Partially-elementary end extensions of countable admissible sets

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Abstract

A result of Kaufmann [Kau] shows that if \( L_\alpha \) is countable, admissible and satisfies \( \Pi_n\)-Collection, then \( \langle L_\alpha, \in \rangle \) has a proper \( \Sigma_{n+1} \)-elementary end extension. This paper investigates to what extent the theory that holds in \( \langle L_\alpha, \in \rangle \) can be transferred to the partially-elementary end extensions guaranteed by Kaufmann’s result. We show that there are \( L_\alpha \) satisfying full separation, powerset and \( \Pi_n\)-Collection that have no proper \( \Sigma_{n+1} \)-elementary end extension satisfying either \( \Pi_n\)-Collection or \( \Pi_{n+3}\)-Foundation. In contrast, we show that if \( A \) is a countable admissible set that satisfies \( \Pi_n\)-Collection and \( T \) is a recursively enumerable theory that holds in \( \langle A, \in \rangle \), then \( \langle A, \in \rangle \) has a proper \( \Sigma_n \)-elementary end extension that satisfies \( T \).

1 Introduction

In [KM], Keisler and Morley prove that every countable model of \( ZF \) has proper elementary end extension. Kaufmann [Kau] refines this result showing that if \( n \geq 1 \) and \( \mathcal{M} \) is a countable structure in the language of set theory that satisfies \( KP + \Pi_n\)-Collection, then \( \mathcal{M} \) has proper \( \Sigma_{n+1} \)-elementary end extension. And, conversely, if \( n \geq 1 \) and \( \mathcal{M} \) is a structure in the language of set theory that satisfies \( KP + V = L \) and has a proper \( \Sigma_{n+1} \)-elementary end extension, then \( \mathcal{M} \) satisfies \( \Pi_n\)-Collection. In particular this shows that for a countable limit ordinal \( \alpha \) and \( n \geq 1 \), \( \langle L_\alpha, \in \rangle \) has a proper \( \Sigma_{n+1} \)-elementary end extension if and only if it satisfies \( \Pi_n\)-Collection. In the context of first-order arithmetic, the McDowell-Specker Theorem [MS] reveals that every model of \( PA \) has a proper elementary end extension. This is refined in Paris and Kirby [PK] where it is shown that if \( n \geq 2 \) and \( \mathcal{M} \) is a countable structure in the language of arithmetic that satisfies \( I\Delta_0 \), then \( \mathcal{M} \) satisfies the arithmetic collection scheme for \( \Sigma_n \)-formulae if and only if \( \mathcal{M} \) has a proper \( \Sigma_n \)-elementary end extension.

A natural question to ask is how much of the theory of \( \mathcal{M} \) satisfying \( KP + \Pi_n\)-Collection can be made to hold in a proper \( \Sigma_{n+1} \)-elementary end extension whose existence is guaranteed by Kaufmann’s result? In particular, is there a proper \( \Sigma_{n+1} \)-elementary end extension of \( \mathcal{M} \) that also satisfies \( KP + \Pi_n\)-Collection? Or, if \( \mathcal{M} \) is transitive, is there a proper \( \Sigma_{n+1} \)-elementary end extension of \( \mathcal{M} \) that satisfies full induction for all set-theoretic formulæ? In section 3 we show that the answers to the latter two of these questions is “no”. For \( n \geq 1 \), there is an \( L_\alpha \) satisfying full separation, powerset...
and $\Pi_n$-Collection that has no proper $\Sigma_{n+1}$-elementary end extension satisfying either $\Pi_n$-Collection or $\Pi_{n+3}$-Foundation. A key ingredient is a generalisation of a result due to Simpson (see [Kau, Remark 2]) showing that if $n \geq 1$ and $\mathcal{M}$ is a structure in the language of set theory satisfying $\text{KP} + V = L$ that has $\Sigma_n$-elementary end extension satisfying enough set theory and with a new ordinal but no least new ordinal, then $\mathcal{M}$ satisfies $\Pi_n$-Collection. Here “enough set theory” is either $\text{KP} + \Pi_{n-1}$-Collection or $\text{KP} + \Pi_{n+2}$-Foundation. In section 4 we obtain a strong converse to this generalisation of Simpson’s result for countable admissible sets using the Barwise Compactness Theorem. We show that if $A$ is a countable admissible set that satisfies $\Pi_n$-Collection and $T$ is a recursively enumerable theory that holds in $\langle A, \in \rangle$, then $\langle A, \in \rangle$ has a $\Sigma_n$-elementary end extension that satisfies $T$ with a new ordinal but no least new ordinal.

2 Background

Let $\mathcal{L}$ be the language of set theory—the language whose only non-logical symbol is the binary relation $\in$. Let $\Gamma$ be a collection of $\mathcal{L}$-formulae.

- **$\Gamma$-Separation** is the scheme that consists of the sentences

\[ \forall \exists \forall w \exists y \forall x (x \in y \iff (x \in w \land \phi(x, \bar{z}))), \]

for all formulae $\phi(x, \bar{z})$ in $\Gamma$. Separation is the scheme that consists of these sentences for every formula $\phi(x, \bar{z})$ in $\mathcal{L}$.

- **$\Gamma$-Collection** is the scheme that consists of the sentences

\[ \forall \exists \forall w ((\forall x \in w \exists y \phi(x, y, \bar{z}) \Rightarrow \exists c (\forall x \in w) (\exists y \in c) \phi(x, y, \bar{z}))), \]

for all formulae $\phi(x, y, \bar{z})$ in $\Gamma$. Collection is the scheme that consists of these sentences for every formula $\phi(x, y, \bar{z})$ in $\mathcal{L}$.

- **$\Gamma$-Foundation** is the scheme that consists of the sentences

\[ \forall \exists (\exists y \phi(x, \bar{z}) \Rightarrow \exists y (\phi(y, \bar{z}) \land (\forall w \in y) \neg \phi(w, \bar{z}))), \]

for all formulae $\phi(x, \bar{z})$ in $\Gamma$. If $\Gamma = \{x \in z\}$, then the resulting axiom is referred to as Set-Foundation. Foundation is the scheme that consists of these sentences for every formula $\phi(x, \bar{z})$ in $\mathcal{L}$.

