SUPERCONDUCTIVITY WITH DEFORMED FERMI SURFACES AND COMPACT STARS

Armen Sedrakian

Institute for Theoretical Physics, Tübingen University, D-72076 Tübingen, Germany

sedrakia@tphys.physik.uni-tuebingen.de

Abstract I discuss the deformed Fermi surface superconductivity (DFS) and some of its alternatives in the context of nucleonic superfluids and two flavor color superconductors that may exist in the densest regions of compact stellar objects.

1. Introduction

The astrophysical motivation to study the superconducting phases of dense matter arises from the importance of pair correlations in the observational manifestations of dense matter in compact stars. If the densest regions of compact stars contain deconfined quark matter it must be charge neutral and in $\beta$ equilibrium with respect to the Urca processes $d \rightarrow u + e + \bar{\nu}$ and $u + e \rightarrow d + \nu$, where $e$, $\nu$, and $\bar{\nu}$ refer to electron, electron neutrino, and antineutrino. The $u$ and $d$ quarks in deconfined matter fill two different Fermi spheres which are separated by a gap of the order of electron chemical potential. At high enough densities (where the typical chemical potentials become of the order of the rest mass of a $s$ quark), strangeness nucleation changes the equilibrium composition of the matter via the reactions $s \rightarrow u + e + \bar{\nu}$ and $u + e \rightarrow s + \nu$. Although the strangeness content of matter affects its $u$-$d$ flavor asymmetry, the separation of the Fermi energies remains a generic feature. The dense quark matter is expected to be a color superconductor (the early work is in Refs. [1]; recent developments are summarized in the reviews [2]).

Accurate description of the matter in this regime requires, first, tools to treat the Lagrangian of QCD in the nonperturbative regime and, second, an understanding of the superconductivity under asymmetry in the population of fermions that pair. The first principle lattice QCD calculations are currently not feasible for the purpose of understanding the physics of compact stars; the effective models that are used rarely incorporate all aspects of the known phenomenology like de-confinement and chiral restoration. Despite of the limitations of current models, a lot can be learned about generic features of possible
phases of dense matter at densities where the perturbation theory fails. This mini-review concentrates on the second issue - the BCS superconductors under asymmetric conditions. Since the subject is of importance in a broader context of metallic superconductors, nucleonic superfluids, and dilute atomic gases, and much of our current understanding comes from the research in these fields, I will describe the relevant physics of non-relativistic superconductors first (Sections 2 and 3). Section 4 discusses the flavor asymmetric condensates in the context of QCD using the effective Nambu-Jona-Lasinio model; the emphasis is on the color superconducting state with deformed Fermi surfaces, but the discussion is sufficiently general to be applied to other non-BCS phases.

A comprehensive coverage of the recent developments is not possible in the present format; the choice of the topics will be thus personal and the list of the references necessarily incomplete.

2. Homogeneous superconducting state

Historically, asymmetric superconductors were studied in the early sixties (shortly after the advent of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity) in the context of metallic superconductors with paramagnetic impurities [3–6]. There is no bulk magnetic field in these systems due to the Meissner effect, however the paramagnetic impurities flip the spins of electrons in the collisions thereby inducing an asymmetry in the populations of spin-up and down fermions (which are assumed to pair in a state of total spin zero). The effect of impurities can be modeled by an average spin-polarizing field which gives rise to a separation of the Fermi levels of the spin-up and down electrons. Weak coupling analysis of the BCS equations revealed a double valued character of the gap as a function of the difference in chemical potentials \( \delta \mu \equiv (\mu_\uparrow - \mu_\downarrow) / 2 \), where \( \mu_\uparrow, \mu_\downarrow \) are the chemical potentials of the spin up/down electrons. The first branch corresponds to a constant value \( \Delta(\delta \mu) = \Delta(0) \) over the asymmetry range \( 0 \leq \delta \mu \leq \Delta(0) \) and vanishes beyond the point \( \delta \mu = \Delta(0) \); the second branch exists in the range \( \Delta(0) / 2 \leq \delta \mu \leq \Delta(0) \) and increases from zero at the lower limit to \( \Delta(0) \) at the upper limit. Only the \( \delta \mu \leq \Delta(0) / \sqrt{2} \) portion of the upper branch is stable (that is, only in this range of asymmetries the superconducting state lowers the grand thermodynamic potential of the normal state). Thus, the dependence of the superconducting state on the shift in the Fermi surfaces is characterized by a constant value of the gap which vanishes at the Chandrasekhar-Clogston limit \( \delta \mu_1 = \Delta(0) / \sqrt{2} \) [3, 4].

