PROPERTIES OF BLOW-UP SOLUTIONS TO A PARABOLIC SYSTEM WITH NONLINEAR LOCALIZED TERMS

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Abstract. This paper deals with blow-up properties of the solution to a semi-linear parabolic system with nonlinear localized sources involved in a product with local terms, subject to the null Dirichlet boundary condition. We investigate the influence of localized sources and local terms on blow-up properties for this system. It will be proved that: (i) when $m, q \leq 1$ this system possesses uniform blow-up profiles. In other words, the localized terms play a leading role in the blow-up profile for this case. (ii) when $m, q > 1$, this system presents single point blow-up patterns, or say that, in this time, local terms dominate localized terms in the blow-up profile. Moreover, the blow-up rate estimates in time and space are obtained, respectively.

1. Introduction and Main Results. In this paper, we consider the following semi-linear parabolic system with nonlinear localized sources accompanied by local terms

$$
\begin{cases}
  u_t = \Delta u + u^m(x, t)v^n(x_0, t), & x \in \Omega, \ t > 0, \\
  v_t = \Delta v + u^p(x_0, t)v^q(x, t), & x \in \Omega, \ t > 0, \\
  u = v = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
$$

(1.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $m, n, p, q$ are nonnegative constants and satisfy $m + n > 0$, $p + q > 0$, and $x_0 \in \Omega$ is a fixed point. Initial data $u_0(x), v_0(x) \in C_0(\Omega)$ are non-negative and nontrivial. Using the methods of [8] and [17] we know that (1.1) has a local non-negative solution, and that the Comparison Principle is true. Moreover, if $m, n, p, q \geq 1$ then the Uniqueness holds.

The blow-up property of the solution to a single equation of the form

$$
\begin{cases}
  u_t - \Delta u = u^m(x, t)v^n(x_0(t), t) - \mu u^q(x, t), & x \in \Omega, \ t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega
\end{cases}
$$

(1.2)

has been discussed by many authors, see [3, 4, 17, 18, 19] and the references therein. In the paper [17], Souplet obtained a sharp critical blow-up exponent for system (1.2). Lately, Souplet [18] introduced a new method to investigate the profile of the

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blow-up solution to system (1.2) with $m = \mu = 0$. He proved that if $p > 1$, then uniformly on compact subsets of $\Omega$ holds
\[
\lim_{t \to T} (T - t)^{\frac{1}{p - 1}} u(x, t) = \lim_{t \to T} (T - t)^{\frac{1}{q - 1}} \|u(t)\|_\infty = (p - 1)^{- \frac{1}{p - 1}}.
\]

In the paper [16], Pedersen and Lin studied the following problem
\[
\begin{aligned}
  u_{it} &= \Delta u + u^p_{i+1}(x_0, t), & x \in \Omega, & t > 0, \\
  u_i(x, 0) &= u_{i0}(x) \geq 0, & x \in \Omega, \\
  u_i = 0, & i = 1, \ldots, k, & u_{k+1} := u_1, & x \in \partial \Omega, & t > 0
\end{aligned}
\]
with $p_i > 1$. They first proved that the solution blows up in finite time if the initial data $u_{i0}(x)$ are large enough, and then derived the blow-up rate of the solution.

Our present work is inspired by papers [16] [17] [18] mentioned just now, and [20] [22]. In the paper [20], Wang discussed the finite time blow-up of the positive solution to the problem below
\[
\begin{aligned}
  u_t &= \Delta u + u^m v^n, & v_t = \Delta v + u^p v^q, & x \in \Omega, & t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, & v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \\
  u = v = 0, & x \in \partial \Omega, & t > 0
\end{aligned}
\]
Let $\lambda$ be the first eigenvalue of $-\Delta$ in $\Omega$ with null Dirichlet boundary condition. His results are:

(i) Assume that
\[
m > 1, \quad n > 0, \quad p = 0, \quad q = 1, \quad \lambda < 1, \quad m \leq 1 + n(1 - \lambda)/\lambda,
\]
\[
or
\quad q > 1, \quad p > 0, \quad n = 0, \quad m = 1, \quad \lambda < 1, \quad q \leq 1 + p(1 - \lambda)/\lambda.
\]
Furthermore, if $m = 1 + n(1 - \lambda)/\lambda$ in (1.4) or $q = 1 + p(1 - \lambda)/\lambda$ in (1.5), it is assumed that $\lambda < 2/3$. Then, for any nontrivial initial data, i.e. $u_0(x) \neq 0, v_0(x) \neq 0$, the solution of (1.3) blows up in finite time.

(ii) If (1.4), (1.5) and the condition that $m \leq 1, \quad q \leq 1$ and $np \leq (1 - m)(1 - q)$ do not hold, then the solution of (1.3) blows up in finite time for large initial data.

In the paper [22], Wang evaluated the blow-up rate of the solution to (1.3) with $\Omega = B_R(0)$. Under some suitable conditions he obtained that
\[
c(T - t)^{-\theta} \leq \max_{0 \leq |x| \leq R} u(\cdot, t) = u(0, t) \leq C(T - t)^{-\theta}, \quad t \in [0, T),
\]
\[
c(T - t)^{-\sigma} \leq \max_{0 \leq |x| \leq R} v(\cdot, t) = v(0, t) \leq C(T - t)^{-\sigma}, \quad t \in [0, T)
\]
for some positive constants $c$ and $C$, here $\theta = (1 + n - q)/(np - (1 - m)(1 - q))$ and $\sigma = (1 + p - m)/(np - (1 - m)(1 - q))$, and $T$ is the blow-up time of $(u, v)$.

The primary purpose of this paper is to explore the influence of localized terms and local terms on the blow-up properties of system (1.1). Our first result is related to some sufficient conditions for that $(u, v)$ blows up in finite time.

**Theorem 1.** If the conditions of the above (i) hold then the solution to (1.1) blows up in finite time for any nontrivial initial data. If the conditions of the above (ii) hold then the solution to (1.1) blows up in finite time for large initial data.

When $m \leq 1$ and $q \leq 1$, we have the following uniform blow-up profiles in the interior, which show that the localized terms $u^p(x_0, t)$ and $v^q(x_0, t)$ play a leading role in the blow-up profile.
Theorem 2. Assume that \((u,v)\) is a classical solution of (1.1), which blows up in finite time \(T\). Let \(m,q \leq 1\), then the following statements hold uniformly on any compact subset of \(\Omega\).

(i) If \(m,q < 1\) and \(np - (1-m)(1-q) > 0\), then
\[
\lim_{t \to T} u(x,t) (T-t)^\theta = \theta^\theta (\sigma/\theta)^{np-1/(1-m)(1-q)}, \quad \lim_{t \to T} v(x,t) (T-t)^\sigma = \sigma^\sigma (\sigma/\theta)^{np-1/(1-m)(1-q)},
\]
where
\[
\theta = (1+n-q)/(np - (1-m)(1-q)), \quad \sigma = (1+p-m)/(np - (1-m)(1-q)).
\]

(ii) If \(m = 1\) or \(q = 1\), then
\[
\lim_{t \to T} |\ln(T-t)|^{-1} \ln u(x,t) = \frac{1+n-q}{np}, \quad \lim_{t \to T} |\ln(T-t)|^{-1} \ln v(x,t) = \frac{1+p-m}{np}.
\]

For the case \(m > 1\) and \(q < 1\), or the case \(m \leq 1\) and \(q > 1\), we do not know how to deal with the blow-up properties of system (1.1). In the following, we focus only on the case of \(m,q > 1\). Let us first make some assumptions:

(H1) \(m,q > 1\), and \(\Omega = B(0;R), x_0 = 0\).

(H2) Initial data \(u_0(x), v_0(x) : \bar{B}(0;R) \to \mathbb{R}^1\) are nonnegative nontrivial, radially symmetric non-increasing continuous functions and vanish on \(\partial B(0;R)\).

(H3) \(u_0(x)\) and \(v_0(x)\) satisfy \(\Delta u_0(x) + u_0^n(x)v_0^n(0) \geq 0\) and \(\Delta v_0(x) + u_0^n(0)v_0^n(x) \geq 0\) in \(B(0;R)\).

Under the above assumption (H2), the solution \((u,v)\) is radially symmetric and non-increasing in \(x\). Therefore, \(u(x,t) = u(r,t), v(x,t) = v(r,t)\) and \(u(0,t) = \max_{\Omega} u(x,t), v(0,t) = \max_{\Omega} v(x,t)\).

Theorem 3. Let assumptions (H1)–(H3) hold. Assume that \((u,v)\) is a classical solution to (1.1) in \(B(0;R) \times (0,T)\), which blows up in finite time \(T\). If \(p \geq m-1 > 0\) and \(n \geq q-1 > 0\), then \(u\) and \(v\) must blow up simultaneously.

Theorem 4. Let assumptions (H1)–(H3) hold. Suppose that \((u,v)\) is a classical solution to (1.1) in \(B(0;R) \times (0,T)\), \(u\) and \(v\) blow up simultaneously in finite time \(T\). Then the parameters \(m,n,p\) and \(q\) must satisfy: (a) \(p \geq m-1\) and \(n \geq q-1\), or (b) \(p < m-1\) and \(n < q-1\).

