Clifford Residues and Charge Quantization

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Abstract

We derive the quantization of action, particle number, and electric charge in a Lagrangian spin bundle over $\mathbb{M} \equiv M \setminus \cup D_J$, Penrose’s conformal compactification of Minkowsky space, with the world tubes of massive particles removed.

Our Lagrangian density, $L_g$, is the spinor factorization of the Maurer-Cartan 4-form $\Omega^4$; it’s action, $S_g$, measures the covering number of the 4 internal $u(1) \times su(2)$ phases over external spacetime $M$. Under $PTC$ symmetry, $L_g$ reduces to the second Chern form $TrK_L \wedge K_R$ for a left $\oplus$ right chirality spin bundle. We prove a residue theorem for $\mathfrak{gl}(2, \mathbb{C})$-valued forms, which says that, when we “sew in” singular loci $D_J$ over which the $u(1) \times su(2)$ phases of the matter fields have some extra twists compared to the 8 vacuum modes, the additional contributions to the action, electric charge, lepton and baryon numbers are all topologically quantized. Because left and right chirality 2-forms are chiral dual, forms are quantized over their dual cycles. Thus it is the interaction $c_2(E)$, with a globally nontrivial magnetic field, that forces electric fields to be topologically quantized over spatial 2 cycles, $\int_{S^2} K_{0r} e^0 \wedge e^r = 4\pi N$.

1 Introduction

Yang-Mills monopoles have topologically quantized magnetic charges because it is the magnetic parts, $K_{jk} e^j \wedge e^k$ ($j, k = 1, 2, 3$), of their $su(2)$-valued spin-curvature 2 forms, $K = d\Omega + \Omega \wedge \Omega$, that “wrap” integrally around spatial 2-cycles. For electric charge to be topologically quantized, the electric field would have to wrap integrally about its dual (spatial) cycle,

$$\int_{S^2} K_{0r} e^0 \wedge e^r = 4\pi N;$$

Gauss’ law.
The instantons and dyons of Yang Mills theories, which possess (anti-) self-dual curvatures, also possess nonzero electric fields, \( K_{ij} = \pm \epsilon^{kl} K_{kl} \), and electric charges that are quantized because their magnetic charges are. However, they live in a Euclidean four-space. Any analogous construction in Minkowsky space, where \( ** = -1 \), must have imaginary (anti-) self-dual curvatures \( \ast K = \pm i K \).

We exhibit a model here in which Left and Right spin curvatures are imaginary chiral dual: \( K_R = \pm i K_L \). Localized chiral dual solutions are dyons with half-integral units of electric and magnetic charges. For \( PT \) antisymmetric \( (PTA) \) solutions, the electric semi-charges add, while the magnetic semi-charges cancel, thus binding together the left and right chiral halves into a bispinor particle.

In the work of Van der Waerden [2], Sachs [3], Penrose [4], and Keller [5], it becomes clear that geometric and Fermionic fields are the integral and half-integral sectors of one unified spin-4 tensor field.

In a companion paper [7] (see also [8]), we exhibited a grand-unified Lagrangian density,

\[
\mathcal{L}_g = i \int_M d\xi^\pm \eta^\pm \wedge d\eta^\mp \wedge \chi \pm \wedge \sigma^\mp d\xi^\pm ,
\]

(sum over all neutral sign combinations) invariant under the group \( E_P \) of passive Einstein transformations; Sachs’ [3] term for the global extension of the Poincaré group to a Friedmann universe. \( E_P \) transformations connect the same physical state in the moving frames of different observers. In the \( PTC \)-symmetric geometrical optics \((g.o.)\) regime in \( \mathbb{M} = \mathbb{M}_g \cup D_J \), outside the singular loci \( D_J \), \( \mathcal{L}_g \) reduces to the Maurer-Cartan 4 form. This gives a natural topological action

\[
S_g = i \int_M Tr \Omega^L \wedge \Omega^R \wedge \Omega^R \wedge \Omega_R \equiv i \int_M \tilde{\mathcal{L}}_g,
\]

which measures the covering number of spin space over spacetime, and comes in quantized units.

\( \mathcal{L}_g \) of (1) is not unique—but its action \( S_g \) does have a desirable feature: The terms in \( S_g \) decompose into effective electroweak, strong, and gravitational potentials and curvatures, together with their proper field actions \( \tilde{\mathcal{L}}_g \). We show here, using spin residues—“winding numbers” of \( gl(2, \mathbb{C}) \)-valued forms about each codimension \( J \), singularity \( D_J \)—that these actions and charges are topologically quantized.

The singular loci \( D_J \) are where \( J = 1, 2, 3, \) or 4 pairs of spin rays cross, forming caustics. Here the \( gl(2, \mathbb{C}) \) phases of \( J \) chiral pairs of spinors, i.e. the local, path-dependent exponents in the geometrical optics \((g.o.)\) ansatz

\[
\psi(x) = e^{\frac{i}{2} (\sigma^a(x) + i \psi^a(x)) \sigma^a \psi(0)} \equiv e^{\frac{i}{2} \psi^a(x) \sigma^a \psi(0)},
\]

cannot be defined. This happens when

1. \( D_J \) contains a zero of \( \psi \equiv \xi_{\pm}, \eta_{\pm}, \zeta_{\pm}, \) or \( \chi_{\pm} \);
2. $\psi$ or $d\psi$ is undefined somewhere in $D_J$, i.e. $D_J$ contains a *singular point* of $\psi$;

3. the phases of each field in $\mathfrak{g}$ are *defined*, but $J$ pairs break away from PTC conjugacy. The transformations that create these states violate the *spin isometry condition*

$$\zeta^\pm \xi_\mp = 1 = \chi^\pm \eta_\mp. \quad (4)$$

4. $J$ of the 4 gradients in $L_g$ become *linearly* dependent in $D_J$, and so fail to span a 4-volume element. The remaining pairs span the $(4 - J)$-surface over which the $J$ broken out fields are quantized, as we show below.

We call the row spinors $\zeta^\mp$ and $\chi^\mp$ in (1) the *Baryonic spinors*. They must be treated as *independent variables* from the *leptonic* (column) spinors $\xi_\pm$ and $\eta_\pm$ in the variation of $L_g$ within each singular domain $D_J$. In the companion paper [7], we identify codimension $J = 1, 2, 3, \text{ and } 4$ *topological defects* in the multi-spinor fields with leptons, bosons, hadrons and their reaction vertices, respectively. Inside the $D_1$, $L_g$ gives Dirac equations coupling each chiral pair of matter fields through nonlinear scatterings with the vacuum fields, thus creating the effective masses of bispinor particles [7].

However, it is not necessary to unravel the detailed structure of these core regions to prove that they carry *integral charges*—electric charge, lepton number, and baryon number—and of *action*, provided that the “inner” solutions for $L_g$ match the “outer” (g.o.) solutions for $\hat{L}_g$ outside the singular domains, i.e. in $M \equiv M_\# \cup D_J$.

Below we prove a $(3 + 1)$-dimensional Clifford residue theorem for Lie-algebra-valued forms, that says each singular domain contributes integral units of action and charge for *any* Lagrangian density that is a natural 4 form. The argument breaks down into four steps:

1. Separate the action into outer (field) and inner (matter) contributions,

$$S_g = \int_M \hat{L}_g + \int_{\cup D_J} L_g = S_F + S_M.$$  

2. Show that the field action for the vacuum spin bundle $\hat{\Psi}$ over the compact base space, $M_\# \equiv S^1 \times S^3$, is topologically quantized.

3. Act on $\hat{\Psi}$ with topologically nontrivial active local Einstein ($E_A$) transformations that may become singular in codimension-$J$ domains $D_J$.

4. Show that the resulting *field actions* and *charges* are all topologically quantized over $M$. 

3
Spin Connections and Maurer-Cartan Forms

We briefly review how spinors factor the “internal” Lie-algebra $gl(2, \mathbb{C})$ of conformal spinors (see Appendix). The affine spin connection $\Omega$ gives the spin-space increment that corresponds to each space-time increment, and vice versa. $\Omega$ is a $gl(2, \mathbb{C})$-valued 1 form that enters into the covariant derivative to assure covariance under coupled internal/external spin transformations in any moving frame.

We specialize below to spacetime and spin frames adapted to a Friedmann universe; an expanding “3 brane” $S_3(T)$ that, at “cosmic” time $T$, is approximately a hypersphere $S^3(a) \subset \mathbb{R}^4$, with radius

$$a(T) = e^{\frac{x}{a\#}} a\# \equiv \gamma a\#. \quad (5)$$

Here $a\#$ is the equilibrium radius [9]; $\gamma$ is the conformal scale factor.

The real radial coordinate $T$ is not directly visible to us as observers embedded in $S_3(T)$. In relativistic kinematics, $T$ is replaced by arctime $x^0 \subset S^1$: the arclength travelled on $\tilde{S}^3$ by a photon, projected down to $S^3(a\#)$, the fiducial three-sphere of stationery radius $a\#$.

Arctime $x^0$ enters [9] as the real part of a complex time coordinate $z^0 \equiv x^0 + iy^0$; cosmic time $T \equiv y^0$ is the imaginary part. We do our local physics in a dilation-invariant way by projecting down to $M\# \equiv S^1 \times S^3(a\#)$, Penrose’s conformal compactification of Minkowsky space, with canonical (Lie-algebra) “coordinates” $x = (x^0, x^1, x^2, x^3)$.

