The Geometry of Metrical Multi-Time Lagrange Spaces

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Abstract

Section 1 contains historical and bibliographical notes upon the problem of geometrization of Lagrangians defined on the tangent bundle or the jet bundle of order one, and emphasizes the original elements of our approach in this direction. The geometrization of a Kronecker $h$-regular Lagrangian function with partial derivatives begins in Section 2 by introduction of notion of metrical multi-time Lagrange space $ML_p^n = (J^1(T, M), L)$ and by proving a theorem of characterization of these spaces. Section 3 constructs the canonical nonlinear connection $\Gamma = (M_{\alpha}^{(i)}(\beta), N_{\alpha}^{(i)})$, naturally induced by the Lagrangian $L = L \sqrt{|h|}$ of the metrical multi-time Lagrange space $ML_p^n$. At the same time, Section 3 offers a geometrical interpretation to the extremals of the Lagrangian $L$. Section 4 proves the theorem of existence and uniqueness of Cartan canonical connection $C$ of a metrical multi-time Lagrange space $ML_p^n$ and studies its torsion and curvature d-tensors.

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1 Introduction

A lot of geometrical models in Mechanics, Physics or Biology are based on the notion of ordinary Lagrangian. In this sense, we recall that a Lagrange space $L^n = (M, L(x, y))$ is defined as a pair which consists of a real, smooth, $n$-dimensional manifold $M$ coordinated by $(x^i)_{i=1}^n$ and a regular Lagrangian $L : TM \to R$. The differential geometry of Lagrange spaces is now used in various fields to study natural phenomena where the dependence on position, velocity or momentum is involved. Also, this geometry gives a model for both the gravitational and electromagnetic field theory, in a very natural blending of the geometrical structure of the space with the characteristic properties of the physical fields.
At the same time, there are many problems in Physics and Variational Calculus in which time dependent Lagrangians (i.e., a smooth real function on $R \times TM$) are involved. A geometrization of time dependent Lagrangians was realized by Miron and Anastasiei in [7], using the configuration bundle $R \times TM \to M$, whose geometrical invariance group (gauge group) is of the form

$$
\begin{align*}
\tilde{t} &= t \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j.
\end{align*}
$$

The attached geometry is called the "Rheonomic Lagrange Geometry". Nevertheless, the main inconvenient of rheonomic Lagrange geometry is determined by the "absolute" character of time $t$ which is emphasized by the structure of its gauge group. In our paper, we remove this inconvenient, replacing the above bundle of configuration with 1-jet fibre bundle $J^1(R, M) \equiv R \times TM \to R \times M$ which is characterized by the gauge group

$$
\begin{align*}
\tilde{t} &= \tilde{t}(t) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{dt} y^j.
\end{align*}
$$

The structure of our gauge group underlines the relativistic role of time, and consequently, we can use the name of "Relativistic Rheonomic Lagrange Geometry".

In the last thirty years, many mathematicians were concerned by the geometrization of a multi-time Lagrangian depending on first order partial derivatives, which is defined on the 1-jet fibre bundle $J^1(T, M)$, where $T$ is a smooth, real, $p$-dimensional "multi-time" manifold coordinated by $(t^\alpha)_{\alpha=1}^p$ and $M$ is a smooth, real, $n$-dimensional "spatial" manifold coordinated by $(x^i)_{i=1}^n$.

One point of view is described by Gotay, Isenberg and Marsden [4] and is known under the name of Multisimplectic Geometry. This paper naturally generalizes the dual Hamiltonian formalism used in the classical mechanics and stands out by its finite dimensional and non metric spatial model $J^1(T, M)$. In contrast, Michor and Rațiu [6] construct their geometrization on the infinite dimensional space of the embeddings $Emb(T, M)$ which is endowed with a metrical structure $G$. The third way is sketched by Miron, Kirkovits and Anastasiei in [8]. They construct a metrical geometry attached to a first order Lagrangian on the finite dimensional total space of the vector bundle $\oplus_{\alpha=1}^p TM \to M$, where the coordinates of $\alpha$-th copy of $TM$ are
denoted \((x^i, x^a)\), and the gauge group of bundle of configuration is of the form

\[
\begin{aligned}
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^a &= \frac{\partial \tilde{x}^i}{\partial x^j} x^j_a.
\end{aligned}
\]  

(1.3)

This geometry relies on a given semi-Riemannian metric \(h_{\alpha\beta}\) on \(R^p\) and a given "\(a\ priori\)" nonlinear connection on \(E = \oplus_{\alpha=1}^p TM\). Starting with a multi-time Lagrangian \(L: \oplus_{\alpha=1}^p TM \rightarrow R\) and using an adapted basis of the nonlinear connection, they introduce a Sasakian-like metric on \(TE\), setting

\[
G = g_{ij} \delta x^i \otimes \delta x^j + G^{(ij)}(\alpha \beta) \delta x^i_a \otimes \delta x^j_{\beta},
\]  

(1.4)

where \(G^{(ij)}(x^k, x^\ell) = \frac{1}{2} \frac{\partial^2 L}{\partial x^k_i \partial x^\ell_j}\) and \(g_{ij}(x^k, x^\ell) = h_{\alpha\beta}G^{(ij)}(\alpha \beta)\). Also, the paper \[8\] use the Lagrangian density \(D = L dt^1 \wedge dt^2 \ldots \wedge dt^p\), developing a multi-time Lagrangian geometry, in the sense of linear connections, torsions and curvatures.

In this paper, we naturally extend the Rheonomic Lagrange Geometry of vector bundle \(R \times TM \rightarrow M\) to the 1-jet fibre bundle \(J^1(T, M) \rightarrow T \times M\). Using the gauge group

\[
\begin{aligned}
\tilde{t}^\alpha &= \tilde{t}^\alpha(t) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^a &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^\beta}{\partial x^j} x^j_{\beta}.
\end{aligned}
\]  

(1.5)

which is more general than that used in the papers \[4, 5\]. In order to develop our geometry, we start "\(a\ priori\)" with a semi-Riemannian metric \(h = h_{\alpha\beta}(t)\) on the temporal manifold \(T\), and we use the following three distinct notions:

i) multi-time Lagrangian function — A smooth function \(L: J^1(T, M) \rightarrow R\).

ii) multi-time Lagrangian (Olver’s terminology) — A local function \(L\) on \(J^1(T, M)\) which transform by the rule \(\hat{L} = L|\det J|\), where \(J\) is the Jacobian matrix of coordinate transformations \(t^\alpha = \tilde{t}^\alpha(t)\). If \(L\) is a Lagrangian function on 1-jet fibre bundle, then \(L = L \sqrt{|h|}\) represent a Lagrangian on \(J^1(T, M)\).

iii) multi-time Lagrangian density (Marsden’s terminology) — A smooth map \(D: J^1(T, M) \rightarrow \Lambda^p(T^*T)\). For example, the entity \(D = L dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p\), where \(L\) is a Lagrangian, represents a Lagrangian density on \(J^1(T, M)\).

In this terminology, we create a geometry attached to a first order Kronecker \(h\)-regular Lagrangian function on \(J^1(T, M)\), which can be called the Metrical Multi-Time Lagrangian Geometry. The condition of Kronecker \(h\)-
regularity imposed to the given multi-time Lagrangian function \( L \) is

\[
G^{(\alpha)(\beta)}(t^\gamma, x^k, x_\alpha^k) = \frac{1}{2} \frac{\partial^2 L}{\partial x_{\alpha}^i \partial x_{\beta}^j} = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x_\gamma^k),
\]

where \( g_{ij}(t^\gamma, x^k, x_\gamma^k) \) is a d-tensor on \( J^1(T, M) \), symmetric, having the rank \( n \), and of constant signature. This condition allows us to build from \( L \) a natural nonlinear connection on \( J^1(T, M) \). At the same time, the condition of Kronecker \( h \)-regularity is required by the following reasons:

1) the construction of Sasakian-like metric

\[
G = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x^i_{\alpha} \otimes \delta x^j_{\beta},
\]

on jet bundle \( J^1(T, M) \), that has the physical meaning of gravitational potential on \( J^1(T, M) \);

2) to find explicitly canonical d-connections, d-torsions and d-curvatures.

We emphasize that the physical aspects of the metrical multi-time Lagrange geometry are exposed in \([9], [10]\). From this point of view, the Maxwell and Einstein equations allow us to appreciate the Metrical Multi-Time Lagrange Geometry like a natural model necessary in the study of physical fields in a general setting.

