One vertex spin-foams with the dipole cosmology boundary

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Abstract

We find all the spin-foams contributing in the first order of the vertex expansion to the transition amplitude of the Bianchi–Rovelli–Vidotto dipole cosmology model. Our algorithm is general and provides spin-foams of arbitrarily given, fixed boundary and a number of internal vertices. We use the recently introduced operator spin-network diagram framework.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Spin-foams are quantum histories of states of the gravitational field according to the spin-foam models of quantum gravity. In the usual formulation, a spin-foam is a 2-complex, whose faces are colored with representations of a given group (depending on a model, e.g., SU(2)) and edges are colored with invariants of the tensor products [1–4], or equivalently with operators if one uses the operator spin-foam framework [5]. The spin-foams encode the data necessary to calculate the transition amplitude between states of loop quantum gravity [4, 6–10], or more generally, the Rovelli boundary transition amplitude [4]. There are a few candidates for the spin-foam models of quantum gravity [11–17, 19]. In their original formulation, they provide discrete path integrals for gravity. In those cases, the 2-complexes are simplicial, i.e. they are the 2-skeletons of complexes dual to a simplicial decomposition of spacetime, and their boundaries are simplicial graphs, i.e. 1-skeletons of complexes dual to a simplicial decompositions of space. In order to provide transition amplitudes between the loop quantum gravity states, the models need to be extended. Important for the compatibility with loop quantum gravity is to admit a sufficiently general class of 2-complexes, such that all the (closed, either abstract or embedded in a 3-manifold) graphs are obtained as their boundaries.
The Engle–Pereira–Rovelli–Livine (EPRL) model was successfully extended to a class of linear 2-complexes [20] (by introducing a so-called extended EPRL vertex) and to a class of combinatorial 2-complexes [21]. In [24], we proposed another possible extension to a class of 2-complexes naturally described in terms of a diagrammatic formalism introduced therein and called operator spin-network diagrams (OSN diagrams). A similar diagrammatic framework for simplicial 2-complexes was introduced before in [23]. An additional advantage of our formalism is that the OSN diagrams do not require either 3D or 4D imagination, they are easy to use and to classify possible spin-foams. We utilize and even improve these technical advantages in the current work.

The extended EPRL vertex was applied by Bianchi, Rovelli and Vidotto to introduce dipole cosmology, a quantum cosmological model which opens a new theory that can be called spin-foam cosmology [25–27]. This application of spin-foams in cosmology gave us the motivation to do the current research. In the original formulation (which we briefly recall in section 1.1), a specific 2-complex was used (we will call it a BRV foam), and its introduction was judged a posteriori by a correct semiclassical limit of the transition amplitude. The 2-complex chosen has one internal vertex, and the calculation of a transition amplitude may be thought to be analogous to first-order calculations in the perturbation approach to quantum field theory. Using the analogy with quantum field theory, the choice of the BRV spin-foam may be interpreted as a choice of a specific interaction vertex. The class of complexes defined by OSN diagrams includes the BRV vertex. However, it also includes a variety of other possible interaction vertices which a priori can not be discarded. A natural question arises, whether all of them are physical. A first check is to calculate a semiclassical limit of a transition amplitude in first order of vertex expansion including contributions from all the vertices. We expect that the transition amplitude in the semiclassical limit is dominated by the BRV transition amplitude, and therefore, a spin-foam cosmology model based on the class of complexes defined by OSN diagrams has a proper semiclassical limit. Let us note that contributions from certain 2-complexes with one internal vertex and the boundary of dipole cosmology were investigated before in [27]. The class of 2-complexes that we consider includes all the 2-complexes from [27] as well as many others.

In this paper, we report on our results of testing the hypothesis stated in the paragraph above. First problem that we solve is to find all possible 2-complexes with a given boundary. We solve this problem in section 3 with the use of the squid sets that we introduce in section 2. The solution that we present in that section is not limited to the dipole cosmology model only. Actually, it applies to the general spin-foam case. The problem that we solved arises naturally in the spin-foam theory, when one calculates contributions from all spin-foams to a transition amplitude between given initial and final states (which is a problem analogous to the scattering problem in quantum field theory).

The algorithm that we present controls the order of vertex expansion; in particular, it may be applied to first-order calculations in dipole cosmology. This is the second problem that we solve. In section 4, we apply the general scheme to the model of dipole cosmology and find all the spin-foams in the class defined by OSN diagrams whose boundary is the boundary graph of dipole cosmology and which have exactly one internal vertex and have no edges connecting this vertex with itself. Those spin-foams contribute to the transition amplitude in the first order of vertex expansion. We further show that a large class of contributions is indeed dominated by the BRV transition amplitude in the approximation of large universe.

4 Strictly speaking in [21], a generalized EPRL model is introduced, and a special case of this model is the extension mentioned.
In order to make the paper more self-contained, in the next two subsections we briefly recall the construction of Bianchi–Rovelli–Vidotto spin-foam cosmology model [25] and the definition of an OSN diagram [24].

1.1. Dipole cosmology: an overview

The idea of the Bianchi–Rovelli–Vidotto model of spin-foam cosmology [25] is to calculate the transition amplitude of the extended EPRL model between coherent states peaked on homogeneous, isotropic geometries. The amplitude is calculated under certain approximations and it is shown that in a classical limit the vacuum Friedmann dynamics is recovered.

First approximation is choosing states supported on a graph $|\Gamma_0|$ which has two nodes connected with four links (figure 1(a)). The graph is called a dipole graph [25]. This approximation is viewed as a certain cutoff of the inhomogeneous degrees of freedom of a metric. The graph has a topological interpretation: the two nodes correspond to two tetrahedra, and the four links correspond to their faces (triangles). Two faces corresponding to a same link are identified. The resulting picture is therefore that of a 3-sphere triangulated with two tetrahedra whose boundaries are identified. An approximation of the universe with two tetrahedra may seem to be crude; however, the Regge-type calculations show that even few tetrahedra may approximate the continuous FRW universe [28]. The geometry is encoded in coherent states (in and out). Each state is of the heat-kernel type:

$$\Psi_{H_\ell}(U_\ell) = \int dg_n \prod_\ell K_\ell(s_\ell(t),U_\ell g_{s_\ell(t)} H_\ell^{-1}),$$

(1)

where $K_\ell$ is an analytic continuation to $SL(2,\mathbb{C})$ of the heat kernel on $SU(2)$, $t$ is a spread of the heat kernel, $n$ runs through the set of nodes of a dipole graph, $\ell$ runs through the set of links of a dipole graph, $s(\ell)$ denotes a source of a link $\ell$ and $t(\ell)$ denotes a target of a link $\ell$. The integration is over two copies of $SU(2)$ group and implements a projection onto gauge-invariant states. The state is peaked at a classical geometry encoded in $SL(2,\mathbb{C})$ matrices $H_\ell$. The homogeneity and isotropy of the geometry is reflected by a special form of the matrices:

$$H_\ell(z_{\text{in/out}}) = u_\ell e^{-iz_{\text{in/out}}/u_\ell} u_\ell^{-1},$$

(2)
where \( u_\ell \in SU(2), z_{in/out} \in \mathbb{C} \) and \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices. Each matrix \( u_\ell \) gives rise to a normalized vector \( \vec{u}_\ell \in \mathbb{R}^3 \) defined by \( \vec{u}_\ell \cdot \vec{u}_\ell = u_\ell \sigma_3 u_\ell^{-1} \). The vectors \( \vec{u}_\ell \) are interpreted as vectors normal to the corresponding triangles bounding the tetrahedron which is associated with a source (target) of the link \( \ell \). The real and imaginary parts of the complex numbers \( z_{in/out} \) are interpreted as the standard scale factor \( a \) of cosmology and its time derivative \( \dot{a} \), i.e. the real part of \( z_{in/out} \) is proportional to \( \dot{a}_{in/out} \) and the imaginary part is proportional to \( a_{in/out} \), \( \Re(z_{in/out}) \sim \dot{a}_{in/out}, \Im(z_{in/out}) \sim a_{in/out} \).

Second approximation is that of large universe, and third approximation is the first order of vertex expansion. The 2-complex used in the BRV model (in the first order of vertex expansion) is an image of a homotopy of two dipole graphs (initial and final) into a point (figure 1(b)). The corresponding transition amplitude is a function of two complex variables \( W(z_{in}, z_{out}) \). In the approximation of large universe, it satisfies a quantum constraint

\[
\dot{H} W(z, z') = 0.
\]  

The corresponding classical constraint turns out to be the Friedmann Hamiltonian constraint.

Our expectation is that the transition amplitude \( \tilde{W}(z, z') \) including contributions from the spin-foams that we find in this paper (in the first order of vertex expansion) converges in a limit of large universe to \( W(z, z') \), and therefore, the full transition function in the approximation of large universe still satisfies the Hamiltonian constraint: \( \dot{H} \tilde{W}(z, z') = 0 \).

