Group theoretic approach and analytical solutions of the compressible Navier–Stokes equations

Dina Razafindralandy, Aziz Handouni
La Rochelle Université – France

Abstract

A group theoretic analysis of the compressible Navier-Stokes equations of an ideal gas are carried out. The 12-dimensional Lie symmetry group is computed. The commutation table and the Levi decomposition of its Lie algebra are presented. The equations are reduced and self-similar one-, two- and three-dimensional solutions are computed. Many of them are graphically illustrated.

1 Introduction

Fluid flows have an important role in engineering science. To understand the complex phenomena that arise in them, computing analytical solutions may be helpful. Indeed, an analytical solution may serve as a simplified model of the flow in a particular configuration. It may also be used to calibrate or tests numerical schemes or turbulent models.

Some methods exist in finding particular solutions of partial differential equations (separation of variables, assuming axisymmetry, ...). Most of them consist in choosing an ansatz and reduce the equations accordingly. However, finding ansätze which effectively reduce the equation is not an easy task. A tool which provides a systematic way of finding the such ansätze is the Lie symmetry-group theory [14, 10].

In incompressible fluid mechanics, the Lie group theory has been used to find analytical solutions of the Navier–Stokes equations [6], and particularly vortex-like ones [8], to study turbulence [20, 12, 16] or to model particular flows such as boundary layers [11] and thin isothermal and non-isothermal shear layers [15].

In this article, we deal with the compressible case where only few exact solutions are available. The reader may refer to some papers such as [4, 5, 19, 17] to find analytical solutions. Most of them are either one-dimensional or axisymmetric.

The aim of the present work is to carry out a group theoretic analysis of the governing equations and to propose a wide class of analytical solutions.

In section 2 the hypotheses on the flow are listed and the velocity-pressure-density formulation of the compressible Navier-Stokes equations is presented. In section 3 the Lie method of symmetry computation is briefly recalled. The Lie
point-symmetry group of the equations and its Lie algebra are then analysed. It will be shown that the Lie-algebra is solvable. The symmetry group will be used to find ansätze and reduce the equations in the subsequent sections. We do not intend to be exhaustive. Rather, some interesting self-similar solutions, such as vortex-like ones, are presented, in order to complete the set of available solutions in the literature. In sections 4, 5 and 6 respectively steady bidimensional, unsteady bidimensional and three-dimensional cases are considered.

2 Model equations

The motion of a fluid is governed by the Navier-Stokes equations [1]:

\[
\begin{align*}
\frac{d\rho}{dt} + \rho \text{ div } \mathbf{u} &= 0 \\
\rho \frac{d\mathbf{u}}{dt} &= \text{ div } \sigma \\
\rho \frac{d\epsilon}{dt} \left( e + \frac{\mathbf{u}^2}{2} \right) &= \text{ div}(\sigma \mathbf{u} + \kappa \nabla T)
\end{align*}
\]

in absence of body force. In these equations, \( \rho, \mathbf{u} = (u,v,w), \epsilon \) and \( T \) are respectively the density, the velocity, the internal energy and the temperature. \( \kappa \) is the thermal diffusion coefficient, taken constant. With the hypothesis of a Newtonian fluid, the stress tensor \( \sigma \) writes:

\[
\sigma = \left( -p - \frac{2\mu}{3} \text{ div } \mathbf{u} \right) \mathbf{l}_d + 2\mu \mathbf{S}.
\]

where \( p \) is the pressure field, \( \mu \) is the dynamic viscosity, \( \mathbf{l}_d \) is the three-dimensional identity matrix and \( \mathbf{S} \) is the strain rate tensor

\[
\mathbf{S} = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}.
\]

The variation of \( \mu \) with the temperature is neglected. Equations (4) can also be formulated as follows

\[
\begin{align*}
\frac{d\rho}{dt} + \rho \text{ div } \mathbf{u} &= 0 \\
\rho \frac{d\mathbf{u}}{dt} &= \text{ div } \sigma \\
\rho \frac{d\epsilon}{dt} &= \sigma : \mathbf{S} + \kappa \Delta T
\end{align*}
\]
where the double dot sign stands for the Frobenius inner product:

\[ \sigma : S = \text{tr}(\tilde{\sigma} \cdot S) = \sum_{i,j} \sigma_{ij} S_{ij}. \]

Assume that the fluid is an ideal gas. We then have:

\[ p = \rho TR \quad (5) \]

where \( R \) is the gas constant. Moreover, the internal energy is proportional to the temperature, that is

\[ e = C_v T \quad (6) \]

where \( C_v \) is the (constant) specific heat at constant volume. Inserting relations (6) and (5) in the energy equation of (4) and using the mass balance equation, we get the density-velocity-pressure formulation (see also \( \text{[18, 21, 7]} \)):

\[
\begin{cases}
    \frac{d\rho}{dt} + \rho \text{div } u = 0 \\
    \rho \frac{d\mathbf{u}}{dt} = \text{div } \sigma \\
    \frac{C_v}{R} \left( \frac{dp}{dt} + p \text{div } \mathbf{u} \right) = \sigma : S + \frac{\kappa}{R} \Delta \left( \frac{p}{\rho} \right)
\end{cases}
\quad (7)
\]

Expliciting material derivatives with the relation

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla), \quad (8) \]

one gets a (five-dimensional) partial differential equation

\[ \mathbf{E} \left( t, \mathbf{x}, \mathbf{u}(2), p(2), \rho(2) \right) = 0 \quad (9) \]

where \( \mathbf{u}(2), p(2) \) and \( \rho(2) \) gather \( u, v, w, p, \rho \) and all of their partial derivatives up to second order. A componentwise expression of equation (9) is given in appendix \( \text{A} \), equation (123).

## 3 Lie symmetry group and Lie algebra

In this section, we study the Lie symmetry group admitted by equations (7) and its Lie algebra. The suited framework to compute Lie symmetry groups is the language of jet space but to simplify the presentation, we avoid its introduction. More details can be found in \( \text{[14, 2, 9, 10]} \).

A transformation

\[ T : (t, \mathbf{x}, u, p, \rho) \mapsto (\hat{t}, \hat{\mathbf{x}}, \hat{u}, \hat{p}, \hat{\rho}) \quad (10) \]

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is called a (point-)symmetry of equations (9) if it transforms any solution of (9) into another solution, that is

$$E(t, x, u(2), p(2), \rho(2)) = 0 \implies E(\hat{t}, \hat{x}, \hat{u}(2), \hat{p}(2), \hat{\rho}(2)) = 0. \quad (11)$$

where $\hat{u}(2), \hat{p}(2)$ and $\hat{\rho}(2)$ represent the transforms $\hat{u}, \hat{p}, \hat{\rho}$ of $u, p, \rho$ and their partial derivatives with respect to $\hat{t}$ and $\hat{x}$ up to second order. Our aim is to find all the local Lie symmetry groups of (7), that are families of symmetries

$$G = \{ T_\varepsilon : (t, x, u, p, \rho) \mapsto (\hat{t}, \hat{x}, \hat{u}, \hat{p}, \hat{\rho}) \mid \varepsilon \in I \subset \mathbb{R}, \ T_\varepsilon \text{ symmetry of (7)} \}$$

having a structure of a local Lie group. For the sake of simplicity, we assume that the group is additive. In particular, $0 \in I$ and $T_0$ is the identity transformation.

Computing local Lie symmetries is generally easier if symmetry condition (11) is linearized at the vicinity of the identity. To this aim, consider the vector field

$$X = \xi_t \partial_t + \xi_x \partial_x + \xi_y \partial_y + \xi_z \partial_z + \xi_u \partial_u + \xi_v \partial_v + \xi_w \partial_w + \xi_p \partial_p + \xi_\rho \partial_\rho \quad (12)$$

where the components

$$\xi^r(t, x, u, p, \rho) = \frac{d\hat{r}}{d\varepsilon} \bigg|_{\varepsilon=0}, \ r = t, x, y, z \quad (13)$$

represent the infinitesimal variation of the independent variables under the action of $G$ and

$$\xi_q(t, x, u, p, \rho) = \frac{d\hat{q}}{d\varepsilon} \bigg|_{\varepsilon=0}, \ q = u, v, w, p, \rho \quad (14)$$

represent those of the dependent variables. Vector field $X$ is called the generator of $G$. According to the Lie group theory [14, 10], if $G$ is a Lie symmetry of (7) then

$$\text{pr}^{(2)}X \cdot E \bigg|_{E=0} = 0 \quad (15)$$

where $\text{pr}^{(2)}X$ is the second-order prolongation of $X$. It writes:

$$\text{pr}^{(2)}X = X + X^{(1)} + X^{(2)}$$

where

$$X^{(1)} = \sum_q \sum_s \xi_s \frac{\partial}{\partial q_s} \quad (16)$$
takes into account the infinitesimal variation of first order partial derivatives under the action of $G$ (see also equation (124)) and

$$X^{(2)} = \sum_q \sum_{r,s} \xi^{rs}_q \frac{\partial}{\partial q_{rs}}$$  \hspace{1cm} (17)

acts on second order derivatives. In (16) and (17), the sums are over all dependent variables $q = u, v, w, p, \rho$ and over all independent variables $r, s = t, x, y, z$.

The coefficients of $X^{(1)}$ and $X^{(2)}$ are linked to those of $X$ by the relations:

$$\xi^s_q = D_s \xi_q - \sum_{m=t,x,y,z} q_m D_s \xi^m_q,$$ \hspace{1cm} (18)

$$\xi^{rs}_q = D_s \xi^r_q - \sum_{m=t,x,y,z} q_{rm} D_s \xi^m_q,$$ \hspace{1cm} (19)

$D_r$ being the total derivation operator with respect to $r$.

