Energy scattering for radial focusing inhomogeneous bi-harmonic Schrödinger equations

Tarek Saanouni

Received: 12 September 2020 / Accepted: 25 March 2021 / Published online: 22 May 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
This note studies the asymptotic behavior of global solutions to the fourth-order Schrödinger equation

\[ i \dot{u} + \Delta^2 u + F(x, u) = 0. \]

Indeed, for both cases, local and non-local source term, the scattering is obtained in the focusing mass super-critical and energy sub-critical regimes, with radial setting. This work uses a new approach due to Dodson and Murphy (Proc Am Math Soc 145(11):4859–4867, 2017).

Mathematics Subject Classification 35Q55

Contents

1 Introduction ............................................... 2
2 Background and main results ...................................... 3
  2.1 Preliminary ............................................. 3
  2.2 Main results ............................................. 6
  2.3 Useful estimates ........................................... 7
3 The Schrödinger problem (1.1) ..................................... 8
  3.1 Variational analysis ......................................... 8
  3.2 Morawetz identity .......................................... 10
  3.3 Scattering criterion ......................................... 14
  3.4 Scattering .............................................. 17
4 The generalized Hartree problem (1.2) ................................. 18
  4.1 Variational analysis ......................................... 18
  4.2 Morawetz identity .......................................... 18
  4.3 Scattering criterion ......................................... 24
  4.4 Scattering .............................................. 26
References .................................................. 26

Communicated by A. Malchiodi.

Tarek Saanouni
t.saanouni@qu.edu.sa ; Tarek.saanouni@ipeiem.rnu.tn
1 Department of Mathematics, College of Sciences and Arts of Uglat Asugour, Qassim University, Buraydah, Kingdom of Saudi Arabia
1 Introduction

This manuscript is concerned with the energy scattering theory of the Cauchy problem for the following inhomogeneous focusing Schrödinger equation

$$\begin{align*}
i\dot{u} + \Delta^2 u - |x|^{2b}|u|^{2(q-1)}u &= 0; \\
u(0,\cdot) &= u_0, \tag{1.1}\end{align*}$$

and the inhomogeneous focusing Choquard equation

$$\begin{align*}
i\dot{u} + \Delta^2 u - (\mathcal{I}_\alpha \ast |\cdot|^b|u|^p)|x|^b|u|^{p-2}u &= 0; \\
u(0,\cdot) &= u_0. \tag{1.2}\end{align*}$$

Here and hereafter $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, for some natural integer $N$. The unbounded inhomogeneous term is $|\cdot|^b$, for some $b < 0$. The source terms satisfy $q > 1$ and $p \geq 2$.

The Riesz-potential is defined on $\mathbb{R}^N$ by

$$\mathcal{I}_\alpha : x \to \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N-\alpha}{2}}2^{\alpha}|x|^{N-\alpha}}, \quad 0 < \alpha < N.$$ 

The fourth-order Schrödinger problem was considered first in [10,11] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with a Kerr non-linearity.

The equation (1.1) enjoys the scaling invariance

$$u_\lambda = \lambda^{\frac{2+4b}{q-1}} u(\lambda^4\cdot, \lambda \cdot), \quad \lambda > 0.$$ 

The homogeneous Sobolev norm invariant under the previous scaling is $\|\cdot\|_{\dot{H}^s_c}$, where

$$s_c := \frac{N}{2} - \frac{2 + b}{q - 1}.$$ 

Similarly, the equation (1.2) satisfies the scaling invariance

$$u_\lambda = \lambda^{\frac{4+2b+\alpha}{2(p-1)}} u(\lambda^4\cdot, \lambda \cdot), \quad \lambda > 0.$$ 

This gives the critical Sobolev index

$$s'_c := \frac{N}{2} - \frac{4 + 2b + \alpha}{2(p-1)}.$$ 

The above problems have the same Sobolev critical index in the limit case $\alpha = 0$. In this note, one focus on the mass super-critical and energy sub-critical regimes $0 < s_c, s'_c < 2$.

To the author knowledge, in the literature, there exist few works dealing with the above inhomogeneous fourth-order Schrödinger equations. Indeed, for a local source term, in the mass-critical case, the existence of non-global solutions with negative energy was investigated in [3]. Moreover, the local well-posedness in the energy space of the problem (1.1) was established recently in [8]. Some well-posedness issues of the inhomogeneous bi-harmonic Choquard problem (1.2) were treated by the author in a submitted paper [6].

The scattering of the global focusing solutions under the ground state threshold exist for the limiting case $b = 0$ in (1.1), was proved [7] using the concentration compactness method due to [12]. This result was revisited recently in [4]. Note that there exist many works dealing with the biharmonic Schrödinger problem with a local source term. On the contrary, there is a few literature which treats the Hartree equation of the fourth order. In fact, some local
and global existence results in $H^s$ for the fourth-order non-linear Schrödinger–Hartree equation with variable dispersion coefficients were obtained in [1]. In addition, in the mass-super-critical and energy-sub-critical regimes, a sharp threshold of global well-posedness and scattering of energy solutions versus finite time blow-up dichotomy was given [16]. For the stationary case, see [2].

The challenge of this work is to extend [17], where the author establishes the scattering of defocusing global solutions to a class of inhomogeneous bi-harmonic Schrödinger equations. Indeed, one treats the scattering in the repulsive regime, which is related to the ground state threshold [9,12]. This gives a more complete idea about the above Schrödinger problems. Moreover, this work is an extension of the recent paper [4] to the case $b \neq 0$.

It is the aim of this note, to investigate the asymptotic behavior of global solutions to both inhomogeneous fourth-order Schrödinger and Choquard equations. Indeed, by use of Morawetz estimates and a scattering criterion in the spirit of [18], one obtains the scattering in the energy space under the ground state threshold. This work follows the ideas of [9] for the NLS equation.

The rest of this paper is organized as follows. The next section contains the main results and some useful estimates. Section three is devoted to prove the scattering of global solutions to the inhomogeneous fourth-order Schrödinger equation (1.1). The last section is consecrated to establish the scattering of global solutions to the inhomogeneous fourth-order Choquard equation (1.2).

Here and hereafter, $C$ denotes a constant which may vary from line to another. Denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\| \cdot \|_r := \| \cdot \|_{L^r}$ and $\| \cdot \| := \| \cdot \|_2$. The inhomogeneous Sobolev space $H^2 := H^2(\mathbb{R}^N)$ is endowed with the norm

$$\| \cdot \|_{H^2} := \left( \| \cdot \|^2 + \| \Delta \cdot \|^2 \right)^{\frac{1}{2}}.$$  

Let us denote also $C_T(X) := C([0, T], X)$ and $X_{rd}$ the set of radial elements in $X$. Moreover, for an eventual solution to (1.1) or (1.2), $T^* > 0$ denotes it’s lifespan. Finally, $x^\pm$ are two real numbers near to $x$ satisfying $x^+ > x$ and $x^- < x$.

## 2 Background and main results

This section contains the contribution of this paper and some standard estimates needed in the sequel.

### 2.1 Preliminary

Take for $R > 0$, $\psi_R := \psi(\cdot R)$, where $0 \leq \psi \leq 1$ is a radial smooth function satisfying

$$\psi \in C^\infty_0(\mathbb{R}^N), \quad supp(\psi) \subset \{|x| < 1\}, \quad \psi = 1 \text{ on } \left\{ |x| < \frac{1}{2} \right\}.$$

The mass-critical and energy-critical exponents for the Schrödinger problem (1.1) are

$$q_* := 1 + \frac{4 + 2b}{N} \quad \text{and} \quad q^* := \left\{ \begin{array}{ll} 1 + \frac{4 + 2b}{N-4}, & \text{if } N \geq 5; \\ \infty, & \text{if } 1 \leq N \leq 4. \end{array} \right.$$
The mass-critical and energy-critical exponents for the Choquard problem (1.2) are

\[ p_* := 1 + \frac{\alpha + 4 + 2b}{N} \quad \text{and} \quad \rho_* := \begin{cases} 1 + \frac{4+2b+\alpha}{N-4} & \text{if } N \geq 5; \\ \infty & \text{if } 1 \leq N \leq 4. \end{cases} \]

Here and hereafter, define the real numbers

\[ B := \frac{Np - N - \alpha - 2b}{2}, \quad A := 2p - B; \]
\[ D := \frac{Nq - N - 2b}{2}, \quad E := 2q - D. \]

For \( u \in H^2 \), take the mass, the action and the constraint of the Schrödinger problem (1.2),

\[ M[u] := \|u\|^2; \]
\[ S[u] := M[u] + E[u] := \|u\|^2_{H^2} - \frac{1}{q} \int_{\mathbb{R}^N} |x|^{2b}|u|^{2q} \, dx; \]
\[ K[u] := \|\Delta u\|^2 - \frac{D}{2q} \int_{\mathbb{R}^N} |x|^{2b}|u|^{2q} \, dx. \]

Similarly, the action and the constraint of the Choquard problem (1.1), read

\[ S[u] := M[u] + E[u] := \|u\|^2_{H^2} - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |x|^b|u|^p)|x|^b|u|^p \, dx; \]
\[ K[u] := \|\Delta u\|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} (I_\alpha * |x|^b|u|^p)|x|^b|u|^p \, dx. \]

**Definitions 2.1** Let us recall that

1. A ground state of (1.1) is a solution to

\[ Q + \Delta^2 Q - |x|^{2b}|Q|^{2(q-1)} Q = 0, \quad 0 \neq Q \in H^2, \quad (2.3) \]

which minimizes the problem

\[ \inf_{0 \neq u \in H^2} \left\{ S[v] \text{ s.t. } K[v] = 0 \right\}. \]

2. A ground state of (1.2) is a solution to

\[ \phi + \Delta^2 \phi - (I_\alpha * |x|^b|\phi|^p)|x|^b|\phi|^{p-2}\phi = 0, \quad 0 \neq \phi \in H^2, \quad (2.4) \]

which minimizes the problem

\[ \inf_{0 \neq u \in H^2} \left\{ S[u] \text{ s.t. } K[u] = 0 \right\}. \]

Let \( \phi \) is a ground state and \( u \) a solution to (1.1) and \( Q \) a ground state and \( v \) a solution to (1.2). The following scale invariant quantities describe the dichotomy of global/non-global existence of solutions [7].

\[ ME[u] := \frac{E[u]^{\frac{s_c}{s_c}} M[u]^{2-s_c}}{E[\phi]^{\frac{s_c}{s_c}} M[\phi]^{2-s_c}}, \quad MG[u] := \frac{\|\Delta u\|^{s_c} \|u\|^{2-s_c}}{\|\Delta \phi\|^{s_c} \|\phi\|^{2-s_c}}; \]
\[ ME[v] := \frac{E[v]^{\frac{s_c}{s_c}} M[v]^{2-s_c}}{E[Q]^{\frac{s_c}{s_c}} M[Q]^{2-s_c}}, \quad MG[v] := \frac{\|\Delta v\|^{s_c} \|v\|^{2-s_c}}{\|\Delta Q\|^{s_c} \|Q\|^{2-s_c}}. \]

Let us recall some local well-posedness results [6,8] about the above inhomogeneous fourth-order Schrödinger problems.
Proposition 2.2 Let \( N \geq 3, \max\{-4, -\frac{N}{2}\} < 2b < 0, \max\{1, 1 + \frac{1+2b}{N}\} < q < q^* \) and \( u_0 \in H^2 \). Then, there exists \( T^* = T^*(\|u_0\|_{H^2}) \) such that (1.1) admits a unique maximal solution
\[
u \in C_{T^*}(H^2).
\]
Moreover,
1. the solution satisfies the conservation laws
\[
\text{Mass} := M[u(t)] = M[u_0], \quad \text{Energy} := \mathcal{E}[u(t)] = \mathcal{E}[u_0].
\]
2. \( u \in L^q_t((0, T^*), W^{2,r}_r) \) for any admissible pair \((q, r)\) in the meaning of Definition 2.10.

