ON THE BOUNDEDNESS OF ANTI-CANONICAL VOLUMES OF SINGULAR FANO 3-FOLDS IN CHARACTERISTIC $p > 5$

OMPROKASH DAS

Abstract. In this article we prove the following version of the Weak-BAB conjecture for 3-folds in char $p > 5$: Fix a DCC set $I \subseteq [0, 1)$ and an algebraically closed field $k$ of characteristic $p > 5$. Let $\mathcal{D}$ be a collection of klt pairs $(X, \Delta)$ satisfying the following properties: (1) $X$ is a projective 3-fold, (2) $\Delta$ is an $\mathbb{R}$-divisor with coefficients in $I$, (3) $K_X + \Delta \equiv 0$, and (4) $-K_X$ is ample. Then the set $\{\text{vol}_X(-K_X) \mid (X, \Delta) \in \mathcal{D} \text{ for some $\Delta$}\}$ is bounded from above.

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1. Introduction

Given a smooth projective variety $X$, the minimal model program predicts that $X$ is birational to a variety $Y$ with canonical singularities such that either $K_Y$ is ample, or $Y$ admits a fibration whose general fibers are Calabi-Yau varieties or Fano varieties. In other words, one may say that, birationally, every variety is in some sense constructed from varieties $X$ with good singularities such that either $K_X$ is ample or $K_X \equiv 0$ or $-K_X$ is ample. So it is quite natural to study such special varieties with the hope of obtaining some sort of classification theory. One such classification is finding the moduli space of given class of varieties. In this article we are interested in the moduli problem.
of Fano varieties i.e., $-K_X$ is ample, of dimension 3 in positive characteristic. The first problem in this direction is proving the boundedness of the moduli functor, i.e., finding a fibration $f : \mathcal{X} \to T$ such that every Fano 3-fold appears as closed fiber of $f$. However, this turns out to be a too general of a question to be true without some restriction on the singularities of $X$; counterexamples are known to exists even in characteristic 0. This leads to the following conjecture of Borisov, Alexeev and Borisov, known as the BAB conjecture.

**Conjecture 1.1 (BAB Conjecture).** Fix a positive integer $n$ and a real number $\varepsilon > 0$. The set of all projective varieties $X$ satisfying the following properties:

1. the dimension of $X$ is $n$,
2. $(X, \Delta)$ has $\varepsilon$-log canonical singularities for some boundary $\mathbb{R}$-divisor $\Delta$,
3. $-(K_X + \Delta)$ is ample,

form a bounded family.

Note that a pair $(X, \Delta)$ is said to have $\varepsilon$-log canonical singularities if the discrepancies satisfy $a(E; X, \Delta) \geq -1+\varepsilon$ for all divisors $E$ over $X$. A necessary condition that follows from the BAB conjecture is that the volumes of $-K_X$ (see Definition 3.1) is bounded from above; this is known as the Weak-BAB conjecture.

**Conjecture 1.2 (Weak-BAB Conjecture).** Fix a positive integer $n$ and a real number $\varepsilon > 0$. Let $\mathcal{D}$ be the set of all log pairs $(X, \Delta)$ satisfying the following properties:

1. $X$ is a projective variety of dimension $n$,
2. $(X, \Delta)$ has $\varepsilon$-log canonical singularities for some boundary $\mathbb{R}$-divisor $\Delta$, and
3. $-(K_X + \Delta)$ is ample.

Then there exists a positive real number $M(n, \varepsilon)$ depending only on $n$ and $\varepsilon > 0$ such that the set $\{ \text{vol}_X(-K_X) \mid (X, \Delta) \in \mathcal{D} \text{ for some } \Delta \}$ is bounded from above by $M(n, \varepsilon)$.

We note that the BAB conjecture (and hence the Weak-BAB conjecture) is known in dimension 2 in arbitrary characteristic due to Alexeev, [Ale94]. In dimension 3 the following partial results on the BAB conjecture were known for a while: The toric Fano 3-fold case by Borisov brothers in [BB92]. In [KMM92], the authors proved the conjecture for smooth Fano varieties in char 0; Kawamata in [Kaw92] proved the conjecture in char 0 for $\mathbb{Q}$-factorial terminal Fano 3-folds of Picard number $\rho(X) = 1$. In [KMMT00], the authors proved the same conjecture for $\mathbb{Q}$-factorial Fano 3-folds with canonical singularities in char 0.
The Weak-BAB conjecture is known in dimension 3 in char 0 when the Picard number \( \rho(X) = 1 \), due to \([\text{Lai16}]\), and the general case due to Jiang, \([\text{Jia14}]\). More recently the BAB conjecture in full generality (and hence the Weak-BAB conjecture) is completely proved in char 0 in every dimension in a series of breakthrough papers by Birkar, \([\text{Bir16a}, \text{Bir16b}]\). On the other hand, very little is known about either of these two conjectures in positive characteristic in dimension 3 or higher. In this article we prove a special case of the Weak-BAB conjecture for 3-folds in characteristic \( p > 5 \). More specifically, using the ideas from \([\text{HMX14}]\) we prove the following results:

**Theorem 1.3.** Fix an algebraically closed field \( k = \overline{k} \) of characteristic \( p > 5 \) and a DCC set \( I \subseteq [0,1) \). Let \( \mathcal{O} \) be the set of all klt pairs \((X, \Delta)\), where

1. \( X \) is a projective 3-fold,
2. the coefficients of \( \Delta \) belong to \( I \),
3. \( K_X + \Delta \equiv 0 \), and
4. \( -K_X \) is big,

then the set

\[ \{ \mathrm{vol}_X(-K_X) \mid (X, \Delta) \in \mathcal{O} \text{ for some } \Delta \} \]

is bounded from above.

We note that the theorem above is a positive characteristic \((p > 5)\) analog of Theorem B in \([\text{HMX14}]\) in dimension 3.

As a corollary we get the following version of the Weak-BAB conjecture.

**Corollary 1.4.** Fix an algebraically closed field \( k = \overline{k} \) of characteristic \( p > 5 \) and a DCC set \( I \subseteq [0,1) \). Let \( \mathcal{O} \) be the set of all klt pairs \((X, \Delta)\), where

1. \( X \) is a projective 3-fold,
2. the coefficients of \( \Delta \) belong to \( I \),
3. \( K_X + \Delta \equiv 0 \), and
4. \( -K_X \) is ample,

then the set

\[ \{ \mathrm{vol}_X(-K_X) \mid (X, \Delta) \in \mathcal{O} \text{ for some } \Delta \} \]

is bounded from above.

We note that, as far as we know, except \([\text{Zhu17}]\), our result is the only result on the Weak-BAB conjecture for 3-folds in positive characteristic. About the paper \([\text{Zhu17}]\), in this article the author proves a boundedness result for the anti-canonical volumes of Fano 3-folds in char \( p > 5 \) under the restrictive assumption that the Seshadri constant of \( -K_X \) is larger than some fixed number. We note that this kind of restriction is not standard in the context of
moduli problems. Seshadri constants typically do not play an important role in the theory of moduli spaces and do not seem to appear naturally in the context of the minimal model program. On the other hand, our hypothesis involving DCC sets is a standard hypothesis which appears quite naturally and frequently in various statements related to the minimal model program and the moduli problem in general, for example, see [HMX14, Corollary 1.7] for a boundedness result of Fano varieties (the BAB conjecture) in characteristic 0 involving DCC sets. In this sense our result is an important first step towards the proof of the boundedness of Fano 3-folds in characteristic \( p > 5 \).

**Idea of the Proof of Theorem 1.3:** The intuitive idea of the proof is the following: Since the coefficients of \( \Delta \) are contained in a fixed DCC set \( I \), we can find an \( \varepsilon > 0 \) depending only on the set \( I \) and satisfying the following properties: if \( (X, \Delta) \in \mathcal{D} \) and \( \Phi \geq 0 \) is an effective \( \mathbb{R} \)-Cartier divisor on \( X \) such that \( K_X + \Phi \equiv 0 \) and \( \Phi \) is contained in the “\( \varepsilon \)-neighborhood” of \( \Delta \), then \( (X, \Phi) \) has klt singularities. Now if the \( \text{vol}_X(-K_X) \) is not bounded above as \( X \) varies in \( \mathcal{D} \), then choose a pair \( (X, \Delta) \in \mathcal{D} \) such that \( \text{vol}_X(\Delta) = \text{vol}_X(-K_X) > n^n \), where \( n = \dim X \). Then we can construct an effective \( \mathbb{R} \)-divisor \( \Psi \sim \mathbb{R} \Delta \) contained in the “\( \varepsilon \)-neighborhood” of \( \Delta \) such that \( (X, \Psi) \) is not klt, which is a contradiction.

**Acknowledgement.** I would like to thank Joe Waldron for answering several of my questions. I would also like to thank Burt Totaro for his valuable comments. My sincerest gratitude goes to Christopher Hacon for answering many questions and fruitful discussions. I would also like to thank Akash Sengupta for pointing out an error in the previous version.

2. Preliminaries

Throughout the article by an arbitrary field \( k \), we mean that the characteristic of \( k \) is either 0 or positive and \( k \) is possibly imperfect; in particular \( k \) is not necessarily algebraically closed.

**Definition 2.1.** Let \( k \) be an arbitrary field. A variety \( X \) over \( k \) is an integral separated scheme of finite type over \( k \). A curve over \( k \) is a variety of dimension 1 over \( k \). A surface over \( k \) is a variety of dimension 2 over \( k \).

