On the non-autonomous Schrödinger-Poisson problems in $\mathbb{R}^3$

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Abstract

In this paper, we study the problem:

\begin{align*}
-\Delta u + u + \lambda K(x) \phi u &= a(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2 & \text{in } \mathbb{R}^3,
\end{align*}

where $\lambda > 0$ and $2 < p < 4$. We require that $K(x)$ and $a(x)$ are nonnegative functions in $\mathbb{R}^3$ and satisfy some suitable assumptions, but not requiring any symmetry property on them. Assuming that $\lim_{|x| \to \infty} K(x) = K_\infty \geq 0$ and $\lim_{|x| \to \infty} a(x) = a_\infty > 0$, we establish some existence results of positive solutions, depending on the parameter $\lambda$. More importantly, we prove the existence of ground state solutions for the case $3.18 \approx 1 + \sqrt{73}/3 < p < 4$.

Keywords: Schrödinger-Poisson systems; Variational methods; Ground state.

1 Introduction

In this paper, we are concerned with the following Schrödinger-Poisson system

\begin{align*}
-\Delta u + u + K(x) \phi u &= f(x,u) & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2 & \text{in } \mathbb{R}^3,
\end{align*}

where $K$ is a nonnegative function in $\mathbb{R}^3$ and $f \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$.

This system arises in Quantum Mechanics: in 1998 Benci and Fortunato [5] proposed it as a model describing the interaction of a charged particle with the electrostatic field.
The unknowns $u$ and $\phi$ of the system represent the wave functions associated to the particle and the electric potential, and the function $K$ is a nonnegative density charge.

If the nonlinear term $f(x, u)$ is replaced with 0, the eigenvalue problem for (1) has been studied in [5] (in the case in which the charged particle lies in a bounded space region $\Omega$) and in [10] (under the action of an external nonzero potential).

In recent years, in order to better simulate the interaction effect among many particles in Quantum Mechanics, Schrödinger-Poisson system with a local nonlinear term $f(x, u)$, i.e., system (1), has begun to receive much attention; see [2, 3, 21, 23, 25] for the autonomous case, see [11, 12, 26, 27, 28, 30] for the non-autonomous case, and see [1, 22, 24] for the so-called semi-classical states.

Let us briefly comment the known results for the nonlinear Schrödinger-Poisson system (1). In [25], the authors use a minimization procedure in an appropriate manifold to find a positive solution (possibly nonradial) for the autonomous system

$$\begin{align*}
-\Delta u + \beta u + \phi u &= |u|^{p-2} u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \lim_{|x| \to +\infty} \phi(x) = 0 \quad \text{in } \mathbb{R}^3,
\end{align*}$$

for $p = \frac{8}{3}$. They obtain a solution of (2) with frequency $\beta > 0$.

Ruiz [21] studies a class of autonomous Schrödinger-Poisson systems

$$\begin{align*}
-\Delta u + u + \lambda \phi u &= |u|^{p-2} u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \lim_{|x| \to +\infty} \phi(x) = 0 \quad \text{in } \mathbb{R}^3,
\end{align*}$$

where $2 < p < 6$ and $\lambda > 0$. By restricting the energy functional $I$ of (3) to the subspace $H_1^1$ of $H^1(\mathbb{R}^3)$ consisting of radially symmetric functions, he gives existence and nonexistence results for (3), depending on the parameters $\lambda$ and $p$. It turns out that $p = 3$ is a critical value for the existence of solutions. His approach is based on minimizing $I$ on a certain manifold. A point worth emphasizing is that Ruiz points out that the usual Nehari manifold $\mathcal{N} = \{u \in H_1^1 \setminus \{0\} : I'(u)u = 0\}$ is not appropriate for it. In order to obtain the existence result, he establishes a new manifold $M = \{u \in H_1^1 \setminus \{0\} : F(u) = 0\}$, where $F(u) = 0$ is nothing but the equation $2I'(u)u = 0$ minus the Pohozaev identity of (3) proved by [13].

From then on, the manifold $M$ or $\overline{M} = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : F(u) = 0\}$ has been applied in the study of some types of nonlinear equations. Azzollini and Pomponio [3] for autonomous Schrödinger-Poisson systems, for example, Li and Ye [17] for Kirchhoff type equations.

Recently, Cerami and Varia [11] study a non-autonomous Schrödinger-Poisson system without any symmetry assumptions

$$\begin{align*}
-\Delta u + u + \lambda K(x) \phi u &= a(x) |u|^{p-2} u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K(x) u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}$$

(SP_\lambda)

where $\lambda > 0$ and $4 < p < 6$. Assuming $K$ and $a$ are nonnegative functions in $\mathbb{R}^3$ satisfying

$$\begin{align*}
\lim_{|x| \to \infty} K(x) &= K_\infty = 0, \\
\lim_{|x| \to \infty} a(x) &= a_\infty > 0,
\end{align*}$$

(4)
and some other suitable assumptions, they proved the existence of positive solutions for \((SP_\lambda)\) by using the Nehari manifold method. Since the condition (1) holds, the authors can make the energy estimates with the help of the solutions of the problem at infinity

\[-\Delta u + u = a_\infty |u|^{p-2}u,\]

which has a unique radial positive ground state \(w_0\) with an exponential decay to zero at infinity.

Later, Varia [27] considers another case of \((SP_\lambda)\), that is,

\[
\lim_{|x| \to \infty} K(x) = K_\infty > 0, \quad \lim_{|x| \to \infty} a(x) = a_\infty > 0.
\]

The author finds a positive ground state solution \(w\) (its energy level is positive) of the problem at infinity

\[
\begin{align*}
-\Delta u + u + \lambda K_\infty \phi u &= a_\infty |u|^{p-2}u & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K_\infty u^2 & \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(4 < p < 6\) if \(\lambda > 0\), and \(2 < p < 6\) if \(\lambda < 0\). With the help of the condition (5) and some other assumptions, using the same method as in [11], the author proves the existence of positive ground state for \((SP_\lambda)\) with \(4 < p < 6\) if \(\lambda > 0\), or \(2 < p < 6\) if \(\lambda < 0\).

As we can see, the results in [11, 27] leave a gap, say, the case \(2 < p < 4\) and \(\lambda > 0\) regardless of \(\lim_{|x| \to \infty} K(x) = K_\infty \geq 0\). We remark that the most important case in applications, \(p = 8/3\), is included in this gap.

Inspired by the above facts, in this work we try to fill this gap and deal with the case in which \(2 < p < 4\) and \(\lambda > 0\).

We point out that both the usual Nehari manifold and the manifold \(M\) established by Ruiz [21] is not good choices for this case. In fact, since \(2 < p < 4\), the energy functional \(I\) constrained on its Nehari manifold is not bounded below (see Appendix). Furthermore, for the non-autonomous system \((SP_\lambda)\), its related Pohozaev equality is as follows

\[
0 = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} [5K(x) + 2(\nabla K(x), x)] \phi_{K,u} u^2 dx - \frac{3}{p} \int_{\mathbb{R}^3} a(x) u^p dx - \int_{\mathbb{R}^3} (\nabla a(x), x) u^p dx,
\]

here we need to assume the functions \(K, a \in C^1(\mathbb{R}^3)\). Obviously, the Pohozaev equality of the non-autonomous case is more complicated than that of the autonomous case (see Theorem 2.2 of [21]).

Our approach consists of minimizing the energy functional \(I\) of \((SP_\lambda)\) on a certain manifold, which is a subset of the Nehari manifold and is a natural constraint of the energy functional \(I\). To the best of our knowledge, this approach is entirely new in the literature. We always assume that the functions \(a, K\) verify, respectively

\[(D1) \ a \text{ is a positive continuous function on } \mathbb{R}^3 \text{ such that } \lim_{|x| \to \infty} a(x) = a_\infty > 0 \text{ uniformly on } \mathbb{R}^3,\]
\[ a_{\text{max}} := \sup_{x \in \mathbb{R}^3} a(x) < \frac{2a_{\infty}}{(4 - p) A(p)^{(p-2)/2}}, \]

where
\[
A(p) = \begin{cases} \left( \frac{2}{4-p} \right)^{1/(p-2)}, & \text{if } 2 < p \leq 3 \\ \frac{1}{2} \left( \frac{2}{4-p} \right)^{2/(p-2)}, & \text{if } 3 < p < 4; \end{cases}
\]

\[(D2) \quad K \in L^\infty(\mathbb{R}^3) \setminus \{0\} \text{ is a non-negative function on } \mathbb{R}^3 \text{ such that}
\lim_{|x| \to \infty} K(x) = K_\infty \geq 0 \text{ uniformly on } \mathbb{R}^3.\]

It is well known that \((SP_\lambda)\) can be easily transformed in a nonlinear Schrödinger equation with a non-local term (see [1, 10, 21] etc.). Briefly, the Poisson equation is solved by using the Lax–Milgram theorem, so, for all \(u \in H^1(\mathbb{R}^3)\), a unique \(\phi_{K,u} \in D^{1,2}(\mathbb{R}^3)\) given by
\[
\phi_{K,u}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y) u^2(y)}{|x-y|} dy,
\]
such that \(-\Delta \phi = Ku^2\) and that, inserted into the first equation, gives
\[
-\Delta u + u + \lambda K(x) \phi_{K,u} u = a(x) |u|^{p-2} u \quad \text{in } \mathbb{R}^3. \tag{E_\lambda}
\]

Moreover, Eq. \((E_\lambda)\) is variational and its solutions are the critical points of the functional defined in \(H^1(\mathbb{R}^3)\) by
\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} K \phi_{K,u}(x) u^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^3} a |u|^p \, dx.
\]

Furthermore, it is easy to prove that \(J_\lambda\) is a \(C^1\) functional with derivative given by
\[
\langle J'_\lambda(u), \varphi \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + u \varphi + \lambda K \phi_{K,u} u \varphi - a |u|^{p-2} u \varphi) \, dx
\]
for all \(\varphi \in H^1(\mathbb{R}^3)\), where \(J'_\lambda\) denotes the Fréchet derivative of \(J_\lambda\). Note that \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a solution of \((SP_\lambda)\) if and only if \(u\) is a critical point of \(J_\lambda\) and \(\phi = \phi_{K,u}\).

Before stating our results we need to introduce some notations and definitions. Denote by \(S\) and \(\overline{S}\) the best constants for the embedding of \(H^1(\mathbb{R}^3)\) and \(D^{1,2}(\mathbb{R}^3)\), respectively, in \(L^6(\mathbb{R}^3)\). Let
\[
B(p) = \frac{A(p)^{(p-2)/2} - 1}{A(p)} \tag{7}
\]
and
\[
\Lambda_0 = \left[ 1 - A(p) \left( \frac{4 - p}{2a_{\infty}} \right)^{2/(p-2)} \right] \left( \frac{a_{\text{max}}}{S^p} \right)^{2/(p-2)} \frac{S^2}{K_{\text{max}}} \tag{8}
\]
where $S_p$ is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$, $a_{\text{max}} = \sup_{x \in \mathbb{R}^3} a(x)$ and $K_{\text{max}} = \sup_{x \in \mathbb{R}^3} K(x)$. In particular, if $K(x) \equiv K_\infty$ and $a(x) \equiv a_\infty$, then the equality (5) becomes as follows

$$
\Lambda_0 = \left[1 - A(p) \left(\frac{4 - p}{2}\right)^{2/(p-2)}\right] \left(\frac{a_\infty}{S_p}\right)^{2/(p-2)} \frac{S^2 S^4}{K^2_\infty}.
$$

Moreover, we denote by $w_0$ be the unique positive solution with $w_0(0) = \max_{x \in \mathbb{R}^3} w_0(x)$ for the following Schrödinger equation

$$
-\Delta u + u = a_\infty |u|^{p-2} u \quad \text{in } \mathbb{R}^3,
$$

and set $\alpha_0^\infty = \frac{p-2}{p} \|w_0\|_{H^1}^2$.

**Definition 1** $u$ is a ground state solution of $(SP_\lambda)$ we mean that $u$ is such a solution of $(SP_\lambda)$ which has the least energy among all nontrivial solutions of $(SP_\lambda)$.

Now, we give our main results.

**Theorem 2** Suppose that $2 < p < 4$, $K(x) \equiv K_\infty > 0$ and $a(x) \equiv a_\infty > 0$. Then for each $0 < \lambda < B(p) \Lambda_0$, Problem $(SP_\lambda)$ has a positive solution $w_\lambda \in H^1(\mathbb{R}^3)$ such that $J_\lambda (w_\lambda) > \alpha_0^\infty$.

**Theorem 3** Suppose that $2 < p < 4$, $K_\infty > 0$ and in addition to conditions (D1) and (D2) hold, we also have

(D3) $\int_{\mathbb{R}^3} (a - a_\infty) w_\lambda^p \geq 0$, $\int_{\mathbb{R}^3} K \phi_{K,w_\lambda} w_\lambda^2 \leq \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty},w_\lambda} w_\lambda^2$ and

$$
\int_{\mathbb{R}^3} K \phi_{K,w_\lambda} w_\lambda^2 - \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty},w_\lambda} w_\lambda^2 < \int_{\mathbb{R}^3} (a - a_\infty) w_\lambda^p,
$$

where $w_\lambda$ is a positive solution obtained in Theorem 2. Then for each $0 < \lambda < B(p) \Lambda_0$, Problem $(SP_\lambda)$ has a positive solution $v_\lambda \in H^1(\mathbb{R}^3)$.

**Theorem 4** Suppose that $2 < p < 4$, $K_\infty \geq 0$ and in addition to conditions (D1)–(D2) hold, we also have

(D4) $\int_{\mathbb{R}^3} (a - a_\infty) w_0^p > 0$.

Then there exists $0 < \Lambda_0 \leq B(p) \Lambda_0$ such that for every $0 < \lambda < \Lambda_0$, Problem $(SP_\lambda)$ has one positive solution $v_\lambda \in H^1(\mathbb{R}^3)$.

