A general sufficient criterion for energy conservation in the Navier–Stokes system

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Communicated by: C. Miao

Funding information
China Postdoctoral Science Foundation, Grant/Award Number: 2020M672196; National Natural Science Foundation of China, Grant/Award Number: 11971446, 12071113, 11601492, 11701145

1 INTRODUCTION

The homogeneous Navier–Stokes equations describing the motion of incompressible fluid in three-dimensional space read

\[
\begin{aligned}
&v_t - \Delta v + \text{div} (v \otimes v) + \nabla \pi = 0, \quad (x, t) \in \Omega \times (0, +\infty), \\
&\text{div} \ v = 0, \\
&v_0 = v(x, 0).
\end{aligned}
\] (1.1)

Here, \(v\) stands for the velocity field of the flow, and \(\pi\) represents the pressure of the fluid, respectively. The initial datum satisfies \(\text{div} \ v_0 = 0\). Usually, one considers the Navier–Stokes equations on the periodic domain \((\Omega = \mathbb{T}^3)\), on smooth bounded domain \(\Omega\) with Dirichlet boundary condition or on the whole space \((\Omega = \mathbb{R}^3)\).

It is well known that Leray–Hopf weak solutions of the Navier–Stokes Equations (1.1) obey the energy inequality

\[
\|v(T)\|_{L^2(\Omega)}^2 + 2 \int_0^T \|\nabla v\|_{L^2(\Omega)}^2 \ dt \leq \|v_0\|_{L^2(\Omega)}^2.
\]
rather than energy equality. The first attempts to determine sufficient conditions implying energy conservation of Leray–Hopf weak solutions in the homogeneous incompressible Navier–Stokes equations were given by Lions\(^1\) and Prodi\(^2\) provided that the criterion
\[
v \in L^4 (0, T; L^4(\Omega))
\]
(1.2)
is satisfied. Then Serrin\(^3\) showed the energy conservation by giving a criterion in a scaling invariant space, that is,
\[
v \in L^p(0, T; L^q(\Omega)), \text{ with } 2/p + d/q \leq 1 \text{ and } q \geq d.
\]
where \(d\) is the spatial dimension. However, the weak solution that satisfy the given criterion will immediately become a classical one. Later, Shinbrot\(^4\) extended Lions’ condition\(^1\) for energy conservation to
\[
v \in L^p(0, T; L^q(\Omega)) \text{ with } 2/p + 2/q = 1, q \geq 4.
\]
(1.3)
It is worth remarking that condition (1.3) is weaker than (1.2) when the dimension \(d \geq 4\), and more importantly, it is true regardless of the dimension of the underlying space. On the other hand, the energy conservation condition (1.3) can be replaced by
\[
v \in L^p(0, T; L^q(\Omega)) \text{ with } 1/p + 3/q = 1, 3 < q < 4;
\]
(1.4)
which was recently obtained by Beirao da Veiga and Yang.\(^5\) Using the Fourier methods, Cheskidov et al\(^6\) gave the following sufficient condition for energy conservation (here, \(A\) denotes the Stokes operator associated to the Dirichlet boundary conditions) that
\[
A^{5/12}v \in L^3 (0, T; L^2(\Omega))
\]
in fact, this criterion is equivalent in terms of scaling to \(v \in L^3(0, T; L^{9/2}(\Omega))\). Very recently, Berselli and Chiodaroli\(^7\) and Zhang\(^8\) obtained energy equality via the following condition:
\[
\nabla v \in L^p (0, T; L^q(\Omega)), \frac{1}{p} + \frac{3}{q} = 2, \frac{3}{2} < q < 9/5 \text{ or } \frac{1}{p} + \frac{6}{3q} = 1, \frac{9}{5} \leq q.
\]
(1.5)
It is worth remarking that the domain \(\Omega\) in most the aforementioned conditional results is the smooth bounded. The first objective of this paper is to show the following sufficient criterion for the weak solutions keeping the energy of the Navier–Stokes equations on the periodic domain \(\mathbb{T}^d\),
\[
v \in L^{\frac{2e}{e-1}} (0, T; L^{\frac{2e}{e-1}} (\mathbb{T}^d)) \text{ and } \nabla v \in L^p (0, T; L^q (\mathbb{T}^d)).
\]
(1.6)
Surprisingly, this result covers the corresponding results of (1.3), (1.4), and (1.5) on periodic domain, and further discussion will be found in Remarks 1.2 and 1.3. Indeed, we will also prove this class for the following compressible Navier–Stokes equations with degenerate viscosities and general pressure law (GNS):
\[
\begin{cases}
\rho_t + \text{div} (\rho v) = 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) + \nabla p(\rho) - \text{div} (\nu(\rho)\nabla v) - \nabla (\mu(\rho) \text{ div } v) = 0,
\end{cases}
\]
(1.7)
with the initial data
\[
\rho(0, x) = \rho_0(x) \text{ and } (\rho v)(0, x) = \rho_0(x)v_0(x), \ x \in \Omega,
\]
(1.8)
where the unknown functions \(\rho\) and \(v\) denote the density of the fluid and velocity of the fluid, respectively; \(\nabla v = 1/2(\nabla v + \nabla v^T)\) stands for the stain tensor. The general pressure \(0 \leq p(\rho) \in C^1(0, \infty)\) with \(p'(\cdot) > 0\) and the viscosity coefficients \(\nu(\rho), \mu(\rho) : (0, \infty) \to [0, \infty]\) are continuous functions of density. We will consider the case of bounded domain with periodic boundary conditions, namely, \(\Omega = \mathbb{T}^d\), where \(d \geq 2\) is the dimension of the domain. It should be noted that when \(\nu(\rho) \equiv \nu, \mu(\rho) \equiv \mu\) and \(p(\rho) = p^\gamma\) with \(\gamma > 1\), then (GNS) will reduce to the classical isentropic compressible Navier–Stokes equations (ICNS):
\[
\begin{cases}
\rho_t + \text{div} (\rho v) = 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) + \nabla p^\gamma - \nu \Delta v - \mu \text{ div } v = 0,
\end{cases}
\]
(1.9)
and when \( v(\rho) = \rho, \mu(\rho) \equiv 0 \) and \( p(\rho) = \rho^\gamma \) with \( \gamma > 1 \), then (GNS) reduces to the compressible Navier–Stokes equations with degenerate viscosity and \( \gamma \)-pressure law (CNSD) as follows:

\[
\begin{align*}
\rho_t + \text{div} (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) + \nabla p &= 0.
\end{align*}
\] (1.10)

The global existence of weak solutions which satisfy the energy inequality has already been known; see Lions\(^9\) and Feireisl et al\(^{10}\) for (ICNS) with constant viscosity coefficients case and Vassure and Yu\(^{11}\) for (CNSD) and Li and Xin\(^{12}\) for (GNS) with degenerate viscosity coefficients case, but the regularity and even the uniqueness of weak solutions are still open problems. Since the weak solutions satisfy the energy inequality rather than equality due to the basic a priori estimates and the lack of regularity, this anomalous dissipation of the energy opens a possibility for an energy sink other than the natural viscous dissipation; however, such a property of real fluids is not expected to exist physically, so it is therefore a famous problem to consider the sufficient criterion for the energy conservation of weak solutions. Roughly speaking, this addresses the question how much regularities are needed for a weak solution to conserve energy, which is also involving the uniqueness of the weak solutions. On the other hand, energy conservation is also one aspect of the Onsager’s conjecture in the context of homogeneous incompressible Euler equations,\(^{13}\) in which Onsager conjectured that the kinetic energy is globally conserved for Hölder continuous solutions with the exponent greater than \( 1/3 \), while an energy dissipation phenomenon occurs for Hölder continuous solutions with the exponent less than \( 1/3 \). For the positive part, the milestone work is due to Constantin et al,\(^{14}\) in which it was proved that the energy of 3D incompressible Euler equations is conserved for every weak solution in \( L^3(0, T; B^\alpha_{2,\infty}) \) with \( \alpha > 1/3 \). On the other hand, Isett resolved the “negative” part of Onsager’s conjecture for 3D incompressible Euler equations,\(^{15}\) where he proved that for any \( \alpha < 1/3 \) there is a nonzero weak solution to the incompressible Euler equations in the class \( \nu \in C^\alpha \) and \( p \in C^\alpha \) such that \( \nu \) is identically 0 outside a finite time interval. In particular, the solution \( \nu \) fails to conserve the energy. We refer the reader to previous works\(^{16–25}\) for recent progress in this direction.