When \(m > 1\) and \(q > 1\), system (1.1) possesses the following single point blow-up patterns, which illustrate that the local terms \(u^m(x,t)\) and \(v^n(x,t)\) dominate the localized terms \(u^p(x_0,t)\) and \(v^n(x_0,t)\) in the blow-up profile.

Theorem 5. Let assumptions (H1) and (H2) be satisfied. If \((u,v)\) is a classical solution of (1.1) which blows up in finite time \(T\), then \(x = 0\) is the only blow-up point of \((u,v)\).

When \(u\) and \(v\) blow up simultaneously, we may estimate the blow-up rate as follows.

Theorem 6. Under the conditions of Theorem 4, there exist constants \(0 < c \leq C\) such that the following statements hold for all \(0 \leq t < T\).

(i) If (a) \(p > m-1\) and \(n > q-1\), or (b) \(p < m-1\) and \(n < q-1\), then
\[
c(T-t)^{-\theta} \leq u(0,t) \leq C(T-t)^{-\theta}, \quad c(T-t)^{-\sigma} \leq v(0,t) \leq C(T-t)^{-\sigma},
\]
where \(\theta\) and \(\sigma\) are defined in Theorem 2.
If $p > m - 1$ and $n = q - 1$, then
\[c|\ln(T - t)| \leq u^{1+p-m}(0, t) \leq C|\ln(T - t)|, \quad c \leq v^n(0, t)\ln v(0, t)\] if $n > q - 1$, then
\[c \leq u^p(0, t)\ln u(0, t) \leq C|\ln(T - t)|, \quad c|\ln(T - t)| \leq v^{1+n-q}(0, t) \leq C|\ln(T - t)|.

Remark 1. If $m = q$, $n = p$ and $u_0(x) = v_0(x)$, then system (1.1) turns into a single equation. From above, we draw a complete conclusion on the blow-up profiles. More precisely, the problem possesses uniform blow-up profiles if and only if $m$, the power of the local term, is less than 1.

Furthermore, for problem (1.1) with suitable initial data, its blow-up rate in space can be evaluated as follows.

Theorem 7. Let assumptions (H1) and (H2) be satisfied. Suppose further that there exists some constant $c > 0$ such that $u_0'(r) \leq -cr$ and $v_0'(r) \leq -cr$ in $[0, R]$. If the classical solution $(u, v)$ of (1.1) blows up in finite time $T$, then
\[u(r, t) \leq Cr^{-\alpha}, \quad v(r, t) \leq Cr^{-\beta}, \quad (r, t) \in (0, R] \times [0, T)
\]
holds for some constant $C > 0$ and for any $\alpha > 2/(m - 1), \beta > 2/(q - 1)$.

There are many known results concerning blow-up properties for parabolic equations, of which the reaction terms are nonlinear nonlocal or of other type. Apart from works mentioned above, we refer to [1, 2, 5, 6, 7, 10–16, 23, 24, 25] and the references therein.

2. Proofs of Theorems 1 and 2. In this section, we prove Theorems 1 and 2.

2.1. Proof of Theorem 1. We use the results of [20] and a comparison argument to prove Theorem 1. Without loss of generality we may assume that $u_0(x), v_0(x) > 0$ in $\Omega$. And hence $u, v > 0$ in $\Omega \times [0, T)$, $T$ being the maximal existence time of $(u, v).

Let $B(x_0, d)$ be the ball centered at $x_0$ with radius $d > 0$ such that $B(x_0, d) \subset \Omega$, and let $(u, v)$ be the solution of the auxiliary problem
\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = u^m(x_0, t)u^n(x_0, t), & x \in B(x_0, d), \ t > 0, \\
\frac{\partial v}{\partial t} - \Delta v = v^p(x_0, t)v^q(x_0, t), & x \in B(x_0, d), \ t > 0, \\
\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0, & x \in \partial B(x_0, d), \ t > 0, \\
u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in B(x_0, d),
\end{cases}
\]

where $u_0(x)$ and $v_0(x)$ are nonnegative smooth symmetric, radially non-increasing functions which are less than $u_0(x)$ and $v_0(x)$ on $B(x_0, 0)$ respectively. Then $u(\cdot, t)$ and $v(\cdot, t)$ are radially symmetric non-increasing. By the comparison principle, $u \geq u, v \geq v$ as long as $(u, v)$ and $(\tilde{u}, \tilde{v})$ exist. Therefore,
\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u \geq u^m(x, t)\tilde{u}^n(x, t), & x \in B(x_0, d), \ t > 0, \\
\frac{\partial v}{\partial t} - \Delta v \geq v^p(x, t)\tilde{v}^q(x, t), & x \in B(x_0, d), \ t > 0, \\
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0, & x \in \partial B(x_0, d), \ t > 0, \\
u(x, 0) = u_0(x), \ v(x, 0) = \tilde{v}_0(x), & x \in \bar{B}(x_0, d).
\end{cases}
\]
When the condition (ii) holds, we choose \((u_0(x), v_0(x))\) and \((\bar{u}_0(x), \bar{v}_0(x))\) are sufficiently large. By the results of [20], \((\bar{u}, \bar{v})\) blows up in the finite time, and so does \((u, v)\).

2.2. Proof of Theorems 2

In what follows we intend to verify Theorem 2. For convenience, denote

\[ f(t) = u^p(x_0, t), \quad F(t) = \int_0^t f(s)ds, \quad g(t) = v^q(x_0, t), \quad G(t) = \int_0^t g(s)ds. \tag{2.1} \]

Before we prove Theorem 2, we claim that if \((u, v)\) is a classical solution of system (1.1) which blows up in finite time \(T\), that is,

\[ \|u(t)\|_{\infty} + \|v(t)\|_{\infty} \rightarrow \infty \quad \text{as} \quad t \rightarrow T, \]

then \(u\) and \(v\) blow up simultaneously. In fact, we have

**Lemma 1.** Assume that \((u, v)\) is a classical solution of (1.1), which blows up in finite time \(T\). Let \(m \leq 1\) and \(q \leq 1\), then

\[ \lim_{t \to T} g(t) = \lim_{t \to T} G(t) = \infty, \quad \lim_{t \to T} f(t) = \lim_{t \to T} F(t) = \infty. \]

Moreover, \(u\) and \(v\) blow up simultaneously.

**Proof.** Since \((u, v)\) blows up in finite time \(T\), it can be deduced that

\[ \|u(t)\|_{\infty} \rightarrow \infty \quad \text{or} \quad \|v(t)\|_{\infty} \rightarrow \infty \quad \text{as} \quad t \rightarrow T. \]

Without loss of generality we may assume that \(\|u(t)\|_{\infty} \rightarrow \infty\) as \(t \rightarrow T\). Suppose on the contrary that \(\lim_{t \to T} g(t) < \infty\). So, from the equation of \(u\) in system (1.1), we know that \(u\) exists globally, since \(0 < m \leq 1\). This is a contradiction. Therefore, \(\lim_{t \to T} g(t) = \infty\).

Combining \(\lim_{t \to T} g(t) = \infty\) and \(g(t) = v^q(x_0, t)\) yields that \(v(x_0, t) \rightarrow \infty\) as \(t \rightarrow T\). Namely, \(u\) and \(v\) blow up simultaneously.

Next, we infer that \(\lim_{t \to T} G(t) = \infty\). Set \(U(t) = \max_{x \in \bar{\Omega}} u(x, t)\), then \(U(t)\) is Lipschitz continuous and

\[ U'(t) \leq U^m(t)g(t) \quad \text{a.e. in} \quad [0, T). \tag{2.2} \]

By integrating (2.2) we get

\[
\begin{cases}
\frac{1}{1-m} U^{1-m}(t) \leq \int_0^t g(s)ds + \frac{U^{1-m}(0)}{1-m} = G(t) + \frac{U^{1-m}(0)}{1-m} & \text{if } m < 1, \\
\ln U(t) \leq \int_0^t g(s)ds + \ln U(0) = G(t) + \ln U(0) & \text{if } m = 1.
\end{cases}
\]

From \(\lim_{t \to T} U(t) = \infty\), it follows that \(\lim_{t \to T} G(t) = \infty\).

Furthermore, because of \(\lim_{t \to T} \|v(t)\|_{\infty} = \infty\) which was showed above, applying similar arguments as above to the equation of \(v\) in system (1.1), it is reasonable that \(\lim_{t \to T} f(t) = \infty\) and \(\lim_{t \to T} F(t) = \infty\).

To illustrate Theorem 2, we start by presenting two lemmas, which show the relationships among \(u, v, F(t)\) and \(G(t)\).

**Lemma 2.** Under the conditions of Theorem 2, the following statements hold uniformly on any compact subset of \(\Omega\).

