Exact soliton-like solutions of the radial Gross–Pitaevskii equation

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Abstract
We construct exact ring soliton-like solutions of the cylindrically symmetric (i.e. radial) Gross–Pitaevskii equation with a potential, using the similarity transformation method. Depending on the choice of the allowed free functions, the solutions can take the form of stationary dark or bright rings whose time dependence is in the phase dynamics only, or oscillating and bouncing solutions, related to the second Painlevé transcendent. In each case the potential can be chosen to be time independent.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Many nonlinear equations of motion allow interesting solutions such as bright and dark solitons, which propagate without dispersion [1, 2]. More generally, we can also consider localized soliton-like stationary solutions, which have the appearance of solitons except for some aspects of dynamics. They are usually accompanied by a spatially inhomogeneous external potential-like term in the equation of motion and often called also solitons, or solitary waves, or for dark soliton cases, kink-like solutions [3].

The nonlinear Schrödinger equation (NLS),

\[ i\psi_t = -\psi_{xx} + V(x,t)\psi + \sigma |\psi|^2 \psi, \tag{1} \]

also known as the Gross–Pitaevskii equation (GPE), is of special interest since it describes the relevant physics in nonlinear optical systems [4] as well as in degenerate quantum gases [5, 6]. For \( V = 0 \) and \( \sigma = -1 \) it is called focusing and allows bright solitons, while for \( \sigma = +1 \) it is called defocusing and allows dark solitons. These solutions can also work to some extent when a potential term \( V \) is added to the nonlinear equation [7]. Equation (1) is one-dimensional, and while the trivial extension to two dimensions by replacing the \( x \)-derivatives by a Laplacian is possible, the corresponding solutions are plane waves without solitonic...
properties [3]. Typically they are not stable: the stationary bright soliton solution survives only under certain conditions and if supported by an external potential [8], and the stationary dark soliton decays into vortices by the snaking instability unless the system is in practice a quasi-one-dimensional one [9] (however, in the presence of supersonic flow, convectively unstable dark soliton solutions of the two-dimensional NLS equation have been found [10]). This applies especially to any static solution of equation (1), which gives the motivation to seek such multidimensional solutions that can be expected to have at least extended lifetimes. The static soliton-like solutions are also interesting more generally, connecting to kinks and domain walls [3], although e.g. in superfluid He the multicOMPONENT structure of the order parameter leads to a rich set of complex structures [11].

Instead of making such a plane-wave extension into two dimensions, one can look at cylindrically symmetric systems, which brings forward the concept of ring solitons. Since their introduction in a nonlinear optics setting [12], the interest has been in dark ring solitons, which have been treated essentially numerically and in the domain of a large radius [13, 14]. Although one can reduce the dynamics into a one-dimensional radial equation [15], exact analytic solutions (either bright or dark) have not been obtained so far. It has been only shown that in the limit of infinitesimal amplitude, such solitons are analytically described by the radial KdV equation [12]. Apart from being simply convenient, exact solutions would allow one to analyse the stability and decay of ring solitons, including the effect of external potentials, in the same fashion as for trivially extended plane-wave solutions in two dimensions.

We have found that using a similarity transformation method, one can actually construct exact analytical ring soliton-like solutions of the radial GPE. They are limited to certain forms of external potential, but even then, they can provide a starting point for further studies of the ring solitons in cylindrically symmetric potentials. In this paper we first describe the similarity transformation in section 2, and then present the solutions in section 3. Then, in section 4 we discuss the similarity solutions associated with the second Painlevé equation. We summarize our work and discuss its implications in section 5.

2. The similarity transformation

The radial GPE in dimensionless form is given by

$$i \psi_T = -\psi_{RR} - \frac{1}{R} \psi_R + V(R, T) \psi + \sigma |\psi|^2 \psi,$$

(2)

where $\psi$ is the complex amplitude of the electric field in a nonlinear optics setting or the macroscopical wavefunction of a Bose–Einstein condensate in an ultracold atomic gas setting. In our approach, we focus on the latter framework, in which case $V(R, T)$ is the external potential. We will mainly consider the defocusing case $\sigma = 1$. We note that equations (1) and (2) are not connected by a general transformation combining point-, gauge- or scale-transformations, which were discussed, e.g., in [16] and [17].

