TORIC KÄHLER–EINSTEIN METRICS AND CONVEX COMPACT POLYTOPES

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Abstract. We show that any compact convex simple lattice polytope is the moment polytope of a Kähler–Einstein orbifold, unique up to orbifold covering and homothety. Using the symplectic approach of Donaldson [12], we extend the Wang–Zhu Theorem [35] giving the existence of a Kähler–Ricci soliton on any toric monotone manifold on any compact convex simple labelled polytope satisfying the combinatoric condition corresponding to monotonicity. We obtain that any compact convex simple polytope $P \subset \mathbb{R}^n$ admits a set of inward normals, unique up to dilatation, such that there exists a symplectic potential satisfying the Guillemin boundary condition (with respect to these normals) and the Kähler–Einstein equation on $P \times \mathbb{R}^n$.

1. Introduction

The question of existence of Kähler–Einstein metrics on compact complex manifold has been subject of intense investigations for the last decades. This problem makes sense on a compact complex manifold $(M^{2n}, J)$ with a given Kähler class $\Omega \in H^2_{dR}(M)$ for which there is $\lambda \in \mathbb{R}$ such that $\lambda \Omega = 2\pi c_1(M)$. The case $\lambda \leq 0$ is non-obstructed and the existence of a Kähler–Einstein metric $(g, \omega)$, with $\omega \in \Omega$ was proved forty years ago [5, 36]. The case $\lambda > 0$ proved to be a more difficult question, conjecturally related to a certain notion of stability [11, 33] and for which there are various known obstructions, notably the Futaki invariant [16]. In the toric case (the Kähler structure is invariant by the Hamiltonian action of a real torus of dimension $n = \dim_{\mathbb{C}} M$), it follows from the Wang–Zhu Theorem [35] which has recently been extended to orbifolds [31], that the only obstruction to the existence of Kähler–Einstein metrics on monotone symplectic toric orbifolds (in the sense that there exists $\lambda > 0$ such that $\lambda[\omega] = c_1(M)$) is the vanishing of the Futaki invariant. Through the toric correspondence, finding such orbifolds is a combinatorial problem on labelled polytopes. In the first part of this paper, we prove that any polytope can be labelled to satisfy these two conditions. To give a precise statement, we now recall the main lines of the correspondence.

Symplectic toric compact orbifolds are classified by rational labelled polytopes via the Delzant–Lerman–Tolman correspondence [10, 24]. A labelled polytope is a pair $(P, \nu)$ where $P$ is a simple bounded convex polytope, open in a $n$–dimensional
vector space $t^*, \nu = \{\nu_1, \ldots, \nu_d\} \subset t$ is a set of vectors, inward to $P$, such that if we denote $F_1, \ldots, F_d$ the facets (codimension 1 face) of $P$, the vector $\nu_k$ is normal to $F_k$ for $k = 1, \ldots, d$ where $d$ is the number of facets. The defining functions of a labelled polytope $(P, \nu)$ are the affine-linear functions $L_1, \ldots, L_d$ on $t^*$ such that $P = \{p \in t^* | L_k(p) > 0\}$ and $dL_k = \nu_k$. A rational labelled polytope $(P, \nu, \Lambda)$ is a labelled polytope $(P, \nu)$ and $\Lambda$ a lattice in $t$ such that $\nu \subset \Lambda$.

Remark 1.1. If $(P, \nu, \Lambda)$ is rational, there are (uniquely determined) positive integers $m_1, \ldots, m_d$ such that $\frac{1}{m_i} \nu_i$ are primitive elements of $\Lambda$. Then $(P, m_1, \ldots, m_d)$ is a rational labelled polytope in the sense of Lerman–Tolman [24].

For a given symplectic toric compact orbifold $(M, \omega, T)$, $t$ is the Lie algebra of the torus $T = t/\Lambda$ and the closure $\overline{P}$ is the image of the moment map. The symplectic properties are encoded in the data $(P, \nu)$. Notably, see [12], monotone symplectic toric orbifolds correspond to what we will call monotone labelled polytopes.

Definition 1.2. We say that $(P, \nu)$ is monotone if there exists $p \in P$ such that $L_1(p) = L_2(p) = \cdots = L_d(p)$. In that case, we call $p$ the preferred point of $(P, \nu)$.

The space of invariant Kähler metrics on $M$ is parameterized by a subspace of convex functions on $P$, the set of symplectic potentials $S(P, \nu)$, see [2, 11, 13], whose definition we precisely recall in [2, 3]. The scalar curvature of the metric $g_u$, associated to $u \in S(P, \nu)$, is given by the Abreu formula

\[ S(u) = -\sum_{i,j=1}^n \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} \]

where $(x_1, \ldots, x_n)$ are coordinates on $t^*$ and $u^{ij} = (\text{Hess } u)^{-1}$, see [1, 2].

The extremal affine function of $(P, \nu)$, denoted $A_{(P, \nu)}$, is an affine-linear function on $t^*$ which corresponds to the Futaki invariant [10] restricted to the (real) Lie algebra of the torus (the symplectic counterpart of the Futaki invariant as introduced in [21]). In particular, $A_{(P, \nu)}$ is constant if and only if the Futaki invariant vanishes on $t$. The extremal affine function is a useful invariant of $(P, \nu)$ since it satisfies

- $g_u$ is extremal, in the sense of Calabi [8], if and only if $S(u) = A_{(P, \nu)}$
- $A_{(P, \nu)}$ is constant, should a constant scalar curvature $T$-invariant compatible Kähler (cscK) metric exist.

In this paper, we prove the following statement.

Theorem 1.3. Given a compact simple convex polytope $\overline{P}$, there exists a set of normals $\nu$, unique up to dilatation, such that $(P, \nu)$ is monotone and has a constant extremal affine function.

In dimension 2, the existence of such labelling follows from elementary considerations [13].

Remark 1.4. As noticed in [11], labelled polytopes with constant extremal affine function are those for which the centers of mass of $(P, d\varpi)$ and $(\partial P, d\sigma_{\nu})$ coincide, where $d\sigma_{\nu}$ is the volume form on $\partial P$ such that $\nu_k \wedge d\sigma_{\nu} = -d\varpi$ on the facet $F_k$. The set of normals $\nu$ given by Theorem 1.3 is characterized by the fact that the preferred point of $(P, \nu)$ (as a monotone labelled polytope) coincides with the center of mass of $(\partial P, d\sigma_{\nu})$. This last characterization was proved by Mabuchi [25].
and used to classify toric complex surfaces admitting a compatible Kähler–Einstein metric: \( \mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( \mathbb{CP}^2 \# 3 \mathbb{CP}^2 \).

We will see that the set of normals \( \nu \) given by Theorem 1.3 can be included in a lattice if and only if \( P \) is a lattice polytope (i.e. whose vertices lie in a lattice) and thus, using the Theorem of Wang–Zhu/Shi–Zhu \([35, 31]\) we get

**Corollary 1.5.** Every (simple convex compact) lattice polytope is the moment polytope of a compact Kähler–Einstein toric orbifold, unique up to dilatation or orbifold covering.

The case where the set of normals \( \nu \) given by Theorem 1.3 is not rational motivates us to extend the Wang–Zhu Theorem for general labelled polytopes. More precisely, Wang and Zhu showed in \([35]\) that any Fano toric manifold \( (M^{2n}, J, T) \) admits a Kähler–Ricci soliton \((g, Z)\), that is, a Kähler metric \( g \) and a holomorphic vector field \( Z \) such that

\[
\rho^g - \frac{1}{2} \lambda \omega = L_Z \omega
\]

where \( \rho^g \) is the Ricci form of \( g \), \( \omega \) the Kähler form \( \omega = g(J \cdot, \cdot) \) and \( \lambda = \frac{1}{2n} \text{Scal} \) with \( \text{Scal} = \int_M \text{Scal} \omega^n / \int_M \omega^n \). In that case, \( 2\pi c_1(M) = \lambda [\omega] \). The vector field \( Z \) is uniquely determined by the data \((M, [\omega], T)\), see \([34]\). In the toric case specifically, denoting \( p \) the preferred point of the monotone labelled polytope \((P, \nu)\) associated to \((M, [\omega], T)\), there is a unique linear function on \( t^* \), \( a \in t \), such that

\[
\int_P e^{2a} (f - f(p)) d\omega = 0
\]

for all \( f \in \text{Aff}(P, \mathbb{R}) \). If a holomorphic vector field \( Z \) satisfies (2), then \( Z = JX_a - iX_a \), see \([35]\). The case \( a = 0 \) implies that \( Z = 0 \) and the Kähler-Ricci soliton of Wang–Zhu is a Kähler–Einstein metric.

According to the work of Donaldson \([12]\), a symplectic potential \( u \in S(P, \nu) \) corresponds to a Kähler–Ricci soliton with respect to \( \lambda > 0 \) and \( a \in t \) if and only if

\[
\frac{1}{2} \log \det(\text{Hess} u)_x + \lambda h(x) = a(x)
\]

where \( h \) is the Legendre transform of \( u \) (seen as a function on \( P \), via the change of variable \( x \mapsto (du)_x \in t \) and the preferred point of \( (P, \nu) \) is the origin.

With this set up, Donaldson gave an alternative proof of the Wang–Zhu Theorem in \([12]\) using estimates of Tian–Zhu \([34, \text{Section 5}]\) the proof of which rely on a modification of the celebrated Yau’s argument \([36]\).

