Explicit presentation of relative Steinberg groups

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Abstract

We find an explicit presentation of relative linear Steinberg groups \( \text{St}(n, R, I) \) for any ring \( R \) and \( n \geq 4 \) by generators and relations as abstract groups. We also prove a similar result for relative simply laced Steinberg groups \( \text{St}(\Phi; R, I) \) for commutative \( R \) and \( \Phi \in \{ A_\ell, D_\ell, E_\ell | \ell \geq 3 \} \).

1 Introduction

Relative linear Steinberg groups \( \text{St}(n, R, I) \) are defined by F. Keune and J.-L. Loday in \([2,3]\) as certain crossed modules over the absolute Steinberg group \( \text{St}(n, R) \), where \( R \) is an associative ring with 1 and \( I \triangleleft R \) is an ideal. They actually considered only the stable case \( n = \infty \), but the definition easily generalizes to the unstable situation. The relative Steinberg groups have explicit presentation by generators and relations, but as groups with an action of \( \text{St}(n, R) \). There is another definition of \( \text{St}(n, R, I) \) by M. Tulenbaev \([6]\) if \( R \) is commutative, but in terms of generators \( X_{vw} \) parametrized by vectors \( v, w \) satisfying some conditions instead of the elementary generators \( z_{ij}(a, p) \). By \([5]\), both definitions give the same groups for commutative rings.

For relative simply laced Steinberg groups \( \text{St}(\Phi; R, I) \) we use a definition from \([5]\). It is well-known that relative Steinberg groups (both linear and simply laced) are generated by the elementary conjugates \( z_{ij}(a, p) = x_{ji}(p)x_{ij}(a) \) as abstract groups (see, for example, \([1]\) lemma 5 for the case of simply-laced groups). By \([5]\) theorem 9, all relations in \( \text{St}(\Phi; R, I) \) come from relative Steinberg groups associated with root subsystems of \( \Phi \) of type \( A_3 \), i.e. they involve a bounded number of roots. We give an explicit list of such relations (theorems \([1]\) and \([2]\) and show, in particular, that they involve only roots of subsystems

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of types $A_1, A_1 \times A_1$, and $A_2$. Our list consists of a finite number of identities parametrized by roots.

To prove our result in the linear case, it is useful to consider a slightly more general groups. It is straightforward to construct absolute and relative Steinberg groups associated with any associative ring $R$ with a complete family of full orthogonal idempotents instead of a matrix algebra $M(n, S)$. Below we use the notation $\text{St}(R)$ and $\text{St}(R, A)$ for such Steinberg groups, where $A$ is an ideal or, more generally, a ring-theoretic crossed module over $R$. They should not be confused with the stable Steinberg groups such as $\lim_{\rightarrow} \text{St}(n, R)$. In the commutative case we denote the base ring by $K$ and its ideal (or a commutative crossed module) by $a$.

As an application, we prove the following: the subgroup of $\text{St}(\Phi; K, a)$ generated by elementary generators $x_i(a)$ is normal and contains the commutator, and similarly for linear groups over arbitrary rings. See theorems 3 and 4 in the last section.

2 Relative linear Steinberg groups

We use the conventions $gh = ghg^{-1}$ and $[g, h] = ghg^{-1}h^{-1}$ for group theoretical operations. If a group $G$ acts on a group $H$ by automorphisms, then we usually denote the action by $\varphi$, i.e. as the conjugation in $H \rtimes G$.

Let $R$ be a unital associative ring. A non-unital $R$-algebra $A$ is a non-unital associative ring and an $R$-$R$-bimodule with the same addition such that

$$p(ab) = (pa)b, \quad (ab)p = a(bp), \quad (ap)b = a(pb)$$

for all $a, b \in A$ and $p \in R$. As in group theory, we say that $d: A \to R$ is a crossed module if $A$ is a non-unital $R$-algebra, $d$ is a homomorphism of non-unital $R$-algebras and $ab = d(a)b = a d(b)$ for all $a, b \in A$. For example, if $I$ is an ideal of $R$, then the inclusion map $d: I \to R$ is a crossed module. Another examples are the ring homotopes used in [12]: if $s \in R$ is a central element, then $R(s) = \{a(s) | a \in R\}$ is a non-unital $R$-algebra with the operations

$$a^{(s)} + b^{(s)} = (a + b)^{(s)}, \quad a^{(s)}b^{(s)} = (asb)^{(s)}, \quad pa^{(s)} = (pa)^{(s)}, \quad a^{(s)}p = (ap)^{(s)},$$

and the ring homomorphism $d: R(s) \to R, a^{(s)} \mapsto as$ is a crossed module. The centrality condition is essential.

If $A$ is a non-unital $R$-algebra, then the semi-direct product $A \rtimes R$ is a ring with an ideal $A$. As an abelian group it is the direct sum of $A$ and $R$, the multiplication is given by

$$(a \oplus p)(b \oplus q) = (ab + aq + pb) \oplus pq.$$ 

In particular, $A$ is a crossed module over $A \rtimes R$.

From now on fix a unital ring $R$ with a complete family of full orthogonal idempotents $e_1, \ldots, e_n$ and a non-unital $R$-algebra $A$. In other words, $e_i^2 = e_i$,
$e_i e_j = 0$ for $i \neq j$; $\sum_{i=1}^{n} e_i = 1$, and $R = Re_i R$ as an ideal. An “isotropic” linear Steinberg group $\text{St}(R)$ is the abstract group with the generators $x_{ij}(p)$ for $1 \leq i, j \leq n$, $i \neq j$, $p \in e_i Re_j$ and the relations

\[
x_{ij}(p)x_{ij}(q) = x_{ij}(p+q); \quad \text{(St1)}
\]

\[
[x_{ij}(p), x_{jk}(q)] = x_{ik}(pq) \quad \text{for} \quad i \neq k; \quad \text{(St2)}
\]

\[
[x_{ij}(p), x_{kl}(q)] = 1 \quad \text{for} \quad j \neq k \quad \text{and} \quad i \neq l; \quad \text{(St3)}
\]

There is a canonical group homomorphism $\text{st}: \text{St}(R) \to R^*$, $x_{ij}(p) \mapsto 1 + p$. If $R = M(n, S)$ is a matrix ring with the diagonal idempotents, then this Steinberg group may be identified with the usual unstable Steinberg group $\text{St}(n, S)$. An unrelativized Steinberg group $\text{St}(A)$ is given by the same presentation, but with $a \in e_i Ae_j$ for the generators $x_{ij}(a)$.

Now we define $\text{St}^\prime(R, A)$ as the group with an action of $\text{St}(R)$ generated by elements $x_{ij}(a)$ for $1 \leq i, j \leq n$, $i \neq j$, $a \in e_i Ae_j$ with the relations (Rel1), (Rel2), (Rel3) as in the unrelativized case and

\[
x_{ij}(p)x_{kl}(a) = x_{kl}(a) \quad \text{for} \quad j \neq k \quad \text{and} \quad l \neq i; \quad \text{(Rel1)}
\]

\[
x_{ij}(p)x_{jk}(a) = x_{ik}(pa)x_{jk}(a) \quad \text{for} \quad i \neq k; \quad \text{(Rel2)}
\]

\[
x_{jk}(p)x_{ij}(a) = x_{ij}(a)x_{ik}(-ap) \quad \text{for} \quad i \neq k. \quad \text{(Rel2')}
\]

Finally, if $d: A \to R$ is a crossed module, then a relative Steinberg group $\text{St}(R, A)$ is the factor-group of $\text{St}^\prime(R, A)$ by

\[
x_{ij}(d(a))g = x_{ij}(a)g \quad \text{for any} \quad g \in \text{St}^\prime(R, A). \quad \text{(Rel3)}
\]

There is a canonical isomorphism $\text{St}^\prime(R, A) \cong \text{St}(A \rtimes R, A)$ and a decomposition $\text{St}(A \rtimes R) \cong \text{St}^\prime(R, A) \rtimes \text{St}(R)$ into a semi-direct product. If $d: A \to R$ is a crossed module, then $\text{St}(R, A) \to \text{St}(R), x_{ij}(a) \mapsto x_{ij}(d(a))$ is a crossed module in the sense of group theory and the sequence

\[
\text{St}(R, A) \to \text{St}(R) \to \text{St}(R/d(A)) \to 1
\]

is exact.

The opposite ring $R^{op}$ also has a complete family of full orthogonal idempotents $e_i^{op}$ and $A^{op}$ is a non-unital $R^{op}$-algebra. The transposition maps are the anti-isomorphisms

\[
\text{St}(R) \to \text{St}(R^{op}), x_{ij}(p) \mapsto x_{ji}(p^{op});
\]

\[
\text{St}^\prime(R, A) \to \text{St}^\prime(R^{op}, A^{op}), x_{ij}(a) \mapsto x_{ji}(a^{op}).
\]

In other words, $x_{ij}(p) \mapsto x_{ji}(-p^{op})$ and $x_{ij}(a) \mapsto x_{ji}(-a^{op})$ are isomorphisms of these groups. If $d: A \to R$ is a crossed module, then $d^{op}: A^{op} \to R^{op}$ is also a crossed module and there is a transposition map $\text{St}(R, A) \to \text{St}(R^{op}, A^{op})$.