Later, Liang\(^{27}\) derived a energy conservation criterion via the gradient of velocity for isentropic Navier–Stokes equations (ICNS) under the following condition:

\[
0 < c_1 \leq \rho \leq c_2 < \infty, \nu \in L^\infty (0, T; L^2(\mathbb{T}^3)), \nabla \nu \in L^2 (0, T; L^2(\mathbb{T}^3)),
\]

then the energy of weak solutions is globally conserved, which means the energy equality holds for any \( t \in [0, T] \). It is worth noting that though the part \( 1/p + 3/q = 1, 3 < q < 4 \) was not mentioned in Nguyen et al,\(^{26}\) it is a direct consequence from interpolation and \( \nu \in L^4(0, T; L^4(\mathbb{T}^3)) \) (see Beirao da Veiga and Yang\(^{5}\) and the corresponding proof in Theorem 1.2).

Later, Liang\(^{27}\) derived a energy conservation criterion via the gradient of velocity for isentropic Navier–Stokes equations (ICNS) under the following condition:

\[
0 < c_1 \leq \rho \leq c_2 < \infty, \nu \in L^\infty (0, T; L^2(\mathbb{T}^3)), \nabla \nu \in L^2 (0, T; L^2(\mathbb{T}^3)),
\]

then the energy of weak solutions is locally conserved, which means the energy equality holds in the sense of distribution in \( (0, T) \).
When the density may contain vacuum, in the spirit of well-known Shinbrot’s criterion,4 Yu28 showed that if a weak solution \((\rho, v)\) of (1.9) satisfies

\[
\sqrt{\rho} v \in L^\infty(0, T; L^2(\Omega)), \quad \forall v \in L^2(0, T; L^2(\Omega)),
\]

\[
0 \leq \rho \leq c < \infty, \quad \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega)),
\]

\[
v \in L^p(0, T; L^q(\Omega)) \quad \text{with} \quad 1/p + 1/q = 5/12 \quad \text{and} \quad v_0 \in L^2(\Omega),
\]

then the energy is globally conserved. Recently, for Equations (1.9), Chen et al29 obtained the energy balance in a bounded domain with physical boundaries under the following condition:

\[
\sqrt{\rho} v \in L^\infty(0, T; L^2(\Omega)), \quad v \in L^2(0, T; H^1(\Omega)),
\]

\[
0 \leq \rho \leq c < \infty, \quad \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega)),
\]

\[
v \in L^p(0, T; L^q(\Omega)), \quad p \geq 4, q \geq 6, \quad \text{and} \quad v_0 \in L^2(\Omega), q_0 > 3.
\]

As mentioned above, the energy conservation for fluid equations addresses the question how much regularities are needed for a weak solution to conserve energy, which will help us further to consider the uniqueness and regularity of the weak solutions. However, up to now, the related results on energy conservation of weak solutions for compressible Navier–Stokes equations are less satisfactory than incompressible ones. For example, compared with the Shinbrot’s condition (1.3) for the incompressible Navier–Stokes equations, the criterion (1.11) obtained by Nguyen et al26 requires additional constraint that \(\sup \sup |h|^{-1/2} \|\rho(\cdot + h, t) - \rho(\cdot, t)\|_{L(\Omega)} < \infty\). Note that, due to the blow-up criteria only via the density for strong solutions to the Cauchy problem of compressible isentropic Navier–Stokes equations in \(\mathbb{R}^3\), under the assumptions on the coefficients of viscosity established in previous studies,30,31 it implies the bound of density to the 3-D compressible Navier–Stokes equations yields strong solutions and the strong solutions are expected to meet energy conservation. Based on this, the second objective of this paper is to remove (1.11)_2 to obtain the persistence of energy. Moreover, when we see the criterion obtained by Liang,27 on the one hand, the restrictions on the indexes \(p\) and \(s\) are “subcritical” other than “critical” and are stronger when \(s > 3\) compared with the result (1.5) for incompressible case. On the other hand, the criterion obtained in Liang27 only implies the energy conserved “locally” not “globally.” Hence, our third objective of this paper is to improve the criterion via the gradient of velocity and to show (1.11)_1 and (1.5) guarantee the energy equality in system (1.7) globally.

Before stating the main results, we introduce the definition of the weak solutions.

**Definition 1.1.** A pair \((\rho, v)\) is called a weak solution to (1.7) with initial data \((\rho_0, v_0)\) if \((\rho, v)\) satisfy

(i) Equations (1.7) hold in \(D'(0, T; \mathbb{T}^d)\) and

\[
P(\rho, \rho|v|^2) \in L^\infty(0, T; L^1(\mathbb{T}^d)), \quad \forall v \in L^2(0, T; L^2(\mathbb{T}^d)),
\]

(ii) \(\rho(\cdot, t) \to \rho_0\) in \(D'(\Omega)\) as \(t \to 0\), that is,

\[
\lim_{t \to 0} \int_{\Omega} \rho(x, t) \varphi(x) dx = \int_{\Omega} \rho_0(x) \varphi(x) dx,
\]

for every test function \(\varphi \in C^\infty_0(\mathbb{T}^d)\).

(iii) \((\rho v)(\cdot, t) \to \rho_0 v_0\) in \(D'(\mathbb{T}^d)\) as \(t \to 0\), that is,

\[
\lim_{t \to 0} \int_{\mathbb{T}^d} (\rho v)(x, t) \psi(x) dx = \int_{\mathbb{T}^d} (\rho_0 v_0)(x) \psi(x) dx,
\]

for every test vector field \(\psi \in C^\infty_0(\mathbb{T}^d)^d\).
(iv) the energy inequality holds

\[ E(t) + \int_0^T \int_{\mathbb{T}^d} [\nu(\rho) |\nabla v|^2 + \mu(\rho)] \, \text{div} \, v^2 \, dx \, dt \leq E(0), \tag{1.18} \]

where \( E(t) = \int_{\mathbb{T}^d} \left[ 1/2 \rho |v|^2 + P(\rho) \right] \, dx \) and \( P(\rho) = \rho \int_1^\rho p(z)/z^2 \, dz \).

We formulate our first result as follows:

**Theorem 1.1.** For any dimension \( d \geq 2 \), let \((\rho, v)\) be a weak solution to the general compressible Navier–Stokes Equations 1.7. Assume that 1 < \( p, q < \infty \) and

\[
0 < c_1 \leq \rho \leq c_2 < \infty, \forall \in L^\infty(0, T; L^2(\mathbb{T}^d)), \nabla \forall \in L^2(0, T; L^2(\mathbb{T}^d)),
\]

\[
v \in L^{2^*}(0, T; L^{2^*}(\mathbb{T}^d)), \nabla v \in L^p(0, T; L^q(\mathbb{T}^d)) \text{ and } \sqrt{\rho_0} v_0 \in L^{2+\delta}
\]

for any \( \delta > 0 \),

then the energy of weak solutions is globally conserved, that is, for any \( t \in [0, T] \)

\[ E(t) + \int_0^T \int_{\mathbb{T}^d} [\nu(\rho) |\nabla v|^2 + \mu(\rho)] \, \text{div} \, v^2 \, dx \, dt = E(0), \tag{1.20} \]

where \( E(t) = \int_{\mathbb{T}^d} \left[ 1/2 \rho |v|^2 + P(\rho) \right] \, dx \) and \( P(\rho) = \rho \int_1^\rho p(z)/z^2 \, dz \).

**Remark 1.1.** We follow the path of Nguyen et al.\(^{26}\) to prove Theorem 1.1. The improvement of their condition (1.11) are threefold. First, Theorem 1.1 removed the additional restriction on the regularity of density. Second, Theorem 1.1 not only covers their result (1.11) but also allows us to derive new criterion (see the following corollary). Third, the regularity of pressure \( p(\rho) \) is relaxed from \( C^2(0, \infty) \) in Nguyen et al.\(^{26}\) to \( C^1(0, \infty) \).

