Color superconductor with a color-sexet sextet condensate

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We analyze color superconductivity of one massive flavor quark matter at moderate baryon density with a spin-zero color-sexet sextet condensate. The most general Higgs-type ground-state expectation value of the order parameter implies complete breakdown of the SU(3) × U(1) symmetry. However, both the conventional fourth-order polynomial effective bosonic description, and the NJL-type fermionic description in the mean-field approximation favor an enhanced SO(3) symmetry of the ground state. We ascribe this finding to the failure of the mean-field approximation and propose that a more sophisticated technique is needed.

PACS numbers: 12.38.Aw

I. INTRODUCTION

Viewing the low-temperature deconfined QCD matter at moderate baryon densities as a BCS-type color superconductor is based on good assumptions (see [1, 2, 3] for original references and [4] for a recent review). First, the only degrees of freedom relevant for the effective field theory description of such a matter are the relativistic colored quark fields with their appropriate Fermi surfaces. The colored gauge fields can be introduced perturbatively, and eventually switched off in the lowest approximation. Second, the quarks interact with each other by an attractive interaction providing for Cooper instability. It is natural to speak of the Higgs phases of QCD [5].

Due to the mere fact that the quarks carry the Lorentz index (spin), color and flavor, the ordered colored-quark phases could be numerous. Which of them is energetically most favorable depends solely upon the numerical values of the input parameters (chemical potentials, and the dimensionful couplings) in the underlying effective Lagrangian. Because there are no experimental data on the behavior of the cold deconfined quark matter available, all generically different, theoretically safe [6] and interesting possibilities should be phenomenologically analyzed. Moreover, one should be prepared to accept the fact that one or both our assumptions can be invalid. In any case there are the low-temperature many-fermion systems which are not the Landau–Fermi liquids, and which become peculiar superconductors [7].

Recently, all distinct forms of the quasiquark dispersion laws corresponding to different sets of 16 matrices in the Lorentz index were systematically derived [8]. Those exhibiting spontaneous breakdown of the rotational symmetry manifested in the anisotropic form of the dispersion law are particularly interesting. Their possible nodes can yield important physical consequences even if the corresponding gaps are numerically small [9].

To have a complete list of different ordered quantum phases of the quark matter it would be good to know what is the pattern of spontaneous breakdown of the color SU(3) if an effective interaction prefers not the standard quark-quark Cooper pairing in the antisymmetric color antitriplet, but rather in the symmetric color sextet. Such a pairing would influence qualitatively not only the quark, but also the gluon spectrum.

Although the explicit analysis presented in this paper is strictly phenomenological we describe here briefly a mechanism which, within QCD and under plausible assumptions, can yield the desired color-sexet sextet diquark condensate. Instabilities of the perturbative QCD in the two-gluon channel discussed in [10] justify contemplating several types of effective colored excitations in the deconfined phase at moderate densities with effective (but in practice theoretically unknown) couplings to both quarks and gluons. According to [10], there should be four types of two-gluon collective excitations: spin-zero color-singlet, spin-zero color octet, spin-one color octet, and spin-two color 27-plet. It is easy to show that exchange of a massive color-octet scalar results in a four-quark interaction

\[ \mathcal{L}_{\text{int}} = G(\bar{\psi} \hat{\lambda} \psi)^2, \]  

with \( G > 0 \), which is necessary for the color-sexet sextet diquark condensation. It is, however, not easy to show which of the exchanges, including the one-gluon one, is eventually the most important. In fact, exchange of the color-singlet scalar would also lead to an attractive interaction in the color-sexet sextet quark–quark channel, but as we aim at a phenomenological analysis and do not attempt to evaluate the effective coupling \( G \), we restrict ourselves in the following to the single interaction term \( G \).

We note that the argument leading to the conjectured colored collective modes excited by two gluon operators is the same as that leading, in the quark sector, to the phenomenologically useful [11] color-antitriplet scalar field with the quantum numbers of a diquark.

The possibility of diquark condensation in the color-symmetric channel has already been investigated in various contexts, for instance, within the color-flavor-locking

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scheme \cite{12}, and as an admixture to the color-antitriplet condensate \cite{13,11}. The algebraic structure of spontaneous symme-
tic fermionic second-quantized quark field, and apply it to the case of color-sextet condensation. Section IV con-
tains a summary and a brief discussion of the obtained results, and comparison of the two approaches.

