COBORDISM OF FOLD MAPS, STABLY FRAMED MANIFOLDS AND IMMERSIONS

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Abstract. We give complete geometric invariants of cobordisms of fold maps with oriented singular set and cobordisms of even codimensional fold maps. These invariants are given in terms of cobordisms of stably framed manifolds and cobordisms of immersions with prescribed normal bundles.

Introduction

We defined and used geometric cobordism invariants of fold maps in [16, 18, 17] which describe the immersion of the singular set of the fold map into the target manifold together with more detailed informations about the tubular neighbourhood of the singular set of the fold map in the source manifold. In this paper we define further invariants which describe the cobordism class of the source manifold and its fold map outside of the singular set as well. In [16] we showed that the cobordism groups of fold maps contain stable homotopy groups of spheres as direct components and in [17] we showed that they also contain stable homotopy groups of the classifying spaces $BO(k)$. In this paper, by using the results of Ando [4, Theorem 0.1, Theorem 3.2] about the existence of fold maps, we show that together with our invariants defined in [16, 18, 17], cobordism groups of manifolds with stable framings (see Section 1.4) give complete cobordism invariants of the cobordism classes of even codimensional fold maps and the cobordism classes of framed fold maps (see Definition 1.4) of arbitrary codimension into the Euclidean space (see Theorem 2.1, Corollary 2.2).

We emphasize that our results in this paper are based on elementary constructions of geometric cobordism invariants [16, 18, 17], constructions of special fold maps [17] and an elementary application (see Section 3) of a simple corollary [4, Theorem 0.1, Theorem 3.2] of the h-principle for fold maps of Ando [4, Theorem 0.5, Theorem 2.1].

Independently from our present paper Sadykov [25] gave a splitting of the cobordism groups of fold maps in terms of homotopy groups of spectra, also in the odd codimension case. These spectra were constructed [23] using the h-principle for fold maps of Ando [4, Theorem 0.5, Theorem 2.1] but with a quite different and much more sophisticated approach than ours. Sadykov also showed that our geometric invariants defined in [16, 17, 18] coincide with direct summands of the splitting given by [25].

For other more sophisticated applications of the h-principle for fold maps of Ando toward cobordisms of fold maps, see, [1, 2, 3, 5, 6].

For further results about cobordisms of fold maps with positive codimension, see, [7, 36]. For cobordisms of negative codimensional singular maps with completely different approach from our present paper, see, [1, 2, 3, 5, 6, 11, 12, 13, 14, 24, 27, 29].

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The paper is organized as follows. In Section 1 we give the basic definitions, in
Section 2 we state our main results and in Section 3 we prove our results about complete
invariants.

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Notations. In this paper the symbol “II” denotes the disjoint union, for any number $x$
the symbol “$[x]$” denotes the greatest integer $i$ such that $i \leq x$, $\varepsilon^1_X$ denotes the trivial line
bundle over the space $X$, $\varepsilon^k$ denotes the trivial line bundle over the point, and the symbols
$\xi^k$, $\eta^k$, etc. usually denote $k$-dimensional real vector bundles. The symbol $T \xi^k$ denotes
the Thom space of the bundle $\xi^k$. The symbol $\text{Im} \xi^k_N(n-k,k)$ denotes the cobordism
group of $k$-codimensional immersions into an $n$-dimensional manifold $N$ whose normal bundles are
induced from $\xi^k$ (this group is isomorphic to the group $\{N, T \xi^k\}$, where $\tilde{N}$ denotes the one point compactification of the manifold $N$ and the symbol $\{X,Y\}$ denotes the group of stable homotopy classes of continuous maps from the space $X$ to the
space $Y$). The symbol $\text{Im} \xi^k(n-k,k)$ denotes the cobordism group of $k$-codimensional
immersions into $\mathbb{R}^n$ whose normal bundles are induced from $\xi^k$ (this group is isomorphic
to $\pi_n^s(T \xi^k)$). The symbol $\pi_n^s(X)$ ($\pi_n^s$) denotes the $n$th stable homotopy group of the
space $X$ (resp. spheres). The symbols $\Omega_m$ and $\mathcal{G}_m$ denote the usual cobordism groups
of oriented and unoriented $m$-dimensional manifolds, respectively. The symbol “id”
denotes the identity map of the space $A$. The symbol $\varepsilon$ denotes a small positive number.
All manifolds and maps are smooth of class $C^\infty$.

