Sobolev Homeomorphisms and Composition Operators

V. Gol’dshtein and A. Ukhlov

ABSTRACT

We study invertibility of bounded composition operators of Sobolev spaces. The problem is closely connected with the theory of mappings of finite distortion. If a homeomorphism \( \varphi \) of Euclidean domains \( D \) and \( D' \) generates by the composition rule \( \varphi^* f = f \circ \varphi \) a bounded composition operator of Sobolev spaces \( \varphi^* : L^1_\infty(D') \to L^1_p(D), p > n - 1 \), has finite distortion and Luzin \( N \)-property then its inverse \( \varphi^{-1} \) generates the bounded composition operator from \( L^1_{p'}(D), p' = p/(p-n+1) \), into \( L^1_\infty(D) \).

Introduction

Let \( \varphi \) be a homeomorphism of Euclidean domains \( D, D' \subset \mathbb{R}^n \). It is known [1] that \( \varphi \) is a quasiconformal mapping if and only if the composition operator \( \varphi^* \) is an isomorphism of Sobolev spaces \( L^1_n(D') \) and \( L^1_n(D) \). If \( \varphi \) generates a bounded composition operator of Sobolev spaces \( L^1_q(D') \) and \( L^1_q(D), q \neq n \), then the inverse homeomorphism \( \varphi^{-1} \) is not necessary generates the bounded composition operator of same spaces. In the more general case homeomorphisms that generate composition operators from \( L^1_p(D') \) to \( L^1_q(D) \), \( 1 \leq q \leq p \leq \infty \), are mappings with bounded \((p,q)\)-distortion. These classes of mappings were introduced in [2] as a natural solution of the change of variable problem in Sobolev spaces. Inverse mappings to homeomorphisms with bounded \((p,q)\)-distortion can be described in the same category of mappings with bounded mean distortion. In [3] these classes of mappings were studied in a relation with Sobolev type embedding theorems for non-regular domains.

We recall, that Sobolev space \( L^1_p(D), 1 \leq p \leq \infty \), consists of locally summable, weakly differentiable functions \( f : D \to \mathbb{R} \) with the finite seminorm:

\[
\| f \|_{L^1_p(D)} = \| |\nabla f| \|_{L^p(D)}, \quad \nabla f = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).
\]

As usually Lebesgue space \( L^p(D), 1 \leq p \leq \infty \), is the space of locally summable functions with the finite norm:

\[
\| f \|_{L^p(D)} = \left( \int_D |f|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
\]

and

\[
\| f \|_{L^\infty(D)} = \text{ess sup}_{x \in D} |f(x)|, \quad p = \infty.
\]

A mapping \( \varphi : D \to \mathbb{R}^n \) belongs to \( L^1_p(D), 1 \leq p \leq \infty \), if its coordinate functions \( \varphi_j \) belong to \( L^1_p(D), j = 1, \ldots, n \). In this case formal Jacobi matrix \( D\varphi(x) = \left( \frac{\partial \varphi_i}{\partial x_j}(x) \right), \ i,j = 1, \ldots, n \), and its determinant (Jacobian) \( J(x, \varphi) = \det D\varphi(x) \) are well defined at almost all points \( x \in D \). The norm \( |D\varphi(x)| \) of the matrix \( D\varphi(x) \) is the norm of the

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corresponding linear operator $D\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix $D\varphi(x)$. We will use the same notation for this matrix and the corresponding linear operator.

Recall that a mapping $\varphi : D \to D'$ is called a mapping with bounded $(p, q)$-distortion $1 \leq q \leq p \leq \infty$, if $\varphi$ belongs to Sobolev space $W_{1, \text{loc}}^1(D)$ and the local $p$-distortion

$$K_p(x) = \inf \{ k : |D\varphi|(x) \leq k|J(x, \varphi)|^{\frac{1}{p}}, \ x \in D \}$$

belongs to Lebesgue space $L_r(D)$, where $1/r = 1/q - 1/p$ (if $p = q$ then $r = \infty$).

Mappings with bounded $(p, q)$-distortion have a finite distortion, i.e. $D\varphi(x) = 0$ for almost all points $x$ that belongs to set $Z = \{ x \in D : J(x, \varphi) = 0 \}$.

Necessity of studying of Sobolev mappings with integrable distortion arises in problems of the non-linear elasticity theory [4, 5]. In these works J. M. Ball introduced classes of mappings, defined on bounded domains $D \subset \mathbb{R}^n$:

$$A^+_{p,q}(D) = \{ \varphi \in W^1_p(D) : \text{adj} \ D\varphi \in L_q(D), \ J(x, \varphi) > 0 \ a. e. \ in \ D \},$$

$p, q > n$, where adj $D\varphi$ is the formal adjoint matrix to the Jacobi matrix $D\varphi$:

$$\text{adj} \ D\varphi(x) \cdot D\varphi(x) = \text{Id} \ J(x, \varphi).$$

The class of mappings with bounded $(p, q)$-distortion is a natural generalization of mappings with bounded distortion and represents a non-homeomorphic case of so-called $(p, q)$-quasiconformal mappings [2, 3, 6, 7]. Such classes of mappings have applications to the Sobolev type embedding problems [7–9].

The following assertion demonstrates a connection between Sobolev spaces and mappings with bounded $(p, q)$-distortion [2]. A homeomorphism $\varphi$ of Euclidean domains $D$ and $D'$ is a mapping with bounded $(p, q)$-distortion, $1 \leq q < p < \infty$, if and only if $\varphi$ generates a bounded operator of Sobolev spaces

$$\varphi^* : L^1_p(D') \to L^1_q(D)$$

by the composition rule $\varphi^* f = f \circ \varphi$. We call $\varphi^*$ a composition operator of Sobolev spaces.

In the frameworks of the inverse operator problem in [6] was proved, that if a homeomorphism $\varphi : D \to D'$ generates a bounded composition operator

$$\varphi^* : L^1_p(D') \to L^1_q(D), \ n - 1 < q \leq p < +\infty,$$

then the inverse mapping $\varphi^{-1} : D' \to D$ generates a bounded composition operator

$$(\varphi^{-1})^* : L^1_q(D) \to L^1_{p'}(D'), \ q' = q/(q - n + 1), \ p' = p/(p - n + 1).$$

The main result of the article concerns to invertibility of a composition operator in the limit case $p = \infty$.