II. HIGGS MECHANISM WITH AN SU(3) SEXTET

Simplifying as much as possible we consider the rela-
tivistic quark matter of one massive flavor (say s-quark matter) in the deconfined phase at moderate baryon den-
sity. We assume that its ground state is characterized by the quark-quark Cooper-pair condensate in the antisym-
metric spin zero state. By Pauli principle this means the symmetric sextet state in SU(3) i.e.,

\[ \langle 0 | \psi_{\alpha i} (C \gamma_5)_{\alpha \beta} \psi_{\beta j} | 0 \rangle \propto \langle \Phi_{ij} \rangle_0, \]

where we insert a dimensionful constant of proportionality to make \( \Phi \) a dimension-one operator. The constant of proportionality can be determined within the mean-field approximation to be \( 3/2G \), see Sec. III.

Treating the \( u \) and \( d \) quarks as nearly degenerate in mass and both much lighter than the \( s \) quark, such a condensate may provide a complement to the usual picture of \( u \) and \( d \) pairing in the color-antitriplet channel \cite{15}.

In an effective Higgs description \( \Phi_{ij} \) is a spin-zero color-sextet order parameter which transforms under the color SU(3) as a complex symmetric matrix,

\[ \Phi \rightarrow U \Phi U^T. \]

The dynamics of \( \Phi \) is governed by the most general Lagrangian invariant under global SU(3) \( \times U(1) \) and space-
time transformations. As the full Lorentz invariance is explicitly broken by the presence of a dense medium, we require that the Lagrangian be invariant under spatial rotations only.

Since we aim at an effective description of the super-
conducting phase, renormalizability is not an issue here, and we have to include all possible interactions built up from the sextet \( \Phi \) that respect the symmetry of the theory.

In accordance with our assumptions, the gauge in-
teraction can be switched on perturbatively by gauging the global SU(3) color symmetry. Formally, we just replace the ordinary derivative of \( \Phi \) with the covariant derivative

\[ D_\mu \Phi = \partial_\mu \Phi - ig A_\mu^a \left( \frac{1}{2} \lambda_a \Phi + \Phi \frac{1}{2} \lambda_a^T \right), \]

where \( A_\mu^a \) is the colored gluon field. The effective Lagrangian thus has the form

\[ \mathcal{L} = \alpha_\varepsilon \text{tr}(D_\mu \Phi)^\dagger D^\mu \Phi + \alpha_m \text{tr}(D_\mu \Phi)^\dagger D^\mu \Phi - V(\Phi) + \ldots, \]

where \( V(\Phi) \) is the most general SU(3) \( \times U(1) \) invariant polynomial in \( \Phi \) and the ellipses stand for other possible terms that involve covariant derivatives and/or gauge field strength tensors \( F_{\mu \nu} \).

A. SU(3) invariants from a sextet

The ground-state expectation value \( \langle \Phi \rangle_0 = \phi \) is at the tree level given by the minimum of the scalar potential \( V(\Phi) \). To proceed with our analysis, we have to specify its concrete form.

Note that the group SU(3) has only three algebraically independent invariant tensors, namely \( \delta_{ij} \), \( \varepsilon_{ijk} \), and \( \varepsilon^{ijk} \), the lower and upper indices transforming under the fundamental representation of SU(3) and its complex conjugate, respectively (see, for example, \cite{16}). As a consequence, the most general SU(3) \( \times U(1) \) invariant built up from a single sextet \( \Phi \) can be constructed from products and sums of \( \det(\Phi^{\dagger} \Phi) \) and \( \text{tr}(\Phi^{\dagger} \Phi)^n \), the symbols “det” and “tr” referring to determinant and trace in the color space, respectively \cite{34}.

Of these polynomials, however, only three are algebra-
ically independent. Indeed, express

\[ \text{tr} \Phi^{\dagger} \Phi = \alpha + \beta + \gamma, \]
\[ \text{tr}(\Phi^{\dagger} \Phi)^2 = \alpha^2 + \beta^2 + \gamma^2, \]
\[ \det \Phi^{\dagger} \Phi = \alpha \beta \gamma, \]

where \( \alpha, \beta, \gamma \) are the eigenvalues of \( \Phi^{\dagger} \Phi \) \cite{35}, and define the symmetric polynomials

\[ \pi_1 = \alpha + \beta + \gamma, \]
\[ \pi_2 = \alpha \beta + \alpha \gamma + \beta \gamma = \frac{1}{2}[- \text{tr}(\Phi^{\dagger} \Phi)^2 + (\text{tr} \Phi^{\dagger} \Phi)^2], \]
\[ \pi_3 = \alpha \beta \gamma. \]

