Transformations and BRST-Charges in 2 + 1 Dimensional Gravitation

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Abstract
Canonical transformations relating the variables of the ADM-, Ashtekar’s and Witten’s formulations of gravity are computed in 2 + 1 dimensions. Three different forms of the BRST-charge are given in the 2+1 dimensional Ashtekar formalism, two of them using Ashtekar’s form of the constraints and one of them using the forms suggested by Witten. The BRST-charges are of different rank.

1 Introduction
There exist two different formalisms describing gravitation in 3 + 1 dimensions: the ADM- and Ashtekar’s formalisms. There is a canonical transformation relating these to each other as shown by A.Ashtekar [1], J.Friedman and I.Jack [2] and M.Henneaux, J.E.Nelson and C.Schomblond [3]. This transformation is built on the properties of the spatial spin connection.

In 2 + 1 dimensions there are already three different formulations: the ADM, Ashtekar’s and Witten’s [4]. These formulations are equivalent if the metric is nondegenerate. For the relations between them see e.g. [5], [6]. One
of the purposes of this paper is to find an explicit canonical transformation between the Ashtekar and the ADM formalisms in \(2 + 1\) dimensions. In this case we can not make use of the spatial spin connection, because the fundamental variables in the Ashtekar’s formalism have \(\text{SO}(1,2)\) indices while the spatial spin connection has \(\text{SO}(2)\) indices. (Section 2).

Since the constraint algebra in Ashtekar’s formulation is different from the one in Witten’s formulation a canonical transformation relating these formulations must take place in the extended phase space and involve ghost fields too [3]. Moreover the constraint algebra of the Ashtekar formalism shows some peculiar features: the Poisson bracket between the Hamiltonian and the vector constraints contains terms of the form: \(\mathcal{H}\mathcal{G}_I\) and \(\mathcal{H}_a\mathcal{G}_I\), where \(\mathcal{H}\) is the Hamiltonian constraint, \(\mathcal{H}_a\) \((a = 1, 2)\) are the vector constraints and the \(\mathcal{G}_I\)'s are the constraints called Gauss’ law. These terms can be interpreted in two different ways: one can either say that \(\mathcal{H}\mathcal{G}_I\) is the Hamiltonian constraint multiplied by a structure function which is ‘accidentally’ a function of Gauss’ law, or that it is Gauss’ law multiplied by the structure function containing the Hamiltonian constraint. These two interpretations give birth to two different forms of the BRST-charge of the theory. Together with the BRST-charge of the Witten’s formalism one has three different forms of the BRST-charge of the same theory. It is possible to obtain one form knowing another one by canonical transformations. The purpose of Section 3 is to show these forms of the BRST-charge explicitly and to compute one of the canonical transformations between them.

It is worth noting that these forms of the BRST-charge while belonging to the same theory are of different rank.

### 2 The transition between the ADM- and -Ashtekar’s formalisms in \(2 + 1\) dimensions

In the ADM formalism the fundamental variables one uses are the spatial metric \(g_{ab}(a, b = 1...d - 1)\) and it’s canonically conjugated momenta \(\Pi^{ab}\). An equivalent description of gravity can be given using as basic variables the triads \(e_{aI}\), where ‘\(a\)’ are the spatial indices and ‘\(I\)’ the internal indices. The internal algebra is \(\text{SO}(3)\) in \(3 + 1\) dimensions and \(\text{SO}(1,2)\) in \(2 + 1\) dimensions. The relation between these variables and the metric is given by:
The indices 'I' are raised and lowered by the Killing metric of the algebra.

The canonical momenta conjugated to the triads are $\Pi_I^a$. One can build the same gravitational theory on these variables as the one built on the $g_{ab}$ and $\Pi^{ab}$, if an extra constraint is introduced.

One can make a canonical transformation starting from the triad variables and obtain a new formalism. See e.g. [8].

$$(e^I_a, \Pi_I^b) \rightarrow (K_I^a, E^b_J)$$

where: $E^b_I = e e^b_I$, $e = \sqrt{g}$, $g$ being the determinant of the spatial metric and $K_a^I$ is related to the extrinsic curvature by:

$$K_{ab} = e_a^I K^I_b$$

The extrinsic curvature is weakly symmetrical.