$M\#$ is a very nice space on which to work, because it is a Lie group:

$$M\# \equiv S^1 \times S^3 \sim U(1) \times SU(2) .$$

$S^3$ has two natural representations of translation, Left ($L$) and Right ($R$), that derive from Left or Right translation in $SU(2)$. These are the two chiralities.

Adding a $u(1)$ generator $\sigma_0$ to each, we obtain $\sigma_\alpha \in u(1) \times su(2)_L$ and $\bar{\sigma}_\alpha \in u(1) \times su(2)_R$, the left and right Lie algebras. These must be viewed as independent generators of chiral $U(1) \times SU(2)$. However, note that $\bar{\sigma}_\alpha$ is the dual Lie algebra to $\sigma_\alpha \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, under the Clifford-Killing form for the Minkowski metric, $\eta_{\alpha\beta} \equiv diag (1, -1, -1, -1)$:

$$\{\sigma_\alpha, \bar{\sigma}_\beta\} \equiv \sigma_\alpha \bar{\sigma}_\beta + \sigma_\beta \bar{\sigma}_\alpha = 2\eta_{\alpha\beta} \sigma_0. \quad (6)$$

$$\sigma^\alpha = \bar{\sigma}_\alpha; \quad \sigma^\alpha \sigma_\rho = -2, \quad (7)$$

is the Lorenz-invariant form.

We may thus define the Clifford product of “spinorized” tangent vectors $a, b \in TM\#$,

$$\bar{b} = b^\beta \bar{\sigma}_\beta : \frac{1}{2} (a b + b a) = \eta_{\alpha\beta} a^\alpha b^\beta \sigma_0 \equiv a_\beta b^\beta \sigma_0. \quad (8)$$
This is the scalar $\sigma_0$ in the Lie algebra times the Minkowsky product of the vectors. Note that the Clifford scalar is picked out by the Trace:

$$\frac{1}{2} Tr(ab) = a_\beta b^\beta = a_0 b^0 - a_1 b^1 - a_2 b^2 - a_3 b^3.$$  \hspace{1cm} (9)

In curved spacetime (A11), the $\eta_{\alpha\beta}$ are replaced by the metric coefficients $g_{\alpha\beta}$.

The columns of spin frames (A5) are a basis for the fundamental $L$ and $R$ chirality spinors $\xi_{\pm}(x)$ and $\eta_{\pm}(x)$ painted on $\mathbb{M}_\#$ by the spinorization maps

$$S: g_{\pm}(x) \equiv \exp \left( \frac{i}{2a_\#} x^\alpha \sigma_\alpha^{\pm} \right) : \mathbb{S}^1 \times \mathbb{S}^3 \to U(1)_{\pm} \times SU(2)_L$$

$$\bar{S}: \bar{g}_{\pm}(x) \equiv \exp \left( \frac{i}{2a_\#} x^\alpha \bar{\sigma}_{\alpha}^{\pm} \right) : \mathbb{S}^1 \times \mathbb{S}^3 \to U(1)_{\pm} \times SU(2)_R,$$  \hspace{1cm} (10)

where $\sigma_\alpha^{\pm} \equiv (\pm \sigma_0, \sigma)$. Their infinitesimal versions are the $L$- and $R$-invariant Maurer-Cartan 1 forms:

$$TS(x) \equiv g_{\pm}^{-1} dg_{\pm}(x) = \frac{i}{2a_\#} \sigma_\alpha^{\pm} e^\alpha(x) : e_{\beta}(x) \to \frac{i}{2a_\#} \sigma_\beta^{\pm}(x)$$

$$\bar{T}\bar{S}(x) \equiv \bar{g}_{\pm}^{-1} d\bar{g}_{\pm}(x) = \frac{i}{2a_\#} \bar{\sigma}_{\alpha}^{\pm} \bar{e}^\alpha(x) : \bar{e}_{\beta}(x) \to \frac{i}{2a_\#} \bar{\sigma}_{\beta}^{\pm}(x).$$  \hspace{1cm} (11)

The Maurer-Cartan 1 forms give the images in the “internal” Lie algebras $u(1)_{\pm} \times su(2)_L$ and $u(1)_{\pm} \times su(2)_R$ of infinitesimal $L$ and $R$ translations on $\mathbb{M}_\#$; i.e. the canonical spin-space increments that accompany a spacetime translation on $\mathbb{M}_\#$.

In the presence of a source, a translation is accompanied by active local spin space increments $\ell(x)$ and $r(x)$ in the reference frame of an observer $O$. $O$ then experiences the vector potentials

$$\Omega_L \equiv \ell^{-1} d\ell = \Omega_{La} \epsilon^a;$$

$$\Omega_{La} = \ell^{-1} \partial_a \ell; \quad \Omega_R \equiv r^{-1} dr.$$  \hspace{1cm} (12)

The Lie-algebra-valued 1 forms, or spin connections $\Omega_L$ and $\Omega_R$ are the Maurer-Cartan 1 forms for local $GL(2,\mathbb{C})$ deformations $\ell(x)$ and $r(x)$ of the canonical maps (10) of spacetime into spin space (see (A10) below). Regular g.o. perturbations do not change the rank of the mapping $\psi$ of physical space to spin space.

### 3 Vector Potentials from Active Local Spin Transformations

Active local ($E_A$) transformations represent both local dilation/boost flows and local $U(1) \times SU(2)$ phase flows in the geometrical optics (g.o.) regime. $E_A$ transformations on the tetrads (A8), (A9) are presented as complexified chiral

$$U(1) \times SU(2) \overset{c}{\rightarrow} GL(2,\mathbb{C})$$
spin transformations on the canonical spin frames:

\[
\ell (z) = \ell (0) L (z) \equiv \ell (0) \exp \left( \frac{i}{\hbar} \left( \varphi^\alpha_L (z) + i \varphi^\alpha_R (z) \right) \sigma_\alpha \right) \equiv \ell (0) \exp \left( \frac{i}{\hbar} \sum (z) \sigma_\alpha \right)
\]

\[
\bar{r} (z) = \bar{R} (z) \bar{r} (0) \equiv \exp \left( \frac{i}{\hbar} \left( \varphi^\alpha_R (z) + i \varphi^\alpha_L (z) \right) \sigma_\alpha \right) \bar{r} (0) \equiv e^{\frac{i}{\hbar} \sum (z) \sigma_\alpha \bar{r} (0)} ,
\]

where we may take \( \ell (0) = \sigma_0 = r (0) \).

In a spin bundle \( E \) with a momentum flow \( y^\beta (x) \), the Cartan moving spin frames (13) are path dependent functions of \( x \). The \( \theta^\alpha_L (x) \) are the coefficients of the anti-Hermitian (aH) matrices \( i \frac{1}{2} \sigma_\alpha \) that generate (local) unitary \( U (1) \times SU (2)_L \) spin transformations. Their differentials are the electroweak vector potentials:

\[
\frac{i}{2} \mathbf{d} \theta^\alpha \sigma_\alpha \equiv W^\beta e^\beta.
\]

The \( \varphi^\alpha_L (x) \) are the coefficients of the Hermitian (H) generators \( \frac{1}{2} \sigma_\alpha \) which give the local dilation/boost flow, and whose differentials are the gravitational potentials,

\[
\frac{i}{2} \mathbf{d} \varphi^\alpha \sigma_\alpha \equiv \Phi^\beta e^\beta.
\]

For example, the Newtonian potential \( \mathbf{d} \varphi^0 (x) \) represents a local contraction of the spatial step corresponding to a fixed increment in the amplitude of the spinor fields. Outside the singular loci, we expect the phase flow to be analytic, so the Cauchy–Riemann equations will hold:

\[
\frac{\partial \varphi^\alpha}{\partial z^\beta} = 0 = \frac{\partial \theta^\alpha}{\partial x^\beta} = \frac{\partial \varphi^\alpha}{\partial y^\beta} = \frac{\partial \theta^\alpha}{\partial x^\beta}.
\]

There the \( gl (2, \mathbb{C}) \) phase factors \( \theta^\alpha (z) \) and \( \varphi^\beta (z) \) in (13) are functions of \( z \), the position-momentum coordinates assigned to a point in phase space by an observer, \( O \). \( E \) transformations thus act on the complexified tetrads

\[
q_\alpha (z) \equiv \ell (z) \otimes_\alpha \bar{r} (z);
\]

\[
z \equiv z^\beta = x^\beta + iy^\beta, \beta = 0, 1, 2, 3;
\]

\[
z^\beta \in \mathbb{C}M \subset T^*M.
\]

The \( z^\beta \) are 4 complex coordinates on the Dirac phase space.