Now we consider some identities in $\text{St}^\prime(R, A)$ on the elements $z_{ij}(a, p) = x_{ji}(p)x_{ij}(a)$ for $a \in e_i Ae_j$ and $p \in e_j Re_i$. In order to write them compactly, we
use the abbreviations

\[ z_{ij,k}(a, b; p) = z_{ij}(a, p) x_{ik}(b) x_{jk}(pb) \]  \hspace{1cm} (Z2)

\[ = x_{ij}(p)(x_{ij}(a) x_{ik}(b)) \];

\[ z_{ij,k}(a, b; p) = z_{jk}(b, p) x_{ik}(a) x_{ij}(-ap) \]  \hspace{1cm} (Z2a)

\[ = x_{ik}(p)(x_{ik}(a) x_{jk}(b)) \];

\[ z_{i\oplus j,k}(a, b; p, q) = z_{i,k|j}(a, -aq; p) z_{j,k|i}(b, -bp; q) \]  \hspace{1cm} (Z4)

\[ = x_{ki}(p)x_{ij}(q)(x_{ik}(a) x_{jk}(b)) \];

\[ z_{i,j\oplus k}(a, b; p, q) = z_{i|j,k}(qa, a; p) z_{j|i,k}(pb, b; q) \]  \hspace{1cm} (Z4a)

\[ = x_{ij}(p)x_{ik}(q)(x_{ij}(a) x_{ik}(b)) \]

for distinct \( i, j, k \).

**Lemma 1.** The elements \( z_{ij}(a, p) \) satisfy the relations

\[ z_{ij}(a + a', p) = z_{ij}(a, p) z_{ij}(a', p); \]  \hspace{1cm} (Add1)

\[ z_{i,j|k}(a + a', b + b'; p) = z_{i,j|k}(a, b; p) z_{i,j|k}(a', b'; p); \]  \hspace{1cm} (Add2)

\[ z_{i|j,k}(a + a', b + b'; p) = z_{i|j,k}(a, b; p) z_{i|j,k}(a', b'; p); \]  \hspace{1cm} (Add2a)

\[ z_{i\oplus j,k}(a + a', b + b'; p, q) = z_{i\oplus j,k}(a, b; p, q) z_{i\oplus j,k}(a', b'; p, q); \]  \hspace{1cm} (Add3)

\[ z_{i,j\oplus k}(a + a', b + b'; p, q) = z_{i,j\oplus k}(a, b; p, q) z_{i,j\oplus k}(a', b'; p, q); \]  \hspace{1cm} (Add3a)

\[ z_{ij}(c-p)(x_{ik}(a) x_{jk}(b)) = x_{ik}(a + cb - cpa) x_{jk}(b + pcb - pcpa); \]  \hspace{1cm} (Conj1)

\[ z_{ij}(c-p)(x_{ki}(a) x_{kj}(b)) = x_{ki}(a + acp + bpcp) x_{kj}(b - ac - bpc); \]  \hspace{1cm} (Conj1a)

\[ [x_{jk}(pa) x_{ik}(a), x_{kj}(b) x_{ki}(-bp)] = z_{ij}(ab, p) \text{ for } i \neq j; \]  \hspace{1cm} (Mult)

\[ [z_{ij}(a, p), z_{kl}(b, q)] = 1 \text{ for } i, j, k, l \text{ distinct}; \]  \hspace{1cm} (Dis)

\[ z_{i\oplus j,k}(a, b; p, q) = z_{i\oplus j,k}(a, b; a, q, p); \]  \hspace{1cm} (Sym)

\[ z_{i,j\oplus k}(a, b; p, q) = z_{i,j\oplus k}(a, b; a, q, p); \]  \hspace{1cm} (Syma)

\[ z_{ij}(c-r)(x_{ik}(a) x_{kj}(b)) = x_{ki}(pc-c.rc) x_{jk}(pc-rc)p) z_{i\oplus j,k}(a + acr + br.cr, b - ac - brc; p, q); \]  \hspace{1cm} (Conj2)

\[ z_{i,j\oplus k}(a, b; p, q) = x_{ik}(eq-c.r) x_{jk}(rcq-rcrp) z_{k,i\oplus j}(a + acr + br.cr, b - ac - brc; p, q); \]  \hspace{1cm} (Conj2a)

\[ z_{i\oplus j,k}(a, qa; r - pq, p) = z_{i,j\oplus k}(-ap, a; q, r). \]  \hspace{1cm} (HW)

in \( \text{St}'(R, A) \). If \( d: A \rightarrow R \) is a crossed module, then they also satisfy a variant of \([\text{Rel3}]\)

\[ z_{ij}(a, p + d(b)) = x_{ji}(b) z_{ij}(a, p) x_{ji}(-b). \]  \hspace{1cm} (Rel4)

**Proof.** The relations \([\text{Add1}], \text{Add2}, \text{Add3}, \text{Conj1}, \text{Dis}, \text{Sym}, \text{Rel4}\)
obvious. The remaining ones may be written as
\[
\begin{align*}
[x_{ji}(p)x_{ik}(a), x_{ji}(p)x_{kj}(b)] &= x_{ji}(p)x_{ij}(ab); \
&\text{(Mult)} \\
&\text{so they hold in } \text{St}'(R,A). \text{ Finally, the relations with the index } t \text{ follow by transposition and the commutativity relations } (\text{St3}), (\text{Add2}), (\text{Add2}^t), (\text{Sym}), (\text{Sym}^t). \text{ The relation } (\text{HW}) \text{ is equivalent to its own transpose modulo these commutativity relations.}
\end{align*}
\]

\section{Generators and relations}

Let $\text{St}^F(R,A)$ be the free group generated by the symbols $z_{ij}(a,p)$, where $1 \leq i, j \leq n$, $i \neq j$, $a \in e_i A e_j$, and $p \in e_j R e_i$. A transposition map for this group is the anti-isomorphism
\[
\text{St}^F(R,A) \rightarrow \text{St}^F(R^\text{op},A^\text{op}), z_{ij}(a,p) \mapsto z_{ji}(a^\text{op},p^\text{op}).
\]

The relations from lemma 1 are formal identities in $\text{St}^F(R,A)$ if we use the notation $x_{ij}(a) = z_{ij}(a,0)$, $\text{Z2}$, $\text{Z2}^t$, $\text{Z4}$, and $\text{Z4}^t$. In the proofs that some identities from lemma 1 hold in a factor-group of $\text{St}^F(R,A)$ we usually consider only the non-transposed versions.

Now let $\overline{\text{St}}(R,A)$ be the factor-group of $\text{St}^F(R,A)$ by (Add1), (Dis), (Conj2), (Conj2$^t$), and (HW). If $d: A \rightarrow R$ is a crossed module, then we define $\text{St}(R,A)$ as the factor-group of $\overline{\text{St}}(R,A)$ by (Rel4).

\textbf{Lemma 2.} The Steinberg relations and all the identities from lemma 1 except (Rel4) hold in $\overline{\text{St}}(R,A)$.

\textit{Proof.} The identity (Conj1$^t$) follows from the case $p = q = 0$ of (Conj2$^t$), and (Mult$^t$) follows from the case $a = q = r = 0$. The Steinberg relations easily follow from (Add1), (Conj2$^t$), and (Dis$.^t$). Also, (Add2$^t$) follows from (Conj1$^t$), (Add1$^t$), and the Steinberg relations.

Substituting $b = ra$, $q = 0$ in (Conj2$^t$) and applying (St3), we get (Sym$^t$). Now (Add3$^t$) follows from (Add2$^t$) and (Sym$^t$).

By lemma 1 there is a canonical homomorphism $\mu : \overline{\text{St}}(R,A) \rightarrow \text{St}'(R,A), z_{ij}(a,p) \mapsto z_{ij}(a,p)$. So there is a sequence
\[
\text{St}^F(R,A) \rightarrow \overline{\text{St}}(R,A) \rightarrow \text{St}'(R,A) \rightarrow \text{GL}(A)
\]

and similarly for the relative groups if $d: A \rightarrow R$ is a crossed module. Here $\text{GL}(A)$ is the group of quasi-invertible elements of $A$, i.e. elements $a \in A$ such
that \( a + b + ab = a + b + ba = 0 \) for some \( b \in A \), and \( \text{st}: x_{ij}(a) \mapsto a \) is \( \text{St}(R) \)-equivariant. The diagonal group

\[
D(R) = \{ r \in R^* \mid e_i r e_j = 0 \text{ for } i \neq j \}
\]

acts on the this sequence and its relativized variant by

\[
r_{z_{ij}}(a, p) = z_{ij}(r a r^{-1}, r p r^{-1}).
\]

Our goal is to show that if \( n \geq 4 \), then \( \text{St}(R) \) acts on \( \text{St}'(R, A) \) and \( \mu \) is an isomorphism of groups with actions of \( \text{St}(R) \). Before we start the proof it is useful to simplify calculations in \( \text{St}'(R, A) \).

Let

\[
\Phi = \{ e_i - e_j \mid 1 \leq i, j \leq n; i \neq j \} \subseteq \mathbb{R}^n
\]

be the root system of type \( A_{n-1} \), so every symbol \( z_{ij}(-, =) \) in \( \text{St}'(R, A) \) is indexed by the root \( e_i - e_j \). A subset \( \Sigma \subseteq \Phi \) is called closed if \( \alpha, \beta \in \Sigma \Rightarrow \alpha + \beta \in \Phi \) imply \( \alpha + \beta \in \Sigma \). A subset \( \Sigma \subseteq \Phi \) is called special closed, if it is closed and does not contain opposite roots. It is well-known that \( \Sigma \) is special closed if and only if it is closed and lies in an open half-space (with the boundary passing through \( 0 \)). We say that an element \( \alpha \) of a special closed \( \Sigma \subseteq \Phi \) is extreme if it is not a sum of two elements of \( \Sigma \). Hence if \( \alpha \in \Sigma \) is extreme, then \( \Sigma \setminus \{ \alpha \} \subseteq \Phi \) is also a special closed subset.