1.2. Definition of OSN-diagrams

In this subsection, we recall the definition of OSN diagrams that we introduced in [24]. In the analogy to spin-foams, which are colored 2-complexes, OSN diagrams are colored graph diagrams. We first recall the definition of graph diagrams and then we recall the definition of coloring which turns a graph diagram into an OSN diagram.

One may think of graph diagrams as a way of building 2-complexes from building blocks which are (suitable) neighborhoods of vertices of the corresponding foam [1, 20]. These neighborhoods are obtained as the images of a homotopy of a graph. When glued together, they form a 2-complex. The way one glues them together is encoded in certain relations.

Strictly speaking, a graph diagram \( (G, \mathcal{R}) \) consists of a set \( G \) of oriented, connected, closed graphs \( \{\Gamma_1, \ldots, \Gamma_N\} \) and a family \( \mathcal{R} \) of relations defined as follows (see figure 2).

- \( \mathcal{R}_{node} \): a symmetric relation in the set of nodes of the graphs which we call the node relation, such that each node \( n \) is either in relation with precisely one \( n' \neq n \) or is unrelated (in the latter case, it is called a boundary node).
- \( \mathcal{R}_{link} \): a family of symmetric relations in the set of links of the graphs which we call collectively the link relation. If a node \( n \) is in relation with a node \( n' \), then we define a bijective map between incoming/outgoing links at \( n \), with outgoing/incoming links at \( n' \); no link is left free neither at the node \( n \) nor at \( n' \); two links identified with each other by the bijection are called to be in the relation \( \mathcal{R}_{link}^{n,n'} \) at the pair of nodes \( n, n' \); a link which intersects \( nh \) twice emerges in the relation twice: once as an incoming link and once as an outgoing link.

In order to be related, two nodes have to satisfy the consistency condition: the number of the incoming/outgoing links at each of them has to coincide with the number of the outgoing/incoming links at the other one (with possible closed links counted twice). Note that two graphs can be treated as one disconnected graph. Thus, to reduce that ambiguity, we assume that all the graphs defining the diagram are connected.

An OSN diagram \( (G = \{\Gamma_1, \ldots, \Gamma_N\}, \mathcal{R}, \rho, P, A) \) is defined by using a compact group \( G \) and coloring a graph diagram \( (G, \mathcal{R}) \) as follows (see figure 2).
The coloring $\rho$ assigns to each link $\ell$ of each graph $\Gamma_I$, $I = 1, \ldots, N$, an irreducible representation of the group $G$:

$$\ell \mapsto \rho_\ell.$$  \hfill (4)

It is assumed that whenever two links $\ell$ and $\ell'$ are mapped to each other by the link relation $\mathcal{R}_{\text{link}}$ at some nodes $n$ and $n'$, then

$$\rho_\ell = \rho_{\ell'}.$$  \hfill (5)

The coloring $P$ assigns to each node $n$ an operator:

$$n \mapsto P_n \in \mathcal{H}_n \otimes \mathcal{H}_n^*.$$  \hfill (6)

Here, $\mathcal{H}_n$ is a Hilbert space defined at each node in the following way:

$$\mathcal{H}_n = \text{Inv} \left( \bigotimes_i \mathcal{H}_{\rho_i}^n \otimes \bigotimes_j \mathcal{H}_{\rho_j} \right) \subset \left( \bigotimes_i \mathcal{H}_{\rho_i}^* \otimes \bigotimes_j \mathcal{H}_{\rho_j} \right)$$  \hfill (7)

where $i/j$ labels the links incoming/outgoing at $n$.

Whenever two nodes $n$ and $n'$ are related by $\mathcal{R}_{\text{node}}$, then (from (5) and (7)) it follows that $\mathcal{H}_n = \mathcal{H}_{n'}^*$ and it is assumed about $P$ that

$$P_n = P_{n'}^*.$$  \hfill (8)

The coloring $A$ assigns to each graph $\Gamma_I$ a tensor, which we call contractor:

$$\Gamma_I \mapsto A_{\Gamma_I} \in \left( \bigotimes_n \mathcal{H}_n \right)^*,$$  \hfill (9)

where $n$ runs through the nodes of $\Gamma_I$.

Each graph $\Gamma_I$ itself defines a contractor, in the sense that there is a natural contraction defined by the graph $\Gamma_I$ and the natural trace operation in $\bigotimes_\ell \mathcal{H}_\ell \otimes \mathcal{H}_\ell^*$ which contains $\bigotimes_n \mathcal{H}_n$, where $n/\ell$ ranges the set of nodes/links of $\Gamma_I$. We denote this natural contractor by $A_{\Gamma_I}^\text{natur}$. However, the natural contraction is often preceded by some additional operations, like the EPRL embedding which gives rise to the EPRL OSN diagrams.
Figure 3. A graph (on the lhs) and its oriented squid set (on the rhs) which consists of four three-leg squids.

Figure 4. (a) The oriented squid of $k^- = 2$ incoming, and $k^+ = 4$ outgoing legs. (b) An example of an oriented squid set.

2. Characterization and construction of the diagrams

We will introduce now a useful characterization of the diagrams. The characterization will allow us to control the structure and the complexity of the diagrams in a clear way. Finally, it will lead to algorithms for construction of diagrams. In this section, we will introduce the first, simpler algorithm. It will be improved to produce diagrams of a given boundary, in the next section.

2.1. Squids

An important element of our characterization of graph diagrams is an oriented squid set. Given a graph (oriented and closed), its oriented squid set is obtained by removing from each link of the graph a point of its entry and shaking the whole thing so that the disconnected parts of each link go apart (figure 3).

Two different graphs may define a single squid set. Therefore, it makes sense to introduce and consider the notion of a squid set on its own. An oriented squid consist of one point called head, $k^+$ legs beginning at the head and $k^-$ legs ending at the head. In other words, it may be considered as the topological space obtained by glueing $k^+ + k^-$ oriented intervals with the head in the suitable way (figure 4(a)). An oriented squid set $S$ is a disjoint finite union of squids (figure 4(b)).
Figure 5. (a) An oriented squid set (lhs) and a unique oriented graph obtained by glueing the legs (rhs). (b) An oriented squid set whose legs can not be glued to give an oriented graph. (c) An oriented squid set (lhs) and three different oriented graphs (rhs), each of which can be obtained by glueing the legs in a suitable way.

A graph $\Gamma$ consists of a squid set $S_\Gamma$ and information about glueing the legs of the squids. Conversely, given a squid set $S$, a graph may be obtained by glueing the end of each outgoing leg with the beginning of arbitrarily chosen incoming leg of either the same or a different squid. However, this procedure neither is unique, nor there is a guarantee that it can be completed. Therefore, a single squid set can be the squid set of either more than one graph, or of an exactly one graph, or of no graph at all (figure 5).

A graph diagram $(\Gamma, R)$ can also be obtained from a squid set $S_\Gamma$ by endowing it with additional structure. The part of the structure has already been described above; it allows us to reconstruct the graph $\Gamma$. The second element of the structure, the node relation $R_{\text{node}}$, is given by indicating a set of pairs $\{\{\lambda_1, \lambda'_1\}, \ldots, \{\lambda_N, \lambda'_N\}\}$ of squids $\lambda_i, \lambda'_i \in S$, whose heads are in the node relation $R_{\text{node}}$. The last element of the graph diagram structure, a link relation, is defined for each pair $\{\lambda_i, \lambda'_i\}$ as a bijection between the incoming/outgoing legs of the squid $\lambda_i$ with the outgoing/incoming legs of the squid $\lambda'_i$ (figure 6).

2.2. The algorithm

This characterization of an arbitrary graph diagram leads to the following algorithm for construction of all the graph diagrams.

(i) Squids: fix a squid set $S$ (figure 7).
(ii) Node relation: choose $N$ pairs $\lambda_i, \lambda'_i$, $i = 1, \ldots, N$, of consistent squids (the numbers of incoming/outgoing legs of unprimed squid in each pair coincides with the numbers of the outgoing/incoming legs in the primed squid); an element $\lambda \in S$ can emerge only once in this set of pairs or not at all (figure 8). The relation $R_{\text{node}}$ is such that the heads of the chosen squids $\lambda_i$ and $\lambda'_i$, $i = 1, \ldots, N$ are related to each other, and to nothing else.
(iii) Link relation: for each pair $\lambda_i, \lambda'_i$ of the squids, $i = 1, \ldots, N$, define a bijective map carrying the incoming/outgoing legs on one squid into the outgoing/incoming legs of the other squid.
Figure 6. The construction of a graph diagram from a squid set: (i) the solid curves are squid legs meeting at squid heads, (ii) the dashed curves connecting the ends of the squid legs define glueing the legs into a (set of connected) graph(s), (iii) the dashed curves connecting the squid heads define a node relation and (iv) the dotted curves define a link relation at each pair of related nodes.

Figure 7. An oriented squid set $\mathcal{S}$.

Figure 8. Three pairs of squids have been chosen to be related by a node relation.

