Remarks on finite-time blow-up in a fully parabolic attraction-repulsion chemotaxis system via reduction to the Keller–Segel system

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June 2, 2021

Abstract. This paper deals with the fully parabolic attraction-repulsion chemotaxis system

\[
\begin{aligned}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \\
    v_t &= \Delta v - v + u, \\
    w_t &= \Delta w - w + u,
\end{aligned}
\]

under homogeneous Neumann boundary conditions and initial conditions, where \( \Omega \) is an open ball in \( \mathbb{R}^n \) (\( n \geq 3 \)), \( \chi, \xi > 0 \) are constants. When \( w = 0 \), finite-time blow-up in the corresponding Keller–Segel system has already been obtained. However, finite-time blow-up in the above attraction-repulsion chemotaxis system has not yet been established except for the case \( n = 3 \). This paper provides an answer to this open problem by using a transformation which leads to a system presenting structural advantages respect to the original.

2010 Mathematics Subject Classification. Primary: 35B44; Secondary: 35Q92, 92C17.

Key words and phrases: chemotaxis; attraction-repulsion; finite-time blow-up.

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†Partially supported by Grant-in-Aid for Scientific Research (C), No. 21K03278.
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1. Introduction

In this paper we consider the fully parabolic attraction-repulsion chemotaxis system

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega, & t > 0, \\
    v_t &= \Delta v - v + u, & x \in \Omega, & t > 0, \\
    w_t &= \Delta w - w + u, & x \in \Omega, & t > 0, \\
    \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu &= 0, & x \in \partial \Omega, & t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & w(x, 0) &= w_0(x), & x \in \Omega,
\end{align*}
\]

where \( \Omega := B(0, R) \subset \mathbb{R}^n \) is an open ball centered at the origin with radius \( R > 0 \); \( \chi, \xi > 0 \) are constants; \( \nu \) is the outward normal vector to \( \partial \Omega \). Moreover, the initial data \( u_0, v_0, w_0 \) are supposed to be radially symmetric and positive functions which satisfy that

\[
(u_0, v_0, w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega).
\]

In biology, the functions \( u, v \) and \( w \) represent the cell density, the concentration of attractive and repulsive chemical substances, respectively. The system (1.1) is one of variations of the chemotaxis system proposed by Keller and Segel [5] (see also Hillen–Painter [4]). The first equation in (1.1) describes the time evolution of the cell density in response to its own chemical attractants and repellents. Specifically, it implies that the cell movement is directed toward a higher concentration of the attractive signal and away from the repulsive signal.

On the other hand, in mathematics, it is important to consider whether a solution of the system (1.1) can blow up or not. In this paper we show finite-time blow-up of a solution to the system (1.1). The classical parabolic–elliptic Keller–Segel system and the parabolic–elliptic–elliptic attraction-repulsion chemotaxis system have been investigated in many literatures on chemotaxis systems (see e.g., Arumugam–Tyagi [1]). Before presenting the main result, we give an overview of known results about some problems related to (1.1).

We first focus on the parabolic–parabolic chemotaxis system

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla v), \\
    v_t &= \Delta v - v + u.
\end{align*}
\]

