A $\frac{3}{2}$-Approximation Algorithm for Tree Augmentation via Chvátal-Gomory Cuts

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Abstract

The weighted tree augmentation problem (WTAP) is a fundamental network design problem. We are given an undirected tree $G = (V, E)$, an additional set of edges $L$ called links and a cost vector $c \in \mathbb{R}^L_{\geq 1}$. The goal is to choose a minimum cost subset $S \subseteq L$ such that $G = (V, E \cup S)$ is 2-edge-connected. In the unweighted case, that is, when we have $c_\ell = 1$ for all $\ell \in L$, the problem is called the tree augmentation problem (TAP).

Both problems are known to be APX-hard, and the best known approximation factors are 2 for WTAP by (Frederickson and JáJá, ’81) and $\frac{3}{2}$ for TAP due to (Kortsarz and Nutov, TALG ’16). In the case where all link costs are bounded by a constant $M$, (Adjiashvili, SODA ’17) recently gave a $\approx 1.96418 + \varepsilon$-approximation algorithm for WTAP under this assumption. This is the first approximation with a better guarantee than 2 that does not require restrictions on the structure of the tree or the links.

In this paper, we improve Adjiashvili’s approximation to a $\frac{3}{2} + \varepsilon$-approximation for WTAP under the bounded cost assumption. We achieve this by introducing a strong LP that combines $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts for the standard LP for the problem with bundle constraints from Adjiashvili. We show that our LP can be solved efficiently and that it is exact for some instances that arise at the core of Adjiashvili’s approach. This results in the improved guarantee of $\frac{3}{2} + \varepsilon$.

For TAP, this is the best known LP-based result, and matches the bound of $\frac{3}{2} + \varepsilon$ achieved by the best SDP-based algorithm due to (Cheriyan and Gao, arXiv ’15).

1 Introduction

The tree augmentation problem (weighted or unweighted) is a fundamental and intensively studied problem in the area of network design, see for example the surveys by Khuller [17] and Kortsarz and Nutov [20]. While already in the unweighted case the problem is known to be APX-hard, the best algorithms for WTAP and TAP achieve approximation factors of 2 and $\frac{3}{2}$ respectively. One of the main open questions about these problems is to improve the quality of approximation algorithms.

Adjiashvili [1] recently managed to push the approximation guarantee for WTAP below 2 in case the link costs are bounded by a constant $M$, which was the first improvement in over 35 years that did not restrict the structure of the tree or the links. His algorithm is based on an LP that strengthens the standard LP for the problem. Letting $\text{cov}(e)$ denote the set of links connecting

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distinct connected components of $G - e$, for each tree edge $e \in E$, the standard LP for WTAP is

$$\min \sum_{\ell \in L} c_\ell x_\ell$$

(1)

s.t. \hspace{1cm}

$$\sum_{\ell \in \text{cov}(e)} x_\ell \geq 1 \quad \text{for all } e \in E,$$

(2)

$$x_\ell \geq 0 \quad \text{for all } \ell \in L.$$  

(3)

This LP is known as the cut LP.

**Our results.** We add to the cut LP all its $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts, thus obtaining a new LP for WTAP that we call the odd-cut LP. The odd-cut LP is key to our approach. It has three extremely useful properties:

- One can solve the odd-cut LP efficiently, even though separating $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts is NP-hard in general [1].
- The odd-cut LP is compatible with the decomposition approach of [1] to split the given instance and LP solution into well-structured independent instances together with their own local LP solutions.
- The odd-cut LP is exact if for a certain choice of root $r \in V[G]$, every link $\ell$ connects either two different connected components of $G - r$ (in which case $\ell$ is called a cross-link) or some node of $G$ to one of its ancestors (in which case $\ell$ is called an up-link).

We prove the last property by establishing that the constraint matrix of the cut LP is an integral binet matrix. These matrices were introduced by Appa and Kotnyek [2] as a generalization of network matrices. Relying on earlier work by Edmonds and Johnson [12], Appa et al. [3] proved that the integer hull of polyhedra of the form \{ $x \mid Ax \geq b$, $x \geq 0$ \} can be described by $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts whenever $A$ is an integral binet matrix and $b$ is an integer vector. This results in the following theorem.

**Theorem 1.** The odd-cut LP is integral for WTAP instances that contain only cross- and up-links.

Although the odd-cut LP alone might be sufficient to obtain a $\frac{3}{2}$-approximation for WTAP (or maybe even better approximation), we combine the odd-cut LP with the bundle constraints from [1], resulting in the odd-cut bundle LP. This last LP is the one that we use in our algorithm.

We follow the decomposition approach of [1]. After splitting the given instance and its optimum LP solution into independent rooted instances and corresponding LP solutions (respectively), at an extra cost of $\varepsilon \text{OPT}$, Adjishvili applies to each one of the local instances two distinct procedures producing feasible solutions, whose cost is bounded in terms of the local LP solution.

One of the two procedures of [1] produces an integer solution of cost at most $c^T x^{\text{in}} + 2c^T x^{\text{cr}} + \varepsilon \text{OPT}$, where $c^T x^{\text{in}}$ is the local LP cost on in-links (defined as all the links that are not cross-links) and $c^T x^{\text{cr}}$ is the local LP cost on cross-links. This is the part of the analysis where bundle constraints are used. We keep this procedure as is in our algorithm.

Using Theorem 1, we improve the other procedure of [1] to obtain an integer solution of cost at most $2c^T x^{\text{in}} + c^T x^{\text{cr}}$. This gives a significant improvement in the approximation factor since
combining both procedures, we can construct an integer solution in each of the local instances of cost at most

\[
\min \left\{ c^T x^{in} + 2c^T x^{cr} + \delta, 2c^T x^{in} + c^T x^{cr} \right\}
\]

\[
\leq \frac{1}{2} \left( c^T x^{in} + 2c^T x^{cr} + \delta \right) + \frac{1}{2} \left( 2c^T x^{in} + c^T x^{cr} \right)
\]

\[
\leq \frac{3}{2} (c^T x^{in} + c^T x^{cr}) + \delta,
\]

where \( \delta \) is a small quantity whose sum across the local instances is at most \( \varepsilon \text{OPT} \). This yields our main result.

**Theorem 2.** For every fixed \( \varepsilon > 0 \) and \( M \in \mathbb{R}_{\geq 1} \), there exists an LP-based polynomial time \( \frac{3}{2} + \varepsilon \)-approximation algorithm for WTAP with link costs in \([1, M]\).

This result is the best known for WTAP with link costs bounded by a constant, and a significant improvement over the previously known \( \approx 1.96418 + \varepsilon \)-approximation \([1]\). For TAP, it is the best LP-based result and matches the results of the SDP-based algorithm of \([5]\), while only being the \( \varepsilon \)-term worse than the best overall algorithm \([22]\). Finally, we point out that while the approximation factors are improving, the proofs are actually getting simpler, which we see as another indication of the power behind our approach.

**Related work.** Frederickson and JáJá \([13]\) showed that WTAP is NP-hard even if the tree has constant diameter and the link costs are either 1 or 2. For TAP, Cheriyan et al. \([6]\) proved that the problem is NP-hard even if the links form a cycle on the leaves of the tree. Kortsarz, Krauthgamer and Lee \([19]\) then showed that even TAP is APX hard, meaning that these problems have no PTAS, unless \( P = NP \).

For WTAP, there are several 2-approximations known, the first given by Frederickson and JáJá \([13]\) and then simplified by Khuller and Thurimella \([18]\). The primal-dual approach of Goe-mans et al. \([14]\) and the iterative rounding algorithm of Jain \([16]\) also give a 2-approximation for this problem. Until recently, the factor of 2 was best known for general trees. Adjiashvili \([1]\) then managed to give a \( \delta + \varepsilon \)-approximation for the case where all link costs are bound by a constant, for any small \( \varepsilon > 0 \) and \( \delta = \frac{3}{2} \left( \frac{23 + 3\sqrt{5}}{121} \right) \approx 1.96418 \).

For TAP, the best known approximation is a \( \frac{5}{3} \)-approximation algorithm by Kortsarz and Nutov \([22]\), which is purely combinatorial. There is also a \( \frac{5}{3} + \varepsilon \)-approximation for any \( \varepsilon > 0 \), given by Cheriyan and Gao \([5]\), which is also combinatorial but whose analysis is based on an SDP relaxation, showing an integrality gap of \( \frac{3}{2} + \varepsilon \) for the SDP. As far as LP-based algorithms are concerned, the state of the art was a \( \frac{7}{3} \)-approximation due to Kortsarz and Nutov \([21]\) until Adjiashvili \([1]\) gave a \( \frac{5}{3} + \varepsilon \)-approximation based on the bundle LP, for any small \( \varepsilon > 0 \).

