\textbf{PT–symetric regularization and the new shape-invariant potentials}

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\textbf{Abstract}

All the well known exactly solvable $s$–wave potentials $V(r), \: r \in (0, \infty)$ may be $\mathcal{PT}$–symmetrically regularized (say, by a “small” imaginary shift of the coordinate axis) and re-defined as acting on the whole real line, $r \to x - i \varepsilon, \: x \in (-\infty, \infty)$. There is no surprise that the new models are still exactly solvable and that their energies remain real. What is an unexpected result? The observed drastic and unpredictable changes in the forms of the new spectra themselves.

PACS 03.65.Ge, 03.65.Fd

text of seminar proposed for ATOMKI, Debrecen;
August 3, 2018, Latex file debrec.tex
1 Introduction

This lecture is mainly intended as a preliminary review and summary of the new exactly solvable models obtained by the author, step-by-step, during the recent months, and accessible already in the form of several short notes.

In essence, the overall structure or “contents” of this text on non-Hermitian models might just parallel the page 297 of the review paper [1]. There, the exactly solvable Hermitian models have been listed and characterized by their so called shape invariance.

The quoted list of potentials further decays in the two separate families. Let us call them $\mathcal{L}$ and $\mathcal{J}$ for our present purposes. They are characterized by the respective Laguerre and Jacobi polynomial forms of their bound states.

In the former family the first subcategory $\mathcal{L}X$ is defined on the whole axis, $x \in (-\infty, \infty)$. It contains just the quadratic oscillator $V^{(QO)}(x)$ and the exponential Morse interaction $V^{(EM)}(x)$.

The second subcategory $\mathcal{L}R$ of the former family consists of the two best known three-dimensional solvable models, viz., of the Coulomb or Kepler $V^{(CK)}(r)$ and the spherical harmonic $V^{(SH)}(r)$, both with $r \in (0, \infty)$ of course.

The latter family $\mathcal{J}$ has been conveniently split in the three subcategories with two elements each. These subcategories are characterized by their ranges of coordinates.

A particle “lives” on a finite interval in the first subcategory $\mathcal{J}Y$. Temporarily, let us skip and ignore this case completely.

The second subcategory $\mathcal{J}X$ is an opposite extreme with coordinates covering the whole real axis, $x \in \mathbb{R}$. We may refer to its elements as Rosen Morse $V^{(RM)}(x)$ and the scarf potential $V^{(SP)}(x)$.

This agrees with the convention accepted in ref. [1]. Similarly, the respective Pöschl-Teller and Eckart forces $V^{(PT)}(x)$ and $V^{(EF)}(x)$ are simply determined as contained in the last, third subcategory $\mathcal{J}R$ of the whole list.

In what follows, I shall try to parallel this classification of the Hermitian solvable
models within the slightly more general, non-Hermitian framework of the so called \( \mathcal{PT} \) symmetric quantum mechanics [2]. In detail I shall only outline some of the most interesting observations, new results and open questions concerning these brand-new and still fairly unusual models.

A unifying principle of all the present generalizations will be the condition of the so called \( \mathcal{PT} \) symmetry [2]. This, in essence, means the commutativity of our Hamiltonians \( H \) with the product of the parity \( \mathcal{P} \) and the operator \( \mathcal{T} \) of time reversal. One has to note that in all the present time-independent models the latter operator symbolizes just a reflection \( \mathcal{T} i = -i \mathcal{T} \) of the complex plane with respect to its real axis.

2 Various types of the \( \mathcal{PT} \)-symmetric potentials

One of the first studies of the \( \mathcal{PT} \) symmetric potentials [3] paid attention to the quartic (i.e., unsolvable) power-law forces and has been motivated by their close formal (and, first of all, Fourier-transformation) connection to their Hermitian power-law analogues.

Similar questions can be, of course, much more easily studied within the domain of solvable interactions. Unfortunately, their explicit descriptions and/or classifications has only been initiated, quite recently, after a few parallel pioneering proposals of the \( \mathcal{PT} \) symmetric quantum mechanics by Andrianov et al [4], Bender et al [2], Cannata et al [5] et al.

2.1 Generalizations within the family \( \mathcal{L} \)

2.1.1 \( \mathcal{LX}(\mathcal{PT}) \)

I shall omit here the detailed analysis of the \( \mathcal{PT} \) symmetrized QO and EM models. My main reason is an idea of keeping this text as short as reasonable, supported by another important fact that the majority of the most relevant comments on the \( \mathcal{PT} \) symmetrized category \( \mathcal{LX}(\mathcal{PT}) \) (and on its two models denoted, naturally, as
$V^{(PTQO)}(x)$ and $V^{(PTEM)}(x)$ are already easily accessible in the respective publications [6] and [7].

2.1.2 $\mathcal{LR}(\mathcal{PT})$

I have mixed reasons why I shall also skip the other two Laguerre-related models $V^{(PTCK)}(x)$ and $V^{(PTSH)}(x)$.

The reason remains similar in the latter case only. Indeed, most of the facts I know about it did already appear in my letter [8]. In contrast to that, I (and, apparently, most of us) do not know virtually anything about the former model $V^{(PTCK)}(x)$ yet. Only some indirect hints have appeared in a few loosely related publications [9]. I strongly believe that the situation will significantly improve soon [10].

2.2 Generalizations within the subcategory $\mathcal{JX}$

In the frame of our present denotation, the respective $\mathcal{PT}$ symmetrized models $V^{(PTSP)}(x)$ and $V^{(PTRM)}(x)$ have been proposed in the recent short communications [11] and [12]. Considerations based on the economy of space force me again to restrain from any re-copied formulae and repeated comments.

Let me confirm this rule of conduct by one slightly discomforting exception. One can really feel puzzled by the latter innocent-looking, smooth and asymptotically vanishing potential which can be always uniformly bounded, $|V^{(PTRM)}(x)| < \delta$. After its deeper and constructive study we were still not that much surprised by its ability of supporting a ground state in the weak coupling regime where, by our assumption, both its couplings were small of the order $O(\delta)$ at least.