Proposition 2.3 Let \( N \geq 3, 0 < \alpha < N \) and \( \max\{-N + \alpha, -4(1 + \frac{\alpha}{N}), N - 8 - \alpha\} < 2b < 0 \). Assume that \( N \geq 5 \) or \( 3 \leq N \leq 4 \) and \( 2\alpha + 4b + N > 0 \). If \( u_0 \in H^2 \) and \( 2 \leq p < p^* \). Then, there exists \( T^* = T^*(\|u_0\|_{H^2}) \) such that (1.2) admits a unique maximal solution
\[
u \in C_{T^*}(H^2).
\]
Moreover,
1. the solutions satisfies the conservation laws
\[
\text{Mass} := M[u(t)] = M[u_0], \quad \text{Energy} := E[u(t)] = E[u_0].
\]
2. \( u \in L^q_t((0, T^*), W^{2,r}_r) \) for any admissible pair \((q, r)\).

Remarks 2.4
1. The regularity condition \( p \geq 2 \), gives the restriction \( N - 8 - \alpha < 2b \). This seems to be technical, because it doesn’t appear in the energy;
2. denote for simplicity \((N, \alpha, b)\) satisfies (C) if, \( 0 < \alpha < N \) and \( \max\{-N + \alpha, -4(1 + \frac{\alpha}{N}), N - 8 - \alpha\} < 2b < 0 \) and \([N \geq 5 \) or \( 3 \leq N \leq 4 \) and \( 2\alpha + 4b + N > 0 \)].

The next inhomogeneous Gagliardo-Nirenberg type inequalities [6] are adapted to the above problems.

Proposition 2.5 Let \( N \geq 1 \).

1. Assume that \( \max\{-4, -N\} < 2b < 0 \) and \( 1 < q < q^* \). Then,
   a. there exists \( C(N, q, b) > 0 \), such that for any \( v \in H^2 \),
   \[
   \int_{\mathbb{R}^N} |x|^{2b}|v|^{2q} \, dx \leq C(N, q, b)\|v\|^q\|\Delta v\|^D;
   \]
   b. moreover,
   \[
   C(N, q, b) = \frac{2q}{E} \left( \frac{E}{D} \right)^{\frac{D}{2}} \|Q\|^{-2(q-1)},
   \]
   where \( Q \) is a solution to (2.3).

2. Let \( 0 < \alpha < N \), \( \max\{-N + \alpha, -4(1 + \frac{\alpha}{N})\} < 2b < 0 \) and \( 1 + \frac{\alpha}{N} < p < p^* \). Then,
   a. there exists \( C(N, p, b, \alpha) > 0 \), such that for any \( u \in H^2 \),
   \[
   \int_{\mathbb{R}^N} (I_{\alpha} * |\cdot|^{b}|u|^p)|x|^b|u|^p \, dx \leq C(N, p, b, \alpha)\|u\|^A\|\Delta u\|^B;
   \]
b. moreover,
\[
C(N, p, b, \alpha) = \frac{2p}{A} \left( \frac{A}{B} \right)^{\frac{b}{2}} \|\phi\|^{-2(p-1)},
\]
where \(\phi\) is a solution to (2.4).

2.2 Main results

This subsection contains the contribution of this note. The first main goal of this manuscript is to prove the following scattering result.

**Theorem 2.6** Let \(N \geq 5, \max\{-4, -\frac{N}{2}\} < 2b < 0\) and \(\max\{q_*, x_0\} < q < q^*\) such that \(q \geq \frac{3}{2}\). Let \(Q\) be a solution to (2.3) satisfying (2.5) and \(u_0 \in H^2\) such that \(E[u_0] \geq 0\) and
\[
\max \left\{ \mathcal{M} E[u_0], \mathcal{M} G[u_0] \right\} < 1.
\]
Take a maximal solution \(u \in C_T^* (H^2)\) of (1.1). Thus, \(u\) is global and there exists \(u_\pm \in H^2\) such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it \Delta_2} u_\pm\|_{H^2} = 0.
\]

**Remarks 2.7** Note that
1. The global existence of solutions under the assumption (2.7) was proved in [6];
2. The condition \(N \geq 5\) is required in the proof of the scattering criterion;
3. The radial assumption is required in one step of the proof of Morawetz estimate and in the scattering criterion;
4. \(x_0\) is the positive root of the polynomial function
\[
(2X - 1)(X - 1) - \frac{2(2 + b)}{N - 4};
\]
5. The set of \(q\) satisfying the above conditions is nonempty because
\[
P(q^*) = 2 \left( \frac{4 + 2b}{N - 4} \right)^2 > 0;
\]
6. The proof avoids the concentration compactness method due to [12] and follows the new approach of [5]. First, using the radial Sobolev embedding, one establishes a localized Morawetz estimate (see Lemma 3.4), which in turn implies an energy evacuation for large time (see Lemma 3.5). Second, one obtains a scattering criterion introduced by Tao [18] (see Proposition 3.6). This suffices to prove the above Theorem.

The second main goal of this manuscript is to prove the following scattering result.

**Theorem 2.8** Let \((N, \alpha, b)\) satisfying (C) and \(\max\{p_*, x_\alpha\} < p < p^*\) such that \(p \geq \max\{2, \frac{3}{2} + \frac{q}{N}\}\). Let \(\phi\) satisfying (2.4), (2.6) and \(u_0 \in H^2\) satisfying
\[
\max \left\{ \mathcal{M} E[u_0], \mathcal{M} G[u_0] \right\} < 1.
\]
Take \(u \in C_T^* (H^2)\) be a maximal solution to (1.2). Then, \(u\) is global and there exists \(u_\pm \in H^2\) such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it \Delta_2} u_\pm\|_{H^2} = 0.
\]
Remarks 2.9 Note that

1. the global existence of solutions under the assumption (2.7) was proved in [6];
2. $x_\alpha$ is the positive root of the polynomial
   $$ (X - 1)(2X - 1) - \frac{4 + 2b + \alpha}{N - 4}; $$
3. the set of $p$ satisfying the above conditions is nonempty because
   $$ P_\alpha(p^*) = 2 \left( \frac{4 + 2b + \alpha}{N - 4} \right)^2 > 0; $$
4. it seems that the local and non-local source terms in the above Schrödinger problems have a similar asymptotic behavior;
5. the proof, which follows the new approach of [5], is based on the energy evacuation for large time in Lemma 4.6 and the scattering criterion in Proposition 4.7.

2.3 Useful estimates

Let us gather some classical tools needed in the sequel.

Definition 2.10 Take $N \geq 1$ and $s \in [0, 2)$. A couple of real numbers $(q, r)$ is said to be $s$-admissible (admissible for $0$-admissible) if

$$ \frac{2N}{N - 2s} \leq r < \frac{2N}{N - 4}, \quad 2 \leq q, r \leq \infty \quad \text{and} \quad N \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{4}{q} + s. $$

Denote the set of $s$-admissible pairs by $\Gamma_s$ and $\Gamma := \Gamma_0$. If $I$ is a time slab, one denotes the Strichartz spaces

$$ S^s(I) := \cap_{(q,r) \in \Gamma_s} L^q(I, L^r) \quad \text{and} \quad S^{s'}(I) := \cap_{(q,r) \in \Gamma_{s'}} L^q(I, L^{r'}). $$

Recall the Strichartz estimates [7,15].

Proposition 2.11 Let $N \geq 1$, $0 \leq s < 2$ and $t_0 \in I \subset \mathbb{R}$, an interval. Then,

1. $\sup_{(q,r) \in \Gamma} \| u \|_{L^q(I, L^r)} \lesssim \| u(t_0) \| + \inf_{(\tilde{q}, \tilde{r}) \in \Gamma} \| \tilde{i} \tilde{u} + \tilde{\Delta}^2 u \|_{L^\tilde{q}(I, L^{\tilde{r}})};$
2. $\sup_{(q,r) \in \Gamma} \| \tilde{\Delta} u \|_{L^q(I, L^r)} \lesssim \| \tilde{\Delta} u(t_0) \| + \| \tilde{i} \tilde{u} + \tilde{\Delta}^2 u \|_{L^2(I, W^{1, \frac{2N}{2N-4}})}; \quad \forall N \geq 3;$
3. $\sup_{(q,r) \in \Gamma_s} \| u \|_{L^q(I, L^r)} \lesssim \| u(t_0) \| + \inf_{(\tilde{q}, \tilde{r}) \in \Gamma_{s'}} \| \tilde{i} \tilde{u} + \tilde{\Delta}^2 u \|_{L^\tilde{q}(I, L^{\tilde{r}})}.$

Let us recall a Hardy-Littlewood-Sobolev inequality [13].

Lemma 2.12 Take $N \geq 1$.

1. Let $0 < \lambda < N$ and $1 < r, s < \infty$ satisfying $2 = \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N}$. Thus,
   $$ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u(x)v(y)}{|x - y|^\lambda} \, dx \, dy \leq C_{N,s,\lambda} \| u \|_r \| v \|_s, \quad \forall u \in L^r, \forall v \in L^s.$$
2. Let $0 < \alpha < N$ and $1 < r, s, q < \infty$ satisfying $1 + \frac{\alpha}{q} = \frac{1}{r} + \frac{1}{s} + \frac{1}{\alpha}$. Thus,
   $$ \| (I_\alpha * u) v \|_{r'} \leq C_{N,s,\alpha} \| u \|_r \| v \|_q, \quad \forall u \in L^s, \forall v \in L^q.$$

Finally, let us give an abstract result.
Lemma 2.13  Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that
\[
X \leq a + bX^\theta \text{ on } [0, T],
\]
where $a, b > 0$, $\theta > 1$, $a < 1 - \frac{1}{\theta}((\theta b)^\frac{1}{1-\theta}$ and $X(0) \leq (\theta b)^\frac{1}{1-\theta}$. Then
\[
X \leq \frac{\theta}{\theta - 1}a \text{ on } [0, T].
\]

Proof  The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (\theta b)^\frac{1}{1-\theta}$ and increasing on $((\theta b)^\frac{1}{1-\theta}, \infty)$. The assumptions imply that $f((\theta b)^\frac{1}{1-\theta}) < 0$ and $f(\frac{\theta}{\theta - 1}a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (\theta b)^\frac{1}{1-\theta}$, we conclude the proof by a continuity argument. □

3 The Schrödinger problem (1.1)

The goal of this section is to prove Theorem 2.6. Let us collect some estimates needed in the proof of the scattering of global solutions to the focusing Schrödinger problem (1.1).