The variable $R$ in equation (2) is strictly positive, so to simplify the treatment it is useful to apply the transformation $x = \ln (R)$ and $t = T$, which results in

$$i \psi_t = -\frac{1}{e^{2x}} \psi_{xx} + V(x, t) \psi + |\psi|^2 \psi.$$

(3)

From now on we will consider the generalized equation

$$i \psi_t = -\alpha_1(x, t) \psi_{xx} + \alpha_2(x, t) \psi + \alpha_3(x, t) |\psi|^2 \psi,$$

(4)

where $\alpha_1-3$ are some functions of $x$ and $t$, and consider our case, equation (2), as the following special case:

$$\alpha_1(x, t) = \frac{1}{e^{2x}}, \quad \alpha_2(x, t) = V(x, t), \quad \alpha_3(x, t) = 1.$$  

(5)
We now use the similarity ansatz [18–20]

$$\psi(x, t) = \rho(x, t) e^{\psi(x, t)} \phi(\eta(x, t)),$$

where $$\rho \in \mathbb{R}^+$$ and $$\{\psi, \eta\} \in \mathbb{R}$$ are some functions to be determined, and $$\phi(\eta)$$ is assumed to satisfy

$$-\phi_{\eta\eta} + g(\eta)\phi_{\eta} + h(\eta)\phi + G \phi^3 = 0,$$

where $$G$$ is a constant and $$g$$ and $$h$$ are arbitrary functions of $$\eta$$. By a simple transformation we can set $$g(\eta) = 0$$ without loss of generality. We have, for example, the following special cases:

$$-\phi_{\eta\eta} + \eta\phi + \frac{2}{3} \phi^3 = 0$$

with $$g = 0$$, $$h = \eta$$, $$G = 2[21]$$ and

$$-\phi_{\eta\eta} + \phi^3 = \mu \phi$$

with $$g = 0$$, $$h = -\mu$$, $$G = 1$$.

Substituting equation (6) in equation (4) we obtain the following set of equations:

$$\frac{\rho^2}{c_2} \xi_{e\eta} = 0,$$

$$\frac{\rho^2}{c_2} + 2 \alpha_1 \left(\frac{\rho^2}{c_2} \xi_{e\eta}\right) = 0,$$

$$\eta_{\xi} + 2 \phi_{\eta} \eta_{\alpha_1} = 0,$$

$$- \alpha_3 \rho^2 + G \eta_{\alpha_1} = 0,$$

$$\alpha_1 \left( \frac{\rho^2}{c_2} - \eta \frac{\rho_{\alpha_1}}{\rho} \right) + \alpha_2 + \phi_e = 0.$$

Assuming equation (5), we obtain from equations (10a)–(10d) the following solutions:

$$\rho = \frac{e^{-\frac{1}{2} x}}{c_1(t)},$$

$$\varphi = \frac{3c_1(t)}{8c_1(t)} e^{2\varphi} + c_2(t),$$

$$\eta = \frac{3}{2 \sqrt{Gc_1(t)}} e^{\frac{\eta}{2}} + c_3,$$

where $$c_{1,2}(t)$$ are arbitrary functions of time and $$c_3$$ is a constant. From equation (10e) we obtain an expression for the potential:

$$\alpha_2 = \frac{1}{9} e^{\frac{1}{2} x} + \frac{h(\eta)}{Gc_1^2(t)} - \frac{3}{16} \frac{c_1^2(t) + 2c_1(t)c_1(t)e^{2\varphi}}{c_1^2(t)} e^{2\varphi} - c_2(t).$$

We will next consider the potentials and solutions that can be obtained with various choices of $$c_{1,2}(t)$$ and $$h(\eta)$$.  

