DIRICHLET SPECTRUM AND HEAT CONTENT

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Abstract. Let $M$ be a complete Riemannian manifold and $D \subset M$ a smoothly bounded domain with compact closure. We use Brownian motion and the classic results on the Stieltjes moment problem to study the relationship between the Dirichlet spectrum of $D$ and the heat content asymptotics of $D$. Central to our investigation is a sequence of invariants associated to $D$ defined using exit time moments. We prove that our invariants determine that part of the spectrum corresponding to eigenspaces which are not orthogonal to constant functions, that our invariants determine the heat content asymptotics associated to the manifold, and that when the manifold is a generic domain in Euclidean space, the invariants determine the Dirichlet spectrum.

1: Introduction

Let $(M,g)$ be a complete Riemannian manifold and suppose that $D \subset M$ is a smoothly bounded domain with compact closure. Let $\text{spec}(D)$ be the spectrum of the Laplace operator acting on functions with Dirichlet boundary conditions. We take the Laplacian to be positive with elements of the spectrum listed in increasing order with multiplicity. We study the relationship between the Dirichlet spectrum of $D$ and the heat content asymptotics of $D$. We recall the required facts:

Let $p_D(x,y,t)$ be the heat kernel associated to $D$, let $dg$ be the volume form associated to the metric, and let

\begin{equation}
    u(x,t) = \int_D p_D(x,y,t)dg(y)
\end{equation}

be the solution to the initial value problem

\begin{equation}
    \frac{1}{2}\Delta u = \frac{\partial u}{\partial t} \quad \text{on } D \times (0, \infty)
\end{equation}

\begin{equation}
    u(x,0) = \begin{cases} 
        1 & \text{if } x \in D \\
        0 & \text{if } x \in \partial D
    \end{cases}
\end{equation}

\begin{equation}
    u(x,t) = 0 \text{ if } x \in \partial D
\end{equation}
Let $q(t)$ be the heat content of $D$ at time $t$:

\begin{equation}
q(t) = \int_D u(x,t) dg.
\end{equation}

It is a theorem of van den Berg and Gilkey [BG] that $q(t)$ admits a small time asymptotic expansion:

\begin{equation}
q(t) \simeq \sum_{n=0}^{\infty} q_n t^{n/2}
\end{equation}

where the coefficients $q_n$ are locally computable geometric invariants of $D$ (cf [G] for a recent survey of results concerning heat content). We will refer to the coefficients occurring on right hand side of (1.4) as the heat content asymptotics of $D$ and we write

\begin{equation}
hca(D) = \{q_n\}_{n=0}^{\infty}.
\end{equation}

We note that, in contrast to the heat trace asymptotics, the heat content asymptotics are not spectral. Our results involve relationships between the sets $\text{spec}(D)$ and $\text{hca}(D)$ for arbitrary complete Riemannian manifolds $(M,g)$ and arbitrary smoothly bounded domains with compact closure. These results arise naturally in the context of probability and help to shed light on a wide range of phenomena (estimates of the principal eigenvalue, comparison theorems, isoperimetric phenomena, etc) tying probability to geometric analysis. The probabilistic tools involved are of two types: those which give a probabilistic representation of the solution of boundary value problems, and those involving the representation of nonnegative sequences of real numbers as moments associated to a distribution function (ie classical moment problems). To concisely state our results, we recall the necessary material:

Let $X_t$ be Brownian motion on $M$. Let $P^x$, $x \in M$, be the family of probability measures charging Brownian paths beginning at $x$, and let $E^x$ be the corresponding collection of expectation operators. Let $\tau$ be the first exit time of $X_t$ from $D$:

\[ \tau = \inf\{t \geq 0 : X_t \notin D\}. \]