Note that the values of \( \pi_1, \pi_2, \pi_3 \) determine those of \( \alpha, \beta, \gamma \) uniquely as the three roots of the cubic equation

\[ x^3 - \pi_1 x^2 + \pi_2 x - \pi_3 = 0. \]

Thus also the values of all \( \text{tr}(\Phi^{\dagger} \Phi)^n = \alpha^n + \beta^n + \gamma^n \) for \( n \geq 3 \) are fixed. Moreover, they can be expressed directly in terms of \( \pi_1, \pi_2, \pi_3 \) as the Taylor coefficients of the generating function

\[ f(t) = \text{tr} \ln(1 + t \Phi^{\dagger} \Phi) = \ln \det(1 + t \Phi^{\dagger} \Phi), \]
which is readily rewritten as
\[ f(t) = \ln(1 + \pi_1 t + \pi_2 t^2 + \pi_3 t^3). \] (5)

We have thus shown that the scalar potential \( V(\Phi) \) can always be expressed as a function of the three independent invariants \( \det(\Phi^\dagger \Phi), tr(\Phi^\dagger \Phi), \) and \( tr(\Phi^\dagger \Phi)^2 \).

B. Symmetry-breaking patterns

We shall now turn to the structure of the ground state. In our effective Higgs approach, the \( SU(3) \times U(1) \) symmetry is spontaneously broken by the ground-state expectation value \( \Phi \) of the field \( \Phi \), which is a constant due to the translation invariance of the ground state. We can exploit the symmetry to give the \( \Phi \) as simple form as possible. In fact, as shown by Schur, any complex symmetric matrix can always be written as
\[ \phi = U \Delta U^T, \]
where \( U \) is an appropriate unitary matrix, and \( \Delta \) is a real, diagonal matrix with non-negative entries. In our case, we set \( \Delta = \text{diag}(\Delta_1, \Delta_2, \Delta_3) \).

Consequently, there are several distinct patterns of spontaneous symmetry breaking possible.

(a) \( \Delta_1 > \Delta_2 > \Delta_3 > 0 \). This ordering can always be achieved by the allowed appropriate real orthogonal transformations. The continuous \( SU(3) \times U(1) \) symmetry is completely broken (only a discrete \((Z_2)^3\) symmetry is left).

(b) Two \( \Delta \)'s are equal, say \( \Delta_1 = \Delta_2 \neq \Delta_3 \). This implies an enhanced \( O(2) \) symmetry in the corresponding \( 2 \times 2 \) block of \( \phi \).

(c) \( \Delta_1 = \Delta_2 = \Delta_3 \neq 0 \). The vacuum remains \( O(3) \) symmetric.

(d) Some of \( \Delta_1 = 0 \). Then there is a residual \( U(1) \) or \( U(2) \) symmetry of the vacuum corresponding to the vanishing entry or entries of \( \Delta \).

The concrete type of the symmetry breaking pattern is determined by the scalar potential \( V(\Phi) \). Note that, having relaxed the renormalizability requirement, we can always choose the potential \( V(\Phi) \) so that it yields as its minimum any desired values of \( \Delta_1, \Delta_2, \Delta_3 \), just take
\[ V(\Phi) = \frac{1}{2} a_1 \left[ tr(\Phi^\dagger \Phi - \pi_1) \right]^2 + \frac{1}{2} a_2 \left[ tr(\Phi^\dagger \Phi - \pi_1^2 + 2\pi_2) \right]^2 + \frac{1}{2} a_3 \left[ \det(\Phi^\dagger \Phi - \pi_3) \right]^2 \]
with all \( a_1, a_2, a_3 \) positive. The \( \pi \)'s here are to be interpreted as vacuum expectation values of the corresponding operators.

C. Higgs mechanism with a quartic potential

Up to now we have repeatedly stressed the fact that we are dealing with an effective theory and therefore we should include in our Lagrangian all possible interactions preserving the \( SU(3) \times U(1) \) symmetry.

Nevertheless, under some specific conditions it is plausible to start up with a renormalizable linear sigma model that is, take a general quartic potential \( V(\Phi) \) and neglect all operators of dimension greater than four. In Sec. IV we will see that this rather restrictive choice is justified when the underlying microscopic interaction is of four-fermion type.

