Solution of Schrödinger equation for Three Dimensional Harmonics Oscillator plus Rosen-Morse Non-central potential using NU Method and Romanovski Polynomials

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Abstract. The energy eigenvalues and eigenfunctions of Schrödinger equation for three dimensional harmonic oscillator potential plus Rosen-Morse non-central potential are investigated using NU method and Romanovski polynomial. The bound state energy eigenvalues are given in a closed form and corresponding radial wave functions are expressed in associated Laguerre polynomials while angular eigen functions are given in terms of Romanovski polynomials. The Rosen-Morse potential is considered to be a perturbation factor to the three dimensional harmonic oscillator potential that causes the increase of radial wave function amplitude and decrease of angular momentum length. Keywords: Schrödinger Equation, Three dimensional Harmonic Oscillator potential, Rosen-morse non-central potential, NU method, Romanovski Polynomials

1. Introduction
The exact analytical solutions of Schrödinger equations for some physical potentials are very essential since the knowledge of wavefunctions and energy contains all possible important information of the physical properties of quantum system. Recently, considerably efforts have been paid to obtain the exact solution of the central and non-central potentials. There are only a few potentials for which the Schrodinger equation can be solved exactly.

In recent years, numerous studies have been made in analyzing the bound state of charged particle moving in a vector potential and a non-central scalar potential, such as an electron moving in a Coulomb field with simultaneously precence of Aharonov-Bohm field [1-3], or/and magnetic monopole [4], Makarov potential [5] or ring-shaped-oscillator potential [6-8], etc. In most of these studies, the eigenvalues and eigenfunctions are obtained using separation variables in spherical coordinate system. Very recently, supersymmetric quantum mechanics with the idea of shape invariance [8-9], factorization method [10-11], and Nikiforov-Uvarov method [12-13] are widely used to derive the energy spectrum and the wave function of a charge particle moving in non-central potential.

Very recently, an alternative method used to solve Schrodinger equation for a class of shape invariant potentials is NU method and Romanovski polynomials [14-16]. NU method was developed by Nikiforov-Uvarov [12] is based on solving the second order linear differential equations by reducing it to
a generalized equation of hypergeometric type by a suitable change of variable. As NU method, finite Romanovski polynomials, which is a traditional method, consist of reducing Schrodinger equation by an appropriate change of the variable to that of very form of generalized hypergeometric equation [17]. The polynomial was discovered by Sir E.J. Routh [18] and rediscovered 45 years later by V. I. Romanovski [19]. The notion “finite” refers to the observation that, for any given set of parameters (i.e. in any potential) only a finite of polynomials appear orthogonal[20]. It seems that the two methods are very similar but they solve Schrodinger equation differently. NU method is applied wider than Romanovski polynomials.

In this paper we will attempt to solve the Schrodinger equation for three dimensional harmonic oscillator potential with simultaneously presence of Rosen-Morse non-central potential using NU method for radial Schrodinger equation and Romanovski polynomials for angular Schrodinger equation. Three dimensional harmonics oscillator is one of exactly solvable potential that used to describe the nuclei, atomic or molecular vibration. Non-central potential composed of spherical harmonics oscillator with square of inverse potential together with ring-shaped non-central potential, or double ring shaped potential have been investigated intensively by some authors [21-23]. The Rosen-Morse potential is trigonometric potential which was proposed by Rosen-Morse [24] in 1935 and was used to describe the diatomic molecular vibration. The approximate bound state solution for trigonometric Rosen-Morse potential have also been studied for l-state solution [15,25], and Coulombic Rosen Morse non-central potential, particularly for part, has been investigated intensively [15].

This paper is organized as follows. In section 2, we review NU method and the finite Romanovski polynomials briefly. In section 3, we find the bound state energy solution and radial wave function expressed in associated Laguerre polynomials, and the angular wave functions in terms of Romanovski polynomials. A brief conclusion is presented in section 5.

2. Review of formulas for NU method and finite Romanovski Polynomials

2.1. Review of Nikiforov-Uvarov Method

The one-dimensional Schrodinger equation of any shape invariant potential can be reduced into hypergeometric or confluent hypergeometric type differential equation by suitable variable transformation[12,26]. The hypergeometric type differential equation, which is solved using Nikiforov-Uvarov method, is presented as

$$\frac{d^2\Psi(s)}{ds^2} + \frac{r(s)}{\sigma(s)} \frac{d\Psi(s)}{ds} + \frac{\sigma(s)}{\sigma'^2} \Psi(s) = 0 \quad (1)$$

where $\sigma(s)$ and $\sigma'(s)$ are polynomials at most in the second order, and $r(s)$ is first order polynomial.