Finally, we point out again that the Lagrangian density used in our study is

\[
D = L \sqrt{|h|} dt^1 \wedge dt^2 \ldots \wedge dt^p.
\]

2 Metrical multi-time Lagrange spaces

Let us consider \( T \) (resp. \( M \)) a "temporal" (resp. "spatial") manifold of dimension \( p \) (resp. \( n \)), coordinated by \((t^\alpha)_{\alpha=1}^p\) (resp. \((x^i)_{i=1}^n\)). Let \( E = J^1(T, M) \to T \times M \) be the jet fibre bundle of order one associated to these manifolds. The bundle of configuration \( J^1(T, M) \) is coordinated by \((t^\alpha, x^i, x_\alpha^i)\), where \( \alpha = 1, p \) and \( i = 1, n \). Note that the terminology used above is justified in \([3], [4]\).

Remarks 2.1 i) Throughout this paper, the indices \( \alpha, \beta, \gamma, \ldots \) run from 1 to \( p \), and the indices \( i, j, k, \ldots \) run from 1 to \( n \).

ii) In the particular case \( T = R \) (i. e., the temporal manifold \( T \) is the usual time axis represented by the set of real numbers), the coordinates \((t^1, x^i, x_1^i)\) of the 1-jet space \( J^1(R, M) \equiv R \times TM \) are denoted \((t, x^i, y^i)\).

We start our study considering a smooth multi-time Lagrangian function \( L : E \to R \), which is locally expressed by \( E \ni (t^\alpha, x^i, x_\alpha^i) \to L(t^\alpha, x^i, x_\gamma^k) \in R. \)
The vertical fundamental metrical d-tensor of $L$ is

$$G_{(i)(j)} = \left(\frac{\partial^2 L}{\partial x_i^\alpha \partial x_j^\beta}\right).$$

Now, let $h = (h_{\alpha\beta})$ be a fixed semi-Riemannian metric on the temporal manifold $T$ and $g_{ij}(t^\gamma, x^k, x_\gamma^k)$ be a d-tensor on $E$, symmetric, of rank $n$, and having a constant signature.

**Definition 2.1** A multi-time Lagrangian function $L : E \to R$ whose vertical fundamental metrical d-tensor is of the form

$$G_{(i)(j)}^{(\alpha)(\beta)}(t^\gamma, x^k, x_\gamma^k) = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x_\gamma^k),$$

is called a Kronecker $h$-regular multi-time Lagrangian function with respect to the temporal semi-Riemannian metric $h = (h_{\alpha\beta})$.

In this context, we can introduce the following

**Definition 2.2** A pair $ML^n_p = (J^1(T, M), L)$, where $p = \text{dim} T$ and $n = \text{dim} M$, which consists of the 1-jet fibre bundle and a Kronecker $h$-regular multi-time Lagrangian function $L : J^1(T, M) \to R$ is called a metrical multi-time Lagrange space.

**Remarks 2.2** i) In the particular case $(T, h) = (R, \delta)$, a metrical multi-time Lagrange space is called a relativistic rheonomic Lagrange space and is denoted $RL^n = (J^1(R, M), L)$.

ii) If the temporal manifold $T$ is 1-dimensional, then, via a temporal reparametrization, we have $J^1(T, M) \equiv J^1(R, M)$. In other words, a metrical multi-time Lagrangian space having dim $T = 1$ is a reparametrized relativistic rheonomic Lagrange space.

**Examples 2.1** i) Suppose that the spatial manifold $M$ is also endowed with a semi-Riemannian metric $g = (g_{ij}(x))$. Then, the multi-time Lagrangian function

$$L_1 : J^1(T, M) \to R, \quad L_1 = h^{\alpha\beta}(t)g_{ij}(x)x_\alpha x_\beta$$

is a Kronecker $h$-regular multi-time Lagrangian function. Consequently, $ML^n_p = (J^1(T, M), L_1)$ is a metrical multi-time Lagrange space. We underline that the multi-time Lagrangian $L_1 = L_1 \sqrt{|h|}$ is exactly the energy multi-time Lagrangian whose extremals are the harmonic maps between the pseudo-Riemannian manifolds $(T, h)$ and $(M, g)$. At the same time, this multi-time Lagrangian is a basic object in the physical theory of bosonic strings.

ii) In above notations, taking $U_{(i)}(t, x)$ as a d-tensor field on $E$ and $F : T \times M \to R$ a smooth map, the more general multi-time Lagrangian
function

\[ L_2 : E \to R, \quad L_2 = h'^\alpha\beta(t)g_{ij}(x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x)x^i_\alpha + F(t, x) \]

is also a Kronecker \( h \)-regular multi-time Lagrangian. The metrical multi-time Lagrange space \( ML^n_p = (J^1(T, M), L_2) \) is called the autonomous metrical multi-time Lagrange space of electrodynamics because, in the particular case \((T, h) = (R, \delta)\), we recover the classical Lagrangian space of electrodynamics \( \mathcal{L} \) which governs the movement law of a particle placed concomitantly into a gravitational field and an electromagnetic one. From physical point of view, the semi-Riemannian metric \( h_{\alpha\beta}(t) \) (resp. \( g_{ij}(x) \)) represents the gravitational potentials of the space \( T \) (resp. \( M \)), the d-tensor \( U^{(\alpha)}_{(i)}(t, x) \) stands for the electromagnetic potentials and \( F \) is a function which is called potential function. The non-dynamical character of spatial gravitational potentials motivates us to use the term "autonomous".

iii) More general, if we consider \( g_{ij}(t, x) \) a d-tensor field on \( E \), symmetric, of rank \( n \) and having constant signature on \( E \), we can define the Kronecker \( h \)-regular multi-time Lagrangian function

\[ L_3 : E \to R, \quad L_3 = h'^\alpha\beta(t)g_{ij}(t, x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x)x^i_\alpha + F(t, x). \]

The pair \( ML^n_p = (J^1(T, M), L_3) \) is a metrical multi-time Lagrange space which is called the non-autonomous metrical multi-time Lagrange space of electrodynamics. Physically, we remark that the gravitational potentials \( g_{ij}(t, x) \) of the spatial manifold \( M \) are dependent of the temporal coordinates \( t^\gamma \), emphasizing their dynamic character.

An important role and, at the same time, an obstruction in the subsequent development of the metrical multi-time Lagrangian theory, is played by the following

**Theorem 2.1** (characterization of metrical multi-time Lagrange spaces)

If we have \( \dim T \geq 2 \), then the following statements are equivalent:

i) \( L \) is a Kronecker \( h \)-regular multi-time Lagrangian function on \( J^1(T, M) \).

ii) The multi-time Lagrangian function \( L \) reduces to a non-autonomous electrodynamics Lagrangian function, that is,

\[ L = h'^\alpha\beta(t)g_{ij}(t, x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x)x^i_\alpha + F(t, x). \]

**Proof.** ii)⇒ i) Obviously.

i)⇒ ii) Suppose that \( L \) is a Kronecker \( h \)-regular multi-time Lagrangian function, that is,

\[ \frac{1}{2} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = h'^\alpha\beta(t^\gamma)g_{ij}(t, x^k, x^k). \]
Firstly, we assume that there are two distinct indices $\alpha$ and $\beta$ in the set \{1, \ldots, p\} such that $h^{\alpha\beta} \neq 0$. Let $k$ (resp. $\gamma$) be an arbitrary element of the set \{1, \ldots, n\} (resp. \{1, \ldots, p\}). Differentiating the above relation by $x^k$ and using the Schwartz theorem, we obtain the equalities

$$\frac{\partial g_{ij}}{\partial x^k} h^{\alpha\beta} = \frac{\partial g_{ij}}{\partial x^\alpha} h^{\gamma\beta} , \quad \forall \alpha, \beta, \gamma \in \{1, \ldots, p\}, \quad \forall i, j, k \in \{1, \ldots, n\}. $$

Contracting by $h^{\gamma\mu}$, we deduce

$$\frac{\partial g_{ij}}{\partial x^k} h^{\alpha\beta} h^{\gamma\mu} = 0, \quad \forall \mu \in \{1, \ldots, p\}. $$

The assumption $h^{\alpha\beta} \neq 0$ implies that $\frac{\partial g_{ij}}{\partial x^k} = 0$ for arbitrary $k$ and $\gamma$, that is, $g_{ij} = g_{ij}(t^\mu, x^m)$.

