Loop Quantum Cosmology: Effective theories and oscillating universes

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Abstract

Despite its great successes in accounting for the current observations, the so called ‘standard’ model of cosmology faces a number of fundamental unresolved questions. Paramount among these are those relating to the nature of the origin of the universe and its early evolution. Regarding the question of origin, the main difficulty has been the fact that within the classical general relativistic framework, the ‘origin’ is almost always a singular event at which the laws of physics break down, thus making it impossible for such an event, or epochs prior to it, to be studied. Recent studies have shown that Loop Quantum Cosmology may provide a non-singular framework where these questions can be addressed. The crucial role here is played by quantum effects, i.e. corrections to the classical equations of motion, which are incorporated in effective equations employed to develop cosmological scenarios.

In this chapter we shall consider the three main types of quantum effects expected to be present within such a framework and discuss some of their consequences for the effective equations. In particular we discuss how such corrections can allow the construction of non-singular emergent scenarios for the origin of the universe, which are past-eternal, oscillating and naturally emerge into an inflationary phase. These scenarios provide a physically plausible picture for the origin and early phases of the universe, which is in principle testable. We pay special attention to the interplay between these different types of correction terms. Given the absence, so far, of a complete derivation of such corrections in general settings, it is important to bear in mind the questions of consistency and robustness of scenarios based on partial inclusion of such effects.
1 Introduction

Cosmology has undergone tremendous advances in recent years, both theoretically and observationally. An important outcome of these developments has been the emergence of a ‘standard’ model of cosmology, with an early phase of accelerated expansion (the so called inflationary phase) followed by a Hot Big Bang expansion phase, which very successfully describes the central features observed by recent high-precision observations. Despite these successes, however, a number of fundamental questions remain. Foremost among these is the unsatisfactory singular origin for the universe predicted by this model, where the laws of physics break down and scientific predictability comes to an end. Another crucial shortcoming is the absence of a successful model of inflation that is properly situated within a fundamental theory of quantum gravity. It has long been hoped that these problems can be successfully resolved once a complete theory of quantum gravity is known. In this chapter we shall concentrate on some recent developments in Loop Quantum Cosmology (LQC), i.e. the specialisation of Loop Quantum Gravity to cosmological backgrounds, which potentially have important consequences for these questions.

Within the framework of LQC (see [1] for a detailed review) space-time is fundamentally described by a discrete structure whose dynamics is governed by difference rather than differential equations. This discreteness is an essential feature on small scales, such as those prevailing in the earliest phases of the universe. According to LQC, as the universe evolves and grows in size, this initial discrete quantum phase is succeeded by a tamer semi-classical phase in which the spacetime is well approximated by classical geometry, but governed by dynamical equations which are corrected by quantisation effects [2]. Finally as the universe grows still larger, the familiar classical continuum picture is approached to a good approximation.

To get a glimpse of possible LQC effects that are likely to be present, we consider the standard isotropic and homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models which are sourced by a scalar field $\phi$ with a potential $V(\phi)$ and a Hamiltonian $H_\phi = \frac{1}{2}a^{-3}p_\phi^2 + a^3V(\phi)$, where $a$ is the scale factor and $p_\phi$ is the scalar field momentum. The scalar field is a common choice for the ‘matter’ source in studies of the early universe, since once inflation begins all other matter sources except the scalar field are rapidly redshifted away. Classically the dynamics is described by the Hamiltonian constraint $H = 0$ which can be written as

$$\frac{1}{2}a^{-3}p_\phi^2 + a^3V(\phi) = \frac{3c^2}{8\pi G}(a^2 + c^2 k)a.$$ (1)

