BROWNIAN MOTION AND RICCI CURVATURE ON AN INFINITE DIMENSIONAL SYMPLECTIC GROUP RELATED TO THE Diffeomorphism GROUP OF THE CIRCLE

MANG WU

ABSTRACT. An embedding of the group Diff(S^1) of orientation preserving diffeomorphims of the unit circle S^1 into an infinite-dimensional symplectic group, Sp(∞), is studied. The authors prove that this embedding is not surjective. A Brownian motion is constructed on Sp(∞). This study is motivated by recent work of H. Airault, S. Fang and P. Malliavin. The Ricci curvature of the infinite-dimensional symplectic group is computed. The result shows that in almost all directions, the Ricci curvature is negative infinity.

1. INTRODUCTION

The group Diff(S^1) of orientation preserving diffeomorphims of the unit circle S^1 has been extensively studied for a long time. One of the goals of the research has been to construct and study the properties of a Brownian motion on this group. In [1] H. Airault and P. Malliavin considered an embedding of Diff(S^1) into an infinite-dimensional symplectic group.

This group, Sp(∞), can be represented as a certain infinite-dimensional matrix group. For such matrix groups, the method of [6, 7] can be used to construct a Brownian motion living in the group. This construction relies on the fact that these groups can be embedded into a larger Hilbert space of Hilbert-Schmidt operators. We use the same method to construct a Brownian motion on Sp(∞). One of the advantages of Hilbert-Schmidt groups is that one can associate an infinite-dimensional Lie algebra to such a group, and this Lie algebra is a Hilbert space. This is not the case with Diff(S^1), as an infinite-dimensional Lie algebra associated with Diff(S^1) is not a Hilbert space with respect to the inner product compatible with the symplectic structure on Diff(S^1).

In the current paper, we describe in detail the embedding of Diff(S^1) into Sp(∞), and construct a Brownian motion on Sp(∞). We also compute the Ricci curvature of Sp(∞). Our motivation comes from an attempt to use this embedding to better understand Brownian motion in Diff(S^1) as studied by H. Airault, S. Fang and P. Malliavin in a number of papers (e.g. [1, 2, 4, 5]). One of the main results of the paper is Theorem 4.6 where we describe the embedding of Diff(S^1) into Sp(∞) and prove that the map is not surjective. Theorem 6.17 gives the construction of a Brownian motion on Sp(∞). In order for this Brownian motion to live in the group we are forced to choose a non-Ad-invariant inner product on the Lie algebra of Sp(∞). This fact has a potential implication for this Brownian motion not to be quasi-invariant for the appropriate choice of the Cameron-Martin subgroup of Sp(∞). This is in contrast to results in [2]. The latter can be explained by the fact that the Brownian motion we construct in Section 6 lives in a subgroup of Sp(∞) whose Lie algebra is much smaller than the full Lie algebra of Sp(∞).

2. THE SPACES H AND H_ω

Definition 2.1. Let H be the space of complex-valued C^∞ functions on the unit circle S^1 with the mean value 0. Define a bilinear form ω on H by

\[ \omega(u, v) = \frac{1}{2\pi} \int_0^{2\pi} uv' d\theta, \quad \text{for any } u, v \in H. \]

Key words and phrases. Diffeomorphism group of the circle, infinite-dimensional symplectic group, Brownian motion, Ricci curvature.
Remark 2.2. By using integration by parts, we see that the form $\omega$ is anti-symmetric, that is, $\omega(u,v) = -\omega(v,u)$ for any $u,v \in H$.

Next we define an inner product $(\cdot, \cdot)_{\omega}$ on $H$ which is compatible with the form $\omega$. First, we introduce a complex structure on $H$, that is, a linear map $J$ on $H$ such that $J^2 = -id$. Then the inner product is defined by $(u,v)_{\omega} = \pm \omega(u,Jv)$, where the sign depends on the choice of $J$. The complex structure $J$ in this context is called the Hilbert transform.

Definition 2.3. Let $\mathbb{H}_0$ be the Hilbert space of complex-valued $L^2$ functions on $S^1$ with the mean value 0 equipped with the inner product

$$(u,v) = \frac{1}{2\pi} \int_0^{2\pi} u \bar{v} \, d\theta, \quad \text{for any } u,v \in \mathbb{H}_0.$$

Notation 2.4. Denote $\hat{e}_n = e^{in\theta}, n \in \mathbb{Z}\{0\}$, and $\mathcal{H}_H = \{\hat{e}_n, n \in \mathbb{Z}\{0\}\}$. Let $\mathbb{H}^+$ and $\mathbb{H}^-$ be the closed subspaces of $\mathbb{H}_0$ spanned by $\{\hat{e}_n : n > 0\}$ and $\{\hat{e}_n : n < 0\}$, respectively. By $\pi^+$ and $\pi^-$ we denote the projections of $\mathbb{H}_0$ onto subspaces $\mathbb{H}^+$ and $\mathbb{H}^-$, respectively. For $u \in \mathbb{H}_0$, we can write $u = u_+ + u_-$, where $u_+ = \pi^+(u)$ and $u_- = \pi^-(u)$.

Definition 2.5. Define the Hilbert transformation $J$ on $\mathcal{H}_H$ by

$$J : \hat{e}_n \mapsto i\text{sgn}(n)\hat{e}_n$$

where $\text{sgn}(n)$ is the sign of $n$, and then extended by linearity to $\mathbb{H}_0$.

Remark 2.6. In the above definition, $J$ is defined on the space $\mathbb{H}_0$. We need to address the issue whether it is well–defined on the subspace $H$. That is, if $J(H) \subseteq H$. We will see that if we modify the space $H$ a little bit, for example, if we let $C^1_0(S^1)$ be the space of complex-valued $C^1$ functions on the circle with mean value zero, then $J$ is not well–defined on $C^1_0(S^1)$. This problem really lies in the heart of Fourier analysis. To see this, we need to characterize $J$ by using the Fourier transform.

Notation 2.7. For $u \in \mathbb{H}_0$, let $\mathcal{F} : u \mapsto \hat{u}$ be the Fourier transformation with $\hat{u}(n) = (u, \hat{e}_n)$. Let $\hat{J}$ be a transformation on $L^2(\mathbb{Z}\{0\})$ defined by $(\hat{J}\hat{u})(n) = i\text{sgn}(n)\hat{u}(n)$ for any $\hat{u} \in L^2(\mathbb{Z}\{0\})$.

The Fourier transformation $\mathcal{F} : \mathbb{H}_0 \to L^2(\mathbb{Z}\{0\})$ is an isomorphism of Hilbert spaces, and $J = \mathcal{F}^{-1} \circ \hat{J} \circ \mathcal{F}$.

Proposition 2.8. The Hilbert transformation $J$ is well–defined on $H$, that is $J(H) \subseteq H$.

Proof. The key of the proof is the fact that functions in $H$ can be completely characterized by their Fourier coefficients. To be precise, let $u \in \mathbb{H}_0$ be continuous. Then $u$ is $C^\infty$ if and only if $\lim_{n \to \infty} n^k \hat{u}(n) = 0$ for any $k \in \mathbb{N}$. From this fact, it follows immediately that $J$ is well–defined on $H$, because $J$ only changes the signs of the Fourier coefficients of a function $u \in H$.

For completeness of exposition, we give a proof of this fact. Though the statement is probably a standard fact in the Fourier analysis, we found it proven only in one direction in [11].

We first assume that $u$ is $C^\infty$. Then $u(\theta) = u(0) + \int_0^\theta u'(t) \, dt$. So

$$\hat{u}(n) = \frac{1}{2\pi} \left( \int_0^{2\pi} \int_0^{2\pi} u'(t) \chi_{[0,\theta]}(dt) e^{-in\theta} \, d\theta \right) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \, d\theta \int_0^\theta u'(t) \, dt = -\frac{1}{2\pi in} \int_0^{2\pi} u'(t) - u'(t)e^{-int} \, dt = \frac{\hat{u}(n)}{in},$$

where we have used Fubini’s theorem and the continuity of $u'$. Now, $u'$ is itself $C^\infty$, so we can apply the procedure again. By induction, we get $\hat{u}(n) = \frac{\hat{u}^{(k)}(n)}{(in)^k}$. But from the general theory of Fourier analysis, $\hat{u}^{(k)}(n) \to 0$ as $n \to \infty$. Therefore $n^k \hat{u}(n) \to 0$ as $n \to \infty$. Therefore
Conversely, assume \( u \) is such that for any \( k, n^n \hat{u}(n) \to 0 \) as \( n \to \infty \). Then the Fourier series of \( u \) converges uniformly. Also by assumption that \( u \) is continuous, the Fourier series converges to \( u \) for all \( \theta \in S^1 \) (see Corollary I.3.1 in [11]). So we can write \( u(\theta) = \sum_{n \neq 0} \hat{u}(n)e^{in\theta} \).

Fix a point \( \theta \in S^1 \),

\[
\frac{d}{dt} \bigg|_{t=\theta} \sum_{n \neq 0} \hat{u}(n)e^{in\theta} = \lim_{t \to \theta} \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{u}(n)\frac{e^{in\theta} - e^{in\theta}}{t - \theta}.
\]

Note that the derivatives of \( \cos nt \) and \( \sin nt \) are all bounded by \( |n| \). So by the mean value theorem, \( |\cos nt - \cos t\theta| \leq |n||t - \theta| \), and \( |\sin nt - \sin n\theta| \leq |n||t - \theta| \). So

\[
\left|\frac{e^{in\theta} - e^{in\theta}}{t - \theta}\right| \leq 2|n|, \quad \text{for any } t, \theta \in S^1.
\]

Therefore, by the growth condition on the Fourier coefficients \( \hat{u} \), we have

\[
\lim_{N \to \infty} \sum_{n=-N}^{N} \hat{u}(n)\frac{e^{in\theta} - e^{in\theta}}{t - \theta}
\]

converges at the fixed \( \theta \in S^1 \) and the convergence is uniform in \( t \in S^1 \). Therefore we can interchange the two limits, and obtain

\[
\left(\sum_{n \neq 0} \hat{u}(n)e^{in\theta}\right)' = \sum_{n \neq 0} \hat{u}(n)ine^{in\theta},
\]

which means we can differentiate term by term. So the Fourier coefficients of \( u' \) are given by \( \hat{u}'(n) = in\hat{u}(n) \). Clearly, \( \hat{u}' \) satisfies the same condition as \( \hat{u} \) : \( n^n \hat{u}'(n) \to 0 \) as \( n \to \infty \). By induction, \( u \) is \( j \)-times differentiable for any \( j \). Therefore, \( u \) is \( C^\infty \).

**Proposition 2.9.** Let \( C^1_0(S^1) \) be the space of complex-valued \( C^1 \) functions on the circle with the mean value zero. Then the Hilbert transformation \( J \) is not well--defined on \( C^1_0(S^1) \), i.e., \( J(C^1_0(S^1)) \nsubseteq C^1_0(S^1) \).

**Proof.** Let \( C(S^1) \) be the space of continuous functions on the circle. In [11], it is shown that there exists a function in \( C(S^1) \) such that the corresponding Fourier series does not converge uniformly [11] Theorem II.1.3], and therefore there exists an \( f \in C(S^1) \) such that \( Jf \notin C^1(S^1) \) [11] Theorem II.1.4]. Now take \( u = f - f_0 \) where \( f_0 \) is the mean value of \( f \). Then \( u \) is a continuous function on the circle with the mean value zero, and \( Ju \) is not continuous.

Using Notation [2,4] let us write \( u = u_+ + u_- \). Then we can use the relation

\[
iu + Ju = 2iu_+ \quad \text{and} \quad iu - Ju = 2iu_-.
\]

to see that \( u_+ \) and \( u_- \) are not continuous. Integrating \( u = u_+ + u_- \), we have

\[
\int_0^\theta u(\theta)d\theta = \int_0^\theta u_+(\theta)d\theta + \int_0^\theta u_-(\theta)d\theta.
\]

Denote the three functions in the above equation by \( v, v_1, v_2 \). By theorem I.1.6 in [11],

\[
\hat{u}(n) = \frac{\hat{v}(n)}{in}, \quad \text{and} \quad \hat{v}(n) = \frac{\hat{u}_+(n)}{in}, v_2(n) = \frac{1}{in}\hat{u}_-(n) \quad \text{for } n \neq 0.
\]

Let \( g = v - v_0 \) where \( v_0 \) is the mean value of \( v \). Then \( g \in C^1_0(S^1) \). Write \( g = g_+ + g_- \) [2,4] Then \( g_+ = v_1 - (v_1)_0 \) and \( g_- = v_2 - (v_2)_0 \) where \( (v_1)_0 \) and \( (v_2)_0 \) are the mean values of \( v_1 \) and \( v_2 \) respectively. Then \( g_+, g_- \notin C^1_0(S^1) \) since \( v_1' = u_+, v_2' = u_- \) are not continuous.

By the relation

\[
iu + Ju = 2iu_+ \quad \text{and} \quad iu - Ju = 2iu_-,
\]

we see that \( Jg \notin C^1_0(S^1) \).

**Notation 2.10.** Define an \( \mathbb{R} \)-bilinear form \( (\cdot, \cdot)_\omega \) on \( H \) by

\[
(u, v)_\omega = -\omega(u, Jv) \quad \text{for any } u, v \in H.
\]
Proposition 2.11. \((\cdot,\cdot)_\omega\) is an inner product on \(H\).

Proof: We need to check that \((\cdot,\cdot)_\omega\) satisfies the following properties (1) \((\lambda u,v)_\omega = \lambda (u,v)_\omega\) for \(\lambda \in \mathbb{C}\); (2) \((v,u)_\omega = (u,v)_\omega\); (3) \((u,u)_\omega > 0\) unless \(u = 0\).

(1) for \(\lambda \in \mathbb{C}\),
\[
(\lambda u, v)_\omega = -\omega(\lambda u, Jv) = -\lambda \cdot \omega(u, Jv) = \lambda \cdot (u,v)_\omega.
\]

To prove (2) and (3), we need some simple facts: \(H^+ = \pi^+(H) \subseteq H\) and \(H^- = \pi^-(H) \subseteq H\), and \(H = H^+ \oplus H^-.\) If \(u \in H^+, v \in H^-,\) then \((u,v) = 0\). If \(u \in H^+,\) then \(\bar{u} \in H^-, Ju = iu, Ju \in H^+.\) If \(u \in H^-\), then \(\bar{u} \in H^+, Ju = -iu, Ju \in H^-\). \(J\bar{u} = J\bar{u} \in H^-\). In particular, if \(u \in H^+,\) then \(u' \in H^+;\) if \(u \in H^-\), then \(u' \in H^-\).

(2) By definition,
\[
(v,u)_\omega = -\omega(v, J\bar{u}) = \omega(J\bar{u}, v) = \frac{1}{2\pi} \int (J\bar{u})v' d\theta
\]
and
\[
(u,v)_\omega = -\omega(u, J\bar{v}) = \omega(J\bar{v}, u) = \frac{1}{2\pi} \int J\bar{v}u' d\theta = \frac{1}{2\pi} \int (Jv)\bar{u}' d\theta.
\]
Write \(u = u_+ + u_-\) and \(v = v_+ + v_-\) as in Notation 2.4. Using the above fact, we can show that the above two quantities are equal to each other.

(3) Write \(u = u_+ + u_-\), then
\[
(u,u)_\omega = \frac{1}{2\pi} \int (-i\bar{u}_+ u'_- + i\bar{u}_- u'_+) d\theta = \sum_{n \neq 0} |n||\hat{u}(n)|^2.
\]
Therefore, \((u,u)_\omega > 0\) unless \(u = 0\). \(\square\)

Definition 2.12. Let \(\mathbb{H}_\omega\) be the completion of \(H\) under the norm \(\| \cdot \|_\omega\) induced by the inner product \((\cdot,\cdot)_\omega\).

Define
\[
\mathcal{B}_\omega = \left\{ e_n = \frac{1}{\sqrt{n}} e^{in\theta}, n > 0 \right\} \cup \left\{ e_n = \frac{1}{i\sqrt{n}} e^{in\theta}, n < 0 \right\}.
\]

Remark 2.13. \(\mathbb{H}_\omega\) is a Hilbert space. Also the norm \(\| \cdot \|_\omega\) induced by the inner product \((\cdot,\cdot)_\omega\) is strictly stronger than the norm \(\| \cdot \|\) induced by the inner product \((\cdot,\cdot)\). So \(\mathbb{H}_\omega\) can be identified as a proper subspace of \(\mathbb{H}_0\). The inner product \((\cdot,\cdot)_\omega\) or the norm induced by it is sometimes called the \(H^{1/2}\) metric or the \(H^{1/2}\) norm on the space \(H\).

One can verify that \(\mathcal{B}_\omega\) is an orthonormal basis of \(\mathbb{H}_\omega\). From the definition of the inner product \((\cdot,\cdot)_\omega\), we have the relation \(\omega(u,v) = (u,\mathcal{B}_\omega)\) for any \(u, v \in H\). This can be used to extend the form \(\omega\) to \(\mathbb{H}_\omega\).

Finally, from the non-degeneracy of the inner product \((\cdot,\cdot)_\omega\), we see that the form \(\omega(\cdot,\cdot)\) on \(\mathbb{H}_\omega\) is also non-degenerate.

3. An Infinite-Dimensional Symplectic Group

Definition 3.1. Let \(\mathcal{B}(\mathbb{H}_\omega)\) be the space of bounded operators on \(\mathbb{H}_\omega\) equipped with the operator norm. For an operator \(A \in \mathcal{B}(\mathbb{H}_\omega)\)

(1) suppose \(A\) is an operator on \(\mathbb{H}_\omega\) satisfying \(\hat{A}u = \mathcal{B}_\omega\) for any \(u \in \mathbb{H}_\omega\), then \(\hat{A}\) is the conjugate of \(A\);
(2) suppose \(A^\dagger\) is an operator on \(\mathbb{H}_\omega\) satisfying \((Au,v)_\omega = (u,A^\dagger v)_\omega\) for any \(u,v \in \mathbb{H}_\omega\), then \(A^\dagger\) is the adjoint of \(A\);
(3) then \(A^T = \hat{A}^\dagger\) is the transpose of \(A\);
(4) suppose \(A^\#\) is an operator on \(\mathbb{H}_\omega\) satisfying \(\omega(Au,v) = \omega(u,A^\# v)\) for any \(u,v \in \mathbb{H}_\omega\), then \(A^\#\) is the symplectic adjoint of \(A\);
(5) \(A\) is said to preserve the form \(\omega\) if \(\omega(Au,Av) = \omega(u,v)\) for any \(u,v \in \mathbb{H}_\omega\).

In the orthonormal basis \(\mathcal{B}_\omega\), an operator \(A \in \mathcal{B}(\mathbb{H}_\omega)\) can be represented by an infinite-dimensional matrix, still denoted by \(A\), with \((m,n)\)th entry equal to \(A_{m,n} = (Ae_m,e_n)_\omega\).
Remark 3.2. If we represent an operator $A \in B(\mathbb{H}_\omega)$ by a matrix $\{A_{m,n}\}_{m,n \in \mathbb{Z}\setminus\{0\}}$, the indices $m$ and $n$ are allowed to be both positive and negative following Definition 2.12 of $\mathcal{B}_\omega$.