In fact, the argument of Donaldson holds for any labelled polytope without any modification modulo the use of Tian–Zhu estimates which hold on a compact complex manifold or orbifold. In order to find appropriate scope for extending these estimates in the case when \((P, \nu)\) is non necessarily rational, we consider \( P \times t \) with its symplectic structure (that is, \( P \times t \subset t^* \times t \cong T^*t \)) and the \( t \)-Hamiltonian action by translation of the second factor, the moment map being the projection on the first factor. The invariant Kähler metric \( g_u \), for a symplectic potential \( u \in S(P, \nu) \), is simply a \( t \)-invariant Kähler metric on \( P \times t \) with specific behavior along \( \partial P \times t \). As introduced in \([15]\), see also \([12, 13]\) and \([21]\) for each vertex \( p \) of \( P \) there is an open toric symplectic manifold \((M_p, \omega_p, T_p)\) depending only on \((P, \nu)\). In the rational case, \((M_p, \omega_p, T_p)\) is a uniformizing chart for the orbifold. The boundary condition...
on symplectic potentials corresponds to the fact that $g_u$ defines a smooth metric on each of the manifolds $(M_p, \omega_p, T_p)$, see [2.3] In Section 4, we notice that the test functions appearing in Donaldson’s proof and on which he used Tian–Zhu estimates, behave as functions defined on the compact set $\overline{P}$ while the boundary condition, suitably interpreted, allows us to apply the (local) computations of Yau [35] and Tian–Zhu [34] on each chart $(M_p, \omega_p, T_p)$.

**Theorem 1.6.** Let $(P, \nu)$ be a monotone labelled polytope with preferred center $0 \in t^*$ and compact closure $\overline{P}$. There exists a solution $u \in S(P, \nu)$ of equation (4), so that $g_u$ is a $t$–invariant Kähler–Ricci soliton on $P \times t$. This solution $u$ is unique in $S(P, \nu)$ up to addition of an affine-linear function and $g_u$ is Kähler–Einstein if and only if $A(P, \nu)$ is constant.

A result of Donaldson [13] implies that the set of normals for which there exists a solution of the Abreu equation is open in the set of inward normals of a fixed polytope, see §5.1. Together with Theorems 1.3 and 1.6, it yields to

**Corollary 1.7.** For each $n$–dimensional polytope $P$, there exists a non-empty open set $E(P)$ of inward normals $\nu$ for which there exists an extremal toric Kähler metric $g_u$ with $u \in S(P, \nu)$. Moreover, $E(P)$ contains a codimension $n$ subset corresponding to cscK metrics and contains the 1–dimensional cone of toric Kähler–Einstein metrics. In particular, if there exists a lattice for which $P$ is rational then there exist extremal toric Kähler orbifolds with moment polytope $\overline{P}$.

A compact toric symplectic orbifold associated to a rational labelled polytope $(P, \nu, \Lambda)$ is a compactification of $P \times T$ where $T = t/\Lambda$ and $P \subset t^*$. Consequently, as a straightforward application of Theorems 1.3 and 1.6, we obtain the existence of singular toric Kähler–Einstein metrics on any smooth compact toric symplectic manifold, see §5.3 and Proposition 5.6 Here, the complex structure and the metric are singular while the symplectic structure is smooth. This differs from the more usual conical singular Kähler–Einstein metrics studied for e.g. [14, 32], where the symplectic form and the metric are singular but not the complex structure.

**Remark 1.8.** Donaldson recently suggested to approach the question of existence of smooth Kähler–Einstein metric using deformation of singular ones, on smooth compact complex manifolds, with conical singularity along a divisor, see [14]. This idea has been implemented since, see [19, 27]. Note that Li adapted the estimates of Wang–Zhu to study singular toric Kähler–Einstein metrics [27].

**Theorem 1.6** provides an alternative proof of the Futaki–Ono–Wang Theorem [17]. This theorem establishes the existence of toric Sasaki–Ricci soliton on contact toric manifolds with a (fixed) Reeb vector field satisfying the two conditions:

– the basic first Chern form of the normal bundle of the Reeb foliation is positive,
– the first Chern class of the contact bundle is trivial.

Compact contact toric manifolds with a fixed Reeb vector field are in one-to-one correspondence with labelled polytopes whose defining functions lie in a lattice and satisfy a certain weaker condition than the Delzant condition, see [6, 23, 25]. In this correspondence as well, a compatible toric Sasaki metric is given by a symplectic potential and the scalar curvature is given by the Abreu formula (1), up to an additive constant depending only on the dimension, see [3]. The hypothesis of the Futaki–Ono–Wang Theorem corresponds to the fact that the associated labelled polytope is monotone, see [17, 29].
2. Labelled polytopes and toric geometry

For the purpose of this paper, we need to slightly reinterpret the geometry associated to a labelled polytope based on the approach [12, 13, 15].

2.1. Symplectic toric orbifolds as compactifications. Let \((M, \omega, T)\) be a compact symplectic toric orbifold, that is, there is a torus \(T \subset \text{Ham}(M, \omega)\) of maximal dimension, \(2 \dim T = \dim M\). We denote \(t = \text{Lie} T\). There is a moment map
\[
x : M \to t^*
\]
which is \(T\)-invariant and uniquely determined, up to addition of a constant, by the relation \(d(x, a) = -\omega(X_a, \cdot)\). The image of the moment map \(P = \text{Im} x\) is a compact convex simple polytope \(P\) to the facets of \(\nu\) and lying in \(\Lambda\), the lattice of circle subgroups of \(T\), and thus makes \((P, \nu)\) a rational labelled polytope with respect to \(\Lambda\) as defined in the introduction. The Delzant–Lerman–Tolman correspondence [10, 24] states that the data \((M, \omega, T)\) characterizes \((M, \omega, T)\) up to a \(T\)-equivariant symplectomorphism.

In [15], Duistermaat and Pelayo gave a way (alternative to the so-called Delzant construction [10]) to build \((M, \omega, T)\) from the combinatorial data of \((P, \nu, \Lambda)\). The facets of \(P\) are still denoted \(F_1, \ldots, F_d \in F(P)\). For \(F \in F(P)\), denote \(I_F \subset \{1, \ldots, d\}\), the set of indices such that \(F = \bigcap_{k \in I_F} F_k\). For example, \(\overline{F} \in F(P)\) and \(I_{\overline{F}} = \emptyset\).

For a vertex \(p, I(p)\) has \(n\) elements and \(\Lambda_p = \text{span}_\mathbb{Z}\{\nu_k | k \in I(p)\}\) is a lattice in \(t\). For a face \(F \in F(P), T_F = \text{span}_\mathbb{R}\{\nu_k | k \in I_F\}/\Lambda_p \cap \text{span}_\mathbb{R}\{\nu_k | k \in I_F\}\) is a subtorus of \(T_p = t/\Lambda_p\) if \(p \in F\).

Given a vertex \(p\) of \(\overline{P}\), we call \(F_p(P)\) the set of faces containing \(p\) and for \(F \in F_p(P)\) we define \(s_p(F) = \{x | x \in E, p \in E, E \subset F\}\) where \(E\) is the interior of the face \(E\) (in \(E\)). In particular, the interior of a vertex is the vertex itself. Set
\[
M_p = \bigcup_{F \in F_p(P)} (s_p(F) \times T_p/T_F)/\sim
\]
where, for \((x, \theta) \in F \times T_p/T_F\) and \((x', \theta') \in F' \times T_p/T_F', (x, \theta) \sim (x', \theta')\) if

1) \(x = x'\), and

2) the equivalence classes of \(\theta\) and \(\theta'\) in \(T_p/T_F\) coincide.

Here, the first condition implies that \(F \cap F' \neq \emptyset\). so \(F \cap F' \in F_p(P)\). The second condition refers to the fact that \(T_p/T_{F \cap F'}\) is the quotient of \(T_p/T_F\) by \(T_{F \cap F'}/T_F\). For an equivariant neighborhood \(U_p\) of \(0 \in \mathbb{C}^n\), the map \(\phi_p : U_p \to M_p\), defined by

\[
\phi_p(z) = \left[\left(p + \frac{1}{2} |z_i|^2 \nu_{k_i}^* \left( e^{2\pi \sqrt{-1} \phi_1}, \ldots, e^{2\pi \sqrt{-1} \phi_n} \right) \right) \right]
\]

\(\phi_p(z)\) is \(P\) the interior of the polytope and \(P\) its closure. In this text, polytopes are always assumed to be convex and simple with compact closure.
where $z = (|z_1| e^{2\pi i \theta_1}, \ldots, |z_n| e^{2\pi i \theta_n})$, is a well-defined (i.e does not depend on the choice of $e^{2\pi i \theta_i}$ when $|z_i| = 0$) equivariant homeomorphism. The chart $(U_p, \phi_p)$ provides a (smooth) differential structure to $M_p$.

Now, the cotangent space of $T_p$ is naturally equipped with an exact symplectic form, the differential of the Liouville 1–form, for which the action of $T_p$ on itself pullbacks to a Hamiltonian action. Given an equivariant trivialization $T^* T_p \simeq \mathfrak{t}^* \times T_p$, the product $P \times T_p$ inherits of the structure of Hamiltonian $T_p$–space whose moment map is simply the projection on the first factor. The chart above extends this structure to give a (non-compact) symplectic toric manifold $(M_p, \omega_p, T_p)$ with moment map $\pi: M_p \to \mathfrak{t}^*$ so that $\text{Im } \pi = \bigcup_{F \in \mathcal{F}(P)} \tilde{F}$, see [12, 24].

When $(P, \nu, \Lambda)$ is rational, $\Lambda_p \subset \Lambda$ for all vertex $p$ and the quotient of $T_p$ by its finite subgroup $\Lambda_p / \Lambda$ is the torus $T = \mathfrak{t} / \Lambda$. The quotient map $q_p: T_p \to T$ gives a way to glue $M_p$ to $\tilde{P} \times T$ providing an orbifold uniformizing chart with structure group $\tilde{\Lambda} / \Lambda_p$.

Doing that on all vertices, we obtain the compact symplectic toric orbifold, $(M, \omega, T)$, associated to $(P, \nu, \Lambda)$ with moment map $\pi: M \to \mathfrak{t}^*$.

**Definition 2.1.** [10] A rational labelled polytope $(P, \nu, \Lambda)$ is Delzant if, for each vertex $p$, $\Lambda_p = \Lambda$. In particular, $(P, \nu, \Lambda)$ is Delzant if and only if the associated symplectic toric orbifold is a manifold (all orbifold structure groups are trivial).