3.1 Variational analysis

Lemma 3.1  Let $N \geq 3, \{-4, -\frac{N}{2}\} < 2b < 0$ and $q_* < q < q^*$. Take $u_0 \in H^2$ satisfying (2.7). Then, there exists $0 < \delta < 1$ such that the solution $u \in C(\mathbb{R}, H^2)$ satisfies
\[
\max \left\{ \sup_{t \in \mathbb{R}} \mathcal{M}E(u(t)), \sup_{t \in \mathbb{R}} \mathcal{M}G(u(t)) \right\} < 1 - \delta.
\]

Proof  Denote $C_{N,q,b} := C(N, q, b)$ given by Proposition 2.5. The inequality $\mathcal{M}E(u_0) < 1$ gives the existence of $\delta > 0$ such that
\[
1 - \delta > \frac{M(u_0)\frac{2-\delta}{\delta}E(u_0)}{M(\phi)\frac{2-\delta}{\delta}E(\phi)} \geq \frac{M(u_0)\frac{2-\delta}{\delta}E(u_0)}{M(Q)\frac{2-\delta}{\delta}E(Q)} \left( \|\Delta u(t)\|^2 - \frac{1}{q} \int_{\mathbb{R}^N} |x|^{2b}|u|^{2q} \, dx \right) \geq \frac{M(u_0)\frac{2-\delta}{\delta}E(u_0)}{M(Q)\frac{2-\delta}{\delta}E(Q)} \left( \|\Delta u(t)\|^2 - \frac{C_{N,p,b}}{q} \|u\|E \|\Delta u(t)\|D \right).
\]

Thanks to Pohozaev identities, one has
\[
E(Q) = \frac{D - 2}{D} \|\Delta Q\|^2 = \frac{D - 2}{E} \|Q\|^2.
\]

Thus,
\[
1 - \delta > \frac{D}{D - 2} \frac{M(u_0)\frac{2-\delta}{\delta}}{M(Q)\frac{2-\delta}{\delta} \|\Delta Q\|^2} \left( \|\Delta u(t)\|^2 - \frac{C_{N,q,b}}{p} \|u\|E \|\Delta u(t)\|D \right)
\]
\[ \frac{D}{D-2} M(u_0) \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2 - \frac{D}{D-2} M(Q) \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2 C_{N,q,b} \frac{q}{q} \| u \|^E \| \Delta u(t) \|^D \]

\[ \frac{D}{D-2} M(u_0) \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2 - \frac{D}{D-2} M(Q) \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2 \frac{2}{E} \left( \frac{E}{D} \right)^{D/2} \| Q \|^{-2(q-1)} \| u \|^E \| \Delta u(t) \|^D \]

\[ \frac{D}{D-2} M(u_0) \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2 - \frac{D}{D-2} E \frac{M(u_0) \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2}{M(Q) \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2} \left( \frac{\| Q \|}{\| \Delta Q \|} \right)^D \| Q \|^{-2(q-1)} \| u \|^E \| \Delta u(t) \|^D \]

Using the equalities \( s_c = \frac{D-2}{q-1} \) and \( D = \left( \frac{\| \Delta Q \|}{\| Q \|} \right)^2 \), one has

\[ 1 - \delta > \frac{D}{D-2} M(u_0) \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2 - \frac{D}{D-2} \frac{2}{E} \left( \frac{\| u_0 \| \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2}{M(\phi) \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2} \left( \frac{\| Q \|}{\| \Delta Q \|} \right)^D \| Q \|^{-2(q-1)} \| u \|^E \| \Delta u(t) \|^D \]

\[ > \frac{D}{D-2} E \frac{2}{E} \frac{\| u_0 \| \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2}{M(Q) \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2} - 2 \left( \frac{\| u_0 \| \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2}{\| Q \| \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2} \right)^D \left( \frac{\| Q \|}{\| \Delta Q \|} \right)^D \| Q \|^{-2(q-1)} \| u \|^E \| \Delta u(t) \|^D \]

Take the real function defined on \([0, 1]\) by \( f(x) := \frac{D}{D-2} x^2 - \frac{2}{D-2} x^D \), with first derivative \( f'(x) = \frac{2D}{D-2} x (1 - x^{D-2}) \). Thus, with the table change of \( f \) and the continuity of \( t \rightarrow X(t) := \frac{\| u_0 \| \left( \frac{2-s_c}{-s_c} \right) \| \Delta u(t) \|^2}{\| Q \| \left( \frac{2-s_c}{-s_c} \right) \| \Delta Q \|^2} \), it follows that \( X(t) < 1 \) for any \( t < T^* \). Thus, \( T^* = \infty \) and there exists \( \epsilon > 0 \) near to zero such that \( X(t) \in f^{-1}([0, 1 - \delta]) = [0, 1 - \epsilon] \). This finishes the proof. \( \square \)

Taking \( \delta' := \frac{\delta D}{2q(1-\delta)} \), one proves a coercivity estimate on centered balls with large radials.

**Lemma 3.2** There exists \( R_0 := R_0(\delta, M(u), Q) > 0 \) such that for any \( R > R_0 \),

\[ \sup_{t \in \mathbb{R}} \| \psi_R u(t) \|^2 - s_c \| \Delta (\psi_R u(t)) \|^s_c < (1 - \delta) \| \phi \|^2 - s_c \| \Delta \phi \|^s_c; \]
\[ \| \Delta(\psi_R u) \|^2 - \frac{D}{2q} \int_{\mathbb{R}^N} |x|^{2b} |\psi_R u|^{2q} \, dx \geq \delta' \int_{\mathbb{R}^N} |x|^{2b} |\psi_R u|^{2q} \, dx. \]

**Proof** A direct computation \cite{4} gives
\[ \| \Delta(\psi_R u) \|^2 - \| \psi_R \Delta u \|^2 \leq C(u_0, \phi) R^{-2}. \] (3.9)

Then, one gets the first point via Lemma 3.1. Now, taking account of Proposition 2.5, one has
\[
\mathcal{E}(\psi_R u) = \| \Delta(\psi_R u) \|^2 - \frac{1}{q} \int_{\mathbb{R}^N} |x|^{2b} |\psi_R u|^{2q} \, dx
\]
\[ \geq \| \Delta(\psi_R u) \|^2 \left( 1 - \frac{C_{N,a,b}}{q} \| \psi_R u \| \| \Delta(\psi_R u) \|^D \right) \]
\[ \geq \| \Delta(\psi_R u) \|^2 \left( 1 - \frac{C_{N,a,b}}{q} \| \psi_R u \|^{2-s_c} \| \Delta(\psi_R u) \|^{s_c} \right)^{-1}. \]

So, with the first point, one writes
\[
\mathcal{E}(\psi_R u) \geq \| \Delta(\psi_R u) \|^2 \left( 1 - (1 - \delta) \frac{2}{E} \left( \frac{E}{D} \right)^{\frac{2}{q}} \| Q \|^{-2(q-1)} \| \| Q \|^{2-s_c} \| \Delta Q \|^{s_c} \right)^{-1} \]
\[ \geq \| \Delta(\psi_R u) \|^2 \left( 1 - (1 - \delta) \frac{2}{E} \left( \frac{E}{D} \right)^{\frac{2}{q}} \right) \]
\[ \geq \| \Delta(\psi_R u) \|^2 \left( 1 - (1 - \delta) \frac{2}{D} \right). \]

This implies that
\[ \| \Delta(\psi_R u) \|^2 - \frac{D}{2q} \int_{\mathbb{R}^N} |x|^{2b} |\psi_R u|^{2q} \, dx \geq \delta^2 \| \Delta(\psi_R u) \|^2 \geq \delta' \int_{\mathbb{R}^N} |x|^{2b} |\psi_R u|^{2q} \, dx. \]

The Lemma is proved. \qed

### 3.2 Morawetz identity

This subsection is devoted to prove a classical Morawetz estimate satisfied by the energy global solutions to the inhomogeneous Schrödinger problem (1.1). One adopts the convention that repeated indexes are summed. Also, if \( f, g \) are two differentiable functions, one defines the momentum brackets by
\[ \{ f, g \}_P := \Re (f \overline{\nabla g} - g \overline{\nabla f}). \]

Let us start with an auxiliary result.

**Lemma 3.3** Take \( N \geq 3, \max\{-\frac{N}{x}, -4\} < 2b < 0, \max\{1, 1 + \frac{2+2b}{N}\} < q \leq q^* \) and \( u \in C_T(H^2) \) be a local solution to (1.1). Let \( a : \mathbb{R}^N \to \mathbb{R} \) be a convex smooth function and the real function defined on \([0, T)\), by
\[ M(t) := 2 \int_{\mathbb{R}^N} \nabla a(x) \Re (\nabla u(t,x) \overline{\delta(t,x)}) \, dx. \]

\( \square \) Springer
Then, the following equalities hold on $[0, T]$.

\[
M' = 2 \int_{\mathbb{R}^N} \left( 2 \partial_{jk} \Delta a \partial_j u \partial_k \bar{u} - \frac{1}{2} (\Delta^3 a)|u|^2 - 4 \partial_{jk} a \partial_i k u \partial_j \bar{u} + \Delta^2 a |\nabla u|^2 + \partial_j a(|x|^{2b}|u|^{2(q-1)} u, u)^j \right) \, dx
\]

\[
= 2 \int_{\mathbb{R}^N} \left( 2 \partial_{jk} \Delta a \partial_j u \partial_k \bar{u} - \frac{1}{2} (\Delta^3 a)|u|^2 - 4 \partial_{jk} a \partial_i k u \partial_j \bar{u} + \Delta^2 a |\nabla u|^2 + \frac{q-1}{q} (\Delta a)|x|^{2b}|u|^{2q} - \frac{1}{q} \nabla a \nabla (|x|^{2b})|u|^{2q} \right) \, dx.
\]

**Proof** Denote the source term $\mathcal{N} := -|x|^{2b}|u|^{2(q-1)} u$ and compute

\[
\partial_t \mathcal{N}(\partial_k u \bar{u}) = \mathcal{N}(\partial_k \bar{\bar{u}}) + \mathcal{N}(\partial_k u \bar{u})
\]

\[
= \Re(i \partial_k \bar{\bar{u}}) \Re(i \partial_k \bar{u}) = \Re(\partial_k \bar{\bar{u}} (-\Delta^2 u - \mathcal{N})) - \Re(\partial_k \bar{\bar{u}} (-\Delta^2 u - \mathcal{N}))
\]

\[
= \Re(i \partial_k \Delta^2 u - \partial_k \bar{\bar{u}} \Delta^2 u) + \Re(\partial_k \mathcal{N} - \partial_k \bar{\bar{u}} \mathcal{N}).
\]

Thus,

\[
M' = 2 \int_{\mathbb{R}^N} \partial_k a \Re(i \partial_k \Delta^2 u - \partial_k \bar{\bar{u}} \Delta^2 u) \, dx - 2 \int_{\mathbb{R}^N} \partial_k a \{\mathcal{N}, u\}_{p} \, dx
\]

\[
= -2 \int_{\mathbb{R}^N} \Delta a \Re(i \partial_k \Delta^2 u) \, dx - 4 \int_{\mathbb{R}^N} \Re(i \partial_k a \partial_k \bar{\bar{u}} \Delta^2 u) \, dx + 2 \int_{\mathbb{R}^N} \partial_k a \{\mathcal{N}, u\}_{p} \, dx
\]

The first equality in the above Lemma follows as in Proposition 3.1 in [14]. For the second equality, it is sufficient to use the identity

\[
\{\mathcal{N}, u\}_{p} = -\frac{q-1}{q} \nabla (|x|^{2b})|u|^{2q} - \frac{1}{q} \nabla (|x|^{2b})|u|^{2q}.
\]

\[\square\]

The main result of this subsection is the following.