Then $u(x,t)$ defined in (1.2) can be written as

\begin{equation}
\label{eq:1.6}
u(x,t) = P^x(\tau > t)
\end{equation}

Given (1.6), it is natural to consider the Laplace transform of the random variable $\tau$ which is determined by the exit time moments of $\tau$. Thus, for $k$ a nonnegative integer, we are led to consider the following nonnegative sequence of real numbers:

\begin{equation}
A_n = \int_D E^x[\tau^n] dg.
\end{equation}

We write

\begin{equation}
\text{mspec}(D) = \{A_n\}_{n=0}^{\infty}
\end{equation}

and we note that $\text{mspec}(D)$ is invariant under the action of the isometry group of $M$ (cf 2.12, 2.14). Our first result is the following.
Theorem 1.1. Let $(M,g)$ be a complete Riemannian manifold, $D \subset M$ a smoothly bounded domain with compact closure. For $\lambda \in \text{spec}(D)$, let $E_\lambda(1)$ be orthogonal projection of the constant function 1 onto the eigenspace corresponding to $\lambda$. Define constants $a_\lambda^2 \in \mathbb{R}$ by

$$a_\lambda^2 = \int_D |E_\lambda(1)|^2 \, dg. \quad (1.9)$$

Let $\text{spec}^*(M)$ be the set whose elements are defined by

$$\text{spec}^*(M) = \{ \lambda \in \text{spec}(M) : a_\lambda^2 \neq 0 \}. \quad (1.10)$$

Then

$$\text{mspec}(D) = \text{mspec}(D') \implies \text{spec}^*(D) = \text{spec}^*(D')$$

and we say that $\text{mspec}(D)$ determines $\text{spec}^*(D)$.

We remark that $\text{spec}^*(D)$ is a set; in particular, it contains no information concerning multiplicities.

To prove Theorem 1.1 we note that the Stieltjes moment problem defined by the sequence $\text{mspec}(D)$ fixes a measure which determines both the set $\text{spec}^*(D)$ and the constants $a_\lambda^2$ defined by (1.9). This information, coupled to Theorem 1.1, determine the heat content asymptotics:

Theorem 1.2. Let $(M,g)$ be a complete Riemannian manifold, $D \subset M$ a smoothly bounded domain with compact closure. Then $\text{mspec}(D)$ determines $\text{hca}(D)$.

From the proof of Theorem 1.1 and Theorem 1.2, we obtain as a corollary the fact that the information contained in $\text{spec}^*(D)$ and the partition of volume $\{a_\lambda^2\}_{\lambda \in \text{spec}^*(D)}$ determines $\text{hca}(D)$ (cf Corollary 3.2).

We remark that results analogous to Theorem 1.1 and Theorem 1.2 hold in the category of graphs and graph Laplacians (cf [MM1]). In this context, there arises a natural Dirichlet series whose values at positive integers gives the analog of invariants defined in (1.1) and whose values at negative integers gives the analog of the heat content asymptotics of the associated graph domain (cf [MM1]). In the context of domains in complete manifolds the same is true; the relevant series arises as the Mellin transform of the heat content which admits a meromorphic extension to the plane. We investigate properties of the meromorphic extension of this Dirichlet series, characterizing the connection between special values, residues, and the invariants of interest (cf Proposition (2.1)).

Theorem 1.1, Theorem 1.2 and related results suggest that control of the moment spectrum may be useful in studying a variety of geometric phenomena including isoperimetric conditions (cf [BS] and [M1] for related results, as well as the survey [M2]), and estimates for higher eigenvalues and spectral gaps. In addition, in the category of weighted graphs and discrete Laplacians, the moment spectrum and heat content asymptotics distinguish analogues of the isospectral nonisometric planar polygons of [BCDS] (cf [MM2]), thus suggesting that heat content and Dirichlet spectrum may provide a good collection of invariants for classifying smoothly bounded domains up to isometry.
From the proof of Theorem 1.1 and Proposition 2.1 it is clear that mspec($M$) contains no information concerning multiplicity, nor does it contain information concerning modes orthogonal to constants. Thus, in the presence of symmetry we expect that mspec($M$) will not provide full information concerning the Dirichlet spectrum of the underlying domain. Our final result indicates that this occurrence is “unusual” when the manifold is a smoothly bounded domain in Euclidean space with compact closure.

Recall, the collection of smoothly bounded domains in Euclidean space with compact closure is naturally a Banach manifold. We recall that a property is generic for a Banach manifold if it holds for a set of second category (ie, it holds for the complement of a countable union of nowhere dense sets). We prove:

**Theorem 1.3.** For generic domains $D$ in $\mathbb{R}^n$, $n \geq 2$, mspec($D$) determines spec($D$).

Theorem 1.3 generalizes to domains in Riemannian manifolds of dimension at least two.

