Study on flow parameters of fractal porous media in the high-velocity fluid flow regime

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Abstract. High-velocity fluid flow, which will result in the region of the wellbore or fracture, is generally in the turbulent flow regime and has drawn tremendous attention in petroleum engineering field. Turbulent factor is the key parameter, which is widely used to describe high-velocity flow in porous media. In this work, a theoretical model for turbulent factor in fractal porous media in the high-velocity fluid flow regime is developed. Moreover, a novel analytical expression for the permeability in porous media based on Wu's resistance model is also derived. Then, the analytical Kozeny-Carman constant with no empirical constant is obtained. The predictions of permeability-porosity relation by the current mathematical models have been validated by comparing with available experimental data. Furthermore, the effects of structural parameters of porous media on the curve of velocity and pressure drop are discussed in detail.

1. Introduction
As stated in the literatures (Gewers and Nichol, 1969; Wong, 1970), Darcy’s law is inadequate to describe the pressure drop resulting from turbulent flow in the turbulent flow regime and the effect of inertia should be added. Many scholars (Gewers and Nichol, 1969; Wong, 1970; Noh and Firoozabadi, 2008) modified Darcy’s law by adding an additional term to account for the pressure drop from high velocity flow, and the expression of the pressure drop is

$$\frac{\Delta p}{L} = \frac{\mu}{K} v + \beta \rho v^2$$ \hspace{1cm} (1)

where $\Delta p$ is pressure drop, $L$ is the length along the macroscopic pressure gradient in porous media, $\mu$ is fluid viscosity, $K$ is permeability of porous media, $v$ is velocity, $\rho$ is fluid density and $\beta$ is turbulent factor.

According to the correlation about the turbulent effect given by Eq. (1), the key for calculating the pressure drop in the turbulent flow regime is determination of the turbulent factor. Thus, many other scholars paid their much attention on estimating the turbulent factor through permeability of porous media in the turbulent flow regime. The relevant correlations are summarized in Table 1. It is seen from...
Tab.1 that these former correlations contain the empirical constants, whose specific physical meanings are unknown, and these values vary significantly with different materials and experimental methods.

Due to the complicated structure of porous media, reports about using analytical methods and numerical methods to study turbulent factor are very scarce. It is expected that theoretical investigation on turbulent factor has great significance for sciences and applications. Until recently, only Wu et al. (2008) provided analytical correlation of turbulent factor based on the average hydraulic radius model and the contracting-expanding channel model. And the expression of turbulent factor can be written as

$$\beta = \frac{3\tau(1-\phi)}{4D_p\phi^\gamma} \left( \frac{3}{2} + \frac{1}{\gamma^4} - \frac{5}{2\gamma^2} \right)$$

(2a)

with

$$\gamma = \frac{1}{1-\sqrt{1-\phi}}$$

(2b)

where $\phi$ is the porosity, $\tau$ is the average tortuosity, $\gamma$ is the ratio of pore radius to the throat radius and $D_p$ is the average diameter of particles. Eq. (2) shows that $\beta$ is related to the porosity, the average tortuosity, the average diameter of particles and the ratio $\gamma$. However, it cannot reflect the relationship between $\beta$ and permeability of porous media directly. The expressions for turbulent factor in Tab.1 suggest that the turbulent factor is inversely proportional to the permeability of the porous media. So, it is of theoretical and practical significances to develop an analytical model which can reflect the relationship between $\beta$ and permeability.

Permeability is a basic parameter for characterizing fluid flow behavior, which has a significant impact on the evaluation of flow performance. However, the results from either experiments or numerical simulations are usually expressed as correlations with one or more empirical constants or as curves which ignore the mechanisms behind the phenomena (Yu et al., 2003). Chang and Yortsos (1990) may be the first to apply the fractal theory to study fractal permeability model. However, their model ignores the influence of tortuosity factor on the permeability. Based on Chang and Yortsos’s model, Acuna et al. (1995) proposed a novel model. However, the determinations of the parameters in the new model are not easy and very tedious. In 1996, Adler concluded that the permeability was a function of porosity and fractal dimension. However, he never gave the quantitative expression of permeability. Pitchumani and Ramakrishnan (1999) proposed the first analytical model for permeability of porous media. However, Yu (2001) concluded that the model was completely in error.

