Uncertainty in time–frequency representations on finite Abelian groups and applications

Felix Krahmer † Götz E. Pfander ‡ Peter Rashkov ‡

Keywords: Uncertainty principles, short time Fourier transformation, Gabor frames, sparsity.

ABSTRACT

Classical and recent results on uncertainty principles for functions on finite Abelian groups relate the cardinality of the support of a function to the cardinality of the support of its Fourier transforms. We use these results and their proofs to obtain similar results relating the support sizes of functions and their short–time Fourier transforms. Further, we discuss applications of our results. For example, we use our results to construct a class of equal norm tight Gabor frames that are maximally robust to erasures and we discuss consequences of our findings to the theory of recovering and storing signals which have sparse time–frequency representations.

1. INTRODUCTION

The uncertainty principle establishes restrictions on how well localized the Fourier transform of a well localized function can be and vice versa. In the case of a function defined on finite Abelian groups, localization can be expressed through the cardinality of the support of the function. This case has recently drawn renewed interest. This is due in part to their relevance for compressed sensing and, in particular, for the recovery of lossy signals under the assumption of restricted spectral content [CRT04].

A classical result on the uncertainty principle for functions defined on finite Abelian groups states that the product of the number of nonzero entries in a nontrivial vector, i.e., nontrivial function on a finite set, and the number of nonzero entries in its Fourier transform is not smaller than the order of the group [DS89]. This result can be improved for cyclic groups of prime order: the sum of the number of nonzero entries in a vector and the number of nonzero entries in its Fourier transform exceeds the order of the group [Tao05]. Further, it has recently been shown that the classical bound can be refined for almost any finite Abelian group [Mes05].

The objective of this paper is to establish results similar to those discussed above for joint time–frequency representations, that is, to obtain restrictions on the cardinality of the support of joint time–frequency representations of functions defined on finite Abelian groups. For example, let us consider the simplest time–frequency representation of a function, namely the one that is given by the tensor product of a function and its Fourier transform. In this case, the classical result on the uncertainty principle for nontrivial functions on finite Abelian groups states that the cardinality of the support of this tensor is at least the order of the group.

†School of Engineering and Science, International University Bremen, 28759 Bremen, Germany.
‡Courant Institute of Mathematical Science, New York University, New York USA.
In the following though, we shall be mostly interested in time–frequency representations given by short–time Fourier transforms. It is well-known that, again, the cardinality of the support of any short–time Fourier transform of a nontrivial function defined on a finite Abelian group is bounded below by the order of the group. As seen below, we can improve this bound by using the subgroup structure of the groups and/or by allowing only well-chosen window functions. For example, we show that for a group with prime order and for almost every window function, the sum of the cardinality of the support of the analyzed function and the cardinality of its short–time Fourier transform exceeds the square of the order of the group (see Theorem 4.5).

In addition to the above, we shall give applications of our results to the theory of so-called Gabor frames and the theory of sparse signal recovery. For example, the results on the cardinality of the support of short–time Fourier transforms can be translated into criteria for the recovery of encoded signals from a channel with erasures.

The paper is organized as follows. In Section 2 we give a brief but self-contained account of the Fourier transformation and of the short–time Fourier transform for functions defined on finite Abelian groups. Section 3 discusses uncertainty principles which relate the cardinality of the support of functions with the cardinality of the support of their Fourier transforms. We start Section 3 with a classical result which is based on standard norm estimates [DS89]. In Section 3.1 we state results based on the minors of Fourier transform matrices and which apply only to functions defined on cyclic groups of prime order [Tao05]. Finite Abelian groups of any order are analyzed in Section 3.2. There, the underlying subgroup structure of finite Abelian groups is used to obtain improvements to the classical uncertainty result discussed above [Mes05].

Section 4 is devoted to uncertainty in the short–time Fourier transformation. Following the organization of Section 3 a discussion of general results is followed by results for functions defined on cyclic groups in Section 4.1. Other finite Abelian groups are covered in Section 4.2. We conclude our discussion of the cardinality of the support set of short–time Fourier transforms in Section 4.3 with a conjecture on the possible cardinalities of the support of short–time Fourier transforms with respect to a random window function. In fact, one of the major difficulties to obtain uncertainty principles for the short–time Fourier transform is its dependence on the chosen window function.

Section 5 is devoted to applications of our findings. In Section 5.1 we give applications of the results of Section 4 to communications engineering. There, we discuss the identification/measurement problem for time–varying operators/channels and the transmission through channels with erasures. In addition, we show the existence of a large class of equal norm tight frames of Gabor type. In Section 5.2 we briefly discuss connections of our work to the recovery of signals which have a sparse representation in a given dictionary.

2. BACKGROUND AND NOTATION

For any finite set $A$ we set $\mathbb{C}^A = \{f : A \rightarrow \mathbb{C}\}$. For $|A| = |B| = n$, $\mathbb{C}^A \cong \mathbb{C}^B \cong \mathbb{C}^n$ as vector spaces, where $|A|$ denotes the cardinality of the set $A$. Further, for $A \subseteq B$, we write $A^c = B\setminus A$ and we define the embedding operator $i_A : \mathbb{C}^A \rightarrow \mathbb{C}^B$ where $i_A f(x) = f(x)$ for $x \in A$ and $i_A f(x) = 0$ for $x \in A^c$. Correspondingly, we define the restriction operator $r_A : \mathbb{C}^B \rightarrow \mathbb{C}^A$. Similarly, every map $S : A \rightarrow B$ induces a map $\tilde{S} : \mathbb{C}^B \rightarrow \mathbb{C}^A$, $\left(\tilde{S} f\right)(a) = f(S(a))$. If $S$ is bijective, then $\tilde{S}$ is bijective as well.
For $M \in \mathbb{C}^{m \times n}$ and $A \subseteq \{0,1,\ldots,n-1\}$ and $B \subseteq \{0,1,\ldots,m-1\}$ we let $M_{A,B}$ denote the $|B| \times |A|$-submatrix of $M$ which represents $r_B \circ M \circ i_A$.

For $f \in \mathbb{C}^A$, we use the now customary notation $\|f\|_0 = |\text{supp} \, f|$ where $\text{supp} \, f = \{a \in A : f(a) \neq 0\}$. Clearly, $\| \cdot \|_0$ is not a norm.

### 2.1. Fourier transforms on finite Abelian groups

Throughout this paper, $G$ denotes a finite Abelian group. The identity element of $G$ is denoted by $e$ or by 0 in case that $G$ is cyclic, i.e., if $G = \mathbb{Z}_n$ for some $n \in \mathbb{N}$. The dual group of characters $\hat{G}$ of $G$ is the set of continuous homomorphisms $\xi \in \mathbb{C}^G$ which map $G$ into the multiplicative group $\mathbb{C}^\times = \{z \in \mathbb{C} : |z| = 1\}$. The set $\hat{G}$ is an Abelian group under pointwise multiplication and, as is customary, we shall write this commutative group operation additively. Note that $G$ is isomorphic to $\hat{G}$. Further, Pontryagin duality implies that $\hat{\hat{G}}$ can be canonically identified with $G$, a fact which is emphasized by writing $\langle \xi, x \rangle = \xi(x)$.

The Fourier transform $\mathcal{F} f = \hat{f} \in \mathbb{C}^{\hat{G}}$ of $f \in \mathbb{C}^G$ is given by

$$\hat{f}(\xi) = \sum_{x \in G} f(x) \overline{\xi(x)} = \sum_{x \in G} f(x) \langle \xi, x \rangle, \quad \xi \in \hat{G}.$$ 

The inversion formula for the Fourier transformation allows us to reconstruct the original function from its Fourier transform. Namely, for $f \in \mathbb{C}^G$ we have

$$f(x) = \frac{1}{|G|} \sum_{\xi \in \hat{G}} \hat{f}(\xi) \langle \xi, x \rangle, \quad x \in G.$$ 

The inversion formula implies that

$$\|f\|_2^2 = \frac{1}{|G|} \sum_{\xi \in \hat{G}} |\hat{f}(\xi)|^2 = \frac{1}{|G|} \|\hat{f}\|_2^2, \quad (1)$$

where $\|f\|_2 := (\sum_{t \in G} |f(t)|^2)^{1/2}$. Further, (1) together with $\|\xi\|_2 = |G|^{1/2}$ for all $\xi \in \hat{G}$ implies that the normalized characters in $\{ |G|^{-1/2} \xi \}_{\xi \in \hat{G}}$ form an orthonormal basis for $\mathbb{C}^G$, and $\sum_{x} \langle \xi, x \rangle = 0$ if $\xi \neq 0$ and $\sum_{x} \langle \xi, x \rangle = 0$ if $x \neq 0$.

Fourier transformations are linear maps and we turn now to a discussion of their matrix representations.

For $n \in \mathbb{N}$ and $\omega = e^{2\pi i/n}$, the discrete Fourier matrix $W_{\mathbb{Z}_n}$ of the cyclic group $\mathbb{Z}_n$ is defined by $W_{\mathbb{Z}_n} = (\omega^{rs})_{r,s=0}^{n-1}$. Identifying $\mathbb{C}^{\mathbb{Z}_n}$ with $\mathbb{C}^n$, we have $\hat{f} = W_{\mathbb{Z}_n} \cdot f$.

For an arbitrary finite Abelian group $G$, we can always choose a representation of $G$ as direct product of cyclic groups $G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_m}$ where $d_1, \ldots, d_m$ can be chosen to be powers of prime numbers. A character in the dual group $G$ is then given by

$$\langle (\xi_1, \xi_2, \ldots, \xi_m), (x_1, x_2, \ldots, x_m) \rangle = \langle \xi_1, x_1 \rangle \langle \xi_2, x_2 \rangle \ldots \langle \xi_m, x_m \rangle,$$
where \((\xi_1, \xi_2, \ldots, \xi_m) \in \hat{\mathbb{Z}}_{d_1} \times \hat{\mathbb{Z}}_{d_2} \times \ldots \times \hat{\mathbb{Z}}_{d_m} \cong \hat{G}\). The discrete Fourier matrix \(W_G\) for \(G = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_m}\) is chosen to be the Kronecker product of the Fourier matrices for the groups \(\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \ldots, \mathbb{Z}_{d_m}\), i.e., \(W_G = W_{d_1} \otimes W_{d_2} \otimes \ldots \otimes W_{d_m}\). For example, we have

\[
W_{\mathbb{Z}_2} = \left(\begin{array}{ccc}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & -1 & -1 \\
1 & 1 & -1 \\
\end{array}\right) \quad \text{and} \quad W_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \left(\begin{array}{ccc}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
\end{array}\right).
\]

Note that for appropriately chosen bijections \(S_1 : \{0, 1, \ldots, |G|-1\} \rightarrow G\) and \(S_2 : \{0, 1, \ldots, |G|-1\} \rightarrow \hat{G}\) we have \(\hat{f} \circ S_2 = W_G(f \circ S_1)\) for \(f \in \mathbb{C}^G\).

### 2.2. Short–time Fourier transforms on finite Abelian groups and Gabor frames

For any \(x \in G\), we define the translation operator \(T_x\) as the unitary operator on \(\mathbb{C}^G\) given by \(T_x f(y) = f(y-x), \ y \in G\). Similarly, we define the modulation operator \(M_\xi\) for \(\xi \in \hat{G}\) as the unitary operator defined by \(M_\xi f = f \cdot \xi\), where here and in the following \(\cdot \) denotes the pointwise product of \(f, g \in \mathbb{C}^G\). Since \(\hat{M}_\xi f = T_\xi \hat{f}\), we refer to \(M_\xi\) also as a frequency shift operator.

We set \(\pi(\lambda) = M_\xi \circ T_x\) for \(\lambda = (x, \xi) \in G \times \hat{G}\). The unitary operators \(\pi(\lambda), \ \lambda \in G \times \hat{G}\) are called time–frequency shift operators.

**Definition 2.1.** The short–time Fourier transformation \(V_g : \mathbb{C}^G \rightarrow \mathbb{C}^{G \times \hat{G}}\) with respect to the window \(g \in \mathbb{C}^G \setminus \{0\}\) is given by

\[
V_g f(x, \xi) = \langle f, \pi(x, \xi)g \rangle = \sum_{y \in G} f(y) g(y-x) \langle \xi, y \rangle, \quad (x, \xi) \in G \times \hat{G},
\]

where \(f \in \mathbb{C}^G\).

The inversion formula for the short–time Fourier transform is

\[
f(y) = \frac{1}{|G| \|g\|^2} \sum_{(x, \xi) \in G \times \hat{G}} V_g f(x, \xi) g(y-x) \langle \xi, y \rangle, \quad y \in G,
\]

(2)

i.e., \(f\) can be composed of time–frequency shifted copies of any \(g \in \mathbb{C}^G \setminus \{0\}\). Further, \(|V_g f|_2 = \sqrt{|G| \|f\|_2 \|g\|_2}\). This equation resembles (1), but the so-called Gabor system \(\{\pi(x, \xi)g\}_{(x, \xi) \in G \times \hat{G}}\) is clearly not an orthonormal basis if \(|G| \neq 1\) since it consists of \(|G|^2\) vectors in a \(|G|\) dimensional space. As a matter of fact, such a Gabor system is an equal norm tight frame which is defined below.

**Definition 2.2.** Let \(G\) be a finite Abelian group and let \(K\) be a finite or countably infinite index set. A family of functions \(\{\varphi_k\} \subset \mathbb{C}^G\) with

\[
A \|f\|_2^2 \leq \sum_k |\langle f, \varphi_k \rangle|^2 \leq B \|f\|_2^2, \quad f \in \mathbb{C}^G,
\]

for positive \(A\) and \(B\) is called a frame for \(\mathbb{C}^G\). \(A\) is called an lower frame bound and \(B\) is called a upper frame bound of the frame \(\{\varphi_k\}\).

A frame is called tight if we can choose \(A = B\). If we can choose \(A = B = 1\), then the frame is called Parseval tight frame. If \(\|\varphi_k\| = C > 0\) for all \(k\), then the frame \(\{\varphi_k\}\) is called equal norm frame and if in addition \(C = 1\), then we have a unit norm frame.
A direct consequence of (2) is

**Proposition 2.3.** For any $g \in \mathbb{C}^G \setminus \{0\}$, the collection $\{\pi(\lambda)g\}_{\lambda \in G \times \hat{G}}$ is an equal norm tight frame for $\mathbb{C}^G$ with frame bound $A = B = |G|\|g\|_2^2$.

The usefulness of frames stems largely from the existence of a reconstruction formula similar to (1) and (2).

