Another look at Zagier’s formula for multiple zeta values involving Hoffman elements

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Abstract
In this paper, we give an elementary account into Zagier’s formula for multiple zeta values involving Hoffman elements. Our approach allows us to obtain direct proof in a special case via rational zeta series involving the coefficient $\zeta(2n)$. This formula plays an important role in proving Hoffman’s conjecture which asserts that every multiple zeta value of weight $k$ can be expressed as a $\mathbb{Q}$-linear combinations of multiple zeta values of the same weight involving 2’s and 3’s. Also, using a similar hypergeometric argument via rational zeta series, we produce a new Zagier-type formula for the multiple special Hurwitz zeta values.

Keywords Multiple zeta values · Zagier’s formula for Hoffman elements · Riemann zeta function · Clausen function · Gauss hypergeometric function

Mathematics Subject Classification Primary 11M06 · 11M32; Secondary 11B65 · 11B68

1 Introduction

Multiple zeta values, usually abbreviated MZV’s (sometimes they are called Euler–Zagier sums), are real numbers defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{0 < n_1 < n_2 < \ldots < n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}},$$

where $k_1, k_2, \ldots, k_r$ are positive integers with $k_r > 1$.

Originally defined by Euler for $r = 2$, the theory of multiple zeta values was developed at the beginning of 90’s independently by Hoffman [12] and Zagier [23,24]. For more details about the theory of multiple zeta functions and their special values we recommend

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the monographs [7, 27]. Also, the multiple zeta values can be defined as nested sums,

\[ \zeta(k_1, k_2, \ldots, k_r) = \sum_{n_r=1}^{\infty} \frac{1}{n_r^{k_r}} \sum_{n_{r-1}=1}^{n_r-1} \frac{1}{n_{r-1}^{k_{r-1}}} \cdots \sum_{n_2=1}^{n_r-1} \frac{1}{n_2^{k_2}} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^{k_1}}. \]

We call the above series a multiple zeta of depth \( r \) and weight \( k \), where \( k = k_1 + k_2 + \ldots + k_r \). Obviously, \( 0 < r < k \) and there are \( \binom{k-2}{r-1} \) multiple zeta values of given weight \( k \) and depth \( r \).

Although they look simple, it seems that these numbers have connections with Galois representation theory with applications in computing Feynman integrals in quantum field theory. Moreover, it is not known if MZV’s are \( \mathbb{Q} \)-linear combinations of powers of \( \pi \) and odd values of the Riemann zeta function. In this regard, there are many conjectures concerning these values. One of them, which was conjectured by Hoffman [13] in 1997, and solved by Brown [5] in 2012, asserts that that every multiple zeta value of a given weight \( k \) can be expressed as a \( \mathbb{Q} \)-linear combination of MZV’s of the same weight involving 2’s and 3’s.

The proof of this conjecture which is of motivic flavor relied on the evaluation of a certain family of MZV’s involving Hoffman elements. In other words, Brown [5] showed that the following family of MZV’s,

\[ H(r, s) = \zeta(2, 2, \ldots, 2, 3, 2, 2, \ldots, 2) \]

can be expressed as a \( \mathbb{Q} \)-linear combination of products \( \pi^{2m} \zeta(2n+1) \), with \( m+n = r+s+1 \).

Unfortunately, Brown could not give an explicit formula for \( H(r, s) \). This gap was filled by Zagier [25] who provided the exact formula and its proof.

In this paper, we give a different approach to Zagier’s formula (for the special case \( s = 0 \)) using rational zeta series involving the values \( \zeta(2n) \). The main idea is to look at the even powers in the Taylor series of arcsin and then to express these rational zeta series as a \( \mathbb{Q} \)-linear combination of powers of \( \pi \) and \( \zeta(2n+1) \). Moreover, a Zagier-type formula for multiple \( t \)-values is provided. In our approach, we use properties of special functions, e.g. the Clausen function whose values are related to the values of the Riemann zeta function. More details are given in the next section.

This paper is organized as follows. In Sect. 2 we collect some basic tools about rational zeta series, the Gauss hypergeometric function and the Taylor series expansion of integer powers of arcsin which are used in the proof of the main result of the Sect. 2. Also, we give a proof of Zagier’s formula for the special case \( s = 0 \) using this rational zeta series approach. In Sect. 3, we give a Zagier type formula for the multiple \( t \)-values using the same approach as in Sect. 3. Finally, in the appendix, we provide some basic properties of the Clausen function which served as the main ingredient in the proof of Lemma 2.6.