Here, $k$ is the curvature parameter, which is zero for a spatially flat universe and takes values $+1$ and $-1$ for positively and negatively curved universes. In the homogeneous and isotropic case considered here this constraint equation reduces to the familiar Friedmann equation (which is the relevant Einstein equation in this case). Thus this constraint relates the matter (here taken to be the scalar field) Hamiltonian, given by the terms on the left hand side of equation (1), to the time derivative of the scale factor on the right hand side of this equation.
An inspection of this equation readily demonstrates the difficulty usually faced in the classical domain, namely the fact that solutions to this equation are singular, all reaching a zero volume as $a \to 0$, at some point in the past (in all isotropic models) or both future and past (in the case of closed models with $k = +1$). At such points the matter Hamiltonian (energy) diverges ($H_\phi \to \infty$), while at the same time $\dot{a} \to \pm \infty$. This represents a space-time singularity which occurs for typical cosmological solutions in general relativity [3, 4]. The presence of such singularities has posed a severe barrier to the formulation of physically meaningful cosmological models within the context of classical general relativity, since at such points the theory itself breaks down and fails to be applicable. Thus general relativity cannot tell us what happens at, or beyond, such a singular point.

It has long been hoped that this barrier would be removed once a complete description is available which involves the quantisation of general relativity. The basic idea is that once the universe shrinks to a small enough size, close to the Planck length $\ell_P = \sqrt{G\hbar/c^3}$ (a fundamental length scale constructed in terms of the constants of nature $G$, Planck’s constant $\hbar$ and the speed of light $c$), the quantisation would result in quantum corrections to the classical cosmological evolution equations which could dramatically change their behaviours and in particular prevent such singularities from forming.

An important outcome of the developments in LQC has been the demonstration that there are indeed several different quantum gravity effects, with their corresponding quantum corrections, which could in principle result in well behaved singularity free cosmological scenarios, with potentially observable consequences. In the following we shall briefly review these effects and discuss some of their consequences for non-singular cosmological model construction. This will be presented from a general viewpoint: We do not focus on precise results in very specific models, but rather discuss qualitative aspects expected more generally.

## 2 Quantum corrections

The Friedmann equation (1) already highlights two sources of singular behaviour which need to be remedied by quantum corrections from loop quantum cosmology, if we are to obtain a non-singular model. These are the kinetic term, $\frac{1}{2} a^{-3} p_\phi^2$, which clearly diverges as $a \to 0$ and the term $\dot{a}$, on the right hand side of this equation, which also diverges at a cosmological singularity.

It is important to recall that these two particular sources of classical singularities are directly related to the potential blow up of the curvature (a function of $\dot{a}$ and $1/a$ in an isotropic and homogeneous geometry) which is the hallmark of singularities in the well known solutions of general relativity. This observation may suggest that only quantum corrections relating to terms which correspond to classical unbounded curvature might be the ones which are of relevance in preventing classical singularities to form. But the celebrated singularity theorems do not even refer to curvature blow-ups, indicating that such a restriction is likely to be misleading. In a general unbiased approach, all possible quantum correction terms must be considered, which are to be derived from the underlying
quantum equations and evaluated in cosmological or black hole scenarios. Only then can one be certain of addressing the problem of classical singularities which arise within the classical relativistic framework in a general and systematic way. In this chapter we present a summary of the present status of these questions.

To proceed we note that LQC corrections do indeed modify the two terms mentioned above, which in this case are responsible for the curvature blow-up. First, the inverse volume $a^{-3}$, when adequately quantised, acquires quantum corrections which become significant as $a$ becomes small. In fact it turns out that in the case of exactly homogeneous and isotropic models considered here, the classical divergence is cut off by quantum effects and this expression remains finite. (We note that in the isotropic case there are cancellations which render the right hand side of Eq. (1) free of inverse powers [5, 6]. This will, however, no longer be the case in less symmetric cases which have corrections of the same type as in the isotropic matter part.) Secondly, in loop quantum cosmology, the term $\dot{a}^2$ on the right hand side of Eq. (1) appears only as the leading term in a power series expansion of a more complicated function of $\dot{a}$. On classical scales, where general relativity is currently probed, $\dot{a}$ is small and the leading term is the only relevant one. However, as the universe approaches its classical singularity, $\dot{a}$ increases without bounds making the higher order terms noticeable at some point. It turns out that this results in a power series in $\dot{a}$, which has $\dot{a}^2$ as its leading term, and which sums to a bounded function which can cut off the potential divergences.