**Proposition 2.4.** Let $\{\varphi_k\}$ be a frame for $\mathbb{C}^G$. Then there exists a so-called dual frame $\{\tilde{\varphi}_k\}$ with

$$f = \sum_k (f, \varphi_k)\tilde{\varphi}_k = \sum_k (f, \tilde{\varphi}_k)\varphi_k, \quad f \in \mathbb{C}^G.$$  \hspace{1cm} (3)

Note that Parseval frames are self-dual, i.e., we can choose $\tilde{\varphi}_k = \varphi_k$ for all $k$.

For additional material on frames and, in particular, Gabor frames we refer to the excellent expositions [Christensen 03], [Grochenig 01], [Koornwinder 06]. The geometry of finite frames is discussed in [Balan, Fickus 03].

For a given group $G$, we shall use again the previously defined enumerations $S_2 : \{0, 1, \ldots, |G| - 1\} \rightarrow G$ and $S_1 : \{0, 1, \ldots, |G| - 1\} \rightarrow G$ which gave rise to the Fourier matrix $W_G$. For $g \in \mathbb{C}^G$ and $x \in G$, we define the $|G|\times|G|$-diagonal matrix

$$D_{x,g} = \begin{pmatrix} g(S_1(0) + x) & 0 & \cdots & 0 \\ g(S_1(1) + x) & g(S_1(G - 1) + x) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & g(S_1(|G| - 1) + x) & g(S_1(|G| - 1) + x) \end{pmatrix}.$$  \hspace{1cm} (4)

Then, the $|G|\times|G|^2$-full Gabor system matrix with respect to $g$ is given by

$$A_{G,g} = (D_{S_1(0),g} \cdot W_G | D_{S_1(1),g} \cdot W_G | \cdots | D_{S_1(|G| - 1),g} \cdot W_G)^*,$$

where $M^*$ denotes the adjoint of the matrix $M$. For example, for $G = \mathbb{Z}_4$,

$$A_{\mathbb{Z}_4,(1,2,3,4)} := \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2i & -2 & -2i & -3 & -3i & 4 & 4i & -4 & 4i & -1 & -1 & -2 & -2 & -2 & -2 \\ 3 & -3 & -3 & -3 & -3 & -3i & 4 & 4i & -4 & 4i & -1 & -1 & -2 & -2 & -2 & -2 \\ 4 & 4i & 4i & -4 & 4i & -1 & -1 & -2 & -2 & -2 & -2 & -2 & 3 & -3i & -3 & -3i \end{pmatrix}.$$

Similarly, for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we have

$$A_{\mathbb{Z}_2 \times \mathbb{Z}_2,(1,2,3,4)} := \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & -2 & 2 & -2 & 1 & -1 & 1 & -1 & 4 & -4 & 3 & -3 & 3 & -3 & 3 & -3 \\ 3 & 3 & -3 & -3 & 4 & 4 & -4 & -4 & 1 & 1 & -1 & -1 & 2 & 2 & -2 & -2 \\ 4 & -4 & -4 & 4 & 3 & -3 & -3 & 3 & 2 & -2 & 2 & 1 & -1 & -1 & 1 \end{pmatrix}.$$  \hspace{1cm} (5)

Using the enumeration $S : \{0, 1, \ldots, |G|^2 - 1\} \rightarrow G \times \hat{G}$ which is given by the lexicographic order that is induced by $S_1$ and $S_2$ on $G \times \hat{G}$, we have $V_gf \circ S = A_{G,g}f$. Therefore, we shall refer to $A_{G,g}$ as short-time Fourier transform matrix with respect to the window $g$. Clearly, the rows of $A_{G,g}$ represent the vectors in the Gabor system $\{\pi(\lambda)g\}_{\lambda \in G \times \hat{G}}$, and [2] implies that $A_{G,g}^*A_{G,g}$ is a multiple of the identity matrix.
3. UNCERTAINTY PRINCIPLES FOR THE FOURIER TRANSFORM ON FINITE ABELIAN GROUPS

The following uncertainty theorem for functions defined on finite Abelian groups is the natural starting point for our discussion [DS89].

**Theorem 3.1.** Let $f \in \mathbb{C}^G \setminus \{0\}$, then $\|f\|_0 \cdot \|\hat{f}\|_0 \geq |G|$.

**Proof.** For $f \in \mathbb{C}^G$, $f \neq 0$, and without loss of generality $\|\hat{f}\|_\infty = 1$, we compute

$$|G| = |G|\|\hat{f}\|_\infty^2 \leq |G| \left(\sum_{x \in G} |f(x)|\right)^2 \leq |G|\|f\|_0 \sum_{x \in G} |f(x)|^2 = |G|\|f\|_0 \|\hat{f}\|_0 = \|f\|_0 \|\hat{f}\|_0.$$ 

□

A complementary result characterizes those $f$ for which the bound in Theorem 3.1 is sharp [DS89, MÖP04].

**Proposition 3.2.**

1. If $k$ divides $|G|$, then there exists $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\hat{f}\|_0 = \frac{|G|}{k}$.

2. If $\|f\|_0 \|\hat{f}\|_0 = |G|$ and $e \in \text{supp } f$, then $\text{supp } f$ is a subgroup of $G$.

3.1. Groups of prime order

The geometric mean of two positive numbers is dominated by their arithmetic mean; hence, Theorem 3.1 implies the weaker inequality

$$\|f\|_0 + \|\hat{f}\|_0 \geq 2\sqrt{|G|}.$$  \hspace{1cm} (5)

If $|G|$ is prime, i.e., if $G$ is a cyclic group of prime order, then (5) and also Theorem 3.1 can be improved significantly [Fre04, Tao05].

**Theorem 3.3.** Let $G = \mathbb{Z}_p$ with $p$ prime. Then $\|f\|_0 + \|\hat{f}\|_0 \geq |G| + 1$ holds for all $f \in \mathbb{C}^G \setminus \{0\}$.

This result is a direct consequence from Chebotarev’s Theorem which states that every minor of the Fourier transform matrix $W_{\mathbb{Z}_p}$, $p$ prime, is nonzero [EI76, SL96, Tao05, Fre04]. In fact, to obtain Theorem 3.3 we only need to combine Chebotarev’s Theorem with

**Proposition 3.4.** Let $M \in \mathbb{C}^{m \times n}$. Then $\|f\|_0 + \|Mf\|_0 \geq m+1$ for all $f \in \mathbb{C}^n$ if and only if every minor of $M$ is nonzero. Moreover, if every minor of $M \in \mathbb{C}^{m \times n}$ is nonzero and $k,l$ are given with $k + l \geq m+1$, then there exists $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$.

**Lemma 3.5.** For $M \in \mathbb{C}^{m \times n}$ and $1 \leq k \leq m$, $1 \leq l \leq n$, there exists $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$ if and only if there exist sets $A \subseteq \{0, \ldots, n-1\}$ and $B \subseteq \{0, \ldots, m-1\}$ with $|A| = k$, $|B| = m - l$, and for all $a \in A$ and $y \in B^c$, we have

$$\text{rank } M_{A \setminus \{a\}, B} = \text{rank } M_{A,B} = \text{rank } M_{A,B \cup \{y\}} - 1 < |A|.$$  \hspace{1cm} (6)
Proof of Proposition 3.4. If \( f \) has no zero minors, then (0) in Lemma 3.5 is equivalent to \(|B| < |A|\), implying that there exists \( f \in \mathbb{C}^n \) with \( \|f\|_0 = k \) and \( \|Mf\|_0 = l \) if and only if \( k + l \geq m + 1 \).

It remains to show that \( \|f\|_0 + \|Mf\|_0 \geq m + 1 \) for all \( f \) implies that \( M \) has no zero minors. To this end, assume that there is a \( d \times d \) submatrix \( M_{A,B} \) of \( M \) with \( \det M_{A,B} = 0 \). Then there exists a nonzero vector \( f' \in \mathbb{C}^A \) such that \( M_{A,B} f' = 0 \). For \( f = i_A f' \), \( \|Mf\|_0 \leq m - d \) and therefore \( \|f\|_0 + \|Mf\|_0 \leq d + m - d = m < m + 1 \). \( \square \)

Theorem 3.3 is a clear improvement to Theorem 3.1 but it applies only to cyclic groups of prime order. In fact, any other finite Abelian group \( G \) has proper subgroups which lead to zero minors in \( W_G \). As example, we display in Table 1 counts on the ranks of square submatrices of \( W_{\mathbb{Z}_5} \) and \( W_{\mathbb{Z}_6} \).

Due to their role in obtaining Theorem 3.3, we shall now collect facts regarding zero and nonzero minors of Fourier matrices in general.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 1 | 25 | 0 | 0 | 0 |
| 2 | 0 | 100 | 0 | 0 |
| 3 | 0 | 0 | 100 | 0 |
| 4 | 0 | 0 | 0 | 25 |
| 5 | 0 | 0 | 0 | 0 |

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 1 | 36 | 36 | 0 | 0 | 0 |
| 2 | 0 | 189 | 48 | 0 | 0 |
| 3 | 0 | 0 | 352 | 36 | 0 |
| 4 | 0 | 0 | 0 | 189 | 0 |
| 5 | 0 | 0 | 0 | 0 | 36 |
| 6 | 0 | 0 | 0 | 0 | 1 |

Table 1. Counts of the numerically computed rank of submatrices of \( W_{\mathbb{Z}_5} \) and \( W_{\mathbb{Z}_6} \). The column index is the size of square submatrices considered, and the row index corresponds to their ranks.

Let \( M \in \mathbb{C}^{n \times n} \) and let \( A, B \subset \{1, 2, \ldots, n\} \) such that \(|A| = |B|\). Then \( \det M_{A,B} \) defines a minor of \( M \), and \( \det M_{A^c,B^c} \) is called its complementary minor.

**Proposition 3.6.**

1. The complementary minor of any zero minor in a Fourier matrix \( W_G \) is also zero.

2. Let \( d_0 > 1 \) be the smallest divisor of \(|G|\). Then for all \( d_0 \leq r \leq n - d_0 \), there exists an \( r \times r \) zero minor of the Fourier matrix \( W_G \). In particular, if \(|G|\) is even, then there exist \( r \times r \) zero minor for \( r = 2, 3, \ldots, |G| - 2 \).

3. Any minor of the Fourier matrix \( W_{\mathbb{Z}_n} \), \( n \in \mathbb{N} \), that contains only adjacent rows or columns is nonzero.

**Proof.** 1. The adjoint of a matrix \( M = (m_{kl}) \) is adj \( M = (m_{kl})' \), where \( m_{kl} = (-1)^{k+l} \det M_{(k),\{l\}}' \) is the cofactor of the element \( m_{kl} \). Then for any sets \( A, B \) of cardinality \( r \), Jacobi’s theorem states that

\[
\det M_{A,B} = (-1)^r \det(\text{adj} M)_{A^c,B^c} \cdot (\det M)^{-1}
\]

Furthermore, adj \( M \cdot M = \det M \cdot I \) \cite{Pra94}.

For any zero minor of \( M = W_G \) on the left hand side of (7), Jacobi’s theorem implies that the right hand side, representing a minor in adj \( W_G \), is zero as well. Since \( W_G \cdot \overline{W_G} = |G| \cdot I \), we have
adj $W_G = \frac{\det(W_G)}{|G|} \cdot W_G$. Thus the corresponding minor in $W_G$ is zero, which implies that also the corresponding minor in $W_G$ is zero.

2. Let $d$ divide $|G|$. Part 1 in Proposition 3.2 allows us to choose $f_d$ such that $\|f_d\|_0 = d$ and $\|\hat{f}_d\|_0 = \frac{|G|}{d}$. Hence, for any $r$ with $d \leq r \leq |G| - \frac{|G|}{d}$ we can pick sets $A \supseteq \text{supp } f_d$ and $B \subseteq (\text{supp } f_d)^c$ such that $|A| = |B| = r$. Then $r_A f_d \in \ker M_{A,B}$ and the $r \times r$ minor det $M_{A,B}$ is zero.

This way, we obtain $r \times r$ zero minors for $d_0 \leq r \leq \frac{|G|}{d_0}(d_0 - 1)$ and for $\frac{|G|}{d_0} \leq r \leq |G| - d_0$, where $d_0$ is the smallest nontrivial divisor of $|G|$. The result follows since $d_0 - 1 \geq 1$.

3. A minor with adjacent columns is a determinant of the type
\[
\det \begin{pmatrix}
\omega^{k_1} & \omega^{k_1+1} & \ldots & \omega^{k_1+k}\n\omega^{k_2} & \omega^{k_2+1} & \ldots & \omega^{k_2+k}\n\vdots & \vdots & \ddots & \vdots \\
\omega^{k_m} & \omega^{k_m+1} & \ldots & \omega^{k_m+k}
\end{pmatrix} = \omega^{k_1+k_2+\ldots+k}\det \begin{pmatrix}1 & \omega^{k_1} & \ldots & \omega^{mk}\1 & \omega^{k_2} & \ldots & \omega^{mk}\\vdots & \vdots & \ddots & \vdots \1 & \omega^{k_m} & \ldots & \omega^{mk}\end{pmatrix}
\]

\[
= \omega^{k_1+k_2+\ldots+k}\prod_{i<j} (\omega^k - \omega^j) \neq 0
\]

The second determinant was evaluated using the formula for Vandermonde determinants and the result does not equal 0, as always $i < j$ and $\omega$ is a primitive $n$-th root of unity. \hfill \square

3.2. Groups of non-prime order

Meshulam improved the bound in the classical uncertainty relation presented in Theorem 3.1 for most finite Abelian groups of non-prime order \[\text{Mes05}\]. He defines for $0 < k \leq |G|$ the function
\[
\theta(G, k) = \min \left\{ \|\hat{f}\|_0 : f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\}.
\]

Note that Theorem 3.3 implies that $\theta(\mathbb{Z}_p, k) = p - k + 1$. The main result in \[\text{Mes05}\] is

**Theorem 3.7.** For $k \leq |G|$, let $d_1$ be the largest divisor of $|G|$ which is less than or equal to $k$ and let $d_2$ be the smallest divisor of $|G|$ which is larger than or equal to $k$. Then
\[
\theta(G, k) \geq \frac{|G|}{d_1d_2}(d_1 + d_2 - k).
\]

Tao realized that this theorem simply states that all possible lattice points $(\|f\|_0, \|\hat{f}\|_0)$ lie in the convex hull of the points $(|H|, |G/H|)$, where $H$ ranges over all subgroups of $G$ \[\text{Mes05}\]. To see this, recall that for any divisor $d$ of $|G|$ exists a subgroup $H$ of $G$ with $d = |H|$. Furthermore, the right hand side of expression (8) is linear between two successive divisors and the slope is increasing when $k$ increases. Hence (8) characterizes the convex hull of the points $(|H|, |G/H|)$. Proposition 3.2 part 1, implies that the vertex points $(|H|, |G/H|)$ are attained.