Secondly, assuming that $h^{\alpha\beta} = 0$, $\forall \alpha \neq \beta \in \{1, \ldots, p\}$, it follows $h^{\alpha\beta} = h^{\alpha} \delta^\beta_\alpha$, $\forall \alpha, \beta \in \{1, \ldots, p\}$. In other words, on $T$ we use an orthogonal system of coordinates. In these conditions, the relations

$$\frac{\partial^2 L}{\partial x^i \partial x^j} = 0, \quad \forall \alpha \neq \beta \in \{1, \ldots, p\}, \quad \forall i, j \in \{1, \ldots, n\};$$

$$\frac{1}{2h^{\alpha}(t)} \frac{\partial^2 L}{\partial x^i \partial x^\alpha} = g_{ij}(t^\mu, x^m, x^m_\mu), \quad \forall \alpha \in \{1, \ldots, p\}, \quad \forall i, j \in \{1, \ldots, n\}$$

are true. Now, if we fixe the indice $\alpha$ in the set \{1, \ldots, p\}, we deduce by first relation that the local functions $\frac{\partial L}{\partial x^\alpha}$ depend just of the coordinates $(t^\mu, x^m, x^m_\alpha)$. Considering $\beta \neq \alpha$ in the set \{1, \ldots, p\}, the second relation implies

$$\frac{1}{2h^{\alpha}(t)} \frac{\partial^2 L}{\partial x^i \partial x^\alpha} = \frac{1}{2h^{\beta}(t)} \frac{\partial^2 L}{\partial x^j \partial x^\beta} = g_{ij}(t^\mu, x^m, x^m_\mu), \quad \forall i, j \in \{1, \ldots, n\}.$$  

Because the first term of the above equality depends just of the coordinates $(t^\mu, x^m, x^m_\alpha)$ while the second term is dependent just of $(t^\mu, x^m, x^m_\beta)$ and $\alpha \neq \beta$, we conclude that $g_{ij} = g_{ij}(t^\mu, x^m)$.

Finally, the relation

$$\frac{1}{2} \frac{\partial^2 L}{\partial x^i \partial x^\beta} = h^{\alpha\beta}(t) g_{ij}(t^\mu, x^k), \quad \forall \alpha, \beta \in \{1, \ldots, p\}, \quad \forall i, j \in \{1, \ldots, n\}$$

implies without difficulties that the multi-time Lagrangian function $L$ is a non-autonomous multi-time Lagrangian function of electrodynamics. \[\blacksquare\]
Corollary 2.2 The vertical fundamental metrical d-tensor of an arbitrary Kronecker $h$-regular multi-time Lagrangian function $L$ is of the form

\[ G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha \partial x_\beta} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & \text{dim } T = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & \text{dim } T \geq 2. \end{cases} \]

Remarks 2.3 i) It is obvious that the preceding theorem is an obstruction in the development of a fertile metrical multi-time Lagrangian geometry. This obstruction will be removed in a subsequent paper by the introduction of a more general notion, that of generalized metrical multi-time Lagrange space \[9\]. The generalized metrical multi-time Lagrange geometry is constructed using just a given vertical Kronecker $h$-regular metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ on the 1-jet space $J^1(T, M)$.

ii) In the case dim $T \geq 2$, the above theorem obliges us to continue the study of the metrical multi-time Lagrangian spaces theory, channeling our attention upon the non-autonomous electrodynamics metrical multi-time Lagrangian spaces.

3 Sprays. Nonlinear connection. Harmonic maps

Let $ML_p^n = (J^1(T, M), L)$, where dim $T = p$, dim $M = n$, be a metrical multi-time Lagrange space whose vertical fundamental metrical d-tensor is

\[ G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x_\alpha \partial x_\beta} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p \geq 2. \end{cases} \]

Note that all subsequent entities with geometrical or physical meaning will be directly obtained from the fundamental vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$. This fact points out the metrical character and the naturalness of the metrical multi-time Lagrangian geometry that we construct. At the same time, the form of the invariance gauge group

\[ \begin{align*}
\tilde{t}^\alpha &= \tilde{t}^\alpha(t^\beta) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}_\alpha^i &= \frac{\partial \tilde{x}^j}{\partial x^\beta} \frac{\partial x^\beta}{\partial t^\alpha} x^j_eta 
\end{align*} \]

of the fibre bundle $J^1(T, M) \rightarrow T \times M$ allows us to look out the metrical multi-time Lagrange geometry as a "parametrized" theory, in Marsden's sense \[1\].
Now, assume that the semi-Riemannian temporal manifold \((T, h)\) is compact and orientable. In this context, we can define the energy functional of the Lagrangian function \(L\), setting 
\[
\mathcal{E}_L : C^\infty(T, M) \rightarrow \mathbb{R}, \quad \mathcal{E}_L(f) = \int_T L(t^\alpha, x^i, x^j_i) \sqrt{|h|} \, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p,
\]
where the smooth map \(f\) is locally expressed by \((t^\alpha) \rightarrow (x^i(t^\alpha))\) and \(x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}\).

The extremals of the energy functional \(\mathcal{E}_L\) verifies the Euler-Lagrange equations for every \(i \in \{1, 2, \ldots, n\}\),
\[
2G^{(\alpha)(\beta)}_{(i)(j)} x^j_\alpha x^i_\beta + \frac{\partial^2 L}{\partial x^j_\alpha \partial x^i_\beta} x^j_\alpha - \frac{\partial L}{\partial x^i_\alpha} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i_\alpha} + \frac{\partial L}{\partial x^i_\alpha} H^\gamma_{\alpha\beta} = 0,
\]
where \(x^j_\alpha = \frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta}\) and \(H^\gamma_{\alpha\beta}\) are the Christoffel symbols of the semi-Riemannian metric \(h_{\alpha\beta}\).

Taking into account the Kronecker \(h\)-regularity of the Lagrangian function \(L\), it is possible to rearrange the Euler-Lagrange equations of Lagrangian \(\mathcal{L} = L \sqrt{|h|}\) in the form
\[
\Delta_h x^k + 2G^k(m^\mu, m^m, m^m) = 0,
\]
where
\[
\Delta_h x^k = h^{\alpha\beta} \{ x^k_\alpha - H^\gamma_{\alpha\beta} x^k_\gamma \},
\]
\[
2G^k = \frac{g^{ki}}{2} \left\{ \frac{\partial^2 L}{\partial x^j_\alpha \partial x^i_\beta} x^j_\alpha - \frac{\partial L}{\partial x^i_\alpha} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i_\alpha} + \frac{\partial L}{\partial x^i_\alpha} H^\gamma_{\alpha\beta} + 2g_{ij} h^{\alpha\beta} H^\gamma_{\alpha\beta} x^j_\gamma \right\}.
\]

By a direct calculation, we deduce that the local geometrical entities
\[
2S^k = \frac{g^{ki}}{2} \left\{ \frac{\partial^2 L}{\partial x^j_\alpha \partial x^i_\beta} x^j_\alpha - \frac{\partial L}{\partial x^i_\alpha} \right\}
\]
\[
2\mathcal{H}^k = \frac{g^{ki}}{2} \left\{ \frac{\partial^2 L}{\partial t^\alpha \partial x^i_\alpha} + \frac{\partial L}{\partial x^i_\alpha} H^\gamma_{\alpha\gamma} \right\}
\]
\[
2\mathcal{J}^k = h^{\alpha\beta} H^\gamma_{\alpha\beta} x^j_\gamma
\]
verify the following transformation rules
\[
2S^p = 2S^r \frac{\partial x^p}{\partial x^r} + h^{\alpha\mu} \frac{\partial x^p}{\partial x^r} \frac{\partial x^l_\alpha}{\partial t^\mu} \frac{\partial x^l_\alpha}{\partial x^j_\alpha},
\]
\[
2\mathcal{H}^p = 2\mathcal{H}^r \frac{\partial x^p}{\partial x^r} + h^{\alpha\mu} \frac{\partial x^p}{\partial x^r} \frac{\partial x^l_\alpha}{\partial t^\mu} \frac{\partial x^l_\alpha}{\partial x^j_\alpha}
\]
\[
2\mathcal{J}^p = 2\mathcal{J}^r \frac{\partial x^p}{\partial x^r} - h^{\alpha\mu} \frac{\partial x^p}{\partial x^r} \frac{\partial x^l_\alpha}{\partial t^\mu} \frac{\partial x^l_\alpha}{\partial t^\nu}.
\]
Consequently, the local entities $2G^p = 2S^p + 2H^p + 2J^p$ modify by the transformation laws

$$2\tilde{G}^r = 2G^p \frac{\partial\tilde{x}^r}{\partial x^p} - h^{\alpha\beta} \frac{\partial x^p}{\partial \tilde{x}^\mu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}.$$  

and therefore the geometrical object $\mathcal{G} = (\mathcal{G}^r)$ is a spatial h-spray [12].