3. The ring soliton-like solutions

There is freedom in choosing the potential function $$\alpha_2$$. For example, the $$1/(9R^2)$$ term can be removed if we choose $$h(\eta) = -\frac{1}{4\eta^2}$$ and $$c_3 = 0$$. Then from equation (12) we obtain a harmonic potential

$$\alpha_2 = -\frac{3}{16} \frac{c_1^2(t) + 2c_1(t)c_1(t)e^{2\varphi}}{c_1^2(t)} e^{2\varphi} - c_2(t).$$

See table 1 for a selection of choices for $$h(\eta)$$.  

Table 1. A range of potentials available by suitable choices of \( h(\eta) \). From equation (7) we then have the equation for \( \phi(\eta) \), which we must solve to construct the solutions \( \psi(x, t) \).

| \( h(\eta) \) | \( c_3 \) | Potential | Equation to solve |
|---------------|--------|-----------|------------------|
| \( h(\eta) \) | \( c_3 \) | \( \frac{1}{2\pi^2} + \frac{h(\eta)}{\alpha c_3^2(T) R^2} \left( \frac{\phi^2}{c_1(T)} + \frac{3 c_2^2(T) + 2c_1(T) \phi(\eta)}{c_1^2(T)} \right) R^2 - \hat{c}_2(T) \) | \(-\phi_{\eta\eta} + h(\eta)\phi + G\phi^3 = 0\) |
| \( \eta \) | \( 0 \) | \( \frac{1}{2\pi^2} - \frac{1}{\alpha c_3^2(T) R^2} \left( \frac{\phi^2}{c_1(T)} + \frac{3 c_2^2(T) + 2c_1(T) \phi(\eta)}{c_1^2(T)} \right) R^2 - \hat{c}_2(T) \) | \(-\phi_{\eta\eta} + \eta\phi + G\phi^3 = 0\) |
| \( \eta^2 \) | \( 0 \) | \( \frac{1}{2\pi^2} - \frac{1}{\alpha c_3^2(T) R^2} \left( \frac{\phi^2}{c_1(T)} + \frac{3 c_2^2(T) + 2c_1(T) \phi(\eta)}{c_1^2(T)} \right) R^2 - \hat{c}_2(T) \) | \(-\phi_{\eta\eta} + \eta^2\phi + G\phi^3 = 0\) |
| \( -\frac{1}{4\pi^2} \) | \( 0 \) | \( -\frac{1}{2\pi^2} \left( \frac{\phi^2}{c_1(T)} + \frac{3 c_2^2(T) + 2c_1(T) \phi(\eta)}{c_1^2(T)} \right) R^2 - \hat{c}_2(T) \) | \(-\phi_{\eta\eta} - \frac{\eta^2}{4\pi^2}\phi + G\phi^3 = 0\) |
| \( -\mu \) | \( c_3 \) | \( \frac{1}{2\pi^2} + \frac{-\mu}{\alpha c_3^2(T) R^2} - \frac{1}{16} \left( \frac{3 c_2^2(T) + 2c_1(T) \phi(\eta)}{c_1^2(T)} \right) R^2 - \hat{c}_2(T) \) | \(-\phi_{\eta\eta} - \mu\phi + G\phi^3 = 0\) |

3.1. The dark ring soliton-like solution, \( h(\eta) = -\mu \)

This choice gives the canonical NLS (no external potential) so that from equation (7) we are now requiring \( \phi(\eta) \) to satisfy

\[
\mu \phi = -\phi_{\eta\eta} + G\phi^3,
\]

where \( \mu > 0 \).

