S-SPACES AND LARGE CONTINUUM

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ABSTRACT. We prove that it is consistent with large values of the continuum that there are no S-spaces. We also show that we can also have that compact separable spaces of countable tightness have cardinality at most the continuum.

1. Introduction

An S-space is a regular hereditarily separable space that is not Lindelöf. If an S-space exists it can be assumed to be a topology on $\omega_1$ in which initial segments are open [12]. The continuum hypothesis implies that S-spaces exist [9] and the existence of a Souslin tree implies that S-spaces exist [15]. Therefore it is consistent with any value of $\mathfrak{c}$ that S-spaces exist. Todorcevic [17] proved the major result that it is consistent with $\mathfrak{c} = \aleph_2$ that there are no S-spaces. He also remarks that this follows from PFA. We prove that it is consistent with arbitrary large values of $\mathfrak{c}$ that there are no S-spaces. Our method adapts the approach used in [17] and incorporates ideas, such as the Cohen real trick in Lemma 2.15, first introduced in [1, 2].

The outline of the proof (of Theorem 4.3) is that we choose a regular cardinal $\kappa$ in a model of GCH. We construct a preparatory mixed support iteration sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ consisting of iterands that are Cohen posets and cardinal preserving subposets of Jensen’s poset for adding a generic cub. Following methods first introduced in [13], but more closely those of [17], the poset $P_\kappa$ is shown to be cardinal preserving. We then extend the iteration sequence to one of length $\kappa + \kappa$ with iterands that are ccc posets of cardinality less than $\kappa$. These iterands are the same as those used in [17]. For cofinally many $\beta < \kappa$, $\dot{Q}_{\kappa+\beta}$ is constructed so as to add an uncountable discrete subset to a $P_\beta$-name of an S-space. The bookkeeping is routine to ensure that

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$P_{\kappa+\kappa}$ forces there are no S-spaces. The challenging part of the proof is to prove that these $Q_\beta$ ($\kappa \leq \beta < \kappa + \kappa$) are ccc in this new setting. In the final section, we use similar techniques to produce a model in which compact separable spaces of countable tightness have cardinality at most $c$.

2. Constructing $P_\kappa$

Throughout the paper we assume that GCH holds and that $\kappa > \aleph_2$ is a regular uncountable cardinal.

**Definition 2.1.** The Jensen poset $\mathcal{J}$ is the set of pairs $(a, A)$ where $a$ is a countable closed subset of $\omega_1$ and $A \supset a$ is an uncountable closed subset of $\omega_1$. The condition $(a, A)$ is an extension of $(b, B) \in \mathcal{J}$ providing $a$ is an end-extension of $b$ and $A \subset B$.

We use $E$ to denote the set $\{\lambda + 2k : \lambda < \kappa$ a limit, $k \in \omega\}$. We also choose a family $I = \{I_\gamma : \gamma \in E\}$ of subsets of $\kappa$ such that,

1. $\gamma \in I_\gamma \subset \gamma + 1$ and $|I_\gamma| \leq \aleph_1$,
2. if $\gamma < \omega_2$, then $I_\gamma = \gamma + 1$,
3. if $\mu \in I_\gamma \cap E$, then $I_\mu \subset I_\gamma$
4. for all $I \in [\kappa]^{\aleph_1}$, the set $\{\gamma : I \subset I_\gamma\}$ is unbounded in $\kappa$.

Say that a set $I \subset \kappa$ is $\mathcal{J}$-saturated if it satisfies that $I_\mu \subset I$ for all $\mu < \gamma \in E$.

**Definition 2.2. A.** We define a mixed support iteration sequence $\langle P_\alpha, Q_\beta : \alpha \leq \kappa$, $\beta < \kappa\rangle$:

1. $P_0 = \emptyset$,
2. $p \in P_\alpha$ is a function with $\text{dom}(p)$, a countable subset of $\alpha$, such that $\text{dom}(p) \cap E$ is finite,
3. for all $p \in P_\alpha$ and $\beta \in \text{dom}(p)$, $p(\beta)$ is a $P_\beta$-name forced by $1_{P_\beta}$ to be an element of $Q_\beta$,
4. the support of a $P_\alpha$-name $\tau$, $\text{supp}(\tau)$, is defined, by recursion on $\alpha$ to be the union of the set $\{\text{supp}(\sigma) \cup \text{dom}(q) : (\sigma, q) \in \tau\}$,
5. for $\alpha \in E$, $Q_\alpha$ is the trivial $P_\alpha$-name for $C_{\omega_1} = \text{Fn}(\omega_1, 2)$ (i.e. each element of $Q_\alpha$ has empty support),
6. for $\alpha \in E$, $Q_{\alpha+1}$ is the subposet of the standard $P_{\alpha+1}$-name for $\mathcal{J}$ consisting of the $P_{\alpha+1}$-names that are forced to have the form $(\dot{a}, A)$ where $\text{supp}(\dot{a}) \subset E \cap I_\alpha$, $\text{supp}(A) \subset \alpha$, and $1_{P_{\alpha+1}}$ forces that $(\dot{a}, A) \in \mathcal{J}$. $Q_{\alpha+1}$ is chosen so as to be sufficiently rich in names in the sense that if $p \in P_{\alpha+1}$ and $\dot{q}$ is a $P_{\alpha+1}$-name
such that $p \forces_{P_\alpha} \dot{q} \in \dot{Q}_{\alpha+1}$, then there is a $\dot{q}_1 \in \dot{Q}_{\alpha+1}$ such that $p \forces \dot{q} = \dot{q}_1$.

**B.** For each $\alpha \in \mathbb{E}$, we let $\dot{C}_\alpha$ denote the $P_{\alpha+2}$-name of the generic subset of $\omega_1$ added by $\dot{Q}_{\alpha+1}$.

**Remark 1.** Since we defined the family $\mathcal{J}$ to have the property that $I_\gamma = \gamma + 1$ for all $\gamma \in \omega_2 \cap \mathbb{E}$, it follows that $P_{\omega_2}$ is isomorphic to that used in [17]. It also follows that for all $\beta \in \omega_2 \cap \mathbb{E}$, $P_{\beta+1} \forces \dot{Q}_{\beta+1}$ is countably closed. We necessarily lose this property for $\omega_2 \leq \beta$ for any family $\mathcal{J}$ satisfying our properties (1)-(4). Nevertheless, our development of the properties of $P_\kappa$ will closely follow that of [17].

**Remark 2.** We prove in Lemma 2.13 that, for each $\alpha \in \mathbb{E}$, $\dot{C}_\alpha$ is forced, as hoped, to be a cub. However, even though, for $\beta \geq \omega_2$, $P_{\beta+1}$ does not force that $\dot{Q}_{\beta+1}$ is countably closed, we make note of subsets of the iteration sequence that have special properties, such as in Lemma 2.9.

For any ordered pair $(a, b)$, let $\pi_0((a, b)) = a$ and $\pi_1((a, b)) = b$. For convenience, for an element $v$ of $V$ and any $\alpha < \kappa$, we identify the usual trivial $P_\alpha$-name for $v$ with $v$ itself. In particular, if $s \in \mathcal{C}_{\omega_1}$ and $\alpha \in \mathbb{E}$, then $s \in \dot{Q}_\alpha$. Similarly, if $(\dot{a}, \dot{A})$ is a pair of the form specified in Definition 2.2(6), then again $(\dot{a}, \dot{A})$ can be regarded as an element of $\dot{Q}_{\alpha+1}$. We will say that a $P$-name $\tau$ for a subset of an ordinal $\lambda$ and poset $P$ is canonical if it is a subset of $\lambda \times P$ (and optionally, if $\{p : (\alpha, p) \in \tau\}$ is an antichain for all $\alpha \in \lambda$). Let $\mathcal{D}_\beta$ denote the set of canonical $P_\beta$-names of closed and unbounded subsets of $\omega_1$.

**Definition 2.3.** For each $\alpha < \kappa$, let $P'_\alpha$ denote the subset of $P_\alpha$, where $p \in P'_\alpha$ providing for all $\beta \in \text{dom}(p) \cap \mathbb{E}$, $p(\beta)$ is, literally, an element of $\mathcal{C}_{\omega_1}$.

**Lemma 2.4.** For all $\alpha \leq \kappa$, $P'_\alpha$ is a dense subset of $P_\alpha$.

**Proof.** Assume $\alpha \leq \kappa$ and that, by induction, $P'_\beta$ is a dense subset of $P_\beta$ for all $\beta < \alpha$. Consider any $p \in P_\alpha$. If $\alpha$ is a limit, choose any $\beta < \alpha$ such that $\text{dom}(p) \cap \mathbb{E} \subseteq \beta$. Choose any $p' \in P'_\beta$ so that $p' < p \upharpoonright \beta$. We then have that $p' \cup p \upharpoonright (\alpha \setminus \beta)$ is a condition in $P_\alpha$ that is below $p$.

Now let $\alpha = \beta + 1$. If $\beta \in \mathbb{E}$, then choose $p' \in P'_\beta$ so that there is an $s \in \mathcal{C}_{\omega_1}$ such that $p' \forces_{P_\beta} p(\beta) = s$. Then the desired extension of $p$ in $P'_\alpha$ is $p' \cup \langle \beta, s \rangle$. Similarly, if $\beta \notin \mathbb{E}$ and $p' \in P'_\beta$ with $p' < p \upharpoonright \beta$, then $p' \cup \langle \beta, p(\beta) \rangle \in P'_\alpha$. \qed

**Proposition 2.5.** If $p \in P_\kappa$ then for every $I \subseteq \kappa$, $p \upharpoonright I \in P_\kappa$ and $p \leq p \upharpoonright I$. 

Proposition 2.7. For each $\alpha \leq \kappa$, the set $P_\alpha(E) \subseteq P_\alpha$ and is ccc.

Definition 2.8. For each $\alpha \in E$, let $Q'_{\alpha+1}$ be the subset of $\dot{Q}_{\alpha+1}$ consisting of those pairs $(\dot{a}, \dot{A})$ as in Definition 2.2(6).

We may note that, for each $(\dot{a}, \dot{A}) \in Q'_{\alpha+1}$, $\dot{a}$ is a $P_{\alpha+1}(I_\alpha \cap E)$-name and $\dot{A}$ is a $P_\alpha$-name that is forced by $1_{P_\alpha}$ to be a cub subset of $\omega_1$. Also, for every $p \in P_{\alpha+1}$, $p \upharpoonright \alpha \Vdash p(\alpha + 1) \in Q'_{\alpha+1}$.

Lemma 2.9. If $\alpha \in E$ and, and $\{(\dot{a}_n, \dot{A}_n) : n \in \omega\} \subseteq Q'_{\alpha+1}$ is a sequence that satisfies, for each $n \in \omega$, $1 \Vdash_{P_{\alpha+1}} (\dot{a}_{n+1}, \dot{A}_{n+1}) \leq (\dot{a}_n, \dot{A}_n)$, then there is a condition $(\dot{a}, \dot{A}) \in Q'_{\alpha+1}$ such that

(1) $1 \Vdash_{P_{\alpha+1}}$ forces that $\dot{a}$ is the closure of $\bigcup\{\dot{a}_n : n \in \omega\}$,
(2) $1_{P_\alpha}$ forces that $\dot{A}$ equals $\bigcap\{\dot{A}_n : n \in \omega\}$,
(3) $1 \Vdash_{P_{\alpha+1}}$ forces that $(\dot{a}, \dot{A}) = \bigwedge\{(\dot{a}_n, \dot{A}_n) : n \in \omega\}$.

Proof. In the forcing extension by a $P_{\alpha+1}$-generic filter $G$, it is clear that $(\text{cl}(\bigcup\{\text{val}_G(\dot{a}_n)\}), \bigcap\{\text{val}_G(\dot{A}_n) : n \in \omega\})$ is the meet in $\mathcal{F}$ of the sequence $\{(\text{val}_G(\dot{a}_n), \text{val}_G(\dot{A}_n)) : n \in \omega\}$. We just have to be careful about the supports of the names for these objects. Each $\dot{a}_n$ is a $P_{\alpha+1}(I_\alpha)$-name and so it is clear that there is a $P_{\alpha+1}(I_\alpha \cap E)$-name, $\dot{a}$, such that $1 \Vdash_{P_{\alpha+1}} \dot{a} = \text{cl}(\bigcup\{\dot{a}_n : n \in \omega\})$. This is the only subtle point. Any $P_\alpha$-name, $\dot{A}$, for $\bigcap\{\dot{A}_n : n \in \omega\}$ is adequate (although we are using that each $\dot{A}_n$ is a $P_\alpha$-name forced by 1 to be a cub). \[\Box\]
When we have a sequence \( \{(\hat{a}_n, \hat{A}_n) : n \in \omega\} \subset \hat{Q}'_{\alpha+1} \) as in the hypothesis of Lemma 2.9, we will use \( \bigwedge \{(\hat{a}_n, \hat{A}_n) : n \in \omega\} \) to denote the element \((\hat{a}, \hat{A})\) in the conclusion of the Lemma.

Let \( <_E \) denote the relation on \( P_\kappa \) defined by \( p_1 <_E p_0 \) providing

1. \( p_1 \leq p_0 \),
2. \( p_1 \upharpoonright E = p_0 \upharpoonright E \),
3. for \( \beta \in \text{dom}(p_0) \), \( 1_{p_\beta} \vdash p_1(\beta) < p_0(\beta) \).