**Definition 2.2.** Let \( k \) be an arbitrary field and \( X \) a normal variety over \( k \). Let \( \Delta \) be an \( \mathbb{R} \)-divisor on \( X \). We say that \( \Delta \) is a boundary divisor if the coefficients of \( \Delta \) belong to the closed interval \([0, 1]\). If \( \Delta \) is a boundary divisor, then a pair \( (X, \Delta) \) is called a log pair if \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. For a log pair \( (X, \Delta) \), we define terminal, canonical, klt, plt, dlt and log canonical or lc...
singlarities as in [Kol13, Definition 2.8]. We emphasize that in this article whenever we talk about singularities of a pair \((X, \Delta)\) we always assume that \(\Delta\) is an effective \(\mathbb{R}\)-divisor.

**Remark 2.3.** Let \(C\) be a normal curve over an arbitrary field \(k\) and \(\Delta\) is a boundary divisor on \(C\). Then \((C, \Delta)\) is log canonical (resp. klt) if and only if the coefficients of \(\Delta\) are less than or equal to 1 (resp. strictly less than 1).

**Remark 2.4.** We note that the Nakai-Moishezon-Klemian criterion for ampleness, Kodaira’s lemma for big divisors, etc. hold on projective varieties defined over arbitrary fields. For more details see [Tan18, Remark 2.3 and 2.4].

**Definition 2.5.** Let \(C\) be a normal, i.e., regular curve over an arbitrary field \(k\). Let \(D = \sum r_i P_i\) be an Weil \(\mathbb{R}\)-divisor on \(C\). Then the degree of \(D\) on \(C\) over \(k\) is defined as

\[
\deg_k(D) := \sum_i [k(P_i) : k] \cdot r_i,
\]

where \(k(P_i)\) is the residue field of the closed point \(P_i \in C\) and \([k(P_i) : k]\) is the extension degree of the fields.

2.1. **DCC sets and adjunction.**

**Definition 2.6.** We say that a subset \(I\) of real numbers satisfies the decreasing chain condition or DCC if every for every decreasing sequence \(\{a_i\} \subseteq I\), i.e., \(a_i \geq a_{i+1}\) for all \(i \geq 1\), there exists a \(N \gg 0\) such that \(a_i = a_{i+1}\) for all \(i \geq N\); equivalently, \(I\) does not contain any infinite strictly decreasing sequence.

Let \(I \subseteq [0, 1]\). We define

\[
I_+ := \{0\} \cup \left\{ j \in [0, 1] \mid j = \sum_{p=1}^{\ell} i_p, \text{ for some } i_1, i_2, \ldots, i_\ell \in I \right\}
\]

and

\[
D(I) := \left\{ a \leq 1 \mid a = \frac{m - 1 + f}{m}, m \in \mathbb{N}, f \in I_+ \right\}.
\]

The following lemma shows some useful properties of DCC sets.

**Lemma 2.7.** Let \(I, I_1, I_2, \ldots, I_n\) be subsets of \(\mathbb{R}_{\geq 0}\).

1. Any subset of a DCC set is a DCC set.
2. If \(I_1, I_2, \ldots, I_n\) all satisfy the DCC, then \(\bigcup_{i=1}^n I_n\) satisfies the DCC.
3. If \(I\) satisfies the DCC and \(r_1, r_2, \ldots, r_k \geq 0\), then \(I' = \{ar_i : a \in I, 1 \leq i \leq k\}\) satisfies the DCC.
Let $I \subseteq [0, 1]$, then $I$ satisfies the DCC if and only if $D(I)$ satisfies the DCC.

Proof. Part (1) is obvious. For part (2), by contradiction assume that there is a strictly decreasing sequence $\{t_m\}$ contained in $\cup_{i=1}^{n} I_i$. Then by going to an infinite subsequence we may assume that all the terms of $\{t_m\}$ are contained in $I_j$ for some fixed $j$ satisfying $1 \leq j \leq n$. This is a contradiction to the DCC property of $I_j$. Part (3) follows from part (1) and (2) by noticing that $I' = \cup_{i=1}^{k} J_i$, where $J_i = \{ar_i : a \in I\}$, and that $J_i$ satisfies DCC for all $i$.

For part (4), by contradiction assume that there is a strictly decreasing sequence $\{r_k\}$ contained in $\sum_{i=1}^{n} I_n$. Let $r_k = \sum_{j=1}^{n} r_{kj}$, where $r_{kj} \in I_j$ for all $k$ and $j$. Since $\{r_{kj}\}_{k \geq 1}$ satisfies DCC for all $j$, by going to subsequences with common indices we may assume that each $\{r_{kj}\}_{k \geq 1}$ is a monotonically increasing sequence for all $j$. It then follows that $r_k \leq r_{k+1}$ for all $k \geq 1$, which is a contradiction, since $r_k > r_{k+1}$ for all $k \geq 1$.

For part (5) let’s define $\mathbb{Z}_{\geq 0} \cdot I = \{nr : n \in \mathbb{Z}_{\geq 0}, r \in I\}$. Then $\text{Span}_N(I) = \mathbb{Z}_{\geq 0} \cdot I + \mathbb{Z}_{\geq 0} \cdot I + \cdots (N \text{ times}) + \mathbb{Z}_{\geq 0} \cdot I$.

Thus (5) will follow from (4) if we can show that $\mathbb{Z}_{\geq 0} \cdot I$ is a DCC set. To that end, by contradiction assume that there is a strictly decreasing sequence $\{n_ir_i\}$ in $\mathbb{Z}_{\geq 0} \cdot I$, i.e.,

$$n_1r_1 > n_2r_2 > n_3r_3 > \cdots .$$

(2.1)

Since $\{r_i\}$ is contained in a DCC set $I$, by going to an infinite subsequence we may assume that $\{r_i\}$ is a monotonically increasing sequence, i.e.,

$$r_1 \leq r_2 \leq r_3 \leq \cdots .$$

(2.2)

Therefore we have $r_i \geq r_1 > 0$ for all $i \geq 1$. From (2.1) we also have that $n_1r_1 \geq n_ir_i$ for all $i \geq 1$. Thus we get that $n_1 \geq n_i$ for all $i \geq 1$. In particular, $\{n_i\}$ is a bounded sequence of positive integers, hence a finite set. Thus by going to an infinite subsequence of (2.1) we may assume that $n_i = n_{i+1}$ for all $i \geq 1$, which gives a contradiction to (2.2).

For part (6) see [MP04, Lemma 4.4].
2.2. Adjunction. In this subsection we collect some results about adjunction to codimension 1 subvarieties.

**Proposition 2.8** (Different). Fix a DCC set $I \subseteq [0, 1]$. Let $(X, \Delta)$ be a log pair defined over an arbitrary field $k$ and $S$ a component of $\lfloor \Delta \rfloor$. Let $S^n \to S$ be the normalization morphism. Then there exists a canonically determined effective $\mathbb{R}$-divisor $\Delta_{S^n} \geq 0$ on $S^n$ such that

$$(K_X + \Delta)|_{S^n} \sim_{\mathbb{R}} K_{S^n} + \Delta_{S^n}.$$ 

Moreover, if $(X, S + \Delta)$ is log canonical outside a codimension 3 closed subset and the coefficients of $\Delta$ belong to $I$, then the coefficients of $\Delta_{S^n}$ belong to $D(I)$. More precisely: write $\Delta = S + \sum_{i \geq 2} d_i D_i$, let $P'$ be a prime Weil divisor on $S^n$ and $P$ its image on $S$. Then there exists $m \in \mathbb{N} \cup \{\infty\}$ depending only on $X, S$ and $P$, there are non-negative integers $l_i \in \mathbb{Z}_{\geq 0}$ depending only on $X, S, D_i$ and $P$ such that the coefficient of $P'$ in $\Delta_{S^n}$ is

$$\frac{m - 1}{m} \sum_{i \geq 2} l_i d_i + \sum_{i \geq 2} l_i d_i.$$ 

**Proof.** The proof is same as the proofs of Proposition 4.1 and 4.2 in [Bir16c]. We note that the proof in [Bir16c] essentially reduces the problem to a computation on excellent surfaces. Since any scheme of finite type over an arbitrary field is an excellent scheme and the local rings of excellent schemes are again excellent, the same proof works in our case. □

**Lemma 2.9** (Easy Adjunction). Let $X$ be a normal variety over an arbitrary field $k$. Let $S$ be a prime Weil divisor and $\Delta \geq 0$ an effective $\mathbb{R}$-divisor on $X$ such that $S$ is not contained in the support of $\Delta$ and $K_X + S + \Delta$ is $\mathbb{R}$-Cartier. Let $S^n \to S$ be the normalization morphism and $(S^n, \Delta_{S^n})$ is defined by adjunction $K_{S^n} + \Delta_{S^n} \sim_{\mathbb{R}} (K_X + S + \Delta)|_{S^n}$.

If $(X, S + \Delta)$ is terminal, canonical, klt, plt or lc, then so is $(S^n, \Delta_{S^n})$, respectively.

**Proof.** It follows from [Kol13, Lemma 4.8]. □

3. Lemmas and Propositions

3.1. The volume.

**Definition 3.1.** Let $X$ be a projective variety of dimension $n$ over an algebraically closed field $k$. Let $D$ be a $\mathbb{R}$-Cartier divisor. Then the volume of $D$ is defined as

$$\text{vol}_X(D) := \limsup_{m \to +\infty} \frac{n! \dim_k H^0(X, \mathcal{O}_X(|mD|))}{m^n}.$$
It is known that $D$ is big if and only if $\text{vol}_X(D) > 0$.

In the following lemmas we establish some perturbation techniques for log pairs $(X, \Delta)$ using a divisor $D$ such that $\text{vol}_X(D) > n^n$.