**Lemma 3.4** Take $N \geq 3$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$ and $q_* < q < q^*$. Let $u_0 \in H^2$ satisfying (2.7). Then, for any $T > 0$, one has

\[
\int_0^T \int_{\mathbb{R}^N} |x|^{2b}|u(t)|^{2q} \, dx \, dt \leq CT^{1/3}.
\]

**Proof** Take a smooth real function such that $0 \leq f'' \leq 1$ and

\[
f : r \rightarrow \begin{cases} \frac{r^2}{T^2}, & \text{if } 0 \leq r \leq \frac{1}{2} ; \\ 1, & \text{if } r \geq 1. \end{cases}
\]

Moreover, for $R > 0$, let the smooth radial function defined on $\mathbb{R}^N$ by $f_R := R^2 f(\frac{1}{R})$. One can check that

\[
0 \leq f''_R \leq 1, \quad f'(r) \leq r, \quad N \geq \Delta f_R.
\]

Let the real function

\[
M_R : t \rightarrow 2 \int_{\mathbb{R}^N} \nabla f_R(x) \Re(\nabla u(t, x) \bar{u}(t, x)) \, dx.
\]
By Morawetz estimate in Lemma 3.3, one has

\[
M_R' = 2 \int_{\mathbb{R}^N} \left( 2 \partial_{j\ell} \Delta f_R \partial_j u \partial_\ell \bar{u} - \frac{1}{2} (\Delta^3 f_R) |u|^2 + \Delta^2 f_R |\nabla u|^2 - \frac{1}{q} \nabla f_R \nabla (|x|^{2b}) |u|^{2q} 
+ \frac{q - 1}{q} (\Delta f_R) |x|^{2b} |u|^{2q} - 4 \partial_{j\ell} f_R \partial_j u \partial_\ell \bar{u} \right) \, dx
\]

\[ = 2 \int_{\mathbb{R}^N} \left( 2 \partial_{j\ell} \Delta f_R \partial_j u \partial_\ell \bar{u} - \frac{1}{2} (\Delta^3 f_R) |u|^2 + \Delta^2 f_R |\nabla u|^2 - \frac{1}{q} \nabla f_R \nabla (|x|^{2b}) |u|^{2q} 
+ 2(N \frac{q - 1}{q} \int_{|x| < \frac{r}{2}} |x|^{2b} |u|^{2q} \, dx - 4 \int_{|x| < \frac{r}{2}} |\Delta u|^2 \, dx - \frac{2b}{q} \int_{|x| < \frac{r}{2}} |x|^{2b} |u|^{2q} \, dx \right) \, dx
\]

\[ + 2 \int_{\{ \frac{r}{2} < |x| < r \}} \left( \frac{q - 1}{q} \Delta f_R |x|^{2b} |u|^{2q} - 4 \partial_{j\ell} f_R \partial_j u \partial_\ell \bar{u} \right) \, dx
\]

Using the estimate \( \| \nabla |^k f_R \|_\infty \lesssim R^{2-k} \), one has

\[
\left| \int_{\mathbb{R}^N} \partial_{j\ell} \Delta f_R \partial_j u \partial_\ell \bar{u} \, dx \right| \lesssim R^{-2};
\]

\[
\left| \int_{\mathbb{R}^N} (\Delta^3 f_R) \, dx \right| \lesssim R^{-4};
\]

\[
\left| \int_{\mathbb{R}^N} \Delta^2 f_R \, dx \right| \lesssim R^{-2}.
\]

Moreover, by the radial setting, one writes

\[
\int_{\{ \frac{r}{2} < |x| < r \}} \partial_{j\ell} f_R \partial_j u \partial_\ell \bar{u} \, dx \geq (N - 1) \int_{\{ \frac{r}{2} < |x| < r \}} \frac{f_R(r)}{r^3} |\partial_r u|^2 \, dx = O(R^{-2}).
\]

Now, by Hardy–Littlewood–Sobolev and Strauss inequalities

\[
\left| \int_{\{ \frac{r}{2} < |x| < r \}} \Delta f_R |x|^{2b} |u|^{2q} \, dx \right| \lesssim \int_{\{ \frac{r}{2} < |x| < r \}} |x|^{2b} |u|^{2(q-1)} |u|^2 \, dx
\]

\[
\lesssim \int_{\{ \frac{r}{2} < |x| < r \}} |x|^{2b-(N-1)(q-1)} |u|^2 \, dx
\]

\[
\lesssim R^{2b-(N-1)(q-1)} \|u\|^2
\]

\[
\lesssim R^{-(N-1)(q-1)-2b}.
\]

With the same way, one has

\[
\left| \int_{\{ \frac{r}{2} < |x| < r \}} \nabla f_R \nabla (|x|^{2b}) |u|^{2q} \, dx \right| \lesssim R^{-(N-1)(q-1)-2b}.
\]
Thus, since $\|\nabla u\|^2 \lesssim \|\Delta u\| \lesssim 1$ and $q > q^*$, one gets

$$M_R' \leq 4 \left( \frac{D}{q} \int_{|x| < \frac{R}{2}} |x|^{2b}|u|^2 \, dx - 2 \int_{|x| < \frac{R}{2}} |\Delta u|^2 \, dx \right) + \mathcal{O}(R^{-2}).$$

So, with Lemma 3.2, via (3.9), one gets

$$\sup_{[0,T]} |M_R| \geq 8 \int_0^T \left( \int_{|x| < \frac{R}{2}} |\Delta(\psi_Ru)|^2 \, dx - \frac{D}{2q} \int_{|x| < \frac{R}{2}} |x|^{2b}|\psi_Ru|^2 \, dx \right) \, dt$$

$$+ \mathcal{O}(R^{-2}) T + \mathcal{O}(R^{-(N-1)(q-1)-2b}) T$$

$$\geq 8\delta' \int_0^T \int_{\mathbb{R}^N} |x|^{2b}|\psi_Ru(t)|^2 \, dx \, dt + \mathcal{O}(R^{-2}) T$$

$$\geq 8\delta' \int_0^T \int_{|x| < \frac{R}{2}} |x|^{2b}|u(t)|^2 \, dx \, dt + \mathcal{O}(R^{-2}) T.$$  

Moreover, with Strauss estimate, one has

$$\int_0^T \int_{|x| > \frac{R}{2}} |x|^{2b}|u(t)|^2 \, dx \, dt \leq \int_0^T \|x|^{2b}|u(t)|^{2q-2}\|_{L_\infty(|x| > \frac{R}{2})} \|u_0\|^2 \, dt$$

$$\lesssim TR^{2b-(q-1)(N-1)}.$$  

Since $q > q^*$ and $N > 2 + b$, one has $2 < (q-1)(N-1)-2b$ and the previous computation give

$$\int_0^T \int_{\mathbb{R}^N} |x|^{2b}|u(t)|^2 \, dx \, dt \leq C \left( \sup_{[0,T]} |M| + T \left( R^{2b-(q-1)(N-1)} + R^{-2} \right) \right)$$

$$\leq C \left( R + TR^{-2} \right).$$

Taking $R = T^{1/3} \gg 1$, one gets the requested estimate

$$\int_0^T \int_{\mathbb{R}^N} |x|^{2b}|u(t)|^2 \, dx \, dt \leq CT^{1/3}.$$  

For $0 < T << 1$, the proof follows with Sobolev injections.  

As a consequence, one has the following energy evacuation.

**Lemma 3.5** Take $N \geq 3$ and $\max\{-4, -\frac{N}{2}\} < 2b < 0$ and $q_* < q < q^*$. Let $u_0 \in H^2_{rd}$ satisfying (2.7). Then, there exists a sequence of real numbers $t_n \to \infty$ such that

$$\lim_n \int_{|x| < R} |u(t_n, x)|^2 \, dx = 0, \quad \text{for all} \ R > 0.$$  

**Proof** Take $t_n \to \infty$. By Hölder estimate

$$\int_{|x| < R} |u(t_n, x)|^2 \, dx \leq R^\frac{N}{q} \left( \int_{|x| < R} |u(t_n, x)|^{2q} \, dx \right)^{\frac{1}{q}}$$

$$\leq R^\frac{N}{q} - \frac{2b}{q} \left( \int_{|x| < R} |x|^{2b}|u(t_n, x)|^{2q} \, dx \right)^{\frac{1}{q}}$$
Indeed, by the previous Lemma
\[
\int_{\mathbb{R}^N} |x|^{2b} |u(t_n)|^{2q} \, dx \to 0.
\]

\[\square\]

### 3.3 Scattering criterion

This section is devoted to prove the next result.

**Proposition 3.6** Let \( N \geq 3, \) \( \max\{-4, -\frac{N}{2}\} < 2b < 0. \) Take \( \max\{q_s, x_0\} < q < q^* \) such that \( q \geq \frac{3}{2}. \) Let \( u \in C(\mathbb{R}, H^2_{rad}) \) be a global radial solution to (1.1). Assume that
\[ 0 < \sup_{t \geq 0} \|u(t)\|_{H^2} := E < \infty. \]

There exist \( R, \epsilon > 0 \) depending on \( E, N, q, b \) such that if
\[
\liminf_{t \to +\infty} \int_{|x| < R} |u(t, x)|^2 \, dx < \epsilon,
\]
then, \( u \) scatters for positive time.