Equation (1) can be solved using separation of variable method which is expressed as

$$\Psi = \phi(s) y(s) \quad (2)$$

By inserting equation (2) into equation (1) we get hypergeometric type equation

$$\sigma \frac{d^2 y}{ds^2} + r \frac{d y}{ds} + \lambda y = 0 \quad (3)$$

and $\phi(s)$ is a logarithmic derivative whose solution obtained from condition

$$\frac{\phi'}{\phi} = \frac{\pi}{\sigma} \quad (4)$$

while the function $\pi(s)$ and the parameter $\lambda$ are defined as

$$\pi = \frac{(\sigma - \tau)}{2} \pm \sqrt{\left(\frac{\sigma - \tau)}{2}\right)^2 - \sigma + k \sigma} \quad (5)$$

$$\lambda = k + \pi \quad (6)$$
The value of $k$ in equation (5) can be found from the condition that the expression under the square root of equation (5) must be square of polynomial which is mostly first degree polynomial and therefore the discriminate of the quadratic expression is zero. A new eigenvalue of equation (3) is

$$\lambda = \lambda_n = -n \tau - \frac{n(n-1)}{2} \sigma \ , \ n = 0, 1, 2$$

(7)

where

$$\tau = \tau + 2\pi$$

(8)

The new energy eigenvalue is obtained using equation (6) and (7).

To generate the energy eigenvalues and the corresponding eigenfunctions, the condition that $\tau < 0$ is required. The solution of the second part of the wave function, $y_n(s)$, which is connected to Rodrigues relation, is given as

$$y_n(z) = \frac{c_n}{w(z)} \frac{d^n}{dz^n} \left( \sigma^n(z)w(z) \right)$$

(9)

where $C_n$ is normalization constant, and the weight function $\rho(s)$ must satisfies the condition

$$\frac{\partial (\sigma w)}{\partial s} = \tau(s)w(s)$$

(10)

The wave function of the system is therefore obtained from equation (4), (9) and (10).

### 2.2. Review of finite Romanovski Polynomials

The hypergeometric type differential equation, which can be solved using finite Romanovski polynomials that was developed by Romanovski [19] is presented as in equation (3) in NU method where

$$\sigma(s) = as^2 + bs + c, \quad \tau = fs + e \quad \text{and} \quad \left\{ 1 - n(n-1) + 2n(1-p) \right\} = \lambda = \lambda_n$$

(11)

Equation (11) is described in the textbook by Nikiforov-Uvarov [12] where it is cast into self adjoint form and its weight function $w(s)$ satisfies the so called Pearson differential equation

$$\frac{d(\sigma w)}{ds} = \tau(s)w(s)$$

(12)

The weight function is obtained by solving the Pearson differential equation, that is

$$w(s) = \exp\left( \int \left( \frac{f - 2a}{as^2 + bs + c} + (e - b) ds \right) \right)$$

(13)

The corresponding polynomials are classified according to the weight function, and are built up from the Rodrigues representation as

$$y_n = \frac{1}{w(s)} \frac{d^n}{ds^n} \left[ (as^2 + bs + c)^e w(s) \right]$$

(14)

For Romanovski polynomial, the values of parameters in equation (13) are:

$$a = 1, \ b = 0, \ c = 1, \ f = 2(1-p) \quad \text{and} \quad e = q \quad \text{with} \quad p > 0$$

(15)

By inserting equation (15) into equation (13) we obtain the weight function

$$w(s) = \exp\left( \int \left( \frac{f - 2a}{as^2 + bs + c} + (e - b) ds \right) \right) = \exp\left( \int \frac{(2-2p-2s+q)}{s^2+1} ds \right)$$

(16)

The polynomial associated with equation (16) are named after Romanovski and will be denoted by $R_n^{(p,q)}(s)$. Due to the decrease of the weight function by $s^{-2p}$ [18], integral of the type

$$\int_{-\infty}^{\infty} w^{(p,q)} \frac{d^n}{ds^n} R_n^{(p,q)}(s) R_n^{(p,q)}(s) ds$$

