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Discrete Connections on the Triangulated Manifolds and difference linear equations

Abstract: Following the authors works [1, 2, 3], we develop a theory of the discrete analogs of the differential-geometrical $GL_n$-connections over the triangulated $n$-manifolds. We study a nonstandard discretization based on the interpretation of DG Connection as the linear first order ("triangle") difference equation in the simplicial complexes acting on the scalar functions of vertices. This theory appeared as a by-product of the new type of discretization for the special Completely Integrable Systems, such as the famous 2D Toda Lattice and corresponding 2D stationary Schrodinger operators. A nonstandard discretization of the 2D Complex Analysis based on these ideas was developed in the recent work [4]. A complete classification theory is constructed here for the Discrete DG Connections based on the mixture of the abelian and nonabelian features.

I. General Definitions: The Discrete DG Connections.

Let $M$ be a $n$-dimensional simplicial complex.

By the Discrete Differentially-Geometrical (DG) Connection we call any set of coefficients $0 \neq b_{T,P} \in k^*, k = R, C,$ assigning a nonzero number to every pair consisting of the $n$-simplex $T$ and its vertex $P \in T$.

Every DG-connection defines a first order difference Triangle Operator $Q$ mapping the space of $k$-valued functions of vertices $\psi_P$ into the space of functions of $n$-simplices:

$$(Q \psi)_T = \sum_{P \in T} b_{T,P} \psi_P$$

Such operators played an important role in the works [1, 2, 3]. For the needs of the theory of discrete DG Connections only the linear Triangle

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\textbf{Equation} is important

\[ Q\psi = 0 \]

well defined up to the Abelian Gauge Transformations

\[ Q \rightarrow f_TQg_p^{-1}, \psi_p \rightarrow g_p\psi_p \]

where \( f \neq 0, g \neq 0 \). Beginning from now we denote vertices by the letters \( i, j, l, \ldots \).

Therefore for every \( n \)-simplex \( T \) with vertices \( i, j \in (i_0, \ldots, i_n) \) only the ratios are essential \( \mu_{ij}^T = \frac{b_{T;i}}{b_{T;j}} \) where \( b_{T;i} \) are the coefficients of DG connection associated to the vertices of the simplex \( T \). We assume beginning from now that the DG Connection is given by the set of nonzero numbers \( \mu_{ij}^T \) for all \( n \)-simplices and pairs of their vertices. Obviously we have

\[ \mu_{ii}^T = 1, \mu_{ij}^T\mu_{ji}^T = 1, \mu_{ij}^T\mu_{jk}^T\mu_{ki}^T = -1 \]

Let \( T, T' \) be a pair of \( n \)-simplices such that \( i, j \in T \cap T' \). We define a Gauge-Invariant Coefficients

\[ \mu_{ij}^T \mu_{ji}^T = \rho_{ij}^{TT'} \]

\textbf{Lemma 1} The whole set of the gauge invariant coefficients \( \rho_{ij}^{TT'} \) can be recovered from the Minimal Subset such that \( T \) and \( T' \) are the closest neighbors, i.e. the intersections \( T \cap T' \) are the \( (n-1) \)-dimensional faces. There are "trivial" sets A and B of relations on these quantities:

A. For every triangle \([ijl] \subset T \cap T' \) we have

\[ \rho_{ij}^{TT'} \rho_{jl}^{TT'} \rho_{li}^{TT'} = 1 \]

B. For every closed path \( T_0T_1 \ldots T_NT_0 \) in the Poincare dual cell subdivision of the triangulated manifold \( M \) where \( n \)-simplices \( T \) define the vertices, all pairs \( T_p \cap T_{p+1} \) define the edges (they are dual to the \( n-1 \)-faces), and the edge \( <ij> \) belongs to all \( T_p \), we have

\[ \prod_{p=1}^{N} \rho_{ij}^{T_pT_{p+1}} = 1 \]
Proof. For the simplicial manifold \( M \) every pair of \( n \)-simplices \( T, T' \) such that \( i, j \in T \cap T' \) can be joined by ”path” \( T_0 = T, T_1, \ldots, T_m = T' \) where \( i, j \in T_k \cap T_{k+1} \) for all \( k = 0, \ldots, m - 1 \), and \( T_k \cap T_{k+1} \) are \((n - 1)\)-simplices for all values of \( k \). We have by definition

\[
\rho^{TT'}_{ij} = \prod_{k=0}^{m-1} \rho_{i_j}^{T_k T_{k+1}}
\]

Our trivial set \( A \) of the relations for these quantities follows from the same relations for the quantities \( \mu^T_{ij}, \mu^{T'}_{ij} \) as above. In order to prove the set of relations \( B \) we point out that any such closed path in the dual cell decomposition can be obtained as a product of elementary paths \( T_0 \ldots T_m \) corresponding to the simplicial stars of every \( n - 2 \)-simplices \( < ij > \subset \sigma \subset St(ij) \). We have here

\[
St(\sigma) = T_0 \cup \ldots \cup T_m
\]

and the relation

\[
\rho_{ij}^{T_0 T_1} \cdots \rho_{ij}^{T_m T_0} = \mu_{ij}^{T_0} \mu_{ji}^{T_1} \mu_{ij}^{T_1} \cdots \mu_{ji}^{T_0} = 0
\]

Lemma is proved.

**Problem**: Is it possible to recover the whole DG connection from the Minimal Subset of the abelian gauge-invariant coefficients \( \rho^{TT'}_{ij} \)? Which invariants of DG connection should be added if it is impossible?

We are going to solve this problem for the 2D and 3D manifolds \( n = 2, 3 \) where the whole set of the additional invariants is easy to find out: let us choose any set of the closed combinatorial ”framed” paths \( \gamma_1, \ldots, \gamma_{b_1}, a_1, \ldots, a_{tor_1} \) representing the basis of the homology group \( H_1(M, Z) \). We define the following abelian gauge-invariant topological quantities:

\[
\mu(\gamma) = \prod_{\gamma} \mu_{T_l T_{l+1}}
\]

for every closed ”framed path” \( \gamma_k \) consisting of edges through the vertices \( 0, 1, \ldots, l, l+1, \ldots, m_k = 0 \), equipped by such triangles \( (n\text{-simplices}) \) \( T_l \) that \( [l, l+1] \subset T_l \).

We are going to prove below the following

**Theorem 1** For any \( n \geq 2 \) the set of invariants

\[
\{ \rho^{TT'}_{ij}, \mu(\gamma_k), \mu(a_s) \}
\]
is complete where \( i \neq j, \rho_{ij}^{TT'}, [ij] \subset T \cap T' \).

For the compact oriented 2-manifolds the only nontrivial relation on these quantities is

\[
\prod_{[ij] \in M} \rho_{ij}^{TT'} = 1, \partial T = [ij] + ..., \partial T' = [ji] + ..., T \cap T' = [ij]
\]

Here \([ij]\) means all edges in the manifold \(M\), the 2-simplices \(T, T'\) are oriented as prescribed by the global orientation.

