Static Analysis of Functionally Graded Coated Plate on Elastic Foundation Based on Levy Method

Abdolreza Toudehdehghan
Department of Mechanical Engineering, INTI International University, Jalan BBN 12/1, Bandar Baru Nilai, 71800 Nilai, Negeri Sembilan, Malaysia.

Email: Abdolreza.toudehdehghan@newinti.edu.my

Abstract. Levy method for bending analysis of composite plate resting on a Winkler elastic foundation hypothesis is presented based on classical plate theory. Two models were considered for a composite coated plate. These two models were named conventional composite model and functionally graded coated model. The conventional composite model was consisting of a Ceramic Homogeneous Coated (HC) layer and metal layer. The Functionally Graded Coated (FGC) model was consisting of Functionally Graded Material (FGM) as a composition of ceramic and metal on a metal layer. The material properties of FGM composite are assumed varying exponentially along the thickness direction. Also, the material properties of the nano-FGM layers are assumed homogeneous. Finally, the effect of the FGC layer, as well as, the elastic stiffness of the foundation is presented and compared with the results of the HC model.

Nomenclature

- $a$: Length of plate
- $b$: Width of plate
- $D$: Flexural rigidity of coated plate
- $x$, $y$, $z$: axes coordinates
- $w$: Total displacement in z-axis
- $w_h$: Homogeneous solution
- $w_p$: Particular solution
- $\nu$: Poisson’s ratio
- $E$: Modulus of elasticity
- $E_{fe}$: Modulus elasticity of FGM
- $E_c$: Modulus elasticity of ceramic
- $E_m$: Modulus elasticity of metal
- $q$: Uniformly distributed load
- $k$: Elastic stiffness of foundation
- $n$: number of the series member
- $n_B$: Constant coefficient in homogeneous solution
- $n_A$: Constant coefficient in homogeneous solution
- $n_\alpha$: Constant number
- $n_\beta$: Constant number
- $g_n(y)$: Fourier coefficient in particular solution
- $f_n(y)$: Fourier coefficient in homogeneous solution
- $p_n(x)$: Fourier coefficient in applied load
- $h$: Thickness of FGM plate
- $E_{fe}$: Modulus elasticity of FGM
1. Introduction

Nowadays, structures that perform multiple tasks have become increasingly accessible with the advancement of human knowledge in the construction of new materials. One of the advanced materials developed by Japanese scientists in the early 1980s is Functionally Graded Materials (FGMs)[1]. In FGMs, the composition of composite varies from one side to the other side based on requirements. For instance, an FGM coating layer has pure ceramic at the surface and pure metal at the other surface, in which the composition of the layer changes gradually from ceramic to metal [2]. In order to describe the effective material properties of the FGM layer, several functions were proposed [3]. However, the well-known functions are such as exponential function, power-law function, and sigmoid function.

Due to the material properties of FGM, the separation between the FGM coating layer and the metal substrate is significantly reduced compared to the homogeneous coating layer as conventional composites [4, 5]. Also, FGMs have several advantages over the conventional composites such as thermal barrier, resistant to corrosion, pitting, etc. with reducing thermal stress and stress concentration in the interface between composite layers [6-9]. Therefore, FGMs have a wide range of applications such as mechanical structures, aerospace, optics, electronics, nuclear, civil structures, and chemicals [10].

Analysis structures on foundations can be studied by geotechnical engineering. There are several foundation models introduced by geotechnical engineers such as Winkler, Umansky, Wieghardt, Shechter, Gorbunov-Posadov, Hetenyi, Shechter, Pasternak, Harr, Giround, Poulos, and Selvadurai which are used in the study of structures on elastic bases [11]. The Winkler model is the first base model and also the popular model used in the analysis of structures on elastic foundations. In the Winkler model, it is assumed the surface of the soil under the loading produces a settlement. The Winkler model is not accurate [11]. Because of the simplicity of this model, many researchers prefer to use this model than other precision models [12-15]. Ghannadiasl and Golmogany [16] considered an Euler-Bernoulli beam with several boundary conditions on the Winkler foundation. They used the B-spline collocation method to analyze the static behavior of the beam under a uniformly distributed load. M. Mofid and M. Noroozi [17] studied the static behavior of simply supported thin plate on a modified Winkler foundation.

