Thermodynamics of Taub-NUT/bolt Black Holes in Einstein-Maxwell Gravity

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Abstract

First, we construct the Taub-NUT/bolt solutions of $(2k+2)$-dimensional Einstein-Maxwell gravity, when all the factor spaces of $2k$-dimensional base space $\mathcal{B}$ have positive curvature. These solutions depend on two extra parameters, other than the mass and the NUT charge. These are electric charge $q$ and electric potential at infinity $V$. We investigate the existence of Taub-NUT solutions and find that in addition to the two conditions of uncharged NUT solutions, there exist two extra conditions. These two extra conditions come from the regularity of vector potential at $r = N$ and the fact that the horizon at $r = N$ should be the outer horizon of the NUT charged black hole. We find that the NUT solutions in $2k + 2$ dimensions have no curvature singularity at $r = N$, when the $2k$-dimensional base space is chosen to be $\mathbb{C}P^{2k}$. For bolt solutions, there exists an upper limit for the NUT parameter which decreases as the potential parameter increases. Second, we study the thermodynamics of these spacetimes. We compute temperature, entropy, charge, electric potential, action and mass of the black hole solutions, and find that these quantities satisfy the first law of thermodynamics. We perform a stability analysis by computing the heat capacity, and show that the NUT solutions are not thermally stable for even $k$’s, while there exists a stable phase for odd $k$’s, which becomes increasingly narrow with increasing dimensionality and wide with increasing $V$. We also study the phase behavior of the 4 and 6 dimensional bolt solutions in canonical ensemble and find that these solutions have a stable phase, which becomes smaller as $V$ increases.

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I. INTRODUCTION

It has been known for quite long time that the area law of entropy holds for black holes or black branes in any dimension $2k + 2$, when the $U(1)$ isometry group associated with the (Euclidean) timelike Killing vector $\partial_\tau$ has a fixed point set on the surfaces of $2k$-codimension $1$. However, in recent years, it has been shown that entropy can be associated with a more general class of spacetimes [2, 3]. In these spacetimes, the $U(1)$ isometry group can have fixed points on surfaces of any even codimension, and the spacetime need not be asymptotically flat or asymptotically anti-de Sitter (AdS). In this more general class, the entropy is not just a quarter the area of the $2k$-dimensional fixed point set. Such situations occur in spacetimes containing NUT-charges. In $2k + 2$ dimensions they not only can have $2k$-dimensional fixed point sets (called bolts), they also have fixed point sets with dimension less than $2k$ (called NUT). Here the orbits of the $U(1)$ isometry group develop singularities, which are the gravitational analogues of Dirac string singularities and are referred to as Misner strings. When the NUT charge is nonzero, the entropy of a given spacetime includes not only the entropies of the $2k$-dimensional bolts, but also those of the Misner strings. In a very basic sense, gravitational entropy can be regarded as arising from the Gibbs-Duhem relation applied to the path-integral formulation of quantum gravity [4]. If one calculates the finite total action $I$ evaluated on the classical solution, then the entropy may be written as

$$S = \beta (\mathcal{M} - \Gamma_i C_i) - I$$

where $C_i$ and $\Gamma_i$ are the conserved charges and their associate chemical potentials respectively.

The original asymptotically locally flat NUT/bolt solutions in four dimensions have been constructed in Ref. [5]. There are known extensions of the Taub-NUT/bolt solutions to the case when a cosmological constant is present. In this case the asymptotic structure is only locally de Sitter (for positive cosmological constant) or AdS (for negative cosmological constant) and the solutions are referred to as Taub-NUT-(A)dS metrics. Generalizations to higher dimensions follow closely the four-dimensional case [6, 7, 8, 9, 10, 11, 12]. Taub-NUT solution of the Einstein equations with multiple NUT parameters has been investigated in [13]. Also, charged Taub-NUT solution of the Einstein-Maxwell equations in four dimensions is known [14], and its generalization to six dimensions has been done in Ref. [15, 16]. The existence of NUT charged solutions of Einstein-Yang-Mills and Einstein-Yang-Mills-Higgs
theory and their thermodynamics have also been considered \[17\]. Recently, the existence of Taub-NUT/bolt solution in Gauss-Bonnet and Gauss-Bonnet-Maxwell gravity have been studied by one of us \[18, 19\]. In this paper we, first, construct the \((2k + 2)\)-dimensional NUT/bolt solutions of Einstein gravity in the presence of electromagnetic field and, second, we calculate their conserved charges through the use of the counterterm method and analyze the thermodynamic behavior of these electrically charged NUT/bolt solutions.

The outline of our paper is as follows. We give a brief review of the counterterm method in Sec. \[\Pi\] In. Sec. \[\Pi\] we construct the \((n + 1)\)-dimensional Taub-NUT/bolt solutions of the Einstein-Maxwell gravity. The thermodynamics of electrically charged NUT solutions in 4-dimensions is investigated in Sec. \[\IV\] and that of 6-dimensional solutions is considered in Sec. \[\V\] Generalization of these subjects to the case of \((2k + 2)\)-dimensional solutions is done in Sec. \[\VI\] We finish our paper with some concluding remarks.

\section{General Formalism}

The gravitational action for Einstein gravity in \((n + 1)\) dimensions in the presence of cosmological constant and electromagnetic field is

\[I_G = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} (\mathcal{R} - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) - \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} K(\gamma), \tag{2}\]

where \(\Lambda = -n(n-1)/(2l^2)\) is the cosmological constant, \(\mathcal{R}\) is the Ricci scalar, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is electromagnetic tensor field and \(A_\mu\) is the vector potential. The first term is the Einstein-Hilbert action and the second term is the Gibbons Hawking boundary term which is chosen such that the variational principle is well-defined. The manifold \(M\) has metric \(g_{\mu\nu}\) and covariant derivative \(\nabla_\mu\). \(K\) is the trace of the extrinsic curvature \(K^{\mu\nu}\) of any boundary(ies) \(\partial M\) of the manifold \(M\), with induced metric(s) \(\gamma_{ij}\).

In order to obtain the Einstein-Maxwell equations by the variation of the volume integral with respect to the fields, one should impose the boundary condition \(\delta A_\mu = 0\) on \(\partial M\). Thus the action \(I_G\) is appropriate to study the grand-canonical ensemble with fixed electric potential \(20\). To study the canonical ensemble with fixed electric charge one should impose the boundary condition \(\delta (n^a F_{ab}) = 0\), and therefore the gravitational action is \(21\)

\[\tilde{I}_G = I_G - \frac{1}{4\pi} \int_{\partial M_\infty} d^n x \sqrt{-\gamma} n_a F^{ab} A_b, \tag{3}\]
where $n_a$ is the normal to the boundary $\partial \mathcal{M}$. Varying the action (2) or (3) with respect to the metric tensor $g_{\mu\nu}$ and electromagnetic tensor field $F_{\mu\nu}$, with appropriate boundary condition, the equations of gravitation and electromagnetic fields are obtained as

$$G_{\mu\nu} - \frac{n(n - 1)}{2l^2} g_{\mu\nu} = 8\pi T^{(em)}_{\mu\nu},$$

(4)

$$\nabla_{\mu} F^{\mu\nu} = 0,$$

(5)

where $G_{\mu\nu}$ is the Einstein tensor and $T^{(em)}_{\mu\nu} = 2F^\rho_\mu F_{\rho\nu} - \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu}$ is the energy-momentum tensor of electromagnetic field.