For the compact oriented n-manifolds the complete set of relations on these quantities can be described in the following way: there are trivial relations \(A\) for every 2-simplex \([ijl]\) = \(\Delta \subset T \cap T'\) where

\[
\rho_{ij}^{TT'} \rho_{jl}^{TT'} \rho_{li}^{TT'} = 1
\]

and the relations \(B\) for every closed path in the Poincare dual cell decomposition corresponding to the boundaries of the dual 2-cells. For the description of the nontrivial relations we choose the set of integral 2-chains

\[
z_1, ..., z_{b_2}, u_1, ..., u_{\text{tor}_1}\}
\]

where \(z_1, ..., z_{b_2}\) is the basis of the group \(H_2(M, Z)\) and \(u_s\) represent the basis of cycles mod \(m_s\), i.e. \(\partial u_s = m_s a_s\). Let all chains \(z, u\) be presented as sums \(\sum \Delta_k\) with "framing" \(T_k\) for every oriented 2-simplex \(\Delta_k\), \(\partial T_k = \Delta_k + ....\). The nontrivial relations have the following form: The product

\[
\prod_{[ij]=\Delta_k \cap \Delta_{k'}} \rho_{ij}^{TT'} = 1
\]

is equal to one for the cycles \(z\); it is equal to \(\mu(\partial u_s)^* = \mu(a_s)^m_s\) for the torsion part \(u_s\).

For \(n = 3\) exactly one global relation between these relations follows from the following identity:

\[
\prod_{[ij] \in M} \rho_{ij}^{TT'} = 1
\]

in the compact oriented 3-manifold \(M\) where orientation in the 3-simplices \(T, T'\) is induced by the one in \(M\), \(\partial T = [ijl], \partial T' = [jil]\).

For \(n \geq 2\) any set of quantities

\[
\rho_{ij}^{TT'}, \mu(\gamma_k), \mu(a_s), k = 1, ..., b_1, s = 1, ..., \text{tor}_1
\]
satisfying to this set of trivial and nontrivial relations can be realized by the discrete DG Connection uniquely up to the abelian gauge transformation.

II. The Nonabelian Curvature.

By definition, the Nonabelian Curvature for the discrete $GL_n$ Connection of the type described above is the obstruction to the existence of full $n$-dimensional space of the local solutions to the triangle equation $Q\psi = 0$ in the whole simplicial stars of the vertices. However, on the manifold $M$ it is enough to consider only the obstructions to the existence of local solutions in the simplicial stars $St(\sigma)$ for all $(n-2)$-simplices $\sigma = [0,1,\ldots,n-2]$. The whole set of vertices in this star contains also the complementary set $p, p = 1, 2, \ldots, m$ where the number $m$ depends on $\sigma$. The $n$-simplices in this star are exactly the following

$$T_p = [\sigma, p-1, p]$$

where $p$ is counted here modulo $m$. Every $n$-simplex $T_p$ contains in its boundary a pair of faces inside of this star:

$$[\sigma, p - 1] \cup [\sigma, p] \subset \partial T_p$$

We start with initial data taking $\psi_0, \ldots, \psi_{n-2}, \psi_p$ as an arbitrary $n$-vector $\eta$. From the equation $Q\psi = 0$ in the simplex $T_{p+1}$ we obtain the value

$$\psi_{p+1} = \sum_{q=0}^{q=n-2} \mu_{q,p+1}^T \psi_q + \mu_{p,p+1}^T \psi_p, q \in \sigma, p = 1, \ldots, m$$

$$\psi_q = \psi_q, q = 0, \ldots, n-2$$

or $\eta \rightarrow A_p(\eta) = \eta'$ where $A_p$ is a lower triangle matrix with $(1, \ldots, 1, \mu_{p,p+1}^T)$ along the diagonal. It has only the last nontrivial row except diagonal which is equal to $(\mu_{0,p+1}, \ldots, \mu_{n-2,p+1}, \mu_{p,p+1})$. For simplicity we omitted in these formulas the simplices $\sigma, T_{p+1}$. Their presence is assumed. The full cyclic product of these matrices gives us by definition a Nonabelian Curvature Operator around the $(n-2)$-simplex $\sigma$:

$$K_{\sigma,p} = A_{p+m-1}A_{p+m-2} \ldots A_{p+1}A_p$$
of the same algebraic form as all matrices $A_s$. Here $s$ is counted modulo $m$. Its diagonal part is equal to

$$1,...1,\mu_\sigma = \prod_{s=p}^{s=p+m-1} -\mu_{s,s+1} = \det K_{\sigma,p}$$

Its last row is equal to

$$\alpha_{\sigma,0,p},...,\alpha_{\sigma,n-2,p},\mu_\sigma$$

We call coefficients $\alpha_{\sigma,q,p}$ and $\mu_\sigma$ the Local Curvature Coefficients where $q \in \sigma, p \in (1,2,...,m)$. We say that our Discrete DG Connection is Locally Unimodular, i.e. belongs to the group $SL_n$ if $\det K(\sigma) = \mu(\sigma) = 1$ for all $\sigma$. The Connection is Locally Flat if $K_\sigma = 1$ for all $\sigma$. These definitions imply immediately the following

**Lemma 2** The local curvature operators are equal to the unit matrix if and only if our linear equation $Q\psi = 0$ has exactly $n$-dimensional space of solutions on the universal covering space. Its coefficients can be computed by formula

$$\alpha_{\sigma,q,p} = \mu_{q,p} + \mu_{p-1,q}\mu_{q,p-1} + ... + \mu_{p-1,q}\mu_{p-2,q-1}...\mu_{p-m+2,q-2}\mu_{q,p-m+1}$$

In particular these coefficients transform as multiplicative 1-chains under the abelian gauge transformations

$$\psi_j = h_j\phi_j, h_j \neq 0, \psi_i \rightarrow \phi_i, \mu_{ij}^T \rightarrow (h_i/h_j)^T\mu_{ij}$$

$$\mu_\sigma \rightarrow \mu_\sigma, \alpha_{\sigma,q,p} \rightarrow (h_q/h_p)\alpha_{\sigma,q,p}$$

We can see now how to organize the simplest gauge invariant expressions.

**Lemma 3** The quantities

$$\alpha_{\sigma,q,p}\mu_{p,q} = \alpha_{\sigma,q,p}^*$$

are gauge invariant. They are connected with each other by the formula

$$\alpha_{\sigma,q,p+1}^* = -\alpha_{\sigma,q,p}^*/(\mu_{pq}\mu_{qp}^{T_{p+1}}) + (1 - \mu_\sigma)$$
All quantities $\alpha^{*}_{\sigma,q,p}$ and $\mu_{\sigma}$ can be expressed through the gauge invariant coefficients $\rho_{ij}^{TT'} = \mu_{ij}^{T} T_{ji}^{T'} T_{T}, T', T' \in St(\sigma)$, by the formula

$$\alpha^{*}_{\sigma,q,p} = \sum_{k=0}^{k=m-1} (-1)^{k} \prod_{j=0}^{j=k} \rho_{p-j,q}^{p_{j},T} \rho_{p-j,q+1}^{T_{j},p}$$

$$\prod_{p=1}^{p=m} \rho_{q_{p}}^{T_{p},p+1} = (-1)^{m} \mu_{\sigma}, q = 0, 1, ..., n - 2$$

so we have $n - 1$ different expressions for the same quantity $\mu_{\sigma}$.

**Corollary 1** For the locally unimodular (i.e. locally $SL_n$) connections the condition $\alpha_{\sigma,q,p} = 0$ does not depend on the initial point $p \in ST(\sigma)$, so the property to be locally flat in the star $ST(\sigma)$ depends on the $n - 2$-simplex $\sigma$ only.