(i) If \(m < 1\) and \(q < 1\) then

\[ u^{1-m}(x, t) \sim (1-m)G(t), \quad v^{1-q}(x, t) \sim (1-q)F(t). \]
(ii) If \( m = 1 \) and \( q < 1 \) then
\[
\ln u(x, t) \sim G(t), \quad v^{1-q}(x, t) \sim (1-q)F(t).
\]

(iii) If \( m = q = 1 \) then
\[
\ln u(x, t) \sim G(t), \quad \ln v(x, t) \sim F(t).
\]

(iv) If \( m < 1 \) and \( q = 1 \) then
\[
u^{1-m}(x, t) \sim (1-m)G(t), \quad \ln v(x, t) \sim F(t),
\]
here the notation \( u \sim v \) means that \( u/v \to 1 \) as \( t \to T \).

Proof. (i) When \( m < 1 \) and \( q < 1 \). Direct computations demonstrate
\[
\frac{1}{1-m} \frac{du^{1-m}}{dt} = \frac{1}{1-m} \Delta u^{1-m} + mu^{-m-1} |\nabla u|^2 + g(t) \geq \frac{1}{1-m} \Delta u^{1-m} + g(t),
\]
\[
\frac{1}{1-q} \frac{dv^{1-q}}{dt} = \frac{1}{1-q} \Delta v^{1-q} + qv^{-q-1} |\nabla v|^2 + f(t) \geq \frac{1}{1-q} \Delta v^{1-q} + f(t).
\]

Consequently, \((\frac{1}{1-m} u^{1-m}, \frac{1}{1-q} v^{1-q})\) is a super-solution of the problem below
\[
\begin{cases}
w_t = \Delta w + g(t), \quad z_t = \Delta z + f(t), & x \in \Omega, \quad 0 < t < T, \\
w(x, 0) = \frac{1}{1-m} u_0^{1-m}(x), \quad z(x, 0) = \frac{1}{1-q} v_0^{1-q}(x), & x \in \Omega, \\
w = z = 0, & x \in \partial \Omega, \quad 0 < t < T,
\end{cases}
\]
where \( g(t) = v^n(x_0, t) \), \( f(t) = u^q(x_0, t) \). Theorem 4.1 of [18] asserts that
\[
\lim_{t \to T} \frac{w(x, t)}{G(t)} = \lim_{t \to T} \frac{\|w(t)\|_\infty}{\|v(t)\|_\infty} = 1, \quad \lim_{t \to T} \frac{z(x, t)}{F(t)} = \lim_{t \to T} \frac{\|z(t)\|_\infty}{\|v(t)\|_\infty} = 1
\]
uniformly on any compact subset of \( \Omega \).

By comparison methods, we obtain that
\[
\frac{1}{1-m} u^{1-m}(x, t) \geq w(x, t), \quad \frac{1}{1-q} v^{1-q}(x, t) \geq z(x, t), \quad (x, t) \in \Omega \times [0, T).
\]

Hence, from (2.4) it follows that, uniformly on any compact subset of \( \Omega \) holds
\[
\begin{cases}
\liminf_{t \to T} \frac{u^{1-m}(x, t)}{(1-m)G(t)} \geq 1, & \liminf_{t \to T} \frac{v^{1-q}(x, t)}{(1-q)F(t)} \geq 1, \\
\liminf_{t \to T} \frac{\|u(t)\|_\infty^{1-m}}{(1-m)G(t)} \geq 1, & \liminf_{t \to T} \frac{\|v(t)\|_\infty^{1-q}}{(1-q)F(t)} \geq 1.
\end{cases}
\]

On the other hand, we know that \( U'(t) \leq U^m(t)g(t) \) and \( V'(t) \leq V^q(t)f(t) \) a.e. in \([0, T)\), where \( U(t) = \max_{x \in \Omega} u(x, t), V(t) = \max_{x \in \Omega} v(x, t) \). In view of \( \lim_{t \to \infty} F(t) = \lim_{t \to \infty} G(t) = \infty \) and \( m, q < 1 \), we see that
\[
\limsup_{t \to T} \frac{U^{1-m}(t)}{(1-m)G(t)} \leq 1, \quad \limsup_{t \to T} \frac{V^{1-q}(t)}{(1-q)F(t)} \leq 1.
\]

So, (2.5) and (2.6) guarantee that, uniformly on any compact subset of \( \Omega \),
\[
\lim_{t \to T} \frac{u^{1-m}(x, t)}{(1-m)G(t)} = \lim_{t \to T} \frac{\|u(t)\|_\infty^{1-m}}{(1-m)G(t)} = 1, \quad \lim_{t \to T} \frac{v^{1-q}(x, t)}{(1-q)F(t)} = \lim_{t \to T} \frac{\|v(t)\|_\infty^{1-q}}{(1-q)F(t)} = 1.
\]

(ii) When \( m = 1 \) and \( q < 1 \). Analogous to case (i), we find that \((\ln u, \frac{1}{1-q} v^{1-q})\) is a super-solution of (2.3) with \( w(x, 0) = \ln u_0(x), z(x, 0) = \frac{1}{1-q} v_0^{1-q}(x) \). Proceeding as case (i) we arrive at the corresponding conclusion.
Under the assumptions of Theorem (2.8), for any given positive constants \( \delta, \varepsilon \) and \( \tau \) satisfying \( 0 < \delta, \varepsilon < 1 \) and \( \tau > 1 \), there exists \( T \) such that for all \( t \in [T, T) \), the following statements hold.

(i) If \( m < 1 \) and \( q < 1 \) then

\[
\begin{align*}
\varepsilon \delta \tau^{-\tau} (1 + p - m) \{(1 - q)F(t)\}^{\frac{1 + n - q}{1 - q}} & \leq \tau \tau^{-\tau} (1 + n - q) \{(1 - m)G(t)\}^{\frac{1 + p - m}{1 - m}}, \\
\varepsilon \delta \tau^{-\tau} (1 + n - q) \{(1 - m)G(t)\}^{\frac{1 + p - m}{1 - m}} & \leq \tau \tau^{-\tau} (1 + p - m) \{(1 - q)F(t)\}^{\frac{1 + n - q}{1 - q}}.
\end{align*}
\]

(ii) If \( m = 1 \) and \( q < 1 \) then

\[
\ln \{\varepsilon \delta \tau^{-\tau}\} + \ln \{p/(1 + n - q)\} + \frac{1 + n - q}{1 - q} \ln \{(1 - q)F(t)\} \leq p \delta G(t),
\]

\[
p \delta G(t) \leq \ln \{\varepsilon \delta^{-1} \tau^{-\tau}\} + \ln \{p/(1 + n - q)\} + \frac{1 + n - q}{1 - q} \ln \{(1 - q)F(t)\}.
\]

(iii) If \( m = q = 1 \) then

\[
\ln \{\varepsilon \delta \tau^{-\tau}\} + n \delta F(t) \leq p \varepsilon G(t), \quad p \delta G(t) \leq \ln \{p \varepsilon/(n \varepsilon \tau)\} + n \varepsilon F(t).
\]

(iv) If \( m < 1 \) and \( q = 1 \) then

\[
\begin{align*}
\ln \{\varepsilon \delta \tau^{-\tau}\} + n \delta F(t) & \leq \ln \{n/(1 + p - m)\} + \frac{1 + p - m}{1 - m} \ln \{(1 - m)G(t)\}, \\
\ln \{\varepsilon \delta F(t)\} + \ln \{n/(1 + p - m)\} & + \frac{1 + p - m}{1 - m} \ln \{(1 - m)G(t)\} \leq n \varepsilon F(t).
\end{align*}
\]

Proof. (i) \( m < 1 \) and \( q < 1 \). Observe that \( F'(t) = f(t) = u^p(x_0, t) \) and \( G'(t) = g(t) = u^q(x_0, t) \). By (i) of Lemma (2.2), we know that for chosen positive constants \( \delta < 1 < \tau \), there exists \( t_0 < T \) such that

\[
\begin{align*}
\{\delta(1 - m)G(t)\}^{p/(1 - m)} & \leq \{\tau(1 - m)G(t)\}^{p/(1 - m)}, \quad t \in [t_0, T), \\
\{\delta(1 - q)F(t)\}^{n/(1 - q)} & \leq \{\tau(1 - q)F(t)\}^{n/(1 - q)}, \quad t \in [t_0, T).
\end{align*}
\]

And thus,

\[
\frac{\{\delta(1 - m)G(t)\}^{p/(1 - m)}}{\{\tau(1 - q)F(t)\}^{n/(1 - q)}} \leq \frac{dF}{dG} \leq \frac{\{\delta(1 - q)F(t)\}^{n/(1 - q)}}{\{\tau(1 - m)G(t)\}^{p/(1 - m)}}, \quad t \in [t_0, T). \tag{2.7}
\]

In view of the right-hand side of (2.7),

\[
\{\delta(1 - q)F(t)\}^{n/(1 - q)} dF \leq \{\tau(1 - m)G(t)\}^{p/(1 - m)} dG, \quad t \in [t_0, T).
\]

Integrating the above yields that for \( t_0 \leq t < T \),

\[
\frac{(1 - q)\{\delta(1 - q)\}^{\frac{n}{1 - q}}}{1 + n - q} F^{1 + n - q}(s)|_{t_0}^{t} \leq \frac{(1 - m)\{\tau(1 - m)\}^{\frac{p}{1 - m}}}{1 + p - m} G^{1 + p - m}(s)|_{t_0}^{t} \leq \frac{(1 - m)\{\tau(1 - m)\}^{\frac{p}{1 - m}}}{1 + p - m} G^{1 + p - m}(t). \tag{2.8}
\]

