Research Article

An Inertial Iterative Algorithm with Strong Convergence for Solving Modified Split Feasibility Problem in Banach Spaces

Huijuan Jia,1,2 Shufen Liu,1 and Yazheng Dang3

1College of Computer Science and Technology, Jilin University, 2699 Qianjin Street, Chaoyang District, Changchun 130012, China
2College of Computer Science and Technology, Henan Polytechnic University, 2001 Century Avenue, Shanyang District, Jiaozuo 454003, China
3School of Business, University of Shanghai for Science and Technology, 516 Jungong Road, Yangpu District, Shanghai 200093, China

Correspondence should be addressed to Yazheng Dang; jgdyz@163.com

Received 24 March 2021; Accepted 30 April 2021; Published 11 May 2021

Academic Editor: Ching-Feng Wen

Copyright © 2021 Huijuan Jia et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we propose an iterative scheme for a special split feasibility problem with the maximal monotone operator and fixed-point problem in Banach spaces. The algorithm implements Halpern’s iteration with an inertial technique for the problem. Under some mild assumptions of the monotonicity of the related mapping, we establish the strong convergence of the sequence generated by the algorithm which does not require the spectral radius of $A^T A$. Finally, the numerical example is presented to demonstrate the efficiency of the algorithm.

1. Introduction

The split feasibility problem (shortly SFP) introduced by Censor and Elfving [1] in 1994 can be defined as follows: find a point $x^*$ satisfying

$$
\begin{align*}
  x^* & \in C, \\
  Ax^* & \in Q,
\end{align*}
$$

where $C$ and $Q$ are nonempty closed-convex subsets of real Hilbert spaces $H^1$ and $H^2$, respectively, and $A: H^1 \rightarrow H^2$ is a bounded linear operator. The SFP has broad application in modelling real-world problems such as the inverse problem in signal processing, radiotherapy, and data compression (for example, see [2–4]). Various algorithms have been invented by several authors for solving the SFP and related optimization problems (for example, see [5–10]). The CQ algorithm is one of the most popular solvers for SFP which was first proposed by Byrne [11], taking an initial point arbitrarily and defining the iterative step as

$$
\begin{align*}
  x^{k+1} & = P_C(x^k - \mu A^T(I - P_Q)Ax^k), \quad \forall k \geq 1,
\end{align*}
$$

where $\mu \in (0, (2/\rho(A^T A)))$, $\rho(A^T A)$ is the spectral radius of $A^T A$, and $P_C$ and $P_Q$ denote the metric projections of $H^1$ onto $C$ and $H^2$ onto $Q$, respectively, that is, $P_C(x) = \arg\min_{y \in C} \|x - y\|$ over all $x \in C$. It was proved that the sequence $\{x_n\}$ generated by (2) converges weakly to a solution of the SFP provided the step size $\mu \in (0, (2/\rho(A^T A)))$. As an extension of this CQ algorithm, several iterative algorithms have been invented for solving SFP in Hilbert spaces and Banach spaces (for example, see [12–17]).

Let $H$ be a real Hilbert space, $F$ be a strictly convex, reflexive smooth Banach space, $J_F$ denotes the duality mapping on $F$, and $C$ and $Q$ be nonempty closed-convex subsets of $H$ and $F$, respectively. The following Halpern’s type iteration algorithm was proposed by Alsulami and Takahashi [13] in 2015. Let $\{t^k\}$ be a sequence in $H$ such that $t^k \rightarrow t \in H$ and $x^1, t^1 \in H$:

$$
\begin{align*}
  v^k & = \lambda_k t^k + (1 - \lambda_k)P_C(x^k - rA^T(I - P_Q)Ax^k), \\
  x^{k+1} & = \alpha_k x^k + (1 - \alpha_k)v^k, \quad k \geq 1,
\end{align*}
$$

where $\lambda_k \in (0, 1)$ and $r \in (0, \rho^{-1}(A^T A))$. It was proved that the sequence $\{x_n\}$ generated by the algorithm converges strongly to a solution of the SFP provided $\min_{1 \leq k \leq n}(1 - \lambda_k) > 0$.
where $0 < r < \infty$ and $\{a_k\} \subset (0, 1)$. It was proved that the sequence $\{x^n_k\}$ defined by (3) converges strongly to a point $\omega_0 \in C \cap A^{-1}(Q)$; for some $\omega_0 = \frac{b}{\sum a_k}, \forall a, b \in \mathbb{R}$ if $0 < r \|A\|^2 < 2$, $\lim_{k \to \infty} \lambda_k = 0$, $\sum_{k=1}^\infty \lambda_k = \infty$, and $0 < a \leq b < 1$.

However, the previous algorithms only use the current point to get the next iteration which can lead slow convergence. Inertial technique, as an accelerated method, was first proposed by Polyak [18] to speed up the convergence rate of smooth convex minimization. Subsequently, F. Alvarez in [19] combined with a proximal method to solve the problem of finding the zero of a maximal monotone operator. The main idea of this method is to make use of two previous iterates in order to update the next iterate. Due to the fact that the presence of the inertial term in an algorithm speeds up the convergence rate, inertial type algorithms have been widely studied by authors [5, 20, 21].

In this paper, we study the following modified SFP in the real Banach space:

Find $x \in F(T) \cap C$ such that

$$Ax \in B^{-1}(0),$$

(4)

where $C$ is a nonempty, closed-convex subset of $E_1$, $T: C \to C$ is a Bregman weak relatively nonexpansive mapping, and $B: E_2 \to 2^{E_2}$ is a maximal monotone operator. $E_1$ and $E_2$ are $p$-uniformly convex and uniformly smooth real Banach spaces, $E_1^*$ and $E_2^*$ be the duals of $E_1$ and $E_2$, respectively, $A: E_1 \to E_2$ be a bounded linear operator, and $A^*: E_2^* \to E_1^*$ be the adjoint of $A$. We shall denote the value of the functional $x^* \in E^*$ at $x \in E$ by $\langle x^*, x \rangle$. Obviously, the modified SFP (4) is more general than (1).

Motivated by the above results, in this paper, we present an inertial algorithm for solving (4) in $p$-uniformly convex and uniformly smooth Banach spaces which have strong convergence. Our algorithm is designed to employ previous iterations $x^k$ and $x^{k-1}$ to obtain the next iterative point; all the implementation process does not compute the spectral radius of $A^T A$, which improves the feasibility of the algorithm.

