A modification of Einstein-Schrödinger theory that contains both general relativity and electrodynamics

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Abstract We modify the Einstein-Schrödinger theory to include a cosmological constant $\Lambda_z$ which multiplies the symmetric metric, and we show how the theory can be easily coupled to additional fields. The cosmological constant $\Lambda_z$ is assumed to be nearly cancelled by Schrödinger's cosmological constant $\Lambda_b$ which multiplies the nonsymmetric fundamental tensor, such that the total $\Lambda = \Lambda_z + \Lambda_b$ matches measurement. The resulting theory becomes exactly Einstein-Maxwell theory in the limit as $|\Lambda_z| \to \infty$. For $|\Lambda_z| \sim 1/(\text{Planck length})^2$ the field equations match the ordinary Einstein and Maxwell equations except for extra terms which are $< 10^{-16}$ of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. Additional fields can be included in the Lagrangian, and these fields may couple to the symmetric metric and the electromagnetic vector potential, just as in Einstein-Maxwell theory. The ordinary Lorentz force equation is obtained by taking the divergence of the Einstein equations when sources are included. The Einstein-Infeld-Hoffmann (EIH) equations of motion match the equations of motion for Einstein-Maxwell theory to Newtonian/Coulombian order, which proves the existence of a Lorentz force without requiring sources. This fixes a problem of the original Einstein-Schrödinger theory, which failed to predict a Lorentz force. An exact charged solution matches the Reissner-Nordström solution except for additional terms which are $\sim 10^{-66}$ of the usual terms for worst-case radii accessible to measurement. An exact electromagnetic plane-wave solution is identical to its counterpart in Einstein-Maxwell theory.

Keywords Einstein-Schrodinger Theory, Einstein-Straus Theory, Cosmological Constant

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1 Introduction

The Einstein-Schrödinger theory is a generalization of vacuum general relativity which allows non-symmetric fields. The theory without a cosmological constant was first proposed by Einstein and Straus\textsuperscript{[1,2,3,4,5]}. Schrödinger later showed that it could be derived from a very simple Lagrangian density if a cosmological constant was included\textsuperscript{[6,7,8]}. Einstein and Schrödinger suspected that the theory might include electrodynamics, but no Lorentz force was found\textsuperscript{[9,10]} when using the Einstein-Infeld-Hoffmann (EIH) method\textsuperscript{[11,12]}. Here we show that a simple modification of the Einstein-Schrödinger theory closely approximates Einstein-Maxwell theory, and the Lorentz force does result from the EIH method, and in fact the ordinary Lorentz force equation results when sources are included. The modification is the addition of a second cosmological term $A_z g_{\mu\nu}$, where $g_{\mu\nu}$ is the symmetric metric. We assume this term nearly cancels Schrödinger’s “bare” cosmological term $A_b N_{\mu\nu}$, where $N_{\mu\nu}$ is the nonsymmetric fundamental tensor. The total cosmological constant $\Lambda = A_b + A_z$ can then match cosmological measurements of the accelerating universe. Our theory is related to one in\textsuperscript{[13]}, but it is roughly the electromagnetic dual of that theory, and it allows coupling to additional fields (sources), and it allows $\Lambda \neq 0$.

The origin of our $A_z$ is unknown. One possibility is that $A_z$ could arise from vacuum fluctuations, an idea that has been discussed by many authors\textsuperscript{[14,15,16,17]}. Zero-point fluctuations are essential to both QED and the Standard-Model, and are the cause of the Casimir force\textsuperscript{[15]} and other effects. Another possibility is that $A_z$ arises dynamically, related to the minimum of a potential of some additional field in the theory. Speculation about the origin of this second cosmological constant is beyond the scope of this paper. Our main goal here is to demonstrate that the theory closely approximates Einstein-Maxwell theory.

Like Einstein-Maxwell theory, our theory can be coupled to additional fields using a symmetric metric $g_{\mu\nu}$ and vector potential $A_\mu$, and it is invariant under a $U(1)$ gauge transformation. The theory does not enlarge the invariance group. When coupled to the Standard Model, the combined Lagrangian is invariant under the usual $U(1) \otimes SU(2) \otimes SU(3)$ gauge group. The usual $U(1)$ gauge term $F_{\mu\nu} F^{\mu\nu}$ is incorporated together with the geometry, and is not explicitly in the Lagrangian. Whether this is a step backwards from Einstein-Maxwell theory coupled to the Standard Model, or whether the $SU(2)$ and $SU(3)$ gauge terms could also be incorporated using non-Abelian fields as in\textsuperscript{[18,19]}, or by using higher space-time dimensions, is speculation beyond the scope of this paper.

This paper is organized as follows. In \textsection2 we discuss the Lagrangian density. In \textsection3-\textsection5 we derive the field equations and quantify how closely they approximate the field equations of Einstein-Maxwell theory. In \textsection6 we derive the ordinary Lorentz force equation by taking the divergence of the Einstein equations when sources are included. In \textsection7 we derive the Lorentz force using the EIH method, which requires no sources in the Lagrangian. In \textsection8 we give an exact charged solution and show that it closely approximates the Reissner-
Nordström solution. In §3 we give an exact electromagnetic plane-wave solution which is identical to its counterpart in Einstein-Maxwell theory.

2 The Lagrangian density

Einstein-Maxwell theory can be derived from a Palatini Lagrangian density,

\[
\mathcal{L}(\Gamma^\lambda_{\rho\tau}, g, A_\nu) = -\frac{1}{16\pi} \sqrt{-g} \left[ g^{\mu\nu} R_{\mu\nu}(\Gamma) + (n-2)A_b \right] + \frac{1}{4\pi} \sqrt{-g} A_{[\alpha,\beta]} g^{\alpha\mu} g^{\beta\nu} A_{[\mu,\nu]} + \mathcal{L}_m(u^\nu, \psi, g_{\mu\nu}, A_\nu \cdots). \tag{1}\]

Here \( A_b \) is a bare cosmological constant. The \( \mathcal{L}_m \) term couples the metric \( g_{\mu\nu} \) and electromagnetic potential \( A_\mu \) to additional fields, such as a hydrodynamic velocity vector \( u^\nu \), spin-1/2 wavefunction \( \psi \), or perhaps the other fields of the Standard Model. The original Einstein-Schrödinger theory allows a non-symmetric \( N_{\mu\nu} \) and \( \tilde{\Gamma}^\lambda_{\rho\tau} \) in place of the symmetric \( g_{\mu\nu} \) and \( \Gamma^\lambda_{\rho\tau} \), and excludes the \( \sqrt{-g} A_{[\alpha,\beta]} g^{\alpha\mu} g^{\beta\nu} A_{[\mu,\nu]} \) term. Our “A-renormalized” Einstein-Schrödinger theory introduces an additional cosmological term \( \sqrt{-g}A_z \),

\[
\mathcal{L}(\tilde{\Gamma}^\lambda_{\rho\tau}, N, \rho\tau) = -\frac{1}{16\pi} \sqrt{-N} \left[ N^{-\mu\nu} R_{\mu\nu}(\tilde{\Gamma}) + (n-2)A_b \right] - \frac{1}{16\pi} \sqrt{-g} (n-2)A_z + \mathcal{L}_m(u^\nu, \psi, g_{\mu\nu}, A_\nu \cdots), \tag{2}\]

where \( A_b \approx -A_z \) so that the total \( A \) matches astronomical measurements, \( A = A_b + A_z \approx 10^{-56}\text{cm}^{-2} \), \tag{3}\]

and the physical metric and electromagnetic potential are defined to be

\[
\sqrt{-g} g^{\mu\nu} = \sqrt{-N} N^{-\mu\nu}, \quad A_\nu = \tilde{A}^\sigma_{[\nu]}/[(n-1)\sqrt{-2A_b}]. \tag{4}\]

Eq. (4) defines \( g^{\mu\nu} \) unambiguously because \( \sqrt{-g} = \left[ -\det(\sqrt{-g} g^{\mu\nu}) \right]^{1/(n-2)} \). Here and throughout this paper we use geometrized units with \( c = G = 1 \), the symbols ( ) and [ ] around indices indicate symmetrization and antisymmetrization, \( g = \det(g_{\mu\nu}) \), \( N = \det(N_{\mu\nu}) \), and \( N^{-\mu\nu} \) is the inverse of \( N_{\mu\nu} \) such that \( N^{-\mu\nu} N_{\nu\mu} = \delta^\mu_\nu \). The dimension is assumed to be \( n=4 \), but “\( n \)” is retained in the equations to show how easily the theory can be generalized. The \( \mathcal{L}_m \) term is not to include a \( \sqrt{-g} A_{[\alpha,\beta]} g^{\alpha\mu} g^{\beta\nu} A_{[\mu,\nu]} \) part but may contain the rest of the Standard Model. In \( \mathcal{L}_m \), \( R_{\mu\nu}(\tilde{\Gamma}) \) is a form of Hermitianized Ricci tensor,

\[
R_{\mu\nu}(\tilde{\Gamma}) = \tilde{\Gamma}^\alpha_{\mu\nu,\alpha} - \tilde{\Gamma}^\alpha_{(\nu,\mu),\alpha} + \tilde{\Gamma}^\alpha_{\nu\mu} \tilde{\Gamma}^\alpha_{(\alpha\sigma)} - \tilde{\Gamma}^\alpha_{\nu\sigma} \tilde{\Gamma}^\alpha_{\sigma\mu} - \tilde{\Gamma}^\tau_{[\tau\nu]} \tilde{\Gamma}^\alpha_{[\nu\rho]} / (n-1). \tag{5}\]

This tensor reduces to the ordinary Ricci tensor when \( \Gamma^\alpha_{[\nu\mu]} = 0 \) and \( \Gamma^\alpha_{\alpha[v,\mu]} = 0 \), as occurs in ordinary general relativity.