In the present study, the Levy method is used to solve the Kirchhoff coated plate on Winkler elastic foundation problem. Two coated models as shown in Figure 1 under uniformly distributed load are considered. The first model is consisting of a ceramic homogeneous layer on top of the homogeneous metal layer. The second model is consisting of the FGM layer on top of the homogeneous metal layer. The geometric constraints of a coated plate are consisting of two opposite sides clamped and another two opposite sides pin. An exponential function represents the material properties of the FGM layer. The elastic coefficient of foundation and Poisson’s ration are assumed constant. By increasing the elastic stiffness of the foundation underneath the coated plate, the foundation becomes a rigid layer, causing the deflection of the plate to decrease dramatically. Also, with increasing elastic base stiffness, the behavior of the HC model and the FGC model is very similar. Another case that is of interest in this study is the phenomenon of singularity between the coated layer and the substrate due to the different material properties of layers in the conventional composite (HC model). Using FGM as the coating layer, the de-bonding phenomenon at the interface of layers is completely eliminated. This research can show the priority of FGM as a coated layer of the plate on a rigid foundation.

![Figure 1. Schematic composite plate on the Winkler base.](image-url)
2. Formulations
The governing equation for the lateral deflection of a thin rectangular Plate on Winkler foundation model subjected to uniformly lateral loads is [18]

\[
\frac{\partial^4 w(x, y)}{\partial x^4} + 2 \frac{\partial^2 w(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y)}{\partial y^4} + \frac{k}{D} w(x, y) = \frac{q(x, y)}{D}
\]

\[D = \frac{E h^3}{12(1 - \nu^2)}\]  \hspace{1cm} (1)

The boundary conditions of coated plate on elastic foundation as shown in Figure 2 are

\[x = 0, a \quad \Rightarrow \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0\]  \hspace{1cm} (2)

\[y = \pm \frac{b}{2} \quad \Rightarrow \quad w = 0, \quad \frac{\partial w}{\partial y} = 0\]  \hspace{1cm} (3)

2.1. Material properties of FGP
The FGP include ceramic and metal. The mechanical properties of the FGC layer vary as an exponential function from the ceramic surface to the other surface which is metal along the FGC thickness. Therefore, the Young modulus of the FGM can be determined as [19, 20]:

\[E_{y} = E_{c} e^{\frac{nh_{c}}{h}} \left(1 - \frac{h}{2} \right)\]  \hspace{1cm} (4)

Where \(-h/2 \leq z \leq h/2\). It should be noted that the Poisson’s ratio is assumed constant [21, 22]. Figure 3 shows the exponential changes of the Young’s modulus in FGP thickness. Assuming that the Young’s modulus of ceramic and metal are 100 GPa and 10 GPa respectively.
2.2. Levy solution of coated plate governing equation

Equation (1) is a nonhomogeneous equation. The general solution of this equation is the summation of homogeneous solution and particular solution. Therefore, the general solution can be written as

$$w(x, y) = w_h(x, y) + w_p(x, y)$$

(5)

The homogeneous solution based on simply supported boundary conditions (Equation (3)) is written as

$$w_h(x, y) = \sum_{n=1}^{\infty} f_n(y) \sin \frac{n\pi x}{a}$$

(6)

Substituting Equation (6) into the homogeneous part of Equation (1) then after mathematically simplifying and considering that the deflections of FGP are symmetric to the x-axis. The homogeneous deflection of FGP is

$$w_h(x, y) = \sum_{n=1}^{\infty} \left[ A_n \left( \sinh \alpha_n y \right) (\sin \beta_n y) + B_n \left( \cosh \alpha_n y \right) (\cos \beta_n y) \right] \sin \frac{n\pi x}{a}$$

