ON THE VANISHING VISCOSITY LIMIT OF A CHEMOTAXIS MODEL

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Abstract. A vanishing viscosity problem for the Patlak-Keller-Segel model is studied in this paper. This is a parabolic-parabolic system in a bounded domain $\Omega \subset \mathbb{R}^n$, with a vanishing viscosity $\varepsilon \to 0$. We show that if the initial value lies in $W^{1,p}$ with $p > \max\{2, n\}$, then there exists a unique solution $(u_\varepsilon, v_\varepsilon)$ with its lifespan independent of $\varepsilon$. Furthermore, as $\varepsilon \to 0$, $(u_\varepsilon, v_\varepsilon)$ converges to the solution $(u, v)$ of the limiting system in a suitable sense.

1. Introduction. In this paper we study the parabolic-parabolic system with a small parameter $\varepsilon > 0$:

$$\begin{cases}
  u_t - \Delta u = -\text{div}(u\nabla f(v)) & \text{in } \Omega^T := \Omega \times (0, T), \\
  v_t - \varepsilon \Delta v = g(u, v) & \text{in } \Omega^T := \Omega \times (0, T).
\end{cases}$$

(1.1)

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $u = u(x, t)$ and $v = v(x, t)$ are unknown functions of $(x, t) \in \Omega \times (0, T)$, the functions $f \in C^2(\mathbb{R})$, $g \in C^2(\mathbb{R}^2)$.

Throughout this paper a homogeneous Neumann boundary condition will be coupled with (1.1),

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T).$$

(1.2)

The initial value is given by

$$\begin{cases}
  u|_{t=0} = u_0(x) & \text{in } \Omega, \\
  v|_{t=0} = v_0(x) & \text{in } \Omega.
\end{cases}$$

(1.3)

The case $f(y) = y$ and $g(x, y) = x$ is a simplified Patlak-Keller-Segel (PKS) model for chemotaxis [7]. This model is used to describe oriented movement of cell populations, guided by a chemical gradient which is produced by the cells themselves. In the biological interpretation, $u(x, t)$ is a (scalar) particle density, $v(x, t)$ denotes the chemical concentration, $\varepsilon$ is the diffusion constant for the signal while the diffusion constant of the species has been scaled to 1.

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When the diffusion of the chemical substance is so small that it is negligible, i.e. \( \varepsilon \to 0 \), we can formally take \( \varepsilon = 0 \) in (1.1), which leads to a parabolic-ODE system:

\[
\begin{align*}
    u_t - \Delta u &= -\operatorname{div} (u \nabla f(v)) \quad \text{in } \Omega^T, \\
    v_t &= g(u,v) \quad \text{in } \Omega^T
\end{align*}
\]

(1.4)

with the no-flux boundary condition

\[
\frac{\partial u}{\partial \nu} - u \frac{\partial}{\partial \nu} f(v) = 0 \quad \text{on } \partial \Omega \times (0,T).
\]

(1.5)

Such a system has been proposed by Othmer and Stevens [13] to model the case where there is no diffusion for the chemicals and chemicals can modify the local environment for succeeding passages.

The PKS model has been considered by many people, see for instance the monograph [16] and references therein. Being a parabolic system, the local well-posedness of (1.1) can be proved by standard methods, see [16, Chapter 3], while it is apparently not the case for (1.4). Hence the local well-posedness problem for (1.4) has attracted a lot of interest, see [4, 9, 20, 21]. Moreover, a similar model was studied in [18] for the one dimensional case and in [11] for the multidimensional cases. In [22], global existence and uniqueness of small smooth solutions are established for the model on bounded domains subject to no-flux boundary condition. In [2, 3], the authors studied the following model on bounded domains

\[
\begin{align*}
    u_t &= \kappa \Delta u - \operatorname{div}[u \chi(v) \nabla v] \quad x \in \mathbb{R}^n, t > 0, \\
    v_t &= -vu.
\end{align*}
\]

(1.6)

It was shown in [3] that, under smallness assumptions on the initial data, there exist global weak solutions to (1.6) and the solution converges to a constant state as time goes to infinity. Finally, the problem to identify (1.4) (when \( f(v) = \log v \)) as the vanishing viscosity limit of (1.1) (as \( \varepsilon \to 0 \)) was also studied in [15, 19], by using a Cole-Hopf transformation and then considering a hyperbolic system.

In this paper, we verify rigourously the convergence from (1.1) to (1.4). More precisely we prove

1. uniform in \( \varepsilon \) well-posedness for system (1.1) in \( W^{1,p}(\Omega) \), if \( p > \max\{2,n\} \);
2. well-posedness for system (1.4) in \( W^{1,p}(\Omega) \), if \( p > \max\{2,n\} \);
3. convergence of solutions to (1.1) to the one of (1.4).

First for (1.1) we prove

**Theorem 1.1.** Suppose the initial value satisfies

\[ u_0 \in W^{1,p}(\Omega), v_0 \in W^{1,p}(\Omega), \text{ for some } p > \max\{2,n\}. \]

There exists a constant \( T_0 > 0 \) such that for all \( \varepsilon \) small, system (1.1) admits a unique solution with

\[ u_\varepsilon \in L^p(0,T_0;W^{1,p}(\Omega)), \quad v_\varepsilon \in L^\infty(0,T_0;W^{1,p}(\Omega)). \]

Furthermore, \( (u_\varepsilon,v_\varepsilon) \) enjoy the following uniform regularity as \( \varepsilon \to 0 \).

(i) For any \( k \in [0,1) \), \( u_\varepsilon \) are uniformly bounded in \( C^{1+\frac{k}{2}}(0,T_0;W^{k,p}(\Omega)) \).

(ii) There exists a constant \( \alpha \in (0,1) \) independent of \( \varepsilon \) such that \( u_\varepsilon \) are uniformly bounded in \( C^{\alpha,\alpha/2}(\Omega^{T_0}) \).

(iii) \( v_\varepsilon \) are uniformly bounded in \( L^\infty(0,T_0;W^{1,p}(\Omega)) \cap C^{\alpha,\alpha/2}(\Omega^{T_0}) \).
For $k \in (0, 1)$, as in [5], a function $u \in W^{k,p}(\Omega)$ if

$$\|u\|_{W^{k,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+kp}} \, dxdy \right)^{1/p} < +\infty.$$  

When working with $W^{1,p}$ solutions, the equations (1.1)-(1.3) are understood in weak sense, i.e. for any $\eta \in C^\infty(\overline{\Omega})$ which vanishes on $\Omega \times \{T\}$, we have

$$\begin{align*}
\int_{\Omega} [-u_x \eta_t + \nabla u_x \cdot \nabla \eta - u_x \nabla f(v_x) \cdot \nabla \eta] - \int_{\Omega} u_0 \eta_0 &= 0, \\
\int_{\Omega} [-v_x \eta_t + \varepsilon \nabla v_x \cdot \nabla \eta - g(u_x, v_x) \eta] - \int_{\Omega} v_0 \eta_0 &= 0,
\end{align*}$$

where $\eta_0 = \eta|_{t=0}$.

The existence part will be proven by Banach’s fixed point theorem. However, different from the treatment in many other problems, the contraction property relies on an $L^\infty$ estimate for the parabolic operator $\partial_t - \Delta + \text{div}(b)$, where $b$ is a vector field in $L^\infty(0, T_0; L^p(\Omega))$. This $L^\infty$ estimate allows us to apply standard $W^{2,p}$ estimate to the first equation in (1.1), and then define a map from $L^p(0, T_0; W^{1,p}(\Omega))$ into itself. We will use Moser iteration to prove this $L^\infty$ estimate. This is the reason we need $p > n$ in the above theorem, see Ladyzhenskaia, Solonnikov and Ural’ceva [8, Theorem 10.1]. Further use of Moser iteration and heat kernel representations then give uniform Hölder continuity of $u_x$ and $v_x$. Finally, we stress that although this $L^\infty$ estimate seems only to be a technical point, (1.1) could not be well-posed in $W^{1,n}$ when $n \geq 2$.

The second result is about (1.4).

**Theorem 1.2.** Suppose the initial value satisfies

$$u_0 \in W^{1,p}(\Omega), v_0 \in W^{1,p}(\Omega), \text{ for some } p > \max\{2, n\}.$$  

There exists a unique local solution $(u, v)$ of (1.4), which satisfies

$$u \in L^p(0, T_0; W^{1,p}(\Omega)), \quad v \in L^\infty(0, T_0; W^{1,p}(\Omega))$$

with $T_0$ being the same as in Theorem 1.1. Furthermore, $(u, v)$ enjoys the following regularity.

(i) For any $k \in [0, 1)$, $u \in C^{1+k}((0, T_0); W^{k,p}(\Omega))$.

(ii) There exists a constant $\alpha \in (0, 1)$ such that $u \in C^{\alpha, \alpha/2}(\overline{\Omega T_0})$.

(iii) $v \in \text{Lip}(0, T_0; L^\infty(\Omega)) \cap C^{\alpha, \alpha/2}(\overline{\Omega T_0})$.

The proof of this theorem is similar to the one for Theorem 1.1. Here the equations (1.4)-(1.5) are also understood in weak sense, i.e. for any $\eta \in C^\infty(\overline{\Omega})$ which vanishes on $\Omega \times \{T\}$, we have

$$\begin{align*}
\int_{\Omega} [-u_x \eta_t + \nabla u \cdot \nabla \eta - u \nabla f(v) \cdot \nabla \eta] - \int_{\Omega} u_0 \eta_0 &= 0, \\
\int_{\Omega} [-v_x \eta_t - g(u, v) \eta] - \int_{\Omega} v_0 \eta_0 &= 0.
\end{align*}$$

For the limiting system (1.4), we also prove

**Theorem 1.3.** If the initial value satisfies

$$u_0 \in W^{2,p}(\Omega), v_0 \in W^{2,p}(\Omega), \text{ for some } p > \max\{2, n\}.$$
Then there exists $T_1 \in (0, T_0]$ such that (1.4) admits a unique local solution with
\[ \tilde{u} \in L^p(0, T_1; W^{2,p}(\Omega)), \quad \tilde{v} \in C(0, T_1; W^{2,p}(\Omega)). \]

Finally, we establish the convergence of $(u_\varepsilon, v_\varepsilon)$ to $(u, v)$ and $(\tilde{u}, \tilde{v})$.

**Theorem 1.4.** (i) If the initial value satisfies
\[ u_0 \in W^{1,p}(\Omega), v_0 \in W^{1,p}(\Omega), \text{ for some } p > \max\{2, n\}. \]

Then as $\varepsilon \to 0$, $u_\varepsilon \to u$ in $C(0, T_0; W^{k,p}(\Omega))$ (for any $k \in [0, 1)$) and in $C(\overline{\Omega^0})$, $v_\varepsilon \to v$ in $C(\overline{\Omega^0})$ and $\ast$-weakly in $L^\infty(0, T_0; W^{1,p}(\Omega))$.

(ii) If the initial value is taken as $(u_0, v_0) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$, $p > \max\{2, n\}$.

Then there exists a constant $C$ such that
\[ \sup_{t \in [0, T_1]} \|v_\varepsilon(t) - \tilde{v}(t)\|_{W^{1,p}(\Omega)}^p + \int_0^{T_1} \|u_\varepsilon(t) - \tilde{u}(t)\|_{W^{1,p}(\Omega)}^p dt \leq C \varepsilon^{p/2}. \]

Notations.
- The Laplacian $\Delta$, when viewed as an operator between two spaces defined on $\Omega$, is always coupled with the homogenous Neumann boundary condition.
- We denote by $\|\cdot\|_p$, $\|\cdot\|_{k,p}$ the norms in $L^p(\Omega)$, $W^{k,p}(\Omega)$, respectively.
- Throughout this paper, $C(t)$ denotes a constant satisfying
\[ \lim_{t \to 0} C(t) = 0. \]

It may be different in different places.