For \( r \in P_\kappa(E) \) and compatible \( p \in P_\kappa \), let \( p \land r \) denote the condition with domain \( \text{dom}(p) \cup \text{dom}(r) \) satisfying \( (p \land r)(\beta) = p(\beta) \land r(\beta) \) for \( \beta \in \text{dom}(r) \) and \( p \land r(\beta) = p(\beta) \) for \( \beta \in \text{dom}(p) \setminus \text{dom}(r) \). For convenience, let \( p \land r \) equal \( p \) if \( r \in P_\kappa \) is not compatible with \( p \).

**Lemma 2.10.** Assume that \( \{p_n : n \in \omega\} \subset P'_\kappa \) is a \( <_E \)-descending sequence. Then there is a \( p_\omega \in P'_\kappa \) such that \( \text{dom}(p) = \bigcup_n \text{dom}(p_n) \) and \( p_\omega <_E p_n \) for all \( n \in \omega \).

**Proof.** We let \( J = \bigcup \{\text{dom}(p_n) : n \in \omega\} \). We define \( p_\omega \upharpoonright \beta \) by induction on \( \beta \in E \) so that \( \text{dom}(p_\omega \upharpoonright \beta) = J \cap \beta \). For limit \( \alpha \), simply \( p_\omega \upharpoonright \alpha = \bigcup_{\beta < \alpha} p_\omega \upharpoonright \beta \). If \( p_\omega \upharpoonright \beta <_E p_n \upharpoonright \beta \) for all \( n \in \omega \) and \( \beta < \alpha \), then we have \( p_\omega \upharpoonright \alpha <_E p_n \upharpoonright \alpha \) for all \( n \in \omega \). Now let \( \alpha = \beta + 2 \) with \( \beta \in E \) and assume that we have defined \( p_\omega \upharpoonright \beta \) as above. If \( \beta \in J \), then let \( p_\omega(\beta) = p_0(\beta) \). If \( \beta + 1 \in J \), then \( 1_{p_{\beta+1}} \) forces that \( \{p_n(\beta + 1) : n \in \omega\} \) is a descending sequence in \( \hat{Q}_{\beta+1} \). We define \( p_\omega(\beta + 1) \) to equal \( \bigwedge \{p_n(\beta + 1) : n \in \omega\} \). It follows by the definition of \( \bigwedge \{p_n(\beta + 1) : n \in \omega\} \), that \( 1_{p_{\beta+1}} \vdash p_\omega(\beta + 1) < p_n(\beta + 1) \) for all \( n \in \omega \). \( \square \)

**Lemma 2.11.** For every \( p_0 \in P'_\kappa \) and dense subset \( D \) of \( P_\kappa \), there is a \( p <_E p_0 \) satisfying that the set \( D \cap \{p \land r : r \in P_\kappa(E)\} \) is predense below \( p \). Moreover, there is a countable subset of \( D \cap \{p \land r : r \in P_\kappa(E)\} \) that is predense below \( p \).

**Proof.** Let \( r_0 = p_0 \upharpoonright E \). There is nothing to prove if \( p_0 \in D \) so assume that it is not. By induction on \( 0 < \eta < \omega_1 \), we choose, if possible, conditions \( p_\eta, r_\eta \) such that, for all \( \zeta < \eta \):

1. \( p_\zeta <_E p_\eta \) and \( r_\zeta < r_0 \),
2. \( p_\zeta \land r_\zeta \in D \),
3. \( (p_\eta \land r_\eta) \perp (p_\zeta \land r_\zeta) \).

Suppose that we have so chosen \( \{p_\zeta, r_\zeta : \zeta < \eta\} \). Let \( L_\eta = \bigcup \{\text{dom}(p_\zeta) : \zeta < \eta\} \). If \( \eta = \beta + 1 \), let \( \bar{p}_\eta = p_\beta \). If \( \eta \) is a limit, then let \( \bar{p}_\eta \) be a condition as in Lemma 2.10 for some cofinal sequence in \( \eta \). If \( \{p_\zeta \land r_\zeta : \zeta < \eta\} \) is predense below \( \bar{p}_\eta \), we halt the induction and set \( p = \bar{p}_\eta \). Otherwise we choose any \( p_\eta <_E \bar{p}_\eta \) and an \( r_\eta \supset r_0 \) so that \( p_\eta \land r_\eta \) in \( D \). The induction
will halt for some $\eta < \omega_1$ since the family $\{r_\zeta : \zeta < \eta\}$ is evidently an antichain in $P_\kappa(E)$. 

\textbf{Corollary 2.12.} For each $\beta \in E$, $P_\beta$ is proper and $P_\beta \cap (E \cap \beta)$ does not add any reals.

\textit{Proof.} Let $P_\beta \in M$ where $M$ is a countable elementary submodel of $H(\kappa^+)$. Let $\{D_n : n \in \omega\}$ be an enumeration of the dense open subsets of $P_\beta$ that are members of $M$. By Lemma 2.11 we have that for each $q \in P_\beta \cap M$ and $n \in \omega$, there is a $\bar{q} \leq_E q$ also in $P_\beta \cap M$ so that $D_n \cap \{\bar{q} \land r : r \in P_\beta(E) \cap M\}$ is predense below $\bar{q}$. Let $M \cap \omega_1 = \delta$. Fix any $p_0 \in P_\beta \cap M$. By a simple recursion, we may construct a $\leq_E$-descending sequence $\{p_n : n \in \omega\} \subset M$ so that, for each $n$, $D_n \cap \{p_{n+1} \land r : r \in P_\beta(E) \cap M\}$ is predense below $p_{n+1}$. By Lemma 2.10 we have the $(P_\beta, M)$-generic condition $p_\omega$. It is clear that for each $P_\beta$-name $\tau \in M$ for a subset of $\omega$, $p_\omega$ forces that $\tau$ is equal to a $P_\beta(E)$-name. This implies that $P_\beta \cap (E \cap \beta)$ does not add reals.

We can now prove that $P_{\beta+2}$ does indeed force that $\dot{C}_\beta$ is a cub.

\textbf{Lemma 2.13.} For each $\beta \in E$, $P_{\beta+2}$ forces that $\dot{C}_\beta$ is unbounded in $\omega_1$.

\textit{Proof.} Let $p \in P_{\beta+2}$ be any condition and let $\gamma \in \omega_1$. By possibly strengthening $p$ we can assume that $p(\beta+1) \in Q_{\beta+1}'$. We find $q < p$ so that $q \not\forces \dot{C}_\beta \setminus \gamma$ is not empty. Let $p, P_{\beta+2}$ be members of a countable elementary submodel $M \prec H(\kappa^+)$. Let $\bar{p} < p \forces \beta$ be $(P_\beta, M)$-generic and let $\dot{D} = \pi_1(p(\beta+1)) \in D_\beta$. Since $p, \dot{D}$ are members of $M$ and $p$ forces that $\dot{D}$ is a cub, it follows that $\bar{p} \forces \delta \in \dot{D}$. It also follows that $\bar{p} \forces \dot{a} \subset \dot{D} \cap \delta$. Let $\dot{a}_1$ be the $P_{\beta+1}$-name that has support equal to the support of the name $\dot{a}$ and satisfies that $1_{P_{\beta+1}} \forces \dot{a}_1 = \dot{a} \cup \{\delta\}$. Let $\dot{E}$ be the $P_\beta$-name for $\dot{D} \cup \{\delta\}$ and notice that, given that $(\dot{a}, \dot{D}) \in Q_{\beta+1}'$, we have that $(\dot{a}_1, \dot{E})$ is also in $Q_{\beta+1}'$. Now let $q \in P_{\beta+2}$ be defined according to $q \forces \beta = \bar{p}$, $q(\beta) = p(\beta)$, and $q(\beta+1) = (\dot{a}_1, \dot{E})$. It is immediate that $q \forces \beta+1 < p \forces \beta+1$. Also, $q \forces \beta+1$ forces that $\dot{a}$ is an initial segment of $\dot{a}_1$, that $\dot{a}_1 \subset \dot{D}$, and that $\dot{E} \subset \dot{D}$. Therefore, $q < p$ and $q \forces \delta \in \dot{C}_\beta$. 

\textbf{Lemma 2.14.} For each $\beta \leq \kappa$, $P_\beta$ satisfies the $\aleph_2$-cc.

\textit{Proof.} We prove the lemma by induction on $\beta$. If $\beta \in E$ and $P_\beta$ satisfies the $\aleph_2$-cc, then it is trivial that $P_{\beta+1}$ does as well. Similarly $P_{\beta+2}$ satisfies the $\aleph_2$-cc since $P_{\beta+1} \times Q_{\beta+1}'$ clearly does, and this poset is dense in $P_{\beta+2}$. The argument for limit ordinals $\beta$ with cofinality
less than \( \omega_2 \) is straightforward, so we assume that \( \beta \) is a limit with cofinality greater than \( \omega_1 \). Let \( \{ p_\gamma : \gamma \in \omega_2 \} \) be a subset of \( P_\beta \). Choose any elementary submodel \( M \) of \( H(\kappa^+) \) such that \( \{ p_\gamma : \gamma \in \omega_2 \} \subseteq M \), \(|M| = \aleph_1\), and \( M^2 \subseteq M \). Let \( M \cap \omega_2 = \lambda \) and let \( I = \text{dom}(p_\lambda) \cap M \) and fix any \( \mu \in M \cap \beta \) so that \( I \subseteq \mu \). For each \( \beta \in E \) such that \( \beta + 1 \in I \), let \( \dot{a}_\beta \in M \) so that \( \pi_0(p_\lambda(\beta + 1)) = \dot{a}_\beta \). That is, \( p_\lambda(\beta) = (\dot{a}_\beta, \dot{D}_\beta) \) for some \( \dot{D}_\beta \in D_\beta \). Clearly the countable sequence \( \{ \dot{a}_\beta : \beta \in I \cap E \} \) is an element of \( M \). Therefore there is a \( \gamma \in M \) so that \( \text{dom}(p_\gamma) \cap \mu = I \) and so that \( \pi_0(p_\gamma(\beta + 1)) = \dot{a}_\beta \) for all \( \beta \in E \) such that \( \beta + 1 \in I \). It follows that \( p_\gamma \not\subseteq p_\lambda \).

Now we discuss the Cohen real trick, which, though simple and powerful, is burdened with cumbersome notation.

**Lemma 2.15.** Let \( \alpha \in E \) and let \( p_0 \in P_{\alpha+2} \in M \) be a countable elementary submodel of \( H(\kappa^+) \) and let \( \delta = M \cap \omega_1 \). There is a \( (P_{\alpha+2}, M) \)-generic condition \( p_1 < p_0 \) satisfying that for all \( P_\alpha \) generic filters satisfying \( p_1 \uparrow \alpha \in G_0 \) and \( Q_\alpha \)-generic filters \( p_1(\alpha) \in G_1 \), the collection, in \( V[G_0 \ast G_1] \),

\[
p_1^\uparrow_\alpha = \{ p(\alpha + 1) : p \in M \cap P_{\alpha+2}, p \uparrow (\alpha + 1) \in G_0 \ast G_1, p_1 < p \}
\]

is \( \text{val}_{G_0 \ast G_1}(Q_{\alpha+1} \cap M) \)-generic over \( V[G_0 \ast (G_1 \uparrow \delta)] \).

Moreover, for any \( P_\alpha \)-name \( \dot{Q} \) of a ccc poset and \( P_\alpha \ast \dot{Q} \)-generic filter \( G_0 \ast G_2 \), \( p_1^\uparrow_\alpha \) is also generic over the model \( V[G_0 \ast G_2][G_1 \uparrow \delta] \).

**Proof.** Let \( \dot{Q} \) be any \( P_\alpha \)-name of a ccc poset. Choose any \( \dot{p}_1 < p_0 \uparrow (\alpha + 1) \) that is \( (M, P_\alpha) \)-generic with \( \dot{p}_1(\alpha) = p(\alpha) \). We will let \( p_1 \uparrow \alpha = \dot{p}_1 \uparrow \alpha \) and then we simply have to choose a value for \( p_1(\alpha + 1) \). We may assume that \( \dot{p}_1 \uparrow E = p_0 \uparrow E \). Let \( \dot{G} \) denote the filter \( (G_0 \ast G_1) \cap P_{\alpha+1}(I_\alpha \cap E) \) and let \( R = (M \cap Q_{\alpha+1}) \setminus \dot{G} \). For \( r \in R \) we may regard \( r \) in the extension \( V[\dot{G}] \) to have the form \((a_r, \dot{A}_r)\), with \( a_r \subseteq \omega_1 \), because, for each \( (\dot{a}, \dot{A}) \in M \cap Q_{\alpha+1} \), \( \dot{a} \) has support contained in \( P_{\alpha+1}(I_\alpha \cap E) \).

We have no such reduction for \( \dot{A} \). We adopt the subordering, \( <_R \), on \( R \) where \((a, \dot{A}) <_R (b, \dot{B}) \) in \( R \) will mean that \( 1_{P_{\alpha+1}} \models a \subseteq \dot{A} \). The fact that \( (a, \dot{A}) \in R \) already means that \( 1_{P_{\alpha+1}} \models a \subseteq \dot{A} \). If \( p \in M \cap P_{\alpha+1} \) and \( (a, \dot{A}) \in R \) is such that \( p \models (a, \dot{A}) \prec (b, \dot{B}) \), then there is an \( (a, \dot{A}) \in R \) such that \( p \models \dot{A} = \dot{A}_1 \) and \((a, \dot{A}) <_R (b, \dot{B}) \).