**Remark 2.2.** Taking a bigger lattice $\Lambda \subset \Lambda'$, corresponds to taking the global quotient by the finite group $\Lambda' / \Lambda$, see [4].

**Remark 2.3.** If $(P, \nu, \Lambda)$ is rational, we can replace $\Lambda_p$ by $\Lambda$ in the definition of $T_F$ and set $|M| = \bigcup_{F \in \mathcal{F}(P)} (F \times T / T_F) / \sim$ with the same equivalence relation as above. The topological space $|M|$ is the underlying topological space of $M$ and is a compactification of $P \times T$. The choice of a labelling specifies an orbifold structure on $|M|$ but $|M|$ does not depend on it.

2.2. Action-angle coordinates. For each labelled polytope $(P, \nu)$, with $P \subset \mathfrak{t}^*$, we consider the symplectic manifold $(P \times \mathfrak{t}, dx \wedge d\theta)$ where the coordinates $(x, \theta)$ on $P \times \mathfrak{t}$ are the two projections. The 2–form $dx \wedge d\theta$ is a symplectic form and corresponds to the symplectic form on the cotangent bundle obtained via any equivariant embedding $P \times \mathfrak{t} \subset \mathfrak{t}^* \times \mathfrak{t} \simeq T^* \mathfrak{t}$. The action of $\mathfrak{t}$ on $P \times \mathfrak{t}$ by translation on the second factor is Hamiltonian with moment map $\pi: P \times \mathfrak{t} \to \mathfrak{t}^*$. When $(P, \nu, \Lambda)$ is rational (we can assume $\Lambda = \text{span}_\mathbb{Z} \{\nu_1, \ldots, \nu_d\}$ in view of Remark 2.2) and corresponds to $(M, \omega, T)$, the pre-image $\tilde{M} = x^{-1}(P)$ is exactly the subset of $M$ where $T$ acts freely. By construction, there is an equivariant symplectomorphism between $(\tilde{M}, \omega|_{\tilde{M}})$ and $(P \times \mathfrak{t}, dx \wedge d\theta)$. Thus, there is a symplectomorphism between $(P \times \mathfrak{t}, dx \wedge d\theta)$ and the universal cover of $(\tilde{M}, \omega|_{\tilde{M}})$ (so the action of $T$ on $\tilde{M}$ is the action of $\mathfrak{t}$ via the exponential).

The coordinates $(x, \theta)$ on $P \times \mathfrak{t}$ are only locally defined on $\tilde{M}$ and are the so-called *action-angle coordinates*. They provide a global identification of $T^* \tilde{M}$ with the trivial bundle (on $\tilde{M}$) of fiber $\mathfrak{t} \oplus \mathfrak{t}^*$. Indeed, one can choose a basis $e_1, \ldots, e_n$ of $\mathfrak{t}$, so that $X_i = X_{e_i}$ are vector fields with Hamiltonian $x_i = (x, e_i)$ and span a $n$–dimensional distribution on $\tilde{M}$. The coordinates $(\theta_1, \ldots, \theta_n)$ satisfy $d\theta_i(X_j) = \delta_{ij}$ and thus $d\theta_i$ are globally defined on $\tilde{M}$, see e.g [9].
Proposition 2.4. [1] For any strictly convex function $u \in C^\infty(P)$, the metric

$$g_u = \sum_{i,j} G_{ij} dx_i \otimes dx_j + H_{ij} d\theta_i \otimes d\theta_j,$$

with $(G_{ij}) = \text{Hess } u$ and $(H_{ij}) = (G_{ij})^{-1}$, is a smooth Kähler structure on $P \times t$ compatible with the symplectic form $\omega_{\tilde{\mathcal{Y}}} = dx \wedge d\theta$. Conversely, any $t$-invariant compatible Kähler structure on $(P \times t, dx \wedge d\theta)$ is of this form.

2.3. The boundary condition. Here again $(P, \nu)$ is a labelled polytope (with $\overline{P}$ compact, convex and simple) and the functions $L_1, \ldots, L_d$ are the affine-linear functions defining $(P, \nu)$ as $dL_k = \nu_k$ and $P = \{ x \in t^* | L_k(x) > 0, k = 1, \ldots, d \}$.

Definition 2.5. A symplectic potential of $(P, \nu)$ is a continuous function $u \in C^0(\overline{P})$ whose restriction to $P$ or to any face’s interior (except vertices), is smooth and strictly convex, and $u - u_o$ is the restriction of a smooth function defined on an open set containing $\overline{P}$ where

$$u_o = \frac{1}{2} \sum_{k=1}^d L_k \log L_k$$

is the Guillemin potential. We denote by $S(P, \nu)$, the set of symplectic potentials.

The Guillemin potential is a symplectic potential corresponding to the Guillemin metric [13]. Denote $\text{Aff}(P, \mathbb{R})$ the space of real valued affine-linear functions on $P$.

Proposition 2.6. [4, 13] The set of smooth compatible toric (orbifold) Kähler metrics on $(M, \omega, T)$ is in one-to-one correspondence with the quotient of $S(P, \nu)$ by $\text{Aff}(P, \mathbb{R})$, acting by addition. The correspondence is explicit and given by [6].

The smooth compactification of a metric is a local issue. Even though $(P, \nu)$ might not be rational (for any lattice), a symplectic potential $u \in S(P, \nu)$ defines, via [6], a Kähler metric $g_u$ on $P \times t$ which is $t$-invariant and thus, for any vertex $p$, defines a Kähler metric, still denoted $g_u$, on $P \times T_p$. The boundary condition implies that $g_u$ is the restriction to $P \times T_p$ of a smooth $T_p$-invariant Kähler metric on $(M_p, \omega_p, T_p)$. Recall that $(M_p, \Lambda_p)$ is an orbifold uniformizing chart near the pre-image of a vertex in the rational case and that smooth orbifold metrics are defined as metrics which may be lifted as smooth metrics on a chart. Note that if $p$ and $p'$ are two vertices lying in the same facet $F_k$ then $Z\nu_k \subset \Lambda_p \cap \Lambda_{p'}$.

Apostolov–Calderbank–Gauduchon–Tomesen-Friedman gave the following alternative description of the boundary condition.

Proposition 2.7. [4] Given a labelled polytope $(P, \nu)$, a strictly convex function $u \in C^\infty(P)$ is a symplectic potential of $(P, \nu)$ if and only if, denoting $H = (\text{Hess } u)^{-1}$,

- $H$ is the restriction to $P$ of a smooth $S^2 t^*$–valued function on $\overline{P}$,
- for every $k = 1, \ldots, d$, for every $y$ in the interior of the facet $F_k$,

$$H_y(\nu_k, \cdot) = 0 \quad \text{and} \quad dH_y(\nu_k, \nu_k) = 2\nu_k,$$

- the restriction of $H$ to the interior of any face $F \subset P$ is a positive definite $S^2(t/4F)^*$–valued function.
2.4. Complex coordinates. For a Kähler structure \((g_u, dx \wedge d\theta, J_u)\) given by \([\mathbf{B}]\) on \(P \times t\), the set \(\{J_uX_1, \ldots, J_uX_n, X_1, \ldots, X_n, \}\) is a frame of real holomorphic commutative vector fields, which gives an identification \(t \oplus \sqrt{-1}t \cong T_{(x, \theta)}(P \times t)\) and (a priori) local holomorphic coordinates \(z = t + \sqrt{-1}\theta\) where \(dt_i = -d\theta_i\).

In the rational case, the complex coordinates \(z = t + \sqrt{-1}\theta\) are only local on \(\hat{M}\) and are given by the exponential map, see \([\mathbf{12}]\). Actually, in this context, for a point \(y \in M\), the tangent space \(T_yM \cong t \oplus \sqrt{-1}t \cong \mathbb{C}^n\) is naturally identified with the universal cover of \(\hat{M}\) where the covering map is just the exponential

\[
\hat{M} \cong t \oplus \sqrt{-1}t / 2\pi \sqrt{-1}A \cong (\mathbb{C}^*)^n.
\]

As explained in the literature see for e.g. \([\mathbf{9, 12}]\), by writing \(dx \wedge d\theta\) in the coordinates \(z\), we find a Kähler potential

\[
(dx \wedge d\theta) = \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial t_i \partial t_j} dt_i \wedge d\theta_j = dd^c \phi.
\]

In the rational case, \(\phi\) is a globally defined function on \(t\) via the identification provided by the exponential near a point of \(\hat{M}\). Changing the base point corresponds to translate \(\phi\) by an affine-linear function of \(t\). The correspondence between the symplectic potential \(u\) and the Kähler potential \(\phi\) is done via the Legendre transform:

\[
u(x) = \langle x, t \rangle - \phi(t)
\]

where \(t\) is the unique point of \(t\) such that \(d\phi_t = x\) or inversely \(x\) is the unique point of \(t^*\) such that \(du_x = t\). The image of the differential of the Kähler potential is the (open) polytope \(P\) (i.e \(P = \text{Im}(t \mapsto d\phi_t)\)).

A symplectic potential \(u \in S(P, \nu)\) provides an identification

\[
\Phi_u : P \times t \rightarrow t \oplus \sqrt{-1}t
\]

via its differential \(du : P \rightarrow t\) (which is a diffeomorphism since \(u\) is strictly convex on \(\hat{M}\)) so the coordinates \(z = t + \sqrt{-1}\theta\) are globally defined on \(P \times t\). In the rational case, this identification fits with the fact that both spaces are identified with the universal cover of \(\hat{M}\).