**Theorem 5** Suppose that $\frac{1 + \sqrt{73}}{3} < p < 4$ and in addition to the conditions (D1)–(D2) hold, we also have

(DK, a) the functions $a, K \in C^1(\mathbb{R}^3)$ which satisfy $(\nabla a, x) \leq 0$ and

$$
\frac{3p^2 - 2p - 24}{2(6 - p)} K + \frac{p - 2}{2} (\nabla K, x) \geq 0.
$$

Let $v_\lambda$ be the positive solution obtained in Theorems 2 and 4. Then $v_\lambda$ is a ground state solution of Problem $(SP_\lambda)$.

This paper is organized as follows. We first outline the preliminaries in Section 2 before proving some submanifolds are nonempty in Section 3 and the proof of Theorem 2 in Section 3. In Sections 4 and 5, we prove that Theorems 3 and 4. In Section 6, we prove that Theorem 5.
2 Preliminaries

Throughout this section, we assume that the conditions (D1) and (D2) hold. Define the Nehari manifold

\[ M_\lambda := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \langle J'_\lambda(u) , u \rangle = 0 \} . \]

Then \( u \in M_\lambda \) if and only if \( \| u \|^2_{H^1} + \lambda \int_{\mathbb{R}^3} K\phi_{K,u} u^2 - \int_{\mathbb{R}^3} a|u|^p = 0 \). Moreover, by the Sobolev inequality,

\[
\| u \|^2_{H^1} \leq \| u \|^2_{H^1} + \lambda \int_{\mathbb{R}^3} K\phi_{K,u} u^2 = \int_{\mathbb{R}^3} a|u|^p \leq S_p^{-p} a_{\max} \| u \|_{H^1}^p
\]

for all \( u \in M_\lambda \), where \( a_{\max} = \sup_{x \in \mathbb{R}^3} a(x) \). Thus,

\[
\int_{\mathbb{R}^3} a|u|^p \geq \| u \|^2_{H^1} \geq \left( \frac{S_p}{a_{\max}} \right)^{2/(p-2)} \text{ for all } u \in M_\lambda. \tag{9}
\]

The Nehari manifold \( M_\lambda \) is closely linked to the behavior of the function of the form \( h_{\lambda,u} : t \to J_\lambda(tu) \) for \( t > 0 \). Such maps are known as fibering maps and were introduced by Drábek-Pohozaev in [14] and are also discussed in Brown-Zhang [9] and Brown-Wu [7, 8]. If \( u \in H^1(\mathbb{R}^3) \), we have

\[
h_{\lambda,u}(t) = \frac{t^2}{2} \| u \|^2_{H^1} + \frac{\lambda}{4} t^4 \int_{\mathbb{R}^3} K\phi_{K,u} u^2 - \frac{1}{p} t^p \int_{\mathbb{R}^3} a|u|^p;
\]

\[
h'_{\lambda,u}(t) = t \| u \|^2_{H^1} + \lambda t^3 \int_{\mathbb{R}^3} K\phi_{K,u} u^2 - t^{p-1} \int_{\mathbb{R}^3} a|u|^p;
\]

\[
h''_{\lambda,u}(t) = \| u \|^2_{H^1} + 3\lambda t^2 \int_{\mathbb{R}^3} K\phi_{K,u} u^2 - (p-1) t^{p-2} \int_{\mathbb{R}^3} a|u|^p.
\]

It is easy to see that

\[
\int_{\mathbb{R}^3} a|tu|^p = \| tu \|^2_{H^1} + \lambda \int_{\mathbb{R}^3} K\phi_{K,tu}(tu)^2 - \int_{\mathbb{R}^3} a|tu|^p
\]

and so, for \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \) and \( t > 0 \), \( h'_{\lambda,u}(t) = 0 \) if and only if \( tu \in M_\lambda \), i.e., positive critical points of \( h_{\lambda,u} \) correspond to points on the Nehari manifold. In particular, \( h'_{\lambda,u}(1) = 0 \) if and only if \( u \in M_\lambda \). Thus, it is natural to split \( M_\lambda \) into three parts corresponding to local minima, local maxima and points of inflection. Accordingly, we define

\[
M^+_\lambda = \{ u \in M_\lambda \mid h''_{\lambda,u}(1) > 0 \} ;
\]

\[
M^0_\lambda = \{ u \in M_\lambda \mid h''_{\lambda,u}(1) = 0 \} ;
\]

\[
M^-_\lambda = \{ u \in M_\lambda \mid h''_{\lambda,u}(1) < 0 \} .
\]

**Lemma 6** Suppose that \( u_0 \) is a local minimizer for \( J_\lambda \) on \( M_\lambda \) and that \( u_0 \notin M^0_\lambda \). Then \( J'_\lambda(u_0) = 0 \) in \( H^{-1}(\mathbb{R}^3) \).
Proof. The proof is essentially the same as that in Brown-Zhang [9, Theorem 2.3] (or see Binding-Drábek-Huang [4]), and we omit it here. ■

For each \( u \in M_\lambda \), we have

\[
\begin{align*}
    h''_{\lambda,u}(1) &= \| u \|_{H^1}^2 + 3\lambda \int_{\mathbb{R}^3} K\phi_{K,u}u^2 - (p - 1) \int_{\mathbb{R}^3} a |u|^p \\
    &= -(p - 2) \| u \|_{H^1}^2 + (4 - p) \lambda \int_{\mathbb{R}^3} K\phi_{K,u}u^2 \\
    &= -2 \| u \|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a |u|^p. 
\end{align*}
\]

Furthermore, we have the following result.

**Lemma 7** Suppose that \( 2 < p < 4 \). Then the energy functional \( J_\lambda \) is coercive and bounded below on \( M_\lambda^- \). Furthermore,

\[
J_\lambda(u) > \frac{p - 2}{4p} \left( \frac{S_p}{\lambda a_{\text{max}}} \right)^{2/(p-2)} \quad \text{for all } u \in M_\lambda^-,
\]

where \( a_{\text{max}} = \sup_{x \in \mathbb{R}^3} a(x) \).

**Proof.** For any \( u \in M_\lambda^- \), using [9] and [11], one has

\[
J_\lambda(u) > \frac{p - 2}{4p} \| u \|_{H^1}^2 \geq \frac{p - 2}{4p} \left( \frac{S_p}{\lambda a_{\text{max}}} \right)^{2/(p-2)},
\]

which shows that \( J_\lambda \) is coercive and bounded below on \( M_\lambda^- \). This completes the proof. ■

The function \( \phi_{K,u} \) possesses certain properties (see [3, 21]).

**Lemma 8** For each \( u \in H^1(\mathbb{R}^3) \), we have the following.

(i) \( \phi_{K,u} \geq 0 \);

(ii) \( \int_{\mathbb{R}^3} K\phi_{K,u}u^2 \leq S^{-2} S^{-4} K_{\text{max}} \| u \|_{H^1}^4 \).

For each \( 2 < p < 4 \) and \( u \in M_\lambda \) with \( J_\lambda(u) < A(p) \frac{p - 2}{2p} \left( \frac{S_p}{\lambda a_{\text{max}}} \right)^{2/(p-2)} \), we have

\[
\frac{A(p)(p - 2)}{2p} \left( \frac{S_p}{\lambda a_{\text{max}}} \right)^{2/(p-2)} > \frac{1}{2} \| u \|_{H^1}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} K\phi_{K,u}u^2 - \frac{1}{p} \int_{\mathbb{R}^3} a |u|^p \\
= \frac{p - 2}{2p} \| u \|_{H^1}^2 - \frac{4 - p}{4p} \lambda \int_{\mathbb{R}^3} K\phi_{K,u}u^2 \\
\geq \frac{p - 2}{2p} \| u \|_{H^1}^2 - \lambda S^{-2} S^{-4} K_{\text{max}} \left( \frac{4 - p}{4p} \right) \| u \|_{H^1}^4.
\]

This implies that if \( 2 < p < 4 \) and \( 0 < \lambda < (p - 2) \left( \frac{4 - p}{2} \right)^{(4-p)/(p-2)} \Lambda_0 \), then there exist two positive numbers \( \hat{D}_1, \hat{D}_2 \) with

\[
\sqrt{A(p)} \left( \frac{S_p}{\lambda a_{\text{max}}} \right)^{1/(p-2)} < \hat{D}_1 < \left( \frac{2S_p}{\lambda a_{\text{max}}(4-p)} \right)^{1/(p-2)} < \sqrt{2} \left( \frac{2S_p}{\lambda a_{\text{max}}(4-p)} \right)^{1/(p-2)} < \hat{D}_2
\]

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such that
\[ \|u\|_{H^1} < \hat{D}_1 \text{ or } \|u\|_{H^1} > \hat{D}_2. \]

Thus,
\[
M^\lambda \left( \frac{A(p)(p-2)}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{\frac{2}{p-2}} \right) := \left\{ u \in M^\lambda \mid J^\lambda(u) < \frac{A(p)(p-2)}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{\frac{2}{2p}} \right\}
\]
\[= M^{(1)}_\lambda \cup M^{(2)}_\lambda, \tag{12} \]

where
\[
M^{(1)}_\lambda := \left\{ u \in M^\lambda \mid J^\lambda(u) < \frac{A(p)(p-2)}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{\frac{2}{2p}} \right\}
\]
and
\[
M^{(2)}_\lambda := \left\{ u \in M^\lambda \mid J^\lambda(u) < \frac{A(p)(p-2)}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{\frac{2}{2p}} \right\}. \]

Then for each \(2 < p < 4\) and \(0 < \lambda < (p-2) \left( \frac{4-p}{2} \right)^{(4-p)/(p-2)} \Lambda_0\),
\[
\|u\|_{H^1} < \hat{D}_1 < \left[ \frac{2S_p^p}{(4-p)a_{\text{max}}} \right]^{\frac{1}{p-2}} \tag{13}
\]

and
\[
\|u\|_{H^1} > \hat{D}_2 > 2 \left[ \frac{2S_p^p}{(4-p)a_{\text{max}}} \right]^{\frac{1}{p-2}} \tag{14}
\]

Thus, by the Sobolev inequality, (11) and (13), we have
\[
h_{\lambda,u}''(1) \leq -2 \|u\|_{H^1}^2 + (4-p) S_p^{-p} a_{\text{max}} \|u\|_{H^1}^p < 0 \text{ for all } u \in M^{(1)}_\lambda.
\]

Moreover, using (14), one has
\[
\frac{1}{4} \|u\|_{H^1}^2 - \frac{4-p}{4p} \int_{\mathbb{R}^3} a |u|^p = J^\lambda(u) < \frac{A(p)(p-2)}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{2/(p-2)} \leq \frac{p-2}{2p} \left( \frac{2S_p^p}{a_\infty (4-p)} \right)^{2/(p-2)} < \frac{p-2}{4p} \|u\|_{H^1}^2 \text{ for all } u \in M^{(2)}_\lambda,
\]
which implies that
\[
2 \|u\|_{H^1}^2 < (4-p) \int_{\mathbb{R}^3} a |u|^p \text{ for all } u \in M^{(2)}_\lambda.
\]

Thus,
\[
h_{\lambda,u}''(1) = -2 \|u\|_{H^1}^2 + (4-p) \int_{\mathbb{R}^3} a |u|^p > 0 \text{ for all } u \in M^{(2)}_\lambda.
\]

Furthermore, we have the following result.
Lemma 9  If $2 < p < 4$ and $0 < \lambda < (p - 2) \left(\frac{4-p}{2}\right)^{(4-p)/(p-2)} \Lambda_{0}$, then $M^{(1)}_{\lambda} \subset M^{(1)}_{\lambda}$ and $M^{(2)}_{\lambda} \subset M^{(2)}_{\lambda}$ are $C^1$ submanifolds and so the submanifolds $M^{(1)}_{\lambda}$ and $M^{(2)}_{\lambda}$ are natural constraints for the functional $J_{\lambda}$.

For $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we define

$$T_{a}(u) = \left( \frac{1}{\int_{\mathbb{R}^3} a \, |u|^p} \right)^{1/(p-2)} \left( \frac{\|u\|^{2}_{H^1}}{\int_{\mathbb{R}^3} a \, |u|^p} \right)^{1/(p-2)}.$$ 

Then we have the following results.