The quotient poset \((R/\dot{G}, <_R)\) is isomorphic to \( C_\omega \). Let \( \psi \in V[\dot{G}] \) be an isomorphism from \( C_{(\delta, \delta+\omega)} \) to \((R/\dot{G}, <_R)\). We regard \( C_{(\delta, \delta+\omega)} \) as the canonical subposet of \( Q_\alpha \) and let \( G^\delta_\alpha \) denote a generic filter for this subposet of \( Q_\alpha \). Now we have, in the extension \( V[G][G^\delta_\alpha] \), a \( <_R \)-filter \( R^\delta_\alpha \subseteq R \) given by \( \{ \psi(\sigma) : \sigma \in G^\delta_\alpha \} \). Let \( a_\omega = \{ \delta \} \cup \bigcup \{ a_r : r \in R^\delta_\alpha \} \).
Note that \( \bar{p}_1 \) forces that \( \delta \in \dot{C} \) for all \( \dot{C} \in M \cap D_\alpha \). By the construction, it follows that we may fix a \( P_{\alpha+1} \)-name, \( \dot{a}_\omega \), for \( a_\omega \), that has support contained in \( I_\alpha \cap E \). Let \( \dot{A}_\omega \) be the \( P_{\alpha+1} \)-name satisfying that \( \bar{p}_1 \) forces that \( \dot{A}_\omega \) equals the intersection of all \( C \in D_\alpha \cap M \) such that \( \dot{a}_\omega \subset \dot{C} \).

It follows that for \( r \in R_\alpha^\delta \) and \( \bar{p} \upharpoonright \alpha + 1 < \bar{p}_1 \), \( \bar{p}(\alpha) \in G_\alpha^\delta \), and \( \bar{p}(\alpha + 1) = r \), we have that \( \bar{p} \wedge r \Vdash \dot{A}_\omega \subset \dot{A}_r \) (and this takes place in \( V[G] \)). We may choose \( \dot{A}_\omega \) so that \( \bar{p} \upharpoonright \dot{A}_\omega = \omega_1 \) for all \( \bar{p} \downarrow \bar{p}_1 \) in \( P_{\alpha+1} \).

It then follows that \( (\dot{a}_\omega, \dot{A}_\omega) \) is an element of \( \dot{Q}_{\alpha+1} \). We now define \( p_1 \) so that \( p_1 \uparrow \alpha + 1 = \bar{p}_1 \) and \( p_1(\alpha + 1) = (\dot{a}_\omega, \dot{A}_\omega) \). The fact that \( p_1 \) is \( (M, P_{\alpha+2}) \)-generic follows from the stronger claim below.

**Claim 3.** Let \( G_0 \) be a \( P_\alpha \)-generic with \( \bar{p}_1 \uparrow \alpha \in G_0 \) and let \( G_1 \) be a \( Q_\alpha \)-generic filter with \( \bar{p}_1(\alpha) \in M \cap G_1 \). Also let \( G_0 \ast G_2 \) be \( P_\alpha \ast \dot{Q} \)-generic. Let \( \sigma \in \dot{C}(\delta, \delta + \omega) \) be arbitrary. Let \( \dot{D} \) be a \( P_{\alpha+1} \ast Q \)-name of a dense subset of \( \text{val}_{G_0 \ast G_1}(\dot{Q}_{\alpha+1} \cap M) \). Then there is a \( \tau \supset \sigma \) such that \( \tau \Vdash p^*_1 \cap \text{val}_{G_0 \ast (G_1 \times G_2)}(\dot{D}) \neq \emptyset \).

**Proof of Claim.** Fix the generic filter \( \tilde{G} \subset G_0 \ast G_1 \) as used in the construction of \( (\dot{a}_\omega, \dot{A}_\omega) \) and let \( \psi: C^\delta_\alpha \rightarrow (R/\tilde{G}, <_R) \) denote the above mentioned isomorphism. Let \( (b, \bar{B}) = \psi(\sigma) \) and, using the density of \( \text{val}_{G_0 \ast (G_1 \times G_2)}(\dot{D}) \), choose \( (a, A) < (b, \text{val}_{G_0 \ast G_1}(\bar{B})) \), so that \( (a, A) \in \text{val}_{G_0 \ast (G_1 \times G_2)}(\tilde{D}) \). By elementarity, choose \( (\dot{a}, \dot{A}) \in M \cap \dot{Q}_{\alpha+1} \) such that \( \text{val}_{G_0 \ast G_1}((\dot{a}, \dot{A})) = (a, A) \). Again by elementarity and using that \( \bar{p}_1 \) is \( (M, P_{\alpha+1}) \)-generic, there is a \( p \in M \cap (G_0 \ast G_1) \) such that \( p \Vdash \dot{A} \subset \bar{B} \). Now choose \( \tau \supset \sigma \) so that \( \psi(\tau) = (a, \dot{A}_1) \) satisfies that \( (a, \dot{A}_1) <_R (b, \bar{B}) \) and \( p \Vdash \dot{A}_1 = \dot{A} \). It follows that \( \tau \Vdash (a, \dot{A}_1) \in \text{val}_{G_0 \ast (G_1 \times G_2)}(\tilde{D}) \). Since \( p_1 \cap \tau \) also forces that \( p_1(\alpha + 1) < (a, \dot{A}_1) \) we have that \( p_1 \cap \tau \Vdash (a, \dot{A}_1) \in p^*_1 \). \( \Box \)

This completes the proof of the Lemma. \( \Box \)

**Lemma 2.16.** Let \( \lambda < \kappa \) with \( \lambda \in E \) and let \( \dot{Q} \) be a \( P_\lambda \)-name of a ccc poset. Then \( P_\kappa \) forces that \( Q \) is ccc.

**Proof.** Let \( G \) be a \( P_\lambda \)-generic filter and let \( \dot{Q} = \text{val}_G(\dot{Q}) \). Since \( P_\kappa \) satisfies the \( \aleph_2 \)-cc, we can assume that \( Q \) is of the form \( (\omega_1, \leq Q) \). We work in the extension \( V[G] \) and we view, for each \( \lambda < \alpha \leq \kappa \), \( P_\alpha = P_\alpha/G \) as a subset of \( P_\alpha \). We prove, by induction on \( \lambda \leq \alpha \in E \), that for any countable elementary submodel \( \{Q, \lambda, P_\alpha\} \in M \) and any \( p \in P_\alpha \cap M \), there is a \( p_M \in P_\alpha \) such that \( (1_Q, p_M) \) is \( (M, Q \times P_\alpha) \)-generic. Note that this inductive hypothesis, i.e. the fact that it is \( (1_Q, p_M) \) that is the generic condition rather than \( (q, p_M) \) for some
other \( q \in Q \), is equivalent to the statement that \( P_\alpha \) preserves that \( Q \) is ccc.

The proof at limit steps follows the standard proof (as in [16]) that the countable support iteration of proper posets is proper. We feel that this can be skipped. So let \( \alpha = \beta + 2 \) for some \( \beta \in E \). Let \( M \) be a suitable countable elementary submodel and let \( p \in P_\alpha \cap M \) (such that \( p \upharpoonright \lambda \in G \)). Let \( M \cap \omega_1 = \delta \). By the inductive hypothesis, we can assume that we have \( \bar{p}_1 \in P_\beta \) so that, \( \bar{p}_1 \upharpoonright \lambda \in G \), \( \bar{p}_1 <_E p \upharpoonright \beta \) and so that \( (1_Q, \bar{p}_1) \) is an \((M, Q \times P_\beta)\)-generic condition. Of course it is also clear that \((1_Q, \bar{p}_1)\) is an \((M, Q \times P_{\beta+1})\)-generic condition. Now let \( p_1 \in P_{\beta+2} \) be chosen as in Lemma 2.15. That is, \( p_1 \) is chosen so that for any \( P_\beta \)-generic filter \( G_\beta \supset G \) with \( p_1 \upharpoonright \beta \in G_\beta \), any \( C_\omega \)-generic \( G_1 \) with \( p_1(\beta) \in G_1 \), and, since \( Q \) is ccc in \( V[G_\beta] \), any \( Q \)-generic filter \( G_Q \), we have that \( p_1^{\upharpoonright \beta} \) is generic over \( V[G_\beta \times (G_1 \times G_Q)] \). Let \( G_{\beta+1} = G_\beta \star G_1 \).

Let \( D \in M \) be any dense open subset of \( P_{\beta+2} \star Q \). Let \( R \) denote \( Q_{\beta+1}/(G_\beta \star G_1) \). It follows that \( D/\langle G_\beta \star G_1 \rangle \) or

\[
E = \{(r, q) : (\exists d \in D) \ (d(\beta+1) \in G_\beta \star G_1 \wedge d = d(\beta+1) \star (r, q))\}
\]

is a dense open subset of \( R \times Q \) and \( E \in M[G_{\beta+1}] \). By standard product forcing theory, we have that for each \( r \in R \), \( E_r = \{q \in Q : (\exists s \in R)(s < r \wedge (s, q) \in E)\} \) is a dense subset of \( Q \). For each \( r \in R \cap M[G_{\beta+1}] \), \( E_r \in M[G_{\beta+1}] \) and so, \( E_r \cap M[G_{\beta+1}] \) is a predense subset of \( Q \). This implies that, for each \( \bar{q} \in Q \), the set \( E(\bar{q}) = \{s \in R \cap M[G_{\beta+1}] : (\exists s, q \in E \cap M[G_{\beta+1}])(\bar{q} \not\in q)\} \) is a dense subset of \( R \cap M[G_{\beta+1}] \). Although \( E(\bar{q}) \) need not be an element of \( M[G_{\beta+1}] \), it is an element of \( V[G_\beta \star (G_1 \upharpoonright \delta)] \). Therefore, by Lemma 2.15, \( E(\bar{q}) \cap p_1^{\upharpoonright \beta} \) is not empty for all \( \bar{q} \in G_Q \). By elementarity, it then follows that \( p_1 \) is an \((M, P_{\beta+2} \star Q)\)-generic condition. \( \square \)

3. S-space tasks

Following [1] and [17] we define a poset of finite subsets of \( \omega_1 \) separated by a cub.

**Definition 3.1.** For a family \( \mathcal{U} = \{U_\xi : \xi \in \omega_1\} \) and a cub \( C \subseteq \omega_1 \), define the poset \( Q(\mathcal{U}, C) \subseteq [\omega_1]^{<\omega_0} \) to be the set of finite sets \( H \subseteq \omega_1 \) such that for \( \xi < \eta \) both in \( H \)

1. \( \xi \notin U_\eta \) and \( \eta \notin U_\xi \),
2. there is a \( \gamma \in C \) such that \( \xi < \gamma \leq \eta \).

\( Q(\mathcal{U}, C) \) is ordered by \( \supset \).

**Definition 3.2.** A family \( \mathcal{U} = \{U_\xi : \xi < \omega_1\} \) is an S-space task if it satisfies:
(1) \( \xi \in U_\xi \in [\omega_1]^{<\kappa_1} \),
(2) every uncountable \( A \subset \omega_1 \) has a countable subset that is not contained in any finite union from the family \( \mathcal{U} \).

**Remark 4.** If \( \mathcal{T} \) is a regular locally count!table topology on \( \omega_1 \) that contains no uncountable free sequence (see Definition 5.1), then each neighborhood assignment \( \{ U_\xi : \xi \in \omega_1 \} \) consisting of open sets with countable closures, is an S-space task. An uncountable \( A \subset \omega_1 \) failing property (2) would contain an uncountable free sequence. Suppose that there is a cub \( C \subset \omega_1 \) such that \( Q(\mathcal{U}, C) \) is ccc. Then, as usual, there is a \( q \in Q(\mathcal{U}, C) \) such that any generic filter including \( q \) is uncountable. If \( G \subset Q(\mathcal{U}, C) \) is a filter (even pairwise compatible), then \( \bigcup G \) is a discrete subspace of \( (\omega_1, \mathcal{T}) \). Of course this cub \( C \) can be assumed to satisfy that if \( \xi < \eta \) are separated by \( C \), then \( \eta \notin U_\xi \). This means that requirement (1) in the definition of \( Q(\mathcal{U}, C) \) can be weakened to only require that \( \xi \notin U_\eta \).

The following result is a restatement of Lemma 1 from [17]. It also uses the Cohen real trick. We present a proof that is more adaptable to the modifications needed for the consistency with \( c > \aleph_2 \).

**Proposition 3.3.** Let \( R \) be a ccc poset and let \( \mathcal{U} = \{ U_\xi : \xi \in \omega_1 \} \) be a sequence of \( R \)-names such that \( \mathcal{U} \) is forced to be an S-space task. Then \( R \times P_2 \) forces that for every \( n \in \omega \), every uncountable pairwise disjoint subfamily \( \mathcal{H} \) of \( Q(\mathcal{U}, C_1) \cap [\omega_1]^n \), has a countable subset \( \mathcal{H}_0 \) satisfying that, for some \( \delta \in \omega_1 \) and all \( F \in [\omega_1 \setminus \delta]^n \), there is an \( H \in \mathcal{H}_0 \) such that \( H \cap \bigcup \{ U_\xi : \xi \in F \} = \emptyset \). In particular, \( R \times P_2 \) forces that \( Q(\mathcal{U}, C_1) \) is ccc.

**Proof.** Of course \( P_2 \) is isomorphic to \( C_{\omega_1} \times \check{\omega} \). Fix any \( n \in \omega \) and let \( \{ H_\xi : \xi \in \omega_1 \} \) be \( R \times P_2 \)-names of pairwise disjoint elements of \( [\omega_1]^n \) \( \mathcal{Q}(\mathcal{U}, C_1) \). Since we can pass to an uncountable subcollection of \( \{ H_\xi : \xi \in \omega_1 \} \) we may assume that for all \( \xi \in \omega_1 \), it is forced that there is a \( \delta \in C_1 \) such that \( \xi < \delta \leq \min(H_\xi) \).