Nevertheless, one cannot help feeling caught our of guard by the observation that the related ground state energy may become also arbitrarily large, $E_0 = O(1/\delta^2)$ [12]. Similar paradoxes offer one of the reasons why the solvable models of the $\mathcal{PT}$ symmetric quantum mechanics [2] deserve a thorough and exhaustive analysis.
2.3 Generalizations within the subcategory \( \mathcal{J} \mathcal{R} \)

Moving to a central point of this paper, let us point out that there exists one key difference between the previous oscillators and the “last two” \( \mathcal{P}\mathcal{T} \) symmetrized models \( V^{(PTEF)}(r) \) and \( V^{(PTPT)}(r) \) since both the Hermitian predecessors of these two forces are, generically, strongly singular in the origin.

In the former model \( V^{(PTEF)}(r) \), its smoothness or regularization has been achieved by a small, local deformation of the integration path \( r = r(t), t \in \mathbb{R} \) near the origin. In addition, a complex rotation of one of the couplings by \( \pi/2 \) has been employed, in our preprint [13], as a guarantee of the \( \mathcal{P}\mathcal{T} \) symmetry.

Such a complexification changes the spectrum and becomes a source of its several unusual features. A detailed inspection reveals, e.g., that an increase of the repulsion can lower (!) the energy. More details will be recollected here in sect. 3 below.

In a less drastic approach to the singular forces we preserved the real values of the couplings in \( V^{(PTPT)}(r) \). We employed the “minimal” \( \mathcal{P}\mathcal{T} \) symmetrization \( r(t) = t - i\varepsilon \) of refs. [3] or [8]. Detailed properties of the resulting “last” solvable model have been described in the preprint [14]. At length they will also be discussed in sect. 4 below.

Quotation marks in the word “last” mean that we introduced one more model very recently [15]. The resulting force \( V^{(PTHS)}(x) \) of a Hulthén-type shape fits nicely
in the following scheme,

\[
\begin{array}{c|c}
\text{symmetric} & \text{symmetric} \\ 
\text{straight–line} & \text{straight–line} \\ 
V^{(PTSH)}(r) & V^{(PTPT)}(r) \\ 
in LR [8] & in JR [14] \\
\end{array}
\]

\[\downarrow r = -i \exp ix \quad \downarrow \sinh r = -i \exp ix \]

\[
\begin{array}{c|c}
\text{periodic} & \text{periodic} \\ 
V^{(PTEM)}(x) & V^{(PTHS)}(x) \\ 
on an arch \ x(t), & on an arch \ x(t), \\ 
in LX [7] & in JX [15] \\
\end{array}
\]

The vertical correspondence originates from the changes of variables and the horizontal arrows indicate the transition between the families \( \mathcal{L} \) and \( \mathcal{J} \). One notices the similarities in the (symmetric or periodic) form of the functions \( V \) as well as in the (straight-line or bent-curve) shape of their domains \( r = r(t) \in \mathbb{C} \) or \( x = x(t) \in \mathbb{C} \).

More details will be provided in our final sect. 5.

3 A generalization of the Eckart oscillator

A deeper understanding of one-dimensional systems may be mediated by an analytic continuation of their real Schrödinger equation

\[
\left[-\frac{d^2}{dr^2} + V(r)\right] \psi(r) = E \psi(r), \quad \psi(\pm \infty) = 0.
\]

For one of the simplest particular models \( V(x) = \omega x^2 + \lambda x^4 \) the loss of hermiticity at complex couplings proved more than compensated by the new insight in its solutions. E.g., its spectrum is given by a single multi-sheeted analytic function of \( \lambda \in \mathbb{C} \) [16]. The same idea has been re-applied to the set of resonances in the cubic well.
$V(x) = \omega x^2 + \lambda x^3$ [17]. In the cubic case it was surprising to notice that the spectrum $E_n(\lambda)$ remains real for the purely imaginary couplings $\lambda = ig$. The rigorous proof of this curious observation dates back to the late seventies [18]. It went virtually unnoticed for more than ten subsequent years. The phenomenon only re-entered the physical scene with Zinn-Justin and Bessis who, tentatively, attributed the strict absence of decay $\text{Im} \ E_n(ig) = 0$ to the mind-boggling real-symmetry-plus-imaginary-antisymmetry of the cubic force in question [19]. They also performed a number of numerical experiments, keeping in mind a paramount importance of this peculiar symmetry in field theory. There, it precisely coincides with the fundamental $\mathcal{PT}$ (i.e., parity-plus-time-reversal) invariance. According to Bender et al [20] this new type of symmetry might even replace the traditional requirement of hermiticity in many phenomenological models.

Within the quantum mechanics itself the parallels between $gx^4$ and $igx^3$ inspired the numerical and semi-classical study of the generalized anharmonic forces $V^{(\delta)}(x) = \omega x^2 + gx^2(ix)^{\delta}$ with a variable real exponent $\delta$ [6]. Within the related $\mathcal{PT}$–symmetric branch of the “classical” quantum mechanics there appeared new perturbation series [21] and quasi-classical approximations [22], a new implementation of supersymmetry [4] and the new types of spectra [23].

Among all the different models with the $\mathcal{PT}$–symmetrically broken parity one may distinguish, roughly speaking, between its “stronger” and “weaker” violation. The former group is formally characterized by the globally, asymptotically deformed paths of integration in eq. (1). An illustration may be provided by the elementary ground-state wave function $\psi(x) = \exp(-ix^3 + bx^2)$ of Bender and Boettcher [24] which ceases to be integrable on the real line of $x$. The integrability is only recovered after we bend both the semi-axes downwards,

$$\{x \gg 1\} \rightarrow \{x = \varrho e^{-i\varphi}\}, \quad \{x \ll -1\} \rightarrow \{x = -\varrho e^{i\varphi}\}$$

with $\varrho \gg 1$ and $0 < \varphi < \pi/3$. The wave function obviously corresponds to the quasi-exactly solvable potential $V(x) = -9x^4 - 12bix^3 + 4b^2x^2 - 6ix$ [24] and mimics the choice of $\delta = 2$ in the family $V^{(\delta)}(x)$. The further growth of $\delta > 2$ would make
both the asymptotical $\varphi$--wedges shrink and rotate more and more downwards in the complex plane.