In general the action $I_G$ or $\tilde{I}_G$ are diverged when evaluated on solutions, as is the Hamiltonian and other associated conserved charges. One way of eliminating these divergences is through the use of background subtraction \cite{22,23}, in which the boundary surface is embedded in another (background) spacetime, and all quasilocal quantities are computed with respect to this background, incorporated into the theory by adding to the action the extrinsic curvature of the embedded surface. Such a procedure causes the resulting physical quantities to depend on the choice of reference background; furthermore, it is not possible in general to embed the boundary surface into a background spacetime. For asymptotically AdS solutions, one can instead deal with these divergences via the counterterm method inspired by AdS/CFT correspondence \cite{24}. This conjecture, which relates the low energy limit of string theory in asymptotically AdS spacetime and the quantum field theory on its boundary, has attracted a great deal of attention in recent years. The equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing computation on the other side. A dictionary translating between different quantities in the bulk gravity theory and their counterparts on the boundary has emerged, including the partition functions of both theories. In the present context this conjecture furnishes a means for calculating the action and conserved quantities intrinsically without reliance on any reference spacetime \cite{25,26,27} by adding additional terms on the boundary that are curvature invariants of the induced metric. Although there may exist a very large number of possible invariants one could add in a given dimension, only a finite number of them are non vanishing as the boundary is taken to infinity. Its many applications include computations of conserved quantities for black holes with rotation, various topologies, rotating black strings with zero curvature horizons and rotating higher genus black branes \cite{28}. Although the asymptotic structure of the NUT charged solutions of Ein-
stein gravity in the presence of a cosmological constant is only locally (A)dS, the counterterm method has been employed to calculate the action and the conserved charges of them. The conserved quantities of asymptotically locally AdS solutions in four dimensions have been calculated in [29], those of higher-dimensional solutions in [30], and those of asymptotically locally dS solutions in [31]. Also the conserved quantities of the asymptotically locally AdS solutions have been calculated in [32]. Usually, the counterterm method applies for the case of a specially infinite boundary, but it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [33]. The general correspondence formula is [24]

\[ \int_{\Psi_0} \mathcal{D}\Psi e^{-I_{\text{AdS}}[\Psi]} = \langle \exp \int d^n x O(x) \Psi_0(x) \rangle, \quad (6) \]

where the functional integral on the left hand side is over all the fields \( \Psi \) whose asymptotic boundary values are \( \Psi_0 \), and \( O \) denotes the conformal operators of the boundary conformal field theory. In the classical limit, the correspondence formula can be written as [34]

\[ I_{\text{AdS}}[\Psi_0] = W_{\text{CFT}}[\Psi_0], \quad (7) \]

where \( I_{\text{AdS}} \) is the classical on-shell action of an AdS field theory, expressed in terms of the field boundary values \( \Psi_0 \), and \( W_{\text{CFT}} \) is the CFT effective action. However, one should expect \( I_{\text{AdS}} \) to be divergent as it stands, because of the divergence of the AdS metric on the AdS horizon. Thus, in order to extract the physically relevant information, the on-shell action has to be renormalized by adding counterterms, which cancel the infinities of \( I_G \) in the absence of matter. This counterterms up to nine dimensions is [35]

\[ I_{\text{ct}} = \frac{1}{8\pi} \int_{\partial M_\infty} d^n x \sqrt{-\gamma} \left\{ \frac{n-1}{l} - \frac{l\Theta(n-3)}{2(n-2)} R \right. \\
- \frac{l^3\Theta(n-5)}{2(n-4)(n-2)^2} \left( R_{ab} R^{ab} - \frac{n}{4(n-1)} R^2 \right) + \frac{l^5\Theta(n-7)}{(n-2)^3(n-4)(n-6)} \left( \frac{3n+2}{4(n-1)} R R_{ab} R^{ab} \\
- \frac{n(n+2)}{16(n-1)^2} R^3 - 2 R^{ab} R_{abcd} R^{cd} + \frac{n-2}{2(n-1)} R^{ab} \nabla_a R \nabla_b R - R^{ab} \Box R_{ab} + \frac{1}{2(n-1)} R \Box R \right) \} \quad (8) \]

where \( R \), \( R_{abcd} \), and \( R_{ab} \) are the Ricci scalar, Riemann and Ricci tensors of the boundary metric \( \gamma_{ab} \), and \( \Theta(x) \) is the step function which is equal to one for \( x \geq 0 \) and zero otherwise. In the presence of matter, one may encounters with some divergencies. For \( n > 4 \) the electromagnetic field will cause a power law divergence in the action which can be removed
by a counterterm of the form \[36\]

\[
I_{\text{ct}}^{\text{em}} = \frac{l}{8\pi} \int d^n x \sqrt{-\gamma} \left\{ \frac{\Theta(n-5)}{32} \frac{(n-8)}{(n-4)} R^2 + \frac{l^2 \Theta(n-7)}{4} \frac{(5n-11)}{48(n-1)(n-2)(n-6)} R F^2 \right. \\
+ \frac{(7n-66)}{12(n-6)(n-2)} R_{b}^{a} F_{ae} F_{bc} + \frac{(n-8)}{12(n-4)^2} (\nabla_a F^{ab})^2 \\
+ \left. \frac{(n-12)}{47(n-4)(n-6)} F^{ab} (\nabla_b \nabla^c F_{ca} - \nabla_a \nabla^c F_{cb}) \right\} \\
\tag{9}
\]

Thus, the total finite action can be written as a linear combination of the gravity term \[2\] and the counterterms \[8\] and \[9\]. Having the total finite action, one can use the Brown and York definition of energy-momentum tensor \[22\] to construct a divergence free stress-energy tensor. This tensor is

\[
T_{ab} = \frac{1}{8\pi} \left( \left( K_{ab} - K^{\gamma}_{ab} \right) - \frac{n-l}{l} \gamma_{ab} + \frac{l}{n-2} (R_{ab} - \frac{1}{2} R \gamma_{ab}) \right. \\
\left. + \frac{l^3 \Theta(n-5)}{(n-4)(n-2)^2} \left[ -\frac{1}{2} \gamma_{ab} (R^c d R_{cd} - \frac{n}{4(n-1)} R^2) - \frac{n}{2(n-2)} R R_{ab} \right. \\
\left. + 2 R_{cd} R^{cde} - \frac{n-2}{2(n-1)} \nabla^a \nabla^b R + \nabla^2 R_{ab} - \frac{1}{2(n-1)} \gamma_{ab} \nabla^2 R \right] \right\} + \delta I_{\text{ct}}^{\text{em}} \frac{\delta \gamma_{ab}}{\delta \gamma_{ab}} \tag{10}
\]

The explicit form of the stress-energy tensor due to the electromagnetic field counterterm will be given in Sec. \[\n\] To compute the conserved charges of the spacetime, we choose a spacelike surface \( \Sigma \) in \( \partial M \) with metric \( \sigma_{ij} \), and write the boundary metric in ADM form:

\[
\gamma_{ab} dx^a dx^a = -N^2 dt^2 + \sigma_{ij} \left( d\varphi^i + V^i dt \right) \left( d\varphi^j + V^j dt \right), \tag{11}
\]

where the coordinates \( \varphi^i \) are the angular variables parameterizing the hypersurface of constant \( r \) around the origin, and \( N \) and \( V^i \) are the lapse and shift functions respectively. The conserved charges associated to a Killing vector \( \xi^a \) is

\[
Q(\xi) = \int_{\Sigma} d^{n-1} x \sqrt{\sigma} u^a T_{ab} \xi^b, \tag{12}
\]

where \( \sigma \) is the determinant of the metric \( \sigma_{ij} \) and \( u^a \) is the normal to the quasilocal boundary hypersurface \( \Sigma \). For boundaries with timelike Killing vector \( (\xi = \partial_t) \) one obtains the conserved mass of the system enclosed by the boundary \( \Sigma \). In the context of AdS/CFT correspondence, the limit in which the boundary \( \Sigma \) becomes infinite \( (\Sigma_\infty) \) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface \( \Sigma \) in to a reference of spacetime is required and the quantities which are computed are intrinsic to the spacetimes.
III. \((2k + 2)\)-DIMENSIONAL TAUB-NUT/BOLT SOLUTIONS IN EINSTEIN-MAXWELL GRAVITY

The Euclidean section of the \((2k + 2)\)-dimensional charged Taub-NUT/bolt spacetime can be written as

\[
ds^2 = F(r)(d\tau + N\mathcal{A})^2 + F^{-1}(r)dr^2 + (r^2 - N^2)d\Xi_B,
\]

where \(\tau\) is the coordinate of the fibers \(S^1\) and \(\mathcal{A}\) is the Kähler form of the base space \(\mathcal{B}\), \(N\) is the NUT charge and \(F(r)\) is a function of \(r\). The metric \(d\Xi_B\) is a \(2k\)-dimensional base space Einstein-Kähler manifold \(\mathcal{B}\).