(17)

will be convergent only if

$$n + n < 2p - 1$$

(18)

This means that only a finite number of Romanovski polynomials are orthogonal.
The differential equation satisfied by Romanovski Polynomial obtained by inserting equations (11) and (15) into equation (3) is

\[ (1 + s^2) \frac{\partial R_n^{(p,q)}}{\partial s} + \left(2s(-p + 1) + q\right) \frac{\partial R_n^{(p,q)}(s)}{\partial s} - \left[p(n - 1) + 2n(1 - p)\right] R_n^{(p,q)}(s) = 0 \]  

(19)

where \( y_n = R_n^{(p,q)}(s) \). The Schrödinger equation of the potential of interest will be reduced into the form which is similar to equation (19) by an appropriate transformation of variable, \( r = f(s) \), and by introducing a new wave function which is given as

\[ \Psi(r) = g_n(s) = (1 + s^2)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2} \tan^{-1}s} D_n^{(\beta,\alpha)}(s) \]  

(20)

The eigen function in equation (20) is the solution of Schrödinger equation for potential interest where

\[ D_n^{(\beta,\alpha)}(s) = R_n^{(p,q)}(s) \]  

(21)

The Romanovski polynomials obtained from equations (14), 15 and (16) is expressed as

\[ R_n^{(p,q)}(s) = D_n^{(\beta,\alpha)}(s) = \frac{1}{(1 + s^2)^{\frac{\beta}{2}}} e^{\frac{-\alpha}{2} \tan^{-1}s} \left\{ (1 + s^2)^{\frac{\beta}{2}} (1 + s^2)\right\}^{1-p} e^{\alpha \tan^{-1}s} \} \]  

(22)

3. Schrödinger equation of three dimensional harmonic oscillator (3D HO) plus Rosen-Morse potential

The three dimensional Schrödinger equation for 3D HO potential with simultaneously the presence of trigonometric Rosen-Morse non-central potentials is expressed as

\[ -\frac{\hbar^2}{2M} \left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r^2} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right] - \frac{\partial^2}{\partial \phi^2} \right\} \Psi(r,\theta,\phi) + \left[ \frac{\hbar^2}{2Mr^2} + \left(1 + \frac{1}{2} \mu \cot \theta\right) \right] \Psi(r,\theta,\phi) = E \Psi(r,\theta,\phi) \]  

(23)

Equation (23) is solved using variable separation method by setting \( \Psi(r,\theta,\phi) = R(r)P(\theta)\phi(\phi) \) so we obtain

\[ \frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{\hbar^2}{2M} \frac{\partial^2}{\partial \theta^2} \sin^2 \theta + \frac{\hbar^2}{2} \frac{\partial^2}{\partial \phi^2} \phi = \frac{1}{P \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{\partial^2}{\partial \phi^2} \phi + \left( \nu + \frac{1}{2} \right) - 2 \mu \cot \theta = \lambda = l(l + 1) \]  

(24)

For azimuthal part we have

\[ \frac{1}{\phi} \frac{\partial^2}{\partial \phi^2} = -m^2 \]  

(25)

therefore the solution of azimuthal part of wave function as usual is

\[ \phi = A_m e^{im\phi} \]  

(26)

From equation (24) we obtain the radial and angular parts of Schrödinger equation given as

\[ \frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{r^2}{\hbar^2} M^2 \omega^2 r^2 + \frac{2M^2}{\hbar^2} E = \frac{1}{P \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{\partial^2}{\partial \phi^2} \phi + \left( \nu + \frac{1}{2} \right) - 2 \mu \cot \theta = l(l + 1) \]  

(27)

From the solution of equation (27) will be obtained the energy spectrum of 3D HO potential and the radial part of wave function while from equation (28) will be obtained the angular wave function and the value of \( l \) which is the orbital quantum number.

