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S-AMP: Approximate Message Passing for General Matrix Ensembles

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Motivation

• **Low complexity and near optimal** inference algorithms for linear observation models

\[ y = Ax + \epsilon, \quad \epsilon \sim \mathcal{N}(\epsilon|0, \sigma^2 I), \quad x \sim \prod_{k=1}^{K} p_k(x_k) \]

• **A**: $N \times K$, known and **drawn from known matrix ensemble**

• $N, K \gg 1$
Loopy BP

\[ p(y_1|(Ax)_1) \quad p(y_2|(Ax)_2) \quad p(y_3|(Ax)_3) \]

\[ m_{n \rightarrow k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l \neq k} m_{l \rightarrow n}(x_l) \, dx_l \]

\[ m_{l \rightarrow n}(x_l) \cong p(x_l) \prod_{m \neq n} m_{m \rightarrow l}(x_l) \]
Loopy BP

\[
p(y_1|(Ax)_1) \quad p(y_2|(Ax)_2) \quad p(y_3|(Ax)_3)
\]

- Local Cavity Argument:
  \[
h_{n,k} = \sum_{l \neq k} A_{nl} x_l, \quad x_l \sim m_{l \rightarrow n}(x_l)
\]
- Due to CLT, \( h_{n,k} \) is approximated by Gaussian.
- This leads Loop BP to Loopy EP

\[
m_{n \rightarrow k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l \neq k} m_{l \rightarrow n}(x_l) \, dx_l
\]

\[
m_{l \rightarrow n}(x_l) \cong p(x_l) \prod_{m \neq n} m_{m \rightarrow l}(x_l)
\]
Loopy EP

\[ m_{n \rightarrow k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l \in \mathcal{K} \setminus k} m_{l \rightarrow n}(x_l) \, dx_l \]

\[ m_{l \rightarrow n}(x_l) = \exp \left( -\frac{\lambda_{l \rightarrow n}}{2} x_l^2 + \gamma_{l \rightarrow n} x_l \right) \]
Loopy EP

Define

\[ q_k(x_k) \approx p_k(x_k) \prod_{n \in \mathcal{N}} m_{n \to k}(x_k) \]

Let \( \tilde{q}_k(x_k) \triangleq N(x_k | \mu_k, \sigma_k^2) \) such that

\[
\begin{align*}
\mu_k &= \mathbb{E}[x_k | q(x_k)] \\
\sigma_k^2 &= \mathbb{V}ar[x_k | q(x_k)]
\end{align*}
\]

Then loopy EP update rule is

\[
\begin{align*}
m_{n \to k}(x_k) &= \int p(y_n | (Ax)_n) \prod_{l \in \mathcal{K} \setminus k} m_{l \to n}(x_l) dx_l \\
m_{l \to n}(x_l) &= \exp \left( -\frac{\lambda_{l \to n}}{2} x_l^2 + \gamma_{l \to n} x_l \right)
\end{align*}
\]

\[
m_{k \to n}(x_k) = \frac{\tilde{q}_k(x_k)}{\prod_{n \in \mathcal{N}} m_{n \to k}(x_k)}
\]
Approximate message passing (AMP) [Donoho et al 2009]

- Assume $A_{nk}$ zero mean-iid, $\overline{A_{nk}^2} = 1/N$, $N, K \to \infty$, $\alpha \equiv N/K$ finite
- Reduces the number of messages to $N + K$ means.

$$
\mu^{t+1} = \eta_t \left( A^T z^t + \mu^t \right)
$$

$$
z^t = y - A\mu^t + \frac{1}{\alpha} \left\langle \eta^t_{t-1} \left( A^T z^{t-1} + \mu^{t-1} \right) \right\rangle z^{t-1}
$$

with $\langle u \rangle \triangleq \sum_{k=1}^{K} u_k / K$.

