Harnack Inequality and Applications for Infinite-Dimensional GEM Processes *

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Abstract

The dimension-free Harnack inequality and uniform heat kernel upper/lower bounds are derived for a class of infinite-dimensional GEM processes, which was introduced in \cite{7} to simulate the two-parameter GEM distributions. In particular, the associated Dirichlet form satisfies the super \textemdash log-Sobolev inequality which strengthens the log-Sobolev inequality derived in \cite{7}. To prove the main results, explicit Harnack inequality and super Poincaré inequality are established for the one-dimensional Wright-Fisher diffusion processes. The main tool of the study is the coupling by change of measures.

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1 Introduction

The GEM distribution appears in population genetics describing the distribution of age-ordered allelic frequencies \cite{6}. Due to the many computational friendly properties of the stick-breaking structure, the GEM distribution and various generalizations are widely used as prior distributions in Bayesian statistics \cite{12}. Below we briefly recall a standard construction of the GEM random variables.

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Let \( \{U_n\}_{n \geq 1} \) be a sequence of independent beta random variables with corresponding parameters \( a_n > 0 \) and \( b_n > 0, \ n \geq 1 \). Set

\[
V_1 = U_1, \ V_n = U_n \prod_{i=1}^{n-1} (1 - U_i), \ \ n \geq 2.
\]

For any \( n \geq 1 \), the joint distribution of \( (V_1, \ldots, V_n) \) is the generalized Dirichlet distribution defined in [3]. The law of \( V = (V_1, V_2, \ldots) \) is a probability on the space

\[
\Delta_\infty := \left\{ x = (x_i)_{i \in \mathbb{N}} \in [0, 1]^\mathbb{N} : \sum_{n=1}^{\infty} x_n \leq 1 \right\}
\]
equipped with the usual \( \sigma \)-field induced by the projections \( \{x \mapsto x_i : i \in \mathbb{N}\} \). Let

\[
W_n = V_1 + \ldots + V_n, \ \ n \geq 1.
\]

Then \( W_n \) is monotonically increasing bounded above by 1. If the parameters satisfy

\[
\sum_{i=1}^{\infty} \frac{a_i}{a_i + b_i} = \infty,
\]
then \( 1 - W_n = (1 - U_1) \cdots (1 - U_n) \) converges monotonically to 0 and the law of \( V \) becomes a probability on space

\[
\Delta := \left\{ x \in [0, 1]^\mathbb{N} : \sum_{i \geq 1} x_i = 1 \right\}.
\]

If \( a_i = 1 - \alpha, b_i = \theta + i\alpha \) for a pair of parameters \( 0 \leq \alpha < 1, \theta + \alpha > 0 \), then the law of \( V \) is the well known two-parameter GEM distribution. The GEM distribution with parameter \( \theta \), coined by Ewens and named after Griffiths, Engen, and McCloskey, corresponds to \( \alpha = 0 \). Under assumption (1.2), the representation (1.1) is also known as the stick-breaking model.

To simulate the GEM distributions using Markov processes, a class of infinite-dimensional diffusion processes on \( \Delta_\infty \) have been constructed in [7]. It was proved in [7] that these processes are symmetric with respect to the corresponding GEM distributions and satisfy the log-Sobolev inequality, so that they converge to the GEM distributions exponentially in both entropy and \( L^2 \). In this paper we derive some stronger properties on these processes, including the uniform heat kernel upper/lower bounds and super log-Sobolev inequalities.

The main idea of the study goes back to [13] using the dimension-free Harnack inequality, and the main tool to establish the Harnack inequality is the coupling by change of measures developed from [1], see the recent monograph [18] for a brief theory on coupling by change of measures and applications.

To recall the GEM processes constructed in [7], let \( \{a_i, b_i\}_{i \geq 1} \) be strictly positive numbers. Then the corresponding GEM process is generated by the following second-order differentiable operator on \( \Delta_\infty \) (note that the factor \( 1/2 \) in the diffusion term is missed in [7]):

\[
\mathcal{L}(x) = \frac{1}{2} \sum_{i,j=1}^{\infty} A_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{\infty} C_i(x) \frac{\partial}{\partial x_i}, \ \ x = (x_1, x_2, \cdots) \in \Delta_\infty,
\]
where

\[ A_{ij}(x) := x_i x_j \sum_{k=1}^{i+j} \left( \delta_{ki} \left(1 - \sum_{l=1}^{k-1} x_l\right) - x_k \right) \left( \delta_{kj} \left(1 - \sum_{l=1}^{k-1} x_l\right) - x_k \right), \]

\[ C_i(x) := x_i \sum_{k=1}^{i} \left( \delta_{ik} \left(1 - \sum_{l=1}^{k-1} x_l\right) - x_k \right) \left( a_k \left(1 - \sum_{l=1}^{k-1} x_l\right) - (a_k + b_k) x_k \right) \frac{1}{x_k \left(1 - \sum_{l=1}^{k} x_l\right)}. \]

Here and in what follows, we set \(\sum_{i=1}^{\infty} = 0\) and \(\prod_{i=1}^{\infty} = 1\) by conventions. Obviously, \(A_{ij}(x)\) and \(C_i(x)\) are well defined if \(x = (x_i)_{i \in \mathbb{N}}\) satisfies \(\sum_{i=1}^{n} x_i < 1\) for all \(n \in \mathbb{N}\). By setting \(\frac{a}{\theta} = 1\), they are defined on the whole space \(\Delta_{\infty}\).

The diffusion process generated by \(\mathcal{L}\) on \(\Delta_{\infty}\) is constructed in [7] using the one-dimensional Wright-Fisher diffusion processes, which solve the following SDEs on \([0, 1]\) for \(i \geq 1\):

\[ \frac{d}{t} x_i(t) = \{a_i - (a_i + b_i) x_i(t)\} dt + \sqrt{x_i(t)(1 - x_i(t))} dB_i(t), \]

where \(\{B_i(t)\}_{i \geq 1}\) are independent one-dimensional Brownian motions. By [11, Theorem 3.2] with \(\sigma(x) := \sqrt{x(1-x)1_{[0,1]}(x)}\) and \(b(x) := a_i - (a_i + b_i) x\), the equation has a unique strong solution which is a diffusion process on \([0, 1]\). For any \(x = (x_i)_{i \in \mathbb{N}} \in [0, 1]^\mathbb{N}\), let \(X^x(t) = (x_1(t), x_2(t), \ldots, x_i(t))\), where \(x_i(t)\) solves (1.3) with \(x_i(0) = x_i \in [0, 1]\). Let \(\tilde{P}_t^{a,b}\) be the corresponding Markov semigroup, i.e.

\[ \tilde{P}_t^{a,b} f(x) = \mathbb{E} f(X^x(t)), \quad t \geq 0, f \in \mathcal{B}_b([0, 1]^\mathbb{N}), x \in [0, 1]^\mathbb{N}, \]

where \(\mathcal{B}_b(\cdot)\) denotes the set of all bounded measurable functions on a measurable space.

It is easy to see that \(x_i(t)\) is reversible with respect to the beta distribution

\[ \pi_{a_i,b_i}(dx) := \frac{\Gamma(2a_i + 2b_i)}{\Gamma(2a_i)\Gamma(2b_i)} 1_{[0,1]}(x)(1-x)^{2a_i-1}(1-x)^{2b_i-1} dx. \]

Define the map \(\Phi : [0, 1]^\mathbb{N} \rightarrow \Delta_{\infty}\) by

\[ \Phi(x) = (\phi_1(x), \phi_2(x), \ldots), \quad \phi_n(x) := x_n \prod_{i=1}^{n-1} (1 - x_i), \quad n \geq 1, x = (x_1, x_2, \ldots) \in [0, 1]^\mathbb{N}. \]

Let \(\Xi_{a,b} = \pi_{a,b} \circ \Phi^{-1}\), where \(\pi_{a,b} := \prod_{i \geq 1} \pi_{a_i,b_i}\). It is clear that \(\Xi_{a,b}\) includes the GEM distributions as special examples: the one-parameter GEM distribution \(\pi_{a}^{GEM} = \Xi_{a,b}\) for \(a_i = \frac{a}{\theta}\) and \(b_i = \frac{\theta - a}{\theta}\); and the two-parameter GEM distribution \(\pi_{a,\theta}^{GEM} = \Xi_{a,b}\) for \(a_i = \frac{1-a}{2}\) and \(b_i = \frac{\theta + a}{2}\).

