LOG ABUNDANCE OF THE MODULI B-DIVISORS OF LC-TRIVIAL FIBRATIONS

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ABSTRACT. We prove that the moduli b-divisor of an lc-trivial fibration from a log canonical pair is log abundant. This result follows from a theorem on the restriction of the moduli b-divisor, which is also obtained by E. Floris and V. Lazić [FL19]. We give an alternative proof in Section 4.3, based on a theory of lc-trivial morphisms. We also prove a theorem on extending a finite cover over a closed subvariety of the same degree, on a variety over an arbitrary field.

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1. INTRODUCTION

We work over an algebraically closed field of characteristic zero unless stated otherwise. In Section 3, we work over an arbitrary field.

Log abundance of the moduli b-divisor. Given a proper surjective morphism $f : X \to Y$ of normal varieties with connected fibres, and a log canonical (lc for short)

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pair \((X, B)\) with \(K_X + B \sim_{\mathbb{R}} 0\) over \(Y\), there exists a canonical decomposition of Kodaira type

\[ K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y + M_Y), \]

where \(B_Y\) and \(M_Y\) are \(\mathbb{R}\)-divisors on \(Y\), called the *discriminant* and *moduli* \(\mathbb{R}\)-divisors. For any birational models \(X' \to X, Y' \to Y\) such that the induced map \(f' : X' \to Y'\) is a morphism, we can similarly define the discriminant and moduli \(\mathbb{R}\)-divisors, hence the discriminant \(\mathbb{R}\)-b-divisor \(B\) and moduli \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-b-divisors \(M\) (See Section 2.1 for definition of b-divisors introduced by V. V. Shokurov). Using the theory of variations of Hodge structure, F. Ambro\cite{Amb04} and O. Fujino and Y. Gongyo\cite{FG14} show that, if \(B\) is a \(\mathbb{Q}\)-divisor, then the moduli b-divisor \(M\) is \(\mathbb{Q}\)-b-Cartier and b-nef, hence \(K + B\) is \(\mathbb{Q}\)-b-Cartier. One can extend this result to the real coefficient case (see Corollary 2.26). Moreover, if we assume further that \(Y\) is complete, and every lc centre is vertical \(Y\), then the moduli b-divisor \(M\) is b-nef and abundant (\cite{Amb04} \cite{FG14}). Using techniques from minimal model theory, O. Fujino and Y. Gongyo \cite{FG14} proved that, the assumption that every lc centre is vertical can be removed. In this paper, we will not directly apply this result, but a similar technique from \cite{FG14} will be used in Section 4.3.

Generalised (polarised) pairs, introduced by C. Birkar and D. Q. Zhang \cite{BZh16}, are originated from the above construction. Notation as above, the pair \((X, B + M)\) with data given by \(M\) is a generalised lc (g-lc for short) generalised pair (see Section 2.1 for definition of generalised pairs and generalised singularities). We call it the *induced generalised pair*. The main purpose of this paper is to study the positivity of the moduli b-divisor \(M\), with respect to the induced generalised pair.

Given a generalised pair \((X, B + M)\) with data \(\mathcal{M}\), and an \(\mathbb{R}\)-b-Cartier b-divisor \(D\) on \(X\), we say \(D\) is b-nef and log abundant if \(D_{X'}\) is nef and log abundant with respect to any sufficiently high log resolution \((X', B' + M')\) (see Definition 2.13).

Our main result is the log abundance of the moduli b-divisor.

**Theorem 1.1** (=Corollary 4.29). Let \(f : X \to Y\) be a surjective morphism between normal complete varieties with connected fibres. Suppose that \((X, B)\) is an lc pair with \(K_X + B \sim_{\mathbb{R}} 0/Y\). Then the moduli b-divisor \(M\) is b-nef and log abundant with respect to the induced generalised pair.

**Extending a finite cover.** In Section 3, we prove the following theorem on extending a finite cover over a closed subvariety to a finite cover of the same degree over the whole variety. Although in this paper, we will only use a special case of Theorem 1.2 the ground field is an algebraically closed field of characteristic zero, we will prove prove it in full generality.

**Theorem 1.2** (=Theorem 3.1). Let \(X\) be a normal variety over an arbitrary field and \(S\) be a closed subvariety. Suppose we are given a finite morphism \(\gamma : \tilde{S} \to S\) from a normal variety. Suppose further that one of the following conditions holds:

1. The residue field \(\kappa(\eta_S)\) at the generic point \(\eta_S\) of \(S\) is perfect.
2. \(X\) is regular at the generic point of \(S\).
Then there exists a finite morphism $\rho : \tilde{X} \to X$ of normal varieties together with a closed subvariety $\tilde{S} \subset \tilde{X}$ satisfying:

\[
\begin{array}{c}
\tilde{S} \xrightarrow{\nu} \tilde{S} \leftarrow \tilde{X} \\
\gamma \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
S \leftarrow X
\end{array}
\]

(1). $\tilde{S}$ is mapped onto $S$ through $\rho$ and the above diagram commutes.
(2). $\nu$ is the normalisation of $\tilde{S}$.
(3). $\deg \rho = \deg \gamma$.

The category of lc-trivial morphisms. One of the difficulties of proving Theorem 1.1 using induction lies in the fact that, the restriction of $f$ to an lc centre does not necessarily have (geometric) connected general fibres. We therefore expand the category of lc-trivial fibrations to that of lc-trivial morphisms, which allows its morphism to have disconnected general fibres (see Definitions 2.25 and 4.1). Moreover, we define the discriminant and moduli b-divisors by proper push-forwards via a finite map (see Definitions 4.2 and 4.6). One can verify its positivity by the following theorem. Quite recently, J. Han and W. Liu also obtained a similar result [HL19, Theorem 4.5].

**Theorem 1.3** (=Theorem 4.5). Let $(X/Z, B + M)$ be a generalised sub-pair over a variety $Z$. Let $f : X \to Y$ be a generically finite surjective morphism of normal varieties over $Z$, such that $K_X + B_X + M_X \sim_\mathbb{K} 0/Y$. Then, there is a generalised sub-pair $(Y/Z, B_Y + M_Y)$ with

$$K_X + B + M \sim_\mathbb{K} f^*(K_Y + B_Y + M_Y)$$

Moreover, if $(X/Z, B+M)$ is $g$-lc (resp. $g$-klt, $g$-sub-lc, $g$-sub-klt), then so is $(Y/Z, B_Y + M_Y)$.

Furthermore, we say an lc-trivial morphism is **good** if its moduli b-divisor behaves exactly the same as the classical one (see Definition 4.8). An important feature of a good lc-trivial morphism is that, it has a “dlt model”, namely a dlt-trivial morphism (see Definition 4.17). Note that an lc-trivial fibration is automatically a good lc-trivial morphism.

**Proposition 1.4** (=Proposition 4.22). Let $f : (X, B) \to Y$ be an lc-trivial morphism from an lc pair. Then, there exist a birational model $\phi : Y' \to Y$ and a log birational contraction $\pi : (X', B') \dashrightarrow (X, B)$ such that the induced map $f' : X \dashrightarrow Y'$ is a morphism, $(X', B')$ is quasi-projective $\mathbb{Q}$-factorial dlt and the induced generalised pair $(Y', B_Y + M_Y)$ is quasi-projective $\mathbb{Q}$-factorial g-dlt.

\[
\begin{array}{c}
X' \xrightarrow{\pi} X \\
\downarrow f' \downarrow \downarrow \downarrow \downarrow \\
Y' \xrightarrow{\phi} Y
\end{array}
\]

In particular, if in addition we suppose $f : (X, B) \to Y$ is good, then $f' : (X', B') \to Y'$ is a $\mathbb{Q}$-factorial dlt-trivial morphism.
Adjunction for fibre space commutes with restriction. In Section 4.3, we establish the following theorem of a dlt-trivial morphism, which asserts that, the dlt-triviality in stable under the restriction; moreover, to construct a moduli b-divisor, adjunction for a projective surjective morphism commutes with restriction to a stratum.

The author was informed by E. Floris and V. Lazíc that they already obtained the same result via a different approach, see [FL19, Proposition 4.4].

Theorem 1.5 (=Theorem 4.27). Let $f: (X, B) \rightarrow Y$ be a dlt-trivial morphism and $T$ be a stratum of the induced dlt pair $(Y, B_Y)$. Suppose $S$ is a stratum of $(X, B)$ saturated over $T$ (for example, $S$ is a divisor; see Definition 4.26). Then,

1. $f|_S: (S, B_S) \rightarrow T$ is a dlt-trivial morphism where $K_S + B_S = (K_X + B)|_S$.
2. If we denote by $(T, B_T + M_T)$ the g-dlt generalised pair with the moduli b-divisor $M|_T$ given by the adjunction formula $K_T + B_T + M_T = (K_Y + B_Y + M_Y)|_T$, and we denote by $(T, C_T + N_T)$ the g-dlt generalised pair with the moduli b-divisor $N$ given by the dlt-trivial morphism $f|_S: (S, B_S) \rightarrow T$. Then, we have

$$M|_T = N.$$ 

The above theorem is another main result of this paper. Combining it with Proposition 1.4, one can easily derive Theorem 1.1.

Relative log abundance of the moduli b-divisor. By Lemma 2.9, we obtain a relative version of Theorem 1.1 as below.

Theorem 1.6 (=Corollary 4.31). Let $f: X \rightarrow Y$ be a proper surjective morphism between normal varieties with connected fibres. Suppose that $(X, B)$ is an lc pair with $K_X + B \sim_{\mathbb{R}} 0/Y$. Suppose further that $Y$ is proper over a variety $Z$. Then the moduli b-divisor $M$ is b-nef and log abundant over $Z$ with respect to the induced generalised pair.

Sketch of proof of Theorems 1.1 and 1.5. We prove Theorem 1.1 by induction on the dimension of $X$. To arrange an inductive argument, we remove the assumption on the connectedness of fibres, and assume $f$ is a good lc-trivial morphism. Thanks to Proposition 1.4, we may assume $f: (X, B) \rightarrow Y$ is dlt-trivial. Now by Theorem 1.5, it is sufficient to prove $M$ is b-nef and abundant. Applying Theorem 1.5 again, we may reduce to the case when $(X, B)$ has no horizontal lc centres. Hence, the conclusion follows from Corollary 2.27.

Next we sketch the proof of Theorem 1.5. First, by induction on dimension, we may assume $S$ is a prime divisor and $T$ is a prime divisor or $Y$. Taking the Stein factorisation and by Proposition 1.4, we may assume $f$ has connected fibres. Replacing $f$ by an appropriate base change obtained by Theorem 1.2, we may assume $f|_S$ has connected fibres. Apply the weak semi-stable reduction (Theorem 2.33) to construct a weakly semi-stable fibre space. Finally, by a similar technique of log Minimal Model Program from [FG14], we obtain the conclusion.

Contents of the paper. In Section 2, we collect definitions, notations and results on log MMP, in the setting of normal varieties. In Section 3, we prove Theorem 1.2.
In Section 4.1, we define (good) lc-trivial morphisms and their moduli b-divisors. We prove Theorem 1.3 to obtain the positivity of the moduli b-divisor. In Section 4.2, we define dlt-trivial morphisms and prove Proposition 1.4. In Section 4.3, we prove Theorems 1.5, 1.1 and 1.6.

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2. Preliminaries

In this section we collect definitions and some important results. Throughout this paper all varieties are over a fixed algebraically closed field of characteristic zero and a divisor refers to an $\mathbb{R}$-Weil divisor unless stated otherwise.

2.1. Notations and definitions. We collect some notations and definitions.

Conventions. We denote by $\mathbb{K}$ the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. A birational model of a normal variety $X$, often denoted by $X'$, means a variety admits a proper and birational morphism to $X$, and a divisor $D$ over $X$ means a divisor on a birational model of $X$.

Contractions. In this paper a contraction refers to a proper morphism $f : X \to Y$ of varieties such that $f_* \mathcal{O}_X = \mathcal{O}_Y$. In particular, $f$ has connected fibres. Moreover, if $X$ is normal, then $Y$ is also normal. A contraction $f$ is small if $f$ does not contract any divisor. A birational map $\pi : X \dashrightarrow Y$ is a birational contraction if the inverse of $\pi$ does not contract divisors. Note that $\pi$ is not necessarily a morphism unless stated otherwise.

Divisors. Let $X$ be a normal variety, and let $M$ be an $\mathbb{R}$-divisor on $X$. We denote the coefficient of a prime divisor $D$ in $M$ by $\text{mult}_D M$. Writing $M = \sum m_i M_i$ where $M_i$ are the distinct irreducible components, the notation $M^{\le a}$ means $\sum_{m_i \ge a} m_i M_i$, that is, we ignore the components with coefficient $< a$. One similarly defines $M^{\ge a}$, $M^{> a}$, and $M^{< a}$.

By a $\mathbb{K}$-rational function we mean a formal product of finitely many rational functions with $\mathbb{K}$-exponent, namely $\varphi := \prod_{i=1}^k \varphi_i^{\alpha_i}$ with $\alpha_i \in \mathbb{K}$ for all $i$. We denote its $\mathbb{K}$-Cartier divisor by $(\varphi) := \sum_{i=1}^k \alpha_i (\varphi_i)$. Given two $\mathbb{K}$-Cartier divisors $D, D'$ on $X$, we say $D, D'$ are $\mathbb{K}$-linearly equivalent, and denote by $D \sim_\mathbb{K} D'$, if there exists a $\mathbb{K}$-rational function $\varphi$ such that $D = D' + (\varphi)$.

Given a proper morphism $f : X \to Z$, we say $D, D'$ are $\mathbb{K}$-linearly equivalent (resp. linearly equivalent, equivalent) over $Z$ and denote by $D \sim_\mathbb{K} D'/Z$ (resp. $D \sim D'/Z$, $D = D'/Z$) if there exists an $\mathbb{R}$-Cartier divisor $D_Z$ on $Z$ such that $D \sim_\mathbb{K} D' + f^* D_Z$ (resp. $D \sim D' + f^* D_Z$, $D = D' + f^* D_Z$).

Very exceptional divisors. Let $f : X \to Y$ be a dominant morphism from a normal variety to a variety, $D$ a divisor on $X$, and $Z \subset X$ a closed subset. We say $Z$ is
horizontal over $Y$ if $f(Z)$ dominates $Y$, and we say $Z$ is vertical over $Y$ if $f(Z)$ is a proper subset of $Y$.

Suppose $f$ is a contraction of normal varieties. Recall that a divisor $D$ is very exceptional$/Y$ if $D$ is vertical$/Y$ and for any prime divisor $P$ on $Y$ there is a prime divisor $Q$ on $X$ which is not a component of $D$ but $f(Q) = P$, i.e. over the generic point of $P$ we have $\text{Supp} f^*P \nsubseteq \text{Supp}D$.

If $\text{codim} f(D) \geq 2$, then $D$ is very exceptional. In this case we say $D$ is $f$-exceptional.

For practical reason, we sometimes use the terminology “very exceptional” even when $f$ is not necessarily a contraction. More precisely, let $f : X \to Y$ be a proper surjective morphism from a normal variety to a variety, and $X \to \tilde{Y} \to Y$ be the Stein factorisation. We say a divisor $D$ is very exceptional over $Y$ if it is very exceptional over $\tilde{Y}$.

**b-divisors.** We recall some definitions regarding b-divisors. Let $X$ be a variety. A $b$-divisor $D$ of $X$ is a family $\{D_{X'}\}_{X'}$ of $\mathbb{R}$-Weil divisors indexed by all birational models $X'$ of $X$, such that $\mu_*(D_{X'}) = D_{X''}$ if $\mu : X'' \to X'$ is a birational contraction.

In most cases we focus on a class of b-divisors but not in full generality. An $\mathbb{K}$-b-Cartier $b$-divisor $M$ is defined by the choice of a projective birational morphism $\overline{X} \to X$ from a normal variety and an $\mathbb{K}$-Cartier divisor $\overline{M}$ on $\overline{X}$ in the way that $M_{X'} = \mu^*\overline{M}$ for any birational model $\pi : X' \to Y$. In this case we say that $\overline{M}$ represents $M$ or $M$ descends to $\overline{X}$.

Given an $\mathbb{K}$-b-Cartier $b$-divisors $M$ on $X$ represented by $M_Y$ and a surjective proper morphism $f : Y \to X$, we define the pull-back of $M$ as the $\mathbb{K}$-b-Cartier $b$-divisors $f^*M$ represented by $f^*\overline{M}$ where $f : Y \to \overline{X}$ is induced by $f$ and $\overline{X} \to X$.

An $\mathbb{R}$-b-Cartier $b$-divisor represented by some $\overline{X} \to X$ and $\overline{M}$ is $b$-nef if $\overline{M}$ is nef. Similarly we define a $b$-nef and abundant $b$-divisor if $\overline{M}$ is nef and abundant.

We say a few words about the relative case. Let $f : X \to Y$ be a proper surjective morphism of varieties and $D_1, D_2$ be $\mathbb{R}$-b-Cartier $b$-divisors on $X$ represented by $\overline{D}_1, \overline{D}_2$ on $\overline{X} \to X$. We say $D$ is $\mathbb{K}$-linearly equivalent over $Y$ and denote by $D_1 \sim_\mathbb{K} D_2/Y$ if there exists birational models $\phi : X' \to \overline{X}$ and $Y' \to Y$ such that the induced map $X' \dashrightarrow Y'$ is a morphism and $\phi^*\overline{D}_1 \sim_\mathbb{K} \phi^*\overline{D}_2/Y'$. The reader may want to check this is well-defined since it is independent of the choice of birational models. In particular, we say $D$ is $\mathbb{K}$-linearly trivial over $Y$ if $D \sim_\mathbb{K} 0/Y$.

**Morphisms induced by base change.** Let $f : X \to Y$ be a dominant morphism from a normal variety to a variety. Given a proper morphism $g : Y' \to Y$ of varieties, we call $f' : X \times_Y Y' \to Y'$ the base change. If $g$ is surjective, then we consider the normalisation $W := (X \times_Y Y')^v$ which is a disjoint union of finitely many normal varieties, denote by $W = \bigsqcup_i W_i$. For each $i$, we say $W_i$ is a main component if $W_i$ dominates both $Z$ and $X$. Recall the basic facts:

1. If either $f$ or $g$ has the connected geometric generic fibre, then the main component is unique. In this case, denoting by $X'$ the normalisation of the unique main component, we say $f' : X' \to Y'$ is the morphism induced by the base change.

2. If either $f$ or $g$ has equidimensional fibres, then every component is a main component.
(3) Combined, if $f$ is equidimensional and $g$ has connected geometric generic fibre, then the fibre product is irreducible and reduced.

**Pairs.** A sub-pair $(X/Z, B)$ consists of a normal variety $X$, a proper morphism $X \to Z$ and an $\mathbb{R}$-divisor $B$ such that $K_X + B$ is $\mathbb{R}$-Cartier. We say $B$ is a pre-boundary. If the coefficients of $B$ are at most 1 we say $B$ is a sub-boundary, and if in addition $B \geq 0$, we say $B$ is a boundary. A sub-pair $(X/Z, B)$ is called a pair if $B$ is a boundary. When $Z$ is not relevant we usually drop it and do not mention it: in this case one can just assume $X \to Z$ is the identity. When $Z$ is a point we also drop it but say the pair is projective.

Let $\phi: W \to X$ be a log resolution of a sub-pair $(X, B)$. Let $K_W + B_W$ be the pullback of $K_X + B$. The log discrepancy of a prime divisor $D$ on $W$ with respect to $(X, B)$ is $1 - \text{mult}_D B_W$ and it is denoted by $a(D, X, B)$. We say $(X, B)$ is a sub-lc pair (resp. sub-klt pair) if $a(D, X, B) \geq 0$ (resp. $> 0$) for every $D$. When $(X, B)$ is a pair we remove the sub and say the pair is lc, etc. Note that if $(X, B)$ is an lc pair, then the coefficients of $B$ necessarily belong to $[0, 1]$.

Let $(X, B)$ be a sub-pair. A non-klt place of $(X, B)$ is a prime divisor $D$ on birational models of $X$ such that $a(D, X, B) \leq 0$. A non-klt center is the image on $X$ of a non-klt place. When $(X, B)$ is lc, a non-klt center is also called an lc center. For definitions and standard results on singularities of pairs we refer to [KM98].

**Generalised pairs.** For the basic theory of generalised polarised pairs (generalised pairs for short) we refer to [BZh16, Section 4]. Below we recall some of the main notions and discuss some basic properties.

A generalised sub-pair consists of

- a normal variety $X$ equipped with a proper morphism $X \to Z$,
- an $\mathbb{R}$-divisor $B$ on $X$, and
- a $b$-$\mathbb{R}$-Cartier $b$-divisor over $X$ represented by some projective birational morphism $\overline{X} \xrightarrow{\phi} X$ and $\mathbb{R}$-Cartier divisor $\overline{M}$ on $X$ such that $\overline{M}$ is nef/$Z$ and $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \phi_* \overline{M}$.

A generalised sub-pair is a generalised pair if $B$ is effective.

We usually refer to the sub-pair by saying $(X/Z, B+M)$ is a generalised sub-pair with data $\overline{M}$. Since a $b$-$\mathbb{R}$-Cartier $b$-divisor is defined birationally, in practice we will often replace $\overline{X}$ with a log resolution (and hence omit it) and replace $\overline{M}$ with its pullback. In this case, we say $(\overline{X}, \overline{B} + \overline{M}) \to X$, where $K_{\overline{X}} + \overline{B} + \overline{M}$ is the pull-back of $K_X + B + M$, is a data log resolution. When $Z$ is not relevant we usually drop it and do not mention it: in this case one can just assume $X \to Z$ is the identity. When $Z$ is a point we also drop it but say the pair is projective. A generalised sub-pair naturally defines a $b$-divisor $K + B + M$ which descends to $X$ so that for any projective birational morphism $X' \xrightarrow{\phi'} X$ we have $\phi'^*(K_X + B + M) = K_{X'} + B_{X'} + M_{X'}$. We call $B$ the pre-boundary $b$-divisor and $M$ the moduli $b$-divisor. Similarly, If the coefficients of $B$ are at most 1 we say $B$ is a sub-boundary $b$-divisor, and if in addition $B \geq 0$, we say $B$ is a boundary $b$-divisor.
Now we define generalised singularities of a generalised pair. Replacing $X$ we can assume $\phi$ is a log resolution of $(X, B)$. We can write

$$K_{X} + B + \overline{M} = \phi^{\ast}(K_{X} + B + M)$$

for some uniquely determined $B$. For a prime divisor $D$ on $X$ the generalized log discrepancy $\alpha(D, X, B + M)$ is defined to be $1 - \operatorname{mult}_{D} \overline{B}$.