- $\eta_t(\kappa_k)$ and $\eta^t_t(\kappa_k)/\tau$ are (in some cases) the mean and variance of (Krzakala et al 2012)

$$
q_k(x_k) \approx p_k(x_k) N(x_k | \kappa_k, 1/\tau)
$$

- Return to $\tau$ when discussing EP and S-AMP
Define
\[ q_k(x_k) \equiv p_k(x_k)m_{N\rightarrow k}(x_k) \]

Let \( \tilde{q}_k(x_k) = N(x_k|\mu_k, \sigma_k^2) \) such that
\[ \mu_k = \mathbb{E}[x_k|q_k(x_k)] \]
\[ \sigma_k^2 = \text{Var}[x_k|q(x_k)] \]

Then EP update rule is
\[ m_{N\rightarrow k}(x_k) = \int p(y|Ax) \prod_{l \in K \setminus k} m_{l \rightarrow N}(x_l) \, dx_l \]
\[ m_{k \rightarrow N}(x_k) = \exp \left( -\frac{\Lambda_{kk}}{2} x_k^2 + \gamma_k x_k \right) \]
S-AMP

• generalizes AMP for arbitrary (orthogonally invariant) matrix ensembles.

\[ \mu^{t+1} = \eta_t \left( A^\dagger z^t + \mu^t \right) \]

\[ z^t = y - A\mu^t + \left( 1 - \frac{1}{S_{A}^{t-1}} \right) z^{t-1} \]

\[ s_{A}^{t-1} \triangleq S_A \left( -\left\langle \eta_{t-1}'(A^\dagger z^{t-1} + \mu^{t-1}) \right\rangle \right) \]

\( S_A \) denotes the S-transform (in free probability theory) of the limiting eigenvalue distribution (LED) of \( A^\dagger A \).

• Indeed when the entries of \( A \) be iid with zero mean variance \( 1/N \):

\[ S_A(\omega) = \frac{1}{1 + \omega/\alpha} \]

which yields AMP iteration steps.
EP→S-AMP: Start with EP Update Rule

Let \( J = A^\dagger A / \sigma^2 \) and \( \theta = A^\dagger y / \sigma^2 \). Define

\[
\Sigma = (\Lambda + J)^{-1}\quad \mu = \Sigma(\theta + \gamma)
\]

Then we have

\[
m_{N\to k}(x_k) = \exp \left\{ -\frac{1}{2} \left( \frac{1}{\Sigma_{kk}} - \Lambda_{kk} \right) x_k^2 + \left( \frac{\mu_k}{\Sigma_{kk}} - \gamma_k \right) x_k \right\}
\]
EP $\rightarrow$ S-AMP: Use AMP Notations

- Let $\tau_k = \frac{1}{\Sigma_{kk}} - \Lambda_{kk}$ and $\kappa_k = \left( \frac{\mu_k}{\Sigma_{kk}} - \gamma_k \right) / \tau_k$.
- Hence we can write
  \[ m_{N \rightarrow k}(x_k) \cong N(x_k | \kappa_k, 1/\tau_k) \]
- Write $q_k(x_k)$ in the form of
  \[ q_k(x_k) = \frac{p_k(x_k)N(x_k | \kappa_k, 1/\tau_k)}{Z(\kappa_k, \tau_k)} \]
- Define
  \[ \eta(\kappa_k; \tau_k) \triangleq \kappa_k + \frac{1}{\tau_k} \frac{\partial \log Z(\kappa_k, \tau_k)}{\partial \kappa_k} \]
  \[ \eta'(\kappa_k; \tau_k) \triangleq \frac{\partial \eta(\kappa_k; \tau_k)}{\partial \kappa_k} \]

where $\eta(\kappa_k; \tau_k)$ and $\eta'(\kappa_k; \tau_k)/\tau_k$ are respectively the mean and the variance of $q_k(x_k)$ [Krzakala et.al. 2012].
EP → S-AMP: Move to ADATAP [Opper and Winther 2001]

- Note that

\[ \mu = (\Lambda + J)^{-1}(\gamma + \theta) \iff (\Lambda + J)\mu = \gamma + \theta \]
EP $\rightarrow$ S-AMP: Move to ADATAP [Opper and Winther 2001]

- Note that
  \[ \mu = (\Lambda + J)^{-1}(\gamma + \theta) \iff (\Lambda + J)\mu = \gamma + \theta \]

- Putting everything together leads EP to
  \[ \mu_k = \eta(\kappa_k; \tau_k) \]
  \[ \kappa_k = \frac{1}{\tau_k\sigma^2} \sum_{n \in \mathcal{N}} A_{nk} \left( y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right) + \mu_k \]
  \[ \tau_k = \frac{1}{\sum_{kk}} - \Lambda_{kk}, \quad \Lambda_{kk} = \frac{\tau_k}{\eta' (\kappa_k; \tau_k)} - \tau_k \]

  exactly coincides ADATAP for the linear observation models.
Define

\[ z_{n,k} \triangleq \frac{1}{\tau_k \sigma^2} \left( y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right). \]