The picture above, while formally correct, is physically irrelevant to many systems as it does not conserve the number of particles.

Consider a BCS superconductor under the action of an external field that produces an asymmetry in the population of the fermions; the effect of such field is to transform the symmetric Hamiltonian \( \mathcal{H} \rightarrow \mathcal{H} - \sigma_z I a^\dagger a \), where \( a^\dagger \)
and $a$ are the creation and destruction operators (in the second quantized form), $\sigma_z$ is the $z$ component of the vector of Pauli matrices, $I$ is the magnitude of the asymmetry (for example, for fermions in a magnetic field $I = \mu_B H$, where $\mu_B$ is the Bohr magneton and $H$ is the field intensity). In a non-relativistic set-up the gap and the densities of spin-up and spin-down species are determined by the equations [7–9]

$$\Delta_k = -\sum_{k'} V_{k,k'} \frac{\Delta_{k'}}{2E_{k'}} \left[ 1 - f(E_{k'}^\uparrow) - f(E_{k'}^\downarrow) \right], \quad (1)$$

$$\rho_{\downarrow(\uparrow)} = \frac{1}{2} \sum_k \left[ \left( 1 + \frac{\xi_k}{E_k} \right) f(E_{k}^\uparrow) + \left( 1 - \frac{\xi_k}{E_k} \right) f(-E_{k}^\downarrow) \right], \quad (2)$$

where $V_{k,k'}$ is the pairing interaction, $f(E)$ is the Fermi distribution function, $E_k = \sqrt{\xi_k^2 + \Delta_k^2}$, $E_{k}^{\uparrow(\downarrow)} = E_k \pm \delta \varepsilon_k$, and the symmetric and anti-symmetric combinations of the single-particle spectra are defined as

$$\xi_k = \frac{1}{2} (\varepsilon_{k\uparrow} + \varepsilon_{k\downarrow}), \quad \delta \varepsilon_k = \frac{1}{2} (\varepsilon_{k\uparrow} - \varepsilon_{k\downarrow}). \quad (3)$$

In the zero temperature limit the Fermi distribution function $f(E)$ reduces to a step function $\theta(-E)$. The single particle spectra $\varepsilon_{k\uparrow(\downarrow)}$ are completely general and may include the differences in the (effective) masses and/or self-energies of the two species. Eqs. (1) and (2) should be solved self-consistently. In the research on metallic superconductors these equations were decoupled and Eq. (1) was solved by parametrizing the asymmetry in terms of the difference in the chemical potentials, with the understanding that once the gap equation is solved the densities of the constituents can be computed a posteriori. However, as the value of $\delta \mu$ is changed so does the density of the system, i.e. such a scheme (while being correct) does not incorporate the particle number conservation. If the particle number conservation is implemented explicitly (by solving Eqs. (1) and (2) self-consistently) the gap becomes a single-valued function of the particle number asymmetry $\alpha = (\rho_{\uparrow} - \rho_{\downarrow})/(\rho_{\uparrow} + \rho_{\downarrow})$ [7–9].

Minimizing the free-energy of an asymmetric superconductor at fixed density and temperature leads to stable solutions for the entire region of density asymmetries where non-trivial solutions of the gap equation exist [8]. This can be seen in Figs. 1 and 2 where the temperature and asymmetry dependence of the pairing gap and the free-energy of a homogenous asymmetric superconductor are shown (the examples here and in Figs. 3-8 below are taken from the studies of the tensor $S$-wave pairing in isospin asymmetric nuclear matter, but the overall picture is generic to all asymmetric superconductors). In particular, we see that for a fixed temperature the gap and the free-energy are single valued functions of the density asymmetry $\alpha$ in a particle number conserving
scheme (contrary to the nonconserving scheme, where double valued solutions appear).