For special classes of trees, there are better approximation guarantees known. For example, Cohen and Nutov \([9]\) described a \( 1 + \ln 2 \approx 1.6931 \)-approximation for WTAP for trees with constant radius. For the special case of TAP where every link ends in two leaves of the tree, Maduel and Nutov \([23]\) gave a \( \frac{17}{12} \)-approximation. In case that the tree has radius 3 or 2, they could strengthen the bound to \( \frac{11}{8} \) and \( \frac{4}{3} \), respectively.

In terms of lower bounds on integrality gaps, Cheriyan et al. \([8]\) conjectured that the cut LP has an integrality gap of \( \frac{5}{3} \) for WTAP. However, this conjecture has been refuted by Cheriyan et al. \([7]\), who proved that the integrality gap of the cut LP is at least \( \frac{5}{3} \), even for TAP.
2 Preliminaries

We begin by restating the definition of the weighted tree augmentation problem (WTAP). Recall that a graph is 2-edge-connected if and only if there are at least two edge-disjoint paths between all pairs of nodes.

| (Weighted) Tree Augmentation Problem (WTAP) |
|---------------------------------------------|
| **Input:** | An undirected tree \( G = (V, E) \), an additional set of edges \( L \) on \( V \) called links, and a cost vector \( c \in \mathbb{R}^L \). |
| **Output:** | A minimum cost set of links \( S \) such that \( G = (V, E \cup S) \) is 2-edge-connected. |

The tree augmentation problem (TAP) is then the special case where \( c_\ell = 1 \) for all \( \ell \in L \). Notice that we do not demand that \( E \) and \( L \) are disjoint; consequently, we allow parallel edges in the union of \( E \) and \( L \). We assume without loss of generality that \( c_\ell > 0 \) for all \( \ell \in L \), as we can always pick links with zero cost. Let \( c_{\text{min}} \) and \( c_{\text{max}} \) be the minimum and maximum link costs, then we scale the link costs by \( 1/c_{\text{min}} \), resulting in \( c_\ell \in [1, c_{\text{max}}/c_{\text{min}}] \) for all \( \ell \in L \). For WTAP, we study the case that link costs are bounded from above by a constant \( M \in \mathbb{R}_{\geq 1} \), i.e., \( c_\ell \leq M \) for all \( \ell \in L \). For a graph \( H \), we denote the set of nodes and edges by \( V[H] \) and \( E[H] \), respectively. For a set of nodes \( V' \subseteq V[G] \), we refer to the set of edges between nodes in \( V' \) by \( E[V'] := \{ e = \{u, v\} \in E[G] \mid u, v \in V' \} \). For a vector \( x \in \mathbb{R}^N \geq 0 \), let \( \text{supp}(x) := \{ i \in N \mid x_i > 0 \} \) denote the support of \( X \). For a subset \( N' \subseteq N \), define \( x(N') := \sum_{i \in N'} x_i \). Furthermore, we set \( |k| := \{1, \ldots, k\} \).

**Links.** We write \( e = \{u, v\} \) for an edge \( e \in E \) connecting nodes \( u \) and \( v \), and we write \( \ell = uv \) for a link connecting nodes \( u \) and \( v \). Since \( G \) is a tree, there is a unique path between two nodes \( u, v \in V[G] \). For an \( \ell = uv \in L \), we refer to this path by \( P_\ell^G \) and call all edges \( e \in P_\ell^G \) covered by \( \ell \), since \( P_\ell^G \) together with \( \ell \) is 2-edge-connected. If \( G \) is clear due to the context, we omit it. For a set \( F \subseteq E[G] \), we define \( \text{cov}(F) \) to be the set of links covering at least one edge of \( F \). For brevity, we write \( \text{cov}(e) \) instead of \( \text{cov}(\{e\}) \). For a set of links \( L' \subseteq L \) and a set of edges \( E' \subseteq E \), we say that \( L' \) covers \( E' \) if every edge \( e \in E' \) is covered by at least one link \( \ell \in L' \). The concept of covering is highly relevant due to the following observation.

**Observation 3.** Given an instance \( (G, L, c) \) of WTAP, a set of links \( S \subseteq L \) is a feasible solution if and only if every edge is covered by a link \( \ell \in S \), i.e., \( \bigcup_{\ell \in S} P_\ell = E[G] \).

**Up-links, in-links, cross-links.** For the following classification of links, we assume that our tree is rooted at an arbitrary node \( w \). For a link \( \ell = uv \in L \), let \( \text{lca}(uv) \in V[G] \) denote the least common ancestor of \( u \) and \( v \) in \( G \). If \( \text{lca}(uv) \notin \{u, v\} \) and \( \text{lca}(uv) = w \), we call \( \ell \) an cross-link. Otherwise, we call \( \ell \) an in-link. We call an in-link \( \ell \) an up-link, if \( \text{lca}(uv) \in \{u, v\} \). Notice that links \( \ell \) with \( w \in \ell \) are up-links by this definition. We denote the set of up-links, cross-links and in-links by \( L^u \), \( L^c \) and \( L^i \), respectively. Figure 2 illustrates these types of links.

In order to analyze the approximation guarantee later on, we will need to study the cost carried by different types of links separately. Let \( x \in \mathbb{R}^L \geq 0 \) be a fractional solution to a WTAP instance.
Then we split $x$ into the parts belonging to cross-links $x^{cr}$ and in-links $x^{in}$ as follows:

$$\begin{align*}
x^{cr}_\ell &= \begin{cases} x_\ell & \ell \in L^{cr} \\
0 & \text{else}
\end{cases},
\quad x^{in}_\ell &= \begin{cases} x_\ell & \ell \in L^{in} \\
0 & \text{else}
\end{cases} \quad \text{for all } \ell \in L.
\end{align*}$$

![Figure 1](image_url)

Figure 1: The picture on the left shows a TAP instance rooted at the white node with edges drawn solidly and links with dashed and dotted lines. The dotted link is a cross-link, the dashed link is an in-link and the dash-dotted links are up-links (and also in-links). The picture in the middle shows the same instance after the contraction of the cross-link, and the picture on the right shows a 4-bundle (the thick edges).

### Contraction.

By contracting an edge $e = \{u, v\}$ we mean that $u$ and $v$ are replaced by a new node $w$. All edges $e' \in E(G)$ with either $u \in e$ or $v \in e$ are modified by replacing $u$ or $v$ with $w$, respectively. The edge $e = \{u, v\}$ is deleted. Links $\ell$ with either $u \in \ell$ or $v \in \ell$ are modified in the same way, but links $\ell = uv$ become self-loops of $w$ instead of being deleted.

Contracting a set of edges $F$ is defined by contracting all edges $e \in F$, in any order. Contracting a link $\ell \in L$ refers to contracting all edges in $P_\ell$. Figure 1 illustrates this.

### Bundles and the bundle LP.

The concept of bundles and the bundle LP is due to [1]. A $\gamma$-bundle in a graph $G$ is the union of $\gamma$ paths in $G$ for an integer $\gamma \in \mathbb{N}$. We denote the set of all $\gamma$-bundles in $G$ by $B_\gamma$. Notice that the paths of a $\gamma$-bundle do not have to be distinct, so $B_\gamma \subseteq B_{\gamma+1}$. Also, if $G$ is a tree, there are at most $\left(\binom{|V(G)|}{2}\right)^\gamma$ distinct paths in $G$. Therefore, we have $|B_\gamma| \leq \left(\binom{|V(G)|}{2}\right)^\gamma$ which is polynomial in the input size if $\gamma$ is a constant. Figure 1 shows a 4-bundle.

The bundle LP (LP$_\gamma$) adds the following constraints for every bundle in $B_\gamma$ for a constant $\gamma \in \mathbb{N}$ to the cut LP (1) – (3):

$$\sum_{\ell \in \text{cov}(B)} c_{\ell x_\ell} \geq \text{OPT}(B) \quad \text{for all } B \in B_\gamma. \quad (4)$$

Above, $\text{OPT}(B)$ is the minimum cost of any set of links in $L$ that covers $B$. These constraints are clearly valid for any integral solution, and we can also compute $\text{OPT}(B)$ efficiently using the following lemma.

**Lemma 4** (Lemma A.1 in [1]). Let $(G, L, c)$ be a WTAP instance, with $k$ being the number of leaves of $G$. An optimal solution to $(G, L, c)$ can be computed in time $n^{kO(1)}$. 

