Evaluations of infinite series involving reciprocal hyperbolic functions

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Abstract This paper presents an approach of summation of infinite series of hyperbolic functions. The approach is based on simple contour integral representations and residue computations with the help of some well known results of Eisenstein series given by Ramanujan, Berndt et al. Several series involving hyperbolic functions are evaluated, which can be expressed in terms of $z = 2F_1(1/2, 1/2; 1; x)$ and $z' = dz/dx$. When a certain parameter in these series equal to $\pi$ the series are summable in terms of $\Gamma$ functions. Moreover, some interesting new consequences and illustrative examples are considered.

Keywords Hyperbolic and trigonometric functions; Riemann zeta function; Jacobian elliptic function; Gamma function; residue theorem; Eisenstein series.

AMS Subject Classifications (2010): 34A25; 33B10; 11M36; 11M41; 11M99.

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1 Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

We begin with some basic notations. As usual, put $(a)_n$ denotes the ascending Pochhammer symbol by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \cdots (a+n-1) \quad \text{and} \quad (a)_0 := 1,$$  \hspace{1cm} (1.1)

where $a \in \mathbb{C}$ is any complex number and $n \in \mathbb{N}_0$ is a nonnegative integer, $\Gamma(z)$ denotes the Gamma function, when $\Re(z) > 0$ then

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt.$$  

The Gaussian or ordinary hypergeometric function $\,_{2}F_{1}(a, b; c; x)$ is defined for $|x| < 1$ by the power series

$$\,_{2}F_{1}(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!} \quad (a, b, c \in \mathbb{C}). \hspace{1cm} (1.2)$$

The complete elliptic integral of the first kind is defined by (Whittaker and Watson [31])

$$K := K(x) := K(k^2) := \frac{\pi}{2} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{\pi}{2} \,_{2}F_{1} \left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right). \hspace{1cm} (1.3)$$

Here $x = k^2$ and $k \ (0 < k < 1)$ is the modulus of $K$. The complementary modulus $k'$ is defined by $k' = \sqrt{1-k^2}$. Furthermore,

$$K' := K(k'^2) = \frac{\pi}{2} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2 \sin^2 \varphi}} = \frac{\pi}{2} \,_{2}F_{1} \left( \frac{1}{2}, \frac{1}{2}; 1; 1-k^2 \right). \hspace{1cm} (1.4)$$

Similarly, the complete elliptic integral of the second kind is denoted by (Whittaker and Watson [31])

$$E := E(x) := E(k^2) := \frac{\pi}{2} \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} d\varphi = \frac{\pi}{2} \,_{2}F_{1} \left( \frac{1}{2}, \frac{1}{2}; 1; 1-k^2 \right). \hspace{1cm} (1.5)$$

In order to better state our main results, we shall henceforth adopt the notations of Ramanujan (see Berndt’s book [7]). Let

$$x := k^2, \quad y := y(x) := \frac{K'}{K}, \quad q := q(x) := e^{-y}, \quad z := z(x) := \frac{2}{\pi} K, \quad z' := \frac{dz}{dx}. \hspace{1cm} (1.6)$$

Using the identity $(a)_{n+1} = a(a+1)_n$, it is easily shown that

$$\frac{d}{dx} \,_{2}F_{1}(a, b; c; x) = \frac{ab}{c} \,_{2}F_{1}(a + 1, b + 1; c + 1; x)$$
and more generally,
\[
\frac{d^n}{dx^n} F_1(a, b; c; x) = \frac{(a)_n (b)_n}{(c)_n} F_1(a + n, b + n; c + n; x) \quad (n \in \mathbb{N}_0).
\]  
(1.7)

Then,
\[
\frac{d^n z}{dx^n} = \frac{(1/2)^2}{n!} F_1 \left( \frac{1}{2} + n, \frac{1}{2} + n; 1 + n; x \right).
\]  
(1.8)

Applying the identity (see Theorem 3.5.4(i) in the section 3.5 of chapter 3 of Andrews, Askey and Roy book [1])
\[
2 F_1 \left( a, b; \frac{a + b + 1}{2}; \frac{1}{2} \right) = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{a + b + 1}{2} \right)}{\Gamma \left( \frac{a + 1}{2} \right) \Gamma \left( \frac{b + 1}{2} \right)},
\]  
(1.9)

an elementary calculation shows that
\[
\frac{d^n z}{dx^n} \bigg|_{x=1/2} = \frac{(1/2)^2 \sqrt{\pi}}{n! \left( \frac{1}{2} + \frac{3}{4} \right)}.
\]  
(1.10)

Setting \( n = 1, 2, 3 \) in equation above yield
\[
z(1/2) = \frac{\Gamma^2 \left( \frac{1}{4} \right)}{2\pi^{3/2}}, \quad z'(1/2) = 4\frac{\sqrt{\pi}}{\Gamma^2 \left( \frac{1}{4} \right)} \quad \text{and} \quad z''(1/2) = \frac{\Gamma^2 \left( \frac{1}{4} \right)}{2\pi^{3/2}},
\]
where we have used the two relations
\[
\Gamma(x + 1) = x\Gamma(x) \quad \text{and} \quad \Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}.
\]

**Definition 1.1** Let \( m \in \mathbb{N} \) be a positive integer and \( p \in \mathbb{Z} \) be a integer. Define

- \( S_{p,m}(y) := \sum_{n=1}^{\infty} \frac{n^p}{\sinh^m(ny)} \), \( \bar{S}_{p,m}(y) := \sum_{n=1}^{\infty} \frac{n^p}{\sinh^m(ny)} (-1)^{n-1} \),
- \( C_{p,m}(y) := \sum_{n=1}^{\infty} \frac{n^p}{\cosh^m(ny)} \), \( \bar{C}_{p,m}(y) := \sum_{n=1}^{\infty} \frac{n^p}{\cosh^m(ny)} (-1)^{n-1} \),
- \( S'_{p,m}(y) := \sum_{n=1}^{\infty} \frac{(2n - 1)^p}{\sinh^m((2n - 1)y/2)} \), \( \bar{S}'_{p,m}(y) := \sum_{n=1}^{\infty} \frac{(2n - 1)^p}{\sinh^m((2n - 1)y/2)} (-1)^{n-1} \),
- \( C'_{p,m}(y) := \sum_{n=1}^{\infty} \frac{(2n - 1)^p}{\cosh^m((2n - 1)y/2)} \), \( \bar{C}'_{p,m}(y) := \sum_{n=1}^{\infty} \frac{(2n - 1)^p}{\cosh^m((2n - 1)y/2)} (-1)^{n-1} \).

By (1.6), then the eight series defined in Definition 1.1 can be rewritten as Eisenstein series

- \( S_{p,m}(y) = 2^m \sum_{n=1}^{\infty} \frac{n^p q^{mn}}{(1 - q^{2n})^m} \), \( \bar{S}_{p,m}(y) = 2^m \sum_{n=1}^{\infty} \frac{n^p q^{mn}}{(1 - q^{2n})^m} (-1)^{n-1} \),
expressed in closed forms for all integral \( m \) hyperbolic functions may be expressed in closed form e.g.
\[
y/\pi = \text{depend on a certain parameter,}
\]
Further, he studied several infinite series involving exponential and hyperbolic function, which depend on a certain parameter, \( y/\pi \). When \( y/\pi \) is the square root of a rational number the sums may be expressed in terms of \( \Gamma \)-functions and other well-known transcendental numbers. For details may be found in [35]. Some other related results for infinite series involving hyperbolic functions may be seen in the works of [2–5, 8, 9, 11, 12, 20, 27, 32]. For instance, Berndt [4, 5] found a lot of identities about infinite series using a certain modular transformation formula that originally stems from the generalized Eisenstein series. There are also a lot of recent contributions on Eisenstein series involving hyperbolic functions (see [19, 28–30]) and the references therein.
However, so far, no one has solved the evaluations of series with \( m, p \geq 2 \) defined in Definition 1.1. The purpose of this paper is to present a method of the following eight series of quadratic hyperbolic functions:

\[
S_{2p,2}(y), \ S'_{2p,2}(y), \ C_{2p,2}(y), \ C'_{2p,2}(y), \ \bar{S}_{2p,2}(y), \ \bar{S}'_{2p-1,2}(y), \ \bar{C}_{2p,2}(y) \text{ and } \bar{C}'_{2p-1,2}(y) \quad (p \in \mathbb{N}).
\]

In sections 3.1 and 3.2, we use residue theorem and asymptotic formulas of trigonometric and hyperbolic functions at the poles to establish four equations involving infinite series of trigonometric and hyperbolic functions (see Theorem 3.1). Then apply the four equations obtained and use several well-known results given by Ramanujan, Berndt et al. to prove that the four alternating sums \( \bar{S}_{2p,2}(y), \ \bar{S}'_{2p-1,2}(y), \ \bar{C}_{2p,2}(y) \) and \( \bar{C}'_{2p-1,2}(y) \) can be expressed in terms of \( z \) and \( z' \) (defined by (1.6)). Furthermore, in section 3.3, we use the results about Eisenstein series (3.52)-(3.59) and (3.63)-(3.66) established by Ramanujan, and apply the transformations (2.23)-(2.25) to prove that the four series \( S_{2p,2}(y), \ S'_{2p,2}(y), \ C_{2p,2}(y) \) and \( C'_{2p,2}(y) \) can be evaluated in terms of \( z \) and \( z' \). In the last fourth section, we give numerous examples of series involving hyperbolic functions. Moreover, we prove that the eight series \( (p \geq k) \)

\[
S_{2p,2k}(y), \ \bar{S}_{2p,2k}(y), \ C_{2p,2k}(y), \ \bar{C}_{2p,2k}(y), \ S'_{2p,2k}(y), \ \bar{S}'_{2p-1,2k}(y), \ \bar{C}'_{2p,2k}(y), \ \bar{C}'_{2p-1,2k}(y)
\]
can be evaluated, and give recurrence formulas.

2 Some Lemmas and Transformations

In this section we give some lemmas and transformations, which will be useful in the development of the main results of this paper.

2.1 Three Lemmas

Lemma 2.1 (Residue Theorem, [14]) Let \( \xi(s) \) be a kernel function and let \( r(s) \) be a function which is \( O(s^{-2}) \) at infinity. Then

\[
\sum_{\alpha \in O} \text{Res}(r(s)\xi(s))_{s=\alpha} + \sum_{\beta \in S} \text{Res}(r(s)\xi(s))_{s=\beta} = 0, \tag{2.1}
\]

where \( S \) is the set of poles of \( r(s) \) and \( O \) is the set of poles of \( \xi(s) \) that are not poles \( r(s) \). Here \( \text{Res}(r(s))_{s=\alpha} \) denotes the residue of \( r(s) \) at \( s = \alpha \). The kernel function \( \xi(s) \) is meromorphic in the whole complex plane and satisfies \( \xi(s) = o(s) \) over an infinite collection of circles \( |s| = \rho_k \) with \( \rho_k \to \infty \).

