VARIETIES OF BICOMMUTATIVE ALGEBRAS

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ABSTRACT. Bicommutative algebras are nonassociative algebras satisfying the polynomial identities of right- and left-commutativity \((x_1x_2)x_3 = (x_1x_3)x_2\) and \(x_1(x_2x_3) = x_2(x_1x_3)\). Let \(\mathfrak{B}\) be the variety of all bicommutative algebras over a field \(K\) of characteristic 0 and let \(F(\mathfrak{B})\) be the free algebra of countable rank in \(\mathfrak{B}\). We prove that if \(V\) is a subvariety of \(\mathfrak{B}\) satisfying a polynomial identity \(f = 0\) of degree \(k\), where \(0 \neq f \in F(\mathfrak{B})\), then the codimension sequence \(c_n(V), n = 1, 2, \ldots\), is bounded by a polynomial in \(n\) of degree \(k - 1\). Since \(c_n(\mathfrak{B}) = 2^n - 2\) for \(n \geq 2\), and \(\exp(\mathfrak{B}) = 2\), this gives that \(\exp(V) \leq 1\), i.e., \(\mathfrak{B}\) is minimal with respect to the codimension growth. Up to isomorphism there are three one-generated two-dimensional bicommutative algebras \(A\) which are nonassociative and with the property \(\dim A^2 = 1\). We present bases of their polynomial identities and show that one of these algebras generates the whole variety \(\mathfrak{B}\).

1. Introduction

Bicommutative algebras are nonassociative algebras over a field \(K\) satisfying the polynomial identities of right- and left-commutativity

\[(x_1x_2)x_3 = (x_1x_3)x_2, \quad x_1(x_2x_3) = x_2(x_1x_3).\]

In the sequel we consider algebras over a field \(K\) of characteristic 0 only. One-sided commutative algebras appeared first in the paper by Cayley [5] in 1857. In the modern language this is the right-symmetric Witt algebra \(W_1^{\text{rsym}}\) in one variable. Maybe the most important examples of one-side commutative algebras are Novikov algebras which are left-commutative and right-symmetric. The latter means that the algebras satisfy the polynomial identity \((x_1,x_2,x_3) = (x_1,x_3,x_2)\) for the associator \((x_1,x_2,x_3) = (x_1x_2)x_3 - x_1(x_2x_3)\). The motivation to study Novikov algebras comes from the needs of the Hamiltonian operator in mechanics and the equations of hydrodynamics, see [11] and [10] for details. Examples of bicommutative algebras are the two-dimensional algebras \(A_{\pi,\varrho}, \pi, \varrho \in K\), generated by an element \(r\) and with multiplication rules

\[(2) \quad r \cdot r^2 = \pi r^2, \quad r^2 \cdot r = \varrho r^2, \quad r^2 \cdot r^2 = \pi \varrho r^2.\]

In is easy to see that up to isomorphism the algebras \(A_{\pi,\varrho}\) coincide with one of the algebras \(A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}\), and \(A_{1,-1}\). The first one is nilpotent of class 3, the second one is associative-commutative, and the latter three algebras are nonassociative.

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The structure of the free bicommutative algebra and the most important numerical invariants of the T-ideal of the polynomial identities were described by Dzhumadil’daev, Ismailov, and Tulenbaev [11], see also the announcement [12]. In [10], jointly with Zhakhayev, we proved that finitely generated bicommutative algebras are weakly noetherian, i.e., satisfy the ascending chain condition for two-sided ideals, and answer into affirmative the finite basis problem for varieties of bicommutative algebras over a field of arbitrary characteristic.

One of the most important measures for the complexity of the polynomial identities of a variety $\mathfrak{V}$ of $K$-algebras is the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \ldots$, where $c_n(\mathfrak{V})$ is the dimension of the multilinear polynomials of degree $n$ in the free algebra $F_n(\mathfrak{V})$ of rank $n$. As a first approximation to the more precise estimate of the growth of the codimensions one studies the behaviour of $\sqrt[n]{c_n(\mathfrak{V})}$. In the special case when

$$\exp(\mathfrak{V}) = \lim_{n \to \infty} \sqrt[n]{c_n(\mathfrak{V})}$$

exists it is called the exponent of $\mathfrak{V}$, see Giambruno and Zaicev [14, 15] who proved that for associative PI-algebras the exponent always exists and is an integer. Following [16] the variety $\mathfrak{V}$ is minimal of a given exponent if $\exp(\mathfrak{W}) < \exp(\mathfrak{V})$ for all proper subvarieties $\mathfrak{W}$ of $\mathfrak{V}$. (In $[8]$ such varieties were called extremal.)

It was shown in [11] that for the variety $\mathfrak{B}$ of all bicommutative algebras

$$c_1(\mathfrak{B}) = 1 \text{ and } c_n(\mathfrak{B}) = 2^n - 2, \quad n = 2, 3, \ldots.$$ 

Hence $\exp(\mathfrak{B}) = 2$. Our first main result is that the variety $\mathfrak{B}$ is minimal of exponent 2. More precisely we show that if $\mathfrak{V}$ is a subvariety of $\mathfrak{B}$ satisfying a polynomial identity $f = 0$ of degree $k$, where $0 \neq f \in F(\mathfrak{B}) = F_\infty(\mathfrak{B})$, then the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \ldots$, is bounded by a polynomial in $n$ of degree $k - 1$. The results of [11] give that the variety $\mathfrak{B}$ is generated by the free algebra $F_2(\mathfrak{B})$ of rank 2. As a consequence we slightly improve this and show that $\mathfrak{B}$ is generated by the free algebra $F_1(\mathfrak{B})$ of rank 1. As a byproduct of our approach, starting with the basis of $F(\mathfrak{B})$ in [11] we give a new proof of the description of the cocharacter sequence $\chi_n(\mathfrak{B})$, $n = 1, 2, \ldots$. Finally we study the polynomial identities of the two-dimensional algebras $A_{n,0}$ with multiplication defined by $\mathfrak{B}$.

We show that the algebra $A_{3,-1}$ generates the whole variety $\mathfrak{B}$. The varieties $\text{var}(A_{0,1})$ and $\text{var}(A_{1,0})$ generated by the algebras $A_{0,1}$ and $A_{1,0}$ are defined as subvarieties of $\mathfrak{B}$ by the polynomial identities $x_1(x_2x_3) = 0$ and $(x_1x_2)x_3 = 0$, i.e., they are equal, respectively, to the varieties of left-nilpotent and right-nilpotent of class 3 bicommutative algebras.