(7)

Where

$$\alpha_n = \sqrt{\frac{1}{2} \left( \frac{n^2 \pi^2}{a^2} + \frac{n^4 \pi^4}{a^4} + \frac{k}{D_{fe}} \right)}$$

(8)

$$\beta_n = \sqrt{\frac{1}{2} \left( \frac{n^2 \pi^2}{a^2} - \frac{n^4 \pi^4}{a^4} + \frac{k}{D_{fe}} \right)}$$

(9)

A single Fourier series is expressed as the particular solution of the general solution (Equation (5)) as follows:

$$w_p(x, y) = \sum_{n=1}^{\infty} g_n(y) \sin \frac{n\pi x}{a}$$

(10)

Also, a single Fourier series is expressed as the applied load in Equation (1) as follows:

$$q(x, y) = \sum_{n=1}^{\infty} p_n(y) \sin \frac{n\pi x}{a}$$

(11)

With considering uniformly constant load in Equation (11), the Fourier coefficient $p_n(y)$ can be determined as:

$$p_n(y) = \frac{4q}{n\pi}, \quad n = 1, 3, 5, ...$$

(12)
Substituting Equation (12), (11), and (10) into Equation (1), the Fourier coefficient \( g_n(y) \) can be determined as:

\[
g_n(y) = \frac{4q}{D_{je} n \pi} \frac{1}{n^4 \pi^4 + \frac{k}{a^4}}
\]

(13)

Substituting Equations (7) and (10) into Equation (5), the total deflection of the FGP becomes

\[
w(x, y) = \sum_{n=1}^{\infty} \left[ A_n \sinh \alpha_n y \times \sin \alpha_n y + B_n \cosh \alpha_n y \times \cos \alpha_n y + g_n(y) \right] \sin \frac{n \pi x}{a}
\]

(14)

By using boundary conditions (Equation (3)), the constant coefficients \( A_n \) and \( B_n \) become

\[
A_n = \frac{g_n}{\zeta' \left( \cosh \frac{\alpha_n b}{2} \right) \left( \cosh \frac{\beta_n b}{2} \right)}
\]

\[
B_n = -A_n \left( \tanh \frac{\alpha_n b}{2} \right) \left( \tanh \frac{\beta_n b}{2} \right) - \frac{g_n}{\cosh \frac{\alpha_n b}{2} \left( \cosh \frac{\beta_n b}{2} \right)}
\]

(15)

Where

\[
\zeta_1 = \frac{\zeta_2 + \zeta_3}{\zeta_4 - \zeta_5} - \left( \tanh \frac{\alpha_n b}{2} \right) \left( \tan \frac{\beta_n b}{2} \right)
\]

\[
\zeta_2 = \alpha_n \cosh \left( \frac{\alpha_n b}{2} \right) \sin \left( \frac{\beta_n b}{2} \right)
\]

\[
\zeta_3 = \beta_n \sinh \left( \frac{\alpha_n b}{2} \right) \cos \left( \frac{\beta_n b}{2} \right)
\]

\[
\zeta_4 = \alpha_n \sin \left( \frac{\alpha_n b}{2} \right) \cos \left( \frac{\beta_n b}{2} \right)
\]

\[
\zeta_5 = \beta_n \cos \left( \frac{\alpha_n b}{2} \right) \sin \left( \frac{\beta_n b}{2} \right)
\]

(16)