The paper is organized as follows. Section 2 reviews some preliminaries. Section 3 gives the inertial iterative algorithm and its convergence analysis. Section 4 gives a numerical experiment. Some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries’ results which will be useful for our convergence analysis in this paper. We denote the strong and weak convergence of the sequence $\{x_n\}$ to a point $x$ by $x^k \to x$ and $x^k \rightharpoonup x$, respectively.

Let $E$ be a real Banach space and $1 < q \leq 2 \leq p < \infty$ and $(1/p) + (1/q) = 1$. Define the modulus of smoothness of $E$ as

$$\rho_E(\tau) = \sup \left\{ \frac{\|x - y\| + \|x + y\|}{2} - 1: \|x\| = 1, \|y\| = \tau \right\},$$

(5)

where $E$ is uniformly smooth if and only if $\lim_{\tau \to 0^+} (\rho_E(\tau)/\tau) = 0$ and $E$ is said to be $q$-uniformly smooth if there exists a constant $D_q > 0$ such that $\rho_E(\tau) \leq D_q \tau^q$.

Define the modulus of convexity of $E$ as

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|^q}{2} : x = y = 1; \epsilon = \|x - y\| \right\},$$

(6)

where $E$ is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2)$ and $E$ is $p$-uniformly convex if there is a constant $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p \epsilon^p$ for all $\epsilon \in (0, 2)$. Every uniformly convex Banach space is strictly convex and reflexive. It is known that if $E$ is $p$-uniformly convex and uniformly smooth, then its dual $E^*$ is $q$-uniformly smooth and uniformly convex.

Definition 1 (see [22]). Let $p > 1$. Define the generalized duality mapping $J_E^p: E \to 2^{E^*}$ as

$$J_E^p = \{ x^* \in E^* : \langle x^*, x \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \}.$$  

(7)

It is known that when $E$ is uniformly smooth, then $J_E^p$ is norm-to-norm uniformly continuous on bounded subsets of $E$, and $E$ is smooth if and only if $J_E^p$ is single valued. $J_E^p$ is said to be weak-to-weak continuous if

$$x^k \rightharpoonup x \Rightarrow \langle J_E^p(x^k), y \rangle \to \langle J_E^p(x), y \rangle,$$

for any $y \in E$.  

(8)

Lemma 1 (see [23]). Let $x$ and $y \in E$. If $E$ is a $q$-uniformly smooth Banach space, then there exists a $D_q > 0$ such that

$$\|x - y\|^q \leq \|x\|^q - q \langle J_E^p(x), y \rangle + D_q \|y\|^q.$$  

(9)

Definition 2 (see [24]). A function $f: E \to \mathbb{R} \cup \{+\infty\}$ is said to be

(i) proper if its effective domain $f = \{x \in E : f(x) < +\infty\}$ is nonempty

(ii) convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \forall \lambda \in (0, 1)$ and $x$ and $y \in D(f)$

(iii) lower semicontinuous at $x_0 \in D(f)$ if $f(x_0) \leq \lim_{x \to x_0} \inf f(x)$
Define Definition 3. Let $f: E \rightarrow R$ be a differentiable and convex function. The Bregman distance denoted as $\Delta_f: \text{dom} f \times \text{dom} f \rightarrow [0, +\infty)$ is defined as

$$\Delta_f(x,y) = f(y) - f(x) - \langle f'(x), y - x \rangle, \quad x, y \in E,$$

(10)

where $f'(x)$ is the value of the gradient $f$ at $x$.

It is worthy to note that the duality mapping $J^p_E$ is actually the gradient of the function $f^p_E(x) = (1/p)\|x\|^p$, for $2 \leq p < \infty$. Hence, if $f = f^p$ in (10), the Bregman distance with respect to $f^p$ now becomes

$$\Delta_p(x,y) = \frac{1}{p} \|y\|^p - \frac{1}{p} \|x\|^p - \langle J^p_E(x), y - x \rangle,$$

$$= \frac{1}{p} \langle \|y\|^p - \|x\|^p \rangle + \langle J^p_E(x), y-x \rangle,$$

(11)

It is generally known that the Bregman distance is not a metric as a result of absence of symmetry, but it possesses some distance-like properties which are stated as follows:

$$\Delta_p(x,y) = \Delta_p(x,z) + \Delta_p(z,y) - \langle J^p_E(x), z - y \rangle,$$

$$\Delta_p(x,y) + \Delta_p(y,x) = \langle J^p_E(x) - J^p_E(y), x - y \rangle.$$

(12)

The relationship between the metric and Bregman distance in the $p$-uniformly convex space is as follows:

$$\|x - y\|^p \leq \Delta_p(x,y) \leq \langle J^p_E(x) - J^p_E(y), x - y \rangle,$$

(13)

where $\tau > 0$ is a fixed number.

Let $C$ be a nonempty closed-convex subset of $E$. The Bregman projection is defined as

$$\Pi_Cx = \arg\min_{y \in C} \Delta_{fp}(y,x), \quad x \in E.$$

(14)

And, the metric projection can be defined similarly as

$$P_Cx = \arg\min_{y \in C} \|y-x\|, \quad x \in E.$$

(15)

The Bregman projection is the unique minimizer of the Bregman distance and can be characterized by a variational inequality [25]:

$$\langle J^p_E(x), z - \Pi_Cx \rangle \leq 0, \quad \forall z \in C,$$

(16)

from which we have

$$\langle J^p_E(x), z - \Pi_Cx \rangle \leq 0, \quad \forall z \in C.$$

(17)

The metric projection which is also the unique minimizer of the norm distance can be characterized by the following variational inequality:

$$\langle J^p_E(x - P_Cx), z - P_Cx \rangle \leq 0, \quad \forall z \in C.$$

(18)

We define the functional $V_p: E \times E \rightarrow [0, \infty]$ associated with $f^p(x) = (1/p)\|x\|^p$ by

$$V_p(x,x) = \frac{1}{p} \|x\|^p - \langle \nabla x, x \rangle + \frac{1}{q} \|x\|^q, \quad x \in E, x \in E^*,$$

(19)

where $V_p(x,x) \geq 0$. It then follows that

$$V_p(x,x) = \Delta_p(x, J^p_E(x)) \quad \forall x \in E, x \in E^*.$$

(20)