We say $(X, B + M)$ is generalised lc or g-lc (resp. generalised klt or g-klt) if for each $D$ the generalised log discrepancy $\alpha(D, X, B + M)$ is $\geq 0$ (resp. $> 0$). We say $(X, B + M)$ is generalised dlt or g-dlt if it is g-lc, $(X, B)$ is dlt, and every generalised non-klt center of $(X, B + M)$ is a non-klt center of $(X, B)$ (note that here we are assuming $(X, B)$ is a dlt pair in the usual sense, in particular, $K_{X} + B$ is assumed to be $\mathbb{R}$-Cartier). If in addition the connected components of $|B|$ are irreducible, we say the pair is generalised plt or g-plt for short. Note that g-sub-lc, g-sub-klt, etc. can be defined similarly when we drop the assumption of the effectivity of $B$.

A generalised non-klt center of a generalised sub-pair $(X, B + M)$ is the image of a prime divisor $D$ over $X$ with $\alpha(D, X, B + M) \leq 0$, and the generalised non-klt locus of the generalised sub-pair is the union of all the generalised non-klt centers. When $(X, B + M)$ is g-lc, a generalised non-klt center is also called a g-lc center.

Given a g-lc generalised pair $(X/Z, B + M)$ with data $\overline{M}$, if $\overline{M}$ is a non-negative $\mathbb{R}$-linear combination of $\mathbb{Q}$-Cartier divisors which are nef over $Z$, then the generalised pair $(X/Z, B + M)$ is an NQC generalised pair. Here, NQC stands for nef $\mathbb{Q}$-Cartier combinations.

Let $(X, B + M)$ be a generalised pair as in (1) and let $\psi : X' \rightarrow X$ be a projective birational morphism from a normal variety. Replacing $\phi$ we can assume $\phi$ factors through $\psi$. We then let $B'$ and $M'$ be the push-downs of $B$ and $M$ on $X'$ respectively. In particular,

$$K_{X'} + B' + M' = \psi^{\ast}(K_{X} + B + M).$$

If $B' \geq 0$, then $(X', B' + M')$ is also a generalised pair with data $\overline{M}$. If $(X', B' + M')$ is $\mathbb{Q}$-factorial g-dlt and if every exceptional prime divisor of $\psi$ appears in $B'$ with coefficients one, then we say $(X', B' + M')$ is a $\mathbb{Q}$-factorial g-dlt model(or blow-up) of $(X, B + M)$. Such models exist if $(X, B + M)$ is generalised lc, by [BZh16, Lemma 4.5].

Two generalised sub-pairs $(X/Z, B + M), (X'/Z, B + M)$ with data $\overline{M}$ are B-birational if there is a common log resolution $X \xleftarrow{\xi} (X/Z, B + \overline{M}) \xrightarrow{\pi} X'$ with $K_{X} + B + \overline{M} = \pi^{\ast}(K_{X'} + B' + M')$. In this case, the rational map $(\pi' \circ \pi^{-1}) : X \dashrightarrow X'$ is a $B$-birational map. If its inverse map does not contract divisors, then we say it is a $B$-birational contraction.

Minimal models. A generalised pair $(Y/Z, B_{Y} + M_{Y})$ with data $M_{Y}$ is a log birational model of a generalised pair $(X/Z, B + M)$ with data $\overline{M}$ if we are given a birational map $\phi : X \dashrightarrow Y$, $B_{Y} = B_{Y}^{\sim} + E$ where $B_{Y}^{\sim}$ is the birational transform of $B$ and $E$ is the reduced exceptional divisor of $\phi^{-1}$, that is, $E = \sum E_{j}$ where $E_{j}$ are the exceptional/X prime divisors on $Y$ and $\overline{M}_{Y} = \overline{M}$.

A log birational model $(X/Z, B + \overline{M})$ is a log smooth model of $(X/Z, B + M)$ if it is log smooth with data $\overline{M}$.
A log birational model $(Y/Z, B_Y + M_Y)$ is a weak log canonical (weak lc for short) model of $(X/Z, B + M)$ if

- $M_Y = M$,
- $K_Y + B_Y + M_Y$ is nef/exceptional, and
- for any prime divisor $D$ on $X$ which is exceptional/$Y$, we have

$$a(D, X, B + M) \leq a(D, Y, B_Y + M_Y).$$

A weak lc model $(Y/Z, B_Y + M_Y)$ is a log minimal model of $(X/Z, B + M)$ if

- $(Y/Z, B_Y)$ is a Q-factorial g-dlt,
- the above inequality is strict on log discrepancies is strict.

A log minimal model $(Y/Z, B_Y+M_Y)$ is good if $K_Y+B_Y+M_Y$ is semi-ample/Z. In this case, $K_Y+B_Y+M_Y$ defines a contraction $g: Y \to W$ such that $K_Y+B_Y+M_Y = g^*A_W$ for some ample/Z divisor $A_W$. We say $W$ is the canonical model of $(X/Z, B + M)$.

On the other hand, a log birational model $(Y/Z, B_Y + M_Y)$ is called a weak Mori fibre space of $(X/Z, B + M)$ if

- there is a $K_Y+B_Y+M_Y$-negative extremal contraction $Y \to T$ with $\dim Y > \dim T$, and
- for any prime divisor $D$ (on birational models of $X$) we have

$$a(D, X, B + M) \leq a(D, Y, B_Y + M_Y)$$

and strict inequality holds if $D$ is on $X$ and contracted/$Y$.

A weak Mori fibre space $(Y/Z, B_Y + M_Y)$ is a Mori fibre space of $(X/Z, B + M)$ if

- $(Y/Z, B_Y + M_Y)$ is a Q-factorial g-dlt.

### 2.2. Nef and abundant divisors.

In this subsection we will introduce the notion of nef and abundant divisor and elementary properties in the setting of $\mathbb{R}$-divisors. Most contents of this subsection are taken from [Nak04] with slight modifications. We write proofs of some results for the reader’s convenience.

**Iitaka dimension and numerical dimension.** Recall the following definitions of Iitaka dimension and numerical dimension. Both integers are birational invariants given by the growth of the quantity of sections. We remind the reader to be careful with $\mathbb{R}$-Cartier divisors.

**Definition 2.1** (Invariant Iitaka dimension). Let $X$ be a normal projective variety, and $D$ be an $\mathbb{R}$-Cartier divisor $D$ on $X$. We define the invariant Iitaka dimension of $D$, denoted by $\kappa_i(X, D)$, as follows (see also [Fuj-book17, Definition 2.5.5]): If there is an $\mathbb{R}$-divisor $E \geq 0$ such that $D \sim_\mathbb{R} E$, set $\kappa_i(X, D) = \kappa(X, E)$. Here, the right hand side is the usual Iitaka dimension of $E$. Otherwise, we set $\kappa_i(X, D) = -\infty$. We can check that $\kappa_i(X, D)$ is well-defined, i.e., when there is $E \geq 0$ such that $D \sim_\mathbb{R} E$, the invariant Iitaka dimension $\kappa_i(X, D)$ does not depend on the choice of $E$. By definition, we have $\kappa_i(X, D) \geq 0$ if and only if $D$ is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor.

Let $X \to Z$ be a projective morphism from a normal variety to a variety, and let $D$ be an $\mathbb{R}$-Cartier divisor on $X$. Then the relative invariant Iitaka dimension of $D$, denoted by $\kappa_i(X/Z, D)$, is defined by $\kappa_i(X/Z, D) = \kappa_i(X, D|_F)$, where $F$ is a very general fibre.
(i.e. the fibre over a very general point) of the Stein factorisation of \( X \to Z \). Note that the value \( \kappa_\varpi(X, D|_F) \) does not depend on the choice of \( F \) (see [HH19, Lemma 2.10]).

**Definition 2.2** (Numerical dimension). Let \( X \) be a normal projective variety, and \( D \) be an \( \mathbb{R} \)-Cartier divisor \( D \) on \( X \). We define the numerical dimension of \( D \), denoted by \( \kappa_\varpi(X, D) \), as follows (see also [Nak04, V, 2.5 Definition]): For any Cartier divisor \( A \) on \( X \), we set

\[
\sigma(D; A) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \left| \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD + A \rfloor))}{m^k} > 0 \right\}
\]

if \( \dim H^0(X, \mathcal{O}_X(\lfloor mD + A \rfloor)) > 0 \) for infinitely many \( m \in \mathbb{Z}_{>0} \), and otherwise we set \( \sigma(D; A) := -\infty \). Then, we define

\[
\kappa_\varpi(X, D) := \max \{ \sigma(D; A) \mid A \text{ is a Cartier divisor on } X \}.
\]

Let \( X \to Z \) be a projective morphism from a normal variety to a variety, and let \( D \) be an \( \mathbb{R} \)-Cartier divisor on \( X \). Then, the relative numerical dimension of \( D \) over \( Z \), denoted by \( \kappa_\varpi(X/Z, D) \), is defined by \( \kappa_\varpi(F, D|_F) \), where \( F \) is a very general fibre of the Stein factorisation of \( X \to Z \). We note that the value \( \kappa_\varpi(F, D|_F) \) does not depend on the choice of \( F \), so the relative numerical dimension is well-defined.

For a collection of basic properties of the invariant Iitaka dimension and the numerical dimension, we refer to [HH19, Remark 2.8].

**Definition 2.3** (Relatively abundant divisor and relatively log abundant divisor). Let \( f : X \to Z \) be a projective morphism from a normal variety to a variety, and \( D \) be an \( \mathbb{R} \)-Cartier divisor on \( X \). We say that \( D \) is abundant over \( Z \) if the equality \( \kappa_\varpi(X/Z, D) = \kappa_\varpi(X/Z, D) \) holds. When \( Z \) is a point, we simply say \( D \) is abundant.

Let \( f : X \to Z \) and \( D \) be as above, and \( (X, B + M) \) be a g-sub-lc generalised sub-pair. We say that \( D \) is \( \pi \)-log abundant (or log abundant over \( Z \)) with respect to \( (X, B + M) \) if \( D \) is abundant over \( Z \) and for any g-lc center \( S \) of \( (X, B + M) \) with the normalization \( S' \to S \), the pullback \( D|_{S'} \) is abundant over \( Z \).

**Remark 2.4.** With notation in Definition 2.3, let \( X_\varpi \) be the geometric generic fibre of the Stein factorisation of \( f \). By flat base change, one can easily verify that \( D \) is abundant over \( Z \) if and only if the restriction \( D|_{X_\varpi} \) is abundant. Moreover, if we denote by \( X_\varpi \) the geometric generic fibres of the Stein factorisation of \( D|_{S_i} \) for each g-sub-lc centre \( S_i \), then, \( D \) is log abundant over \( Z \) if and only if \( D|_{X_\varpi} \) is abundant and \( D|_{X_\varpi} \) is abundant for every \( i \).

**Nef and abundant divisors.** Recall that, given a a projective morphism \( f : X \to Z \) from a normal variety to a variety, an \( \mathbb{R} \)-Cartier divisor \( D \) is semi-ample if there exist a proper surjective morphism \( g : X \to Y \) over \( Z \) and an ample divisor \( D_Y \) of \( Y \) such that \( D \sim_\mathbb{R} g^*D_Y \).

**Lemma 2.5.** Notation as above, let \( D \) be an \( \mathbb{R} \)-Cartier divisor.

1. \( D \) is semi-ample if and only if \( D \) is a convex combination of semi-ample \( \mathbb{Q} \)-divisors.
2. let \( D' \) be another \( \mathbb{R} \)-Cartier divisor. If \( D, D' \) are semi-ample, then so is \( D + D' \).
Proof. (1) can be proved by an elementary argument from convex geometry. (2) is a direct consequence of (1). \qed

Lemma 2.6 (Nef and abundant divisor, cf.[Nak04, V. 2.3 Lemma]). Let \( f : X \to Z \) be a projective morphism from a normal variety to a variety, and let \( D \) be an \( \mathbb{R} \)-Cartier divisor on \( X \). Then, the following conditions are equivalent:

1. \( D \) is nef and abundant over \( Z \).
2. there exist a birational model \( \pi : X' \to X \), a surjective morphism \( g : X' \to Y \) of smooth quasi-projective varieties over \( Z \), and a nef and big/\( \mathbb{R} \)/ divisor \( B \) of \( Y \) such that \( \pi^* D \sim_B g^* B \).

Proof. First we prove (1) implies (2). By [Nak04, V. 2.3 Lemma(1)] and Remark 2.4, there exist a non-empty open subset \( U_Z \subset Z \), a birational model \( \pi : X' \to X \), a surjective morphism \( g : X' \to Y \) of smooth quasi-projective varieties over \( Z \), and an \( \mathbb{R} \)-Cartier divisor \( B \) of \( Y \) such that \( (\pi|_{U'_Z})^*(D|_{U'}) \sim (g|_{U'_Y})^*(B|_{U_Y}) \), where \( U, U' \) and \( U_Y \) are the inverse images of \( U_Z \), and \( B|_{U_Y} \) is nef and big over \( Z \). Applying the equidimensional reduction, there exist birational models \( \pi' : X'' \to X' \), \( \phi : Y' \to Y'' \) so that the induced morphism \( g' : X'' \to Y'' \) is equidimensional. Let \( \varphi \) be the \( \mathbb{R} \)-rational function such that \( \pi'^* D + (\varphi) - g'^* B = 0 \) over \( U \). Since \( g' \) is equidimensional, by an easy argument (for example, see [Hu17, Proof of Lemma 3.9]), there exists a divisor \( \Delta \) on \( Y' \) so that \( (\pi \circ \pi')^* D + (\varphi) = g'^* (\phi^* B + \Delta) + E \) where \( E \geq 0 \) is very exceptional/\( Y' \). Hence, by the negativity lemma [Bir12, Lemma 3.3], we deduce \( E = 0 \). Replacing \( Y \) with \( Y' \), \( B \) with \( \phi^* B + \Delta \) and \( X \) with a resolution of \( X'' \), we complete the proof.

We prove (2) implies (1). This follows directly from [Nak04, V. 2.7 Proposition]. \qed

Remark 2.7. One can regard (2) as an alternative definition of nef and abundant \( \mathbb{R} \)-Cartier divisor (see [Amb05][FG12]). Note that by our definition of abundant divisor, notation as above, one cannot expect \( \pi^* D \sim_Q g^* B \) as in [Nak04, V. 2.3 Lemma(1)] unless \( D \sim_Q E/\mathbb{U}_Z \) for some divisor \( E \geq 0 \) and a non-empty open subset \( U_Z \subset Z \) (by [HH19, Lemma 2.10]). Nevertheless, if \( D \geq 0 \) is effective, then we can find \( B \) so that \( \pi^* D = g^* B \).

Lemma 2.8. Let \( f : X \to Z \) be a proper surjective morphism from a normal variety to a variety, \( g : Y \to X \) be a proper surjective morphism of normal varieties, and \( D \) be an \( \mathbb{R} \)-Cartier divisor. Then, \( D \) is nef and abundant over \( Z \) if and only if \( f^* D \) is nef and abundant over \( Z \).

Proof. The lemma follows directly from [Nak04, V. 2.7. Proposition]. \qed

Lemma 2.9 (From global to local). Let \( f : X \to Z \) be a surjective morphism from a normal projective variety to a variety and \( D \) be a nef and abundant \( \mathbb{R} \)-Cartier divisor. Then, \( D \) is nef and abundant over \( Z \).

Proof. By Lemma 2.6, there exist a smooth birational model \( \pi : X' \to X \) and an effective divisor \( E \) such that \( \pi^* D = A_m + \frac{1}{m} E \) with \( A_m \) semi-ample for every \( m \in \mathbb{N} \). By Lemma 2.5, \( A_m \) is semi-ample over \( Z \). Let \( Y_m, Y_{mk} \) be the normal varieties over \( Z \) given by \( A_m, A_{mk} \) respectively, for some integer \( k \geq 2 \), and \( Y \) be a common resolution of them. Replacing \( X' \) we may assume \( g : X' \to Y \) is a morphism. It follows that
Let \( f : X \to Z \) be a projective morphism from a normal variety to a variety, and let \( D_1, D_2 \) be two \( \mathbb{R} \)-Cartier divisors on \( X \). If both \( D_1, D_2 \) are nef and abundant, then so is \( D_1 + D_2 \).

**Proof.** By Lemma 2.6 there exist a birational model \( \pi : X' \to X \), surjective morphisms \( g_1 : X' \to Y_1, g_2 : X' \to Y_2 \) and nef and big/Z divisors \( B_1, B_2 \) on \( Y_1, Y_2 \) respectively, such that \( \pi^*D_1 = g_1^*Y_1, \pi^*D_2 = g_2^*Y_2 \). Let \( g : X' \to Y \) be the Iitaka fibration. Note that \( g_1, g_2 \) are birational to the Iitaka fibrations of \( D_1, D_2 \) respectively. Replacing \( X' \) and \( Y \) we may assume \( g_1, g_2 \) factor through \( g \) which implies the lemma. \( \square \)

**Lemma 2.11.** Let \( f : X \to Z \) be a projective morphism from a normal variety to a variety, and let \( D \) be an \( \mathbb{R} \)-Cartier divisor on \( X \). Suppose \( D \) is a non-negative \( \mathbb{R} \)-linear combination of finitely many \( \mathbb{Q} \)-Cartier divisors which are nef and abundant over \( Z \). Then, \( D \) is nef and abundant over \( Z \). Moreover, the converse holds if there exists a boundary divisor \( B \) such that \((X, B)\) is lc and \( D \sim_{\mathbb{R}} K_X + B \).

**Proof.** The first statement follows immediately from the previous lemma. We prove the second statement. Let \( \pi : (X', B') \to X \) be a \( \mathbb{Q} \)-factorial dlt model. Since \( D \) is nef and abundant over \( Z \), by Lemma 2.6, there exist a birational model \( \pi' : X'' \to X' \), a surjective morphism \( g : X'' \to Y \) of smooth quasi-projective varieties over \( Z \), and a nef and big/Z divisor \( G \) of \( Y \) such that \((\pi \circ \pi')^*D \sim_{\mathbb{R}} g^*G \).

Now let \( \{D_i\}_{i=1}^n \) be the set of all components of \( \pi^*D \) and \( \{G_i\}_{i=1}^m \) be the set of all components of \( G \). Write

\[
D + \sum_{j=1}^k a_j(\varphi_j) = K_X + B = \sum_{j=1}^q \ell_j L_j; \quad (\pi \circ \pi')^*D + \sum_{j=1}^l b_j(\phi_j) = \sum_{j=1}^p c_j g^*G_j
\]

Consider the following \( \mathbb{R} \)-linear space

\[
\mathcal{L} := \left\{ \sum_{j=1}^n \alpha_j D_j | \sum_{j=1}^n \alpha_j D_j + \sum_{j=1}^k \mathbb{R}(\varphi_j) \text{ intersects } \sum_{j=1}^q \mathbb{R} \pi^*L_j \right\}
\]

which is a rational \( \mathbb{R} \)-linear subspace of \( \sum_{j=1}^n \mathbb{R} D_j \). Moreover, by [Bir11, Proposition 3.2], there is a rational polytope \( \mathcal{N} \) containing \( B' \) such that \( K_X + \Delta \) is nef/Z for every \( \Delta \in \mathcal{N} \). Since \( G \) is big, there exists a rational polytope \( \mathcal{P} \) containing \( G \). Given an element \( D' \) of \( \mathcal{L} \), because the conditions that \( \pi^*D' + \sum_{j=1}^l \mathbb{R}(\phi_j) \) intersects \( g^*\mathcal{P} \) and that \( D' + \sum_{j=1}^k \mathbb{R}(\varphi_j) \) intersects \( \sum_{j=1}^q \mathbb{R} \pi^*L_j \n K_X + \mathcal{N} \) are rationally polyhedral, we deduce \( \pi^*D \) can be expressed as a convex combination of nef and abundant/Z \( \mathbb{Q} \)-divisors which in turn implies the lemma by construction. \( \square \)

**Remark 2.12.** We note that the converse statement of the previous lemma does not hold unconditionally by an easy example. With a little more effort, the above lemma can be generalised to NQC generalised pairs.

**Definition 2.13** (b-nef and abundant divisors and b-nef and log abundant divisors). Let \( f : X \to Z \) be a proper morphism from a normal variety to a variety, \((X, B + M)\) be
a g-sub-lc generalised sub-pair with the moduli b-divisor $M$, and $D$ be an $\mathbb{R}$-b-Cartier b-divisor on $X$. We say $D$ is b-nef and log abundant if for any data log resolution $(\mathcal{X}, \mathcal{B} + M) \to X$ to which $D, M$ descends, we have $D_{\mathcal{X}}$ is nef and log abundant. Note that the definition is independent of the choice of $(\mathcal{X}, \mathcal{B} + M)$.

2.3. Nakayama-Zariski decompositions. Nakayama [Nak04] defined a decomposition $D = P_\sigma(D) + N_\sigma(D)$ for any pseudo-effective $\mathbb{R}$-Cartier divisor $D$ on a smooth projective variety. We refer to this as the Nakayama-Zariski decomposition. We call $P_\sigma$ the positive part and $N_\sigma$ the negative part. We can extend it to the singular case as follows. Let $X$ be a normal projective variety and $D$ be a pseudo-effective $\mathbb{R}$-Cartier divisor on $X$. We define $P_\sigma(D)$ by taking a resolution $f : W \to X$ and letting $P_\sigma(D) := f_*P_\sigma(f^*D)$. An $\mathbb{R}$-Cartier divisor $D$ is called pseudo-movable if $D = P_\sigma(D)$.