**Proof** By Lemma 3.1, \( u \) is bounded in \( H^2. \) Take \( \epsilon > 0 \) near to zero and \( R(\epsilon) >> 1 \) to be fixed later. Define the Strichartz norm
\[
\| \cdot \|_{S^s(I)} := \sup_{(q, r) \in \Gamma_s} \| \cdot \|_{L^q(I), L^r}.
\]

Let us estimate the non-linear part of the solution with Strichartz norms.

**Lemma 3.7** Take \( N \geq 3, \) \( \max\{-4, -\frac{N}{2}\} < 2b < 0. \) Take \( q_s < q < q^* \) and \( u \in C(\mathbb{R}, H^2) \) be a global solution to (1.1). Then, there exists \( \theta \in (0, 2q - 1) \) such that
\[
\|u - e^{i\Delta} u_0\|_{S^{s_c}(I)} \lesssim \|u\|^{\theta}_{L^\infty(I, H^2)} \|u\|^{2q-1-\theta}_{L^q(I, L^r)},
\]
for a certain \((a, r) \in \Gamma_{s_c}.\)

**Proof** Take the real numbers
\[
a := \frac{2(2q - \theta)}{2 - s_c}, \quad d := \frac{2(2q - \theta)}{2 + (2q - 1 - \theta) s_c};
\]
\[
r := \frac{2N(2q - \theta)}{(N - 2s_c)(2q - \theta) - 4(2 - s_c)}.
\]
The condition \( \theta = 0^+ \) gives via some computations that the previous pairs satisfy the admissibility conditions. Indeed, \( a > \frac{4}{2 - s_c} \) and \( r < \frac{2N}{N - 4}. \) Moreover,
\[
r = \frac{2N(2q - \theta)}{(N - 2s_c)(2q - \theta) - 4(2 - s_c)} > \frac{2N}{N - 2s_c}.
\]
Thus,
\[
(a, r) \in \Gamma_{s_c}, \quad (d, r) \in \Gamma_{-s_c}, \quad (2q - 1 - \theta)d' = a.
\]
Take two real numbers $r_1, \mu$ satisfying

$$1 = \frac{2}{\mu} + \frac{\theta}{r_1} + \frac{2q - \theta}{r}.$$ 

This gives

$$\frac{2N}{\mu} = N - \frac{N\theta}{r_1} - \frac{N(2q - \theta)}{r}$$

$$= N - \frac{N\theta}{r_1} - \frac{(N - 2s_e)(2q - \theta) - 4(2 - s_e)}{2}$$

$$= -2b - \frac{N\theta}{r_1} + \frac{2 + b}{q - 1}.$$ 

Using Hardy-Littlewood-Sobolev and Hölder estimates, one has

$$\|N\|_{L^d'(I, L^{r'}([x]<1))} \lesssim \|x|^b\|^2_{L^{\mu}([x]<1)} \|u\|^\theta_{L^\infty(I, L^r')} \|u\|_{L^r'}^{2q-1-\theta} \|L^d'(I).$$

Then, choosing $r_1 := \frac{2N}{N-4}$, one gets because $q < q^*$,

$$\frac{2N}{\mu} + 2b = \frac{\theta}{2(q - 1)}(4 + 2b - (q - 1)(N - 4)) > 0.$$ 

So, $|x|^b \in L^\mu([x] < 1)$. Then, by Sobolev injections, one gets

$$\|N\|_{L^d'(I, L^{r'}([x]<1))} \lesssim \|x|^b\|^2_{L^{\mu}([x]<1)} \|u\|^\theta_{L^\infty(I, L^r')} \|u\|_{L^r'}^{2q-1-\theta} \|L^d'(I)$$

Moreover, on the complementary of the unit ball, one takes $r_1 := 2$, so because $q > q^*$, one has

$$\frac{2N}{\mu} + 2b = \frac{\theta}{2(q - 1)}(4 + 2b - N(q - 1)) < 0.$$ 

So, $|x|^b \in L^\mu([x] > 1)$. The proof is achieved via Strichartz estimates. \hfill \Box

The following result is the key to prove the scattering criterion.

**Proposition 3.8** Let $N \geq 3$, $\max\{-4, -\frac{N}{2}\} < 2b < 0$. Take $\max(q_s, x_0) < q < q^*$ such that $q \geq \frac{3}{2}$. Take $u_0 \in H^2_{1,q}$ satisfying (2.7). Then, for any $\varepsilon > 0$, there exist $T$, $\mu > 0$ such that the global solution to (1.1) satisfies

$$\|e^{i(-T)} \Delta^2 u(T)\|_{L^\mu((T, \infty), L^r')} \lesssim \varepsilon^\mu.$$ 

**Proof** One keeps notations of the proof of Lemma 3.7. Let $0 < \beta << 1$ and $T > \varepsilon^{-\beta} > 0$. By the integral formula

$$e^{i(-T)} \Delta^2 u(T) = e^{i\Delta^2 u_0} + i \int_0^T e^{i(-s)} \Delta^2 \left[|x|^{2b}|u|^{2(q-1)}u\right]ds$$

$$= e^{i\Delta^2 u_0} + i \left( \int_0^{T-\varepsilon^{-\beta}} + \int_{T-\varepsilon^{-\beta}}^T \right) e^{i(-s)} \Delta^2 \left[|x|^{2b}|u|^{2(q-1)}u\right]ds$$

$$:= e^{i\Delta^2 u_0} + F_1 + F_2.$$
The term $F_2$. By Lemma 3.7, one has
\[
\|F_2\|_{L^q((T,\infty),L^r)} \lesssim \|u\|_{L^\infty((T,\infty),H^2)}^{2q-1-\theta} \|u\|_{L^q(T,\infty)}^{2q-1-\theta} \lesssim \|u\|_{L^q(T,\infty)}^{2q-1-\theta}.
\]
Now, by the assumptions of the scattering criterion, one has
\[
\int_{\mathbb{R}^N} \psi_R(x)|u(T,x)|^2 \, dx < \epsilon^2.
\]
Moreover, a computation with use of (1.1) and the properties of $\psi$ give
\[
\left| \frac{d}{dt} \int_{\mathbb{R}^N} \psi_R(x)|u(t,x)|^2 \, dx \right| \lesssim R^{-1}.
\]
Then, for any $T - \epsilon^{-\beta} \leq t \leq T$ and $R > \epsilon^{-2-\beta}$, yields
\[
\|\psi_R u(t)\| \leq \left( \int_{\mathbb{R}^N} \psi_R(x)|u(T,x)|^2 \, dx + C \frac{T-t}{R} \right)^{1/2} \leq C \epsilon.
\]
Since $2 < r < \frac{2N}{N-2}$, there exists $\lambda$, $\lambda' \in (0, 1)$ such that, for $R > \epsilon^{\frac{-2N}{(N-1)\lambda'}}$,
\[
\|u\|_{L^\infty((T,\infty),L^r)} \lesssim \|\psi_R u\|_{L^\infty((T,\infty),L^r)} + \|(1 - \psi_R)u\|_{L^\infty((T,\infty),L^r)}
\]
\[
\lesssim \|\psi_R u\|_{L^\infty((T,\infty),L^2)}^{\lambda'} \|\psi_R u\|_{L^\infty((T,\infty),L^{\frac{2N}{N-2}})}^{1-\lambda'}
\]
\[
+ \|(1 - \psi_R)u\|_{L^\infty((T,\infty),L^\infty)}^{\lambda'} \|(1 - \psi_R)u\|_{L^\infty((T,\infty),L^2)}^{1-\lambda'}
\]
\[
\lesssim \epsilon^{\lambda'} + R^{\lambda'(N-1)} \lesssim \epsilon^\lambda.
\]
The third inequality above is thanks to Strauss estimate via the fact that $\psi_R - 1 = 0$ on $\{|x| < \frac{R}{2}\}$. So,
\[
\|F_2\|_{L^q((T,\infty),L^r)} \lesssim \epsilon^{-(2q-1-\theta)\frac{\beta}{\theta}} \|u\|_{L^\infty((T-\epsilon^{-\beta},T),L^r)}^{2q-1-\theta} \lesssim \epsilon^{-(2q-1-\theta)\frac{\beta}{\theta}} \epsilon^{\lambda(2q-1-\theta)} \lesssim \epsilon^{(2q-1-\theta)(\lambda-\frac{\beta}{\theta})}.
\]
• The term $F_1$. Take $\frac{1}{r} = \lambda''(\frac{1}{r} + \frac{\beta}{N})$, $\lambda'' \in (0, 1)$. By interpolation
\[
\|F_1\|_{L^q((T,\infty),L^r)} \lesssim \|F_1\|_{L^\infty((T,\infty),L^\infty)}^{\lambda''} \|F_1\|_{L^q((T,\infty),L^{\frac{2N}{N-2}})}^{1-\lambda''}
\]
\[
\lesssim \|e^{(1-(T-\epsilon^{-\beta}))\Delta^2} u(T-\epsilon^{-\beta}) - e^{i\Delta^2 u_0}\|^\lambda''_{L^q((T,\infty),L^{\frac{2N}{N-2}})} \|F_1\|_{L^\infty((T,\infty),L^\infty)}^{1-\lambda''}
\]
\[
\lesssim \|F_1\|_{L^q((T,\infty),L^\infty)}^{1-\lambda''}.
\]
Recall the free Schrödinger operator decay
\[
\|e^{it\Delta^2} \| \leq C \frac{t^{\frac{\beta}{2} \frac{1}{2} - \frac{1}{r} \cdot r'}}, \quad \forall r \geq 2.
\]
By Proposition 2.5 via the previous estimate, one has
\[
\left\| F_1 \right\|_\infty \lesssim \int_0^{T - \varepsilon} \frac{1}{(t - s)^{N/4}} \left\| |x|^{2b} |u|^{2q - 1} \right\|_1 ds
\]
\[
\lesssim \int_0^{T - \varepsilon} \frac{1}{(t - s)^{N/4}} \left\| u(s) \right\|_{H^2}^{2q - 1} ds
\]
\[
\lesssim (t - T + \varepsilon)^{1 - \frac{N}{4}}.
\]
In the second line one uses the condition \( q \geq \frac{3}{2} \). Thus, if \( \frac{N}{4} > 1 + \frac{1}{a} \), it follows that
\[
\left\| F_1 \right\|_{L^u((T, \infty), L')} \lesssim \left\| F_1 \right\|_{L^u((T, \infty), L^\infty)}^{1 - \lambda''} \left( \int_T^\infty (t - T + \varepsilon)^{1 - \frac{N}{4}} dt \right)^{1 - \lambda''}
\]
\[
\lesssim e^{(1 - \lambda'')(\frac{N}{4} - 1 - \frac{1}{a})}.
\]
The condition \( \frac{N}{4} > 1 + \frac{1}{a} \) reads
\[N \geq 5 \quad \text{and} \quad (2q - 1)(q - 1) > \frac{2(2 + b)}{N - 4}.
\]
Take the polynomial function
\[P(X) := (2X - 1)(X - 1) - \frac{2(2 + b)}{N - 4} := 2(X - x_0)(X - x_1), \quad x_1 > 0.
\]
So using the inequalities \( q \geq \frac{3}{2} \) and \( q > x_1 \), one gets
\[
\left\| F_1 \right\|_{L^u((T, \infty), L')} \lesssim e^c, \quad c > 0.
\]
One concludes the proof by collecting the previous estimates. \( \square \)

Now, one proves the scattering criterion. Taking account of Duhamel formula, there exists \( \gamma > 0 \) such that
\[
\left\| e^{i \Delta^2} u(T) \right\|_{L^u((0, \infty), L')} = \left\| e^{i(-T)\Delta^2} u(T) \right\|_{L^u((0, \infty), L')} \lesssim e^{\mu}.
\]
So, with Lemma 3.7 via the absorption result Lemma 2.13, one gets
\[
\left\| u \right\|_{L^u((0, \infty), L')} \lesssim e^{\mu}.
\]
With Lemma 3.7, one gets for \( u_+ := e^{-iT\Delta^2} u(T) + \int_T^\infty e^{-is\Delta^2} \mathcal{N} ds \),
\[
\left\| u(t) - e^{it\Delta^2} u_+ \right\|_{H^2} \lesssim \left\| u \right\|_{L^\infty((t, \infty), H^2)} \left\| u \right\|_{L^u((t, \infty), L')}^{2q - 1 - \theta} \to 0.
\]
This finishes the proof. \( \square \)

### 3.4 Scattering

Theorem 2.6 about the scattering of energy global solutions to the focusing problem (1.1) follows with Proposition 3.6 via Lemma 3.5.
4 The generalized Hartree problem (1.2)

This section is devoted to prove Theorem 2.8.

