Zipf’s law from a Fisher variational-principle

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Abstract

Zipf’s law is shown to arise as the variational solution of a problem formulated in Fisher’s terms. An appropriate minimization process involving Fisher information and scale-invariance yields this universal rank distribution. As an example we show that the number of citations found in the most referenced physics journals follows this law.

Key words: Fisher information, scale-invariance, Zipf’s law

1. Introduction

This work discusses the application of Fisher’s information measure to some scale-invariant phenomena. We thus begin our considerations with a brief review of the pertinent ingredients.

1.1. Scale-invariant phenomena

The study of scale-invariant phenomena has unravelled interesting and somewhat unexpected behaviours in systems belonging to disciplines of different nature, from physical and biological to technological and social sciences. Indeed, empirical data from percolation theory and nuclear multufacturation \(^2\) reflect scale-invariant behaviour, and so do the abundances of genes in various organisms and tissues \(^3\), the frequency of words in natural languages \(^4\), scientific collaboration networks \(^5\), the Internet traffic \(^6\), Linux packages links \(^7\), as well as electoral results \(^8\), urban agglomerations \(^9\) and firm sizes all over the world \(^11\).

The common feature in these systems is the lack of a characteristic size, length or frequency for an observable \(k\) at study. This lack generally leads to a power law distribution \(p(k)\), valid in most of the domain of definition of \(k\),

\[
p(k) \sim 1/k^{1+\gamma},
\]

with \(\gamma \geq 0\). Special attention has been paid to the class of universality defined by \(\gamma = 1\), which corresponds to Zipf’s law in the cumulative distribution or the rank-size distribution \(2, 3, 4, 6, 7, 8, 10, 11, 12\). Recently, Maillart et al. \(^7\) have studied the evolution of the number of links to open source software projects in Linux packages, and have found that the link distribution follows Zipf’s law as a consequence of stochastic proportional growth. In its simplest formulation, the stochastic proportional growth model, or namely the geometric Brownian motion, assumes the growth of an element of the system to be proportional to its size \(k\), and to be governed by a stochastic Wiener process. The class \(\gamma = 1\) emerges from the condition of stationarity, i.e., when the system reaches a dynamic equilibrium \(12\). Together with geometric Brownian motion, there is a variety of models arising in different fields that yield Zipf’s law and other power laws on a case-by-case basis \(9, 10, 12, 13, 14\), as preferential attachment \(6\) and competitive cluster growth \(15\) in complex networks, used to explain many of the scale-free properties of social, technological and biological networks.

1.2. Fisher’s information measure

Much effort has recently been devoted to Fisher’s information measure (FIM), usually denoted as \(I\). The work of Frieden and co-workers \(16, 17, 18, 19, 20, 21, 22, 23, 24, 25\), Silver \(26\), and Plastino et al. \(27, 28, 29, 30\), among many others, has shed much light upon the manifold physical applications of \(I\). As a small sample we mention that Frieden and Soffer have shown that FIM provides a powerful variational principle, called EPI (extreme physical information) that yields the canonical Lagrangians of theoretical physics \(24\). Additionally, \(I\) has been proved to characterize an arrow of time with reference to the celebrated Fokker-Planck equation \(28\). Moreover, there exist interesting relations that connect FIM and the relative Shannon information measure invented by Kullback \(31, 32\). These can be shown to have some bearing on the time evolution of arbitrary systems governed by quite general continuum equations \(29, 30\). Additionally, a rather general \(I\)-based H theorem has recently been proved \(33, 34\). As for Hamiltonian systems \(35\), EPI allows to describe the behaviour of complex systems, as the allometric or power laws found in biological sciences \(36\). The pertinent list could be extended quite a bit. \(I\) is then an important quantity, involved in many aspects of the theoretical description of nature.
For our present purposes it is of the essence to mention that Frieden et al. [37] have also shown that equilibrium and non-equilibrium thermodynamics can be derived from a principle of minimal Fisher information, with suitable constraints (MFI). Here I is specialized to the particular but important case of translation families, i.e., distribution functions whose form does not change under translational transformations. In this case, Fisher measure becomes

\[ I(F) = c_k \int_{k_1}^{k_2} dk F(k) |\frac{\partial}{\partial \theta} \ln F(k)\|^2. \]