For any \( \alpha = e_i - e_j \in \Phi \) we use the notation \( A_{\alpha} = e_i A e_j \), so \( x_{\alpha}(A_{\alpha}) = \{ x_{ij}(a) \mid a \in A_{\alpha} \} \) is a subgroup of \( \text{St}(A) \), \( \text{St}'(R, A) \), or \( \text{St}(R, A) \) depending on the context. If \( \Sigma \subseteq \Phi \) is a special closed subset, then the restriction of \( \text{St}(A) \to \text{GL}(A) \) to the subgroup \( \prod_{\alpha \in \Sigma} x_{\alpha}(A_{\alpha}) \) is injective, where the product is taken in any order. Moreover, the product map

\[
\prod_{\alpha \in \Sigma} x_{\alpha} : \prod_{\alpha \in \Sigma} A_{\alpha} \to \prod_{\alpha \in \Sigma} x_{\alpha}(A_{\alpha})
\]

is one-to-one.

In the following result we use the notation \( g \{ h \} \) in order to distinguish \( h \in \text{St}(A) \) with its image in \( \text{St}'(R, A) \).

**Lemma 3.** Let \( \Sigma \subseteq \Phi \) be a special closed subset. Then for any \( g \in \prod_{\alpha \in \Sigma} x_{\alpha}(R_{\alpha}) \leq \text{St}(R) \) there is a unique homomorphism \( g \{ - \}_\Sigma : \text{St}(A) \to \text{St}'(R, A) \) such that

\[
\begin{align*}
g_{x_{ij}(p)}(x_{ij}(a))_{\Sigma} &= g \{ x_{ij}(a) \}_{\Sigma} \text{ for } l \neq i \text{ and } j \neq k; \\
g_{x_{kj}(p)}(x_{ij}(a))_{\Sigma} &= g \{ x_{kj}(pa) x_{ij}(a) \}_{\Sigma} \text{ for } j \neq k; \\
g_{x_{ij}(p)}(x_{ij}(a))_{\Sigma} &= g \{ x_{ij}(a) x_{ij}(-ap) \}_{\Sigma} \text{ for } l \neq i; \\
x_{ij}(a)_{\Sigma} &= x_{ij}(a); \\
x_{ij}(p)_{\Sigma} &= z_{ij}(a, p).
\end{align*}
\]

If \( \Sigma' \subseteq \Phi \) is a special closed subset contained in \( \Sigma \) and \( g \in \prod_{\alpha \in \Sigma'} x_{\alpha}(R_{\alpha}) \), then \( g \{ - \}_{\Sigma'} = g \{ - \}_\Sigma \), so the index \( \Sigma \) in the notation is redundant.
Proof. We use induction on the size of $\Sigma$; the case $\Sigma = \emptyset$ is obvious. Suppose that for all proper $\Sigma' \subseteq \Sigma$ such that $\Sigma'$ is a special closed subset of $\Phi$ we already have the homomorphisms from the statement. First of all, we define $g\{\cdot\}_\Sigma$ on the free group $\text{St}^F(A)$ generated by all symbols $x_{ij}(a)$ for $1 \leq i, j \leq n, i \neq j$, and $a \in e_i A e_j$

Take any root $\alpha = e_i - e_j \in \Phi$. Suppose that $\Sigma$ contains an extreme root $\beta_1 = e_k - e_i \neq -\alpha$, then any $g \in \prod_{\beta \in \Sigma} x_{\beta}(R_\beta)$ has a decomposition $g = g_1 x_{k_1}(p)$ for $g_1 \in \prod_{\beta \in \Sigma} x_{\beta}(R_\beta)$, where $\Sigma' = \Sigma \setminus \beta_1$. Let

$$g\{x_{ij}(a)\}_\Sigma = g_1\{x_{ij}(a)\}_{\Sigma'} \text{ for } l \neq i \text{ and } j \neq k;$$

$$g\{x_{ij}(a)\}_\Sigma = g_1\{x_{kj}(pa) x_{ij}(a)\}_{\Sigma'} \text{ for } l = i \text{ and } j \neq k;$$

$$g\{x_{ij}(a)\}_\Sigma = g_1\{x_{ij}(a) x_{id}(-ap)\}_{\Sigma'} \text{ for } l \neq i \text{ and } j = k.$$

We have to check that this definition is independent on $\beta_1$. Let $\beta_2 \in \Sigma \setminus \{-\alpha, \beta_1\}$ be another extreme root. If $\alpha \neq -\beta_1 - \beta_2$, then the definitions of $g\{x_{ij}(a)\}_\Sigma$ via $\beta_1$ and via $\beta_2$ coincide. Indeed, the roots $\alpha, \beta_1, \beta_2$ lie in a common special closed subset $T \subseteq \Phi$, so $g\{x_{ij}(a)\}_\Sigma = g\{h\}_{\Sigma''}$ for some canonical $g_3 \in \prod_{\beta \in \Sigma''} x_{\beta}(R_\beta)$ and $h \in \prod_{\beta \in T} x_{\beta}(A_\beta)$, where $\Sigma'' = \Sigma \setminus \{\beta_1, \beta_2\}$.

Hence we may assume that $\alpha + \beta_1 + \beta_2 = 0$ and there are no other extreme roots of $\Sigma$, so $\Sigma = \{\beta_1, \beta_2, \beta_1 + \beta_2\}$. Without loss of generality, $\alpha = e_i - e_k$, $\beta_1 = e_j - e_i$, and $\beta_2 = e_k - e_j$. In this case we use (HW):

$$x_{ki}(r-pq) x_{kj}(p) x_{ik}(qa) x_{kj}(a)\{\beta_1\}_\Sigma = x_{ji}(q) x_{ji}(r) x_{ij}(a) x_{ij}(a)\{\beta_2\}_\Sigma.$$

It remains to consider the case when there is no such $\beta_1$. This means that $\Sigma = \{-\alpha\}$, so we just define $x_{ji}(p) x_{ij}(a)\{\beta_1\}_\Sigma = z_{ij}(a, p)$.

By construction, $g\{h\}_\Sigma$ is independent on $\Sigma$. The uniqueness claim is also trivial.

Now let us check that $g\{-\}_\Sigma$ factors through the Steinberg relations (i.e. maps them to identities in $\text{St}(R, A)$). For (St1) this is obvious, so we only consider the Steinberg relation for $[x_{\alpha_1}(a), x_{\alpha_2}(b)]$, where $\alpha_1$ and $\alpha_2$ are linearly independent roots. If there is an extreme root $\beta \in \Sigma$ such that $\alpha_1, \alpha_2, \beta$ lie in a common special closed subset of $\Phi$, then the claim follows from the construction of $g\{-\}_\Sigma$ via $\beta$. Otherwise let $\Psi$ be the root subsystem generated by $\alpha_1, \alpha_2, \beta$, and it is necessary of rank 2. If $\Psi$ is of type $A_1 \times A_1$, then we just apply [Dis].

If $\Psi$ is of type $A_2$ and $\alpha_1 + \alpha_2 \notin \Phi$, then we apply [Sym] or [Sym1]. So without loss of generality $\alpha_1 = e_i - e_j$, $\alpha_2 = e_j - e_k$, and $\Sigma \subseteq \{-\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$. Now we do a calculation using [Con2]:

$$x_{ji}(p) x_{kj}(q) x_{ki}(r)\{x_{ij}(a) x_{jk}(b)\}_\Sigma = x_{ji}(p)\{x_{kj}(ra)\}_\Sigma z_{ij}(a, p) x_{ki}(r-pq) x_{kj}(q)\{x_{jk}(b)\}_\Sigma$$

$$= x_{ki}(r-pq) x_{kj}(q)\{x_{ik}(ab) x_{jk}(b + pab)\}_\Sigma x_{ji}(p)\{x_{kj}(ra)\}_\Sigma z_{ij}(a, p)$$

$$= x_{ji}(p) x_{ij}(q) x_{ki}(r)\{x_{ik}(ab) x_{jk}(b) x_{ij}(a)\}_\Sigma.$$

Finally, we have to prove the first three identities from the statement. Let $\alpha \in \Phi$ and $\beta \in \Sigma$ be roots appearing in the identity such that $\alpha \neq -\beta$. Then
there is an extreme $\gamma \in \Sigma$ such that $\alpha, \beta, \gamma$ lie in a common special closed subset of $\Phi$. So we apply the construction of $\Phi(\{-\}) \Sigma$ via $\gamma$ and use the induction hypothesis. \hfill \Box

Hence in the group $\text{St}^\prime(R, A)$ we have

$$
\begin{align*}
z_{i,j}(a, b; p) &= x_{ij}(p)\{x_{ij}(a) x_{ik}(b)\}; \\
z_{i}(a, b; p) &= x_{i}(p)\{x_{ik}(a) x_{ik}(b)\}; \\
z_{i}(a, b; p, q) &= x_{i}(p) x_{i}(q)\{x_{ij}(a) x_{jk}(b)\}; \\
z_{i,j}(a, b; p, q) &= x_{ij}(p) x_{i}(q)\{x_{ij}(a) x_{ik}(b)\}.
\end{align*}
$$

Also, the diagonal group acts on the homomorphisms from lemma 3 in the obvious way.

The next result allows us to identify $\text{St}^\prime(R, A)$ with $\text{St}(A \times R, A)$, so in all identities from lemma 1 we may assume that $p, q, r \in A \times R$ instead of $p, q, r \in R$, and in lemma 2 the element $g$ may be taken in $\prod_{\in \Sigma} x_{\alpha}(A_{\alpha} \times R_{\alpha}) \subseteq \text{St}(A \times R)$.