**Remark 2.8.** The boundary condition on symplectic potentials is equivalent to the common asymptotic behavior of Kähler potentials. Using the identification \(du : P \rightarrow t\) or the inverse \(d\phi : t \rightarrow P\) one can express the boundary condition on \(u\) as asymptotic behavior of the Kähler potential \(\phi\) (recall that \(\frac{\partial^2 \phi}{\partial t_i \partial t_j}(t) = \langle H_{ij}(x)\rangle\) whenever \(du_x = t\)). The Guillemin potential \(u_\circ\) gives \(du_\circ = \frac{1}{2} \sum_{k=1}^d (\log L_k + 1)\nu_k\). In particular, the normals determine the rate of divergence of \(du_\circ\) when \(x \rightarrow \partial P\).

A classical approach adopted in Kähler geometry of compact manifolds, is to fix a complex structure \(J\) on a compact manifold \(M\) and a Kähler class \(\Omega \in H^{1,1}(M, \mathbb{R})\) and study the space of compatible Kähler structures \((g, \omega, J)\) with \(\omega \in \Omega\). (This is equivalent to fix \(\omega\) instead, by Moser’s Theorem.) This approach makes sense in our setting as well even though the cohomology of \(P \times t\) is trivial in view of the following simple fact.

**Lemma 2.9.** Let \((M, \omega, g, J, T)\) be a compact toric Kähler orbifold. Two real closed \((1,1)\)-forms \(\beta, \beta'\) on \(M\) corresponding respectively to potentials \(f, h \in C^\infty(P)\) (i.e \(\beta = dd^c f\), \(\beta' = dd^c f'\) on \(\hat{M}\)) are cohomologous if and only if \(f - h \in C^\infty(\hat{P})\).
For this point of view it is more convenient to work directly on \( \mathfrak{t} \oplus \sqrt{-1} \mathfrak{t} \). Distinct symplectic potentials \( u, u_o \in S(P, \nu) \) lead to distinct Kähler structures on \( \mathfrak{t} \oplus \sqrt{-1} \mathfrak{t} \)
\[
((\Phi^{-1}_u)^*g_{u_o}, \omega_o = (\Phi^{-1}_u)^*dx \wedge d\theta) \quad \text{and} \quad ((\Phi^{-1}_u)^*g_u, \omega = (\Phi^{-1}_u)^*dx \wedge d\theta)
\]
compatible with the same complex structure. Denoting \( \phi \) and \( \phi_o \) the Legendre transform of \( u \) and \( u_o \), respectively, we have \( \omega - \omega_o = dd^c(\phi - \phi_o) \). Going back on \( P \times \mathfrak{t} \), using \( \Phi^{-1}_u \) we get
\[
\text{(10)} \quad (g_{u_o}, dx \wedge d\theta, J_{u_o}) \quad \text{and} \quad ((\Phi_{u_o} \circ \Phi^{-1}_u)^*g_u, (\Phi_{u_o} \circ \Phi^{-1}_u)^*dx \wedge d\theta, J_{u_o}).
\]
The map \( \Phi^{-1}_u \circ \Phiu \) is a \( \mathfrak{t} \)-invariant smooth diffeomorphism of \( \mathcal{T} \times \mathfrak{t} \) thanks to the boundary condition on \( u \) and \( u_o \). In particular, the function \( x \mapsto (\phi - \phi_o)(d(u_o)x) \) is the restriction to \( P \) of a smooth function on \( \mathcal{T} \). In the rational case, this fact is a particular case of Lemma 2.4.

Remark 2.10. For a given symplectic potential \( u \in S(P, \nu) \), the potential of the Ricci form associated to \( f_u \) has been computed to be \( F(x) = \frac{n}{2} \log \det (\text{Hess } u)_x \), see [7]. Thus, using Lemma 2.4, \( (M, \omega) \) is monotone with constant \( \lambda > 0 \) if and only if for any symplectic potential \( u \in S(P, \nu) \), \( F - \lambda \phi \in C^\infty(\mathcal{P}) \) where \( \phi(x) = \phi((du_o)x) \). This condition makes sense in the non rational case as well and is equivalent to the fact that \((P, \nu)\) is monotone in the sense of Definition 1.2, see [12].

Lemma 2.11. Let \( P \) be a polytope of dimension \( n \), there is a \((n+1)\)-dimensional cone of normals \( \nu \) for which \((P, \nu)\) is monotone. Moreover, this cone is parameterized by \( \mathbb{R}_{>0} \times \mathfrak{t} \) via the map
\[
(\lambda, p) \mapsto \left\{ \frac{\lambda \nu_1}{L_1(p)}, \ldots, \frac{\lambda \nu_d}{L_d(p)} \right\}
\]
where \( \nu \) is any given set of normals for \( P \) with defining function \( L_1, \ldots, L_d \).

2.5. The curvature and the extremal affine function. Fixing any euclidian volume form \( dx \wedge d\theta \) on \( \mathfrak{t}^* \), Donaldson [11] pointed out that the \( L^2(P, dx \wedge d\theta) \)-projection of the scalar curvature \( S(u) \), given by [1], on the space of affine linear functions, \( \text{Aff}(P, \mathbb{R}) \), is independent of the choice of \( u \in S(P, \nu) \). The resulting projection \( A_{(P, \nu)} \in \text{Aff}(P, \mathbb{R}) \) is the extremal affine function of \((P, \nu)\) mentioned in the introduction. More precisely, integrating [1] by part, the boundary condition of symplectic potentials gives
\[
\frac{1}{2} \int_P S(u) x_i d\sigma = \int_{\partial P} x_i d\sigma = \frac{1}{2} \int_{\partial P} Z_i(P, \nu)
\]
where \( d\sigma \), when restricted to any facet \( F_k \), is a \((n-1)\)-form defined by \( \nu_k \wedge d\sigma = -d\omega \).

Choose a basis \( (e_1, \ldots, e_n) \) of \( \mathfrak{t} \), this provides a basis \( x_0 = 1 \), \( x_1 = \langle e_1, \cdot \rangle \), \( x_1 = \langle e_1, \cdot \rangle \), \( x_n = \langle e_n, \cdot \rangle \) of \( \text{Aff}(P, \mathbb{R}) \). The extremal affine function of \((P, \nu)\) is \( A_{(P, \nu)} = \sum_{i=0}^n A_i x_i \) where the vector \( A = (A_0, \ldots, A_n) \in \mathbb{R}^{n+1} \) is the unique solution of the linear system:
\[
\sum_{j=0}^n W_{ij}(P) A_j = 2Z_i(P, \nu), \quad i = 0, \ldots, n
\]
where \( \sum_{j=0}^n W_{ij}(P) \) and \( Z_i(P, \nu) = \int_{\partial P} x_i d\sigma = \int_{\partial P} x_i d\sigma \).
3. Proof of Theorem 1.3

The proof of Theorem 1.3 relies on the following lemma.

Lemma 3.1. Let \((P, \nu)\) be a labelled polytope and let \(d\varpi\) be a volume form on \(t^*\). The linear map \(\Phi : \text{Aff}(P, \mathbb{R}) \rightarrow \mathfrak{t}\) defined as

\[
\Phi(P, d\varpi)(f) = \sum_{k=1}^{d} \left( \int_{E_k} f d\sigma_v \right) \nu_k
\]

(12)

does not depend on the set of normals \(\nu\) and \(\Phi(P, d\varpi)(f) = 0\) as soon as \(f\) is constant. Moreover, seen as an endomorphism of \(\mathfrak{t}\),

\[
\Phi(P, d\varpi) = -\text{vol}(P, d\varpi) \text{Id}
\]

(13)

Proof. The first claim (no dependence on \(\nu\)) is straightforward. Suppose that \(\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''\) such that \(F = \mathcal{P}' \cap \mathcal{P}''\) is a facet in both \(P'\) and \(P''\). Note that \(P'\) and \(P''\) induce opposite orientations on \(F\). If we choose \(v \in \mathfrak{t}\) a normal vector to \(F\) inward to \(P'\) and denote \(d\sigma'\) the form on \(F\) such that \(v \wedge d\sigma' = -d\varpi\) then \(-v \in \mathfrak{t}\) is inward to \(P''\) and \(-v \wedge -d\sigma' = -d\varpi\). Therefore,

\[
\Phi(P, d\varpi)(f) = \Phi(P', d\varpi)(f) + \Phi(P'', d\varpi)(f).
\]

By using a triangulation, that is, \(n\)-simplices \(P_1, \ldots, P_N\) such that \(\mathcal{P} = \bigcup_{\alpha=1}^{N} \mathcal{P}_\alpha\) and \(P_\alpha \cap P_\beta = \emptyset\) if \(\alpha \neq \beta\) we get

\[
\Phi(P, d\varpi)(f) = \sum_{\alpha=1}^{N} \Phi(P_\alpha, d\varpi)(f).
\]

(14)

Consider the simplex \(\Sigma = \{x \in \mathbb{R}^n | x_i > 0, \sum_{i=1}^{n} x_i < 1\}\) together with the set of normals \(e = \{e_1, \ldots, e_n, e_0 = -\sum_{i=1}^{n} e_i\}\). Let \(f \in \text{Aff}(\Sigma, \mathbb{R})\), we have

\[
\Phi(\Sigma, dx_1 \wedge \cdots \wedge dx_n)(f) = \sum_{i=1}^{n} \left( \int_{E_i} f d\sigma - \int_{E_0} f d\sigma \right) e_i
\]

(15)

where for \(k = 0, \ldots, n\), \(E_k\) denotes the facet normal to \(e_k\).

Thus, \(\int_{E_i} d\sigma = \int_{E_0} d\sigma = \frac{1}{(n-1)!}\) implies that \(\Phi(\Sigma, dx_1 \wedge \cdots \wedge dx_n)(f) = 0\) as soon as \(f\) is constant.

