Constructive martingale representation in functional Itô calculus: a local martingale extension

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Abstract
The constructive martingale representation theorem of functional Itô calculus is extended, from the space of square integrable martingales, to the space of local martingales. The setting is that of an augmented filtration generated by a Wiener process.

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1 Introduction

Consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which there lives an \(n\)-dimensional Wiener process \(W\). Let \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) denote the augmentation under \(\mathbb{P}\) of the filtration generated by \(W\) until the constant terminal time \(T < \infty\). One of the main results of Itô calculus is the martingale representation theorem which in the present setting is as follows: Let \(M\) be a RCLL local martingale relative to \((\mathbb{P}, \mathcal{F})\), then there exists a progressively measurable \(n\)-dimensional process \(\varphi\) such that

\[
M(t) = M(0) + \int_0^t \varphi(s)'dW(s), \quad 0 \leq t \leq T, \quad \text{and} \quad \int_0^T |\varphi(t)|^2 dt < \infty \quad \text{a.s.}
\]

In particular, \(M\) has continuous sample paths a.s.

Considerable effort has in the literature been made in order to find explicit formulas for the integrand \(\varphi\), i.e. in order to find constructive representations of martingales, mainly using Malliavin calculus, see e.g. [8, 13, 14, 18] and the references therein. The recently developed functional Itô calculus includes a new type of constructive representation of square integrable martingales due to Cont and Fourniè see e.g. [1, 3, 4, 5]. The main result of the present paper is an extension of this result to local martingales.

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The organization of the paper is as follows. Section 2 is based on [1] and contains a brief and heuristic account of the relevant parts of functional Itô calculus including the constructive martingale representation theorem for square integrable martingales. Section 3 contains the local martingale extension of this theorem.

**Remark 1.1** Many of the applications that rely on martingale representation are in mathematical finance. A particular application that may benefit from the local martingale extension of the present paper is optimal investment theory, in which the discounted (using the state price density) optimal wealth process is a (not necessarily square integrable) martingale, see e.g. [8, ch. 3]. In particular, using our main result, Theorem 3.5, it would be possible to derive an explicit formula for the optimal portfolio in terms of the vertical derivative of the discounted optimal wealth process. Similar explicit formulas for optimal portfolios based on the Malliavin calculus approach to constructive martingale representation have, under restrictive assumptions, been studied extensively, see e.g. [2, 6, 7, 10, 11, 15, 16, 17].

2 Constructive representation of square integrable martingales

Denote an $n$-dimensional sample path by $\omega$. Denote a sample path stopped at $t$ by $w_t$, i.e. let $w_t(s) = \omega(t \wedge s), 0 \leq s \leq T$. Consider a real-valued functional of sample paths $F(t, \omega)$ which is non-anticipative (essentially meaning that $F(t, \omega) = F(t, \omega_t)$). The horizontal derivative at $(t, \omega)$ is defined by

$$D F(t, \omega) = \lim_{h \to 0} \frac{F(t + h, \omega_t) - F(t, \omega_t)}{h}.$$ 

The vertical derivative at $(t, \omega)$ is defined by $\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, ..., n)'$, where

$$\partial_i F(t, \omega) = \lim_{h \to 0} \frac{F(t, \omega_t + h e_i I_{[0,T]}) - F(t, \omega_t)}{h}.$$ 

Higher order vertical derivatives are obtained by vertically differentiating vertical derivatives. 

One of the main results of functional Itô calculus is the functional Itô formula, which just is the standard Itô formula with the usual time and space derivatives replaced by the horizontal and vertical derivatives. If the functional $F$ is sufficiently regular (regarding e.g. continuity and boundedness of its derivatives), which we write as $F \in C^{1,2}_b$, then the functional Itô formula holds, see [1, ch. 5,6]. We remark that [12] contains another version of this result.

Using the functional Itô formula it easy to see that if $Z$ is a martingale satisfying

$$Z(t) = F(t, W_t) \ dt \times d\mathbb{P}\text{-a.e.}, \text{ with } F \in C^{1,2}_b,$$

then, for every $t \in [0, T]$,

$$Z(t) = Z(0) + \int_0^t \nabla_\omega F(s, W_s) \ dW(s) \ a.s.$$
We may therefore define the vertical derivative with respect to the process $W$ of a martingale $Z$ satisfying (1) as the $dt \times d\mathbb{P}$-a.e. unique process $\nabla_W Z$ given by

$$\nabla_W Z(t) = \nabla \varphi(t, W_t), \ 0 \leq t \leq T.$$  \hspace{1cm} (2)

Let $C_{b}^{1,2}(W)$ be the space of processes $Z$ which allow the representation in (1). Let $L^2(W)$ be the space of progressively measurable processes $\varphi$ with $E \left[ \int_0^T \varphi(s) \varphi(s) ds \right] < \infty$. Let $\mathcal{M}^2(W)$ be the space of square integrable martingales with initial value 0. Let $D(W) = C_{b}^{1,2}(W) \cap \mathcal{M}^2(W)$.