The second group of the $\mathcal{PT}$ symmetric examples with a “weaker” parity breakdown does not leave the real axis of $x$ at all (i.e., $\varphi \equiv 0$, cf., e.g., [25]). This admits the more natural physical interpretation of the real physical coordinates. Such a form of the $\mathcal{P}$--violation has been also implemented in several numerical and perturbative models. Their subclass which possesses elementary solutions is particularly instructive since it incorporates all the so called shape invariant one-dimensional models of the ordinary quantum mechanics [12].

In both the groups of examples an overall $\mathcal{PT}$ symmetry of the Hamiltonian is, presumably, responsible for its real and discrete spectrum [6]. Cannata et al [5] were the first to notice that one of the various limits $\delta \rightarrow \infty$ of the power-law models with $\varphi \rightarrow \pi/2 - \mathcal{O}(1/\delta)$ becomes exactly solvable in terms of Bessel functions. This re-attracted attention to the related strongly deformed contours [26]. More recently, the same merit of an indirect formal parallel to the Hermitian square well has been also found for the standard real contour. The related $\delta \rightarrow \infty$ wave functions even degenerated to the Laguerre polynomials [7].

The latter unexpected emergence of the new exactly solvable model within the generalized, $\mathcal{PT}$--symmetric quantum mechanics encouraged our present study. Indeed, exactly solvable models are obviously best suited for analyses of methodical questions. In particular, the class of the $\mathcal{PT}$--symmetrized shape invariant oscillators [12] does not seem to differ too much from its Hermitian counterpart. For an explicit analysis of the details of this correspondence one may simply recall the numerous explicit formulae available, e.g., in Table 4.1 of the review [1] or in the original factorization constructions [27] and their Lie-algebraic [28], operator [29] or supersymmetric [30] re-interpretations.

Seemingly, one cannot expect any interesting new developments in the exactly solvable context. Fortunately, in the light of our recent remark on the spherical harmonic oscillator [8] non-trivial innovations may be expected in the domain of singular forces. Indeed, within the $\mathcal{PT}$--symmetric quantum mechanics it is possible
to avoid some isolated singularities by a local deformation of the integration path. In particular, a strong repulsion in the origin (so popular in some phenomenological models [31] but fully impenetrable in one dimension) may be readily controlled by a suitable choice of the cut.

In the present text we intend to re-attract attention to the singular forces. In eq. (1) we shall use the asymptotically real path of integration which is only locally deformed and non-Hermitian. We shall show that this innovative approach enables us to regularize the one-dimensional models via their suitable $\mathcal{PT}$ symmetrization.

### 3.1 Eckart model

Our particular attention will be paid to the exceptional $s-$wave potential

$$V^{(\text{Eck})}(x) = \frac{A(A - 1)}{\sinh^2 x} - 2B \frac{\cosh x}{\sinh x}$$

with the strongly singular core. Usually attributed to Eckart [32], this model is solvable on the half-line with $x \in (0, \infty)$ and, conventionally, $A > 1/2$ and $B > A^2$ [1]. Its fixed value of the angular momentum $\ell = 0$ is in effect a non-locality which lowers its practical relevance in three dimensions. Here, we shall study its $\mathcal{PT}-$symmetrized version with the purely imaginary coupling $B = i\beta$. Besides the obvious relevance of such an exceptional complexified model with a strong singularity in quantum mechanics, an independent encouragement of our study is also provided by its obvious phenomenological and methodical appeal in the context of field theory, especially in connection with the so called Klauder phenomenon [33].

The local deformation of the integration path will enable us to forget about the strong singularity in the origin. This deformation will also admit the presence of the so called irregular components in $\psi(x) \sim x^{1-A}$ near $x = 0$. They would be, of course, unphysical in the usual formalism [34].
3.2 Solution revisited

For all these reasons we have to re-analyze the whole Schrödinger equation anew. Our initial choice of the appropriate variables

\[ \psi(x) = (y - 1)^u(y + 1)^v \varphi \left( \frac{1 - y}{2} \right), \quad y = \frac{\cosh x}{\sinh x} = 1 - 2z \]

is still dictated by the arguments of Lévali [30]. Then we insert \( V^{(Eck)}(x) \) in eq. (1) and our change of variables leads to its new form

\[ z(1 - z) \varphi''(z) + [c - (a + b + 1)z] \varphi'(z) - ab \varphi(z) = 0 \] (2)

where

\[ c = 1 + 2u, \quad a + b = 2u + 2v + 1, \quad ab = (u + v)(u + v + 1) + A(1 - A) \] (3)

and

\[ 4v^2 = 2B - E, \quad 4u^2 = -2B - E. \] (4)

Our differential equation is of the Gauss hypergeometric type and its general solution is well known [35],

\[ \varphi(z) = C_1 \cdot _2F_1(a, b; c; z) + C_2 \cdot z^{1-c} _2F_1(a + 1 - c, b + 1 - c; 2 - c; z). \] (5)

The first thing we notice is that our parameters \( a \) and \( b \) are merely functions of the sum \( u + v \) and vice versa, \( u + v = (a + b - 1)/2 \). The immediate insertion then gives the rule \((a - b)^2 = (2A - 1)^2\) and we may eliminate

\[ a = b \pm (2A - 1). \] (6)

We assume that our solutions obey the standard oscillation theorems [36] and become compatible with the boundary conditions in eq. (1) at a discrete set of energies, i.e., if and only if the infinite series \( _2F_1 \) terminate. Due to the complete \( a \leftrightarrow b \) symmetry, we only have to distinguish between the two possible choices of \( C_2 = 0 \) and \( C_1 = 0 \).

