LICHNEROWICZ TYPE COHOMOLOGY ATTACHED TO A FUNCTION

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Abstract. In this paper we define a new cohomology of a smooth manifold called Lichnerowicz type cohomology attached to a function. Firstly, we study some basic properties of this cohomology as dependence on the function, singular forms, relative cohomology, Mayer-Vietoris sequence, homotopy invariance and next a regular case is studied. Also, the case when the manifold is locally conformally Kähler is considered. The notions are introduced using techniques from the study of two cohomologies of a smooth manifold: the Lichnerowicz cohomology and the cohomology attached to a function. Finally, a twisted cohomology attached to a function which is related to these two cohomologies is defined and studied.

1. Introduction

Let us consider an $n$-dimensional smooth manifold $M$ and $\theta$ be a closed 1-form on $M$. Denote by $\Omega^r(M)$ the set of all $r$-differential forms on $M$ and consider the operator $d_\theta : \Omega^r(M) \to \Omega^{r+1}(M)$ defined by $d_\theta = d - \theta \wedge$, where $d$ is the usual exterior derivative.

Since $d\theta = 0$, we easily obtain that $d_\theta^2 = 0$. The differential complex

\begin{equation}
0 \longrightarrow \Omega^0(M) \xrightarrow{d_\theta} \Omega^1(M) \xrightarrow{d_\theta} \ldots \xrightarrow{d_\theta} \Omega^n(M) \longrightarrow 0
\end{equation}

is called the Lichnerowicz complex of $M$; its cohomology groups $H^\bullet_\theta(M)$ are called the Lichnerowicz cohomology groups of $M$.

This is the classical Lichnerowicz cohomology, also known in literature as Morse-Novikov cohomology, motivated by Lichnerowicz’s work [7] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic geometry manifolds, see [1, 6]. We also notice that Vaisman in [15] studied it under the name of ”adapted cohomology” on locally conformally Kähler manifolds.

We notice that, locally, the Lichnerowicz cohomology complex becomes the de Rham complex after a change $\varphi \mapsto e^f \varphi$ with $f$ a smooth function which satisfies $df = \theta$, namely $d_\theta$ is the unique differential in $\Omega^\bullet(M)$ which makes the multiplication by the smooth function $e^f$ an isomorphism of cochain complexes $e^f : (\Omega^\bullet(M), d_\theta) \to (\Omega^\bullet(M), d)$. For more about this cohomology see for instance [11 8 6 11 15].

On the other hand, in [9], Monnier gave the definition and basic properties of a new cohomology attached to a function. The definition is the following:
If $f$ is a smooth function on a smooth manifold $M$, then we can define the linear operator $d_f : \Omega^r(M) \to \Omega^{r+1}(M)$ by

$$d_f \varphi = fd\varphi - rdf \wedge \varphi, \forall \varphi \in \Omega^r(M).$$

It is easy to see that $d_f^2 = 0$, and, so we have a differential complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_f} \Omega^1(M) \xrightarrow{d_f} \cdots \xrightarrow{d_f} \Omega^n(M) \longrightarrow 0$$

which is called the differential complex attached to the function $f$ of $M$; its cohomology groups $H^\bullet_f(M)$ are called the cohomology groups attached to the function $f$ of $M$. This cohomology was considered for the first time in [8] in the context of Poisson geometry, and more generally, Nambu-Poisson geometry.

We also notice that $d_f$ is an antiderivation, namely

$$d_f(\varphi \wedge \psi) = d_f \varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge d_f \psi, \forall \varphi, \psi \in \Omega^\bullet(M),$$

while $d_\theta$ is not an antiderivation, and it satisfies

$$d_\theta(\varphi \wedge \psi) = d_\theta \varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge d_\theta \psi, \forall \varphi, \psi \in \Omega^\bullet(M).$$

The main goal of this paper is to make a connection between these two cohomologies, giving a positive answer to the question posed by Monnier in [9]: We can apply the techniques used on the Witten complex (Lichnerowicz complex) to differential complex attached to a function and conversely?

A first link is given in the following proposition. If $f$ is a positive real valued smooth function on $M$ then we have

**Proposition 1.1.** Let $\varphi$ be an $r$-form defined on some neighborhood $U$ of $M$ such that $d_f \varphi = 0$. Then there exists an $(r-1)$-form $\psi$ defined on a neighborhood $U' \subset U$ such that $\varphi \wedge df = d\psi \wedge df$, namely $\varphi$ is $d$-exact (modulo $df$).

**Proof.** Let $\varphi$ as in hypothesis. From $d_f \varphi = 0$ because $f \neq 0$ we easily obtain

$$d_\theta \varphi = 0, \quad \theta = d(\log f^r).$$

But the operator $d_\theta$ satisfies a Poincaré Lemma, see the proof of Proposition 3.1. from [15], and thus there exists an $(r-1)$-form defined on $U' \subset U$ such that

$$\varphi = d_\theta \psi = d\psi - d(\log f^r) \wedge \psi = d\psi - r \frac{df}{f} \wedge \psi = d\psi - (r-1) \frac{df}{f} \wedge \psi - \frac{df}{f} \wedge \psi = \frac{1}{f} (f d\psi - (r-1)df \wedge \psi) - \frac{df}{f} \wedge \psi = \frac{1}{f} (df \psi - df \wedge \psi).$$

Now by above discussion we obtain $f \varphi \wedge df = d_f \psi \wedge df = f d\psi \wedge df$ and because $f \neq 0$ we finally get $\varphi \wedge df = d\psi \wedge df$. \qed
Another links between these two cohomologies are detailed in the next sections of the paper which is organized as follows:

In the second section we define a new cohomology so called \textit{Lichnerowicz type cohomology attached to a function} of a smooth manifold. The notions are introduced by combining results from Lichnerowicz cohomology and from cohomology attached to a function. Firstly, we prove that this cohomology is isomorphic to Lichnerowicz cohomology of singular forms (Proposition 2.2), we discuss how the cohomology varies when the function \( f \) changes (Proposition 2.3) and how it depends on the class of \( \theta \) (Proposition 2.4). In particular, we show that if the function \( f \) does not vanish, then the Lichnerowicz type cohomology attached to a function is isomorphic to Lichnerowicz cohomology (Corollary 2.2). Next we study a relative cohomology associated to our cohomology and we will show that it is possible to write a Mayer-Vietoris exact sequence (Theorem 2.1). We also give an appropriate notion of homotopy, but it is an open question whether the cohomology is homotopy invariant in general.