**Corollary 2** For the generic connections such that $\alpha_{\sigma,p} \neq 0$ for all $\sigma, p$, all data $\rho_{ij}^{TT'}$ can be reconstructed from the gauge invariant coefficients of the local nonabelian curvature $\alpha^{*}_{\sigma,q,p}, \mu_{\sigma}$.

Proof of the lemma. Starting with the equation expressing $\alpha_{\sigma,q,p}$ through the collection of $\mu$-s in the previous lemma, we multiply both its sides by the quantity $\mu_{p,q}^{T_{p}}$. After the elementary manipulations, we are coming to the expressions for $\alpha^{*}$ which easily implies all statements of this lemma. Let us avoid here these elementary calculations.

For the proof of the first corollary, we point out that the condition $\mu_{\sigma} = 1$ implies that

$$\alpha^{*}_{\sigma,q,p+1} = \alpha^{*}_{\sigma,q,p}/\rho_{p,p+1}^{T_{p}}$$

where always $\rho \neq 0$, so our corollary obviously follows.

In order to prove second corollary, let us point out that under the assumption of this corollary, we represent $\rho$ as a ratio of the nonzero numbers

$$\rho_{p,p+1}^{T_{p}} = \alpha^{*}_{\sigma,q,p}/\{\alpha^{*}_{\sigma,q,p+1} - 1 + \mu_{\sigma}\}$$

This formula proves our corollary.

**III. The Abelian (Framed) and Nonabelian Holonomy Groups.**
The **Abelian Framed Holonomy Representation** we define for the **Framed Combinatorial Paths** starting and ending in the same point. By definition, a **Framed Combinatorial Path** is a sequence of edges equipped by the \( n \)-simplices containing these edges

\[
\gamma = < i_0, i_1, ..., i_k, i_{k+1}, ..., i_N = i_0 | T_0, T_1, ..., T_k, ..., T_{N-1} >
\]

where \([i_k, i_{k+1}] \subset T_k\). The Abelian Framed Holonomy \( \mu(\gamma) \) for the framed path \( \gamma \) is equal to the product

\[
\mu(\gamma) = \prod_k (-\mu_{T_k}^{i_k, i_{k+1}})
\]

There is a natural multiplication of the framed combinatorial paths \( \gamma_1 \gamma_2 \) and a whole associative semigroup of them \( \Omega = \Omega_{fr}(M, i_0) \). We have an **Abelian Framed Holonomy Representation**

\[
\Omega_{fr}(M, i_0) \rightarrow k^*
\]

For every framed path there is a natural **Inverse Framed Path** \( \gamma^{-1} \) consisting of the same edges and \( n \)-simplices but the order of passing them is reversed. The inverse framed path leads to the inverse framed holonomy. We factorize this semigroup by the relations

\[
<i, i | T > = < i, j | T > < j, i | T > = 1
\]

\[
<i j | T > < j k | T > < k i | T > = 1
\]

for the vertices \( i, j, k \) belonging to the same \( n \)-simplex \( T \). Our relations mean exactly that such pieces can be removed from any path if you meet these vertices one after another as the closest neighbors. We call factor by these relations of the semigroup \( \Omega_{fr}(M, i_0) \) by the **Framed Fundamental Group**

\[
\pi_{fr}^1(M, i_0) = \Omega_{fr}(M, i_0)/\langle\text{Relations}\rangle
\]

For the Abelian Framed Holonomy we need only the factor-group by the commutation relations

\[
\pi_{fr}^1(M, i_0) / \langle aba^{-1}b^{-1} \rangle = H_{fr}^1(M)
\]

We call this factor a **Framed Homology Group** written in the multiplicative form.
Lemma 4 The framed homology group is generated by the framed paths $<i,j,i|T,T'>$ and by the arbitrary closed framed paths $\gamma_s$ whose image in the ordinary homology group (the unframed part) generates it.

Remark 1 For the choosing generators of the framed fundamental group we have to fix for every vertex $j$ a framed path $\delta_i$ joining the vertices $0$ and $i$. As usually, we consider the set of closed paths $(\delta_i) <i,j,i|TT'> (\delta_i^{-1})$ as the additional generators in the framed fundamental group. In this work we are dealing with the abelian case only.

The Abelian Framed Holonomy leads to the representation

$$H_1^{fr}(M) \to k^*$$

The ordinary homology we obtain as a factor-group

$$H_1^{fr}/\langle ij|T > < ji|T' > \rangle = H_1(M)$$

with factorized holonomy representation

$$\mu : H_1(M) \to k^*/\{\rho_{ij}^{TT'}\}$$

for all $i,j,T,T'$.

In order to define a Full Nonabelian Holonomy Representation we introduce an important notion of a Thick Path as a sequence of the oriented $n$-simplices $\kappa = < T_1, ..., T_N >$ such that the next one is attached to the previous one along the common $n-1$-dimensional face $\Delta$ where they induce the opposite orientations. For the Irreducible Thick Path the intersections $T_k \cap T_{k+1} = \Delta_k$ should be exactly equal to the $n-1$-dimensional faces for all $k = 1, ..., N-1$, i.e. $T_k \neq T_{k+1}$. By definition the Closed Thick Paths with period $N$ are defined by the condition that the last $n-1$-dimensional out-simplex $\Delta_N$ coincides with the initial simplex $\Delta_0$. There is also a notion of the Periodic Thick Paths where these sequences are infinite and periodic. We can obviously multiply closed thick paths with the same initial and final $n-1$-simplex $\Delta_0$. The notions of the inverse thick path and of the trivial (empty) thick path are natural. Therefore we are coming to the Associative Semigroup of the Closed Thick Paths $\Omega^{thick}(M, \Delta_0^{n-1})$.

Let us construct a geometric model of the Abstract Thick Paths. We start with the standard linear $n-1$-simplex $\Delta_0 = [0,1,\ldots,n-1] \subset R^{n-1}$.
multiplied by the real line $R$ going along the $n$-th axis $x_n$. Our abstract thick path $\kappa_A$ will be defined by the word $A = a_{i_q}^r ... a_{i_0}^r$ of any length $N = \sum_{s=0}^{s=q} r_s$ in the free associative semigroup with $n$ generators $a_0, ..., a_{n-1}$. As a first step, we take a vertex $i_0$ of the initial $n-1$-simplex $\Delta_0$ for $x_n = 0$ and shift it along the $n$th axis on the positive distance. We get new vertex $i'_0$. Now we construct a linear $n$-simplex $T_1$ with vertices $[0, ..., n - 1, i'_0]$. It contains in the boundary an original in-simplex $\Delta_0 = [0, ..., n - 1]$ and a new linear $n-1$-simplex $\Delta_1 = [0, ..., i'_0, ..., n - 1]$ (the first out-simplex) where exactly one vertex $i_0$ is replaced by the shifted one $i'_0$. Now taking the out-simplex $\Delta_1$ as a new in-simplex instead of $\Delta_0$, we perform this operation once more: we take one of the vertices of the out-simplex $i'_1 \in \Delta_1$ and shift it along the corresponding axis up on the level higher than $i'_0$. It may be the same vertex (it should appear exactly $r_0$ times here as in the word $A$), or another one if we already passed all $r_0$ steps. After that we construct a linear $n-1$-simplex $\Delta_2$ as an out-simplex for the $n$-simplex $T_2$ and so on. Finally we are coming to the realization of the whole word $A$ as a "prism" over $\Delta_0$ consisting of the $n$-simplices such that all their vertices are located on the $x_n$-lines $R_k$ over the original vertices $k \in \Delta^{n-1}$ with monotonically increasing heights $x_n$ except of the vertices of the initial $\Delta_0$. This is an abstract model of thick path with combinatorics defined by the word $A$ consisting of the linear $n$-simplices with vertices located in the union of the $n$ "angle lines" only. For every number $k = 0, 1, ..., n - 1$ there is a subsequence of $n$-simplices in the thick path $T_{(k)} \subset \kappa_A$ with shifts up along the coordinate $x_n$ over the vertex with number $k$ in the vertices $j_l = l \in R_k, l = 0, 1, 2, ..., N_k$ such that $\sum_{k=0}^{n-1} N_k = N$, and $N_k = \sum_s r_s$ where $i_s = k$. We call these sequences $<j_0, j_1, ..., j_{N_k}, T_{(k)}^0, ..., T_{(k)}^N>$ the abstract angle framed paths.