3.1 Solution of radial Schrödinger equation

By setting

\[ \frac{2M}{\hbar^2} E = \frac{M^2 \omega^2}{\hbar^2} = \gamma^2 \quad \text{and} \quad R = \frac{X}{r} \]  

(29)

and inserting it into equation (27) we get
Making a change variable $r^2 = x$ in equation (30) and change it into equation which is given as

$$x \frac{\partial^2 \chi}{\partial x^2} + \frac{1}{2} \frac{\partial \chi}{\partial x} - \left( \gamma^2 x^2 - \epsilon^2 x + l(l+1) \right) \frac{\chi}{4x} = 0$$

By comparing equation (1) and (31) we have

$$\sigma = x, \quad \tilde{r} = \frac{1}{2}, \quad \tilde{\sigma} = -\left\{ \frac{l(l+1) + \gamma^2 x^2 - \epsilon^2 x}{4} \right\}$$

From equations (5) and (32) we get

$$\pi = \frac{1}{4} \pm \frac{1}{\sqrt{16}} \frac{l(l+1) + \gamma^2 x^2 - \epsilon^2 x + kx}{4}$$

The value of $k$ is obtained from the condition that quadratic expression under the square root in equation (33) has to be completely square of first degree of polynomial. From this condition equation (33) is rewritten as

$$\pi = \frac{1}{4} \pm \frac{\epsilon^2}{2} \left\{ x + \frac{4k - \epsilon^2}{2\gamma^2} \right\}$$

and the discriminate of the quadratic expression under the square root that has to be zero is given as

$$\left( k - \frac{\epsilon^2}{4} \right)^2 - \left( \frac{\gamma^2}{4} \right)^2 \left( l + \frac{1}{2} \right)^2 = 0$$

From equation (35) we obtain the value of $k$ as

$$k_1 = \frac{\epsilon^2}{4} + \frac{\gamma}{2} \left( l + \frac{1}{2} \right) \quad \text{or} \quad k_2 = \frac{\epsilon^2}{4} - \frac{\gamma}{2} \left( l + \frac{1}{2} \right)$$

By imposing the condition that $\tau < \mathcal{Q}$ then from equation (34) and (35) we get

$$\tau_1 = -\frac{\gamma x}{2} \frac{l}{2} \quad \text{for } k_1 \quad \text{or} \quad \tau_2 = -\frac{\gamma x}{2} \frac{l+1}{2} \quad \text{for } k_2$$

By using equation (6), (8), (32) and (37) we obtain

$$\lambda_1 = \frac{\epsilon^2}{4} + \frac{\gamma}{2} \left( l - \frac{1}{2} \right) \quad \text{for } k_1 \quad \text{or} \quad \lambda_2 = \frac{\epsilon^2}{4} - \frac{\gamma}{2} \left( l + \frac{3}{2} \right) \quad \text{for } k_2$$

$$r_1 = -\gamma x \left( l - \frac{1}{2} \right) \quad \text{for } k_1 \quad \text{or} \quad r_2 = -\gamma x \left( l + \frac{3}{2} \right) \quad \text{for } k_2$$

From equations (7), (32) and (39) we obtain the same values of $\lambda_n$ either for $k_1$ or $k_2$, that is

$$\lambda = \lambda_n = -n, \quad (-\gamma) = m_r$$

The energy eigenvalue obtained by equating equations (38) and (40) is given as

$$\frac{\epsilon_1^2}{4} = \gamma \left( n, -\left( l - \frac{1}{2} \right) \right) \quad \text{for } k_1 \quad \text{or} \quad \frac{\epsilon_2^2}{4} = \gamma \left( n, + \left( l + \frac{3}{4} \right) \right) \quad \text{for } k_2$$

To have physical meaning, the choice of the values are $k_2, \tau_2, \lambda_2$ and $\lambda_{n_2}$ therefore the energy spectrum of 3D HO plus Rosen-Morse non-central potential is obtained from equation (41) which is given as

$$E = \hbar \omega \left( 2n_r + l + \frac{3}{2} \right)$$
where \( n_r \) is radial quantum number, \( l \) is orbital quantum number and its values depend on the parameters of Rosen-Morse non-central potential. The orbital quantum number obtained from the solution of angular Schrödinger equation in section 3.2 is expressed as

\[
I = \sqrt{\frac{\mu^2}{\sqrt{2n_s}+n_r+\frac{1}{2}}} - \frac{1}{2}
\]  

(43)

We can see from equation (43) that for fixed values of principal quantum number, \( n \), and radial quantum number \( n_r \), the values of \( l = l' \) is not fixed since it depends on the Rosen Morse's parameter.