4.1 Variational analysis

In this sub-section, one collects some estimates needed in the proof of the scattering of global solutions to the focusing Choquard problem (1.2).

Lemma 4.1 Take \((N, \alpha, b)\) satisfying \((C)\) and \(p_* < p < p^*\) such that \(p \geq 2\). Let \(u_0 \in H^2\) satisfying (2.8). Then, there exists \(0 < \delta < 1\) such that the solution \(u \in C(\mathbb{R}, H^2)\) satisfies

\[
\max \left\{ \sup_{t \in \mathbb{R}} ME(u(t)), \sup_{t \in \mathbb{R}} MG(u(t)) \right\} < 1 - \delta.
\]

Let us give a coercivity estimate on centered balls with large radials.

Lemma 4.2 There exists \(R_0 := R_0(\delta, M(u), \phi) > 0\) such that for any \(R > R_0\),

\[
\sup_{t \in \mathbb{R}} \|\psi_R u(t)\|^2 \leq \frac{B}{2p} \int_{\mathbb{R}^N} \left( I_{2a} \ast | \cdot |^b |\psi_R u|^{p} \right) |x|^b |\psi_R u|^p \, dx \geq \delta' \|\psi_R u\|^{2p}_{2Np}. \frac{1}{N+a+2b}
\]

Remark 4.3 The two previous results follow like Lemmas 3.1–3.2 via Proposition 2.5.

4.2 Morawetz identity

This subsection is devoted to prove a classical Morawetz estimate satisfied by the energy global solutions to the inhomogeneous Choquard problem (1.2).

Proposition 4.4 Take \((N, \alpha, b)\) satisfying \((C)\), \(p_* < p < p^*\) such that \(p \geq 2\) and \(u \in C_T(H^2)\) be a local solution to (1.1). Let \(a : \mathbb{R}^N \to \mathbb{R}\) be a convex smooth function and the real function defined on \([0, T)\), by

\[
M : t \to 2 \int_{\mathbb{R}^N} \nabla a(x) \nabla u(t, x) \bar{u}(t, x) \, dx.
\]

Then, the following equality holds on \([0, T)\),

\[
M' = 2 \int_{\mathbb{R}^N} \left( 2\partial_{jk} \Delta a \partial_j u \partial_k \bar{u} - \frac{1}{2} (\Delta^2 a) |u|^2 - 4 \partial_{jk} a \partial_{ij} u \partial_{ij} \bar{u} \right) \frac{B}{2p} \int_{\mathbb{R}^N} \left( I_{2a} \ast \cdot |b| |\psi_R u|^{p} \right) |x|^b |\psi_R u|^p \, dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^N} \left( \Delta a \nabla u \right) \frac{B}{2p} \int_{\mathbb{R}^N} \left( I_{2a} \ast \cdot |b| |\psi_R u|^{p} \right) |x|^b |\psi_R u|^p \, dx
\]

\[
- \frac{2}{p} \int_{\mathbb{R}^N} \partial_k a \partial_k \left( |x|^b [I_{2a} \ast \cdot |b| |\psi_R u|^{p}] |u|^p \right) \, dx.
\]
Then, there is \( \gamma > 0 \) such that \( p_* < p < p^* \) such that \( p \geq 2 \). Let \( u_0 \in H_{r_d}^2 \) satisfying (2.8). Then, there is \( \gamma > 0 \), such that for any \( T > 0 \), one has
\[
\int_0^T \|u(t)\|_{2p}^{2p} dt \leq CT^{1/(1+\gamma)}.
\]

**Proof** One keeps the notations \( f, f_R \) and \( M_R \) defined in the proof of Lemma 3.4. By Morawetz estimate in Proposition 4.4, one has
\[
M'_R = 2 \int_{\mathbb{R}^N} \left( 2\partial_j \Delta f_R \partial_j u \partial_k u \bar{u} - \frac{1}{2} (\Delta^3 f_R) |u|^2 \right)
\]
\[
-4 \partial_{jk} f_R \partial_i u \partial_j \tilde{u} + \Delta^2 f_R |\nabla u|^2 \, dx
\]
\[
-2 \left( -1 + \frac{2}{p} \right) \int_{\mathbb{R}^N} \Delta f_R (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx
\]
\[
+ \frac{2}{p} \int_{\mathbb{R}^N} \partial_k f_R \partial_k (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx
\]
\[
= 2 \int_{\mathbb{R}^N} \left( 2 \partial_{jk} \Delta f_R \partial_j u \partial_k \tilde{u} - \frac{1}{2} (\Delta^3 f_R) |u|^2 \right.
\]
\[
- \frac{2}{p} \partial_k f_R \partial_k (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx + \Delta^2 f_R |\nabla u|^2 \big) \, dx
\]
\[
+ 2 \left( \left( -1 + \frac{2}{p} \right) \int_{|x| < \frac{R}{2}} (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx - 4 \int_{|x| < \frac{R}{2}} |\Delta u|^2 \, dx \right)
\]
\[
+ 2 \left( \left( -1 + \frac{2}{p} \right) \int_{\frac{R}{2} < |x| < R} \Delta f_R (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx
\]
\[
- 4 \int_{\frac{R}{2} < |x| < R} \partial_{jk} f_R \partial_i u \partial_j \tilde{u} \, dx \big) .
\]

Using the estimate \( \|\nabla f_R\|_\infty \lesssim \lesssim R^{-2} \), one has
\[
\left| \int_{\mathbb{R}^N} \partial_{jk} \Delta f_R \partial_j u \partial_k \tilde{u} \, dx \right| \lesssim R^{-2};
\]
\[
\left| \int_{\mathbb{R}^N} (\Delta^3 f_R) |u|^2 \, dx \right| \lesssim R^{-4};
\]
\[
\left| \int_{\mathbb{R}^N} \Delta^2 f_R |\nabla u|^2 \, dx \right| \lesssim R^{-2}.
\]

Moreover, by the radial setting, one writes
\[
\int_{\{ \frac{R}{2} < |x| < R \}} \partial_{jk} f_R \partial_i u \partial_j \tilde{u} \, dx \geq (N - 1) \int_{\{ \frac{R}{2} < |x| < R \}} \frac{f_R(r)}{r^2} |\partial_r u|^2 \, dx = \mathcal{O}(R^{-2}).
\]

Now, by Hardy–Littlewood–Sobolev and Strauss inequalities
\[
\left| \int_{\{ \frac{R}{2} < |x| < R \}} \Delta f_R (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx \right| \lesssim \|u\|^p_{L^{\frac{2Np}{\alpha+N+2b}} \{ |x| > \frac{R}{2} \}}
\]
\[
\lesssim \left( \int_{|x| > \frac{R}{2}} |u(x)|^{\frac{2Np}{\alpha+N+2b}} |u(x)|^2 \, dx \right)^{\frac{\alpha+N+2b}{2N}}
\]
\[
\lesssim \|u\|^p_{L^\infty(|x| > \frac{R}{2})} \left( \int_{\mathbb{R}^N} |u(x)|^2 \, dx \right)^{\frac{\alpha+N+2b}{2N}}
\]
\[
\lesssim R^{\frac{B(N-1)}{N}}.
\]

Thus, since \( \|\nabla u\|^2 \lesssim \|\Delta u\| \lesssim 1 \), one gets denoting \( -\gamma := \max\{ -2, -\frac{B(N-1)}{N} \} \),
\[
M'_R \leq 2 \left( \left( -1 + \frac{2}{p} \right) \int_{|x| < \frac{R}{2}} (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx - 4 \int_{|x| < \frac{R}{2}} |\Delta u|^2 \, dx \right)
\]
\[
- \frac{4}{p} \int_{\mathbb{R}^N} \partial_k f_R \partial_k (I_\alpha \ast | \cdot |^b |u|^p) |x|^b |u|^p \, dx + \mathcal{O}(R^{-\gamma}).
\]
Now, let us define the sets
\[ \Omega := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ s.t } R/2 < |x| < R \} \cup \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ s.t } R/2 < |y| < R \}; \]
\[ \Omega' := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ s.t } |x| > R, |y| < R/2 \} \cup \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ s.t } |x| < R/2, |y| > R \}. \]

Consider the term
\[
(I) := \int_{\mathbb{R}^N} \nabla f_R \left( \frac{x}{|x|^2} I_0 \ast |b|u|^p \right) |x|^p |u|^p \, dx
= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\nabla f_R(x) - \nabla f_R(y))(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p \, dx \, dy
= \left( \int_{\Omega} + \int_{\Omega'} + \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} + \int_{|x| > \frac{R}{2}, |y| > \frac{R}{2}} \right) (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p \, dx \, dy).
\]

Compute
\[
(a) := \int_{\Omega'} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \int_{|x| > R, |y| < \frac{R}{2}} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
+ \int_{|y| > R, |x| < \frac{R}{2}} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \int_{|x| > R, |y| < \frac{R}{2}} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \int_{|x| > R, |y| < \frac{R}{2}} (y(y - x) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy.
\]

Moreover,
\[
(b) := \frac{1}{2} \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \frac{1}{2} \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} (x - y)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \frac{1}{2} \int_{|x| < \frac{R}{2}, |y| < \frac{R}{2}} (I_0(x - y)|y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \frac{1}{2} \int_{\mathbb{R}^N} (I_0 \ast |b|^p |\psi_R u|^p |x|^b |\psi_R u|^p ) \, dx.
\]