Ashtekar’s formalism uses the fundamental canonical variables $A_a^I$ and $E^b_I$, where $A_a^I$ is related to the space component of the spacetime spin connection:

$$A_a^I = f^I_J K^a_{JK}$$

There exists a canonical transformation relating the formalism using the variables $K_a^I$ and $E^b_I$ to Ashtekar’s formalism.

The Hamiltonian constraint pushes the spacelike hypersurface forward in time and the change of the spatial metric of the hypersurface is given by the extrinsic curvature:

$$\{g_{ab}, \mathcal{H}[M]\} = 2MK_{ab} \quad (2)$$

where $\mathcal{H}[M] \equiv \int M\mathcal{H}$. If one introduces the notation:

$$q^{ab} = E^a_I E^b_I$$

eq. (2) gives:

$$\{q^{ab}, \mathcal{H}[N]\} = -2M g^{-1}(q^{ac} q^{bd} K_{cd} - q^{ab} q^{cd} K_{cd})$$
with \( N = M/\sqrt{g} \). On the other hand one knows that in Ashtekar’s formalism the Hamiltonian constraint looks like:

\[
\mathcal{H} = \frac{1}{2} f_{IK} E_{I}^{a} E_{J}^{b} F_{ab}^{K}
\]

(4)

where \( F_{ab}^{K} = \partial_{a} A_{b}^{K} - \partial_{b} A_{a}^{K} + f_{LM}^{K} A_{a}^{L} A_{b}^{M} \) is the curvature field strength. One can use this form of the Hamiltonian constraint to compute explicitly:

\[
\{ q^{cd}, \mathcal{H}[N] \} = -2N f^{IJ} E_{I}^{a} E_{J}^{b} d_{a} d_{b}^{d} L
\]

(5)

where \( d_{a} \) denotes the covariant derivative on the phase space:

\[
D_{a} E_{I}^{b} = \partial_{a} E_{I}^{b} + f_{IKM} A_{a}^{K} E_{b}^{M}
\]

and “( )” denotes symmetrization with factor 1/2 included.

Comparing these two relations one can get the transformation rule between \( K_{a}^{M} \) and \( A_{a}^{M} \):

\[
K_{a}^{M} = f_{L}^{M} I E_{I}^{b} q_{ad} D_{b} d_{d}^{L} - f_{L}^{I} J E_{J}^{b} E_{a}^{M} E_{f} D_{b} E_{L}
\]

(6)

where \( E_{a}^{M} \) is defined as an ‘inverse’ to \( E_{a}^{M} : E_{a}^{M} E_{M}^{b} = \delta_{a}^{b} \). The ‘momenta’ \( E_{I}^{a} \) are of course the same for both formalisms:

\[
E_{I}^{b} = E_{I}^{b}
\]

(7)

If one wants to check whether this transformation expressed by equation 6 and 7 is a canonical one it is straightforward to obtain:

\[
\{ K_{a}^{I}(x), E_{I}^{b}(x') \} = \delta_{a}^{b} \delta_{I}^{I} \delta(x - x')
\]

To get however \( \{ K_{a}^{I}(x), K_{b}^{I}(x') \} \) is a very lengthy calculation.

In 3 + 1 dimensions one can obtain a much simpler relation between \( A_{a}^{I} \) and \( K_{a}^{I} \) using the spatial spin connection \( \Gamma_{a}^{I} \) which is defined to satisfy:

\[
\nabla_{a} e_{I}^{b} - \Gamma_{Ia}^{J} e_{J}^{b} = 0
\]

with \( \Gamma_{a}^{I} = f^{JKa} \Gamma_{JKa} \). The relation between \( A_{a}^{I} \) and \( K_{a}^{I} \) becomes:

\[
A_{a}^{I} = iK_{a}^{I} + \Gamma_{a}^{I}
\]

(8)
See for example [1],[2],[3]. The relation between the "momenta" of these formulations is of course the same as given in equation [4]. The transformation given by equations [5] and [6] is a canonical one since the spatial spin connection can be expressed as the functional derivative of a functional depending on the "momenta":

\[ \Gamma'_b = \frac{\delta F(E)}{\delta E^b_I} \]

where:

\[ F(E) = 2i \int dx E^a_I \Gamma'_a \]

and the transformations of the form: \( A \to A + \partial_E f(E), \ E \to E \) are canonical.

