Inducing critical phenomena in spin chains through sparse alternating fields

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We analyze the phase diagram of the exact ground state (GS) of spin-$s$ chains with ferromagnetic $XXZ$ couplings under $n$-alternating field configurations, i.e., sparse alternating fields having nodes at $n-1$ contiguous sites. It is shown that such systems can exhibit a non-trivial magnetic behavior, which can differ significantly from that of the standard alternating case and enables mechanisms for controlling their magnetic and entanglement properties. The fully aligned phase, whose border in field space can be determined analytically, becomes reachable only above a threshold value of the coupling anisotropy $J_z/J$, which depends on $n$ but is independent of the system size. Below this value the maximum attainable magnetization becomes much smaller. We then show that the GS can exhibit significant magnetization plateaus, persistent for large systems, at which the scaled magnetization $m$ obeys the quantization rule $2ns(1 - m) = \text{integer}$, remarkably coinciding with the Oshikawa, Yamanaka and Affleck (OYA) criterion. We also identify the emergence of field induced spin polymerization, which explains the presence of such plateaus. Entanglement and field induced frustration effects are also analyzed.

I. INTRODUCTION

One of the distinct hallmarks of cooperative behavior in interacting many-body quantum systems are the critical properties and phase transitions that arise when some control parameter is varied, for which entanglement theory has provided a deep understanding. In this scenario, the emergence of notable phenomena such as site frustration and magnetization plateaus is typically associated with antiferromagnetic systems with competing interactions and non-trivial geometries. However, little is known of the critical properties that could be induced even in simple systems through general non-uniform magnetic fields or couplings. Such possibility becomes increasingly feasible due to the recent remarkable advances in quantum control technologies, which have enabled the simulation of finite spin systems with tunable couplings and fields. In particular, the paradigmatic $XXZ$ model has been realized in different systems, which include cold atoms in optical lattices, superconducting Josephson junctions, photon coupled micro-cavities, trapped ions, quantum dots, etc. It has also been employed for implementing quantum information protocols.

Here we will show, employing the $XXZ$ model, that the application of sparse periodic alternating fields in a ferromagnetic chain of arbitrary spin results in novel ground state phase diagrams, which display non-trivial magnetization plateaus and entanglement properties. In the first place, the boundary in field space of the fully aligned phase, which determines the onset of GS entanglement, can be determined analytically and implies a threshold value of the coupling anisotropy, below which the maximum attainable magnetization becomes much smaller. It is then shown that such sparse fields can induce other non-trivial magnetization plateaus, persistent for large sizes, as verified through DMRG calculations. These plateaus satisfy a criterion normally associated with more complex antiferromagnetic systems, which can here be explained through field induced polymerization effects. We also analyze other aspects like field induced frustration, single-spin magnetization and pairwise entanglement, whose results support the polymerization based picture.

The model and the $n$-alternating field configuration are described in section II with the boundary of the fully aligned phase and the conditions under which it can be reached discussed in II A. GS magnetization diagrams are then discussed in II B while pairwise entanglement in II C. The appendices contain the derivation of analytic expressions and the exact solution of the limit case of a XX chain under the present field configurations. Conclusion are finally drawn in III.

II. SPARSE ALTERNATING FIELD CONFIGURATIONS

We consider a cyclic chain of $N$ spins $s$ interacting through first-neighbor $XXZ$ couplings in a non-uniform magnetic field along the $z$ axis. The Hamiltonian reads

$$H = -\sum_{j=1}^{N} [\hbar \sigma_j^z S_j^z + J(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + J_z S_j^z S_{j+1}^z],$$

(1)

where $\hbar$, $S_j^\mu$ are the field and spin components at site $j$ (with $N + 1 \equiv 1$) and $J$, $J_z$ the coupling strengths. As $[H, S_z^\dagger] = 0$, with $S_z = \sum_j S_j^z$ the total spin along the $z$ axis, its eigenstates can be characterized by the total magnetization $M = S_z$. We can set $J > 0$, as the spectrum and entanglement properties of $H$ are the same for $\pm J$. They are also identical for $(\{\hbar^2\}, M)$ and $(\{-\hbar^2\}, -M)$.

We will here examine the $n$-alternating field configura-
The standard alternating (A) case \((h_1, h_2, h_1, h_2, \ldots)\) corresponds to \(n = 1\), while \(n = 2\) defines the “next-alternating” (NA) case \((h_1, 0, h_2, 0, \ldots)\) and \(n = 3\) the “next-next-alternating” (NNA) case \((h_1, 0, 0, h_2, 0, 0, \ldots)\). We set \(N = 2nK\), with \(K\) the number of cells with \(2n\) spins. We first present general results valid \(\forall n\) and then focus on these three cases.

### A. Border of the aligned phase

A first basic question which arises for \(n \geq 2\) is if such sparse fields are sufficient to induce a completely aligned (and hence separable) GS with maximum magnetization \(|M| = Ns\). For \(n = 1\) such phase will obviously always arise for sufficiently strong fields \(h_1, h_2\) of the same sign, but for \(n \geq 2\) it will not be attainable without the aid of a finite value of \(J_z\), i.e., \(J_z > J_z^c(n) \geq 0\), as shown below.

