GLOBAL (IN TIME) SOLUTIONS TO THE 3D-NAVIER-STOKES EQUATIONS ON $\mathbb{R}^3$

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Abstract. In two recent papers ([GZ1] [GZ2]), we provided solutions to the well-known unsolved problem of constructing sufficiency classes of functions in $H[\mathbb{R}^3]^3$ and $V[\mathbb{R}^3]^3$, which would allow global, in time, strong solutions to the three-dimensional Navier-Stokes equations. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. In both previous papers, our solution was restricted to functions defined on a bounded open domain of class $C^3$ contained in $\mathbb{R}^3$. In this paper, we study this problem for functions defined on all of $\mathbb{R}^3$. We prove that, under appropriate conditions, there exists a positive constant $a$ and a number $u_+$, depending only on the domain, the viscosity, the body forces and the eigenvalues of the “Hermite” Stokes operator (defined below) such that, for all functions in a dense set $\mathcal{D}$ contained in the closed ball $B(\mathbb{R}^3)$ of radius $(1/2)u_+$ in $H[\mathbb{R}^3]^3$, the Navier-Stokes equations have unique strong solutions in $C^1((0,\infty), H[\mathbb{R}^3]^3)$.

Introduction

Let $L^2[\mathbb{R}^3]^3$ be the real Hilbert space of square integrable functions on $\mathbb{R}^3$ with values in $\mathbb{R}^3$, and let $H_0[\mathbb{R}^3]^3$ be the completion of the set of functions in

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\{ u \in C^\infty_0[\mathbb{R}^3]^3 \mid \nabla \cdot u = 0 \} \text{ which vanish at infinity with respect to the inner product of } L^2[\mathbb{R}^3]^3, \text{ and let } V_0[\mathbb{R}^3]^3 \text{ be the completion of the above functions which vanish at infinity with respect to the inner product of } H_0^1[\mathbb{R}^3], \text{ the functions in } H_0^1[\mathbb{R}^3]^3 \text{ with weak derivatives in } (L^2[\mathbb{R}^3])^3. \text{ The global in time classical Navier-Stokes initial-value problem (on } \mathbb{R}^3 \text{ and all } T > 0) \text{ is to find functions } u : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ and } p : [0, T] \times \mathbb{R}^3 \to \mathbb{R} \text{ such that }

\begin{align*}
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= f(t) \text{ in } (0, T) \times \mathbb{R}^3, \\
\nabla \cdot u &= 0 \text{ in } (0, T) \times \mathbb{R}^3 \text{ (in the weak sense)}, \\
\lim_{\|x\| \to \infty} u(t, x) &= 0 \text{ on } (0, T) \times \mathbb{R}^3, \\
u(0, x) &= u_0(x) \text{ in } \mathbb{R}^3.
\end{align*}

(1)

The equations describe the time evolution of the fluid velocity } u(x, t) \text{ and the pressure } p \text{ of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient } \nu \text{ in terms of a given initial velocity } u_0(x) \text{ and given external body forces } f(x, t). \text{ (Note that our third condition, } \lim_{\|x\| \to \infty} u(t, x) = 0 \text{ on } (0, T) \times \mathbb{R}^3, \text{ is natural in this case since it is well-known that } H^k_0[\mathbb{R}^3]^3 = H^k[\mathbb{R}^3]^3 \text{ (see Stein } [S] \text{ or } [SY].)

\textbf{Purpose}

Let } P \text{ be the (Leray) orthogonal projection of } (L^2[\mathbb{R}^3])^3 \text{ onto } H_0^1[\mathbb{R}^3]^3 \text{ and define the Stokes operator by: } A u =: -P \Delta u, \text{ for } u \in D(A) \subset H_0^2[\mathbb{R}^3]^3, \text{ the domain of } A. \text{ Let } B u =: 1/2P(-\Delta + |x|^2)u \text{ for } u \in D(B). \text{ We call } B \text{ the Hermite-Stokes operator. The purpose of this paper is to prove that there exists a number } u_+, \text{ depending only on } A, B, f, \nu \text{ and } \mathbb{R}^3, \text{ such that, for all functions in } D = D(A) \cap B(\mathbb{R}^3), \text{ where}
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$B(\mathbb{R}^3)$ is the closed ball of radius $u_+$ in $H_0(\mathbb{R}^3)^3$, the Navier-Stokes equations have unique strong solutions in $u \in L_\infty^\infty([0, \infty); V_0(\mathbb{R}^3)^3] \cap C^1([0, \infty); H_0(\mathbb{R}^3)^3)$.