The first result on unboundedness of solutions to (1.2) was established by Winkler [14]. After that, Winkler [15] succeeded in showing that a solution of (1.2) blows up in finite time under the condition that for all \( m > 0, A > 0 \), the initial data belongs to \( \mathcal{B}(m, A) \) which is the set of radially symmetric positive functions \( (\varphi, \psi) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \) satisfying \( \int_\Omega \varphi = m, \| \psi \|_{W^{1,2}(\Omega)} \leq A \) and \( F(\varphi, \psi) \leq -K(m, A) \) with some \( K(m, A) > 0 \), where \( F \) is the energy functional. Moreover, it was shown in [15] that \( \mathcal{B}(m, A) \) is dense in the space of all radially symmetric positive functions in \( C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \) with respect to the topology in \( L^p(\Omega) \times W^{1,2}(\Omega) \) for all \( p \in (1, \frac{2n}{n+2}) \). Also, some related works which derive lower bound of blow-up time can be found in [2, 8, 9].
Secondly, we turn our eyes into the parabolic–elliptic chemotaxis system with signal-dependent sensitivity,
\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla v), \\
    0 &= \Delta v - v + u.
\end{align*}
\] (1.3)
As to this system, Nagai [10] derived that a radially symmetric solution blows up in finite time under some condition for the energy function and the moment of \(u\) in two or more space dimensions. After that, in the two-dimensional setting, Nagai [11] proved that if \(\int_{\Omega} u_0(x) |x-x_0|^2 \, dx\) for \(x_0 \in \Omega\) is sufficiently small and \(\int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi}\) holds, then there exists a non-radial solution which blows up in finite time.

We now shift our attention to the parabolic–elliptic–elliptic version of the attraction-repulsion chemotaxis system
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \\
    0 &= \Delta v + \alpha u - \beta v, \\
    0 &= \Delta w + \gamma u - \delta w,
\end{align*}
\] (1.4)
where \(\chi, \xi, \alpha, \beta, \gamma, \delta > 0\) are constants. Existence of a solution which blows up in finite time was studied by [7, 12, 16]. More precisely, in the two-dimensional setting, Tao and Wang [12] derived finite-time blow-up under the conditions that \(\int_{\Omega} u_0(x) |x-x_0|^2 \, dx\) for \(x_0 \in \Omega\) is sufficiently small and that
\[
\begin{align*}
    (i) &\quad \chi \alpha - \xi \gamma > 0, \quad \delta = \beta \quad \text{and} \quad \int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi \alpha - \xi \gamma}; \\
    (ii) &\quad \chi \alpha - \xi \gamma > 0, \quad \delta \geq \beta \quad \text{and} \quad \int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi \alpha - \xi \gamma}; \\
    (iii) &\quad \chi \alpha \delta - \xi \gamma \beta > 0, \quad \delta < \beta \quad \text{and} \quad \int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi \alpha \delta - \xi \gamma \beta}.
\end{align*}
\]
The idea in [12] is to reduce (1.4) to the form (1.3) by introducing the linear combination \(z := \chi v - \xi w\). Also, in the two-dimensional setting, Li and Li [7] extended the above (i) to the following two conditions:
\[
\begin{align*}
    (i) &\quad \chi \alpha - \xi \gamma > 0, \quad \delta = \beta \quad \text{and} \quad \int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi \alpha - \xi \gamma}; \\
    (ii) &\quad \chi \alpha - \xi \gamma > 0, \quad \delta \geq \beta \quad \text{and} \quad \int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi \alpha - \xi \gamma}; \\
    (iii) &\quad \chi \alpha \delta - \xi \gamma \beta > 0, \quad \delta < \beta \quad \text{and} \quad \int_{\Omega} u_0(x) \, dx > \frac{8\pi}{\chi \alpha \delta - \xi \gamma \beta}.
\end{align*}
\]
After that, Yu, Guo and Zheng [16] improved the above (iii) by replacing \(\chi \alpha \delta - \xi \gamma \beta\) with \(\chi \alpha - \xi \gamma\) in the first and third conditions in (iii) and filled the gap between the above (ii) and (iii); note that any relationship between \(\delta\) and \(\beta\) is no longer necessary. On the other hand, in the two dimensional setting, Viglialoro [13] provided an explicit lower bound of blow-up time for the system (1.4).

For the fully parabolic attraction-repulsion system with positive parameters \(\alpha, \beta, \gamma, \delta\), i.e., the fully parabolic version of (1.4), Lankeit [6] succeeded in establishing existence of radially symmetric solutions blowing up at some finite time under the condition that \(\chi \alpha - \xi \gamma > 0\) without any restriction on \(\beta, \delta\) in the three-dimensional setting.