2. Preliminaries

We fix a field $K$ of characteristic 0. All algebras, vector spaces, and tensor products will be over $K$. Traditionally, one states the results on polynomial identities and cocharacter sequences in the language of representation theory of the symmetric group $S_n$. Instead, we shall work with representation theory of the general linear group $GL_d = GL_d(K)$. Then using the approach developed by Berele [4] and the author [7] we shall translate easily the results in terms of representations of $S_n$. We start with the necessary background on representation theory of $GL_d$ acting canonically on the $d$-dimensional vector space $KX_d$ with basis $X_d = \{x_1, \ldots, x_d\}$. We refer, e.g., to [29] for general facts and to [9] for applications in the spirit of the problems considered here. All $GL_d$-modules which appear in
this paper are completely reducible and are direct sums of irreducible polynomial modules. The irreducible polynomial representations of $GL_d$ are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$. We denote by $W(\lambda) = W_d(\lambda)$ the corresponding irreducible $GL_d$-module. The action of $GL_d$ on $KX_d$ is extended diagonally on the tensor algebra of $KX_d$ and, up to isomorphism, all $W(\lambda)$ can be found there. The tensor algebra of $KX_d$ is isomorphic, also as a $GL_d$-module, to the free associative algebra $K\langle X_d \rangle = K\langle x_1, \ldots, x_d \rangle$. Since the diagonal action of $GL_d$ on the tensor algebra is not affected on the parentheses, we may work also in the absolutely free algebra $K\{ X_d \}$ and in the relatively free algebra $F_d(\mathfrak{V})$ of any variety $\mathfrak{V}$.

The module $W(\lambda) \subset K\{ X_d \}$ is generated by a unique, up to a multiplicative constant, multihomogeneous element $w_\lambda$ of degree $\lambda = (\lambda_1, \ldots, \lambda_d)$, i.e., homogeneous of degree $\lambda_k$ with respect to each variable $x_k$, called the highest weight vector of $W(\lambda)$. In order to state the characterization of the highest weight vectors we recall that for an algebra $R$ the linear operator $\delta : R \to R$ is a derivation of $KX_d$ and, up to isomorphism, all $W(\lambda)$ can be found there. The tensor algebra of $KX_d$ is isomorphic, also as a $GL_d$-module, to the free associative algebra $K\langle X_d \rangle = K\langle x_1, \ldots, x_d \rangle$. Since the diagonal action of $GL_d$ on the tensor algebra is not affected on the parentheses, we may work also in the absolutely free algebra $K\{ X_d \}$ and in the relatively free algebra $F_d(\mathfrak{V})$ of any variety $\mathfrak{V}$.

Lemma 2.1. (see, e.g., [3]) Let $1 \leq i < j \leq d$ and let $\Delta_{x_i \to x_j}$ be the derivation of $K\{ X_d \}$ defined by $\Delta_{x_i \to x_j} (x_j) = x_i$, $\Delta_{x_i \to x_j} (x_k) = 0$, $k \neq j$. If $w(X_d) = w(x_1, \ldots, x_d) \in K\{ X_d \}$ is multihomogeneous of degree $\lambda_k$ with respect to $x_k$, then $w(X_d)$ is a highest weight vector for some $W(\lambda)$ if and only if $\Delta_{x_i \to x_j} (w(X_d)) = 0$ for all $i < j$. Equivalently, $w(X_d)$ is a highest weight vector for $W(\lambda)$ if and only if

$$g_{ij}(w(X_d)) = w(X_d), \quad 1 \leq i < j \leq d,$$

where $g_{ij}$ is the linear operator of the $KX_d$ which sends $x_j$ to $x_i + x_j$ and fixes the other $x_k$.

If $W_i$, $i = 1, \ldots, m$, are $m$ isomorphic copies of the $GL_d$-module $W(\lambda)$ and $w_i \in W_i$ are highest weight vectors, then the highest weight vector of any submodule $W(\lambda)$ of the direct sum $W_1 \oplus \cdots \oplus W_m$ has the form $\xi_1 w_1 + \cdots + \xi_m w_m$ for some $\xi_i \in K$. Any $m$ linearly independent highest weight vectors can serve as a set of generators of the $GL_d$-module $W_1 \oplus \cdots \oplus W_m$. The algebra $F_d(\mathfrak{V})$ decomposes as a $GL_d$-module as

$$F_d(\mathfrak{V}) = \bigoplus_\lambda m_\lambda(\mathfrak{V}) W(\lambda),$$

where the summation runs on all partitions $\lambda$ in not more than $d$ parts and the nonnegative integer $m_\lambda(\mathfrak{V})$ is the multiplicity of $W(\lambda)$ in the decomposition of $F_d(\mathfrak{V})$. The canonical multigrading of $F_d(\mathfrak{V})$ which counts the degree of each variable in $X_d$ agrees with the action of $GL_d$ in the following way. Let

$$H(F_d(\mathfrak{V}), T_d) = H(F_d(\mathfrak{V}), t_1, \ldots, t_d)$$

$$= \sum_{n_i \geq 0} \dim F_d^{(n)}(\mathfrak{V}) T_d^n = \sum_{n_i \geq 0} \dim F_d^{(n)}(\mathfrak{V}) t_1^{n_1} \cdots t_d^{n_d}$$
be the Hilbert series of \( F_d(\mathfrak{g}) \) as a multigraded vector space, where \( F_{d}^{(n)}(\mathfrak{g}) \) is the multihomogeneous component of \( F_d(\mathfrak{g}) \) of degree \( n = (n_1, \ldots, n_d) \). Then

\[
H(F_d(\mathfrak{g}), T_d) = \sum_{\lambda} m_{\lambda}(\mathfrak{g}) s_{\lambda}(T_d) = \sum_{\lambda} m_{\lambda}(\mathfrak{g}) s_{\lambda}(t_1, \ldots, t_d),
\]

where \( s_{\lambda}(T_d) \) is the Schur function corresponding to the partition \( \lambda \).

There is another group action which is important for the theory of algebras with polynomial identities. The symmetric group \( S_n \) acts on the vector space \( P_n(\mathfrak{g}) \) of the multilinear polynomials of degree \( n \) in \( F_n(\mathfrak{g}) \) by

\[
\sigma(f(x_1, \ldots, x_n)) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n, f \in P_n(\mathfrak{g}).
\]

The \( S_n \)-character of \( P_n(\mathfrak{g}) \) is called the \( S_n \)-cocharacter of \( \mathfrak{g} \). It is known that the decomposition of the \( n \)-th cocharacter

\[
\chi_n(\mathfrak{g}) = \sum_{\lambda \vdash n} m_{\lambda}(\mathfrak{g}) \chi_{\lambda},
\]

where \( \chi_{\lambda} \) is the irreducible \( S_n \)-character indexed with the partition \( \lambda \) of \( n \), is determined by the Hilbert series of \( F_n(\mathfrak{g}) \). The multiplicities \( m_{\lambda}(\mathfrak{g}) \) are the same for \( F_n(\mathfrak{g}) \) in \([3]\) and for \( \chi_n(\mathfrak{g}) \) in \([4]\). Finally, we recall a special case of the Young rule (and of the Littlewood-Richardson rule) for the product of two Schur functions \( s_{(p)}(T_d) \) and \( s_{(q)}(T_d) \) (and also for the tensor product \( W(p) \otimes W(q) \) of the \( GL_d \)-modules \( W(p) \) and \( W(q) \)). We assume that \( p \geq q \). The case \( p < q \) is similar.

\[
s_{(p)}(T_d)s_{(q)}(T_d) = \sum_{k=0}^{q} s_{(p+q-k,k)}(T_d),
\]

\[
W(p) \otimes W(q) \cong \bigoplus_{k=0}^{q} W(p + q - k, k).
\]

We shall need also estimates for the degree of the irreducible \( S_n \)-characters.