1. Preliminaries

1.1. Fold maps. Let $n \geq 1$ and $q \geq 0$. Let $Q^{n+q}$ and $N^n$ be smooth manifolds of
dimensions $n+q$ and $n$ respectively. Let $p \in Q^{n+q}$ be a singular point of a smooth map
$f : Q^{n+q} \to N^n$. The smooth map $f$ has a fold singularity of index $\lambda$ at the singular
point $p$ if we can write $f$ in some local coordinates around $p$ and $f(p)$ in the form

$$f(x_1, \ldots, x_{n+q}) = (x_1, \ldots, x_{n-1}, -x_2^2 - \cdots - x_{n+\lambda-1}^2 + x_{n+\lambda}^2 + \cdots + x_{n+q}^2)$$

for some $\lambda$ $(0 \leq \lambda \leq q+1)$ (the index $\lambda$ is well-defined if we consider that $\lambda$ and $q+1-\lambda$
represent the same index).

A smooth map $f : Q^{n+q} \to N^n$ is called a fold map if $f$ has only fold singularities.

A smooth map $f : Q^{n+q} \to N^n$ has a definite fold singularity at a fold singularity $p \in Q^{n+q}$ if $\lambda = 0$ or $\lambda = q+1$, otherwise $f$ has an indefinite fold singularity of index $\lambda$
at the fold singularity $p \in Q^{n+q}$.

Let $S_{\lambda}(f)$ denote the set of fold singularities of index $\lambda$ of $f$ in $Q^{n+q}$. Note that
$S_{\lambda}(f) = S_{q+1-\lambda}(f)$. Let $S_f$ denote the set $\bigcup S_{\lambda}(f)$.

Note that the set $S_f$ is an $(n-1)$-dimensional submanifold of the manifold $Q^{n+q}$.

Note that each connected component of the manifold $S_f$ has its own index $\lambda$ if we
consider that $\lambda$ and $q+1-\lambda$ represent the same index.

Note that for a fold map $f : Q^{n+q} \to N^n$ and for an index $\lambda$ $(0 \leq \lambda \leq \lfloor q/2 \rfloor)$
the codimension one immersion $f|_{S_{\lambda}(f)} : S_{\lambda}(f) \to N^n$ of the singular set of index $\lambda$
$S_{\lambda}(f)$ has a canonical framing (i.e., trivialization of the normal bundle) by identifying
canonical the set of fold singularities of index $\lambda$ of the map $f$ with the fold germ
$(x_1, \ldots, x_{n+q}) \mapsto (x_1, \ldots, x_{n-1}, -x_2^2 - \cdots - x_{n+\lambda-1}^2 + x_{n+\lambda}^2 + \cdots + x_{n+q}^2)$, see, for example,
If \( f : Q^{n+q} \to N^n \) is a fold map in general position, then the map \( f \) restricted to the singular set \( S_f \) is a general positional codimension one immersion into the target manifold \( N^n \).

Since every fold map is in general position after a small perturbation, and we study maps under the equivalence relation cobordism (see Definition \[\text{Definition 1.3}\]), in this paper we can restrict ourselves to studying fold maps which are in general position. Without mentioning we suppose that a fold map \( f \) is in general position.