**Remark 1.2.** At first glance, energy conservation criteria (1.19) based on a combination of velocity and its gradient are more complicated than (1.22) and (1.23); however, (1.19) together with natural energy estimates \( v \in L^\infty(0, T; L^2(\mathbb{T}^3)), \nabla v \in L^2(0, T; L^4(\mathbb{T}^3)) \) leads to (1.22) and (1.23) in the following corollary.

**Remark 1.3.** By small modification of proof in Theorem 1.1, the results in Theorem 1.1 also hold for homogenous incompressible Navier–Stokes Equations (1.1), that is, \( v \in L^{2\frac{2}{p-2}} \left( 0, T; L^{2\frac{q}{q-1}} (\mathbb{T}^3) \right) \) and \( \nabla v \in L^p \left( 0, T; L^q (\mathbb{T}^3) \right) \) means the energy equality in the classical homogenous incompressible Navier–Stokes equations. The special case \( p = q = 2 \) reduces to the famous Lions’ energy conservation criterion (1.2). As mentioned in latter remark, this result covers (1.3)–(1.5); hence, roughly speaking, this unifies the known energy conservation criteria via the velocity and its gradient in incompressible Navier–Stokes equations. After we finished this paper, we learnt that a special case that \( p = 3, q = 9/5 \) and away from 1/2-Hölder continuous curve in time for general energy equality in the homogeneous Navier–Stokes Equations (1.1) in \( \mathbb{R}^3 \) was considered in Shvydkoy.\(^{32}\)

**Remark 1.4.** The new ingredient in the proof of this theorem is the application of the following inequality

\[ \| \nabla ((\rho \nu^\rho / \rho^\rho)) \|_{L^q(\mathbb{T}^d)} \leq C \| \nabla v \|_{L^p(\mathbb{T}^d)}. \tag{1.21} \]

This helps us to pass to the limit of pressure term only with the positive bounded density, which removes the additional restriction of the density (1.11)\(^2\) in Nguyen et al.\(^{26}\). For the proof of (1.21), we refer the readers to Lemma 2.3 (see also page 7 of Liang\(^{27}\)).

**Remark 1.5.** One can consider Theorem 1.1 and Corollary 1.1 on smooth bounded domain. Combining the framework for bounded domain in Nguyen et al.\(^{26}\) and the proof here, one only needs to deal with the boundary terms caused by integrations by parts. Fortunately, these additional terms are the lower order terms.
Remark 1.6. In dimension $d = 2$, the Gagliardo–Nirenberg inequality guarantees that

$$\|v\|_{L^p(0,T;L^q(\mathbb{T}^3))} \leq C\|v\|_{L^1(0,T;L^2(\mathbb{T}^3))}^{1/2} \|\nabla v\|_{L^2(0,T;L^2(\mathbb{T}^3))}^{1/2} \leq C.$$  

Therefore, according to Theorem 1.1, the bounded density with positive lower bound and natural energy yield the energy conservation of the weak solutions.

Taking the natural energy of weak solutions into account, one immediately derives the following corollary.

**Corollary 1.1.** When the dimension $d = 3$, if the weak solutions $(\rho, v)$ to the compressible Navier–Stokes Equation (1.7) satisfy one of the following two conditions:

1. $0 < c_1 \leq \rho \leq c_2 < \infty$, $v \in L^\infty(0, T; L^2(\mathbb{T}^3))$, $\nabla v \in L^2(0, T; L^2(\mathbb{T}^3))$ and $\sqrt{\rho_0 v_0} \in L^{2+\delta}(\mathbb{T}^3)$ for any $\delta > 0$, 

   $$v \in L^p(0, T; L^{q}(\mathbb{T}^3)) \text{ with } \begin{cases} \frac{2}{p} + \frac{2}{q} = 1, q \geq 4, \\ \frac{1}{p} + \frac{3}{q} = 1, 3 < q < 4; \end{cases}$$  

   \hspace{1cm} (1.22)

2. $0 < c_1 \leq \rho \leq c_2 < \infty$, $v \in L^\infty(0, T; L^2(\mathbb{T}^3))$, $\nabla v \in L^2(0, T; L^2(\mathbb{T}^3))$ and $\sqrt{\rho_0 v_0} \in L^{2+\delta}(\mathbb{T}^3)$ for any $\delta > 0$,

   $$\nabla v \in L^p(0, T; L^{q}(\mathbb{T}^3)) \text{ with } \begin{cases} \frac{1}{p} + \frac{3}{q} = 2, & \frac{3}{2} < q < \frac{9}{5}, \\ \frac{1}{p} + \frac{6}{5q} = 1, & \frac{9}{5} \leq q, \end{cases}$$  

   \hspace{1cm} (1.23)

then the energy is globally conserved, that is, for any $t \in [0, T]$,

$$\mathcal{E}(t) + \int_0^T \int_{\mathbb{T}^3} \left[ |\nabla v|^2 + |\mu(\rho)| \div v|^2 \right] dx dt = \mathcal{E}(0),$$  

\hspace{1cm} (1.24)

where $\mathcal{E}(t) = \int_{\mathbb{T}^3} \left[ 1/2|v|^2 + P(\rho) \right] dx$ and $P(\rho) = \rho \int_1^\rho \frac{z}{z^2} dz$.

Remark 1.7. Compared with the result (1.11) obtained by Nguyen et al., conditions 1.22 only required the density is bounded from below and above. Hence, result (1.22) is an improvement of (1.11) in Nguyen et al.

Remark 1.8. We extend the energy conservation criteria (1.3)–(1.5) from incompressible Navier–Stokes equations to general compressible Navier–Stokes equations with no vacuum.

Remark 1.9. In contrast with (1.12), the generalization in (1.23) is threefold: first, to improve the corresponding results in (1.23); second, to consider the more general equations; third, we can get the energy conservation up to the initial time $t = 0$.

Remark 1.10. It seems that a new strategy for studying the energy equality of fluid equations is to firstly establish a conservation criterion based on a combination of velocity and its gradient, which may be applied to other incompressible and compressible fluid equations. A successful application can be found in Wei et al.

Remark 1.11. In the forthcoming work, the energy conservation criterion for the weak solutions of compressible Navier–Stokes equations allowing vacuum will be considered.
Finally, in Nguyen et al,\textsuperscript{26} one can establish the results parallel to Theorem 1.1 and Corollary 1.1 for the non-homogenous incompressible Navier–Stokes equations below:

\[
\begin{aligned}
\rho_t + \text{div} (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) - \text{div} (\nu(\rho) \nabla v) + \nabla \pi &= 0, \\
\text{div} v &= 0, \\
(\rho, v)|_{t=0} &= (\rho_0, u_0),
\end{aligned}
\]  

(1.25)

and we leave this to the interested readers.

The remainder of this paper is organized as follows. Section 2 is devoted to the auxiliary lemmas involving mollifier and the key inequality (1.21). In Section 3, we first present the proof of Theorem 1.1. Then, based on Theorem 1.1, we complete the proof of Corollary 1.1.