The borders of the aligned phase can be obtained by determining the fields at which the GS undergoes the magnetization transition \(|M| = Ns \to Ns - 1\). As shown in Appendix A, they correspond to hyperbola branches \(\forall n \geq 1\): The GS has \(M = Ns\) if

\[
(h_1 + \beta_n)(h_2 + \beta_n) > \alpha_n^2, \quad (3)
\]

provided \(h_1 + h_2 > 0\) and \(h_1(2) > -\beta_n\), and \(M = -Ns\) if

\[
(h_1 - \beta_n)(h_2 - \beta_n) > \alpha_n^2, \quad (4)
\]

provided \(h_1 + h_2 < 0\) and \(h_1(2) < -\beta_n\), where

\[
\alpha_n = \frac{2h_s}{(j_z + h_s)^n - (j_z - h_s)^n}, \quad j_z > j_z^c(n), \quad (5)
\]

\[
\beta_n = \frac{h_s}{(j_z + h_s)^n + (j_z - h_s)^n} = \sqrt{\alpha_n^2 + h_s^2}, \quad (6)
\]

(see Appendix A), with \(j = 2sJ, j_2 = 2sJ_z\) and

\[
h_s = \sqrt{j_z^2 - j^2}, \quad (7)
\]

the factorizing field \([34]\). Thus, \(\alpha_n, \beta_n\) depend on the scaled coupling strengths \(j, j_2\) and \((\alpha_n/j, \beta_n/j)\) just on the coupling anisotropy \(j_z/j\) but are strictly independent of the total number of cells \(K\). Eqs. \((5)-(6)\) hold (and are real) for both \(h_s\) real \((j_z > j)\) or imaginary \((j_z < j)\), but in the latter the denominator in \((3)-(4)\) vanishes for

\[
j_z \to j_z^c(n) = j \cos(\pi/n), \quad n \geq 2,
\]

implying divergence of \(\alpha_n\) and \(\beta_n\) for \(j_z \to j_z^c(\frac{n}{2})\). The fully aligned phase becomes then unreachable for \(j_z \leq j_z^c(n)\). Here \(-j \cos(\pi/n)\) represents the lowest energy of the \(n - 1\) intermediate spins at \(j_z = 0\) with magnetization \(M_{n-1} = (n-1)s - 1\) (see Appendices A-B).

In the standard alternating case \(n = 1\), Eqs. \((3)-(4)\) lead to

\[
\alpha_1 = j, \quad \beta_1 = j_z, \quad (9)
\]

being verified that the aligned phase is reachable \(\forall j, j_z\) for sufficiently strong \(h_1, h_2\). However, in the NA case \(n = 2\), they imply

\[
\alpha_2 = j^2/2j_z, \quad \beta_2 = j_z - j^2/2j_z, \quad j_z > 0, \quad (10)
\]

which diverge for \(j_z \to j_z^c(2) = 0\): Increasingly stronger fields are here required to reach the aligned phase as \(j_z\) decreases, diverging in the XX limit \(j_z = 0\). For \(j_z \leq 0\) it becomes unreachable (see Appendix C). And in the NNA case \(n = 3\),

\[
\alpha_3 = j^3/4j_z^2 - j^2, \quad \beta_3 = j_z 4j_z^2 - 3j^2/4j_z^2 - j^2, \quad j_z > j/2, \quad (11)
\]

which diverge already for \(j_z \to j_z^c(3) = j/2\). Here the aligned GS cannot be reached for \(j_z \leq j/2\) (Figs. \([23]\)).

Eqs. \((3)-(6)\) also entail that if \(j_z^c(n) < j_z < j\), full alignment requires application of non-zero fields. For \(h_1 = h_2 = h\), they imply

\[
|h| > h_z^c(n) = \alpha_n - \beta_n, \quad (12)
\]
with $h^2(n) > 0$ for $j^c_1(n) < j_2 < j$. And if $h_2 = 0$, a single field $|h_1| > -h^2_2 / \beta_n = h^2_1(2n)$ is sufficient provided $\beta_n > 0$, which is equivalent to $j_2 > j^c_2(2n)$ ($h_2 = 0$ in the $n$-alternating configuration is equivalent to $h_2 = h_1$ in the $2n$-alternating case).

In contrast, for $j_2 > j$ the GS is fully aligned already at zero field $\forall n$, and lower magnetizations $|M|$ arise only for fields of opposite sign beyond the factorization point determined by the field (7) (8): Eqs. (8) intersect now at two critical points $h_1 = -h_2 = \pm h_s \forall n$, where all magnetization plateaus coalesce and the GS becomes $2N s + 1$-fold degenerate, with the GS subspace spanned by a continuous set of completely separable symmetry-breaking product states. These points arise $\forall n$ (Fig. 2 right, Fig. 3 bottom) and are independent of $K$ and $n$.

For $j_2 > j$ and $n > 1$, $\alpha_n$ becomes rapidly small for large $n$ ($\alpha_n \approx 2h_s \beta_n (j_2 + h_s)^n$) or large $j_2/j$ ($\alpha_n \approx j_2 \beta_n$), implying $\beta_n \approx h_s$ and hence non-alignment ($|M| < N s$) just for $|h_1| \geq h_s$ for $i = 1, 2$ (and $h_1 h_2 < 0$), as seen in the bottom panels of Fig. 3. We also note from (9) and (10) that the antiparallel and parallel critical fields $A$–$12$ fully determine $\alpha_n, \beta_n$ and hence the whole border of the aligned phase:

$$\alpha_n = \frac{1}{2} h^2_1(n) - \frac{h^2_2}{2h^2_1(n)}.$$  \hspace{1cm} (13)

The borders $3$–$12$ also indicate the onset of GS entanglement, and correspond to an entanglement transition. As shown in Appendices A–B, the $|M| = N s - 1$ GS is a linear combination of $W$-like states and pair entanglement reaches there full range, with the reduced state of two spins depending just on their positions $i, j$ within the cell but not on their distance. The associated concurrences are just (see Appendix B)

$$C_{ij} = 2|w_i w_j| / K,$$  \hspace{1cm} (14)

where $\sum_{2n}^{2n} |w_i|^2 = 1$, and saturate the monogamy relations.