(iv) Glueing: glue the end of each outgoing leg with the beginning of exactly one incoming leg of either the same or a different squid (figure 9).

(v) Given the data (i)–(iii) perform all possible options of the glueing (iv)—see figure 10.

A graph diagram $\mathcal{D}$ resulting from (i)–(iv) consists of a graph

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N,$$

the disjoint union of connected graphs $\Gamma_I$, $I = 1, \ldots, N$, obtained by the glueing (iv), a node relation given by (ii) and a link relation provided by (iii). We denote it briefly by
Figure 9. One of the graph diagrams that can be constructed from $\mathcal{S}$ (the link relation is omitted).

Figure 10. Three different graph diagrams obtained by different choices of link relation in figure 9. In total, there are eight different graph diagrams for this choice of a node relation and many more for arbitrary choice of a node relation.

$\mathcal{D} = (\{\Gamma_1, \ldots, \Gamma_N\}, \mathcal{R})$. It is turned into an OSN diagram by coloring the links, the nodes and the connected components of $\Gamma$ according to section 1.2.

2.3. Discussion

2.3.1. Colorings. The colorings may be constrained by the geometry of the graph diagram and the relations. In order to control this constraint, one may from the beginning fix the coloring of the boundary links (i.e. the legs of the squids whose heads are boundary nodes—the nodes which are unrelated by the node relation) by representations, and then allow only such gluings that agree with the coloring (i.e. two legs may be glued if and only if they are colored by the same representation).

2.3.2. Reorientation of the diagrams. Graphs used in the definition of OSN diagram are oriented, and the orientation is relevant for the node and link relations. However, the operators evaluated from the OSN diagram are invariant with respect to consistent changes of the orientation accompanied with the dualization of the representation colors.
Suppose \( (\Gamma'_1, \ldots, \Gamma'_N, \mathcal{R}', \rho', P', A') \) is an OSN-diagram obtained from a given OSN diagram \( (\Gamma_1, \ldots, \Gamma_N, \mathcal{R}, \rho, P, A) \) by flipping the orientation of some of the links \( \ell_1, \ldots, \ell_k \), leaving the same node relations and leaving the same link relations, setting

\[
\rho'_{\ell_{i-1} \ell_i} = \rho^*_{\ell_i \ell_{i+1}}, \quad i = 1, \ldots, k, \tag{10}
\]

leaving

\[
\rho'_{\ell} = \rho_{\ell} \quad \text{for each unflipped link } \ell, \quad \text{and } P' = P \text{ as well as } A' = A.
\]

Note that the transformation of the orientations and the labeling \( \rho \mapsto \rho' \) preserve the Hilbert spaces \( \mathcal{H}_n \), where \( n \) ranges the set of the nodes of the diagram graph. This property makes the choice \( P' = P \) and \( A' = A \) possible. The OSN diagram \( (\Gamma'_1, \ldots, \Gamma'_N, \mathcal{R}', \rho', P', A') \) can be considered as a reoriented OSN diagram \( (\Gamma_1, \ldots, \Gamma_N, \mathcal{R}, \rho, P, A) \). The reorientation of any OSN diagram does not change the Hilbert spaces and the resulting operator.

2.3.3. Advantages and a drawback. The characterization and construction presented above allows us to control the complexity of the diagrams by the following measures: the number of internal edges, the number of disconnected components of the graphs and the complexity of each disconnected component.

However, there is one drawback. From the point of view of the physical application, the boundary part of the diagram (consisting of the squids whose heads are not in the node relation with any other head—they form the boundary graph) describes either the initial and final states or, more generally, the surface state. The remaining part of the diagram consists of the pairs of the related nodes (internal edges) and describes the interaction. The two pieces of this information are entangled in the presented characterization. The reason is that the diagrams obtained with the algorithm mentioned above from a given squid set endowed with the node and link relations (the data (i)–(iii)) by implementing all the possible glueings (step (iv)), in general, have different boundary graphs (the graphs share a squid set, but have different graph structures). Therefore, we do not control in that way the boundary of the diagram, i.e. the initial–final/boundary Hilbert space. We improve the characterization and the algorithm in the next section.

3. OSN diagrams with fixed boundary

In this section, we improve the construction and the algorithm introduced in the previous section. The improved algorithm provides all the OSN diagrams which have a same arbitrarily fixed boundary graph. Finally, we will provide the algorithm to calculate the EPRL transition amplitude for a fixed boundary state being of the Livine–Speziale (LS) form.
3.1. The idea and the trick

In [24, see section 6.2], for every graph \( \Gamma \), we introduced the graph diagram corresponding to the static spin-foam, i.e. the spin-foam describing the trivial evolution of the graph. Let us denote this diagram by \( \mathcal{D}_\Gamma \) and refer to it as the static graph diagram of \( \Gamma \). The boundary of \( \mathcal{D}_\Gamma \) is the disjoint union \( \partial \mathcal{D}_\Gamma = \Gamma \cup \bar{\Gamma} \), where \( \bar{\Gamma} \) is a graph obtained by switching the orientation of each of the links of \( \Gamma \) (figure 11).

Given a coloring of the links of the graph \( \Gamma \) by representations, we endow the static graph diagram \( \mathcal{D}_\Gamma \) with the natural colorings (see [24, section 6.2]: (i) the coloring \( \rho \) of the links of the graphs in \( \mathcal{D}_\Gamma \) is the one induced by the coloring of the links of \( \Gamma \); (ii) the operator coloring \( P \) colors with the identity operators; (iii) finally, all the contractor coloring \( A \) assigns to each \( n\theta \) graph of \( \mathcal{D}_\Gamma \) the natural trace contractor \( A^{Tr} \). The result is the static OSN diagram. The corresponding operator is the identity in the Hilbert space given by the coloring of the links of \( \Gamma \). The technical definition of static diagram will be recalled in the next subsection.

The diagrams that we will construct with the improved algorithm introduced below contain the boundary \( \Gamma \) together with its static diagram \( \mathcal{D}_\Gamma \) as a subdiagram. Given \( \Gamma \), our construction will provide all the diagrams of this type, i.e. all the diagrams, modulo the static subdiagram. In terms of the spin-foam formalism, we construct all the spin-foams bounded by an arbitrarily fixed graph \( \Gamma \). In each of the spin-foams, the neighborhood of \( \Gamma \) is homeomorphic to the cylinder \( \Gamma \times [0, 1] \).

The key trick behind our construction is the following observation: given a graph \( \Gamma \) of the squid set \( S_\Gamma \), and an arbitrary graph diagram \( \mathcal{D}_{\text{int}} \) called interaction diagram, whose boundary graph \( \partial \mathcal{D}_{\text{int}} \) has the same squid set

\[
S_{\partial \mathcal{D}_{\text{int}}} = S_\Gamma, \tag{12}
\]

we can combine the diagram \( \mathcal{D}_{\text{int}} \) with the static diagram \( \mathcal{D}_\Gamma \) into the new graph diagram \( \mathcal{D}_\Gamma \# \mathcal{D}_{\text{int}} \) such that the graph \( \Gamma \) becomes its boundary. We achieve that by defining the graph of \( \mathcal{D}_\Gamma \# \mathcal{D}_{\text{int}} \) to be the disjoint union of the graph of \( \mathcal{D}_\Gamma \) with the graph of \( \mathcal{D}_{\text{int}} \) and extending

Figure 11. A static diagram. (a) A given graph \( \Gamma \) consisting of two connected components. (b) The corresponding static graph diagram \( \mathcal{D}_\Gamma \) built of \( \theta \) graphs and suitably defined node and link relations. (c) The scheme of building a static foam from the diagram. (d) The boundary graph \( \partial \mathcal{D}_\Gamma \) of the resulting foam is the disjoint union of \( \Gamma \) and \( \bar{\Gamma} \).
the node and link relations of the component diagrams such that each squid of the boundary \( \partial D_{\text{int}} \) is related to the corresponding squid of \( \tilde{\Gamma} \) (a part of the boundary of \( D_{\Gamma} \))—see figure 12.

The identification of the squid sets \( S_{\Gamma} \) and \( \partial D \) is defined modulo symmetries of \( S_{\Gamma} \) (exchanging identical squids). In the consequence, the graphs \( \tilde{\Gamma} \) and \( \partial D_{\text{int}} \) may admit more than one way of relating their squid. In the algorithm below, the identification (a bijection) between the squid sets will be given, and the freedom will be in the glueing of the legs.

Below we implement this idea in detail, beginning with recalling the exact definition of the static diagrams.

### 3.2. Static diagrams

First, we recall the definition of the static diagram of a graph \( \Gamma' \). We use the squid set \( S_{\Gamma} \). For each squid \( \lambda \in S_{\Gamma} \), we introduce a \( \theta \)-like graph \( \tilde{\theta}_{\lambda} \) as follows (figure 13).