On the other hand, for each \(i = 1, \ldots, n\), the invertible affine-linear map

\[
\psi_i : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
x \mapsto (x_1, \ldots, x_{i-1}, 1 - \sum_{j=1}^{n} x_j, x_{i+1}, \ldots, x_n)
\]

reverses the orientation of \(\mathbb{R}^n\), sends \(E_i\) to \(E_0\) and satisfies \(\psi_i e_0 = e_i\). It follows that \(\psi_i^* d\sigma|_{E_0} = -d\sigma|_{E_i}\) and in particular \(\int_{E_0} f d\sigma = -\int_{E_i} \psi_i^* (f d\sigma) = \int_{E_i} (f \circ \psi_i) d\sigma\). Thus, by writing \(f\) in coordinates \(f(x) = f_0 + \sum_{j=1}^{n} f_j x_j\), equation (15) becomes

\[
\Phi(\Sigma, dx_1 \wedge \cdots \wedge dx_n)(f) = \sum_{i,j=1, j \neq i}^{n} f_j \left( \int_{E_i} x_j d\sigma - \int_{E_0} x_j d\sigma \right) e_i - \sum_{i=1}^{n} f_i e_i \int_{E_0} x_i d\sigma
\]

\[
= -\sum_{i=1}^{n} f_i e_i \int_{E_0} x_i d\sigma.
\]
Observe that for any \( i = 1, \ldots, n \) the \((n - 1)\)-form \((-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n \) vanishes identically when restricted to the facets \( E_1, \ldots, E_n \) and coincides with \( x_i \, d\sigma \) on \( E_0 \). Hence, we have
\[
\int_{E_0} x_i \, d\sigma = (-1)^{i+1} \int_{\Sigma} d(x_i \, dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n) = \int_{\Sigma} dx_1 \wedge \cdots \wedge dx_n
\]
and, seen as an endomorphism of \( \mathbb{R}^n \), \( \Phi_{(\Sigma, dx_1 \wedge \cdots \wedge dx_n)} = -\text{vol}(\Sigma, dx_1 \wedge \cdots \wedge dx_n)\text{Id} \).

There is only one class of affinely equivalent \( n \)-simplices: for any \( n \)-simplex \( P_\alpha \subset \mathfrak{t}^* \) there is an affine-linear map \( \phi_\alpha : \mathfrak{t}^* \to \mathbb{R}^n \) such that \( \phi_\alpha(P_\alpha) = \Sigma \). Hence,
\[
\Phi_{(P_\alpha, d\omega)} = \phi_\alpha^* \circ \Phi_{(\Sigma, (\phi_\alpha^{-1})^*d\omega)} \circ (\phi^{-1})^*
\]
(16)
\[
= -\frac{(\phi_\alpha^{-1})^*d\omega}{dx_1 \wedge \cdots \wedge dx_n} \text{vol}(\Sigma, dx_1 \wedge \cdots \wedge dx_n)\text{Id}
\]
\[
= -\text{vol}(P_\alpha, d\omega)\text{Id}.
\]
The lemma follows then from (13).

Let \( P \) be a \( n \)-dimensional polytope in a vector space \( \mathfrak{t}^* \) of dimension \( n \) and denote by \( F_1, \ldots, F_d \) its facets. Up to translation, one can assume that \( P \) contains the origin and denote \( \nu \) the (unique) set of inward normals for which the defining functions of \((P, \nu)\) satisfies \( L_1(0) = \cdots = L_d(0) = 1 \). To prove Theorem 1.3 we have to show that there exists only one point \( p \in P \) such that \( (P, \nu(r)) \) has a constant extremal affine function, see Lemma 2.11.

Notice that any set of normals \( \nu(r) \) on \( P \) corresponds to a \( r = (r_1, \ldots, r_d) \in \mathbb{R}^d_{>0} \) via the variation of normals:
\[
\nu(r) = \left\{ \frac{1}{r_1} \nu_1, \ldots, \frac{1}{r_d} \nu_d \right\}.
\]

Choose a basis \( e_1, \ldots, e_n \) of \( \mathfrak{t} \) and corresponding coordinates \((x_1, \ldots, x_n)\) on \( \mathfrak{t}^* \). By considering the linear system (11), we know that, for a given set of normals \((P, \nu(r))\) has a constant extremal affine function if and only if
\[
A_{(P, \nu(r))} = A_0 = \frac{Z_0(P, \nu(r))}{W_0(P)}
\]
which happens if and only if
(17) \[
W_0(P)Z_0(P, \nu(r)) = W_0(P)Z_i(P, \nu(r)) \quad i = 1, \ldots, n.
\]
Since \((d\sigma_{\nu(r)})|_{F_k} = r_k(d\sigma_{\nu})|_{F_k}\), the system of equations (17) is linear in \( r \) and reads
(18) \[
W_i(P)\left( \sum_l \int_{F_l} d\sigma_{r_l} \right) = W_0(P) \sum_l \int_{F_l} x_i d\sigma_{r_l} \quad i = 1, \ldots, n.
\]

**Notation:** Indices \( i \) and \( j \) run from 1 to \( n \) while indices \( k \) and \( l \) run from 1 to \( d \).

Observe that, since \( \langle \nu_k, x \rangle = -1 \) on \( F_k \), the vector field \( X = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \) satisfies
\[
\iota_X d\omega = d\sigma_{\nu}, \quad \text{div} X = n \quad \text{and} \quad \text{div} x_j X = (n + 1)x_j.
\]
In particular, \( W_0 = \frac{1}{n} \sum_k \int_{F_k} d\sigma_{\nu}, \) \( W_{i0} = \frac{1}{n+1} \sum_k \int_{F_k} x_i d\sigma_{\nu} \) and the system of equations (17) for \((P, \nu(r))\) becomes,
\[ \sum_{k,l} \left( n \int_{F_k} x_i \, d\sigma - (n + 1) \int_{F_k} \, d\sigma \int_{F_l} x_i \, d\sigma \right) r_l = 0 \quad i = 1, \ldots, n. \]

Since we seek monotone polytopes, by Lemma 2.11, we can restrict our attention to those \( r = (r_1, \ldots, r_d) \in \mathbb{R}^d_0 \) such that there exists \( p \in M \) with

\[ r = (L_1(p), \ldots, L_d(p)). \]

The system of equations (19) becomes, \( i = 1, \ldots, n, \)

\[ \sum_{k,l} \left( n \int_{F_k} x_i \, d\sigma - (n + 1) \int_{F_k} \, d\sigma \int_{F_l} x_i \, d\sigma \right) \left( \langle \nu_l, p \rangle + 1 \right) = 0. \]

Lemma 3.1 implies that \( \sum_{k=1}^d \int_{F_k} d\sigma \nu_k = 0 \), thus equation (20) reads

\[ \sum_l \int_{F_l} x_i \, d\sigma \langle \nu_l, p \rangle = -\frac{1}{n+1} \sum_l \int_{F_l} x_i \, d\sigma. \]

The left hand side is just \( \langle \Phi(P, d\omega)(e_i^*), p \rangle \) where \( \Phi(P, d\omega) = -\text{vol}(P, d\omega) \text{Id} \) thanks to Lemma 3.1. This ensures that there is a unique solution \( p \in t^* \) of the linear system (21) given as

\[ p = \frac{1}{(n+1)\text{vol}(P, d\omega)} \left( \sum_l \int_{F_l} x_1 \, d\sigma, \ldots, \sum_l \int_{F_l} x_n \, d\sigma \right) \]

\[ = \frac{n}{(n+1)\int_{\partial P} \, d\sigma} \left( \int_{\partial P} x_1 \, d\sigma, \ldots, \int_{\partial P} x_n \, d\sigma \right). \]

This solution lies in a segment between the origin and the center of mass of \((\partial P, d\sigma)\) and thus lies in \( P \). This concludes the proof of Theorem 1.3.

**Corollary 3.2.** The labelled polytope \((P, \nu)\) is monotone with preferred point \( p_\nu \) and has a constant extremal affine function \( A_{(P,\nu)} \) if and only if

\[ p_\nu = \frac{1}{\int_P \, d\omega} \left( \int_P x_1 \, d\omega, \ldots, \int_P x_n \, d\omega \right) = \frac{1}{\int_{\partial P} \, d\sigma} \left( \int_{\partial P} x_1 \, d\sigma, \ldots, \int_{\partial P} x_n \, d\sigma \right). \]

Another simple corollary of Theorem 1.3 is that the linear space of \( r \in \mathbb{R}^d \) such that \( A_{(P,\nu(r))} \) is constant meets the interior of the positive quadrant of \( \mathbb{R}^d \) and thus:

**Corollary 3.3.** Given a polytope \( P \), there is a cone of dimension \( d - n \) of inward normals \( \nu \) such that the extremal affine function \( A_{(P,\nu)} \) is constant.

### 4. Proof of Theorem 1.6

**Remark 4.1.** In what follows, any function \( f \) defined on \( P \) or on \( \mathcal{F} \) is identified with its pull-back on \( \mathcal{F} \times t \) which is also denoted \( f \), that is, we identify \( t \)-invariant functions on \( \mathcal{F} \times t \) and functions on \( \mathcal{F} \). On suitable subsets, we even identify \( f \) with the corresponding \( T_p \)-invariant function on the chart \((M_p, \omega_p, T_p)\) of a vertex \( p \).
Fixing an orientation on $t^\ast$, a lattice $\Lambda \subset t$ naturally provides a volume form, say $d\varpi_\Lambda$, on $t^\ast$ since $\text{GL}(\Lambda) \subset \text{SL}(t)$. With the dual volume form $d\varpi_\Lambda^\ast$ on $t$, given by the dual lattice, the volume of the torus $T = t/\Lambda$ is $(2\pi)^n$. Given a rational labelled polytope $(P, \nu, \Lambda)$ associated to the compact symplectic orbifold $(M, \omega, T)$ with moment map $x : M_p \to t^\ast$ and an integrable function $f$ on $U \subset P$, by Fubini’s Theorem we have

$$\int_U f d\varpi_\Lambda = \frac{1}{(2\pi)^n} \int_{x^{-1}(U)} f \frac{\omega_\nu}{n!}$$

as shown in [18]. Note that if $x^{-1}(U)$ is covered by the orbifold uniformizing chart $(M_p, \Lambda/\Lambda_p, \psi_p)$, see (24) we get

$$\int_U f d\varpi_\Lambda = \frac{1}{(2\pi)^n} \int_{x^{-1}(U)} f \frac{\omega_\nu}{n!} = \frac{1}{|\Lambda/\Lambda_p|} \frac{1}{(2\pi)^n} \int_{\psi_p^{-1}(x^{-1}(U))} f \frac{\omega_\nu}{n!} = \frac{1}{\int_{T_p} d\varpi_\Lambda} \int_{\psi_p^{-1}(x^{-1}(U))} f \frac{\omega_\nu}{n!}.$$

(24)

If we consider only a labelled polytope $(P, \nu)$, there is no preferred lattice but we can arbitrarily choose a volume form $d\varpi = dx_1 \wedge \cdots \wedge dx_n$. Formula (23) still holds: for an integrable function $f$ defined on a neighborhood $U$ of a vertex $p$ of $P$, denoting $x : M_p \to t^\ast$ the moment map of the chart $(M_p, \omega_p, T_p)$,

$$\int_U f d\varpi = \frac{1}{c_p} \int_{x^{-1}(U)} f \frac{\omega_\nu}{n!}$$

where $c_p = \int_{T_p} d\varpi^\ast \in \mathbb{R}_{>0}$ is a constant depending on $p$, on $\nu$ and on the volume form $d\varpi$. The value (25) does not depend on the vertex $p$.