It is helpful to decompose \( \tilde{\Gamma}^\alpha_{\nu\mu} \) into a new connection \( \tilde{\Gamma}^\alpha_{\nu\mu} \), and \( A_\sigma \) from \( \mathcal{L}_m \),

\[
\tilde{\Gamma}^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu A_\nu - \delta^\alpha_\nu A_\mu) \sqrt{-2A_b}, \tag{6}\]

where \( \tilde{\Gamma}^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\nu\mu} + (\delta^\alpha_\mu \tilde{\Gamma}^\sigma_{[\nu\sigma]} - \delta^\alpha_\nu \tilde{\Gamma}^\sigma_{[\sigma\mu]}) / (n-1). \tag{7}\]
By contracting (7) on the right and left we see that \( \hat{\Gamma}_{\nu\mu}^\alpha \) has the symmetry
\[
\hat{\Gamma}_{\nu\mu}^\alpha = \hat{\Gamma}_{(\nu\alpha)}^\alpha = \hat{\Gamma}_{\alpha\nu}^\alpha,
\] (8)
so it has only \( n^3 - n \) independent components. Substituting (6) into (5) gives
\[
\mathcal{R}_{\nu\mu}(\hat{\Gamma}) = \mathcal{R}_{\nu\mu}(\hat{\Gamma}) + 2A_{[\nu,\mu]}\sqrt{-2A_b}.
\] (9)

Using (9), the Lagrangian density (2) can be written in terms of \( \tilde{\Gamma}_{\nu\mu}^\alpha \) and \( A_\sigma \),
\[
\mathcal{L}(\tilde{\Gamma}_{\nu\mu}^\alpha, N_{\nu\tau}) = -\frac{1}{16\pi} \sqrt{-N} \left[ N^{-\mu\nu}(\tilde{\mathcal{R}}_{\nu\mu} + 2A_{[\nu,\mu]}\sqrt{-2A_b}) + (n-2)A_b \right]
- \frac{1}{16\pi} \sqrt{-g} (n-2)A_z + \mathcal{L}_m(u^\nu, \psi, g_{\nu\nu}, A_\sigma \ldots). 
\] (10)

Here \( \tilde{\mathcal{R}}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\hat{\Gamma}) \), and from (8) we have
\[
\tilde{\mathcal{R}}_{\nu\mu} = \tilde{\Gamma}_{\nu\mu,\alpha} - \tilde{\Gamma}_{\alpha\nu\mu} + \tilde{\Gamma}_{\nu\mu,\sigma} \tilde{\Gamma}^{\sigma\alpha} - \tilde{\Gamma}_{\nu\mu} \tilde{\Gamma}_{\alpha\sigma}. 
\] (11)

From (8), \( \tilde{\Gamma}_{\nu\mu}^\alpha \) and \( A_\nu \) fully parameterize \( \tilde{\Gamma}_{\nu\mu}^\alpha \) and can be treated as independent variables. It is simpler to calculate the field equations by setting \( \delta\mathcal{L}/\delta\tilde{\Gamma}_{\nu\mu}^\alpha = 0 \) and \( \delta\mathcal{L}/\delta A_\nu = 0 \) instead of setting \( \delta\mathcal{L}/\delta \tilde{\Gamma}_{\nu\mu}^\alpha = 0 \), so we will follow this method.

To do quantitative comparisons of this theory to Einstein-Maxwell theory we will need to use some value for \( A_z \). One possibility is that \( A_z \) results from zero-point fluctuations[14, 15, 16, 17], in which case using (3) we get
\[
A_b \approx -A_z \sim C_z \omega_c l_p^2 \sim 10^{66} \text{cm}^{-2},
\] (12)
\[
\omega_c = \text{(cutoff frequency)} \sim 1/l_p, 
\] (13)
\[
C_z = \frac{1}{2\pi}(\text{fermion spin states} - \text{boson spin states}) \sim \frac{60}{2\pi}
\] (14)

where \( l_p = \text{(Planck length)} = 1.6 \times 10^{-33} \text{cm} \). We will also consider the limit \( \omega_c \to \infty, |A_z| \to \infty, A_b \to \infty \) as in QED, and we will prove that
\[
\lim_{|A_z| \to \infty} \left( \text{A-renormalized Einstein-Schrödinger theory} \right) = \left( \text{Einstein-Maxwell theory} \right). 
\] (15)

The Hermitianized Ricci tensor (5) has the following invariance properties
\[
\mathcal{R}_{\nu\mu}(\hat{\Gamma}^T) = \mathcal{R}_{\mu\nu}(\hat{\Gamma}) 
\] (T = transpose), (16)
\[
\mathcal{R}_{\nu\mu}(\hat{\Gamma}_{\nu\mu,\alpha} + \delta^\alpha_{[\nu,\mu,\tau]}) = \mathcal{R}_{\nu\mu}(\hat{\Gamma}) \quad \text{for an arbitrary } \phi(x^\sigma). 
\] (17)

From (16, 17), the Lagrangians (2, 10) are invariant under charge conjugation,
\[
Q \to -Q, \ A_\sigma \to -A_\sigma, \ \hat{\Gamma}_{\nu\mu}^\alpha \to \hat{\Gamma}_{\mu\nu}^\alpha, \ \hat{\Gamma}_{\nu\mu}^\alpha \to \hat{\Gamma}_{\mu\nu}^\alpha, \ N_{\nu\mu} \to N_{\mu\nu}, \ N^{-\mu\nu} \to N^{-\nu\mu}, 
\] (18)

and also under an electromagnetic gauge transformation
\[
\psi \to \psi e^{i\phi}, \ A_\alpha \to A_\alpha - \frac{\hbar}{Q} \phi_\alpha, \ \hat{\Gamma}_{\rho\tau} \to \hat{\Gamma}_{\rho\tau}^{\alpha}, \ \hat{\Gamma}_{\rho\tau} \to \hat{\Gamma}_{\rho\tau}^{\alpha} + \frac{2\hbar}{Q} \delta^\alpha_{[\rho,\tau]} \sqrt{-2A_b}. 
\] (19)
assuming that $\mathcal{L}_m$ is invariant. With $A_b > 0$, $A_z < 0$ as in (12) then $\tilde{\Gamma}^\alpha_{\nu\mu}$, $\tilde{\Gamma}^\alpha_{\nu\mu}$, $N_{\nu\mu}$ and $N^{-\nu\mu}$ are all Hermitian, $\tilde{R}_{\nu\mu}$ and $R_{\nu\mu}(\tilde{F})$ are Hermitian from (16), and $g_{\nu\mu}$, $A_\sigma$ and $\mathcal{L}$ are real from (4,10).

In this theory the metric (4) is used for measuring space-time intervals, for calculating geodesics, and for raising and lowering of indices. The covariant derivative “;”, is always done using the Christoffel connection formed from $g_{\mu\nu}$, $\Gamma^\alpha_{\nu\mu} = \frac{1}{2} g^{\alpha\sigma} (g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\nu\mu,\sigma})$.

(20)

We will see that taking the divergence of the Einstein equations using (20,4) gives the ordinary Lorentz force equation. The electromagnetic field is defined in terms of the potential (4)

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \quad (21)$$

However, we will also define another field $f^{\mu\nu}$

$$\sqrt{-g} f^{\mu\nu} = \sqrt{-N} N^{-[\nu\mu]} A_\nu^{-1/2}/\sqrt{2} i. \quad (22)$$

Then from (4), $g^{\mu\nu}$ and $f^{\mu\nu} \sqrt{2} i A_\nu^{-1/2}$ are parts of a total field,

$$(\sqrt{-N}/\sqrt{-g}) N^{-\nu\mu} = g^{\mu\nu} + f^{\mu\nu} \sqrt{2} i A_\nu^{-1/2}. \quad (23)$$

We will see that the field equations require $f^{\mu\nu} \approx F_{\mu\nu}$ to a very high precision.

The definitions (4) of $g_{\mu\nu}$ and $A_\nu$ in terms of the “fundamental” fields $N_{\rho\tau}, \tilde{\Gamma}^\rho_{\rho\tau}$ may seem unnatural from an empirical viewpoint. On the other hand, our Lagrangian density (2) seems simpler than (1) of Einstein-Maxwell theory, it contains fewer fields, and these fields have no symmetry restrictions. However, these are all very subjective considerations. It is much more important that our theory closely matches Einstein-Maxwell theory, and hence measurement.

Note that there are many nonsymmetric generalizations of the Ricci tensor besides the Hermitianized Ricci tensor $R_{\nu\mu}(\tilde{F})$ from (5) and the ordinary Ricci tensor $R_{\nu\mu}(\tilde{F})$. For example, we could form any weighted average of $\tilde{R}_{\nu\mu}(\tilde{F}), R_{\nu\mu}(\tilde{F}), R_{\nu\mu}(\tilde{F}^T)$ and $R_{\nu\mu}(\tilde{F}^T)$, and then add any linear combination of the tensors $\tilde{\Gamma}^\alpha_{[\nu\mu]}, \tilde{\Gamma}^\alpha_{[\nu(\nu\mu)]}, \tilde{\Gamma}^\alpha_{[\nu\mu]}, \tilde{\Gamma}^\alpha_{[\nu\mu]}, \tilde{\Gamma}^\alpha_{[\nu\mu]}$, and $\tilde{\Gamma}^\alpha_{[\nu\mu]}$. All of these generalized Ricci tensors would be linear in $\tilde{\Gamma}^\alpha_{[\nu\mu]}$, quadratic in $\tilde{\Gamma}^\alpha_{[\nu\mu]}$, and would reduce to the ordinary Ricci tensor when $\Gamma^\alpha_{[\nu\mu]} = 0$ and $\Gamma^\alpha_{[\nu\mu]} = 0$ as occurs in ordinary general relativity. Even if we limit the tensor to only four terms, there are still eight possibilities. We assert that invariance properties like (16,17) are the most sensible way to choose among the different alternatives, not criteria such as the number of terms in the expression.

Finally, let us discuss some notation issues. We use the symbol $\Gamma^\alpha_{[\nu\mu]}$ for the Christoffel connection (20) whereas Einstein and Schrödinger used it for our $\tilde{\Gamma}^\alpha_{[\nu\mu]}$ and $\tilde{\Gamma}^\alpha_{[\nu\mu]}$ respectively. We use the symbol $g_{\mu\nu}$ for the symmetric metric (4) whereas Einstein and Schrödinger used it for our $N_{\mu\nu}$, the nonsymmetric fundamental tensor. Also, to represent the inverse of $N_{\mu\nu}$ we use $N^{-\nu\mu}$ instead.
of the more conventional $N^{\alpha\sigma}$, because this latter notation would be ambiguous when using $g^{\mu\nu}$ to raise indices. While our notation differs from previous literature on the Einstein-Schrödinger theory, this change is required by our explicit metric definition, and it is necessary to be consistent with the much larger body of literature on Einstein-Maxwell theory.