Using this definition we “devise” the following identity:

\[ z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k}. \]
EP→ S-AMP: Apply Adaptive Damping

Define

$$z_{n,k} \triangleq \frac{1}{\tau_k \sigma^2} \left( y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right).$$

Using this definition we “devise” the following identity:

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k}.$$

Doing so leads to

$$\mu_k = \eta \left( \sum_{n \in \mathcal{N}} A_{nk} z_{n,k} + \mu_k; \tau_k \right)$$

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k}$$

$$\tau_k = \frac{1}{\sum_{kk} - \Lambda_{kk}}, \quad \Lambda_{kk} = \frac{\tau_k}{\eta' (K_k; \tau_k)} - \tau_k$$

This equations can be thought as a finite size interpretation of AMP.
EP → S-AMP: Invoke Self Averaging Ansatz [Opper and Winther 2001]

• We can recover self-averaging matrix ensembles $\tau_k \rightarrow \tau$:

$$\Sigma_{kk} = \left[(\Lambda + J)^{-1}\right]_{kk} = \frac{\partial}{\partial \Lambda_{kk}} \ln \det(\Lambda + J)$$

• by using

$$\frac{1}{K} \ln \det(\Lambda + J) \rightarrow \frac{1}{K} \mathbb{E}_J \left[\ln \det(\Lambda + J)\right] \quad \text{for} \quad K \rightarrow \infty$$
We can recover self-averaging matrix ensembles \( \tau_k \rightarrow \tau \):
\[
\Sigma_{kk} = [(\Lambda + J)^{-1}]_{kk} = \frac{\partial}{\partial \Lambda_{kk}} \ln \det(\Lambda + J)
\]

by using
\[
\frac{1}{K} \ln \det(\Lambda + J) \rightarrow \frac{1}{K} \mathbb{E}_J [\ln \det(\Lambda + J)] \quad \text{for} \quad K \rightarrow \infty
\]

Doing so leads \( \tau_k \) to \( \tau \) that is the solution of
\[
\sigma^2 \tau = \frac{1}{\sigma^2} R_A \left( -\frac{\langle \eta' (A^\dagger z + \mu; \tau) \rangle}{\sigma^2 \tau} \right)
\]

\( R_A \) is the R-transform (in free probability theory) of the LED of \( A^\dagger A \) and
\[
z = y - A \mu + (1 - \sigma^2 \tau) z
\]
EP → S-AMP: Move to S-transform

• Recall that

\[ \sigma^2 \tau = R_A \left( \frac{-< \eta' (A^\dagger z + \mu; \tau)}{\sigma^2 \tau} \right) \]

• By invoking the fact [Haagerup and Larsen 2001]

\[ S_A(\omega) = \frac{1}{R_A(\omega S_A(\omega))} \]

• we have

\[ \sigma^2 \tau = \frac{1}{S_A(-< \eta' (A^\dagger z + \mu; \tau)>)} \]
EP → S-AMP: Move to S-transform

• Recall that

\[ \sigma^2 \tau = R_A \left( \frac{-< \eta'(A^\dagger z + \mu; \tau)}{\sigma^2 \tau} \right) \]

• By invoking the fact [Haagerup and Larsen 2001]

\[ S_A(\omega) = \frac{1}{R_A(\omega S_A(\omega))} \]

• we have

\[ \sigma^2 \tau = \frac{1}{S_A \left(-< \eta'(A^\dagger z + \mu; \tau)\right)} \]

• this completes the mapping at "fixed points":

\[ \mu = \eta \left( A^\dagger z + \mu; \tau \right) \]

\[ z = y - A\mu + \left(1 - \frac{1}{S_A}\right)z \]

\[ s_A = S_A \left(-\langle \eta'(A^\dagger z + \mu; \tau)\rangle\right) \]
What is S-AMP?

