Twistor-Like Formulation of Heterotic Strings

by

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Summary: In this talk new formulations of the Green–Schwarz heterotic strings in $D$ dimensions that involve commuting spinors, are reviewed. These models are invariant under $n$–extended, world sheet supersymmetry as well as under $N = 1$, target space supersymmetry where $n \leq D - 2$ and $D = 3, 4, 6, 10$. The world sheet supersymmetry replaces $n$ components (and provides a geometrical meaning) of the $\kappa$–symmetry in the Green–Schwarz approach. The models in $D = 10$ for $n = 1, 2, 8$ are discussed explicitly.

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1. Introduction

In the Neveu-Schwarz-Ramond (N.S.R.) approach [1] to superstrings the action is invariant under super–reparametrization of the world sheet (w.s.) i.e. it is superconformal invariant in superconformal gauge. The covariant quantization of these models is well understood and is consistent in ten dimensions. However the target space (t.s.) supersymmetry is not manifest and is recovered through the G.S.O. projection [2]. This justifies the large amount of work devoted to the alternative Green-Schwarz (G.S.) approach [3], where the t.s. supersymmetry is manifest. The G.S. approach, even at the classical level, can be formulated only in the special dimensions $D = 3, 4, 6, 10$. In the world sheet the action is invariant under reparametrization and the role of the w.s. supersymmetry is played here by a new local symmetry called $\kappa$–symmetry [4]. However the geometrical meaning of $\kappa$-symmetry in G.S. models is rather obscure. Moreover its gauge fixing requires an infinite chain of ghosts for ghosts (in technical terms, $\kappa$–symmetry is infinitely reducible).

This feature, strongly connected to the mixing of first class and second class constraints which cannot be disentangled covariantly, is the source of the difficulties of the covariant quantization in the G.S. approach. $\kappa$–symmetry is also present in
superparticle [5] (and supermembrane [6]) models. A consistent covariant quantization is now available for superparticles [7] but the problem is still open in the case of superstrings (and supermembranes). Then it is useful to look at superparticles and G.S. superstrings (or supermembranes) from different points of view. In particular it has been suggested [8]–[11] that twistors [12] could play a rôle here.

In an important paper, Sorokin et al. [13] have proposed a new formulation for superparticles in $D = 3, 4$. Subsequently this formulation, or related ones, have been worked out both for superparticles [14]–[21] and G.S. superstrings [22]–[32] in the special dimensions $D = 3, 4, 6, 10$. As in previous approaches, these formulations involve commuting spinors i.e. twistor-like variables. In addition, they show both manifest target space supersymmetry and $n$-extended world line/world sheet supersymmetry ($1 \leq n \leq D - 2$). The supersymmetry of the world manifold replaces $n$ components of the $\kappa$-symmetry and therefore provides a geometrical meaning of that symmetry. At least at the classical level, the maximal extended models, with $n = D - 2$, are of special interest since, in this case, the whole $\kappa$–symmetry is replaced by (extended) supersymmetry of the world manifold.

The hope is that twistor models could help in solving the problem of covariant quantization of G.S. superstrings. Significant steps in that direction have been done by Berkovits [31],[32]. In any case these models are useful to clarify the G.S. approach and the geometrical meaning of the $\kappa$–symmetry.

In the next two sections I shall review briefly the G.S. approach of the heterotic string (sect. 2) and I shall discuss some properties of commuting spinors (twistors) (sect. 3). Then, in section 4, I shall describe the new twistor–like formulations of the heterotic strings in $D = 10$ with $n = 1, 2$ and 8.

2. The G.S. Approach

The heterotic string, in the G.S. approach, describes the embedding of the two dimensional world sheet $\mathcal{M}(2|0)$ on the target superspace $\mathcal{M}(D|2(D - 2))$ where $D = 3, 4, 6, 10$.

The world sheet is parametrized locally by the coordinates $\xi^i \equiv (\xi^+, \xi^-)$. The w.s. zweinbeins $e^\pm_i$ have vanishing torsion, so that the Lorentz connection can be expressed in terms of $e^\pm_i(\xi)$. The w.s. differential is $d = d\xi^i\partial_i = e^+D_+ + e^-D_-$. 

The target superspace has $D$ bosonic dimensions and $2(D - 2)$ fermionic dimensions and is parametrized locally by the w.s. scalar, string coordinates, $Z^M \equiv (X^m, \theta^\mu)$ where $X^m (m = 0, 1, \ldots, D - 1)$ are t.s. vectors and $\theta^\mu (\mu = 1, \ldots, 2(D - 2)$ are Majorana, t.s. spinors (Weyl-Majorana in $D = 10$). The field content of the model is completed by a set of heterotic fermions $\psi_r (r = 1, \ldots, N)$ which are left–handed,
w.s. Weyl-Majorana spinors. In $D = 10$ they are essential, at the quantum level, to cancel the left–handed conformal anomaly (then $N = 32$).