These two effects have already been explored in a number of phenomenological models, some of which we shall describe below. In addition to these effects, however, there is a third, potentially more important, source of corrections which is not so easy to visualise or to derive systematically. According to general considerations of effective actions, such corrections in the isotropic and homogeneous settings are expected to occur as higher derivatives of the scale factor (such as $\ddot{a}$, ...) in the Friedmann equation. The inclusion of these corrections is crucial for a reliable derivation of effective equations which need to include the complete set of quantum corrections. Thus a full consideration of loop quantum corrections would, in addition to inverse volume corrections, also include higher powers as well as higher derivatives of $\dot{a}$. Since these extra terms are also divergent at a cosmological singularity, they must be included in any comprehensive analysis. In the following we shall briefly discuss these three sources of quantum corrections to the effective equations.

### 2.1 Inverse volume effects

In the context of homogeneous and isotropic cosmological models, the main reason for the occurrence of classical singularities is the presence of the divergent inverse volume term in the matter Hamiltonian. An outcome of isotropic LQC, based on general techniques developed in loop quantum gravity [1, 8], has been to show that any well-defined quantization of the classical inverse volume is finite [9]. (Although the case of anisotropic models is different, the differences are not problematic and are well-understood [10].) In LQC settings, the effective equations do not include the inverse volume operator itself but only expectation values or the corresponding eigenvalue function, which can be well approximated by
a continuous function $D(q)a^{-3}$, to replace $a^{-3}$, where the function \[ D(q) = \left\{ \begin{array}{ll}
\frac{3}{2l}q^{1-l} [(l+2)^{-1}((q+1)^{l+2} - |q-1|^{l+2})] \\
- \frac{1}{1+l}q ((q+1)^{l+1} - \text{sgn}(q-1)|q-1|^{l+1}) \end{array} \right\}^{3/(2-2l)} \]
is expressed in terms of the dimensionless ratio $q = a^2/a_*^2$. Here, $a_*^2 = \ell_P^2 j/3$ and $j$ and $l$ are parameters that arise in the process of quantisation. An important feature of this quantisation scheme is that these parameters are not uniquely specified: $l$ lies in the range zero to one and $j$ takes half-integer values. It turns out, however, that both these parameters would be more restricted if analogous operators were derived without assuming symmetries. Current constraints on $l$ are $l_n = 1 - 1/2n$ where $n$ takes integer values \[13\]. The restrictions on $j$ are less clear since its value depends not only on how the operator itself is derived but also on the precise form of states used to implement symmetries \[14, 15\]. (Arguments for $j = 1/2$ have been put forward \[6, 16\], but nevertheless larger values of $j$ can occur in homogeneous models to capture more general inhomogeneous features \[15\].) It is therefore important to bear in mind that the quantisation procedures employed involve ambiguities due in part to non-uniqueness of such parameters. The function $D$ codifies the semi-classical corrections to the evolution equations and has a number of important qualitative features, as can be seen from Fig. 1. It tends to zero as $a \to 0$, increases for small values of $a$, peaks at intermediate values of $a$ ($a \sim a_*$) and as $a$ increases it eventually approaches the classical expectation value of one from above. (Note that a fixed $a_*$ in combination with the rescaling freedom of $a$ in flat models induced by changing spatial coordinates is only an apparent problem which is solved when viewed in the appropriate inhomogeneous context; see \[14\]. The rescaling freedom is simply an accidental property of specific classical solutions which is broken by quantum effects (as it would be by spatial curvature or inhomogeneities). The new parameter $a_*$ is then related to the underlying scale of the discrete quantum gravity state responsible for the breaking of rescaling freedom.)