To construct the GEM diffusion process using the map \(\Phi\) and \(x_i(t), i \geq 1\), we observe that

\[ \Phi : [0, 1]^\mathbb{N} \rightarrow \Delta_{\infty} = \left\{ x \in [0, 1]^\mathbb{N} : \sum_{i=1}^{n} x_i < 1 \quad \forall \ n \in \mathbb{N} \right\} \]
is a bijection with inverse

\[ \Psi(x) = (\psi_1(x), \psi_2(x), \cdots), \quad \psi_n(x) := \frac{x_n}{1 - \sum_{i=1}^{n-1} x_i} \in [0,1), \quad n \geq 1, \ x \in \tilde{\Delta}_\infty. \]

Due to this fact, if \( i \geq 1 \) \( b_i \geq \frac{1}{2} \) has been assumed in [7] so that \( x_i(t) \in [0,1) \) for all \( t > 0 \) and \( i \geq 1 \). In this case, for any \( x \in \Phi([0,1]^N) \), \( X^x(t) := \Phi(X^{\Psi(x)}(t)) \) is a Markov process on \( \tilde{\Delta}_\infty \).

Moreover, according to [7, §3], this Markov process is generated by \( \mathcal{L} \) on \( \tilde{\Delta}_\infty \); that is, the Markov semigroup

\[ P_{t}^{a,b} f(x) := \mathbb{E} f(\Phi(X^{\Psi(x)}(t))) = \bar{P}_{t}^{a,b}(f \circ \Phi)(\Psi(x)), \quad f \in \mathcal{B}_b(\tilde{\Delta}_\infty), t \geq 0, x \in \tilde{\Delta}_\infty, \]

where \( \mathcal{B}_b(\cdot) \) denotes the set of all bounded measurable real functions on a measurable space, is associated to the symmetric Dirichlet form \((\mathcal{E}_{a,b}, \mathcal{D}(\mathcal{E}_{a,b}))\), which is the closure of the following pre-Dirichlet form on \( L^2(\Xi_{a,b}) \):

\[ \mathcal{E}_{a,b}(f,g) := -\int_{\tilde{\Delta}_\infty} f \mathcal{L} g \mathbb{d}\Xi_{a,b} = \frac{1}{2} \int_{\tilde{\Delta}_\infty} \left( \sum_{i,j \geq 1} a_{ij}(\partial_i f)(\partial_j g) \right) \mathbb{d}\Xi_{a,b} \quad f, g \in \mathcal{F}C_b^\infty, \]

where \( \mathcal{F}C_b^\infty \) is the set of all \( C_b^\infty \)-cylindrical functions on \([0,1]^N\).

To extend the above construction for all \( b_i > 0 \) for which \( x_i(t) \) may hit 1, we extend \( \Psi \) to \( \tilde{\Delta}_\infty \) by setting \( \frac{0}{0} = 1 \), i.e., \( \psi_n(x) = 1 \) provided \( \sum_{i=1}^{n-1} x_i = 1 \) (this implies \( x_n = 0 \) for \( x \in \tilde{\Delta}_\infty \)).

Then

\[ \Psi(\tilde{\Delta}_\infty) = E := \{ x = (x_i)_{i \in N} \in [0,1]^N : \text{if } x_i = 1 \text{ for some } i \in \mathbb{N}, \text{ then } x_j = 1 \text{ for all } j \geq i \}, \]

and \( \Phi : E \to \tilde{\Delta}_\infty \) is a bijection with inverse \( \Psi \). In this case we can prove that \( P_t^{a,b} \) given in (1.4) for \( \tilde{\Delta}_\infty \) in place of \( \Delta_\infty \), i.e.

\[ P_{t}^{a,b} f(x) := \mathbb{E} f(\Phi(X^{\Psi(x)}(t))) = \bar{P}_{t}^{a,b}(f \circ \Phi)(\Psi(x)), \quad f \in \mathcal{B}_b(\tilde{\Delta}_\infty), t \geq 0, x \in \tilde{\Delta}_\infty, \]

is also a Markov semigroup. Indeed, since \( \Phi \circ \Psi(x) = x \) for \( x \in \tilde{\Delta}_\infty \), \( P_0^{a,b} \) is the identity operator. Moreover, for any \( t > 0 \) and any \( x \in [0,1]^N \), we have

\[ \mathbb{P}(X^x(t) \notin E) \leq \mathbb{P}(x_i(t) = 1 \text{ for some } i \in \mathbb{N}) = 0, \]

so that \( \Psi \circ \Phi(X^x(t)) = X^x(t) \) \( \mathbb{P} \)-a.s. Thus, by (1.3) and the semigroup property of \( \bar{P}_t^{a,b} \),

\[ P_{t}^{a,b}P_{s}^{a,b} f(x) = \mathbb{E} \left[ (P_{s}^{a,b} f) \circ \Phi(X^{\Psi(x)}(t)) \right] = \mathbb{E} \left[ (\bar{P}_{s}^{a,b} f \circ \Phi) \circ \Psi \circ \Phi(X^{\Psi(x)}(t)) \right] = \mathbb{E} \left[ \bar{P}_{s}^{a,b} f \circ \Phi \Phi(X^{\Psi(x)}(t)) \right] = \bar{P}_{t+s}^{a,b} f(\Psi(x)) = P_{t+s}^{a,b} f(x), \quad s, t > 0, f \in \mathcal{B}_b(\tilde{\Delta}_\infty), x \in \tilde{\Delta}_\infty. \]

So, \( Y^x(t) \) is a Markov process on \( \tilde{\Delta}_\infty \) for any \( x \in \tilde{\Delta}_\infty \). Moreover, as shown in [7, §3] that \( P_{t}^{a,b} \) is associated to the symmetric Dirichlet form \((\mathcal{E}_{a,b}, \mathcal{D}(\mathcal{E}_{a,b}))\) on \( L^2(\Xi_{a,b}) \).
It is now the position to state the main results in the paper. Let
\[ K_{a,b} = \frac{1}{4} \left( \sqrt{(4a-1)(4b-1) + 2(a+b) - 1} \right), \quad a, b > 0; \]
\[ \rho(s, t) = \int_{s \land t}^{s \lor t} \frac{dr}{\sqrt{r(1-r)}}, \quad s, t \in [0, 1]. \]

**Theorem 1.1.** Assume \( a_i \land b_i \geq \frac{1}{4} \) for all \( i \geq 1 \). Then for any positive \( f \in \mathcal{B}_b(\bar{\Delta}_\infty) \) and \( p > 1 \), the following Harnack inequality holds:
\[
(P_t f)^p(x) \leq (P_t f^p(y)) \exp \left[ \frac{p}{p-1} \sum_{i=1}^{\infty} \rho(\psi_i(x), \psi_i(y))^2 K_{a_i,b_i} \right], \quad x, y \in \bar{\Delta}_\infty, t > 0,
\]
where when \( K_{a_i,b_i} = 0 \) we set \( \frac{K_{a_i,b_i}}{\exp[2K_{a_i,b_i}t] - 1} = \frac{1}{2t} \).

The following is a consequence of Theorem 1.1.

**Corollary 1.2.** Assume \( a_i \land b_i \geq \frac{1}{4} \) for large \( i \geq 1 \). If
\[
\lim_{i \to \infty} \frac{a_i + b_i}{\log i} = \infty,
\]
then:

1. \( \Xi_{a,b} \) is the unique invariant probability measure of \( P_t^{a,b} \), and for any \( t > 0 \), \( P_t^{a,b} \) has a symmetric density \( p_t^{a,b}(x, y) \) with respect to \( \Xi_{a,b} \) such that
\[
C^{-1} e^{-c_0 \gamma(t)} \leq \inf_{x,y \in \bar{\Delta}_\infty} p_t^{a,b}(x, y) \leq \sup_{x,y \in \bar{\Delta}_\infty} p_t^{a,b}(x, y) \leq C e^{c_0 \gamma(t)}, \quad t > 0
\]
holds for some constant \( C \geq 1 \) and \( c_0 := 2\rho(0, 1) \), where
\[ \gamma(t) := \sum_{i=1}^{\infty} \frac{K_{a_i,b_i}}{\exp[K_{a_i,b_i}t] - 1} < \infty, \quad t > 0. \]

If \( \inf_{i \geq 1} (a_i \land b_i) \geq \frac{1}{4} \) then (1.8) holds for \( C = 1 \).