**Definition 1.1** (Framed fold map). We say that a fold map \( f : Q^{n+q} \to N^n \) is framed if the codimension one immersion \( f|_{S_f} : S_f \to N^n \) of the singular set \( S_f \) has a framing (i.e., trivialization of the normal bundle) such that for each index \( \lambda \) with \( 0 \leq \lambda \leq [q/2] \) the framing of \( f|_{S_\lambda(f)} : S\lambda(f) \to N^n \) coincides with the canonical framing. For odd \( q \), the framing of the immersion of the singular set of index \((q + 1)/2\) can be arbitrary.

Note that if we have a framed fold map \( f \) into an oriented manifold \( N^n \), then the singular set \( S_f \) has a natural orientation which gives at any point of \( f(S_f) \) (together with the framing of the immersion of \( S_f \)) the orientation of the target manifold \( N^n \).

**Definition 1.2** (Oriented fold map). A fold map \( f : Q^{n+q} \to N^n \) is oriented if there is a chosen consistent orientation of every fiber at their regular points.

For example, a fold map \( f : Q^{n+q} \to N^n \) between oriented manifolds is canonically oriented.

Note that an oriented fold map \( f : Q^{n+q} \to N^n \) with odd \( q \) may not have a framing in the sense of Definition \[\text{Definition 1.3}\]

1.2. **Equivalence relations of fold maps.**

**Definition 1.3** (Cobordism). Two fold maps \( f_i : Q^{n+q}_i \to N^n \) (\( i = 0, 1 \)) of closed \((n + q)\)-dimensional manifolds \( Q^{n+q}_i \) \((i = 0, 1)\) into an \( n \)-dimensional manifold \( N^n \) are cobordant if

a) there exists a fold map \( F : X^{n+q+1} \to N^n \times [0, 1] \) of a compact \((n + q + 1)\)-dimensional manifold \( X^{n+q+1} \),

b) \( \partial X^{n+q+1} = Q_0^{n+q} \amalg Q_1^{n+q} \) and

c) \( F|_{Q_0^{n+q} \times [0, \varepsilon]} = f_0 \times \text{id}_{[0, \varepsilon]} \) and \( F|_{Q_1^{n+q} \times (1 - \varepsilon, 1]} = f_1 \times \text{id}_{[1 - \varepsilon, 1]} \), where \( Q_0^{n+q} \times [0, \varepsilon) \) and \( Q_1^{n+q} \times (1 - \varepsilon, 1] \) are small collar neighbourhoods of \( \partial X^{n+q+1} \) with the identifications \( Q_0^{n+q} = Q_0^{n+q} \times \{0\} \) and \( Q_1^{n+q} = Q_1^{n+q} \times \{1\} \).

If the fold maps \( f_i : Q^{n+q}_i \to N^n \) \((i = 0, 1)\) are oriented, we say that they are oriented cobordant (or shortly cobordant if it is clear from the context) if they are cobordant in the above sense via an oriented fold map \( F \), such that the orientations are compatible on the boundary.

We call the map \( F \) a cobordism between \( f_0 \) and \( f_1 \).

This clearly defines an equivalence relation on the set of (oriented) fold maps of closed \((n + q)\)-dimensional manifolds into an \( n \)-dimensional manifold \( N^n \).

We denote the set of (oriented) cobordism classes of (oriented) fold maps of closed \((n + q)\)-dimensional manifolds into an \( n \)-dimensional manifold \( N^n \) by \( \text{Cob}_{\text{N}, f}(n + q, -q) \) (resp. \( \text{Cob}_0^{\text{N}, f}(n + q, -q) \)). When \( N^n = \mathbb{R}^n \), we denote it by \( \text{Cob}_f(n + q, -q) \) (resp. \( \text{Cob}_0^{f}(n + q, -q) \)). We note that we can define a commutative semigroup operation in the usual way on the set of cobordism classes \( \text{Cob}_{\text{N}, f}(n + q, -q) \) (resp. \( \text{Cob}_0^{\text{N}, f}(n + q, -q) \)) by
the disjoint union. If the target manifold \( N^n \) is the Euclidean space \( \mathbb{R}^n \) (or more generally if \( N^n \) has the form \( \mathbb{R}^1 \times M^{n-1} \) for some \((n-1)\)-dimensional manifold \( M^{n-1} \)), then the elements in the semigroup \( \text{Cob}_{N,f}(n+q, -q) \) (resp. \( \text{Cob}_{N,f}^O(n+q, -q) \)) have their inverses: namely, compose them with a reflection in a hyperplane (in \( \{0\} \times M^{n-1} \) in general, see \cite{8}). Hence the semigroups \( \text{Cob}_{N,f}(n+q, -q) \) (resp. \( \text{Cob}_{N,f}^O(n+q, -q) \)) are in this case actually groups.