Using equation (11c), we can write down \( \phi(\eta) = \sqrt{\mu} \tanh \left( \sqrt{\frac{\mu}{2}} (\eta - \eta_0) \right) \), the kink solution of equation (14) with \( G = 1 \), as

\[
\phi = \sqrt{\mu} \tanh \left[ \sqrt{\frac{\mu}{2}} \left( \frac{3}{2} \left( \frac{\phi^2}{c_1(T)} - \frac{3\phi_0}{c_1(T)} \right) \right) \right],
\]

where we have chosen \( c_3 = \eta_0 - \frac{3}{2\sqrt{2}} \phi_0^2 \) to match with the soliton centre \( \phi_0 \) at time \( t = t_0 \).

Therefore, using equations (6), (11a), (11b) and (15), we have constructed the solution

\[
\psi(R, T) = \frac{\sqrt{\mu}}{c_1(T) R^2} \tanh \left[ \frac{3}{2\sqrt{2}} \sqrt{\mu} \left( \frac{R^2}{c_1(T)} - \frac{R_0^2}{c_1(T_0)} \right) \right] \exp \left( i \left( \frac{3 c_1(T)}{8 c_1(T)} R^2 + c_2(T) \right) \right)
\]

of the original radial GPE, equation (2) (\( \sigma = 1 \)), with the potential

\[
V(R, T) = \frac{1}{9 R^2} + \frac{-\mu}{c_1(T) R^2} - \frac{3}{16} \left( \frac{c_1^2(T) + 2c_1(T) \phi(\eta)}{c_1(T)} \right) R^2 - \hat{c}_2(T).
\]

Let us select \( c_1(T) = 1 \) and \( c_2(T) = -T \). Then from equation (17) we get (see figure 1)

\[
V(R) = \frac{1}{9 R^2} - \mu \frac{1}{R^2} + 1
\]

and from equation (16)

\[
\psi(R, T) = R^{-\frac{1}{2}} \sqrt{\mu} \tanh \left[ \frac{3}{2\sqrt{2}} \sqrt{\mu} \left( R^2 - R_0^2 \right) \right] e^{-i T}.
\]

where \( \mu > 0 \) (see figure 2).

We note that it is possible to rewrite the \( 1/(9R^2) \) term in the potential as a fractional vorticity of \( \frac{1}{3} \) and include the full Laplacian. Then

\[
\psi(R, T, \theta) = R^{-\frac{1}{2}} \sqrt{\mu} \tanh \left[ \frac{3}{2\sqrt{2}} \sqrt{\mu} \left( R^2 - R_0^2 \right) \right] \exp \left( -i \left( T - \frac{\theta}{3} \right) \right)
\]

(20)
solves
\[ i\psi_T = -\psi_{RR} - \frac{1}{R}\psi_R - \frac{1}{R^2}\psi_{\theta\theta} + V(R)\psi + |\psi|^2\psi, \]
(21)

where
\[ V(R) = -\mu \frac{1}{R^2} + 1. \]
(22)

3.2. The stability of the dark solution, \( h(\eta) = -\mu \)

Direct propagation of \( \psi(x, y, t) \) given by a two-dimensional extension of the GPE (1) in rectangular coordinates \( x, y \) shows that the solution given in equation (19) is long-lived, until it decays into a vortex–antivortex necklace by the snake instability. The decay is much faster if instead of the potential in equation (18) we use e.g.
\[ \tilde{V}(R) = \frac{A}{R^3} - \frac{B}{R}. \]
(23)
Figure 3. Results of solving the Bogoliubov equations (24) for the exact solution (19) of equation (2) ($\sigma = 1$) with the potential (18). Here $\lambda = 1$. Shown are the decay times $1/\text{Im}(\epsilon)$, where $\epsilon$ is the Bogoliubov eigenvalue. There is only a single imaginary eigenvalue per $q$ and the corresponding amplitude is localized at the notch of the ring dark soliton-like solution. For low radii ($R_S \lesssim 1.5003$) the only unstable mode is $q = 0$. For radii $4.613 \lesssim R_S \lesssim 8.620$ the primary decay channel is the quadrupole, or $q = 2$, mode, while all higher modes are dynamically stable. As the radius is increased the higher modes become unstable as well. The solution (19) is always formally unstable with respect to the $q = 0$ mode, although the instability times are very long for large radii.

where $A$ and $B$ are chosen so that $V(R)$ and $\tilde{V}(R)$ agree up to the first (linear) order when expanded at $R = R_0$.

