Figure-eight choreographies of the equal mass three-body problem with Lennard-Jones-type potentials

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Abstract

We report on figure-eight choreographic solutions to a system of three identical particles interacting through a potential of Lennard-Jones-type \(1/r_{12} - 1/r_{11}\), where \(r\) is distance between the particles. By numerical search, we found there are a multitude of such solutions. A series of them are close to a figure-eight solutions to a homogeneous system with no \(1/r_{12}\) term in the potential. The rest are very different, and have several points with large curvatures in their figure-eight orbits at which particles are repelled. Here figure-eight choreographies are periodic motions whose shape is symmetric in both horizontal and vertical axes, starting with an isosceles triangle configuration and going back to an isosceles triangle configuration with opposite direction through Euler configuration. Thus the lobe of such a figure-eight may be complex in shape, and need not be convex.

Keywords: three-body problem, figure-eight choreographies, Lennard-Jones-type potentials

(Some figures may appear in colour only in the online journal)

1. Introduction

Choreographic motion of \(N\) bodies is a periodic motion on a closed orbit in which \(N\) identical bodies chase each other on the orbit with equal time-spacing. Moore [1] found a remarkable figure-eight three-body choreographic solution under homogeneous interaction potential \(1/r^n\) by numerical calculations, where \(r\) is a distance between bodies. Chenciner and Montgomery [2] gave a rigorous proof of its existence for \(a = 1\), i.e. for Newtonian gravity.
Sbano [3], and Sbano and Southall [4], subsequently studied mathematically \(N\)-body choreographic solutions under an inhomogeneous potential

\[
u(r) = \frac{1}{r^6} - \frac{1}{r^3},
\]

a model potential between atoms called Lennard-Jones-type potential. For a system under homogeneous potential as \(-1/r^6\), if there exists a periodic solution with period \(T\), there exist scaled solutions for any period \(T\). However, solutions to inhomogeneous potential like (1) cannot be scaled. Sbano and Southall [4] proved that there exist at least two \(N\)-body choreographic solutions for sufficiently large period \(T\), and there exists no solution for small period \(T\).

Choreographic three-body motion on the lemniscate [5] is another example for the figure-eight choreography under inhomogeneous potential though potential \(1/2 \ln r - (\sqrt{3}/24)r^2\) is very strange. To our knowledge, there is no other study on the figure-eight choreography under inhomogeneous potential.

In this paper, we study figure-eight choreographic solutions to a system of three identical bodies interacting through Lennard-Jones-type potential (1) in classical mechanics, by numerical calculation. In section 2, we investigate the figure-eight choreographic solution under homogeneous potential \(-1/r^3\)—the attractive term of Lennard-Jones-type potential (1)—and consider a general construction of figure-eight choreography, independent of the potential energy. In section 3, we define figure-eight choreographic solutions under Lennard-Jones-type potential (1) and show a series of solutions that coincide with the solution to the homogeneous system when \(T \to \infty\). We investigate another series of solutions that was predicted by Sbano and Southall in section 4. Section 5 is a summary and discussion. Our numerical results in this paper were calculated by Mathematica 10.4 at its default precision.

2. Construction of figure-eight choreography

We consider solutions to an equation of motion,

\[
\ddot{q}_i = -\frac{\partial U}{\partial q_i}, \quad i = 0, 1, 2,
\]

for a system of three identical bodies interacting through a potential \(U\) where \(q_i(t) = (x_i(t), y_i(t))\) is a position vector of body \(i\) in a plane of the motion, and dot represents a differentiation in \(t\).

In this section, we take homogeneous potential, a power law potential as \(U\),

\[
U^{(a)} = -\sum_{i>j} \frac{1}{r_i^a},
\]

where \(r_{ij} = |q_i - q_j|\) is a distance between body \(i\) and \(j\). For \(a = 1\), there exists a figure-eight choreographic solution [1, 2], and we have obtained solutions for \(a = 2, 3, \ldots, 14\) numerically.