Chuasuk et al. [26] proved the following inequality:

$$V_p(x,x) + \langle J^p_E(x), x - y \rangle \leq V_p(x,x + y) \quad \forall x \in E, x \in E^*.$$

(21)

Furthermore, $Vp$ is convex in the second variable, and thus, for all $z \in E, \{x_i\}_{i=1}^N \subseteq \{1,0\},$ and $\sum_{i=1}^N t_i = 1$, we have (see [23])

$$\Delta_p \left( z, \sum_{i=1}^N t_i J^p_E(x_i) \right) = V_p \left( z, \sum_{i=1}^N t_i J^p_E(x_i) \right) \leq \sum_{i=1}^N t_i \Delta_p(z, x_i).$$

(22)

Let $C$ be a nonempty, closed, and convex subset of a smooth Banach space $E$, and let $T: C \rightarrow C$. A point $x^* \in C$ is called an asymptotic fixed point of $T$ if a sequence $\{x_n\}_{n \in \mathbb{N}}$ exists in $C$ and converges weakly to $x^*$ such that

$$\lim_{n \rightarrow \infty} \|x_n - T(x_n)\| = 0.$$ We denote the set of all asymptotic fixed points of $T$ by $\Phi(T).$ Moreover, a point $x^* \in C$ is said to be a strong asymptotic fixed point of $T$ if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $C$ which converges strongly to $x^*$ such that

$$\lim_{k \rightarrow \infty} \|x_n - T(x_n)\| = 0.$$ We denote the set of all strong asymptotic fixed points of $T$ by $\Phi(T).$ It follows from the definitions that $\Phi(T) \subseteq \Phi(T) \subseteq \Phi(T).$

Definition 4 (see [27]). Let $T$ be a mapping such that $T: C \rightarrow E$. $T$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x$ and $y \in C$

(ii) quasi-nonexpansive if $\|Tx - y^*\| \leq \|x - y^*\|$ such that $F(T) \neq \emptyset, \forall x \in C,$ and $y^* \in F(T)$

Definition 5 (see [28]). Let $T: C \rightarrow E$ be a mapping. $T$ is said to be

(1) Bregman nonexpansive if

$$\Delta_p(Tx, Ty) \leq \Delta_p(x, y), \quad \forall x$ and $y \in C.$$

(23)

(2) Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \quad \forall x \in C$ and $y^* \in F(T).$

(24)
(3) Bregman weak relatively nonexpansive if \( \widetilde{F}(T) \neq \emptyset, \widetilde{F}(T) = F(T) \), and
\[
\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \quad \forall x \in C \text{ and } y^* \in F(T).
\] (25)

(4) Bregman relatively nonexpansive if \( F(T) \neq \emptyset, \widetilde{F}(T) = F(T) \), and
\[
\Delta_p(y^*, Tx) \leq \Delta_p(y^*, x), \forall x \in C \text{ and } y^* \in F(T).
\] (26)

From the definitions, it is evident that the class of Bregman quasi-nonexpansive maps contains the class of Bregman weak relatively nonexpansive maps. The class of Bregman weak relatively nonexpansive maps contains the class of Bregman relatively nonexpansive maps.

Let \( E \) be a smooth, strictly convex, and reflexive Banach space and \( C \) be a smooth, strictly convex, and reflexive Banach space. Let \( F \) be a monotone, we can obtain (30) from (31) and
\[
\|ax + (1 - \alpha)y\| \leq \alpha\|x\| + (1 - \alpha)\|y\| - W_q(\alpha)g(\|x - y\|),
\] (35)

where \( W_q = a^q(1 - \alpha) + \alpha(1 - \alpha)^q \) and \( B_r = \{ x \in E : \|x\| \leq r \} \).

3. Inertial Iteration Algorithm and Its Strong Convergence

In this section, we present our inertial iterative algorithm for solving the modified SFP (4) in Banach spaces. We also prove its strong convergence under some suitable conditions.

3.1. Inertial Iteration Algorithm. Now, we give our inertial iterative algorithm.

Algorithm 3.1. Suppose \( \Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) = \emptyset \). Let \( \{ \alpha_k \} \subset (0, 1) \), \( 0 < \liminf_k \gamma_k \leq \limsup_k \gamma_k < 1 \), \( x^0 \) and \( x^1 \in C = H_1 \), and \( \{ \theta_k \} \subset (0, 1) \) be a real sequence, and \( r_k > 0 \). Assuming \( x^{k-1} \) and \( x^k \) have been constructed, we calculate the next iterate \( x^{k+1} \) via the following formulas:

\[
w^k = J_{E_i}^{p} \left[ J_{E_i}^{p} \left( x^k \right) + \theta_k \left( J_{E_i}^{p} \left( x^k \right) - J_{E_i}^{p} \left( x^{k-1} \right) \right) \right],
\] (36a)

\[
u^k = \Pi_i J_{E_i}^{p} \left[ J_{E_i}^{p} \left( w^k \right) - \gamma_k \mu_k A^* J_{E_i}^{p} \left( 1 - Q_{r_k}^{p} \right) A w^k \right],
\] (36b)

\[
u^k = J_{E_i}^{p} \left[ \alpha_k J_{E_i}^{p} \left( v^k \right) + (1 - \alpha_k) J_{E_i}^{p} \left( T v^k \right) \right],
\] (36c)

\[
C_k = \left\{ u \in E_i : \langle J_{E_i}^{p} \left( w^k \right) - J_{E_i}^{p} \left( u^k \right), u \rangle \leq \frac{1}{\eta} \left( \| w^k \|^p - \| u^k \|^p \right) \right\},
\] (36d)

\[
H_k = \left\{ u \in E_i : \langle x^k - u, J_{E_i}^{p} \left( x^1 \right) - J_{E_i}^{p} \left( x^k \right) \rangle \geq 0 \right\},
\] (36e)

\[
x^{k+1} = \Pi_{C_i \cap H_k} x^k, \quad \forall k \in N,
\] (36f)

where
\[
\mu_k = \left( D_q \left\| A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k \right\|^q \right)^{(1/(q - 1))}, \quad \gamma_k \in (0, 1).
\]

We can see that during the iteration, it does not require to compute the spectral radius of \(ATA\).

### 3.2. Convergence

Suppose \( \Gamma = F(T) \cap C \cap A^{-1} (B^1 (0)) \neq \emptyset \). Now, we prove the following lemmas which will be used to establish the strong convergence.