The next proposition collects some simple facts about operations on $B(\mathbb{H}_\omega)$ introduced in Definition 3.1.

Proposition 3.3. Let $A, B \in B(\mathbb{H}_\omega)$. Then

1. $\bar{e}_n = i \bar{e}_{-n}$, $\bar{J} e_n = \text{isgn}(n) \bar{e}_n$, $(\bar{e}_n)' = \bar{i} \bar{e}_n$;
2. $(\bar{A})_{m,n} = \overline{A_{n,-m}}$;
3. $(A^\dagger)_{m,n} = A_{n,m}$;
4. $\bar{A}^\dagger = A^\dagger$, and $(A^T)_{m,n} = A_{n,-m}$;
5. If $A = \tilde{A}$, then $(A^\#)_{m,n} = \text{sgn}(mn) \overline{A_{n,m}}$;
6. $\tilde{A} B = \bar{A} \bar{B}$, $(AB)^\dagger = B^\dagger \bar{A}$, $(AB)^T = B^T A^T$, $(AB)^\# = B^\# \bar{A}$;
7. If $A$ is invertible, then $\tilde{A} A^T A \tilde{A}$ are all invertible, and $(\tilde{A})^{-1} = \overline{A^{-1}}$, $(A^T)^{-1} = (A^{-1})^T$, $(\tilde{A}^\dagger)^{-1} = (A^{-1})^\#$;
8. $(\pi^\dagger)_{m,n} = \frac{1}{2} (\delta_{mn} + \text{sgn}(m) \delta_{nm})$, $(\pi^-)_{m,n} = \frac{1}{2} (\delta_{mn} - \text{sgn}(m) \delta_{nm})$, $\pi^\dagger = \pi^-$, $\pi^- = \pi^+$, $(\pi^\dagger)^T = \pi^+$, $(\pi^-)^T = \pi^-$;
9. $J_{m,n} = \text{isgn}(m) \delta_{mn}$, $\bar{J} = J$, $J = i(\pi^+ - \pi^-)$, $J^T = -J$, $J^\dagger = -J$, $J^2 = -\text{id}$;
10. $(A^\#)_{m,n} = \text{sgn}(mn) A_{n,-m}$.

Proof. All of these properties can be checked by straightforward calculations. We only prove (10).

$$(A^\#)_{m,n} = (\bar{A}^\dagger \bar{e}_n, \bar{e}_m) = -\omega(\bar{A}^\# \bar{e}_n, \bar{J} \bar{e}_m) = \omega(\bar{J} \bar{e}_m, \bar{A}^\# \bar{e}_n)$$

\[
= \omega(\bar{J} \bar{e}_m, \bar{e}_n) = -\omega(\bar{e}_n, \bar{J} \bar{A} \bar{e}_m) = -\omega(\bar{e}_n, J(-J \bar{A} \bar{e}_m)) \]

\[
= -\omega(\bar{e}_n, J(-J \bar{A} \bar{e}_m))
\]

where in the last equality we used property (6), $\bar{A} B = \bar{A} B$, and property (9), $J = J$, so that $-J \bar{A} \bar{e}_m = -J \bar{A} \bar{e}_m = -J \bar{A} \bar{e}_m = -J \bar{A} \bar{e}_m$. Therefore,

$$(A^\#)_{m,n} = -\omega(\bar{e}_n, J(-J \bar{A} \bar{e}_m)) = (\bar{e}_n, -J \bar{A} \bar{e}_m) = -J \bar{A} \bar{e}_m = -\omega(\bar{e}_n, J(-J \bar{A} \bar{e}_m))$$

\[
= -\omega(\bar{e}_n, J(-J \bar{A} \bar{e}_m)) = \text{sgn}(mn) (\bar{e}_n, A \bar{e}_m) = \text{sgn}(mn) (\bar{A} \bar{e}_m, \bar{e}_n) = \text{sgn}(mn) \overline{A_{n,m}}
\]

Notation 3.4. For $A \in B(\mathbb{H}_\omega)$, let $a = \pi^+ A \pi^+$, $b = \pi^+ A \pi^-$, $c = \pi^- A \pi^+$, and $d = \pi^- A \pi^-$, where $a : \mathbb{H}_\omega^+ \rightarrow \mathbb{H}_\omega^+$, $b : \mathbb{H}_\omega^- \rightarrow \mathbb{H}_\omega^+$, $c : \mathbb{H}_\omega^+ \rightarrow \mathbb{H}_\omega^-$, $d : \mathbb{H}_\omega^- \rightarrow \mathbb{H}_\omega^-$. Then $A = a + b + c + d$ can be represented as the following block matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $A, B \in B(\mathbb{H}_\omega)$, then the block matrix representation for $AB$ is exactly the multiplication of block matrices for $A$ and $B$.

Proposition 3.5. Suppose $A \in B(\mathbb{H}_\omega)$ with the matrix $\{A_{m,n}\}_{m,n \in \mathbb{Z}\setminus\{0\}}$. Then the following are equivalent

1. $A = \tilde{A}$;
2. if $u = \bar{u}$, then $Au = \overline{A u}$;
3. $A_{m,n} = \overline{A_{n,-m}}$ (3.2);
4. as a block matrix, $A$ has the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.
Proposition 3.6. Let $A \in B(\mathbb{H}_\omega)$. The following are equivalent:

1. $A$ preserves the form $\omega$;
2. $\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$;
3. $\omega(A\tilde{e}_m, \tilde{A} \tilde{e}_n) = \omega(\tilde{e}_m, \tilde{e}_n)$ for any $m, n \neq 0$;
4. $A^TJA = J$;
5. $\sum_{k \neq 0} \text{sgn}(mk)A_{k,m}A_{-k,-n} = \delta_{m,n}$ for any $m, n \neq 0$.

Proof. Equivalence of (1), (3) and (4) follows from Proposition 3.3 and Notation 3.4. First we show that (1) is equivalent to (2).

[(1)⇒(2)]. If $u = \tilde{u}$, then $Au = \tilde{A} \tilde{u} = \overline{A u}$.

[(2)⇒(1)]. Let $u = \tilde{e}_n + \overline{\tilde{e}_n}$, and $v = \overline{\tilde{e}_n} + \overline{\tilde{e}_n}$. Then $u, v$ are real-valued functions on the circle. Using Proposition 3.3, we have $\tilde{e}_n = i \tilde{e}_n$, and therefore $Au = \overline{A u} = \overline{Av}$ imply

$$A\tilde{e}_n + i \tilde{A} \tilde{e}_n = A\overline{\tilde{e}_n} - i \overline{\tilde{A} \tilde{e}_n}$$

$$A\tilde{e}_n - i \tilde{A} \tilde{e}_n = -A\overline{\tilde{e}_n} - i \overline{\tilde{A} \tilde{e}_n}.$$ 

Solving the above two equations for $A\tilde{e}_n$, we have

$$A\tilde{e}_n = -i \overline{A e}_n = \overline{\tilde{A} n} = \tilde{A} \tilde{e}_n$$

with this being true for any $n \neq 0$, and so $A = \tilde{A}$.

If we further assume that $A = \tilde{A}$, then the following two are equivalent to the above:

(I) $a^T\tilde{a} - b^T\tilde{b} = \pi^-$ and $a^T\overline{\tilde{b}} - \overline{\tilde{b}}^Ta = 0$;

(II) $\sum_{k \neq 0} \text{sgn}(mk)A_{k,m}A_{-k,-n} = \delta_{m,n}$ for any $m, n \neq 0$.

Proof. Equivalence of (1) and (2) follows directly from Definition 3.1. Let us check the equiavlency of (2) and (4). First assume that (2) holds. By Remark 2.13, we have $\omega(u, v) = (u, Jv)_\omega$, and therefore

$$\omega(Au, Av) = (Au, \overline{J \tilde{A} v})_\omega = (u, A^T \overline{J \tilde{A} v})_\omega.$$ 

By assumption, $\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$. So by the non-degeneracy of the inner product $(\cdot, \cdot)_\omega$, we have $A^T \overline{J \tilde{A} v} = J \tilde{v}$ for any $v \in \mathbb{H}_\omega$. By definition of $\tilde{A}$, we have $\overline{Av} = \tilde{A} \tilde{v}$. So $A^T \overline{J \tilde{A} v} = J \tilde{v}$ for any $v \in \mathbb{H}_\omega$, or $A^T \overline{J \tilde{A} v} = J$. Taking conjugation of both sides and using $\bar{J} = J$, we see that $A^TJA = J$.

Every step above is reversible, therefore we have implication in the other direction as well.

Now we check the equivalency of (3) and (5). First, by Remark 2.13, $\omega(u, v) = (u, Jv)_\omega$ and Proposition 3.3

$$\omega(\tilde{e}_m, \tilde{e}_n) = (\tilde{e}_m, J \tilde{e}_n)_\omega = -\text{sgn}(m)\delta_{m,-n}.$$ 

On the other hand, by the continuity of the form $\omega(\cdot, \cdot)$ in both variables, we have

$$\omega(A\tilde{e}_m, \tilde{A} \tilde{e}_n) = \omega \left( \sum_k A_{k,m} \tilde{e}_k, \sum_l A_{l,n} \tilde{e}_l \right)$$

$$= \sum_{k,l} A_{k,m} A_{l,n} (\text{sgn}(k)) \delta_{k,-l} = - \sum_k \text{sgn}(k) A_{k,m} A_{-k,n}.$$ 

Now assuming $\omega(A\tilde{e}_m, \tilde{A} \tilde{e}_n) = \omega(\tilde{e}_m, \tilde{e}_n)$, we have

$$- \sum_k \text{sgn}(k) A_{k,m} A_{-k,n} = -\text{sgn}(m)\delta_{m,-n}, \text{ for any } m, n \neq 0.$$ 

By multiplying by $\text{sgn}(m)$ both sides, and replacing $-n$ with $n$, we get (5). Conversely, note that every step above is reversible, therefore we have implication in the other direction.

We have proved equivalence of (1)–(5). Now assume $A = \tilde{A}$. To prove equivalence of (4) and (I), just notice that as block matrices, $A, A^T$ and $J$ have the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} a^T & b^T \\ b^T & a^T \end{pmatrix}, \text{ and } i \begin{pmatrix} \pi^+ & 0 \\ 0 & -\pi^- \end{pmatrix}.$$ 

Equivalence of (5) and (II) follows from the relation $A_{-k,-n} = \overline{A_{k,n}}$. □
Proposition 3.7. Let \( A \in B(\mathbb{H}_\omega) \). If \( A \) preserves the form \( \omega \), then the following are equivalent:

1. \( A \) is invertible.
2. \( AJA^T = J \).
3. \( A^T \) preserves the form \( \omega \).
4. \( \sum_k \text{sgn}(mk)A_{m,k}A_{-n,-k} = \delta_{m,n} \) for any \( m, n \neq 0 \).

If we further assume that \( A = \tilde{A} \), then the following are equivalent to the above:

1. \( a\omega^T - b\pi^T = \pi^- \) and \( b\omega^T = \pi^- \).
2. \( \sum_k \text{sgn}(mk)A_{m,k}A_{n,k} = \delta_{m,n} \) for any \( m, n \neq 0 \).

Proof. We will use several times the fact that if \( A \) preserves \( \omega \), then \( A^TJA = J \).

\[ [(1) \Rightarrow (2)] \] Multiplying on the left by \( (A^T)^{-1} \) and multiplying on the right by \( A^{-1} \) both sides, we get \( J = (A^T)^{-1}JA^{-1} \), and so \( (A^{-1})^TA^{-1} = J \). Taking inverse of both sides, and using \( J^{-1} = -J \), we have \( A^TJA = J \).

\[ [(2) \Rightarrow (1)] \] As \( J \) is injective, so is \( A^TJA \), and therefore \( A \) is injective. On the other hand, by assumption \( AJA^T = J \). As \( J \) is surjective, so \( AJA^T \) is surjective too. This implies that \( A \) is surjective, and therefore \( A \) is invertible.

Equivalence of (2) and (3) follows from \( (A^T)^T = A \) and Proposition 3.6.

Equivalence of (3) and (4) follows directly from Proposition 3.6 and the fact that \( (A^T)_{m,n} = A_{-n,-m} \).

Now assume that \( A = \tilde{A} \). Then equivalence of (3) and (I) can be checked by using multiplication of block matrices as in the proof of Proposition 3.6. Finally (4) is equivalent to (II) as if \( A = A \), then \( A_{-m,-n} = \tilde{A}_{m,n} \). \( \square \)

Corollary 3.8. Let \( A \in B(\mathbb{H}_\omega) \) and \( A = \tilde{A} \). Then the following are equivalent:

1. \( A \) preserves the form \( \omega \) and is invertible;
2. \( A^#A = A^#A = \text{id} \);

Proof. By Proposition 3.3

\[ (A^#A)_{m,n} = \sum_{k \neq 0} (A^#)_{m,k}A_{k,n} = \sum_{k \neq 0} \text{sgn}(mk)A_{k,n}A_{k,m} \]

\[ (AA^#)_{m,n} = \sum_{k \neq 0} A_{m,k}(A^#)_{k,n} = \sum_{k \neq 0} \text{sgn}(nk)A_{m,k}A_{n,k} \]

Therefore, by (II) in Proposition 3.6 and (II) in Proposition 3.7, we have equivalence. \( \square \)

Definition 3.9. Define a (semi)norm \( \| \cdot \|_2 \) on \( B(\mathbb{H}_\omega) \) such that for \( A \in B(\mathbb{H}_\omega) \), \( \| A \|_2 = \text{Tr}(b^\dagger b) = \| b \|_{HS} \). where \( b = \pi^+A\pi^- \). That is, the norm \( \| A \|_2 \) is just the Hilbert-Schmidt norm of the block \( b \).

Definition 3.10. An infinite-dimensional symplectic group \( \text{Sp}(\infty) \) is the set of bounded operators \( A \) on \( H \) such that

1. \( A \) is invertible;
2. \( A = A \);
3. \( A \) preserves the form \( \omega \);
4. \( \| A \|_2 < \infty \).

Remark 3.11. If \( A \) is a bounded operator on \( H \), then \( A \) can be extended to a bounded operator on \( \mathbb{H}_\omega \). Therefore, we can equivalently define \( \text{Sp}(\infty) \) to be the set of operators \( A \in B(\mathbb{H}_\omega) \) such that

1. \( A \) is invertible;
2. \( A = A \);
3. \( A \) preserves the form \( \omega \);
4. \( \| A \|_2 < \infty \).
5. \( A \) is invariant on \( H \), i.e., \( A(H) \subseteq H \).

Remark 3.12. By Corollary 3.8, the definition of \( \text{Sp}(\infty) \) is also equivalent to
(1) \( A = \tilde{A} \);
(2) \( A^*A = A A^* = id \);
(3) \( \|A\|_2 < \infty \).

**Proposition 3.13.** \( Sp(\infty) \) is a group.

**Proof.** First we show that if \( A \in Sp(\infty) \), then \( A^{-1} \in Sp(\infty) \). By the assumption on \( A \), it is easy to verify that \( A^{-1} \) satisfies (1), (2), (3) and (5) in Remark 3.11. We need to show that \( A^{-1} \) satisfies the condition (4), i.e. \( \|A^{-1}\|_2 < \infty \). Suppose

\[
A = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} a' & b' \\ \bar{b'} & \bar{a'} \end{pmatrix},
\]

where by our assumptions all blocks are bounded operators, and in addition \( b \) is a Hilbert-Schmidt operator. We want to prove \( b' \) is also a Hilbert-Schmidt operator. \( AA^{-1} = I \) and \( A^{-1}A = I \) imply that

\[
ab' = -b\bar{a}, \quad a'b + \bar{b}b' = 1.
\]

The last equation gives \( a'ab' + b'\bar{b}b' = b' \), and so

\[
b' = d'ab' + b'\bar{b}b' = d'a\bar{a} + b'\bar{b}
\]

which is a Hilbert-Schmidt operator as \( b \) and \( \bar{b} \) are Hilbert-Schmidt. Therefore \( \|A^{-1}\|_2 < \infty \) and \( A^{-1} \in Sp(\infty) \).

Next we show that if \( A, B \in Sp(\infty) \), then \( AB \in Sp(\infty) \). By the assumption on \( A \) and \( B \), it is easy to verify that \( AB \) satisfies (1), (2), (3) and (5) in Remark 3.11. We need to show that \( AB \) satisfies the condition (4), i.e. \( \|AB\|_2 < \infty \). Suppose

\[
A = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix},
\]

where all blocks are bounded, and \( \|b\|_{HS}, \|d\|_{HS} < \infty \). Then

\[
AB = \begin{pmatrix} ac + bd & ad + bc \\ bc + \bar{a}d & bd + \bar{a}c \end{pmatrix}.
\]

Then

\[
\|AB\|_2^2 = \|ad + bc\|_{HS} \leq \|ad\|_2 + \|bc\|_{HS} < \infty,
\]

since both \( ad \) and \( bc \) are Hilbert-Schmidt operators. Therefore \( \|AB\|_2 < \infty \) and \( AB \in Sp(\infty) \). \( \square \)

### 4. Symplectic Representation of \( Diff(S^1) \)

**Definition 4.1.** Let \( Diff(S^1) \) be the group of orientation preserving \( C^\infty \) diffeomorphisms of \( S^1 \). \( Diff(S^1) \) acts on \( H \) as follows

\[
(\phi. u)(\theta) = u(\phi^{-1}(\theta)) - \frac{1}{2\pi} \int_0^{2\pi} u(\phi^{-1}(\theta)) d\theta.
\]

Note that if \( u \in H \) is real-valued, then \( \phi. u \) is real-valued as well.

**Proposition 4.2.** The action of \( Diff(S^1) \) on \( H \) gives a group homomorphism

\[
\Phi : Diff(S^1) \to Aut H
\]

defined by \( \Phi(\phi)(u) = \phi. u \), for \( \phi \in Diff(S^1) \) and \( u \in H \), where \( Aut H \) is the group of automorphisms on \( H \).