The proof of Theorem 3.7 in \[\text{Mes05}\] is inductive and uses three facts: first, it uses Theorem 3.3 as induction seed, and second, it uses the submultiplicativity of the right hand side of (8). That is, if we denote this right hand side by $u(n, k)$ for $n = |G|$, then it uses that $u(n, k) \leq u(\frac{n}{d}, t)u(d, s)$ for $d$ dividing $n$ and $st \leq k$. The third ingredient is

**Proposition 3.8.** Let $H$ be a subgroup of $G$. For $k \leq |G|$ there exist $s \leq q$, $t \leq p$ with $st \leq k$ and
\[
\theta(G, k) \geq \theta(H, s)\theta(G/H, t).
\]

8
Meshulam’s proof of Proposition 3.8 is heavy on algebraic notation and does not give good insight from the point of view of Fourier analysis. For this reason, and for completeness sake, we give a streamlined version of Meshulam’s proof of Proposition 3.8. See also [LM05] for an elegant and non-inductive proof of Theorem 3.7.

But first, note that if \( G \cong H \times G/H \), then Proposition 3.8 can be proven using the fact that then \( \hat{G} \cong \hat{H} \times \hat{G}/\hat{H} \). and, therefore, \( \hat{f} \) can be calculated by performing two partial Fourier transforms. For example, such argument can be applied to \( G = \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \), \( \gcd(m, n) = 1 \), and \( H = \mathbb{Z}_m \times \{ e \} \). Even simpler is the special case discussed in Proposition 3.9. We state and prove this result to illustrate the main idea used to prove Proposition 3.8.

**Proposition 3.9.** Let \( A_1 \subseteq G_1 \) and \( A_2 \subseteq G_2 \) and \( f \in \mathbb{C}^{G_1 \times G_2} \) be given with \( \text{supp} \ f \subseteq A_1 \times A_2 \). Then \( ||\hat{f}||_0 \geq \theta(G_1, |A_1|) \theta(G_2, |A_2|) \).

**Proof.** We picture \( f \) as a \( |G_1| \times |G_2| \) matrix and note that \( \text{supp} \ f \subseteq A_1 \times A_2 \) implies that \( f \) has exactly \( |G_2 \setminus A_2| \) zero columns and \( |A_2| \) columns with at least \( |G_1 \setminus A_1| \) zeros.

The function \( \mathcal{F}_1 f \) is obtained by applying the \( G_1 \)–Fourier transformation to each column. Hence, \( \mathcal{F}_1 f \) has \( |G_2 \setminus A_2| \) zero columns and, at most, \( |G_1| - \theta(G_1, |A_1|) \) zeros in the remaining \( A_2 \) columns. It is easy to see that in the scenarios which leads to the weakest bound for \( ||\hat{f}||_0 \), we have \( |G_1| - \theta(G_1, |A_1|) \) zeros in each of these \( |A_2| \) columns and that they are lined up to form \( |G_1| - \theta(G_1, |A_1|) \) zero rows in \( \mathcal{F}_1 f \). In this case, the remaining \( \theta(G_1, |A_1|) \) rows contain exactly \( |G_2 \setminus A_2| \) zeros, i.e., \( |A_2| \) nonzero elements.

Now, we calculate \( \mathcal{F} f \) by taking a \( G_2 \)–Fourier transform along each row of \( \mathcal{F}_1 f \). As a result, \( |G_1| - \theta(G_1, |A_1|) \) zero rows remain, and in the other \( \theta(G_1, |A_1|) \) rows, at least \( \theta(G_2, |A_2|) \) zeros are present. We conclude that

\[
||\hat{f}||_0 \geq \theta(G_1, |A_1|) \theta(G_2, |A_2|).
\]

The property that the \( G = G_1 \times G_2 \)–Fourier transformation “splits” into a \( G_1 \)–Fourier transformation and a \( G_2 \)–Fourier transformation is the basis of the simple proof of Proposition 3.9. In the proof of Proposition 3.8 we shall see that the general case follows from small adjustments to the arguments used to prove Proposition 3.9.

**Proof of Proposition 3.8.** Let \( H = \{ x_i \} \) be a subgroup of \( G \) and, abusing notation, we let \( \{ x_i \} \) be a set of coset representatives of the quotient group \( G/H \). Then each element in \( G \) has a unique representation as \( x_i + x_j \). We let \( H^\perp \) denote the characters \( \{ \xi_j \in \hat{G} : \xi_j(H) = 1 \} \). \( H^\perp \) is a subgroup of \( \hat{G} \), and we denote by \( \{ \xi_j \} \) a set of coset representatives of the quotient group \( \hat{G}/H^\perp \). Every element \( \xi \in \hat{G} \) has a unique decomposition as \( \xi_i + \xi_j \).

The Pontryagin duality theorem implies \( \hat{G}/H^\perp \cong \hat{H} \). This allows us to assign a character \( \xi_i^t \in \hat{H} \) to each \( \xi_i \in \hat{G}/H^\perp \) with \( \xi_i^t + \xi_j^t = (\xi_i + \xi_j)^t \). Further, \( \langle \xi_i, x_i \rangle_G = \langle \xi_i^t, x_i \rangle_H \) for all \( x_i \in H \) and all \( \xi_i \in \hat{G}/H^\perp \). Similarly, we use \( G/H \cong H^\perp \) to assign to each \( \xi_j \) an element \( \xi_j^t \in G/H \) with \( \langle \xi_j, x_j \rangle_G = \langle \xi_j^t, x_j + H \rangle_{G/H} \) for all \( x_j \).

1In particular, in the case \( G = \mathbb{Z}_{mn} \), \( \gcd(m, n) = 1 \), \( \mathbb{Z}_m \cong \mathbb{Z}_n \) and \( \mathbb{Z}_m^\perp \cong \mathbb{Z}_n \).
For \( f \in \mathbb{C}^G \) and any \( \xi = \xi_i + \xi_j \in \hat{G} \), we calculate

\[
\hat{f}(\xi) = \hat{f}(\xi_i + \xi_j) = \sum_{x_i, x_j} f(x_i + x_j) \overline{\langle \xi_i + \xi_j, x_i + x_j \rangle}_G \\
= \sum_{x_i, x_j} f(x_i + x_j) \overline{\langle \xi_i, x_i \rangle}_G \overline{\langle \xi_j, x_j \rangle}_G \\
= \sum_{x_i} \left( \sum_{x_j} f(x_i + x_j) \overline{\langle \xi_i, x_i \rangle}_H \right) \overline{\langle \xi_j, x_j \rangle}_G \overline{\langle \xi_j, x_j + H \rangle}_{G/H}
\]

where the last equality follows since \( \xi_j \in H^\perp \) implies \( \langle \xi_j, x_i \rangle_G = 1 \).

We set \( f_1(\xi_i', x_j) := \sum_{x_i \in H} f(x_i + x_j) \overline{\langle \xi_i', x_i \rangle}_H \), which, for fixed \( x_j \), is the \( H \)-Fourier transform \( \mathcal{F}_H \) on the coset \( x_j + H \) in \( G \), and \( f_2(\xi_i', x_j) = f_1(\xi_i, x_j) \overline{\langle \xi_i, x_i \rangle}_G \). Further \( f_1 \) and \( f_2 \) have the same support sets. We summarize that \( \hat{f} \) can be obtained from \( f \) via two partial Fourier transformations and an enclosed unitary multiplication operator, as illustrated in Figure 1.

Let us now fix \( f \in \mathbb{C}^G \) with \( \|f\|_0 \leq k \) and \( \|\hat{f}\|_0 = \theta(G, k) \).

Let \( t := |\{ x_j : \text{supp } f \cap (x_j + H) \neq \emptyset \}|. \) Note that the support of \( f \) contains at most \( k \) elements which are distributed among \( t \) cosets of \( H \). Hence, there must be a coset \( x_{j_0} + H \) which contains \( s' \leq s = \lceil \frac{k}{t} \rceil \) elements of \( \text{supp } f \). Therefore,

\[
\|f_2(\cdot, x_{j_0})\|_0 = \|\mathcal{F}_H f(\cdot + x_{j_0})\|_0 \geq \theta(H, s') \geq \theta(H, s)
\]

This implies that \( \Xi = \{ \xi_i \in \hat{G}/H^\perp : f_2(\xi_i', \cdot) \neq 0 \} \) satisfies \( |\Xi| \geq \theta(H, s) \). In fact, the definition of \( t \) implies that for \( \xi_i \in \Xi \), we have \( 0 < \text{supp } f_2(\xi_i', \cdot) \leq t \). We conclude

\[
\theta(G, k) = \|\hat{f}\|_0 = \sum_{\xi_i} \|\mathcal{F}_{G/H} f_2(\xi_i', \cdot)\|_0 \geq \sum_{\xi_i \in \Xi} \|\mathcal{F}_{G/H} f_2(\xi_i', \cdot)\|_0 \geq \theta(H, s) \theta(G/H, t).
\]

Next, we discuss the question whether the inequality \( [\mathbf{S}] \) in Theorem 3.7 is sharp, or, more precisely, we shall check whether for some given Abelian group \( G \) and \((k, l)\) chosen with \( l \geq \theta(G, k) \geq \frac{|G|}{d_1 + d_2 - k} \) there exists a function \( f \in \mathbb{C}^G \) with \( \|f\|_0 = k \) and \( \|\hat{f}\|_0 = l \). This question has been discussed earlier for \( G = \mathbb{Z}_6 \) and \( G = \mathbb{Z}_9 \) in [FKLM03].

The following affirmative partial result follows from the proof of Proposition 4.5 in [Kut03].

**Proposition 3.10.** If \( 0 < k, l \leq |G| \) satisfy \( l + k \geq |G|+1 \), then there exists a function \( f \in \mathbb{C}^G \) with \( \|f\|_0 = k \) and \( \|\hat{f}\|_0 = l \).

The numerical results collected in Figure 3 and Figure 4 are based on an idea in [FKLM03] and on Lemma 3.5. They show that the set of all possible pairs \((\|f\|_0, \|\hat{f}\|_0)\) is nontrivial in general. The computations that lead to these results are quite involved. For example, the computations showing that there is no function (vector) on \( \mathbb{Z}_{16} \) with six nonzero entries and whose Fourier transform has nine nonzero entries include the calculation of the singular values of \( \left( \begin{array}{c} 16 \\ 9 \end{array} \right) \left( \begin{array}{c} 16 \\ 5 \end{array} \right) = 49969920 \) nine by six matrices.
Figure 1. Illustration of the proof of Proposition 3.8 for $G = \mathbb{Z}_{10} \times \mathbb{Z}_6$ and $k = 17$. The function $f_2$ is obtained by the application of $H$–Fourier transformations to the rows of $f$ which is succeeded by an unitary multipicatation operator. To calculate $\hat{f}$ we apply $G/H$–Fourier transformations to the columns of $f_2$. For clarity, we choose synthetic support sets of $f$, $f_2$, and $\hat{f}$. Here $t = 6$ and $s = \lfloor \frac{17}{6} \rfloor = 2$.

In addition, we give all possible pairs $(\|f\|_0, \|\hat{f}\|_0)$ for the group $G = \mathbb{Z}_6$ and give a partial result for the groups $G = \mathbb{Z}_{2p}$ for, $p \geq 5$ prime. Their proofs are included in the appendix.

**Proposition 3.11.** For $1 \leq k, l \leq 6$ exists $f \in \mathbb{C}^{\mathbb{Z}_6}$ with $\|f\|_0 = k$ and $\|\hat{f}\|_0 = l$ if and only if $kl \geq 6$ and $(k, l) \neq (3, 3)$.

The following result for $\mathbb{Z}_{2p}$, $p \geq 5$ prime, shows that the bound in Theorem 3.7 is not sharp, a fact that was observed for the case $G = \mathbb{Z}_8$ in [FKLM05].

**Proposition 3.12.** For $p \geq 5$ prime there exists no $f \in \mathbb{C}^{\mathbb{Z}_{2p}}$ with $\|f\|_0 = 3$ and $\|\hat{f}\|_0 = p-1$.

![Color coding legend](image)

Figure 2. Color coding which is used in Figures 3–9 to describe subsets of $\mathbb{N}^2$ or $\mathbb{N}^3$. The color determines whether a given value is in the set under discussion. Y-pr indicates that there is proof that the corresponding value is in the set considered. Y-nu implies that there is numerical evidence that the value is in the set and Y-co indicates that we conjecture that the value is in the set. N-pr indicates that there is proof that the corresponding value is not in the set, and N-nu and N-co are defined accordingly. The color adjacent to ? implies that no judgement is made here.
Figure 3. The set \( \{ (\| f \|_0, \| \hat{f} \|_0), f \in \mathbb{C}^G \setminus \{ 0 \} \} \) for all Abelian groups of non-prime order less than or equal to 12. If \( kl < |G| \), then no \( f \) exists with \( (k, l) = (\| f \|_0, \| \hat{f} \|_0) \) by Theorem 3.1. If \(|G|\) divides \( kl \), or if \( k + l \geq |G| + 1 \) then exists \( f \) with \( (k, l) = (\| f \|_0, \| \hat{f} \|_0) \) by Proposition 3.2 and Proposition 3.4. The color code used is described in Figure 2.
4. UNCERTAINTY PRINCIPLES FOR SHORT–TIME FOURIER TRANSFORMS ON FINITE ABELIAN GROUPS

We now turn to discuss minimum support conditions on time-frequency representations of elements in \( \mathbb{C}^G \), in particular, for the short–time Fourier transform of a function \( f \in \mathbb{C}^G \) with respect to a window \( g \in \mathbb{C}^G \).

The simplest joint time-frequency representation of \( f \) is given by the tensor product \( f \otimes \hat{f} \). Similarly, in electrical engineering the so-called Rihaczek distribution, \( R : G \times \hat{G} \rightarrow \mathbb{C} \), which is given by 
\[
Rf(x, \omega) = f(x)\hat{f}(\omega) \langle \omega, x \rangle,
\]
is considered. Theorem 3.1 implies that 
\[
\|Rf\|_0 = \|f \otimes \hat{f}\|_0 = \|f\|_0 \|\hat{f}\|_0 \geq |G|.
\]
Figure 5 lists all possible pairs \((\|f\|_0, \|Rf\|_0)\) for \( f \in \mathbb{C}^Z_4 \).