Any thick path can be realized in the manifold $M$ with the initial oriented $n-1$-simplex $\Delta_0$ is given. An initial oriented simplex $\Delta_0$ uniquely determine the irreducible thick path with given combinatorics. All $n$-simplices $T_k$ should be attached to the previous oriented $n-1$ out-simplex $\Delta_{k-1}$ inducing in it the right orientation, and $T_k \neq T_{k-1}$ in the irreducible case. For the realization of the reducible thick path we should indicate the "turning points" in the sequence of simplices. An irreducible thick path will be determined by the combinatorics of the word $A$ only. Topology of the triangulated manifold $M$ determines which paths with the initial face $\Delta = \Delta_0$ are in fact closed. The set of closed thick paths started in the face $\Delta_0$ we denote by
\( \Omega^{\text{thick}}(M, \Delta_0) \). For the closed thick path the corresponding angle framed paths will not necessarily be closed in the manifold \( M \):

**Lemma 5** There is a natural homomorphism into the permutation group \( S_n \)

\[
P : \Omega^{\text{thick}}(M, \Delta_0) \rightarrow S_n
\]

induced by the permutation of the vertices of the \( n-1 \)-simplex after the identification of the initial in-simplex \( \Delta_0 \) and the last out-simplex in the closed thick path.

The kernel of this representation will be denoted \( \ker P = \Omega^{\text{thick}}_0(M, \Delta_0) \). Let a closed thick path \( \kappa \in \Omega^{\text{thick}}(M, \Delta_0) \) in the manifold \( M \) be given starting and ending in the \( n-1 \)-face \( \Delta_0 \), and let 0, 1, ..., \( n-1 \) be its vertices. For any Discrete DG connection with coefficients \( \mu_{ij}^T \), we define a **Nonabelian Holonomy Representation** along the closed thick path \( \kappa \) with permutation \( P_\kappa \):

Starting from the initial data \( \eta = (\psi_0, ..., \psi_{n-1}) \in R^n_{\Delta_0} \), we step by step calculate the values of function \( \psi \) in the vertices of the thick path \( \kappa \) from the equation \( Q\psi = 0 \). For the thick path \( \kappa = (T_N \ldots T_1) \) it leads to the linear map

\[
\tilde{K}_\kappa : R^n_{\Delta_0} \rightarrow R^n_{\Delta_N} = \tilde{K}_{T_N} \ldots \tilde{K}_{T_1}
\]

where

\[
\tilde{K}_T : R^n_{\Delta} \rightarrow R^n_{\Delta'}
\]

is the one-step map from the **in-face** into the **out-face** for the \( n \)-simplex \( T \) provided by the DG connection.

By the **Nonlinear Holonomy Map** we call a product

\[
K_\kappa = P_\kappa \tilde{K}_\kappa
\]

So the correspondence \( \kappa \rightarrow K_\kappa \) generates a holonomy representation

\[
K : \Omega^{\text{thick}}(M, \Delta_0) \rightarrow GL_n(k)
\]

**Lemma 6** The Holonomy Representation is a Homomorphism

\[
\Omega^{\text{thick}}(M, \Delta_0) \rightarrow GL_n(k)
\]

\[
K_{\kappa_2}K_{\kappa_1} = K_{\kappa_2}K_{\kappa_1}
\]
For the proof of this lemma we point out that $K = P\tilde{K}$. From the definition of $\tilde{K}$ we have:

$$(P_{\kappa_1}^{-1}\tilde{K}_{\kappa_2}P_{\kappa_1})\tilde{K}_{\kappa_1} = \tilde{K}_{\kappa}$$

because the basis of vertices of the $n-1$-simplex $\Delta_0$ is shifted by the permutation $P_{\kappa_1}$ after passing the first closed path $\kappa_1$. Therefore we obtain finally

$$K_{\kappa_2\kappa_1} = (P_{\kappa_2}P_{\kappa_1})P_{\kappa_1}^{-1}\tilde{K}_{\kappa_2}P_{\kappa_1} \tilde{K}_{\kappa_1} = K_{\kappa_2}K_{\kappa_1}$$

Lemma is proved.

For the unique nontrivial closed thick path in the simplicial star $\kappa \subset ST(\sigma)$ surrounding $n-2$-dimensional simplex $\sigma$, this holonomy map reduces to the "Nonabelian Curvature Map" $K_{\sigma,p}$ defined in the previous paragraph where $\Delta = [\sigma, p]$. This simplest path corresponds to the most elementary word $A = a_j^m$ rotating $n$-simplices around the $n-2$-simplex $\sigma$ opposite to the vertex $j \in \Delta$.

**Lemma 7** For every closed thick path $\kappa_A$ determined by the word $A$ and initial $n-1$-simplex $\Delta_0$ the determinant of the Nonabelian Holonomy Map has a form

$$\det K_{\Delta_0} = (-1)^N \prod_{k=0}^{k=n-1} \mu_k(\kappa_A)$$

where

$$\mu_k(\kappa_A) = \prod_{l=0}^{l=N_k} -\mu_{i_l,i_{l+1}}$$

is the product along the "angle" axis $R_k$ going up with the variable $x_n$ from the vertex corresponding to $k \in \Delta_0$ in the abstract model of the thick path. The quantities $\mu_k(\kappa_A)$ are equal to the Abelian Framed Holonomy Representation of the framed paths

$$\gamma_{A,k} = <j_0, ..., j_{N_k} | T_{0}^{k}, ..., T_{N_{k-1}}^{k} >$$

called the "angle" paths of the thick path $\kappa_A$

$$\mu_k(\kappa_A) = \mu(\gamma_{A,k})$$

The product of all angle paths is closed.

If all local curvature operators $K_{\sigma,p}$ are equal to the unit matrix for all $n-2$-simplices $\sigma$, then the Nonabelian Holonomy depends on the class of thick path in the fundamental group $\pi_1(M)$ only.
Lemma 8  All matrix elements $\alpha_{ij}(\kappa, \Delta_0)$ of the operators $K = P \tilde{K}$ of the Nonabelian Holonomy transform under the abelian gauge transformations as one-dimensional multiplicative cochains $\alpha_{ij} \rightarrow h_i/h_j \alpha_{ij}$ where $i, j$ are the vertices of the initial $n-1$-simplex $\Delta_0$ where

\[ K \rightarrow HKH^{-1}, H = \text{diag}(h_1, \ldots, h_{n-1}) \]

For the local Nonabelian Curvature Operators we have $P = 1, \Delta_0 = [\sigma, p]$ where $\sigma$ is an arbitrary $n-2$-simplex, and $p = 1, \ldots, m$ is a vertex in its simplicial star $p \in ST(\sigma)$. All gauge-invariant polynomial in the variables $\alpha_{ij}(\kappa, A), \mu^{ij}_{TT}$ can be expressed as polynomials from the framed abelian holonomy of the closed paths in $M$.