**Lemma 4.** The natural homomorphism $\text{St}^\prime(R, A) \to \text{St}(A \times R, A)$, $z_{ij}(a, p) \mapsto z_{ij}(a, 0 \oplus p)$ is bijective.

**Proof.** Denote this homomorphism by $\zeta$. The inverse homomorphism is given by

$$
\zeta: \text{St}^F(A \times R, A) \to \text{St}^\prime(R, A), z_{ij}(a, p_A \oplus p_R) \mapsto x_{ij}(p_A)z_{ij}(a, p_R).
$$

Clearly, $\zeta$ factors through $\text{Add}$, $\text{Dis}$, and $\text{Rel}$. Also,

$$
\zeta(\zeta_{i,j}(a, b; p_A \oplus p_R, q_A \oplus q_R)) = x_{i,j}(p_A) x_{i,j}(q_A)\zeta_{i,j}(a, b; p_R, q_R).
$$

For $\text{(Con2)}$ if the last two arguments of $\zeta_{i,j,k}$ from the left hand side are in $R$, then the image of the identity under $\zeta$ holds in $\text{St}^\prime(R, A)$ by a triple application of $\text{(Con2)}$. The general case reduces to this special case and $\text{(Con1')}$ by the formula for $\zeta(\zeta_{i,j,k}(a, b; p_A \oplus p_R, q_A \oplus q_R))$ above.

Finally, for $\text{(HW)}$ we do the following calculation in $\text{St}^\prime(R, A)$ using lemma 3:

$$
\zeta(\zeta_{i,j,k}(a, q_A a + q_R a; (r_A \oplus r_R) - (p_A \oplus p_R)(q_A \oplus q_R), p_A \oplus p_R))
= x_{i,j}(q_A) x_{i,j}(r_A - p_A q_R) x_{i,j}(p_A) x_{i,j}(q_R) x_{i,j}(r_R)\{x_{i,j}(a)\}
= \zeta(\zeta_{i,j}(a, b; p_A - p_R, a; q_A \oplus q_R, r_A \oplus r_R)).
$$

Hence $\zeta: \text{St}(A \times R, A) \to \text{St}^\prime(R, A)$ is well-defined. Clearly, it is the inverse of $\zeta$. \hfill \Box

Actually, the identities $\text{(Con2)}$ and $\text{(Con2')}\gamma$ in the definition of $\text{St}(R, A)$ may be replaced by

$$
\begin{align*}
z_{i,j}(c, r)\zeta_{i,j,k}(a, b; p, q) &= z_{i,j,k}(a + c b - q r a, b + r c b - r c r a; p + d(q r + q c r), q - d(p c + q c r)); \\
z_{i,j}(c, r)\zeta_{k,i,j}(a, b; p, q) &= z_{k,i,j}(a + a c r + b r c r, b - a c r - b r c r; p + d(q c - c r p), q + d(r c q - c r p)).
\end{align*}
$$
Indeed, they follow from the ordinary (Conj2) and (Conj2') by lemma [4]. Conversely, taking \( p = q = 0 \) in (Conj2') we obtain (Conj1), and the Steinberg relations formally follow from (Add1), (Conj1), (Conj1'), and (Dis). The identity (Conj2) follows from a combination of (Conj2'), (Conj1), the Steinberg relations, and (Ref).

4 Root elimination: construction

From now on we fix a crossed module \( d: A \to R \). All our further results have versions for the groups \( \text{St} \) associated with any non-unital \( R \)-algebra \( A \) by lemma [4].

In order to construct an action of \( \text{St}(R) \) on \( \text{St}(R, A) \) we use the root elimination technique from [12]. Let \( \Psi \subseteq \Phi \) be a root subsystem (necessarily closed) with the span \( \mathbb{R} \Psi \). We denote the image of \( \Phi \setminus \Psi \) in \( \mathbb{R}^n/\mathbb{R} \Psi \) by \( \Phi/\Psi \) and the map \( \Phi \setminus \Psi \to \Phi/\Psi \) by \( \pi_\Psi \). Let \( e_\ell \sim e_j \) if \( i = j \) or \( e_i - e_j \in \Psi \), this is an equivalence relation on our family of idempotents in \( R \) since \( \Psi \) is closed and \( \Psi = \Psi \). For an equivalence class \( \{e_i\}_{i \in I} \) let \( e_I = \sum_{i \in I} e_i \), these elements also form a complete family of full orthogonal idempotents. It follows that the set \( \Phi/\Psi \) itself is a root system of type \( A_{n-1-m} \) with a suitable dot product, where \( m \) is the rank of \( \Psi \), it parametrizes the root subgroups for a new family of idempotents. Let us denote the groups associated with the new family by \( \text{St}(A, \Phi/\Psi), \text{St}^\circ(R, A; \Phi/\Psi), \text{St}(R, A; \Phi/\Psi), \text{St}(R, A; \Phi/\Psi), \text{St}(R, A; \Phi/\Psi), \) and \( \text{D}(R, \Phi/\Psi) \). For the groups associated with the old family of idempotents we add a parameter \( \Phi \) in the notation.

There is a one-to-one correspondence between root subsystems of \( \Phi \) containing \( \Psi \) and root subsystems of \( \Phi/\Psi \) given by the direct and inverse images under \( \pi_\Psi \). If \( \Xi \subseteq \Phi/\Psi \) is a root subsystem, then the root systems \( \Phi/\Xi \) and \( \Phi/\pi_\Psi^{-1}(\Xi) \) are canonically isomorphic. We write \( \Phi/\alpha \) instead of \( \Phi/\{-\alpha, \alpha\} \).

For a root \( \alpha = e_i - c_m \) consider the homomorphism

\[ F_\alpha: \text{St}(A, \Phi/\alpha) \to \text{St}(A, \Phi), \]

\[ x_{ij}(a) \mapsto x_{ij}(a) \text{ for } i, j \neq \infty; \]

\[ x_{\infty j}(a) \mapsto x_{ij}(e_i a) x_{mj}(e_m a); \]

\[ x_{i\infty}(a) \mapsto x_{il}(ae_i) x_{im}(ae_m); \]

where \( e_\infty = e_i + e_m \) is the new idempotent. Clearly, it is well-defined. Composing these homomorphisms for various roots, we obtain homomorphisms \( F_\Phi: \text{St}(A, \Phi/\Psi) \to \text{St}(A, \Phi) \) for any root subsystem \( \Psi \subseteq \Phi \).

The following lemma is just a generalization of (Conj2) and (Conj2').

**Lemma 5.** Let \( \Psi \subseteq \Phi \) be a root subsystem, \( \Sigma \subseteq \Phi/\Psi \) be a special closed subset. Then for all elements \( f \in \{z_\alpha(A, R_{-\alpha}) \mid \alpha \in \Psi \} \leq \text{St}(R, A; \Phi), g \in \prod_{\beta \in \Sigma} x_{\beta}(R_{\beta}) \leq \text{St}(R, \Phi/\Psi), h \in \text{St}(A, \Phi/\Psi) \) we have

\[ f F_\Phi(g) \{F_\Phi(h)\} f^{-1} = F_\Phi^{(st(f))} \{F_\Phi^{(st(f))} h\} \in \text{St}(R, A; \Phi), \]

where \( st(f) \) is the image of \( f \) in \( \text{GL}(R) \).
Proof. Note that \( \text{st}(f) \) actually lies in \( D(R, \Phi/\Psi) \), so it acts on the unrelativized Steinberg groups associated with \( \Phi/\Psi \). Also, \( \pi^{-1}_\Phi(\Sigma) \subseteq \Phi \) is a special closed subset, so the left hand side is defined. Without loss of generality, \( \Psi = \{ -\alpha, \alpha \} \) for a root \( \alpha = e_l - e_m \) and \( f = z_{lm}(a, p) \). By the proof of lemma 3 applied to \( \Phi/\alpha \), the element \( F_\alpha(x_j(q)) \{ \Phi, F_\alpha(x_j(b)) \} \) for \( e_i - e_j \in \Phi/\alpha \) and such a decomposition preserves the action of \( \text{st}(f) \) on the arguments, so it remains to consider the case \( \Sigma = \{ e_j - e_i \} \), \( g = x_{ji}(q) \), and \( h = x_{ij}(b) \). Let \( e_\infty = e_l + e_m \) be the new idempotent. If \( l, j \neq \infty \), then the identity follows from (Dis). Else this is precisely Conj2 or Conj2. \( \square \)

Now we define \( F_\Psi \) for the Steinberg groups \( \text{St} \). For a root \( \alpha = e_l - e_m \) consider the homomorphism

\[
F_\alpha : \text{St}^F(R, A; \Phi/\alpha) \to \text{St}(R, A; \Phi);
\]

\[
z_{ij}(a, p) \mapsto z_{ij}(a, p) \quad \text{for} \ i, j \neq \infty;
\]

\[
z_{\infty j}(a, p) \mapsto z_{l\infty, j}(e_\alpha, e_\infty a ; p, e_m); \quad z_{i\infty}(a, p) \mapsto z_{i, l\infty}(a, e_\alpha, e_\infty a ; e_l, e_m);
\]

where \( e_\infty = e_l + e_m \) is the new idempotent. The next lemma implies that \( F_\Psi : \text{St}(R, A; \Phi/\Psi) \to \text{St}(R, A; \Phi) \) is well-defined for any root subsystem \( \Psi \subseteq \Phi \).