In 2 + 1 dimensions one can not obtain a similar relation to [5] and [6]; the spatial spin connection is not useful in this case because it has SO(2) indices, while the fundamental variables carry SO(1,2) indices.

### 3 The BRST-Charge

There exist two different sets of constraints in 2+1 dimensional gravity which give the same description if the spatial metric is nondegenerate [5], [6].

The first set of constraints obey the constraint algebra of general relativity and it is determined by geometrical considerations. See for example [7]. In 2+1 dimensions this happens in the same way as in 3+1 dimensions. The spatial translations are generated by the vector-constraints. Using Ashtekar’s formulation these constraints can be expressed as

\[ \mathcal{H}_a = E^b_I F_{ab}^I \approx 0 \] (9)

The timelike translations are generated by the Hamiltonian constraint:

\[ \mathcal{H} = \frac{1}{2} f^{IJ}_K E^a_I E^b_J F_{ab}^K \approx 0 \] (10)

We also have the first class constraint called Gauss’ law:

\[ \mathcal{G}_I \equiv D_a E^a_I = (\partial_a \delta^I_J + f^J_K A^K_a) E^a_J \approx 0 \] (11)

The algebra satisfied by these constraints looks like:

\[ \{ \mathcal{H}_a[N^a], \mathcal{H}_0[M^b] \} = \mathcal{H}_a[\mathcal{L}_N M^a] + \mathcal{G}_I [M^a N^b F_{ab}^I] \] (12)
\{H_a[N^a], H[M]\} = H[L_N M] + G_I[M N^a f^I_{JK} E^b_I F^K_{ab}] \tag{13}

\{H[N], H[M]\} = \pm H_a[(N \partial_b M - M \partial_b N) q^{ba}] \tag{14}

\{G_I[N^I], G_J[N^J]\} = G_I[f^I_{JK} N^J N^K] \tag{15}

where \(H_a[N^a] = f H_a N^a\), etc. \(q^{ba}\) is the spatial metric and the sign in the third equation is \(-\) for a space-time with Euclidean signature and it is \(+\) for Lorentzian signature.

The other set of constraints which can be used in 2 + 1 dimensions was suggested by Witten \[4\]

\[G_I = D_a E_I^a \approx 0 \tag{16}\]

\[\Psi^I = \epsilon^{ab} F^I_{ab} \approx 0 \tag{17}\]

The algebra satisfied by these constraints:

\[\{G_I[N^I], G_J[N^J]\} = G_I[f^I_{JK} N^J M^K] \tag{18}\]

\[\{\Psi^I[N^I], \Psi^J[M^J]\} = \Psi^I[f_{IJK} N^J M^K] \tag{19}\]

\[\{\Psi^I[N^I], \Psi^J[M_J]\} = 0 \tag{20}\]

As it is described in \[6\] these two sets of constraints are related by:

\[H_a = \frac{\epsilon_{ab}}{2} E^b_I \Psi^I \tag{21}\]

\[H = \frac{1}{4} f^I_{JK} E_J^a E_K^b \Psi^I \tag{22}\]

or using a symbolical notation:

\[\mathcal{H}_\mu = M_{\mu I} \Psi^I \tag{23}\]

where \(\mu = 0, 1, 2\) and \(\mathcal{H}_0 \equiv \mathcal{H}\). This notation is a very comfortable one but one shouldn’t confuse it with the usual 3-vector notation, because the quantity \(\{\mathcal{H}, H_a\}\) is not a 3-vector. So \(\mu\) is not a vector index. We have then the ‘matrix’ \(M\) with:

\[M_{aI} = \frac{\epsilon_{ab}}{2} E^b_I \tag{24}\]

\[M_{0I} = \frac{\epsilon_{ab}}{4} f^I_{JK} E_J^a E_K^b \Psi^I = \frac{1}{2} f^I_{JK} E_J^a M_{aK} \tag{25}\]
This matrix is invertible whenever \( \det q^{ab} \neq 0 \), and one can write

\[
\Psi^I = M^aI \mathcal{H}_a + M^0I \mathcal{H}
\]

