ABSTRACT REACTION-DIFFUSION SYSTEMS WITH m-COMpletely ACcretive DIFFUSION OPERATORS AND MEASURABLE REACTION RATES

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(Dedicated to Michel Pierre on the occasion of his 60th birthday)

Abstract. We consider reaction-diffusion systems with merely measurable reaction terms to cover the possibility of discontinuities. Solutions of such problems are defined as solutions to appropriate differential inclusions which, in an abstract form, lead to evolution inclusions of the form

\[ u' \in -Au + F(t,u) \quad \text{on } [0,T], \quad u(0) = u_0, \]

where \( A \) is \( m \)-accretive and \( F \) is of upper semicontinuous type. While such problems, in general, can exhibit non-existence of solutions, the present paper shows that especially for \( m \)-completely accretive \( A \), and under reasonable assumptions on \( F \), mild solutions do exist.

1. Introduction. Consider the prototype model

\[ \partial_t u_i + A_i(D)u_i = g_i(u) \quad \text{in } (0,T) \times \Omega, \quad \partial\nu u_i |_{\partial \Omega} = 0, \quad u_i(0,\cdot) = u_{0,i}. \quad (1) \]

Here \( u = (u_1, \ldots, u_m) \) denotes the vector of concentrations of the involved chemical species, the differential operator \( A_i(D)u_i \) models the diffusion of \( u_i \) and \( g(u_1, \ldots, u_m) \) the reaction kinetics. The domain \( \Omega \subset \mathbb{R}^n \) is assumed to be bounded with smooth boundary. Such reaction-diffusion systems can be considered as abstract evolution problems of the type

\[ u_i' + A_i u_i \ni f_i(u) \quad \text{on } [0,T], \quad u_i(0) = u_{0,i}. \quad (2) \]

in appropriate function space settings \( X = L^p(\Omega; \mathbb{R}^m) \) for \( 1 \leq p < \infty \). The right-hand side in (2) is related to \( g \) from (1) by means of

\[ f(u)(x) := g(u(x)) \quad \text{for } x \in \Omega, \quad (3) \]

i.e. \( f \) is the so-called Nemitsky operator associated with \( g \).

The purpose of the present work is to study the existence of mild solutions of (2) under conditions which not only cover the case of nonlinear diffusion, e.g. governed by the filtration equation or the \( p \)-Laplacian, but allow also for discontinuous

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reaction terms. Such discontinuous rate functions occur for instance in modeling so-called zero-order reactions (see the references in [19]), in porous medium combustion in the limiting case of large activation energy (see [29], [34]; cf. also [23] for conditions that guarantee unique solvability for particular initial values), in climate modeling where the discontinuity is due to a jump of the planetary co-albedo as a function of the temperature (see [18]), and in the instantaneous limit case of irreversible concurring reactions (see §7 in [13]). Discontinuous right-hand sides also appear in mechanical applications, for example due to dry friction effects (see [12], [17] and the references given there). More general areas in which partial differential inclusions of the type considered here appear are parabolic variational inequalities [21] including hysteresis [37] and closed loop (distributed) control systems [22].

The case of discontinuous right-hand sides requires a more general solution concept in which the function $g$ is replaced by a multivalued "regularization" $G$, thus relaxing the imposed equality into an inclusion. Concerning discontinuous ordinary differential equations, see [24]. In the autonomous case and with $g$ defined on $D(g)$, a typical choice for such $G$ is given by

$$G(y) = \bigcap_{\delta > 0} \text{conv} g(B_\delta(y) \cap D(g)).$$

Let us note that a discontinuous "law" $g$ will often be an approximation of a more complicated, possibly continuous or locally Lipschitz model. If the latter corresponds to $h$ and if the graph of $h$ is, in a certain sense, sufficiently close to the graph of $g$, then a given solution of $y' = h(y)$ with initial value $y(0) = y_0$ will be close to some solution of $y' \in G(y)$, $y(0) = y_0$ with $G$ from above. If $g$ is only assumed to be measurable, the following slightly different definition due to A. F. Filippov, namely

$$G(y) = \bigcap_{\delta > 0} \bigcap_{N: \lambda_m(N) = 0} \text{conv} g((B_\delta(y) \cap D(g)) \setminus N)$$

seems sometimes more appropriate since values of $g$ on Lebesgue null sets often should play no role. But note that the set of all solutions to a given initial value problem will in general depend on the definition of $G$. In particular, uniqueness of solutions – if it holds at all – has to be proven separately. For more details see [24] and also §A.1 in [16].

In the general non-autonomous case and in an abstract formulation, this leads to evolution problems of type

$$u' \in -Au + F(t,u) \quad \text{on } J = [0, T], \quad u(0) = u_0$$

with $G$ defined in one of the ways mentioned above. Note that – in contrast to (1) – a componentwise formulation (i.e., separately for all $u_i$) is no longer possible, since the sets $G(y)$ will in general not be Cartesian products. Under the reasonable assumption that $g$ is locally bounded it is easy to check that $G : \mathbb{R}^m_+ \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ is upper semicontinuous with compact convex values and $G(y_0) = \{g(y_0)\}$ if $g$ is continuous at $y_0$; see Example 1.2 in [16].

In the general non-autonomous case and in an abstract formulation, this leads to evolution problems of type

$$u' \in -Au + F(t,u) \quad \text{on } J = [0, T], \quad u(0) = u_0,$$
where the multivalued forcing term $F: J \times X \to 2^X$ is defined as

$$F(t, u) := \{ w \in X : w(x) \in G(t, u(x)) \text{ a.e. on } \Omega \}$$

with domain of definition $D(F) := \{(t, u) \in J \times X : F(t, u) \neq \emptyset \}$. The notation in (5) allows for $A$ to be a set-valued map as well, which appears in certain applications. Instead of $u' \in -Au + F(t, u)$, the notation $(u' + Au) \cap F(t, u) \neq \emptyset$ is also in use; cf. [38]. By a solution of (5) we mean a continuous $u : J = [0, T] \to X$ such that $u$ is a mild solution of the quasi-autonomous problem

$$u' + Au \ni w(t) \quad \text{on } J, \quad u(0) = u_0,$$

with some $w \in L^1(J; X)$ satisfying $w(t) \in F(u(t))$ a.e. on $J$.

The generalization to discontinuous reaction kinetics leads to a substantial difficulty which is related to properties of the quasi-autonomous solution operator $S : L^1(J; X) \to C(J; X)$, defined as

$$Sw := u,$$

where $u$ is the unique mild solution of (7) for right-hand side $w$; (8) note that the initial value is fixed. This operator is non-expansive which follows from the standard integral inequalities for mild (integral) solutions; cf. section 3. While this is helpful to prove existence of solutions of (2) with continuous $f$, this alone is not enough in the multivalued case even under strong compactness assumptions on $A$. Indeed, the following question occurs: given a certain sequence of (approximate) solutions $u_k = Sw_k$, where $w_k \rightharpoonup w$ (weak convergence) in $L^1(J; X)$ and $u_k \to u$ in $C(J; X)$, does $Sw = u$ hold? In other words, under which assumptions does $S : L^1(J; X) \to C(J; X)$ have a weakly×strongly-closed graph?

In general, the answer is “no”, even if $-A$ generates a compact semigroup. Moreover, as has been shown in [11], this problem can in fact cause non-existence of solutions for the evolution inclusion (5) even in finite dimensions. Only under extra assumptions, this defect cannot occur. First, the answer to the question above is affirmative in the linear case, when $-A$ is the generator of a $C_0$-semigroup. In fact this also holds for operators of the type $A + g$, where $-A$ is the generator of a $C_0$-semigroup and $g : X \to X$ is continuous and accretive; see the proof of Theorem 3.5 in [13]. Second, $w_k \rightharpoonup w$ in $L^1(J; X)$ and $Sw_k \to u$ in $C(J; X)$ implies $Sw = u$ if $X^*$ is uniformly convex. This includes the case when $A$ is the subdifferential of a proper convex lower semicontinuous (lsc for short) functional in a Hilbert space.

In this paper we focus on another case, namely the one of $m$-completely accretive operators in so-called normal Banach spaces. The concept of $m$-completely accretive operators in normal Banach spaces has been introduced in [6]. It turned out that this particular class of $m$-accretive operators in a normal Banach space shares some properties which are similar to those known for $m$-accretive operators in Hilbert spaces or, more generally, uniformly convex spaces. The same remark applies to mild solutions of the Cauchy problem associated with an $m$-completely accretive operator in a normal Banach space. In [6] it has been shown, for example, that mild solutions of the homogeneous Cauchy problem, i.e. (7) with $w = 0$, are already strong solutions in case $u_0 \in D(A)$ (at least if the normal Banach space satisfies some extra convergence condition, which holds, for example, in the normal Banach space $L^1$). Later on this result has been extended in [26] to mild solutions of the inhomogeneous Cauchy problem.

The problem of existence of (local mild) solutions for (5) in infinite dimensions and for perturbations of upper semicontinuous type has been studied, e.g., in [2], [35], [11] and Chapter 3 of [38]. These references treat the case when $F$ is defined
on all of \( \overline{D(A)} \), while existence of solutions in closed subsets of \( X \) has been studied in \[9\] and \[14\]. For both cases, further details and results as well as additional references are included in \[13\].

Abstract formulations of reaction-diffusion systems of type (5) with specific diffusion operators of nonlinear filtration type have been considered e.g. in \[19\] and \[10\]. The present paper builds on \[10\] and extends the latter study to more general classes of admissible diffusion operators instead of the filtration equation.

Let us note that for general – but single-valued – nonlinear reaction-diffusion systems, the survey article \[27\] is still worth reading, while latest developments about global existence questions in case of RD-systems with linear diffusion and control of mass are reviewed in \[31\].

2. Preliminaries. For notations and definitions not explained in the sequel we refer to \[9\] and \[16\] for multivalued maps, and to \[3\], \[7\], \[28\] concerning accretive references are included in \[13\].