To write the action, in flat background, let us define

$$\mathcal{E}_\pm^a = D_\pm X^a - (D_\pm \theta \Gamma^a \theta); \quad \mathcal{E}_\alpha^a = D_\pm \theta^\alpha$$

$$B_{\alpha\beta}^{(0)} = 0 = B_{a\beta}^{(0)} = -B_{\beta a}^{(0)} = - (\Gamma_a)_{\beta\gamma} \theta^\gamma$$

where $(\Gamma^a)_{\beta\gamma} = (\gamma^a C)_{\beta\gamma}$ and $(\Gamma^a)^{\beta\gamma} = (C^{-1} \gamma^a)_{\beta\gamma}$ are symmetric in $\beta, \gamma$ ($\gamma^a$ are Dirac matrices and $C$ is the charge conjugation matrix). Moreover we shall write $\mathcal{E}_\pm^A = (\mathcal{E}_\pm^a, \mathcal{E}_\pm^\alpha)$. Then the G.S. action is

$$I = \int e^+ \wedge e^- \{ \mathcal{E}_+^a \mathcal{E}_{-a} + \mathcal{E}_+^A \mathcal{E}_-^B B_{BA}^{(0)} + i \sum_{r=1}^N (\psi_r D_- \psi_r) \}$$

(2.1)

The first term contains the kinetic action, the second is the Wess-Zumino term and the third term is the heterotic action. One should notice that the Virasoro condition $\mathcal{E}_+^a \mathcal{E}_{-a} = 0$ is among the field equations of this action (it is obtained by varying $e^+_i(\xi)$).

In addition to w.s. reparametrization, Weyl and Lorentz local transformations, the action (1) is invariant under the local $\kappa$–symmetry:

$$\delta \kappa \theta^\alpha = \mathcal{E}_-^a (\Gamma_a)^{\alpha\beta} K_\beta; \quad \delta \kappa X^a = (\delta \kappa \theta \Gamma^a \theta)$$

$$\delta \kappa e^i_+ = 4(\mathcal{E}_+^a K_a) e^i_-; \quad \delta \kappa e^i_- = 0 = \delta \kappa \psi_r$$

(2.2)

where $K_\alpha$ are the anticommuting, $\kappa$–symmetry gauge parameters ($K_\alpha$ is a commuting ghost in the BRS version of the $\kappa$–symmetry).

Since on shell $\mathcal{E}_-^a \Gamma_a$ is a projector, there are secondary gauges that affect $K_\alpha$ and involve secondary ghosts and so on, so that $\kappa$–symmetry is infinitely reducible. Then half of the $2(D - 2)$ components of $K_\alpha$ are pure gauge and $\kappa$–symmetry contains only $(D - 2)$ parameters.

It is easy to extend this model in presence of a SUGRA-SYM background [33] (G.S., $\sigma$–model). The tangent target space geometry is described in terms of the supervielbeins $E^A = dZ^M E^A_M(Z)$, the Lorentz superconnection $\Omega_A^B = dZ^M \Omega_{MA}^B(Z)$, the gauge superconnection $A = dZ^M A_M(Z)$ and the two–superform $B = dZ^M dZ^N B_{MN}(Z)$. The curvatures of these superforms are respectively the torsion, $T^A$, the Lorentz curvature, $R_A^B$, the gauge curvature, $F$, and the $B$–curvature, $H = dB$. The intrinsic components of the w.s. pull–back of $E^A$ are denoted $E^A_{\pm}$. The flat case is recovered in the limit $E^A \to \mathcal{E}^A; \ B \to B^{(0)}; \ A \to 0$. 
In these notations the action is

\[ I = \int e^+ \wedge e^- \{ \phi E^a E_{+a} + E^A E^B B_{BA} + i \psi (D_+ - E^B A_B) \psi \} \quad (2.3) \]

where \( \psi \) is the column vector \((\psi_r)\) and \( \phi(Z) \) is the dilaton superfield.

Now the Virasoro condition is

\[ E^a_+ E^-_a = 0 \quad (2.4) \]

and the \( \kappa \)-symmetry transformations are

\[ \delta_\kappa Z^M E^\alpha_M = w^\alpha; \quad \delta_\kappa Z^M E^a_M = 0; \quad \delta_\kappa e^i_+ = 0 \]
\[ \delta_\kappa e^i_- = 4e^i_-(E^a_+ K_\alpha) + \ldots; \quad \delta_\kappa \psi = w^\beta A_\beta \psi \quad (2.5) \]

where \( w^\alpha = E^a_-(\Gamma_a)^{\alpha\beta} K_\beta \) and the dots represent terms involving curvature components. However \( I \) is invariant under \( \kappa \)-symmetry only when the SUGRA-SYM background constraints [34],[9] are imposed:

\[ T^a_{\alpha\beta} - 2\Gamma^a_{\alpha\beta} = 0 = T^\gamma_{\alpha\beta} \]
\[ H_{\alpha\beta\gamma} = 0 = H^a_{\alpha\beta\gamma} - \phi(\Gamma_a)_{\beta\gamma} \quad (2.6) \]

In D=10, eqs.(6) are just the constraints that lead to the field equations for the decoupled, \( N = 1 \), Supergravity and Super Yang-Mills theory (the coupling arises from \( \sigma \)-model quantum corrections [35],[36]).