2. \( P_t^{a,b} \) is strong Feller with respect to the metric
\[ d(x, y) := \left( \sum_{i=1}^{\infty} i^{-2} \rho(\psi_i(x), \psi_i(y))^2 \right)^{\frac{1}{2}}, \quad \xi, \eta \in \bar{\Delta}_\infty. \]

3. Let \( \lambda = \inf_{i \geq 1} (a_i + b_i) \). Then there exists a constant \( c > 0 \) such that
\[
\sup_{x,y \in \bar{\Delta}_\infty} |p_t^{a,b}(x, y) - 1| \leq c e^{-\lambda t}, \quad t > 1.
\]
Remark 1.1  (1) If $a, b$ satisfies (1.2), then $\Xi_{a, b}$ is fully supported on the simplex $\Delta_\infty$, so that due to Corollary 1.2(1) we have $Y(t) \in \Delta_\infty$ $P$-a.s. for any $t > 0$ and any starting point $Y(0) \in \Delta_\infty$.

(2) It is well known that the uniform heat kernel upper bound $Ce^{c_0 \gamma(t)}$ of the heat kernel implies the super log-Sobolev inequality (see [17, Theorem 5.1.7] or [4, Theorem 2.2.3])

\begin{equation}
\Xi_{a, b}(f^2 \log f^2) \leq r\mathcal{E}_{a, b}(f, f) + \log C + c_0 \gamma(r), \quad r > 0, f \in \mathcal{P}(\mathcal{E}_{a, b}), \Xi_{a, b}(f^2) = 1,
\end{equation}

as well as the super Poincaré inequality (see [17, Theorem 3.3.15] or [15, Theorem 4.5])

\begin{equation}
\Xi_{a, b}(f^2) \leq r\mathcal{E}_{a, b}(f, f) + \beta(r)\Xi_{a, b}(|f|^2), \quad r > 0, f \in \mathcal{P}(\mathcal{E}_{a, b})
\end{equation}

for

\[
\beta(r) := C \inf_{t>0} \frac{r}{t} \exp \left[ c_0 \gamma(t) + \frac{t}{r} - 1 \right], \quad r > 0.
\]

This strengthens the log-Sobolev inequality derived in [7].

(3) Theorem 1.3 is stronger than the uniform ergodicity (also called strong ergodicity):

\[
\sup_{x \in \Delta_\infty} \|P_t^{a, b}(x, \cdot) - \Xi_{a, b}\|_{\text{var}} \leq Ce^{-\lambda t}, \quad t \geq 0
\]

for some constant $C > 0$, where $\| \cdot \|_{\text{var}}$ is the total variational and

\[
P_t^{a, b}(x, dy) := p_t^{a, b}(x, y)\Xi_{a, b}(dy)
\]

is the transition probability kernel of the infinite-dimensional diffusion process $Y(t)$.

(4) We also like to mention that by using explicit formula of the heat kernel, the super log-Sobolev inequality has been presented in [8, Theorem 4.1] for the infinite-many-neutral-alleles diffusion processes associated to the Poisson-Dirichlet distributions, which are the image of the corresponding GEM distributions of the descending order statistic.

To illustrate the above results, we consider below a special case where $a_i + b_i \geq b$ for some constant $b > 0$. This covers the two-parameter GEM case where $a_i = \frac{1}{2}$ and $b_i = \frac{b + a_i}{2}$ for some constants $\alpha \in (0, \frac{1}{2})$ and $\theta \geq \frac{1}{2} - \alpha$.

Corollary 1.3. Assume $\inf_{i \geq 1} b_i \geq \frac{1}{2}$, $a_i \geq \frac{1}{4}$ for large enough $i \geq 1$, and $a_i + b_i \geq b$ for some constant $b > 0$ and all $i \geq 1$. Then there exists a constant $C > 0$ such that

\begin{equation}
e^{-ct^{-2}} \leq p_t^{a, b} \leq e^{ct^{-2}}, \quad t > 0,
\end{equation}

and

\begin{equation}
\sup_{x, y \in \Delta_\infty} |p_t^{a, b}(x, y) - 1| \leq e^{ct^{-2} - \lambda t}, \quad t > 0,
\end{equation}

where $\lambda := \inf_{i \geq 1}(a_i + b_i)$. Consequently, (1.9) with $\gamma(r) = \frac{C}{r}$ and (1.10) with $\beta(r) = \exp \left[ \frac{C}{r} \right]$ hold for some constant $C > 0$.

The remainder of the paper is organized as follows. In Section 2 we establish the Harnack inequality and super Poincaré inequality for the Wright-Fisher diffusion processes, which are used in Section 3 to prove Theorem 1.1 and Corollaries 1.2, 1.3.
2 Functional inequalities for the Wright-Fisher diffusion processes

For $a, b > 0$, consider the following SDEs on $[0, 1]$:

\begin{equation}
\mathrm{d}x(t) = \{a - (a + b)x(t)\} \mathrm{d}t + \sqrt{x(t)(1-x(t))} \mathrm{d}B(t),
\end{equation}

where $B(t)$ is a one-dimensional Brownian motion. Let $P_t^{a,b}$ be the Markov semigroup of the solution. Then $P_t^{a,b}$ is symmetric with respect to $\pi_{a,b}$ and, see e.g. [5, §9], has a density $p_t^{a,b}(x, y)$ with respect to $\pi_{a,b}$.

In this section we investigate the Harnack inequality for $P_t^{a,b}$ and the super Poincaré inequality for the associated Dirichlet form

\[ E_{a,b}(f, f) := \frac{1}{2} \int_0^1 x(1-x)f''(x)^2 \, d\pi_{a,b}, \quad f \in \mathcal{D}(E_{a,b}), \]

where $\mathcal{D}(E_{a,b})$ is the completion of $C^1([0, 1])$ under the corresponding $E_{a,b}$-norm. These inequalities imply heat kernel estimates and will be applied in the next section to prove Theorem 1.1 and Corollaries 1.2-1.3.

We will see in Remark 2.1(2) and the proof of Theorem 2.2 that the Harnack inequality (2.2) we present below implies the sharp super Poincaré inequality for $a \wedge b \geq \frac{1}{4}$, and the sharp super Poincaré inequality for $a \wedge b \leq \frac{1}{4}$ will be proved using isoperimetric constants.

2.1 Harnack inequality and heat kernel estimates

For any $x \in [0, 1]$ and $r > 0$, let $B_r(x, r) = \{y \in [0, 1] : \rho(x, y) < r\}$.

**Theorem 2.1.** Let $a \wedge b \geq \frac{1}{4}$. Then for any $p > 1$ and positive $f \in \mathcal{B}_b([0, 1])$, the following Harnack inequality holds:

\begin{equation}
(P_t^{a,b}f)^p(x) \leq (P_t^{a,b}f^p(y)) \exp \left[ \frac{pK_{a,b} \rho(x,y)^2}{(p-1)(\exp[2K_{a,b} t] - 1)} \right], \quad x, y \in [0, 1], t > 0.
\end{equation}

Consequently, the heat kernel $p_t^{a,b}$ satisfies

\begin{equation}
\exp \left[ -\frac{2K_{a,b} \rho(x,y)^2}{\exp[K_{a,b} t] - 1} \right] \leq p_t^{a,b}(x, y) \leq \exp \left[ \frac{2K_{a,b} \rho(0,1)^2}{\exp[K_{a,b} t] - 1} \right], \quad t > 0, x, y \in [0, 1].
\end{equation}

**Proof.** (a) We first observe that (2.3) follows from (2.2). Let $p = 2$ and $P = P_{1/2}$. (2.2) implies

\begin{equation}
(Pf)^2(x) \leq (Pf^2(y))e^{\Psi(x,y)}, \quad x, y \in [0, 1], 0 \leq f \in \mathcal{B}_b([0, 1]),
\end{equation}

where $\Psi(x,y) := \frac{2K_{a,b} \rho(x,y)^2}{\exp[K_{a,b} t] - 1}$. So, applying [13, Theorem 1.4.1(5)] with $\Phi(r) = r^2$ and using the symmetry of $p(x,y) := p_t^{a,b}(x,y)$, we obtain

\[ p_t^{a,b}(x,y) = \int_{[0,1]} p(x,z)p(y,z)\pi_{a,b}(\mathrm{d}z) \geq e^{-\Psi(x,y)}, \]