5
For a $\gamma$-bundle $B$ in $G$, we know that $B$ is a forest with at most $2\gamma$ leaves. In order to apply this lemma, we contract all edges in $E[G] \setminus B$ to obtain an equivalent tree with at most $2\gamma$ leaves. Then we can apply Lemma 4 to obtain an optimal solution.

We will later consider the case where we have the set of $\gamma$-bundles $\mathcal{B}_\gamma$ of a graph $G$, and then contract edges in $E[G]$ to obtain a subgraph $G'$. In this case, we are interested in the set of edges in $G'$ that are a $\gamma$-bundle in $G$. We refer to the set of edges in $G'$ that are $\gamma$-bundles in $G$ by $\mathcal{B}_\gamma(G')$. Notice that due to the contractions, it is possible that a $\gamma$-bundle in $G'$ is not a $\gamma$-bundle in $G$.

3 A Stronger LP

In this section, we introduce the odd-cut bundle LP ($\text{LP}_{\text{odd}}^\gamma$) which strengthens the bundle LP used in [1]. We use our LP in the algorithmic framework described in Section 4 to obtain better approximation guarantees. The additional constraints that we need are Chvátal-Gomory cuts obtained from the cut LP.

3.1 Deriving and Separating the Odd-Cut Constraints

**Derivation.** Consider an LP of the form $\min \{ c^T x \mid Ax \geq b, \ x \in \mathbb{Z}^n \}$, with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. A Chvátal-Gomory cut [8, 15] is a constraint of the form $\lambda^T Ax \geq \lceil \lambda^T b \rceil$, where the vector of multipliers $\lambda \in \mathbb{R}_+^m$ is chosen in such a way that $\lambda^T A \in \mathbb{Z}^n$. Clearly, any such constraint is valid for the integer solutions of the LP. It is well known that the cuts obtained for $\lambda \in \{0, 1\}^m$ imply the cuts obtained for larger multipliers, hence one can assume $\lambda \in \{0, 1\}^m$ without loss of generality. (A proof of this fact and more background on the Chvátal-Gomory cuts can be found, for instance, in [10].) In case we restrict further $\lambda$ to be in $(0, \frac{1}{2})^m$, we obtain a $(0, \frac{1}{2})$-Chvátal-Gomory cut [4]. These cuts are a proper specialization of the Chvátal-Gomory cuts, and are precisely the cuts that we use here.

A $(0, \frac{1}{2})$-Chvátal-Gomory cut for the cut LP is any constraint of the form

$$\sum_{e \in E[G]} \lambda_e x(\text{cov}(e)) + \sum_{\ell \in L} \mu_{\ell} x_{\ell} \geq \left\lceil \sum_{e \in E[G]} \lambda_e \right\rceil \quad (5)$$

where $\lambda \in \{0, \frac{1}{2}\}^{E[G]}$ and $\mu \in \{0, \frac{1}{2}\}^L$ are such that the coefficients in the left-hand side are all integral. Notice that for any fixed $\lambda$, there is a unique $\mu$ that achieves this.

Let $K := \text{supp}(\lambda) = \{ e \in E[G] \mid \lambda_e = \frac{1}{2} \}$. Since $G$ is a tree, there exists a (not necessarily connected) set $S \subseteq V[G]$ such that $K = \delta_G(S)$. Notice that the right-hand side of (5) is $\lceil |\delta_G(S)|/2 \rceil$, and that the cut is redundant whenever $|\delta_G(S)|$ is even.

Let $\pi(S)$ denote the multiset of links $\ell$ such that $P_\ell$ intersects $\delta_G(S)$, in which the multiplicity of $\ell$ is defined as $\lfloor \frac{1}{2} |P_\ell \cap \delta_G(S)| \rfloor$ (see Figure 2). Now, assuming that $|\delta_G(S)|$ is odd, we can rewrite the constraint (5) as

$$x(\pi(S)) \geq \frac{|\delta_G(S)| + 1}{2} \quad (6)$$

Let $S$ denote the collection of all sets $S$ such that $|\delta_G(S)|$ is odd, and let the odd-cut LP be the LP resulting from the cut LP after adding the odd-cut constraint (4) for each set $S \in S$:
Figure 2: A set $S = S_1 \cup S_2 \cup S_3$ of vertices and the corresponding connected components it induces on tree $G$. Edges $e \in \delta_G(S)$ are blue. Links $\ell \in \delta_L(S)$ are in green. The other links are in red. For each link $\ell$, the figure gives its multiplicity in $\pi(S)$.

\[
\min \sum_{\ell \in L} c_{\ell} x_{\ell},
\]

s.t. $x(\pi(S)) \geq \frac{|\delta_G(S)| + 1}{2}$ for all $S \in \mathcal{S}$,

\[
x_{\ell} \geq 0 \quad \text{for all } \ell \in L.
\]

We remark that the first set of constraints includes in particular the covering constraint $x(\text{cov}(e)) \geq 1$ for each tree edge $e \in E[G]$. Indeed, if $S$ denotes any of the two connected components arising when $e$ is deleted from $G$, we have $\delta_G(S) = \{e\}$ and thus $S \in \mathcal{S}$. The corresponding odd-cut constraint is the covering constraint.

**Separation.** A nice fact that is key to our approach is that separation of the $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts arising from an IP $\min \{c^T x \mid Ax \geq b, \ x \in \mathbb{Z}^n\}$ (here we take $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$) can be done in polynomial time whenever the matroid represented over $GF(2)$ by $(I \bar{A})$ is graphic (or co-graphic), where $\bar{A} := A \pmod{2}$ is the parity matrix of $A$. This was proved by Caprara and Fischetti [4], and implies directly that the constraints of the odd-cut LP can be separated in polynomial time.

A more direct way to prove this is to rewrite inequality (6) as the $T$-cut constraint $x(\delta_L(S)) + y(\delta_G(S)) \geq 1$ after introducing the slack variables $y_e := x(\text{cov}(e)) - 1 \geq 0$ for $e \in E[G]$, and then solve the separation problem with the algorithm of Padberg and Rao [24].

To see this, let $S \in \mathcal{S}$ and let $H$ be the graph obtained from adding all links in $L$ to $G$. Assuming

\footnote{Taking $T$ to be the set of odd-degree nodes of the tree $G$.}
The odd-cut

Theorem 1. For every binet matrix $A$ and every vector $b \in \mathbb{Z}^m$, the integer hull of the polyhedron $P := \{ x \in \mathbb{R}^n \mid Ax \geq b, \ x \geq 0 \}$ is described by its $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut.

Proof. Let $(G, L, c)$ denote the WTAP instance and let $r$ denote the chosen root. For a directed edge $e = (u, v)$, let $z = z(e) \in \mathbb{R}^{V[G] \setminus \{r\}}$ denote the (truncated) incidence vector of $e$ (since the row corresponding to root $r$ is removed), defined as $z_a := 1$ if $a = u$, $z_a := -1$ if $a = v$ and $z_a = 0$ otherwise, for $a \in V[G] \setminus \{r\}$.

Direct all the edges of $G$ away from the root and let $R \in \mathbb{R}^{(V[G] \setminus \{r\}) \times E[G]}$ denote the (truncated) incidence matrix of the resulting directed graph $\overrightarrow{G}$. Every directed edge $e = (u, v)$ of $\overrightarrow{G}$ has a corresponding column $z(e)$ in $R$.

Now, we define a matrix $S \in \mathbb{R}^{(V[G] \setminus \{r\}) \times L}$ encoding the links of WTAP instance. Each link $\ell = uv \in L$ has a corresponding column $S_{\ell}$ in $S$. If $\ell = uv$ is an up-link with $u = \text{lca}(uv)$, we let $S_{\ell}$ be the incidence vector $z(u, v)$ of the directed edge $(u, v)$. If $\ell = uv$ is a cross-link, then $u, v \neq r$ and the corresponding column $S_{\ell}$ has $S_{\ell}^{uv} := 1$ if $a = \{u, v\}$ and $S_{\ell}^{uv} := 0$ otherwise.

Consider the matrix $M = (S R)$. Since $\sum_{a \neq r} |S_{\ell}^{uv}| \leq 2$ for all $\ell \in L$ and $\sum_{a \neq r} |R_{ar}| \leq 2$ for all $e \in E[G]$, we see that $M$ is the incidence matrix of a bidirected graph. Moreover, $M$ has $R$ as a basis. Thus $B := R^{-1}S$ is a binet matrix. We claim that $B \in \mathbb{R}^{(V[G] \setminus \{r\}) \times L} \succeq \mathbb{R}^{E[G] \times L}$ is the constraint matrix $A$ of the cut LP $\min\{c^T x \mid Ax \geq 1, \ x \geq 0\}$.