**Lemma 3.2.** Let $X$ be a proper variety of dimension $n$ defined over an algebraically closed field $k$ of arbitrary characteristic. Let $M$ be a big $\mathbb{Q}$-Cartier divisor on $X$ such that $\text{vol}_X(M) > n^n$ and $x \in X$ is a smooth closed point. Then for every $\varepsilon > 0$ there exists an effective $\mathbb{Q}$-Cartier divisor $D = D(x, \varepsilon)$ such that $D \sim \mathbb{Q} M$ and

$$\text{mult}_x D \geq n - \varepsilon.$$ 

**Proof.** The following proof is based on the proof of [Kol97, Lemma 6.1].

Let $t > 0$ be a positive integer such that $tM$ is a Cartier divisor. Let $m_x \subseteq \mathcal{O}_X$ be the ideal sheaf of $\{x\} \subseteq X$. For a positive integer $s > 0$ consider the following exact sequence

$$0 \longrightarrow m_x^s \otimes \mathcal{O}_X(tM) \longrightarrow \mathcal{O}_X(tM) \longrightarrow (\mathcal{O}_X/m_x^s) \otimes \mathcal{O}_X(tM) \cong \mathcal{O}_X/m_x^s \longrightarrow 0.$$ 

Then observe that

$$h^0(X, m_x^s \otimes \mathcal{O}_X(tM)) > 0 \quad \text{if} \quad h^0(X, \mathcal{O}_X(tM)) > h^0(X, \mathcal{O}_X/m_x^s)).$$

Let $\{x_1, x_2, \ldots, x_n\}$ be a local coordinate system around $x \in X$, i.e., it is a $k(x)$-basis of the vector space $m_x/m_x^2$. Then we have

$$h^0(X, \mathcal{O}_X/m^s) = \dim_k k[x_1, x_2, \ldots, x_n]/(x_1, x_2, \ldots, x_n)^s = \left(\frac{n + s - 1}{n}\right) = \frac{s^n}{n!} + O(s^{n-1}).$$

Since $\text{vol}_X(M) > n^n$, from the definition of volume it follows that $h^0(X, \mathcal{O}_X(tM)) > \frac{(nt)^n}{n!}$ for infinitely many values of $t$ which are sufficiently large and divisible. Thus from (3.2) we see that $h^0(X, \mathcal{O}_X(tM)) > h^0(X, \mathcal{O}_X/m_x^s)$ for $s, t \gg 0$ and satisfying $tn > s$, where $n = \dim X$. In particular, from (3.1) it follows that $h^0(X, m_x^s \otimes \mathcal{O}_X(tM)) > 0$ for $s, t \gg 0$ and satisfying $tn > s$.

Choose $t \gg 0$ such that the open interval $(t(n - \varepsilon), tn)$ contains a positive integer, say $s > 0$, i.e., $t(n - \varepsilon) < s < tn$, i.e., $n - \varepsilon < s/t < n$. Let $D(s, t, x)$ be the divisor of zeros of a non-zero global section of $m_x^s \otimes \mathcal{O}_X(tM)$ and $D(x, \varepsilon) = D(s, t, x)/t$. Then $\text{mult}_x D(x, \varepsilon) \geq s/t > n - \varepsilon$. \hfill \square

**Lemma 3.3.** Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$. Let $(X, \Delta)$ be a log pair and $D$ a $\mathbb{R}$-Cartier divisor such that $\text{vol}_X(D) > n^n$. Then for every smooth closed point $x \in X$ contained in the support of $\Delta$, there exists an effective $\mathbb{R}$-Cartier divisor $\Pi \sim \mathbb{R} D$ passing through $x \in X$ such that $(X, \Delta + \Pi)$ is not klt at $x \in X$. 
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Proof. First assume that $D$ is a $\mathbb{Q}$-Cartier divisor. Set $\varepsilon = \text{mult}_x \Delta > 0$. Then by Lemma 3.2 there exists an effective $\mathbb{Q}$-Cartier divisor $0 \leq \Pi \sim_\mathbb{Q} D$ such that $\text{mult}_x \Pi > n - \varepsilon$.

Claim: $(X, \Delta + \Pi)$ is not klt at $x \in X$.

Proof of the claim. Let $f : Y \to X$ be the blow-up of $X$ at $x$ and $E$ the unique exceptional divisor. Since $x \in X$ is a smooth closed point, locally near $x$ we have $K_Y = f^*K_X + (n - 1)E$, and $a(E; X, \Delta + \Pi) = (n - 1 - \text{mult}_x \Delta - \text{mult}_x \Pi) = (n - 1 - \varepsilon - \text{mult}_x \Pi) < -1$; hence $(X, \Delta + \Pi)$ is not klt.

Now consider the case when $D$ is an $\mathbb{R}$-Cartier divisor. Since volume is a continuous function (see [Laz04, Theorem 2.2.44]), there exists an effective $\mathbb{Q}$-Cartier divisor $D' \geq 0$ sufficiently close to $D$ such that $D \geq D'$ and $\text{vol}_X(D') > n^a$. Then by what we have just proved there exists an effective $\mathbb{Q}$-Cartier divisor $\Pi' \sim_\mathbb{Q} D'$ such that $(X, \Delta + \Pi')$ is not klt. Let $D = D' + E'$ and $\Pi = \Pi' + E'$. Then $\Pi \sim_\mathbb{R} D$ and $(X, \Delta + \Pi)$ is not klt.

The following two lemmas (3.4 and 3.5) shows what kind of properties of the total space of a fibration $f : X \to Y$ transfer to its generic fiber $X_\eta$.

Lemma 3.4. [BCZ18, Lemma 2.20] Let $f : X \to Y$ be a dominant morphism of finite type between two integral schemes of finite type over a field $k$ of arbitrary characteristic. Let $\eta$ be the generic point of $Y$, and $X_\eta$ the generic fiber. Then the following statements hold:

1. $X_\eta$ is an integral scheme.
2. $K(X_\eta) \cong K(X)$.
3. If $x'$ is a point in $X_\eta$ and $x$ its image in $X$ through the set theoretic inclusion $X_\eta \subseteq X$, then $\mathcal{O}_{X_\eta,x'} \cong \mathcal{O}_{X,x}$.

In particular, if $X$ is normal (resp. regular, resp. $\mathbb{Q}$-factorial), then $X_\eta$ normal (resp. regular, resp. $\mathbb{Q}$-factorial).

Lemma 3.5. [DW, Corollary 2.2] Let $f : X \to Y$ be a dominant morphism between two varieties with $X$ normal. Let $\eta$ be the generic point of $Y$ and $X_\eta$ the generic fiber. Further assume that $(X, \Delta)$ is a pair such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $(X, \Delta)$ has terminal, canonical, klt, plt, dlt or lc singularities, then the pair $(X_\eta, \Delta|_{X_\eta})$ has terminal, canonical, klt, plt, dlt or lc singularities, respectively.
Lemma 3.6. Let \((X, \Delta)\) be a klt pair of dimension 3 and \(\Phi\) an \(\mathbb{R}\)-divisor such that the pair \((X, \Phi)\) is not-log canonical. Then there exists a \(\lambda \in (0, 1)\) such that \((X, (1 - \lambda)\Delta + \lambda\Phi)\) is log canonical but not klt.

Proof. Let \(f : Y \to X\) be a log resolution of \((X, \Delta + \Phi)\). Consider the pair \((X, (1 - t)\Delta + t\Phi)\). Let \(\{E_i\}\) be the set of all exceptional divisors of \(f\) and also the strict transform of the components of \(\Delta\) and \(\Phi\). Let \(a_i(t), a_i(E_i; X, \Delta)\) and \(a_i(E_i; X, \Phi)\) be the discrepancy of \(E_i\) with respect to the pairs \((X, (1 - t)\Delta + t\Phi)\), \((X, \Delta)\) and \((X, \Phi)\), respectively. Then it is clear that \(a_i(t) = (1 - t)a_i(E_1; X, \Delta) + ta_i(E_i; X, \Delta)\). Therefore \(a_i(t)\) is either a constant polynomial or linear polynomial in \(t\). Note that \(a_i(t)\) will be a constant polynomial if and only if \(a_i(E; X, \Delta) = a_i(E; X, \Phi)\). Since \((X, \Delta)\) is klt and \((X, \Phi)\) is not log canonical, if follows that there exists at least one \(i\) such that \(a_i(t)\) is a polynomial of degree 1. Moreover, from the same hypothesis it also follows that \(a_i(0) > -1\) for all \(i\) and \(a_j(1) < -1\) for some \(j\). Without loss of generality assume that \(a_i(t)\) is a polynomial of degree 1 for all \(i\). Then from the graphs of \(a_i(t)\)'s (which are straight lines) it is clear that there exists a \(t = \lambda \in (0, 1)\) such that \(a_i(\lambda) \geq -1\) for all \(i\) and \(a_j(\lambda) = -1\) for at least one \(j\). In particular, \((X, (1 - \lambda)\Delta + \lambda\Phi)\) is log canonical but not klt for some \(\lambda \in (0, 1)\).

\[\square\]

Lemma 3.7. Let \((X, \Delta)\) be a log canonical pair of dimension 2 defined over an algebraically closed field \(k\). Then \(X\) is \(\mathbb{Q}\)-Gorenstein, i.e., \(K_X\) is \(\mathbb{Q}\)-Cartier.