It is worth to remark that, given a normal variety $X$ projective/$\mathbb{Z}$ and a pseudo-effective divisor $D/\mathbb{Z}$, Nakayama [Nak04] also defines the relative Nakayama-Zariski decomposition $D = P_\sigma(D/\mathbb{Z}) + N_\sigma(D/\mathbb{Z})$. However, this notion may bring some subtle difficulties outside a neighborhood of generic fibre (see [Nak04, III, 4.3. Lemma]). So we will NOT use this notion in the article.

Asymptotic vanishing orders. We collect basic definitions from [Nak04] but in the setting of normal varieties instead of smooth varieties.

Suppose that $D$ is a divisor on a normal projective variety $X$ with $\kappa_\nu(D) \geq 0$ and $\Gamma$ is a prime divisor over $X$ with its corresponding discrete valuation $\text{mult}_\Gamma$. We define the asymptotic vanishing order of $D$ along $\Gamma$ as

$$o_\Gamma(D) := \inf\{\text{mult}_\Gamma(L)| L \in |E|_\mathbb{Q}\},$$

where $0 \leq E \sim_\mathbb{R} D$ and $|E|_\mathbb{K}$ denotes the sets of effective divisors $L$ satisfying $L \sim_\mathbb{K} E$ and the asymptotic fixed part as $F(D) := \sum_{\Gamma}o_\Gamma(D)\Gamma$ where $\Gamma$ runs over all prime divisors on $X$. One may easily verify the following elementary lemma.

**Lemma 2.14.** The above definition is independent of the choice of $E$. Moreover, the asymptotic vanishing order coincides with the number derived similarly from $|E|_\mathbb{R}$ instead of $|E|_\mathbb{Q}$.

**Proof.** Replacing $X$ with a resolution we can assume $X$ is smooth and $\Gamma$ is a prime divisor on $X$. Since the first assertion can be implied by the latter, it suffices to show the latter assertion. Let $E' \sim_\mathbb{R} E$ be another effective divisor and write $E' + \sum_i a_i(f_i) = E$ where $a_i \in \mathbb{R}$ and $f_i$’s are rational functions. It is enough to show that

$$\inf\{\text{mult}_\Gamma(L)| L \in |E|_\mathbb{Q}\} \leq \text{mult}_\Gamma E'.$$

By [Fuj-book17, Proof of Lemma 2.5.6], for any sufficiently divisible positive integer $n$, there is an injection

$$H^0(X, \mathcal{O}_X([nE'])) \hookrightarrow H^0(X, \mathcal{O}_X([((n + 1)E']$$

given by a rational function $1/g_n$. Moreover, by the construction of $g_n$, we have that $(g_n)$ is a $\mathbb{Q}$-linear combination of $(f_i)$’s and $\lim_{n \to \infty} \frac{1}{n+1}(g_n) = \sum_i a_i(f_i)$. Now set $E_n = E + \frac{1}{n+1}(g_n)$. We obtain $0 \leq E_n \sim_\mathbb{Q} E$ and $\lim_{n \to \infty} \text{mult}_\Gamma E_n = \text{mult}_\Gamma E'$ which in turn implies the inequality ($\dagger$).

\[\square\]
Note that when $D$ is a $\mathbb{Q}$-Cartier divisor, the definition here coincides with the classical definition due to the previous lemma and a standard argument from convex geometry.

Let us fix an ample divisor $A$. We define the numerical vanishing order of a pseudo-effective divisor $D$ along $\Gamma$ as

$$\sigma_\Gamma(D) := \lim_{\epsilon \to 0} o_\Gamma(D + \epsilon A).$$

[Nak04, III, 1.7. Lemma] verifies that the definition above is independent of the choice of $A$. Readers may notice that the variety here is not required to be smooth. Since we work with discrete valuations, by passing to a resolution, the argument from [Nak04] still works. Moreover, $\sigma_\Gamma$ defines a lower semi-continuous convex function on the pseudo-effective cone $\text{PE}(X) \subset N^1(X)$, and $\sigma_\Gamma = o_\Gamma$ is continuous on the big cone $\text{Big}(X)$. Hence we have $\sigma_\Gamma(D) := \lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon E)$ for any pseudo-effective divisor $E$.

By an easy calculation one deduces that the negative part of the Nakayama-Zariski decomposition of $D$ has the coefficients which are numerical asymptotic vanishing orders: $N_\sigma(D) = \sum_\Gamma \sigma_\Gamma(D) \delta_\Gamma$, where $\Gamma$ runs over all prime divisors on $X$. Note that in general neither $N_\sigma(D)$ nor $P_\sigma(D)$ are $\mathbb{R}$-Cartier.

2.4. Minimal model theory for generalised pairs. In this subsection we review the log minimal model program (log MMP) on a generalised log canonical divisor from [BZh16]. We will use standard results of Log MMP (cf. [KM98][BCHM10]).

Log MMP with scaling. Let $(X/Z, B + C + M) = (X_\mathbb{Z}, B_1 + C_1 + M_1)$ be a $\text{g-lc}$ generalised pair with data $\overrightarrow{M}$ such that $K_{X_\mathbb{Z}} + B_1 + C_1 + M_1$ is nef$/\mathbb{Z}$, $B_1 \geq 0$, and $C_1 \geq 0$ is $\mathbb{R}$-Cartier. Suppose $X \to Z$ is projective, $X$ is $\mathbb{Q}$-factorial klt and $C$ is ample$/\mathbb{Z}$. Then,

\begin{itemize}
  \item[(*)] for any $s \in (0, 1)$ there is a boundary $\Delta \sim_{\mathbb{R}} B + sA + M/Z$ such that $(X, \Delta + (1 - s)C)$ is klt.
\end{itemize}

So, by a standard minimal model theory, either $K_{X_\mathbb{Z}} + B_1 + M_1$ is nef$/\mathbb{Z}$ or there is an extremal ray $R_1/Z$ such that $(K_{X_\mathbb{Z}} + B_1 + M_1) \cdot R_1 < 0$ and $(K_{X_\mathbb{Z}} + B_1 + \lambda_1 C_1) \cdot R_1 = 0$ where

$$\lambda_1 := \inf\{t \geq 0 | K_{X_\mathbb{Z}} + B_1 + tC_1 + M_1 \text{ is nef$/\mathbb{Z}$}\}.$$

Now, if $K_{X_\mathbb{Z}} + B_1 + M_1$ is nef$/\mathbb{Z}$ or if $R_1$ defines a Mori fibre structure, we stop. Otherwise, $R_1$ gives a divisorial contraction or a log flip $X_1 \dashrightarrow X_2$. We now consider $(X_2/Z, B_2 + \lambda_1 C_2 + M_2)$ where $B_2 + \lambda_1 C_2 + M_2$ is the birational transform of $B_1 + \lambda_1 C_1 + M_1$ and continue. By continuing this process, we obtain a sequence of numbers $\lambda_i$ and a log MMP$/Z$ which is called the log MMP$/Z$ on $K_{X_\mathbb{Z}} + B_1 + M_1$ with scaling of $C_1$. Note that by definition $\lambda_i \geq \lambda_{i+1}$ for every $i$, and we usually put $\lambda = \lim_{i \to \infty} \lambda_i$.

Recall that, a Cartier divisor $D$ on $X/Z$ is called movable over $Z$ if $g_*\mathcal{O}_X(D) \neq 0$ and if the cokernel of the natural homomorphism $g^*g_*\mathcal{O}_X(D) \otimes \mathcal{O}_X(-D) \to \mathcal{O}_X$, called the base ideal of $|D/Z|$, has a support of codimension $\geq 2$, where $g : X \to Z$ is a proper morphism (see [Fuj-book17, Definition 2.4.3]). In the geometric context, when $Z$ is quasi-projective (for instance, $Z$ is affine), by a theorem of Bertini (see [Stack20, Lemma 0FD5]), $D$ is movable$/Z$ if and only if the relative base locus $\text{Bs}(|D/Z|)$ has
codimension $\geq 2$. An $\mathbb{R}$-Cartier divisor $D$ on $X$ is called movable over $Z$ if it is a non-negative $\mathbb{R}$-linear combination of Cartier movable over $Z$ divisors.

With notation as paragraph one, in general, when $K_X + B + M$ is pseudo-effective, we do not know whether the above log MMP terminates, but we know that in some step of the MMP we reach a model $Y$ on which $K_Y + B_Y + M_Y$, the push-down of $K_X + B + M$, is a pseudo-movable over $Z$ divisor (i.e. a limit of movable over $Z$ divisors): indeed, if the MMP terminates, then the claim is obvious; otherwise the MMP produces an infinite sequence $X_i \rightarrow X_{i+1}$ of flips and a decreasing sequence $\lambda_i$ of numbers in $(0, 1]$ such that $K_{X_i} + B_i + \lambda_i C_i + M_i$ is nef over $Z$; by [BZh16, Lemma 4.4(2)](cf. [BCHM10][Bir12, Theorem 1.9]), $\lim \lambda_i = 0$; in particular, if $Y := X_1$, then $K_Y + B_Y + M_Y$ is the limit of the movable over $Z$ $\mathbb{R}$-divisors $K_Y + B_Y + \lambda_i C_Y + M_Y$.

**Log MMP on very exceptional divisors.** With the discussion above on reaching a pseudo-movable model, one can obtain the following lemmas of log MMP on very exceptional divisors. Notation as above, recall the basic fact that, if $K_X + B + M \equiv E$ for some $\mathbb{R}$-Cartier divisor, then by the negativity Lemma, the equation $K_X + B_i + M_i \equiv E_i$ holds for each $i$ since $X_i$ is $\mathbb{Q}$-factorial.

**Lemma 2.15** (cf. [Bir12, Theorem 3.4]). Let $(X/Z, B + M)$ be a $g$-lc pair with data $\overline{M}$ such that $K_X + B + M \equiv E/Z$ with $E \geq 0$ very exceptional over $Z$. Suppose $X \rightarrow Z$ is projective and $X$ is $\mathbb{Q}$-factorial klt. Then, any log MMP over $Z$ on $K_X + B + M$ with scaling of an ample over $Z$ divisor terminates with a model $Y$ on which $K_Y + B_Y + M_Y \equiv E_Y = 0/Z$.

**Proof.** Since the question is local, by shrinking $Z$ we may assume $Z$ is affine. With notation as above, we run a log MMP over $Z$ on $K_X + B + M$ with scaling of an ample over $Z$ divisor $C$. The only divisors that can be contracted are the components of $E$ hence $E$ remains very exceptional over $Z$ during the MMP. By the discussions above, we reach a model $Y$ on which $K_Y + B_Y + M_Y \equiv E_Y$, the push-down of $K_X + B + M$ and $E$, is pseudo-movable over $Z$. Since $K_Y + B_Y + M_Y$ is nef on the very general curves of $S/Z$ for any component of $E_Y$, and $E_Y$ is very exceptional over $Z$, by the negativity lemma [Bir12, Lemma 3.3], we deduce $E_Y = 0$. $\square$

With a little more work we can obtain a slightly generalised result. Notation as below, we note that, by [Nak04, V. 1.12, Corollary], for a very general fibre $F$, the condition $E^h|_F = N_e((K_X + B + M)|_F)$ is equivalent to that $\kappa_\sigma((K_X + B + M)|_F) = 0$. So it does not depend on the choice of $F$, and is preserved under MMP.

**Lemma 2.16.** Let $(X/Z, B + M)$ be a $g$-lc pair with data $\overline{M}$ such that $K_X + B + M \equiv E/Z$ with $E \geq 0$ very exceptional over $Z$. Writing $E = E^h + E^v$ where $E^h$ denotes the horizontal over $Z$ part and $E^v$ denotes the vertical over $Z$ part, we suppose that

- $X \rightarrow Z$ is projective and $X$ is $\mathbb{Q}$-factorial klt,
- $E^h|_F = N_e((K_X + B + M)|_F)$, where $F$ is a very general fibre of the Stein factorisation of $X \rightarrow Z$, and
- $E^v$ is very exceptional over $Z$.

Then, any log MMP over $Z$ on $K_X + B + M$ with scaling of an ample over $Z$ divisor terminates with a model $Y$ on which $K_Y + B_Y + M_Y \equiv E_Y = 0/Z$. 

Proof. With a similar argument as above, we run a log MMP/\mathbb{Z} and reach a model \( Y \) on which \( K_Y + B_Y + M_Y \equiv E_Y \), the push-down of \( K_X + B + M \) and \( E \), is pseudo-movable/\mathbb{Z}.

It follows that, for a very general fibre, we have \( E_Y|_F = E_Y^b|_F \) is pseudo-movable, and thus one infers \( E_Y^b = 0 \) which in turn implies that \( E_Y \) is very exceptional/\mathbb{Z}. So the lemma is proved by Lemma 2.15. \(\square\)

The same arguments as in the previous lemma imply:

**Lemma 2.17** ([HL18, Proposition 3.8], cf. [Bir12, Theorem 3.5]). Let \((X/Z, B + M)\) be a g-lc pair with data \(\overline{M}\) such that \( X \to Z \) is projective, \( X \) is Q-factorial klt, and \( K_X + B + M \equiv Q = Q_+ - Q_-/\mathbb{Z} \) where \( Q_+ \), \( Q_- \geq 0 \) have no common components and \( Q_+ \) is very exceptional/\mathbb{Z}. Then, any log MMP/\mathbb{Z} on \( K_X + B + M \) with scaling of an ample/\mathbb{Z} divisor contracts \( Q_+ \) after finitely many steps.

**Definition 2.18** (G-dlt model, [HL18, Definition 2.2]). A g-lc generalised pair \((X/Z, B + M)\) with data \(\overline{M}\) is generalised dlt or g-dlt if \( X \) is quasi-projective and there is an open subset \( U \subset X \) containing the generic points of all g-lc centres with \((U, B|_U + M|_U)\) log smooth.

**Lemma 2.19** (G-dlt modification, [HL18, Proposition 3.9]). Let \((X/Z, B + M)\) be a g-lc generalised pair with data \(\overline{M}\). Then, there exists a birational model \( \pi : X' \to X \) such that \((X', B' + M')\) be a Q-factorial g-dlt generalised pair with data \(\overline{M}\), where \( K_{X'} + B' + M' = \pi^*(K_X + B + M) \), and that \( a(E, X, B + M) = 0 \) for every exceptional prime divisor \( E \).

*Proof.* The lemma is almost direct consequence of Lemma 2.17. It only remains to check that \((X_i, B_i + M_i)\) being g-dlt is preserved under MMP. Indeed, this follows directly from the fact that, the locus contracted by the MMP does not contain any g-lc centre. (For a detailed argument, see [HL18, Lemma 3.7].) \(\square\)

2.5. Lc-trivial fibrations. We collect some definitions and results from [Amb99] [Amb04] [Amb05] with slight modifications.

**Pre-discriminant divisors.** Let \( f : X \to Y \) be a proper surjective morphism of normal varieties and \((X, B)\) is lc over the generic point \( \eta_Y \) of \( Y \). For a prime divisor \( P \subset Y \). By shrinking \( Y \) around the generic point of \( P \), we assume that \( P \) is Cartier. We set

\[
 b_P = \max\{t \in \mathbb{Q}|(X, B + t f^*P)\text{ is sub-lc over the generic point of } P\}
\]

and set

\[
 B_Y = \sum_P (1 - b_P) P,
\]

where \( P \) runs over prime divisors on \( Y \). Then it is easy to see that \( B_Y \) is well defined since \( b_P = 1 \) for all but a finite number of prime divisors and it is called the *pre-discriminant divisor*.

If \( K_X + B \sim_R f^* D \) for some divisor \( D \) on \( Y \), then we set

\[
 M_Y = D - K_Y - B_Y
\]
and call $M_Y$ the pre-moduli divisor. We note that the pre-discriminant divisor (resp. pre-moduli divisor) is called the discriminant divisor (resp. moduli divisor) in literatures such as [Amb99][Amb04] etc.. The pre-discriminant (resp. pre-moduli) divisor is said to be the discriminant (resp. moduli) divisor when $f$ has connected fibres.

We collect some basic properties of the pre-discriminant divisors.

**Remark 2.20 ([Amb99, Remark 3.1, Example 3.1]).** (1) If $f' : (X', B') \to (X, B) \to Y$ is the map induced by a proper surjective generically finite morphism $\gamma$ and $K_{X'} + B' = \gamma^*(K_X + B)$, then $B'_Y = B_Y$. In other words, for computing $B_Y$ we are free to replace $(X, B)$ by any alteration $(X', B')$.

(2). (Additivity.) If $D$ is an $\mathbb{R}$-Cartier divisor on $Y$, then $K_X + B + f^*D$ is again lc over $\eta_Y$ and $(X, B + f^*D)$ gives the pre-discriminant part $B_Y + D$.

(3). (Finite maps.) Assume that $f$ is a finite map and $K_X + B$ is the pull-back of the divisor $K_Y + B_Y$. If $P$ is a prime divisor on $X$, $Q = f(P)$ and $w = \text{mult}_P(f^*Q)$, then

$$b_Q = a(Q; Y, B_Y) = a(P; X, B)/w = b_P/w.$$  

Let $\phi : Y' \to Y$ be a birational contraction from a normal variety $Y'$. Let $X'$ be a resolution of the main component of $X \times_Y Y'$ which dominates $Y'$. The induced morphism $\pi : X' \to X$ is birational, and $K_{X'} + B' = \pi^*(K_X + B)$. Let $B_{Y'}$ be the pre-discriminant of $K_{X'} + B'$ on $Y'$. Since the definition of the pre-discriminant is divisorial and $\phi$ is an isomorphism over codimension one points of $Y$, by Remark 2.20(1) we have $B_Y = \phi_*B_{Y'}$. This means that there exists a unique b-divisor $B$ of $Y$ such that $B_{Y'}$ is the pre-discriminant on $Y'$ of the induced fibre space $f' : (X', B') \to Y'$, for every birational model $Y'$ of $Y$. We call $B$ the pre-discriminant b-divisor. We define the pre-moduli b-divisor $M$ in a similar way. The pre-discriminant (resp. pre-moduli) b-divisor is said to be discriminant (resp. moduli) when $f$ has connected fibres.

**Lc-trivial fibrations.** Recall that the discrepancy b-divisor $A = A(X, B)$ of a pair $(X, B)$ is the b-divisor of $X$ with the trace $A_Y$ defined by the formula

$$K_Y = f^*(K_X + B) + A_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $A^* = A^*(X, B)$ by

$$A^*_Y = \Sigma_{a_i > 1}a_iA_i$$

for

$$K_Y = f^*(K_X + B) + \Sigma a_iA_i,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Note that $A(X, B) = A^*(X, B)$ when $(X, B)$ is sub-klt.

By the definition, we have $\mathcal{O}_X([A^*(X, B)]) = \mathcal{O}_X$ if $(X, B)$ is lc (see [Fuj12, Lemma 3.19]). We also have $\mathcal{O}_X([A(X, B)]) = \mathcal{O}_X$ when $(X, B)$ is klt.

**Definition 2.21** (cf. [FG14, Definition 3.2] [Amb04, Definition 2.1]). A $\mathbb{K}$-lc-trivial (resp. $\mathbb{K}$-klt-trivial) fibration $f : (X, B) \to Y$ consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a sub-pair $(X, B)$ satisfying the following properties:

(1) $(X, B)$ is sub-lc (resp. sub-klt) over the generic point of $Y$;
(2) \( \text{rank} f_*\mathcal{O}_X([\mathbb{A}^*(X, B)]) = 1; \)

(3) There exists an \( \mathbb{R} \)-Cartier divisor \( D \) on \( Y \) such that

\[ K_X + B \sim_K f^*D. \]

We briefly sketch the results for \( \mathbb{Q} \)-lc-trivial fibrations. Note that we allow \( D \) to be \( \mathbb{R} \)-Cartier in the definition of a \( \mathbb{Q} \)-lc-trivial fibration. In fact, by modifying \( D \) to a \( \mathbb{Q} \)-Cartier divisor, one can easily reduce to the classical situation. Thanks to the important results [Amb04, Theorem 2.5][FG14, Theorem 3.6] obtained by the theory of variations of (mixed) Hodge structure, the moduli b-divisor \( M \) of a \( \mathbb{Q} \)-lc-trivial fibration is \( \mathbb{Q} \)-b-Cartier and b-nef. Hence \( K + B \) is \( \mathbb{R} \)-b-Cartier.

If we assume further that \( Y \) is complete, the geometric generic fiber \( X_\eta = X \times_Y \text{Spec}(k(Y)) \) is a projective variety and \( (X_\eta, B_\eta) \) is klt, where \( B_\eta = B|_{X_\eta} \). Then by [Amb04, Theorem 3.3][FG14, Theorem 3.10] the moduli b-divisor \( M \) is b-nef and abundant.

Moreover, by [FG14, Theorem 1.1], the moduli b-divisor \( M \) of a \( \mathbb{Q} \)-lc-trivial fibration from an lc pair is b-nef and abundant. As we mentioned in the introduction, we will not directly apply [FG14, Theorem 1.1] in this article, but we will use their strategy in Section 4.3.

The following lemma is elementary.

**Lemma 2.22.** Let \( f : (X, B) \to Y \) be an \( \mathbb{R} \)-lc-trivial (resp. \( \mathbb{R} \)-klt-trivial) fibration. Then, \( B \) is a convex combination of \( \mathbb{Q} \)-divisors \( B_i \) such that \( f : (X, B_i) \to Y \) is \( \mathbb{Q} \)-lc-trivial (resp. \( \mathbb{Q} \)-klt-trivial).

**Proof.** We only prove for the \( \mathbb{R} \)-lc-trivial fibrations. Let \( f : (X, B) \to Y \) be an \( \mathbb{R} \)-lc-trivial fibration, \( \varphi = \prod_{i=1}^k \varphi_i^{a_i} \) be an \( \mathbb{R} \)-rational function so that \( K_X + B + (\varphi) = f^*D \).

Let \( \mathcal{L} \subset \text{CDiv}_R(Y) \) be a finite dimensional rational linear subspace containing \( D \), \( \mathcal{L} \subset \text{CDiv}_R(X) \) be a rational polytope containing \( B \) such that, for every \( \Delta \in \mathcal{L} \), we have \( (X, \Delta) \) is a sub-pair which is sub-lc over the generic point of \( Y \). Now we consider the rational polytope

\[ \mathcal{P} := \{ \Delta \in \mathcal{L} | \Delta + \sum_{i=1}^k \mathbb{R}(\varphi_i) \text{ intersects } f^*\mathcal{V} \} \]

For every \( \Delta \in \mathcal{P} \), we have further \( K_X + \Delta \sim_{\mathbb{R}} 0/Y \). It is obvious that \( B \in \mathcal{P} \).