**Lemma 2.2.** The degree \( d_{\lambda} \) of the irreducible \( S_n \)-character \( \chi_{\lambda}, \lambda = (\lambda_1, \lambda_2) \vdash n \), is a polynomial in \( n \) of degree \( \lambda_2 \).

**Proof.** By the hook formula

\[
d_{\lambda} = \frac{n!}{\prod h_{ij}},
\]

where \( h_{ij} \) is the length of the hook at the \((i, j)\)-position of the Young diagram of \( \lambda \). For \( \lambda = (\lambda_1, \lambda_2) \vdash n \) the lengths of the hooks of the first row are equal, reading them from right to left, to

\[
1, 2, \ldots, n - 2\lambda_2, n - 2\lambda_2 + 2, \ldots, n - \lambda_2 + 1
\]

and those of the second row are \( 1, 2, \ldots, \lambda_2 \). Hence

\[
d_{\lambda} = \frac{n(n - 1) \cdots (n - \lambda_2 + 1)(n - 2\lambda_2 + 1)}{\lambda_2!},
\]

which is a polynomial of degree \( \lambda_2 \) in \( n \). \( \square \)

Let \( \mathfrak{B} \) be the variety of all bicommutative algebras. We assume that the free bicommutative algebras \( F = F(\mathfrak{g}) \) and \( F_d = F_{d}(\mathfrak{g}) \) are freely generated, respectively, by the sets \( X = \{x_1, x_2, \ldots \} \) and \( X_d = \{x_1, \ldots, x_d \} \). By \([11]\) the basis
of the square $F_d^2$ of the algebra $F_d$ as a $K$-vector space consists of the following polynomials:

$$u_{i,j} = x_{i_1} \cdots (x_{i_{p-1}} ((\cdots ((x_{i_p} x_{j_1}) x_{j_2}) \cdots) x_{j_q})) \cdots,$$

where $p, q \geq 1$, $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq i_p \leq d$, $1 \leq j_1 \leq \cdots \leq j_q \leq d$. For any permutations $\sigma \in S_p$ and $\tau \in S_q$ the element $u_{i,j}$ from (6) satisfy the equality

$$u_{i,j} = x_{i_{\sigma(1)}} \cdots (x_{i_{\sigma(p-1)}} ((\cdots ((x_{i_{\sigma(p)}} x_{j_{\tau(1)}}) x_{j_{\tau(2)}}) \cdots) x_{j_{\tau(q)}})) \cdots.$$

The properties and the multiplication rules of $F_d(\mathcal{B})$ from [11] are restated in [10] in the following way. We consider the polynomial algebra $K[Y_d, Z_d] = K[y_1, \ldots, y_d, z_1, \ldots, z_d]$ in $2d$ commuting and associative variables. We associate to each monomial $u_{i,j}$ in (3) the monomial

$$\psi(u_{i,j}) = y_{i_1} \cdots y_{i_p} z_{j_1} z_{j_2} \cdots z_{j_q} \in K[Y_d, Z_d]$$

and extend $\psi$ by linearity to a linear map $\psi : F_d^2 \to K[Y_d, Z_d]$. The image $\psi(F_d^2)$ is spanned by all monomials

$$Y_\alpha^\beta Z_d^\gamma = y_1^{\alpha_1} \cdots y_d^{\alpha_d} z_1^{\beta_1} \cdots z_d^{\beta_d}, \quad |\alpha| = \sum_{i=1}^d \alpha_i > 0, |\beta| = \sum_{j=1}^d \beta_j > 0.$$

Then we define an algebra $G_d$ generated by $X_d$ with basis

$$X_d \cup \{ Y_\alpha^\beta Z_d^\gamma | |\alpha|, |\beta| > 0 \}$$

and multiplication rules

$$x_i x_j = y_i z_j,$$

$$x_i (Y_\alpha^\beta Z_d^\gamma) = y_i Y_\alpha^\beta Z_d^\gamma,$$

$$(Y_\alpha^\beta Z_d^\gamma) x_j = Y_\alpha^\beta Z_d^\gamma z_j,$$

$$(Y_\alpha^\beta Z_d^\gamma)(Y_\gamma^\delta Z_d^\epsilon) = Y_\alpha^{\alpha+\gamma} Z_d^{\beta+\delta}.$$

The algebras $F_d$ and $G_d$ are isomorphic both as algebras and as multigraded vector spaces with isomorphism $\psi : F_d \to G_d$ which sends $x_i \in F_d$ to $x_i \in G_d$ and acts on $F_d^2$ in the same way as the linear map $\psi : F_d^2 \to K[Y_d, Z_d]$ defined above.

### 3. Free Bicommutative Algebras

In this section we give an alternative proof of the formula for the cocharacter sequence of $\mathcal{B}$ given in [11] and describe the highest weight vectors of the irreducible $GL_d$-submodules of $F_d = F_d(\mathcal{B})$.