For large asymmetries the dependence of the gap on the temperature shows the re-entrance phenomenon - the pairing correlations are restored (e. g. for $\alpha = 0.1$ in Fig. 1) as the temperature is increased and a second (lower) critical temperature appears. The re-entrance in the superconducting state with increasing temperature can be attributed to the smearing of the Fermi surfaces which increases the phase-space overlap between the quasi-particles that pair. Increasing the temperature further suppresses the pairing gap due to the thermal excitation of the system very much the same way as in the symmetric superconductors. Clearly, the pairing gap has a maximum at some intermediate temperature. The values of the two critical temperatures are controlled by different mechanisms: the superconductivity is destroyed with decreasing temperature at a lower critical temperature when the smearing of the Fermi surfaces becomes insufficient to maintain the phase space coherence. The upper critical temperature is the analog of the BCS critical temperature and corresponds to a transition to the normal state because of the thermal excitation of the system. At low temperatures the transition from the normal to the superconducting state is of the first order, while at the temperatures near the critical temperature - of the second order. The order of the phase transition changes from the first to the second as the temperature is increased. Another aspect of the asymmetric superconducting state is the gapless nature of the excitations;
in analogy to the non-ideal Bose gas where only part of the particles are in the zero-momentum ground state, in the asymmetric superconductors not all the pairs are gapped (see e.g. [9]). The presence of gapless excitations affects the dynamical properties of superconductors - the heat capacity, thermal conductivity, photon and sound absorption, etc.

3. Superconducting phases with broken space symmetries

3.1 LOFF phase

Larkin and Ovchinnikov [10] and, independently, Fulde and Ferrell [11] (LOFF) discovered in 1964 that the superconducting state can sustain asymmetries beyond the Chandrasekhar-Clogston limit if one pairs electrons with nonzero center-of-mass momentum. The weak coupling result for the critical shift in the Fermi surfaces for LOFF phase is $\delta \mu_2 = 0.755 \Delta(0)$ $> \delta \mu_1 = 0.707 \Delta(0)$. Since the condensate wave-function depends on the center-of-mass momentum of the pair its Fourier transform will vary in the configuration space giving rise to a lattice structure with finite share modulus. This spatial variation of the order parameter in the configuration space implies that the condensate breaks both the rotational and translational symmetries. There are thus additional massless Goldstone collective excitations associated with the broken global symmetries (in excess of other collective excitations that are present in the symmetric phase).

Consider again non-relativistic fermions. Their BCS spectrum (for homogeneous systems) is isotropic; when the polarizing field drives apart the Fermi surfaces of spin-up and down fermions the phase space overlap is lost, the pair correlations are suppressed, and eventually disappear at the Chandrasekhar-Clogston limit. The LOFF phase allows for a finite center-of-mass momentum of Cooper pairs $\vec{Q}$ and the quasiparticle spectrum is of the form

$$\varepsilon_{\uparrow\downarrow}^{0}(\vec{Q}, \vec{q}) = \frac{1}{2m} \left( \frac{\vec{Q}}{2} \pm \vec{q} \right)^2 - \mu_{\uparrow\downarrow} + \text{selfenergy terms}. \quad (4)$$

Thus, the LOFF phase requires a positive $\propto Q^2$ increase in the kinetic energy of the quasiparticles which makes it less favorable than the BCS state. However, the anisotropic term $\propto \vec{Q} \cdot \vec{q}$ (which can be interpreted as a dipole deformation of the isotropic spectrum) modifies the phase space overlap of the fermions and promotes pairing. The LOFF phase becomes stable when the loss in the kinetic energy that is needed to move the condensate is smaller than the gain in the potential pairing energy due to an increase in the phase-space overlap. The magnitude of the total momentum is a (variational) parameter for a minimization of the ground state of the system. The dependence of the pairing gap and the free-energy of a LOFF superconductor on the total momentum of the condensate and the density asymmetry is shown in Figs. 3 and 4 [12].
The self-consistent solution of Eqs. (1) and (2) leads to a single valued pairing gap and stable superconducting state for arbitrary finite momentum of the condensate, in particular $Q \rightarrow 0$ limit is consistent with the earlier discussion of homogeneous asymmetric BCS condensates. For large enough asymmetries the minimum of the free-energy moves from the $Q = 0$ line to intermediate values of $Q$, i.e., the ground state of the system corresponds to a condensate with nonzero center-of-mass momentum of Cooper pairs. Note that for the near critical range of asymmetries the condensate exists only in the LOFF state and its dependence on the total momentum shows the re-entrance behavior seen in the temperature dependence of the homogeneous superconductors. Clearly, a single wave-vector condensate is an approximation; in general the LOFF phase can acquire a complicated lattice structure. A large number of lattice structures were studied in Refs. [13, 14] in the Ginzburg Landau regime, were it was found that the face-centered cubic lattice has the lowest energy. The LOFF phase obtains additional collective excitations (Goldstone modes) due to the breaking of the rotational and translational continuous space symmetries [15]. Identifying the order of the phase transition from the LOFF to the normal state is a complex problem and depends, among other things, on the preferred lattice structure (see Ref. [16] and references therein).
3.2 DFS phase