It suffices to show that, there exists a convex combination \( B = \sum_j \alpha_j B_j \) of \( \mathbb{Q} \)-divisors \( B_j \in \mathcal{P} \) with \( \text{rank} f_*\mathcal{O}_X([\mathbb{A}^*(X, B)])) = 1 \). To this end, pick a log resolution \( \pi : \widetilde{X} \to X \) of \( (X, \sum_j B_j) \) where every element of \( \mathcal{P} \) is supported by \( \sum_j B_j \). Note that the proofs of [Fuj12, Lemma 3.19 and 3.20] still work for \( \mathbb{R} \)-sub-boundaries. Hence, by shrinking \( Y \), we may assume \( (X, \Delta) \) is sub-lc for every \( \Delta \in \mathcal{P} \), and we have

\[ f_*\mathcal{O}_X([\mathbb{A}^*(X, \Delta)]) = f_*\pi_*\mathcal{O}_{\widetilde{X}}(\sum_{a_i \neq -1} [a_i]A_i) \]

where \( K_{\widetilde{X}} = \pi^*(K_X + \Delta) + \sum a_iA_i \). Consider the rational sub-polytope

\[ \mathcal{Q} = \{ \Delta \in \mathcal{P} | [\mathbb{A}^*(X, \Delta)] \leq [\mathbb{A}^*(X, B)] \}. \]
Then, for any $B_j \in \mathcal{Q}$, we have $\text{rank} f_* \mathcal{O}_X([A^*(X, B_j)]) = 1$ which completes the proof.

\textbf{Remark 2.23.} The converse direction of the previous lemma is not true in general. See the example below.

\textbf{Example 2.24.} Let $\pi : X \to \mathbb{P}^2$ be the blow-up at two points $p_1, p_2 \in \mathbb{P}^2$, $E$ be one of the exceptional curve. Let $L_0$ be the line passing through both two points, $L_{i,j}$ be general lines passing through $p_i$, for $i = 1, 2, j = 1, 2$, and let $L'_0, L'_{i,j}$ be the birational transforms on $X$. Let $mA$ be a general member of $|mL|$, where $L$ is a general line, and $A'$ be the birational transform of $A$ on $X$.

Set $B_1 = 3A' - E$ and $B_2 = \frac{3}{2}A' + \frac{3}{4}(L'_{1,1} + L'_{1,2}) + \frac{3}{4}(L'_{2,1} + L'_{2,2}) - \frac{1}{2}L'_0$. One can easily check that $K_X + B_1 \sim_0 0, K_X + B_2 \sim_0 0$ with $h^0(X, \mathcal{O}_X([-B_1])) = 0, h^0(X, \mathcal{O}_X([-B_2])) = 0$, but $h^0(X, \mathcal{O}_X(E + L'_0)) = 1$.

Next we introduce the notion of lc-trivial fibration.

\textbf{Definition 2.25.} An lc-trivial (resp. klt-trivial) fibration $f : (X, B) \to Y$ consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a sub-pair $(X, B)$ such that $B$ is a convex combination of divisors $B_i$ such that $f : (X, B_i) \to Y$ is $\mathbb{R}$-lc-trivial (resp. $\mathbb{R}$-klt-trivial).

By Lemma 2.22, an lc-trivial fibration is a convex combination of $\mathbb{Q}$-lc-trivial fibrations. We immediately obtain the following corollary.

\textbf{Corollary 2.26.} With notation in Definition 2.25, letting $B$ be the discriminant b-divisor, we have $B$ is a convex combination of $\mathbb{Q}$-b-divisors $B_i$ such that $K + B_i$ is $\mathbb{Q}$-b-Cartier. In particular, $K + B$ is $\mathbb{R}$-b-Cartier. In the same way, the moduli b-divisor $M$ is a convex combination of $\mathbb{Q}$-b-Cartier and $b$-nef $\mathbb{Q}$-divisors. In particular, $(Y, B_Y + M_Y)$ with data $M$ is an $\text{NQC}$ generalised sub-pair.

The results from [Amb04][FG14] can be generalised in the same way as below.

\textbf{Corollary 2.27 (cf.[Amb04, Theorem 3.3][FG14, Theorem 3.10]).} Let $f : (X, B) \to Y$ be an lc-trivial fibration. Suppose that $Y$ is complete, the geometric generic fiber $X_Y^\gamma = X \times_Y \text{Spec}(k(Y))$ is a projective variety and $(X_Y, B_Y)$ is klt, where $B_Y = B|_{X_Y}$. Then, the moduli b-divisor $M$ is a convex combination of $\mathbb{Q}$-b-Cartier and $b$-nef and abundant $\mathbb{Q}$-divisors. In particular, $M$ is $\mathbb{R}$-b-Cartier and $b$-nef and abundant.

\textit{Proof.} The arguments are analogous to the first paragraph in the proof of 2.22. The last assertion follows from 2.11. \qed

The category of lc-trivial (resp. klt-trivial) fibrations is closed under base change (see [Amb05, Proposition 3.1][FG14, Section 3.3]). More precisely, given an lc-trivial fibration $f : (X, B) \to Y$ and a surjective proper morphism $\gamma : Y' \to Y$, the \textit{induced lc-trivial fibration} $f' : (X', B') \to Y'$ is given by the normalization $X'$ of the main component of $X \times_Y Y'$ and $K_X' + B' = \rho^*(K_X + B)$ where $\rho : X' \to X$ is the induced morphism.
Lemma 2.28 (cf. [Amb05, Proposition 3.1]). Let $f : (X, B) \to Y$ be an lc-trivial fibration. Let $\gamma : Y' \to Y$ be a surjective proper morphism from a normal variety $Y'$, and let $f' : (X', B') \to Y'$ be an lc-trivial fibration induced by base change

$$(X, B) \xrightarrow{\rho} (X', B') \xrightarrow{\gamma} Y'$$

Let $M$ and $M'$ be the corresponding moduli b-divisors. Then $\gamma^* M = M'$.

2.6. Weak semi-stable reductions. In this article we apply the theorems of the weak toroidalisation and the weak semi-stable reduction developed by Abramovich, Denef and Karu [ADK13][AK00]. See also [Amb04, Section 4], [Kaw15, Theorem 2].

The following theorem is [ADK13, Theorem 1.1] which is a slight generalisation of [AK00, Theorem 2.1]. Note that condition (2) below is similar to [ADK13, Theorem 1.1] condition (4) but with an extra condition on $\Delta_Y$. The extra condition will not cause any trouble so the proof from [ADK13] is still sufficient. For the reader’s convenience, we give a sketch of proof with an emphasis on the exceptional loci. For a detailed proof we refer to [ADK13].

Theorem 2.29 (Weak toroidalisation). Let $f : X \to Y$ be a proper surjective morphism of normal varieties and $Z \subset X, Z_Y \subset Y$ be proper closed subsets. Then there exist proper birational morphisms $\pi, \phi$ and a commutative diagram

$$(X, \Delta) \xrightarrow{\pi} X \supset Z \xrightarrow{f} Y \xrightarrow{\phi} (Y, \Delta_Y)$$

satisfying the following conditions:

1. $(X, \Delta), (Y, \Delta_Y)$ are quasi-projective log smooth.
2. $\pi^{-1} Z \cup \text{Ex}(\pi) \subseteq \Delta$ and $\phi^{-1} Z_Y \cup \text{Ex}(\phi) \subseteq \Delta_Y$.
3. $\Delta'' = f^{-1} \Delta_Y$ where $\Delta''$ denotes the vertical $Y$ part of $\Delta$.

Furthermore, in the language of toroidal geometry, we may require:

(3*) $(X, \Delta), (Y, \Delta_Y)$ are toroidal varieties and $f$ is a toroidal morphism.

Sketch of Proof. Note that the theorem is trivial by Hironaka’s log resolution without the condition (3*). In order to prove (3*), we follow the strategy of [AK00, Section 2] and [ADK13]. Replacing $X, Y$ we may assume $X, Y$ are projective and $Z, Z_Y$ are Cartier divisors. We prove by induction on the relative dimension $n$ of $f$. If $n = 0$, then by the flattening lemma and strong log resolution we may construct a finite morphism $\overline{f} : (X, \Delta) \to (Y, \Delta_Y)$ ramified over the snc divisor $\Delta_Y$ which satisfies the conditions (1)-(3), except that $(X, \Delta)$ is not necessarily log smooth (see [ADK13, Sec.3.4.2]). It is toroidal by Abhyankar’s lemma [ADK13, Lemma 3.3]. Replacing $(X, \Delta)$ with a toroidal resolution we immediately obtain the conclusion.
By induction we suppose Theorem 2.29 holds in relative dimension \( n - 1 \). Replacing \( X, Y, Z, Z_Y \) we may factorise \( f : X \xrightarrow{g} P \xrightarrow{h} Y \) with relative dimension \( n - 1 \) of \( h \) (see [ADK13, 3.5]) and assume \( f^{-1}Z_Y \subset Z \). Now we construct a commutative diagram

\[
\begin{array}{c}
\hat{X} \\
\mu \downarrow \\
\hat{P} \\
\gamma \downarrow \\
\bar{P} \\
\phi \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
X \\
X/G \\
\phi \\
P \\
Y \\
\end{array}
\]

such that \( f = h \circ g \), the horizontal maps \( \mu' := \pi \circ \mu, \gamma' := \phi \circ \gamma \) are Galois alterations and \( \hat{g} : \hat{X} \to \hat{P} \) is a nodal family of curves. Moreover, there are finitely many disjoint sections \( \sigma_i : \hat{P} \to \hat{X} \) of \( \hat{g} \) into the smooth locus of \( \hat{X} \) such that \( G \) permutes the sections \( \sigma_i \) and the locus \( \hat{Z} = \mu'^{-1}Z \subset (\bigcup \sigma_i(\hat{P})) \) for some proper closed subset \( \hat{D} \subset \hat{P} \) (see [dJ97, Theorem 2.4, Remark 2.5]). Note that \( \text{Ex}(\pi) \) is vertical over \( \hat{P}/G \) which in turn implies that \( \mu^{-1}\text{Ex}(\pi) \) is vertical over \( \hat{P} \). So, replacing \( Z \) with \( \pi^{-1}Z \cup \text{Ex}(\pi) \) and \( X, P, \hat{D} \) accordingly we may assume \( X = \hat{X}/G, P = \hat{P}/G \). Let \( Z_P \subset P \) be the union of \( \gamma(\hat{D}) \) and the loci over which \( Z, P \) or \( X \) are not smooth. Applying the inductive assumption to \( h : (P, Z_P) \to (Y, Z_Y) \), we obtain a diagram as follows:

\[
\begin{array}{c}
\overline{P}, \Delta_{\overline{P}} \\
\overline{Y}, \Delta_{\overline{Y}} \\
\end{array}
\begin{array}{c}
\phi \downarrow \\
\pi \\
\end{array}
\begin{array}{c}
P \\
Y \\
\end{array}
\]

where \( \overline{h} : (\overline{P}, \Delta_{\overline{P}}) \to (\overline{Y}, \Delta_{\overline{Y}}) \) satisfies the conditions (1)-(3*). Since \( g \) is equidimensional, if we let \( \overline{X} \) be the normalisation of \( X \times_P \overline{P} \) and \( \overline{Z} \) be the union of the inverse image of \( Z \) and the exceptional locus \( \text{Ex}(\pi) \), then by the inductive assumption we deduce \( \overline{g}(\text{Ex}(\pi)) \subset \Delta_{\overline{P}} \) which in turn implies that \( \overline{Z}^h = \pi^{-1}Z^h \) where \( \overline{Z}^h \) denotes the union of the horizontal/\( \overline{P} \) components of \( \overline{Z} \).

Replacing \( \hat{P}, \hat{X} \) with the normalisations of the fibre products and the morphisms \( \gamma, \mu \) accordingly, by Abhyankar’s lemma again, we see \( (\hat{P}, \Delta_{\hat{P}} = \gamma^{-1}\Delta_{\overline{P}}) \) and \( (\hat{X}, \hat{\Delta} = (\gamma \circ \hat{g})^{-1}\Delta_{\overline{P}} + \sum \sigma_i(\hat{P})) \) are toroidal. Since \( \overline{g}(\overline{Z}^h) \subset \Delta_{\overline{P}} \), we deduce \( \mu^{-1}\overline{Z} \subset \hat{\Delta} \).

Finally we use the argument from [ADK13, Sec.3.9] to modify \( \hat{X} \) to a \( G \)-equivariant toroidal variety and then \( \overline{X} \) accordingly. Writing \( U_{\hat{P}} = \hat{P}/\Delta_{\hat{P}} \), because \( (\hat{g}^{-1}U_{\hat{P}}, \hat{\Delta}|_{\hat{g}^{-1}U_{\hat{P}}} = \sum \sigma_i(U_{\hat{P}})) \) is log smooth over \( U_{\hat{P}} \) and \( G \subset \text{Aut}(\hat{U} \subset \hat{X}) \) where \( \hat{U} = \hat{X}/\hat{\Delta} = \hat{g}^{-1}U_{\hat{P}}/\sigma_i(U_{\hat{P}}) \), it follows that the \( G \)-equivariant resolution \( b : (\hat{U}, \hat{\Delta}) \to (\hat{U}, \hat{\Delta}) \) preserves \( \hat{U} \), hence \( \text{Ex}(b) \subset \hat{\Delta} \). Indeed, \( b \) is the composite of two normalised blow-ups and both procedures do not intersect with \( \hat{U} \) ([ADK13, Sec.3.9.1, 3.9.5, 3.9.6]). Letting \( (\overline{X}, \overline{\Delta}) = (\hat{X}/G, \hat{\Delta}/G) \) and \( \psi : \overline{X} \to \overline{X} \) be the induced modification, we see
Ex(ψ) ⊂ \Delta' and (X', \Delta') \to (\overline{P}, \Delta_P) is toroidal ([ADK13, Sec.2.3]). Replacing (X, \Delta) with a toroidal resolution of (X, \Delta) we complete the proof. □

Remark 2.30. Since we may suppose both Z and Z_Y are Cartier divisors, it is obvious that π^{-1}Z and π^{-1}Z_Y are snc divisors by Krull’s Hauptidealsatz. However, in general one cannot resolve the loci Ex(π) and Ex(φ) to snc divisors toroidally unless each component of Ex(π) and Ex(φ) is a toroidal stratum (see [ADK13, 2.5]). Hence, we have no idea if one can further require the exceptional loci to be snc divisors.

Remark 2.31. The argument for Theorem 2.29 also applies to varieties over a non-closed field of characteristic zero. For definition of toroidal varieties over a non-closed field, we refer to [ADK13, Sec.2.2]. Note that, a variety (X, ∆) with a reduced divisor is toroidal if the base change (X_k, ∆_k) over its algebraic closure k ⊆ \overline{k} is toroidal, where (X_k, ∆_k) is a pair of schemes of finite type over k. Let us say a few words about resolution in arbitrary characteristic. Recall the fact: Smoothness, non-singularity, and geometric regularity (=regularity if the residue field is perfect) are locally equivalent (cf.[Stack20, Lemma 038X, Lemma 01V7, Lemma 0381]). Moreover, given a variety over a perfect field, the smooth locus coincides with the regular locus ([Stack20, Lemma 0B8X]). In the proof of Theorem 2.29, we applies strong log resolution in the first paragraph to settle the case of relative dimension zero, which remains open in positive characteristic; In the last paragraph, the claim that, the quotient \((\bar{X}/G, \bar{\Delta}/G)\) is toroidal, fails in positive characteristic (see [AdJ97, Sec.0.3.2]).

We also note that, since the arguments of Theorems 2.32 and 2.33 from a toroidal morphism is mostly combinatorial, we might expect them to work over an arbitrary field.

**Theorem 2.32 (Equidimensional reduction).** Let \(f : X \to Y\) be a proper surjective morphism of normal varieties and \(Z \subset X\) be a proper closed subset. Then there exists a commutative diagram

\[
\begin{array}{ccc}
(X', \Delta') & \xrightarrow{\pi} & X \\
\downarrow{f'} & & \downarrow{f} \\
(Y', \Delta_Y) & \xrightarrow{\phi} & Y
\end{array}
\]

which consists of a \(Q\)-factorial klt quasi-projective variety \(X'\) with a reduced divisor \(\Delta'\) whose components are normal, a smooth quasi-projective variety \(Y'\) with a snc divisor \(\Delta_Y\), a projective morphism \(f' : X' \to Y'\) and projective birational morphisms \(\pi : X' \to X\), \(\phi : Y' \to Y\) and which satisfy the following conditions:

1. \((X', \Delta')\) is lc, and writing \(\Delta' = \sum_{j \in J} \Delta_j'\) for the sum of prime divisors, every lc centre is an irreducible component of the intersection locus \(\bigcap_{j \in J} \Delta_j'\) for some \(J \subseteq I\).

2. \(\pi^{-1}Z \subseteq \Delta'\) is a divisor, \(\Delta' = f'^{-1}\Delta_Y\), where \(\Delta'\) denotes the vertical/Y' part of \(\Delta'\).

3. All the fibers of \(f'\) have the same dimension. In particular, \(f'\) is flat. Furthermore, if \(Z\) is vertical/Y, then \((X', \Delta'')\) also satisfies the conditions listed above.
Sketch of Proof. For the detailed argument we refer to [AK00] and we briefly sketch it. By Theorem 2.29 and Remark 2.30 there exist proper birational morphisms $\pi, \phi$ and a commutative diagram

$$
\begin{array}{ccc}
(X, \Delta) & \xrightarrow{\pi} & X \supset Z \\
\downarrow{f} & & \downarrow{f} \\
(Y, \Delta_Y) & \xrightarrow{\phi} & Y
\end{array}
$$

satisfying the following conditions:

- $(X, \Delta), (Y, \Delta_Y)$ are quasi-projective smooth toroidal varieties.
- $f$ is a toroidal morphism.
- $Z := \pi^{-1}Z \subseteq \Delta$ is a snc divisor.

Next by [AK00, Proposition 4.4] there exist birational projective morphisms $\pi': X' \to \overline{X}$ and $\phi': Y' \to \overline{Y}$ corresponding to projective subdivisions such that the induced map $f': X' \to Y'$ is an equidimensional toroidal morphism. Moreover, by [AK00, Remark 4.5] and replacing $X'$, we may assume $X'$ is quasi-smooth (i.e. each cone $\sigma_x$ is simplicial, or each local toric model has only abelian quotient singularities.), hence $\mathbb{Q}$-factorial. By a standard theory of toric varieties, we see $X'$ is klt and $(X', \Delta')$ is lc where $\Delta' = \pi'^{-1}\Delta$. Moreover, writing $\Delta' = \sum_j \Delta'_j$ for the sum of prime divisors, every lc centre is an irreducible component of the intersection locus $\bigcap_j \Delta'_j$. Note that $\pi'^{-1}Z$ has pure codimension one by Krull’s Hauptidealsatz. Letting $\Delta_Y' = \phi'^{-1}\Delta_Y, \pi = \pi \circ \pi'$ and $\phi = \phi \circ \phi'$ we conclude the theorem. For the flatness of $f'$, see [AK00, Remark 4.6].

**Theorem 2.33** (Weak semi-stable reduction). *With the notation above, if we suppose further that $f$ has connected fibres, then there exists a commutative diagram

$$
\begin{array}{ccc}
(X'', \Delta'') & \xrightarrow{\mu} & (X', \Delta') & \xrightarrow{\pi} & X \\
\downarrow{f''} & & \downarrow{f'} & & \downarrow{f} \\
(Y'', \Delta_{Y''}) & \xrightarrow{\gamma} & (Y', \Delta_{Y'}) & \xrightarrow{\phi} & Y
\end{array}
$$

which consists of another $\mathbb{Q}$-factorial klt quasi-projective variety $X''$ with a reduced divisor $\Delta''$ whose components are normal, a smooth quasi-projective variety $Y''$ with a snc divisor $\Delta_{Y''}$, a finite surjective morphism $\gamma: Y'' \to Y'$ and the induced morphism $f'': X'' \to Y''$ and which satisfy the following conditions:

1. $(X'', \Delta'')$ is lc, and writing $\Delta'' = \sum_{j \in I} \Delta''_j$ for the sum of prime divisors, every lc centre is an irreducible component of the intersection locus $\bigcap_{j \in J} \Delta''_j$ for some $J \subseteq I$.
2. $\Delta'' = \mu^{-1}\Delta'$, $\Delta_{Y''} = \gamma^{-1}\Delta_{Y'}$, and $\Delta'' = f'^{-1}\Delta_{Y'}$ where $\Delta''$ denotes the vertical/Y part of $\Delta'$.
3. All the fibers of $f''$ are equidimensional and reduced. In particular, $f''$ is flat.

Furthermore, if $Z$ is vertical/Y, then $(X'', \Delta_{Y''})$ also satisfies the conditions listed above.
Sketch of Proof. Since \( f' : (X', \Delta') \to (Y', \Delta_Y) \) is an equidimensional toroidal morphism, by [AK00, Proposition 5.1] (applying Kawamata’s covering trick, see [Kaw81]) there exists a finite surjective morphism \( \gamma : Y'' \to Y' \) so that the induced morphism \( f'' : (X'', \Delta'') \to (Y'', \Delta_{Y''}) \) is an equidimensional toroidal morphism with reduced fibres. By the construction of [AK00, Proposition 5.10] one immediately obtained that \( (X'', \Delta'') \) is quasi-smooth and \( (Y'', \Delta_{Y''}) \) is log smooth. □

Remark 2.34. The condition on the connectedness of fibres can be removed. But note that in this case, \( X' \times_Y Y'' \) could possibly have multiple main components and after taking normalisation \( X'' \) is a disjoint union of finitely many \( \mathbb{Q} \)-factorial klt varieties. See [Amb04, Theorem 4.3] for the semi-stable reduction in codimension one for a proper surjective morphism with disconnected general fibres.