We thus take up a general quartic potential \( V(\Phi) \),
\[ V(\Phi) = -a \, tr(\Phi^\dagger \Phi + b \, tr(\Phi^\dagger \Phi)^2 + c \, (tr(\Phi^\dagger \Phi))^2, \] (6)
where the minus sign at \( a \) suggests spontaneous symmetry breaking at the tree level. Varying \( V(\Phi) \) with respect to \( \Phi^\dagger \), we derive a necessary condition for the vacuum expectation value \( \phi \),
\[ -a \phi + 2b \phi \phi^\dagger \phi + 2c \phi \, tr(\phi^\dagger \phi) = 0. \] (7)
A simple observation of (7) reveals that, should the matrix \( \phi \) be non-singular, we can divide by it and arrive at the condition
\[ 2b \phi^\dagger \phi = a - 2c \, tr(\phi^\dagger \phi). \]
Thus, unless \( b = 0 \), \( \phi^\dagger \phi \) and hence also \( \Delta \) must be proportional to the identity matrix.

Moreover, even when \( \phi \) is singular, it can be replaced with the real diagonal matrix \( \Delta \) and we see from (7) that all non-zero entries \( \Delta_i \) satisfy the equation
\[ 2b \Delta_i^2 = a - 2c \, tr(\Delta^2). \]
Thus all non-zero \( \Delta \)'s develop the same value.

Which of the suggested solutions of (7) represents the absolute minimum of the potential depends on the input parameters \( a, b, c \), which must be inferred from the underlying theory \([36]\). We therefore stop the Higgs-like analysis here with the simple conclusion that under fairly general circumstances the quartic potential can be minimized by a matrix \( \Delta \) proportional to the unit matrix, thus leading to an interesting symmetry-breaking pattern (see the paragraphs (c) above and below).

D. Gluon mass spectrum

Let us now switch on the gauge interaction perturbatively. Due to the spontaneous symmetry breaking some of the gluons acquire non-zero masses via the Higgs mechanism. At the lowest order of the power expansion in the effective theory, the mass matrix of the gluons follows from the scalar field kinetic terms in \( \Phi \) upon replacing \( \Phi \) with \( \phi \).

Now, recalling the particular form of the covariant derivative in \( \Phi \), we arrive at the following gluon mass squared matrix:
The subscripts \( e, m \) distinguish between the temporal ("electric") and spatial ("magnetic") components of the gluon field.

Let us briefly comment on the above mentioned four types of symmetry breaking patterns.

(a) \( \Delta_1 > \Delta_2 > \Delta_3 > 0 \). The \( SU(3) \times U(1) \) symmetry is completely broken, therefore there are nine massless Nambu–Goldstone modes. Eight of them are eaten by the gluons, which thus acquire non-zero unequal masses (with an appropriate diagonalization in the \((A^3, A^0)\) block). There is one physical Nambu–Goldstone boson corresponding to the broken global \( U(1) \) baryon number symmetry of the underlying theory. Going to the unitary gauge, we can transform away eight of the original twelve degrees of freedom and parameterize the sextet field \( \Phi \) as

\[
\Phi(x) = \frac{1}{\sqrt{2}} e^{i\theta(x)} \left( \begin{array}{ccc}
\Delta_1(x) & 0 & 0 \\
0 & \Delta_2(x) & 0 \\
0 & 0 & \Delta_3(x)
\end{array} \right),
\]

the \( \Delta \)'s representing three massive radial modes and \( \theta \) the Nambu–Goldstone mode.

(b) \( \Delta_1 = \Delta_2 \neq \Delta_3 \). One gluon is left massless, corresponding to the Gell-Mann matrix \( \lambda_2 \) which generates the \( SO(2) \) symmetry of the ground state.

(c) \( \Delta_1 = \Delta_2 = \Delta_3 \neq 0 \). There are three massless gluons corresponding to the generators \( \lambda_3, \lambda_5, \lambda_7 \) of the \( SO(3) \) subgroup of \( SU(3) \). All other gluons receive equal masses so that the symmetry breaking \( SU(3) \to SO(3) \) is isotropic.

(d) Some of \( \Delta_i = 0 \). There is always an unbroken global \( U(1) \) symmetry that arises from a combination of the original baryon number \( U(1) \) and the diagonal generators of \( SU(3) \), hence all Nambu–Goldstone modes that stem from the symmetry breaking are absorbed into the gauge bosons.

E. Interpretation of the results

So far in this section, we have worked out the usual Higgs mechanism for the case that the scalar field driving the spontaneous symmetry breaking transforms as a sextet under the color \( SU(3) \). However, one must exercise some care when applying the results to the physical situation under consideration, that is, color superconductivity. In the very origin of possible problems lies the fact that \( \Phi \) is not an elementary dynamical field but rather a composite order parameter.

Anyway, our analysis of symmetry breaking patterns still holds as for this purpose one can regard \( \Phi \) as simply a shorthand notation for the condensate in Eq. (4).

