The Champernowne constant is not Poissonian

Ísabel Pirsic *, Wolfgang Stockinger †

Abstract

We say that a sequence $(x_n)_{n \in \mathbb{N}}$ in $[0, 1)$ has Poissonian pair correlations if

$$\lim_{N \to \infty} \frac{1}{N} \# \{1 \leq l \neq m \leq N : \|x_l - x_m\| \leq \frac{s}{N} \} = 2s$$

for every $s \geq 0$. In this note we study the pair correlation statistics for the sequence of shifts of $\alpha$, $x_n = \{2^n \alpha\}$, $n = 1, 2, 3, \ldots$, where we choose $\alpha$ as the Champernowne constant in base 2. Throughout this article $\{\cdot\}$ denotes the fractional part of a real number. It is well known that $(x_n)_{n \in \mathbb{N}}$ has Poissonian pair correlations for almost all normal numbers $\alpha$ (in the sense of Lebesgue), but we will show that it does not have this property for all normal numbers $\alpha$, as it fails to be Poissonian for the Champernowne constant.

1 Introduction and main result

The concept of Poissonian pair correlations has its origin in quantum mechanics, where the spacings of energy levels of integrable systems were studied. See for example [1] and the references cited therein for detailed information on that topic. Rudnik and Sarnak first studied this concept from a purely mathematical point of view and over the years the topic has attracted wide attention, see e.g., [5, 9, 10, 11, 12]. Recently, Aistleitner, Larcher and Lewko (see [2]) could give a strong link between the concept of Poissonian pair correlations and the additive energy of a finite set of integers, a notion that plays an important role in many mathematical fields, e.g., in additive combinatorics. Roughly speaking, they proved that if the first $N$ elements of an increasing sequence of distinct integers $(a_n)_{n \in \mathbb{N}}$, have an arbitrarily small energy saving, then $(\{a_n \alpha\})_{n \in \mathbb{N}}$ has Poissonian pair correlations for almost all $\alpha$. This result recovers the metric Poissonian pair correlation property for lacunary sequences as well. In this paper the authors also raised the question if an increasing sequence of distinct integers with maximal additive energy can have Poissonian pair correlations for

*The author is supported by the Austrian Science Fund (FWF), Project P27351-N26.
†The author is supported by the Austrian Science Fund (FWF), Project F5507-N26, which is a part of the Special Research Program Quasi-Monte Carlo Methods: Theory and Applications.
almost all $\alpha$. Jean Bourgain could show that the answer to this question is negative, see the appendix of [2] for details and a second problem which was also solved by Bourgain. Recently, the results of Bourgain have been further extended, see [1, 7, 8].

Let $\| \cdot \|$ denote the distance to the nearest integer. A sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers in $[0, 1)$ has Poissonian pair correlations if

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| \leq \frac{s}{N} \right\} = 2s \quad (1)$$

for every $s \geq 0$. Due to a result by Grepstad and Larcher [4] (see also [3, 14]), we know that a sequence which satisfies property (1), is also uniformly distributed in $[0, 1)$, i.e., it satisfies

$$\lim_{N \to \infty} \frac{1}{N} \# \{ 1 \leq n \leq N : x_n \in [a, b) \} = b - a$$

for all $0 \leq a < b \leq 1$. Note that the other direction is not necessarily correct. For instance the Kronecker sequence $(\{n\alpha\})_{n \in \mathbb{N}}$, does not have this property for any real $\alpha$; a fact that is related to the famous Three Gap Theorem, see [13]. Poissonian pair correlation is a typical property of a sequence. Random sequences, i.e., almost all sequences, have the Poissonian pair correlation property. Nevertheless, until now we do not know any single of a sequence which has Poissonian pair correlations.