Here, we consider only the cases where all the factor spaces of \(\mathcal{B}\) have positive curvature. Thus, the base space \(\mathcal{B}\) may be the product of \(\mathbb{C}P^k\) spaces for all values of \(k\). For completeness, we give the 1-forms and the metrics of these factor spaces. The 1-forms and the metrics of \(\mathbb{C}P^k\) is

\[
\mathcal{A}_k = 2(k + 1) \sin^2 \xi_k (d\psi_k + \frac{1}{2k} \mathcal{A}_{k-1}),
\]

\[
d\Sigma_k^2 = 2(k + 1) \left\{d\xi_k^2 + \sin^2 \xi_k \cos^2 \xi_k (d\psi_k + \frac{1}{2k} \mathcal{A}_{k-1})^2 + \frac{1}{2k} \sin^2 \xi_k d\Sigma_{k-1}^2 \right\}
\]

where \(\mathcal{A}_{k-1}\) is the Kähler potential of \(\mathbb{C}P^{k-1}\). In Eqs. (14) and (15) \(\xi_k\) and \(\psi_k\) are the extra coordinates corresponding to \(\mathbb{C}P^k\) with respect to \(\mathbb{C}P^{k-1}\). The metric \(\mathbb{C}P^k\) is normalized such that, Ricci tensor is equal to the metric, \(R_{\mu\nu} = g_{\mu\nu}\). The 1-form and the metric of \(\mathbb{C}P^1\) \((S^2)\) is

\[
\mathcal{A}_1 = 4 \sin^2 \xi_1 d\psi_1
\]

\[
d\Sigma_1^2 = 4 \left(d\xi_1^2 + \sin^2 \xi_1 \cos^2 \xi_1 d\psi_1^2\right)
\]

and those of \(\mathbb{C}P^k\) can be constructed through the use of Eqs. (14) and (15).

The gauge potential has the form

\[
A = h(r)(d\tau + N\mathcal{A}),
\]

where \(h(r)\) is a function of \(r\). The electromagnetic field equation for the metric with vector potential is

\[
(r^2 - N^2)^2h''(r) + 2kr(r^2 - N^2)h'(r) - 4kN^2h(r) = 0
\]
where prime denotes a derivative with respect to \( r \). The solution of Eq. (19) may be expressed in terms of hypergeometric function \(_2F_1([a,b],[c],z)\) in a compact form as

\[
h(r) = \frac{qr}{(r^2 - N^2)^k} + V(2k - 1)N^{2k} _2F_1\left([-\frac{1}{2},-k],[\frac{1}{2}],\frac{r^2}{N^2}\right)
\]

where \( V \) and \( q \) are two arbitrary constants which correspond to electric potential at infinity and charge respectively. To find the function \( F(r) \), one may use any components of Eq. (4). The simplest equation is the \( tt \) component of these equations which can be written as

\[
r F'(r) + \left(\frac{(2k - 1)r^2 + N^2}{r^2 - N^2}\right) F(r) - \frac{l^2 + (2k + 1)(r^2 - N^2)}{l^2} = \frac{(r^2 - N^2)}{k} h'^2(r) - \frac{4N^2}{(r^2 - N^2)^k} h^2(r),
\]

with the solution

\[
F(r) = \frac{r}{(r^2 - N^2)^k} \int^r \frac{(s^2 - N^2)^{k-1}}{kl^2s^2} \left\{ l^2(s^2 - N^2)^2 h'^2(s) - 4l^2N^2kh^2(s) \\
+k(2k + 1)(s^2 - N^2)^2 + kl^2(s^2 - N^2) \right\} ds - \frac{mr}{(r^2 - N^2)^k},
\]

where \( h(r) \) is given in Eq. (20). One may note that the above solution reduces to the \((2k + 2)\)-dimensional solution given in [30] in the absence of electromagnetic field \((h(r) = 0)\).

### A. NUT Solutions

The solutions of Eq. (21) describe NUT solutions, if

(I) \( F(r_+ = N) = 0 \).

(II) \( \beta = 4\pi/F'(r_+) = 4\pi(k + 1)N \).

(III) \( h(r_+ = N) = 0 \).

(IV) \( F(r) \) should have no positive roots at \( r > N \).

The first condition comes from the fact that all the extra dimensions should collapse to zero at the fixed point set of \( \partial/\partial\tau \), the second one ensures that there is no conical singularity with a smoothly closed fiber at \( r = N \). Of course, if one use the Misner’s argument [37], the period of time coordinate is found to be different from \( 4\pi(k + 1)N \) when the space is singular [7, 15]. Here we consider the period of Euclidean time coordinate by the elimination of conic singularities. The third condition comes from the regularity of vector potential at \( r = N \) and the fourth one comes up since \( r = N \) should be the outer horizon. Condition (III) gives a relation between \( q \) and \( V \) as

\[
q_n = -\frac{2\sqrt{\pi}N^{2k-1}\Gamma(k + 1)}{\Gamma(k - \frac{1}{2})} V_n
\]
Using the first two conditions with Eq. (23), one finds that Einstein-Maxwell gravity in even dimensions admits NUT solutions with any base space when the mass parameter is fixed to be

\[ m_n = N^{2k-1}\{2B(k)[l^2 - 2(k+1)N^2] + 4D(k)V_n^2\} \tag{24} \]

where \( B(k) \) and \( D(k) \) are

\[ B(k) = \frac{\Gamma(\frac{3}{2} - k)\Gamma(k+1)}{(2k-1)\sqrt{\pi}l^2}, \tag{25} \]

\[ D(k) = \frac{(-1)^{k-1}\sqrt{\pi}(2k-1)(k-1)\Gamma(k+1)}{k\Gamma(k-\frac{1}{2})}. \tag{26} \]

Finally, we should apply the fourth condition. As we will see in the next section, the fourth condition is not necessary in 4 dimensions. However, this condition restrict the value of \( V_n \) to be less than a critical value \( V_{\text{crit}} \) in dimensions higher than four. To find \( V_{\text{crit}} \) we proceeds as follows. We define the function \( g_{\text{nut}}(r) \) as the numerator of \( F_{\text{nut}}(r) = F(q = q_n, m = m_n, r)/(r - N) \) which is positive at \( r = N \), and solve the system of two equations

\[ \begin{cases} g_{\text{nut}}(r) = 0 \\ g'_{\text{nut}}(r) = 0 \end{cases} \tag{27} \]

for the unknown \( V \) and \( r \). The \( V \) obtained by this method is the critical value \( V_{\text{crit}} \). We consider the solutions of system of two equations (27) in the following sections for various dimensions.