On the other hand, the norm of any derivative $|\nabla^{g_\nu} \nabla^{g_\nu} \cdots \nabla^{g_\nu} \psi|_{g_\nu}$ is a smooth function on $\overline{P}$ as soon as $\psi$ is a smooth $t$-invariant function on $P \times t$. Indeed, $|\nabla^{g_\nu} \nabla^{g_\nu} \cdots \nabla^{g_\nu} \psi|_{g_\nu}$ is then a $T_p$-invariant function on $M_p$ for each vertex $p$ of $P$ and thus, a smooth function on the image of the moment map of $(M_p, \omega_p, T_p)$ which is $\cup_{F \in \mathcal{F}_p(P)} F$, see (24). These smooth continuations (one for each vertex) coincide when overlapping and thus $|\nabla^{g_\nu} \nabla^{g_\nu} \cdots \nabla^{g_\nu} \psi|_{g_\nu} \in C^\infty(\overline{P})$.

The above comments provide a scope for extending standard norms on functional spaces on $P$. Namely, given $u \in S(P, \nu)$ and $d\varpi$, we take the pointwise norms of the derivatives on $P \times t$ (or on $M_p$ if applicable) with respect to the Kähler metric $g_\nu$ while we integrate over $\overline{P}$ using the volume form $d\varpi$. Therefore, we define $L^p$-norms, $C^k$-norms, Hölder norms on suitable spaces of functions on $\overline{P}$ giving rise to the definition of $L^p$-space, $C^k$-space and Hölder space. These spaces do not depend on $d\varpi$ and coincide respectively with their ($T$-invariant) namesake on toric Kähler orbifolds in the rational case. Moreover, even when $(P, \nu)$ is non rational, they behave as if they were defined on a Kähler compact manifold: Sobolev inequalities, Hölder inequalities, Schauder estimates (for smooth operators on $\overline{P}$), Kondrakov Theorem hold as well, see [20]. Because (25) is (24) in the rational case, a lot of proofs (particularly in Donaldson’s work [11, 12, 13]) are formally the same.

**Remark 4.2.** The boundary condition on symplectic potentials given by [14], recalled in [23] implies that $(\text{Hess} \, u)^{-1}$ is the restriction of a smooth $S^2 t^\ast$–valued function...
on $\overline{P}$. In particular, with notation \((15)\),
\begin{equation}
\Delta^{g_u} = \sum_{i,j=1}^{n} H_{ij} \frac{\partial^2}{\partial x_j \partial x_i} + \frac{\partial H_{ij}}{\partial x_j} \frac{\partial}{\partial x_i}
\end{equation}
is a smooth operator on $C^\infty(\overline{P})$ and, as an example of what has been said above, for $f \in C^\infty(\overline{P})$, $g_u(\nabla g_u f, \nabla g_u f) = \sum_{i,j=1}^{n} H_{ij} f_i f_j \in C^\infty(\overline{P})$.

4.2. Donaldson’s proof of the theorem of Wang and Zhu. In \cite{12}, Donaldson gave an alternative proof of Wang–Zhu’s Theorem by turning the problem of finding Ricci–Kähler solitons into solving a PDE defined on symplectic potentials of a labelled\(^3\) polytope $(P, \nu)$ associated to the toric manifold $(M, \omega_o, T)$. More precisely, given a monotone labelled polytope $(P, \nu)$ with preferred point $0 \in \mathbb{R}^+$ and a smooth function $A \in C^\infty(\overline{P})$, we look for a symplectic potential $u$ such that
\begin{equation}
L_u - h_u = A(x) + \langle x, \gamma \rangle
\end{equation}
where $L_u = \log \det \nabla^2 u$, $h_u(x) = \langle x, \nabla u \rangle - u(x)$ and $\gamma \in \mathfrak{t}$ is the unique critical point of the functional $\gamma \mapsto \int_P e^A e^\gamma dx_1 \ldots dx_n$. The symplectic potential $u$ defines a Kähler–Ricci soliton if $A = 0$ (assuming $\lambda = 1$).

Donaldson used the continuity method to show that there always exists a solution to the equation \((26)\) by following the steps:

1. Consider a path $A_s$ where $A_0 = L_{u_o} - h_{u_o}$ for a symplectic potential $u_o$ and $A_1 = A$ is an arbitrary smooth function. Hence, the set
$$S = \{ s \in [0,1] \mid \exists u \text{ such that } L_u - h_u = A_s + \langle x, \gamma_s \rangle \}$$
is non empty. The proof of uniqueness of the solution (up to addition of an affine-linear functions) and that $S$ is open in $[0,1]$ relies on convex analysis on the space of symplectic potentials.

2. The fact that $S$ is closed in $[0,1]$ is equivalent to the fact that the set of solutions of equation \((26)\) with $A = A_s$ for some $s \in [0,1]$ is closed in the $C^\infty$-topology. By considering $\phi$, the Legendre transform of $u$, equation \((26)\) reads
$$\log \det (\nabla^2 \phi) t - \phi(t) = \langle (d\phi)_t \rangle + \langle (d\phi)_t, \gamma \rangle$$
so that $\psi = \phi - \phi_o$ satisfies
\begin{equation}
(\omega_o + d\tilde{\psi})^n = e^{\psi + JX_\gamma} \psi + \hat{A}(d\psi) \omega_o^n
\end{equation}
where $\hat{A}(d\psi_t) = A(d\psi_t + (d\phi_o)_t) - A_0((d\phi_o)_t)$.

3. Then, Donaldson explained that, in this situation, a $L^\infty$–bound on $\psi$ would lead to a bound on all higher derivatives and referred to Tian–Zhu \cite{31} Section 5 which provides $C^2$ and $C^3$ a priori estimates on the space of solutions of a Monge–Ampère equation with parameter (corresponding to a path to the Kähler–Ricci soliton equation) on compact Kähler manifolds or compact Kähler orbifolds as used in \cite{31}.

4. The only remaining step is to prove the existence of a $L^\infty$–bound on $\psi$. By a standard argument of convex analysis, such a bound is equivalent to a $L^\infty$–bound

\(^3\)In \cite{12}, the labelling is encoded in a volume form on the boundary.
on $u$. Hence, the proof is concluded by getting a bound on the set of symplectic potentials $u$ such that $|L_u - h_u| < C$ and normalized so that their derivative vanishes at 0. This last step only involves the labelled polytope’s data.

In conclusion, the only part of Donaldson’s proof which does not clearly extend to non necessarily rational labelled polytopes is the step 3 above.

4.3. The estimates of Tian and Zhu. Let $(P, \nu)$ be a monotone labelled polytope with preferred point $p \in P$ and $a \in \mathfrak{t}$ be defined by (3). In this section we need to work from the complex point of view, thus on $t \oplus i$ it as explained in §2.4. We fix a symplectic potential $u_o \in \mathcal{S}(P, \nu)$, thus the Kähler structure $(g_o, \omega_o, J)$ on $t \oplus i$ (J denote the endomorphism of the tangent bundle induced by $i$), we denote $\rho_o$ its Ricci form and $\phi_o$ the Legendre transform of $u_o$. Note that, for $Z = JX_a - iX_a$, the function $\theta_Z = -2\langle x, a \rangle$ satisfies

$$\mathcal{L}_Z \omega_o = dd^c \theta_Z$$

and that there exists a function $f_o \in C^\infty(P)$ such that

$$\rho_o - \lambda \omega_o = dd^c f_o$$

see Lemma 2.9. Another symplectic potential $u \in \mathcal{S}(P, \nu)$ gives another Kähler structure $(g, \omega, J)$ on $t \oplus i$. We denote by $\rho$ its Ricci form and $\phi$ the Legendre transform of $u$. Finally, we denote $\psi(t) = \phi(t) - \phi_o(t)$ and, thus, $\omega - \omega_o = dd^c \psi$.

If $(g, \omega, J)$ is a Kähler–Ricci soliton with respect to $Z = JX_a - iX_a$ in the sense that it satisfies (2) with $\lambda > 0$, one can write

$$-dd^c \log \frac{\omega^n}{\omega_o^n} = dd^c (\theta_Z - f_o + \lambda \psi + Z.\psi).$$

Indeed, this is equation (2) with $\rho - \rho_o = -dd^c \log \frac{\omega^n}{\omega_o^n}$ and $n \lambda \int_M d\omega = \int_{\partial P} d\sigma$.