Let $T$ be a theory in a language that includes $\mathcal{L}$. Let $\Gamma$ be a class of $\mathcal{L}$-formulae. A formula is $\Gamma$ in $T$ or $\Gamma^T$ if it is provably equivalent in $T$ to a formula in $\Gamma$. A formula is $\Delta_n$ in $T$ or $\Delta_n^T$ if it is both $\Sigma_n^T$ and $\Pi_n^T$.

- **$\Delta_n$-Separation** is the scheme that consists of the sentences

\[ \forall \exists (\forall w (\phi(v, \bar{z}) \iff \psi(v, \bar{z})) \Rightarrow \forall w \exists y \forall x (x \in y \iff (x \in w \land \phi(x, \bar{z})))), \]

for all $\Sigma_n$-formulae $\phi(x, \bar{z})$ and $\Pi_n$-formulae $\psi(x, \bar{z})$.

- **$\Delta_n$-Foundation** is the scheme that consists of the sentences

\[ \forall \exists (\forall w (\phi(x, \bar{z}) \iff \psi(x, \bar{z})) \Rightarrow (\exists x \phi(x, \bar{z}) \Rightarrow \exists y (\phi(y, \bar{z}) \land (\forall w \in y) \neg \phi(w, \bar{z})))), \]

for all $\Sigma_n$-formulae $\phi(x, \bar{z})$ and $\Pi_n$-formulae $\psi(x, \bar{z})$. 

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Following [Mat01], we take Kripke-Platek Set Theory (KP) to be the $\mathcal{L}$-theory axiomatised by: Extensionality, Emptyset, Pair, Union, $\Delta_0$-Separation, $\Delta_0$-Collection and $\Pi_1$-Foundation. Note that this differs from [Bar75, Fri], which defines Kripke-Platek Set Theory to include Foundation. The theory KPI is obtained from KP by adding the axiom Infinity, which states that a superset of the von Neumann ordinal $\omega$ exists. We use $M^-$ to denote the theory that is obtained from KP by replacing $\Pi_1$-Foundation with Set-Foundation and removing $\Delta_0$-Collection, and adding an axiom $TCo$ asserting that every set is contained in a transitive set. The theory $M$ is obtained from $M^-$ by adding Powerset. Zermelo Set Theory ($\mathcal{Z}$) is obtained for $M$ by removing $TCo$ and adding Separation.

The theory KP proves $TCo$ (see, for example, [Bar75 I.6.1]). The following are some important consequences of fragments of the collection scheme over the theory $M^-$:

- The proof of [Bar75 1.4.4] generalises to show that, in the theory $M^-$, $\Pi_n$-Collection implies $\Sigma_{n+1}$-Collection.
- [FLW] Lemma 4.13 shows that, over $M^-$, $\Pi_n$-Collection implies $\Delta_{n+1}$-Separation.
- It is noted in [FLW] Proposition 2.4 that if $T$ is $M^- + \Pi_n$-Collection, then the classes $\Sigma^T_{n+1}$ and $\Pi^T_{n+1}$ are closed under bounded quantification.

Let $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ be an $\mathcal{L}$-structure. If $a \in M$, then we will use $a^*$ to denote the set $\{x \in M \mid \mathcal{M} \models (x \in a)\}$, as long as $\mathcal{M}$ is clear from the context. Let $\Gamma$ be a collection of $\mathcal{L}$-formulae. We say $X \subseteq M$ is $\Gamma$ over $\mathcal{M}$ if there is a formula $\phi(x, \vec{z})$ in $\Gamma$ and $\vec{a} \in M$ such that $\{x \in M \mid \mathcal{M} \models \phi(x, \vec{a})\}$. In the special case that $\Gamma$ is all $\mathcal{L}$-formulae, we say that $X$ is a definable subclass of $\mathcal{M}$. A set $X \subseteq M$ is $\Delta_n$ over $\mathcal{M}$ if it is both $\Sigma_n$ over $\mathcal{M}$ and $\Pi_n$ over $\mathcal{M}$.

A structure $\mathcal{N} = \langle N, \in^\mathcal{N} \rangle$ is an end extension of $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$, written $\mathcal{M} \subseteq_e \mathcal{N}$, if $\mathcal{M}$ is a substructure of $\mathcal{N}$ and for all $x \in M$ and for all $y \in N$, if $\mathcal{N} \models (y \in x)$, then $y \in M$. An end extension $\mathcal{N}$ of $\mathcal{M}$ is proper if $\mathcal{M} \neq \mathcal{N}$. We say that $\mathcal{N}$ is a $\Sigma_n$-elementary end extension of $\mathcal{M}$, and write $\mathcal{M} \triangleleft_{e,n} \mathcal{N}$, if $\mathcal{M} \subseteq_e \mathcal{N}$ and $\Sigma_n$ properties are preserved between $\mathcal{M}$ and $\mathcal{N}$.

As shown in [Bar75 Chapter II], the theory KP is capable of defining Gödel’s constructible universe ($L$). For all sets $X$,

$\text{Def}(X) = \{Y \subseteq X \mid Y$ is a definable subclass of $\langle X, \in \rangle\}$,

which can be seen to be a set in the theory KP using a formula for satisfaction in set structures such as the one described in [Bar75 Section III.1]. The levels of $L$ are then defined by the recursion:

$L_0 = \emptyset$ and $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$ if $\alpha$ is a limit ordinal,

$L_{\alpha+1} = L_\alpha \cup \text{Def}(L_\alpha)$, and

$L = \bigcup_{\alpha \in \text{Ord}} L_\alpha$.

The function $\alpha \mapsto L_\alpha$ is total and $\Delta^\mathcal{L}_1$. The axiom $V = L$ asserts that every set is the member of some $L_\alpha$. A transitive set $M$ such that $\langle M, \in \rangle$ satisfies KP is said to
be an admissible set. An ordinal $\alpha$ is said to be an admissible ordinal if $L_\alpha$ is an admissible set.

Let $T$ be a $\mathcal{L}$-theory. A transitive set $M$ is said to be a minimum model of $T$ if $\langle M, \in \rangle \models T$ and for all transitive sets $N$ with $\langle N, \in \rangle \models T$, $M \subseteq N$. For example, $L_{\omega_1^\omega}$ is the minimum model of KPI. Gostanian [Gos] §1 shows that all sufficiently strong subsystems of ZF and $ZF^-$ obtained by restricting the separation and collection schemes to formulae in the Lévy classes have minimum models. In particular:

**Theorem 2.1** (Gostanian [Gos]) Let $n \in \omega$. The theory $Z + \Pi_n$-Collection has a minimum model. Moreover, the minimum model of this theory satisfies $V = L$.