**Theorem A.** Let a homeomorphism $\varphi : D \to D'$ has finite distortion, Luzin $N$-property (the image of a set measure zero is a set measure zero) and generates a bounded composition operator

$$\varphi^* : L^1_{\infty}(D') \to L^1_q(D), \ q > n - 1.$$

Then the inverse mapping $\varphi^{-1} : D' \to D$ generates a bounded composition operator

$$(\varphi^{-1})^* : L^1_q(D) \to L^1_1(D'), \ q' = q/(q - n + 1).$$
The invertibility problem for composition operators in Sobolev spaces is closely connected with a regularity problem for invertible Sobolev mappings. The regularity problem for mappings which are inverse to Sobolev homeomorphisms was studied by many authors. In article [10] was proved that if a mapping \( \varphi \in W^{1}_{n, \text{loc}}(D) \) and \( J(x, \varphi) > 0 \) for almost all points \( x \in D \), then \( \varphi^{-1} \) belongs to \( W^{1}_{1, \text{loc}}(D') \).

The assumption that \( \varphi \) has finite distortion cannot be dropped out. Indeed, consider the function \( g(x) = x + u(x) \) on the real line, where \( u \) is the standard Cantor function. Let \( f = g^{-1} \). Then the derivative \( f' = 0 \) on the set of positive measure and \( h^{-1} \) fails to be absolutely continuous. In this case we can prove only that the inverse homeomorphism has a finite variation on almost all lines [11]. In work [11] was obtained the following result: if a homeomorphism \( \varphi : D \to D' \) belongs to the Sobolev space \( L^{1}_{\rho, \text{loc}}(D) \), \( p > n - 1 \), then the inverse mapping \( \varphi^{-1} : D' \to D \) has a finite variation on almost all lines (belongs to \( \text{BVL}(D') \)).

In work [12] the local regularity of plane homeomorphisms that belong to Sobolev space \( W^{1}_{1}(D) \) was studied. For the case of space \( \mathbb{R}^{n} \), \( n \geq 3 \), recent work [13] contains the following result for domains in \( \mathbb{R}^{n} \), \( n \geq 3 \): if the norm of the derivative \( |D \varphi| \) belongs to Lorentz space \( L^{n-1,1}(D) \) and a mapping \( \varphi : D \to D' \) has finite distortion, then the inverse mapping belongs to Sobolev space \( W^{1}_{1, \text{loc}}(D') \) and has finite distortion. Recall that

\[
L^{n-1}(D) \subset L^{n-1,1}(D) \subset \bigcap_{p>n-1} L^{p}(D).
\]

Note, that results about regularity of mappings inverse to Sobolev homeomorphisms follows from Theorem A. Indeed, substituting in the norm inequality for the inverse operator coordinate functions \( x_{j} \in L^{1}_{\rho, \text{loc}}(D) \) we see that \( \varphi^{-1} \) belongs to \( L^{1}_{1, \text{loc}}(D') \).

The suggested method of investigation is based on a relation between Sobolev mappings, composition operators of spaces of Lipschitz functions and a change of variable formula for weakly differentiable mappings.

1. Composition operators in Sobolev spaces

A locally integrable function \( f : D \to \mathbb{R} \) is absolutely continuous on a straight line \( l \) having non-empty intersection with \( D \) if it is absolutely continuous on an arbitrary segment of this line which is contained in \( D \). A function \( f : D \to \mathbb{R} \) belongs to the class \( \text{ACL}(D) \) (absolutely continuous on almost all straight lines) if it is absolutely continuous on almost all straight lines parallel to any coordinate axis.

Note that \( f \) belongs to Sobolev space \( L^{1}_{1}(D) \) if and only if \( f \) is locally integrable and it can be changed by a standard procedure on a set of measure zero (changed to its Lebesgue values at any point where the Lebesgue values exist) so , that a modified function belongs to \( \text{ACL}(D) \), and its partial derivatives \( \frac{\partial f}{\partial x_{i}}(x) \), \( i = 1, \ldots, n \), exist almost everywhere and are integrable in \( D \). From this point we will use such modified functions only. Note that first weak derivatives of the function \( f \) coincide almost everywhere with the usual partial derivatives (see, e.g., [14] ).

A mapping \( \varphi : D \to \mathbb{R}^{n} \) belongs to the class \( \text{ACL}(D) \), if its coordinate functions \( \varphi_{j} \) belong to \( \text{ACL}(D) \), \( j = 1, \ldots, n \).
We will use the notion of approximate differentiability. Let $A$ be a subset of $\mathbb{R}^n$. Density of set $A$ at a point $x \in \mathbb{R}^n$ is the limit
\[
\lim_{r \to 0} \frac{|B(x,r) \cap A|}{|B(x,r)|}.
\]
Here by symbol $|A|$ we denote Lebesgue measure of the set $A$.

A linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is called an approximate differential of a mapping $\varphi : D \to \mathbb{R}^n$ at point $a \in D$, if for every $\varepsilon > 0$ the density of the set
\[
A_\varepsilon = \{ x \in D : |\varphi(x) - \varphi(a) - L(x-a)| < \varepsilon |x-a| \}
\]
at point $a$ is equal to one.

A point $y \in \mathbb{R}^n$ is called an approximate limit of a mapping $\varphi : D \to \mathbb{R}^n$ at a point $x$, if the density of the set $D \setminus \varphi^{-1}(W)$ at this point is equal to zero for every neighborhood $W$ of the point $y$.

For a mapping $\varphi : D \to \mathbb{R}^n$ we define approximate partial derivatives
\[
ap \frac{\partial \varphi_i}{\partial x_j}(x) = \lim_{t \to 0} \frac{\varphi_i(x+te_j) - \varphi_i(x)}{t}, \quad i, j = 1, \ldots, n.
\]

Approximate differentiable mappings are closely connected with Lipschitz mappings. Recall, that a mapping $\varphi : D \to \mathbb{R}^n$ is a Lipschitz mapping, if there exists a constant $K < +\infty$ such that
\[
|\varphi(x) - \varphi(y)| \leq K|x-y|
\]
for every points $x, y \in D$.

The value
\[
||\varphi| \text{ Lip}(D)|| = \sup_{x,y\in D} \frac{|\varphi(x) - \varphi(y)|}{|x-y|}
\]
we call the norm of $\varphi$ in the space $\text{Lip}(D)$.

The next assertion describes this connection between approximate differentiable mappings and Lipschitz mappings in details [15].