**Lemma 5.** \( \{x^k\} \) generated by (36a)-(36f) is well-defined.

\[
\Delta_p(x^*, u^k) = \Delta_p(x^*, \Pi_C J_{E_k}^p \left[ J_{E_k}^p (w^k) - \gamma_k \mu_k A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k \right]),
\]

\[
\leq \Delta_p(x^*, J_{E_k}^p \left[ J_{E_k}^p (w^k) - \gamma_k \mu_k A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k \right]),
\]

\[
= \frac{\|x^k\|^p}{\alpha} - \langle J_{E_k}^p (w^k), x^* \rangle + \langle \gamma_k \mu_k A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k, x^* \rangle + \frac{1}{q} \left\| J_{E_k}^p (w^k) \right\|^q,
\]

\[
\leq \frac{\|x^k\|^p}{\alpha} - \langle J_{E_k}^p (w^k), x^* \rangle + \gamma_k \mu_k \langle J_{E_k}^p (I - Q_{r_k}^B) Aw^k, Ax^* \rangle + \frac{1}{q} \left\| J_{E_k}^p (w^k) \right\|^q
\]

\[

\leq \frac{\|x^k\|^p}{\alpha} - \langle J_{E_k}^p (w^k), x^* \rangle + \frac{1}{q} \left\| J_{E_k}^p (w^k) \right\|^q + \gamma_k \mu_k \langle J_{E_k}^p (I - Q_{r_k}^B) Aw^k, Ax^* - Aw^k \rangle
\]

\[
+ \frac{D_q \gamma_k \mu_k^2 q}{q} \left\| A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k \right\|^q,
\]

\[
= \Delta_p(x^*, w^k) + \mu_k \gamma_k \langle J_{E_k}^p (I - Q_{r_k}^B) Aw^k, Ax^* - Q_{r_k}^B Aw^k + Q_{r_k}^B Aw^k - Aw^k \rangle
\]

\[
+ \frac{D_q \gamma_k \mu_k^2 q}{q} \left\| A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k \right\|^q,
\]

\[
= \Delta_p(x^*, w^k) + \gamma_k \mu_k \langle J_{E_k}^p (I - Q_{r_k}^B) Aw^k, Ax^* - Q_{r_k}^B Aw^k \rangle
\]

\[
- \gamma_k \mu_k \langle J_{E_k}^p (I - Q_{r_k}^B) Aw^k, Aw^k - Q_{r_k}^B Aw^k \rangle
\]

\[
+ \frac{D_q \gamma_k \mu_k^2 q}{\alpha} \left\| A^* J_{E_k}^p (I - Q_{r_k}^B) Aw^k \right\|^q,
\]

\[
= \Delta_p(x^*, w^k) + \delta(x^*, u^k) + \alpha_k \Delta_p(x^*, v^k) + \alpha_k \Delta_p(x^*, T^k),
\]

\[
\leq \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, T^k),
\]

\[
\leq \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, v^k),
\]

\[
= \Delta_p(x^*, v^k).
\]
where the second inequality is from Lemma 2. Furthermore, from (32), we have
\begin{align*}
\Delta_p(x^*, v^k) &\leq \Delta_p(x^*, w^k) \\
&\quad - \gamma_k \mu_k \langle f''_E(I - Q_{\Gamma_k}) Aw^k, (I - Q_{\Gamma_k}) Aw^k \rangle \\
&\quad + \frac{D_q \gamma_k^{(p-1)}}{q} \| A^* f'_E(I - Q_{\Gamma_k}) Aw^k \|_p^p.
\end{align*}
(40)

By the definitions of \( \mu_k \), we have
\begin{equation}
\Delta_p(x^*, v^k) \leq \Delta_p(x^*, w^k)
\end{equation}
and
\begin{align*}
- \gamma_k \mu_k \langle (I - Q_{\Gamma_k}) Aw^k \rangle^p \\
&\quad + \frac{D_q \gamma_k^{(p-1)}}{q} \| A^* f'_E(I - Q_{\Gamma_k}) Aw^k \|_p^p.
\end{align*}
(41)

Since \( \gamma_k^{(p-1)} \in (0, 1) \), we have
\begin{equation}
\Delta_p(x^*, v^k) \leq \Delta_p(x^*, w^k).
\end{equation}
(42)

Thus,
\begin{equation}
\Delta_p(x^*, u^k) \leq \Delta_p(x^*, w^k),
\end{equation}
(43)

which implies
\[ \langle f''_E(u^k) - f''_E(u^k), u^k \rangle \leq (1/q) \| u^k \|_p^p - \| u^k \|_p^p. \]

So \( \Gamma \subset C_{x^*} \), for all \( k \in \mathbb{N} \). Since \( x^{k+1} = \Pi_{C_0 \cap H_k}(x^*) \), then
\[ \langle f''_E(x^1), v - x^{k+1} \rangle \leq 0, \quad \forall v \in C_0 \cap H_k \subset C. \]

Therefore, \( C_0 \cap H_k \) is nonempty, and thus, \( x^{k+1} = \Pi_{C_0 \cap H_k}(x^*) \) is well-defined.

**Lemma 6.** Let \( \{ x^k \} \) be a sequence generated by Algorithm 3. Then,
\begin{align*}
(1) \lim_{k \to \infty} \| x^{k+1} - x^k \| &= 0 \\
(2) \lim_{k \to \infty} \| x^k - u^k \| &= 0 \\
(3) \lim_{k \to \infty} \| T x^k - v^k \| &= 0 \\
(4) \lim_{k \to \infty} \| x^k - v^k \| &= 0 \\
(5) \lim_{k \to \infty} \| (I - Q_{\Gamma_k}) Aw^k \| &= 0
\end{align*}

**Proof**

(1) Let \( z \in \Gamma \). Since \( \Gamma \subset C_0 \cap H_k \), \( \forall k \geq 1 \), and \( x_{k+1} = \Pi_{C_0 \cap H_k}(x^*) \), we have
\[ \Delta_p(x^{k+1}, x^1) \leq \Delta_p(z, x^1), \quad \forall k \geq 1. \]
(44)

Thus, \( \{ \Delta_p(x^{k+1}, x^1) \} \) is bounded.