Lemma 10  Suppose that $2 < p < 4$ and $0 < \lambda < (p - 2) \left(\frac{4-p}{2}\right)^{(4-p)/(p-2)} \Lambda_{0}$. Then for each $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ with

$$\int_{\mathbb{R}^3} a \, |u|^p \geq \frac{a_{\max}}{s^p} \left[ 1 - A(p) \left( \frac{4-p}{2a_{\infty}} \right)^{2/(p-2)} \right]^{(p-2)/2} \|u\|_{H^1}^p,$$

there exists $\tilde{t}_{a,\lambda}^{(0)}(u) > \left(\frac{p}{4-p}\right)^{1/(p-2)} T_{a}(u)$ such that

$$\inf_{t \geq \tilde{t}_{a,\lambda}^{(0)}} J_{\lambda}(tu) = \inf_{T_{a}(u) < t < \tilde{t}_{a,\lambda}^{(0)}} J_{\lambda}(tu) < 0.$$

Proof. For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $t > 0$,

$$J_{\lambda}(tu) = \frac{t^2}{2} \|u\|_{H^1}^2 + \frac{\lambda t^4}{4} \int_{\mathbb{R}^3} K\varphi_{K,u} u^2 - \frac{t^p}{p} \int_{\mathbb{R}^3} a \, |u|^p$$

$$= t^4 \left[ g(t) + \frac{\lambda}{4} \int_{\mathbb{R}^3} K\varphi_{K,u} u^2 \right] = h_{\lambda,u}(t),$$

where $g(t) = \frac{t^{-2}}{2} \|u\|_{H^1}^2 - \frac{t^{p-4}}{p} \int_{\mathbb{R}^3} a \, |u|^p$. Clearly, $J_{\lambda}(tu) = 0$ if and only if $g(t) + \frac{\lambda}{4} \int_{\mathbb{R}^3} K\varphi_{K,u} u^2 = 0$. It is easy to claim that $g(\tilde{t}_{a}) = 0$, $\lim_{t \to 0^+} g(t) = \infty$ and $\lim_{t \to \infty} g(t) = 0$, where $\tilde{t}_{a} = \left(\frac{p}{2}\right)^{1/(p-2)} T_{a}(u)$. Moreover,

$$g'(t) = -t^{-3} \|u\|_{H^1}^2 + \frac{(4-p)}{p} \frac{t^{p-5}}{p} \int_{\mathbb{R}^3} a \, |u|^p$$

$$= t^{-3} \left[ \frac{(4-p)}{p} \frac{t^{p-2}}{p} \int_{\mathbb{R}^3} a \, |u|^p - \|u\|_{H^1}^2 \right],$$

which implies that $g(t)$ is decreasing on $0 < t < \left(\frac{p}{4-p}\right)^{1/(p-2)} T_{a}(u)$ and is increasing on $t > \left(\frac{p}{4-p}\right)^{1/(p-2)} T_{a}(u)$, and so

$$\inf_{t > 0} g(t) = g \left( \left(\frac{p}{4-p}\right)^{1/(p-2)} T_{a}(u) \right)$$

$$= -\frac{p-2}{2(4-p)} \left( \frac{2 \|u\|_{H^1}^2}{(4-p) \int_{\mathbb{R}^3} a \, |u|^p} \right)^{-2/(p-2)} \|u\|_{H^1}^2.$$
Thus, by Lemma 8 (ii) and the Sobolev inequality, for each $u \in H^{1}(\mathbb{R}^{3}) \setminus \{0\}$ with

$$
\int_{\mathbb{R}^{3}} a |u|^p \geq \frac{a_{\text{max}}}{S_p^p} \left[ 1 - A(p) \left( \frac{(4 - p) a_{\text{max}}}{2a_{\infty}} \right)^{2/(p-2)} \right] \frac{(p-2)/2}{\|u\|_{H^1}^p}
$$

and $0 < \lambda < (p - 2) \left( \frac{4 - p}{2} \right)^{(4-p)/(p-2)} \Lambda_0$, we have

$$
\inf_{t \geq 0} g(t) = g \left( \left( \frac{p}{4 - p} \right)^{2/(p-2)} T_a(u) \right)
$$

$$
= - \frac{p - 2}{2 (4 - p)} \left( \frac{(4 - p) \int_{\mathbb{R}^3} a |u|^p}{2 \|u\|_{H^1}^2} \right)^{\frac{2}{p-2}} \|u\|_{H^1}^2
$$

$$
\leq - \frac{p - 2}{2 (4 - p)} \left( \frac{(4 - p) a_{\text{max}}}{2 S_p^p} \right)^{\frac{2}{p-2}} \left( 1 - A(p) \left( \frac{(4 - p) a_{\text{max}}}{2a_{\infty}} \right)^{2/(p-2)} \right) \|u\|_{H^1}^4
$$

$$
< - \frac{\lambda}{4} \cdot \frac{K^2_{\text{max}}}{S^2 S_4} \|u\|_{H^1}^4 \leq - \frac{\lambda}{4} \int_{\mathbb{R}^3} K \phi_{K,u} u^2,
$$

which implies that there are

$$
0 < \tilde{\tau}^{(1)}_{a,\lambda}(u) < \left( \frac{p}{4 - p} \right)^{1/(p-2)} T_a(u) < \tilde{\tau}^{(0)}_{a,\lambda}(u) \quad (16)
$$

such that $g \left( \tilde{\tau}^{(j)}_{a,\lambda}(u) \right) + \frac{\lambda}{4} \int_{\mathbb{R}^3} K \phi_{K,u} u^2 = 0$ for $j = 0, 1$, i.e.,

$$
J_{\lambda} \left( \tilde{\tau}^{(j)}_{a,\lambda}(u) u \right) = 0 \text{ for } j = 0, 1.
$$

Then we can conclude that for $0 < \lambda < (p - 2) \left( \frac{4 - p}{2} \right)^{(4-p)/(p-2)} \Lambda_0$ and $u \in H^{1}(\mathbb{R}^{3}) \setminus \{0\}$ with

$$
\int_{\mathbb{R}^{3}} a |u|^p \geq \frac{a_{\text{max}}}{S_p^p} \left[ 1 - \left( \frac{(4 - p) a_{\text{max}}}{2a_{\infty}} \right)^{1/(p-2)} \right] \frac{p/(p-2)}{\|u\|_{H^1}^p},
$$

one has

$$
J_{\lambda} \left( \left( \frac{p}{4 - p} \right)^{\frac{1}{p-2}} T_a(u) u \right)
$$

$$
= \left[ \left( \frac{p}{4 - p} \right)^{\frac{1}{p-2}} T_a(u) \right]^{4} \left[ g \left( \left( \frac{p}{4 - p} \right)^{\frac{1}{p-2}} T_a(u) \right) + \frac{\lambda}{4} \int_{\mathbb{R}^3} K \phi_{K,u} u^2 \right]
$$

$$
< 0,
$$

and so

$$
\inf_{t \geq 0} J_{\lambda}(tu) < 0. \quad (17)
$$
Note that
\[ h'_{\lambda,u}(t) = 4t^3 \left[ g(t) + \frac{\lambda}{4} \int_{\mathbb{R}^3} K \phi_{K,u} u^2 \right] + t^4 g'(t). \]
This shows that
\[ h'_{\lambda,u}(t) < 0 \text{ for all } t \in \left( \overline{t^{(1)}_{a,\lambda}(u)}, \left( \frac{p}{4-p} \right)^{1/(p-2)} T_a(u) \right] \]
and
\[ h'_{\lambda,u}(\overline{t^{(0)}_{a,\lambda}(u)}) > 0. \]
Therefore,
\[ \inf_{0 \leq t \leq t^{+}_{\lambda}(u)} J_{\lambda}(tu) = \inf_{t \geq 0} J_{\lambda}(tu) < 0. \]
This completes the proof. ■

**Lemma 11** Suppose that \(2 < p < 4\) and \(0 < \lambda < B(p) A_0\). Then for each \(u \in H^1(\mathbb{R}^3) \setminus \{0\}\) with
\[
\int_{\mathbb{R}^3} a |u|^p \geq \frac{a_{\text{max}}^p}{S_p} \left[ 1 - \left( \frac{(4-p) a_{\text{max}}^2}{2a_\infty^2} \right)^{\frac{1}{p-2}} \right] \|u\|_{H^1}^p,
\]
there exist \(t^{+}_{\lambda}(u), t^{-}_{\lambda}(u) > 0\) satisfying
\[
T_a(u) < t^{-}_{\lambda}(u) < \sqrt{A(p) T_a(u)} < \left( \frac{2}{4-p} \right)^{1/(p-2)} T_a(u) < t^{+}_{\lambda}(u)
\]
such that \(t^{\pm}_{\lambda} u \in M^{\pm}_{\lambda}\). Furthermore,
\[ J_{\lambda}(t^{-}_{\lambda} u) = \sup_{0 \leq t \leq t^{+}_{\lambda}} J_{\lambda}(tu) \]
and
\[ J_{\lambda}(t^{+}_{\lambda} u) = \inf_{t \geq t^{+}_{\lambda}} J_{\lambda}(tu) = \inf_{t \geq 0} J_{\lambda}(tu) < 0. \]

**Proof.** Define
\[ f(t) = t^{-2} \|u\|_{H^1}^2 - t^{p-4} \int_{\mathbb{R}^3} a |u|^p \text{ for } t > 0. \]
Clearly, \(tu \in M_{\lambda}\) if and only if \(f(t) + \lambda \int_{\mathbb{R}^3} K \phi_{K,u} u^2 = 0\). Moreover, it is easy to see that \(f(T_a(u)) = 0\), \(\lim_{t \to 0^+} f(t) = \infty\) and \(\lim_{t \to \infty} f(t) = 0\). Since \(2 < p < 4\) and
\[ f'(t) = t^{-3} \left( -2 \|u\|_{H^1}^2 + (4-p) t^{p-2} \int_{\mathbb{R}^3} a |u|^p \right), \]
we conclude that \( f(t) \) is decreasing on \( 0 < t < \left( \frac{2}{4-p} \right)^{1/(p-2)} T_a(u) \) and is increasing on \( t > \left( \frac{2}{4-p} \right)^{1/(p-2)} T_a(u) \). Thus,

\[
\inf_{t>0} f(t) = f \left( \left( \frac{2}{4-p} \right)^{1/(p-2)} T_a(u) \right).
\]

For \( 0 < \lambda < B(p) A_0 \) and \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \) with

\[
\int_{\mathbb{R}^3} a |u|^p \geq a_{\max}^{p} \frac{a_{\max}^{2}}{S_p^{p}} \left[ 1 - \left( \frac{(4-p) a_{\max}^{2}}{2a_{\infty}^{2}} \right)^{\frac{1}{p-2}} \right] \|u\|_{H^1}^{p-2} \|u\|_{H^1}.
\]

using Lemma \( \S(ii) \) and the Sobolev inequality, one has

\[
\begin{align*}
\int_{\mathbb{R}^3} a |u|^p &\geq \frac{a_{\max}}{S_p^{p}} \left[ 1 - \left( \frac{(4-p) a_{\max}^{2}}{2a_{\infty}^{2}} \right)^{\frac{1}{p-2}} \right] \|u\|_{H^1}^{p-2} \|u\|_{H^1} \\
&= -\frac{p-2}{4-p} \left( \frac{2 \|u\|_{H^1}^{2}}{(4-p) \int_{\mathbb{R}^3} a |u|^p} \right)^{-\frac{1}{p-2}} \|u\|_{H^1}^{2} \\
&\leq -\frac{p-2}{4-p} \left( \frac{(4-p) a_{\max}^{2}}{2S_p^{p}} \right)^{\frac{2}{p-2}} \left[ 1 - A(p) \left( \frac{(4-p) a_{\max}^{2}}{2a_{\infty}^{2}} \right)^{\frac{2}{p-2}} \right] \|u\|_{H^1}^{4} \\
&< -\lambda \frac{K_{\max}^{2}}{S^2 S_4} \|u\|_{H^1}^{4} \leq -\lambda \int_{\mathbb{R}^3} K \phi_{K,u} u^2.
\end{align*}
\]

Moreover, for each \( 2 < p < 4 \), by (7), we have

\[
T_a(u) < \sqrt{A(p)} T_a(u) < \left( \frac{2}{4-p} \right)^{1/(p-2)} T_a(u)
\]

and

\[
B(p) \leq (p-2) \left( \frac{4-p}{2} \right)^{(1-p)/(p-2)}.
\]
Combining the above two inequalities with (18), for $0 < \lambda < B(p) \Lambda_0$ we have

$$f \left( \sqrt{A(p)} T_a(u) \right) T_a(u) = \sqrt{A(p)} \left( \frac{\|u\|_{H^1}^2}{\int_{\mathbb{R}^3} a |u|^p} \right)^{1/(p-2)}$$

$$= - \frac{A(p)^{(p-2)/2} - 1}{A(p)} \left( \frac{\int_{\mathbb{R}^3} a |u|^p}{\|u\|_{H^1}^2} \right)^{2/(p-2)} \|u\|_{H^1}^2$$

$$\leq - \left( \frac{A(p)^{(p-2)/2} - 1}{A(p)} \right) \left( \frac{a_{\text{max}}}{S_p^p} \right)^{2/(p-2)} \left( \frac{4 - p}{2} \right)^{2/(p-2)} \left( \frac{4 - p}{2} \right)^{2/(p-2)} \left( \frac{4 - p}{2} \right)^{2/(p-2)} \Lambda_0 \|u\|_{H^1}^4$$

$$= - B(p) \frac{K_{\text{max}}^2}{S^2 S^4} \Lambda_0 \|u\|_{H^1}^4$$

$$< - \lambda \frac{K_{\text{max}}^2}{S^2 S^4} \|u\|_{H^1}^4 \leq - \lambda \int_{\mathbb{R}^3} K \phi_{K,u} u^2.$$ 

Thus, there exist

$$T_a(u) < t^-_{\lambda} := t^-_{\lambda}(u) < \sqrt{A(p)} T_a(u) < \left( \frac{2}{4 - p} \right)^{1/(p-2)} T_a(u) < t^+_{\lambda} := t^+_{\lambda}(u)$$

such that $f \left( t^\pm_{\lambda} \right) + \lambda \int_{\mathbb{R}^3} K \phi_{K,u} u^2 = 0$, that is $t^\pm_{\lambda} u \in M_{\lambda}$. Moreover,

$$h''_{\lambda,t^-_{\lambda} u}(1) = -2 \|t^-_{\lambda} u\|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a |t^-_{\lambda} u|^p = (t^-_{\lambda})^5 f'(t^-_{\lambda}) < 0$$

and

$$h''_{\lambda,t^+_{\lambda} u}(1) = -2 \|t^+_{\lambda} u\|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a |t^+_{\lambda} u|^p = (t^+_{\lambda})^5 f'(t^+_{\lambda}) > 0,$$

which show that $t^\pm_{\lambda} u \in M_{\lambda}^\pm$ and

$$h'_{\lambda,t^\pm_{\lambda} u}(t) = t^3 \left( f(t) + \lambda \int_{\mathbb{R}^3} K \phi_{K,u} u^2 \right).$$

This implies that $h'_{\lambda,t^\pm_{\lambda} u}(t) > 0$ for all $t \in (0, t^-_{\lambda}) \cup (t^+_{\lambda}, \infty)$ and $h'_{\lambda,t^\pm_{\lambda} u}(t) < 0$ for all $t \in (t^-_{\lambda}, t^+_{\lambda})$. Thus,

$$J_{\lambda}(t^-_{\lambda} u) = \sup_{0 \leq t \leq t^+_{\lambda}} J_{\lambda}(tu) \text{ and } J_{\lambda}(t^+_{\lambda} u) = \inf_{t \geq t^+_{\lambda}} J_{\lambda}(tu),$$

and so $J_{\lambda}(t^+_{\lambda} u) < J_{\lambda}(t^-_{\lambda} u)$. Moreover, by Lemma [10] we have

$$J_{\lambda}(t^+_{\lambda} u) = \inf_{t \geq 0} J_{\lambda}(tu) < 0.$$
This completes the proof. □

By (12) and Lemma 9, we can define

$$\alpha^\lambda_\lambda = \inf_{u \in \mathcal{M}^{(1)}} J_\lambda (u) = \inf_{u \in \mathcal{M}^{(1)}} J_\lambda (u)$$

and

$$\alpha^+_{\lambda} = \inf_{u \in \mathcal{M}^{(2)}} J_\lambda (u) = \inf_{u \in \mathcal{M}^{(2)}} J_\lambda (u).$$

Adopting the idea of Ni and Takagi [20], we have the following result.