**Lemma 3.1.** As a multigraded vector space the square $F_d^2$ of the free bicommutative algebra $F_d$ is isomorphic to the tensor product $\omega(K[Y_d]) \otimes \omega(K[Z_d])$, where $\omega$ is the augmentation ideal of the polynomial algebra, i.e., the ideal of polynomials without constant term. As a $GL_d$-module $F_d^2$ is isomorphic to the direct sum of tensor products

$$\bigoplus_{p,q \geq 1} W(p) \otimes W(q).$$
Proof. We identify the monomial $Y_d^{\alpha}Z_d^\beta \in K[Y_d, Z_d]$ with $Y_d^{\alpha} \otimes Z_d^\beta \in K[Y_d] \otimes K[Z_d]$. Then the first part of the lemma is simply a restatement of the fact that the image of $E_d^2$ under the action of $\psi$ has a basis $\{Y_d^{\alpha}Z_d^\beta \mid |\alpha|, |\beta| > 0\}$. The second part of the lemma holds because the $GL_d$-module $K[Y_d]^{(p)}$ of the homogeneous polynomials of degree $p$ in $K[Y_d]$ is isomorphic to $W(p)$ and similarly for $K[Z_d]^{(q)}$. □

**Proposition 3.2.**\[11\] The cocharacter sequence of the variety $\mathcal{B}$ of all bicommutative algebras is

$$
\chi_n(\mathcal{B}) = \sum_{(\lambda_1, \lambda_2) \vdash n} m_{(\lambda_1, \lambda_2)}(\mathcal{B}) \chi_{(\lambda_1, \lambda_2)},
$$

where

$$
m_{(1)}(\mathcal{B}) = 1,
$$

$$
m_{(n)}(\mathcal{B}) = n - 1, \quad n > 1,
$$

$$
m_{(\lambda_1, \lambda_2)}(\mathcal{B}) = n - 2\lambda_2 + 1, \quad \lambda_2 > 0.
$$

**Proof.** The multiplicities of the irreducible $S_n$-characters in the cocharacter sequence \[11\] and of the irreducible $GL_d$-modules of the homogeneous component $E_d^{(n)}$ of degree $n$ of the free algebra $F_d$ in \[3\] are the same for $d \geq n$. Hence we may work in $F_d$ instead of with $\chi_n(\mathcal{B})$. Since the case $n = 1$ is trivial, we shall assume that $n > 1$. By Lemma \[3.1\] and the Young rule \[5\], we derive that the only nontrivial multiplicities $m_{\lambda}(\mathcal{B})$ are for $\lambda = (\lambda_1, \lambda_2)$. Then $m_{\lambda}(\mathcal{B})$ is equal to the number of tensor products $W(p) \otimes W(q)$ which contain an isomorphic copy of $W(\lambda)$ as a submodule. For $\lambda = (n)$ there are $n - 1$ possibilities

$$W(1) \otimes W(n - 1), W(2) \otimes W(n - 2), \ldots, W(n - 1) \otimes W(1),$$

i.e., $m_{(n)}(\mathcal{B}) = n - 1$. For $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_2 > 0$ the possibilities are

$$W(\lambda_2) \otimes W(n - \lambda_2), W(\lambda_2 + 1) \otimes W(n - \lambda_2 - 1), \ldots, W(n - \lambda_2) \otimes W(\lambda_2),$$

which gives $m_{(\lambda_1, \lambda_2)}(\mathcal{B}) = n - 2\lambda_2 + 1$. □

The action of $GL_d$ on the $d$-dimensional vector space $KX_d$ induces a similar action on $KY_d$ and $KZ_d$ which is extended diagonally on the square $G_d^2$ of the algebra $G_d$.

**Lemma 3.3.** The following polynomials $w^{(k)}_\alpha$ form a maximal linearly independent system of highest weight vectors of the $GL_d$-submodules $W(\lambda)$ in $G_d^2$:

$$w^{(j)}_{(n)} = y_1^n z_1^{n-j}, \quad j = 1, 2, \ldots, n - 1,$$

$$w^{(j)}_\lambda = y_1^{j}(y_1 z_2 - y_2 z_1)^{\lambda_2} z_1^{\lambda_1 - \lambda_2 - j}, \quad j = 0, 1, \ldots, \lambda_1 - \lambda_2, \quad \text{if } \lambda_2 > 0.$$

**Proof.** For a fixed $\lambda$ the elements \[11\] are linearly independent because are nonzero and of pairwise different degree in $y_1$. They are of degree $\lambda_1$ with respect to $y_1, z_1$ and of degree $\lambda_2$ with respect to $y_2, z_2$. By Proposition \[3.2\], the multiplicities of $W(n)$ and $W(\lambda)$, $\lambda = (\lambda_1, \lambda_2) \vdash n$, in $G_d^2$ are, respectively,

$$m_{(n)}(\mathcal{B}) = n - 1 \quad \text{and} \quad m_{(\lambda_1, \lambda_2)}(\mathcal{B}) = n - 2\lambda_2 + 1 = \lambda_1 - \lambda_2 + 1.$$

Hence their number coincides with the number of polynomials in \[11\]. Now, it is sufficient to show that all $w^{(j)}_\lambda$ are highest weight vectors. Applying Lemma \[2.1\]
this is obvious for \( w^{(j)}_{(k)} \). Let \( \lambda_2 > 0 \). The analogue \( \Delta_{y_2 \to y_1, z_2 \to z_1} \) of the derivation \( \Delta_{x_2 \to x_1} \), acting on \( K[Y_d, Z_d] \), sends \( y_1, z_1 \) to 0 and \( y_2, z_2 \) to \( y_1, z_1 \), respectively. Obviously

\[
\Delta_{y_2 \to y_1, z_2 \to z_1}(w^{(j)}_{(\lambda)}) = \lambda_2 y_1^j (y_1 z_2 - y_2 z_1)^{\lambda_2 - j} z_1^{\lambda_1 - \lambda_2 - j} \Delta_{y_2 \to y_1, z_2 \to z_1}(y_1 z_2 - y_2 z_1) = 0
\]

and all \( w^{(j)}_{(\lambda)} \) are highest weight vectors.

\[\square\]

4. Subvarieties

In this section we assume that \( \mathfrak{M} \) is a proper subvariety of \( \mathfrak{B} \) and \( \mathfrak{M} \) satisfies a nontrivial polynomial identity \( f = 0 \) of degree \( k \), where \( 0 \neq f(X_d) \in F = F(\mathfrak{B}) \)

Since the case \( k = 1 \) is trivial we shall assume that \( k \geq 2 \). In the sequel we shall work mainly in the isomorphic copies \( G \) and \( G_d \) of the algebras \( F \) and \( F_d \) instead of \( F \) and \( F_d \). Identifying \( F \) and \( F_d \) with their isomorphic copies, we shall denote the corresponding elements with the same symbols. In particular, if \( f(X_d) \in F_d^2 \) we shall write \( f(Y_d, Z_d) \in G_d^2 \) and vice versa. Since the \( GL_d \)-module generated by \( f \) contains an irreducible submodule \( W(\lambda) \), there exists a highest weight vector \( w_\lambda \) such that the polynomial identity \( w_\lambda = 0 \) follows from the polynomial identity \( f = 0 \). Hence we may assume that \( \mathfrak{M} \) satisfies some polynomial identity \( w_\lambda(x_1, x_2) = 0 \) for \( \lambda \vdash k \). Then \( w_\lambda(Y_2, Z_2) \in G_2 \) is a linear combination of the highest weight vectors in \( \mathfrak{B} \) and for some \( \xi_j \in K \)

\[
w^{(k)} = \sum_{j=1}^{k-1} \xi_j y_1^j z_1^{k-j}, \text{ for } \lambda = (k),
\]

\[
w^{(k)} = (y_1 z_2 - y_2 z_1)^{\lambda_2} \sum_{j=0}^{\lambda_1 - \lambda_2} \xi_j y_1^j z_1^{\lambda_1 - \lambda_2 - j}, \text{ for } \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0.
\]