Yu et al. (2002) rigorously derived a fractal permeability model based on Hagen-Poiseulle equation and the model can be expressed as

$$K = \frac{\pi L^{1-D_t} D_f \left( 2r_{max} \right)^{3+D_t}}{128A \left( 3 + D_t - D_f \right)}$$

(3a)

where $K$ is permeability, $D_f$ is pore fractal dimension, $D_t$ is tortuosity fractal dimension, $r_{max}$ is the maximum radius. $L$ is the length of capillaries and $A$ is the total area of a unit cell. Xu and Yu (2008) further expressed the permeability as
Eq. (3b) denotes that the permeability is a function of the fractal dimensions $D_T$ and $D_f$, porosity and the maximum pore radius. However, Hagen-Poiseulle equation, used in the Eqs. (3a) and (3b), is inadequate to describe the permeability for turbulent flow in the high-velocity flow regime.

At present, the high-velocity flow mechanism and the characteristics behavior of high-velocity flow in porous media are also not well determined. In this paper, we attempt to establish theoretical models for turbulent factor and permeability based on the fractal theory and Wu's resistance model (Wu and Yu, 2007; Wu et al., 2008).

2. Fractal characteristics of porous media

As stated in the literatures (Katz and Thompson, 1985; Hansen and Skjelletorp, 1988; Anderson et al., 1996; Yu et al., 2002; Yu and Li, 2001; Yu and Cheng, 2002), the microstructures and pore size distributions of porous media have the fractal characters undoubtedly. According to the fractal approach, the cumulative size distribution of pores whose sizes are greater than or equal to the size $r$ follows the fractal scaling law

$$N(\xi \geq r) = \left(\frac{r_{\text{max}}}{r}\right)^{D_f}$$

where $N$ is the number of pores, $\xi$ is the length scale and $r$ is the pore size. The fractal dimension for pore size distribution is in the range of $0 < D_f < 2$ in two dimensions. $r$ and $r_{\text{max}}$ are the pore radius and the maximum pore radius respectively. The total number of pores from the smallest radius $r_{\text{min}}$ to the largest radius $r_{\text{max}}$ can be obtained as (Yu and Li, 2001)

$$N = \left(\frac{r_{\text{max}}}{r_{\text{min}}}\right)^{D_f}$$

Differentiating Eq. (4) with respect to $r$ results in the number of pores whose sizes are within the infinitesimal range $r$ to $r+dr$ as

$$-dN = D_f r_{\text{max}}^{D_f} r^{-(D_f+1)} dr$$

Then, the probability density function for pore size distribution in fractal porous media can be obtained (Yu and Li, 2001)

$$f(r) = D_f r_{\text{min}}^{D_f} r^{-(D_f+1)}$$

Yu and Li (2001) proposed that fractal dimension can be determined according to porosity $\phi$ and the ratio $r_{\text{min}}/r_{\text{max}}$ in porous media

$$D_f = d - \frac{\ln \phi}{\ln \left(r_{\text{min}}/r_{\text{max}}\right)}$$

where $d$ is the Euclidean dimension.

A bundle of tortuous capillaries with variable cross-sectional areas is used to represent the flow paths in a porous sample. The relationship between the capillary length and capillary size also exhibits the self-similar fractal scaling law (Yu et al., 2002; Yu and Cheng, 2002)

$$L(r) = (2r)^{1-D_T} L_0^{D_T}$$

where $1 < D_T < 2$ is the fractal dimension for tortuosity in two dimensions. Based on Eq. (8), the tortuosity (Bear, 1975; Dullien, 1979) can then be rewritten as

$$\tau = L(r)/L_0 = (2r)^{1-D_T} L_0^{D_T-1}$$

The average tortuosity can be also calculated as (Yu, 2008)

$$\bar{\tau} = \int_{r_{\text{min}}}^{r_{\text{max}}} \tau fdr = 2^{1-D_T} r_{\text{min}}^{1-D_T} D_f L_0^{D_T-1} \frac{L_0^{D_T-1}}{D_T + D_f - 1}$$

where the representative length of the capillary can be written as (Xu and Yu, 2008; Xu et al., 2013)
\[ L_0 = r_{\text{max}} \sqrt[2]{\frac{\pi D_{\text{f}}}{2D_k}} \frac{1-\varphi}{\varphi} \]  

Equations (4)-(10) form the theoretical base of the present work.