In figure 1, \(q_0(t)\) of the figure-eight choreographic solution for \(a = 6\) is shown. The two perpendicular mirror symmetric axes of the figure-eight are taken as \(x\)- and \(y\)-axes, thus the origin is the center of mass, \(\sum q_i = 0\). Points labeled by \((k \mod 12)\) are the positions \(q_0(kT/12)\) when \(T\) is the period of the motion and \(k\) is integer. Because of equal time spacing of choreography,

\[
q_{i+1}(t) = q_i(t + T/3), \quad i = 0, 1, 2,
\]

they are also the positions \(q_1((k - 4)T/12)\) and \(q_2((k + 4)T/12)\). Here and hereafter it is assumed that subscripts are modulo 3.
In the time interval $T/12$, the figure-eight solution takes two special configurations alternately, Euler configuration when one body is in the origin, and an isosceles triangle configuration when one body is on the $x$-axis. In figure 1, three bodies take the isosceles triangle configuration at $t = 2kT/12$ and Euler configuration at $t = (2k + 1)T/12$. The dashed triangle in figure 1 is the isosceles triangle at $t = 0$ and the dashed line segment the successive Euler configuration at $t = T/12$. In the following two subsections, we show necessary conditions for these two special configurations at $t = 0$ and $T/12$. Then, using these, we give a general construction of the figure-eight solutions in subsection 2.3.

2.1. Isosceles triangle configuration

In the isosceles triangle configuration at $t = 0$, we denote the position of the body in the first quadrant labeled by 0, $q(0)$, as $(x_0, y_0)$. Then that in the forth quadrant labeled by 8, $q_2(0)$, is $(x_0, -y_0)$ by definition, and that on the $x$-axis labeled by 4, $q_1(0)$, is $(-2x_0, 0)$ by $\sum q_i = 0$. Thus, the positions of bodies in the isosceles triangle configuration at $t = 0$ are written by $q(0)$ as

$$q_0(0) = (x_0, y_0), \quad q_1(0) = (-2x_0, 0), \quad q_2(0) = (x_0, -y_0).$$  \hspace{1cm} (5)

In this configuration, the velocity vector $\dot{q}_0(0)$—indicated by the arrow at the point labeled 4 in figure 1—is parallel to the $y$-axis, since the figure-eight is symmetric in the $x$-axis and its curvature is continuous. By the same symmetry, tangent lines at $q_0(0)$ and $q_2(0)$ are symmetric in the $x$-axis, thus meet at a point $c$ in the $x$-axis.

Here the total angular momentum $\sum \dot{q}_i \times q_i$ is zero, since the areas of the left and right lobe of the figure-eight orbit are equal. The total linear momentum $\sum \dot{q}_i$ is also zero, since $\sum q_i = 0$.

By the zero total angular and linear momentum, we have $|\sum q_i \times \dot{q}_i| = |\sum (q_i - c) \times \dot{q}_i| = |q_0(0) - c| \times |\dot{q}_0(0)| = |q_0(0) - c|$. Hence $\dot{q}_0(0) = 0$ or $c = q_0(0)$. Supposing $\dot{q}_0(0) = 0$ holds, either $\dot{q}_0(0) = \dot{q}_2(0) = 0$ or $\dot{q}_0(0) = (0, y_0) = -\dot{q}_2(0) \neq 0$ is deduced, which leads to a motion keeping the isosceles triangle, $q_0 = (x(t), y(t)), q_1 = (-2x(t), 0)$ and $q_2 = (x(t), -y(t))$, with $(x(0), y(0)) = (x_0, y_0)$. Therefore $c = q_0(0)$; that is, $\dot{q}_0(0)$ and $\dot{q}_2(0)$ are parallel to the edges of the isosceles triangle as shown in figure 1. Thus, for $\sum \dot{q}_i = 0$ and clockwise motion in the left lobe, the velocity vectors of bodies in the isosceles triangle configuration at $t = 0$ are written by $(x_0, y_0)$ and $v > 0$ as

![Figure 1. Figure-eight choreographic solution for homogeneous potential $U^{(6)}$. Points labeled by $(k \mod 12)$ are the positions of bodies $q_0(kT/12)$, $q_1((k - 4)T/12)$ and $q_2((k + 4)T/12)$, where $T$ is the period of the motion and $k$ is integer. Arrows are their velocity vectors.](image-url)
\[ \dot{q}_0(0) = \frac{\nu(q_0(0) - q_0(0))}{|q_0(0) - q_0(0)|}, \]
\[ \dot{q}_1(0) = (0, \frac{2\nu}{\sqrt{1 + (3x_0/y_0)^2}}), \]
\[ \dot{q}_2(0) = \frac{\nu(q_2(0) - q_2(0))}{|q_2(0) - q_2(0)|}. \] (6)