In the continuous limit \((\Delta k \to dk)\), the Fisher information measure is cast as

\[ I(F) = c_k \int_{k_1}^{k_2} dk F(k) |\frac{\partial}{\partial \theta} \ln F(k)\|^2. \]

Instead of using translation invariance à la Frieden-Soffer [24], we will appeal to scaling invariance [38] so that we can anticipate some new physics. All members of the family \( F(k/\theta) \) possess identical shape —there are no characteristic size, length or frequency for the observable \( k \)— namely \( dk F(k/\theta) = dk' F(k') \) under the transformation \( k' = k/\theta \).

To deal with this new symmetry it is convenient to change to the new coordinate \( u = \ln k \) and parameter \( \Theta = \ln \theta \). Why? Because then the scale invariance becomes again translational invariance, and we are entitled to use one essential result of [34], namely, that MFI leads to a Schrödinger-like equation. Note that the new coordinate \( u' = \ln k' \) transforms as \( u' = u - \Theta \). Defining \( f(u) = F(e^u) \) and taking into account the fact that the Jacobian of the transformation is \( |dx/du| = e^u \) and \( \partial f/\partial \theta = e^{-\Theta} \partial f/\partial \Theta \), the Fisher information measure acquires now the form

\[ I(F) = c_k e^{-2\Theta} \int_{u_1}^{u_2} du e^u f(u) \left( \frac{\partial \ln f(u)}{\partial u} \right)^2, \]

where \( u_1 = \ln k_1 \), and the factor \( e^{-2\Theta} \) guarantees the invariance of the associated Cramer-Rao inequality as shown in [38].

For reasons that will become apparent below, we will apply the MFI without any constraint. This is tantamount to posing no bound to the physical “sizes” that characterize the system. The extremization of Fisher information with no constraints (\( \mu_i = 0 \)) is written as

\[ \delta \left\{ \int_{u_1}^{u_2} du e^u f(u) \left( \frac{\partial \ln f(u)}{\partial u} \right)^2 \right\} = 0. \]

Introducing \( f(u) = e^{-\Theta^2} \), and varying with respect to \( \Psi \) and \( \partial \Psi/\partial u \) as in [37], one is easily led to a (real) Schrödinger-like equation of the form

\[ \left[ -\frac{\partial^2}{\partial u^2} + 1 \right] \Psi(u) = 0. \]

Notice that the lack of normalization constraints implies zero eigenvalue, since the Lagrange multiplier associated with the normalization is the energy eigenvalue [37]. At this point we introduce boundary conditions to guarantee convergence of the Fisher measure [3] and thus compensate for the lack of constraints in 6. We impose \( \lim_{u \to \infty} \Psi(u) = 0 \) and \( \Psi(u_1) = \sqrt{N} \), where \( N \) is an dimensionless constant the meaning of which will become clear later. The solution to 7 with these boundary conditions is \( \Psi(u) = \sqrt{N} e^{-|u-u_1|/2} \), which leads to \( f(u) = N e^{-|u-u_1|} \) and to the density distribution

\[ F(k) \Delta k = N_k e^{|k-k_1|/2} \Delta k, \]

with \( N = 1 \) for a density normalized to unity. This distribution is just the Zipf’s law [1] of universal class \( y = 1 \) of Refs. [2] [3] [4] [6] [7] [2] [11] [12]. This result is remarkable: Zipf’s law has been here derived from first principles.