Due to \( \lim_{t \to T} F(t) = \infty \) and \( q < 1 \), for given constant \( 0 < \varepsilon < 1 \), there exists \( t_0 : t_0 \leq t_0 < T \) such that \( F^{(1 + n - q)/(1 - q)}(t_0) \leq (1 - \varepsilon)F^{(1 + n - q)/(1 - q)}(t) \) for all \( t \in [t_0, T) \). Hence, from (2.8) it can be deduced that for \( t_0 \leq t < T \),

\[
\varepsilon \delta \tau^{-\tau} (1 + p - m) \{(1 - q)F(t)\}^{\frac{1 + n - q}{1 - q}} \leq \tau \tau^{-\tau} (1 + n - q) \{(1 - m)G(t)\}^{\frac{1 + p - m}{1 - m}}. \tag{2.9}
\]
Application of similar analysis as above to the left-hand side of (2.7) guarantees that there exists $t^*_i < T$ such that for $t^*_i \leq t < T$,

$$
\varepsilon \delta \tau^{\frac{n}{m+1}} (1 + n - q)/(1 - m) G(t) \leq \tau^{\frac{n}{m+1}} (1 + p - m)/(1 - q) F(t).
$$

(2.10)

Set $T = \max \{ t_0, t^*_i \}$, then (2.9) and (2.10) ensure (i) of Lemma 3.

Analogous to case (i), we can draw the other conclusions of Lemma 3.

**Proof of Theorem 2.** Choose $(\delta_i)_{i=1}^{\infty}, (\varepsilon_i)_{i=1}^{\infty}, \{\tau_i\}_{i=1}^{\infty}$ satisfying $0 < \delta_i, \varepsilon_i < 1$ and $\tau_i > 1$ with $\delta_i, \varepsilon_i, \tau_i \to 1$. Putting $(\delta, \varepsilon, \tau) = (\delta_i, \varepsilon_i, \tau_i)$ in Lemma 3, we get $\tilde{T}_i < T$ such that the corresponding (i)–(iv) of Lemma 3 hold for all $\tilde{T}_i \leq t < T$.

(i) $m < 1$ and $q < 1$. From (i) of Lemma 2, it follows that for such sequences $(\delta_i)_{i=1}^{\infty}$ and $\{\tau_i\}_{i=1}^{\infty}$, there exists $t_i \leq \tilde{T}_i$ with $t_i \to T$ as $i \to \infty$ such that

$$
\delta_i(1 - m) G(t) \leq u(x, t) \leq \tau_i(1 - m) G(t) \quad \text{for } t_i \leq t < T.
$$

(2.11)

Denote $T_i^* = \max \{ t_i, \tilde{T}_i \}$, then (2.11) and (i) of Lemma 3 assert that for $T_i^* \leq t < T$,

$$
F'(t) \geq \delta_i^{\frac{n}{m+1}} (1 - m) G(t)^{\frac{p}{1 - m}} \quad \text{and} \quad F'(t) \leq \tau_i^{\frac{n}{m+1}} (1 - m) G(t)^{\frac{p}{1 - m}}.
$$

(2.12)

Notice that

$$
1 - \frac{p\theta}{\sigma(1 - q)} = -\frac{np - (1 - m)(1 - q)}{(1 - q)(1 + p - m)} = -\frac{1}{\sigma(1 - q)} < 0.
$$

Integrating (2.12) and (2.13) from $t$ from $T$ and using of $\lim_{t \to T} F(t) = \infty$, we obtain that, for $T_i^* \leq t < T$,

$$
C_i^{-1} \sigma(\theta / \sigma)^{-\frac{n}{m+1}} \leq (T - t)^{1/[\sigma(1 - q)]} \leq C_i^{-1} \sigma(\theta / \sigma)^{-\frac{n}{m+1}},
$$

(2.14)

where

$$
c_i = \delta_i^{\frac{n}{m+1}} (\delta / \tau_i)^{1 - m \frac{p}{1 + p - m}} \varepsilon_i^{\frac{n}{m+1}}, \quad C_i = \tau_i^{\frac{n}{m+1}} (\tau / \delta_i)^{1 - m \frac{p}{1 + p - m}} \varepsilon_i^{\frac{n}{m+1}}.
$$

Since $C_i \to 1$ and $C_i \to 1$ on account of $\delta_i, \varepsilon_i, \tau_i \to 1$, and $T_i^* \to T$ as $i \to \infty$, by letting $t \to T$ in (2.14) we find

$$
((1 - q) F(t))^{1/(1 - q)} \sim \sigma^\theta (\theta / \sigma)^{n/[np - (1 - m)(1 - q)]} (T - t)^{-\sigma}.
$$

(2.15)

Similar as above, it can be inferred that

$$
((1 - m) G(t))^{1/(1 - m)} \sim \theta^\theta (\sigma / \theta)^{n/[np - (1 - m)(1 - q)]} (T - t)^{-\theta}.
$$

(2.16)

From (i) of Lemma 2, (2.15) and (2.16), we know that

$$
\lim_{t \to T} u(x, t)(T - t)^{\theta} = \theta^\theta (\sigma / \theta)^{n/[np - (1 - m)(1 - q)]}
$$

and

$$
\lim_{t \to T} v(x, t)(T - t)^{\sigma} = \sigma^\theta (\theta / \sigma)^{n/[np - (1 - m)(1 - q)]}.
$$

(ii) $m = 1$ or $q = 1$. We divide this case into three subcases: (1) $m = 1$, $q < 1$; (2) $m = q = 1$; and (3) $m < 1$, $q = 1$. We first discuss subcase (1) $m = 1$ and $q < 1$. 

Analogous as the beginning of the proof of case (i), it follows from (ii) of Lemma 3 and (ii) of Lemma 2 that for $T^*_i < T$,

$$
G'(t) \geq \delta^{n/(1-q)}_i [(1-q)F(t)]^{n/(1-q)}
\geq \delta^{n/(1-q)}_i \{\varepsilon_i(1+n-q)(p\delta_i)^{-1}\tau_i^{-n/(1-q)}A^{n/(1+n-q)} \exp\left\{\frac{np\delta_i}{1+n-q}G(t)\right\}
= (\delta_i/\tau_i)^{n/(1+n-q)} \{\varepsilon_i p^{-1}(1+n-q)\}^{\frac{n}{1+n-q}} \exp\left\{\frac{np\delta_i}{1+n-q}G(t)\right\},
G'(t) \leq (\tau_i/\delta_i)^{n/(1+n-q)} \{(pe_i)^{-1}(1+n-q)\}^{\frac{n}{1+n-q}} \exp\left\{\frac{np\tau_i}{1+n-q}G(t)\right\}.
$$

And hence, for $T^*_i < t < T$,

$$
\exp\left\{\frac{-np\delta_i}{1+n-q}G(t)\right\}G'(t) \geq (\delta_i/\tau_i)^{n/(1+n-q)} \{\varepsilon_i p^{-1}(1+n-q)\}^{\frac{n}{1+n-q}} \exp\left\{\frac{-np\delta_i}{1+n-q}G(t)\right\},
\exp\left\{\frac{-np\tau_i}{1+n-q}G(t)\right\}G'(t) \leq (\tau_i/\delta_i)^{n/(1+n-q)} \{(pe_i)^{-1}(1+n-q)\}^{\frac{n}{1+n-q}} \exp\left\{\frac{-np\tau_i}{1+n-q}G(t)\right\}.
$$

Define $A = -\ln(np) + (1-q)\ln(1+n-q)/(1+n-q)$. Integrating (2.17) from $t$ to $T$ and using $\lim_{t \to T} G(t) = \infty$, we deduce that for $t \in [T^*_i, T)$,

$$
\frac{1}{T_i} \{c_i + |\ln(T-t)|\} \leq \frac{n}{1+n-q}G(t) \leq \frac{1}{\delta_i} \{\hat{C}_i + |\ln(T-t)|\},
$$

where

$$
c_i = A - \frac{n^2 + (1-q)(1+n-q)}{(1-q)(1+n-q)} \ln \tau_i + \frac{n}{1+n-q} \ln \{p\varepsilon_i \delta_i^{n/(1-q)}\},
\hat{C}_i = A - \frac{n^2 + (1-q)(1+n-q)}{(1-q)(1+n-q)} \ln \delta_i + \frac{n}{1+n-q} \ln \{p\varepsilon_i \delta_i^{n/(1-q)}\}.
$$

By joining (2.18) and (ii) of Lemma 3 it follows that for $T^*_i \leq T < T$,

$$
\frac{\delta_i}{\tau_i} \{c_i + |\ln(T-t)|\} \leq \frac{n}{1-q} \ln \{1-q\}F(t) \leq \frac{\tau_i}{\delta_i} \{C_i + |\ln(T-t)|\}.
$$

where

$$
c_i = \hat{c}_i - \frac{n\tau_i}{\delta_i(1+n-q)} \ln \{p\varepsilon_i (1+n-q)^{1/(1-q)}\},
C_i = \hat{C}_i - \frac{n\delta_i}{\tau_i(1+n-q)} \ln \{p(1+n-q)^{-1}(1+n-q)^{1/(1-q)}\}.
$$