Furthermore,
\[
(c) := \int_{\frac{R}{2} < |x| < R} \int_{\mathbb{R}^N} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= \int_{\frac{R}{2} < |x| < R, |y - x| > \frac{R}{2}} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
+ \int_{\frac{R}{2} < |x| < R, |y - x| < \frac{R}{2}} (\nabla f_R(x)(x - y) \frac{I_0(x - y)}{|x - y|^2} |y|^b |u(y)|^p |x|^b |u(x)|^p ) \, dx \, dy
= O\left( \int_{\frac{R}{2} < |x| < R} (I_0 \ast |b|^p |u|^p |x|^b |u|^p ) \, dx \right).
\]
Moreover, since for large $R > 0$ on $\{|x| > R, |y| < \frac{R}{2}\}$, $|x - y| \simeq |x| > R \gg \frac{R}{2} > |y|$, one has

\[
(a) = \int_{\{|x| > R, |y| < \frac{R}{2}\}} (y - x) \left( \frac{I_\alpha(x - y)}{|x - y|^2} \right) \frac{|y|^b |u(y)|^p |x|^b |u(x)|^p} {\|x\|^{\frac{N}{N-a}}} dx dy
\]

\[
\lesssim \int_{\{|x| > R, |y| < \frac{R}{2}\}} \left( \frac{|y|^b |u(y)|^p |x|^b |u(x)|^p} {\|x\|^{\frac{N}{N-a}}} \right) dx dy
\]

\[
\lesssim \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( I_\alpha(x - y) \chi_{|x| > R} \right) \frac{|y|^b |u(y)|^p |x|^b |u(x)|^p} {\|x\|^{\frac{N}{N-a}}} dx dy
\]

\[
\lesssim \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( I_\alpha(x - y) \chi_{|x| > R} \right) \frac{|y|^b |u(y)|^p |x|^b |u(x)|^p} {\|x\|^{\frac{N}{N-a}}} dx dy.
\]

Taking account of Hardy–Littlewood–Sobolev inequality, Hölder and Strauss estimates and Sobolev injections via the fact that $p_\ast < p < p^\ast$, write

\[
(a) \lesssim \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( I_\alpha(x - y) \chi_{|x| > R} \right) \frac{|y|^b |u(y)|^p |x|^b |u(x)|^p} {\|x\|^{\frac{N}{N-a}}} dx dy
\]

\[
\lesssim \|u\|^{p \frac{2Np}{N-a+2b}} \left( \int_{\{|x| > R\}} \frac{|u|^p}{\|x\|^{\frac{N}{N-a}}} \right)^{\frac{N}{N-a+2b}}
\]

\[
\lesssim \left( \int_{\{|x| > R\}} |u|^2 (|x|^{\frac{N-a-1}{2}} \|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}})^{\frac{1}{N-a+2b}} dx \right)^{\frac{N}{N-a+2b}}
\]

\[
\lesssim \|u\|^{\frac{N-a+2b}{N-a+2b}} \frac{1}{\frac{2N}{R^{(\frac{N-a}{a})}}} (\|u\|^{\frac{1}{2}} \|\nabla u\|)^{\frac{1}{2N-a}}
\]

\[
\lesssim R^{-\frac{B(N-1)}{N}}.
\]

Take the quantity

\[
(II) := \int_{\mathbb{R}^N} \nabla f_R(I_\alpha \ast |\cdot|^b |u|^p) x |x|^{b-2} |u|^p dx
\]

\[
= \int_{|x| < \frac{R}{2}} (I_\alpha \ast |\cdot|^b |u|^p) |x|^b |u|^p dx + \int_{\frac{R}{2} < |x| < R} \nabla f_R(I_\alpha \ast |\cdot|^b |u|^p) x |x|^{b-2} |u|^p dx.
\]

Now, using the properties of $f_R$, one has

\[
\int_{\frac{R}{2} < |x| < R} \nabla f_R(I_\alpha \ast |\cdot|^b |u|^p) x |x|^{b-2} |u|^p dx \lesssim \int_{\frac{R}{2} < |x| < R} \frac{|\nabla f_R(x)|}{|x|} (I_\alpha \ast |\cdot|^b |u|^p) x |x|^{b-2} |u|^p dx
\]

\[
\lesssim \int_{\frac{R}{2} < |x| < R} (I_\alpha \ast |\cdot|^b |u|^p) x |x|^{b-2} |u|^p dx
\]

\[
\lesssim R^{-\frac{B(N-1)}{N}}.
\]

So,

\[
(II) = \int_{|x| < \frac{R}{2}} (I_\alpha \ast |\cdot|^b |u|^p) x |x|^{b-2} |u|^p dx + \mathcal{O}(R^{-\gamma}).
\]
Then,
\[
M' \leq 2 \left( 1 - \frac{2}{p} \right) N \int_{|x| < \frac{R}{2}} (I_a \ast |\cdot|^b |u|^p |u|^p) \, dx - 4 \int_{|x| < \frac{R}{2}} |\Delta u|^2 \, dx \\
- \frac{4}{p} \left( (\alpha - N)(I) + b(II) \right)
\]
\[
\leq 2 \left( 1 - \frac{2}{p} \right) N \int_{|x| < \frac{R}{2}} (I_a \ast |\cdot|^b |\psi_R u|^p |\psi_R u|^p) \, dx - 4 \int_{|x| < \frac{R}{2}} |\Delta u|^2 \, dx \\
- \frac{4}{p} \left( b - \frac{N - \alpha}{2} \right) \int_{\mathbb{R}^N} (I_a \ast |\cdot|^b |\psi_R u|^p) |x|^b |\psi_R u|^p \, dx + O(R^{-\gamma})
\]
\[
\leq \frac{4B}{p} \int_{|x| < \frac{R}{2}} (I_a \ast |\cdot|^b |\psi_R u|^p) |x|^b |\psi_R u|^p \, dx - 8 \int_{|x| < \frac{R}{2}} |\Delta u|^2 \, dx + O(R^{-\gamma}).
\]
So, with Lemma 4.2, via (3.9), one gets
\[
\sup_{[0,T]} |M| \geq 8 \int_{0}^{T} \left( \int_{|x| < \frac{R}{2}} |\Delta(\psi_R u)|^2 \, dx - \frac{B}{2p} \int_{|x| < \frac{R}{2}} (I_a \ast |\psi_R u|^p) |\psi_R u|^p \, dx \right) \, dt + O(R^{-\gamma}) \, T
\]
\[
\geq 8 \delta' \int_{0}^{T} \|\psi_R u(t)\|_{L^\frac{2Np}{N + \alpha + 2\delta}}^2 \, dt + O(R^{-\gamma}) \, T
\]
\[
\geq 8 \delta' \int_{0}^{T} \|u(t)\|_{L^\frac{2Np}{N + \alpha + 2\delta}}^2 \, dt + O(R^{-\gamma}) \, T.
\]
Moreover, with Strauss estimate, arguing like in (4.10), one has
\[
\|u(t)\|_{L^\frac{2Np}{N + \alpha + 2\delta}}^2 \leq R^{-\frac{2B(N-1)}{N}}.
\]
The previous calculus gives
\[
\int_{0}^{T} \|u(t)\|_{L^\frac{2Np}{N + \alpha + 2\delta}}^2 \, dt \leq C \left( \sup_{[0,T]} |M| + TR^{-\gamma} \right)
\]
\[
\leq C \left( R + TR^{-\gamma} \right).
\]
Taking \( R = T^{1/(1+\gamma)} \gg 1 \), one gets the requested estimate. For \( 0 < T \ll 1 \), the proof follows with Sobolev injections.

As a consequence, one has the following energy evacuation.

**Lemma 4.6** Take \((N, \alpha, b)\) satisfying (C). Let \( p_* < p < p^* \) such that \( p \geq 2 \) and \( u_0 \in H^2_{rd} \) satisfying \( (2.8) \). Then, there exists a sequence of real numbers \( t_n \to \infty \) such that the global solution to (1.2) satisfies
\[
\lim_{n} \int_{|x| < R} |u(t_n, x)|^2 \, dx = 0, \quad \text{for all } R > 0.
\]

**Proof** Take \( t_n \to \infty \). By Hölder estimate
\[
\int_{|x| < R} |u(t_n, x)|^2 \, dx \leq R^{\frac{2B}{N}} \|u(t_n)\|_{L^\frac{2Np}{N + \alpha + 2\delta}}^2 \to 0.
\]
Indeed, by the previous Lemma
\[
\|u(t_n)\|_{L^\frac{2Np}{N + \alpha + 2\delta}} \to 0.
\]
4.3 Scattering criterion

This subsection is devoted to prove the next result.

**Proposition 4.7** Take \((N, \alpha, b)\) satisfying (C). Let \(\max\{p_*, x_\alpha\} < p < p^*\) such that \(p \geq \max\{2, \frac{3}{2} + \frac{\alpha}{N}\}\). Let \(u \in C(\mathbb{R}, H^2_{rd})\) be a global radial solution to (1.2). Assume that

\[
0 < \sup_{t \geq 0} \|u(t)\|_{H^2} := E < \infty.
\]

There exist \(R, \epsilon > 0\) depending on \(E, N, p, b, \alpha\) such that if

\[
\lim_{t \to +\infty} \int_{|x| < R} |u(t, x)|^2 \, dx < \epsilon,
\]

then, \(u\) scatters for positive time.

**Proof** By Lemma 4.1, \(u\) is bounded in \(H^2\). Take \(\epsilon > 0\) near to zero and \(R(\epsilon) \gg 1\) to be fixed later. Let us give a technical result.

**Lemma 4.8** Let \((N, b, \alpha)\) satisfying (C) and \(p_* < p < p^*\) satisfying \(p \geq 2\). Then, there exists \(\theta \in (0, 2p - 1)\) such that the global solution to (1.2) satisfies

\[
\|u - e^{i\Delta} u_0\|_{S_c^c(I)} \lesssim \|u\|^\theta_{L^{\infty}(I, H^2)} \|u\|_{L^2(I, L^2)}^{2p - 1 - \theta},
\]

for certain \((a, r) \in \Gamma_{s'_c}\).

**Proof** Take the real numbers

\[
a := \frac{2(p - \theta)}{2 - s'_c}, \quad d := \frac{2(p - \theta)}{2 + (2p - 1 - \theta)s'_c},
\]

\[
r := \frac{2N(2p - \theta)}{(N - 2s'_c)(2p - \theta) - 4(2 - s'_c)}.
\]

The condition \(\theta = 0^+\) gives via previous computation

\((a, r) \in \Gamma_{s'_c}, \quad (d, r) \in \Gamma_{-s'_c}, \quad \text{and} \quad (2p - 1 - \theta)d' = a\).