Following the paper [12], we can offer a geometrical interpretation to the equations

$$\Delta_h x^l + 2G^l(t, x, x^k, x^j) = 0, \quad \forall \ l \in \{1, \ldots, n\},$$

via the harmonic map equations of a spatial spray, if the spatial h-spray $\mathcal{G}$ is the $h$-trace of a spatial spray $G$.

In the particular case $\text{dim} \ T = 1$, every spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ is the $h$-trace of a spatial spray, namely $G = (G^{(l)}_{(1)1})$, where $G^{(l)}_{(1)1} = h^{11}G^l$. In other words, the equality $\mathcal{G}^l = h^{11}G^{(l)}_{(1)1}$ is true.

On the other hand, in the case $\text{dim} \ T \geq 2$, the characterization theorem of the Kronecker $h$-regular Lagrangians functions ensures us that

$$L = h^{\alpha\beta}(t)g_{ij}(t, x) x^i_\alpha x^j_\beta + U^{(\alpha)}(i) x^i_\alpha + F(t, x).$$

In this particular situation, by computations, the expressions of $S^l$, $H^l$ and $J^l$ reduce to

$$2S^l = h^{\alpha\beta} \Gamma^l_{jk} x^j_\alpha x^k_\beta + \frac{g^{li}}{2} \left[ U^{(\alpha)}(i) x^i_\alpha - \frac{\partial F}{\partial x^l} \right]$$

$$2H^l = -h^{\alpha\beta} H^\gamma_{\alpha\beta} x^l_\gamma + \frac{g^{li}}{2} \left[ 2h^{\alpha\beta} \frac{\partial g_{ij}}{\partial x^l} x^j_\beta + \frac{\partial U^{(\alpha)}(i)}{\partial x^\mu} + U^{(\alpha)}(i) H^\gamma_{\alpha\gamma} \right]$$

$$2J^l = h^{\alpha\beta} H^\gamma_{\alpha\beta} x^l_\gamma,$$

where

$$\Gamma^l_{jk} = \frac{g^{li}}{2} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

are the generalized Christoffel symbols of the "multi-time" dependent metric $g_{ij}$, and

$$U^{(\alpha)}(i) = \frac{\partial U^{(\alpha)}(i)}{\partial x^\mu} - \frac{\partial U^{(\alpha)}(i)}{\partial x^\mu}.$$

Consequently, the expression of the spatial h-spray $\mathcal{G} = (\mathcal{G}^l)$ becomes

$$2\mathcal{G}^p = 2S^p + 2H^p + 2J^p = h^{\alpha\beta} \Gamma^l_{jk} x^j_\alpha x^k_\beta + 2T^l,$$
where

\[ 2T^i = \frac{g^{i\alpha}}{2} \left[ 2\partial_{\alpha}^\beta \frac{\partial g_{ij}}{\partial \alpha} x^j_{\beta} + U_{(i)}^{(\alpha)} x^i_{\alpha} + \partial U_{(i)}^{(\alpha)} / \partial \alpha + U_{(i)}^{(\alpha)} H^\gamma_{\alpha} \frac{\partial F}{\partial x^\gamma} \right]. \tag{3.8} \]

The geometrical object \( T = (T^i) \) is a \( d \)-tensor field on \( E = J^1(T, M) \). It follows that \( T \) can be written as the \( h \)-trace of the \( d \)-tensor \( T^{(l)}_{(\alpha)\beta} = \frac{h_{\alpha\beta}}{p} T^l \), where \( p = \dim T \), that is, \( T^i = h_{\alpha\beta} T^{(l)}_{(\alpha)\beta} \). Of course, this writing is not unique but it is a natural extension of the case \( \dim T = 1 \).

Finally, we conclude that the spatial \( h \)-spray \( G = (G^l) \) is the \( h \)-trace of the spatial spray

\[ G^{(l)}_{(\alpha)\beta} = \frac{1}{2} \Gamma^l_{jk} x^j_{\alpha} x^k_{\beta} + T^{(l)}_{(\alpha)\beta}, \tag{3.9} \]

that is, \( G^l = h_{\alpha\beta} G^{(l)}_{(\alpha)\beta} \).

**Theorem 3.1** The extremals of the energy functional \( E_L \) attached to a Kronecker \( h \)-regular Lagrangian function \( L \) on \( J^1(T, M) \) are harmonic maps \([12]\) of the spray \( (H, G) \) defined by the temporal components

\[ H^{(i)}_{(\alpha)\beta} = \begin{cases} -\frac{1}{2} H^1_{11} x^i_{\alpha}, & p = 1 \\ -\frac{1}{2} H^\gamma_{\alpha\beta} x^i_{\gamma}, & p \geq 2 \end{cases} \]

and the local spatial components \( G^{(l)}_{(\alpha)\beta} = \)

\[ \begin{dcases} \frac{h_{11}\delta^{ik}}{4} \left[ \frac{\partial^2 L}{\partial x^j\partial y^k} y^j - \frac{\partial L}{\partial x^k} + \frac{\partial^2 L}{\partial y^k} + \frac{\partial L}{\partial x^k} H^1_{11} + 2h_{11} H^1_{11} g_{kl} y^l \right], & p = 1 \\ \frac{1}{2} \Gamma^l_{jk} x^j_{\alpha} x^k_{\beta} + T^{(l)}_{(\alpha)\beta}, & p \geq 2, \end{dcases} \]

where \( p = \dim T \).

**Definition 3.1** The spray \( (H, G) \) constructed in the preceding theorem is called the canonical spray attached to the metrical multi-time Lagrange space \( ML^n \).

In the sequel, using the canonical spray \( (H, G) \) of the metrical multi-time Lagrange space \( ML^n \), one naturally induces \([12]\) a nonlinear connection \( \Gamma \) on \( E = J^1(T, M) \), defined by the temporal components

\[ M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} = \begin{cases} -H^1_{11} y^i, & p = 1 \\ -H^\gamma_{\alpha\beta} x^i_{\gamma}, & p \geq 2, \end{cases} \tag{3.10} \]
and the spatial components

\[
N^{(i)}_{(\alpha)j} = \frac{\partial G^i}{\partial x_j} h_{\alpha\gamma} = \begin{cases} 
  h_{11} \frac{\partial G^i}{\partial y^j}, & p = 1 \\
  \Gamma^i_{jk} \frac{\partial x^k}{\partial \gamma} + \frac{g^{ik}}{2} \frac{\partial g_{jk}}{\partial \alpha} + \frac{g^{ik}}{4} h_{\alpha\gamma} U_{(k)j}, & p \geq 2,
\end{cases}
\]

where \( G^i = h^\alpha\beta G^{(i)}_{\alpha\beta} \). The nonlinear connection \( \Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}) \) is called the canonical nonlinear connection of the metrical multi-time Lagrange space \( ML^n_p \).