### 2 | NOTATIONS AND SOME AUXILIARY LEMMAS

First, we introduce some notations used in this paper. For \( p \in [1, \infty] \), the notation \( L^p(0, T; X) \) stands for the set of measurable functions on the interval \((0, T)\) with values in \( X \) and \( \|f(\cdot, t)\|_X \) belonging to \( L^p(0, T) \). The classical Sobolev space \( W^{k,p}(\mathbb{T}^d) \) is equipped with the norm \( \|f\|_{W^{k,p}(\mathbb{T}^d)} = \sum_{a=0}^{k} \|D^a f\|_{L^p(\mathbb{T}^d)} \). The space \( C_0^\infty(\mathbb{T}^d) \) is the bounded smooth functions on \( \mathbb{T}^d \). \( c_1, c_2 \) and \( C \) are positive constants. For simplicity, we denote by

\[
\int_0^T \int_{\mathbb{T}^d} f(x, t) dx dt = \int_0^T \int f(x) dx \quad \text{and} \quad \|f\|_{L^p(0, T; X)} = \|f\|_{L^p(X)}.
\]

Let \( \eta_\varepsilon : \mathbb{R}^d \to \mathbb{R} \) be a standard mollifier, that is, \( \eta(x) = C_0 e^{-\frac{|x|^2}{\varepsilon^2}} \) for \( |x| < 1 \) and \( \eta(x) = 0 \) for \( |x| \geq 1 \), where \( C_0 \) is a constant such that \( \int_{\mathbb{R}^d} \eta(x) dx = 1 \). For \( \varepsilon > 0 \), we define the rescaled mollifier \( \eta_\varepsilon(x) = 1/\varepsilon^d \eta(x/\varepsilon) \). For any function \( f \in L^1_{\text{loc}}(\Omega) \), its mollified version is defined as

\[
f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^d} f(x - y) \eta_\varepsilon(y) dy, \quad x \in \Omega_\varepsilon,
\]

where \( \Omega_\varepsilon = \{ x \in \Omega : d(x, \partial \Omega) > \varepsilon \} \).

We first recall the results involving the mollifier established in Nguyen et al.\textsuperscript{26}

**Lemma 2.1** (Nguyen et al.\textsuperscript{26}). Suppose that \( f \in L^p(0, T; L^q(\mathbb{T}^d)) \). Then for any \( \varepsilon > 0 \), there holds

\[
\|\nabla f^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} \leq C \varepsilon^{-1} \|f\|_{L^p(0, T; L^q(\mathbb{T}^d))},
\]

(2.1)

and if \( p, q < \infty \)

\[
\lim_{\varepsilon \to 0} \varepsilon \|\nabla f^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} = 0.
\]

Moreover, if \( 0 < c_1 \leq g \leq c_2 < \infty \), then there holds, for any \( \varepsilon > 0 \),

\[
\|\nabla(f^\varepsilon/g^\varepsilon)\|_{L^p(0, T; L^q(\mathbb{T}^d))} \leq C \varepsilon^{-1} \|f\|_{L^p(0, T; L^q(\mathbb{T}^d))},
\]

(2.2)

and if \( p, q < \infty \)

\[
\lim_{\varepsilon \to 0} \varepsilon \|\nabla(f^\varepsilon/g^\varepsilon)\|_{L^p(0, T; L^q(\mathbb{T}^d))} = 0.
\]

(2.3)

The next lemma with \( p = q, p_1 = q_1, p_2 = q_2 \) was proved in Nguyen et al.\textsuperscript{26} We generalize it by extending the integral norms with different exponents in space and time.
Lemma 2.2. Let $1 \leq p, q, p_1, p_2, q_1, q_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. Assume $f \in L^p(0, T; W^{1,q_1}(\mathbb{T}^d))$ and $g \in L^q(0, T; L^{q_2}(\mathbb{T}^d))$. Then for any $\epsilon > 0$, there holds

$$
\|(fg)^{T} - f^Tg^T\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C\epsilon \|f\|_{L^p(0,T;W^{1,q_1}(\mathbb{T}^d))}\|g\|_{L^q(0,T;L^{q_2}(\mathbb{T}^d))}.
$$

(2.4)

Moreover, if $p_2, q_2 < \infty$ then

$$
\lim_{\epsilon \to 0} \epsilon^{-1} \|(fg)^{T} - f^Tg^T\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0.
$$

(2.5)

Proof. Thanks to the fact observed in Constantin et al.\(^{14}\) and the ideas in Lions,\(^{35}\) we know that

$$
(fg)^{T} - f^Tg^T = R^T - (f^T - f)(g^T - g).
$$

(2.6)

where

$$
R^T(x, t) := \int_{\mathbb{R}^d} (f(y, t) - f(x, t))(g(y, t) - g(x, t)) \eta_t(x - y)dy.
$$

Using the triangle's inequality, it yields that

$$
\|(fg)^{T} - f^Tg^T\|_{L^r(\mathbb{T}^d)} \leq C \left( \|R^T\|_{L^r(\mathbb{T}^d)} + \|(f-f^T)(g-g^T)\|_{L^s(\mathbb{T}^d)} \right).
$$

(2.7)

Let $B(x, \epsilon) = \{ y \in \mathbb{T}^d; |x - y| < \epsilon \}$, then by means of Hölder's inequality and direct computation, we see that

$$
|R^T| \leq \int_{B(x, \epsilon)} 1/\epsilon^d |f(y) - f(x)| |g(y) - g(x)| dy
\leq C \left( 1/\epsilon^d \int_{B(x, \epsilon)} |f(y) - f(x)|^s_1 dy \right)^{1/s_1} \left( 1/\epsilon^d \int_{B(x, \epsilon)} |g(y) - g(x)|^s_2 dy \right)^{1/s_2}
\leq C \epsilon \left( 1/\epsilon^d \int_{B(x, \epsilon)} \int_0^1 |\nabla f(x + (y-x)\omega)|^s_1 d\omega dy \right)^{1/s_1} \left( 1/\epsilon^d \int_{B(x, \epsilon)} |g(x + \omega \epsilon)|^s_2 d\omega + |g(x)|^s_2 \right)^{1/s_2}
\leq C \epsilon \left( \int_{\mathbb{R}^d} |\nabla f(x-z)|^s_1 \int_0^1 1_{B_0(x, s\epsilon)}(z)/(\epsilon s)^d d\omega dz \right)^{1/s_1} \left( \int_{\mathbb{R}^d} |g(x-z)|^s_2 1_{B_0(x, \epsilon d)}(z)/(\epsilon d)^d dz \right)^{1/s_2}
\leq C \epsilon \left( |\nabla f|^s_1 \ast J_{s_1}/(\epsilon s)^d + |g|^s_2 + J_{s_2} \right)^{1/s_2},
$$

(2.8)

where $s_1 \leq q_1, s_2 \leq q_2$ with $1/s_1 + 1/s_2 = 1$, $J_{s_1} = \int_0^1 1_{B_0(\epsilon s)}(z)/(\epsilon s)^d ds \geq 0$, $J_{s_2} = 1_{B_0(\epsilon d)}(z)/(\epsilon d)^d \geq 0$ and $\int_{\mathbb{R}^d} \int_0^1 1_{B_0(\epsilon d)}(z)/(\epsilon d)^d dz = \int_{\mathbb{R}^d} 1_{B_0(\epsilon d)}(z)/(\epsilon d)^d dz = \text{measure}(B_0(1))$.

Then in view of the Minkowski inequality, we conclude that

$$
\|R^T\|_{L^r} \leq C \epsilon \|(|\nabla f|^s_1 \ast J_{s_1})^{1/s_1} (|g|^s_2 + J_{s_2})^{1/s_2}\|_{L^r}
\leq C \epsilon \left( \|(\nabla f)^{s_1} \ast J_{s_1}\|_{L^{s_1}} \|\nabla g\|_{L^{s_2}} \right) \|\nabla f\|_{L^{s_1}} \|g\|_{L^{s_2}}.
$$

(2.9)
Furthermore, one has
\[
| (f^x - f)(g^x - g) | \leq \int |(f(y) - f(x))| \eta_s(x - y) dy \int |(g(y) - g(x))| \eta_s(x - y) dy
\]
\[
\leq C \varepsilon \left( 1/\varepsilon^d \int_{B(x, \varepsilon)} 0^1 | \nabla f(x + (y - x)s) | ds dy \right) \left( 1/\varepsilon^d \int_{B(x, \varepsilon)} | g(y) - g(x) | dy \right)
\]
\[
\leq C \varepsilon \left( 1/\varepsilon^d \int_{B(x, \varepsilon)} 0^1 | \nabla f(x + (y - x)s) |^{1/s_1} ds dy \right)^{1/s_1} \left( 1/\varepsilon^d \int_{B(x, \varepsilon)} | g(y) - g(x) |^{2/s_2} dy \right)^{1/s_2}.
\]

Along the same lines of derivation of (2.8) and (2.9), we arrive at
\[
\|(f^x - f)(g^x - g)\|_{L^2} \leq C \varepsilon \| \nabla f \|_{L^p} \| g \|_{L^{2s}}.
\]

In combination with (2.6), (2.9) and (2.11) and using the Hölder inequality with respect to time, we can deduce the result (2.4).