In summary, finite-time blow-up has been shown for the parabolic–elliptic–elliptic attraction-repulsion chemotaxis system (1.4). However, finite-time blow-up in the fully parabolic system (1.1) has not been obtained yet except for the case \(n = 3\).
The purpose of this paper is to give an answer to the above open problem, that is, to establish finite-time blow-up in the fully parabolic system (1.1). The strategy for proving finite-time blow-up is to apply the method in [15] to the system (1.1) via the linear combination of the solution components \( v, w \) such that \( z := \chi v - \xi w \). After this transformation, the analysis reduces to citing a well-known result from the literature, but a new information about blow-up in (1.1) is obtained.

2. Main results and their proofs

In this section we give two main theorems. The first one asserts finite-time blow-up in the system (1.1). The statement and proof read as follows.

**Theorem 2.1.** Let \( \Omega := B(0, R) \subset \mathbb{R}^n \) be an open ball centered at the origin with radius \( R > 0 \). Let \( m > 0, A > 0 \) and \( \chi > \xi \). Then there exist constants \( T = T(m, A) > 0 \) and \( K = K(m, A) > 0 \) such that if \((u_0, v_0, w_0)\) belongs to the set

\[
\mathcal{C}(m, A) := \left\{(u_0, v_0, w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \midight.
\]

\[
u_0 \text{ and } \chi v_0 - \xi w_0 \text{ are radially symmetric and positive in } \overline{\Omega}
\]

\[
\text{with } \int_{\Omega} u_0 = m, \|\chi v_0 - \xi w_0\|_{W^{1,2}(\Omega)} \leq A \text{ and } G(u_0, v_0, w_0) \leq -K \right\},
\]

(2.1)

where \( G \) is the energy functional defined as

\[
G(u_0, v_0, w_0) := \frac{1}{2} \int_{\Omega} |\nabla (\chi v_0 - \xi w_0)|^2 + \frac{1}{2} \int_{\Omega} (\chi v_0 - \xi w_0)^2
\]

\[
- (\chi - \xi) \int_{\Omega} u_0 (\chi v_0 - \xi w_0) + (\chi - \xi) \int_{\Omega} u_0 \ln u_0,
\]

(2.2)

then the corresponding solution \((u, v, w)\) of the system (1.1) blows up before or at time \( T \).

**Remark 2.1.** For \( n = 3 \), the above result is covered by [6, Theorem 1.1 with \( \beta = \delta = 1 \)].

Before proving the above theorem, we give the local solvability to clarify blow-up.

**Lemma 2.2 ([12, Lemma 3.1]).** Let \((u_0, v_0, w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)\). Then there exists \( T_{\max} \in (0, \infty) \) such that (1.1) possesses a unique classical solution \((u, v, w)\) such that

\[
u, v, w \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),
\]

and

\[
\text{if } T_{\max} < \infty, \text{ then } \lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]