Thanks to Theorem 2.29 ([ADK13, Theorem 1.1]), we may slightly generalise the weak semi-stable reduction for practical reason. The additional condition below is in spirit closer to that in Ambro’s semi-stable reduction in codimension one [Amb04, Theorem 4.3]. In this article we will not deal with the case that \( X'' \) is disconnected.

Lemma 2.35. Let \( f : X \to Y \) be a proper surjective morphism of normal varieties and \( Z \subset X, Z_Y \subset Y \) be proper closed subsets. With notation from Theorem 2.32 and 2.33, we may further require the condition: the loci \( \pi^{-1}Z \cup \operatorname{Ex}(\pi) \subseteq \Delta' \) and \( \phi^{-1}Z_Y \cup \operatorname{Ex}(\phi) \subseteq \Delta_{Y'} \). In particular, the loci \( (\pi \circ \mu)^{-1}Z \cup \operatorname{Ex}(\pi \circ \mu) \subseteq \Delta'' \) and \( (\phi \circ \gamma)^{-1}Z_Y \cup \operatorname{Ex}(\phi \circ \gamma) \subseteq \Delta_{Y''} \).

Proof. Notation as in proofs of Theorem 2.32 and 2.33, by Theorem 2.29, we may assume \( \tilde{f} : (\overline{X}, \overline{\Delta}) \to (\overline{Y}, \overline{\Delta_Y}) \) satisfies the conditions listed in Theorem 2.29. Since \( \pi', \phi' \) correspond to projective subdivisions, one can easily check \( \pi^{-1}Z \cup \operatorname{Ex}(\pi) \subset \Delta' \) and \( \phi^{-1}Z_Y \cup \operatorname{Ex}(\phi) \subset \Delta_{Y'} \) are divisors. Also, since \( \mu, \gamma \) are finite, there are no exceptional loci. Hence we conclude the lemma. □

3. Extending a finite cover from a subvariety

The main purpose of this section is to prove the following theorem on extending a finite cover over a closed subvariety to a finite cover of the same degree over the whole variety. Although in this paper, we will only use a special case the ground field is an algebraically closed field of characteristic zero, we prove it in full generality. Throughout this section, all varieties are over an arbitrary field.

Theorem 3.1. Let \( X \) be a normal variety and \( S \) be a closed subvariety. Suppose we are given a finite morphism \( \gamma : \tilde{S} \to S \) from a normal variety. Suppose further that one of the following conditions holds:

1. The residue field \( \kappa(\eta_S) \) at the generic point \( \eta_S \) of \( S \) is perfect.
2. \( X \) is regular at the generic point of \( S \).
Then there exists a finite morphism \( \rho : \tilde{X} \to X \) of normal varieties together with a closed subvariety \( \tilde{S} \subset \tilde{X} \) satisfying:

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\nu} & \tilde{X} \\
\downarrow{\gamma} & & \downarrow{\rho} \\
S & \hookrightarrow & X
\end{array}
\]

(1) \( \tilde{S} \) is mapped onto \( S \) through \( \rho \) and the above diagram commutes.
(2) \( \nu \) is the normalisation of \( \tilde{S} \).
(3) \( \deg \rho = \deg \gamma \).

We need some knowledge from commutative algebra. We begin with the following elementary lemma. For a domain \( A \), we denote by \( \kappa(p) \) the residue field at \( p \), and by \( K(A) \) the field of fractions.

**Lemma 3.2.** Let \( A \subset B \) be an integral extension of domains, where \( A \) is integrally closed. Let \( q \subset B \) be a prime ideal, \( p = q \cap A \), and \( A/p \subset B/q \) be the induced inclusion of domains. Suppose further that one of the following conditions holds:

1. \( \kappa(q) \) is generated by one element over \( \kappa(p) \).
2. \( A \) is Noetherian, and \( B \) is finite and locally flat over \( A \) at \( p \).
3. \( A \) is regular at \( p \), and \( [K(B) : K(A)] < +\infty \).

Then, we have \([\kappa(q) : \kappa(p)] < +\infty \) and the inequality

\[ [\kappa(q) : \kappa(p)] \leq [K(B) : K(A)]. \]

**Proof.** Let \( S = A \setminus p \) be the multiplicative set. Replacing \( A, B \) with \( S^{-1}A, S^{-1}B \), we may assume \( A \) is a local domain. In particular, \( A/p \) is a field, denote by \( k \), and \( B/q \) is integral over \( k \) which is also a field by \([AM69, Proposition 5.7] \).

**Case 1.** We suppose that \( B/q \) is generated by one element \( b \) over \( k \). Lift \( b \) to any element \( b \in B \). Let \( \alpha(X) \in K(A)[X] \) be the minimal polynomial of \( b \in K(B) \) over \( K(A) \). Since \( A \) is integrally closed, by \([AM69, Proposition 5.15] \), we deduce \( \alpha(X) \in A[X] \). Thus we have \( K(A) \subset K(A)[b] \subset K(B) \) with \( [K(A)[b] : K(A)] = \deg(\alpha) \). On the other hand, the reduction mod \( p \) gives \( \alpha(X) \in A/p[X] \) which still satisfies \( \alpha(b) = 0 \) in \( B/q \). In particular, if \( \beta(X) \in k[X] \) is the minimal polynomial of \( b \), then \( \beta(X) \) is divisible by \( \beta(X) \), hence \( \deg(\beta) \leq \deg(\alpha) \). Therefore, we deduce

\[ [K(B) : K(A)] \geq [K(A)[b] : K(A)] \geq [B/q : k]. \]

**Case 2.** We suppose that \( A \) is Noetherian and \( B \) is finite and flat over \( A \). Since \( A \) is Noetherian and \( B \) is finite over \( A \), by \([Stack20, Lemma 0OFP] \), we deduce \( B \) is of finite presentation over \( A \). Moreover, because \( B \) is flat over \( A \), by \([Stack20, Lemma 0ONX] \), \( B \) is a free \( A \)-module, and \( \dim_{K(p)} B \otimes_A K(p) = \dim_{K(A)} B \otimes_A K(A) \). Since \( B \) is integral over \( A \), by \([AM69, Propositions 3.5 and 5.7] \), we have \( B \otimes_A K(A) = K(B) \). On the other hand, \( \dim_{K(p)} B \otimes_A K(p) = \dim_{K(p)} B/pB \geq \dim_{K(p)} B/q \). We established the inequality.

**Case 3.** We suppose that \( A \) is regular at \( p \), hence a regular local ring. We prove by induction on the height of \( p \). We first assume \( \text{ht}(p) = 1 \). Due to \([AM69, Proposition 9.2] \), \( A \) is a discrete valuation ring, and let \( \pi \in A \) be a uniformiser. By \([Stack20, \]
Lemma 031F], we deduce $B$ is a semi-local domain with finitely many maximal ideals $q_i$ lying over $p$. Moreover, $[\kappa(q_i) : \kappa(p)] < +\infty$ for each $i$. Let $M = B/\pi B$. We claim that $M$ is a $B$-module of finite length. Indeed, since $B \subset K(A)^{\oplus n}$, where $n = [K(B) : K(A)]$, we apply [Stack20, Lemma 00PE] to obtain that

\[(1) \quad \text{length}_A M \leq n \cdot \text{length}_A (A/\pi A) = n\]

is finite which proves the claim by [Stack20, Lemma 00IX]. Hence, applying [Stack20, Lemma 02M0], we immediately deduce

\[(2) \quad \text{length}_A M = \sum_i [\kappa(q_i) : \kappa(p)] \text{length}_{B_{q_i}} M_{q_i} \geq [\kappa(q) : \kappa(p)].\]

Combining (1) and (2), we conclude the result.

Now we suppose $\text{ht}(p) = d$, and Lemma 3.2 holds in height $\leq d - 1$. Given a regular sequence $\{x_1, \ldots, x_d\}$ of $A$, by [Stack20, Lemma 00NP], $A/(x_1)$ is a regular local ring. In particular, it is a domain by [Stack20, Lemma 00PG], which in turn implies that $(x_1)$ is prime. So, by [AM69, Theorem 5.10], there is a prime ideal $r \subset B$ such that $r \cap A = (x_1)$. Apply the inductive assumption to $A/(x_1) \subset B/r$ to deduce $[\kappa(r) : \kappa((x_1))] \leq [K(B) : K(A)]$ and $[\kappa(q) : \kappa(p)] \leq [\kappa(r) : \kappa((x_1))]$, hence $[\kappa(q) : \kappa(p)] \leq [K(B) : K(A)]$. 

\[\square\]

Remark 3.3. Notation as above, from equations (1) and (2), and by the inductive argument, we obtain a stronger inequality $\sum_i [\kappa(q_i) : \kappa(p)] \leq [K(B) : K(A)]$ in Case 3. If the equality holds, then for each $1 \leq i \leq d$, there is a unique prime ideal $r_i \subset B$ such that $r_i \cap A = (x_i)$ and that $[\kappa(r_i) : \kappa((x_i))] = [K(B) : K(A)]$.

Remark 3.4. In the proof of Case 3, although we do not assume $B_p$ to be Noetherian explicitly, we can deduct it (if $\dim A_p = 1$) by Krull-Akizuki [Stack20, Lemma 00IX]. Thus, $M$ is Artinian hence the finiteness of $\text{length}_{B_p} M$. If we assume further that $B_p$ is integrally closed (which is reasonable by Krull-Akizuki’s theorem), then $A_p \hookrightarrow B_p$ is an extension of discrete valuation rings, hence achieving a refined inequality [Stack20, Lemma 09E5]. Furthermore, assuming $K(B)/K(A)$ is finite separable, one infers the fundamental identity [Stack20, Remark 09E8]. In particular, this identity becomes a simple form as in [Stack20, Lemma 09EB] when $K(B)/K(A)$ is finite Galois. For a general form of this identity in algebraic geometry, see [Ful83, Examples 4.3.6 and 4.3.7]. One can derive the inequality in Case 3 by this identity and the fact that the multiplicity $e_V Y = 1$ if and only if $Y$ is regular at $V$ (see [Ful83, Example 4.3.5(d)]). Finally, we remark that, a similar inequality can be established for an extension of valuation rings; see [Stack20, Lemma 0ASH].

Recall that a ring $A$ is Japanese if for every finite extension $L$ of its field of fractions $K$, the integral closure of $A$ in $L$ is a finitely generated $A$-module, and it is a Nagata ring if it is Noetherian, and or every prime ideal $p$ the ring $A/p$ is Japanese. Recall a basic fact that, a Nagata ring $A$ is universally Japanese, and any finite type $A$-algebra is Nagata (see [Stack20, Proposition 0334]). Every quasi-excellent ring is a Nagata ring (see [Stack20, Lemma 07QV]), so in particular almost all Noetherian rings that occur in algebraic geometry are Nagata rings. See [Stack20, Proposition 0335] for examples.
Remark 3.5. We collect some important cases in which the inequality of the previous lemma holds. With notation as in Lemma 3.2, we assume further that $A$ is Nagata. In this situation, we may replace $B$ with its integral closure in the field of fractions, so we may further assume $B$ is finite over $A$. In particular, it automatically holds that $[K(B) : K(A)] < +\infty$ and $[\kappa(q) : \kappa(p)] < +\infty$.

If the residue field extension $\kappa(q)/\kappa(p)$ is separable, then by the primitive element theorem, the condition (1) is satisfied. Furthermore, if $\kappa(p)$ is perfect, then $\kappa(q)/\kappa(p)$ is separable, and by [Stack20, Lemma 09H2], $\kappa(q)$ is perfect.

If the height of $p$ is one, then $A$ is regular at $p$ as $A$ is Noetherian and integrally closed ([AM69, Proposition 9.2]). Note that $ht(q) = 1$ and $B$ is integrally closed, so $B$ is regular at $q$.

Remark 3.6. Strikingly enough, Lemma 3.2 does not hold true if we drop all the three assumptions listed. See the following counter-example taken from MathOverflow.

Example 3.7. Let $R_m = \mathbb{F}_p[t_i, x_i]_{i \in \mathbb{N}_m}$ where $\mathbb{N}_m := \{1, 2, \ldots, m\}$. Let $\sigma : R_m \to R_m$ be the order $p$ automorphism sending $t_i$ to $t_i$ and $x_i$ to $x_i + t_i$. Let $S_m \subset R_m$ be the subring of the fixed elements under the action. Then, by [AM69, Proposition 7.8 and Corollary 7.7], $R_m$ and $S_m$ are Noetherian integrally closed domains and $R_m$ is integral over $S_m$. In particular, $R_m$ is the integral closure of $S_m$ in the extension of field of fractions, and the field extension has degree $p$ by Galois theory.

Let $q \subset R_m$ be the prime ideal $q = (t_i)_{i \in \mathbb{N}_m}$. This lies over the prime ideal $p = S_m \cap q$. Consider an element $f \in R_m$. We can write $f$ as

$$f = f_0 + \sum_i t_if_i + \text{terms of higher orders}$$

where $f_0, f_i$ are polynomials of $x_j$ and all other terms have order higher than one in $t_k$’s. Now if $f \in S$, then $\sigma(f) = f$. Note that

$$\sigma(f) = f_0 + \sum_i t_i(f_i + \partial f_0/\partial x_i) + \text{terms of higher orders}.$$  

This means that $\partial f_0/\partial x_i$ is identically zero. In other words, we see that $f_0 \in \mathbb{F}_p[x_i^p]$. Hence we see that $\kappa(p) = K(S_m/p) = \mathbb{F}_p(x_i)_{i \in \mathbb{N}_m}$. Since $\kappa(q) = \mathbb{F}_p(x_i)_{i \in \mathbb{N}_m}$, we deduce $[\kappa(q) : \kappa(p)] = p^m > p = [K(R_m) : K(S_m)].$

Let $R_\infty = \mathbb{F}_p[t_i, x_i]_{i \in \mathbb{N}}$. We may construct $S_\infty$ in the same way. Then, $R_\infty$ and $S_\infty$ are integrally closed domains and $R_\infty$ is integral over $S_\infty$, but they are not Noetherian. Let $q \subset R_\infty$ be the prime ideal $q = (t_i)_{i \in \mathbb{N}}$ and $p = S_\infty \cap q$. One can easily calculate that $[K(R_m) : K(S_m)] = p$ but $[\kappa(q) : \kappa(p)] = +\infty$.

Example 3.8. By making minor changes to the previous example, we may construct a counter-example, as Remark 3.6 desired, in characteristic 0. Let $T_m = \mathbb{Z}[\zeta, t_i, x_i]_{i \in \mathbb{N}_m}$ where $\mathbb{N}_m := \{1, 2, \ldots, m\}$ and $\zeta$ is the $p$-th root of unity. Let $\sigma : T_m \to T_m$ by $\sigma|_{\mathbb{Z}[\zeta]} = 1_{\mathbb{Z}[\zeta]}$, $\sigma(t_i) = t_i$ and $\sigma(x_i) = \zeta x_i + t_i$ be an automorphism of order $p$. Let $U_m \subset T_m$ be the subring of the fixed elements under the action. Then, $T_m$ and $U_m$ are Noetherian integrally closed domains and $T_m$ is integral over $U_m$ with $[K(T_m) : K(U_m)] = p$.

Let $q = (1 - \zeta, t_i)_{i \in \mathbb{N}_m}$ and $p = q \cap U_m$ be prime ideals. By similar arguments, one can verify that $[\kappa(q) : \kappa(p)] = p^m$. 
Lemma 3.9. Let $A \subset B$ be a finite extension of domains, $p \subset A$ be a prime ideal and $q \subset B$ be a prime ideal such that $q \cap A = p$. Suppose $[K(B) : K(A)] = [\kappa(q) : \kappa(p)]$ and $A$ is regular at $p$. Then, $B$ is regular at $q$.

Proof. Replacing $A, B$ with $A_p, B_p$, we may assume $A$ is a regular local ring. By Remark 3.3, there is a unique prime ideal $q$ lying over $p$, which in turn implies, by [AM69, Corollary 5.8], $B$ is a local domain.

We prove by induction on the height of $p$. First assume $ht(p) = 1$. Since $\dim B = 1$, and by Krull-Akizuki [Stack20, Lemma 00PG], $B$ is Noetherian, it suffices to show $B$ is integrally closed. Let $B \subset \overline{B}$ be the integral closure in the field of fractions. By Remark 3.3, there is a unique prime ideal $\overline{q} \subset \overline{B}$ such that $\overline{q} \cap B = q$. In particular, $\overline{B}$ is a discrete valuation ring. Let $\pi$ be the uniformiser of $\overline{B}$. By [Stack20, Lemma 09E5], the ramification index of $\overline{B}$ over $A$ is one, hence $\pi \in q$. So, $q$ is principal which in turn implies $B$ is integrally closed by [AM69, Proposition 9.2].

Now we assume $ht(p) = d$ and suppose Lemma 3.9 holds in height $\leq d - 1$. Let $\{x_1, \ldots, x_d\} \subset p$ be a regular sequence of $A$. By [Stack20, Lemma 00NP], $A/(x_1)$ is a regular local ring and $(x_1)$ is prime. By Remark 3.3, there exists a unique prime ideal $\mathfrak{r}_1$ such that $\mathfrak{r}_1 \cap A = (x_1)$, and that $[\kappa(\mathfrak{r}_1) : \kappa((x_1))] = [K(B) : K(A)]$. Moreover, by Remark 3.3 and the discussion above, the homomorphism of discrete valuation rings $A_{(x_1)} \rightarrow B_{(x_1)} = B_{\mathfrak{r}_1}$ has the ramification index one, hence $\mathfrak{r}_1 B_{\mathfrak{r}_1} = (x_1)$. In particular $\mathfrak{r}_1 = (x_1)$. Apply the inductive assumption to $A/(x_1) \subset B/\mathfrak{r}_1$. We deduce $B/\mathfrak{r}_1$ is regular. Since $B$ is finite over $A$, we see $B$ is Noetherian. Therefore, by [Stack20, Lemma 00NU], $B$ is a regular local ring. The lemma is proved. \qed

Lemma 3.10. Let $A$ be an integrally closed Nagata ring, $p \subset A$ be a prime ideal and $C \supset A/p$. Suppose $C$ is an integrally closed domain finite over $A/p$. Suppose further that one of the following conditions holds:

1. The residue field $\kappa(p)$ is perfect.
2. $A$ is regular at $p$.

Then there exists an integrally closed domain $B \supset A$ together with a prime ideal $\mathfrak{r} \subset B$ satisfying:

\begin{align*}
B \xrightarrow{q} B/\mathfrak{r} \xrightarrow{i} C \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
A \xrightarrow{p} A/p
\end{align*}

1. $\mathfrak{r} \cap A = p$ and the above diagram commutes.
2. $C$ is the integral closure of $B/\mathfrak{r}$ in its field of fractions.
3. $B$ is finite over $A$ with $[K(B) : K(A)] = [K(C) : K(A/p)]$.

Proof. Since $C$ is a finitely generated $A/p$-algebra, writing $C = A/p[y_1, y_2, \ldots, y_k]$, we consider the inclusions of fields

$$K(A/p) \subseteq K(A/p)(y_1) \subseteq K(A/p)(y_1, y_2) \subseteq \cdots \subseteq K(A/p)(y_1, y_2, \ldots, y_k) = K(C).$$
We will prove the lemma inductively. If all the inclusions above are equalities, then there is nothing to prove as $K(A/p) = K(C)$. So we assume at least one inclusion is strict. After re-indexing we may assume the inclusion $K(A/p) \subset K(A/p)(y_1)$ is strict.

Let $C_1 = A/p[y]$ be the sub-ring of $C$ generated by $y = y_1$ over $A/p$. Let
\[
\overline{a}(X) = X^n + \overline{a}_1X^{n-1} + \ldots + \overline{a_n} \in K(A/p)[X]
\]
be the minimal polynomial of $y$. Since all coefficients $\overline{a}_i \in K(A/p)$, there is an element $\overline{f} \in A/p$ such that $\overline{a}_i \in (A/p)\overline{f}$. For each $i$, we pick $a_i \in A$, and pick $f \in A\setminus p$ so that $\overline{a}_i$'s and $\overline{f}$ are the reductions of $a_i$'s and $f$ mod $p$ respectively. We therefore construct a polynomial $\alpha(X) = X^n + a_1X^{n-1} + \ldots + a_n \in A_f[X]$, which gives a surjective homomorphism $\phi : A_f[X] \to A_f[w] := A_f[X]/(\alpha)$ and then induces a surjective homomorphism $\rho : A_f[w] \to C_{1,7}$.

Because $C_1$ is a domain and so is the localisation of $C_1$, the kernel of $\rho$ is a prime ideal, denote by $a$. It is obvious that $a\cap A_f = pA_f$. Let $q_{min} \subseteq a$ be a minimal prime ideal, $q_{min}\cap A_f = p_{min}$, and $Q_{min} = \phi^{-1}q_{min}$. By definition $Q_{min}$ is a minimal prime ideal containing $\alpha$, and by assumption $A_f[X]$ is Noetherian. So, by Krull’s Hauptidealsatz [AM69, Corollary 11.17], we deduce that the height $ht(Q_{min}) = 1$. On the other hand, $Q_{min}$ contracts to $p_{min}$ in $A_f$. We claim $\alpha \notin p_{min}A_f[X]$. In fact, if we assume the opposite, then we deduce that, after reduction mod $pA_f$, we have $\overline{\alpha} = 0 \in A_f/pA_f[X] = (A/p)\overline{f}[X]$ which is a contradiction. Therefore we have $Q \supseteq p_{min}A_f[X]$, hence $ht(Q_{min}) = ht(p_{min}) + 1$ which in turn implies that $p_{min} = (0)$.

Now we have a surjective homomorphism $q_U : U_1 := A_f[w]/q_{min} \to C_{1,7}$. Moreover, since $q_{min}\cap A_f = (0)$, it in turn produces a finite domain extension $A_f \subset U_1$. We denote by $x \in U_1$ the image of $w$. Because $A_f$ is integrally closed and $x$ is integral over $A_f$, by [AM69, Proposition 5.15], it turns out that the minimal polynomial of $x$ over $K(A)$, denoted by $\alpha'(X) = X^m + a'_1X^{m-1} + \ldots + a'_m$, has all its coefficients $a'_i \in A_f$ and $m$ divides $n$. Since the minimal polynomial of an algebraic element over a field is unique, by $q_U(\alpha'(x)) = 0$, one can easily obtain $\overline{a}'_i = \overline{a}_i$ and $m = n$, where $\overline{a}'_i = q_U(a_i)$. It follows that $U_1 = A_f[X]/(\alpha')$ and $[K(U_1) : K(A)] = [K(C_1) : K(A)]$.