Lemma 6. The homomorphism \( F_\alpha : \text{St}(R, A; \Phi/\alpha) \to \text{St}(R, A; \Phi) \) is well-defined for any root \( \alpha \). If \( \alpha, \beta \in \Phi \) are linearly independent and \( \Psi = \Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta) \), then

\[
F_\alpha \circ F_{\pi_\alpha(\beta)} = F_\beta \circ F_{\pi_\beta(\alpha)} : \text{St}(R, A; \Phi/\Psi) \to \text{St}(R, A; \Phi).
\]

Proof. Clearly, \( F_\alpha \) factors through (Add1), (Dis), and (Rel3). Let \( \alpha = e_l - e_m \) and \( e_\infty = e_l + e_m \) be the new idempotent. Below all the indices are distinct.

We have

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(x_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
\text{and the transposed versions directly by definitions.}
\]

Lemma 7 implies that \( F_\alpha \) factors through (Conj2). For (HW) we consider all nontrivial cases, i.e. the cases involving the index \( \infty \):

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
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\]

\[
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\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
F_\alpha(z_{i\infty, j, k}(a, b; p, q)) = \frac{x_{ik}(p)}{\alpha} x_{jk}(q) x_{ij}(a) x_{jk}(b);
\]

\[
\text{and the transposed versions directly by definitions.}
\]
The last claim follows from the formulas for $F_\alpha(z_{i\oplus j,k}(a,b;p,q))$, where the indices may coincide with $\infty$.

There is also a useful formula

$$F_\Psi(\{g\{h\}) = F_\Psi(\{F_\Psi(h)\}) \in \overline{St}(R,A),$$

where $\Psi \subseteq \Phi$ is a root subsystem, $g = \prod_{a \in \Sigma} x_{a}(R_\alpha) \leq St(R,\Phi/\Psi)$ for some special closed $\Sigma \subseteq \Phi/\Psi$, and $h \in St(A,\Phi/\Psi)$.

**Lemma 7.** Let $\alpha \in \Phi$ be a root and suppose that $n \geq 3$. Then $F_\alpha : \overline{St}(R,A;\Phi/\alpha) \to \overline{St}(R,A;\Phi)$ is surjective.

**Proof.** Let $\alpha = e_l - e_m$ and $e_\infty = e_l + e_m$. We have to show that every generator $z_{ij}(a,p)$ of $St(R,A;\Phi)$ lies in the image of $F_\alpha$. If $i, j, l, m$ are distinct, then $z_{ij}(a,p) = F_\alpha(z_{ij}(a,p))$. If $i \in \{l,m\}$ and $j \notin \{l,m\}$, then $z_{ij}(a,p) = F_\alpha(z_{\infty k}(a,p))$. The case $i \notin \{l,m\}$ and $j \in \{l,m\}$ follows by transposition.

If $\{i,j\} = \{l,m\}$, then choose an index $k$ different from $i$ and $j$. We have

$$z_{ij}(-ap,q) = F_\alpha(z_{\infty k}(a+qa,p-pq)x_{\infty k}(-a-qa))$$

by $(\text{HW})$ for $a \in e_i Ae_k$ and $p \in e_k Re_j$. It follows that $z_{ij}(a,p)$ lies in the image if $a \in e_i Ae_k Re_j$. But the idempotent $e_k$ is full, hence $e_i Ae_k Re_j = e_i Ae_j$. $\square$

5  Root elimination: bijectivity

**Lemma 8.** Let $\alpha = e_l - e_m \in \Phi$ be a root and suppose that $n \geq 4$. Then $F_\alpha : \overline{St}(R,A;\Phi/\alpha) \to \overline{St}(R,A;\Phi)$ is an isomorphism. Also, $F_\alpha(\text{st}(x_{m}(p))F_\alpha^{-1}(x_{lm}(a))) = z_{lm}(a,p)$ in $St(R,A;\Phi)$.

**Proof.** Let $e_\infty = e_l + e_m$. We have to find the preimages $z_{ij}(a,p) \in \overline{St}(R,A;\Phi/\alpha)$ of $z_{ij}(a,p)$ and to prove the defining identities for them. Note that the elements $x_{lm}(p)$ and $x_{ml}(q)$ for $p, q \in R$ act on $\overline{St}(R,A;\Phi/\alpha)$ via their images in $D(R;\Phi/\alpha)$. Let

$$z_{ij}(a,p) = z_{ij}(a,p) \text{ if } i, j, l, m \text{ are distinct, } a \in e_i Ae_j, p \in e_j Re_i;$$

$$z_{ij}(a,p) = z_{\infty k}(a,p) \text{ if } i \in \{l,m\}, j \notin \{l,m\}, a \in e_i Ae_j, p \in e_j Re_i;$$

$$z_{ij}(a,p) = z_{\infty k}(a,p) \text{ if } i \notin \{l,m\}, j \in \{l,m\}, a \in e_i Ae_j, p \in e_j Re_i;$$

$$\bar{x}_{ij}(a,p) = z_{\infty k}(-a,p)x_{\infty k}(a) \text{ if } \{i,j\} = \{l,m\}, k \notin \{l,m\}, a \in e_i Ae_k, p \in e_k Re_j;$$

$$\bar{x}_{ij}(a,q) = z_{\infty k}(a,q)x_{\infty k}(-a) \text{ if } \{i,j\} = \{l,m\}, k \notin \{l,m\}, q \in e_j Re_k, a \in e_k Ae_j.$$

The idea is to find elements $\bar{x}_{ij}(a)$ for $\{i,j\} = \{l,m\}$ such that $\bar{x}_{ij}^k(a,p) = \bar{x}_{ij}(ap)$ and $\bar{x}_{ij}(a,q) = \bar{x}_{ij}(qa)$, and then define $z_{ij}(a,r) = \text{st}(x_{ij}(r)\bar{x}_{ij}(a))$. All calculations below are in $\overline{St}(R,A;\Phi/\alpha)$.

Let $\{i,j\} = \{l,m\}$. It is easy to see that $\bar{x}_{ij}^k(a,p)$ acts on $\overline{St}(R,A;\Phi/\alpha)$ by conjugation in the same way as its image in $D(R;\Phi/\alpha)$. Indeed, these actions
coincide on the image of $F_\beta$ for $\beta = e_\infty - e_k$ by lemma $[\text{II}]$ and $F_\beta$ is surjective by lemma $[\text{IV}]$. In particular, $[\overline{x}^k_{ij}(a, p), \overline{x}^k_{ij}(b, q)] = 1$ for all $k, o \notin \{l, m\}$. Also,

$$\overline{x}^k_{ij}(a, p) \overline{x}^k_{ij}(b, p) = z^\infty(-a, p) z^\infty(-b, p) x^\infty(b) x^\infty(a) = \overline{x}^k_{ij}(a + b, p).$$

By the same argument, the factors in the definitions of $\overline{x}^k_{ij}$ and $k \overline{x}^\ast_{ij}$ commute.

If $o \notin \{k, l, m\}$, $a \in e_i R e_k$, $p \in e_k R e_o$, $q \in e_o R e_j$, $r \in e_k R e_j$, then $qa = ra = 0$. We apply $(\text{HW})$ to get

$$\overline{x}^k_{ij}(a, r + pq) = z^\infty(-a, r + pq) x^\infty(a) = z^\infty(-ap, q) z^\infty(-a, r) x^\infty(ap) x^\infty(a) = \overline{x}^k_{ij}(ap, q) \overline{x}^k_{ij}(a, r).$$

In other words, $\overline{x}^k_{ij}(a, pq) = \overline{x}^\circ_{ij}(ap, q)$ and $\overline{x}^k_{ij}(a, r + pq) = \overline{x}^k_{ij}(a, r) \overline{x}^k_{ij}(a, pq)$ if $p \in e_k R e_o$, $q \in e_o R e_j$, and $k \neq o$. Since the idempotent $e_o$ is full, $\overline{x}^k_{ij}(a, r + r') = \overline{x}^k_{ij}(a, r) \overline{x}^k_{ij}(a, r')$ for all $r, r' \in e_k R e_j$. It follows that $\overline{x}^k_{ij}(a, pq) = \overline{x}^k_{ij}(ap, q)$ for $p \in e_k R e_k$ and $q \in e_k R e_j$.

Since $e_k$ is a full idempotent, the homomorphism $e_i(A e_k \otimes e_k R e_o) \rightarrow e_i(A e_j), a \otimes p \rightarrow ap$ is bijective. Hence there is a unique homomorphism $\overline{x}_{ij} : e_i(A e_j) \rightarrow \text{ST}(R, A; \Phi)$ such that $\overline{x}^k_{ij}(a, p) = \overline{x}_{ij}(ap)$. Clearly, $\overline{x}^k_{ij}(a, p) = \overline{x}_{ij}(a, p)$ for all indices $o \notin \{l, m\}$.

Let $k, o$ be distinct indices not in $\{i, j\} = \{l, m\}$, $q \in e_i R e_k$, $a \in e_k R e_o$, $p \in e_o R e_j$, so $pq = 0$. Applying $(\text{HW})$ with $r = 0$ we get

$$k \overline{x}_{ij}(q, ap) = x_{k\infty}(-ap) x_{k\infty}(ap, q) = z^\infty(-ap) x^\infty(qa) = \overline{x}^o_{ij}(qa, p).$$

Since $k \overline{x}_{ij}(q, a)$ is also additive on $q$ and $a$, we get $k \overline{x}_{ij}(q, a) = \overline{x}_{ij}(qa)$. Let $\tilde{z}_{ij}(a, r) = s^t(x_{ji}(r) r) \overline{x}_{ij}(a)$, where $st(x_{ji}(r))$ is the image of $x_{ji}(r)$ in $D(R, \Phi)$. $\tilde{z}_{ij}(a, r)$ is also additive on $q$ and $a$, we get $k \tilde{z}_{ij}(q, a) = \tilde{x}_{ij}(qa)$. Let $\tilde{z}_{ij}(a, r) = s^t(x_{ji}(r) r) \tilde{x}_{ij}(a)$. When $i, j \in \{l, m\}$, then the element $G_\alpha(\tilde{z}_{ij}(a, p))$ acts on $\text{ST}(R, A; \Phi)$ in the same way as $\overline{x}_{ij}(a, p) = x_{\infty}(a, p)$, $x_{\infty}(a, p) = \overline{x}_{ij}(a, p)$. We have to check that $G_\alpha$ factors through the defining identities of $\text{ST}(R, A; \Phi)$. This is clear for $\text{Add1}$, $\text{Disc}$, $\text{Rec1}$, and the Steinberg relations. Since the definition of $G_\alpha$ is invariant under transposition, it suffices to consider only the non-transposed identities.