In other words, we claim that $A = R^{-1}S$, or equivalently $RA = S$. This actually is clear: For every link $\ell = uv$, the sum of the columns $z(e)$ of $R$ that correspond to the tree edges $e = (u, v)$ that are in $P_\ell$ is the corresponding column of $S$. 

\[ x(\pi(S)) \geq \frac{1}{2} + 1 \]

\[ \iff \frac{1}{2} \sum_{e \in \delta_G(S)} x(\text{cov}(e)) \geq \frac{1}{2} + 1 \]

\[ \iff \sum_{e \in \delta_G(S)} (x(\text{cov}(e)) - 1) + x(\delta_L(S)) \geq 1 \]

\[ \iff (x, y) (\delta_H(S)) \geq 1 \]

Above, the second form is the original expression of the odd-cut constraint as a $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut, see (\[\].)

3.2 Exactness of the Odd-Cut LP when all the In-links are Up-links

An integer matrix $M$ is said to be the incidence matrix of a bidirected graph if $\sum_{j} |M_{ij}| \leq 2$ for every fixed column index $j$. A binet matrix is any matrix of the form $B = S^{-1}R$ where $M = (S R)$ is the incidence matrix of a bidirected graph with full row-rank and $R$ is a basis of $M$. Binet matrices are a generalization of network matrices and were introduced by Appa and Kotnyek [2].

Appa et al. [3] proved the following result, extending results of Edmonds and Johnson [11, 12] for incidence matrices of bidirected graphs.

Theorem 5. For every binet matrix $A \in \mathbb{Z}^{m \times n}$ and every vector $b \in \mathbb{Z}^m$, the integer hull of the polyhedron $P := \{ x \in \mathbb{R}^n \mid Ax \geq b, \ x \geq 0 \}$ is described by its $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts.

\[ x(\pi(S)) \geq \frac{1}{2} + 1 \]

\[ \iff \frac{1}{2} \sum_{e \in \delta_G(S)} x(\text{cov}(e)) \geq \frac{1}{2} + 1 \]

\[ \iff \sum_{e \in \delta_G(S)} (x(\text{cov}(e)) - 1) + x(\delta_L(S)) \geq 1 \]

\[ \iff (x, y) (\delta_H(S)) \geq 1 \]

Above, the second form is the original expression of the odd-cut constraint as a $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut, see (\[\].)
By Theorem 5, the cut LP becomes integral after one round of \(\{0, \frac{1}{2}\}\)-Chvátal Gomory cuts. That is, the odd-cut LP is integral.

We point out that for these WTAP instances, there is a combinatorial algorithm that finds an optimum solution in strongly polynomial time. This follows from our proof of Theorem 1 and an algorithm of Edmonds and Johnson [11, 12], see also [3].

3.3 The Odd-Cut Bundle LP

As its name indicates, the odd-cut bundle LP (LP\(_{\text{odd}}\)) contains all of the constraints of the odd-cut LP and additionally the bundle constraints for a constant \(\gamma \in \mathbb{N}\). As before, let \(S\) denote the collection of all sets \(S \subseteq V[G]\) such that \(|\delta_G(S)|\) is odd, let \(B_\gamma\) be the set of all \(\gamma\)-bundles, and let \(\text{OPT}(B)\) the cost of an integral optimal solution for the WTAP instance obtained from contracting all edges not in \(B\) (with respect to the given costs \(c\)). Then, the odd-cut bundle LP is given by:

\[
\begin{align*}
\min \sum_{\ell \in L} c_\ell x_\ell, \\
\text{s.t. } & x(\pi(S)) \geq \frac{|\delta_G(S)| + 1}{2} \text{ for all } S \in S, \\
& \sum_{\ell \in \text{cov}(B)} c_\ell x_\ell \geq \text{OPT}(B) \text{ for all } B \in B_\gamma, \\
& x_\ell \geq 0 \text{ for all } \ell \in L.
\end{align*}
\]

From Lemma 4 and the discussion above, we obtain the following result.

**Lemma 6.** For any constant \(\gamma \in \mathbb{N}\), the odd-cut bundle LP can be solved in polynomial time.

4 Decomposition of LP solutions

In this section, we will discuss how we use solutions of the odd-cut bundle LP to solve a WTAP instance approximately. We use the approach of [1], with two major differences:

- We use the odd-cut bundle LP (LP\(_{\text{odd}}\)) with \(\gamma = \lceil \frac{28M}{\varepsilon^2} \rceil\) instead of the bundle LP as basis for the decomposition and subsequent rounding.

- The additional \(\{0, \frac{1}{2}\}\)-Chvátal-Gomory cuts in the odd-cut bundle LP compared to the bundle LP allow us to round instances with only up-links and cross-links without increasing cost. Since in-links can be split into two up-links, this yields a rounding that increases the cost incurred by in-links by a factor of 2 and leaves the cost of cross-links unchanged. This replaces a rounding based on a reduction to edge cover in [1] that increases the cost of in-links by a factor of 2\(\lambda\) and the cost of cross-links by a factor \(\frac{4}{3} \lambda - 1\) for some \(\lambda > 1\).

For the decomposition, we will assume that we are given a shadow-complete WTAP instance \((G, L, c)\) and a fractional solution \(x\) to the odd-cut bundle LP (LP\(_{\text{odd}}\)) for \(\gamma = \lceil \frac{28M}{\varepsilon^2} \rceil\). An instance \((G, L, c)\) is shadow-complete, if for every link \(\ell \in L\) all its shadows — i.e., links \(\ell'\) with \(P_{\ell'} \subseteq P_\ell\) — are also in \(L\). \(\square\) We assume that the cost of a shadow \(\ell'\) of \(\ell\) fulfills \(c_{\ell'} \leq c_\ell\) — otherwise, we can

\(\square\)We can assume without loss of generality that the instance is shadow-complete — otherwise, we can add the missing shadows with a cost equal to the cheapest original link of which they are a shadow. Should these links appear in any solution, we can replace them with the original link (which covers even more edges) at no cost.
One of the key properties is that the pairs $\beta$ for a set of pairs $x$ was shown in [1] for the bundle $LP$ these properties were shown in [1] for the bundle $LP$. Then, we decompose $G$ and $x$ by splitting repeatedly along $\alpha$-thin edges, which are defined as follows.

For an edge $e = \{u, v\}$, let $G^u$ and $G^v$ be the sub-trees of $G$ that are obtained by deleting $e$, with $G^u$ containing $u$ and $G^v$ containing $v$, respectively. An edge $e = \{u, v\}$ is $\alpha$-thin with respect to $x$ for an $\alpha \in \mathbb{R}_{\geq 0}$ if the costs of links on both sides of the edge are at least $\alpha$ (we ignore links covering $e$ here):

$$
\sum_{\ell \in L, \ell \subseteq V[G^u]} c_{\ell} x_\ell \geq \alpha,
\sum_{\ell \in L, \ell \subseteq V[G^v]} c_{\ell} x_\ell \geq \alpha.
$$

(7)

If $G$ has an $\alpha$-thin edge $e = \{u, v\}$ with respect to $x$, we split $G$ and $x$ along $e$:

We obtain $x^u$ and $x^v$ by splitting all links covering $e$ in $x$ into two parts, the part in $G^u$ and the part in $G^v$:

$$
x^u_{e=pq} := \begin{cases} 
x_\ell & \text{if } p, q \in V[G^u] \setminus \{u\}, 
0 & \text{if } p \in V[G^u] \text{ or } q \in V[G^v], 
\sum_{\ell \in \text{cov}(e)} c_{\ell} x_\ell & \text{if } p \in V[G^u], q = u, \text{ for all } \ell \in L.
\end{cases}
$$

$x^v$ is defined symmetrically. Notice that $\frac{c_{e^p}}{c_{e^u}} \geq 1$: this term ensures that the cost in the sub-trees does not decrease – this is important for the bundle constraints. Figure 3 depicts this splitting. If $G^u$ contains an $\alpha$-thin edge with respect to $x^u$, we repeat this procedure for $G^u$ and $x^u$, and similarly for $G^v$ and $x^v$ if $G^v$ contains an $\alpha$-thin edge with respect to $x^v$. At the end, we obtain a set of pairs $\{(G^i, x^i) \mid i \in [k]\}$ such that no $G^i$ contains an $\alpha$-thin edge with respect to $x^i$. We refer to the contraction of $E^h$ and the repeated edge-splitting as $\alpha$-thin edge decomposition of $G$ and $x$. Notice that we have $\supp(x^i) \cap \supp(x^j) = \emptyset$ for $i \neq j$ – this allows us to round each $(G^i, x^i)$ independently from each other later on.