Just as \( q (t) \) is the complex position-momentum vector for the harmonic oscillator \( q (t) = i \omega q (t) \) as a first order system, the \( q_\alpha \) are complex vectors in the position-momentum frame bundle \( \mathbb{C}M \). This complex structure, along with the antisymmetric inner product

\[
\langle \ell_1, \ell_2 \rangle \equiv |\ell| = \ell_1^T \ell_2 \equiv \ell \ell^1, \ell_2 ,
\]

gives a symplectic structure on \( T^*\mathbb{M} \). The norm of a spin frame is its determinant (16), the area in phase space that it spans.
The canonical spin connections on $\mathbb{M}_\#$ are obtained for $y^\beta = 0$; they are the Maurer-Cartan 1 forms (11) on the Lie groups $U(1)_\pm \times SU(2)_L$ and $U(1)_\pm \times SU(2)_R$:

$$\hat{\Omega}_L = g^{-1}_- d g_+ = \frac{i}{2a_#} \sigma^\alpha \gamma_\alpha, \quad \hat{\Omega}_R = g^{-1}_+ d g_- = \frac{i}{2a_#} \bar{\sigma}^\alpha \gamma_\alpha. \quad (17)$$

It is important to note that Lagrangian (1) contains wedge products of right and left Lie algebra-valued forms:

$$\hat{\Omega}_L^\alpha \wedge \hat{\Omega}_R^\beta = \frac{1}{2a_#} (\sigma_0 e^0 + \bar{\sigma}_j e^j) \wedge \frac{1}{2a_#} (\sigma_0 e^0 + \sigma_k e^k)$$

$$= -\frac{1}{2a_#} \sigma_j \left[ e^0 \wedge e^j + \frac{i}{2} e^j e^k \wedge e^k \right], \quad (18)$$

for the vacuum spin connections (17) on $\mathbb{M}_\#$. Note that $\Omega^L \wedge \Omega_L$ includes both magnetic $(e^k \wedge e^l)$ and electric $(e^0 \wedge e^j)$ components. No electric components would have appeared without the $P$-conjugation in (13).

The wedge product of two left and two right Maurer-Cartan 1 forms makes the Maurer-Cartan 4 form, the scalar in the Lie algebra times the volume form:

$$\Omega^4 \equiv \frac{1}{2} Tr \Omega^L \wedge \Omega_L \wedge \Omega^R \wedge \Omega_R; \quad (19)$$

$$\Omega^4 \equiv \left( \frac{1}{2a_#} \right)^4 \frac{i}{16a_#} \frac{1}{2} Tr (e^0 \wedge e^1 \wedge e^2 \wedge e^3) \equiv \frac{i}{16a_#} d^4 V. \quad (20)$$

The $\frac{1}{2} Tr$ picks out the scalar component, $\sigma_0$.

By definition, all integrands must be scalars, i.e. multiples of the Clifford unit, $\sigma_0$. This is especially clear in curved space, where the Clifford-algebra frame $\sigma(z)$ varies from point to point. It is a standard calculation to check that \( \int \Omega^4 \) is invariant with respect to the full conformal group of nonsingular local $E$ transformations, $e^\alpha(z) = \Lambda^\alpha_\beta e^\beta(z)$, of the 1 forms and their “internal” representations (13), (A10) on spinor and spin-vector fields. \( \int \Omega^4 \) is also $P$, $T$, and $C$ invariant. Furthermore, scalar functions, $f(x) \Omega^4$, of the Maurer-Cartan 4 form are the only 4 forms that can be invariantly integrated! This is because all natural 4 forms are scalar multiples of the volume form, (13).

Our Lagrangian density, $\Omega^4$, of (13) is the invariant measure on the Einstein group

$$E = \mathbb{C}(U(1) \times SU(2))^4 = GL(2, \mathbb{C})^4. \quad (21)$$

Its integral gives the covering number $W$ of the group manifold $E$ over spacetime, $\mathbb{M}$. Local extrema of the action integral (1) over $\mathbb{M}$ are achieved \( \hat{\Omega}_L \), \( \hat{\Omega}_R \) when all 4 pairs are PTC symmetric. From (17),

$$S_F \equiv \int_{\mathbb{M}} L_g \frac{PTC}{2} \int_{\mathbb{M}} Tr \Omega^L \wedge \Omega_L \wedge \Omega^R \wedge \Omega_R \equiv \int_{\mathbb{M}} \Omega^4 \equiv -16\pi^3 W. \quad (22)$$
For \( M_\# = S^1 \times S^3 \),

\[
i \int_{S^1 \times S^3} \hat{\Omega}^4 \equiv -16\pi^3.
\] (23)

Spin frames are the fundamental degree-1 maps of spin space over \( M_\# \).

When singularities of map (11) that assigns spin space increments to spacetime increments are present, we simply restrict \( TS \) to the regular region \( \bar{M} \equiv M_\# \setminus \cup D_J \) where all 4 spin connections are defined. The singular loci \( D_J \) are the supports of matter fields in this model.

The global spin connections \( \hat{\Omega} \) provide a minimum vacuum energy (23). But they have another dramatic effect. When wedge products \( \hat{\Omega}^{4-J} \) multiply local perturbations \( \tilde{\Omega}^J \), they effectively quantize their Hodge dual fields over Poincaré dual cycles \( \gamma^{4-J} \). This happens because products of Clifford-algebra-valued forms require both their Clifford and Hodge duals to make the Clifford scalar \( \sigma_0 \) times the volume element (20). This leads to a residue theorem below that classifies the topological obstructions \( D_J \) to relaxation of the field energy \( V_F = -S_F \), to the global minimum \( 16\pi^3 \) of (23).

### 4 Clifford Residues and deRham Cohomology

The charges in nature—electric charge, mass, baryon number, etc.—are detected by integrating far fields in the regular region, outside the supports \( B_3 \) of their respective current 3 forms \( *J(x) \). If the same far field could be produced by an active local spin transformation, \( T_A(x) \in E_A \), acting on the vacuum fields around \( B_3 \), then the same charge would be detected within \( B_3 \). What happens inside our singular domains \( D_J \) is that the diagonal (PTC-symmetric) subalgebra breaks back to the full Lie algebra of independent \( L \times R \) spin transformations:

\[
gl(2, \mathbb{C})_{PTC} \xrightarrow{D_J} gl(2, \mathbb{C})_L \oplus gl(2, \mathbb{C})_R
\] (24)

for each of the \( J = 1, 2, 3, \) or 4 chiral pairs that break away from PTC conjugacy.

We now remove open neighborhoods \( B_J \) containing each singular locus \( D_J \), and consider the effect on action integral (22).

Uhlenbeck’s theorem and Taubes patching assure us that we can replace any vector potential singular inside a domain \( D_J \) by a regular connection, and change the action by an integral multiple of \( 8\pi^2 \). We prove an analogous result for spin bundles below.

Suppose \( \ell(z) \) of (15) is a section of the (left) \( gl(2, \mathbb{C}) \) spin-frame bundle over the Dirac phase space \( \mathcal{CM} \subset \mathbb{C}^4 \), with the singular loci removed [1]. We may write \( \ell(z) \) in polar form as

\[
\ell(z) = \ell(0) \exp \left( \frac{i}{2} \theta L^\alpha (z) - \frac{1}{8} \varphi L^\alpha (z) \right) \sigma_\alpha \\
\equiv \ell(0) \exp \frac{i}{2} \xi L^\alpha (z) \sigma_\alpha,
\] (25)
just as we may write a complex function \( w(z) \) of one complex variable as
\[
w(z) = w(0) \exp(i\theta(z) - \varphi(z))
\equiv w(0) \exp i\varsigma(z),
\]
with the phase \( \varsigma(z) \) complex.

When phase singularities are present, \( \theta \) becomes path-dependent. But \( \varphi \) does not, provided \( w(z) \) is single-valued. The phase advance around a 1-cycle \( \gamma \) parametrized by \( t \), enclosing \( N \) zeroes and \( M \) poles of \( w \), is the logarithmic residue
\[
\int_{\gamma} w^{-1}(z) dw(z) \equiv \int_{\gamma} w^{-1} \left( \frac{dw}{dt} \right) dt = i \int_{\gamma} d\theta(z) = i2\pi m,
\]
where \( m \equiv N - M \). It detects the winding number of the \( u(1) \) phase about singularities by integrating about 1-cycles that lie completely within the regular region.

The analog for spin bundles \( E \) is obtained by integrating \( gl(2, \mathbb{C}) \)-valued \( m \) forms about cycles \( \gamma_m \) that lie completely within the regular region \( \mathbb{C}M \). On \( \gamma_m, z \equiv (z^0, z^1, z^2, z^3) \) is parameterized by \( t^\alpha \).

We define the integral of a \( gl(2, \mathbb{C}) \)-valued \( m \) form \( \omega^m \) on and \( m \)-chain \( \gamma_m \) as the integral of its scalar component,
\[
\int_{\gamma_m} \omega^m \equiv \frac{1}{2} \int_{\gamma_m} Tr\omega^m.
\]
Note (A11) that products of left and right \( gl(2, \mathbb{C}) \)-valued forms make the Clifford scalar, \( \sigma_0 \). We call such products Clifford-algebra-valued forms.

