Asymptotic vacua with higher derivatives

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Abstract

We study limits of vacuum, isotropic universes in the full, effective, four-dimensional theory with higher derivatives. We show that all flat vacua as well as general curved ones are globally attracted by the standard, square root scaling solution at early times. Open vacua asymptote to horizon-free, Milne states in both directions while closed universes exhibit more complex logarithmic singularities, starting from initial data sets of a smaller dimension.

1 Introduction

Vacuum states are not trivially obtainable for simple isotropic universes in general relativity, and one has to go beyond them to anisotropic, or more general inhomogeneous cosmologies for a vacuum to start making sense [1]. However, isotropic vacua are very common in effective theories with higher derivatives, see e.g., [2]-[7]. Such classical vacua are usually thought of as acquiring a physical significance when viewed as possible low-energy manifestations of a

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more fundamental superstring theory, although their treatment shows an intrinsic interest quite independently of the various quantum considerations.

In this paper, we consider the asymptotic limits towards singularities of vacuum universes coming from effective theories with higher derivatives. Such a study is related to the existence and stability of an inflationary stage at early times in these contexts, and also to the intriguing possibilities of solutions with no particle horizons. For flat vacua, we find the general asymptotic solution with an early-time singularity. This result is then extended to cover general curved vacuum isotropic solutions and we give the precise form of the attractor of all such universes with a past singularity. We also obtain special asymptotic states valid specifically for open or closed vacua starting from lower-dimensional initial data.

The plan of this paper is as follows. In the next section, we derive the basic cosmological equations and show that they form an autonomous dynamical system in suitable variables. Then, in Sections 3-5, we apply asymptotic methods to study the early and late evolution of these isotropic cosmologies. In particular, we study separately the flat and curved subcases and show that there exist certain properties valid asymptotically irrespective of the influence of curvature, while other limits, coined here ‘Milne states’, have a very sensitive dependence on the sign of the constant curvature slices.

## 2 The vacuum field

We consider a vacuum, FRW universe with scale factor $a(t)$ determined by the Robertson-Walker metric of the form

$$g_4 = -dt^2 + a^2 g_3.$$  \hspace{1cm} (2.1)

Each slice is given the 3-metric

$$g_3 = \frac{1}{1 - kr^2}dr^2 + r^2g_2.$$  \hspace{1cm} (2.2)

$k$ being the (constant) curvature normalized to take the three values 0, +1 or −1 for the complete, simply connected, flat, closed or open space sections respectively, and the 2-dimensional sections are such that

$$g_2 = d\theta^2 + \sin^2 \theta d\phi^2.$$  \hspace{1cm} (2.3)
Our general higher order action is (we set $8\pi G = c = 1$, and the sign conventions are those of [9])

$$S = \frac{1}{2} \int_{\mathcal{M}^4} \mathcal{L}(R) d\mu_g, \quad \mathcal{L}(R) = R + \beta R^2 + \gamma \text{Ric}^2 + \delta \text{Riem}^2,$$

(2.4)

where $\beta, \gamma, \delta$ are constants. We consider a family of metrics $\{g_s : s \in \mathbb{R}\}$, and denote its compact variation by $\hat{g}_{\mu\nu} = (\partial g/\partial s)_{s=0}$. Since in four dimensions we have the Gauss-Bonnet identity,

$$\hat{S}_{\text{GB}} = \int_{\mathcal{M}^4} (R_{\text{GB}}^2 d\mu_g) = 0, \quad R_{\text{GB}}^2 = R^2 - 4\text{Ric}^2 + \text{Riem}^2,$$

(2.5)

in the derivation of the field equations through a $g$-variation of the action (2.4), only terms up to $\text{Ric}^2$ will matter. Below we focus in the case where $\mathcal{M}^4$ is a homogeneous and isotropic universe with metric (2.1), in which case we have a second useful identity,

$$\int_{\mathcal{M}^4} \left( (R^2 - 3\text{Ric}^2) d\mu_g \right) = 0,$$

(2.6)

which further enables us to include the contribution of the $\text{Ric}^2$ term into the coefficient of $R^2$, altering only the arbitrary constants. Hence, the field equations read as follows:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{\xi}{6} \left[ 2 R R^{\mu\nu} - \frac{1}{2} R^2 g^{\mu\nu} - 2 (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma}) \nabla_{\rho} \nabla_{\sigma} R \right] = 0,$$