Now, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces is

\[ u^0 = \frac{1}{N}, \quad u^r = 0, u^i = \frac{\mathcal{V}^i}{N}, \tag{28} \]

where \( N \) and \( \mathcal{V}^i \) are the lapse and shift functions respectively. The electric field is \( E^\mu = g^{\mu\nu}F_{\nu\rho}u^\rho \), and the electric charge can be found by calculating the flux of the electric field at infinity, yielding

\[ Q_k = (2k-1)(4\pi)^{k-1}q. \tag{29} \]

The electric potential \( \Phi \), measured at infinity with respect to the horizon, is defined by

\[ \Phi = A_\mu \chi^\mu \bigg|_{r \to \infty} - A_\mu \chi^\mu \bigg|_{r = r_+}, \tag{30} \]

where \( \chi = \partial_r \) is the null generator of the horizon. We find

\[ \Phi_n = (-1)^{k+1}V_n. \tag{31} \]
B. Bolt Solutions

The conditions for having a regular bolt solution are (I) \( F(r = r_b) = 0 \), (II) \( F'(r_b) = [(k + 1)N]^{-1} \) and (III) \( h(r_b) = 0 \) with \( r_b > N \). Condition (II) follows from the fact that we want to avoid a conical singularity at the bolt, together with the fact that the period of \( \tau \) will still be \( 4\pi(k + 1)N \). The first and third conditions gives

\[
 m_b = \sum_{i=0}^{k} \binom{k}{i} \left( \frac{(-1)^i N^{2i} r_b^{2i-2} - 1}{2k-2i-1} \right) + \frac{(2k + 1)}{l^2} \sum_{i=0}^{k+1} \binom{k+1}{i} \left( \frac{(-1)^i N^{2i} r_b^{2i-1}}{2k-2i+1} \right) 
\]

\[
 + \int_{r_b}^{r_{\infty}} \frac{(r^2 - N^2)^{k-1}}{kr^2} [(r^2 - N^2)^2 h^2(r) - 4N^2 k h^2(r)] dr. \tag{32}
\]

where \( V_b \) is the solution of \( h(r_b) = 0 \). The second condition together with Eq. (32) gives the following equation for \( r_b \)

\[
 kl^2 r_b^3 - k(2k + 1) N \left[ (2k + 1)(r_b^2 - N^2) + l^2 r_b^2 \right] - (k + 1)(2k - 1)^2 l^2 V_b^2 N(r_b^2 - N^2) = 0. \tag{33}
\]

Equation (33) for fixed values of \( l \) and \( V_b \) is a quartic equation in \( r_b \) and has two real solutions greater than \( N \) provided \( N < N_{\text{max}} \) and one real solution greater than \( r_b \) if \( N = N_{\text{max}} \). Numerical calculation shows that \( N_{\text{max}} \) decreases as \( V \) increases in various dimensions. We will give the equation which is satisfied by \( N_{\text{max}} \) in four and six dimensions in Secs. IV and V respectively. The electric charge is the same as the NUT solutions, Eq. (29), while the electric potential may be obtained by use of Eq. (30) as

\[
 \Phi_b = -\frac{(-1)^k q r_b}{(2k-1)N^{2k}} F_1 \left( \left[ -\frac{1}{2}, -k \right], \left[ -\frac{1}{2} \right], r_b^2 \right). \tag{34}
\]

IV. THERMODYNAMICS OF 4-DIMENSIONAL SOLUTIONS

In four dimensions, the functions \( h(r) \) and \( F(r) \) given in Eqs. (20) and (22) are

\[
 h(r) = q r \frac{r^2 + N^2}{r^2 - N^2} + V \frac{r^2 + N^2}{r^2 - N^2}, \tag{35}
\]

\[
 F(r) = \frac{r^4 + (l^2 - 6N^2)r^2 - ml^2r + (4N^2V^2 + N^2 - q^2)l^2 - 3N^4}{l^2(r^2 - N^2)}. \tag{36}
\]

A. NUT Solutions

Since \( m_n \) and \( q_n \) of Eqs. (24) and (23) become

\[
 m_n = 2Nl^2(l^2 - 4N^2), \tag{37}
\]
\[ q_n = -2NV_n, \quad (38) \]

the function \( F(r) \) for NUT solution can be written as

\[
F_n(r) = \frac{r^2(r + N) + l^2(r - N) - N^2(5r - 3N)}{(r + N)l^2}.
\]

It is notable that \( F_n(r) \) is independent of \( q \) and \( V \), and therefore there is no restriction on \( V_n \) from the fourth condition. The electric charge and potential are

\[
Q = q, \quad \Phi_n = -\frac{q}{2N}.
\]

Using Eqs. (2) and (8), the Euclidean actions in the grand-canonical and canonical ensembles can be calculated as

\[
I_n = \frac{4\pi N^2}{l^2}(l^2 - 2N^2 + 2l^2V^2), \quad (41)
\]
\[
\tilde{I}_n = \frac{4\pi N^2}{l^2}(l^2 - 2N^2 - 2l^2V^2). \quad (42)
\]

Also the total mass may be calculated as

\[
\mathcal{M}_n = \frac{m_n}{2}, \quad (43)
\]

and as in the case of uncharged solutions \[30\] the total angular momentum is zero. The Gibbs and Helmholz free energies are

\[
G(T, \Phi) = \frac{I}{\beta} = \frac{32(2\Phi^2 + 1)l^2\pi^2T^2 - 1}{512l^2\pi^3T^3}, \quad (44)
\]
\[
F(T, Q) = \frac{\tilde{I}}{\beta} = -\frac{1}{512} \frac{1 - 32l^2\pi^2T^2 + 1024Q^2l^2\pi^4T^4}{l^2\pi^3T^3}, \quad (45)
\]

where \( T \) is the Hawking temperature which is inverse of \( \beta_n = 8\pi N \). The entropy can be obtained as

\[
S_n = -\left( \frac{\partial G}{\partial T} \right)_\Phi = -\left( \frac{\partial F(T, Q)}{\partial T} \right)_Q = \frac{4\pi N^2}{l^2}(l^2 - 6N^2 + 2l^2V^2). \quad (46)
\]

The entropy can also be obtained through the use of Gibbs-Duhem relation \[\Pi\], where \( \mathcal{C}_i = Q \) and \( \Gamma_i = \Phi \). As one can see from Eq. \[46\], the entropy is positive for 1) \( l^2 > 6N^2 \) and also 2) \( l^2 < 6N^2 \) provided \( |V_n| > (2l)^{-1}\sqrt{12N^2 - 2l^2} \). It is worth to mention that the second case
occurs only in the case of 4-dimensional NUT solution in the presence of electromagnetic field.

The Smarr-type formula for mass versus extensive quantities $S$ and $Q$ may be written as

$$M_n = \sqrt{Z(-4Z + l^2)} \frac{l^2}{l^2},$$

(47)

where $Z$ is

$$Z = \frac{l}{12}(1 \pm \sqrt{l^2 + 12Q^2 - 6S/\pi})$$

(48)

One may then regard the parameters $S$ and $Q$ as a set of extensive parameters for the mass $M(S, Q)$ and define the intensive parameters conjugate to $S$ and $Q$, which are the temperature and the electric potential respectively. One obtains

$$T_n = \left(\frac{\partial M_n}{\partial S}\right)_Q = \frac{1}{8\pi N},$$

$$\Phi_n = \left(\frac{\partial M_n}{\partial Q}\right)_S = -\frac{q}{2N}$$

(49)

for $+$ and $-$ sign in Eq. (48) provided $12N^2 > l^2$ and $12N^2 < l^2$ respectively. Equations (49) show that the quantities $T_n$ and $\Phi_n$ coincide with those which was calculated in Sec. (III). Thus the thermodynamic quantities calculated in Sec. (III) for Nut solution, satisfy the first law of thermodynamics $dM = TdS + \Phi dQ$.