Preliminaries

In terms of notation and convention, we follow Sell and You [SY]. In order to simplify notation, we let $H$ denote $H_0(\mathbb{R}^3)^3$ and $V$ denote $V_0(\mathbb{R}^3)^3$. Our use of the Fourier transform follows the definition of Rudin [RU]:

$$\hat{h}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i k \cdot x} h(x) \, dx,$$

so that no factors of $2\pi$ appear in the transform pairs. In order to simplify our proofs, we always assume that all functions $u, v$ are in $D(A)$ and, as in [GZ2], we take $c = \max\{c_i\}$, where $c_i$ is one of the nine positive constants that appear on pages 363-367 in [SY]. It will also be convenient to use the fact that the norms of $V$ and $V^{-1}$ are equivalent in their respective graph norms relative to $H$.

The Stokes Operator

It is known that $A$ is a nonnegative linear operator which generates an analytic contraction semigroup. It follows that the fractional powers $A^{1/2}$ and $A^{-1/2}$ are well defined. Moreover, it is also known (cf., [SY], [T]) that the norms $\|A^{1/2}u\|_H$ and $\|A^{-1/2}u\|_H$ are equivalent to the corresponding norms induced by the Sobolev space $(H^1(\mathbb{R}^3))^3$, so that:

$$\|u\|_V = \|A^{1/2}u\|_H \text{ and } \|u\|_{V^{-1}} = \|A^{-1/2}u\|_H.$$

In addition, $A$ is an isomorphism from $D(A) \xrightarrow{\text{onto}} D(A^{-1})$. Furthermore, the embeddings $V \to H \to V^{-1}$ are continuous, and it is easy to see that $A^{-1}$ is the projection of an operator represented by the Riesz potential, mapping $D(A^{-1})$
onto $D(A)$ (see Stein [S]). Applying the Leray projection to equation (1), with $C(u, u) = P(u \cdot \nabla)u$, we can recast equation (1) in the standard form:

$$
\partial_t u = -\nu A u - C(u, u) + P f(t) \quad \text{in } (0, T) \times \mathbb{R}^3,
$$

$$
\nabla \cdot u = 0 \quad \text{in } (0, T) \times \mathbb{R}^3,
$$

$$
\lim_{\|x\| \to \infty} u(t, x) = 0 \quad \text{on } (0, T) \times \mathbb{R}^3,
$$

$$
u u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^3
$$

where we have used the fact that the orthogonal complement of $H[\mathbb{R}^3]$ relative to $(L^2[\mathbb{R}^3])^3$ is $\{v : v = \nabla q, q \in (H^1[\mathbb{R}^3])^3\}$ to eliminate the pressure term (see Galdi [GA] or [SY, T1, T2]). Theorem 1 below will be used to get our basic estimate in Theorem 3. This result is a simple extension of the bounded domain case first proved by Constantin and Foias [CF].

**Theorem 1.** Let $\alpha_i, 1 \leq i \leq 3$, satisfy $0 \leq \alpha_1 \leq 3$, $0 \leq \alpha_2 \leq 2$, $0 \leq \alpha_3 \leq 3$, with $\alpha_1 + \alpha_2 + \alpha_3 \geq 3/2$ and

$$(\alpha_1, \alpha_2, \alpha_3) \notin \{(3/2, 0, 0), (0, 3/2, 0), (0, 0, 3/2)\}.$$

Then there is a positive constant $c = c(\alpha_i)$ such that

$$
|\langle C(u, v), w \rangle_{\mathcal{H}}| \leq c \left\| A^{\alpha_1/2} u \right\|_{\mathcal{H}} \left\| A^{(1+\alpha_2)/2} v \right\|_{\mathcal{H}} \left\| A^{\alpha_3/2} w \right\|_{\mathcal{H}}.
$$

We shall make use of the following interpolation inequality: (see Sell and You [SY], page 363)

$$
\left\| A^\gamma u \right\|_{\mathcal{H}} \leq c \left\| A^{\alpha} u \right\|_{\mathcal{H}}^{\theta} \left\| A^{\beta} u \right\|_{\mathcal{H}}^{(1-\theta)}
$$

for all $u \in D(A^\alpha)$, where $\gamma = \theta \alpha + (1-\theta)\beta$, $\alpha, \beta, \gamma \in \mathbb{R}$, $0 \leq \theta \leq 1$ and $\beta \leq \alpha$. 