We observe that \( x^{k+1} \in H_k \), and by (17), we have
\[ \langle f''_E(x^1) - f''_E(x^1), x^k - x^{k+1} \rangle \leq 0. \]
(45)

Also, by (18), we have
\[ \Delta_p(x^{k+1}, x^1) \leq \Delta_p(x^{k+1}, x^1) - \Delta_p(x^1, x^1), \quad \forall k \geq 1, \]
(46)

that is, \( \Delta_p(x^1, x^k) \leq \Delta_p(x^{k+1}, x^1) - \Delta_p(x^{k+1}, x^k) \).

Thus, \( \Delta_p(x^1, x^1) \leq \Delta_p(x^{k+1}, x^1) \); therefore, \( \{ \Delta_p(x^1, x^1) \} \) is a bounded monotone nondecreasing sequence. Hence, \( \lim_{k \to \infty} \Delta_p(x^1, x^1) \) exists. From (46), we have \( \lim_{k \to \infty} \Delta_p(x^{k+1}, x^k) = 0 \). Thus, using Lemma 2.8, we get
\[ \lim_{k \to \infty} \| x^{k+1} - x^k \| = 0. \]
(47)

(2) By the uniform continuity of \( f''_E \) on bounded subsets of \( E_1 \), from (47) we have
\[ \lim_{k \to \infty} \| f''_E(x^{k+1}) - f''_E(x^k) \| = \lim_{k \to \infty} \| f''_E(x^1) - f''_E(x^{k+1}) \| = 0. \]
(48)

From (36a), we obtain
\[ u^k = f''_E(x^1) + \theta_k (f''_E(x^1) - f''_E(x^{k+1})). \]
(49)

Then,
\[ f''_E(u^k) = f''_E(x^1) + \theta_k (f''_E(x^1) - f''_E(x^{k+1})), \]
(50)

which gives
\[ \| f''_E(u^k) - f''_E(x^1) \| = \theta_k \| f''_E(x^1) - f''_E(x^{k+1}) \|. \]
(51)

Therefore,
\[ \lim_{k \to \infty} \| f''_E(u^k) - f''_E(x^1) \| = 0. \]
(52)

Since \( f''_E \) is also uniformly continuous on bounded subsets of \( E_1 \), we have
\[ \lim_{k \to \infty} \| x^k - u^k \| = 0. \]
(53)

(3) From (47) and (53), we obtain
\[ \| x^{k+1} - w^k \| = \| x^{k+1} - x^k + x^k - w^k \| \leq \| x^{k+1} - x^k \| + \| x^k - w^k \| \quad \text{as } k \to \infty. \]
(54)

Note that, from the construction of \( C_0 \), we have that
\[ \Delta_p(x^{k+1}, u^k) \leq \Delta_p(x^{k+1}, w^k) \quad \text{as } k \to \infty. \]
(55)

Thus, \( \Delta_p(x^{k+1}, u^k) \to 0 \) as \( k \to \infty \); also from Lemma 3, we have
\[ \lim_{k \to \infty} \| x^{k+1} - u^k \| = 0. \]
(56)
Similarly, \( \|x^k - u^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - u^k\| \); it then follows that
\[
\lim_{n \to \infty} \|x^k - u^k\| = 0. \tag{57}
\]
Using Lemma 4, from (36c) we have
\[
\lim_{k \to \infty} \|u^k - u^k\| = 0. \tag{58}
\]

It follows from (53) and (57) that
\[
\Delta_p(x^*, u^k) = \Delta_p \left( x^*, J_{E^k}^{p} \left( \alpha_k J_{E^k}^{p} (v^k) + (1 - \alpha_k) J_{E^k}^{p} (T v^k) \right) \right),
\]
\[
= V_p(x^*, \alpha_k J_{E^k}^{p} (v^k) + (1 - \alpha_k) J_{E^k}^{p} (T v^k)),
\]
\[
= \frac{1}{p} \|x^*\|^p - \alpha_k \langle J_{E^k}^{p} (v^k), x^* \rangle - (1 - \alpha_k) \langle J_{E^k}^{p} (T v^k), x^* \rangle + \frac{1}{q} \|\alpha_k J_{E^k}^{p} v^k + (1 - \alpha_k) J_{E^k}^{p} T v^k\|^q,
\]
\[
\leq \frac{1}{p} \|x^*\|^p - \alpha_k \langle J_{E^k}^{p} (v^k), x^* \rangle - (1 - \alpha_k) \langle J_{E^k}^{p} (T v^k), x^* \rangle + \frac{1}{q} \|\alpha_k v^k + (1 - \alpha_k) T v^k\|^p
\]
\[
- \frac{W_q(\alpha_k)}{q} g \left( \|J_{E^k}^{p} (v^k) - J_{E^k}^{p} (T v^k)\| \right),
\]
\[
= \alpha_k \left( \frac{1}{p} \|x^*\|^p - \langle J_{E^k}^{p} (v^k), x^* \rangle + \frac{1}{q} \|v^k\|^q \right) + (1 - \alpha_k) \left( \frac{1}{p} \|x^*\|^p - \langle J_{E^k}^{p} (T v^k), x^* \rangle + \frac{1}{q} \|T v^k\|^q \right)
\]
\[
- \frac{W_q(\alpha_k)}{q} g \left( \|J_{E^k}^{p} (v^k) - J_{E^k}^{p} (T v^k)\| \right),
\]
\[
= \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, T(v^k)) - \frac{W_q(\alpha_k)}{q} g \left( \|J_{E^k}^{p} (v^k) - J_{E^k}^{p} (T v^k)\| \right). \tag{59}
\]

Since \( T \) is Bregman weak relatively nonexpansive, we have
\[
\Delta_p(x^*, u^k) \leq \alpha_k \Delta_p(x^*, v^k) + (1 - \alpha_k) \Delta_p(x^*, v^k) - \frac{W_q(\alpha_k)}{q} g \left( \|J_{E^k}^{p} (v^k) - J_{E^k}^{p} (T v^k)\| \right),
\]
\[
= \Delta_p(x^*, v^k) - \frac{W_q(\alpha_k)}{q} g \left( \|J_{E^k}^{p} (v^k) - J_{E^k}^{p} (T v^k)\| \right),
\]
\[
\leq \Delta_p(x^*, u^k) - \frac{W_q(\alpha_k)}{q} g \left( \|J_{E^k}^{p} (v^k) - J_{E^k}^{p} (T v^k)\| \right). \tag{60}
\]
Hence, from (12) and (13), we get
\[
\frac{W_q(\alpha_k)}{q} g\left(\|J^p_{E_1}(v^k) - J^p_{E_1}(Tv^k)\|\right) \leq \Delta_p(x^*, w^k) - \Delta_p(x^*, u^k),
\]
\[
= \Delta_p(u^k, w^k) + \langle J^p_{E_1}(u^k) - J^p_{E_1}(w^k), v^k - w^k \rangle,
\]
\[
\leq \langle J^p_{E_1}(u^k) - J^p_{E_1}(w^k), v^k - w^k \rangle + \langle J^p_{E_1}(x^*) - J^p_{E_1}(u^k), u^k - w^k \rangle,
\]
\[
= \langle J^p_{E_1}(x^*) - J^p_{E_1}(w^k), u^k - w^k \rangle.
\]  
(61)