Lemma 2.2 ([32, 33]) Let \( n \) be a integer, then the following asymptotic formulas hold:

\[
\pi \cot(\pi s) \overset{s \to n}{=} \frac{1}{s - n} - 2 \sum_{k=1}^{\infty} \zeta(2k) (s - n)^{2k-1}, \tag{2.2}
\]

\[
\frac{\pi}{\sin(\pi s)} \overset{s \to n}{=} (-1)^n \left( \frac{1}{s - n} + 2 \sum_{k=1}^{\infty} \zeta(2k) (s - n)^{2k-1} \right), \tag{2.3}
\]

\[
\pi \coth(\pi s) \overset{s \to ni}{=} \frac{1}{s - ni} - 2 \sum_{k=1}^{\infty} (-1)^k \zeta(2k) (s - ni)^{2k-1}, \tag{2.4}
\]

\]
\[
\frac{\pi}{\sinh(\pi s)} \rightarrow (-1)^n \left( \frac{1}{s-ni} + 2 \sum_{k=1}^{\infty} (-1)^k \zeta(2k) (s-ni)^{2k-1} \right),
\]  
(2.5)

\[
\pi \tan(\pi s) \rightarrow \frac{1}{s-2n-1} + 2 \sum_{k=1}^{\infty} \zeta(2k) \left( s-\frac{2n-1}{2} \right)^{2k-1},
\]  
(2.6)

\[
\pi \tanh(\pi s) \rightarrow \frac{1}{s-\frac{2n-1}{i}} - 2 \sum_{k=1}^{\infty} (-1)^k \zeta(2k) \left( s-\frac{2n-1}{2} i \right)^{2k-1},
\]  
(2.7)

\[
\pi \cos(\pi s) \rightarrow (-1)^n \left\{ \frac{1}{s-\frac{2n-1}{2}} + 2 \sum_{k=1}^{\infty} \zeta(2k) \left( s-\frac{2n-1}{2} \right)^{2k-1} \right\},
\]  
(2.8)

\[
\pi \cosh(\pi s) \rightarrow (-1)^n i \left\{ \frac{1}{s-\frac{2n-1}{2} i} + 2 \sum_{k=1}^{\infty} (-1)^k \zeta(2k) \left( s-\frac{2n-1}{2} i \right)^{2k-1} \right\},
\]  
(2.9)

where \( \zeta(s) \) and \( \bar{\zeta}(s) \) denote the Riemann zeta function and alternating Riemann zeta function by

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1) \quad \text{and} \quad \bar{\zeta}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (\Re(s) \geq 1).
\]

When \( s = 2m \ (m \in \mathbb{N}) \) is an even, Euler proved the famous formula (\( [1] \))

\[
\zeta(2m) = \frac{(-1)^{m-1} B_{2m}}{2 (2m)!} (2\pi)^{2m},
\]  
(2.10)

where \( B_{2m} \) is Bernoulli number.

**Lemma 2.3** ([7]) For any \( p \in \mathbb{N} \), the four series

\[
S_{2p-1,1}(y), \ C_{2p-1,1}(y), \ C_{2p-2,1}(y) \text{ and } C'_{2p-2,1}(y)
\]

can be expressed in terms of \( z \) (Here \( z \) is defined by (1.6)).

**Proof.** On the one hand, by theorems of Hermite, which may be found in Cayley’s book [13] or see Berndt’s book [7] and paper [9], for \( u = 2Kt/\pi \) and \( |u| < K' \),

\[
\text{smu} = u - (1 + k^2) \frac{u^3}{3!} + (1 + 14k^2 + k^4) \frac{u^5}{5!} - (1 + 135k^2 + 135k^4 + k^6) \frac{u^7}{7!} + \cdots,
\]  
(2.11)

\[
\text{cnu} = 1 - \frac{u^2}{2!} + (1 + 4k^2) \frac{u^4}{4!} - (1 + 44k^2 + 16k^4) \frac{u^6}{6!} + (1 + 408k^2 + 912k^4 + 64k^6) \frac{u^8}{8!} + \cdots,
\]  
(2.12)

\[
\text{dmu} = 1 - k^2 \frac{u^2}{2!} + k^2(4 + k^2) \frac{u^4}{4!} - k^2(16 + 44k^2 + k^4) \frac{u^6}{6!} + \cdots.
\]
\[ + k^2 (64 + 912k^2 + 408k^4 + k^6) u^8 + \cdots, \quad (2.13) \]

where \(sn\), \(cn\) and \(dn\) denote the Jacobian elliptic functions (The power series expansions of the functions \(sn\), \(cn\), \(dn\) and related functions are easily found. They are given to quite high orders by Hancock [17]). On the other hand, by results of Jacobi, which may be found in Whittaker and Watson’s treatise [31] (or see Guo, Tu and Wang [16]), for \(u = 2Kt/\pi\) and \(|u| < K'\),

\[
\begin{align*}
\text{sn} &= \frac{2\pi}{Kk} \sum_{n=0}^{\infty} \frac{q^{(2n+1)/2} \sin(2n+1)t}{1 - q^{2n+1}} \\
&= \frac{2\pi}{Kk} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1} \sin(2n+1)t}{(2j+1)!} \\
&= \frac{\pi}{Kk} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi u/2K)^{2j+1}}{(2j+1)!} S_{2j+1,1}(y), \quad (2.14) \\
\text{cn} &= \frac{2\pi}{Kk} \sum_{n=0}^{\infty} \frac{q^{n+1/2} \cos(2n+1)t}{1 + q^{2n+1}} \\
&= \frac{2\pi}{Kk} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j} \cos(2n+1)t}{(2j)!} \\
&= \frac{\pi}{Kk} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi u/2K)^{2j}}{(2j)!} C_{2j,1}(y), \quad (2.15) \\
\text{dn} &= \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n \cos(2nt)}{1 + q^{2n}} \\
&= \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{j=0}^{\infty} \frac{(-1)^j (2t)^{2j}}{(2j)!} \sum_{n=1}^{\infty} \frac{n^{2j} q^n}{1 + q^{2n}} \\
&= \frac{\pi}{2K} + \frac{\pi}{K} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi u/K)^{2j}}{(2j)!} C_{2j,1}(y), \quad (2.16)
\end{align*}
\]

and, from Greenhill’s treatise [15], we have

\[
\text{sn}^2 u = \left(1 - \frac{E}{K}\right) \frac{1}{k^2} - \frac{\pi^2}{K^2 k^2} \sum_{n=1}^{\infty} \frac{n \cos(n \pi u/K)}{\sinh(ny)} \\
= \left(1 - \frac{E}{K}\right) \frac{1}{k^2} - \frac{\pi^2}{K^2 k^2} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi u/K)^{2j}}{(2j)!} S_{2j+1,1}(y), \quad (2.17)
\]

where \(K\) and \(E\) are represented by \((1.3)\) and \((1.5)\), respectively.

From \((2.14)\), a straightforward calculation yields

\[
\text{sn}^2 u = u^2 - (1 + k^2) \frac{u^4}{3} + (2 + 13k^2 + 2k^4) \frac{u^6}{45} + \cdots. \quad (2.18)
\]

Then, combining \((2.11)-(2.18)\) and applying the notations of \((1.6)\), by comparing coefficients of \(u^j\) in equations \((2.11)-(2.18)\) in the right hand side, we may deduce the desired results. Thus, the proof of the lemma is therefore complete. \(\square\)
In the Entries 15, 16 and 17 of Chapter 17 of [7], Berndt gave some explicit formulas for

\[ S_{2p-1,1}(y) \quad (p = 2, 3, 4, 5), \quad S'_{2p-1,1}(y) \quad (p = 1, 2, 3, 4, 5, 6), \]
\[ C_{2p-2,1}(y) \quad (p = 1, 2, 3, 4, 5), \quad C'_{2p-2,1}(y) \quad (p = 1, 2, 3, 4, 5). \]

Note that from [7], we can also obtain that

\[ S_{1,1}(y) = \frac{1}{4}(xz^2 - 2x(1 - x)zz'). \]

2.2 Three Transformations

In the Chapters 17 and 18 of Berndt’s book [7], he described some procedures in the theory of elliptic functions by which “new” formulas can be produced from “old” formulas. Here, we need to cite the three transformations:

\[
\begin{align*}
\text{Given} & \quad \Omega(x, e^{-y}, z) = 0 \Rightarrow \Omega(1 - x, e^{-\pi y/2}, yz/\pi) = 0, \quad (2.19) \\
\text{Given} & \quad \Omega(x, e^{-y}, z) = 0 \Rightarrow \Omega\left(\frac{x}{x-1}, -e^{-y}, z\sqrt{1-x}\right) = 0, \quad (2.20) \\
\text{Given} & \quad \Omega(x, e^{-y}, z) = 0 \Rightarrow \Omega\left(\left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2, e^{-2y}, \frac{1}{2}z(1 + \sqrt{1-x})\right) = 0, \quad (2.21)
\end{align*}
\]

where \(\Omega(x, e^{-y}, z)\) is an arbitrary function of \(x, e^{-y}\) and \(z\). In fact, if add \(z'\) in the left hand side of transformations (2.19)-(2.21), namely, change \(\Omega(x, e^{-y}, z)\) to the form

\[ \Omega(x, e^{-y}, z, z') = 0, \quad (2.22) \]

(\(\Omega(x, e^{-y}, z, z')\) is an arbitrary function of \(x, e^{-y}, z\) and \(z'\)) after some elementary manipulation we can achieve the following three new transformations.

**Theorem 2.4** Given the formula (2.22), we can deduce the formulas

\[
\begin{align*}
\Omega\left(1 - x, e^{-\pi y/2}, yz/\pi, \frac{1}{\pi}\left(\frac{1}{x(1-x)}z - yz'\right)\right) = 0, \quad (2.23) \\
\Omega\left(\frac{x}{x-1}, -e^{-y}, z\sqrt{1-x}, (1-x)^{3/2}\left(\frac{z}{2} - (1-x)z'\right)\right) = 0, \quad (2.24) \\
\Omega\left(\left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2, e^{-2y}, \frac{1}{2}z(1 + \sqrt{1-x}), \frac{1 + \sqrt{1-x}}{4(1 - \sqrt{1-x})}\left(-\frac{z}{2} + (\sqrt{1-x} + 1)\right)z'\right) = 0. \quad (2.25)
\end{align*}
\]

**Proof.** Suppose that \(x_1, y_1, z_1\) and \(z'_1\) is another set of parameters such that

\[ \Omega(x_1, e^{-y_1}, z_1, z'_1) = 0. \]

According to (2.19)-(2.21), in order to prove (2.23)-(2.25) we only need to prove that

\[ z'_1 = \frac{1}{\pi}\left(\frac{1}{x(1-x)}z - yz'\right), \quad \text{if} \quad x_1 = 1 - x, \]

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\[ z'_1 = (1 - x)^{3/2} \left( \frac{z}{2} - (1 - x)z' \right), \text{ if } x_1 = \frac{x}{x - 1}, \]
\[ z'_1 = \frac{(1 + \sqrt{1 - x})^3}{4(1 - \sqrt{1 - x})} \left( - \frac{z}{2} + (\sqrt{1 - x} + 1 - x) \right) z', \text{ if } x_1 = \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right)^2. \]