In the former case with the convenient \( b = -N \) (= non-positive integer) the resulting numbers \( a + b \) and \( u + v \) prove both real. Using the definition of \( B \) the
difference \( u - v = -i\beta/(u + v) \) comes out purely imaginary. The related terminating wave function series (5), i.e.,

\[
\psi(x) = \left(\frac{1}{\sinh x}\right)^{u+v} e^{(v-u)x} \cdot \varphi[z(x)]
\]

is asymptotically normalizable if and only if \( u + v > 0 \). This condition fixes the sign in eq. (6) and gives the explicit values of all the necessary parameters,

\[
a = 2A - N - 1, \quad u + v = A - N - 1, \quad u - v = -i \frac{\beta}{A - N - 1}.
\]

For all the non-negative integers \( N \leq N_{\text{max}} < A - 1 \) the spectrum of energies is obtained in the following closed form,

\[
E = -\frac{1}{2} \left( u^2 + v^2 \right) = -(A - N - 1)^2 + \frac{\beta^2}{(A - N - 1)^2}, \quad N = 0, 1, \ldots, N_{\text{max}}.
\]

The normalizable wave functions become proportional to Jacobi polynomials,

\[
\varphi[z(x)] = \text{const.} \cdot P_N^{(u/2,v/2)}(\coth x).
\]

Before we start a more thorough discussion of this result we have shortly to return to the second option with \( C_1 = 0 \) in eq. (5). Curiously enough, this does not bring us anything new. Although the second Gauss series terminates at the different \( b = c - 1 - N \), the factor \( z^{1-c} \) changes the asymptotics and one only reproduces the former solution. All the differences prove purely formal. In the language of our formulae one just replaces \( u \) by \(-u\) in (and only in) both equations (7) and (8). No change occurs in the polynomial (10).

### 3.3 Spectrum

The new spectrum of energies seems phenomenologically appealing. The separate \( N \)-th energy remains negative if and only if the imaginary coupling stays sufficiently weak, \( \beta^2 < (A - N - 1)^4 \). Vice versa, the highest energies may become positive, with \( E = E_{N_{\text{max}}} \) growing extremely quickly whenever the value of the coupling \( A \) approaches its integer lower estimate \( 1 + N_{\text{max}} \) from above. In this way, even a weak \( \mathcal{PT} \) symmetric force \( V^{(Eck)}(x) \) is able to produce a high-lying normalizable
excitation. This feature does not seem connected to the presence of the singularity as it closely parallels the similar phenomenon observed for the $\mathcal{PT}$ symmetric Rosen-Morse oscillator which remains regular in the origin [12]. Also, in a way resembling harmonic oscillators the distance of levels in our model is safely bounded from below. Abbreviating $D = A - N - 1 = A_{\text{effective}} > 0$ its easy estimate

$$E_N - E_{N-1} = (2D + 1)\left(1 + \frac{\beta^2}{D^2(D + 1)^2}\right) > 1$$

(useful, say, in perturbative considerations) may readily be improved to $E_N - E_{N-1} > \beta^2/D^2$ at small $D \ll 1$, to $E_N - E_{N-1} > 2D$ at large $D \gg 1$ and, in general, to an algebraic precise estimate obtainable, say, via MAPLE [37].

Let us emphasize in the conclusion that the formulae we obtained are completely different from the usual Hermitian $s$-wave results as derived, say, by Lévai [30]. He has to start from the regularity in the origin which implies an opposite sign in eq. (6). This must end up with the constraint $B > 0$. Moreover, the size of $B$ would limit the number of bound states. In the present $\mathcal{PT}$ symmetric setting, a few paradoxes emerge in this comparison. Some of them may be directly related to the repulsive real core in our $V^{(Eck)}(x)$ with imaginary $B$. Thus, one may notice that the increase of the real repulsion lowers the $N$-th energy. In connection with that, the number of levels grows with the increase of coupling $A$. In effect, the new bound-state levels emerge as decreasing from the positive infinity (!). At the same time, the presence of the imaginary $B = i\beta$ shifts the whole spectrum upwards precisely in the manner known from the non-singular models.

4 A generalization of the Pöschl-Teller potential

Among all the exactly solvable models in quantum mechanics the one-dimensional Schrödinger equation (1) with one of the most elementary bell-shaped potentials $V^{(bs)}(r) = G/\cosh^2 r$ is particularly useful. Its applications range from the analyses of stability and quantization of solitons [38] to phenomenological studies in atomic and molecular physics [39], chemistry [40], biophysics [41] and astrophysics [42]. Its
appeal involves the solvability by different methods [30] as well as a remarkable role in the scattering [43]. Its bound-state wave functions represented by Jacobi polynomials offer one of the most elementary illustrations of properties of the so-called shape invariant systems [1]. The force $V^{(bs)}(r)$ is encountered in the so-called $\mathcal{PT}$ symmetric quantum mechanics [2] where it appears as a Hermitian super-partner of a complex “scarf” model [11].

Curiously enough, it is not too difficult to extend the exact solvability of the potential $V^{(bs)}(r)$ to all its “spiked” (often called Pöschl-Teller [44]) shape invariant generalizations

$$V^{(PT)}(r) = -\frac{A(A + 1)}{\cosh^2 r} + \frac{B(B - 1)}{\sinh^2 r}. \quad (11)$$

In a way resembling the preceding section, our new one-dimensional Schrödinger eq. (1) is also too singular at $B(B - 1) \neq 0$. The force $V^{(PT)}(r)$ must be confined to the semi-axis, $r \in (0, \infty)$. This makes the “improved” Pöschl-Teller model (11) much less useful in practice since its higher partial waves are not solvable.