**Lemma 12** Suppose that $2 < p < 4$ and $0 < \lambda < B (p) \Lambda_0$. Then for each $u \in \mathcal{M}^{(1)}_\lambda$, there exist $\sigma > 0$ and a differentiable function $t^* : B (0; \sigma) \subset H^1 (\mathbb{R}^3) \to \mathbb{R}^+$ such that $t^* (0) = 1$, $t^* (v) (u - v) \in \mathcal{M}^{(1)}_\lambda$ for all $v \in B (0; \sigma)$ and

$$\langle (t^*)' (0), \varphi \rangle = \frac{2 \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + u \varphi) + 4 \lambda \int_{\mathbb{R}^3} K \phi_K, u \varphi - p \int_{\mathbb{R}^3} a |u|^{p^p - 2} u \varphi}{\|u\|_{H^1}^2 - (p - 1) \int_{\mathbb{R}^3} a |u|^p}$$

for all $\varphi \in H^1 (\mathbb{R}^3)$.

**Proof.** For all $u \in \mathcal{M}^{(1)}_\lambda$, we define a function $F_u : \mathbb{R} \times H^1 (\mathbb{R}^3) \to \mathbb{R}$ by

$$F_u (t, v) = \langle J'_\lambda (t (u - v)), t (u - v) \rangle$$

$$= t^2 \int_{\mathbb{R}^3} [\nabla (u - v)^2 + (u - v)^2] + t^4 \lambda \int_{\mathbb{R}^3} K \phi_K, u^2$$

$$- tv \int_{\mathbb{R}^3} a |u - v|^p.$$

Thus, $F_u (1, 0) = \langle J'_\lambda (u), u \rangle = 0$ and

$$\frac{d}{dt} F_u (1, 0) = 2 \|u\|_{H^1}^2 + 4 \lambda \int_{\mathbb{R}^3} K \phi_K, u^2 - p \int_{\mathbb{R}^3} a |u|^p$$

$$= -2 \|u\|_{H^1}^2 - (p - 4) \int_{\mathbb{R}^3} a |u|^p < 0.$$

According to the implicit function theorem, there exist $\sigma > 0$ and a differentiable function $t^* : B (0; \sigma) \subset H^1 (\mathbb{R}^3) \to \mathbb{R}$ such that $t^* (0) = 1$,

$$\langle (t^*)' (0), \varphi \rangle = \frac{2 \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + u \varphi) + 4 \lambda \int_{\mathbb{R}^3} K \phi_K, u \varphi - p \int_{\mathbb{R}^3} a |u|^{p^p - 2} u \varphi}{\|u\|_{H^1}^2 - (p - 1) \int_{\mathbb{R}^3} a |u|^p}$$

and

$$F_u (t^* (v), v) = 0 \text{ for all } v \in B (0; \sigma),$$

which is equivalent to

$$\langle J'_\lambda (t^* (v) (u - v)), t^* (v) (u - v) \rangle = 0 \text{ for all } v \in B (0; \sigma).$$
Furthermore, using the continuity of the maps $t^*$,
\[
h''_{\lambda,t^*(v)(u-v)}(1) = -2 \|t^*(v)(u-v)\|_{H^1}^2 - (p - 4) \int_{\mathbb{R}^3} a |t^*(v)(u-v)|^p < 0,
\]
and
\[
J_\lambda(t^*(v)(u-v)) < A(p) \left( \frac{S_p}{a_\infty} \right)^{2/(p-2)}
\]
still hold if $\sigma$ is sufficiently small. Therefore, $t^*(v)(u-v) \in M^{(1)}_\lambda$ for all $v \in B(0; \sigma)$. This completes the proof. \qed

Then we have the following result.

**Proposition 13** Suppose that $2 < p < 4$ and $0 < \lambda < B(p) \Lambda_0$. Then there exists a sequence \( \{u_n\} \subset M^{(1)}_\lambda \) such that
\[
J_\lambda(u_n) = \alpha^- + o(1) \quad \text{and} \quad J'_\lambda(u_n) = o(1) \quad \text{in} \quad H^{-1}(\mathbb{R}^3).
\]

**Proof.** Applying Lemma 7 and the Ekeland variational principle [15], there exists a minimizing sequence \( \{u_n\} \subset M^{(1)}_\lambda \) such that
\[
J_\lambda(u_n) < \alpha^- + \frac{1}{n} \tag{19}
\]
and
\[
J_\lambda(u_n) \leq J_\lambda(w) + \frac{1}{n} \|w - u_n\|_{H^1} \quad \text{for all} \quad w \in M^{(1)}_\lambda. \tag{20}
\]
Applying Lemma 12 with $u = u_n$, there exists a function $t_n^* : B(0; \delta_n) \to \mathbb{R}$ for some $\delta_n > 0$ such that $t_n^*(w)(u_n - w) \in M^{(1)}_\lambda$. Let $0 < \delta < \delta_n$ and $u \in H^1(\mathbb{R}^3)$ with $u \neq 0$. We set $w_\delta = \frac{\delta u}{\|u\|_{H^1}}$ and $z_\delta = t_n^*(w_\delta)(u_n - w_\delta)$. Since $z_\delta \in M^{(1)}_\lambda$, we deduce from (20) that
\[
J_\lambda(z_\delta) - J_\lambda(u_n) \geq -\frac{1}{n} \|z_\delta - u_n\|_{H^1}.
\]
By the mean value theorem, we obtain
\[
\langle J'_\lambda(u_n), z_\delta - u_n \rangle + o(\|z_\delta - u_n\|_{H^1}) \geq -\frac{1}{n} \|z_\delta - u_n\|_{H^1}.
\]
Therefore,
\[
\langle J'_\lambda(u_n), -w_\delta \rangle + (t_n^*(w_\delta) - 1) \langle J'_\lambda(u_n), u_n - w_\delta \rangle \geq -\frac{1}{n} \|z_\delta - u_n\|_{H^1} + o(\|z_\delta - u_n\|_{H^1}). \tag{21}
\]
Now we observe that $t_n^*(w_\delta)(u_n - w_\delta) \in M^{(1)}_\lambda$ and, consequently, we derive from (21) that
\[
-\delta \langle J'_\lambda(u_n), \frac{u_n}{\|u_n\|_{H^1}} \rangle + \frac{(t_n^*(w_\delta) - 1)}{t_n^*(w_\delta)} \langle J'_\lambda(z_\delta), t_n^*(w_\delta)(u_n - w_\delta) \rangle + (t_n^*(w_\delta) - 1) \langle J'_\lambda(u_n) - J'_\lambda(z_\delta), u_n - w_\delta \rangle \geq -\frac{1}{n} \|z_\delta - u_n\|_{H^1} + o(\|z_\delta - u_n\|).
We rewrite the above inequality in the following form

$$
\left\langle J'_\lambda (u_n), \frac{u_n}{\|u_n\|_{H^1}} \right\rangle \leq \frac{\|z_\delta - u_n\|_{H^1}}{\delta n} + \frac{o\left(\|z_\delta - u_n\|_{H^1}\right)}{\delta} + \frac{(t^*_n(w_\delta) - 1)}{\delta} \langle J'_\lambda (u_n) - J_\lambda (z_\delta), u_n - w_\delta \rangle.
$$

(22)

We can find a constant $C > 0$, independent of $\delta$ such that

$$
\|z_\delta - u_n\|_{H^1} \leq \delta + C \left(\|t^*_n(w_\delta) - 1\| \right)
$$

and

$$
\lim_{\delta \to 0} \frac{|t^*_n(w_\delta) - 1|}{\delta} \leq \|\langle (t^*_n)'(0) \| \leq C
$$

for a fixed $n$. Let $\delta \to 0$ in (22) and using the fact that $\lim_{\delta \to 0} \|z_\delta - u_n\|_{H^1} = 0$, we obtain

$$
\left\langle J'_\lambda (u_n), \frac{u_n}{\|u_n\|_{H^1}} \right\rangle \leq \frac{C}{n},
$$

which implies that

$$
J_\lambda (u_n) = o(-1) + o(1) \quad \text{and} \quad J'_\lambda (u_n) = o(1) \quad \text{in} \quad H^{-1}(\mathbb{R}^3).
$$

This completes the proof. ■

3 The problem at infinity

In this section, we assume that $K(x) \equiv K_\infty > 0$ and $a(x) \equiv a_\infty > 0$. Now we consider the problem at infinity related to Eq. $(E_\lambda)$ as follows

$$
- \Delta u + u + \lambda K_\infty \phi_u(x)u = a_\infty |u|^{p-2} u.
$$

$(E_\infty)$

All solutions of Eq. $(E_\infty)$ are critical points of the functional $J_\infty^\lambda \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ defined by

$$
J_\infty^\lambda (u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,u}(x)u^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^3} a_\infty |u|^p \, dx.
$$

Define

$$
M_\infty^\lambda := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \left\langle (J_\infty^\lambda)'(u), u \right\rangle = 0 \},
$$

where $(J_\infty^\lambda)'$ denotes the Fréchet derivative of $J_\infty^\lambda$. Then $u \in M_\infty^\lambda$ if and only if $\|u\|_{H^1}^2 + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,u}u^2 - \int_{\mathbb{R}^3} a_\infty |u|^p = 0$. Note that $M_\infty^\lambda = M_\lambda$ with $K \equiv K_\infty$ and $a \equiv a_\infty$. Furthermore, define $M_\infty^{\lambda,(j)} = M_\lambda^{(j)}$ with $K \equiv K_\infty$ and $a \equiv a_\infty$ for $j = 1, 2$.

If $w_0$ is the unique positive solution of Eq. $(E_0^\infty)$, then

$$
\|w_0\|_{H^1}^2 = \int_{\mathbb{R}^3} a_\infty |w_0|^p \, dx = \left( \frac{\mathcal{S}_p}{a_\infty} \right)^{2/(p-2)},
$$

where

$$
\mathcal{S}_p := \min_{\{u \in H^1_0(\mathbb{R}^3) \mid \|u\|_{H^1} = 1\}} \left\{ \int_{\mathbb{R}^3} |u|^p \, dx \right\}.
$$

16
and
\[ \int_{\mathbb{R}^3} a_\infty |w_0|^p = \frac{a_\infty}{S_p^p} \| w_0 \|_{H^1}^p > \frac{a_\infty}{S_p^p} \left( 1 - A(p) \left( \frac{4 - p}{2} \right)^{2/(p-2)} \right)^{(p-2)/2} \| w_0 \|_{H^1}^p. \]

By Lemma \[11\] for each \( 2 < p < 4 \) and \( 0 < \lambda < B(p) \Lambda_0 \) there exist two constants \( t^{\infty,-}_\lambda \), and \( t^{\infty,+}_\lambda \) satisfying
\[ 1 < t^{\infty,-}_\lambda < \sqrt{A(p)} < t^{\infty,+}_\lambda, \]
such that \( t^{\infty,\pm}_\lambda w_0 \in M^{\infty,\pm}_\lambda \), where \( M^{\infty,\pm}_\lambda = M^\pm_\lambda \) with \( K \equiv K_{\infty} \) and \( a \equiv a_\infty \). Furthermore,
\[ J^{\infty}_\lambda (t^{\infty,-}_\lambda w_0) = \sup_{0 \leq t \leq t^{\infty,+}_\lambda} J^{\infty}_\lambda (tw_0), \]
and
\[ J^{\infty}_\lambda (t^{\infty,+}_\lambda w_0) = \inf_{t \geq t^{\infty,-}_\lambda} J^{\infty}_\lambda (tw_0) = \inf_{t \geq 0} J^{\infty}_\lambda (tw_0) < 0. \]

It is easy to see that
\[ J^{\infty}_\lambda (t^{\infty,-}_\lambda w_0) = \frac{1}{2} \| t^{\infty,-}_\lambda w_0 \|_{H^1}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty}, t^{\infty,-}_\lambda w_0} (t^{\infty,-}_\lambda w_0)^2 - \frac{1}{p} \int_{\mathbb{R}^3} a_\infty |t^{\infty,-}_\lambda w_0|^p \]
\[ = \frac{(t^{\infty,-}_\lambda)^2}{4} \left[ 1 - \frac{4 - p}{p} (t^{\infty,-}_\lambda)^{p-2} \right] \| w_0 \|_{H^1}^2 \]
\[ < A(p) \frac{p - 2}{2p} \| w_0 \|_{H^1}^2 \]
\[ = A(p) \frac{p - 2}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{2/(p-2)}, \]
which implies that \( t^{\infty,-}_\lambda w_0 \in M^{\infty,(1)}_\lambda \), i.e., \( M^{\infty,(1)}_\lambda \) is nonempty.