By formula (1.7) and definition of \( z \), then
\[ z' = \frac{1}{4} \mathbf{2F1} \left( \frac{3}{2}, \frac{3}{2}; 2; x \right). \tag{2.26} \]

Hence, if \( x_1 = 1 - x \),
\[ z'_1 = \frac{1}{4} \mathbf{2F1} \left( \frac{3}{2}, \frac{3}{2}; 2; 1 - x \right) = - \frac{d}{dx} \mathbf{2F1} \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - x \right) \]
\[ = - \frac{d}{dx} \left( \frac{yz}{\pi} \right) = \frac{1}{\pi} \left( \frac{1}{x(1 - x)z} - yz' \right), \tag{2.27} \]

if \( x_1 = \frac{x}{x - 1} \),
\[ z'_1 = \frac{1}{4} \mathbf{2F1} \left( \frac{3}{2}, \frac{3}{2}; 2; \frac{x}{x - 1} \right) = -(x - 1)^2 \frac{d}{dx} \mathbf{2F1} \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{x}{x - 1} \right) \]
\[ = -(x - 1)^2 \frac{d}{dx} \left( \sqrt{1 - xz} \right) = (1 - x)^{3/2} \left( \frac{z}{2} - (1 - x)z' \right). \tag{2.28} \]

if \( x_1 = \left( (1 - \sqrt{1 - x})/(1 + \sqrt{1 - x}) \right)^2 \),
\[ z'_1 = \frac{1}{4} \mathbf{2F1} \left( \frac{3}{2}, \frac{3}{2}; 2; \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right)^2 \right) \]
\[ = \frac{(1 + \sqrt{1 - x})^3 \sqrt{1 - x} \frac{d}{dx} \mathbf{2F1} \left( \frac{1}{2}, \frac{1}{2}; 1; \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right)^2 \right)}{2(1 - \sqrt{1 - x})} \]
\[ = \frac{(1 + \sqrt{1 - x})^3 \sqrt{1 - x} \frac{d}{dx} (z(1 + \sqrt{1 - x}))}{4(1 - \sqrt{1 - x})} \]
\[ = \frac{(1 + \sqrt{1 - x})^3}{4(1 - \sqrt{1 - x})} \left( - \frac{z}{2} + (\sqrt{1 - x} + 1 - x) \right) z', \tag{2.29} \]

where we have used the three relations (see [6, 7])
\[ \frac{dy}{dx} = - \frac{1}{x(1 - x)z^2}, \quad 2 \mathbf{F1} \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{x}{x - 1} \right) = \sqrt{1 - xz} \]
and
\[ 2 \mathbf{F1} \left( \frac{1}{2}, \frac{1}{2}; 1; \left( \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right)^2 \right) = \frac{1}{2} z(1 + \sqrt{1 - x}). \]

Thus, the proofs of (2.23)-(2.25) are completed. □

It should be emphasized that the reference [7] also contains many other types of transformations.
3 Main Theorems and Proofs

In this section, we will prove our main results by using the residue theorem, transformations (2.23)-(2.25) and some well-known results of Ramanujan, Berndt et al. First, we need to give the following two definitions.

**Definition 3.1** Let $s$ be a complex number and $a, b, \theta$ be reals with $|\theta| < 2b\pi$ and $a, b \neq 0$. Define

$$F_1(s, \theta; a, b) := \frac{\pi^3 \cosh(\theta s)}{\sin(a\pi s) \sinh(b\pi s)},$$

$$F_2(s, \theta; a, b) := \frac{\pi^3 \sinh(\theta s)}{\cos(a\pi s) \cosh(b\pi s)},$$

$$F_3(s, \theta; a, b) := \frac{\pi^3 \cosh(\theta s)}{\sin(a\pi s) \cosh(b\pi s)},$$

$$F_4(s, \theta; a, b) := \frac{\pi^3 \sinh(\theta s)}{\cos(a\pi s) \sinh(b\pi s)}.$$
Similarly, many other asymptotic expansions can also be established. For example,
\[
\left( \frac{\pi}{\sin(a\pi s)} \right)^3 = (-1)^n \left( \frac{1}{(as-n)^3} + 3\frac{\zeta(2)}{as-n} + \frac{51}{4} \frac{\zeta(4)}{(as-n)^2} + o(as-n) \right), \ s \to n/a.
\]

### 3.1 Some Theorems and Proofs

**Theorem 3.1** For reals \(a, b, \theta\) and \(|\theta| < 2b\pi\), \(a, b \neq 0\),

\[
b^2 \pi^2 \sum_{n=1}^{\infty} \frac{\cosh(\theta n/a)}{\sinh^2(bn\pi/a)} (-1)^n + \frac{\theta^2}{4} + \frac{\zeta(2)}{2} (a^2 - 2b^2) = 0,
\]

\[
b^2 \pi^2 \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta/2a)}{\cosh^2((2n-1)\pi b/2a)} (-1)^n - \frac{\theta^2}{4} + \frac{\zeta(2)}{2} (a^2 - 2b^2) = 0,
\]

\[
b^2 \pi^2 \sum_{n=1}^{\infty} \frac{\cos((2n-1)\theta/2a)}{\sinh((2n-1)\pi a/2b)} (-1)^n - \frac{\theta^2}{4} + \frac{\zeta(2)}{2} (a^2 - 2b^2) = 0,
\]

**Proof.** Now we are ready to prove Theorem 3.1. In the context of this paper, the proof is based on the functions \(F_j(s, \theta; a, b)\) \((j = 1, 2, 3, 4)\) and the usual residue computation. It is clear that the functions \(F_i(s, \theta; a, b)\) \((i = 1, 2, 3, 4)\) are meromorphic in the entire complex plane with some simple poles, see Table 3.1. (Here \(n\) is positive integer).

| poles            | functions            |
|------------------|----------------------|
| \(0, \pm n/a, \pm n/b\) | \(F_1(s, \theta; a, b)\) |
| \(\pm (2n-1)/2a, \pm (2n-1)i/2b\) | \(F_2(s, \theta; a, b)\) |
| \(0, \pm n/a \pm (2n-1)i/2b\) | \(F_3(s, \theta; a, b)\) |
| \(0, \pm (2n-1)/2a, \pm n/b\) | \(F_4(s, \theta; a, b)\) |

**TABLE 3.1.** Poles of functions \(F_i(s, \theta; a, b)\)

Since the proofs of identities (3.6)-(3.8) are similar to the proof of (3.5), so we only prove the formula (3.5) in here. First, we note that the function \(F_1(s, \theta; a, b)\) only have poles at the
Therefore, the residue of the pole of order three at 0 is found to be
\[ \text{Res}(F_1(s, \theta; a, b))_{s=\pm n/a} = (-1)^n \frac{n^2 \pi^2}{a} \frac{\cosh(n\theta/a)}{\sinh^2(bn\pi/a)}. \] (3.9)

From (3.2), at a imaginary number \( \pm ni/b \), the pole has order two and the residue is
\[ \text{Res}(F_1(s, \theta; a, b))_{s=\pm ni/b} = \frac{\pi}{b^2} \frac{d}{ds} \left( \frac{\cosh(\theta s)}{\sin(a\pi s)} \right) |_{s=\pm ni/b} \]
\[ = \frac{\theta \pi}{b^2} \frac{\sin(n\theta/b)}{\sinh(\pi n/b)} + a^2 \frac{\cos(n\theta/b) \cosh(\pi n/b)}{\sinh(\pi n/b)}. \] (3.10)

Furthermore, applying (2.3) and (2.5) we deduce the asymptotic expansion
\[ F_1(s, \theta; a, b) = \cosh(\theta s) \left( \frac{1}{ab^2 s} + \left( \frac{a}{b^2} - \frac{2}{a} \right) \frac{\zeta(2)}{s} + o(1) \right), \quad s \to 0. \]

Therefore, the residue of the pole of order three at 0 is found to be
\[ \text{Res}(F_1(s, \theta; a, b))_{s=0} = \frac{1}{2} \frac{d^2}{ds^2} \left\{ \cosh(\theta s) \left( \frac{1}{ab^2 s} + \left( \frac{a}{b^2} - \frac{2}{a} \right) \frac{\zeta(2)}{s} + o(s^3) \right) \right\} |_{s=0} \]
\[ = \frac{\theta^2}{2ab^2} + \zeta(2) \left( \frac{a}{b^2} - \frac{2}{a} \right). \] (3.11)

By Lemma 2.1, summing the three contributions (3.9)-(3.11) yields the desired result (3.5). This completes the proof. Similarly, consider functions \( F_j(s, \theta; a, b) (j = 2, 3, 4) \). Then using residue theorems and asymptotic formulas (3.1)-(3.4), by calculations similar to those above, we can deduce the evaluations (3.6)-(3.8). \( \square \)

**Theorem 3.2** For reals \( a, b, \theta \) and \(|\theta| < 2b\pi\), \( a, b \neq 0 \),

\[
\frac{\pi^2 \theta}{a^2} \sum_{n=1}^{\infty} \frac{\cosh(n\theta/a)}{\sinh^3(\pi n/b/a)} - 2 \frac{\pi^3 b}{a^2} \sum_{n=1}^{\infty} \frac{\sinh(n\theta/a) \cosh(\pi n b/a)}{\sinh^3(\pi n/b/a)} - \frac{\pi^2 \theta}{b^2} \sum_{n=1}^{\infty} \frac{\cos(n\theta/b)}{\sinh^2(\pi n a/b)} \\
+ 2 \frac{\pi^3}{b^2} \sum_{n=1}^{\infty} \frac{\sin(n\theta/b) \cosh(\pi n a/b)}{\sinh^3(\pi n a/b)} + \theta \left( \frac{1}{b^2} - \frac{1}{a^2} \right) \zeta(2) + \frac{\theta^3}{12a^2 b^2} = 0, \quad (3.12)
\]

\[
\frac{\theta}{a^2} \sum_{n=1}^{\infty} \frac{\cosh((2n - 1)\theta/2a)}{\cosh^2((2n - 1)b\pi/2a)} - \frac{\pi b}{a^2} \sum_{n=1}^{\infty} \frac{\sinh((2n - 1)\theta/2a) \sinh((2n - 1)b\pi/2a)}{\cosh^3((2n - 1)b\pi/2a)} \\
- \frac{\theta}{b^2} \sum_{n=1}^{\infty} \frac{\cos((2n - 1)\theta/2b)}{\cosh^2((2n - 1)a\pi/2b)} + \frac{\pi a}{b^2} \sum_{n=1}^{\infty} \frac{\sin((2n - 1)\theta/2b) \sinh((2n - 1)a\pi/2b)}{\cosh^3((2n - 1)a\pi/2b)} = 0, \quad (3.13)
\]

\[
\frac{\theta}{a^2} \sum_{n=1}^{\infty} \frac{\cosh(n\theta/a)}{\cosh^2(\pi n b/a)} - \frac{\pi b}{a^2} \sum_{n=1}^{\infty} \frac{\sinh(n\theta/a) \sinh(\pi n b/a)}{\cosh^3(\pi n b/a)} + \frac{\theta}{b^2} \sum_{n=1}^{\infty} \frac{\cos((2n - 1)\theta/2b)}{\sinh^2((2n - 1)a\pi/2b)} \\
- 2 \frac{\pi a}{b^2} \sum_{n=1}^{\infty} \frac{\sin((2n - 1)\theta/2b) \cosh((2n - 1)a\pi/2b)}{\sinh^3((2n - 1)a\pi/2b)} + \frac{\theta}{2a^2} = 0. \quad (3.14)
\]
Proof. The proofs of (3.12)-(3.14) are similar to the proof of (3.5). We only prove (3.12). From definition of $H_1(s, \theta; a, b)$, it is easy to see that the function only have poles at the 0, $\pm n/a$ and $\pm ni/b$, where $n$ is a positive integer. Applying (3.1) and (3.2), we find that at $\pm n/a$ and $\pm ni/b$, the poles have order two and the residues are

\[
\text{Res}(H_1(s, \theta; a, b))_{s=\pm n/a} = \frac{\pi^2}{a^2} \left\{ \frac{\theta \cosh(n\theta/a)}{\sin^2(\pi bn/a)} - 2\pi b \frac{\sin(n\theta/a) \cosh(\pi bn/a)}{\sinh^3(\pi bn/a)} \right\},
\]