- \( \tilde{\lambda} \): we introduce a new squid \( \tilde{\lambda} \) obtained by flipping the orientation of each of the legs of \( \lambda \).
- \( \theta_{\lambda} \): glue each outgoing/incoming leg \( \ell_{\lambda,i} \) of \( \lambda \) with the corresponding incoming/outgoing leg \( \ell_{\bar{\lambda},i} \) of \( \tilde{\lambda} \) to obtain a link \( \ell_{\lambda,i} \circ \ell_{\bar{\lambda},i} \circ \ell_{\lambda,i} \) connecting the head \( n_{\lambda} \) of the squid \( \lambda \) with the head \( n_{\bar{\lambda}} \) of the squid \( \tilde{\lambda} \); the result is a closed graph \( \theta_{\lambda} \).
- \( \tilde{\theta}_{\lambda} \): on each link \( \tilde{\ell}_{\lambda,i} \circ \ell_{\bar{\lambda},i} \circ \ell_{\lambda,i} \), we introduce an extra node \( n_{\lambda,i} \); the resulting graph is denoted \( \tilde{\theta}_{\lambda} \).

The links of the graph \( \tilde{\theta}_{\lambda} \) are just the legs \( \ell_{\lambda,i}, \bar{\ell}_{\lambda,i}, i = 1, 2, ..., \) of the squids \( \lambda \) and \( \bar{\lambda} \), respectively.
The result of this procedure is a family of the graphs $\tilde{\theta}_\lambda$, one per each squid $\lambda \in S_\Gamma$. The disjoint union $\bigsqcup_\lambda \tilde{\theta}_\lambda$ is the graph of the static graph diagram $D_\Gamma$. The node and the link relations in $\bigsqcup_\lambda \tilde{\theta}_\lambda$ are induced by the structure of the graph $\Gamma$—see figure 14.

To define the node relation, note that the set of the nodes of the graph $\bigsqcup_\lambda \tilde{\theta}_\lambda$ consists of the nodes of the following two types.

- The first type is the heads of the squids $\lambda \in S_\Gamma$ (denoted by $n_\lambda$) and the heads of the conjugate squids $\bar{\lambda}$ (denoted by $n_{\bar{\lambda}}$, ...)—those nodes are left unrelated by the node relation, i.e. they become the boundary nodes.
- The ends of the legs of the squids $\lambda \in S_\Gamma$ (glued with the legs of the conjugate squids $\bar{\lambda}$) denoted by $n_{\ell_\lambda,i}$. Whenever $\ell_{\lambda,i} \circ \ell'_{\lambda',i'}$ is a link of $\Gamma$, then the nodes $n_{\ell_{\lambda,i}}$ and $n_{\ell'_{\lambda',i'}}$ are in the node relation.

The link relation is defined as follows.

- For every pair of the nodes $n_{\ell_{\lambda,i}}$ and $n_{\ell'_{\lambda',i'}}$ related above by the node relation, each node is bivalent. The link relation is defined to pair the links $\ell_{\lambda,i}$ and $\ell'_{\lambda',i'}$, as well as the links $\ell_{\lambda,i}$ and $\ell'_{\lambda',i'}$ (see the previous item).

Given a static graph diagram $D_\Gamma$, the natural coloring consists of irreducible representations freely assigned to the links, the operators $P_{n_\lambda} = \text{id} : \mathcal{H}_{n_\lambda} \rightarrow \mathcal{H}_{n_\lambda}$ for every head $n_\lambda$ and every squid $\lambda$, and $P_{n_{\ell_{\lambda,i}}, n'_{\ell'_{\lambda',i'}}} = \text{id} : \mathcal{H}_{n_{\ell_{\lambda,i}}} \rightarrow \mathcal{H}_{n'_{\ell'_{\lambda',i'}}}$.

Finally, the contractor assigned to each component graph $\tilde{\theta}_\lambda$ is the natural $A^\Gamma_\theta$. 

**Figure 13.** The procedure of creating a $\theta$-like graph from a squid $\lambda$.

**Figure 14.** (a) A static diagram. (b) Its boundary graph.
3.3. The improved algorithm

We present now our algorithm for the construction of the OSN diagrams whose unoriented boundary is an arbitrarily fixed unoriented graph $|\Gamma|$.

(i) $\Gamma$: choose an orientation of each link of $|\Gamma|$, the result is an oriented graph $\Gamma$ (figure 15).

(ii) $S_{\text{int}}$: to the squid set $S_\Gamma$ of the graph $\Gamma$ add $N$ pairs of squids $\lambda_1, \lambda'_1, \ldots, \lambda_N, \lambda'_N$, such that for each pair the number of incoming/outgoing legs of one squid equals the number of outgoing/incoming legs of the other one (figure 16). Denote the resulting squid set $S_{\text{int}}$. Introduce a node relation by relating the head of $\lambda_i$ with the head of $\lambda'_i$, $i = 1, \ldots, N$.

(iii) $D_{\text{int}}$: to the squid set $S_{\text{int}}$ with the chosen set of pairs of the squids $\lambda_1, \lambda'_1, \ldots, \lambda_N, \lambda'_N$ and the node relation apply steps (iii) and (iv) of the algorithm of section 2.2. Denote the resulting graph diagram $D_{\text{int}}$ (figure 17).

(iv) $D_\Gamma \# D_{\text{int}}$: use the static graph diagram $D_\Gamma$ of the graph $\Gamma$ and construct the union of the diagrams as it was explained above (figure 18).

(v) Coloring: define arbitrary coloring of the diagram $D_\Gamma \# D_{\text{int}}$, which turns it into operator a spin-network diagram (figure 19).
Figure 17. Step (iii) of the improved algorithm: gluing of a graph diagram $D_{int}$ (one of several possible) from the squid set $S_{int}$ presented in figures 16(a) and (b), respectively. The dotted lines mark the gluing of the legs of the squids.

Figure 18. Step (iv) of the improved algorithm: the graph diagram $D_{int}$ of figures 17(a) and (b) combined with the static diagram $D_{\Gamma}$ of the graph $\Gamma$ of figure 15, into the final graph diagram $D_{\Gamma} \# D_{int}$.

(vi) Consider all possible: orientations of $[\Gamma]$, $N$-tuples of pairs of squids added to $S_{int}$, ways of connecting the legs of $S_{int}$, link relations for each $\lambda_i, \lambda'_i$, colorings in step (v).

Note that it would be insufficient to fix one orientation of the boundary. \textit{A priori} all the orientations have to be taken into account. On the other hand, in general, the algorithm will possibly give OSN diagrams related by the reorientation (see the previous section). In specific
cases, the redundancy should be reduced. The presented construction allows us to control the level of complexity of resulting diagram by the level of complexity of the diagram $D_{\text{int}}$. The complexity can be measured by the number of pairs of the nodes related by the node relation, i.e. the number of internal edges. The simplest case is zero internal edges, i.e. the squid set

$$S_{\text{int}} = S_\Gamma.$$  

In that case, all the graph used to define graph diagram $D_{\text{int}}$ becomes the boundary graph $\partial D_{\text{int}}$.

A general example of the interaction graph diagram $D_{\text{int}}$ is given by a graph $\Gamma_{\text{int}}$ and a node relation $R_{\text{node}}^{N_{\text{int}}}$ consisting of exactly $N$ pairs of nodes. Increasing the number $N$, we increase the complexity of the diagram $D_{\Gamma}$.$\#D_{\text{int}}$.

In the previous subsection, we have recalled the structure of colorings that turn graph diagrams into OSN diagrams. In the case of the graph diagrams constructed in this subsection, without lack of generality, it is sufficient to consider colorings of the diagrams $D_{\Gamma}$.$\#D_{\text{int}}$ provided by our algorithm, which reduced to the static graph diagram $D_\Gamma$ provides the static OSN diagram defined above. A way to control the freedom in the colorings by representations is to fix a coloring of the links of the boundary graph $\Gamma$.

### 3.4. EPRL amplitude

We will calculate the amplitudes for boundary states being the LS semicoherent states $|j_\ell, u_{\ell,i(t)}, u_{\ell,i(t)}\rangle$. They form an overcomplete basis [14], so it is enough to perform the calculation just for them.
In this subsection, the nodes of the interaction diagram $D_m$ will be denoted by the Roman numbers I, II, . . . to distinguish them from $D_\Gamma$ nodes, denoted by $n_i$.

