We consider attractor varieties arising in the construction of dyonic black holes in Calabi-Yau compactifications of IIB string theory. We show that the attractor varieties are constructed from products of elliptic curves with complex multiplication for $N = 4, 8$ compactifications. The heterotic dual theories are related to rational conformal field theories. The emergence of curves with complex multiplication suggests many interesting connections between arithmetic and string theory. This paper is a brief overview of a longer companion paper entitled “Arithmetic and Attractors” [hep-th/9807087].
1. Introduction

This paper is a short introduction to [1]. We have written it as a separate note with the hope that the present paper is something that people might actually read. Further explanations, more precise statements, other results, and more complete references can be found in [1].

2. The attractor equations

The attractor mechanism, discovered by Ferrara, Kallosh and Strominger in [2], is a fascinating phenomenon that combines supersymmetric black holes, dynamical systems, and, as we show here, number theory. Put briefly, in constructing spherically symmetric dyonic black holes in $d = 4, \mathcal{N} = 2$ supergravity coupled to abelian vectormultiplets, one finds that the radial evolution of vectormultiplet scalars is described by a dynamical system. Under good conditions, (leading to a black hole with a smooth horizon of positive area), the vectormultiplets flow to a fixed point in their target space. The fixed point is determined by the dyonic charge $\hat{\gamma}$ of the black hole, where $\hat{\gamma}$ is a vector in the lattice $\Lambda$ of electric and magnetic charges of the $\mathcal{N} = 2$ abelian gauge theory. Suppose the supergravity is the low-energy limit of type IIB superstring theory compactified on a Calabi-Yau (CY) 3-fold $X$. Then, the vectormultiplet moduli space is identified with $M$, the moduli space of complex structures on $X$, and the charge lattice is $\Lambda = H^3(X; \mathbb{Z})$. The equation for the fixed point complex structure, called the attractor equation for the charge $\hat{\gamma} \in H^3(X; \mathbb{Z})$, is a condition on the Hodge structure of $X$:

$$\hat{\gamma} = \hat{\gamma}^{3,0} + \hat{\gamma}^{0,3}. \quad (2.1)$$

If solutions to (2.1) exist then they are isolated points in $M$.

Equation (2.1) also turns out to be related to a nonperturbative statement about BPS states [3]. Indeed, $\hat{\gamma}$ specifies a superselection sector $\mathcal{H}_{\hat{\gamma}}$ in the Hilbert space of states for the IIB compactification on $X$. The central charge of the supersymmetry algebra in the superselection sector $\mathcal{H}_{\hat{\gamma}}$ may be expressed in terms of a nowhere vanishing holomorphic $(3,0)$ form $\Omega$ on $X$. If $\gamma$ is Poincaré dual to $\hat{\gamma}$ the central charge $Z(\Omega; \gamma)$ satisfies:

$$|Z(\Omega; \gamma)|^2 \equiv \frac{|\int_{\gamma} \Omega|^2}{i \int \Omega \wedge \overline{\Omega}}. \quad (2.2)$$
The BPS states of charge \( \hat{\gamma} \) are the single-particle states in \( \mathcal{H}_{\hat{\gamma}} \) saturating the Bogomolnyi bound. They might or might not exist, depending on \( \hat{\gamma} \). If \( \gamma \) does support a BPS state then the mass of the BPS state in Planck units is just (2.2). Note that, because of monodromy, we must consider the target space of the vector multiplet scalars to be in the universal cover \( \tilde{\mathcal{M}} \). Note too that the mass depends only on the point \( z \in \tilde{\mathcal{M}} \) in Teichmüller space and not on the choice of \( \Omega \). The relation between BPS masses and the attractor equation (2.1) follows from the:

**Theorem 3.** \( |Z(z; \gamma)|^2 \) has a stationary point at \( z = z_* (\gamma) \in \tilde{\mathcal{M}} \), with \( Z(z_*; \gamma) \neq 0 \) iff \( \hat{\gamma} \) has Hodge decomposition \( \hat{\gamma} = \hat{\gamma}^{3,0} + \hat{\gamma}^{0,3} \). Moreover, if such a stationary point exists in the interior of \( \tilde{\mathcal{M}} \) then it is a local minimum of \( |Z(z; \gamma)|^2 \).