7
which implies the desired lower bound estimate in (2.3). Next, by [18] Theorem 1.4.1(6), (2.4) implies
\[
(Pf)^2(x) \leq \frac{1}{\int_0^1 e^{-\Psi(x,y) \pi_{a,b}(dy)}} \pi_{a,b}(f^2) \leq 1.
\]
Taking \( f(z) = \frac{p(x,z)}{\sqrt{p_t(x,x)}} \), we arrive at
\[
p_t^{a,b}(x,x)^2 = (Pf(x))^2 \leq \frac{1}{\int_0^1 e^{-\Psi(x,y) \pi_{a,b}(dy)}} \leq \exp \left[ \frac{2K_{a,b} \rho(0,1)^2}{\exp[K_{a,b}t] - 1} \right], \quad x \in [0,1].
\]
This implies the desired upper bound estimate in (2.3) since
\[
p_t^{a,b}(x,x)^2 = \left( \int_{[0,1]} p(x,z) p(y,z) \pi_{a,b}(dz) \right)^{\frac{1}{2}} \left( \int_{[0,1]} p(y,z) \pi_{a,b}(dz) \right)^{\frac{1}{2}} = \sqrt{p_t^{a,b}(x,x)p_t^{a,b}(y,y)}.
\]
(b) next, we prove the Harnack inequality (2.2) using coupling by change of measures. Let \( T > 0 \) and \( x, y \in [0,1] \) be fixed. Without loss of generality, we assume that \( y > x \). Let \( x(t) \) solve (2.1) for \( x(0) = x \), and let \( y(t) \) solve the following equation on \([0,1]\) with reflection with \( y(0) = y \):
\[
\begin{align*}
dy(t) &= \{a - (a + b) y(t)\} dt + \sqrt{y(t)(1-y(t))} dB(t) \\
&\quad - 1_{[0,\tau]}(t) \sqrt{y(t)(1-y(t))} \xi(t) dt,
\end{align*}
\]
where \( \tau := \inf\{t \geq 0 : x(t) = y(t)\} \) is the coupling time and
\[
\xi(t) := \frac{\rho(x,y) \exp[K_{a,b}t]}{\int_0^T \exp[2K_{a,b}t] dt}, \quad t \geq 0.
\]
Below, we prove the inequality
\[
d\rho(x(t),y(t)) \leq -\{K_{a,b} \rho(x(t),y(t)) + \xi(t)\} dt, \quad t \in [0,\tau)
\]
by using Itô’s formula for \( \rho(x(t),y(t)) \), see (2.7) below. To avoid the singularity of \( \rho(x,y) \) for \( x < y \) at \( x = 0 \) and \( y = 1 \), one may prove (2.6) in a similar way by applying Itô’s formula to \( \rho_\varepsilon(x(t),y(t)) := \int_x^{y(t)} \frac{ds}{s(1+\varepsilon(s-x)^{-1})} \) for \( \varepsilon > 0 \) and finally letting \( \varepsilon \to 0 \).

Obviouslly, we have \( x(t) < y(t) \) for \( t < \tau \), and \( x(t) = y(t) \) for \( t \geq \tau \). Consequently, \( y(t) > 0 \) and \( x(t) < 1 \) for \( t < \tau \). Therefore, by Itô’s formula we obtain
\[
d\rho(x(t),y(t))
\begin{align*}
&= \left\{ \frac{4a - 1 - 2(2a + 2b - 1)y(t)}{4\sqrt{y(t)(1-y(t))}} - \frac{4a - 1 - 2(2a + 2b - 1)x(t)}{4\sqrt{x(t)(1-x(t))}} - \xi(t) \right\} dt
\end{align*}
\]
for \( t \in [0, \tau) \). Since \( y(t) > x(t) \) for \( t \in [0, \tau) \), we have
\[
\frac{4a - 1 - 2(2a + 2b - 1)y(t)}{4\sqrt{y(t)(1 - y(t))}} - \frac{4a - 1 - 2(2a + 2b - 1)x(t)}{4\sqrt{x(t)(1 - x(t))}} = \frac{1}{4} \int_{x(t)}^{y(t)} \frac{d}{ds} \left( \frac{4a - 1 - 2(2a + 2b - 1)s}{\sqrt{s(1 - s)}} \right) ds = -\frac{1}{8} \int_{x(t)}^{y(t)} \frac{4a - 1 + 4(b - a)s}{\{s(1 - s)\}^{3/2}} ds \\
\leq -c \int_{x(t)}^{y(t)} \frac{ds}{\sqrt{s(1 - s)}} = -c\rho(x(t), y(t)), \quad t \in [0, \tau),
\]
where
\[
(2.9) \quad c := \inf_{s \in (0, 1)} \frac{4a - 1 + 4(b - a)s}{8s(1 - s)} = K_{a,b}.
\]
Since \( (2.9) \) is trivial when \( a \land b = \frac{1}{3} \), we only prove it for \( a \land b > \frac{1}{3} \). In this case we have
\[
\frac{4a - 1 + 4(b - a)s}{8s(1 - s)} \rightarrow \infty \text{ as } s \rightarrow 0 \text{ or } 1, \text{ so that the inf is reached in } (0, 1). \text{ It is easy to see that in } (0, 1) \text{ we have } \frac{d}{ds} \left( \frac{4a - 1 + 4(b - a)s}{8s(1 - s)} \right) = 0 \text{ if and only if }
\[
4(b - a)s^2 + 2(4a - 1)s - (4a - 1) = 0,
\]
so that the inf is reached at
\[
s_0 = \sqrt{(4a - 1)(4b - 1) - (4a - 1)} = \frac{4a - 1}{4a - 1 + \sqrt{(4a - 1)(4b - 1)}}.
\]
Thus,
\[
c = \frac{4a - 1 + 4(b - a)s_0}{8s_0(1 - s_0)} = \frac{1}{4} \left( \sqrt{(4a - 1)(4b - 1) + 2a + 2b - 1} \right) = K_{a,b}.
\]
Combining \( (2.7), (2.8) \) and \( (2.9) \), we prove \( (2.6) \). Consequently,
\[
\rho(x(t), y(t)) \leq \rho(x, y)e^{-K_{a,b}t} - \int_0^t e^{-K_{a,b}(t-s)} \xi(s) ds = \frac{\rho(x, y)e^{-K_{a,b}t} \int_0^T e^{2K_{a,b}s} ds}{\int_0^T e^{2K_{a,b}s} ds}, \quad t \in [0, \tau).
\]
This implies \( \tau \leq T \), so that \( x(T) = y(T) \).
Now, rewrite \( (2.5) \) as
\[
dy(t) = \{a - (a + b)y(t)\} dt + \sqrt{y(t)(1 - y(t))} \, dB(t),
\]
where, by Girsanov’s theorem,
\[
\tilde{B}(t) := B(t) - \int_0^{\tau} \xi(s) ds, \quad t \geq 0.
\]
is a one-dimensional Brownian motion under the probability measure \( dQ := RdP \) for
\[
R := \exp \left[ \int_0^T \xi(t)dB(t) - \frac{1}{2} \int_0^T \xi(t)^2dt \right].
\]
So, by the weak uniqueness of the solution to (2.1), we have
\[
P_T^{a,b}f(y) = \mathbb{E}_Q f(y(T)) = \mathbb{E}[Rf(y(T))].
\]
Combining this with \( x(T) = y(T) \) observed above, we obtain
\[
(P_T^{a,b}f(y))^p = (\mathbb{E}[Rf(y(T))])^p = (\mathbb{E}[Rf(x(T))])^p
\leq (\mathbb{E}f^p(x(T)))(\mathbb{E}R_{\frac{p-1}{p-1}}^p)^{p-1} = (P_T^{a,b}f^p(x))(\mathbb{E}R_{\frac{p-1}{p-1}}^p)^{p-1}.
\]
This implies (2.2) since, by the definitions of \( R, \xi \) and the fact that \( \tau \leq T \),
\[
\mathbb{E}R_{\frac{p-1}{p-1}}^p \leq e^{\frac{p}{2(p-1)} \int_0^T \xi(t)^2dt} \mathbb{E} \left[ \frac{p}{p-1} \int_0^T \xi(t)dB(t) - \frac{p^2}{2(p-1)^2} \int_0^T \xi(t)^2dt \right]
= \exp \left[ \frac{p}{2(p-1)^2} \int_0^T \xi(t)^2dt \right] = \exp \left[ \frac{pK_{a,b}p(x,y)^2}{(p-1)^2(\exp[2K_{a,b}T] - 1)} \right].
\]

Remark 2.1. (1) From the proof we see that the condition \( a \wedge b \geq \frac{1}{4} \) is more or less essential for the desired explicit Harnack inequality using coupling by change of measures. This condition might be dropped using a localization argument as in [2], which, however, will lead to a less explicit Harnack inequality.