2.2. Euler configuration

In the Euler configuration at \( t = T/12 \), body 0 at \( q_{0}(T/12) \) labeled by 1 in figure 1 is at the origin. Body 1 at \( q_{1}(T/12) \) labeled by 5 is in the second quadrant and body 2 at \( q_{2}(T/12) \) labeled by 9 in the forth quadrant, where \( q_{2}(T/12) = -q_{1}(T/12) \) by \( \sum q_i = 0 \). Thus, the positions of bodies in the Euler configuration at \( t = T/12 \) are written by \( q_0 = 0, q_1 = -q_0 \). (7)

In this configuration, velocity vectors \( \dot{q}_0 \) and \( \dot{q}_2 \) are parallel as shown by arrows at points labeled by 5 and 9 in figure 1, since the figure-eight is symmetric in both \( x \)- and \( y \)-axes. Thus, writing \( \dot{q}_1 = a\dot{q}_0 \) we have, by zero total angular momentum,
\[ \sum_i q_i \times \dot{q}_i = 0 \] where \( q_1 \neq 0 \). If either \( \dot{q}_0 = 0 \) or \( q_1 |q_1| = 0 \) holds, one dimensional motion along a line connecting 0 and 1 is deduced. Thus, \( a = 1 \) and \( \dot{q}_1 = \dot{q}_2 \). Therefore, for \( \sum q_i = 0 \), the velocity vectors of bodies in the Euler configuration at \( t = T/12 \) are written by \( \dot{q}_0 \) as
\[ \dot{q}_1 = \dot{q}_2 = -\dot{q}_0/2. \] (8)

These velocity vectors are shown by arrows at points labeled by 5, 9 and 7 in figure 1.

2.3. Construction of figure-eight choreography

Inversely, we suppose that a three body motion \( q_i(t) \) satisfies isosceles triangle configuration (5) and (6) at \( t = 0 \), and Euler configuration (7) and (8) at some \( t = t_0 > 0 \). Because the Euler conditions (7) and (8) at \( t = t_0 \) and equation of motion (2), are invariant under inversions of \( x \), \( y \) and \( t \) with exchange of bodies 1 and 2, we have
\[ q_i(t + t_0) = -q_{i+2}(t_0 - t). \] (9)

By setting \( t = t_0 \) in the equation (9) and in its derivative in \( t \), we obtain \( q_i(2t_0) = -q_{i+2}(0) \) and \( \dot{q}_i(2t_0) = \dot{q}_{i+2}(0) \) at \( t = 2t_0 \), which are the inversions of \( x \) with cyclic permutation of subscript \( i \) to \( i + 2 \) in the isosceles triangle conditions (5) and (6) at \( t = 0 \). Since the equation of motion (2) is invariant under these transformations we have
\[ q_i(t + 2t_0) = (x_{i+2}(t_0 - t), y_{i+2}(t_0 - t)). \] (10)

Using the equation (10) twice we obtain the choreographic relation (4) with \( T/3 = 4t_0 \), thus the motion is choreographic and periodic with period \( 12t_0 \).

Equations (9), (10),
\[ q_i(t + 3t_0) = (x_{i+2}(t_0 - t), -y_{i+2}(t_0 - t)) \] (11)
obtained by substitution of equation (9) into (10), and \( q_i(t) \) itself, give one third of the orbits, \( q_i(t) \) for \( 0 \leq t \leq 4t_0 \), by the four possible inversions in \( x \) and/or \( y \) of the orbit for \( 0 \leq t \leq t_0 \) of the body \( 2i, i + 2, 2i + 1 \) and \( i \). The rest of the orbits, \( q_i(t) \) for \( 4t_0 \leq t \leq 8t_0 \) and for \( 8t_0 \leq t \leq 12t_0 \)
are given by shifting the subscripts $i$ by one and two respectively, by choreographic relation. Each of the four subscripts, $2i, i+2, 2i+1$ and $i$, yields a set $\{0,1,2\}$ when $i$ is shifted by zero, one or two. Thus the orbit $q_i(t)$ for $0 \leq t \leq 12t_0$ is symmetric in both $x$- and $y$-axes.