3. Twistors

Twistors are a deep mathematical concept introduced by Penrose [12]. Roughly speaking a twistor is a couple of commuting spinors \((\lambda^\alpha, w_\alpha)\) and a point in the twistor–space corresponds to a light–line in space time. An ambitious program is to study Q.F.T. (in particular quantum gravity) in twistor space \((\lambda, w)\) rather than in the usual phase space \((X, P)\).

However we shall use the word “twistor” in a more superficial way. For us, a twistor (strictly speaking: an half–twistor) is simply a commuting Majorana spinor \( \lambda^\alpha \) in the special dimensions \( D = 3, 4, 6, 10 \) where it has \( 2(D-2) \) real components. Moreover, due to the fundamental \( \Gamma \)-matrix identity
\[ \Gamma^a_{\alpha\beta} \Gamma^a_{\alpha\gamma\delta} + \Gamma^a_{\beta\gamma} \Gamma^a_{\alpha\delta} + \Gamma^a_{\gamma\alpha} \Gamma^a_{\alpha\beta\delta} = 0 \] (3.1)

which holds in these (and only these) dimensions, the vector \( v^a = (\lambda \Gamma^a \lambda) \) is light-like: \( v^a v_a = 0 \). Therefore, if the momentum \( p^a \) of a (super)particle is written as \( p^a = (\lambda \Gamma^a \lambda) \), the massless condition \( p^a p_a = 0 \) is automatically satisfied. Similarly, in G.S. superstrings, the constraint \( E^a_+ = (\lambda \Gamma^a \lambda) \) implies the Virasoro condition \( E^a_+ E_{-a} = 0 \). In addition, since in Majorana representation \( \Gamma^0 = 1 \), one has

\[ v^0 = (\lambda \Gamma^0 \lambda) > 0 \] (3.2)

so that the light-like vector \( v^a \) points in the future light-cone.

It is worthwhile to report at this point a remark of Bengtsson et al.\[10\]. In first quantized models, covariance requires both positive and negative energy states. But supersymmetry \( \{Q,Q\} = H! \) requires positive energy. This is at the origin of the difficulty for the covariant quantization of superparticle and G.S. superstring models. Owing to eq. (2), quantization in twistor-space could be the key to overcome this problem.

In any case, coming back to our discussion, we have shown that twistors describe light-like vectors that point in the future light-cone. However a light-like vector \( v^a \), with \( v^0 > 0 \) and modulo a scaling, parametrizes the sphere \( S^{D-2} \). In fact one has \( \sum_{i=1}^{D-1} v_i^2 = v_0^2 \). On the other hand, a twistor \( \lambda^\alpha \), modulo a scaling, parametrizes the sphere \( S^{2(D-2)-1} \). In fact, since \( \Gamma^0 = 1 \), one has \( \sum_{\alpha=1}^{2(D-2)} (\lambda^\alpha)^2 = v^0 > 0 \)

Then the problem is how to make the twistor description equivalent to the light-like vector one i.e. how to get the sphere \( S^{D-2} \) parametrized by twistors.

The problem would be solved if it was possible to consider twistors modulo gauge transformations that belong to the cosets \( \frac{S^{2(D-2)-1}}{S^{D-2}} \). These cosets are \( Z_2, S^1, S^3 \) and \( S^7 \) respectively, in the magic dimensions \( D = 3, 4, 6, 10 \) (an isomorphism known as Hopf fibration). Moreover the Hopf fibration is related to the division algebras of real, complex, quaternionic and octonionic numbers, i.e. \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \). Then a simple idea is to consider twistors valued in the division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) respectively in \( D = 3, 4, 6, 10 \). However Table I shows that an obstruction has to be waited in the most relevant dimension \( D = 10 \). Indeed in this case \( S^7 \) is not a group and the division algebra of octonions is not associative.

Recently a way has been found to overcome this \( D = 10 \) barrier. The method is based on the isomorphism \[19],[20\]
where $\text{Spin}(1, D-1)$ is the covering of the Lorentz group in D dimensions, $SO^\uparrow(1, 1)$ is the orthochronous Lorentz group in $D = 2$, $K$ is the abelian group of the special conformal transformations in $D - 2$ dimensions (conformal boosts) and $\times$ denotes a semidirect product. The explicit construction of this isomorphism in terms of twistors is the following [21]. Consider $D - 2$ twistors $\lambda^\alpha_q$ ($q = 1, ... D - 2$) such that

i) $(\lambda_p \Gamma^a \lambda_q) = v^a \delta_{qp}$

ii) $\lambda^\alpha_q$ are defined modulo $SO^\uparrow(1, 1) \otimes SO(D - 2)$ gauge transformations;

iii) The $(D - 2) \times 2(D - 2)$ matrix $\lambda^\alpha_q$ has highest rank: $\text{rank} ||\lambda|| = D - 2$.