Let $B_1$ be the integral closure of $A$ in $U_1$, and consider the extension of domains $A \subset B_1$. Since $U_1$ is finite over $A_f$, by [AM69, Proposition 5.12] we deduce $U_1 = B_1.f$. Furthermore, if we denote by $C''_1$ the image of $B_1$ in $C_{1,7}$ through $q_U$, then we have $C''_1 = C_{1,7}$. Therefore, we have $K(B_1) = K(U_1)$ and $K(C''_1) = K(C_1)$. We note that every element of $C''_1$ is integral over $A/p$, hence integral over $C$ which in turn implies that it is an element of $C$. So, we have $C''_1 \subset C$. Let $q_1 : B_1 \to C''_1$ be the surjective homomorphism induced by $q_U$. We obtain a commutative diagram.

\[
\begin{array}{ccc}
B_1 & \xrightarrow{q_1} & C''_1 \\
\downarrow & & \downarrow \\
A & \xrightarrow{p} & A/p
\end{array}
\]

Next we let $\overline{B}_1$ be the integral closure of $B_1$ in its field of fractions. Since $A$ is a Nagata ring, by [AM69, Corollary 5.4], we see $\overline{B}_1$ is the integral closure of $A$ in $K(B_1)$, and therefore it is finite over $A$. Moreover, $\overline{B}_1$ is also a Nagata ring. By [AM69,
Theorem 5.11], there exists a prime ideal $\mathfrak{r}_1$ such that $\mathfrak{r}_1 \cap B_1 = r_1$ where $r_1$ is the kernel ideal of $q_1$. Letting $C_1 = B_1/\mathfrak{r}_1$, we consider the commutative diagram

$$
\begin{array}{c}
B_1 \ar[r]^{q_1} \ar[u] & C'_1 \ar[u] \\
\downarrow \ar[r]^{\rho_U} & \downarrow \ar[r] & \downarrow \\
A \ar[r]^{p} & A/p
\end{array}
$$

of which all horizontal arrows are surjective homomorphisms. Suppose one of the following conditions holds:

1. The residue field $\kappa(p)$ is perfect.
2. $A$ is regular at $p$.

By Lemma 3.2 and Remark 3.5, we deduce

$$
[K(B_1) : K(A)] = [K(C_1) : K(A)] 
\geq [K(C_1) : K(A/p)] = [K(C'_1) : K(C'_1)][K(C'_1) : K(A/p)].
$$

Hence we achieve the equation $[K(C_1) : K(C'_1)] = 1$ and

$$
[K(B_1) : K(A)] = [K(C_1) : K(A/p)].
$$

By assumption $C$ is integrally closed. Hence the integral closure of $C_1$ in its field of fractions lies in $C$ which in turn implies that $C_1$ is a sub-ring of $C$. Also note that, in the first case, $\kappa(\mathfrak{r}_1)$ is perfect; in the second case, by Lemma 3.9, $B_1$ is regular at $\mathfrak{r}_1$. Therefore, we may continue the procedure above for $B_1, C_1$ instead of $A, A/p$. Because $[K(C) : K(C_1)] < [K(C) : K(A/p)]$, after finitely many steps we obtain the required integrally closed domain $B$. \qed

Proof of Theorem 3.1. Let $U \subseteq X$ be an affine open subset such that $U \cap S \neq \emptyset$, and let $S_U, \tilde{S}_U$ be affine open subsets $S \cap U, \tilde{S} \cap \gamma^{-1}S_U$ of $S, \tilde{S}$ respectively. By the previous lemma, there exists a finite morphism $\rho_U : \tilde{U} \to U$ with a closed subvariety $\tilde{S}_U \subset \tilde{U}$ which induces a normalisation $\nu_U : \tilde{S}_U \to \tilde{S}_U$. Moreover, we have $\deg \rho_U = \deg \gamma$.

Let $X \hookrightarrow X^c$ be an open immersion to a complete normal variety and let $S^c$ be the Zariski closure of $S$ in $X^c$. Thus, there exists an open immersion $\tilde{S} \hookrightarrow \tilde{S}^c$ such that the induced map $\gamma^c : \tilde{S}^c \to S^c$ is generically finite dominant. Replacing $\gamma^c$ by the Stein factorisation we can assume it is still finite. Replacing $X, S, \tilde{S}$ with $X^c, S^c, \tilde{S}^c$, we assume they are complete and consider the following commutative diagram:

$$
\begin{array}{c}
\tilde{S} \ar[r] \ar[dr]_{\gamma} & \tilde{S}_U^c \ar[d] \ar[r] & \tilde{U} \ar[d] \\
\downarrow & S \ar[r]_{\rho_U} & X
\end{array}
$$
Let \( \tilde{U} \hookrightarrow \tilde{X} \) be an open immersion to a complete normal variety such that the induced map \( \rho : \tilde{X} \to X \) is a proper surjective morphism and let \( \tilde{S} \) be the Zariski closure of \( \tilde{S}_U \) in \( \tilde{X} \). Replacing \( \tilde{X} \) by the Stein factorisation we can assume further that \( \rho \) is a finite morphism. Since \( \tilde{S} \to S \) is a finite morphism and the induced map \( \nu : \tilde{S} \to \tilde{S} \) is birational, we conclude that \( \nu \) the normalisation. \( \square \)

**Remark 3.11.** One can arrange a scheme-theoretic argument so that Theorem 3.1 can be generalised to a quasi-compact and quasi-separated normal Nagata scheme \( X \) with an integral closed subscheme \( S \), instead of varieties. Indeed, with notation above, by Nagata compactification \([\text{Stack20, Theorem 0F41}]\), we may compactify \( \rho_U \) to a proper morphism \( \rho : \tilde{X} \to X \). Apply the normalisation \([\text{Stack20, Lemma 035L}]\) and the Stein factorisation \([\text{Stack20, Theorem 03H0}]\) to reduce to the case when \( \tilde{X} \) is normal and \( \rho \) is finite, hence the conclusion.

**4. LC-TRIVIAL MORPHISMS AND MODULI B-DIVISORS**

In this section we generalise the notion of lc-trivial fibrations to proper surjective morphisms, and then we define the discriminant b-divisor and moduli b-divisor. Under these constructions we study the positivity of the moduli b-divisor of a class of lc-trivial morphisms, namely good lc-trivial morphisms.

### 4.1. Lc-trivial morphisms

**Definition 4.1.** An lc-trivial (resp. klt-trivial) morphism \( f : (X, B) \to Y \) consists of a proper surjective morphism \( f : X \to Y \) between normal varieties and a sub-pair \( (X, B) \) satisfying the following properties:

1. \( K_X + B \sim_\mathbb{K} 0/Y \),
2. If we denote by \( X \dashrightarrow \tilde{Y} \to Y \) the Stein factorisation, then \( \tilde{f} : (X, B) \to \tilde{Y} \) is an lc-trivial (resp. klt-trivial) fibration (see Definition 2.25).

To study these objects, we want to establish a canonical bundle formula of Kodaira type. Unfortunately, the classical definition of the discriminant divisor usually disables the positivity of the moduli b-divisor (see \([\text{Amb99, Remark 3.2}]\)). We therefore introduce the following definition of the discriminant divisor of a proper generically finite surjective morphism. The construction is from \([\text{HL19, Theorem 4.5}]\) and \([\text{FG12, Lemma 1.1}]\).

**Definition 4.2** (See \([\text{HL19, Theorem 4.5}]\) and \([\text{FG12, Lemma 1.1}]\)). Let \((X/Z, B+M)\) be a generalised sub-pair, \( B \) be its pre-boundary \( \mathbb{R}-b\)-divisor, \( M \) be its moduli \( \mathbb{R}-b\)-Cartier b-divisor and \( f : X \to Y \) be a proper surjective generically finite morphism over \( Z \) between normal varieties such that

\[
K_X + B + M \sim_\mathbb{K} 0/Y.
\]

We define the generalised sub-pair \((Y, B_Y + M_Y)\) by specifying the traces on the higher models of \( Y \) as follows: for any birational morphism \( \phi : Y' \to Y \), consider the following...
Lemma 4.4. Let $f : X \to Y$ be a finite surjective morphism and $D$ be an $\mathbb{R}$-Cartier divisor on $X$. Let $\phi : Y' \to Y$, $\pi : X' \to X$ be birational models and $f' : X' \to Y'$ be a finite surjective morphism such that $\phi \circ f' = f \circ \pi$. Suppose $X, X', Y$ and $Y'$ are $\mathbb{Q}$-factorial. Then, we have

$$f'_* \pi^* D = \phi^* f_* D.$$ 

Proof. If $\phi$ is a small contraction, then there is nothing to prove. Suppose the set of prime exceptional/$Y$ divisors is non-empty, and we pick an element $E_Y$. Let $\sum_k E_k$ be a reduced divisor whose components are all prime divisors mapped onto $E_Y$. It suffices to show the equation near the point $y$. Since the push-forward and the pull-back of $\mathbb{R}$-Cartier divisors are $\mathbb{R}$-linear, we may assume $D$ is a prime divisor. Moreover, since the question is local, shrinking $Y$ and the others accordingly, we may assume $X, Y$ are affine.

$$\sum_k E_k \subset X' \xrightarrow{f'} X \quad \sum_k E_k \subset Y' \xrightarrow{\phi} Y$$

Take a $\mathbb{Q}$-rational function $\varphi$ so that $D' = D + (\varphi)$ does not contain any image of $\sum_k E_k$ in its support. This is possible because there are only finitely many such images. In
particular, $f_*D'$ does not contain the image of $E_Y$ in its support by the definition of proper push-forwards. So, near the point $y$ we have $f'_*\pi^*D' = 0 = \phi^*f_*D'$. It remains to verify $f'_*\pi^*(\varphi) = \phi^*f_*(\varphi)$ which follows from the fact that the proper push-forward of a principal divisor of a non-zero rational function $(\varphi)$ is that of its norm $N_{K(X)/K(Y)}(\varphi)$ (see [Ful83, Proposition 1.4]).

Quite recently, J. Han and W. Liu obtained a similar result [HL19, Theorem 4.5] as below.

**Theorem 4.5.** Let $(X/Z, B + M)$ be a generalised sub-pair over a variety $Z$. Let $f : X \to Y$ be a generically finite surjective morphism of normal varieties over $Z$, such that $K_X + B_X + M_X \sim_{\mathbb{Q}} 0/Y$. Then, there is a generalised sub-pair $(Y/Z, B_Y + M_Y)$ with

$$K_X + B + M \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y)$$

Moreover, if $(X/Z, B + M)$ is g-lc (resp. g-klt, g-sub-lc, g-sub-klt), then so is $(Y/Z, B_Y + M_Y)$.

**Proof.** We use the notation in Definition 4.2. By Lemma 4.4, we infer $M^Y$ descends to some model $Y' \to Y$. By flattening lemma (a special case of Theorem 2.32) we may suppose the induced morphism $f' : X' \to Y'$ is finite, $X'$ is a $\mathbb{Q}$-factorial kl birational model of $X$ and $(Y', B_{Y'})$ is log smooth. Hence, we deduce $M_{Y'} = M^Y_{Y'}$ is nef $/Z$ which completes the first part of the theorem.

To show the second part, we simply notice that, by definition $B_{Y'} = \frac{1}{\deg_f}(R' + B')$ and apply [FG12, Proof of Lemma 1.1] to conclude the result. \hfill \Box

With the above construction we are ready to define the moduli b-divisors of lc-trivial morphisms.

**Definition 4.6.** Let $f : (X,B) \to Y$ be an lc-trivial morphism, $X \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{\gamma} Y$ be the Stein factorisation, $\varphi$ be a $\mathbb{R}$-rational function, $D$ be an $\mathbb{R}$-Cartier divisor on $Y$ and

$$K_X + B + (\varphi) = \tilde{f}^*(K_{\tilde{Y}} + B_{\tilde{Y}} + M_{\tilde{Y}}) = f^*D$$

with the pre-boundary b-divisor $\tilde{B}$ and the moduli $\mathbb{R}$-b-Cartier b-divisor $\tilde{M}$. By Definition 4.2, Remark 4.3 and Theorem 4.5 we define a pre-boundary b-divisor $B$ and the moduli $\mathbb{R}$-b-Cartier b-divisor $M$ satisfying

$$K_X + B + (\varphi) = f^*(K_Y + B_Y + M_Y)$$

and $(Y, B_Y + M_Y)$ is a generalised sub-pair where $B_Y, M_Y$ are the traces of $B, M$ on $Y$. We call $B$ the **discriminant b-divisor** and $M$ the **moduli b-divisor**.

**Remark 4.7.** (1). We remark that the discriminant b-divisor $B$ (resp. the moduli b-divisor $M$) defined above does not in general coincide with that in [Amb99] and the pre-discriminant (resp. pre-moduli b-divisor) in Section 2.6. As pointed out in [Amb99, Remark 3.2] the classical construction does not guarantee the b-nefness of $M$.

(2). Recall Definition 2.25 of lc-trivial fibration. Writing $B = \sum_i r_i B_i$ as a convex combination such that $\tilde{f} : (X, B_i) \to \tilde{Y}$ is $\mathbb{Q}$-lc-trivial for each $i$, and given $b_i$ with $b_i(K_F + B_{i,F}) \sim 0$ where $F$ is a general fibre and $K_F + B_{i,F} = (K_X + B_i)|_F$ for each $i$
and fixed canonical divisors, we may set $\varphi = \prod_i \varphi_i^{r_i/b_i}$, where $\varphi_i$’s are rational functions. Moreover, the rational functions $\varphi_i$’s are uniquely determined up to rational functions on $Y$. Hence, so are the b-divisors $M$ and $\bar{M}$.

(3). If $(X, B)$ is lc (resp. klt, sub-lc, sub-klt), then by Theorem 4.5, $(Y, B_Y + M_Y)$ is a g-lc (resp. g-klt, g-sub-lc, g-sub-klt) generalised pair. Moreover, if $(Y, B_Y + M_Y)$ is g-sub-lc, then every g-sub-lc centre of $(Y, B_Y + M_Y)$ is dominated by a sub-lc centre of $(X, B)$ by a calculation of the pull-back formula.

**Definition 4.8.** With the notation of Definition 4.6, we say an lc-trivial morphism $f : (X, B) \to Y$ is good if the equation $\bar{M} = \gamma^* M$ holds. In particular, every lc-trivial fibration is good by definition.

**Lemma 4.9.** Let $f : X \to Y$ be a generically finite proper surjective morphism between normal varieties, and let $D$ be an $\mathbb{R}$-b-Cartier divisor on $X$ with $D \sim_\mathbb{K} 0/Y$ (resp. $D \sim 0/Y$, $D = 0/Y$). Then, we have

$$D \sim_\mathbb{K} (\text{resp. } \sim, =) f^*(\frac{1}{\deg f} f_* D).$$

**Proof.** Note that $f_* D$ is $\mathbb{R}$-b-Cartier b-divisor by Lemma 4.4 and thus the pull-back is well-defined. Replacing $X, Y$ we can assume both $D, f_* D$ descend to $X, Y$ respectively. The rest can be argued analogously as in Remark 4.3. $\Box$

**Remark 4.10.** (1). With the notation of Definition 4.6, by the above lemma we see that an lc-trivial morphism being good is equivalent to $\bar{M} = 0/Y$.

(2). With the notation of Definition 4.2, if $M \sim_\mathbb{K} 0/Y$, then by Lemma 4.9 we have $M \sim_\mathbb{K} f^* M^Y$. In addition, $M$ is b-nef and abundant (resp. b-semi-ample, b-ample) over $Z$ if and only if $M^Y$ is also.

**Proposition 4.11.** Let $f : (X, B) \to Y$ be a good lc-trivial morphism and $X \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{\tau} Y$ be a factorisation of $f$ such that $\tau$ is generically finite. Then, $\tilde{f} : (X, B) \to \tilde{Y}$ is also a good lc-trivial morphism with

$$\tau^* M = \bar{M}$$

where $M$ and $\bar{M}$ are the moduli b-divisors on $Y$ and $\tilde{Y}$ respectively. In particular, $\tilde{f}$ is a good lc-trivial morphism. Moreover, if $Y$ is proper over $Z$, then $M, \bar{M}$ are b-nef over $Z$.

**Proof.** Let $\tilde{f} : X \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{\varrho} \tilde{Y}$ and $f : X \xrightarrow{f} Y \xrightarrow{\gamma} Y$ be the corresponding Stein factorisations respectively. Consider the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow f & & \downarrow \gamma \\
Y & \xrightarrow{\tau} & Y
\end{array}$$

By the rigidity lemma, the induced map $\tilde{\tau}$ is a morphism. Because $\varrho$ and $\tau$ are generically finite, one infers that $\tilde{\tau}$ is generically finite. Since $\tilde{f}$ is a contraction, one infers that $\tilde{\tau}$ is birational which in turn implies that $\tilde{f} : (X, B) \to \tilde{Y}$ is also an lc-trivial
morphism. Since \( f \) is good, we have \( \tilde{M} = \gamma^* M \) and \( \hat{M} = \frac{1}{\deg \gamma} \varphi^* \tilde{M} \) where \( \tilde{M} \) and \( \hat{M} \) are the moduli b-divisors given by \( \tilde{f} \) and \( \tilde{f} \) respectively. Note that \( \tilde{M} = \hat{M} \) as \( \tilde{\tau} \) is birational. By Lemma 4.9 we immediately obtain that \( \tilde{M} = \varphi^* \hat{M} \) which proves the first assertion.

The second assertion follows from [Amb04, Theorem 2.5]. \( \square \)

Remark 4.12 (Base change of lc-trivial morphisms). Because the fibre product of two generically finite morphisms usually have multiple main components, the category of lc-trivial (resp. klt-trivial) morphisms is NOT closed under generically finite base change. However, since base change of a finite morphism is also finite, we can partially recover “base change of an lc-trivial morphism” via the following construction: given an lc-trivial morphism \( f : (X, B) \to Y \), a generically finite proper surjective morphism \( \mu : V \to Y \) of normal varieties and denoting by \( \tilde{V} \to \tilde{Y} \to Y \) the Stein factorisation, we have the following commutative diagram.

\[
\begin{array}{ccc}
W & \longrightarrow & V \times_Y X \\
\downarrow \tilde{h} & & \downarrow \tilde{f} \\
\tilde{V} & \longrightarrow & \tilde{V} \times_{\tilde{Y}} \tilde{Y}
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \varsigma & & \downarrow \gamma \\
V & \longrightarrow & \mu
\end{array}
\]

Note that in general the main components of \( V \times_Y Y' \) is neither unique nor reduced as \( \gamma \) is finite. We choose one of the main components and let \( \tilde{V} \) be its normalisation. By the universal property of the fibre product we see \( V \times_Y X \simeq V \times_Y \tilde{Y} \times_{\tilde{Y}} X \), so \( \tilde{g} \) has geometrically connected fibres (see [Stack20, Lemma 0555]) and there exists a unique main component of \( V \times_Y X \) which dominates the previously chosen main component (see [Stack20, Lemma 01JT, Lemma 054Q]). We write \( W \) for its normalisation. Hence \( \tilde{\gamma} \) induces a contraction \( \tilde{h} \): Indeed, the generic fibre of \( \tilde{f} \) is geometrically normal by [Stack20, Lemma 0380], hence the geometric connectedness of the generic fibre of \( \tilde{h} \).

Moreover, \( \varsigma \) is finite surjective. In particular, \( h : W \to \tilde{V} \to V \) is the Stein factorisation of the composite morphism \( h \).

Note that \( W \to X \) is generically finite. We write \( K_W + B_W \) for the pull-back of \( K_X + B \). In order to show \( h \) is an lc-trivial morphism, it remains to check that \( \tilde{h} \) is an lc-trivial fibration with respect to \( (W, B_W) \). By the universal property of the fibre product, it follows that \( \tilde{h} \) is the morphism induced by the generically finite base change \( \tilde{\mu} : \tilde{V} \to \tilde{Y} \) which in turn implies that \( \tilde{h} \) is an lc-trivial fibration since the category of lc-trivial fibrations is closed under generically finite base change. By Lemma 2.28 ([Amb05, Proposition 3.1]) we have an equation of moduli b-divisors \( \tilde{M}_h = \tilde{\mu}^* \tilde{M} \) where \( \tilde{M}, \tilde{M}_h \) are defined by the lc-trivial fibrations \( \tilde{f}, \tilde{h} \) respectively. Since \( M = \frac{1}{\deg \gamma} \varphi^* \tilde{M} \) and \( M_h = \frac{1}{\deg \varsigma} \varsigma^* \tilde{M} \) where \( M, M_h \) are defined by the lc-trivial morphisms \( f, h \) respectively,
we have
\[ M = \frac{1}{\deg \mu} \mu_M. \]

Assume further that \( f : (X, B) \to Y \) is good. By definition we have \( \tilde{M} = \gamma^* M \), and by Lemma 4.9 we deduce \( \tilde{M}_h = \varsigma^* M_h \), which in turn implies that \( h \) is good and \( M_h = \mu^* M \).

Thanks to the Stein factorisation and Lemma 2.28 ([Amb05, Proposition 3.1]), we may slightly generalise the above discussion as follows. The arguments are analogous, so we omit the proof.

**Proposition 4.13** (cf. [Amb05, Proposition 3.1]). Let \( f : (X, B) \to Y \) be a good lc-trivial morphism, \( \gamma : Y' \to Y \) be a surjective proper morphism from a normal variety \( Y' \), and \( f' : (X', B') \to Y' \) be an lc-trivial morphism induced by base change.

\[
\begin{array}{ccc}
(X, B) & \xrightarrow{\rho} & (X', B') \\
\downarrow f & & \downarrow f' \\
Y & \xleftarrow{\gamma} & Y'
\end{array}
\]

Let \( M \) and \( M' \) be the corresponding moduli b-divisors. Then, \( f' : (X', B') \to Y' \) is good and
\[ \gamma^* M = M'. \]

**Remark 4.14.** It is worth remarking that Proposition 4.13 fails in general if we abandon the assumption of goodness. See the counter-example below.