To motivate our next step recall that the LOFF spectrum can be viewed as a dipole \( \propto P_1(x) \) perturbation of the spherically symmetrical BCS spectrum, where \( P_l(x) \) are the Legendre polynomials, and \( x \) is the cosine of the angle between the particle momentum and the total momentum of the Cooper pair. The \( l = 1 \) term in the expansion about the spherically symmetric form of Fermi surface corresponds to a translation of the whole system, therefore it preserves the spherical shapes of the Fermi surfaces. We now relax the assumption that the Fermi surfaces are spherical and describe their deformations by expanding the spectrum in spherical harmonics [17, 18]

\[
\varepsilon_{\uparrow(\downarrow)}(\vec{Q}, \vec{q}) = \varepsilon_{\uparrow(\downarrow)}^0(\vec{Q}, \vec{q}) + \sum_l \epsilon_{l,\uparrow(\downarrow)} P_l(x),
\]

where the coefficients \( \epsilon_l \) for \( l \geq 2 \) describe the deformation of the Fermi surfaces which break the rotational \( O(3) \) symmetry down to \( O(2) \). The \( O(2) \) symmetry axis is chosen spontaneously and clearly need not coincide with the direction of the total momentum (this subsection assumes \( Q = 0 \)). A single-component and spatially homogeneous system of non-interacting particles fills the states within its Fermi sphere homogeneously. In Fermi liquids the homogeneous filling prescription is extrapolated to (arbitrarily strongly) interacting quasiparticles. It is by no means obvious that such a prescription should re-

Figure 5. The dependence of the pairing gap in the DFS phase on the density asymmetry and the total momentum of the condensate [17].

Figure 6. The dependence of the free energy of the DFS phase on the same parameters as in Fig. 5 [17].
Figure 7. A projection of the Fermi surfaces on a plane parallel to the axis of the symmetry breaking. The concentric circles correspond to the two populations of spin/isospin-up and down fermions in spherically symmetric state ($\delta \epsilon = 0$), while the deformed figures correspond to the state with relative deformation $\delta \epsilon = 0.64$. The density asymmetry is $\alpha = 0.35$.

main valid for two or multi-component systems which interact, for example, by pairing forces. The expansion (5) is an example of a non-Fermi-liquid prescription for filling the particle states within a volume bounded by a (deformed) Fermi surface; the deformations are stable if they lower the ground state energy of the system with respect to the undeformed state. Note that in solids the Fermi surfaces are rarely spherical while their topology is dictated by the form of and interactions with the ion lattice. Note also that one should distinguish between the spontaneous deformation of Fermi-surfaces and explicit breaking of rotational symmetry by external fields. In the latter case the initial Lagrangian contains term(s) that explicitly break the symmetry and the resulting anisotropy of the self-energies can be interpreted as a deformation of the Fermi surfaces [19]. These type of deformations are interaction induced and are unrelated to the spontaneous deformations that appear even if the interaction is O(3) symmetric.