The fact that KP is able to define satisfaction in set structures also facilitates the definition of formulae expressing satisfaction, in the universe, for formulae in any given level of the Lévy hierarchy.

**Definition 2.1** The formula $Sat_{\Delta_0}(q,x)$ is defined as

\[(q \in \omega) \land (q = \lceil \phi(v_1, \ldots, v_m) \rceil \text{ where } \phi \text{ is } \Delta_0) \land (x = \langle x_1, \ldots, x_m \rangle) \land \\
\exists N (\bigcup N \subseteq N \land (x_1, \ldots, x_m \in N) \land (\langle N, \in \rangle \models \phi(x_1, \ldots, x_m))).\]

We can now inductively define formulae $Sat_{\Sigma_n}(q,x)$ and $Sat_{\Pi_n}(q,x)$ that express satisfaction for formulae in the classes $\Sigma_n$ and $\Pi_n$.

**Definition 2.2** The formulae $Sat_{\Sigma_n}(q,x)$ and $Sat_{\Pi_n}(q,x)$ are defined recursively for $n > 0$. $Sat_{\Sigma_{n+1}}(q,x)$ is defined as the formula

\[\exists \vec{y} \vec{k} \exists \vec{b} \left( (q = \lceil \exists \vec{u}\phi(\vec{u}, v_1, \ldots, v_l) \rceil \text{ where } \phi \text{ is } \Pi_n) \land (x = \langle x_1, \ldots, x_l \rangle) \land \\
\land (b = \langle \vec{y}, x_1, \ldots, x_l \rangle) \land (k = \lceil \phi(\vec{u}, v_1, \ldots, v_l) \rceil) \land Sat_{\Pi_n}(k, b) \right);
\]

and $Sat_{\Pi_{n+1}}(q,x)$ is defined as the formula

\[\forall \vec{y} \vec{k} \forall \vec{b} \left( (q = \lceil \forall \vec{u}\phi(\vec{u}, v_1, \ldots, v_l) \rceil \text{ where } \phi \text{ is } \Sigma_n) \land (x = \langle x_1, \ldots, x_l \rangle) \land \\
\land (b = \langle \vec{y}, x_1, \ldots, x_l \rangle) \land (k = \lceil \phi(\vec{u}, v_1, \ldots, v_l) \rceil) \Rightarrow Sat_{\Sigma_n}(k, b) \right).
\]

**Theorem 2.2** Suppose $n \in \omega$ and $m = \max\{1, n\}$. The formula $Sat_{\Sigma_n}(q,x)$ (respectively $Sat_{\Pi_n}(q,x)$) is $\Sigma_m^{KP}$ (respectively $\Pi_m^{KP}$). Moreover, $Sat_{\Sigma_n}(q,x)$ (respectively $Sat_{\Pi_n}(q,x)$) expresses satisfaction for $\Sigma_n$-formulae (or $\Pi_n$-formulae, respectively) in the theory KP, i.e., if $M \models KP$, $\phi(v_1, \ldots, v_k)$ is a $\Sigma_n$-formula, and $x_1, \ldots, x_k$ are in $M$, then for $q = \lceil \phi(v_1, \ldots, v_k) \rceil$, $M$ satisfies the universal generalisation of the following formula:

\[x = \langle x_1, \ldots, x_k \rangle \Rightarrow (\phi(x_1, \ldots, x_k) \iff Sat_{\Sigma_n}(q,x)).\]

\[\square\]

Friedman [Fri] Section 2] classifies the countable ordinals that can appear as the order type of the ordinals of a standard part of a nonstandard model of KP. The key ingredient in Friedman’s classification is the fact that every countable admissible set has an end extension with no least new ordinal that satisfies KP.

**Theorem 2.3** (Friedman [Fri] Theorem 2.2) Let $M$ be a countable admissible set. Let $T$ be a recursively enumerable $\mathcal{L}$-theory such that $\langle M, \in \rangle \models T$. Then there exists $\mathcal{N} = \langle N, e^\mathcal{N} \rangle$ such that $\langle M, \in \rangle \subseteq e \mathcal{N}$, $\mathcal{N} \models T$ and $\text{Ord}^\mathcal{N} \setminus \text{Ord}^{\langle M, \in \rangle}$ is nonempty and has no least element. \(\square\)
Barwise [Bar75, Appendix] introduces the machinery of admissible covers to apply
infinitary compactness arguments, such as the one used in the proof of Theorem 2.3, to
nonstandard countable models. The proof of [Bar75, Theorem A.4.1] shows that for any
countable model $\mathcal{M}$ of KP $+$ Foundation and for any recursively enumerable $\mathcal{L}$-theory $T$
that holds in $\mathcal{M}$, $\mathcal{M}$ has proper end extension that satisfies $T$. By calibrating [Bar75, Appendix], Ressayre [Res, Theorem 2.15] shows that this result also holds for countable
models of KP $+$ $\Sigma_1$-Foundation.

**Theorem 2.4** Let $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$ be a countable model of KP $+$ $\Sigma_1$-Foundation. Let $T$
be a recursively enumerable theory such that $\mathcal{M} \models T$. Then there exists $\mathcal{N} \models T$
such that $\mathcal{M} \subseteq_e \mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$. $\blacksquare$

Kaufmann [Kau] identifies necessary and sufficient conditions for models of $\mathcal{M}^-$ to
have proper $\Sigma_n$-elementary end extensions.

**Theorem 2.5** (Kaufmann [Kau, Theorem 1]) Let $n \geq 1$. Let $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$ be a model
of KP. Consider

- (I) there exists $\mathcal{N} = \langle N, \in^\mathcal{N} \rangle$ such that $\mathcal{M} \prec_e \mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$;
- (II) $\mathcal{M} \models \Pi_n$-Collection.