**Remarks 3.1**

i) Considering the particular case \((T, h) = (R, \delta)\), the canonical nonlinear connection \( \Gamma = (0, N^{(i)}_{(\alpha)j}) \) of the relativistic rheonomic Lagrange space \( RL^n = (J^1(R, M), L) \) reduces to the canonical nonlinear connection of the Lagrange space \( L^n = (M, L) \).

ii) In the case of an autonomous electrodynamics metrical multi-time Lagrange space (i.e., \( g_{ij}(t^\gamma, x^k) = g_{ij}(x^k) \)), the generalized Christoffel symbols \( \Gamma^i_{jk}(t^\mu, x^m) \) of the metrical d-tensor \( g_{ij} \) reduce to the classical ones \( \gamma^i_{jk}(x^m) \) and the canonical nonlinear connection becomes \( \Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}) \), where

\[
M^{(i)}_{(\alpha)\beta} = \begin{cases} 
  -H^i_{11} y^i, & p = 1 \\
  -H^i_{\alpha\beta} x^i, & p \geq 2,
\end{cases}
\]

\[
N^{(i)}_{(\alpha)j} = \begin{cases} 
  \gamma^i_{jk} y^k + \frac{g^{ik}}{4} h_{11} U_{(k)j}, & p = 1 \\
  \gamma^i_{jk} x^k + \frac{g^{ik}}{4} h_{\alpha\gamma} U_{(k)j}, & p \geq 2.
\end{cases}
\]

## 4 Cartan canonical \( h \)-normal \( \Gamma \)-linear connection. d-Torsions and d-curvatures

Suppose that \( J^1(T, M) \) is endowed with a nonlinear connection \( \Gamma \) defined by the temporal components \( M^{(i)}_{(\alpha)\beta} \) and the spatial components \( N^{(i)}_{(\alpha)j} \). Let \( \left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0} \right\} \subset \mathcal{X}(E) \) and \( \{ dt^\alpha, dx^i, \delta x^i_\alpha \} \subset \mathcal{X}^*(E) \) be the adapted bases
of the nonlinear connection $\Gamma$, where

$$
\begin{align*}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} - M^{(j)}_{(\beta)\alpha} \frac{\partial}{\partial x^j_{\beta}}, \\
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i} \frac{\partial}{\partial x^j_{\beta}}, \\
\delta x^i_\alpha &= dx^i_\alpha + M^{(i)}_{(\alpha)\beta} dt^\beta + N^{(i)}_{(\alpha)j} dx^j.
\end{align*}
\tag{4.1}
$$

Using the notations

$$
\mathcal{X}(\mathcal{H}_T) = \text{Span} \left\{ \frac{\delta}{\delta t^\alpha} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span} \left\{ \frac{\partial}{\partial x^i_\alpha} \right\},
$$

$$
\mathcal{X}^*(\mathcal{H}_T) = \text{Span} \{ dt^\alpha \}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span} \{ dx^i \}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span} \{ \delta x^i_\alpha \},
$$

we obtain without difficulties the following

**Proposition 4.1**

i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as

$$
\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}).
$$

ii) The Lie algebra $\mathcal{X}^*(E)$ of covector fields decomposes as

$$
\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}).
$$

Let us consider $h_T$, $h_M$ and $v$ the canonical projections of the above decompositions.

**Definition 4.1** A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$ is called a $\Gamma$-linear connection on $E$ if $\nabla h_T = 0$, $\nabla h_M = 0$, $\nabla v = 0$.

In the adapted basis $\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha} \right\} \subset \mathcal{X}(E)$, a $\Gamma$-linear connection $\nabla$ on $E$ is defined by nine local coefficients,

$$
\nabla \Gamma = \{ C^{\alpha}_{\beta\gamma}, C^k_{\alpha\gamma}, C^{(k)(\beta)}_{(\alpha)\gamma}, L^\alpha_{\beta\gamma}, L^k_{\alpha\gamma}, C^{(k)(\beta)}_{(\alpha)\gamma}, C^{k(\gamma)}_{(\alpha)\beta}, C^{(k)(\beta)(\gamma)}_{(\alpha)\beta} \},
$$

introduced by

$$
\begin{align*}
\nabla \frac{\delta}{\delta t^\beta} &= \frac{\partial}{\partial t^\beta} \frac{\delta}{\delta t^\beta}, \quad \nabla \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} \frac{\delta}{\delta x^i}, \quad \nabla \frac{\partial}{\partial x^i_\alpha} &= \frac{\partial}{\partial x^i_\alpha} \frac{\partial}{\partial x^i_\alpha}, \\
\nabla \frac{\partial}{\partial t^\beta} &= \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau}, \quad \nabla \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau}, \quad \nabla \frac{\partial}{\partial x^i_\alpha} &= \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau},
\end{align*}
\tag{4.2}
$$
Remark 4.1 The transformation rules of the above connection coefficients are completely described in [12].

Example 4.1 If \( h_{\alpha\beta} \) (resp. \( g_{ij} \)) is a semi-Riemannian metric on the temporal (resp. spatial) manifold \( T \) (resp. \( M \)), \( H_{\alpha\beta}^{\gamma} \) (resp. \( \gamma^k_{ij} \)) are its Christoffel symbols and \( \Gamma_0 = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}) \), where \( M^{(i)}_{(\alpha)\beta} = -H_{\alpha\beta}^{\gamma}x^i_{\gamma} \), \( N^{(i)}_{(\alpha)j} = \gamma^j_{ik}\bar{e}_{\alpha} \), is the canonical nonlinear connection on \( T \) attached to the metric pair \( (h_{\alpha\beta}, g_{ij}) \), then the following set of local coefficients [12]

\[
B \Gamma_0 = (\bar{G}_{\alpha\beta}^\gamma, 0, G_{(\alpha)(i)\gamma}, 0, I_{ij}, L_{(\alpha)(ij)}^{(k)(\beta)}, 0, 0, 0),
\]

where \( \bar{G}_{\alpha\beta}^\gamma = H_{\alpha\beta}^{\gamma}, G_{(\alpha)(i)\gamma} = -\delta^\gamma_i H_{\alpha\gamma}, I_{ij} = \gamma_{ij}^k \) and \( L_{(\alpha)(ij)}^{(k)(\beta)} = \delta_{ij}^\gamma \gamma_{ij}^k \), is a \( \Gamma_0 \)-linear connection which is called the Berwald \( \Gamma_0 \)-linear connection of the metric pair \( (h_{\alpha\beta}, g_{ij}) \).

We recall that a \( \Gamma \)-linear connection \( \nabla \) on \( E \), defined by the local coefficients [12] induces a natural linear connection on the d-tensors set of the jet fibre bundle \( E = J^1(T, M) \), which is characterized by a collection of local derivative operators like \( \nabla /\gamma \), \( \nabla |_p \) and \( \nabla |_{(p)} \). The previous local operators are called the \( T \)-horizontal covariant derivative, \( M \)-horizontal covariant derivative and vertical covariant derivative. The detailed expressions of these derivative operators are completely described in [13].

The study of the torsion \( T \) and curvature \( R \) d-tensors of an arbitrary \( \Gamma \)-linear connection \( \nabla \) on \( E \) was made in [14]. In this context, we proved that the torsion d-tensor is determined by twelve effective local torsion d-tensors, while the curvature d-tensor of \( \nabla \) is determined by eighteen local d-tensors.

Now, let \( h_{\alpha\beta} \) be a fixed pseudo-Riemannian metric on the temporal manifold \( T \), \( H_{\alpha\beta}^{\gamma} \) its Christoffel symbols and \( J = J^{(i)}_{(\alpha)(\beta)} = \frac{\partial}{\partial x^i}, dt^\alpha \otimes dx^j \), where \( J^{(i)}_{(\alpha)(\beta)} = h_{\alpha\beta} \delta^i_j \), the normalization d-tensor [12] attached to the metric \( h_{\alpha\beta} \). The big number of torsion and curvature d-tensors which characterize a general \( \Gamma \)-linear connection on \( E \) determines us to consider the following [14].

Definition 4.2 A \( \Gamma \)-linear connection \( \nabla \) on \( E = J^1(T, M) \), defined by the local coefficients

\[
\nabla \Gamma = (\bar{G}_{\alpha\beta}^\gamma, G_{(\alpha)(i)\gamma}, \bar{L}_{\beta j}, L_{(\alpha)(ij)}^{(k)(\beta)}, \bar{G}_{\beta(j)}^{\alpha}, \bar{C}_{(\alpha)(i)j}^{(k)(\beta)}, \bar{C}_{(\alpha)(i)j}^{k(\gamma)}),
\]

that verify the relations \( \bar{G} = G, \bar{L}_{\beta j} = 0, \bar{G}_{\beta(j)}^{\alpha} = 0 \) and \( \nabla J = 0 \), is called a h-normal \( \Gamma \)-linear connection.