Proof of thislemmas presents no difficulties. For the proof let us point out that our matrix elements $\alpha_{ij}$ can be expressed as the sums of the products of the quantities $\mu^{il}_{ij}$ along the paths easily visible as the paths monotonically going up (see Fig) in the abstract model of thick path, starting in the simplex $\Delta_0$ and ending in the upper $n-1$-simplex (who coincides with $\Delta_0$ for the closed paths in the manifold.) Therefore after the gauge transformations only the boundaries of the paths will give contribution, so only the ends $i, j$ remain in the final answer. At the same time, all gauge invariant expressions presented as polynomials of the path integrals of the quantities $\mu$ can be expressed through the holonomy of the closed paths (no free ends can be left for the gauge invariant expression). Our lemma is proved.

Example 1  Let us consider an interesting example of the Canonical Connection on the triangulated manifolds $M$ where all connection coefficients are equal to one in every simplex $b_T = 1$. We have also $\mu^{il}_{ij} = -1$. This connection has been considered in [4]. It appeared also in the work [5] in the different terminology for the needs of the coloring problem. Its image belongs to the group $S_{n+1}$ but we normally realize this group linearly $S_{n+1} \subset GL_n$ using the imbedding $R^n \subset R^{n+1}$ as a subspace invariant under the permutation of all coordinates $\psi_i$ where $\sum \psi_i = 0, i = 0, \ldots, n-1$. Exactly that corresponds to the canonical connection in the work [4]. Starting from any $n-2$-simplex $\Delta_0$, we construct a Coloring of the Vertices by the $n+1$ colors $u_0, \ldots, u_n$ along the thick path with combinatorics corresponding to the word $A$. Assigning to the initial vertices of the in-simplex $\Delta_0$ the colors $u_0, \ldots, u_{n-1}$, we can see that the final coloring of the vertices of the out-simplex is uniquely defined.
by two factors: by the combinatorics of the word $A$ and by the realization of this final out-simplex in the manifold $M$, i.e. by the permutation $P(\kappa)$. We denote a nonabelian holonomy map associated with this connection by the
\[ K^\text{can}_\kappa = P_n \tilde{K}^\text{can}_\kappa \]

**Lemma 9** For any closed thick path $\kappa$ with combinatorics corresponding to the word $A = a_i^r a_j^q ... a_i^{r_0}$ the resulting permutation corresponding to the holonomy of the canonical connection is given by the formula
\[ K^\text{can}_\kappa = P_\kappa \tau^r_{i_0,n} ... \tau^q_{i_0,n} \]
where $i = 0,1,...,n-1$ and $\tau_{i,n}$ is a permutation of the points $i,n$ only, $\tau^2_{i,n} = 1$. The permutation $P$ in this formula permutes only the numbers $0,1,...,n-1$ leaving the index $n$ invariant.

Proof of this lemma follows immediately from the definition of the abstract model of the thick path.

III. Solution of the Reconstruction Problem for the case $n = 2$

**Flat connections. The case $n \geq 2$.**

Consider now any oriented trangulated 2-manifold $M$ with the vertices $i,j,...$, 2-simplices $T, T', ...$ and with the discrete DG Connection. Our field is $k = R, C$ only.

**Theorem 2** All coefficients $\mu^T_{ij}$ of the discrete DG Connection over the field $k$ can be recovered up to abelian gauge transformation from the abelian framed holonomy representation
\[ \mu : H^r_1(M) \to k^* \]
where the framed abelian holonomy image of the group $H^r_1(M)$ is generated by the elements $\rho_{ij} = \mu^T_{ij} \mu^T_{ji} = <i,j,i|TT'>^*$ and by the images of the generators of the ordinary homology group $\mu(\gamma_s), \mu(a_q) \in k^*$.

Proof. Let us describe the reconstruction process. Introduce the new quantities by the following formula:
\[ \lambda_{ij} = -\mu^T_{ij} / \sqrt{\rho^T_{ij}}, \partial T = [ij] + ..., \partial T' = [ji] + ... \]
where some specific value of the square root is chosen. If our manifold is oriented, we choose the orientation of the triangles $T, T'$ to be the same as the global orientation of manifold, and $\partial T = [ij] + \ldots$. In that case we forget about the indices $TT'$ in the formulas, so we have $\lambda_{ij} = -\mu_{ij}/\sqrt{\rho_{ij}} = \sqrt{-\mu_{ij}/\mu_{ji}}$ where
\[
\rho_{ij} = \rho_{ji}, \lambda_{ij} = \lambda_{ji}^{-1}
\]
Nonuniqueness of choosing the square root we resolve by choosing square roots separately defining $\sqrt{-\mu_{ij}}$ in every triangle $T$ with requirement
\[
\sqrt{-\mu_{ij}} \sqrt{-\mu_{jl}} \sqrt{-\mu_{li}} = 1
\]
. We shall return to these details later.

**Lemma 10** For the coboundary of the $k^*$-valued multiplicative cochain $\lambda = (\lambda_{ij})$ defined by the formula
\[
d\lambda(T) = \lambda(\partial T) = \lambda_{ij}\lambda_{jl}\lambda_{li}, T = [ijl]
\]
we have
\[
d\lambda(T) = (\rho_{ij}\rho_{ji}\rho_{li})^{-1/2} = \rho^{-1/2(T)}
\]
Therefore for every finite triangulated domain $D$ in the manifold $M$ following integral formula is true expressing the integral of the "Curvature" $(\rho(T))^{-1/2}$ along the domain $D$ through the framed abelian holonomy of the boundary curves $\partial D = \bigcup_q \gamma_q$ with framing by the triangles looking in the external direction to the domain $D$:
\[
\prod_{T \in D} (\mu_{ij}^T)^{-1/2} = (\mu(\partial D))^{-1/2}
\]
In particular, for the compact oriented manifold $M$ we have
\[
\prod_{T \in M} (\rho(T))^{-1/2} = 1
\]
The proof of this lemma follows immediately from definition of the quantities $\rho_{ij}$ and $\lambda_{ij}$ taking into account the equality $\mu_{ij}\mu_{ji}\mu_{li} = -1$ and our agreement that $\mu_{ij}^T = \mu_{ij}$ for the right orientation leading to the conclusion that $\mu_{ji} = \mu_{ji}^T$ where $T' \neq T$, and $\rho_{ij} = \mu_{ij}\mu_{ji} = \rho_{ji}$. Our Lemma is proved.
As a corollary, we are coming to the following conclusion: knowing $\rho_{ij}$ we can reconstruct an unknown cochain $\lambda$. After that we define our DG Connection by the formula

$$\mu_{ij} = -\lambda_{ij} \sqrt{\rho_{ij}}$$

By definition, this is a solution of our system. A cochain $\lambda$ is nonunique: any cocycle $\delta$ may be used to change it: $\lambda' = \lambda \delta$, i.e. $\lambda'_{ij} = \lambda_{ij} \delta_{ij}$ where $\delta_{ij} \delta_{jl} \delta_{li} = 1$ for ever triangle $[ijl]$. It is obvious that there is nothing except the set of all 1-cocycles $\delta$ and all possible changes of signs in the definition of the square roots of $\rho_{ij}$ what may lead to the same set of the data $\{\rho_{ij}\}$. Making an arbitrary abelian Gauge transformations $\mu_{ij} \rightarrow \mu'_{ij} = (h_i/h_j) \mu_{ij}$ we change $\lambda$ by the cocycle $\delta_{ij} = h_i/h_j$, i.e. by the coboundary. Changing signs of the square roots $(\rho_{ij})^{1/2} \rightarrow -(\rho_{ij})^{1/2}$ we change $\lambda$ by the same signs, so the resulting value of $\mu$ remains unchanged.