If \( f(X_d) \in F_d \) is multihomogeneous then its partial linearization \( \text{lin}_{x_i} f(X_d) \) in \( x_i \) is the component of degree \( \deg_{x_i} \lambda - 1 \) with respect to \( x_i \) of the polynomial \( f(x_1, \ldots, x_i + x_{d+1}, \ldots, x_d) \in F_{d+1} \). If \( \Delta_{x_i \to x_{d+1}} \) is the derivation of \( F_{d+1} \) which sends \( x_i \) to \( x_{d+1} \) and the other \( x_j \) to 0, then

\[
\text{lin}_{x_i} f(X_d) = (\text{lin}_{x_i} f)(X_{d+1}) = \Delta_{x_i \to x_{d+1}}(f(X_d)).
\]

If \( u \in F \) then \( (\text{lin}_{x_i} f)(x_1, \ldots, x_d, u) \) can be expressed in terms of derivations as

\[
(\text{lin}_{x_i} f)(x_1, \ldots, x_d, u) = \Delta_{x_i \to u}(f(X_d)),
\]

where \( \Delta_{x_i \to u} \) is the derivation of \( F \) sending \( x_i \) to \( u \) and all other generators \( x_j \) to 0. The action of the analogue of \( \Delta_{x_i \to x_{d+1}} \) on \( G^2 \) is clear: It sends \( y_i \) and \( z_i \), respectively, to \( y_{d+1} \) and \( z_{d+1} \) and all other variables \( y_j \) and \( z_j \) to 0. We denote this derivation by \( \Delta_{y_i \to y_{d+1}, z_i \to z_{d+1}} \). Now we shall translate the action of \( \delta_{x_i \to u} \) on \( F^2 \), \( u \in F^2 \), in the language of \( G \) and the usual partial derivatives.

**Lemma 4.1.** Let \( u \in K[Y, Z] \) be in the image \( G^2 \subset K[Y, Z] \) of \( F^2 \). Let \( \Delta_{y_i, z_i \to u} \) be the derivation of \( K[Y, Z] \) which sends the variables \( y_i, z_i \) to \( u \) and the other variables to 0. If \( f(X_d) \in F^2 \) is multihomogeneous, then the image of \( (\text{lin}_{x_i} f)(x_1, \ldots, x_d, u) \) in \( G^2 \) is

\[
\Delta_{y_i, z_i \to u}(f) = \left( \frac{\partial f}{\partial y_i} + \frac{\partial f}{\partial z_i} \right) u.
\]
Proof. It is sufficient to consider the case when \(f\) and \(u\) are monomials and \(i = 1\):
\[
f = (y_1^\alpha z_1^\beta v_1 v_2, v_1 = y_2^\gamma z_2^\delta, v_2 = z_2^\epsilon v_2, \alpha + \beta \geq 1, |\alpha| > 0, |\beta| > 0, \\
u = Y_d^\gamma Z_d^\delta, |\gamma| > 0, |\delta| > 0.
\]
Then
\[
\Delta_{y_1 \rightarrow y_{d+1}, z_1 \rightarrow z_{d+1}}(f) = \frac{\partial f}{\partial y_1} y_{d+1} + \frac{\partial f}{\partial z_1} z_{d+1}.
\]
In virtue of (11) we may assume that the preimage \(\psi^{-1}(\frac{\partial f}{\partial y_1} y_{d+1})\) in \(F_{d+1}^2\) is of the form \(\alpha_1(\cdots(x_{d+1} x_{j_1})\cdots)\), where the dots before and after \((x_{d+1} x_{j_1})\) correspond to the beginning and the end of the element in the form (10). Since \(u \in F_d^2\) we obtain that
\[
\psi(\alpha_1(\cdots(u x_{j_1})\cdots)) = \alpha_1(\cdots(Y_d^\gamma Z_d^\delta x_{j_1})\cdots) = \frac{\partial f}{\partial y_1} u.
\]
Similarly
\[
\psi(\beta_1(\cdots(x_{i_1} u)\cdots)) = \beta_1(\cdots(y_{i_1} Y_d^\gamma Z_d^\delta)\cdots) = \frac{\partial f}{\partial z_1} u.
\]
\(\square\)

Lemma 4.2. If \(0 \neq f \in W(\lambda_1, \lambda_2) \subset F(\mathfrak{B})\), then all polynomial identities \(w_{(\mu_1, \mu_2)}^{(j)} = 0\) with \(\mu_2 \geq \lambda_1\) are consequences of the polynomial identity \(f = 0\).

Proof. As commented in the beginning of the section, we may assume that \(f = w_\lambda\) is a highest weight vector in \(W(\lambda_1, \lambda_2) \subset F_2\). Hence, working in \(G_2\) instead of in \(F_2\), \(w_\lambda\) has the form (12), i.e.,
\[
w_\lambda = \sum_{j \geq p} \xi_j w_{\lambda_1}^{(j)} = (y_1 z_2 - y_2 z_1)^{\lambda_2} \sum_{j \geq p} \xi_j y_1^{j-1} z_1^{\lambda_1 - \lambda_2 - j}, \xi \neq 0.
\]
First, let \(p > 0\), i.e., \(w_\lambda\) is divisible by \(y_1^p\). The partial linearizations of the identity \(w_\lambda = 0\) are its consequences. Hence \(\Delta_{y_1 \rightarrow y_{2}, z_1 \rightarrow z_2}(w_\lambda) = 0\) which has the form
\[
(y_1 z_2 - y_2 z_1)^{\lambda_2} \sum_{j \geq p} \xi_j (y_1^{j-1} z_1^{\lambda_1 - \lambda_2 - j} + (\lambda_1 - \lambda_2 - j) y_1^{j-1} z_1^{\lambda_1 - \lambda_2 - j-1} z_2) = 0
\]
is also a consequence of \(w_\lambda = 0\) and the same holds for
\[
w_{(\lambda_1, \lambda_2+1)} = \Delta_{y_1 \rightarrow y_{2}, z_1 \rightarrow z_2}(w_\lambda) z_1 - (\lambda_1 - \lambda_2) w_\lambda z_2
\]
\[
= -(y_1 z_2 - y_2 z_1)^{\lambda_2+1} \sum_{j \geq p-1} (j+1) \xi_j z_1^{\lambda_1 - \lambda_2 - j-1} = 0.
\]
We obtained that \(w_{(\lambda_1, \lambda_2+1)} = 0\) is a consequence of \(w_{(\lambda_1, \lambda_2)} = 0\). It is divisible by \(y_1^{p-1}\) but is not divisible by \(y_1^p\). Continuing in this way we shall reach a consequence
\[
w_{(\lambda_1, \lambda_2+p)} = (y_1 z_2 - y_2 z_1)^{\lambda_2+p} \sum_{j \geq p} \xi_j z_1^{\lambda_1 - \lambda_2 - j} = 0.
\]
Now the consequence
\[
y_1 \Delta_{y_1 \rightarrow y_{2}, z_1 \rightarrow z_2}(w_{(\lambda_1, \lambda_2+p)}) - y_2(\lambda_1 - \lambda_2 + p) w_{(\lambda_1, \lambda_2+p)} = 0
\]
is of the form \(w_{(\lambda_1, \lambda_2+p+1)} = 0\) and is divisible by \(z_1^{\lambda_1 - \lambda_2 - p-1}\) only. Continuing the process we shall obtain as a consequence
\[
w_{(\lambda_1, \lambda_1)} = (y_1 z_2 - y_2 z_1)^{\lambda_1} = w_{(\lambda_1, \lambda_1)}^{(0)}.
\]
Since all $w^{(j)}_{(\mu_1, \mu_2)}$ with $\mu_2 \geq \lambda_1$ are divisible by $w^{(0)}_{(\lambda_1, \lambda_1)}$ and hence are its consequences, we complete the proof. \hfill \Box