In practice, the deformation parameters $\epsilon_l$ ($l \geq 2$) are determined from the minimization of the free-energy of the system in full analogy to the total momentum $Q$ of a Cooper pair. And they can be determined in a volume conserv-
Superconductivity with deformed Fermi surfaces and compact stars

ing manner by solving Eqs. (1) and (2) simultaneously (the resulting phase is abbreviated as the DFS phase). It is convenient to work with dimensionless deformation parameters corresponding to relative and conformal deformations defined as $\delta \epsilon = (\epsilon_{2, \uparrow} - \epsilon_{2, \downarrow})/2\mu$ and $\epsilon = (\epsilon_{2, \uparrow} + \epsilon_{2, \downarrow})/2\mu$, where $\mu$ is the chemical potential in the symmetric phase. The dependence of the pairing gap and the free-energy of the DFS phase on asymmetry and the relative deformation (at zero conformal deformation) is shown in Fig. 5 and 6, respectively. Although the density asymmetry ($\alpha$) changes in the interval $[-1; 1]$ in general, the symmetry of the equations with respect to the indices labeling the species reduces the range of $\alpha$ to $[0; 1]$. The relative deformation is not bounded and can assume both positive and negative values. Fig. 7 shows a typical configuration of deformed Fermi surface which lowers the ground state energy below the non-deformed state. For $\alpha = 0$ Eqs. (1), (2) and (3) are symmetrical under interchange of the sign of $\delta \epsilon$ and the critical deformation for which the pairing vanishes is the same for prolate/oblate deformations. For finite $\alpha$ and the positive range of $\delta \epsilon$, the maximum value of the gap is attained at constant $\delta \epsilon$; for negative $\delta \epsilon$ the maximum increases as a function of the deformation and saturates around $\delta \epsilon \simeq 1$. As for the LOFF phase, the re-entrance phenomenon sets in for large asymmetries as $\delta \epsilon$ is increased from zero to finite values. And the mechanism by which the superconductivity is revived is based on the same phase-space argument, but involves a deformation of the Fermi surfaces rather than a motion of the condensate. Unlike the LOFF phase, the DFS phases does not break the translational symmetry of a superconductor (there are still additional collective excitations generated by the broken continuous rotational symmetry).

3.3 DFS vs LOFF

Which patterns of symmetry breaking are the most favorable if the Cooper pairs move with a finite center-of-mass momentum and the Fermi surfaces are allowed to be deformed? To answer this question we use the set-up of the previous sections and choose to work with the spectrum (5) at finite values of $Q$ and $\delta \epsilon$ [18]. Figure 8 displays the difference between the free energies of the superconducting and normal states $\delta \mathcal{F}$ normalized to its value in the asymmetric BCS state $\delta \mathcal{F}_{00} = \delta \mathcal{F}(Q = 0, \delta \epsilon = 0)$. Since the energy of the pair interactions scales as the square of the pairing gap, the shape of the $\delta \mathcal{F}$ surface closely resembles that of the pairing gap (see for details Ref. [18]). The asymmetric BCS state is the stable ground state of the system ($\delta \mathcal{F} < 0$), however it corresponds to a saddle point - perturbations for finite $\delta \epsilon$ and $Q$ are unstable towards evolution to lower energy states. For the pure LOFF phase ($\delta \epsilon = 0$) the ground state corresponds to finite momentum $Q \sim 0.5$ (in units of Fermi-momentum $p_F$). For the pure DFS phase ($Q = 0$) there are two
Figure 8. The dependence of the free energy of the combined DFS and LOFF phases on the center-of-mass momentum of the pairs $Q$ (in units of Fermi momentum $p_F$) and the relative deformations $\delta \epsilon$ for a fixed density asymmetry [18].

The minima corresponding to $\delta \epsilon \simeq -0.8$ and $\delta \epsilon \simeq 0.55$, i.e., prolate and oblate deformations of the minority and majority Fermi spheres, respectively. In general the position of the minimum of $\delta F$ in the $\delta \epsilon$-$Q$ plane (passing through the minima of the limiting cases) prefers either large deformations or large finite momenta. The absolute minimum energy state corresponds to $\delta \epsilon \simeq 0.55$ and $Q = 0$; that is, while the LOFF phase is a local minimum state, it is unstable towards evolution to a pure DFS phase with oblate and prolate deformations of the majority and minority Fermi spheres. Further work will be needed to clarify how universal are these features. In particular, the assumption of a single wave-vector LOFF phase should be relaxed.

3.4 Alternatives

To complete our discussion of non-relativistic superfluids let us briefly mention some of the alternatives to the LOFF and DFS phases. One possibility is that the system prefers a phase separation of the superconducting and normal phases in real space, such that the superconducting phase contains particles with the same chemical potentials, i.e. is symmetric, while the normal phase remains asymmetric [20, 21].