### 3.4.1. LS semicoherent states

Let us recall the definition of LS semicoherent states. Consider a graph $\Gamma$ (we assume $\Gamma = \Gamma_\text{in} \sqcup \Gamma_\text{out}$). For each link $\ell \in \Gamma^{(1)}$, we pick a triple: a half-integer spin label $j_\ell$ and two unit vectors $\vec{u}_{s}(\ell), \vec{u}_{t}(\ell)$. Given a unit vector $\vec{u}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, one can define an $SU(2)$ element $u$ such that

$$u \sigma_3 u^{-1} = \vec{u} \cdot \vec{\sigma}$$

Thus, for each link $\ell$, we have a spin label $j_\ell$ and two $SU(2)$ elements $u_{s}(\ell), u_{t}(\ell)$. Let us introduce a shortcut notation for $SU(2)$ elements:

$$SU(2) \ni h \mapsto |h\rangle_j := D_j(h) |j\rangle = \sum_m D_j(h)_m |j, m\rangle$$

and

$$\langle h_j| := |h\rangle_j^\dagger.$$  \hspace{1cm} (16)

The non-gauge-invariant LS semicoherent states are defined thus as follows:

$$\psi_{\text{NGI}}^{j_\ell, u_{s}(\ell), u_{t}(\ell)}(U_\ell) = \prod_\ell \langle u_{s}(\ell)|_{j_\ell} U_\ell |u_{t}(\ell)\rangle_{j_\ell}$$ \hspace{1cm} (17)

and the gauge-invariant states are

$$\psi_{\text{GI}}^{j_\ell, u_{s}(\ell), u_{t}(\ell)}(U_\ell) = \int_{SU(2)^N} dh_n \prod_\ell \langle u_{s}(\ell)|_{j_\ell} h_n^{-1} U_\ell h_n |u_{t}(\ell)\rangle_{j_\ell}.$$ \hspace{1cm} (18)

In Dirac notation, we will denote the LS states as $\psi_{\text{GI}}^{j_\ell, u_{s}(\ell), u_{t}(\ell)} = \langle j_\ell, u_{s}(\ell), u_{t}(\ell)\rangle$.

### 3.4.2. The EPRL map

The algorithm that we present in the next subsection is valid for both the Euclidean EPRL model and the Lorentzian EPRL model (in the convention introduced in [19]). In the EPRL model, the contractor (or the vertex amplitude) is defined by the so-called (Lorentzian/Euclidean) EPRL map $Y$ [19].

The EPRL map is defined as follows: given a vector $|j, m\rangle$ of the spin $j$ representation of $SU(2)$, the map $Y$ acts as

$$Y : |j, m\rangle \mapsto |y, j, j\rangle : j, m\rangle$$ \hspace{1cm} (19)

in the Lorentzian case, and

$$Y : |j, m\rangle \mapsto C_{m_+ m_-}^m |j_+, m_+\rangle \otimes |j_-, m_-\rangle \quad \text{for} \quad j_\pm = \frac{|1 \pm y|}{2} j$$ \hspace{1cm} (20)

in the Euclidean case.

The EPRL contractor is built from generalized Wigner matrices of the elements of group $G$ (see [24]):

$$G \ni g \mapsto \tilde{D}^{(j)}(g)_m^n := \langle j, m\rangle Y^\dagger g Y |j, n\rangle.$$ \hspace{1cm} (21)
3.4.3. The EPRL–Feynman rules. Assume that each graph of $\mathcal{D}_{int}$ is either colored by the EPRL contractor or is the $\theta^T$ graph. Then, for the LS boundary state, one can easily write down the amplitude of the OSD using the following Feynman-like rules.

(i) Labeling (for example see figure 20(a)):

(a) Each link $\ell \in \theta_n$ at the boundary static diagram $\mathcal{D}_{\Gamma}$ label by $|u_{\ell,n}\rangle_j$—these are the external legs of the diagram. For this procedure, we neglect the fact that the links in the $\mathcal{D}_{\Gamma}$ are split into two parts and we treat them as one link.

(b) Each link $\ell'$, that is in the face relation with one of the boundary links $\ell$, labeled by the spin $j_{\ell'} = j_{\ell}$.

(c) Each link $\ell_f$, that is not in the face relation of the boundary, labeled by an arbitrary spin $j_f$—the same for all equivalence class $[\ell_f]$.

(d) Each node $I$ of each graph colored by $A^{\text{EPRL}}$ label by a $G$ element $g_I$ (where $G = SL(2, \mathbb{C})$ in the Lorentzian case and $G = SO(4)$ in the Euclidean case).

(ii) The terms for the external legs are obtained as follows.

(a) Pick a node $n$ in the boundary graph and a link $\ell$ outgoing from it.

(b) Go to $\theta_n$ in the boundary static diagram and pick the link corresponding to $\ell$. Write for it

$$\langle u_{\ell,n}, Y^{|g_I^{-1}|} \rangle_{j_{\ell}} \cdots$$

(c) Follow the link relation of $\ell$. Whenever it leads to a node $I$ of a graph colored by $A^{\text{EPRL}}$, write $Y^{|g_I^{-1}}$:

$$\langle u_{\ell,n}, Y^{|g_I^{-1}|} \rangle_{j_{\ell}} \cdots$$

(d) Then, whenever leaving a graph colored by $A^{\text{EPRL}}$ at node II, write $g_{II} Y$:

$$\langle u_{\ell,n}, Y^{|g_I^{-1}|} g_{II} Y \rangle_{j_{\ell}} \cdots$$

Figure 20. An illustration to section 3.4. (a) A labeling, as in step (i). To make the figure more legible, only those elements of the labeling and the link relations are written that are used in the following examples. (b) The elements of the diagram that contribute to the boundary term $\langle u_{1,j_1} Y^{|g_I^{-1}|} \rangle_{j_1} Y^{|g_I^{-1}|} \langle u_{1,j_1} \rangle_{j_1}$. (c) Another boundary term, this time including only the $\theta$-graphs: $\langle u_{1,j_2} \rangle_{j_2}$. (d) The internal face’s term $\sum_{j_{II}} [\langle Y^{|g_I^{-1}|} \rangle_{j_{II}} Y^{|g_{II}^{-1}|} \langle u_{1,j_1} \rangle_{j_1} Y^{|g_{II}^{-1}|} \langle u_{1,j_1} \rangle_{j_1}]$. 

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(e) Repeat it for each graph passed—if it is colored by $A^{EPRl}$. But if the graph passed is colored by $A^{Tr}$ write the identity or just do not write anything:

$$\langle u_{\ell,n}\rangle \ Y^1 g_{1}^{-1} g_{u} Y 1 Y^1 g_{v}^{-1} g_{v1} Y \cdots .$$

(f) When reaching again the boundary static graph, end the formula with $|u_{\ell,n'}\rangle_{\ell'}$, where $\ell'$ is the appropriate boundary link (it does not need to be $\ell$):

$$\langle u_{\ell,n}\rangle_{\ell'} Y^1 g_{1}^{-1} g_{u} Y Y^1 g_{v}^{-1} g_{v1} Y Y^1 g_{v2} Y |u_{\ell,n'}\rangle_{\ell'}.$$  

(22)

Thanks to the consistency relations (5), we are sure that $j_{\ell'} = j_{\ell}$.

One can find two examples in figures 20(b) and (c).

In fact, recalling our notation (15), all the elements of formula (22) are in fact the trace of the generalized Wigner matrices:

$$\langle \mathcal{D}^{i_1}(u_{\ell})^{j_{i}}_{m} \mathcal{D}^{j_{i}}(g_{u}^{-1} g_{u1})^{m_{i}}_{m} \mathcal{D}^{j_{i}}(g_{v}^{-1} g_{v1})^{m_{i}}_{m} \delta^{m_{i}}_{m} \cdots \mathcal{D}^{j_{i}}(u_{\ell})^{m_{i}}_{m} \rangle.$$  

(23)

Repeat this procedure for all links in the boundary graphs and multiply the terms.

(iii) Terms for internal faces.

For the remaining links do as follows.

(a) Pick a link $\ell_f$ that has not been used yet and write $\langle j_{f}, m \rangle$ for it (where $m \in \{-j_{f}, -j_{f} + 1, \ldots, j_{f} \}$).

(b) Repeat steps (c)–(e) of the procedure for external links.

(c) When reaching the link $\ell_f$ again, close the formula with $\langle j_{f}, m \rangle$.

(d) Sum over $m$:

$$\sum_{m=-j_{f}}^{j_{f}} \langle j_{f}, m \rangle Y^1 g_{1}^{-1} g_{u} Y \cdots Y^1 g_{v}^{-1} g_{v1} Y |j_{f}, m \rangle.$$  

(24)

The resulting term is in fact the $SU(2)$ trace of the generalized Wigner matrices:

$$\text{Tr}_{j_{f}}[\mathcal{D}^{j_{f}}(g_{u}^{-1} g_{u1})^{m} \cdots \mathcal{D}^{j_{f}}(g_{v}^{-1} g_{v1})^{m}].$$  

(25)

An example can be found in figure 20(d).

(iv) Face amplitudes: multiply each term obtained in step (iii) by a face amplitude $A_f = 2j_f + 1$.

Multiply each term obtained in step (ii) by boundary face amplitude $A = \sqrt{2j_f + 1}$.

(v) Summation and integration:

(a) Sum with respect to all possible combination of $j_f$’s:

$$\sum_{(j_f)} \cdots .$$

(b) For each node $I$ of the graphs colored by the $A^{EPRl}$ contractor integrate over $G$ with respect to the Haar measure $d g_I$:

$$\int_{G} \prod_{I_{1 \in I}} d g_{I_{1}} \cdots ,$$

where $N$ is the number of nodes in $\Gamma^{(0)}_{\text{int}}$.