If $\mathcal{M} \models V = \text{L}$, then (I) $\Rightarrow$ (II). If $\mathcal{M}$ is countable, then (II) $\Rightarrow$ (I). $\blacksquare$

It should be noted that Kaufmann proves that (II) implies (I) in the above under
the weaker assumption that $\mathcal{M}$ is a resolvable model of $\mathcal{M}^-$. A model $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$ of
$\mathcal{M}^-$ is resolvable if there is a function $F$ that is $\Delta_1$ over $\mathcal{M}$ such that for all $x \in M$,
there exists $\alpha \in \text{Ord}^M$ such that $x \in F(\alpha)$. The function $\alpha \mapsto L_\alpha$ witnesses the fact
that any model of KP $+$ $V = \text{L}$ is resolvable.

## 3 Admissible sets admitting topless partially elementary
end extensions

The next result is a generalisation of the result, due to Simpson, that that is mentioned
in [Kau, Remark 2]. The proof of this generalisation is based on Enayat’s proof of a
refinement of Simpson’s result (personal communication) that corresponds to the specific
case of the following theorem when $n = 1$ and $\mathcal{M}$ is transitive.

**Theorem 3.1** Let $n \geq 1$. Let $\mathcal{M} = \langle M, \in^\mathcal{M} \rangle$ be a model of KP $+$ $V = \text{L}$. Suppose
$\mathcal{N} = \langle N, \in^\mathcal{N} \rangle$ is such that $\mathcal{M} \prec_e \mathcal{N}$, $\mathcal{N} \models \text{KP}$ and $\text{Ord}^N \setminus \text{Ord}^M$ is nonempty and
has no least element. If $\mathcal{N} \models \Pi_{n-1}$-Collection or $\mathcal{N} \models \Pi_{n+2}$-Foundation, then $\mathcal{M} \models \Pi_n$-Collection.

**Proof** Assume that $\mathcal{N} = \langle N, \in^\mathcal{N} \rangle$ is such that

- (I) $\mathcal{M} \prec_e \mathcal{N}$;
- (II) $\mathcal{N} \models \text{KP}$;
- (III) $\text{Ord}^\mathcal{N} \setminus \text{Ord}^\mathcal{M}$ is nonempty and has no least element.
Note that, since $\mathcal{M} \preceq_{e,1} \mathcal{N}$ and $\mathcal{M} \models V = L$, for all $\beta \in \text{Ord}^\mathcal{N} \setminus \text{Ord}^\mathcal{M}$, $M \subseteq (L_\beta^\mathcal{N})^*$. We need to show that if either $\Pi_{n-1}$-Collection or $\Pi_{n+2}$-Foundation hold in $\mathcal{N}$, then $\mathcal{M} \models \Pi_n$-Collection. Let $\phi(x, y, z)$ be a $\Pi_n$-formula. Let $\vec{a}, b \in M$ be such that

$$\mathcal{M} \models (\forall x \in b)(\exists y \phi(x, y, \vec{a})).$$

So, for all $x \in b^*$, there exists $y \in M$ such that

$$\mathcal{M} \models \phi(x, y, \vec{a}).$$

Therefore, since $\mathcal{M} \preceq_{e,n} \mathcal{N}$, for all $x \in b^*$, there exists $y \in M$ such that

$$\mathcal{N} \models \phi(x, y, \vec{a}).$$

Now, define $\theta(\beta, \xi, b, \vec{a})$ to be the formula

$$(\forall x \in b)(\exists y \in L_\beta)(\forall w \in L_\xi)\psi(w, x, y, \vec{a}).$$

If $\Pi_{n-1}$-Collection holds in $\mathcal{N}$, then $\theta(\beta, \xi, b, \vec{a})$ is equivalent to a $\Sigma_{n-1}$-formula. Without $\Pi_{n-1}$-Collection, $\theta(\beta, \xi, b, \vec{a})$ can be written as a $\Pi_{n+2}$-formula. Therefore, $\Pi_{n-1}$-Collection or $\Pi_{n+2}$-Foundation in $\mathcal{N}$ will ensure that there is a least $\beta_0 \in \text{Ord}^\mathcal{N}$ such that $\mathcal{N} \models \theta(\beta_0, \xi, b, \vec{a})$. Moreover, by (1), $\beta_0 \in M$. Therefore,

$$\mathcal{N} \models (\forall x \in b)(\exists y \in L_{\beta_0})(\forall w \in L_\xi)\psi(w, x, y, \vec{a}).$$

So, for all $x \in b^*$, there exists $y \in (L_{\beta_0}^\mathcal{M})^*$, for all $w \in (L_\xi^\mathcal{N})^*$,

$$\mathcal{N} \models \psi(w, x, y, \vec{a}).$$

Which, since $\mathcal{M} \preceq_{e,n} \mathcal{N}$, implies that for all $x \in b^*$, there exists $y \in (L_{\beta_0}^\mathcal{M})^*$, for all $w \in M$,

$$\mathcal{M} \models \psi(w, x, y, \vec{a}).$$

Therefore, $\mathcal{M} \models (\forall x \in b)(\exists y \in L_{\beta_0})\phi(x, y, \vec{a})$. This shows that $\Pi_n$-Collection holds in $\mathcal{M}$. □

Enayat uses a specific case of Theorem 3.1 to show that the $(L_{\omega^2 \times}, \in)$ has no proper $\Sigma_1$-elementary end extension that satisfies KP (personal communication). We now turn generalising this result to show that for all $n \geq 1$, the minimum model of $\mathbb{Z} + \Pi_n$-Collection has no proper $\Sigma_{n+1}$-elementary end extension that satisfies either KP + $\Pi_{n+2}$-Foundation or KP + $\Pi_n$-Collection. However, by Theorem 2.3, for all $n \geq 1$, the minimum model of $\mathbb{Z} + \Pi_n$-Collection does have a proper $\Sigma_{n+1}$-elementary end extension.

The following result follows from [M], Theorem 4.4:
Theorem 3.2 Let $n \geq 1$. The theory $M + \Pi_{n+1}$-Collection + $\Pi_{n+2}$-Foundation proves that there exists a transitive model of $Z + \Pi_n$-Collection. \qed

Corollary 3.3 Let $n \geq 1$. Let $M$ be the minimal model of $Z + \Pi_n$-Collection. Then there is an instance of $\Pi_{n+1}$-Collection that fails in $\langle M, \in \rangle$. \qed

Theorem 3.4 Let $n \geq 1$. Let $M$ be the minimal model of $Z + \Pi_n$-Collection. Then $\langle M, \in \rangle$ has a proper $\Sigma_{n+1}$-elementary end extension, but neither

(I) a proper $\Sigma_{n+1}$-elementary end extension satisfying $KP + \Pi_{n+3}$-Foundation, nor

(II) a proper $\Sigma_{n+1}$-elementary end extension satisfying $KP + \Pi_n$-Collection.