(2.7)

where we have set $\xi = 2(3\beta + \gamma + \delta)$. We note that because of the form of the coefficient $\xi$, some ‘memory’ of the original fully quadratic theory (2.4) remains, and the final effective action leading to the field equations (2.7) is not equivalent to an $R + \zeta R^2$ action with $\zeta$ arbitrary (a use of the latter action would mean taking into account only the algebraic dependence of the quadratic curvature invariants).

Eq. (2.7) naturally splits into 00- and $ii$-components ($i = 1, 2, 3$), but only the 00-component of (2.7) will be used below. Using the metric (2.1), the field equation (2.7) takes the form

$$\frac{k + \dot{a}^2}{a^2} + \xi \left[ 2 \frac{\ddot{a} \dot{a}}{a^3} + 2 \frac{\ddot{a} \dot{a}^2}{a^4} - \frac{\dot{a}^2}{a^2} - 3 \frac{\dot{a}^4}{a^4} - 2 \frac{k \dot{a}^2}{a^4} + \frac{k^2}{a^4} \right] = 0,$$

(2.8)

which after setting $H = \dot{a}/a$ for the Hubble expansion rate leads to our basic cosmological equation in the form

$$\ddot{H} = \frac{1}{2} \frac{\dot{H}^2}{H} - 3H \dot{H} + \kappa \frac{\dot{a}^2}{a^2} - \frac{1}{2} \frac{k^2}{a^4} - \frac{1}{12} \frac{1}{H} - \frac{\kappa}{12 \epsilon a^2} \frac{1}{H},$$

(2.9)

where now we have put $\epsilon = \xi/6$. Setting $x = H$, $y = \dot{H}$ and $z = a^{-2}$, Eq. (2.9) can be written as an autonomous dynamical system of the general form

$$\dot{x} = f_{\text{VAC}}(x), \quad x = (x, y, z),$$

(2.10)
that is

\[
\dot{x} = y \\
\dot{y} = \frac{y^2}{2x} - 3xy + kxz - \frac{k^2z^2}{2x} - \frac{x}{12\epsilon} - \frac{kz}{12\epsilon x} \\
\dot{z} = -2xz
\] (2.11)
equivalent to the vacuum, 3-dimensional vector field \( f_{\text{VAC}} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto f_{k,\text{RAD}}(x, y, z) \) with

\[
f_{\text{VAC}}(x, y, z) = \left( y, \frac{y^2}{2x} - 3xy + kxz - \frac{k^2z^2}{2x} - \frac{x}{12\epsilon} - \frac{kz}{12\epsilon x}, -2xz \right).
\] (2.12)

This field completely describes the dynamical evolution of a vacuum, flat or curved, FRW universe in the gravity theory defined by the full quadratic action (2.4). In the following, we shall assume that \( x \neq 0 \), that is we consider only non-static universes.

### 3 The unique flat vacuum

When \( k = 0 \), the vacuum field (2.12) becomes

\[
f_{0,\text{VAC}}(x, y, z) = \left( y, \frac{y^2}{2x} - 3xy - \frac{x}{12\epsilon}, -2xz \right),
\] (3.1)
and the system (2.11) reads,

\[
\dot{x} = y \\
\dot{y} = \frac{y^2}{2x} - 3xy - \frac{x}{12\epsilon} \\
\dot{z} = -2xz
\] (3.2)