Changing $\lambda$ by the cocycle $\delta$ nonhomologous to zero, we also change $\mu$ by the same $\delta$, i.e. $\mu \rightarrow \mu \delta = \mu'$. Therefore our framed abelian holonomy along the closed contours will be changed by the integrals

$$\mu(\gamma) \rightarrow \mu(\gamma) \prod_{[ij] \in \gamma} \delta_{ij}$$

Now we are fixing $\tau$ by the requirement of the theorem that the framed abelian holonomy is prescribed along some basis $\gamma_k$ of the homology group $H_1(M)$. Our theorem is proved.

**Lemma 11** Let $\lambda'^{TT}_{ij} = \mu'^{T}_{ij}/\sqrt{\rho'^{TT}_{ij}}$ and $\lambda'^{TT}_{ij} \rightarrow \lambda'^{TT}_{ij} \delta_{ij}$, $\mu'^{T}_{ij} \rightarrow \mu'^{T}_{ij} \delta_{ij}$ where $\delta$ is an ordinary multiplicative cocycle. Then the framed abelian holonomy is changed by the "multiplicative integral" of $\delta$ along the same closed paths. The operators of Nonabelian Holonomy along the Thick Paths are changed in the following way:

$$K_\kappa \rightarrow C^{-1} HK_\kappa H^{-1}$$

where $C = \delta_\kappa$ is a value of the 1-cocycle $\delta$ on the element $\kappa \in \pi_1(M)$ of the fundamental group, $\Delta_0$ is an initial $n-1$-simplex of the thick path $\kappa$ and $H = \text{diag}(\hat{h}_0, ..., \hat{h}_{n-1})$ is the diagonal matrix whose entries are well-defined up to the common nonzero multiplier, and $\hat{h}_i/\hat{h}_j = \delta_{ij}$. 

Proof. The framed abelian part of this lemma is obvious from definitions of \( \lambda \) and \( \delta \). In order to prove nonabelian part, we consider this Discrete DG Connection on the special abelian covering \( \pi : \hat{M} \to M \) such that our cocycle became exact \( \pi^*\delta = dh \). Consider any closed thick path \( \kappa \) starting and ending in the \( n-1 \)-simplex \( \Delta_0 \) of the manifold \( M \) and realizing a generator \( \gamma \in H_1(M) \). Without any losses of generality, we may think that our cocycle \( \delta \) has nontrivial “multiplicative integral” along this basic element only, and that our covering is cyclic. On the covering space \( \hat{M} \) we choose a covering \( n-1 \)-simplex \( \Delta_0 \). After that we get a unique covering thick path \( \hat{\kappa} \) starting in \( \Delta \). This path is a covering path over the closed path \( \kappa \) with period \( N \), i.e. it consists of the sequence of \( n \)-simplices \( T_1,...,T_N,T_N,... \) such that \( \pi(T_i) = \pi(T_{N+i}) \), and \( T_{N-1} \cap T_N \in (\pi)^{-1}(\Delta_0) \). The monodromy map on the covering space \( R = \hat{\gamma} : \hat{M} \to M \) transforms thick covering path into itself, and \( R(T_1) = T_N, R(\Delta) = T_{N-1} \cap T_N \). Consider the function \( \hat{h} \) in the covering thick path. It is nonperiodic: we have \( R^*(\hat{h}) = C\hat{h} \) where \( C = \prod_{\gamma} \delta_{ij} \) by definition. In the covering path our DG Connections both are periodic and gauge (abelian) equivalent to each other but the equivalence is nonperiodic. According to the lemma 6 (above) we can see that the matrix elements of our Nonabelian Holonomy Operator transform by the following formula
\[
\alpha_{ij} \to \frac{\hat{h}_i}{R^*(\hat{h}_j)}\alpha_{ij}, i,j \in \Delta_0
\]
because our thick path starts at \( \Delta \) and ends at \( R(\Delta) \) in the covering space. At the same time, we have \( R^*(\hat{h}) = C\hat{h} \). This is exactly the statement of our lemma. Lemma is proved.

**Theorem 3** For every data \( \rho_{ij}^{TT'} \) on the orientable 2-manifold \( M \) corresponding to flat \( GL_2 \) connections (i.e. with trivial nonabelian local curvature), there exist exactly one \( SL_2 \)-connection up to abelian gauge transformation and changing sign.

**Remark 2** The existence of \( SL_n \)-connection follows from the same arguments also for all \( n \geq 2 \), but the uniqueness for \( n > 2 \) will be proved later using some additional arguments not presented yet (see below).

Proof of this theorem follows from the previous lemma: we may change determinant on the Nonabelian Holonomy Group multiplying \( \rho_{ij} \) by the 1-cocycle \( \delta_{ij} \). Let us point out that after making determinant equal to 1, we may also change sign of the holonomy.
The first Chern Number: Let now $n = 2$. Consider the case $k = C$ and assume that $|\arg[\rho^{TT}_ij]| < \pi/2$. We define an integer-valued cohomology class $c_1 \in H^2(M, Z)$ by the formula for the cochain:

$$c_1(T) = \frac{1}{2\pi i} \arg[(\rho(T))^{-1/2}]$$

From the equality

$$\prod_T (\rho(T))^{-1/2} = 1$$

for the closed oriented manifold we obtain the result

$$\sum_{T \in M} c_1(T) = r \in Z$$

Our condition permits us to make such unique choice that

$$|\arg[\rho(T)^{-1/2}]| < \pi$$

in all cases. With this agreement we are coming to the well-defined integer number.

IV. Multidimensional Discrete DG Connections.

Consider now any $n > 2$. We expect that all DG Connection can be reconstructed from the framed abelian representation. Let a closed oriented triangulated manifold $M$ be given, and $s_k$ are the numbers of $k$-simplices. We know a few number of general relations for these numbers:

$$s_{n-1} = \frac{n + 1}{2} s_n$$

$$\sum_{k=0}^{k=n} (-1)^k s_k = \eta(M)$$

where $\eta(M)$ is the Euler characteristic. In particular, $\eta = 0$ for the odd dimensions $n = 2t + 1$. Let us present the numbers $s_k$ in the form:

$$s_k = \frac{(n + 1)!}{(k + 1)!(n - k)!m_k} s_n$$
The meaning of this is following: every \( n \)-simplex has exactly \((n + 1)!/(k + 1)! \) \((n - k)! \) faces of the dimension \( k \). If every \( k \)-simplex belongs exactly to \( m_k \) \( n \)-dimensional simplices, one can deduce that this number \( s_k \) can be computed exactly as it is written here. For example, we have always \( m_{n-1} = 2 \) in the manifolds, \( 3 \leq m_{n-2} < m_{n-3} < ... < m_0 \). In general, this formula gives definition of the numbers \( m_k \) as some sort of the mean value of the number of \( n \)-simplices containing a random \( k \)-simplex.