### 4.1 Regularization

We may repeat that the impossibility of using eq. (11) in three dimensions (or on the whole axis in one dimension at least) is felt unfortunate because the singular potentials themselves are frequently needed in methodical considerations [33] and in perturbation theory [45]. They are encountered in phenomenological models [46] and in explicit computations [31] but not too many of them are solvable [47].

We feel mainly inspired by the pioneering letter [6] where Bender and Boettcher modified the harmonic oscillator $V^{(HO)}(r) = r^2$ by a complex downward shift of its axis of coordinates,

$$r = x - i\varepsilon, \quad x \in (-\infty, \infty). \quad (12)$$

The $\mathcal{PT}$ symmetry of their model $V^{(BB)}(x) = V^{(HO)}(x - i\varepsilon) = x^2 - 2i\varepsilon x - c^2$ means its invariance with respect to the simultaneous reflection $x \to -x$ and complex conjugation $i \to -i$. Various other complex interactions have been subsequently tested and studied within this framework.
Our study [8] of the three-dimensional $\mathcal{PT}$ symmetric harmonic oscillator offers the details of our present key idea. The shift (12) has been employed there as a source of a *regularization* of the strongly singular centrifugal term. As long as

$$\frac{1}{(x - i \varepsilon)^2} = \frac{(x + i \varepsilon)^2}{(x^2 + \varepsilon^2)^2}$$

at any $\varepsilon \neq 0$, this term remains nicely bounded in a way which is uniform with respect to $x$. Without any difficulties one may work with $V^{(RHO)}(x) = r^2(x) + \ell(\ell + 1)/r^2(x)$ on the whole real line of $x$. In what follows the same idea will be applied to the regularized Pöschl-Teller-like potential

$$V^{(RPT)}(x) = V^{(PT)}(x - i \varepsilon), \quad 0 < \varepsilon < \pi/2.$$  

This potential is a simple function of the Lévai’s [30] variable $g(r) = \cosh 2r$. As long as $g(x - i \varepsilon) = \cosh 2x \cos 2\varepsilon - i \sinh 2x \sin 2\varepsilon$, the new force is $\mathcal{PT}$ symmetric on the real line of $x \in (-\infty, \infty)$,

$$V^{(RPT)}(-x) = [V^{(RPT)}(x)]^*.$$  

Due to the estimates

$$|\sinh^2(x - i \varepsilon)|^2 = \sinh^2 x \cos^2 \varepsilon + \cosh^2 x \sin^2 \varepsilon = \sinh^2 x + \sin^2 \varepsilon$$
and

$$|\cosh^2(x - i \varepsilon)|^2 = \sinh^2 x + \cos^2 \varepsilon$$

the regularity of $V^{(RPT)}(x)$ is guaranteed for all its parameters $\varepsilon \in (0, \pi/2)$.

### 4.2 Solutions

In a way paralleling the three-dimensional oscillator the mere analytic continuation of the $s$–wave bound states does not give the complete solution. One must return to the original differential equation (1). There we may conveniently fix $A + 1/2 = \alpha > 0$ and $B - 1/2 = \beta > 0$ and write

$$\left(-\frac{d^2}{dx^2} + \frac{\beta^2 - 1/4}{\sinh^2 r(x)} - \frac{\alpha^2 - 1/4}{\cosh^2 r(x)}\right) \psi(x) = E \psi(x), \quad r(x) = x - i \varepsilon. \quad (13)$$

This is the Gauss differential equation

$$z(1 + z) \varphi''(z) + [c + (a + b + 1)z] \varphi'(z) + ab \varphi(z) = 0 \quad (14)$$
in the new variables

$$\psi(x) = z^\mu(1 + z)^\nu \varphi(z), \quad z = \sinh^2 r(x)$$

and

$$\frac{d^2 \varphi}{dz^2} + \left(c + \frac{a + b + 1}{z}\right) \frac{d \varphi}{dz} + \frac{ab}{z^2} \varphi = 0$$

with $\frac{d \varphi}{dz} = \mu\nu \psi(x)$. The parameter $z$ runs from $z = 0$ to $z = \infty$, from which we are led to use the modified Bessel functions of first kind $I_{\mu}(z)$.
using the suitable re-parameterizations

\[ \alpha^2 = (2\nu - 1/2)^2, \quad \beta^2 = (2\mu - 1/2)^2, \]

\[ 2\mu + 1/2 = c, \quad 2\mu + 2\nu = a + b, \quad E = -(a - b)^2. \]

In the new notation we have the wave functions

\[ \psi(x) = \sinh^{\tau\beta+1/2}[r(x)] \cosh^{\sigma\alpha+1/2}[r(x)] \varphi[z(x)] \quad (15) \]

with the sign ambiguities \( \tau = \pm 1 \) and \( \sigma = \pm 1 \) in \( 2\mu = \tau\beta + 1/2 \) and \( 2\nu = \sigma\alpha + 1/2 \). This formula contains the general solution of hypergeometric eq. (14),

\[ \varphi(z) = C_1 \ _2F_1(a, b; c; -z) + C_2 z^{1-c} \ _2F_1(a + 1 - c, b + 1 - c; 2 - c; -z). \quad (16) \]

The solution should obey the complex version of the Sturm-Liouville oscillation theorem [36]. In the case of the discrete spectra this means that we have to demand the termination of our infinite hypergeometric series. This suppresses an asymptotic growth of \( \psi(x) \) at \( x \to \pm \infty \).