Our main interest below is to study the behaviour of the universe described by (3.1), (3.2) near the initial singularity, taken here to lie at \( t = 0 \) (it is really arbitrary, however, and we could have placed it at any \( t_0 \) and used the variable \( \tau = t - t_0 \) instead of \( t \)). Following the method of asymptotic splittings of Refs. [10, 11, 12], we find that of the \( 2^3 - 1 = 7 \) possible asymptotic decompositions that the field (3.1) possesses, there is only one that leads to a fully acceptable dominant balance, namely,

\[
f_{0,\text{VAC}} = f_{0,\text{VAC}}^{(0)} + f_{0,\text{VAC}}^{(\text{sub})},
\] (3.3)
with with dominant part
\[ f_{0,\text{VAC}}^{(0)}(x) = \left( y, \frac{y^2}{2x} - 3xy, -2xz \right), \] (3.4)
and subdominant part
\[ f_{0,\text{VAC}}^{(\text{sub})}(x) = \left( 0, -\frac{x}{12\epsilon}, 0 \right). \] (3.5)
We will eventually construct asymptotic series solutions which encode information about
the leading order behaviour of all solutions as well as their generality (number of arbitrary
constants) near the spacetime singularity at \( t = 0 \). For any given dominant asymptotic
solution of the system \((3.2)\), we call the pair \((a, p)\) a dominant balance of the vector field
\( f_{0,\text{VAC}} \), where \( a = (\theta, \eta, \rho) \in \mathbb{C}^3 \) are constants and \( p = (p, q, r) \in \mathbb{Q}^3 \), and we look for a
leading order behaviour of the form
\[ x(t) = at^p = (\theta t^p, \eta t^q, \rho t^r). \] (3.6)
Such behaviours correspond to the possible asymptotic forms of the integral curves of the
vacuum field \( f_{0,\text{VAC}} \), as we take it to a neighborhood of the singularity. Substituting the
forms \((3.6)\) into the dominant system \((\dot{x}, \dot{y}, \dot{z})(t) = f_{0,\text{VAC}}^{(0)} \) and solving the resulting nonlin-
ear algebraic system to determine the dominant balance \((a, p)\) as an exact, scale invariant
solution, leads to the unique flat-vacuum balance \( B_{0,\text{VAC}} \in \mathbb{C}^3 \times \mathbb{Q}^3 \), with
\[ B_{0,\text{VAC}} = (a, p) = \left( \left( \frac{1}{2}, -\frac{1}{2}, \rho \right), (-1, -2, -1) \right). \] (3.7)
In particular, this means that the vector field \( f_{0,\text{VAC}}^{(0)} \) is a scale-invariant system, cf. \[10, 11, 12].

Further, we need to show that the term \((3.5)\) in the basic decomposition \((3.3)\) of the
flat-vacuum field \((3.1)\) is itself weight-homogeneous with respect to the flat-vacuum balance
\((3.7)\) for this splitting to be finally acceptable. For this we need to check that this candidate
subdominant part is indeed subdominant. Using the balance \( B_{0,\text{VAC}} \) defined by Eq. \((3.7)\),
we find that
\[ \frac{f_{0,\text{VAC}}^{(\text{sub})}(at^p)}{tp^{-1}} = f_{0,\text{VAC}}^{(\text{sub})}(a) t^2 = \left( 0, -\frac{t^2}{24\epsilon}, 0 \right). \] (3.8)
Taking the limit as \( t \to 0 \), this goes to zero asymptotically provided that the form \( f_{0,\text{VAC}}^{(\text{sub})}(a) \)
is different from zero. This happens only when \( \epsilon \neq 0 \), that is for all cases except when
\[ 3\beta + \gamma + \delta = 0. \]

We conclude that this basic decomposition is acceptable asymptotically in all higher order gravity theories when this constraint holds true. A physical example that is excluded in this analysis and consequently needs a separate treatment is the so-called conformally invariant Bach-Weyl gravity cf. [13]. Note that the same constraint appears in the stability analysis of the purely radiation universes in these theories, cf. Ref. [8].