The Hermite-Stokes Operator

The operator $\hat{B} = 1/2(-\Delta + |x|^2)$ is the three-dimensional version of the standard harmonic oscillator operator, which generates the Hermite functions (products of the Hermite polynomials by $e^{-x^2/2}$) as eigenfunctions for the eigenvalue problem on $\mathbb{R}$, (see Hermite [HR], Appell and Kamé de Fériet [AK], and Magnus, Oberhettinger and Soni [MOS]). It is easy to show directly, by separation of variables, that the solution to the 3-dimensional problem is the product of the solutions to the 1-dimensional problem, while the eigenvalues for the 3-dimensional Hermite polynomials are the sums of those for the 1-dimensional polynomials. Furthermore, $\hat{B}$, and hence $B = \mathcal{F}\hat{B}$, is positive with a compact inverse, while $A$ has an unbounded inverse on $\mathcal{H}_0(\mathbb{R}^3)^3$. It turns out that $\hat{B}$ is “natural” for $\mathbb{R}^3$ in the sense that it is the only positive self-adjoint (sectorial) operator of lowest degree that is invariant under both rotations and Fourier transformations. (This is actually true for $\mathbb{R}^n$, $n \geq 1$.)

We will have need of the fact that every function $h(t) \in \mathcal{H}$ has an expansion in terms of the eigenfunctions of $B$ so that, for example, $B^{-\beta}h(t) = \sum_{k=1}^{\infty} \lambda_k^{-\beta}h_k(t)e^k(x)$ and, from here, it is easy to see that $\|B^{-\beta}h(t)\|_{\mathcal{H}} \leq \lambda_1^{-\beta}\|h(t)\|_{\mathcal{H}}$, where $\lambda_1^{-1}$ is the largest eigenvalue of $B^{-1}$. We also need the following result for our basic Theorem.

Lemma 2. $D(A) = D(B)$.

Proof. If we define a norm on $D(A)$ by $\|u\|_A = \|Au\|_\mathcal{H}$, then $(D(A), \| \cdot \|_A)$ is a Hilbert space. Now note that the Fourier transform $\mathcal{F}(\cdot)$ is an isometric isomorphism on $(D(A), \| \cdot \|_A)$ to $\left(D(\mathbb{P}|x|^2), \| \cdot \|_A\right)$, since $\|Au\|_\mathcal{H} = \|\mathcal{F}(Au)\|_\mathcal{H}$. 
\[ \|P|\xi|^2 \tilde{u} \|_H \]. It is now easy to see that \( D(A) = D(P|\xi|^2) \). From this, it follows that \( D(A) = D(B) \).

\[ \square \]

It follows from the above lemma that \((AB)^{-\delta}\) is bounded for \( \delta > 0 \). The following estimate is equation 61.24.1 on page 366 in Sell and You [SY]. If we set \( \alpha_1 = 1, \alpha_2 = 1/2, \) and \( \alpha_3 = 0 \) in Theorem 1, along with the interpolation inequality, we get that

\[ \langle C(u, v), w \rangle_H \leq c \|A^{1/2}u\|_H \|Av\|_H \|w\|_H. \quad (4) \]

**Theorem 3.** Let \( u, v, w \in H \), and let \( \varepsilon > 0 \) be arbitrary. Then, for \( \delta = 1/4 + \varepsilon/2 \), we have that:

\[ \langle (AB)^{-(1+\delta)}C(u, v), w \rangle_H \leq c\lambda_1^{-(1+\delta)} \|u\|_H \|v\|_H \|w\|_H. \quad (5) \]

**Proof.** Using the self-adjoint property of \( A \), and integration by parts, we have

\[ \langle A^{-\beta}C(u, v), h \rangle_H = \langle C(u, v), A^{-\beta}h \rangle_H = -\langle C(u, A^{-\beta}h), v \rangle_H. \]

It now follows from Theorem 1 that:

\[ \langle A^{-\beta}C(u, v), h \rangle_H \leq c \|A^{\alpha_1/2}u\|_H \|A^{-\beta+(1+\alpha_2)/2}h\|_H \|A^{\alpha_3/2}v\|_H. \]

If we set \( \beta = 1 + \delta, \alpha_1 = \alpha_3 = 0 \), we have

\[ \langle A^{-(1+\delta)}C(u, v), h \rangle_H \leq c \|u\|_H \|v\|_H \|A^{(\alpha_2-1-2\delta)/2}h\|_H. \]

With \( \delta = 1/4 + \varepsilon/2 \), we get that, for the last term to reduce to \( \|h\|_H \), we can set \( \alpha_2 = 3/2 + \varepsilon \). It follows that the conditions of Theorem 1 are satisfied if \( 3/2 + \varepsilon < 2 \).