Define
\[ \alpha^{\infty,-}_\lambda = \inf_{u \in M^{\infty,(1)}_\lambda} J^{\infty}_\lambda (u) = \inf_{u \in M^{\infty,-}_\lambda} J^{\infty}_\lambda (u) \text{ for } 2 < p < 4, \]
and
\[ \alpha^{\infty,+}_\lambda = \inf_{u \in M^{\infty,(2)}_\lambda} J^{\infty}_\lambda (u) = \inf_{u \in M^{\infty,+}_\lambda} J^{\infty}_\lambda (u) \text{ for } 2 < p < 4. \]

Then by Lemma \[7\] and Theorem \[21\] in Appendix, for each \( 2 < p < 4 \) we obtain
\[ \frac{p - 2}{4p} \left( \frac{S_p^p}{a_{\text{max}}} \right)^{2/(p-2)} \leq \alpha^{\infty,-}_\lambda < A(p) \frac{p - 2}{2p} \left( \frac{S_p^p}{a_\infty} \right)^{2/(p-2)}, \quad (23) \]
and
\[ \alpha^{\infty,+}_\lambda = -\infty. \]

Next, we consider the energy functional \( J^{\infty}_0 \) in \( H^1 (\mathbb{R}^3) \) associated to Eq. \( (E^{\infty}_0) \),
\[ J^{\infty}_0 (u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) - \frac{1}{p} \int_{\mathbb{R}^3} a_\infty |u|^p. \quad (24) \]
Consider the minimizing problem:

\[ \inf_{u \in M_0^\infty} J_0^\infty (u) = \alpha_0^\infty, \]

where

\[ M_0^\infty = \{ u \in H^1 (\mathbb{R}^3) \setminus \{0\} \mid \langle (J_0^\infty)' (u), u \rangle = 0 \}. \]

We now begin the proof of Theorem 2. Let \( \{u_n\} \subset M_0^\infty \), (1) \( \lambda \) be a sequence satisfying

\[ J_0^\infty (u_n) = \alpha_0^\infty, \quad - \lambda + o(1). \]

Applying the concentration-compactness principle of P. L. Lions [18, 19] (or see Azzollini and Pomponio [3, Lemma 2.6]), for each \( \theta > 0 \) there exist a positive constant \( R = R(\theta) \) such that

\[ \int_{|B_N(0; R)|^c} |\nabla u_n (x)|^2 + u_n^2 (x) < \theta \text{ uniformly for } n \geq 1. \]

We define the new sequence of functions \( v_n := u_n (\cdot - z_n) \in H^1 (\mathbb{R}^3) \). It is easy to see that \( \phi_{v_n} = \phi_{u_n} (\cdot - z_n) \), which implies that \( \{v_n\} \subset M_0^\infty \) and \( J_\lambda^\infty (v_n) = \alpha_\lambda^\infty - o(1) \).

Moreover, by (26), for each \( \theta > 0 \) there exists a positive constant \( R = R(\theta) \) such that

\[ \int_{|B_N(0; R)|^c} |\nabla v_n (x)|^2 + v_n^2 (x) < \theta \text{ uniformly for } n \geq 1. \]

Since \( \{v_n\} \) is also bounded in \( H^1 (\mathbb{R}^3) \), we can assume that there exist a subsequence \( \{v_n\} \) and \( w_\lambda \in H^1 (\mathbb{R}^3) \) such that

\[ v_n \rightharpoonup w_\lambda \text{ weakly in } H^1 (\mathbb{R}^3); \]

\[ v_n \rightarrow w_\lambda \text{ strongly in } L^r_{\text{loc}} (\mathbb{R}^3) \text{ for } 2 \leq r < 6; \]

\[ v_n \rightarrow w_\lambda \text{ a.e. in } \mathbb{R}^3. \]

Using (27) – (29) and Fatou’s Lemma, for any \( \theta > 0 \) and \( n \geq 1 \) large enough, there exists a constant \( R > 0 \) such that

\[ \int_{\mathbb{R}^3} |v_n - w_\lambda|^p \leq \int_{B^R(0,R)} |v_n - w_\lambda|^p + \int_{[B^R(0,R)]^c} |v_n - w_\lambda|^p \]

\[ \leq \theta + S_p^{-p} \left[ \int_{[B^R(0,R)]^c} (|\nabla v_n (x)|^2 + v_n^2 (x)) + \int_{[B^R(0,R)]^c} (|\nabla w_\lambda (x)|^2 + w_\lambda^2 (x)) \right] \]

\[ \leq (1 + 2S_p^{-p}) \theta, \]

which implies that for every \( r \in (2, 6) \),

\[ v_n \rightarrow w_\lambda \text{ strongly in } L^r (\mathbb{R}^3). \]
Since $\phi : L^{12/5}(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ is a continuous function, it follows from (31) that
\[ \phi_{K_{\infty},v_n} \to \phi_{K_{\infty},w_{\lambda}} \text{ in } D^{1,2}(\mathbb{R}^3), \tag{32} \]
and
\[ \int_{\mathbb{R}^3} \phi_{K_{\infty},v_n} v_n^2 \to \int_{\mathbb{R}^3} \phi_{K_{\infty},w_{\lambda}} w_{\lambda}^2. \tag{33} \]
Since $\{v_n\} \subset \mathcal{M}_{\lambda}^{\infty,1}$, using (31) and (33), we have
\[ \int_{\mathbb{R}^3} a_{\infty} |w_{\lambda}|^p \geq \left( \frac{S_p}{a_{\infty}} \right)^{2/(p-2)} > 0, \]
which implies that $w_{\lambda} \neq 0$ and
\[ \int_{\mathbb{R}^3} a_{\infty} |w_{\lambda}|^p - \lambda \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty},w_{\lambda}} w_{\lambda}^2 \geq \|w_{\lambda}\|_{H^1}^2 > 0. \]
Now we prove that
\[ v_n \to w_{\lambda} \text{ strongly in } H^1(\mathbb{R}^3). \]
Suppose the contrary. Then
\[ \|w_{\lambda}\|_{H^1} < \liminf \|v_n\|_{H^1}. \tag{34} \]
By Lemma 11 there is a unique $t_{\lambda}^- > 0$ such that $t_{\lambda}^- w_{\lambda} \in \mathcal{M}_{\lambda}^{\infty,-}$ and $(h_{\lambda,w_{\lambda}}')' (t_{\lambda}^-) = 0$. So it follows from (31), (33) and (34) that $(h_{\lambda,v_n}')' (t_{\lambda}^-) > 0$ for $n$ sufficiently large. Since $v_n \in \mathcal{M}_{\lambda}^{\infty,1}$, we have $(h_{\lambda,v_n}')' (1) = 0$ and it is clear from Lemma 11 that $h_{\lambda,v_n}$ is increasing on $(t_{\lambda}^-,1)$ for $n$ sufficiently large. Hence, $h_{\lambda,v_n}(t_{\lambda}^-) < h_{\lambda,v_n}(1)$ for $n$ sufficiently large, which implies that
\[ J_{\lambda}^\infty(t_{\lambda}^- v_n) < J_{\lambda}^\infty(v_n) \text{ for } n \text{ sufficiently large.} \]
Thus, using (31), (33) and (34),
\[ J_{\lambda}^\infty(t_{\lambda}^- w_{\lambda}) < \liminf J_{\lambda}^\infty(t_{\lambda}^- v_n) \leq \liminf J_{\lambda}^\infty(v_n) = \alpha_{\lambda}^{\infty,-}, \]
which is a contradiction. Hence $v_n \to w_{\lambda}$ strongly in $H^1(\mathbb{R}^3)$, which implies that
\[ J_{\lambda}^\infty(v_n) \to J_{\lambda}^\infty(w_{\lambda}) = \alpha_{\lambda}^{\infty,-} \text{ as } n \to \infty. \]
Thus, $w_{\lambda}$ is a minimizer for $J_{\lambda}^\infty$ on $\mathcal{M}_{\lambda}^{\infty,-}$. Since $|w_{\lambda}| \in \mathcal{M}_{\lambda}^{\infty,1} \subset \mathcal{M}_{\lambda}^{\infty,-}$ and $J_{\lambda}^\infty(|w_{\lambda}|) = J_{\lambda}^\infty(w_{\lambda}) = \alpha_{\lambda}^{\infty,-}$, we may assume that $w_{\lambda}$ is a positive solution of Eq. (E$_{\lambda}$) by Lemma 6. Since $(4-p) \int_{\mathbb{R}^3} a_{\infty} |w_{\lambda}|^p < 2 \|w_{\lambda}\|_{H^1}^2$ and $t_{a_{\infty}}(w_{\lambda}) w_{\lambda} \in \mathcal{M}_0^{\infty}$, where
\[ \left( \frac{4-p}{2} \right)^{1/(p-2)} < t_{a_{\infty}}(w_{\lambda}) := \left( \frac{\|w_{\lambda}\|_{H^1}^2}{\int_{\mathbb{R}^3} a_{\infty} |w_{\lambda}|^p} \right)^{1/(p-2)} < 1, \tag{35} \]
using Lemma 11 we have
\[ \alpha_{\lambda}^{\infty,-} = J_{\lambda}^\infty(w_{\lambda}) > J_{\lambda}^\infty(t_0(w_{\lambda}) w_{\lambda}) \geq \alpha_{0}^{\infty} + \lambda |t_0(w_{\lambda})|^4 \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty},w_{\lambda}} w_{\lambda}^2 > \alpha_{0}^{\infty}. \tag{36} \]
This completes the proof.
4 Proof of Theorem 3

First, we define the Palais-Smale (or simply (PS)) sequences, (PS)-values, and (PS)-conditions in $H^1(\mathbb{R}^3)$ for $J_\lambda$ as follows.

Definition 14 (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a (PS)$_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $J_\lambda$ if $J_\lambda(u_n) = \beta + o(1)$ and $J'_\lambda(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^3)$ as $n \to \infty$.

(ii) $J_\lambda$ satisfies the (PS)$_\beta$-condition in $H^1(\mathbb{R}^3)$ if every (PS)$_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $J_\lambda$ contains a convergent subsequence.

Proposition 15 Let $\{u_n\}$ be a (PS)$_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $J_\lambda$. There exist a subsequence $\{u_n\}$, $m \in \mathbb{N}$, sequences $\{x_n^i\}_{i=1}^m$ in $\mathbb{R}^3$, and functions $v_0 \in H^1(\mathbb{R}^3)$, and $0 \neq w_i \in H^1(\mathbb{R}^3)$, for $1 \leq i \leq m$ such that

(i) $|x_n^i| \to \infty$ and $|x_n^i - x_n^j| \to \infty$ as $n \to \infty$, for $1 \leq i \neq j \leq m$;

(ii) $-\Delta v_0 + v_0 + \lambda K\phi_{K,v_0} v_0 = a |v_0|^{p-2} v_0$ in $\mathbb{R}^3$;

(iii) $-\Delta w^i + w^i + \lambda K_\infty \phi_{K_\infty,w} w = a_\infty |w^i|^{p-2} w^i$ in $\mathbb{R}^3$;

(iv) $u_n = v_0 + \sum_{i=1}^m w^i (\cdot - x_n^i) + o(1)$ strongly in $H^1(\mathbb{R}^3)$;

(v) $J_\lambda(u_n) = J_\lambda(v_0) + \sum_{i=1}^m J_\lambda^\infty(w^i) + o(1)$.

Proof. Similar to the argument in Lions [18, 19].

Corollary 16 Suppose that $\{u_n\} \subset M_\lambda^-$ is a (PS)$_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $J_\lambda$ with $0 < \beta < \alpha^\infty_\lambda$. Then there exists a subsequence $\{u_n\}$ and a non-zero $u_0$ in $H^1(\mathbb{R}^3)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ and $J_\lambda(u_0) = \beta$. Furthermore, $(u_0, \phi_{u_0})$ is a non-zero solution of Eq. $(E_\lambda)$.

By Theorem 2, we conclude that Eq. $(E_\lambda^\infty)$ admits a positive solution $w_\lambda(x) \in M_\lambda^\infty$ (up to translation) such that

$$J_\lambda^\infty(w_\lambda) = \alpha^\infty_\lambda < A(p) \frac{p-2}{2p} \left( \frac{S_p}{a_\infty} \right)^{2/(p-2)},$$

and $\int_{\mathbb{R}^3} a_\infty |w_\lambda|^p < \frac{2}{4-p} \|w_\lambda\|_{H^1}^2$. Furthermore, we define

$$\left( \frac{4-p}{2a_{\text{max}}} \right)^{1/(p-2)} \leq T_a(w_\lambda) := \left( \frac{\|w_\lambda\|_{H^1}^2}{\int_{\mathbb{R}^3} a |w_\lambda|^p} \right)^{1/(p-2)}.$$

Then we have the following results.
Lemma 17 Suppose that $2 < p < 4$ and $0 < \lambda < B(p) \Lambda_0$. Then there exists $t^\infty_\lambda > \left(\frac{2}{4-p}\right)^{1/(p-2)} t_{a,\infty}(w_\lambda) > 1$ such that

$$J^\infty_\lambda (w_\lambda) = \sup_{0 \leq t \leq t^\infty_\lambda} J^\infty_\lambda (tw_\lambda) = \alpha^{\infty,-}_\lambda,$$

where $t_{a,\infty}(w_\lambda)$ is defined as (35).