We now move on to check our asymptotic solutions in terms of their consistency with the overall approximation scheme we are using (cf. [10, 11, 12] for more details and proofs). We construct a series representation of the asymptotic solutions valid locally around the initial singularity, so that it is dominated by the dominant balance solutions we have built so far. The degree of generality of these final series solutions will depend on the number of arbitrary constants in them, and as explained in [10], the arbitrary constants of any (particular or general) solution first appear in those terms in the asymptotic series solution whose coefficients \( c_i \) have indices \( i = \rho s \), where \( \rho \) is a non-negative \( \mathcal{K} \)-exponent and \( s \) denotes the least common multiple of the denominators of the set of all subdominant exponents and those of all the \( \mathcal{K} \)-exponents with positive real parts (in our case, \( s = 2 \)). These exponents are numbers belonging to the spectrum of the so-called Kovalevskaya matrix given by

\[ \mathcal{K} = D f_{0,VAC}^{(0)}(a) - \text{diag}(p). \]  

Hence, the \( \mathcal{K} \)-exponents depend on the dominant part of the vector field as well as the dominant balance. In the present case, the Kovalevskaya matrix is

\[ \mathcal{K}_{0,VAC} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1/2 & 0 \\ -2\rho & 0 & 0 \end{pmatrix}, \]  

with spectrum

\[ \text{spec}(\mathcal{K}_{0,VAC}) = \{-1, 0, 3/2\}. \]  

The number of non-negative \( \mathcal{K} \)-exponents equals the number of arbitrary constants that appear in the series expansions. There is always the \( -1 \) exponent that corresponds to an arbitrary constant, the position of the singularity, and because the \( \text{spec}(\mathcal{K}_{0,VAC}) \) in our case possesses two non-negative eigenvalues, the balance \( \mathcal{B}_{0,VAC} \) indeed corresponds to the dominant behaviour of a general solution having the form of a formal series and valid locally around the initial singularity. To find it, we substitute the Fuchsian series expansions (no
constant first term, rational exponents)

\[ x(t) = \sum_{i=0}^{\infty} c_1 i t^{1/2 - 1}, \quad y(t) = \sum_{i=0}^{\infty} c_2 i t^{2/2 - 2}, \quad z(t) = \sum_{i=0}^{\infty} c_3 i t^{3/2 - 1}, \quad (3.12) \]

where \( c_{10} = 1/2, \ c_{20} = -1/2, \ c_{30} = \rho \), in the original system \((3.2)\) and we are led to various recursion relations that determine the unknowns \( c_1, c_2, c_3 \) term by term. Further algebraic manipulations lead to the final series representation of the solution in the form:

\[ x(t) = \frac{1}{2} t^{-1} + c_{13} t^{1/2} - \frac{1}{36\epsilon} t + \cdots \quad (3.13) \]

The corresponding series expansion for \( y(t) \) is given by the first time derivative of the above expression, while the corresponding series expansion for \( z(t) \) is given by

\[ z(t) = \rho t^{-1} - \frac{4\rho c_{13}}{3} t^{1/2} + \frac{1}{36\epsilon} t + \cdots \quad (3.14) \]

Finally, we arrive at the following asymptotic form of the scale factor around the singularity:

\[ a(t) = \alpha t^{1/2} + \frac{2c_{13}\alpha}{3} t^2 - \frac{\alpha}{72\epsilon} t^{5/2} + \frac{4\alpha c_{13}^2}{9} t^{7/2} + \cdots, \quad (3.15) \]

where \( \alpha \) is a constant of integration and \( \alpha^{-2} = \rho \).

As a final test for admission of this solution, we use Fredholm’s alternative to be satisfied by any admissible solution. This leads to a compatibility condition for the positive eigenvalue \( 3/2 \) and the associated eigenvector,

\[ v^\top \cdot \left( \mathcal{K} - \frac{j}{s} I \right) c_j = 0, \quad (3.16) \]

where \( I \) denotes the identity matrix, and we have to satisfy this at the \( j = 2 \) level. This gives the following orthogonality constraint,

\[ (2, 1, -\frac{8\rho}{3}) \cdot \begin{pmatrix} -\frac{1}{2} c_{13} + c_{23} \\ c_{13} - 2c_{23} \\ -2\rho c_{13} - \frac{3}{2} c_{33} \end{pmatrix} = 0. \quad (3.17) \]

It is immediate to check that this is indeed satisfied, thus leading to the conclusion that \((3.13)-(3.14)\) corresponds to a valid asymptotic solution around the singularity.