Proof. Since \((X, \Delta)\) is log canonical, \((X, 0)\) is numerically log canonical (see [KM98, 4.1] for the definition). Then by [FT12, Proposition 6.3(b)], \((X, 0)\) is log canonical, i.e., \(K_X\) is \(\mathbb{Q}\)-Cartier.

\[\square\]

4. Log canonical thresholds

Definition 4.1. Let \(I\) and \(J\) be two sets such that \(I \subseteq [0, 1]\) and \(J \subseteq \mathbb{R}_{\geq 0}\). Let \(\mathcal{T}_n(I)\) be the set of all log pairs \((X, \Delta)\) of dimension \(n\) over some field \(k\) such that \((X, \Delta)\) is log canonical and the coefficients of \(\Delta\) belong to \(I\). Let \(M\) be an effective \(\mathbb{R}\)-Cartier divisor on \(X\). Then we define

\[
\text{lct}(X, \Delta; M) = \sup\{t \in \mathbb{R} \mid (X, \Delta + tM) \text{ is log canonical}\},
\]

and

\[
\text{LCT}_n(I, J) = \{\text{lct}(X, \Delta; M) \mid (X, \Delta) \in \mathcal{T}_n(I) \text{ and the coefficients of } M \text{ belong to } J\}.
\]

Conjecture 4.2 (\(\text{LCT}_n(I, J)\)). If \(I\) and \(J\) satisfy DCC, then \(\text{LCT}_n(I, J)\) satisfy ACC.
This conjecture is known in every dimension $n$ over algebraically closed field of characteristic 0 due to [HMX14]. Over algebraically closed field of characteristic $p > 5$, it is known in dimension at most 3 due to [Bir16c].

**Theorem 4.3** (dlt-Model). Let $(X, \Delta)$ be a log canonical pair of dimension at most 3 defined over a field $k$. If $\text{dim} \, X = 3$, then we further assume that $k$ is an algebraically closed field of char $p > 5$; otherwise we assume that $k$ is an arbitrary field.

Then there exits a birational morphism $f : (Y, \Delta_Y) \to (X, \Delta)$ extracting only exceptional divisors of discrepancy $a(E; X, \Delta) = -1$ such that $\Delta_Y$ is an effective $\mathbb{R}$-divisor, $Y$ is $\mathbb{Q}$-factorial, $(Y, \Delta_Y)$ has dlt singularities, and

$$K_Y + \Delta_Y = f^*(K_X + \Delta).$$

**Proof.** When $\text{dim} \, X = 3$, it is Theorem 1.6 in [Bir16c]. When $\text{dim} \, X = 2$, this follows from a standard application of MMP (see [Kol13, Corollary 1.36]) by noticing that MMP for excellent surfaces is known, thanks to Tanaka (see [Tan18, Theorem 1.1]).

5. LOG CANONICAL_THRESHOLDS IN DIMENSION ONE

In this section we establish various ACC-type results for curves over arbitrary fields. These results are an important part of our argument in the main technical result, Theorem 7.1 in Section 7.

First we need the following very useful lemma.

**Lemma 5.1.** [BCZ18, Lemma 3.2] Let $C$ be a regular curve over an arbitrary field $k$. If $\deg_k K_C < 0$ and $\ell = H^0(C, \mathcal{O}_C)$, then $C$ is a conic over $\ell$ and $\deg_\ell K_C = -2$. Furthermore, if $\text{char}(\ell) > 2$, then $C_\ell = C \times_\ell \ell \cong \mathbb{P}^1_\ell$.

**Lemma 5.2** (ACC for log canonical thresholds for curves). With the notations as in Definition 4.1, the ACC for log canonical thresholds holds in dimension 1 over arbitrary fields, i.e., if $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}_{\geq 0}$ are two DCC sets then, LCT$_1(I, J)$ satisfies the ACC.

**Remark 5.3.** Here we do not fix the base field $k$, i.e., the base field $k$ may vary as $C$ varies.

**Proof.** On the contrary assume that it is not true, then there exist a sequence of pairs $(C_i, \Delta_i) \in \Sigma_1(I)$ and effective $\mathbb{R}$-divisors $M_i$ with coefficients in $J$ such that $t_i = \text{lct}(C_i, \Delta_i; M_i)$ is a strictly increasing sequence. Since $C_i$ is a regular curve and $t_i$ is a log canonical threshold, it follows that one of the points
in the support of $M_i$ has coefficient 1 in $\Delta_i + t_i M_i$. Let $\Delta_i = \sum a_{ij} x_{ij}$ and $M_i = \sum b_{ij} y_{ij}$, where $x_{ij}, y_{ij}$ are closed points of $C_i$. Without loss of generality we may assume that $a_{i1} + t_i b_{i1} = 1$ for all $i \geq 1$. Since $\{a_{i1}\}$ and $\{b_{i1}\}$ are both contained in DCC sets, replacing them by a subsequence with common indices we may assume that they are both monotonically increasing sequence. Then $t_i b_{i1} = 1 - a_{i1}$ is monotonically decreasing. Since $\{t_i\}$ is strictly increasing it follows that $\{t_i b_{i1}\}$ is strictly increasing, hence a contradiction. 

\[ \square \]

### Proposition 5.4

Fix a DCC set $I \subseteq [0, 1]$. Then there is a finite subset $I_0 \subseteq I$ with the following properties:

If $(C, \Delta)$ is a log pair such that

1. $C$ is a regular curve over some arbitrary field $k$,
2. the coefficients of $\Delta$ belong to $I$, and
3. $K_C + \Delta \equiv 0$,

then the coefficients of $\Delta$ belong to $I_0$.

### Remark 5.5

Here we do not fix the base-field $k$, i.e., the base field $k$ may vary as $C$ varies.

**Proof.** First we note that $(C, \Delta)$ is log canonical, since $C$ is a curve and $\Delta$ is a boundary divisor. Now it is enough to show that the coefficients of $\Delta$ belong to an ACC set. If not then assume that there is a strictly increasing sequence of coefficients, say

\[ a_{i1} < a_{21} < \cdots < a_{i1} < \cdots \]

where $\Delta_i = \sum_j a_{ij} x_{ij}$ and $(C_i, \Delta_i)$ is a pair as in the hypothesis.

Suppose that $C_i$ is defined over the field $k_i$, and let $H^0(C_i, \mathcal{O}_{C_i}) = \ell_i$. Then $\ell_i$ is a finite extension of $k_i$. From Lemma 5.1 it follows that $C_i$ is a conic over $\ell_i$ and $\deg_{\ell_i} K_{C_i} = -2$. Therefore $K_{C_i} + \Delta_i \equiv 0$ implies that $\deg_{\ell_i} (K_{C_i} + \Delta_i) = [l_i : k_i] \cdot \deg_{k_i} (K_{C_i} + \Delta_i) = 0$; in particular we have

\[ -2 + \sum_j n_{ij} a_{ij} = 0, \]

where $n_{ij} > 0$ are positive integers (see Definition 2.5).

Then $n_{i1} a_{i1} + \sum_{j \geq 2} n_{ij} a_{ij} = 2$. We claim that $\{n_{i1} : i \geq 1\}$ is a bounded set. Indeed, if not then there is an unbounded subsequence $\{n_{ik1}\}_{k \geq 1}$. Since $\{a_{i1}\}_{i \geq 1}$ is contained in a DCC set, it has a non zero minimum, say $\min \{a_{i1} : i \geq 1\} = a > 0$. Then for $k \gg 0$ we have $n_{ik1} a_{ik1} \geq n_{ik1} a > 2$, which contradicts equation (5.2).
Claim 5.6. \( \{ \sum_{j \geq 2} n_{ij}a_{ij} \}_{i \geq 1} \) is a DCC set.

Proof of the Claim. Since \( \{ n_{i1} \} \) is a bounded sequence, from (5.2) it follows that \( \sum_{j \geq 2} n_{ij}a_{ij} \) is also a bounded sequence. Note that if we can show that the number of prime components of the divisors \( \Delta_i \) is bounded, then the claim will follow from Lemma 2.7 part (5). To that end let \( N_i \) be the number of prime components of \( \Delta_i \) for all \( i \). Let \( a > 0 \) be the minimum of the set \( \{ a_{ij} : i \geq 1, j \geq 2 \} \). Then \( \sum_{j \geq 2} n_{ij}a_{ij} \geq N_i a > 0 \), since \( n_{ij} \geq 1 \) for all \( i, j \).

Now if \( \{ N_i \} \) is unbounded, then \( \{ N_i a \} \) is unbounded, which contradicts the boundedness of \( \sum_{j \geq 2} n_{ij}a_{ij} \). In particular, the claim follows from Lemma 2.7 part (5) and (1).

\[ \square \]

Theorem 5.7. Fix a DCC set \( I \subseteq \mathbb{R}_{\geq 0} \). Then there exists \( 0 < \varepsilon < 1 \) with the following properties: if \( (C, \Theta) \) and \( (C, \Theta') \) are two log pairs of dimension 1 defined over some arbitrary field \( k \) such that the coefficients of \( \Theta \) belong to \( I \), and

\( (1 - \varepsilon)\Theta \leq \Theta' \leq \Theta \),

then \( (C, \Theta) \) is log canonical if and only if \( (C, \Theta') \) is log canonical. Moreover, if \( (C, \Theta') \) is log canonical and \( K_C + \Theta' \equiv 0 \), then \( \Theta' = \Theta \).

Remark 5.8. Here we do not fix the base field \( k \), i.e., the base field \( k \) may vary as \( C \) varies.