Proof. Let

$$b^\infty_\lambda(t) = t^{-2} \|w_\lambda\|^2_{H^1} - t^{p-4} \int_{\mathbb{R}^3} a_\infty |w_\lambda|^p$$

for $t > 0$. Clearly,

$$b^\infty_\lambda(1) + \lambda \int_{\mathbb{R}^3} K\infty \phi_{K\infty,w_\lambda} w_\lambda^2 = 0$$

for all $0 < \lambda < \frac{2\pi}{2} \Lambda_0$. It is easy to see that $b^\infty_\lambda (t_{a,\infty}(w_\lambda)) = 0$, $\lim_{t \to 0^+} b^\infty_\lambda(t) = \infty$ and $\lim_{t \to \infty} b^\infty_\lambda(t) = 0$. Since $2 < p < 4$ and

$$(b^\infty_\lambda)'(t) = -2t^{-3} \|w_\lambda\|^2_{H^1} + (4-p) t^{p-5} \int_{\mathbb{R}^3} a_\infty |w_\lambda|^p$$

we have $b^\infty_\lambda(t)$ is decreasing on $0 < t < \left(\frac{2}{4-p}\right)^{1/(p-2)} t_{a,\infty}(w_\lambda)$ and is increasing on $t > \left(\frac{2}{4-p}\right)^{1/(p-2)} t_{a,\infty}(w_\lambda)$. By (35) and (39), we have

$$\inf_{t > 0} b^\infty_\lambda(t) < -\lambda \int_{\mathbb{R}^3} K\infty \phi_{K\infty,w_\lambda} w_\lambda^2,$$

which implies that there exists $t^\infty_\lambda > \left(\frac{2}{4-p}\right)^{1/(p-2)} t_{a,\infty}(w_\lambda) > 1$ such that

$$b^\infty_\lambda (t^\infty_\lambda) + \lambda \int_{\mathbb{R}^3} K\infty \phi_{K\infty,w_\lambda} w_\lambda^2 = 0.$$

Similar to the argument in the proof of Lemma 11, we obtain

$$J^\infty_\lambda (w_\lambda) = \sup_{0 \leq t \leq t^\infty_\lambda} J^\infty_\lambda (tw_\lambda) = \alpha^{\infty,-}_\lambda.$$

This completes the proof. $$\blacksquare$$

Lemma 18 Suppose that $2 < p < 4$ and conditions (D1) − (D3) hold. Then for every $0 < \lambda < B(p) \Lambda_0$ there exist two constants $t^{(i)}_\lambda$, $i = 1, 2$ satisfying

$$T_a (w_\lambda) < t^{(1)}_\lambda < \sqrt{A(p)T_a (w_\lambda)} < \left(\frac{2}{4-p}\right)^{1/(p-2)} T_a (w_\lambda) < t^{(2)}_\lambda,$$
such that $t^{(i)}_{\lambda} w_{\lambda} \in M^{(i)}_{\lambda}$ for $i = 1, 2$ and

$$J_\lambda \left( t^{(1)}_{\lambda} w_{\lambda} \right) = \sup_{0 \leq t \leq t^{(2)}_{\lambda}} J_\lambda (tw_{\lambda}) < \alpha^{\infty-},$$

and

$$J_\lambda \left( t^{(2)}_{\lambda} w_{\lambda} \right) = \inf_{t \geq t^{(1)}_{\lambda}} J_\lambda (tw_{\lambda}).$$

**Proof.** Let

$$b_\lambda (t) = t^{-2} \left\| w_{\lambda} \right\|_{H^1}^2 - t^{p-4} \int_{\mathbb{R}^3} a \left| w_{\lambda} \right|^p \text{ for } t > 0.$$  

Clearly, $tw_{\lambda} \in M_\lambda$ if and only if $b_\lambda (t) + \lambda \int_{\mathbb{R}^3} K \phi_{K,w_{\lambda}} w_{\lambda}^2 = 0$. It is easy to see that $b_\lambda (T_a (w_{\lambda})) = 0$, limit$_{t \to 0^+}$ $b_\lambda (t) = \infty$ and limit$_{t \to \infty}$ $b_\lambda (t) = 0$. Since $2 < p < 4$ and

$$b'_\lambda (t) = t^{-3} \left( -2 \left\| w_{\lambda} \right\|_{H^1}^2 + (4 - p) t^{p-2} \int_{\mathbb{R}^3} a \left| w_{\lambda} \right|^p \right),$$

we have $b (t)$ is decreasing on $0 < t < \left( \frac{2}{2 - p} \right)^{1/(p-2)} T_a (w_{\lambda})$ and is increasing on $t > \left( \frac{2}{2 - p} \right)^{1/(p-2)} T_a (w_{\lambda})$. It follows from the condition (D3) that $T_a (w_{\lambda}) \leq T_{a_{\infty}} (w_{\lambda}) < 1$ and $b_\lambda (t) \leq b_\lambda^\infty (t)$, where $b_\lambda^\infty$ is as in (38). Moreover, again using the condition (D3) and (10), one has

$$\inf_{t > 0} b_\lambda (t) = b_\lambda \left( \left( \frac{2}{2 - p} \right)^{1/(p-2)} T_a (w_{\lambda}) \right) \leq - \frac{p - 2}{2 - p} \left( \frac{4 - p}{2} \right)^{2/(p-2)} \left\| w_{\lambda} \right\|_{H^1}^2 \left( \frac{\left\| w_{\lambda} \right\|_{H^1}^2}{\int_{\mathbb{R}^3} a_{\infty} \left| w_{\lambda} \right|^p} \right)^{-2/(p-2)}$$

$$= \inf_{t > 0} b_\lambda^\infty (t) \leq - \lambda \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty},w_{\lambda}} w_{\lambda}^2 \leq - \lambda \int_{\mathbb{R}^3} K \phi_{K,w_{\lambda}} w_{\lambda}^2.$$

Thus, there are $T_a (w_{\lambda}) < t^{(1)}_{\lambda} < \left( \frac{2}{2 - p} \right)^{1/(p-2)} T_a (w_{\lambda}) < t^{(2)}_{\lambda}$ such that

$$b_\lambda \left( t^{(i)}_{\lambda} \right) + \lambda \int_{\mathbb{R}^3} K \phi_{K,w_{\lambda}} w_{\lambda}^2 = 0 \text{ for } i = 1, 2,$$

that is $t^{(i)}_{\lambda} w_{\lambda} \in M_{\lambda}$ for $i = 1, 2$. Moreover,

$$h^{\mu}_{\lambda,t^{(1)}_{\lambda} w_{\lambda}} (1) = - 2 \left\| t^{(1)}_{\lambda} w_{\lambda} \right\|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a \left| t^{(1)}_{\lambda} w_{\lambda} \right|^p = \left( t^{(1)}_{\lambda} \right)^5 b' (t^{(1)}_{\lambda}) < 0$$

and

$$h^{\mu}_{\lambda,t^{(2)}_{\lambda} w_{\lambda}} (1) = - 2 \left\| t^{(2)}_{\lambda} w_{\lambda} \right\|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a \left| t^{(2)}_{\lambda} w_{\lambda} \right|^p = \left( t^{(2)}_{\lambda} \right)^5 b' (t^{(2)}_{\lambda}) > 0.$$
This shows that \( t_\lambda^{(1)} w_\lambda \in M^-_\lambda \) and \( t_\lambda^{(2)} w_\lambda \in M^+_\lambda \). Since

\[
 t_\lambda^{(1)} < \left( \frac{2}{4 - p} \right)^{\frac{1}{p - 2}} T_\alpha (w_\lambda) \leq \left( \frac{2}{4 - p} \right)^{\frac{1}{p - 2}} t_{a_\infty} (w_\lambda) < \min \left\{ \left( \frac{2}{4 - p} \right)^{\frac{1}{p - 2}}, t_\lambda^\infty \right\},
\]

where \( t_\lambda^\infty \) is as in Lemma \[17\] it follows from Lemma \[17\] and the condition \((D3)\) that for every \( 0 < \lambda < B (p) \Lambda_0 \),

\[
 J_\lambda \left( t_\lambda^{(1)} w_\lambda \right) = J_\lambda^\infty \left( t_\lambda^{(1)} w_\lambda \right) - \frac{[t(1)_\lambda]^p}{p} \int_{\mathbb{R}^3} (a - a_\infty) w_\lambda^p + \frac{\lambda}{4} \left[ t(1)_\lambda \right]^4 \left( \int_{\mathbb{R}^3} K \phi_{K,w_\lambda} w_\lambda^2 - \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty,w_\lambda}} w_\lambda^2 \right)
\]

\[
 \leq \sup_{0 \leq t \leq t_\lambda^\infty} J_\lambda^\infty (tw_\lambda) - \frac{[t(1)_\lambda]^p}{p} \int_{\mathbb{R}^3} (a - a_\infty) w_\lambda^p + \frac{\lambda}{4} \left[ t(1)_\lambda \right]^4 \left( \int_{\mathbb{R}^3} K \phi_{K,w_\lambda} w_\lambda^2 - \int_{\mathbb{R}^3} K_{\infty} \phi_{K_{\infty,w_\lambda}} w_\lambda^2 \right)
\]

\[
 < \alpha_\lambda^{\infty,-} < A (p) \frac{2}{2p} \left( \frac{S_p}{a_\infty} \right)^{2/(p-2)},
\]

which implies that \( t_\lambda^{(1)} w_\lambda \in M^{(1)}_\lambda \) and \( J_\lambda \left( t_\lambda^{(1)} w_\lambda \right) < \alpha_\lambda^{\infty,-} \). Moreover,

\[
 h_{\lambda,w_\lambda} (t) = t^3 \left( b_\lambda (t) + \lambda \int_{\mathbb{R}^3} K \phi_{K,w_\lambda} w_\lambda^2 \right),
\]

which implies that \( h_{\lambda,w_\lambda} (t) > 0 \) for all \( t \in (0, t_\lambda^{(1)}) \cup (t_\lambda^{(2)}, \infty) \) and \( h_{\lambda,w_\lambda} (t) < 0 \) for all \( t \in (t_\lambda^{(1)}, t_\lambda^{(2)}) \). Thus,

\[
 J_\lambda \left( t_\lambda^{(1)} w_\lambda \right) = \sup_{0 \leq t \leq t_\lambda^{(2)}} J_\lambda (tw_\lambda) \text{ and } J_\lambda \left( t_\lambda^{(2)} w_\lambda \right) = \inf_{t \geq t_\lambda^{(1)}} J_\lambda (tw_\lambda),
\]

this shows that \( J_\lambda \left( t_\lambda^{(2)} w_\lambda \right) \leq J_\lambda \left( t_\lambda^{(1)} w_\lambda \right) < \alpha_\lambda^{\infty,-} \) and so \( t_\lambda^{(2)} w_\lambda \in M^{(2)}_\lambda \). This completes the proof. ■

**We now begin the proof of Theorem \[3\]** By Proposition \[13\] there exists a sequence \( \{u_n\} \subset M^{(1)}_\lambda \) satisfying

\[
 J_\lambda (u_n) = \alpha^-_\lambda + o (1) \text{ and } J_\lambda' (u_n) = o (1) \text{ in } H^{-1} (\mathbb{R}^3).
\]

Thus, by Corollary \[16\] and Lemmas \[18\] and \[19\] we can conclude that Eq. \((E_\lambda)\) has a non-zero solution \( v_\lambda \in M^-_\lambda \) such that \( J_\lambda (v_\lambda) = \alpha^-_\lambda \). Thus, \( v_\lambda \) is a minimizer for \( J_\lambda \) on \( M^-_\lambda \). Since \( |v_\lambda| \in M^{(1)}_\lambda \subset M^-_\lambda \) and \( J_\lambda (|v_\lambda|) = J_\lambda (v_\lambda) = \alpha^-_\lambda \), by Lemma \[6\] we may assume that \( v_\lambda \) is a positive solution of Eq. \((E_\lambda)\).
5 Proof of Theorem 4

In the following the assumptions of Theorem 4 hold. First, we consider the minimizing problem:

$$\inf_{u \in M} J_\infty^\infty (u) = \alpha_0^\infty.$$ 

It is known that Eq. \((E_\infty^\infty)\) has a unique positive solution \(w_0(x)\) (up to translation) such that \(J_\infty^\infty (w_0) = \alpha_0^\infty = \frac{p-2}{2p} \left( \frac{S_p}{a_\infty} \right)^{2/(p-2)}\) and \(w_0(0) = \max_{x \in \mathbb{R}^3} w_0(x)\) (see [16, 29]). Define

$$\left( \frac{a_\infty}{a_{\text{max}}} \right)^{1/(p-2)} < T_a(w_0) := \left( \frac{\|w_0\|^2_{H^1}}{\int_{\mathbb{R}^3} a |w_0|^p} \right)^{1/(p-2)} < 1. \quad (41)$$

By the condition \((D4)\), we have

$$\int_{\mathbb{R}^3} a |w_0|^p > \int_{\mathbb{R}^3} a_\infty |w_0|^p = \frac{a_\infty}{S_p} \|w_0\|^p_{H^1}$$

$$> \frac{a_{\text{max}}}{S_p} \left( 1 - A(p) \left( \frac{(4-p)a_{\text{max}}}{2a_\infty} \right)^{2/(p-2)} \right)^{(p-2)/2} \|w_0\|^p_{H^1}.$$ 

Then we have the following result.

**Lemma 19** Suppose that \(2 < p < 4\) and conditions \((D1)-(D3)\) hold. Then there exists a positive number \(\Lambda_0 \leq B(p) \Lambda_0\) such that for every \(\lambda < \Lambda_0\) there are \(T_a(w_0) < t_\lambda^- < \sqrt{A(p)T_a(w_0)} < \left( \frac{2}{4-p} \right)^{1/(p-2)} T_a(w_0) < t_\lambda^+\)

such that \(t_\lambda^\pm w_0 \in M_\lambda^\pm\) and

$$J_\lambda \left( t_\lambda^- w_0 \right) = \sup_{0 \leq t \leq t_\lambda^-} J_\lambda (tw_0) < \begin{cases} a_\lambda^\infty, & \text{if } K_\infty > 0, \\ \alpha_0^\infty, & \text{if } K_\infty = 0, \end{cases}$$

and

$$J_\lambda \left( t_\lambda^+ w_0 \right) = \inf_{t \geq t_\lambda^-} J_\lambda (tw_0) < 0.$$ 

**Proof.** By Lemma 11 for each \(0 < \lambda < B(p) \Lambda_0\), there are \(T_a(w_0) < t_\lambda^- < \sqrt{A(p)T_a(w_0)} < \left( \frac{2}{4-p} \right)^{1/2(p-2)} T_a(w_0) < t_\lambda^+\)

such that \(t_\lambda^\pm w_0 \in M_\lambda^\pm\), and

$$J_\lambda \left( t_\lambda^- w_0 \right) = \sup_{0 \leq t \leq t_\lambda^-} J_\lambda (tw_0),$$

$$J_\lambda \left( t_\lambda^+ w_0 \right) = \inf_{t \geq t_\lambda^-} J_\lambda (tw_0).$$
Note that by Lemmas 7 and 19, there are positive constants which implies that there exists a positive number and for all .