Consequently three body motion $q_i(t)$, which satisfies conditions (5)–(8), is the choreography satisfying the choreographic relations (4) with period $T = 12t_0$. The orbit is figure-eight in shape in the sense that it is symmetric in both $x$- and $y$-axes and passes through the origin. This result holds for the other potential $U$ if it is invariant under inversions of $x$, $y$ and $t$. Note that the set of initial conditions $q_i(0)$ and $\dot{q}_i(0)$ in equations (5) and (6) are determined by three parameters $(x_0, y_0, v)$.

For the figure-eight solution under homogeneous potential (3) with $a = 6$ shown in figure 1, the three parameters are

$$(x_0, y_0, v) = (x_0, 0.985945x_0, 0.234675x_0^{-3})$$

(12)

where $x_0 > 0$ and its period is $T = 61.2000x_0^4$.

3. Figure-eight choreography under Lennard-Jones-type potential

We consider, then, motions under Lennard-Jones-type potential

$$U^{(12,6)} = \sum_{i>j} u(r_{ij}),$$

(13)

numerically. We define the figure-eight choreography as a motion starting with the isosceles triangle configurations (5) and (6), and going through the Euler configuration (7) and (8).

We start numerical integration at $t = 0$ with the isosceles triangle initial conditions (5) and (6) determined by a set of parameters $(x_0, y_0, v)$. We stop integration if three bodies are aligned in a line, that is,

$$(q_1 - q_0) \times (q_2 - q_0) = 0$$

(14)

is satisfied. We denote this instant as $t_f$. Then we investigate the outer product

$$P(x_0, y_0, v) = \dot{q}_2 \times \dot{q}_1,$$

(15)

and the difference

$$D(x_0, y_0, v) = (q_1 - q_0)^2 - (q_2 - q_0)^2,$$

(16)

at $t = t_f$.

Thus the Euler configurations (7) and (8) are written as $P(x_0, y_0, v) = 0$ and $D(x_0, y_0, v) = 0$, since $D(x_0, y_0, v) = 0$ with (14) leads to $\dot{q}_0(t_f) = 0$, and $P(x_0, y_0, v) = 0$ with the zero total angular momentum assured by the isosceles triangle conditions (5) and (6) leads to $\dot{q}_i(t_f) = \dot{q}_0(t_f)$, according to the argument in section 2.2. Therefore, if these conditions are satisfied the solution has choreographic properties (4) with period $T = 12t_f$, and has figure-eight symmetry, as stated in section 2.

In figure 2, curves in the $(y_0,v)$-plane for $x_0 = 0.75$ with $P(x_0, y_0, v) = 0$ and $D(x_0, y_0, v) = 0$ are shown by dashed and solid curves respectively. The crossing points of the dashed and solid curves are the parameters $(y_0,v)$ for figure-eight choreographic solutions.

We discuss the solution $(x_0, y_0, v) = (0.75, 0.725966, 0.522742)$ labeled by $\alpha$ in figure 2. This solution is very close to the solution to the homogeneous system (12) shown in figure 1. If we plot orbits of two solutions in the same figure, it is difficult to distinguish them at the usual printing resolution.
We follow this solution, changing $x_0$ moderately, and so obtain a series of solutions. In figure 3, a set of $(x_0, y_0)$ for the series of this solution $\alpha$ is shown by solid curve. Point labeled by $\alpha$ is the solution $\alpha$. The dashed line, $y_0 = 0.985945 x_0$ is the solution to the homogeneous system (12). For $x_0 \geq 0.75$, the solid curve is almost on this line as noted above.

The other dashed curve in figure 3 is a boundary of a region of strong repulsive force,

$$\min(2y_0, \sqrt{9x_0^2 + y_0^2}) \leq r'_0 = 1,$$

(18)

in which two bodies in the isosceles triangle configuration feel strong repulsive force, where $r'_0$ is a distance $r$ which makes the potential (1) zero. See figure 4. Deep in the region (18) no solution is expected to exist, since it is difficult for the system to take the isosceles triangle configuration against the strong repulsive force. Actually, in figure 2 no crossing point can be seen deep in the region (18), that is, $y_0 < r'_0/2 = 0.5$.

Dashed line and dashed curve intersect at $x_0 = 0.507$. Thus this series of solutions cannot exist for $x_0 < 0.507$, and the solid curve turns and goes along the boundary of the region (18). The smallest $x_0$ for this series of solutions is about $x_0 = 0.6812$, with $y_0 = 0.617578$.

