Einstein like \((\varepsilon)\)-para Sasakian manifolds

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Abstract. Einstein like \((\varepsilon)\)-para Sasakian manifolds are introduced. For an \((\varepsilon)\)-para Sasakian manifold to be Einstein like, a necessary and sufficient condition in terms of its curvature tensor is obtained. The scalar curvature of an Einstein like \((\varepsilon)\)-para Sasakian manifold is obtained and it is shown that the scalar curvature in this case must satisfy certain differential equation. A necessary and sufficient condition for an \((\varepsilon)\)-almost paracontact metric hypersurface of an indefinite locally Riemannian product manifold to be \((\varepsilon)\)-para Sasakian is obtained and it is proved that the \((\varepsilon)\)-para Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature is always Einstein like.

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1 Introduction

In 1976, Sato [9] introduced an almost paracontact structure on a differentiable manifold, which is an analogue of the almost contact structure [8, 4] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, Takahashi [11] studied almost contact manifolds equipped with associated pseudo-Riemannian metrics. The indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as \((\varepsilon)\)-almost contact metric manifolds and \((\varepsilon)\)-Sasakian manifolds, respectively [2, 5]. Also, in 1989, Matsumoto [6] replaced the structure vector field \(\xi\) by \(-\xi\) in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field \(\xi\) is always timelike. Because of these circumstances, the authors in [12] introduced \((\varepsilon)\)-almost paracontact structures by associating a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, where the structure vector field \(\xi\) is spacelike or timelike according as \(\varepsilon = 1\) or \(\varepsilon = -1\).

In [10], Sharma introduced and studied Einstein like para Sasakian manifolds. Motivated by his study, in this paper we introduce and study Einstein like \((\varepsilon)\)-almost paracontact metric manifolds. The paper is organized as follows. Section 2 contains some preliminaries about \((\varepsilon)\)-para Sasakian manifolds. In section 3, we give the definition of an Einstein like \((\varepsilon)\)-almost paracontact metric manifold and give some basic properties. For an \((\varepsilon)\)-para Sasakian manifold to be Einstein like, we also find a necessary and sufficient condition in terms of its curvature tensor. We also find the scalar curvature of an Einstein like \((\varepsilon)\)-para Sasakian manifold and show that the scalar curvature in this case must satisfy certain differential equation. In section 4, we find a necessary and sufficient condition for an \((\varepsilon)\)-almost paracontact metric hypersurface of an indefinite locally Riemannian product manifold to be \((\varepsilon)\)-para Sasakian. Finally we prove that an \((\varepsilon)\)-para Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature is always Einstein like.

2 Preliminaries

Let \(M\) be an \(n\)-dimensional almost paracontact manifold [9] equipped with an almost paracontact structure \((\varphi, \xi, \eta)\) consisting of a tensor field \(\varphi\) of type \((1,1)\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying

\[
\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0.
\]

By a semi-Riemannian metric [7] on a manifold \(M\), we understand a non-degenerate symmetric tensor field \(g\) of type \((0,2)\). In particular, if its index is 1, it becomes a Lorentzian metric [1]. Throughout the paper we assume that \(X, Y, Z, U, V, W \in \mathfrak{X}(M)\), where \(\mathfrak{X}(M)\) is the Lie algebra of vector fields in
M, unless specifically stated otherwise. Let $g$ be a semi-Riemannian metric with index($g$) = $\nu$ in an $n$-dimensional almost paracontact manifold $M$ such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y),$$

(2.1)

where $\varepsilon = \pm 1$. Then $M$ is called an $(\varepsilon)$-almost paracontact metric manifold equipped with an $(\varepsilon)$-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ [12]. In particular, if index($g$) = 1, then an $(\varepsilon)$-almost paracontact metric manifold is a Lorentzian almost paracontact manifold. In particular, if the metric $g$ is positive definite, then an $(\varepsilon)$-almost paracontact metric manifold is the usual almost paracontact metric manifold [9]. The equation (2.1) is equivalent to

$$g(X, \varphi Y) = g(\varphi X, Y)$$

along with

$$g(X, \xi) = \varepsilon\eta(X).$$

(2.2)