For both cases, further details and results as well as additional in \[9\] and \[14\]. For every \( \epsilon > 0 \) compact values; as a prototype one may think of the following situation:

\[
F(x) = \begin{cases} \lambda(x) & \text{if } x \in \Omega \cap A \setminus \emptyset, \\
\emptyset & \text{otherwise},
\end{cases}
\]

\[
A(x) \subseteq \text{compact}
\]

In general, \( \text{usc} \) is stronger than \( \epsilon\)-\( \delta \)-usc, but both concepts coincide if \( F \) has compact values; let us also note that compactness of \( \text{gr}(F) \) implies that \( F \) is \( \text{usc} \) with compact values. In applications one also has to consider multivalued maps having only weakly compact values; as a prototype one may think of the following situation:

\[
X = L^p(\Omega) \quad \Omega = [a,b] \subset \mathbb{R}, \quad p \in [1,\infty) \quad \text{and} \quad F : \Omega \to 2^X \setminus \emptyset \quad \text{given by}
\]

\[
F(u) = \{ w \in X : w(x) \in \text{Sgn}(u(x)) \text{ a.e. on } \Omega \},
\]

where \( \text{Sgn}((0) = \rho/|\rho| \text{ if } \rho \neq 0 \) and \( \text{Sgn}(0) = [-1,1] \). In such cases another concept is more natural. We call \( F \) weakly \( \text{usc} \) if \( F^{-1}(A) \) is closed for all weakly closed \( A \subseteq X \). Evidently \( \text{usc} \) is stronger than weakly \( \text{usc} \) and simple examples show that a weakly \( \text{usc} \) \( F \) with compact convex values may fail to be \( \text{usc} \). The following fact about weakly \( \text{usc} \) maps is Proposition 2 in \[11\].

**Proposition 1.** Let \( X \) be a Banach space, \( \Omega \neq \emptyset \) a subset of another Banach space and \( F : \Omega \to 2^X \setminus \emptyset \) have weakly compact values. Then the following holds.

(a) If \( F \) is \( \epsilon\)-\( \delta \)-usc then \( F \) is weakly usc.

(b) If the values of \( F \) are also convex, then \( F \) is weakly usc iff \( (x_n) \subseteq \Omega \) with \( x_n \to x_0 \in \Omega \) and \( y_n \in F(x_n) \) implies \( y_{n_k} \to y_0 \) (weak convergence) with \( y_0 \in F(x_0) \) for some subsequence \( (y_{n_k}) \) of \( (y_n) \).

Let \( \Omega \subseteq \mathbb{R}^n \) be Lebesgue measurable and \( \lambda_n \) denote the Lebesgue measure on \( \mathbb{R}^n \). Then \( F : \Omega \to 2^X \setminus \emptyset \) is called measurable if \( F^{-1}(V) \) is Lebesgue measurable for every open \( V \subseteq X \), which is satisfied if \( F^{-1}(A) \) is Lebesgue measurable for every closed \( A \subseteq X \). If \( X \) is separable and \( F \) has measurable values then \( F \) admits a measurable selection, and if \( \Omega \) is also bounded and the values of \( F \) are compact then \( F \) is Lusin, i.e. given \( \epsilon > 0 \) there is a closed \( \Omega_\epsilon \subseteq \Omega \) with \( \lambda_n(\Omega \setminus \Omega_\epsilon) \leq \epsilon \) such that \( F|_{\Omega_\epsilon} \) is continuous, where continuity is understood with respect to the Hausdorff metric. Proofs of these facts can be found in Chapter 1 of \[16\].
The next result is Proposition 2.1 in [10].

**Proposition 2.** Let \( G : \mathbb{R}_+^m \to 2^{\mathbb{R}^m} \setminus \emptyset \) be usc with compact convex values, \( \Omega \subset \mathbb{R}^n \) be measurable and bounded and let \( X = L^p(\Omega)^m \) with \( p \in [1, \infty) \). Then \( F : X \to 2^X \), defined by

\[
F(u) = \{ v \in X : v(x) \in G(u(x)) \text{ a.e. on } \Omega \},
\]

has nonempty, weakly compact and convex values for all \( u \in L^\infty(\Omega)^m \). Moreover, \( F \) is \( \epsilon, \delta \)-usc on every \( \cdot, \infty \)-bounded subset of \( X_+ := \{ u \in X : u_i \geq 0, i = 1, \ldots, m \} \).

While it is rather obvious that, in the situation of Proposition 2, \( F \) is weakly usc on \( \cdot, \infty \)-bounded subsets, such \( F \) need not be usc as shown in [10].

One way to prove existence of solutions of (5) on closed sets and for Carathéodory right-hand sides is a reduction to the almost usc case. Here \( F : D(F) \subset J \times X \to 2^X \setminus \emptyset \) is called almost usc if for every \( \epsilon > 0 \) there exists a closed \( J_\epsilon \subset J \) with \( \lambda_1(J \setminus J_\epsilon) \leq \epsilon \) such that the restriction of \( F \) to \( D(F) \cap [J_\epsilon \times X] \) is usc. If corresponding restrictions of \( F \) are \( \epsilon, \delta \)-usc, the \( F \) is called almost \( \epsilon, \delta \)-usc. Now the key point for the reduction to the almost usc case is a consequence of the next lemma which is contained in Theorem 1 in [32].

**Lemma 2.1.** Let \( X \) be a separable Banach space and \( K : J = [0, T] \subset \mathbb{R} \to 2^X \). Let \( F : gr(K) \to 2^X \) with closed values be such that \( gr(F(t, \cdot)) \) is closed in \( K(t) \times X \) for every \( t \in J \). Then there exists \( F_0 : gr(K) \to 2^X \) with closed values \( F_0(t, x) \subset F(t, x) \) such that

(a) If \( I \subset J \) is measurable and \( u, v : I \to X \) are measurable with \( v(t) \in F(t, u(t)) \) on \( I \) then \( v(t) \in F_0(t, u(t)) \) a.e. on \( I \).

(b) For every \( \epsilon > 0 \) there is a closed \( J_\epsilon \subset J \) with \( \lambda_1(J \setminus J_\epsilon) \leq \epsilon \) such that

\( F_0|_{[J_\epsilon \times X] \cap gr(K)} \) has closed graph.

Note that \( F_0 \equiv \emptyset \) is not excluded. We will apply this idea on the level of \( G \), i.e. in a finite dimensional space, and there we are able to obtain an almost usc \( G_0 \) such that \( \emptyset \neq G_0(t, x) \subset G(t, x) \) and the solution sets remain the same.

We call a multivalued map \( G : D(G) \subset J \times X \to 2^X \) locally integrably bounded if for every bounded \( B \subset X \) there is \( c_B \in L^1(J) \) such that \( \|G(t, y)\| \leq c_B(t) \) a.e. on \( J \) for every \( y \in B \) such that \( (t, y) \in D(G) \); here \( \|G(t, y)\| := \sup|z| : z \in G(t, y)\).
Lemma 2.2. Let $X$ be a real Banach space, $A : D(A) \subset X \to 2^X \setminus \emptyset$ be $m$-accretive, $I = [0, T) \subset \mathbb{R}$ and $K : I \to 2^X \setminus \emptyset$ with $K_A(t) := K(t) \cap D(A) \neq \emptyset$ on $I$ such that

$$\left(I + \lambda A\right)^{-1} K(t) \subset K(t) \quad \text{for all } \lambda > 0, \ t \in I. \tag{9}$$

Moreover, assume that $\rho(\cdot, K(\cdot))$ is lsc from the left and $T_K(\cdot, \cdot)$ is lsc with convex values. Then $\rho(\cdot, K_A(\cdot))$ is lsc from the left, and $v \in T_K^A(t, u)$ for $(t, u) \in gr(K_A)$ implies $v \in T_K^A(t, u)$.

(b) We now turn to basic facts about $m$-completely accretive operators; cf. [6]. Given a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ we denote by $M(\Omega)$ the set of real-valued measurable functions on $(\Omega, \Sigma, \mu)$. We consider the space

$$L_0(\Omega) = \{u \in M(\Omega) : \int_{\Omega} (|u| - k)^+ d\mu \leq \infty \text{ for any } k > 0\}$$

which is a closed linear subspace of

$$L^1(\Omega) + L^\infty(\Omega) = \{u \in M(\Omega) : \int_{\Omega} (|u| - k)^+ d\mu < \infty \text{ for some } k > 0\}$$

equipped with the norm

$$\|u\|_{1+\infty} = \inf_{k > 0} \left\{ k + \int_{\Omega} (|u| - k)^+ d\mu \right\},$$

and thus $u_n \to u$ in $L_0(\Omega)$ if and only if $\int_{\Omega} (|u_n - u| - k)^+ d\mu \to 0$ for all $k > 0$, and in this case there exists a subsequence $(u_{n_k})_k$ such that $u_{n_k} \to u$ almost everywhere on $\Omega$.

For elements $u, v \in M(\Omega)$ we write $u \ll v$ if

$$\int_{\Omega} j(u) d\mu \leq \int_{\Omega} j(v) d\mu \text{ for any } j \in J_0,$$

where $J_0 = \{j : \mathbb{R} \to [0, \infty) : j \text{ convex, lower semicontinuous with } j(0) = 0\}$.

A Banach space $(X, \| \cdot \|_X)$ with $X \subset M(\Omega)$ is called normal if

$$u \in M(\Omega), v \in X, u \ll v \Rightarrow u \in X \text{ and } \|u\|_X \leq \|v\|_X.$$

In this sense a normal Banach space is compatible with the order relation $\ll$.

Examples of normal Banach spaces are $(L_0(\Omega), \| \cdot \|_{1+\infty})$, the usual Lebesgue spaces $(L^p(\Omega), \| \cdot \|_p)$ for $1 \leq p \leq \infty$, $(L^1(\Omega) + L^\infty(\Omega), \| \cdot \|_{1+\infty})$, $L^1(\Omega) \cap L^\infty(\Omega)$ endowed with the norm $\| \cdot \|_{1\cap\infty} = \max\{\| \cdot \|_1, \| \cdot \|_\infty\}$, as well as Orlicz spaces $L_N(\Omega)$ endowed with the corresponding Luxembourg norm. For any normal Banach space $X$ one has the continuous injection $L^1(\Omega) \cap L^\infty(\Omega) \hookrightarrow X \hookrightarrow L^1(\Omega) + L^\infty(\Omega)$.

In the remainder of this section, $X$ always denotes a normal Banach space contained in $L_0(\Omega)$. An operator $A$ in $X$, i.e. a mapping $A$ from $X$ to $2^X$ which is identified with its graph $\{(x, y) : x \in X, y \in Ax\} \subset X \times X$, is called completely accretive if

$$u - \tilde{u} \ll u - \tilde{u} + \lambda(v - \tilde{v})$$

for all $(u, v), (\tilde{u}, \tilde{v}) \in A$ and $\lambda > 0$. As $X$ is normal it follows that a completely accretive operator is also accretive in the classical sense. An equivalent characterization
of complete accretivity is given by:

\[
\int_{\Omega} p(u - \tilde{u})(v - \tilde{v})d\mu \geq 0
\]

for all \((u, v), (\tilde{u}, \tilde{v}) \in A\) and any \(p \in \mathcal{P}_0\), where

\[
\mathcal{P}_0 = \{ p \in C^1(\mathbb{R}) : 0 \leq p' \leq 1, \text{supp } p' \text{ is compact, } 0 \notin \text{supp } p \}.
\]

In terms of the resolvent \(J_{\lambda}^A = (I + \lambda A)^{-1}\) of \(A\), complete accretivity of \(A\) means that \(J_{\lambda}^A : R(I + \lambda A) \to X\) is not only a contraction in the norm of \(X\), but also an \(N\)-contraction for every normal functional \(N : X \to (-\infty, \infty]\), i.e. for every functional \(N : X \to (-\infty, \infty]\) which satisfies

\[
u, v \in X, \, u \ll v \Rightarrow N(u) \leq N(v).
\]

In particular, resolvents of completely accretive operators are contractive in any \(L^p\)-norm, \(1 \leq p \leq \infty\), and also order-preserving. An operator \(A\) in \(X\) is called \(m\)-completely accretive if \(A\) is completely accretive and \(R(I + \lambda A) = X\) for \(\lambda > 0\). Recall that an \(m\)-accretive operator with order preserving resolvents in a Banach lattice is called \(m\)-\(T\)-accretive. Evidently, \(m\)-completely accretive operators belong to this class.