Moreover, by the condition (D3) and (41), one has

\[
J_\lambda (t_\lambda^- w_0) = \inf_{t \geq t_\lambda^-} J_\lambda (tw_0) = \inf_{t \geq 0} J_\lambda (tw_0) < 0.
\]

We now begin the proofs of Theorems 4. Let \( \{u_n\} \subset M^{(1)}_\lambda \) be a sequence satisfying

\[
J_\lambda (u_n) = \alpha^-_\lambda + o(1) \quad \text{and} \quad J'_\lambda (u_n) = o(1) \quad \text{in} \quad H^{-1}(\mathbb{R}^3).
\]

Note that by Lemmas 7 and 19,

\[
0 < \frac{p - 2}{4p} \left( \frac{S_p}{a_\infty} \right)^{2/(p-2)} \leq \alpha^-_\lambda < \begin{cases} 
\alpha^-_\lambda \to 0^- & \text{if} \, K \geq 0, \\
\alpha^-_0 & \text{if} \, K \leq 0.
\end{cases}
\]

Since \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \), we can assume that there exists \( u_0 \in H^1(\mathbb{R}^3) \) such that

\[
u \begin{align*}
&u_n \rightharpoonup u_0 \quad \text{weakly in} \quad H^1(\mathbb{R}^3); \\
&u_n \rightarrow u_0 \quad \text{strongly in} \quad L^r_{\text{loc}}(\mathbb{R}^3) \quad \text{for} \ 2 \leq r < 6; \\
&u_n \rightarrow u_0 \quad \text{a.e. in} \quad \mathbb{R}^3.
\end{align*}
\]

First, we claim that \( u_0 \neq 0 \). Suppose otherwise, that is \( u_0 \equiv 0 \). Since \( \{u_n\} \subset M^{(1)}_\lambda \) and \( \alpha^-_\lambda > 0 \), we deduce from the Sobolev imbedding theorem that \( \|u_n\|_{H^1} > \nu > 0 \) for some constant \( \nu \) and for all \( n \). Applying the concentration-compactness principle of P. L. Lions [18] [19], there are positive constants \( R, \theta \) and a sequence \( \{z_n\} \subset \mathbb{R}^3 \) such that

\[
\int_{B^3(0,R)} |u_n(x + z_n)|^p \geq \theta \quad \text{for} \ n \text{ sufficiently large.}
\]
We will show that \( \{ z_n \} \) is an unbounded sequence in \( \mathbb{R}^3 \). Suppose the contrary, then we can assume that \( z_n \to z_0 \) for some \( z_0 \in \mathbb{R}^3 \). By (11) and the Sobolev inequality,

\[
\int_{B^3(z_0;R)} |u_0|^p \geq \theta,
\]

this contradicts \( u_0 \equiv 0 \). Thus, \( \{ z_n \} \) is an unbounded sequence in \( \mathbb{R}^3 \). Set \( \tilde{u}_n(z) = u_n(z + z_n) \). Then

\[
J_0^\infty (\tilde{u}_n) \to \alpha_\lambda^- \text{ and } (J_0^\infty)'(\tilde{u}_n) = o(1) \text{ in } H^{-1}(\mathbb{R}^3).
\]

Since \( \{ \tilde{u}_n \} \) is bounded in \( H^1(\mathbb{R}^3) \), we may assume that there exists \( \tilde{u}_0 \in H^1(\mathbb{R}^3) \) such that

\[
\tilde{u}_n \rightharpoonup \tilde{u}_0 \text{ weakly in } H^1(\mathbb{R}^3) \text{ and } (J_0^\infty)'(\tilde{u}_0) = 0. \tag{47}
\]

From (26), we have \( \tilde{u}_0 \geq 0 \) and \( \tilde{u}_0 \not\equiv 0 \) in \( \mathbb{R}^3 \). Thus, \( \tilde{u}_0 \in M_0^\infty \) and \( \alpha_0^- \leq J_0^\infty (\tilde{u}_0) \leq \alpha_\lambda^- \), a contradiction. Thus, \( u_0 \not\equiv 0 \) and \( J_\lambda (u_0) = 0 \) in \( H^{-1}(\mathbb{R}^3) \), which implies that \( u_0 \) is a nontrivial solution of Eq. \( (E_\lambda) \). Since \( \{ u_n \} \subset M_\lambda^{(1)} \) and

\[
\| u_n \|_{H^1} < \hat{D}_1 < \left[ \frac{2 S_p}{(4 - p) a_{\max}} \right]^{1/(p-2)} \text{ for all } n = 1, 2, \ldots,
\]

using Fatou’s lemma, we have

\[
\| u_0 \|_{H^1} \leq \liminf \| u_n \|_{H^1} \leq \hat{D}_1 < \left[ \frac{2 S_p}{(4 - p) a_{\max}} \right]^{1/(p-2)}.
\]

Thus, by (11) and the Sobolev inequality,

\[
h'_{u_0} (1) = -2 \| u_0 \|^2_{H^1} + (4 - p) \int_{\mathbb{R}^3} a |u_0|^p \\
\leq -2 \| u_0 \|^2_{H^1} + (4 - p) a_{\max} S_p^{-p} \| u_0 \|^p_{H^1} \\
< 0,
\]

which implies that \( u_0 \in M_\lambda^- \) and \( J_\lambda (u_0) \geq \alpha_\lambda^- \). Now we will show that \( u_n \to u_0 \) strongly in \( H^1(\mathbb{R}^3) \). Suppose the contrary, then there exists \( \xi > 0 \) such that \( \| u_n - u_0 \|_{H^1} > \xi \).

Let \( v_n = u_n - u_0 \). Then by (28) – (30),

\[
v_n &\rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^3); \tag{48}
\]

\[
v_n \to 0 \text{ strongly in } L^r_{loc}(\mathbb{R}^3) \text{ for } 2 \leq r < 6; \tag{49}
\]

\[
v_n \to 0 \text{ a.e. in } \mathbb{R}^3.
\]

Moreover, by the conditions \( (D1) - (D2) \) and (25),

\[
\| v_n \|^2_{H^1} + \int_{\mathbb{R}^3} K_\infty \phi K_\infty v_n v_n^2 = \int_{\mathbb{R}^3} a_\infty |v_n|^p + o(1).
\]

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Thus, by Brezis-Lieb lemma [6] and [30] Lemma 2.2, we have

\[ J_\lambda (u_n) = \frac{1}{2} \| u_n \|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^3} K \phi_{K,u_n} u_n^2 - \frac{1}{p} \int_{\mathbb{R}^3} a |u_n|^p \]
\[ = \frac{1}{2} \| u_0 \|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^3} K \phi_{K,u_0} u_0^2 - \frac{1}{p} \int_{\mathbb{R}^3} a |u_0|^p \]
\[ + \frac{1}{2} \| v_n \|_{H^1}^2 + \frac{1}{4} \int_{\mathbb{R}^3} K_\infty \phi_{K,\infty} v_n^2 - \frac{1}{p} \int_{\mathbb{R}^3} a_\infty |v_n|^p + o(1) \]
\[ \geq \begin{cases} \alpha^- + \alpha^-_\infty + o(1), & \text{if } K_\infty > 0, \\ \alpha^- + \alpha^-_0 + o(1), & \text{if } K_\infty = 0, \end{cases} \]

which implies that

\[ \lim_{n \to \infty} J_\lambda (u_n) = \alpha^- \geq \begin{cases} \alpha^- + \alpha^-_\infty, & \text{if } K_\infty > 0, \\ \alpha^- + \alpha^-_0, & \text{if } K_\infty = 0, \end{cases} \]

a contradiction. Therefore, \( u_n \to u_0 \) strongly in \( H^1 (\mathbb{R}^3) \) and \( J_\lambda (u_0) = \alpha^- \). Since \( |u_0| \in M^- \) and \( J_\lambda (|u_0|) = J_\lambda (u_0) = \alpha^- \), by Lemma [6] we may assume that \( u_0 \) is a positive solution of Eq. \( (E_\lambda) \). This completes the proof.

6 Ground State solutions

Lemma 20 Suppose that \( \frac{1 + \sqrt{74}}{3} < p < 4 \) and the condition \( (D_{K,a}) \) holds. Let \( u_0 \) is a non-trivial solution of Eq. \( (E_\lambda) \). Then \( u_0 \in M^- \).

Proof. Since \( u_0 \) is a non-trivial solution of Eq. \( (E_\lambda) \),

\[ \int_{\mathbb{R}^3} |\nabla u_0|^2 + \int_{\mathbb{R}^3} u_0^2 + \lambda \int_{\mathbb{R}^3} K \phi_{K,u_0} u_0^2 - \int_{\mathbb{R}^3} a |u_0|^p = 0. \]

(51)

It is easy to obtain the Pohozaev identity related to Eq. \( (E_\lambda) \) as follows,

\[ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_0|^2 + \frac{3}{2} \int_{\mathbb{R}^3} u_0^2 + \frac{5}{4} \lambda \int_{\mathbb{R}^3} K \phi_{K,u_0} u_0^2 + \lambda \frac{2}{3} \int_{\mathbb{R}^3} (\nabla K, x) \phi_{K,u_0} u_0^2 \\
= \int_{\mathbb{R}^3} a |u_0|^p + \frac{1}{p} \int_{\mathbb{R}^3} (\nabla a, x) |u_0|^p, \]

(52)

(or see the proof of Theorem 1.3 in [13]). If we make \( \frac{51}{p} \times \frac{2}{p} \) minus \( \frac{52}{p} \), then we obtain

\[ \frac{6 - p}{2p} \int_{\mathbb{R}^3} |\nabla u_0|^2 - \frac{3(p - 2)}{2p} \int_{\mathbb{R}^3} u_0^2 \\
= \frac{5p - 12}{4p} \lambda \int_{\mathbb{R}^3} K \phi_{K,u_0} u_0^2 + \lambda \frac{2}{3} \int_{\mathbb{R}^3} (\nabla K, x) \phi_{K,u_0} u_0^2 - \frac{1}{p} \int_{\mathbb{R}^3} (\nabla a, x) |u_0|^p, \]

which implies that

\[ \int_{\mathbb{R}^3} |\nabla u_0|^2 = \frac{3(p - 2)}{2p} \int_{\mathbb{R}^3} u_0^2 + \lambda \int_{\mathbb{R}^3} \left( \frac{5p - 12}{2(6 - p)} K + \frac{1}{2} (\nabla K, x) \right) \phi_{K,u_0} u_0^2 \\
- \frac{1}{p} \int_{\mathbb{R}^3} (\nabla a, x) |u_0|^p. \]

(53)
By (11), (53) and the condition \((D_{K,a})\), one has

\[
\begin{align*}
\mathcal{H}_{x,u_0}^{\lambda} (1) &= -(p - 2) \| u_0 \|_{H^1}^2 + (4 - p) \lambda \int_{\mathbb{R}^3} K \phi_{K,a_0} u_0^2 \\
&= -(p - 2) \int_{\mathbb{R}^3} |\nabla u_0|^2 - (p - 2) \int_{\mathbb{R}^3} u_0^2 + (4 - p) \lambda \int_{\mathbb{R}^3} K \phi_{K,a_0} u_0^2 \\
&= \frac{-2p(p - 2)}{6 - p} \int_{\mathbb{R}^3} u_0^2 - \lambda \int_{\mathbb{R}^3} \left( \frac{3p^2 - 2p - 24}{2(6 - p)} K + \frac{p - 2}{2} \langle \nabla K, x \rangle \right) \phi_{K,a_0} u_0^2 \\
&\quad + \frac{p - 2}{p} \int_{\mathbb{R}^3} \langle \nabla a, x \rangle |u_0|^p \\
&< 0.
\end{align*}
\]

This completes the proof. □

We now begin the proof of Theorem 5. Let \(v_\lambda\) be a positive solution of Eq. \((E_\lambda)\) such that \(v_\lambda \in M^-_\lambda\) and \(J_\lambda (v_\lambda) = \alpha^-_\lambda\). Then by Lemma 20, \(v_\lambda\) be a ground state solution of Eq. \((E_\lambda)\).