Furthermore, if \( q_2 < \infty \), let \( \{g_n\} \in C^\infty_0(\mathbb{T}^d) \) with \( g_n \to g \) strongly in \( L^{q_2} \). Thus, by density arguments, we find that
\[
\|(fg)^x - f^x g^x\| \leq C \|[f(g - g_n)]^x + (fg_n)^x - f^x (g - g_n)^x\|_{L^p} + \|f(g_n)^x - f^x g_n^x\|_{L^p}
\]
\[
\leq C (\|f\|_{L^p} \|g - g_n\|_{L^{q_2}} + \varepsilon^2 \| \nabla f \|_{L^p} \| \nabla g_n \|_{L^{q_2}}),
\]

which means
\[
\varepsilon^{-1}\|(fg)^x - f^x g^x\|_{L^p} \leq C (\|\nabla f\|_{L^p} \|g - g_n\|_{L^{q_2}} + \varepsilon \| \nabla f \|_{L^p} \| \nabla g_n \|_{L^{q_2}});
\]

hence, as \( \varepsilon \to 0 \) and \( n \to \infty \), we can obtain that
\[
\varepsilon^{-1}\|(fg)^x - f^x g^x\|_{L^p} \leq C \left( \int_0^T (\| \nabla f \|_{L^p} \|g - g_n\|_{L^{q_2}} + \varepsilon \| \nabla f \|_{L^p} \| \nabla g_n \|_{L^{q_2}})^p dt \right)^{1/p}
\]
\[
\leq C \left( \int_0^T (\| \nabla f \|_{L^p} \|g - g_n\|_{L^{q_2}})^p dt \right)^{1/p} + C \varepsilon \left( \int_0^T (\| \nabla f \|_{L^p} \| \nabla g_n \|_{L^{q_2}})^p dt \right)^{1/p}
\]
\[
\leq C \| \nabla f \|_{L^p} \|g - g_n\|_{L^{q_2}} + \varepsilon \| \nabla f \|_{L^p} \| \nabla g_n \|_{L^{q_2}} \to 0.
\]

Then, we have completed the proof of Lemma 2.2. □

The next lemma is the key to remove (1.11).2

**Lemma 2.3.** Assume that \( 0 < \rho \leq \rho(x, t) \leq \bar{\rho} < \infty \) and \( \nu \in W^{1,p}(\mathbb{T}^d) \) with \( 1 \leq p \leq \infty \). Then
\[
\| \partial ((\nu \rho^x) / \rho^x) \|_{L^p(\mathbb{T}^d)} \leq C \| \nabla \nu \|_{L^p(\mathbb{T}^d)}.
\]

**Proof.** By direct computation, one has
\[
\partial ((\nu \rho^x) / \rho^x) = (\partial (\nu \rho^x) - \nu \partial \rho^x) / \rho^x - ((\nu \rho^x - \rho^x \nu) \partial \rho^x / (\rho^x)^2) := I_1 + I_2.
\]
Let $B(x, \epsilon) = \{ y \in \mathbb{T}^d; |x - y| < \epsilon \}$, in light of the Hölder inequality, we have

$$|I_1| \leq C \left| \int \rho(y) (\nu(y) - \nu(x)) \nabla \eta_\epsilon(x - y)dy \right| \leq C \|\rho\|_{L^\infty} \left( \int_{\mathbb{R}^d} |\nu(y) - \nu(x)| \right) \left( \int_{\mathbb{R}^d} 1/\epsilon^d \nabla \eta \left( \frac{x - y}{\epsilon} \right) \right) dy \leq C \left( 1/\epsilon^d \int_{B(x, \epsilon)} |\nu(y) - \nu(x)|^p/\epsilon^p dy \right)^{1/p}.$$  \hspace{1cm} (2.17)

Then using the mean value theorem, one can obtain

$$1/\epsilon^d \int_{B(x, \epsilon)} |\nu(y) - \nu(x)|^p/\epsilon^p dy \leq C \int_{B(x, \epsilon)} \int_0^1 |\nabla \nu(x + (y - x)s)|^p/\epsilon^p ds dy \leq C \int_0^1 \int_{B(0, 1)} |\nabla \nu(x + s\epsilon \omega)|^p d\omega ds,$$  \hspace{1cm} (2.18)

where $J_\epsilon(z) = \int_0^1 1_{B(0, \epsilon)}(z)/\epsilon^d ds \geq 0$ and it’s easy to check that $\int_{\mathbb{R}^d} J_\epsilon dz =$ measure of $B(0, 1)$. Next, to estimate $I_2$, due to the Hölder inequality again, one deduces

$$|I_2| = \left| \int \rho(y) (\nu(y) - \nu(x)) \eta_\epsilon(x - y) dy \int \rho(y) \nabla \eta_\epsilon(x - y) dy/ \left( \int \rho(y) \eta_\epsilon(x - y) dy \right)^2 \right| \leq C \|\rho\|_{L^\infty} \left( \int_{\mathbb{R}^d} |\nu(y) - \nu(x)| \right) \left( \int_{\mathbb{R}^d} 1/\epsilon^d \nabla \eta \left( \frac{x - y}{\epsilon} \right) \right) dy \leq C \left( 1/\epsilon^d \int_{B(x, \epsilon)} |\nu(y) - \nu(x)|^p/\epsilon^p dy \right)^{1/p}.$$  \hspace{1cm} (2.19)

Therefore, by the same arguments as in (2.18), in combination with (2.16)–(2.19), we have

$$|I_1| + |I_2| \leq C (|\nabla \nu|^p * J_\epsilon)^{1/p}. \hspace{1cm} (2.20)$$

Then from the Minkowski inequality, we arrive at

$$\| \partial \left( (\rho \nu)^f / \rho^f \right) \|_{L^p(\mathbb{T}^d)} \leq C \|\nabla \nu\|_{L^p(\mathbb{T}^d)} J_\epsilon^{1/p} \|_{L^p} \leq C \|\nabla \nu\|_{L^p} \|J_\epsilon\|_{L^p}^{1/p} \leq C \|\nabla \nu\|_{L^p}.$$  \hspace{1cm} (2.21)

Then we have completed the proof of lemma 2.3. \hfill \Box

### 3. PROOF OF THEOREM 1.1 AND COROLLARY 1.1

In this section, we first present the proof of Theorem 1.1. Then, making use of interpolation and the natural energy, we prove Corollary 1.1 by the results of Theorem 1.1.

**Proof of Theorem 1.1.** Let $\phi(t)$ be a smooth function compactly supported in $(0, +\infty)$. Multiplying (1.7)$_2$ by $(\phi(t)(\rho \nu)^f / \rho^f)^\gamma$, then integrating it over $(0, T) \times \mathbb{T}^d$, we have

$$\int_0^T \int (\phi(t) \rho \nu)^f / \rho^f \left[ \partial_t (\rho \nu)^f + \text{div} \ (\rho \nu \otimes \nu)^f + \nabla p(\rho)^f - \nabla (\rho \nu \text{div} \nu)^f - \nabla (\mu(\rho) \text{div} \nu)^f \right] = 0. \hspace{1cm} (3.1)$$
We will rewrite every term of the last equality to pass to the limits of $\epsilon$. For the first term in (3.1), a straightforward calculation and (1.7), yields that

$$\int_0^T \int \phi(t)(\rho v)^r / \rho^r \partial_t (\rho v)^r = \int_0^T \int \phi(t) \left[ 1/2 \partial_t \left( (\rho v)^r \right)^2 / \rho^r + 1/2 \partial_t \rho^r \right] (\rho v)^r / (\rho^r)^2 \right]$$

(3.2)

For the second term in (3.1), by virtue of the integration by parts, it gives that

$$\int_0^T \int \phi(t)(\rho v)^r / \rho^r \text{ div } (\rho v \otimes v)^r = -\int_0^T \int \phi(t) \nabla ((\rho v)^r / \rho^r) \left[ (\rho v \otimes v)^r - (\rho v)^r \otimes v^r \right]$$

