge choice. This means that a four-dimensional Riemannian manifold obeying the self-duality equation (A.16) is a gravitational instanton which is a Ricci-flat, Kähler manifold or called a CY 2-fold [21]. Thus we have verified the result in [2,22].

In six dimensions, Eq. (A.2) is similarly written as
\[ f_{ab}^e - \phi_{[a} \delta_{b]}^e = \pm \frac{1}{2} T_{ab}^{cd} (f_{cd}^e - \phi_{[c} \delta_{d]}^e). \]  
(A.17)
Contracting $T_f^{abc}$ on both sides of Eq. (A.2) leads to the result

$$g_{ab} + \frac{1}{2} I^{cd} g_{cd} b I_{ba} = \mp \frac{1}{2} T_a b c d g_{cd}. \tag{A.18}$$

Using Eqs. (A.6) and (A.8), the above equation can be written as

$$(p + 1) \phi_a + \chi_a = \mp \frac{1}{2} T_a b c d f_{bcd} \tag{A.19}$$

where $\chi_a \equiv \frac{1}{2} I^{cd} f_{cd} b I_{ba}$ and it can be inverted as

$$f_{[abc]} = \pm \frac{1}{2} T_{abc} d ((p + 1) \phi_d + \chi_d) = 2 \omega_{[abc]} \tag{A.20}.$$}

Contracting $\frac{1}{2} I^{ab} I_{eg}$ on both sides of Eq. (A.17) and using the identity $\frac{1}{4} T^{cd} I_{cd} = I_{ab}$, we get the relation

$$\phi_a = - \chi_a. \tag{A.21}$$

Therefore the relation (A.20) reduces to

$$f_{[abc]} = \pm \frac{p}{2} T_{abc} d \phi_d = 2 \omega_{[abc]}. \tag{A.22}$$

Using this result together with the relation (A.5), the self-duality equation (A.17) takes the form

$$\omega_{ab} + \phi_{[a} \delta^c_{b]} + \frac{p}{4} T_{ab} e f \phi_f = \pm \frac{1}{2} T^{cd} \left( \omega_{cd} + \phi_{[c} \delta^e_{d]} + \frac{p}{4} T_{cd} e f \phi_f \right). \tag{A.23}$$

An important step is to find a combination $\phi_{[a} \delta^c_{b]} + \zeta T_{ab} e f \phi_f$ which automatically obeys the same type of self-duality equations, i.e.,

$$\phi_{[a} \delta^c_{b]} + \zeta T_{ab} e f \phi_f = \pm \frac{1}{2} T^{cd} \left( \phi_{[c} \delta^e_{d]} + \zeta T_{cd} e f \phi_f \right). \tag{A.24}$$

It turns out that, unlike the four-dimensional case, such a combination exists only for the $(-)$-equation with $\zeta = \frac{1}{2}$. It may not be surprising since this case only corresponds to a Ricci-flat, Kähler manifold, i.e., a CY 3-fold. But the sign for the self-duality equation depends on the parity of the symplectic matrix $I_{ab}$ in Eq. (A.1). If we flip the orientation of a six-dimensional manifold by choosing a symplectic matrix $I_{ab}$ different from (2.5), instead the $(+)$-equation only may admit such a combination. In this case a CY 3-fold will be defined by the $(+)$-equations with the different choice of $I_{ab}$. Subtracting (A.24) from (A.23) with both the lower $(-)$-sign gives us the result

$$\omega_{ab} + \frac{p - 2}{4} T_{ab} e f \phi_f = - \frac{1}{2} T^{cd} \left( \omega_{cd} + \frac{p - 2}{4} T_{cd} e f \phi_f \right). \tag{A.25}$$

Therefore, as in the four-dimensional case, the choice $p = 2$ leads to the desired self-duality equation for spin connections:

$$\omega_{ab} = - \frac{1}{2} T^{cd} \omega_{cd}. \tag{A.26}$$
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