Thus one effect of quantum corrections on the evolution Eq. (1) is to replace the inverse volume in this equation by $D(q)a^{-3}$. We shall see that these features induce radically different behaviours in LQC models compared to the corresponding classical ones. A simple way to see some of the qualitative effects of these modifications is to rewrite the quantum corrected equations in the form of standard Einstein equations in terms of an effective perfect fluid with a variable equation of state \[17\] (see also \[19\]). In that case the increasing behaviour of $D(q)$ for small values of $a$ has the consequence of making the effective matter Hamiltonian an increasing function of the scale factor. Thus for small values of $a$ the energy increases with volume, implying that in these regimes the pressure of the fluid, defined as the negative derivative of energy with respect to volume, is negative. The presence of such a negative pressure could in principle provide a mechanism for resisting the collapse of the universe into a singularity. As we shall see below, this picture has indeed been shown to occur in a number of isotropic scenarios where the inverse volume correction has been included. This correction on its own, however, is not sufficient to ensure singularity.
Figure 1: In isotropic loop quantum cosmology, functions with inverse powers of the scale factor are finite instead of diverging at $a = 0$ as happens classically. This is captured in a correction function $D(q)$ where $q = a^2/a_*^2$ is defined in terms of a characteristic scale $a_*$ due to quantum gravity. A plot of the function $D$ against $q$ shows how $D$ and hence the quantised inverse volume deviate from the classical result (dashed) and approach zero as $a \to 0$. 
avoidance in more general settings. In such cases other corrections, such as those discussed below, are also likely to be required.

2.2 Higher order corrections

In homogeneous and isotropic models inverse volume effects occur mainly (but not only) in the matter Hamiltonians. As was mentioned above, it is expected on general grounds that in addition to these effects there also exist higher order curvature corrections. (See [18]; such terms have also been computed by a WKB analysis [19, 20].) In the case of isotropic models these terms take the form of a subseries of higher powers of \( \dot{a} \), i.e. \( \sum_n c_n \dot{a}^n \), with coefficients \( c_n \) which may depend on \( a \) (but not on derivatives). Although different possibilities exist depending on the precise quantization, the qualitative form is known to be \( a^{-2} \dot{a}^2 (1 + O(\ell_P^2 \dot{a}^2 / a^2 c^2)) \) which, for a specific example, sums to a closed form function

\[
\frac{c^2}{2\ell_P^2} \left( 1 - \sqrt{1 - \frac{4\ell_P^2 \dot{a}^2}{a^2 c^2}} \right) \sim \frac{\dot{a}^2}{a^2} - \frac{\ell_P^2 \dot{a}^4}{c^2 a^4} + \cdots
\]

(3)

(see also Sec. 3.3). Despite the remaining quantization ambiguities, qualitative properties are quite robust and follow from very general considerations in the full setting of loop quantum gravity which allow only specific types of variables, namely holonomies and fluxes, to be quantized in a way which allows the general covariance and background independence of general relativity to be realized after quantization. In our case, the form of this function replacing the classical \( \dot{a}^2 / a^2 \) already provides a way of seeing how such quantum corrections can lead to the boundedness of the curvature, by noting that the reality of the above expression imposes an upper limit on the curvature related term \( \dot{a}^2 \). The coefficients \( c_n \) of the terms \( c_n \dot{a}^n \) in this subseries, most of which are exceedingly small yet important for the final result, can in principle be determined exactly from the quantum theory. Note, however, that the precise bounded function requires the taking into account of all the infinite number of higher order terms in the subseries.

An important point to bear in mind in this connection is that there are additional quantum corrections, discussed in the following subsection, which are likely to be larger than almost all these terms in the series, independently of the regime being studied. As a result the precise form of the above function, and in particular its ability to provide an upper bound for curvature, is reliable only if one can show that all these other quantum corrections are absent. Otherwise, the analysis needs to be repeated by including all these other terms.

2.3 Higher derivative corrections

A potentially far more important set of corrections concerns terms which involve higher derivatives. In contrast to the two types of corrections discussed above, terms of this type arise in the effective equations more indirectly. Intuitively, they can be understood as capturing the essential dynamical behaviour of a quantum system described by a wave
function [21]. Unlike a classical mechanical system with a finite number of degrees of freedom, a quantum mechanical state or wave function requires the specification of infinitely many variables. In addition to expectation values of basic operators $\hat{q}$ and $\hat{p}$, corresponding to position and momentum, which in a semiclassical state can be identified with the classical variables, there are independent variables which represent the infinitely many different ways the wave function can be deformed from a Gaussian form. These include the spread and various forms of distortions of the wave function.