We recall that the sequence $(\{2^n\alpha\})_{n \in \mathbb{N}}$ has the Poissonian property for almost all $\alpha$. In this note, we study the distribution of the pair correlations of the sequence $(\{2^n\alpha\})_{n \in \mathbb{N}}$, where $\alpha$ is the Champernowne constant in base 2, i.e., $\alpha = 0.1101101011011\ldots_2$. It is a well known fact that the Champernowne constant in base 2 is normal to base 2. Moreover we know that the sequence $(\{2^n\alpha\})_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if $\alpha$ is normal, see e.g., [6]. If we want to investigate, whether the distribution of the pair correlations for some explicit given sequence is Poissonian, the sequence has to be uniformly distributed modulo 1. Therefore, if we investigate the distribution of the spacings between the sequence elements of $(\{2^n\alpha\})_{n \in \mathbb{N}}$, the only reasonable choice for $\alpha$ is a normal number. We obtain the following result.

**Theorem 1** The sequence $(\{2^n\alpha\})_{n \in \mathbb{N}}$, where $\alpha$ is the Champernowne constant in base 2, i.e., $\alpha = 0.110110010111011\ldots_2$ does not have Poissonian pair correlations.

The work on this paper was initiated by the conjecture of G. Larcher (mentioned during a personal discussion) that all normal numbers are Poissonian, due to the lacunarity of $2^n$. To make it more tangible why this conjecture is reasonable, we recall that Kronecker sequences are not Poissonian for any $\alpha$ and $(\{\alpha^n\})_{n \in \mathbb{N}}$, $d \geq 2$ is Poissonian for almost all $\alpha$, whereby it is known that $\alpha$ has to satisfy some Diophantine condition, see e.g., [12]. Hence, one would
expect the sequence \(\{2^n\alpha\}_{n\in\mathbb{N}}\) to have the Poissonian property for all normal numbers \(\alpha\), as it shows less structure than the Kronecker and polynomial sequences. The motivation to study the sequence described in Theorem 1 was to find the first explicit example of a sequence having Poissonian pair correlations. At least our result allows to immediately deduce that the sequence \(\{2^n\alpha\}_{n\in\mathbb{N}}\) cannot have Poissonian pair correlations for all normal numbers \(\alpha\).

To prove Theorem 1 we basically use elementary counting techniques from combinatorics. We give a short outline of the proof. Let \(e, d\), be two integers, where \(d = 2^e\) is understood to be very large compared to \(e\). Further, we set \(s = 1\) and \(N = 2d + e\) in (1). The reason for choosing \(N\) in such a manner is the following. The Champernowne constant is the concatenation of the numbers which have a digit expansion in base 2 of length 1, 2, 3, \ldots, \(d\), \ldots and so forth. In order to account for all blocks of words of length 1, \ldots, \(d\), we have to choose \(N\) large enough, i.e., in our case at the beginning of the block of words having length \(d + 1\). Note that the length of the block containing the words of length \(d\) is \(2d + e - 1\) and for a very large \(e\) all previous blocks of words with length 1, \ldots, \(d - 1\) have in total approximately this length. We then count the occurrence of bit patterns (in the block of words of length \(d\)), which correspond to shifts of \(\alpha\) having a distance < \(1/N\) (we will henceforth abbreviate to simply saying that the patterns have this distance). If we have two patterns which match in the first \(d + e\) bits or which are of the form

\[
\underbrace{a_1a_2\ldots01\ldots1}_{d+e} b_1b_2b_3\ldots \\
\underbrace{c_1c_2c_3\ldots}_{d+e} < b_1b_2b_3\ldots,
\]

and there is an overlap of \(e\) bits to the word length \(d\). The overlap \(e\) is understood to be small.

First, we investigate the type of pattern where the first \(e\) bits match the last ones, i.e., \(a_i = a_{d+i}\) for \(i = 1, \ldots, e\). We denote the index before the start of a
possible matching word by \( z \geq 0 \), i.e., if a match occurs then there is an \( n \) such that
\[
a_{z+1}a_{z+2} \cdots = c_{n,1}c_{n,2} \cdots
\]
and at least one of
\[
a_za_{z+1} \cdots = c_{n-1,d}c_{n-1,d-1} \cdots,
\]
\[
a_{z+d+1}a_{z+d+2} \cdots = c_{n+1,1}c_{n+1,2} \cdots.
\]

**Basic Fact (BF1):** for a match to occur, \( a_z \) must not equal \( a_{z+d} \) since these bits correspond to the least significant bits of consecutive digit expansions \( c_{n-1}, c_n \).