Equal spin (isospin, flavor) pairing is another option, if the interaction between the same spin particles is attractive [22–24]. Since the separation of the
Superconductivity with deformed Fermi surfaces and compact stars

Fermi surfaces does not affect the spin-1 pairing on each Fermi surface, an asymmetric superconductor evolves into a spin-1 superconducting state (rather than a non-superconducting state) as the asymmetry is increased. Therefore, the spin-1 pairing is the limiting state for very large asymmetries. If the states corresponding to different Fermi surfaces are characterized by spin (as is the case in the metallic superconductors) the pairing interaction in a spin-1 state should be $P$ wave and the transition is from the $S$ to the $P$ wave pairing. For larger number of discrete quantum numbers that characterize the fermions (say spin and isospin) the transition may occur between different $S$ wave phases (e.g. from isospin singlet to the isospin triplet state in nuclear matter).

4. Flavor asymmetric quark condensates

This section deals with the color superconductors and describes a straightforward formalism for extending the discussion of the previous sections to relativistic systems. Below it will be assumed that the superconducting phase is chirally symmetric and particles are interacting only via a pairing force (self-energy and vertex renormalization are ignored). The flavor asymmetric color superconducting quark matter appears in the context of the two-flavor pairing (2SC-phase) described by the order parameter \[ \Delta \propto \langle \psi^T (x) C \gamma_5 \tau_2 \lambda_2 \psi(x) \rangle, \] (6)\]

where $C = i \gamma^2 \gamma^0$ is the charge conjugation operator, $\tau_2$ is the second component of the Pauli matrix acting in the $SU(2)_f$ flavor space, $\lambda_A$ is the antisymmetric Gell-Mann matrix acting in the $SU(3)_c$ color space. The Ansatz for the order parameter implies that the color $SU(3)_c$ symmetry is reduced to $SU(2)_c$ since only two of the quark colors are involved in the pairing while the third color remains unpaired. The effective Lagrangian density of the Nambu-Jona-Lasinio model that describes our system is of the form

\[
\mathcal{L}_{\text{eff}} = \bar{\psi}(x) (i \gamma^\mu \partial_\mu) \psi(x) + G_1 (\psi^T C \gamma_5 \tau_2 \lambda_A \psi(x)) (\psi^T C \gamma_5 \tau_2 \lambda_A \psi(x)).
\] (7)

The partial densities and the gap equation for the up and down paired quarks can be found from the fixed points of the thermodynamic potential density $\Omega$ \[ \frac{\partial \Omega}{\partial \Delta} = 0, \quad - \frac{\partial \Omega}{\partial \mu_f} = \rho_f; \] (8)

the flavor index $f = u, d$ refers to up ($u$) and down ($d$) quarks. For the Lagrangian density defined by Eq. (6) and the pairing channel Ansatz (6), the finite temperature thermodynamical potential $\Omega$ per unit volume is

\[
\Omega = -2 \sum_{p,ij} \left\{ 2p + \frac{1}{\beta} \log \left[ f(x_{ij}) \right]^{-1} + E_{ij} + \frac{2}{\beta} \log \left[ f(s_{ij} E_{ij}) \right]^{-1} \right\} + \frac{\Delta^2}{4G_1}.
\] (9)
where the indices $i, j = (+, −)$ sum over the four branches of the paired and unpaired quasiparticle spectra defined, respectively, as

$$\xi_{±±} = (p ± µ) ± δµ$$

and

$$E_{±±} = \sqrt{(p ± µ)^2 + |∆|^2 ± δµ},$$

where $δµ = (µ_u − µ_d)/2$ and $µ = (µ_u + µ_d)/2$ with $µ_u$ and $µ_d$ being the chemical potentials of the up and down quarks; $s_{+j} = 1$ and $s_{−j} = \text{sgn}(p − µ)$ and $f(ξ_{ij})$ are the Fermi distribution functions. The variations of the thermodynamic potential (9) provide the gap equation

$$\Delta = 8G_1 \sum_p \left\{ \frac{Δ}{E_{++} + E_{++}} \left[ \tanh \left( \frac{βE_{++}}{2} \right) + \tanh \left( \frac{βE_{−−}}{2} \right) \right] + \text{ex} \right\},$$