Fortunately, from a practical point of view, not all these infinite number of variables are expected to be important observationally. Thus in practice one can usually restrict considerations to a finite, and small, number of such deformations. It should, however, be borne in mind that even when such variables are not observed directly, they may still have important indirect dynamical effects, even on the expectation values. As a wave packet moves, it generically spreads and deforms in ways which also affect the evolution of the expectation values. All these infinite number of quantum variables are in principle dynamical and, via their back-reaction on the motion of what can be considered as the classical variables, lead to quantum corrections.

A simple way of illustrating this is by considering an an-harmonic oscillator with the Hamiltonian operator $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{q}) = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 + \frac{1}{3}\lambda\hat{q}^3$, where $V$ is the potential, $m$ is the mass, $\omega$ is the frequency and $\lambda$ is the nonlinear coupling constant. The equation of motion for the expectation value $\langle \hat{p} \rangle$ can then be written, using the commutator $[\hat{p}, \hat{H}] = \hat{p}\hat{H} - \hat{H}\hat{p}$, as

$$ \frac{d}{dt}\langle \hat{p} \rangle = \frac{1}{i\hbar}\langle [\hat{p}, \hat{H}] \rangle = -m\omega^2\langle \hat{q} \rangle - \lambda\langle \hat{q}^2 \rangle = -m\omega^2\langle \hat{q} \rangle - \lambda\langle \hat{q} \rangle^2 - \lambda\Delta q^2 $$

where $\Delta q = \sqrt{\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2}$. As can be seen, the classical equation of motion $\dot{p} = -V'(q)$ is changed by the addition of an extra term, containing $\Delta q$, which would classically be zero and corresponds to quantum fluctuations. Moreover, the rate of change of $\Delta q$ with respect to time, $d\Delta q/dt$, is itself non-zero as the quantum state spreads during its evolution. These variables, together with the infinite number of others which characterize a wave function, are thus coupled to each other. (A systematic approximation in adiabatic regimes where fluctuations do not change rapidly reproduces low energy effective actions [22, 23].) This complicated back-reaction effect is captured by correction terms in effective equations which, from an effective action point of view, in general require higher time derivatives.

The above discussion indicates that such higher derivative correction terms are the most difficult to compute since, in a Hamiltonian formulation, they require all the details of a dynamical semiclassical quantum state through $\Delta q(t)$ and other quantum parameters. At present, they are known precisely only for a free, massless scalar in a spatially flat isotropic universe. In itself this is a very simple system which is not very interesting from a dynamical point of view. Nevertheless, it is closely related to an exactly solvable system [24, 25] which therefore allows it to be used to set up a general perturbation theory. This can enable the study of more realistic models by including a matter potential or inhomogeneities and proceeding perturbatively to obtain the resulting higher derivative
terms which are required for a complete effective analysis. This question is currently under investigation, with some initial results described in Sec. 3.3 below. Clearly the construction of more realistic cosmological scenarios within loop quantum cosmology, which would require the inclusion of such quantum back-reaction terms (to be distinguished from classical back-reaction of developing inhomogeneities), will need to await the outcome of these investigations.

2.4 Combined effects of multiple corrections

The above discussions highlight the fact that the derivation of full effective equations within the loop quantum cosmology framework is likely to include three different types of quantum corrections. These corrections fall into two categories. First, there are those that are direct consequences of quantum geometry and already occur in static settings for which no equations of motion need to be solved. The inverse volume and higher degree corrections discussed above belong to this category. On the other hand the higher derivative terms, which are due to genuine quantum effects and constitute the most important corrections, occur in generic dynamical quantum systems. While effects in the first category become important only when the universe enters high curvature or strong quantum regimes, and subside quickly once it exits such regimes, effects belonging to the second category are active at all times. Even when they are tiny, they can still add up during the long evolution times which are the hallmark of cosmological scenarios, and this cumulative effect can change the dynamical behaviour of the universe dramatically.