**BF2:** As a first consequence of BF1, \( z \) must be zero or greater than \( e \) since otherwise \( a_z = a_{z+d} \) and similarly \( z \) must be at most \( d \), else \( a_{z-d} = a_z \), i.e., \( z \in \{0, e+1, \ldots, d\} \).

**BF3:** Furthermore, for a match with \( z > 0 \) to occur it is necessary that \( a_{z+1} = 1 \) and at least one zero occurs in the sequence \( a_{e+1} \ldots a_z \). This excludes subpatterns of the forms
\[
a_{e+1} \ldots a_{z+1} = 1 \ldots 10 \text{ or } 1 \ldots 11,
\]
which cannot occur due to the fact that in this case \( c_n = c_{n-1} + 1 \) has carries affecting \( a_d \ldots a_{d+e} \).

We now make case distinctions according to the number \( k \) of ones in the ‘middle block’ \( a_{e+1} \ldots a_d \).

If \( k = 0 \) the pattern can occur in the full block only if \( z = 0 \) and \( a_{z+1} = c_{n,1} = 1 \) which cannot happen in the middle block or if \( a_1 = 0 \).

If \( k = 1 \) this type of pattern (or ‘meta-pattern’) can occur in the case \( a_{e+1} = 1 \) only if \( a_1 = 1 \) and \( z = 0 \); BF3 forbids \( z > 0 \) and for \( z = 0 \) again \( a_{z+1} = a_1 = 1 \) is necessary. If the 1 appears later in the middle block, again \( z = 0 \) is possible, if \( a_1 = 1 \) or \( z = j \) if \( j + 1 \) is the index of the 1. This gives

- \( 2^{e-1} + 2^{e-1}(d - e - 1) \) patterns occurring only one time
- \( 2^{e-1}(d - e - 1) \) patterns occurring two times.

Let us also look at the case \( k = 2 \): first, \( a_{e+1} = a_{e+2} = 1 \) by BF2 again necessitates \( z = 0 \) and \( a_1 = 1 \), and can occur only in one match. If \( a_{e+1} = 1 \neq a_{e+2} \) there are one or two possible matches in dependence of \( a_1 = 0 \) or 1. Finally, two or three possible matches can happen if both ones occur later in the middle block. The tally thus is:

- one match: \( 2^{e-1}(1 + (d - e - 2)) \) patterns
- two matches: \( 2^{e-1}((d - e - 2) + (d - e - 1)) \) patterns
- three matches: \( 2^{e-1}(d - e - 1) \) patterns

We can now present the general case \( 2 < k < d - e \):
• $a_1 = 0, a_{e+1} = 1, a_{e+2} = 0$: we have $2^{e-1} \binom{d-e-2}{k-1}$ patterns having $k-1$ matches.

• $a_1 = 0, a_{e+1} = a_{e+2} = 1$: let $a_{e+1} = \cdots = a_{e+j} = 1 \neq a_{e+j+1}$, i.e., there are $j$ consecutive ones at the start of the middle block followed by a zero. Then there are $k-j$ ones left to distribute on $d-e-j-1$ places. We have a match for each of those ones, so there are $2^{e-1} \binom{d-e-j-1}{k-j}$ patterns having $k-j$ matches, where $j = 1, \ldots, k$.

• $a_1 = 0, a_{e+1} = 0$: we have $2^{e-1} \binom{d-e-1}{k}$ patterns having $k$ matches.

• $a_1 = 1, a_{e+1} = 0$: we have $2^{e-1} \binom{d-e-1}{k}$ patterns having $k+1$ matches.