**Theorem 1.** Let $\varphi : D \to \mathbb{R}^n$ be a measurable mapping. Then the following assertions are equivalent:
1) The mapping $\varphi : D \to \mathbb{R}^n$ is approximate differentiable almost everywhere in $D$.
2) The mapping $\varphi : D \to \mathbb{R}^n$ has approximate partial derivatives $\text{ap} \frac{\partial \varphi_i}{\partial x_j}$, $i, j = 1, \ldots, n$ almost everywhere in $D$.
3) There exists a collection of closed sets $\{A_k\}_{k=1}^\infty$, $A_k \subset A_{k+1} \subset D$, such that a restriction $\varphi|_{A_k}$ is a Lipschitz mapping on the set $A_k$ and
\[
|D \setminus \bigcup_{k=1}^\infty A_k| = 0.
\]

If a mapping $\varphi : D \to D'$ has approximate partial derivatives $\text{ap} \frac{\partial \varphi_i}{\partial x_j}$ almost everywhere in $D$, $i, j = 1, \ldots, n$, then the formal Jacobi matrix $D\varphi(x) = (\text{ap} \frac{\partial \varphi_i}{\partial x_j}(x))$, $i, j = 1, \ldots, n$, and its Jacobian determinant $J(x, \varphi) = \det D\varphi(x)$ are well defined at almost all points of
The norm $|D\varphi(x)|$ of the matrix $D\varphi(x)$ is the norm of the linear operator determined by the matrix in Euclidean space $\mathbb{R}^n$.

In the theory of mappings with bounded mean distortion additive set functions play a significant role. Let us recall that a nonnegative mapping $\Phi$ defined on open subsets of $D$ is called a finitely quasiadditive set function [16] if

1) for any point $x \in D$, there exists $\delta$, $0 < \delta < \text{dist}(x, \partial D)$, such that $0 \leq \Phi(B(x, \delta)) < \infty$ (here and in what follows $B(x, \delta) = \{y \in \mathbb{R}^n : |y - x| < \delta\}$);

2) for any finite collection $U_i \subset U \subset D$, $i = 1, \ldots, k$ of mutually disjoint open sets the following inequality $\sum_{i=1}^{k} \Phi(U_i) \leq \Phi(U)$ takes place.

Obviously, the last inequality can be extended to a countable collection of mutually disjoint open sets from $D$, so a finitely quasiadditive set function is also countable quasi additive.

If instead of the second condition we suppose that for any finite collection $U_i \subset D$, $i = 1, \ldots, k$ of mutually disjoint open subsets of $D$ the equality

$$\sum_{i=1}^{k} \Phi(U_i) = \Phi(U)$$

takes place, then such set function is said to be finitely additive. If the last equality can be extended to a countable collection of mutually disjoint open subsets of $D$, then such set function is said to be countable additive.

A nonnegative mapping $\Phi$ defined on open subsets of $D$ is called a monotone set function [16] if $\Phi(U_1) \leq \Phi(U_2)$ under the condition, that $U_1 \subset U_2 \subset D$ are open sets.

Note, that a monotone (countable) additive set function is the (countable) quasiadditive set function.

Let us reformulate an auxiliary result from [16] in a convenient for this study way.

**Proposition 1.** Let a monotone finitely additive set function $\Phi$ be defined on open subsets of the domain $D \subset \mathbb{R}^n$. Then for almost all points $x \in D$ the volume derivative

$$\Phi'(x) = \lim_{\delta \to 0, B_\delta \ni x} \frac{\Phi(B_\delta)}{|B_\delta|}$$

is finite and for any open set $U \subset D$, the inequality

$$\int_U \Phi'(x) \, dx \leq \Phi(U)$$

is valid.

A nonnegative finite valued set function $\Phi$ defined on a collection of measurable subsets of an open set $D$ is said to be absolutely continuous if for every number $\varepsilon > 0$ can be found a number $\delta > 0$ such that $\Phi(A) < \varepsilon$ for any measurable sets $A \subset D$ from the domain of definition of $\Phi$, which satisfies the condition $|A| < \delta$.

Let $E$ be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. Define Lebesgue space $L_p(E)$, $1 \leq p \leq \infty$, as a Banach space of locally summable functions $f : E \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid L_p(E)\| = \left( \int_E |f|^p(x) \, dx \right)^{1/p}, \quad 1 \leq p < \infty,$$
A function $f$ belongs to the space $L_{p,\text{loc}}(E)$, $1 \leq p \leq \infty$, if $f \in L_p(F)$ for every compact set $F \subset E$.

For an open subset $D \subset \mathbb{R}^n$ define the seminormed Sobolev space $L_p^1(D)$, $1 \leq p \leq \infty$, as a space of locally summable, weakly differentiable functions $f : D \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L_p^1(D)\| = \|\nabla f \mid L_p(D)\|, \quad 1 \leq p \leq \infty.$$  

Here $\nabla f$ is the weak gradient of the function $f$, i.e. $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$,

$$\int_D f \frac{\partial \eta}{\partial x_i} \, dx = - \int_D \frac{\partial f}{\partial x_i} \eta \, dx, \quad \forall \eta \in C_0^\infty(D), \quad i = 1, \ldots, n.$$  

As usual $C_0^\infty(D)$ is the space of infinitely smooth functions with a compact support.

Note, that smooth functions are dense in $L_p^1(D)$, $1 \leq p < \infty$ (see, for example [14], [17]). If $p = \infty$ we can assert only that for arbitrary function $f \in L_p^1(D)$ there exists a sequence of smooth functions $\{f_k\}$ converges locally uniformly to $f$ and $\|f_k \mid L_p^1(D)\| \to \|f \mid L_p^1(D)\|$ (see [17]).

The Sobolev space $W_p^1(D)$, $1 \leq p \leq \infty$, is a Banach space of locally summable, weakly differentiable functions $f : D \to \mathbb{R}$, equipped with the following norm:

$$\|f \mid W_p^1(D)\| = \|f \mid L_p(D)\| + \|f \mid L_p^1(D)\|.$$  

A function $f$ belongs to the space $L_{p,\text{loc}}^1(D)$ ($W_{p,\text{loc}}^1(D)$), $1 \leq p \leq \infty$, if $f \in L_p^1(K)$ ($f \in W_p^1(K)$) for every compact subset $K \subset D$. The Sobolev space $L_p^1(D)$ is the closure of the space $C_0^\infty(D)$ in $L_p^1(D)$.

A mapping $\varphi : D \to D'$ belongs to Lebesgue class $L_p^1(E)$ if its coordinate functions $\varphi_j$, $j = 1, \ldots, n$ belong to $L_p^1(E)$. A mapping $\varphi : D \to D'$ belongs to Sobolev class $W_p^1(D)$ ($L_p^1(D)$) if its coordinate functions $\varphi_j$, $j = 1, \ldots, n$, belong to $W_p^1(D)$ ($L_p^1(D)$).