**Corollary 4.3.** If $0 \neq f \in F$ is of degree $k$ then all identities $w^{(j)}_{(\mu_1, \mu_2)} = 0$ with $\mu_2 \geq k$ follow from the identity $f = 0$.

**Proof.** The statement follows immediately from Lemma 4.2 because if $(\lambda_1, \lambda_2) \vdash k$, then $\lambda_1 \leq k$. \hfill \Box

**Lemma 4.4.** The polynomial identity $w^{(0)}_{(k,k)} = (y_1 z_2 - y_2 z_1)^k = 0$ has as consequences all identities

$$(y_1 z_1)^k(y_1 - z_1)^k w^{(j)}_{\mu} = 0$$

for all $\mu = (\mu_1, \mu_2)$ and all $j = 0, 1, \ldots, \mu_1 - \mu_2$.

**Proof.** We apply the derivation $\Delta_{y_2, z_2 \rightarrow y_1, z_1}$ and obtain as a consequence of the identity $w^{(0)}_{(k,k)} = 0$ the identity

$$\Delta_{y_2, z_2 \rightarrow y_1, z_1}(w^{(0)}_{(k,k)}) = y_1 z_1 \left( \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) (y_1 z_2 - y_2 z_1)^k$$

$$= k y_1 z_1 (y_1 z_2 - y_2 z_1)^{k-1} \left( \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2} \right) (y_1 z_2 - y_2 z_1)$$

$$= k y_1 z_1 (y_1 - z_1)(y_1 z_2 - y_2 z_1)^{k-1} = 0.$$

Continuing in this way we obtain

$$\Delta^k_{y_2, z_2 \rightarrow y_1, z_1}(w^{(0)}_{(k,k)}) = k! (y_1 z_1)^k (y_1 - z_1)^k = 0$$

which gives that $(y_1 z_1)^k(y_1 - z_1)^k w^{(j)}_{\mu} = 0$ for all $\mu$ and all $j$. \hfill \Box

**Corollary 4.5.** The variety $\mathfrak{F}$ is generated by its one-generated free algebra $F_1(\mathfrak{F})$.

**Proof.** If $\var{F_1(\mathfrak{F})} \neq \mathfrak{F}$, then by Lemma 4.2 the algebra $F_1(\mathfrak{F})$ satisfies some identity $w^{(0)}_{(k,k)}$, and by Lemma 4.4 satisfies the identity $(y_1 z_1)^k(y_1 - z_1)^k = 0$ in one variable. This means that $(y_1 z_1)^k(y_1 - z_1)^k = 0$ in $F_1(\mathfrak{F})$ which is impossible. \hfill \Box

The following theorem is the first main result of our paper.

**Theorem 4.6.** If $\mathfrak{V}$ is a proper subvariety of the variety $\mathfrak{F}$ of all bicommutative algebras such that $\mathfrak{V}$ satisfies a polynomial identity $f = 0$ of degree $k$, $0 \neq f \in F(\mathfrak{V})$, then $c_n(\mathfrak{V})$ is bounded by a polynomial of degree $k - 1$.

**Proof.** Let

$$(13) \lambda_\mu(\mathfrak{V}) = \sum_{\lambda \vdash n} m_\lambda(\mathfrak{V}) \chi_\lambda, \quad n = 1, 2, \ldots,$$

be the cocharacter sequence of $\mathfrak{V}$. By Proposition 3.2, the summation in (13) runs on $\lambda = (\lambda_1, \lambda_2) \vdash n$. By Corollary 4.3 we obtain that $m_{(\lambda_1, \lambda_2)}(\mathfrak{V}) = 0$ for $\lambda_2 \geq k$.

If $\lambda_1 - \lambda_2 \leq 3k - 1$, then

$$m_{(\lambda_1, \lambda_2)}(\mathfrak{V}) \leq m_{(\lambda_1, \lambda_2)}(\mathfrak{V}) \leq \lambda_1 - \lambda_2 + 1 \leq 3k.$$

Now, let $\lambda_1 - \lambda_2 \geq 3k$. By Lemma 4.4, the variety $\mathfrak{V}$ satisfies the identities

$$w_j = (y_1 z_1)^k(y_1 - z_1)^k(y_1 z_2 - y_2 z_1)^{\lambda_2} y_1^{\lambda_1} z_1^{\lambda_1 - \lambda_2 - 3k - j} = 0, \quad j = 0, 1, \ldots, \lambda_1 - \lambda_2 - 3k.$$
over a field of characteristic 0 is either linear (and bounded by 
\( n \))

\[ A \]

bicommutative algebras exponentially as \( 2^{k} \).