3 The Einstein equations

To set $\delta \mathcal{L}/\delta(\sqrt{-NN^{-1}\nu\mu}) = 0$ we need some initial results. Using (4) and the identities $\det(sM) = s^n \det(M)$, $\det(M^{-1}) = 1/\det(M)$ gives

$$\sqrt{-N} = (-\det((\sqrt{-NN^{-1}\nu\mu}))^{1/(n-2)},$$

(24)

$$\sqrt{-g} = (-\det((\sqrt{-g} g^{\nu\mu}))^{1/(n-2)} = (-\det((\sqrt{-NN^{-1}(\nu\mu)})^{1/(n-2)}).$$

(25)

Using (24,25,4) and the identity $\partial(\det(M^{-1}))/\partial M_{\nu\mu} = M_{\nu\mu}^{-1}$ gives

$$\sqrt{-N} = \frac{N_{\nu\mu}}{(n-2)}, \quad \sqrt{-g} = \frac{g_{\nu\mu}}{(n-2)}.$$  

(26)

Setting $\delta \mathcal{L}/\delta(\sqrt{-NN^{-1}\nu\mu}) = 0$ using (10,26) gives the field equations,

$$\tilde{\mathcal{R}}_{\nu\mu} + 2A_{[\nu,\mu]} \sqrt{-2}A_b + A_b N_{\nu\mu} + A_z g_{\nu\mu} = 8\pi S_{\nu\mu},$$

(27)

where $S_{\nu\mu}$ and the energy-momentum tensor $T_{\nu\mu}$ are defined by

$$S_{\nu\mu} \equiv 2 \frac{\delta \mathcal{L}_m}{\delta (\sqrt{-NN^{-1}\nu\mu})} = 2 \frac{\delta \mathcal{L}_m}{\delta (\sqrt{-g} g^{\nu\mu})},

\quad T_{\nu\mu} \equiv S_{\nu\mu} - \frac{1}{2} g_{\nu\mu} S_{\alpha\alpha}, \quad S_{\nu\mu} = T_{\nu\mu} - \frac{1}{(n-2)} g_{\nu\mu} T_{\alpha\alpha}.$$  

(28)

(29)

The second equality in (28) results because $\mathcal{L}_m$ in (2) contains only the metric $\sqrt{-g} g^{\mu\nu} = \sqrt{-NN^{-1}(\mu\nu)}$ from (4), and not $\sqrt{-NN^{-1}(\mu\nu)}$. Taking the symmetric and antisymmetric parts of (27) and using (21) gives

$$\tilde{\mathcal{R}}_{(\nu\mu)} + A_b N_{(\nu\mu)} + A_z g_{\nu\mu} = 8\pi \left( T_{\nu\mu} - \frac{1}{(n-2)} g_{\nu\mu} T_{\alpha\alpha} \right),$$

(30)

$$N_{[\nu\mu]} = F_{\nu\mu} \sqrt{2} i A_b^{-1}/2 - \tilde{\mathcal{R}}_{[\nu\mu]} A_b^{-1}.$$  

(31)

Also from the curl of (31) we get

$$\tilde{\mathcal{R}}_{[\nu\mu,\sigma]} + A_b N_{[\nu\mu,\sigma]} = 0.$$  

(32)

To put (30) into a form which looks more like the ordinary Einstein equations, we need some preliminary results. The definitions (4,22) of $g_{\nu\mu}$ and $f_{\nu\mu}$
can be inverted exactly to give $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$. An expansion in powers of $A_b^{-1}$ will better serve our purposes, and is derived in Appendix A.

\begin{equation}
N_{(\nu\mu)} = g_{\nu\mu} - 2\left(f^{\sigma}_{\nu\alpha} f^{\alpha}_{\mu \rho} - \frac{1}{2(n-2)} g_{\nu\mu} f^{\rho\sigma} f^{\sigma}_{\rho \rho}\right) A_b^{-1} + (f^4)A_b^{-2} \ldots \tag{33}
\end{equation}

\begin{equation}
N_{[\nu\mu]} = f_{\nu\mu} \sqrt{2} i A_b^{-1/2} + (f^3)A_b^{-3/2} \ldots \tag{34}
\end{equation}

Here the notation $(f^3)$ and $(f^4)$ is for terms like $f_{\nu\alpha} f^{\rho\sigma} f^{\alpha}_{\mu \rho}$ and $f_{\nu\alpha} f^{\rho\sigma} f^{\sigma}_{\rho \rho} f^{\mu}_{\mu}$. Let us consider the size of these higher order terms relative to the leading order terms, for worst-case fields accessible to measurement. In geometrized units an elementary charge has

\begin{equation}
Q_e = \epsilon \sqrt{G} = \sqrt{\frac{e^2}{hc}} = \sqrt{\alpha} l_P = 1.38 \times 10^{-34}\text{cm} \tag{35}
\end{equation}

where $\alpha = e^2/\hbar c$ is the fine structure constant and $l_P = \sqrt{G \hbar / c^3}$ is the Planck length. If we assume that charged particles retain $f^1_0 \sim Q/r^2$ down to the smallest radii probed by high energy particle physics experiments ($10^{-17}\text{cm}$) we have from $\left(33,34\right)$,

\begin{equation}
|f^1_0|^2 / A_b \sim (Q_e / (10^{-17})^2) / A_b \sim 10^{-66}. \tag{36}
\end{equation}

Here $|f^1_0|$ is assumed to be in some standard spherical or cartesian coordinate system. If an equation has a tensor term which can be neglected in one coordinate system, it can be neglected in any coordinate system, so it is only necessary to prove it in one coordinate system. The fields at $10^{-17}\text{cm}$ from an elementary charge would be larger than near any macroscopic charged object, and would also be larger than the strongest plane-wave fields. Therefore the higher order terms in $\left(33,34\right)$ must be $< 10^{-66}$ of the leading order terms, so they will be completely negligible for most purposes.

In $\left(38\right)$we will calculate the connection equations resulting from \(\delta \mathcal{L} / \delta \tilde{\Gamma}^\alpha_{\nu\mu} = 0\). Solving these equations gives $\left(38,39\right)$, which can be abbreviated as

\begin{equation}
\tilde{\Gamma}^\alpha_{(\nu\mu)} = \Gamma^\alpha_{\nu\mu} + \mathcal{O}(A_b^{-1}), \quad \tilde{\Gamma}^\alpha_{[\nu\mu]} = \mathcal{O}(A_b^{-1/2}), \tag{37}
\end{equation}

\begin{equation}
\tilde{G}_{\nu\mu} = G_{\nu\mu} + \mathcal{O}(A_b^{-1}), \quad \tilde{R}_{[\nu\mu]} = \mathcal{O}(A_b^{-1/2}), \tag{38}
\end{equation}

where $\Gamma^\alpha_{\nu\mu}$ is the Christoffel connection $\left(20\right)$, $\tilde{\mathcal{R}}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\tilde{\Gamma})$, $\tilde{R}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\Gamma)$ and

\begin{equation}
\tilde{G}_{\nu\mu} = \tilde{\mathcal{R}}_{(\nu\mu)} = -\frac{1}{2} g_{\nu\mu} \tilde{R}^\rho_{\rho}, \quad \tilde{G}_{\nu\mu} = \tilde{R}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R. \tag{39}
\end{equation}

In $\left(38\right)$ the notation $\mathcal{O}(A_b^{-1})$ and $\mathcal{O}(A_b^{-1/2})$ indicates terms like $f^{\sigma}_{\nu\alpha} f^{\alpha}_{\mu\rho} A_b^{-1}$ and $f_{[\nu\mu,\alpha]} A_b^{-1/2}$.

From the antisymmetric part of the field equations $\left(31\right)$ and $\left(34,38\right)$ we get

\begin{equation}
f_{\nu\mu} = F_{\nu\mu} + \mathcal{O}(A_b^{-1}). \tag{40}
\end{equation}
So \( f_{\nu\mu} \) and \( F_{\nu\mu} \) only differ by terms with \( A_b \) in the denominator, and the two become identical in the limit as \( A_b \to \infty \). Combining (30) with its contraction, and substituting (39,33,3) gives the Einstein equations

\[
\tilde{G}_{\nu\mu} = 8\pi T_{\nu\mu} - \Lambda_b \left( N_{(\nu\mu)} - \frac{1}{2} g_{\nu\mu} N_{\rho}^{\rho} \right) + A_z \left( \frac{n}{2} - 1 \right) g_{\nu\mu} + (f^4)A_b^{-1}, \quad (41)
\]

From (28,29) we see that \( T_{\nu\mu} \) will be the same as in ordinary general relativity, for example when we include classical hydrodynamics or spin-1/2 fields as in [21,22]. Therefore from (40), equation (42) differs from the ordinary Einstein equations only by terms with \( A_b \) in the denominator, and it becomes identical to the ordinary Einstein equations in the limit as \( A_b \to \infty \) (with an observationally valid total \( \Lambda \)). In §5 we will examine how close the approximation is for \( A_b \) from (12).

### 4 Maxwell’s equations

Setting \( \delta\mathcal{L}/\delta A_\tau = 0 \) and using (10,22) gives

\[
0 = \frac{\sqrt{2}}{2\sqrt{-g}} (\sqrt{-N} N^{\nu[\omega]}),_\omega - 4\pi j^\tau = \frac{(\sqrt{-g} f^\omega)_\omega}{\sqrt{-g}} - 4\pi j^\tau, \quad (43)
\]

where

\[
j^\tau = \frac{-1}{\sqrt{-g}} \left[ \frac{\partial \mathcal{L}_m}{\partial A_\tau} - \left( \frac{\partial \mathcal{L}_m}{\partial A_{\tau,\omega}} \right),_\omega \right]. \quad (44)
\]

From (43,21) we get Maxwell’s equations,

\[
f^{\omega\tau},_\omega = 4\pi j^\tau, \quad (45)
\]

\[
F_{[\nu\mu,\alpha]} = 0. \quad (46)
\]

where \( f_{\nu\mu} = F_{\nu\mu} + O(A_b^{-1}) \) from (40). From [21,22] we see that \( j^\mu \) will be the same as in ordinary general relativity, for example when we include classical hydrodynamics or spin-1/2 fields as in [21,22]. From [40], we see that equations (45,46) differ from the ordinary Maxwell equations only by terms with \( A_b \) in the denominator, and these equations become identical to the ordinary Maxwell equations in the limit as \( A_b \to \infty \). In §5 we will examine how close the approximation is for \( A_b \) from (12).

Because \( \mathcal{L}_m \) in (2) couples to additional fields only through \( g_{\mu\nu} \) and \( A_\mu \), any equations associated with additional fields will be the same as in ordinary general relativity. For example in the spin-1/2 case, setting \( \delta\mathcal{L}/\delta \bar{\psi} = 0 \) will give the ordinary Dirac equation in curved space as in [21,22]. It would be interesting to investigate what results if one includes \( f_{\mu\nu}, N_{\mu\nu} \) or \( \tilde{\Gamma}_{\alpha}^{\mu} \) in \( \mathcal{L}_m \).
although there does not appear to be any empirical reason for doing so. A continuity equation follows from \(14\) regardless of the type of source,

\[
j^\rho_\nu = \frac{1}{4\pi} f^\tau_\rho [\tau_\nu] = 0.
\] (47)

Note that the covariant derivative in \(14,47\) is done using the Christoffel connection \(29\) formed from the symmetric metric \(1\).