The remaining part of this paper is organized as follows. In Section 2 we provide several technical tools, in particular, an $L^\infty$ estimate which is crucial for our proof. In Section 3 we construct solutions $(u_\varepsilon, v_\varepsilon)$ to (1.1). This is the existence part of Theorem 1.1. Further properties of $(u_\varepsilon, v_\varepsilon)$ are established in Section 4. Section 5 and Section 6 are devoted to the proof of existence part in Theorem 1.2 and Theorem 1.3 respectively. Finally, we study the convergence of $(u_\varepsilon, v_\varepsilon)$ in Section 7.

2. Preliminary. In this section, we present some tools which will be used in the proof of Theorem 1.1.

**Lemma 2.1.** Let $w$ be the solution of
\[ \begin{cases} \frac{\partial w}{\partial t} - \varepsilon \Delta w = \phi & \text{in } \Omega \times (0, T), \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T), \\ w|_{t=0} = w(0) & \text{in } \Omega, \end{cases} \]

where $\varepsilon > 0$. If
\[ \phi \in L^p(0, T; W^{1,p}(\Omega)), \quad w(0) \in W^{1,p}(\Omega). \]

Then for each $t \in (0, T)$, $w(t) \in W^{1,p}(\Omega)$ with the estimate
\[ \|w(t)\|_{1,p} \leq \|w(0)\|_{1,p} + \int_0^t \|\phi(s)\|_{1,p} ds. \]
Proof. By the semigroup theory [14], the solution of (2.1) is given by

$$w(t) = e^{\varepsilon t \Delta} w(0) + \int_0^t e^{\varepsilon(t-s)\Delta} \phi(s) ds.$$  

(2.3)

By [14, Theorem 6.13], we have

$$\|e^{\varepsilon t \Delta}\|_{L(W^{k,p}(\Omega),W^{k,p}(\Omega))} \leq 1, \quad \forall \ v \geq 0, \ \varepsilon > 0, \ t > 0.$$  

Using Minkovski inequality, we have

$$\|w(t)\|_{1,p} \leq \|e^{\varepsilon t \Delta} w(0)\|_{1,p} + \int_0^t \|e^{\varepsilon(t-s)\Delta} \phi(s)\|_{1,p} ds$$

$$\leq \|w(0)\|_{1,p} + \int_0^t \|\phi(s)\|_{1,p} ds. \quad \square$$

Next, let us recall some a priori estimates for the equation

$$\begin{cases}
\partial_t w - \Delta w = -\text{div} \vec{\psi} & \text{in } \Omega \times (0,T), \\
\frac{\partial w}{\partial \nu} - \vec{\psi} \cdot \nu = 0 & \text{on } \partial \Omega \times (0,T), \\
|w|_{t=0} = 0 & \text{in } \Omega.
\end{cases}$$

(2.4)

where $\vec{\psi} = (\psi_1, \ldots, \psi_n)$. The existence of a weak solution will follow once we establish suitable a priori estimates, see [12, Theorem 6.39].

By [12, Theorem 6.38 and Proposition 7.18], we have

**Proposition 1.** Let $p \in (1, \infty)$ and $w \in L^p(0,T;W^{1,p}(\Omega))$ be the weak solution of (2.4). If $\vec{\psi} \in L^p(0,T;L^p(\Omega))$, then for any $t \in (0,T)$, we have

$$\int_0^t \|w(s)\|_{1,p}^p ds \leq C \int_0^t \|\vec{\psi}(s)\|_{1,p}^p ds,$$

where $C$ depends on $n$, $p$, $t$ and it is increasing in $t$.

**Proposition 2.** Let $p \in (1, \infty)$ and $w \in L^p(0,T;W^{2,p}(\Omega))$ be the solution of (2.4). If $\vec{\psi} \in L^p(0,T;W^{1,p}(\Omega))$, for any $t \in (0,T)$, we have

$$\int_0^t \|w(s)\|_{2,p}^p ds \leq C \int_0^t \|\vec{\psi}(s)\|_{1,p}^p ds,$$

where $C$ depends on $n$, $p$, $t$ and it is increasing in $t$.

**Remark 1.** It is not expected that the constant $C$ in these two propositions goes to 0 as $T \to 0$. Therefore using only these two propositions are not sufficient to construct contraction maps and solutions to (1.1).

The following is an $L^\infty$ estimate, which will be crucial for our following proofs.

**Lemma 2.2.** Let $w$ be the solution of

$$\begin{cases}
\partial_t w - \Delta w = -\text{div}(w\vec{a}) - \text{div} \vec{b} & \text{in } \Omega \times (0,T), \\
\frac{\partial w}{\partial \nu} - w\vec{a} \cdot \nu - \vec{b} \cdot \nu = 0 & \text{on } \partial \Omega \times (0,T), \\
w|_{t=0} = 0 & \text{in } \Omega,
\end{cases}$$

(2.5)
where $\vec{a} = (a_1, \cdots, a_n)$ and $\vec{b} = (b_1, \cdots, b_n)$ belong to $L^\infty(0, T; L^p(\Omega))$ for some $p > \max\{2, n\}$. Then we have
\[
\sup_{t \in [0, T]} \|w(t)\|_\infty \leq C \sup_{t \in [0, T]} \|\vec{b}(t)\|_p,
\]
where $C$ depends on $\Omega$, $\|\vec{a}\|_{L^\infty(0, T; L^p(\Omega))}$, $\|\vec{b}\|_{L^\infty(0, T; L^p(\Omega))}$, $p$, $n$, $T$ and it is increasing in $T$.

**Proof.** We use Moser iteration to prove this $L^\infty$ estimate.

**Step 1.** Multiplying the first equation of (2.5) by $w$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} w(t)^2 + \int_{\Omega} |\nabla w(t)|^2 = \int_{\Omega} w(t) \vec{a}(t) \cdot \nabla w(t) + \int_{\Omega} \vec{b}(t) \cdot \nabla w(t)
\]
(2.6)

Using Young inequality and Hölder inequality, we get
\[
I \leq \frac{1}{4} \|\nabla w(t)\|_2^2 + \|w(t)\|_{\frac{2p}{p-2}}^2 \|\vec{a}(t)\|_p^2.
\]
(2.7)

For $n > 2$, we make use of interpolation inequality
\[
\|w(t)\|_{\frac{2n}{n-2}} \leq \|w(t)\|_\theta \|w(t)\|_1^{1-\theta},
\]
which holds for $\theta = 1 - n/p$. By Sobolev embedding theorem, there exists a constant $C_S$ such that
\[
\|w(t)\|_{\frac{2n}{n-2}} \leq C_S \|w(t)\|_{1,2}.
\]

For $n \leq 2$, we also have $\|w(t)\|_{\frac{2p}{p-2}} \leq \|w(t)\|_\theta \|w(t)\|_1^{1-\theta}$ for some $\theta \in (0, 1)$. The treatment is similar to the $n > 2$ case, so we will assume $n > 2$.

The second term in the right-hand side of (2.7) is estimated from above by
\[
\|\vec{a}(t)\|_p^2 \|w(t)\|_{\frac{2p}{p-2}}^2 \leq \|\vec{a}(t)\|_2^2 \|w(t)\|_2^2 \|w(t)\|_{\frac{2p}{p-2}}^{2(1-\theta)}
\]
\[
\leq \frac{1}{2C_S} \|w(t)\|_{\frac{2n}{n-2}}^2 + C(p, n) \|w(t)\|_2^2 \|\vec{a}(t)\|_p^{2/\theta}
\]
\[
\leq \frac{1}{2} \|w(t)\|_{1,2}^2 + C(p, n) \|w(t)\|_2^2 \|\vec{a}(t)\|_p^{2/\theta}.
\]

Concerning $II$, by Cauchy inequality and Hölder inequality, we get
\[
II \leq \frac{1}{4} \|\nabla w(t)\|_2^2 + \|\vec{b}(t)\|_p^2 \|\Omega\|^{\frac{p-2}{p}}.
\]

Putting these back into (2.6), we get
\[
\frac{d}{dt} \|w(t)\|_2^2 \leq C(p, n)(\|\vec{a}(t)\|_p^{2/\theta} + 1) \|w(t)\|_2^2 + 2 \|\vec{b}(t)\|_p^2 \|\Omega\|^{\frac{p-2}{p}}.
\]

By Gronwall’s inequality, we have
\[
\|w(t)\|_2^2 \leq C \int_0^t \|\vec{b}(s)\|_p^2 ds,
\]
where $C$ depends on $\Omega$, $\|\vec{a}\|_{L^\infty(0, T; L^p(\Omega))}$, $p$, $n$, $t$ and is increasing in $t$. Therefore $w \in L^\infty(0, T; L^2(\Omega))$ and
\[
\sup_{t \in [0, T]} \|w(t)\|_2 \leq CT \sup_{t \in [0, T]} \|\vec{b}(t)\|_p.
\]
(2.8)
Step 2. For any odd number $q \geq 3$, multiplying the first equation of (2.5) by $w^{q}$, we get

$$\frac{d}{dt} \int_{\Omega} w(t)^{q+1} + q \int_{\Omega} w(t)^{q-1} |\nabla w(t)|^2$$

$$= q \int_{\Omega} w(t)^{q} \nabla w(t) \cdot \vec{a}(t) + q \int_{\Omega} w(t)^{q-1} \nabla w(t) \cdot \vec{b}(t).$$

Using Young's inequality, we have

$$\text{III} \leq \frac{q}{4} \int_{\Omega} w(t)^{q-1} |\nabla w(t)|^2 + q \left( \int_{\Omega} |w(t)|^{(q+1) \frac{p}{p-2}} \right)^{\frac{p-2}{p}} \|\vec{a}(t)\|_p^2$$

and

$$\text{IV} \leq \frac{q}{4} \int_{\Omega} w(t)^{q-1} |\nabla w(t)|^2 + q \left( \int_{\Omega} |w(t)|^{(q-1) \frac{p}{p-2}} \right)^{\frac{p-2}{p}} \|\vec{b}(t)\|_p^2.$$ 

Hence

$$\frac{d}{dt} \|w(t)\|_{q+1}^2 + \|\nabla w(t)\|_{\frac{p+1}{q+1}}^2 \leq (q+1)^2 \|w(t)\|_{(q+1) \frac{p}{p-2}}^2 \|\vec{a}(t)\|_p^2$$

$$+ (q+1)^2 \|w(t)\|_{(q-1) \frac{p}{p-2}}^2 \|\vec{b}(t)\|_p^2. \quad (2.9)$$

Now, for $n > 2$, we make use of Gagliardo-Nirenberg-Sobolev inequality in the form of

$$\|h\|_s \leq C_{\text{GNS}} \left( \|\nabla h\|_2^{1-1/r} \|h\|_1^{1/r} + \|h\|_1 \right), \quad (2.10)$$

where $1/r = 2n/(s(n+2)) - (n-2)/(n+2)$, $C_{\text{GNS}} \geq 1$ is a constant independent of $r \in (s, r_0)$ if $r_0 > s$ is prescribed in advance and $s \in \left[ \frac{2n}{n-2} \right]$.

For $n \leq 2$, we have $\|h\|_s \leq C_{\text{GNS}} \left( \|\nabla h\|_2^{1-1/s} \|h\|_1^{1/s} + \|h\|_1 \right)$, for any $s \geq 1$. The treatment is similar to the $n > 2$ case, so we will assume $n > 2$.