2 Zagier’s formula for Hoffman elements

The formula for \( H(r, s) \) was provided by Zagier [25] in the following

**Theorem 2.1** The following formula holds true:

\[ H(r, s) := \zeta(\underbrace{2, 2, \ldots, 2}_{r}, 3, \underbrace{2, 2, \ldots, 2}_{s}) = 2 \sum_{k=1}^{r+s+1} (-1)^k c_{r,s} \zeta(2k+1) \zeta(\underbrace{2, 2, \ldots, 2}_{r+s+1-k}), \]

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Another look at Zagier’s formula for multiple zeta values

where \( c_{r,s}^k = \binom{2k}{2r+2} - \left( 1 - \frac{1}{2^{2k}} \right) \binom{2k}{2r+1} \) and \( r, s \geq 0 \) are integers.

As we mentioned in the introduction, this formula served as the missing ingredient in Brown’s proof of Hoffman’s conjecture. Moreover, this played a key role in proving the zig-zag conjecture of Broadhurst [6] by Brown and Schnetz.

Also, it is worth to mention that Brown’s theorem (Hoffman’s conjecture) implies that if \( Z_k \) is the \( \mathbb{Q} \)-vector space spanned by all multiple zeta values of weight \( k \), then

\[
\dim Z_k \leq d_k,
\]

where \( d_k \) is the coefficient of \( x^k \) in the powers series expansion of \( \frac{1}{1 - x^2 - x^3} \).

The proof given by Zagier [25] is elementary, but indirect. In other words, the proof of Theorem 2.1 is given by computing the associated generating functions of both sides in a closed form, and then showing they are entire functions of exponential growth that agree at sufficiently many points to force their equality. However, his proof was simplified by Li in [16].

The special case when \( s = 0 \) of Theorem 2.1 reads as following

**Theorem 2.2** We have

\[
H(r, 0) := \zeta(2, 2, \ldots, 2, 3) = 2 \sum_{k=1}^{r+1} (-1)^k c_{r,0}^k \zeta(2k + 1) \zeta(2, 2, \ldots, 2),
\]

where \( c_{r,0}^k = \binom{2k}{2r+2} - \left( 1 - \frac{1}{2^{2k}} \right) 2k \), and \( r \geq 0 \).

Before we delve into the proof of the above theorem, let us recall the famous Euler’s formula for \( \zeta(2) \) which asserts that

\[
\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

One of the many proofs of this result relies on the power series expansion of \( \arcsin^2(x) \) near \( x = 0 \),

\[
\arcsin^2(x) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n \left( \frac{2n}{n} \right) x^{2n}}, \quad |x| < 1,
\]

and by using Wallis’ integral formula after we substitute \( x = \sin t \). More generally, if we integrate (1) we obtain

\[
\sum_{n=1}^{\infty} \frac{2^{2n}}{n^3 \left( \frac{2n}{n} \right) x^{2n}} = 4 \int_0^y \frac{\arcsin^2(x)}{x} \, dx = 2x^2 \cdot {}_4F_3 \left( 1, 1, 1, 1; \frac{3}{2}, 2, 2; x^2 \right),
\]

where \( {}_p F_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}, |x| < 1 \), and \( (a)_n = a(a+1) \ldots (a+n-1) \) is the Pochhammer symbol.

For instance, when \( y = \frac{1}{2} \) we have

\[
\sum_{n=1}^{\infty} \frac{1}{n^3 \left( \frac{2n}{n} \right)} = 4 \int_0^{\frac{1}{2}} \frac{\arcsin^2(x)}{x} \, dx = -2 \int_0^{\frac{\pi}{2}} z \log \left( 2 \sin \frac{z}{2} \right) \, dz
\]
$$= -\frac{\zeta(3)}{3} - \frac{\pi \sqrt{3}}{72} \left( \Psi \left( \frac{1}{3} \right) - \Psi \left( \frac{2}{3} \right) \right),$$

where $\Psi(z)$ is the trigamma function. Also, for $y = 1$ we have the special value

$$\int_0^1 \frac{\arcsin^2 x}{x} \, dx = \frac{\pi^2 \log 2}{4} - \frac{7}{8} \zeta(3).$$