(2) We will see in the proof of Theorem 2.2 below that the Harnack inequality (2.2) also implies the heat kernel upper bound

\[
\sup_{x,y \in [0,1]} p_t^{a,b}(x,y) \leq \frac{c_{a,b}}{t^{2(a\vee b)}}, \quad t \in (0,1]
\]
for some constant \( c_{a,b} > 0 \), which is much better than (2.3) in short time. Next, by repeating the argument in the proof of Lemma 2.3 in [9], we see that the Harnack inequality (2.2) implies the following Gaussian type upper bound estimate: for any \( \delta > 2 \) there exists a constant \( C(\delta) > 0 \) such that

\[
p_t^{a,b}(x,y) \leq \frac{C(\delta) \exp\left[ -\frac{\rho(x,y)^2}{\delta} + C(\delta)t \right]}{\sqrt{\pi_{a,b}(B_{\sqrt{2}\rho}(x,\sqrt{t}))^{\pi_{a,b}(B_{\sqrt{2}\rho}(y,\sqrt{t}))}}, \quad t > 0, x, y \in [0,1],
\]
where we have used the fact that \( \sqrt{2}\rho \), rather than \( \rho \), is the intrinsic distance induced by the diffusion process. Moreover, according to [10] Theorem 7.2 which works for the present
case by using the transform \( x \mapsto \frac{x}{2} + \frac{1}{2} \) which maps \([-1, 1]\) therein onto the present \([0, 1]\), there exists constants \(c_1, c_2, c'_1, c'_2 > 0\) such that

\[
(2.12) \quad \frac{c'_1 \exp[-c'_2 \frac{\rho(x,y)^2}{t}]}{\sqrt{\pi_{a,b}(B_r(x, \frac{\sqrt{t}}{2})) \pi_{a,b}(B_r(y, \frac{\sqrt{t}}{2}))}} \leq p_{t}^{a,b}(x, y) \leq \frac{c_1 \exp[-c_2 \frac{\rho(x,y)^2}{t}]}{\sqrt{\pi_{a,b}(B_r(x, \frac{\sqrt{t}}{2})) \pi_{a,b}(B_r(y, \frac{\sqrt{t}}{2}))}}
\]

holds for all \( t \in (0, 1], x, y \in [0, 1] \). However, all these estimates can not be extended to infinite-dimensions.

(3) The leading term of the heat kernel \( p_t^{a,b}(x, y) \) has been figured out in the last display in [5] §9 as follows for \( b_0 = 2a \) and \( b_1 = 2b \) (since the reference measure used there is \( dy \) rather than the invariant measure \( \pi_{a,b}(dy) \), we multiply the factor \( y^{1-2a}(1 - y)^{1-2b} \):

\[
p_t^{a,b}(x, y) \sim \frac{1}{t^{2a}} e^{\frac{x+y}{t}} \psi_{2a} \left( \frac{x+y}{t^2} \right) + \frac{1}{t^{2b}} e^{\frac{y-x}{t}} \psi_{2b} \left( \frac{y-x}{t^2} \right),
\]

where \( \sqrt{s_l} := \sin^{-1} \sqrt{s}, \sqrt{s_r} := \sin^{-1} \sqrt{1-s} \) for \( s \in [0, 1] \), and

\[
\psi_b(x) := \sum_{j=0}^{\infty} \frac{x^j}{j! \Gamma(j+b)}, \quad x, b > 0.
\]

This suggests

\[
\sup_{x, y \in [0, 1]} p_t^{a,b}(x, y) \geq \max_{x \in \{t, 1-t\}} p_t^{a,b}(x, x) \geq \frac{c}{t^{(2a) \vee (2b)}}, \quad t \in (0, 1]
\]

for some constant \( c > 0 \), so that the above uniform heat kernel estimate (2.10) implied by the Harnack inequality is sharp for \( a \wedge b \geq \frac{1}{4} \). See Corollary 2.3 below for a sharp uniform heat kernel estimate also for \( a \wedge b \leq \frac{1}{4} \) using the super Poincaré inequality, which is of order \( t^{-(2a) \vee (2b)} \).

### 2.2 Super Poincaré inequality and heat kernel estimates

According to [14], the Dirichlet form \((\mathcal{E}_{a,b}, \mathcal{D}(\mathcal{E}_{a,b}))\) is said to satisfy the super Poincaré inequality if there exists a function \( \beta : (0, \infty) \to (0, \infty) \) such that

\[
(2.13) \quad \pi_{a,b}(f^2) \leq r \mathcal{E}_{a,b}(f, f) + \beta(r) \pi_{a,b}(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}_{a,b}).
\]

As \( \mathcal{D}(\mathcal{E}_{a,b}) \) is the closure of \( C^1([0, 1]) \) under the associated Dirichlet norm, one only needs to verify the inequality for \( f \in C^1([0, 1]) \).

**Theorem 2.2.** There exists a constant \( c = c(a, b) > 0 \) such that the following super Poincaré inequality

\[
(2.14) \quad \pi_{a,b}(f^2) \leq r \mathcal{E}_{a,b}(f, f) + \left( 1 \vee \frac{c}{r^{2(2a) \vee (2b)}} \right) \pi_{a,b}(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}_{a,b}).
\]

On the other hand, the super Poincaré inequality (2.13) implies

\[
(2.15) \quad \liminf_{r \to 0} \beta(r) r^{\frac{1}{2(2a) \vee (2b)}} > 0,
\]

so that (2.14) is sharp for small \( r > 0 \).
Proof. (1) The proof of (2.14) consists of the following four steps.

(1a) It is easy to see that the generator \( L_{a,b} := \frac{1}{2}x(1-x)\partial_x^2 + (a-(a+b)x)\partial_x \) has a spectral gap \( \lambda_1 = a + b \) with the first eigenvalue \( u(x) := x - \frac{a}{a+b} \). Then the Poincaré inequality

\[
\pi_{a,b}(f^2) \leq \frac{1}{a+b} \mathcal{E}_{a,b}(f,f) + \pi_{a,b}(f)^2, \quad f \in \mathcal{D}(\mathcal{E}_{a,b})
\]

holds. Thus, for the first assertion it suffices to prove (2.14) for small \( r > 0 \), say \( r \in (0, 1] \).

(1b) To prove (2.14) for \( r \in (0, 1] \), we first consider \( a \wedge b \geq \frac{1}{4} \) and prove (2.10) using the Harnack inequality (2.2). Since \( K_{a,b} \geq 0 \), we have

\[
\frac{K_{a,b}}{\exp[2K_{a,b}')} - 1 \leq \frac{1}{2t^2}, \quad t > 0.
\]

So, by (2.2) with \( p = 2 \) we obtain

\[
(P^a_b)^2(x) \exp \left[ - \frac{\rho(x, y)^2}{t} \right] \leq P^a_b f^2(y), \quad t > 0, x, y \in [0, 1], f \in \mathcal{B}_b([0, 1]).
\]

Let \( B_\rho(x, r) = \{ y \in [0, 1] : \rho(x, y) \leq r \} \) for \( x \in [0, 1] \) and \( r > 0 \). This implies

\[
(P^a_b)^2(x) \exp \left[ - \frac{\rho(x, y)^2}{t} \right] \pi_{a,b}(dy) \leq \int_{B_\rho(x, \sqrt{2t})} P^a_b f^2(y) \pi_{a,b}(dy) \leq 1, \quad \pi_{a,b}(f^2) \leq 1.
\]

Taking

\[
f(z) = \frac{p^a_b(x, z)}{\sqrt{p^a_b(x, x)}}, \quad z \in [0, 1],
\]

we arrive at

\[
p^a_b(x, x) \leq \frac{e^2}{\pi_{a,b}(B_\rho(x, \sqrt{2t}))}, \quad x \in [0, 1], t > 0.
\]

Similar but less explicit estimates can be derived from (2.11) or (2.12). We intend to prove

\[
\inf_{x \in [0, 1]} \pi_{a,b}(B_\rho(x, \sqrt{t})) \geq c_0 t^{2(a\wedge b)}, \quad t \in [0, 1]
\]

for some constant \( c_0 > 0 \), so that (2.10) follows from (2.17).