Consequently, (2.18) and (2.19) guarantee that for $T^*_i \leq T < T$,

$$
\frac{\hat{c}_i + |\ln(T-t)|}{\tau_i |\ln(T-t)|} \leq \frac{npG(t)}{(1-q)F(t)} \leq \frac{\hat{C}_i + |\ln(T-t)|}{\delta_i |\ln(T-t)|},
\frac{\delta_i (c_i + |\ln(T-t)|)}{\tau_i |\ln(T-t)|} \leq \frac{n\ln\{1-q\}F(t)}{(1-q)\ln(T-t)} \leq \frac{\tau_i (C_i + |\ln(T-t)|)}{\delta_i |\ln(T-t)|}.
$$

Note that $\hat{c}_i, \hat{C}_i \to A + n\ln p/(1+n-q)$ and $c_i, C_i \to -\ln(np) + \ln(1+n-q)$ because of $\delta_i, \tau_i, \varepsilon_i \to 1$ as $i \to \infty$. By letting $i \to \infty$ in (2.20), we get

$$
\lim_{t \to T} \ln \{1-q\}F(t)/\ln(T-t)^{-1} = \frac{1-q}{n}, \quad \lim_{t \to T} G(t)/\ln(T-t)^{-1} = \frac{1+n-q}{np}.
$$

(2.21)
As \( v^{1-q}(x, t) \sim (1-q)F(t) \) uniformly on compact subsets of \( \Omega \), we claim that, uniformly on compact subsets of \( \Omega \) there holds
\[
\ln v(x, t) \sim \frac{1}{1-q} \ln \{(1-q)F(t)\}. \tag{2.22}
\]
Therefore, it can be deduced from (ii) of Lemma 2, (2.21) and (2.22) that uniformly on compact subsets of \( \Omega \),
\[
\ln u(x,t) \sim G(t) \sim \frac{1+n-q}{np} |\ln(T-t)|, \quad \ln v(x, t) \sim \frac{1}{n} |\ln(T-t)|.
\]
And thereby, uniformly on any compact subset of \( \Omega \),
\[
\lim_{t \to T} |\ln(T-t)|^{-1} \ln u(x, t) = (1+n-q)/(np), \quad \lim_{t \to T} |\ln(T-t)|^{-1} \ln v(x, t) = 1/n.
\]
Finally, we can verify subcase (2) and (3) by similar means of subcase (1) and case (i). So, we complete the proof of Theorem 2. \( \square \)

3. Proofs of Theorems 3–7. In this section, we pay attention to system (1.1) with \( m, q > 1 \). By the results of [8] and [20], applying standard methods we find from assumptions (H2)–(H3) that the following results are true.

1. \( u(x, t) > 0 \) and \( v(x, t) > 0 \) in \( B(0; R) \times (0, T) \), here \( T > 0 \) is the maximal existence time of the solution \( (u, v) \) to problem (1.1).

2. \( u(x, t) = u(r, t), \ v(x, t) = v(r, t) \) and \( u_r(r, t) \geq 0, \ v_r(r, t) \leq 0 \) in \( (0, R) \times (0, T) \).

3. \( u_t \geq 0 \) and \( v_t \geq 0 \) for \( (x, t) \in B(0; R) \times (0, T) \).

To prove Theorems 3–7 we begin with giving an elementary lemma, which will play an important part in the following.

Lemma 4. Let assumptions (H1)–(H3) be satisfied. Suppose that \((u, v)\) is a classical solution of problem (1.1) which blows up in finite time \( T \), then for some \( t_1 < T \), there exists a positive constant \( \varepsilon \leq 1 \) such that
\[
u_t(x, t) \geq \varepsilon u^m(x, t)u^n(0, t), \quad \nu_t(x, t) \geq \varepsilon u^p(0, t)v^q(x, t), \quad x \in B(0; R), \quad t_1 \leq t < T. \tag{3.1}
\]
In addition,
\[
u_t(0, t) \leq u^m(0, t)v^n(0, t), \quad v_t(0, t) \leq u^p(0, t)v^q(0, t), \quad 0 < t < T. \tag{3.2}
\]
Proof. By the result (2) listed above,
\[
u(0, t) = \max_{x \in B(0; R)} u(x, t), \quad v(0, t) = \max_{x \in B(0; R)} v(x, t), \quad 0 < t < T.
\]
Hence, \( \Delta u(0, t) \leq 0, \Delta v(0, t) \leq 0 \) for any \( 0 < t < T \). And thus,
\[
u_t(0, t) \leq u^m(0, t)v^n(0, t), \quad v_t(0, t) \leq u^p(0, t)v^q(0, t), \quad 0 < t < T,
\]
which is just the assertion (3.2).

Next, we infer the assertion (3.1) and proceed our discussion as that in [22]. Since \((u, v)\) blows up in finite time \( T \), and \( u_t, v_t \geq 0 \) for all \((x, t) \in B(0; R) \times (0, T)\), it can be deduced that for any \( t_0 : 0 < t_0 < T, \ u_t(x, t_0) \neq 0 \) or \( v_t(x, t_0) \neq 0 \) in \( B(0; R) \).
Otherwise, \((u, v)\) cannot blow up in finite time. Denote \(\varphi = u_t, \psi = v_t\), then
\[
\begin{cases}
\varphi_t = \Delta \varphi + m u^m(x, t) v^n(0, t) \varphi \\
+ m u^m(x, t) v^n(0, t) \psi(0, t), & x \in B(0; R), \ t_0 \leq t < T,
\psi_t = \Delta \psi + p u^{p-1}(0, t) v^q(0, t) \varphi(0, t) \\
+ q u^p(0, t) v^{q-1}(x, t) \psi, & x \in B(0; R), \ t_0 \leq t < T,
\varphi(x, t) = \psi(x, t) = 0, & |x| = R, \ t_0 \leq t < T,
\varphi(x, t_0) \geq \varphi(x, t_0), \neq 0, \ \psi(x, t_0) \geq \varphi(x, t_0), \neq 0, & x \in B(0; R).
\end{cases}
\]

The maximal principle shows that
\[
\varphi(x, t) > 0, \ \psi(x, t) > 0, \ \forall \ (x, t) \in B(0; R) \times (t_0, T),
\]
where \(\eta\) is the unit outward normal. By the standard method it follows that for any \(t_1 : t_0 < t_1 < T\), there exists \(0 < \varepsilon \leq 1\) such that
\[
\varphi(x, t_1) \geq \varepsilon u^m(x, t_1) v^n(0, t_1), \ \psi(x, t_1) \geq \varepsilon u^p(0, t_1) v^q(x, t_1), \ x \in \overline{B}(0; R),
\]
i.e., for \(t = t_1\) and \(x \in \overline{B}(0; R),
\]
\[
\Delta u + u^m(x, t) v^n(0, t) \geq \varepsilon u^m(x, t) v^n(0, t), \ \Delta v + u^p(0, t) v^q(x, t) \geq \varepsilon u^p(0, t) v^q(x, t).
\]

Set \(w(x, t) = u_t(x, t) - \varepsilon u^m(x, t) v^n(0, t), \ z(x, t) = v_t(x, t) - \varepsilon u^p(0, t) v^q(x, t)\). By using the ideas of [9, 21], we are sure that \(w(x, t) \geq 0, z(x, t) \geq 0\). Indeed,
\[
w_t - \Delta w = (u_t - \Delta u)_t - \varepsilon m u^{m-1}(x, t) v^n(0, t)(u_t - \Delta u) - \varepsilon n u^m(x, t) v^n(0, t) u_t + \varepsilon (m - 1) u^{m-2}(x, t) v^n(0, t) |\nabla u|^2
\]
\[
\geq (u_t - \Delta u)_t - \varepsilon m u^{m-1}(x, t) v^n(0, t)(u_t - \Delta u) - \varepsilon n u^m(x, t) v^n(0, t) u_t
\]
\[
\geq m u^{m-1}(x, t) v^n(0, t) w, \ x \in B(0; R), \ t_1 < t < T,
\]
\[
z_t - \Delta z \geq q u^p(0, t) v^{q-1}(x, t) z, \ x \in B(0; R), \ t_1 < t < T,
\]
\[
w(x, t_1) = \Delta u(x, t_1) + u^m(x, t_1) v^n(0, t_1) - \varepsilon u^m(x, t_1) v^n(0, t_1) \geq 0, \ x \in B(0; R),
\]
\[
z(x, t_1) = \Delta v(x, t_1) + u^p(0, t_1) v^q(x, t_1) - \varepsilon u^p(0, t_1) v^q(x, t_1) \geq 0, \ x \in B(0; R),
\]
\[
w(x, t) = z(x, t) = 0, \ |x| = R, \ t_1 < t < T.
\]
The maximum principle implies that \(w \geq 0, z \geq 0\). Therefore,
\[
u_t(x, t) \geq \varepsilon u^m(x, t) v^n(0, t), \ v_t(x, t) \geq \varepsilon u^p(0, t) v^q(x, t), \ x \in B(0; R), \ t_1 \leq t < T,
\]
which means the assertion (3.1) is true.