Infinitesimal symmetry condition (15) applied to (7) leads to system of partial differential equations on the $\xi^r$ and $\xi_q$. After solving this system, one finds that $X$ generates a Lie symmetry of (7) if it is a linear combination of the following vector fields (see appendix A):

$$X_1 = \frac{\partial}{\partial t}, \hspace{1cm} X_2 = \frac{\partial}{\partial x}, \hspace{1cm} X_3 = \frac{\partial}{\partial y}, \hspace{1cm} X_4 = \frac{\partial}{\partial z},$$

$$X_5 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \hspace{1cm} X_6 = t \frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \hspace{1cm} X_7 = t \frac{\partial}{\partial z} + \frac{\partial}{\partial w},$$

$$X_8 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} + v \frac{\partial}{\partial w} - w \frac{\partial}{\partial v}, \hspace{1cm} X_9 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} + w \frac{\partial}{\partial u} - u \frac{\partial}{\partial w},$$

$$X_{10} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u},$$

$$X_{11} = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w} - 2p \frac{\partial}{\partial p},$$

$$X_{12} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2 \rho \frac{\partial}{\partial \rho}.$$
The generic element $T_\varepsilon$ of the one-dimensional Lie symmetry group generated by each of these vector fields can be computed by solving the equations

$$\begin{cases}
\frac{d\hat{r}}{d\varepsilon} = \xi^r(\hat{t}, \hat{x}, \hat{u}, \hat{p}, \hat{\rho}), & r = t, x, y, z, \\
\frac{d\hat{q}}{d\varepsilon} = \xi_q(\hat{t}, \hat{x}, \hat{u}, \hat{p}, \hat{\rho}), & q = u, v, w, p, \rho,
\end{cases} \quad (20)$$

$$\hat{r}(\varepsilon = 0) = r, \quad \hat{q}(\varepsilon = 0) = q.$$ 

These groups combines into the 12-dimensional Lie symmetry group $G$ of equations (7), generated by the following point transformations:

- time translations, obtained from $X_1$:
  $$ (t, x, u, p, \rho) \mapsto (t + \varepsilon, x, u, p, \rho), \quad (21) $$

- space translations, encoded by $X_2, X_3,$ and $X_4$:
  $$ (t, x, u, p, \rho) \mapsto (t, x + \epsilon, u, p, \rho), \quad (22) $$

- Galilean transformations, corresponding to $X_5, X_6, X_7$:
  $$ (t, x, u, p, \rho) \mapsto (t, x + \epsilon t, u + \epsilon, p, \rho), \quad (23) $$

- rotations, induced by $X_8, X_9$ and $X_{10}$:
  $$ (t, x, u, p, \rho) \mapsto (t, Rx, Ru, p, \rho), \quad (24) $$

- and the two scale transformations, generated respectively by $X_{11}$ and $X_{12}$:
  $$ (t, x, u, p, \rho) \mapsto (e^{\varepsilon t} t, e^\varepsilon x, e^{-\varepsilon} u, e^{-2\varepsilon} p, \rho), \quad (25) $$
  $$ (t, x, u, p, \rho) \mapsto (t, e^\varepsilon x, e^\varepsilon u, p, e^{-2\varepsilon} \rho). \quad (26) $$

In these expressions, $\varepsilon, \epsilon$ and $R$ are respectively arbitrary scalar, vector and 3D rotation matrix.

Vector fields $X_i, i = 1, \cdots, 12$, constitute a basis of the 12-dimensional Lie algebra $\mathfrak{g}$ of the Lie symmetry group $G$. The commutation table of $\mathfrak{g}$ is presented on Table 1. It shows that the subalgebra

$$ \mathfrak{g}^{\text{rad}} = \text{span}(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_{11}, X_{12}) \quad (27) $$
Table 1: Commutation table of $\mathfrak{g}$

|     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|---------|
| $X_1$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 2$X_1$| 0     |           |           |         |
| $X_2$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | -$X_4$| $X_3$ | $X_2$     | $X_2$     | $X_2$   |
| $X_3$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | $X_4$ | 0     | -$X_2$    | $X_3$     | $X_3$   |
| $X_4$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | -$X_3$| $X_2$ | 0         | $X_4$     | $X_4$   |
| $X_5$ | -$X_2$| 0     | 0     | 0     | 0     | 0     | 0     | -$X_7$| $X_6$ | -$X_5$    | $X_5$     |         |
| $X_6$ | -$X_4$| 0     | 0     | 0     | 0     | 0     | 0     | $X_7$ | 0     | -$X_5$    | -$X_6$    | $X_6$   |
| $X_7$ | -$X_4$| 0     | 0     | 0     | 0     | 0     | 0     | -$X_6$| $X_5$ | 0         | -$X_7$    | $X_7$   |
| $X_8$ | 0     | 0     | -$X_4$| $X_3$ | 0     | -$X_7$| $X_6$ | 0     | -$X_{10}$| $X_9$     | 0         | 0       |
| $X_9$ | 0     | $X_4$ | 0     | -$X_2$| $X_7$ | 0     | -$X_5$| $X_{10}$| 0     | -$X_8$    | 0         | 0       |
| $X_{10}$ | 0     | -$X_3$| $X_2$ | 0     | -$X_6$| $X_5$ | 0     | -$X_9$| $X_8$ | 0         | 0         | 0       |
| $X_{11}$ | -2$X_1$| -$X_2$| -$X_3$| -$X_4$| $X_5$ | $X_6$ | $X_7$ | 0     | 0     | 0         | 0         | 0       |
| $X_{12}$ | 0     | -$X_2$| -$X_3$| -$X_4$| -$X_5$| -$X_6$| -$X_7$| 0     | 0     | 0         | 0         | 0       |

is solvable. Indeed, the derived series terminates in the zero algebra:

\[
[g^{\text{rad}}, g^{\text{rad}}] = \text{span}(X_1, X_2, X_3, X_4, X_5, X_6, X_7),
\]

\[
[[g^{\text{rad}}, g^{\text{rad}}], g^{\text{rad}}] = \text{span}(X_2, X_3, X_4),
\]

\[
[[[g^{\text{rad}}, g^{\text{rad}}], g^{\text{rad}}], g^{\text{rad}}] = \{0\}.
\]  

(28)

The subalgebra $g^s = \text{span}(X_8, X_9, X_{10})$ of infinitesimal rotations is semi-simple. The Levi decomposition of $\mathfrak{g}$ is then

\[
\mathfrak{g} = g^{\text{rad}} \oplus g^s
\]

(29)

$g^{\text{rad}}$ being the radical.

In the next sections, successive reductions are applied to the equations and some self-similar solutions are computed. We begin with steady bidimensional solutions.

### 4 Steady bidimensional solutions

In this section, we take $w = 0$ and look for solutions invariant under both $X_1$ and $X_4$ that are steady and bidimensional solutions. We write:

\[
u(t, x, y) = u_1(x, y), \quad v(t, x, y) = v_1(x, y), \quad p(t, x, y) = p_1(x, y),
\]

\[
\rho(t, x, y) = \rho_1(x, y).
\]

(30)
The reduced equations write:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 v_1}{\partial y} = 0, \\
\rho_1 \left( \frac{u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} \right) + \frac{\partial \rho_1}{\partial x} = \frac{\mu}{3} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial x \partial y} + \frac{3 \partial^2 u_1}{\partial y^2} \right), \\
\rho_1 \left( \frac{v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} \right) + \frac{\partial \rho_1}{\partial y} = \frac{\mu}{3} \left( \frac{\partial^2 u_1}{\partial x \partial y} + \frac{4 \partial^2 v_1}{\partial y^2} + \frac{3 \partial^2 v_1}{\partial x^2} \right), \\
C_v \left( \frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 v_1}{\partial y} \right) = \sigma + \frac{\kappa}{R} \left( \frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} \right).
\end{array} \right.
\]  

(31)

To find solutions, we reduce these new equations further. The Lie algebra of equations (31) is spanned by:

\[
\begin{aligned}
Y_1 &= \frac{\partial}{\partial x}, \\
Y_2 &= \frac{\partial}{\partial y}, \\
Y_3 &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1}, \\
Y_4 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} - 2p_1 \frac{\partial}{\partial p_1}, \\
Y_5 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + u_1 \frac{\partial}{\partial u_1} + v_1 \frac{\partial}{\partial v_1} - 2p_1 \frac{\partial}{\partial p_1}.
\end{aligned}
\]  

(32)

4.1 Reduction with $Y_1$

Reduction under vector field $Y_1$ suggests a solution in the form

\[
\begin{aligned}
u_1(x, y) &= u_2(y), \\
v_1(x, y) &= v_2(y), \\
p_1(x, y) &= p_2(y), \\
\rho_1(x, y) &= \rho_2(y).
\end{aligned}
\]  

(33)

Inserting these expressions in equations (31), it follows:

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\rho_2 v_2)' = 0, \\
\rho_2 v_2 u_2' = \mu u_2', \\
3\rho_2 v_2' + p_2' = 4\mu v_2', \\
3C_v (p_2 v_2)' = R(4\mu v_2'^2 - 3v_2^2 p_2 + 3\mu u_2'^2) + 3\kappa (p_2 / \rho_2)''.
\end{array} \right.
\]  

(34)

In equations (34) and in the rest of the document, a prime symbol is used to designate a derivation of a function depending on a single variable. One solution of (34) is

\[
\begin{aligned}
\rho_2(y) &= \frac{a}{v_2(y)}, \\
u_2(y) &= u_3 + u_4 e^{\mu y}, \\
v_2(y) &= v_3 e^{\mu y}, \\
p_2(y) &= p_3 e^{\mu y}.
\end{aligned}
\]  

(35)
where $a$, $u_3$, $u_4$ and $p_3$ are constants linked by the relations
\[
p_3 = \frac{av_3}{3} \quad \text{and} \quad u_4^2 = \left( \frac{2C_v}{3R} - \frac{4\kappa}{3R\mu} - 1 \right) v_3^2. \quad \text{(36)}
\]

We deduce the following class of solutions of (7):
\[
\begin{align*}
    u(t,x,y) &= u_3 \pm \sqrt{2C_v v_3^2 - 4\kappa - 3R\mu v_3^2} e^{ay} v_3, \\
    v(t,x,y) &= v_3 e^{ay}, \\
    p(t,x,y) &= \frac{av_3}{3} e^{ay}, \\
    \rho(t,x,y) &= a e^{-ay}. 
\end{align*}
\quad \text{(37)}
\]

It can be observed that the pressure and the density are respectively proportional and inversely proportional to $v$.

When $u_3 = 0$, the flow is parallel, as can be observed in Figure 1, left. Indeed, in a suitable orthogonal frame,
\[
    u = u_4 e^{\frac{a}{3}(cx + dy)}, \quad v = 0
\]
for some constants $u_4$, $c$ and $d$. This solution exhibits an exponential growth of norm of the velocity in both $x$ and $y$ directions.

Some other simple solutions belonging to class (37) are plotted in Figure 1.