**Example 2** As a simplest example we take sphere \( M = S^n = \partial \Delta_{n+1} \) as a boundary of the \((n + 1)\)-simplex. In this case \( m_{n-1} = k + 1 \).

Let us try to count a number of parameters in our reconstruction problem.

**Lemma 12** The number \((\mu)\) of independent quantities \( \mu^{TT'}_{ij} \) modulo the abelian gauge transformations is equal to \( ns_n - s_0 + 1 = (\mu) \).

Proof. In every \( n \)-simplex \( T = [0, 1, ..., n] \) we have exactly \( n \)-dimensional manifold of the quantities \( \mu^T_{ij} \) with (multiplicative) basis \( \mu^T_{0i} \) according to the relations indicated in their definition above. Different \( n \)-simplices are completely independent. Applying the abelian gauge transformations we extract exactly \( s_0 - 1 \) parameter because the constant function leads to the trivial gauge transformation. This argument implies the statement of our lemma.

For every \( n \geq 2 \) we define the number \((\rho)\) equal to the (multiplicative) dimension of the set of all quantities \( \rho^{TT'}_{ij} \) plus \( b = b_k \) where \( b \) is the rank of the Betti number \( b_1 \) for the real positive holonomy representation \( k^* = R_+ \), and it is equal to the larger number \( b_1^* = b_1 + \text{torsion}_1 \) for \( k^* = C^* \). We present it by the expression

\[
(\rho) = (n - 1)s_{n-1} - (n - 2)s_{n-2} + b + R
\]

where \( R \) is the remaining part.

In order to explain meaning of this phrase, let us point out that by the same reason there exist exactly \( n - 1 \)-dimensional space of quantities \( \rho^{TT'}_{ij} \) in every \( n - 1 \)-simplex \( T \cap T' \). However, they are not independent for different pairs of neighbors \( T, T' \). According to the lemmas above (see local nonabelian curvature) there are \( n - 1 \) different expressions for the quantity \( \mu(\sigma) = \det K_{\sigma,p} \) in the star \( St(\sigma) \) of every \( n - 2 \)-simplex \( \sigma \) depending on the vertex
of this simplex $\sigma$. It is very probable that the number of remaining relations does not depend on triangulation (at least for $n = 3$. What is important is that these relations are not independent for $n > 2$ in general as we shall see below. Therefore some unknown number $R$ enters our calculation making the answer totally unclear.

**Example 3** Consider the simplest case $n = 2$ where everything is already known. The number $(\mu)$ is described by this formula for all $n \geq 2$. For the number $(\rho)$ we have $(\rho) = s_1 - 1 + b$ where $b = 2g$ and $R = -1$ because there exist a nontrivial global relation $\prod_{ij; T, T' \in M} \rho^T_{ij} = 1$. Taking into account the relations $s_2 = 3/2s_3, \eta(M) = s_2 - s_1 + s_0 = 2 - 2g$, we are coming to the equality

$$(\mu) = (\rho)$$

for the closed oriented 2-manifolds. The neighboring oriented pairs $TT'$ are chosen such that $\partial T = [ij] + ..., \partial T' = [ji] + ....$. It corresponds to the fact established above that for the reconstruction of the DG Connection we have to include in the data also values of the framed abelian holonomy on the set of $2g$ closed paths presenting the basis of the $H_1(M)$ except all $\rho_{ij}$.

**The case $n \geq 3$**

**Example 4** For the case $n = 3$ we have $(\mu) = 3s_3 - s_0 + 1$ and $s_2 = 2s_3, s_3 - s_2 + s_1 - s_0 = 0$. Therefore we obtain as a corollary of the relation $3s_3 - s_0 + 1 = 2s_2 - s_1 + b + R$ that

$$R + b = 1$$

For the homological 3-spheres we have $b = 0$ and $R = 1$.

We are going to prove below that the framed abelian holonomy uniquely determines the DG Connection up to the abelian gauge transformation for all $n \geq 3$. Therefore the dimension $\rho^{\text{realizable}}$ generated by all $\rho^T_{ij}$ generated by the DG Connections is equal for $n \geq 3$ to the number $(\mu) - b$. For example, we have for $n = 3$:

$$\rho^{\text{realizable}} = 3s_3 - s_0 + 1 - b = 2s_2 - s_1 + R$$
where $R = 1 - b$. The space of all functions with formal properties of the quantities like $\rho_{ij}^{TT'}$ and general local relations indicated above for $n = 3$ has dimension equal to $2s_2 - s_1$ plus something depending on the topology of the manifold $M$ only. Looking on the right-hand side of this relation, we expect to find out exactly one global dependence between the $s_1$ already known general "local" relations for these quantities in the simplicial stars of all $n - 2$-simplices, plus exactly $b$ "global" relations depending on the 1-cycles in the $n$-manifold $M$ as a minimal possibility. We shall describe these relations below for the closed oriented 3-manifolds. Let now $n \geq 3$.

**Theorem 4** Let $k^* = C^*$.

I. For every integral 2-cycle $z = \sum \Delta_k, a_j \in Z$ presented as a sum of the oriented 2-simplices in the $n$-manifold $M$ there is a relation between the quantities $\rho_{ij}^{TT'} \in C^*$: Let an arbitrary "framing" be chosen along this 2-cycle, i.e. with every 2-simplex $\Delta_k \subset z$ we associate an $n$-simplex $T$ such that $\partial T = \Delta_k + ...$. This relation has a form

$$\prod_{[ij] \in \Delta_k \cap \Delta_{k'} \subset z} \rho_{ij}^{T_k T_{k'}} = 1$$

II. For every 1-dimensional "torsion cycle" $a \in H_1(M, Z)$ of the order $r$ (i.e. $a^r = 1$ in the multiplicative form) there is a relation between the abelian framed holonomy $\mu(a)$ along the framed path $a$ and the quantities $\rho_{ij}^{TT'}$. Let an integral chain (simply, a formal sum of the oriented 2-simplices) $u = \sum_k \Delta_k$ with some framing $T_k, \partial T_k = \Delta_k + ...$ is given such that $\partial u = a^r$. The relation has a form

$$\prod_{[ij] \in \Delta_k \cap \Delta_{k'} \subset u} \rho_{ij}^{T_k T_{k'}} = \mu(\partial u) = \mu(a)^r$$

, and $(\partial u)$ is a boundary 1-chain with induced framing. The product along the chain $u$ is defined representing our chain $u$ through the pairs of 2-simplices $\Delta_k, \Delta_{k'}$ whose intersection is exactly the edge $[ij]$ entering them with the opposite orientations.