In the third section we consider the regular case, i.e., the case where the function \( f \) does not have singularities in a neighborhood of \( S = f^{-1}(\{0\}) \). In a similar manner with the study from [9] concerning to cohomology attached to a function, we can relate our cohomology with the Lichnerowicz cohomology of \( M \) and of \( S \) (Theorem 3.1). Also in this regular case, we prove a homotopy invariance (Proposition 3.1).

In the four section we study some aspects concerning to Lichnerowicz type cohomology attached to a function of locally conformally Kähler, briefly l.c.K., manifolds and we introduce three cohomological invariants of these manifolds. Also we consider another cohomology attached to a function of l.c.K. manifolds related to our cohomology (Corollaries 4.1, 4.2).

In the last section, following an argument inspired from [15], we define a twisted cohomology attached to a function, which is connected to these two cohomologies.

**2. Lichnerowicz Type Cohomology Attached to a Function**

In this section we define a Lichnerowicz type cohomology attached to a function and we study several properties of this new cohomology in relation with some classical properties of Lichnerowicz cohomology and of cohomology attached to a function.

For this purpose we consider again \( \theta \in \Omega^1(M) \) be a closed 1-form on \( M \). If \( f \) is a smooth function on \( M \), let us remark that \( df(f\theta) = 0 \). Then if we replace \( d \) by \( df \) and \( \theta \) by \( f\theta \) in the definition of Lichnerowicz operator \( d_\theta \), then we obtain the following operator:

\[
d_{f,\theta}\varphi = df\varphi - f\theta \wedge \varphi, \quad \varphi \in \Omega^r(M).
\]

Taking into account that \( df(f\theta) = 0 \), then an easy calculation using (1.3) leads to \( d^2_{f,\theta} = 0 \). Thus, we obtain the differential complex

\[
0 \longrightarrow \Omega^0(M) \xrightarrow{d_{f,\theta}} \Omega^1(M) \xrightarrow{d_{f,\theta}} \ldots \xrightarrow{d_{f,\theta}} \Omega^n(M) \longrightarrow 0
\]

which is called the \textit{Lichnerowicz type complex attached to the function} \( f \) of \( M \); its cohomology groups \( H^*_\theta(M) \) are called the Lichnerowicz type cohomology groups attached to the function \( f \) of \( M \).

**Remark 2.1.** The operator \( d_{f,\theta} \) may be defined in the form

\[
d_{f,\theta}\varphi = df\varphi - rdf \wedge \varphi, \quad \varphi \in \Omega^r(M).
\]

Using the definition of \( d_{f,\theta} \) by direct calculus we obtain
Proposition 2.1. If \( f, g \in C^\infty(M) \) then

(i) \( d_{f+g,\theta} = d_f + d_g, d_{f,0} = d_f, d_{0,\theta} = 0, d_{-f,\theta} = -d_f, \).
(ii) \( d_{f,\theta} = fd_{\theta} + gd_{f,\theta} - fgd_\theta, d_{1,\theta} = d_\theta, d_\theta = \frac{1}{2}(fd_{1,\theta} + \frac{1}{f}d_f) \).
(iii) \( d_{f,\theta}(\varphi \land \psi) = d_f \varphi \land \psi + (-1)^{\deg \varphi} \land d_f \psi. \)

Also, if \( \theta_1 \) and \( \theta_2 \) are two closed 1-forms on \( M \) then

\[
d_{f,\theta_1+\theta_2}(\varphi \land \psi) = d_{f,\theta_1} \varphi \land \psi + (-1)^{\deg \varphi} \land d_{f,\theta_2} \psi,
\]

which says that the wedge product induces the map

\[
\wedge : H^{r_1}_{f,\theta_1}(M) \times H^{r_2}_{f,\theta_2}(M) \to H^{r_1+r_2}_{f,\theta_1+\theta_2}(M).
\]

Corollary 2.1. The wedge product induces the following homomorphism

\[
\wedge : H^r_{f,\theta}(M) \times H^r_{f,-\theta}(M) \to H^r_{f}(M).
\]

In the following we prove some basic properties of this new cohomology.

2.1. Singular \( r \)-forms. According to [9] a form \( \varphi \in \Omega^r(M \setminus S) \) is called a singular \( r \)-form if the form \( f^r \varphi \) can be extended to a smooth \( r \)-form on \( M \). We denote the space of singular \( r \)-forms by \( \Omega^r_f(M) \).

If \( \varphi \in \Omega^r_f(M) \) is a singular \( r \)-form then \( d_\theta \varphi \) is a singular \( (r+1) \)-form, and so \( d_\theta \varphi = d\varphi - \theta \land \varphi \) is a singular \( r+1 \)-form. In fact we have

\[
f^{r+1}d_\theta \varphi = d_\theta(f^{r+1}\varphi) - (r+1)df \land f^r \varphi,
\]

so \( f^{r+1}d_\theta \varphi \) also extend to a smooth form on \( M \). Therefore we obtain a chain complex \((\Omega^*_{f}(M),d_\theta)\) called the Lichnerowicz complex of singular forms. Similar to Proposition 2.4 from [9] we have

Proposition 2.2. The cohomology of \((\Omega^*_{f}(M),d_\theta)\) is isomorphic to \( H^*_{f,\theta}(M) \).

Proof. Define a map of chain complexes \( \chi : (\Omega^*_{f}(M),d_\theta) \to (\Omega^*_{f}(M),d_{f,\theta}) \) by setting

\[
\chi^r : \Omega^r_{f}(M) \to \Omega^r(M), \chi^r(\varphi) = f^r \varphi.
\]

By direct calculus we obtain

\[
d_{f,\theta}(f^r \varphi) = f^{r+1}d_\theta \varphi \tag{2.3}
\]

and so \( \chi \) induces an isomorphism in cohomology. \( \square \)

2.2. Dependence on the function \( f \). As in the case of the cohomology \( H^*_{f}(M) \), a natural question to ask about the cohomology \( H^*_{f,\theta}(M) \) is how it depends on the function \( f \). Similar with the Proposition 3.2. from [9], we explain this fact for our cohomology. We have

Proposition 2.3. If \( h \in C^\infty(M) \) does not vanish, then cohomologies \( H^*_{f,\theta}(M) \) and \( H^*_{f,h,\theta}(M) \) are isomorphic.