(3.3)

To deal with the second term on the right hand-side of above equality (3.3), it follows from the integration by parts once again that

$$-\int_0^T \int \phi(t) \nabla ((\rho v)^r / \rho^r) (\rho v)^r \otimes v^r = \int_0^T \int \phi(t) \left( \text{ div } v^r |(\rho v)^r|^2 / \rho^r + 1/2 v^r / \rho^r \nabla |(\rho v)^r|^2 \right)$$

(3.4)

Then inserting (3.4) into (3.3), we have

$$\int_0^T \int \phi(t)(\rho v)^r / \rho^r \text{ div } (\rho v \otimes v)^r = -\int_0^T \int \phi(t) \nabla ((\rho v)^r / \rho^r) \left[ (\rho v \otimes v)^r - (\rho v)^r \otimes v^r \right]$$

(3.5)

For the pressure term in (3.1), together with the integration by parts, one has

$$\int_0^T \int \phi(t)(\rho v)^r / \rho^r \nabla (p(\rho))^r = \int_0^T \int \phi(t)(\rho v)^r / \rho^r \nabla \left[ (p(\rho))^r - p(\rho^r) \right] + \int_0^T \int \phi(t)(\rho v)^r / \rho^r \nabla p(\rho^r)$$

(3.6)
Using the mass equation (1.7), the second term on the right hand-side of (3.6) can be rewritten as

$$
\int_0^T \int \phi(t)(\rho v) / \rho \, \nabla p(\rho^*) = \int_0^T \int \phi(t)(\rho v) / \rho \, \nabla p(\rho^*) \, dz \, dx \, dt
$$

$$
\int_0^T \int \phi(t)(\rho v) / \rho \, \nabla p(\rho^*) = \int_0^T \int \phi(t)(\rho v) / \rho \, \nabla p(\rho^*) + \int_0^T \frac{1}{\rho^*} p(\rho^*) \, dz \, dx \, dt
$$

(3.7)

where $P(\rho^*) = \rho^* \int_1^{\rho^*} p(z) / z^2 \, dz$.

Finally, for the viscous terms in (3.1), using the integration by parts, we have

$$
- \int_0^T \int \phi(t)(\rho v) / \rho \, \nabla (\mu(\rho) \, v) = \int_0^T \int \phi(t)(- \nabla (\mu(\rho) \, v) / \rho) - \nabla (\mu(\rho) \, v) / \rho)
$$

(3.8)

and

$$
- \int_0^T \int \phi(t)(\rho v) / \rho \, \nabla (\mu(\rho) \, v) = \int_0^T \int \phi(t)(- \nabla (\mu(\rho) \, v) / \rho)
$$

(3.9)

Then substituting (3.2), (3.5)–(3.9) into (3.1), we see that

$$
\int_0^T \int \phi(t) \partial_t \left( \frac{1}{2} \left| \rho v \right|^2 / \rho + P(\rho^*) \right) - \int_0^T \int \phi(t) \left( \nabla (\mu(\rho) \, v) / \rho \right) + \int_0^T \int \phi(t) \nabla (\mu(\rho) \, v) / \rho
$$

(3.10)

Next, we need to prove that the terms on the right hand-side of (3.10) tend to zero as $\varepsilon \to 0$.

Firstly, it follows from Lemma 2.1 and Lemma 2.2 that

$$
\| \nabla (\nu(\varepsilon) \, v) \|_{L^2(L^2)} \leq C \varepsilon^{-1} \| \nabla v \|_{L^2(L^2)},
$$

$$
\limsup_{\varepsilon \to 0} \varepsilon \| \nabla (\nu(\varepsilon) \, v) \|_{L^2(L^2)} = 0,
$$

$$
\| (\nu(\varepsilon) \, v - \nu^* v) \|_{L^2(L^2)} \leq C \varepsilon \| \nu \|_{L^\infty(L^2)} \| v \|_{L^2(W^{1,2})}.
$$

(3.11)

Moreover, due to the Hölder inequality, we can obtain that

$$
\left| \int_0^T \int \phi(t) \nabla (\nu(\varepsilon) \, v) (\nu(\varepsilon) \, v - \nu^* v) / \rho^* \right| \leq C \| \nabla (\nu(\varepsilon) \, v) \|_{L^2(L^2)} \| (\nu(\varepsilon) \, v - \nu^* v) / \rho^* \|_{L^2(L^2)}
$$

$$
\leq C \varepsilon \| \nabla (\nu(\varepsilon) \, v) \|_{L^2(L^2)} \| \nu \|_{L^\infty(L^2)} \| v \|_{L^2(W^{1,2})}.
$$

(3.12)

As a consequence, in combination with (3.11) and (3.12), we have

$$
\limsup_{\varepsilon \to 0} \left| \int_0^T \int \phi(t) \nabla (\nu(\varepsilon) \, v) (\nu(\varepsilon) \, v - \nu^* v) / \rho^* \right| = 0.
$$
Likewise, there also holds

\[
\limsup_{\epsilon \to 0} \left| \int_0^T \int \phi(t) \nabla (\mu(\rho) \text{ div } v) \left( (\rho v)^\epsilon - \rho^\ell v^\ell \right) / \rho^\ell \right| = 0. \tag{3.13}
\]

Next, by means of the triangle inequality, the Hölder inequality and Lemma 2.3, we obtain

\[
\int_0^T \int \phi(t) \text{ div } [(\rho v)^\epsilon / \rho^\ell] \left[ ( (p(\rho))^\epsilon - p(\rho^\ell) \right] \leq \int_0^T \int \phi(t) \text{ div } [(\rho v)^\epsilon / \rho^\ell] \left| ( (p(\rho))^\epsilon - p(\rho^\ell) \right| + \int_0^T \int \phi(t) \text{ div } [(\rho v)^\epsilon / \rho^\ell] \left| p(\rho) - p(\rho^\ell) \right| \\
\leq C \text{ div } [(\rho v)^\epsilon / \rho^\ell] \| (p(\rho))^\epsilon - p(\rho) \|_{L^1(L^2)} + \| (p(\rho) - p(\rho^\ell) \|_{L^1(L^2)} \\
\leq C \| \nabla v \|_{L^1(L^2)} (\| (p(\rho))^\epsilon - p(\rho) \|_{L^1(L^2)} + \| p^\ell \|_{L^\infty(L^2)} \| \rho - \rho^\ell \|_{L^1(L^2)}), \tag{3.14}
\]

which implies that

\[
\limsup_{\epsilon \to 0} \int_0^T \int \phi(t) \text{ div } ((\rho v)^\epsilon / \rho^\ell) \left( (p(\rho))^\epsilon - p(\rho^\ell) \right) = 0.
\]

At this stage, it is enough to show

\[
\limsup_{\epsilon \to 0} \int_0^T \int \phi(t) \nabla ((\rho v)^\epsilon / \rho^\ell) \left[ (\rho v \otimes v)^\epsilon - (\rho v)^\ell \otimes v^\ell \right] \\
+ \limsup_{\epsilon \to 0} \int_0^T \int \phi(t) [\rho^\ell v^\ell - (\rho v)^\ell \nabla ((\rho v)^\epsilon / \rho^\ell)] = 0, \tag{3.15}
\]

under the hypothesis

\[
v \in L^\infty \left( L^{\frac{2n}{n-1}} \right) \text{ and } \nabla v \in L^p \left( L^q \right). \tag{3.16}
\]

To do this, applying Lemma 2.2, we obtain that

\[
\| (\rho \otimes v)^\epsilon - (\rho v)^\ell \otimes v^\ell \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)} \leq C \| v \|_{L^2(W^{1,\infty})} \| \rho v \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)},
\]

\[
\| \nabla ((\rho v)^\epsilon / \rho^\ell) \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)} \leq C \| v \|_{L^2(W^{1,\infty})} \| \rho v \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)}, \tag{3.17}
\]

\[
\limsup_{\epsilon \to 0} \| \nabla ((\rho v)^\epsilon / \rho^\ell) \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)} = 0.
\]