In figure 5, orbits $q_i(t)$ in the $y_0 \geq 0.617578$ branch of this series are shown for $x_0 = 0.6812$, 0.75, 1.0, 1.5. They are almost scalable. In figure 6, those in the $y_0 < 0.617578$ branch are shown. They are gourd-shaped figure-eights with necks slightly inside of $x = \pm x_0$. Of width about 2$y_0$. On both sides of the isosceles triangle configuration, two bodies are repelled since
In default precision calculation by Mathematica we cannot get this gourd-shaped figure-eight for $x_0 > 2.5$. This limit is probably due to the inaccuracy in numerical integration, and will exist for any large $x_0$. 

2$r_0 < r_0$, where $r_0 = 2^{1/6} = 1.12$ is the distance $r$ which makes the potential (1) minimum. See figure 4.

Figure 3. A series of solutions $\alpha$. The solid curve is a set of $(x_0, y_0)$ for the series of solutions $\alpha$. Points labeled by $\alpha$ and $\alpha'$ are the solutions labeled by the same labels in figure 2. The dashed line is the solution to the homogeneous system (17). The other dashed curve is a boundary of the region of strong repulsive force (18).

Figure 4. Lennard-Jones-type potential (1) has minimum at $r = r_0 = 2^{1/6} = 1.12$ and is zero at $r = r_0' = 1$. 

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In figure 2, solutions to be discussed are labeled by the Greek alphabets or those with prime, where the same letters mean that the solutions belong to the same series of solution. The solution $\alpha'$ in figure 2, whose orbit is shown at the top in figure 6, thus belongs to the series of solutions $\alpha$, as shown by the point $\alpha'$ in figure 3.

4. Other figure-eight choreography

We investigate the other series of solutions labeled by $\beta$–$\epsilon$ in figure 2. To distinguish these series we adopt total energy.
In figure 7, sets of \((x_0, E)\) for the series of solutions \(\alpha-\epsilon\) are shown in \((x_0, E)\)-plane.

In figures 8, 10, 12 and 13, orbits are shown for the series of solutions \(\beta, \gamma, \delta\) and \(\epsilon\) respectively. In these figures, three orbits are displayed as follows: the orbit for the smallest \(x_0\) is placed in figure (a). For large \(x_0\), the orbit in the higher \(E\) branch is placed in figure (b) and that in the lower \(E\) branch in figure (c).

4.1 Points of large curvature in orbits

We notice points of large curvature in the orbits shown in figures 6, 8, 10, 12 and 13. The large curvature occurs if distance between two bodies is less than \(r_0\). We call a closed interval in \(t\) which satisfies \(r_{ij}(t) \leq r_0\) a collisional interval between body \(i\) and \(j\). We then define the number of collisions \(n_{ij}\) by the number of collisional intervals between body \(i\) and \(j\) in \(0 \leq t < T\) with periodic boundary conditions. The number of collisions body \(i\) experiences is

\[
E(x_0, y_0, \nu) = \sum_i \frac{1}{2} q_i^2 + U
\]

instead of \(y_0\). In figure 7, sets of \((x_0, E)\) for the series of solutions \(\alpha-\epsilon\) are shown in \((x_0, E)\)-plane.

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\[
n_i = \sum_{j \neq i} n_{ij}
\]

Since \(n_0 = n_1 = n_2\), we use \(n_0\) as the number of collisions. Note that the term ‘collision’ here does not mean the usual collision of two point bodies \(i\) and \(j\) without size—as in classical mechanics—at which \(r_{ij} = 0\), but a repulsion which is considered as collision of two balls with radius \(r_0/2\) where \(r_{ij}\) cannot be zero due to the strong repulsive force.

In figure 9, \(r_{01}(t)\) and \(r_{02}(t)\) for the orbit in figure 8(a) are shown. We can count \(n_{01} = 4\) and \(n_{02} = 4\) then \(n_0 = n_{01} + n_{02} = 8\). Similarly, we can count \(n_0\) for the other orbits. They seem to be conserved within the same series of solutions. The counts \(n_0\) for series of solutions \(\beta-\epsilon\) are 8, 8, 16 and 24 respectively.

However, for the series of solutions \(\alpha\) the conservation of \(n_0\) does not hold. The count \(n_0\) for the orbits shown in figure 6 is 4, and \(n_0\) for those in figure 5 is 0; \(n_0\) changes around \(x_0 = 0.72\). Rather than \(n_0\), however, the number of local minimums in \(r_{01}(t)\) and \(r_{02}(t)\) for the orbits in figure 5 is 4; these are lowered and change to four collisional intervals.