Proof The fact that $\langle M, \in \rangle$ has a proper $\Sigma_{n+1}$-elementary end extension follows from Theorem 2.5. Let $\mathcal{N} = \langle N, \in^N \rangle$ be such that $\mathcal{N} \models KP$, $N \neq M$ and $\langle M, \in \rangle \preceq_{e,n+1} \mathcal{N}$. Since $M$ is the minimal model of $Z + \Pi_n$-Collection, $\langle M, \in \rangle \models \neg \sigma$ where $\sigma$ is the sentence

$$\exists x (x \text{ is transitive } \land \langle x, \in \rangle \models M + \Sigma_{n+1}$-Separation + $\Pi_n$-Collection).$$

Since $\sigma$ is $\Sigma_1^{KP}$ and $\langle M, \in \rangle \preceq_{e,1} \mathcal{N}$, $\mathcal{N} \models \neg \sigma$. Since $\mathcal{N} \models KP$ and $M \neq N$, $\text{Ord}^N \setminus \text{Ord}^{\langle M, \in \rangle}$ is nonempty. If $\gamma$ is the least element of $\text{Ord}^N \setminus \text{Ord}^{\langle M, \in \rangle}$, then

$$\mathcal{N} \models ((L, \in) \models Z + \Pi_n$-Collection),$$

which contradicts the fact that $\mathcal{N} \models \neg \sigma$. Therefore, $\text{Ord}^N \setminus \text{Ord}^{\langle M, \in \rangle}$ is nonempty and contains no least element. Therefore, by Theorem 2.8 and Corollary 3.3 there must be both an instance of $\Pi_n$-Collection and an instance of $\Pi_{n+3}$-Foundation that fails in $\mathcal{N}$. \qed

4 Building partially elementary end extensions

In this section we show that if $M$ is a countable admissible set that satisfies $\Pi_n$-Collection, then we can build proper $\Sigma_n$-elementary end extensions of $\langle M, \in \rangle$ that satisfy as much of the first-order theory $\langle M, \in \rangle$ as we want them to. More specifically, we show that if $M$ is a countable admissible set with $\langle M, \in \rangle \models \Pi_n$-Collection and $T$ is recursively enumerable with $\langle M, \in \rangle \models T$, then there exists a proper $\Sigma_n$-elementary end extension of $\langle M, \in \rangle$ that satisfies $T$.

We construct proper $\Sigma_n$-elementary end extensions of admissible sets using an appropriate version of the Barwise Compactness Theorem. In order to present this construction, it is convenient to introduce a family of class theory extensions of KP. Closely related class theory extensions of KP have been used in [FT] to present Barwise Compactness Arguments and [JS] to study extensions of KP obtained by adding fixed point axioms. Let $\mathcal{L}^c$ be the first-order language of class theory— the language obtained from $\mathcal{L}$ adding a unary relation $S$ that distinguishes sets from classes. To simplify the presentation of $\mathcal{L}^c$-formulae, we will treat $\mathcal{L}^c$ as a two-sorted language with sorts sets (referred to using lower case Roman letters $w, x, y, z, \ldots$) that are the elements of the domain that satisfy $S$ and classes (referred to using upper case Roman letters $W, X, Y, Z \ldots$) that are any element of the domain. Therefore, $\exists x(\cdot \cdot \cdot)$ is an abbreviation for $\exists x(S(x) \land \cdot \cdot \cdot)$, $\forall x(\cdot \cdot \cdot)$ is an abbreviation for $\forall x(S(x) \Rightarrow \cdot \cdot \cdot)$, $\exists x(x = X)$ is an abbreviation for $S(X)$, etc. We say that an $\mathcal{L}^c$-formula, $\phi$, is elementary if $\phi$ contains only atomic formulae in
the form $x \in Y$, $x \in y$ and $S(Y)$ and all of the quantifiers in $\phi$ are restricted to sets. In other words, an elementary $\mathcal{L}^c$-formula is a formula that does not contain the symbol = and only contains set variables with the possible exception of subformulae in the form $x \in Y$ where $Y$ is a free (class) variable. The collection $\Delta^c_0$ is the smallest class of elementary $\mathcal{L}^c$-formulae that contains all atomic formulae, is closed under the connectives of propositional logic, and quantification in the form $\forall x \in y$ and $\exists x \in y$ where $x$ and $y$ are distinct variables. The classes $\Sigma^c_n$ and $\Pi^c_n$ are the classes of elementary formulae defined inductively from the class $\Delta^c_0$ in the usual way.

- KP$^c$ is the $\mathcal{L}^c$-theory with axioms:
  
  \[
  \forall X \forall Y (X \in Y \Rightarrow \exists x(x = X));
  \]
  
  (Extensionality$^c$) $\forall X \forall Y (X = Y \iff \forall x(x \in X \iff x \in Y));$
  
  (Pairing$^c$) $\forall x \forall y \exists z \forall w (w \in z \iff w = x \lor w = y);$ 

  (Union$^c$) $\forall x \exists y \forall z (z \in y \iff (\exists w \in x)(z \in w));$

  (\$\Delta^c_0\$-Separation) for all $\Delta^c_0$-formulae, $\phi(x, \bar{Z}),$

  \[
  \forall \bar{Z} \forall w \exists y \forall x (x \in y \iff (x \in w) \land \phi(x, \bar{Z}));
  \]

  (\$\Delta^c_0\$-Collection) for all $\Delta^c_0$-formulae, $\phi(x, y, \bar{Z}),$

  \[
  \forall \bar{Z} \forall w ((\forall x \in w) \exists y \phi(x, y, \bar{Z}) \Rightarrow \exists c (\forall x \in w)(\exists y \in c) \phi(x, y, \bar{Z}));
  \]

  (\$\Pi^c_1\$-Foundation) for all $\Pi^c_1$-formulae, $\phi(x, \bar{Z}),$

  \[
  \forall \bar{Z} (\exists x \phi(x, \bar{Z}) \Rightarrow \exists y (\phi(y, \bar{Z}) \land (\forall w \in y) \neg \phi(w, \bar{Z}))) ;
  \]