Note that $g(\xi, \xi) = \varepsilon$, that is, the structure vector field $\xi$ is never lightlike. An $(\varepsilon)$-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ is called an $(\varepsilon)$-para Sasakian structure if

$$(\nabla_X \varphi)Y = -g(\varphi X, \varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X,$$

(2.3)

where $\nabla$ is the Levi-Civita connection with respect to $g$. A manifold endowed with an $(\varepsilon)$-para Sasakian structure is called an $(\varepsilon)$-para Sasakian manifold. In an $(\varepsilon)$-para Sasakian manifold we have

$$\nabla \xi = \varepsilon \varphi,$$

(2.4)

$$\Phi(X, Y) \equiv g(\varphi X, Y) = \varepsilon g(\nabla_X \xi, Y) = (\nabla_X \eta)Y.$$  

(2.5)

For more details we refer to [12].

3 Einstein like $(\varepsilon)$-para Sasakian manifolds

We begin with the following definition analogous to Einstein like para Sasakian manifolds [10].

**Definition 3.1** An $(\varepsilon)$-almost paracontact metric manifold is said to be Einstein like if its Ricci tensor $S$ satisfies

$$S(X, Y) = ag(X, Y) + bg(\varphi X, Y) + c\eta(X)\eta(Y)$$

(3.1)

for some real constants $a$, $b$ and $c$.

**Proposition 3.2** In an Einstein like $(\varepsilon)$-almost paracontact metric manifold, we have

$$S(\varphi X, Y) = ag(\varphi X, Y) + bg(\varphi X, \varphi Y),$$

(3.2)

$$S(X, \xi) = \varepsilon a\eta(X) + c\eta(X),$$

(3.3)

Moreover, if the manifold is $(\varepsilon)$-para Sasakian, then

$$\varepsilon a + c = 1 - n,$$

(3.4)

$$r = na + b\text{trace}(\varphi) + \varepsilon c,$$

(3.5)

where $r$ is the scalar curvature.

**Proof.** The equations (3.2) and (3.3) are obvious. In an $(\varepsilon)$-para Sasakian manifold, it follows that $S(X, \xi) = -(n - 1)\eta(X)$, which in view of (3.3) implies (3.4). Now, let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame. Then from (3.1), we have

$$r = \sum_{i=1}^{n} \{\varepsilon_i a g(e_i, e_i) + \varepsilon_i b g(\varphi e_i, e_i) + \varepsilon_i c g(\xi, e_i) g(\xi, e_i)\},$$

which gives (3.5). ■

**Theorem 3.3** In an Einstein like $(\varepsilon)$-para Sasakian manifold, the scalar curvature $r$ satisfies the following two differential equations

$$b\xi r - 2c\varepsilon = 2\varepsilon (1 - n)\left(b^2 - c^2 - cn\right).$$

(3.6)
\textbf{Proof.} From (3.1), it follows that the Ricci operator $Q$ satisfies
\begin{equation}
QX = aX + b\varphi X + \varepsilon c \eta (X) \xi. \tag{3.7}
\end{equation}
Differentiating (3.7), we find
\begin{equation}
(\nabla_Y Q) X = b (\nabla_Y \varphi) X + \varepsilon c (\nabla_Y \eta) (X) \xi + \varepsilon c \eta (X) \nabla_Y \xi.
\end{equation}
Using (2.3), (2.5) and (2.4) in the above equation we get
\begin{equation}
(\nabla_Y Q) X = -\varepsilon b \eta (X) Y + \varepsilon c \eta (X) \varphi Y
- (bg (X,Y) - 2\varepsilon b \eta (X) \eta (Y) - \varepsilon cg (\varphi X, Y)) \xi. \tag{3.8}
\end{equation}
Now, using (3.8) we have
\begin{equation}
(\text{div} Q) X = \{\varepsilon (1 - n) b + c \text{trace}(\varphi)\} \eta (X). \tag{3.9}
\end{equation}
From (3.5) and (3.4) we get
\begin{equation}
r = b \text{trace}(\varphi) - \varepsilon (n - 1)(c + n) \tag{3.10}
\end{equation}
Using $Xr = 2(\text{div} Q) X$ and (3.10) in (3.9) we get (3.6). $\blacksquare$