**Definition 1.4** (Framed cobordism). Two (oriented) framed fold maps

\[
f_i: Q_i^{n+q} \to N^n
\]

\((i = 0, 1)\) of closed \((n+q)\)-dimensional manifolds \( Q_i^{n+q} \) \((i = 0, 1)\) into an \( n \)-dimensional manifold \( N^n \) are **(oriented) framed cobordant** if they are (oriented) cobordant in the sense of Definition \ref{definition} by a framed fold map \( F: X^{n+q+1} \to N^n \times [0, 1] \) such that the framing of the immersion \( F|_{S_F}: S_F \to N^n \times [0, 1] \) restricted to the immersion \( f_i|_{S_{f_i}}: S_{f_i} \to N^n \times \{i\} \) coincides with the framing of the fold map \( f_i \) \((i = 0, 1)\).

Let us denote the (oriented) framed cobordism semigroup of (oriented) framed fold maps of \((n+q)\)-dimensional manifolds into \( N^n \) by \( \text{Cob}_{N,f,f_i}(n+q, -q) \) (resp. \( \text{Cob}_{N,f,f_i}^O(n+q, -q) \)).

Note that for even codimension \( q = 2k \) \((k \geq 0)\) the (oriented) framed cobordism semigroup \( \text{Cob}_{N,f,f_i}(n+q, -q) \) (resp. \( \text{Cob}_{N,f,f_i}^O(n+q, -q) \)) is naturally isomorphic to the (oriented) cobordism semigroup \( \text{Cob}_{N,f}(n+q, -q) \) (resp. \( \text{Cob}_{N,f}^O(n+q, -q) \)).

1.3. Cobordism invariants of fold maps. As an imitation of lifting positive codimensional singular maps \cite{8} Section 7, Proof of Theorem 2], we defined and used geometric invariants of cobordisms of fold maps (for the definitions and notations, see, \cite{17} Section 2]), namely the homomorphisms

\[
\xi_{\lambda,q}^N: \text{Cob}_{N,f}^O(n+q, -q) \to \text{Imm}_N^{\text{e}B(O(\lambda) \times O(q+1-\lambda))}(n-1, 1)
\]

for \( 0 \leq \lambda < (q+1)/2 \) and

\[
\xi_{(q+1)/2,q}^N: \text{Cob}_{N,f}^O(n+q, -q) \to \text{Imm}_N^{\text{e}B}(n-1, 1)
\]

for \( q \) odd and \( \lambda = (q+1)/2 \). These homomorphisms map a cobordism class of a fold map \( f \) into the cobordism class of the immersion of its fold singular set \( S_\lambda(f) \) of index \( \lambda \) with normal bundle induced from the target of the universal fold germ bundle of index \( \lambda \). In the case of oriented fold maps (e.g., in the case of oriented manifolds \( Q^{n+q} \) and \( N^n \)), we have the analogous homomorphisms

\[
\xi_{\lambda,q}^{O,N}: \text{Cob}_{N,f}^O(n+q, -q) \to \text{Imm}_N^{\text{e}B(O(\lambda) \times O(q+1-\lambda))}(n-1, 1)
\]

for \( 0 \leq \lambda < (q+1)/2 \) and

\[
\xi_{(q+1)/2,q}^{O,N}: \text{Cob}_{N,f}^O(n+q, -q) \to \text{Imm}_N^{\text{e}B}(n-1, 1)
\]

for \( q \) odd and \( \lambda = (q+1)/2 \) as well. We used these homomorphisms in \cite{16} \cite{18} \cite{17} in order to describe cobordisms of fold maps.