**Proof.** Let \( u \in H \), then \( \phi. u \) is a \( C^\infty \) function with the mean value 0, and so \( \phi. u \in H \). It is also clear that \( \phi. (u + v) = \phi. u + \phi. v \) and \( \phi. (\lambda u) = \lambda \phi. u \). So \( \Phi \) is well-defined as a map from \( Diff(S^1) \) to \( \text{End} H \), the space
of endomorphisms on $H$. Now let us check that $\Phi$ is a group homomorphism. Suppose $\phi, \psi \in \text{Diff}(S^1)$ and $u \in H$, then

$$\Phi(\phi \psi)(u)(\theta) = u((\phi \psi)^{-1}(\theta)) - \frac{1}{2\pi} \int_0^{2\pi} u((\phi \psi)^{-1}(\theta)) d\theta$$

$$= u((\psi^{-1} \phi^{-1})(\theta)) - \frac{1}{2\pi} \int_0^{2\pi} u((\psi^{-1} \phi^{-1})(\theta)) d\theta.$$  

On the other hand,

$$\Phi(\phi)\Phi(\psi)(u)(\theta) = \Phi(\phi) \left[ u(\psi^{-1}(\theta)) - \frac{1}{2\pi} \int_0^{2\pi} u(\psi^{-1}(\theta)) d\theta \right]$$

$$= \Phi(\phi) \left[ u(\psi^{-1}(\theta)) - \frac{1}{2\pi} \int_0^{2\pi} u(\psi^{-1}(\theta)) d\theta \right]$$

So $\Phi(\phi \psi) = \Phi(\phi) \Phi(\psi)$. In particular, the image of $\Phi$ is in the $\text{Aut}H$. □

**Lemma 4.3.** Any $\phi \in \text{Diff}(S^1)$ preserves the form $\omega$, that is, $\omega(\phi u, \phi v) = \omega(u, v)$ for any $u, v \in H$.

**Proof.** By Definition 4.1, $\phi u = u(\psi) - u_0, \phi v = v(\psi) - v_0$, where $\psi = \phi^{-1}$ and $u_0, v_0$ are the constants. Then

$$\omega(\phi u, \phi v) = \omega(u(\psi) - u_0, v(\psi) - v_0)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} [u(\psi(\theta)) - u_0][v(\psi(\theta)) - v_0]' d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(\psi)'(\psi) v(\psi)' d\theta - \frac{1}{2\pi} \int_0^{2\pi} u_0 v(\psi(\theta)) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(\psi)'(\psi) d\psi$$

$$= \omega(u, v).$$

□

We are going to prove that a diffeomorphism $\phi \in \text{Diff}(S^1)$ acts on $H$ as a bounded linear map, and that $\Phi(\phi)$ is in $\text{Sp}(\infty)$. The next lemma is a generalization of a proposition in a paper of G. Segal [12].

**Lemma 4.4.** Let $\psi \neq id \in \text{Diff}(S^1)$ and $\phi = \psi^{-1}$. Let

$$I_{n,m} = (\psi, e^{im\theta}, e^{in\theta}) = \frac{1}{2\pi} \int_0^{2\pi} e^{im\phi - in\theta} d\theta.$$  

Then

1. $\sum_{n>0,m<0} |n| |I_{n,m}|^2 < \infty$, and $\sum_{m>0,n<0} |n| |I_{n,m}|^2 < \infty$.  
2. For sufficiently large $|m|$ there is a constant $C$ independent of $m$ such that

$$\sum_{n \neq 0} |n| |I_{n,m}|^2 < C|m|. \quad (4.1)$$

**Proof.** Let

$$m_{\phi'} = \min\{\phi'(|\theta|) | \theta \in S^1\}; \quad \text{and} \quad M_{\phi'} = \max\{\phi'(|\theta|) | \theta \in S^1\}.$$  

Since $\phi$ is a diffeomorphism, we have $0 < m_{\phi'} < M_{\phi'} < \infty$.

Take four points $a, b, c, d$ on the unit circle such that $a$ corresponds to $m_{\phi'}$ in the sense $\tan(a) = m_{\phi'}$, $b$ corresponds to $M_{\phi'}$, $c$ is opposite to $a$, i.e., $c = a + \pi$, $d$ is opposite to $b$, i.e., $d = b + \pi$. The four points on the circle are arranged in the counter-clockwise order, and $0 < a < b < \phi$, $\pi < c < d < \frac{3}{2}\pi$. [Details of proof here.]

□
Let $\tau \in S^1$ such that $\tau \neq \frac{3\pi}{4}, \frac{5\pi}{4}$. Define a function $\phi_\tau$ on $S^1$ by
\[
\phi_\tau(\theta) = \frac{\cos \tau \cdot \phi(\theta) - \sin \tau \cdot \theta}{\cos \tau - \sin \tau}.
\]
We will show that if $\tau \in (b, c)$ or $\tau \in (d, a)$, then $\phi_\tau$ is an orientation preserving diffeomorphism of $S^1$, where $(b, c)$ is the open arc from the point $b$ to the point $c$, and $(d, a)$ is the open arc from the point $d$ to the point $a$.

Clearly $\phi_\tau$ is a $C^\infty$ function on $S^1$. Also, $\phi_\tau(0) = 0$ and $\phi_\tau(2\pi) = 2\pi$. Taking derivative with respect to $\theta$, we have
\[
\phi'_\tau(\theta) = \frac{\cos \tau \cdot \phi'(\theta) - \sin \tau \cdot \theta}{\cos \tau - \sin \tau}.
\]
By the choice of $\tau$, we can prove that $\phi'_\tau(\theta) > 0$. Therefore, $\phi_\tau$ is an orientation preserving diffeomorphism as claimed.

Let $m, n \in \mathbb{Z}\setminus\{0\}$. Let $\tau_{mn} = \text{Arg}(m + in)$, i.e., the argument of the complex number $m + in$, considered to be in $[0, 2\pi]$. Then we have $m\phi - n\theta = (m - n)\phi_{mn}$.

If $\tau_{mn} \in (b, c)$, then $\phi_{-mn}$ is a diffeomorphism. Let $\psi_{-mn} = \phi_{mn}^{-1}$. Then
\[
I_{n,m} = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\phi_{mn}} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} \psi_{mn}'(\theta) d\theta,
\]
where the last equality is by change of variable. On integration by parts $k$ times, we have
\[
I_{n,m} = \left(\frac{1}{i(m-n)}\right)^k \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} \psi_{mn}'^{(k+1)}(\theta) d\theta.
\]
Let $\alpha = [\alpha_0, \alpha_1]$ be a closed arc contained in the arc $(b, c)$. Let $S_\alpha$ be the set of all pairs of nonzero integers $(m, n)$ such that $\alpha_0 < \tau_{mn} < \alpha_1$, where $\tau_{mn} = \text{Arg}(m + in)$. We are going to consider an upper bound of the sum $\sum_{(m,n) \in S_\alpha} |n||I_{n,m}|^2$.

For the pair $(m, n)$, if $|m - n| = p$, the condition $\alpha_0 < \tau_{mn} < \alpha_1$ gives us both an upper bound and a lower bound for $n$:
\[
\frac{m\phi}{m\phi' - 1} p \leq n \leq \frac{M\phi'}{M\phi' - 1} p.
\]
So $|n| \leq C_1 p$ where $C_1$ is a constant which does not depend on the pair $(m, n)$. Also, the number of pairs $(m, n) \in S_\alpha$ such that $|m - n| = p$ is bounded by $C_2 p$ for some constant $C_2$. Let $C_3 = \max \{|\psi_\tau^{(k+1)}(\theta)| : \theta \in S^1, \tau \in [\alpha_0, \alpha_1]\}$. Then
\[
|I_{n,m}| \leq C_3 \left|\frac{1}{i(m-n)}\right|^k \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} d\theta = C_3 p^{-k}.
\]
Therefore,
\[
\sum_{(m,n) \in S} |n||I_{n,m}|^2 = \sum_p \sum_{(m,n) \in S_\alpha, |m-n| = p} |n||I_{n,m}|^2 \leq \sum_p C_1 p \cdot C_3^2 p^{-2k} \cdot C_2 p = C_\alpha \sum_p p^{-2k},
\]
where the constant $C_\alpha$ depends on the arc $\alpha$.

Similarly, for a closed arc $\beta = [\beta_0, \beta_1]$ contained in the arc $(d, a)$, we have
\[
\sum_{(m,n) \in S_\beta} |n||I_{n,m}|^2 \leq C_\beta \sum_p p^{-2k},
\]
where the constant $C_\beta$ depends on the arc $\beta$. 
Now let \( \alpha = \left[ \frac{3}{2}, \pi \right] \), and \( \beta = \left[ \frac{3}{2} \pi, 2\pi \right] \). Then \( \alpha \) is contained in \((b, c)\) and \( \beta \) is contained in \((d, a)\). We have

\[
\sum_{n > 0, m < 0} |n||l_{n,m}|^2 = C_\alpha \cdot \sum_p p^{-(2k-2)} < \infty
\]

and

\[
\sum_{n < 0, m > 0} |n||l_{n,m}|^2 = C_\beta \cdot \sum_p p^{-(2k-2)} < \infty,
\]

which proves (1) of the lemma.

To prove (2), we let \( \alpha = [\alpha_0, \alpha_1] \) be a closed arc contained in the arc \((b, c)\) such that \( b < \alpha_0 < \frac{3}{2} \pi \) and \( \pi < \alpha_1 < c \), and \( \beta = [\beta_0, \beta_1] \) be a closed arc contained in the arc \((d, a)\) such that \( d < \beta_0 < \frac{3}{2} \pi \) and \( 0 < \beta_1 < a \). Then we have

\[
\sum_{(m,n) \in S_\alpha} |n||l_{n,m}|^2 + \sum_{(m,n) \in S_\beta} |n||l_{n,m}|^2 \leq C_{\alpha \beta}
\]

for some constant \( C_{\alpha \beta} \).

Let \( m > 0 \) be sufficiently large, and \( N_m \) be the largest integer less than or equal to \( m \tan(\alpha_0) \),

\[
\sum_{0 < n \leq N_m} |l_{n,m}|^2 \leq \sum_{n \neq 0} |l_{n,m}|^2.
\]

Note that \( l_{n,m} \) is the \( n \)th Fourier coefficient of \( \psi e^{im\theta} \). Therefore,

\[
\sum_{n \neq 0} |l_{n,m}|^2 = \| \psi e^{im\theta} \|_{L^2}^2
\]

which is bounded by a constant \( K \). Therefore,

\[
\sum_{0 < n \leq N_m} |n||l_{n,m}|^2 \leq Km \tan(\alpha_0).
\]

On the other hand,

\[
\sum_{n < 0} |n||l_{n,m}|^2 + \sum_{n > N_m} |n||l_{n,m}|^2 \leq \sum_{(m,n) \in S_\alpha} |n||l_{n,m}|^2 + \sum_{(m,n) \in S_\beta} |n||l_{n,m}|^2 = C_{\alpha \beta}.
\]

Therefore,

\[
\sum_{n \neq 0} |n||l_{n,m}|^2 \leq C_{\alpha \beta} + K m \tan(\alpha_0) \leq m C_+,
\]

where \( C_+ \) can be chosen to be, for example, \( K \tan(\alpha_0) + C_{\alpha \beta} \), which is independent of \( m \).

Similarly, for \( m < 0 \) with sufficiently large \( |m| \)

\[
\sum_{n \neq 0} |n||l_{n,m}|^2 \leq m C_-.
\]

Let \( C = \max\{C_+, C_-\} \). Then we have, for sufficiently large \( |m| \),

\[
\sum_{n \neq 0} |n||l_{n,m}|^2 \leq |m| C,
\]

which proves (2) of the lemma. \( \square \)

**Lemma 4.5.** For any \( \psi \in \text{Diff}(S^1) \), \( \Phi(\psi) \in \mathcal{B}(H) \), the space of bounded linear maps on \( H \). Moreover,

\[
\|\Phi(\psi)\| \leq C, \quad \|\Phi(\psi)\|_2 \leq C,
\]

where \( C \) is the constant in Equation 4.1.
Proof. First observe that the operator norm of $\Phi(\psi)$ is

$$||\Phi(\psi)|| = \sup \{ ||\psi.u||_{\omega} \mid u \in H, ||u||_{\omega} = 1 \}.$$  

For any $u \in H$, let $\hat{u}$ be its Fourier coefficients, that is $\hat{u}(n) = (u, \hat{e}_n)$, and let $\tilde{u}$ be defined by $\tilde{u} = (u, \tilde{e}_n)_{\omega} (2.10, 2.12)$. It can be verified that the relation between $\hat{u}$ and $\tilde{u}$ is: if $n > 0$, then $\tilde{u}(n) = \sqrt{n} \hat{u}(n)$; if $n < 0$, then $\tilde{u}(n) = i \sqrt{|n|} \hat{u}(n)$. We have

$$||u||_{\omega}^2 = (u, u)_{\omega} = (\hat{u}, \hat{u})_{\ell^2} = \sum_{n \neq 0} |\hat{u}(n)|^2 = \sum_{n \neq 0} n ||\hat{u}(n)||^2.$$  

Let $\phi = \psi^{-1}$. We have $u(\phi) = \sum_{m \neq 0} \hat{u}(m) e^{im\phi}$. Using the notation $I_{n,m}$ (4.4), we have

$$||\psi.u||_{\omega}^2 = \sum_{n \neq 0} |n||\psi.u(n)||^2 = \sum_{n \neq 0} |n| \left| \frac{1}{2\pi} \int_0^{2\pi} u(\phi(\theta)) e^{-in\theta} d\theta \right|^2$$

$$= \sum_{n \neq 0} |n| \left| \sum_{m \neq 0} \hat{u}(m) \frac{1}{2\pi} \int_0^{2\pi} e^{im\phi} e^{-in\theta} d\theta \right|^2$$

$$= \sum_{n \neq 0} |n| \left| \sum_{m \neq 0} \hat{u}(m) I_{n,m} \right|^2$$

$$\leq \sum_{m,n \neq 0} |n||\hat{u}(m)||^2 |I_{n,m}|^2 = \sum_{m \neq 0} |\hat{u}(m)|^2 \sum_{n \neq 0} |n||I_{n,m}|^2$$

$$= \sum_{|m| < M_0} |\hat{u}(m)|^2 \sum_{n \neq 0} |n||I_{n,m}|^2 + \sum_{|m| > M_0} |\hat{u}(m)|^2 \sum_{n \neq 0} |n||I_{n,m}|^2,$$

where the constant $M_0$ in the last equality is a positive integer large enough so that we can apply part (2) of Lemma 4.4. It is easy to see that the first term in the last equality is finite. For the second term we use Lemma 4.4

$$\sum_{|m| > M_0} |\hat{u}(m)|^2 \sum_{n \neq 0} |n||I_{n,m}|^2 \leq C \sum_{|m| > M_0} |\hat{u}(m)|^2 |m| \leq C.$$  

Thus for any $u \in H$ with $||u||_{\omega} = 1$, $||\psi.u||_{\omega}$ is uniformly bounded. Therefore, $\Phi(\psi)$ is a bounded operator on $H$.

Now we can use Lemma 4.4 again to estimate the norm $||\Phi(\psi)||_2$

$$||\Phi(\psi)||_2 = \sum_{n > 0, m < 0} |((\psi, \hat{e}_m, \hat{e}_n)_{\omega}|^2 = \sum_{n > 0, m < 0} |\psi, \hat{e}_m, \hat{e}_n)_{\omega}|^2$$

$$= \sum_{n > 0, m < 0} |n||I_{n,m}|^2 < \infty.$$  

Theorem 4.6. $\Phi : Diff(S^1) \to Sp(\infty)$ is a group homomorphism. Moreover, $\Phi$ is injective, but not surjective.

Proof. Combining Lemma 4.3 and Lemma 4.5, we see that for any diffeomorphism $\psi \in Diff(S^1)$ the map $\Phi(\psi)$ is an invertible bounded operator on $H$, it preserves the form $\omega$, and $||\Phi(\psi)||_2 < \infty$. In addition, by our remark after Definition 4.1, $\psi.u$ is real-valued, if $u$ is real-valued. Therefore, $\Phi$ maps $Diff(S^1)$ into $Sp(\infty)$.

Next, we first prove that $\Phi$ is injective. Let $\psi_1, \psi_2 \in Diff(S^1)$, and denote $\phi_1 = \psi_1^{-1}, \phi_2 = \psi_2^{-1}$. Suppose $\Phi(\psi_1) = \Phi(\psi_2)$, i.e. $\psi_1.u = \psi_2.u$, for any $u \in H$. In particular, $\psi_1.e^{i\theta} = \psi_2.e^{i\theta}$. Therefore

$$e^{i\phi_1} - C_1 = e^{i\phi_2} - C_2,$$

where $C_1 = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi_1} d\theta$, and $C_2 = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi_2} d\theta$. Note that $e^{i\phi_1}$ and $e^{i\phi_2}$ have the same image as maps from $S^1$ to $C$. This implies $C_1 = C_2$, since otherwise $e^{i\phi_1} = e^{i\phi_2} + (C_1 - C_2)$ and $e^{i\phi_1}$ and $e^{i\phi_2}$ would have had
different images. Therefore, we have $e^{i\phi_1} = e^{i\phi_2}$. But the function $e^{i\tau} : S^1 \to S^1$ is an injective function, so $\phi_1 = \phi_2$. Therefore $\psi_1 = \psi_2$, and so $\Phi$ is injective.

To prove that $\Phi$ is not surjective, we will construct an operator $A \in \text{Sp}(\infty)$ which can not be written as $\Phi(\psi)$ for any $\psi \in \text{Diff}(S^1)$. Let the linear map $A$ be defined by the corresponding matrix $\{A_{m,n}\}_{m,n \in \mathbb{Z}}$ with the entries

\[
A_{1,1} = A_{-1,-1} = \sqrt{2} \\
A_{1,-1} = i, A_{-1,1} = -i \\
A_{m,m} = 1, \text{ for } m \neq \pm 1
\]

with all other entries being 0.

First we show that $A \in \text{Sp}(\infty)$. For any $u \in H$, we can write $u = \sum_{n \neq 0} \bar{u}(n) \tilde{e}_n$. Then $A$ acting on $u$ changes only $\tilde{e}_1$ and $\tilde{e}_{-1}$. Therefore, $Au \in H$, and clearly $A$ is a well–defined bounded linear map on $H$ to $H$. Moreover, $\|A\| < \infty$. It is clear that $A_{m,n} = \overline{A_{-m,-n}}$, and therefore $A = \overline{A}$ by Proposition 3.3. Moreover, $A$ preserves the form $\omega$ by part(II) of Proposition 3.6 as

\[
\sum_{k \neq 0} \text{sgn}(mk)A_{k,m} \overline{A_{k,n}} = \delta_{m,n}.
\]

Finally, $A$ is invertible, since $\{A_{k,m}\}_{m,n \in \mathbb{Z}}$ is, with the inverse $\{B_{k,m}\}_{m,n \in \mathbb{Z}}$ given by

\[
B_{1,1} = B_{-1,-1} = \sqrt{2} \\
B_{1,-1} = -i, B_{-1,1} = i \\
B_{m,m} = 1, \text{ for } m \neq \pm 1
\]

with all other entries being 0. Next we show that $A \neq \Phi(\psi)$ for any $\psi \in \text{Diff}(S^1)$. First observe that if we look at any basis element $\tilde{e}_1 = e^{i\theta}$ as a function from $S^1$ to $\mathbb{C}$, then the image of this function lies on the unit circle. Clearly, when acted by a diffeomorphism $\phi \in \text{Diff}(S^1)$, the image of the function $\phi \cdot e^{i\theta}$ is still a circle with radius 1. But if we consider $A \tilde{e}_1$ as a function from $S^1$ to $\mathbb{C}$, we will show that the image of the function $A \tilde{e}_1 : S^1 \to \mathbb{C}$ is not a circle. Therefore, $A \neq \Phi(\psi)$ for any $\psi \in \text{Diff}(S^1)$. Indeed, by definition of $A$ we have

\[
A \tilde{e}_1 = \sqrt{2} \tilde{e}_1 - i \tilde{e}_{-1}.
\]

Let us write it as a function on $S^1$

\[
A \tilde{e}_1(\theta) = \sqrt{2} e^{i\theta} - e^{-i\theta} = (\sqrt{2} - 1) \cos \theta + i(\sqrt{2} + 1) \sin \theta,
\]

and then we see that the image lies on an ellipse, which is not the unit circle

\[
\frac{x^2}{(\sqrt{2} - 1)^2} + \frac{y^2}{(\sqrt{2} + 1)^2} = 1.
\]

5. THE LIE ALGEBRA ASSOCIATED WITH $\text{DIFF}(S^1)$

Let $\text{diff}(S^1)$ be the space of smooth vector fields on $S^1$. Elements in $\text{diff}(S^1)$ can be identified with smooth functions on $S^1$. The space $\text{diff}(S^1)$ is a Lie algebra with the following Lie bracket

\[
[X,Y] = XY' - X'Y, \quad X, Y \in \text{diff}(S^1),
\]

where $X'$ and $Y'$ are derivatives with respect to $\theta$.