Thus, it suffices to assume that \( \varepsilon < 1/2 \), which we will do in the rest of the paper.
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without comment. Our proof is completed by taking $h = B^{-\beta}w$, and the fact that

$$\|B^{-\beta}w\|_H \leq \lambda_1^{-\beta} \|w\|_H.$$  \hfill □

**Example 4.** If we use Theorem 1, with $\alpha_1 = 5/4$, $\alpha_2 = 1/4$, and $\alpha_3 = 0$, along with the interpolation inequality, and the fact that $\|A^{1/2}u\|_H \leq \|Au\|_H$ we have that, for all $u, v \in D(A)$,

$$\|C(u, v)\|_H \leq c \|A^{1/2}u\|^{3/4}_H \|Au\|^{1/4}_H \|A^{1/2}v\|^{3/4}_H \|Av\|^{1/4}_H$$

$$\leq c \|Au\|_H \|Av\|_H.$$

(6)

A better estimate is possible, but for our use, equation (6) will suffice.

**Definition 5.** We say that the operator $J(\cdot, t)$ is (for each $t$)

1) 0-Dissipative if $\langle J(u, t), u \rangle_H \leq 0$.

2) Dissipative if $\langle J(u, t) - J(v, t), u - v \rangle_H \leq 0$.

3) Strongly dissipative if there exists an $\alpha > 0$ such that

$$\langle J(u, t) - J(v, t), u - v \rangle_H \leq -\alpha \|u - v\|^2_H.$$

4) Uniformly dissipative if there exists a strictly monotone increasing function $a(t)$ with $a(0) = 0$, $\lim_{t \to \infty} a(t) = \infty$, and:

$$\langle J(u, t) - J(v, t), u - v \rangle_H \leq -a(\|u - v\|_H) \|u - v\|_H.$$

Note that, if $J(\cdot, t)$ is a linear operator, definitions 1) and 2) coincide. Theorem 6 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35, page 887 in, Vol. IIB], while Theorem 7 is from Miyadera [M, p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]).
Theorem 6. Let $\mathbb{B}[\mathbb{R}^3]$ be a closed, bounded, convex subset of $\mathbb{H}[\mathbb{R}^3]$. If $J(\cdot, t) : \mathbb{B}[\mathbb{R}^3] \to \mathbb{H}[\mathbb{R}^3]$ is closed and strongly dissipative for each fixed $t \geq 0$ then, for each $b \in \mathbb{B}[\mathbb{R}^3]$, there is a $u \in \mathbb{B}[\mathbb{R}^3]$ with $J(u, t) = b$ (e.g., the range, $\text{Ran}[J(\cdot, t)] \supset \mathbb{B}[\mathbb{R}^3]$).

Theorem 7. Let $\{A(t), t \in I = [0, \infty)\}$ be a family of operators defined on $\mathbb{H}[\mathbb{R}^3]$ with domains $D(A(t)) = D$, independent of $t$. We assume that $D = D \cap \mathbb{B}[\mathbb{R}^3]$ is a closed convex set (in an appropriate topology):

1. The operator $A(t)$ is the generator of a contraction semigroup for each $t \in I$.
2. The function $A(t)u$ is continuous in both variables on $I \times D$.

Then, for every $u_0 \in D$, the problem $\partial_t u(t, x) = A(t)u(t, x)$, $u(0, x) = u_0(x)$, has a unique solution $u(t, x) \in C^1(I; D)$.