It has been shown in [21] that $D - 2$ twistors that fulfill these conditions parametrize the sphere $S^{D-2}$.

This construction is the geometrical basis of the maximally extended twistor–like models of superparticle and heterotic strings. These models have $(D - 2)$–extended world sheet supersymmetries and the twistors arise as the superpartners of $\theta^\alpha$. Of course one can construct intermediate models with a smaller number of twistors i.e. a smaller number of w.s. supersymmetries.

**4. The twistor approach to G.S. heterotic strings**

In this section I shall present the twistor–like formulations of G.S. heterotic strings in $D$ dimensions, with $N = (n, 0)$–extended, w.s. supersymmetry and $n \leq D - 2$. These models describe the embedding of the superworld sheet $\mathcal{M}(2|n)$ into the target superspace $\mathcal{M}(D|2(D - 2))$. I shall consider explicitly only the case
with \( D = 10 \) and \( n = 1, 2, 8 \). The superworld sheet is parametrized locally by the supercoordinates \( \xi^I = (\xi^{(+)}, \xi^{(-)}, \eta^{(q)}) \) where \( \eta^{(q)} \) are real Grassmann parameters and \( q = 1, \ldots, n \). The supervielbeins are \( e^A \equiv (e^+, e^-, e^q) \) so that the differential is \( d = e^+ D_+ + e^- D_- + e^q D_q \). The w.s. tangent space structure group is \( SO^\uparrow(1, 1) \otimes SO(n) \) i.e. the product of the orthochronous \( D = 2 \) Lorentz group and the group \( SO(n) \) that acts on the \( e^q \)s. We shall call \( \Delta \) the covariant differential with respect to the Lorentz and the \( SO(n) \) group. The w.s. torsion \( T^A = \Delta e^A \) is constrained as follows:

\[
T^- = \sum_q e^q \wedge e^q; \quad T^+ = 0; \quad T^q = e^+ \wedge e^- T^q_{-+}
\]

As in the G.S. approach, the target fields are the string supercoordinates and the heterotic fermions but now they are w.s. superfields:

\[
\hat{Z}^M(\xi^{(\pm)}, \eta^{(q)}) \equiv (\hat{X}^m, \hat{\theta}^\mu); \quad \hat{\psi}(\xi^{(+)}, \eta^{(q)}) \equiv (\hat{\psi}_r)
\]

The geometry of the target space is described in terms of the supervielbeins \( \hat{E}^A = E^A(\hat{Z}) \), the superconnections \( \hat{\Omega}^A_B = \Omega^B_A(\hat{Z}) \), \( \hat{A} = A(\hat{Z}) \) and the two-superform \( \hat{B} = B(\hat{Z}) \). The SUGRA-SYM constraints (2.6) are imposed as w.s. superfields. The intrinsic components of the pull back of \( \hat{E}^A \) on the superworld sheet are \( \hat{E}^A_{\pm}, \hat{E}^A_q \) and we shall write

\[
(\hat{E}^A_{\pm}, \hat{E}^A_q)|_{\eta^{(q)}=0} = (E^A_{\pm}, \lambda^A_q)
\]

One should notice that \( \lambda^A_q \) are commuting spinors (twistors).

In order to formulate twistor–string models with \((n, 0)\)–extended, w.s. supersymmetry, two key ingredients are needed. The first ingredient is to impose a suitable constraint to enforce the fundamental twistor relation \( E^a_\pm = (\lambda^a \lambda) \). Of course this constraint must be written as a superfield equation in order to preserve the w.s. supersymmetry. It turns out that the right constraint is

\[
\hat{E}^a_q = 0 \quad (q = 1, \ldots, n)
\]

This constraint means that the embedding of the superworld sheet in the target superspace is such that the odd part of the tangent superworld sheet lies entirely within the odd part of the tangent target superspace. Eq. (1), developed in power of \( \eta \), gives the component constraints:
\[ \lambda_q^a = 0 \] (4.2)

\[ (\lambda_p \Gamma^a \lambda_q) = \delta_{pq} E^a_\alpha \] (4.3)

together with further constraints, irrelevant here.

If \( n = D - 2 \), eq. (3) is just eq. (3.3). Moreover, since here the structure group is \( S^\uparrow(1, 1) \otimes SO(D - 2) \), \( \lambda_q^a \) transform under gauge transformations that belong to this group. Then, by assuming that the rectangular matrix \( \lambda_q^a \) has highest rank, we meet the situation described at the end of section 3: the twistors \( \lambda_q^a \) parametrize the sphere \( S^{D-2} \) and therefore describe the light-like vector \( E^a_\alpha \) with \( E^0_\alpha > 0 \).