Remark. For each graph, the last integration gives multiplication by the volume of the group $G$. In the Euclidean case, this volume is equal to 1 thanks to normalization of the Haar measure.

However, in the Lorentzian, the integration goes over a noncompact group and the volume is infinite. Thus, to avoid this infinity, remember to drop one integration for each graph [18, 21].
The resulting amplitude is therefore

\[
A_D = \int \prod_{\ell \in \Gamma_{\text{int}}} dg_\ell \sum_{j_{\ell}} \prod_{f} A_f(j_f) \text{Tr}_{j_f} \left[ Y^\dagger g_1^{-1} g_2 Y \cdots Y^\dagger g_{\text{int}}^{-1} g_{\text{int}v} Y \right]
\times \prod_{\ell} A_{\ell}(j_{\ell}) \langle u_{\ell,n} | j_{\ell} | Y^\dagger g_1^{-1} g_2 Y \cdots Y^\dagger g_{\text{int}}^{-1} g_{\text{int}v} Y | u_{\ell,n}' \rangle_{j_{\ell}}. \tag{26}
\]

In the next section, we apply the algorithm to construct all the OSN diagrams of the Bianchi–Rovelli–Vidotto dipole cosmology.

## 4. Diagrams with boundary given by dipole cosmology graph

In this section, we apply the algorithm presented above to a specific example of a boundary graph, namely the one given by the dipole cosmology model of Bianchi and co-workers [25]. The same algorithm could be applied to the graphs of many-node/many-link approaches to spin-foam cosmology [26].

The dipole cosmology model is an attempt to test the behavior of the spin-foam transition amplitudes in the limit of the homogeneous and isotropic boundary states. The boundary state of this model is supported on the so-called dipole graph (figure 21) which consists of two disjoint components, four-valent theta graphs representing the initial and the final geometry, respectively. The transition amplitude is calculated under several approximations. One of the approximations is the vertex expansion, i.e. at the first order one considers contribution of the spin-foams with four internal edges and one interaction vertex only. Some of these spin-foams have already been found in [27]; however, we find all of them.

### 4.1. The improved algorithm in the dipole cosmology case

We apply now the algorithm of the previous section to construct all the OSN diagrams whose boundary is fixed (modulo an orientation) to be the graph \( |\Gamma| \) which consists of two disjoint four-valent theta graphs \( |\Gamma_{\text{int}}| \) (figure 21),

\[
|\Gamma| = |\Gamma_{\text{int}}| \cup |\Gamma_{\text{int}}|,
\]

and which correspond to 1-vertex spin-foams.

In terms of the improved algorithm of the previous section, this assumption means that the interaction diagram \( D_{\text{int}} \) consists of one graph \( \Gamma_{\text{int}} \), whereas the node and link relations
Figure 22. Construction of the graph diagram. (a) Step (i): choose an orientation of each link. (b) Step (ii): construct the squid set $S_\Gamma$. (c) Step (iii): construct an interaction graph $D_{\text{int}} = \Gamma_{\text{int}}$; an example is depicted.

are trivial. That is its squid set equals the squid set of the boundary (the initial plus the final) graph. Specifically, for this dipole cosmology example and with the 1-vertex assumption, the improved algorithm from section 3.3 reads

(i) $\Gamma$: choose an orientation of each link of each of the two graphs $|\Gamma_{\text{av}}|$ (figure 22(a)).

(ii) $S_{\text{int}} = S_\Gamma$: for the interaction squid set take the squid set $S_\Gamma$ of the graph $\Gamma$; this means that in point (ii) of the general algorithm presented in section 3.3, we set $N = 0$—in other words, we consider the first order of the vertex and edge expansion; the interaction squid set consists of four four-valent squids; two of them are oriented freely—their orientation determines the orientation of the remaining two squids and defines the orientation of $\Gamma$ (figure 22(b)).

(iii) $\Gamma_{\text{int}}$: glue each incoming/outgoing leg of each squid of $S_{\text{int}}$ with an outgoing/incoming leg of another (or the same) squid of $S_{\text{int}}$. In the next subsection, we construct and list all the possible (unoriented) interaction graphs (they are depicted later, in figure 29). $\Gamma_{\text{int}}$ is obtained by assigning orientation to each link of one such graph (figure 22(c)). Together with the trivial node and link relations, $\Gamma_{\text{int}}$ defines an interaction graph $D_{\text{int}}$.

(iv) $D_\Gamma \# D_{\text{int}}$: use the static graph diagram $D_\Gamma$ of the graph $\Gamma$ (figure 23(a)) and construct the union of the diagrams as it was explained above (figure 23(b)).

(v) Coloring: define arbitrary coloring of the diagram $D_\Gamma \# D_{\text{int}}$ which turns it into an OSN diagram.

(vi) Consider all the possible: orientations of the two independent squids of $S_\Gamma$, ways of connecting the legs of the four squids, all possible node relations between the nodes of the interaction graph and corresponding nodes of static diagram, all possible link relations between the links of the interaction graph and the corresponding links of the static diagram.

4.2. All the possible interaction graphs

In this subsection, we construct all the possible interaction graphs $\Gamma_{\text{int}}$. We obtain each interaction graph $\Gamma_{\text{int}}$ by assigning an orientation to each link of an (unoriented) graph $|\Gamma_{\text{int}}|$ defined by the following two properties:

- each graph $|\Gamma_{\text{int}}|$ has exactly four nodes,
Figure 23. Construction of the graph diagram. Step (iv): the static graph diagram $D_\Gamma$ ((a) the dotted lines denote the link relations) is attached to the diagram $D_{\text{int}}$ (b), and the final diagram $D_\Gamma \# D_{\text{int}}$ is obtained.

- each node of $|\Gamma_{\text{int}}|$ is precisely four-valent.

We find below all possible graphs $|\Gamma_{\text{int}}|$. We depicted the resulting graphs in figure 29. Note that in order to obtain an interaction graph $\Gamma_{\text{int}}$, we assign to each link of a graph $|\Gamma_{\text{int}}|$ an orientation consistent with the orientation of the boundary (and the squid set). Given a graph $|\Gamma_{\text{int}}|$ and (oriented) boundary graph $\Gamma$, such a choice of compatible orientation may be impossible. For example, take graph 1 from figure 29 as a graph $|\Gamma_{\text{int}}|$. It is not possible to choose an orientation of links of this graph compatible with orientation of the boundary graph from figure 22(a). This is because the boundary graph figure 22(a) has a node with three outgoing and one incoming link and such a structure of a node is not possible for graph 1 from figure 29 (since each link of this graph forms a loop, the number of incoming links and outgoing links needs to be equal at every node). Note that there is a distinguished graph $|\Gamma_{\text{int}}|$—the graph 20 from figure 29 used in [25]. This graph may be oriented in a way compatible with any boundary graph $\Gamma$. The natural question which arises is whether for every graph from figure 29 there is a boundary graph such that orientation of $|\Gamma_{\text{int}}|$ may be chosen to be compatible with this boundary graph. The answer is affirmative. It may be shown that the orientation of links of each graph $|\Gamma_{\text{int}}|$ may be chosen to be compatible with a boundary graph oriented such that at every node a number of incoming links equals to a number of outgoing links.

We now present in details the construction of unoriented graphs possessing exactly four nodes, all of which are four-valent, i.e. all possible graphs $|\Gamma_{\text{int}}|$. It is well known that each unoriented graph may be encoded in the adjacency matrix. It is a symmetric matrix $A \in \text{Sym}(n)$ with the number of columns/rows $n$ equal to the number of the vertices of this graph. The entries $A_{ij}$ are equal to the numbers of links connecting node $i$ with node $j$, with a specification that links forming closed loops (corresponding to diagonal entries) are counted twice. An example of such matrix and the corresponding graph is given in figure 24.

However, given a graph, there are many corresponding matrices, because for each permutation $\sigma \in S_n$ the matrices

$$
(\sigma \circ A)_{ij} := A_{\sigma(i)\sigma(j)}
$$

(28)
and $A_{ij}$ define the same graph. There is a natural bijective correspondence between graphs with $n$ vertices and orbits, elements of $\text{Sym}(n)/S_n$. In our case, graphs have four nodes. We are therefore interested in $4 \times 4$ matrices. The condition that each node is four-valent corresponds to an assumption that the sum of numbers in each row/column is equal to 4:

$$\forall i \sum_{j=1}^{4} A_{ij} = 4.$$  \hfill (29)

The set of the possible interaction graphs $G_{int}$ is therefore characterized by the moduli space:

$$\left\{ A \in \text{Sym}(4) : \forall i \sum_{j=1}^{4} A_{ij} = 4 \right\} / S_4.$$  \hfill (30)

First, we introduce a parametrization of the space of symmetric matrices satisfying (29), and then, we find the moduli space using Wolfram’s Mathematica 8.0.