Now let \(A\) be an \(m\)-completely accretive operator in a normal Banach space \(X\). Then, as \(A\) is in particular \(m\)-accretive in \(X\), by the general theory of accretive operators and nonlinear semigroups (cf., e.g., [7], [3]) it follows that, for every \(w \in L^1(J; X)\) with \(J = [0, T]\) and \(u_0 \in \overline{D(A)}\), the Cauchy problem

\[
(CP) \quad u' + Au \geq w \text{ on } J, \; u(0) = u_0,
\]

has a unique mild solution \(u \in C(J; X)\). Moreover, if \(\tilde{u} \in C(J; X)\) is the mild solution of \((CP)\) with initial data \(\tilde{u}_0 \in \overline{D(A)}\) and right-hand side \(\tilde{w} \in L^1(J; X)\), then

\[
\|u(t) - \tilde{u}(t)\|_X \leq \|u(s) - \tilde{u}(s)\|_X + \int_s^t [u(\tau) - \tilde{u}(\tau), w(\tau) - \tilde{w}(\tau)] d\tau, \tag{11}
\]

for all \(0 \leq s \leq t \leq T\), where the bracket \([\cdot, \cdot]\) is defined by

\[
[x, y] := \lim_{\lambda \to 0^+} \frac{\|x + \lambda y\|_X - \|x\|_X}{\lambda}.
\]

The contraction principle (11) shows that the mild solution depends continuously on the data: if \((u_0)_n \subset \overline{D(A)}, u_{0n} \to u_0 \text{ in } X, (w_n)_n \subset L^1(J; X)\) and \(w_n \to w\) in \(L^1(J; X)\), then the mild solutions \(u_n\) of (CP) with data \(u_{0n}, w_n\) converge in \(C(J; X)\) to the unique mild solution of the Cauchy problem (CP) with data \(u_0, w\).

For many applications it would be useful to have the same result of continuous dependence of the mild solution on the data under the weaker assumption that \(w_n\) converges to \(w\) only weakly in \(L^1(J; X)\). Unfortunately, such a result does not hold in general. The situation is even worse: if the right-hand sides \(w_n\) are only weakly convergent to \(w\) in \(L^1(J; X)\), then even if one is able to prove, by some compactness argument, that the corresponding mild solutions \(u_n\) converge in \(C(J; X)\) to a function \(u\), then it may happen that the limit function \(u\) is not the mild solution of (CP) with data \(u_0, w\). If \(X^*\) is uniformly convex, this problem does not arise. In this case the duality mapping \(J : X \to X^*\) is single-valued and uniformly continuous on bounded sets, and \([x, y] = \langle J(x), y \rangle_{X^*} \text{ for } x, y \in X\), and therefore, even if \(w_n\) converges only weakly to \(w\) in \(L^1(J; X)\) one can easily
pass to the limit in the right-hand side of (11) (with \( u, w \) replaced by \( u_n, w_n \)), and this proves that \( u \) is a mild solution to (CP) with data \( u_0, w \).

We will prove a similar result in the case where \( A \) is \( m \)-completely accretive in a normal Banach space \( X \). A key point in the proof is the fact that mild solutions of the Cauchy problem (CP) for an \( m \)-completely accretive operator do not only satisfy the comparison principle (11) for the norm in \( X \), but a whole family of comparison principles for normal functionals. In particular, for any smooth function \( j \in J_0 \) with \( j' \in P_0 \) we have

\[
\int_{\Omega} j(u(t)-\tilde{u}(t))d\mu \leq \int_{\Omega} j(u(s)-\tilde{u}(s))d\mu + \int_{s}^{t} \int_{\Omega} j'(u(\tau)-\tilde{u}(\tau))(w(\tau)-\tilde{w}(\tau))d\mu d\tau \quad (12)
\]

for all \( 0 \leq s \leq t \leq T \).

This is a well-known result which easily follows from the fact that, by definition, a mild solution \( u \) of (CP) with data \( u_0, w \) is the uniform limit of \( \epsilon \)-discrete approximate solutions \( u^\epsilon \). Here, by an \( \epsilon \)-discrete approximate solution of (CP) we mean a piecewise constant function \( u^\epsilon \) with \( u^\epsilon(0) = u_0, u^\epsilon(t) = u_k \) for \( t \in (t_{k-1}, t_k] \), \( k = 1, \ldots, m \), where \( 0 = t_0 < t_1 < \ldots < t_m < T \) with \( t_k - t_{k-1} < \epsilon \) for \( k = 1, \ldots, m \), \( T - t_m < \epsilon \), and the \( u_k \) solve the following discrete scheme obtained by implicit Euler discretization of (CP)

\[
\frac{u_k - u_{k-1}}{t_k - t_{k-1}} + Au_k \geq w_k \quad \text{for } k = 1, \ldots, m, \quad (13)
\]

with \( \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|w_k - w(t)\|dt < \epsilon \). Let \( \tilde{u}^\epsilon \) be an \( \epsilon \)-discrete approximate solution of (CP) with data \( \tilde{u}_0, \tilde{w} \) for the same time discretization of \( J = [0, T] \) and for discrete data \( \tilde{w}_1, \ldots, \tilde{w}_m \) satisfying \( \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|\tilde{w}_k - \tilde{w}(t)\|dt < \epsilon \) and denote the corresponding discrete solutions of (13) by \( \tilde{u}_k, k = 1, \ldots, m \). Then, by (10), for \( k = 1, \ldots, m \),

\[
\int_{\Omega} j'(u_k - \tilde{u}_k) \left( w_k - \tilde{w}_k - \frac{u_k - u_{k-1}}{t_k - t_{k-1}} + \frac{\tilde{u}_k - \tilde{u}_{k-1}}{t_k - t_{k-1}} \right) d\mu \geq 0.
\]

As \( j(r) - j(s) \leq j'(r)(r - s) \) for \( r, s \in \mathbb{R} \), by convexity of \( j \), one easily deduces that

\[
\int_{\Omega} j(u_k - \tilde{u}_k)d\mu \leq \int_{\Omega} j(u_{k-1} - \tilde{u}_{k-1})d\mu + (t_k - t_{k-1}) \int_{\Omega} j'(u_k - \tilde{u}_k)(w_k - \tilde{w}_k)d\mu
\]

for all \( k \). Then summation over \( k = l, \ldots, n \) where \( l, n \) are such that \( t \in (t_{n-1}, t_n] \), and \( s \in (t_{l-1}, t_l) \), passing to the limit with \( \epsilon \to 0 \) yields (12).

3. The quasi-autonomous solution operator. If \( A \) is \( m \)-accretive, which will always be the case below, the concepts of mild and integral solutions of (7) coincide. Here a function \( u \in C(J; X) \) with \( u(0) = u_0 \) is called an integral solution of (7) if

\[
|u(t) - x| \leq |u(s) - x| + \int_{s}^{t} |u(\tau) - x, w(\tau) - y| d\tau \quad \text{for all } 0 \leq s \leq t \leq T, (x, y) \in A.
\]

The latter implies (11) which then yields the basic integral inequalities

\[
|u(t) - \tilde{u}(t)| \leq |u(s) - \tilde{u}(s)| + \int_{s}^{t} |w(\tau) - \tilde{w}(\tau)| d\tau \quad \text{for } 0 \leq s \leq t \leq T, \quad (14)
\]
whenever \( u \) and \( \tilde{u} \) are mild solutions of (7) for right-hand sides \( w, \tilde{w} \in L^1(J; X) \) and initial values \( u_0, \tilde{u}_0 \), respectively. If the initial values coincide then, in particular,

\[
|(Sw)(t) - (S\tilde{w})(t)| \leq \int_0^t |w(s) - \tilde{w}(s)|ds \quad \text{for} \quad t \in J;
\]

hence \( S : L^1(J; X) \to C(J; X) \) is a nonexpansive mapping.

We now consider the case of \( m \)-completely accretive operators in a normal Banach space. In this situation, the graph of \( S \) is weakly strongly closed, which is contained in

\textbf{Theorem 3.1.} Let \( A \) be an \( m \)-completely accretive operator in a normal Banach space \( X \subset L_0(\Omega) \), (\( u_{0n} \)) \( \subset \overline{D(A)} \) and (\( w_{0n} \)) \( \subset L^1(J; X) \), \( J = [0, T] \). For \( n \in \mathbb{N} \), let \( u_n \) be the mild solution of (CP) with data \( w_{0n}, w_n \). Assume that \( u_{0n} \) converges to some \( u_0 \) in \( X \), \( w_n \) converges weakly to a function \( w \) in \( L^1(J; X) \) and \( u_n \) converges to some function \( u \) in \( C(J; X) \). Then \( u \) is the mild solution of (CP) with data \( u_0, w \).

\textbf{Proof.} Note first that \( L^1(J; X) \) is continuously embedded in \( L^1(J; L_0(\Omega)) \). Denote by \( v = \lambda_1 \otimes \mu \) the product measure on the product \( \sigma \)-field \( L_1 \otimes \Sigma \), where \( L_1 \) denotes the \( \sigma \)-algebra of all Lebesgue measurable subsets of \( J \) and \( \lambda_1 \) is the one-dimensional Lebesgue measure. As usual we identify an abstract function \( v \in L^1(J; L_0(\Omega)) \) with the real-valued \( \nu \)-measurable function \( v \) defined by \( v(t, x) = v(t)(x) \) for almost all \( t \in J \) and \( x \in \Omega \). With this identification we also have the continuous embedding \( L^1(J; L_0(\Omega)) \hookrightarrow L_0(Q) \) where \( Q = J \times \Omega \). Clearly, we then also have \( C(J; X) \hookrightarrow L_0(Q) \). Therefore, it follows from the assumptions that

\[
\begin{align*}
    u_n &\to u \quad \text{strongly in} \ L_0(Q), \\
    w_n &\to w \quad \text{weakly in} \ L_0(Q).
\end{align*}
\]

Now consider a smooth function \( j \in \mathcal{J}_0 \) with \( j' \in \mathcal{P}_0 \). Let \( \hat{u} \) denote the unique mild solution of (CP) with data \( u_0, w \). According to (12) we have

\[
\int_\Omega j(u_n(t) - \hat{u}(t))d\mu \\
\leq \int_\Omega j(u_n(t) - w_n(t))d\mu + \int_0^t \int_\Omega j'(u_n(\tau) - \hat{u}(\tau))(w_n(\tau) - w(\tau))d\mu d\tau
\]

for all \( 0 \leq t \leq T \), for any \( n \in \mathbb{N} \).