We say that a mapping $\varphi : D \to D'$ generates a bounded composition operator

$$\varphi^* : L_p^1(D') \to L_q^1(D), \quad 1 \leq q \leq p \leq \infty,$$

if for every function $f \in L_p^1(D')$ the composition $f \circ \varphi \in L_q^1(D)$ and the inequality

$$\|\varphi^* f \mid L_q^1(D)\| \leq K \|f \mid L_p^1(D')\|$$

holds.

**Theorem 2.** A homeomorphism $\varphi : D \to D'$ between two domains $D, D' \subset \mathbb{R}^n$ generates a bounded composition operator

$$\varphi^* : L_{1,\infty}^1(D') \to L_q^1(D), \quad 1 < q < +\infty,$$

if and only if $\varphi$ belongs to the Sobolev space $L_q^1(D)$. 

Proof. Necessity. Substituting in the inequality
\[ \| \varphi^* f | L^1_q(D) \| \leq K \| f | L^1_{\infty}(D') \| \]
the test functions \( f_j(y) = y_j \in L^1_{\infty}(D'), j = 1, ..., n \) we see that \( \varphi \) belongs to \( L^1_q(D) \).

Sufficiency. Let a function \( f \in L^1_{\infty}(D') \cap C^\infty(D') \). Then
\[ \| \varphi^* f | L^1_q(D) \| = \left( \int_D |\nabla (f \circ \varphi)|^q \, dx \right)^{\frac{1}{q}} \leq \left( \int_D |D\varphi|^q |\nabla f|^q (\varphi(x)) \, dx \right)^{\frac{1}{q}} \]
\[ \leq \left( \int_D |D\varphi|^q \, dx \right)^{\frac{1}{q}} \| f | L^1_{\infty}(D') \| = \| \varphi | L^1_q(D) \| \cdot \| f | L^1_{\infty}(D') \| . \]
For arbitrary function \( f \in L^1_{\infty}(D') \) consider a sequence of smooth functions \( f_k \in L^1_{\infty}(D') \) such that
\[ \lim_{k \to \infty} \| f_k | L^1_{\infty}(D') \| = \| f | L^1_{\infty}(D') \| \]
and \( f_k \) converges locally uniformly to \( f \) in \( D' \). Then, the sequence \( \varphi^* f_k \) converges locally uniformly to \( \varphi^* f \) in \( D \) and is a bounded sequence in \( L^1_q(D) \). Since the space \( L^1_q(D), 1 < q < \infty \), is a reflexive space there exists a subsequence \( f_{k_l} \in L^1_q(D) \) which weakly converges to \( f \in L^1_q(D) \) and
\[ \| \varphi^* f | L^1_q(D) \| \leq \lim_{l \to \infty} \inf \| \varphi^* f_{k_l} | L^1_q(D) \| . \]
So, passing to limit when \( l \) tends to +\( \infty \) in the inequality
\[ \| \varphi^* f_{k_l} | L^1_q(D) \| \leq K \| f_{k_l} | L^1_{\infty}(D') \| \]
we obtain
\[ \| \varphi^* f | L^1_q(D) \| \leq K \| f | L^1_{\infty}(D') \| . \]

The next theorem gives a "localization" property of the composition operator on spaces of functions with compact support and/or its closure in \( L^1_{\infty} \).

Theorem 3. Let a homeomorphism \( \varphi : D \to D' \) between two domains \( D, D' \subset \mathbb{R}^n \) generates a bounded composition operator
\[ \varphi^* : L^1_{\infty}(D') \to L^1_q(D), 1 \leq q < +\infty. \]
Then there exists a bounded monotone countable additive function \( \Phi(A') \) defined on open bounded subsets of \( D' \) such that for every function \( f \in L^1_{\infty}(A') \) the inequality
\[ \int_{\varphi^{-1}(A)} |\nabla (f \circ \varphi)|^q \, dx \leq \Phi(A') \text{esssup}_{y \in A'} |\nabla f|^q (y) \]
holds.
**Proof.** Let us define \( \Phi(A') \) by the following way \([2, 6]\)

\[
\Phi(A') = \sup_{f \in \hat{L}^1_\infty(A')} \left( \frac{\|\varphi^* f \ | \ L^1_q(D)\|}{\|f \ | \ L^\infty_\infty(A')} \right)^q,
\]

Let \( A'_1 \subset A'_2 \) be bounded open subsets of \( D' \). Extending functions of space \( \hat{L}^1_\infty(A'_1) \) by zero onto the set \( A'_2 \), we obtain an inclusion \( L^\infty_\infty(A'_1) \subset L^\infty_\infty(A'_2) \). Obviously

\[
\|f \ | \ L^1_\infty(A'_1)\| = \|f \ | \ L^1_\infty(A'_2)\|
\]

for every \( f \in L^\infty_\infty(A'_1) \). By the following inequality

\[
\Phi(A'_1) = \sup_{f \in \hat{L}^1_\infty(A'_1)} \left( \frac{\|\varphi^* f \ | \ L^1_q(D)\|}{\|f \ | \ L^\infty_\infty(A'_1)} \right)^q = \sup_{f \in \hat{L}^1_\infty(A'_2)} \left( \frac{\|\varphi^* f \ | \ L^1_q(D)\|}{\|f \ | \ L^\infty_\infty(A'_2)} \right)^q \leq \sup_{f \in \hat{L}^1_\infty(A'_2)} \left( \frac{\|\varphi^* f \ | \ L^1_q(D)\|}{\|f \ | \ L^\infty_\infty(A'_2)} \right)^q = \Phi(A'_2),
\]

the set function \( \Phi \) is monotone.