Remark 4.2 Taking into account the local covariant \( T \)-horizontal \( |_\gamma \), \( M \)-horizontal \( |_k \) and vertical \( |_{(p)} \) covariant derivatives induced by \( \nabla \), the condition \( \nabla J = 0 \) is equivalent to

\[
J^{(i)}_{(\alpha)(\beta)j/\gamma} = 0, \quad J^{(i)}_{(\alpha)(\beta)j} = 0, \quad J^{(i)}_{(\alpha)(\beta)j}^{(k)} = 0.
\]
Theorem 4.2 The coefficients of a $h$-normal $\Gamma$-linear connection $\nabla$ verify the identities

\[
\begin{align*}
\bar{C}^\gamma_{\alpha \beta} &= H^\gamma_{\alpha \beta}, & \bar{L}^\alpha_{\beta j} &= 0, & \bar{C}^{\alpha(\gamma)}_{\beta(j)} &= 0, \\
G^{(k)(\beta)}_{\alpha(i)\gamma} &= \delta^\beta_\alpha G^k_{i\gamma} - \delta^k_{\beta(j)} H^\beta_{\alpha \gamma}, & L^{(k)(\beta)}_{\alpha(i)j} &= \delta^\beta_\alpha L^k_{ij}, & C^{(k)(\beta)(\gamma)}_{\alpha(i)j(j)} &= \delta^\beta_\alpha C_{k(i)(j)}^m.
\end{align*}
\]

Remarks 4.3

i) The preceding theorem implies that a $h$-normal $\Gamma$-linear on $E$ is determined just by four effective coefficients

\[
\nabla \Gamma = (H^\gamma_{\alpha \beta}, \ G^k_{i\gamma}, \ L^k_{ij}, \ C_{k(i)(j)}^m).
\]

ii) In the particular case $(T, H) = (R, \delta)$, a $\delta$-normal $\Gamma$-linear connection identifies to the notion of $N$-linear connection used in [7].

Example 4.2 The canonical Berwald $\Gamma_0$-linear connection associated to the metric pair $(h_{\alpha \beta}, g_{ij})$ is a $h$-normal $\Gamma_0$-linear connection defined by the local coefficients $B\Gamma_0 = (H^\gamma_{\alpha \beta}, 0, \delta^k_{i(j)}, 0)$.

Note that, in the particular case of a $h$-normal $\Gamma$-linear connection $\nabla$, the torsion d-tensors $T^m_{\alpha \beta}, \ T^m_{\alpha j}$ and $P^m_{\alpha(j)}$ vanish. Thus, the torsion d-tensor $T$ of $\nabla$ is determined by the following nine local d-tensors [13]

| $h_T$ | $h_M$ | $\nu$ |
|-------|-------|-------|
| $h_T h_T$ | 0 | 0 | $R^{m}_{(\mu)\alpha \beta}$ |
| $h_M h_T$ | 0 | $T^m_{\alpha j}$ | $R^m_{(\mu)\alpha j}$ |
| $h_M h_M$ | 0 | $T^m_{ij}$ | $R^m_{(\mu)ij}$ |
| $\nu h_T$ | 0 | 0 | $P^{m(\beta)}_{(\mu)\alpha(j)}$ |
| $\nu h_M$ | 0 | $P^{m(\beta)}_{\alpha(i)}$ | $P^{m(\beta)}_{\alpha(i)}$ |
| $\nu \nu$ | 0 | 0 | $S^{(m)(\alpha)(\beta)}_{\alpha(i)}$ |

(4.3)

where

\[
\begin{align*}
P^{m(\beta)}_{(\mu)\alpha(j)} &= \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^j} - \delta^\beta_\mu C^{m}_{j\alpha} + \delta^m_{j} H_{\mu \alpha}, & \quad P^{m(\beta)}_{(\mu)\alpha(i)} &= \frac{\partial N^{(m)}_{(\mu)i}}{\partial x^j} - \delta^\beta_\mu L_{ji}^m, \\
R^{m}_{(\mu)\alpha \beta} &= \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial t^\beta} - \frac{\partial M^{(m)}_{(\mu)\beta}}{\partial t^\alpha}, & \quad R^{m}_{(\mu)\alpha j} &= \frac{\partial M^{(m)}_{(\mu)\alpha}}{\partial x^j} - \delta^m_{j} N_{(\mu)i}, \\
R^{m}_{(\mu)ij} &= \frac{\partial N^{(m)}_{(\mu)i}}{\partial x^j} - \frac{\partial N^{(m)}_{(\mu)j}}{\partial x^i}, & \quad S^{(m)(\alpha)(\beta)}_{\alpha(i)} &= \delta^\alpha_\mu C^{m}_{i\beta} - \delta^\beta_\mu C^{m}_{j\alpha}.
\end{align*}
\]
\[ T^m_{\alpha j} = -G^m_{j\alpha}, \quad T^m_{ij} = L^m_{ij} - L^m_{ji}, \quad P^m_{i(j)\beta} = C^m_{i(j)\beta}. \]

**Remark 4.4** For the Berwald \( \Gamma^0 \)-linear connection associated to the metrics \( h_{\alpha\beta} \) and \( g_{ij} \), all torsion d-tensors vanish, except

\[ R^{(m)}_{(\mu)\alpha\beta} = -H^\gamma_{\mu\alpha\beta}x_j^m, \quad R^{(m)}_{(\mu)ij} = r^{m}_{ij}x^\mu, \]

where \( H^\gamma_{\mu\alpha\beta} \) (resp. \( r^m_{ij} \)) are the curvature tensors of the metric \( h_{\alpha\beta} \) (resp. \( g_{ij} \)).

The number of the effective curvature d-tensors of a \( h \)-normal \( \Gamma \)-linear connection \( \nabla \) reduces from eighteen to seven. The local d-tensors of the curvature d-tensor \( R \) of \( \nabla \) are represented in the table [11]

| \( h_T h_T \) | \( h_T h_M \) | \( h_M h_T \) | \( h_M h_M \) | \( vh_T \) | \( vh_M \) | \( vv \) |
|--------|--------|--------|--------|--------|--------|--------|
| \( H^\alpha_{\eta j\gamma} \) | \( R^l_{ij\beta \gamma} \) | \( R^l_{i(j)\alpha\beta \gamma} = \delta^\alpha_{\eta} R^l_{ij\beta \gamma} + \delta^\beta_{\eta} H^\alpha_{\eta j\gamma} \) | \( R^l_{i j(k)\beta \gamma} \) | \( P^l_{i j(k)\beta \gamma} \) | \( P^l_{i j(k)\beta \gamma} \) | \( S^l_{i j(k)\beta \gamma} \) |

where

\[
\begin{align*}
H^\alpha_{\eta j\gamma} &= \frac{\partial H^\alpha_{\eta j\gamma}}{\partial \gamma} + H^\mu_{\eta j\gamma} H^\alpha_{\mu\beta}, \\
R^l_{ij \beta \gamma} &= \delta G^l_{ij \beta \gamma} - \delta G^l_{ij \beta \gamma} + G^m_{ij \beta \gamma} - C^l_{ij \beta \gamma} \delta^\alpha_{\eta} R^m_{ij \beta \gamma} + C^l_{i j(k) \beta \gamma}, \\
R^l_{i j(k) \beta \gamma} &= \frac{\partial L^l_{ik}}{\partial \eta} + \frac{\partial L^l_{ik}}{\partial \beta} + L^l_{ik} L^m_{mk} - L^l_{ik} L^m_{mk} + C^l_{i j(k) \beta \gamma}, \\
P^l_{i j(k) \beta \gamma} &= \frac{\partial C^l_{i j(k) \beta \gamma}}{\partial \gamma} - C^l_{i j(k) \beta \gamma} + C^l_{j(k)} P^m_{i j(k) \beta \gamma}, \\
P^l_{i j(k) \beta \gamma} &= \frac{\partial C^l_{i j(k) \beta \gamma}}{\partial \gamma} - C^l_{i j(k) \beta \gamma} + C^l_{j(k)} P^m_{i j(k) \beta \gamma}, \\
S^l_{i j(k) \beta \gamma} &= \frac{\partial C^l_{i j(k) \beta \gamma}}{\partial \gamma} - \frac{\partial C^l_{i j(k) \beta \gamma}}{\partial \beta} + C^m_{i j(k) \beta \gamma} C^m_{i j(k) \beta \gamma} - C^m_{i j(k) \beta \gamma} C^m_{i j(k) \beta \gamma}.
\end{align*}
\]
Remark 4.5 In the case of the Berwald $\Gamma_0$-linear connection associated to the metric pair $(h_{\alpha\beta}, g_{ij})$, all curvature d-tensors vanish, except $H^\gamma_{\alpha\beta\gamma}$ and $R^l_{ijk} = r^l_{ijk}$, where $r^l_{ijk}$ is the curvature tensor of the metric $g_{ij}$.