The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates, is usually performed by analyzing the behavior of the entropy near equilibrium. The local stability in any ensemble requires that $S(Y)$ be a concave function of its extensive variables or that its Legendre transformation is a convex function of the intensive variables. The stability can also be studied by the behavior of the energy $M(X)$ which should be a convex function of its extensive variable. In our case the mass $M(S, Q)$ is a function of entropy and charge. The number of thermodynamic variables depends on the ensemble that is used. In the canonical ensemble, the charge is a fixed parameter, and therefore the positivity of the heat capacity $C_Q = T(\partial S/\partial T)_Q$ is sufficient to ensure local stability. The heat capacity $C_Q$ at constant charge is

$$C_n = \frac{8\pi N^2}{l^2}(12N^2 - l^2)$$

(50)

which is positive provided $l^2 < 12N^2$. Thus, the NUT solution is stable in two cases. The first case is when $6N^2 < l^2 < 12N^2$ for which both the entropy and the heat capacity are
positive. The second case is when \( l^2 < 6N^2 \) provided \(|V_n| > (2l)^{-1} \sqrt{12N^2 - 2l^2} \). This second case is a stable phase for charged NUT solutions, which does not occur in the absence of electromagnetic field. That is, by turning on the electromagnetic field, the stable phase of the four-dimensional NUT solution gets a new zone.

**B. Bolt Solutions**

In 4 dimensions, the bolt mass, charge and electric potential may be written from Eqs. 32, 35, and 34 as

\[
m_b = \frac{r_b^4 + (l^2 - 6N^2)r_b^2 + N^2(l^2 - 3N^2)}{l^2 r_b} - \frac{V^2(r_b^2 - N^2)^2}{r_b^3},
\]

\[
q_b = -\frac{V(r_b^2 + N^2)}{r_b},
\]

\[
\Phi_b = -\frac{qr_b}{r_b^2 + N^2}.
\]

where \( r_b \) is the solution of the following equation

\[
6Nr_b^4 - l^2r_b^3 + 2N(l^2 - 3N^2 + l^2V^2)r_b^2 - 2l^2V^2N^3 = 0.
\]

For fixed values of \( l \) and \( V \), Eq. 54 has two real solutions greater than \( N \) provided \( N < N_{\text{max}} \) and one real solution \( r_b > N \) for \( N = N_{\text{max}} \) where \( N_{\text{max}} \) is the smallest root of the following equation

\[
62208N^{10} - 82944(1 - V^2)l^2N^8 + (41472V^4 - 27648V^2 + 41904)l^4N^6
\]

\[
+ (9216V^6 + 9216V^4 - 4464V^2 - 9648)l^6N^4 + (768V^8 + 3072V^6
\]

\[
+ 3024V^4 + 1632V^2 + 912)l^8N^2 - (16V^6 + 48V^4 + 21V^2 + 16)l^{10} = 0
\]

Thus, we have bolt solution for \( N \leq N_{\text{max}} \). For example, for \( l = 1 \) and \( V = 1 \), the maximum value of \( N_{\text{max}} \) is 0.1033. It is a matter of numerical calculation to show that \( N_{\text{max}} \) decreases as \( V \) increases. The finite actions in grand-canonical and canonical ensembles are

\[
I_b = -\pi \frac{r_b^4 - l^2r_b^2 + N^2(3N^2 - l^2)]r_b^2 - (r_b^4 + 4N^2r_b^2 - N^4)l^2V^2}{(3r_b^2 - 3N^2 + l^2)r_b^2 + (r_b^2 - N^2)l^2V^2},
\]

\[
\tilde{I}_b = -\pi \frac{r_b^4 - l^2r_b^2 + N^2(3N^2 - l^2)]r_b^2 + (3r_b^4 + N^4)l^2V^2}{(3r_b^2 - 3N^2 + l^2)r_b^2 + (r_b^2 - N^2)l^2V^2}.
\]
FIG. 1: Entropy (bold line) and heat capacity (solid line) of 4-dimensional bolt solution for \(l = 1\) and \(V = 0\) in the allowed range of \(N\).

while the entropy and specific heat are calculated as

\[
S_b = \pi \frac{3r_b^2 + r_b^2(l^2 - 12N^2) + N^2(l^2 - 3N^2)}{(3r_b^2 - 3N^2 + l^2)r_b^2 + (r_b^2 - N^2)l^2V^2}r_b^2 + \pi \frac{[r_b^2 + 4N^2r_b^2 - N^2]l^2V^2}{(3r_b^2 - 3N^2 + l^2)r_b^2 + (r_b^2 - N^2)l^2V^2},
\]

(58)

\[
C_b = [4N(r_b^2 + N^2)]^2V^2 + 36Nr_b^4 - 5l^2r_b^3 + 4N^2r_b^2(2l^2 - 3N^2) - l^2N^2r_b^2l^2r_b^2]^{-1} \times \\
2N\pi - (48N^3r_b^6l^4 + 6Nr_b^8l^4 + 6N^9l^4 + 48N^7r_b^2l^4 - 108N^5r_b^4l^4)V^4 - (264N^3r_b^8l^2 \\
+ 36N^4r_b^{10}l^2 + 12N^6r_b^8l^4 + 216N^7l^2r_b^4 - 45N^2r_b^7l^4 + 60N^9l^2r_b^2 - 76N^5r_b^4l^4 - 5r_b^3l^4 \\
+ 5N^6r_b^3l^4 - 16N^7r_b^2l^4 - 960N^5l^2r_b^6 + 45N^4r_b^5l^4 + 112N^3r_b^6l^4)V^2 + 264N^3r_b^8l^2 \\
+ 15r_b^{11}l^2 + 360N^7r_b^6 - 213N^2r_b^9l^2 - 36N^4r_b^{10}l^2 - 396N^5r_b^8 - 6N^3r_b^6l^4 + 72N^7l^2r_b^4 \\
- 12N^5l^2r_b^6 - 10N^5r_b^4l^4 + 5r_b^9l^4 + 7N^4r_b^5l^4 - 63N^4l^2r_b^7 + 1368N^3r_b^{10} - 54Nr_b^{12} \\
- 126N^9r_b^4 - 16N^3r_b^6l^4 + 12N^2r_b^7l^4 - 27N^6l^2r_b^5],
\]

In Figs. 1, 2 and 3 the entropy and specific heat are plotted for \(l = 1, V = 0, V = 1\) and \(V = 3\) up to \(N_{\text{max}}\) for outer horizons of the bolt solutions respectively. As one can see from these figures, the bolt solution is stable in the range \(0 < N < N_{\text{st}} < N_{\text{max}}\), while \(N_{\text{st}}\) is less than \(N_{\text{max}}\) for \(V \neq 0\), and equal to \(N_{\text{max}}\) for \(V = 0\). That is, the uncharged solution is stable in the whole allowed range \(0 < N < N_{\text{max}}\), while the stable phase becomes smaller as \(V\) increases.
FIG. 2: Entropy (bold line) and heat capacity (solid line) of 4-dimensional bolt solution for \( l = 1 \) and \( V = 1 \) in the allowed range of \( N \).

FIG. 3: Entropy (bold line) and heat capacity (solid line) of 4-dimensional bolt solution for \( l = 1 \) and \( V = 3 \) in the allowed range of \( N \).