This theorem is very important to our considerations because, while supergravity is only an approximation at large charges, the minimization principle applies to an exact formula for BPS states. Hence it is sensible to think about the exact nature of the attractor varieties (2.1) even for small charges.

This theorem also raises the question of whether attractor points are *global* minima of the BPS mass (2.2). In [1], section 9.2, we give an example of a compactification and charges \( \gamma \) such that the equation (2.1) has more than one distinct solution in \( \tilde{\mathcal{M}} \) (not related by the duality group.) If our example is correct then it might present an interesting twist on the D-brane interpretation of black hole entropy [6]. In any case, the example shows that the the dynamical system of black hole entropy [3]. In any case, the example shows that the the dynamical system of [2] defined by a charge \( \gamma \) can have more than one basin of attraction \( B \). The entropy is determined by the data \( (\gamma, B) \). We refer to this data as an *area code*.

3. Some solutions to the attractor equations

In this section we describe solutions to the attractor equations for \( d = 4 \) compactifications with \( \mathcal{N} = 8, 4 \) supersymmetry.

Let \( X \) be a 3-dimensional complex torus. Compactification of IIB superstrings on \( X \) leads to \( d = 4, \mathcal{N} = 8 \) supergravity. There are 28 independent abelian gauge fields and the electric/magnetic charge lattice \( \Lambda \) is a rank 56 module for an integral form of the maximally split form of \( E_7 \), often called \( E_7 (\mathbb{Z}) \) in the physics literature [7][8]. This form of \( E_7 \) preserves a certain quartic form \( I_4 (\gamma) \) on \( \Lambda \) [9][10]. Perturbative string theory and D-branes give a model for \( \Lambda \) as the lattice \( H^{\text{odd}} (X; \mathbb{Z}) \oplus II^{6,6} \oplus II^{6,6} \) where \( II^{r,s} \) is the
even unimodular lattice of signature \((-1)^r, (+1)^s\). In particular, \(H^3(X; \mathbb{Z})\) is a submodule of \(\Lambda\). By considerations of \(R\)-symmetry the compactification moduli can be divided into “vectormultiplets” and “hypermultiplets” (in the terminology of \(\mathcal{N} = 2\) representations) and the attractor mechanism fixes the values of the vectormultiplets, leaving the hypermultiplets arbitrary \([1]\). Chief among the vectormultiplets are the complex structure moduli.

By \(U\)-duality we can take \(\hat{\gamma} \in H^3(X; \mathbb{Z})\) and the complex structure is then fixed by (2.1). The solution to the equation, described in \([1]\), shows that \(X\) is isogenous to a product of 3 elliptic curves:

\[
X_\gamma \cong E_{\tau(\gamma)} \times E_{\tau(\gamma)} \times E_{\tau(\gamma)}
\]

where for \(\tau\) in the upper half plane,

\[
E_{\tau(\gamma)} \equiv \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})
\]

and in (3.1) \(\tau(\gamma) = i\sqrt{I_4(\gamma)}\). The remaining vectormultiplet moduli are related to the values of 2-form, 4-form, and 6-form potentials. The remaining attractor equations state that these are of type \((1, 1)\) in the complex structure fixed by (2.1).

Let us now turn to compactifications on \(X = K3 \times T^2\). The attractor equations for \(\mathcal{N} = 4\) have been written in \([12]\). Once again, the moduli must be decomposed into vector- and hyper- multiplets. Once again, we focus on the complex structure moduli.