The cases of the remaining identities where at most one index is in $\{l, m\}$ are obvious, in the remaining ones two of the indices $i, j, k$ coincide with $l, m$. Let $D_{ij}$ be the subgroup of $\text{ST}(R, A; \Phi)$ generated by all $z_{ij}(a, p)$ and $z_{ji}(b, q)$, $\Psi = \{\pm(e_i - e_j), \pm(e_i - e_k), \pm(e_j - e_k)\} \subseteq \Phi$ be a root subsystem, $e^\infty = e_i + e_j + e_k$ be the corresponding idempotent. Choose an index $o \notin \{i, j, k\}$.

If $\{j, k\} = \{l, m\}$, then

$$G_\alpha(z_{i[j,k]}(a, b; p, q)) = 1 + q x^\infty(p) x^\infty(a) \overline{x}_{jk}(b) x^\infty(-bp) = 1 + q G_\alpha(z_{ij}(a, b; p, q)).$$

(G1)
Hence under this assumption and for $g \in D_{ij}$, $u = st(g) \in GL(R)$, $r \in e_i Re_{o}$, $a \in e_o Re_k$, $p \in e_k Re_i$ we have

$$G_a(z_{ik}(ra,p)) = F_{\Psi_a}(u(1+p)[x_{\infty}(r)[x_{\infty}(a)]x_{\infty}(-a))]$$

$$= 1 + pu^{-1} e_i F_{\Psi_a}(x_{\infty}(r)[x_{\infty}(a)]x_{\infty}(-a))$$

$$= 1 + pu^{-1} e_i \{x_{\infty}(a) - apu^{-1} e_i\}x_{\infty}(-a)\}$$

Similarly, if $g \in D_{ij}$, $u = st(g)$, $r \in e_j Re_o$, $b \in e_o Ae_k$, $q \in e_k Re_j$, then

$$G_a(z_{jk}(rb,q)) = F_{\Psi_a}(u[1+p][x_{\infty}(r)[x_{\infty}(b)]x_{\infty}(-b))]$$

$$= 1 + qu^{-1} e_j F_{\Psi_a}(x_{\infty}(r)[x_{\infty}(b)]x_{\infty}(-b))$$

It follows that $G_a$ factors through $\{\text{Conj1}\}$ and, consequently, through $\{\text{Add2}\}$. Now we prove that $G_a$ factors through $\{\text{Sym}\}$ in the form

$$[z_{ik}(a,p)] x_{ij}(-aq), z_{jk}(b,q) x_{ji}(-bp) x_{ki}(-qbp) = 1.$$  

If $\{j,k\} = \{l,m\}$, then for $r \in e_j Ae_o$, $b \in e_o Ae_k$, $q = 0$ we have

$$G_a(z_{ik}(a,p) x_{jk}(rb) x_{ji}(-rbp)) = x_{\infty}(r) x_{\infty}(a) x_{\infty}(b) x_{\infty}(-b)$$

and the general case follows by using that $G_a$ factors through $\{\text{Add2}\}$ and applying a conjugation by $1 + q \in GL(R)$. The case $\{i,k\} = \{l,m\}$ follows by symmetry. If $\{i,j\} = \{l,m\}$, then for $r \in e_j Re_o$ and $b \in e_o Re_k$ we have

$$G_a(z_{jk}(rb,q) x_{ji}(-rbp) x_{ki}(-qrbp)) = x_{\infty}(q) x_{\infty}(b) x_{\infty}(-b)$$

so

$$G_a(z_{i\oplus j,k}(a,b;p,q)) = z_{\infty}(a + b, p + q) \quad \text{(G4)}$$

and $G_a$ factors through the remaining case of $\{\text{Sym}\}$. Consequently, it factors through $\{\text{Add3}\}$.

Now it is easy to see that $G_a$ factors through $\{\text{Conj2}\}$ in the form

$$G_a(z_{i\oplus j,k}(a,b;p,q)) = G_a(z_{i\oplus j,k}(e_i u(a+b), e_j u(a+b); (p+q)u^{-1} e_i, (p+q)u^{-1} e_j))$$
for \( g \in D_{ij} \) and \( u = \text{st}(g) \). If \( \{i, j\} = \{l, m\} \), this follows from (G4), and the other two cases follows from (G2) and (G3) by using symmetry and that \( G_\alpha \) factors through \( \text{Ad}_d \).

It remains to prove that \( G_\alpha \) factors through \( \text{HW} \). The cases with \( p = 0 \) or \( q = 0 \) formally follow from the other relations, and the cases \( \{i, j\} = \{l, m\} \), \( \{j, k\} = \{l, m\} \) follow from them by conjugation using \( \text{C1} \) and \( \text{C2} \). Finally, if \( \{i, k\} = \{l, m\} \), \( s \in e_iR e_o \), \( a \in e_oR e_k \), then

\[
G_\alpha(z_{ij,k}(sa, qsa; r - pq, p)) = 1 + r - pq(x_{ij}(qsa, p) x_{ik}(sa) x_{kj}(-sap)) = 1 + r - pq(x_{ij}(p) x_{io}(a) x_{jo}(q) x_{io}(a)) x_{io}(a) x_{io}(a) = F_{\Psi/\alpha}(1 + p + q + r)(x_{io}(a) x_{io}(a) x_{io}(a)) = 1 + r(x_{ij}(q) x_{io}(a) x_{io}(a) x_{io}(a)) = G_\alpha(z_{ij,k}(sa, qsa; r - pq, p)).
\]

Hence \( G_\alpha: \text{St}(R; A; \Phi) \to \text{St}(R, A; \Phi/\alpha) \) is well-defined. But \( F_\alpha \) is surjective by lemma 4 and obviously \( G_\alpha \circ F_\alpha \) is the identity, so \( F_\alpha \) is an isomorphism with the inverse \( G_\alpha \). The second claim follows from \( G_\alpha(z_{lm}(a, p)) = 1 + p G_\alpha(x_{lm}(a)) \).

Now we are ready to prove our first main result.

**Theorem 1.** Let \( R \) be a ring with a complete family of \( n \) full orthogonal idempotents, \( A \) be a non-unital \( R \)-algebra. Then the homomorphism \( \mu: \text{St}(R, A) \to \text{St}'(R, A) \) is surjective for \( n \geq 3 \) and bijective for \( n \geq 4 \). If \( d: A \to R \) is a crossed module, then the induced homomorphism \( \text{St}(R, A) \to \text{St}(R, A) \) is also surjective or bijective.

**Proof.** By lemma 4 it suffices to consider the case of a crossed module. Suppose that \( n \geq 3 \) and take \( g \in \text{St}(R, A) \). Surjectivity means that \( x_{ij}(p) \mu(g) \in \text{St}(R, A) \) lies in the image of \( \mu \) for all generators \( x_{ij}(p) \in \text{St}(R) \). But this follows from lemma 7 applied to the root \( e_i - e_j \).

Now suppose that \( n \geq 4 \). We construct an action of \( \text{St}(R) \) on \( \text{St}(R, A) \) as follows. For a root \( \alpha = e_i - e_j \) and \( p \in e_iR e_j \) let

\[
x_{ij}(p) F_\alpha(g) = F_\alpha(1 + p, g),
\]

this gives an automorphism of \( \text{St}(R, A) \) by lemma 8. Hence we have a homomorphism \( \text{Ad}: \text{St}(R, A) \to \text{Aut}(\text{St}(R, A)) \), where \( \text{St}^\ast(R) \) is the free group with generators \( x_{ij}(p) \).

If \( \Psi \subseteq \Phi \) is a root subsystem of rank at most 2, then any element from \( \langle x_\alpha(R_\alpha) \mid \alpha \in \Psi \rangle \leq \text{St}(R) \) acts on the image of \( F_\Phi \) independently on its decomposition into a product of generators with roots in \( \Psi \), also \( F_\Phi: \text{St}(R, A; \Phi/\Psi) \to \text{St}(R, A; \Phi) \) is surjective by lemma 7. Hence \( \text{Ad} \) is well-defined on \( \text{St}(R) \).
Obviously, $\mu: \overline{\text{St}}(R, A) \to \text{St}(R, A)$ is $\text{St}(R)$-equivariant. Also, there is a homomorphism

$$\nu: \text{St}(R, A) \to \overline{\text{St}}(R, A), \theta_{ij}(a) \mapsto \theta_{ij}(a).$$

By the second claim of lemma $\Box \nu \circ \mu$ is the identity. Hence $\mu$ is an isomorphism.  