Let $X \in \text{diff}(S^1)$, and $\rho_t$ be the corresponding flow of diffeomorphisms. We define an action of $\text{diff}(S^1)$ on $H$ as follows: for $X \in \text{diff}(S^1)$ and $u \in H$, $X.u$ is a function on $S^1$ defined by

\[
(X.u)(\theta) = \frac{d}{dt} \bigg|_{t=0} \left[ (\rho_{t_\cdot}u)(\theta) \right],
\]

where $\rho_t$ acts on $u$ via the representation $\Phi : \text{Diff}(S^1) \to \text{Sp}(\infty)$. 
Indeed, recall that a basis element $\tilde{m}$ is contained in diff$_1$. By Proposition 5.1 and a simple verification depending on the signs of $m$, $n$ we see that

\[ X_l e^{in\theta} = -i ne^{in\theta} \cos(l\theta) = -\frac{1}{2} \ln \left[ e^{i(n+l)\theta} + e^{i(n-l)\theta} \right] \]

\[ Y_k e^{in\theta} = -i ne^{in\theta} \sin(k\theta) = -\frac{1}{2} n \left[ e^{i(n+k)\theta} - e^{i(n-k)\theta} \right] . \]

Indeed, recall that a basis element $\tilde{e}_n \in \mathcal{B}_\omega$ has the form

\[ \tilde{e}_n = \begin{cases} \frac{1}{\sqrt{n}} e^{i\theta} & n > 0 \\ \frac{1}{i\sqrt{|n|}} e^{i\theta} & n < 0. \end{cases} \]

Suppose $m, n > 0$

\[ X_l \tilde{e}_n = \frac{1}{\sqrt{n}} X_l e^{in\theta} = -\frac{1}{2} \sqrt{n} \left[ e^{i(n+l)\theta} + e^{i(n-l)\theta} \right]. \]
and
\[ \langle e^{j(n+1)\theta} \tilde{e}_m, \tilde{e}_m \rangle_\omega = \sqrt{m} \delta_{m-n,k}; \quad \langle e^{j(n-1)\theta} \tilde{e}_m, \tilde{e}_m \rangle_\omega = \sqrt{m} \delta_{n-m,l}. \]
Therefore,
\[ (X_t)_{m,n} = (X_t \tilde{e}_n, \tilde{e}_m)_{\omega} = (-i) \frac{1}{2} \sqrt{mn} (\delta_{m-n,l} + \delta_{n-m,l}). \]
All other cases can be verified similarly.

**Remark 5.4.** Recall that \( \mathbb{H}_\omega \) is the completion of \( H \) under the metric \( \langle \cdot, \cdot \rangle_\omega \). The above calculation shows that the trigonometric basis \( X_t, Y_k \) of \( \text{diff}(S^1) \) act on \( \mathbb{H}_\omega \) as unbounded operators. They are densely defined on the subspace \( H \subseteq \mathbb{H}_\omega \).

6. BROWNIAN MOTION ON \( \text{Sp}(\infty) \)

**Notation 6.1.** As in [1], let \( \text{sp}(\infty) \) be the set of infinite-dimensional matrices \( A \) which can be written as block matrices of the form
\[ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \]
such that \( a + a^\dagger = 0, b = b^T \), and \( b \) is a Hilbert-Schmidt operator.

**Remark 6.2.** The set \( \text{sp}(\infty) \) has a structure of Lie algebra with the operator commutator as a Lie bracket, and we associate this Lie algebra with the group \( \text{Sp}(\infty) \).

**Proposition 6.3.** Let \( \{A_{m,n}\}_{m,n \in \mathbb{Z} \setminus \{0\}} \) be the matrix corresponding to an operator \( A \). Then any \( A \in \text{sp}(\infty) \) satisfies (1) \( A_{m,n} = \overline{A_{-m,-n}} \); (2) \( A_{m,n} + \overline{A_{n,m}} = 0 \) for \( m, n > 0 \); (3) \( A_{m,n} = A_{-n,-m} \), for \( m > 0, n < 0 \).

Moreover, \( A \in \text{sp}(\infty) \) if and only if (1) \( A = A^\dagger \); (2) \( \pi^+ A \pi^- \) is Hilbert-Schmidt; (3) \( A + A^\# = 0 \).

**Proof.** The first part follows directly from definition of \( \text{sp}(\infty) \). Then we can use this fact and the formula for the matrix entries of \( A^\# \) in Proposition [3,3] to prove the second part.

**Definition 6.4.** Let \( HS \) be the space of Hilbert-Schmidt matrices viewed as a real vector space, and \( \text{sp}_{HS} = \text{sp}(\infty) \cap HS \).

The space \( HS \) as a real Hilbert space has an orthonormal basis
\[ \mathscr{B}_{HS} = \{e^{Re}_{mn} : m,n \neq 0\} \cup \{e^{Im}_{mn} : m,n \neq 0\}, \]
where \( e^{Re}_{mn} \) is a matrix with \( (m,n) \)-th entry 1 all other entries 0, and \( e^{Im}_{mn} \) is a matrix with \( (m,n) \)-th entry i all other entries 0.

The space \( \text{sp}_{HS} \) is a closed subspace of \( HS \), and therefore a real Hilbert space. According to the symmetry of the matrices in \( \text{sp}_{HS} \), we define a projection \( \pi : HS \to \text{sp}_{HS} \), such that
\[ \begin{align*}
\pi(e^{Re}_{mn}) &= \frac{1}{2} (e^{Re}_{mn} - e^{Re}_{-m,-n} - e^{Re}_{-n,-m}), \quad \text{if} \quad \text{sgn}(mn) > 0, \\
\pi(e^{Im}_{mn}) &= \frac{1}{2} (e^{Im}_{mn} + e^{Im}_{-m,-n} - e^{Im}_{-n,-m}), \quad \text{if} \quad \text{sgn}(mn) > 0, \\
\pi(e^{Re}_{mn}) &= \frac{1}{2} (e^{Re}_{mn} + e^{Re}_{-m,-n} + e^{Re}_{-n,-m}), \quad \text{if} \quad \text{sgn}(mn) < 0, \\
\pi(e^{Im}_{mn}) &= \frac{1}{2} (e^{Im}_{mn} + e^{Im}_{-m,-n} - e^{Im}_{-n,-m}), \quad \text{if} \quad \text{sgn}(mn) < 0.
\end{align*} \]

**Notation 6.5.** We choose \( \mathscr{B}_{sp}_{HS} = \pi(\mathscr{B}_{HS}) \) to be the orthonormal basis of \( \text{sp}_{HS} \).

Clearly, if \( A \in \text{sp}_{HS} \), then \( |A|_{\text{sp}_{HS}} = |A|_{HS} \).

**Definition 6.6.** Let \( W_t \) be a Brownian motion on \( \text{sp}_{HS} \) which has the mean zero and covariance \( Q \), where \( Q \) is assumed to be a positive symmetric trace class operator on \( H \). We further assume that \( Q \) is diagonal in the basis \( \mathscr{B}_{sp}_{HS} \).
Remark 6.7. \( Q \) can also be viewed as a positive function on the set \( \mathcal{B}_{\mathfrak{sp}_{\text{HS}}} \), and the Brownian motion \( W_t \) can be written as
\[
W_t = \sum_{\xi \in \mathcal{B}_{\mathfrak{sp}_{\text{HS}}}} \sqrt{Q(\xi)} B_t^\xi \xi,
\]
where \( \{B_t^\xi\}_{\xi \in \mathcal{B}_{\mathfrak{sp}_{\text{HS}}}} \) are standard real-valued mutually independent Brownian motions.

Our goal now is to construct a Brownian motion on the group \( \text{Sp}(\infty) \) using the Brownian motion \( W_t \) on \( \mathfrak{sp}_{\text{HS}} \). This is done by solving the Stratonovich stochastic differential equation
\[
\delta X_t = X_t \delta W_t.
\]
This equation can be written as the following Itô stochastic differential equation
\[
dX_t = X_t dW_t + \frac{1}{2} X_t dt,
\]
where \( D = \text{Diag}(D_m) \) is a diagonal matrix with entries
\[
D_m = -\frac{1}{4} \text{sgn}(m) \sum_k \text{sgn}(k) [Q^R_{mk} + Q^I_{mk}]
\]
with \( Q^R_{mk} = Q(\pi(e^{Re})) \) and \( Q^I_{mk} = Q(\pi(e^{Im})) \).

Notation 6.8. Denote by \( \mathfrak{sp}_{\text{HS}}^Q = Q^{1/2}(\mathfrak{sp}_{\text{HS}}) \) which is a subspace of \( \mathfrak{sp}_{\text{HS}} \). Define an inner product on \( \mathfrak{sp}_{\text{HS}}^Q \) by \( \langle u, v \rangle_{\mathfrak{sp}_{\text{HS}}^Q} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_{\mathfrak{sp}_{\text{HS}}} \). Then \( \mathcal{B}_{\mathfrak{sp}_{\text{HS}}^Q} = \{\xi = Q^{1/2}\xi : \xi \in \mathcal{B}_{\mathfrak{sp}_{\text{HS}}} \} \) is an orthonormal basis of the Hilbert space \( \mathfrak{sp}_{\text{HS}}^Q \).

Notation 6.9. Let \( L_2^0 \) be the space of Hilbert-Schmidt operators from \( \mathfrak{sp}_{\text{HS}}^Q \) to \( \mathfrak{sp}_{\text{HS}} \) with the norm
\[
|\Phi|_{L_2^0}^2 = \sum_{\xi \in \mathcal{B}_{\mathfrak{sp}_{\text{HS}}^Q}} |\Phi_\xi|_{\mathfrak{sp}_{\text{HS}}}^2 = \sum_{\xi, \xi' \in \mathcal{B}_{\mathfrak{sp}_{\text{HS}}}} Q(\xi) |\langle \Phi_\xi, \xi' \rangle_{\mathfrak{sp}_{\text{HS}}}|^2 = \text{Tr}[\Phi Q \Phi^*],
\]
where \( Q(\xi) \) means \( Q \) evaluated at \( \xi \) as a positive function on \( \mathcal{B}_{\mathfrak{sp}_{\text{HS}}} \).

Lemma 6.10. If \( \Psi \in L(\mathfrak{sp}_{\text{HS}}, \mathfrak{sp}_{\text{HS}}^Q) \), a bounded linear operator from \( \mathfrak{sp}_{\text{HS}} \) to \( \mathfrak{sp}_{\text{HS}}^Q \), then \( \Psi \) restricted on \( \mathfrak{sp}_{\text{HS}}^Q \) is a Hilbert-Schmidt operator from \( \mathfrak{sp}_{\text{HS}}^Q \) to \( \mathfrak{sp}_{\text{HS}} \) and \( |\Psi|_{L_2^0} \leq \text{Tr}(Q)|\Psi|^2 \), where \( ||\Psi|| \) is the operator norm of \( \Psi \).

Proof: \[
|\Psi|_{L_2^0}^2 = \sum_{\xi \in \mathfrak{sp}_{\text{HS}}^Q} |\Psi_\xi|_{\mathfrak{sp}_{\text{HS}}}^2 \leq ||\Psi||^2 \sum_{\xi \in \mathfrak{sp}_{\text{HS}}^Q} |\xi|_{\mathfrak{sp}_{\text{HS}}}^2
= ||\Psi||^2 \sum_{\xi \in \mathfrak{sp}_{\text{HS}}} |Q^{1/2}\xi, Q^{1/2}\xi\rangle_{\mathfrak{sp}_{\text{HS}}} = ||\Psi||^2 \sum_{\xi \in \mathcal{B}_{\mathfrak{sp}_{\text{HS}}}} |\langle Q^{1/2}\xi, \xi \rangle_{\mathfrak{sp}_{\text{HS}}} = ||\Psi||^2 \text{Tr}(Q)\]
\]
\[\square\]

Notation 6.11. Define \( B : \mathfrak{sp}_{\text{HS}} \to L_2^0 \) by \( B(Y)A = (I + Y)A \) for \( A \in \mathfrak{sp}_{\text{HS}}^Q \), and \( F : \mathfrak{sp}_{\text{HS}} \to \mathfrak{sp}_{\text{HS}} \) by \( F(Y) = \frac{1}{2}(I + Y)D \).

Note that \( B \) is well–defined by Lemma 6.10. Also \( D \in \mathfrak{sp}_{\text{HS}} \), and so \( F(Y) \in \mathfrak{sp}_{\text{HS}} \) and \( F \) is well–defined as well.

Theorem 6.12. The stochastic differential equation
\[
dY_t = B(Y_t)dW_t + F(Y_t)dt
\]
\[Y_0 = 0\]
\[(6.5)\]
has a unique solution, up to equivalence, among the processes satisfying
\[ P \left( \int_0^T |Y_t|^2_{\text{SpHS}} \, ds < \infty \right) = 1. \]

Proof. To prove this theorem we will use Theorem 7.4 from the book by G. DaPrato and J. Zabczyk \[^3\] as it has been done in \[^6\][^7\]. It is enough to check

1. \( B \) is a measurable mapping,
2. \( |B(Y_1) - B(Y_2)|_{L^2_0} \leq C_1|Y_1 - Y_2|_{\text{SpHS}} \) for \( Y_1, Y_2 \in \text{SpHS} \);
3. \( |B(Y)|^2_{L^2_0} \leq K_1(1 + |Y|^2_{\text{SpHS}}) \) for any \( Y \in \text{SpHS} \);
4. \( F \) is a measurable mapping,
5. \( |F(Y_1) - F(Y_2)|_{\text{SpHS}} \leq C_2|Y_1 - Y_2|_{\text{SpHS}} \) for \( Y_1, Y_2 \in \text{SpHS} \);
6. \( |F(Y)|^2_{\text{SpHS}} \leq K_2(1 + |Y|^2_{\text{SpHS}}) \) for any \( Y \in \text{SpHS} \).

Proof of 1. By the proof of 2, \( B \) is a continuous mapping, therefore it is measurable.

Proof of 2.
\[
|B(Y_1) - B(Y_2)|_{L^2_0}^2 = \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} |(Y_1 - Y_2)\xi|^2_{\text{SpHS}} = \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} Q(\xi)|Y_1 - Y_2\xi|^2_{\text{SpHS}} \\
\leq \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} Q(\xi)\|\xi\|^2|Y_1 - Y_2|^2_{\text{SpHS}} \leq \max_{\xi \in \mathcal{R}_Q \text{SpHS}} \|\xi\|^2 \left( \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} Q(\xi) \right) |Y_1 - Y_2|^2_{\text{SpHS}} \\
= \text{Tr}Q \left( \max_{\xi \in \mathcal{R}_Q \text{SpHS}} \|\xi\|^2 \right)|Y_1 - Y_2|^2_{\text{SpHS}} = C_1^2|Y_1 - Y_2|^2_{\text{SpHS}},
\]
where \( \|\xi\| \) is the operator norm of \( \xi \), which is uniformly bounded for all \( \xi \in \mathcal{R}_Q \text{SpHS} \).

Proof of 3.
\[
|B(Y_1)|^2_{L^2_0} = \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} |(I + Y)\xi|^2_{\text{SpHS}} = \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} Q(\xi)|I + Y\xi|^2_{\text{SpHS}} \\
\leq |(I + Y)\xi|^2_{\text{SpHS}} \sum_{\xi \in \mathcal{R}_Q \text{SpHS}} Q(\xi)\|\xi\|^2 \leq (1 + |Y|^2_{\text{SpHS}}) \cdot K_1.
\]

Proof of 4. By the proof of 5, \( F \) is a continuous mapping, therefore it is measurable.

Proof of 5.
\[
|F(Y_1) - F(Y_2)|_{\text{SpHS}} = \left| \frac{1}{2}(Y_1 - Y_2)D \right|_{\text{SpHS}} \leq \left| \frac{1}{2}D \right| |Y_1 - Y_2|_{\text{SpHS}}
\]

Proof of 6.
\[
|F(Y)|^2_{\text{SpHS}} = \left| \frac{1}{2}(I + Y)D \right|_{\text{SpHS}}^2 \leq \left| \frac{1}{2}D \right|^2 |I + Y|^2_{\text{SpHS}} \leq K_2(1 + |Y|^2_{\text{SpHS}}).
\]

Notation 6.13. Let \( B^\# : \text{SpHS} \to L^2_0 \) be the operator \( B^\#(Y)A = A^\#(I + Y) \), and \( F^\# : \text{SpHS} \to \text{SpHS} \) be the operator \( F^\#(Y) = \frac{1}{2}D^\#(Y + I) \).

Proposition 6.14. If \( Y_t \) is the solution to the stochastic differential equation
\[
dx_t = B(X_t)\,dW_t + F(X_t)\,dt \\
X_0 = 0,
\]
where \( B \) and \( F \) are defined in Notation \[^6\][^11\], then \( Y_t^\# \) is the solution to the stochastic differential equation
\[
dx_t = B^\#(X_t)\,dW_t + F^\#(X_t)\,dt \\
X_0 = 0,
\] (6.6)
where $B^\#$ and $F^\#$ are defined in Notation 6.13.

**Proof.** This follows directly from the property $(AB)^\# = B^\# A^\#$ for any $A$ and $B$, which can be verified by using part (5) of Proposition 5.3.