The most apparent deviation from the standard Higgs mechanism is the presence of non-trivial normalization constants at the kinetic terms in Eq. (4). This is due to the compositeness of the field \( \Phi \) [18, 19].

Further, the power expansion of the effective Lagrangian [4] can be reliable as long as the expansion parameter is sufficiently small. In the standard Ginzburg–Landau theory, this is only true near the critical temperature. It is, however, plausible to think of a zero-temperature effective field theory for the superconducting phase. We therefore understand our Lagrangian as such an effective expansion in terms of the Nambu–Goldstone modes [20, 21] generalized by inclusion of modes of the modulus of the order parameter in [19, 22]. In ordinary superconductivity, the Nambu–Goldstone mode is the Bogolyubov–Anderson mode, and the modulus mode is the Abrahams–Tsuneto mode [23].

Our last remark points to the above calculated masses of gluons generated by the Higgs mechanism. To specify the scale of the masses one would have to know the normalization coefficients \( \alpha_{e,m} \). These are unknown parameters of the effective theory and have to be determined from the matching with the microscopic theory. At zero temperature, they are roughly

\[ \alpha_{e,m} \propto \mu^2/\phi^2, \]

and as a result, both electric and magnetic masses are found to be of order \( g\mu \), where \( \mu \) is the baryon chemical potential. Their physical origin is, however, very different. The electric (Debye) mass is non-zero even in the normal state i.e., above the critical temperature, due to polarization effects in the quark medium. On the other hand, the magnetic (Meissner) mass arises purely as a consequence of the spontaneous symmetry breaking. It
is thus zero at the critical point and increases as the temperature is lowered, to become roughly equal in order of magnitude to the Debye mass at \( T = 0 \).

Unfortunately, this is not the end of the story. As pointed out by Rischke who calculated the gluon masses microscopically for the two-flavor color superconductor \[10\], the lowest order kinetic term alone does not give correct ratios of gluon masses of different adjoint colors. It is therefore not of much help to just try to adjust the normalization of the kinetic term. As a remedy to this problem, it is necessary to make use of higher order contributions to the gluon masses.

In the two-flavor color superconductor with a color-antitriplet condensate, there is only one generically different higher order contribution that can change the ratios of the gluon masses from those given by the lowest order kinetic term (see Ref. \[10\], Eq. 153). This reflects the symmetry of the problem: the order parameter (conventionally chosen to point in the direction of the third color) leaves unbroken an \( SU(2) \) subgroup of the original color \( SU(3) \). Under the unbroken subgroup, the gluons of colors 4–7 transform as a complex doublet and thus have to receive equal masses, possibly different from the mass of gluon 8. The most general gluon mass matrix is thus specified by two parameters.

In our case of a color-sextet condensate, the \( SU(3) \) symmetry can be completely broken and we thus expect that there are in general no relations among the eight gluon masses. We do not go into details here, but just list the kinetic terms of order four in the field \( \Phi \), which give gluon mass ratios different from the lowest order values:

\[
\langle \text{tr}(\Phi^\dagger D_i \Phi) \rangle^2, \\
\text{tr} \left[ (D_\mu \Phi)^\dagger (D^\mu \Phi) \Phi^\dagger \Phi \right], \\
\text{tr} \left[ \Phi^\dagger (D_\mu \Phi) \Phi^\dagger (D^\mu \Phi) \right] + \text{h.c.},
\]

and analogously the terms contributing to the electric gluon masses.

In our Lagrangian the \( SU(3) \times U(1) \) symmetry is realized linearly and these terms are found ‘by inspection’. It would be inappropriate to repeat the analysis using the non-linearly realized effective Lagrangian along the lines of \[24\], analyzing the color-antitriplet case. The kinetic terms should follow from symmetry considerations, albeit again with theoretically undetermined coefficients.

Finally we note that as the Debye masses of all gluons are non-zero in the normal state, one might expect that in the superconducting phase they remain non-zero even for those gluons which correspond to unbroken symmetries, in contrast to the conclusions of the effective theory discussed. However, as shown by Rischke for the two-flavor color superconductor, the “unbroken” electric gluons have, somewhat surprisingly, zero Debye mass at \( T = 0 \). This is because the quark colors they interact with are bound in the condensate and hence there are no low energy levels to be excited by long-wavelength chromoelectric fields.