3. Results and Discussions

As described in the previous section, two coated plate models are considered in this study. In order to solve the equations of coated plates on elastic foundation, a single series equation with two terms are used. The first coated plate model is consisting of the homogeneous ceramic as a coated layer on top of the metal layer (HC). The second coated plate model is consisting of the FGM as a coated layer on top of the metal layer (FGC). The geometrical dimensions, material properties of the ceramic layer, metal layer, FGM layer and elastic coefficient of foundation which used in static analysis of the coated rectangular plates on an elastic foundation are presented in Table 1. Figure 4 demonstrates the deflections of two coated models along the width of the coated plates with clamped-clamped boundary conditions under considering the various range of the elastic stiffness. With increasing the elastic stiffness (close to reality [18]) at the base, the difference deflection in two coated models are disappeared. Also, Figure 4 shows that the deflections along the width of plate and displacements at the constraints are dramatically reduced. Figure 5 shows the deflections of the two coated plate models along the half-length of the plates which the constraints are pin – pin. No displacement happens at pin
constraints. The behavior of two coated models plate with increasing the elastic stiffness of the
foundation is very close. Also, the deflections dramatically decrease with increasing the elastic stiffness.
It can be deduced from Figures 4 and 5 that the mechanical behavior of the two coated plate models on
the elastic foundation is quite similar to considering the real elastic stiffness \( k > 100 \text{MPa} \). Figures 6
and 7 show the 3D deflections of the coated plate based on the data in Table 1 for the conventional
coated plate model (HC) and the FGM coated plate model (FGC), respectively.

**Table 1.** Dimensions, material properties and elastic constant of foundation

| Parameter | Quantity | Parameter | Quantity |
|-----------|----------|-----------|----------|
| \( h_m \) (m) | 0.006 | \( V_{12} = V_{21} \) | 0.3 |
| \( h_c \) (m) | 0.004 | \( k \) (Pa) | 50000 |
| \( h_{fe} \) (m) | 0.004 | \( E_m \) (GPa) | 150 |
| \( a \) (m) | 1 | \( E_c \) (GPa) | 250 |
| \( b \) (m) | 1 | \( F \) (N/m) | 22000 |

![Figure 4](image.png)

*Figure 4.** Deflection at the center of coated plates along the width based on various elastic coefficients.*
Figure 5. Deflection of half plates along the length at the center with considering various elastic coefficients.

Figure 6. Deflection of homogeneous coated plate subjected to uniformly distributed load on elastic foundation.
Figures 8 and 9 show the stress at the center of the plate along the thickness of the coated plate, taking into account the minimum and maximum elastic stiffness values, respectively. Both figures show that using the FGM as a coating layer, the singularity phenomenon is completely eliminated between the coated layer and the substrate layer. Therefore, in the FGC model de-bonding phenomenon at the interface between layers will not occur. Figure 5 clearly shows that when the elastic stiffness is about $k = 0.05MPa$. The stress increased at the outer surfaces of the coated layer and substrate layer. This problem is eliminated by increasing the elastic stiffness to $k = 100MPa$ as shown in Figure 6.

Figure 8. Stress across the thickness of the coated plates in two coated models on elastic foundation ($k = 0.05MPa$).
Figure 9. Stress across the thickness of the coated plates in two coated models on elastic foundation ($k = 100\text{MPa}$).

4. Conclusion
The thin rectangular coated plate on Winkler foundation was solved by using single series equation for homogeneous solution, particular solution, and distributed loading on plate. Two terms of series equations are used to find the solution. Two models of coated plates are defined. In the first model, the coated layer is homogeneous ceramic (HC). The second one has FGM as a coated layer (FGC). The material properties of the FGM layer are defined based on exponential function. The results show that the priority of the FGC plate on a rigid enough foundation ($k > 10\text{MPa}$).