Tian and Zhu studied the Monge–Ampère equation (30) using the continuity method on the modified equation (31) imposing the normalization

$$\int_M e^{f_o} \omega^n = \int_M \omega_o^n, \quad \int_M e^{\theta_Z + Z.\psi} \omega^n = \int_M e^{f_o - \psi} \omega^n = \int_M \omega_o^n.$$

We will prove that their estimates still hold in our generalized setting, precisely

**Lemma 4.3.** Fix $0 \leq s \leq 1$. If $\psi = \phi - \phi_o \in C^\infty(t)$ is a solution of

$$\omega^n = e^{f_o - \theta_Z - Z.\psi} \omega_o^n$$

where $\phi, \phi_o$ are the Legrendre transforms of $u, u_o \in \mathcal{S}(P, \nu)$ and $\phi_o$ is a potential for $\omega_o$ then a $C^0$ bound on $\psi$ provides $C^2$ and $C^3$ bounds on $\psi$.

In view of the discussion of §4.1 Lemma 4.3 is enough to fulfill step 3 of Donaldson’s proof.

**Remark 4.4.** Note that the normalization (29) only affects $\psi$ up to an additive constant, so the condition $\text{Im}(t \mapsto \langle d\phi_o \rangle_t) = \text{Im}(t \mapsto \langle d\phi_o \rangle_t) = P$ is not over determined.

**Proof of Lemma 4.3.** Yau [36] produced estimates for solutions of the Monge–Ampère equation

$$\omega^n = e^{f_o} \omega_o^n$$
involved in the Calabi conjecture where \( F \) is smooth and satisfies \( \int_M e^F \omega^n = \int_M \omega^n \).
He normalized the solutions by imposing \( \int_M \psi \omega^n = 0 \). We follow Yau’s convention sign for the Laplacian.

Equation (30) is (31) with \( F \) replaced by \( f_o - \theta Z - s \psi - Z. \psi \). Thus Tian–Zhu [34] had to adapt Yau’s approach. A key ingredient for this adaptation is a result of Zhu [37] giving an a priori bound on \( |Z. \psi| \) on compact Kähler manifolds as soon as \( (\operatorname{Im} Z). \psi = 0 \) and \( \omega_o + dd^c \psi > 0 \). The proof of Zhu use classification of complex surfaces and cannot be directly adapted to our generalized setting. However, we only need a weaker result: since \( \psi(t) = \phi(t) - \phi_o(t) \) and \( Z = JX_a - iX_a \), with respect to the coordinates \( t + i \theta \), we have

\[
\frac{\left| (Z. \psi) t \right| |d \psi(JX_a - iX_a) t|}{\sum_{i=1}^{n} a_i \left( \frac{\partial \phi_o}{\partial t_i} - \frac{\partial \phi_o}{\partial t_i} \right)} = \left| \left\langle a, x \right\rangle - \left\langle a, x_o \right\rangle \right| \\
\leq \max \{|a, x| \mid x \in \mathcal{P}| - \min \{|a, x_o| \mid x_o \in \mathcal{P}|}.
\]

Apart from Zhu’s Lemma, the arguments of Tian–Zhu are essentially local, using the compactness of the manifold only to get bounds on various continuous functions (depending on \( (\omega_o, g_{o}) \)) appearing in the equations. Applying the principle explained in §4.1 is then enough to claim that the estimates hold in our setting.

We present below the details for getting the second order estimate.

As a first step, a local computation shows that, for solutions of (31), a priori bounds on \( |F| \) and \( |\Delta^{g_o} \psi| \) give a priori bounds on \( |dd^c \psi|_{g_o} \), see [29] Proposition 5.3.4. Hence, for solutions of (30), it is sufficient to bound \( |\Delta^{g_o} \psi| \) and \( |s \psi + Z. \psi| \).

For the latter, in our case, a \( C^0 \)-bound on \( \psi \) is given (see §4.2) and a bound on \( |Z. \psi| \) follows from (32). It remains to find a bound on \( |\Delta^{g_o} \psi| \). We only have to find an upper bound to \( \Delta^{g_o} \psi \) since

\[
0 < \operatorname{tr}_{\omega_o}(\omega_o + dd^c \psi) = n + \Delta^{g_o} \psi
\]
where \( \operatorname{tr}_{\omega_o} \) is the trace with respect to \( \omega_o \). Tian and Zhu computed that

\[
\Delta^g((n + \Delta \psi)(-c(\psi)) \geq \exp(-c(\psi))(c + \inf_{i \neq j} R_{i \bar{j}l})(n + \Delta \psi) \left( \sum_i \frac{1}{1 + \psi_{i}} \right) \\
+ \exp(-c(\psi)) \left( \Delta(f_o - \theta Z - \lambda \psi - Z. \psi) - n^2 \inf_{i \neq j} R_{i \bar{j}l} - cn(n + \Delta \psi) \right)
\]

at any point \( p \), where \( \Delta = \Delta^{g_o} \), \( R_{i \bar{j}l} \) are components of the curvature tensor of the metric \( g_o \) with respect to holomorphic coordinates, say \( z \), chosen at \( p \) so that \( (g_o)_{ij} = \delta_{ij} \) and \( \psi_{i \bar{j}} = \delta_{ij} \psi_{i} \) (this convention is used in local computation mentioned above).

Note that each function appearing in the right hand side of (33) is smooth on \( \mathcal{P} \). In particular, \( x \mapsto R_{i \bar{j}l}(t_x) \) defines a smooth function on \( \mathcal{P} \) as one can see easily using the boundary condition on symplectic potentials as they are stated in Proposition 2.7. Indeed, changing the variables from \( t \) to \( x \), one gets
\[ R_{ii\overline{l}} = -\frac{\partial^2 (g_{ij})}{\partial z_i \partial \overline{z}_l} + \sum_{p,q} g_p^q \frac{\partial (g_{ij})}{\partial z_{i}} \frac{\partial (g_{kl})}{\partial \overline{z}_k}. \]

(34)

Now, let \( p \in \mathbb{P} \) be a point where the function \( \exp(-c\psi)(n + \Delta \psi) \) attains its maximum. Then, using (32) and the compactness of \( \mathbb{P} \), we can show, as Tian and Zhu did, that at this point \( p \), there exist \( C_1, C_2 > 0 \) such that

\[ \Delta(-f_{s} + \theta_Z + \lambda \psi + Z \psi) \leq C_1 + C_2(n + \Delta \psi). \]

Inserting this into (33) and using some local formulas (following [36]), there are constants \( C_3, C_4, C_5 \) independent of \( \psi \) such that

\[ \Delta^g((n + \Delta \psi) \exp(-c\psi)) \geq - \exp(-c\psi)(C_3 + C_4(n + \Delta \psi)) + C_5 \exp(-c\psi + \frac{s}{n-1}\psi)(n + \Delta \psi)^{n/(n-1)}. \]

(36)

Then, Yau applied the Maximum Principle: the left hand side of (36) is the Laplacian of a function at its maximum so it must be negative. This argument holds if the maximum is not attained on the boundary of a manifold (which was obviously the case in Yau and Tian–Zhu setting). Actually, it works in our setting as well: if \( p \in F \subset \partial \mathbb{P} \) where \( F \) is a face containing a vertex, say \( q \), then the left hand side of (36) is the Laplacian (of a smooth metric \( g_n \)) of a function defined on \( M_q \) (see [23]) which attains a local maximum at \( p \) so it must be negative. Hence, we get the Tian–Zhu estimate in our generalized setting:

\[ (n + \Delta \psi_s) \leq C(1 + \exp(-s \inf_M \psi_s)) \exp(-c(\psi_s + \inf_M \psi_s)) \]

(37)

for constants \( c, C \) independent of \( \psi \).

Following the same argument, the \( C^3 \)–estimate of Tian and Zhu holds as well. □

5. Applications

5.1. Extremal Kähler equation. Consider the extremal Kähler equation

\[ S(u) = A(P, \nu) \]

(38)

for \( u \in S(P, \nu) \) where \( S(u) \) is the scalar curvature of \( g_u \), given by Abreu’s formula [1]. If \( u \) is solution of (38), then \( g_u \) is an extremal Kähler metric in the sense of Calabi [8]. In [13], Donaldson proved that the cokernel of the linearisation of the map \( u \mapsto S(u) \) in \( C^\infty(\mathbb{P}) \) is the set of affine linear functions on \( \mathbb{P} \). In particular, denoting \( \mathcal{N}(P) \) the cone of normals inward to \( P \), the linearisation of the extension of the scalar curvature map

\[ S : \bigcup_{\nu \in \mathcal{N}(P)} S(P, \nu) \longrightarrow C^\infty(\mathbb{P}) \]

(39)

\[ u \mapsto S(u) \]

is surjective on \( \{ A(P, \nu) \mid \nu \in \mathcal{N}(P) \} \cap (C^\infty(\mathbb{P}) / \text{Aff}(P, \mathbb{R})) \). Note that \( \bigcup_{\nu \in \mathcal{N}(P)} S(P, \nu) \) is path connected. Thus, the set \( E(P) \) of inward normals \( \nu \) for which there exists

\[ ^4 \text{The argument is stated for } n = 2 \text{ but holds in general.} \]
an extremal Kähler metric $g_u$ with $u \in \mathcal{S}(P, \nu)$ is open in $\mathbb{N}(P)$. Gathering this with Theorem 1.3 Theorem 1.6 and Corollary 3.3 we get Corollary 1.7

5.2. Lattice polytopes and Kähler–Einstein orbifolds.

**Lemma 5.1.** Let $(P, \nu)$ be a monotone labelled polytope with a constant extremal affine function. Then, span$_{\mathbb{Z}}\{\nu\}$ is a lattice if and only if $P$ is a lattice polytope.