3.1. Proofs of Theorems 3 and 4. In this subsection, we prove Theorems 3 and 4.

**Proof of Theorem 3.** Assume on the contrary that \(u\) blows up in finite time \(T\) and \(v\) is bounded in \(B(0; R) \times (0, T)\). By (3.1) and (3.2) in Lemma 4, we have
\[
\varepsilon u^m(0, t) v^n(0, t) \leq u_t(0, t) \leq u^m(0, t) v^n(0, t), \ t \in [t_1, T),
\]
\[
\varepsilon u^p(0, t) v^q(0, t) \leq v_t(0, t) \leq u^p(0, t) v^q(0, t), \ t \in [t_1, T).
\]
As \(v\) is nonnegative and bounded in \(B(0; R) \times (0, T)\), we claim that \(v(0, t) \geq c > 0\), where \(c\) is a constant. Indeed, let \(w\) be the solution of the heat equation \(w_t = \Delta w\).
with null Dirichlet boundary condition and \( w(x, 0) = v_0(x) \), then the comparison principle asserts that \( v \geq w \) in \( B(0; R) \times (0, T) \). Since \( 0 \in B(0; R) \) and \( v_0(x) \geq 0 \), \( \forall x \) for any fixed \( t_1 \in (0, T) \), there exists some constant \( c = c(t_1, T) > 0 \) such that
\[
w(0, t) \geq c \quad \text{for all } t_1 \leq t \leq T,
\]
and so does \( v \). Without loss of generality, we assume that \( t_1 = 0 \), thus \( v(0, t) \geq c > 0 \) for all \( t \in [0, T] \). Thereafter, there exist positive constants \( C_1 \geq C_2 \) and \( C_3 \geq C_4 \) such that for \( t \in [t_1, T) \),
\[
C_2 u_m^{m}(0, t) \leq u_t(0, t) \leq C_1 u_m^{m}(0, t), \quad C_4 u^{p}(0, t) \leq v_t(0, t) \leq C_3 u^{p}(0, t). \tag{3.3}
\]
Due to \( m > 1 \) and \( \lim_{t \to T} u(0, t) = \infty \), integrating the first inequality of (3.3) yields
\[
C_2(m-1)(T-t) \leq u^{1-m}(0, t) \leq C_1(m-1)(T-t), \quad t_1 \leq t < T.
\]
Consequently, for \( t \in [t_1, T) \),
\[
C_4[ C_1(m-1)(T-t) ]^{p/(1-m)} \leq v_t(0, t) \leq C_3[ C_2(m-1)(T-t) ]^{p/(1-m)}.
\]
As \( p \geq m-1 > 0 \), it can be deduced that \( \lim_{t \to T} v(0, t) = \infty \). This is a contradiction. Therefore, Theorem 3 is completed.

**Proof of Theorem 4.** By (3.1) and (3.2) in Lemma 4, we have
\[
\frac{\varepsilon u^{m-p}(0, t)}{v^{q-n}(0, t)} \leq \frac{d}{dt}(0, t) \leq \frac{u^{m-p}(0, t)}{v^{q-n}(0, t)}, \quad \forall t \in [t_1, T). \tag{3.4}
\]
In view of the right-hand side of (3.4),
\[
\varepsilon u^{m-p}(0, t)du(0, t) \leq v^{n-q}(0, t)dv(0, t), \quad t \in [t_1, T). \tag{3.5}
\]
When \( p \geq m-1 \), suppose on the contrary that \( n < q - 1 \). By integrating (3.5), we see that
\[
\ln u(0, s)|_{t_1}^{t} \leq \frac{1}{\varepsilon(1+n-q)} v^{1+n-q}(0, s)|_{t_1}^{t} \leq \frac{v^{1+n-q}(0, t_1)}{\varepsilon(q-1-n)} \quad \text{if } p = m-1,
\]
\[
\frac{1}{1+p-m} u^{1+p-m}(0, s)|_{t_1}^{t} \leq \frac{v^{1+n-q}(0, t_1)}{\varepsilon(q-1-n)} \quad \text{if } p > m-1.
\]
Since \( \lim_{t \to T} u(0, t) = \infty \), taking \( t \to T \) in the above leads to a contradiction. Consequently, \( n \geq q - 1 \).

When \( n \geq q - 1 \), by using of analogous arguments, we can show \( p \geq m-1 \).

Similarly, we may conclude (b) of Theorem 4.

**3.2. Proof of Theorem 5.** We adopt the ideas of [9] to verify Theorem 5.

**Proof of Theorem 5.** Assume on the contrary that \( (u, v) \) blows up at another point \( x^* \neq 0 \). Furthermore, we may think without loss of generality that \( u \) blows up at the point \( x^* \) as \( t \to T \), i.e., \( \limsup_{t \to T} u(x^*, t) = \infty \). Set \( r^* = |x^*| \), then \( r^* > 0 \). Since \( u(x, t) = u(r, t) \) is non-increasing in \( r \), \( \limsup_{t \to T} u(r, t) = \infty \) for any \( r \in [0, r^*] \) with \( r = |x| \).

Let \( a \) be a fixed number satisfying \( a = r^*/3 \) and
\[
B_a^+(0; R) = B(0; R) \cap \{ x \in \mathbb{R}^N | x_1 > a \} = \{ x \in B(0; R) | x_1 > a \}.
\]
Define
\[
J(x, t) = u_{x_1}(x, t) + c(x_1)u^{m}(x, t), \quad (x, t) \in B_a^+(0; R) \times [0, T),
\]
where \( 1 < m_0 < m \), and
\[
c(x_1) = \varepsilon (x_1 - a)^2
\]
with \( \varepsilon > 0 \) is a small constant to be determined.
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Straightforward computation gives

$$J_t - \Delta J = (u_t - \Delta u)_{x_1} + m_0 c(x_1) u^{m_0-1}(u_t - \Delta u) - c'(x_1) u^{m_0}$$

$$- 2m_0 u^{m_0-1} c'(x_1) u_{x_1} - m_0 (m_0 - 1) c(x_1) u^{m_0-2} |\nabla u|^2$$

$$\leq \{ m u^{m-1}(x, t) v^n(0, t) - 4\varepsilon_0 |x_1 - a| u^{m-1} \} J$$

$$- c(x_1) u^m \{ (m - m_0) u^{m-1}(x, t) v^n(0, t)$$

$$- 4\varepsilon_0 |x_1 - a| u^{m-1} + 2(x_1 - a)^2 \}$$

$$\leq b J - c(x_1) u^m \{ (m - m_0) u^{m-1}(x, t) v^n(0, t) - 4\varepsilon_0 Ru^{m_0-1} + 2R^{-2} \},$$

(3.6)

where

$$b \equiv m u^{m-1}(x, t) v^n(0, t) - 4\varepsilon_0 |x_1 - a| u^{m-1}.$$ 

Remember that $v(r, t) > 0$ in $[0, R] \times (0, T)$ and $v(0, t) = \max_{0 \leq r \leq R} v(r, t)$ for $t \in [0, T)$, then $v(0, t) > 0$ for some constant $c$. By $m_0 < m$, there exists $\varepsilon_1 > 0$ so small that for $0 < \varepsilon \leq \varepsilon_1$,

$$(m - m_0) u^{m-1}(x, t) v^n(0, t) - 4\varepsilon_0 Ru^{m_0-1} + 2R^{-2} \geq 0$$

in $B_0^+(0; R) \times (0, T)$. Consequently, from (3.6) and (3.7) follows

$$J_t - \Delta J - b J \leq 0, \quad (x, t) \in B_0^+(0; R) \times (0, T).$$

(3.8)

Moreover, since $u_0 \leq 0, \not\equiv 0$ (otherwise, $u_0 \equiv 0$, which contradicts the assumption on $u_0$), by the standard methods one can deduce that $u_i(r, t) < 0$ provided that $r \not= 0$ and $t > 0$. And thus, $u_{x_1}(x, t) < 0$ for $(x, t) \in B_0^+(0; R) \times (0, T)$. Replacing $[0, T)$ by $[t^*, T)$ for some $t^* \in (0, T)$ in the following discussion, we may assume that $u_{x_1}(x, t) < 0$ holds on $B_0^+(0; R) \times [0, T)$. Hence,

$$J(x, t) = u_{x_1}(x, t) < 0, \quad (x, t) \in \partial B_0^+(0; R) \times (0, T).$$

(3.9)

On the other hand,

$$J(x, 0) = u_{x_1}(x, 0) + c(x_1) u_0^{m_0}(x),$$

$$\leq u_{x_1}(x, 0) + \varepsilon R^2 \max_{x \in B(0; R)} v_0^{m_0}(x)$$

$$\leq 0, \quad \forall x \in B_0^+(0; R)$$

provided that $0 < \varepsilon \leq \varepsilon_2$ for some sufficiently small $\varepsilon_2$.