Moreover, using (1), one can also derive Euler’s 1775 famous formula [2] for Apery’s constant,

$$\zeta(3) = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}.$$

Similarly, the Taylor series expansions for $\arcsin^4(x)$ and $\arcsin^6(x)$ near $x = 0$ are given by

$$\arcsin^4 x = \frac{3}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{1}{m^2} \cdot \frac{1}{2^{2n}n^2(2n-1)!} x^{2n},$$

and

$$\arcsin^6 x = \frac{45}{4} \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n-1} \sum_{p=1}^{m-1} \frac{1}{m^2} \cdot \frac{1}{p^2} \right) \frac{4^n}{2^{2n}n^2} x^{2n}.$$

As it has already been highlighted in [4, 9, 15, 22] one can obtain the Taylor series for even integer powers of arcsine by comparing the coefficients of like powers of $\lambda$ in the formulas

$$\cos(\lambda \arcsin x) = \sum_{n=0}^{\infty} \frac{\zeta(2n)}{n!} \cdot (2\arcsin x)^{2n+2}.$$

Similarly, Chu and Zheng [9] gave explicit closed form for integer powers of arcsin in terms of elementary symmetric functions. In other words, let

$$e_r(x_1, x_2, \ldots, x_r) = \prod_{i=1}^{r} (1 + tx_i) = \sum_{m} e_r(x_1, x_2, \ldots, x_r) \cdot t^m.$$
Proposition 2.3 [Hoffman–Zagier] We have
\[\zeta(2, 2, \ldots, 2) = \frac{\pi^{2r}}{(2r + 1)!}.\]

Proof Substituting \(x = \sin t\) in formula (4) we have
\[
\frac{t^{2r}}{(2r)!} = \frac{1}{4^r} \sum_{n=1}^{\infty} \frac{4^n}{n^2} \sin^{2n} t \sum_{1 \leq n_1 < n_2 < \ldots < n_r-1 < n} \frac{1}{n_1^2 n_2^2 \ldots n_r^2}.
\]

Integrating from 0 to \(\frac{\pi}{2}\), and using Wallis’ formula,\(\int_0^{\frac{\pi}{2}} \sin^{2n} t \, dt = \frac{\pi^{2n}}{2^{2n+1}}\), we have
\[
\frac{\pi^{2r}}{(2r + 1)!} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{1 \leq n_1 < n_2 < \ldots < n_r-1 < n} \frac{1}{n_1^2 n_2^2 \ldots n_r^2} = \zeta(2, 2, \ldots, 2)
\]
and we are done. \(\square\)

Remark Using (5), the above formula can be expressed in terms of elementary symmetric polynomials as
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} e_r \left(1, \frac{1}{2^2}, \ldots, \frac{1}{(n-1)^2}\right) = \frac{\pi^{2r+2}}{(2r + 3)!}.
\]

In what follows we will derive Theorem 2.2 in a direct and more elementary way using rational zeta series involving the coefficient \(\zeta(2n)\). We have

Theorem 2.4 We have
\[
H(r, 0) := \zeta(2, 2, \ldots, 2, 3)
\]
\[
= -4(2r + 3) \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)(2n + 2r + 3)2^{2n}} \zeta(2, 2, \ldots, 2, r+1).
\]

Proof of Theorem 2.4 Multiply Eq. (4) by \(\frac{(2r)!}{x}\) and integrate from 0 to \(\sin t\) to obtain
\[
\int_0^{\sin t} \frac{\arcsin^{2r}(x)}{x} \, dx = \frac{(2r)!}{4^r} \sum_{n=1}^{\infty} \frac{4^n}{2n^3} \sum_{1 \leq n_1 < n_2 < \ldots < n_r-1 < n} \frac{1}{n_1^2 n_2^2 \ldots n_r^2}.
\]

By the substitution \(x = \sin u\) in the integral on the left-hand side, we have
\[
\int_0^{\sin t} \frac{\arcsin^{2r}(x)}{x} \, dx = \int_0^t u^{2r} \cot u \, du
\]
\[
= \frac{(2r)!}{4^n} \sum_{n=1}^{\infty} \frac{4^n}{2n^3} \sum_{1 \leq n_1 < n_2 < \ldots < n_r-1 < n} \frac{1}{n_1^2 n_2^2 \ldots n_r^2}.
\]