To get more insight on the stability we consider the Bogoliubov–de-Gennes equations for low-lying modes $\epsilon$ [2, 3, 5]. In dimensionless natural units ($\mu = \hbar = 2m = 1$) they are

\begin{equation}
\left( \frac{\mathcal{L}}{\partial \psi_0^2} \right) \left( \begin{array}{c} \psi_q^2 \\ \bar{\psi}_q^2 \end{array} \right) = \epsilon \left( \begin{array}{c} u_q \\ -v_q \end{array} \right),
\end{equation}

where we have defined the Bogoliubov amplitudes $u$ and $v$ for which

\begin{equation}
\psi(r, T) = e^{-iT} \left[ \psi_0(r) + [u(r) e^{-i\epsilon T} + \bar{v}(r) e^{i\epsilon T}] \right],
\end{equation}

by [22]

\begin{equation}
\left( \begin{array}{c} u(r) \\ v(r) \end{array} \right) = e^{iq\theta} \left( \begin{array}{c} u_q(R) \\ v_q(R) \end{array} \right)
\end{equation}

representing a partial wave of angular momentum $q$ relative to the condensate, and

\begin{equation}
\mathcal{L} \equiv - \left( \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} - \frac{q^2}{R^2} \right) + \frac{1}{9R^2} - \frac{\lambda}{R^3} + 2|\psi_0|^2,
\end{equation}

and $\psi_0(R, T)$ is given by equation (19) (with the substitution $\mu \to \lambda$). If $\epsilon$ is imaginary, $u$ and $v$ are normalized according to

\begin{equation}
\int dR (|u(r)|^2 - |v(r)|^2) = 0.
\end{equation}

The instability time is given by $T_d = 1/\text{Im}(\epsilon)$, and if all the eigenvalues are real, the condensate is dynamically stable. The results of numerically solving equation (24) are shown in figure 3. There is only one (purely) imaginary eigenvalue, and the Bogoliubov amplitudes are localized at the radius $R_S$. Equation (24) is solved numerically using Lagrange functions, choosing a Laguerre discrete-variable representation [23, 24], which is particularly efficient for radial
Figure 4. The amplitudes $u$ and $v$ corresponding to the only imaginary Bogoliubov mode, and the wavefunction $\psi_0$ for $R_S = 50.0$, $\lambda = 1$, $q = 8$, and $\theta = 0$ at $T = 0$. The inset shows the Bogoliubov spectrum $\epsilon$. The amplitudes $u$ and $v$ are localized around the notch at $R_S = 50.0$. We have chosen one possible normalization satisfying (28).

Figure 5. The amplitudes $u$ and $v$ of figure 4 added to the wavefunction $\psi_0$ at $\theta = \pi/8$. The instability mode is directly related to the snake instability, having typical behaviour around the ring (for $q = 8$). The inset shows how the Bogoliubov mode affects the location of the notch around the circumference.

geometry. The abscissas can be found using Newton’s method [25], and the resulting linear eigenvalue problem is solved by Hessenberg QR iteration [26].

The minimum of the potential (18) with $\lambda = 1$ occurs at $R_0 = \left(\frac{1}{4}\right)^{\frac{1}{2}} \approx 0.44$. Therefore, the soliton-like solution (19) is stable if $R_S$ is around (a healing length away from) the minimum of the potential. Another region of stability appears to be the limit $R_S \to \infty$ (see figure 3), although it is approached rather slowly. In this limit we obtain the one-dimensional GPE with a constant potential, which can be removed by redefining the zero of energy.