![Figure 3: We split along $e = \{u, v\}$. For the dashed link $\ell = pq$ we have $x^u_{e^p} = x^v_{e^q} = 0$, since it covers $e$. In order to ensure that the coverage of edges in $G^u$ and $G^v$ does not decrease, we increase $x^u_{p_u}$ and $x^v_{q_v}$ by $\frac{c_{e^p}}{c_{e^u}} x_\ell$ and $\frac{c_{e^q}}{c_{e^v}} x_\ell$, respectively.](figure3.png)

In the next lemma, we list important properties that the decomposition $(G^i, x^i), i \in [k]$ has – these properties were shown in [1] for the bundle $LP$, but they also apply to the odd-cut bundle $LP$. One of the key properties is that the pairs $(G^i, x^i)$ are $\beta$-simple. A pair $(G^i, x^i)$ is called $\beta$-simple for $\beta \in \mathbb{N}$, if there exists a $\beta$-center $v \in V[G^i]$ such that removal of $v$ decomposes $G^i$ into a forest of $t$ trees $K_1, \ldots, K_t$ such that for all $j \in [t]$:
1. the cost of links in $K_j$ is bounded by $\beta$:

$$\sum_{\ell \in L, \ell \subseteq V[K_j]} c_\ell x_\ell \leq \beta,$$

(8)

2. $K_j$ has at most $\beta$ leaves.

Notice that the trees $K_1, \ldots, K_t$ contain neither the $\beta$-center $v$ nor edges incident to $v$.

**Lemma 7** (Section 3.1 in [1]). Let $(G, L, c)$ be a WTAP instance with $c_\ell \leq M$ for all $\ell \in L$ and a constant $M \in \mathbb{R}_{\geq 1}$, and let $x \in \mathbb{R}^L_{\geq 0}$ be a feasible solution to the odd-cut bundle LP ($\text{LP}^\text{odd}_\gamma$) for $\gamma = \lceil \frac{28M}{\varepsilon^2} \rceil$. Let $E^h$ be the heavily-covered edges of $G$ with respect to $x$, $L^h$ be a set of links that covers $E^h$, $\overline{G}$ be the graph obtained from $G$ by contracting $L^h$ and $(G^i, x^i)$, $i \in [k]$ the $\frac{4M}{\varepsilon^2}$-thin edge decomposition of $\overline{G}$ and $x$. Let $E^s$ be the set of edges along which we have split in the $\frac{4M}{\varepsilon^2}$-thin edge decomposition. Then

1. $x^i$ is a feasible solution to the odd-cut LP for $G^i$, and it fulfills

$$\sum_{\ell \in \text{cov}(B)} c_\ell x_\ell \geq \text{OPT}(B) \quad \text{for all } B \in \mathcal{B}^G_i$$

(9)

where $\mathcal{B}^G_i$ is the collection of all edge sets in $E[G^i]$ that are a $\gamma$-bundle in $G$.

2. $\text{supp}(x^i) \cap \text{supp}(x^j) = \emptyset$ for $i, j \in [k], i \neq j$,

3. every $(G^i, x^i), i \in [k]$ is $\frac{4M}{\varepsilon^2}$-simple and we have

$$\sum_{\ell \in L, \ell \subseteq V[K_j]} c_\ell x_\ell \leq \frac{4M}{\varepsilon^2},$$

(10)

for the trees $K_j, j \in [t]$ that are created by removing a $\beta$-center $v$ from $G^i$.

4. $\sum_{i \in [k]} c^T x^i \leq (1 + \varepsilon) c^T x$,

5. we can efficiently compute sets of links $L^h, L^s \subseteq L$ covering $E^h, E^s$, respectively, with $c(L^h) \leq \varepsilon c^T x$ and $c(L^s) \leq O(\varepsilon) \sum_{i \in [k]} c^T x^i$.

The proof of this lemma can be found in Appendix A and is based on [1] Section 3.1. Properties 1 and 2 allow us to round each $x^i$ individually, and property 3 makes it possible to employ a rounding algorithm from [1] Lemma 3.8. Properties 4 and 5 ensure that the decomposition costs us only a factor of $(1 + O(\varepsilon))$, and that the edges not contained in a $G^i$ can also be covered at cost $O(\varepsilon)\text{OPT}$. Thus, the approximation guarantee is dominated by how well we can round the individual $x^i$ solutions.

### 5 Rounding the Solution

Given a WTAP instance $(G, L, c)$, the overall algorithm begins by solving the odd-cut bundle LP ($\text{LP}^\text{odd}_\gamma$) for $\gamma = \lceil \frac{28M}{\varepsilon^2} \rceil$. It then computes a solution for $L^h$ for the heavily-covered edges $E^h$, and...
contracts $L^h$ to obtain a new graph $G$ and decomposes $G$ and the solution $x$ into $\frac{40M}{\epsilon^2}$-simple pairs $(G^i, x^i), i \in [k]$, as described in Section 3. In the process, it also computes a set of links $L^a$ covering $E^a$ as defined in Lemma 7.

Afterwards, we round each $(G^i, x^i)$ individually using the best of the following lemmas — the first one performs well for pairs where cross-links carry much of the cost and is based on the integrality of the odd-cut LP for instances with only cross- and up-links, while the second performs well for pairs where in-links carry much of the cost and is due to [1]; it relies on the bundle constraints. The output of the algorithm is then the union of $L^h$ and $L^a$ as computed by Lemma 7 with the rounded solutions for all $(G^i, x^i)$.

**Lemma 8.** Let $x$ be a feasible solution to the odd-cut LP for $G$. Let $G$ be rooted at any node $r \in V[G]$, then we can compute in polynomial time a solution $S \subseteq L$ covering $E[G]$ with cost at most

$$c(S) \leq 2c^T x^{in} + c^T x^{cr}.$$  \hfill (11)

**Proof.** We define a new feasible solution for the odd-cut LP for $G$ by the applying to the following modification to all in-links with mass that are not up-links, i.e., all $\ell \in \text{supp}(x) \cap (L^{in} \setminus L^{up})$. Let $\ell = uv$ and $w := \text{lca}(uv)$. Then we increase $x_{uw}$ and $x_{wv}$ by $x_{\ell}$, and set $x_{\ell} := 0$, and we call the resulting solution $y$.

By construction, we have $\text{supp}(x) \cap (L^{in} \setminus L^{up}) = \emptyset$ — that means that the support of $y$ contains only up- and cross-links. At the same time, we know that $c^T y \leq 2c^T x^{in} + c^T x^{cr}$ since we doubled the mass of in-links. Notice finally that $y$ is feasible for the odd-cut LP, since the total mass of links covering each edge did not change by the modification.

Now resolve the odd-cut LP with the set of links restricted to $\text{supp}(y)$. We know by Theorem 1 that the odd-cut LP is integer for this set of links, allowing us to obtain an integral solution $z$ with $c^T z \leq 2c^T x^{in} + c^T x^{cr}$. \hfill \Box

The idea of the rounding algorithm from 1 is as follows. Given a $\frac{40M}{\epsilon^2}$-simple pair $(H, x)$ with a $\beta$-center $r$, we replace all cross-links $\ell = uv$ by two up-links $ur$ and $rv$. After that, we split $H$ at $r$ into trees $H_1, \ldots, H_m$ such that $r$ is part of all trees. Since we no longer have cross-links, we can split $x$ into feasible solutions $x^1, \ldots, x^m$ for the bundle LP for $H_1, \ldots, H_m$ with $\text{supp}(x^i) \cap \text{supp}(x^j) = \emptyset$ if $i \neq j$. Every sub-tree $H_1, \ldots, H_m$ of $H$ has at most $\frac{40M}{\epsilon^2} + 1$ leaves (the root causes the additional +1); it can then be shown that every $H_j$ was a $\left\lceil \frac{28\epsilon}{M} \right\rceil$-bundle in the original graph, allowing us to use a bundle constraint to bound the rounding cost.