V. THERMODYNAMICS OF 6-DIMENSIONAL SOLUTIONS

In six dimensions the base space \( \mathcal{B} \) can be the 4-dimensional space \( \mathbb{C}P^2 \) or the product of two 2-dimensional spaces \( S^2 \times S^2 \). From Eqs. (20) and (22) the functions \( h(r) \) and \( F(r) \) may be written as

\[
h(r) = \frac{qr}{(r^2 - N^2)^2} - \frac{V(r^4 - 6r^2N^2 - 3N^4)}{(r^2 - N^2)^2}. \tag{59}
\]

\[
F(r) = \frac{3r^6 + (l^2 - 15N^2)r^4 + 3N^2(15N^2 - 2l^2)r^2 + 3N^4(5N^2 - l^2)}{l^2(r^2 - N^2)^2} - \frac{mr}{(r^2 - N^2)^2} - \frac{3r^2 - N^2}{2(r^2 - N^2)^2} q^2 - \frac{4N^2r^6 + 15N^2r^4 - 9N^4(r^2 - N^2)}{(r^2 - N^2)^4} V^2 - \frac{16N^2r^3}{(r^2 - N^2)^4} q V. \tag{60}
\]
The mass and charge parameters of the NUT solutions may be obtained from Eqs. (24) and (23) as
\[ m_n = -\frac{8N^3}{3l^2}(l^2 - 6N^2 + 9l^2V_n^2), \] (61)
\[ q_n = 8N^3V_n. \] (62)

Substituting the above parameters in Eq. (60) one obtains
\[ F_n(r) = \frac{(r - N)}{3(r + N)^4l^2}[-12N^2V_n^2l^2(r - N) + r^3(3r^2 + 24N^2 + l^2) + 5Nr^2(3r^2 + l^2) + rN^2(-27N^2 + 7l^2) + 3N^3(-5N^2 + l^2)]. \] (63)

It is worth to note that the Taub-NUT solution has not curvature singularity at \( r = N \) for \( B = \mathbb{CP}^2 \), while for \( B = S^2 \times S^2 \) the metric has curvature singularity. Since \( F_n(r) \) depends on \( V_n \), the fourth condition for NUT solution restricts the allowed values of \( V_n \). As we discussed in Sec. III, one may obtained the critical value of \( V_{\text{crit}} \) by solving the system of two equations (27). To be more clear, we obtain \( V_{\text{crit}} \) for asymptotically flat \((l \to \infty)\) NUT solutions. The system of two equations (27) becomes
\[
\begin{align*}
(r + 3N)(r + N)^2 - 12N^2(r - N)V_{\text{crit}}^2 &= 0, \\
(3r + 7N)(r + N) - 12N^2V_{\text{crit}}^2 &= 0
\end{align*}
\]
with the following solution for \( V_{\text{crit}} \)
\[ V_{\text{crit}} = \sqrt{\frac{66 + 30\sqrt{5}}{6}} \approx 1.92. \] (64)

For arbitrary values of \( l \), one may find the critical value of \( V \) numerically. For \( l = 1 \) and \( N = 1 \) the critical value of potential which is obtained by the above method is \( V_{\text{crit}} = 4.67 \). This can be seen in Fig. 4 which shows the function \( F_n(r) \) as a function of \( r \) for various values of \( V \) including \( V = V_{\text{crit}} \). The electric charge and potential are
\[ Q_n = 12\pi q, \] (65)
\[ \Phi_n = \frac{q}{8N^3}. \] (66)

The Euclidean actions can be calculated by use of Eqs. (2), (8) and (9) as
\[ I_n = \frac{8\pi N^3}{3l^2}(l^2 - 4N^2 - 18l^2V^2), \] (67)
FIG. 4: $F_n(r)$ versus $r$ for $V = 3 < V_{\text{crit}}$ (point), $V = V_{\text{crit}} = 4.67$ (line) and $V = 6 > V_{\text{crit}}$ (bold-line).

It is notable to mention that the electromagnetic field will cause a power law divergence $16\pi V^2 N^2 r$ in action which is removed by the counterterm $[9]$. The stress-energy tensor corresponding to the counterterm $[9]$ in six dimension is

$$T^{(\text{em})}_{ab} = \frac{1}{4\pi} (F^c_{\ a} F_{cb} - \frac{1}{4} F_{cd} F^{cd} g_{ab})$$

The mass due to this counterterm is

$$\mathcal{M}^{(\text{em})}_{\text{ct}} = 16\pi N^2 V^2 r,$$ \hspace{1cm} (69)

which removes the power law divergence of the mass due to electromagnetic field. The total mass is

$$\mathcal{M}_n = 4\pi m_n$$ \hspace{1cm} (70)

The Smarr-type formula for mass is

$$\mathcal{M}_n = \frac{6144\pi^2 Z^4 - 1024\pi^2 l^2 Z^3 - l^2 Q^2}{96\pi Z^{3/2} l^2},$$ \hspace{1cm} (71)

where $Z = N^2$ is a function of $Q$ and $S$ which may be obtained by the following equation

$$640\pi^2 Z^4 - 96\pi^2 l^2 Z^3 - Sl^2 Z - \frac{3}{16} l^2 Q^2 = 0.$$ \hspace{1cm} (72)

The entropy and specific heat are

$$S_n = \frac{32\pi^2 N^4}{l^2} (20N^2 - 3l^2 - 54l^2 V^2),$$ \hspace{1cm} (73)

$$C_n = \frac{384\pi^2 N^4}{l^2} (l^2 - 10N^2 - 9l^2 V^2).$$ \hspace{1cm} (74)
As one can see from Eq. 73 for $|V| < (6l)^{-1} \sqrt{(40/3)N^2 - 2l^2}$ and $|N| \geq \sqrt{0.15l}$, the entropy is positive, but $C_n$ is negative in this range. Thus, $S_n$ and $C_n$ are not positive simultaneously, and the NUT black hole is completely unstable in this ensemble as in the case of uncharged solution 30.

B. Bolt Solutions

The parameters $m_b$, $q_b$ and $\beta_b$ are

$$m_b = \frac{1}{3l^2r_b}[3r_b^6 + (l^2 - 15N^2)r_b^4 - 3N^2(2l^2 - 15N^2)r_b^2 - 3N^4(l^2 - 5N^2)] \quad (75)$$

$$q_b = \frac{V(r_b^4 - 6N^2r_b^2 - 3N^4)}{r_b}, \quad (77)$$

$$\beta_b = \frac{8\pi l^2r_b^3}{2(5r_b^2 - 5N^2 + l^2)r_b^2 + 9(r_b^2 - N^2)l^2V^2}. \quad (78)$$

where $r_b$ is real root of the following equation

$$30Nr_b^4 - 2l^2r_b^3 + 3(2l^2 - 10N^2 + 9l^2V^2)Nr_b^2 - 27N^3l^2V^2 = 0. \quad (79)$$

Equation (54) has two real solutions greater than $N$ provided $N < N_{\text{max}}$ and one real solution $r_b > N$ for $N = N_{\text{max}}$ where $N_{\text{max}}$ is the smallest root of the following equation

$$9 \times (10)^5 N^{10} + (10)^4 \times (324V^2 - 72)l^2N^8 + (10)^3 \times (4374V^4 - 648V^2 + 217)l^4N^6$$

$$+ (10)^2 \times (26244V^6 + 5832V^4 - 999V^2 - 294)l^6N^4 + (10) \times (59049V^8 + 52488V^6$$

$$+ 14823V^4 + 2052V^2 + 156)l^8N^2 - (729V^6 + 486V^4 + 81V^2 + 8)l^{10} = 0. \quad (80)$$

As in the case of 4-dimensional solutions, $N_{\text{max}}$ decreases as $V$ increases.