**Lemma 6.15.** Let $U$ and $H$ be real Hilbert spaces. Let $\Phi : U \to H$ be a bounded linear map. Let $G : H \to H$ be a bounded linear map. Then

$$Tr_H(G\Phi \Phi^*) = Tr_U(\Phi^* G \Phi)$$

**Proof.**

$$Tr_H(G\Phi \Phi^*) = \sum_{i,j \in H,k \in U} G_{ij} \Phi_j(\Phi^*)_{ki} = \sum_{i,j \in H,k \in U} G_{ij} \Phi_j \Phi_k$$

$$Tr_U(\Phi^* G \Phi) = \sum_{i,j \in H,k \in U} (\Phi^*)_{ki} G_{ij} \Phi_j \Phi_k.$$ 

Therefore $Tr_H(G\Phi \Phi^*) = Tr_U(\Phi^* G \Phi)$.

**Lemma 6.16.**

$$\sum_{\xi \in \mathcal{B}_{spHS}} (Q^{1/2} \xi)(Q^{1/2} \xi)^\# = -D$$

**Proof.** If $\xi \in \mathcal{B}_{spHS}$, then $\xi \in sp(\infty)$, so $\xi^\# = -\xi$. We will use the fact that

$$(e^{Re}_{ik}(\xi^*)_{pq})_{ij} = \delta_{ip}\delta_{jk}\delta_{pq}$$

where $e^{Re}_{ij}$ is the matrix with the $(i, j)$th entry being 1 and all other entries being zero. Using this fact, we see

1. For $\xi = \frac{1}{2}(e^{Re}_{mn} - e^{Re}_{nn} + e^{Re}_{m,-n} - e^{Re}_{n,-m})$ with $\text{sgn}(mn) > 0$,

$$\langle Q^{1/2} \xi \rangle \langle Q^{1/2} \xi \rangle^\# = -\frac{1}{4} Q^{Re}_{nn} [-e^{Re}_{mn} - e^{Re}_{nn} - e^{Re}_{m,-n} - e^{Re}_{n,-m}]$$

2. For $\xi = \frac{1}{2}(e^{Im}_{mn} + e^{Im}_{nn} - e^{Im}_{m,-n} - e^{Im}_{n,-m})$ with $\text{sgn}(mn) > 0$,

$$\langle Q^{1/2} \xi \rangle \langle Q^{1/2} \xi \rangle^\# = -\frac{1}{4} Q^{Im}_{nn} [-e^{Im}_{mn} - e^{Im}_{nn} - e^{Im}_{m,-n} - e^{Im}_{n,-m}]$$

3. For $\xi = \frac{1}{2}(e^{Re}_{mn} - e^{Re}_{n,-m} + e^{Re}_{m,-n} + e^{Re}_{n,m})$ with $\text{sgn}(mn) < 0$,

$$\langle Q^{1/2} \xi \rangle \langle Q^{1/2} \xi \rangle^\# = -\frac{1}{4} Q^{Re}_{mn} [e^{Re}_{mn} + e^{Re}_{nn} + e^{Re}_{m,-n} + e^{Re}_{n,-m}]$$

4. For $\xi = \frac{1}{2}(e^{Im}_{mn} + e^{Im}_{nn} - e^{Im}_{m,-n} - e^{Im}_{n,-m})$ with $\text{sgn}(mn) < 0$,

$$\langle Q^{1/2} \xi \rangle \langle Q^{1/2} \xi \rangle^\# = -\frac{1}{4} Q^{Im}_{mn} [e^{Im}_{mn} + e^{Im}_{nn} + e^{Im}_{m,-n} + e^{Im}_{n,-m}]$$

Each of the above is a diagonal matrix. The lemma can be proved by looking at the diagonal entries of the sum.

**Theorem 6.17.** Let $Y_t$ be the solution to Equation (6.5). Then $Y_t + I \in \text{Sp}(\infty)$ for any $I > 0$ with probability 1.

**Proof.** The proof is adapted from papers by M. Gordina [2, 7]. Let $Y_t$ be the solution to Equation (6.5) and $Y_t^\#$ be the solution to Equation (6.6). Consider the process $Y_t = (Y_t, Y_t^\#)$ in the product space $\text{spHS} \times \text{spHS}$. It satisfies the following stochastic differential equation

$$dY_t = (B(Y_t), B^\#(Y_t^\#))dW + (F(Y_t), F^\#(Y_t^\#))dt.$$ 

Let $G$ be a function on the Hilbert space $\text{spHS} \times \text{spHS}$ defined by $G(Y_1, Y_2) = \Lambda((Y_1 + I)(Y_2 + I))$, where $\Lambda$ is a nonzero linear real bounded functional from $\text{spHS} \times \text{spHS}$ to $\mathbb{R}$. We will apply Itô’s formula to $G(Y_t) = G(Y_t, Y_t^\#)$. Then $(Y_t + I)(Y_t^\# + I) = I$ if and only if $\Lambda((Y_t + I)(Y_t^\# + I) - I) = 0$ for any $\Lambda$. 


In order to use Itô’s formula we must verify that $G$ and the derivatives $G_t, G_Y, G_{YY}$ are uniformly continuous on bounded subsets of $[0, T] \times sp_{HS} \times sp_{HS}$, where $G_Y$ is defined as follows

$$G_Y(Y)(S) = \lim_{\epsilon \to 0} \frac{G(Y + \epsilon S) - G(Y)}{\epsilon}$$

for any $Y, S \in sp_{HS} \times sp_{HS}$ and $G_{YY}$ is defined as follows

$$G_{YY}(Y)(S \otimes T) = \lim_{\epsilon \to 0} \frac{G(Y + \epsilon T)(S) - G(Y)(S)}{\epsilon}$$

for any $Y, S, T \in sp_{HS} \times sp_{HS}$. Let us calculate $G_t, G_Y, G_{YY}$. Clearly, $G_t = 0$. It is easy to verify that for any $S = (S_1, S_2) \in sp_{HS} \times sp_{HS}$

$$G_Y(Y)(S) = \Lambda(S_1 (Y_2 + I) + (Y_1 + I)S_2)$$

and for any $S = (S_1, S_2) \in sp_{HS} \times sp_{HS}$ and $T = (T_1, T_2) \in sp_{HS} \times sp_{HS}$

$$G_{YY}(Y)(S \otimes T) = \Lambda(S_1 T_2 + T_1 S_2).$$

So the condition is satisfied.

We will use the following notation

$$G_Y(Y)(S) = \langle \tilde{G}_Y(Y), S \rangle_{sp_{HS} \times sp_{HS}}$$

$$G_{YY}(Y)(S \otimes T) = \langle \tilde{G}_{YY}(Y)S, T \rangle_{sp_{HS} \times sp_{HS}},$$

where $\tilde{G}_Y(Y)$ is an element of $sp_{HS} \times sp_{HS}$ corresponding to the functional $G_Y(Y)$ in $(sp_{HS} \times sp_{HS})^*$ and $\tilde{G}_{YY}(Y)$ is an operator on $sp_{HS} \times sp_{HS}$ corresponding to the functional $G_{YY}(Y) \in ((sp_{HS} \times sp_{HS}) \otimes (sp_{HS} \times sp_{HS}))^*$.

Now we can apply Itô’s formula to $G(Y_t)$

$$G(Y_t) - G(Y_0) = \int_0^t \langle \tilde{G}_Y(Y_s), (B(Y_s)dW_s, B^*(Y_s^#)dW_s) \rangle_{sp_{HS} \times sp_{HS}} ds$$

$$\quad + \int_0^t \frac{1}{2} \text{Tr}_{sp_{HS} \times sp_{HS}} \left[ \tilde{G}_{YY}(Y_s) \left( B(Y_s)Q^{1/2}, B^*(Y_s^#)Q^{1/2} \right) \right] ds.$$

Let us calculate the three integrands separately. The first integrand is

$$\langle \tilde{G}_Y(Y_s), (B(Y_s)dW_s, B^*(Y_s^#)dW_s) \rangle_{sp_{HS} \times sp_{HS}}$$

$$= (B(Y_s)dW_s)(Y_s^# + I) + (Y_s + I)(B^*(Y_s^#)dW_s)$$

$$= (Y_s + I)dW_s(Y_s^# + I) + (Y_s + I)dW_s(Y_s^# + I) = 0.$$

The second integrand is

$$\langle \tilde{G}_Y(Y_s), (F(Y_s), F^*(Y_s^#)) \rangle_{sp_{HS} \times sp_{HS}}$$

$$= F(Y_s)(Y_s^# + I) + (Y_s + I)F^*(Y_s^#)$$

$$= \frac{1}{2}(Y_s + I)D(Y_s^# + I) + \frac{1}{2}(Y_s + I)D^*(Y_s^# + I)$$

$$= \frac{1}{2}(Y_s + I)(D + D^*)(Y_s^# + I)$$

$$= (Y_s + I)D(Y_s^# + I),$$

where we have used the fact that $D = D^*$, since $D$ is a diagonal matrix with all real entries.
The third integrand is
\[
\frac{1}{2} \operatorname{Tr}_{\mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}} \left[ \bar{G}_{YY}(Y_s) \left( B(Y_s)Q^{1/2}, B^\#(Y_s^\#)Q^{1/2} \right) \left( B(Y_s)Q^{1/2}, B^\#(Y_s^\#)Q^{1/2} \right)^* \right]
\]
\[
= \frac{1}{2} \operatorname{Tr}_{\mathfrak{sp}_{\text{HS}}} \left[ \left( B(Y_s)Q^{1/2}, B^\#(Y_s^\#)Q^{1/2} \right)^* \bar{G}_{YY}(Y_s) \left( B(Y_s)Q^{1/2}, B^\#(Y_s^\#)Q^{1/2} \right) \right]
\]
\[
= \frac{1}{2} \sum_{\xi \in \mathfrak{sp}_{\text{HS}}} G_{YY}(Y_s) \left( \left( B(Y_s)Q^{1/2} \xi, B^\#(Y_s^\#)Q^{1/2} \xi \right) \otimes \left( B(Y_s)Q^{1/2} \xi, B^\#(Y_s^\#)Q^{1/2} \xi \right) \right)
\]
\[
= \sum_{\xi \in \mathfrak{sp}_{\text{HS}}} \left( B(Y_s)Q^{1/2} \xi \right) \left( B^\#(Y_s^\#)Q^{1/2} \xi \right)
\]
\[
= \sum_{\xi \in \mathfrak{sp}_{\text{HS}}} (Y + I) \left( Q^{1/2} \xi \right) \left( Q^{1/2} \xi \right)^\# (Y + I)
\]
\[
= -(Y + I)D(Y + I),
\]
where the second equality follows from Lemma 6.15 and the last equality follows from Lemma 6.16.

The above calculations show that the stochastic differential of \(G\) is zero. So \(G(Y_t) = G(Y_0) = \Lambda(I)\) for any \(t > 0\) and any nonzero linear real bounded functional \(\Lambda\) on \(\mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}\). This means \(\langle Y_t + I \rangle (Y_t + I) = I\) almost surely for any \(t > 0\). Similarly we can show \(\langle Y_t^+ + I \rangle (Y_t^+ + I) = I\) almost surely for any \(t > 0\). Therefore \(Y_t + I, Y_t^+ + I \in \text{Sp}(\infty)\) almost surely for any \(t > 0\). \(\square\)

7. Ricci Curvature of \(\text{Sp}(\infty)\)

In [8, 9], Gordina computed the Ricci curvatures of the quotient space \(\text{Diff}(S^1)/S^1\) and several Hilbert-Schmidt groups. Using this method, I computed the Ricci curvatures of the symplectic group \(\text{Sp}(\infty)\). The result shows that for most of the directions the Ricci curvatures of \(\text{Sp}(\infty)\) are not bounded from below.

Let \(G\) be a finite dimensional Lie group, and \(\mathfrak{g}\) its Lie algebra. Let \(\langle \cdot, \cdot \rangle_\mathfrak{g}\) be an inner product on \(\mathfrak{g}\). Then \(\langle \cdot, \cdot \rangle_\mathfrak{g}\) defines a unique left-invariant metric on the Lie group \(G\) compactible with the Lie group structure. In [10], J. Milnor studied the Riemannian geometry of Lie groups. For \(x, y, z \in \mathfrak{g}\), the Levi-Civita connection \(\nabla_x\) is given by
\[
\langle \nabla_x y, z \rangle_\mathfrak{g} = \frac{1}{2} \left( \langle [x, y], z \rangle_\mathfrak{g} - \langle y, [x, z] \rangle_\mathfrak{g} + \langle [z, x], y \rangle_\mathfrak{g} \right) \quad (7.1)
\]
The Riemann curvature tensor \(R_{xy}\) is given by
\[
R_{xy} = \nabla_x \nabla_y - \nabla_y \nabla_x + \nabla_{[x, y]} \quad (7.2)
\]
For any orthogonal \(x, y \in \mathfrak{g}\), the sectional curvature \(K(x, y)\) is given by
\[
K(x, y) = \langle R_{xy}(x), y \rangle_\mathfrak{g} \quad (7.3)
\]
Let us choose an orthonormal basis \(\{\xi_i\}_{i=1}^N\) of \(\mathfrak{g}\), where \(N\) is the dimension of the Lie group \(G\). Let \(x \in \mathfrak{g}\). Then the Ricci curvature \(\text{Ric}(x)\) is given by
\[
\text{Ric}(x) = \sum_{i=1}^N K(x, \xi_i) = \sum_{i=1}^N \langle R_{x\xi_i}(x), \xi_i \rangle_\mathfrak{g} \quad (7.4)
\]
The group \(\text{Sp}(\infty)\) and its Lie algebra \(\mathfrak{sp}(\infty)\) are defined in Section 3 and Section 6. Basically, elements in both the Lie group \(\text{Sp}(\infty)\) and the Lie algebra \(\mathfrak{sp}(\infty)\) are block matrices of the form:
\[
\begin{pmatrix}
a & b \\
b & \bar{a}
\end{pmatrix}
\]
where each of the blocks is an infinite-dimensional matrix. The blocks $a$ and $\bar{a}$ are complex conjugate with each other. The blocks $b$ and $\bar{b}$ are also complex conjugate with each other, are required to be a Hilbert-Schmidt matrices. For a block matrix to be an element of $\text{Sp}(\infty)$, it is also required that the matrix is invertible, and preserve a certain symplectic form. For a block matrix to be an element of $\mathfrak{sp}(\infty)$, it is required that $a + a^\dagger = 0$ or $a^T + \bar{a} = 0$, which means the block $a$ is conjugate skew-symmetric, and $b = b^T$, which means the block $b$ is symmetric.

To write the block matrix explicitly, we index the matrix by $\{A_{mn}\}_{m,n \in \mathbb{Z}\{0\}}$. An entry in block $a$ has $m,n > 0$; an entry in block $\bar{a}$ has $m,n < 0$; an entry in block $b$ has $m > 0, n < 0$; an entry in block $\bar{b}$ has $m < 0, n > 0$. The condition that blocks $a$ and $b$ are conjugate to blocks $\bar{a}$ and $\bar{b}$ can be expressed as $A_{m,n} = \overline{A_{-m,-n}}$. The condition $a + a^\dagger = 0$ or $a^T + \bar{a} = 0$ can be expressed as $A_{mn} + \overline{A_{mn}} = 0$ where $m,n > 0$ or $m,n < 0$. The condition $b = b^T$ can be expressed as $A_{m,n} = A_{-n,-m}$ where $m > 0, n < 0$.

To find Ricci curvature, we need to choose a metric for the Lie algebra $\mathfrak{sp}(\infty)$. Let us define a sequence of positive numbers

$$\{\lambda_i \in \mathbb{R}^+ | \lambda_i = \lambda_{-i}, i \in \mathbb{Z}\{0\}\}$$

The sequence $\{\lambda_i\}$ will serve as parameters to fine tune the metric that we are going to choose.

Remark 7.1. Let us first consider the space $HS$ of Hilbert-Schmidt matrices. The Hilbert space $HS$, if viewed as a complex Hilbert space, has a canonical inner product given by:

$$\langle A, B \rangle = \text{Tr}(AB^\dagger) = \text{Tr}(AB^T), \quad A, B \in HS$$

If viewed as real Hilbert space, $HS$ has a canonical inner product given by: for $A, B \in HS$, writing $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_1, A_2, B_1, B_2$ are matrices with real value entries, then

$$\langle A, B \rangle = \text{Tr}(A_1B_1^\dagger) + \text{Tr}(A_2B_2^\dagger)$$

Let $e_{ab}$ be the infinite-dimensional matrix with 1 in the entry $(a,b)$, and 0 in all other entries, where $a, b$ are indices of the matrix such that $a, b \in \mathbb{Z}\{0\}$. Then the above canonical inner product on $HS$ viewed as real Hilbert space is equivalent to choosing the set

$$\{e_{ab}, ie_{ab} | a, b \in \mathbb{Z}\{0\}\}$$

as an orthonormal basis.

Definition 7.2. Let

$$\xi_{ab} = 2\lambda_a\lambda_b e_{ab} \quad (7.5)$$

We define an inner product $\langle \cdot, \cdot \rangle_{HS}$ on $HS$ by choosing the following set

$$\{\xi_{ab}, i\xi_{ab} | a, b \in \mathbb{Z}\{0\}\} \quad (7.6)$$

as an orthonormal basis for the real Hilbert space $HS$.

Remark 7.3. If we set the parameter $\lambda_i = 1/\sqrt{2}$ for all $i \in \mathbb{Z}\{0\}$ in Definition 7.2 we can recover the canonical inner product of $HS$ (remark 7.1) as a real Hilbert space.

The Lie algebra $\mathfrak{sp}(\infty)$ may contain unbounded operators. For simplicity, we consider the subspace $\mathfrak{sp}_{HS} = \mathfrak{sp}(\infty) \cap HS$. Now we can choose orthonormal set of the space $\mathfrak{sp}_{HS}$ according to the symmetry of matrices in the Lie algebra $\mathfrak{sp}(\infty)$.

Definition 7.4. Let

$$\mu^{Re}_{ab} = \lambda_a \lambda_b (e_{a,b} - e_{-a,-b} - e_{-b,a}) \quad a > b > 0$$

$$\mu^{Im}_{ab} = \lambda_a \lambda_b (ie_{a,b} - ie_{-a,-b} - ie_{-b,a}) \quad a \geq b > 0$$

$$\nu^{Re}_{ab} = \lambda_a \lambda_b (e_{a,b} + e_{-a,-b} + e_{-b,a}) \quad a \geq -b > 0$$

$$\nu^{Im}_{ab} = \lambda_a \lambda_b (ie_{a,b} + ie_{-b,a} - ie_{-a,-b} - ie_{b,a}) \quad a \geq -b > 0$$
Let \( A^{Re} = \{ \mu_{ab}^{Re} | a > b > 0 \} \), \( A^{Im} = \{ \mu_{ab}^{Im} | a \geq b > 0 \} \), \( B^{Re} = \{ v_{ab}^{Re} | a \geq -b > 0 \} \), \( B^{Im} = \{ v_{ab}^{Im} | a \geq -b > 0 \} \), and \( \mathcal{B}_\lambda = A^{Re} \cup A^{Im} \cup B^{Re} \cup B^{Im} \).