Two collisions may occur simultaneously. In figure 11, \(r_{01}(t)\) and \(r_{02}(t)\) for the orbit in figure 10(c) are shown. The collisional intervals in \(r_{01}(t)\) and \(r_{02}(t)\) overlap at \(t = T/12\) and
and represent simultaneous collisions between body 0 and 1, and between 0 and 2 in the Euler configuration, where body 0 is at the origin and is in collision with bodies 1 and 2 from opposite sides.

If there is no simultaneous collision, the collisional segment, which we define here as the segment of the orbit corresponding to the collisional interval in $t$, includes the points with large curvatures whose accelerations are toward the outside of the figure-eight lobe. For simultaneous collision, as we can see from the orbit near the origin in figure 10(c), the collisional segment does not include the point of large curvature, and looks smooth.

Except for the series of the solutions $\alpha$ and $\beta$, all series of the solutions, i.e. series $\gamma$–$\epsilon$, have simultaneous collisions. At large $x_0$ in higher $E$ branches, the solutions shown in figures 10(b), 12(b) and 13(b), have four collisional segments surrounding the origin. At the smallest $x_0$, solutions shown in the same figures (a) still have the four separate collisional segments. At

Figure 8. Orbits for series of solution $\beta$. Figures in parentheses are $(x_0, y_0, v, E)$. (a) $(0.726, 0.766, 265, 0.302, 694; 0.0274, 632)$. (b) $(1.0, 0.956, 733, 0.144, 241; 0.002, 638, 43)$. (c) $(1.0, 1.241, 130, 0.071, 7890; 0.000, 697, 053)$.

Figure 9. Distances between bodies for the orbit shown in figure 8 (a). Black curve is $r_{01}(t)$ and gray curve is $r_{02}(t)$. Horizontal axis is $12t/T$. Dashed line shows $r_0$.
large $x_0$ in lower $E$ branches, the four collisional segments are merged into two smooth collisional segments at the origin shown in the same figures (c). Though the collisional segments at the origin look smooth, the minimums of $r_{01}(t)$ and $r_{02}(t)$ do not occur simultaneously and they still happen at slightly different times. They seem to occur simultaneously at $x_0 \to \infty$.

4.2. Behaviors and range of the total energy

There are three series of solutions—$\alpha$, $\beta$, and $\gamma$—in the $E \geq 0$ region in figure 7. In figure 2, the $E = 0$ curve is indicated by the dotted curve, and we can see there are five solutions $\alpha$, $\alpha'$, $\beta$, $\beta'$, and $\gamma$ above the $E = 0$ curve, i.e. in the $E \geq 0$ region. We explored such maps as figure 2

Figure 10. Orbits for a series of solution $\gamma$. Figures in parentheses are $(x_0, y_0, v, E)$. (a) $(0.6007, 0.748371, 0.371779; 0.0622734)$. (b) $(0.8, 1.081836, 0.126051; 0.00561328)$. (c) $(0.8, 1.136739, 0.0749665; −0.00519619)$.

Figure 11. Distances between bodies for the orbit shown in figure 10(c). Black curve is $r_{01}(t)$ and gray curve is $r_{02}(t)$. Horizontal axis is $12t/T$. Dashed line shows $r_0$. 

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for various \(x_0\) with various range of \(y_0\) and \(v\), and concluded numerically that these series of solutions \(\alpha\), \(\beta\) and \(\gamma\) are the only series of solutions in the \(E > 0\) region.

The whole series of solutions \(\alpha\) and \(\beta\) exist in the region \(E > 0\), whereas the lower energy branch for the series of solution \(\gamma\) crosses the \(E = 0\) once at \(x_0 = 0.671188\) in figure 7. We found numerically that this is the only solution with \(E = 0\). The orbit for the \(E = 0\) solution is shown in figure 14 together with its parameters; this is a member of series \(\gamma\) and resembles the solution shown in figure 10(c).

Thus around \(E = 0\), there are six solutions for \(E > 0\) and one solution for \(E = 0\). For \(E < 0\) there will be many—probably infinitely many—solutions, though only five are shown in figure 7.