Proof. For each \( r \in \mathbb{N} \), consider the linear isomorphism

\[
\Phi^r : \Omega^r(M) \to \Omega^r(M), \Phi^r(\varphi) = \frac{\varphi}{h^r}. \tag{2.4}
\]

If \( \varphi \in \Omega^r(M) \), one checks easily that

\[
\Phi^{r+1}(d_{f,h,\theta} \varphi) = d_{f,\theta}(\Phi^r(\varphi)). \tag{2.5}
\]
Indeed, we have
\[
\Phi^{r+1}(d_{f,h,\theta}\varphi) = \Phi^{r+1}(d_{f,h}\varphi - fh\vartheta \wedge \varphi) = \Phi^{r+1}(d_{f,h}\varphi) - \Phi^{r+1}(fh\vartheta \wedge \varphi) = d_f(\Phi^r(\varphi)) - \vartheta \wedge \Phi^r(\varphi) = d_f(\theta^\ast(\varphi)),
\]
where we have used the relation \(\Phi^{r+1}(d_{f,h}\varphi) = d_f(\Phi^r(\varphi))\) from [9]. Thus \(\Phi\) induces an isomorphism between cohomologies \(H^\ast_{f,\theta}(M)\) and \(H^\ast_{f,h,\theta}(M)\).

**Corollary 2.2.** If the function \(f\) does not vanish, then \(H^\ast_{f,\theta}(M)\) is isomorphic to the Lichnerowicz cohomology \(H^\ast_0(M)\).

*Proof.* We take \(h = \frac{1}{f}\) in the above proposition. \(\square\)

We also have

**Corollary 2.3.** If \(f\) and \(g\) are smooth functions on \(M\) such that \(S = f^{-1}(0) = g^{-1}(0)\) and \(f = g\) on some neighborhood of \(S\), then \(H^\ast_{f,\theta}(M) \cong H^\ast_{g,\theta}(M)\).

### 2.3. Dependence on the class of \(\theta\).

Another natural question to ask about the cohomology \(H^\ast_{f,\theta}(M)\) is if it depends on the class of \(\theta\) as in the case of Lichnerowicz cohomology \(H^\ast_0(M)\). We have

**Proposition 2.4.** The Lichnerowicz type cohomology attached to a function \(f\) depends only on the class of \(\theta\). In fact, we have the isomorphism
\[
H^\ast_{f,\theta}(M) \cong H^\ast_{f,\theta - d\sigma}(M).
\]

*Proof.* By direct calculus we easily obtain \(d_{f,\theta}(e^\sigma\varphi) = e^\sigma d_{f,\theta - d\sigma}\varphi\), where \(\sigma\) is a smooth function and thus the map \([\varphi] \mapsto [e^\sigma\varphi]\) establishes an isomorphism between cohomologies \(H^\ast_{f,\theta - d\sigma}(M)\) and \(H^\ast_{f,\theta}(M)\). \(\square\)

### 2.4. Relative cohomology.

The relative de Rham cohomology was first defined in [2] p. 78. Also, a relative vertical cohomology of real foliated manifolds can be found in [12]. In [9] is given a relative cohomology for \(H^\ast_{f}(M)\) and in [1] is studied a relative cohomology for \(H^\ast_0(M)\). In this subsection we construct a similar version for our combined cohomology \(H^\ast_{f,\theta}(M)\).

Let \(\mu : M \to M'\) be a morphism between two smooth manifolds. Taking into account the standard relation \(d\mu^* = \mu^* d'\), (here \(d'\) denotes the exterior derivative on \(M'\)), we obtain
\[
d_{\mu^*} f \mu^* = \mu^* d_f, f \in C^\infty(M').
\]

Indeed, for \(\varphi \in \Omega^*(M')\), we have
\[
d_{\mu^*} f (\mu^* \varphi) = \mu^* f d(\mu^* \varphi) - r d(\mu^* f) \wedge \mu^* \varphi = \mu^* f (\mu^* \varphi) - r (\mu^* d_f \varphi) \wedge \mu^* \varphi = (\mu^* f \varphi) - \mu^* (r d_f \varphi) \wedge \varphi = \mu^* d_f \varphi.
\]

The relation (2.6) says that we have the homomorphism
\[
\mu^* : H^\ast_f(M') \to H^\ast_{\mu^* f}(M), \mu^*[\varphi] = [\mu^* \varphi].
\]
Now, taking into account (2.6) we obtain
\[(2.7) \quad d_{\mu^* f,\mu^* \theta} \mu^* = \mu_* d_{f,\theta}\]
for any smooth function \(f \in C^\infty(M')\) and for any closed 1-form \(\theta \in \Omega^1(M')\). Indeed, for \(\varphi \in \Omega^r(M')\), we have
\[
d_{\mu^* f,\mu^* \theta}(\mu^* \varphi) = d_{\mu^* f}(\mu^* \varphi) - \mu^* f \mu^* \theta \wedge \mu^* \varphi = \mu_* d' f \varphi - \mu^* (f \theta \wedge \varphi) = \mu^* (d_{f,\theta} \varphi).
\]
The relation (2.7) says that we have the homomorphism
\[(2.8) \quad \mu^* : H^r_{f,\theta}(M') \to H^r_{\mu^* f,\mu^* \theta}(M), \mu^*[\varphi] = [\mu^* \varphi].\]
If \(\mu\) is a diffeomorphism then \(H^r_{f,\theta}(M') \cong H^r_{\mu^* f,\mu^* \theta}(M)\).

We define the differential complex
\[
\cdots \to \Omega^r(\mu) \xrightarrow{\tilde{d}_{f,\theta}} \Omega^{r+1}(\mu) \xrightarrow{\tilde{d}_{f,\theta}} \cdots
\]
where
\[
\Omega^r(\mu) = \Omega^r(M') \oplus \Omega^{r-1}(M), \text{ and } \tilde{d}_{f,\theta}(\varphi, \psi) = (d_{f,\theta} \varphi, \mu^* \varphi - d_{\mu^* f,\mu^* \theta} \psi).
\]
Taking into account \(d_{f,\theta}^2 = d_{\mu^* f,\mu^* \theta}^2 = 0\) and (2.7) we easily verify that \(\tilde{d}_{f,\theta}^2 = 0\).

Denote the cohomology groups of this complex by \(H^*_{f,\theta}(\mu)\).