**Remark 7.5.** It is easy to verify that matrices in the set \( \mathcal{B}_\lambda \) all belong to the space \( \mathfrak{sp}_{HS} \). So \( \mathcal{B}_\lambda \) is a subset of \( \mathfrak{sp}_{HS} \) and \( \mathfrak{sp}(\infty) \). Also, by definition of \( \xi_{ab} \) (equation [7.5]), it is easy to verify

\[
\begin{align*}
\mu_{ab}^{Re} &= \frac{1}{2}(\xi_{a,b} - \xi_{b,a} + \xi_{a,-b} - \xi_{-b,a}) \\
\mu_{ab}^{Im} &= \frac{1}{2}(i\xi_{a,b} - i\xi_{b,a} + i\xi_{a,-b} - i\xi_{-b,a}) \\
v_{ab}^{Re} &= \frac{1}{2}(\xi_{a,b} + \xi_{b,a} + \xi_{a,-b} + \xi_{-b,a}) \\
v_{ab}^{Im} &= \frac{1}{2}(i\xi_{a,b} + i\xi_{b,a} - i\xi_{a,-b} - i\xi_{-b,a})
\end{align*}
\]

**Definition 7.6.** We define an inner product \( \langle \cdot, \cdot \rangle_{sp} \) on both \( \mathfrak{sp}_{HS} \) and \( \mathfrak{sp}(\infty) \) by choosing the set \( \mathcal{B}_\lambda \) as an orthonormal set.

**Remark 7.7.** We note that the inner product on \( \mathfrak{sp}_{HS} \) and \( \mathfrak{sp}(\infty) \) is equivalent to the subspace inner product induced from the inner product on \( HS \) defined in Definition [7.2]. Therefore, for \( x, y \in \mathfrak{sp}_{HS} \), \( \langle x, y \rangle_{HS} = \langle x, y \rangle_{sp} \).

**Remark 7.8.** For \( \mu_{ab}^{Re} \), the indices satisfy \( a > b > 0 \), which means the entry is in the strict upper triangular block. For \( \mu_{ab}^{Im} \), the indices satisfy \( a \geq b > 0 \), which means the entry is in the upper triangular block including the diagonal. For \( v_{ab}^{Re} \), the indices satisfy \( a \geq -b > 0 \), which means the entry is in the other upper triangular block including the diagonal.

**Definition 7.9.** Using Ricci curvature formula (Equation [7.4]) for \( \mathfrak{sp}(\infty) \) and \( \mathfrak{sp}(\infty) \), we define, for \( x \in \mathfrak{sp}(\infty) \),

\[
\text{Ric}(x) = \sum_{\xi \in \mathcal{B}_\lambda} K(x, \xi) = \sum_{\xi \in \mathcal{B}_\lambda} \langle R_x \xi (x), \xi \rangle_{sp}
\]

(7.8)

By definition of \( \mathcal{B}_\lambda \), the above sum will break into four parts:

\[
\text{Ric}(x) = \sum_{a > b > 0} K(x, \mu_{ab}^{Re}) + \sum_{a \geq b > 0} K(x, \mu_{ab}^{Im}) + \sum_{a \geq -b > 0} K(x, v_{ab}^{Re}) + \sum_{a \geq -b > 0} K(x, v_{ab}^{Im})
\]

(7.9)

For computational reason, we define the following truncated Ricci curvature:

\[
\text{Ric}^N(x) = \sum_{N > a > b > 0} K(x, \mu_{ab}^{Re}) + \sum_{N \geq a \geq b > 0} K(x, \mu_{ab}^{Im}) + \sum_{N \geq a \geq -b > 0} K(x, v_{ab}^{Re}) + \sum_{N \geq a \geq -b > 0} K(x, v_{ab}^{Im})
\]

(7.10)

We have \( \text{Ric}(x) = \lim_{N \to \infty} \text{Ric}^N(x) \).

In the rest of the paper, we will compute the following Ricci curvatures via the corresponding truncated Ricci curvatures:

\[
\text{Ric}(\mu_{ab}^{Re}), \text{Ric}(\mu_{ab}^{Im}), \text{Ric}(v_{ab}^{Re}), \text{Ric}(v_{ab}^{Im})
\]

All of these computations boil down to matrix multiplications. The following lemma is an important tool to the computation of Ricci curvature.

**Lemma 7.10.** We have the following Levi-Civita connection formula, where \( \delta \) is the Kronecker delta:

\[
\begin{align*}
\nabla_{\xi_{ab}} \xi_{cd} &= \delta_{bc} \lambda^2_{ac} \xi_{db} - \delta_{da} \lambda^2_{ac} \xi_{cb} - \delta_{cb} \lambda^2_{ac} \xi_{da} + \delta_{db} \lambda^2_{ac} \xi_{ca} - \delta_{ac} \lambda^2_{db} \xi_{bd} \\
\nabla_{\xi_{ab}} i_{\xi_{cd}} &= -\delta_{bc} \lambda^2_{ac} \xi_{d} + \delta_{da} \lambda^2_{ac} \xi_{c} + \delta_{cb} \lambda^2_{ac} \xi_{d} + \delta_{db} \lambda^2_{ac} \xi_{c} + \delta_{ac} \lambda^2_{db} \xi_{d} \\
\nabla_{i_{\xi_{ab}}} \nabla_{\xi_{cd}} &= \delta_{bc} \lambda^2_{ac} \xi_{d} - \delta_{da} \lambda^2_{ac} \xi_{c} + \delta_{db} \lambda^2_{ac} \xi_{c} - \delta_{ac} \lambda^2_{db} \xi_{d}
\end{align*}
\]
Proof. We have

\[ \xi_{ab,cd} = 2\lambda_c^2 \delta_{bc} \xi_{ad} \]

So

\[ [\xi_{ab}, \xi_{cd}] = \xi_{ab,cd} - \xi_{cd,ab} = 2\lambda_c^2 \delta_{bc} \xi_{ad} - 2\lambda_a^2 \delta_{ad} \xi_{cb} \]

In the following, \( \langle \cdot, \cdot \rangle_{HS} \) stands for \( \langle \cdot, \cdot \rangle_{HS} \). Using orthonormality,

\[
2 \langle \nabla_{\xi_{ab}} \xi_{cd}, \xi_{ef} \rangle = \langle [\xi_{ab}, \xi_{cd}], \xi_{ef} \rangle - \langle [\xi_{cd}, \xi_{ef}], \xi_{ab} \rangle + \langle [i \xi_{ef}, \xi_{ab}], \xi_{cd} \rangle = 0
\]

Therefore,

\[
\nabla_{\xi_{ab}} \xi_{cd} = \delta_{bc} \lambda_c^2 \xi_{ad} - \delta_{da} \lambda_a^2 \xi_{bc} - \delta_{cd} \lambda_d^2 \xi_{ab} + \delta_{db} \lambda_b^2 \xi_{ac} + \delta_{bd} \lambda_d^2 \xi_{ca} - \delta_{ac} \lambda_a^2 \xi_{bd}
\]

Similarly,

\[
2 \langle \nabla_{i \xi_{ab}} i \xi_{cd}, \xi_{ef} \rangle = \langle [i \xi_{ab}, i \xi_{cd}], \xi_{ef} \rangle - \langle [i \xi_{cd}, i \xi_{ef}], i \xi_{ab} \rangle + \langle [i \xi_{ef}, i \xi_{ab}], i \xi_{cd} \rangle = 0
\]

Therefore,

\[
\nabla_{i \xi_{ab}} i \xi_{cd} = -\delta_{bc} \lambda_c^2 i \xi_{ad} + \delta_{da} \lambda_a^2 i \xi_{bc} - \delta_{cd} \lambda_d^2 i \xi_{ab} + \delta_{db} \lambda_b^2 i \xi_{ac} + \delta_{bd} \lambda_d^2 i \xi_{ca} - \delta_{ac} \lambda_a^2 i \xi_{bd}
\]

Similarly,

\[
2 \langle \nabla_{i \xi_{ab}} i \xi_{cd}, i \xi_{ef} \rangle = \langle [i \xi_{ab}, i \xi_{cd}], i \xi_{ef} \rangle - \langle [i \xi_{cd}, i \xi_{ef}], i \xi_{ab} \rangle + \langle [i \xi_{ef}, i \xi_{ab}], i \xi_{cd} \rangle = 0
\]

Therefore,

\[
\nabla_{i \xi_{ab}} i \xi_{cd} = \delta_{bc} \lambda_c^2 i \xi_{ad} - \delta_{da} \lambda_a^2 i \xi_{bc} - \delta_{cd} \lambda_d^2 i \xi_{ab} + \delta_{db} \lambda_b^2 i \xi_{ac} + \delta_{bd} \lambda_d^2 i \xi_{ca} - \delta_{ac} \lambda_a^2 i \xi_{bd}
\]

Similarly,

\[
2 \langle \nabla_{i \xi_{ab}} i \xi_{cd}, i \xi_{ef} \rangle = \langle [i \xi_{ab}, i \xi_{cd}], i \xi_{ef} \rangle - \langle [i \xi_{cd}, i \xi_{ef}], i \xi_{ab} \rangle + \langle [i \xi_{ef}, i \xi_{ab}], i \xi_{cd} \rangle = 0
\]

Therefore,

\[
\nabla_{i \xi_{ab}} i \xi_{cd} = \delta_{bc} \lambda_c^2 i \xi_{ad} - \delta_{da} \lambda_a^2 i \xi_{bc} - \delta_{cd} \lambda_d^2 i \xi_{ab} + \delta_{db} \lambda_b^2 i \xi_{ac} + \delta_{bd} \lambda_d^2 i \xi_{ca} - \delta_{ac} \lambda_a^2 i \xi_{bd}
\]

Similarly,
Remark 7.11. Once we have the above lemma, we can use equation (7.7) to change the basis elements of $\text{sp}(\infty)$ into the basis elements of $\text{HS}$, and then use formula (7.1), (7.2), (7.3) and (7.4) to compute the Ricci curvature of $\text{Sp}(\infty)$. But since each basis $\mu^{\text{Re}}_{ab}, \mu^{\text{Im}}_{ab}, v^{\text{Re}}_{ab}$, and $v^{\text{Im}}_{ab}$ has four terms, and each connection formula in the above lemma has six terms, the combination will be huge. For example, the sectional curvature
\begin{align*}
K(\mu^{\text{Re}}_{ab}, \mu^{\text{Re}}_{cd}) &= \langle R_{\mu^{\text{Re}}_{ab}, \mu^{\text{Re}}_{cd}}(\mu^{\text{Re}}_{ab}), \mu^{\text{Re}}_{cd} \rangle \\
&= -\langle \nabla_{\mu^{\text{Re}}_{ab}, \mu^{\text{Re}}_{cd}}(\mu^{\text{Re}}_{ab}), \nabla_{\mu^{\text{Re}}_{ab}, \mu^{\text{Re}}_{cd}}(\mu^{\text{Re}}_{ab}) \rangle
\end{align*}
will have 21,504 terms. Therefore, I use a computer program to facilitate the computation.

Theorem 7.12. Let $a, b \in \mathbb{Z}\setminus\{0\}$.
For $a > b > 0$,
\[
\text{Ric}^N(\mu^{\text{Re}}_{ab}) = \frac{1}{16} \left[ -24 \lambda^4_a - 24 \lambda^4_b + 48 \lambda^2_a \lambda^2_b - 12 \lambda^2_a \sum_{d=1}^{a-1} \lambda^2_d + 8 \lambda^2_a \sum_{d=1}^{b-1} \lambda^2_d + 8 \lambda^2_b \sum_{d=1}^{a-1} \lambda^2_d \right.
\]
\[
- 12 \lambda^2_b \sum_{d=1}^{b-1} \lambda^2_d + 8 \sum_{d=1}^{a-1} \lambda^4_d + 8 \sum_{d=1}^{b-1} \lambda^4_d - 16 N \lambda^4_a - 16 N \lambda^4_b - 12 \lambda^2_a \sum_{c=a+1}^{N} \lambda^2_c + 8 \sum_{c=a+1}^{N} \lambda^4_c + 8 \sum_{c=a+1}^{N} \lambda^4_c \right].
\]
For $a > b > 0$,
\[
\text{Ric}^N(\mu^{\text{Im}}_{ab}) = \frac{1}{16} \left[ -40 \lambda^4_a - 40 \lambda^4_b - 32 \lambda^2_a \lambda^2_b - 12 \lambda^2_a \sum_{d=1}^{a-1} \lambda^2_d + 8 \lambda^2_a \sum_{d=1}^{b-1} \lambda^2_d + 8 \lambda^2_b \sum_{d=1}^{a-1} \lambda^2_d \right.
\]
\[
- 12 \lambda^2_b \sum_{d=1}^{b-1} \lambda^2_d + 8 \sum_{d=1}^{a-1} \lambda^4_d + 8 \sum_{d=1}^{b-1} \lambda^4_d - 16 N \lambda^4_a - 16 N \lambda^4_b - 12 \lambda^2_a \sum_{c=a+1}^{N} \lambda^2_c + 8 \sum_{c=a+1}^{N} \lambda^4_c + 8 \sum_{c=a+1}^{N} \lambda^4_c \right].
\]
For $a = b > 0$,
\[
\text{Ric}^N(\mu^{\text{Im}}_{ab}) = 0.
\]
For $a > -b > 0$,
\[
\text{Ric}^N(v^{\text{Re}}_{ab}) = \frac{1}{16} \left[ -40 \lambda^4_a - 40 \lambda^4_b - 48 \lambda^2_a \lambda^2_b - 12 \lambda^2_a \sum_{d=1}^{a-1} \lambda^2_d + 8 \lambda^2_a \sum_{d=1}^{b-1} \lambda^2_d + 8 \lambda^2_b \sum_{d=1}^{a-1} \lambda^2_d \right.
\]
\[
- 12 \lambda^2_b \sum_{d=1}^{b-1} \lambda^2_d + 8 \sum_{d=1}^{a-1} \lambda^4_d + 8 \sum_{d=1}^{b-1} \lambda^4_d - 16 N \lambda^4_a - 16 N \lambda^4_b - 12 \lambda^2_a \sum_{c=a+1}^{N} \lambda^2_c + 8 \sum_{c=a+1}^{N} \lambda^4_c + 8 \sum_{c=a+1}^{N} \lambda^4_c \right].
\]
For $a = -b > 0$,
\[
\text{Ric}^N(v^{\text{Re}}_{ab}) = \frac{1}{16} \left[ -192 \lambda^4_a - 32 \sum_{d=1}^{a-1} \lambda^4_d - 192 N \lambda^4_a - 32 \sum_{c=a+1}^{N} \lambda^4_c \right].
\]
For $a > -b > 0$,

\[
R^N(v_{ab}^\text{Im}) = \frac{1}{16} \left[ -40\lambda_a^4 - 40\lambda_b^4 - 32\lambda_a^2\lambda_b^2 - 12\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 8\lambda_a^2 \sum_{d=1}^{b-1} \lambda_d^2 - 8\lambda_b^2 \sum_{d=1}^{a-1} \lambda_d^2 - 12\lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 - 8\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 8\lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 - 32\lambda_a^2 \lambda_b^2 - 12\lambda_a^2 \lambda_b^2 - 16\lambda_a^2 - 16\lambda_b^2 - 12\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12\lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 - 8\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 - 8\lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12\lambda_a^2 - 12\lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12\lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 \right].
\]

For $a = -b > 0$,

\[
\text{Ric}^N(v_{ab}^\text{Im}) = 0.
\]

**Corollary 7.13.** If we set the parameter $\lambda_i = 1/\sqrt{2}$, for all $i \in \mathbb{Z} \setminus \{0\}$, then we recover the canonical inner product on the space $\mathcal{H}S$ (remark 7.3). In this case, we have

\[
\begin{align*}
\text{Ric}^N(\mu_{ab}^\text{Re}) &= -\frac{3}{8}N - \frac{1}{8}, & \text{for } a > b > 0; \\
\text{Ric}^N(\mu_{ab}^\text{Im}) &= -\frac{7}{8}N - \frac{11}{8}, & \text{for } a > b > 0; \\
\text{Ric}^N(\mu_{ab}^\text{Im}) &= 0, & \text{for } a = b > 0; \\
\text{Ric}^N(v_{ab}^\text{Re}) &= -\frac{7}{8}N - \frac{13}{8}, & \text{for } a > -b > 0; \\
\text{Ric}^N(v_{ab}^\text{Re}) &= -\frac{7}{2}N - \frac{5}{2}, & \text{for } a = -b > 0; \\
\text{Ric}^N(v_{ab}^\text{Im}) &= -\frac{7}{8}N - \frac{11}{8}, & \text{for } a > -b > 0; \\
\text{Ric}^N(v_{ab}^\text{Im}) &= 0, & \text{for } a = -b > 0.
\end{align*}
\]

**Remark 7.14.** By the above corollary, we see that for most of the basis element $\xi \in \mathcal{B}_\lambda$, we have $\text{Ric}(\xi) = \lim_{N \to \infty} \text{Ric}^N(\xi) = -\infty$. 