### 2: Mellin transforms and Dirichlet series

As in the introduction, let $(M, g)$ be a complete Riemannian manifold and suppose that $D \subset M$ is a smoothly bounded domain with compact closure. Given continuous functions, $f, h$, on $D$, denote the natural pairing by

$$\langle f, h \rangle = \int_D fhdg.$$  

Let $\Delta$ be the Laplace operator and suppose that spec($D$) is the Dirichlet spectrum associated to $D$.

**Definition 2.1.** Given $\lambda \in$ spec($D$), let $E_\lambda(1)$ be orthogonal projection of the constant function 1 onto the eigenspace associated to $\lambda$. Let $a_\lambda^2$ be the nonnegative real number defined by

$$a_\lambda^2 = \langle E_\lambda(1), E_\lambda(1) \rangle.$$  

We call the set whose elements are given by $a_\lambda^2$ as $\lambda$ runs through spec($D$) a spectral partition of volume and we write

$$vp(D) = \{a_\lambda^2\}_{\lambda \in \text{spec}(D)}.$$  

As suggested by Definition 2.1, the set $vp(D)$ partitions the volume amongst eigenspaces. In particular

$$\sum_{\lambda \in \text{spec}^+(D)} a_\lambda^2 = \text{vol}(D)$$  

where spec$^+(D)$ is as in (1.10). For $s$ complex, Re($s$) $\geq 0$, we define

$$\zeta_D(s) = \sum_{\lambda \in \text{spec}^+(D)} a_\lambda^2 \left(\frac{2}{\lambda}\right)^s.$$
We show that $\zeta_D(s)$ is closely related to the heat content $q(t)$ defined in (1.3).

Starting with the heat kernel written in terms of the spectral data, we have

\begin{equation}
q(t) = \sum_{\lambda \in \text{spec}^\ast(D)} a_\lambda^2 e^{\frac{-\lambda t}{2}}.
\end{equation}

We note that $q(t)$ is continuous and bounded on $[0, \infty)$. As mentioned in the introduction, $q(t)$ admits a small time asymptotic expansion given in (1.4). For complex $s$, $\text{Re}(s) > 0$, the Mellin transform of $q(t)$ is defined by

\begin{equation}
\mathcal{M}Q(s) = \int_0^\infty q(t)t^s dt.
\end{equation}

Using (2.5) we see that for $\text{Re}(s) > 0$,

\begin{equation}
\mathcal{M}Q(s) = \Gamma(s)\zeta_M(s)
\end{equation}

where $\zeta_D(s)$ is given by (2.3) and $\Gamma(s)$ is the gamma function. By the standard theory of regularized series (cf [JL]), $\mathcal{M}Q(s)$ admits a meromorphic extension to the plane with poles restricted to lie at the negative half-integers. In addition, the poles are simple with residues given by

\begin{equation}
\text{Residue}_{s = -\frac{N}{2}} \mathcal{M}Q(s) = q_N
\end{equation}

where $q_N$ is as in (1.4) (ie the residues are given by the heat content asymptotics). This proves the first part of

**Proposition 2.1.** For $\text{Re}(s) > 0$, let $\zeta_D(s)$ be defined as in (2.3). Then $\Gamma(s)\zeta_D(s)$ extends meromorphically to the complex plane with poles restricted to lie at the negative half integers. In addition, the poles are simple and for $N$ a natural number,

\begin{equation}
\Gamma(N)\zeta_M(N) = \frac{1}{N}A_N
\end{equation}

\begin{equation}
\text{Residue}_{s = -\frac{N}{2}} \Gamma(s)\zeta_M(s) = q_N
\end{equation}

where $A_N$ is given by (1.7) and $q_N$ is given by (1.4).

**Proof.** The claim (2.9) follows immediately from (2.6) and (2.7). To see that (2.8) holds, let $X_t$ be Brownian motion on $M$, and let $\tau$ be the first exit time from $D$. Let

\begin{equation}
h(x, s) = \mathbb{E}^x[e^{-s\tau}].
\end{equation}

Then $h$ is the unique solution of the Dirichlet problem

\begin{equation}
\frac{1}{2} \Delta h - sh = 0 \text{ on } D \times (0, \infty) \quad \quad h = 1 \text{ on } D \times \{0\}.
\end{equation}
Expanding \( h(x, s) \) using power series and using (2.11), we see that the exit time moments can be defined by recursive solution of Poisson problems. More precisely, suppose that

\[
\frac{1}{2} \Delta u_1 + 1 = 0 \quad \text{on } D
\]
\[
u_1 = 0 \quad \text{on } \partial D
\]

and

\[
\frac{1}{2} \Delta u_k + ku_{k-1} = 0 \quad \text{on } D
\]
\[
u_k = 0 \quad \text{on } \partial D.
\]