(3.15)

\[
\text{Res}(H_1(s, \theta; a, b))_{s=\pm ni/b} = \frac{\pi^2}{b^2} \left\{ \frac{\theta \cos(n\theta/b)}{\sin^2(\pi an/b)} + 2a \frac{\sin(n\theta/b) \cosh(\pi an/b)}{\sinh^3(\pi an/b)} \right\}.
\]

(3.16)

Moreover, a simple calculation gives

\[
H_1(s, \theta; a, b) = \frac{\theta}{a^2 b^2 s^3} + 2\zeta(2) \left( \frac{1}{b^2} - \frac{1}{a^2} \right) \frac{\theta}{s} + \frac{\theta^3}{6a^2 b^2 s} + o(1), \quad s \to 0.
\]

(3.17)

Hence, the residue of the pole of order three at 0 is found to be

\[
\text{Res}(H_1(s, \theta; a, b))_{s=0} = 2\zeta(2) \theta \left( \frac{1}{b^2} - \frac{1}{a^2} \right) + \frac{\theta^3}{6a^2 b^2}.
\]

(3.18)

Thus, by residue theorem and using (3.15), (3.16) and (3.18), we obtain the desired result.

Similarly, consider $H_2(s, \theta; a, b)$ and $H_3(s, \theta; a, b)$, proceeding in a similar manner and using the calculations above, the evaluations (3.13) and (3.14) can be established.

\[\square\]

### 3.2 Evaluations of $\bar{S}_{2p,2}(y)$, $\bar{S}_{2p-1,2}(y)$, $\bar{C}_{2p,2}(y)$ and $\bar{C}_{2p-1,2}(y)$

In Theorem 3.1, differentiating (3.5), (3.7) 2p times and (3.6), (3.8) 2p − 1 times with respect to $\theta$, respectively, then setting $\theta = 0$ and letting $b\pi/a = \alpha$ and $a\pi/b = \beta$, after elementary calculations and simplifications, we can get the following theorem.

**Theorem 3.3** Let $p$ be a positive integer and $\alpha, \beta$ be real numbers such as $\alpha \beta = \pi^2$, then

\[
\alpha^{2p} \bar{S}_{2p,2}(\alpha) - 2p(-1)^{p-1}\pi^{2p-2}\beta \bar{S}_{2p-1,1}(\beta)
\]

\[
- (-1)^p \pi^{2p-2} \beta^2 \sum_{n=1}^{\infty} \frac{n^{2p} \cosh(n\beta)}{\sinh^2(n\beta)} - \delta_p = 0,
\]

(3.19)

\[
\alpha^{2p} \bar{C}_{2p-1,2}(\alpha) - (4p - 2)(-1)^p \pi^{2p-1}\bar{C}_{2p-2,1}(\beta)
\]

\[
+ (-1)^p \pi^{2p-1} \beta \sum_{n=1}^{\infty} \frac{(2n - 1)^{2p-1} \sinh((2n - 1)\beta/2)}{\cosh^2((2n - 1)\beta/2)} = 0,
\]

(3.20)

\[
\alpha^{2p+1} \bar{C}_{2p,2}(\alpha) - (-1)^p \frac{\pi^{2p}}{2^{2p-2}} \bar{S}_{2p-1,1}(\beta)
\]

\[
+ (-1)^p \frac{\pi^{2p}}{2^{2p-2}} \beta \sum_{n=1}^{\infty} \frac{(2n - 1)^{2p} \cosh((2n - 1)\beta/2)}{\sinh^2((2n - 1)\beta/2)} = 0,
\]

(3.21)

\[
\alpha^{2p} \bar{S}_{2p-1,2}(\alpha) + 2^{2p-1}(2p - 1)(-1)^p \pi^{2p-1}\bar{C}_{2p-2,1}(\beta)
\]

\[
+ 2^{2p-1}(-1)^{p-1} \pi^{2p-1} \beta \sum_{n=1}^{\infty} \frac{n^{2p-1} \sinh(n\beta)}{\cosh^2(n\beta)} - \chi_p = 0.
\]

(3.22)

where $\delta_1 = 1/2, \chi_1 = \pi$ and $\delta_p = \chi_p = 0$ if $p \geq 2$.
Note that using the relation
\[
\frac{dy}{dx} = -\frac{1}{x(1-x)z^2},
\]
we can find that
\[
\sum_{n=1}^{\infty} \frac{n^2 p \cosh(ny)}{\sinh^2(ny)} = x(1-x)z^2 \frac{d}{dx} S_{2p-1,1}(y),
\] (3.23)
\[
\sum_{n=1}^{\infty} \frac{(2n-1)^2 p^{-1} \sinh((2n-1)y/2)}{\cosh^2((2n-1)y/2)} = 2x(1-x)z^2 \frac{d}{dx} C_{2p-1,2}(y),
\] (3.24)
\[
\sum_{n=1}^{\infty} \frac{(2n-1)^2 p \cosh((2n-1)y/2)}{\sinh^2((2n-1)y/2)} = 2x(1-x) \frac{d}{dx} S_{2p-1,1}(y),
\] (3.25)
\[
\sum_{n=1}^{\infty} \frac{n^{2p-1} \sinh(ny)}{\cosh^2(ny)} = x(1-x)z^2 \frac{d}{dx} C_{2p-1,2}(y).
\] (3.26)

Hence, from Lemma 2.3 with \( \beta = y \) and \( \alpha = \pi^2/y \), Theorem 3.3, formulas (3.23)-(3.26) and transformation (2.23), we arrive at the conclusion.

**Theorem 3.4** For any \( p \in \mathbb{N} \), the four series
\[ S_{2p,2}(y), \ S'_{2p-1,2}(y), \ C_{2p,2}(y) \text{ and } C'_{2p-1,2}(y) \]
can be expressed in terms of \( z \) and \( z' \).

For example, setting \( p = 2 \) and \( \beta = y \) in (3.19) yields
\[
\bar{S}_{4,2}(\pi^2/y) = \sum_{n=1}^{\infty} \frac{n^4(-1)^{n-1}}{\sinh^2(\pi n^2/y)} = \frac{xy^5}{8\pi^6} \left\{ 4x(1-x)yz' + (1-x)yz^2 - 4 \right\}.
\] (3.27)

Then, applying the transformation (2.23) to (3.27), we obtain the result
\[
\bar{S}_{4,2}(y) = \sum_{n=1}^{\infty} \frac{n^4(-1)^{n-1}}{\sinh^2(ny)} = \frac{1}{8}(1-x)z^5 \left[ z - 4(1-x)z' \right].
\] (3.28)

In next Corollary 3.5, we have given all explicit evaluations for
\[ \bar{S}_{2p,2}(y), \ S'_{2p-1,2}(y), \ C_{2p,2}(y) \text{ and } C'_{2p-1,2}(y) \]
with \( p = 1, 2, 3, 4, 5 \).

**Corollary 3.5** We have
\[
\bar{S}_{2,2}(y) = \frac{1}{8}x(1-x)z^2 \left[ 4x(1-x)(z')^2 + 4(1-x)zz' - z^2 \right],
\] (3.29)
\[
\bar{S}_{4,2}(y) = \frac{1}{8}x(1-x)z^5 \left[ z - 4(1-x)z' \right],
\] (3.30)
\[
\bar{S}_{6,2}(y) = \frac{1}{8}x(1-x)z^7 \left[ 6(1-x)(2-x)z' + (2x-3)z \right],
\] (3.31)
3.3 Evaluations of $S$ 

In this subsection, we use Ramanujan’s results of Eisenstein series to evaluate the four series and give all explicit evaluations for $1 \leq p \leq 5$. In Ramanujan’s notation, the three relevant Eisenstein series are defined for $|q| < 1$ by

$$P(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n},$$
$$Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n},$$
\[ R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \] (3.51)

The functions \( P, \) \( Q \) and \( R \) were thoroughly studied in a famous paper [18] by Ramanujan. By Entry 13 in the Chapter 17 of Ramanujan’s third notebook (Berndt [7]), we have

\[
\begin{align*}
P(q) &= (1 - 5x)z^2 + 12x(1 - x)zz', \\
P(q^2) &= (1 - 2z)z^2 + 6x(1 - x)zz', \\
Q(q) &= z^4(1 + 14x + x^2), \\
Q(q^2) &= z^4(1 - x + x^2), \\
R(q) &= z^6(1 + x)(1 - 34x + x^2), \\
R(q^2) &= z^6(1 + x)(1 - x/2)(1 - 2x).
\end{align*}
\]
(3.52) \( \cdots \) (3.57)

In [7], \( L(q) = P(q), \) \( M(q) = Q(q) \) and \( N(q) = R(q) \) in the present notation. Ramanujan [18] proved the general result

\[ \sum_{n=1}^{\infty} \frac{n^p q^n}{(1 - q^n)^2} = \sum \mathcal{C}_{l,m,n} P^l Q^m R^n, \] (3.58)

and gave all explicit formulas for \( p = 2, 4, 6, 8, 10, 12, 14, \) where \( \mathcal{C}_{l,m,n} \) is a constant and \( m, n \) are non-negative integers with \( l \leq 2 \) and \( 2l + 4m + 6n = p + 2. \) Since \( q = e^{-y}, \) then the left hand side of Eisenstein series (3.58) can be written as

\[ \sum_{n=1}^{\infty} \frac{n^p q^n}{(1 - q^n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n^p}{\sinh^2(ny/2)} = \frac{1}{4} S_{2p,2}(y/2). \] (3.59)

Hence, the series \( S_{2p,2}(y/2) \) \((p \in \mathbb{N})\) can be expressed in terms of \( z \) and \( z'. \)

**Theorem 3.6** For positive integer \( p, \) the four series

\( S_{2p,2}(y), \) \( S'_{2p,2}(y), \) \( C_{2p,2}(y) \) and \( C'_{2p,2}(y) \)

can be expressed in terms of \( z \) and \( z'. \)

**Proof.** Using the transformation (2.25) to (3.59), then \( S_{2p,2}(y/2) \Rightarrow S_{2p,2}(y), \) which implies that \( S_{2p,2}(y) \) can be represented by \( z \) and \( z'. \) Then, according to Definition 1.1, we deduce the relations

\[
\begin{align*}
S'_{p,m}(y) &= S_{p,m}(y/2) - 2^p S_{p,m}(y), \\
C'_{p,m}(y) &= C_{p,m}(2y) - 2^p C_{p,m}(2y).
\end{align*}
\] (3.60) \( \cdots \) (3.61)

Moreover, applying the transformation (2.24) to \( \sinh^2((2n - 1)y/2), \) then

\( \sinh^2((2n - 1)y/2) \to -\cosh^2((2n - 1)y/2). \)

Hence, the series \( S'_{2p,2}(y) \) and \( C'_{2p,2}(y) \) can be evaluated by \( z \) and \( z'. \) Finally, using (3.61) and transformation (2.25), we prove that \( C_{2p,2}(2y) \) can be represented by \( z \) and \( z'. \) Then employ the relation

\[ C_{p,m}(y) = C'_{p,m}(2y) + 2^p C_{p,m}(2y) \] (3.62)
to complete the proof of Theorem 3.6.