**Proof.** (of the theorem.)

The method of computing Ricci curvature and truncated Ricci curvature is stated in Definition 7.9. Ricci curvature is defined in terms of sectional curvature, which can be expressed in terms of Riemann tensor and the inner product of the Lie algebra. Riemann tensor is defined in terms of Levi-Civita connection. The formula of Levi-Civita connection is the content of Lemma 7.10. So the method of computing Ricci curvature is straightforward. But there are huge number of terms. Therefore, I used a computer program to facilitate the computation.
\[ \text{Ric}^N(\mu_{ab}^{Re}) \]
\[ = \sum_{N \geq c > d > 0} K(\mu_{ab}^{Re}, \mu_{cd}^{Re}) + \sum_{N \geq c \geq d > 0} K(\mu_{ab}^{Re}, \mu_{cd}^{Im}) \]
\[ + \sum_{N \geq c \geq d - 1 > 0} K(\mu_{ab}^{Re}, \nu_{cd}^{Re}) + \sum_{N \geq c \geq d - 1 > 0} K(\mu_{ab}^{Re}, \nu_{cd}^{Im}) \]
\[ = \sum_{N \geq c > d > 0} K(\mu_{ab}^{Re}, \mu_{cd}^{Re}) + \sum_{N \geq c \geq d > 0} K(\mu_{ab}^{Re}, \mu_{cd}^{Im}) \]
\[ + \sum_{N \geq c \geq d - 1 > 0} K(\mu_{ab}^{Re}, \nu_{cd}^{Re}) + \sum_{N \geq c \geq d - 1 > 0} K(\mu_{ab}^{Re}, \nu_{cd}^{Im}) \]
\[ := \sum_{N \geq c > d > 0} A_{\text{upper}} + \sum_{N \geq c \geq d > 0} A_{\text{diagonal}} \]

We have
\[ A_{\text{upper}} = \frac{1}{16} \left[ -16 \delta_{a,c} \lambda_a^4 - 24 \delta_{a,c} \delta_{a,d} \lambda_a^4 + 24 \delta_{a,c} \delta_{b,d} \lambda_b^4 - 16 \delta_{a,d} \lambda_a^4 \right. \]
\[ + 24 \delta_{a,d} \delta_{b,c} \lambda_a^4 + 8 \delta_{a,c} \delta_{a,d} \lambda_a^2 \lambda_a^2 + 8 \delta_{a,b} \delta_{b,d} \lambda_a^2 \lambda_b^2 - 12 \delta_{a,b} \lambda_a^2 \lambda_c^2 \]
\[ + 8 \delta_{b,d} \lambda_a^2 \lambda_c^2 - 12 \delta_{a,c} \lambda_a^2 \lambda_d^2 + 8 \delta_{a,c} \lambda_d^2 \lambda_a^2 - 24 \delta_{a,c} \delta_{b,d} \lambda_b^4 - 24 \delta_{a,d} \delta_{b,c} \lambda_a^2 \lambda_b^2 \]
\[ - 16 \delta_{b,c} \lambda_b^4 - 24 \delta_{b,c} \delta_{b,d} \lambda_a^4 - 16 \delta_{a,b} \lambda_b^4 + 8 \delta_{a,b} \lambda_b^4 \lambda_c^4 \]
\[ + 8 \delta_{a,c} \lambda_b^2 \lambda_d^2 + 8 \delta_{a,c} \lambda_a^4 + 8 \delta_{a,c} \lambda_c^4 + 8 \delta_{a,c} \lambda_d^4 + 8 \delta_{b,c} \lambda_d^4 \]

and
\[ A_{\text{diagonal}} = \frac{1}{16} \left[ -16 \delta_{a,c} \lambda_a^4 - 16 \delta_{a,c} \delta_{a,d} \lambda_a^4 + 18 \delta_{a,c} \delta_{b,d} \lambda_a^4 - 12 \delta_{a,d} \lambda_a^4 \right. \]
\[ + 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 + 12 \delta_{a,c} \delta_{a,d} \lambda_a^2 \lambda_a^2 + 12 \delta_{a,d} \delta_{b,c} \lambda_a^2 \lambda_b^2 + 18 \delta_{a,d} \delta_{b,c} \lambda_a^2 \lambda_c^2 \]
\[ + 12 \delta_{b,c} \lambda_a^4 - 18 \delta_{a,c} \lambda_a^2 \lambda_d^2 + 12 \delta_{a,c} \lambda_d^2 \lambda_a^2 + 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 12 \delta_{a,d} \lambda_a^2 \lambda_b^2 - 18 \delta_{a,d} \lambda_a^2 \lambda_c^2 \]
\[ + 12 \delta_{a,c} \lambda_b^2 \lambda_d^2 - 18 \delta_{a,c} \lambda_b^2 \lambda_d^2 + 6 \delta_{a,d} \lambda_c^4 + 6 \delta_{b,d} \lambda_c^4 + 6 \delta_{a,d} \lambda_c^4 + 6 \delta_{a,c} \lambda_d^4 + 6 \delta_{b,c} \lambda_d^4 \]

So
\[ \sum_{N \geq c > d > 0} A_{\text{upper}} = \frac{1}{16} \left[ -16(a - 1) \lambda_a^4 + 24 \lambda_a^4 - 16(N - a) \lambda_a^4 - 12 \lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 \right. \]
\[ + 8 \lambda_a^2 \sum_{c=b+1}^{N} \lambda_c^2 - 12 \lambda_a^2 \sum_{d=1}^{b-1} \lambda_d^2 + 8 \lambda_a^2 \sum_{d=1}^{b-1} \lambda_d^2 + 24 \lambda_b^4 \]
\[ - 16(b - 1) \lambda_b^4 + 16(N - b) \lambda_b^4 + 8 \lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12 \lambda_b^2 \sum_{c=b+1}^{N} \lambda_c^2 + 8 \lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 \]
\[ - 12 \lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 + 8 \sum_{c=a+1}^{N} \lambda_c^4 + 8 \sum_{c=b+1}^{N} \lambda_c^4 + 8 \sum_{d=1}^{b-1} \lambda_d^4 + 8 \sum_{d=1}^{b-1} \lambda_d^4 \]

and
\[
\sum_{N \geq c = d > 0} A_{\text{diagonal}} = \frac{1}{16} \left[ -12 \lambda^4_d - 16 \lambda^4_d - 12 \lambda^2_d - 18 \lambda^4_a + 12 \lambda^2_a \lambda^2_d - 18 \lambda^4_a + 12 \lambda^2_a \lambda^2_b - 12 \lambda^4_b \\
- 16 \lambda^4_d - 12 \lambda^4_b + 12 \lambda^2_a \lambda^2_b - 18 \lambda^4_b + 12 \lambda^2_a \lambda^2_b - 18 \lambda^4_b + 6 \lambda^4 + 6 \lambda^4 + 6 \lambda^4_b \right]
\]

Therefore, for \( a > b > 0 \),
\[
\text{Ric}^{N}(\mu^{Re}_{ab}) = \sum_{N \geq c = d > 0} A_{\text{upper}} + \sum_{N \geq c = d > 0} A_{\text{diagonal}}
\]
\[
= \frac{1}{16} \left[ -24 \lambda^4_d - 24 \lambda^4_b + 48 \lambda^2_a \lambda^2_b - 12 \lambda^2_a \sum_{d=1}^{a-1} \lambda^2_d + 8 \lambda^2_a \sum_{d=1}^{b-1} \lambda^2_d + 8 \lambda^2_b \sum_{d=1}^{a-1} \lambda^2_d \\
- 12 \lambda^4_d \sum_{d=1}^{b-1} \lambda^2_d + 8 \sum_{d=1}^{b-1} \lambda^2_d - 16 N \lambda^4_a - 16 N \lambda^4_b - 12 \lambda^2_a \sum_{c=a+1}^{N} \lambda^2_c \\
+ 8 \sum_{c=b+1}^{N} \lambda^2_c \sum_{c=a+1}^{N} \lambda^2_c - 12 \lambda^2_b \sum_{c=b+1}^{N} \lambda^2_c + 8 \sum_{c=a+1}^{N} \lambda^4_c + 8 \sum_{c=b+1}^{N} \lambda^4_c \right]
\]

Next,
\[
\text{Ric}^{N}(\mu^{Im}_{ab})
\]
\[
= \sum_{N \geq c = d > 0} K(\mu^{Im}_{ab}, \mu^{Re}_{cd}) + \sum_{N \geq c = d > 0} K(\mu^{Im}_{ab}, \mu^{Im}_{cd})
\]
\[
+ \sum_{N \geq c = d > 0} K(\mu^{Re}_{ab}, \mu^{Re}_{cd}) + \sum_{N \geq c = d > 0} K(\mu^{Im}_{ab}, \mu^{Im}_{c,d})
\]
\[
= \sum_{N \geq c = d > 0} K(\mu^{Im}_{ab}, \mu^{Re}_{cd}) + \sum_{N \geq c = d > 0} K(\mu^{Im}_{ab}, \mu^{Im}_{cd})
\]
\[
+ \sum_{N \geq c = d > 0} K(\mu^{Re}_{ab}, \mu^{Re}_{c,d}) + \sum_{N \geq c = d > 0} K(\mu^{Im}_{ab}, \mu^{Im}_{c,d})
\]
\[
= \sum_{N \geq c = d > 0} \left[ K(\mu^{Im}_{ab}, \mu^{Re}_{cd}) + K(\mu^{Im}_{ab}, \mu^{Im}_{cd}) + K(\mu^{Re}_{ab}, \mu^{Re}_{c,d}) + K(\mu^{Im}_{ab}, \mu^{Im}_{c,d}) \right]
\]
\[
+ \sum_{N \geq c = d > 0} \left[ K(\mu^{Im}_{ab}, \mu^{Re}_{cd}) + K(\mu^{Im}_{ab}, \mu^{Im}_{cd}) + K(\mu^{Re}_{ab}, \mu^{Re}_{c,d}) + K(\mu^{Im}_{ab}, \mu^{Im}_{c,d}) \right]
\]
\[
:= \sum_{N \geq c = d > 0} B_{\text{upper}} + \sum_{N \geq c = d > 0} B_{\text{diagonal}}
\]

We have
\[
B_{\text{upper}} = \frac{1}{16} \left[ + 32 \delta_{a,b} \delta_{a,c} \lambda^4_d + 32 \delta_{a,b} \delta_{a,d} \lambda^4_a - 16 \delta_{a,c} \lambda^4 - 24 \delta_{a,c} \delta_{a,d} \lambda^4 - 8 \delta_{a,c} \delta_{b,d} \lambda^4 \\
- 16 \delta_{a,d} \lambda^4_a + 8 \delta_{a,d} \lambda^4_a - 8 \delta_{a,e} \delta_{a,d} \lambda^4_a + 16 \delta_{a,c} \delta_{b,d} \lambda^4_a + 16 \delta_{a,d} \delta_{b,c} \lambda^4_a \\
- 8 \delta_{b,c} \delta_{a,d} \lambda^4_b + 40 \delta_{b,c} \delta_{a,d} \lambda^4_b - 12 \delta_{a,d} \delta_{b,c} \lambda^4_c - 8 \delta_{b,c} \delta_{d,c} \lambda^4_b + 40 \delta_{a,b} \delta_{c,d} \lambda^4_a \\
- 12 \delta_{a,c} \lambda^4_a - 8 \delta_{a,c} \delta_{a,d} \lambda^4_a + 8 \delta_{a,c} \delta_{b,d} \lambda^4_a + 8 \delta_{a,d} \delta_{b,c} \lambda^4_a - 16 \delta_{a,c} \lambda^4 - 24 \delta_{a,c} \lambda^4_b \\
- 16 \delta_{b,c} \delta_{a,d} \lambda^4_b - 8 \delta_{a,d} \delta_{b,c} \lambda^4_c - 12 \delta_{b,c} \delta_{a,d} \lambda^4_c - 16 \delta_{a,b} \delta_{a,c} \lambda^4_d + 8 \delta_{a,c} \lambda^4_d + 8 \delta_{b,c} \lambda^4_d \right]
\]
and

\[
B_{\text{diagonal}} = \frac{1}{16} \left[ + 24 \delta_{a,b} \delta_{a,c} \lambda_a^4 - 32 \delta_{a,b} \delta_{a,d} \lambda_a^4 + 24 \delta_{a,b} \lambda_a^4 \delta_{a,c} \lambda_a^4 - 12 \delta_{a,c} \lambda_a^4 - 16 \delta_{a,c} \lambda_a^4 - 12 \delta_{a,c} \lambda_a^4 - 16 \delta_{a,c} \lambda_a^4 - 12 \delta_{a,c} \lambda_a^4 - 16 \delta_{a,c} \lambda_a^4 + 6 \delta_{a,b} \lambda_a^4 \delta_{a,d} \lambda_a^4 + 20 \delta_{a,b} \delta_{a,d} \lambda_a^4 \lambda_a^4 + 20 \delta_{a,b} \delta_{a,d} \lambda_a^4 \lambda_a^4 - 18 \delta_{a,b} \delta_{a,d} \lambda_a^4 \lambda_a^4 - 18 \delta_{a,b} \delta_{a,d} \lambda_a^4 \lambda_a^4 - 18 \delta_{a,b} \delta_{a,d} \lambda_a^4 \lambda_a^4 - 18 \delta_{a,b} \delta_{a,d} \lambda_a^4 \lambda_a^4 \right]
\]

For \( a > b > 0 \),

\[
\sum_{N \geq c > d > 0} B_{\text{upper}} = \frac{1}{16} \left[ - 16(a - 1) \lambda_a^4 + 8 \lambda_a^4 - 16(N - a) \lambda_a^4 + 16 \lambda_a^4 \lambda_b^4 - 12 \lambda_a^4 \sum_{c=a+1}^{N} \lambda_c^4 - 16(b - 1) \lambda_b^4 \right]
\]

For \( a = b > 0 \),

\[
\sum_{N \geq c > d > 0} B_{\text{upper}} = \frac{1}{16} \left[ + 32(a - 1) \lambda_a^4 + 32(N - a) \lambda_a^4 - 16(a - 1) \lambda_a^4 - 16(N - a) \lambda_a^4 \right]
\]

For \( a > b > 0 \),

\[
\sum_{N \geq c > d > 0} B_{\text{diagonal}} = \frac{1}{16} \left[ - 12 \lambda_a^4 - 16 \lambda_a^4 - 12 \lambda_a^4 - 18 \lambda_a^4 - 12 \lambda_a^4 \lambda_b^4 - 18 \lambda_a^4 - 12 \lambda_a^4 \lambda_b^4 - 12 \lambda_a^4 \lambda_b^4 + 6 \lambda_a^4 + 6 \lambda_a^4 + 6 \lambda_a^4 + 6 \lambda_a^4 \right]
\]

For \( a = b > 0 \),

\[
\sum_{N \geq c > d > 0} B_{\text{diagonal}} = 0
\]
Therefore, for $a > b > 0$,
\[
\text{Ric}^N(\mu^{Im}_{ab}) = \sum_{N \geq c, d \geq 0} B_{upper} + \sum_{N \geq c, d \geq 0} B_{diagonal}
\]
\[
= \frac{1}{16} \left[ -40\lambda^4_d - 40\lambda^4_b - 32\lambda^2_d \lambda^2_b - 12\lambda^2_d \sum_{d=1}^{a-1} \lambda^2_d - 8\lambda^2_a \sum_{d=1}^{b-1} \lambda^2_d - 8\lambda^2_b \sum_{d=1}^{a-1} \lambda^2_d \right]
\]
\[
-12\lambda^2_b \sum_{d=1}^{b-1} \lambda^2_d + 8 \sum_{d=1}^{a-1} \lambda^2_d + 8 \sum_{d=1}^{b-1} \lambda^2_d - 16N\lambda^4_d - 16N\lambda^4_b - 12\lambda^2_b \sum_{c=a+1}^{N} \lambda^2_c
\]
\[
-8\lambda^2_a \sum_{c=b+1}^{N} \lambda^2_c - 8\lambda^2_b \sum_{c=a+1}^{N} \lambda^2_c - 12\lambda^2_b \sum_{c=b+1}^{N} \lambda^2_c + 8 \sum_{c=a+1}^{N} \lambda^4_c + 8 \sum_{c=b+1}^{N} \lambda^4_c \right]
\]
and for $a = b > 0$,
\[
\text{R}^N(\mu^{Im}_{ab}) = \sum_{N \geq c, d \geq 0} B_{upper} + \sum_{N \geq c, d \geq 0} B_{diagonal} = 0
\]

Next, we compute $\text{R}^N(\nu^{Re}_{a,b})$ for $a \geq -b > 0$. Replacing $b$ with $-b$, it’s equivalent to computing $\text{R}^N(\nu^{Re}_{a,-b})$ for $a \geq b > 0$.

\[
\text{Ric}^N(\nu^{Re}_{a,-b}) = \sum_{N \geq c, d \geq 0} K(\nu^{Re}_{a,-b}, \nu^{Re}_{c,d}) + \sum_{N \geq c, d \geq 0} K(\nu^{Re}_{a,-b}, \mu^{Im}_{c,d})
\]
\[
+ \sum_{N \geq c, d \geq 0} K(\nu^{Re}_{a,-b}, \nu^{Re}_{c,-d}) + \sum_{N \geq c, d \geq 0} K(\nu^{Re}_{a,-b}, \nu^{Im}_{c,-d})
\]
\[
= \sum_{N \geq c, d \geq 0} \left[ K(\nu^{Re}_{a,-b}, \mu^{Im}_{c,d}) + K(\nu^{Re}_{a,-b}, \nu^{Im}_{c,d}) + K(\nu^{Re}_{a,-b}, \nu^{Re}_{c,-d}) + K(\nu^{Re}_{a,-b}, \nu^{Im}_{c,-d}) \right]
\]
\[
+ \sum_{N \geq c, d \geq 0} \left[ K(\nu^{Re}_{a,-b}, \mu^{Im}_{c,d}) + K(\nu^{Re}_{a,-b}, \nu^{Re}_{c,-d}) + K(\nu^{Re}_{a,-b}, \nu^{Im}_{c,-d}) \right]
\]
\[
:= \sum_{N \geq c, d \geq 0} C_{upper} + \sum_{N \geq c, d \geq 0} C_{diagonal}
\]

We have
\[
C_{upper} = \frac{1}{16} \left[ -160\delta_{a,b} \delta_{a,c} \lambda^4_d + 480\delta_{a,b} \delta_{a,c} \delta_{a,d} \lambda^4_a - 160\delta_{a,b} \delta_{a,d} \lambda^4_a - 16\delta_{a,c} \lambda^4_a
\]
\[
-24\delta_{a,c} \lambda^4_a + 8\delta_{a,d} \lambda^4_a - 16\delta_{a,d} \lambda^4_a + 8\delta_{a,c} \delta_{a,d} \lambda^4_a - 8\delta_{a,c} \delta_{a,d} \lambda^4_a
\]
\[
-8\delta_{a,d} \delta_{a,d} \lambda^4_a - 24\delta_{a,b} \delta_{a,d} \lambda^4_a = 12\delta_{a,d} \lambda^4_a - 8\delta_{a,d} \lambda^4_a
\]
\[
-24\delta_{a,b} \delta_{a,d} \lambda^2_d - 12\delta_{a,c} \lambda^2_d - 8\delta_{a,c} \Delta^2_d - 8\delta_{a,c} \delta_{a,d} \lambda^2_d - 8\delta_{a,c} \delta_{a,d} \lambda^2_d - 8\delta_{a,c} \delta_{a,d} \lambda^2_d
\]
\[
-16\delta_{a,c} \lambda^2_d - 24\delta_{a,b} \delta_{a,d} \lambda^2_d - 16\delta_{a,d} \lambda^2_d + 8\delta_{a,c} \delta_{a,d} \lambda^2_d + 8\delta_{a,c} \delta_{a,d} \lambda^2_d
\]
\[
-16\delta_{a,b} \lambda^2_d - 24\delta_{a,b} \delta_{a,d} \lambda^2_d - 8\delta_{a,b} \Delta^2_d - 12\delta_{a,b} \lambda^2_d + 12\delta_{a,b} \lambda^2_d
\]
and

\[ C_{\text{diagonal}} = \frac{1}{16} \left[ -120 \delta_{a,b} \delta_{a,c} \lambda_a^4 + 480 \delta_{a,b} \delta_{a,c} \delta_{a,d} \lambda_a^4 - 120 \delta_{a,b} \delta_{a,d} \lambda_a^4 - 12 \delta_{a,c} \lambda_a^4 - 16 \delta_{a,d} \lambda_a^4 - 6 \delta_{a,d} \delta_{b,d} \lambda_a^4 - 6 \delta_{a,d} \delta_{c,d} \lambda_a^4 + 4 \delta_{a,c} \delta_{b,d} \lambda_a^4 \right. \\
\left. + 4 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 36 \delta_{a,b} \delta_{a,d} \lambda_a^4 - 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 18 \delta_{a,d} \delta_{b,c} \lambda_a^4 - 36 \delta_{a,b} \delta_{a,d} \lambda_a^4 - 12 \delta_{a,b} \delta_{a,d} \lambda_a^4 - 12 \delta_{b,c} \delta_{b,c} \lambda_a^4 - 12 \delta_{b,c} \delta_{b,c} \lambda_a^4 - 12 \delta_{b,c} \delta_{b,c} \lambda_a^4 - 12 \delta_{b,c} \delta_{b,c} \lambda_a^4 - 12 \delta_{b,c} \delta_{b,c} \lambda_a^4 \\
\right. \\
\left. + 12 \delta_{a,b} \delta_{a,d} \lambda_a^4 + 6 \delta_{a,d} \lambda_a^4 + 6 \delta_{a,d} \lambda_a^4 + 12 \delta_{a,b} \delta_{a,c} \lambda_a^4 + 6 \delta_{a,c} \lambda_a^4 + 6 \delta_{a,c} \lambda_a^4 \right] \\
\]