This line of reasoning can be easily carried over to our case, since due to the diagonal nature of the matrix \( \Delta \), one can immediately check which quark colors participate in the condensate. We thus conjecture that the naive expectation that the Debye masses of the gluons of the unbroken symmetry are zero, is correct at zero temperature, as long as the colors that the gluon interacts with both have non-zero gap \( \Delta_i \). This is the case, for instance, for the gluons of the \( SO(2) \) and \( SO(3) \) ground state symmetries discussed before (see paragraphs (b) and (c) above).

To provide a waterproof verification of this conjecture, on should carry out a microscopic calculation similar to that of \[10\].

III. FERMIONIC BCS-TYPE DESCRIPTION

In the previous section we used an effective Higgs-like theory to treat the kinematics of color superconductivity with a color-sextet condensate. The construction of the effective Lagrangian is based solely on the \( SU(3) \times U(1) \) symmetry. Such an approach is thus pretty convenient to extract as much information about the kinematics as possible, but fails to explain the very fact of Cooper pair formation. To understand the dynamics of color superconductivity, we need a microscopic description of the quark system.

As is well known from BCS theory of superconductivity, fermions (quarks in our case) will tend to form Cooper pairs if there is an attractive effective two-body interaction between them. As is usual in attempts to describe the behavior of deconfined QCD matter, we employ the Nambu–Jona-Lasinio model and look for the di-quark condensate as a constant self-consistent solution to the equations of motion.

Because the excitation spectrum of cold strongly coupled deconfined QCD matter at moderate baryon density is not known, the effective quark–quark interaction relevant for color superconductivity can only be guessed. In any case the excitations of such a matter are of two sorts:

1. Colored quasiparticles excited by the primary quantum fields with modified dispersion laws.

2. Collective excitations, which can be in principle both colored and colorless, and are excited by the appropriate polynomials of the primary quantum fields.

We want to argue in favor of possible existence of massive color-octet spin-zero collective modes excited by two gluon operators \[10\], the exchange of which produces the desired effective four-quark interaction attractive in the color-sextet quark–quark channel. The (naive) point is that the QCD-induced force between two gluons, which can in general be in any of

\[
8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27,
\]

is attractive is the color-octet spin-zero configuration.
Inspired by this argument, we choose for our NJL-type analysis a four-quark interaction which mimics the exchange of an intermediate color-octet scalar particle. As we note below, however, we could have as well included interactions with Lorentz vectors or tensors. Nonetheless, the Lorentz structure of the interaction does not play almost any role in our calculation, and we therefore restrict to the single interaction term \( \Sigma \) suggested above.

Our effective Lagrangian for one massive quark flavor thus reads
\[
\mathcal{L} = \bar{\psi}(i\partial - m + \mu \gamma_0)\psi + G(\bar{\psi}\lambda\psi)^2, \tag{8}
\]
where the arrow over Gell-Mann \( \lambda \)-matrices implies appropriate summation over adjoint \( SU(3) \) indices. Otherwise, Lorentz and color indices are suppressed.

We treat the model Lagrangian \( \Sigma \) in the mean-field approximation. As this is a standard way of dealing with NJL-type models, we sketch only the main steps. Detailed account of the techniques used can be found, for example, in the recent paper by Alford et al. \( [3] \).

To extract the color-sextet condensate, we split our Lagrangian into a free and interacting part \( \mathcal{L}_0 \) and \( \mathcal{L}_\text{int} \), respectively,
\[
\mathcal{L}_0' = \bar{\psi}(i\partial - m + \mu \gamma_0)\psi + \frac{1}{2}\bar{\psi}\Delta(C\gamma_5)\psi^T - \frac{1}{2}\psi^T\Delta^\dagger(C\gamma_5)\psi,
\]
\[
\mathcal{L}_\text{int} = -\frac{1}{2}\bar{\psi}\Delta(C\gamma_5)\psi^T + \frac{1}{2}\psi^T\Delta^\dagger(C\gamma_5)\psi + G(\bar{\psi}\lambda\psi)^2,
\]
where \( \Delta \) is the desired gap parameter which, as shown in the preceding section, can be sought in the form of a real diagonal non-negative matrix in the color space. We introduce the standard Nambu–Gorkov doublet notation,
\[
\Psi(p) = \begin{pmatrix} \bar{\psi}(p) \\ \psi^T(-p) \end{pmatrix},
\]
in which the calculation of the free propagator amounts to inverting a \( 2 \times 2 \) matrix,
\[
S^{-1}(p) = \begin{pmatrix} \phi - m + \mu \gamma_0 & \Delta(C\gamma_5) \\ -\Delta^\dagger(C\gamma_5) & \phi + m - \mu \gamma_0 \end{pmatrix}.
\]
The explicit form of the propagator has been given by several authors, see, for instance, \( [23, 26] \).