6 Relative simply laced Steinberg groups

In this section $K$ is a commutative unital ring and $a$ is a non-unital commutative $K$-algebra (satisfying the identity $ap = pa$ for $a \in a$ and $p \in K$). Let $\Phi$ be a crystallographic reduced irreducible simply laced root system, i.e. either $A_\ell$ for $\ell \geq 1$, or $D_\ell$ for $\ell \geq 4$, or $E_\ell$ for $\ell \in \{6, 7, 8\}$. A corresponding Steinberg group $\text{St}(\Phi, K)$ is the abstract group with generators $x_\alpha(p)$ for $\alpha \in \Phi$, $p \in K$ and the relations

$$x_\alpha(p)x_\alpha(q) = x_\alpha(p + q); \quad (\text{St}1)$$
$$[x_\alpha(p), x_\beta(q)] = x_{\alpha + \beta}(N_{\alpha\beta} p q) \text{ for } \alpha + \beta \in \Phi; \quad (\text{St}2)$$
$$[x_\alpha(p), x_\beta(q)] = 1 \text{ for } \alpha + \beta \notin \Phi \cup \{0\}. \quad (\text{St}3)$$

Here $N_{\alpha\beta} \in \{-1, 1\}$ are the so-called structure constants, they are determined by the corresponding Chevalley group scheme over $\text{Spec}(\mathbb{Z})$ up to a choice of parametrizations of the root subgroups. The unrelativized Steinberg group $\text{St}(\Phi, a)$ is defined in the same way, but with the parameters in $a$.

We define a group $\text{St}'(\Phi; K, a)$ as the group with an action of $\text{St}(\Phi, K)$ generated by the elements $x_\alpha(a)$ for $\alpha \in \Phi$, $a \in a$ satisfying the Steinberg relations and

$$x_\alpha(p)x_\beta(a) = x_\beta(a) \text{ for } \alpha + \beta \notin \Phi \cup \{0\}; \quad (\text{Rel}1)$$
$$x_\alpha(p)x_\beta(a) = x_\beta(a) x_{\alpha + \beta}(N_{\alpha\beta} a p) \text{ for } \alpha + \beta \in \Phi. \quad (\text{Rel}2)$$

If $d: a \to K$ is actually a crossed module, then the relative Steinberg group $\text{St}(\Phi; K, a)$ is the factor-group of $\text{St}'(\Phi; K, a)$ by

$$x_\alpha(d(a))g = x_\alpha(a) g x_\alpha(-a) \text{ for any } g. \quad (\text{Rel}3)$$

As in the linear case, there is an isomorphism $\text{St}'(\Phi; K, a) \cong \text{St}(\Phi; a \rtimes K, a)$ and a decomposition $\text{St}(\Phi; a \rtimes K) \cong \text{St}'(\Phi; K, a) \rtimes \text{St}(\Phi, K)$. If $d: a \to K$ is a crossed module, then $\text{St}(\Phi; K, a) \to \text{St}(\Phi, K), x_\alpha(a) \mapsto x_\alpha(d(a))$ is a group-theoretical crossed module and the sequence

$$\text{St}(\Phi; K, a) \to \text{St}(\Phi, K) \to \text{St}(\Phi, K/d(a)) \to 1$$
is exact. Again as in the linear case, we use the elements

\[
\begin{align*}
\alpha(a, p) &= x^{\alpha(p)} x^{\alpha(a)}; \\
\beta(a, b; p) &= \alpha(a, p) x^{\alpha} x^{\beta} (N - \alpha, \beta b p) \\
&= x^{\alpha(p)} (x^{\alpha(a)} x^{\alpha}) (x^{\beta(b)}) \quad \text{for } \alpha, \beta \in \Phi; \\
\alpha\beta(a, b; p, q) &= \alpha(a - \beta) (a, N - \beta, \alpha a q p) \beta(a, \beta, \alpha (b, N - \alpha, \beta b p; q) \\
&= x^{\alpha(p)} x^{\beta(q)} (x^{\alpha(a)} x^{\beta(b)}) \quad \text{for } \alpha, \beta \in \Phi.
\end{align*}
\]

in the group \(St'(\Phi; K, a)\), where \(\alpha, \beta, \alpha - \beta \in \Phi\).

In the case \(\Phi = A_{\ell}\), the ring \(M(\ell + 1, a)\) is naturally a non-unital \(M(\ell + 1, K)\)-algebra. The ring \(M(\ell + 1, K)\) has a canonical family of full orthogonal idempotents \(e_1, \ldots, e_{\ell+1}\). There are canonical isomorphisms

\[
\begin{align*}
St(A_{\ell}, K) &\to St(M(\ell + 1, K), x_{e_i - e_j}(p) \mapsto x_{ij}(pe_{ij}); \\
St(A_{\ell}, a) &\to St(M(\ell + 1, a), x_{e_i - e_j}(a) \mapsto x_{ij}(ae_{ij}); \\
St'(A_{\ell}; K, a) &\to St'(M(\ell + 1, K), M(\ell + 1, a), z_{a - e_j}(a, p) \mapsto z_{ij}(ae_{ij}, pe_{ij})
\end{align*}
\]

If \(d: a \to K\) is a crossed module, then \(d: M(\ell + 1, a) \to M(\ell + 1, K)\) is also a crossed module, so there is a canonical isomorphism

\[St(A_{\ell}; K, a) \cong St(M(\ell + 1, K), M(\ell + 1, a)).\]

Of course, \(St(M(\ell + 1, K)) = St(M(\ell + 1, K), A_{\ell})\) in our notation from section 9.

**Lemma 9.** The structure constants satisfy the relations

\[
N_{\alpha\beta} = -N_{\beta\alpha} = -N_{-\alpha, \beta} = N_{-\beta, \alpha} = N_{\beta, \alpha} = N_{-\alpha, \beta, \alpha}
\]

if \(\alpha, \beta, \alpha + \beta \in \Phi\).

**Proof.** This is precisely [7] (14.2)–(14.6)] (recall that \(N_{\alpha\beta} \in \{-1, 1\}\) in our case).

Let \(G\) be the family of all root subsystems of \(\Phi\) of types \(A_1\) and \(A_3\) ordered by inclusion. For every \(\Psi \in G\) there are natural \(St(\Psi, K)\)-equivariant homomorphisms

\[St'(\Psi; K, a) \to St'(\Phi; K, a), \quad St(\Psi; K, a) \to St(\Phi; K, a)\]

given by \(x_{\alpha}(a) \mapsto x_{\alpha}(a)\).

**Lemma 10.** If the rank of \(\Phi\) is at least 3, then

\[St'(\Phi; K, a) = \text{colim}_{\Psi \in G} St'(\Psi; K, a), \quad St(\Phi; K, a) = \text{colim}_{\Psi \in G} St(\Psi; K, a).\]

Here both colimits are taken in the category of groups.
Proof. By \cite{5} theorem 9] the second equality holds if a is an ideal of K. Hence the first equality holds for arbitrary non-unital commutative K-algebra a. To prove the second equality in full generality, note that St'(Ψ; K, a) is generated by the elements \( z_\alpha(a, p) \) (for example, by \cite{3} lemma 4). Hence St(Ψ; K, a) is the factor-group of St'(Ψ; K, a) by the identities \( x_\alpha(d(a))z_\alpha(b, p) = x_\alpha(a)z_\alpha(b, p)x_\alpha(-a) \) and \( z_\alpha(b, p + d(a)) = x_\alpha(a)z_\alpha(b, p)x_\alpha(-a) \). Both of them already hold in St(\{−\alpha, α\}; K, a).

In our second main result we actually do not need all the identities from the next lemma. But they are useful to prove that various homomorphisms from the relative Steinberg group are well-defined as in the proof of lemma 8.

Lemma 11. The elements \( z_\alpha(a, p) \) satisfy the relations

\[
\begin{align*}
z_\alpha(a + a', p) &= z_\alpha(a, p)z_\alpha(a', p); \quad \text{(Add1)} \\
z_\alpha(z\alpha(b, p)) &= z_\alpha(a, b; p)z_\alpha(a', b; p); \quad \text{(Add2)} \\
z_\alpha\oplus(b, p, q) &= z_\alpha\oplus(a, b, p, q); \quad \text{(Add3)} \\

\end{align*}
\]

\[
\begin{align*}
[x_\alpha(-\beta)(\alpha, b)]x_\alpha(a, p), x_\beta(b)x_\alpha(-\delta b)] &= z_\alpha(a, p)z_\alpha(b, p); \quad \text{for } \alpha + \beta \in \Phi; \quad \text{(Mult)} \\
[z_\alpha(a, p), z_\alpha(b, p)] &= 1 \text{ for } \alpha \perp \beta; \quad \text{(Dis)} \\
z_\alpha\oplus(a, b, p, q) &= z_\alpha\oplus(a, b; p, q); \quad \text{(Sym)} \\

\end{align*}
\]

\[
\begin{align*}
&x_\alpha(a, p), x_\beta(b)x_\alpha(-\delta b)]z_\alpha(a, p)z_\alpha(b, p); \quad \text{for } \alpha + \beta \in \Phi \\
&z_\alpha(a, b, p, q) = z_\alpha(a, b; p, q); \quad \text{(Conj2)} \\
&z_\alpha(a, b, p, q) = z_\alpha(a, b; p, q); \quad \text{(Conj2')}
\end{align*}
\]

in St'(R, A), where \( \varepsilon = N_{\alpha\beta} \) is a structure constant. If \( d: a \to K \) is a crossed module, then they also satisfy

\[
\begin{align*}
&z_\alpha(a, p + d(b)) = x_\alpha(a, -b)); \quad \text{(Rel4)} \\
&z_\alpha(a, b; p, q) = z_\alpha(a, b; p, q); \quad \text{(Conj2')}
\end{align*}
\]

Proof. This directly follows from lemma 8.
relations \((\text{Add}1), (\text{Dis}), (\text{Conj}2), (\text{HW})\). If \(d: a \to K\) is a crossed module, then \(\text{St}(\Phi; K, a)\) is the factor-group of \(\text{St}'(\Phi; K, a)\) by \((\text{Rel}4)\). In this presentation of \(\text{St}(\Phi; K, a)\) the axiom \((\text{Conj}2)\) may be replaced by \((\text{Conj}2')\).