For \( a > b > 0 \),

\[ \sum_{N \geq c > d > 0} C_{\text{upper}} = \frac{1}{16} \left[ -16(a - 1) \lambda_a^4 + 8 \lambda_a^4 - 16(N - a) \lambda_a^4 - 12 \lambda_a^2 \sum_{c=a+1}^N \lambda_c^2 \right. \\
\left. - 8 \lambda_a^2 \sum_{c=b+1}^N \lambda_c^2 - 12 \lambda_a^2 \sum_{c=b+1}^N \lambda_d^2 - 8 \lambda_a^2 \sum_{c=d+1}^N \lambda_d^2 + 8 \lambda_b^4 \right. \\
\left. - 16(b - 1) \lambda_b^4 - 16(N - b) \lambda_b^4 - 8 \lambda_b^4 \sum_{c=b+1}^N \lambda_c^2 - 12 \lambda_b^4 \sum_{c=b+1}^N \lambda_d^2 - 8 \lambda_b^4 \sum_{c=d+1}^N \lambda_d^2 \\
\right. \\
\left. - 12 \lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 + 8 \sum_{d=1}^{c-a+1} \lambda_d^4 + 8 \sum_{d=1}^{c-b+1} \lambda_d^4 + 8 \sum_{d=1}^{a-1} \lambda_d^4 \right] \\
\]

For \( a = b > 0 \),

\[ \sum_{N \geq c > d > 0} C_{\text{upper}} = \frac{1}{16} \left[ -160(a - 1) \lambda_a^4 - 160(N - a) \lambda_a^4 - 16(a - 1) \lambda_a^4 - 16(N - a) \lambda_a^4 \\
\right. \\
\left. - 24 \lambda_a^2 \sum_{c=a+1}^N \lambda_c^2 - 12 \lambda_a^2 \sum_{c=a+1}^N \lambda_d^2 - 8 \lambda_a^2 \sum_{c=d+1}^N \lambda_d^2 - 4 \lambda_a^2 \sum_{c=a+1}^N \lambda_d^2 \\
\right. \\
\left. - 8 \lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 16(a - 1) \lambda_a^4 - 16(N - a) \lambda_a^4 - 8 \lambda_a^4 \sum_{c=a+1}^N \lambda_c^2 \\
\right. \\
\left. - 12 \lambda_a^2 \sum_{a+1}^N \lambda_c^2 - 8 \lambda_a^2 \sum_{a+1}^N \lambda_d^2 - 12 \lambda_a^2 \sum_{a+1}^N \lambda_d^2 - 12 \lambda_a^2 \sum_{a+1}^N \lambda_d^2 \\
\right. \\
\left. + 16 \sum_{c=a+1}^N \lambda_c^4 + 8 \sum_{a+1}^N \lambda_d^4 + 8 \sum_{a+1}^N \lambda_d^4 \right] \\
\]

For \( a > b > 0 \),

\[ \sum_{N \geq c = d > 0} C_{\text{diagonal}} = \frac{1}{16} \left[ -12 \lambda_a^4 - 16 \lambda_a^4 - 12 \lambda_a^4 - 18 \lambda_a^4 - 12 \lambda_a^2 \lambda_b^2 - 18 \lambda_a^4 - 12 \lambda_a^2 \lambda_b^2 - 12 \lambda_b^4 \\
\right. \\
\left. - 16 \lambda_b^4 - 12 \lambda_a^4 - 12 \lambda_a^2 \lambda_b^2 - 18 \lambda_b^4 - 12 \lambda_a^2 \lambda_b^2 - 18 \lambda_a^4 + 6 \lambda_a^4 + 6 \lambda_a^4 + 6 \lambda_a^4 \right] \\
\]

For \( a = b > 0 \),

\[ \sum_{N \geq c = d > 0} C_{\text{diagonal}} = 0 \]
Therefore, for $a > -b > 0$,

\[
\text{Ric}^N(v_{ab}^{Re}) = \sum_{N \geq c > d > 0} C_{upper} + \sum_{N \geq c = d > 0} C_{diagonal}
\]

\[
= \frac{1}{16} \left[ -40 \lambda^4_a - 40 \lambda^4_b - 48 \lambda^2_a \lambda^2_b - 12 \lambda^2_a \sum_{d=1}^{a-1} \lambda^2_d - 8 \lambda^2_b \sum_{d=1}^{b-1} \lambda^2_d - 8 \lambda^2_b \sum_{d=1}^{a-1} \lambda^2_d - 12 \lambda^2_b \sum_{d=1}^{b-1} \lambda^2_d + 8 \sum_{d=1}^{a-1} \lambda^2_d + 8 \sum_{d=1}^{b-1} \lambda^2_d - 16 N \lambda^4_a - 16 N \lambda^4_b - 12 \lambda^2_a \sum_{c=a+1}^{N} \lambda^2_c - 8 \lambda^2_a \sum_{c=b+1}^{N} \lambda^2_c \right]
\]

and, for $a = -b > 0$,

\[
\text{R}^N(v_{ab}^{Re}) = \sum_{N \geq c > d > 0} C_{upper} + \sum_{N \geq c = d > 0} C_{diagonal}
\]

\[
= \frac{1}{16} \left[ -192 \lambda^4_a - 32 \sum_{d=1}^{a-1} \lambda^4_d - 192 N \lambda^4_a - 32 \sum_{c=a+1}^{N} \lambda^4_c \right]
\]

Next, we compute $R^N(v_{a,-b}^{Im})$ for $a \geq -b > 0$. Replacing $b$ with $-b$, it’s equivalent to computing $R^N(v_{a,-b}^{Im})$ for $a \geq b > 0$.

\[
\text{Ric}^N(v_{a,-b}^{Im}) = \sum_{N \geq c > d > 0} K(v_{a,-b}^{Im}, H_{cd}^{Re}) + \sum_{N \geq c > d > 0} K(v_{a,-b}^{Im}, H_{cd}^{Im})
\]

\[
+ \sum_{N \geq c > d > 0} K(v_{a,b}^{Im}, v_{cd}^{Re}) + \sum_{N \geq c > d > 0} K(v_{a,b}^{Im}, v_{cd}^{Im})
\]

\[
= \sum_{N \geq c > d > 0} K(v_{a,-b}^{Im}, H_{cd}^{Re}) + \sum_{N \geq c > d > 0} K(v_{a,-b}^{Im}, H_{cd}^{Im})
\]

\[
+ \sum_{N \geq c > d > 0} K(v_{a,b}^{Im}, v_{cd}^{Re}) + \sum_{N \geq c > d > 0} K(v_{a,b}^{Im}, v_{cd}^{Im})
\]

\[
= \sum_{N \geq c > d > 0} \left[ K(v_{a,-b}^{Im}, \mu_{cd}^{Re}) + K(v_{a,-b}^{Im}, \mu_{cd}^{Im}) + K(v_{a,b}^{Im}, v_{cd}^{Re}) + K(v_{a,b}^{Im}, v_{cd}^{Im}) \right]
\]

\[
+ \sum_{N \geq c > d > 0} \left[ K(v_{a,-b}^{Im}, \mu_{cd}^{Re}) + K(v_{a,-b}^{Im}, \mu_{cd}^{Im}) + K(v_{a,b}^{Im}, v_{cd}^{Re}) + K(v_{a,b}^{Im}, v_{cd}^{Im}) \right]
\]

\[
:= \sum_{N \geq c > d > 0} D_{upper} + \sum_{N \geq c = d > 0} D_{diagonal}
\]

We have

\[
D_{upper} = \frac{1}{16} \left[ +32 \delta_{a,b} \delta_{a,c} \lambda^4 + 32 \delta_{a,b} \delta_{a,d} \lambda^4 - 16 \delta_{a,c} \lambda^4 - 24 \delta_{a,c} \delta_{a,d} \lambda^4
\]

\[
+ 8 \delta_{a,c} \delta_{a,d} \lambda^4 - 16 \delta_{a,d} \lambda^4 + 8 \delta_{a,d} \delta_{a,c} \lambda^4 - 8 \delta_{a,c} \delta_{a,d} \lambda^4 - 16 \delta_{a,c} \delta_{a,d} \lambda^2
\]

\[
+ 16 \delta_{a,d} \delta_{a,c} \lambda^2 - 8 \delta_{a,d} \delta_{a,c} \lambda^2 + 40 \delta_{a,b} \delta_{a,d} \lambda^2 \delta^2 - 12 \delta_{a,d} \lambda^2 \delta^2 - 8 \delta_{a,b} \lambda^2 \delta^2
\]

\[
+ 40 \delta_{a,b} \delta_{a,d} \lambda^2 \delta^2 - 12 \delta_{a,c} \lambda^2 \delta^2 - 8 \delta_{a,b} \lambda^2 \delta^2 + 8 \delta_{a,c} \delta_{a,d} \lambda^2 + 8 \delta_{a,b} \delta_{a,d} \lambda^2
\]

\[
- 16 \delta_{a,c} \lambda^2 + 24 \delta_{a,b} \delta_{a,d} \lambda^2 - 16 \delta_{a,d} \lambda^2 + 8 \delta_{a,b} \lambda^2 \delta^2 - 12 \delta_{a,d} \lambda^2 \delta^2 - 8 \delta_{a,c} \lambda^2 \delta^2
\]

\[
- 12 \delta_{a,c} \lambda^2 \delta^2 - 16 \delta_{a,b} \delta_{a,d} \lambda^4 + 8 \delta_{a,d} \lambda^4 - 16 \delta_{a,b} \delta_{a,d} \lambda^4 + 8 \delta_{a,c} \lambda^4 + 8 \delta_{a,b} \lambda^4 + 8 \delta_{a,c} \lambda^4
\]
and

\[
D_{\text{diagonal}} = \frac{1}{16} \left[ +24 \delta_{a,b} \delta_{a,c} \lambda^4_{a} - 32 \delta_{a,b} \delta_{a,c} \lambda^4_{a} + 24 \delta_{a,b} \delta_{a,c} \lambda^4_{a} - 12 \delta_{a,c} \lambda^4_{a} \\
-16 \delta_{a,c} \lambda^4_{a} + 6 \delta_{a,b} \delta_{a,c} \lambda^4_{a} - 12 \delta_{a,b} \lambda^4_{a} + 6 \delta_{a,b} \delta_{a,c} \lambda^4_{a} + 20 \delta_{a,c} \delta_{a,b} \lambda^2_{a} \lambda^2_{b} \\
+20 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} + 60 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} - 18 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} - 12 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} + 60 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} \\
-18 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} - 12 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} + 6 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} + 6 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} - 12 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} \\
-18 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} - 12 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} + 6 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} - 12 \delta_{a,b} \delta_{a,c} \lambda^2_{a} \lambda^2_{b} + 6 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} + 6 \delta_{a,b} \lambda^2_{a} \lambda^2_{b} \right]
\]

For \( a > b > 0 \),

\[
\sum_{N \geq c > d > 0} D_{\text{upper}} = \frac{1}{16} \left[ -16(a-1) \lambda^4_{a} + 8 \lambda^4_{a} - 16(N-a) \lambda^4_{a} + 16 \lambda^2_{a} \lambda^2_{b} \\
-12 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} - 8 \lambda^2_{a} \sum_{c=b+1}^{N} \lambda^2_{c} - 12 \lambda^4_{a} \sum_{c=a+1}^{N} \lambda^2_{c} - 8 \lambda^2_{a} \sum_{c=b+1}^{N} \lambda^2_{c} + 12 \lambda^4_{a} \sum_{c=a+1}^{N} \lambda^2_{c} + 8 \lambda^2_{a} \sum_{c=b+1}^{N} \lambda^2_{c} \\
-16(b-1) \lambda^4_{b} - 16(N-b) \lambda^4_{b} - 8 \lambda^2_{b} \sum_{c=a+1}^{N} \lambda^2_{c} - 12 \lambda^4_{b} \sum_{c=a+1}^{N} \lambda^2_{c} - 8 \lambda^2_{b} \sum_{c=b+1}^{N} \lambda^2_{c} - 12 \lambda^4_{b} \sum_{c=b+1}^{N} \lambda^2_{c} \\
-8 \lambda^2_{b} \sum_{d=1}^{b-1} \lambda^2_{d} + 12 \lambda^4_{b} \sum_{d=1}^{b-1} \lambda^2_{d} + 8 \sum_{c=a+1}^{N} \lambda^4_{c} + 8 \sum_{c=b+1}^{N} \lambda^4_{c} + 8 \sum_{d=1}^{b-1} \lambda^2_{d} + 8 \sum_{d=1}^{b-1} \lambda^2_{d} \right]
\]

For \( a = b > 0 \),

\[
\sum_{N \geq c > d > 0} D_{\text{upper}} = \frac{1}{16} \left[ +32(a-1) \lambda^4_{a} + 32(N-a) \lambda^4_{a} - 16(a-1) \lambda^4_{a} - 16(N-a) \lambda^4_{a} \\
+40 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} - 12 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} - 8 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} + 40 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} - 12 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} \\
-8 \lambda^2_{a} \sum_{d=1}^{a-1} \lambda^2_{d} - 16(a-1) \lambda^4_{a} - 16(N-a) \lambda^4_{a} - 8 \lambda^2_{a} \sum_{c=a+1}^{N} \lambda^2_{c} - 12 \lambda^2_{a} \sum_{d=1}^{a-1} \lambda^2_{d} - 16 \sum_{c=a+1}^{N} \lambda^4_{c} \\
+8 \sum_{c=a+1}^{N} \lambda^4_{c} + 8 \sum_{c=a+1}^{N} \lambda^4_{c} - 8 \sum_{d=1}^{a-1} \lambda^2_{d} + 8 \sum_{d=1}^{a-1} \lambda^2_{d} \right]
\]

For \( a > b > 0 \),

\[
\sum_{N \geq c > d > 0} D_{\text{diagonal}} = \frac{1}{16} \left[ -12 \lambda^4_{a} - 16 \lambda^4_{a} - 12 \lambda^4_{a} - 18 \lambda^4_{a} - 12 \lambda^2_{a} \lambda^2_{b} + 18 \lambda^4_{a} - 12 \lambda^2_{a} \lambda^2_{b} \\
-12 \lambda^4_{b} - 16 \lambda^4_{b} - 12 \lambda^4_{b} - 12 \lambda^2_{a} \lambda^2_{b} - 18 \lambda^4_{b} - 12 \lambda^2_{a} \lambda^2_{b} - 18 \lambda^4_{b} + 6 \lambda^4_{a} + 6 \lambda^4_{b} + 6 \lambda^4_{a} + 6 \lambda^4_{b} \right]
\]

For \( a = b > 0 \),

\[
\sum_{N \geq c > d > 0} D_{\text{diagonal}} = 0
\]
Therefore, for $a > -b > 0$,
\[
\text{Ric}^N(\nu^{ab}) = \sum_{N \geq c > d > 0} D_{\text{upper}} + \sum_{N \geq c = d > 0} D_{\text{diagonal}}
\]
\[
= \frac{1}{16} \left[ -40\lambda_a^4 - 40\lambda_b^4 - 32\lambda_a^2\lambda_b^2 - 12\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 8\lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 - 8\lambda_b^2 \sum_{d=1}^{a-1} \lambda_d^2 \\
-12\lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 + 8 \sum_{d=1}^{a-1} \lambda_d^2 + 8 \sum_{d=1}^{b-1} \lambda_d^2 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 \\
-8\lambda_a^2 \sum_{c=b+1}^{N} \lambda_c^2 - 8\lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12\lambda_b^2 \sum_{c=b+1}^{N} \lambda_c^2 + 8 \sum_{c=a+1}^{N} \lambda_c^4 + 8 \sum_{c=b+1}^{N} \lambda_c^4 \right]
\]
and, for $a = -b > 0$,
\[
\text{Ric}^N(\nu^{ab}) = \sum_{N \geq c > d > 0} D_{\text{upper}} + \sum_{N \geq c = d > 0} D_{\text{diagonal}} = 0
\]

Remark 7.15. Taking $N \to \infty$, we see that the Ricci curvature is negative infinity in most directions.

REFERENCES

1. Airault, H. and Malliavin, P.: Regularized Brownian motion on the Siegel disk of infinite dimension, \textit{Ukraïn. Mat. Zh.}, \textbf{52} (2000) 1158–1165.
2. Airault, H. and Malliavin, P.: Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry, \textit{J. Funct. Anal.}, \textbf{241} (2006) 99–142.
3. DaPrato, G. and Zabczyk, J.: \textit{Stochastic Equations in Infinite Dimensions}, Cambridge University Press, Encyclopedia of mathematics and its applications, 1992.
4. Fang, S.: Canonical Brownian motion on the diffeomorphism group of the circle, \textit{J. Funct. Anal.} \textbf{196} (2002) 162–179.
5. Fang, S. and Luo, D.: Flow of homeomorphisms and stochastic transport equations, \textit{Stoch. Anal. Appl.}, \textbf{25} (2007) 1079–1108.
6. Gordina, M.: Holomorphic functions and the heat kernel measure on an infinite-dimensional complex orthogonal group, \textit{Potential Anal.}, \textbf{12} (2000) 325–357.
7. Gordina, M.: Heat kernel analysis on infinite dimensional groups, in: \textit{Infinite dimensional harmonic analysis III}, (2005) 71–81 World Sci. Publ., Hackensack, NJ.
8. Gordina, M.: \textit{Riemannian geometry of Diff(S^1)/S^1}, Journal of Functional Analysis, \textbf{239}, Issue 2, 2006, pp.611-630.
9. Gordina, M.: \textit{Hilbert-Schmidt groups as infinite-dimensional Lie groups and their Riemannian geometry}, Journal of Functional Analysis, \textbf{227}, 2005, pp.245-272.
10. Milnor, J.: \textit{Curvatures of left invariant metrics on Lie groups}, Advances in Math., \textbf{21}, 1976, pp.293-329.
11. Katznelson, Y.: \textit{An Introduction to Harmonic Analysis}, Cambridge University Press, Cambridge Mathematical Library, 2004.
12. Segal, G.: Unitary Representations of some Infinite Dimensional Groups, \textit{Commun. Math. Phys.}, \textbf{80} (1981) 301–341.

\textit{E-mail address: mangwu@math.ucr.edu}