3. Fractal models for flow parameters

According to Wu et al. (2008), the total pressure drop is the sum of the viscous energy loss and kinetic energy losses along the flow paths, i.e.

\[ \frac{\Delta p}{L_0} = \frac{72 \mu \bar{\tau} (1-\varphi)^2}{D_p \varphi^3} + \left( \frac{3}{2} + \frac{1}{\gamma^4} - \frac{5}{2\gamma^2} \right) \frac{3\bar{\tau} (1-\varphi)}{4D_p \varphi^3} \rho \gamma^2 \]  

(11)

The relation between particle diameter and the maximum pore radius \( r_{\text{max}} \) can be expressed as (Xu and Yu, 2008)

\[ D_p = 2r_{\text{max}} \sqrt{(1-\varphi)}/\varphi \]  

(12)

Combining Eqs. (1), (11) and (12) yield the permeability \( K \) and turbulent factor \( \beta \):

\[ K = \frac{r_{\text{max}}^2 \varphi^2}{18\bar{\tau} (1-\varphi)} \]  

(13)

\[ \beta = \left( \frac{3}{2} + \frac{1}{\gamma^4} - \frac{5}{2\gamma^2} \right) \frac{3\bar{\tau} (1-\varphi)}{8r_{\text{max}}^2 \varphi^2} \sqrt{\frac{1-\varphi}{\varphi}} \]  

(14)

Eq. (13) indicates that the permeability is related to porosity, maximum pore size and the average tortuosity. Eq. (14) reveals that the turbulent factor \( \beta \) is explicitly related to porosity, maximum pore size, the ratio of pore diameter to the throat diameter and the average tortuosity. Inserting Eq. (13) into Eq. (14) yields

\[ \beta = \frac{C_f}{K^{0.5} \varphi^{1.5}} \]  

(15)

where the coefficient \( C_f = \left( \frac{3}{2} + \frac{1}{\gamma^4} - \frac{5}{2\gamma^2} \right) \sqrt{2\bar{\tau}} \frac{1}{16} \) is related to the ratio of pore diameter to the throat diameter and the average tortuosity.

According to Bear (1975), the relation between permeability and porosity can be written as

\[ K = \frac{\varphi^3 D_p^2}{36k (1-\varphi)^2} \]  

(16)

where \( k \) is the Kozeny-Carman (KC) constant. Then, the KC constant can also be written as

\[ k = 2\bar{\tau} = 2^{2-D_1} \frac{r_{\text{min}}^{D_1} D_p L_{\text{cap}}^{D_1-1}}{D_1 + D_1 - 1} \]  

(17)

It can be found from Eq. (17) that the KC constant \( k \) is a function of the minimum pore radius \( r_{\text{min}} \) and the representative length of the capillary \( L_0 \) as well as fractal dimensions \( D_1 \) and \( D_1 \).

4. Results and discussions

The measured permeability-porosity relation by Shao et al. (2012) and that predicted by Yu et al. (2002) Eq. (3a), Xu and Yu (2008) Eq. (3b) and the proposed model Eq. (13) are compared in Fig.1. Because the relevant parameter values of porous media were not entirely reported in the experiment of Shao et al. (2012), the tortuosity fractal dimension \( D_1 = 1.34 \) and the value of \( r_{\text{min}} / r_{\text{max}} = 0.01 \) are chosen for predicting the permeability-porosity relation by the fractal model (Yu et al., 2002; Xu and Yu, 2008) and the proposed model. It can be clearly seen in Fig.1 that the proposed model calculated permeability-porosity relation is in excellent agreement with that of the fractal model (Yu et al., 2002; Xu and Yu, 2008) and
the experimental data. The predicted values are almost in the middle region between the highest and lowest values of experimental data. This verifies the proposed permeability model for porous media.