Clearly the construction of any realistic cosmological scenario needs to include all these corrections simultaneously. An important issue to bear in mind in this connection is that the relative sizes of these effects are regime dependent, and in general largely unknown at present. As a result, so far the consequences of different loop quantum cosmological corrections have often been studied individually by judicious choices of suitable parameters which single out a specific correction term as the dominant one. For instance, inverse volume effects can be studied in isolation by choosing a large value for the parameter $j$ in the function $D(q)$, which has the effect of shifting the peak in Fig. 1 to larger values of $a$, corresponding to larger volumes. In this way, inverse volume corrections will still be relevant even for relatively large universes, where the other curvature corrections can be ignored. Similarly, the effects of higher order corrections can be studied in isolation by including a free scalar field as the only matter source and choosing its energy, determined by $p_\phi$, to be very large [26, 27]. An analogous procedure, however, is not currently known for genuine quantum effects (higher derivative terms) since they are quantum fluctuations which are determined by the quantum dynamics and cannot be tuned easily. The first two effects, however, can be and have been studied in this way, revealing possible physical effects concerning the questions of origin and early inflationary phases of the universe.

In general, therefore, given this multiplicity of effects, care must be taken in constructing realistic cosmological scenarios within loop quantum cosmology. In particular, two important questions need to be borne in mind. The first concerns the consistency of the set of corrections taken into account. For example, often regimes which require higher order
correction terms would also necessitate the inclusion of higher derivative terms. Thus, unless higher derivative terms can be shown to be absent, keeping only the higher order terms would not be consistent within the perturbation theory employed. Such higher derivative terms have been proved to be absent in spatially flat isotropic models sourced by a free massless scalar [24], but do arise in more general settings of physical interest.

Secondly, one needs to ensure the robustness of results obtained by employing models which take into account only a subset of such corrections. In particular, only results which can be shown to be robust with respect to the addition of corrections which have been ignored, as well as changes in the ambiguity parameters, can be treated as realistic. So far, on the other hand, robustness has most often been studied within the more narrow settings of individual types of corrections. An exception is the ongoing perturbative analysis of [45, 46] as described in Sec. 3.3. This is particularly important in connection with higher derivative corrections, whose dynamical consequences are likely to be far more complicated than those corresponding to higher order terms. This is mainly due to the fact that such higher derivative terms would change the character of the evolution equations by changing their order. In fact ignoring such higher derivative terms amounts to singular perturbations of the evolution equations, with potentially important dynamical consequences. Their physical significance is given by capturing state properties. Only these corrections provide a controlled generic implementation of quantum aspects such as the spreading or deformations of states.

Thus the construction of realistic cosmological scenarios in loop quantum cosmology will need to take into account all three types of corrections. Particular attention needs to be paid to the question of robustness of models which include these corrections only partially. As an example we note that even though inverse volume and higher order corrections have often been found to enhance one another, there are specific settings where they compete against each other, such as when a universe sourced by a free massless scalar enters the deep quantum regime [24].

In the following section we shall summarise some of the phenomenological scenarios constructed using various corrections. Clearly, in view of the potential difficulties discussed above it is important to bear in mind that details and even qualitative aspects of such models may change once the complete set of quantum corrections are taken into account.

3 Non-singular and oscillating universes

According to classical singularity theorems, gravitational collapse generically results in a singularity [3, 4]. These theorems are mainly statements in differential geometry which use Einstein’s equations only in one place, in order to relate the Ricci curvature to the energy-momentum. The use of Einstein equations is nevertheless crucial since it allows the geodesic behaviour, corresponding to trajectories of test masses, to be related to the positive energy conditions. In particular for ordinary matter and reasonable types of sources of energy encountered classically, this would result in focusing of the geodesic families, i.e. gravitational attraction in physical terms. It is this focusing which is essential
for the generic singularity theorems to hold. Thus the singularity theorems rely both on Einstein’s equations as well as the energy conditions that are imposed.