M-Dissipative Conditions

Let us assume that $f(t) \in L^\infty([0, \infty); \mathbb{H})$ and is Lipschitz continuous in $t$, with $\|f(t) - f(\tau)\|_{\mathbb{H}} \leq d |t - \tau|^{\theta}$, $d > 0$, $0 < \theta < 1$. With $\delta$ as in Theorem 3, we can rewrite equation (3) in the form:

$$\begin{align*}
&\partial_t u = \nu(AB)^{1+\delta} J(u, t) \text{ in } (0, T) \times \Omega, \\
&(7) \quad J(u, t) = -B^{-(1+\delta)} A^{-\delta} u - \nu^{-1}(AB)^{-(1+\delta)} C(u, u) + \nu^{-1}(AB)^{-(1+\delta)} Pf(t).
\end{align*}$$

Approach

We begin with a study of the operator $J(\cdot, t)$, for fixed $t$, and seek conditions depending on $A, B, \nu$, and $f(t)$ which guarantee that $J(\cdot, t)$ is m-dissipative for each $t$. Clearly $J(\cdot, t) : D[(AB)^{(1+\delta)}] \overset{onto}{\longrightarrow} D[(AB)^{(1+\delta)}]$ and, since $\nu(AB)^{(1+\delta)}$
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is a closed positive (m-accretive) operator (so that \(-(AB)^{(1+\delta)}\) generates a linear contraction semigroup), we expect that \( \nu(AB)^{(1+\delta)}J(\cdot,t) \) will be m-dissipative for each \( t \).

**Theorem 8.** For \( t \in I = [0, \infty) \) and, for each fixed \( u \in \mathbb{H} \), \( J(u,t) \) is Lipschitz continuous, with \( \|J(u,t) - J(u,\tau)\|_\mathbb{H} \leq d'|t-\tau|^\theta \), where \( d' = d\nu^{-1}a^{-(1+\delta)} \), \( d \) is the Lipschitz constant for the function \( f(t) \) and \( a^{-(1+\delta)} = \left\|(AB)^{-(1+\delta)}\right\|_\mathbb{H} \).

**Proof.** For fixed \( u \in \mathbb{H} \),

\[
\|J(u,t) - J(u,\tau)\|_\mathbb{H} = \nu^{-1} \left\|(AB)^{-(1+\delta)}(Pf(t) - Pf(\tau))\right\|_\mathbb{H} \\
\leq d\nu^{-1}a^{-(1+\delta)}|t-\tau|^\theta = d'|t-\tau|^\theta.
\]

\( \square \)

**Main Results**

**Theorem 9.** Let \( f = \sup_{t \in \mathbb{R}^+} \|Pf(t)\|_\mathbb{H} < \infty \), then there exists a positive constant \( u_+ \), depending only on \( f, A, B \) and \( \nu \) such that, for all \( u \) with \( \|u\|_\mathbb{H} \leq u_+ \), \( J(\cdot,t) \) is strongly dissipative.

**Proof.** The proof of our first assertion has two parts. First, we require that the nonlinear operator \( J(\cdot,t) \) be 0-dissipative, which gives us an upper bound \( u_+ \) in terms of the norm (e.g., \( \|u\|_\mathbb{H} \leq u_+ \)). We then use this part, and the fact that \( \|u\|_\mathbb{H} \leq \|Au\|_\mathbb{H} \), to show that \( J(\cdot,t) \) is strongly dissipative on the closed ball, \( B_+ = \{u \in \mathbb{H} : \|Au\|_\mathbb{H} \leq (1/2)u_+\} \).
Part 1) From equation (5), we consider the expression

$$\langle J(u, t), (AB)^{-\delta}u \rangle_H = -\langle B^{-1}(AB)^{-\delta}u, (AB)^{-\delta}u \rangle_H$$

$$+ \nu^{-1} \left< -(AB)^{-(1+\delta)} C(u, u) + (AB)^{-(1+\delta)} \mathbb{P} f(t), (AB)^{-\delta}u \right>_H$$

$$= -\left\| B^{-1/2}(AB)^{-\delta}u \right\|_H^2 - \nu^{-1} \left< (AB)^{-(1+\delta)} C(u, u), (AB)^{-\delta}u \right>_H + \nu^{-1} \left< (AB)^{-(1+\delta)} \mathbb{P} f(t), (AB)^{-\delta}u \right>_H$$

It follows that

$$\langle J(u, t), (AB)^{-\delta}u \rangle_H \leq -\left\| B^{-1/2}(AB)^{-\delta}u \right\|_H^2 + \nu^{-1} \left< C((AB)^{-(1+\delta)}u, (AB)^{-\delta}u \right>_H$$

$$+ \nu^{-1} a^{-(1+\delta)} f \left\| (AB)^{-\delta}u \right\|_H$$

$$\leq -\left\| B^{-1/2}(AB)^{-\delta}u \right\|_H^2 + ca^{-\delta} \nu \left( \nu \lambda_1 \right)^{1+\delta} \left\| u \right\|_H^3 + \nu^{-1} a^{-(1+2\delta)} f \left\| u \right\|_H.$$