In particular, if \( u_0 \geq 0 \), then \( u(\cdot, t) \geq 0 \) for all \( t \in (0, T_{\max}) \). Moreover, if \( u_0, v_0, w_0 \) are radially symmetric in \( \overline{\Omega} \), then so are \( u, v, w \).
Proof of Theorem 2.1. We introduce the linear combination of the variables \((v, w)\) such that
\[
z := \chi v - \xi w. \]
Then the system (1.1) is rewritten as
\[
\begin{aligned}
&u_t = \Delta u - \nabla \cdot (u \nabla z), \quad x \in \Omega, \ t \in (0, T_{max}), \\
z_t = \Delta z - z + (\chi - \xi)u, \quad x \in \Omega, \ t \in (0, T_{max}), \\
\nabla u \cdot \nu = \nabla z \cdot \nu = 0, \quad x \in \partial \Omega, \ t \in (0, T_{max}), \\
u(x, 0) = u_0(x), \ z(x, 0) = z_0(x), \ x \in \Omega,
\end{aligned}
\]
(2.3)
where \(z_0 := \chi v_0 - \xi w_0\) and \(\chi > \xi\). We also define the energy functional for (2.3) as
\[
F(u, z) := \frac{1}{2} \int_{\Omega} |\nabla z|^2 + \frac{1}{2} \int_{\Omega} z^2 - (\chi - \xi) \int_{\Omega} uz + (\chi - \xi) \int_{\Omega} u \ln u.
\]
Then the functional \(F\) satisfies the energy inequality
\[
\frac{d}{dt} F(u(\cdot, t), z(\cdot, t)) \leq -D(u(\cdot, t), z(\cdot, t)) \quad \text{for all} \ t \in (0, T_{max}), \tag{2.4}
\]
where \(D\) is the dissipation rate defined as
\[
D(u, z) := \int_{\Omega} z_t^2 + (\chi - \xi) \int_{\Omega} u \cdot |\nabla \ln u - \nabla z|^2
\]
and \(T_{\text{max}} \in (0, \infty]\) is the maximal existence time of the solution \((u, z)\) to the system (2.3).
Indeed, in order to confirm (2.4), multiplying the second equation in (2.3) by \(z_t\), we have
\[
\int_{\Omega} z_t^2 + \frac{d}{dt} \left[ \frac{1}{2} \int_{\Omega} |\nabla z|^2 + \frac{1}{2} \int_{\Omega} z^2 - (\chi - \xi) \int_{\Omega} u(z - \ln u) \right] = (\chi - \xi) \int_{\Omega} u_t(\ln u - z).
\]
This together with
\[
\int_{\Omega} u_t(\ln u - z) = - \int_{\Omega} u \cdot |\nabla \ln u - \nabla z|^2
\]
implies (2.4). Now we shall verify the conditions for blow-up in [15, Theorem 1.1] which asserts that for all \(m > 0\) and \(A > 0\) there exist \(T = T(m, A) > 0\) and \(K = K(m, A) > 0\) such that if \((u_0, z_0)\) belongs to the set
\[
B(m, A) := \left\{ (u_0, z_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \right| \begin{array}{l}
u0, z_0 \text{ are radially symmetric and positive in } \overline{\Omega} \\
with \int_{\Omega} u_0 = m, \ \|z_0\|_{W^{1,2}(\Omega)} \leq A \text{ and } F(u_0, z_0) \leq -K \end{array}\right\}, \tag{2.5}
\]
then the corresponding solution \((u, z)\) of (2.3) blows up before or at time \(T\), provided that \(\chi - \xi = 1\). Recalling that \(z = \chi v - \xi w, \ z_0 = \chi v_0 - \xi w_0\) and
\[
F(u_0, z_0) = \mathcal{G}(u_0, v_0, w_0),
\]
we see by the assumption of Theorem 2.1 that all the above conditions for blow-up are satisfied at least in the case $\chi - \xi = 1$. In the case $\chi - \xi > 0$, we can develop the proof similarly. Indeed, as in [15, Theorem 5.1], we first find $c_1 = c_1(m, A, n) > 0$ such that

$$F(\tilde{u}, \tilde{z}) \geq -c_1 \left( D^0(\tilde{u}, \tilde{z}) + 1 \right)$$

(2.6)

with $\theta := \frac{n+2}{n+4}$ for all $(\tilde{u}, \tilde{z}) \in \{(u, z) \in C^1(\bar{\Omega}) \times C^2(\bar{\Omega}) \mid u, z \text{ are radially symmetric and positive with } \nabla z \cdot \nu = 0 \text{ on } \partial \Omega \text{ and } \int_{\Omega} u = m, \int_{\Omega} z \leq M, z(x) \leq B|x|^{-\kappa} \text{ for all } M, B > 0 \text{ and } \kappa > n - 2\}$. In particular, (2.6) can be applied to $(\tilde{u}, \tilde{z}) := (u(\cdot, t), z(\cdot, t))$ for each $t \in (0, T_{\text{max}})$ with some suitable $M, B > 0$. Setting