![Orbits for series of solutions δ](image1)

**Figure 12.** Orbits for series of solutions \(\delta\). Figures in parentheses are \((x_0, y_0, v; E)\).
(a) (0.6501, 0.597985, 0.304229; −0.143858), (b) (0.8, 0.827038, 0.126408; −0.0330865), (c) (0.84, 0.848830, 0.0757119; −0.0387688).

![Orbits for series of solutions ε](image2)

**Figure 13.** Orbits for series of solutions \(\epsilon\). Figures in parentheses are \((x_0, y_0, v; E)\).
(a) (0.7074, 0.579781, 0.204620; −0.211945), (b) (0.91, 0.803912, 0.0857343; −0.0497687), (c) (0.91, 0.811359, 0.0540501; −0.0521151).
The range of the total energy for the figure-eight solutions is
\[-\frac{5546}{10924} \leq E \leq 0.295542.\] (21)

The upper limit is found in three series of solutions $\alpha$, $\beta$ and $\gamma$ in $E_0$, and is given by a solution $(x_0, y_0, v; E) = (0.686512, 0.639267, 0.646723; 0.295542)$ in the series of solution $\alpha$. The lower limit is the minimum of the potential energy $U$ either for the isosceles triangle configuration
\[
\min_{x_0, y_0}(u(2y_0) + 2u(\sqrt{9x_0^2 + y_0^2})) = 3u(r_0) = -\frac{3}{4} = -0.75,
\] (22)
at $(x_0, y_0) = (2^{-5/6}/\sqrt{3}, 2^{-5/6})$, or for the Euler configuration
\[
\min_r(2u(r) + u(2r)) = -\frac{5546}{10924} = -0.507781
\] (23)
at $r = (2731/1376)^{1/6} = 1.121$.

When $x_0 \to \infty$ the total energies $E$ for all series in figure 7 seem to go toward zero. For a homogeneous system with potential (3), by substituting (12) into (5) and (6), $E$ for a figure-eight solution is given by
\[
E = \frac{0.0467827}{x_0^6}.
\] (24)

For the series of solutions $\alpha$, from the plot of $E x_0^6$ against $x_0$, we can see $E x_0^6$ tends to a constant value, and we obtain asymptotic forms for both branches as
\[
E \to \frac{0.047}{x_0^6} \quad \text{or} \quad \frac{0.035}{x_0^6},
\] (25)
by fitting at $x_0 = 2, 2.05$ and $2.1$. For the other series, since we do not have numerical solutions for $x_0 > 1.2$ yet, we cannot investigate the asymptotic behaviors of $E$ precisely.

In figure 15, the total energy $E$ for all series of the solutions $\alpha$–$\epsilon$ are plotted against the period $T$ instead of $x_0$ in the ($T, E$)-plane. We expect a minimum period $T$ within a series of solutions will be smaller for a series with simpler shape of orbit. Actually in figure 15, we can see a trend that the series with smaller number of collisions $n_0$, which may be index of complexity of the shape of the orbit, have smaller minimum period. Because the series $\alpha$ has

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**Figure 14.** Orbit for a solution with $E = 0$. $(x_0, y_0, v; E) = (0.671188, 0.893818, 0.188131; 6.8 \times 10^{-9})$. 

The range of the total energy for the figure-eight solutions is
\[-\frac{5546}{10924} \leq E \leq 0.295542.\] (21)

The upper limit is found in three series of solutions $\alpha$, $\beta$ and $\gamma$ in $E_0$, and is given by a solution $(x_0, y_0, v; E) = (0.686512, 0.639267, 0.646723; 0.295542)$ in the series of solution $\alpha$. The lower limit is the minimum of the potential energy $U$ either for the isosceles triangle configuration
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\[
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For the series of solutions $\alpha$, from the plot of $E x_0^6$ against $x_0$, we can see $E x_0^6$ tends to a constant value, and we obtain asymptotic forms for both branches as
\[
E \to \frac{0.047}{x_0^6} \quad \text{or} \quad \frac{0.035}{x_0^6},
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by fitting at $x_0 = 2, 2.05$ and $2.1$. For the other series, since we do not have numerical solutions for $x_0 > 1.2$ yet, we cannot investigate the asymptotic behaviors of $E$ precisely.