Write \(\Omega^{3,0} = \Omega^{2,0} \wedge dz\) where \(z \mod(\mathbb{Z} + \tau\mathbb{Z})\) is the flat coordinate for the \(T^2\) factor of \(X\). Choosing a basis for \(H^1(T^2; \mathbb{Z})\) we can identify \(H^3(K3 \times T^2; \mathbb{Z}) \cong H^2(K3; \mathbb{Z}) \oplus H^2(K3; \mathbb{Z})\) so that \(\hat{\gamma} \cong p \oplus q\). The attractor equations become:

\[
2\text{Im} \bar{C} \Omega^{2,0} = p \quad 2\text{Im} \bar{C} \tau \Omega^{2,0} = q
\]

for some complex constant \(C\). This can be solved directly with the result that \(\Omega^{2,0} = C(q - \bar{\tau}p)\) where \(C\) is a constant.

\[
\tau = \tau(p, q) \equiv \frac{p \cdot q + \sqrt{D_{p,q}}}{p^2}
\]

and \(D_{p,q} \equiv (p \cdot q)^2 - p^2 q^2\). These equations determine the complex structure of the K3 surface uniquely, by the global Torelli theorem. Since \(\Omega^{2,0} = C(q - \bar{\tau}p)\) the Neron-Severi lattice has rank \(\rho = 20\) and therefore the transcendental lattice has rank 2. We call such maximally algebraic K3 surfaces “attractive K3 surfaces.” They have been completely classified \([13]\), and the classification reveals that the attractor variety is again related to a
product of three isogenous elliptic curves. To explain this we need to pause and recall a few definitions.

An integral binary quadratic form is a matrix

\[ Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \] (3.5)

where \( a, b, c \in \mathbb{Z} \). The discriminant of the form is \( D = -4 \det Q = b^2 - 4ac \). The form is primitive if \( \gcd(a, b, c) = 1 \), positive if \( Q > 0 \) (i.e., iff \( D < 0, a > 0 \)) and even if \( a, c \) are even. To a positive quadratic form we associate an element \( \tau \) of the upper halfplane via:

\[ ax^2 + bxy + cy^2 \equiv a|x - \tau y|^2. \] (3.6)

Two forms \( Q, Q' \) are said to be properly equivalent if there is an \( s \in SL(2, \mathbb{Z}) \) such that:

\[ s \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} s^{tr} = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}. \] (3.7)

This induces the standard fractional linear action on \( \tau \).

Returning to K3 surfaces, if \( S \) is an attractive K3 surface with transcendental lattice \( T_S \) of rank 2 then we may choose a \( \mathbb{Z} \)-basis \( (t_1, t_2)_{\mathbb{Z}} = T_S \) and associate to it the positive even quadratic form:

\[ \begin{pmatrix} t_1^2 & t_1 \cdot t_2 \\ t_1 \cdot t_2 & t_2^2 \end{pmatrix} \] (3.8)

Two oriented bases for \( T_S \) map to properly equivalent forms. By the global Torelli theorem the complex structure is determined by \( T_S \) so there is a well-defined injective map from attractive K3 surfaces to equivalence classes of positive even quadratic forms. It turns out that the map is in fact surjective \[13\]. The proof begins with the torus \( A_Q = E_{\tau_1} \times E_{\tau_2} \) where

\[ \tau_1 = \frac{-b + \sqrt{D}}{2a}, \quad \tau_2 = \frac{b + \sqrt{D}}{2}. \] (3.9)

One first forms the Kummer variety and then takes a branched double-cover.

Applying this to our case, to a charge \( \hat{\gamma} = p \oplus q \) we associate the form of discriminant \( D_{p,q} \):

\[ Q_{p,q} \equiv \frac{1}{2} \begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix} \] (3.10)

The transcendental lattice of the attractor K3 has a basis with Gram matrix \( 2Q_{p,q} \). Therefore, the attractor variety \( X_{p,q} \) is related to a product of three elliptic curves:

\[ X_{p,q} \xrightarrow{2:1} Km \left( E_{\tau(p,q)} \times E_{\tau'(p,q)} \right) \times E_{\tau(p,q)} \] (3.11)
with
\[ \tau'(p, q) = -\frac{p \cdot q + \sqrt{D}}{2} \] (3.12)

Similar considerations apply to the attractor varieties in the FHSV model. The double cover of the Enriques surface in that model is an attractive K3 surface. It is now determined by \( p, q \in \mathcal{I}^{10,2} \).