From [18], we can find that

\[
\sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^n)^2} = \frac{Q(q) - P^2(q)}{288}, \tag{3.63}
\]

\[
\sum_{n=1}^{\infty} \frac{n^4 q^n}{(1-q^n)^2} = \frac{P(q)Q(q) - R(q)}{720}, \tag{3.64}
\]

\[
\sum_{n=1}^{\infty} \frac{n^6 q^n}{(1-q^n)^2} = \frac{Q^2(q) - P(q)R(q)}{1008}, \tag{3.65}
\]

\[
\sum_{n=1}^{\infty} \frac{n^8 q^n}{(1-q^n)^2} = \frac{P(q)Q^2(q) - Q(q)R(q)}{720}, \tag{3.66}
\]

\[
\sum_{n=1}^{\infty} \frac{n^{10} q^n}{(1-q^n)^2} = \frac{3Q^3(q) + 2R^2(q) - 5P(q)Q(q)R(q)}{1584}. \tag{3.67}
\]

Using (3.63)-(3.67), we can get some examples of \( S_{2p,2}(y) \), \( S'_{2p,2}(y) \), \( C_{2p,2}(y) \) and \( C'_{2p,2}(y) \).

**Corollary 3.7** We have

\[
S_{2,2}(y/2) = \frac{x(1-x)}{3} z^2 \left[ z^2 - (1-x)(z')^2 \right], \tag{3.68}
\]

\[
S_{4,2}(y/2) = \frac{x(1-x)x^5}{30} \left[ 2(1 + 14x + x^2)z' + (7 + x)z \right], \tag{3.69}
\]

\[
S_{6,2}(y/2) = \frac{x(1-x)^2}{42} z^7 \left[ 2(1 - 33x - 33x^2 + 3x^3)z' + (11 + 22x - x^2)z \right], \tag{3.70}
\]

\[
S_{8,2}(y/2) = \frac{x(1-x)(1 + 14x + x^2)z^9}{30} \left[ 2(1 + 14x + x^2)z' + (7 + x)z \right], \tag{3.71}
\]

\[
S_{10,2}(y/2) = \frac{x(1-x)z^{11}}{66} \left\{ -10(1 + x)(1 - 20x - 474x^2 - 24x^3 + 4x^4)z' \right\}, \tag{3.72}
\]

\[
S_{2,2}(y) = \frac{x(1-x)}{24} z^2 \left[ z^2 - 4(1-2x)z' - 12x(1-x)(z')^2 \right], \tag{3.73}
\]

\[
S_{4,2}(y) = \frac{x(1-x)x^5}{120} \left[ 4(1 - x + x^2)z' + (2x - 1)z \right], \tag{3.74}
\]

\[
S_{6,2}(y) = \frac{x(1-x)^2}{168} z^7 \left[ 2(2 - x)(1 + 2x - 1)z' - (2x^2 - 2x - 1)z \right], \tag{3.75}
\]

\[
S_{8,2}(y) = \frac{x(1-x)(1 - x + x^2)z^9}{120} \left[ 4(1 - x + x^2)z' + (2x - 1)z \right], \tag{3.76}
\]

\[
S_{10,2}(y) = \frac{x(1-x)z^{11}}{264} \left\{ -10(2 - 5x + 2x^2 + 2x^3 - 5x^4 + 2x^5)z' \right\}, \tag{3.77}
\]

\[
S'_{2,2}(y) = \frac{x(1-x)^3}{6} \left[ z + 2(1 + x)z' \right], \tag{3.78}
\]

\[
S'_{4,2}(y) = \frac{x(1-x)^5}{30} \left[ -2(7 - 22x + 7x^2)z' + (11 - 7x)z \right], \tag{3.79}
\]

\[
S'_{6,2}(y) = \frac{x(1-x)^7}{42} \left[ 2(31x^3 - 15x^2 - 15x + 31)z' + (31x^2 - 10x - 5)z \right], \tag{3.80}
\]
Corollary 3.8 We have

\[ S'_{8,2}(y) = \frac{x(1-x)z^9}{30} \left\{ -(254 - 568x + 372x^2 - 568x^3 + 254x^4)z' + (71 - 93x + 213x^2 - 127x^3)z \right\}, \]  \hspace{1cm} (3.81)

\[ S'_{10,2}(y) = \frac{x(1-x)z^{11}}{66} \left\{ 10(511 - 1261x + 1006x^2 + 1006x^3 - 1261x^4 + 511x^5)z' + (-1261 + 2012x + 3018x^2 - 5044x^3 + 2555x^4)z \right\}. \]  \hspace{1cm} (3.82)

**Proof.** These results follow from (3.52)-(3.57), (3.59), (3.60) and (3.63)-(3.67). \qed

Proof. Apply the transformation (2.24) to (3.78)-(3.82) to obtain (3.83)-(3.87), respectively.

Then apply the transformation (2.25) to (3.83)-(3.87) to obtain (3.88)-(3.92), respectively. \qed
Corollary 3.9 We have

\[
C_{2,2}(y) = \frac{x(1-x) z^3}{24} [2(2-x) z' - z],
\]

(3.93)

\[
C_{4,2}(y) = \frac{x(1-x) z^5}{480} [2(-8+8x+7x^2) z' + (4+7x) z],
\]

(3.94)

\[
C_{6,2}(y) = \frac{x(1-x) z^7}{2688} [2(32-48x+78x^2-31x^3) z' - (16-52x+31x^2) z],
\]

(3.95)

\[
C_{8,2}(y) = \frac{x(1-x) z^9}{7680} \left\{ 2\left(-128+256x+96x^2-224x^3+127x^4\right) z' \right\},
\]

(3.96)

\[
C_{10,2}(y) = \frac{x(1-x) z^{11}}{67584} \left\{ 10(512-1280x+1568x^2-1072x^3+1294x^4-511x^5) z' \right\},
\]

(3.97)

\[
C_{2,2}(y) = \frac{x z^3}{192} [2(1-x)(2-x+6\sqrt{1-x}) + (x-1-3\sqrt{1-x}) z],
\]

(3.98)

\[
C_{4,2}(y) = \frac{x z^5}{15360} \left\{ 2(1-x) \left[-8 - 120\sqrt{1-x} + 8(8+60\sqrt{1-x}+7x) \right] z' \right\},
\]

(3.99)

\[
C_{6,2}(y) = \frac{x z^7}{344064} \left\{ 2(1-x) \left[32(1+63\sqrt{1-x})-48x(1+42\sqrt{1-x}) \right] z' \right\},
\]

(3.100)

\[
C_{8,2}(y) = \frac{x z^9}{3932160} \left\{ 2(1-x) \left[-128(1+255\sqrt{1-x})+64(4+765\sqrt{1-x})+127x^4 \right] z' \right\},
\]

(3.101)

\[
C_{10,2}(y) = \frac{x z^{11}}{138412032} \left\{ 10(1-x) \left[512(1+1023\sqrt{1-x})-256x(5+4092\sqrt{1-x}) \right] z' \right\},
\]

(3.102)

Proof. The ten desired formulas follow without difficulty from (3.61) and (3.62) with the help of Corollaries 3.5 and 3.8. \( \square \)

3.4 Three Differential Equations

In Theorem 3.2, expanding both sides of (3.12)-(3.14) in powers of \( \theta \), then equating coefficients of \( \theta^{2p+1} \) on both sides and letting \( b \pi /a = \alpha \) and \( a \pi /b = \beta \), and noting that the elementary relations

\[
\frac{d}{dx} \left(x^{2p+1} S_{2p,2}(x)\right) = (2p+1)x^{2p} \sum_{n=1}^{\infty} \frac{n^{2p}}{\sinh^2(nx)} - 2x^{2p+1} \sum_{n=1}^{\infty} \frac{n^{2p+1} \cosh(nx)}{\sinh^3(nx)},
\]

\[
\frac{d}{dx} \left(x^{2p+1} C_{2p,2}(x)\right) = (2p+1)x^{2p} \sum_{n=1}^{\infty} \frac{(2n-1)^{2p}}{\cosh^2((2n-1)x/2)}
\]


\[-x^{2p+1} \sum_{n=1}^{\infty} \frac{(2n-1)^{2p+1} \sinh((2n-1)x/2)}{\cosh^3((2n-1)x/2)},\]

and

\[
\frac{d}{dx} \left( x^{2p+1} C_{2p,2}(x) \right) = (2p+1)x^{2p} \sum_{n=1}^{\infty} \frac{n^{2p}}{\cosh^2(nx)} - 2x^{2p+1} \sum_{n=1}^{\infty} \frac{n^{2p+1} \sinh(nx)}{\cosh^3(nx)},
\]

\[
\frac{d}{dx} \left( x^{2p+1} S'_{2p,2}(x) \right) = (2p+1)x^{2p} \sum_{n=1}^{\infty} \frac{(2n-1)^{2p}}{\sinh^2((2n-1)x/2)} - x^{2p+1} \sum_{n=1}^{\infty} \frac{(2n-1)^{2p+1} \cosh((2n-1)x/2)}{\sinh^3((2n-1)x/2)},
\]

we can obtain the following three differential equations.