Our series solution \((3.13)-(3.14)\) has three arbitrary constants, \( c_{13}, \rho \) and another one corresponding to the arbitrary position of the singularity (taken here to be zero without loss
of generality), and is therefore a local expansion of the general solution around the initial singularity. Since the leading order coefficients are real, by a theorem of Goriely and Hyde, cf. [11], we conclude that there is an open set of initial conditions for which the general solution blows up at the finite time (initial) singularity at \( t = 0 \). This proves the stability of our solution in the neighborhood of the singularity.

We can see that the transformation \( c_{13} = 3c'_{13}/2\alpha \) and \( \epsilon = -k/6 \) in the series expansion (3.15) leads to a form obtained by setting \( \zeta = 0 \) in the series expansion found for the flat, radiation case of Ref. [14] (cf. Eq. (21) in that reference, where the term \( 12\xi \theta^3 \) was mistakenly written as \( 24\xi \theta^3 \) there).

4 Curved vacua

As we have seen in the previous section when \( k = 0 \) and we have a flat, vacuum FRW model in the fully quadratic theory of gravity defined by the action (2.4), the vector field \( f_{0,\text{VAC}} \) has one admissible asymptotic solution near the initial singularity, namely, the form (3.13)-(3.14). In this family, all flat vacua are asymptotically dominated (or ‘attracted’) at early times by the form \( a(t) \sim t^{1/2} \), thus proving the stability of this solution in the flat case.

When \( k \neq 0 \), and we have the situation of a vacuum, curved family of FRW universes, the vacuum field \( f_{\text{VAC}} \) has more terms than those present in the flat case, namely, those that contain \( k \) in (2.12). Below we shall use the suggestive notation \( f_{k,\text{VAC}} \) instead of \( f_{\text{VAC}} \) to signify that we are dealing with non-flat vacua. The field \( f_{k,\text{VAC}} \) (or the basic system (2.11)) can decompose precisely in \( 2^6 - 1 = 63 \) different ways. Of these 63 decompositions, there are only three that eventually lead to fully acceptable dominant balances, while the rest 60 decompositions fail to lead to an acceptable picture for various different asymptotic reasons. The only acceptable asymptotic splittings of the vector field \( f_{k,\text{VAC}} \) of the general form \( f_{k,\text{VAC}} = f_{k,\text{VAC}}^{(0)} + f_{k,\text{VAC}}^{(\text{sub})} \), have dominant parts

\[
\begin{align*}
\mathbf{f}_{k,\text{VAC}}^{(0)} &= \left( y, \frac{y^2}{2x} - 3xy, -2xz \right), \\
\mathbf{f}_{k,\text{VAC}}^{(0)} &= \left( y, -3xy + kxz, -2xz \right), \\
\mathbf{f}_{k,\text{VAC}}^{(0)} &= \left( y, \frac{y^2}{2x} - 3xy + kxz - \frac{k^2 z^2}{2x}, -2xz \right).
\end{align*}
\]
while their subdominant parts are given respectively by the forms,

\[
f^{(\text{sub})}_{k,\text{VAC}1} = \left(0, kxz - \frac{k^2 z^2}{2x} - \frac{x}{12\epsilon} - \frac{kz}{12x\epsilon}, 0\right),
\]
(4.4)

\[
f^{(\text{sub})}_{k,\text{VAC}2} = \left(0, \frac{y^2}{2x} - \frac{k^2 z^2}{2x} - \frac{x}{12\epsilon} - \frac{kz}{12x\epsilon}, 0\right),
\]
(4.5)

\[
f^{(\text{sub})}_{k,\text{VAC}3} = \left(0, -\frac{x}{12\epsilon} - \frac{kz}{12x\epsilon}, 0\right).
\]
(4.6)