Using the technique used to obtain Theorem 3.1, we obtain the well-known result

**Proposition 4.1.** For \( f, g \in \mathbb{C}^G \setminus \{0\} \), we have 
\[
\|V_g f\|_0 \geq |G| \text{ for } f = g = \delta.
\]

**Proof.** Clearly \( \|V_0 \delta\|_0 = |G| \). For \( f, g \in \mathbb{C}^G \setminus \{0\} \), 
\[
|G| \|f\|_2^2 \|g\|_2^2 = \|V_g f\|_2^2 \leq \|V_g f\|_0 \|V_g f\|_\infty^2 \leq \|V_g f\|_0 \|f\|_2^2 \|g\|_2^2
\]
and the result follows. \( \Box \)

We shall now seek lower bounds on \( \|V_g f\|_0 \) depending on \( \|f\|_0, \|\hat{f}\|_0, \|g\|_0, \) and \( \|\hat{g}\|_0 \).

**Proposition 4.2.** For \( f, g \in \mathbb{C}^G \setminus \{0\} \), we have 
\[
\|V_g f\|_0 \geq \max\{ \theta(G, \|g\|_0) \theta(G, \|\hat{f}\|_0), \theta(G, \|f\|_0) \theta(G, \|\hat{g}\|_0) \}, \tag{9}
\]

\[\]
For the Abelian group $G = \mathbb{Z}_4$ all possible pairs $(\|f\|_0, \|Rf\|_0)$ are colored red, those pairs that are not achieved by some $f \in \mathbb{C}^{2^4}$ are colored blue in accordance with the color code given in Figure 2.

and, therefore,

$$\|V_g f\|_0 \geq \frac{1}{2} \left( \theta(G, \|g\|_0) \theta(G, \|\hat{f}\|_0) + \theta(G, \|f\|_0) \theta(G, \|\hat{g}\|_0) \right), \tag{10}$$

and

$$\|V_g f\|_0 \geq \sqrt{\theta(G, \|f\|_0) \theta(G, \|\hat{f}\|_0) \theta(G, \|g\|_0) \theta(G, \|\hat{g}\|_0)}. \tag{11}$$

**Proof.** We shall prove $\|V_g f\|_0 \geq \theta(G, \|f\|_0) \theta(\widehat{G}, \|\hat{g}\|_0)$. Then (10) follows from $\|V_g f\|_0 = \|V_f \hat{f}\|_0$ and $\theta(G, k) = \theta(\widehat{G}, k)$ for any $k$, or, alternatively from $\|V_g f\|_0 = \|V_f \hat{g}\|_0$. Further, (9) implies (10) and (11) since the maximum of two positive numbers dominates their arithmetic and geometric means.

To see (9), observe first that the so-called symplectic Fourier transformation $F_s = R \circ F^{-1}_G \circ F_G$, i.e., the composition of a Fourier transformation $F_G$ on $G$, an inverse Fourier transformation $F^{-1}_G$ on $\widehat{G}$, and the axis transformation $R : F \mapsto F \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ obeys the same uncertainty principle as the Fourier transformation on the group $G \times \widehat{G}$. For $f, g \in \mathbb{C}^G$, we calculate

$$F_s V_g f(r, \rho) = \sum_{x \in G} \sum_{\xi \in \hat{G}} V_g f(x, \xi) \langle \rho, x \rangle \langle \xi, r \rangle = \sum_{x \in G} \sum_{\xi \in \hat{G}} \sum_{t \in G} f(t) g(t-x) \langle \xi, t \rangle \langle \rho, x \rangle \langle \xi, r \rangle$$

$$= \sum_{x \in G} \sum_{\xi \in \hat{G}} f(t) g(t-x) \langle \rho, x \rangle \sum_{\xi \in \hat{G}} \langle \xi, r-t \rangle = |G| \sum_{x \in G} f(r) g(r-x) \langle \rho, x \rangle$$

and note that $\text{supp } F_s V_g f = \text{supp } f \times \text{supp } \hat{g}$. Proposition 3.9 implies that $\|V_g f\|_0 = \|F_s^{-1}(F_s V_g f)\|_0 \geq \theta(G, \|f\|_0) \theta(\widehat{G}, \|\hat{g}\|_0)$. \hfill $\square$

For $G = \mathbb{Z}_6$, we list in Table 2 the lower bounds on $\|V_g f\|_0$ given by (9) for different values of $\|f\|_0$, $\|\hat{f}\|_0$, $\|g\|_0$ and $\|\hat{g}\|_0$.

**Corollary 4.3.** For $f, g \in \mathbb{C}^{2^p} \setminus \{0\}$, $p$ prime,

$$\|V_g f\|_0 \geq \max \{ (p+1-\|g\|_0)(p+1-\|\hat{f}\|_0), (p+1-\|f\|_0)(p+1-\|\hat{g}\|_0) \}$$

14
We set $A, b$ then Table 2.

and $\|V_g f\|_0 \geq (p+1)^2 - \frac{1}{2}(p+1)\|f\|_0 + \|\hat{f}\|_0 + \|g\|_0 + \|\hat{g}\|_0 + \frac{1}{2} (\|\hat{f}\|_0\|g\|_0 + \|f\|_0\|\hat{g}\|_0)$. Now, we give an improvement to the lower bound on $\|V_g f\|_0$ that is given in Corollary 4.3

**Proposition 4.4.** For $f, g \in \mathbb{C}^{Z_p} \setminus \{0\}$, $p$ prime,

$$\|V_g f\|_0 \geq \begin{cases} p(p+1) - \|f\|_0\|g\|_0 & \text{if } \|f\|_0 + \|g\|_0 > p; \\ p(p+1) - (p+1)(\|f\|_0)(p+1-\|g\|_0) & \text{if } \|f\|_0 + \|g\|_0 \leq p. \end{cases}$$

**Proof.** Note that for all $x \in G$, $V_g f(x, \cdot) = \langle f, \pi(x, \cdot)g \rangle$ represents the Fourier transform of a vector of the form $f T_x \hat{g}$, i.e.,

$$V_g f(x, \xi) = \langle f, \pi(x, \xi)g \rangle = \sum y \hat{g}(y-x)\langle \xi, x \rangle = \hat{f} T_x \hat{g}(\xi) \quad x \in G, \xi \in \hat{G}.$$  

As long as $f T_x \hat{g} \neq 0$, Theorem 3.3 applies and so $\|f T_x \hat{g}\|_0 + \|\hat{f} T_x \hat{g}\|_0 \geq p + 1$. For $K := \{x : f T_x \hat{g} \neq 0\}$ we get

$$\|V_g f\|_0 = \sum_{x \in K} \|\hat{f} T_x \hat{g}\|_0 \geq |K|(p+1) - \sum_{x \in K} \|f T_x \hat{g}\|_0 = |K|(p+1) - \|f\|_0\|g\|_0,$$

where $\sum_{x \in K} \|f T_x \hat{g}\|_0 = \|f\|_0\|g\|_0$ follows from a simple counting argument.

We shall now estimate $|K|$ using the Cauchy-Davenport inequality, which states that for non-empty subsets $A$ and $B$ of $\mathbb{Z}_p$, $p$ prime, $|A+B| \geq \min(|A|+|B|-1, p)$, where $A+B = \{a+b : a \in A, b \in B\}$ [Káro05]. Now $K = \{x : f T_x \hat{g} \neq 0\} = \{x : (\text{supp } \hat{g}) + x \cap \text{supp } f \neq \emptyset\} = \text{supp } f - \text{supp } \hat{g}$. We set $A = \text{supp } f, B = \text{supp } \hat{g}$, and obtain $|K| = |\text{supp } f - \text{supp } \hat{g}| \geq \min(\|f\|_0 + \|g\|_0 - 1, p)$. If $\|f\|_0 + \|g\|_0 \geq p+1$, then $|K| = p$ and, hence, $\|V_g f\|_0 \geq p(p+1) - \|f\|_0\|g\|_0$. If $\|f\|_0 + \|g\|_0 \leq p$, then $|K| \geq \|f\|_0 + \|g\|_0 - 1$ and so

$$\|V_g f\|_0 \geq (\|f\|_0 + \|g\|_0 - 1)(p+1) - \|f\|_0\|g\|_0 = p(p+1) - (p+1)\|f\|_0\|g\|_0.$$

The lower bound on $\|V_g f\|_0$ given in Proposition 4.4 is illustrated for $G = \mathbb{Z}_5$ in Table 3. To establish results similar to Proposition 3.11 for the short-time Fourier transformations for a given group $G$ is quite tedious since it requires to check all combinations of $\|f\|_0$ and $\|g\|_0$. For the case $G = \mathbb{Z}_4$, however, we have assembled all possible and impossible combinations in Figure 7. A derivation of the entries can be found in the appendix.
Figure 6. The set \( \{ (\|f\|_0, \|g\|_0, \|V_g f\|_0), f, g \in \mathbb{C}^G \setminus \{0\} \} \) for \( G = \mathbb{Z}_5 \). The color coding is based on Figure 2 and justified by Proposition 4.4 and Theorem 4.5.

Figure 7. Same as Figure 6 for \( G = \mathbb{Z}_3 \).

4.1. Groups of prime order

In the following, we shall fix the window \( g \) and vary only the analyzed function \( f \). The main result in this section is

**Theorem 4.5.** There exists \( g \in \mathbb{C}^Z_p \), \( p \) prime, such that for all \( f \in \mathbb{C}^Z_p \)

\[
\|f\|_0 + \|V_g f\|_0 \geq p^2 + 1. \tag{12}
\]

Moreover, for \( 1 \leq k \leq p \) and \( 1 \leq l \leq p^2 \) with \( k + l \geq p^2 + 1 \) there exists \( f \) with \( \|f\|_0 = k \) and \( \|V_g f\|_0 = l \).

Figure 8. The set \( S_g = \{ (\|f\|_0, \|V_g f\|_0), f \in \mathbb{C}^G \setminus \{0\} \} \) for appropriately chosen \( g \in \mathbb{C}^G \setminus \{0\} \) and \( G = \mathbb{Z}_5 \) or \( G = \mathbb{Z}_7 \). The color coding is based on Figure 2 and justified by Theorem 4.5.

We picture this result for \( G = \mathbb{Z}_5 \) and \( G = \mathbb{Z}_7 \) in Figure 8. Note that Theorem 4.5 follows from Proposition 3.4 together with Theorem 4 from [LPW05] which we state as

**Theorem 4.6.** For almost every \( g \in \mathbb{C}^Z_p \), \( p \) prime, we have that every minor of \( A_{Z_p,g} \) is nonzero.

**Outline of a proof of Theorem 4.6.** It suffices to show that each square submatrix \( (A_{Z_p,g})_{A,B} \) has determinant nonzero for almost every \( g \).
To this end, choose \( A \subseteq G \) and \( B \subseteq G \times \hat{G} \) with \( |A| = |B| \) and set \( P_{A,B}(z) = \det(A_{z_p,z})_{A,B} \), \( z = (z_0, z_1, \ldots, z_{p-1}) \). To show that \( P_{A,B} \neq 0 \), we shall locate a term in the polynomial in standard form which has a nonzero coefficient. To construct this term, we determine first the maximal possible exponent of \( z_0 \) in one of the terms of \( P \) that are not trivially zero. Next, we determine the maximal exponent that \( z_1 \) can have in a monomial where the maximal exponent of \( z_0 \) is attained and so on.

Using generalized Vandermonde determinants, it can then be shown that the coefficient of this “maximal” term within \( P_{A,B} \) can be expressed as a product of different minors of the discrete Fourier matrix \( W_{Z_p} \). For \( p \) prime, all these minors are nonzero, so the polynomial \( P \) has a nonzero coefficient for this “maximal term”, hence is not identically 0, and nonzero almost everywhere. We have \( P = \prod_{A,B:|B|=|A|} P_{A,B} \neq 0 \), which implies that for \( g \notin Z_P = \{z: P(z) = 0\} \), every minor of \( A_{Z_p,g} \) is nonzero. Clearly, since \( P \neq 0 \), \( Z_P \) has Lebesgue measure 0. \( \square \)

Clearly, this proof of Theorem 4.6 is based on Chebotarev’s Theorem. Also, Chebotarev’s Theorem and therefore Theorem 3.3 can be obtained as a corollary to Theorem 4.6 as shown in the Appendix.

It is easy to see that if \( g \in \mathbb{C}^{Z_p} \) satisfies (12) then \( \|g\|_0 = \|\hat{g}\|_0 = p \), i.e., \( g(x) \neq 0 \) for all \( x \in G \) and \( \hat{g}(\xi) \neq 0 \) for all \( \xi \in \hat{G} \) [LPW05]. Further, we have

**Proposition 4.7.** There exists a \( g \in \mathbb{C}^{Z_p} \), \( p \) prime, with \( |g(x)| = 1 \) for all \( x \in G \) and which satisfies the conclusions of Theorem 4.6.

**Proof.** Theorem 4.6 implies that all minors of \( A_{G,g} \) are nonzero polynomials in the polynomial ring \( \mathbb{C}[z_0, \ldots, z_{n-1}] \). Let \( P \) be the product of all these minor polynomials, which, by assumption, is nonzero. We have to show that \( P(\hat{g}) \neq 0 \) for some \( g \in \mathbb{C}^{Z_p} \) with \( |g(x)| = 1 \) for all \( x \in G \).

This follows since the only polynomial \( P \) with \( P(g) = 0 \) whenever \( |g(x)| = 1 \) for all \( x \in G \) is trivial, i.e., \( P \equiv 0 \), which we show below using induction over the number of variables \( n \).