There are only five nonisomorphic algebras

\[ \text{Proposition 5.1.} \]

\[ k \] polynomial of degree \( k \)

Let \( V \) be the subvariety of \( \mathcal{B} \) defined by the polynomial identity of right nilpotency

\[ ((x_{1}x_{2})x_{3}) \cdots x_{k} = 0. \]

It is easy to see that the image in \( G \) of the T-ideal \( T(\mathcal{Y}) \) of the identities of \( \mathcal{Y} \) is generated as an ordinary two-sided ideal by the products

\[ y_{i_{1}}z_{i_{2}} \cdots z_{i_{k}}. \]

Hence if \( \mu_{2} \geq k - 1 \), then all \( y_{(\mu_{1}, \mu_{2})}^{(j)} \) belong to this T-ideal and

\[ w_{(n-2k+4)}^{(n-k+2, k-2)}(y_{1}z_{2} - y_{2}z_{1})^{k-2} \]

does not belong to this ideal. Hence \( c_{n}(\mathcal{Y}) \geq d_{(n-k+2, k-2)} \) which is a polynomial of degree \( k - 2 \). We do not know whether there exists a variety \( \mathcal{Y} \subset \mathcal{B} \) satisfying a polynomial identity in one variable of degree \( k \) such that \( c_{n}(\mathcal{Y}) \) grows as a polynomial of degree \( k - 1 \).

5. Two-dimensional algebras

The classification of all two-dimensional algebras can be traced back to the two-dimensional part of the classification project in the seminal book by B. Peirce published lithographically in 1870 in a small number of copies for distribution among his friends and then reprinted posthumously in 1881 with addenda of his son C.S. Peirce. The complete classification over any field was finished by Petersson in 2000. The paper contains also the history of the classification. See as well [18] for the contributions of Peirce and [17, 23]. Concerning the polynomial identities of two-dimensional algebras, Giambruno, Mishchenko, and Zaicev proved that the growth of the codimension sequence \( c_{n}(A) \) of such an algebra \( A \) over a field of characteristic 0 is either linear (and bounded by \( n + 1 \)) or grows exponentially as \( 2^{n} \). In this section we shall study the polynomial identities of the bicommutative algebras \( A_{\pi, \delta} \) with multiplication given in (2).

\[ \text{Proposition 5.1.} \]

There are only five nonisomorphic algebras \( A_{\pi, \delta} \):

\[ A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}, A_{1,-1}. \]
Proof. Let $\pi = 0$, $\varrho \neq 0$. If we replace the generator $r$ by $r_1 = \varrho r$, then

$$r^2 \cdot r = \varrho r^2, \quad \varrho^3 r^2 \cdot r_1 = \varrho \varrho^2 r_1^2, \quad r_1^2 \cdot r_1 = r_1^2,$$

i.e., $A_{0,\varrho} \cong A_{0,1}$. Similarly, $A_{\pi,0} \cong A_{1,0}$. If $\pi = \varrho \neq 0$, then the change of the generator $r$ with $r_1 = \pi r$ gives that

$$r^2 \cdot r = \pi r^2, \quad \pi^3 r_1^2 \cdot r_1 = \pi \pi^2 r_1^2, \quad r_1^2 \cdot r_1 = r_1^2, \quad r_1 \cdot r_1^2 = r_1^2,$$

and $A_{\pi,\pi} \cong A_{1,1}$. Finally, let $\pi \neq \varrho$ be different from 0. We fix solutions $\xi$ and $\eta$ of the linear system

$$\pi(\xi + \eta) = 1, \quad \varrho(\xi + \eta) = -1.$$

Then $r_1 = \xi r + \eta r^2$ satisfies the conditions

$$r_1^2 = (\xi + \eta)(\xi + \eta)r^2 = -\frac{1}{\varrho} r^2,$$

$$r_1 \cdot r_1^2 = -\frac{1}{\varrho}(\xi + \eta^2)r^2 = -\frac{1}{\varrho} \pi(\xi + \eta \varrho)r^2 = \pi(\xi + \eta \varrho)r_1^2 = r_1^2,$$

$$r_1^2 \cdot r_1 = \varrho(\xi + \eta \varrho)r_1^2 = -r_1^2,$$

i.e., $A_{\pi,\varrho} \cong A_{1,-1}$.

Obviously, the five algebras $A_{0,0}, A_{1,1}, A_{0,1}, A_{1,0}, A_{1,-1}$ are pairwise nonisomorphic: $A_{0,0}$ is nilpotent of class 3, $A_{1,1}$ is associative-commutative, $A_{0,1}$ is left-nilpotent of class 3 but not right-nilpotent, $A_{1,0}$ is right-nilpotent of class 3 but not left-nilpotent, $A_{1,-1}$ is noncommutative and neither left- nor right-nilpotent.

The algebra $A_{0,0}$ satisfies the identities

$$x_1 x_2 = x_2 x_1, \quad (x_1 x_2)x_3 = 0.$$

Obviously they form a basis of its polynomial identities and the cocharacter sequence of $A_{0,0}$ is

$$\chi_1(A_{0,0}) = \chi_{(1)}, \quad \chi_2(A_{0,0}) = \chi_{(2)}, \quad \chi_n(A_{0,0}) = 0, \quad n = 3, 4, \ldots.$$

Similarly, one basis of the polynomial identities of the algebra $A_{1,1}$ consists of

$$x_1 x_2 = x_2 x_1, \quad (x_1 x_2)x_3 = x_1(x_2 x_3)$$

and the cocharacter sequence is

$$c_n(A_{1,1}) = \chi_{(n)}, \quad n = 1, 2, \ldots.$$

The next theorem gives bases for the polynomial identities and the cocharacter sequences of the other three algebras.

**Theorem 5.2.** (i) As subvarieties of the variety $\mathcal{B}$ of all bicommutative algebras the varieties $\text{var}(A_{0,1})$ and $\text{var}(A_{1,0})$ generated by the algebras $A_{0,1}$ and $A_{1,0}$ are defined by the identities of left-nilpotency $x_1(x_2x_3) = 0$ and right-nilpotency $(x_1x_2)x_3 = 0$, respectively. Their cocharacter and codimension sequences coincide and are

$$\chi_1(A_{0,1}) = \chi_1(A_{1,0}) = \chi_{(1)}, \chi_n(A_{0,1}) = \chi_{(n)} + \chi_{(n-1,1)}, \quad n = 2, 3, \ldots,$$

$$c_n(A_{0,1}) = c_n(A_{1,0}) = n, \quad n = 1, 2, \ldots.$$

(ii) The algebra $A_{1,-1}$ generates the whole variety $\mathcal{B}$. 
The origins in $F = F(\mathcal{B})$ of the polynomials $w^{(j)}_{x}$ from (11) have the form
\[ w^{(j)}_{(\lambda)}(x_1) = x_1 \cdots (x_1 - (x_1 x_1) \cdots x_1) \cdots, \]
with $j$ times $n - j$ times
\[ w^{(j)}_{(\lambda_1, \lambda_2)}(x_1, x_2) = x_1 \cdots x_1 ((x_1 x_2 \cdots x_2) \cdots) \cdots. \]
Obviously $w^{(j)}_{(\lambda_1, \lambda_2)}$ follows from $x_1 (x_2 x_3) = 0$ for $\lambda = (n), j = 2, \ldots, n - 1, n \geq 3,$ for $\lambda = (n-1, 1), j = 1, \ldots, n - 2,$ and for $\lambda = (\lambda_1, \lambda_2), \lambda_2 \geq 2.$ On the other hand $w^{(1)}_{(\lambda)}(r) = r^2 \neq 0,$ $w^{(0)}_{(n-1, 1)}(r, r^2) = -r^2 \neq 0.$ This shows that the identities of $A_{0,1}$ follow from $x_1 (x_2 x_3) = 0,$ $\chi_1(A_{0,1}) = \chi_1,$ $\chi_n(A_{0,1}) = \chi_n,$ $A_{0,1} = n = 2, 3, \ldots,$ and $c_n(A_{0,1}) = n = 1, 2, \ldots.$ The proof for $A_{1,0}$ is similar.