We apply (2.10) with $h(t) = w(t)^{(q+1)/2}$ and $s = \frac{2p}{p-2}$, which gives

$$(q+1)^2 \|w(t)\|_{(q+1) \frac{p}{p-2}}^2 \|\vec{a}(t)\|_p^2 = (q+1)^2 \|\vec{a}(t)\|_p^2 \|h(t)\|_s^2$$

$$\leq 2(q+1)^2 C_{\text{GNS}} \|\vec{a}(t)\|_p^2 \left( \frac{1}{4} \|\nabla h(t)\|_2^2 \right)^{(1-1/r)} \left( 4^{1-1/r} \|h(t)\|_1^{2/r} + \|h(t)\|_2^2 \right)$$

$$\leq \frac{1}{4} \|\nabla h(t)\|_2^2 + C4^{n \frac{(p-2)}{p}} \left( |C_{\text{GNS}}(q+1)| \frac{p(n+2)}{p-n} \|\vec{a}(t)\|_p \frac{p(n+2)}{p-n} + 1 \right) \|h(t)\|_1^2.$$ 

Without loss of generality, we can assume $\|h(t)\|_1 > 1$. Next, we apply (2.10) with $h(t) = w(t)^{(q+1)/2}$ and $s' = \frac{2q-1}{q+1} \frac{2p}{p-2}$, which gives

$$(q+1)^2 \|w(t)\|_{(q-1) \frac{p}{p-2}}^2 \|\vec{b}(t)\|_p^2 = (q+1)^2 \|\vec{b}(t)\|_p^2 \|h(t)\|_{s'}^2$$

$$\leq \frac{1}{4} \|\nabla h(t)\|_2^2 + C4^{n \frac{(p-2)}{p}} \left( |C_{\text{GNS}}(q+1)| \frac{p(n+2)}{p-n} \|\vec{b}(t)\|_p \frac{p(n+2)}{p-n} + 1 \right) \|h(t)\|_1^2.$$ 

Here we have used $\|h\|_{s'} \leq C(\Omega) \|h\|_s$.

Thus for any $t \in [0, T]$, by (2.9), we get

$$\frac{d}{dt} \|w(t)\|_{q+1}^2 + \frac{1}{2} \|\nabla w(t)\|_{\frac{p+1}{q+1}}^2$$

$$\leq C(q+1) \frac{p(n+2)}{p-n} \left( \|\vec{b}(t)\|_p \frac{p(n+2)}{p-n} + \|\vec{a}(t)\|_p \frac{p(n+2)}{p-n} + 1 \right) \|w(t)\|_{\frac{q+1}{q}}^2.$$
where $C$ depends on $C_{GNS}$, $n$ and $p$.

From this differential inequality, we deduce that for any odd number $q \geq 3$,

$$
\sup_{t \in [0,T]} \|w(t)\|_q^{q+1} \leq C(q + 1) \sup_{t \in [0,T]} \left( \|\tilde{a}(t)\|_{p}^{p(n+2)} + \|\tilde{b}(t)\|_{p}^{p(n+2)} + 1 \right) \sup_{t \in [0,T]} \|w(t)\|_{\frac{q+1}{q}}^{q+1}
$$

Take $q + 1 = 2^k + 1$ and denote

$$
\Phi_k = \sup_{t \in [0,T]} \int_{\Omega} w(t)^{2^k} \, dx.
$$

It satisfies

$$
\Phi_{k+1} \leq C 2^{\frac{p(n+2)}{p-n}} (k+1) \sup_{t \in [0,T]} \left( \|\tilde{a}(t)\|_{p}^{p(n+2)} + \|\tilde{b}(t)\|_{p}^{p(n+2)} + 1 \right) \Phi_k^2, \forall k \geq 1.
$$

This implies

$$
\Phi_{k+1} \leq C 2^{k-1} \left( \sup_{t \in [0,T]} \|\tilde{a}(t)\|_{p}^{p(n+2)} + \sup_{t \in [0,T]} \|\tilde{b}(t)\|_{p}^{p(n+2)} + 1 \right) 2^{k-1} \sum_{t=1}^{k} \frac{2^{(n+2)(t+1)2^{k-t}}}{k} \Phi_1^{1/2}
$$

and hence we obtain

$$
\sup_{t \in [0,T]} \|w(t)\|_{2^{k+1}} \leq \Phi_{k+1}^{\frac{1}{2^{k+1}}}
$$

Letting $k \to +\infty$, by (2.8), we obtain

$$
\sup_{t \in [0,T]} \|w(t)\|_{\infty} \leq C \left( \sup_{t \in [0,T]} \|\tilde{a}(t)\|_{p}^{p(n+2)} + \sup_{t \in [0,T]} \|\tilde{b}(t)\|_{p}^{p(n+2)} + 1 \right) \sup_{t \in [0,T]} \|w(t)\|_2
$$

where $C$ depends on $C_{GNS}$, $\Omega$, $\|\tilde{a}\|_{L^\infty(0,T;L^p(\Omega))}$, $\|\tilde{b}\|_{L^\infty(0,T;L^p(\Omega))}$, $p$, $n$ and $T$. 

Next, we need some estimates on $\|g(u,v)\|_{1,p}$ and $\|f(v)\|_{1,p}$. Since $\nabla(f(y) - f(0)) = \nabla f(y)$, without loss of generality, we may assume $f(0) = 0$. Denoting $K = \max\{D^a g(0,0), f'(0)\}$, for $|\alpha| \leq 1$ and take $L$ to be the maximal of the Lipschitz constants for $g$, $\nabla g$, $f$ and $f'$, we have

**Lemma 2.3.** For any $u, v \in W^{1,p}(\Omega)$ with $p > \max\{2, n\}$, we have

$$
\|g(u,v)\|_{1,p} \leq C \left( \|u\|_{1,p}^{2} + \|v\|_{1,p}^{2} + 1 \right),
\|f(v)\|_{1,p} \leq C \left( \|v\|_{1,p}^{2} + 1 \right),
$$

where $C$ depends on $n$, $\Omega$, $K$ and $L$. 

Proof. For any $u, u', v, v' \in W^{1,p}(\Omega)$, by the Lipschitz continuity of $f$ and $g$, we get

$$|D^\alpha g(u, v)| \leq L(|u| + |v|) + K,$$

(2.12)

$$|f(v)| \leq L|v|, \quad |f'(v)| \leq L|v| + K.$$  

It follows from (2.12) that

$$\|g(u, v)\|_p \leq \|Lu + Lv + K\|_p$$

and

$$\|f(v)\|_p \leq L\|v\|_p.$$  

Next for $j = 1, \ldots, n$, by Sobolev embedding theorem and Young’s inequality,

$$\|\partial_x g(u, v)\|_p = \|g_1 u_{x_j} + g_2 v_{x_j}\|_p$$

$$\leq \|g_1 u_{x_j}\|_p + \|g_2 v_{x_j}\|_p$$

$$\leq C (\|u\|_{1,p}^2 + \|v\|_{1,p}^2 + 1)$$

and

$$\|\partial_x f(v)\|_p = \|f'(v) v_{x_j}\|_p \leq C (\|v\|_{1,p}^2 + 1),$$

where $C$ is a constant determined by $\Omega$, $K$ and $L$. These relations are summarized as (2.11).

To prove Theorem 1.1, we also need a generalized Gronwall’s inequality.

**Lemma 2.4.** Let $\eta(t)$ be nonnegative, absolutely continues function on $[0, T]$, which satisfies for a.e. $t \in (0, T)$ the differential inequality

$$\eta(t) \leq a(t) + b(t) \int_0^t k(s) \eta^m(s) ds,$$

where $m > 1$ is a positive integer and $a(t), b(t), k(t)$ are positive, continuous functions on $[0, T]$. Then for $0 \leq t \leq \alpha_m$,

$$\eta(t) \leq b(t) \left\{ \frac{b(0)^{m-1}}{a(0)^{m-1}} - (m - 1) \int_0^t \left( \frac{b(s)^m}{a(s)^m} \left( \frac{a(s)}{b(s)} \right)' + k(s) b^m(s) \right) ds \right\}^{\frac{1}{m}},$$

where $\alpha_m$ is defined to be the supreme of those $t \in [0, T]$ satisfying

$$\frac{b(0)^{m-1}}{a(0)^{m-1}} - (m - 1) \int_0^t \left( \frac{b(s)^m}{a(s)^m} \left( \frac{a(s)}{b(s)} \right)' + k(s) b^m(s) \right) ds > 0.$$

**Proof.** Let

$$\psi(t) = a(t) + b(t) \int_0^t k(s) \eta^m(s) ds,$$

which satisfies $\psi(0) = a(0), \psi(t) \geq \eta(t)$ and

$$\psi'(t) = a'(t) + b'(t) \int_0^t k(s) \eta^m(s) ds + b(t) k(t) \eta^m(t)$$

$$= a'(t) + b'(t) \left( \psi(t) - a(t) \right) + b(t) k(t) \eta^m(t).$$

Hence

$$\left( \frac{\psi(t)}{b(t)} \right)^{-m} \left( \frac{\psi(t)}{b(t)} \right)' \leq \left( \frac{a(t)}{b(t)} \right)^{-m} \left( \frac{a(t)}{b(t)} \right)' + k(t) b^m(t).$$
Integrating this inequality in \((0,t)\), we get
\[
\frac{1}{1-m} \left( \frac{\psi(t)}{b(t)} \right)^{1-m} - \left( \frac{\psi(0)}{b(0)} \right)^{1-m} \leq \int_0^t \left( \frac{b(s)^m}{a(s)^m} \left| \frac{a(s)}{b(s)} \right|' + k(s)b^m(s) \right) ds.
\]
Then for \(t \in [0, \alpha_m]\), we have
\[
\eta(t) \leq \psi(t) \leq b(t) \left\{ \frac{b(0)^{m-1}}{a(0)^{m-1}} - (m-1) \int_0^t \left( \frac{b(s)^m}{a(s)^m} \left| \frac{a(s)}{b(s)} \right|' + k(s)b^m(s) \right) ds \right\}^{\frac{1}{m-1}}. \quad \Box
\]

3. Proof of Theorem 1.1. Now we are ready to prove Theorem 1.1.

**Proof.** We will apply Banach’s fixed point theorem on \(L^p(0, T; W^{1,p}(\Omega))\). The proof is divided into three steps.

**Step 1.** Given a function \(z_\varepsilon \in L^p(0, T; W^{1,p}(\Omega))\), we solve the problem
\[
\partial_t v_\varepsilon - \varepsilon \Delta v_\varepsilon = g(z_\varepsilon, v_\varepsilon), \quad v_\varepsilon(x, t)|_{t=0} = v_0 \tag{3.1}
\]
with homogeneous Neumann boundary. By the semigroup theory, \(v_\varepsilon\) is given by
\[
v_\varepsilon(t) = e^{\varepsilon t \Delta} v_0 + \int_0^t e^{\varepsilon (t-s) \Delta} g(z_\varepsilon(s), v_\varepsilon(s)) ds.
\]
Applying Lemma 2.1 and Lemma 2.3, we get
\[
\|v_\varepsilon(t)\|_{1,p} \leq \|v_0\|_{1,p} + \int_0^t \|g(z_\varepsilon(s), v_\varepsilon(s))\|_{1,p} ds
\]
\[
\leq \|v_0\|_{1,p} + C \int_0^t \left( \|z_\varepsilon(s)\|_{1,p}^2 + \|v_\varepsilon(s)\|_{1,p}^2 \right) ds + Ct.
\]
By taking \(a(t) = \|v_0\|_{1,p} + C \int_0^t \|z_\varepsilon(s)\|_{1,p}^2 ds + Ct, k(t) = C(K, L), b(t) = 1\) and \(m = 2\), Lemma 2.4 gives
\[
\|v_\varepsilon(t)\|_{1,p} \leq \left\{ \frac{1}{a(t)} - Ct \right\}^{-1} = \frac{a(t)}{1 - Ct a(t)}.
\]
If \(\int_0^T \|z_\varepsilon(s)\|_{1,p}^p ds \leq M (M \text{ to be determined later})\) and \(T\) is so small that \(CT a(T) < 1/2\), because \(p > 2\), we have
\[
\sup_{t \in [0, T]} \|v_\varepsilon(t)\|_{1,p} \leq 2 \|v_0\|_{1,p} + C \int_0^T \|z_\varepsilon(s)\|_{1,p}^2 ds + CT
\]
\[
\leq 2 \|v_0\|_{1,p} + CT^{(p-2)/p} \left\{ \int_0^T \|z_\varepsilon(s)\|_{1,p}^p ds \right\}^{2/p} + CT. \tag{3.2}
\]
In fact, because \(e^{t \Delta} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)\) is a strongly continuous semigroup for \(t \geq 0\) (see [6]), \(v_\varepsilon \in C([0, T_0]; W^{1,p}(\Omega))\).