### 4.2 Reduction with $Y_3$

Self-similar solutions under the infinitesimal rotation $Y_3$ write:
\[
\begin{align*}
    u_1 &= u_2(r) \cos \theta - v_2(r) \sin \theta, \quad v_1 = u_2(r) \sin \theta + v_2(r) \cos \theta, \\
    p_1 &= p_2(r), \quad \rho_1 = \rho_2(r) 
\end{align*}
\quad \text{(38)}
\]
where \((r, \theta)\) are the polar coordinates. The equations are reduced into

\[
\begin{cases}
(r\rho^2 u_2)' = 0 \\
\rho_2 \left( u_2 u'_2 - \frac{v_2^2}{r} \right) + p'_2 = 4\mu \left( u''_2 + \frac{u'_2}{r} - \frac{u_2}{r^2} \right) \\
\rho_2 \left( u_2 v'_2 - \frac{u_2 v_2}{r} \right) = \mu \left( v''_2 + \frac{v'_2}{r} - \frac{v_2}{r^2} \right) \\
\frac{Cv}{R} \left( (p_2 u_2)' + \frac{p_2 u_2}{r} \right) = E + \frac{\kappa}{R} \left( (p_2/p_2)' + \frac{(p_2/p_2)'}{r} \right)
\end{cases}
\]  

(39)

with

\[E = -\frac{p_2 (ru_2)'}{r} + \frac{\mu}{3r^2} \left( 4u_2^2 + 3v_2^2 - 4ru_2 u_2' - 6rv_2 v_2' + 4r^2 u_2'^2 + 3r^2 v_2'^2 \right).\]

These equations admit the following infinitesimal symmetry

\[r \frac{\partial}{\partial r} - 2u_2 \frac{\partial}{\partial u_2} - 2v_2 \frac{\partial}{\partial v_2} + \rho \frac{\partial}{\partial \rho} - 3\mu \frac{\partial}{\partial p_2}.
\]  

(40)

A self-similar solution under this symmetry verify:

\[u_3 = r^2 u_2(r), \quad v_3 = r^2 v_2(r), \quad p_3 = r^3 p_2(r), \quad \rho_3 = \frac{\rho_2(r)}{r}. \]  

(41)

where \(u_3, v_3, p_3\) and \(\rho_3\) are constants. Inserting (41) into equations (39) leads to the following algebraic relations on these constants:

\[
\begin{cases}
\rho_3 u_3 = -\mu \\
\rho_3 v_3^2 + 3p_3 + 2\mu u_3 = 0 \\
3p_3(4C_\nu \mu + R\mu - 16\kappa) = \mu R\rho_3 (28u_3^2 + 27v_3^2).
\end{cases}
\]  

(42)

The velocity components verify:

\[u(r, \theta) = \frac{u_3 \cos \theta - v_3 \sin \theta}{r^2}, \quad v(r, \theta) = \frac{u_3 \sin \theta + v_3 \cos \theta}{r^2}.
\]

If \(e_r\) and \(e_\theta\) designate the unitary radial and angular vectors, then the solution, in polar coordinates, is

\[\mathbf{u}(r, \theta) = \frac{1}{r^2} (u_3 e_r + v_3 e_\theta), \quad p(r, \theta) = \frac{p_3}{r^3}, \quad \rho(r, \theta) = \rho_3 r.
\]  

(43)

This solution represents a steady vortex flow. It is sketched in Figure 2. The constant \(u_3\) being negative, the origin is a sink. The sign of \(v_3\) determines the direction of rotation. The velocity magnitude increases as \(r^{-2}\) towards the sink.
4.3 Reduction with $Y_4$

A basis of invariants under the scale transformation generated by $Y_4$ is

$$\eta = \frac{y}{x}, \quad u_2(\eta) = xu_1(x,y), \quad v_2(\eta) = xv_1(x,y),$$

$$p_2(\eta) = x^2p_1(x,y), \quad \rho_2(\eta) = \rho_1(x,y).$$

The reduced equations are

$$\begin{cases}
(\rho_2 v_2 - \rho_2 u_2 \eta)' = 0 \\
\rho_2 (v_2 u_2' - u_2 (v_2 \eta))' - p_2' \eta - 2p_2 + \frac{\mu}{3} (v_2 \eta - 3u_2 - 4u_2 \eta^2)'' = 0 \\
\rho_2 (v_2 v_2' - u_2 (v_2 \eta))' + p_2' + \frac{\mu}{3} (u_2 \eta - 4v_2 - 3v_2 \eta^2)'' = 0 \\
C_v (p_2 (v_2 - u_2 \eta))' - 2C_v p_2 u_2 + R p_2 (v_2 - u_2 \eta)' - R \mu S_2 + \\
k \left( \frac{p_2}{\rho_2} \right)'' \eta^2 + 6 \left( \frac{p_2}{\rho_2} \right)' \eta + 6 \frac{p_2}{\rho_2} = 0
\end{cases}$$

(45)

where

$$S_2 = \frac{4}{3} (v_2' - u_2' \eta - u_2)^2 + \frac{4}{3} v_2' u_2 + (u_2' - v_2 \eta - v_2)^2.$$ 

The solution of system (45) is

$$u_2(\eta) = \frac{(\rho_3 - v_3 \eta) u_3}{\rho_3 (1 + \eta^2)}, \quad v_2(\eta) = u_2(\eta) \eta + \frac{v_3}{\rho_2(\eta)},$$

$$p_2(\eta) = \frac{(\rho_3^2 + v_3^2) u_2(\eta)}{2(v_3 \eta - \rho_3)}, \quad \rho_2(\eta) = \frac{\rho_3 - v_3 \eta}{(1 + \eta^2) u_2(\eta)}.$$ 

(46)
where $u_3$ and $v_3$ are constants and

$$
\rho_3 = \frac{-2\kappa + 4\mu R}{C_v}.
$$

(47)

As a result,

$$
u(t, x, y) = \frac{(\rho_3 x - v_3 y)u_3}{(x^2 + y^2)\rho_3}, \
v(t, x, y) = \frac{(\rho_3 y + v_3 x)u_3}{(x^2 + y^2)\rho_3}, \
p(t, x, y) = \frac{-\rho_3^2 + v_3^2}{2(x^2 + y^2)\rho_3}u_3, \
\rho(t, x, y) = \frac{\rho_3}{u_3}
$$

(48)

This is an incompressible solution, representing also a vortex, but with a velocity magnitude proportional to $r^{-1}$ towards the origin. If $u_3 > 0$, it models a swirling source and when $u_3 < 0$, the origin is a sink. These two cases are represented in Figure 3 when $|u_3| = 1$ and $v_3/\rho_3 = 1$.

Figure 3: Vortex sink (left) and source (right). $\|u\| \propto r^{-1}$

### 4.4 Reduction with a linear combination of $Y_1$ and $Y_2$

Invariant solutions under a linear combination $-bY_1 + aY_2$, for some constants $a$ and $b$, can be written as follows:

$$
u_1(x, y) = u_2(\eta), \quad v_1(x, y) = v_2(\eta), \
p_1(x, y) = p_2(\eta), \quad \rho_1(x, y) = \rho_2(\eta)
$$

(49)

where the self-similarity variable is

$$
\eta = ax + by.
$$
Relations (49) transform equations (39) into

\[
\begin{cases}
  a(\rho_1 u_1') + b(\rho_1 v_1') = 0 \\
  3\rho_1 (au_1 u'_1 + bv_1 u'_1) + 3ap'_1 = \mu((4a^2 + 3b^2)u''_1 + abv'_1) \\
  3\rho_1 (au_1 v'_1 + bv_1 v'_1) + 3bp'_1 = \mu((4b^2 + 3a^2)v''_1 + abu'_1) \\
  \frac{C_v}{R}(a(p_1 u_1) + b(p_1 v_1)') = \sigma : S + (a^2 + b^2)\frac{\kappa}{R}(p_1/\rho_1)''
\end{cases}
\]

(50)

One solution can easily be found if \(u_2\) and \(v_2\) are linear. In this case

\[
\begin{align*}
  u_2(\eta) &= u_3 \eta, \\
  v_2(\eta) &= -\frac{a}{b}u_3 \eta, \\
  p_2(\eta) &= p_3
\end{align*}
\]

(51)

\[
\rho_2(\eta) = \frac{-2\kappa p_3 b^2}{\mu u_3^2 R \eta^2 (a^2 + b^2) + 2\kappa p_3 (\rho_3 - \rho_4 \eta^2)}
\]

where \(u_3\), \(p_3\), \(\rho_3\) and \(\rho_4\) are constants. We get.

\[
\begin{align*}
  u(t, x, y) &= (ax + by)u_3, \\
  v(t, x, y) &= -\frac{a}{b}(ax + by)u_3, \\
  p(t, x, y) &= p_3,
\end{align*}
\]

(52)

\[
\begin{align*}
  \rho(t, x, y) &= \frac{-2\kappa p_3 b^2}{\mu u_3^2 R (ax + by)^2 (a^2 + b^2) + 2\kappa p_3 (\rho_3 - \rho_4 (ax + by)^2)}.
\end{align*}
\]

This solution represents a parallel, but direction-changing flow, with a uniform pressure. It is represented in Figure 4 in the case \(a > 0\) and \(b > 0\). When \(a = 0\) the flow is parallel to \(x\)-axis.

In the next section, we seek unsteady solutions.
5 Unsteady bidimensional solutions

In the case of unsteady bidimensional flow, the equations are

\[
\begin{cases}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \\
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} = \frac{\mu}{3} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{3 \partial^2 u}{\partial y^2} \right) \\
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} = \frac{\mu}{3} \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{4 \partial^2 v}{\partial y^2} + \frac{3 \partial^2 v}{\partial x^2} \right) \\
C_v \left( \frac{\partial p}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} \right) = \sigma : \mathbf{S} + \frac{\kappa}{R} \left( \frac{\partial^2 p/\rho}{\partial x^2} + \frac{\partial^2 p/\rho}{\partial y^2} \right)
\end{cases}
\] (53)

The Lie algebra of these equations is spanned by \(X_1, X_2, X_3, X_5, X_6, X_{10}, X_{11}\) and \(X_{12}\) (without the terms in \(\frac{\partial}{\partial z}\) and \(\frac{\partial}{\partial w}\)). Let us begin with solutions homogeneous in \(x\) direction, i.e. invariant under \(X_2\).