5 The connection equations

Setting \(\delta \mathcal{L}/\delta F^\alpha_\nu = 0\) with a Lagrange multiplier term \(\Omega^\mu F^{\alpha \mu}_\nu\) to enforce the symmetry \(8\), and using \(10,43\) gives

\[
(\sqrt{-N} N^{-\nu\rho\tau})_{,\beta} + \tilde{\Gamma}^\tau_{\alpha\beta} \sqrt{-N} N^{-\nu\rho\tau} + \tilde{\Gamma}^\rho_{\beta\sigma} \sqrt{-N} N^{\nu\sigma\tau} - \tilde{\Gamma}^\rho_{\beta\alpha} \sqrt{-N} N^{\nu\rho\tau} \\
= \frac{8\pi \sqrt{2} i}{(n-1 \Lambda_b^{1/2})} \sqrt{-g} j^{[\nu \rho]} \delta_\beta. \] (48)

These are the connection equations, analogous to \(g^{\nu\rho;\beta} = 0\) in the symmetric case. Note that we can also derive Ampere's law \(13\) by antisymmetrizing and contracting these equations. From the definition of matrix inverse, \(N^{-\nu\rho\tau} = (1/N) \partial N/\partial N_{\tau\rho}\) and \(N^{-\nu\rho\tau} N_{\tau\mu} = \delta^\mu_\rho\) we get the identity

\[
(\sqrt{-N})_{,\beta} N_{\tau\rho,\sigma} = \frac{\sqrt{-N}}{2} N^{-\nu\rho\tau} N_{\tau\rho,\sigma} = -\frac{\sqrt{-N}}{2} N^{-\nu\rho\tau,\sigma} N_{\tau\rho}. \] (49)

Contracting \(48\) with \(N_{\tau\rho,\sigma}\) using \(8,49\), and dividing this by \((n-2)\) gives,

\[
(\sqrt{-N})_{,\beta} \tilde{\Gamma}^\alpha_{\alpha\beta} \sqrt{-N} = -\frac{8\pi \sqrt{2} i}{(n-1)(n-2) \Lambda_b^{1/2}} \sqrt{-g} j^{[\nu \rho]} N_{[\nu \rho]}. \] (50)

Multiplying \(48\) by \(-N_{\nu\rho} N_{\tau\mu}\) and using \(50\) gives

\[
N_{\nu\rho, \beta} - \tilde{\Gamma}^\alpha_{\nu\beta} N_{\nu\alpham} - \tilde{\Gamma}^\alpha_{\mu\beta} N_{\nu\alphan} = \frac{-8\pi \sqrt{2} i}{(n-1) \Lambda_b^{1/2}} \sqrt{-g} \left( N_{\nu[\alpha} N_{\beta]\mu} + \frac{N_{[\alpha\beta]} N_{\nu\mu}}{(n-2)} \right) j^{\alpha}. \] (51)

Equation \(51\) together with \(50,38\) are often used to define the Einstein-Schrödinger theory, particularly when \(T_{\mu\nu} = 0, j^\alpha = 0\).

Equations \(48\) or \(51\) can be solved exactly \(23,24\), similar to the way \(g^{\nu\rho;\beta} = 0\) can be solved to get the Christoffel connection. An expansion in powers of \(A_b^{-1}\) will better serve our purposes, and such an expansion is derived in Appendix E of \(21\), and is also stated without derivation in \(25\),

\[
\tilde{\Gamma}^\alpha_\nu = \Gamma^\alpha_\nu + \gamma^\alpha_\nu, \] (52)

\[
\gamma^\alpha_\nu = -2 \left[ f^\tau_\nu f^\alpha_\mu,\tau + f^{\alpha\tau} f_\tau(\nu_\mu) + \frac{1}{4(n-2)} ((f^{\rho\sigma} f_{\nu\rho})_\nu f^\alpha_\mu - 2(f^{\rho\sigma} f_{\sigma\rho})_\nu f^\alpha_\mu) \right. \\
+ \left. \frac{4\pi}{(n-2)} j^{\tau} \left( f^{\rho\sigma} g_{\nu\rho} + \frac{2}{(n-1)} f_\rho(\nu \delta^\nu_\mu) \right) \right] A_b^{-1} + (f^{\nu\mu}) A_b^{-2} \ldots. \] (53)

\[
\gamma^\alpha_\nu = \left[ \frac{1}{2} f_\nu \nu_\alpha + f^{\alpha\mu \nu} - f^{\alpha \nu \mu} + \frac{8\pi}{(n-1)} j^{\nu \delta^\alpha_\mu} \right] \sqrt{2} i A_b^{-1/2} + (f^{\nu\mu}) A_b^{-3/2} \ldots. \] (54)
In (52), $\Gamma^\alpha_{\nu\mu}$ is the Christoffel connection (20). The notation $(f^3)$ and $(f^4)$ refers to terms like $f^\alpha f^\beta f^\gamma f^\nu_{\nu\mu}$ and $f^\alpha f^\beta f^\gamma f^\rho_{\nu\mu \nu\mu}$ as in (53,54). As in (53,54), we see from (56) that the higher order terms in (53,54) must be $<10^{-56}$ of the leading order terms, so they will be completely negligible for most purposes.

Extracting $\mathcal{T}_{\nu\beta}$ of (52) from the Hermitianized Ricci tensor (11) gives,

$$\mathcal{R}_{[\nu\mu]}(\hat{\Gamma}) = \mathcal{R}_{\nu\mu}(\Gamma) + \tau^\alpha_{\nu\alpha} - \tau^\alpha_{\nu\alpha} - \tau^\alpha_{\nu\alpha} + \tau^\alpha_{\nu\alpha} - \tau^\alpha_{\nu\alpha},$$

$$\mathcal{R}_{[\nu\mu]}(\hat{\Gamma}) = \tau^\alpha_{[\nu\alpha]} - \tau^\alpha_{[\nu\alpha]} + \tau^\alpha_{[\nu\alpha]} + \tau^\alpha_{[\nu\alpha]} + \tau^\alpha_{[\nu\alpha]}.$$

Substituting (52,53,55) into (55), and using (59) gives

$$(\hat{G}_{\nu\mu} - G_{\nu\mu}) =$$

$$-2f^\gamma (\nu, f^\mu)_{\alpha} + 2f^\alpha f^\gamma (\nu, f^\mu)_{\alpha} - f^\alpha (\nu, f^\mu)_{\alpha} + f^\alpha (\nu, f^\mu)_{\alpha} + \frac{1}{2} f^\alpha (\nu, f^\mu)_{\alpha}$$

$$- g_{\nu\mu} f^\beta f^\gamma (\nu, f^\mu)_{\alpha} - \frac{1}{4} g_{\nu\mu} (f^\rho f^\sigma, f^\beta f^\gamma (\nu, f^\mu)_{\alpha}) - \frac{3}{4} g_{\nu\mu} (f^\sigma f^\rho, f^\beta f^\gamma (\nu, f^\mu)_{\alpha})$$

$$+ 8\pi f^\sigma f^\rho (\nu, f^\mu)_{\alpha} - \frac{32\pi^2}{(n-1)} j^\nu j^\mu + \frac{16\pi^2}{(n-1)} g_{\nu\mu} j^\rho + (f^4) \bar{A}_{b}^{-1} \ldots.$$ (57)

From (42) we can define an “effective” energy momentum tensor $\hat{T}_{\nu\mu}$ which applies when $G_{\nu\mu}$ is used in the Einstein equations and $\mathcal{L}_{m} = 0$,

$$8\pi \hat{T}_{\nu\mu} = 2 \left( f^\nu f^\rho (\nu, f^\mu)_{\alpha} - \frac{1}{4} g_{\nu\mu} f^\gamma (\nu, f^\mu)_{\alpha} \right) - (\hat{G}_{\nu\mu} - G_{\nu\mu}).$$ (58)

Substituting (51,55) into (56) gives

$$\hat{R}_{[\nu\mu]} = \left( \frac{3}{2} f^\alpha (\nu, f^\mu)_{\alpha} + 2 f^\gamma f^\gamma (\nu, f^\mu)_{\alpha} - 2 f^\gamma f^\gamma (\nu, f^\mu)_{\alpha} - \frac{8\pi(n-2)}{(n-1)} j^\nu j^\mu \right) \sqrt{2} \bar{A}_{b}^{-1/2} \ldots$$ (59)

As we have noted in (43) and (44) the $\bar{A}_{b}$ in the denominator of (57,59) causes our Einstein and Maxwell equations (42,45,46) to become the ordinary Einstein and Maxwell equations in the limit as $\omega \rightarrow \infty, |A_{z}| \rightarrow \infty, \bar{A}_{b} \rightarrow \infty$, and it also causes the relation $f_{\nu\mu} \approx F_{\nu\mu}$ from (40) to become exact in this limit. Let us examine how close these approximations are when $\bar{A}_{b} \sim 10^{66} cm^{-2}$ as in (42).

We will start with the Einstein equations (42). Let us consider worst-case values of the $O(\bar{A}_{b}^{-1})$ terms in (57) and compare these to the ordinary electromagnetic term in (58). If we assume that charged particles retain $f^1_{1,0} \sim Q/r^2$ down to the smallest radii probed by high energy particle physics experiments ($10^{-17} cm$) we have,

$$|f^1_{1,0} f^1_{0,1} | \sim 4/\bar{A}_{b} (10^{-17})^2 \sim 10^{-32},$$

$$|f^1_{1,0} f^1_{0,1} | / \bar{A}_{b} \sim 6/\bar{A}_{b} (10^{-17})^2 \sim 10^{-32}.$$ (60,61)

So for electric monopole fields, terms like $f^\gamma f^\gamma f^\alpha f^\alpha f_{\nu\mu} A^{-1}_{b}$ and $f^\alpha f^\gamma f^\rho_{\nu\mu \nu\mu}$ in (57) must be $<10^{-32}$ of the ordinary electromagnetic term in (58). And
regarding $j^\tau$ as a substitute for $(1/4\pi)f^{\omega\tau}$ from (35), the same is true for the $j^\tau$ terms. For an electromagnetic plane-wave in a flat background space

$$A_\mu = A\epsilon_\mu \sin(k_\alpha x^\alpha), \quad \epsilon^\alpha \epsilon_\alpha = -1, \quad k^\alpha k_\alpha = k^\alpha \epsilon_\alpha = 0,$$

(62)

$$f_{\nu\mu} = 2A[\mu,\nu] = 2A\epsilon_{[\mu} k_{\nu]} \cos(k_\alpha x^\alpha), \quad j^\tau = 0.$$  

(63)

Here $A$ is the magnitude, $k^\alpha$ is the wavevector, and $\epsilon^\alpha$ is the polarization. Substituting (62–63) into (57), all of the terms vanish for a flat background space. Also, for the highest energy gamma rays known in nature ($10^{12}$eV, $10^{14}$Hz) we have from (12),

$$|f^{1/0};[1/0]|/A_b \sim (E/\hbar c)^2/A_b \sim 10^{-16},$$

(64)

$$|f^{1/0};[1/0]|/A_b \sim (E/\hbar c)^2/A_b \sim 10^{-16}.$$  

(65)

So for electromagnetic plane-wave fields, even if some of the terms in (57) were non-zero because of spatial curvatures, they must still be $<10^{-16}$ of the ordinary electromagnetic term in (35). Therefore even for the most extreme worst-case fields accessible to measurement, the extra terms in the Einstein equations (42) must all be $<10^{-16}$ of the ordinary electromagnetic term.

Now let us look at the approximation $f_{\nu\mu} \approx F_{\nu\mu}$ from (40), and Maxwell’s equations (45–46). From the covariant derivative commutation rule, the cyclic identity (42) must all be worst-case fields accessible to measurement, the extra terms in the Einstein equations (42) must all be $<10^{-16}$ of the ordinary electromagnetic term.