**Step 2.** With \(v_\varepsilon\) defined as in (3.1), we study the equation
\[
\partial_t u_\varepsilon - \Delta u_\varepsilon = -\text{div}(u_\varepsilon \nabla f(v_\varepsilon)), \quad u_\varepsilon(x, t)|_{t=0} = u_0, \tag{3.3}
\]
Take the decomposition \(u_\varepsilon = u_{\varepsilon 1} + u_{\varepsilon 2}\), where \(u_{\varepsilon 1}\) is the solution of
\[
\begin{cases}
\partial_{\varepsilon 1} - \Delta u_{\varepsilon 1} = -\text{div}(u_{\varepsilon 1} \nabla f(v_{\varepsilon})), \quad u_{\varepsilon 1}(x, t)|_{t=0} = u_0, \\
\partial u_{\varepsilon 1} = 0 & \text{on } \partial \Omega \times (0, T), \\
u_{\varepsilon 1}(T) = u_0 & \text{in } \Omega.
\end{cases} \tag{3.4}
\]
By the semigroup theory, we have
\[ \|u_{12}(t)\|_{1,p} \leq \|u_0\|_{1,p}, \quad \forall t > 0. \]  
(3.6)

Next, by Lemma 2.3, we have
\[ \|\nabla f(v_z)\|_{L^\infty(0,T;L^p(\Omega))} \leq C \sup_{t \in [0,T]} (\|v_z(t)\|_{1,p}^2 + 1) \]
and by Sobolev embedding theorem
\[ \|u_{11}\nabla f(v_z)\|_{L^\infty(0,T;L^p(\Omega))} \leq C\|u_0\|_{1,p} \sup_{t \in [0,T]} (\|v_z(t)\|_{1,p}^2 + 1). \]

Thus we can apply Lemma 2.2 and (3.2) to (3.5), which gives a constant \( C \) depending on \( \|v_0\|_{1,p}, \|u_0\|_{1,p}, M \) and \( T \), such that
\[ \sup_{t \in [0,T]} \|u_{12}(t)\|_{L^\infty(\Omega)} \leq C. \]  
(3.7)

Combining (3.6) and (3.7), we get
\[ \sup_{t \in [0,T]} \|u_{11}(t)\|_{\infty} \leq \sup_{t \in [0,T]} \|u_{11}(t)\|_{\infty} + \sup_{t \in [0,T]} \|u_{12}(t)\|_{\infty} \leq \|u_0\|_{1,p} + C. \]  
(3.8)

By Proposition 1, Sobolev embedding theorem and (3.2), we have
\[ \int_0^t \|u_{12}(s)\|_{1,p}^{\|p} ds \leq C \int_0^t \|u_{11}(s)\|_{1,p}^{\|p} ds \]
\[ \leq C \int_0^t \|u_z(s)\|_{\infty}^{\|p} \|\nabla f(v_z(s))\|_{1,p}^{\|p} ds \]
\[ \leq C(\|u_0\|_{1,p} + 1)^p \sup_{s \in [0,T]} \|f(v_z(s))\|_{1,p}^{\|p} t \]
\[ \leq C(\|u_0\|_{1,p} + 1)^p \sup_{s \in [0,T]} (\|v_z(s)\|_{1,p}^2 + 1)^p t \]
\[ \leq C(\|u_0\|_{1,p} + 1)^p \left\{ \|v_0\|_{1,p}^{\|p} + T \|z_0(s)\|_{1,p}^{\|p} + T^{2(p-2)} \left( \int_0^T \|z_0(s)\|_{1,p}^{\|p} ds \right)^2 \right\} t. \]  
(3.9)

Combining (3.6) and (3.9) leads to
\[ \int_0^t \|u_z(s)\|_{1,p}^{\|p} ds \leq (\|u_0\|_{1,p} + C) t, \]  
(3.10)

where \( C \) is the coefficient of \( t \) in (3.9).

**Step 3.** By the previous two steps, we can define a map \( B : L^p(0,T;W^{1,p}(\Omega)) \rightarrow L^p(0,T;W^{1,p}(\Omega)) \) as \( Bz = u_z \). By (3.10), there exist \( M \) and \( T \) such that if \( \int_0^T \|z_0(s)\|_{1,p}^{\|p} ds \leq M \), it holds that
\[ B : \mathcal{B}_M \rightarrow \mathcal{B}_M, \]
where
\[ \mathcal{B}_M := \{ \varphi \in L^p(0,T;W^{1,p}(\Omega)) : \|\varphi\|_{L^p(0,T;W^{1,p}(\Omega))} \leq M, \varphi(0) = u_0 \}. \]  
(3.11)
We show when $T$ is small enough, $B$ is a contraction on $\mathcal{M}$. To this aim, consider two functions $z_1^\varepsilon, z_2^\varepsilon \in \mathcal{M}$ and denote the images by $u_1^\varepsilon = Bz_1^\varepsilon, u_2^\varepsilon = Bz_2^\varepsilon$, respectively. The corresponding solutions of the equation (3.1) are denoted by $v_1^\varepsilon$ and $v_2^\varepsilon$, respectively.

By the semigroup theory, it is directly deduced that

\[
\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p} \leq \int_0^t \|g(z_1^\varepsilon(s), v_1^\varepsilon(s)) - g(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_{1,p} ds
\]

\[
\leq L \int_0^t (\|z_1^\varepsilon(s) - z_2^\varepsilon(s)\|_p + \|v_1^\varepsilon(s) - v_2^\varepsilon(s)\|_p) ds
\]

\[
+ \int_0^t \|\nabla g(z_1^\varepsilon(s), v_1^\varepsilon(s)) - \nabla g(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_p ds.
\]

By (3.2) and (3.11), we find a constant $\chi$, depending only on $L$, $M$, $K$ and $\|v_0\|_{1,p}$ such that

\[
\sup_{t \in [0,T]} \|v_i^\varepsilon(t)\|_{1,p} \leq \chi, \ i = 1, 2.
\]

(3.12)

Since $p > 2$, we have

\[
\int_0^t \|\nabla g(z_1^\varepsilon(s), v_1^\varepsilon(s)) - \nabla g(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_p ds
\]

\[
= \int_0^t \|g_1(z_1^\varepsilon(s), v_1^\varepsilon(s))\nabla z_1^\varepsilon(s) + g_2(z_1^\varepsilon(s), v_1^\varepsilon(s))\nabla v_1^\varepsilon(s)
\]

\[
- g_1(z_2^\varepsilon(s), v_2^\varepsilon(s))\nabla z_2^\varepsilon(s) - g_2(z_2^\varepsilon(s), v_2^\varepsilon(s))\nabla v_2^\varepsilon(s)\|_p ds
\]

\[
\leq \int_0^t \left[\|g_1(z_1^\varepsilon(s), v_1^\varepsilon(s)) - g_1(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_p \nabla z_1^\varepsilon(s)\|_p
\]

\[
+ \|g_1(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_p \nabla z_2^\varepsilon(s)\|_p
\]

\[
+ \|g_2(z_1^\varepsilon(s), v_1^\varepsilon(s)) - g_2(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_p \nabla v_1^\varepsilon(s)\|_p
\]

\[
+ \|g_2(z_2^\varepsilon(s), v_2^\varepsilon(s))\|_p \nabla v_2^\varepsilon(s)\|_p\right] ds
\]

\[
\leq \int_0^t \left\{\|L(z_1^\varepsilon(s) - z_2^\varepsilon(s)) + \|v_1^\varepsilon(s) - v_2^\varepsilon(s)\| \|\nabla z_1^\varepsilon(s)\|_{1,p} + \|v_1^\varepsilon(s)\|_{1,p}\right\}
\]

\[
+ (L \chi + L\|z_2^\varepsilon(s)\|_{1,p} + K)\|\nabla z_1^\varepsilon(s) - z_2^\varepsilon(s)\|_{1,p} + \|v_1^\varepsilon(s) - v_2^\varepsilon(s)\|_{1,p}\right\} ds
\]

\[
\leq C(t^{(p-2)/p} + t^{(p-1)/p}) \left\{ \int_0^t \|z_1^\varepsilon(s) - z_2^\varepsilon(s)\|_{1,p}^p ds \right\}^{1/p}
\]

\[
+ C(t^{(p-2)/p} + t^{(p-1)/p}) \left\{ \int_0^t \|v_1^\varepsilon(s) - v_2^\varepsilon(s)\|_{1,p}^p ds \right\}^{1/p},
\]

where $g_1(z_1^\varepsilon, v_1^\varepsilon) = \frac{\partial g(z_1^\varepsilon, v_1^\varepsilon)}{\partial z_1^\varepsilon}$, $g_2(z_1^\varepsilon, v_1^\varepsilon) = \frac{\partial g(z_1^\varepsilon, v_1^\varepsilon)}{\partial v_1^\varepsilon}$, for $i = 1, 2$ and $C$ is a constant determined by $\chi$, $p$, $K$, $M$ and $L$.

Applying Gronwall’s inequality, we obtain

\[
\sup_{t \in [0,T]} \|v_i^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p} \leq Ce^{(T^{p-1} + T^{p-1}) (T^{2p-1} + T^{p-2})} \int_0^T \|z_1^\varepsilon(s) - z_2^\varepsilon(s)\|_{1,p} ds.
\]

(3.14)
Concerning \( u_1^\varepsilon - u_2^\varepsilon \), it satisfies
\[
(u_1^\varepsilon - u_2^\varepsilon)_t - \Delta(u_1^\varepsilon - u_2^\varepsilon) = -\text{div}(u_1^\varepsilon \nabla f(v_1^\varepsilon)) - \text{div}(u_2^\varepsilon \nabla f(v_2^\varepsilon))
\]
- \text{div}(u_1^\varepsilon \nabla f(v_1^\varepsilon)) - \text{div}(u_2^\varepsilon \nabla f(v_2^\varepsilon)).