5.1 Reduction with \(X_2\)

\(X_2\)-invariant solutions are of the form

\[
\begin{align*}
& u(t, x, y) = u_1(t, y), \quad v(t, x, y) = v_1(t, y), \\
& p(t, x, y) = p_1(t, y), \quad \rho(t, x, y) = \rho_1(t, y).
\end{align*}
\] (54)

In this case, equations (53) reduce into

\[
\begin{cases}
\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 v_1}{\partial y} = 0 \\
\rho_1 \left( \frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial y} \right) = \mu \left( \frac{\partial^2 u_1}{\partial y^2} \right) \\
\rho_1 \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial y} \right) = -p_1 \frac{\partial v_1}{\partial y} + \frac{4 \mu}{3} \left( \frac{\partial^2 v_1}{\partial y^2} \right) \\
C_v \left( \frac{\partial p_1}{\partial t} + \frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 v_1}{\partial y} \right) = -p_1 \frac{\partial v_1}{\partial y} + \frac{4 \mu}{3} \left( \frac{\partial v_1}{\partial y} \right)^2 + \mu \left( \frac{\partial u_1}{\partial y} \right)^2 + \frac{\kappa}{R} \left( \frac{\partial^2 p_1/\rho_1}{\partial y^2} \right)
\end{cases}
\] (55)

These equations admit the following infinitesimal symmetries

\[
\begin{align*}
Z_1 &= \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial y}, \quad Z_3 = \frac{\partial}{\partial u_1}, \quad Z_4 = t \frac{\partial}{\partial y} + \frac{\partial}{\partial v_1}
\end{align*}
\]
\[
Z_5 = 2t \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} - u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} - 2p_1 \frac{\partial}{\partial p_1},
\]
\[
Z_6 = y \frac{\partial}{\partial y} + u_1 \frac{\partial}{\partial u_1} + v_1 \frac{\partial}{\partial v_1} - 2p_1 \frac{\partial}{\partial p_1}.
\]

A basis of invariants under \(Z_4\) is
\[
v_2(t) = v_1(t, y) - \frac{y}{t}, \quad u_1(t, y) = u_2(t),
\]
\[
p_1(t, y) = p_2(t), \quad \rho_1(t, y) = \rho_2(t).
\]

The corresponding solution for \(t > 0\) is
\[
\begin{align*}
  u(t, x, y) &= u_3, \\
  v(t, x, y) &= \frac{v_3 + y}{t}, \\
  p(t, x, y) &= \frac{4\mu + p_3}{3t} t^{-R/C_v}, \\
  \rho(t, x, y) &= \frac{\rho_3}{t},
\end{align*}
\]

where \(u_3, v_3, p_3\) and \(\rho_3\) are constants. The pressure and density are uniform but time-dependent. The flow is sketched in Figure 5 for a fixed \(t > 0\) and \(v_3 = 0\). The \(x\)-axis can be seen as a wall. When \(u_3 = 0\), the flow is parallel to the \(y\)-axis.

![Figure 5: Solution (57). Left: \(u_3 = 0\). Right: \(u_3 > 0\)](image)

Invariants under the vector field \(Z_5\) are
\[
\eta = \frac{y}{\sqrt{t}}, \quad u_2(\eta) = yu_1(t, y), \quad v_2(\eta) = yv_1(t, y),
\]
\[
p_2(\eta) = y^2 p_1(t, y), \quad \rho_2(\eta) = \rho_1(t, y).
\]
These invariants reduce equations (55) into
\[
\begin{align*}
-\eta^3 \rho'_2 + 2(v_2 \rho'_2 + v'_2 \rho_2) \eta - 2 \rho_2 v_2 &= 0 \\
-\eta^3 \rho_2 u'_2 + 2 \rho_2 v_2(u'_2 \eta - u_2) + 2\mu(-\eta^2 u'_2 + 2\eta u'_2 - 2u_2) &= 0 \\
-\eta^3 \rho_2 v'_2 + 2 \rho_2 (v_2 v'_2 \eta - v'_2) + 2\eta p'_2 - 4p_2 + \frac{8\mu}{3}(-\eta^2 v'_2 + 2\eta v'_2 - 2v_2) &= 0 \\
\frac{C_0}{R}(-\eta^3 p'_2 + 2 v_2 p'_2 \eta - 6 v_2 p_2 + 2\eta p_2 v'_2) &= 2p_2(-\eta v'_2 + v_2) \\
+2\mu(\eta^2 u'_2 - 2\eta u_2 u'_2 + u_2^2) + \frac{8\mu}{3}(\eta^2 v'_2 - 2\eta v'_2 v_2 + v_2^2) + \frac{2\kappa \eta^4}{R} \left( \frac{p_2}{\eta^2 \rho_2} \right)'' &= 0
\end{align*}
\]

A solution of these equations can be found with the ansatz:
\[
u_2(\eta) = u_3 \eta^2, \quad v_2(\eta) = v_3 \eta^2, \quad p_2(\eta) = p_3 \eta^2, \quad \rho_2 = \rho_3 \eta^{-2}
\] (59)

where \(u_3, v_3, p_3\) and \(\rho_1\) are constants. Inserting these relations into the equations implies relations on these constants and leads to the following parallel flow:

\[
u(t,x,y) = \frac{u_3 y}{t}, \quad \nu(t,x,y) = \frac{v_3 y}{t},
\]

\[
\begin{align*}
u(t,x,y) &= \frac{\mu \rho_3 R(4 + 3u_3^2)}{3t(\rho_3 R - 2\kappa)}, \quad \rho(t,x,y) = \frac{\rho_3 t}{y^2}
\end{align*}
\]

(60)

The flow is plotted in Figure 6, left.

---

Figure 6: Left: solution (60). Right: solution (64)

Self-similar solutions of equations (55) under \(Z_6\) have the form

\[
u_1(t,y) = y u_2(t), \quad v_1(t,y) = y v_2(t),
\]

\[
u_1(t,y) = p_2(t), \quad \rho_1(t,y) = \rho_2(t)/y^2.
\]

(61)
This change of variables simplifies the equations into

\[
\begin{aligned}
-\rho'_2 + v_2\rho_2 &= 0 \\
\frac{3C_v}{R}(p'_2 + p_2v_2) &= -3p_2v_2 + \mu(3u_2^2 + 4v_2^2) + \frac{6\kappa p_2}{R\rho_2} \\
v'_2 + v_2^2 &= 0 \\
3C_v(p'_2 + p_2v_2) &= -3p_2v_2 + \mu(3u_2^2 + 4v_2^2) + \frac{6\kappa p_2}{R\rho_2}
\end{aligned}
\]  

(62)

One solution of equations (62) is

\[
\begin{aligned}
u_2(t) &= u_3, & v_2(t) &= 0, & \rho_2(t) &= \rho_3, \\
p_2(t) &= p_3 \exp\left(\frac{2\kappa t}{\rho_3 C_v}\right) - \frac{\rho_3 \mu R u_3^2}{2\kappa},
\end{aligned}
\]  

(63)

where \(u_3, p_3\) and \(\rho_3\) are arbitrary constants. We get the following self-similar solution under \(X_2\) and \(Z_6\):

\[
\begin{aligned}
u(t, x, y) &= u_3 y, & p(t, x, y) &= p_3 \exp\left(\frac{2\kappa t}{\rho_3 C_v}\right) - \frac{\rho_3 \mu R u_3^2}{2\kappa}, \\
\rho(t, x, y) &= \frac{\rho_3}{y^2}.
\end{aligned}
\]  

(64)

This solution is graphically presented in Figure 6 (right). If one limits to \(y \in [0, h]\) for some constant \(h\), it may represent a Couette flow, with a uniform but time-dependent pressure field and a density depending on the wall distance.

Another solution of (62) is

\[
\begin{aligned}
u_2(t) &= \frac{u_3}{t + v_3}, & v_2(t) &= \frac{1}{t + v_3}, & \rho_2(t) &= (t + v_3)\rho_3, \\
p_2(t) &= p_3 \exp\left(\frac{2\kappa - \rho_3(C_v + R)}{\rho_3 C_v}\ln(t + v_3)\right) + \frac{\rho_3 \mu R \mu (4 + 3u_3^2)}{3(t + v_3)(\rho_3 R - 2\kappa)}.
\end{aligned}
\]  

where \(u_3, p_3\) and \(\rho_3\) are constants. Hence,

\[
\begin{aligned}
u(t, x, y) &= \frac{u_3 y}{t + v_3}, & v(t, x, y) &= \frac{y}{t + v_3}, & \rho(t, x, y) &= \frac{(t + v_3)\rho_3}{y^2}, \\
p(t, x, y) &= p_3 \exp\left(\frac{2\kappa - \rho_3(C_v + R)}{\rho_3 C_v}\ln(t + v_3)\right) + \frac{\rho_3 \mu R \mu (4 + 3u_3^2)}{3(t + v_3)(\rho_3 R - 2\kappa)}.
\end{aligned}
\]  

(65)

This solution has the same profile as solution (60) which is plotted in the left part of Figure 6, but with an algebraic time evolution of the pressure instead of a hyperbolic one.

Still in the bidimensional case, we reduce equations (53) under infinitesimal Galilean transformation \(X_5\) and under infinitesimal scale transformations.
5.2 Galilean transformation

Self-similar solutions of (53) under $X_5$ can be expressed as follows

\[ u(t, x, y) = u_1(t, y) + \frac{x}{t}, \quad v(t, x, y) = v_1(t, y), \]

\[ p(t, x, y) = p_1(t, y), \quad \rho(t, x, y) = \rho_1(t, y). \]  \quad (66)

The reduced equations read:

\[
\begin{align*}
\frac{\partial p_1}{\partial t} + \frac{p_1}{t} + \frac{\partial p_1 v_1}{\partial y} &= 0 \\
\rho_1 \left( \frac{\partial u_1}{\partial t} + \frac{u_1}{t} + v_1 \frac{\partial u_1}{\partial y} \right) &= \mu \left( \frac{\partial^2 u_1}{\partial y^2} \right) \\
\rho_1 \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial y} \right) + \frac{\partial p_1}{\partial y} &= \frac{4\mu}{3} \frac{\partial^2 v_1}{\partial y^2} \\
C_v \left( \frac{\partial p_1}{\partial t} + \frac{p_1}{t} + \frac{\partial p_1 v_1}{\partial y} \right) &= \sigma : S + \kappa \left( \frac{\partial^2 p_1 / \rho_1}{\partial y^2} \right)
\end{align*}
\]

with

\[
\frac{\sigma : S}{\mu} = -\frac{p_1}{\mu} \left( \frac{1}{t} + \frac{\partial v_1}{\partial y} \right) + \left( \frac{\partial u_1}{\partial y} \right)^2 + \frac{4}{3} \left( \frac{\partial v_1}{\partial y} \right)^2 + \frac{4}{3t^2} - \frac{4}{3t} \frac{\partial v_1}{\partial y}. \quad (68)
\]

The symmetries of (67) are generated by

\[
W_1 = \frac{\partial}{\partial y}, \quad W_2 = \frac{1}{t} \frac{\partial}{\partial u_1}, \quad W_3 = t \frac{\partial}{\partial y} + \frac{\partial}{\partial v_1},
\]

\[
W_4 = 2t \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} - u_1 \frac{\partial}{\partial u_1} - v_1 \frac{\partial}{\partial v_1} - 2 \frac{\partial p_1}{\partial p_1}, \quad (69)
\]

\[
W_5 = y \frac{\partial}{\partial y} + u_1 \frac{\partial}{\partial u_1} + v_1 \frac{\partial}{\partial v_1} - 2 \frac{\partial p_1}{\partial p_1}.
\]

Solutions of (67) which are invariant under $W_1$ are $y$-independant. Solving equations (67) with this constraint leads to

\[ u_1(t, y) = \frac{u_2}{t}, \quad v_1(t, y) = v_2, \quad p_1(t, y) = p_2 t^{(-1 - R/C_v)} + \frac{4\mu}{3t}, \]

\[ \rho_1(t, y) = \frac{\rho_2}{t}. \]  \quad (70)
where $u_2$, $v_2$, $p_2$ and $\rho_2$ are constants. Consequently,

$$
\begin{align*}
    u(t, x, y) &= \frac{x + u_2}{t}, \quad v(t, x, y) = v_2, \\
    p(t, x, y) &= p_2 t^{(-1 - R/C_v)} + \frac{4\mu}{3t}, \quad \rho(t, x, y) = \frac{\rho_2}{t}.
\end{align*}
$$

(71)

This solution is a rotation of solution (57). It is presented in Figure 7.