The case for $R_S = 50.0$, $\lambda = 1$, and $q = 8$ is shown in more detail in figures 4 and 5. Because the ring suffers from the snake instability in general, and because there is only
one mode with dynamical instability (per \( q \)), they must be related. That this is indeed so is confirmed in figures 4 and 5. The amplitudes \( u \) and \( v \) corresponding to the imaginary eigenvalue are localized at the notch. Here the azimuthal symmetry was broken by choosing \( (26) \), that is, an overall phase of 0.

3.3. The bright ring soliton-like solution, \( h(\eta) = -\mu \)

When \( G = -1 \), equation (14) has also a bright soliton-like solution \( \phi(\eta) = \sqrt{2}\sqrt{\mu} \sech[\sqrt{-\mu}(\eta - \eta_0)] (\mu < 0) \). Similarly as in section 3.1 (now \( \alpha_3 = G = -1 \) so nothing changes), we arrive at the solutions

\[
\psi(R, T) = \frac{\sqrt{-2\mu}}{c_1(T)R^2} \text{sech} \left[ \frac{3}{2} \sqrt{-\mu} \left( \frac{R^2}{c_1(T)} - \frac{R_0^2}{c_1(T_0)} \right) \right] \exp \left[ i \left( \frac{3c_1(T)}{8c_1(T)} R^2 + c_2(T) \right) \right]
\]

(29)

with the potential (17) \( (\mu < 0) \) and

\[
\psi(R, T) = R^{-\frac{3}{2}} \sqrt{-2\mu} \sech \left[ \frac{3}{2} \sqrt{-\mu} (R^2 - R_0^2) \right] e^{-iT}
\]

(30)

with the potential (18), where \( \mu < 0 \) (see figure 6), of the original radial GPE, equation (2) \( (\sigma = -1) \).

4. Painlevé II, \( h(\eta) = \eta \)

We are now requiring \( \phi(\eta) \) to satisfy

\[-\phi_{\eta\eta} + \eta \phi + 2\phi^3 = 0. \quad (31)\]

This means \( g(\eta) = 0, h(\eta) = \eta \), and \( G = 2 \). As we can see in table 1, equation (31) arises as the similarity reduction of equation (2) \( (\sigma = 1) \) with

\[
V(R, T) = \frac{1}{9R^2} - \frac{3}{16} \frac{c_1^2(T) + 2c_1(T)c_2(T)}{c_1^2(T)} R^2 + \frac{3}{2c_1^2(T)} - \dot{c}_2(T). \quad (32)
\]
This is similar to what happens when the modified KdV equation,
\[
v_t - 6v^2v_x + v_{xxx} = 0, \quad (33)
\]
is transformed with a similarity reduction
\[
v(x, t) = \frac{w(z)}{(3t)^{\frac{1}{3}}}, \quad z = \frac{x}{(3t)^{\frac{1}{3}}} \quad (34)
\]
such that \( w(z) \) satisfies PII:
\[
w_{zz} = 2w^3 + zw + \alpha, \quad (35)
\]
where \( \alpha \) is a constant of integration [27].

Interestingly, we can eliminate time dependence from the potential \( (32) \) without sacrificing nontrivial time dependence in the solution, unlike what happened in sections 3.1 and 3.3, by choosing
\[
c_1(T) = c_4 \cosh [2\sqrt{k}(T + c_5)]^{\frac{1}{3}}, \quad (36)
\]
\[
\dot{c}_2(T) = \frac{3}{2^7 c_1^3(T)} - c_6, \quad (37)
\]
where \( k \neq 0, c_4, c_5, \) and \( c_6 \) are constants.