The first decomposition (4.1) has identical dominant part as the flat splitting of the previous section, hence identical dominant balance, namely, its asymptotic balance is \(\mathcal{B}_{k,\text{VAC}1} \in \mathbb{C}^3 \times \mathbb{Q}^3\), with

\[
\mathcal{B}_{k,\text{VAC}1} = (a, p) = \left(\left(\frac{1}{2}, -\frac{1}{2}, \rho\right), \left(-1, -2, -1\right)\right),
\]
(4.7)

In particular, this means that the vector field \(f^{(0)}_{k,\text{VAC}1}\) is a scale-invariant system. However, its subdominant part (4.4) is different, and we need to show that the higher order terms (4.4) in the basic decomposition of the vacuum field are themselves weight-homogeneous with respect to the balance (4.7) for this to be an acceptable one. To prove this, we first split the subdominant part (4.3) by writing

\[
f^{(\text{sub})}_{k,\text{VAC}1}(x) = f^{(1)}_{k,\text{VAC}1}(x) + f^{(2)}_{k,\text{VAC}1}(x) + f^{(3)}_{k,\text{VAC}1}(x),
\]
(4.8)

where

\[
f^{(1)}_{k,\text{VAC}1}(x) = (0, kxz, 0), \quad f^{(2)}_{k,\text{VAC}1}(x) = \left(0, -\frac{k^2 z^2}{2x} - \frac{x}{12\epsilon}, 0\right), \quad f^{(3)}_{k,\text{VAC}1}(x) = \left(0, -\frac{kz}{12x\epsilon}, 0\right),
\]
(4.9)

and using the balance \(\mathcal{B}_{k,\text{VAC}1}\) defined by Eq. (4.7), we find that

\[
\frac{f^{(1)}_{k,\text{VAC}1}(at^p)}{t^{p-1}} = f^{(1)}_{k,\text{VAC}1}(a)t = \left(0, \frac{k\rho}{2} t, 0\right),
\]
(4.10)

\[
\frac{f^{(2)}_{k,\text{VAC}1}(at^p)}{t^{p-1}} = f^{(2)}_{k,\text{VAC}1}(a)t^2 = \left(0, \left(-k^2 \rho^2 - \frac{1}{24\epsilon}\right) t^2, 0\right),
\]
(4.11)

\[
\frac{f^{(3)}_{k,\text{VAC}1}(at^p)}{t^{p-1}} = f^{(3)}_{k,\text{VAC}1}(a)t^3 = \left(0, -\frac{k\rho}{6\epsilon} t^3, 0\right).
\]
(4.12)

Hence, taking the limit as \(t \to 0\), we see that these forms go to zero asymptotically provided that \(f^{(i)}_{k,\text{VAC}1}(a), i = 1, 2, 3\) are all different from zero, which happens only when \(\epsilon \neq 0\), that is for all cases except when \(3\beta + \gamma + \delta = 0\). Since the subdominant exponents

\[
q^{(0)} = 0 < q^{(1)} = 1 < q^{(2)} = 2 < q^{(3)} = 3,
\]
(4.13)
are ordered, we conclude that the subdominant part (4.4) is weight-homogeneous as promised. Further, since the Kovalevskaya matrix and its spectrum are identical to the flat vacuum case, we arrive at the following asymptotic series representation for the decomposition (4.1):

\[ x(t) = \frac{1}{2} \ t^{-1} - \frac{k\rho}{2} + c_{13} \ t^{1/2} - \left( \frac{k^2 \rho^2}{4} + \frac{1}{36\epsilon} \right) \ t + \cdots, \]  
\hspace{1cm} (4.14)

while the corresponding series expansion for \( y(t) \) is given by the first time derivative of the above expression, and that for \( z(t) \) is given by

\[ z(t) = \rho \ t^{-1} - k\rho^2 - \frac{4\rho \ c_{13}}{3} \ t^{1/2} + \left( \frac{k^2 \rho^2 (1+2\rho)}{4} + \frac{1}{36\epsilon} \right) \ t + \cdots. \]  
\hspace{1cm} (4.15)