In a deeper analysis let us first put \( C_2 = 0 \). We may satisfy the termination condition by the non-positive integer choice of \( b = -N \). This implies that \( a = N + 1 + \sigma\alpha + \tau\beta \) is real and that our wave function may be made asymptotically (exponentially) vanishing under certain conditions. Inspection of the formula (15) recovers that the boundary condition \( \psi(\pm \infty) = 0 \) will be satisfied if and only if

\[ 1 \leq 2N + 1 \leq 2N_{\text{max}} + 1 < -\sigma\alpha - \tau\beta. \]

The closed Jacobi polynomial representation of the wave functions follows easily,

\[ \varphi[z(x)] = C_1 \frac{N!\Gamma(1 + \tau\beta)}{\Gamma(N + 1 + \tau\beta)} \mathcal{P}_{N}^{(\tau\beta,\sigma\alpha)}[\cosh 2r(x)]. \]

The final insertions of parameters define the spectrum of energies,

\[ E = -(2N + 1 + \sigma\alpha + \tau\beta)^2 < 0. \quad (17) \]

Now we have to return to eq. (16) once more. A careful analysis of the other possibility \( C_1 = 0 \) does not recover anything new. The same solution is obtained,
with $\tau$ replaced by $-\tau$. We may keep $C_2 = 0$ and mark the two independent solutions by the sign $\tau$. Once we define the maximal integers $N^{(\sigma,\tau)}_{\text{max}}$ which are compatible with the inequality
\[
2N^{(\sigma,\tau)}_{\text{max}} + 1 < -\sigma \alpha - \tau \beta
\]we get the constraint $N \leq N^{(\sigma,\tau)}_{\text{max}}$. The set of our main quantum numbers is finite.

### 4.3 Paradoxes

Let us now compare our final result (17) with the known $\varepsilon = 0$ formulae for $s$ waves [30]. An additional physical boundary condition must be imposed in the latter singular limit [48]. This condition fixes the unique pair $\sigma = -1$ and $\tau = +1$. Thus, the set of the $s$–wave energy levels $E_N$ is not empty if and only if $\alpha - \beta > 1$. In contrast, all our $\varepsilon > 0$ potentials acquire a uniform bound $|V^{(RPT)}(x)| < \text{const} < \infty$. Due to their regularity, no additional constraint is needed. Our new spectrum $E^{(\sigma,\tau)}_N$ becomes richer. For the sufficiently strong couplings it proves composed of the three separate parts,
\[
\begin{align*}
E^{(-,-)}_N < 0, & \quad 0 \leq N \leq N^{(-,-)}_{\text{max}}, \quad \alpha + \beta > 1, \\
E^{(-,+)}_N < 0, & \quad 0 \leq N \leq N^{(-,+)}_{\text{max}}, \quad \alpha > \beta + 1, \\
E^{(+,-)}_N < 0, & \quad 0 \leq N \leq N^{(+,-)}_{\text{max}}, \quad \beta > \alpha + 1.
\end{align*}
\]
The former one is non-empty at $A + B > 1$ (with our above separate conventions $A > -1/2$ and $B > 1/2$). Concerning the latter two alternative sets, they may exist either at $A > B$ or at $B > A + 2$, respectively. We may summarize that in a parallel to the $\mathcal{PT}$ symmetrized harmonic oscillator of ref. [8] we have the $N^{(-,+)}_{\text{max}} + 1$ quasi-odd or “perturbed”, analytically continued $s$–wave states (with a nodal zero near the origin) complemented by certain additional solutions.

In the first failure of a complete analogy the number $N^{(-,-)}_{\text{max}} + 1$ of our quasi-even states proves systematically higher than $N^{(-,+)}_{\text{max}} + 1$, especially at the larger “repulsion” $\beta \gg 1$. This is a certain paradox, strengthened by the existence of another quasi-odd family which behaves very non-perturbatively. Its members (with
the ground state $\psi_0^{(+,-)}(x) = \cosh^{A+1}[r(x)] \sinh^{1-B}[r(x)]$ etc do not seem to have any $s-$wave analogue. They are formed at the prevalent repulsion $B > A+2$ which is even more counter-intuitive. The exact solvability of our example enables us to understand this apparent paradox clearly. In a way characteristic for many $\mathcal{PT}$ symmetric systems some of the states are bound by an antisymmetric imaginary well. The whole history of the $\mathcal{PT}$ symmetric models starts from the purely imaginary cubic force [19] after all. A successful description of its perturbative forms $V(x) = \omega x^2 + i\lambda x^3$ is not so enigmatic [18], especially due to its analogies with the real and symmetric $V(x) = \omega x^2 + \lambda x^4$ [17]. The similar mechanism creates the states with $(\sigma, \tau) = (+, -)$ in the present example. A significant novelty of our new model $V^{(RPT)}(x)$ lies in the dominance of its imaginary component at the short distances, $x \approx 0$. Indeed, we may expand our force to the first order in the small $\epsilon > 0$. This gives the approximation

$$\frac{1}{\sinh^2(x - i\epsilon)} = \frac{\sinh^2(x + i\epsilon)}{(\sinh^2 x + \sin^2 \epsilon)^2} = \frac{1}{\sinh^2 x} + 2i\epsilon \frac{\cosh x}{\sinh^3 x} + \mathcal{O}(\epsilon^2). \quad (20)$$

We see immediately the clear prevalence of the imaginary part at the short distances, especially at all the negligible $A = \mathcal{O}(\epsilon^2)$.

An alternative approach to the above paradox may be mediated by a sudden transition from the domain of a small $\epsilon \approx 0$ to the opposite extreme with $\epsilon \approx \pi/2$. This is a shift which changes $\cosh x$ into $\sinh x$ and vice versa. It intertwines the role of $\alpha$ and $\beta$ as a strength of the smooth attraction and of the singular repulsion, respectively. The perturbative/non-perturbative interpretation of both our quasi-odd subsets of states becomes mutually interchanged near both the extremes of $\epsilon$.