Figure 7: Solution (71) with $t > 0$ and $u_2 = 0$

If instead of $W_1$, we consider $W_3$ then the self-similar solutions of equations (67) are of the form:

$$
\begin{align*}
    u_1(t, y) &= u_2(t), \quad v_1(t, y) = v_2(t) + \frac{y}{t}, \quad p_1(t, y) = p_2(t), \quad \rho_1(t, y) = \rho_2(t).
\end{align*}
$$

With these relations, the solutions to equations (67) is

$$
\begin{align*}
    u_2 &= \frac{u_3}{t}, \quad v_2 = \frac{v_3}{t}, \quad \rho_2 = \frac{\rho_3}{t}, \quad p_2 = \frac{4R\mu}{3(2R + C_v)t} + p_3 t^{-2 - 2R/C_v}
\end{align*}
$$

where $u_3, v_3, p_3$ and $\rho_3$ are arbitrary scalars. The corresponding solution of (7) is

$$
\begin{align*}
    u(t, x, y) &= \frac{x + u_3}{t}, \quad v(t, x, y) = \frac{v_3 + y}{t}, \\
    p(t, x, y) &= \frac{4R\mu}{3(2R + C_v)t} + p_3 t^{-2 - 2R/C_v}, \quad \rho(t, x, y) = \frac{\rho_3}{t}.
\end{align*}
$$

(72)

It represents a source flow around the point $(-u_3, -v_3)$. The velocity field is plotted in Figure 8.

To find self-similar solutions under infinitesimal scale transformation $W_5$, we set:

$$
\begin{align*}
    u_1(t, y) &= u_2(t)y, \quad v_1(t, y) = v_2(t)y, \\
    p_1(t, y) &= p_2(t), \quad \rho_1(t, y) = \frac{\rho_2(t)}{y^2}.
\end{align*}
$$

(73)
It follows from the reduced equations that

\[ v_2(t) = \frac{\delta}{t + v_3}, \quad u_2(t) = \frac{u_3}{(t + v_3)^2}, \quad \rho_2(t) = \frac{(t + v_3)^2 \rho_3}{t} \]  

(74)

where \( u_3 \) and \( \rho_3 \) are constants and \( \delta = 0 \) or 1. Thus,

\[ u(t, x, y) = \frac{u_3 y}{t(t + v_3)^2} + \frac{x}{t^2} \quad v(t, x, y) = \frac{y \delta}{t + v_3} \quad \rho(t, x, y) = \frac{(t + v_3)^2 \rho_3}{ty^2} \]

(75)

The pressure \( p(t) \) is the solution of the ordinary differential equation:

\[ \frac{C_v}{R} \frac{\partial p}{\partial t} + \frac{(C_v + R)(2t + v_3)^2 \rho_3 - 2\kappa t^2}{\rho_3 Rt(t + v_3)^2} p = \mu \frac{3u_3^2 + 4(t^2 + v_3t + v_3^2)}{3t^2(t + v_3)^2}. \]

(76)

If \( u_3 = 0 \) and \( \delta = 0 \), the velocity field is similar to Figure 8 but 90 degrees rotated. In other cases, the flow is graphically presented in Figure 9.

Another solution of equations (67) can be found by setting \( v_1 \) constant and
$u_1$ linear in $y$. This leads to the following solution of (7)

$$
\begin{align*}
    u(t, x, y) &= \frac{x + y + u_2}{t} - v_2, \\
    v(t, x, y) &= v_2, \\
    \rho(t, x, y) &= \frac{1}{t} \frac{1}{(y - v_2 t)^2 \rho_3 + (y - v_2 t) \rho_4 + \rho_5}
\end{align*}
$$

(77)

where $u_2, v_2, \rho_3, \rho_4$ and $\rho_5$ are constants. $p(t)$ is the solution of

$$
\frac{C_v}{R^{1/2}} \rho' + \frac{C_v + R - 2 \kappa \rho_2 t^2}{R t} p - \frac{7 \mu}{t^2} = 0.
$$

(78)

In the next subsection, we calculate bidimensional solutions of (7) which are self-similar under scale transformations.

5.3 Scale transformations

Infinitesimal symmetry $X_{11}$ suggests a change of variables:

$$
\begin{align*}
    u(t, x, y) &= \frac{u_1(\chi, \eta)}{\sqrt{t}}, \\
    v(t, x, y) &= \frac{v_1(\chi, \eta)}{\sqrt{t}}, \\
    p(t, x, y) &= \frac{p_1(\chi, \eta)}{t}, \\
    \rho(t, x, y) &= \rho_1(\chi, \eta) \\
    \text{with} \\
    \chi &= \frac{x}{\sqrt{t}}, \\
    \eta &= \frac{y}{\sqrt{t}}
\end{align*}
$$

(79)
The equations of the new variables are:

\[
\begin{cases}
-\frac{\chi}{2} \frac{\partial \rho_1}{\partial \chi} - \frac{\eta}{2} \frac{\partial \rho_1}{\partial \eta} + \frac{\partial \rho_1 u_1}{\partial \chi} + \frac{\partial \rho_1 v_1}{\partial \eta} = 0 \\
\rho_1 \left( -\frac{\chi}{2} \frac{\partial u_1}{\partial \chi} - \frac{\eta}{2} \frac{\partial u_1}{\partial \eta} - \frac{1}{2} u_1 + u_1 \frac{\partial u_1}{\partial \chi} + v_1 \frac{\partial u_1}{\partial \eta} \right) = -\frac{\partial \rho_1}{\partial \chi} + \frac{\mu_3}{3} \left( 4 \frac{\partial^2 u_1}{\partial \chi^2} + \frac{\partial^2 v_1}{\partial \chi \partial \eta} + 3 \frac{\partial^2 u_1}{\partial \eta^2} \right) \\
\rho_1 \left( -\frac{\chi}{2} \frac{\partial v_1}{\partial \chi} - \frac{\eta}{2} \frac{\partial v_1}{\partial \eta} - \frac{1}{2} v_1 + u_1 \frac{\partial v_1}{\partial \chi} + v_1 \frac{\partial v_1}{\partial \eta} \right) = -\frac{\partial \rho_1}{\partial \eta} + \frac{\mu_3}{3} \left( 4 \frac{\partial^2 v_1}{\partial \chi^2} + \frac{\partial^2 u_1}{\partial \chi \partial \eta} + 3 \frac{\partial^2 v_1}{\partial \eta^2} \right) \\
\frac{C_v}{R} \left( -\frac{\chi}{2} \frac{\partial \rho_1}{\partial \chi} - \frac{\eta}{2} \frac{\partial \rho_1}{\partial \eta} - \rho_1 + \frac{\partial \rho_1 u_1}{\partial \chi} + \frac{\partial \rho_1 v_1}{\partial \eta} \right) = -p \left( \frac{\partial u_1}{\partial \chi} + \frac{\partial v_1}{\partial \eta} \right) + \mu S' + \kappa \left( \frac{\partial^2 u_1}{\partial \chi^2} + \frac{\partial^2 v_1}{\partial \eta^2} \right) \frac{p_1}{\rho_1}
\end{cases}
\]

where

\[
S' = \frac{4}{3} \left[ \left( \frac{\partial u_1}{\partial \chi} \right)^2 + \left( \frac{\partial v_1}{\partial \eta} \right)^2 - \frac{\partial u_1}{\partial \chi} \frac{\partial v_1}{\partial \eta} \right] + \left( \frac{\partial u_1}{\partial \eta} + \frac{\partial v_1}{\partial \chi} \right)^2.
\]

A solution to these equations is

\[
\begin{align*}
&u_1(\chi, \eta) = u_2 \chi, \quad v_1(\chi, \eta) = 0, \quad \rho_1(\chi, \eta) = \frac{\rho_2}{\chi^2}, \\
&p_1(\chi, \eta) = \frac{4\mu_s^2 \rho_2 R}{3\rho_2 R - 6\kappa}
\end{align*}
\]

for some constants \(u_2\) and \(\rho_2\). We deduce that

\[
\begin{align*}
u(t, x, y) &= \frac{u_2 x}{t}, & v(t, x, y) &= 0, \\
p(t, x, y) &= \frac{4\mu_s^2 \rho_2 R}{3(\rho_2 R - 2\kappa)t}, & \rho(t, x, y) &= \frac{\rho_2 t}{x^2}.
\end{align*}
\]

To get self-similar solutions of the bidimension equations under \(X_{12}\), one makes the change of variables:

\[
\begin{align*}
&\eta = \frac{y}{x}, \quad u_1(t, \eta) = \frac{u(t, x, y)}{x}, \quad v_1(t, \eta) = \frac{v(t, x, y)}{x}, \\
p_1(t, \eta) &= p(t, x, y), \quad \rho_1(t, \eta) = x^2 \rho(t, x, y).
\end{align*}
\]
The equations become:

\[
\begin{align*}
\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial \eta} \left[ \rho_1 (v_1 - u_1 \eta) \right] = 0 \\
\rho_1 \left[ \frac{\partial u_1}{\partial t} + u_1^2 + (v_1 - u_1 \eta) \frac{\partial u_1}{\partial \eta} \right] - \eta \frac{\partial \rho_1}{\partial \eta} = \mu \left( \frac{4 \eta^2}{3} + 1 \right) \frac{\partial^2 u_1}{\partial \eta^2} - \frac{\eta \partial^2 v_1}{\partial \eta^2} \\
\rho_1 \left[ \frac{\partial v_1}{\partial t} + u_1 v_1 + (v_1 - u_1 \eta) \frac{\partial v_1}{\partial \eta} \right] + \frac{\partial p_1}{\partial \eta} = -\mu \frac{\eta \partial^2 u_1}{3 \partial \eta^2} + \mu \left( \frac{4}{3} + \eta^2 \right) \frac{\partial^2 v_1}{\partial \eta^2} \\
\frac{C_v}{R} \left[ \frac{\partial p_1}{\partial \eta} + (v_1 - \eta u_1) \frac{\partial p_1}{\partial \eta} \right] = -\left( \frac{C_v}{R} + 1 \right) p_1 \left( u_1 - \eta \frac{\partial u_1}{\partial \eta} - \frac{\partial v_1}{\partial \eta} \right) + \mu \frac{\kappa}{R} \left( 1 + \eta^2 \right) \frac{\partial^2 p_1}{\partial \eta^2} \frac{p_1}{\rho_1} - 2 \eta \frac{\partial p_1}{\partial \eta} \frac{p_1}{\rho_1} + 2 \frac{p_1}{\rho_1}
\end{align*}
\]

\[(83)\]

where

\[
D = -\frac{2}{3} \left( u_1 - \eta \frac{\partial u_1}{\partial \eta} + \frac{\partial v_1}{\partial \eta} \right)^2 + 2 \left( u_1 - \eta \frac{\partial u_1}{\partial \eta} \right)^2 + 2 \left( \frac{\partial v_1}{\partial \eta} \right)^2 + \left( \frac{\partial u_1}{\partial \eta} - \eta \frac{\partial v_1}{\partial \eta} + v_1 \right)^2
\]