We may now state and prove a residue theorem for Clifford-algebra-valued Forms (see (12), (13) for the \( \gamma_m = S_{n-1} \) case):

**Theorem 1:** The Clifford residues,
\[
\int_{\gamma_1} \ell^{-1} d\ell = i2\pi m,
\]
\[
\int_{\gamma_3} r^{-1} d\ell \wedge d\ell^{-1} \wedge r^{-1} d\ell = 8\pi^2 m_1 m_2 m_3,
\]
and
\[
\int_{\gamma_4} d\ell^{-1} \wedge r^{-1} d\ell \wedge d\ell^{-1} \wedge r^{-1} d\ell = i16\pi^3 m m_1 m_2 m_3
\]
for bundles of \( gl(2, \mathbb{C}) \) spin frames over \( \mathbb{C}M \) are quantized about 1-cycles \( \gamma_1 \), 3-cycles \( \gamma_3 \), and 4-cycles \( \gamma_4 \). The periods \( m, m_1, m_2, \) and \( m_3 \) are integers that are invariant under nonsingular PTC-symmetric local deformations,
\[
\ell'(z) = \ell L(z),
\]
\[
r'(z) = R(z) r = L^{-1}(z) r,
\]
provided that \( \ell'(z) \) and \( r'(z) \) remain single-valued about \( \gamma \).
**Proof:** Using the identity

\[
\exp \left( \frac{i}{2} \xi^a \sigma_a \right) = \cos \frac{\xi}{2} + i \left( \sin \frac{\xi}{2} \right) 2^{a} \sigma_a,
\]

we calculate from (29) that

\[
\ell^{-1} d\ell'(z) = \left( \frac{i}{2} d\theta^\alpha(z) - \frac{i}{2} d\varphi^\alpha(z) \right) \sigma_a \\
\equiv \frac{i}{2} d\kappa^\alpha(z) \sigma_a.
\]

Assuming that the column spinors in \( \ell(z) \) are single-valued on \( \gamma_1 \), the \( U(1) \) phase advance, \( \frac{1}{2} \Delta \theta^0 \), about \( \gamma_1 \) must be an integral multiple of \( 2\pi \):

\[
\int_{\gamma_1} \ell^{-1} d\ell = \int_{\gamma_1} \frac{i}{2} d\kappa_0 \sigma_0 = \int_{\gamma_1} \left( \frac{i}{2} d\theta_0 - \frac{i}{2} d\varphi_0 \right) \sigma_0 \\
= \frac{i}{2} \left[ \Delta \theta^0 \right]_{\gamma_1} - \frac{1}{2} \left[ \Delta \varphi^0 \right]_{\gamma_1} = i2\pi m.
\]

Integral (31) is the period about a homology 1-cycle, \( \gamma_1 \subset H_1(C\mathbb{M}) \), of the nonexact differential 1 form,

\[\Omega \equiv \ell^{-1} d\ell \in H^1(C\mathbb{M}),\]

which belongs to the first deRham cohomology class of \( C\mathbb{M} \). Its period, \( m \), is invariant under both homologous deformations, \( \gamma_1' \in H_1(M) \), of the cycle, and nonsingular \( E \) perturbations, \( \ell'(z) = \ell(L(z)) \), of the spin frame. If \( \gamma_1 \) is parametrized by time \( t \), integral (31) measures the difference \( \Delta \theta^0 \) of the \( U (1) \) phase shifts between paths—or the phase shift along a path that winds around the worldtube \( D_3 \times I \) of a massive particle. Integral (31) then gives the Bohr-Sommerfeld quantization conditions.

In the spinfluid regime, where the dilation/boost flow \( y^\alpha = y^\alpha(x^\alpha) \) is a path-dependent function of 4 position \( x^\alpha \), we could choose the \( x^\alpha \equiv \ell^\alpha \) as our integration parameters. Alternatively, we could choose spherical-polar coordinates and parametrize the spatial 3-ball \( D_3 \) by \((r, \theta, \varphi)\). \( D_3 \) compactifies to a 3-cycle \( \gamma_3 \) when the perturbed fields must match the vacuum fields on its boundary.

We get scalar-valued 3 forms in (27) from terms in \( \sigma_\tau e^\tau \land \sigma_\theta e^\theta \land \sigma_\varphi e^\varphi = i\sigma_\tau e^\tau \land e^\theta \land e^\varphi \). For example, suppose \( \gamma_3 \) contains a radially symmetric \( SU(2) \) "hedgehog" monopole, i.e. a diagonal map from physical space \( M_\# \) to \( \sigma \)-space:

\[
\ell(x) = e^{\frac{i}{2}\theta^0(x)\sigma_\theta e^\theta f(r)\hat{\tau} \hat{\sigma}} \\
r(x) = e^{\frac{i}{2}\theta^0(x)\sigma_\theta e^\theta f(r)\hat{\tau} \hat{\sigma}}.
\]

Then

\[
\int_{\gamma_3} r^{-1} dr \land dr^{-1} r \land \ell^{-1} d\ell = -i \int_{f(r) \times S_2(\theta, \varphi)} d \left[ f(r) \tilde{\sigma}_r e^\tau \land \tilde{\sigma}_\theta e^\theta \land \tilde{\sigma}_\varphi e^\varphi \right] \\
= 2\pi n \cdot 4\pi m = 8\pi^2 M,
\]

where \( \hat{\tau} \cdot \sigma \equiv \sigma_r \), and \( f \) has \( n \) radial cycles over \( I(r) \).
More generally, the $SU(2)$ monopole may have angular dependence as well. Then

$$
\int_{\gamma_3} r^{-1}dr \wedge d\tau^{-1}r \wedge \ell^{-1}d\ell = -i \int_{\gamma_3 \times \partial \mathcal{D}} d\left[f(r) g(\theta, \varphi) \cdot \sigma\right]
$$

(33)

where $M$ must be an integer for $r$ to be single valued on spatial 2-surfaces $\mathcal{S}_2(\theta, \varphi) = \partial \mathcal{D}$ enclosing the support of the monopole fields.

Integral (33) is quantized because it is the period of the 3 form $\Omega^3 \equiv \frac{1}{2} Tr \Omega^3$ over the 3-cycle $\gamma_3$. Similarly, the quantization of

$$
\int_{\gamma_4} d\ell^{-1} \ell^{-1}d\tau \wedge d\tau^{-1}r \wedge \ell^{-1}d\ell = (i2\pi n) \left(8\pi^2 m\right) = i16\pi^3 nm = i16\pi^3 N
$$

(34)

about 4-cycles $\gamma_4 = \gamma_1 \times \gamma_3$ follows from the fact that only terms like

$$
\sigma_0 e^0 \wedge \bar{\sigma}_1 e^1 \wedge \bar{\sigma}_2 e^2 \wedge \sigma_3 e^3 = i\sigma_0 d^4V
$$

can make a scalar-valued 4 form. The integer $N$ is the action contained in $\gamma_4$. ■

Heuristically, the reason for this quantization is easy to see: the $u(1) \times su(2)$ phase gradients of four independent spinor fields must be stretched over the four orthogonal spacetime directions in order for $L_g$ of (1) to reproduce the 4-volume element. Integrals of these gradients are quantized over the “vacuum” $\mathcal{M} \equiv \mathcal{M}_g \cup \mathcal{D}$ and over localized $E_A$ perturbations, provided that these patch smoothly into the vacuum phase distribution outside $\mathcal{D}$. Such perturbations may only integral units to the action. These integers are invariant under “small” $E$ transformations (connected to the identity), and may change value by integer amounts only for the “large” $E_A$ transformations associated with introducing another singularity.

On an expanding deformed space we may write our topological Lagrangian $\mathcal{L}_T$ in intrinsic coordinates as

$$
\mathcal{L}_T = \frac{i}{2} Tr \Omega^L \wedge \Omega^R \wedge \Omega_L
$$

(35)

The $\omega \equiv \omega_\alpha E^\alpha$ are intrinsic spin-connection 1 forms in the coordinate frame of a co-moving observer and $\left|-g\right|^{\frac{1}{2}}$ is his 4-volume expansion factor:

$$
e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \left|-g\right|^{\frac{1}{2}} E^0 \wedge E^1 \wedge E^2 \wedge E^3.
$$

Noting that PTC symmetry gives invariance of the trace,

$$
R(x) = L^{-1}(x) \Longrightarrow \mathcal{L}_T = Tr L^{-1}(x) \Omega^4 L(x) = \mathcal{L}_T,
$$

(36)

we have:
Corollary 1: The action

\[ S_T \equiv \int_M \mathcal{L}_T \]  

is invariant under the group \( E_P \) of passive Einstein transformations which connects tetrads and spinors that could represent the same physical state to observers using different external/internal coordinate/spin frames. These include the proper \( E \) transformations (A15), cosmic expansion, plus the \( P, T, \) and \( C \) reversals which preserve PTC symmetry:

\[
(q_\alpha, \xi_\pm, \chi_\pm) \xrightarrow{P} (\bar{q}_\alpha, \eta_\pm, \zeta_\pm), \quad (\xi_\pm, \eta_\pm) \xrightarrow{T} (\chi_\pm, \zeta_\pm), \quad (\xi_\pm, \eta_\pm) \xrightarrow{C} (\xi_\mp, \eta_\mp). \tag{38}
\]

Furthermore, since all nonzero 4 forms are proportional to the volume element, with a local scale factor that may be taken up into \( |-g|^\frac{1}{2} \) of (35), we have:

\[
\text{Corollary 2: Any Lagrangian density on the multiply-connected space } M = M_\# \cup D_J \text{ that is a natural (i.e. } E_P\text{-invariant) 4 form must be locally proportional to } L_T \text{ of (35).}
\]

\[
\text{Corollary 2 apparently relieves us mortals of the task of guessing the “real” grand-unified field Lagrangian, and gives us license to employ } L_T \text{ as our Lagrangian density outside the singular loci. The problem is that we mortals apparently cannot experience a } T \text{ reversed world, and so cannot know } \Omega^L \text{ and } \Omega^R \text{ of (35), nor the contribution to } |-g|^\frac{1}{2} \text{ from } \dot{y}^0 = \frac{\dot{a}}{a}, \text{ the rate of cosmic expansion! The best we can do is to substitute } (\Omega^L, \Omega^R) \text{ for } (\Omega_L, \Omega_R) \text{ and use the static approximation}
\]

\[
\mathcal{L}_S = \frac{i}{2} Tr \Omega_L \wedge \Omega_L \wedge \Omega_R \wedge \Omega_R \]  

as our Lagrangian density. The action integral

\[
S_S = \frac{i}{2} \int_M Tr \Omega_0 e^0 \wedge \Omega_1 e^1 \wedge \Omega_2 e^2 \wedge \Omega_3 e^3 \]  

may be done in either the extrinsic polar 1 forms \( \Omega_\alpha e^\alpha \) on \( M \) or in the intrinsic 1 forms \( \omega_\alpha E^\alpha \) on our dilated spacetime \( M' \). \( S_S \) is invariant with respect to static dilations \( \gamma = \frac{a}{a'} \) (i.e. scale invariant) but cannot pick up \( \dot{y}^0 \), the dilation rate. \( S_S \) agrees with the topological action, \( S_T \), in the \( T \)-symmetric (static) case.
5 Dual Residues and Charge Quantization

The global spin connections $\hat{\Omega}$ provide the minimum vacuum energy. But they have another dramatic effect. When wedge products $\hat{\Omega}^{4-J}$ multiply perturbations $\tilde{\Omega}^J$, they effectively quantize these over Poincaré dual cycles $\ast D^J \equiv B_{4-J}$. This happens because products of “polarized” Clifford-algebra-valued forms require both their Clifford and Hodge duals to make the Clifford scalar $\sigma_0$ times the volume element—and so contribute to the action.