**Lemma 9** (Lemma 3.8 in [1]). Let $x$ be a feasible solution to the odd-cut bundle LP ($L^\text{odd}_\gamma$) for $G$, with $\gamma = \left\lceil \frac{28\epsilon}{M} \right\rceil$. Given a $\frac{40M}{\epsilon^2}$-simple pair $(H, x)$ from the $\frac{40M}{\epsilon^2}$-thin decomposition of $G$ and $x$ that is rooted at a $\frac{40M}{\epsilon^2}$-center $r$, we can compute in polynomial time a set of links $S \subseteq L$ covering $E[H]$ with cost at most

$$c(S) \leq c^T x^{in} + 2c^T x^{cr} + |V^h \cap V[H]|$$  \hfill (12)

with $V^h$ being the set of nodes created by the contractions of the links in $L^h$.

For completeness, we give a proof of this lemma in Appendix B. Together, these two lemmas allow us to proof the following theorem.

**Theorem 2.** For every fixed $\epsilon > 0$ and $M \in \mathbb{R}_{\geq 1}$, there exists an LP-based polynomial time $\frac{3}{2} + \epsilon$-approximation algorithm for WTAP with link costs in $[1, M]$. 

Proof. For constant $\gamma$, the odd-cut bundle $\text{LP}^\text{odd}$ can be solved in polynomial time due to Lemma 3. The decomposition can be performed in polynomial time, as can the computation of $L^h$ and $L^s$ in Lemma 7 and the rounding schemes of Lemmas 8 and 9. Since we cover all $E[G]$ as well as $E_h$ and $E_s$, we produce a feasible solution due to Observation 3.

What is left to analyze the approximation ratio of the algorithm. By Lemma 7, we know that $c(L^h \cup L^s) \leq O(\varepsilon) \cdot \text{OPT}$. For a pair $(H, x)$ in the decomposition, we know by Lemmas 8 and 9 that the cost for its covering $S$ can be bounded by

$$c(S) \leq \min(2c^\top x^h + c^\top x^s, c^\top x^h + 2c^\top x^s + |V^h \cap V[H]|)$$

$$\leq \frac{1}{2} (2c^\top x^h + c^\top x^s + c^\top x^h + 2c^\top x^s + |V^h \cap V[H]|)$$

$$\leq \frac{3}{2} c^\top x^h + \frac{3}{2} c^\top x^s + \frac{1}{2} |V^h \cap V[H]|.$$

Thus, the cost of the computed coverings $S_1, \ldots, S_k$ for $(G^1, x^1), \ldots, (G^k, x^k)$ is bounded by

$$\sum_{i=1}^k c(S_i) \leq |V^h| + \sum_{i=1}^k \frac{3}{2} c^\top x^i = |V^h| + \frac{3}{2} c^\top x.$$

Since we have $|V^h| \leq |L^h| \leq c(L^h) \in O(\varepsilon) \text{OPT}$, the claim follows (every link costs at least 1, and we need at least one link for a node in $V^h$).

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A Proof of Lemma 7

In this section, we prove that the properties of the decomposition in Lemma 7 also hold if we use the odd-cut bundle LP as a basis for the decomposition instead of the bundle LP. The proof ideas are all due to [1]. Notice that we use \( \gamma = \left\lceil \frac{28M}{\varepsilon^2} \right\rceil \) and \( \beta = \frac{10M}{\varepsilon^2} \) instead of \( \gamma = \left\lceil \frac{56M}{\varepsilon^2} \right\rceil \) and \( \beta = \frac{12M}{\varepsilon^2} \) in [1]; this is the result of slightly improved estimations. In particular, we use that:

- Links covering an edge \( e = \{u, v\} \) can cover at most one leaf in the subtrees \( G_u \) and \( G_v \). This argument allows us to reduce \( \beta \) from \( \frac{12M}{\varepsilon^2} \) to \( \frac{10M}{\varepsilon^2} \).
- The cost of links completely in a sub-tree of a \( \frac{10M}{\varepsilon^2} \)-simple pair is actually bounded by \( \frac{4M}{\varepsilon^2} \), even though the pair might not be \( \frac{4M}{\varepsilon^2} \)-simple due to the number of leaves of the sub-trees.
- The previous point allows us bound \( |V^h| \) by \( \frac{6M}{\varepsilon^2} \); together with the improved \( \beta = \frac{10M}{\varepsilon^2} \) we obtain \( \gamma = \left\lceil \frac{28M}{\varepsilon^2} \right\rceil \).

Details can be found in the proof below.

Lemma 7 (Section 3.1 in [1]). Let \((G, L, c)\) be a GTAP instance with \( c_\ell \leq M \) for all \( \ell \in L \) and a constant \( M \in \mathbb{R}_{\geq 1} \), and let \( x \in \mathbb{R}^L \geq 0 \) be a feasible solution to the odd-cut bundle LP (LP_{odd}^γ) for \( \gamma = \left\lceil \frac{28M}{\varepsilon^2} \right\rceil \). Let \( E^h \) be the heavily-covered edges of \( G \) with respect to \( x \), \( L^h \) be a set of links that covers \( E^h \), \( G \) be the graph obtained from \( G \) by contracting \( L^h \) and \( (G^i, x^i) \), \( i \in [k] \) the \( \frac{4M}{\varepsilon^2} \)-thin edge decomposition of \( G \) and \( x \). Let \( E^o \) be the set of edges along which we have split in the \( \frac{4M}{\varepsilon^2} \)-thin edge decomposition. Then

1. \( x^i \) is a feasible solution to the odd-cut LP for \( G^i \), and it fulfills

\[
\sum_{\ell \in \text{cov}(B)} c_\ell x^i_\ell \geq \text{OPT}(B) \quad \text{for all } B \in \mathcal{B}^G_i
\]

where \( \mathcal{B}^G_i \) is the collection of all edge sets in \( E(G^i) \) that are a \( \gamma \)-bundle in \( G \).

2. \( \text{supp}(x^i) \cap \text{supp}(x^j) = \emptyset \) for \( i, j \in [k], i \neq j \).

3. every \( (G^i, x^i), i \in [k] \) is \( \frac{10M}{\varepsilon^2} \)-simple and we have

\[
\sum_{\ell \in L, \ell \subseteq V(K_j)} c_\ell x^i_\ell \leq \frac{4M}{\varepsilon^2}
\]

for the trees \( K_j, j \in [t] \) that are created by removing a \( \beta \)-center \( v \) from \( G^i \).
We now prove the five properties in the order in which they were stated.

4. $\sum_{i \in [k]} c^\top x^i \leq (1 + \varepsilon) c^\top x$.

5. We can efficiently compute sets of links $L^h, L^s \subseteq L$ covering $E^h, E^s$, respectively, with $c(L^h) \leq \varepsilon c^\top x$ and $c(L^s) \leq O(\varepsilon) \sum_{i \in [k]} c^\top x^i$.

Proof. We now prove the five properties in the order in which they were stated.

1. Feasibility: We start with a solution $x$ that is feasible for the odd-cut bundle LP for $G$, then we contract the links in $L^h$ to obtain a new graph $\overline{G}$. Since we do not modify the solution during the contraction, $x$ fulfills (3) for $B \in \mathcal{B}^G_\gamma$ by definition of $\mathcal{B}^G_\gamma$, because $\mathcal{B}^G_\gamma$ is the set of edges in $G'$ that are $\gamma$-bundles in $G$.

The odd-cut constraints for $E[\overline{G}]$ are

$$\sum_{x \in L} \left[\frac{1}{2}|P \cap \delta_{\overline{G}}(S)|\right] x_L \geq \frac{|\delta_{\overline{G}}(S)| + 1}{2}$$

for all $S \in \mathcal{S}_{\overline{G}}$

where $\mathcal{S}_{\overline{G}}$ being the collection of all sets $S$ such that $|\delta_{\overline{G}}(S)|$ is odd. Let $\mathcal{S}_G$ being the collection of all sets $S$ such that $|\delta_G(S)|$ is odd. For a set $S \in \mathcal{S}_{\overline{G}}$ let $S' \in \mathcal{S}_G$ be the set with $\delta_{\overline{G}}(S) = \delta_G(S')$. The existence of $S'$ is guaranteed since $\overline{G}$ is a contraction of the tree $G$. Since we do not modify the solution during the contraction, the existence of $S'$ for every $S$ guarantees that the odd-cut constraints are fulfilled.

Now assume that we have a graph $H$ and a solution $x$ which is feasible for the odd-cut bundle LP for $G$, and fulfills (3) for all $B \in \mathcal{B}^H_\gamma$. We now show that splitting an edge $e = \{u, v\} \in E[H]$ results in two pairs $(H^u, x^u)$, $(H^v, x^v)$ such that $x^u$, $x^v$ are feasible for the odd-cut bundle LP for $H^u, H^v$ and fulfill (3) for all $B \in \mathcal{B}^H_\gamma$ and $B \in \mathcal{B}^H_\gamma$, respectively.