In the last line, we used our estimate from Theorem 3. We now choose the first eigenvalue $\lambda_n$, $n \geq 1$, and number $\omega$ such that

(1) $\lambda_n^{-1/2} a^{-\delta} \left\| u \right\|_H \leq \left\| B^{-1/2}(AB)^{-\delta}u \right\|_H \leq \lambda_1^{-1/2} a^{-\delta} \left\| u \right\|_H,$

(2) $\lambda_1^{-\omega/2} a^{-\delta} \left\| u \right\|_H \leq \left\| B^{-1/2}(AB)^{-\delta}u \right\|_H \leq \lambda_1^{-1/2} a^{-\delta} \left\| u \right\|_H,$

and let $\lambda_0^{-1} = \max\{\lambda_n^{-1}, \lambda_1^{-\omega}\}$. It then follows that $-\lambda_0^{-1} a^{-2\delta} \left\| u \right\|_H^2 \geq -\left\| B^{-1/2}(AB)^{-\delta}u \right\|_H^2$. Thus, J(\cdot, t) will be 0-dissipative if

$$-\lambda_0^{-1} a^{-2\delta} \left\| u \right\|_H^2 + ca^{-\delta} \nu \left( \nu \lambda_1 \right)^{1+\delta} \left\| u \right\|_H^3 + \nu^{-1} a^{-(1+2\delta)} f \left\| u \right\|_H \leq 0,$$

so that

(8) $a^{-\delta} \left\| u \right\|_H \left[ c(\nu \lambda_1^{1+\delta})^{-1} \left\| u \right\|_H^2 - \lambda_0^{-1} a^{-\delta} \left\| u \right\|_H + (\nu a^{1+\delta})^{-1} f \right] \leq 0.$

Since $\left\| u \right\|_H > 0$, we have that J(\cdot, t) is 0-dissipative if

$$c(\nu \lambda_1^{1+\delta})^{-1} \left\| u \right\|_H^2 - \lambda_0^{-1} a^{-\delta} \left\| u \right\|_H + (\nu a^{1+\delta})^{-1} f \leq 0.$$
Solving, we get that

$$u_\pm = \frac{\nu \lambda_1^{1+\delta}}{2c \lambda_0 a^2} \left\{ 1 \pm \sqrt{1 - \frac{(4c \lambda_0^2 f)}{\nu^2 a (1-\delta) \lambda_1^{1+\delta}}} \right\} = \frac{\nu \lambda_1^{1+\delta}}{2c \lambda_0 a^2} \left\{ 1 \pm \sqrt{1 - \gamma} \right\},$$

where $\gamma = \frac{(4c \lambda_0^2 f)}{\nu^2 a (1-\delta) \lambda_1^{1+\delta}}$. Since we want real distinct solutions, we must require that

$$\gamma = \frac{(4c \lambda_0^2 f)}{\nu^2 a (1-\delta) \lambda_1^{1+\delta}} < 1 \implies \nu^2 a (1-\delta) \lambda_1^{1+\delta} > 4c \lambda_0^2 f \implies \nu > 2 \lambda_0 a^{-\frac{(1-\delta)/2}{\lambda_1^{1+\delta}/2} (cf)^{1/2}}.$$

It follows that, if $Pf \neq 0$, then $u_- < u_+$, and our requirement that $J$ is 0-dissipative implies that, since our solution factors as $(\|u\| - u_+)(\|u\| - u_-) \leq 0$, we must have that:

$$\|u\| - u_+ \leq 0, \quad \|u\| - u_- \geq 0.$$

First observe that terms of the form $(AB)^{-\delta}u$ are dense. Then note that $J(u, t)$ is closed, and the dissipative nature of an operator is determined on a dense set.