Then

\[
E^x[x^k] = u_k(x).
\]

Thus, if \( \phi_\lambda \) is a normalized eigenfunction, we have

\[
\langle u_k, \phi_\lambda \rangle = -\frac{2}{\lambda} \langle u_k, \frac{1}{2} \Delta \phi_\lambda \rangle
\]
\[
= \frac{2}{\lambda} k \langle u_{k-1}, \phi_\lambda \rangle.
\]

Writing

\[
1 = \sum_{\lambda \in \text{spec}(\mathcal{M})} \langle 1, \phi_\lambda \rangle \phi_\lambda,
\]

the proposition follows.

3: Proof of Theorem 1.1 and Theorem 1.2

We begin with a corollary of Proposition 2.1:

**Corollary 3.1.** For \( A_n \) as defined in (1.7), set

\[
\mu_n = \frac{A_n}{n!}.
\]

Then the collection \( \{\mu_n\} \) satisfies Carleman’s condition:

\[
\sum \mu_{2n}^{-\frac{1}{2n}} = \infty.
\]

**Proof.** From Proposition (2.1), \( \mu_n = \zeta_D(n) \). Thus,

\[
\mu_n = \sum_{\lambda \in \text{spec}^*(D)} a_\lambda^2 \left(\frac{2}{\lambda}\right)^n
\]
\[
\leq \left(\frac{2}{\lambda}\right)^n \text{vol}(D).
\]
Proof of Theorem 1.1. Let \( \mu_n \) be as defined in (3.1) and note that \( \mu_n > 0 \) for all nonnegative integers \( n \). Define a bounded, nondecreasing function, \( \psi : [0, \infty) \to [0, \infty) \) by

\[
\psi(x) = \sum_{\lambda \in \text{spec}^*(M)} a^2_\lambda 1_{\left[1/\lambda, \infty\right)}(x)
\]

where \( 1_{\left[1/\lambda, \infty\right)}(x) \) is the indicator function of the interval \( \left[1/\lambda, \infty\right) \). Then \( \psi \) solves the Stieltjes moment problem for the moments \( \mu_n \):

\[
\mu_n = \int_0^\infty x^n d\psi.
\]

Recall the classic result of Carleman (cf [A]):

**Theorem.** Suppose \( \{\mu_n\} \) is a sequence of nonnegative real numbers. If (3.2) holds, then the Stieltjes moment problem for the sequence \( \{\mu_n\} \) is determined.

By Corollary 3.1, the moments satisfy Carleman’s condition and thus the unique solution of the Stieltjes Moment Problem is given by (3.3). Thus, the sequence \( \{A_n\} \) determines both the set \( \text{spec}^*(M) \) (the discontinuities of \( \psi(x) \)), as well as the collection of jumps, \( \text{vp}(D) \). This proves Theorem 1.1.

From the proof of Theorem 1.1 we immediately conclude:

**Corollary 3.1.** Let \( D \subset M \) be a smoothly bounded domain with compact closure. Then \( m\text{spec}(D) \) determines \( \text{vp}(D) \).

*Proof of Theorem 1.2.** Theorem 1.2 follows immediately from Theorem 1.1, Corollary 3.1, and (2.4).

Finally, we give a relationship between Dirichlet spectrum and heat content asymptotics.

**Corollary 3.2.** Let \( D \subset M \) be a smoothly bounded domain with compact closure. Then \( \text{spec}^*(D) \cup \text{vp}(D) \) determines \( \text{hca}(D) \).

*Proof.* This is immediate from the definition of \( \zeta_D(s) \) and Proposition 2.1. In fact, from Corollary 3.1 and (2.4) it is clear that the heat content (not just the asymptotics) is determined.

### 4: Proof of Theorem 1.3

In this section we consider \( C^k \)-domains with compact closure in \( \mathbb{R}^n, \ n \geq 2 \).