On the other hand, using \(u \cot u = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} u^{2n}, |u| < \pi\), we derive
\[
\int_0^t u^{2r} \cot u \, du = -2 \int_0^t \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} u^{2n+2r-1} \, du = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \frac{t^{2n+2r}}{(2n + 2r)}.
\]
Therefore, we obtain
\[-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \cdot \frac{t^{2n+2r}}{(2n + 2r)} = \frac{(2r)!}{4^r} \sum_{n=1}^{\infty} \frac{4^n}{2n^3} \left(\frac{2n}{n}\right) \sin^{2n} t \sum_{n_1 < n_2 < \ldots < n_r < n} \frac{1}{n_1 n_2^2 \ldots n_r^2}.\]

Finally, integrating from 0 to \(\frac{\pi}{2}\) and using Wallis’ integral formula, \(\int_0^{\pi/2} \sin^{2n} t dt = \frac{\pi^{2n}}{2^{2n} n!}\), we obtain
\[-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \frac{t^{2n+2r+1}}{2^{2n+2r+1}(2n + 2r + 1)} (2n + 2r + 1) = \frac{(2r)!}{4^r} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{n_1 < n_2 < \ldots < n_r < n} \frac{1}{n_1 n_2^2 \ldots n_r^2} \frac{\pi^{2r}}{(2r)!} = \zeta(2, 2, \ldots, 2, 3).\]

or equivalently,
\[-4 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r)(2n + 2r + 1)} \frac{\pi^{2r}}{(2r + 1)!} \frac{2^{2n}}{(2n + 2r + 2)(2n + 2r + 3) \ldots (2n + 2r + 2s + 3) 2^{2n}} = \zeta(2, 2, \ldots, 2, 3).\]

Now, using \(\zeta(2, 2, \ldots, 2) = \frac{\pi^{2r}}{(2r + 1)!}\) (Proposition 2.3), we obtain the conclusion of the theorem. \(\square\)

**Remark** We believe that for \(s > 0\) integer, the following equality is true.

**Conjecture** We have
\[H(r, s) = -4\pi^{2r+2s+2} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)(2n + 2r + 3) \ldots (2n + 2r + 2s + 3) 2^{2n}}. \quad (6)\]

Now, we state the following

**Theorem 2.5** (Orr, [19]) For \(p \in \mathbb{N}\), and \(|z| < 1\),
\[\int_0^{\pi z} x^p \cot x dx = (\pi z)^p \sum_{k=0}^{p} \frac{p!(1)^{k+1}}{(p - k)! (2\pi z)^k} \text{Cl}_{k+1}(2\pi z) + \delta_{[\frac{p}{2}], \frac{p}{2}} \frac{\pi^{2p}}{2^p} \zeta(p + 1),\]
where \(\text{Cl}_{k+1}\) is the \((k + 1)\)-Clausen function.

For more basic properties of the Clausen functions, see the appendix.

The missing ingredient for our main purpose is given by

**Lemma 2.6** We have the following equality
\[-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + p) 2^{2n}} = \log 2 + \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{p!(1)^{k+1}}{(p - 2k)! (2\pi)^{2k}} \zeta(2k + 1) + \delta_{\lfloor p/2 \rfloor, p/2} \frac{p!(1)^{p/2}}{\pi^p} \zeta(p + 1).\]

In this moment, we have all the ingredients to prove Zagier’s result in the special case \((s = 0)\),
Proof of Theorem 2.2} From Theorem 2.3 we only need to relate the rational zeta series
\[ \sum_{n=0}^{\infty} \frac{\zeta(2n)(2n + 2r + 2)(2n + 2r + 3)2^{2n}}{(2n + 2r + 2)(2n + 2r + 3)2^{2n}} \]
to a \( \mathbb{Q} \)-linear combinations of powers of \( \pi \) and odd zeta values.

Indeed, by applying Lemma 2.6 (for \( p = 2r + 2 \) and \( p = 2r + 3 \), we have

\[-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)(2n + 2r + 3)2^{2n}} = -2 \left( \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)2^{2n}} - \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 3)2^{2n}} \right) \]

\[= \sum_{k=1}^{r+1} \frac{(2r + 2)!(-1)^k(4^k - 1)\zeta(2k + 1)}{(2r + 2 - 2k)!(2\pi)^{2k}} + \delta_{r+1,r+1} \frac{(2r + 2)!(-1)^r+1\zeta(2r + 3)}{\pi^{2r+2}} \]