Recall that splitting an edge $e = \{u, v\}$ results in $x^u$ (and analogously $x^v$) in the following way:

$$x^u_{\ell=pq} := \begin{cases} x_L & \text{if } p, q \in V[G^u] \setminus \{u\}, \\ 0 & \text{if } p \in V[G^v] \text{ or } q \in V[G^v], \\ x_L + \sum_{\ell' \in \text{cov}(e) \colon p \in \ell'} \frac{c_{\ell'}}{c_{\ell}} x_{\ell'} & \text{if } p \in V[G^u], q = u, \end{cases} \quad \text{for all } \ell \in L.$$

This definition differs from (1) in the additional $\frac{c_{\ell'}}{c_{\ell}} x_{\ell'}$ term that ensures that bundle constraints remain feasible in the case that the shadows to which mass is shifted to are cheaper than the links where the mass originates from. Due to this, we have that for every edge $e \in E[G^u]$

$$\sum_{\ell \in \text{cov}(e)} x^u_{\ell} \geq \sum_{\ell \in \text{cov}(e)} x_L, \quad \text{and} \quad \sum_{\ell \in \text{cov}(e)} c_{\ell} x^u_{\ell} \geq \sum_{\ell \in \text{cov}(e)} c_{\ell} x_L$$

by definition of the splitting; the same applies to any edge set $F \subseteq E[G^u]$:

$$\sum_{\ell \in \text{cov}(F)} x^u_{\ell} \geq \sum_{\ell \in \text{cov}(F)} x_L, \quad \text{and} \quad \sum_{\ell \in \text{cov}(F)} c_{\ell} x^u_{\ell} \geq \sum_{\ell \in \text{cov}(F)} c_{\ell} x_L.$$

This implies that the left hand sides of (9) can only get larger in $G^u$, while the right hand sides stay the same.
For the odd-cut constraints, consider a set \( S \subseteq V[G^u] \) such that \( |\delta_{G^u}(S)| \) is odd, and let \( \mathcal{S}_{G^u} \) be the collection of these sets. The important fact here is that the mass of links covering an edge in \( E[G^u] \) does not decrease. We define \( \alpha_{\ell,S} := \frac{1}{2}|P\ell \cap \delta_{G^u}(S)| \) for all \( \ell \in L, S \in \mathcal{S}_{G^u} \), and note that \( \alpha_{\ell,S} = \alpha_{\ell',S} \) for \( \ell = pu \in L, \ell' = pq \in \text{cov}(\ell) \) yielding

\[
\sum_{\ell \in L} \left[ \frac{1}{2}|P\ell \cap \delta_{G^u}(S)| \right] x^u
\]

\[
= \sum_{\ell \in L, \ell \subseteq V[G^u] \setminus \{u\}} \alpha_{\ell,S} \cdot x^u + \sum_{\ell = pu \in L} \alpha_{\ell,S} \left( x^u + \sum_{\ell' \in \text{cov}(\ell)} \frac{c_{\ell'}}{c_\ell} x^{\ell'} \right) + \sum_{\ell \in L, \ell \subseteq \text{cov}(\ell)} \alpha_{\ell,S} \cdot 0
\]

\[
\geq \sum_{\ell \in L, \ell \subseteq V[G^u] \setminus \{u\}} \alpha_{\ell,S} \cdot x^u + \sum_{\ell = pu \in L} \alpha_{\ell,S} x^u + \sum_{\ell = pu \in L} \sum_{\ell' \in \text{cov}(\ell)} \alpha_{\ell,S} \cdot x^{\ell'}
\]

\[
\geq \sum_{\ell \in L, \ell \subseteq V[G^u] \setminus \{u\}} \alpha_{\ell,S} \cdot x^u + \sum_{\ell = pu \in L} \sum_{\ell' \in \text{cov}(\ell)} \alpha_{\ell,S} \cdot x^{\ell'}
\]

\[
\geq \frac{|\delta_{G^u}(S)| + 1}{2} \quad \text{for all } S \in \mathcal{S}_{G^u}.
\]

This proves adherence to the odd-cut constraints. Analogous statements apply to \( G^v \) and \( x^v \). This argumentation can now be used inductively on the splitting process – this completes the proof of this property.

2. **Disjointness**: By definition of the edge splitting, we know that \( V[G^i] \) and \( V[G^j] \) are disjoint for all \( i \neq j, i, j \in [k] \). Furthermore, for any link \( \ell \in \text{supp}(x^i) \), we have \( \ell \subseteq V[G^j] \), proving this claim.

3. **Simplicity**: Here, we have to show that every \((G^i, x^i)\) is \( \frac{10M}{\varepsilon^2} \)-simple, which requires that there is a node \( v \) for every \( G^i \) whose removal decomposes \( G^i \) into trees \( K_1, \ldots, K_t \) with two properties:

- \( K_j \) fulfills \( \sum_{\ell \in L, \ell \subseteq V[K_j]} c_{\ell x^\ell} \leq \frac{10M}{\varepsilon^2} \) for all \( j \in [t] \),
- \( K_j \) has at most \( \frac{10M}{\varepsilon^2} \) leaves for all \( j \in [t] \).

Furthermore, we have to prove that

\[
\sum_{\ell \in L, \ell \subseteq V[K_j]} c_{\ell x^\ell} \leq \frac{4M}{\varepsilon^2} \quad \text{for all } j \in [t].
\]

Notice that this implies the first property of \((G^i, x^i)\) being \( \frac{10M}{\varepsilon^2} \)-simple.

Let \((H, x) := (G^i, x^i)\) for some \( i \in [k] \). We know that every edge \( e = \{u, v\} \in E[H] \) satisfies

\[
\sum_{\ell \in L, \ell \subseteq V[H^u]} c_{\ell x^\ell} < \frac{4M}{\varepsilon^2} \quad \text{and/or} \quad \sum_{\ell \in L, \ell \subseteq V[H^v]} c_{\ell x^\ell} < \frac{4M}{\varepsilon^2}.
\]

We distinguish two cases:
(a) In this case, there exists an edge \( e = \{u, v\} \in E[H] \) with:

\[
\sum_{\ell \in L, \ell \subseteq V[H^u]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2} \quad \text{and} \quad \sum_{\ell \in L, \ell \subseteq V[H^v]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2}.
\]

Thus, choosing \( u \) or \( v \) decomposes \( H \) into subtrees \( K_1, \ldots, K_t \) with

\[
\sum_{\ell \in L, \ell \subseteq V[K_i]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2}
\]

for all \( i \in [t] \).

(b) Otherwise, we have for all edges \( e = \{u, v\} \in E[H] \), that

\[
\text{either } \sum_{\ell \in L, \ell \subseteq V[H^u]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2} \quad \text{or} \quad \sum_{\ell \in L, \ell \subseteq V[H^v]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2}. \tag{14}
\]

We orient every edge \( e = \{u, v\} \) based on whether \( \sum_{\ell \in L, \ell \subseteq V[H^u]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2} \) holds: if yes, we orient it from \( u \) to \( v \), otherwise from \( v \) to \( u \). Since \( H \) is a tree, there is a node \( v \in V[H] \) with \( \text{outdeg}(v) = 0 \). Removing \( v \) again decomposes \( H \) into sub-trees \( K_1, \ldots, K_t \) with

\[
\sum_{\ell \in L, \ell \subseteq V[K_i]} c_{\ell}x_{\ell} < \frac{4M}{\varepsilon^2}
\]

for all \( i \in [t] \).

This completes the first part of the proof. Now we bound the number of leaves for a tree \( K_j, j \in [t] \).

Now consider a tree \( K_j, j \in [t] \), and let \( e \) be the edge that connected \( K_j \) to the \( \beta \)-center in \( G_i \) that was removed. The number of leaves that a tree \( K_j \) has can be bounded by the following argument. Every link \( \ell = uv \) has cost at least 1 and can contribute to the covering of at most 2 leaf edges of \( K_j \) if \( u, v \in V[K_j] \); if either \( u \) or \( v \) is not in \( V[K_j] \) then \( \ell \) can cover at most one of \( K_j \) leaf edge. Finally, if both \( u, v \notin V[K_j] \) it cannot cover leaf edges of \( K_j \) at all.

We get the following bound for the number of leaves \( |\text{Leaves}(K_j)| \) of \( K_j \) (since \( e \notin E^h \)):

\[
|\text{Leaves}(K_j)| \leq 2 \cdot \sum_{\ell \in L, \ell \subseteq V[K_j]} x_{\ell} + \sum_{\ell \in \text{cov}(e)} x_{\ell} \leq 2 \cdot \frac{4M}{\varepsilon^2} + \frac{2}{\varepsilon} \leq \frac{10M}{\varepsilon^2}.
\]

This completes the proof of this property.