**Corollary 3** For the closed oriented 3-manifolds $M$ we have $b_2 = b_1$ by the Poincare duality. Therefore we have $b = b_2 + \text{torsion}_1$, and the number of relations is equal to the rank $b$ of the topological part of the framed abelian holonomy.
The proof of this theorem follows from the integral formula:

**Lemma 13** For every integral 2-chain $w = \sum_k \Delta_k$ with given framing of the 2-simplices following "Integral Formula" is true:

$$\prod_{[ij] \in \Delta \cap \Delta' \subset w} \rho^{TT'}_{ij} = \prod_{[ij] \in \partial w} -\mu^T_{ij},$$

**The first Chern class**

As a by-product of this theorem we define a first Chern class $c_1 \in H^2(M, Z)$ in the same way as for $n = 2$ above: we put

$$(c_1, z) = (2\pi)^{-1} \sum_{[ij] \in \Delta \cap \Delta'} \arg[\rho^{T_k T_{k'}}_{ij}] \in Z$$

for every cycle $z$; For the cycles modulo finite order we define

$$(c_1, u) = (2\pi)^{-1} \sum \arg \rho^{T_k T_{k'}}_{ij} \in Z/rZ$$

as in the theorem above. The topological properties of these quantities in the discrete case should be especially discussed. We avoid this discussion here.

**Reconstruction of the Connection for $n \geq 3$**

Let us start with the case $n = 3$. Consider now a three-dimensional oriented manifold $M$. For every simplicial star $St(\sigma)$ of one dimensional simplex $\sigma = (ij) \subset M$ we reconstruct the quantities $\mu^T_{ij}$ up to the unknown constant $\delta_{ij}, T \in St(\sigma)$, from the equations

$$\rho^T_{ij} = \mu^T_{ij}, T = 1, 2, \ldots q(\sigma)$$

These equations are solvable because the trivial set of relations B is satisfied (see Lemma 1). Here the simplices $T_p \in St(\sigma)$ are numerated in the natural cyclic order modulo $q(\sigma)$ where $T_{q+1} = T_1$. So we know the quantities

$$\mu^T_{ij} = \delta_{ij}$$
where $\delta_{ij}\delta_{ji} = 1$, if solution $\mu_{ij}^T = (\mu_{ji}^T)^{-1}$ exists. Let us consider the necessary equation

$$\mu_{ij}^T \mu_{ji}^T \mu_{ii}^T = -1$$

We are coming to the conclusion that our problem is solvable if and only if following three requirements are satisfied: 1. The quantity

$$\bar{\mu}_{ij}^T \bar{\mu}_{ji}^T \bar{\mu}_{ii}^T = \bar{\mu}[\Delta]^T$$

is in fact cochain depending only on the oriented simplex $[ijl] = \Delta$, i.e. $\bar{\mu}[ijl][\bar{\mu}[jil]] = 1$; 2. This cochain is closed; 3. This cochain is exact.

Proof of the Statement 1: We define this quantity on the oriented manifolds $M$ for the 3-simplices $T$ of the right orientation such that $\partial T = [ijl] + ...$. This agreement makes it well-defined as a function of oriented simplices. In order to prove that $\bar{\mu}[ijl] = \bar{\mu}[jil]^{-1}$, we use the identity

$$\rho_{ij}^{TT'} \rho_{ji}^{TT'} \rho_{ii}^{TT'} = 1$$

for the pair of oriented simplices such that $\Delta = T \cap T'$. We know that $\rho_{ij}^{TT'} = \bar{\mu}_{ij}^T \bar{\mu}_{ji}^T$. This equality immediately implies our result. The statement 1 is proved.

Proof of the Statement 2. For every 3-simplex $T$ we consider a full product of the quantities $\bar{\mu}[\Delta_s]$ along the boundary simplices $\Delta_s, s = 1, 2, 3, 4$. It is easy to see that for every edge $ij \subset T$ we have exactly two multipliers in this product equal to $\bar{\mu}_{ij}^T$ and $\bar{\mu}_{ji}^T$. We use here the result of the statement 1 expressing everything through the quantities $\bar{\mu}^T$ for the faces of any orientations (other 3-simplices are not needed). This property implies our result. The statement 2 is proved.

Proof of the Statement 3. We know already that this is a multiplicative cocycle with values in $C^*$. As everybody knows, its exactness requires the conditions formulated in the Theorem above where the homological relations for the connection coefficients were found. Therefore the statement 3 follows from the homological arguments.

Using this result, we easily reconstruct our connection $\mu_{ij}^T$: we take any solution to the equation

$$d\delta = \bar{\mu}[\Delta]$$

and put $\mu_{ij}^T = \delta_{ji} \bar{\mu}_{ij}^T$. This solution satisfies to all requirements and define exactly the same local part of the framed abelian holonomy representation.
This set of quantities can be choose modulo closed 1-cocycle $\delta_{ij} \to \delta'_{ij}$ such that their ratio is closed. This degree of freedom should be used in order to make the proper adjustment of the global part of framed abelian holonomy along the basis of the first homology group. The exact part of this cocycle is responsible for the abelian gauge transformations. These arguments finish our problem.

Nonorientable case can be easily reduced to the oriented one using the orientable 2-covering in the same way as for the case $n = 2$. For the manifolds $M$ of all dimensions $n \geq 3$ we develop the same reconstruction process as for $n = 3$.

Step 1. Consider the simplicial star $St(ij)$ of the edge $ij$. Solve the equations

$$\rho_{ij}^{TT'} = \tilde{\mu}_{ij}^{T} \tilde{\mu}_{ji}^{T'}$$

in the star. This problem can be solved uniquely up to unknown constant $\delta$:

$$\mu_{ij}^{T} = \delta_{ij} \tilde{\mu}_{ij}^{T}, \delta_{ij} \delta_{ji} = 1$$

This is true because $\rho_{ij}^{TT'}$ is a 1-cocycle in the dual cell decomposition of the star $St(ij)$ where $T$ are the vertices (see the relations B in Lemma 1). So our condition for the solvability of that intermediate problem is $H^1(St(ij), k^*) = 1$. It is certainly true in the manifolds.

Step 2. In order to solve the equation $\mu_{ij}^{T} \mu_{jl}^{T} \mu_{li}^{T} = -1$ for every $n$-simplex $T$ we need to prove that this quantity

$$\tilde{\mu}_{ij}^{T} \tilde{\mu}_{jl}^{T} \tilde{\mu}_{li}^{T} = \tilde{\mu}^{T}[ijl]$$

is in fact a well-defined multiplicative cocycle independent on $T$. This statement follows from the requirement

$$\rho_{ij}^{TT'} \rho_{jl}^{TT'} \rho_{li}^{TT'} = 1$$

where $\rho_{ij}^{TT'} = \tilde{\mu}_{ij}^{T} \tilde{\mu}_{ji}^{T'}$. So we conclude that $\tilde{\mu}^{T}[ijl] \tilde{\mu}^{T'}[jil] = 1$ for every pair $T, T'$. The proof that this 2-cochain is closed is the same as for the case $n = 3$.

Step 3. In order to prove that this cochain is exact we use an analog of the same relations for the quantities $\rho$ integrated along the cycles (see the Theorem above). This theorem gives us the set of relations which leads to the property of any cocycle to be exact in the elementary homological algebra.

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After that all arguments coincide with the case $n = 3$. Our reconstruction process is finished.
In particular, we see that the Uniqueness Theorem for all $n \geq 2$ follows from our results:

**Theorem 5** The framed abelian holonomy representation determines completely the Discrete $\text{GL}_n$-Connection $\{\mu_{ij}^T\}$ on the triangulated $n$-manifold $M$ up to the abelian gauge transformation. The set of conditions on the data of the framed abelian holonomy representation found in this work is necessary and sufficient for the reconstruction.

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