\[= \sum_{k=1}^{r+1} \frac{(2r + 3)!(-1)^k(4^k - 1)\zeta(2k + 1)}{(2r + 2 - 2k)!(2\pi)^{2k}} - \sum_{k=1}^{r+1} \frac{(2r + 3)!(-1)^k(4^k - 1)\zeta(2k + 1)}{(2r + 2 - 2k)!(2\pi)^{2k}} \]

\[= \sum_{k=1}^{r+1} \left[ \frac{(2r + 2)!(-1)^k(4^k - 1)}{(2\pi)^{2k}} \delta_{r+1,k} \right] \zeta(2k + 1) \]

\[= \sum_{k=1}^{r+1} \left( -1 \right)^k \left( \frac{1}{2^{2k}} \right) \delta_{r+1,k} \zeta(2k + 1) \]

\[= \sum_{k=1}^{r+1} \left( -1 \right)^k \left( \frac{1}{2^{2k}} \right) \delta_{r+1,k} \zeta(2k + 1) \]

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\[= \sum_{k=1}^{r+1} \left( -1 \right)^k \left( \frac{1}{2^{2k}} \right) \delta_{r+1,k} \zeta(2k + 1) \]

\[= \frac{1}{2(r + 3)} \sum_{k=1}^{r+1} \frac{(2r + 3)!}{(2r + 2)!(2r + 1 - k)!} \left( -1 \right)^k \]

\[\times \left[ \frac{2k\delta_{r+1,k}}{(2^{r+1})} - \left( 1 - \frac{1}{4^k} \right) 2k \right] \zeta(2k + 1) \]
\[
\frac{1}{(2r + 3) \zeta(2, 2, \ldots, 2)} \sum_{k=1}^{r+1} (-1)^k \zeta(2, 2, \ldots, 2) c_{r, 0}^k \zeta(2k + 1),
\]
where obviously \(2k_{r+1,k} \frac{c_{r+1}}{2r+1} = (2r+2)\) for \(1 \leq k \leq r + 1\).

By Theorem 2.4, we have

\[
H(r, 0) := \zeta(2, 2, \ldots, 2, 3) = 2(2r + 3) \cdot \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)(2n + 2r + 3)2^{2n}} \cdot \zeta(2, 2, \ldots, 2),
\]

which will give us

\[
H(r, 0) = 2(2r + 3) \cdot \frac{1}{(2r + 3) \zeta(2, 2, \ldots, 2)} \sum_{k=1}^{r+1} (-1)^k \zeta(2, 2, \ldots, 2) c_{r, 0}^k \zeta(2k + 1) \cdot \zeta(2, 2, \ldots, 2),
\]

which finally reduces to

\[
H(r, 0) = 2 \sum_{k=1}^{r+1} (-1)^k c_{r, 0}^k \zeta(2, 2, \ldots, 2) \zeta(2k + 1),
\]

which is exactly what we wanted to prove. \(\Box\)

### 3 A Zagier type formula for multiple t-values

In a similar fashion with the multiple zeta values, we also define the multiple Hurwitz zeta values and multiple t-values,

\[
\zeta(k_1, k_2, \ldots, k_r; a_1, a_2, \ldots, a_r) = \sum_{1 \leq n_1 < n_2 < \ldots < n_r} \frac{1}{(n_1 + a_1)^{k_1} (n_2 + a_2)^{k_2} \ldots (n_r + a_r)^{k_r}}
\]

and

\[
t(k_1, k_2, \ldots, k_r) = 2^{-(k_1+k_2+\ldots+k_r)} \zeta(k_1, k_2, \ldots, k_r; -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})
\]

\[
= \sum_{1 \leq n_1 < n_2 < \ldots < n_r} \frac{1}{(2n_1 - 1)^{k_1} (2n_2 - 1)^{k_2} \ldots (2n_r - 1)^{k_r}}.
\]

Unfortunately, there are not many identities known in the literature about multiple Hurwitz zeta values and multiple t-values. For more details see [14,18,21,26] among others.