A solution of \[(83)\] is

\[
\begin{align*}
\rho_1(t, \eta) = \frac{1}{t}, \quad v_1(t, \eta) = \frac{\eta}{t}, \quad p_1(t, \eta) = p_2(t), \quad \rho_1(t, \eta) = \rho_2
\end{align*}
\]

\[(84)\]

where \( \rho_2 \) is a constant, and \( p_2(t) \) is the solution of

\[
3 C_v \rho_2^2 p_2'(t) + 6 t (C_v \rho_2 - \kappa t + \rho_2 R) p_2(t) - 4 \mu \rho_2 R = 0
\]

\[(85)\]

With the original variables, we get:

\[
\begin{align*}
\frac{u(t, x, y)}{t} = \frac{x}{t}, \quad \frac{v(t, x, y)}{t} = \frac{y}{t}, \\
\frac{p(t, x, y)}{t} = p_2(t), \quad \rho(t, x, y) = \frac{\rho_2}{x^2}
\end{align*}
\]

\[(86)\]

The pressure \( p_2(t) \) can be written in terms of Whittaker \( M \) functions \[13\]. For instance, when \( C_v \rho_2 = 1 \), if we call \( b = \rho_2 R \) then

\[
p_2(t) = \frac{2 b \mu e^{\kappa t} M_{b, b+1/2}(2 \kappa t)}{3(2b + 1) \kappa t^2(2\kappa t)^b} + \frac{e^{2\kappa t}}{t^{2+2b}}
\]

\[(87)\]

\( \rho_3 \) being a constant.
Other solutions of \[ (83) \] can be obtained knowing that these equations admit the following infinitesimal symmetry:

\[
\left(1 + \eta^2\right) \frac{\partial}{\partial \eta} + (\eta u_1 - v_1) \frac{\partial}{\partial u_1} + (\eta v_1 + u_1) \frac{\partial}{\partial v_1} - 2 \rho_1 \eta \frac{\partial}{\partial \rho_1}.
\]

It suggests the change of variables:

\[
u_1(t, \eta) = u_2(t) \eta + v_2(t), \quad v_1(t, \eta) = v_2(t) \eta - u_2(t), \quad p_1(t, \eta) = p_2(t), \quad \rho_1(t, \eta) = \frac{\rho_2(t)}{1 + \eta^2},
\]

(88)

\]
corresponding to a velocity field

\[
u(t, r, \theta) = v_2(t) r e_r + u_2(t) r e_\theta.
\]

Inserting these relations into equations (83), we obtain:

\[
v_2(t) = \frac{f'(t)}{2f(t)}, \quad u_2(t) = \pm \sqrt{v_2^2(t) + v'_2(t)}, \quad \rho_2(t) = \rho_3
\]

\[
p_2(t) = \frac{\mu R}{3} \frac{h(t)}{f(t)^{1+R/C_v}} \int \frac{f(t)^{R/C_v-1}f'(t)^2}{h(t)} dt
\]

with \( f(t) = at^2 + 2bt + 2, \) \( a, b \) and \( \rho_3 \) being constants such that \( 2a \geq b^2 \). It follows that

\[
u(t, r, \theta) = \frac{at + b}{at^2 + 2bt + 2} re_r \pm \frac{\sqrt{2a - b^2}}{at^2 + 2bt + 2} re_\theta, \quad \rho(t, r, \theta) = \frac{\rho_3}{r^2},
\]

\[
p(t, r, \theta) = \frac{4\mu R e^{4\kappa t/C_v \rho_3}}{(at^2 + 2bt + 2)^{R/C_v + 1}} \int \frac{(at^2 + 2bt + 2)^{R/C_v - 1}(at + b)^2}{3C_v e^{4\kappa t/C_v \rho_3}} dt.
\]

\]

If \( a = b^2/2 \) then the flow is radial, with \( u = \frac{r}{t + 2/b} e_r \). The velocity field is similar to that of solution (72), plotted in Figure 8 for a fixed \( t \) but the pressure and density fields are different. The evolution of the flow with time can be visualized in Figure 11 in the case \( b = 0, a = 1 \) and \( u_2(t) > 0 \).

In the following section, we compute some three-dimensional solutions of equations (7).

6 3D-solutions

We first consider solutions invariant under \( X_1, X_2 \) and \( X_4 \). Such solutions depend only on \( y \):

\[
u(t, x, y, z) = u_1(y), \quad v(t, x, y, z) = v_1(y), \quad w(t, x, y, z) = w_1(y), \quad p(t, x, y, z) = p_1(y), \quad \rho(t, x, y, z) = \rho_1(y).
\]

\]

(90)
Figure 11: Solution (89) with \( b = 0 \) and \( a = 1 \), at \( t = -1 \) (top left), \( t = 0 \) (top right), \( t = 1 \) (bottom left) and \( t = 10 \) (bottom right)

The reduced equations are:

\[
\begin{align*}
(\rho_1 v_1)' &= 0, \\
\rho_1 v_1 u_1' &= \mu u_1'', \\
3\rho_1 v_1 v_1' + p_1 &= 4\mu v_1'', \\
\rho_1 v_1 w_1' &= \mu w_1'', \\
3C_v(p_1 v_1)' &= R(4\mu v_1'^2 - 3v_1' p_1 + 3\mu u_1'^2 + 3\mu w_1'^2) + 3\kappa(p_2/\rho_2)''.
\end{align*}
\]

We find the following solution which is an extension of (37):

\[
\begin{align*}
u(t, y) &= u_3 \pm \sqrt{\frac{(2C_v\mu - 4\kappa - 3R\mu)v_3^2 - 3R\mu w_3^2}{3R\mu}} e^{ay/\mu}, \\
v(t, y) &= v_3 e^{ay/\mu}, \\
w(t, x, y, z) &= w_3 e^{ay/\mu}, \\
p(t, y) &= \frac{av_3}{3} e^{ay/\mu}, \\
\rho(t, y) &= \frac{a}{v_3} e^{-ay/\mu},
\end{align*}
\]

In these expressions, \( a, u_3, v_3 \) and \( w_3 \) are arbitrary constants.
The generator \((X_2 + X_5) + (X_3 + X_6)\) leads to the ansatz:

\[
\begin{align*}
  u(t, x, y, z) &= u_1(t, z) + \frac{xu_0}{tu_0 + a}, \\
  v(t, x, y, z) &= v_1(t, z) + \frac{yv_0}{tv_0 + b} \\
  w(t, x, y, z) &= w_1(t, z), \\
  p(t, x, y, z) &= p_1(t, z), \\
  \rho(t, x, y, z) &= \rho_1(t, z)
\end{align*}
\] (93)

where \(a, b, u_0\) and \(v_0\) are arbitrary constants. Equations (7) reduce into:

\[
\begin{align*}
  \frac{\partial \rho_1}{\partial t} + w_1 \frac{\partial \rho_1}{\partial z} + \rho_1 \delta &= 0, \\
  \rho_1 \left( \frac{\partial u_1}{\partial t} + \frac{u_0u_1}{u_0t + a} + u_1 \frac{\partial u_1}{\partial z} \right) &= \mu \frac{\partial^2 u_1}{\partial z^2}, \\
  \rho_1 \left( \frac{\partial v_1}{\partial t} + \frac{v_0v_1}{v_0t + b} + v_1 \frac{\partial v_1}{\partial z} \right) &= \mu \frac{\partial^2 v_1}{\partial z^2}, \\
  \rho_1 \left( \frac{\partial w_1}{\partial t} + w_1 \frac{\partial w_1}{\partial z} \right) + \frac{\partial p_1}{\partial z} &= 4 \mu \frac{\partial^2 w_1}{\partial z^2}, \\
  \frac{C_v}{R} \left( \frac{\partial p_1}{\partial t} + w_1 \frac{\partial p_1}{\partial z} \right) &= \left( \frac{C_v}{R} + 1 \right) \delta p_1 + \mu S_2 + \frac{\kappa}{R} \frac{\partial^2 p_1}{\partial \rho^2}
\end{align*}
\] (94)

where \(\delta = \frac{u_0}{u_0t + a} + \frac{v_0}{v_0t + b} + \frac{\partial w_1}{\partial z}\)

is the divergence of \(u\) and

\[
S_2 = -\frac{2}{3} \delta^2 + \frac{2u_0^2}{(tu_0 + a)^2} + \frac{2v_0^2}{(tv_0 + b)^2} + \left( \frac{\partial u_1}{\partial z} \right)^2 + \left( \frac{\partial v_1}{\partial z} \right)^2 + 2 \left( \frac{\partial w_1}{\partial z} \right)^2.
\]

The infinitesimal symmetries of these equations are

\[
\begin{align*}
  R_1 &= \frac{\partial}{\partial z}, \\
  R_2 &= t \frac{\partial}{\partial z} + \frac{\partial}{\partial u}, \\
  R_3 &= \frac{1}{u_0t + a} \frac{\partial}{\partial u}, \\
  R_4 &= \frac{1}{v_0t + b} \frac{\partial}{\partial v}, \\
  R_5 &= z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2 \rho \frac{\partial}{\partial \rho}.
\end{align*}
\] (95)

Without loss of generality, assume that \(u_0 = v_0 = 1\).