In figure 15, the total energy $E$ for all series of the solutions $\alpha$–$\epsilon$ are plotted against the period $T$ instead of $x_0$ in the ($T, E$)-plane. We expect a minimum period $T$ within a series of solutions will be smaller for a series with simpler shape of orbit. Actually in figure 15, we can see a trend that the series with smaller number of collisions $n_0$, which may be index of complexity of the shape of the orbit, have smaller minimum period. Because the series $\alpha$ has
the simplest shape of orbit, we conjecture that the minimum of $T$ among all solutions is the minimum of $T$ in the series $\alpha$; we then find

$$14.5 \leq T.$$  \hfill (26)

5. Summary and discussions

We have studied figure-eight choreographic solutions to a system of three identical particles interacting through a potential of Lennard-Jones-type (13). The lobes of such figure-eight solutions can be complex shapes, though they need to be convex in the Newtonian three-body problem [6].

By numerical search, we found there are a multitude of such solutions. A series of solutions $\alpha$ tend to a figure-eight solution to a system with homogeneous potential (3) when $x_0 \to \infty$. The rest are very different, and have several points with large curvatures, collisional segments, in their figure-eight orbits. These results coincide with the theorem by Sbano and Southall [4] introduced in section 1. According to their theorem, there exists a lower bound in the period $T$ of the figure-eight solution, which we conjectured in equation (26).

In this paper, we have defined the figure-eight choreography as a motion starting with the isosceles triangle configurations (5) and (6), and going to the Euler configuration (7) and (8). In our numerical calculations, however, we stop integration in collinear configurations, which means that no collinear configuration other than the Euler configurations (7) and (8) is allowed. This artificial limitation is introduced by the simplicity of our numerical calculation in a first treatment, and should be removed in future.

In figure 2, we can see very fine structure of the solid curves, $P(x_0, y_0, v) = 0$, in the $v < 0.2$ region. If we magnify this region we see a lot of crossing points of solid and dashed curves, which means there are many figure-eight solutions. An orbit for one of these points labeled by $\zeta$ in figure 2 is shown in figure 16. At $x = -1.5$, the curve is not continuous because of insufficient precision of numerical integration. In order to explore these complex solutions more elaborate numerical calculations are necessary. We expect there are infinitely many solutions outside the region (18). Since the solutions originate from the repulsions between bodies their orbits must include two points whose distance is less than $r_0$. 

Figure 15. The set $(T, E)$ of the solutions $\alpha$ in the $(T, E)$-plane.
Though $\max_{i}(x_i(t)) = 2x_0$ is satisfied in the usual figure-eight shape, such as shown in figure 1, and in all orbits shown in this paper except for figure 16, this is not true in general. We can imagine a counter example from the orbit shown in figure 16.

The importance of the three-body problem under Lennard-Jones-type potential in physical chemistry and of choreographies eludes us at present. However, choreographies found in this paper may play some role in molecular dynamics. In molecular dynamics [7], motions of molecules or atoms are calculated in classical mechanics sometimes under Lennard-Jones-type potential, that is, as $N$-body problems under Lennard-Jones-type potential. Thus we notice that in such $N$-body problems some three bodies could form bound states in positive energies such as choreographies with positive energies in $\alpha$, $\beta$ and part of $\gamma$ series we found, which might have some effect in molecular dynamics.

Investigation of the figure-eight choreographies under various other inhomogeneous potentials, such as Lennard-Jones-type potential with different powers $\rho(r) = (r^{b} - r^{a})$, $b > a$,

Buckingham potential

$$e^{-r} - 1/r^{b},$$

Morse potential

$$(1 - e^{-a(r-r_0)})^2,$$

or screened Coulomb potential

$$-e^{-ar}/r,$$

etc, are also interesting. Does the qualitative nature of our numerical results change dramatically? For example, do we still see just one figure-eight with convex lobes? Do we get an infinite number coalescing with more and more kinks? These questions should be studied in future, and our numerical method exploring the figure-eight solutions explained in section 3 will be applied for such potentials, as an effective numerical method.

Our numerical method has two good features, compared to the other numerical methods used to find figure-eight orbits, such as Moore’s relaxation method [1] or the truncated Fourier series method used by Simó [8]. Firstly, our method can find solutions with non-minimal action functional. This is important since Sbano and Southall proved that at least one figure-eight solution is a mountain-pass critical point of the action functional for sufficiently large period $T$ in Lennard-Jones-type potentials [4]. Secondly, in our method, solutions are visualized by contour maps of $P(x_0, y_0, v) = 0$ and $D(x_0, y_0, v) = 0$ such as figure 2. This is most helpful in exploring the solutions and understanding the relationships between solutions.
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