It follows that, for $u_- \leq \|u\| \leq u_+$, $(J(u, t), u)_{\mathbb{H}} \leq 0$. (It is clear that, when $Pf(t) = 0, u_- = 0, u_+ = \nu(c \lambda_0 a^\delta)^{-1} \lambda_1^{1+\delta}$.)
Part 2): Now, for any \( u, v \in H \) with \( \max(\|Au\|_H, \|Av\|_H) \leq (1/2)u_+ \), we have that
\[
\langle J(u, t) - J(v, t), (AB)^{-\delta}(u - v) \rangle_H = -\left\| B^{-1/2}(AB)^{-\delta}(u - v) \right\|_H^2
\]
\[
- \nu^{-1} \left\langle (AB)^{-1}\delta[C(u, u - v) + C(v, u - v)], (AB)^{-\delta}(u - v) \right\rangle_H
\]
\[
\leq -\lambda_0^{-1}a^{-2\delta} \|u - v\|_H^2 + ca^{-\delta}\nu^{-1}(1+\delta) \|u - v\|_H^2 (\|u\|_H + \|v\|_H)
\]
\[
\leq -\lambda_0^{-1}a^{-2\delta} \|u - v\|_H^2 + \frac{1}{2}a^{-\delta}\nu^{-1}(1+\delta) \|u - v\|_H^2 \left\{ \frac{1}{4}\nu \lambda_1^{1+\delta}(c^{-1}a^{-\delta} \lambda_0^{-1}) \left( 1 + \sqrt{1 - \gamma} \right) \right\}
\]
\[
= -\alpha \|u - v\|_H^2, \quad \alpha = \frac{1}{2}\lambda_0^{-1}a^{-2\delta} \left\{ 1 - \sqrt{1 - \gamma} \right\}.
\]

\[\square\]

Theorem 10. The operator \( A(t) = \nu A^{(1+\delta)}J(\cdot, t) \) is closed, uniformly dissipative and jointly continuous in \( u \) and \( t \). Furthermore, for each \( t \in \mathbb{R}^+ \) and \( \beta > 0 \), \( \text{Ran}[I - \beta A(t)] \supset \mathbb{B}[\Omega] \), so that \( A(t) \) is m-dissipative on \( D \).

Proof. Since \( J(\cdot, t) \) is strongly dissipative and closed on \( B \), it follows from Theorem 6 that \( \text{Ran}[J(\cdot, t)] \supset B \).

To show that \( A(t) = \nu (AB)^{(1+\delta)}J(\cdot, t) \) is uniformly dissipative for \( u, v \in \mathbb{B}_+ \), we have
\[
\langle A(t)u - A(t)v, (u - v) \rangle_H = -\nu \left\| A^{1/2}(u - v) \right\|_H^2
\]
\[
- \langle (1/2)[C(u - v, u) + C(u - v, v), (u - v) \rangle_H.
\]
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Now, from equation (4),

$$\langle A(t)u - A(t)v, u - v \rangle_{\mathbb{H}} \leq -\nu \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}}^2 + \frac{1}{2} c \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \left\{ \left\| Au \right\|_{\mathbb{H}} + \left\| Av \right\|_{\mathbb{H}} \right\}$$

We now use $-\lambda_0^{-1}a^{-\delta} \left\{ \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \right\} \geq -\left\| A^{1/2}(u - v) \right\|_{\mathbb{H}}$, and the fact that the first eigenvalue of $B$ is $1/2$, so that $\lambda_1^{1+\delta} < 1$, to get:

$$\langle A(t)u - A(t)v, u - v \rangle_{\mathbb{H}} \leq -\nu \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}}^2 + \frac{1}{2} c \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}} \left\{ \left\| Au \right\|_{\mathbb{H}} + \left\| Av \right\|_{\mathbb{H}} \right\}$$

If we set $a \left( \left\| (u - v) \right\|_{\mathbb{H}} \right) = -\frac{1}{2} \nu \lambda_0^{-1}a^{-\delta} \left[ -1 + \sqrt{1 - \gamma} \right] \left\| A^{1/2}(u - v) \right\|_{\mathbb{H}}$, we have that:

$$\langle A(t)u - A(t)v, u - v \rangle_{\mathbb{H}} \leq -a \left( \left\| (u - v) \right\|_{\mathbb{H}} \right) \left\| (u - v) \right\|_{\mathbb{H}}.$$
with \( t_n, t \in I \) and \( t_n \to t \). Then (see equation (6))

\[
\| A(t_n)u_n - A(t)u \|_\mathbb{H} \leq \| A(t_n)u - A(t)u \|_\mathbb{H} \]