• $a_1 = 1, a_{e+1} = 1$: let $a_{e+1} = \cdots = a_{e+j} = 1 \neq a_{e+j+1}$, i.e., there are $j$ consecutive ones at the start of the middle block followed by a zero. Then there are $k-j$ ones left to distribute on $d-e-j-1$ places. We have a match for each of those ones plus one attributed to $z = 0$, i.e., there are $2^{e-1} \binom{d-e-j-1}{k-j}$ patterns having $k-j+1$ matches, where $j = 1, \ldots, k$.

The case $k = d-e$ has only patterns matching just once.

Taking all together we get the following formula for the number of pairs $c_{n_1, i_1}, c_{n_2, i_2}$ such that there is a match in (at least) $w$ bits. Note that the pairs are ordered.

$$2^e \sum_{k=1}^{d-e-1} \binom{k-1}{k-j} \binom{d-e-j-1}{k-j} + \sum_{j=0}^{k-1} \binom{k-j+1}{k-j} \binom{d-e-j-1}{k-j}$$

$$= 2^e \sum_{k=1}^{d-e-1} \sum_{j=0}^{k-1} (k-j)^2 \binom{d-e-j-1}{k-j}$$

$$= 2^e (d-e) \sum_{k=0}^{d-e-1} k \frac{((d-e-k+1)k+1)}{(d-e-k+1)(d-e-k+2)} \binom{d-e-1}{k}$$

$$\geq 2^e (d-e) \sum_{k=0}^{d-e-1} k^2 \binom{d-e-1}{k}$$

$$= 2^e (d-e) \sum_{k=0}^{d-e-1} \frac{k^2}{d-e} \binom{d-e-1}{k}$$

If we divide this amount of pairs by $N = 2^{d+e}$ (recall that we set $d = 2^e$) and consider $e \to \infty$, we obtain $+\infty$ in the limit and deduce that the pair correlations distribution cannot be asymptotically Poissonian. We note that the second equality was obtained by Mathematica [13].

For the sake of completeness, we study two further types of patterns. We will see that these two structures of patterns yield a negligible amount of pairs. The
next type of pattern is where the matching \( c_n \) ends in a string of ones, inducing a chain of carries for \( c_{n+1} \). I.e., there are \( j_0, j_1, 1 \leq j_0 \leq e < j_1 \leq d - 1 \) such that
\[
a_{j_0}a_{j_0+1} \ldots a_ea_{e+1} \ldots a_ja_{j+1} = 01 \ldots 11 \ldots 10
\]
and \( a_i = a_{d+i} \) for \( 1 \leq i < j_0 \), \( a_i = 1 - a_{d+i} \) for \( j_0 \leq i \leq e \). Again, a possible matching \( c_n \) can obviously not start with an index earlier than \( e + 1 \) since then inevitably mismatches \( a_i \neq a_{d+i} \) that cannot be accounted for by carries occur. But then each of the consecutive ones can be taken as start of a \( c_n \)-block, i.e., \( z = e, \ldots, j_1 - 1 \) are all possible, giving \( j_1 - e \) matches. For the case \( j_1 = d \) there are \( d - e \) matches as well, plus an additional one if \( a_1 = 1 \). Both subcases have \( 2^{j_0-1} \) according patterns for \( j_0 \geq 2 \) and additionally there is one further case for \( j_0 = 1 \) with \( d - e \) matches, the pattern \( 01^a0^e-1 \). The number of ordered pairs thus equals:
\[
2 \left( \sum_{j_0=1}^{e} \sum_{j_1=e+1}^{d-1} \binom{j_1-e}{2} 2^{j_0+d-j_1-2} + \left( \binom{d-e}{2} + \binom{d-e+1}{2} \right) \sum_{j_0=2}^{e} 2^{j_0-2} + \binom{d-e}{2} \right)
= \frac{2^e-1}{2^{e-1}} (2^d - 2^e) - (d-e)2^e < 2^{d+1}.
\]

The next type of pattern, which only yields a negligible amount of relevant pairs, is the one where \( a_1 \ldots a_e = 1 \ldots 1 \) and \( a_{e+1} \ldots a_{z+1} = 1 \ldots 1 \), where \( z \in \{ e, \ldots, d-1 \} \). As a consequence thereof \( a_{d+1} \ldots a_{d+e} = 0 \ldots 0 \). Hence, we have
\[
\sum_{i=3}^{d-e} \binom{i-1}{2} = \frac{1}{6} (d-e-2)(d-e-1)(d-e)
\]
pairs with distance \(< 1/N\). 