Substituting (64) into the field equations (61) and using (59–66) we get

$$2f_{\mu} = F_{\mu\alpha} + \left(\theta_{[\tau,\alpha]} \varepsilon_{\nu\mu} \tau^\alpha + f^{\alpha\tau} C_{\alpha\tau\nu\mu} + \frac{2(n-2)A}{(n-1)} f_{\nu\mu} \right.$$

$$+ \frac{8\pi(n-2)}{(n-1)} f_{[\nu,\mu]} + (f^3) A_b^{-1} \ldots$$

(67)

where $\varepsilon_{\tau\nu\mu\alpha} = (\text{Levi–Civita tensor})$, $C_{\alpha\tau\nu\mu} = (\text{Weyl tensor})$, and

$$\theta_{\tau} = \frac{1}{4} f_{[\nu,\mu]} \varepsilon^\tau_{\nu\mu\alpha}, \quad f_{[\nu,\mu]} = -\frac{2}{3} \theta_{\tau} \varepsilon_{\nu\mu\alpha}.$$  

(68)

The $\theta_{[\tau,\alpha]} \varepsilon_{\nu\mu} \tau^\alpha A_b^{-1}$ term in (67) is divergenceless so that it has no effect on Ampere’s law (35). The $f_{\nu\mu} A/A_b$ term is $\sim 10^{-122}$ of $f_{\nu\mu}$ from (32). The $(f^3) A_b^{-1}$ term is $< 10^{-66}$ of $f_{\nu\mu}$ from (35). The largest observable values of the Weyl tensor might be expected to occur near the Schwarzschild radius, $r_s = 2Gm/c^2$, of black holes, where it takes on values around $r_s/r^3$. The largest value of $r_s/r^3$ would occur near the lightest black holes, which would be of about one solar mass, where from (12),

$$\frac{C_{0101}}{A_b} \sim \frac{1}{A_b r_s^2} = \frac{1}{A_b} \left(\frac{c^2}{2Gm_\odot}\right)^2 \sim 10^{-77}.$$  

(69)
And regarding \( j^\tau \) as a substitute for \((1/4\pi)f^{\omega_\tau}_\omega\) from (43), the \( j_{\nu\mu}A^{-1}_\nu \) term is \(< 10^{-32}\) of \( f_{\nu\mu} \) from (71). Therefore, the last four terms in (71) must all be \(< 10^{-32}\) of \( f_{\nu\mu} \). Consequently, even for the most extreme worst-case fields accessible to measurement, the extra terms in Maxwell’s equations (45,46) must be \(< 10^{-32}\) of the ordinary terms.

The divergenceless term \( \theta_{\nu\alpha}[\epsilon_{\nu\mu}^\tau\alpha A^{-1}_\mu] \) of (67) should also be expected to be \(< 10^{-32}\) of \( f_{\nu\mu} \) from (60,61,65). However, we need to consider the possibility where \( \theta_\nu \) changes extremely rapidly. Taking the curl of (67), the \( F_{\nu\mu} \) and \( j_{\nu\mu} \) terms drop out, and we get something similar to the Proca equation (20,27).

\[
\theta_\rho = \left( -\theta_\rho,\sigma \right) + \frac{1}{2} \epsilon_{\rho}^{\sigma\nu\mu} (f^{\alpha\tau}C_{\alpha\tau[\nu\mu],\sigma}) + \frac{(3n-7)\Lambda}{(n-1)} \theta_\rho + (f^{3\rho}) \left( \frac{1}{2\Lambda_b} \right) \ldots (70)
\]

Here the constraint \( \theta_\nu = 0 \) results from the definition (63) and we are using a \((1,-1,-1,-1)\) metric signature. Eq. (70) suggests that \( \theta_\rho \) Proca-wave solutions might exist in this theory. Assuming that the magnitude of \( C_{\alpha\tau\nu\mu} \) is roughly proportional to \( \theta_\rho \) for such waves, and assuming that \( f_{\nu\mu} \) going according to (67) with \( F_{\nu\mu} = 0 \), the extra terms in (70) could perhaps be neglected in the weak field approximation. Using (70) and \( \Lambda_b \approx -\Lambda_z = C_z l_P^2 \) from (12), such Proca-wave solutions would have an extremely high minimum frequency

\[
\omega_{\text{proca}} = \sqrt{2\Lambda_b} \approx \sqrt{2C_z} \omega_c l_P \sim 10^{43} \text{rad/s}, (71)
\]

where the cutoff frequency \( \omega_c \) and \( C_z \) come from (13,14).

There are several points to make about (70,71). 1) A particle associated with a \( \theta_\rho \) field would have mass \( \hbar\omega_{\text{proca}} \) which is much greater than could be produced by particle accelerators, and so it would presumably not conflict with high energy physics experiments. 2) We have recently shown that \( \sin[kr - \omega t] \) Proca-wave solutions do not exist in the theory, using an asymptotically flat Newman-Penrose \( 1/r \) expansion similar to \( 28,29 \). However, it is still possible that wave-packet solutions could exist. 3) Substituting the \( k = 0 \) flat space Proca-wave solution \( \theta_\rho = (0,1,0,0)\sin[\omega_{\text{proca}} t] \) and \( F_{\nu\mu} = 0 \) into (67,68,57), and assuming a flat background space gives \( T_{\rho\sigma} = -2/\Lambda_b < 0 \). This suggests that Proca-wave solutions might have negative energy, but because \( \sin[kr - \omega t] \) solutions do not exist, and because of the other approximations used, this calculation is extremely uncertain. 4) With a cutoff frequency \( \omega_c \sim 1/l_P \) from (13) we have \( \omega_{\text{proca}} > \omega_c \) from (71,13,14), so Proca-waves would presumably be cut off. More precisely, (71) says that Proca-waves would be cut off if \( \omega_c > 1/(l_P\sqrt{2C_z}) \). Whether \( \omega_c \) is caused by a discreteness, uncertainty or foaminess of spacetime near the Planck length \( 30,31,32,33,34 \), or by some other effect, the same \( \omega_c \) which cuts off \( \Lambda_z \) in (12) should also cut off very high frequency electromagnetic and gravitational waves, and Proca-waves. 5) If wave-packet Proca-wave solutions do exist, and they have negative energy, it is possible that \( \theta_\rho \) could function as a kind of built-in Pauli-Villars field. Pauli-Villars regularization in quantum electrodynamics requires a negative energy Proca field with a mass \( \hbar\omega_{\text{proca}} \) that goes to infinity as \( \omega_c \to \infty \), as we have from (71,6). As mentioned initially, it might be more correct to
take the limit of this theory as $\omega_c \to \infty$, $|A_z| \to \infty$, $A_b \to \infty$, as in quantum electrodynamics. In this limit (70,71) require that $\theta \to 0$ or $\omega_{Proca} \to \infty$, and the theory becomes exactly Einstein-Maxwell theory as in (15). Finally, we should emphasize that Proca-wave solutions are only a possibility suggested by equation (70). Their existence and their possible interpretation are just speculation at this point. We are continuing to pursue these questions.

6 The Lorentz force equation

A generalized contracted Bianchi identity for this theory can be derived using only the connection equations (48) and the symmetry (8) of $\tilde{\Gamma}^{\alpha}_{\nu\mu}$.

\[
(\sqrt{-N} N^{\sigma\nu} \tilde{R}_{\sigma\lambda} + \sqrt{-N} N^{\lambda\nu} \tilde{R}_{\sigma\lambda})_{,\nu} - \sqrt{-N} N^{\lambda\sigma} \tilde{R}_{\sigma\nu,\lambda} = 0.
\] (72)

This identity can also be written in terms of $g^{\rho\tau}$, $f^{\rho\tau}$ and $\tilde{G}^{\nu\mu}$ from (4,22,39),

\[
\tilde{G}^{\nu\sigma} = \left(\frac{3}{2} f^{\sigma\nu} \tilde{R}_{[\rho,\nu]} + f^{\rho\sigma} ; \alpha \tilde{R}_{[\sigma\nu]}\right) \sqrt{2} i A_b^{-1/2}.
\] (73)

The identity was originally derived\[3, 7\] assuming $j^{\nu} = 0$ in (48). The derivation for $j^{\nu} \neq 0$ was first done\[25\] by applying an infinitesimal coordinate transformation to an invariant integral, and it is also done in Appendix B of\[21\] using a much different direct computation method. Clearly (72,73) are generalizations of the ordinary contracted Bianchi identity $2(\sqrt{-g} R^{\nu,\lambda})_{,\nu} - \sqrt{-g} g^{\rho\sigma} R_{\sigma\nu,\lambda} = 0$ or $G^{\nu\sigma} = 0$, which is also valid in this theory.

Another useful identity\[13\] can be derived using only the definitions (4,22)

\[
\left(N^{(\mu}_{\nu}) - \frac{1}{2} \delta^{\mu}_{\nu} N^{\rho}_{\rho}\right) ; \mu = \left(\frac{3}{2} f^{\sigma\nu} N_{[\rho,\nu]} + f^{\rho\sigma} ; \alpha N_{[\sigma\nu]}\right) \sqrt{2} i A_b^{-1/2}.
\] (74)

The ordinary Lorentz force equation results from taking the divergence of the Einstein equations (11) using (73,74,75,81,74,21)

\[
T^{\nu\sigma} = \frac{1}{8\pi} \left[ \tilde{G}^{\nu\sigma} + A_b \left(N^{(\mu}_{\nu}) - \frac{1}{2} \delta^{\mu}_{\nu} N^{\rho}_{\rho}\right) ; \mu \right] F_{\nu\sigma} j^{\nu}.
\] (75)

Note that the covariant derivatives in (73,74,75) are all done using the Christoffel connection (20) formed from the symmetric metric (4).

7 The Einstein-Infeld-Hoffmann Equations of motion

For Einstein-Maxwell theory, the EIH method allows the equations of motion to be derived directly from the electro-vac field equations. For neutral particles the method has been verified to Post-Newtonian order\[11\], and in fact it was the method first used to derive the Post-Newtonian equations of motion\[35\]. For charged particles the method has been verified to Post-Coulombian order\[12,36,37\], meaning that it gives the same result as the Darwin Lagrangian\[27\]. In\[10\] we derived the exact Lorentz force equation for
this theory by including source terms in the Lagrangian. Here we derive the Lorentz force using the EIH method because it requires no source terms, and also to show definitely that the well known negative result of \cite{9,10} for the unmodified Einstein-Schrödinger theory does not apply to the present theory.

We will only cover the bare essentials of the EIH method which are necessary to derive the Lorentz force. We will also only calculate the equations of motion to Newtonian/Coulombian order, because this is the order where the Lorentz force first appears.