Let \( A'_i, \ i \in \mathbb{N}, \) be open disjoint subsets at the domain \( D' \), \( A'_0 = \bigcup_{i=1}^{\infty} A'_i \). Choose arbitrary functions \( f_i \in \hat{L}^1_\infty(A'_i) \) with following properties

\[
\|\varphi^* f_i \ | \ L^1_q(D)\| \geq (\Phi(A'_i) (1 - \frac{\varepsilon}{2^i}))^{\frac{1}{q}} \|f_i \ | \ L^1_\infty(A'_i)\|
\]

and

\[
\|f_i \ | \ L^1_\infty(A'_i)\| = 1,
\]

while \( i \in \mathbb{N} \). Here \( \varepsilon \in (0, 1) \) is a fixed number. Letting \( g_N = \sum_{i=1}^{N} f_i \) we obtain

\[
\|\varphi^* g_N \ | \ L^1_q(D)\| \geq \left( \sum_{i=1}^{N} (\Phi(A'_i) (1 - \frac{\varepsilon}{2^i})) \|f_i \ | \ L^1_\infty(A'_i)\|^{\frac{1}{q}} \right) \geq \left( \sum_{i=1}^{N} (\Phi(A'_i) - \varepsilon \Phi(A'_0)) \right)^{\frac{1}{q}} \|g_N \ | \ L^1_\infty(\bigcup_{i=1}^{N} A'_i)\|
\]

since sets, on which the gradients \( \nabla \varphi^* f_i \) do not vanish, are disjoint. From the last inequality follows that

\[
\Phi(A'_0)^{\frac{1}{q}} \geq \sup_{g_N \ | \ L^\infty_\infty(\bigcup_{i=1}^{N} A'_i)} \left( \frac{\|\varphi^* g_N \ | \ L^1_q(D)\|}{\|g_N \ | \ L^\infty_\infty(\bigcup_{i=1}^{N} A'_i)\|} \right) \geq \left( \sum_{i=1}^{N} (\Phi(A'_i) - \varepsilon \Phi(A'_0)) \right)^{\frac{1}{q}}.
\]
Here the upper bound is taken over all above-mentioned functions
\[
g_N \in L^1_\infty \left( \bigcup_{i=1}^{N} A_i' \right).
\]
Since both \( N \) and \( \varepsilon \) are arbitrary, we have finally
\[
\sum_{i=1}^{\infty} \Phi(A'_i) \leq \Phi \left( \bigcup_{i=1}^{\infty} A'_i \right).
\]

The validity of the inverse inequality can be proved in a straightforward manner. Indeed, choose functions \( f_i \in \tilde{L}^1_{\infty}(A'_i) \) such that \( \|f_i| \tilde{L}^1_{\infty}(A'_i)\| = 1 \).

Letting \( g = \sum_{i=1}^{\infty} f_i \) we obtain
\[
\| \varphi^* g \mid L^1_q(D) \| \leq \left( \sum_{i=1}^{\infty} \Phi(A'_i) \| f_i \mid \tilde{L}^1_{\infty}(A'_i) \|^{q} \right)^{1/q} = \left( \sum_{i=1}^{\infty} \Phi(A'_i) \right)^{\frac{1}{q}} \| g \mid \tilde{L}^1_{\infty} \left( \bigcup_{i=1}^{\infty} A'_i \right) \|,
\]
since sets, on which the gradients \( \nabla \varphi^* f_i \) do not vanish, are disjoint. From this inequality follows that
\[
\Phi \left( \bigcup_{i=1}^{\infty} A'_i \right) \leq \sup \left\{ \| \varphi^* g \mid L^1_q(D) \| : g \in \tilde{L}^1_{\infty} \left( \bigcup_{i=1}^{\infty} A'_i \right) \right\} \leq \left( \sum_{i=1}^{\infty} \Phi(A'_i) \right)^{\frac{1}{q}},
\]
where the upper bound is taken over all functions \( g \in L^1_{\infty} \left( \bigcup_{i=1}^{\infty} A'_i \right) \).

By the definition of the set function \( \Phi \) we have
\[
\| \varphi^* f \mid L^1_q(D) \|^p \leq \Phi(A') \| f \mid \tilde{L}^1_{\infty}(A') \|^q,
\]
Since the support of the function \( f \circ \varphi \) is contained in the set \( \varphi^{-1}(A') \) we have
\[
\int_{\varphi^{-1}(A)} |\nabla (f \circ \varphi) |^q dx \leq \Phi(A') \text{esssup}_{y \in A'} |\nabla f |^q(y).
\]

Theorem proved.

We recall some basic facts about \( p \)-capacity. Let \( G \subset \mathbb{R}^n \) be an open set and \( E \subset G \) be a compact set. For \( 1 \leq p \leq \infty \) the \( p \)-capacity of the ring \((E, G)\) is defined as
\[
cap_p(E, G) = \inf \left\{ \int_G |\nabla u|^p : u \in L^1_p(G) \cap C_0^\infty(G), u \geq 1 \text{ on } E \right\}.
\]
Functions \( u \in L^1_p(G) \cap C_0^\infty(G), u \geq 1 \text{ on } E \), are called admissible functions for ring \((E, G)\).

We need the following estimate of the \( p \)-capacity [18].

**Lemma 1.** Let \( E \) be a connected closed subset of an open bounded set \( G \subset \mathbb{R}^n, n \geq 2 \), and \( n - 1 < p < \infty \). Then
\[
cap_p^{n-1}(E, G) \geq c \frac{(diam E)^p}{|G|^p-n+1},
\]
where a constant $c$ depends on $n$ and $p$ only.

For readers convenience we will prove this fact.

**Proof.** Let $d$ be diameter of set $E$. Without loss of generality we can suggest, that $d = \text{dist}(0, a)$ for some point $a = (0, ..., 0, a_n)$. For arbitrary number $t$, $0 < t < d$, denote by $P_t$ the hyperplane $x_n = t$.

In the subspace $x_n = 0$ we consider the unit $(n - 2)$-dimensional sphere $S^{n-2}$ with the center at the origin and fix an arbitrary point $z \in E \cap P_t$. For every point $y \in S^{n-2}$ denote by $R(y)$ the supremum of numbers $r_0$ such that $z + ry \in G$ while $0 \leq r \leq r_0$. Then for every admissible function $f \in C^\infty_0(G)$ the following inequality holds. Applying Hölder inequality to the right side of the last inequality, we have

$$1 = f(z) - f(z + R(y)y) \leq \int_0^{R(y)} |\nabla f(z + ry)| \, dr = \int_0^{R(y)} (|\nabla f(z + ry)|r^n)^{\frac{n-2}{p}} \, dr$$

Multiplying both sides of this inequality on $((p - 1)/(p - n + 1))^{1-p} \cdot (R(y))^{n-p-1}$ and integrating by $y \in S^{n-2}$, we obtain

$$\left(\frac{p - 1}{p - n + 1}\right)^{p-1} \int_{S^{n-2}} (R(y))^{p-n+1} \, dy \leq \int_{S^{n-2}} dy \int_0^{R(y)} |\nabla f(z + ry)|r^{n-2} \, dr \leq \int_{P_t} |\nabla f|^p \, dz.$$  

For the lower estimate of the left integral we use again Hölder inequality. Denote by $\omega_{n-2}$ the $n-2$-dimensional area of sphere $S^{n-2}$. By simple calculations we get

$$\omega_{n-2}^p = \left(\int_{S^{n-2}} dy\right)^p \leq \left(\int_{S^{n-2}} (R(y))^{n-p-1} \, dy\right)^{n-1} \left(\int_{S^{n-2}} (R(y))^{n-1} \, dy\right)^{p+1-n} \leq ((n-1)m_{n-1}(G \cap P_t))^{p-n+1} \int_{S^{n-2}} (R(y))^{n-p-1} \, dy \leq (n-1)m_{n-1}(G \cap P_t)^{p-n+1} (n-1)^{n-1}.$$  

Here $m_{n-1}(A)$ is $(n-1)$-Lebesgue measure of the set $A$.