4. Attractors are arithmetic

The first connection between arithmetic and supersymmetric black holes is related to questions about \( U(\mathbb{Z}) \)-duality orbits of charges. In the \( \mathcal{N} = 4 \) compactifications the charges \( (p, q) \) form a doublet of the \( S \)-duality group. Therefore the action of \( S \)-duality on (3.10) is the action (3.7). Again we must pause and recall some general results.

Quite generally, for a fixed value of \( D \) there are a finite number of \( SL(2, \mathbb{Z}) \) equivalence classes of forms (3.5) of discriminant \( D \). If \( Q \) is primitive, these classes are uniquely associated to ideal classes in the quadratic imaginary field

\[ K_D \equiv \mathbb{Q}[\sqrt{D}] \equiv \{ r_1 + r_2 \sqrt{D} : r_1, r_2 \in \mathbb{Q} \} \] (4.1)

(In our case \( D < 0 \) and \( D = 0, 1 \text{mod} 4 \).) Moreover, an important result in number theory states that the number of classes \( h(D) \) is finite and grows for large \( |D| \) like \( |D|^{1/2} \).

Returning to black holes, we apply these standard results to the form \( Q_{p,q} \). Curiously, for large \( |D_{p,q}| \), where the supergravity approximation is accurate, the near-horizon metric of the black hole only depends on \( |D_{p,q}| \), and in particular the entropy is \( S = A/4 = \pi \sqrt{-D_{p,q}} \), where \( A \) is the area of the horizon (4.4). A consequence is that, if \( (p, q) \mathbb{Z} \subset \mathcal{I}^{6,22} \) is primitive, and if \( Q_{p,q} \) is primitive, then the ideal classes of \( K_{D_{p,q}} \) are in 1-1 correspondence with the \( U \)-duality classes of black holes of fixed area \( A \). Moreover, for large \( A \) the number of such classes grows like \( A \). Pursuing this line of thought for other models leads to some interesting issues in the arithmetic of lattices, only briefly touched on in (4).

The second (and more substantial) connection to arithmetic begins with the remark that if \( \tau \) is quadratic imaginary then \( E_\tau \) has the property of “complex multiplication.” \( E_\tau \) is a group so we can consider its endomorphism algebra, \( \text{End}(E_\tau) \). In general this algebra is \( \mathbb{Z} \), with \( n \in \mathbb{Z} \) acting as \( z \to nz \). However, if \( \tau \) is quadratic imaginary so that
$a\tau^2 + b\tau + c = 0$, for $a, b, c \in \mathbb{Z}$ then the endomorphism algebra of $E_\tau$ is larger than $\mathbb{Z}$ because:

$$\omega \cdot (\mathbb{Z} + \tau \mathbb{Z}) \subset \mathbb{Z} + \tau \mathbb{Z} \quad \text{for} \quad \omega = \frac{D + \sqrt{D}}{2}. \quad (4.2)$$

Under these circumstances we say that “$E_\tau$ has complex multiplication by $z \to \omega z$.”

When $E_\tau$ has complex multiplication the curve has special arithmetic properties. Assume for simplicity that $j \neq 0, (12)^3$. Then we can choose a Weierstrass model:

$$y^2 = 4x^3 - c(x + 1)$$

$$c = \frac{27j}{j - (12)^3} \quad (4.3)$$

Now, one of the truly amazing properties of the $j$ function, essentially the first main theorem of complex multiplication, is that if $\tau$ is quadratic imaginary then $j(\tau)$ is an algebraic integer. Here are a few amusing examples (Many more are available in [15], and elsewhere):

$$j(i) = (12)^3 = 1728$$

$$j(2i) = (66)^3 = 287496$$

$$j(3i) = 76771008 + 44330496\sqrt{3} \quad (4.4)$$

$$j(4i) = 2^3 \cdot 3^3 \cdot 5 \cdot 181 \cdot (210319) + 2 \cdot 3^7 \cdot 7^2 \cdot 11^2 \cdot 19 \cdot 59\sqrt{2}$$