**Theorem 3.10** Let \( p \) be a non-integer and \( \alpha, \beta \) be real numbers such as \( \alpha \beta = \pi^2 \), then

\[
\beta^{p-1} \frac{d}{d\alpha} \left( \alpha^{2p+1} S_{2p,2}(\alpha) \right) + (-\alpha)^{p-1} \frac{d}{d\beta} \left( \beta^{2p+1} S_{2p,2}(\beta) \right) + \delta_p = 0, \tag{3.103}
\]

\[
\beta^{p-1} \frac{d}{d\alpha} \left( \alpha^{2p+1} C'_{2p,2}(\alpha) \right) + (-\alpha)^{p-1} \frac{d}{d\beta} \left( \beta^{2p+1} C'_{2p,2}(\beta) \right) = 0, \tag{3.104}
\]

\[
2^{2p} \beta^{p-1} \frac{d}{d\alpha} \left( \alpha^{2p+1} C_{2p,2}(\alpha) \right) - (-\alpha)^{p-1} \frac{d}{d\beta} \left( \beta^{2p+1} S'_{2p,2}(\beta) \right) + \lambda_p = 0, \tag{3.105}
\]

where \( \delta_0 = (1/\alpha - 1/\beta)/6, \lambda_0 = 1/2\beta, \delta_1 = 1/2, \lambda_1 = 0 \) and \( \delta_p = \lambda_p = 0 \) if \( p \geq 2. \)

Setting \( p = 0 \) in (3.103)-(3.105) and noting that the fact

\[
\frac{d\beta}{d\alpha} = -\frac{\beta}{\alpha},
\]

we can get

\[
\frac{d}{d\alpha} \left( \alpha S_{0,2}(\alpha) + \beta S_{0,2}(\beta) - \frac{1}{6}(\alpha + \beta) \right) = 0, \tag{3.106}
\]

\[
\frac{d}{d\alpha} \left( \alpha C'_{0,2}(\alpha) + \beta C'_{0,2}(\beta) \right) = 0, \tag{3.107}
\]

\[
\frac{d}{d\alpha} \left( \alpha C_{0,2}(\alpha) - \beta S'_{0,2}(\beta) + \alpha/2 \right) = 0. \tag{3.108}
\]

Hence, by integrating yield the three identities

\[
\alpha S_{0,2}(\alpha) + \beta S_{0,2}(\beta) - \frac{1}{6}(\alpha + \beta) + 1 = 0, \tag{3.109}
\]

\[
\alpha C'_{0,2}(\alpha) + \beta C'_{0,2}(\beta) = 1, \tag{3.110}
\]

\[
\alpha C_{0,2}(\alpha) - \beta S'_{0,2}(\beta) + \frac{\alpha}{2} = 1, \tag{3.111}
\]

where we have used these well-known results ([6, 7, 21])

\[
S_{0,2}(\pi) = \frac{1}{6} - \frac{1}{2\pi}, \quad C'_{0,2}(\pi) = \frac{1}{2\pi}, \quad C_{0,2}(\pi) = -\frac{1}{2} + \frac{1}{2\pi} + \frac{\Gamma^4(1/4)}{16\pi^3}.
\]

\[\text{20}\]
and

\[ S'_{0,2}(\pi) = \frac{1}{2\pi} + \frac{\Gamma^4(1/4)}{16\pi^3}. \]

The formula (3.109) can be found in Berndt [6] by another method. However, when \( p \geq 1 \), we have been unable, so far, to make any progress with the three differential equations (3.103)-(3.105).

4 Examples and Further Results

4.1 Examples

By (1.6) if \( x = 1/2 \), then

\[ y = \pi, \quad z\left(\frac{1}{2}\right) = \frac{\Gamma^2(1/4)}{2\pi^{3/2}}, \quad z'\left(\frac{1}{2}\right) = \frac{4\sqrt{\pi}}{\Gamma^2(1/4)}. \]

Hence, from the preceding discussion, it is easy to see that the eight series

\[ S_{2p,2}(\pi), \quad S'_{2p,2}(\pi), \quad C_{2p,2}(\pi), \quad C'_{2p,2}(\pi), \quad S_{2p-1,2}(\pi), \quad C_{2p,2}(\pi) \text{ and } C'_{2p-1,2}(\pi) \]

can be evaluated in terms of \( \Gamma \) functions. Moreover, by Mathematica computations, we can find that \( (p \in \mathbb{N}) \)

\[ S_{4p-2,2}(\pi) = a_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p}} + \eta_p, \tag{4.1} \]
\[ S_{4p,2}(\pi) = b_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p+1}}, \tag{4.2} \]
\[ C_{4p-2,2}(\pi) = c_p \frac{\Gamma^{8p-4}(1/4)}{\pi^{6p-2}} + d_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p}}, \tag{4.3} \]
\[ C_{4p,2}(\pi) = e_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p+1}} + f_p \frac{\Gamma^{8p+4}(1/4)}{\pi^{6p+3}}, \tag{4.4} \]
\[ S'_{4p-2,2}(\pi) = g_p \frac{\Gamma^{8p-4}(1/4)}{\pi^{6p-2}} + h_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p}}, \tag{4.5} \]
\[ S'_{4p,2}(\pi) = i_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p+1}} + j_p \frac{\Gamma^{8p+4}(1/4)}{\pi^{6p+3}}, \tag{4.6} \]
\[ C'_{4p-2,2}(\pi) = k_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p}}, \tag{4.7} \]
\[ C'_{4p,2}(\pi) = l_p \frac{\Gamma^{8p}(1/4)}{\pi^{6p+1}}, \tag{4.8} \]
\[ S_{2p,2}(\pi) = m_p \frac{\Gamma^{4p}(1/4)}{\pi^{3p+1}} + n_p \frac{\Gamma^{4p+4}(1/4)}{\pi^{3p+3}} + \eta_p, \tag{4.9} \]
\[ C_{2p,2}(\pi) = o_p \frac{\sqrt{2}\Gamma^{4p}(1/4)}{\pi^{3p+1}} + p_p \frac{\sqrt{2}\Gamma^{4p+4}(1/4)}{\pi^{3p+3}}, \tag{4.10} \]
\[ S'_{2p-1,2}(\pi) = q_p \frac{\Gamma^{4p-2}(1/4)}{\pi^{3p-1/2}} + r_p \frac{\Gamma^{4p+2}(1/4)}{\pi^{3p+3/2}}, \tag{4.11} \]
$$C'_{2p-1,2}(\pi) = s_p \frac{\sqrt{2} \Gamma^{4p-2}(1/4)}{\pi^{3p-1/2}} + t_p \frac{\sqrt{2} \Gamma^{4p+2}(1/4)}{\pi^{3p+3/2}}, \tag{4.12}$$

where $\eta_1 = -\frac{1}{8\pi^2}$, $\eta_p = 0$, $p \geq 2$ and the quantities

$$a_p, \ b_p, \ c_p, \ d_p, \ e_p, \ f_p, \ g_p, \ h_p, \ i_p, \ j_p, \ k_p, \ l_p, \ m_p, \ n_p, \ o_p, \ p_p, \ q_p, \ r_p, \ s_p, \ t_p \in \mathbb{Q}$$

are rational numbers. From Corollaries 3.5, 3.7-3.9 with the help of Mathematica, we calculated a few items of these quantities above, see Table 3.9. The closed form of $C'_{2,2}(\pi)$ can be found in Corollary 3.9 of Berndt’s [10] with Bialek and Yee or Berndt’s book [2] with Andrews. Moreover, from Table 3.1, we give a conjecture.

**Conjecture 4.1** For positive integer $p$, the following relations hold:

$$c_p = \frac{1}{24p-2}g_p, \quad d_p = \frac{1}{24p-2}h_p = \frac{(-1)^p}{26p-3}k_p, \quad e_p = \frac{-1}{24p}l_p = \frac{(-1)^p}{26p}m_p, \quad f_p = \frac{1}{24p}n_p. \tag{4.13}$$

From Example 4.1, we have

$$S_{2,2}(\frac{\pi}{2}) = -\frac{1}{2\pi^2} + \frac{\Gamma^4(1/4)}{16\pi^4} + \frac{\Gamma^8(1/4)}{192\pi^6},$$

$$S_{4,2}(\frac{\pi}{2}) = \frac{11\Gamma^8(1/4)}{640\pi^7} + \frac{\Gamma^{12}(1/4)}{1024\pi^9},$$

$$S_{6,2}(\frac{\pi}{2}) = \frac{9\Gamma^{12}(1/4)}{1024\pi^{10}} + \frac{29\Gamma^{16}(1/4)}{57344\pi^{12}},$$

$$S_{8,2}(\frac{\pi}{2}) = \frac{363\Gamma^{16}(1/4)}{40960\pi^{13}} + \frac{33\Gamma^{20}(1/4)}{65536\pi^{15}},$$

$$S_{10,2}(\frac{\pi}{2}) = \frac{945\Gamma^{20}(1/4)}{65536\pi^{16}} + \frac{4761\Gamma^{24}(1/4)}{5767168\pi^{18}},$$

$$S_{12,2}(\frac{\pi}{2}) = \frac{1179927\Gamma^{24}(1/4)}{34078720\pi^{19}} + \frac{8289\Gamma^{24}(1/4)}{4194304\pi^{21}},$$

$$C_{2,2}(2\pi) = \frac{(2\sqrt{2} + 1)\Gamma^4(1/4)}{512\pi^4} - \frac{(3\sqrt{2} + 1)\Gamma^8(1/4)}{12288\pi^6},$$

$$C_{4,2}(2\pi) = \frac{-3(20\sqrt{2} + 1)\Gamma^8(1/4)}{327680\pi^7} + \frac{(5\sqrt{2} + 1)\Gamma^{12}(1/4)}{524288\pi^9},$$

$$C_{6,2}(2\pi) = \frac{9(22\sqrt{2} + 1)\Gamma^{12}(1/4)}{8388608\pi^{10}} - \frac{3(217\sqrt{2} - 1)\Gamma^{16}(1/4)}{469762048\pi^{12}},$$

$$C_{8,2}(2\pi) = \frac{-21(1560\sqrt{2} - 1)\Gamma^{16}(1/4)}{5368709120\pi^{13}} + \frac{3(995\sqrt{2} + 11)\Gamma^{20}(1/4)}{8589934592\pi^{15}},$$

$$C_{10,2}(2\pi) = \frac{945(362\sqrt{2} + 1)\Gamma^{20}(1/4)}{137438953472\pi^{16}} + \frac{9(190971\sqrt{2} + 17)\Gamma^{24}(1/4)}{12094627905536\pi^{18}},$$

$$C'_{2,2}(2\pi) = \frac{-(2\sqrt{2} - 1)\Gamma^4(1/4)}{128\pi^4} + \frac{(3\sqrt{2} - 1)\Gamma^8(1/4)}{3072\pi^6},$$

$$C'_{4,2}(2\pi) = \frac{3(20\sqrt{2} - 1)\Gamma^8(1/4)}{20480\pi^7} - \frac{(5\sqrt{2} - 1)\Gamma^{12}(1/4)}{32768\pi^9},$$