From the strong convergence in \( L_0(Q) \) it follows that there exists a subset \( M \subset Q \) of finite measure and a subsequence of \( (u_n)_n \), still denoted by \( (u_n)_n \) in the following, such that

\[
\begin{align*}
    j'(u_n - \hat{u}) &= 0 \quad \text{almost everywhere on} \ Q \setminus M, \\
    j'(u_n - \hat{u}) &\to j'(u - \hat{u}) \quad \text{almost everywhere on} \ Q.
\end{align*}
\]

For convenience of the reader we recall the arguments used to prove the first claim. First note that as \( j' \in \mathcal{P}_0 \) there exists \( k > 0 \) such that \( j'(r) = 0 \) for all \( r \in [-k, k] \).

As \( u_n \to u \) in \( L_0(Q) \), one has \( \int_Q (|u_n - u| - k/2)^+ d\nu \to 0 \). Consequently, there exists a subsequence, still denoted by \( (u_n)_n \), such that \( \sum_n \int_Q (|u_n - u| - k/2)^+ d\nu < \infty \).

Set

\[
M := \cup_n \{ |u_n - \hat{u}| > k \} := \cup_n \{ (t, x) \in Q_1 \mid |u_n(t, x) - \hat{u}(t, x)| > k \}.
\]

As

\[
\{ |u_n - \hat{u}| > k \} \subset \left\{ |u_n - u| > \frac{3k}{4} \right\} \cup \left\{ |u - \hat{u}| > \frac{k}{4} \right\}
\]

\[
\int_\Omega j(u_n(t) - \hat{u}(t))d\mu \\
\leq \int_\Omega j(u_n(t) - w_n(t))d\mu + \int_0^t \int_\Omega j'(u_n(\tau) - \hat{u}(\tau))(w_n(\tau) - w(\tau))d\mu d\tau
\]

for all \( 0 \leq t \leq T \), for any \( n \in \mathbb{N} \).

From the strong convergence in \( L_0(Q) \) it follows that there exists a subset \( M \subset Q \) of finite measure and a subsequence of \( (u_n)_n \), still denoted by \( (u_n)_n \) in the following, such that

\[
\begin{align*}
    j'(u_n - \hat{u}) &= 0 \quad \text{almost everywhere on} \ Q \setminus M, \\
    j'(u_n - \hat{u}) &\to j'(u - \hat{u}) \quad \text{almost everywhere on} \ Q.
\end{align*}
\]

For convenience of the reader we recall the arguments used to prove the first claim. First note that as \( j' \in \mathcal{P}_0 \) there exists \( k > 0 \) such that \( j'(r) = 0 \) for all \( r \in [-k, k] \).

As \( u_n \to u \) in \( L_0(Q) \), one has \( \int_Q (|u_n - u| - k/2)^+ d\nu \to 0 \). Consequently, there exists a subsequence, still denoted by \( (u_n)_n \), such that \( \sum_n \int_Q (|u_n - u| - k/2)^+ d\nu < \infty \).
and as
\[ \nu\left(\left\{|u_n - u| > \frac{3k}{4}\right\}\right) \leq \frac{4}{k} \int_{\{|u_n - u| > \frac{3k}{4}\}} (|u_n - u| - \frac{k}{2}) d\nu \]
\[ \leq \frac{4}{k} \int_{Q} (|u_n - u| - \frac{k}{2})^+ d\nu \]
we have
\[ \nu(M) \leq \nu\left(\bigcup_n \left\{|u_n - u| > \frac{3k}{4}\right\}\right) + \nu\left(\left\{ |u - \tilde{u}| > \frac{k}{4}\right\}\right) \]
\[ \leq \frac{4}{k} \sum_n \int_{Q} (|u_n - u| - \frac{k}{2})^+ d\nu + \nu\left(\left\{ |u - \tilde{u}| > \frac{k}{4}\right\}\right) \]
\[ < \infty \]
and the claim is proved.

As a consequence, in the second integral on the right-hand side of (15) we may replace \(Q\) by the finite measure set \(M\). As \(j'(u_n - \tilde{u}) \rightarrow j'(u - \tilde{u})\) almost everywhere on \(M\), \((j'(u_n - \tilde{u}))_n\) is uniformly bounded in \(L^\infty(Q)\) and \((w_n - w) \rightarrow 0\) weakly in \(L_0(M) \equiv L^1(M)\) (as \(M\) is a set of finite measure), it follows by a classical result of integration theory (which involves Egorov’s theorem and uniform integrability of weakly convergent sequences in \(L^1\)) that
\[ \int_0^t \int_{\Omega} j'(u_n(\tau) - \tilde{u}(\tau))(w_n(\tau) - w(\tau))d\mu d\tau \]
\[ = \int_{M \cap (0,t) \times \Omega} j'(u_n(\tau) - \tilde{u}(\tau))(w_n(\tau) - w(\tau))d\nu \rightarrow 0. \]

Consequently, if we pass to the limit in (15) we find
\[ \int_{\Omega} j(u(t) - \tilde{u}(t))d\mu \leq 0 \quad \text{for all } t \in J, \]
and thus \(j(u(t) - \tilde{u}(t)) = 0\) a.e. on \(\Omega\), for all \(t \in J\). As the absolute value function \(j(r) = |r|\) may be approximated by a sequence of smooth functions \((j_n)_n \subset J_0\) with \(j'_n \in P_0\), it follows that \(u(t) = \tilde{u}(t)\) almost everywhere on \(\Omega\), for all \(t \in J\), hence \(u\) is indeed the mild solution of (CP) with data \(u_0, w\).

4. Existence and viability for abstract evolutions. We here consider the initial value problem (5) in the abstract situation when \(A\) is \(m\)-accretive in a real Banach space. To avoid problems concerning the continuation of local mild solutions, we impose the growth condition
\[ \|F(t, u)\| = \sup \{|y| : y \in F(t, u)\} \leq c(t)(1 + |u|) \quad \text{on } D(F) \quad (16) \]
for some \(c \in L^1(J)\). While this condition of at most linear growth appears rather restrictive, it should be noted that \(F\) can possibly be restricted to a smaller domain \(D(F)\) on which this condition is then fulfilled. Indeed in case (b) below, \(D(F)\) will be the graph of a multivalued map \(K : J \rightarrow 2^X \setminus \emptyset\) and this will lead to a rather “thin” domain for \(F\), e.g. an \(L^\infty\)-bounded set.

Below it will always be assumed that the values of \(F\) are at least weakly relatively compact. Then the following criterion for weak relative compactness in \(L^1(J; X)\) applies to the set of integrable selections of \(F\), i.e. to
\[ F(u) := \{ w \in L^1(J; X) : w(t) \in F(t, u(t)) \text{ a.e. on } J \}. \quad (17) \]
Lemma 4.1. Let $X$ be a Banach space, $J = [0,T] \subset \mathbb{R}$ and $W \subset L^1(J;X)$ be uniformly integrable. Suppose that there exist weakly relatively compact sets $C(t) \subset X$ such that $w(t) \in C(t)$ a.e. on $J$, for all $w \in W$. Then $W$ is weakly relatively compact in $L^1(J;X)$.

This is Corollary 2.6 in [20] specialized to Lebesgue measure.

(a) Existence without constraints.

Evidently $u$ is a mild solution of (5) iff $u \in C(J;X)$ is a fixed point of $H := S \circ F$, where $Sw$ denotes the unique mild solution of (7) corresponding to $w \in L^1(J;X)$ for fixed $u_0 \in \overline{D(A)}$ as defined in (8), and $F(u)$ is given by (17). Exploitation of (16) immediately yields a closed bounded convex subset of $C(J;X)$ which is invariant under $H$. In order to obtain a fixed point of $H$, additional assumptions are of course needed and in the multivalued case a further difficulty occurs, since the graph of $H$ need not be closed in general. For $m$-completely accretive operators, $\text{gr}(S)$ is weakly×strongly-closed due to Theorem 3.1, and then the following result applies under appropriate compactness conditions. Below, we call a subset $W \subset L^1(J;X)$ integrably bounded if there exists $\varphi \in L^1(J)$ such that $|w(t)| \leq \varphi(t)$ a.e. on $J$ for all $w \in W$.

Lemma 4.2. Let $J = [0,T] \subset \mathbb{R}$, $X$ be a real Banach space and $A$ be $m$-accretive such that $S : L^1(J;X) \rightarrow C(J;X)$ has weakly×strongly-closed graph and maps integrably bounded into relatively compact sets. Let $F : J \times \overline{D(A)} \rightarrow 2^X \setminus \emptyset$ with weakly compact convex values be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in \overline{D(A)}$, $F(t, \cdot)$ is weakly usc for almost all $t \in J$ and (16) is satisfied. Then (5) has a mild solution for every $u_0 \in \overline{D(A)}$.

Proof. We only sketch the main arguments, since the proof is very similar to the one of Lemma 3.3 in [13], and most of the arguments are also contained in [11]. First of all, $F$ can be extended to all of $J \times X$ such that the properties remain unchanged. This can be achieved if we let $\tilde{F}(t,x) = F(t,Px)$ on $J \times X$ with $P : X \rightarrow \overline{D(A)}$ given by $Px = J_{\lambda(x)}x$, where we define $\lambda(x) = \rho(x, \overline{D(A)})$ on $X$ and $J_0x := x$ on $\overline{D(A)}$. Next, using the characterization of Proposition 1, the assumptions on $F$ imply, as in the proof of Theorem 1 in [11], that $F$ is weakly usc with nonempty weakly compact convex values. Using (16) and (14), we obtain a bounded closed convex set $M_0 \subset C(J;X)$ which is invariant under $H$. Since $W := F(M_0)$ is integrably bounded, the set $M := \text{cl} \, H(M_0)$ is a compact convex subset of $C(J;X)$ due to the compactness property of $S$ and $M$ is invariant under $H$, too. It is then important to observe that the sets $F(u)$ are decomposable, which means that

$$\chi_A \, w + (1 - \chi_A) \overline{w} \in F(u)$$

for $w, \overline{w} \in F(u)$ and measurable $A \subset J$.

