Influence of a surface plates inhomogeneity on a translation oscillations of a drop

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Abstract. Natural and forced oscillations of a sandwiched fluid drop are investigated. A drop is cylindrical in equilibrium, surrounded by another liquid and bounded axially by two parallel solid planes. Hocking’s boundary conditions hold on the contact line: the velocity of the contact line motion is proportional to the deviation of the contact angle from its equilibrium value. The Hocking’s parameter (so-called wetting parameter) is the proportionality coefficient in this case. This parameter is considered as a function of coordinates, i.e. solid plates have nonuniform surface. The translation vibration force is perpendicular to the symmetry axis of the drop. The solution of the boundary value problem is found using Fourier series of Laplace operator eigen functions. Both translation mode and other azimuthal modes are excited because energy passes from first azimuthal modes to others due to nonuniform surfaces.

1. Introduction
Motion of triple contact line is one of the most important and well-known in the multiphase hydrodynamics [1-5]. The most frequently used condition for contact line velocity is the one applied by Hocking [6] for investigation of standing waves between two vertical walls. This condition assumes a linear relationship between the velocity of the contact line motion and the contact angle

\[ \frac{\partial \zeta^*}{\partial t} = \Lambda^* \mathbf{k} \cdot \nabla \zeta^*, \]

where \( \zeta^* \) is the deviation of the interface from the equilibrium position, \( \mathbf{k} \) is the external normal to the solid surface, \( \Lambda^* \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter) having the dimension of the velocity. There are two important limit of the boundary condition (1): (a) \( \zeta^* = 0 \) – the requirement of the fixed contact line (pinned-end edge condition), (b) \( \mathbf{k} \cdot \nabla \zeta^* = 0 \) – the constant contact angle. The Hocking parameter was constant in all papers in which this condition was used [7-17]. Note that in the article [18] it was suggested to consider this parameter as complex.

In the present article, we consider the azimuthal oscillations of cylindrical fluid drop which surrounded by other ideal liquid. We assume that Hocking parameter is considered as a function of coordinates. The natural and forced translational oscillations of cylindrical drop for case of uniform plates were investigated in [9, 17].
2. Problem formulation
Following [9], consider the oscillations of an incompressible fluid drop of density $\rho_i^*$ and kinematic viscosity $\nu_i^*$ surrounded by other fluid of density $\rho_e^*$ and kinematic viscosity $\nu_e^*$. The system is bounded by two parallel solid surfaces which are separated by a distance $h^*$ (see Figure 1). The vessel is closed at infinity. In equilibrium the drop has circle cylindrical form with radius $R_0^*$. The equilibrium contact angle $\theta_0$ is equal to 0.5\pi. The system is affected by a vibration field with amplitude $A^*$ and frequency $\omega^*$. The vibration force is directed perpendicularly the symmetry axis of the drop. We assume that the frequency is high to neglect the dissipative effects caused by acoustic radiation and viscous dissipation, i.e. $\omega^* R_0^* \ll c$ and $\delta = (\nu^*/\omega^*)^{1/2} \ll R_0^*$, where $c$ is the sound velocity, $\delta$ is the viscous boundary-layer thickness. The amplitude of external force is considered small in the sense that $A^* \ll R_0^*$.

Figure 1. Problem geometry.

We work in cylindrical coordinates $r^*$, $\alpha^*$ and $z^*$ because of the problem symmetry. Let the lateral surface of the drop be described by $r^* = r_0^* + \zeta^*(\alpha^*, z^*, t^*)$, where $\zeta^*$ is the surface deviation from equilibrium. The liquid motion is irrotational in the accepted approximations, which makes it convenient to introduce the velocity potential. Thus, the dynamics of the liquid is described by the Bernoulli and Laplace equations. We use the following quantities as the measurement units: $R_0^*$ for length, $h^*$ for height, $\left( (\rho_i^* + \rho_e^*) R_0^3 \sigma^{-1} \right)^{1/2}$ for time, $A^* \left( (\rho_i^* + \rho_e^*) R_0^3 \sigma^{-1} \right)^{1/2}$ for velocity, $A^* R_0^* \left( (\rho_i^* + \rho_e^*) R_0^3 \sigma^{-1} \right)^{-1/2}$ for velocity potential, $A^* \sigma (R_0^*)^{-2}$ for pressure, $A^*$ for surface deviation. The amplitude of oscillations is considered small $\varepsilon = A^*/ R_0^*$, which allows us to linearize the governing equations and simplify the boundary conditions. Thus, the dimensionless linear boundary value problem is determined by

$$\Delta \phi_j = 0 \quad p_j = -\rho_j \left( \frac{\partial \phi_j}{\partial t} - \omega^* \rho_j e^{\lambda^* t} \cos \alpha^* \right), \quad j = i, e,$$  \hspace{1cm} (2)

$$r = 1: \quad \left[ \frac{\partial \phi}{\partial r} \right] = 0, \quad \left[ p \right] = \zeta + \frac{\partial^2 \zeta}{\partial \alpha^2} + b^* \frac{\partial^2 \zeta}{\partial z^2}, \quad \frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial r},$$  \hspace{1cm} (3)

$$z = \pm 1/2: \quad \frac{\partial \phi}{\partial z} = 0,$$  \hspace{1cm} (4)

$$r = 1, \quad z = \pm 1/2: \quad \frac{\partial \zeta}{\partial t} = +\lambda^*(\alpha^*) \frac{\partial \zeta}{\partial z}, \quad \lambda^*(\alpha) = \lambda_0 |\sin(k \cos(\alpha))|,$$  \hspace{1cm} (5)