- In summary

\[
\begin{align*}
\mu^{t+1} &= \eta_t \left( A^\dagger z^t + \mu^t \right) \\
z^t &= y - A\mu^t + \left( 1 - \frac{1}{S_{A^{t-1}}} \right) z^{t-1} \\
S_{A^{t-1}}^{t-1} &\triangleq S_{A^\dagger A} \left( -\langle \eta'_{t-1}(A^\dagger z^{t-1} + \mu^{t-1}) \rangle \right)
\end{align*}
\]

where \( \eta_t(x^t) = \eta(x^t; \tau^t) \) and

\[
\tau^t = \frac{1}{\sigma^2 S_A \left( -\langle \eta'(A^\dagger z^t + \mu^t; \tau^t) \rangle \right)}
\]

- Oops, S-AMP includes a hard fixed point equation.
- As a matter of fact we don’t know what is the best update rule for \( \tau^t \)
A Variant of S-AMP

• By making analogy with the state evolution formula [Bayati and Montani 2011]

\[ \mu^{t+1} = \eta \left( A^\dagger z^t + \mu^t; \tilde{\tau}^t \right) \]

\[ z^t = y - A\mu^t + \left( 1 - \frac{1}{s^{t-1}_A} \right) z^{t-1} \]

\[ s^{t-1}_A \triangleq S_A \left( -\langle \eta' (A^\dagger z^{t-1} + \mu^{t-1}; \tilde{\tau}^{t-1}) \rangle \right) \]

where \( \tilde{\tau}^t \) is updated by using the solution

\[ \tilde{\tau}^t = \frac{1}{\sigma^2 S_A \left( -\tilde{\tau}^t / \tilde{\tau}^{t-1} \langle \eta' (A^\dagger z^{t-1} + \mu^{t-1}; \tilde{\tau}^{t-1}) \rangle \right)} \]
A Variant of S-AMP

• By making analogy with the state evolution formula [Bayati and Montari 2011]

\[
\begin{align*}
\mu^{t+1} &= \eta \left( A^\dagger z^t + \mu^t; \tilde{\tau}^t \right) \\
z^t &= y - A\mu^t + \left( 1 - \frac{1}{s_{A}^{t-1}} \right) z^{t-1} \\
s_{A}^{t-1} &\triangleq S_{A} \left( - \langle \eta' \left( A^\dagger z^{t-1} + \mu^{t-1}; \tilde{\tau}^{t-1} \right) \rangle \right)
\end{align*}
\]

where \( \tilde{\tau}^t \) is updated by using the solution

\[
\bar{\tau}^t = \frac{1}{\sigma^2 S_{A} \left( - \frac{\tilde{\tau}^t}{\tilde{\tau}^{t-1}} \langle \eta' \left( A^\dagger z^{t-1} + \mu^{t-1}; \tilde{\tau}^{t-1} \right) \rangle \right)}
\]

• i.e.

\[
\bar{\tau}^t = \frac{1}{\sigma^2 R_{A}} \left( - \frac{\langle \eta' \left( A^\dagger z^{t-1} + \mu^{t-1}; \tilde{\tau}^{t-1} \right) \rangle}{\sigma^2 \bar{\tau}^{t-1}} \right)
\]
• A random row orthogonal ensemble defined as

\[ A = \alpha^{-\frac{1}{2}} P_{\alpha} O, \quad \alpha \leq 1 \]

where \( P_{\alpha} \) is the \( N \times K \) matrix with entries \( [P_{\alpha}]_{ij} = \delta_{ij}, \forall ij \).

• In this case we have

\[
S_A(z) = \frac{1 + z}{1 + z/\alpha}
\]

\[
R_A(z) = \frac{z - \alpha + \sqrt{(\alpha - z)^2 + 4\alpha^2z}}{2\alpha z}
\]
Simulation Results

- Let $p_k(x_k) = (1 - \rho)\delta(x_k) + \rho N(x_k|0,1)$, with $\rho \in (0,1)$.
- For the closed-forms of $\eta_t(\cdot)$ and $\eta'_t(\cdot)$, see [Krzakala et.al. 2012].

- S-AMP for the row orthogonal matrix ensemble (solid curves) and the iid zero-mean ensemble (dashed curves).
- Confidence intervals (CIs) are also shown for $\alpha = 1/3$.
- We set $\sigma^2 = -20$ dB and $\rho = 0.1$, and $K = 1200$.
- The numbers in the plot are the predictions of replica theory [Kabashima and Vekapera 2014]