To define our parametrization in a transparent way, we introduce a triple ($K_4, d, m$) (see figure 25):

- the complete graph $K_4$ on four nodes—the skeleton of a 4-simplex (we denote by $K_4^{(0)}$ the set of its nodes and by $K_4^{(1)}$ the set of its links);
- labeling of its nodes

$$d : K_4^{(0)} \ni n \mapsto d_n \in \{0, 2, 4\}$$  \hfill (31)
such that for all four nodes $n_1, n_2, n_3, n_4$ of the graph $K_4$ the numbers $d_{n_1}, d_{n_2}, d_{n_3}, d_{n_4}$ satisfy the generalized triangle inequalities:

$$\forall i \quad d_{ni} \leq \sum_{j \neq j} d_{nj}; \quad (32)$$

- labeling of its links

$$m : K_4^{(1)} \ni \ell \mapsto m_{\ell} \in \{0, 1, 2, 3, 4\} \quad (33)$$

such that

$$\forall n \in K_4^{(0)} \quad \sum_{\ell \in K_4^{(1)} \setminus \{\ell \neq \ell\}} m_{\ell} = d_n. \quad (34)$$

The condition that $d_n, n \in K_4^{(0)}$ satisfy the generalized triangle inequalities (32) ensures the existence of at least one labeling $m^5$

To each triple $(K_4, d, m)$ corresponds a (multi)graph $\Gamma_{(K_4, d, m)}$, defined in the following way (see also an example in figure 26):

- it has the same set of nodes $\Gamma_{(K_4, d, m)} = K_4^{(0)}$;
- for each pair $(n, n')$ of different nodes, there are exactly $m_{\ell}$ links of $\Gamma_{(K_4, d, m)}$ connecting the nodes $n$ and $n'$, where $\ell$ is the link of $K_4$ connecting $n$ with $n'$;
- at each node $n$, there are precisely $(4 - d_n)/2$ links each of which makes a loop connecting $n$ with itself (figure 27).

Alternatively, one may read from $(K_4, d, m)$ the corresponding adjacency matrix. Simply choose some ordering of nodes (in our case it is a labeling of nodes with numbers $\{1, 2, 3, 4\}$) and let $\ell_{ij} = \ell_{ji}, i \neq j$ be the link in $K_4$ connecting nodes $n_i$ and $n_j$. Now

- terms $4 - d_n$ correspond to diagonal entries of the adjacency matrix, i.e. $A_{nn} := 4 - d_n$;
- terms $m_{\ell_{ij}}$ correspond to off-diagonal entries, i.e. $A_{ij} := m_{\ell_{ij}}$ for $i \neq j$.

This correspondence is depicted in figure 28. Note that having known the numbers $d_n$, 5 This well-known fact is used, e.g., in the representation theory of SU(2) to construct of the invariants of the tensor product $H_{d_1/2} \otimes H_{d_2/2} \otimes H_{d_3/2} \otimes H_{d_4/2}$, where $\dim H_j = 2j + 1$, and $d_{n_1} + \cdots + d_{n_4} \in 2\mathbb{N}$ by the construction.
In this subsection, we discuss in more detail the diversity of the resulting graph diagrams. As we explained in the previous subsection, there are exactly 20 interaction graphs. However, the number of the graph diagrams resulting from the procedure described above is different. When \(d_1, d_2, d_3, d_4\) are becoming larger, this method becomes considerably faster than the direct method. It has the additional advantage that it is easily applicable to a more general case studied in [26] where the four nodes are not necessarily four-valent.

We next find orbits of action of permutation group \(S_4\) on the set of those solutions. To this end, we used Mathematica 8.0. The resulting graphs are depicted in figure 29.

Note that one could further restrict the number of matrices considered by requiring that the sequence \((d_1, d_2, d_3, d_4)\) is monotonous and considering only orbits under action of \(S_4/H\), where \(H\) is the subgroup, which does not change the sequence \((d_1, d_2, d_3, d_4)\). This remark enables one to do the calculation without using a computer. On the other hand, one could write a program which does not use the parametrization we introduced—e.g., one could generate matrices with entries taking values in the set \([0, 1, 2, 3, 4]\) (with even numbers on diagonal) and choose only those which satisfy equation (29) (a direct method). We have chosen the method that we presented here, because it gives better understanding of the structure of the graphs considered, it is less laborious than calculation by hand and the version that we used is easier to implement than the direct method. It has the additional advantage that it is easily applicable to a more general case studied in [26] where the four nodes are not necessarily four-valent. When \(d_1, d_2, d_3, d_4\) are becoming larger, this method becomes considerably faster than the direct method.

4.3. Possible graph diagrams and an interesting observation

As we explained in the previous subsection, there are exactly 20 interaction graphs. However, the number of the graph diagrams resulting from the procedure described above is different. In this subsection, we discuss in more detail the diversity of the resulting graph diagrams.

Given an oriented interaction graph \(D_{\text{int}}\) and a static diagram \(D_{\text{r}}\), there may be more than one graph diagrams \(D_{\text{r}} \# D_{\text{int}}\). The ambiguity is in the choice of the node relation and the link relations.
Figure 29. The list of all the possible interaction graphs in the first order of the vertex expansion (modulo orientations). Note that the spin-foams considered in [27] have the interaction graphs of type 18, 19 or 20.
• **The ambiguity in node relation.** It exists if an oriented interaction graph $\Gamma_{\text{int}}$ has two nodes, say $n_1$ and $n_2$, such that the number of the incoming/outgoing links at $n_1$ is equal to the number of the incoming/outgoing links at $n_2$. Then, for every node relation between the nodes of the interaction graph and the corresponding nodes of the static diagram, there is another, different node relation obtained by switching the nodes $n_1$ and $n_2$—see figure 30.

• **The ambiguity in link relations.** Having settled down the node relation, there are still many possible link relations. The only condition that each link relation needs to satisfy is that the incoming/outgoing link at each node in the interaction graph is in relation with the outgoing/incoming link at the corresponding node in the static diagram (see figure 31).

Furthermore, some colorings of the boundary links may be incompatible with some interaction graphs—it may happen that the amplitude is zero for every coloring of a given
interaction graph. In order to see how this limits the number of possible interaction graphs, consider the coloring of boundary graph depicted in figure 32(a) and the OSN diagram in figure 32(b) (node relations and the corresponding coloring with operators are omitted for clarity). It is straightforward to see that the amplitude is non-zero only if the representations...
$\rho_2$ and $\rho_3$ and, respectively, the representations $\rho_1$ and $\rho_5$ are equal ($\rho_2 = \rho_3$, $\rho_1 = \rho_5$). Importantly, note that, because there are links forming closed loops in the interaction graph, the amplitude is non-zero only if among representations $\rho_1, \rho_2, \rho_3, \rho_4$ or among representations $\rho_5, \rho_6, \rho_7, \rho_8$ there is a pair of equal representations. In addition, since the interaction graph is connected, the amplitude is non-zero only if there is a pair of equal representations $\rho_i = \rho_j$, such that $i \in \{1, 2, 3, 4\}$, $j \in \{5, 6, 7, 8\}$. These two conditions do not depend on the choice of node and link relations but on the structure of interaction graph only. This example shows that there are colorings of the boundary graph which are not compatible with the given interaction graph.

A similar analysis may be performed for other interaction graphs. It leads to an interesting conclusion. There is a distinguished interaction graph, which is not limited by the coloring in the way described above—it is the graph 20 from figure 29 used in [25]. The corresponding amplitude is non-zero even if all eight links of the boundary graph are labeled with pairwise different representations. In this generic case, all other interaction graphs give identically zero amplitude. We expect therefore that for a generic boundary state (which is a linear combination of spin-network states of all possible spins), the graph diagram with this interaction graph gives a major contribution. This conclusion needs however further justification.
4.4. Transition amplitudes

In this subsection, we show that the transition amplitudes of some diagrams are exponentially suppressed. We do the calculation for the Euclidean EPRL vertex amplitude.

In dipole cosmology, the boundary state (1) is considered in the large \( j \) limit. According to [22], it can be approximated by a linear combination of LS states:

\[
\langle j_\ell, u_\ell, u_\ell | \Psi_{H_\ell} \rangle \sim \prod_\ell e^{-\frac{(j_\ell - j_0)^2}{2\Delta_j}} e^{i\xi_{j_\ell}}
\]

(36)

(for some \( j_0^\ell \) and \( \Delta j_\ell \)), where \( u_{L,\ell} = u_{L,\ell(t)} = u_\ell \). Thus, in our case, it is enough to perform the calculation for the LS boundary state of this special form.

As an example, let us consider the interaction graph 16 of figure 29. The diagram is shown in figure 33(a). We will compare the amplitude of this diagram with the amplitude of the diagram of type 20 shown in figure 33(b). Nodes and links of the diagrams are labeled as in the figure.