\textbf{Theorem 3.4} 
In an Einstein like $(\varepsilon)$-para Sasakian manifold, if $\text{trace}(\varphi)$ is constant then
\begin{equation}
\text{trace}(\varphi) = \frac{\varepsilon (n - 1) b}{c}. \tag{3.11}
\end{equation}
\textbf{Proof.} Using $Xr = 2(\text{div} Q) X$ in (3.9), we get
\begin{equation}
dr = 2(\varepsilon (1 - n) b + c \text{trace}(\varphi)) \eta. \tag{3.12}
\end{equation}
Since $\text{trace}(\varphi)$ is constant, from (3.5), it follows that $r$ is constant. Hence (3.12) gives (3.11). $\blacksquare$

From now on in this section the $\text{trace}(\varphi)$ will be assumed to be constant.

\textbf{Theorem 3.5} 
An $(\varepsilon)$-para Sasakian manifold with constant $\text{trace}(\varphi)$ is Einstein like if and only if the $(0, 2)$-tensor field $C^1_\varepsilon (\varphi R)$ is a linear combination of $g$, $\Phi$ and $\eta \otimes \eta$ formed with constant coefficients.

\textbf{Proof.} In an $(\varepsilon)$-para Sasakian manifold the curvature tensor $R$ satisfies [12]
\begin{equation}
R (X,Y) \varphi Z = \varphi R (X,Y) Z + \varepsilon \Phi (Y,Z) X - \varepsilon \Phi (X,Z) Y - 2\varepsilon \Phi (Y,Z) \eta (X) \xi + 2\varepsilon \Phi (X,Z) \eta (Y) \xi
- \varepsilon g (Y,Z) \varphi X + \varepsilon g (X,Z) \varphi Y + 2\eta (Y) \eta (Z) \varphi X - 2\eta (X) \eta (Z) \varphi Y.
\end{equation}

Then we have
\begin{equation}
S (Y, \varphi Z) = C^1_\varepsilon (\varphi R) (Y,Z) + \varepsilon (n - 2) \Phi (Y,Z) + (2\eta (Y) \eta (Z) - \varepsilon g (Y,Z)) \text{trace}(\varphi). \tag{3.13}
\end{equation}
Since in an $(\varepsilon)$-para Sasakian manifold, it follows that [12] $S (X, \varphi Y) = S (\varphi X, Y)$, and also it can be verified that $C^1_\varepsilon (\varphi R) (Y,Z) = C^1_\varepsilon (\varphi R) (Z,Y)$; therefore the equation (3.13) is consistent. Now, if the manifold is Einstein like then from (3.2), (3.13) and (3.11), it follows that
\begin{equation}
C^1_\varepsilon (\varphi R) = \frac{b}{c} (c + n - 1) g + (a - \varepsilon (n - 2)) \Phi - \frac{\varepsilon}{c} (c + 2b (n - 1)) \eta \otimes \eta, \tag{3.14}
\end{equation}
which shows that $C^1_\varepsilon (\varphi R)$ is a linear combination of $g$, $\Phi$ and $\eta \otimes \eta$ formed with constant coefficients. The converse is easy to follow. $\blacksquare$

\textbf{Corollary 3.6} 
In an Einstein like $(\varepsilon)$-para Sasakian manifold with constant $\text{trace}(\varphi)$, the $(0, 2)$-tensor field $C^1_\varepsilon (\varphi R)$ is parallel along the vector field $\xi$.

\textbf{Proof.} Since in an Einstein like $(\varepsilon)$-para Sasakian manifold $\nabla_\xi \Phi = 0$ and $\nabla_\xi \eta = 0$, therefore from (3.14) we conclude that $C^1_\varepsilon (\varphi R)$ is parallel along the vector field $\xi$. $\blacksquare$

\textbf{Theorem 3.7} 
In an Einstein like $(\varepsilon)$-para Sasakian manifold, we have
\begin{equation}
\mathcal{L}_\xi S = 2a \varepsilon \Phi + 2b \varepsilon (g - \varepsilon \eta \otimes \eta). \tag{3.15}
\end{equation}
Proof. In an \((\varepsilon)\)-para Sasakian manifold, we obtain
\[
\mathcal{L}_\xi \eta = \nabla_\xi \eta = 0, \quad \mathcal{L}_\xi \Phi = 2\varepsilon (g - \eta \otimes \eta), \quad \mathcal{L}_\xi g = 2\varepsilon \Phi.
\] (3.16)
Now, taking Lie derivative of \(S\) in the direction of \(\xi\) in (3.1) and using (3.16), we obtain (3.15). ■