Therefore,
\[
\psi(R, T) = \rho(R, T) e^{i\varphi(R, T)} \phi(\eta(R, T)), \quad (38)
\]
where
\[
\rho(R, T) = \frac{R^{-\frac{1}{3}}}{c_4 \cosh [2\sqrt{k}(T + c_5)]^{\frac{1}{3}}}, \quad (39a)
\]
\[
\varphi(R, T) = \frac{\sqrt{k}}{2} \tanh [2\sqrt{k}(T + c_5)] R^2 + c_7 - c_6(T + c_5) + \frac{3\tanh [2\sqrt{k}(T + c_5)]}{\sqrt{k} 2^{\frac{1}{3}} c_4^3}, \quad (39b)
\]
\[
\eta(R, T) = \frac{3}{2^7 c_4 \cosh [2\sqrt{k}(T + c_5)]^{\frac{1}{3}}} R^{\frac{1}{3}}, \quad (39c)
\]
and where \( c_7 \) is a constant, solves equation (2) \( (\sigma = 1) \) with the potential \( (32) \)
\[
V(R) = \frac{1}{9R^2} - kR^2 + c_6, \quad (40)
\]
if \( \phi(\eta) \) solves equation (31), the second Painlevé equation.

If \( k < 0 \), we obtain from equation (36)
\[
c_1(T) = c_4 \cos [2\sqrt{|k|}(T + c_5)]^{\frac{1}{3}}. \quad (41)
\]
Then equation (38) with
\[
\rho(R, T) = \frac{R^{-\frac{1}{3}}}{c_4 \cos [2\sqrt{|k|}(T + c_5)]^{\frac{1}{3}}}, \quad (42a)
\]
\[
\varphi(R, T) = -\frac{\sqrt{|k|}}{2} \tan [2\sqrt{|k|}(T + c_5)] R^2 + c_7 - c_6(T + c_5) + \frac{3\tan [2\sqrt{|k|}(T + c_5)]}{\sqrt{|k|} 2^\frac{1}{3} c_4^3}, \quad (42b)
\]
\[
\eta(R, T) = \frac{3}{2^7 c_4 \cos [2\sqrt{|k|}(T + c_5)]^{\frac{1}{3}}} R^{\frac{1}{3}} \quad (42c)
\]
solves equation (2) with the binding trap potential
\[
V(R) = \frac{1}{9R^2} + |k| R^2 + c_6. \tag{43}
\]

Any nontrivial real solution of equation (31) with the boundary condition \(\phi(\eta) \to 0\) as \(\eta \to \infty\) is asymptotic to \(vAi(\eta)\) for some nonzero real constant \(v\), where \(Ai\) denotes the Airy function [28]. The choice \(|v| = 1\) corresponds to the Hastings–McLeod solution [29], and the choice \(|v| < 1\) to the Segur–Ablowitz solution [30]. Since by equations (39c) and (42c) we have \(\eta \geq 0\), the solution is well approximated by taking the Airy function for \(\phi(\eta)\) (provided \(v\) is small enough):
\[
\phi(\eta) = vAi(\eta). \tag{44}
\]
From equations (38), (42a), (42c) and (44), it follows that \(R|\psi|^2 \to 0\) for all \(R\) as \(T \to (l + \frac{1}{2}) \pi\),
\[
|v| = 0.9 \quad \text{and} \quad c_4 = 3/2 \tag{45}
\]
where \(l \in \mathbb{Z}\). We have made density plots corresponding to \(v = 0.9\) and \(c_4 = 3/2\) for various values of \(k\) (see figure 7). Note that unlike for the dark soliton-like solutions, here the solution is actually dynamical and evolving in time, describing a scattering (one-time or periodic) of the bright solution from the central potential.