**B. Magnetization**

A second fundamental question which arises is if magnetization plateaus with $|M| < N s$ of significant width do also emerge. For large systems the GS will indeed possess such plateaus (Fig. 4), at which the scaled magnetization $m = M/(N s)$ obeys the quantization rule

$$2ns(1 - m) = q,$$  \hspace{1cm} (15)

with $q$ integer. This result can be readily understood by considering the situation where one of the fields ($h_1$) is sufficiently strong so that the spin chain can be viewed approximately as $K$ polymerized subsystems consisting of $2n - 1$ spins-$s$ with a field $h_2$ at the central site (Fig. 1), separated by fully aligned spins. When $h_2$ is varied the polymer GS magnetizations $M_{2n-1}$ will be $(2n - 1)s - q$ with $q$ integer, starting from $q = 0$ when $j_2 > j^c_2(n)$. Therefore, the total GS magnetization will be $K \{s + [(2n - 1)s - q] \}$, entailing then (16) and meaning that the plateaus in $m$ reflect essentially the polymer magnetizations. Eq. (15) coincides, remarkably, with the OYA criterion (21), which has been successfully used in antiferromagnetic chains in uniform fields. Intermediate magnetizations arise then in the transition regions between these plateaus and imply no definite magnetization at the single cell level.

In Fig. 3 we show representative results for the GS magnetization in a small spin $1/2$ chain. In the standard alternating case $n = 1$ (left panels), the GS reaches all magnetizations for any anisotropy $j_2/j$, with the fully aligned $|M| = N s/2$ sectors separated from the $M = 0$ plateau by a narrow band containing all intermediate magnetizations. In contrast, in a NA $n = 2$ configuration (center), it is first verified that for $j_2 = j^c_2(2) = 0$, the GS cannot be fully aligned. Moreover, it has strictly
This can be understood again from the strong field limit $M = 0$ for all fields, as can be rigorously shown through its Jordan-Wigner fermionization (see Appendix C). And for $j_z > 0$ this configuration exhibits a noticeable behavior, showing wide $0 \leq |M| \leq N/4$ sectors in addition to the aligned phases, with the $|M| = N/4$ plateau persisting for large $N$ (see below). Finally, in the NNA $n = 3$ case (right) it is again verified that if $j_z \leq j_z(3) = j/2$, the GS cannot be fully aligned (top panel), reaching instead a maximum magnetization $|M| = 2sK = N/6$ for $j_z = 0$ (and also $0 < j_z < j/2$ if $s = 1/2$): For strong parallel fields, spins with field become aligned while those without form essentially entangled dimers with zero magnetization, entailing $|M| = 2sK$. And when $j_z > j/2$, the magnetization diagram becomes similar to that of the $n = 1$ case, although with a much wider transition sector between the $M = 0$ and $|M| = N/2$ plateaus.

Previous results imply that the threshold $j_z(n)$ of the aligned phase is actually a critical point below which a whole interval of magnetizations cease to be reachable. This can be understood again from the strong field limit $h_1, h_2 \to \infty$, where spins with field are fully aligned while those without form essentially isolated chains of $n - 1$ spins, with effective fields $sJ_z$ at the endpoints: For $n = 2$ and $j_z \to 0^+$, all magnetizations $M \geq 0$ (and not just $N/2$ and $N/2 - 1$) of the whole chain become degenerate at strong fields, since the $M_1 = \pm 1/2$ states of each of the $2K$ single spins without field become degenerate, remaining just $M = 0$ for $j_z \leq 0$. Similarly, for $n = 3$ and $j_z \to j/2$, all chain magnetizations $M \geq 2sK = N/6$ become degenerate at strong fields, since each pair without field may have magnetizations $M_2 = 1$ or 0, degenerate precisely at $j_z = j/2$.

In Fig. 4 we show the GS scaled magnetization for a chain of $N = 120$ spins, obtained with density matrix renormalization (DMRG) [55–57, 64]. In the $n = 1$ case the transition region from $M = 0 \to N/2$ is again quite narrow (top left), in agreement with [15], since here the “polymer” formed for large $h_1$ consists of just one spin, whose lower state may have only two magnetizations: $M_1 = s$ and $-s$ (see inset), i.e. $q = 0$ and $q = 2s$, leading just to $|m| = 1, 0$ plateaus. For $n = 2$ and $j_z > 0$, the GS possesses plateaus at $|m| = 1, 1/2$ (top right), reflecting the magnetizations $M_3 = 3/2, 1/2$ ($q = 0, 1$) of the trimer formed by the three spins trapped between two aligned spins. Moreover, the trimer cannot reach $M_3 = -1/2$ (except for large $h_2 \approx -h_1$) entailing no wide $m = 0$ plateau. For $n = 3$, however, pentamer magnetization $M_5$ does reach $-1/2$, entailing a large $m = 0$ plateau, in addition to the aligned phase $|m| = 1$ ($M_5 = 5/2, q = 0$) and smaller intermediate plateaus at $|m| = 2/3, 1/3$ ($M_5 = 3/2, 1/2, q = 1, 2$, bottom left). Such persistent plateaus also occur for higher spins, as seen for $n = 1$ and $n = 2$ (bottom right), where $|m| = 1, 3/4, 1/2, 1/4$, following the trimer magnetizations $M_3 = 3, 2, 1, 0$.

Fig. 5 shows the single spin magnetization $\langle S^z_i \rangle$ of the first four spins in the chains of Fig. 3. For $s = 1/2$, $1/2 - |\langle S^z_i \rangle|$ is also a measure of the entanglement of spin $i$ with the rest of the chain (i.e., of the mixedness of the single spin reduced state [65]), with $|\langle S^z_i \rangle| = 0$ (1/2) implying maximum (zero) $i$–rest entanglement.

The spins with field will align with the field direction as $h_1$ increases, leading for $n = 1$ to type-a (b) spin configurations for strong parallel (antiparallel) fields. However, those without field ($n \geq 2$) exhibit a more complex behavior. For $n = 2$ and $j_z = 0$, the total GS magnetization $M$ vanishes $\forall h_1, h_2$, implying that these spins become antialigned for $h_1 = h_2$, leading to a type-b Néel
configuration, but have zero magnetization \(\langle S_i^z \rangle = 0\) for \(h_1 = -h_2\), entailing a type-d configuration. This configuration also holds for \(j_z > 0\) if \(h_1 = -h_2\) (and \(|h_i| > h_3\) if \(j_z > j\)), since \(M\) still vanishes, implying that these spins become frustrated, as the attractive \(S_i^z S_{i+1}^z\) coupling cannot be satisfied with both adjacent spins. This is a clear example of field-induced frustration, and entails maximum i-rest entanglement, mostly saturated with neighboring zero field spins. On the other hand, for large \(h_1 = h_2\) and \(j_z > 0\), they become aligned (type-a).