Symmetry \(R_1\) reduces equations (94) into:

\[
\begin{align*}
  (t + a)(t + b) \rho'_2 + \rho_2 &= 0, \\
  (t + a)u'_2 + u_2 &= 0, \\
  (t + b)v'_2 + v_2 &= 0, \\
  w'_2 &= 0, \\
  C_v p'_2 &= R \sigma : S
\end{align*}
\] (96)
where
\[ u_2(t) = u_1(t, z), \quad v_2(t) = v_1(t, z), \quad w_2(t) = w_1(t, z), \quad p_2(t) = p_1(t, z), \quad \rho_2(t) = \rho_1(t, z). \] (97)

The resolution of the equations gives the bidimensional solution
\[
\begin{align*}
  u(t, x, y, z) &= \frac{x + u_3}{t + a}, & v(t, x, y, z) &= \frac{y + v_3}{t + b}, \\
  w(t, x, y, z) &= w_3, & \rho(t, x, y, z) &= \frac{\rho_3}{h(t)}, \\
  p(t, x, y, z) &= \frac{4R\mu}{3C_v} h(t)^{-R/C_v - 1} \int h(t) + (a - b)^2 \, dt \quad (98)
\end{align*}
\]

where \( u_3, v_3, w_3 \) and \( \rho_3 \) are arbitrary constants and \( h(t) = (t + a)(t + b) \).

In the particular case where \( a = b \), the expression of \( p \) in (98) simplifies into
\[
p(t, x, y, z) = \frac{4R\mu}{3(2R + C_v)} t + p_3 t^{-2R/C_v - 2},
\]

\( p_3 \) being a constant.

An invariant solution of equations (94) under \( R_2 \) has the following form:
\[
\begin{align*}
  u_1(t, z) &= u_2(t), & v_1(t, z) &= v_2(t), & w_1(t, z) &= w_2(t) + \frac{zw_0}{tw_0 + c}, \\
  p_1(t, z) &= p_2(t), & \rho_1(t, z) &= \rho_2(t). \quad (99)
\end{align*}
\]

Assume that \( w_0 = 1 \). A straightforward solution of (94), when \( a = 0 \), is
\[
\begin{align*}
  u(t, x, y, z) &= \frac{x + u_3}{t}, & v(t, x, y, z) &= \frac{y + v_3}{t + b}, \\
  w(t, x, y, z) &= \frac{z + w_3}{t + c}, & \rho(t, x, y, z) &= \frac{\rho_3}{h(t)}, \\
  p(t, x, y, z) &= \frac{4R\mu}{3C_v} h(t)^{-1} \int \frac{t^2(b - c)^2 + bc(t + b)(t + c)}{h(t)^{1 - \frac{R}{C_v}} b}. \quad (100)
\end{align*}
\]

where \( u_3, v_3, w_3, \rho_3 \) are constants and \( h(t) = t(t + b)(t + c) \).

Another solution of (94) in the form (99), with \( a = b = c = t_0 \), is
\[
\begin{align*}
  u(t, x, y, z) &= \frac{x + u_3}{t + t_0}, & v(t, x, y, z) &= \frac{y + v_3}{t + t_0}, \\
  w(t, x, y, z) &= \frac{z + w_3}{t + t_0}, & \rho(t, x, y, z) &= \frac{\rho_3}{(t + t_0)^3}, \\
  p(t, x, y, z) &= p_3 (t + t_0)^{-3 - \frac{R}{C_v}}. \quad (102)
\end{align*}
\]
where $u_3$, $v_3$, $w_3$, $\rho_3$ are constants. In (100) and (102), the flow is fully three-dimensional. It is purely radial around the source point $(u_3, v_3, w_3)$. The pressure and the density are uniform but time-dependent.

A solution of (94) invariant under $R_5$ takes the form:

$$
\begin{align*}
  u_1(t, z) &= u_2(t)z, & v_1(t, z) &= v_2(t)z, & w_1(t, z) &= w_2(t)z, \\
  p_1(t, z) &= p_2(t), & \rho_1(t, z) &= \rho_2(t)z^{-2}.
\end{align*}
$$

Inserting these expressions into (94) gives:

$$
\begin{align*}
  u_2(t) &= \frac{u_3}{(t + a)(t + c)}, & v_2(t) &= \frac{v_3}{(t + b)(t + c)}, & w_2(t) &= \frac{1}{t + c}, \\
  \rho_2(t) &= \frac{(t + c)\rho_3}{(t + a)(t + b)}
\end{align*}
$$

where $u_3$, $v_3$ and $\rho_3$ are arbitrary constants. The pressure reads:

$$
p_2(t) = \frac{\mu R h(t)^{-\frac{n}{\nu}} - h_c(t)f(t)}{3C_v} \int h(t)^{-\frac{n}{\nu}} g(t) h_c(t)f(t) t.
$$

with

$$
\begin{align*}
  h(t) &= (t + a)(t + b)(t + c), & h_c(t) &= (t + c)^{2\frac{(a + c)(b - c)}{\rho_3 C_v}},
\end{align*}
$$

$$
f(t) = \exp \left( \frac{\kappa t(t + 2a + 2b - 2c)}{\rho_3 C_v} \right)
$$

and

$$
g(t) = 4(a^2 + b^2 + c^2 - ab - bc - ca)t^2 \\
  + 4[a^2(b + c) + b^2(c + a) + c^2(a + b) - 6abc]t \\
  + 4[a^2(b - c) + b^2(c - a) + c^2(a - b)] \\
  + 3[v_3^2(t + a)^2 + u_3^2(t + b)^2].
$$

At last,

$$
\begin{align*}
  u(t, x, y, z) &= \frac{u_3z + x(t + c)}{(t + a)(t + c)}, & v(t, x, y, z) &= \frac{v_3z + y(t + c)}{(t + b)(t + c)}, \\
  w(t, x, y, z) &= \frac{z}{t + c}, & p(t, x, y, z) &= p_2(t), \\
  \rho(t, x, y, z) &= \frac{\rho_3(t + c)}{(t + a)(t + b)z^2}.
\end{align*}
$$
The projection of $u$ on $xy$-planes are sketched in Figure 12 for $c = 0$, $u_3 = v_3 = 1$ at $t = 1$. The center-point, at which the velocity is vertical, is located at

$$x = \frac{-u_3 z}{t + c}, \quad y = \frac{-v_3 z}{t + c}. \quad (109)$$

As can be observed, the $xy$-plane-projected flow is radial around the center-point defined by equation (109) when $a = b = 1$. It is not the case any longer if $a \neq b$. The value of $c$ changes the position of the center-point (109) but not the shape of the $xy$-projections. It however has more influence on the projection of $u$ on $xz$- or $yz$-plane. It can be stated in Figure 13.

It is noteworthy that the generator

$$\frac{\partial}{\partial z} + f(t) \frac{\partial}{\partial p} \quad (110)$$

is an infinitesimal symmetry of the four first equations of (94) for any regular
function \( f \). It leads to the ansatz

\[
\begin{align*}
    u_1(t, z) &= u_2(t), \\
    v_1(t, z) &= v_2(t), \\
    w_1(t, z) &= w_2(t), \\
    p_1(t, z) &= f(t)z + p_2(t), \\
    \rho_1(t, z) &= \rho_2(t).
\end{align*}
\] (111)

From the three first equations of (94), this ansatz allows to obtain:

\[
\begin{align*}
    u_2 &= \frac{u_3}{t + a}, \\
    v_2 &= \frac{v_3}{t + b}, \\
    \rho_2 &= \frac{\rho_3}{(t + a)(t + b)}, \\
    f(t) &= \frac{-\rho_3 w_2(t)}{(t + a)(t + b)}
\end{align*}
\] (112)

where \( u_3, v_3 \) and \( \rho_3 \) are arbitrary constants The last equation of (94) becomes:

\[
-3\rho_3[C_w w_2(t)(t + a)(t + b) + Rw_2(t)(a + b + 2t)]z + \Phi(t)
\] (113)
where \( \Phi(t) \) is the \( z \)-independent rest of the equation. As seen, this equation still contains the variable \( z \). It has not been reduced because vector field (110) is not a symmetry of the last equation of (94). Cancelling the coefficient of \( z \) and \( \Phi(t) \), we get:

\[
\begin{equation}
\begin{aligned}
w_2(t) &= w_3 \int h(t) \frac{\partial}{\partial z} t \\
\text{with} & \quad h(t) = (t + a)(t + b),
\end{aligned}
\end{equation}
\]

(114)

\( w_3 \) being a constant, and

\[
\begin{equation}
\begin{aligned}
p_2(t) &= h(t)^{-}\frac{\partial}{\partial z} \int \left[ \rho_3 w_3 w_2(t) + \frac{4R\mu}{3C_v} g(t) h(t)^{\frac{\partial}{\partial z} - 1} \right] t \\
\end{aligned}
\end{equation}
\]

(115)

with

\[
\begin{equation}
g(t) = (t + a)(t + b) + (a + b)^2.
\end{equation}
\]

(116)

It follows that:

\[
\begin{align}
u(t, x, y, z) &= \frac{x + u_3}{t + a}, & v(t, x, y, z) &= \frac{y + v_3}{t + b}, \\
w(t, x, y, z) &= w_3 \int h(t)^{-}\frac{\partial}{\partial z} t, & \rho(t, x, y, z) &= \frac{\rho_3}{h(t)}, \\
p(t, x, y, z) &= p_2(t) - \rho_3 w_3 h(t)^{1-\frac{\partial}{\partial z}}.
\end{align}
\]

(117)

Lastly, consider the symmetry generator \( X_1 - aX_2 - bX_3 - cX_4 \) where \( a, b \) and \( c \) are constants. A basis of invariants under this vector field is

\[
\begin{align}
\xi &= t + ax + by + cz, \\
u_1 &= u, & v_1 &= v, & w_1 &= w, & p_1 &= p, & \rho_1 &= \rho.
\end{align}
\]

The equations can then be reduced with the ansatz

\[
\begin{align}
u(t, x, y, z) &= u_1(\xi), & v(t, x, y, z) &= v_1(\xi), & w(t, x, y, z) &= w_1(\xi), \\
p(t, x, y, z) &= p_1(\xi), & \rho(t, x, y, z) &= \rho_1(\xi).
\end{align}
\]

Indeed, system (7) become

\[
\begin{align}
(\rho_1(1 + D))' &= 0 \\
\rho_1(1 + D)u_1' + ap_1' - \mu \left( \frac{a}{3} D + A v_1 \right)'' &= 0 \\
\rho_1(1 + D)v_1' + bp_1' - \mu \left( \frac{b}{3} D + A v_1 \right)'' &= 0 \\
\rho_1(1 + D)w_1' + cp_1' - \mu \left( \frac{c}{3} D + A w_1 \right)'' &= 0 \\
C_v p_1'(1 + D) + (C_v + R)p_1 D' + R\mu S_2 - \kappa A \left( \frac{p_1}{\rho_1} \right)'' &= 0
\end{align}
\]

(118)
where \( A = a^2 + b^2 + c^2 \), \( D \) is the function of \( \xi \) defined by
\[
D = au_1 + bv_1 + cw_1
\]
and
\[
S_2 = \frac{2}{3}(D')^2 + 2(au'_1)^2 + 2(bv'_1)^2 + 2(cw'_1)^2
\]
\[+ (bu'_1 + av'_1)^2 + (cu'_1 + bw'_1)^2 + (cv'_1 + bw'_1)^2.\]

After integration, the first four equations of (119) become
\[
\rho_1(1 + D) = \rho_2
\]
\[
\rho_2 u_1 + ap_1 - \mu \left( \frac{a}{3} D + Au_1 \right)' = \rho_2 u_2
\]
\[
\rho_2 v_1 + bp_1 - \mu \left( \frac{b}{3} D + Av_1 \right)' = \rho_2 v_2
\]
\[
\rho_2 w_1 + cp_1 - \mu \left( \frac{c}{3} D + Aw_1 \right)' = \rho_2 w_2
\]
for some constants \( \rho_2, u_2, v_2 \) and \( w_2 \). These equations can be used to express \( p_1, v_1 \) and \( w_1 \) as functions of \( u_1 \). Using the last equation of (119) and going back to the original variables, we get a traveling wave solution of (7):
7 Conclusion

The infinitesimal Lie symmetries of the compressible Navier-Stokes equations were computed. The corresponding Lie group action were presented. From the commutation table, the Levi decomposition of the Lie algebra were presented.