Now, let us consider $ML^n_p = (J^1(T, M), L)$ a metrical multi-time Lagrangian space and

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^\alpha_{\gamma} \partial x^\beta_{\gamma}} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t)g_{ij}(t^\gamma, x^k), & p \geq 2 \end{cases}$$

its vertical fundamental metrical d-tensor. Let $\Gamma = (M^i_{(\alpha)\beta}, N_{(\alpha)ij})$ be the canonical nonlinear connection of the metrical multi-time Lagrange space $ML^n_p$.

The main result of this paper is the theorem of existence of the Cartan canonical $h$-normal linear connection $C\Gamma$ which allow the natural subsequent development of the metrical multi-time Lagrange theory of physical fields [10].

Theorem 4.3 (existence and uniqueness of Cartan canonical connection)
On the metrical multi-time Lagrange space $ML^n_p = (J^1(T, M), L)$ endowed with its canonical nonlinear connection $\Gamma$ there is a unique $h$-normal $\Gamma$-linear connection

$$CT = (H^\gamma_{\alpha\beta}, G^k_{\gamma j}, L^i_{jk}, C^{i(\gamma)}_{j(k)})$$

having the metrical properties

i) $g_{ij|k} = 0$, $g_{ij}^{(\gamma)}_{(k)} = 0$,

ii) $G^k_{\gamma j} = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}$, $L^k_{ij} = L^k_{ji}$, $C^{i(\gamma)}_{j(k)} = C^{i(\gamma)}_{k(j)}$.

Proof. Let $CT = (\tilde{G}^\gamma_{\alpha\beta}, \tilde{G}^k_{\gamma j}, \tilde{L}^i_{jk}, \tilde{C}^{i(\gamma)}_{j(k)})$ be $h$-normal $\Gamma$-linear connection whose coefficients are defined by $\tilde{G}^\gamma_{\alpha\beta} = H^\gamma_{\alpha\beta}$, $\tilde{G}^k_{\gamma j} = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}$, and

$$L^i_{jk} = \frac{g^{im}}{2} \left( \frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right),$$

$$C^{i(\gamma)}_{j(k)} = \frac{g^{im}}{2} \left( \frac{\delta g_{jm}}{\delta x^\gamma} + \frac{\delta g_{km}}{\delta x^\gamma} - \frac{\delta g_{jk}}{\delta x^\gamma} \right).$$

By computations, one easily verifies that $CT$ satisfies the conditions i and ii.

Conversely, let us consider a $h$-normal $\Gamma$-linear connection

$$\tilde{CT} = (\tilde{G}^\gamma_{\alpha\beta}, \tilde{G}^k_{\gamma j}, \tilde{L}^i_{jk}, \tilde{C}^{i(\gamma)}_{j(k)})$$
which satisfies $i$ and $ii$. It follows that
\[
\tilde{G}^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}, \quad \text{and} \quad \tilde{G}^{\gamma}_{jk} = \frac{g^{ki}}{2} \frac{\delta g_{ij}}{\delta t^\gamma}.
\]

The condition $g_{ijk} = 0$ is equivalent to
\[
\frac{\delta g_{ij}}{\delta x^k} = g_{mj} \tilde{L}^m_{ik} + g_{im} \tilde{L}^m_{jk}.
\]

Applying a Christoffel process to the indices \{i, j, k\}, we find
\[
\tilde{L}^i_{jk} = \frac{g^{im}}{2} \left( \frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right).
\]

By analogy, using the relations $C^{i(\gamma)}_{j(k)} = C^{i(\gamma)}_{k(j)}$ and $g_{ij}^{(\gamma)} |_{(k)} = 0$, following a Christoffel process applied to the indices \{i, j, k\}, we obtain
\[
\tilde{C}^{i(\gamma)}_{j(k)} = \frac{g^{im}}{2} \left( \frac{\partial g_{jm}}{\partial x^\gamma_{(\gamma)}} + \frac{\partial g_{km}}{\partial x^j_{(\gamma)}} - \frac{\partial g_{jk}}{\partial x^m_{(\gamma)}} \right).
\]

In conclusion, the uniqueness of the Cartan canonical connection $C^T$ is clear. ■

**Remarks 4.6**

i) Replacing the canonical nonlinear connection $\Gamma$ by a general one, the previous theorem holds good.

ii) In the particular case $(T, h) = (R, \delta)$, the Cartan canonical $\delta$-normal $\Gamma$-linear connection of the relativistic rheonomic Lagrange space $RL^n = (J^1(R, M), L)$ reduces to the Cartan canonical connection of the Lagrange space $L^n = (M, L)$, constructed in [3].

iii) As a rule, the Cartan canonical connection of a metrical multi-time Lagrange space $ML^n_p$ verifies also the metrical properties 
\[
h_{\alpha\beta/\gamma} = h_{\alpha\beta|k} = h_{\alpha\beta}^{(\gamma)} |_{(k)} = 0 \quad \text{and} \quad g_{ij/\gamma} = 0.
\]

iv) In the case $\dim T \geq 2$, the coefficients of the Cartan connection of a metrical multi-time Lagrange space reduce to
\[
\tilde{G}^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}, \quad \tilde{C}^{\gamma}_{jk} = \frac{g^{ki}}{2} \frac{\partial g_{ij}}{\partial t^\gamma}, \quad L^i_{jk} = \Gamma^i_{jk}, \quad C^{i(\gamma)}_{j(k)} = 0.
\]

v) Particularly, the coefficients of the Cartan connection of an autonomous metrical multi-time Lagrange space of electrodynamics (i.e., $g_{ij}(t, x^k, x^k) = g_{ij}(x^k)$) are the same with those of the Berwald connection, namely, $CT = (H^{\alpha\beta}_{\alpha\beta}, 0, \gamma_{jk}, 0)$. Note that the Cartan connection is a $\Gamma$-linear connection, where $\Gamma$ is the canonical nonlinear connection of the metrical multi-time Lagrangian space while the Berwald connection is a $\Gamma_0$-linear connection, $\Gamma_0$ being the canonical nonlinear connection associated to the metric pair $(h_{\alpha\beta}, g_{ij})$. Consequently, the Cartan and Berwald connections are distinct.
Theorem 4.4 The torsion d-tensor \( T \) of the Cartan canonical connection of a metrical multi-time Lagrange space is determined by the local components

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & h_T & h_M & v \\
\hline
 h_T h_T & 0 & 0 & 0 & 0 \\
 h_M h_T & 0 & 0 & T_m^{(1)} & T_m^{(1)} \\
 h_M h_M & 0 & 0 & 0 & 0 \\
 v h_T & 0 & 0 & P_m^{(1)} & P_m^{(1)} \\
 v h_M & 0 & 0 & 0 & 0 \\
 v v & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

(4.6)

where,

i) for \( p = \dim T = 1 \), we have

\[
\begin{align*}
T_m^{(1)} &= -C_j^{m1} , \quad P_m^{(1)} = C_m^{m1} , \quad F_m^{(1)} (1,1) = -C_m^{m1} , \\
N_m^{(1)} &= \frac{\partial N_m^{(1)}}{\partial y^i} - \frac{\delta x^j}{\delta x^i} , \\
R_m^{(1)} &= \frac{\partial N_m^{(1)}}{\partial t} + H_1^{m1} \left[ N_m^{(1)} - \frac{\partial N_m^{(1)}}{\partial y^k} y^k \right] .
\end{align*}
\]

ii) for \( p = \dim T \geq 2 \), denoting

\[
\begin{align*}
F_m^{(\mu)} &= \frac{g^{m \mu}}{2} \left[ \frac{\partial g_{\mu \nu}}{\partial \nu} + \frac{1}{2} h_{\nu \lambda} U^{(\lambda)}_{(\mu \nu)} \right] , \\
H_{\mu \lambda \beta} &= \frac{\partial H_{\mu \lambda}}{\partial \beta} - \frac{\partial H_{\mu \beta}}{\partial \lambda} + H_{\mu \beta} H_{\eta \lambda} - H_{\mu \lambda} H_{\eta \beta} , \\
\gamma_{m \nu \beta} &= \frac{\partial \gamma_{m \nu}}{\partial \beta} + \Gamma_{m \nu \beta} \gamma_{m \nu} - \Gamma_{m \nu} \Gamma_{m \beta} ,
\end{align*}
\]