In contrast, for \(n = 3\) the two contiguous spins without field tend to form an entangled dimer, leading for \(j_z = 0\) to a type-e configuration \(\langle S_i^z \rangle \approx 0\) for \(i = 2, 3\) if \(h_1 = h_2\) and a type-f configuration if \(h_1 = -h_2\), here slightly polarized towards b. In this case there is actually a spin configuration transition when \(0 < j_z < j_z^c(3) = j/2\), where \(\langle S_i^z \rangle\) changes sign at the central spins and the polarization evolves from type-b to type-c, crossing exactly type-f. For \(j_z > j/2\), these central spins remain significantly entangled for antiparallel fields, polarized towards type-c, while for parallel fields they become increasingly aligned as \(|h_i|\) and hence \(|M|\) increases. Previous behaviors can also be seen at the bottom panels for \(j_z > j\), which depict the “evolution” of \(\langle S_i^z \rangle\) with \(\theta = \tan^{-1}(-h_2/h_1)\) between the fully aligned phases.

C. Pairwise Entanglement

We show in Fig. 6 illustrative results for the pairwise entanglement measured through the concurrence \(C_{ij}\), in the chains of Fig. 3 for \(j_z/j = 0.75\). It is first verified that in the \(n = 3\) NNA case, the two contiguous spins with zero field \((C_{23},\text{top right})\) are highly entangled in the \(M = 0\) plateau, since the spins form there essentially a type-f dimerized configuration (see bottom row of Fig. 5). Accordingly, the concurrence \(C_{55}\) of a non-contiguous pair with zero field spins (bottom right) vanishes in this plateau. In contrast, the latter becomes significant in the \(|M| = 4\) and \(|M| = 2\) plateaus \((|m| = 2/3, 1/3)\), where the intermediate field \(h_2\) is weak, in agreement with the pentamerization argument.

On the other hand, in the \(n = 2\) NA case, \(C_{23}\) (spin without field and spin with field \(h_2\), top left) is clearly significant in the \(|M| = N/4\) plateaus emerging for small \(|h_2|\), and small or zero in the same plateaus emerging for small \(|h_1|\) and strong \(|h_2|\), supporting the trimerization argument. This is verified in \(C_{24}\) (bottom left), which is also significant (zero) when \(C_{23}\) is large (small) in these plateaus, entailing essentially no entanglement between trimers. \(C_{24}\) is also non-negligible at the \(M = 0\) plateau, where nearest spins with no field become entangled due to the field induced frustration. It is also confirmed that all concurrences are finite at the \(|M| = Ns - 1\) band, in agreement with Eq. (14).

III. CONCLUSIONS

We have shown that \(n\)-alternating field configurations can lead to novel GS phase diagrams which differ significantly from those of the standard alternating case. They can exhibit non-trivial magnetization plateaus associated with field induced frustration and polymerization phenomena which persist for large sizes, as verified by DMRG calculations, and which satisfy a quantization rule compatible with the OYA criterion. Exact analytic size-independent expressions for the border of the fully aligned phase were also derived, and imply a critical \(n\)-dependent threshold anisotropy below which a whole interval of GS magnetizations become unreachable even for arbitrarily strong fields. These results open new possibilities for applications of finite chains with simple interactions under controllable fields, such as entanglement tuning and plateaus formation at rational values of the scaled magnetization, and pave the way to study the emergence of critical phenomena induced through non-uniform fields within more general architectures and couplings.
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Appendix A: Border of the aligned phase in the \( n \)-alternating spin-\( s \) XYZ system

We prove here Eqs. (3)–(5) and (7). We consider the Hamiltonian (1) in a cyclic \( n \)-alternating field configuration (2), with \( N = 2nK \) spins. The fully aligned states \( |M = \pm Ns \rangle \) are trivial eigenstates of \( H \) \( \forall n \), with energies

\[
E_{\pm Ns} = -K s \pm (h_1 + h_2) + nj_z, \tag{A1}
\]

where \( j_z = 2sJ_z \), which are degenerate for \( h_1 + h_2 = 0 \) and independent of \( J \). On the other hand, the \( M = Ns - 1 \) eigenstate of lowest energy can be obtained by diagonalizing \( H \) in the invariant subspace spanned by the \( 2n \) \( W \)-like states (here \( S^z = S^z_1 - iS^y_1 \))

\[
|W_i \rangle = \frac{1}{\sqrt{2sN}} \sum_{i=0}^{K-1} S^{-2n-1} |Ns \rangle, \quad i = 1, \ldots, 2n, \tag{A2}
\]

having the form

\[
|Ns - 1 \rangle = \sum_{i=1}^{2n} w_i |W_i \rangle, \tag{A3}
\]

with \( \sum_i |w_i|^2 = 1 \). The ensuing energy difference is

\[
E_{Ns-1} - E_{Ns} = j_z + \lambda_n, \tag{A4}
\]

with \( \lambda_n \) the lowest eigenvalue of a \( 2n \times 2n \) matrix \( H_n \) depending just on \( h_1, h_2 \) and \( j = 2sJ \), of elements

\[
(H_n)_{kl} = \delta_{kl}(h_1 \delta_{k,l+1} - h_2 \delta_{k+1,l}) - j_n \delta_{k,l+1}, \tag{A5}
\]

where \( \delta_{k,2n+1} = \delta_{k,1} \), \( \delta_{k,0} = \delta_{k,2n} \) and \( j_n = j/2 \) for \( n \geq 2 \) while \( j_1 = j \). Then, for \( h_1 + h_2 > 0 \), a fully aligned GS requires \( j_z + \lambda_n > 0 \), i.e. \( H_n + j_z \mathbb{1} \) positive definite, implying

\[
D_n = \text{Det} [H_n + j_z \mathbb{1}] > 0. \tag{A6}
\]

The border of the aligned phase is then determined by the equation \( D_n = 0 \).