The amplitude of the diagram of figure 33(b) was computed in [25]. It was shown that the main contribution to the amplitude comes from the terms with \( j_\ell = j \) for \( \ell = 1, \ldots, 4 \) and \( j_\ell = j' \) for \( \ell = 5, \ldots, 8 \). Since we consider this amplitude as a reference, it is enough to consider only such terms; thus, we choose this special coloring of the links.
The amplitude of the diagram figure 33(b) reads

$$A_{20} = \left[ \int_{SO(4)^2} d\mathbf{g}_N \sum_{\ell=1}^4 \langle \mathbf{u}_{\ell} \rangle Y_0^{\gamma} \mathbf{g}_N \mathbf{Y} | \mathbf{u}_{\ell} \rangle \right] \times \left[ \int_{SO(4)^2} d\mathbf{g}_S \sum_{\ell=5}^8 \langle \mathbf{u}_{\ell} \rangle Y_{\ell}^{\gamma} \mathbf{g}_S \mathbf{Y} | \mathbf{u}_{\ell} \rangle \right]$$

$$= \left[ \int_{SO(4)} d\mathbf{g} \sum_{\ell=1}^4 \langle \mathbf{u}_{\ell} \rangle Y_0^{\gamma} \mathbf{g} \mathbf{Y} | \mathbf{u}_{\ell} \rangle \right] \left[ \int_{SO(4)} d\mathbf{g} \sum_{\ell=5}^8 \langle \mathbf{u}_{\ell} \rangle Y_{\ell}^{\gamma} \mathbf{g} \mathbf{Y} | \mathbf{u}_{\ell} \rangle \right] \left( \int_{SO(4)} d\mathbf{g} \right)^2$$

(37)

Each term \( \int_{SO(4)} d\mathbf{g} \sum_{\ell=1}^4 \langle \mathbf{u}_{\ell} \rangle Y_0^{\gamma} \mathbf{g} \mathbf{Y} | \mathbf{u}_{\ell} \rangle \) is the coherent regular tetrahedron of LS, which was shown in [14] to be equal to \( N_0 j^{-3} \) in large \( j \) limit. Thus, the amplitude \( A_{20} \) decays polynomially when the spins \( j \) and \( j' \) are growing.

The amplitude of the diagram figure 33(a) is considered below using the algorithm presented in section 3.4. We will not calculate it explicitly. It will be enough to show the large \( j \) behavior of it.

Thanks to very simple structure of the diagram, each term of the amplitude is of the form

$$\langle \mathbf{u}_{\ell} \rangle Y_0^{\gamma} \mathbf{g}_N \mathbf{Y} | \mathbf{u}_{\ell} \rangle_{ji}.$$  

(38)

Some examples are described in figure 34.

The most interesting example is figure 34(d). In this case, the \( G \) group elements cancel:

$$\langle \mathbf{u}_{\ell} \rangle Y_0^{\gamma} \mathbf{g}_N^{-1} \mathbf{g}_N \mathbf{Y} | \mathbf{u}_{\ell} \rangle_{ji} = \langle \mathbf{u}_{\ell} \rangle Y_0^{\gamma} \mathbf{Y} | \mathbf{u}_{\ell} \rangle_{ji} = \langle \mathbf{u}_{\ell} \rangle | \mathbf{u}_{\ell} \rangle_{ji},$$

so the contribution can be easily calculated:

$$\langle \mathbf{u}_{\ell} \rangle | \mathbf{u}_{\ell} \rangle_{ji} = \langle j, j | \mathbf{u}_{\ell}^{-1} \mathbf{u}_{\ell} | j, j \rangle = \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) \mathbf{u}_{\ell}^{-1} \mathbf{u}_{\ell} \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right)^{2j}.$$  

(40)

From the definition (14) of the group elements \( \mathbf{u} \), one can find the matrix element

$$\left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) \mathbf{u}_{\ell}^{-1} \mathbf{u}_{\ell} \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) = \cos \frac{\theta_0}{2} \cos \frac{\theta_1}{2} + \sin \frac{\theta_0}{2} \sin \frac{\theta_1}{2} e^{i(\phi_0 - \phi_1)} =: \alpha_{6-7},$$  

(41)

and using some trigonometric identities, one can estimate its modulus

$$|\alpha_{6-7}|^2 = \frac{1}{4} \cos^2 \frac{\theta_0}{2} \cos^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_0}{2} \sin^2 \frac{\theta_1}{2} e^{i(\phi_0 - \phi_1)}$$

$$\leq \frac{1}{4} \cos^2 (\theta_0 - \theta_1) \leq 1,$$  

(42)

(43)

where the first inequality is strong for all pairs \( \phi_0 \neq \phi_1 \) (because \( \phi \in [0, 2\pi) \)) and the second inequality is strong for all pairs \( \phi_0 \neq \phi_1 \) (because \( \phi \in [0, \pi] \)). Thus,

$$|\alpha_{6-7}| \leq 1, \quad \bar{u}_6 \neq \bar{u}_7,$$  

(44)

$$|\alpha_{5-6}| = 1, \quad \bar{u}_6 = \bar{u}_7.$$  

(45)

So in the large \( j \) limit, the term \( \langle \mathbf{u}_{\ell} \rangle | \mathbf{u}_{\ell} \rangle_{ji} = |\alpha_{6-7}|^2 \) is suppressed unless \( \bar{u}_6 = \bar{u}_7 \).

In the boundary state (1), the group elements \( \mathbf{u}_{\ell,n} \) represent the directions \( \bar{u}_{\ell,n} \) perpendicular to the faces of the quantum polyhedron represented by the node \( n \). In the case of dipole cosmology, where the nodes are four-valent, the group elements \( \mathbf{u}_{\ell,n} \) represent the directions
Figure 34. Examples of terms appearing in the amplitude $A_{16c}$: (a) $\langle u_1 | j_1 \ Y^+ \ g_N^{-1} \ gS \ Y \ | u_1 \rangle_{j_1}$, (b) $\langle u_2 | j_2 \ Y^+ \ g_N^{-1} \ gS \ Y \ | u_3 \rangle_{j_3}$, (c) $\langle u_4 | j_4 \ Y^+ \ g_N^{-1} \ gS' \ Y \ | u_5 \rangle_{j_5}$, (d) $\langle u_6 | j_6 \ Y^+ \ g_N^{-1} \ gN \ Y \ | u_7 \rangle_{j_7}$. 

$\langle u_1 | j_1 \ Y^+ \ g_N^{-1} \ gS \ Y \ | u_1 \rangle_{j_1}$

$\langle u_2 | j_2 \ Y^+ \ g_N^{-1} \ gS \ Y \ | u_3 \rangle_{j_3}$

$\langle u_4 | j_4 \ Y^+ \ g_N^{-1} \ gS' \ Y \ | u_5 \rangle_{j_5}$

$\langle u_6 | j_6 \ Y^+ \ g_N^{-1} \ gN \ Y \ | u_7 \rangle_{j_7}$
perpendicular to the faces of a quantum tetrahedron. A quantum tetrahedron with two faces overlapping is highly non-classical. Therefore for a semiclassical tetrahedron, $\vec{u}_{\ell,\theta} \neq \vec{u}_{\ell',\theta}$ for $\ell \neq \ell'$. As a result the amplitude of this diagram is proportional to $\alpha_{6-7}$. Since the reference amplitude $A_{20}$ behaves polynomially, in the $j \gg 1$ limit the amplitude $A_{16}$ is negligible when compared to $A_{20}$.

This scenario appears whenever there is a loop in the diagram. Thus, each diagram with graphs 1–16 is exponentially suppressed in the large $j$ limit. Moreover, the polynomial behavior of $A_{20}$ was also shown in the Lorentzian EPRL model [26]; thus, our result is valid also in that case.

5. Summary, conclusions and outlook

We presented a general algorithm for finding all spin-foams with given boundary graph. This algorithm is very well suited for finding perturbative series of the scattering-like amplitudes in the spin-foam theory. That is given any in and out spin-network states, one can easily find all the diagrams connecting these states up to arbitrary 'order'. The order in our framework is given by two numbers $V$ and $N$, where $V$ is the number of physically relevant graphs in $D_{\text{int}}$ (i.e. different than $\theta_{\text{R}}$) and $N$ is the number of 'extra internal edges' (i.e. the number of pairs of squids added to $S_{\Gamma}$ in step (ii) of our algorithm)—see section 3.3. For the in and out states of the form of LS states, we presented a Feynman-like rules for constructing the corresponding transition amplitude—see section 3.4.

We applied our algorithm to the dipole cosmology model [25]. All the diagrams giving a contribution to the amplitude in the zeroth order of the edge expansion are found and described in section 4.3. However, in a generic case, only one of them is not suppressed, i.e. the one used in [25].

The calculation that we presented illustrates an application of OSN diagrams. The diagrams help to find all terms contributing to the physical transition amplitudes. Moreover, some properties of amplitudes, like the vanishing of some terms (see section 4.4) can be read directly from the diagram. The strength of this formalism lies in simplifying the classification of 2-complexes with given properties (such as order of vertex expansion or structure of boundary graph).

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