Let \( x \in [0, \frac{1}{2}] \) and take \( t_0 = \rho(\frac{1}{2}, \frac{3}{4}) \). Then there exists a unique \( y_t \in (x, \frac{3}{4}) \) such that

\[
\sqrt{t} \wedge t_0 \leq \rho(x, y_t) = \int_x^{y_t} \frac{ds}{\sqrt{s(1-s)}} \leq 2 \int_x^{y_t} \frac{ds}{\sqrt{s}} = 4(\sqrt{y_t} - \sqrt{x}).
\]

So,

\[
B_\rho(x, \sqrt{t}) \supset B_\rho(x, \sqrt{t_0} \wedge t) \supset [x, (\sqrt{x} + \frac{1}{4} \sqrt{t \wedge t_0})^2].
\]
Combining this with \((\sqrt{x} + \frac{1}{4}\sqrt{t \wedge t_0})^2 \leq y_t^2 \leq \frac{3}{4}\), and noting that \(4a \geq 1\), we obtain

\[
\int_{B_\rho(x, \sqrt{t})} s^{2a-1}(1 - s)^{2b-1} ds \geq \frac{1}{4(2b-1)^{+}} \int_{(\sqrt{x} + \frac{1}{4}\sqrt{t \wedge t_0})^2} s^{2a-1} ds
\]

\[
= \frac{1}{2a4^{(2b-1)^{+}}} \left( (\sqrt{x} + \frac{1}{4}\sqrt{t \wedge t_0})^{4a} - (\sqrt{x} + \frac{1}{8}\sqrt{t \wedge t_0})^{4a} \right)
\]

\[
\geq \frac{\sqrt{t \wedge t_0}}{4^{1+(2b-1)^{+}}} (\sqrt{x} + \frac{1}{8}\sqrt{t \wedge t_0})^{4a-1} \geq \frac{(t \wedge t_0)^{2a}}{8^{4a-1}4^{1+(2b-1)^{+}}}
\]

Therefore,

\[
\inf_{x \in [0, \frac{1}{2}]} \pi_{a,b}(B_\rho(x, \sqrt{t})) \geq c_1 t^{2a}, \quad t \in [0, 1]
\]

holds for some constant \(c_1 > 0\). Similarly, we have

\[
\inf_{x \in [\frac{1}{2}, 1]} \pi_{a,b}(B_\rho(x, \sqrt{t})) \geq c_2 t^{2b}, \quad t \in [0, 1]
\]

for some constant \(c_2 > 0\). Combining them together we prove (2.18), and hence (2.10) as observed above.

Now, according to [17] Theorem 3.3.15 or [15] Theorem 4.5, (2.10) implies the super Poincaré inequality (2.13) for

\[
\beta(r) := \inf_{t > 0} \left\{ \frac{r}{t} e^{\frac{r}{t} - 1} \| p^{a,b}_r \|_\infty \right\} \leq \| p^{a,b}_r \|_\infty \leq \frac{c_0}{r^{2(a+b)}}, \quad r \in (0, 1].
\]

That is, (2.14) holds for \(r \in (0, 1]\).

(1c) Next, we consider the case that \(a \vee b \leq \frac{1}{4}\), and prove (2.13) for small \(r > 0\) using isoperimetric constants. Let \(\mu_a(dx) = 1_{[0, \frac{1}{2}]}(x)x^{2a-1}dx\). Let \(\mu^\partial_a\) be the boundary measure induced by \(\mu_a\) under the intrinsic metric \(\rho\). We have

\[
(2.19) \quad \mu^\partial_a(\{x\}) := \lim_{\varepsilon \to 0} \frac{\mu_a(\{y : \rho(y, x) \leq \varepsilon\})}{2\varepsilon} \geq c_1 x^{2a-\frac{1}{2}}, \quad x \in (0, 1/2)
\]

for some constant \(c_1 > 0\). Now, for any set \(A \subset [0, \frac{1}{2}]\) with \(\mu_a(A) \in (0, \mu_a([0, \frac{1}{2}]))\), let \(\partial_0 A\) be the set of boundary points of \(A\) included in \((0, \frac{1}{2})\). Then \(\partial_0 A \neq \emptyset\). It follows from (2.19) and \(2a - \frac{1}{2} \leq 0\) that

\[
k(s) := \inf_{A \subset [0, \frac{1}{2}], \mu_a(A) \leq s} \frac{\mu^\partial_a(\partial_0 A)}{\mu_a(A)} \geq \frac{c_1}{s}, \quad 0 < s < \mu_a([0, 1/2]).
\]

So,

\[
k^{-1}(2r^{-1/2}) \geq c_2 \sqrt{r}
\]

holds for some constant \(c_2 > 0\) and small \(r > 0\). Therefore, according to [17] Theorem 3.4.16], the super Poincaré inequality

\[
(2.21) \quad \mu_a(f^2) \leq r \int_0^{\frac{1}{r}} x f'(x)^2 \mu_a(dx) + \frac{c}{\sqrt{r}} \mu_a(|f|)^2, \quad r \in (0, 1], f \in C_b^1([0, 1/2])
\]
holds for some constant \( c > 0 \). In case the book [17] is not easy to find, we present below a brief proof of the assertion, see also the proof of Theorem 3.4(1) in [14] where the last term in the first display should be changed into \( \frac{2k(r)}{r} \). In fact, let \( f \in C^1_b([0, \frac{1}{2}]) \) with \( \mu_a(|f|) = 1 \). We have \( \mu_a(f^2 > t) \leq t^{-1/2} \) so that by the coarea formula,

\[
\int_0^{\frac{1}{2}} \sqrt{x(1-x)} |(f^2)'(x)| \mu_a(dx) = \int_0^\infty \mu_a^0(\{f^2 = t\} \setminus \{0, 1/2\})dt \geq k(s) \int_{s^{-2}}^\infty \mu_a(f^2 > t)dt \\
\leq k(s)\mu_a(f^2) - k(s) \int_0^{s^{-2}} \frac{dt}{\sqrt{t}} = k(s)\mu_a(f^2) - \frac{2k(s)}{s}, \quad s \in (0, \mu_a([0, 1/2]).
\]

Combining this with

\[
\int_0^{\frac{1}{2}} \sqrt{x(1-x)} |(f^2)'(x)| \mu_a(dx) \leq 2\sqrt{\mu_a(f^2)} \left( \int_0^{\frac{1}{2}} x(1-x)f'(x)^2 \mu_a(dx) \right)^\frac{1}{2} \\
\leq \frac{2}{k(s)} \int_0^{\frac{1}{2}} x(1-x)f'(x)^2 \mu_a(dx) + \frac{k(s)}{2s} \mu_a(f^2),
\]

we prove

\[
\mu_a(f^2) \leq \frac{4}{k(s)^2} \int_0^{\frac{1}{2}} x(1-x)f'(x)^2 \mu_a(dx) + \frac{4}{s}, \quad \mu_a(|f|) = 1, s \in (0, \mu_a([0, 1/2]).
\]

Taking \( s = k^{-1}(2r^{-1/2}) \) in this inequality and using (2.20), we prove (2.21) for small \( r > 0 \). Consequently,

\[
\pi_{a,b}(f^21_{[0, \frac{1}{2}]}(x) \leq r \int_0^{\frac{1}{2}} (1-x)f'(x)^2 \pi_{a,b}(dx) + \frac{c}{\sqrt{r}} \pi_{a,b}(|f|1_{[0, \frac{1}{2}]} \leq r, \quad r \in (0, 1]
\]

holds for some constant \( c > 0 \) and all \( f \in C^1_b([0, 1]) \).