The case \( n = 1 \) follows since any nonzero polynomial in one variable has only finitely many zeros, i.e., only \( P \equiv 0 \) vanishes for all \( z \in S^1 = \{z: |z| = 1\} \). Next, we consider a polynomial \( P \) of \( n \) variables which we regard as a polynomial in \( z_{n-1} \) with coefficients in the polynomial ring \( \mathbb{C}[z_0, \ldots, z_{n-2}] \), i.e.,

\[
P(z_{n-1}) = Q_m(z_0, \ldots, z_{n-2})z_{n-1}^m + Q_{m-1}(z_0, \ldots, z_{n-2})z_{n-1}^{m-1} + \cdots + Q_0(z_0, \ldots, z_{n-2})
\]

For any fixed \( (c_0, \ldots, c_{n-2}) \in (S^1)^{n-1} \) we have

\[
Q_m(c_0, \ldots, c_{n-2})z_{n-1}^m + Q_{m-1}(c_0, \ldots, c_{n-2})z_{n-1}^{m-1} + \cdots + Q_0(c_0, \ldots, c_{n-2}) = 0
\]

for all \( z_{n-1} \in S^1 \), hence, all its coefficients \( Q_k(c_0, \ldots, c_{n-2}) \), \( k = 0, \ldots, m \) vanish. In other words, we have that \( Q_k \in \mathbb{C}[z_0, \ldots, z_{n-2}] \), \( k = 0, \ldots, m \) vanish on \( (S^1)^{n-1} \), which, by induction hypothesis, implies that all \( Q_k \equiv 0 \) and therefore \( P \equiv 0 \). \( \square \)

Table 3 together with Lemma 3.3 show that the condition “\( G = Z_p \) with \( p \) prime” is necessary for the existence of \( g \in \mathbb{C}^G \) satisfying (12).

**Proposition 4.8.** If \( |G| \) is not prime, then \( A_{G,g} \) has zero minors for all \( g \in \mathbb{C}^G \).

17
Proof. Let \(|G| = k \cdot m\), \(k, m \neq 1\). We consider only \(G = \mathbb{Z}_{km}\), the general case follows since the Fourier matrix \(W_G\) for any non-cyclic \(G\) is a Kronecker product of Fourier matrices of cyclic groups.

For a primitive \(|G|\)-th root of unity \(\omega\), we have \((\omega^k)^m = \omega^{|G|} = 1\), so the discrete Fourier matrix \(W_G\) has a 1 in its \((k, m)\)-entry. Now the matrix given by the first \(|G|\) columns of \(A_{G,g}\) results from \(W_G\) by multiplying the \(i\)-th row by \(c_i\). So the minor given by the columns 0 and \(k\) and the rows 0 and \(m\) of \(A\) is det \(\begin{pmatrix} c_0 & c_m \\ c_0 & c_m \end{pmatrix}\) = 0. Hence \(A_{G,g}\) has a zero minor.

\[
\square
\]

4.2. Groups of non-prime order

Recall Proposition 4.1, namely, the fact that for any \(G\) and \(0 < k \leq |G|\) we have

\[\min_{g \in \mathbb{C}^G \setminus \{0\}} \min \{ \| V_g f \|_0 : f \in \mathbb{C}^G \text{ and } 0 < \| f \|_0 \leq k \} = |G|,\]

and

\[\max_{g \in \mathbb{C}^G \setminus \{0\}} \max \{ \| V_g f \|_0 : f \in \mathbb{C}^G \text{ and } 0 < \| f \|_0 \leq k \} = |G|^2.\]

Certainly, \(\| V_g f \|_0 = |G|\) is a rare event. In fact, it is reasonable to assume that \(\| V_g f \|_0 = |G|^2\) for almost every pair \((f, g)\). We shall now address the question whether for an appropriately chosen window \(g\), we can achieve \(\| V_g f \|_0 \geq l\) for some \(|G| < l \leq |G|^2\).

To this end, we define for \(1 \leq k \leq |G|\)

\[
\phi(G, k) := \max_{g \in \mathbb{C}^G \setminus \{0\}} \min \{ \| V_g f \|_0 : f \in \mathbb{C}^G \text{ and } 0 < \| f \|_0 \leq k \}. \tag{13}
\]

Using this notation, Theorem 4.5 indicates that \(\phi(\mathbb{Z}_p, k) = p^2 - k + 1\) for \(p\) prime. Taking max and min is justified due to the compactness of the unit ball in \(\mathbb{C}^G\). In fact, we have

**Proposition 4.9.** For almost every \(g \in \mathbb{C}^G\), \(\min_{0 < \| f \|_0 \leq k} \| V_g f \|_0 = \phi(G, k)\) for all \(k \leq |G|\).

In the following, we set \(Q_{A,B}(z) = \det(A_{G,z})_{A,B} = (A_{G,z})_{A,B} \in z_{|G|-1}, \) for \(A \subseteq G\) and \(B \subseteq G \times \hat{G}\). \(Q_{A,B}\) is a homogeneous polynomial in \(z_0, z_1, \ldots, z_{|G|-1}\) of degree \(2|A|\).

**Lemma 4.10.** The vector \(g \in \mathbb{C}^G\) satisfies \(\min_{0 < \| f \|_0 \leq k} \| V_g f \|_0 \geq l\) if and only if \(Q_{A,B}(g) \neq 0\) for all \(A \subseteq G\) with \(|A| = k\) and all \(B \subseteq G \times \hat{G}\) with \(|B| = |G|^2 - l + 1\).

| \(k\) | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| 1   | 125| 0  | 0  | 0  | 0  | 0  |
| 2   | 0  | 3000| 0  | 0  | 0  | 0  |
| 3   | 0  | 0  | 23000| 0  | 0  | 0  |
| 4   | 0  | 0  | 0  | 63250| 0  | 0  |
| 5   | 0  | 0  | 0  | 0  | 53130| 0  |

Table 3. Count of numerically computed ranks of minors of \(A_{\mathbb{Z}_k,g}\) and \(A_{\mathbb{Z}_p,g}\) for randomly generated \(g\). Columns correspond to the dimension of square submatrices and rows to the rank of submatrices considered.

| \(l\) | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| 1   | 216| 216| 0  | 0  | 0  | 0  |
| 2   | 0  | 9234| 1368| 0  | 0  | 0  |
| 3   | 0  | 0  | 141432| 2106| 0  | 0  |
| 4   | 0  | 0  | 0  | 881409| 0  | 0  |
| 5   | 0  | 0  | 0  | 0  | 2261952| 0  |
| 6   | 0  | 0  | 0  | 0  | 0  | 1947792|
Proof. Fix $A \subseteq G$ with $|A| = k$ and $g \in \mathbb{C}^G$. Then $g$ satisfies $\|V_g f\|_0 \geq l$ for all $f$ with $\text{supp} \ f \subseteq A$ if and only if $\langle f \mid \pi(\lambda)g \rangle_A = \langle f \mid \pi(\lambda) \rangle_A \neq 0$ for at least $l$ elements $\lambda \in G \times \hat{G}$ for all $f$ with $\text{supp} \ f \subseteq A$, i.e., for at most $|G|^2 - l$ vectors in $\{\pi(\lambda)g\}$ we have $\langle f \mid \pi(\lambda)g \rangle_A = 0$ for $\text{supp} \ f \subseteq A$. This is equivalent to $\{\pi(\lambda)g \}_x \in A$ spans $\mathbb{C}^A$ whenever $|B| = |G|^2 - l + 1$. That is, if and only if rank $(A_{G,g})_{A,B} = |A|$ for all $B$ with $|B| = |G|^2 - l + 1$. But this is equivalent to $Q_{A,B}(g) \neq 0$ for all $|B| = |G|^2 - l + 1$. The result follows since for each $f$ with $\|f\|_0 \leq k$ exists $A \subseteq G$ with $|A| = k$ and $\text{supp} \ f \subseteq A$. □

Proof of Proposition 4.11. Lemma 4.10 and $\min_{0 < \|f\|_0 \leq k} \|V_g f\|_0 \geq \phi(G, k)$, for some $g_k \in \mathbb{C}^G \setminus \{0\}$ imply that $Q_{A,B} \neq 0$ for all pairs $A \subseteq G$ and $B \subseteq G \times \hat{G}$ with $|B| = |G|^2 - \phi(G, |A|) + 1$. Hence, $Q = \bigcap_{A,B: |B| = \phi(G, |A|) + 1} Q_{A,B} \neq 0$. This implies that $Q(g) \neq 0$ for almost every $g \in \mathbb{C}^G$ and therefore, for almost every $g \in \mathbb{C}^G$ we have $\min_{0 < \|f\|_0 \leq k} \|V_g f\|_0 \geq \phi(G, k)$ for all $k \leq |G|$.

To obtain bounds on $\phi(G, k)$ for groups of non-prime order, we shall apply Meshulam’s strategy to the function $\phi$.

Proposition 4.11. Let $H$ be a subgroup of the finite Abelian group $G$. For $k \in \mathbb{N}$ exist $s, t \in \mathbb{N}$ with $st \leq k$ such that

$$\phi(G, k) \geq \phi(H, s)\phi(G/H, t)$$

(14)

Proof. In the following, we express the short–time Fourier transformation for functions defined on $G$ as two consecutive short–time Fourier transformations. We apply again the notation from the proof of Theorem 3.7, i.e., $H = \{x_i\} = \{y_i\}$ and $\{y_j\}$ is a set of coset representatives of the quotient group $G/H$. As before $H^\perp = \{\xi_j \in \hat{G} : \xi_j(H) = 1\}$ and $\{\xi\}$ is a set of coset representatives of $\hat{G}/H^\perp$.

Set

$$\phi_H(G, k) = \max_{g_1 \in \mathbb{C}^H, g_2 \in \mathbb{C}^{G/H}} \min_{f \in \mathbb{C}^G} \left\{ \|V_{g_1 \otimes g_2} f\|_0 : f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\},$$

where $g_1 \otimes g_2(x_i + x_j) = g_1(x_i)g_2(x_j + H)$. Clearly $\phi(H, k) \geq \phi_H(G, k)$, so (14) follows from $\phi_H(G, k) \geq \phi(H, s)\phi(G/H, t)$, which we shall show below. First, note that a similar argument as is used in Proposition 4.9 gives that for almost every pair $(g_1, g_2)$,

$$\phi_H(G, k) = \min_{0 < \|f\|_0 \leq k} \|V_{g_1 \otimes g_2} f\|_0, \quad 1 \leq k \leq |G|.$$ 

Therefore, we can pick $g_1$ and $g_2$ so that for all possible $k, s, t$

$$\phi_H(G, k) = \min_{0 < \|f\|_0 \leq k} \|V_{g_1 \otimes g_2} f\|_0, \quad \phi(H, s) = \min_{0 < \|f\|_0 \leq s} \|V_{g_1} f\|_0, \quad \phi(G/H, t) = \min_{0 < \|f\|_0 \leq t} \|V_{g_2} f\|_0. \quad (15)$$

We fix $x = x_i + x_j$ and $\xi = \xi_i + \xi_j$, and compute as in the proof of Proposition 3.8

$$V_{g_1 \otimes g_2} f(x, \xi) = \sum_{y_j} \sum_{y_i} f(y_i + y_j) g_1(y_i - x_i) g_2(y_j - x_j + H) \langle \xi_i, y_i \rangle_H \langle \xi_j, y_j \rangle \delta_{G/H} \langle \xi_i, y_i \rangle H_{G/H}$$

$$= \sum_{y_j} g_2(y_j - x_j + H) \langle \xi_j, y_j \rangle \delta_{G/H} \langle \xi_j, y_j + H \rangle \langle \xi_i, y_i \rangle \sum_{y_i} f(y_i + y_j) g_1(y_i - x_i) \langle \xi_i, y_i \rangle H_{G/H}$$

19
as expected, the computational results are better than those given in (17), since the tensor approach obtained using \( \| \{ y \coset \partial \text{short–time Fourier transform of } T_{\xi} \} \) where we used \( \| g \|_v \) from Proposition 4.13.

Let \( (H, s) = (n, k) \), i.e., \( v(a, b) = v(a, c, d) \geq v(ac, bd) \). We proceed by induction on \( |G| = n \). Suppose (16) holds for \( |G| = 1, \ldots, n-1 \). If \( n \) is prime, then Proposition 4.11 implies \( v(n, k) = n(1 + n - k) < n^2 - k + 1 = \phi(\Z_p, k) \) for all \( k \). Else, we choose a nontrivial divisor \( d \) of \( n \), and let \( H \) be a subgroup of \( G \) of order \( d \). By Proposition 4.11 there exist \( s, t \) with \( 1 \leq s \leq d, \leq t \leq \min \{ \frac{k}{2}, \frac{n}{2} \} \) such that \( \phi(G, k) \geq \phi(H, s) \phi(G/H, t) \). Therefore, \( \phi(G, k) \geq \phi(d, s) v(\frac{n}{2}, t) \geq v(n, st) \geq \phi(n, k) \).

For the case \( G = \Z_{pq} \), we can improve this estimate by finding the convex hull of all pairs \( (|H|, |G/H|) \) for all subgroups \( H \) of \( G \) as in [Mes05].

**Proposition 4.13.** Let \( G = \Z_{pq} \) with \( q < p \) and \( p, q \) prime. Then

\[
\phi(G, k) \geq \begin{cases} p^2(q^2 - k + 1) & \text{if } k < q; \\ (p^2 - \frac{k}{q} + 1)(q^2 - q + 1) & \text{else.} \end{cases}
\]

The proof of Proposition 4.13 is included in the appendix. At \( k = q \), the two lower bounds in (17) coincide and lead to what a geometric argument shows to be the optimal value that can be obtained using \( g = g_1 \odot g_2 \). So the two straight lines give a convex hull similar to [Mes05]. However, as expected, the computational results are better than those given in (17), since the tensor approach cannot be used to find optimal bounds for \( \phi(G, k) \). See Table 4 for an illustration of (17) for \( G = \Z_6 \).
Table 4. Lower bounds for $\|V_gf\|_0$ given by Theorem 4.12 and Proposition 4.13 for $G = \mathbb{Z}_6$ and almost every $g \in \mathbb{C}^{\mathbb{Z}_6}$.

|        | 1     | 2     | 3     | 4     | 5     | 6     |
|--------|-------|-------|-------|-------|-------|-------|
| Theorem 4.12 | 36    | 18    | 12    | 10    | 8     | 6     |
| Proposition 4.13 | 36    | 26    | 25    | 23    | 22    | 20    |

4.3. Outlook

For $|G|$ prime, Theorem 4.5 characterizes all pairs $(\|f\|_0, \|V_gf\|_0)$, $f \in \mathbb{C}^G$ which are achieved for almost every window function $g \in \mathbb{C}^G$. Below, we conjecture a similar classification result for general finite Abelian Groups.