For \( 0 \leq \lambda \leq (q+1)/2 \), we define the (oriented) framed cobordism invariants

\[
\xi_{\lambda,q}^N: \text{Cob}_{N,f,f_i}^O(n+q, -q) \to \text{Imm}_N^{\text{e}B(O(\lambda) \times O(q+1-\lambda))}(n-1, 1)
\]
and

\[ \xi_{\lambda,q}^{Q,N} : \text{Cob}^{O}_{N,f,fr}(n + q, -q) \to \text{Imm}^{\varepsilon^{1}_{BS(O(\lambda) \times O(q+1-\lambda))}}_{N}(n-1,1) \]

in the analogous way.

1.4. Framed cobordism of manifolds.

**Definition 1.5** (Stably $(n-1)$-framed manifolds and stably $(n-1)$-framed cobordism). For $n > 0, q \geq 0$ an $(n + q)$-dimensional manifold $Q^{n+q}$ is stably $(n-1)$-framed if the vector bundle $TQ^{n+q} \oplus \varepsilon_{Q^{n+q}}^{2}$ has $n + 1$ sections that are linearly independent at every point of $Q^{n+q}$ (shortly, we say that it has $n + 1$ independent sections).

Let $Q^{n+q}_i$ be closed (oriented) stably $(n-1)$-framed $(n+q)$-dimensional manifolds, i.e., the vector bundles $TQ^{n+q}_i \oplus \varepsilon_{Q^{n+q}_i}^{2}$ have $n + 1$ independent sections $e^1_i, \ldots, e^{n+1}_i$ ($i = 0, 1$). We say that the manifolds $Q^{n+q}_0$ and $Q^{n+q}_1$ are stably (oriented) $(n-1)$-framed cobordant if

a) they are (oriented) cobordant in the usual sense by an (oriented) $(n+q+1)$-dimensional manifold $W^{n+q+1}$,

b) one of the two trivial line bundles in the direct sum $TQ^{n+q}_0 \oplus \varepsilon_{Q^{n+q}_0}^{2}$ (resp. $TQ^{n+q}_1 \oplus \varepsilon_{Q^{n+q}_1}^{2}$) corresponds to the inward (resp. outward) normal section of the boundary of $W^{n+q+1}$,

c) the vector bundle $TW \oplus \varepsilon_{W}^{1}$ has $n + 1$ independent sections $f^1, \ldots, f^{n+1}$,

d) the sections $f^j$ restricted to the boundary $Q^{n+q}_0 \cup Q^{n+q}_1$ of $W^{n+q+1}$ coincide with the sections $e^j_i$ ($j = 1, \ldots, n + 1$ and $i = 0, 1$).

We denote the set of (oriented) stably $(n-1)$-framed cobordism classes of closed (oriented) stably $(n-1)$-framed $(n+q)$-dimensional manifolds by $C_{n+q}(n)$ (resp. $C^O_{n+q}(n)$) which is an abelian group with the disjoint union as operation.

By [26] Lemma 3.1, there is a homomorphism

\[ \sigma^{O}_{n,q} : \text{Cob}^{O}_{f,fr}(n + q, -q) \to C^{O}_{n+q}(n), \]

which maps a cobordism class of a framed fold map $g : Q^{n+q} \to \mathbb{R}^n$ into the stably $(n-1)$-framed cobordism class of the source manifold $Q^{n+q}$ with the stable framing obtained by [26] Lemma 3.1.\[1\]

2. Results

In this section, the target manifold $N^n$ is always assumed to be $\mathbb{R}^n$. Note that the group

\[ \text{Imm}^{\varepsilon^{1}_{BS(O(\lambda) \times O(q+1-\lambda))}}_{N}(n-1,1) \]