It turns out that $\{\nabla_W Z : Z \in D(W)\}$ is dense in $L^2(W)$ and that $D(W)$ is dense in $\mathcal{M}^2(W)$, see [1, ch. 7]. Using this it is possible to show that the vertical derivative operator $\nabla_W (\cdot)$ admits a unique extension to $\mathcal{M}^2(W)$, in the following way: For $Y \in \mathcal{M}^2(W)$ the (weak) vertical derivative $\nabla_W Y$ is the unique element in $L^2(W)$ satisfying

$$E[Y(T)Z(T)] = E \left[ \int_0^T \nabla_W Y(t) \nabla_W Z(t) dt \right]$$  \hspace{1cm} (3)

for every $Z \in D(W)$, where $\nabla_W Z$ is defined in (2). Using the result above it is possible to prove the following constructive martingale representation theorem, see [1, ch. 7].

**Theorem 2.1 (Cont and Fournié)** For any square integrable martingale $Y$ relative to $(\mathbb{P}, \mathcal{F})$ and every $t \in [0, T]$,

$$Y(t) = Y(0) + \int_0^t \nabla_W Y(s) \, dW(s) \ a.s.$$  

3 Constructive representation of local martingales

This section contains an extension of the vertical derivative $\nabla_W (\cdot)$ and the constructive martingale representation in Theorem 2.1 to local martingales. Let $\mathcal{M}^{loc}(W)$ denote the space of local martingales relative to $(\mathbb{P}, \mathcal{F})$ with initial value zero and RCLL sample paths. Before extending the definition of the vertical derivative to $\mathcal{M}^{loc}(W)$ we recall the definition of a local martingale.

**Definition 3.1** $M$ is said to be a local martingale if there exists a sequence of non-decreasing stopping times $\{\theta_n\}$ with $\lim_{n \to \infty} \theta_n = \infty$ a.s. such that the stopped local martingale $M(\cdot \wedge \theta_n)$ is a martingale for each $n \geq 1$.

**Theorem 3.2 (Definition of $\nabla_W (\cdot)$ on $\mathcal{M}^{loc}(W)$)**

- There exists a progressively measurable $dt \times d\mathbb{P}$-a.e. unique extension of the vertical derivative $\nabla_W (\cdot)$ from $\mathcal{M}^2(W)$ to $\mathcal{M}^{loc}(W)$, such that, for $M \in \mathcal{M}^{loc}(W)$,

$$M(t) = \int_0^t \nabla_W M(s) \, dW(s), \ 0 \leq t \leq T, \quad \text{and} \quad \int_0^T |\nabla_W M(t)|^2 dt < \infty \ a.s.$$  \hspace{1cm} (4)
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- Specifically, for $M \in \mathcal{M}^{\text{loc}}(W)$ the vertical derivative $\nabla_W M$ is defined as the progressively measurable $dt \times dP$-a.e. unique process satisfying

$$\nabla_W M(t) = \lim_{n \to \infty} \nabla_W M_n(t) \ dt \times dP$$

where $\nabla_W M_n$ is the vertical derivative of $M_n := M(\cdot \wedge \tau_n) \in \mathcal{M}^2(W)$ and $\tau_n$ is given by

$$\tau_n = \theta_n \wedge \inf \{ s \in [0, T] : |M(s)| \geq n \} \wedge T$$

where $\{\theta_n\}$ is an arbitrary sequence of stopping times of the kind described in Definition 3.1.

**Remark 3.3** Note that if $M$ in Theorem 3.2 satisfies $M(t) = \int_0^t \gamma(s)'dW(s), 0 \leq t \leq T$ a.s. for some process $\gamma$, then $\gamma = \nabla_W M$ $dt \times dP$-a.e. It follows that the extended vertical derivative $\nabla_W M$ defined in Theorem 3.2 does not depend (modulo possibly on a null set $dt \times dP$) on the particulars of the chosen stopping times $\{\theta_n\}$.

**Proof.** The martingale representation theorem implies that there, for $M \in \mathcal{M}^\infty(W)$, exists a progressively measurable process $\varphi$ satisfying

$$M(t) = \int_0^t \varphi(s)'dW(s), \ 0 \leq t \leq T,$$

and $\int_0^T |\varphi(t)|^2 dt < \infty$ a.s. (7)

Therefore, if we can prove that

$$\lim_{n \to \infty} \nabla_W M_n(t) = \varphi(t) \ dt \times dP$$

then it follows that there exists a progressively measurable process, denote it by $\nabla_W M$, which is $dt \times dP$-a.e. uniquely defined by (5) and satisfies $\nabla_W M(t) = \varphi(t) \ dt \times dP$-a.e., which in turn implies that the integrals of $\nabla_W M$ and $\varphi$ coincide in the way that (7) implies (4). All we have to do is therefore to prove that (8) holds.