we have

\[
\begin{align*}
T_m^{(\alpha \beta)} &= -C_{\beta \alpha} , \quad P_m^{(\beta)} (\mu \alpha) = -\delta_{\beta \gamma} C_m^{m \alpha} , \quad R_m^{(\beta)} (\mu \alpha) = -H_{\mu \beta \gamma} x_m^{m} , \\
R_m^{(\alpha \beta)} &= \frac{\partial N_m^{(\alpha \beta)}}{\partial \beta} + \frac{g^{m \mu}}{2} H_{\mu \alpha} \left[ \frac{\partial g_{\nu \beta}}{\partial \nu} + \frac{1}{2} h_{\nu \lambda} U^{(\lambda)}_{(\nu \beta)} \right] , \\
R_m^{(\alpha \beta \gamma)} &= \gamma_{m \alpha \beta} x_m^{m} + \left[ F_m^{(\mu \beta \alpha)} - F_m^{(\mu \alpha \beta)} \right] .
\end{align*}
\]
Remark 4.7 In the case of autonomous metrical multi-time Lagrange space of electrodynamics (i.e., \( g_{ij}(t^\gamma, x^k) = g_{ij}(x^k) \)), all torsion d-tensors of the Cartan connection vanish, except

\[
R^{(m)}_{(\mu)\alpha\beta} = -H^{\gamma}_{\mu\alpha\beta} x^m_{\gamma}, \quad R^{(m)}_{(\mu)\alpha\beta} = -\frac{h_{\mu\beta} g^{mk}}{4} \left[ H^{\gamma}_{\alpha\beta} U^{(\gamma)}_{(k)ij} + \frac{\partial U^{(\eta)}_{(k)ij}}{\partial \alpha} \right],
\]

\[
P^{(m)}_{(\mu)ij} = r^{m}_{ijk} x^{k} + \frac{h_{\mu\gamma} g^{mk}}{4} \left[ U^{(\eta)}_{(k)ij} + U^{(\eta)}_{(k)ji} \right],
\]

where \( H^{\gamma}_{\mu\alpha\beta} \) (resp. \( r^{m}_{ijk} \)) are the curvature tensors of the semi-Riemannian metric \( h_{\alpha\beta} \) (resp. \( g_{ij} \)).

Theorem 4.5 The curvature d-tensor \( \mathbf{R} \) of the Cartan canonical connection is determined by the local components

| \( h_T \) | \( h_M \) | \( v \) |
|---|---|---|
| \( p = 1 \) | \( p \geq 2 \) | \( p = 1 \) | \( p \geq 2 \) | \( p = 1 \) | \( p \geq 2 \) |
| \( h_T h_T \) | 0 | \( R^{(m)}_{ij\gamma} \) | 0 | \( R^{(m)}_{ij\beta} \) | 0 | \( R^{(m)}_{ij\beta} \) | 0 |
| \( h_M h_T \) | 0 | 0 | \( R^{(m)}_{ijkl} \) | \( R^{(m)}_{ij\beta} \) | \( R^{(m)}_{ij\gamma} \) | 0 | \( R^{(m)}_{ij\beta} \) | 0 |
| \( v h_T \) | 0 | 0 | \( P^{(m)}_{ij(k)} \) | 0 | \( P^{(m)}_{ij(k)} \) | 0 | \( P^{(m)}_{ij(k)} \) | 0 |
| \( v h_M \) | 0 | 0 | \( P^{(m)}_{ij(k)} \) | 0 | \( P^{(m)}_{ij(k)} \) | 0 | \( P^{(m)}_{ij(k)} \) | 0 |
| \( vu \) | 0 | 0 | \( S^{(m)}_{ij(k)} \) | \( S^{(m)}_{ij(k)} \) | \( S^{(m)}_{ij(k)} \) | \( S^{(m)}_{ij(k)} \) | \( S^{(m)}_{ij(k)} \) | \( S^{(m)}_{ij(k)} \) |

where \( R^{(m)}_{(\eta)(i)(j)\beta} = \delta^{(m)}_{\eta} R^{(m)}_{ij\beta} + \delta^{(m)}_{\eta} R^{(m)}_{ij\beta} \), \( R^{(m)}_{(\eta)(i)(j)\beta} = \delta^{(m)}_{\eta} R^{(m)}_{ij\beta} \), \( R^{(m)}_{(\eta)(i)(j)\beta} = \delta^{(m)}_{\eta} R^{(m)}_{ij\beta} \) and

(i) for \( p = \text{dim} T = 1 \), we have

\[
R^{(l)}_{ijl} = \frac{\delta G^{(l)}_{ijl}}{\delta x^k} - \frac{\delta L^{(l)}_{ijl}}{\delta t} + \frac{L_{ijl}^{(m)}}{\delta x^k} - L_{ijkl}^{(m)} C^{(m)}_{ikl} + C^{(m)}_{ikl} P^{(m)}_{ijl},
\]

\[
R^{(l)}_{ijl} = \frac{\delta L^{(l)}_{ijl}}{\delta x^k} - \frac{\delta L^{(l)}_{ijl}}{\delta t} + \frac{L_{ijkl}^{(m)}}{\delta x^k} - \frac{L_{ijkl}^{(m)}}{\delta x^k} + C^{(m)}_{ikl} P^{(m)}_{ijl},
\]

\[
P^{(l)}_{ii(k)} = \frac{\partial C^{(l)}_{ii(k)}}{\partial y^k} - C^{(l)}_{ii(k)} C^{(l)}_{ii(k)} + C^{(l)}_{ii(k)} P^{(m)}_{ii(k)},
\]

\[
P^{(l)}_{ij(k)} = \frac{\partial C^{(l)}_{ij(k)}}{\partial y^k} - C^{(l)}_{ij(k)} C^{(l)}_{ij(k)} + C^{(l)}_{ij(k)} P^{(m)}_{ij(k)},
\]

\[
S^{(l)}_{ij(k)} = \frac{\partial C^{(l)}_{ij(k)}}{\partial y^k} - \frac{\partial C^{(l)}_{ij(k)}}{\partial y^k} + C^{(l)}_{ij(k)} C^{(l)}_{ij(k)} - C^{(l)}_{ij(k)} C^{(m)}_{ij(k)}.
\]
For $p = \dim T \geq 2$, we have

$$H^{\alpha}_{\eta\beta\gamma} = \frac{\partial H^{\alpha}_{\eta\beta}}{\partial t^{\gamma}} + \frac{\partial H^{\alpha}_{\gamma\eta}}{\partial t^{\beta}} + H^{\alpha}_{\eta\gamma} H^{\alpha}_{\mu\beta} - H^{\alpha}_{\eta\beta} H^{\alpha}_{\mu\gamma},$$

$$R^{l}_{i\beta\gamma} = \frac{\delta G^{l}_{i\beta}}{\delta t^{\gamma}} - \frac{\delta G^{l}_{i\gamma}}{\delta t^{\beta}} + G^{m}_{i\beta} G^{l}_{m\gamma} - G^{m}_{i\gamma} G^{l}_{m\beta},$$

$$R^{l}_{i\beta k} = \frac{\delta G^{l}_{i\beta}}{\delta x^{k}} - \frac{\delta \Gamma^{l}_{ik}}{\delta t^{\beta}} + \Gamma^{m}_{i\beta} \Gamma^{l}_{mk} - \Gamma^{m}_{ik} \Gamma^{l}_{mk},$$

$$R^{l}_{i\beta k} = r^{l}_{i\beta k} = \frac{\partial \Gamma^{l}_{i\beta}}{\partial x^{k}} - \frac{\partial \Gamma^{l}_{i\beta}}{\partial x^{k}} + \Gamma^{m}_{i\beta} \Gamma^{l}_{mk} - \Gamma^{m}_{ik} \Gamma^{l}_{mk}.$$  

**Remark 4.8** In the case of an autonomous metrical multi-time Lagrange space of electrodynamics (i.e., $g_{ij}(t^{\gamma}, x^{k}, x^{k}_{\gamma}) = g_{ij}(x^{k})$), all curvature d-tensors of the Cartan canonical connection vanish, except $H^{\alpha}_{\eta\beta\gamma}$ and $R^{l}_{i\beta k} = r^{l}_{i\beta k}$, that is, the curvature tensors of the semi-Riemannian metrics $h_{\alpha\beta}$ and $g_{ij}$.

**Open problem.** The development of an analogous metrical multi-time Lagrangian geometry on $J^2(T, M)$ is in our attention.

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