The vacuum fields can thus be used to “probe” inside the singular loci to produce new invariants—integrals of Hodge dual fields over Poincaré-dual cycles. These are the charges. We prove they are quantized below.

To account for the polarization of local $J$-fields form $\tilde{\Omega}^J (x)$ by the vacuum spin connections $\hat{\Omega}^{4-J}$, we write each spin connection as a perturbation $\tilde{\Omega} (x)$ added onto the global “vacuum” distribution $\hat{\Omega}$:

$$\Omega = \hat{\Omega} + \tilde{\Omega}.$$  

The perturbed action $\tilde{S}_T$ will then have contributions from products of the $(4-J)$ vacuum connections, $J = 1, 2, 3, \text{ or } 4$, and $J$ perturbed fields inside each codimension—$J$ singular domain.

If the perturbed fields $\tilde{\Omega}^J$ agree with the vacuum fields outside the singular domain,

$$\ast D^J \equiv B_{4-J} \subset \gamma_{4-J},$$  

then their contributions to the action must be quantized by Theorem 1.

The action contributed by each domain $B_{4-J}$ is

$$\tilde{S}_T = \frac{i}{2} \int_{B_{4-J} \times I_J} \text{Tr} \tilde{\Omega}^J (x) \wedge \hat{\Omega}^{4-J} = -16\pi^3 m_J,$$  

where $I_J$ is a cycle parametrized by the $J$ variables $\ast x$ orthogonal to $B_{4-J}$.

Now if the perturbed fields $\tilde{\Omega}^J (x)$ for $x \in B_{4-J}$ are independent of $\ast x$, the integral over $I_J$ may be factored out of (43). The result is that the dual current $(4-J)$ forms $\ast \tilde{\Omega}^J$ become quantized over their supports $B_{4-J}$. Thus we have

**Theorem 2:** The dual residues $\int_{B_{4-J}} \ast \tilde{\Omega}^J \equiv Q_J$ are quantized, provided the perturbed fields agree (up to a trivial gauge transformation) with the vacuum fields outside a support $B_{4-J}$.

**Proof:** The proof is a calculation which we outline below for each case.

**J = 1 Case:** The 3 vacuum spin connections create the “polarized” 3-volume forms

$$\hat{\Omega}^3 = \frac{1}{16a_\#} \sigma_\alpha e^\alpha e^\beta \wedge e^\gamma \wedge e^\delta \equiv \frac{2}{3} a_\# \sigma_\alpha e^\alpha,$$  

where

$$a_\# = \frac{\sqrt{2\pi}}{3\epsilon}.$$
against which the 1 form field perturbations $\tilde{\Omega}(x) \equiv \tilde{\Omega}^\beta(x)\sigma_\beta$ are integrated. Each “vacuum polarization” \(^44\) picks out its own internal direction $\sigma_\alpha$ in the trace. The resulting contribution,

$$\tilde{S}_T \equiv \frac{2}{3} \int_{\gamma_1 \times B_3} \tilde{\Omega}_\alpha e^\alpha \wedge *e^\alpha = -16\pi^3 M, \quad (45)$$

to the action is quantized: $M = \Delta W$ is an integer, if we require the perturbation to produce an integral change in the covering number $W$ of internal (spin) space over external spacetime, $\mathbb{M}$.

When the perturbations $\tilde{\Omega}^J(x)$ are time independent, integrating \(^{45}\) over $x^0 \in [0, 4\pi]$ gives the quantized charge

$$\int_{B_3} *J(x) \equiv \int_{B_3} J_0(x) e^1 \wedge e^2 \wedge e^3 = 8\pi^2 Q. \quad (46)$$

The charge $Q$ appears as the integral of the current 3 form $*J(x)$ dual to the 1 form perturbation \(^{46}\).

$$\tilde{\Omega}(x) \equiv J(x) \equiv J_\alpha(x) e^\alpha, \quad (47)$$

produced when 1 chiral pair of “matter spinors” break away from PTC symmetry inside $B_3$. This creates a bispinor Fermion. The quantized charge \(^{46}\) is then the Noether charge under complex time translation,

$$\int_{B_3} *J = \int_{B_3} i(d\theta^0 + id\phi^0) e^1 \wedge e^2 \wedge e^3 = 8\pi^2 (i q - m). \quad (48)$$

Elsewhere (Noether), we identify the real (external) part, $m$, of the time-translation charge with the mass and the imaginary (internal) part with the electric charge of a bispinor Fermion, in the $J = 1$ case.

Precisely the dual situation arises in the

$J = 3$ Case: Here integration against 1 vacuum connection $\tilde{\Omega}$ quantizes the integrals of 3 forms around 1-cycles, or orbits $\gamma_1$. We identify the quantized integrals of 3-form densities in the $J = 3$ case as particle energy-momenta $P$. Integrating over $\gamma_1$ then gives the Bohr-Sommerfeld condition that quantizes the energy-momentum 1 forms of particles around orbits $\gamma_1$. This is the

$J = 4$ Case: The quantization of action

$$\int_{\gamma_1} \tilde{\Omega} \int_{B_3} \tilde{\Omega}^3 \equiv \int_{\gamma_1} (P_0 e^0 - P_\beta e^\beta) = -16\pi^3 N. \quad (49)$$

We examine the most interesting case below, the

$J = 2$ Case: Here 2 forms become quantized over their dual 2 cycles. This gives quantization of electric flux—Gauss’law—after converting $\omega \wedge \tilde{\Omega}$ to the field 2 form $K \equiv d\Omega + \Omega \wedge \tilde{\Omega}$, then integrating by parts. ■
6 Chern Classes for Bispinor Bundles

Using expressions (A20) for the spin curvatures, we may rewrite the $T$-symmetric ($T_S$) part, (44), of the action as

$$S_S = \frac{i}{2} \int_M \text{Tr} \Omega_L \wedge \Omega_L \wedge \Omega_R \wedge \Omega_L$$

$$\equiv \frac{i}{2} \int_M \text{Tr} (K_L - d\Omega_L) \wedge (K_R - d\Omega_R).$$  

(50)

Using the Bianchi identity $dK = K \wedge \Omega - \Omega \wedge K$, upon integration by parts (50) may be written as

$$S_S = \frac{i}{2} \int_M \text{Tr} K_L \wedge K_R + \int_M \text{Tr} \Omega_L \wedge (K_L + K_R) \wedge \Omega_R + \sum M_J,$$  

(51)

where the $M_J$ are some Chern-Simons-type integrals about the boundaries of the singular domains. We showed [7], [11], that the term in $(K_L + K_R)$, the $PT$-symmetric (neutral) part of the net spin curvature, contains the Palantini action for gravitation. It vanishes in the $PT$ antisymmetric ($PT_A$) case. There $S_S$ is stationarized at

$$\hat{S}_A = \frac{i}{2} \int_M \text{Tr} K_L \wedge K_R \equiv -16\pi^3 C_2$$  

(52)

for the $PT_A, u(1) \times su(2)$ phase perturbations associated with electroweak potentials and charges.

$C_2$ is the second Chern number [14] for the chiral bispinor bundle $\psi : M \rightarrow L \oplus R$ under the Clifford-Killing form (3), (A11) for the Minkowsky metrics. This requires wedge products of left and right Lie-algebra-valued 2 forms to make an $E_P$-invariant 4 form, since the passive Einstein transformations include reciprocal Lorenz boosts on left and right spinors.

The chiral version of the second Chern form is thus the wedge product of the left-and-right $u(1) \times su(2)$-valued spin-curvature 2 forms,

$$K_L \equiv (K^\chi_L)_{\sigma} e^\alpha \wedge e^\beta,$$

$$K_R \equiv (K^\rho_R)_{\bar{\sigma}} e^\gamma \wedge e^\delta.$$

The $PT_A$ part (52) of the action (50) is quantized because it is the second Chern number of a bispinor bundle. It resembles the Yang-Mills action $\int \text{Tr} (F \wedge *F)$, with Hodge $*$ replaced by $P$ reversal. This resemblance is deeper than it appears, due to equation (A3).