The dominant part (20) of our present model leaves its asymptotics comparatively irrelevant. In contrast to many other $\mathcal{PT}$ symmetric models as available in the current literature our potential vanishes asymptotically,

$$V^{(RPT)}(x) \to 0, \quad x \to \pm \infty.$$ 

An introduction and analysis of continuous spectra in the $\mathcal{PT}$ symmetric quantum mechanics seems rendered possible at positive energies. This question will be left open here. In the same spirit of a concluding remark we may also touch the problem
of the possible breakdown of the $\mathcal{PT}$ symmetry. This has recently been studied on the background of the supersymmetric quantum mechanics [4]. In our present solvable example the violation of the $\mathcal{PT}$ symmetry is easily mimicked by the complex choice of the couplings $\alpha$ and $\beta$. Due to our closed formulae the energies will still stay real, provided only that $\text{Im} (\sigma \alpha + \tau \beta) = 0$.

5 $\mathcal{PT}$ symmetric Hulthén potential

Quantum mechanics is often forced to formulate its predictions numerically. Its exactly solvable models are scarce. Only their subclass in one dimension is broader and, in this sense, privileged and exceptional. It involves the harmonic oscillator and Morse potentials (with bound states expressible in terms of the Laguerre polynomials) as well as several other, less popular models solvable in terms of the polynomials of Jacobi etc (cf., e.g., review [1] for more details).

One encounters a high degree of analogy between the real and complex forces for the exactly solvable models based on the use of Jacobi polynomials. An appropriate exactly solvable $\mathcal{PT}$ symmetrization of both the one-dimensional shape-invariant models with the “canonical” Rosen-Morse and scarf-hyperbolic shapes has been recently proposed and analyzed in refs. [11].

As we already noticed, a word of warning comes from the phenomenologically most appealing Coulomb interaction. Up to now, its only available $\mathcal{PT}$ symmetric simulation proves merely partially solvable [9]. In this section we intend to offer a partial remedy. We shall derive and describe a complete and exact solution for an appropriate $\mathcal{PT}$ symmetric complexification of the singular phenomenological Hulthén potential [43] which is known to mimic very well the shape of the Coulombic force in the vicinity of its singularity.
5.1 Method: The Liouvillean change of variables

In the first step let us recollect that in the spirit of the old Liouville’s paper [49] the change of the (real) coordinates (say, \( r \leftrightarrow \xi \)) in Schrödinger equation

\[
\left[ -\frac{d^2}{dr^2} + W(r) \right] \chi(r) = -\kappa^2 \chi(r) \quad (21)
\]

may sometimes mediate a transition between two different potentials. It is easy to show [50] that once we forget about boundary conditions one simply has to demand the existence of the invertible function \( r = r(\xi) \) and its few derivatives \( r'(\xi), r''(\xi), \ldots \) in order to get the explicit correspondence between the two bound state problems, viz., original (21) and the new Schrödinger equation with the known wave functions

\[
\Psi(\xi) = \frac{\chi[r(\xi)]}{\sqrt{r'(\xi)}} \quad (22)
\]

generated by the new interaction

\[
V(\xi) - E = [r'(\xi)]^2 \left\{ W[r(\xi)] + \kappa^2 \right\} + \frac{3}{4} \left[ \frac{r''(\xi)}{r'(\xi)} \right]^2 - \frac{1}{2} \left[ \frac{r''''(\xi)}{r'(\xi)} \right]. \quad (23)
\]

One might recall the well known mapping between the Morse and harmonic Laguerre-related oscillators as one of the best known explicit illustrations. For it, the necessary preservation of the correct physical boundary conditions is very straightforward to check [43].

In the Jacobi-polynomial context the Liouvillean changes of variables have been applied systematically to all the Hermitian models (cf. Figure 5.1 in the review [1] or ref. [51] for a more detailed illustration). A similar thorough study is still missing for the \( \mathcal{PT} \) symmetric models within the same class.

In the present letter we shall try to fill the gap. For the sake of brevity we shall only restrict our attention to the \( \mathcal{PT} \) symmetric initial eq. (21) with the Pöschl-Teller potential studied and solved exactly in our recent preprint [14],

\[
W(r) = \frac{\beta^2 - 1/4}{\sinh^2 r} - \frac{\alpha^2 - 1/4}{\cosh^2 r}, \quad r = x - i\varepsilon, \quad x \in (-\infty, \infty) \quad (24)
\]

The normalizable solutions are proportional to the Jacobi polynomials,

\[
\chi(r) = \sinh^{\tau\beta+1/2} r \cosh^{\sigma\alpha+1/2} r \ P_{n}^{(\tau\beta,\sigma\alpha)}(\cosh 2r)
\]
at all the negative energies \(-\kappa^2 < 0\) such that
\[
\kappa = \kappa_n^{(\sigma, \tau)} = -\sigma\alpha - \tau\beta - 2n - 1 > 0.
\]
These bound states are numbered by \(n = 0, 1, \ldots, n_{max}^{(\sigma, \tau)}\) and by the generalized parities \(\sigma = \pm 1\) and \(\tau = \pm 1\).

We may note that our initial \(\mathcal{PT}\) symmetric model (21) remains manifestly regular provided only that its constant downward shift of the coordinates \(r = r(x) = x - i\varepsilon\) remains constrained to a finite interval, \(\varepsilon \in (0, \pi/2)\). In this sense our initial model (24) is closely similar to the shifted harmonic oscillator. At the same time, one still misses an analogue of a transition to its Morse-like final partner \(V(\xi)\). In a key step of its present construction let us first pick up the following specific change of the axis of coordinates,
\[
\sinh r(x) = -ie^{i\xi}, \quad \xi = v - iu.
\]
(25)
The main motivation of such a tentative assignment lies in the related shift and removal of the singularity (sitting at \(r = 0\)) to infinity \((u \rightarrow +\infty)\). In fact, one cannot proceed sufficiently easily in an opposite direction, i.e., from a choice of a realistic \(V(\xi)\) to a re-constructed \(r(\xi)\). This is due to the definition (23) containing the third derivatives and, hence, too complicated to solve.