Using the Hölder inequality and Lemma 2.1, we find

\[
\int_0^T \int \phi(t) \nabla \left( ((\rho v)^\epsilon / \rho^\ell) \left[ (\rho v \otimes v)^\epsilon - (\rho v)^\ell \otimes v^\ell \right] \right) \\
\leq C \| \nabla \left( ((\rho v)^\epsilon / \rho^\ell) \right) \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)} \| (\rho v \otimes v)^\epsilon - (\rho v)^\ell \otimes v^\ell \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)} \\
\leq C \| v \|_{L^2(W^{1,\infty})} \| \rho v \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)} \\
\leq C \| v \|_{L^2(W^{1,\infty})} \| \rho v \|_{L^\frac{2n}{n-1} \left( L^{\frac{2n}{n-1}} \right)}, \tag{3.18}
\]

which in turn gives

\[
\limsup_{\epsilon \to 0} \left| \int_0^T \int \phi(t) \nabla \left( ((\rho v)^\epsilon / \rho^\ell) \left[ (\rho v \otimes v)^\epsilon - (\rho v)^\ell \otimes v^\ell \right] \right) \right| = 0.
\]
Now, we turn our attentions to the term \( \int_0^T \int \phi(t)[\rho v^\varepsilon - (\rho v)^\varepsilon / \rho^\varepsilon \nabla ((\rho v)^\varepsilon / \rho^\varepsilon) \). Since \( \rho v \in L^{\frac{p}{p-1}}(L^{\frac{2q}{q}}) \), we derive from Lemma 2.1 that

\[
\lim_{\varepsilon \to 0} \epsilon \| \nabla ((\rho v)^\varepsilon / \rho^\varepsilon) \|_{L^{\frac{p}{p-1}}(L^{\frac{2q}{q}})} = 0. \tag{3.19}
\]

In addition, we conclude from Lemma 2.2 that

\[
\| \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \|_{L^p(L^q)} \leq C \epsilon \| v \|_{L(W^{1,q})} \| \rho \|_{L^p}. \tag{3.20}
\]

Then, together with the Hölder's inequality and (3.20), we have,

\[
\left| \int_0^T \int \phi(t)[\rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon / \rho^\varepsilon \nabla ((\rho v)^\varepsilon / \rho^\varepsilon)] \right| \leq C \| \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \|_{L^p(L^q)} \| (\rho v)^\varepsilon / \rho^\varepsilon \|_{L^{\frac{2q}{q}}} \epsilon \| \nabla ((\rho v)^\varepsilon / \rho^\varepsilon) \|_{L^{\frac{2q}{q}}} \\
\leq C \epsilon \| \nabla ((\rho v)^\varepsilon / \rho^\varepsilon) \|_{L^{\frac{2q}{q}}} \leq C \epsilon \| \nabla ((\rho v)^\varepsilon / \rho^\varepsilon) \|_{L^{\frac{2q}{q}}},
\tag{3.21}
\]

which together with (3.19) yields that

\[
\lim_{\varepsilon \to 0} \sup \left| \int_0^T \int \phi(t)[\rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon / \rho^\varepsilon \nabla ((\rho v)^\varepsilon / \rho^\varepsilon)] \right| = 0.
\]

Collecting all the above estimates, using the integration by parts with respect to \( t \) and letting \( \varepsilon \to 0 \), for any \( \phi(t) \in D(0, T) \), we have

\[
- \int_0^T \int \frac{1}{2} \rho v^2 + P(\rho) + \int_0^T \int \phi(t) (v(\rho)|\nabla v|^2 + \mu(\rho)|\text{div} v|^2) = 0. \tag{3.22}
\]

The next objective is to get the energy equality up to the initial time \( t = 0 \). First, we claim that for any \( t_0 \geq 0 \),

\[
\lim_{t \to t_0} \left\| \sqrt{\rho}(t) \right\|_{L^1(\mathbb{T}^d)} = \left\| \sqrt{\rho}(t_0) \right\|_{L^1(\mathbb{T}^d)} \quad \text{and} \quad \lim_{t \to t_0} \left\| P(\rho)(t) \right\|_{L^1(\mathbb{T}^d)} = \left\| P(\rho)(t_0) \right\|_{L^1(\mathbb{T}^d)}. \]

In fact, by the energy estimates (1.15) and the weak continuity of \( \rho \) and \( \rho v \) in (1.16) and (1.17), we have

\[
\rho v \in L^\infty (0, T; L^2(\mathbb{T}^d)) \cap H^1(0, T; W^{-1,1}(\mathbb{T}^d)) \leftrightarrow C ([0, T]; L^2_{\text{weak}}(\mathbb{T}^d)), \quad \text{and} \quad \sqrt{\rho} \in L^\infty (0, T; L^\infty (\mathbb{T}^d)) \cap H^1 (0, T; H^{-1} (\mathbb{T}^d)) \leftrightarrow C ([0, T]; L^l_{\text{weak}} (\mathbb{T}^d)), \quad \text{for any} \ l \in (1, +\infty)
\tag{3.23}
\]

Due to the convexity of \( \rho \mapsto P(\rho) \), we have

\[
\int_{\mathbb{T}^d} P(\rho)(t_0) \leq \lim_{t \to t_0} \int_{\mathbb{T}^d} P(\rho)(t), \quad \text{for any} \ t_0 \geq 0. \tag{3.24}
\]
Meanwhile, using the natural energy (1.15), (1.18), (3.23), and (3.24), we have

\[
0 \leq \lim_{t \to 0^+} \int |\sqrt{\rho v} - \sqrt{\rho_0 v_0}|^2 \, dx \\
= 2 \lim_{t \to 0^+} \left( \int \left( \frac{1}{2} \rho |v|^2 + P(\rho) \right) \, dx - \int \left( \frac{1}{2} \rho_0 |v_0|^2 + P(\rho_0) \right) \, dx \right) \\
+ 2 \lim_{t \to 0^+} \left( \int \rho_0 v_0 \left( \sqrt{\rho_0 v_0} - \sqrt{\rho v} \right) \, dx + \int (P(\rho_0) - P(\rho)) \, dx \right) \\
\leq 2 \lim_{t \to 0^+} \int \rho_0 v_0 \left( \sqrt{\rho_0 v_0} - \sqrt{\rho v} \right) \, dx \\
= 0,
\]

where the last equality sign comes from

\[
2 \lim_{t \to 0^+} \int \sqrt{\rho_0 v_0} \left( \sqrt{\rho_0 v_0} - \sqrt{\rho v} \right) \, dx = 2 \lim_{t \to 0^+} \int \sqrt{\rho_0 v_0} / \sqrt{\rho} \left( \sqrt{\rho} \sqrt{\rho_0 v_0} - \rho v \right) \, dx \\
\leq 2 \lim_{t \to 0^+} \int \sqrt{\rho_0 v_0} / \sqrt{\rho} \left( \sqrt{\rho} \sqrt{\rho_0 v_0} - \rho_0 v_0 \right) \, dx \\
+ 2 \lim_{t \to 0^+} \int \sqrt{\rho_0 v_0} / \sqrt{\rho} (\rho_0 v_0 - \rho v) \, dx \\
\leq 2 \lim_{t \to 0^+} \int \rho_0 v_0 \left( \sqrt{\rho} - \sqrt{\rho_0} \right) \, dx \\
+ 2 \lim_{t \to 0^+} \int \sqrt{\rho_0 v_0} (\rho_0 v_0 - \rho v) \, dx \\
= 0,
\]

where we used (1.15), (3.23) and \( \sqrt{\rho_0 v_0} \in L^{2+\delta} \) for any \( \delta > 0 \). Then we have

\[
\sqrt{\rho v}(t) \to \sqrt{\rho v}(0) \text{ strongly in } L^2(\mathbb{T}^d) \text{ as } t \to 0^+.
\]

Similarly, one has the right temporal continuity of \( \sqrt{\rho v} \) in \( L^2(\mathbb{T}^d) \), hence, for any \( t_0 \geq 0 \), we infer that

\[
\sqrt{\rho v}(t) \to \sqrt{\rho v}(t_0) \text{ strongly in } L^2(\mathbb{T}^d) \text{ as } t \to t_0^+.
\]