The spin curvature 2 forms $K_L$ and $K_R$ are infinitesimal $L$ and $R$ spin transformations; they output infinitesimal $u(1) \times su(2)$ holonomy operators about the boundaries of their input two-cells. They thus naturally decompose into imaginary self-dual (left) and anti-self-dual (right) parts:

$$*K_L = iK_L \quad *K_R = -iK_R,$$  

(53)

since $\beta \rightarrow i\beta$ takes us from $SO(4)$ in equations (A2), (A3) to $SO(1,3)$.
Furthermore, the unperturbed spin curvatures of canonical connections (57),

\[ \hat{K}_{L\pm} = -\frac{1}{a^2} \sigma_j \left\{ \frac{i}{2} \epsilon^{\ell \mu \nu} e^\ell \wedge e^\mu \pm i e^0 \wedge e^j \right\} \]
\[ = (iB_L + E_L) \cdot \sigma \]
\[ \hat{K}_{R\pm} = -\frac{1}{a^2} \bar{\sigma}_j \left\{ \frac{i}{2} \epsilon^{\ell \mu \nu} e^\ell \wedge e^\mu \mp i e^0 \wedge e^j \right\} \]
\[ = (iB_R + E_R) \cdot \bar{\sigma}, \]  

are chiral dual:

\[ \hat{K}_{R\mp} = i \hat{K}_{L\pm}. \]  

The “vacuum fields” (54) are global dyons, with equal electric and magnetic fields distributed over \( S_3(a_\#) \).

\( PT_A, A \) perturbations \( \kappa \), for which \( R^{\mp} = (L_\pm)^{-1} \), preserve the metric tensor (A11), and therefore the Hodge \( * \)-operator. They thus preserve chiral duality conditions (55). For these, our \( U(1) \times SU(2) \) action on \( M_\# = S_1 \times S_3 \) maps to an \( R \times SU(2) \) Yang-Mills action on \( R_4 \):

\[ i \frac{8}{7} \int_{M_\#} Tr \kappa_{L\pm} \wedge \kappa_{R\mp} = -i \frac{8}{7} \int_{R_4} Tr \kappa \wedge * \bar{\kappa}. \]  

We may thus pull back the t'-Hooft/Jackiw-Noel-Rebbi multi-instanton solutions [14] on \( R_4 \) to obtain localized multi-dyon solutions on \( M_\# \). The “global dyon” (54) centered at \( 0 \in R_4 \) combines with a local dyon centered at \( N \in S_3(a_\#) \), the north pole of our reference three sphere of radius \( \lambda = a_\# \), to produce radial spin curvatures of:

\[ \kappa_{L\pm} = \left[ \frac{2 \lambda^2}{(r^2 + \lambda^2)^2} \right] \sigma_r \left[ i e^0 \wedge e^r \pm e^0 \wedge e^r \right], \]
\[ \kappa_{R\pm} = \left[ \frac{2 \lambda^2}{(r^2 + \lambda^2)^2} \right] \bar{\sigma}_r \left[ i e^0 \wedge e^r \mp e^0 \wedge e^r \right]. \]  

Here we use spherical-polar coordinates

\[ e^0 \wedge e^r = dx^0 \wedge dr, \quad e^0 \wedge e^r = r^2 \sin \theta d\theta \wedge d\phi \]

in both physical space and spin space.

We suggest that the opposite nonAbelian magnetic fields in (57) could bind \( L- \) and \( R- \) chirality spinors into charged bispinor Fermions. Each contributes an action of

\[ i \frac{2}{7} \int_{M_\# \setminus 0} \kappa_{L\pm} \wedge \kappa_{R\pm} = -16 \pi^3 T^2 \equiv -16 \pi^3 C_2 \]

proportional to the square of the charge. But there is also a contribution from the interaction of each localized charge with the global fields (54)! We show below how the interaction with the vacuum magnetic fields in (54) quantizes the flux of the electric field through any 2-surface that encloses a charge.
7 Topological Quantization of Electric Flux

To derive the quantization of electric flux from the quantization of action, expand the $PT_A$ parts of each spin curvature as the sum of the vacuum fields $\hat{\kappa}$ of (74) and the perturbations $\tilde{\kappa}$ due to local sources:

$$\hat{\kappa}_L \pm \hat{\kappa}_R \pm \tilde{\kappa}_L \pm \tilde{\kappa}_R \equiv \kappa_{\alpha\beta} \sigma^\lambda \epsilon^\alpha \wedge \epsilon^\beta.$$  

(60)

Substituting ansatz (60) into action (56), we obtain the cross terms

$$S_c = -\frac{1}{2a^2_0} \int \left[ \hat{\kappa}_{0j}^i + \epsilon^j_k \hat{\kappa}_{kl}^i L_0 \right] e^0 \wedge e^1 \wedge e^2 + P$$  

(61)

between the local dyon fields and the vacuum fields (since only terms in $\sigma_j \bar{\sigma}_j$ will contribute to the trace).

In the $PT_A$ case, the magnetic fields cancel, but the local electric fields add:

We get a charged bispinor particle with a net “radial hedgehog” electric field:

$$\tilde{\kappa}_{L0j} + \tilde{\kappa}_{R0j} = \tilde{\kappa}_{o3j}.$$  

(62)

Inserting (62) into (61), we obtain the local $\wedge$ global interaction energy density

$$V_c = \frac{1}{2a^2_0} \tilde{\kappa}_{0j} e^0 \wedge e^1 \wedge e^2 \wedge e^3.$$  

(63)

Note that it is the vacuum magnetic fields in (74)—the spin curvatures of the canonical degree-1 maps (10) of $SU(2)$ over $S^3(a_\#)$—that endows potential energy to each “radial hedgehog” configuration of electric fields $\tilde{\kappa}_{o3j} e^0 \wedge e^3$ floating within it.

For example, suppose that the local electric field $E_3(x^1, x^2) = \tilde{\kappa}_{o3}^3(x^1, x^2)$ is in the 3 direction in both physical space and spin space, but its amplitude depends on the coordinates $(x^1, x^2)$ on a spatial 2 surface, $S_{12}$. We may then separate the $PT_A$ part of action (61) into the product of integrals over $S_{12}$ and over its normal coordinates $x^0 \in S_1(a_\#)$ and $x_3 \in S_1(a_\#)$:

$$S_c = -\frac{1}{a_0} \int_{S_{12}} E_3(x^1, x^2) e^1 \wedge e^2 \int_{S_1(a_\#) \times S_1(a_\#)} e^0 \wedge e^3 = -4\pi^2 \int_{S_{12}} E_3(x^1, x^2) e^1 \wedge e^2 = -16\pi^3 N.$$  

(64)

$S_c$ is quantized over the normal surface $S_{12}$ supporting the perturbation $E_3(x_1, x_2)$, via conditions (74), (72), (74). We have thus derived a version of Gauss’ law

$$\int_{S_{12}} E_3(x^1, x^2) e^1 \wedge e^2 = 4\pi N,$$

where $N$ is an integer, because the action (72) must change in integral steps $\Delta C_2$ for each localized “bubble” of field patched into the vacuum.
For a sphere $S^2(\theta, \varphi)$ of radius $r$ surrounding a charge with radial electric field $E_r(\theta, \varphi) = \tilde{\kappa}_0 r$, (61) gives

$$\int_{S^2(\theta, \varphi)} E_r(\theta, \varphi) r^2 \sin \theta d\theta \wedge d\varphi = 4\pi N. \quad (65)$$

More generally, (61) integrates the spin-space component of the field normal under spinorization map (11) to the spatial area element:

$$\int_{S^2} E \cdot dA = 4\pi N, \quad (66)$$

where $S^2$ is any 2-surface enclosing the charge. This is Gauss’ law.

Quantization of the normal flux of the electric field over a closed spatial 2-surface thus follows directly from the quantization of the topological action (52). It is the vacuum magnetic fields in (54) that convert the integral of the electric field 2 form $\kappa_0 e^0 \wedge e^j$ into the integral of the dual 2 form $\kappa_0 e^0 \wedge e^\varphi$ over a spatial homology cycle:

$$\int_{S^2} \star \tilde{\kappa} = 4\pi N,$$

Gauss’ law.

After accounting for the action of the homogeneous field (54), and the action (61) of localized charges immersed in this field, there is the remaining contribution of the product of 2 perturbed fields

$$S_K = i \frac{1}{8} \int_M Tr \tilde{\kappa}_L \wedge \tilde{\kappa}_R = -\frac{1}{8} \int_M Tr \tilde{\kappa}_L \wedge \star \tilde{\kappa}_L. \quad (67)$$

This is the Yang-Mills/Weinberg-Salam “field action,” which is usually added by hand to couple sources to their fields.

Note that the action (67) is quadratic in the charges, whether it comes from products (61) of the local field interacting with the global background field, with another charge, or with itself. This offers an explanation not only of why charge has the units of action, but an estimate of the unit of charge divided by the unit of action, i.e. of the fine-structure constant, $\alpha$.

8 The Fine Structure Constant

Our spin connections, curvatures, and actions above have all been geometrical objects in the bundle of spin frames over $T^*M$. No physical units like $e$ or $\hbar$ have explicitly appeared. The topologically quantized electric charge and action appeared as “covering numbers” of internal space over external spacetime 2- and 4-cycles, respectively. However, since the relative increment $\alpha$ to the action introduced by adding a single charge in (14) is dimensionless, we might as well compute it in our geometric units.
The action $S_c$ of (61), due to a unit electric charge immersed in the global magnetic field, is $16\pi^3$. As in (67) [16], we must multiply this by $\frac{1}{4}$ to obtain the Maxwell/Yang Mills field action produced by a single charge. If we take $\hbar$ as the physical unit of action, we obtain
\[
\alpha \equiv \frac{e^2}{\hbar} = \frac{12}{4\pi^3} \approx \frac{1}{124}
\] as the number of units of action produced by adding a unit charge to the vacuum fields.