For each \( (r, p) \in R \times P_2 \) and \( H \in [\omega_1]^n \), let \( \Gamma_\xi(H, (r, p)) \) be the set \( \{ s \in R : (\exists q \in P_2)((s, q) < (r, p) \& (s, q) \models H = H_\xi) \} \). In other words, \( \Gamma_\xi(H, (r, p)) \) is not empty if and only if \( (r, p) \models H \neq H_\xi \). We say that \( \Gamma_\xi(H, (r, p)) \) is \( \omega_1 \)-full simply if it is not empty.

Now we define what it means for \( \Gamma_\xi(H, (r, p)) \) to be \( \omega_1 \)-full for \( H \in [\omega_1]^{n-1} \). We require that there is a set \( \{ \check{\eta}_\zeta : \zeta \in \omega_1 \} \) of canonical \( R \)-names such that \( r \models \check{\eta}_\zeta \in \omega_1 \setminus \zeta \) and for \( (\eta, s) \in \check{\eta}_\zeta \), \( s \leq r \) and satisfies that \( \Gamma_\xi(H \cup \{ \eta \}, (s, p)) \) is \( \omega_1 \)-full. It is worth noting that \( (r, p) \) has been changed to \( (s, p) \) rather than to some \( (s, q) \) with \( q < p \). This definition
generalizes to $H \in [\omega_1]^1$. We say that $\Gamma_\xi(H,(r,p))$ is $\omega_1$-full if there is a set of canonical $\bar{R}$-names $\{\check{\eta}_\zeta : \zeta \in \omega_1\}$ such that, for each $\zeta \in \omega_1$, $r \Vdash \check{\eta}_\zeta \in (\omega_1 \setminus \zeta)$, and for $(\eta,s) \in \check{\eta}_\zeta$, $s \leq r$ and $\Gamma_\xi(H \cup \{\eta\},(s,p))$ is $\omega_1$-full.

Claim 5. Suppose that $\Gamma_\xi(\emptyset,(r,p))$ is $\omega_1$-full and that $M \prec H(\kappa^+)$ is countable and $\{\xi,\mathcal{U},R,(r,p)\} \in M$. Then for any $\bar{r} < r \in R$ and finite $F \subset \omega_1 \setminus M$, there are $(s,q), H \in M$ such that

1. $(s,q) < (r,p) \in R \times P_2$,
2. $H \cap \bigcup \{\check{U}_\zeta : \zeta \in F\}$ is empty,
3. $(s,q) \Vdash H \models \check{H}_\xi = H,$
4. $s \not\in \bar{r}$.

Proof of Claim: Let $\check{W}_F = \bigcup \{\check{U}_\zeta : \zeta \in F\}$. Since $R \in M \prec H(\kappa^+)$ is ccc and forces that $\mathcal{U}$ is an S-space task, it follows that for each $\bar{R}$-name $\check{A} \in M$ for an uncountable subset of $\omega_1$, the set $\check{A} \cap M$ is forced to not be contained in $\check{W}_F$. By induction on $1 \leq i \leq n$, we choose $(\eta_i,s_i) \in (\omega_1 \times R) \cap M$ and $\bar{r}_i < s_i$ so that $\bar{r}_i \Vdash \eta_i \notin \check{W}_F$, $s_i \leq s_j$ and $\bar{r}_i \leq \bar{r}_j$ for $j < i$, and $\Gamma_\xi(\{\eta_j : 1 \leq j < i\},(s_i,p))$ is $\omega_1$-full.

Let $\bar{r}_0 = \bar{r}$, $(s_0,q_0) = (r,p)$, $\emptyset = \{\eta_j : 1 \leq j < 1\}$ and we assume by induction that, at stage $i$, $\Gamma(\{\eta_j : 1 \leq j < i\},(s_i,p))$ is $\omega_1$-full. Fix any sequence $\{\check{\eta}_\zeta : \omega \leq \zeta \in \omega_1\} \in M$ witnessing that $\Gamma_\xi(\{\eta_j : j < i\},(s_i,p))$ is $\omega_1$-full. We have that $\{\check{\eta}_\zeta : \omega \leq \zeta \in \omega_1\} \in M$ is an $R$-name for an uncountable subset of $\omega_1$. It follows that $\bar{r}_{i-1}$ forces that there is a $\zeta \in M$ such that $\check{\eta}_\zeta \notin \check{W}_F$. We find an extension $\bar{r}_{i+1}$ of $\bar{r}_i$ so that we may choose $\zeta \in M$ and $(\eta,s) \in \check{\eta}_\zeta$ such that $\eta \notin \check{W}_F$, $\bar{r}_{i+1} < s \leq s_i$. Therefore we set $(\xi_i,s_{i+1},q_{i+1}) = (\eta,s,q)$ and this completes the construction.

Setting $H = \{\xi_i : 1 \leq i \leq n\}$ and $(s,q) = (s_n,q_n)$ completes the proof of the Claim. \qed

Claim 6. If $\Gamma_\xi(H,(r,p))$ is not $\omega_1$-full, there is an $s < r$ in $R$ and a $\zeta < \omega_1$ such that $\Gamma_\xi(H \cup \{\eta\},(s,p))$ is not $\omega_1$-full for all $\zeta < \eta \in \omega_1$.

Proof of Claim: Since $\Gamma_\xi(H,(r,p))$ is not $\omega_1$-full, there is some $\zeta \in \omega_1$ so that the suitable nice name $\check{\eta}_\zeta$ does not exist. It follows immediately that $\check{\eta}_\gamma$ does not exist for all $\zeta < \gamma \in \omega_1$. In addition, since $\check{\eta}_\zeta$ fails to exist, it is because $\Gamma_\xi(H \cup \{\eta\},(s',r))$ is $\omega_1$-full for all $s' \not\in s$. \qed

Claim 7. For every $(r,p) \in R \times P_2$, there is a $\delta$ so that $\Gamma_\delta(\emptyset,(r,p))$ is $\omega_1$-full.

Proof of Claim: Let $M_0$ be a countable elementary submodel of $H(\kappa^+)$ so that $\{\mathcal{U},(r,p),R\} \in M_0$. Choose any $p_1 <_E p$ (i.e. $p_1(0) = p(0)$ and
\[ p_1(0) \models p_1(1) < p(1) \] that is \((M_0, P_2)\)-generic. Notice that \((r, p_1)\) is therefore \((M, R \times P_2)\)-generic since \(R\) is ccc. Let \(\delta_0 = M_0 \cap \omega_1\). Choose any continuous \(\varepsilon\)-chain \(\{M_\alpha : 0 < \alpha < \omega_1\}\) of countable elementary submodels of \(H(\kappa^+)\) such that \(p_1 \in M_1\). For each \(\alpha \in \omega_1\), let \(\delta_\alpha = M_\alpha \cap \omega_1\). We did not actually have to choose \(p_1\) before choosing \(M_1\) of course. Let \(C\) be the cub \(\{\delta_\alpha : \alpha \in \omega_1\}\) and let \(p_2 \in P_2\) be a common extension of \(p_1\) and \((\emptyset, (\emptyset, \delta_0 \cup (C \setminus \delta_0)))\) (or equivalently \(p_2(0) \leq p_1(0)\) and \(p_2(0) \models p_2(1) \leq (\pi_0(p_1(1)), \pi_1(p_1(1)) \cap C)\)). It follows that \(p_2 \models C \setminus \delta_0 \subset C\).

Assume \(\Gamma_{\delta_0}(\emptyset, (r, p))\) is not \(\omega_1\)-full. Choose \(s_0 < r\) and \(\zeta_0 \in \omega_1\) as in Claim 5. By elementarity we may assume that \(s_0, \zeta_0\) are in \(M_1\).

Now choose any \(\bar{s}_0 < s_0\) so that there is a \(q_0 < p_1\) and an \(H \in [\omega_1 \setminus \delta_0]^n\) such that \((\bar{s}_0, q_0) \models H_{\delta_0} = H\). Of course this implies that \(\Gamma_{\delta_0}(H, (r, p))\) is not \(\omega_1\)-full. Let \(H\) be enumerated in increasing order \(\{\eta_i : 1 \leq i \leq n\}\).

Since \((\bar{s}_0, q) \models H_{\delta_0} \in Q(U, \bar{C}_1)\), we can assume that \(q\) has already determined the members of \(\bar{C}_1\) that separate the elements of \(\{\delta_0\} \cup H\). In other words, there is a set \(\{\alpha_i : 1 \leq i \leq n\} \subset \omega_1\) so that \(\{\delta_{\alpha_i} : 1 \leq i \leq n\} \subset \pi_0(q(1)) \subset C\) such that, for each \(1 \leq i < n\), \(\delta_0 \leq \delta_{\alpha_{i-1}} \leq \eta_i\). Therefore, \(\{\eta_j : 1 \leq j < i\} \in M_{\alpha_i}\) for all \(i < n\) and \(\Gamma_{\delta_0}(\{\eta_j : 1 \leq j \leq n\}, (r, p))\) is \(\omega_1\)-full. Clearly, for all \(s' < \bar{s}_0\), \(\Gamma_{\delta_0}(\{\eta_j : 1 \leq j \leq n\}, (s', p))\) is also \(\omega_1\)-full.

By the choice of \(s_0\) and \(\zeta_0\), we have that \(\Gamma_{\delta_0}(\{\eta_1\}, (s_0, p)) \in M_{\alpha_2}\) is not \(\omega_1\)-full. We note that \(\bar{s}_0\) is \((M_{\alpha_2}, R)\)-generic condition. There is therefore, by Claim 5 a \(\zeta_1 \in M_{\alpha_2}\) and a pair \(\bar{s}_1 < s_1\) so that \(s_1 \in M_{\alpha_2}\), \(\bar{s}_1 < s_0\) and \(\Gamma_{\delta_0}(\{\eta_1, \eta\}, (s_1, p))\) is not \(\omega_1\)-full for all \(\eta > \zeta_1\). Following this procedure we can recursively choose a pair of descending sequences \(\{s_i : 1 \leq i \leq n\} \subset R\) and \(\{\bar{s}_i : 1 \leq i \leq n\} \subset R\) so that

1. \(s_{i-1} \in M_{\alpha_i}\) and \(\bar{s}_i < s_i\),
2. \(\Gamma_{\delta_0}(\{\eta_1, \ldots, \eta_i\}, (s_i, p))\) is not \(\omega_1\)-full.

We now have a contradiction that completes the proof. We noted above that since \(\bar{s}_n < s_0\), \(\Gamma_{\delta_0}(\{\eta_1, \ldots, \eta_n\}, (\bar{s}_n, p))\) is \(\omega_1\)-full. However since \(\bar{s}_n < s_n\), this contradicts that \(\Gamma_{\delta_0}(\{\eta_1, \ldots, \eta_n\}, (s_n, p))\) is not \(\omega_1\)-full. \(\square\)

Now we complete the proof of the Proposition. Consider any countable elementary submodel \(M\) as in Claim 5 and let \(\delta = M \cap \omega_1\). Let \(p_1\) be a condition as in Lemma 2.13 applied to the case \(\alpha = 0\). Let \(G_R\) be any \(R\)-generic filter and let \(G_1 \subset C_{\omega_1}\) be any generic filter, which is generic over the model \(V[G_R]\). Pass to the extension \(V[G_R]\).
Fix any $F \in [\omega_1 \setminus \delta]^n$. It follows from Claim 5 and Claim 6 that the set $\mathcal{W}_F$ of those $(t, (\dot{b}, \dot{B})) \in M \cap (C_{\omega_1} \ast \dot{J})$ for which

$$\exists \xi \in \delta)(\exists s \in G_R) (s \Vdash H \cap \mathcal{W}_F = \emptyset \& (s, (t, (\dot{b}, \dot{B}))) \Vdash H = \dot{H}_\xi)$$

is a dense subset of $M \cap (C_{\omega_1} \ast \dot{J})$. The proof is that Claim 6 provides a potential $\xi \in M$ to strive for, and Claim 5 provides an $(s, q)$ to yield an element of $\mathcal{W}_F$.

It then follows easily that, in the extension $V[G_R \times G_1]$, the set

$$\text{val}_{G_1 \ast \delta}(\mathcal{W}_F) = \{\text{val}_{G_1}( (\dot{b}, \dot{B}) ) : (\exists t \in G_1)((t, (\dot{b}, \dot{B}))) \in \mathcal{W}_F \}$$

is a dense subset of $\text{val}_{G_1}(M \cap \dot{J})$ which is an element of $V[G_R \times (G_1 \upharpoonright \delta)]$. Since $p_1$ forces that the generic filter meets $\text{val}_{G_1 \ast \delta}(\mathcal{W}_F)$, this completes the proof.

For any $\alpha \leq \kappa$ and subset $I \subset \alpha$, we will say that a $P_\alpha$-name $\dot{E}$ is a $P_\alpha(I)$-name if it is a $P_\alpha(I)$-name in the usual recursive sense. This definition makes technical sense even if $P_\alpha(I)$ is not a complete subposet of $P_\alpha$.

**Corollary 3.4.** Let $\lambda \in E$ and let $\dot{R}_0$ be a $P_\lambda(I_\lambda)$-name that is forced by $P_\lambda$ to be ccc poset. Let $\dot{R}$ be a $P_\lambda$-name of a ccc poset such $1_{P_\lambda}$ forces that $\dot{R}_0 \subset_c \dot{R}$. Assume that $\mathcal{U} = \{\dot{U}_\xi : \xi \in \omega_1\}$ is a sequence of $P_\lambda(I_\lambda) \times \dot{R}_0$-names of subsets of $\omega_1$ such that $P_\lambda \ast \dot{R}$ forces that $\mathcal{U}$ is an $S$-space task. Then the $P_{\lambda + 2}$-name $Q(\mathcal{U}, \dot{C}_\lambda)$ satisfies that $P_{\lambda + 2}$ forces that $\dot{R} \times Q(\mathcal{U}, \dot{C}_\lambda)$ is ccc.