From Eqs. \( \text{(A5)}-\text{(A6)} \) it follows that \( D_n \) has the form

\[
D_n = a_n h_1 h_2 + b_n (h_1 + h_2) + c_n, \tag{A7}
\]

with \( a_n, b_n, c_n \) field-independent. Eq. \( \text{(A6)} \) then leads to Eq. (3), with

\[
\alpha_n^2 = \frac{b^2_n - a_n c_n}{a^2_n}, \quad \beta_n = \frac{b_n}{a_n}. \tag{A8}
\]

Eq. (4) follows by symmetry.

For \( n = 1 \), \( H_n \) is a \( 2 \times 2 \) matrix and a trivial calculation yields \( a_1 = 1 \), \( b_1 = j_z \) and \( c_1 = j_z^2 - j^2 \), which leads to \( \alpha_1 = j \), \( \beta_1 = j_z \), i.e. Eq. \( \text{(A8)} \) (in this case \( \lambda_n = \lambda_1 = \frac{h_1 + h_2}{2} - \sqrt{(\frac{h_1 - h_2}{2})^2 + j^2} \), and Eq. (3) can be directly obtained from the condition \( j_z + \lambda_1 > 0 \). For general \( n \), evaluation of \( D_n \) yields

\[
a_n = d_{n-1}^2, \quad b_n = d_{2n-1}, \tag{A9}
\]

and \( c_n = d_{2n} - \frac{j^2}{4} d_{2n-2} - 2 \frac{j^4}{4n} \), where \( d_n = \text{Det}(M_n) \) is the determinant of an \( n \times n \) Toeplitz tri-diagonal matrix of elements \( (M_n)_{ij} = j_z \delta_{ij} - \frac{1}{2} \delta_{i,j+1} \). It then satisfies

\[
d_{n+1} = j_z d_n - (j/2) d_{n-1}, \tag{A10}
\]

for \( n \geq 1 \), with \( d_1 = j_z, d_0 = 1, i.e., (d_{n+1}^2) = A^n(j_z) \), with \( A = (j_z^2 - j^4/4) \). Hence, for any \( n \geq 1 \), diagonalization of \( A \) yields

\[
d_n = \frac{(j_z + h_2)^{n+1} - (j_z - h_2)^{n+1}}{2^{n+1} h_2}, \tag{A11}
\]

with \( h_2 = \sqrt{j^2 - j_z^2} \). Eqs. \( \text{(A9)}-\text{(A11)} \) then lead to \( \alpha_n = 2(j/2)^n/d_{n-1} \) and hence to Eq. (5), with \( \beta_n = d_{2n-1}/d_{n-1} \) then given by Eq. \( \text{(4)} \).

The lowest eigenvalue of \( H_n \) satisfies \( \lambda_n \leq -j \cos(\pi/n) \), with the upper bound reached for \( h_1, h_2 \to \infty \). Thus, the aligned state stability condition \( j_z + \lambda_n > 0 \) requires \( j_z > j \cos(\pi/n) = j_z^*(n) \). This threshold value of \( j_z \) represents the lowest eigenvalue of the \((n-1) \times (n-1)\) block of \( H_n \) associated with the \( n-1 \) spins with no field, which is tridiagonal with eigenvalues \( -j \cos(\pi k/n) \), \( k = 1, \ldots, n-1 \). Hence, \( -j \cos(\pi/n) \) represents the lowest energy of the \( n-1 \) spins trapped between the two aligned spins in the limit \( h_1, h_2 \to \infty \), for \( j_z = 0 \) and magnetization \((n-1)s-1\). While Eq. \( \text{(A6)} \) is in principle a necessary condition for stability of the \( M = Ns \) aligned GS, it turns out to be sufficient for \( j_z > j_z^*(n) \) and \( h_1 + h_2 > 0 \), since in this case the GS magnetization decreases in steps of length 1 as the fields \( h_1, h_2 \) decrease from \( +\infty \). The only exception occurs for \( j_z > j \) along the segment \( h_1 = -h_2 = h \) between the factorizing fields (see bottom panels in Fig. 3), where the aligned states \( M = \pm Ns \) are degenerate GS’s if \( |h| \leq h_2 \) and all GS magnetizations plateaus merge if \( |h| = h_2 \).

Appendix B: Reduced states in the \( M = Ns - 1 \) GS

From the form \( \text{(A2)} \) of the states \( |W_i \rangle \), it becomes apparent that the reduced state of any two distinct spins \( k \neq l \) in the \( |M = Ns - 1 \rangle \) GS \( \text{(A3)} \) will depend just on their positions \( i, j \) within the cell each spin belongs, but not on their absolute distance \( |k - l| \). Since the reduced
state will also commute with the total spin $S_k^z = S_k^x + S_k^y$ of the pair, it will be given, for $M = Ns - 1$, by

$$\rho_{ij} = \begin{pmatrix} 1 - \frac{|w_i|^2 + |w_j|^2}{K} & 0 & 0 & 0 \\ 0 & \frac{|w_i|^2}{K} & \frac{w_i w_j^*}{K} & 0 \\ 0 & \frac{w_i w_j^*}{K} & \frac{|w_j|^2}{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (B1)$$

in the subspace spanned by the states \{\{ss\}, |s, s-1\rangle, |s-1, s\rangle\}, where $|s-1\rangle = \frac{1}{\sqrt{2s}} |s\rangle$. Eq. (B1) is valid for any $s$ and $i, j = 1, \ldots, n$. It can then be always considered as a mixed state of an effective two-qubit system, as just states $|s\rangle$ and $|s-1\rangle$ are involved at each spin. A similar expression holds for the reduced state in the $M = -Ns + 1$ GS in the corresponding subspace.