Also, let us recall the Gauss hypergeometric function which is defined for \(|x| < 1\) by the power series

\[
_{2}F_{1}(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!},
\]
where \((q)_n = q(q + 1) \cdots (q + n - 1)\) is the Pochhammer symbol. As we have already seen in the previous section, the odd integer powers from the Taylor series of \(\arcsin\) can be obtained by comparing the coefficients like powers of \(\lambda\) in the formulas:

\[
\sin(\lambda \arcsin(x)) = \lambda x \cdot {}_2F_1\left(\frac{1 + \lambda}{2}, \frac{1 - \lambda}{2}; \frac{1}{2}; x^2\right).
\]

This implies the following Taylor series for the odd integer powers of \(\arcsin\) function,

\[
\arcsin^{2r+1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n + 1)4^n} \cdot x^{2n+1} \cdot \sum_{0 \leq n_1 < n_2 < \ldots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2}.
\]

Again, we begin with the much simpler formula which is given by

**Proposition 3.1** ([18,26]) We have the following evaluation

\[
t(2, 2, \ldots, 2) = \frac{\pi^{2r}}{2^{2r}(2r)!}.
\]

**Proof** Similarly to the proof of Proposition 2.3, substitute \(x = \sin t\) in Eq. (7) to obtain

\[
\frac{t^{2r+1}}{(2r + 1)!} = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n + 1)4^n} \cdot \sin^{2n+1} t \cdot \sum_{0 \leq n_1 < n_2 < \ldots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2}.
\]

Again, integrating from 0 to \(\frac{\pi}{2}\) and using Wallis’ formula in the form, \(\int_0^{\frac{\pi}{2}} \sin^{2n+1} t dt = \frac{1}{2n+1} \cdot \frac{\pi^{2n+2}}{(2n)!} n \geq 0\), we have

\[
\frac{\pi^{2r+2}}{2^{2r+2}(2r + 1)! (2r + 2)} = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} \cdot \sum_{0 \leq n_1 < n_2 < \ldots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2} = t(2, 2, \ldots, 2),
\]

and the conclusion follows immediately. \(\square\)

Next, we evaluate \(t(2, 2, \ldots, 2, 3)\) in terms of rational zeta series involving \(\zeta(2n)\). This formula will be similar to the one produced in Theorem 2.4.

**Theorem 3.2** We have

\[
T(r) := t(2, 2, \ldots, 2, 3) = -4(r + 1) \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)(2n + 2r + 2)2^{2n} \cdot \frac{t(2, 2, \ldots, 2)}{r + 1}}.
\]

**Proof** Dividing by \(x\) and integrating from 0 to \(\sin t\) in formula (6), we have

\[
\int_0^{\sin t} \frac{\arcsin^{2r+1}(x)}{x} dx = (2r + 1)! \sum_{n=0}^{\infty} \frac{(2n)!}{(2n + 1)2^{2n} \cdot \sum_{0 \leq n_1 < n_2 < \ldots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2}}.
\]

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By the substitution \( x = \sin u \) in the integral, we have

\[
\int_0^t u^{2r+1} \cot u \, du = (2r + 1)! \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2n)}{(2n + 1)^2} 4^n \sum_{0 \leq n_1 < n_2 < \cdots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2}.
\]

On the other hand, using \( u \cot u = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{2^n} u^{2n} \), \( |u| < \pi \), we derive

\[
\int_0^t u^{2r+1} \cot u \, du = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{2^n} \cdot \frac{t^{2n+2r+1}}{2n + 2r + 1}.
\]

Therefore, we obtain

\[
-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{2^n} \cdot \frac{t^{2n+2r+1}}{2n + 2r + 1} = (2r + 1)! \sum_{n=0}^{\infty} \frac{(2n)}{(2n + 1)^2} 4^n \sin^{2n+1} t \sum_{0 \leq n_1 < n_2 < \cdots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2}.
\]

Finally, integrating from 0 to \( \frac{\pi}{2} \) and using Wallis’ integral formula in the form,

\[
\int_0^{\frac{\pi}{2}} \sin^{2n+1} t \, dt = \frac{1}{2n+1} \cdot \frac{\pi^{2n}}{(2n)!}, \quad n \geq 0,
\]

we have

\[
\frac{\pi^{2r+2}}{(2r + 1)! 2^{2r+2}} \left( -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)(2n + 2r + 2)2^{2n}} \right) = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^3} \sum_{0 \leq n_1 < n_2 < \cdots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i + 1)^2}.
\]

Now, using the fact that \( t(2, 2, \ldots, 2) = \frac{\pi^{2r}}{2^{2r}(2r)!} \) (Proposition 3.1) we obtain the desired result. \( \square \)