**Proof.** Let $G_\lambda$ be a $P_\lambda$-generic filter and pass to the extension $V[G_\lambda]$. Let $\dot{R} = \text{val}_{G_\lambda}(\dot{R})$ and observe that we may now regard $\mathcal{U}$ as a family of $R$-names of subsets of $\omega_1$ that is forced to be an $S$-space task. We would like to simply apply Lemma 3.3, but unfortunately, $P_{\lambda + 2}$ is not isomorphic to $P_\lambda \ast P_2$. Naturally the difference is that $\dot{Q}_{\lambda + 1}$ is a proper subset of $\dot{J}$. It will suffice to identify the three key places in the proof of Lemma 3.3 that depended on consequences of the properties of $\dot{J}$ and to verify that the consequences also hold for $\dot{Q}_{\lambda + 1}$. The first was in the proof of Claim 4 where we selected a condition $p_2(1) \in \dot{J}$ that satisfied that $\pi_1(p_2(1))$ was forced to be a subset of $C \cup \delta_0$ for the cub $C$. Since, in this proof, $C$ will be an cub set in the model $V[G_\lambda]$, it follows from condition (6) of Definition 2.2 this can be done. The next property of $P_2$ that we used was that Lemma 2.15 holds, but of course this also holds for $P_{\lambda + 2}$. The third is in the proof and statement of Claim 5. When choosing the pair $(s, q)$ in $\dot{R} \times P_2$ we require that it satisfies condition (2) in Claim 5. In the current situation, each $\dot{U}_\zeta$ is not simply
an $R$-name but rather it is a $P_\lambda(I_\lambda) \ast \bar{R}_0$-name. Therefore, there is a $P_\lambda(I_\lambda)$-name for a suitable $q$ so that $(s, q) \models H \cap \bigcup \{ \dot{U}_\zeta : \zeta \in F \}$ is empty. This causes no difficulty since $P_\lambda(I_\lambda)$-names for elements of $Q_{\lambda+1}$ are, in fact, elements of $Q_{\lambda+1}$. That is, a choice for $(s, q)$ in $R \times (\dot{Q}_\lambda \ast \dot{Q}_{\lambda+1})$ can be made in $V[G_\lambda]$ as required in Claim 5. \hfill \Box

4. BUILDING THE FINAL MODEL

In this section we present the construction of the iteration sequence of length $\kappa + \kappa$ extending that of Definition 2.2 that will be used to prove the main theorem.

We introduce more terminology.

Definition 4.1. Fix any $\mu \leq \lambda \leq \kappa$ and define $Q(\lambda, \mu)$ to be the set of all iterations $q$ of the form $\langle P_\alpha^q, \dot{Q}_\beta^q : \alpha \leq \lambda + \mu, \beta < \lambda + \mu \rangle \in H(\kappa^+)$ satisfying that

(1) $\langle P_\alpha^q, \dot{Q}_\beta^q : \alpha \leq \lambda, \beta < \lambda \rangle$ is our sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ from Section 2,

(2) for all $\lambda \leq \beta < \lambda + \mu$, $\dot{Q}_\beta^q \in H(\kappa)$ is a $P_\beta^q$-name of a ccc poset,

(3) for all $\alpha \leq \mu$ and $p \in P_\alpha^q$, $p \upharpoonright \lambda \in P_\lambda^q$ and $\text{dom}(p) \setminus \lambda$ is finite,

(4) if $\lambda < \kappa$, then $q \in H(\kappa)$.

For $q \in Q(\lambda, \mu)$, let $q(\kappa)$ denote the element of $Q(\kappa, \mu)$ where $\dot{Q}_{\kappa+\beta}^q = \dot{Q}_{\kappa+\beta}^q$ for all $\beta < \mu$.

Lemma 4.2. Let $\mu < \kappa$ and let $q \in Q(\kappa, \mu)$ and let $U = \{ \dot{U}_\xi : \xi \in \omega_1 \}$ be a sequence of $P_{\kappa+\mu}^q$-names. Assume that $P_{\kappa+\mu}^q$ forces that $U$ is an S-space task. Let $\bar{M}$ be an elementary submodel of $H(\kappa^+)$ of cardinality $\aleph_1$ that is closed under $\omega$-sequences and contains $\{ U, q \}$. Choose any $\lambda \in E \cap \kappa$ so that $\bar{M} \cap \kappa \subset I_\lambda$. Then $P_{\kappa+\mu}^q$ forces that $Q(U, C_\lambda)$ is ccc.

Proof. Since $\mu \in \bar{M}$, it follows that $\mu \leq \lambda$. Furthermore, by the assumptions on $q \in Q$ and $q \in \bar{M}$, it follows that there is a $\gamma \in \bar{M} \cap \kappa$ such that $\dot{Q}_\beta$ is a $P_\gamma$-name for all $\kappa \leq \beta < \kappa + \mu$. In addition, for each $\beta \in \bar{M} \cap \mu$, $\dot{Q}_\beta$ is a $P_\gamma(\bar{M} \cap \gamma)$-name. Since $\gamma < \lambda$, there is a $P_\lambda$-name, $\bar{R}$, of a finite support iteration of length $\mu$ such that $P_\kappa \ast \bar{R}$ is isomorphic to $P_{\kappa+\mu}^q$. More precisely, the $\beta$-th iterand for $\bar{R}$ is the name $\dot{Q}_{\kappa+\beta}$. Similarly, let $\bar{R}_0$ be the set of conditions in $\bar{R}$ with support contained in $\bar{M} \cap \mu$ and values taken in $\bar{M} \cap \dot{Q}_{\kappa+\beta}$ for each $\beta$ in the support. Then we have that $1_{P_\lambda} \models \bar{R}_0 \subset \bar{\epsilon} \bar{R}$. By minor re-naming, we may treat $U$ as a sequence of $P_\lambda(I_\lambda) \ast \bar{R}_0$-names. Since $P_{\kappa+\mu}^q$ forces that $U$ is an S-space task, it follows that $P_\lambda \ast \bar{R}$ also forces that $U$ is
an S-space task. By Corollary 3.4, $P_{\lambda+2}$ forces that $\dot{R} \times Q(\mathcal{U}, \dot{C}_\lambda)$ is ccc. By Lemma 2.16, $P_\kappa$ forces that $\dot{R} \times Q(\mathcal{U}, \dot{C}_\lambda)$ is ccc. Since $P^q_{\alpha+\mu}$ is isomorphic to $P_\kappa \times \dot{R}$, this completes the proof. □

**Theorem 4.3.** Let $\kappa > \aleph_2$ be a regular cardinal in a model of GCH. There is an iteration sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa + \kappa, \beta < \kappa + \kappa \rangle$ such that $P_{\kappa+\kappa}$ forces that there are no S-spaces and, for all $\mu < \kappa$, $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa + \mu, \beta < \kappa + \mu \rangle$ is in $Q(\kappa, \mu)$. It therefore follows that $P_{\kappa+\kappa}$ is cardinal preserving and forces that $\kappa^{<\kappa} = \kappa = \mathfrak{c}$.

The iteration can be chosen so that, in addition, Martin’s Axiom holds in the extension.

**Proof.** Fix a sequence $\mathcal{I} = \{ I_\gamma : \gamma \in \kappa \}$ as described in the construction of the sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$. Also let $Q(\lambda, \mu)$ for $\mu \leq \lambda < \kappa$ be defined as in Definition 4.1.

We introduce still more notation. For all $\alpha \leq \lambda < \kappa$, let $P^\lambda_\alpha$ simply denote $P_\alpha$ and $\dot{Q}^\lambda_\alpha = \dot{Q}_\alpha$. Also for any $\mu \leq \lambda < \kappa$ and sequence $q' = \langle \dot{Q}'_\beta : \beta < \mu \rangle \in H(\kappa)$, let $Q^\lambda_{\alpha+\beta}(q')$ denote $Q'_\beta$. By recursion on $\alpha < \mu$, let $P^\lambda_{\alpha+\mu}(q')$ denote the limit of the iteration sequence $\langle P^\lambda_{\alpha}(q'), \dot{Q}^\lambda_{\alpha}(q') : \zeta < \alpha, \beta < \alpha \rangle$ so long as this sequence (and its limit) is in $Q(\lambda, \alpha)$. Say that a sequence $q' = \langle \dot{Q}'_\beta : \beta < \lambda \rangle \in H(\kappa)$ is suitable if for all $\alpha \in E \cap \lambda+1$, $\langle P^\lambda_{\alpha}(q'), \dot{Q}^\lambda_{\alpha}(q') : \zeta \leq \alpha, \beta < \alpha \rangle$ is in $Q(\lambda, \alpha)$. We state for reference two properties of suitable sequences.

**Fact 1.** If $\lambda$ is a limit ordinal, then $\langle \dot{Q}'_\beta : \beta < \lambda \rangle \in H(\kappa)$ is suitable so long as $\langle \dot{Q}'_\beta : \beta < \mu \rangle$ is suitable for all $\mu < \lambda$.

**Fact 2.** If $q' = \langle \dot{Q}'_\beta : \beta < \lambda \rangle \in H(\kappa)$ is suitable, then $\langle \dot{Q}'_\beta : \beta < \lambda + 1 \rangle$ is suitable for any $P^\lambda_{\lambda+\lambda}(q')$-name $\dot{Q}'_\lambda$ of a ccc poset of cardinality at most $\aleph_1$.

Now that we have this cumbersome, but necessary, notation out of the way, the proof of the theorem is a routine consequence of the prior results. Let $\sqsubset$ be a well ordering of $H(\kappa)$ in type $\kappa$. We recursively define a sequence $\langle \dot{Q}'_\beta : \beta < \kappa \rangle$ and a 1-to-1 sequence $\langle U_\beta : \beta < \kappa \rangle$.

One inductive assumption is that every initial segment of $\langle \dot{Q}'_\beta : \beta < \kappa \rangle$ is a suitable sequence. The list $\{ U_\beta : \beta < \kappa \}$ will contain the list the potential S-space tasks as we deal with them.

Let $\lambda < \kappa$ and assume that $\langle \dot{Q}'_\beta, U_\beta : \beta < \lambda \rangle \in H(\kappa)$ has been chosen. If $\lambda \notin E$, then $\dot{Q}'_\lambda$ is the trivial poset and $U_\lambda = \lambda$. Now let $\lambda \in E$ and let $q' = \langle \dot{Q}'_\beta : \beta < \lambda \rangle$. Consider the set of all $P^\lambda_{\lambda+\lambda}(q')$-names $U = \{ \dot{U}_\xi : \xi \in \omega_1 \}$ that are forced to be S-space tasks. Consider only those $U$ for
which there is an elementary submodel $\bar{M}$ of $H(\kappa^+)$ as in Lemma 4.2. More specifically, such that $\bar{M} \cap \lambda \subset \mathcal{I}_\lambda$, $\{\mathcal{U}, P^{\kappa+1}(\mathcal{q})\} \in \bar{M}$, $|\bar{M}| = \aleph_1$, and $\bar{M}^\omega \subset \bar{M}$. The final requirement of such $\mathcal{U}$ is that they are not in the set $\langle \mathcal{U}_\beta : \beta < \lambda \rangle$. If any such $\mathcal{U}$ exist, then let $\mathcal{U}_\lambda$ be the $\subseteq$-minimal one. Loosely, $\mathcal{U}_\lambda$ is the $\subseteq$-minimal $S$-space task that has not yet been handled and can be handled at this stage. Otherwise, let $\mathcal{U}_\lambda = \lambda$ (so as to preserve the 1-to-1 property). Now we choose $\hat{Q}'_\lambda$. If $\mathcal{U}_\lambda = \lambda$, then $\hat{Q}'_\lambda$ is the trivial poset. Otherwise, of course, $\hat{Q}'_\lambda$ is the $P^{\lambda+2}(\mathcal{q})$-name for $Q(\mathcal{U}_\lambda, \hat{C}_\lambda)$. By Lemma 4.2 and Fact 2, $\langle \hat{Q}'_\beta : \beta \leq \lambda \rangle$ is suitable.

This completes the recursive construction of the suitable sequence $\mathcal{q}' = \langle \hat{Q}'_\beta : \beta < \kappa \rangle$ and the listing $\langle \mathcal{U}_\beta : \beta < \kappa \rangle$. It remains only to prove that if $\mathcal{U} = \{\hat{U}_\xi : \xi \in \omega_1\}$ is a $P^{\kappa+1}(\mathcal{q})$-name of an $S$-space task, then there is an $\alpha < \kappa$ such that $\mathcal{U} = \mathcal{U}_\alpha$. Fix any such $\mathcal{U}$ and elementary submodel $\bar{M} \prec H(\kappa^+)$ such that $\{\mathcal{U}, P^{\kappa+1}(\mathcal{q})\} \in \bar{M}$, $|\bar{M}| = \aleph_1$, and $\bar{M}^\omega \subset \bar{M}$. Let $\Lambda$ be the set of $\lambda \in \kappa$ such that $\bar{M} \cap \kappa \subset \mathcal{I}_\lambda$. Let $\gamma$ be the order type of the set of predecessors of $\mathcal{U}$ in the well ordering $\subseteq$. Choose any $\lambda \in \Lambda$ such that the order type of $\Lambda \cap \lambda$ is greater than $\gamma$. Note that $\Lambda \subset E$. For every $\mu \in \Lambda \cap \lambda$, $\mathcal{U}$ would have been an appropriate choice for $\mathcal{U}_\mu$ and if not chosen, then $\mu \neq \mathcal{U}_\mu \subset \mathcal{U}$. Since the sequence is 1-to-1, there is therefore a $\mu \in \Lambda \cap \lambda$ such that $\mathcal{U} = \mathcal{U}_\mu$.