**Example 4.15.** Let \( h : S \to \mathbb{P}^1 \) be a minimal rational elliptic surface with the nontrivial moduli part \( M_{\mathbb{P}^1} \sim \mathbb{Q} c\{ x \} \), the discriminant divisor \( B_{\mathbb{P}^1} \sim \mathbb{Q} b\{ x \} \), \( p : Q \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the projection from a smooth quadric and let \( g : (X, S_1 + S_2) \to Q \) be an lc-trivial fibration induced by base change, where \( S_1, S_2 \) are two copies of \( S \) over two general lines \( L_1, L_2 \). Note that \( N^1(Q) \) is generated by two lines \( L^\perp, L^\parallel \) with distinct first Chern classes \( \alpha, \beta \). Also note that \( L_1, L_2 \) have the same first Chern class \( c_1(L_1) = c_1(L_2) = \beta = c_1(L^\parallel) \). Furthermore, we let \( \gamma : Q \to \mathbb{P}^2 \) be the projection from a point outside \( Q \) in \( \mathbb{P}^3 \) which is finite of degree 2. Consider the lc-trivial morphism \( f = \gamma \circ g : (X, S_1 + S_2 + G) \to \mathbb{P}^2 \), where
\[
G \sim_{\mathbb{Q}} \begin{cases} (2 - b - c)g^* L^\perp, & b + c < 2 \\ (b + c - 2)g^* L^\parallel, & b + c \geq 2 \end{cases}
\]
and the corresponding moduli divisors \( M_Q, M_{\mathbb{P}^2} \) on \( Q, \mathbb{P}^2 \) respectively. One can easily calculate \( M_Q \sim_{\mathbb{Q}} c L^\perp \) and \( M_{\mathbb{P}^2} \sim_{\mathbb{Q}} \xi L \) where \( L \) is a general line in \( \mathbb{P}^2 \). Due to Lemma 4.4, it is obvious that \( M_Q, M_{\mathbb{P}^2} \) represent their moduli b-divisors and \( \kappa(M_Q) = 1 \neq 2 = \kappa(M_{\mathbb{P}^2}) \).

This is an easy example of an lc-trivial morphism which is not good.

As we discussed in Remark 4.12, we may construct an lc-trivial fibration \( f' : X \to Q \) induced by base change \( \gamma \). One simply notices \( f' = g \) and \( M_Q \sim_{\mathbb{K}} \gamma^* M_{\mathbb{P}^2} \).
Remark 4.16 (Comparison of sub-lc centres). By the construction of the discriminant b-divisor, as we discussed in Remark 4.7(3), if we suppose \((X, B)\) is sub-lc, then every g-sub-lc centre of the induced generalised sub-pair is dominated by some sub-lc centre. Indeed, with the notation of Definition 4.6, if an irreducible closed subset \(T\) of \((Y, B_Y + M_Y)\) is a g-sub-lc centre, then every irreducible component \(\tilde{T}\) of its inverse image \(\gamma^{-1}T\) is a g-sub-lc centre of the induced generalised sub-pair \((\tilde{Y}, B_{\tilde{Y}} + M_{\tilde{Y}})\).

Assume further that \(f : (X, B) \to Y\) is good. From the equation \(K_{\tilde{Y}'} + B_{\tilde{Y}'} = \gamma'^* (K_Y + B_Y)\) on sufficiently high birational models \(\tilde{Y}', Y'\), one can easily check that every vertical sub-lc centre of \((X, B)\) is mapped onto a g-sub-lc centre of \((\tilde{Y}, B_{\tilde{Y}} + M_{\tilde{Y}})\), and hence it is mapped onto a g-sub-lc centre of \((Y, B_Y + M_Y)\).

4.2. Dlt-trivial morphisms. In this subsection we introduce the notion of dlt-trivial morphism. This class of algebraic fibre spaces can be regarded as “dlt models” of good lc-trivial morphisms.

Definition 4.17. An lc-trivial morphism \(f : (X, B) \to Y\) of normal varieties is called a dlt-trivial morphism if

- \(X\) and \(Y\) are quasi-projective,
- \(f : (X, B) \to Y\) is good,
- \((X, B)\) is dlt, and
- the induced generalised pair \((Y, B_Y + M_Y)\) is g-dlt.

In addition, a dlt-trivial morphism is called \(\mathbb{Q}\)-factorial if both \(X\) and \(Y\) are \(\mathbb{Q}\)-factorial, and is called a dlt-trivial fibration if \(f\) has connected fibres.

Remark 4.18. Given a dlt-trivial morphism, notation as above, there exist birational models \(\pi : X' \to X, \phi : Y' \to Y\) together with a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{\phi} & Y
\end{array}
\]

satisfying the following properties:

1. \(f : (X, B) \to Y\) is a good lc-trivial morphism of quasi-projective varieties.
2. \(X', Y'\) are non-singular quasi-projective varieties endowed with snc divisors \(\Sigma_{X'}, \Sigma_{Y'}\) on \(X', Y'\) respectively.
3. \(f'^{-1}(\Sigma_{Y'}) = \Sigma_{X'}^v\), where \(\Sigma_{X'}^v\) denotes the vertical/Y' part of \(\Sigma_{X'}\).
4. \(B'\) and \(B_{Y'}, M_{Y'}\) are supported by \(\Sigma_X\) and \(\Sigma_Y\) respectively, where \(K_{X'} + B' = \pi'^*(K_X + B), B_{Y'} = B_{Y'}, M_{Y'} = M_{Y'}\) and \(B, M\) are the discriminant and the moduli b-divisor defined by the lc-trivial morphism.
5. \(B'\) and \(B_{Y'}\) are sub-boundaries; \(B\) and \(B_Y = B_Y\) are effective divisors.
6. \(\pi, \phi\) are isomorphisms at the generic point of every (g-)sub-lc centre.
7. \(M_{Y'}\) represents the moduli b-divisor \(M\).

The reader may easily check that the assumptions listed above is an alternative definition of a dlt-trivial morphism.
The rest of the subsection will be devoted to proving that every good lc-trivial morphism possesses a “dlt model”, namely an equivalent dlt-trivial morphism, in the sense of B-birational equivalence.

**Lemma 4.19.** Let $f : (X, B) \to Y$ be an lc-trivial morphism from an lc pair and $(Y, B_Y + M_Y)$ be the induced g-lc generalised pair. Let $\phi : Y' \to Y$ be a g-dlt model and $\pi : (\overline{X}, \overline{B}) \to X$ be a quasi-projective log smooth model of $(X, B)$ such that $\overline{f} : \overline{X} \to Y'$ is a morphism. Then, $(\overline{X}/Y', \overline{B})$ has a good minimal model $(X'/Y', B')$.

Moreover, $(X', B')$ is a log birational model of $(X, B)$ and the induced map $(X', B') \dashrightarrow (X, B)$ is a B-birational contraction.

**Proof.** Let $f : X \to \tilde{Y} \to Y$ and $\overline{f} : \overline{X} \to \overline{Y} \to Y'$ be the Stein factorisations. By the rigidity lemma, the induced map $\tilde{\phi} : \tilde{Y}' \to \tilde{Y}$ is a birational morphism. Replacing $Y, Y'$ with $\tilde{Y}, \tilde{Y}'$ we may assume $f$ is a contraction. Note that it no longer holds that $(Y', B_{Y'} + M_{Y'})$ is g-dlt. However, by Remark 4.16 we may assume that $a(\Gamma, Y, B_Y + M_Y) = 0$ for every exceptional/Y prime divisor $\Gamma$ on $Y'$. By Remark 4.7(3), replacing $(\overline{X}, \overline{B})$ with a suitable blow-up, we may assume every exceptional/Y prime divisor $\Gamma$ on $Y'$ is dominated by a component of $S \subset [\overline{B}]$ with $a(S, X, B) = 0$.

We write $E := K_{\overline{X}} + \overline{B} - \pi^*(K_X + B) \geq 0$ and $E = E^h + E^v$ where $E^h$ denotes the horizontal/Y part and $E^v$ denotes the vertical/Y part. Since $E^h$ is exceptional/X, we see

$$E^h|_F = N_\phi((K_{\overline{X}} + \overline{B}^\geq_0)|_F)$$

where $F$ is a general fibre.

Since $E^v$ is exceptional/X, it is very exceptional/Y by definition. We claim that it is also very exceptional/Y’. To this end, let $Z'$ be an irreducible closed subset of $\overline{f}(E^v) \subset Y'$, and let $0 \leq E_{Z'} \leq E^v$ be the effective divisor supported by all components of $E^v$ mapped onto $Z'$ with the same coefficients. Write $E^v = \sum Z_i E_{Z_i}$. If $\text{codim}_Y Z' \geq 2$, then $E_{Z'}$ is very exceptional/Y’. If $\text{codim}_Y Z' = 1$ and $\text{codim}_Y \phi(Z') = 1$, then it is also very exceptional/Y’ as $E_{Z'}$ is very exceptional/\phi(Z’). It remains to prove when $Z'$ is an exceptional/Y divisor. Recall our assumption that $Z'$ is dominated by a component of $S \subset [\overline{B}]$ with $a(S, X, B) = 0$. Obviously $S \notin \text{Supp}E_{Z'}$. Therefore $E_{Z'}$ is again very exceptional/Y’. Since $E^v$ is the sum of $E_{Z'}$’s, the claim is proved.

Hence, by Lemma 2.16, we can run a log MMP/Y’ on $K_{\overline{X}} + \overline{B}$ which terminates with a good minimal model. We thus obtain the first assertion. Moreover, since the divisor contracted by the process of above log MMP is exactly $E$, one may easily infer that $(X', B')$ is a log birational model of $(X, B)$, and the induced map $(X', B') \dashrightarrow (X, B)$ is a B-birational contraction. \hfill \Box

**Remark 4.20.** (1). In the previous lemma, if $f$ is good, then every vertical lc centre of $(X, B)$ is mapped onto a g-lc centre of $(Y, B_Y + M_Y)$ by Remark 4.16. However,
a g-lc centre of \((Y, B_Y + M_Y)\) is not necessarily an lc centre of \((Y, B_Y)\). Hence a vertical lc centre of \((X, B)\) is not necessarily mapped onto an lc centre of \((Y, B_Y)\) unless \((Y, B_Y + M_Y)\) is g-dlt. This is our motivation to introduce the notion of dlt-trivial morphisms.

(2). One should not expect in general that the rational map \(X' \rightarrow X\) is a morphism as the example below indicates.

**Example 4.21.** Let \((X, B)\) be a log terminal pair and the diagram below be a \(K_X\)-flip.

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^+ \\
\downarrow{f} & & \downarrow{f^+} \\
Z & \xrightarrow{f} & Y
\end{array}
\]

Suppose \(\phi\) is also a \(K_X + B\)-flop. Then, for any common log crepant model \((W, B_W)\) of \((X, B)\) and \((X^+, B^+)\), we have \((W, B_W)\) is sub-klt but not klt.

The next theorem guarantees that any good lc-trivial morphism from an lc pair has a \(\mathbb{Q}\)-factorial “dlt model”.

**Proposition 4.22.** Let \(f : (X, B) \rightarrow Y\) be an lc-trivial morphism from an lc pair. Then, there exist a birational model \(\phi : Y' \rightarrow Y\) and a log birational model \((X', B')\) of \((X, B)\) with a \(B\)-birational contraction \(\pi : (X', B') \rightarrow (X, B)\) such that the induced map \(f' : X \rightarrow Y'\) is a morphism, \((X', B')\) is quasi-projective \(\mathbb{Q}\)-factorial dlt and the induced generalised pair \((Y', B_Y', + M_{Y'})\) is quasi-projective \(\mathbb{Q}\)-factorial g-dlt.

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{\phi} & Y
\end{array}
\]

In particular, if in addition we suppose \(f : (X, B) \rightarrow Y\) is good, then \(f' : (X', B') \rightarrow Y'\) is a \(\mathbb{Q}\)-factorial dlt-trivial morphism.

**Proof.** Pick a log resolution \(\overline{\pi} : \overline{X} \rightarrow X\) of \((X, B)\) and a proper closed subset \(Z \subset Y\) such that \((\overline{X}, \overline{B})\) is log smooth over \(Y \setminus Z\), where \((\overline{X}, \overline{B})\) is a log smooth model of \((X, B)\). Now pick a log resolution \(\overline{\phi} : \overline{Y} \rightarrow Y\) of \((Y, \text{Supp}(B_Y + M_Y) \cup Z)\) to which the moduli b-divisor \(\overline{M}\) descends. Replacing \(\overline{\pi}\) we can assume the induce map \(\overline{f} : \overline{X} \rightarrow \overline{Y}\) is a morphism.

By construction and possibly further replacing \(\overline{\pi}\), there exist simple normal crossing divisors \(\Sigma_{\overline{X}}, \Sigma_{\overline{Y}}\) on \(\overline{X}, \overline{Y}\) respectively, so that \(\overline{f} : (\overline{X}, \Sigma_{\overline{X}}) \rightarrow (\overline{Y}, \Sigma_{\overline{Y}})\) satisfies conditions (2)-(4) and (7) in Remark 4.18. Note that \(\phi^{-1}(\text{Supp}(B_Y + M_Y)) \cup \text{Ex}(\phi) \subseteq \Sigma_{\overline{Y}}\) is snc. Let \((\overline{Y}, B_{\overline{Y}} + M_{\overline{Y}})\) be a log smooth model of \((Y, B_Y + M_Y)\). Run a log MMP/\(Y\) on \(K_{\overline{Y}} + B_{\overline{Y}} + M_{\overline{Y}}\) with scaling of some ample/\(Y\) divisor which terminates with a good minimal model \((Y', B_{Y'} + M_{Y'})\) by Lemma 2.15 and [Bir12, Theorem 4.1(iii)]. Replacing \(\overline{Y}\) and accordingly \(\overline{f}, \overline{\pi}\) we can assume \(\overline{\phi} : \overline{Y} \rightarrow Y'\) is a morphism satisfying the condition (6) in Remark 4.18. Now we run a log MMP/\(Y'\) on \(K_{\overline{X}} + \overline{B}\) with scaling of some ample/\(Y'\) divisor which terminates with a good minimal model \((X'/Y', B')\) by Lemma 4.19. Moreover, the induced map \((X', B')\) is a log birational model of \((X, B)\).
Replacing \( X \) we can assume \( \pi : X \rightarrow X' \) is a morphism satisfying the condition (6) in Remark 4.18. It is obvious that \( B', B_Y \) are effective. So the condition (5) is satisfied and we proved the first statement.

In addition, if \( f : (X, B) \rightarrow Y \) is good, then \( f' : (X', B') \rightarrow Y' \) satisfies the condition (1) in Remark 4.18, and it is therefore a dlt-trivial morphism. \( \square \)

4.3. Log abundance of the moduli b-divisor. The main goal of this subsection is to prove the following theorem which is a generalisation of [FG14, Theorem 1.1]. Recall from Definition 2.13 that, a b-nef and log abundant b-divisor \( D \) is represented by a nef and log abundant divisor \( D_X \) on a data log resolution \( (\overline{X}, \overline{B + M}) \rightarrow X \), to which \( D, M \) descends.

**Theorem 4.23.** Let \( f : (X, B) \rightarrow Y \) be a good lc-trivial morphism from an lc pair to a complete variety. Then, the moduli b-divisor \( M \) is b-nef and log abundant with respect to the induced generalised pair.

We begin with an elementary lemma.

**Lemma 4.24.** Let \( f : X \rightarrow Y \) be a contraction of quasi-projective normal varieties. Suppose the following conditions holds:

1. (Weakly semi-stable) There exist reduced divisors \( \Delta, \Delta_Y \) on \( X, Y \) respectively, such that \( f : (X, \Delta) \rightarrow (Y, \Delta_Y) \) satisfies the conditions listed in Theorem 2.33.
2. (Lc-trivial) There exists a sub-boundary \( B \) such that \( f : (X, B) \rightarrow Y \) is an lc-trivial fibration.
3. (Resolved) Let \( B \) be the discriminant b-divisor, \( M \) be the moduli b-divisor with traces \( B_Y, M_Y \) on \( Y \). We have that, \( B \) is supported by \( \Delta \), \( B_Y, M_Y \) are supported by \( \Delta \), and \( M \) is represented by \( M_Y \).

Then, we have:

1. Given a prime divisor \( Q \) of \( Y \) such that \( (Y, Q + \Delta_Y) \) is log smooth, if \( Q \not\subseteq \Delta_Y \), then \( b_Q = 1 \) and \((X, B + f^*Q)\) is sub-lc (for definition of \( b_Q \), see Section 2.5).
2. Given a vertical prime divisor \( S \subset B = 1 \) with its image \( T \), and any prime divisor \( Q \) of \( T \), if \( Q \not\subseteq \Delta_T = (\Delta_Y - T) \mid_T \), then \( b_Q = 1 \).

**Proof.** (i). Set \( b := \sup \{ t | (X, B + tf^*Q) \text{ is sub-lc} \} \). By condition (3), \( f^*Q \) does not contain any sub-lc centre in its support, hence \( b > 0 \). Let \( S \) be the newly constructed sub-lc centre of \((X, B + bf^*Q)\). Since \( f \) is a resolved quasi-trivial morphism, the image \( T \) of \( S \) is a sub-lc center of \((Y, B_Y + bQ)\). Because \( Q \) does not contain any sub-lc centre of \((Y, B_Y)\) in its support, we deduce that \( T \) is also newly constructed, hence \( b = 1 \). In particular, we have \( b_Q = 1 \).

(ii). Shrinking \( Y \) and \( X \) accordingly, we may assume \( Q \) is smooth and its support does not contain any sub-lc centre of \((Y, B_Y)\). Moreover, we may assume \( Y \) is affine. By a theorem of Bertini, there is a prime divisor \( P \subset Y \) containing \( Q \) such that \((Y, P + \Delta_Y)\) is log smooth. Indeed, one simply notices that intersecting non-transversally defines a proper linear subspace of \( V \) in [Stack20, Lemma 0FD5]. Hence, by (i), we deduce \((X, \Delta + f^*P)\) is sub-lc which in turn implies \( b_Q = 1 \). \( \square \)
Definition 4.25 (Filtration of strata in codimension one). Let \((X, B)\) be a sub-dlt pair and \(S\) be a stratum. An ascending sequence of strata

\[ S = S_0 \subset S_1 \subset \ldots \subset S_n = X \]

called a filtration of strata in codimension one if the codimension of \(S_i\) in \(S_{i+1}\) is one for all \(i\).

Definition 4.26. Let \(f : (X, B) \to (Y, B_Y)\) be a surjective morphism of sub-dlt pairs, and \(T\) a stratum of \(Y\). A stratum \(S\) of \((X, B)\) is said to be saturated over \(B\) if

\(\star\) there exist a filtration of strata in codimension one of \(T\):

\[ T = T_0 \subset T_1 \subset \ldots \subset T_m = Y, \]

and a filtration of strata in codimension one of \(S\):

\[ S = S_0 \subset S_1 \subset \ldots \subset S_n = X \]

such that \(f(S) = T\) and for every \(0 \leq j \leq m\), there is some \(j \leq i \leq n\) with \(f(S_i) = T_j\).

A similar technique from [FG14, Theorem 1.1] will be used in the proof of the next theorem. But we will not directly apply [FG14, Theorem 1.1]. See also

Theorem 4.27. Let \(f : (X, B) \to Y\) be a dlt-trivial morphism and \(T\) be a stratum of the induced dlt pair \((Y, B_Y)\). Suppose \(S\) is a stratum of \((X, B)\) saturated over \(T\). Then,

1. \(f\big|_S : (S, B_S) \to T\) is a dlt-trivial morphism where \(K_S + B_S = (K_X + B)|_S\).
2. If we denote by \((T, B_T + M_T)\) the g-dlt generalised pair with the moduli b-divisor \(M|_T\) given by the adjunction formula \(K_T + B_T + M_T = (K_Y + B_Y + M_Y)|_T\), and we denote by \((T, C_T + N_T)\) the g-dlt generalised pair with the moduli b-divisor \(N\) given by the dlt-trivial morphism \(f\big|_S : (S, B_S) \to T\). Then, we have

\[ M|_T = N. \]

Proof. We will prove by induction on the dimension of \(X\). We divide the proof into several steps.

Step 1. If \(\dim X = 1\), then there is nothing to prove. Suppose \(\dim X = n\) and by induction we may assume Theorem 4.27 holds in dimension \(\leq n - 1\). By definition of \(S\), there is a filtration of strata in codimension one of \(T\):

\[ T = T_0 \subset T_1 \subset \ldots \subset T_m = Y, \]

and a filtration of strata in codimension one of \(S\):

\[ S = S_0 \subset S_1 \subset \ldots \subset S_n = X \]

such that for every \(j\), there is some \(j \leq i\) with \(f(S_i) = T_j\). Let \(Z\) be the image of \(S_{n-1}\) on \(Y\), \(K_{S_{n-1}} + B_{S_{n-1}} = (K_X + B)|_{S_{n-1}}\) and \((Z, \Delta_Z + P_Z)\) be the induced g-lc generalised pair with data \(P_Z\). Note that either \(Z = Y\) or \(Z = T_{m-1}\). If we suppose \(f|_{S_{n-1}} : (S_{n-1}, B_{S_{n-1}}) \to Z\) is a dlt-trivial fibration and we denote by \((T, \Delta_T + P_T)\) the g-dlt generalised pair with data \(P_T\) given by the adjunction formula \(K_T + \Delta_T + P_T = (K_Z + \Delta_Z + P_Z)|_T\) and denote by \(P\) the b-divisor represented by \(P_Z\), then \(P|_T\) is the
b-divisor represented by $P\eta$, and hence, by Theorem 4.27 in dimension $n - 1$, we have $f|_S : (S, B_S) \to T$ is also a dlt-trivial morphism and

$$P|_T = N.$$ 

Therefore it suffice to show that $f|_{S_{n-1}} : (S_{n-1}, B_{S_{n-1}}) \to Z$ is a dlt-trivial morphism and $P = M|_Z$. Replacing $S, T$ with $S_{n-1}, Z$, we may assume $S$ is a prime divisor on $X$ and $T = Y$ or $T$ is a prime divisor on $Y$.