Proof. Let’s consider the log canonical thresholds of the pairs \( (C,0) \) with respect to \( \Delta \) where the coefficients of \( \Delta \) are contained in \( I \). By Lemma 5.2 these log canonical thresholds satisfy the ACC. Then there exist an \( \varepsilon' > 0 \) such that the open interval \( (1 - \varepsilon', 1) \) does not contain any log canonical threshold. Fix an \( \varepsilon > 0 \) such that \( 0 < \varepsilon < \varepsilon' \). Let \( (C, \Theta) \) and \( (C, \Theta') \) are two log pairs with coefficients of \( \Theta \) in \( I \) and \( (1 - \varepsilon)\Theta \leq \Theta' \leq \Theta \). Clearly if \( (C, \Theta) \) is log canonical, the so is \( (C, \Theta') \). So assume that \( (C, \Theta') \) is log canonical. Then \( (C, (1 - \varepsilon)\Theta) \) is log canonical. Therefore \( \text{lct}((C,0); \Theta) \geq 1 - \varepsilon > 1 - \varepsilon' \). Since the interval \( (1 - \varepsilon', 1) \) does not contain any log canonical thresholds, \( \text{lct}((C,0); \Theta) \geq 1 \); in particular \( (C, \Theta) \) is log canonical.

For the second part by contradiction assume that the conclusion is false. Then there is a strictly decreasing sequence \( \{ \varepsilon_i > 0 \} \) with \( \lim \varepsilon_i = 0 \) which satisfies the following properties: For each \( i \geq 1 \), there are log pairs \( (C_i, \Theta_i) \) and \( (C_i, \Theta'_i) \) such that the coefficients of \( \Theta_i \) belong to \( I \) and \( K_{C_i} + \Theta'_i \equiv 0 \), but

\[ (1 - \varepsilon_i)\Theta_i \leq \Theta'_i < \Theta_i \].

\[ (5.3) \]
Write $\Theta_i = \sum d_{ij} D_{ij}$ and $\Theta'_i = \sum d'_{ij} D_{ij}$.

There are two cases depending on whether the coefficients of $\Theta'_i$ belongs to a fixed DCC set or not.

**Case I:** Assume that the coefficients of $\Theta'_i$ are in a fixed DCC set $J \subseteq \mathbb{R}_{\geq 0}$. Then by Proposition 5.4 the coefficients of $\Theta'_i$ are contained in a finite subset $J_0 \subseteq J$. Relabeling the indices of $D_{ij}$ if necessary we may assume from (5.3) that

\[(5.4) \quad (1 - \varepsilon_i)d_{i1} \leq d'_{i1} < d_{i1} \quad \text{for all } i \geq 1.\]

Now by going to subsequences with common indices we may assume that $\{d_{i1}\}$ is monotonically increasing and $\{d'_{i1}\}$ is a constant sequence. Let $\lim d_{i1} = d$. Then from (5.4) we get

$$\lim(1 - \varepsilon_i)d_{i1} \leq \lim d'_{i1} \leq \lim d_{i1} \Rightarrow d \leq d'_{i1} \leq d, \quad \text{i.e., } d = d'_{i1}. $$

Then we have $d = \lim d_{i1} \geq d_{i1} > d'_{i1} = d$, a contradiction.

**Case II:** The coefficients of $\Theta'_i$ are not in a fixed DCC set. Thus by relabeling indices if necessary we may assume that $\{d_{i1}\}$ is not contained in a DCC set. Then going to subsequences with common indices we may assume that $\{d'_{i1}\}$ is strictly decreasing and $\{d_{i1}\}$ is monotonically increasing. Note that in this case we can only say that $(1 - \varepsilon_i)d_{i1} \leq d'_{i1} \leq d_{i1}$; strict inequality may not hold here. Let $\lim d_{i1} = d$ and $\lim d'_{i1} = d'$. Then $d_{i1} \leq d$ and $d'_{i1} > d'$ for all $i \geq 1$. Thus $d \geq d_{i1} \geq d'_{i1} > d' = \lim d'_{i1} \geq \lim(1 - \varepsilon_i)d_{i1} = d$, a contradiction.

\[\square\]

6. Log canonical thresholds in dimension two and three

In this section we prove some results on the log canonical thresholds in dimension two and three and some other ACC-type results in positive characteristic over arbitrary fields.

**Theorem 6.1.** With notations as in Definition 4.1, ACC for log canonical thresholds hold in dimension 2 over arbitrary fields, i.e., if $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}_{\geq 0}$ satisfies DCC, then $\text{LCT}_2(I, J)$ satisfies ACC.

**Remark 6.2.** Here we do not fix the base field $k$, i.e., the base field $k$ may vary as the surfaces vary.

**Proof.** This theorem is proved for surfaces defined over algebraically closed field in [Bir16c, Proposition 11.2]. In what follows we show that a similar
By contradiction assume that there is a sequence of log canonical pairs \((X_i, \Delta_i)\) of dimension 2 and effective \(\mathbb{R}\)-Cartier divisors \(M_i \geq 0\) with coefficients of \(\Delta_i\) in \(I\) and the coefficients of \(M_i \in J\) such that \(t_i := \lct(M_i; (X_i, \Delta_i))\) forms a strictly increasing sequence of real numbers. Now there are two cases based on the dimension of the log canonical centers.

**Case I:** For infinitely many \(i\), \((X_i, \Delta_i + t_i M_i)\) has a log canonical center of dimension 1 contained in the \(\text{Supp} M_i\). In this case by going to an infinite subsequence we may assume that for all \(i \geq 1\), \((X_i, \Delta_i + t_i M_i)\) has a log canonical center of dimension 1 contained in the \(\text{Supp} M_i\). In particular, one of the coefficients of \(\Delta_i + t_i M_i\) is 1 for every \(i\). Let \(\Delta = \sum a_{ij} D_{ij}\) and \(M_i = \sum b_{ij} E_{ij}\). Without loss of generality assume that \(a_{i1} + t_i b_{i1} = 1\). Since \(\{a_{11}\}\) and \(\{b_{11}\}\) are contained in DCC sets, by going to subsequences with common indices we may assume that \(\{a_{11}\}\) and \(\{b_{11}\}\) are both monotonically increasing. Then from \(t_i b_{i1} = 1 - a_{11}\) we see that the left hand side is strictly increasing sequence, while the right hand side is a monotonically decreasing sequence, hence a contradiction.

**Case II:** For infinitely many \(i\), \((X_i, \Delta_i + t_i M_i)\) has a log canonical center \(P_i\) of dimension 0 contained in the support of \(M_i\). By going to an infinite subsequence we may assume that for all \(i \geq 1\), \((X_i, \Delta_i + t_i M_i)\) has a log canonical center \(P_i\) of dimension 0 contained in the support of \(M_i\). Let \(f_i : Y_i \rightarrow X_i\) be a dlt-model of \((X_i, \Delta_i + t_i M_i)\) (see Theorem 4.3). Then there is an exceptional divisor \(E_i\) of discrepancy \(-1\) which intersects the strict transform of \(M_i\) and \(f_i(E_i) = P_i\). Write

\[ K_{Y_i} + E_i + \Gamma_i = f_i^*(K_{X_i} + \Delta_i + t_i M_i). \]

Since the coefficients of \(M_i\) are in a DCC set and \(t_i\) is a strictly increasing sequence, it is easy to see that the coefficients of \(t_i M_i\) are contained in a DCC set. Thus the coefficients of \(\Delta_i + t_i M_i\) are in a DCC set by Lemma 2.7, part (4); in particular the coefficients of \(\Gamma_i + E_i\) are in a DCC set. Since \((Y_i, \Gamma_i + E_i)\) is dlt, \(E_i\) is a regular curve by [BCZ18, Lemma 3.4]. Then by adjunction we have \(K_{E_i} + \Gamma_{E_i} = (K_{Y_i} + E_i + \Gamma_i)|_{E_i}\), and the coefficients of \(\Gamma_{E_i}\) are in a DCC set by Proposition 2.8. Let \(\ell_i = H^0(E_i, \mathcal{O}_{E_i})\). Then \(\deg_{\ell_i}(K_{E_i} + \Gamma_{E_i}) = |\ell_i : k_i| (K_{Y_i} + E_i + \Gamma_i) \cdot k_i E_i = 0\), where \(X_i\) is defined over \(k_i\). This implies that \(\deg_{\ell_i} K_{E_i} < 0\). Thus by Lemma 5.1 \(E_i\) is a conic, and \(\deg_{\ell_i} K_{E_i} = -2\). Then by Proposition 5.4 the coefficients of \(\Gamma_{E_i}\) are contained in a finite set, say \(I_0 \subseteq [0, 1]\), since \(K_{E_i} + \Gamma_{E_i} \equiv 0\) as \(\deg_{\ell_i}(K_{E_i} + \Gamma_{E_i}) = 0\). By our construction and Proposition 2.8, for each \(i \geq 1\) there is a number \(1 - \frac{1}{m_i} + t_i \frac{\sum_j n_{ij} b_{ij}}{m_i}\)
contained in \( I_0 \) which is a coefficient of \( \Gamma_{E_i} \), where \( b_{ij} \)'s are the coefficients of \( \Gamma_i \) and thus contained in a DCC set. Since \( I_0 \) is a finite set, we may assume that 
\[
1 - \frac{1}{m_i} + t_i \sum_{j} n_{ij} b_{ij} = b \in I_0 \text{ for all } i.
\]
Then \( t_i \sum_j n_{ij} b_{ij} = m_i (b - 1) + 1 \). If \( b < 1 \), then \( m_i = m_{i+1} \) for all \( i \gg 0 \), otherwise the right hand side is negative. If \( b = 1 \), then \( t_i \sum_j n_{ij} b_{ij} = 1 \) for all \( i \). In either case we have \( t_i \sum_j n_{ij} b_{ij} = b' \) for some constant \( b' > 0 \) and for all \( i \geq 1 \). Then \( t_i \sum_j n_{ij} b_{ij} = m_i (b' - 1) + 1 \). If \( b' < 1 \), then \( m_i = m_{i+1} + 1 \) for all \( i \gg 0 \), otherwise the right hand side is negative. If \( b' = 1 \), then \( t_i \sum_j n_{ij} b_{ij} = 1 \) for all \( i \). In either case we have \( t_i \sum_j n_{ij} b_{ij} = b' \) for some constant \( b' > 0 \) and for all \( i \geq 1 \). Now as in the proof of the Claim 5.6 we can prove that the number of components of \( \Gamma_{E_i} \) is bounded. Then \( t_i \sum_j n_{ij} b_{ij} \) satisfies DCC by Lemma 2.7, part(5). In particular, \( \frac{1}{t_i} \) satisfies DCC, hence \( t_i \) satisfies ACC, which is a contradiction.

\[\square\]

**Theorem 6.3.** [Bir16c, Proposition 11.7] Let \( k \) be a fixed algebraically closed field of characteristic \( p > 0 \), and \( I \subseteq [0,1] \) a DCC set. Then there exists a finite subset \( I_0 \subseteq I \) with the following properties:

If \( (X,\Delta) \) is a log pair such that

1. \( X \) is a projective variety of dimension 2 over \( k \),
2. \( (X,\Delta) \) is log canonical,
3. the coefficients of \( \Delta \) belong to \( I \), and
4. \( K_X + \Delta \equiv 0 \),

then the coefficients of \( \Delta \) belong to \( I_0 \).