22
| $p$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $a_p$ | $1 \cdot 3 \cdot 7$ | $1 \cdot 11 \cdot 2^{13}$ | $27 \cdot 2^{26}$ | $19 \cdot 2^{32}$ | $49329$ |
| $b_p$ | $1 \cdot 3 \cdot 5 \cdot 7$ | $1 \cdot 5 \cdot 65 \cdot 2^{13}$ | $85 \cdot 2^{26}$ | $5 \cdot 2^{32}$ | $392931$ |
| $c_p$ | $1 \cdot 2^{16}$ | $22^{26}$ | $2^{36}$ | $2^{46}$ | $746175969$ |
| $d_p$ | $-1 \cdot 3 \cdot 2^{29}$ | $-153 \cdot 11 \cdot 2^{29}$ | $1701 \cdot 2^{39}$ | $-12677553$ | $19 \cdot 2^{49}$ |
| $e_p$ | $-3 \cdot 3 \cdot 5 \cdot 2^{11}$ | $21 \cdot 11 \cdot 2^{11}$ | $18711 \cdot 2^{11}$ | $-5544693 \cdot 5 \cdot 2^{11}$ | $-201573603$ |
| $f_p$ | $1 \cdot 2^{24}$ | $33289 \cdot 7660737$ | $2^{34} \cdot 2^{44}$ | $2^{34} \cdot 2^{54}$ | $18845414721$ |
| $g_p$ | $1 \cdot 2^{10}$ | $945 \cdot 480249$ | $2^{10} \cdot 2^{22}$ | $286 \cdot 2^{28}$ | $746175969$ |
| $h_p$ | $-1 \cdot 3 \cdot 7 \cdot 2^{27}$ | $153 \cdot 11 \cdot 2^{27}$ | $-1701 \cdot 2^{25}$ | $12677553$ | $19 \cdot 2^{31}$ |
| $i_p$ | $3 \cdot 5 \cdot 2^{27}$ | $18711 \cdot 65 \cdot 2^{27}$ | $85 \cdot 2^{25}$ | $201573603$ | $5 \cdot 2^{31}$ |
| $j_p$ | $1 \cdot 2^{16}$ | $33289 \cdot 7660737$ | $2^{22} \cdot 2^{28}$ | $2^{34}$ | $18845414721$ |
| $k_p$ | $1 \cdot 3 \cdot 7 \cdot 2^{25}$ | $153 \cdot 11 \cdot 2^{25}$ | $1701 \cdot 2^{18}$ | $12677553$ | $19 \cdot 2^{22}$ |
| $l_p$ | $3 \cdot 5 \cdot 2^{25}$ | $18711 \cdot 65 \cdot 2^{25}$ | $85 \cdot 2^{17}$ | $201573603$ | $5 \cdot 2^{24}$ |
| $m_p$ | $1 \cdot 2^{5}$ | $-1 \cdot 9 \cdot 2^{11}$ | $-9 \cdot 2^{11}$ | $945$ | $2^{17}$ |
| $n_p$ | $-1 \cdot 2^{9}$ | $1 \cdot 2^{11}$ | $2^{12}$ | $2^{17}$ | $-27$ |
| $o_p$ | $-1 \cdot 2^{15}$ | $3 \cdot 2^{15}$ | $-99 \cdot 2^{15}$ | $819 \cdot 2^{15}$ | $-171045$ |
| $p_p$ | $1 \cdot 2^{14}$ | $-5 \cdot 2^{15}$ | $93 \cdot 2^{15}$ | $-2985 \cdot 2^{15}$ | $156249$ |
| $q_p$ | $1 \cdot 2^{12}$ | $-3 \cdot 2^{12}$ | $45 \cdot 2^{12}$ | $-1071 \cdot 2^{12}$ | $44793$ |
| $r_p$ | $0 \cdot 2^{7}$ | $1 \cdot 2^{9}$ | $-5 \cdot 2^{9}$ | $243 \cdot 2^{9}$ | $-2565$ |
| $s_p$ | $-1 \cdot 2^{5}$ | $3 \cdot 2^{5}$ | $-15 \cdot 2^{5}$ | $189 \cdot 2^{5}$ | $-3969$ |
| $t_p$ | $1 \cdot 2^{10}$ | $-1 \cdot 2^{10}$ | $7 \cdot 2^{10}$ | $-87 \cdot 2^{10}$ | $1809$ |

**TABLE 4.1.** Coefficients
Example 4.2  We have
\[
\sum_{n=1}^{\infty} \frac{n^2 \cosh(n\pi)}{\sinh^2(n\pi)} = -\frac{1}{8\pi^2} + \frac{\Gamma^4(1/4)}{32\pi^4} + \frac{\Gamma^8(1/4)}{512\pi^6},
\]
\[
\sum_{n=1}^{\infty} \frac{n^4 \cosh(n\pi)}{\sinh^2(n\pi)} = \frac{\Gamma^8(1/4)}{128\pi^7} + \frac{\Gamma^{12}(1/4)}{2048\pi^9},
\]
\[
\sum_{n=1}^{\infty} \frac{n^6 \cosh(n\pi)}{\sinh^2(n\pi)} = \frac{9\Gamma^{12}(1/4)}{2^{11}\pi^{10}} + \frac{\Gamma^{16}(1/4)}{2^{12}\pi^{12}},
\]
\[
\sum_{n=1}^{\infty} \frac{n^8 \cosh(n\pi)}{\sinh^2(n\pi)} = \frac{9\Gamma^{16}(1/4)}{2^{11}\pi^{13}} + \frac{33\Gamma^{16}(1/4)}{2^{17}\pi^{15}},
\]
\[
\sum_{n=1}^{\infty} \frac{n^{10} \cosh(n\pi)}{\sinh^2(n\pi)} = \frac{945\Gamma^{20}(1/4)}{2^{17}\pi^{16}} + \frac{27\Gamma^{24}(1/4)}{2^{16}\pi^{18}},
\]
\[
\sum_{n=1}^{\infty} \left(\frac{2n-1}{2}\right)^2 \cosh((2n-1)/2) \over \sinh^2((2n-1)/2) = \sqrt{2}\Gamma^4(1/4) / 8\pi^4 + \sqrt{2}\Gamma^8(1/4) / 128\pi^6,
\]
\[
\sum_{n=1}^{\infty} \left(\frac{2n-1}{2}\right)^4 \cosh((2n-1)/2) \over \sinh^2((2n-1)/2) = \sqrt{2}\Gamma^8(1/4) / 32\pi^7 + \sqrt{2}\sqrt{2}\Gamma^{12}(1/4) / 1024\pi^9,
\]
\[
\sum_{n=1}^{\infty} \left(\frac{2n-1}{2}\right)^6 \cosh((2n-1)/2) \over \sinh^2((2n-1)/2) = \frac{99\sqrt{2}\Gamma^{12}(1/4)}{2^{10}\pi^{10}} + \frac{93\sqrt{2}\Gamma^{16}(1/4)}{2^{13}\pi^{12}},
\]
\[
\sum_{n=1}^{\infty} \left(\frac{2n-1}{2}\right)^8 \cosh((2n-1)/2) \over \sinh^2((2n-1)/2) = \frac{819\sqrt{2}\Gamma^{16}(1/4)}{2^{10}\pi^{13}} + \frac{2985\sqrt{2}\Gamma^{20}(1/4)}{2^{16}\pi^{15}},
\]
\[
\sum_{n=1}^{\infty} \left(\frac{2n-1}{2}\right)^{10} \cosh((2n-1)/2) \over \sinh^2((2n-1)/2) = \frac{171045\sqrt{2}\Gamma^{20}(1/4)}{2^{15}\pi^{16}} + \frac{156249\sqrt{2}\Gamma^{24}(1/4)}{2^{19}\pi^{18}}.
\]

Example 4.3  We have
\[
\sum_{n=1}^{\infty} \frac{n^3 \cosh(n\pi)}{\sinh^3(n\pi)} (-1)^{n-1} = \frac{1}{16\pi^3} + \frac{3\Gamma^4(1/4)}{128\pi^5} - \frac{3\Gamma^8(1/4)}{1024\pi^7} + \frac{\Gamma^{12}(1/4)}{8192\pi^9},
\]
\[
\sum_{n=1}^{\infty} \frac{n^5 \cosh(n\pi)}{\sinh^4(n\pi)} (-1)^{n-1} = -\frac{5\Gamma^8(1/4)}{512\pi^8} + \frac{5\Gamma^{12}(1/4)}{4096\pi^{10}} - \frac{\Gamma^{16}(1/4)}{32768\pi^{12}},
\]
\[
\sum_{n=1}^{\infty} \frac{n^7 \cosh(n\pi)}{\sinh^5(n\pi)} (-1)^{n-1} = \frac{63\Gamma^{12}(1/4)}{8192\pi^{11}} - \frac{7\Gamma^{16}(1/4)}{8192\pi^{13}} + \frac{13\Gamma^{20}(1/4)}{524288\pi^{15}},
\]
\[
\sum_{n=1}^{\infty} \frac{n^9 \cosh(n\pi)}{\sinh^6(n\pi)} (-1)^{n-1} = -\frac{81\Gamma^{16}(1/4)}{8192\pi^{14}} + \frac{297\Gamma^{20}(1/4)}{262144\pi^{16}} - \frac{17\Gamma^{24}(1/4)}{524288\pi^{18}},
\]
4.2 Further Results

From [21], we can find that

\[
\sum_{n=1}^{\infty} \frac{n^4 \cosh(n\pi)}{\sinh^3(n\pi)} (-1)^{n-1} = \frac{10395\Gamma^2(1/4)}{524288\pi^{17}} - \frac{297\Gamma^3(1/4)}{131072\pi^{19}} + \frac{2169\Gamma^2(1/4)}{33554432\pi^{21}},
\]

\[
\sum_{n=1}^{\infty} \frac{n^3 \sinh(n\pi)}{\cosh^3(n\pi)} (-1)^{n-1} = -\frac{3\sqrt{2}\Gamma^4(1/4)}{2\pi^5} + \frac{3\sqrt{2}\Gamma^8(1/4)}{2^{10}\pi^7} - \frac{\sqrt{2}\Gamma^6(1/4)}{2^{14}\pi^9},
\]

\[
\sum_{n=1}^{\infty} \frac{n^5 \sinh(n\pi)}{\cosh^3(n\pi)} (-1)^{n-1} = \frac{15\sqrt{2}\Gamma^8(1/4)}{2^{11}\pi^8} - \frac{25\sqrt{2}\Gamma^6(1/4)}{2^{15}\pi^{10}} + \frac{13\sqrt{2}\Gamma^6(1/4)}{2^{19}\pi^{12}},
\]