The ensuing concurrence \[61\] becomes $C_{ij} = 2\langle |p_{ij}\rangle_{23}\rangle$ and is then given by Eq. [A1]. These concurrences saturate the monogamy relations \[62\,\, 63\], namely

$$\sum_{l \neq i} C_{li}^2 = 4\frac{|w_i|^2}{K} \left(1 - \frac{|w_i|^2}{K}\right) = C_{i,\text{rest}}^2, \quad (B2)$$

where $C_{i,\text{rest}}^2 = 2(1 - \text{Tr} \rho_i^2)$ is the concurrence of single spin $i$ with the rest of the chain, with

$$\rho_i = \begin{pmatrix} 1 - \frac{|w_i|^2}{K} & 0 & 0 & 0 \\ 0 & \frac{|w_i|^2}{K} & \frac{w_i w_j^*}{K} & 0 \\ 0 & \frac{w_i w_j^*}{K} & \frac{|w_j|^2}{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (B3)$$

the reduced state of spin $i$ in the state \[\Lambda3\].

The corresponding negativity \[67\] is, setting $\gamma_{ij} = 1 - (|w_i|^2 + |w_j|^2)/K$,

$$N_{ij} = \frac{1}{2} \left(\sqrt{\gamma_{ij}^2 + 4|w_i w_j|^2} - \gamma_{ij}\right), \quad (B4)$$

with $N_{ij} \to C_{ij}^2/2$ for large $K$.

Due to the symmetry $w_{n+1+i} = w_{n+1-i}$, valid for $i = 1, \ldots, n-1$ under cyclic conditions, the coefficients $w_i$ in \[\Lambda3\] can be obtained by diagonalizing an effective $(n + 1) \times (n + 1)$ matrix $H^*$. Altogether there are just $(n + 1)$ distinct coefficients $w_i$, and hence just $(n + 1)(n + 2)/2$ distinct pairwise concurrences and negativities for general $h_1, h_2$ in the $|M| = Ns - 1$ GS.

Appendix C: Exact solution of the XX chain in $n$-alternating field configurations

When $J_z = 0$, the XXZ model reduces to the XX model. For $s = 1/2$, the ensuing Hamiltonian can be mapped exactly to a bilinear fermionic form in the annihilation $c_j^\dagger$ and creation $c_j$ operators by means of the Jordan-Wigner transformation \[2\], $c_j^\dagger = S_j^x \exp(-i\pi \sum_{k=1}^{j-1} S_k^z S_k^z)$ for each value of the fermionic number parity

$$P = \exp(i\pi \mathbf{N}) = \sigma = \pm 1, \quad (C1)$$

where $\mathbf{N} = \sum_{j=1}^N c_j^\dagger c_j = S^2 + N/2$ is the fermion number operator. This leads to

$$H = -\sum_j \left[h_j (c_j^\dagger c_j - 1/2) - \eta_j \frac{J}{2} (c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1})\right], \quad (C2)$$

where, for cyclic conditions, $\eta_j = 1 \forall j$ and $\eta_j = 1 (-1)$ for $j \leq N-1 (j = N)$. After a discrete Fourier transform of the fermion operators, it can be expressed as a sum of $K 2n \times 2n$ matrices $H_k$:

$$H = -\sum_{k=1}^{K-\delta_{s,1/2}} c_k^\dagger \cdot H_k c_k - \epsilon, \quad (C3)$$

with $c_k^\dagger = (c_k^\dagger, c_k^{N+(2n)}, \ldots, c_k^{(2n-1)+N/(2n)})$, $D_k$ a diagonal matrix of elements $(D_k)_{ii} = -J \cos(\omega_k + \pi(i-1)/n)$, $A$ a circulant matrix specified by the vector $(h^+, h^-, h^+, h^+, \ldots)$, and

$$h^\pm = \frac{h_1 \pm h_2}{2}, \quad \epsilon = \frac{Nh^+}{2}, \quad \omega_k = 2\pi k/N. \quad (C6)$$

Eq. (C3) shows that the Fourier transformed $n$-alternating field configuration leads to off diagonal hopping terms specifying the allowed momentum values.

Due to the parity dependence of the energy levels, the number of GS magnetization transitions is associated to the number of times the single particle energies change sign \[70\]. Hence, field values at which single particle energies vanish can be determined by solving

$$\text{Det} (H_k) = 0, \quad (C7)$$

with $k = 1/2, 1, \ldots, K$.

For standard alternating fields $n = 1$, Eq. (C4) becomes

$$H_k = \begin{pmatrix} h^+ - J \cos \omega_k & h^- \\ h^- & h^+ + J \cos \omega_k \end{pmatrix}, \quad (C8)$$

yielding the well known single particle energies \[68\,\, 74\]

$$\lambda_k^\pm = h^+ \pm \sqrt{(h^-)^2 + J^2 \cos^2 \omega_k}. \quad (C9)$$

In this case

$$\text{Det} (H_k) = h_1 h_2 - J^2 \cos^2 \omega_k, \quad (C10)$$

and Eq. (C7) determines $N/2$ hyperbolas in the $(h_1, h_2)$ field space, meaning that the GS will then exhibit definite magnetization plateaus ranging from $|M| = 0$ to $|M| = N/2$. In particular, for $k = N/2$ the lowest $\sigma = -1$ parity level becomes negative and we recover exactly the
hyperbola $h_1 h_2 = j^2$ of the $N/2 \to N/2 - 1$ transition, in agreement with Eq. (3)–(5) for $n = 1$ and $j_z = 0$. For $n \geq 2$ the expressions for the eigenvalues are more involved.