Theorem 3.8 In an Einstein like \((\varepsilon)\)-para Sasakian manifold with constant trace\((\varphi)\), we have
\[
\mathcal{L}_\xi (C^1_1(\varphi R)) = \frac{2\varepsilon b}{c} (c + n - 1) \Phi + 2\varepsilon (a - \varepsilon (n - 2)) (g - \eta \otimes \eta).
\] (3.17)
Proof. Taking Lie derivative of \(C^1_1(\varphi R)\) in the direction of \(\xi\) in (3.14) and using (3.16), we get (3.17). ■

4 \((\varepsilon)\)-para Sasakian hypersurfaces

Let \(\tilde{M}\) be a real \((n + 1)\)-dimensional manifold. Suppose \(\tilde{M}\) is endowed with an almost product structure \(J\) and a semi-Riemannian metric \(\tilde{g}\) satisfying
\[
\tilde{g}(JX, JY) = \tilde{g}(X, Y)
\] (4.1)
for all vector fields \(X, Y\) in \(\tilde{M}\). Then we say that \(\tilde{M}\) is an indefinite almost product Riemannian manifold. Moreover, if on \(\tilde{M}\) we have
\[
(\tilde{\nabla}_X J)Y = 0
\] (4.2)
for all \(X, Y \in \mathfrak{X}(\tilde{M})\), where \(\tilde{\nabla}\) is the Levi-Civita connection with respect to \(\tilde{g}\), we say that \(\tilde{M}\) is an indefinite locally Riemannian product manifold.

Now, let \(M\) be an orientable non-degenerate hypersurface of \(\tilde{M}\). Suppose now that \(N\) is the normal unit vector field of \(M\) such that \(\tilde{g}(N, N) = \varepsilon\) and
\[
JN = \xi \in \mathfrak{X}(M).
\] (4.3)
Let
\[
JX = \varphi X + \eta(X) N.
\] (4.4)

Proposition 4.1 The set \((\varphi, \xi, \eta, g)\) is an \((\varepsilon)\)-almost paracontact metric structure, where \(g\) is the induced metric on \(M\).

Proof. We have
\[
X = J^2 X = \varphi^2 X + \eta(\varphi X) N + \eta(X) \xi,
\]
where (4.4) and (4.3) are used. Equating tangential and normal parts we get \(\varphi^2 = I - \eta \otimes \xi\) and \(\eta \circ \varphi = 0\), respectively. We also have
\[
N = J^2 N = J\xi = \varphi \xi + \eta(\xi) N,
\]
where (4.4) and (4.3) are used. Equating tangential and normal parts we get \(\varphi \xi = 0\) and \(\eta(\xi) = 1\), respectively. Finally, we have \(g(X, Y) = \tilde{g}(JX, JY)\), which in view of (4.4) gives (2.1). ■

The Gauss and Weingarten formulas are given respectively by
\[
\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon g(AX, Y)N, \\
\tilde{\nabla}_X N = -AX,
\]
where \(\nabla\) is the Levi-Civita connection with respect to the semi-Riemannian metric \(g\) induced by \(\tilde{g}\) on \(M\) and \(A\) is the shape operator of \(M\).

Proposition 4.2 The \((\varepsilon)\)-almost paracontact metric structure on \(M\) satisfies
\[
(\nabla_X \varphi)Y = \eta(Y)AX + \varepsilon g(AX, Y)\xi, \\
(\nabla_X \eta)Y = -\varepsilon g(AX, \varphi Y), \\
\nabla_X \xi = -\varphi AX,
\]
(4.7)
(4.8)
(4.9)
Proof. Using (4.4), (4.3), (4.5) and (4.6) in \((\nabla_X J) Y = 0\), we get
\[
0 = (\nabla_X \varphi) Y - \eta(Y) AX - h(X, Y) \xi + ((\nabla_X \eta) Y) N + h(X, \varphi Y) N
\]
Equating tangential and normal parts we get (4.7) and (4.8), respectively. Eq. (4.8) implies (4.9). ■

Now we obtain the following theorem of characterization for \((\varepsilon)\)-para Sasakian hypersurfaces.