We may assume further that $f$ has connected fibres. Indeed, let $f : X \to \tilde{Y} \to Y$ be the Stein factorisation which in turn induces a factorisation of $f|_S : S \to \tilde{T}_\nu$ where $\tilde{T}$ is the image of $S$ on $\tilde{Y}$ with is normalisation $\tilde{T}_\nu$. Since $\gamma|_{\tilde{T}_\nu}$ is finite and $f|_S$ is an lc-trivial morphism, by the rigidity lemma, we deduce that $\tilde{f}_S$ is an lc-trivial morphism. If we fix an $\mathbb{R}$-rational function $\varphi$, then by Remark 4.7(2) the moduli b-divisors $M, \tilde{M}$ on $Y, \tilde{Y}$ are uniquely determined. Since $\varphi$ can be chosen sufficiently general, we may assume $\varphi|_S$ is well-defined, and $M, \tilde{N}$ on $T, \tilde{T}_\nu$ are uniquely determined. By construction we have $N = \frac{1}{\deg T} \gamma_{T, \ast} \tilde{N}$ where the notation $\gamma_T := \gamma|_{\tilde{T}_\nu}$. Thus, by Lemma 4.9, it suffices to prove $M|_{\tilde{T}_\nu} = \tilde{N}$ and $\tilde{f}_S$ is good.

Thanks to Proposition 4.22 there exist a birational model $\phi : \tilde{Y}' \to \tilde{Y}$ and a log birational model $(X', B')$ of $(X, B)$ with a $B$-birational contraction $\pi : (X', B') \dashrightarrow (X, B)$ such that the induced map $f' : (X', B') \to \tilde{Y}'$ is a $\mathbb{Q}$-factorial dlt-trivial fibration. Let $\tilde{T}', S'$ be the birational transforms of $T, S$, and $\tilde{M}', \tilde{N}'$ be the moduli b-divisors given by $f', f'|_{S'}$ respectively. Because the restricted map $(S', B_{S'}) \dashrightarrow (S, B_S)$ is $B$-birational where $B_{S'}$ is given by $K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}$, then we have $\tilde{M} = \tilde{M}', \tilde{N} = \tilde{N}'$. Replacing $X, Y, f$ with $X', \tilde{Y}', \tilde{f}'$ and the rest accordingly we may assume $f$ is a contraction.

**Step 2.** In this step we will modify $X, Y$ and $f$ so that one can alter the restricted morphism $f|_S : S \to T$ to a contraction. Indeed, let $f|_S : S \to \tilde{T}$ be the Stein factorisation. Since $\zeta$ is finite, by Theorem 3.1, there exists a finite morphism $\rho : \tilde{Y} \to Y$ together with a prime divisor $\tilde{T}$ such that $\nu : \tilde{T} \to \tilde{T}$ is the normalisation.

By abuse of notation, we will freely use the notations $T_i, S_i$ and $Z$ afterwards. Now write $Y_1 = \tilde{Y}, T_1 = \tilde{T}$. Let $f_1 : (X_1, B_1) \to Y_1$ be the lc-trivial fibration induced by the finite base change $\rho$ and let $\varphi : X_1 \to X$ be the induced morphism. By a diagram chase, given any main component $S_1$ of the normalisation of $S \times_T T_1$, the following diagram
commutes where the notations $g_{S_1} := g|_{S_1}$, $f_{S_1} := f|_{S_1}$ and $\rho_{T_1} := \rho|_{T_1} = \zeta$.

Since $\nu : T_1 \to \hat{T}$ is the normalisation, there is a main component $S_1$ such that the induced morphism $f_{S_1}$ is a fibration and $f_{S_1} = \hat{f}_S$.

Let $f_1 : (X_1, B_1) \to Y_1, f_{S_1} : (S_1, B_{S_1}) \to T_1$ be the induced lc-trivial fibrations of $f, f|_S$ under base changes $\rho, \zeta$, and let $M_1, N_1$ be the moduli b-divisors defined by $f_1, f_{S_1}$ respectively. According to Remark 4.12, one infers that $M_1 = \rho^* M$ and $\frac{1}{\deg \zeta} \zeta_1 N_1 = N$. So, by Lemma 4.9, it is enough to prove $M_1|_{T_1} = N_1$ which implies the goodness of $f|_S$.

**Step 3.** We continue modifying $X_1, Y_1$ and $f_1$ so that we will apply weak semi-stable reduction. Since $(Y_1, B_{Y_1} + M_{Y_1}), (T_1, B_{T_1} + M_{T_1})$ are g-sub-lc, there exist birational morphisms $\sigma : X'_1 \to X_1, \psi : Y'_1 \to Y_1$ satisfying the following conditions:

1. $(X'_1, B'_1)$ and $(Y'_1, B_{Y'_1} + M_{Y'_1})$ are log smooth where $K_{X'_1} + B'_1 = \sigma^*(K_{X_1} + B_1)$ and $K_{Y'_1} + B_{Y'_1} + M_{Y'_1} = \psi^*(K_{Y_1} + B_{Y_1} + M_{Y_1})$.
2. $M_{Y'_1}$ represents the moduli b-divisor $M_1$, in particular, $M_{Y'_1}|_{T'_1}$ represents the moduli b-divisor $M_1|_{T_1}$, where $T'_1$ is the birational transform of $T_1$.
3. $N_{T'_1}$ represents the moduli b-divisor $N_1$, where $N_{T'_1} = N_{1,T'_1}$.

Replacing $X_1, Y_1$ and $f_1$ with $X'_1, Y'_1$ and $f'_1$, we may assume $f_1 : (X_1, B_1) \to (Y_1, B_{Y_1} + M_{Y_1})$ is a contraction of quasi-projective sub-dlt pairs satisfying the conditions (1)-(3).

Now let $Z_Y \subset Y_1$ be a proper closed subset such that $(X_1, B_1)$ is log smooth over $Y_1 \setminus Z_Y$ and $Z = f_1^{-1}(\text{Supp}(B_{Y_1} + M_{Y_1}) \cup Z_Y) \cup \text{Supp} B_1$ and apply Theorem 2.32 and 2.33 to construct a commutative diagram

\[
\begin{array}{ccc}
(X_2, \Delta_2) & \xrightarrow{\mu} & X_1 \\
\downarrow f_2 & & \downarrow f_1 \\
(Y_2, \Delta_{Y_2}) & \xrightarrow{\eta} & Y_1
\end{array}
\]

which consists of a $\mathbb{Q}$-factorial klt quasi-projective variety $X_2$ with a reduced divisor $\Delta_2$ whose components are normal, a smooth quasi-projective variety $Y_2$ with a snc divisor $\Delta_{Y_2}$, a projective morphism $f_2 : X_2 \to Y_2$ and generically finite projective surjective morphisms $\mu : X_2 \to X_1, \eta : Y_2 \to Y_1$ and which satisfy the following conditions:

(i). $(X_2, \Delta_2)$ is lc.

(ii). $\mu^{-1}Z \subseteq \Delta_2$ is a reduced divisor, $\Delta_2'' = f_2^{-1} \Delta_{Y_2}$ where $\Delta''_2$ denotes the vertical/Y/2 part of $\Delta_2$. 

(iii). All the fibers of $f_2$ are equidimensional and reduced.

There exist a g-sub-lc centre $T_2 \subseteq B_{Y_2}^{-1}$ and a sub-lc centre $S_2 \subseteq B_2^{-1}$ dominant over $T_2$ such that $\sigma|_{T_2} : T_2 \to T_1, \mu|_{S_2} : S_2 \to S_1$ are generically finite projective surjective since $S_1$ is a prime divisor and $T_1$ is a prime divisor or the whole variety $Y_1$. Moreover, the induced morphism $f_2|_{S_2} : (S_2, B_{S_2}) \to T_2$ is an lc-trivial fibration where $B_{S_2}$ is defined by $K_{S_2} + B_{S_2} = (K_{X_2} + B_2)|_{S_2}$. Consider the following commutative diagram where the notations $f_{S_2} := f_2|_{S_2}, \eta_{T_2} = \eta|_{T_2}$ and $\mu_{S_2} = \mu|_{S_2}$.

Moreover, by Lemma 2.28 ([Amb05, Proposition 3.1]), we see $M_2 = \eta^*M_1$ and $N_2 = (\eta_{T_2})^*N$ where $M_2$ and $N_2$ are the moduli b-divisors given by $f_2 : (X_2, B_2) \to Y_2$ and $f_{S_2} : (S_2, B_{S_2}) \to T_2$ respectively, and hence $M_2|_{T_2} = (\eta_{T_2})^*M|_{T_1}$. By conditions (2), (3), we deduce that $N_2$ is represented by $N_{T_2} = (\eta_{T_2})^*N_{T_1}$ and $M_2$ is represented by $M_{Y_2} = \eta^*M_{Y_1}$. So it suffices to show that $N_{T_2} = M_{Y_2}|_{T_2}$.

**Step 4.** The final step is to establish $N_{T_2} = M_{Y_2}|_{T_2}$ which completes the whole proof. By Lemma 2.35, we have $\mu^{-1}\text{Supp} B_1 \cup \text{Ex}(\mu) \subseteq \Delta_2$ and $\eta^{-1}(\text{Supp}(B_{Y_1} + M_{Y_1})) \cup \text{Ex}(\eta) \subseteq \Delta_{Y_2}$. Hence, $(X_2, B_2)$ is lc with $B_2$ supported by $\Delta_2$, and $(Y_2, B_{Y_2} + M_{Y_2})$ is log smooth with $B_{Y_2} + M_{Y_2}$ supported by $\Delta_{Y_2}$. Because by construction we have

$$(B_2 + \sum_{P \in \Delta_{Y_2}} b_PF_2^*P)^>0 \leq \Delta_2,$$

where by the notation $P \in \Delta_{Y_2}$ we mean $P$ is a component of the reduced divisor $\Delta_{Y_2}$, we see $(X_2, (B_2 + \sum_{P \in \Delta_{Y_2}} b_PF_2^*P)^>0)$ is sub-lc. Now we pick a sufficiently small number $\epsilon > 0$, write

$$\Theta_2 := B_2 + \sum_{P \in \Delta_{Y_2}} b_PF_2^*P$$

and run a log MMP/$Y_2$ on

$$K_{X_2} + \Xi_2 := K_{X_2} + \Theta^>0 + \epsilon \sum E_i$$

where $E_i$s are all prime divisors on $X_2$ with $f_2(E_i) \in \Delta_{Y_2}$ and the multiplicities $\text{mult}_{E_i} \Theta^>0 < 1$. Because $-B_1^0|_{F_1} = N_{\sigma}((K_{X_1} + B_1^>0)|_{F_1})$ on a general fibre $F_1$ of $f_1$, there exists a horizontal/$Y_2$ divisor $D^h := -B_2^{h<0} \geq 0$ satisfying

$$D^h|_{F_2} = -B_2^{h<0}|_{F_2} = -B_2^0|_{F_2} = N_{\sigma}((K_{X_2} + \Xi_2)|_{F_2})$$
where $\mathcal{B}^{y}_{2}$ denotes the horizontal $/Y_2$ part of $\mathcal{B}_2$ and $F_2$ denotes a general fibre of $f_2$. Moreover, since $\epsilon$ is sufficiently small, we see $\Xi_2 \leq \Delta_2$ which in turn implies that $(X_2, \Xi_2)$ is lc. One can easily check that the vertical $/Y_2$ divisor

$$D^v := -\Theta^v; < 0 + \epsilon \sum_i E_i \geq 0$$

is very exceptional $/Y_2$. Now we have

$$K_{X_2} + \Xi_2 \sim_{\mathbb{R}} D/Y_2$$

where $D := D^h + D^v$. Therefore, by Lemma 2.16 the above MMP $/Y_2$ contracts all components of $D$ and terminates with a good minimal model $(X_3/Y_2, \Xi_3)$ where $\Xi_3$ is the birational transform of $\Xi_2$. Consider the commutative diagram where the notation $f_3$ denotes the induced morphism.

![Diagram](image)

Note that $\Xi_3 = B_3 + \sum_{P \in \Delta_{Y_2}} b_P f_3^* P$, and that the rational map given by the MMP $(X_2, \Theta_2) \dasharrow (X_3, \Xi_3)$ is B-birational. Also note that $K_X + B_3 \sim_{\mathbb{R}} 0/Y_2$, the pair $(X_3, \Xi_3)$ is $\mathbb{Q}$-factorial lc and the vertical $/Y_2$ part of $\Xi_3$ is a reduced divisor. In addition, since $f_2$ is weakly semi-stable, it holds that

$$\Xi_3^v = (B_3 + \sum_{P \in \Delta_{Y_2}} b_P f_3^* P)^v = \sum_{P \in \Delta_{Y_2}} f_3^* P = f_3^* \Delta_{Y_2}$$

where $\Xi_3^v$ denotes the vertical $/Y_2$ part of $\Xi_3$. Because $\text{mult}_{S_2} B_2 = 1$, the rational map $X_2 \dasharrow X_3$ does not contract $S_2$. Let $S_3$ be the birational transform of $S_2$ with its normalisation $v : S_3^v \rightarrow S_3$. Therefore, by the adjunction formula (for example, see [Kaw07]) one infers that

$$\Xi_3^v \geq (B_3 + \sum_{P \in \Delta_{Y_2}} b_P f_3^* P - S_3)|_{S_3^v} \geq \sum_{Q \in \Delta_{T_2}} f_{S_3^v} Q = f_{S_3}^* \Delta_{T_2}$$

where $f_{S_3} := (f_3 \circ v)|_{S_3^v} : S_3^v \rightarrow T_2$, the boundary divisor $\Xi_3^v$ is defined by $K_{S_3^v} + \Xi_3^v = (K_{X_3} + \Xi)|_{S_3^v}$ and $\Delta_{T_2} := (\Delta_{Y_2} - T_2)|_{T_2}$ is a snc divisor on $T_2$.

Writing $C_{T_2}$ for the discriminant divisor of $f_{S_2} : (S_2, B_{S_2}) \rightarrow T_2$, since $f_{S_2}$ has equidimensional and reduced fibres, by Lemma 4.24, we have $\text{Supp}C_{T_2} \subseteq \Delta_{T_2}$, hence $C_{T_2} = \sum_{Q \in \Delta_{T_2}} (1 - b_Q)Q$ where $b_Q$ is given by the lc threshold over the generic point of
Q. Since \((S_2, B_{S_2} + \sum_{Q \in \Delta_{T_2}} b'_Q(f_2|_{S_2})^*Q)\) is sub-lc, where \(b'_Q := b_P\) if \(Q\) is an irreducible component of \(P|_{T_2}\) for some \(P \in \Delta_{Y_2}\), we deduce \(b_Q \geq b'_Q\), hence \(C_{T_2} \leq (B_{Y_2} - T_2)|_{T_2}\).

On the other hand, writing \(K_{S_3} + B_{S_3} = (K_{X_3} + B_3)|_{S_3'}\), because
\[
\Xi_{S_3} = B_{S_3'} + \sum_{Q \in \Delta_{T_2}} b'_Q f_{S_3}^*Q,
\]
by \((\clubsuit)\) it in turn holds that
\[
B_{S_3'} \geq \sum_{Q \in \Delta_{T_2}} (1 - b'_Q) f_{S_3}^*Q.
\]

We therefore deduce \(b_Q \leq b'_Q\), hence \(C_{T_2} \geq (B_{Y_2} - T_2)|_{T_2}\). So we conclude \(C_{T_2} = B_{T_2} = (B_{Y_2} - T_2)|_{T_2}\). Because \(K_{T_2} + B_{T_2} + M_{Y_2}|_{T_2} = K_{T_2} + C_{T_2} + N_{T_2}\), we obtain the equation \(N_{T_2} = M_{Y_2}|_{T_2}\). 

\(\square\)

**Remark 4.28.** By the inductive argument and the previous theorem, it is easy to see that, for any stratum \(T\) of \((Y, B_Y)\), there exists a stratum \(S\) of \((X, B)\) saturated over \(T\).

**Proof of Theorem 4.23.** By Proposition 4.22, replacing \(f : (X, B) \to Y\) we may assume it is a dlt-trivial morphism. Let \((Y, B_Y + M_Y)\) be the induced g-dlt generalised pair. It suffices to show that, for every lc centre \(T\) of \((Y, B_Y)\), the moduli b-divisor \(M|_T\) is b-nef and abundant. Thanks to Theorem 4.27 and the above remark, there exists a stratum \(S\) of \((X, B)\) with \(f(S) = T\) such that, writing \(K_S + B_S = (K_X + B)|_S\), the morphism \(f|_S : (S, B_S) \to T\) is dlt-trivial and \(M|_T = N\) where \(N\) is the moduli b-divisor of \(f|_S\). So, it is sufficient to prove \(M\) is b-nef and abundant when \(S = X\) and \(T = Y\).

To this end, let \(W\) be a minimal horizontal/Y lc centre of \((X, B)\). Again by Theorem 4.27, writing \(K_W + B_W = (K_X + B)|_W\), the morphism \(f|_W : (W, B_W) \to Y\) is dlt-trivial and \(M = P\) where \(P\) is the moduli b-divisor of \(f|_W\). Replacing \((X, B)\) with \((W, B_W)\) we may assume \([B]\) has no horizontal/Y component. Finally, the result follows from Corollary 2.27 and Remark 4.10.

Since an lc-trivial fibration is a good lc-trivial morphism, we immediately obtain the corollary.

**Corollary 4.29.** Let \(f : X \to Y\) be a surjective morphism between normal complete varieties with connected fibres. Suppose that \((X, B)\) is an lc pair with \(K_X + B \sim_R 0/Y\). Then the moduli b-divisor \(M\) is b-nef and log abundant with respect to the induced generalised pair.

The following theorem is a relative version of Theorem 4.23.

**Theorem 4.30.** Let \(f : (X, B) \to Y\) be a good lc-trivial morphism from an lc pair. Suppose that \(Y\) is proper over a variety \(Z\). Then, the moduli b-divisor \(M\) is b-nef and log abundant over \(Z\) with respect to the induced generalised pair.
Proof. Since the definition of relative abundance (see Definition 2.3) only concerns a general fibre, the question is local. So, we may assume $Z$ is affine.

We will follow a standard procedure to compactify $(X,B)$ and $Y$ so that we may reduce the theorem to the projective case. First, let $Z \hookrightarrow Z^c, Y \hookrightarrow Y^c$ and $X \hookrightarrow X^c$ be open immersions to projective varieties such that the induced map $Y^c \dashrightarrow Z^c$ and $X^c \dashrightarrow Y^c$ are proper morphisms. Now take a resolution $\pi^c : \overline{X}^c \rightarrow X^c$ so that the induced morphism $\pi : \overline{X} \rightarrow X$ is a log resolution of $(X,B)$. Let $(\overline{X}, \overline{B})$ be a log smooth model of $(X,B)$, and $(\overline{X}', \overline{B}')$ be a log smooth dlt pair where $\overline{B}'$ is given by the closure of $B$ in $\overline{X}'$. Next, we run a log MMP/$Y^c$ on $K_{\overline{X}} + \overline{B}$ with scaling of some ample/$Y^c$ divisor. Because $K_{\overline{X}} + \overline{B} \sim_\mathbb{R} 0$ on $Y$, by [AdJ97, Theorem 3.4], the log MMP contracts $E$ after finitely many steps and we reach a model $\overline{X}'_1$ on which $K_{\overline{X}_1'} + \overline{B}_1' \sim_\mathbb{R} 0/Y$. Because the generic point of every lc centre of $(\overline{X}_1, \overline{B}_1)$ is mapped into $Y$, also noting that $K_{\overline{X}_1'} + \overline{B}_1'$ is a convex combination of $\mathbb{Q}$-divisors which are $\mathbb{R}$-linearly trivial over $Y$, by [Bir12, Theorem 1.4][HX13, Theorem 1.1] (see also [Ha19, Theorem 1.2] [Hu17, Theorem 3.2] for a generalised result for an arbitrary $\mathbb{R}$-boundary), the above log MMP terminates with a good minimal model $g_1^c : (X_1^c, B_1^c) \rightarrow Y_1^c/Y^c$.

Let $X_1 = X_1^c \times_{Z^c} Z$, $B_1 = B_1'|_{X_1}$ and $Y_1 = Y_1^c \times_{Z^c} Z$. Note that the induced map $(X_1,B_1) \dashrightarrow (X,B)$ is B-birational. Hence, they induce the same discriminant b-divisor $\mathcal{B}$ and moduli b-divisor $\mathcal{M}$ on $Y$. Since Definition 4.1 and 4.8 is local near the generic point, one can easily check that $f_1 : (X_1,B_1) \rightarrow Y$ is a good lc-trivial morphism. Writing $\mathcal{M}$ for the moduli b-divisor of $g_1^c$, since $\gamma : Y_1^c \rightarrow Y^c$ is generically finite, we see $\mathcal{M}|_{Y_1} = \gamma^* \mathcal{M}$. By Theorem 4.23, the moduli b-divisor $\mathcal{M}$ is b-nef and log abundant with respect to the induced generalised pair. So, by Lemma 2.9, the moduli b-divisor $\mathcal{M}$ is b-nef and log abundant over $Z^c$. Since the definition of relative abundance only concerns a general fibre, $\mathcal{M}|_{Y_1} = \mathcal{M}$ is b-nef and log abundant over $Z$. Finally, by Remarks 4.10 and 4.16, $\mathcal{M}$ is b-nef and log abundant over $Z$ with respect to the induced generalised pair.

As an immediate consequence, we obtain a relative version of Corollary 4.29.

Corollary 4.31. Let $f : X \rightarrow Y$ be a proper surjective morphism between normal varieties with connected fibres. Suppose that $(X,B)$ is an lc pair with $K_X + B \sim_\mathbb{R} 0/Y$. Suppose further that $Y$ is proper over a variety $Z$. Then the moduli b-divisor $\mathcal{M}$ is b-nef and log abundant over $Z$ with respect to the induced generalised pair.

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