Take two real numbers \(r_1, \mu\) satisfying

\[
1 + \frac{\alpha}{N} = \frac{2}{\mu} + \frac{\theta}{r_1} + \frac{2p - \theta}{r}.
\]

This gives

\[
\frac{2N}{\mu} = \alpha + N - \frac{N\theta}{r_1} - \frac{N(2p - \theta)}{r}
\]

\[
= \alpha + N - \frac{N\theta}{r_1} - \frac{(N - 2s'_c)(2p - \theta) - 4(2 - s'_c)}{2}
\]

\[
= -2b - \frac{N\theta}{r_1} + \theta \frac{4 + 2b + \alpha}{2(p - 1)}.
\]

Using Hardy–Littlewood–Sobolev and Hölder estimates, one has

\[
\|\mathcal{N}\|_{L^{s'_c}(I, L^r(|x| < 1))} \lesssim \|x|^{b/2}_{L^{\infty}(|x| < 1)} \|u\|^\theta_{L^{\infty}(I, L^2)} \|u\|_{L^2(I, L^2)}^{2p - 1 - \theta} \|u\|_{L^{s'_c}(I)}.
\]
Then, choosing \( r_1 := \frac{2N}{N-4} \), one gets because \( p < p^* \),
\[
\frac{2N}{\mu} + 2b = \frac{\theta}{2(p-1)}(4 + 2b + \alpha - (p-1)(N-4)) > 0.
\]
So, \( |x|^b \in L^\mu(\{|x| < 1\}) \). Then, by Sobolev injections, one gets
\[
\|\mathcal{N}\|_{L^\theta(L^\infty(L^\mu(\{|x| < 1\})))} \lesssim \| |x|^b \|_{L^\mu(\{|x| < 1\})}^2 \|u\|_{L^\infty(L^\mu(\{|x| < 1\})^\theta)} \|u\|_{L^{2p-1-\theta}(I)}^{2p-1-\theta} \|L^\theta(\{x\})\|.
\]
Moreover, on the complementary of the unit ball, one takes \( r_1 := 2 \), so because \( p > p^* \), one has
\[
\frac{2N}{\mu} + 2b = \frac{\theta}{2(p-1)}(4 + 2b + \alpha - N(p-1)) < 0.
\]
So, \( |x|^b \in L^\mu(\{|x| > 1\}) \). The proof is achieved via Strichartz estimates. \( \square \)

The following result is the key to prove the scattering criterion.

**Proposition 4.9** Take \((N, \alpha, b)\) satisfying \(\mathcal{C}\). Let \(\max\{p^*, x_\alpha\} < p < p^* \) such that \(p \geq \max\{2, \frac{3}{2} + \frac{\alpha}{N}\} \). Let \(u_0 \in H^2 \) satisfying (2.8). Then, for any \(\varepsilon > 0\), there exist \(\Gamma, \mu > 0\) such that the global solution to (1.2) satisfies
\[
\|e^{i(-T)\Delta^2} u(T)\|_{L^\mu((T, \infty), L^{\infty})} \lesssim \varepsilon^\mu.
\]

**Proof** One keeps notations of the proof of Lemma 4.8. Let \(0 < \beta \ll 1\) and \(T > \varepsilon^{-\beta} > 0\). By the integral formula
\[
e^{i(-T)\Delta^2} u(T) = e^{i\Delta^2} u_0 + i \int_0^T e^{i(-s)\Delta^2} [(I_\alpha * | \cdot | b|u|^p)|x|^b|u|^{p-2}u] \, ds
\]
\[
e^{i\Delta^2} u_0 + i \left( \int_0^{T-\varepsilon^{-\beta}} + \int_{T-\varepsilon^{-\beta}}^T \right) e^{i(-s)\Delta^2} [(I_\alpha * | \cdot | b|u|^p)|x|^b|u|^{p-2}u] \, ds
\]
\[
e^{i\Delta^2} u_0 + F_1 + F_2.
\]
• The linear term. Since \((a, x_\frac{N_p}{N+p-4}, \frac{N_p}{N+p-4}) \in \Gamma\), by Strichartz estimate and Sobolev injections, one has
\[
\|e^{i\Delta^2} u_0\|_{L^\mu((T, \infty), L^{\infty})} \lesssim \||\n\|^e^{i\Delta^2} u_0\|_{L^\mu((T, \infty), L^\infty)} \lesssim \|u_0\|_{L^2}.
\]
• The term \(F_2\). By Lemma 4.8, one has
\[
\|F_2\|_{L^\mu((T, \infty), L^{\infty})} \lesssim \|u\|_{L^\infty((T, \infty), H^1)} \|u\|_{L^\mu((T, \infty), L^{\infty})}^{2p-1-\theta} \lesssim \|u\|_{L^{\mu((T, \infty), L^{\infty})}}^{2p-1-\theta}.
\]
Following lines in the proof of Proposition 3.8, there exists \(\lambda, \lambda' \in (0, 1)\) such that, for \(R > \varepsilon^{-\lambda'(N-1)/\alpha} \),
\[
\|F_2\|_{L^\mu((T, \infty), L^{\infty})} \lesssim \varepsilon^{(2p-1-\theta)(\lambda - \frac{\beta}{\alpha})}.
\]
• The term \(F_1\). Take \(\frac{1}{\tilde{r}} = \lambda'' \left(\frac{1}{r} + \frac{s_L}{N}\right)\) and \(\lambda'' \in (0, 1)\). By interpolation
\[
\|F_1\|_{L^\mu((T, \infty), L^{\infty})} \lesssim \|F_1\|^{\lambda''}_{L^\mu((T, \infty), L^{\infty})} \|F_1\|^{-\frac{1}{\tilde{r}}}_{L^{\mu((T, \infty), L^{\infty})}} \|F_1\|^{\frac{1}{\tilde{r}}}_{L^{\mu((T, \infty), L^{\infty})}}\]
\[ \|e^{i(T-\epsilon^{-\beta})\Delta^2}u(T-\epsilon^{-\beta}) - e^{i\Delta^2}u_0\|_{L^2((T,\infty),L^\infty(T,\infty),L^\infty)} \leq \|e^{i(T-\epsilon^{-\beta})\Delta^2}u(T-\epsilon^{-\beta}) - e^{i\Delta^2}u_0\|_{L^2((T,\infty),L^\infty(T,\infty),L^\infty)} \]

With the free Schrödinger operator decay, for \( T \leq t \) and \( p \geq \frac{3}{2} + \frac{\alpha}{N} \), one gets via Proposition 2.5,

\[ \left\| F_1 \right\|_{L^p((T,\infty),L^\infty)} \lesssim \int_0^t (t-s)^{-\frac{N}{4}} \left\| u(s) \right\|_{H^2}^{2p-1} ds \lesssim (t-T+\epsilon^{-\beta})^{\frac{1}{2}-\frac{N}{4}}. \]

Thus, if \( \frac{N}{4} > 1 + \frac{1}{\alpha} \), it follows that

\[ \left\| F_1 \right\|_{L^p((T,\infty),L^\infty)} \lesssim \left( \int_T^\infty (t-T+\epsilon^{-\beta})^{a[1-\frac{N}{4}]} dt \right)^{\frac{1-\lambda''}{a}} \lesssim \epsilon^{(1-\lambda'')\beta[\frac{N}{4}-\frac{1}{2}]} \]

The condition \( \frac{N}{4} > 1 + \frac{1}{\alpha} \) is equivalent to \( N \geq 5 \) and \( (p-1)(2p-1) > \frac{4+2b+\alpha}{N-4} \). Taking the polynomial function

\[ P_\alpha(X) := (X-1)(2X-1) - \frac{4+2b+\alpha}{N-4} := (X-x_\alpha)(X-x_\alpha), \quad x_\alpha > 0, \]

the previous inequality is equivalent to \( p > x_\alpha \). Thus, one concludes the proof by collecting the previous estimates.

The proof of the scattering criterion follows like the case of an inhomogeneous local source term.

\[ \square \]

### 4.4 Scattering

Theorem 2.8 about the scattering of energy global solutions to the focusing problem (1.2) follows with Proposition 4.7 via Lemma 4.6.

### References

1. Anquet, C., Villamizar-Roa, J.: On the management fourth-order Schrödinger–Hartree equation. Evol. Equ. Contr. Theor. 9(3), 865–889 (2020)
2. Cao, D., Dai, W.: Classification of nonnegative solutions to a bi-harmonic equation with Hartree type nonlinearity. Proc. R. Soc. Edinb. Sect. A : Math. 149(4), 979–994 (2019)
3. Cho, Y., Ozawa, T., Wang, C.: Finite time blowup for the fourth-order NLS. Bull. Korean Math. Soc. 53(2), 615–640 (2016)
4. Dinh, V.D.: Dynamics of radial solutions for the focusing fourth-order nonlinear Schrödinger equations. Nonlinearity 34(2), 776 (2021)
5. Dodson, B., Murphy, J.: A new proof of scattering below the ground state for the 3D radial focusing cubic NLS. Proc. Am. Math. Soc. 145(11), 4859–4867 (2017)
6. Ghanmi, R., Saanouni, T.: A note on the inhomogeneous fourth-order Schrödinger equation, submitted
7. Guo, Q.: Scattering for the focusing $L^2$-supercritical and $H^2$-subcritical bi-harmonic NLS equations. Commun. Part. Differ. Equ. 41(2), 185–207 (2016)
8. Guzman, C.M., Pastor, A.: On the inhomogeneous bi-harmonic nonlinear schrödinger equation: local, global and stability results. Nonlinear Anal.: Real World Appl. 56, 103–174 (2020)
9. Holmer, J., Roudenko, S.: A sharp condition for scattering of the radial 3D cubic non-linear Schrödinger equations. Commun. Math. Phys. 282, 435–467 (2008)
10. Karpman, V.I.: Stabilization of soliton instabilities by higher-order dispersion: fourth-order nonlinear Schrödinger equation. Phys. Rev. E. 53(2), 1336–1339 (1996)
11. Karpman, V.I., Shagalov, A.G.: Stability of soliton described by nonlinear Schrödinger type equations with higher-order dispersion. Physica D 144, 194–210 (2000)
12. Kenig, C., Merle, F.: Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. Acta Math. 201(2), 147–212 (2008)
13. Lieb, E.: Analysis, 2nd ed., Graduate Studies in Mathematics, Vol. 14. American Mathematical Society, Providence (2001)
14. Miao, C., Wu, H., Zhang, J.: Scattering theory below energy for the cubic fourth-order Schrödinger equation. Math. Nachrichten. 288(7), 798–823 (2015)
15. Pausader, B.: Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case. Dyn. Partial Differ. Equ. 4(3), 197–225 (2007)
16. Saanouni, T.: Non-linear bi-harmonic Choquard equations. Commun. Pure Appl. Anal. 19(11), 5033–5057 (2020)
17. Saanouni, T.: Scattering for radial defocusing inhomogeneous bi-harmonic Schrödinger equations. Potential Anal. (2021)
18. Tao, T.: On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation. Dyn. Partial Differ. Equ. 1(1), 1–48 (2004)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.