(ii) By Corollary 4.5 it is sufficient to show that the algebra $A_{1, -1}$ does not satisfy any identity in one variable. Let
\[ w^{(j)}_{(\lambda)}(y_1, z_1) = \sum_{j=1}^{n-1} \xi_j w^{(j)}_{(\lambda)}(y_1, z_1), \quad \xi_j \in K, \]
be a polynomial in $G$ which corresponds to a homogeneous polynomial identity $f(x_1) = 0$ in one variable and of degree $n \geq 2,$ $0 \neq f(x_1) \in F(\mathcal{B}).$ We shall evaluate $f(x_1)$ on all $\gamma r + \delta r^2 \in A_{1, -1}, \gamma, \delta \in K.$ Since
\[ (\gamma r + \delta r^2)^2 = (\gamma^2 - \delta^2) r^2, \]
\[ (\gamma r + \delta r^2) \cdot (\gamma r + \delta r^2)^2 = (\gamma - \delta)(\gamma^2 - \delta^2) r^2, \]
\[ (\gamma r + \delta r^2)^2 \cdot (\gamma r + \delta r^2) = - (\gamma + \delta)(\gamma^2 - \delta^2) r^2, \]
we obtain that the evaluation of the proimage of $w^{(j)}_{(\lambda)}(y_1, z_1)$ on $\gamma r + \delta r^2$ is equal to
\[ (-1)^{n-j-1}(\gamma^2 - \delta^2)(\gamma - \delta)^{j-1}(\gamma + \delta)^{n-j-1} r^2 = (-1)^{n-1}(\delta - \gamma)^j(\delta + \gamma)^{n-j}. \]
Hence
\[ f(\gamma r + \delta r^2) = (-1)^{n-1} w^{(n)}(\gamma - \delta, \delta + \gamma) r^2 = 0. \]
When $\gamma$ and $\delta$ run on the whole field $K$ the same holds for $\delta - \gamma$ and $\delta + \gamma.$ Therefore the polynomial $w^{(n)}(y_1, z_1)$ vanishes evaluated on the infinite field $K$ and hence is identically equal to 0. This means that $A_{1, -1}$ does not satisfy any polynomial identity in one variable and hence generates the whole variety $\mathcal{B}.$

The following easy lemma gives an upper bound for the codimensions of a finite dimensional algebra. It was established in a more general form for graded algebras in [2]. We include the proof for self-containedness of the exposition and to correct the misprint $c_n(A) \leq \dim^n(A)$ instead of $c_n(A) \leq \dim^{n+1}(A)$.

**Lemma 5.3.** If $A$ is a finite dimensional algebra then
\[ c_n(A) \leq \dim^{n+1}(A), \quad n = 1, 2, \ldots. \]
Proof. Let \( \dim(A) = m \) and let \( A \) have a basis \( \{r_1, \ldots, r_m\} \). We consider the multilinear identity
\[
 f(x_1, \ldots, x_n) = \sum_{(\sigma)} \xi(\sigma)(x_{\sigma(1)} \cdots x_{\sigma(n)}) = 0, \quad \xi(\sigma) \in K,
\]
where the summation runs on all permutations \( \sigma \in S_n \) and all possible bracket decompositions. Clearly, \( f(x_1, \ldots, x_n) = 0 \) is a polynomial identity for \( A \) if and only if \( f(r_{i_1}, \ldots, r_{i_n}) = 0 \) for all possible choices of the basis elements \( r_{i_1}, \ldots, r_{i_n} \). Let
\[
 f(r_{i_1}, \ldots, r_{i_n}) = \sum_{j=1}^{m} f_j(r_{i_1}, \ldots, r_{i_n}) r_j,
\]
where \( f_j(r_{i_1}, \ldots, r_{i_n}) \in K \) are linear functions in the coefficients \( \xi(\sigma) \). Considering \( \xi(\sigma) \) as unknowns, we obtain the linear homogeneous system
\[
 f_j(r_{i_1}, \ldots, r_{i_n}) = 0, \quad r_{i_1}, \ldots, r_{i_n} \in \{r_1, \ldots, r_m\}, j = 1, \ldots, m.
\]
The system has \( n!C_n \) unknowns, where \( C_n \) is the \( n \)-th Catalan number (equal to the number of the bracket decompositions). Since the codimension \( c_n(A) \) is equal to the rank of the system and the system has \( m^{n+1} \) equations, its rank is less or equal to \( m^{n+1} \) and the same holds for the \( n \)-th codimension \( c_n(A) \).

\( \square \)

Remark 5.4. It was shown in [13] that if the two-dimensional algebra \( A \) has a one-dimensional nilpotent ideal, then \( c_n(A) \leq n + 1 \). The algebras \( A_{0,1} \) and \( A_{1,0} \) satisfy this condition and Theorem 5.2 (i) shows that their codimensions are very close to the upper bound. For the algebra \( A_{1,-1} \) the results in [13] give that
\[
 \frac{2^n}{n^2} \leq c_n(A_{1,-1}) \leq 2^{n+1}.
\]
The bound \( c_n(A_{1,-1}) \leq 2^{n+1} \) can be improved if we consider two-dimensional algebras \( A \) with the property that \( \dim(A^2) = 1 \). In the proof of Lemma 5.3 the algebra \( A_{1,-1} \) has a basis \( \{r, r^2\} \) and the values of \( f(r^{i_1}, \ldots, r^{i_n}) \), \( i_k = 1, 2 \), belong to the ideal \( A_{1,-1}^2 \). Hence, solving the linear system (14) we have to follow only the coefficient of \( r^2 \). Since the number of the equations is \( 2^n \), we obtain \( c_n(A_{1,-1}) \leq 2^n \). By [11] and Theorem 5.2 (ii) we have that \( c_n(A_{1,-1}) = c_n(\mathbb{B}) = 2^n - 2 \). Again, this is very close to the upper bound \( 2^n \).

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