**Theorem 4.3** Let \(M\) be an orientable hypersurface of an indefinite locally Riemannian product manifold. Then \(M\) is an \((\varepsilon)\)-para Sasakian manifold if and only if the shape operator is given by
\[
A = -\varepsilon I + \varepsilon \eta \otimes \xi.
\]

**Proof.** Let \(M\) be an \((\varepsilon)\)-para Sasakian manifold. By using (2.4) and (4.9) we get
\[
AX = -\varepsilon X + \varepsilon \eta(X) \xi + \eta(AX) \xi
\]
In particular, we have \(A\xi = \eta(A\xi) \xi\). Thus, we have
\[
\eta(AX) = \varepsilon g(\xi, AX) = \varepsilon g(A\xi, X) = \varepsilon g(\eta(A\xi) \xi, X) = \eta(A\xi) \eta(X).
\]
Using this in (4.11) we get
\[
A = -\varepsilon I + (\varepsilon + \eta(A\xi)) \eta \otimes \xi.
\]
Now, we use (4.13) in (4.7) to find
\[
(\nabla_X \varphi) Y = -\varepsilon \eta(Y) X + 2 \varepsilon \eta(X) \eta(Y) \xi + 2 \eta(A\xi) \eta(X) \eta(Y) \xi - g(X, Y) \xi.
\]
From (4.14) and (2.3) we get \(\eta(A\xi) = 0\), which when used in (4.13) yields (4.10).

Conversely, using (4.10) in (4.7) we see that \(M\) is \((\varepsilon)\)-para Sasakian manifold. ■

Now, assume that the indefinite almost product Riemannian manifold \(\tilde{M}\) is of almost constant curvature [13] so that its curvature tensor \(\tilde{R}\) is given by
\[
\tilde{R}(X, Y, Z, W) = k \{ \tilde{g}(Y, Z) \tilde{g}(X, W) - \tilde{g}(X, Z) \tilde{g}(Y, W) + \tilde{g}(JY, Z) \tilde{g}(JX, W) - \tilde{g}(JX, Z) \tilde{g}(JY, W) \}
\]
for all vector fields \(X, Y, Z, W\) on \(\tilde{M}\). If \(M\) is an \((\varepsilon)\)-para Sasakian hypersurface, then in view of (4.10) and (4.15) the Gauss equation becomes
\[
R(X, Y, Z, W) = (k - 1) \{ g(Y, Z) g(X, W) - g(X, Z) g(Y, W) \}
+ k \{ g(\varphi Y, Z) g(\varphi X, W) - g(\varphi X, Z) g(\varphi Y, W) \}
- \varepsilon g(Y, Z) \eta(X) \eta(W) - \varepsilon g(X, W) \eta(Y) \eta(Z)
+ \varepsilon g(X, Z) \eta(Y) \eta(W) + \varepsilon g(Y, W) \eta(X) \eta(Z).
\]
After calculating \(R(X, Y) \xi\) from (4.16) and comparing the resulting expression with [12]
\[
R(X, Y) \xi = \eta(X) Y - \eta(Y) X,
\]
we find that \(k = 2 - \varepsilon\). With this value of \(k\), from (4.16), we obtain
\[
S = ((2 - \varepsilon)(n - 2) - n) g + (2 - \varepsilon) \text{trace}(\varphi) \Phi + \varepsilon(4 - \varepsilon - n) \eta \otimes \eta.
\]
Thus we have proved the following:

**Theorem 4.4** An \((\varepsilon)\)-para Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature \((2 - \varepsilon)\) is Einstein like.

**Remark 4.5** A hypersurface is called a quasi-umbilical hypersurface [3] if
\[
h(X, Y) = \alpha g(X, Y) + \beta u(X) u(Y),
\]
where \(\alpha\) and \(\beta\) are some smooth functions and \(u\) is a 1-form. From (4.10) we see that the \((\varepsilon)\)-para Sasakian hypersurface is quasi-umbilical.
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