Remark 1 In the proof we have only studied the case, where a fixed bit pattern of length \( w \) overlaps two words of length \( d \). Of course, an overlap of the pattern with three words might also occur, but these cases yield a small number of pairs with prescribed distance. Therefore we have omitted the exact study of these structures. If the relative number of pairs in the block of words of length \( d \) would have given a number less than \( 2s \), then a study of the occurrence of the pattern in the block of words of length \( d-1 \) (and so forth) would have been necessary.

Remark 2 The techniques from above can of course be adapted to any other base \( b \), i.e., we can conclude that the Champernowne constant in base \( b \geq 2 \) (note that the Champernowne constant in base \( b \) is normal to base \( b \)) is not Poissonian.
3 Open Problems and Outlook

In this section we first want to state an open problem, which involves the notion of weak pair correlations (introduced by Steinerberger in [14]), a concept that relaxes the requirements of [11].

We state the following open problem.

Problem 1: Does the sequence \((x_n)_{n\in\mathbb{N}} = \{(2^n\alpha)\}_{n\in\mathbb{N}}\), where \(\alpha\) is the Champernowne constant in base 2, satisfy the notion of weak pair correlation, i.e., is there an \(0 < \beta < 1\), such that

\[
\lim_{N \to \infty} \frac{1}{N^{2-\beta}} \#\{1 \leq l \neq m \leq N : \|x_l - x_m\| \leq \frac{s}{N^{\beta}}\} = 2s
\]

for every \(s \geq 0\)?

Further, we still need to find an explicit example of a sequence with Poissonian pair correlations and maybe criteria which relax the definition of Poissonian pair correlations, e.g., that it possibly suffices to show that [11] holds for \(s \in \mathbb{N}\) only. A possible approach would be to modify the Champernowne constant in a certain way, e.g., by shifts, such that we avoid the situation that we have too many patterns where the first and last \(c\) bits match.

4 Appendix

Though the here presented results are not needed for the proof of Theorem 1, they give additional interesting information about the pair correlation structure of the Champernowne constant and therefore we add them as appendix.

In the previous section we have counted the occurrence of a bit pattern \(a_1 \ldots a_w\) in the full block of words of length \(d\). Now, we consider patterns of the form \(b := a_1 a_2 \ldots a_w b_1 b_2 b_3 \ldots = a_1 a_2 \ldots a_j 01 \ldots 1 b_1 b_2 b_3 \ldots\) and \(c := a'_1 a'_2 \ldots a'_w c_1 c_2 c_3 \ldots = a_1 a_2 \ldots a_j 10 \ldots 0 c_1 c_2 c_3 \ldots\), with \(b_1 b_2 b_3 \ldots > c_1 c_2 c_3 \ldots\).

These two types of bit words also have a distance less than \(1/N\).

First, we study the case \(b_1 = 1\) and \(c_1 = 0\), and then generalize our investigations. Therefore, we are aiming at counting the occurrences of bit blocks of the form \(B := a_1 a_2 \ldots a_j 01 \ldots 11\) (in the full block of words of length \(d\)) and the ones of the form \(C := a_1 a_2 \ldots a_j 10 \ldots 00\).
**Corollary 1** The patterns of the form B and C yield for \( j = d \) at least
\[
2^{d-e-1}(d-e-5)
\]
pairs with distance less than \( 1/N \). For \( j > d \) we obtain at least
\[
2^{-1-e}(2^e - 2)(2^{2+e} + 2^3d + 2^{1+e}d - 2^{d+e} - 2^{1+e} - 2^{2+d})
\]
pairs with distance less than \( 1/N \).