The electric potential is

$$\Phi_b = -\frac{qr_b}{r_b^2 - 6N^2r_b^2 - 3N^4}. \quad (81)$$

The finite action, entropy and heat capacity can be obtained as

$$I_b = \frac{8\pi^2 r_b^2[-3r_b^6 + (l^2 + 5N^2)r_b^4 + 3N^2(5N^2 - 2l^2)r_b^2 + 3N^4(5N^2 - l^2)]}{32(5r_b^2 - 5N^2 + l^2)r_b^2 + 9(r_b^2 - N^2)l^2V^2}$$

$$+ 12\pi l^2V^2 \frac{(r_b^6 - 15N^2r_b^4 - 21N^4r_b^2 + 3N^6)}{2(5r_b^2 - 5N^2 + l^2)r_b^2 + 9(r_b^2 - N^2)l^2V^2}; \quad (82)$$
and $N < N_m$ in the allowed range of $N_t^2$. While $r=3$ for outer horizons of the bolt solutions, respectively. As one can see from these figures, the bolt solution is stable in the range $0 < N < N_{st} < N_{max}$, while $N_{st}$ is less than $N_{max}$ for $V \neq 0$, and equal to $N_{max}$ for $V = 0$. That is, the uncharged

$S_b = \frac{-8\pi^2}{3} [15r_b^6 + (3l^2 - 65N^2)r_b^4 + 3N^2(55N^2 - 6l^2)r_b^2 + 9N^4(5N^2 - l^2)]r_b^2$ + $\frac{2(5r_b^2 - 5N^2 + l^2)r_b^2 + 9(r_b^2 - N^2)l^2V^2}{2r_b^2(5r_b^2 - 5N^2 + l^2) + 9(r_b^2 - N^2)l^2V^2}$, \hspace{1cm} (83)

$C_b = 3\pi^2 N_l^2 l^2(27l^2 N^5 + 54N^3r_b^2l^2 - 9NI^2r_b^4) V^2 - 90N^5r_b^2 + 24N^3r_b^2l^2 - r_b^2N^4 - 8N^2r_b^4 + 220N^3r_b^4 - 10l^2r_b^3N^2 - 50N^3r_b^6 + 3r_b^4l^2]\text{^{-1}}[-3600N^{11}r_b^4 - 440N^4r_b^6l^4 + 9800N^6l^2r_b^9 + 900N^10r_b^12 + 20N^8r_b^14 - 156N^8r_b^14 - 14880N^7l^2r_b^8 - 288N^3r_b^10l^4 + 18000N^5r_b^10l^2 + 1152N^7l^2r_b^14 - 1680N^3r_b^12 + 4600N^4r_b^14 + 160N^2r_b^14 - 8280N^9r_b^6l^4 - 576N^6r_b^7l^4 + 324N^9l^4r_b^4 + 840N^5r_b^8l^4 + 4020N^8r_b^6l^4 + 200N^2r_b^14 + 180N^2r_b^13l^2 - 60r_{b,15}^2 - 12r_{b,13}l^4 + 23100N^9l^8 + 500N^7b^10 - 204800N^7r_{b,10} + 62500N^5r_{b,12} + 400N^3r_{b,14} + 9900N^3r_{b,14} + 6000N^{11}r_{b,12} + (3136N^7r_{b,14} - 20412N^{9}l^4r_{b,12} + 7938N^8r_{b,14} + 900N^2r_{b,14} - 6156N^6r_{b,12} - 12312N^{5}r_{b,12}l^4 - 6264N^3r_{b,14}l^4 + 378N^{10}r_{b,14}l^4 + 5364N^4r_{b,14}l^4 + 180N^2r_{b,12}l^4 - 1944N^{11}r_{b,12}l^4 + 11340N^{13}r_{b,12}l^4 - 386100N^9r_{b,12}l^2 + 1746N^2r_{b,11}l^4 + 159840N^7l^2r_{b,8} + 116640N^{11}l^2r_{b,4} - 52380N^{5}r_{b,10}l^2 - 24560N^3l^2r_{b,12} - 124320N^{3}l^2r_{b,12}l^4)V^2 + (19440N^7r_{b,14}l^4 - 10692N^3r_{b,10}l^4 - 14823N^5r_{b,12}l^4 - 117369N^9l^4r_{b,4} + 37908N^{11}r_{b,2}l^4 + 405N^5r_{b,12}l^4 + 2187N^{13}l^4)V^4]. \hspace{1cm} (84)$
FIG. 6: Entropy (bold line) and heat capacity (solid line) of 6-dimensional bolt solution for $l = 1$ and $V = 1$ in the allowed range of $N$.

FIG. 7: Entropy (bold line) and heat capacity (solid line) of 4-dimensional bolt solution for $l = 1$ and $V = 3$ in the allowed range of $N$.

solution is stable in the whole allowed range $0 < N < N_{\text{max}}$, while the stable phase becomes smaller as $V$ increases.

VI. THERMODYNAMICS OF $(2k + 2)$-DIMENSIONAL CHARGED TAUB-NUT SOLUTIONS

Using Eqs. (2), (8) and (9), the Euclidean actions can be calculated as

$$I_{N_k} = -J(k)[l^2 - 2kN^2 + 2(-1)^{k+1}(2k - 1)^2l^2V^2],$$

(85)

where $J(k)$ is a function of $k$ as

$$J(k) = \frac{(4\pi)^{k+1}(k + 1)\Gamma\left(\frac{1}{2} - k\right)\Gamma(k + 1)}{16\pi^{3/2}l^2}N^{2k}.$$  (86)

Also the total mass may be calculated as

$$M_n = \frac{k(4\pi)^{k-1}}{2}m_n.$$  (87)
As in the case of uncharged solutions [30], the total angular momentum is zero. The entropy can also be obtained through the use of Gibbs-Duhem relation (1), where $C_i = Q$ and $\Gamma_i = \Phi$

\[ S_n = -J(k)[(2k - 1)l^2 - 2k(2k + 1)N^2 + 2(2k - 1)^3l^2V^2] \]  

(88)

The Smarr-type formula for mass versus extensive quantities $S$ and $Q$ may be written as

\[ \mathcal{M}_n = k(4\pi)^{k-1}B(k)Z^{(k-1/2)}[l^2 - 2(k + 1)Z] + \frac{2kD(k)}{(4\pi)^kZ^{(k-1/2)}}\{\frac{\Gamma(k - \frac{1}{2})}{\Gamma(k + 1)(2k - 1)}\}^2Q^2, \]  

(89)

where $Z$ is the solution of the following equation:

\[ S_n + \frac{(4\pi)^{k+1}(k + 1)\Gamma(\frac{1}{2} - k)\Gamma(k + 1)}{16\pi^{3/2}l^2}\{(2k - 1)l^2 - 2k(2k + 1)Z + \frac{(2k - 1)^2}{2\pi}\frac{\Gamma(k - \frac{1}{2})}{(4\pi)^k\Gamma(k + 1)Z^{(k-1/2)}}\}^2Q^2\}Z^k = 0. \]  

(90)

One may then regard the parameters $S$ and $Q$ as a set of extensive parameters for the mass $\mathcal{M}(S, Q)$ and define the intensive parameters conjugate to $S$ and $Q$, which are the temperature and the electric potential respectively. One obtains

\[ T_n = \left(\frac{\partial\mathcal{M}_n}{\partial S}\right)_Q = \frac{1}{4(k + 1)\pi N}, \]

\[ \Phi_n = \left(\frac{\partial\mathcal{M}}{\partial Q}\right)_S = V_n. \]  

(91)

Equations (91) show that the quantities $T_n$ and $\Phi_n$ coincide with those which was calculated in Sec. (III). Thus the thermodynamic quantities calculated in Sec. (III) for Nut solution, satisfy the first law of thermodynamics. The specific heat capacity may be obtained as

\[ C_n = 2J(k)[k(2k - 1)l^2 - 2k(2k + 1)(2k + 1)N^2 - 2(k - 1)(2k - 1)^3l^2V^2]. \]  

(92)

Since $J(k)$ will produce a minus sign for odd $k$, Eqs. (88) and (92) show that the entropy and heat capacity are positive for odd $k$ provided

\[ \frac{2k(2k + 1)}{(2k - 1)}N^2 - 2(2k - 1)^2l^2V^2 < l^2 < \frac{2k(2k + 1)(2k + 1)N^2 + 2(k - 1)(2k - 1)^3l^2V^2}{k(2k - 1)} \]

Hence, for odd $k > 1$, the NUT solutions will be thermally stable in the above range, which becomes increasingly narrow with increasing dimensionality and wide with increasing $V$. 21
For $k = 1$, as we see in Sec. IV, there exists another stable phase. For even $k$, however, no minus sign is produced by $J(k)$, meaning that the entropy will be positive for

$$l^2 < \frac{2k(2k+1)}{(2k-1)}N^2 - 2(2k-1)^2l^2V^2$$

and the specific heat will be positive for

$$l^2 > \frac{2k(k+1)(2k+1)N^2 + 2(k-1)(2k-1)^3l^2V^2}{k(2k-1)}$$

Since the second value will always be larger than the first, this means there is no range in which both will be positive, and thus for all even $k$, the NUT solutions will be thermally unstable.