From (58), we have
\[
\lim_{k \to \infty} \frac{W_q(\alpha_k)}{q} g\left(\|J^p_{E_1}(v^k) - J^p_{E_1}(Tv^k)\|\right) = 0,
\]  
(62)
which implies
\[
\lim_{k \to \infty} g\left(\|J^p_{E_1}(v^k) - J^p_{E_1}(Tv^k)\|\right) = 0.
\]  
(63)
By the property of mapping \(g\), we obtain
\[
\lim_{k \to \infty} \|J^p_{E_1}(v^k) - J^p_{E_1}(Tv^k)\| = 0.
\]  
(64)
Since \(J^p_{E_1}\) is uniformly continuous on bounded subsets of \(E_1\), we have
\[
\lim_{k \to \infty} \|T(v^k) - v^k\| = 0.
\]  
(65)

(4) From Algorithm 3.1, we have that
\[
J^p_{E_1}(u^k) - J^p_{E_1}(v^k) = (1 - \alpha_k)(J^p_{E_1}(Tv^k) - J^p_{E_1}(v^k)).
\]  
(66)

Since \(0 < \liminf_{k \to \infty} \alpha_k \leq \limsup_{k \to \infty} \alpha_k < 1\) and from (65), we have
\[
\lim_{k \to \infty} \|J^p_{E_1}u^k - J^p_{E_1}v^k\| = 0.
\]  
(67)
Hence,
\[
\lim_{k \to \infty} \|u^k - v^k\| = 0.
\]  
(68)
It is easy to obtain
\[
\lim_{k \to \infty} \|u^k - v^k\| = 0,
\]  
(69)
(5) From (41), we have
\[
\sqrt[\gamma_k]{h_k} \left(1 - \sqrt[\gamma_k]{h_k}^{-1}\right) \|I - Q^B_{E_1}Aw^k\|^p \leq \Delta_p(x^*, w^k) - \Delta_p(x^*, v^k).
\]  
(70)
From (13) and (16), we get
\[
\gamma_k h_k \left(1 - \sqrt[\gamma_k]{h_k}^{-1}\right) \|I - Q^B_{E_1}Aw^k\|^p \leq \Delta_p(v^k, w^k) + \langle J^p_{E_1}x^* - J^p_{E_1}v^k, v^k - w^k \rangle,
\]
\[
\leq \langle J^p_{E_1}v^k, v^k - w^k \rangle + \langle J^p_{E_1}x^* - J^p_{E_1}v^k, v^k - w^k \rangle,
\]
\[
= \langle J^p_{E_1}x^* - J^p_{E_1}w^k, v^k - w^k \rangle.
\]  
(71)

It follows from (69) that
\[
\lim_{k \to \infty} \|I - Q^B_{E_1}Aw^k\|^p = 0.
\]  
(72)
Then,
\[
\lim_{k \to \infty} \|I - Q^B_{E_1}Aw^k\| = 0.
\]  
(73)
Now, we present the following strong convergence theorem for Algorithm 3.1.

**Theorem 1.** Suppose \(\Gamma = F(T) \cap C \cap A^{-1}(B^{-1}(0)) \neq \emptyset\). Then, the sequence \(\{x_n\}\) generated by Algorithm 3.1 converges strongly to \(u \in \Gamma\), where \(u = \Pi_{\Gamma}x_1\).

**Proof.** We have known in Lemma 6 (1) that
\[
\lim_{k \to \infty} \Delta_p(x^k, x^k) \text{ exists. Now, we show that } x^k \to x \in \Gamma.
\]  
Let \(m \text{ and } k \in \mathbb{N}\), then from Lemma 2, we have
\[
\Delta_p(x^m, x^k) \leq \Delta_p(x^m, x^1) - \Delta_p(x^k, x^1) \to 0.
\]  
(74)
Therefore, by Lemma 3, we get that \( \|x^m - x^k\| \to 0 \) as \( m \) and \( k \to \infty \). Thus, \{\{x^k\}\} is a Cauchy sequence in \( C \). Since \( C \) is closed and convex, it implies that there exists \( x \in C \) such that \( x^k \to x \) as \( k \to \infty \). Since \( \|x^k - \nu^k\| \to 0 \), \( \|T\nu^k - \nu^k\| \to 0 \), and \( T \) is a Bregman weak relatively nonexpansive mapping, then \( x \in \tilde{F}(T) \). More so, since \( \|x^k - u^k\| \to 0 \), then \( u^k \to x \) and by the linearity of \( A \), we have \( Au^k \to A\tilde{x} \). Also from (73), \( Q_{\nu^k}^B Au^k \to A\tilde{x} \). Since \( Q_{\nu^k}^B \) is a resolvent metric of \( B \) for \( r_k > 0 \), then for all \( k \in \mathbb{N} \), we have
\[
\frac{j_{E_k}(Au^k - Q_{\nu^k}^B(Au^k))}{r_k} \in BQ_{\nu^k}^B(Au^k) \tag{75}
\]
So for all \( (s, s^*) \in B \), we have
\[
0 \leq \langle s - Q_{\nu^k}^B(Au^k), s^* - \frac{j_{E_k}(Au^k - Q_{\nu^k}^B(Au^k))}{r_k} \rangle \tag{76}
\]
It follows from (73) that for all \( (s, s^*) \in B \), we have
\[
0 \leq \langle s^* - 0, s - A\tilde{x} \rangle \tag{77}
\]
Since \( B \) is maximal monotone, then it implies that \( A\tilde{x} \in B^{-1}(0) \); hence, \( \tilde{x} \in A^{-1}B^{-1}(0) \). Therefore, \( \tilde{x} \in \Gamma \).