Boundary value problem (2)–(5) contains five dimensionless parameters: aspect ratio $-b = R_0^*/h^*$, the Hocking constant (wetting parameter) $-\lambda_0 = \lambda^* b \left( (\rho_i^* + \rho_e^*) R_0^3 \sigma^{-1} \right)^{1/2}$, the frequency $-\omega^*$.
\[ \omega = \omega^* \left( \left( \rho_e^* + \rho_i^* \right) R^* \sigma^{-1 \frac{1}{2}} \right), \text{ the density of the external liquid } - \rho_e^* = \rho_e^* \left( \rho_e^* + \rho_i^* \right)^{-1}, \text{ the density of the drop liquid } - \rho_i^* = \rho_i^* \left( \rho_e^* + \rho_i^* \right)^{-1}, \rho_e + \rho_i = 1. \text{ The function } \lambda (\alpha) \text{ describes} \]

3. Natural oscillations

To investigate the problem, it is convenient to begin with a consideration of the natural oscillations of a cylindrical drop. In further analysis our interest deals with the odd azimuthal oscillation modes due to the function \( \lambda (\alpha) \) and even vertical modes due to the external force. The system of equations and boundary conditions (2)-(5) can be written without vibrational force. Representing the fields of the velocity potentials and surface deviation as

\[ \varphi_e (r, \alpha, z, t) = \text{Re} \left[ i \Omega \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m+l,n} F_{e}^{i} (r) \cos (2 \pi n \alpha) \cos ((2m+1) \alpha) e^{i \omega t} \right] \]  
\[ \varphi_e (r, \alpha, z, t) = \text{Re} \left[ i \Omega \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m+l,n} F_{e}^{i} (r) \cos (2 \pi n \alpha) \cos ((2m+1) \alpha) e^{i \omega t} \right] \]  
\[ \zeta (z, \alpha, t) = \text{Re} \left[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m+l,n} \cos (2 \pi n \alpha) + d_{m+l,n} G_{m+l} (z) \cos ((2m+1) \alpha) e^{i \omega t} \right], \]

where \( \Omega \) is an eigen frequency, \( F_{e}^{i} (r) = r^{2m+1}, F_{e}^{i} (r) = l_{2m+1} (2 \pi n \alpha) \) for \( n \geq 1 \), \( G_{2m+1} (z) = \text{sh} \left( b^{-1} z \sqrt{(2m+1)^2} - 1 \right) \) for \( m \geq 1 \).

Substituting solutions (6)–(8) into Eqs. (2)–(5), we obtain a spectral-amplitude problem of complex algebraic equations series which eigenvalues are the values of the natural oscillation frequency \( \Omega \). This problem is not given because of their complexity and bulkiness. The equations have complex solutions (were solved numerically with the usage of the two-dimensional secant method), this leads to oscillation damping due to the dissipation on the contact line.

Figures 2 and 3 show the real part of \( \text{Re} (\Omega) \) (oscillation frequency) and imaginary part \( \text{Im} (\Omega) \) (damping ratio) of the complex natural frequency \( \Omega \) for the oscillation modes with azimuthal number 1 and 3 and zero vertical number, i.e. \( \Omega_{1,0} \) and \( \Omega_{3,0} \), respectively. The frequency decreases as the capillary parameter increases (see Figures 2a, 3a), and translation frequency \( \text{Re} (\Omega_{1,0}) \) vanishes for a critical value \( \lambda_0 \) for any parameters (see [9, 17] for details). That is, such "bending" oscillations exist only for sufficiently small \( \lambda_0 \) and are due to strong interaction of the drop with the plate. At higher values \( \lambda_0 \), the restoring force acting on the drop on the side of the substrate is not sufficient for the occurrence of oscillatory motion. In the case of a fixed edge angle, the drop freely slides over the substrate and there aren’t oscillations which accompanied by the motion of the drop center. The damping decrement \( \text{Im} (\Omega_{1,0}) \) has a branching point of the solution, after which one of the branches of the solution increases very rapidly with growth of \( \lambda_0 \) (Figure 2b). This solution is responsible for the attenuation of free translational oscillations. In general case, the damping decrement \( \text{Im} (\Omega) \) has maximum for finite value \( \lambda_0 \) and tends to zero for limit cases \( \lambda_0 \to 0 \) and \( \lambda_0 \to \infty \) (Figure 3b). The frequencies values are almost no different from theirs equilibrium values for finite and large wavenumber \( k \) (Figure 2,3). However the case of pinned contact line is realized for small \( k \) as \( \lambda (\alpha) \to \lambda_0 k | \cos (\alpha) | \) for \( k \ll 1 \).
For the azimuthal modes $\Omega_{0,m} (b = 1, \rho_i = 0.7)$ there are interval of values $\lambda_0$ on which the frequencies $\Omega_{0,m} (b = 1, \rho_i = 0.7)$ vanish (Figure 4a). Length of this interval increases with growth of the parameter $b$ (compare Figure 3a with Figure 4a). This effect is associated with dissipation during to the interaction of the contact line with the plates as the increase of parameter $b$ corresponds to an increase in the drop radius so contact line length (see [9] for details). The similar effect was obtained for volume oscillations of cylindrical bubble [19]. However heterogeneity of plate surface changes the magnitude of dissipation and can lead to the disappearance of this effect. The frequency $\Omega_{3,0}$ does not go to zero for any values of parameter $\lambda_0$ at $k = 1$ (Figure 4). The azimuthal mode $\Omega_{3,0}$ has very difficult behaviour at $k = 10$ which needs a future research.