In the A.A \( n = 2 \) case, the determinant of the \( H_k \) is

\[
    \text{Det} (H_k) = \frac{J^4}{4} \sin^2 (2\omega_k),
\]

which becomes zero only for $k = N/4$ and leads to at least one identically zero single particle energy. The latter means that there is no single particle energy which changes sign as the fields are varied and indicates that there should be no GS magnetization transition. Furthermore, we have proved the following lemma:

**Lemma 1.** The GS of a finite XX spin system in a $n = 2$ next-alternating field configuration is a nondegenerate half-filled state with definite magnetization $M = 0$, $\forall h_1, h_2$.

**Proof:** We first start by comparing the number of energy levels with negative single particle energies within each parity $\sigma$ and their ensuing lowest energy state $E_\sigma$. Since $\text{Det} [H_k] = J^2 \sin^2 (2\omega_k) \geq 0 \forall k$, then each matrix $H_k$ is either positive (or negative) semi-definite, or it has two positive and two negative eigenvalues. However, since the determinant of any leading principal minor connecting $k$ with $k + 2 = -J^2 \cos (\omega_k)$, $H_k$ cannot be positive nor negative semi-definite. In the $\sigma = 1$ subspace, $\text{Det} [H_k] > 0 \forall k = \{1/2, \ldots, K - 1/2\}$, entailing that there are always $N/2$ negative single particle energies, whereas for $\sigma = -1$ there are $N/2 - 1$, as one of the eigenvalues of $H_{N/4}$ is identically zero. Due to this small, albeit important, difference in the number of negative energy levels, $E_1 < E_{-1} \forall h_1, h_2$. While this result can be numerically verified, for $h_2 = \pm h_1 = \pm h$ a series expansion of the energy difference between the lowest state within each parity $\Delta E = E_{-1} - E_1$ shows that $\Delta E > 0 \forall h$. Likewise, for strong fields a second order perturbation treatment in the couplings shows that the $M = 0$ eigenstate is the GS $\forall J$. $\square$

[1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999); S. Sachdev, Nature Physics 4, 173 (2008).
[2] E. Lieb, T. Schultz, D. Mattis, Ann. Phys. 16, 407 (1961).
[3] M. Vojta, Rep. Prog. Phys. 66, 2069 (2003).
[4] N. Laflencie, I. Affleck, and M. Berciu, J. Stat. Mech. P12001 (2005).
[5] Z. Wang, T. Lorenz, D. I. Gorbunov, P. T. Cong, Y. Kohama, S. Niesen, O. Breunig, J. Engelmayr, A. Herman, J. Wu, K. Kindo, J. Wosnitza, S. Zherlitsyn, and A. Loidl, Phys. Rev. Lett. 120, 207205 (2018).
[6] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) 416, 608 (2002).
[7] T.J. Osborne, M.A. Nielsen, Phys. Rev. A 66, 032110 (2002).
[8] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[9] A.R. Its, B-Q Jin, V.E. Korepin, J. Phys. A Math. Gen. 38, 2975 (2005).
[10] L. Amico, R. Fazio, A. Osterloh and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
[11] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. 82, 277 (2010).
[12] J. Stasinska, B. Rogers, M. Paternostro, G. De Chiara, and A. Sanpera, Phys Rev A 89, 032330 (2014).
[13] N. Blanc, J. Trinh, L. Dong, X. Bai, A. A. Aczel, M. Mourigal, L. Bentsen, T. Siegrist, and A. P. Ramirez, Nature Physics 14, 273 (2018).
[14] C. Lacroix, P. Mendels, and F. Mila, *Introduction to Frustrated Magnetism: Materials, Experiments, Theory* (Springer, Berlin, 2013).
[15] F. Michaud, T. Coletta, S. R. Manmana, J. D. Picon, and F. Mila, Phys. Rev. B 81, 014407 (2010).
[16] A. Honecker, J. Schuhenburg, and J. Richter, J. Phys.: Condens. Matter 16, S749 (2004).
[17] A. Tanaka, K. Totsuka, and X. Hu, Phys. Rev. B 79, 064412 (2009).
[18] M. Takigawa, and F. Mila, *Magnetization plateaus* (Springer Series in Solid-State Sciences, Vol. 164 (Springer, 2011), Chap. 10, pp. 241267.
[19] C. A. Lamas, S. Capponi, and P. Pujol, Phys. Rev. B 84, 115125 (2011); F. Elias, M. Arlego, and C. A. Lamas, Phys. Rev. B 95, 214426 (2017).
[20] H. Hu, C. Cheng, Z. Xu, H.-G. Luo, and S. Chen, Phys. Rev. B 90, 035150 (2014).
[21] M. Oshikawa, M. Yamanaka, I. Affleck, Phys. Rev. Lett. 78, 1984 (1997).
[22] H. Zhang, C. A. Lamas, M. Arlego, W. Brenig, Phys. Rev. B 93, 235150 (2016).
[23] C. K. Majumdar, and D. K. Ghosh, J. Math. Phys. 10, 1399 (1969).
[24] S. Nishimoto, N. Shibata, and C. Hotta, Nat. Commun. 4, 2287 (2013).
[25] P.C. Alcaraz, A.L. Malvezzi, J. Phys. A 28, 1521 (1995).
[26] M. Asoudeh, V. Karimipour, Phys. Rev. B 71, 022308 (2005).
[27] G.-F. Zhang, S.-S. Li, Phys. Rev. A 72, 034302 (2005).
[28] N. Canosa, R. Rossignoli, and J.M. Matera, Phys. Rev. B 81, 054415 (2010).
[29] M. Cerezo, R. Rossignoli, and N. Canosa, Phys. Rev. B 92, 224422 (2015); Phys. Rev. A 94, 042335 (2016).
[30] M. Cerezo, R. Rossignoli, N. Canosa, and E. Rios, Phys. Rev. Lett. 119, 226005 (2017).
[31] T. Chanda, T. Das, D. Sadhukhan, A. K. Pal, A. Sen De, and U. Sen, Phys. Rev. A 94, 042310 (2016).
[32] M. Lewenstein, A. Sanpera, V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-Body Systems* (Oxford University Press, NY, 2012).
[33] I. M. Georgescu, S. Ashhab, and F. Nori, Rev. Mod. Phys. 85, 153 (2014).
[34] *Principles and Methods of Quantum Information Technologies*, ed. by Y. Yamamoto and K. Semba (Springer, New York, 2016).
[35] J.D. Johnson, M. McCarty, Phys. Rev. A 6, 1613 (1972).
[36] C.N. Yang and C.P. Yang, Phys. Rev. 150, 321 (1966).
[37] S.-J. Gu, H.-Q. Lin, and Y.-Q. Li, Phys. Rev. A 68,
[38] F. C. Alcaraz, S. R. Salinas, and W. F. Wreszinski, Phys. Rev. Lett. 75, 930 (1995); F. C. Alcaraz, A. Saguia, and M. S. Sarandy, Phys. Rev. A 70, 032333 (2004).