**Proof.** We start by studying the occurrence of the first pattern. Note that we first consider the case, where the first and the last \( e \) bits of \( a_1a_2...a_j01...1 \)
match. In the following, we distinguish several distinct cases, depending on the position of the index \( j \). Later, we will see that the only relevant cases are the ones where \( j \geq d \). Thus, we will examine only those in more detail.

- **\( d > j > e \):**
  - Here, we have the structure \( a_1...a_{e+1} = 1...1 \) and \( a_{j+1}a_{j+2}...a_{d+e} = 01...1 \), i.e., the first and the last \( e \) digits match and \( a_{e+1} = 1 \) since \( b_1 = 1 \).
  - If \( a_{e+1} \neq 1 \), then we have the structure \( a_1...a_ea_{e+1} = 1...10 \) and again \( a_{j+1}a_{j+2}...a_{d+e} = 01...1 \). Such a pattern occurs, if there exists \( e+1 \leq z < j \) such that \( a_{e+1}a_{e+2}...a_z = 01...1 \).

- **\( j = d \):**
  - First, let \( a_{e+1} = 1 \) and due to \( j = d \), \( a_1a_2...a_e = a_{d+1}a_{d+2}...a_{d+e} = 01...1 \). Let \( k \) be the number of ones in the block \( a_{e+1}...a_d \). Then, we obtain
    \[
    \sum_{k=2}^{d-e-1} \sum_{l=1}^{k-1} (k-l) \binom{d-e-l-1}{k-l}
    \]
    matches.
  - Consider now \( a_{e+1} = 0 \). If there exists \( z \leq d \) with \( a_{e+1}a_{e+2}...a_z = 01...1 \), then this case yields
    \[
    \sum_{k=1}^{d-e-1} \sum_{l=1}^{k} l \binom{d-e-l-2}{k-l}
    \]
    matches.

- **\( j > d \):**
  - Let \( a_{e+1} = 1 \). We have the structure (the first and last \( e \) digits are again equal) \( a_1...a_e = a_{d+1}...a_{j}a_{j+1}...a_{d+e} = a_{d+1}...a_j0...1 \).
    In total there are
    \[
    2^{j-d-1} \left[ \sum_{k=2}^{d-e-1} \left( \sum_{l=1}^{k} (k-l) \binom{d-e-l-1}{k-l} + (k-l+1) \binom{d-e-l-1}{k-l} \right) \right]
    \]
matches.

– Let $a_{e+1} = 0$. Here, we get (similar to above)

$$2^{j-d} \left( \sum_{k=1}^{d-e-1} \sum_{l=1}^{k} \binom{d-e-l-2}{k-l} \right)$$

matches.

• $j < e$: In this case the first and the last $e$ digits cannot match.

• $j = e$: Here, we have $a_{e+1}a_{e+2} \ldots a_{d+e} = 01 \ldots 1$. Therefore (in case that the first and last $e$ digits match) $a_1 \ldots a_e = 1 \ldots 1$. Such a pattern appears $d - e$ times.

In the following we study the structure of the second pattern mentioned at the beginning. We distinguish the following cases.

• $d > j > e$: Here, we have $a_{j+1}a_{j+2} \ldots a_{d+e} = 10 \ldots 0$. If the first and last $e$ digits match, $a_1 \ldots a_e = 0 \ldots 0$ has to hold and $a_{e+1} = 0$.

• $j = d$: Here, we have $a_1a_2 \ldots a_e = a_{d+1}a_{d+2} \ldots a_{d+e} = 10 \ldots 0$, and $a_{e+1} = 0$.

• $j > d$: Here we have, assuming that the first and last $e$ bits match $a_1 \ldots a_{e} = a_{d+1} \ldots a_{j+1}a_{j+2} \ldots a_{d+e} = a_{d+1} \ldots a_{j}1 \ldots 0$.