Then, employing once more the basic inequalities (14) for mild solutions of (7), the values of $H$ turn out to be contractible; cf. the proof of Theorem 1 in [11] and note that $H$ will in general not have convex values. Finally, since $\text{gr}(S)$ is weakly×strongly-closed, the graph of $H$ is closed. Indeed, given $(u_k)$ with $u_k \rightarrow u \in C(J;X)$ and $w_k \in F(u_k)$, we may assume $w_{km} \rightarrow w$ in $L^1(J;X)$ by Proposition 1 where $w \in F(u)$. If also $v_k \rightarrow v \in C(J;X)$ for $v_k \in H(u_k)$ then $v = Sw$ and, hence, $v \in H(u)$.

Consequently, a fixed point theorem for usc maps with closed contractible values (for instance the Corollary given in [25]) applies and yields the desired solution. □
We now have the subsequent existence result for (5), where we require the following compactness property of $S$:

$$\{(Sw_k)(t) : k \geq 1\} \text{ is relatively compact in } X \text{ for any } t \in (0,T) \text{ and any sequence } (w_k) \subset L^1(J;X) \text{ with } |w_k(t)| \leq \varphi(t) \text{ for } \varphi \in L^1(J).$$

(18)

Let us note that (18) holds if the semigroup generated by $-A$ is compact, but is a weaker condition. In fact, the difference is important in concrete applications. For instance, in case of $Au = -\Delta \phi(u)$, condition (18) holds whenever $\phi$ is continuous and strictly increasing, which is not enough to have a compact semigroup.

**Theorem 4.3.** Let $J = [0,T] \subset \mathbb{R}$, $X$ be a real Banach space and $A$ be $m$-accretive such that $S : L^1(J;X) \to C(J;X)$ has weakly*-strongly-closed graph and satisfies (18). Let $F : J \times \overline{D(A)} \to 2^X \setminus \emptyset$ with weakly compact convex values be such that $F(\cdot,x)$ has a strongly measurable selection for every $x \in \overline{D(A)}$, $F(t,\cdot)$ is weakly usc for almost all $t \in J$ and (16) is satisfied. Then (5) has a mild solution for every $u_0 \in \overline{D(A)}$.

**Proof.** In order to apply Lemma 4.2, we show that $S : L^1(J;X) \to C(J;X)$ maps integrably bounded into weakly compact sets. For this purpose, let $\varphi \in L^1(J)$ and $(w_k) \subset L^1(J;X)$ with $|w_k(t)| \leq \varphi(t)$. We let $u_k := Sw_k$. Evidently $(u_k)$ has relatively compact sections due to the compactness assumption (18). Let $\{S(t)\}_{t \geq 0}$ denote the semigroup generated by $-A$. Then, given $0 \leq s \leq t, t \leq T$, inequality (14) implies

$$|u_k(t) - u_k(t)| \leq |S(t-s)u_k(s) - S(t-s)u_k(s)| + \int_s^t \varphi(\tau) d\tau + \int_s^T \varphi(\tau) d\tau$$

$$\leq |S(|t-t|)u_k(s) - u_k(s)| + \int_s^t \varphi(\tau) d\tau + \int_s^T \varphi(\tau) d\tau.$$  

Now if $(u_k)$ is not equicontinuous, then $|u_k(t_k) - u_k(\tilde{t}_k)| \geq \epsilon_0 > 0$ with $t_k \to t, \tilde{t}_k \to t$ and $t = 0$ is not possible. Since $(u_k(s))$ is relatively compact for every $s \in J$, the estimate above yields the contradiction $\epsilon_0 \leq 2 \int_s^t \varphi(\tau) d\tau$ for all $s \in [0,t)$. Therefore $(u_k)$ is equicontinuous with relatively compact sections, thus relatively compact in $C(J;X)$ by Arzela/Ascoli. Application of Lemma 4.2 ends the proof. \hfill $\square$

**Remark 1.** The fixed point approach used above relies heavily on the fact that the associated quasi-autonomous evolution equation (7) admits a mild solution for every right-hand side in $L^1(J;X)$, but the latter is, in general, only valid for $m$-accretive $A$. In case $A$ is accretive and satisfies a range condition, without being $m$-accretive, solvability of (7) and, hence, of (5) is more difficult because the solutions need to stay in $\overline{D(A)}$ which no longer holds automatically but needs to be accounted for. Existence results for accretive operators which satisfy the (weak) range condition can be found, e.g., in [30], [15] and the references therein.

**b) Existence with time-dependent constraints.**

We first need further notation for this general case of time-dependent constraints, i.e. side conditions of the type $u(t) \in K(t)$ for given closed sets $K(t)$. We call a multivalued map $K : J \to 2^X$ a tube. Such a tube $K(\cdot)$ is called weakly positively invariant (or viable) for $u' \in -Au + F(t,u)$ if, for every $t_0 \in [0,T)$ and every initial
value \( u_0 \in K_A(t_0) \), the evolution problem

\[
 u' = -Au + F(t, u) \quad \text{on } [t_0, T], \quad u(t_0) = u_0
\]

has a mild solution such that \( u(t) \in K(t) \) on \([t_0, T] \). In order that solutions obey such constraints, some conditions are required which become active when a solution touches the boundary of \( K_A(t) \). Adapted from necessary conditions in the case of singlevalued forcing terms, i.e. \( F(t, u) = \{ f(t, u) \} \), we will require the following kind of "subtangential condition" (cf. [9]):

\[
 F(t, x) \cap T^A_K(t, x) \neq \emptyset \quad \text{for all } (t, x) \in \text{gr}(K_A) \text{ with } t < T,
\]

where \( T^A_K \) is defined by

\[
 T^A_K(t, x) = \{ z \in X : \lim_{h \to 0+} h^{-1} \rho(S_z(h)x, K_A(t + h)) = 0 \}. 
\]

Here \( S_z(\cdot) \) denotes the semigroup generated by \(-A_z\) with \( A_z x := Ax - z \) on \( D(A_z) = D(A) \). In the special case \( A = 0 \) this becomes

\[
 T_K(t, x) = \{ z \in X : \lim_{h \to 0+} h^{-1} \rho(x +hz, K(t + h)) = 0 \},
\]

and if, in addition, \( K(t) \equiv K \) holds then \( T^A_K(t, x) = T_K(x) \) is the Bouligand contingent cone with respect to \( K \) at the point \( x \); for the latter see, e.g., §4 in [16]. For \( K(t) \equiv K \), the set \( T^A_K(t, x) \) does not depend on \( t \) and we write \( T^A_K(x) \) then.

In the case of time-dependent constraints and with closed \( K(t) \), it is natural to assume that \( \text{gr}(K_A) \) is closed from the left, by which we mean that

\[
 (t_n) \subset J \text{ with } t_n \not\to t \text{ and } x_n \in K_A(t_n) \text{ with } x_n \to x \implies x \in K_A(t). \tag{23}
\]

Constraints of the type \( u(t) \in K(t) \) on \( J \) will be incorporated into the evolution inclusion (5) be means of defining \( F \) only on the graph of the tube \( K \). In this way it is avoided to first extend a given right-hand side \( F \) to all of \( J \times X \) in a way that preserves all the relevant properties of \( F \), to then show existence of a solution \( u \) without caring about the constraints and, finally, to prove that this solution respects the given constraints. In fact, after extending \( F \) to all of \( J \times X \) there might be solutions leaving the tube, while others satisfy \( u(t) \in K(t) \).

We also need the following simple fact which is Proposition 1 in [14]. In the sequel, given \( 0 \leq c \in L^1(J), \hat{c} \) always refers to the function provided by Proposition 4 below; note that the usual identification of functions which are equal almost everywhere is not done here.

**Proposition 4.** Let \( J = [0, T] \subset \mathbb{R} \) and \( c : J \to \mathbb{R}_+ \) with \( c \in L^1(J) \). Then there exists \( \hat{c} : J \to \mathbb{R}_+ \cup \{ \infty \} \) with \( \hat{c} \in L^1(J) \) having the following property: For every \( t_0 \in J \) there is \( \delta = \delta(t_0) > 0 \) such that

\[
 c(t_0) \leq \hat{c}(t) \quad \text{for all } t \in (t_0 - \delta, t_0 + \delta) \cap J.
\]

In presence of constraints, existence results usually have to be based on suitable approximate solutions. Note that the fixed-point approach from above is not possible, since the fixed-point map \( S \circ F \) maps a continuous function \( v \) to the solution \( u \) of the quasi-autonomous problem (7), the right-hand side \( w(t) \) being a selection of \( F(\cdot, \cdot, w(\cdot)) \). In this situation, the fact that \( v \) satisfies \( v(t) \in K(t) \) on \( J \) does not imply that \( u \) obeys the same constraints.

The next result is Lemma 1 in [14], and it also applies in the present multivalued case since no regularity of \( f \) is assumed. Below, \( u(\cdot; w) \) is used as a synonym for \( S w \).
Lemma 4.4. Let \( A \) be \( m \)-accretive in a real Banach space \( X \), \( J = [0, T] \subset \mathbb{R} \) and \( K : J \to 2^X \) with closed values be such that \( K_A(0) \neq \emptyset \) and \( Q = \text{gr}(K_A) \) is closed from the left. Suppose that \( f : Q \to X \) satisfies \( |f(t, x)| \leq c(t) \) on \( Q \) for some \( c \in L^1(J) \) and the subtangential condition
\[
f(t, x) \in T^A_K(t, x) \quad \text{for all } (t, x) \in Q \text{ with } t < T.
\]
Let \( J_0 \subset J \) be closed and \( Q_0 = [J_0 \times X] \cap Q \). Then, given \( u_0 \in K_A(0) \) and \( \epsilon > 0 \), there is a strongly measurable \( w : J \to X \) such that
\[
|w(t)| \leq \hat{c}(t) \quad \text{a.e. on } J,
\]
\[
w(t) \in f([J_{t, \epsilon} \ast B_{\gamma \epsilon}(u(t; w))] \cap Q_0) \quad \text{on } J_0,
\]
\[
w(t) \in f([J_{t, \epsilon} \ast B_{\gamma \epsilon}(u(t; w))] \cap Q) \quad \text{on } J
\]
with \( \gamma = 1 + T \), where \( J_{t, \epsilon} = [t - \epsilon, t] \).