**Proof.** Let \(\overline{\text{St}}(\Phi; K, a)\) be the abstract group from the statement with a canonical homomorphism
\[
\overline{\text{St}}(\Phi; K, a) \to \text{St}'(\Phi; K, a)
\]
from lemma 11. It is bijective if \(\Phi = A_3\) by theorem 1.

Let us show that \(\overline{\text{St}}(\Phi; K, a) = \text{colim}_{\Psi \in G} \text{St}'(\Psi; K, a)\) canonically maps to \(\overline{\text{St}}(\Phi; K, a)\) if \(\Psi\) is of type \(A_3\). It suffices to show that if \(\Psi, \Psi' \subseteq \Phi\) are of type \(A_2\), then the two images of \(g\) in \(\overline{\text{St}}(\Psi; K, a)\) coincide. By \([5, \text{lemma } 2]\) we may assume that \(\Psi \cap \Psi'\) is of type \(A_2\). Then the image of \(g\) in \(\overline{\text{St}}(\Psi \cap \Psi'; K, a)\) lies in the image of \(\overline{\text{St}}(\Psi \cap \Psi'; K, a) \to \text{St}'(\Psi \cap \Psi'; K, a)\) by theorem 1. This implies the claim.

Hence \(\overline{\text{St}}(\Phi; K, a) \to \text{St}'(\Phi; K, a)\) is an isomorphism. The proof for any of the two presentations of \(\text{St}(\Phi; K, a)\) is the same. 

The theorem gives a slight strengthening of lemma 11: all relations between the generators \(z_{ij}(a, p)\) actually comes from root subsystems of ranks 1 and 2 (i.e. \(A_1, A_1 \times A_1\), and \(A_2\)).

## 7 Factoring out transvections

In this section we study the factor-groups of \(\text{St}(R, A)\) and \(\text{St}(\Phi; K, a)\) by the images of \(\text{St}(A)\) and \(\text{St}(\Phi; a)\).

In the linear case the group \(\text{St}(A)\) usually cannot be a crossed module over \(\text{St}(R)\) satisfying natural properties: its image in \(\text{GL}(R)\) is a subgroup of
\[
\{ g \in \text{GL}(R) \mid e_i g e_j \in A \text{ for } i \neq j, e_i g e_i \in A^2 \},
\]
so this image does not contain \(x_{ij}(p) x_{ij}(a)\) for generic \(p\) and \(a\). Similarly, the unrelativized Steinberg group is not invariant under root elimination, i.e. \(F_{\Phi}: \text{St}(A, \Phi/\Psi) \to \text{St}(A, \Phi)\) is not surjective in general even if the rank of \(\Phi/\Psi\) is large.

However, it turns out that at least the image of \(\text{St}(A)\) in \(\text{St}(R, A)\) is normal. In the following theorem we use \(\overline{\text{St}}(R, A)\) instead of \(\text{St}(R, A)\) to cover the case \(n = 3\).

**Theorem 3.** Let \(R\) be a ring with a complete family of \(n \geq 3\) full orthogonal idempotents, \(d: A \to R\) be a crossed module. Then the image of \(\text{St}(A) \to \overline{\text{St}}(R, A), x_{ij}(a) \to x_{ij}(a)\) is normal with abelian factor-group. This factor-group is the abelian group with generators \(z_{ij}(a, p)\) (the images of \(z_{ij}(a, p)\)) and rela-
Proof. Let $N \leq \overline{\text{St}}(R, A)$ be the image of $\text{St}(A)$, i.e. the subgroup generated by all $x_{ij}(a)$. First of all, we have to show that $N$ is normal. Up to transposition this follows from $\text{Dis}$, $\text{Conj}$, $\text{Mult}$, and the following identities (they are special cases of $\text{Conj2}$):

$$z_{ij}(a, r) = z_{ij}(a, p) + z_{ij}(b, q) - z_{ij}(-c, r) = z_{ik}(a, p) + z_{jk}(b, q);$$
$$z_{ik}(a, r - pq) + z_{jk}(qa, p) = z_{ij}(-ap, q) + z_{ik}(a, r).$$

We use the additive notation for $\overline{\text{St}}(R, A)/N$. The identity (FT1) is the same as $\text{Add1}$. By (Mult), $z_{ij}(ab, p) \in N$ for $a \in E_i A$ and $b \in A e_j$. Since $e_k$ is a full idempotent, (FT3) follows. The identity (FT4) follows from (Rel4).

By (FT1) and (FT3), the images of $\text{Conj}$ and $\text{HW}$ in $\overline{\text{St}}(R, A)/N$ are

$$z_{ij}(c, r) + z_{ik}(a, p) + z_{jk}(b, q) - z_{ij}(-c, r) = z_{ik}(a, p) + z_{jk}(b, q);$$
$$z_{ik}(a, r - pq) + z_{jk}(qa, p) = z_{ij}(-ap, q) + z_{ik}(a, r).$$

Hence any two generators $z_{ij}(a, p)$ and $z_{kj}(b, q)$ commute unless $\{i, j\} = \{k, l\}$. Taking $r = pq$ in the image of $\text{HW}$, we get (FT5). Hence the image of $\text{HW}$ may be written as

$$z_{ik}(a, r - pq) + z_{ik}(a, pq) = z_{ik}(a, r),$$

and (FT2) follows. The commutativity of $\overline{\text{St}}(R, A)/N$ follows from (FT2), (FT5), and the already proved cases of commutativity.

Conversely, it is easy to see that the images of $\text{Add1}$, $\text{Dis}$, $\text{Conj}$, $\text{HW}$, $\text{Rel4}$ follow from (FT1)–(FT5). ☐

**Theorem 4.** Let $K$ be a commutative unital ring, $d: a \rightarrow K$ be any of the root systems $\mathbf{A}_i, D_i, E_i$ with $l \geq 3$. Then the image of $\text{St}(\Phi, a) \rightarrow \text{St}(\Phi; K, a), x_\alpha(a) \rightarrow x_\alpha(a)$ is normal with abelian factor-group. This factor-group is the abelian group with generators $z_\alpha(a, p)$ (the images of $z_\alpha(a, p)$) and relations

$$z_\alpha(a + b, p) = z_\alpha(a, p) + z_\alpha(b, p);$$
$$z_\alpha(a, p + q) = z_\alpha(a, p) + z_\alpha(a, q);$$
$$z_\alpha(ab, p) = 0 \text{ for } a, b \in a;$$
$$z_\alpha(a, d(b)) = 0;$$
$$z_{\alpha + \beta}(a, pq) = z_\alpha(ap, q) + z_\beta(qa, p) \text{ if } \alpha + \beta \in \Phi.$$
Proof. This follows from lemma \[10] and theorem \[3]. The structure constants in all summands in \( (FT5) \) are the same, so they may be omitted by \( (FT1) \) and \( (FT2) \). □

In conclusion let us list several further problems.

- Give an explicit presentation of relative unitary Steinberg groups and relative Steinberg groups of type \( F_4 \).

- Give an explicit presentation of birelative Steinberg groups such as \( \text{St}(\Phi; K, a) \otimes \text{St}(\Phi; K, b) \). Here \( \otimes \) means the non-abelian tensor product of crossed modules over \( \text{St}(\Phi, K) \). If \( \Phi \) is simple laced of rank at least 2, then by \[10\] the image of this tensor product in the simply connected Chevalley group \( G(\Phi, R) \) (i.e. the commutator \( [E(\Phi; K, a), E(\Phi; K, b)] \)) is generated by the image of \( \text{St}(\Phi; K, a \otimes b) \) and the elementary commutators \( y_{\alpha}(a, b) = [x_{\alpha}(a), x_{-\alpha}(b)] \) for \( a \in a \) and \( b \in b \) (they are the images of \( x_{\alpha}(a) \otimes x_{-\alpha}(b) \)). Moreover, there are nice relations between \( y_{\alpha}(a, b) \) modulo \( E(\Phi; K, ab) \). See also \[8, 9, 11\] for another such results for elementary groups.

- Calculate explicitly the relative linear Steinberg group \( \text{St}(R, A) \) and the kernel \( K_2(R, A) = \text{Ker}(\text{St}(R, A) \to \text{GL}(A)) \) if \( A \) is contained in the Jacobson radical of \( A \rtimes R \), and similarly for simply laced Steinberg groups. Such a description exists for the relative \( K_2 \)-functor over a commutative ring \( (4) \) and modulo some subgroup for the relative matrix linear Steinberg group over any ring \( (11) \) in the case \( A^2 = 0 \).

- Find some natural factor-group of \( \text{St}(A) \) with a structure of a crossed module over \( \text{St}(R, A) \), and similarly for simply laced Steinberg groups. By theorems \[3\] and \[4\] the image of the unrelativized Steinberg group is normal in the relative Steinberg group, so at least some such factor-group exists (i.e. the image itself), although we do not know its explicit presentation. The unrelativized Steinberg group cannot be itself a crossed module in general: consider the field \( \mathbb{F}_2 \) with a crossed module \( 0: V \to \mathbb{F}_2 \) over it for some finite dimensional vector space \( V_{\mathbb{F}_2} \). The group \( \text{St}(A; \mathbb{F}_2, V) \) is finite for \( \ell \geq 2 \) (for example, by \[4\]), hence every finitely generated crossed module over it is of polynomial growth. But the unrelativized group \( \text{St}(A; V) \) (with zero product on \( V \)) is the \( \frac{\ell(\ell+1)}{2} \)-th direct power of the free product \( V * V \), so it has exponential growth if \( \dim(V) > 1 \).

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