If we regrade the complex \(\Omega^r(M)\) as \(\Omega^r(M) = \Omega^{r-1}(M)\), then we obtain an exact sequence of differential complexes
\[(2.9) \quad 0 \to (\Omega^r(M), d_{\mu^* f,\mu^* \theta}) \xrightarrow{\alpha} (\Omega^r(\mu), \tilde{d}_{f,\theta}) \xrightarrow{\beta} (\Omega^r(M'), \tilde{d}_{f,\theta}) \to 0
\]
with the obvious mappings \(\alpha\) and \(\beta\) given by \(\alpha(\psi) = (0, \psi)\) and \(\beta(\varphi, \psi) = \varphi\), respectively. From (2.9) we have an exact sequence in cohomologies
\[
\cdots \to H^{r-1}_{\mu^* f,\mu^* \theta}(M) \xrightarrow{\alpha^*} H^{r}_{f,\theta}(\mu) \xrightarrow{\beta^*} H^{r}_{f,\theta}(M') \xrightarrow{\delta^*} H^{r}_{\mu^* f,\mu^* \theta}(M) \to \cdots
\]
It is easily seen that \(\delta^* = \mu^*\). Here \(\mu^*\) denotes the corresponding map between cohomology groups. Let \(\varphi \in \Omega^r(M')\) be a \(\tilde{d}_{f,\theta}\)-closed form, and \((\varphi, \psi) \in \Omega^r(\mu)\). Then \(\tilde{d}_{f,\theta}(\varphi, \psi) = (0, \mu^* \varphi - d_{\mu^* f,\mu^* \theta} \psi)\) and by the definition of the operator \(\delta^*\) we have
\[
\delta^*[\varphi] = [\mu^* \varphi - d_{\mu^* f,\mu^* \theta} \psi] = [\mu^* \varphi] = \mu^*[\varphi].
\]
Hence we finally get a long exact sequence
\[(2.10) \quad \cdots \to H^{r-1}_{\mu^* f,\mu^* \theta}(M) \xrightarrow{\alpha^*} H^{r}_{f,\theta}(\mu) \xrightarrow{\beta^*} H^{r}_{f,\theta}(M') \xrightarrow{\mu^*} H^{r}_{\mu^* f,\mu^* \theta}(M) \to \cdots
\]
We have

**Proposition 2.5.** If the manifolds \(M\) and \(M'\) are of the \(n\)-th and \(n'\)-th dimension, respectively, then

(i) \(\beta^* : H^{n+1}_{f,\theta}(\mu) \to H^{n+1}_{f,\theta}(M')\) is an epimorphism,

(ii) \(\alpha^* : H^n_{\mu^* f,\mu^* \theta}(M) \to H^n_{f,\theta}(\mu)\) is an epimorphism,

(iii) \(\beta^* : H^r_{f,\theta}(\mu) \to H^r_{f,\theta}(M')\) is an isomorphism for \(r > n + 1\),

(iv) \(\alpha^* : H^r_{\mu^* f,\mu^* \theta}(M) \to H^r_{f,\theta}(\mu)\) is an isomorphism for \(r > n'\),
sequence of cochain complexes

\[ H^r_{f,\theta}(\mu) = 0 \text{ for } r > \max\{n + 1, n'\} \]

2.5. A Mayer-Vietoris sequence. Since the differentials \( d_{f,\theta} \) commutes with the restrictions to open subsets, one can construct, in the same way as for the de Rham cohomology, see \([2, 13]\), a Mayer-Vietoris exact sequence, namely:

Suppose \( M \) is the union of two open subsets \( U, V \). Then the following is a short exact sequence of cochain complexes

\[ 0 \to (\Omega^\bullet(M), d_{f,\theta}) \overset{\alpha}{\to} (\Omega^\bullet(U) \oplus \Omega^\bullet(V), d_{f,\theta|U} \oplus d_{f,\theta|V}) \overset{\beta}{\to} \]

\[ \tilde{\Omega}^\bullet = (\Omega^\bullet(U \cap V), d_{f,\theta|U \cap V}) \to 0 \]

where \( \alpha(\varphi) = (\varphi|_U, \varphi|_V) \) and \( \beta(\varphi, \psi) = \varphi|_{U \cap V} - \psi|_{U \cap V} \). So we obtain the following Mayer-Vietoris sequence:

**Theorem 2.1.** If \( \mathcal{U} = \{U, V\} \) is an open cover of \( M \), we have the long exact sequence

\[ \ldots \to H^r_{f,\theta}(M) \overset{\alpha}{\to} H^r_{f,\theta|U}(U) \oplus H^r_{f,\theta|V}(V) \overset{\beta}{\to} \]

\[ H^r_{f,\theta|U \cap V}(U \cap V) \overset{\delta}{\to} H^{r+1}_{f,\theta}(M) \to \ldots \]

where \( \alpha_*([\varphi]) = ([\varphi|_U], [\varphi|_V]), \beta_*([\varphi], [\psi]) = [\varphi|_{U \cap V} - \psi|_{U \cap V}], \delta([\sigma]) = [d_{f,\theta}|_{U \cap V} \cdot \sigma] = -[d_{f,\theta}|_{U \cap V} \cdot \sigma] \). Here \( \{\lambda_U, \lambda_V\} \) is a partition of unity subordinate to \( \{U, V\} \) and the forms under consideration are assumed to be extended by 0 to the whole \( M \).

2.6. Homotopy invariance.

**Definition 2.1.** \([9]\). Let \( M \) and \( M' \) two smooth manifolds and \( f \in C^\infty(M) \) and \( f' \in C^\infty(M') \). A **morphism** from the pair \((M, f)\) to the pair \((M', f')\) is a pair \((\phi, \alpha)\) formed by a morphism \( \phi : M \to M' \) and a real valued function \( \alpha : M \to \mathbb{R} \), such that \( \alpha \) does not vanish on \( M \) and \( \phi^* f' = f' \circ \phi = \alpha f \).

We will say that the pairs \((M, f)\) and \((M', f')\) are equivalent if there exists a morphism \( \Phi = (\phi, \alpha) \) between these two pairs where \( \phi \) is a diffeomorphism. This notion of equivalence between the pairs is sometimes called "contact equivalence" in singularity theory. In \([9]\) is proved that a morphism \( \Phi = (\phi, \alpha) \) from the pair \((M, f)\) to the pair \((M', f')\) induces a chain map \( \Phi^* : \Omega^\bullet(M', d_{f'}) \to \Omega^\bullet(M, d_f) \) defined by

\[ \Phi^* : \Omega^\bullet(M') \to \Omega^\bullet(M), \Phi^*(\varphi) = \frac{\phi^* \varphi}{\alpha^f} \]

and this map induces an homomorphism in cohomology \( \Phi^* : H^\bullet_{f'}(M') \to H^\bullet_f(M) \). If \( \Phi \) is diffeomorphism then \( H^\bullet_{f'}(M') \) and \( H^\bullet_f(M) \) are isomorphic.

Now, taking into account that for any \( \varphi \in \Omega^\bullet(M') \) we have \( \Phi^*(d_{f'} \varphi) = d_f(\Phi^*(\varphi)) \) by direct calculus we obtain

\[ \Phi^*(d_{f'} \varphi) = d_{f,\varphi^f}(\Phi^*(\varphi)) \]

for any closed 1-form \( \theta \) on \( M' \) and \( \varphi \in \Omega^\bullet(M') \).