It should be clear that we can ensure that Martin’s Axiom holds in the extension by making minor adjustments to the choice of $\hat{Q}'_\beta$ for $\beta \notin \mathcal{E}$ in the sequence $\langle \hat{Q}'_\beta : \beta < \kappa \rangle$ together with some additional bookkeeping.

5. **Moore-Mrowka tasks**

The Moore-Mrowka problem asks if every compact space of countable tightness is sequential. A space has countable tightness if the closure of a set is equal to the union of the closures of all its countable subsets. A space is sequential providing that each subset is closed so long as it contains the limits of all its converging (countable) subsequences. To illustrate that a sequential space has countable tightness, note that a space has countable tightness if a set is closed so long as it contains the closures of all of its countable subsets. Say that a compact non-sequential space of countable tightness is a Moore-Mrowka space.

Results on the Moore-Mrowka problem have closely resembled those of the S-space problem. In particular, there are proofs that PFA implies there are no Moore-Mrowka spaces that have many similarities to the proof that PFA implies there are no S-spaces. While it is independent with CH as to whether Moore-Mrowka spaces exist [5], it is known that
♦ implies there are (Cohen indestructible) Moore-Mrowka spaces of cardinality $\aleph_1$ [14]. In addition, ♦ implies there is a separable compact space of countable tightness with cardinality $2^{\aleph_1}$ (greater than $c$) [8]. It is also known that the addition of $\aleph_2$ Cohen reals over a model of ♦ + $\aleph_2 < 2^{\aleph_1}$ results in a model in which there is a compact separable space of countable tightness that has cardinality greater than $c$ [6]. Of course these spaces are Moore-Mrowka spaces since every separable sequential space has cardinality at most $c$.

Here are two open problems and a third that we solve in the affirmative in this section.

**Question 5.1.** Is it consistent with $c > \aleph_2$ that every compact space of countable tightness is sequential?

**Question 5.2.** Is it consistent with $p > \aleph_2$ that there is a Moore-Mrowka space?

**Question 5.3.** Is it consistent with $c > \aleph_2$ that every separable Moore-Mrowka space has cardinality at most $c$?

A Moore-Mrowka task mentioned in the title of the section is similar to an S-space task. The difference will be that rather than using the poset $Q(U, C)$ to force an uncountable discrete subset, we will hope to force an uncountable (algebraic) free sequence. We define these notions and indicate their relevance.

**Definition 5.1.** A sequence $\{x_\alpha : \alpha \in \omega_1\}$ is a free sequence in a space $X$ if, for every $\delta < \omega_1$, the initial segment $\{x_\alpha : \alpha \in \delta\}$ and the final segment $\{x_\beta : \beta \in \omega_1 \setminus \delta\}$ have disjoint closures.

A sequence $\{x_\alpha, U_\alpha, W_\alpha : \alpha \in \omega_1\}$ is an algebraic free sequence in a space $X$ providing

1. $x_\alpha \in U_\alpha$ and $W_\alpha$ are open sets with $\overline{U_\alpha} \subset W_\alpha$,
2. For every $\alpha < \delta \in \omega_1$, $x_\delta \notin W_\alpha$ and there is a finite $H \subset \delta + 1$ such that $\{x_\eta : \eta \leq \delta\} \subset \bigcup \{U_\beta : \beta \in H\}$.

Free sequences were introduced by Arhangelskii. Algebraic free sequences were introduced by Todorcevic in a slightly different formulation. The advantage of an algebraic free sequence is that the only reference to the (second order) closure property is with the pairs $U_\alpha, W_\alpha$. If $\{x_\alpha, U_\alpha, W_\alpha : \alpha \in \omega_1\}$ is an algebraic free sequence, then the set $\{x_{\alpha+1} : \alpha < \omega_1\}$ is a free sequence. This follows from the fact that for all $\delta \in \omega_1$, there is a finite $H \subset \delta + 1$ satisfying that $\{x_\alpha : \alpha \leq \delta\} \subset U_H = \bigcup \{U_\alpha : \alpha \in H\}$ and $\{x_\beta : \delta < \beta \in \omega_1\}$ is disjoint from $W_H = \bigcup \{W_\alpha : \alpha \in H\}$. The free sequence property now follows from
the fact that $U_H$ and $X \setminus W_H$ have disjoint closures. This was crucial in Balogh’s proof \cite{Balogh} that PFA implies there are no Moore-Mrowka spaces.

**Proposition 5.2** (\cite{Balogh}). A compact space has countable tightness if and only if it contains no uncountable free sequence.

**Definition 5.3.** A sequence $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ is a Moore-Mrowka task if, for all $\alpha \in \omega_1$, $\alpha \in A_\alpha \subset \alpha + 1$, and

1. for all $\beta < \alpha$ there is a $\gamma$ such that $A_\gamma \cap \{\beta, \alpha\} = \{\alpha\}$, and
2. for all uncountable $A \subset \omega_1$, there is a $\delta \in \omega_1$ such that for all $\beta \in \omega_1 \setminus \delta$, $(A \cap \delta) \cap \bigcap_{\gamma \in H} A_\gamma$ is not empty for all finite $H \subset \{\gamma : \beta \in A_\gamma\}$.

The idea behind a Moore-Mrowka task is that we identify $\omega_1$ with a set of points in space $X$ and so that there is a collection $\{U_\alpha, W_\alpha : \alpha \in \omega_1\}$ that is a neighborhood assignment for those points. For each $\alpha$, $\overline{U_\alpha} \subset W_\alpha$ and $W_\alpha \cap \omega_1$ is also contained in $\alpha + 1$. Then we set $A_\alpha = U_\alpha \cap \omega_1$. Condition (1) is trivial to arrange but condition (2) is a $\diamondsuit$-like condition. A distinction with S-space task is that the non-existence of a Moore-Mrowka task extracted from a compact space of countable tightness does not imply that the space is sequential. The similarity with S-space task is that we will use a Moore-Mrowka task to generically introduce an algebraic free sequence.

**Definition 5.4.** Let $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ be a Moore-Mrowka task and let $C \subset \omega_1$ be a cub. The poset $\mathcal{M}(\mathcal{A}, C)$ is the set of finite subsets of $\omega_1 \setminus \min(C)$ that are separated by $C$. For each $H \in \mathcal{M}(\mathcal{A}, C)$ and each $\beta \in H$, let $A(H, \beta)$ be the intersection of the family $\{A_\gamma : \gamma \in H, \beta \in A_\gamma\}$. We define $H < K$ from $\mathcal{M}(\mathcal{A}, C)$ providing $H \supset K$ and for each $\alpha \in H \cap \max(K)$, $\alpha \in A(K, \min(K \setminus \alpha))$.

**Lemma 5.5.** Let $\lambda \in E$ and let $\dot{R}_0$ be a $P_\lambda(I_\lambda)$-name that is forced by $P_\lambda$ to be ccc poset. Let $\dot{R}$ be a $P_\lambda$-name of a ccc poset such $1_{P_\lambda}$ forces that $\dot{R}_0 \subset_c \dot{R}$. Assume that $\mathcal{A} = \{\dot{A}_\xi : \xi \in \omega_1\}$ is a sequence of $P_\lambda(I_\lambda) \times \dot{R}_0$-names of subsets of $\omega_1$ such that $P_\lambda \times \dot{R}$ forces that $\mathcal{A}$ is a Moore-Mrowka task. Then the $P_{\lambda+2}$-name $\mathcal{M}(\mathcal{U}, \dot{C}_\lambda)$ satisfies that $P_{\lambda+2}$ forces that $\dot{R} \times \mathcal{M}(\mathcal{U}, \dot{C}_\lambda)$ is ccc.

**Proof.** The proof proceeds much as it did in Lemma 3.3 and Corollary 3.4 for S-space tasks. To show that a poset of the form $\mathcal{M}(\mathcal{A}, C)$ is ccc, it again suffices to prove that, for each $n \in \omega$, there is no uncountable antichain consisting of pairwise disjoint sets of cardinality $n$. So we consider an arbitrary family of pairwise disjoint sets of cardinality $n$. Fix $P_{\lambda+2} \times \dot{R}$-names $\{\dot{H}_\xi : \xi \in \omega_1\}$ for a set of pairwise disjoint elements...
of $\mathcal{M}(\mathcal{A}, \dot{\mathcal{C}}_\lambda) \cap [\omega_1]^n$. Following Lemma 3.3, we may assume that, for
each $\xi \in \omega_1$, it is forced that $\xi < \min(\bar{H}_\xi)$ and that $\{\xi\} \cup \bar{H}_\xi$ is also
separated by $\dot{\mathcal{C}}_\lambda$. We prove that no condition forces this to be an
antichain.

Let $M$ be a countable elementary submodel containing all the above
and let $p_1 \in P_{\lambda+2}$ be chosen as in Lemma 2.15 so that $p_1$ is $(M, P_{\lambda+2})$-
generic and so that $p_1(\lambda) \in M$. Let $p_1 \upharpoonright \lambda \in G_\lambda$ be a $P_\lambda$-generic filter
and pass to the extension $V[G_\lambda]$. Let $R = \text{val}_{G_\lambda}(\hat{R})$ and let $G_1 \subset C_{\omega_1}$
so that $p_1 \upharpoonright \lambda + 1 \in G_\lambda \ast G_1$ is $P_{\lambda+1}$-generic. Let $\delta = M \cap \omega_1$. We
will prove that $p_1$ forces that $\bar{H}_\delta$ is compatible with some element of
$\{\bar{H}_\eta : \eta \in \delta\}$.

For each $\zeta \in \omega_1$, let, in $V[G_\lambda]$, $\dot{J}_\xi$ denote the $R$-name for the set
$\{\gamma : \zeta \in \dot{A}_\gamma\}$ and, for each finite $F \subset \omega_1$, also let $\dot{A}_F$ denote the $R$-
name for $\bigcap_{\eta \in F} \dot{A}_\eta$. We leave the reader to check that it suffices to prove
that $p_1$ forces that for each finite $F \subset \dot{J}_{\min(\bar{H}_\delta)}$, there is an $\eta < \delta$ such
that $\bar{H}_\eta \subset \dot{A}_F$. For each $\zeta \in \omega_1$ and finite $F \subset \omega_1$, we will let $J_\zeta$ and $A_F$
declare the $\text{val}_{G_\lambda}(\dot{J}_\xi)$ and $\text{val}_{G_\lambda}(\dot{A}_F)$ respectively. Also, for the remainder
of the proof we will treat each $\bar{H}_\xi$ as the canonical $R \times (Q_\lambda \ast \dot{Q}_{\lambda+1})$-
name obtained from the evaluation of the original $P_{\lambda+2} \ast \hat{R}$-name by
$G_\lambda$. For each $\xi \in \omega_1$ and $H \in [\omega_1]^n$, let $\Gamma_\xi(H)$ be the (possibly empty)
set of conditions in $R \times (Q_\lambda \ast \dot{Q}_{\lambda+1})$ that force $H$ to equal $\bar{H}_\xi$.

We need an updated version of $\omega_1$-full. Say that a countable set $B$
in $V[G_\lambda][G_R]$, is $\mathcal{A}$-large if there is a $\gamma \in \omega_1$ such that $B \cap \dot{A}_F \neq \emptyset$
for all $\beta \in \omega_1 \setminus \gamma$ and finite $F \in J_\beta$. We may interpret this as that $\mathcal{B}$
contains $\omega_1 \setminus \gamma$.

For $\xi \in \omega_1$ and $(r, p) \in R \times (Q_\lambda \ast \dot{Q}_{\lambda+1})$, let $\Gamma_\xi(H, (r, p))$ be the set of
conditions in $\Gamma_\xi(H)$ that are below $(r, p)$. In other words, $\Gamma_\xi(H, (r, p))$
is not empty if and only if $(r, p) \not\Vdash H \neq \bar{H}_\xi$. Similarly, for each $0 < i < n$
and $H \in [\omega_1]^n$, let $\Gamma_\xi(H, (r, p)) = \bigcup\{\Gamma_\xi(H \cup \{\eta\}, (r, p)) : \eta \in \omega_1\}.$
For $H \in [\omega_1]^n$, say that $\Gamma_\xi(H, (r, p))$ is full if $\Gamma_\xi(H, (r, p))$ is not empty
for all $\bar{r} \leq r$. For $0 < i < n$ and $H \in [\omega_1]^{n-i}$, say that $\Gamma_\xi(H, (r, p))$ is
full if there is a $R$-name $\hat{B}$ that is forced to be an $\mathcal{A}$-large set of $\eta \in \omega_1$
and, for each $\eta$ and $s \Vdash \eta \in \hat{B}$, $\Gamma_\xi(H \cup \{\eta\}, (s, p))$ is full.

Claim 8. Suppose that $\xi, r, p \in M[G_\lambda]$ and that $\Gamma_\xi(\emptyset, (r, p))$ is full.
Suppose also that $\bar{r} \in R$ forces that $F$ is a finite subset of $\dot{J}_\xi$ for some
$\delta \leq \xi \in \omega_1$. Then there are $(s, q), H \in M[G_\lambda]$ and $\bar{s} < \bar{r}$ such that

1. $(s, q) < (r, p)$ in $R \times (Q_\lambda \ast \dot{Q}_{\lambda+1})$, 
2. $\bar{s} < s$, 
3. $\bar{s} \Vdash H \subset \dot{A}_F$, 


(4) \((s, q) \Vdash \dot{H}_\xi = H\).