1.3. Goals and motivation

Scale-invariant phenomena are generally addressed by appeal to ad-hoc models (see the references citing in 1.1). In spite of the success of these models, the intrinsic complexity involved therein makes their study at a macroscopic level a rather difficult task. One sorely misses a general formulation of the thermodynamics of scale-invariant physics, which is not quite established yet. It is our goal here to show, in such a vein, that thermodynamics of scale-invariant physics, which is not quite thermodynamics is generated in [34].

2. Minimum Fisher Information approach (MFI)

The Fisher information measure \( I \) for a system described by a set of coordinates \( q \) and physical parameters \( \theta \), has the form [34]

\[ I(F) = \int_{\Omega} dq F(q|\theta) \sum_{ij} c_{ij} \frac{\partial}{\partial \theta_i} \ln F(q|\theta) \frac{\partial}{\partial \theta_j} \ln F(q|\theta), \quad (2) \]

where \( F(q|\theta) \) is the density distribution in a configuration space \( q \) of volume \( \Omega \) conditioned by the physical parameters \( \theta \). The constants \( c_{ij} \) account for dimensionality, and take the form \( c_{ij} = c_{i|j} \) if \( q_i \) and \( q_j \) are uncorrelated. The equilibrium state of the system minimizes \( I \) subject to prior conditions, like the normalization of \( F \) or any constraint on the mean value of an observable \( \langle A_i \rangle \) [57]. The MFI is then written as a variation problem of the form

\[ \delta \left\{ I(F) - \sum_i \mu_i \langle A_i \rangle \right\} = 0, \quad (3) \]

where \( \mu_i \) are appropriate Lagrange multipliers.

2.1. One-dimensional system with discrete coordinate

Because of the nature of the systems to be addressed we consider now a one-dimensional system with a physical parameter \( \theta \) and a discrete coordinate \( k = k_1, k_2, \ldots, k_n \) where \( k_{n+1} - k_i = \Delta k \) for a certain value of the interval \( \Delta k \). This scenario arises, for instance, in the case of nuclear multifragmentation [2], the abundances of genes [3], the frequency of words [4], scientific collaboration networks [5], the Internet traffic [6], Linux packages links [7], electoral results [8], urban agglomerations [9,10], firm sizes [11,12], etc.
3. Applications

A common representation of empirical data is the so-called rank-plot or Zipf plot [4, 10, 13], where the jth element of the system is represented by its size, length or frequency \( k_j \) against its rank, sorted from the largest to the smallest one. This process just renders the inverse function of the ensuing cumulative distribution, normalized to the number of elements. We call \( r \) the rank that ranges from 1 to \( N \). Thus, the constant \( N \) arising from the boundary conditions is the total number of elements considered in building up the distribution [8], as will be illustrated in the examples bellow. This rank-distribution takes the form

\[
k(r) = \frac{N k^1}{r}
\]

which yields a straight line in a logarithmic representation with slope \(-1\).

In Fig. 1a we depict the known behavior [12] of the rank size distribution for the top 100 largest cities of the United States [39], which shows a slope near \(-1\) (\( \gamma = 1 \)) in the logarithmic representation of the rank-plot.

We have also studied the system formed by the most referenced physics journals [40], using their total number of cites as coordinate \( k \). If a journal receives more cites due to its popularity, it becomes even more popular and, therefore, receives still more cites, etc. Under such conditions, proportional growth and scale invariance are expected, as we depict in Fig. 1b, where the slope’s value can be regarded as illustrating the universality of the underlying law.

4. Conclusions

We have here shown that Zipf’s law results from the scaling invariance of the Crammer-Rao inequality derived in [35]. This entails that the relevant probability distribution, usually called the rank-distribution, has to be size-invariant. Consequently, it should be derivable from a minimization process in which Fisher’s information measure is the protagonist. No constraints are needed in the concomitant variational problem because, a priori, our sizes have no upper bound. A physical analogy is the non-normalizability of plane waves. The universal character of our demonstration thus resides in the universal form to be minimized (Fisher’s), with no constraints.

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