For models with \( n = 1 \), eq. (1) is sufficient to write the action in a manifestly supersymmetric form i.e. as a full superspace integral. On the contrary if \( n > 1 \), the action term \( I^{(B)} \) that involves the two–form \( B \) cannot be written naively as a full superspace integral without introducing new auxiliary superfields.

The second ingredient needed to write a supersymmetric action \( I^{(B)} \), if \( n > 1 \), is an interesting property of the two superform \( \hat{B} \), that holds if the twistor constraint, eq. (1), and the SUGRA–SYM constraints (the superfield version of eq. (2.6)), are satisfied. Indeed in this case there exists a modified two–superform \( \tilde{B} \)

\[ \tilde{B} = \hat{B} + \frac{1}{n} \epsilon^+ \wedge \epsilon^- \sum_q \hat{E}^A_q \hat{E}^B_q \hat{E}^C_q \hat{H}_{CAB} \] (4.4)

such that the pull–back on the superworld sheet \( \mathcal{M} \) of \( d\tilde{B} \) vanishes

\[ d\tilde{B}|_{\mathcal{M}} = 0 \] (4.5)

This fact allows to write \( I^{(B)} \) in two equivalent ways.

**I^0** way:* Due to eq. (5), under the infinitesimal super–reparametrization \( \xi^I \rightarrow \xi^I + \epsilon^I \)

\[ \delta_\epsilon \tilde{B} = (d\epsilon \tilde{B} + i_\epsilon d\tilde{B})|_{\mathcal{M}} = d\epsilon \tilde{B}|_{\mathcal{M}} \] (4.16)

so that, if \( \mathcal{M}_0 \) is the slide of \( \mathcal{M} \) at \( \eta^{(q)} = 0 = d\eta^{(q)} \), the action

* This is the approach of refs. [27], [28]. However, since the author of refs. [27], [28] was not yet aware of the Weil triviality of \( \hat{B} \), he imposed an additional, unnecessary constraint on \( \hat{B} \), to recover the w.s. supersymmetry.
is super–reparametrization invariant, even if it is not a full superspace integral. The same mechanism\cite{37}, called Weil triviality, is operative in S.Y.M. theories and permits in that context to apply the descent equation method to get the consistent chiral anomaly of these models.

**\(I^{(B)}\) way:** Eq. (5) implies that locally \(\tilde{B}|_{\mathcal{M}} = d\tilde{Q}|_{\mathcal{M}}\). Then \(I^{(B)}\) can be written as a full superspace integral simply by imposing this condition as a constraint:

\[
I^{(B)} = \int_{\mathcal{M}} \hat{P}^{IJ} (\hat{B}_{IJ} - \partial_I \hat{Q}_J) \tag{4.7}
\]

where the lagrangian multipliers \(\hat{P}^{IJ}\) are Grassmann–antisymmetric w.s. super-fields. \(I^{(B)}\) is invariant under the local gauge transformations (Bianchi gauge)

\[
\delta \hat{P}^{IJ} = \partial_K \hat{\Lambda}^{KIJ} \tag{4.8}
\]

where the gauge parameters \(\hat{\Lambda}^{KIJ}\) are Grassmann–antisymmetric in their indices. This Bianchi gauge allows to prove the equivalence between \(I^{(B)}\) and \(I^{(B)}\).

Now let us discuss more explicitly the \(D=10, (n,0)\)–extended, twistor–string models for \(n = 1, 2, 8\).

**4.1. \(D=10, N=(1,0),\) heterotic, twistor–string model\cite{22},\cite{27}\**

Here the superworld sheet has only one odd dimension and we shall write \((\hat{E}^A_{\pm}, \hat{E}^A_1)_{\eta=0} = (E^A_{\pm}, \lambda^A)\). Now the twistor constraint is

\[
\hat{E}_1^a = 0
\]

which implies

\[
\lambda^a = 0; \quad E_{-}^a = (\lambda \Gamma^a \lambda) \tag{4.9}
\]

**This is the approach of refs. \cite{21},\cite{30}, firstly proposed in a different context in ref. \cite{38}.**
As said before, the action can be written as a full superspace integral without resort to the property of Weil triviality.

The superspace action is

$$I = \int d^2 \xi \, d\eta \, sdet \, e^\left\{ \hat{P}_a \hat{E}_1^a + \hat{E}_1^A \hat{E}_1^B \hat{B}_{BA} + i\hat{\psi} D_1 \hat{\psi} \right\}$$

(4.10)

where the superfield $\hat{P}_a = P_a + \eta \beta_a$ are lagrangian multipliers, $\hat{\psi}_r = \psi_r + \eta \chi_r$ are the heterotic fermions and

$$D \hat{\psi} = e^+ D_+ \hat{\psi} + e^- D_- \hat{\psi} + e^1 D_1 \hat{\psi} = d \hat{\psi} - \hat{E}_1^B \hat{A}_B \hat{\psi}$$

is the covariant differential of $\hat{\psi}$ in presence of the background, gauge superconnection $\hat{A}$. The first term in the r.h.s. of eq. (10) imposes the twistor constraint, the second one contains the Wess-Zumino term and the last one is the heterotic action.