• $j = e$: This scenario cannot happen, as in this case it should be $a_{e+1}a_{e+2} \ldots a_{d+e} = 10 \ldots 0$.

It remains to check if above cases allow starting and end blocks of the form $a_1 \ldots a_e = a_1 \ldots a_{m-1}01 \ldots 1$ and $a_{d+1} \ldots a_{d+e} = a_1 \ldots a_{m-1}10 \ldots 0$, respectively. If $d > j > e$, or $j \leq e$, then this cannot happen. The case $j \geq d$ allows starting and end blocks of this form for the second pattern.

Above, we have investigated how often (and for which cases) one of the two patterns $B$ and $C$ occurs. It remains to analyse how many matches of the respective patterns agree in the first $j$ digits. We will see that the only relevant cases are where $j \geq d$.

• $d > j > e$: If we investigate this case for the patterns $B$ and $C$, we see that the first $j$ digits do not agree. I.e., we do not obtain pairs having distance less than $1/N$.

• $j \leq e$: In this case we also do not get pairs with the prescribed distance.

• $j = d$: Here we have for the pattern $B$ the structure $a_1a_2 \ldots a_{e} = a_{d+1}a_{d+2} \ldots a_{d+e} = 01 \ldots 1$. For the pattern $C$ we have the
feasible structure \( a_1a_2 \ldots a_e = 01 \ldots 1 \) and \( a_{d+1}a_{d+2} \ldots a_{d+e} = 10 \ldots 0 \). I.e., we obtain

\[
2 \sum_{k=2}^{d-e-1} \sum_{l=1}^{k-1} (l-1)(k-l) \binom{d-e-l-1}{k-l}
\]
pairs with distance < 1/N. Note that the last equation can be simplified to (2).

- \( j > d \): Here, we therefore get

\[
2 \sum_{k=2}^{d-e-1} \sum_{l=1}^{k-1} (l-1)(k-l) \binom{d-e-l-1}{k-l}
\]

pairs with distance < 1/N. Summation for \( d + 1 \leq j \leq d + e - 1 \) yields the claimed formula (3).

Let now \( n \in \mathbb{N}, n \geq 2 \), with \( b_1 \ldots b_n = b_1 \ldots b_{n-1}1 \) and \( c_1 \ldots c_n = c_1 \ldots c_{n-1}0 \).

To present this general case of above considerations, we give an example for \( j = d \) and \( n = 3 \):

Let \( n = 3 \). As before we have for the first pattern \( a_1a_2 \ldots a_e = a_{d+1}a_{d+2} \ldots a_{d+e} = 01 \ldots 1 \). For the second pattern we have the feasible structure \( a_1a_2 \ldots a_e = 01 \ldots 1 \) and \( a_{d+1}a_{d+2} \ldots a_{d+e} = 10 \ldots 0 \). To guarantee \( b_1b_2b_3 > c_1c_2c_3 \), we have possibilities of the form \( b_1b_2b_3 = 111 \land c_1c_2c_3 = 110 \) and so forth. Since a carry is necessary for the second pattern (the end block is the complement of the starting block), the only relevant options are \( b_1 = c_1 = 1 \). The remaining \( n-2 \) digits can be chosen arbitrarily from \( \{0,1\} \) (\( b_n \) and \( c_n \) have to be 1 and 0, respectively). If we consider for example \( b_1b_2b_3 = 111 \land c_1c_2c_3 = 110 \), then we also prescribe a structure for \( a_{e+1}a_{e+2}a_{e+3} \). In this case \( a_{e+1}a_{e+2}a_{e+3} = 110 \) has to hold. Alternative structures for \( a_{e+1}a_{e+2}a_{e+3} \), like 100, 101, 111 are not possible. The first pattern allows as many shifts as we have ones at the beginning of the block \( a_{e+1} \ldots a_d \). The second pattern appears once (01 \ldots 1 is at the end of a word of length \( d \), and 110 \ldots 10 \ldots 0 is the start of a new word of length \( d \)).