To obtain a mild solution from approximate solutions, some compactness property is required. The following condition is weaker than (18), since here \( F \) is only defined on \( \text{gr}(K_A) \) and \( F(\text{gr}(K_A)) \) can be "thin" in the sense that some compactness holds for the set of all \( Sw \) with \( w \in F(v) \) for certain \( v \). The condition we impose is
\[
(w_k) \subset L^1(J; X) \text{ with } w_k(t) \in F(\text{gr}(K_A)) \text{ a.e. on } J \text{ such that } |w_k(t)| \leq \varphi(t) \text{ for } \varphi \in L^1(J) \text{ and } w_k \to w \text{ in } L^1(J; X) \implies
\]
\[
Sw_k \to Sw \text{ in } C(J; X) \text{ for every fixed initial value in } K_A(0).
\]

We now have

Theorem 4.5. Let \( J = [0, T] \subset \mathbb{R} \) be a real Banach space and \( A \) be \( m \)-accretive such that \( S : L^1(J; X) \to C(J; X) \) satisfies (28). Let \( K : J \to 2^X \setminus \emptyset \) be such that \( K_A(t) := K(t) \cap D(A) \) on \( J \) has \( \text{gr}(K_A) \) closed from the left. Let \( F : \text{gr}(K_A) \to 2^X \setminus \emptyset \) with closed convex values be almost \( \epsilon\)-\( \delta \)-usc, satisfying the growth condition (16) and the subtangential condition (20). Suppose that \( F \) maps bounded subsets of \( \text{gr}(K_A) \) into weakly relatively compact sets. Then (5) has a mild solution for every \( u_0 \in K_A(0) \).

Proof. In order to employ Lemma 4.4, we consider \( F \) on a certain smaller tube \( \hat{K}_A(\cdot) \) such that \( \|F(t, x)\| \leq \hat{c}(t) \) on \( \text{gr}(K_A) \) for some \( \hat{c} \in L^1(J) \). For this purpose fix \( x_0 \in D(A) \) with \( |u_0 - x_0| \leq 1 \), let \( r(\cdot) \) be the solution of
\[
r' = 1 + \hat{c}(t)(1 + r + |S(t)x_0|) \quad \text{on } J, \quad r(0) = 1
\]
and define \( \hat{K}(\cdot) \) by \( \hat{K}(t) := K(t) \cap \overline{B}_{r(t)}(S(t)x_0) \) on \( J \). Evidently, \( u_0 \in \hat{K}(0) \), \( \text{gr}(\hat{K}_A) \) is closed from the left and \( \|F(t, x)\| \leq \hat{c}(t) \) on \( \text{gr}(\hat{K}_A) \) for \( \hat{c} = \gamma c \) with sufficiently large \( \gamma > 1 \). As in [14], it is now straightforward to check that (20) also holds for \( \hat{K} \) instead of \( K \). Hence all assumptions of Theorem 4.5 are also satisfied if \( K \) is replaced by \( \hat{K} \).

Given a sequence \( \epsilon_k \searrow 0 \), there are closed \( J_k \subset J \) such that \( \lambda_1(J \setminus J_k) \leq \epsilon_k \) and \( F|[J_k \times X]|\text{gr}(K_A) \) is \( \epsilon\)-\( \delta \)-usc. We may also assume \( J_k \subset J_{k+1} \) for all \( k \geq 1 \). Applying Lemma 4.4 with \( f \) being any selection of \( F \), there are \( w_k \in L^1(J; X) \) such that \( |w_k(t)| \leq \hat{c}(t) \) a.e. on \( J \),
\[
w_k(t) \in F([J_{t, \epsilon_k} \ast B_{\gamma \epsilon_k}(u(t; w_k))] \cap \text{gr}(K_A)) \quad \text{on } J
\]
and
\[
w_k(t) \in F([J_{t, \epsilon_k} \ast B_{\gamma \epsilon_k}(u(t; w_k))] \cap \text{gr}(K_A) \cap [J_k \times X]) \quad \text{on } J_k
\]
with $\gamma = 1 + T$, where $J_{t,\epsilon_k} = [t - \epsilon_k, t]$.

Since $|u(t; w_k)| \leq |u_0| + \int_0^T \epsilon dt$ for every $k$, we have

$$w_k(t) \in F([J \times B_R(u_0)] \cap \text{gr}(K_A))$$

on $J$ for all $k$ with some large $R > 0$. Hence the values $w_k(t)$ are from a weakly relatively compact set, thus by Lemma 4.1 we may assume that $w_k \rightharpoonup w$ in $L^1(J; X)$, by selecting a subsequence. Then $u(\cdot; w_k) \rightarrow u(\cdot; w) =: u$ in $C(J; X)$ due to (28), and therefore (30) yields $w_k \in M_{\delta,m}$ for all large $k$, where

$$M_{\delta,m} = \{ v \in L^1(J; X) : v(t) \in \text{conv} F([J_t, \delta \times B_\delta(u(t))] \cap \text{gr}(K_A)) \cap [J_m \times X] \text{ a.e. on } J_m \}.$$

Consequently, $w \in M_{\delta,m}$ for all $\delta > 0$ and $m \geq 1$, since $M_{\delta,m} \subset L^1(J; X)$ is closed convex and, hence, weakly closed. Exploiting the fact that $F([J_m \times X] \cap \text{gr}(K_A))$ is $\epsilon$-$\delta$-usc, this implies $w(t) \in F(t, u(t))$ a.e. on $J_m$, since the $F(t, u)$ are also convex. This holds for any $m \geq 1$, hence $w(t) \in F(t, u(t))$ a.e. on $J$, and therefore $u$ is a mild solution of (5).

**Remark 2.** It would be interesting to know if the condition $w_k(t) \in F(\text{gr}(K_A))$ in (28) can be replaced by $w_k(t) \in F(t, K_A(t))$. This would be possible if appropriate approximate solutions are obtained which do not have the deviation in the $t$-variable introduced via $J_{t,\epsilon}$. For single-valued continuous $f$ instead of $F$ and for increasing (with respect to inclusion) $K(\cdot)$ this is possible and has been shown in [14]. In the present case it is open.

5. **Application to RD-systems with measurable reaction rates.** We finally come back to a non-autonomous version of the prototype model (1), i.e. we consider the system

$$\partial_t u_i + A_i(D)u_i = g_i(t, u) \quad \text{in } (0, T) \times \Omega, \quad \partial_\nu u_i|_{\partial \Omega} = 0, \quad u_i(0, \cdot) = u_{0,i} \quad (31)$$

with measurable right-hand side $g : (0, T) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$. As before, this system of partial differential equations is rewritten as the partial differential inclusion system

$$\partial_t u + A(D)u \in G(t, u) \quad \text{in } (0, T) \times \Omega, \quad \partial_\nu u_i|_{\partial \Omega} = 0, \quad u_i(0, \cdot) = u_{0,i}. \quad (32)$$

In (32), the right-hand side is defined as

$$G(t, y) = \bigcap_{\delta > 0} \text{conv} g(t, B_\delta(y) \cap \mathbb{R}^m_+). \quad (33)$$

Below we call $u$ a solution of (32) and, hence, of (31), if $u$ is a mild solution of (5) with $F$ defined by (6). Note that the existence result below would also hold for other definitions of $G$ as long as the $G(\cdot, x)$ admit measurable selections and the $G(t, \cdot)$ are usc, but possibly resulting in a different solution set.

(a) **Existence of non-negative solutions.**

It is well known that invariance techniques can be useful to obtain qualitative properties of solutions like nonnegativity, global existence and asymptotic stability, existence of periodic solutions etc.; cf. [33]. In the simplest case, the basic idea of this approach is as follows. Suppose that $C$ is a compact convex subset of $\mathbb{R}^m$ which is (weakly) positively invariant for the ordinary differential equation

$$y' = g(t, y) \quad \text{on } \mathbb{R}_+$$

and
associated to (31), and consider $K = \{u \in X : u(x) \in C \text{ a.e. on } \Omega\}$ with $X = L^p(\Omega)^m$, say. Then, for example, $K$ is (weakly) positively invariant for $u_t = D\Delta u + g(t,u)$ with $D = \text{diag}(d_1, \ldots, d_m)$, given that $K$ is positively invariant for $u_t = D\Delta u$. Of course the latter condition implies severe restrictions on the structure of the sets $C$. In particular, in case of different diffusion coefficients ($d_i \neq d_k$ if $i \neq k$) it turns out that $C$ has to be a “rectangle” such that $0 \in C$, i.e. $C = [a,b] \subset \mathbb{R}^m$ with $a_k \leq 0 \leq b_k$; here $[a,b]$ is short for $\prod_{k=1}^m \{a_k, b_k\}$ in case $[a,b] \subset \mathbb{R}^m$.

Below, this approach is generalized and adapted to the model problem under consideration. For this purpose we look for a tube $C(t) = [0, y(t)]$ with $y : [0, \tau) \to \mathbb{R}^m_+$ such that $C(\cdot)$ is bounded on bounded intervals and weakly positively invariant for $y' \in G(t,y)$. We then consider

$$K(t) := \{u \in X : u(x) \in C(t) \text{ a.e. on } \Omega\},$$

and if $K(\cdot)$ turns out to be viable for the abstract RD-system (5) corresponding to (32), then (5) has a solution on $[0, \tau)$ provided that $u_0 \in K_A(0)$. A good candidate for $y(\cdot)$ would of course be the maximal $- \text{ with respect to the partial (componentwise) ordering } \leq \text{ on } \mathbb{R}^m_+ \text{ – solution of } y' \in G(t,y)$ with initial value $y_0 = (|u_{0,1}|, \ldots, |u_{0,m}|)$. But such a maximal solution need not exist unless $G$ has a quasimonotone maximal selection $\hat{g}$, i.e. $\hat{g} : \mathbb{R}^m_+ \to \mathbb{R}^m$ such that $\hat{g}_k(t, \cdot)$ is increasing in $y_j$ for all $j \neq k$ and $\hat{g}(t,y) \in G(t,y) \subset \hat{g}(t,y) - \mathbb{R}^m_+$ on $\mathbb{R}^m_+ \times \mathbb{R}^m_+$.

Still, one may consider the smallest quasimonotone function above $G$. So, let

$$\begin{align*}
\bar{g}_k(t,y) &:= \max\{z_k : z \in G(t,y)\} \quad \text{on } \mathbb{R}^m_+, \\
\hat{g}_k(t,y) &:= \max\{\bar{g}_k(t,z) : 0 \leq z \leq y, z_k = y_k\} \quad \text{on } \mathbb{R}^m_+, \\
\hat{G}(t,y) &:= \{z \in \mathbb{R}^m : \exists \bar{z} \in G(t,y) \text{ for some } \bar{z} \}
\end{align*}$$

In this situation we get the existence of a (local) maximal $- \text{ solution of } y' \in \hat{G}(t,y)$ on $\mathbb{R}^m_+$, $y(0) = y_0$. (35)

For $y_0 \geq 0$ this solution is nonnegative, provided that for every $y \geq 0$ with $y_k = 0$ there is $z \in \hat{G}(t,y)$ such that $z_k \geq 0$. Since $\hat{G}(t,y) \supset G(t,y)$ on $\mathbb{R}^m_+$, this property will always hold if $g$ satisfies $\hat{g}_k(t,y) \geq 0$ whenever $y_k = 0$, which is a natural assumption if $g$ models a chemical reaction. We call such $g$ quasi-positive and we say that $g$ is locally integrably bounded, if for every bounded $B \subset \mathbb{R}^m_+$ there is $c = c_B \in L^1_{\text{loc}}(\mathbb{R}^m_+)$ such that $|g(t,y)| \leq c_B(t)$ a.e. on $\mathbb{R}^m_+$ for all $y \in B$.