In components, eq. (10) becomes

$$I = \int d^2 \xi \, d\eta \, sdet \, e^{\left\{ P'_a (E_1^a - \lambda \Gamma^a \lambda) + \phi E_1^a E_1^a - E_1^a E_1^a B_{BA} + i \sum_r \psi_r D_1 \psi \right\}}$$

(4.11)

To get eq. (11) the SUGRA-SYM constraints have been used, the non propagating fields $\lambda^a$ and $\chi_r$ have been eliminated through their field equations and the shift $P_a = P'_a - \phi (\lambda \Gamma^a \lambda)$ has been performed.

The action (10) is manifestly invariant under w.s. reparametrization, Weyl, Lorentz and local $N = (1,0)$ supersymmetry. It is also invariant under the two–fold reducible, residual $\kappa$–symmetry with only 7 superfield parameters i.e. 7+7 parameters (the 8th one having been absorbed by the w.s. supersymmetry).

These $\kappa$–transformations are

$$\delta_\kappa \hat{Z}_M^a \hat{E}_M^\alpha = \hat{w}^a = (\hat{E}_1 \Gamma^a \hat{E}_1) \Gamma^\alpha_\beta \hat{K}_\beta - 2(\hat{E}_1 \hat{K}) \hat{E}_1^\alpha,$$

$$\delta_\kappa \hat{Z}_M^a \hat{E}_M^a = 0; \quad \delta_\kappa \hat{P}^a = 4i(\hat{w} \Gamma^a \hat{E}_1); \quad \delta_\kappa \hat{\psi} = \hat{w}^a \hat{A}_a \hat{\psi}$$

Moreover the action (10) is also invariant under the local symmetry ($\beta$–symmetry)

$$\delta P'^a = \beta (E_-^a + \lambda \Gamma^a \lambda); \quad \delta e_+ = -\beta e_+$$

From this symmetry one can show that the field equations from the action (11) are the same as in the G.S. approach. Indeed, by varying $\lambda^a$ one get $P'_a \Gamma^a \lambda = 0$ so that
\[ P'{}^a = \gamma(\lambda \Gamma^a \lambda) = \frac{1}{2} \gamma(E^a + \lambda \Gamma^a \lambda) \]

and one can choose \( \gamma = 0 \) by gauge fixing the \( \beta \)-symmetry. This (1,0)–model does not seem suitable for quantization, or, to say better, the natural quantum version of this model is the...N.S.R. string. To be more explicit, let us consider the model in flat target space and let us impose the following, non covariant gauge fixing for the (super) \( \kappa \)-symmetry:

\[ \theta^\alpha - Q^a \Gamma_a^{\alpha\beta} N_\beta = 0 \]  
\[ \lambda^\alpha - \Lambda^a \Gamma_a^{\alpha\beta} N_\beta = 0 \]

where \( N_\alpha \) is a constant, \( D = 10, W\)–M spinor and the new fields \( \Lambda^a, Q^a \) are restricted by the conditions \( \Lambda^a \Lambda_a = 0 = \Lambda^a Q_a \). Then \( \Lambda^a \) and \( Q^a \) have 9 components each so that eqs. (12), (13) correspond to 7+7 conditions. If this gauge is imposed partially to fix only the superpartners of \( K_\alpha \) and one integrate over \( \lambda^\alpha, \Lambda^a \) and lagrangian multipliers, one gets formally the G.S. string action.

But now let us impose the full gauge fixing (12), (13) then integrate over \( \theta^\alpha, \lambda^\alpha, \Lambda^a \) and Lagrangian multipliers and perform the change of variables [29]

\[ \bar{X}^a = X^a + i Q^a (n \cdot Q) \]
\[ \bar{\phi}^a = 2 \sqrt{n \cdot E_-} \left[ Q^a + \frac{1}{2} \frac{(n \cdot Q)(Q \cdot D_- Q)}{(n \cdot E_-)} \right] \]

where \( n^a = (N \Gamma^aN) \). Then from eq. (10) one recovers the action for the N.S.R. string. Two comments are in order:

i) Since here the \( \kappa \)-symmetry is (two-fold) reducible, one must fix also the secondary gauges. If this is done properly, the factors \( n \cdot E_- \) that arise from the path integrations cancel exactly.

ii) Since from eq. (9), \( E_-^0 > 0 \), the G.S. and N.S.R. models obtained through the procedure outlined above, have their phase spaces halved with respect to those of the usual formulations. For further details see ref. [29].
4.2. D=10, N=(2,0), heterotic, twistor–string model [27],[31],[32]