Similarly, when \( b \leq \frac{1}{4} \), we have

\[
\pi_{a,b}(f^21_{[\frac{1}{2}, 1]} \leq r \int_{\frac{1}{2}}^1 (1-x)f'(x)^2 \pi_{a,b}(dx) + \frac{c}{\sqrt{r}} \pi_{a,b}(|f|1_{[\frac{1}{2}, 1]} \leq r, \quad r \in (0, 1]
\]

for some constant \( c > 0 \) and all \( f \in C^1_b([0, 1]) \). Combining them together we prove (2.14) with \( r \in (0, 1] \) for \( a \lor b \leq \frac{1}{4} \).

(1d) Finally, let \( a \land b < \frac{1}{4} \) but \( a \lor b \geq \frac{1}{4} \), for instance, we assume that \( a < \frac{1}{4} \) and \( b \geq \frac{1}{4} \). In this case we have \( x^{2a-1} \geq x^{-\frac{1}{2}} \) for \( x \in (0, 1] \), but \( x^{2a-1} \leq 2^{\frac{1}{2}-2a}x^{-\frac{1}{2}} \) for \( x \in [\frac{1}{2}, 1] \). So, by (2.14) for \( a \land b \geq \frac{1}{4} \) we obtain

\[
\pi_{a,b}(f^21_{[\frac{1}{2}, 1]} \leq c_1 \pi_{a,b}^0(f^2) \leq r \rho_{\frac{1}{2}, b}(f, f) + \frac{c_2}{\rho_{a,b}} \pi_{a,b}^0(|f|)^2 \\
\leq c_3 \rho_{a,b}(f, f) + \frac{c_3}{\rho_{a,b}} \pi_{a,b}(|f|)^2 \quad r \in (0, 1], f \in C^1_b([0, 1])
\]

14
for some constants $c_1, c_2, c_3 > 0$. Combining this with (2.22), we prove (2.14) for $r \in (0, 1]$. 

(2) To prove the second assertion, let (2.13) hold for some $b$. Take $f(x) = (\varepsilon - x)^+$ for $\varepsilon \in (0, 1/2)$. Then there exists constants $c_1, c_2 > 0$ such that

$$
\pi_{a,b}(f^2) \geq c_1 \varepsilon^{2a+2}, \quad \pi_{a,b}(f) + \delta_{a,b}(f, f) \leq c_2 \varepsilon^{2a+1}, \quad \varepsilon \in (0, 1/2).
$$

So, by (2.13) we obtain

$$
\beta(r) \geq \frac{1}{c_3^2} \sup_{\varepsilon \in (0, \frac{1}{2})} \left( \frac{c_1}{\varepsilon^{2a}} - \frac{r c_2}{\varepsilon^{2a+1}} \right) = c_3 r^{-2a}
$$

for some constant $c_3 > 0$ and small $r > 0$. Therefore,

$$
\lim inf_{r \to 0} \beta(r) r^{2a} \geq c_3 > 0.
$$

Similarly, by taking $f(x) = (x + \varepsilon - 1)^+$ in (2.13) we obtain $\lim inf_{r \to 0} \beta(r) r^{2b} > 0$; while (2.13) with $f(x) := (x + \varepsilon - \frac{1}{2})^+ \land (\frac{1}{2} + \varepsilon - x)^+$ implies $\lim inf_{r \to 0} \beta(r) r^{1/2} > 0$. In conclusion, (2.15) holds.

We would like to indicate that when $a \land b > \frac{1}{4}$, the desired super Poincaré inequality can also be proved using isoperimetric constants. However, the argument we used is more straightforward and it stresses the sharpness of the Harnack inequality (2.2).

**Corollary 2.3.** There exist constants $c_1, c_2 > 0$ such that

$$
(2.24) \quad \sup_{x,y \in [0, 1]} |p_{t}^{a,b}(x, y) - 1| \leq \frac{c_1 e^{-(a+b)t}}{(t \land 1)^{\frac{1}{2} \sqrt{2a} \sqrt{2b}}}, \quad t > 0;
$$

$$
(2.25) \quad \sup_{x,y \in [0, 1]} p_{t}^{a,b}(x, y) \geq \frac{c_2}{t^{\frac{1}{2} \sqrt{2a} \sqrt{2b}}}, \quad t \in (0, 1].
$$

**Proof.** (1) Proof of (2.2). By [17, Theorem 3.3.15 (2)] or [15, Theorem 4.5], (2.14) implies (2.26) for some constant $c > 0$. So, it suffices to prove (2.24) for $t \geq 2$. By the Poincaré inequality (1.16), we have

$$
\|P_{t}^{a,b} - \pi_{a,b}\|_{2 \to 2} \leq e^{-(a+b)t}, \quad t \geq 0,
$$

where, for any $p, q \geq 1$, $\| \cdot \|_{p \to q}$ stands for the operator norm from $L^p(\pi_{a,b})$ to $L^q(\pi_{a,b})$. Combining this with (2.26) we obtain

$$
\|P_{t}^{a,b} - \pi_{a,b}\|_{2 \to \infty} \leq \|P_{1}^{a,b} - \pi_{a,b}\|_{2 \to \infty} \|P_{t-1}^{a,b} - \pi_{a,b}\|_{2 \to 2} \leq C e^{-(a+b)t}, \quad t \geq 1
$$
for some constant $C > 0$. Therefore, by the symmetry of $P_{t}^{a,b}$ in $L^{2}(\pi_{a,b})$, this implies

$$\sup_{x,y \in [0,1]} p_{t}^{a,b}(x, y) - 1 = \| P_{t}^{a,b} - \pi_{a,b} \|_{1} \leq \| P_{t}^{a,b} - \pi_{a,b} \|_{2}. \| P_{t}^{a,b} - \pi_{a,b} \|_{2} \rightarrow \infty$$

$$= \| P_{t}^{a,b} - \pi_{a,b} \|_{2}^{2} \leq C^{2} e^{-(a+b)t}, \ t \geq 2.$$

Therefore, (2.24) holds also for $t \geq 2$.

(2) To prove (2.25), we use again [17] Theorem 3.3.15] or [15] Theorem 4.5] that (2.13) holds for

$$\beta(r) := \inf_{t > 0} \left\{ \frac{r}{t} e^{t-1} \| P_{t}^{a,b} \|_{\infty} \right\} \leq \| P_{t}^{a,b} \|_{\infty}.$$

Combining this with the second assertion in Theorem 2.2, we obtain

$$\lim_{t \rightarrow 0} \inf \| P_{t}^{a,b} \|_{\infty}^{2(a)\lor(2b)\lor \frac{1}{2}} > 0,$$

which implies (2.26) for some constant $c_{2} > 0$.

3 Proofs of Theorem 1.1 and Corollaries 1.2-1.3

Proof of Theorem 1.1. By (1.5) and Theorem 2.1 we have

$$\left( P_{t}^{a,b} f(x) \right)^{p} = \left( P_{t}^{a,b} (f \circ \Phi)(\Psi(x)) \right) = \left( \prod_{i=1}^{\infty} P_{t}^{a_{i},b_{i}} f(\Psi(x)) \right)^{p}$$

$$\leq \left\{ \left( \prod_{i=1}^{\infty} P_{t}^{a_{i},b_{i}} \right)^{p} \exp \left[ \sum_{i=1}^{\infty} \frac{pK_{a_{i},b_{i}}}{(p-1)(\exp[2K_{a_{i},b_{i}}] - 1)} \right] \right\}$$

$$= (P_{t}^{a,b} f^{p}(y)) \exp \left[ \frac{p}{p-1} \sum_{i=1}^{\infty} \frac{K_{a_{i},b_{i}}}{\exp[2K_{a_{i},b_{i}}] - 1} \right].$$

Thus, (1.6) holds.