**Conjecture 4.14.** For every finite Abelian group $G$ and almost every $g \in \mathbb{C}^G$, we have

$$\{(\|f\|_0, \|V_gf\|_0), f \in \mathbb{C}^G \setminus \{0\}\} = \{(\|\hat{f}\|_0 + |G|^2 - |G|), f \in \mathbb{C}^G \setminus \{0\}\}.$$

This conjecture is illustrated in Figure 9. As noted earlier, the numerical testing based on the rank of submatrices of $A_{G,g}$ is very cost intensive since the number of submatrices that have to be considered grows combinatorially.

|        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|
| $\mathbb{Z}_4$ |        |        |        |        |        |        |
| $\mathbb{Z}_6$ |        |        |        |        |        |        |
| $\mathbb{Z}_8$ |        |        |        |        |        |        |

**Figure 9.** Same as Figure 8 for $G = \mathbb{Z}_4$, $G = \mathbb{Z}_6$, and $G = \mathbb{Z}_8$. The color coding from Figure 2 is applied in accordance with Conjecture 4.14 and numerical experiments based on Lemma 3.5.

Consequences of this conjecture are discussed in Section 5. Here, we state some preliminary observations regarding Conjecture 4.14.

For example, the technique used to prove Theorem 4.5 possesses certain degrees of freedom, that is, we only need to show that a particular product of minors is nonzero. Nevertheless, these degrees of freedom do not allow us to prove Conjecture 4.14. For example, for $G = \mathbb{Z}_4$, we can choose the $4 \times 4$ submatrix

$$M(z) = \left(A_{\mathbb{Z}_4}, (z_0, z_1, z_2, z_3)\right)_{\{0,1,4,12\},\{0,1,2,3\}} = \begin{pmatrix} z_0 & z_0 & z_3 & z_1 \\ z_1 & -z_1 & z_0 & z_2 \\ z_2 & z_2 & z_1 & z_3 \\ z_3 & -z_3 & z_2 & z_0 \end{pmatrix}$$
In this submatrix, none of the monomials that is “maximal” in the sense described above, namely the monomials \( z_3^3 z_2^0 \), \( z_3^2 z_2 z_1^0 \), \( z_3^3 z_2 z_1^0 \), and \( z_3^2 z_2 z_1^2 \), has a nonzero coefficient in the polynomial \( P(z_0, z_1, z_2, z_3) = \det M(z) = -2z_3^2 z_1^2 - 2z_2^3 z_2^3 - 2z_3^3 z_3^3 - 4z_0 z_1 z_2 z_3 \neq 0 \).

Using Proposition 3.6, we derive a partial result on nonzero minors of \( A_{\mathbb{Z}, g} \).

**Proposition 4.15.** For every \( n \), any minor of the full Gabor system matrix \( A_{\mathbb{Z}, g} \), where the columns corresponding to each fixed translation are adjacent with respect to modulation is nonzero for almost every \( g \). The same holds for a minor corresponding to a submatrix of size \( n \times n \), where the columns corresponding to each fixed modulation are adjacent with respect to translation.

5. APPLICATIONS

We shall now turn to applications of the results stated in Section 4 to communications engineering and, in the subsequent section, to the problem of recovering sparse signals from incomplete data.

5.1. Gabor frames, erasures, and the identification of operators

We are interested in transmitting information in the form of the entries of a vector \( f \in \mathbb{C}^G \) over a channel in such a way that recovery of the information at the receiver is robust to errors introduced by the channel. In particular, we will focus on two problems. First, we shall discuss transmission over a channel with erasure, i.e., some of the vector entries may be lost during transmission. Second, we discuss the so-called identification problem for another class of operators, namely, of linear time-variant operators which play a central role in wireless and mobile communications. Clearly, knowledge of the operator at hand would help to counteract disturbances that were caused during transmission.

We begin with a brief discussion of the recovery of information from a vector that suffered erasures. Rather then sending the information in raw form, i.e., sending vector entries one-by-one, information is being coded prior to transmission. For example, we can choose a frame \( \{\varphi_k\}_{k \in K} \) for \( \mathbb{C}^G \) and send the coefficients \( \langle f, \varphi_k \rangle \), \( k \in K \). If none of the transmitted coefficients are lost, the receiver can use a dual frame \( \{\tilde{\varphi}_k\} \) of \( \{\varphi_k\} \) and recover \( f \) using (3). In fact, even if some coefficients are lost and only \( \langle f, \varphi_k \rangle \) is received for \( k \in K' \subset K \), then the information can still be recovered if and only if \( \{\varphi_k\}_{k \in K'} \) remains a frame. This necessitates that \(|K'| \geq |G| = \dim \mathbb{C}^G| \).

**Definition 5.1.** A frame \( F = \{\varphi_k\}_{k \in K} \) in \( \mathbb{C}^G \) is maximally robust to erasures if the removal of any \( l \leq |K| - |G| \) vectors from \( F \) leaves a frame.

Similarly, we give

**Definition 5.2.** A set of \( m \) vectors in \( \mathbb{C}^G \) is in general position, if any collection of at most \( |G| \) of these vectors are linearly independent.

Before giving slight generalizations of results from [LPW05] on Gabor frames that are maximally robust to erasure in Theorem 5.4, we introduce some vocabulary and notation regarding the previously mentioned operator identification problem.

**Definition 5.3.** A linear space of operators \( \mathcal{H} \) mapping \( \mathbb{C}^A \) to \( \mathbb{C}^B \) is called identifiable with identifier \( g \in \mathbb{C}^A \) if the linear map \( \varphi_g : \mathcal{H} \rightarrow \mathbb{C}^B \), \( H \mapsto Hg \) is injective, i.e., if \( Hg \neq 0 \) for all \( H \in \mathcal{H}\setminus\{0\} \).
Time-variant communication channels, for example, multipath channels in wireless telephony, are often modeled through a combination of translation operators (time-shift, delay) and modulation operators (frequency shifts that are caused by the Doppler effect). Therefore, identification of $H_\Lambda = \{ \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda), \ c_\lambda \in \mathbb{C} \}$ for $\Lambda \subseteq G \times \hat{G}$ is quite relevant.

**Theorem 5.4.** For $g \in \mathbb{C}^G \setminus \{0\}$, the following are equivalent:

1. Every minor of $A_{G,g}$ of order $|G|$ is nonzero.
2. The vectors from the Gabor system $\{ \pi(\lambda)g \}_{\lambda \in G \times \hat{G}}$ are in general position.
3. The Gabor system $\{ \pi(\lambda)g \}_{\lambda \in G \times \hat{G}}$ is an equal norm tight frame which is maximally robust to erasures.
4. For all $f \in \mathbb{C}^G \setminus \{0\}$ we have $\|V_gf\|_0 \geq |G|^2 - |G| + 1$.
5. For all $f \in \mathbb{C}^G$, $V_gf(\lambda)$, and, therefore, $f$, is completely determined by its values on any set $\Lambda$ with $|\Lambda| = |G|$.
6. $H_\Lambda$ is identifiable by $g$ if and only if $|\Lambda| \leq |G|$

For $|G|$ prime, Theorem 5.4 ensures the existence of $g \in \mathbb{C}^G$ which satisfy parts 1-5 in Theorem 5.4. A verification of Conjecture 1.14 would also confirm the existence of $g \in \mathbb{C}^G$ satisfying Theorem 5.4 part 4 and therefore Theorem 5.4 parts 1-5 for general finite Abelian groups.

**Remark 5.5.** To our knowledge, the only known equal norm tight frames that are maximally robust to erasures are so-called harmonic frames (see Conclusions in [CK03]). Harmonic frames for $\mathbb{C}^n$ with $m \geq n$ elements are obtained by deleting uniformly $m - n$ components of the characters of $\mathbb{Z}_m$ [CK03]. Similarly, Theorem 1.6 together with Theorem 1.7 provides us with a large class of equal norm tight frames with $p^2$ elements in $\mathbb{C}^n$ for $n \leq p$. Namely, we can choose $g \in (S^1)^p$ and remove $p - n$ components of the equal norm tight frame $\{ \pi(\lambda)g \}_{\lambda \in G \times \hat{G}}$ in order to obtain an equal norm tight frame which is maximally robust to erasure. Note that this frame is not a Gabor frame proper. Reducing the number of vectors in the frame to $m \leq p^2$ leaves an equal norm frame which is maximally robust to erasure but which might not be tight. This holds for harmonic frames too and with the restriction to $p$ prime, we have shown the existence of Gabor frames which share the usefulness of harmonic frames when it comes to transmission of information through erasure channels.

Background and more details on frames and erasures can be found in [CK03, GK01, SH03] and the references cited therein.

### 5.2. Signals with sparse representations

In Section 5.1 we discussed the recovery of signals or operators from $|G|$ known complex numbers. Here, we will use the functions $\phi$ and $\theta$ which were defined in Section 3.2 and Section 1.2 to refine some of these findings. That is, we show that a function/signal which can be represented as a linear combination of a small number of pure frequencies or of a small number of time-frequency shifts of
a fixed function \( g \), can be recovered from fewer than \(|G|\) of its values. Our brief discussion is based on the most basic ideas and results from the theory of sparse signal recovery.

There exist a number of entry points to the theory of sparse signal recovery. Here, we shall consider dictionaries \( D = \{ g_0, g_1, \ldots, g_{n-1} \} \) of \( n \) vectors in \( \mathbb{C}^n \), or equivalently, in \( \mathbb{C}^G \). For \( k \leq n = |G| \) we shall examine the sets

\[
\Sigma_k^D = \{ f \in \mathbb{C}^n : f = M_D c = \sum_r c_r g_r, \text{ with } \| c \|_0 \leq k \}.
\]

The central question is: how many values of \( f \in \Sigma_k^D \) need to be known (or stored), in order that \( c \in \mathbb{C}^N \) with \( f = \sum_r c_r g_r \) and \( \| c \|_0 \leq k \), and therefore \( f \), is uniquely determined by the known data?

To this end, we set

\[
\psi(D, k) = \min \left\{ \| f \|_0 : f \in \Sigma_k^D \right\},
\]

and observe the following well known result.

**Proposition 5.6.** Any \( f \in \Sigma_k^D \) is fully determined by any choice of \( n - \psi(D, 2k) + 1 \) values of \( f \).

Note that unlike in Theorem 5.4, we do not assume knowledge of the set \( \text{supp} \ c \) for \( c \) with \( M_D c = f \), \( \| f \|_0 \) in Proposition 5.4 and in the following.

**Proof.** Assume that for some \( B \subset \mathbb{C}^n \) with \( |B| = n - \psi(D, 2k) + 1 \), two coefficient vectors \( c_1, c_2 \in \mathbb{C}^N \) exist that satisfy \( r_B M_D c_1 = r_B f = r_B M_D c_2 \) and \( \| c_1 \|_0, \| c_2 \|_0 \leq k \). Then \( \| c_2 - c_1 \|_0 \leq 2k \) with \( \| M_D(c_2 - c_1) \|_0 \leq n - |B| = n - (n - \psi(D, 2k) + 1) = \psi(D, 2k) - 1 \), a contradiction.

A classical dictionary for \( \mathbb{C}^G \) is \( D_G = \{ \xi \}_{\xi \in G} \), where \( G \) is a finite Abelian group. Then

\[
\psi(D, k) = \min \left\{ \| f \|_0 : f \in \Sigma_k^D \right\} = \min \left\{ \| \hat{f} \|_0 : \| f \|_0 \leq k \right\} = \theta(G, k).
\]

This equality together with Proposition 5.6 demonstrates the relevance of the results cited in Section 3 for the recovery of signals with limited spectral content. For example, Theorem 3.7 shows that for any finite Abelian group of order 16 we have \( \theta(G, 6) \geq 3 \). In fact, our computations that are illustrated in Figure 4 show that \( \theta(G, 6) = 4 \) for \(|G| = 16\), and, hence, any \( f \in \Sigma_3^{D_G} = \{ f : \| \hat{f} \|_0 \leq 3 \} \) can be recovered from any choice of \( |G| - \theta(G, 2 \cdot 3) + 1 = 16 - 4 + 1 = 13 \) values of \( f \). For \( f \in \Sigma_3^{D_{Z_4}} \), on the other side, Theorem 3.3 implies that \( f \) is already fully determined by \( |Z_{17}| - \theta(Z_{17}, 2 \cdot 3) + 1 = 17 - (17 - 6 + 1) + 1 = 6 \) of its values.

The results in Section 4 which involve the function \( \phi \) are relevant to determine vectors which have sparse representations in the dictionary \( D_{A_G,g} \) which consists of the columns of \( A_{G,g} \). In fact, we have \( F \in \Sigma_k^{D_{A_G,g}} \) if and only if \( F = V_g f \) for some \( f \in \mathbb{C}^G \) with \( \| f \|_0 \leq k \) and, therefore,

\[
\psi(D_{A_G,g}, k) = \min \left\{ \| V_g f \|_0 : \| f \|_0 \leq k \right\} = \phi(G, k).
\]

For \(|G| \) prime for example, this leads to the following short-time Fourier transform version of Theorem 1.1 in [CRT04].

**Theorem 5.7.** Let \( g \in \mathbb{C}^G \), \( p \) prime, satisfy the conclusion of Theorem 4.3. Then any \( f \in \mathbb{C}^G \) with \( \| f \|_0 \leq \frac{1}{2} |\Lambda| \), \( \Lambda \subset \mathbb{Z}_p \times \mathbb{T}_p \) is uniquely determined by \( \Lambda \) and \( r_{\Lambda} V_g f \).
In terms of sparse representations, the Gabor frame dictionary \( \{ \pi(\lambda)g \}_{\lambda \in G \times \hat{G}} \) of time–frequency shifts of a prototype vector \( g \), i.e., the dictionary consisting of the rows of \( A_{G,g} \), appears to be more interesting. Rudimentary numerical experiments based on Lemma 3.5 give some indication that for any Abelian group \( G \), and almost every \( g \in \mathbb{C}^G \), we have for \( k \leq |G| \\
\psi(\{ \pi(\lambda)g \}_{\lambda \in G \times \hat{G}}, k) = \theta(G, k).
\)

For \( |G| \) prime, Theorem 4.6 implies that \( \psi(\{ \pi(\lambda)g \}_{\lambda \in G \times \hat{G}}, k) = p - k + 1 = \theta(G, k) \), and analogous to Theorem 5.7 we obtain

**Theorem 5.8.** Let \( g \in \mathbb{C}^{\mathbb{Z}_p^p} \), \( p \) prime, satisfy the conclusion of Theorem 4.6. Then any \( f \in \mathbb{C}^{\mathbb{Z}_p^p} \) with \( f = \sum_{\lambda \in A} c_\lambda \pi(\lambda)g, \ A \subset \mathbb{Z}_p \times \mathbb{Z}_p \) is uniquely determined by \( B \) and \( r_B f \) whenever \( |B| \geq 2|A| \).

Note that similar to before, the recovery of \( f \) from \( 2|A| \) samples of \( f \) in Theorem 5.8 does not require knowledge of \( \Lambda \).