Fig.1 Comparison between the proposed model and the former model and available experimental data

Fig.2 Relations between KC constant and porosity of porous media

Fig.3 Relations between permeability and turbulence factor

Fig.4 Relations between porosity and turbulence factor
Using the formulas of $D_f$ by Eq. (7), the average tortuosity by Eq.(9), the permeability by Eq.(13) and the turbulent factor by Eq. (15), we can find that the curve of velocity versus pressure drop by Eq. (1) is a function of the porosity only after tortuosity fractal dimension $D_T=1.34$ and the value of $r_{min}/r_{max}=0.01$ are given. Fig.5 shows the velocity versus pressure drop at different porosities. The figure demonstrates that the pressure drop increases with the velocity, and this is expected. Fig.5 also reveals that pressure drop decreases with the porosity under a given velocity. This may be attributed to the decrease of the flow resistance, which is resulted from the increase in porosity.

![Fig.5 The velocity versus pressure drop at different porosities](image1)

![Fig.6 The velocity versus pressure drop at different $r_{min}/r_{max}$](image2)

Fig.6 shows the velocity versus pressure drop at different radius ratios $r_{min}/r_{max}$. In the proposed model, the pore fractal dimension $D_f=1.1$ and tortuosity fractal dimension $D_T=1.34$. The figure indicates that the pressure drop increases with the decrease of $r_{min}/r_{max}$. This is because decreased $r_{min}/r_{max}$ implies increases of percent of small pores, and the flow resistance will be increased. Then, the pressure drop will be increased for a given velocity.

![Fig.7 The velocity versus pressure drop at different pore fractal dimensions](image3)

![Fig.8 The velocity versus pressure drop at different tortuosity fractal dimensions](image4)

Fig.7 gives the schematic of velocity versus pressure drop at different pore fractal dimensions. In the proposed model, the ratio $r_{min}/r_{max}$ of the minimum pore diameter to the maximum pore diameter $r_{min}/r_{max}=0.01$ and the tortuosity fractal dimension $D_T=1.34$. The figure indicates that the smaller pressure drop corresponds to the larger pore fractal dimension. This is because increased pore fractal dimension implies increases of porosity, and the flow resistance decreases with the increase of porosity. Fig.8 illustrates the effect of tortuosity fractal dimension $D_T$ on the curve of velocity versus pressure drop. In this study, the ratio $r_{min}/r_{max}$ of the minimum pore diameter to the maximum pore diameter $r_{min}/r_{max}$ for different samples is taken to be 0.01 and the pore fractal dimension $D_f=1.1$. It can be seen
from Fig. 8 that the pressure drop increases with the increased tortuosity fractal dimension. The main reason is that the increased tortuosity fractal dimension corresponds to the higher tortuous capillary channel, which will enhance the flow resistance.

5. Conclusions
In this paper, a theoretical model for flow parameters (turbulent factor and permeability) of fractal porous media in high-velocity flow regime has been proposed based on Wu’s resistance model as well as fractal geometry. In addition, the analytical Kozeny-Carman constant with no empirical constant has also been obtained. With the theoretical study of the model in this paper, following conclusion can be drawn:

1. The present model predictions for permeability-porosity relation of porous media have the similar variation trend with available experimental data. This verifies the validity of the present fractal analysis of the permeability of porous media. The permeability is a function of porosity, maximum pore size and the average tortuosity.
2. The turbulent factor $\beta$ is explicitly related to porosity, maximum pore size, the ratio of pore diameter to the throat diameter and the average tortuosity.
3. The KC constant $k$ is a function of the minimum pore radius $r_{min}$ and the representative length of the capillary $L_0$ as well as fractal dimensions $D_f$ and $D_T$.
4. The pressure drop decreases with the porosity $\phi$, the radius ratio $r_{min}/r_{max}$ and the pore fractal dimension $D_f$, whereas it increases with the tortuosity fractal dimension $D_T$.
5. Compared with empirical correlation, our model can reveal the more physical mechanisms that affect the flow parameters in porous media in the high-velocity flow regime.

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