Denote by $u(t) = m_{n-1}(G \cap P_t)$. Using the last estimate we obtain

$$\int_{P_t} |\nabla f|^p \, dz \geq \left(\frac{p - 1}{p - n + 1}\right)^{1-p} (n-1)^{\frac{n-p-1}{n-1}} \omega_{n-2}^{\frac{p}{n-1}} (u(t))^{\frac{n-p-1}{n-1}}.$$  

After integrating by $t \in (0, d)$ we have

$$\int_G |\nabla f|^p \, dx \geq \left(\frac{p - 1}{p - n + 1}\right)^{1-p} (n-1)^{\frac{n-p-1}{n-1}} \omega_{n-2}^{\frac{p}{n-1}} \int_0^d (u(t))^{\frac{n-p-1}{n-1}} \, dt.$$
By Hölder inequality

\[
d^p = \left( \int_0^d \, dt \right)^p \leq \left( \int_0^d u(t) \, dt \right)^{p-n+1} \left( \int_0^d (u(t))^{\frac{n-1}{n-p}} \, dt \right)^{n-1} \\
\leq |G|^{p-n+1} \left( \int_0^d (u(t))^{\frac{n-1}{n-p}} \, dt \right)^{n-1}.
\]

Therefore

\[
\int_G |\nabla f|^p \, dx \geq \left( \frac{p-1}{p-n+1} \right)^{1-p} (n-1)^{\frac{n-1}{n-p}} \omega_{n-2} \left( \frac{d^p}{|G|^{p-n+1}} \right)^{\frac{1}{n-1}}.
\]

Since \( f \) is an arbitrary admissible function the required inequality is proved.

Let us define a class \( \text{BVL} \) of mappings with finite variation. A mapping \( \varphi : D \to \mathbb{R}^n \) belongs to the class \( \text{BVL}(D) \) (i.e., has finite variation on almost all straight lines) if it has finite variation on almost all straight lines \( l \) parallel to any coordinate axis: for any finite number of points \( t_1, \ldots, t_k \) that belongs to such straight line \( l \)

\[
\sum_{i=0}^{k-1} |\varphi(t_{i+1}) - \varphi(t_i)| < +\infty.
\]

For a mapping \( \varphi \) with finite variation on almost all straight lines, the partial derivatives \( \partial \varphi_i / \partial x_j, i, j = 1, \ldots, n \), exists almost everywhere in \( D \).

**Theorem 4.** [11] Let a homeomorphism \( \varphi : D \to D' \) generates a bounded composition operator

\[
\varphi^* : L^1_{\infty}(D') \to L^1_q(D), \quad q > n - 1.
\]

Then the inverse homeomorphism \( \varphi^{-1} : D' \to D \) belongs to the class \( \text{BVL}(D') \).

For readers convenience we reproduce here a slightly modified proof of this fact.

**Proof.** Take an arbitrary \( n \)-dimensional open parallelepiped \( P \) such that \( \overline{P} \subset D' \) and its edges are parallel to coordinate axis. Let us show that \( \varphi^{-1} \) has finite variation on almost all intersection of \( P \) and straight lines parallel to \( x_n \)-axis.

Let \( P_0 \) be the projection of \( P \) on the subspace \( x_n = 0 \), and let \( I \) be the projection of \( P \) on the coordinate axis \( x_n \). Then \( P = P_0 \times I \). The monotone countable-additive function \( \Phi \) determines a monotone countable additive function of open sets \( A \subset P_0 \) by the rule \( \Phi(A, P_0) = \Phi(A \times I) \). For almost all points \( z \in P_0 \), the quantity

\[
\overline{\Phi}(z, P_0) = \lim_{r \to 0} \left[ \frac{\Phi(B^{n-1}(z, r), P_0)}{r^{n-1}} \right]
\]

is finite [19] (here \( B^{n-1}(z, r) \) is the \( (n - 1) \)-dimensional ball of radius \( r > 0 \) centered at the point \( z \)).

The \( n \)-dimensional Lebesgue measure \( \Psi(U) = |\varphi^{-1}(U)| \), where \( U \) is an open set in \( D' \), is a monotone countable additive function and, therefore, also determines a monotone
countable additive function $\Psi(A, P_0) = \Psi(A \times I)$ defined on open sets $A \subset P_0$. Hence $\Psi'(z, P_0)$ is finite for almost all points $z \in P_0$.

Choose an arbitrary point $z \in P_0$ where $\Psi'(z, P_0) < +\infty$ and $\Psi'(z, P_0) < +\infty$. On the section $I_z = \{z\} \times I$ of the parallelepiped $P$, take arbitrary mutually disjoint closed intervals $\Delta_1, \ldots, \Delta_k$ with lengths $b_1, \ldots, b_k$ respectively. Let $R_i$ denote the open set of points for which distances from $\Delta_i$ smaller than a given $r > 0$:

$$R_i = \{x \in G : \text{dist}(x, \Delta_i) < r\}.$$ 

Consider the ring $(\Delta_i, R_i)$. Let $r > 0$ be selected so that $r < cb_i$ for $i = 1, \ldots, k$, where $c$ is a sufficiently small constant. Then the function $u_i(x) = \text{dist}(x, \Delta_i)/r$ is an admissible for ring $(\Delta_i, R_i)$.

By Theorem 3 we have the estimate

$$||\varphi^* u_i | L^1_q(D)||^q \leq \Phi(A')||u_i | L^\infty(A')||^q$$

for every function $u_i$, $i = 1, \ldots, k$.

Hence, for every ring $(\Delta_i, R_i)$, $i = 1, \ldots, k$, the inequality

$$\text{cap}_q^{1/2} (\varphi^{-1}(\Delta_i), \varphi^{-1}(R_i)) \leq \Phi(R_i)^{1/2} \text{cap}_\infty(\Delta_i, R_i)$$

holds.

