A Note on the Set-Theoretic Representation of
Arbitrary Lattices

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Abstract

Every lattice is isomorphic to a lattice whose elements are sets of
sets and whose operations are intersection and the operation \( \lor^* \) defined
by \( A \lor^* B = A \cup B \cup \{ Z : (\exists X \in A)(\exists Y \in B) X \cap Y \subseteq Z \} \). This
representation spells out precisely Birkhoff’s and Frink’s representation of
arbitrary lattices, which is related to Stone’s set-theoretic representation
of distributive lattices. (AMS Subject Classification, 1991: 06B15)

As a generalization of his representation theory for Boolean algebras, Stone has
developed in [4] a representation theory for distributive lattices. This representa-
tion theory has set-theoretic and topological aspects. Set-theoretically, every
distributive lattice \( L \) is isomorphic to a set lattice \( L^* \), i.e. a lattice whose el-
ements are sets and whose operations are intersection and union. In Stone’s
representation, the elements of \( L^* \) are certain subsets of the set \( F(L) \) of prime
filters of \( L \). Topologically, \( F(L) \) can be viewed as a \( T_0 \)-space with the elements
of \( L^* \) constituting a subbasis.

Following ideas of Priestley’s [3], Urquhart has developed in [5] the topo-
logical aspects of this representation theory to cover arbitrary bounded lattices.
However, Birkhoff and Frink had already in [1] (section 6) a simple set-theoretic
representation for arbitrary lattices, also inspired by Stone, but different from
Urquhart’s representation.

In the Birkhoff-Frink representation, every lattice \( L \) is isomorphic to a lattice
\( L^* \) whose elements are sets of sets, whose meet operation is intersection and
whose join operation is a set-theoretic operation \( \lor^* \) unspecified by Birkhoff and
Frink. The elements of \( L^* \) are certain subsets of a set \( F(L) \), which may be
either the set of all filters of \( L \), or the set of all principal filters of \( L \), or any set
of filters of \( L \) that for every pair of distinct elements of \( L \) has a filter containing
one element of the pair but not the other. Stone’s set-theoretic representation
for distributive lattices may be viewed as a special case of the Birkhoff-Frink
representation: if for a distributive lattice \( L \) we take \( F(L) \) to be the set of all
prime filters of \( L \), then \( \lor^* \) collapses into set-theoretic union.

The aim of this note is to make precise some details of the Birkhoff-Frink
representation, which doesn’t seem to be very well known. We shall explicitly
characterize the operation \( \lor^* \) when \( F(L) \) is the set of all filters of \( L \), or of all
principal filters of \( L \). The interest of this exercise is in applications that may be found in the models of nondistributive nonclassical logics, where the semantic clause for disjunction may be derived from the operation \( \vee^* \).

Let \( L = \langle D, \wedge, \vee \rangle \) be an arbitrary lattice, and let \( F(L) = \{ X : X \text{ is a filter of } L \} \). For every \( a \in D \), let \( f(a) = \{ X \in F(L) : a \in X \} \). Let now \( D^* = \{ f(a) : a \in D \} \), and let

\[
\begin{align*}
  f(a) \wedge^* f(b) &= f(a) \cap f(b), \\
  f(a) \vee^* f(b) &= f(a) \cup f(b) \cup \{ Z \in F(L) : (\exists X \in f(a))(\exists Y \in f(b))X \cap Y \subseteq Z \}.
\end{align*}
\]

The second of these equalities corresponds to the semantic clause for disjunction introduced in [2] (section 3.2), which has since found its way into a number of papers on models of substructural logics.

In the proof of the following proposition we assume for \( a \in D \) that \( [a] = \{ b \in D : a \leq b \} \); that is, \( [a] \) is the principal filter generated by \( a \).