The value (68) does not agree very well with the observed value, $\alpha \approx \frac{1}{137}$. Either the mathematical model for charge quantization presented here fails to capture “real world” physics, or there is a “real world” correction to this model.

But expression (40), which we derived for the static ($T$-symmetric) case, does require a correction. When the radius $a(a_0)$ of our Friedmann universe $S^3(a(a_0))$ is expanding with Minkowsky time $x^0$, we need to include the factor $\dot{y}^0$ in the metric tensor. This shows up in $|g|^{\frac{1}{2}}$ of (35), but not in our static scale factor $\gamma^{-4}$ of (41).

This correction arises because our intrinsic tetrads are co-moving with the Friedmann flow. We thus experience [16] a Euclidean boost—a tilt of our cotangent frame into the radial ($y^0$) direction:
\[
\begin{bmatrix}
\dot{y}^0 \\
|d\mathbf{x}|
\end{bmatrix}' = \begin{bmatrix}
\cos \lambda & \sin \lambda \\
-\sin \lambda & \cos \lambda
\end{bmatrix}
\begin{bmatrix}
\dot{y}^0 \\
|d\mathbf{x}|
\end{bmatrix},
\]
or
\[
(d\mathbf{y} + idx)' = e^{i\lambda} (d\mathbf{y} + idx),
\]
where $\lambda \equiv \tan^{-1} \left( \frac{\dot{y}^0}{\dot{a}} \right) \equiv \tan^{-1} \dot{y}^0$.

The Minkowsky-time 1 form, $e^0 = dx^0 = |d\mathbf{x}| \equiv dx$, must suffer the same contraction as the spacelike increment to preserve $c = 1$, and special relativity. Thus our real Minkowsky 4-volume element $V$ suffers the contraction
\[
d^4V' \equiv \text{Re} (e^0 \wedge e^1 \wedge e^2 \wedge e^3)' = \cos 4\lambda (e^0 \wedge e^1 \wedge e^2 \wedge e^3) \equiv (\cos 4\lambda) d^4V
\]
\[
\Rightarrow d^4V = (\cos 4\lambda)^{-1} d^4V',
\]
when projected to the static reference sphere $S^3 (a_\#)$.

If we, as co-moving observers, could somehow deduce the value $\lambda = \tan^{-1} \dot{y}^0$ of this radial tilt of our cotangent space, we could use (70) to correct our static approximation (41) for cosmic expansion. But we can, because the spacelike—or SU (2)—component of what we observe to be a lightlike translation changes when our spatial hypersurface is tilted with respect to the invariant null direction! It is precisely this tilt that gave [16] our correction to the Weinberg angle $\theta_W$. This required a value of $\dot{y}^0 \approx 0.16$ to match the current best value of 28.5° for $\theta_W$. Inserting $\dot{y}^0 = 0.16$ into (70), we obtain $(\cos 4\lambda)^{-1} \approx 1.11$ as the correction factor (70) for our co-expanding 4-volume element. This gives a corrected action of
\[
(1.11) 124 \approx 124 + 13 = 137
\]
for a unit charge moving with the Friedmann flow. We can interpret the additional 13 units as its “kinetic energy” with respect to the stationery reference sphere $\mathbb{S}^3(a\#)$.

From action (71), we obtain

$$\alpha' \approx \frac{1}{137}$$

for the fine structure constant, as measured by a co-moving observer. This is close to the measured value of $(137.037)^{-1}$.

### 9 Conclusions and Open Questions

From a class of Lagrangian densities which reduce to the Maurer-Cartan 4 form $\Omega^4$ in the PTC-symmetric limit, we have derived the quantization of action and charge. These are simply the covering numbers of the internal phases in chiral spin bundles over 4-cycles and 2-cycles in the multiply-connected external spacetime $\mathbb{M} \equiv \mathbb{M}_\# \cup D_j$. It is electric flux that is quantized over spatial surfaces $S_2 = \partial D_3$ surrounding a charge, because the vacuum magnetic fields $\tilde{\kappa}_{kl}$ convert the electric flux $\tilde{\kappa}_{0j}$ to quantized action. We thus have a realization of a “dual topological field theory” [15], [17], [18], in which Hodge star is replaced by a duality operation between internal Lie algebras. This is none other than the one induced by Clifford product (A11), in which the tetrads in (A10) are dyads in some fundamental, global spinor fields.

Thus, the metric tensor (A11) needed to contract two spin-1 tensors (vectors) is itself a spin-2 tensor. Any natural $E_p$-invariant 4 form—e.g. a Lagrangian density—must be the Clifford-scalar part of a spin-4 tensor,

$$\mathcal{L}_g \in \otimes^8 \subset \Lambda^4.$$  \hfill (72)

We have shown here that the simplest realization (1) of such a natural Lagrangian (2) gives quantized actions and charges.

When we add one unit of charge to the vacuum fields (54), we increment the action by $\sim 137$ units, as measured in our intrinsic frame, co-moving with cosmic expansion. We thus derive a value of $\alpha \sim (137)^{-1}$ for the fine structure constant.

Is this a numerical coincidence, or is there some relevance to fundamental physics in the mathematical structure we have developed here? More basically, do the cosmological background fields $K$ really exist, and do they play a fundamental role in charge quantization? These questions await further investigation.

### 10 Appendix

Recall [1], [13], [12] that

$$\text{chiral } SO(4) \equiv \text{Spin } 4 \sim SU(2)_L \times SU(2)_R / \mathbb{Z}_2$$
presents a point \((a^0, \mathbf{a}) \in \mathbb{R}^4\) as the “quaternion,” \(q\), and its quaternionic conjugate, \(\bar{q}\):

\[
q = a^0 \sigma_0 + i \mathbf{a} \cdot \mathbf{\sigma} \equiv a^0 \sigma_0 + ia^j \sigma_j, \\
\bar{q} = a^0 \sigma_0 + i \mathbf{a} \cdot \mathbf{\sigma} \equiv a^0 \sigma_0 + ia^j \bar{\sigma}_j;
\]

\[j = 1, 2, 3.\]  \hspace{1cm} (A1)

The infinitesimal \(so(4)\) isometries of \(S_3\),

\[
\delta \begin{bmatrix} a^0 \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{\beta}^T \\ \mathbf{\beta} & [\mathbf{\alpha}] \end{bmatrix} \begin{bmatrix} a^0 \\ \mathbf{a} \end{bmatrix},
\]

are presented on the position quaternion, \(q\), as

\[
q' = Lq\bar{R} = e^{i2(\mathbf{\alpha} + \mathbf{\beta}) \cdot \mathbf{\sigma}} q e^{i2(\mathbf{\alpha} - \mathbf{\beta}) \cdot \bar{\mathbf{\sigma}}},
\]

with

\[
[\mathbf{\alpha}] = \begin{bmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{bmatrix}.
\]  \hspace{1cm} (A2)

\(\mathbf{\sigma}\) and \(\bar{\mathbf{\sigma}}\) generate the left and right Lie algebras—which must be viewed as completely independent in chiral \(so(4)\), giving 6 generators in all. Pure left-spin transformations \(\mathbf{\alpha} = \mathbf{\beta}\) correspond to self-dual 2 forms, under the usual identification of skew-symmetric matrices \([\mathbf{\alpha}]\) with 2 forms \([10]\). Pure right transformations \(\mathbf{\alpha} = -\mathbf{\beta}\) correspond to anti-self-dual 2 forms.

Dilations (e.g. of a Friedmann universe) may be included by adding a scalar generator \(\sigma_0\); complexification of which gives an internal \(U(1)\) phase shift. There are 4 representations, \(\text{exp} \left( \frac{i}{2} y^0 \theta (z) - \frac{i}{2} \bar{x}^0 \bar{\theta} (z) \right) \sigma_0\), of translations in complex-time \(z^0 \equiv x^0 + iy^0\), distinguished by the sign of the internal \(u(1)\) phase advance with logradius \(y^0\), \(\text{sgn} \left( \frac{\partial \theta}{\partial y^0} \right)\), which we identify with the charge of the field, and by the dilation behavior, \(\text{sgn} \left( \frac{\partial \bar{\theta}}{\partial x^0} \right)\), which distinguishes leptonic (light) from baryonic (heavy) spinors. These combine with the two chiralities to give 8 fundamental spinor representations \(\mathbf{8}\) of the spin isometry group, or Einstein group, \(E\); the globalization of the Poincaré group to a Friedmann universe. These make up the Cartan moving spin frames

\[
\ell^\pm, u_\pm, r^\pm, v_\pm,
\]

(A4)

pairwise. Each spin frame contains two basis spinors with opposite helicity: the fundamental null modes of the Dirac operators.