For the scale factor, we find

\[ a(t) = \alpha \ t^{1/2} - \frac{k\rho\alpha}{2} \ t^{3/2} + \frac{2c_{13}\alpha}{3} \ t^2 - \left( \frac{k^2 \rho^2 \alpha}{8} + \frac{\alpha}{72\epsilon} \right) \ t^{5/2} + \cdots, \]  
\hspace{1cm} (4.16)

where \( \alpha \) is a constant of integration and \( \alpha^{-2} = \rho \). This series (4.14) has three arbitrary constants, \( \rho, c_{13} \) and a third one corresponding to the arbitrary position of the singularity, and is therefore a local expansion of the general solution around the initial singularity. The transformation \( c_{13} = 3\dot{c}_{13}/2\alpha \) and \( \epsilon = k/6 \) in the series expansion (4.16) leads to the form which is obtained by setting \( \zeta = 0 \) in the series expansion found for the curved, radiation case, cf. Eq. (4.13) of [8]. In addition, by setting \( k = 0 \) we are lead to the form (3.13) found for the flat vacuum.

We note that because of the square root, limits can only be taken in the backward direction, \( t \downarrow 0 \), in the solution (4.16), another way of expressing the curious fact that this solution (along with Eq. (3.15) found in the previous Section) is only valid at early times and corresponds to a past singularity.

5 Milne states

We now move on to the analysis of the last two decompositions, namely, those with dominant parts (4.2) and (4.3). We show below that these lead to particular solutions for \( k = -1 \) and \( k = +1 \). In the case of open universes, \( k = -1 \), and the dominant parts take the forms

\[ f^{(0)}_{-1, \text{VAC}_2} = (y, -3xy - xz, -2xz), \]  
\hspace{1cm} (5.1)

\[ f^{(0)}_{-1, \text{VAC}_3} = \left( y, \frac{y^2}{2x} - 3xy - xz - \frac{z^2}{2x}, -2xz \right), \]  
\hspace{1cm} (5.2)
with subdominant parts given by

\[ f^{(sub)}_{-1, VAC_2} = \left( 0, \frac{y^2}{2x} - \frac{z^2}{2x} - \frac{x}{12\epsilon} + \frac{z}{12x\epsilon}, 0 \right), \]  

\[ f^{(sub)}_{-1, VAC_3} = \left( 0, -\frac{x}{12\epsilon} + \frac{z}{12x\epsilon}, 0 \right), \]  

respectively. These two forms lead to the same acceptable asymptotic balance

\[ B_{-1, VAC_{2,3}} = (a, p) = ((1, -1, 1), (-1, -2, -2)), \]  

while the structure of the \( \mathcal{K} \)-matrices is

\[ \mathcal{K}_{-1, VAC_2} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ -2 & 0 & 0 \end{pmatrix}, \quad \text{spec}(\mathcal{K}_{-1, VAC_2}) = \{-1, -1, 2\}, \]  

and

\[ \mathcal{K}_{-1, VAC_3} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & -2 \\ -2 & 0 & 0 \end{pmatrix}, \quad \text{spec}(\mathcal{K}_{-1, VAC_3}) = \{-1, -2, 2\}. \]  

Since we are interested in the behaviour of solutions near singularities, we set the arbitrary constants corresponding to the negative eigenvalues equal to zero, and we are led to the following form for \( x(t) \), common for both decompositions,

\[ x(t) = t^{-1} + c_{12} t - \left( \frac{c_{12} - 18\epsilon c_{12}^2}{60\epsilon} \right) t^3 + \ldots. \]  

The corresponding series expansion for \( y(t) \) is given by the first time derivative of the above expression, while the corresponding series expansion for \( z(t) \) is given by

\[ z(t) = t^{-2} - c_{12} + \left( \frac{c_{12} \left( 42\epsilon c_{12} + 1 \right)}{120\epsilon} \right) t^2 + \ldots. \]  