Finally, we show that \( \tilde{x} = \Pi_{\Gamma} x^1 \). Suppose there exists \( y \in \Gamma \) such that \( y = \Pi_{\Gamma} x^1 \). Then,
\[
\Delta_p(\tilde{x}, x^1) \leq \Delta_p(\tilde{x}, x^1) \tag{78}
\]
We have shown in Lemma 5 that \( \Gamma \subset C_k \) and \( \forall k \geq 1 \), then \( \Delta_p(x^k, x^1) \leq \Delta_p(\tilde{x}, x^1) \). By the lower semicontinuity of the norm, we have
\[
\Delta_p(\tilde{x}, x^1) = \frac{\|x\|}{q} - \langle j_{E_k}^p \tilde{x}, x^1 \rangle + \frac{\|x^1\|^p}{p} \leq \liminf_{k \to \infty} \left\{ \frac{\|x\|}{q} - \langle j_{E_k}^p x^k, x^1 \rangle + \frac{\|x^1\|^p}{p} \right\}, \tag{79}
\]
\[
= \liminf_{k \to \infty} \Delta_p(\tilde{x}, x^1), \leq \limsup_{k \to \infty} \Delta_p(\tilde{x}, x^1) \leq \Delta_p(\tilde{x}, x^1). \tag{78}
\]
Combining (78) and (79), we have \( y = \tilde{x} \). This completes the proof.

4. Numerical Example

In this section, we present one numerical example to compare the performance of Algorithm 3.1 with Algorithm (3).

Example 1. Let \( E_1 = E_2 = \mathbb{R}^n \) and \( A \) be a \( n \times n \) randomly generated matrix. Let \( C = \{x \in \mathbb{R}^n : \langle a, x \rangle \geq 1 \} \), where \( a = (-5, -5, 0, 0, \ldots, 0) \in \mathbb{R}^n \). Then,
\[
\Pi_C(x) = P_C(x) = \frac{1 - \langle a, x \rangle}{\|a\|_2^2} a + x. \tag{80}
\]
Let \( B : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) be defined by \( B(x) = \{2x\} \), and \( T = P_C \). We take \( \theta_0 = (3/7)k, r_k = (1/2k) \), and \( \alpha_k = (k/3k + 1) \).

We select \( x^0 = (-1, -1, -1, -1, \ldots, -1) \in \mathbb{R}^n \) and \( x^1 = (-2, -2, -2, -2, \ldots, -2) \in \mathbb{R}^n \) as initial points and select \( r = 1 = y_k \) in the two algorithms. The iterative process of Algorithm 3.1 for the example is as follows:
\[
\begin{cases}
u^k = x^k + \frac{3}{7k} (x^k - x^{k-1}); \\
u^k = P_C(w^k - \mu_k A^* (I - Q_{\nu^k}^B) A w^k), \\
u^k = \frac{k}{3k + 1} \nu^k + \frac{2k + 1}{3k + 1} P_C(\nu^k), \\
k \in C_k = \{u \in E_1 : \langle w^k - u, u \rangle \leq \left( \|w^k\|^2 - \|u\|^2 \right), \\
H_k = \{u \in E_1 : \langle x^k - u, x^1 - x^k \rangle \geq 0 \}, \\
x^{k+1} = P_{C_k \cap H_k}(x^1), \forall k \in \mathbb{N},
\end{cases}
\]
where \( \mu_k \) is chosen as defined by (37) and \( Q_{\nu^k}^B(Au^k) = (k/k + 1)Au^k Q \) for all \( k \geq 1 \). We choose various values of \( n \) as follows:

- Case I: \( n = 10 \), Case II: \( n = 20 \), Case III: \( n = 40 \), and Case IV: \( n = 50 \), and use \( \left( \|x^{k+1} - x^k\|^2 / \|x^1 - x^k\|^2 \right) < 10^{-6} \) as the stopping criterion. We, thus, plot the graph of \( \|x^{k+1} - x^k\|^2 \) against the number of iteration in each case and compare the computation results of our algorithm with Algorithm (3) of Aladraji and Takahashi [13]. The computation results can be seen in Figure 1 and Table 1.

From Table 1, we can find that Algorithm 3.1 performs better in terms of number of iterations and CPU time taken for computation than Algorithm (3). From Figure 1, we can see that the error generated by Algorithm 3.1 in the previous iterative steps is ascending; then, it goes down quickly in the latter iterative steps and converges to zero, which is just the effect of the inertial technique, while the error generated by Algorithm (3) always decreases and converges slowly to zero. The results manifest that inertial technique is an effective method for improving the convergence.
Table 1: Computation result for the example.

| Case | CPU time (sec.) | Iter. |
|------|----------------|-------|
| I: \( n = 10 \) | 0.1960 | 0.157 | 247 | 56 |
| II: \( n = 20 \) | 0.2030 | 0.1630 | 315 | 94 |
| III: \( n = 40 \) | 0.2840 | 0.2280 | 422 | 156 |
| IV: \( n = 50 \) | 0.3260 | 0.2490 | 472 | 183 |

Figure 1: Example cases. (a) Top-Left Case I \( n = 10 \); (b) Top-Right Case II \( n = 20 \); (c) Bottom-Left Case III \( n = 40 \); (d) Bottom-Right Case IV \( n = 50 \).
5. Conclusion

In this paper, we introduced an inertial iterative algorithm for approximating a common solution of the split feasibility problem, monotone inclusion problem, and fixed-point problem for the class of Bregman weak relative non-expansive mapping in $p$-uniformly convex and uniformly smooth Banach spaces. Our algorithm is designed in such a way that its implementation does not require to compute the spectral radius of $A^TA$. We also proved a strong convergence theorem under some suitable conditions. Finally, a numerical example is given to test the accuracy and efficiency of our algorithm. The results in this paper improve and extend many related results in the literature.

Data Availability

No data were used to support this study.

Disclosure

Opinions expressed and conclusions arrived are those of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 61872126), Henan Province Key Science and Technology Project (Grant no. 192102210123), and Young Backbone Teachers in Universities of Henan Province (Grant no. 2019GGJS061).