Let \( k > n + 2 \) and let \( \mathcal{B} \) be the collection of \( C^k \)-domains in \( \mathbb{R}^n \) with compact closure. Recall, \( \mathcal{B} \) is a Banach manifold: Given \( b \in \mathcal{B} \), we identify \( b \) with its boundary, \( \partial b \). The tubular neighborhood theorem identifies a neighborhood of \( \partial b \) in \( \mathbb{R}^n \) with sections of the normal bundle to the boundary of \( b \), denoted \( C^k(\partial b, N\partial b) \). Pairing with the outward pointing unit normal vector gives an isomorphism between \( C^k(\partial b, N\partial b) \) and \( C^k(\partial b) \). We identify domains near \( b \) by identifying their boundaries as those obtained by flow in the normal bundle.
normal direction prescribed by elements of $C^k(\partial b)$. More precisely, if $\nu$ is the outward pointing unit normal vector along the boundary of $b$ and $f \in C^k(\partial b)$, then for $\epsilon$ small enough, the set
\begin{equation}
\partial b_\epsilon = \{ y \in \mathbb{R}^n : y = \sigma + \epsilon f(\sigma)\nu(\sigma), \quad \sigma \in \partial b \}
\end{equation}

bounds a $C^k$-domain in $\mathbb{R}^n$ and $\epsilon \to b_\epsilon$ where $b_\epsilon$ is the domain bounded by $\partial b_\epsilon$ is a smooth curve in $\mathcal{B}$ passing through $b$ at $\epsilon = 0$. This provides an identification of a neighborhood of $b \in \mathcal{B}$ with a neighborhood of $0$ in $C^k(\partial b)$, which shows that $\mathcal{B}$ is a Banach manifold. In addition, the construction indicates that that there is a natural choice for the tangent space of $b \in \mathcal{B}$:
\begin{equation}
T_b \mathcal{B} \simeq C^k(\partial b).
\end{equation}

It is a theorem of Uhlenbeck ([U] also [CV]) that the collection of $C^k$-domains $b \in \mathcal{B}$ for which all Dirichlet eigenvalues have multiplicity one is open and dense in $\mathcal{B}$. In the sequel, we adopt Uhlenbeck’s approach to establish a generic property useful for our purposes. We begin by recalling the necessary notation.

Let $H_k(\mathbb{R}^n)$ be the Sobolev space of functions on $\mathbb{R}^n$ with distributional derivatives up through order $k$ which are $L^2$. Let $H_{k,0}(\mathbb{R}^n) \subset H_k(\mathbb{R}^n)$ be the closure of the space of smooth functions on $\mathbb{R}^n$, and let $S_k(\mathbb{R}^n)$ be the unit ball in $H_k(\mathbb{R}^n)$. Let $\phi : S_k(\mathbb{R}^n) \times \mathbb{R} \times \mathcal{B} \to H_{k-2}(\mathbb{R}^n)$ be defined by $\phi(u, \lambda, b) = \Delta_b u - \lambda u$ where $\Delta_b$ is the Dirichlet Laplacian on $b$. Let $Q \subset S_k(\mathbb{R}^n) \times \mathbb{R} \times \mathcal{B}$ be defined by $Q = \phi^{-1}(0)$. Then $Q$ is the collection of domains, their Dirichlet spectrum, and their corresponding normalized Dirichlet eigenfunctions. It is a corollary of the Sard-Smale theorem that $Q$ is a Banach submanifold of $H_k(\mathbb{R}^n) \times \mathbb{R} \times \mathcal{B}$. The tangent space at a point $(u, \lambda, b) \in Q$ is given by
\begin{equation}
T_{(u, \lambda, b)} \simeq \left\{ (v, \eta, f) \in H_{k,0}(b) \times \mathbb{R} \times T_b \mathcal{B} : \int_b uv = 0, \quad (\Delta_b + \lambda)v + \eta u + D_u \phi(f) = 0 \right\}
\end{equation}

where $D_u \phi$ is the derivative of the function $\phi$ with respect to $\mathcal{B}$.

**Definition.** Let $b \in \mathcal{B}$. We say that $b$ has Property M if for all $\lambda \in \text{spec}(b)$,
\begin{equation}
\langle \mathcal{E}_\lambda(1), \mathcal{E}_\lambda(1) \rangle \neq 0
\end{equation}

where $\mathcal{E}_\lambda$ is projection on the eigenspace corresponding to $\lambda$.