The radial wave functions are calculated using equations (4), (9), (10), (37), and (39). The first part of the wave function obtained using equations (4), (32) and (37) is given as

\[
\phi(r) = (x) \frac{n_r}{\sqrt{2n_s+1}} e^{-r}
\]  

(44)

and by using equations (10), (32), and (39) we get the weight function which is given as

\[
w(x) = \rho(x) = x^{1/2} e^{-r}
\]  

(45)

The second part of radial wave function obtained using equations (9) and (45) is given as

\[
y_n(x) = c_{n_r} \frac{d^{n_r}}{w(x) dx^n_r} \left( \sigma_{n_r} (x) w(x) \right) = c_{n_r} \frac{d^{n_r}}{x^{1/2} e^{-r} dx^n_r} \left( x^{1/2} e^{-r} \right)
\]  

(46)

By setting \( \beta x = z \) in equation (46), we get the second part of radial wave function and the total wave function in terms of associated Laguerre polynomials

\[
y_{n_r}(x) = C_{n_r} \left( \gamma \right)^{n_r/2} L_{n_r}^{(1/2)} (z) \quad \text{and} \quad \chi_{n_r} = B_{n_r} z^{(l'+1)/2} e^{-z} L_{n_r}^{(1/2)} (z)
\]  

(47)

The radial wave functions with and without the presence of Rosen-Morse potential and its related energy spectra for \( n_l = n_r \) are listed in Table 1. The graph of radial wave functions, \( R_{n,m\gamma n} (r) \), for \( n_r = n_l = 1 \), \( m = l \) and different values of \( \nu \) and \( \mu \) are shown in Figures 1 and 2. As shown in Table 1 and Figures 1-2, the two terms of Rosen Morse potential, \( \csc^2 \theta \) and \( \cot \theta \), give different effect on the radial wave function. The \( \csc^2 \theta \) term causes the increase of the energy and the radial wave function's amplitude, as shown in Figure 1, while \( \cot \theta \) term causes the decrease in energy and radial wave function’s amplitude, as shown in Figure 2. The normalization factor \( B_{n_r} \) in equation (47) can be obtained from the normalization condition of radial wave function which is expressed as

\[
\int \chi_{n_r} (r) \chi_{n_r} (r) dr = \delta_{n_r, n_r}
\]  

(48)

By using equations (47) and (48) we get

\[
B_{n_r} = \left( \frac{2\sqrt{\gamma n_r}}{n_r + l + 1/2} \right)^{1/2}
\]  

(49)

By the absent of Rosen Morse potential the radial wave function becomes the wave function of 3D HO.

Table 1. The second part of and the radial wave functions for \( n_l = n_r \) and the energy for various values of \( l' \).

| No | \( n_l \) | \( m \) | \( \nu \) | \( \mu \) | \( l' \) | \( y_{n_r} \) | \( R_{n,m\gamma n} (\gamma) \) | \( E_n (\hbar \omega) \) |
|----|---------|------|------|------|------|--------|-----------------|------|
| 1  | 2       | 1    | 0    | 0    | 3    | \( C_{l'} (24.75 - 11\gamma^2 + \gamma^2 r^2) \) | \( C_{l'} e^{-r^2/2} \left( 24.75 - 11\gamma^2 + \gamma^2 r^2 \right) \) | 8.5  |

6
The solution of angular Schrödinger equation

By setting \( P = \frac{Q}{\sqrt{\sin \theta}} \) in equation (28) then equation (28) becomes

\[
\begin{align*}
2 & \quad 2 & \quad 1 & \quad 2 & \quad 0 & \quad 4.65 & \quad C_2 \left\{ (43.97) - 14.3r^2 + y^2 r^4 \right\} + 10.15 \\
3 & \quad 2 & \quad 1 & \quad 0 & \quad 2 & \quad 2.95 & \quad C_3 \left\{ (24.25) - 10.9r^2 + y^2 r^4 \right\} + 8.45 \\
4 & \quad 2 & \quad 1 & \quad 2 & \quad 2 & \quad 4.64 & \quad C_2 \left\{ (43.84) - 14.3r^2 + y^2 r^4 \right\} + 10.14 \\
5 & \quad 3 & \quad 0 & \quad 0 & \quad 0 & \quad 3 & \quad C_3 \left\{ (160.88) - (107.25)y^2 \right\} + 10.5 \\
6 & \quad 3 & \quad 1 & \quad 0 & \quad 0 & \quad 4 & \quad C_4 \left\{ (268.13) - (146.25)y^2 \right\} + 11.5 \\
7 & \quad 3 & \quad 1 & \quad 2 & \quad 0 & \quad 5.65 & \quad C_5 \left\{ (533.19) - (223.72)y^2 \right\} + 13.15 \\
8 & \quad 3 & \quad 1 & \quad 0 & \quad 2 & \quad 3.98 & \quad C_6 \left\{ (265.62) - (145.41)y^2 \right\} + 11.48 \\
9 & \quad 3 & \quad 1 & \quad 2 & \quad 2 & \quad 5.64 & \quad C_7 \left\{ (531.21) - (223.2)y^2 \right\} + 13.14
\end{align*}
\]