Take the decomposition \( u_1^\varepsilon - u_2^\varepsilon \) as \( u_1^\varepsilon - u_2^\varepsilon = U_{\varepsilon 1} + U_{\varepsilon 2} \), where \( U_{\varepsilon 1} \) is the solution of
\[
\begin{align*}
\partial_t U_{\varepsilon 1} - \Delta U_{\varepsilon 1} &= -\text{div}(u_1^\varepsilon \nabla f(v_1^\varepsilon)) \quad \text{in } \Omega \times (0, T), \\
\frac{\partial U_{\varepsilon 1}}{\partial \nu} - u_1^\varepsilon \nabla f(v_1^\varepsilon) \cdot \nu &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
U_{\varepsilon 1}|_{t=0} &= 0 \quad \text{in } \Omega.
\end{align*}
\]

Direct calculation gives
\[
\|\nabla (f(v_1^\varepsilon(t))) - f(v_2^\varepsilon(t))\|_p \\
= \|((f'((v_1^\varepsilon(t))) - f'((v_2^\varepsilon(t))))\nabla v_1^\varepsilon(t) + f'(v_2^\varepsilon(t))(\nabla v_1^\varepsilon(t) - \nabla v_2^\varepsilon(t))\|_p \\
\leq L\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p}\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p} + (L\|v_2^\varepsilon(t)\|_{1,p} + K)\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p} \\
\leq (2L + K)\|v_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p}.
\]

Then by Proposition 1, we have
\[
\int_0^T \| U_{\varepsilon 1}(s) \|_{1,p}^p \, ds \leq C \int_0^T \| u_1^\varepsilon(s) \|_{1,p} \| \nabla f(v_1^\varepsilon(s)) - \nabla f(v_2^\varepsilon(s)) \|_{1,p} \, ds \\
\leq C \int_0^T \| u_1^\varepsilon(s) \|_{1,p} \| 2L + K \|_{1,p} \| u_1^\varepsilon(s) - v_2^\varepsilon(s) \|_{1,p} \, ds \\
\leq C(2L + K)\sup_{s\in[0,T]}\|v_1^\varepsilon(s) - v_2^\varepsilon(s)\|_{1,p}^p \\
\tag{3.15}
\]

Next, \( U_{\varepsilon 2} \) is the solution of
\[
\begin{align*}
\partial_t U_{\varepsilon 2} - \Delta U_{\varepsilon 2} &= -\text{div}((u_1^\varepsilon - u_2^\varepsilon) \nabla f(v_2^\varepsilon)) \quad \text{in } \Omega \times (0, T), \\
\frac{\partial U_{\varepsilon 2}}{\partial \nu} - (u_1^\varepsilon - u_2^\varepsilon) \nabla f(v_2^\varepsilon) \cdot \nu &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
U_{\varepsilon 2}|_{t=0} &= 0 \quad \text{in } \Omega.
\end{align*}
\]

By Lemma 2.2 and (3.8), there exists a constant \( C \) such that
\[
\| u_1^\varepsilon - u_2^\varepsilon \|_{L^\infty(0,T;L^\infty(\Omega))} \leq C \sup_{t\in[0,T]}\|u_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p}.
\]

Combining this estimate with Proposition 1 and (3.12), we deduce that
\[
\int_0^t \| U_{\varepsilon 2}(s) \|_{1,p}^p \, ds \leq C \int_0^t \| u_1^\varepsilon(s) - u_2^\varepsilon(s) \|_{1,p}^p \| \nabla f(v_2^\varepsilon(s)) \|_{1,p}^p \, ds \\
\leq CT \sup_{t\in[0,T]}\|u_1^\varepsilon(t) - v_2^\varepsilon(t)\|_{1,p}^p, \\
\tag{3.16}
\]

where \( C \) depends on \( M, L, K \) and \( \chi \).

Combining (3.14), (3.15) and (3.16), we get
\[
\int_0^T \| u_1^\varepsilon(s) - u_2^\varepsilon(s) \|_{1,p}^p \, ds \leq C e^{(T^p + T^{p-1})} (T^{2p} + T^{p-1}) \int_0^T \| z_1^\varepsilon(s) - z_2^\varepsilon(s) \|_{1,p}^p \, ds.
\tag{3.17}
\]

Hence the mapping \( B \) is a contraction on \( \mathcal{B}_M \) when \( T \) is small enough.
Finally, we apply Banach’s fixed point theorem to get a unique fixed point of $B$, which means there exists a constant $T_0 > 0$, such that for any $\varepsilon > 0$, system (1.1) with (1.2) and (1.3) admits a unique weak solution $(u_\varepsilon, v_\varepsilon)$ in $[0, T_0]$.

**Remark 2.** The above construction gives the solution $(u_\varepsilon, v_\varepsilon)$ which satisfies, instead of the Neumann boundary condition (1.2), the following boundary condition (in the weak sense as in (1.7))

$$
\begin{align*}
\frac{\partial u_\varepsilon}{\partial \nu} - u_\varepsilon \frac{\partial}{\partial \nu} f(v_\varepsilon) &= 0 \quad \text{on} \quad \partial \Omega \times (0, T_0), \\
\frac{\partial v_\varepsilon}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega \times (0, T_0).
\end{align*}
$$

However, for each $\varepsilon$ fixed, standard parabolic regularity theory implies that $u_\varepsilon$ and $v_\varepsilon$ are smooth enough in $\Omega \times (0, T_0)$. Hence these boundary conditions are satisfied in the classical sense. Then the Neumann boundary condition (1.2) is also satisfied by $(u_\varepsilon, v_\varepsilon)$.

4. **Further properties of $(u_\varepsilon, v_\varepsilon)$**. In the following, $(u_\varepsilon, v_\varepsilon)$ denotes the solution constructed in the previous section. Since for each $\varepsilon > 0$ fixed, (1.1) is a standard parabolic equation, $(u_\varepsilon, v_\varepsilon)$ is smooth in $\Omega \times (0, T_0)$ and continuous up to $t = 0$. However, such regularity may not be uniform in $\varepsilon$.

In this section we establish some regularity for $(u_\varepsilon, v_\varepsilon)$, which is uniform as $\varepsilon \to 0$.

First, we prove a uniform Hölder continuity in $t$ of $u_\varepsilon(t)$ with respect to $W^{k,p}(\Omega)$ norms, for any $k \in [0, 1)$.

Before proceeding to the proof, we recall a lemma about the mapping properties of the heat semigroup, which is taken from Taylor [17].

**Lemma 4.1.** *(Taylor [17, p. 317])* For any $0 < t \leq 1$, $p \geq q$, $s \geq r$, the heat semigroup associated to $\Delta$ satisfies

$$
e^{t\Delta} : W^{r,q}(\Omega) \to W^{s,p}(\Omega) \text{ with norm } C t^{-\beta},$$

where

$$
-\beta = -\frac{n}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{1}{2} (s - r).
$$

**Proposition 3.** For any $k \in [0, 1)$, there exists a constant $C$ such that

$$
\|u_\varepsilon(t + \Delta t) - u_\varepsilon(t)\|_{k,p} \leq C(\Delta t)^{(1-k)/2}, \quad \forall t \in [0, T_0), \quad \Delta t \in (0, T_0 - t).
$$

**Proof.** Applying the semigroup theory, we have

$$
u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \text{div}(u_\varepsilon(s) \nabla f(v_\varepsilon(s))) ds.
$$

By Minkovski inequality, for any $u \in L^p(0, t; W^{k,p}(\Omega))$, we have

$$
\left\| \int_0^t u(s) ds \right\|_{k,p} = \left( \int_{\Omega} \left( \int_0^t u(x,s) ds \right)^p dx \right)^{1/p} + \left( \int_{\Omega} \int_{\Omega} \frac{|\int_0^t u(x,s) ds - \int_0^t u(y,s) ds|^p}{|x - y|^{n+kp}} dxdy \right)^{1/p}
$$
\[
\leq \int_0^t \left( \int_{\Omega} |u(x,s)|^p dx \right)^{1/p} ds + \int_0^t \left( \int_{\Omega \times \Omega} \frac{|u(x,s) - u(y,s)|^p}{|x-y|^{n+1}} dxdy \right)^{1/p} ds
\]
\[
\leq \int_0^t \|u(s)\|_{k,p} ds.
\]

Hence by Lemma 4.1, we have
\[
\|e^{(t+\Delta t)\Delta} u_0 - e^{t\Delta} u_0\|_{k,p} = \left\| \int_t^{t+\Delta t} \Delta e^{\tau \Delta} u_0 d\tau \right\|_{k,p}
\]
\[
\leq C \int_t^{t+\Delta t} \tau^{-\frac{1+k}{p}} \|u_0\|_{1,p} d\tau
\]
\[
\leq C(\Delta t)^{(1-k)/2}.
\]

Next,
\[
\left\| \int_0^{t+\Delta t} e^{(t+\Delta t-s)\Delta} \text{div}(u_x(s)\nabla f(v_x(s))) ds - \int_0^t e^{(t-s)\Delta} \text{div}(u_x(s)\nabla f(v_x(s))) ds \right\|_{k,p}
\]
\[
\leq \int_t^{t+\Delta t} \|e^{(t+\Delta t-s)\Delta} \text{div}(u_x(s)\nabla f(v_x(s)))\|_{k,p} ds
\]
\[
+ \int_0^t \|e^{(t+\Delta t-s)\Delta} - e^{(t-s)\Delta}\text{div}(u_x(s)\nabla f(v_x(s)))\|_{k,p} ds.
\]

The first term in the right-hand side of (4.1) is estimated by applying Lemma 4.1, which gives
\[
I \leq C \int_t^{t+\Delta t} (t + \Delta t - s)^{-(1+k)/2} \|u_x(s)\nabla f(v_x(s))\|_p ds \leq C(\Delta t)^{(1-k)/2},
\]  
(4.2)

where C is determined by k, L, M and \(\chi\).

For the second term J, since \(u_x \nabla f(v_x) \in L^\infty(0,T;L^p(\Omega))\), we have
\[
J \leq C \int_0^t \int_t^{t+\Delta t} \|\Delta e^{(t-s)\Delta} \text{div}(u_x(s)\nabla f(v_x(s)))\|_{k,p} d\tau ds
\]
\[
\leq C \int_0^t \int_t^{t+\Delta t} (t - s)^{-\frac{3+k}{2}} \|u_x(s)\nabla f(v_x(s))\|_p d\tau ds
\]  
(4.3)
\[
\leq C((\Delta t)^{1-k/2} + t^{(1-k)/2} - (t + \Delta t)^{(1-k)/2}) \leq C(\Delta t)^{(1-k)/2},
\]

where C is determined by k, L, M and \(\chi\).

Combining (4.1), (4.2) and (4.3), we finish the proof.

Combining this proposition with Sobolev embedding theorem, we obtain the uniform Hölder continuity of \(u_x\).

**Corollary 1.** There exists an \(\alpha \in (0,1)\) depending on \(n\) and \(p\) such that \(u_x\) are uniformly bounded in \(C^{\alpha,\alpha/2}(\Omega^n)\).

Note that interior Hölder continuity of \(u_x\) can also be proved by utilizing Moser iteration and Moser’s Harnack inequality, see [8, Theorem 10.1].

Next, we show that \(v_x\) are uniformly Hölder continuous in \(t\).
Proposition 4. There exist two constants $\alpha \in (0,1)$ and $C$ independent of $\varepsilon$ such that for any $x \in \Omega$ and $0 \leq t_1 < t_2 < T_0$,

$$|v_\varepsilon(x,t_2) - v_\varepsilon(x,t_1)| \leq C|t_2 - t_1|^\alpha.$$ 

Proof. Without loss of generality assume $t_1 = 0$.