  (\$\Delta^c_1\$-CA) for all $\Sigma^c_1$-formulae, $\phi(x, \bar{Z})$, and for all $\Pi^c_1$-formulae, $\psi(x, \bar{W}),$

  \[
  \forall \bar{Z} \forall \bar{W} (\forall x (\phi(x, \bar{Z}) \iff \psi(x, \bar{W})) \Rightarrow \exists X \forall y (y \in X \iff \phi(x, \bar{Z})) ) .
  \]

Friedman’s class theory $\text{Adm}^c$ \cite[Definition 1.14]{Fri} differs from $\text{KP}^c$ by including the full scheme of foundation for $\mathcal{L}^c$-formulae instead of $\Pi^c_1$-Foundation. It should also be noted that the theory $\text{KP}^c$ utilised in \cite{JS} includes full $\in$-induction for all $\mathcal{L}^c$-formulae. We have chosen to include only $\Pi^c_1$-Foundation in $\text{KP}^c$ in order to ensure that $\text{KP}^c$ is a conservative extension of $\text{KP}$, which here only includes $\Pi^c_1$-Foundation.

The theory $\text{KP}^c$ proves that the class of elementary $\mathcal{L}^c$-formulae that are equivalent to a $\Sigma^c_1$-formula is closed under quantification that is bounded by a set variable and, similarly, the class of elementary $\mathcal{L}^c$-formulae that are equivalent to a $\Pi^c_1$-formula is also closed under quantification that is bounded by a set variable. The usual argument showing that $\text{KP}$ proves $\Sigma^c_1$-Collection adapts to show that $\text{KP}^c$ proves $\Sigma^c_1$-Collection.

We will also be interested in extensions of $\text{KP}^c$ that are obtained by strengthening the class comprehension scheme. For $n > 1$, define:

- \(\Delta^c_n\)-CA for all $\Sigma^c_n$-formulae, $\phi(x, \bar{Z})$, and for all $\Pi^c_n$-formulae, $\psi(x, \bar{W}),$

  \[
  \forall \bar{Z} \forall \bar{W} (\forall x (\phi(x, \bar{Z}) \iff \psi(x, \bar{W})) \Rightarrow \exists X \forall y (y \in X \iff \phi(x, \bar{Z})) ) .
  \]
Lemma 4.1 Let $\Delta^*_n$ denote the smallest class of $\mathcal{L}$-formulae that contains the atomic formulae in the form $x \in y$, is closed under the connectives of propositional logic, and quantification in the form $\forall x \in y$ and $\exists x \in y$ where $x$ and $y$ are distinct variables.

Note that the class $\Delta^*_0$, when viewed as a class of $\mathcal{L}^c$-formulae, is just the class of $\Delta^*_0$-formulae in which all variables are restricted to sets.

Lemma 4.1 Let $\phi(\vec{z})$ be a $\Delta_0$-formula. There is a $\Delta^*_0$-formula $\phi'(\vec{z})$ such that

\[ \text{Extensionality} \vdash \forall \vec{z}(\phi(\vec{z}) \iff \phi'(\vec{z})). \]

Proof Replace any subformula in the form $x = y$ with $(\forall w \in x)(w \in y) \land (\forall w \in y)(w \in x)$. $\Box$

Theorem 4.2 Let $n \in \omega$. Let $\mathcal{M} = \langle M, \in^M, S^M \rangle$ be a model of $\text{KP}^c + \Delta_{n+1}\text{-CA}$. Then $\mathcal{M}_{\text{Set}} = \langle S^M, \in^M \rangle$ satisfies $\text{KP} + \Pi^c_n\text{-Collection} + \Pi^c_{n+1}\text{-Foundation}.$

Proof Let $\mathcal{M}_{\text{Set}} = \langle S^M, \in^M \rangle$. It is immediate that $\mathcal{M}_{\text{Set}}$ satisfies Extensionality, Emptyset, Pair and Union. Lemma 4.1 and the scheme of $\Delta^*_0$-Separation in $\mathcal{M}$ imply that $\mathcal{M}_{\text{Set}}$ satisfies $\Delta^*_0$-Separation. Similarly, employing Lemma 4.1 shows that $\Delta^*_0$-Collection in $\mathcal{M}$ implies $\Delta_0$-Collection in $\mathcal{M}_{\text{Set}}$, and $\Pi^c_1$-Foundation in $\mathcal{M}$ implies $\Pi^c_1$-Foundation in $\mathcal{M}_{\text{Set}}$. This shows that the theorem holds when $n = 0$. Therefore, assume that $n > 0$. We need to verify that $\Pi^c_n$-Collection and $\Pi^c_{n+1}$-Foundation hold in $\mathcal{M}_{\text{Set}}$. Let $V_{\Pi_n} \in M$ be such that

\[ \langle \ulcorner \phi(x) \urcorner, a \rangle \in V_{\Pi_n} \text{ if and only if } \mathcal{M}_{\text{Set}} \models \text{Sat}_{\Pi_n}(\ulcorner \phi(x) \urcorner, a). \]

Let $V_{\Sigma_n} \in M$ be such that

\[ \langle \ulcorner \phi(x) \urcorner, a \rangle \in V_{\Sigma_n} \text{ if and only if } \mathcal{M}_{\text{Set}} \models \text{Sat}_{\Sigma_n}(\ulcorner \phi(x) \urcorner, a). \]

Note that $\Delta_{n+1}$-CA in $\mathcal{M}$ ensures that the classes $V_{\Pi_n}$ and $V_{\Sigma_n}$ exist. To see that $\mathcal{M}_{\text{Set}}$ satisfies $\Pi^c_n$-Collection, let $\phi(x, y, \vec{z})$ be a $\Pi^c_n$-formula. Let $b, \vec{a} \in S^M$ be such that

\[ \mathcal{M}_{\text{Set}} \models (\forall x \in b)(\exists y \phi(x, y, \vec{a})). \]

Consider $\theta(x, y, \vec{a}, v, V_{\Pi_n})$ defined by:

\[ \exists u(u = \langle x, y, \vec{a} \rangle \land (v, u) \in V_{\Pi_n}). \]

Note that, by Lemma 4.1, $\theta(x, y, \vec{a}, v, V_{\Pi_n})$ is equivalent to a $\Sigma^c_1$-formula. Moreover, $\mathcal{M} \models (\forall x \in b)(\exists y \theta(x, y, \vec{a}, \ulcorner \phi(x, y, \vec{z}) \urcorner, V_{\Pi_n}).$

Therefore, by $\Sigma^c_1$-Collection in $\mathcal{M}$,

\[ \mathcal{M} \models \exists c(\forall x \in b)(\exists y \in c)\theta(x, y, \vec{a}, \ulcorner \phi(x, y, \vec{z}) \urcorner, V_{\Pi_n}). \]
Therefore,
\[ M_{\text{Set}} \models \exists c(\forall x \in b)(\exists y \in c)\phi(x, y, \bar{a}). \]

This shows that \( M_{\text{Set}} \) satisfies \( \Pi_n \)-Collection.