**Proof of Claim:** There is an \(R \times Q_\lambda\)-name \(\dot{B}_0 \in M[G_\lambda]\) that is forced to be an \(\mathcal{A}\)-large subset of \(\delta\) and witnesses that \(\Gamma_\xi(\emptyset, (r, p))\) is full. Therefore there are \(\eta < \delta\) and \(r' < \bar{r}\) such that \(\bar{r}_1 \Vdash \eta \in \dot{B}_0 \cap A_F\). There is no loss to assuming, by elementarity, that \(\bar{r}_1\) extends some \(r_1 \in M[G_\lambda]\) such that \(r_1 \Vdash \eta \in \dot{B}_0\). Since \(r_1 \Vdash \eta \in \dot{B}_0\), we have that \(\Gamma_\xi(\{\eta\}, (r_1, p))\) is full. Following a recursion of length \(n\), there is an \(\bar{r}_n < \bar{r}\) in \(R\), an \(H = \{\eta_1, \ldots, \eta_n\} \in M[G_\lambda]\), and an \(\bar{r}_n < r_n \in M[G_\lambda]\) such that \(\bar{r}_n \Vdash H \subseteq A_F\) and \(\Gamma_\xi(H, (r_n, p))\) is full. Since \(\bar{r}_n < r_n\), \(\Gamma_\xi(H, (r_n, p))\) is not empty. Therefore there is a pair \((\bar{s}, \bar{q}) < (r, p)\) forcing that \(H = H_\xi\). By elementarity, since \(\xi, H, p \in M[G_\lambda]\), the set of \(\{s \in R \cap M : (\exists q)((s, q) < (r, p) \& (s, q) \Vdash H = H_\xi)\}\) is predense below \(r_n\). Therefore there is an \((s, q) < (r, p) \in M[G_\lambda]\) with \(s \notin \bar{r}_n\) such that \((s, q) \Vdash H = H_\xi\). Let \(\bar{s}\) be any extension of \(s, \bar{r}_n\). \(\Box\)

**Claim 9.** For every \((r, p) \in R \times (Q_\lambda \star \dot{Q}_{\lambda + 1})\), there is a \(\delta\) so and a \(r_0 < r\) such that \(\Gamma_\delta(\emptyset, (r_0, p))\) is full.

**Proof of Claim:** Let \((r, p) \in M_0\) be a countable elementary submodel of \(H(\kappa^+)\)[\(G_\lambda\)] so that \(\{\mathcal{A}, R, P_{\lambda + 2}\} \in M_0\). Choose any \((\bar{r}, \bar{p}) < (r, p)\) that is an \((M_0, R \times (Q_\lambda \star \dot{Q}_{\lambda + 1}))\)-generic condition. Let \(\delta_0 = M_0 \cap \omega_1\). Choose any continuous \(\varepsilon\)-chain \(\{M_\alpha : 0 < \alpha < \omega_1\}\) of countable elementary submodels of \(H(\kappa^+)\)[\(G_\lambda\)] such that \(\{M_0, (\bar{r}, \bar{p})\} \in M_1\).

For each \(\alpha \in \omega_1\), let \(\delta_\alpha = M_\alpha \cap \omega_1\). Let \(C^*\) be the cub \(\{\delta_\alpha : \alpha \in \omega_1\}\). Choose any extension \((r_n, p_n)\) of \((\bar{r}, \bar{p})\) such that \(\pi_1(p_2(\lambda + 1)) \subseteq C^* \cup \delta_0\), and so that there is an \(H = \{\xi_1, \ldots, \xi_n\} \in [\omega_1]^n\) with \((r_n, p_n) \Vdash H = H_0\). Of course this implies that \(\Gamma_{\delta_0}(H, (r_n, p)) \supseteq \Gamma_{\delta_0}(H, (r_n, p_n))\) is actually full. Okay, then \(H_{n-1} = \{\xi_1, \ldots, \xi_{n-1}\}\) is in \(M_{\alpha_n}\). Let’s take the \(R\)-name \(\dot{E}_{n-1}\) to the set of \(\{\eta, \bar{r}\}\) such that \(\Gamma_{\delta_0}(\{\eta\} \cup H_{n-1}, (\bar{r}, p))\) is full. The condition \(r_n\) forces that \(\dot{E}_{n-1}\) is uncountable. Since \(\mathcal{A}\) is a Moore-Mrowka task in \(V[G_\lambda \star G_R]\), \(r_n\) forces that \(\dot{E}_{n-1} \in M_{\alpha_n}\) contains an \(\mathcal{A}\)-large set. By elementarity and the fact that \(r_n\) is \((M_{\alpha_n}, R)\)-generic, there is an \(r_{n-1}\) in \(M_{\alpha_n}\) that forces \(\dot{E}_{n-1}\) contains an \(\mathcal{A}\)-large set. Therefore, for such an \(r_{n-1} \in M_{\alpha_n}\), we have that \(\Gamma_{\delta_0}(H_{n-1}, (r_{n-1}, p))\) is full. This recursion continues as above and for each \(i < n\), there is an \(r_i \in M_{\alpha_i}\) such that \(\Gamma_{\delta_0}(\{\xi_j : j < i\}, (r_i, p))\) is full. Setting \(\delta = \delta_0\), this completes the proof of the Claim. \(\Box\)

Following the proof of Corollary 3.4 we can complete the proof using that \(p_1\) satisfied the conclusion of Lemma 2.15. Using Claim 9 it follows from Claim 8 that in \(V[G_\lambda][G_R]\), for each \(\dot{\delta} \leq \xi \in \omega_1\) and finite \(F \subset J_\xi\), the set \(W_F\) consisting of those \(p \in M[G_\lambda] \cap (Q_\lambda \star \dot{Q}_{\lambda + 1})\) for which there
is a \( s \in G_R \) and \( \xi \in \delta \) such that \((\bar{s}, p) \models \hat{H}_\xi \subset A_F\), is a dense subset of \( M[\lambda] \cap (Q_\lambda \ast Q_{\lambda+1})\). By the genericity of \(((G_1) \upharpoonright \delta) \ast (p^1_\lambda)\) over the model \( V[G_1 \ast R]\) as in Lemma 2.15 it meets \( \mathcal{W}_F\). It follows that \( p_1 \) forces that there is a \( \xi \in \delta \) such that \( \hat{H}_\xi \subset A_F\). Applying this fact to \( \zeta = \min(H_\delta) \) completes the proof. \( \square \)

Now we show that Moore-Mrowka tasks will arise that will allow us to prove there is a minor additional condition that we can place on the construction of \( P_\kappa^\kappa \) (assuming an extra \( \lozenge \)-principle) that will force there are no separable Moore-Mrowka spaces of cardinality greater than \( c \). Let \( S^\kappa_1 \) denote the set of \( \lambda \in \kappa \) that have cofinality \( \omega_1 \). We will assume there is a \( \lozenge (S^\kappa_1) \)-sequence.

We begin with this Lemma.

**Lemma 5.6** \((c^c = c)\). Let \( X \) be a separable Moore-Mrowka space of cardinality greater than \( c \). Let \( X \in M \) be an elementary submodel of \( H(\theta) \) for some sufficiently large \( \theta \) such that \(|M| = c\) and \( M^\mu \subset M \) for all \( \mu < c \). For any point \( z \in X \setminus X \) there is a sequence \( \{B_\eta : \eta < c\} \) of countable subsets of \( M \cap X \) satisfying, for all \( \eta < \zeta < c \),

1. \( \overline{B_\eta} \) contains \( B_\zeta \cup \{z\} \)
2. for all \( A \subset M \cap X \) with \( z \in \overline{A} \), there is an \( \alpha < c \) such that \( \overline{A} \) contains \( B_\alpha \).

**Proof.** Since \( X \) is separable, we can let \( B_0 \in M \) be any countable dense subset. Fix an enumeration \( \{S_\zeta : \zeta < c\} \) of all the countable subsets of \( M \cap X \) that have \( z \) in their closure. Let \( \mathcal{W} \in M \) be a base for the topology. Assume we have chosen \( \{B_\xi : \xi < \eta\} \) for some \( \eta < c \). Assume, by induction, that \( B_\xi \) is also a subset of \( S_\xi \). The set \( S_\eta \cup \{\overline{B_\xi} : \xi < \eta\} \) is an element of \( M \) and every member contains \( z \). Let \( K_\eta \) denote the intersection of this family. Choose any neighborhood \( U \in \mathcal{W} \) of \( z \). Since \( z \in W \cap K_\eta \), it follows from elementarity that \( M \cap W \cap K_\eta \) is non-empty. Therefore, \( z \) is in the closure of some countable \( B_\eta \subset M \cap K_\eta \). This completes the inductive construction of the family. We simply have to verify that property (2) holds. Let \( z \in \overline{A} \) for some \( A \subset M \cap X \). By countable tightness, there is an \( \eta \) such that \( S_\eta \subset A \). Therefore \( \overline{A} \supset B_\eta \). \( \square \)

**Remark 10.** A compact separable space of cardinality at most \( c \) will have a \( G_\delta \)-dense set of points of character less than \( c \). Therefore, in a model with \( p = c \), any such space has the property that the sequential closure of any subset is countably compact. In particular, in such a model a Moore-Mrowka space necessarily has weight at least \( c \) and will have a countably compact subset that is not closed. A space is said to be C-closed if it has no such subspace, see [7][10].
Definition 5.7. Say that a sequence \( \langle y_\alpha, U_\alpha, W_\alpha : \alpha < \kappa \rangle \) is a \( \kappa \)-MM sequence of a space \( X \) if

1. \( U_\alpha, W_\alpha \) are open in \( X \) and \( y_\alpha \in U_\alpha \subset \overline{U_\alpha} \subset W_\alpha \),
2. \( y_\alpha \notin U_\alpha \) for all \( \alpha < \gamma < \kappa \),
3. for all \( \beta < \alpha < \kappa \), \( U_\gamma \cap \{ y_\beta, y_\alpha \} = \{ y_\alpha \} \) for some \( \alpha \leq \gamma < \kappa \),
4. for every \( A \subset \kappa \), there is a countable \( B \subset A \) and a \( \gamma < \kappa \) such that the closure of \( \{ y_\alpha : \gamma < \alpha < \kappa \} \) is either contained in the closure of \( \{ y_\beta : \beta \in B \} \) or is disjoint from the closure of \( \{ y_\alpha : \alpha \in A \} \).

Theorem 5.8. Let \( \langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa + \kappa, \beta < \kappa + \kappa \rangle \) be an iteration sequence in the sense of Theorem 4.3. In particular, assume that for all \( \mu < \kappa \) there is a \( q_\mu \in Q(\mu, \mu) \) satisfying that \( P_{\kappa+\lambda} \) is equal to \( P_{\kappa+\mu}^{q_\mu(\kappa)} \).

Let \( \dot{X} \) be a \( P_{\kappa+\kappa} \)-name of a compact separable space of countable tightness. Assume also that \( \langle \dot{y}_\alpha, \dot{U}_\alpha, \dot{W}_\alpha : \alpha < \kappa \rangle \) is forced to be a \( \kappa \)-MM sequence of \( \dot{X} \). Then there is a cub \( C_\dot{X} \subset \kappa \) such that for each \( \lambda \in C_\dot{X} \cap S^\kappa_1 \), there is an injection \( f_\lambda : \omega_1 \to \lambda \) such that \( A = \langle \dot{A}_\eta : \eta < \omega_1 \rangle \), where \( \dot{A}_\eta = \{ \xi : y_{f_\lambda(\xi)} \in \dot{U}_{f_\lambda(\eta)} \} \), is forced by \( P_{\lambda+\lambda}^{q_\lambda} \) to be a Moore-Mrowka task.

Proof. We may assume, since it is forced to be compact and separable, that \( \dot{X} \) is a \( P_{\kappa+\kappa} \)-name of a closed subspace of \( [0,1]^\kappa \). Let \( G \) be a \( P_{\kappa+\kappa} \)-generic filter so that we may make some observations about \( \dot{X} \) and the \( \kappa \)-MM sequence \( \langle \dot{y}_\alpha, \dot{U}_\alpha, \dot{W}_\alpha : \alpha < \kappa \rangle \). There is a point \( z \in \text{val}_G(\dot{X}) \) that is a \( \kappa \)-accumulation point of \( \{ y_\alpha : \alpha < \kappa \} \). We check that \( z \) is the unique such point. If \( U, W \) are open neighborhoods of \( z \) with \( \overline{U} \subset W \), then \( A = \{ \alpha < \kappa : y_\alpha \in U \} \) is cofinal in \( \kappa \). By condition (4) of the \( \kappa \)-MM property, there is a countable \( B \subset A \) so that the closure of \( \{ y_\beta : \beta \in B \} \) contains \( \{ y_\alpha : \text{sup}(B) < \alpha < \kappa \} \). It thus follows that that \( \{ y_\alpha : \text{sup}(B) < \alpha < \kappa \} \) is contained in \( W \) and shows that \( X \setminus W \) contains no \( \kappa \)-accumulation points of \( \{ y_\alpha : \alpha < \kappa \} \). Now assume that \( z \) is in the closure of \( \{ y_\beta : \beta \in A \} \) for some \( A \subset \kappa \). Since the second clause of condition (4) of the \( \kappa \)-MM property fails, it follows that there is a countable \( B \subset A \) such that the closure of \( \{ y_\beta : \beta \in B \} \) contains a final segment of \( \{ y_\alpha : \alpha < \kappa \} \). We will be interested in the subspace \( X_\lambda = \{ x \upharpoonright \lambda : x \in X \} \) of \( [0,1]^\lambda \). Since this space is a continuous image of \( X \), it also has countable tightness. Let \( \dot{z} \) be a canonical \( P_{\kappa+\kappa} \)-name for \( z \).