Finally, we arrive at the following asymptotic form for the scale factor \( a(t) \) around the singularity:

\[ a(t) = a t + \frac{\alpha c_{12}}{2} t^3 - \frac{\alpha \left( c_{12} - 18\epsilon c_{12}^2 \right)}{240\epsilon} t^5 + \ldots, \]  

where \( \alpha = \pm 1 \) as dictated by the definition \( z(t) = 1/a(t)^2 \). This solution has therefore two arbitrary constants, \( c_{12} \) and a second one corresponding to the arbitrary position of the singularity (taken here to be zero without loss of generality), and is therefore a local expansion
of a particular solution around the singularity. Since the time singularity can be approached here from either the past or the future direction, we conclude that it represents a 2-parameter family of past, or future Milne states for these open vacua. This is also reminiscent of the Frenkel-Brecher horizonless solutions, cf. [15], with the important difference that their solutions are matter-filled an possibly valid only in the past direction.

On the other hand, when \( k = +1 \), the decomposition (4.2) does not lead to an acceptable dominant balance, but (4.3) does, namely,

\[
\mathbf{f}^{(0)}_{+1, \text{VAC}_3} = \left( y, \frac{y^2}{2x} - 3xy + xz - \frac{z^2}{2x}, -2xz \right),
\]

with subdominant part

\[
\mathbf{f}^{(\text{sub})}_{+1, \text{VAC}_3} = \left( 0, -\frac{x}{12\epsilon} - \frac{z}{12x\epsilon}, 0 \right),
\]

and we obtain

\[
\mathbf{B}_{+1, \text{VAC}_3} = (a, p) = ((1, -1, 3), (-1, -2, -2)).
\]

The corresponding \( \mathcal{K} \)-matrix is

\[
\mathcal{K}_{+1, \text{VAC}_3} = \begin{pmatrix}
1 & 1 & 0 \\
10 & -2 & -2 \\
-6 & 0 & 0 \\
\end{pmatrix}, \quad \text{spec}(\mathcal{K}_{+1, \text{VAC}_3}) = \{-1, -2\sqrt{3}, 2\sqrt{3}\},
\]

and we expect particular solutions in this case with the given leading order, however, due to the irrational Kowalevskaya exponents the resulting series will contain logarithmic terms.

6 Conclusion

In this paper we have considered the possible singular behaviours and asymptotic limits of vacuum isotropic universes in the fully quadratic gravity theory which apart from the Einstein term contains terms proportional to a linear combination of \( R^2, \text{Ric}^2 \) and \( \text{Riem}^2 \). Taking into account various asymptotic conditions that have to hold in order to have admissible solutions, we are left with only three possible asymptotic decompositions of the vacuum vector field near the singular state.

It turns out that a prominent role in the early asymptotic evolution of both flat and curved vacua in this theory is played by a scaling form that behaves as \( t^{1/2} \) near the initial
singularity. Using various asymptotic and geometric arguments, we were able to built a solution of the field equations in the form of a Fuchsian formal series expansion compatible with all other constraints, dominated asymptotically to leading order by this solution and having the correct number of arbitrary constants that makes it a general solution of the field equations. In this way, we conclude that this exact solution is an early time attractor of all homogeneous and isotropic vacua of the theory, thus proving stability against such ‘perturbations’.

For open vacua, there is a 2-parameter family of Fuchsian solutions that is dominated asymptotically by the Milne form both for past and future singularities. In the case of closed models, we have logarithmic solutions coming from a manifold of initial conditions with smaller dimension than the full phase space but dominated asymptotically by the same $a(t) \sim t$ form.

It is instructive to also comment on our present results in connection with the recent results of \cite{8} on the stability of radiation, curved universes for the same class of theories. We have shown that at early times both radiation and vacuum, flat or curved universes are past-attracted by the ‘universal’ $t^{1/2}$ asymptote and this is the most dominant feature in all these models. However, the behaviour of open vacua in these theories is more complex, for they allow universes that emerge from initial data sets of smaller dimension and valid for both early and late times. These universes asymptote to the Milne form during their early and late evolution toward singularities. Closed vacua, on the other hand, develop in time as more complex solutions that are characterized by logarithmic formal series, but asymptotically their leading order is described again by simple singularities similar to the open case treated here.

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