Finally, we need to verify that \( M_{\text{Set}} \) satisfies \( \Pi_{n+1} \)-Foundation. To this end, let \( \phi(x, \bar{z}) \) be a \( \Pi_{n+1} \)-formula. Therefore, \( \phi(x, \bar{z}) \) can be written as \( \forall y \psi(y, x, \bar{z}) \) where \( \psi(y, x, \bar{z}) \) is \( \Sigma_n \). Let \( \bar{a} \in S^M \). Consider \( \theta(y, x, \bar{a}, V_{\Sigma_n}) \) defined by:
\[
\forall u(u = \langle y, x, \bar{a} \rangle \Rightarrow (v, u) \in V_{\Sigma_n}).
\]

Note that \( \theta(y, x, \bar{a}, V_{\Sigma_n}) \) is equivalent to a \( \Pi_1 \)-formula. Therefore, \( \Pi_1 \)-Foundation, the class
\[
\{x \mid \forall y \theta(y, x, \bar{a}, \bar{r} \psi(y, x, \bar{z})\}
\]
is either empty or has an \( e^M \)-least element in \( M \). Therefore, the class \( \{x \mid \phi(x, \bar{a})\} \)
is either empty or has an \( e^M \)-least element in \( M_{\text{Set}} \). This shows that \( M_{\text{Set}} \) satisfies \( \Pi_{n+1} \)-Foundation. □

Conversely, if \( M \) is a model of KP + \( \Pi_n \)-Collection + \( \Pi_{n+1} \)-Foundation, then one can adjoin the classes that are \( \Delta_{n+1} \) over \( M \) to \( M \) to obtain a model of \( \text{KP}^c + \Delta_{n+1} \)-CA.

**Theorem 4.3** Let \( n \in \omega \). Let \( M = \langle M, e^M \rangle \) be a model of KP + \( \Pi_n \)-Collection + \( \Pi_{n+1} \)-Foundation. Let
\[
\mathcal{X} = \{X \subseteq M \mid X \text{ is } \Delta_{n+1} \text{ over } M\}\setminus\{a^* \mid a \in M\}
\]
and \( e' = e^M \cup \in \in \{M \times \mathcal{X}\} \).

Then \( \langle M \cup \mathcal{X}, e', M \rangle \models \text{KP}^c + \Delta_{n+1} \)-CA.

**Proof** Note that \( \langle M \cup \mathcal{X}, e', M \rangle \) clearly satisfies \( \forall X \forall Y(X \in Y \Rightarrow \exists x(x = X)) \), Extesionality\(^c \), Pairing\(^c \) and Union\(^c \). Let \( \phi(\bar{x}, Z_0, \ldots, Z_{m-1}) \) be a \( \Delta_0^c \)-formula and let \( A_0, \ldots, A_{m-1} \in M \cup \mathcal{X} \). Since for all \( i \in m \), the formula \( y \in A_i \) can be expressed as a \( \Delta_{n+1} \)-formula with parameters from \( M \), there exists a \( \Sigma_{n+1} \)-formula \( \psi(\bar{x}, \bar{z}) \) and a \( \Pi_{n+1} \)-formula \( \theta(\bar{x}, \bar{z}) \), and \( \bar{a} \in M \) such that for all \( \bar{x} \in M \),
\[
\langle M \cup \mathcal{X}, e', M \rangle \models \phi(\bar{x}, A_0, \ldots, A_{m-1})
\]
if and only if \( M \models \psi(\bar{x}, \bar{a}) \) if and only if \( M \models \theta(\bar{x}, \bar{a}) \).

Therefore, \( \Delta_0^c \)-Separation in \( \langle M \cup \mathcal{X}, e', M \rangle \) follows from \( \Delta_{n+1} \)-Separation in \( M \), \( \Delta_0^c \)-Collection in \( \langle M \cup \mathcal{X}, e', M \rangle \) follows from \( \Sigma_{n+1} \)-Collection in \( M \), and \( \Pi_1 \)-Foundation in \( \langle M \cup \mathcal{X}, e', M \rangle \) follows from \( \Pi_{n+1} \)-Foundation in \( M \). Similarly, if \( \phi(\bar{x}, \bar{Z}) \) is a \( \Sigma_{n+1} \)-formula (\( \Pi_{n+1} \)-formula) and \( \bar{A} \in M \cup \mathcal{X} \), then there exists a \( \Sigma_{n+1} \)-formula (\( \Pi_{n+1} \)-formula, respectively), \( \psi(x, \bar{z}) \) and \( \bar{a} \in M \) such that for all \( x \in M \),
\[
\langle M \cup \mathcal{X}, e', M \rangle \models \phi(x, \bar{A}) \text{ if and only if } M \models \psi(x, \bar{a}).
\]

Therefore, \( \langle M \cup \mathcal{X}, e', M \rangle \) satisfies \( \Delta_{n+1} \)-CA. □

We will build \( \Sigma_n \)-elementary end extensions of countable admissible sets satisfying \( \Pi_n \)-Collection using an appropriate version of the Barwise Compactness Theorem.

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**Definition 4.2** Let $A$ be a countable admissible set. Let $\mathcal{L}'$ be obtained from $\mathcal{L}$ by adding constant symbols $\bar{a}$ for each $a \in A$. Let $\mathcal{L}''$ be obtained from $\mathcal{L}'$ by adding constant symbols $c_n$ for each $n \in \omega$. Write $\mathcal{L}''_A$ for the fragment of $\mathcal{L}''_{\omega_1}$ that is coded in $A$. We will identified an $\mathcal{L}''_A$-theory, $T$, with the set of codes of sentences in $T$.