The function $u_i(x) = \text{dist}(x, \Delta_i)/r$ is admissible for ring $(\Delta_i, R_i)$ and we have the upper estimate

$$\text{cap}_\infty(\Delta_i, R_i) \leq |\nabla u_i| = \frac{1}{r}.$$ 

Applying the lower bound for the capacity of the ring (Lemma 1), we obtain

$$\left(\frac{(\text{diam} \varphi^{-1}(\Delta_i))^{q/(n-1)}}{(\text{len} \varphi^{-1}(R_i))^{(q-n+1)/(n-1)}}\right)^{1/2} \leq c_1 \Phi(R_i)^{1/2} \cdot \frac{1}{r}.$$ 

This inequality gives

$$\text{diam} \varphi^{-1}(\Delta_i) \leq c_2 \left(\frac{|\varphi^{-1}(R_i)|}{r^{n-1}}\right)^{q-1} \cdot \left(\frac{\Phi(R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$ 

Summing over $i = 1, \ldots, k$ we obtain

$$\sum_{i=1}^k \text{diam} \varphi^{-1}(\Delta_i) \leq c_2 \sum_{i=1}^k \left(\frac{|\varphi^{-1}(R_i)|}{r^{n-1}}\right)^{q-1} \cdot \left(\frac{\Phi(R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$ 

Hence

$$\sum_{i=1}^k \text{diam} \varphi^{-1}(\Delta_i) \leq c_2 \left(\sum_{i=1}^k \frac{|\varphi^{-1}(R_i)|}{r^{n-1}}\right)^{q-1} \cdot \left(\sum_{i=1}^k \frac{\Phi(R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$ 

Using the Besicovitch type theorem [20] for the estimate of the value of the function $\Phi$ in terms of the multiplicity of a cover, we obtain

$$\sum_{i=1}^k \text{diam} \varphi^{-1}(\Delta_i) \leq c_3 \left(\frac{|\varphi^{-1}(\bigcup_{i=1}^k R_i)|}{r^{n-1}}\right)^{q-1} \cdot \left(\frac{\Phi(\bigcup_{i=1}^k R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$ 

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Hence
\[
\sum_{i=1}^{k} \text{diam } \varphi^{-1}(\Delta_i) \leq c_3 \left( \frac{|\varphi^{-1}(B^n_{r^{-1}}(z, r), P_0)|}{r^{n-1}} \right)^{\frac{q-n+1}{q}} \cdot \left( \frac{\Phi(B^n_{r^{-1}}(z, r), P_0)}{r^{n-1}} \right)^{\frac{n-1}{q}}.
\]

Because \( \overline{\Psi}(z, P_0) < +\infty \) and \( \overline{\Psi}(z, P_0) < +\infty \) we obtain finally
\[
\sum_{i=1}^{k} \text{diam } \varphi^{-1}(\Delta_i) < +\infty.
\]

Therefore \( \varphi^{-1} \in \text{BVL}(D') \).

Theorem proved.

2. Invertibility of composition operators

Let us recall the change of variable formula for Lebesgue integral [21]. Let a mapping \( \varphi : D \to \mathbb{R}^n \) be such that there exists a collection of closed sets \( \{A_k\}_{k=1}^{\infty} \), \( A_k \subset A_{k+1} \subset D \) for which restrictions \( \varphi|_{A_k} \) are Lipschitz mapping on sets \( A_k \) and
\[
\left| D \setminus \bigcup_{k=1}^{\infty} A_k \right| = 0.
\]

Then there exists a measurable set \( S \subset D \), \( |S| = 0 \) such that the mapping \( \varphi : D \setminus S \to \mathbb{R}^n \) has Luzin \( N \)-property and the change of variable formula
\[
\int_{E} f \circ \varphi(x) |J(x, \varphi)| \, dx = \int_{\mathbb{R}^n \setminus \varphi(S)} f(y) N_f(E, y) \, dy
\]
holds for every measurable set \( E \subset D \) and every nonnegative Borel measurable function \( f : \mathbb{R}^n \to \mathbb{R} \). Here \( N_f(y, E) \) is the multiplicity function defined as the number of preimages of \( y \) under \( f \) in \( E \).

If a mapping \( \varphi \) possesses Luzin \( N \)-property (the image of a set of measure zero has measure zero), then \( |\varphi(S)| = 0 \) and the second integral can be rewritten as the integral on \( \mathbb{R}^n \). Note, that if a homeomorphism \( \varphi : D \to D' \) belongs to the Sobolev space \( W^{1}_{n, \text{loc}}(D) \) then \( \varphi \) has Luzin \( N \)-property and the change of variable formula holds [22].

If a mapping \( \varphi : D \to \mathbb{R}^n \) belongs to the Sobolev space \( W^{1}_{1, \text{loc}}(D) \) then by [21] there exists a collection of closed sets \( \{A_k\}_{k=1}^{\infty} \), \( A_k \subset A_{k+1} \subset D \) for which restrictions \( f|_{A_k} \) are Lipschitz mapping on sets \( A_k \) and
\[
\left| D \setminus \bigcup_{k=1}^{\infty} A_k \right| = 0.
\]

Hence for such mappings the previous change of variable formula is correct.

Like in [23] (see also [13]) we define a measurable function
\[
\mu(y) = \begin{cases} 
\left( \frac{|\text{adj } D\varphi(x)|}{|J(x, \varphi)|} \right)_{x = \varphi^{-1}(y)} & \text{if } x \in D \setminus S \text{ and } J(x, \varphi) \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]
Because the homeomorphism \( \varphi \) has finite distortion the function \( \mu(y) \) is well defined almost everywhere in \( D' \).

The following lemma was proved (but does not formulated) in [13] under an additional assumption that \( |D\varphi| \) belongs to the Lorentz space \( L_{n-1,n}(D) \).

**Lemma 2.** Let a homeomorphism \( \varphi : D \to D' \), \( \varphi(D) = D' \) belongs to the Sobolev space \( L_q^1(D) \) for some \( q > n - 1 \). Then the function \( \mu \) is locally integrable in the domain \( D' \).