Next, it follows from (1.18) that

\[
\lim_{t \to t_0^+} \mathcal{E}(t) \leq \mathcal{E}(t_0).
\]

This and (3.28) imply

\[
\lim_{t \to t_0^+} \|P(\rho)(t)\|_{L^1(\mathbb{T}^d)} \leq \|P(\rho)(t_0)\|_{L^1(\mathbb{T}^d)}.
\]

Notice from (1.15) and the mass equation (1.7), that

\[
P(\rho) \in L^\infty_0 \left(0, T; L^\infty(\mathbb{T}^d) \right) \cap H^1 \left(0, T; H^{-1}(\mathbb{T}^d) \right) \hookrightarrow C \left([0, T]; L^2_{\text{weak}}(\mathbb{T}^d) \right).
\]

Hence, (3.24), (3.30) and (3.31) guarantee

\[
\lim_{t \to t_0^+} \|P(\rho)(t)\|_{L^1(\mathbb{T}^d)} = \|P(\rho)(t_0)\|_{L^1(\mathbb{T}^d)}.
\]
Before we go any further, it should be noted that (3.22) remains valid for function \( \phi \) belonging to \( W^{1,\infty} \) rather than \( C^1 \), and then for any \( t_0 > 0 \), we redefine the test function \( \phi \) as \( \phi_\tau \), for some positive \( \tau \) and \( \alpha \) such that \( \tau + \alpha < t_0 \), that is,

\[
\phi_\tau(t) = \begin{cases} 
0, & 0 \leq t \leq \tau, \\
\frac{t}{\tau/\alpha} - \tau, & \tau \leq t \leq \tau + \alpha, \\
1, & \tau + \alpha \leq t \leq t_0, \\
0, & t_0 - \tau/\alpha \leq t \leq t_0 + \alpha, \\
0, & t_0 + \alpha \leq t.
\end{cases}
\]

(3.33)

Then substituting this test function into (3.22), we arrive at

\[
- \int_\tau^{\tau+\alpha} \int 1/2 \rho v^2 + P(\rho) \, dx + \int_0^{t_0} \int_{\tau}^{\tau+\alpha} (1/2 \rho v^2 + P(\rho)) \, dx dt \\
+ \int_\tau^{t_0+\alpha} \phi_\tau(v(\rho) ||Dv||^2 + \mu(\rho)) \, div v^2) = 0.
\]

(3.34)

Taking \( \alpha \to 0 \) and using the fact that \( \int_0^1 (v(\rho)||Dv||^2 + \mu(\rho)) \, div v^2 \) is continuous with respect to \( t \) and the Lebesgue point theorem, we deduce that

\[
- \int (1/2 \rho v^2 + P(\rho)) (\tau) dx + \int (1/2 \rho v^2 + P(\rho)) (t_0) dx \\
+ \int_0^{t_0} \int (v(\rho)||Dv||^2 + \mu(\rho)) \, div v^2) \, dx dt = 0.
\]

(3.35)

Finally, letting \( \tau \to 0 \), using the continuity of \( \int_0^1 (v(\rho)||Dv||^2 + \mu(\rho)) \, div v^2 \), (3.28) and (3.32), we can obtain

\[
\int (1/2 \rho v^2 + P(\rho)) (t_0) dx + \int_0^{t_0} \int (v(\rho)||Dv||^2 + \mu(\rho)) \, div v^2 \) \, dx dt = \int (1/2 \rho v_0^2 + P(\rho_0)) \, dx.
\]

(3.36)

Then we have completed the proof of Theorem 1.1.

\[\square\]

We are in a position to prove Corollary 1.1.

**Proof of Corollary 1.1.**

(1) The natural energy gives \( v \in L^2(0, T; H^1(\mathbb{T}^3)) \). Choosing \( p = q = 2 \) in (1.19), we immediately prove that the condition \( v \in L^4(0, T; L^4(\mathbb{T}^3)) \) yields energy equality.

It is worth remarking that the rest proof in (1.22) can be reduced to this special case. Next, we first deal with the case (1.22) in Corollary 1.1 with \( q \geq 4 \) and \( \frac{2}{p} + \frac{2}{q} = 1 \). The Gagliardo–Nirenberg inequality guarantees that

\[
\|v\|_{L^{q}(0,T,L^{q}(\mathbb{T}^{3}))} \leq C\|v\|_{L^{q}(0,T,L^{q}(\mathbb{T}^{3}))}^{\frac{4-q}{2q}} \|v\|_{L^{p}(0,T,L^{p}(\mathbb{T}^{3}))}^{\frac{q}{2q}} \leq C.
\]

(3.37)

From the result just proved, we obtain energy equality via (1.22) with \( q \geq 4 \).

Then we consider (1.22) with \( 3 < q < 4 \) and \( \frac{1}{p} + \frac{3}{q} = 1 \). Using the Gagliardo–Nirenberg inequality again, we know that

\[
\|v\|_{L^{q}(0,T,L^{q}(\mathbb{T}^{3}))} \leq C\|v\|_{L^{q}(0,T,L^{q}(\mathbb{T}^{3}))}^{\frac{4-q}{2q}} \|v\|_{L^{p}(0,T,L^{p}(\mathbb{T}^{3}))}^{\frac{q}{2q}} \leq C.
\]

(3.38)

We finish the proof of (1.22).
(2) Now, we focus on the proof of (1.23). Indeed, note that $v \in L^p(0, T; W^{1,q}(\mathbb{T}^3))$, therefore, according to Theorem 1.1, it suffices to derive $v \in L^{12/7}(0, T; L^{20/7}(\mathbb{T}^3))$ from (1.23). For $q \geq 9/5$, by the Gagliardo–Nirenberg inequality, we get

$$\|v\|_{L^{12/7}(\mathbb{T}^3)} \leq C\|v\|_{L^p(\mathbb{T}^3)}^{2/9} \|\nabla v\|_{L^q(\mathbb{T}^3)}^{6/9},$$

(3.39)

Thanks to $\frac{1}{p} + \frac{6}{5q} = 1$, we further infer that

$$\|v\|_{L^{12/7}(0, T; L^{20/7}(\mathbb{T}^3))} \leq C\|v\|_{L^p(0, T; L^q(\mathbb{T}^3))}^{2/9} \|\nabla v\|_{L^q(0, T; L^q(\mathbb{T}^3))}^{6/9} \leq C.$$  

(3.40)

In light of Theorem 1.1, we have proved (1.23) for $q \geq 9/5$.

Finally, for $3/2 < q < 9/5$, it follows the Gagliardo–Nirenberg inequality that

$$\|v\|_{L^{12/7}(\mathbb{T}^3)} \leq C\|v\|_{L^p(\mathbb{T}^3)}^{2/9} \|\nabla v\|_{L^q(\mathbb{T}^3)}^{6/9},$$

(3.41)

Thanks to $\frac{1}{p} + \frac{3}{q} = 2$, we further have

$$\|v\|_{L^{12/7}(0, T; L^{20/7}(\mathbb{T}^3))} \leq C\|v\|_{L^p(0, T; L^q(\mathbb{T}^3))}^{2/9} \|\nabla v\|_{L^q(0, T; L^q(\mathbb{T}^3))}^{6/9} \leq \left(\|\nabla v\|_{L^q(0, T; L^q(\mathbb{T}^3))} + \|v\|_{L^q(0, T; L^q(\mathbb{T}^3))}\right)^{\frac{2}{9}} \|\nabla v\|_{L^q(0, T; L^q(\mathbb{T}^3))}^{\frac{6}{9}},$$

(3.42)

then we conclude the desired result from Theorem 1.1. The proof of this corollary is completed.

\[\square\]

ACKNOWLEDGEMENTS

Wang was partially supported by the National Natural Science Foundation of China under grant (No. 11971446, No. 12071113, and No. 11601492). Ye was partially supported by the National Natural Science Foundation of China under grant (No. 11701145) and China Postdoctoral Science Foundation (No. 2020M672196).

CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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How to cite this article: Wang Y, Ye Y. A general sufficient criterion for energy conservation in the Navier-Stokes system. Math Meth Appl Sci. 2023;46(8):9268-9285. doi:10.1002/mma.9051