References

[1] Y. Censor and T. Elfving, “A multiprojection algorithm using Bregman projections in a product space,” Numerical Algorithms, vol. 8, no. 2, pp. 221–239, 1994.
[2] Q. H. Ansari and A. Rehan, “Split feasibility and fixed point problems,” in Nonlinear Analysis: Approximation Theory, Optimization and Application, Q. H. Ansari, Ed., Springer, New York, NY, USA, pp. 281–322, 2014.
[3] Y. Censor, T. Elfving, N. Kopf, and T. Bortfeld, “The multiple-sets split feasibility problem and its applications for inverse problems,” Inverse Problems, vol. 21, no. 6, pp. 2071–2084, 2005.
[4] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, “A unified approach for inversion problems in intensity-modulated radiation therapy,” Physics in Medicine and Biology, vol. 51, no. 10, pp. 2353–2365, 2006.
[5] H. A. Abass, K. O. Aremu, L. O. Jolaoso, and O. T. Mewomo, “An inertial forward-backward splitting method for approximating solutions of certain optimization problems,” Journal of Nonlinear Functional Analysis, vol. 2020, Article ID 6, 20 pages, 2020.
[6] Y. Z. Dang, J. Sun, and S. Zhang, “Double projection algorithms for solving the split feasibility problems,” Journal of Industrial & Management Optimization, vol. 4, no. 15, pp. 2023–2034, 2019.
[7] Y. Z. Dang, Z. H. Xue, and Y. Gao, “Fast self-adaptive regularization iterative algorithm for solving split feasibility problem,” Journal of Industrial & Management Optimization, vol. 13, no. 5, pp. 1–15, 2017.
[8] Q. Yang, “The relaxed CQ algorithm solving the split feasibility problem,” Inverse Problems, vol. 20, no. 4, pp. 1261–1266, 2004.
[9] Y. Yao, W. Jigang, and Y.-C. Liou, “Regularized methods for the split feasibility problems,” Abstract and Applied Analysis, vol. 2012, 13 pages, Article ID 140679, 2012.
[10] B. Qu and N. Xiu, “A note on the CQ algorithm for the split feasibility problem,” Inverse Problems, vol. 21, no. 5, pp. 1655–1665, 2005.
[11] C. Byrne, “A unified treatment of some iterative algorithms in signal processing and image reconstruction,” Inverse Problems, vol. 20, no. 1, pp. 103–120, 2004.
[12] W. Takahashi, “The split feasibility problems in Banach spaces,” Journal of Nonlinear and Convex Analysis, vol. 15, no. 6, pp. 1349–1355, 2014.
[13] O. S. Iyiola and Y. Shehu, “A cyclic iterative method for solving multiple sets split feasibility problems in Banach Spaces,” Quaestiones Mathematicae, vol. 39, no. 7, pp. 959–975, 2016.
[14] S. M. Altsalami and W. Takahashi, “Iterative methods for the split feasibility problems in Banach spaces,” Journal of Nonlinear and Convex Analysis, vol. 16, pp. 585–596, 2015.
[15] M. Raiesi, G. Zanami Eskandani, and M. Eslamian, “A general algorithm for multiple-sets split feasibility problem involving resolvents and Bregman mappings,” Optimization, vol. 67, no. 2, pp. 309–327, 2018.
[16] M. Raiesi, “Split common null point and common fixed point problems between Banach spaces and Hilbert spaces,” Mediterranean Journal of Mathematics, vol. 14, 2017.
[17] M. Eslamian, “Split common fixed point and common null point problem,” Mathematical Methods in the Applied Sciences, vol. 40, no. 18, pp. 7410–7424, 2017.
[18] B. T. Polyak, “Some methods of speeding up the convergence of iteration methods,” USSR Computational Mathematics and Mathematical Physics, vol. 4, no. 5, pp. 1–17, 1964.
[19] F. Alvarez and H. Attouch, “An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping,” Set-Valued Analysis, vol. 9, no. 1/2, pp. 3–11, 2001.
[20] L. O. Jolaoso, T. O. Alakoya, A. Taiwo, and O. T. Mewomo, “Inertial extragradient method via viscosity approximation approach for solving equilibrium problem in Hilbert space,” Optimization, vol. 70, no. 2, pp. 387–412, 2020.
[21] T. O. Alakoya, L. O. Jolaoso, and O. T. Mewomo, “Modified inertial subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problems,” Optimization, vol. 70, no. 3, pp. 545–574, 2020.
[22] T. O. Alakoya, L. O. Jolaoso, and O. T. Mewomo, “A general iterative method for finding common fixed point of finite family of demicontractive mappings with accretive variational inequality problems in Banach spaces,” Nonlinear Studies, vol. 27, no. 1, pp. 1–24, 2020.
[23] Z.-B. Xu and G. F. Roach, “Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces,” Journal of Mathematical Analysis and Applications, vol. 157, no. 1, pp. 189–210, 1991.
[24] A. Taiwo, T. O. Alakoya, and O. T. Mewomo, “Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach
spaces,” *Numerical Algorithms*, vol. 86, no. 4, pp. 1359–1389, 2020.

[25] F. Schöpfer, T. Schuster, and A. K. Louis, “An iterative regularization method for the solution of the split feasibility problem in Banach spaces,” *Inverse Problems*, vol. 24, no. 5, p. 055008, 2008.

[26] P. Chuasuk, A. Farajzadeh, and A. Kaewcharoen, “An iterative algorithm for solving split feasibility problems and fixed point problems in p-uniformly convex and smooth Banach spaces,” *Journal of Computational Analysis and Applications*, vol. 28, no. 1, pp. 49–66, 2020.

[27] Y. Shehu, “Iterative methods for split feasibility problems in certain Banach spaces,” *Journal of Nonlinear and Convex Analysis*, vol. 16, no. 12, pp. 1–15, 2015.

[28] S. Semmes, Lecture Note on an Introduction to Some Aspects of Functional Analysis, 2: Bounded Linear Operators, p. 14, Rice University, Houston, TX, USA, 2008, http://maths.rice.edu.

[29] S. Suantai, Y. Shehu, and P. Cholamjiak, “Nonlinear iterative methods for solving the split common null point problem in Banach spaces,” *Optimization Methods and Software*, vol. 34, no. 4, pp. 853–874, 2019.

[30] O. T. Mewomo and F. U. Ogbuisi, “Convergence analysis of an iterative method for solving multiple-set split feasibility problems in certain Banach spaces,” *Quaestiones Mathematicae*, vol. 41, no. 1, pp. 129–148, 2018.