Actually \( \det q^{ab} = 8 \det M \). Using these relations one can rewrite the constraint-algebra of the first (Ashtekar’s) set of constraints in the following way:

\[
\{ \mathcal{H}_a[N^a], \mathcal{H}_b[P^b] \} = \mathcal{H}_a[\mathcal{L}_N P^a] + \mathcal{G}_I \mathcal{H}_I[\frac{(M^{-1})^{\mu I} \epsilon^{ab} P^a N^b}{2}] \tag{26}
\]

\[
\{ \mathcal{H}_a[N^a], \mathcal{H}[P] \} = \mathcal{H}[\mathcal{L}_N P] + \mathcal{H}_I \mathcal{G}_I[\frac{(M^{-1})^{\mu K} PN^a f^{IJ} E^b K E^b \epsilon^{ab}}{2}] \tag{27}
\]

The other equations remain the same. One can notice a very interesting feature of this algebra: the presence of terms including \( \mathcal{G}_I \mathcal{H}_I \). This leads to the possibility of two different forms of the BRST-charge of the theory.

Introducing a Grassmann-odd ghost-variable to each constraint of a theory one can always find a nilpotent quantity called the BRST-charge \([10]\). A general form of the BRST-charge was given for example in \([11]\).

\[
Q = \Psi^a \eta^a + \sum_{i=1}^N U^{a_1...a_i} \mathcal{P}_{a_1}...\mathcal{P}_{a_i} \tag{28}
\]

where \( N \) is the rank of the theory, \( \Psi \)-s are the constraints and \( \mathcal{P}_{a_i} \) are the ghost-momenta canonical to the \( \eta^a \) ghost-variables. Knowing the constraint-algebra the coefficients \( U^{a_1...a_i} \) can be computed. The first ones are especially simple:

If \( \{ \Psi_a, \Psi_b \} = C_{ab} \Psi_c \) then \( U^c = -\frac{1}{2} C_{ab} \eta^a \eta^b \) and from here:

\[
Q = \Psi^a \eta^a - \frac{1}{2} C_{ab} \eta^a \eta^b \mathcal{P}_c + ...
\]

If we now look at the algebra of the constraints \( \mathcal{H}_\mu \) (equations \([26] \) and \([27] \)) we notice that the coefficients \( U^c \) can be written in two different ways.

The term

\[
\mathcal{H}_\mu \mathcal{G}_I[\frac{1}{2} \epsilon^{ab} (M^{-1})^{\mu I} P^a N^b]
\]

can be understood either as:

**CHOICE 1:** \( C^I_{ab} \mathcal{G}_I \), that is the Poisson bracket between the two constraints equals to Gauss’ law multiplied by a coefficient that contains a constraint: \( C^I_{ab} = \frac{1}{2} \epsilon_{ab} \mathcal{H}_\mu (M^{-1})^{\mu I} \) or
CHOICE 2: $C_{ab}^\mu \mathcal{H}_\mu$ which means that the P.B. is a constraint $\mathcal{H}_\mu$ times a coefficient containing Gauss’ law. This coefficient is: $C_{ab}^\mu = \frac{1}{2} \epsilon_{ab} G_I (M^{-1})^{\mu I}$

This means that there are two possible forms of the BRST-charge, depending on which way one wants to understand the algebra.

The algebra of the 2 + 1 dimensional gravity is of course the simplest if one uses the constraints suggested by Witten \[4\]. This gives the simplest form to the BRST-charge too:

$$Q^W = \Psi^I \tilde{\eta}_I + G_I \tilde{\rho}^I - f^I_K \tilde{\eta}_K \tilde{\rho}^K \tilde{\mathcal{P}}_J - \frac{1}{2} f^I_{JK} \tilde{\rho}^J \tilde{\rho}^K \tilde{\mathcal{P}}_I$$

(30)

where $\tilde{A}^I_a$ and $\tilde{E}^a_I$ are the phase-space variables, $\tilde{\mathcal{P}}_I$-the momentum canonical to the ghost $\tilde{\rho}^I$, $\tilde{\Pi}^I$ -the momentum canonical to the ghost $\tilde{\eta}_I$.