Set $\varepsilon = \min\{1, \varepsilon_1, \varepsilon_2\}$, then (3.8) and (3.10) hold. Application of the maximum principle to (3.8) and (3.10) ensures that

$$J(x, t) \leq 0, \quad (x, t) \in B_0^+(0; R) \times (0, T).$$

Namely,

$$- u^{m_0} u_{x_1} \geq c(x_1), \quad (x, t) \in B_0^+(0; R) \times (0, T).$$

(3.11)

Take $y = (2a, 0, 0, \cdots, 0)$ and $z = (r^*, 0, 0, \cdots, 0)$, then $y, z \in B_0^+(0; R)$. Integrating (3.11) yields that

$$0 < \int_y^z c(x_1) dx_1 \leq \frac{1}{m_0 - 1} u^{1-m_0}(z, t), \quad 0 < t < T.$$

The fact that $\lim_{t \to T} u(z, t) = \infty$ and $m_0 > 1$ leads to a contradiction. Therefore, $u$ blows up only at a single point $x = 0$, and so does the solution $(u, v)$ of system (1.1). Consequently, we conclude Theorem 5.
3.3. **Proof of Theorem 6.** In this subsection, we deduce Theorem 6 by introducing a lemma first, which shows the relationship between \( u(0,t) \) and \( v(0,t) \).

**Lemma 5.** Under the conditions of Theorem 4, for any given \( 0 < \delta < 1 \), there exists \( t_1 \leq T_0 < T \) such that the following statement holds for all \( t \in [T_0, T) \).

(i) If (a) \( p > m - 1 \) and \( n > q - 1 \), or (b) \( p < m - 1 \) and \( n < q - 1 \), then

\[
\delta \sigma v^{1+n-q}(0,t) \leq \theta u^{1+p-m}(0,t), \quad \delta \varepsilon u^{1+p-m}(0,t) \leq \sigma v^{1+n-q}(0,t).
\]

(ii) If \( p > m - 1 \) and \( n = q - 1 \), then

\[
\delta \varepsilon (1 + p - m) \ln v(0,t) \leq u^{1+p-m}(0,t), \quad \delta \varepsilon u^{1+p-m}(0,t) \leq (1 + p - m) \ln v(0,t).
\]

(iii) If \( p = m - 1 \) and \( n > q - 1 \), then

\[
\delta \varepsilon v^{1+n-q}(0,t) \leq (1 + n - q) \ln u(0,t), \quad \delta \varepsilon (1 + n - q) \ln u(0,t) \leq v^{1+n-q}(0,t).
\]

(iv) If \( p = m - 1 \) and \( n = q - 1 \), then

\[
\delta \varepsilon \ln v(0,t) \leq \ln u(0,t), \quad \delta \varepsilon \ln u(0,t) \leq \ln v(0,t).
\]

**Proof.** (i) (a) \( p > m - 1 \) and \( n > q - 1 \). One can deduce from the right-hand side of (3.4) that

\[
\frac{1}{1 + p - m} u^{1+p-m}(0,s) \bigg|_{t_1}^t \leq \frac{1}{\varepsilon (1 + n - q)} v^{1+n-q}(0,s) \bigg|_{t_1}^t \leq \frac{v^{1+n-q}(0,t)}{\varepsilon (1 + n - q)}. \tag{3.12}
\]

Notice that \( \lim_{t \to T} u(0,t) = \infty \) and \( p > m - 1 \), for given \( 0 < \delta < 1 \), there exists \( t_2 : t_1 \leq t_2 < T \) such that \( u^{1+p-m}(0,t_1) \leq (1 - \delta) u^{1+p-m}(0,t) \) for all \( t_2 \leq t < T \), and thus (3.12) ensures

\[
\frac{\delta}{1 + p - m} u^{1+p-m}(0,t) \leq \frac{1}{\varepsilon (1 + n - q)} v^{1+n-q}(0,t) \quad \text{for } t \in [t_2, T).
\]

On the other hand, application of similar analysis to the left-hand side of (3.4) derives that for given \( 0 < \delta < 1 \), there exists \( t_2 : t_1 \leq t_2 < T \) such that

\[
\delta \varepsilon (1 + p - m) v^{1+n-q}(0,t) \leq (1 + n - q) u^{1+p-m}(0,t) \quad \text{for } t \in [t_2, T).
\]

Define \( T_0 = \max\{t_2, t_2^*\} \), then we come to the conclusion (i) from (3.13), (3.14) and the definitions of \( \theta \) and \( \sigma \).

Analogously, we can demonstrate other cases.

**Proof of Theorem 6.** (i) We need only prove the case (a) \( p > m - 1 \) and \( n > q - 1 \), since the case (b) \( p < m - 1 \) and \( n < q - 1 \) can be treated similarly. Combining the first inequality of (3.2) with (i) of Lemma 5, we see that

\[
u_t(0,t) \leq \{\theta/(\delta \sigma)\}^{n/(1+n-q)} u^{m+n/\sigma/\theta}(0,t), \quad t \in [T_0, T).
\]

Since

\[
1 - m - n \sigma/\theta = 1 - m - \frac{n(1 + p - m)}{1 + n - q} = \frac{(1 - m)(1 - q) - np}{1 + n - q} = -\frac{1}{\theta} < 0
\]

and \( \lim_{t \to T} u(0,t) = \infty \), by integrating (3.15) we find that there exists a constant \( c_1 > 0 \) such that

\[
u(0,t) \geq c_1 (T - t)^{-\theta}, \quad T_0 \leq t < T.
\]

Applying the above arguments to the first inequality of (3.1) and using (i) of Lemma 5 show that there exists a constant \( C_1 > 0 \) such that

\[
u(0,t) \leq C_1 (T - t)^{-\theta}, \quad T_0 \leq t < T.
\]
Consequently, it follows from (3.16), (3.17) and (i) of Lemma 5 that there exist positive constants \(0 < c_2 \leq C_2\) such that
\[
c_2(T - t)^{-\sigma} \leq v(0, t) \leq C_2(T - t)^{-\sigma}, \quad T_0 \leq t < T.
\] (3.18)

Let \(c = \min\{c_1, c_2\}\) and \(C = \max\{C_1, C_2\}\), then (3.16)–(3.18) imply the desired conclusion (i) of Theorem 6.

(ii) \(p > m - 1\) and \(n = q - 1\). By joining the second inequality of (3.2) and (ii) of Lemma 5, we have
\[
v_t(0, t) \leq \{(1 + p - m)/(\delta \varepsilon)\}^{p/(1 + p - m)}(\ln v(0, t))^{p/(1 + p - m)}v^q(0, t), \quad T_0 \leq t < T.
\] (3.19)

Observe that \(\lim_{t \to T} v(0, t) = \infty\). From (3.19) follows
\[
\int_{v(0, t)}^{\infty} s^{-q}(\ln s)^{-p/(1 + p - m)}ds \leq \{(1 + p - m)/(\delta \varepsilon)\}^{p/(1 + p - m)}(T - t), \quad T_0 \leq t < T.
\] (3.20)

Similarly, in view of the second inequality of (3.1) and (ii) of Lemma 5 we have
\[
\int_{v(0, t)}^{\infty} s^{-q}(\ln s)^{-p/(1 + p - m)}ds \geq \varepsilon(\delta \varepsilon(1 + p - m))^{p/(1 + p - m)}(T - t), \quad T_0 \leq t < T.
\] (3.21)

Since
\[
\lim_{t \to T} \frac{\int_{v(0, t)}^{\infty} s^{-q}(\ln s)^{-p/(1 + p - m)}ds}{v^{1-q}(0, t)(\ln v(0, t))^{-p/(1 + p - m)}} \equiv \lim_{v(0, t) \to \infty} \frac{\int_{v(0, t)}^{\infty} s^{-q}(\ln s)^{-p/(1 + p - m)}ds}{v^{1-q}(0, t)(\ln v(0, t))^{-p/(1 + p - m)}}
\]
\[
= \lim_{v(0, t) \to \infty} \frac{-v^{-q}(\ln v)^{-p/(1 + p - m)}}{1 - q} = \lim_{v(0, t) \to \infty} \frac{-1}{1 - q} = \frac{1}{q - 1},
\]
i.e.,
\[
(q - 1) \int_{v(0, t)}^{\infty} s^{-q}(\ln s)^{-p/(1 + p - m)}ds \sim v^{1-q}(0, t)(\ln v(0, t))^{-p/(1 + p - m)}.
\] (3.22)

Therefore, by (3.22) there exists \(T_1 < T\) such that for all \(t \in [T_1, T)\),
\[
\frac{v^{1-q}(0, t)}{2(\ln v(0, t))^{p/(1 + p - m)}} \leq (q - 1) \int_{v(0, t)}^{\infty} s^{-q}(\ln s)^{-p/(1 + p - m)}ds \leq \frac{2v^{1-q}(0, t)(\ln v(0, t))^{p/(1 + p - m)}}{(q - 1)}. \] (3.23)

Let \(T^* = \max\{T_0, T_1\}\), then it can be deduced from (3.20), (3.21) and (3.23) that there exist some positive constants \(c \leq C\) such that
\[
c(T - t)^{-1} \leq v^{p-1}(0, t)(\ln v(0, t))^{p/(1 + p - m)} \leq C(T - t)^{-1}, \quad T^* \leq t < T. \] (3.24)

In addition, on account of the first inequalities of (3.1) and (3.2), and \(n = q - 1\),
\[
\varepsilon v^{q-1}(0, t) \leq u^{-m}(0, t)u_t(0, t) \leq v^{q-1}(0, t), \quad T^* \leq t < T. \] (3.25)
We need only to verify the estimate of $u$; see that there exist some positive constants $c_1$ such that

$$c_1(T - t)^{-1} \leq u^{p-m}(0, t)u_0(0, t) \leq C_1(T - t)^{-1}, \quad T^* \leq t < T. \quad (3.26)$$

Due to $1 + p - m > 0$, integration of (3.20) in $[T^*, t)$ yields that

$$c_1|\ln(T - t)| \leq u^{1+p-m}(0, t) \leq C_1|\ln(T - t)|, \quad T^* \leq t < T \quad (3.27)$$

for some positive constants $c \leq C$. Therefore, we come to the conclusion (ii) of Theorem 6 from (3.24), (3.27) and $n = q - 1$.