In the mean-field approximation, \( \Delta \) is determined from a single one-loop Feynman graph. Regulating the quadratic divergence with a three-momentum cutoff \( \Lambda \) and evaluating explicitly the Wick-rotated integral over the temporal component of the loop momentum, we finally arrive at the gap equation
\[
1 = 2G \int_0^\Lambda \frac{d^3\vec{p}}{(2\pi)^3} \left( \frac{1}{E_+} + \frac{1}{E_-} \right), \tag{9}
\]
where \( E_\pm \) represent the positive energies given by the dispersion relations of the quasiquark excitations,
\[
E_\pm^2 = \left( \sqrt{\vec{p}^2 + m^2} \pm \mu \right)^2 + |\Delta|^2.
\]

A few remarks to the gap equation \( \Sigma \) are in order. First, in the loop integral, we have ignored a term proportional to \( \mu \gamma_0 \) which generates the operator \( \bar{\psi}(C\gamma_5)\gamma_0\bar{\psi}^T \) that breaks Lorentz invariance. In fact, we should have expected such a term to appear, since Lorentz invariance is explicitly broken by the presence of the chemical potential in the Lagrangian \( \Sigma \). For our treatment of color superconductivity at non-zero chemical potential to be fully consistent, we would have to include such operators into our Lagrangian from the very beginning and solve a coupled set of gap equations for both Lorentz invariant and non-invariant condensates \( [27] \). Here, for the sake of simplicity, we ignore this difficulty and neglect the secondary effects of Lorentz-invariance breaking induced by the chemical potential.

Second, the gap equation \( \Sigma \) is understood as a matrix equation in the color space. Its matrix structure is, however, trivial. In fact, we get three separated identical equations for the diagonal elements \( \Delta_1, \Delta_2, \Delta_3 \) of the gap matrix. This means that, at least at the level of the mean-field approximation, our model favors an enhanced \( SO(3) \) symmetry of the ground state — the gaps for all three colors are the same. This is apparently not a peculiar consequence of our particular choice of interaction in \( \Sigma \), but holds for any \( SU(3) \)-invariant four-fermion interaction. The only effect of adding also the Lorentz vector or tensor channel interactions, for example, would be in the modification of the effective coupling constant \( G \). The Lorentz structure of the interaction does not play any role and the resulting form of the gap equation is a consequence of the identity \( \overline{\lambda}\Delta\lambda^T = 4\Delta/3 \), which holds for any diagonal matrix \( \Delta \). We will return to the discussion of this point in the next section where we will comment on a correspondence between the bosonic and fermionic approaches.

Third, the extension of the gap equation to non-zero temperatures is easy. We can either first calculate the thermodynamical potential \( \Omega \) and then minimize it with respect to \( \Delta \) or, alternatively, proceed in the same manner as before and derive a self-consistency condition for the thermal Green function \( [38] \). Performing the sum over Matsubara frequencies in the last step, the result is
\[
1 = \frac{2}{3} G \int_0^\Lambda \frac{d^3\vec{p}}{(2\pi)^3} \left( \frac{1}{E_+} \tanh \frac{E_+}{2T} + \frac{1}{E_-} \tanh \frac{E_-}{2T} \right).
\]
This gap equation can be used for the study of temperature dependence of the gap and, in particular, for finding the critical temperature at which the \( SU(3) \) symmetry is restored \( [28] \).
IV. SUMMARY AND DISCUSSION

Let us briefly summarize our results. First we developed the Higgs mechanism for a color sextet and found out that although the underlying symmetry allows for a complete spontaneous breakdown, for a generic quartic scalar potential the pattern $SU(3) \to SO(3)$ is preferred.

After then, we used the Nambu–Jona-Lasinio model to calculate the gaps $\Delta_1, \Delta_2, \Delta_3$ self-consistently in the mean-field approximation and our result was in accord with the preceding Higgs-type analysis.

This is, of course, not only a coincidence, but follows from a general correspondence between four-fermion-interaction models and linear sigma models provided by the Hubbard–Stratonovich transformation.

Let us sketch the main idea. In the path integral formalism, one first introduces an auxiliary scalar integration variable which has no kinetic term and couples to the fermion via the Yukawa interaction. The action now becomes bilinear in the fermion variables and one can integrate them out explicitly. The logarithm of the fermion determinant gives rise to a kinetic term of the scalar field and the model hence becomes equivalent to the linear sigma model, up to a choice of the renormalization prescription.