To see that Property M is generic, we define a function $I : Q \to \mathbb{R}$ by
\begin{equation}
I(u, \lambda, b) = \int_b u(x)dx
\end{equation}

where $dx$ denotes Lebesgue measure. We note that $I$ is clearly $C^k$, and thus $I^{-1}(\mathbb{R}^n \setminus \{0\})$ is open. We will show that $D_u I$, the derivative of $I$ with respect to domain variations, is always surjective.

To see that this is the case fix $(u, \lambda, b) \in Q$. An infinitesimal variation of the domain $b$ is given by fixing an element $f \in C^k(\partial b)$. We denote by $\delta u$ the corresponding infinitesimal
change in $u$ and by $\delta \lambda$ the corresponding infinitesimal change in $\lambda$. A straightforward computation then gives:

\begin{equation}
D_b I(\delta u, \delta \lambda, f) = \int_b \delta u(x) dx.
\end{equation}

Using Hadamard’s classic results on the variation of Green’s functions for perturbed domains (cf [H], [GS]), we have an expression for $\delta u(x)$:

\begin{equation}
\delta u(x) = -\int_{\partial b} f(\sigma) \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial G}{\partial \nu}(x, \sigma) d\sigma
\end{equation}

where $d\sigma$ is the induced surface measure on the boundary and $G$ is the Green’s function for $D$. As pointed out by Uhlenbeck for a similar computation, it is a corollary of unique continuation that $\frac{\partial u}{\partial \nu}(\sigma) \frac{\partial G}{\partial \nu}(x, \sigma)$ is not identically zero. We conclude that we can find $f$ such that $D_b I(\delta u, \delta \lambda, f) \neq 0$. As a corollary, we obtain

**Theorem 4.1.** Let $\mathcal{B}$ be the Banach manifold of $C^k$-domains with compact closure. Then Property M is generic for $\mathcal{B}$.

**Proof of Theorem 1.3.** From Theorem 1.1, we know that $\text{spec}^*(D)$ is determined by $\text{mspec}(D)$. By Uhlenbeck’s theorem, for a dense open set of domains, all eigenvalues have multiplicity one. By Theorem 4.1, for domains with all eigenvalues of multiplicity one, it is generically the case that $\text{spec}^*(D) = \text{spec}(D)$. This concludes the proof of Theorem 1.3.

**References**

[A] N. Akhiezer, *The Classical Moment Problem*, Hafner, New York, 1965.

[BG] M. van den Berg and P. Gilkey, *Heat content asymptotics of a Riemannian manifold with boundary*, Jour. Funct. Anal. 120 (1994), 48–71.

[BS] A. Burchard and M. Schmuckenschläger, *Comparison theorems for exit times*, GAFA 11 (2001), 651-692.

[CV] Y. Colin de Verdiere, *Multiplicités des valeurs propres Laplacians discret et Laplacians continus*, Rend. di Math. 13 (1993), 433–460.

[G] P. Gilkey, *Heat content asymptotics*, in: Geometric Aspects of Partial Differential Equations, Contemp. Math. 242 (1999), AMS, Providence, RI, 125-134.

[GS] P. R. Garabedian and M. Schiffer, *Convexity of domain functionals*, Jour. d’Anal. Math. 2 (1952), 281–369.

[H] J. Hadamard, *Mémoire sur les problème d’analyse relatif à l’équilibre des plaques élastique encastrées*, Mémoires des savantes étrangers 33 (1908).

[JL] J. Jorgenson and S. Lang, *Basic Analysis of Regularized Series and Products*, LNM 1564, Springer-Verlag, New York, 1991.

[M1] P. McDonald, *Isoperimetric conditions, Poisson problems and diffusions in Riemannian manifolds*, Potential Analysis 16 (2002), 115-138.

[M2] P. McDonald, *Recent results in geometric analysis involving probability*, In: Recent Advances in Applied Probability (to appear).

[MM1] P. McDonald and R. Meyers, *Diffusions on graphs, Poisson problems and spectral geometry*, Trans. AMS (to appear).
[MM2] P. McDonald and R. Meyers, *Isospectral graphs and random walks* (in preparation).

[U] K. Uhlenbeck, *Generic properties of eigenfunctions*, Amer. J. Math 98 (1976), 1059–1078.

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