4. Cost Increase: We have to show that

\[
\sum_{i \in [k]} c^T x^i \leq (1 + \varepsilon) c^T x.
\]

Consider an edge-splitting of a pair \((H, x) := (G^i, x^i)\) along an edge \( e = \{u, v\} \) into \( H^u, H^v, x^u \) and \( x^v \). We have \( c^T x \leq c^T x^u + c^T x^v + \sum_{\ell \in \text{cov}(e)} c_{\ell}x_{\ell} \) by definition of the edge-splitting. Thus,
5. Remaining Edges: We need to show the existence of link sets $L^h, L^s$ that cover $E^h$ and $E^s$, respectively, with $c(L^h) \leq \epsilon c^T x$ and $c(L^s) \leq O(\epsilon) \sum_{i \in [k]} c^T x^i$.

Consider the graph $G'$ obtained by contracting all edges in $E[G]\setminus E^h$. Since $E^h := \{ e \in E[G] \mid x(\text{cov}(e)) \geq \frac{2}{\epsilon} \}$ we know that $y := \frac{\epsilon}{2} x$ is a feasible solution to the cut LP for $G'$. The cut LP is known to have an LP gap of at most 2 due to various 2-approximation algorithms, e.g., [14, 16] or [11] Proposition 2.1; applying this to $y$ yields a set of links $L^h$ with

$$c(L^h) \leq 2c^T y = 2\frac{\epsilon}{2} c^T x = \epsilon c^T x.$$ 

This shows the first part of the claim. For the second part, remember that all edges $e \in E^s$ used for splitting were $\frac{4M}{\epsilon}$-thin edges, i.e.,

$$c^T x^i = \sum_{e \in L_i, e \in V(G')} c_e x^i \geq \frac{4M}{\epsilon^2} \quad \text{for all } i \in [k].$$

Since $|E^s| = k - 1$, any inclusion-wise minimal covering of $E^s$ costs at most $(k - 1)M$ and we have

$$(k - 1)M = \frac{(k - 1)M}{\sum_{i \in [k]} c^T x^i} \sum_{i \in [k]} c^T x^i \leq \frac{(k - 1)M}{\frac{4M}{\epsilon^2}} \sum_{i \in [k]} c^T x^i \leq \frac{\epsilon^2}{4} \sum_{i \in [k]} c^T x^i \in O(\epsilon) \sum_{i \in [k]} c^T x^i.$$
B  Proof of Lemma 9

Lemma 9 (Lemma 3.8 in [1]). Let \( x \) be a feasible solution to the odd-cut bundle LP (LP_{odd}^\gamma) for \( G \), with \( \gamma = \lceil \frac{28M}{\varepsilon^2} \rceil \). Given a \( \frac{40M}{\varepsilon^2} \)-simple pair \( (H, x) \) from the \( \frac{4M}{\varepsilon^2} \)-thin decomposition of \( G \) and \( x \) that is rooted at a \( \frac{40M}{\varepsilon^2} \)-center \( r \), we can compute in polynomial time a set of links \( S \subseteq L \) covering \( E[H] \) with cost at most

\[
c(S) \leq c^T x^m + 2c^T x^s + |V^h \cap V[H]|
\]

with \( V^h \) being the set of nodes created by the contractions of the links in \( L^h \).

Proof. Consider the following rounding procedure. Given \( (H, x) \), let \( r \) be a \( \frac{40M}{\varepsilon^2} \)-center of \( (H, x) \). Let \( H_1, \ldots, H_m \) be the sub-trees of \( H \) created by removing \( r \). For a sub-tree \( H_i \), let \( e_i \) be the edge that connects \( H_i \) to \( r \) in \( H \). We define new sub-trees \( \overline{H}_i, i \in [m] \) by adding \( e_i \) and \( r \) to \( H_i \). We now create a new solution \( y \) that contains no cross-links by splitting all cross-links at \( r \) into two up-link shadows by defining:

\[
y_{\ell=uv} := \begin{cases} x_\ell & \ell \in \text{supp}(x^m), r \notin \ell \\ x_\ell + \sum_{c' \in \text{supp}(x^s)} x_{c'} & u = r, v \in V[H] \setminus \{r\} \\ 0 & \text{else} \end{cases}
\]

for all \( \ell \in L \).

Notice that we have \( c^T y \leq c^T x^m + 2c^T x^s \). Since \( y \) contains no cross-links, it is the union of \( m \) disjoint solutions, one for each \( \overline{H}_i, i \in [m] \). We will now focus on rounding the solution \( z \) for a tree \( \overline{H} \in \{\overline{H}_i \mid i \in [m]\} \) with \( z \) being defined by

\[
z_\ell := \begin{cases} y_\ell & \ell \subseteq V[\overline{H}] \\ 0 & \text{else} \end{cases}
\]

for all \( \ell \in L \).

We now make the following claim.

Claim 10. If \( z \) is a feasible solution to the bundle LP

\[
\text{OPT}(E[\overline{H}]) \leq c^T z + |V^h \cap V[\overline{H}]|.
\]

Proof. Let \( e \) be the edge connecting \( \overline{H} \) to \( r \) in \( H \). Since \( (H, x) \) is \( \frac{10M}{\varepsilon^2} \)-simple and \( e \notin E^h \), we know that \( c^T z \leq \frac{10M}{\varepsilon^2} x(\text{cov}(e))M \leq \frac{12M}{\varepsilon^2} \). Due to Lemma 8 property 3 we even have that \( c^T z \leq \frac{4M}{\varepsilon^2} + x(\text{cov}(e))M \leq \frac{6M}{\varepsilon^2} \).

Since \( z \) fulfills the cut constraints, we know that \( \text{OPT}(E[\overline{H}]) - c^T z \leq c^T z \leq \frac{6M}{\varepsilon^2} \). Thus, if \( |V^h \cap V[\overline{H}]| \geq \frac{M}{\varepsilon^2} \), we are done. We therefore assume that \( |V^h \cap V[\overline{H}]| < \frac{6M}{\varepsilon^2} \). In this case, we want to employ the bundle constraints that \( z \) fulfills, meaning that we have to analyze whether \( \overline{H} \) is a \( \lceil \frac{28M}{\varepsilon^2} \rceil \)-bundle in \( G \).

Due to \( (H, x) \) being \( \frac{10M}{\varepsilon^2} \)-simple, we know that the number of leaves of \( \overline{H} \) is bounded by \( \frac{10M}{\varepsilon^2} + 1 \). We now split \( \overline{H} \) at all nodes with degree at least 3. The number of nodes with degree at least 3 is bounded by the number of leaves. As a consequence, we obtain a decomposition into \( Q_1, \ldots, Q_t \) paths with \( t \leq 2 \cdot \left( \frac{10M}{\varepsilon^2} + 1 \right) \).

Each \( Q_{i\cdot}, i \in [t] \) is the union of paths in \( G \), separated by components that were contracted into a node of \( V^h \). If a \( Q_i \) is the union of \( k \) paths, there need to be \( k - 1 \) nodes in the interior of the
path that by construction cannot be part of other $Q_i$, since $Q_i$ has no degree 3 or higher nodes in its interior. Thus, each node of $V^h$ is only capable of splitting a single path in $\overline{H}$ into two paths in $G$. Thus, we have at most $2 \cdot \left(\frac{10M}{\varepsilon^2} + 1\right)$ paths $Q_1, \ldots, Q_t$ in $G$ that are splitted into two paths in $G$ at most $\frac{6M}{\varepsilon^2}$ times. Therefore, $\overline{H}$ is the union of at most $2 \cdot \left(\frac{10M}{\varepsilon^2} + 1\right) + \frac{6M}{\varepsilon^2} \leq \frac{28M}{\varepsilon^2}$ paths in $G$ and is therefore a $\lceil \frac{28M}{\varepsilon^2} \rceil$-bundle in $G$.

Thus, we have by the corresponding bundle-constraint:

$$c^T z = \sum_{t \in \text{cov}(\overline{H})} c_t z_t \geq \sum_{t \in \text{cov}(\overline{H})} c_t x_t \geq \text{OPT}(E[\overline{H}])$$

which completes the proof of this claim.

Claim 10 completes the proof, except for the run-time. However, we can use Lemma 4 to compute solutions for $\gamma$-bundles, and since $\text{OPT}(E[\overline{H}]) \in O(M)$, we can use simple enumeration in the case of many contracted nodes.