This is thus a rank 1 form of the BRST-charge.

There is a theorem due to Batalin and Fradkin \[9\] that says: if a BRST-charge can be expressed in two different forms there exists a canonical transformation in the extended phase space relating the variables used in the two expressions. This means that one should be able to find a canonical transformation that relates $\tilde{A}^I_a$,$ \tilde{E}^a_I$,$ \tilde{\rho}^I$, $\tilde{\mathcal{P}}_I$, $\tilde{\eta}_I$ and $\tilde{\Pi}^I$ to the variables $A^I_a, E^a_I, \rho^I_{, I}, \mathcal{P}_I$, $\eta_I$ and $\Pi^I$ used in the expression of the two forms of $Q^{(A)}$. The first natural thing is to try to find a set of canonical transformations starting with:

$$\tilde{\eta}_I = M_{\mu I} \eta^\mu$$

(31)

This gives of course:

$$\tilde{\Pi}^I = (M^{-1})^{\mu I} \Pi^\mu$$

(32)

The other relations between these variables are:

$$\tilde{A}^I_a = A^I_a - f^I_J K (M^{-1})^{\mu J} M_{aK} \Pi^\mu \eta^b - \frac{\epsilon_{ab}}{2} (M^{-1})^{bI} \Pi^\mu \eta^a$$

(33)

while the other three variables remain unchanged: $\tilde{E}^a_I = E^a_I$, $\tilde{\rho}^I = \rho^I_{, I}$ and $\tilde{\mathcal{P}}_I = \mathcal{P}_I$.

One should stress again that $\mu$ is not a vector index in this case. The BRST-charge computed by substituting the variables with tilde in $Q^{(W)}$ by $E^a_I, A^I_a$, etc. is:

$$Q^{(A2)} = \mathcal{H} \eta + \mathcal{H}_a \eta^a + G_I \rho_I - \frac{\epsilon_{ab}}{2} (M^{-1})^{bI} \Pi^\mu \eta^a$$

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This corresponds to the "second choice" of the coefficients on page 3 and it is of rank 1. One can of course compute the form of the BRST-charge corresponding to the first choice. This is done in the simplest way as in [12]. We get:

\[ Q^{A1} = G^I \rho^I + H_\mu [\eta^\mu] + (M^{-1})^{\mu I} \frac{\epsilon^{ab}}{4} \mathcal{P}_I \eta^a \eta^b \]

\[ + (M^{-1})^{\mu I} \frac{\epsilon^{ab}}{2} f^{jK} E^{b}_j \mathcal{P}_K \eta^a \]

\[ + (M^{-1})^{\mu I} \frac{\epsilon^{ab}}{4} f^{jK} \mathcal{P}_j \mathcal{P}_K \eta^a \eta^b \]

\[ - \frac{1}{2} f^{IJ} K^L \rho^I \rho^J \mathcal{P}_L - \Pi d \eta^c \partial_c \eta^d - \Pi \eta^c \partial_c \eta + \Pi \partial_c \eta^c \eta \]

\[ - q^{cd} \Pi d \eta \partial_c \eta - 2 E^I_b \mathcal{P}_I \eta \partial_c \eta^c \]

As we see this corresponds to rank 2. There exists of course a transformation between \( Q^{A1} \) and \( Q^{A2} \) and another one between \( Q^{A1} \) and \( Q^{W} \) but they are very complicated.

We obtained three different forms of the BRST-charge of the same theory, two of them being of rank 1 (\( Q^W \) and \( Q^{A2} \)) and one of them rank 2 (\( Q^{A1} \)). This is an interesting example of the fact that a theory can have BRST-charges of different rank depending on how one formulates the constraints so one can change the rank of a theory by reformulating the constraints [3, 10].

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