(or \[1\text{Imm}^{\varepsilon^{1}_{BS(O(\lambda) \times O(q+1-\lambda))}}_{N}(n-1,1) \]), which is the target of the homomorphism $\xi^{O,N}_{\lambda,q}$ (resp. $\xi^{O,N}_{\lambda,q}$), is naturally identified with the group

\[ \pi^{s}_{n-1} \oplus \pi^{s}_{n-1}(B(O(\lambda) \times O(q+1-\lambda))) \]

(resp. $\pi^{s}_{n-1} \oplus \pi^{s}_{n-1}(BS(O(\lambda) \times O(q+1-\lambda)))$).

The main results of this section are the following.

\[1\text{In [26] Lemma 3.1} \] $n$ independent sections for $TQ^{n+q} \oplus \varepsilon_{Q^{n+q}}^{1}$ are constructed.
Theorem 2.1. Let $n > 0, q \geq 0$. Then, the homomorphisms

$$\sigma_{n,q}^O \oplus \bigoplus_{1 \leq \lambda \leq (q+1)/2} \xi_{\lambda,q}^O : \text{Cob}_{f,fr}^O(n + q, -q) \rightarrow \mathcal{C}_{n+q}^O(n) \oplus \bigoplus_{1 \leq \lambda \leq (q+1)/2} \pi_{n-1}^s \oplus \pi_{n-1}^s(B(S(O(\lambda) \times O(q + 1 - \lambda)))$$

and

$$\sigma_{n,q} \oplus \bigoplus_{1 \leq \lambda \leq (q+1)/2} \xi_{\lambda,q} : \text{Cob}_{f,fr}^O(n + q, -q) \rightarrow \mathcal{C}_{n+q}(n) \oplus \bigoplus_{1 \leq \lambda \leq (q+1)/2} \pi_{n-1}^s \oplus \pi_{n-1}^s(B(O(\lambda) \times O(q + 1 - \lambda)))$$

denoted by $\mathcal{O}_{n,q}$ and $\mathcal{O}_{n,q}$, respectively, are injective. In other words, for closed (oriented) manifolds $Q^{n+q}$ ($i = 0, 1$) two framed fold maps $f_i : Q_i^{n+q} \rightarrow \mathbb{R}^n$ ($i = 0, 1$) are (oriented) framed cobordant if and only if

$$\mathcal{O}_{n,q}([f_0]) = \mathcal{O}_{n,q}([f_1]).$$

Corollary 2.2. For $k \geq 0$, the homomorphism $\mathcal{O}_{n,2k}$ gives a complete invariant of the (oriented) cobordism group $\text{Cob}^O_f(n + 2k, -2k)$ of fold maps.

Remark 2.3. By Corollary 2.2 the homomorphisms

$$\mathcal{O}_{n,0} : \text{Cob}^O_f(n, 0) \rightarrow \mathcal{C}_n^O(n)$$

and

$$\mathcal{O}_{n,0} : \text{Cob}^O_f(n, 0) \rightarrow \mathcal{C}_n(n)$$

are injective, and by Ando [4, Theorem 3.2] they are surjective as well. Hence, we have $\text{Cob}^O_f(n, 0) = \mathcal{C}_n^O(n)$ and $\text{Cob}^O_f(n, 0) = \mathcal{C}_n(n)$. Note that since the group $\mathcal{C}_n^O(n)$ is isomorphic to $\pi_n^s$, we obtain another argument for the isomorphism $\text{Cob}^O_f(n, 0) = \pi_n^s$ (for the original proof, see Ando [11, 15]).

3. Proof

In this section, we prove Theorem 2.1.