In fact, much more is true. The field extension $\hat{K}_D \equiv K_D(j(\tau)) = \mathbb{Q}[\tau, j(\tau)]$ is a certain Galois extension of the quadratic imaginary field known as a “class field.” It follows after a little work that in general

the attractor varieties are arithmetic varieties, defined over number fields associated to the field of definition of the periods.

Moreover, in the FHSV model, at an attractor point determined by $p_0, q_0 \in \mathcal{H}^{10,2}(2)$, the BPS mass-squared spectra are, up to an overall constant, integers reflecting the arithmetic of $K_D$: They are norms of certain ideals in the ideal class determined by $Q_{p_0, q_0}$. Thus, BPS mass-generating functions, such as occur in various quantum corrections to low energy effective actions, are generalizations of $L$-functions and $\zeta$-functions of $K_D$.

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1 A quite different connection between class field theory and string theory and conformal field theory was proposed by Witten in [14].

2 It is possible that in some examples one must take a further finite abelian extension of the class field itself.
5. Rational theories are attractive

The attractive K3 surfaces singled out by the attractor mechanism also have an interesting interpretation in 8-dimensional $F$-theory compactifications. The moduli space of the $F$-theory compactification is (neglecting the heterotic dilaton, and discrete identifications) a Grassmannian of spacelike 2-planes:

$$\mathcal{B}^{18,2} \equiv Gr^+_2 (II^{18,2} \otimes \mathbb{R}). \quad (5.1)$$

Regarding this space as a space of projection operators the attractor equations state that

$$p_L = q_L = 0. \quad (5.2)$$

where the subscript indicates the “leftmoving projection” onto the 18 dimensional negative definite subspace of $\mathbb{R}^{18,2}$. On the other hand, identifying (5.1) as Narain moduli space, the equations (5.2) are the equations defining the rational torus compactifications of the heterotic string, so heterotic rational conformal field theories (RCFT’s) correspond to IIB compactification on attractor varieties. There are at least three interesting aspects of this connection.

First, following through $F$-theory duality one gets some new insight into the role of the Mordell-Weil group in an $F$-theory compactification. The torsion-free part of the Mordell-Weil group is an even lattice and therefore generates a chiral vertex operator algebra. This chiral algebra is the enhanced chiral algebra of the heterotic compactification, and is related to the algebra of BPS states [17].

Second, this connection suggests that there is a more general relation between heterotic RCFT compactifications and type II attractor compactifications. The realization in 8-dimensional compactifications is too trivial to be really useful. Nevertheless, an extension to lower dimensional compactifications might be very useful. Such a connection realizes in a modest way part of a dream of Friedan and Shenker [18]: a dense set of arithmetic points in Calabi-Yau moduli space corresponds to rational conformal field theories. Indeed, the attractive K3 surfaces constitute an isolated, but dense, set of points in the moduli space of complex structures of algebraic K3 surfaces.

Third, this connection suggests an interesting arithmetic property of the K3 mirror map. The natural coordinates in heterotic compactification are the standard flat coordinates $y = (T, U, \vec{A})$ parametrizing flat triples $(G, B, \vec{A})$ on $T^2$ where $G$ is a metric, $B$ a 2-form, and $\vec{A}$ an $E_8 \times E_8$ connection. The RCFT compactifications of the heterotic string...
are just those such that $y$ is quadratic imaginary. On the other hand, on the IIB side, the natural coordinates on (5.1) are the $\vec{\alpha}, \vec{\beta}$ coefficients in the Weierstrass model of an elliptically fibered K3 with section [19][20]:

$$ZY^2 = 4X^3 - f_8(s,t)XZ^2 - f_{12}(s,t)Z^3$$

$$f_8(s,t) = \alpha_{-4}s^8 + \cdots + \alpha_{+4}t^8$$

$$f_{12}(s,t) = \beta_{-6}s^{12} + \cdots + \beta_{+6}t^{12}$$

The $F$-map, or $K3$ mirror map, is the map from an appropriate equivalence class of $y$ to an appropriate equivalence class $[(\vec{\alpha}, \vec{\beta})]$ (the latter class is that following from changes of coordinates in (5.3)). This map should have properties quite analogous to those of the $j$-function: quadratic imaginary $y$’s should map to arithmetic values of $(\vec{\alpha}, \vec{\beta})$. In [1] we verify that this is indeed the case for some families of K3 surfaces.

6. Attempts to understand general $\mathcal{N} = 2$ attractors

It is also of great interest to extend the above results to more general CY compactifications. Naturally, given the examples of $T^6$ and $K3 \times T^2$ one leaps to the conclusion that all attractors are arithmetic. This leap is formulated as three “arithmetic attractor conjectures” in section 8.2 of [1]. These conjectures state that at an attractor point the periods and the algebraic complex structure coordinates are both arithmetic, and the relation between them generalizes the relation associating to quadratic imaginary $\tau \in K_D$ the value $j(\tau)$ in a class field. If something like the attractor conjectures is actually true, it would be yet another example of the unreasonable effectiveness of physics in mathematics, for it could be the beginning of a solution to one of the more stubborn Hilbert problems.

Unfortunately, it is not easy to find concrete examples verifying the attractor conjectures. In the context of toric constructions of Calabi-Yau 3-folds these conjectures are related to difficult questions about the arithmetic values of certain (GKZ) hypergeometric functions generalizing the Appel and Lauricella functions. In section 8.3 of [1] we do manage to describe one slightly nontrivial example. It involves the two-parameter family of CY 3-folds of degree 8 in $\mathbb{P}^{1,1,2,2,2}$ studied in [22]. The attractor conjectures can be verified along a special divisor in the complex structure moduli space. Admittedly, the

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3 A somewhat different arithmetic aspect of mirror maps was discussed by Lian and Yau in [21]. This involves the nature of the coefficients of mirror maps in a $q$-expansion.
example does not really leave the world of quadratic imaginary fields and is a bit too special to constitute major evidence for the attractor conjectures. Happily, one spinoff of the example is the demonstration that there exist charges with two attractor points, as mentioned above.

7. Further speculations: Galois groups and height functions

Regrettably, the connection between attractor varieties and complex multiplication lends itself to rampant speculation. We have suppressed most of this but two further speculations do seem irresistible.

First, in $\mathcal{N} = 4, 8$ compactification since the algebraic complex structure moduli are valued in a class field $\hat{K}_D$ of $K_D$ the Galois group $\text{Gal}(\hat{K}_D/K_D)$ acts on the vector multiplet attractor moduli corresponding to blackholes of entropy $\pi\sqrt{-D}$ and permutes them. Recall that these points correspond to $U$-duality inequivalent backgrounds. Nevertheless, although $U$-duality-inequivalent, they are “unified” by the Galois action. So, does the absolute Galois group play an analogous unifying role for other compactifications?

Second, the exponential growth of the dimension of the space of BPS states, which accounts for black hole entropy in the discussion of Strominger and Vafa [6], is a little reminiscent of the theory of height functions. Since the $\mathcal{N} = 8$ attractor varieties are isogenous to products of elliptic curves we can explore this relation in a semi-quantitative way, and an attempt in this direction is described in [1]. The result, which we stress is preliminary, is that at least for some charges the black hole entropy $S = \pi \sqrt{I_4(\gamma)}$ compares favorably with the logarithmic Faltings height $h(X_\gamma)$ of the abelian variety (3.1). Using results of Faltings and Silverman we give a rough estimate in [1] suggesting that $h(X_\gamma/\hat{K}) \sim \kappa \log(S/\pi)$ where we expect $\kappa$ to be a simple rational number of order one, but could not determine it.

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