### 6. APPENDIX

#### 6.1. Proof of Lemma 3.5

If \( f \in \mathbb{C}^n \) with \( \|f\|_0 = k \) and \( \|Mf\|_0 = l \), then \( A = \text{supp} f \) and \( B^c = \text{supp} Mf \) satisfy \( 0 \neq r_A f \in \text{ker} M_{A,B} \), so \( \text{rank} M_{A,B} < |A| \). Moreover, for \( a \in A \), \( \text{supp} f = A \) implies \( f \notin \{ g : \|g\|_0 < |A| \} \subseteq i_{A \setminus \{a\}} \text{ker} M_{A \setminus \{a\},B} \) and, hence,

\[
\{ \in A \text{ ker } M_{A,B} \setminus i_{A \setminus \{a\}} \text{ ker } M_{A \setminus \{a\},B} \}.
\]

So \( \text{dim ker } M_{A,B} \geq \text{dim ker } M_{A \setminus \{a\},B} + 1 \). We conclude that for all \( a \in A \),

\[
\text{rank } M_{A \setminus \{a\},B} \leq \text{rank } M_{A,B} = |A| - \text{dim ker } M_{A,B} \leq |A| - \text{dim ker } M_{A \setminus \{a\},B} - 1 = \text{rank } M_{A \setminus \{a\},B}
\]

which implies \( \text{rank } M_{A \setminus \{a\},B} = \text{rank } M_{A,B} \). Also, \( \text{supp } Mf = B^c \), so for \( y \in B^c \), \( Mf(y) \neq 0 \). Therefore, \( f \notin \text{ker } M_{A,B \cup \{y\}} \) and so \( f \in i_A \text{ ker } M_{A,B \cup \{y\}} \). This implies

\[
\text{rank } M_{A,B} = |A| - \text{ker } M_{A,B} < |A| - \text{ker } M_{A,B \cup \{y\}} = \text{rank } M_{A,B \cup \{y\}}.
\]

The submatrices considered differ only by one column, so the rank can increase at most by one and we get \( \text{rank } M_{A,B} = \text{rank } M_{A,B \cup \{y\}} - 1 \).

Suppose now that \( A \subseteq \{0, \ldots, n-1\} \) and \( B \subseteq \{0, \ldots, m-1\} \) with \( |A| = k \) and \( |B| = m - l \) satisfy (6). This implies \( \text{dim ker } M_{A,B} \geq 1 \) and that for any \( a \in A \),

\[
\text{dim ker } M_{A \setminus \{a\},B} = |A| - 1 - \text{rank } M_{A \setminus \{a\},B} = |A| - 1 - \text{rank } M_{A,B} = \text{dim ker } M_{A,B} - 1.
\]

So \( i_{A \setminus \{a\}} \text{ ker } M_{A \setminus \{a\},B} \subseteq i_A \text{ ker } M_{A,B} \), and there exists \( f_a \in i_A \text{ ker } M_{A,B} \setminus i_{A \setminus \{a\}} \text{ ker } M_{A \setminus \{a\},B} \), so \( f_a(a) \neq 0 \) and \( f_a(x) = 0 \) for \( x \notin A \) and \( \text{supp } Mf_a \cap B = \emptyset \).

Similarly, (6) implies also that for any \( y \in B^c \) we have \( i_A \text{ ker } M_{A,B \cup \{y\}} \subseteq i_A \text{ ker } M_{A,B} \), so there exists \( g_y \) such that \( Mg_y(y) \neq 0 \) while \( Mg_y(b) = 0 \) for all \( b \in B \).

To conclude this proof, we enumerate the vectors \( f_a, a \in A \) and \( g_y, y \in B^c \) and choose a linear combination

\[
f = \sum_{a \in A} c_a f_a + \sum_{y \in B^c} c_y g_y = \sum_{r=0}^{k+l-1} d_r h_r \quad (18)
\]
with the property that \( \text{supp} \ f = \bigcup_{a \in A} \text{supp} \ f_a = A \) and \( \text{supp} \ Mf = \bigcup_{y \in B^c} \text{supp} \ Mg_y = B^c \).

By construction we have \( \text{supp} \ f \subseteq A \) and \( \text{supp} \ Mf \subseteq B^c \). To get the reverse inequality, we assume without loss of generality that \( \min_{x \in \text{supp} \ h_r} |h_r(x)| = 1 \) for all \( r \), and choose \( d_r = N^{2r} \), where \( N - 1 \geq \|h_r\|_\infty, \|Mh_r\|_\infty, \|Mh_r\|_\infty^1 \) for \( r = 0, 1, \ldots, k+l-1 \). Since \( f_{a_0}(a_0) \neq 0 \) we can find \( s = \max \{ r : h_r(a_0) \neq 0 \} \). Then

\[
|f(a_0)| = \left| \sum_{r=0}^{s} d_r h_r(a_0) \right| \geq |N^{2s} h_s(a_0)| - \left| \sum_{r=0}^{s-1} N^{2r} h_r(a_0) \right| \geq N^{2s} - (N-1) \sum_{r=0}^{s-1} (N^2)^r = N^{2s} - \frac{N^{2s} - 1}{N-1} > 0,
\]

so \( a_0 \in \text{supp} \ f \).

Similarly, \( Mg_{y_0}(y_0) \neq 0 \) for fixed \( y_0 \in B^c \) implies that for \( s = \max \{ r : Mh_r(y_0) \neq 0 \} \) we have

\[
|Mf(y_0)| = \left| \sum_{r=0}^{s} d_r Mh_r(y_0) \right| \geq |N^{2s} Mh_s(y_0)| - \left| \sum_{r=0}^{s-1} N^{2r} Mh_r(y_0) \right| \geq \frac{N^{2s}}{N-1} - \frac{N^{2s} - 1}{N+1} > 0.
\]

We conclude that \( \text{supp} \ f = A \) and \( \text{supp} \ Mf = B^c \).

### 6.2. Proof of Proposition 3.11

Theorem 3.1 and Proposition 3.10 cover all cases but \((k, l) = (2, 4), (3, 3), (4, 2)\). For \( \omega = e^{2\pi i/6} \), we have \( F(1, -1, 0, 1, -1, 0) = (0, 0, 1-\omega^2, 0, 1-\omega^4, 0) \), and only the case \((k, l) = (3, 3)\) remains to be excluded.

The assumption \( \|f\|_0 = 3 \) leads to three different cases.

**Case 1.** If \( f = (c_0, 0, c_2, 0, c_4, 0) \) then \( \hat{f}(\xi) = \hat{f}(\xi + 3) \) and if \( f = (0, c_1, 0, c_3, 0, c_5) \) then \( \hat{f}(\xi) = -\hat{f}(\xi + 3) \). In either case, \( \|\hat{f}\|_0 \) is even and cannot be 3.

**Case 2.** If two entries whose indices differ by 3 are both nonzero, then the support of the Fourier transform cannot be 3 either. To see this, consider without loss of generality, \( f = (c_0, *, *, c_3, *, *) \). Then, for \( c_k \), located at position \( k \), being the third nonzero entry, we have

\[
\hat{f} = (c_0+c_3+c_k, c_0-c_3+\omega^k c_k, c_0+c_3+\omega^{2k} c_k, c_0-c_3+\omega^{3k} c_k, c_0+c_3+\omega^{4k} c_k, c_0-c_3+\omega^{5k} c_k) \tag{19}
\]

If three coordinates of \( \hat{f} \) are 0, then two of the respective sums in (19) contain either both \( c_0 + c_3 \) or both \( c_0 - c_3 \). Without loss of generality, we assume that \( \hat{f}(l_1) = c_0+c_3+\omega^{l_1k} c_k \neq 0 \neq c_0+c_3+\omega^{l_2k} c_k = \hat{f}(l_2) \), \( l_1 < l_2 \). Since \( c_k \neq 0 \) we have \( \omega^{l_1k} = \omega^{l_2k} \) and \( \omega^{(l_2-l_1)k} = 1 \). Since \( k = 1, 2, 4 \) or 5, we must have \( l_1 = l_2 \), but that is a contradiction, as of two entries with distance 3, one must contain the summand \( c_3 - c_0 \) and one \( c_0 + c_3 \).

**Case 3.** If all three nonzero entries are adjacent, then \( \hat{f} \) must have three adjacent entries as well, as otherwise, we could just exchange the roles of \( f \) and \( \hat{f} \) and return to Case 1 or Case 2. Without loss of generality we assume \( f = (c_0, c_1, c_2, 0, 0, 0) \). A modulation in \( f \) results in a translation in \( \hat{f} \), so without loss of generality, we can also assume the first three entries of \( \hat{f} \) to be 0. Hence,

\[
\begin{pmatrix}
1 & 1 & 1 & c_0 \\
1 & \omega & \omega^2 & c_1 \\
1 & \omega^2 & -\omega & c_2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & -\omega
\end{pmatrix}
= 0 \quad \text{but} \quad \det\begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & -\omega
\end{pmatrix}
= -1 \neq 0
\]

and, therefore, \( f = 0 \).
6.3. Proof of Proposition 3.12

The group \( \mathbb{Z}_{pq} \) has \((p-1)(q-1)\) automorphisms, each of them mapping one of the \((p-1)(q-1)\) elements of order \( pq \) to 1. The \( p-1 \) automorphisms on the group \( \mathbb{Z}_{2p} = \{0, 1, 2, \ldots, 2p-1\} \) will allow us to consider only \( f \) with well-"concentrated" nonzero entries.

Every automorphism \( \sigma \) on \( \mathbb{Z}_{pq} \) induces an automorphism \( \tilde{\sigma} \) on the character group \( \widehat{\mathbb{Z}}_{pq} \), which satisfies \( \langle \tilde{\sigma}(\xi), x \rangle = \langle \xi, \sigma^{-1}(x) \rangle \). Further,

\[
\hat{f} \circ \sigma(\xi) = \frac{1}{pq} \sum_{x \in \mathbb{Z}_{pq}} f(\sigma(x))(\xi, x) = \frac{1}{pq} \sum_{y \in \mathbb{Z}_{pq}} f(y)(\xi, \sigma^{-1}(y)) = \frac{1}{pq} \sum_{y \in \mathbb{Z}_{pq}} f(\tilde{\sigma}(\xi), y) = \hat{f}(\tilde{\sigma}(\xi))
\]

Let \( f \in C^{\mathbb{Z}_{2p}}, p \geq 5 \) prime, be given with \( \|f\|_0 = 3 \). Then at least two of the addresses of the non-zero elements have the same parity. By a translation of \( f \) we can move those elements to positions \( 0, 2k \), where \( k \in \mathbb{Z}_{2p} \). The support of \( \hat{f} \) is not affected by this. If \( k \) is odd, then \( k \) is a generator of \( \mathbb{Z}_{2p} \) and we choose \( \sigma_1 \) with \( \sigma_1(k) = 1 \). If \( k \) is even, then \( p + k \) is odd and we pick \( \sigma_1 \) with \( \sigma_1(p+k) = 1 \). In either case \( \sigma_1(2k) = 2 \). The corresponding automorphism \( \tilde{\sigma}_1 \) in \( \widehat{\mathbb{Z}}_{2p} \) will affect \( \text{supp} \hat{f} \), but \( \|f\|_0 \) does not change.

Let the third non-zero element have address \( r \). If \( \sigma_1(r) \neq p + 1 \), then there are either \( p-1 \) adjacent zeroes among the addresses \( 3, 4, p+1 \) or among \( p+1, \ldots, 2p-1 \).

In case that \( \sigma_1(r) = p + 1 \), then we apply another automorphism \( \sigma_2 \) in a similar way as above. If \( \frac{p+1}{2} \) is a generator for \( \mathbb{Z}_{2p} \), then \( \sigma_2(\frac{p+1}{2}) = 1, \sigma_2(2) = \sigma_2(4\frac{p+1}{2}) = 4\sigma_2(\frac{p+1}{2}) = 4, \) and \( \sigma_2(p+1) = 2\sigma_2(\frac{p+1}{2}) = 2 \). Otherwise, we choose \( \sigma_2 \) such that \( \sigma_2(p + \frac{p+1}{2}) = 1 \), so \( \sigma_2(p+1) = 2\sigma_2(p + \frac{p+1}{2}) = 2 \) and \( \sigma_2(2) = 2\sigma_2(p+1) = 4 \). In both cases, \( \text{supp} (f \circ \sigma_2 \circ \sigma_1) = \{0, 2, 4\} \), so the vector contains a string of at least \( p-1 \) consecutive zeros on addresses \( 5, \ldots, 2p-1 \).

The following lemma from [DS89] implies that \( \|f \circ \sigma \|_0 > p-1 \) and, therefore, \( \|\hat{f}\|_0 \geq p \).

**Lemma 6.1.** If \( \hat{f} \) has \( N \) nonzero elements, then \( f \) cannot have \( N \) consecutive zeros.

6.4. Justification of Figure 7

Let \( \omega = e^{2\pi i/3} \). For \( \|f\|_0 = 1 \), we calculate

\[
V_{(a,b,c)}(d,0,0) = (a\overline{\alpha}, \omega^2 a\overline{\alpha}, \omega a\overline{\alpha}, a\overline{\alpha}, \omega^2 a\overline{\alpha}, \omega a\overline{\alpha}, d\overline{\beta}, \omega^2 d\overline{\beta}, \omega d\overline{\beta})
\]

So in any case, \( \|V_{g} f\|_0 \geq 3 \|g\|_0 \), which justifies all cases involving \( \|f\|_0 = 1 \) or \( \|g\|_0 = 1 \).

For the case \( \|f\|_0 = 2 \) and \( \|g\|_0 = 2 \), we note \( \|V_{(1,1,0)}(1, -1, 0)\|_0 = 8 \) and \( \|V_{(1,1,0)}(1, 10, 0)\|_0 = 9 \), which justifies the two red fields. Now assume that there are \( f \) and \( g \) with \( \|f\|_0 = 2 \) and \( \|V_{g} f\|_0 \leq 7 \). Then \( V_{g} f \) has at least two zero entries. Note that the scalar product of \( f \) and another vector with support size 2 can only vanish, if \( \text{supp} f = \text{supp} g \). So the zero entries in \( V_{g} f \) must correspond to the same translation. If we set without loss of generality \( f = (a, b, 0), g = (c, d, 0) \), then zeros at two different modulations \( M_{j_1} \) and \( M_{j_2} \) imply \( a\overline{c} + \omega^{j_1} b\overline{d} = 0 = a\overline{c} + \omega^{j_2} b\overline{d} \), which clearly admits no nontrivial solution.