**Proof.** Using the change of variable formula for Lebesgue integral [21] and Luzin \( N \)-property of \( \varphi \) we have the following equality

\[
\int_{D'} \mu(y) \, dy = \int_{D' \setminus \varphi(S)} \mu(y) \, dy = \int_{D \setminus S} |\mu(\varphi(x))|J(x, \varphi)| \, dx = \int_{D} |\text{adj} \, D\varphi|(x) \, dx.
\]

Applying Hölder inequality, we obtain that for every compact subset \( F' \subset D' \)

\[
\int_{F'} \mu(y) \, dy \leq \int_{F} |\text{adj} \, D\varphi|(x) \, dx \leq C \int_{F} |D\varphi|^{n-1}(x) \, dx,
\]

where \( F' = \varphi(F) \). Therefore, \( \mu \) belongs to \( L_{1,\text{loc}}(D') \), since \( \varphi \) belongs to \( L_q^1(D) \), \( q > n - 1 \), and as consequence \( \varphi \in L_{n-1,\text{loc}}^1(D) \).

**Theorem 5.** Let a homeomorphism \( \varphi : D \to D' \), \( \varphi(D) = D' \), has finite distortion, Luzin \( N \)-property (the image of a set measure zero is a set measure zero) and generates a bounded composition operator

\[
\varphi^* : L_{\infty}^1(D') \to L_q^1(D), \quad q > n - 1.
\]

Then the inverse homeomorphism \( \varphi^{-1} : D' \to D \) has integrable first weak derivatives and induces a bounded composition operator

\[
(\varphi^{-1})^* : L_{q'}^1(D) \to L_q^1(D'), \quad q' = q/(q - n + 1).
\]

**Proof.** We prove that \( \varphi^{-1} \in \text{ACL}(D') \). Since absolute continuity is the local property, it is sufficient to prove that the mapping \( \varphi^{-1} \) belongs to \( \text{ACL} \) on every compact subset of \( D' \). Consider arbitrary cube \( Q' \subset D' \), \( \overline{Q'} \subset D \), with edges parallel to coordinate axes, and \( Q = \varphi^{-1}(Q') \). For \( i = 1, \ldots, n \) we will use a notation: \( Y_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \),

\[
F_i(x) = (\varphi_1(x), \ldots, \varphi_{i-1}(x), \varphi_{i+1}(x), \ldots, \varphi_n(x))
\]

and \( Q_i' = \text{intersection of the cube } Q' \text{ with a line } Y_i = \text{const.} \)

Using the change of variable formula and the Fubini theorem [24] we obtain the following estimate

\[
\int_{F_i(Q)} H^{n-1}(dY_i) \int_{Q_i'} \mu(y) \, H^1(dy) = \int_{Q'} \mu(y) \, dy = \int_{Q} |\text{adj} \, D\varphi|(x) \, dx < +\infty.
\]
Hence for almost all \( Y_i \in F_i(Q) \)
\[
\int_{Q'_i} \mu(y) \, H^1(dy) < +\infty.
\]

Let \( \text{ap} \, J \varphi(x) \) be an approximate Jacobian of the trace of the mapping \( \varphi \) on the set \( \varphi^{-1}(Q'_i) \) [24]. Consider a point \( x \in Q \) in which there exists a non-generated approximate differential \( \text{ap} \, Df(x) \) of the mapping \( \varphi : D \to D' \). Let \( L : \mathbb{R}^n \to \mathbb{R}^n \) be a linear mapping induced by this approximate differential \( \text{ap} \, Df(x) \). We denote by the symbol \( P \) the image of the unit cube \( Q_0 \) under the linear mapping \( L \) and by \( P_i \) the intersection of \( P \) with the image of the line \( x_i = 0 \). Let \( d_i \) be a length of \( P_i \). Then
\[
d_i \cdot |\text{adj} \, Df_i(x)| = |Q_0| = |J(x, \varphi)|.
\]
So, since \( d_i = \text{ap} \, J \varphi(x) \) we obtain that for almost all \( x \in Q \setminus Z, Z = \{x \in D : J(x, \varphi) = 0\} \), we have
\[
\text{ap} \, J \varphi(x) = \frac{|J(x, \varphi)|}{|\text{adj} \, Df_i(x)|}.
\]

So, we have for arbitrary compact set \( A' \subset Q'_i \), and for almost all \( Y_i \subset F_i(Q) \), the following inequality:
\[
H^1(\varphi^{-1}(A')) \leq \int_{\varphi^{-1}(A')} \frac{|\text{adj} \, D\varphi|(x)}{|\text{adj} \, Df_i|(x)|} \, H^1(dx)
\]
\[
= \int_{\varphi^{-1}(A')} \frac{|\text{adj} \, D\varphi|(x)}{|J(x, \varphi)|} \cdot \frac{|J(x, \varphi)|}{|\text{adj} \, Df_i|(x)|} \, H^1(dx) = \int_{\varphi^{-1}(A')} \mu(\varphi(x)) \text{ap} \, J \varphi(x) \, H^1(dx).
\]

By using the change of variable formula for the Lebesgue integral [24, 25] we obtain
\[
H^1(f^{-1}(A')) \leq \int_{A'} \mu(y) \, H^1(dy) < +\infty.
\]

Therefore, the mapping \( \varphi^{-1} \) is absolutely continuous on almost all lines in \( D' \) and is a weakly differentiable mapping.

Since the homeomorphism \( \varphi \) has Luzin \( N \)-property then preimage of a set positive measure is a set positive measure. Hence, the volume derivative of the inverse mapping
\[
J_{\varphi^{-1}}(y) = \lim_{r \to 0} \frac{|\varphi^{-1}(B(y, r))|}{|B(y, r)|} > 0
\]
almost everywhere in \( D' \). So \( J(y, \varphi^{-1}) \neq 0 \) for almost all points \( y \in D \). Integrability of the \( q' \)-distortion follows from the inequality
\[
|D\varphi^{-1}(y)| \leq |D\varphi(x)|^{n-1}/|J(x, \varphi)|
\]
which holds for almost all points \( y = \varphi(x) \in D' \).
Indeed, with the help of the change of variable formula, we have

$$\int_{D'} \left( \frac{|D\varphi^{-1}(y)|^{q'}}{|J(y, \varphi^{-1})|} \right)^{\frac{1}{q'-1}} \ dy = \int_{D'} \left( \frac{|D\varphi^{-1}(y)|}{|J(y, \varphi^{-1})|} \right)^{\frac{q'}{q'-1}} |J(y, \varphi^{-1})| \ dy$$

$$\leq \int_{D} \left( \frac{|D\varphi^{-1}(\varphi(x))|}{|J(\varphi(x), \varphi^{-1})|} \right)^{\frac{q'}{q'-1}} dx \leq \int_{D} |D\varphi(x)|^{q} dx < +\infty,$$

since by Theorem 2 $\varphi$ belongs to $L^{1}_{q}(D)$.

The boundedness of the composition operator follows from integrability of the $p'$-distortion [2]. The theorem proved.

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