Thus \( \Phi \) induces an homomorphism in Lichnerowicz type cohomology attached to a function \( \Phi^* : H^\bullet_{f',\theta}(M') \to H^\bullet_{f,\varphi^f}(M) \). If \( \Phi \) is diffeomorphism then \( H^\bullet_{f',\theta}(M') \) and \( H^\bullet_{f,\varphi^f}(M) \) are isomorphic.

**Remark 2.2.** For \( \alpha = 1 \) then we obtain the homomorphism from (2.8).
Definition 2.2. ([9]). A homotopy from the pair \((M, f)\) to the pair \((M', f')\) is given by two smooth maps
\[ h : M \times [0, 1] \to M', \ a : M \times [0, 1] \to \mathbb{R}, \]
such that for each \(t \in [0, 1]\), we have a morphism
\[ H_t \equiv (h_t, a_t) : (M, f) \to (M', f') \]
(i.e., \(a\) does not vanish, \(f' \circ h(x, t) = a(x, t)f(x)\)), where \(h_t = h(\cdot, t), \ a_t = a(\cdot, t)\).

Now, if \(H = (h, a)\) is a homotopy from \((M, f)\) to \((M', f')\), from above discussion we obtain a map at cohomology level
\[ H^*_f : H^*_f(M') \to H^*_{f, h \cdot a}(M). \]
For the Lichnerowicz cohomology \(H^*_f(M)\) the problem of homotopy invariance is solved by Lemma 1.1 from [9]. For the cohomology attached to a function \(H^*_f(M)\) the problem of homotopy invariance is the following: given a homotopy \(H\), from \((M, f)\) to \((M', f')\), is it true that \(H^*_f = H^*_{f'}\) at the cohomology level? This problem is partial solved in [9] namely: If the complements of the zero level sets of \(f\) and \(f'\) are dense sets, then in degree zero we do have \(H^*_f = H^*_{f'} : H^*_f(M) \to H^*_{f'}(M')\). For higher degree, a partial result in the regular case is also given in [9]. For our Lichnerowicz type cohomology attached to a function \(H^*_f, h \cdot a\) this problem is still open, but in the next section we prove a homotopy invariance in the regular case.

2.7. Examples.

Example 2.1. Consider \(M = \mathbb{R}^2 - \{(−1, 0), (1, 0)\}\) and let \(\theta\) and \(\eta\) be a generator of \(H^1_{BR}(M)\) supported in \((-\infty, 0) \times \mathbb{R}\) and \(U := (0, \infty) \times \mathbb{R}\), respectively. Then taking into account that \(d\eta = 0\) and the fact that \(\eta|_U\) cannot be \(d|_{|U} - \text{exact}\), see [3], we easily obtain that \(d_{f, \theta}(f\eta) = 0\) and \(f|U \eta|_U\) cannot be \(d|_{U, \theta|_U} - \text{exact}\), for a smooth function on \(M\). Using Mayer-Vietoris sequence for the cohomology \(H^*_f, \theta\) from (2.11) one can show that \(f\eta\) generates \(H^1_{f, \theta}(M)\).

Example 2.2. Suppose we have two manifolds \(M_1, M_2\) and two closed 1-forms \(\theta_1\) and \(\theta_2\), on \(M_1\) and \(M_2\), respectively. Let \(\theta := \text{pr}_1^*\theta_1 + \text{pr}_2^*\theta_2 \in \Omega^1(M_1 \times M_2)\) which is also closed. Then one defines a mapping
\[ \Psi : \Omega^k(M_1) \times \Omega^l(M_2) \to \Omega^{k+l}(M_1 \times M_2), \quad \Psi(\varphi, \psi) = \text{pr}_1^*\varphi \wedge \text{pr}_2^*\psi. \]

Then if we consider two smooth functions \(f_1\) and \(f_2\) on \(M_1\) and \(M_2\), respectively such that \(f := \text{pr}_1^*f_1 = \text{pr}_2^*f_2\) then by direct calculus we obtain that
\[ d_{f, \theta}(\Psi(\varphi, \psi)) = \Psi(d_{f_1, \theta_1}\varphi, \psi) + (-1)^{\text{deg}\, \varphi}\Psi(\varphi, d_{f_2, \theta_2}\psi) \]
and hence we have an induced mapping
\[ H^*_{f_1, \theta_1}(M_1) \otimes H^*_{f_2, \theta_2}(M_2) \to H^*_{f, \theta}(M_1 \times M_2). \]
3. The regular case

In this section we study the regular case i.e., the case where the function \( f \) does not have singularities in a neighborhood of its zero set (i.e. 0 is a regular value). The subset \( S = f^{-1}(\{0\}) \) is then an embedded submanifold of \( M \). We also assume that \( S \) is connected. In this case, the cohomology attached to a function \( H^*_f(M) \) is related with the de Rham cohomologies \( H^*_{dR}(M) \) and \( H^*_{dR}(-1)(S) \), see [9]. Similarly we can relate in this case the Lichnerowicz type cohomology attached to a function \( H^*_{f,\theta}(M) \) with the Lichnerowicz cohomologies \( H^*_{\theta}(M) \) and \( H^*_{i\theta}(-1)(S) \), where \( i : S \to M \) is the natural inclusion. Also in this regular case, we prove a homotopy invariance.

**Theorem 3.1.** If 0 is a regular value of \( f \) then, for each \( r \geq 1 \), there is an isomorphism

\[
H^*_{f,\theta} \cong H^*_{\theta}(M) \oplus H^*_{i\theta}(-1)(S),
\]

for any closed 1-form \( \theta \) on \( M \).

**Proof.** The proof it follows in a similar manner with the proof of Theorem 4.1 from [9] and we need to briefly recall some preliminary results.

Let \( U \subset U' \) be tubular neighborhoods of \( S \). We may assume that \( U = S \times ]-\varepsilon,\varepsilon[ \) and \( U' = S \times ]-\varepsilon',\varepsilon'[ \), with \( \varepsilon' > \varepsilon \), and that

\[
f|_{U'} : S \times ]-\varepsilon',\varepsilon[ \to \mathbb{R}, (x,t) \mapsto t.
\]

Let us consider the projection \( \pi : U' \to S \) and \( \rho : \mathbb{R} \to \mathbb{R} \) be a smooth function which is 1 on \([-\varepsilon,\varepsilon]\) and has support contained in \([-\varepsilon',\varepsilon']\). Note that the function \( \rho \circ f \) is 1 on \( U \), and we can assume that the function \( \rho \circ f \) vanishes on \( M \setminus U' \).