VII. CONCLUSION

In this paper, we first, constructed asymptotically AdS Taub-NUT/bolt solutions in $(2k + 2)$-dimensional Einstein-Maxwell gravity with curved base spaces. We found that the function $F(r)$ of the metric does not depend on the specific form of the base factors on which one constructs the circle fibration. In the presence of electromagnetic field, there exist two extra parameters, in addition to the mass and the NUT charge, namely; the electric charge $q$ and the potential at infinity $V$. We found that in order to have NUT charged black holes in Einstein-Maxwell gravity, in addition to the two conditions of uncharged NUT solutions, there exists two other conditions. The first extra condition comes from the regularity of vector potential at $r = N$ which gives a relation between $q$ and $V$. Indeed, the existence of the parameter $V$ enables us to get a regularity condition on the one-form potential which is identical to that required to obtain a NUT solution. If one of these parameters vanishes then the other one should be equal to zero and the solution reduces to the uncharged solution. The second extra condition comes from the fact that the horizon at $r = N$ should be the outer horizon of the black hole. Indeed, Einstein-Maxwell gravity admits NUT black holes provided the potential parameter is less than a critical value $V_{\text{crit}}$, which may be obtained by solving the system of two equations (27). In any dimension larger than four, the mass parameter $m$ which is fixed by these four NUT conditions depends on the fundamental constant $l$ and parameters $N$ and $V$ (or $q$), while in 4-dimensions the mass parameter does not depend on $V$ (or $q$). We also found that the NUT solutions have no
curvature singularity at $r = N$ when the metric of the base space is chosen to be $\mathbb{C}P^k$, and have curvature singularity for other curved base spaces. Then, we obtained the bolt solutions of Einstein-Maxwell gravity in various dimensions, and gave the equations which can be solved for the horizon radius of the bolt solution. We found that in order to have bolt solution, the NUT parameter should be smaller than a maximum value $N_{\text{max}}$ which decreases as $V$ increases.

Second, we investigated the thermodynamics of NUT solutions in any dimensions. We calculated the conserved quantities and the Euclidean actions of the NUT/bolt solutions through the use of counterterms renormalization procedure. Also we obtained the charge and electric potentials of the solutions in an arbitrary dimension, and calculated the entropy through the use of Gibbs-Duhem relation. We obtained a Smarr-type formula for the mass as a function of the extensive parameters $S$ and $Q$, calculated the temperature and electric potential, and showed that these quantities satisfy the first law of thermodynamics. Then, we studied the phase behavior of the charged NUT solutions in canonical ensemble by calculating the heat capacity, and found that the NUT solutions are not thermally stable for even $k$, while there exists a stable phase for odd $k$, which becomes increasingly narrow with increasing dimensionality and wide with increasing $V$. We also studied the phase behavior of bolt solutions in 4 and 6 dimensions in canonical ensemble and found that these solutions have a stable phase, which becomes smaller as $V$ increases.

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[1] G. W. Gibbons and W. Hawking, Comm. Math. Phys. 66, 291 (1979).
[2] S. W. Hawking, C. J. Hunter and D. N. Page, Phys. Rev. D 59, 044033 (1999).
[3] R. B. Mann, Phys. Rev. D 60, 104047 (1999).
[4] R. B. Mann, Found. Phys. 33, 65 (2003).
[5] A. H. Taub, Annal. Math. 53, 472 (1951); E. Newman, L. Tamburino, and T. Unti, J. Math. Phys. 4, 915 (1963).
[6] F. A. Bais and P. Batenburg, Nucl. Phys. B 253, 162 (1985).
[7] D. N. Page and C. N. Pope, Class. Quant. Grav. 4, 213 (1987).
[8] M. M. Akbar and G. W. Gibbons, Class. Quant. Grav. 20, 1787 (2003).
[9] M. Taylor-Robinson, hep-th/9809041.
[10] A. Awad and A. Chamblin, Class. Quant. Grav. 19, 2051 (2002).
[11] R. B. Mann and C. Stelea, Class. Quant. Grav. 21, 2937 (2004).
[12] D. Astefanesei, R. B. Mann and E. Radu, Phys. Lett. B 620, 1 (2005); D. Astefanesei, R. B. Mann and E. Radu, J. High Energy Phys. 01, 049 (2005).
[13] R. B. Mann and C. Stelea, Phys. Lett. B 634, 448 (2006).
[14] D. R. Brill, Phys. Rev. 133, B845 (1964).
[15] R. B. Mann and C. Stelea, Phys. Lett. B 632, 537 (2006).
[16] A. Awad, Class. Quant. Grav. 23, 2849 (2006).
[17] E. Radu, Phys. Rev. D 67, 084030 (2003); Y. Brihaye, E. Radu, Phys. Lett. B 615, 1 (2005).
[18] M. H. Dehghani and R. B. Mann, Phys. Rev. D 72, 124006 (2005).
[19] M. H. Dehghani and S. H. Hendi, Phys. Rev. D 73, 0840021 (2006).
[20] M. M. Caldarelli, G. Cognola and D. Klemm, Class. Quantum Grav. 17, 399 (2000).
[21] S. W. Hawking and S. F. Ross, Phys. Rev. D 52, 5865 (1995).
[22] J. D. Brown and J. W. York, Phys. Rev. D 47, 1407 (1993).
[23] J. D. Brown, J. Creighton and R. B. Mann, Phys. Rev. D 50, 6394 (1994).
[24] J. Maldacena, Adv. Theor. Math. Phys., 2, 231 (1998); E. Witten, ibid. 2, 253 (1998); O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept., 323, 183 (2000).
[25] M. Hennigson and K. Skenderis, J. High Energy Phys. 7, 023 (1998).
[26] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999).
[27] S. Nojiri and S. D. Odintsov, Phys. Lett. B 444, 92 (1998); S. Nojiri, S. D. Odintsov and S. Ogushi, Phys. Rev. D 62, 124002 (2000).
[28] M. H. Dehghani, Phys. Rev. D 66, 044006 (2002); ibid. 65, 124002 (2002); M. H. Dehghani and A. Khodam-Mohammadi, ibid. 67, 084006 (2003).
[29] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D 59, 064010 (1999); R. Emparan, C. V. Johnson, R. C. Myers, Phys. Rev. D 60, 104001 (1999).
[30] R. Clarkson, L. Fatibene and R. B. Mann, Nucl. Phys. B 652, 348 (2003).
[31] R. B. Mann and C. Stelea, Phys. Rev. D 72, 084032 (2005).
[32] R. G. Cai and L. M. Cao, J. High Energy Phys. 3, 083 (2006).

[33] M. H. Dehghani and R. B. Mann, Phys. Rev. D 64, 044003 (2001); M. H. Dehghani, *ibid.* 65, 104030 (2002); M. H. Dehghani and H. KhajehAzad, Can. J. Phys. 81, 1363 (2003).

[34] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).

[35] P. Kraus, F. Larsen and R. Siebelink, Nucl. Phys. B 563, 259 (1999).

[36] M. Taylor-Robinson, hep-th/0002125 (2000).

[37] C. W. Misner, J. Math. Phys. 4, 924 (1963).