Thus, we get the following corollary.
Corollary 2 The patterns of the form $b$ and $c$ yield for $j = d$ at least ($k$ denotes the number of ones in the block $a_{e+n+1} \ldots a_d$ and $l$ the number of ones at the beginning of such a block)

$$2 \sum_{n=2}^{d-e} 2^{n-2} \sum_{k=1}^{d-e-n} \sum_{l=1}^{k} \binom{d-e-n-l-1}{k-l}$$

pairs with distance $< 1/N$. For $j > d$ we obtain at least

$$2 \sum_{j=d+1}^{d+e-1} 2^{j-d} \left[ \sum_{n=2}^{d-e} 2^{n-2} \sum_{k=1}^{d-e-n} \sum_{l=1}^{k} \binom{d-e-n-l-1}{k-l} \right] \sum_{j=d+1}^{d+e-1} 2^{j-d} \sum_{n=2}^{d-e} 2^{n-2}$$

pairs with distance $< 1/N$.

Acknowledgements. We would like to thank Gerhard Larcher for his valuable comments, suggestions and his general assistance during the writing of this paper.

References

[1] I. Aichinger, C. Aistleitner, G. Larcher, On Quasi-Energy-Spectra, Pair Correlations of Sequences and Additive Combinatorics, arXiv:1708.08590v1, 2017.

[2] C. Aistleitner, G. Larcher, M. Lewko, Additive energy and the Hausdorff dimension of the exceptional set in metric pair correlation problems. With an appendix by Jean Bourgain, Israel J. Math. To appear. Available at https://arxiv.org/abs/1606.03591.

[3] C. Aistleitner, T. Lachmann, and F. Pausinger, Pair correlations and equidistribution, arXiv:1612.05495, 2017. To appear in: Journal of Number Theory.

[4] S. Grepstad and G. Larcher, On Pair Correlation and Discrepancy, Arch. Math. 109, 143-149, 2017.

[5] D. R. Heath-Brown, Pair correlation for fractional parts of $\alpha n^2$, Math. Proc. Cambridge Philos. Soc., 148(3):385-407, 2010.

[6] L. Kuipers and H. Niederreiter, Uniform distribution of sequences, Wiley, New York, 1974.
[7] T. Lachmann and N. Technau, *On Exceptional Sets in the Metric Poissonian Pair Correlations problem*, arXiv:1708.08599, 2017.

[8] G. Larcher. *Remark on a result of Bourgain on Poissonian pair correlation*, Preprint, 2017.

[9] R. Nair and M. Pollicott, *Pair Correlations of sequences in higher dimensions*, Israel J. Math. 157 (2007), 219238.

[10] Z. Rudnick and A. Zaharescu, *A metric result on the pair correlation of fractional parts of sequences*, Acta Arith., 89(3):283-293, 1999.

[11] Z. Rudnick, P. Sarnak, and A. Zaharescu, *The distribution of spacings between the fractional parts of $n^2\alpha$*, Invent. Math., 145(1):37-57, 2001.

[12] Z. Rudnick and P. Sarnak, *The pair correlation function of fractional parts of polynomials*, Comm. Math. Phys., 194(1):61-70, 1998.

[13] V. T. Sós, *On the distribution mod 1 of the sequence na*, Ann. Univ. Sci. Budapest. Rolando Eőtvös, Sect. Math., 1:127-134, 1958.

[14] S. Steinerberger, *Localized Quantitative Criteria for Equidistribution*, arXiv:1701.08323, 2017.

[15] Wolfram Research, Inc., Mathematica, Version 11.2, Champaign, IL (2017).

Authors Addresses:
Isabel Pirsic and Wolfgang Stockinger, Institut für Finanzmathematik und Angewandte Zahlentheorie, Johannes Kepler Universität Linz, Altenbergerstrae 69, A-4040 Linz, Austria.

Email: isabel.pirsic(at)jku.at, wolfgang.stockinger(at)jku.at