Self-similar solutions were computed from the symmetries of the equations and successive reductions. These solutions represents many types of model flows. One can cite for example flows representing bidimensional vortices, evolving as $r^{-1}$ or $r^{-2}$ from a well or toward a sink point. Bidimensional solutions depending exponentially on $y$ could also be obtained. One can also cite the three dimensional vortex-like and traveling wave solutions.

Note that, since a Lie-symmetry takes a solution into another one, many other solutions from those presented here can be obtained, by composing them by transformations (21)–(26).

In the present analysis, we did not intend to be exhaustive. Many other combinations of the presented infinitesimal generators may lead to completely new solutions.

A Components of the equations and the infinitesimal generators

The componentwise expression of equations (7) is

$$
\begin{align*}
\rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) &= 0 \\
\rho(u_t + uu_x + vv_y + wu_z) &= -p_x + \mu \left( \frac{4u_{xx} + v_{xy} + w_{xz}}{3} + u_{yy} + u_{zz} \right) \\
\rho(v_t + uv_x + vv_y + wv_z) &= -p_y + \mu \left( \frac{u_{xy} + 4v_{yy} + w_{yz}}{3} + v_{xx} + v_{zz} \right) \\
\rho(w_t + uw_x + vv_y + wu_z) &= -p_z + \mu \left( \frac{u_{xz} + w_{yz} + 4w_{zz}}{3} \right) + w_{xx} + w_{yy} \\
\frac{C_v}{R} (p_t + up_x + vp_y + wp_z) &= - \left( \frac{C_v}{R} + 1 \right) p(u_x + v_y + w_z) \\
&- \frac{2}{3} \mu (u_x + v_y + w_z)^2 \\
&+ \mu \left[ 2 \left( u_x^2 + v_y^2 + w_z^2 \right) + (u_y + v_x)^2 + (u_z + w_x)^2 + (v_z + w_y)^2 \right] \\
&+ \frac{\kappa}{R} \left[ \frac{p_{xx} + p_{yy} + p_{zz}}{\rho} - 2 p_x \rho_x + p_y \rho_y + p_z \rho_z \right] \\
&- p \frac{\rho_{xx} + \rho_{yy} + \rho_{zz}}{\rho^2} + 2 p \frac{\rho^2_x + \rho^2_y + \rho^2_z}{\rho^3} .
\end{align*}
$$

(123)

The first-order-derivative part $X^{(1)}$ of the prolonged vector field $pr^{(2)}X$
writes:

\[
X^{(1)} = \xi_t^u \frac{\partial}{\partial u} + \xi_t^v \frac{\partial}{\partial v} + \xi_t^w \frac{\partial}{\partial w} + \xi_t^\rho \frac{\partial}{\partial \rho} \\
+ \xi_v^u \frac{\partial}{\partial u} + \xi_v^v \frac{\partial}{\partial v} + \xi_v^w \frac{\partial}{\partial w} + \xi_v^\rho \frac{\partial}{\partial \rho} \\
+ \xi_w^u \frac{\partial}{\partial u} + \xi_w^v \frac{\partial}{\partial v} + \xi_w^w \frac{\partial}{\partial w} + \xi_w^\rho \frac{\partial}{\partial \rho} \\
+ \xi_\rho^u \frac{\partial}{\partial u} + \xi_\rho^v \frac{\partial}{\partial v} + \xi_\rho^w \frac{\partial}{\partial w} + \xi_\rho^\rho \frac{\partial}{\partial \rho}.
\]

(124)

The components of \(X^{(1)}\) can be expressed as functions of the \(\xi^r\) and \(\xi_q\) from relation (18). For example,

\[
\xi^r_p = D_t \xi^r_\rho - \rho_t D_t \xi^r - \rho_x D_t \xi^x - \rho_y D_t \xi^y - \rho_z D_t \xi^z
\]

\[
= \frac{\partial \xi^r_\rho}{\partial t} + u_t \frac{\partial \xi^r_\rho}{\partial u} + v_t \frac{\partial \xi^r_\rho}{\partial v} + w_t \frac{\partial \xi^r_\rho}{\partial w} + p_t \frac{\partial \xi^r_\rho}{\partial p} + \rho_t \frac{\partial \xi^r_\rho}{\partial \rho}
\]

\[
- \rho_x \left( \frac{\partial \xi^r_x}{\partial t} + u_t \frac{\partial \xi^r_x}{\partial u} + v_t \frac{\partial \xi^r_x}{\partial v} + w_t \frac{\partial \xi^r_x}{\partial w} + p_t \frac{\partial \xi^r_x}{\partial p} + \rho_t \frac{\partial \xi^r_x}{\partial \rho} \right)
\]

\[
- \rho_y \left( \frac{\partial \xi^r_y}{\partial t} + u_t \frac{\partial \xi^r_y}{\partial u} + v_t \frac{\partial \xi^r_y}{\partial v} + w_t \frac{\partial \xi^r_y}{\partial w} + p_t \frac{\partial \xi^r_y}{\partial p} + \rho_t \frac{\partial \xi^r_y}{\partial \rho} \right)
\]

\[
- \rho_z \left( \frac{\partial \xi^r_z}{\partial t} + u_t \frac{\partial \xi^r_z}{\partial u} + v_t \frac{\partial \xi^r_z}{\partial v} + w_t \frac{\partial \xi^r_z}{\partial w} + p_t \frac{\partial \xi^r_z}{\partial p} + \rho_t \frac{\partial \xi^r_z}{\partial \rho} \right)
\]

(125)

The second-order-derivative part \(X^{(2)}\) can be written as a sum over the dependent variables \(q = u, v, w, p, \rho:\)

\[
X^{(2)} = \sum_q \left( \xi_q^t \frac{\partial}{\partial q} + \xi_q^x \frac{\partial}{\partial q_x} + \xi_q^y \frac{\partial}{\partial q_y} + \xi_q^z \frac{\partial}{\partial q_z} + \xi_q^z \frac{\partial}{\partial q_{zz}} \\
+ \xi_q^t \frac{\partial}{\partial q_t} + \xi_q^x \frac{\partial}{\partial q_x} + \xi_q^y \frac{\partial}{\partial q_y} + \xi_q^z \frac{\partial}{\partial q_z} + \xi_q^z \frac{\partial}{\partial q_{zz}} \right),
\]

that is

\[
X^{(2)} = \xi_u^t \frac{\partial}{\partial u} + \xi_u^x \frac{\partial}{\partial u_x} + \xi_u^y \frac{\partial}{\partial u_y} + \xi_u^z \frac{\partial}{\partial u_z} \\
+ \xi_u^x \frac{\partial}{\partial u_x} + \xi_u^y \frac{\partial}{\partial u_y} + \xi_u^z \frac{\partial}{\partial u_z} \\
+ \xi_u^y \frac{\partial}{\partial u_y} + \xi_u^z \frac{\partial}{\partial u_z} + \xi_u^z \frac{\partial}{\partial u_{zz}}.
\]

(126)
Its coefficients can be computed from relation (19). An example of second order coefficient is:

\[\xi_{xx} = D_x \xi_x u - u_x D_x \xi_t - u_{xx} D_x \xi_y - u_{xx} D_x \xi_z.\]

After developing the total derivatives, one finds the expression of \(\xi_{xx}^x\) as a function the partial derivatives of the \(\xi_r\) and \(\xi_q\) up to second order.

To compute symmetries, \(pr^{(2)}X\) is applied on equation (123). For instance, the action of \(pr^{(2)}X\) on the first component of equation (123) gives:

\[\xi^t + \xi_u \rho_x + u \xi^x + \xi_v \rho_y + v \xi^y + \xi_w \rho_z + w \xi^z + \rho (u_x + v_y + w_z) + \rho (\xi^x + \xi^y + \xi^z) = 0.\]

It is a polynomial equation on the jet variables \(t, x, u_{(2)}, p_{(2)}, \rho_{(2)}\), the coefficients being functions of the \(\xi_r\) and \(\xi_q\) and their derivatives. The same procedure is applied to the four last components of equation (123) (and to their differential consequences, see [14]). The application of the infinitesimal symmetry condition (15) leads then to a system of polynomial equations on \(u, v, w, p, \rho, u_t, u_x, \cdots\) (up to second order derivatives). Equating that the coefficients of these polynomials are zero, and with the help of a symbolic software [3], one gets a system of partial differential equations on the \(\xi_r\) and \(\xi_q\), the solution of which is

\[
\begin{align*}
\xi^t &= c_1 t + 2c_{11}t, & \xi_p &= -2c_{11}p, & \xi_\rho &= -2c_{12}p \\
\xi^x &= c_2 x + c_5 t + c_9 z - c_{10} y + c_{11} x + c_{12} x \\
\xi^y &= c_3 y + c_6 t - c_8 z + c_{10} x + c_{11} y + c_{12} y \\
\xi^z &= c_4 z + c_7 t + c_8 y - c_9 x + c_{11} z + c_{12} z \\
\xi_u &= c_5 + c_9 w - c_{10} v - c_{11} u + c_{12} u \\
\xi_v &= c_6 - c_8 w + c_{10} u - c_{11} v + c_{12} v, \\
\xi_w &= c_7 + c_8 v - c_9 u - c_{11} w + c_{12} w
\end{align*}
\]

In these expressions, the \(c_i\) are arbitrary scalars. Therefore,

\[X = \sum_{i=1}^{12} c_i X_i\]

where the \(X_i\) are given in section [3].

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