We shall see below that we are quite lucky with our purely trial and error choice of eq. (25). Firstly, we already clearly see that the real line of \(x\) becomes mapped upon a manifestly \(\mathcal{PT}\) symmetric curve \(\xi = v - iu\) in accordance with the compact and invertible trigonometric rules
\[
\sinh x \cos \varepsilon = e^u \sin v,
\]
\[
\cosh x \sin \varepsilon = e^u \cos v,
\]
i.e., in such a way that
\[
v = \arctan \left( \frac{\tanh x}{\tan \varepsilon} \right) = v(x) \in \left( v(\langle \infty \rangle), v(\langle \infty \rangle) \right) \equiv \left( -\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \right),
\]
\[
u = u(x) = \frac{1}{2} \ln \left( \sinh^2 x + \sin^2 \varepsilon \right).
\]
This relationship is not too different from the Morse-harmonic equivalence studied in ref. [7]. Our present path of \(\xi\) is a very similar down-bent arch which starts in its
left imaginary minus infinity, ends in its right imaginary minus infinity while its top
lies at \( x = v = 0 \) and \(-u = -u(0) = \ln 1 / \sin \varepsilon > 0 \). The top may move towards the
singularity in a way mimicked by the diminishing shift \( \varepsilon \to 0 \). Indeed, although the
singularity originally occurred at the finite value \( r \to 0 \), it has now been removed
upwards, i.e., in the direction of \(-u \to +\infty \).

5.2 Consequences

The first consequence of our particular change of variables (25) is that it does not
change the asymptotics of the wave functions. As long as \( r' (\xi) = i \tanh r (\xi) \) the
transition from eq. (21) to (1) introduces just an inessential phase factor in \( \Psi (\xi) \).
This implies that the normalizability (at a physical energy) as well as its violations
(off the discrete spectrum) are both in a one-to-one correspondence.

The explicit relation between the old and new energies and couplings is not too
complicated. Patient computations reveal its closed form. With a bit of luck, the
solution proves non-numerical. The new form of the potential and of its binding
energies is derived by the mere insertion in eq. (23),

\[
V (\xi) = \frac{A}{(1 - e^{2\xi})^2} + \frac{B}{1 - e^{2\xi}}, \quad E = \kappa^2.
\]  

(26)

At the imaginary \( \xi \) and vanishing \( A = 0 \) this interaction coincides with the Hulthén
potential.

In the new formula for the energies one has to notice their positivity. This is
extremely interesting since the potential itself is asymptotically vanishing at both
ends of its integration path. One may immediately recollect that a similar paradox
has already been observed in a few other \( \mathcal{PT} \) symmetric models with an asymptotic
decrease of the potential to minus infinity [24, 52].

The exact solvability of our modified Hulthén potential is not yet guaranteed at
all. A critical point is that the new couplings depend on the old energies and, hence,
on the discrete quantum numbers \( n, \sigma \) and \( \tau \) in principle. This could induce an
undesirable state-dependence into our new potential. Vice versa, the closed solvabil-
ity of the constraint which forbids this state-dependence will be equivalent to the
solvability at last.

A removal of the obstacle means in effect a transfer of the state-dependence (i.e., of the $n-$, $\sigma-$ and $\tau-$dependence) in

$$ A = A(\alpha) = 1 - \alpha^2, \quad C = (A + B) = \kappa^2 - \beta^2 $$

from $C$ to $\beta$. To this end, employing the known explicit form of $\kappa$ we may re-write

$$ C = C(\sigma, \tau, n) = (\sigma \alpha + 2n + 1)(\sigma \alpha + 2n + 1 + 2\tau \beta). \quad (27) $$

This formula is linear in $\tau \beta$ and, hence, its inversion is easy and defines the desirable state-dependent quantity $\beta = \beta(\sigma, \tau, n)$ as an elementary function of the constant $C$. The new energy spectrum acquires the closed form

$$ E = E(\sigma, \tau, n) = A + B + \frac{1}{4} \left[ \sigma \alpha + 2n + 1 - \frac{A + B}{\sigma \alpha + 2n + 1} \right]^2. \quad (28) $$

Our construction is complete. The range of the quantum numbers $n$, $\sigma$ and $\tau$ remain the same as above.

In the light of our new result we may now split the whole family of the exactly solvable $\mathcal{P}\mathcal{T}$ symmetric models in the two distinct categories. The first one “lives” on the real line and may be represented or illustrated not only by the popular Laguerre-solvable harmonic oscillator [8] but also by our initial Pöschl-Teller Jacobi-solvable force (24).

The second category requires a narrow arch-shaped path of integration which all lies confined within a vertical strip. It contains again both the Laguerre and Jacobi solutions. The former ones may be represented by the complex Morse model of ref. [7]. Our present new Hulthén example offers its Jacobi solvable counterpart. The scheme becomes, in a way, complete.

The less formal difference between the two categories may be also sought in their immediate physical relevance. Applications of the former class may be facilitated by a limiting transition which is able to return them back on the usual real line. In contrast, the second category may rather find its most useful place in the methodical considerations concerning, e.g., field theories and parity breaking [53]. Within the
quantum mechanics itself an alternative approach to the second category might also parallel studies [5] of the “smoothed” square wells in non-Hermitian setting.

In the conclusion let us recollect that the $\mathcal{PT}$ symmetry of a Hamiltonian replaces and, in a way, generalizes its usual hermiticity. This is the main reason why there existed a space for a new solvable model among the singular interactions. An exactly solvable example with an “intermediate” (i.e., hyperbola-shaped) arc of coordinates remains still to be discovered. Indeed, this type of a deformed contour has only been encountered in the “quasi-solvable” (i.e., partially numerical) model of ref. [24]) and in the general unsolvable forces studied by several authors by means of the perturbative [18], numerical [54] and WKB [55] approximative techniques.

**Acknowledgement**

Partially supported by the grant Nr. A 1048004 of the Grant Agency of the Academy of Sciences of the Czech Republic.
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