**Proof.** One direction is straightforward: if $\nu$ spans a lattice $\Lambda$ there is a toric Kähler–Einstein orbifold with moment polytope $P$ and the general theory tells us that there is $\lambda > 0$ such that $\lambda P$ is a lattice polytope. Conversely, if $p_\nu$ is the preferred point of $(P, \nu)$ and if $P$ lies in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, then, Corollary 3.2 implies that $p_\nu$ lies in $\mathbb{Q}^n$. In particular, $p_\nu$ and $\mathbb{Z}^n$ span a lattice, say $\Lambda^*$, and we can assume now that $p_\nu = 0$ (by translation) and that the set of vertices of $P$ lies in $\Lambda^*$ and, up to a dilatation of $\nu$, $L_l = \langle \nu_l, \cdot \rangle + 1, l = 1, \ldots, d$. For any normal $\nu_l \in \nu$ and any vertex $q$, there is an integer $r_{q,l}$ such that $p - r_{q,l}q \in \Lambda^* \cap F_l$ so that $0 = L_l(p - r_{q,l}q) = 1 - r_{q,l}(q, \nu_l)$. Taking the least common multiple $m$ of $\{r_{q,l}\}$, the set $\{mr_1, \ldots, mr_l\}$ is included in the dual lattice of $\Lambda^*$.

**Corollary 5.2.** If $P$ is a polytope with vertices in $\Lambda^*$ and, $\nu$ is a set of normals given by Theorem 1.3 then there exists a real number $s > 0$ such that span$_{\mathbb{Z}}\{s\nu\}$ is a sublattice of $\Lambda$, the dual lattice of $\Lambda^*$.

5.3. Singular Kähler–Einstein metrics. Consider a Delzant labelled polytope $(P, \eta, \Lambda)$, see Definition 2.1. The associated compact symplectic toric orbifold $(M, \omega, T)$ is a smooth manifold (all the orbifold structure groups are trivial). Denote the moment map $x : M \to t^*$ and recall that $T = t/\Lambda$.

The construction of $(M, \omega, T)$ of Duistermaat–Pelayo [15], recalled in Section 2, allows us to see $M$ as a smooth compactification of $P \times T$. The underlying topological space to $M$ only depends on $P$ but the differential structure depends on $\eta$, see [21]. Moreover, there is an equivariant symplectomorphism between the open subset of $M$ where the torus acts freely, say $\tilde{M} = x^{-1}(P)$, and $P \times T$.

Consider another set $\nu$ of normals inward to $P$ and a symplectic potential $u \in \mathcal{S}(P, \nu)$. The Kähler metric $g_u$ defined by (6) is a $t$-invariant smooth Kähler metric on $P \times t$ and, thus, defines a smooth Kähler metric, still denoted $g_u$, on $P \times T$, compatible with the symplectic form $dx \wedge d\theta$. Hence, via the equivariant symplectic embedding

$$(P \times T, dx \wedge d\theta, T) = (\tilde{M}, \omega|_{\tilde{M}}, T) \subset (M, \omega, T),$$

$g_u$ is a smooth Kähler metric on $\tilde{M}$ compatible with the symplectic form $\omega|_{\tilde{M}}$. This metric $g_u$ is not the restriction of a smooth metric on $M$ unless $\nu = \eta$.

**Remark 5.3.** If $\nu$ span a sub-lattice of $\Lambda$, the Kähler structure $(g_u, \omega|_{\tilde{M}}, J_u)$ compactifies smoothly (in the orbifold sense) on the orbifold associated to $(P, \nu, \{\Lambda\}')$.

We now describe the singular behavior of $g_u$ along the toric submanifolds corresponding to the pre-image of the facets of $P$. For each facet $F_k$, there is a real number $a_k > 0$ such that

$$a_k \nu_k = \eta_k.$$
The type of the singularity along \( x^{-1}(\hat{F}_k) \) only depends on \( a_k \) and to understand which types of singularity may occur we only need to study the possibilities on a sphere. To see this, we can use the alternative definition of \( S(P,\nu) \) of \( \text{[4]} \), recalled in \( \text{[2.3]} \).

Let \( P = (0,2) \subset \mathbb{R} \) and \( \nu_1 = \frac{1}{a} \eta_1 \) where \( \eta_1 \) is the generator of \( S^1 \). On \( (0,2) \times S^1 \) the metric \( g_u \) is

\[
g_u = \frac{dx \otimes dx}{2ax + \mathcal{O}(x^2)} + \left( 2ax + \mathcal{O}(x^2) \right) d\theta \otimes d\theta.
\]

Using polar coordinates near the south pole \( x^{-1}(0) \) (with the period \( 2\pi \) given by \( \eta_1 \) as in \( \text{[5]} \)), we have \( x = \frac{1}{a} r^2 \) and \( \text{(41)} \) is

\[
g_u = \frac{1}{a} \left( dr \otimes dr + a^2 r^2 dt \otimes dt \right) - \frac{r^2 dr \otimes dr}{a(a + \mathcal{O}(r^2))} + \mathcal{O}(r^4) d\theta \otimes d\theta
\]
as computed in \( \text{[4]} \). The last two terms are smooth and vanish at \( r = 0 \). Therefore

- if \( a < 1 \), \( g_u \) has a singularity of conical type and angle \( 2a\pi \),
- if \( a = 1 \), \( g_u \) is smooth,
- if \( a > 1 \), \( g_u \) has a singularity characterized by a large angle \( 2a\pi > 2\pi \).

Consequently, we obtain

**Proposition 5.4.** Let \( (M,\omega,T) \) be a smooth compact symplectic toric manifold associated to the Delzant labelled polytope \( (P,\eta,\Lambda) \). For any set \( \nu \) of normals inward to \( P \), the symplectic potentials in \( S(P,\nu) \) define \( T \)-invariant, compatible, Kähler metrics on the open dense subset where the torus acts freely via formula \( \text{(41)} \). The behavior of these metrics along the pre-image of the interior of the facet \( \hat{F}_k \) only depends on the real number \( a_k > 0 \), defined by \( a_k \nu_k = \eta_k \), as in \( \text{[13]} \).

**Remark 5.5.** One could interpret the Kähler metric \( g_u \) has a singular Kähler metric on the smooth algebraic toric manifold underlying \( (M,\omega,T) \), see \( \text{[24]} \). Moreover, the singularity of \( \Phi_{a_k} \circ \Phi_{\nu}^{-1} \) (see notation \( \text{[10]} \)) when \( u \in S(P,\nu) \) and \( u_\omega \in S(P,\eta) \) lies along the boundary \( \partial P \) and only depends on the ratio of \( \eta \) and \( \nu \). Consequently, one could interpret the Kähler metric \( g_u \) as a singular Kähler metric in the usual sense (smooth complex structure and singular symplectic structure).

**Proposition 5.6.** Let \( (M,\omega,T) \) be a smooth compact symplectic toric manifold associated to the Delzant labelled polytope \( (P,\eta,\Lambda) \). Fix \( \nu \), a set of inward normals such that \( (P,\nu) \) is monotone and has a constant extremal affine function equals to \( 2n \).

For any \( \lambda > 0 \), there exists a \( T \)-invariant Kähler–Einstein metric \( g_\lambda \) smooth on the open dense subset where the torus acts freely, compatible with \( \omega \) and with scalar curvature equals to \( 2n\lambda \). The type of singularity of \( g_\lambda \) along the pre-image of the interior of the facet \( \hat{F}_k \) is one of the 3 cases of \( \text{[13]} \) with \( a_k \) defined by \( \lambda \nu_k = \eta_k \).

In particular, for \( \lambda \) small enough, the singularity along the pre-image of the interior of any facet is of conical type.

**Remark 5.7.** Proposition 5.4 gives a way to construct plenty of singular (but rather uninteresting) metrics. For instance, a dilatation of the set of normals, \( s\nu \) with \( s > 0 \), corresponds to multiplying the volume (with respect to \( g_u \in S(P,s\nu) \)) of the orbits of \( T \) in \( M \) by a factor \( s^2 \). To avoid this phenomenon, we can, for instance,
normalize the average of the scalar curvature but it may not be the better way to do it, as suggested in Remark 5.8 below.

5.4. Singular Kähler–Einstein metrics on the first Hirzebruch surface. Let $P$ be the convex hull of the points $(1, 0), (1, 1), (2, 2), (2, 0)$ in $\mathbb{R}^2$. Consider the two sets of inward normals:

\[
\eta = \left\{\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \eta_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\},
\]

\[
\nu(C) = \left\{\nu_1 = C \frac{7}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \nu_2 = C \frac{7}{8} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nu_3 = C \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nu_4 = C \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.
\]

One can check that $(P, \eta)$ satisfies the Delzant condition and corresponds to the first Hirzebruch surface $\mathbb{P}(\mathcal{O} + \mathcal{O}(-1))$. On the other hand, $(P, \nu(C))$ is monotone and has constant extremal affine function. Actually, see [22], we explicitly know the form of the Kähler–Einstein metric on quadrilaterals in terms of the inverse of the Hessian of the potential: using notation [6], it reads

\[
(H_{ij}) = \frac{x_1}{x_1^2 - x_2} \left( \begin{array}{cc} A(x_1) + B(x_2/x_1) & (x_2/x_1)A(x_1) + x_1B(x_2/x_1) \\ (x_2/x_1)A(x_1) + x_1B(x_2/x_1) & (x_2/x_1)^2A(x_1) + x_1^2B(x_2/x_1) \end{array} \right)
\]

with

\[
A(x) = -2C \frac{(x-1)(x-2)(2+3x)}{7},
\]

\[
B(y) = -2Cy(y-1).
\]

Remark 5.8. Normalizing the average scalar curvature of $(P_2, \nu)$ to be equal to the one of $(P_2, \eta)$ gives $C = 81/77$ and induces worst singularities than $C = 1$. The case $C = 1$ gives a singularity of angle $2\pi 5/7$ along the infinite section of $\mathbb{P}(\mathcal{O} + \mathcal{O}(-1))$ which is precisely the angle of singularity obtained by Székelyhidi [30] in the limit case of a construction (using Calabi ansatz) of metrics on $\mathbb{P}(\mathcal{O} + \mathcal{O}(-1))$ satisfying $\text{Ric}(\omega) \geq \frac{2}{3} \omega$, see also [26] [27].

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