(iii) $p = m - 1$ and $n > q - 1$, the conclusion (iii) of Theorem 6 can be derived in the similar way as case (ii).

(iv) $p = m - 1$ and $n = q - 1$. It follows from (3.1), (3.2) and (iv) of Lemma 5 that

$$\varepsilon u^{m+n-\delta}(0, t) \leq u_t(0, t) \leq u^{m+n/(\varepsilon \delta)}(0, t), \quad T_0 \leq t < T.$$ 

Recall that \(\lim_{t \to T^+} u(0, t) = \infty\), integrating the above yields that for $T_0 \leq t < T$, 

$$(T - t)[(m-1)\varepsilon + n]/(\varepsilon \delta)]^{-\frac{1}{m+n-\delta}} u(0, t) \leq \varepsilon [m+n\delta - 1](T - t)^{-\frac{1}{m+n-\delta}}.$$ 

Consequently, for some positive constants $c \leq C$, there exists $T_0 \leq T^* < T$ such that

$$c_1|\ln(T - t)| \leq \ln u(0, t) \leq C|\ln(T - t)|, \quad T^* \leq t < T. \quad (3.28)$$

Moreover, from (3.28) and (iv) of Lemma 5 we see that there exist some positive constants $c \leq C$ such that

$$c_1|\ln(T - t)| \leq \ln v(0, t) \leq C|\ln(T - t)|, \quad T^* \leq t < T. \quad (3.29)$$

Therefore, (3.28) and (3.29) imply the conclusion (iv) of Theorem 6.

3.4. Proof of Theorem 7. Similar as in subsection 3.2, we still apply the ideas of [9] to proceed our discussion for Theorem 7.

Proof of Theorem 7. We need only to verify the estimate of $u$, since the estimate of $v$ can be obtained analogously.

Set

$$J(r, t) = u_r(r, t) + c(r)u^{m_0}(r, t), \quad (r, t) \in [0, R] \times [0, T),$$

where $1 < m_0 < m$, and

$$c(r) = \varepsilon r^{1+\delta}$$

with any constant $\delta > 0$ and small constant $\varepsilon > 0$ to be defined.

Direct calculation for $J$ shows that in $(0, R) \times (0, T)$,

$$J_t = \frac{N-1}{r} J_r - J_{rr}$$

$$= (u_t - \frac{N-1}{r} u_r - u_{rr})_r + m_0 c(r) u^{m_0-1}(u_t - \frac{N-1}{r} u_r - u_{rr})$$

$$- (N - 1)r^{-2} u_r - (N - 1)r^{-1} c'(r) u^{m_0} - c''(r) u^{m_0}$$

$$- 2m_0 c(r) u^{m_0-1} u_r - m_0(m_0 - 1)c(r) u^{m_0-2}|u_r|^2$$

$$\leq \{m u^{m-1}(r, t)v^n(0, t) - (N - 1)r^{-2} - 2\varepsilon m_0(1 + \delta)r^\delta u^{m_0-1}\} u_r$$

$$+ m_0 c(r) u^{m+m_0-1}(r, t)v^n(0, t) - (N - 1)r^{-1} c'(r) u^{m_0} - c''(r) u^{m_0}$$

$$\leq b J - c(r) u_m \{m - m_0\} u^{m-1}(r, t)v^n(0, t)$$

$$+ \delta(N + \delta) R^{-2} - 2\varepsilon m_0(1 + \delta) R^\delta u^{m_0-1},$$

\[\blacksquare\]
where
\[ b = m u^{m-1}(r, t)v^n(0, t) - (N - 1)r^{-2} - 2\varepsilon m_0 (1 + \delta) r^\delta u^{m_0 - 1}. \]
Note that \( v(r, t) > 0 \) in \([0, R] \times [0, T]\) and \( v(0, t) = \max_{0 \leq r \leq R} v(r, t) \) for \( t \in [0, T] \), then \( v(0, t) > c_1 > 0 \) for some constant \( c_1 \). Consequently, by \( 1 < m_0 < m \), we know that there exists \( \varepsilon_1 > 0 \) small enough such that for \( 0 < \varepsilon \leq \varepsilon_1 \),
\[
(m - m_0)u^{m-1}(r, t)v^n(0, t) + \delta(N + \delta) R^{-2} - 2\varepsilon m_0 (1 + \delta) R^\delta u^{m_0 - 1} \geq 0
\]
in \((0, R) \times (0, T)\). Thus, from (3.30) and (3.31) follows
\[
J_r - \frac{N - 1}{r} J_r - J_{rr} - b J \leq 0, \quad (r, t) \in (0, R) \times (0, T).
\] (3.32)
In addition, as \( u(r, t) > 0 \) for \((r, t) \in (0, R) \times (0, T)\) and \( u(R, t) = 0 \) for all \( t \in (0, T) \), the strong maximum principle for parabolic equations guarantees that \( u_r(R, t) < 0 \) for \( t \in (0, T) \). Hence,
\[
J(0, t) = u_r(0, t) = 0, \quad J(R, t) = u_r(R, t) < 0, \quad t \in (0, T).
\] (3.33)
For \( t = 0 \),
\[
J(r, 0) = u_r'(r) + \varepsilon r^{1+\delta} u_0^{m_0}(r) \leq -cr + \varepsilon r R^{\delta} u_0^{m_0}(0) \leq 0, \quad r \in (0, R),
\] (3.34)
provided that \( \varepsilon \leq \varepsilon_2 = cR^{-\delta}u_0^{m_0}(0) \).
Therefore, choose \( \varepsilon = \min\{1, \varepsilon_1, \varepsilon_2\} \), then (3.32) and (3.34) hold. Application of the maximum principle to (3.32)–(3.34) asserts that
\[
J(r, t) \leq 0, \quad (r, t) \in (0, R) \times (0, T).
\]
That is,
\[
- u^{m_0 - 1}u_r \geq \varepsilon r^{1+\delta}, \quad (r, t) \in (0, R) \times (0, T).
\] (3.35)
Integrating this inequality we obtain that
\[
u(r, t) \leq \left( \frac{\varepsilon(m_0 - 1)}{2 + \delta} r^{2+\delta} + u^{1-m_0}(0, t) \right)^{-1/(m_0 - 1)}
\]
\[
\leq \left( \frac{\varepsilon(m_0 - 1)}{(2 + \delta)} \right)^{-1/(m_0 - 1)} r^{-\frac{2+\delta}{m_0-1}}, \quad (r, t) \in (0, R) \times [0, T).
\] (3.36)
Since \( \delta > 0 \) is arbitrary and \( 1 < m_0 < m \), it is obvious that \((2 + \delta)/(m_0 - 1) > 2/(m - 1) \), and \((2 + \delta)/(m_0 - 1) > 2/(m - 1) \) can be made arbitrarily close to \(2/(m - 1) \). Consequently, (3.36) implies the assertion of \( u \). So, we conclude Theorem 7.

**Remark 2.** Looking through the whole Section 3, we find that, for the following coupled equations with the same initial and boundary conditions as system (1.1)
\[ u_t = \Delta u + u^m(0, t)v^n(x, t), \quad v_t = \Delta v + u^p(0, t)v^q(0, t), \quad x \in B(0; R), \quad t > 0, \]
the assertions of Theorems 3.4 and 6 are still true only if we keep assumption (H2), and replace assumptions (H1) and (H3) by (A1) and (A3) respectively,
(A1) \( n > 1 \) and \( p > 1 \).
(A3) \( u_0(x) \) and \( v_0(x) \) satisfy \( \Delta u_0(x) + u_0^m(0) v_0^n(x) \geq 0 \) and \( \Delta v_0(x) + u_0^p(0) v_0^q(0) \geq 0 \) in \( B(0; R) \).

Analogously, we can deal with the below equations coupled by the same initial-boundary conditions as system (1.1)
\[ u_t = \Delta u + u^m(0, t)v^n(x, t), \quad v_t = \Delta v + u^p(0, t)v^q(x, t), \quad x \in B(0; R), \quad t > 0. \]
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