If \( \psi \) is a form on \( S \), we will denote by \( \overline{\psi} \) the form \( \rho(f)\pi^*\psi \). Notice that from

\[
d\overline{\psi} = \rho(f)\pi^*(d\psi) + \rho'(f)d\pi\psi
\]

it easily follows that

\[
df \wedge d\overline{\psi} = df \wedge d\overline{\psi}.
\]

Now we notice that for the closed 1-form \( \theta \) on \( M \) we have

\[
\theta|_{U'} \wedge \overline{\psi} = (i \circ \pi)^*\theta \wedge \overline{\psi} = \rho(f)\pi^*(i^*\theta) \wedge \pi^*\psi = \rho(f)\pi^*(i^*\theta \wedge \psi) = \overline{i^*\theta \wedge \psi},
\]

for any form \( \psi \) on \( S \).

In the sequel we denote by \( \zeta \) the linear application

\[
\zeta : \Omega^r(M) \oplus \Omega^{r-1}(S) \to \Omega^r(M), \quad \zeta(\varphi, \psi) = f^r \varphi + f^{r-1}df \wedge \overline{\psi}.
\]

If \( (\varphi, \psi) \in \Omega^r(M) \oplus \Omega^{r-1}(S) \), with \( d_\theta \varphi = 0 \) and \( d_{i\theta} \psi = 0 \), then using (3.2) and (3.3), we find

\[
df,\theta(\zeta(\varphi, \psi)) = f^{r+1}d_\theta \varphi - f^r df \wedge (d\overline{\psi} - \theta|_{U'} \wedge \overline{\psi}) = f^{r+1}d_\theta \varphi - f^r df \wedge d_{i\theta} \psi = 0.
\]

Similarly, one checks that if \( \varphi \in \Omega^{r-1}(M) \) and \( \psi \in \Omega^{r-2}(S) \), then

\[
\zeta(d_\theta \varphi, d_{i\theta} \psi) = df,\theta(f^{-1} \varphi - f^{-2} df \wedge \overline{\psi}).
\]

We conclude that \( \zeta \) induces a map at the level of cohomology

\[
\zeta^* : H^*_\theta(M) \oplus H^*_{i\theta}(-1)(S) \to H^*_{f,\theta}(M), \quad \zeta^*([\varphi], [\psi]) = [\zeta(\varphi, \psi)].
\]

Finally, according to [9], \( \zeta \) is bijective for all \( r \geq 1 \) and so the theorem follows. \( \square \)
Example 3.1. Let \( S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \) be the 1-sphere and \( f : S^1 \to \mathbb{R} \) the function \( f(x_1, x_2) = x_1 \), so that \( S = f^{-1}(\{0\}) \) is the equator. Then taking into account that \( H^0_f(S^1) = 0 \), for any closed, non-exact 1-form \( \theta \) on \( S^1 \), see Example 1.6 from \[3\], we obtain \( H^0_{f,\theta}(S^1) = 0 \).

In the regular case we have the following homotopy invariance:

Proposition 3.1. Let \( U \) and \( W \) be tubular neighborhoods of \( S_f = f^{-1}(0) \) and \( S_g = g^{-1}(0) \), respectively. We assume that \( f \) and \( g \) do not have singularities on \( U \) and \( W \). If \( H_t \) is a homotopy from \( (U, f) \to (W, g) \), then the induced linear applications between the cohomology spaces are the same: \( H^*_1 = H^*_0 \).

Proof. We can assume that \( U = S_f \times ] - \varepsilon, \varepsilon [ \) and \( W = S_g \times ] - \varepsilon, \varepsilon [ \), with

\[
f(x, \rho) = \rho, \ g(y, \tau) = \tau.
\]

We denote by \( \Psi_f \) and \( \Psi_g \) the linear maps:

\[
\Psi_f : H^r_{h_1^*(\theta|_W)}(U) \oplus H^r_{h_1^*(\theta|_W)}(U) \to H^r_{f, h_1^*(\theta|_W)}(U), \quad \Psi_f([\varphi], [\psi]) = [\rho^* \varphi + \rho^* d\rho \wedge \psi],
\]

\[
\Psi_g : H^r_{\theta|_W}(W) \oplus H^r_{\theta|_W}(W) \to H^r_{g, \theta|_W}(W), \quad \Psi_g([\varphi], [\psi]) = [\tau^* \varphi + \tau^* d\tau \wedge \psi],
\]

which by the previous theorem, are isomorphisms.

Now, we set \( K_t^* = \Psi_f^{-1} \circ H_t^* \circ \Psi_g \), for every \( t \in [0, 1] \). If \( ([\varphi], [\psi]) \in H^r_{\theta|_W}(W) \oplus H^r_{\theta|_W}(W) \), then by a similar calculus as in the proof of Proposition 4.12 from \[9\] we have

\[
H^*_t(\Psi_g([\varphi], [\psi])) = [\rho^* h_1^* \varphi + \rho^* d(\log |a_t|) \wedge h_1^* \psi + \rho^* d\rho \wedge h_1^* \psi].
\]

Now taking into account \( d_{h_1^*(\theta|_W)} h_1^* \psi = h_1^* (d_{\theta|_W} \psi) = 0 \), we conclude that

\[
K_t^*([\varphi], [\psi]) = \left( [h_1^* \varphi + d_{h_1^*(\theta|_W)} (\log |a_t| h_1^* \psi)], [h_1^* \psi] \right) = \left( [h_1^* \varphi], [h_1^* \psi] \right).
\]

Since the Lichnerowicz cohomology is homotopy invariant, see Lemma 1.1 from \[3\], we have \( K_1^* = K_0^* \) and it follows that \( H^*_1 = H^*_0 \). \( \square \)

4. LICHNEROWICZ TYPE COHOMOLOGY ATTACHED TO A FUNCTION OF LOCALLY CONFORMALLY KÄHLER MANIFOLDS

The applications of Lichnerowicz cohomology to locally conformally Kähler or locally conformally symplectic manifolds, are well known in literature see for instance \[11, 13, 5, 6, 14, 15\]. In this section we study some aspects of Lichnerowicz cohomology attached to a function for locally conformally Kähler manifolds.

Definition 4.1. (\[14\]). A locally conformally Kähler, briefly l.c.K., manifold is an Hermitian manifold \( (M, g) \) for which an open covering \( \{U_i\} \) exists, and for each \( i \) a smooth function \( \sigma_i : U_i \to \mathbb{R} \) such that \( g = e^{-\sigma_i}(g|_{U_i}) \) is a Kähler metric on \( U_i \), called a locally conformally Kähler metric.

Accordingly, \[14, 15\], the manifold \( M \) has a Kähler metric which is locally conformally with a Hermitian metric. It is easy to see that \( \theta|_{U_i} = d\sigma_i \) defines a global closed 1-form, and that \( (M, \omega) \) has the characteristic property

\[
d\omega = \theta \wedge \omega,
\]
where $\omega$ is the Hermitian form of $(M, g)$. If we take $U_i = M$, then $(M, \omega)$ is called globally conformally Kähler manifold. The form $\theta$ is called the Lee form of $(M, \omega)$. It is exact iff $(M, \omega)$ is globally conformal Kähler manifold.