$$y(t) := -F(u(\cdot, t), z(\cdot, t)), \quad t \in [0, T_{\text{max}}],$$

we next combine (2.6) with (2.4) and consequently we have

$$y'(t) \geq c_2y^\frac{1}{\sigma}(t) \quad \text{for all } t \in (0, T_{\text{max}})$$

with some $c_2 = c_2(m, A, n) > 0$. This yields that

$$y(t) \geq y(0) \cdot \left( 1 - \frac{1}{\theta}c_2y^\frac{1}{\sigma}(0) t \right)^{-\frac{\sigma}{1-\sigma}} \quad \text{for all } t \in (0, T_{\text{max}}).$$

In particular, this leads us to the conclusion $T_{\text{max}} < \infty$. \qed

We next give and show the second main theorem, which guarantees that if the set $\mathcal{C}(m, A)$ defined in (2.1) equipped with a suitable topology, then it is dense in the space of radially symmetric positive functions.

**Theorem 2.3.** Let $\Omega := B(0, R) \subset \mathbb{R}^n$ ($n \geq 3$) be an open ball centered at the origin with radius $R > 0$. Let $p \in (1, \frac{2n}{n+2})$. Then for all $m > 0$, $A > 0$, the set $\mathcal{C}(m, A)$ defined in (2.1) is dense in the space

$$Y := \left\{ (u, v, w) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \mid u \text{ and } \chi v - \xi w \text{ are radially symmetric and positive in } \bar{\Omega} \right\}$$

with respect to the topology in $L^p(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, that is, for all $(u_0, v_0, w_0) \in \mathcal{C}(m, A)$ and all $\varepsilon > 0$ there exists $(u_\varepsilon, v_\varepsilon, w_\varepsilon) \in Y$ such that

$$\|u_0 - u_\varepsilon\|_{L^p(\Omega)} + \|v_0 - v_\varepsilon\|_{W^{1,\infty}(\Omega)} + \|w_0 - w_\varepsilon\|_{W^{1,\infty}(\Omega)} < \varepsilon.$$

In particular, the solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of the system (1.1) with initial data $(u_\varepsilon, v_\varepsilon, w_\varepsilon)|_{t=0} = (u_0, v_0, w_0)$ blows up in finite time.

**Proof of Theorem 2.2.** Let $n \geq 3$, $p \in (1, \frac{2n}{n+2})$, $m > 0$ and $A > 0$. Then we see from [15, Theorem 1.2] that the set $\mathcal{B}(m, A)$ defined in (2.5) is dense in the space

$$X := \left\{ (u, z) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \mid u, z \text{ are radially symmetric and positive in } \bar{\Omega} \right\},$$

and that for all $(u_0, z_0) \in X$ and all $\varepsilon > 0$ there exists $(u_{0\varepsilon}, z_{0\varepsilon}) \in X$ such that the solution $(u_\varepsilon, z_\varepsilon)$ of the system (2.3) with initial data $(u_\varepsilon, z_\varepsilon)|_{t=0} = (u_{0\varepsilon}, z_{0\varepsilon})$ blows up in finite time, which concludes the proof. \qed
**Open problem.** Consider the fully parabolic attraction-repulsion chemotaxis system with positive parameters $\alpha, \beta, \gamma, \delta$:

\[
\begin{aligned}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v - \beta v + \alpha u, \quad x \in \Omega, \quad t > 0, \\
  w_t &= \Delta w - \delta w + \gamma u, \quad x \in \Omega, \quad t > 0.
\end{aligned}
\]

This cannot be reduced to the Keller–Segel system as in (2.3) via the transformation $z = \chi u - \xi w$ in the case $\beta \neq \delta$.

In this case, Lankeit [6] established finite-time blow-up in the three dimensional-setting. However, for $n \geq 4$, finite-time blow-up is left as an open problem.

**Note.** After the completion of this paper, we confirmed that Fujie and Suzuki [3, Remark 1.5] derived Theorem 2.1 by the same transformation $z = \chi u - \xi w$.

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