The EIH method assumes the “slow motion approximation”, meaning that \( v/c \ll 1 \). The fields are expanded in the form \cite{11,12,36,37},

\[
g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}\gamma_{\rho\sigma}/2, \tag{76}
\]

\[
\gamma_{00} = 2\gamma_{00}\lambda^2 + 4\gamma_{00}\lambda^4 \ldots \tag{77}
\]

\[
\gamma_{0k} = 3\gamma_{0k}\lambda^3 + 5\gamma_{0k}\lambda^5 \ldots \tag{78}
\]

\[
\gamma_{ik} = 4\gamma_{ik}\lambda^4 \ldots \tag{79}
\]

\[
A_0 = 2A_0\lambda^2 + 4A_0\lambda^4 \ldots \tag{80}
\]

\[
A_k = 3A_k\lambda^3 + 5A_k\lambda^5 \ldots \tag{81}
\]

\[
f_{0k} = 2f_{0k}\lambda^2 + 4f_{0k}\lambda^4 \ldots \tag{82}
\]

\[
f_{ik} = 3f_{ik}\lambda^3 + 5f_{ik}\lambda^5 \ldots \tag{83}
\]

where \( \lambda \sim v/c \) is the expansion parameter, the order of each term is indicated with a left subscript \cite{9}, \( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \), and Latin indices run from 1-3. The field \( \gamma_{\mu\nu} \) (often called \( \bar{h}_{\mu\nu} \) in other contexts) is used instead of \( g_{\mu\nu} \) only because it simplifies the calculations. Because \( \lambda \sim v/c \), when the expansions are substituted into the Einstein and Maxwell equations, a time derivative counts the same as one higher order in \( \lambda \). The general procedure is to substitute the expansions, and solve the resulting field equations order by order in \( \lambda \), continuing to higher orders until a desired level of accuracy is achieved. At each order in \( \lambda \), one of the \( \gamma_{\mu\nu} \) terms and one of the \( f_{\mu\nu} \) terms will be unknowns, and the equations will involve known results from previous orders because of the nonlinearity of the Einstein equations.

The expansions \( 76-83 \) use only alternate powers of \( \lambda \) essentially because the Einstein and Maxwell equations are second order differential equations \cite{35}, although for higher powers of \( \lambda \), all terms must be included to predict radiation \cite{12,36,37}. Because \( \lambda \sim v/c \), the expansions have the magnetic components \( A_k \) and \( f_{ik} \) due to motion at one order higher in \( \lambda \) than the electric components \( A_0 \) and \( f_{0i} \). As in \cite{12,36,37}, \( f_{0k} \) and \( f_{ik} \) have even and odd powers of \( \lambda \) respectively. This is the opposite of \cite{9,10} because we are assuming a direct definition of the electromagnetic field \( f^{\alpha\rho} = \varepsilon^{\alpha\rho\sigma\mu}N_{[\sigma\mu]}/2 \) assumed in \cite{9,10}.

The field equations are assumed to be of the standard form

\[
G_{\mu\nu} = 8\pi T_{\mu\nu} \quad \text{where} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}, \tag{84}
\]

or

\[
R_{\mu\nu} = 8\pi S_{\mu\nu} \quad \text{where} \quad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}. \tag{85}
\]
However, with the EIH method we solve a sort of quasi-Einstein equations,

$$0 = \tilde{G}_{\mu\nu} - 8\pi \tilde{T}_{\mu\nu},$$

(86)

where

$$\tilde{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} R_{\alpha\beta}, \quad \tilde{T}_{\mu\nu} = S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} S_{\alpha\beta}. $$

(87)

Here the use of $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$ is not an approximation because (85) implies whether $\tilde{G}_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ are defined with $\eta_{\mu\nu}$ or $g_{\mu\nu}$. Note that the references use many different notations in (86): instead of $\tilde{G}_{\mu\nu}$ others use $\Pi_{\mu\nu}/2 + \Lambda_{\mu\nu}$, $\Phi_{\mu\nu}/2 + A_{\mu\nu}$ or [LS : $\mu\nu$] and instead of $8\pi \tilde{T}_{\mu\nu}$ others use $-2S_{\mu\nu}$, $-A'_{\mu\nu}$, $-A_{\mu\nu}$ or [RS : $\mu\nu$].

The equations of motion result as a condition that the field equations (86) have acceptable solutions. In the language of the EIH method, acceptable solutions are those that contain only “pole” terms and no “dipole” terms, and this can be viewed as a requirement that the solutions should resemble Reissner-Nordström solutions asymptotically. To express the condition of solvability we must consider the integral of the field equations (86) over 2D surfaces $S$ surrounding each singularity,

$$lC_\mu = \frac{1}{2\pi} \int_S (l\tilde{G}_{\mu k} - 8\pi l\tilde{T}_{\mu k}) n_k dS.$$

(88)

Here $n_k$ is the surface normal and $l$ is the order in $\lambda$. Assuming that the divergence of the Einstein equations (84) vanishes, and that (86) has been solved to all previous orders, it can be shown[11] that in the current order

$$(l\tilde{G}_{\mu k} - 8\pi l\tilde{T}_{\mu k})|_{k=0} = 0. $$

(89)

Here and throughout this section “$|$” represents ordinary derivative[11]. From Green’s theorem, (89) implies that $lC_\mu$ in (88) will be independent of surface size and shape[11]. The condition for the existence of an acceptable solution for $4\gamma_{ik}$ is simply

$$4C_i = 0,$$

(90)

and these are also our three $O(\lambda^4)$ equations of motion[11]. The $C_0$ component of (88) causes no constraint on the motion[11] so we only need to calculate $\tilde{G}_{ik}$ and $\tilde{T}_{ik}$.

At this point let us introduce a Lemma from [11] which is derived from Stokes’s theorem. This Lemma states that

$$\int_S F_{(\ldots)kl} n_k dS = 0 \quad \text{if} \quad F_{(\ldots)kl} = -F_{(\ldots)lk},$$

(91)

where $F_{(\ldots)kl}$ is any antisymmetric function of the coordinates, $n_k$ is the surface normal, and $S$ is any closed 2D surface which may surround a singularity. The equation $4C_i = 0$ is a condition for the existence of a solution for $4\gamma_{ik}$ because
\(4\gamma_{ik}\) is found by solving the \(O(\lambda^4)\) field equations [80], and \(4C_i\) is the integral of these equations. However, because of the Lemma [91], it happens that the \(4\gamma_{ik}\) terms in \(\mathcal{G}_{ik}\) integrate to zero in [88], so that \(4C_i\) is actually independent of \(4\gamma_{ik}\). In fact it is a general rule that \(C_i\) for one order can be calculated using only results from previous orders [11], and this is a crucial aspect of the EIH method. Therefore, the calculation of the \(O(\lambda^4)\) equations of motion [90] does not involve the calculation of \(4\gamma_{ik}\), and we will see below that it also does not involve the calculation of \(3f_{ik}\) or \(4f_{ok}\).

The \(4\mathcal{G}_{ik}\) contribution to [88] is derived in [11]. For two particles with masses \(m_1, m_2\) and positions \(\xi_1, \xi_2\), the \(O(\lambda^3)\) term from the integral over the first particle is

\[
\mathcal{G}_i = \frac{1}{2\pi} \int 4\mathcal{G}_{ik} n_k dS = -4 \left\{ m_1 \xi_1^i - m_1 m_2 \frac{\partial}{\partial \xi_1^i} \left( \frac{1}{r} \right) \right\},
\]

where

\[
r = \sqrt{(\xi_1^i - \xi_2^i)(\xi_1^i - \xi_2^i)}.
\]

If there is no other contribution to [88], then [90] requires that \(\mathcal{G}_i = 0\) in [92], and the particle acceleration will be proportional to a \(\nabla (m_1 m_2 / r)\) Newtonian gravitational force. These are the EIH equations of motion for vacuum general relativity to \(O(\lambda^4)\), or Newtonian order.

Because our effective energy momentum tensor [88] is quadratic in \(f_{\mu\nu}\), and the expansions [77,83] begin with \(\lambda^2\) terms, the \(O(\lambda^2) - O(\lambda^3)\) calculations leading to [92] are unaffected by the addition of the electromagnetic terms to the vacuum field equations. However, the \(8\pi 4\mathcal{T}_{ik}\) contribution to [88] will add to the \(4\mathcal{G}_{ik}\) contribution. To calculate this contribution, we will assume that our singularities in \(f_{\nu\mu}\) are simple moving Coulomb potentials, and that \(\theta^\rho = 0, \Lambda = 0\). Then from [67,82,83] we see that \(2\mathcal{F}_{0k} = 2f_{0k}\), and from inspection of the extra terms in our Maxwell equations [45,46,67] and Proca equation [71], we see that these equations are both solved to \(O(\lambda^3)\). Because [88] is quadratic in \(f_{\mu\nu}\), we see from [82,83] that only \(2f_{0k}\) can affect the \(O(\lambda^3)\) equations of motion. Including only \(2f_{0k}\), our \(f_{\mu\nu}\) is then a sum of two Coulomb potentials with charges \(Q_1, Q_2\) and positions \(\xi_1^i, \xi_2^i\) of the form

\[
2A_\mu = (2\varphi, 0, 0, 0), \quad 2f_{0k} = 2 A_{[k|0]} = -2\varphi_{|k},
\]

\[
2\varphi = \psi^1 + \psi^2, \quad \psi^1 = Q_1 / r_1, \quad \psi^2 = Q_2 / r_2,
\]

\[
r_a = \sqrt{(x^a - \xi_1^a)(x^a - \xi_2^a)}, \quad a = 1...2.
\]

Because [88] is quadratic in both \(f_{\mu\nu}\) and \(g_{\mu\nu}\), and the expansions [77,83] start at \(\lambda^2\) in both of these quantities, no gravitational-electromagnetic interactions will occur at \(O(\lambda^4)\). This allows us to replace covariant derivatives with ordinary derivatives, and \(g_{\mu\nu}\) with \(\eta_{\mu\nu}\) in [88]. This also allows us to replace \(\mathcal{T}_{\mu\nu}\) from [80,87] with [88]. Keeping only \(O(\lambda^3)\) terms when [91] is
substituted, the spacial part of (58) gives,

\[ 8\pi^2 \tilde{T}_{sm} = 2 \left( f_s^0 f_{0m} - \frac{1}{2} \eta_{sm} f^r_0 f_{0r} \right) + \left( 2 f^a_0 f_{0(s|m)} + f^a_s f_{0|m}^a + f^0_a f^a_0 |m| - \frac{1}{2} \eta_{sm} (f^r_0 f_{0r})^a |a| \right) \Lambda^{-1}_b. \tag{97} \]

Note that \( \varphi \) from (95) obeys Gauss's law,

\[ \varphi |a| a = 0. \tag{98} \]

Substituting (94) into (97) and using (98) gives

\[ 8\pi^2 \tilde{T}_{sm} = -2 \left( \varphi |s| \varphi |m| + \frac{1}{2} \eta_{sm} \varphi |r| \varphi |r| \right) + \left( 2 \varphi |s| \varphi |m| - \frac{1}{2} \eta_{sm} (\varphi |r| \varphi |r|)^a |a| \right) \Lambda^{-1}_b \]

\[ = -2 \left( \varphi |s| \varphi |m| + \frac{1}{2} \eta_{sm} \varphi |r| \varphi |r| \right) - 2 \left( \varphi |s| \varphi |m| + \varphi |r| \varphi |r| |s|m |m| \right) \Lambda^{-1}_a. \tag{99} \]

From (91), the second group of terms in (100) integrates to zero in (88), so it can have no effect on the equations of motion. The first group of terms in (100) is what one gets with Einstein-Maxwell theory \( [12, 36, 37] \), so at this stage we have effectively proven that the theory predicts a Lorentz force.