The Neumann heat kernel of $\partial_t - \varepsilon \Delta$ is $G(x,y,\varepsilon(t-s))$. Thus

$$v_\varepsilon(x,t) = \int_\Omega G(x,y,\varepsilon t)v_0(y)dy - \int_0^t \int_\Omega G(x,y,\varepsilon(t-s))g(u_\varepsilon(y,s), v_\varepsilon(y,s))dyds.$$ 

By [10, Theorem 10.2], for each $x \in \Omega$, we have

$$\int_\Omega G(x,y,\varepsilon t)dy = 1.$$ 

Therefore, for any $x \in \Omega$,

$$|v_\varepsilon(x,t) - v_0(x)| = \left|\int_\Omega G(x,y,\varepsilon t)v_0(y)dy - \int_\Omega G(x,y,\varepsilon t)v_0(x)dy - \int_0^t \int_\Omega G(x,y,\varepsilon(t-s))g(u_\varepsilon(y,s), v_\varepsilon(y,s))dyds\right|$$

$$\leq \left|\int_\Omega G(x,y,\varepsilon t)|v_0(y) - v_0(x)|dy\right|$$

$$+ \left|\int_0^t \int_\Omega G(x,y,\varepsilon(t-s))g(u_\varepsilon(y,s), v_\varepsilon(y,s))dyds\right|.$$ 

First

$$I \leq \int_{B(x,t^\frac{4}{n})} G(x,y,\varepsilon t)|v_0(y) - v_0(x)|dy + \int_{\Omega \setminus B(x,t^\frac{4}{n})} G(x,y,\varepsilon t)|v_0(y) - v_0(x)|dy.$$ 

Since $W^{1,p}(\Omega) \hookrightarrow C^{1-\frac{2}{n}}(\Omega)$, we have

$$I_1 \leq C \int_{B(x,t^\frac{4}{n})} G(x,y,\varepsilon t)|y-x|^{1-\frac{2}{n}}dy \leq Ct^{\frac{4}{n}(1-\frac{2}{n})}.$$ 

Furthermore, by [1], we have the Gaussian bounds on heat kernel

$$\frac{e^{-c|x-y|^2/(t-s)}}{c(t-s)^\frac{n}{2}} \leq G(x,y,t-s) \leq c e^{-c|x-y|^2/(t-s)/(t-s)^\frac{n}{2}},$$

where $c$ is a constant. Therefore

$$I_2 \leq C\|v_0\|_\infty \int_{R^n \setminus B(x,t^\frac{4}{n})} (\varepsilon t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{c(t-s)}}dy$$

$$\leq C \int_{R^n \setminus B(0, Cc^{-\frac{1}{2}}(t^\frac{4}{n})^{-\frac{1}{2}})} e^{-|z|^2}dz$$

$$\leq Ce^{-Ce^{-1}t^{-\frac{1}{2}}} \leq Ct^\frac{1}{2},$$

where the constant $C$ depends on $n$. 

Next, since \( g(x, y) \) is a \( C^2 \) lipschitz-continuous mapping, we have

\[
\left| \int_0^t \int_\Omega G(x, y, \varepsilon(t - s)) g(u_\varepsilon(y, s), v_\varepsilon(y, s)) dyds \right|
\leq \int_0^t \int_{\mathbb{R}^n} c[\varepsilon(t - s)]^{-\frac{\alpha}{2}} e^{-\frac{|x - y|^2}{\varepsilon(t - s)}} (L\|u_\varepsilon(s)\|_\infty + L\|v_\varepsilon(s)\|_\infty + K) dyds
\leq Ct,
\]

where \( C \) is a constant determined by \( n, L, K, \|u_0\|_{1,p} \) and \( \|v_0\|_{1,p} \). These relations are summarized as the conclusion.

Since \( v_\varepsilon \in L^\infty(0, T; W^{1,p}(\Omega)) \), combining this proposition with Sobolev embedding theorem we obtain

**Corollary 2.** There exists an \( \alpha \in (0, 1) \) such that \( v_\varepsilon \) are uniformly bounded in \( C^{\alpha,\alpha/2}(\Omega_{T_0}) \).

5. **Proof of Theorem 1.2: Well-posedness.**

**Proof.** We apply Banach's fixed point theorem on the Banach space \( L^p(0, T; W^{1,p}(\Omega)) \). The proof is divided into three steps.

**Step 1.** Given a function \( z \in L^p(0, T; W^{1,p}(\Omega)) \), we solve

\[
\partial_t v = g(z, v)
\]

with the initial value (1.3). This is equivalent to the integral equation

\[
v(x, t) = v_0(x) + \int_0^t g(z(x, s), v(x, s)) ds, \quad \forall x \in \Omega.
\]

By Minkovski inequality,

\[
\|v(t)\|_{1,p} \leq \|v_0\|_{1,p} + C \int_0^t (\|z(s)\|_{2,p}^2 + \|v(s)\|_{2,p}^2) ds + Ct.
\]

Similar to (3.2), we have

\[
\sup_{t \in (0, T)} \|v(t)\|_{1,p} \leq 2\|v_0\|_{1,p} + CT + CT(p-2)/p \left\{ \int_0^T \|z(s)\|_{1,p}^p ds \right\}^{2/p}.
\]

Furthermore, it is easy to deduce that \( v \in C(0, T_0; W^{1,p}(\Omega)) \).

**Step 2.** With \( v \) defined as in (5.1), we solve the equation for \( u \). Similar to (3.10), we get

\[
\int_0^t \|u(s)\|_{1,p}^p ds \leq \left( \|u_0\|_{1,p}^p + C \right) t,
\]

where \( C \) is the same as (3.10) with \( \|z\|_{L^p(0, T; W^{1,p}(\Omega))} \) replaced by \( \|z\|_{L^p(0, T; W^{1,p}(\Omega))} \).

**Step 3.** Define a map \( T : L^p(0, T; W^{1,p}(\Omega)) \rightarrow L^p(0, T; W^{1,p}(\Omega)) \) as \( T(z) = u \). Choose \( M \) as in (3.11). By (5.3), if \( \|z\|_{L^p(0, T; W^{1,p}(\Omega))} \leq M \) and \( T \) is small enough, it holds that

\[
T : \mathcal{B}_M \rightarrow \mathcal{B}_M,
\]

where

\[
\mathcal{B}_M := \{ \varphi \in L^p(0, T; W^{1,p}(\Omega)) : \|\varphi\|_{L^p(0, T; W^{1,p}(\Omega))} \leq M, \varphi(0) = u_0 \}.
\]

We show when \( T \) is small enough, the mapping \( T \) is a contraction on \( \mathcal{B}_M \). Consider two functions \( z^1, z^2 \in \mathcal{B}_M \) and denote the images by \( u^1 = Tz^1, u^2 = Tz^2 \),

\[
\left| \int_0^t \int_\Omega G(x, y, \varepsilon(t - s)) g(u^1(y, s), v^1(y, s)) dyds \right|
\leq \int_0^t \int_{\mathbb{R}^n} c[\varepsilon(t - s)]^{-\frac{\alpha}{2}} e^{-\frac{|x - y|^2}{\varepsilon(t - s)}} (L\|u_\varepsilon(s)\|_\infty + L\|v_\varepsilon(s)\|_\infty + K) dyds
\leq Ct,
\]

as required. \( \square \)
respectively. The corresponding solutions of the equation (5.1) are denoted by \( v^1 \) and \( v^2 \). By (5.4), we have

\[
\sup_{t \in [0,T]} \|v^i(t)\|_{W^{1,p}(\Omega)} \leq \chi, \tag{5.5}
\]

where \( \chi \) is the constant in (3.12). Similar to (3.14), we have

\[
\sup_{t \in [0,T]} \|v^1(t) - v^2(t)\|_{W^{1,p}(\Omega)} \leq C e^{(T^p + T^p - 1)} (T^{2p - 1} + T^{p - 1}) \int_0^T \|z^1(s) - z^2(s)\|_{W^{1,p}} ds.
\]

It follows that

\[
\int_0^T \|u^1(s) - u^2(s)\|_{W^{1,p}} ds \leq C e^{(T^p + T^p - 1)} (T^{2p} + T^{p - 1}) \int_0^T \|z^1(s) - z^2(s)\|_{W^{1,p}} ds,
\]

where the constant \( C \) is the same as in (3.17). Hence the mapping \( T \) is a contraction on \( \mathfrak{B}_M \) with the same \( T_0 \) as in Theorem 1.1.

Applying Banach’s fixed point theorem, we get the unique fixed point of \( T \), which is the unique weak solution \( (u, v) \) of (1.4) in \([0, T_0]\).

6. Proof of Theorem 1.3. In order to prove Theorem 1.3, we need the following lemma. Denoting \( K_1 = \max\{f'(0), f''(0)\} \), we have

**Lemma 6.1.** For any \( v, v' \in W^{2,p}(\Omega) \) with \( p > n \), we have

\[
\|f(v) - f(v')\|_{L^p} \leq C \|v - v'\|_{L^p},
\]

where \( C \) is a constant dependent on \( \|v\|_{L^p}, \|v'\|_{L^p}, K_1 \) and \( L \).

**Proof.** Since the Lipschitz constant of \( f \) is \( L \),

\[
\|f(v) - f(v')\|_{L^p} \leq L \|v - v'\|_{L^p}.
\]

Applying Sobolev embedding theorem and Young inequality, for \( i = 1, \cdots, n \), we have

\[
\|f(v)_{x_i} - f(v')_{x_i}\|_p = \|f'(v)v_{x_i} - f'(v')v'_{x_i}\|_p
\]

\[
= \|(f'(v) - f'(v'))v_{x_i} + (v_{x_i} - v'_{x_i})f'(v')\|_p
\]

\[
\leq C \|v - v'\|_{W^{1,p}},
\]

where \( C \) depends on \( L, \|v\|_{W^{1,p}}, \|v'\|_{W^{1,p}} \) and \( K_1 \).

Next, for \( i, j = 1, \cdots, n \), we have

\[
\|f(v)_{x_i x_j} - f(v')_{x_i x_j}\|_p = \|f''(v)v_{x_i x_j} - f''(v')v'_{x_i x_j} + f'(v)v_{x_i}v'_{x_j} - f'(v')v'_{x_i}v'_{x_j}\|_p
\]

\[
\leq \|(f''(v) - f''(v'))v_{x_i x_j} + [(v_{x_i} - v'_{x_i})v_{x_j} + (v_{x_j} - v'_{x_j})v'_{x_i}]f''(v')
\]

\[
+ (f'(v) - f'(v'))v_{x_i}v'_{x_j} + (v_{x_i} - v'_{x_i})f'(v')\|_p
\]

\[
\leq C \|v - v'\|_{L^2},
\]

where \( C \) depends on \( L, \|v\|_{L^2}, \|v'\|_{L^2} \) and \( K_1 \). These relations are summarized as the conclusion.

Denoting \( K_2 = \max\{g(0,0), g_i(0,0), g_{ij}(0,0)\} \), for \( i, j = 1, 2 \), we have

**Lemma 6.2.** For any \( u, u' \in L^p(0,T;W^{2,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \) and \( v, v' \in L^\infty(0,T;W^{2,p}(\Omega)) \) with \( p > \max\{2, n\} \), suppose there exists a constant \( C \) such that

\[
\sup_{s \in [0,T]} \|u(s) - u'(s)\|_\infty \leq C \sup_{s \in [0,T]} \|v(s) - v'(s)\|_{1,p}.
\]
Then we have
\[
\int_0^t \|g(u(s), v(s)) - g(u'(s), v'(s))\|_{2,p} ds \leq C(t) \left\{ \int_0^t \|u(s) - u'(s)\|_{2,p} ds \right\}^{1/p} + C(t) \sup_{s \in [0,T]} \|v(s) - v'(s)\|_{1,p}.
\]

Proof. Direct calculation gives
\[
\int_0^t \|g(u(s), v(s)) - g(u'(s), v'(s))\|_p ds \leq L \left( \int_0^t \|u(s) - u'(s)\|_p ds + \int_0^t \|v(s) - v'(s)\|_p ds \right).
\]

Similar to (3.13), we have
\[
\int_0^t \|\nabla g(u(s), v(s)) - \nabla g(u'(s), v'(s))\|_p ds \leq C(t) \left\{ \int_0^t \|u(s) - u'(s)\|_{2,p} ds \right\}^{1/p} \\sup_{s \in [0,T]} \|v(s) - v'(s)\|_{1,p}^{1/p}.
\]

On the other hand, for \(i, j = 1, \ldots, n\), we have
\[
\int_0^t \|\nabla^2 g(u(s), v(s)) - \nabla^2 g(u'(s), v'(s))\| ds \leq C(t) \left\{ \int_0^t \|u(s) - u'(s)\|_{2,p} ds \right\}^{1/p} \\sup_{s \in [0,T]} \|v(s) - v'(s)\|_{1,p}^{1/p} + C(t) \left\{ \int_0^t \|v(s) - v'(s)\|_{2,p} ds \right\}^{1/p}.
\]

These relations are summarized as the conclusion. \(\square\)

Now it is time to prove Theorem 1.3.