**Theorem 6.4.** Fix a DCC set \( I \subseteq \mathbb{R}_{\geq 0} \). Then there exists an \( 0 < \varepsilon < 1 \) with the following properties: If \( (S,\Theta) \) and \( (S,\Theta') \) are two \( \mathbb{Q} \)-Gorenstein log pairs of dimension 2 over some arbitrary fields such that the coefficients of \( \Theta \) belong to \( I \), and

\[
(1 - \varepsilon)\Theta \leq \Theta' \leq \Theta,
\]

then \( (S,\Theta) \) is log canonical if and only if \( (S,\Theta') \) is log canonical. Moreover, if all the surfaces are defined over some fixed algebraically closed field \( k = \overline{k} \) and \( (S,\Theta') \) is log canonical and \( K_S + \Theta' \equiv 0 \), then \( \Theta' = \Theta \).

**Remark 6.5.** In the first part we do not fix the base field, i.e., the base field may vary as the surfaces vary.

**Proof.** First note that if \( (S,0) \) is not log canonical, then the above statement is vacuously true. So assume that \( (S,0) \) is log canonical. Now consider the log canonical thresholds of the pairs \( (S,0) \) with respect to \( \Delta \) with coefficients of \( \Delta \) in \( I \). By Theorem 6.1 these log canonical thresholds satisfy the ACC. Thus there exist an \( \varepsilon' > 0 \) such that the open interval \( (1 - \varepsilon', 1) \) does not contain any log canonical threshold. Fix an \( \varepsilon > 0 \) such that \( 0 < \varepsilon < \varepsilon' \). Let \( (S,\Theta) \) and \( (S,\Theta') \) be two log pairs with coefficients of \( \Theta \) in \( I \) and \( (1 - \varepsilon)\Theta \leq \Theta' \leq \Theta \). Clealy if \( (S,\Theta) \) is log canonical, the so is \( (S,\Theta') \). So assume that \( (S,\Theta') \) is
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log canonical. Then $(S, (1 - \varepsilon)\Theta)$ is log canonical. Therefore $\text{lct}((S, 0); \Theta) \geq 1 - \varepsilon > 1 - \varepsilon'$. Since the interval $(1 - \varepsilon', 1)$ does not contain any log canonical thresholds, $\text{lct}((S, 0); \Theta) \geq 1$; in particular $(S, \Theta)$ is log canonical.

For the second part by contradiction assume that the conclusion is false. Then there is a strictly decreasing sequence $\{\varepsilon_i > 0\}$ with $\lim \varepsilon_i = 0$ which satisfies the following properties: For each $i \geq 1$, there are log canonical pairs $(S_i, \Theta_i)$ and $(S_i, \Theta'_i)$ such that the coefficients of $\Theta_i$ belong to $I$ and $K_S + \Theta'_i \equiv 0$, but

$$ (1 - \varepsilon_i)\Theta_i \leq \Theta'_i < \Theta_i. $$

We remark that the rest of the proof works exactly as in the proof of Theorem 5.7 by replacing the use of Proposition 5.4 by Theorem 6.3 and noticing the fact that the dimension of the ambient varieties were never used in the proof of Theorem 5.7.

\begin{proof}
A similar proof as in the proof of the first part of the Theorem 6.4 works here by noticing the fact that the ACC for log canonical thresholds is known for 3-folds defined over an algebraically closed field of char $p > 5$ due to Birkar, [Bir16c, Theorem 1.10].
\end{proof}

**Theorem 6.6.** Let $k$ be a fixed algebraically closed field of characteristic $p > 5$, and $I \subseteq \mathbb{R}_{\geq 0}$ a DCC set. Then there exists an $0 < \varepsilon < 1$ with the following properties: If $(X, \Theta)$ and $(X, \Theta')$ are two $\mathbb{Q}$-factorial log pairs of dimension 3 over $k$ such that the coefficients of $\Theta$ belong to $I$, and

$$ (1 - \varepsilon)\Theta \leq \Theta' \leq \Theta, $$

then $(X, \Theta)$ is log canonical if and only if $(X, \Theta')$ is log canonical.

**Proof.** A similar proof as in the proof of the first part of the Theorem 6.4 works here by noticing the fact that the ACC for log canonical thresholds is known for 3-folds defined over an algebraically closed field of char $p > 5$ due to Birkar, [Bir16c, Theorem 1.10].

\begin{proof}

\end{proof}

**Theorem 6.7.** Fix a DCC set $I \subseteq \mathbb{R}_{\geq 0}$. Let $\mathcal{S}$ be the collection of log pairs $(X, \Delta)$ of dimension at most 3 satisfying the following properties:

1. if the dimension of $X$ is at most 2, then they are all defined over arbitrary fields and $X$ is $\mathbb{Q}$-Gorenstein.
2. if dimension $X$ is 3, then $X$ is defined over some fixed algebraically closed field $k = \overline{k}$ or char $p > 5$, and $X$ is $\mathbb{Q}$-factorial,
3. $(X, \Delta)$ is log canonical, and
4. the coefficients of $\Delta$ belong to $I$.

then there exists an $0 < \varepsilon < 1$ satisfying the following properties: If $(X, \Theta)$ and $(X, \Theta')$ are two log pairs such that the coefficients of $\Theta$ belong to $I$ and

$$ (1 - \varepsilon)\Theta \leq \Theta' \leq \Theta, $$

...
then \((X, \Theta)\) is log canonical if and only if \((X, \Theta')\) is log canonical. Moreover, if \(\dim X \leq 2\), \((X, \Theta')\) is log canonical and \(K_X + \Theta' \equiv 0\), and \(X\) is defined over some fixed algebraically closed field \(k = \overline{k}\) whenever \(\dim X = 2\), then \(\Theta' = \Theta\).

**Proof.** It follows from Theorem 5.7, Theorem 6.4 and Theorem 6.6 by choosing the minimum of three values of \(\varepsilon > 0\). □

### 7. Main Theorem

First we prove the following main technical result.

**Theorem 7.1.** Fix an algebraically closed field \(k\) of characteristic \(p > 5\), and a DCC set \(I \subseteq [0, 1]\). Let \(\mathcal{D}\) be the set of all klt pairs \((X, \Delta)\), where \(X\) is a projective 3-fold, \(K_X + \Delta \equiv 0\), \(\Delta\) is big and its coefficients belong to \(I\).

Then there exists a constant \(0 < \varepsilon < 1\) satisfying the following properties:

If \(\Phi \geq 0\) is an effective \(\mathbb{R}\)-divisor such that \(K_X + \Phi \equiv 0\) and \(\Phi \geq (1 - \delta)\Delta\) for some \(0 < \delta < \varepsilon\) and \((X, \Delta) \in \mathcal{D}\), then \((X, \Phi)\) has klt singularities.

**Remark 7.2.** The statement above and its proof given below are both based on the proof of Lemma 6.1 in [HMX14]. However, we note that our proof is significantly more involved than that of [HMX14], having to do with the failure of Bertini’s theorem for base-point free linear systems and other related results in positive characteristic. In a key argument in the proof of [HMX14, Lemma 6.1], the authors reduce the problem to a Mori fiber space \(g : Y \to Z\) and then restrict everything to the general fibers of \(g\) which reduces the problem to a lower dimension. We face several challenges at this stage in positive characteristic. The first problem is that in positive characteristic the general fibers \(F\) of a given fibration may have very bad singularities, they could in fact be non-reduced schemes, even if the fibration is a Mori fiber space; this completely destroys the hope of using general fibers. On top of that, even if we know that the general fibers \(F\) of \(g : Y \to Z\) are reduced, irreducible and normal varieties, we still can not guarantee that \(F\) has good MMP singularities via adjunction from \(Y\); this has to do with the failure of generic smoothness for fibrations in positive characteristic. In order to circumvent these issues, we work with the generic fiber \(Y_\eta\) of \(g : Y \to Z\) instead of general fibers \(F\). The generic fiber \(Y_\eta\) is now a normal integral scheme over the function field \(K(Z)\) of \(Z\). However, this comes with a new set of challenges, since \(Y_\eta\) is defined over \(K(Z)\) which is an imperfect field, lots of standard results are either not known for varieties over imperfect field or they are known to fail. Fortunately, in the recent years lots of progress has been made towards understanding the birational geometry of surfaces over imperfect fields, especially the minimal model program, mostly due to Tanaka, see [Tan18]. We are able to use his
results to our advantage to prove some ACC-type results for surfaces over arbitrary fields (see Section 6). In our proof we also have to deal with curves over imperfect fields and we need various ACC-type results on them as well, which are developed in Section 5.