Proof of Theorem 2.1. Let $f : Q^{n+q} \rightarrow \mathbb{R}^n$ be a framed fold map. If

$$\xi_{\lambda,q}([f]) \in \text{Im} \text{mm}^2_{B(O(\lambda) \times O(q + 1 - \lambda))}(n - 1, 1)$$

is zero for $1 \leq \lambda \leq (q + 1)/2$, then by gluing the null-cobordisms of the fold singularity bundles [17] to the fold map $f$, we obtain a framed fold map $F : W^{n+q+1} \rightarrow \mathbb{R}^n \times [0, 1]$, where

1. $W^{n+q+1}$ is a compact $(n + q + 1)$-dimensional manifold with boundary $Q^{n+q} \cup \partial_{n+q}^q$,
2. $F|_{Q^n \times [0, \varepsilon)} = f \times \text{id}_{[0, \varepsilon]}$, where $Q^{n+q} \times [0, \varepsilon)$ is a small collar neighbourhood of $Q^{n+q} \subset W^{n+q+1}$ in $W^{n+q+1}$ with the identification $Q^{n+q} = Q^{n+q} \times \{0\}$ and
3. $F$ is a submersion into $\mathbb{R}^n \times (0, 1)$ near $P^{n+q}$.
If \( \sigma_{n,q}(f) \) is zero, then since the manifolds \( Q^{n+q} \) and \( P^{n+q} \) are stably \((n-1)\)-framed cobordant by the manifold \( W^{n+q+1} \) and framing \( \mathfrak{f} \) obtained by [26] Lemma 3.1 from the framed fold map \( F \), the stably \((n-1)\)-framed manifold \( P^{n+q} \) is also zero in the stable \((n-1)\)-framed cobordism group \( C_{n+q}(n) \). Hence by gluing a stably \((n-1)\)-framed null-cobordism of \( P^{n+q} \) to \( W^{n+q+1} \), we obtain a compact \((n+q+1)\)-dimensional manifold \( X^{n+q+1} \) with boundary \( Q^{n+q} \) such that the bundle \( TX^{n+q+1} \oplus e^1_{X^{n+q+1}} \) has an \((n+1)\)-framing which coincides with the framing \( \mathfrak{f} \) on \( W^{n+q+1} \).

Since \( \mathbb{R}^n \times [0,1] \) is contractible, we can extend the map \( F \) to a continuous map \( G: X^{n+q+1} \to \mathbb{R}^n \times [0,1] \). Note that the \((n+1)\)-framing of the bundle \( TX^{n+q+1} \oplus e^1_{X^{n+q+1}} \) gives a fiberwise epimorphism \( H \) of the bundle \( TX^{n+q+1} \oplus e^1_{X^{n+q+1}} \) into the tangent bundle \( T(\mathbb{R}^n \times [0,1]) \) covering the continuous map \( G \) by mapping the \( n+1 \) frames to the canonical bases of \( T(\mathbb{R}^n \times [0,1]) \) at any points of \( X^{n+q+1} \). We may suppose that near the submanifold \( P^{n+q} \) of \( X^{n+q+1} \) the continuous map \( G \) coincides with the fold map \( F \), and by the construction of the \((n+1)\)-framing \( \mathfrak{f} \) of the bundle \( TW^{n+q+1} \oplus e^1_{W^{n+q+1}} \) (see [26] Lemma 3.1 and also [3] Lemma 3.1) the bundle homomorphism \( H|_{TX^{n+q+1}}: TX^{n+q+1} \to T(\mathbb{R}^n \times [0,1]) \) is given by the differential of the fold map \( F \) (which is a submersion) near \( P^{n+q} \).

Hence similarly to [4] Proof of Theorem 3.2] using the relative h-principle for fold maps [4] Theorem 0.5, Theorem 2.1], we see that there is a framed fold map \( g: X^{n+q+1} \to \mathbb{R}^n \times [0,1] \) which coincides with \( F \) on \( W^{n+q+1} \) and whose boundary \( \partial g \) coincides with the framed fold map \( f \). Hence the framed fold map \( f \) is framed null-cobordant. The oriented case is proved in a similar way.

\[ \square \]

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