The following version of the Barwise Compactness Theorem appears as [Fri, Theorem 1.12]:

**Theorem 4.4** (Barwise Compactness) Let $A$ be a countable admissible set and let $X \subseteq \mathcal{P}(A)$ such that $\langle A \cup X, \in, A \rangle \models \text{KP}^c$. If $T$ is an $\mathcal{L}''_A$-theory with $T \in A \cup X$ and for all $T_0 \subseteq T$ with $T_0 \in A$, $T_0$ has a model, then $T$ has a model. $\blacksquare$

The proof of the next result is based on the proof of [Fri, Theorem 2.2].

**Theorem 4.5** Let $n \in \omega$. Let $A$ be a countable admissible set such that $\langle A, \in \rangle \models \Pi_n^\text{-Collection}$. Let $S$ be a recursively enumerable $\mathcal{L}$-theory such that $\langle A, \in \rangle \models S$. Then there exists $M = \langle M, \in^M \rangle$ such that

(i) $M \models S$;

(ii) $\langle A, \in \rangle \prec_n M$;

(iii) $\text{Ord}^M \setminus A$ has no least element.

**Proof** Let $\mathcal{X} = \{X \subseteq A \mid X$ is a $\Delta_{n+1}$ subset of $A\}\setminus A$.

By Theorem 4.3, $\langle A \cup \mathcal{X}, \in, A \rangle \models \text{KP}^c + \Delta_{n+1}\text{-CA}$. Let $T_0$ be $\mathcal{L}''_A$-theory with axioms:

(I) $S$;

(II) for all $a \in A$,

$$\forall x \left( x \in \bar{a} \iff \bigvee_{b \in a} x = \bar{b} \right);$$

(III) for all $\alpha \in \text{Ord}^{\langle A, \in \rangle}$, $\bar{\alpha} \in c_0$;

(IV) for all $\Pi_n$-formulae, $\phi(x_1, \ldots, x_m)$ and for all $a_1, \ldots, a_m \in A$ such that $\langle A, \in \rangle \models \phi(a_1, \ldots, a_m)$,

$$\phi(\bar{a_1}, \ldots, \bar{a_m}).$$

Note that $T_0 \in A \cup \mathcal{X}$ and for all $T' \subseteq T_0$ with $T' \in A$, $T'$ has a model. Therefore, by Theorem 4.4, $T_0$ is consistent. Let $\langle \phi_n \mid k \in \omega \rangle$ be an enumeration of the $\mathcal{L}''_A$-sentences. Let $m \in \omega$ and suppose that $T_{3m} \supseteq T_0$ has been defined, contains only finitely many of the constant symbols $c_k$ and is consistent. Define $T_{3m+1} = \{ T_{3m} \cup \{ \phi_m \} \}$ if $T_{3m} \cup \{ \phi_m \}$ is consistent

Define:

$$T_{3m+1} = \begin{cases} T_{3m} \cup \{ \phi_m \} & \text{if } T_{3m} \cup \{ \phi_m \} \text{ is consistent} \\ T_{3m} \cup \{ \neg \phi_m \} & \text{otherwise} \end{cases}$$

Define:

$$\text{if } \neg \phi_m \in T_{3m+1} \text{ and } \phi_m = \bigwedge \Gamma, \text{ then let}$$
\[ T_{3m+2} = T_{3m+1} \cup \{ \neg \psi \} \text{ for some } \psi \in \Gamma \text{ with } T_{3m+1} \cup \{ \neg \psi \} \text{ consistent}; \]

if \( \neg \phi_m \in T_{3m+1} \) and \( \phi_m = \forall x \psi(x) \), then let

\[ T_{3m+2} = T_{3m+1} \cup \{ \neg \psi(c_k) \} \text{ for some } c_k \text{ not appearing in } T_{3m+1}; \]

\[ T_{3m+2} = T_{3m+1} \text{ otherwise.} \]

Define:

if for some \( \alpha \in \text{Ord}^{(A, \in)} \), \( T_{3m+2} \cup \{ \neg(\bar{\alpha} \in c_m) \} \) is consistent, then let

\[ T_{3m+3} = T_{3m+2} \cup \{ \neg(\bar{\alpha} \in c_m) \}; \]

otherwise let \( T_{3m+3} = T_{3m+2} \cup \{ (c_k \in c_m) \} \cup \{ \bar{\alpha} \in c_k \mid \alpha \in \text{Ord}^{(A, \in)} \} \)

where \( c_k \) does not appear in \( T_{3m+2} \).

Now, \( T_{3m+3} \supseteq T_{3m} \supseteq T_0 \) and \( T_{3m+3} \) only contains finitely many of the constant symbols \( c_k \). Moreover, by Theorem 4.4, \( T_{3m+3} \) is consistent. Now, let

\[ T = \bigcup_{m \in \omega} T_m. \]

Therefore, \( T \) is consistent. Now, the terms of \( T \) form a Henkin model \( M = \langle M, \in^M \rangle \) satisfying (i)-(iii). \( \Box \)

Combined with Theorem 3.1, we obtain the following characterisation of admissible \( L_\alpha \) that have proper \( \Sigma_n \)-elementary end extensions with no least new ordinal satisfying any recursively enumerable fragment of the theory of \( \langle L_\alpha, \in \rangle \).

**Corollary 4.6** Let \( L_\alpha \) be countable and admissible. Then the following are equivalent:

(I) For any recursively enumerable \( L \)-theory \( T \), there exists \( M \models T \) with \( \langle L_\alpha, \in \rangle \prec_{e,n} M \) and \( \text{Ord}^M \setminus \alpha \) is nonempty and has no least element.

(II) \( \langle L_\alpha, \in \rangle \models \Pi_n^{\text{-Collection}} \).

\( \Box \)

We suspect that Barwise’s admissible cover machinery [Bar75, Appendix] may useful in shedding light on the following question:

**Question 4.1** Is there a version of Theorem 4.5 that holds for all countable models of \( \text{KP} + \Pi_n^{\text{-Collection}} + \Pi_{n+1}^{\text{-Foundation}} \)?

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