**Remark** If we define

\[
T(r, s) = t([2]^r, 3, [2]^s) = t(2, 2, \ldots, 2, 3, 2, 2, \ldots, 2),
\]

we believe that the following holds true:

**Conjecture**

\[
T(r, s) = \frac{-2}{(2r + 1)!} \left( \frac{\pi}{2} \right)^{2r+2s+2} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)(2n + 2r + 2) \ldots (2n + 2r + 2s + 2)2^{2n}}. \quad (8)
\]

In what follows, we derive a Zagier-type formula for the multiple \( t \)-values in the spirit of Theorem 2.2.
Theorem 3.3 We have

\[ T(r) = \sum_{k=1}^{r+1} (-1)^{k+1} d_{r,0}^k \frac{1}{2^{2k}} \zeta(2k+1) \cdot \frac{1}{2} (2, 2, \ldots, 2), \]

where \( d_{r,0}^k = \left( \frac{2k}{2r+1} \right) + \left( 1 - \frac{1}{2^{2k}} \right) 2k, \) and \( r \geq 0 \) integer.

Proof Our intention is to use Lemma 2.6. Indeed, put \( p = 2r + 1 \) and \( p = 2r + 2 \) and by subtracting, this gives us

\[
\begin{align*}
-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)(2n + 2r + 2)2^{2n}} &= -2 \left( \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)2^{2n}} - \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)2^{2n}} \right) \\
&= 2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 2)2^{2n}} - 2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)2^{2n}} \\
&= \sum_{k=1}^{r+1} \frac{(2r + 1)!(-1)^{k} (4k - 1) \zeta(2k + 1)}{(2r + 1 - 2k)! (2\pi)^{2k}} - \sum_{k=1}^{r+1} \frac{(2r + 2)!(-1)^{k} (4k - 1) \zeta(2k + 1)}{(2r + 2 - 2k)! (2\pi)^{2k}} \\
&\quad - \delta_{r+1,k} \cdot \frac{(2r + 2)!(-1)^{k}}{\pi^{2k}} \zeta(2k + 1) \\
&= \sum_{k=1}^{r+1} \left[ \frac{(-1)^{k} (4k - 1)}{(2\pi)^{2k}} \left( \frac{(2r + 1)!}{(2r + 1 - 2k)!} - \frac{(2r + 2)!}{(2r + 2 - 2k)!} \right) \right] \\
&\quad - \delta_{r+1,k} \cdot \frac{(2r + 2)!(-1)^{k}}{\pi^{2k}} \zeta(2k + 1) \\
&= \sum_{k=1}^{r+1} \left[ \frac{(-1)^{k} (4k - 1)}{(2\pi)^{2k}} (2k)! \left( \frac{(2r + 1)}{(2k + 1)} - \frac{(2r + 2)}{(2k + 2)} \right) \right] \\
&\quad - \delta_{r+1,k} \frac{(2r + 2)!(-1)^{k}}{\pi^{2k}} \zeta(2k + 1) \\
&= \sum_{k=1}^{r+1} \frac{(-1)^{k} (2k)!}{\pi^{2k}} \left[ \left( 1 - \frac{1}{4k} \right) \left( \frac{(2r + 1)}{(2k + 1)} \right) - \delta_{r+1,k} \right] \zeta(2k + 1) \\
&= \sum_{k=1}^{r+1} \frac{(-1)^{k+1} (2k)!}{\pi^{2k}} \left( \frac{2r + 1}{2k - 1} \right) \left( \frac{\delta_{r+1,k}}{(2r+1)(2k-1)} + \left( 1 - \frac{1}{4k} \right) \right) \zeta(2k + 1) \\
&= \sum_{k=1}^{r+1} \frac{(-1)^{k+1} (2k)!}{\pi^{2k}} \cdot \frac{(2r + 1)!}{(2r + 2 - 2k)!} \left( \frac{\delta_{r+1,k}}{(2r+1)(2k-1)} + \left( 1 - \frac{1}{4k} \right) \right) \zeta(2k + 1)
\end{align*}
\]
\[
\frac{1}{2r + 2} \sum_{k=1}^{r+1} (-1)^{k} \left( \frac{2r + 2 + 2r + 2\pi^2}{\pi^2 r^2} \right) \cdot \frac{\pi^{2(r+1-k)}}{(2r + 1 - k)!)2^{2(r+1-k)}} \cdot \frac{1}{2^{2k}} \\
\cdot \left[ \frac{(2k)\delta_{r+1,k}}{(2k+1)} + \left( 1 - \frac{1}{4^k} \right) 2k \right] \zeta(2k + 1) \\
= \frac{1}{(2r + 2)r(2, 2, \ldots, 2)} \sum_{k=1}^{r+1} (-1)^{k+1} \\
\times \left[ \left( \frac{2k}{2r + 1} \right) + \left( 1 - \frac{1}{4^k} \right) 2k \right] \frac{1}{2^{2k}} \zeta(2k + 1) t(2, 2, \ldots, 2).
\]