[39] N. Canosa, R. Rossignoli, Phys. Rev. A 73, 022347 (2006); E. Ríos, R. Rossignoli, and N. Canosa, J. Phys. B 50, 095501 (2017).

[40] O. Breunig, M. Garst, A. Klümper, J. Rohrkamp, M. Turbull, T. Lorenz, Phys. Rev. Lett. 111, 187202 (2013).

[41] J. Ren, Y. Wang, and W.-L. You, Phys. Rev. A 97, 042318 (2018).

[42] J. Reisons, E. Mascarenhas, and V. Savona, Phys. Rev. B 96, 165137 (2017).

[43] C. Noh, and D. G. Angelakis, Rep. Prog. Phys. 80, 016401 (2017).

[44] Y.P. Shim, S. Oh, X. Hu, M. Friesen, Phys. Rev. Lett. 106, 180503 (2011).

[45] G. Xu, G. Long, Sci. Rep. 4, 6814 (2014).

[46] Y. Salathé et al, Phys. Rev. X 5, 021027 (2015).

[47] I. Arrazola, J.S. Pedernales, L. Lamata, E. Solano, Sci. Rep. 6, 30534 (2016).

[48] R. Toskovic et al, Nat. Phys. 12, 656 (2016).

[49] O.V. Marchukov, A.G. Volosniev, M. Valiente, D. Petrosyan, N.T. Zinner, Nat. Comms. 13070 (2016); A.G. Volosniev, D. Petrosyan, M. Valiente, D.V. Fedorov, A.S. Jensen, N.T. Zinner, Phys. Rev. A 91, 023620 (2015).

[50] S. Whitlock, A. W. Glaetzle, and P. Hannaford, J. Phys. B: At. Mol. Opt. Phys. 50, 074001 (2017).

[51] T.L. Nguyen, et al, Phys. Rev. X 8, 011032 (2018).

[52] O. Breunig, M. Garst, A. Klimper, J. Rohrkamp, M. Turbull, T. Lorenz, Sci. Adv. 3, 3773 (2017).

[53] S.C. Benjamin, S. Bose, Phys. Rev. A 70, 032314 (2004).

[54] A. Bayat, S. Bose, Phys. Rev. A 81, 012304 (2010); L. Banchi, A. Bayat, P. Verrucchi, S. Bose, Phys. Rev. Lett. 106, 140501 (2011).

[55] S. R. White, Phys. Rev. B 48, 10 345 (1993).

[56] A. Kolezhuk, R. Roth, and U. Schollwöck, Phys. Rev. Lett. 77, 5142 (1996); Phys. Rev. B 55, 8928 (1997).

[57] U. Schollwöck, Rev. Mod. Phys. 77, 259 (2003); Ann. Phys. 326, 96 (2011).

[58] The sign of $J$ can be changed by local rotations of angle $\pi$ around the $z$ axis at even (or odd) sites.

[59] These cases are linked by a global rotation of angle $\pi$ around the $x$ axis, which leaves the coupling unchanged.

[60] W. Dür, G. Vidal, J.I. Cirac, Phys. Rev. A 62, 062314 (2000).

[61] S. Hill, W.K. Wootters, Phys. Rev. Lett. 78, 5022 (1997); W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[62] V. Coffman, J. Kundu, W.K. Wootters, Phys. Rev. A 61, 052306 (2000).

[63] T.J. Osborne, F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).

[64] We usually perform 20 sweeps keeping up to 900 states.

[65] In a pure state with definite total magnetization $M$ along $z$, the single spin reduced state $\rho_i$ commutes with $S^z_i \forall s$. For $s = 1/2$ its eigenvalues are then $1/2 \pm \langle S^z_i \rangle$, and its purity $\text{Tr} \rho_i^2 = \frac{1}{2} + 2\langle S^z_i \rangle^2$.

[66] A. Böttcher, S.M. Grudsky, Spectral Properties of Banded Toeplitz Matrices, SIAM (2005).

[67] G. Vidal, R.F. Werner, Phys. Rev. A 65, 032314 (2002); K. Zyczkowski, P. Horodecki, A. Sanpera, M. Lewenstein, Phys. Rev. A 58, 883 (1998).

[68] J.H. H. Perk, H.W. Capel, M.J. Zuilhof, Th.J. Siskens, Phys. A 81, 319 (1975).

[69] K. Okamoto, K. Yasumura, J. Phys. Soc. J. 59, 993 (1990).

[70] N. Canosa, R. Rossignoli, Phys. Rev. A 75 032350 (2007).

[71] S. Deng, G. Ortiz, L. Viola, EPL 84, 67008 (2008).

[72] U. Divakaran, A. Dutta, D. Sen, Phys. Rev. B 78, 144301 (2008).

[73] A. De Pasquale, P. Facchi, Phys. Rev. A 80 032102 (2009).

[74] A. Dutta, G. Aeppli, B.K. Chakrabarti, U. Divakaran, T.F. Rosenbaum, D. Sen, Quantum phase transitions in transverse field spin models: From statistical physics to quantum information Cambridge Univ. Press, UK (2015).