For the case \( \|f\|_0 = 2 \) and \( \|g\|_0 = 3 \) which is equivalent to the case \( \|f\|_0 = 3 \) and \( \|g\|_0 = 2 \), we note that \( \|V_{(1,1,1)}(1, -1, 0)\|_0 = 6 \), \( \|V_{(2,-4,8)}(2, 1, 0)\|_0 = 7 \), \( \|V_{(1,2,3)}(2, -1, 0)\|_0 = 8 \) and
\[ \|V_{(1,2,3)}(1,2,0)\|_0 = 9, \] which justifies the four red fields. Now assume, there are \( f \) and \( g \) with \( \|f\|_0 = 2 \), \( \|g\|_0 = 3 \) and \( \|V_g f\|_0 \leq 5 \). Then \( V_g f \) has at least four zero entries, in particular two that correspond to the same translation. Without loss of generality, we assume that this is the zero-translation and that \( f \) is supported in the first two coordinates, i.e., \( f = (a,b,0) \), \( g = (c,d,e) \). Then we get as before \( a\bar{c} + \bar{\omega}^a b\bar{d} = 0 = a\bar{c} + \bar{\omega}^b c\bar{d} \) which has no nontrivial solutions.

For the case \( \|f\|_0 = 3 \) and \( \|g\|_0 = 3 \), we note that \( \|V_{(1,1,1)}(1,1,1)\|_0 = 3 \), \( \|V_{(1,1,1)}(1,1,-2)\|_0 = 6 \), \( \|V_{(1,2,5)}(10,5,2)\|_0 = 7 \), \( \|V_{(1,2,3)}(-5,1,1)\|_0 = 8 \) and \( \|V_{(1,2,3)}(1,2,3)\|_0 = 9 \), which justifies the five red fields. Multiplying \( f \) or \( g \) by a constant does not change \( \|V_g f\|_0 \), so we can normalize \( f(0) = g(0) = 1 \). Hence we can set \( f = (1,a,b) \), \( g = (1,c,d) \). Then again, \( \|V_g f\|_0 \leq 5 \) implies that \( V_g f \) has two zero entries that correspond to the same translation and we shall assume without loss of generality and for the remainder of this section that those appear at \( x = 0 \) and \( \xi = 1,2 \), i.e., we have

\[
1 + \omega a\bar{c} + \bar{\omega}^2 b\bar{d} = 0 = 1 + \omega^2 a\bar{c} + \omega b\bar{d}
\]

and hence \( b\bar{d} = a\bar{c} = 1 \) and \( g = (1,\frac{1}{a},\frac{1}{b}) \).

Before continuing, we state

**Lemma 6.2.** Let \( S \) be a shearing on \( \mathbb{C}^{2^a \times 2^b} \), i.e., \( S \) translates the \( (x = 1) \)-row of an element in \( \mathbb{C}^{2^a \times 2^b} \) by 1 and the \( (x = 2) \)-row by 2. Then given \( f, g \in \mathbb{C}^{2^a} \), there exist \( \tilde{f}, \tilde{g} \in \mathbb{C}^{2^a} \), such that \( \text{supp}(V_g \tilde{f}) \) is the image of \( \text{supp}(V_g f) \) under \( S \).

**Proof** Suppose, two vectors \( f = (u,v,w) \) and \( g = (x,y,z) \) are given, and consider the vectors \( \tilde{f} = (u,v,\omega w) \) and \( \tilde{g} = (x,y,\omega z) \). Then

\[
V_g \tilde{f}(0,\xi) = u_x + \bar{\omega}^x v_y + \bar{\omega}^{x+2} (\omega w) \bar{v}_x = u_x + \bar{\omega}^x v_y + \bar{\omega}^{x+2} \bar{v}_x = V_g f(0,\xi),
\]

\[
V_g \tilde{f}(1,\xi) = u_y + \bar{\omega}^y v_x + \bar{\omega}^{y+2} (\omega w) \bar{v}_y = u_y + \bar{\omega}^{y+1} v_x + \bar{\omega}^{y+2} \bar{v}_y = V_g f(1,\xi + 1),
\]

and

\[
V_g \tilde{f}(2,\xi) = u_z + \bar{\omega}^z v_x + \bar{\omega}^{z+2} (\omega w) \bar{v}_z = \omega (u_z + \bar{\omega}^{z+2} v_x + \bar{\omega}^{z+2} \bar{v}_z) = \omega V_g f(2,\xi + 2).
\]

As a multiplication by \( \omega \) does not change the support, we get the sheared image of the original support set as desired.

We now use Lemma 6.2 to show that in the case \( \|f\|_0 = \|g\|_0 = 3 \), no support size of 4 is possible. In fact this would imply that the short-time Fourier transform has five zeroes, so there is a second row with two zeroes (without loss of generality the row \( x = 1 \)). By shearing we can move them to \( \xi = 1,2 \) without changing the first row, i.e.,

\[
\frac{1}{a} + \bar{\omega}^\frac{a}{b} + \bar{\omega}^2 b = 0 = \frac{1}{a} + \bar{\omega}^{\frac{a}{b}} + \bar{\omega} b.
\]

This implies \( \frac{1}{a} = \frac{a}{b} = b \) and hence \( a = 1, \omega = a = \omega^2 \), and \( b = \bar{\omega} \) accordingly. This reduces to the example for \( \|V_g f\|_0 = 3 \) given above. Thus, \( \|V_g f\|_0 = 4 \) is impossible.

For a support size of 5, we can use the same argument to exclude that the remaining two zeroes occur at the same \( x \). So in addition to the two zeros for \( x = 0 \), we can have zeroes at \( x = 1,2 \) and either \( \xi = 0 \) for both or \( \xi = 1 \) for both. All other combinations can be reduced to these two by shearing and conjugation (using \( \omega^2 = \bar{\omega} \)).

28
These two cases correspond to solving

\[ a + \omega^k \frac{b}{a} + \omega^{2k} \frac{b}{a} = 0 = \frac{1}{a} + \omega^k \frac{b}{a} + \omega^{2k} b \]

for \( k = 0, 1 \). These equations can be solved exactly using Mathematica. The only solutions are modulations of shearings of the solution with \( \|V_R f\|_0 = 3 \) considered above. So again, it follows that a short–time Fourier transform with support size 5 is not possible.

6.5. Proof of Cheboratev’s Theorem \[3.3\] based on Theorem\[r\]theorem:LaPfaWa.

Fix \( A, \tilde{A} \subseteq \mathbb{Z}_p \) with \( |A| = |\tilde{A}| \). We have to show that the restricted Fourier transformation \( \mathcal{F}_{A \rightarrow \tilde{A}} : \mathbb{C}^A \rightarrow \mathbb{C}^{\tilde{A}} \) is an isomorphism. For \( g \) such that \( A_{z_p,g} \) has no zero minors, define \( M_g : \mathbb{C}^p \rightarrow \mathbb{C}^p \) to be the pointwise multiplication operator with the vector \( g \). Since \( g \) has no zero components, \( M \) is an isomorphism, and, moreover, \( M_g \) restricts to an isomorphism on \( \mathbb{C}^A \). Set \( B = \{0\} \times \tilde{A} \). Therefore, \( V_g : \mathbb{C}^A \rightarrow \mathbb{C}^B \) is an isomorphism since \( |B| = |\tilde{A}| \). The result follows since the restricted Fourier transformation \( \mathcal{F}_{A \rightarrow \tilde{A}} \) is nothing but \( P \circ V_g \circ M_g \) where \( P \) is the projection of \( B = \{0\} \times \tilde{A} \) onto \( \tilde{A} \).

6.6. Proof of Proposition \[4.13\]

Proposition \[4.13\] implies that there exists \( s, t \) such that \( st \leq k \) and \( \phi(G, k) \geq \phi(H, s)\phi(G/H, t) \). For \( G = \mathbb{Z}_{pq} \) and \( |H| = p \), we have \( \phi(H, s) = p^2 - s + 1 \) and \( \phi(G/H, t) = q^2 - t + 1 \). As \( st \leq k \), we can find \( \tilde{t} \in \mathbb{R} \) such that \( q \geq \tilde{t} \geq t \) and \( p \geq \frac{k}{\tilde{t}} \geq s \). Hence,

\[ \phi(G, k) \geq (p^2 - s + 1)(q^2 - t + 1) \geq (p^2 - \frac{k}{\tilde{t}} + 1)(q^2 - \tilde{t} + 1). \]

So \( \phi(G, k) \) must exceed the minimum of \( M(u) = (p^2 - \frac{k}{u} + 1)(q^2 - u + 1) \), where \( u \) ranges from \( \frac{k}{p} \) to \( q \) since \( \frac{k}{p} \leq u \leq q \) is assumed. We have \( M'(u) = -(p^2 + 1) + \frac{k(q^2 + 1)}{u^2} = 0 \) if and only if \( u = \pm \sqrt{\frac{kq^2 + 1}{p^2}} \).

As \( M(u) \rightarrow -\infty \) for \( u \rightarrow 0^+ \) and \( u \rightarrow \infty \), the only positive extremum is a maximum and the minimum is attained in a boundary point. A simple calculation gives that \( M(q) \leq M \left( \frac{k}{p} \right) \).

For \( k < q \), the condition \( 1 \leq s, 1 \leq t \), implies that \( t \) ranges only from 1 to \( k \). The same arguments as used above show again that the minimum is attained at a boundary point and that \( M(1) \geq M(k) \).

6.7. Proof of Proposition \[4.15\]

As in the proof of Theorem \[4.6\], choose \( A \subseteq G \) and \( B \subseteq G \times \hat{G} \) with \( |A| = |B| \) and set \( P_{A,B}(z) = \det(A_{z_m,z})_{A,B} \), \( z = (z_0, z_1, \ldots, z_{n-1}) \). In that proof, we identified a “maximal” term within \( P_{A,B} \), the coefficient of which can be expressed as a product of different minors of the discrete Fourier matrix \( W_{z_m} \). Each of these minors arise from the columns of \( P_{A,B}(z) \) that correspond to a specific translation. By assumption, these columns are adjacent with respect to modulation in \( A_{z_m,z} \).

So each of these minors is a minor of the DFT matrix corresponding to adjacent columns, where each row is multiplied by some factor \( z_i \). Using the multilinearity of the determinant, we can pull
the factors outside. By Proposition 3.6 we conclude that these minors of the DFT-matrix are nonzero, hence also their product. So the "maximal" term has a nonzero coefficient.

To obtain the dual statement, take the Fourier transform of each column of $A_{\mathbb{Z}_n, g}$. By linearity, the resulting matrix can have no size-$n$ zero minors either, as that would mean that one column of the corresponding submatrix is a linear combination of other columns. As $\hat{M}_\xi T_x g = T_x M_{-\xi} \hat{g}$, the resulting matrix will correspond to $A_{\mathbb{Z}_n, \hat{g}}$, except that modulations and translations have exchanged their roles. So modulation adjacency becomes translation adjacency, which implies the dual statement.

Acknowledgment. We would like to thank Norbert Kaiblinger, Franz Luef, and Ewa Matusiak for sharing with us the results of their thorough discussions on the uncertainty principle for functions on finite Abelian groups. Further, we thank Michael Stoll for offering advice on algebraic geometry issues that are relevant to our work, and Dan Alistarh and Sergiu Ungureanu for writing some of Matlab code used.

References

[BF03] J.J. Benedetto and M. Fickus. Finite normalized tight frames. *Adv. Comput. Math.*, 18(2-4):357–385, 2003.

[Chr03] O. Christensen. *An Introduction to Frames and Riesz bases*. Birkhäuser, Boston, 2003.

[CK03] P.G. Casazza and J. Kovačević. Equal-norm tight frames with erasures. *Advances in Computational Mathematics*, 18(2-4):387 – 430, February 2003.

[CRT04] E. Candes, J. Romberg, and T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. preprint, 2004.

[DS89] D. Donoho and P. Stark. Uncertainty principles and signal recovery. *SIAM Journal on Applied Mathematics*, 49:906–931, 1989.

[EI76] R. J. Evans and I. M. Isaacs. Generalized Vandermonde determinants and roots of unity of prime order. *Proc. Amer. Math. Soc.*, 58:51–54, 1976.

[FKLM05] H. G. Feichtinger, N. Kaiblinger, F. Luef, and E. Matusiak. On the uncertainty principle on $\mathbb{C}_N$. personal communications, 2005.

[Fre04] P.E. Frenkel. Simple proof of Chebotarev’s theorem on roots of unity. Preprint. math.AC/0312398, 2004.

[GK01] V.K. Goyal and J. Kovačević. Quantized frame expansions with erasures. *Appl. Comp. Harm. Analysis*, 10:203–233, 2001.

[Grö01] K.H. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.

[Kár05] Gy. Károlyi. Cauchy-Davenport theorem in group extensions. *L’Enseignement Mathématique*, 5:239–254, 2005.

[Kat76] Y. Katznelson. *An Introduction to Harmonic Analysis*. Dover, New York, 1976.
[KC06] J. Kovačević and A. Chebira. Life beyond bases: The advent of the frames. *Signal Processing Magazine*, 2006. to appear.

[Kut03] G. Kutyniok. A weak qualitative uncertainty principle for compact groups. *Illinois J. Math.*, 47(3):709–724, 2003.

[LM05] F. Luef and E. Matusiak. A general additive uncertainty principle for finite abelian groups. preprint, 2005.

[LPW05] J. Lawrence, G.E. Pfander, and D. Walnut. Linear independence of Gabor systems in finite dimensional vector spaces. To appear in *J. Fourier Anal. Appl.*, 2005.

[Mes05] R. Meshulam. An uncertainty inequality for finite abelian groups. *European J. of Combin.*, 2005.

[MÖP04] E. Matusiak, M Özaydin, and T. Przebinda. The Donoho-Stark uncertainty principle for a finite abelian group. *Acta Math. Univ. Comenian. (N.S.)*, 73(2):155–160, 2004.

[Pra94] V. V. Prasolov. *Problems and theorems in linear algebra*. American Mathematical Society, Providence R.I., 1994.

[SH03] T. Strohmer and R.W. Heath, Jr. Grassmanian frames with applications to coding and communications. *Appl. Comp. Harm. Analysis*, 14(3):257–275, May 2003.

[SL96] P. Stevenhagen and H. W. Lenstra, Jr. Chebotarëv and his density theorem. *Math. Intelligencer*, 18(2):26–37, 1996.

[Tao05] T. Tao. An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.*, 12:121–127, 2005.