Let \( M \prec H(\kappa^+) \) so that \( \text{sup}(M \cap \kappa) = \lambda \in S^\kappa_1 \) and \( M^\omega \subset M \). We note that it follows from Corollary 2.14 and the fact that \( P_{\kappa+\kappa}/P_\kappa \) is ccc, that every countable subset of \( M \cap \kappa \) in \( V[G] \) has a name in \( M \). Assume also that \( \dot{z}, \dot{X}, P_{\kappa+\kappa} \) and the \( \kappa \)-MM sequence are elements of
Choose any continuous $\varepsilon$-increasing sequence $\{M_\eta : \eta \in \omega_1\}$ of countable elementary submodels of $M$ such that $Y_\lambda = \bigcup\{M_\eta \cap \lambda : \eta \in \omega_1\}$ is cofinal in $\lambda$. Define $f_\lambda$ so that $f_\lambda(\eta) = \sup(M_\eta \cap \lambda)$. It should be clear that to show that $\mathcal{A}$, as in the statement of the Theorem, is forced by $P^{\mathcal{A}_\lambda}_{\lambda+\lambda}$ to be a Moore-Mrowka task it is sufficient to check that condition (2) of Definition 5.3 is forced to hold. Let $\hat{A}$ be any $P^{\mathcal{A}_\lambda}_{\lambda+\lambda}$-name of an uncountable subset of $\omega_1$. We may regard $P^{\mathcal{A}_\lambda}_{\lambda+\lambda}$ as a complete subposet of $P_{\kappa+\kappa}$ and so consider $\text{val}_G(\hat{A})$ in $V[G]$. In the space $X_\lambda$, it is clear that $z \upharpoonright \lambda$ is in the closure of the set $\{y_{f_\lambda(\eta)} : \eta \in A\}$. Therefore, there is a countable $B \subset A$ such that $z \upharpoonright \lambda$ is in the closure of the set $\bar{y}(f_\lambda(B)) = \{y_{f_\lambda(\eta)} : \eta \in B\}$. Now $B$ is a countable subset of $M \cap \lambda$, and so there is a $P^{\mathcal{A}_\lambda}_{\lambda+\lambda}$-name $\hat{B}$ in $M$ such that $\text{val}_G(\hat{B})$ is $B$. Now we can apply elementarity (using that $f_\lambda \upharpoonright B \in M$) and observe that $\hat{z}$ is forced to be in the closure of $\{\bar{y}_{f_\lambda(\beta)} : \beta \in \hat{B}\}$. Moreover, by elementarity and the $\kappa$-MM property, there is a $\gamma < \kappa \cap M$ such that the closure of $\bar{y}(f_\lambda(\hat{B}))$ is forced to contain $\{\bar{y}_{\alpha} : \gamma < \alpha < \kappa\}$. For each $\gamma < \alpha < \kappa$, $\bar{y}(f_\lambda(\hat{B}))$ is forced to meet $\bigcap_{\varsigma \in H} \hat{U}_\varsigma$ for all finite $H \subset \{\varsigma : \alpha \in \hat{U}_\varsigma\}$. Of course there is an $\delta \in \omega_1$ such that $\gamma < f_\lambda(\delta)$. This completes the proof that, for all $\beta \in \omega_1 \setminus \delta$, $\hat{A} \cap \delta$ is forced to meet $\bigcap_{\varsigma \in H} \hat{A}_\varsigma$ for all finite $H \subset \{\varsigma : \beta \in \hat{A}_\varsigma\}$. 

\begin{theorem}
It is consistent with Martin’s Axiom and $\mathfrak{c} > \aleph_2$ that there are no $S$-spaces and that compact separable spaces of countable tightness have cardinality at most $\mathfrak{c}$.
\end{theorem}

\begin{proof}
Let $\kappa > \aleph_2$ be a regular cardinal in a model of GCH. Using an iteration sequence as in Theorem 4.3 it follows from Theorem 5.8 and Lemma 5.6 that it suffices to ensure that for each $X$ and $\kappa$-MM-sequence as in Theorem 5.8 there is a $\lambda \in C_X \cap S_1^\kappa$ so that $I_\lambda$ is chosen suitably and so that $\bar{Q}_{\kappa+\lambda}$ is chosen to be $M(\mathcal{A}, C_\lambda)$ for a sequence $\mathcal{A}$ as identified in Theorem 5.8. This is a somewhat routine application of $\diamondsuit(S_1^\kappa)$.

Since $S_1^\kappa$ is stationary, we may assume that $\diamondsuit(S_1^\kappa)$ holds in $V$. There are many equivalent formulations of $\diamondsuit(S_1^\kappa)$ and we choose this one: There is a sequence $\langle h_\alpha : \alpha \in S_1^\kappa \rangle$ satisfying

\begin{enumerate}
  \item for each $\alpha \in S_1^\kappa$, $h_\alpha : \alpha \times \alpha \to \alpha$ is a function,
  \item for all functions $h : \kappa \times \kappa \to \kappa$, the set $\{\alpha \in S_1^\kappa : h_\alpha \subseteq h\}$ is stationary.
\end{enumerate}

We will also have to recursively define our sequence $\mathcal{J} = \{I_\gamma : \gamma \in E\}$ since special choices will have to be made for indices in $S_1^\kappa$ and which, due to conditions (3) and (4) impact all the subsequent choices. To
assist with the condition (4) of the requirements on \( J \), we choose an enumeration \( \{ J_\xi : \xi \in \kappa \} \) of \([\kappa]^{\aleph_1}\) as follows. Let \( D \subset \kappa \) be a cub consisting of \( \lambda \) such that \( \mu + \mu^{\aleph_1} < \lambda \) for all \( \mu < \lambda \). For each \( \mu \in D \), the list \( \{ J_\xi : \mu \leq \xi < \mu^{\aleph_1} \} \) is an enumeration of \([\mu]^{\aleph_1}\).

Say that a sequence \( J_\lambda = \{ I_\gamma : \gamma \in E \cap \lambda \} \subset [\lambda]^{\aleph_1} \) is an acceptable sequence if it satisfies the properties (1), (2), and (3) that we assume for the sequence \( J \) in section 2, and, it also satisfies that, for each \( \xi < \mu \in \lambda \) such that \( \mu + \mu^{\aleph_1} < \lambda \), there is a \( \zeta \in E \cap \mu + \mu^{\aleph_1} \) such that \( J_\xi \subset I_\zeta \). If \( \{ J_\lambda : \lambda \in D \} \) is an increasing sequence of acceptable sequences, then the union, \( J \), satisfies the requirements of section 2. Similarly, once we have chosen an acceptable sequence \( J_\lambda \), we will assume that the sequence \( \langle P_\alpha, \hat{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle \) is defined as in Definition 2.2 using the sequence \( J_\lambda \).

In a similar fashion, we relativize the definition of \( Q(\lambda, \mu) \) from Definition 4.1. Given an acceptable sequence \( J_\lambda \), say that a sequence \( q' = \{ \hat{Q}_\beta : \beta < \lambda \} \in H(\kappa) \) is \( J_\lambda \)-suitable providing (as in Theorem 4.3), by induction on \( \beta < \lambda \), \( \hat{Q}_\beta^\lambda(q) = \hat{Q}'_\beta \) is a \( P_{\lambda+\beta}^\lambda(q) \)-name of a ccc poset, where \( P_\alpha^\lambda(q) = P_\alpha \) for \( \alpha \leq \lambda \) and, for \( \beta > 0 \), \( P_{\lambda+\beta}^\lambda(q) \) is the usual poset from the iteration sequence \( \langle P_\alpha^\lambda(q), \hat{Q}_\beta^\lambda(q) : \alpha \leq \beta, \zeta < \beta \rangle \).

Let \( f \) be any function from \( \kappa \) onto \( H(\kappa) \). We recursively choose our sequences \( \{ J_\lambda : \lambda \in D \} \) and \( \{ \hat{Q}'_\gamma : \gamma \in \kappa \} \). The critical inductive assumptions are, for \( \lambda \in D \),

1. \( J_\lambda \) extends \( J_\mu \) for all \( \mu \in D \cap \lambda \),
2. \( J_\lambda \) is acceptable,
3. \( \{ \hat{Q}'_\gamma : \gamma < \lambda \} \) is \( J_\lambda \)-suitable.

Now let \( \lambda \in D \) and assume we have constructed, for each \( \mu \in D \cap \lambda \), \( J_\mu \) and and \( \{ \hat{Q}'_\gamma : \gamma < \mu \} \). If \( D \cap \lambda \) is cofinal in \( \lambda \), then we simply let \( J_\lambda = \bigcup \{ J_\mu : \mu \in D \cap \lambda \} \) and there is nothing more to do. Otherwise, let \( \mu \) be the maximum element of \( D \cap \lambda \).

Case 1: \( \mu \notin S_1^\kappa \). First choose any acceptable \( J_\lambda \supset J_\mu \). Choose \( \{ \hat{Q}'_\beta : \mu \leq \beta < \lambda \} \) by induction as follows. For \( \mu < \beta \notin E \), let \( q \) denote \( \{ \hat{Q}'_\gamma : \gamma < \beta \} \). Let \( \zeta < \kappa \) be minimal so that \( \hat{Q}'_\beta = f(\zeta) \) is a \( P_{\mu+\beta}(q) \)-name of a ccc poset that is not in the list \( \{ \hat{Q}'_\gamma : \gamma < \beta \} \). For \( \mu \leq \beta \in E \), choose, if possible minimal \( \zeta < \kappa \) so that \( f(\zeta) \) is equal to \( Q(U, \hat{C}_\beta) \) for some S-space task that is not yet handled and let \( \hat{Q}'_\beta = f(\zeta) \). Otherwise, let \( \hat{Q}'_\beta = C_\omega \).

The verification of the inductive hypotheses in Case 1 is routine. We also note that if the induction continues to \( \kappa \), then \( P_{\kappa+\kappa}^\kappa(\{ \hat{Q}'_\beta : \beta < \kappa \}) \) will force that there are no S-spaces and that Martin’s Axiom holds.
Case 2: \( \mu \in S^*_1 \). Let \( q \) denote \( \{ \dot{Q}_\beta' : \beta < \mu \} \). Now we decode the element \( h_\mu \) from the \( \diamond \)-sequence. If there is any \( (\alpha, \xi) \in \mu \times \mu \) such that \( f(h_\mu(\alpha, \xi)) \) is not a \( P_{\mu+\mu}(q) \)-name, then proceed as in Case 1. For each \( \alpha \in \mu \), if \( f(h_\mu(\alpha, 0)) \) is not a name of a finite subset of \( \mu \), then proceed as in Case 1, otherwise let \( \dot{F}_\alpha = f(h_\mu(\alpha, 0)) \). Similarly, if there is an \( \alpha \in \mu \) such that \( f(h_\mu(\alpha, 1)) \) is not a name of a positive rational number, then proceed as in Case 1, otherwise let \( \dot{\epsilon}_\alpha = f(h_\mu(\alpha, 1)) \). If there is an \( \alpha \in \mu \) and a \( \xi > 1 \) such that \( f(h_\mu(\alpha, \xi)) \) is not a name of an element of \([0, 1]\), then proceed as in Case 1, otherwise let

\[
\dot{y}_\alpha(\xi) = \begin{cases} 
  f(h_\mu(\alpha, \xi + 2)) & \text{if } \xi < \omega \\
  f(h_\mu(\alpha, \xi)) & \text{if } \omega \leq \xi < \mu
\end{cases}
\]

It now follows that \( \dot{y}_\alpha \) is a name of an element of \([0, 1]^\mu \) and let the name \( \{ \dot{x} \in [0, 1]^\mu : (\forall \beta \in \dot{F}_\alpha)[x(\beta) - \dot{y}_\alpha(\beta)] < \dot{\epsilon}_\alpha \} \) be denoted by \( \hat{U}_\alpha \).

Now we ask if there is a function \( f_\mu : \omega_1 \to \mu \) as in Theorem 5.8. In particular, if there is an \( I \in [\mu]^{\aleph_1} \) and such a function \( f_\mu : \omega_1 \to \mu \) such that the sequence \( A = \{ \dot{A}_\eta : \eta \in \omega_1 \} \) as defined in the statement of Theorem 5.8 satisfies that \( P_{\mu+\mu}(q) \) forces that \( A \) is a Moore-Mrowka task and each \( \dot{A}_\alpha \) is a \( P_{\mu+\mu}(q)(I) \star \dot{R}_0 \)-name in the sense of Lemma 5.5. If all this holds, then choose an appropriate \( I_\mu \) so that \( I \subset I_\mu \) and define \( \dot{Q}'_\mu \) to be \( \mathcal{M}(\mathcal{A}, \dot{C}_\mu) \). For the remaining choices proceed as in Case 1.

The construction of \( P_{\kappa+\kappa} = P_{\kappa+\kappa}(q) \) where \( q = \{ \dot{Q}'_\beta : \beta < \kappa \} \) is complete. As explained at the beginning of the proof, it follows from Lemma 5.6 and Theorem 5.8 and that the fact that \( D \) is a cub, that separable Moore-Mrowka spaces in this model have cardinality at most \( c \).

\[ \square \]

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