Now, if $(M, \omega)$ is an l.c.K. manifold with $\theta$ its Lee form, then due to (4.1) we have $d_0 \omega = 0$. Therefore, $\omega$ represents a cohomology class in the Lichnerowicz complex $(\Omega^\bullet(M), d_0)$.

**Definition 4.2.** The cohomology class $[\omega] \in H^2_\theta(M)$ of $\omega$ is called the Lichnerowicz class of the l.c.K. manifold $(M, \omega)$.

This invariant is called in [11] the Morse-Novikov class of l.c.K. manifolds.

Now if we consider a nonvanishing smooth function $f$ on the l.c.K. manifold $(M, \omega, \theta)$ then by (2.3) we have

$$d_{f,\theta}(f^2 \omega) = f^3 d_\theta \omega = 0$$

which say that $f^2 \omega$ defines a cohomology class in the Lichnerowicz type complex attached to the function $(\Omega^\bullet(M), d_{f,\theta})$.

**Definition 4.3.** The cohomology class $[f^2 \omega] \in H^2_{f,\theta}(M)$ of $f^2 \omega$ is called the Lichnerowicz class attached to the function $f$ of the l.c.K. manifold $(M, \omega)$.

For an $n$-dimensional complex manifold $M$, consider $\Omega^{p,q}(M)$ the set of all $(p, q)$-forms on $M$ and the decomposition of the exterior derivative $d = \partial + \bar{\partial}$. Then we have the decomposition $d_f = \partial_f + \bar{\partial}_f$, where

$$\partial_f : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \partial_f \varphi = f \partial \varphi - (p + q) \partial f \wedge \varphi,$$

$$\bar{\partial}_f : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M), \quad \bar{\partial}_f \varphi = f \bar{\partial} \varphi - (p + q) \bar{\partial} f \wedge \varphi.$$

Also taking into account the decomposition $\theta = \theta^{1,0} + \theta^{0,1}$, consider the Hodge components of the Lichnerowicz differential $d_\theta = d - \theta \wedge$ as

$$d_\theta = \partial_\theta + \bar{\partial}_\theta, \quad \partial_\theta = \partial - \theta^{1,0} \wedge, \quad \bar{\partial}_\theta = \bar{\partial} - \theta^{0,1} \wedge.$$

The Bott-Chern cohomology of the differential complex attached to a function and of Lichnerowicz complex of an $n$-dimensional complex manifold $M$ are defined in a usual way.

Now, for a complex manifold $M$, similar Lichnerowicz type cohomology attached to a function $f$ of Bott-Chern type can be defined. We have a decomposition of $d_{f,\theta}$ into

$$d_{f,\theta} = \partial_{f,\theta} + \bar{\partial}_{f,\theta}, \quad \partial_{f,\theta} = \partial_f - f \theta^{1,0} \wedge, \quad \bar{\partial}_{f,\theta} = \bar{\partial}_f - f \theta^{0,1} \wedge.$$

Now, from $d_{f,\theta}^2 = 0$ we obtain

$$d_{f,\theta}^2 = \bar{\partial}_{f,\theta} \partial_{f,\theta} + \bar{\partial}_{f,\theta} \partial_{f,\theta} = 0.$$

The differential complex

$$\ldots \to \Omega^{p-1,q-1}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q}(M) \xrightarrow{\partial_{f,\theta} \partial_{f,\theta}} \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q+1}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q+1}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \ldots$$

is called the Bott-Chern-Lichnerowicz complex attached to the function $f$ of $M$ and the corresponding Bott-Chern-Lichnerowicz cohomology groups attached to the function $f$ of bidegree $(p, q)$ are given by

$$H^{p,q}_{f,\theta, BC}(M) = \frac{\text{Ker}\{\Omega^{p,q}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p+1,q}(M)\} \cap \text{Ker}\{\Omega^{p,q}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q+1}(M)\}}{\text{Im}\{\Omega^{p-1,q-1}(M) \xrightarrow{\partial_{f,\theta} \bar{\partial}_{f,\theta}} \Omega^{p,q}(M)\}}.$$
Thus, for any l.c.K. manifold \((M, \omega, \theta)\) we have three cohomological invariants attached to the function \(f\):

- the Lee class attached to the function \(f\), \([f\theta] \in H^1_f(M)\);
- the Lichnerowicz class attached to the function \(f\), \([f^2\omega] \in H^2_{f,\theta}(M)\);
- the Bott-Chern-Lichnerowicz class attached to the function \(f\), \([f^2\omega] \in H^{1,1}_{f,\theta,BC}(M)\).

We notice that an important problem to solve is to study some obstructions corresponding to these invariants.

Now, using an argument inspired from \([5, 6]\), we briefly present another cohomology associated to an l.c.K. manifold which is connected with the Lichnerowicz type cohomology attached to a function of l.c.K. manifolds. Let \((M, \omega, \theta)\) be an l.c.K. manifold with \(\theta\) its Lee form. We consider the closed 1-forms \(\theta\) and such that the Lee form. Suppose that \(\omega\) is defined by

\[
\theta_0 = m\theta \quad \text{and} \quad \theta_1 = (m + 1)\theta, \quad m \in \mathbb{R}.
\]

Denote by \(H^*_{f,\theta_0}(M)\) and \(H^*_{f,\theta_1}(M)\) the Lichnerowicz type cohomologies attached to a function \(f\) of the complexes \((\Omega^\bullet(M), df, \theta_0)\) and \((\Omega^\bullet(M), df, \theta_1)\), respectively.

Let \(\hat{\Omega}^r(M) = \Omega^r(M) \oplus \Omega^{r-1}(M)\) and \(\hat{\delta}_f : \hat{\Omega}^r(M) \to \hat{\Omega}^{r+1}(M)\) be the differential operator defined by

\[
\hat{\delta}_f(\varphi, \psi) = (df, \theta_1, \varphi - f^2\omega \wedge \psi, -df, \theta_0, \psi).
\]

Using \([4, 1]\) and \(df(f^2\omega) = f^3d\omega\), by direct calculus it follows \(\hat{\delta}_f^2 = 0\). Thus, we can consider the complex \((\hat{\Omega}^\bullet(M), \hat{\delta}_f)\) and the associated cohomology \(\hat{H}^\bullet_f(M)\). We have the following result which relates \(\hat{H}^\bullet_f(M)\) with Lichnerowicz type cohomologies attached to a function \(H^*_{f,\theta_0}(M)\) and \(H^*_{f,\theta_1}(M)\).