Let us recall some results about stopping times and martingales. The stopped local martingale $M(\cdot \wedge \theta_n)$ is a martingale for each $n$, by Definition 3.1. Stopped RCLL martingales are martingales. The minimum of two stopping times is a stopping time. Using these results we obtain that

$$M(\cdot \wedge \theta_n \wedge \inf \{ s \in [0, T] : |M(s)| \geq n \} \wedge T) = M(\cdot \wedge \tau_n)$$

is a martingale, for each $n$. Moreover, $M$ is by the standard martingale representation result a.s. continuous. Hence, we may define a sequence of, a.s. continuous, martingales $\{M_n\}$ by

$$M_n = M(\cdot \wedge \tau_n) = \int_0^{\tau_n} \varphi(s)'dW(s) \ a.s.$$
where the last equality follows from (4). Now, use the definition of \( \tau_n \) in (6) to see that

\[
|M_n(t)| = \left| \int_0^{t \land \tau_n} \varphi(s)'dW(s) \right| \leq n \text{ a.s.}
\]

for any \( t \) and \( n \), and that in particular \( M_n \) is, for each \( n \), a square integrable martingale. Moreover, (9) implies that \( M_n \) satisfies

\[
M_n(t) = \int_0^t I_{(s \leq \tau_n)} \varphi(s)'dW(s), \quad 0 \leq t \leq T \text{ a.s.} \quad (10)
\]

Since each \( M_n \) is a square integrable martingale we may use Theorem 2.1 on \( M_n \), which together with (10) implies that

\[
M_n(t) = \int_0^t \nabla_W M_n(s)'dW(s) = \int_0^t I_{(\tau_n \leq s)} \varphi(s)'dW(s), \quad 0 \leq t \leq T \text{ a.s.} \quad (11)
\]

where \( \nabla_W M_n \) is the vertical derivative of \( M_n \) with respect to \( W \) (defined in (9)) and where we also used the continuity of the Itô integrals. The equality of the two Itô integrals in (11) implies that

\[
\nabla_W M_n(t) = I_{(t \leq \tau_n)} \varphi(t) \ dt \times d\mathbb{P}\text{-a.e.} \quad (12)
\]

The local martingale property of \( M \) implies that \( \lim_{n \to \infty} \theta_n = \infty \) a.s. Using this and the definition of \( \tau_n \) in (6) we conclude that for almost every \( \omega \in \Omega \) and each \( t \in [0,T] \) there exists an \( N(\omega, t) \) such that

\[
n \geq N(\omega, t) \Rightarrow \sup_{0 \leq s \leq t} |M(\omega, s)| \leq n \text{ and } t \leq \theta_n(\omega) \Rightarrow t \leq \tau_n(\omega) \Rightarrow t \land \tau_n = t. \quad (13)
\]

Hence, it follows from (12) and (13) that there exists an \( N(\omega, t) \) such that

\[
n \geq N(\omega, t) \Rightarrow \nabla_W M_n(\omega, t) = \varphi(\omega, t) \ dt \times d\mathbb{P}\text{-a.e.}
\]

which means that (8) holds.

If \( M \) is a RCLL local martingale then \( M - M(0) \in \mathcal{M}^{\infty}(W) \), which implies that \( \nabla_W (M - M(0)) \) is defined in Theorem 3.2. This observation allows us to extend the definition of the vertical derivative to RCLL local martingales not necessarily starting at zero in the following obvious way.

**Definition 3.4** The vertical derivative of a local martingale \( M \) relative to \( (\mathbb{P}, \mathcal{F}) \) with RCLL sample paths is defined as the progressively measurable \( dt \times d\mathbb{P}\)-a.e. unique process \( \nabla_W M \) satisfying

\[
\nabla_W M(t) = \nabla_W (M - M(0))(t), \quad 0 \leq t \leq T, \quad (14)
\]

where \( \nabla_W (M - M(0))(t) \) is defined in Theorem 3.2.

The following result is an immediate consequence of Theorem 3.2 and Definition 3.4.

**Theorem 3.5 (Main result)** If \( M \) is a local martingale relative to \( (\mathbb{P}, \mathcal{F}) \) with RCLL sample paths, then

\[
M(t) = M(0) + \int_0^t \nabla_W M(s)'dW(s), \quad 0 \leq t \leq T, \quad \text{and} \quad \int_0^T |\nabla_W M(t)|^2 dt < \infty \text{ a.s.}
\]

where \( \nabla_W M(s) \) is defined in Definition 3.4.
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