7 Appendix

Throughout this section, we assume that the conditions \((D_1)\) and \((D_2)\) hold. We denote by \(w_0\) be the unique positive solution with \(w_0(0) = \max_{x \in \mathbb{R}^3} w_0(x)\) for the following nonlinear Schrödinger equation:

\[
-\Delta u + u = a_\infty |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^3.
\]  

\((E_0^\infty)\)

If \(K(x) \equiv K_\infty\) and \(a(x) \equiv a_\infty\), then

\[
T_a (w_0) = T_{a_\infty} (w_0) = \left( \frac{\| w_0 \|_{H^1}^2}{\int_{\mathbb{R}^3} a_\infty |w_0|^p} \right)^{1/(p-2)} = 1,
\]

and

\[
\int_{\mathbb{R}^3} a_\infty |w_0|^p = \frac{a_\infty}{S_p} \| w_0 \|_{H^1}^p.
\]  

\((54)\)

Moreover, by Lemmas 10 and 11 there exists \(\left( \frac{p}{4-p} \right)^{1/(p-2)} < t^+_\lambda < \hat{t}_0 (w_0)\) such that

\[
J_\lambda^\infty (t^+_\lambda w_0) = \inf_{t \in (\frac{p}{4-p})^{1/(p-2)} < t < \hat{t}_0 (w_0)} J_\lambda^\infty (tw_0) = \inf_{t \geq 0} J_\lambda^\infty (tw_0) < 0.
\]

For \(R > 1\), we define a function \(\psi_R \in C^1 (\mathbb{R}^3, [0, 1])\) such that

\[
\psi_R (x) = \begin{cases} 
1 & |x| < \frac{R}{2}, \\
0 & |x| > \frac{R}{2}.
\end{cases}
\]

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and $|\nabla \psi_R| \leq 1$ in $\mathbb{R}^3$. Let $u_R(x) = w_0(x) \psi_R(x)$. Then

$$\int_{\mathbb{R}^3} |u_R|^p \to \int_{\mathbb{R}^3} |w_0|^p \text{ as } R \to \infty,$$

$$\|u_R\|_{H^1} \to \|w_0\|_{H^1} \text{ as } R \to \infty,$$

and

$$\int_{\mathbb{R}^3} K_\infty \phi_{K_R} u_R^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K_\infty u_R^2(x) u_R^2(y)}{|x-y|} dxdy \to \int_{\mathbb{R}^3} K_\infty \phi_{K_R} w_0^2 \text{ as } R \to \infty.$$

Since $J_\lambda^\infty \in C^1 (H^1(\mathbb{R}^3), \mathbb{R})$, by (54) - (56), there exists $R_0 > 0$ such that

$$\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p > \frac{2a_\infty}{ps_p} \|u_{R_0}\|_{H^1}^p$$

and

$$J_\lambda^\infty (t^+_\lambda u_{R_0}) < 0.$$

Let

$$u_{R_0,N}^{(i)}(x) = w_0(x + iN^3e) \psi_R(x + iN^3e)$$

for $e \in \mathbb{S}^2$ and $i = 1, 2, \ldots, N$, where $N^3 > 2R_0$. Then by the condition (D1),

$$\left\| u_{R_0,N}^{(1)} \right\|^2_{H^1} = \left\| u_{R_0} \right\|^2_{H^1} \text{ for all } N,$$

$$\int_{\mathbb{R}^3} a \left| u_{R_0,N}^{(1)} \right|^p \to \int_{\mathbb{R}^3} a_\infty \left| u_{R_0} \right|^p \text{ as } N \to \infty,$$

$$\int_{\mathbb{R}^3} K_\phi \phi_{K_{R_0,N}} u_{R_0,N}^{(1)}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x) K(y) \left[ u_{R_0,N}^{(1)}(x) \right]^2 \left[ u_{R_0,N}^{(1)}(y) \right]^2}{|x-y|} dxdy$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x-N^3e) K(y-N^3e) u_{R_0}^2(x) u_{R_0}^2(y)}{|x-y|} dxdy$$

$$\to K_\infty^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{R_0}^2(x) u_{R_0}^2(y)}{|x-y|} dxdy \text{ as } N \to \infty.$$

Moreover, if $a \geq a_\infty$ and $K \leq K_\infty$, then there exists $N_0 > 0$ with $N_0^3 > 2R_0$ such that for every $N \geq N_0$,

$$\int_{\mathbb{R}^3} a \left| u_{R_0,N}^{(i)} \right|^p \geq \int_{\mathbb{R}^3} a_\infty \left| u_{R_0} \right|^p > \frac{2a_\infty}{ps_p} \|u_{R_0}\|_{H^1}^p = \frac{2a_\infty}{ps_p} \left\| u_{R_0,N}^{(i)} \right\|_{H^1}^p$$

and

$$\inf_{t \geq 0} J_\lambda \left( t u_{R_0,N}^{(i)} \right) \leq J_\lambda \left( t^+_\lambda u_{R_0,N}^{(i)} \right) < J_\lambda^\infty \left( t^+_\lambda u_{R_0} \right).$$

for all $e \in \mathbb{S}^2$ and $i = 1, 2, \ldots, N$, since $iN^3 \geq N^3$.  

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Let
\[ w_{R_0,N}(x) = \sum_{i=1}^{N} u_{R_0,N}^{(i)}. \]

Observe that \( w_{R_0,N} \) is a sum of translation of \( u_{R_0} \), and if \( N^3 \geq N_0^3 > 2R_0 \) the summands have disjoint support. In such case we have:
\[
\|w_{R_0,N}\|_{H^1}^2 = N \|u_{R_0}\|_{H^1}^2, \tag{58}
\]
\[
\int_{\mathbb{R}^3} a |w_{R_0,N}|^p = \sum_{i=1}^{N} \int_{\mathbb{R}^3} a |u_{R_0,N}^{(i)}|^p \tag{59}
\]
and
\[
\int_{\mathbb{R}^3} K\phi_{K,w_{R_0,N}}w_{R_0,N}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x) K(y) w_{R_0,N}^2(x) w_{R_0,N}^2(y) \frac{dx dy}{|x-y|}
\]
\[
= \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x) K(y) \left[u_{R_0,N}^{(i)}\right]^2(x) \left[u_{R_0,N}^{(i)}\right]^2(y) \frac{dx dy}{|x-y|}
\]
\[
+ \sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x) K(y) \left[u_{R_0,N}^{(i)}\right]^2(x) \left[u_{R_0,N}^{(j)}\right]^2(y) \frac{dx dy}{|x-y|}. \tag{60}
\]

Now we compute:
\[
\sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x) K(y) \left[u_{R_0,N}^{(i)}\right]^2(x) \left[u_{R_0,N}^{(j)}\right]^2(y) \frac{dx dy}{|x-y|} \leq \frac{N^2 - N}{N^3 - 2R_0} K_{\text{max}}^2 \left(\int_{\mathbb{R}^3} w_0^2(x) \, dx\right)^2,
\]
which implies that
\[
\sum_{i \neq j}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x) K(y) \left[u_{R_0,N}^{(i)}\right]^2(x) \left[u_{R_0,N}^{(j)}\right]^2(y) \frac{dx dy}{|x-y|} \to 0 \text{ as } N \to \infty. \tag{61}
\]

Then following the idea of Ruiz [23], we have the following result.

**Theorem 21** Suppose that \( 2 < p < 4 \) and \( 0 < \lambda < B(p) \Lambda_0 \). Let \( K(x) \leq K_{\infty} \) and \( a(x) \geq a_\infty \). Then
\[
\alpha_\lambda^+ = \inf_{u \in M_\lambda^{(2)}} J_{\lambda,K,a} (u) = \inf_{u \in M_\lambda^{(1)}} J_\lambda (u) = -\infty.
\]

**Proof.** For \( N \in \mathbb{N} \) and let
\[
f_N(t) = t^{-2} \left\|w_{R_0,N}\right\|_{H^1}^2 - t^{p-4} \int_{\mathbb{R}^3} a |w_{R_0,N}|^p \text{ for } t > 0
\]

and
\[ f_\infty(t) = t^{-2} \|u_{R_0}\|_{H^1}^2 - t^{p-4} \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p \text{ for } t > 0. \]

Then by (58) and (59),
\[
\begin{align*}
  f_N(t) &= t^{-2} N \|u_{R_0}\|_{H^1}^2 - t^{p-4} \sum_{i=1}^N \int_{\mathbb{R}^3} a |u_{R_0,N}^{(i)}|^p \\
  &\leq t^{-2} N \|u_{R_0}\|_{H^1}^2 - t^{p-4} N \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p = N f_\infty(t) .
\end{align*}
\]

(62)

Clearly, \( t w_{R_0,N} \in M_\lambda \) if and only if \( f_N(t) + \lambda \int_{\mathbb{R}^3} K_\mu w_{R_0,N}^2 w_{R_0,N}^2 = 0 \). It is easy to see that \( f_\infty(t_0(u_{R_0})) = 0 \), \( \lim_{t \to 0^+} f_\infty(t) = \infty \) and \( \lim_{t \to \infty} f_\infty(t) = 0 \), where
\[
t_0(u_{R_0}) = \left( \frac{\|u_{R_0}\|_{H^1}^2}{\int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p} \right)^{1/(p-2)}.
\]

Since \( 2 < p < 4 \) and
\[
(f_\infty)'(t) = -2t^{-3} \|u_{R_0}\|_{H^1}^2 + (4 - p) t^{p-5} \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p ,
\]
we have \( f_\infty \) is decreasing on \( 0 < t < \left( \frac{2 \|u_{R_0}\|_{H^1}^2}{(4 - p) \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p} \right)^{1/(p-2)} \) and is increasing on \( t > \left( \frac{2 \|u_{R_0}\|_{H^1}^2}{(4 - p) \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p} \right)^{1/(p-2)} \). Then by (57),
\[
\inf_{t > 0} f_\infty(t) = f_\infty \left( \left( \frac{2 \|u_{R_0}\|_{H^1}^2}{(4 - p) \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p} \right)^{1/(p-2)} \right) \\
= -\frac{p - 2}{2(4 - p)} \left( \frac{(4 - p) \int_{\mathbb{R}^3} a_\infty |u_{R_0}|^p}{2 \|u_{R_0}\|_{H^1}^2} \right)^{2/(p-2)} \|u_{R_0}\|_{H^1}^2 \\
< -\frac{p - 2}{2(4 - p)} \left( \frac{(4 - p)^2 a_\infty S_p^{-p} \|u_{R_0}\|_{H^1}^2}{2 \|u_{R_0}\|_{H^1}^2} \right)^{2/(p-2)} \|u_{R_0}\|_{H^1}^2 \\
= -\frac{p - 2}{2(4 - p)} \left( \frac{(4 - p)^2 a_\infty}{p S_p^p} \right)^{2/(p-2)} \|u_{R_0}\|_{H^1}^4 .
\]

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Moreover, by Lemma 8 and (62), for any $0 < \lambda < B(p) \Lambda_0$, we have

\[
\inf_{t > 0} f_N(t) \leq f_N \left( \left( \frac{2 \left\| u_{R_0} \right\|_{H^1}^2}{(4 - p) \int_{\mathbb{R}^3} a_\infty \left| u_{R_0} \right|^p} \right)^{1/(p-2)} \right)
\]

\[
\leq - \lambda \frac{p - 2}{2(4 - p)} \left( \frac{4 - p}{p} \right)^{(p-2)/2} \left\| u_{R_0} \right\|_{H^1}^2
\]

\[
< - \lambda N \frac{p - 2}{2} \left( \frac{4 - p}{p} \right)^{(p-2)/2} \left\| u_{R_0} \right\|_{H^1}^4
\]

\[
< - \lambda N \frac{p}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K^2 u_{R_0}^2(x) u_{R_0}^2(y)}{|x - y|} dx dy,
\]

using (61), we can conclude that

\[
\inf_{t > 0} f_N(t) < - \lambda \left[ N \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K^2 u_{R_0}^2(x) u_{R_0}^2(y)}{|x - y|} dx dy + \sum_{i \neq j} N \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K^2 u_{R_0}^{(i)}(x) u_{R_0}^{(j)}(y)}{|x - y|} dx dy \right]
\]

\[
= - \lambda \int_{\mathbb{R}^3} K \phi_{K,w_{R,N}} w_{R,N}^2 \quad \text{for } \lambda \text{ sufficiently large.}
\]

Thus, for every $0 < \lambda < B(p) \Lambda_0$, there exist

\[
1 < t_{\lambda,N}^{(1)} < \left( \frac{2 \left\| u_{R_0} \right\|_{H^1}^2}{(4 - p) \int_{\mathbb{R}^3} a_\infty \left| u_{R_0} \right|^p} \right)^{1/(p-2)} < t_{\lambda,N}^{(2)}
\]

such that $f_N\left(t_{\lambda,N}^{(i)}\right) + \lambda \int_{\mathbb{R}^3} K \phi_{K,w_{R,N}} w_{R,N}^2 = 0$ for $i = 1, 2$ and for all $N \in \mathbb{N}$, that is $t_{\lambda,N}^{(i)} w_{R,N} \in M_\lambda$ for $i = 1, 2$ and for all $N \in \mathbb{N}$. Moreover,

\[
h''_{\lambda t_{\lambda,N}^{(1)}} w_{R,N} (1) = -2 \left\| t_{\lambda,N}^{(1)} w_{R,N} \right\|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a \left| t_{\lambda,N}^{(1)} w_{R,N} \right|^p dx
\]

\[
= \left( t_{\lambda,N}^{(1)} \right)^5 f_N' \left( t_{\lambda,N}^{(1)} \right) < 0,
\]

and

\[
h''_{\lambda t_{\lambda,N}^{(2)}} w_{R,N} (1) = -2 \left\| t_{\lambda,N}^{(2)} w_{R,N} \right\|_{H^1}^2 + (4 - p) \int_{\mathbb{R}^3} a \left| t_{\lambda,N}^{(2)} w_{R,N} \right|^p dx
\]

\[
= \left( t_{\lambda,N}^{(2)} \right)^5 f_N' \left( t_{\lambda,N}^{(2)} \right) > 0.
\]

Therefore, we can conclude that $t_{\lambda,N}^{(1)} w_{R,N} \in M^-_\lambda$ and $t_{\lambda,N}^{(2)} w_{R,N} \in M^+_\lambda$. Moreover, by (58) – (61), we have

\[
J_\lambda \left( t_{\lambda,N}^{(2)} w_{R,N} \right) = \inf_{t > 0} J_\lambda(t w_{R,N}) \leq J_\lambda \left( t_{\lambda,N}^{(1)} w_{R,N} \right)
\]

\[
\leq N J_\lambda^\infty \left( t_{\lambda,N}^{(1)} u_{R_0} \right) + C_0 \quad \text{for some } C_0 > 0,
\]

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and so \( J_{\lambda} \left( t_{\lambda,N}^{(2)}, w_{R,N} \right) \to -\infty \) as \( N \to \infty \). Therefore,

\[
\alpha_{\lambda}^+ = \inf_{u \in M_{\lambda}^{(2)}} J_{\lambda}(u) = \inf_{u \in M_{\lambda}^+} J_{\lambda}(u) = -\infty.
\]

This completes the proof. ■

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