**Proof of Theorem 7.1.** We fix an $0 < \varepsilon < 1$ as obtained in the Theorem 6.7. Now by contradiction assume that there is a pair $(X, \Phi)$ such that $\Phi \geq (1-\delta)\Delta$ for some $0 < \delta < \varepsilon$, $K_X + \Phi \equiv 0$ and $(X, \Phi)$ is not klt. Note that $\Phi > (1-\delta)\Delta$, since $(X, \Delta)$ is klt, and $\Phi$ is also big, since $\Delta$ is. Now we want to modify $\Phi$ so that $(X, \Phi)$ becomes log canonical but the other properties of $\Phi$ are preserved. If $(X, \Phi)$ is already log canonical, then there is nothing to do. So assume that $(X, \Phi)$ is not log canonical. Then by Lemma 3.6 there exists a $\lambda \in (0, 1)$ such that $(X, (1-\lambda)\Delta + \lambda \Phi)$ is log canonical but not klt. Observe that $((1-\lambda)\Delta + \lambda \Phi) > (1-\delta)\Delta$ and $K_X + (1-\lambda)\Delta + \lambda \Phi = (1-\lambda)(K_X + \Delta)$. Thus by replacing $(1-\lambda)\Delta + \lambda \Phi$ by $\Phi$ we may assume that $(X, \Phi)$ is log canonical but not klt. Let $f : Y \to X$ be a dlt-model of $(X, \Phi)$, whose existence is guaranteed by [Bir16c, Theorem 1.6]. We write

(7.1) $K_Y + \Psi = f^*(K_X + \Phi)$,

and

(7.2) $K_Y + \Gamma + \sum a_i S_i = f^*(K_X + \Delta)$,

where $[\Psi] = \sum S_i$ and $a_i < 1$ for all $i$, and $\Gamma \geq 0$ is an effective $\mathbb{R}$-divisor such that $\text{Supp} \Gamma \subseteq \text{Supp}(f^{-1}_*\Delta)$ and $\Gamma$ and $[\Psi]$ do not share any common component.

Since $K_Y + \Phi \equiv 0$, we have $K_Y + \Psi \equiv 0$. Thus by [BW17, Theorem 1.7] running a $(K_Y + \Psi - S_1)$-MMP we end up with a Mori fiber space $g : W \to Z$. Let $\phi : Y \dashrightarrow W$ be induced birational map. Since $K_Y + \Psi \equiv 0$, every step of this MMP is $S_1$-positive, in particular, $\phi_*S_1$ is $g$-ample and hence $\phi_*S_1$ is not contracted by $g$. Observe that since $K_X + \Psi \equiv 0$ and $K_Y + \Gamma + \sum a_i S_i \equiv 0$ (as $K_X + \Delta \equiv 0$), these relations are preserved at every step of the $(K_Y + \Psi - S_1)$-MMP and eventually we have $K_W + \phi_*\Psi \equiv 0$ and $K_W + \phi_*\Gamma + \sum a_i \phi_*S_i \equiv 0$; this follows from [KM98, Theorem 3.7(4)] which in our case can be obtained from the cone theorem and base-point free theorem as in [BW17, Theorems 1.1 and 1.2]. It also follows from [KM98, Lemma 3.38] that $(W, \phi_*\Psi)$ is log canonical, however, note that it is not necessarily dlt.

Next we want to establish an inequality that $\Psi > (1-\varepsilon)\Gamma + \sum S_i$; this we will be used heavily in the rest of the proof. To that end we first show that $\Gamma \neq 0$. Indeed, if $\Gamma = 0$, then $\text{Supp}(f^{-1}_*\Delta) \subseteq \text{Supp}[\Psi]$. Since $\Phi > (1-\varepsilon)\Delta$,
this implies that every component of $\Delta$ appears in $\Phi$ with coefficient 1. In particular, $\Phi > \Delta$; but then $\Phi \equiv \Delta$ implies that $\Phi = \Delta$, which is a contradiction, since $(X, \Delta)$ is klt and $(X, \Phi)$ is not klt. Therefore $\Gamma > 0$, and then from the inequality $\Phi > (1-\varepsilon)\Delta$ and (7.1) and (7.2) it follows that $\Psi > (1-\varepsilon)\Gamma + \sum S_i$.

In the following discussion we separate three cases based on the relative dimension of $g : W \to Z$. However, first we claim that we may assume that $\phi_*\Gamma$ is not contracted by $g : W \to Z$. Indeed, if $\phi_*\Gamma$ is contracted by $g$, then $\phi_*\Gamma \equiv_g 0$, since $\rho(W/Z) = 1$. Again since $\rho(W/Z) = 1$, any non-zero effective divisor on $W$ which is not contracted by $g$ is $g$-ample. In particular, from the discussion above it follows that $\sum \phi_* S_i$ is $g$-ample. Then $K_W + \sum \phi_* S_i \equiv_g K_W + \phi_* \Gamma + \sum \phi_* S_i \equiv \sum (1-a_i) \phi_* S_i$ is $g$-ample, since $a_i < 1$ for all $i$. But then we have $K_W + \phi_* \Psi \equiv 0$ and $[\Psi] = \sum S_i$, which is a contradiction. Therefore we may assume that $\phi_* \Gamma$ is not contracted by $g : W \to Z$ in the following discussion.

**Case I:** Relative dimension of $g : W \to Z$ is 1. Let $F$ be the generic fiber of $g$. Then Replacing $Y, \Gamma$ and $\Psi$ by $F$ and the restriction of $\phi_* \Gamma$ and $\phi_* \Psi$ to $F$, may assume that $Y$ is a regular curve over an imperfect field $\ell = K(Z)$, where $K(Z)$ is the function field of $Z$ (see Lemma 3.4 and 3.5). Further notice that $S_1$ and $\Gamma$ are non-zero effective divisors on $Y$, and $H^0(Y, \mathcal{O}_Y) = \ell$, since $g_* \mathcal{O}_W = \mathcal{O}_Z$, and $\deg_{\ell} K_Y < 0$, since $K_Y + \Psi \equiv 0$ and $\Psi > 0$. Thus by Lemma 5.1 $Y$ is a conic over $\ell$ and $\deg_{\ell} K_Y = -2$. Now $K_Y + \Gamma + \sum a_i S_i \equiv 0$ and $a_i < 1$ for all $i$, so $\deg_{\ell}(K_Y + \Gamma + \sum S_i) > 0$. By construction we also have $\Psi > (1-\varepsilon)\Gamma + \sum S_i$. Thus for some $0 < \eta < \varepsilon$ we get that $\deg_{\ell}(K_Y + (1-\eta)\Gamma + \sum S_i) = \deg_{\ell}(K_Y + \Psi) = 0$. But then we have (7.3)

$$(1-\varepsilon) \left( \Gamma + \sum S_i \right) \leq (1-\varepsilon) \Gamma + \sum S_i \leq (1-\eta) \Gamma + \sum S_i \leq \left( \Gamma + \sum S_i \right).$$

Since $(Y, \Psi)$ is log canonical by Lemma 3.5 and $\Psi > (1-\varepsilon)\Gamma + \sum S_i$, $(Y, (1-\varepsilon)\Gamma + \sum S_i)$ is also log canonical. Then from (7.3) and Theorem 6.7 we get that $(Y, \Gamma + \sum S_i)$ is log canonical. In particular, then $(Y, (1-\eta)\Gamma + \sum S_i)$ is log canonical. Now observe that the coefficients of $\Gamma + \sum S_i$ are contained in $I \cup \{1\}$, which is a DCC set, hence by Theorem 6.7 $(1-\eta)\Gamma + \sum S_i = \Gamma + \sum S_i$, which is a contradiction, since $\deg_{\ell}((1-\eta)\Gamma + \sum S_i) = 0$ and $\deg_{\ell}(K_Y + \Gamma + \sum S_i) > 0$.

**Case II:** Relative dimension of $g : W \to Z$ is 2. Let $F$ be the generic fiber of $g$. Then replacing $Y, \Gamma$ and $\Psi$ by $F$ and the restriction of $\phi_* \Gamma$ and $\phi_* \Psi$ to $F$, may assume that $Y$ is a normal $\mathbb{Q}$-factorial surface over an imperfect field $\ell = K(Z)$, where $K(Z)$ is the function field of $Z$ (see Lemma 3.4 and 3.5). We note that $S_1$ is an ample divisor, $\Gamma$ is a non-zero effective divisor (hence
ample), and \( H^0(Y, \mathcal{O}_Y) = \ell \) and the Picard number \( \rho(Y) = 1 \) (see \cite[Lemma 6.6]{Tan15}). We also note that \((Y, \Psi)\) is a log canonical pair by Lemma 3.5.