Proof. We apply Banach’s fixed point theorem on the Banach space \(L^p(0, T; W^{2,p}(Ω))\). The proof is divided into three steps.

**Step 1.** By Theorem 1.2, there exists a unique solution \(u \in L^p(0, T_0; W^{1,p}(Ω))\), \(v \in L^\infty(0, T_0; W^{1,p}(Ω))\). For any \(T \leq T_0\), take a function \(\tau \in L^p((0, T; W^{2,p}(Ω)))\) satisfying
\[
\tau(x, t) = u(x, t), \text{ in } L^p(0, T; W^{1,p}(Ω)) \cap L^\infty(0, T; L^\infty(Ω))
\]
and solution \(\tilde{v}(x, t)\) to
\[
\frac{\partial \tilde{v}}{\partial t} = g(\tau(x, t), \tilde{v}(x, t)) \text{ in } Ω^T.
\] (6.1)

Then \(\tilde{v}(x, t) = v(x, t)\) in \(L^\infty(0, T; W^{1,p}(Ω))\) and it satisfies (5.5).

The equation (6.1) is equivalent to the integral equation
\[
\tilde{v}(x, t) = v_0(x) + \int_0^t g(\tau(x, s), \tilde{v}(x, s)) ds, \quad \forall x \in Ω.
\]

Direct calculation gives
\[
\int_0^t \|g(\tau(s), \tilde{v}(s))\|_p ds \leq L \left( \int_0^t \|\tau(s)\|_p ds + \int_0^t \|\tilde{v}(s)\|_p ds \right) + K_2 |Ω|^{1/p} t.
\]
For $0 \leq i, j \leq n$, by Hölder inequality, we have
\[
\int_0^t \|g_1(\tau(s), \tilde{v}(s))r_i(s) + g_2(\tau(s), \tilde{v}(s))\tilde{v}_i(s)\|_p ds \\
\leq (L\|\tau + \tilde{v}\|_{L^\infty(\Omega \times (0,t))} + K_2)(\int_0^t \|\tau(s)\|_{1,p} ds + \int_0^t \|\tilde{v}(s)\|_{1,p} ds)
\]
and
\[
\int_0^t \|g_{11}(\tau(s), \tilde{v}(s))r_i(s)r_j(s) + g_{12}(\tau(s), \tilde{v}(s))r_i(s)\tilde{v}_j(s) + g_{11}(\tau(s), \tilde{v}(s))\tilde{v}_i(s)r_j(s) + g_{12}(\tau(s), \tilde{v}(s))\tilde{v}_i(s)\tilde{v}_j(s)\|_p ds \\
\leq C\int_0^t (\|\tau(s) + \tilde{v}(s)\|_{\infty} + K_2)\left(\|\tau(s)\|_{1,p} \|\tau(s)\|_{2,p} + (\|\tilde{v}(s)\|_{1,p} + 1)\|\tau(s)\|_{2,p} + \|\tilde{v}(s)\|_{1,p} \|\tilde{v}(s)\|_{2,p} + (\|\tau(s)\|_{1,p} + 1)\|\tilde{v}(s)\|_{2,p}\right) ds
\]
\[
\leq C(t^{(p-2)/p} + t^{(p-1)/p})\left(\int_0^t \|\tau(s)\|_{2,p}^p ds \right)^{1/p} \\
+ C(t^{(p-2)/p} + t^{(p-1)/p})\left(\int_0^t \|\tilde{v}(s)\|_{2,p}^p ds \right)^{1/p} + C(t).
\]
Hence
\[
\|\tilde{v}(t)\|_{2,p} \leq \|v_0\|_{2,p} + \int_0^t \|g(\tau(s), \tilde{v}(s))\|_{2,p} ds \\
\leq \|v_0\|_{2,p} + C(t^{(p-2)/p} + t^{(p-1)/p})\left(\int_0^t \|\tau(s)\|_{2,p}^p ds \right)^{1/p} \\
+ C(t^{(p-2)/p} + t^{(p-1)/p})\left(\int_0^t \|\tilde{v}(s)\|_{2,p}^p ds \right)^{1/p} + C(t).
\]
By Gronwall’s inequality,
\[
\|\tilde{v}(t)\|_{2,p}^p \leq \left( C e^{(Tp + Tp^r)(T^{p-1} + T^{p-2})} + 1 \right) \|v_0\|_{2,p}^p \\
+ C e^{(Tp + Tp^r)(T^{2p-1} + T^{p-2})} \int_0^t \|\tau(s)\|_{2,p}^p ds + C(t).
\]
(6.2)

**Step 2.** With $\tilde{v}$ defined as in (6.1), we study the equation for $\tilde{u}$. Take the decomposition $\tilde{u} = \kappa_1 + \kappa_2$, where $\kappa_1$ is the solution of
\[
\begin{cases}
(\kappa_1)_t - \Delta \kappa_1 = 0 & \text{in } \Omega \times [0,T], \\
\frac{\partial \kappa_1}{\partial n} = 0 & \text{on } \partial \Omega \times [0,T], \\
\kappa_1(x,0) = u_0 & \text{in } \Omega
\end{cases}
\]
and $\kappa_2$ is the solution of
\[
\begin{cases}
(\kappa_2)_t - \Delta \kappa_2 = -\text{div}(u \nabla f(v)) & \text{in } \Omega \times [0,T], \\
\frac{\partial \kappa_2}{\partial n} - u \nabla f(v) \cdot \nu = 0 & \text{on } \partial \Omega \times [0,T], \\
\kappa_2(x,0) = 0 & \text{in } \Omega
\end{cases}
\]
(6.3)

By the semigroup theory, we have
\[
\|\kappa_1(t)\|_{2,p} \leq \|u_0\|_{2,p}, \quad \forall t > 0.
\]
On the other hand, applying Proposition 2, Sobolev embedding theorem and Lemma 6.1 to (6.3), we get
\[
\int_0^t \| \kappa_2(s) \|_{2,p}^p \, ds \leq C \int_0^t \| \tilde{u}(s) \nabla f(\tilde{v}(s)) \|_{1,p}^p \, ds
\]
\[
\leq C \int_0^t \left[ \| \tilde{u}(s) \|_\infty^p \| f(\tilde{v}(s)) \|_{1,p}^p + \| \tilde{u}(s) \|_{1,p}^p \| f(\tilde{v}(s)) \|_{2,p}^p + \| \tilde{u}(s) \|_\infty \| f(\tilde{v}(s)) \|_{2,p}^p \right] \, ds
\]
\[
\leq C(t) \left( \sup_{s \in [0,t]} \| \tilde{v}(s) \|_{2,p}^p + 1 \right),
\]
where \( C \) depends \( \Omega, p, n \) and \( \chi \). Combining this with (6.2) leads to
\[
\int_0^t \| \tilde{u}(s) \|_{2,p}^p \, ds \leq \| u_0 \|_{2,p}^p t + C(t) \| v_0 \|_{2,p}^p + C(t) \int_0^t \| \tau(s) \|_{2,p}^p \, ds + C(t). \tag{6.4}
\]

**Step 3.** By the previous two steps, we can define a map \( H : L^p(0,T;W^{2,p}(\Omega)) \to L^p(0,T;W^{2,p}(\Omega)) \) as \( H \tau = \tilde{u} \). Choose \( M \) as in (3.9). By (6.4), if \( \| \tau \|_{L^p(0,T;W^{2,p}(\Omega))} \leq M \) and \( T \) is small enough, it holds that
\[
H : \mathfrak{B}_M \to \mathfrak{B}_M,
\]
where
\[
\mathfrak{B}_M := \{ \varphi \in L^p(0,T;W^{2,p}(\Omega)) : \| \varphi \|_{L^p(0,T;W^{2,p}(\Omega))} \leq M, \ \varphi(0) = u_0 \}. \tag{6.5}
\]

We show when \( T \) is small enough, the mapping \( H \) is a contraction on \( \mathfrak{B}_M \). To this aim, consider two functions \( \tau_1, \tau_2 \in \mathfrak{B}_M \) and denote the images by \( u'_1 = H \tau_1, \ u'_2 = H \tau_2 \). The corresponding solutions of the equation (6.1) are denoted by \( v'_1 \) and \( v'_2 \), respectively. By (6.2) and (6.5), we find a constant \( \lambda \), depending on \( M \) and \( \| v_0 \|_{W^{2,p}(\Omega)} \), such that
\[
\sup_{t \in [0,T]} \| v'_i(t) \|_{2,p} \leq \lambda, \ i = 1, 2. \tag{6.6}
\]
By the semigroup theory, Lemma 6.2 and (3.14), if \( T \) is small enough, we deduce that, for all \( t \in [0,T] \),
\[
\| v'_1(t) - v'_2(t) \|_{2,p} \leq C \int_0^t \| g(\tau_1(s), v'_1(s)) - g(\tau_2(s), v'_2(s)) \|_{2,p} \, ds
\]
\[
\leq C(t) \left\{ \int_0^t \| \tau_1(s) - \tau_2(s) \|_{2,p}^p \, ds \right\}^{1/p} + C(t) \left\{ \int_0^t \| v'_1(s) - v'_2(s) \|_{2,p}^p \, ds \right\}^{1/p},
\]
where \( C \) depends on \( L, M, \chi, p \) and \( \lambda \). Applying Gronwall’s inequality, we get
\[
\| v'_1(t) - v'_2(t) \|_{2,p} \leq C(t) \int_0^t \| \tau_1(s) - \tau_2(s) \|_{2,p}^p \, ds. \tag{6.7}
\]

Next we come to the estimate of \( u'_1 - u'_2 \), which satisfies\n\[
(u'_1 - u'_2)_t - \Delta (u'_1 - u'_2) = -\text{div}(u'_1 \nabla f(v'_1) - u'_2 \nabla f(v'_2)) = -\text{div}(u'_1 - u'_2) \nabla f(v'_1) - \text{div}(u'_2 \nabla f(v'_1) - f(v'_2)).
\]
Take the decomposition \( u'_1 = u'_2 \) as \( u'_1 - u'_2 = \kappa'_1 + \kappa'_2 \), where \( \kappa'_1 \) is the solution of
\[
\begin{cases}
\partial_t \kappa'_1 - \Delta \kappa'_1 = -\text{div}((u'_1 - u'_2) \nabla f(v'_1)) & \text{in } \Omega \times (0,T), \\
\frac{\partial \kappa'_1}{\partial \mathbf{n}} - (u'_1 - u'_2) \nabla f(v'_1) \cdot \mathbf{n} = 0 & \text{on } \partial \Omega \times (0,T), \\
\kappa'_1|_{t=0} = 0 & \text{in } \Omega.
\end{cases}
\]
By (6.6), Lemma 6.1 and Sobolev embedding theorem, we have
\[ \int_0^t \| \kappa_2'(s) \|^p_{2,p} ds \leq C \int_0^t \left\| (u'_1(s) - u'_2(s)) \nabla f(v'_1(s)) \right\|^p_{1,p} ds \]
\[ \leq C \int_0^t \| u'_1(s) - u'_2(s) \|^p_{1,p} \| f(v'_1(s)) \|^p_{2,p} ds \]
\[ \leq C t \sup_{s \in [0,t]} \| v'_1(s) - v'_2(s) \|^p_{2,p}, \] (6.8)
where \( C \) is a constant dependent of \( M, K_1, \lambda, L \) and \( \chi \).