For completeness we will finish the derivation. First, we see from (100) that \( 4 \tilde{T}_{sm}|_a = 0 \). This is to be expected because of (89), and it means that the \( 8\pi^2 \tilde{T}_{sm} \) contribution to the surface integral (88) will be independent of surface size and shape. This also means that only 1/distance² terms such as \( \eta_{sm}/r^2 \) or \( x_s x_m / r^4 \) can contribute to (88). The integral over a term with any other distance-dependence would depend on the surface radius, and therefore we know beforehand that it must vanish or cancel with other similar terms [11]. Now, \( \varphi |s| = \psi^1 |s| + \psi^2 |s| \) from (95). Because \( \psi^1 |s| \) and \( \psi^2 |s| \) both go as 1/distance², but are in different locations, it is clear from (100) that contributions can only come from cross terms between the two. Including only these terms gives,

\[ 8\pi^2 \tilde{T}^c_{sm} = -2 (\psi^1 |s| \psi^2 |m| + \psi^2 |s| \psi^1 |m| + \eta_{sm} \psi^1 |r| \psi^2 |r|). \tag{101} \]

Some integrals we will need can be found in [11]. With \( \psi = 1/\sqrt{x^s x^s} \) we have,

\[ \frac{1}{4\pi} \int_0^1 \psi |m| n_m dS = -1, \quad \frac{1}{4\pi} \int_0^1 \psi |a| n_m dS = -\frac{1}{3} \delta_{sm}. \tag{102} \]

Using (101, 102, 105) and integrating over the first particle we get,

\[ \frac{1}{2\pi} \int_1 \left[ -8\pi \tilde{T}_{sm} \right] n_m dS = \frac{1}{2\pi} \int_1 2 (\psi^1 |s| \psi^2 |m| + \psi^2 |s| \psi^1 |m| + \eta_{sm} \psi^1 |r| \psi^2 |r|) n_m dS \]

\[ = 4 Q_1 \psi^2_s (\xi_1) (-\frac{1}{3} - 1 + \frac{1}{3}) = -4 Q_1 \psi^2_s (\xi_1). \tag{104} \]
Using (90, 102, 104, 105) we get
\begin{align}
0 &= 4C_{i} = -4 \left\{ m_{1} \xi_{i} - m_{1} m_{2} \frac{\partial}{\partial \xi_{1}} \left( \frac{1}{r} \right) \right\} - 4Q_{1} \psi_{0}^{2} (\xi_{1}) \quad (105) \\
&= -4 \left\{ m_{1} \xi_{i} - m_{1} m_{2} \frac{\partial}{\partial \xi_{1}} \left( \frac{1}{r} \right) + Q_{1} Q_{2} \frac{\partial}{\partial \xi_{1}} \left( \frac{1}{r} \right) \right\}, \quad (106)
\end{align}

where
\begin{equation}
r = \sqrt{(\xi_{1}^{2} - \xi_{2}^{2})(\xi_{1}^{2} - \xi_{3}^{2})}. \quad (107)
\end{equation}

These are the EIH equations of motion for this theory to $O(\lambda^{4})$, or Newtonian/Coulombian order. These equations clearly exhibit the Lorentz force, and in fact they match the $O(\lambda^{4})$ equations of motion of Einstein-Maxwell theory.

### 8 An exact electric monopole solution

Here we give an exact charged solution for this theory which closely approximates the Reissner-Nordström solution \cite{38, 39} of Einstein-Maxwell theory. A MAPLE program \cite{40} which checks the solution and the derivation \cite{41} are available. The solution is

\begin{align}
ds^{2} &= \tilde{c} adt^{2} - \frac{1}{\tilde{c} a} dr^{2} - \tilde{c} r^{2} d\theta^{2} - \tilde{c} r^{2} \sin^{2} \theta d\phi^{2}, \\
f^{10} &= \frac{Q}{\tilde{c} r}, \quad \sqrt{-N} = r^{2} \sin \theta, \quad \sqrt{-g} = \tilde{c} r^{2} \sin \theta, \\
F_{01} &= -A_{0} = \frac{Q}{r^{2}} \left[ 1 + \frac{4M}{A_{0} r^{3}} - \frac{4A}{3A_{0}} + 2 \left( \tilde{c} - 1 - \frac{Q^{2} \dot{\tilde{V}}}{A_{0} r^{4}} \right) \left( 1 - \frac{A}{A_{0}} \right) \right], \quad (110) \\
a &= 1 - \frac{2M}{r} - \frac{A r^{3}}{3} + \frac{Q^{2} \dot{\tilde{V}}}{r^{2}} \left( 1 - \frac{A}{A_{0}} \right), \quad (111)
\end{align}

where (') means $\partial/\partial r$, and $\tilde{c}$ and $\dot{\tilde{V}}$ are very close to one for ordinary radii,

\begin{align}
\tilde{c} &= \sqrt{1 - \frac{2Q^{4}}{A_{0} r^{4}}} = 1 - \frac{Q^{2}}{A_{0} r^{4}} \cdots - \frac{(2i)!}{i! (2i+1) A_{0} r^{4}} \left( \frac{2Q^{2}}{A_{0} r^{4}} \right)^{i}, \\
\dot{\tilde{V}} &= \frac{r A_{0}}{Q^{2}} \left( \int r^{2} \tilde{c} \, dr - \frac{r^{3}}{3} \right) = 1 + \frac{Q^{2}}{10A_{0} r^{4}} \cdots + \frac{(2i)!}{i! (i+1)! 4^{i} (2i+1)} \left( \frac{2Q^{2}}{A_{0} r^{4}} \right)^{i},
\end{align}

and the nonzero connections are

\begin{align}
\tilde{F}_{00}^{1} &= \frac{a d'}{2} - \frac{4a^{2} Q^{2}}{A_{0} r^{5}}, \quad \tilde{F}_{0}^{0} = \tilde{F}_{0}^{0} = \frac{a'}{2a}, \quad \tilde{F}_{11}^{1} = \frac{-a'}{2a}, \\
\tilde{F}_{12}^{3} &= \tilde{F}_{21}^{3} = \tilde{F}_{13}^{3} = \tilde{F}_{31}^{3} = \frac{1}{r}, \\
\tilde{F}_{22}^{3} &= -a r, \quad \tilde{F}_{33}^{3} = -a r \sin^{2} \theta, \quad \tilde{F}_{23}^{3} = \tilde{F}_{32}^{3} = \cot \theta, \quad \tilde{F}_{33}^{2} = -\sin \theta \cos \theta, \\
\tilde{F}_{02}^{2} &= -\tilde{F}_{20}^{2} = \tilde{F}_{03}^{3} = -\tilde{F}_{30}^{3} = -\frac{2a \sqrt{2} i Q}{\sqrt{A_{0} r^{3}}}, \quad \tilde{F}_{10}^{1} = -\tilde{F}_{01}^{1} = -\frac{2a \sqrt{2} i Q}{\sqrt{A_{0} r^{3}}}.
\end{align}
The solution matches the Reissner-Nordstrøm solution except for terms which are negligible for ordinary radii. To see this, first recall that $\Lambda/\Lambda_b \sim 10^{-122}$ from (3.12), so the $\Lambda$ terms are all extremely tiny. Ignoring the $\Lambda$ terms and keeping only the $O(\Lambda^{-1})$ terms in (110,111,112,113) gives

$$F_{01} = \frac{Q}{r^2} \left[ 1 + \frac{4M}{\Lambda_b r^3} - \frac{4Q^2}{\Lambda_b r^4} \right] + O(\Lambda^{-2}),$$

(115)

$$A_0 = \frac{Q}{r} \left[ 1 + \frac{M}{\Lambda_b r^3} - \frac{4Q^2}{5\Lambda_b r^4} \right] + O(\Lambda^{-2}),$$

(116)

$$a = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \left[ 1 + \frac{Q^2}{10\Lambda_b r^4} \right] + O(\Lambda^{-2}),$$

(117)

$$\tilde{c} = 1 - \frac{Q^2}{\Lambda_b r^4} + O(\Lambda^{-2}).$$

(118)

For the smallest radii probed by high-energy particle physics we get from (3.16),

$$\frac{Q^2}{\Lambda_b r^4} \sim 10^{-66}.$$ 

(119)

The worst-case value of $M/\Lambda_b r^3$ might be near the Schwarzschild radius $r_s$ of black holes where $r = r_s = 2M$ and $M/\Lambda_b r^3 = 1/2\Lambda_b r_s^2$. This value will be largest for the lightest black holes, and the lightest black hole that we can expect to observe would be of about one solar mass, where we have

$$\frac{M}{\Lambda_b r^3} \sim \frac{1}{2\Lambda_b} \left( \frac{c^2}{2GM_\odot} \right)^2 \sim 10^{-77}.$$ 

(120)

From (119,120,3.12) we see that our electric monopole solution (108,111) has a fractional difference from the Reissner-Nordstrøm solution [38, 39] of at most $10^{-66}$ for worst-case radii accessible to measurement. Clearly our solution does not have the deficiencies of the Papapetrou solution [42, 43] in the original theory, and it is almost certainly indistinguishable from the Reissner-Nordstrøm solution experimentally. Also, when this solution is expressed in Newman-Penrose tetrad form, it can be shown to be of Petrov Type-D [24]. And of course the solution reduces to the Schwarzschild solution for $Q = 0$. And from (115,118) we see that the solution goes to the Reissner-Nordstrøm solution exactly in the limit as $\Lambda_b \to \infty$.