In terms of the NJL model the interpretation of the correspondence is a little bit different. Here one cannot carry out the usual renormalization program and the choice of an ultraviolet regulator becomes physically significant. So in the effective scalar field action the operators with dimension four or less are dominant since they are generated with divergent coefficients. The quadratic divergences cancel due to the gap equation in the underlying NJL model but the logarithmic ones remain.

One thus receives an a posteriori justification for the choice of the linear sigma model as the starting point for the Higgs-type analysis in subsection II.C. On the other hand, one should bear in mind that these conclusions are valid only in the mean-field approximation that we employed.

In terms of the effective scalar field $\Phi$, the true vacuum is determined by the absolute minimum of the full quantum effective potential which is no longer restricted to contain operators of dimension four or less.

In the NJL model, going beyond the mean-field approximation could destroy the simple structure of the one-loop gap equation. Generally, the resulting set of algebraic equations for $\Delta_1, \Delta_2, \Delta_3$ must be permutation invariant since permutations of diagonal elements of the matrix $\Delta$ belong to the symmetry group $SU(3)$ of the theory. For four-fermion interactions the $SU(3)$ structure of an arbitrary Feynman graph can be investigated making use of the Fierz identities in the color space. One gets three coupled, but still rather simple equations for the three gaps. It is then perhaps a matter of numerical calculations to decide whether these equations possess asymmetric solutions and whether they are more energetically favorable than those with $\Delta_1 = \Delta_2 = \Delta_3$.

We suspect that asymmetric solutions implying a complete breakdown of the $SU(3) \times U(1)$ symmetry can also be obtained from interactions that mimic many-body forces (six-fermion or more). The correspondence with linear sigma model via the Hubbard–Stratonovich transformation is then lost and it could hopefully suffice to stay at the level of the mean-field approximation, thus requiring much less manual work than in the previous case.

Investigations in the two directions mentioned above are already in progress.

Acknowledgments

The authors are greatly indebted to Dirk Rischke for his kind and insightful remarks which resulted in the discussion in subsection II.E. They are also grateful to Michael Buballa for critical comments on an early draft of the paper. T.B. would like to thank W. Grimus for bringing the reference to his attention. J.H. acknowledges with pleasure Uwe-Jens Wiese for several useful and pleasant discussions.

This work was supported in part by grant GACR 202/02/0847. The work of T.B. was also in part supported by a graduate fellowship of the Faculty of Mathematics and Physics, Charles University.
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[34] The $U(1)$ symmetry guarantees that the $SU(3)$ invariants $\det \Phi$ and $\det \Phi^\dagger$ always come with the same power.
[35] Here and in the following, we act as if $\Phi^\dagger \Phi$ were a c-number matrix and not an operator one. We can do so since all components of $\Phi^\dagger \Phi$ commute with one another. The constants $\alpha, \beta, \gamma$ are to be understood as eigenvalues of the $3 \times 3$ matrix $\Phi^\dagger \Phi$, disregarding its operator nature. Our goal is the generating function (5) which yields purely algebraic relations among $\det(\Phi^\dagger \Phi)$ and $\text{tr}(\Phi^\dagger \Phi)^n$, without any reference to $\alpha, \beta, \gamma$. Our conclusions about (in)dependence of various polynomials are thus valid for c-number as well as operator matrices.
[36] Similar analyses within a more general class of models have been performed in [30, 31].
[37] Without further knowledge, we can only constrain the values of $b$ and $c$ by the requirement of boundedness of $V(\Phi)$ from below. It is clear that at least one of these parameters must be positive, positivity of both being, of course, the safest choice. The sizes of the two quartic interaction terms are restricted by the inequalities $\text{tr}(\Phi^\dagger \Phi)^2 \leq (\text{tr} \Phi^\dagger \Phi)^2 \leq 3 \text{tr}(\Phi^\dagger \Phi)^2$ where the equality sign in the left and right hand side inequality occurs when only one $\Delta_i$ is non-zero and $\Delta_1 = \Delta_2 = \Delta_3$, respectively. The potential is thus bounded from below if and only if, $b$ is positive and $c > -b/3$, or $c$ is positive and $b > -c$.
[38] Once we have found the gap equation, we can obtain the thermodynamical potential by integrating it over the gap parameter (for further details see [32]).
[39] Formally, we imagine this as adding two- and more-loop graphs, with full fermion propagators inserted in place of the free ones, to the right hand side of the gap equation (9). Physically this means that the effective coupling constant $G$ may depend on the order parameter $\Delta$.