The function \( \kappa_2' \) is the solution of
\[ \begin{align*}
\partial_s \kappa_2' - \Delta \kappa_2' & = -\text{div}(u'_2 \nabla (f(v'_1) - f(v'_2))) \quad \text{in } \Omega \times (0,T), \\
\partial \kappa_2' - u'_2 & \cdot \nabla (f(v'_1) - f(v'_2)) \cdot \nu = 0 \quad \text{on } \partial \Omega \times (0,T), \\
\kappa_2'|_{t=0} & = 0 \quad \text{in } \Omega.
\end{align*} \]
It follows from Lemma 6.1 that
\[ \int_0^t \| \kappa_2'(s) \|^p_{2,p} ds \leq C \int_0^t \left\| u'_2(s) \nabla (f(v'_1(s)) - f(v'_2(s))) \right\|^p_{1,p} ds \]
\[ \leq C \int_0^t \left\| u'_2(s) \right\|^p_{1,p} \left\| f(v'_1(s)) - f(v'_2(s)) \right\|^p_{2,p} ds \]
\[ \leq C \sup_{s \in [0,t]} \left\| v'_1(s) - v'_2(s) \right\|^p_{2,p}, \] (6.9)
where \( C \) depends on \( M \).

Combining (6.7), (6.8) and (6.9), we get
\[ \int_0^t \| u'_1(s) - u'_2(s) \|^p_{2,p} ds \leq C \sup_{s \in [0,t]} \left\| v'_1(s) - v'_2(s) \right\|^p_{2,p} \]
\[ \leq C(t) \int_0^t \| \tau_1(s) - \tau_2(s) \|^p_{2,p} ds. \]
Hence the mapping \( H \) is a contraction on \( \mathfrak{B}_M \) when \( T \) is small enough.

Finally, we apply Banach’s fixed point theorem to get a unique fixed point of \( H \), which means that there exists a unique weak solution \((\bar{u}, \bar{v})\) of the limiting equations (1.4) in \([0, T_1] \).

7. Proof of Theorem 1.4. In this section, we study how the solution \((u_\varepsilon, v_\varepsilon)\) of system (1.1) converges to the solution \((u, v)\) of the limiting system (1.4), as \( \varepsilon \to 0 \).

If \( u_0 \in W^{1,p}(\Omega), v_0 \in W^{1,p}(\Omega) \) with \( p > \max\{2,n\} \), by Theorem 1.1 and Arzela-Ascoli theorem, after passing to a subsequence we may assume
- \( u_\varepsilon \to \bar{u} \) weakly in \( L^p(0,T_0; W^{1,p}(\Omega)) \), strongly in \( C(0,T_0; W^{k,p}(\Omega)) \) (for any \( k \in [0,1) \)) and in \( C(\Omega^{T_0}) \);
- \( v_\varepsilon \to \bar{v} \) strongly in \( C(\Omega^{T_0}) \) and \( *\)-weakly in \( L^\infty(0,T_0; W^{1,p}(\Omega)) \).

Then \( f(v_\varepsilon) \to f(\bar{v}), g(u_\varepsilon, v_\varepsilon) \to g(\bar{u}, \bar{v}) \) strongly in \( C(\Omega^{T_0}), \nabla f(v_\varepsilon) \to \nabla f(\bar{v}) \) weakly in \( L^p(0,T_0; L^p(\Omega)) \).

With these convergence in hand, passing to the limit in (1.7) leads to (1.8), that is, \((\bar{u}, \bar{v})\) is a solution of (1.4) in the weak sense. In particular, since \( g(\bar{u}, \bar{v}) \in C^{\alpha,\alpha}(\Omega^{T_0}) \), the second equation in (1.4) implies that the distributional derivative \( \bar{v}_t \) is a \( C^{\alpha,\alpha}(\Omega^{T_0}) \) function and this equation can be understood in the classical sense.
Since \((\tilde{u}, \tilde{v})\) enjoys the same regularity with \((u, v)\), by the uniqueness of solutions to (1.4), we deduce that \((\tilde{u}, \tilde{v}) \equiv (u, v)\) in \(\Omega^0\). In particular, the limit of \((u_\varepsilon, v_\varepsilon)\) is independent of the choice of subsequences and we get the full convergence of \((u_\varepsilon, v_\varepsilon)\) to \((u, v)\) as \(\varepsilon \to 0\).

Finally, by these convergence, \((u, v)\) inherits the regularity of \((u_\varepsilon, v_\varepsilon)\). Furthermore, since \(\partial_t v \in L^\infty(0, T_0; L^\infty(\Omega))\), we get the Lipschitz regularity in \(t\) for \(v\). This gives the regularity of \((u, v)\) in Theorem 1.2.

Next assuming the initial value \(u_0 \in W^{2,p}(\Omega), v_0 \in W^{2,p}(\Omega)\) with \(p > \max\{2, n\}\), we give an explicit convergence rate of \((u_\varepsilon, v_\varepsilon)\) to \((\tilde{u}, \tilde{v})\).

By Lemma 4.1, we have

**Lemma 7.1.** For \(0 < t \leq 1/\varepsilon\), \(p \geq q, s \geq r\),

\[
e^{\varepsilon t \Delta} : W^{r,q}(\Omega) \to W^{s,p}(\Omega) \text{ with norm } C(t \varepsilon)^{-\beta},
\]

where the constant \(\beta\) is the same as in Lemma 4.1.

Now let us prove Theorem 1.4.

**Proof.** By the semigroup theory, we have

\[
\|v_\varepsilon(t) - \tilde{v}(t)\|_{1,p} \\
\leq \left\| e^{\varepsilon t \Delta} v_0 - v_0 \right\|_{1,p} + \left\| \int_0^t e^{\varepsilon (t-s) \Delta} g(u_\varepsilon(s), v_\varepsilon(s)) ds - \int_0^t g(u(s), \tilde{v}(s)) ds \right\|_{1,p} \\
\leq \left\| e^{\varepsilon t \Delta} v_0 - v_0 \right\|_{1,p} + \left\| \int_0^t e^{\varepsilon (t-s) \Delta} \left( g(u_\varepsilon(s), v_\varepsilon(s)) - g(u(s), \tilde{v}(s)) \right) ds \right\|_{1,p} \\
+ \left\| \int_0^t \left( e^{\varepsilon (t-s) \Delta} (\tilde{u}(s), \tilde{v}(s)) - g(u(s), \tilde{v}(s)) \right) ds \right\|_{1,p}.
\]

(7.1)

First, by Minkowski inequality and Lemma 7.1, we have

\[
I_1 = \left\| \int_0^t \Delta e^{\varepsilon \tau} v_0 d\tau \right\|_{1,p} \leq \int_0^t \| \Delta e^{\varepsilon \tau} v_0 \|_{1,p} d\tau \\
\leq \int_0^t \tau^{-1/2} \| v_0 \|_{2,p} d\tau \leq \| v_0 \|_{2,p} (t \varepsilon)^{1/2}.
\]

Next, similar to (3.13), the second term in the right-hand side of (7.1) is estimated from above by

\[
I_2 \leq \int_0^t \left\| (g(u_\varepsilon(s), v_\varepsilon(s)) - g(\tilde{u}(s), \tilde{v}(s))) \right\|_{1,p} ds \\
\leq C(t^{(p+1)/p} + t^{(p-2)/p}) \left\{ \int_0^t \| u_\varepsilon(s) - \tilde{u}(s) \|_{1,p}^p ds \right\}^{1/p} \\
+ C(t^{(p+1)/p} + t^{(p-2)/p}) \left\{ \int_0^t \| v_\varepsilon(s) - \tilde{v}(s) \|_{1,p}^p ds \right\}^{1/p}.
\]


Finally, to estimate $I_3$, similar to Lemma 6.2, note that

$$\int_0^t \|g(\tilde{u}(t), \tilde{v}(t))\|_{2,p} ds \leq C(t),$$

where $C(t)$ depends on $p$, $M$, $\lambda$, $K_2$ and $\Omega$. By Lemma 7.1, we get

$$I_3 = \left\| \int_0^t \int_0^{\epsilon(t-s)} \Delta e^{\Delta t} g(\tilde{u}(s), \tilde{v}(s)) d\tau ds \right\|_{1,p}$$

$$\leq \int_0^t \int_0^{\epsilon(t-s)} \|\Delta e^{\Delta t} g(\tilde{u}(s), \tilde{v}(s))\|_{1,p} d\tau ds$$

$$\leq C \int_0^t \int_0^{\epsilon(t-s)} \tau^{-1/2} \|g(\tilde{u}(s), \tilde{v}(s))\|_{2,p} d\tau ds$$

$$\leq C \varepsilon^{1/2} \int_0^t (t-s)^{1/2} \|g(\tilde{u}(s), \tilde{v}(s))\|_{2,p} ds$$

Thus

$$\|v_\varepsilon(t) - \tilde{v}(t)\|_{1,p}^p \leq C(t^{p-1} + t^{p-2}) \int_0^t \|u_\varepsilon(s) - \tilde{u}(s)\|_{1,p}^p ds$$

$$+ C(t^{p-1} + t^{p-2}) \int_0^t \|v_\varepsilon(s) - \tilde{v}(s)\|_{1,p}^p ds + C(t) \varepsilon^{p/2}.$$

By Gronwall’s inequality, for any $0 < T \leq T_1$, we have

$$\sup_{t \in [0,T]} \|v_\varepsilon(t) - \tilde{v}(t)\|_{1,p}^p \leq C e^{(T^{p-1})} (T^{2p-1} + T^{p-1}) \int_0^T \|u_\varepsilon(s) - \tilde{u}(s)\|_{1,p}^p ds$$

$$+ C(T) \varepsilon^{p/2},$$

where $C$ is the same as in (3.14).

Concerning $u_\varepsilon - \tilde{u}$, it satisfies

$$\begin{cases}
(\partial_t - \Delta)(u_\varepsilon - \tilde{u}) = -\text{div}(u_\varepsilon \nabla f(v_\varepsilon)) + \text{div}(\tilde{u} \nabla f(\tilde{v})) & \text{in } \Omega \times (0, T_1),

\frac{\partial}{\partial \nu} (u_\varepsilon - \tilde{u}) + \tilde{u} \nabla f(\tilde{v}) \cdot \nu - u_\varepsilon \nabla f(v_\varepsilon) \cdot \nu = 0 & \text{on } \partial \Omega \times (0, T_1),

u_\varepsilon(0) - \tilde{u}(0) = 0 & \text{on } \Omega.
\end{cases}$$

By (3.12) and (5.5) with the same $\chi$ in (3.12), we have

$$\sup_{t \in [0,T]} \|v_\varepsilon(t)\|_{1,p} \leq \chi, \quad \sup_{t \in [0,T]} \|\tilde{v}(t)\|_{1,p} \leq \chi.$$

Similar to step 3 in the proof of Theorem 1.1, for any $0 < T \leq T_1$,

$$\int_0^T \|u_\varepsilon(s) - \tilde{u}(s)\|_{1,p}^p ds \leq C e^{(T^{p-1})} (T^{2p} + T^{p-1}) \int_0^T \|u_\varepsilon(s) - \tilde{u}(s)\|_{1,p}^p ds$$

$$+ C(T) \varepsilon^{p/2}.$$
From this inequality, we get a constant $C(t)$ such that
\[ \| v_\varepsilon(t) - \bar{v}(t) \|_{1,p}^p + \int_0^t \| u_\varepsilon(s) - \bar{u}(s) \|_{1,p}^p ds \leq C(t) \varepsilon^{p/2}. \]

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