This maximal solution of (35) is the appropriate candidate for the construction of the tubes $C(t)$ and $K(t)$, up to some regularization. This approach yields

**Theorem 5.1.** Let $\Omega \subset \mathbb{R}^m$ be open bounded, $X = L^p(\Omega)^m$ with $m \geq 1$, $p \in [1, \infty)$ and $A$ be $m$-$T$-accretive in $X$ with $0 \in A(0)$ such that $(I + \lambda A)^{-1}u \leq u$ if $\lambda > 0, u \equiv \alpha \in \mathbb{R}^m_+$. Suppose also that if $w_k$ is bounded in $L^\infty(J; L^\infty(\Omega)^m)$ with $w_k \to w$ in $L^1(J, X)$ then $u(\cdot; w_k) \to u(\cdot; w)$ in $C(J, X)$ for every fixed initial value in $L^\infty(\Omega)^m \cap D(A)$. Let $g : \mathbb{R}^m_+ \to \mathbb{R}^m$ be $L^1 \otimes B_m$-measurable, quasi-positive and locally integrably bounded. Then (31) has a local solution for every $u_0 \in L^\infty(\Omega)^m \cap D(A)$. This solution exists at least on $[0, \tau)$, where $\tau$ is the maximal time of existence of the maximal (with respect to $\leq$) solution $\hat{y}$ of (35) with $\hat{G}$ defined in (34), and initial value $y_0 = (|u_{0,1}|, \ldots, |u_{0,m}|)$.

*Proof.* Given $u_0$ as above and $y_0 := (|u_{0,1}|, \ldots, |u_{0,m}|)$, let us first show that (35) has a maximal (with respect to $\leq$) solution on $[0, \tau)$ for a $\tau > 0$. Note first
that $G$, defined by (33), is such that all $G(\cdot, y)$ have a measurable selection. The map $G$ is locally integrably bounded, hence the $G(t, \cdot)$ are usc with compact convex values. Let $\hat{G}$ be given by (34) and note that $\hat{G}$ has the same properties as $G$.

By Proposition 3 there is an almost usc $\hat{G}_0 : \mathbb{R}^{m+1}_+ \to \mathbb{R}^m$ with compact convex values such that $\emptyset \neq \hat{G}_0(t, y) \subset \hat{G}(t, y)$ on $\mathbb{R}^{m+1}_+$. Moreover, the solution sets for $y' \in \hat{G}(t, y)$ and $y' \in \hat{G}_0(t, y)$ coincide. Application of Theorem 1 of [8] to $\hat{G}(t, y) = \hat{G}(t, y^+)$ with $y^+_i \coloneqq \max\{y_i, 0\}$ then yields the (local) existence of a maximal (with respect to $\leq$) solution; note that $\hat{y}(t, y) := \hat{g}(t, y^+)$ is quasimonotone in $y$ with respect to $\mathbb{R}^m_+$ and $\hat{G}(t, y) = \hat{G}(t, y)$ on $\mathbb{R}^{m+1}_+$. This maximal solution can be extended up to a non-continuable maximal solution $\tilde{y}$, defined on a maximal interval of existence $[0, \tau)$ with appropriate $\tau > 0$.

In general, $\tilde{y}$ is only absolutely continuous and therefore, in order to be able to check the subdifferential condition for $F$ with respect to $K$, we regularize $\hat{G}$ and $\tilde{y}$ by approximation. Let $J = [0, \tau - h]$ for small $h > 0$. First, given $\epsilon > 0$ there is a closed $J_\epsilon \subset J$ with $\lambda_1(J \setminus J_\epsilon) \leq \epsilon$ such that $\hat{G}_{0|J_\epsilon \times \mathbb{R}^m}$ is usc. We may assume that, at the same time, $\hat{G}_{0|J_\epsilon \times \mathbb{R}^m}$ is usc, where $\hat{G}_0 \subset G$ is almost usc and given by Lemma 3. Let $P_\epsilon(t)$ denote the metric projection onto $J$, and define $G_\epsilon(t, y) := \text{conv} G(P_\epsilon(t), y)$ as well as $\check{G}_\epsilon(t, y) := \text{conv} \hat{G}(P_\epsilon(t), y)$. Again by Theorem 1 of [8], the differential inclusion (35) with $G_\epsilon$ instead of $G$ has a (local) maximal (with respect to $\leq$) solution which has a non-continuable extension to $\tilde{y}_\epsilon : [0, \tau_\epsilon) \to \mathbb{R}^m_+$. Furthermore, we have $\limsup_{\epsilon \to 0^+} \tau_\epsilon \geq \tau$, since any sequence $(\tilde{y}_{\epsilon_k})$ for $\epsilon_k \to 0^+$ has a subsequence which converges to a solution of (35). Moreover, since solutions of (5) for $F_{\epsilon}$ corresponding to $G_\epsilon$ will -- up to subsequences -- converge to a solution of (5) for $F$, it suffices to consider $G_\epsilon$ instead of $G$, i.e. the case of usc $G$.

A further approximation is required. Due to Lemma 2.2 in [16] there are locally Lipschitz multivalued maps $H^j : \mathbb{R}^{m+1}_+ \to 2^{\mathbb{R}^m} \setminus \emptyset$ such that

$$
\tilde{G}(t, y) \subset H^j_+(t, y) \subset H^j(t, y) \subset \text{conv} \hat{G}(B_{r_j}(t, y) \cap \mathbb{R}^{m+1}_+) \quad \text{on } \mathbb{R}^{m+1}_+ \text{ for all } i \geq 1
$$

(36)

with $r_j = 3^{1-j}$. If we let $h^j(t, y) = \max H^j(t, y)$, the maximum taken component-wise, then $h^j$ is locally Lipschitz, hence the initial value problem

$$
y' = h^j(t, y) \quad \text{on } \mathbb{R}_+, \quad y(0) = y_0
$$

has a unique solution $y^j(\cdot)$ with $[0, \tau_j]$ as the maximal interval of existence. Moreover, $y^j(\cdot) \geq 0$ since $h^j$ is quasi-positive. As before, $\limsup_{j \to \infty} \tau_j \geq \tau$ and we only need to show the existence of a solution of (5) on $[0, \tau_j]$. We may assume $\tau_j > \tau - h$ and let $C_j(t) = [0, y^j(t))$ as well as

$$
K_j(t) = \{u \in X : u(x) \in C_j(t) \text{ a.e. on } \Omega\}.
$$

The remainder of the proof is very close to step 2 of the proof of Theorem 4.1 in [10]; we will not reproduce all details but recall the main facts. First,

$$
T_{C_j}(t, y) = \{z \in \mathbb{R}^m : z_k \geq 0 \text{ if } y_k = 0, z_k \leq y^j_k(t) \text{ if } y_k = y^j_k(t)\} \quad \text{on } \text{gr } (C^j),
$$

hence $T_{C_j}(\cdot, \cdot)$ is lsc with closed convex values since $y^j$ is continuously differentiable. Using the fact that $z \in T_{C_j}(t, y)$ implies $\lim_{h \to 0^+} h^{-1} \rho(y + hz, C_j(t + h)) = 0$ here, it follows that

$$
T_{K_j}(t, u) = \{w \in X : w(x) \in T_{C_j}(t, u(x)) \text{ a.e. on } \Omega\} \quad \text{on } \text{gr } (K^j).
$$
This directly shows that all $T_{K^j}(t,u)$ are closed and convex. It also implies that $T_{K^j}(\cdot,\cdot)$ is lsc on $gr(K^j)$.

Let $F : gr(K^j) \to 2^X$ be defined by

$$F(t,u) := \{ w \in X : w(x) \in G_0(t,u(x)) \text{ a.e. on } \Omega \}. $$

By Proposition 2, $F$ is almost $\epsilon,\delta$-usc on $gr(K^j)$, since the latter is a bounded subset of $L^\infty(J;L^\infty(\Omega))$. Since $G_0$ is locally integrably bounded, the values of $F$ are weakly relatively compact and also closed convex. On $gr(K^j)$, the map $F$ is actually bounded. It remains to verify $F(t,u) \cap T_{K^j}(t,u) \neq \emptyset$ for every $t \in [0,\tau-h)$ and $u \in K^j(t)$. Again, up to obvious modifications, this is included in the proof of Theorem 4.1 in [10]. Evidently, the distance $\rho(\cdot, K^j(\cdot))$ is lsc due to $y^j \in C^1$. By the assumptions on $A$, the resolvents of $A$ leave the sets $K^j(t)$ invariant. Hence Proposition 2.2 applies, showing that $F$ satisfies the subtangential condition with respect to $K^j$, and $gr(K^j)$ is closed from the left. Furthermore, the solution operator $S$ associated with $A$ satisfies the compactness assumption (28), due to the fact that $gr(K^j)$ is bounded in $L^\infty(J;L^\infty(\Omega))$. Therefore, all assumptions of Theorem 4.5 are fulfilled, hence (5) has a mild solution on $[0, \tau-h]$. Since $h > 0$ was arbitrary small, the result follows.

(b) Examples of admissible diffusion operators.