**Proposition 4.1.** Let \((M, \omega)\) be an l.c.K. manifold with \(\theta\) its Lee form. Suppose that \(i^r : \Omega^r(M) \to \hat{\Omega}^r(M)\) and \(\pi^r_2 : \hat{\Omega}^r(M) \to \Omega^{r-1}(M)\) are homomorphisms of \(C^\infty(M, \mathbb{R})\)-modules defined by

\[
\pi^r_2(\varphi, \psi) = \psi,
\]

for \(\varphi \in \Omega^r(M)\) and \(\psi \in \Omega^{r-1}(M)\). Then:

i) The mappings \(i^r\) and \(\pi^r_2\) induce an exact sequence of complexes

\[
0 \to (\Omega^\bullet(M), df, \theta_0) \xrightarrow{i^r} (\hat{\Omega}^\bullet(M), \hat{\delta}_f) \xrightarrow{\pi^r_2} (\Omega^\bullet-1(M), df, \theta_1, \omega) \to 0.
\]

ii) This exact sequence induces a long exact cohomology sequence

\[
\ldots \to H^r_{f,\theta_0}(M) \xrightarrow{(i^r)^*} \hat{H}^r_f(M) \xrightarrow{\pi^r_2} H^{r-1}_{f,\theta_0}(M) \xrightarrow{-\delta^{r-1}} H^{r+1}_{f,\theta_1}(M) \to \ldots,
\]

where the connector homomorphism \(-\delta^{r-1}\) is defined by

\[
(-\delta^{r-1})[\varphi] = [\varphi \wedge f^2\omega], \quad \forall [\varphi] \in H^{r-1}_{f,\theta_0}(M).
\]

From the above proposition, we obtain

**Corollary 4.1.** Let \((M, \omega)\) be an l.c.K. manifold with \(\theta\) its Lee form and such that the Lichnerowicz type cohomology groups attached to a function \(H^r_{f,\theta_0}(M)\) and \(H^r_{f,\theta_1}(M)\) have finite dimension, for all \(r\). Then, the cohomology group \(\hat{H}^r_f(M)\) has also finite dimension, for all \(r\), and

\[
\hat{H}^r_f(M) \cong \frac{H^r_{f,\theta_1}(M)}{\text{Im} \delta^{r-2}} \oplus \ker \delta^{r-1},
\]
where $\delta^r : H^r_{f,\theta_0}(M) \to H^{r+2}_{f,\theta_1}(M)$ is the homomorphism given by (4.11).

**Corollary 4.2.** Let $(M, \omega)$ be an l.c.K. manifold with $\theta$ its Lee form such that the dimensions of the cohomology groups $H^r_{f,\theta_0}(M)$ and $H^r_{f,\theta_1}(M)$ are finite, for all $r$. Suppose that $f^2 \omega$ is $d_{f,\theta}$-exact, that is, there exists a 1-form $\omega'$ on $M$ satisfying $f^2 \omega = d_f \omega' - f \theta \wedge \omega'$. Then, for all $r$, we have

$$\bar{H}^r_f(M) \cong H^r_{f,\theta_1}(M) \oplus H^r_{f,\theta_0}(M).$$

**5. A twisted cohomology attached to a function**

In this section we consider another operator which is related in terms of operators $d_\theta$ and $d_f$. Using this new operator we obtain a twisted cohomology attached to a function.

Let us consider again a closed 1-form $\theta$ on $M$ and a smooth function $f$ on $M$. If we replace $d$ by $d_\theta$ in the definition of exterior derivative attached to a function $d_f$, we obtain the operator

$$d_{\theta,f} : \Omega^r(M) \to \Omega^{r+1}(M), \quad d_{\theta,f}\varphi = f d_\theta \varphi - r d_\theta f \wedge \varphi.$$  

**Remark 5.1.** The operators $d_{f,\theta}$ and $d_{\theta,f}$ are related by $d_{\theta,f}\varphi = d_{f,\theta}\varphi + r f \theta \wedge \varphi$ for any $\varphi \in \Omega^r(M)$.

Using this definition, by direct calculus we obtain

**Proposition 5.1.** The operator $d_{\theta,f}$ has the following properties:

i) $d_{0,f} = d_f$, $d_{\theta,0} = 0$, $d_{\theta,f+g} = d_{\theta,f} + d_{\theta,g}$;

ii) $d_{\theta,1} = d_\theta + r \theta \wedge$, $d_{\theta,f g} = f d_\theta g + g d_\theta f - f g d_{\theta,1}$, $d_{\theta,1} = \frac{1}{2} \left( \frac{1}{d_\theta} + \frac{1}{d_\theta} \right)$;

iii) $d_{\theta,f}(\varphi \wedge \psi) = d_{\theta,f} \varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge d_{\theta,f} \psi + f \theta \wedge \varphi \wedge \psi$;

iv) $d_{\theta,f}^2 = f \theta \wedge d_f$.

Since $d_{\theta,f}^2 \neq 0$, we notice that $d_{\theta,f}$ defines a twisted cohomology, [16], of differential forms on $M$, which is given by

$$H^*_w,\theta,f(M) = \frac{\text{Ker} d_{\theta,f}}{\text{Im} d_{\theta,f} \cap \text{Ker} d_{\theta,f}}$$

and called twisted cohomology attached to the function $f$ of $M$. It is isomorphic to the cohomology of the cochain complex $(\Omega^*_{\theta,f}(M), d_{\theta,f})$ consisting of the differential forms $\varphi \in \Omega^*(M)$ satisfying $d_{\theta,f}^2 \varphi = f \theta \wedge d_f \varphi = 0$.

The complex $\Omega^*_{\theta,f}(M)$ admits a subcomplex $\overline{\Omega}^*_{\theta,f}(M)$, namely, the ideal generated by $f \theta$. On this subcomplex, $d_{\theta,f} = d_f$, which means that it is a subcomplex of the complex attached to the function $f$ of $M$. Hence, one has the homomorphisms

$$a : H^r(\overline{\Omega}^*_{\theta,f}(M)) \to H^r_{w,\theta,f}(M), \quad b : H^r(\overline{\Omega}^*_{\theta,f}(M)) \to H^r_f(M).$$

Now, we can easily construct a homomorphism

$$c : H^r_{w,\theta,f}(M) \to H^{r+1}_f(M).$$

Indeed, if $[\varphi] \in H^r_{w,\theta,f}(M)$, where $\varphi$ is $d_{\theta,f}$-closed form, then we put $c([\varphi]) = [f \theta \wedge \varphi]$, and this produces the homomorphism from (5.4). We notice that the existence of $c$ gives some relation between $d_{\theta,f}$ and the cohomology attached to a function of $M$. 

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