\[
\sum_{n=1}^{\infty} \frac{n^7 \sinh(n\pi)}{\cosh^3(n\pi)} (-1)^{n-1} = -\frac{693\sqrt{2}\Gamma^6(1/4)}{2^{17}\pi^{11}} + \frac{65\sqrt{2}\Gamma^6(1/4)}{2^{20}\pi^{13}} - \frac{293\sqrt{2}\Gamma^6(1/4)}{2^{24}\pi^{15}},
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-1)^2 \cosh((2n-1)\pi/2)}{\sinh^3((2n-1)\pi/2)} (-1)^{n-1} = \frac{\Gamma^2(1/4)}{4\pi^{7/2}} + \frac{\Gamma^6(1/4)}{512\pi^{15/2}},
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-1)^2 \sinh((2n-1)\pi/2)}{\cosh^3((2n-1)\pi/2)} (-1)^{n-1} = -\frac{\sqrt{2}\Gamma^6(1/4)}{23\pi^{7/2}} + \frac{\sqrt{2}\Gamma^6(1/4)}{25\pi^{11/2}},
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-1)^4 \sinh((2n-1)\pi/2)}{\cosh^3((2n-1)\pi/2)} (-1)^{n-1} = \frac{3\sqrt{2}\Gamma^6(1/4)}{2^{4}\pi^{13/2}} - \frac{\sqrt{2}\Gamma^6(1/4)}{2^{6}\pi^{17/2}} + \frac{\sqrt{2}\Gamma^6(1/4)}{2^{11}\pi^{21/2}},
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-1)^6 \sinh((2n-1)\pi/2)}{\cosh^3((2n-1)\pi/2)} (-1)^{n-1} = -\frac{45\sqrt{2}\Gamma^6(1/4)}{2^{7}\pi^{19/2}} + \frac{21\sqrt{2}\Gamma^6(1/4)}{2^{9}\pi^{23/2}} - \frac{5\sqrt{2}\Gamma^6(1/4)}{2^{12}\pi^{27/2}},
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-1)^8 \sinh((2n-1)\pi/2)}{\cosh^3((2n-1)\pi/2)} (-1)^{n-1} = \frac{189\sqrt{2}\Gamma^6(1/4)}{2^{7}\pi^{25/2}} - \frac{87\sqrt{2}\Gamma^6(1/4)}{2^{9}\pi^{29/2}} + \frac{157\sqrt{2}\Gamma^6(1/4)}{2^{11}\pi^{33/2}},
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-1)^{10} \sinh((2n-1)\pi/2)}{\cosh^3((2n-1)\pi/2)} (-1)^{n-1} = -\frac{19845\sqrt{2}\Gamma^6(1/4)}{2^{11}\pi^{31/2}} + \frac{9045\sqrt{2}\Gamma^6(1/4)}{2^{13}\pi^{35/2}} - \frac{1035\sqrt{2}\Gamma^6(1/4)}{2^{15}\pi^{39/2}}.
\]

4.2 Further Results

From [21], we can find that

\[
\frac{1}{(2s-1)!} \int dy^{2s-2} \frac{1}{\sinh^2(y)} = \sum_{k=1}^{s} \frac{A_{2s,2k}}{\sinh^{2k}(y)},
\]

\[
\frac{1}{(2s-1)!} \int dy^{2s-2} \frac{1}{\cosh^2(y)} = \sum_{k=1}^{s} (-1)^{k+1} \frac{A_{2s,2k}}{\cosh^{2k}(y)},
\]

where the coefficients \( A_{2s,2k} \) (\( A_{2s,2s} := 1 \)) have the following recurrence relation, for \( s \geq 1 \) and \( 1 \leq k \leq s \):

\[
A_{2s+2,2k} = \frac{1}{2s(2s+2)} ( (2k-1)(2k-2)A_{2s,2k-2} + 4k^2A_{2s,2k} ) .
\]

Some values of \( A_{2s,2k} \) see Table 5 of [21]. Using the notations (4.14) and (4.15), one obtains for \( s \geq 1 \) and \( p \geq 0 \),

\[
\frac{1}{(2s-1)!} \int dy^{2s-2} S_{p,2}(y) = \sum_{k=1}^{s} A_{2s,2k} S_{p+2s-2,2k}(y),
\]

\[25\]
\[
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} \bar{S}_{p,2}(y) = \sum_{k=1}^{s} A_{2s,2k} \bar{S}_{p+2s-2,2k}(y), \\
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} S_{p,2}(y) = \sum_{k=1}^{s} (-1)^{k+1} A_{2s,2k} S_{p+2s-2,2k}(y), \\
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} C_{p,2}(y) = \sum_{k=1}^{s} (-1)^{k+1} A_{2s,2k} C_{p+2s-2,2k}(y), \\
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} \bar{S}_{p,2}'(y) = \frac{1}{2^{2s-2}} \sum_{k=1}^{s} A_{2s,2k} \bar{S}_{p+2s-2,2k}'(y), \\
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} S_{p,2}'(y) = \frac{1}{2^{2s-2}} \sum_{k=1}^{s} A_{2s,2k} S_{p+2s-2,2k}'(y), \\
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} C_{p,2}'(y) = \frac{1}{2^{2s-2}} \sum_{k=1}^{s} (-1)^{k+1} A_{2s,2k} C_{p+2s-2,2k}'(y), \\
\frac{1}{(2s-1)!} \frac{d^{2s-2}}{dy^{2s-2}} \bar{C}_{p,2}'(y) = \frac{1}{2^{2s-2}} \sum_{k=1}^{s} (-1)^{k+1} A_{2s,2k} \bar{C}_{p+2s-2,2k}'(y).
\]

From the recurrence relations (4.16)-(4.23), we obtain that

\[
S_{p+2k-2,2k}(y) \in \mathbb{Q} \left[ \frac{d^{2k-2}}{dy^{2k-2}} S_{p,2}(y), \frac{d^{2k-4}}{dy^{2k-4}} S_{p+2,2}(y), \ldots, \frac{d^{2}}{dy^{2}} S_{p+2k-4,2}(y), S_{p+2k-2,2}(y) \right],
\]

\[
\bar{S}_{p+2k-2,2k}(y) \in \mathbb{Q} \left[ \frac{d^{2k-2}}{dy^{2k-2}} \bar{S}_{p,2}(y), \frac{d^{2k-4}}{dy^{2k-4}} \bar{S}_{p+2,2}(y), \ldots, \frac{d^{2}}{dy^{2}} \bar{S}_{p+2k-4,2}(y), \bar{S}_{p+2k-2,2}(y) \right],
\]

\[
C_{p+2k-2,2k}(y) \in \mathbb{Q} \left[ \frac{d^{2k-2}}{dy^{2k-2}} C_{p,2}(y), \frac{d^{2k-4}}{dy^{2k-4}} C_{p+2,2}(y), \ldots, \frac{d^{2}}{dy^{2}} C_{p+2k-4,2}(y), C_{p+2k-2,2}(y) \right],
\]

\[
\bar{C}_{p+2k-2,2k}(y) \in \mathbb{Q} \left[ \frac{d^{2k-2}}{dy^{2k-2}} \bar{C}_{p,2}(y), \frac{d^{2k-4}}{dy^{2k-4}} \bar{C}_{p+2,2}(y), \ldots, \frac{d^{2}}{dy^{2}} \bar{C}_{p+2k-4,2}(y), \bar{C}_{p+2k-2,2}(y) \right].
\]

Hence, all series of the forms \((p \geq k)\)

\[
S_{2p,2k}(y), \bar{S}_{2p,2k}(y), C_{2p,2k}(y), \bar{C}_{2p,2k}(y), S'_{2p,2k}(y), \bar{S}'_{2p,2k}(y), C'_{2p,2k}(y), \bar{C}'_{2p,2k}(y)
\]

can be evaluated. So, when \(x = 1/2, y = \pi\), then these series can be evaluated by Gamma function and \(\pi\). Next, we give some specific cases by using Mathematica.

\[
S_{4,4} = \sum_{n=1}^{\infty} \frac{n^4}{\sinh^4(n\pi)} = -\frac{1}{32\pi^3} \frac{\Gamma^8(1/4)}{1920\pi^7} + \frac{\Gamma^8(1/4)}{1024\pi^8} + \frac{\Gamma^{16}(1/4)}{393216\pi^{12}},
\]

26
\[ S_{4,4}(\pi) = \sum_{n=1}^{\infty} \frac{n^4(-1)^{n-1}}{\sinh^4(n\pi)} = \frac{1}{32\pi^4} + \frac{\Gamma^4(1/4)}{64\pi^6} + \frac{\Gamma^8(1/4)}{192\pi^7} - \frac{3\Gamma^8(1/4)}{1024\pi^8} - \frac{\Gamma^{12}(1/4)}{3072\pi^9} + \frac{\Gamma^{12}(1/4)}{4096\pi^{10}} - \frac{\Gamma^{16}(1/4)}{393216\pi^{12}}, \]

\[ C_{4,4}(\pi) = \sum_{n=1}^{\infty} \frac{n^4}{\cosh^4(n\pi)} = -\frac{\Gamma^4(1/4)}{128\pi^6} - \frac{\Gamma^8(1/4)}{5120\pi^7} + \frac{\Gamma^8(1/4)}{1024\pi^8} + \frac{\Gamma^{12}(1/4)}{24576\pi^9} - \frac{\Gamma^{12}(1/4)}{8192\pi^{10}}, \]

\[ \bar{C}_{4,4}(\pi) = \sum_{n=1}^{\infty} \frac{n^4(-1)^{n-1}}{\cosh^4(n\pi)} = \frac{\sqrt{2}\Gamma^4(1/4)}{64\pi^6} + \frac{\sqrt{2}\Gamma^8(1/4)}{256\pi^7} - \frac{3\sqrt{2}\Gamma^8(1/4)}{1024\pi^8} - \frac{5\sqrt{2}\Gamma^{12}(1/4)}{24576\pi^9} + \frac{\sqrt{2}\Gamma^{12}(1/4)}{8192\pi^{10}} - \frac{\sqrt{2}\Gamma^{16}(1/4)}{262144\pi^{12}} \]

and

\[ S'_{4,4}(\pi) = \sum_{n=1}^{\infty} \frac{(2n-1)^4}{\sinh^4((2n-1)\pi/2)} = \frac{\Gamma^4(1/4)}{8\pi^6} - \frac{\Gamma^8(1/4)}{320\pi^7} + \frac{\Gamma^8(1/4)}{64\pi^8} - \frac{\Gamma^{12}(1/4)}{1536\pi^9} + \frac{\Gamma^{12}(1/4)}{512\pi^{10}}, \]

\[ C'_{4,4}(\pi) = \sum_{n=1}^{\infty} \frac{(2n-1)^4}{\cosh^4((2n-1)\pi/2)} = \frac{\Gamma^8(1/4)}{80\pi^7} - \frac{\Gamma^8(1/4)}{32\pi^8}, \]

\[ S'_{3,4}(\pi) = \sum_{n=1}^{\infty} \frac{(2n-1)^3(-1)^{n-1}}{\sinh^4((2n-1)\pi/2)} = \frac{\Gamma^2(1/4)}{4\pi^{9/2}} + \frac{\Gamma^6(1/4)}{16\pi^{11/2}} - \frac{\Gamma^{10}(1/4)}{192\pi^{15/2}} + \frac{3\Gamma^{10}(1/4)}{512\pi^{17/2}}, \]

\[ C'_{3,4}(\pi) = \sum_{n=1}^{\infty} \frac{(2n-1)^3(-1)^{n-1}}{\cosh^4((2n-1)\pi/2)} = \frac{\sqrt{2}\Gamma^2(1/4)}{8\pi^{9/2}} + \frac{\sqrt{2}\Gamma^6(1/4)}{16\pi^{11/2}} - \frac{3\sqrt{2}\Gamma^6(1/4)}{64\pi^{13/2}} - \frac{\sqrt{2}\Gamma^{10}(1/4)}{384\pi^{15/2}}. \]

These results obtained can be checked in Mathematica or Maple. It is possible that closed form representations of some other infinite series involving hyperbolic functions can be proved using techniques of the present paper.

Acknowledgments. We thank the anonymous referee for suggestions which led to improvements in the exposition.

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