(i) A typical example of an $m$-completely accretive operator in $L^1(\Omega)$, $\Omega$ a bounded open subset of $\mathbb{R}^N$, is the operator $A$ induced in $L^1(\Omega)$ by the $p$-Laplacian which is formally given by the differential expression $-\Delta_p(u) = -\text{div}(|Du|^{p-2}Du)$ ($1 < p < \infty$) with standard boundary conditions. More generally, the operator induced in $L^1(\Omega)$ by a Leray-Lions type operator of the form $-\text{div} a(x,Du)$ with $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ being a Carathéodory function which is monotone in the second variable and satisfies the following coercivity and growth estimate:

$$\exists \lambda > 0 \text{ such that } a(x,\xi) \cdot \xi \geq \lambda |\xi|^p \text{ a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N,$$

$$\exists c > 0, a_0 \in L^p(\Omega) \text{ such that } |a(x,\xi)| \leq a_0(x) + c|\xi|^{p-1} \text{ a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N$$

is $m$-completely accretive in $L^1(\Omega)$ for standard boundary conditions (see e.g. [4] for the corresponding operator with homogeneous Dirichlet boundary condition and $1 < p < N$; see e.g. [1] for the case of Neumann or more general boundary conditions). Moreover, $0 \in A(0)$ and thus, clearly, $J_A^\alpha u \leq u$ for $\alpha \in \mathbb{R}_+$. As the operator $A$ associated with the $p$-Laplacian and homogeneous Dirichlet, Neumann or mixed Dirichlet-Neumann boundary condition is homogeneous of order $p-1$, the semigroup generated by $-A$ is equicontinuous (see [5]). Moreover, resolvents of $A$ are compact, at least if $p \geq 2$ (this follows from the a-priori-estimates for renormalized, respectively entropy solutions of the diffusion equation with right-hand side in $L^1$ proved in [4], respectively [1]). For $1 < p < 2$ and for the more general Leray-Lions type operators as above one can still at least prove the following weaker compactness property for the solution operator: let $J = [0,T]$ and $Q = J \times \Omega$. If $(w_n)_n$ is bounded in $L^\infty(Q)$ and $w_n \rightharpoonup w$ in $L^1(J;L^1(\Omega))$, then $S(w_n) \rightharpoonup S(w)$ in $C(J;L^1(\Omega))$. This follows from the fact that for $L^\infty$-data (i.e. $L^\infty$ right-hand side and $L^\infty$-initial data) mild solutions are weak solutions. Moreover, the set of weak solutions corresponding to a set of data bounded in $L^\infty$ satisfies a uniform bound in $L^\infty(Q)$ and in $V = \{ u \in L^p(J;W^{1,p}(\Omega)); u_t \in L^p(J;(W^{1,p}(\Omega)^*)) \}$, thus, by Aubin-Lions lemma, is relatively compact in $L^1(J;L^{2-\epsilon}(\Omega))$ for any $0 < \epsilon \leq 1$. By the contraction principle in $L^{2-\epsilon}(\Omega)$ the relative compactness of the solution set in
A further class of admissible diffusion operators are operators of the type \( A + B \) with \( A \) as in (i) above and \( B \) being the operator induced by a maximal monotone graph \( \beta \). More precisely, we have the following result.

**Lemma 5.2.** Let \( A \) be an \( m \)-completely accretive operator in \( L^p(\Omega) \), \( 1 \leq p < \infty \), \((\Omega, \Sigma, \mu)\) a finite measure space. Assume \( 0 \in A(0) \) and that, for fixed \( u_0 \in D(A) \), the solution operator \( S \) of the Cauchy problem for \( A \) has the following compactness property: if \( (w_n)_n \) is bounded in \( L^\infty(Q) \) for \( Q = J \times \Omega \) with \( J = [0, T] \) and \( w_n \rightharpoonup w \) in \( L^2(J; L^2(\Omega)) \), then \( S(w_n) \to S(w) \) in \( C(J; L^1(\Omega)) \). Moreover, let \( B \) be the realization in \( L^p(\Omega) \) of a maximal monotone graph \( \beta \subset \mathbb{R}^2 \) with \( 0 \in \beta(0) \). Then \( A + B \) is \( m \)-completely accretive in \( L^p(\Omega) \), \( 0 \in (A + B)(0) \) and \( J_{A+B}^\lambda u \leq u \) for \( u \equiv \alpha \in \mathbb{R}_+ \) and the solution operator for the evolution problem for \( A + B \) satisfies the same compactness property as the one for \( A \).

**Proof.** By the results of [6], \( A + B \) is \( m \)-completely accretive, and as \( 0 \in (A + B)(0) \) by assumption, the property \( J_{A+B}^\lambda u \leq u \) for \( u \equiv \alpha \in \mathbb{R}_+ \) follows immediately from complete accretivity. In order to prove the compactness property for the solution operator corresponding to \( A + B \), according to Theorem 3.1, it is sufficient to show that, for fixed initial data \( u_0 \in \overline{D(A + B)} \), the solution operator for the evolution problem for \( A + B \) has the property: if \( w_n \rightharpoonup w \) in \( L^2(J; L^2(\Omega)) \), then there is a subsequence \( (w_{n'}) \) such that \( S(w_{n'}) \) converges strongly in \( C(J; L^p(\Omega)) \) to some function. Without loss of generality (the general case follows from density arguments and the contraction principle for mild solutions) we may assume that \( w_n \in W^{1,1}(J; L^p(\Omega)) \) and \( u_0 \in D(A + B) \) with \( \beta^0(u_0) \in L^\infty(\Omega) \) (where \( \beta^0 \) denotes the minimal section of the graph \( \beta \)). For such data we know that the mild solution of the Cauchy problem for \( A + B \) and data \( u_0, f_n \) is actually a strong solution; cf. [26]. Note also that, by the results of nonlinear semigroup theory, this strong solution \( u_n \) is the limit in \( C(J; L^p(\Omega)) \) as \( \lambda \to 0 \) of the strong solution \( u_\lambda^\lambda \) of the Cauchy problem for \( A + B_\lambda \) with data \( u_0, w_n \). Here \( B_\lambda \) denotes the usual Yosida approximation of \( B \). Clearly, \( B_\lambda \) is the operator induced in \( L^p(\Omega) \) by \( \beta_\lambda \), the Yosida approximation of the maximal monotone graph \( \beta \). For \( j \in J_0 \) with \( j' \in \mathcal{P}_0 \), by complete accretivity of \( A \), we have

\[
\int_{\Omega} j(u_n^\lambda(T)) \, d\mu + \int_0^T \int_{\Omega} \beta_\lambda(u_n^\lambda) j'(u_n^\lambda) \, d\mu \, d\tau \leq \int_0^T \int_{\Omega} w_n j'(u_n^\lambda) \, d\mu \, d\tau + \int_{\Omega} j(u_0) \, d\mu.
\]

Let \( k > 0 \). Choosing \( j = j_\varepsilon \) such that \( j'_\varepsilon \) remains uniformly bounded and \( j'_\varepsilon(r) \to \text{sign}^+(\beta_\lambda(r) - k) \) for all \( r \in \mathbb{R} \) as \( \varepsilon \to 0 \), in the limit (exploiting the fact that the first integral on the left-hand side is non-negative) we find

\[
\int_0^T \int_{\Omega} \beta_\lambda(u_n^\lambda) \text{sign}_0^+(\beta_\lambda(u_n^\lambda) - k) \, d\mu \, d\tau \leq \int_0^T \int_{\Omega} w_n \text{sign}_0^+(\beta_\lambda(u_n^\lambda) - k) \, d\mu \, d\tau + \int_{\Omega} (\beta_\lambda(u_0) - k)^+ \, d\mu.
\]
Subtracting $\int_0^T \int_\Omega k \text{sign}^+(\beta_\lambda(u_n^\lambda) - k)$ on both sides, taking into account that $\beta_\lambda(r) \leq \beta^0(r)$ for $r \geq 0$, $\lambda > 0$, we obtain
\[
\int_0^T \int_\Omega (\beta_\lambda(u_n^\lambda) - k)^+ \, d\mu \, d\tau \leq \int_0^T \int_\Omega (w_n - k)^+ \, d\mu \, d\tau + \int_\Omega (\beta^0(u_0) - k)^+ \, d\mu.
\]
(37)

In a similar way one can prove
\[
\int_0^T \int_\Omega (\beta_\lambda(u_n^\lambda) + k)^- \, d\mu \, d\tau \leq \int_0^T \int_\Omega (w_n + k)^- \, d\mu \, d\tau + \int_\Omega (\beta^0(u_0) + k)^- \, d\mu.
\]
(38)

Therefore, it follows that $\| \beta_\lambda(u_n^\lambda) \|_{L^\infty(Q)} \leq \| w_n \|_{L^\infty(Q)} + \| \beta^0(u_0) \|_{L^\infty(\Omega)}$, and thus, as $(w_n)_n$ is bounded in $L^\infty(Q)$, $\beta_\lambda(u_n^\lambda)_{\lambda}$ is also bounded in $L^\infty(Q)$, for each fixed $n$. Therefore there exists a subsequence such that $\beta_\lambda(u_n^\lambda)$ converges weak* in $L^\infty(Q)$ and, as $\Omega$ is a set of finite measure, also weakly in $L^p(Q) \simeq L^p(J; L^p(\Omega))$ to some function $b_\lambda \in L^\infty(Q)$. As $u_n^\lambda \to u_n$ in $C(J; L^p(\Omega))$, by classical arguments for maximal monotone graphs, $b_n \in \beta(u_n)$ a.e. on $Q$. Using the compactness property of the solution operator for the evolution problem for $A$ and Theorem 3.1 we deduce that $u_n$ is the mild (and, in fact, even strong) solution of $\frac{d}{dt} u_n + Au_n \ni b_n + w_n$ on $(0, T)$ with $u_n(0) = u_0$. From (37) and (38) and the weak convergence of $\beta_\lambda(u_n^\lambda)$ to $b_n$ in $L^1(Q)$ it follows that $b_n$ satisfies the same estimates (37) and (38), and therefore also $(b_n)_n$ is weakly relatively compact in $L^p(J; L^p(\Omega))$. Therefore there exists a subsequence such that $b_{n'}$ converges weakly in $L^1(J; L^p(\Omega))$ to some function $b$. Using now again the compactness property of the solution operator for $A$ and Theorem 3.1 it follows that $u_{n'}$ converges in $C(J; L^p(\Omega))$ to the mild solution $u$ of the Cauchy problem $u' + Au \ni w - b$ on $(0, T)$, $u(0) = u_0$.

(c) Existence of solutions to reaction-diffusion equations with $m$-completely accretive diffusion operators and measurable reaction rates.

Combining the above results about $m$-completely accretive operators with the invariance approach to reaction-diffusion inclusions, we obtain

Corollary 1. Let $\Omega \subset \mathbb{R}^m$ be open bounded and $A = (A_1, \ldots, A_m)$ an operator in $X = L^1(\Omega)^m$ with $A_i$ $m$-completely accretive in $L^1(\Omega)$ such that $0 \notin A(0)$. Suppose that, for all $t \in J := [0, T]$ and fixed initial value in $L^\infty(\Omega)^m_+ \cap D(A)$, the set $\{(Sw_k(t) : k \geq 1\}$ is relatively compact whenever $(w_k)$ is bounded in $L^\infty(J \times \Omega)^m$ with $w_k \to w$ in $L^1(J \times \Omega)^m$. Let $g : J \times \mathbb{R}^m_+ \to \mathbb{R}^m$ be $L^1 \otimes B_{m}$-measurable, quasi-positive and locally integrably bounded. Then (31) has a local (non-negative) solution for every $u_0 \in L^\infty(\Omega)^m_+ \cap D(A)$. This solution exists at least on $[0, \tau)$, where $\tau$ is the maximal time of existence of the maximal (with respect to $\leq$) solution $\hat{y}$ of (35) with $\hat{G}$ defined in (34) and initial value $y_0 = (|u_{0,1}|, \ldots, |u_{0,m}|, \infty)$.

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