The only significant difference between our electric monopole solution and the Reissner-Nordstrøm solution occurs on the Planck scale. From (108,112), the surface area of the solution is [44].

$$\left( \text{surface area} \right) = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} = 4\pi r^2\tilde{c} = 4\pi r^2 \sqrt{1 - \frac{2Q^2}{\Lambda_b r^4}}.$$ 

(121)

The origin of the solution is where the surface area vanishes, so in our coordinates the origin is not at $r = 0$ but rather at

$$r_0 = \sqrt{Q(2/\Lambda_b)^{1/4}}.$$ 

(122)
From (35) we have \( r_0 \sim l_P \sim 10^{-33} \text{cm} \) for an elementary charge, and \( r_0 \ll 2M \) for any realistic astrophysical black hole. For \( Q/M < 1 \) the behavior at the origin is hidden behind an event horizon nearly identical to that of the Reissner-Nordström solution. For \( Q/M > 1 \) where there is no event horizon, the behavior at the origin differs markedly from the simple naked singularity of the Reissner-Nordström solution. For the Reissner-Nordström solution all of the relevant fields have singularities at the origin, with \( g_{00} \sim Q^2/r^2 \), \( A_0 = Q/r \), \( F_{01} = Q/r^2 \), \( R_{00} \sim 2Q^4/r^6 \) and \( R_{11} \sim 2/r^2 \). For our solution the metric has a less severe singularity at the origin, with \( g_{11} \sim \sqrt{r}/\sqrt{r - r_0} \). Also, the fields \( N_{\mu\nu}, N^{-\nu\mu}, \sqrt{-N}, A_\nu, \sqrt{-g}f^{\nu\mu}, \sqrt{-\bar{g}}f_{\nu\mu}, \sqrt{-g}g_{\nu\mu}, \) and the functions “a” and \( \bar{V} \) all have finite nonzero values and derivatives at the origin, because it can be shown that \( \bar{V}(r_0) = \sqrt{2}[\Gamma(1/4)]^2/6\sqrt{\pi}−2/3 = 1.08137 \). The fields \( F_{\nu\mu}, \Gamma^\alpha_{\nu\mu} \) and \( \sqrt{-g} R_{\nu\mu} \) are also finite and nonzero at the origin, so if we use the tensor density form of the field equations [47], there is no ambiguity as to whether the field equations are satisfied at this location.

9 An exact electromagnetic plane-wave solution

Here we give an exact electromagnetic plane-wave solution for this theory which is identical to the electromagnetic plane-wave solution in Einstein-Maxwell theory, usually called the Baldwin-Jeffery solution [45, 46, 47, 48]. We will not do a full derivation, but a MAPLE program [40] which checks the solution is available. We present the solution in the form of an pp-wave solution [47], and a gravitational wave component is included for generality. The solution is expressed in terms of null coordinates \( x, y, u = (t-z)/\sqrt{2}, v = (t+z)/\sqrt{2} \),

\[
g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & H & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sqrt{-g}f^{\mu\nu} = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & \dot{f}_x \\ 0 & 0 & 0 & \dot{f}_y \\ 0 & 0 & 0 & \dot{f}_y \\ -\dot{f}_x & -\dot{f}_y & 0 & 0 \end{pmatrix}, \quad (123)
\]

\[
f_{\mu\nu} = 2A_{[\nu,\mu]} = 2A_{[\mu,\nu]} = \sqrt{2} \begin{pmatrix} 0 & 0 & -\dot{f}_x & 0 \\ 0 & 0 & -\dot{f}_y & 0 \\ \dot{f}_x & \dot{f}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sqrt{-g} = \sqrt{-N} = 1 \quad (124)
\]

where

\[
k_\mu = (0,0,-1,0), \quad A_\mu = (0,0,A,0), \quad A = -\sqrt{2}(x\dot{f}_x + y\dot{f}_y), \quad (125)
\]

\[
H = 2\dot{H} + A^2
\]

\[
= 2(h_x x^2 + h_x x y - h_x y^2) + 2(f_x^2 + f_y^2)(x^2 + y^2), \quad (126)
\]

\[
\dot{H} = h_x x^2 + h_x x y - h_x y^2 + (y\dot{f}_x - x\dot{f}_y)^2. \quad (127)
\]

and the nonzero connections are

\[
\dot{F}^3_{33} = \frac{1}{2} \frac{\partial H}{\partial x}, \quad \dot{F}^3_{33} = \frac{1}{2} \frac{\partial H}{\partial y}, \quad \dot{F}^4_{33} = \frac{1}{2} \frac{\partial H}{\partial u} - \frac{1}{A_0} \frac{\partial (f_x^2 + f_y^2)}{\partial u},
\]
\[
\tilde{F}_{43}^4 = \frac{1}{2} \frac{\partial H}{\partial x} - \frac{2i}{\sqrt{\Lambda_b}} \frac{\partial \tilde{f}_x}{\partial u}, \quad \tilde{F}_{31}^4 = \frac{1}{2} \frac{\partial H}{\partial x} + \frac{2i}{\sqrt{\Lambda_b}} \frac{\partial \tilde{f}_x}{\partial u},
\]
\[
\tilde{F}_{23}^4 = \frac{1}{2} \frac{\partial H}{\partial y} - \frac{2i}{\sqrt{\Lambda_b}} \frac{\partial \tilde{f}_y}{\partial u}, \quad \tilde{F}_{32}^4 = \frac{1}{2} \frac{\partial H}{\partial y} + \frac{2i}{\sqrt{\Lambda_b}} \frac{\partial \tilde{f}_y}{\partial u}.
\]

(129)

Here \(h_+ (u), h_\times (u)\) characterize the gravitational wave component, \(\tilde{f}_x (u), \tilde{f}_y (u)\) characterize the electromagnetic wave component, and all of these are arbitrary functions of the coordinate \(u = (t - z)/\sqrt{2}\).

The solution above has been discussed extensively in the literature on Einstein-Maxwell theory \([45, 46, 47, 48]\) so we will not interpret it further. It is the same solution which forms the incoming waves for the Bell-Szekeres colliding plane-wave solution \([48]\), although the full Bell-Szekeres solution does not satisfy our theory because the electromagnetic field is not null after the collision.

10 Conclusions

The Einstein-Schrödinger theory is modified to include a cosmological constant \(\Lambda_z\) which multiplies the symmetric metric. This is assumed to be nearly cancelled by Schrödinger’s “bare” cosmological constant \(\Lambda_b\) which multiplies the nonsymmetric fundamental tensor, such that the total cosmological constant \(\Lambda = \Lambda_b + \Lambda_z\) matches measurement. The resulting theory closely approximates Einstein-Maxwell theory for \(|\Lambda_z| \sim 1/(\text{Planck length})^2\), and it becomes exactly Einstein-Maxwell theory in the limit as \(|\Lambda_z| \to \infty\).

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A Solution for \(N_{\nu\mu}\) in terms of \(g_{\nu\mu}\) and \(f_{\nu\mu}\)

Here we invert the definitions (4.22) of \(g_{\nu\mu}\) and \(f_{\nu\mu}\) to obtain (33,34), the approximation of \(N_{\nu\mu}\) in terms of \(g_{\nu\mu}\) and \(f_{\nu\mu}\). First let us define the notation

\[
\hat{f}^{\nu\mu} = f^{\nu\mu} \sqrt{2} i \Lambda_b^{-1/2}.
\]

(130)

We assume that \(|\hat{f}^{\nu\mu}| \ll 1\) for all components of the unitless field \(\hat{f}^{\nu\mu}\), and find a solution in the form of a power series expansion in \(\hat{f}^{\nu\mu}\). Lowering an index on (23) gives

\[
(\sqrt{-\tilde{\mathcal{N}}} / \sqrt{-g}) N^{\nu\mu} \alpha = \delta^{\nu\mu}_\alpha - \hat{f}^{\nu\mu}_\alpha.
\]

(131)

Let us consider the tensor \(\hat{f}^{\nu\mu}_\alpha\). Because \(g_{\nu\alpha}\) is symmetric and \(\hat{f}^{\nu\mu}\) is antisymmetric, it is clear that \(\hat{f}^{\nu\mu}_\alpha = 0\). Also because \(f_{\alpha\sigma} f^{\nu\mu}_\sigma\) is symmetric it is clear that

\[
\hat{f}^{\nu\mu}_\alpha = 0.
\]

This content seems to be from a technical or scientific paper, discussing solutions to Einstein-Maxwell equations with a focus on gravitational and electromagnetic waves. The text includes mathematical expressions and detailed explanations of theoretical concepts. Without additional context or a specific question, I can't provide a more specific breakdown or analysis of the content. However, the document appears to be discussing solutions to specific equations and their implications in the context of gravitational and electromagnetic theories.
\[ \dot{f}^\nu f^\rho_{\nu} \dot{f}^\rho_{\nu} = 0. \] In matrix language therefore \( \text{tr}(\dot{f}) = 0, \text{tr}(\dot{f}^3) = 0, \) and in fact \( \text{tr}(\dot{f}^p) = 0 \) for any odd \( p. \) Using the well-known formula \( \det(e^M) = \exp(\text{tr}(M)) \) and the power series \( \ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4 \ldots \) we then get \cite{72},

\[ \ln(\det(I - \dot{f})) = \text{tr}(\ln(I - \dot{f})) = -\dot{f}^\sigma f^\sigma_{\rho}/2 + (\dot{f}^4) \ldots \]  

(132)

Here the notation (\( \dot{f}^4 \)) refers to terms like \( \dot{f}^\sigma f^\sigma_{\rho} f^\rho_{\sigma} f^\sigma_{\nu} \). Taking \( \ln(\det()) \) on both sides of (131) using (132) and the identities \( \det(sM) = s^p \det(M), \det(M^{-1}) = 1/\det(M) \) gives

\[ \ln \left( \sqrt{-g} \right) = \frac{1}{(n-2)} \ln \left( \frac{N^{(n/2-1)}}{g^{(n/2-1)}} \right) = - \frac{1}{2(n-2)} \dot{f}^\sigma f^\sigma_{\rho} + (\dot{f}^4) \ldots \]  

(133)

Taking \( e^{\frac{2}{x}} \) on both sides of (133) and using \( e^{\frac{2}{x}} = 1 + x + x^2/2 \ldots \) gives

\[ \sqrt{-g} = 1 - \frac{1}{2(n-2)} \dot{f}^\sigma f^\sigma_{\rho} + (\dot{f}^4) \ldots \]  

(134)

Using the power series \( (1-x)^{-1} = 1 + x + x^2 + x^3 \ldots, \) or multiplying (131) term by term, we can calculate the inverse of (131) to get \cite{72}

\[ (\sqrt{-g}/\sqrt{-g}) N^{\nu}_{\mu} = \delta^\nu_{\mu} + \dot{f}^\nu_{\mu} + \dot{f}^\rho_{\nu} \dot{f}^\rho_{\mu} + \dot{f}^\rho_{\mu} \dot{f}^\rho_{\nu} + (\dot{f}^4) \ldots \]  

(135)

\[ N_{\nu\mu} = (\sqrt{-g}/\sqrt{-g}) (g_{\nu\mu} + \dot{f}_{\nu\mu} + \dot{f}_{\mu\nu} \dot{f}^\sigma_{\mu} + \dot{f}_{\nu\sigma} \dot{f}^\rho_{\sigma} \dot{f}^\rho_{\mu} + (\dot{f}^4) \ldots. \]  

(136)

Here the notation (\( \dot{f}^4 \)) refers to terms like \( \dot{f}^{\nu\sigma} \dot{f}^{\nu\sigma} \dot{f}^{\nu\mu} \). Since \( \dot{f}^{\nu\sigma} \dot{f}^{\nu\sigma} \) is symmetric and \( \dot{f}^{\nu\sigma} \dot{f}^{\nu\mu} \) is antisymmetric, we obtain from (135) the final result (33,34).

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