Since now the structure group is $SO^1(1,1) \otimes SO(2)$ and $SO(2)$ is isomorphic to $U(1)$, one can choose a single complex, grassmann parameter $\eta$ (with its conjugate $\bar{\eta}$) as coordinate of the odd dimensions of the superworld sheet. As for the w.s. pull back of the target supervielbeins $\hat{E}^A$, we shall write

$$(\hat{E}^A_\pm, \hat{E}^A_1, \hat{E}^A_1)_{\eta=0} = (E^A_\pm, \lambda^A, \bar{\lambda}^A).$$

$\lambda^a$ are complex, commuting, t.s. Weyl–Majorana spinors. Now the twistor constraint, in superfield language, is

$$\hat{E}^a_1 = 0 = \hat{E}^a_1$$

In components, it yields

$$\lambda^a = 0 = \bar{\lambda}^a$$

$$(\lambda \Gamma^a \lambda) = 0 = (\bar{\lambda} \Gamma^a \bar{\lambda}) \quad (4.14)$$

$$E^a_- = (\lambda \Gamma^a \bar{\lambda}) \quad (4.15)$$

A complex, commuting spinor $\lambda^a$ in D=10 that satisfies eq. (4.14) is called a pure spinor. It is equivalent to a couple of twistors: $\lambda^a = \lambda^a_1 + i\lambda^a_2$ such that

$$(\lambda_1, \Gamma^a \lambda_1) = (\lambda_2 \Gamma^a \lambda_2) \quad ; \quad (\lambda_1 \Gamma^a \lambda_2) = 0$$

For a pure spinor, $(\lambda \Gamma^a \bar{\lambda})$ is a light–like vector and eq. (15) implies the Virasoro condition.

As in the $N = (1,0)$ case, the action consists of 3 terms:

$$I = I^{(C)} + I^{(B)} + I^{(h)}.$$

$I^{(C)}$ imposes the twistor constraint. It is:

$$I_c = \int d^2 \xi d\eta d\bar{\eta} \left\{ \hat{P}_a \hat{E}^a_1 - \hat{P}_a \hat{E}^a_1 \right\} \quad (4.16)$$
where the lagrangian multipliers $\hat{P}_a$ are complex, w.s. superfields.

$I^{(B)}$ is the action term which involves the 2-superform $B$.

According to our previous discussion, it can be written as

$$I^{(B)} = \int_{\mathcal{M}_0} \tilde{B} \tag{4.17}$$

where $\mathcal{M}_0$ is the slide of $\mathcal{M}$ at $\eta = 0 = d\eta$ and

$$\tilde{B} = \hat{B} + e^+ \wedge e^- \hat{E}_1 \hat{E}_1^B E_+^C \hat{H}_{CBA} = \hat{B} + e^+ e^- \phi E_a^a E_{+a}$$

where the twistor and SUGRA–SYM constraints have been used.

As a consequence of the Weil triviality of $\tilde{B}$, $I^{(B)}$ is super–reparametrization invariant, even if it is not a full superspace integral. Moreover

$$I^{(B)} = \int_{\mathcal{M}_0} [e^+ \wedge e^- \phi E_a^a E_{+a} + B] \tag{4.18}$$

coincides with the G.S. action (without the heterotic fermions).

$I^{(h)}$ is the heterotic action. Consider a set of 16, complex, covariantly chiral fermions $\hat{\psi}_r (r = 1, \ldots, 16)$:

$$\mathcal{D}_1 \hat{\psi}_r = 0 = \bar{\mathcal{D}}_1 \hat{\psi}_r$$

Then the heterotic action is

$$I^{(h)} = i \int d^2 \xi d\eta d\tilde{\eta} \sum_r \hat{\psi}_r \hat{\psi}_r = i \int d^2 \xi \sum_r (\bar{\psi}_r D_+ \psi_r + \bar{\chi}_r \chi_r)$$

where $\chi_r = \mathcal{D}_1 \hat{\psi}_r \big|_{\eta=0}$

Now the $\kappa$–symmetry has six superparameters since two of the eight parameters in the G.S. approach, here are absorbed by the $N = (2, 0)$ supersymmetry.

As shown in [31],[32] the $N = (2, 0)$ model is well suited to be quantized and provides a consistent and useful, semicovariant quantization scheme for the G.S. (heterotic) superstring. $\kappa$–symmetry can be gauge fixed by killing six components of $\hat{\theta}^a$. More precisely, a spinor of $SO(1, 9)$ can be decomposed under $U(4)$ as follows
and $\kappa$–symmetry is gauge fixed by gauging to zero the $U(4)$ irrep. $6$ of $\hat{\theta}^\alpha$. $SO(1,9)$ is broken to $U(4)$ but the $N=(2,0)$ supersymmetry is preserved and can be fixed in superconformal gauge.