\[
= \| \mathbb{P}f(t_n) - \mathbb{P}f(t) \|_\mathbb{H} + \| \nu A(u_n - u) + C(u_n - u_n) + C(u, u_n - u) \|_\mathbb{H}
\]

\[
\leq d |t_n - t|^{\alpha} + \nu \| A(u_n - u) \|_\mathbb{H} + \| C(u_n - u, u_n) + C(u, u_n - u) \|_\mathbb{H}
\]

\[
\leq d |t_n - t|^{\alpha} + \nu \| A(u_n - u) \|_\mathbb{H} + c \| A(u_n - u) \|_\mathbb{H} \{ \| Au_n \|_\mathbb{H} + \| Au \|_\mathbb{H} \}
\]

\[
\leq d |t_n - t|^{\alpha} + \nu \| A(u_n - u) \|_\mathbb{H} + 2c \| A(u_n - u) \|_\mathbb{H} u_+.
\]

It follows that \( A(t)u \) is continuous in both variables. \( \square \)

Since \( \mathbb{B}_+ \) is the closure of \( \mathbb{D} = D(A) \cap \mathbb{B} \) equipped with the restriction of the graph norm of \( A \) induced on \( D(A) \), it follows that \( \mathbb{B}_+ \) is a closed, bounded, convex set. We now have:

**Theorem 11.** For each \( T \in \mathbb{R}^+ \), \( t \in (0, T) \) and \( u_0 \in \mathbb{D} \subset \mathbb{B} \), the global in time Navier-Stokes initial-value problem in \( \mathbb{R}^3 \):

\[
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t) \text{ in } (0, T) \times \mathbb{R}^3,
\]

\[
\nabla \cdot u = 0 \text{ in } (0, T) \times \mathbb{R}^3,
\]

\[
(9)
\]

\[
\lim_{\|x\| \to \infty} u(t, x) = 0 \text{ on } (0, T) \times \mathbb{R}^3,
\]

\[
uu(0, x) = u_0(x) \text{ in } \mathbb{R}^3,
\]

has a unique strong solution \( u(t, x) \), which is in \( L^2_{loc}([0, \infty); \mathbb{H}^2] \) and in \( L^\infty_{loc}([0, \infty); \mathbb{V}) \cap C^1([0, \infty); \mathbb{H}] \).

**Proof.** Theorem 7 allows us to conclude that, when \( u_0 \in \mathbb{D} \), the initial value problem is solved and the solution \( u(t, x) \) is in \( C^1([0, \infty); \mathbb{D}] \). Since \( \mathbb{D} \subset \mathbb{H}^2 \), it follows that
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$u(t, x)$ is also in $V$, for each $t > 0$. It is now clear that, for any $T > 0$,

$$
\int_0^T \|u(t, x)\|^2_{H^2} \, dt < \infty, \quad \text{and} \quad \sup_{0 < t < T} \|u(t, x)\|^2_{V} < \infty.
$$

This gives our conclusion. □

DISCUSSION

It is known that, if $u_0 \in V$, and $f(t)$ is $L^\infty([0, \infty), H]$ then there is a time $T > 0$ such that a weak solution with this data is uniquely determined on any subinterval of $[0, T)$ (see Sell and You, page 396, [SY]). Thus, we also have that:

**Corollary 12.** For each $t \in \mathbb{R}^+$ and $u_0 \in D$ the Navier-Stokes initial-value problem on $\mathbb{R}^3$:

$$
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p &= f(t) \quad \text{in} \quad (0, T) \times \mathbb{R}^3, \\
\nabla \cdot u &= 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^3, \\
\lim_{\|x\| \to \infty} u(t, x) &= 0 \quad \text{on} \quad (0, T) \times \mathbb{R}^3, \\
u_0(t) &= u_0(x) \quad \text{in} \quad \mathbb{R}^3.
\end{align*}
$$

has a unique weak solution $u(t, x)$, which is in $L^2_{loc}([0, \infty); H^2]$ and in $L^\infty_{loc}([0, \infty); V] \cap C^1([0, \infty); H]$.

Since we require that our initial data be in $H^2$, the conditions for the Leray-Hopf weak solutions are not satisfied. However, it was an open question as to whether these solutions developed singularities, even if $u_0 \in C_0^\infty$ (see Giga [G] and references therein). The above Corollary shows that it suffices that $u_0(x) \in H^2$ to insure that the solutions develop no singularities.
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