**Proposition 1.** The following equalities hold:

\[
\begin{align*}
  (1.1) \quad & f(a) \wedge^* f(b) = f(a \wedge b), \\
  (1.2) \quad & f(a) \vee^* f(b) = f(a \vee b).
\end{align*}
\]

**Proof.** The proof of (1.1) is quite straightforward, and we only need to consider the proof of (1.2). So suppose \( Z \in f(a) \vee^* f(b) \). If \( a \in Z \) or \( b \in Z \), then, since \( Z \) is a filter, \( a \vee b \in Z \). If, on the other hand, for some \( X \) and \( Y \) we have that \( a \in X, b \in Y \) and \( X \cap Y \subseteq Z \), then, since \( X \) and \( Y \) are filters, \( a \vee b \in X \cap Y \), and so \( a \vee b \in Z \). For the converse, suppose \( Z \in f(a \vee b) \), that is \( a \vee b \in Z \). If \( c \in [a \vee b] \), then \( a \vee b \leq c \), and, since \( Z \) is a filter, \( c \in Z \). So \( [a \vee b] \subseteq Z \), but, since \( [a] \cap [b] = [a \vee b] \), we have that \( [a] \cap [b] \subseteq Z \). Hence for some \( X \), namely \( [a] \), and some \( Y \), namely \( [b] \), we have that \( a \in X, b \in Y \) and \( X \cap Y \subseteq Z \), and so we have proved (1.2).

Since it is quite easy to see that \( f : D \rightarrow D^* \) is one-one and onto, we obtain that \( L = \langle D, \wedge, \vee \rangle \) is isomorphic to \( L^* = \langle D^*, \wedge^*, \vee^* \rangle \).

Note that we obtain the isomorphism of \( L \) with \( L^* \) also when \( F(L) \) is taken to be the set of all principal filters of \( L \), and not the set of all filters of \( L \). Another alternative, yielding again the isomorphism of \( L \) with \( L^* \), is to replace \( \vee^* \) by the operation \( \vee^{**} \) defined by

\[ A \vee^{**} B = \{ Z : (\exists X \in A)(\exists Y \in B)X \cap Y \subseteq Z \}. \]

We have preferred to work with \( \vee^* \), rather than with the more simply defined operation \( \vee^{**} \), which coincides with \( \vee^* \) on \( D^* \) as it was defined up to now, in order to be able to connect smoothly the isomorphism of \( L \) and \( L^* \) with Stone’s representation theory. This connection is made by the following proposition.

**Proposition 2.** If \( L \) is a distributive lattice and \( F(L) \) is the set of all prime filters of \( L \), then \( f(a) \vee^* f(b) = f(a) \cup f(b) \).
Proof. Suppose \( Z \in f(a) \lor^* f(b) \). As in the proof of the previous proposition, it follows that \( a \lor b \in Z \). Since \( Z \) is prime, \( a \in Z \) or \( b \in Z \), that is \( Z \in f(a) \cup f(b) \). The converse, namely, \( f(a) \cup f(b) \subseteq f(a) \lor^* f(b) \), is trivial.

This trivial converse can, however, be blocked if \( \lor^* \) is replaced by \( \lor^{**} \). Indeed, suppose \( a \in Z \); then we must show that for some prime filters \( X \) and \( Y \) we have that \( a \in X \), \( b \in Y \) and \( X \cap Y \subseteq Z \). The prime filter \( X \) can be \( Z \), but, since \( b \) may be the least element of \( L \), there is no guarantee that there is a prime, i.e. proper, filter \( Y \) such that \( b \in Y \).

To conclude, we note that for the sake of symmetry we can define \( f(a) \land^* f(b) \) either as \( f(a) \cap f(b) \cap \{ Z \in F(L) : (\exists X \in f(a))(\exists Y \in f(b)) X \cup Y \subseteq Z \} \), or as \( \{ Z \in F(L) : (\exists X \in f(a))(\exists Y \in f(b)) X \cup Y \subseteq Z \} \); both of these sets are equal to \( f(a) \cap f(b) \). In these new definitions of \( \land^* \), unions of filters occur where in the definitions of \( \lor^* \) and \( \lor^{**} \) we had intersections. Then remark that the set of filters \( F(L) \), which is a semilattice with \( \cap \), is not necessarily closed under \( \cup \).

References

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