where ex abbreviates a second term which follows from the first one via a simultaneous interchange of the signs; the partial densities of the up/down quarks are

$$ρ_u/d = \sum_{p,j=±} \frac{2f(ξ_{−j}) - 2f(ξ_{±j})}{1 ± \frac{ξ_{j−} + ξ_{j+}}{E_{j−} + E_{j+}}} \tanh \left( \frac{βE_{j−}}{2} \right) ± \left( 1 ± \frac{ξ_{j−} + ξ_{j+}}{E_{j−} + E_{j+}} \right) \tanh \left( \frac{βE_{j+}}{2} \right),$$

where and the upper/lower sign corresponds to the $u/d$-quarks. The free energy $F$ is related to the thermodynamic potential $Ω$ by the relation $F = Ω + µ_uρ_u + µ_dρ_d$ and, as already discussed for non-relativistic superconductors, the energy should be minimized at constant temperature and density of the matter at various flavor asymmetries defined as $α ≡ (ρ_d − ρ_u)/(ρ_d + ρ_u)$ [25]. Eqs. (9) and (10) are the (ultra)relativistic counterparts of Eqs. (1) and (2) which, apart from the relativistic form of the spectrum, include the contribution of anti-particles.

To obtain the selfconsistent solutions we employ a three-dimensional momentum space cut-off $|p| < Λ$ to regularize the divergent integrals. The phenomenological value of the coupling constant $G_1$ in the $⟨qq⟩$ Cooper channel is related to the coupling constant in the $⟨q\bar{q}⟩$ di-quark channel by the relation $G_1 = N_c/(2N_c − 2)G$; the latter coupling constant and the cut-off are fixed by adjusting the model to the vacuum properties of the system [26, 27]. Figure 9 summarizes the main features of the color superconducting DFS phase [25]. The physically relevant regime of flavor asymmetries which is likely to occur in the charge-neutral matter under $β$ equilibrium is $0.1 ≤ α ≤ 0.3$ [28–31]. The dependence of the color superconducting gap (left panel) and the free energy difference between the superconducting state and normal state (right panel) are shown as a function of deformation parameter $ε_A$ for several flavor asymmetries at the baryonic density $ρ_B = 0.31$ fm$^{-3}$ and temperature $T = 2$ MeV.
Figure 9. The color superconducting gap (left panel) and the free energy (right panel) as a function of relative deformation parameter $\varepsilon_A$ for the values of the flavor asymmetry $\alpha = 0$ (solid lines), 0.1 (dashed lines), 0.2 (short dashed lines) and 0.3 (dashed dotted lines) at density $\rho_B = 0.31$ fm$^{-3}$ and temperature $T = 2$ MeV [25].

The properties of the asymmetric superconductors have been an exciting subject since the advent of the BCS theory of superconductivity more than four decades ago. While the early studies were motivated by the effects of the paramagnetic impurities on the superconducting state and the possible coexistence of the ferromagnetic and superconducting phases in metallic superconductors,
the recent work on this subject has been motivated by the need to understand the nucleonic superfluids, the colored quark superconductors and the dilute trapped atomic gases.

This mini-review focused on several non-BCS phases that may be featured by the asymmetric superconductors. The main points are summarized below:

- For small asymmetries, the superconducting state is homogeneous and the order parameter preserves the space symmetries. For most of the systems of interest the number conservation should be implemented by solving equations for the gap function and the densities of species self-consistently. In such a scheme the physical quantities are single valued functions of the asymmetry and temperature, contrary to the double valued results obtained in the non-conserving schemes.

- For large enough asymmetries the homogeneous state becomes unstable towards formation of either the LOFF phase - a superconducting state with nonzero center-of-mass momentum of the Cooper pairs, or the DFS phase - a superconducting state which requires a quadrupole deformation of Fermi surfaces. A combined treatment of these phases in non-relativistic systems shows that while the LOFF phase corresponds to a local minimum, the DFS phase has energy lower that the LOFF phase. These phases break either the rotational, the translational or both symmetries.

- The temperature dependence of the pairing gap for the homogeneous, LOFF and DFS superconducting phases shows the phenomenon of re-entrance: the superconducting state is revived at finite temperatures. There are two critical temperatures for the phase transitions from the normal to the superconducting state and back as the temperature is increased from zero to finite values.