On the other hand, we know by Theorem 3.2 that

\[
T(r) = 2(r + 1) \left( -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2r + 1)(2n + 2r + 2)2^{2n}} \right) \cdot t(2, 2, \ldots, 2),
\]

and finally, we obtain

\[
T(r) = \sum_{k=1}^{r+1} (-1)^{k+1} \left[ \left( \frac{2k}{2r + 1} \right) + \left( 1 - \frac{1}{4^k} \right) 2k \right] \frac{1}{2^{2k}} \zeta(2k + 1) t(2, 2, \ldots, 2).
\]

\[ \square \]

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Appendix

The Clausen function (integral) is defined by

\[
\text{Cl}_2(\theta) = -\int_0^\theta \log \left( 2 \sin \frac{t}{2} \right) dt = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^2},
\]

and its Taylor series expansion is given by

\[
\frac{\text{Cl}_2(\theta)}{\theta} = 1 - \log |\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n + 1)} \left( \frac{\theta}{2\pi} \right)^{2n}, |\theta| < 2\pi.
\]

The higher order Clausen functions are

\[
\text{Cl}_{2m}(\theta) = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^{2m+1}}, \quad \text{Cl}_{2m+1}(\theta) = \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2m+1}}.
\]

Using the properties of the Riemann zeta function, we have the following particular values:

\[
\text{Cl}_{2m}(\pi) = 0, \quad \text{Cl}_{2m+1}(\pi) = -\frac{(4^m - 1)\zeta(2m + 1)}{4^m}
\]
Another look at Zagier’s formula for multiple zeta

and
\[ \text{Cl}_{2m} \left( \frac{\pi}{2} \right) = \beta(2m), \quad \text{Cl}_{2m+1} \left( \frac{\pi}{2} \right) = -\frac{(4^m - 1)\zeta(2m + 1)}{2^{4m+1}}, \]

where \( \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} \), \( \text{Res} > 0 \) is the Dirichlet beta function.

Moreover,
\[
\frac{d}{d\theta} \text{Cl}_{2m}(\theta) = \text{Cl}_{2m-1}(\theta), \quad \frac{d}{d\theta} \text{Cl}_{2m+1}(\theta) = -\text{Cl}_{2m}(\theta),
\]
and
\[
\int_0^\theta \text{Cl}_{2m}(x)\,dx = \zeta(2m + 1) - \text{Cl}_{2m+1}(\theta), \quad \int_0^\theta \text{Cl}_{2m-1}(x)\,dx = \text{Cl}_{2m}(\theta).
\]

Proof of the Lemma 2.6 (see also [19]). Theorem 2.4 for \( z = \frac{1}{2} \) gives us
\[
\int_0^{\frac{\pi}{2}} x^p \cot x\,dx = \left( \frac{\pi}{2} \right)^p \left( \log 2 + \sum_{k=1}^{[p/2]} p!(-1)^k \frac{(4^k - 1)}{(p - 2k)!} (2\pi)^{2k} \zeta(2k + 1) \right) + \delta_{[p/2]} \frac{p!(-1)^{[p/2]} \zeta(p + 1)}{2^p}.
\]

On the other hand, since \( \cot x = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \cdot x^{2n-1}, \ |x| < \pi \), by integration and Fubini’s theorem, we obtain
\[
\int_0^{\frac{\pi}{2}} x^p \cot x\,dx = \int_0^{\frac{\pi}{2}} x^p \left( -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \cdot x^{2n-1} \right)\,dx = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \int_0^{\frac{\pi}{2}} x^{2n+p-1}\,dx
\]
\[ \text{or equivalently,} \]
\[
\int_0^{\frac{\pi}{2}} x^p \cot x\,dx = -2 \left( \frac{\pi}{2} \right)^p \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + p)2^n},
\]
and the lemma follows immediately. \( \square \)

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