To match the standard convention for chiral bispinors on the conformal compactification \(\mathcal{M}_\# = S^1 \times S^3 (a_\#)\) of Minkowsky space \(\mathbb{R}^4\), we write the
leptonic spin frames \( \ell^\pm (x) \) and \( r^\pm (x) \) columnwise as the \( GL(2, \mathbb{C}) \) matrices

\[
\ell^\pm (x) \equiv \left[ \begin{array}{c} \ell_1 (x) \\ \ell_2 (x) \end{array} \right] ^\pm = \sigma_0 \exp \left( \frac{\pm i x^0 \sigma_0 + x^j \sigma_j}{2 \sigma_0} \right) \equiv \sigma_0 \tilde{g}_{\pm} (x)
\]

\[
r^\pm (x) \equiv \left[ \begin{array}{c} r_1 (x) \\ r_2 (x) \end{array} \right] ^\pm = \bar{\sigma}_0 \exp \left( \frac{\pm i x^0 \bar{\sigma}_0 + x^j \bar{\sigma}_j}{2 \bar{\sigma}_0} \right) \equiv \bar{\sigma}_0 \bar{g}_{\pm} (x).
\]

We also write the right spin frame row-wise as

\[
\tilde{\ell} (x) \equiv \left[ \begin{array}{c} \tilde{\ell}_1 (x) \\ \tilde{\ell}_2 (x) \end{array} \right] \equiv \ell^T (x) \epsilon^T,
\]

where \( \epsilon \equiv i \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

The overbar indicates space (\( P \)) reversal, or Dirac conjugation. We have the Lie-algebra isomorphism:

\[
\bar{\sigma}_\alpha \sim \epsilon^{-1} (\sigma_\alpha)^T \epsilon = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3).
\]

The moving spin frames \( \ell (x) \) and \( r (x) \) factor the moving tetrads. These are the spin-1 tensors

\[
q_\alpha (x) = \sigma_\alpha^A \ell_A (x) \otimes r^B (x) \equiv \ell \otimes_\alpha \tilde{r}:
\]

\[
q_0 (x) \equiv \ell_1 (x) \otimes \tilde{\ell}_2 (x) - \ell_2 (x) \otimes \tilde{\ell}_1 (x) \equiv \ell \otimes_0 \tilde{r},
q_1 (x) \equiv \ell_1 (x) \otimes \tilde{\ell}_1 (x) + \ell_2 (x) \otimes \tilde{\ell}_2 (x) \equiv \ell \otimes_1 \tilde{r},
q_2 (x) \equiv i \left( \ell_1 (x) \otimes \tilde{\ell}_2 (x) - \ell_2 (x) \otimes \tilde{\ell}_1 (x) \right) \equiv \ell \otimes_2 \tilde{r},
q_3 (x) \equiv \ell_1 (x) \otimes \tilde{\ell}_2 (x) + \ell_2 (x) \otimes \tilde{\ell}_1 (x) \equiv \ell \otimes_3 \tilde{r};
\]

\[
\bar{q}_\alpha (x) = r (x) \otimes_\alpha \tilde{\ell} (x).
\]

The matrix representations \( q_\alpha (x) \) and \( \bar{q}_\alpha (x) \) of the moving tetrads \( q_\alpha (x) \) and \( \bar{q}_\alpha (x) \) have the matrix elements of the Pauli spin matrices \( \sigma_\alpha \) and \( \bar{\sigma}_\alpha \) with respect to the moving spin frames \( \ell (x) \) and \( r (x) \).

Under complex \( E \) transformations \( \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \), the matrix representations of the tetrads with respect to the original basis \( (\ell (0), \tilde{r} (0)) \) are,

\[
q'_\alpha (z) = L (z) q_\alpha (0) \tilde{R} (z) = e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}} e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}} e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}} e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}},
\]

\[
\bar{q}'_\alpha (z) = \bar{R} (z) q_\alpha (0) \bar{L} (z) = e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}} e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}} e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}} e^{\frac{\pm i \zeta^* (z) \sigma_\alpha}{2 \sigma_0}},
\]

where \( \zeta^* (z) \equiv \theta^* (z) + i \varphi^* (z) \). These obey the anti-commutation relations

\[
\bar{q}'_\alpha q_\beta + q_\alpha q'_\beta \equiv \{ q'_\alpha, q_\beta \} = \{ R \bar{q}_\alpha \bar{L}, L q_\beta \bar{R} \} = 2 g_{\alpha \beta} \sigma_0
\]
of the complexified Clifford algebra of (A10). The metric tensor in (A11) is derived from the tetrads (A8) and (A9)—which are in turn derived from the 8 fundamental global spinor fields, the dynamical variables in the theory.

$L$ and $R$ chirality spinors are coupled through the Dirac operators

$$D \equiv iq^a \partial_a, \quad \bar{D} \equiv i\bar{q}^a \bar{\partial}_a.$$  

(A12)

These are the translation invariant derivations, or Lie-algebra-valued vector fields dual to the Maurer-Cartan forms (11).

Covariant derivatives $\nabla_\alpha$ automatically appear in the Dirac operators (A12) by differentiating the Cartan moving spin frames in

$$\partial_\alpha \xi \equiv \partial_\alpha (\ell \xi) = \ell (\partial_\alpha + \Omega_\alpha) \xi \equiv \xi \nabla_\alpha \xi.$$  

The Dirac equations for a bispinor particle are [7], [3]:

$$D \xi \equiv iq^a (\partial_\alpha + \Omega_{L\alpha}) \xi = \frac{1}{2a} \eta$$  

$$\bar{D} \eta \equiv i\bar{q}^a (\partial_\alpha + \Omega_{R\alpha}) \eta = \frac{1}{2a} \xi$$  

(A13)

in the chiral representation [9].

To preserve Einstein covariance of the Dirac equations (A13), we must write all our matter fields with respect to the same moving spin frames that factor the spacetime tetrads (A8):

$$\xi_\pm (x) \equiv \ell^\pm (x) \xi_{\pm} (x) \equiv \ell^\pm (x) \left( \lambda_\pm + \tilde{\xi}_\pm (x) \right) \xrightarrow{g.o.} \tilde{r}^\pm (x) \lambda_\pm$$  

$$\eta_\pm (x) \equiv r^\pm (x) \eta_{\pm} (x) \equiv r^\pm (x) \left( \rho_\pm + \tilde{\eta}_\pm (x) \right) \xrightarrow{g.o.} \tilde{r}^\pm (x) \rho_\pm.$$  

(A14)

$\lambda_\pm$ and $\rho_\pm$ are the homogeneous background, or vacuum, values of the “leptonic spinors,” $\xi_\pm$ and $\eta_\pm$, $\tilde{\xi}_\pm$ and $\tilde{\eta}_\pm$ are their localized envelope modulations. These constitute electrons ($\tilde{\xi}_- \oplus \tilde{\eta}_-$), positrons ($\tilde{\xi}_+ \oplus \tilde{\eta}_+$) and neutrinos ($\tilde{\xi}_+ \oplus \tilde{\eta}_-$) in this model [11]. The expressions $\xrightarrow{g.o.}$ hold in the geometrical optics (g.o.) regime where no two rays of the same spinor field cross; thus the phase advance along paths is well-defined.

Constant $gl(2, \mathbb{C})$ phase shifts generate the group of spacetime isometries, or passive Einstein transformations, $E_P$. These connect the spin frames that represent the same state to different observers:

Spatial translations: \[ \Delta \theta^j_L = \Delta \theta^j_R = \frac{\Delta x^j}{a_s} \]

Boosts: \[ \Delta \varphi^j_L = \Delta \varphi^j_R = \frac{\Delta y^j}{a_s} \]

Arctime translations: \[ \Delta \theta^0_L = - \Delta \theta^0_R = \pm \frac{\Delta t^0}{a_s} \]

Rotations: \[ \Delta \theta^j_L = - \Delta \theta^j_R \]  

(A15)

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The conformal dual spinor to $\xi_-$,

$$\xi^- \equiv \xi_+^T \gamma \epsilon \equiv \xi^- \ell^-,$$

where $\xi^- \equiv \xi_+^T \epsilon$ and $\ell^- \equiv \epsilon^{-1} (\ell^-)^T \gamma \epsilon = (\ell^+)^{-1}, \quad \text{(A16)}$

is defined so that $E$ transformations (A15) along with

$$\text{Cosmic Expansion: } \triangle \varphi^0_L = \triangle \varphi^0_R = \frac{\Delta y^0}{a^#}, \quad \text{(A17)}$$

are spin isometries. The $E$ invariance of the $GL(2, \mathbb{C})$ matrix product

$$\ell^- \ell^+ = \sigma_0 \quad \text{(A18)}$$

is what assures that the inner product $\xi^+ \xi^-$ is $E$ invariant.

The spin connections (12) may thus be written as

$$\Omega^L_{\pm} = \tilde{\ell}^\pm \tilde{d} \tilde{\ell}^\pm; \quad \Omega^L_{\pm} = \tilde{\ell}^\pm \tilde{d} \tilde{\ell}^\pm; \quad \Omega^R_{\pm} = (\tilde{d} \tilde{\ell}^\pm) \tilde{\ell}^\mp. \quad \text{(A19)}$$

In curved spacetime, where $\tilde{d} \tilde{d} \neq 0$, these possess spin curvatures

$$\tilde{\ell}^\pm \tilde{d} \tilde{\ell}^\pm = K^\pm_{L} = \left( \tilde{d} \tilde{\Omega}_L + \tilde{\Omega}_L \wedge \tilde{\Omega}_L \right)^\pm; \quad \tilde{\ell}^\mp \tilde{d} \tilde{\ell}^\mp = K^\pm_{R} = \left( \tilde{d} \tilde{\Omega}_R + \tilde{\Omega}_R \wedge \tilde{\Omega}_R \right)^\pm. \quad \text{(A20)}$$

Effective spin connections (A19) and curvatures (A20) appear in the g.o. regime for each $PTC$-symmetric pair of spinor fields in our Lagrangian 4 form (1). It is products of terms like these that give the topological forms (35), (52) for the action in the $PTC$-symmetric regime.

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