Since \( Y \) is a surface let’s rename \( S_i \)’s by \( C_i \). Then \( C_1 \) is ample on \( Y \). Furthermore, since \( \rho(Y) = 1 \), any non-zero effective divisor is ample; in particular \( \sum C_i \) is ample. Thus \( K_Y + \Gamma + \sum C_i \equiv \sum (1 - a_i)C_i \) is ample, since \( a_i < 1 \) for all \( i \). Since \( \Psi > (1 - \varepsilon)\Gamma + \sum C_i \), \( K_Y + \Psi \equiv 0 \) and \( \rho(Y) = 1 \), there exists an \( 0 < \eta < \varepsilon \) such that \( K_Y + (1 - \eta)\Gamma + \sum C_i \equiv 0 \). We also note that \((Y, (1 - \varepsilon)\Gamma + \sum C_i)\) is log canonical, since \((1 - \varepsilon)\Gamma + \sum C_i < \Psi . \) Let \( C_n^1 \to C_1 \) be the normalization morphism. We have the following adjunction equations

\[
(7.4a) \quad (K_Y + (1 - \varepsilon)\Gamma + \sum C_i)|_{C_n^1} = K_{C_n^1} + \Theta_1,
\]

\[
(7.4b) \quad (K_Y + (1 - \eta)\Gamma + \sum C_i)|_{C_n^1} = K_{C_n^1} + \Theta,
\]

\[
(7.4c) \quad (K_Y + \Gamma + \sum C_i)|_{C_n^1} = K_{C_n^1} + \Theta.
\]

Note that a priori it is not clear whether the coefficients of \( \Theta \) are in the DCC set \( D(I \cup \{1\}) \), since we do not know whether \((Y, \Gamma + \sum C_i)\) is log canonical or not. In particular, some of the coefficients of \( \Theta \) could potentially be larger than 1. However, we claim that \((Y, \Gamma + \sum C_i)\) is indeed log canonical, and thus by Proposition 2.8 the coefficients of \( \Theta \) are in the DCC set \( D(I \cup \{1\}) \).

The proof goes as follows: we have the following inequalities

\[
(1 - \varepsilon) (\Gamma + \sum C_i) \leq (1 - \varepsilon)\Gamma + \sum C_i \leq \Gamma + \sum C_i.
\]

Since \((Y, (1 - \varepsilon)\Gamma + \sum C_i)\) is log canonical, by Theorem 6.7, \((Y, \Gamma + \sum C_i)\) is log canonical.

Now from \((7.4)\) we get that

\[
(1 - \varepsilon)\Theta \leq \Theta_1 \leq \Theta_2 \leq \Theta,
\]

where the first inequality follows from the fact that the coefficients of \( \Theta \) belong to \( D(I \cup \{1\}) \) and the following inequality

\[
(7.5) \quad t \left( \frac{m - 1 + f}{m} \right) \leq \left( \frac{m - 1 + tf}{m} \right) \quad \text{for any } t \leq 1.
\]

Now by adjunction (see Lemma 2.9) \((C_n^1, \Theta_1)\) is log canonical. Since the coefficients of \( \Theta \) are in a DCC set, by Theorem 6.7 \((C_n^1, \Theta)\) is log canonical; in particular \((C_n^1, \Theta_2)\) is log canonical; hence again by Theorem 6.7, \( \Theta_2 = \Theta \), since \( K_{C_n^1} + \Theta_2 \equiv 0 \). But this is a contradiction, since \( K_{C_n^1} + \Theta \) is ample, as it is the pullback of an ample divisor by finite morphism.
Case III: Relative dimension of $g : W \to Z$ is 3, i.e., $\dim Z = 0$. Then replacing $Y, \Gamma$ and $\Psi$ by $W, \phi \ast \Gamma$ and $\phi \ast \Psi$ we may assume that $Y$ is a projective 3-fold over the algebraically closed base field $k$, Picard number $\rho(X) = 1$ and $S_1$ is an ample divisor on $Y$ and $\Gamma$ is a non-zero effective divisor (hence ample).

Now since $\rho(Y) = 1$, every non-zero effective divisor is ample. In particular, $K_Y + \Gamma + \sum S_i \equiv \sum (1 - a_i)S_i$ is ample, since $a_i < 1$ for all $i$. Also, since $K_Y + \Psi \equiv 0, \Psi > (1 - \varepsilon)\Gamma + \sum S_i$ and $\rho(X) = 1$, it follows that there exists $0 < \eta < \varepsilon$ such that $K_Y + (1 - \eta)\Gamma + \sum S_i \equiv 0$. We note that $(Y, (1 - \varepsilon)\Gamma + S_i)$ is log canonical, since $(1 - \varepsilon)\Gamma + S_i \leq \Psi$. Let $S_1^n \to S_1$ be the normalization morphism. We have the following adjunction equations

$$(K_Y + (1 - \varepsilon)\Gamma + \sum S_i) |_{S_1^n} = K_{S_1^n} + \Theta_1,$$

$$(K_Y + (1 - \eta)\Gamma + \sum S_i) |_{S_1^n} = K_{S_1^n} + \Theta_2, \quad \text{and}$$

$$(K_Y + \Gamma + \sum S_i) |_{S_1^n} = K_{S_1^n} + \Theta.$$

As in the proof of Case II using Theorem 6.7 we see that $(Y, \Gamma + S_i)$ is log canonical; in particular the coefficients of $\Theta$ are in the DCC set $D(I \cup \{1\})$ by Proposition 2.8. The rest of the arguments are verbatim to the Case II via Lemma 3.7 and Theorem 6.7.

\[\Box\]

Proof of Theorem 1.3. Let $\varepsilon > 0$ be a constant given by Theorem 7.1. We claim that

$$\operatorname{vol}_X(-K_X) = \operatorname{vol}_X(\Delta) \leq \frac{27}{\varepsilon^3} \quad \text{for all } (X, \Delta) \in \mathcal{D}.$$ 

By contradiction assume that there is a $(X, \Delta) \in \mathcal{D}$ such that $\operatorname{vol}_X(X, \varepsilon \Delta) > 3^3$. Since the volume is a continuous function (see [Laz04, Theorem 2.2.44]), we have

$$\operatorname{vol}_X(\eta \Delta) > 3^3 \quad \text{for some } 0 < \eta < \varepsilon.$$ 

Note that $(X, (1 - \eta)\Delta)$ klt, since $(X, \Delta)$ is klt. Let $x \in X$ be a smooth closed point of $X$ contained in the support of $\Delta$. Then by Lemma 3.3 there exists an effective $\mathbb{R}$-divisor $\Pi$ passing through $x$ such that $\Pi \sim_{\mathbb{R}} \eta \Delta$ and $(X, (1 - \eta)\Delta + \Pi)$ is not klt. This is a contradiction to Theorem 7.1, since $0 < \eta < \varepsilon$.

\[\Box\]

Proof of Corollary 1.4. The proof follows immediately from Theorem 1.3. 

\[\Box\]
References

[Ale94] V. Alexeev, *Boundedness and $K^2$ for log surfaces*, Internat. J. Math. 5(6), 779–810 (1994).

[BB92] A. A. Borisov and L. A. Borisov, *Singular toric Fano three-folds*, Mat. Sb. 183(2), 134–141 (1992).

[BCZ18] C. Birkar, Y. Chen and L. Zhang, *Iitaka $C_{n,m}$ conjecture for 3-folds over finite fields*, Nagoya Math. J. 229, 21–51 (2018).

[Bir16a] C. Birkar, *Anti-pluricanonical systems on Fano varieties*, ArXiv e-prints (March 2016), 1603.05765.

[Bir16b] C. Birkar, *Singularities of linear systems and boundedness of Fano varieties*, ArXiv e-prints (September 2016), 1609.05543.

[Bir16c] C. Birkar, *Existence of flips and minimal models for 3-folds in char $p$*, Ann. Sci. Éc. Norm. Supér. (4) 49(1), 169–212 (2016).

[BW17] C. Birkar and J. Waldron, *Existence of Mori fibre spaces for 3-folds in char $p$*, Adv. Math. 313, 62–101 (2017).

[DW] O. Das and J. Waldron. *On the abundance formula for 3-folds in characteristic $p > 5$*, To appear in Math. Z.

[FT12] O. Fujino and H. Tanaka, *On log surfaces*, Proc. Japan Acad. Ser. A Math. Sci. 88(8), 109–114 (2012).

[HMX14] C. D. Hacon, J. McKernan and C. Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) 180(2), 523–571 (2014).

[Jia14] C. Jiang, *Boundedness of anti-canonical volumes of singular log Fano threefolds*, ArXiv e-prints (November 2014), 1411.6728.

[Kaw92] Y. Kawamata, *Boundedness of $\mathbf{Q}$-Fano threefolds*, in *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 439–445, Amer. Math. Soc., Providence, RI, 1992.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[KKM92] J. Kollár, Y. Miyaoka and S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. 36(3), 765–779 (1992).

[KKMT00] J. Kollár, Y. Miyaoka, S. Mori and H. Takagî, *Boundedness of canonical $\mathbf{Q}$-Fano 3-folds*, Proc. Japan Acad. Ser. A Math. Sci. 76(5), 73–77 (2000).

[Kol97] J. Kollár, *Singularities of pairs*, in *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287, Amer. Math. Soc., Providence, RI, 1997.

[Kol13] J. Kollár, *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 2013, With a collaboration of Sándor Kovács.

[Lai16] C.-J. Lai, *Bounding volumes of singular Fano threefolds*, Nagoya Math. J. 224(1), 37–73 (2016).

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.
[MP04] J. McKernan and Y. Prokhorov, *Threefold thresholds*, Manuscripta Math. **114**(3), 281–304 (2004).

[Tan15] H. Tanaka, *Behavior of canonical divisors under purely inseparable base changes*, ArXiv e-prints (February 2015), 1502.01381.

[Tan18] H. Tanaka, *Minimal model program for excellent surfaces*, Ann. Inst. Fourier (Grenoble) **68**(1), 345–376 (2018).

[Zhu17] Z. Zhuang, *Weak boundedness of Fano threefolds with large Seshadri constants in characteristic $p > 5$*, ArXiv e-prints (November 2017), 1711.02803.

**Department of Mathematics, University of California, Los Angeles, 520 Portola Plaza, Math Sciences Building 6363.**

*E-mail address:* omprokash@gmail.com, das@math.ucla.edu