Figure 1. The graph of 1st state of Radial wave function for 3D HO- Rosen Morse system with \( \csc^2 \theta \) term

- \( R_{1201}(r) = C_2 r^{1.55} e^{-\frac{r^2}{2}} (5.15 - r^2) \)
- \( R_{1601}(r) = C_2 r^{1.05} e^{-\frac{r^2}{2}} (9.56 - r^2) \)

Figure 2. The graph of 1st state of Radial wave function for 3D HO- Rosen Morse system with \( \cot \theta \) term:

- \( R_{1102}(r) = C_2 r^{1.55} e^{-\frac{r^2}{2}} (4.05 - r^2) \)
- \( R_{1106}(r) = C_2 r^{1.05} e^{-\frac{r^2}{2}} (4.45 - r^2) \)
- \( R_{1010}(r) = C_2 r^{0.55} e^{-\frac{r^2}{2}} (3.02 - r^2) \)

3.2 The solution of angular Schrödinger equation

By setting \( P = \frac{Q}{\sqrt{\sin \theta}} \) in equation (28) then equation (28) becomes
To solve equation (50) we introduce a new variable \( \cot \theta = s \) and equation (50) change into

\[
(1 + s^2) \frac{d^2 Q}{ds^2} + 2s \frac{dQ}{ds} - \left[ \left( v + 1 \right) + \frac{1}{4} \right] Q = 0
\]

(51)

Equation (51) is solved in terms of Romanovski polynomial by setting

\[
Q(\theta) = g_s(s) \left[ \left( 1 + s^2 \right)^{\frac{\alpha}{2}} e^{-\frac{\alpha}{2} \tan^2} D_n(\beta, \alpha)(s) \right]
\]

(52)

By inserting equation (52) into equation (51) we obtain

\[
(1 + s^2) \frac{d^2 D}{ds^2} + \left[ 2s(\beta + 1) - \alpha \right] \frac{dD}{ds} - \left[ \frac{\beta \alpha - \frac{\alpha}{4} + \beta^2 - 2\mu - \left( l(l + 1) + \frac{1}{4} \right)}{1 + s^2} \right] D = 0
\]

(53)

Equation (53) reduces to differential equation satisfied by Romanovski polynomials

\[
(1 + s^2) \frac{d^2 D}{ds^2} + \left[ 2s(\beta + 1) - \alpha \right] \frac{dD}{ds} - \left[ \frac{\beta \alpha - \frac{\alpha}{4} + \beta^2 - 2\mu - \left( l(l + 1) + \frac{1}{4} \right)}{1 + s^2} \right] D = 0
\]

(54)

when

\[
\beta \alpha - \frac{\alpha}{4} + \beta^2 - 2\mu - \left( l(l + 1) + \frac{1}{4} \right) = 0
\]

(55)

By comparing equations (10) and (54) we obtain

\[
(\beta + 1) = -p + 1; \alpha = -q \quad \text{and} \quad v(v + 1) + m^2 - \frac{1}{4} - \beta^2 - \beta = n(n - 1) + 2n(1 - p)
\]

(56)

From equation (55) we have

\[
-\frac{\alpha}{4}^2 + \beta^2 - \left( l(l + 1) + \frac{1}{4} \right) = 0; \beta \alpha - 2\mu = 0
\]

(57)

By using equation (56) together with equation (57) and by imposing the condition that \( p > 0 \) we get

\[
\beta = \frac{-\sqrt{v(v + 1) + m^2 - n - \frac{1}{2}}}{\sqrt{v(v + 1) + m^2 + n + \frac{1}{2}}}
\]