4. Forced oscillations

Here we consider the problem of forced oscillations. The solution in this case is similar to solution (6)–(8) with exception for the time dependence: oscillation frequency is the forced frequency $\omega$.

The dependence of the surface oscillations amplitude in the layer centre $\zeta_0$ and at the upper plate $\zeta_k$, the form of contact line and the contact angle deviation on the upper plate on the forced frequency $\omega$ is given in Figures 5, 6 for different values of the Hocking parameter $\lambda_0$ and of the aspect ratio $b$. Note that the curves have a resonant shape in the limiting case $\lambda_0 \to 0$. In other limiting case $\lambda_0 \to \infty$ oscillations amplitude tends to $|\rho_i - \rho_s|$. The amplitude of the contact line oscillations is finite for any values of the parameter $\lambda_0$. Results for homogenous plates can be seen in [13, 17] more details.
The translation mode is excited by the first (Figures 5a, 6a,b) for any wavenumber $k$. This mode exists at $\lambda_0 = 1$ and $b = 1$ and the first resonance peak corresponds to main translation frequency (Figures 5a,b). Drop makes longitudinal periodic oscillations in this case (Figures 6a,b), the edge angle varies widely (Figure 6c). Note that there are “antiresonance” frequencies such that the contact line amplitude goes to zero. Contact angle also does not change at these points (Figure 5c). As was shown above, the first mode disappears with increasing parameter $\lambda_0$ and the third mode becomes the main resonance at $\lambda_0 = 5$ and $b = 3$ (Figures 5d,e). However, the third mode also can disappear at the aspect ratio $b$ and wavenumber $k$ are changed (Figures 3, 4). Figure 5d shown how the amplitude of the contact line changes with increasing of wavenumber $k$. Value of $k$ plays the role of an effective wetting parameter as $\lambda_0$. Thus the main frequency of third mode goes to zero as wavenumber $k$.

**Figure 5.** Deviation of contact line at upper plate (a, d), drop surface at center layer (b, e) and contact angle at upper plate vs frequency ($\rho = 0.7$, $k = 0.1$ – solid, $k = 1$ – dotted, $k = 10$ – dashed. (a)-(c) – $b = 1$, $\lambda_0 = 1$, (d)-(f) – $b = 3$, $\lambda_0 = 5$.

**Figure 6.** Evolution of drop surface (a), contact line (b) and contact angle (c) at upper plate ($b = 1$, $\rho = 0.7$, $\omega = 3.4$). $T = 2\pi \omega^{-1}$ – oscillation period, (a)-(b) $t = 0$ – solid line, $t = 0.125T$ – dashed, $t = 0.25T$ – dotted, $t = 0.375T$ – dot-and-dashed.
growths (see Figures). As results the resonance peak vanishes and the amplitude of surface oscillations goes to $\rho_1 - \rho_c$.

5. Conclusion
The behaviour of cylindrical drop between solid plates has been considered taking into account the dynamics of the contact angle under translational vibrations. The solid plates have nonuniform surfaces by function $\lambda(\alpha) = \lambda_0 \left| \sin(k \cos(\alpha)) \right|$. The main purpose of this paper is to propose a method for studying the drop forced oscillation on an inhomogeneous substrates and determining $\lambda$. The investigation of natural oscillations has shown that wavenumber $k$ plays the role of effective wetting parameter. Also the main frequency of azimuthal mode may not disappear for different values of $k$, in contrast to the homogeneous case.

For small values of the parameter $\lambda_0$, i.e. with a weak energy dissipation, the amplitude of the surface forced oscillations is large and tends to infinity in the limit $\lambda_0 \to 0$. The amplitude of the contact line oscillations is finite for any values of $\lambda_0$. There are "antiresonant" frequencies, i.e. such vibration frequencies, for which the contact line does not move for any values of $\lambda_0$, and the contact angle does not change. Moreover the change of a drop volume (i.e. aspect ratio $b$) can suppress resonances.

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6. References
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