- The color superconducting DFS-phase, which is treated in a four-fermion contact interaction model, is preferred to the homogeneous 2SC state for asymmetries that are typical to matter under $\beta$-equilibrium.

Acknowledgments

I would like to thank Umberto Lombardo, Herbert Muther, Philippe Nozieres, Peter Schuck, and Hans Schulze for their contribution to the research reported here. This work was supported by a grant provided by the SFB 382 of the DFG.
References

[1] B. C. Barrois, Nucl. Phys. B129, 390 (1977); S. C. Frautschi, in “Hadronic matter at extreme energy density”, edited by N. Cabibbo and L. Sertorio (Plenum Press, 1980); D. Bailin and A. Love, Phys. Rept. 107, 325 (1984) and references therein.

[2] K. Rajagopal and F. Wilczek, hep-ph/0011333; M. G. Alford, hep-ph/0102047; T. Schaefer, hep-ph/0304281; D. H. Rischke, nucl-th/0305030; C. D. Roberts and S. M. Schmidt, nucl-th/0005064; R. Casalbuoni and G. Nardulli, hep-ph/0305069.

[3] A. M. Clogston, Phys. Rev. Lett. 9, 266 (1962).

[4] B. S. Chandrasekhar, Appl. Phys. Lett. 1, 7 (1962).

[5] G. Sarma, Phys. Chem. Solids 24, 1029 (1963).

[6] L. P. Gor'kov and A. I. Rusinov, Zh. Eksp. Teor. Fiz. 46, 1363 (1964) [Sov. Phys. JETP 19, 922 (1964)].

[7] A. Sedrakian, T. Alm, and U. Lombardo, Phys. Rev. C 55, R582 (1996).

[8] A. Sedrakian and U. Lombardo, Phys. Rev. Lett. 84, 602 (2000).

[9] U. Lombardo, P. Nozieres, P. Schuck, H.-J. Schulze and A. Sedrakian, Phys. Rev. C 64, 064314 (2001).

[10] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 47, 1136 (1964) [Sov. Phys. JETP 20, 762 (1965)].

[11] P. Fulde and R. A. Ferrell, Phys. Rev. 135, A550 (1964).

[12] A. Sedrakian, Phys. Rev. C 63, 025801 (2001).

[13] J. A. Bowers and K. Rajagopal, Phys. Rev. D 66, 065002 (2002).

[14] J. A. Bowers, Ph. D. thesis, hep-ph/0305301.

[15] R. Casalbuoni, R. Gatto, M. Mannarelli and G. Nardulli, Phys. Rev. D 66, 014006 (2002).

[16] R. Casalbuoni and G. Tonini, hep-ph/0310128.

[17] H. Muther and A. Sedrakian, Phys. Rev. Lett. 88, 252503 (2002).

[18] H. Muther and A. Sedrakian, Phys. Rev. C 67, 015802 (2003).

[19] E. Nakano, T. Maruyama and T. Tatsumi, Phys. Rev. D 68, 105001 (2003).

[20] P. F. Bedaque, Nucl. Phys. A 697, 569 (2002).

[21] P. F. Bedaque, H. Caldas, G. Rupak, cond-mat/0306694.

[22] T. Shafer, Phys. Rev. D 62, 094007 (2002).

[23] M. Buballa, J. Hosek and M. Oertel, Phys. Rev. Lett. 90, 182002 (2003).

[24] M. Alford, J. Bowers, J. Cheyne and G. Cowan, Phys. Rev. D 67, 054018 (2003).

[25] H. Muther and A. Sedrakian, Phys. Rev. D 67, 085024 (2003).

[26] T. M. Schwarz, S. P. Klevansky and G. Papp, Phys. Rev. C 60, 055205 (1999).

[27] M. Buballa, J. Hosek and M. Oertel, Phys. Rev. D 65, 014018 (2002).

[28] K. Iida and G. Baym, Phys. Rev. D 63, 074018 (2001).

[29] A. W. Steiner, S. Reddy and M. Prakash, Phys. Rev. D 66, 094007 (2002).

[30] D. Blaschke, S. Fredriksson, H. Grigorian and A. M. Oztas, nucl-th/0301002.

[31] M. Huang and I. Shovkovy, Nucl. Phys. A 729, 835 (2003).