The limiting potential with \(k = 0\) can be obtained by choosing \(c_1(T) = c_4(T + c_5)^\frac{3}{2}\), \(c_2(T) = \frac{3}{2^2 c_1(T)^\frac{3}{2}} - c_6\). \tag{46}
Now equation (38)
\[\rho(R, T) = \frac{R^{-\frac{1}{2}}}{c_4(T + c_5)^\frac{3}{2}}, \tag{48a}\]
\[\varphi(R, T) = \frac{1}{T + c_5} \left[\frac{R^2}{4} - c_6(T + c_5)^2 - \frac{3\sqrt{2}}{8c_4^3}\right] + c_7, \tag{48b}\]
\[\eta(R, T) = -\frac{3}{2\sqrt{2}c_4} \left(\frac{R}{T + c_5}\right)^\frac{3}{2}, \tag{48c}\]
solves equation (2) \((\sigma = 1)\) with the potential (see figure 7(f))
\[
V(R) = \frac{1}{9R^2} + c_6. \tag{49}
\]
We note that the \(k = 0\) case is also obtained by choosing \(c_1(T) = \text{const.}\) and \(c_2(T) = \frac{3}{2^2 c_1^\frac{3}{2}} - c_6\), which leads to
\[\rho(R) = \frac{1}{c_1 R^\frac{3}{2}}, \tag{50a}\]
\[\varphi(T) = \left(\frac{3}{2^2 c_1^\frac{3}{2}} - c_6\right) T + c_7, \tag{50b}\]
\[\eta(R) = \frac{3}{2\sqrt{2}c_1} R^\frac{3}{2}, \tag{50c}\]
\[V(R) = \frac{1}{9R^2} + c_6. \tag{50d}\]
In this case the equation may be called a ‘nonlinear Bessel’ equation.
Figure 7. Density plots of $R|\psi|^2$ as given by equation (38). In all of the plots $\nu = 0.9$ and $c_4 = 3/2^4$. In (a) and (b) the radius of the centre of mass is oscillating and the condensate is breathing between a point and an extended state. The magnitude of $k$ determines the period of the oscillations. In (c), (d) and (e) there occurs a reflection at the repulsive core potential around $R = 0$ of an incoming density from infinity. The magnitude of $k$ determines the spread and velocity of the incoming and outgoing rings and the duration of the reflection dynamics. $c_5$ determines when the reflection occurs in time. In (f) is shown the limit $k \to 0$ of potentials (40) and (43), the limiting behaviour of oscillations as the period goes to infinity. Note that the colouring is not to scale.

(a) $k = -0.1$ and $c_5 = -\frac{1}{4\pi^2}$. (b) $k = -0.02$ and $c_5 = -\frac{3}{4\pi^2}$. (c) $k = 2.0$ and $c_5 = 0$. (d) $k = 0.01$ and $c_5 = 0$. (e) $k = 0.001$ and $c_5 = 0$. (f) $k = 0$ and $c_5 = 0$.

5. Summary

We have obtained exact localized solutions of the radial Gross–Pitaevskii equation describing cylindrically symmetric systems. We have concentrated on solutions that may have implications for studies of ring solitons in quantum gases, but it should be noted that still other solutions are possible with different choices of $h(\eta)$ and the integration ‘constants’ $c_{1,2}(T)$.

The form of the potentials corresponding to exact solutions suggests that confinement of cold atoms in traps which are repulsive at the core should be studied further, especially in the context of ring soliton stability. Previous work and our own numerical simulations have shown that in general the ring dark soliton is ultimately destroyed by the snaking instability, and the ring collapses into a vortex necklace [13]. Having a solution with a potential that has both a repulsive core, and a minimum at finite $R$, however, suggests that e.g. toroidal traps (also known as ring traps) might provide better stability for the ring dark soliton, and results of numerical investigations testing this idea will be reported in a later paper. Such potentials are
experimentally feasible [31–33]. We will also discuss elsewhere the possibility of creating dark solitons in toroidal traps using time-averaged potentials and/or adiabatic passage techniques [34–37].

An additional aspect to such studies comes from the observation that some of our solutions allow one to choose freely the location of the dark or bright solution \(R_0\), as this location does not depend on the parameters of the potential. Thus the long-time survival of the solution is not connected to locating the soliton-like structure at the minimum of some confining potential. It means that one may study ring soliton dynamics before the decay into vortices takes place, and look for similarities to such behaviour in one-dimensional systems [38]. Another feature is to consider the case of a bright ring soliton in such traps, as it has not been studied very much in the past. Such considerations form the starting point for further work on ring solitons and soliton-like structures, including their stability and dynamics.

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