The relevant point is that both the left and the right conformal anomaly vanishes in this gauge. Let us recall that for a conformal field with left (right) conformal weight $j_L (j_R)$ the left (right) conformal anomaly is proportional to $\eta_L(n_R)$ where $n = 6j^2 - 6j + 1$. For our model in this gauge, one has:

| Field content                                      | $n_R$ : | $n_L$ : |
|---------------------------------------------------|---------|---------|
| 4 chiral and 4 antichiral bosonic superfields      | 4 x 3   | 4 x 2   |
| 2 chiral and 2 antichiral fermionic superfields    | -2 x 3  | +2      |
| 16 complex heterotic fermions                      | 0       | +16     |
| 1 reparametrization ghost                          | -26     | -26     |
| 2 supersymmetry ghosts                             | +2 x 11 | 0       |
| 1 vector ghost                                     | -2 = 0  | 0 =     |
|                                                   | 0       | 0       |

To appreciate this result one should notice that for the G.S. heterotic string in semilight cone gauge [39], a simple calculation of the right conformal anomaly, gives $n_R = 10 + 4 - 26 = -12$. This “anomaly” turns out to be a trivial cocycle [40] in this gauge so that it can be avoided perturbatively by subtracting suitable counterterms. Therefore it does not implies an inconsistency and no problems arise in $\sigma$–model perturbative calculations. However, due to this cocycle, higher stringly loop (i.e. higher genus) calculations become impraticable. On the contrary, since in the Berkovits quantization scheme of the $N=(2,0)$ model the conformal anomaly vanishes, higher genus calculations of string amplitudes become possible and in fact have been done successfully [32].

4.3. $D=10$, $N=(8,0)$, heterotic, twistor–string model [28],[30]

As in the previous cases, the action consists of 3 terms:

$$I = I^{(C)} + I^{(B)} + I^{(h)}$$
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$I^{(C)}$ impose the twistor constraint

\[ \hat{E}_q^a = 0 \quad q = 1, ... 8 \quad (4.19) \]

and is given by

\[ I^{(C)} = \int d^2 \xi d^8 \eta \sum_{q=1}^{8} \hat{P}_a^q \hat{E}_q^a \quad (4.20) \]

In components, the twistor constraint yields

\[ \lambda_q^a = 0 \quad (4.21) \]

\[ (\lambda_p \Gamma^a \lambda_q) = \delta_{pq} E^a_- \quad (4.22) \]

together with further constraints, irrelevant here. As for $I^{(B)}$, it can be written in two equivalent ways in terms of the modified two–superform [30]

\[ \hat{B} = \hat{B} + e^+ \wedge e^- \frac{1}{8} \sum_q \hat{E}_q^A \hat{E}_q^B \hat{E}_q^C \hat{H}_{CBA} \]

that, under the twistor and SUGRA–SYM constraints, satisfies eq. (5). They are: [28], [30]

\[ I^{(B)} = \int_{\mathcal{M}_0} d\hat{B}, \quad (4.23) \]

where $\mathcal{M}_0$ is the slide of $\mathcal{M}$ at $\eta^{(q)} = 0 = d\eta^{(q)}$, or

\[ I'^{(B)} = \int d^2 \xi d^8 \eta \hat{P}^{IJ} \left( \hat{B}_{IJ} - \partial_I \hat{Q}_J \right), \quad (4.24) \]

where $\hat{P}^{IJ}$ are Grassmann–antisymmetric, w.s. superfields.

Under the twistor and SUGRA–SYM constraints, in both cases one recover from eqs. (20) and (23) (or (24)) the G.S. action (without heterotic fermions). Here the relevant point is that the action (20) is invariant under a set of local abelian transformations that involve $\hat{P}_a^q$ [30]. They are similar to the Bianchi gauge transformations,
eq. (8), and generalize the $\beta$–symmetry which is present in the $(1,0)$–model. From the Bianchi gauge transformations of $P^{IJ}$ one can reduces eq. (25) to eq. (24) and from these involving $P^a$ one can show that the lagrangian multipliers do not alter the field equations of the G.S. approach. The same remark applies to the $N = (2,0)$ case.

The problem is with the heterotic action $I^{(h)}$: unfortunately, up to now, nobody has succeeded to write an $N$–extended w.s. supersymmetric version of $I^{(h)}$ for $N > (2,0)$ (in the proposal of ref. [28], half of the heterotic fermions have negative norm).

5. Conclusions

We have show that the introduction of commuting spinors (twistors) leads to new formulations of superparticles and heterotic string models that exhibit $n$–extended world sheet supersymmetry as well as target space supersymmetry.

The twistor approach has clarified the geometrical meaning of $\kappa$–symmetry, which appears as (extended) world sheet supersymmetry in disguise. Moreover it seems useful to clarify to some extent, the relation between G.S. and N.S.R. formulations. At the quantum level, the $N = (2,0)$–model yields a new promising method to calculate G.S. superstring amplitudes at higher stringly loops.

There are, of course, open problems.

A consistent treatment of the heterotic fermions for $N > (2,0)$ is still lacking. Moreover it would be interesting to extend these twistor–like formulations to supermembranes and non–heterotic superstrings. In this direction work is in progress.

Finally the big problem is to find new consistent quantization schemes for $N > (2,0)$ models, in particular for the $D = 10, N = (8,0)$ model.

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