References
[1] S. Chakraverty and K. K. Pradhan, Vibration of functionally graded beams and plates: Academic Press, 2016.
[2] H.-S. Shen, Functionally graded materials: nonlinear analysis of plates and shells: CRC press, 216.
[3] A. Toudehdehghan, J. Lim, K. E. Foo, M. Ma'arof, and J. Mathews, "A brief review of functionally graded materials," in MATEC Web Conferences, 2017.
[4] S.-H. Chi and Y.-L. Chung, "Mechanical behavior of functionally graded material plates under transverse load—Part I: Analysis," International Journal of Solids and Structures, vol. 43, pp. 3657-3674, 2006.
[5] H. Shodja, H. Haftbaradaran, and M. Asghari, "A thermoelasticity solution of sandwich structures with functionally graded coating," Composites science and technology, vol. 67, pp. 1073-1080, 2007.
[6] F. Ramirez, P. R. Heyliger, and E. Pan, "Static analysis of functionally graded elastic anisotropic plates using a discrete layer approach," Composites Part B: Engineering, vol. 37, pp. 10-20, 2006.
[7] A. Toudehdehghan, M. M. Rahman, and F. Tarlochan, "Mechanical and Thermal Analysis of Classical Functionally Graded Coated Beam," in E3S Web of Conferences, 2018, p. 01033.
[8] A. Toudehdehghan, M. M. Rahman, and F. Tarlochan, "Analyzing The Behavior of Classical Functionally Graded Coated Beam," in MATEC Web of Conferences, 2017, p. 03009.
[9] M. Zaki, F. Tarlochan, and S. Ramesh, "Two dimensional elastic deformations of functionally graded coated plates with clamped edges," Composites Part B: Engineering, vol. 45, pp. 1010-1022, 2013.
[10] D. Jha, T. Kant, and R. Singh, "A critical review of recent research on functionally graded plates," *Composite Structures*, vol. 96, pp. 833-849, 2013.

[11] E. Tsudik, *Analysis of structures on elastic foundations*: J. Ross Publishing, 2012.

[12] L. FRÝBA, "History of Winkler foundation," *Vehicle system dynamics*, vol. 24, pp. 7-12, 1995.

[13] A. Ghannadiasl and M. Mofid, "An analytical solution for free vibration of elastically restrained Timoshenko beam on an arbitrary variable Winkler foundation and under axial load," *Latin American Journal of Solids and Structures*, vol. 12, pp. 2417-2438, 2015.

[14] S. Binesh, "Analysis of beam on elastic foundation using the radial point interpolation method," *Scientia Iranica*, vol. 19, pp. 403-409, 2012.

[15] D. A. Dillard, B. Mukherjee, and R. C. Batra, "Reflections on the 150th Anniversary of Winkler’s Foundation and its Profound Influence on the Field of Adhesion," in *40th Annual Meeting of the Adhesion Society*, 2017.

[16] A. GHANNADIASL and M. Z. GOLMOGANY, "Analysis of Euler-Bernoulli Beams with arbitrary boundary conditions on Winkler foundation using a B-spline collocation method," *Engineering Transactions*, vol. 65, pp. 423–445, 2017.

[17] M. Mofid and M. NOUROUZI, "A plate on Winkler foundation with variable coefficient," 2009.

[18] S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of plates and shells*: McGraw-hill, 1959.

[19] A. Chakraborty, S. Gopalakrishnan, and J. Reddy, "A new beam finite element for the analysis of functionally graded materials," *International Journal of Mechanical Sciences*, vol. 45, pp. 519-539, 2003.

[20] M. Aydogdu and V. Taskin, "Free vibration analysis of functionally graded beams with simply supported edges," *Materials & design*, vol. 28, pp. 1651-1656, 2007.

[21] A. Toudedehghian and M. M. Rahman, "Comparison conventional coated beam with functionally graded coated beam," *International Journal of Engineering and Technology (UAE)*, vol. 7, pp. 713-721, 2018.

[22] Y. Hao, W. Zhang, J. Yang, and S. Li, "Nonlinear dynamic response of a simply supported rectangular functionally graded material plate under the time-dependent thermalmechanical loads," *Journal of Mechanical Science and Technology*, vol. 25, p. 1637, 2011.