(58)

\[
l = \ell = \frac{\sqrt{v(v + 1) + m^2 + n + \frac{1}{2}}^2 - \frac{\mu^2}{\left( v(v + 1) + m^2 + n + \frac{1}{2} \right)^2} - \frac{1}{2}}{\mu^2}
\]

(59)

The weight function obtained from equations (7), (56), (58) and (59) is given as

\[
w(\beta, \alpha, \sigma, \sigma_s) = \left( 1 + s^2 \right)^{\frac{\alpha}{2}} e^{-\sigma \tan^2 s}
\]

(61)

Using equations (13) and (61) we obtain the Romanovski polynomials given as

\[
R_n(\beta, \alpha)(s) = \frac{1}{\left( 1 + s^2 \right)^{\frac{\alpha}{2}} e^{-\sigma \tan^2 s}} \int (1 + s^2)^{\frac{\alpha}{2}} e^{-\sigma \tan^2 s} \left( 1 + s^2 \right)^{\frac{\alpha}{2}} e^{-\sigma \tan^2 s}
\]

(62)

The angular wave function obtained from equations (11) and (62) is given as

\[
Q_n(\theta = \cot^{-1} s) = g_n(s) = \left( 1 + s^2 \right)^{\frac{\alpha}{2}} e^{-\frac{\alpha}{2} \tan^2 s} R_n(\beta, \alpha)(s)
\]

(63)

The polar wave function obtained from equation (63) is given as
Since the values of $\beta_{n_2}$ and $\alpha_{n_2}$ are n-dependence therefore the orthogonal integral of Romanovski polynomial is not produced from the orthogonality integral of wave function thus the polynomial is infinity [18]. The orthogonality integral of the wave function expressed in equation (76) is given as

$$
\int P_{n_1}(\theta) P_{n_r}(\theta) \sin \theta d \theta = \delta_{n_1,n_r}
$$

The effect of Rosen Morse potential to the angular wave function is shown in Figures 3-5. Figures 3(a)-5(a) show the graph of three dimensional polar representations of the absolute value of $Y_{l,m}^n$ and Figures 3(b)-5(b) show the polar diagrams of $Y_{l,m}^n$ which is obtained from equations (26) and (64). Figure (3) shows the angular wave function with the absent of RM potential, Figure (4) shows $\csc \theta$ term causes the increase of the absolute value of the angular wave function, Figure (5) shows $\cot \theta$ term causes the decrease of the absolute value of the angular wave function in the interval of $0 < \theta < \frac{\pi}{2}$ but increasing in the $\frac{\pi}{2} < \theta < \pi$ interval.

The total un-normalized wave function for the n level is given as

$$
\psi(r, \theta, \phi) = e^{i n \phi} \left\{ \frac{r^{-\frac{1}{2}}}{2} e^{i n \alpha_{n_2} \tan^{-1}(\cot \theta)} \right\}
$$

By the absent of Rosen-Morse potential the wave function in equation (79) reduces to the three dimensional spherical harmonics oscillator wave function.

4. Conclusion

The Schrodinger equation for three dimensional harmonic oscillator potential with simultaneously the presence of Rosen-Morse non-central potential is solved approximately using NU method and Romanovski polynomial. The energy spectrum of non-central system is expressed in the closed form and is reduced to the energy spectrum for three dimensional harmonics oscillator potential by the absent of
Rosen-Morse potential. The presence of Rosen Morse non-central potential causes the energy of 3D HO increases. The effect of the two of potential terms to the energy is different. The $\csc^2 \theta$ term ($\nu$ values) causes the increase of energy, but the $\cot \theta$ term ($\mu$ values) causes the energy of 3D HO decreases. In the same way with the energy, the radial wave function amplitude increases by the increase of $\nu$ values, but decreases by the increase of $\mu$ values. The approximate radial wave function is expressed in term of associated Laguerre polynomials and reduces to the wave function of three dimensional harmonics oscillator for the absent of Rosen-Morse potential. The angular wave functions are expressed in terms of Romanovski polynomials, and reduce to associated Legendre polynomials by the absent of trigonometric Rosen-Morse non-central potential. The $\cot \theta$ term ($\mu$ values) causes the length of the vector of angular function decreases and is shifted to the smaller values of polar angle, the larger is the value of $\mu$ the faster is the decrease of the vector length and the polar angle for small values of $n_l$, but this effect reduces when the values of $n_l$ get larger.

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