Proof of Corollary 1.2. (a) Let $i_{0} \geq 0$ such that $a_{i} \land b_{i} \geq \frac{1}{4}$ for $i > i_{0}$. It is easy to see that (1.7) implies $\gamma(t) < \infty$ for all $t > 0$. By (2.22), there exists a constant $c_{1} \geq 1$ such that

$$c_{1}^{-1} e^{-c_{1}t^{-1}} \leq P_{t}^{a_{i},b_{i}}(x, y) \leq c_{1} e^{c_{1}t^{-1}}, \ t > 0, 1 \leq i \leq i_{0}.$$

Combining this with (2.3) we obtain

$$C^{-1} e^{-c_{1}t} \leq \prod_{i=1}^{\infty} P_{t}^{a_{i},b_{i}}(x_{i}, y_{i}) \leq CE^{c_{1}t}, \ t > 0, x = (x_{1}, x_{2}, \ldots), y = (y_{1}, y_{2}, \ldots) \in [0,1]^{N}$$

for some constant $C \geq 1$, and $C = 1$ if $i_{0} = 0$. So, according to (1.4) and the definition of $\Xi_{a,b}$, the density of $P_{t}^{a,b}$ with respect to $\Xi_{a,b}$ exists and is given by

$$P_{t}^{a,b}(x, y) = \prod_{i=1}^{\infty} P_{t}^{a_{i},b_{i}}(\psi_{i}(x), \psi_{i}(y)), \ t > 0, x, y \in \Delta_{\infty}.$$


Thus, (1.8) holds.

Next, since $P_t^{a,b}$ is symmetric in $L^2(\Xi_{a,b})$, $\Xi_{a,b}$ is its invariant probability measure. Moreover, (1.8) implies the Harnack inequality

$$(P_t^{a,b} f(x))^p \leq (P_t^{a,b} f(y))^p C_{a,b}^p e^{2p\gamma(t)}, \quad t > 0, p > 1, x, y \in \bar{\Delta}_\infty$$

for all positive $f \in \mathcal{B}_b(\bar{\Delta}_\infty)$. Then, according to [18, Theorem 1.4.1(3)] or [16, Proposition 3.1(3)], $P_t^{a,b}$ has a unique invariant probability measure. Therefore, the proof of (1) is finished.

(b) Since for any $a, b > 0$ the semigroup $P_t^{a,b}$ has a continuous density with respect to $\pi_{a,b}$ (see [5, §9] as mentioned in Remark 2.1), it is strong Feller with respect to the metric $\rho$. So, due to the first equality in (1.1), it is strong Feller with respect to the metric $d$. Moreover, by the symmetry of $P_t^{a,b}$, the strong Feller property of $P_t^{a,b}$ with respect to $d$ is not affected by changing finite many $(a_i, b_i)$. Thus, without loss of generality, we may and do assume that $a_i \wedge b_i \geq \frac{1}{4}$ for all $i \geq 1$. In this case, (1.7) implies

$$C(t) := \sup_{i \geq 1} \frac{q^2 K_{a_i, b_i}}{\exp[2K_{a_i, b_i}^i]} < \infty, \quad t > 0.$$ 

Then (1.6) yields

$$(P_t^{a,b} f(x))^p \leq (P_t^{a,b} f(y))^p \exp\left[\frac{pC(t)d(x, y)^2}{p - 1}\right], \quad t > 0, x, y \in \bar{\Delta}_\infty.$$ 

According to [18] Theorem 1.4.1(1) or [16] Proposition 3.1(1), $P_t^{a,b}$ is strong Feller with respect to the metric $d$, i.e. (2) holds.

(c) Finally, as in [7, Theorem 3.1], the Poincaré inequality

$$\Xi_{a,b}(f^2) \leq \frac{1}{\lambda} \delta_{a,b}(f, f), \quad f \in \mathcal{D}(\delta_{a,b}), \Xi_{a,b}(f) = 0$$

holds. So,

$$\|P_t^{a,b} - \Xi_{a,b}\|_2 \leq e^{-\lambda t}, \quad t \geq 0,$$

where $\| \cdot \|_2$ is the $L^2$-norm with respect to $\Xi_{a,b}$. On the other hand, (1) implies

$$\|P_{\frac{t}{2}}^{a,b} - \Xi_{a,b}\|_{1 \rightarrow \infty} < \infty.$$ 

Moreover, by the symmetry of $P_t^{a,b}$, we have

$$\|P_{\frac{t}{2}}^{a,b} - \Xi_{a,b}\|_{L^1(\Xi_{a,b}) \rightarrow L^\infty(\Xi_{a,b})} = \|P_{\frac{t}{2}}^{a,b} - \Xi_{a,b}\|_{L^2(\Xi_{a,b}) \rightarrow L^\infty(\Xi_{a,b})} \leq \|P_{\frac{t}{2}}^{a,b} - \Xi_{a,b}\|_{2 \rightarrow \infty},$$

where $\| \cdot \|_{2 \rightarrow \infty}$ is defined as $\| \cdot \|_{1 \rightarrow \infty}$ using $\Xi_{a,b}(f^2) \leq 1$ in place of $\Xi_{a,b}(|f|) \leq 1$, i.e. for a linear operator $P$ on $L^2(\Xi_{a,b})$,

$$\|P\|_{2 \rightarrow \infty} := \sup_{\Xi_{a,b}(f^2) \leq 1} \sup_{x \in \bar{\Delta}_\infty} |Pf(x)|.$$
Therefore,
\[
\| P_{t}^{a,b} - \Xi_{a,b} \|_{1 \to \infty} \leq \| P_{t}^{a,b} - \Xi_{a,b} \|_{L^1(\Xi_{a,b}) \to L^2(\Xi_{a,b})} \| P_{t}^{a,b} - \Xi_{a,b} \|_{2 \to \infty}
\]
\[
\leq \| P_{t}^{a,b} - \Xi_{a,b} \|_{2}^{2} \| P_{t}^{a,b} - \Xi_{a,b} \|_{2}^{2} \| P_{t}^{a,b} - \Xi_{a,b} \|_{2} \leq c e^{-\lambda t}
\]
holds for some constant \( c > 0 \) and all \( t \geq 1 \). Now, for any \( t \geq 1 \) and \( \varepsilon > 0 \), with \( f(z) := p_{\varepsilon}^{a,b}(z, y) \) this implies
\[
p_{t,\varepsilon}^{a,b}(x, y) = (P_{t}^{a,b} f)(x) \leq ce^{-\lambda t} \Xi_{a,b}(f) = ce^{-\lambda t}, \quad x, y \in \bar{\Delta}_{\infty}.
\]
Therefore, the proof of (3) is finished.

Proof of Corollary 1.3. By Corollary 1.2(3) and Remark 1.1(2), it suffices to prove \( \gamma(t) \leq \frac{c}{t^2} \) for some constant \( c > 0 \) and \( t \in (0, 1] \) as in this case the estimate holds for all \( t > 0 \), so that
\[
\inf_{t > 0} \left( \frac{r}{t} \exp \left[\frac{c_0\gamma(t) + \frac{t}{r} - 1}{t} \right] \right) \leq \inf_{t > 0} \left( \frac{r}{t} \exp \left[\frac{c_0c}{t^2} + \frac{t}{r} - 1 \right] \right) \leq \exp \left[\frac{C}{r^2} \right], \quad r > 0
\]
holds for some constant \( C > 0 \) by taking \( t = r^{\frac{1}{4}} \) for \( r < 1 \).

By the definition of \( K_{a_i, b_i} \) and the condition \( a_i + b_i \geq bi \) for some constant \( b > 0 \), there exist \( i_0 \in \mathbb{N} \) and a constant \( c_1 > 0 \) such that \( K_{a_i, b_i} \geq c_1 i \) for \( i \geq i_0 \). Since \( K_{a_i, b_i} \geq 0 \) and \( e^{s} \geq \frac{s}{2} e^{s/2} \) holds for \( s \geq 0 \), we get
\[
\gamma(t) \leq \frac{i_0}{t} + \sum_{i > i_0} \frac{c_1 i}{\exp[ci t]} - 1 \leq \frac{i_0}{t} + \sum_{i > i_0} \frac{2}{t} e^{-c_1 i t/2}
\]
\[
\leq \frac{i_0}{t} + \frac{2}{t} \int_{1}^{\infty} e^{-c_1 s/2} ds = \frac{i_0}{t} + \frac{4}{c_1 t^2} \leq \frac{c}{t^2}, \quad t \in (0, 1]
\]
for some constant \( c > 0 \). The proof is finished.

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