### 3.1 Bounces from inverse volume corrections

The loop quantum corrections discussed above have implications for both ingredients of the singularity theorems. Inverse volume corrections change the matter Hamiltonian and thus can, effectively, break some of the positive energy conditions in quantum gravity regimes. This easily follows from the discussion in previous sections where we noted that the pressure corresponding to the effective fluid becomes negative as we approach the singularity. (In some cases, the effective fluid even behaves like a ‘phantom’ fluid with an equation of state \( w < -1 \).) This amounts to the violation of the null energy condition, independently of the form of the scalar potential \( V(\phi) \), thus removing an important obstacle to singularity avoidance – but without introducing non-physical concepts such as a negative kinetic energy term which is otherwise used to model a ‘phantom fluid’ \[28\]. Similarly, higher order and higher derivative terms would change Einstein’s equations themselves even without considering the form of matter. Although it is not as easy to see, these corrections can change the relation between focusing and positive energy conditions to the extent that even classical matter satisfying all positive energy conditions can start to de-focus geodesics once they enter a strong quantum regime. Thus, just before a classical singularity could be reached, the dynamics of space-time is changed in a way which allows it to evade the devastating consequences predicted by the classical singularity theorems.

Indeed, one can intuitively understand quantum effects as being associated with the emergence of repulsive contributions to the gravitational force on small scales, which would otherwise always be attractive classically \[29\]. This is again easier to see for inverse volume corrections. We can view Eq. (1), taking for simplicity the spatially flat case \( (k = 0) \), as the energy equation of a classical mechanical system with kinetic term \( \dot{a}^2 a \) and an \( a \)-dependent potential \( W(a) = -\frac{1}{2} D(a) p_\phi^2 / a^3 - a^3 V(\phi) \). For a complete analysis we would have to know the dynamical behaviour of \( \phi \). This, however, could be avoided by assuming that the potential \( V(\phi) \) vanishes, which would imply that \( p_\phi \) is constant. Then, \( W(a) \propto -D(a) / a^3 \), which allows the gravitational force to be readily determined as \( F(a) = -W'(a) \propto -3D(a) / a^4 + D'(a) / a^3 \). Classically \( D(a) = 1 \) and the force is negative, pointing to smaller volumes and thus implying attraction. As a result a classical universe, once it starts collapsing, has no way of avoiding a singularity since gravity only enhances the collapse. In a quantum regime, however, \( D(a) \neq 1 \) and in fact increases for small \( a \). The second term in the force is thus positive in these regimes and can dominate the first term. Thus, the gravitational force becomes repulsive and may prevent the total collapse (or drive an inflationary, accelerated expansion phase). These effects thus allow the possibility of a non-singular bounce for a collapsing universe instead of a singular origin.

In fact the universe could be eternal. To see this, consider a positively curved isotropic universe satisfying Eq. (1) with \( k = 1 \). A bounce corresponds to a turning point (a minimum) of the scale factor, treated as a function of time. For a bounce to occur, there must thus exist a solution to Eq. (1) which at some point satisfies \( \dot{a} = 0 \) and \( \ddot{a} > 0 \). This
requires a balance at this point between the right hand side of the equation and the matter Hamiltonian on the left hand side. Such a bounce is impossible to occur in approaching a singularity within a classical framework since in that case as \( a \to 0 \) the matter energy on the left hand side diverges. (Solutions to \( \dot{a} = 0 \) are possible but have \( \ddot{a} < 0 \), thus corresponding to a recollapse.) The effect of including the correction term \( D(q) \) on the left hand side of the Eq. (1) is that on very small scales the energy density decreases, even during the collapse phase, in such a way that it is eventually balanced by the term involving \( a \) on the right hand side. Moreover, \( \dot{a} > 0 \) is then related to the presence of a negative pressure. (See [30, 17] for detailed discussions. Similar bounces result even for spatially flat models when the potential can become negative [31].)

Since positively curved universes can recollapse after an expanding phase, the presence of the bounce allows oscillations, i.e. bounces followed by recollapses, to occur. A transparent way of seeing this is to employ a dynamical systems approach [17] (see also [32]). Assuming for simplicity a constant potential, the effective evolution equations can in this case be expressed as a two-dimensional autonomous system in the form [17]

\[
\dot{H} = \left( \frac{8\pi G}{c^2} V - 3H^2 \right) \left( 1 - \frac{A}{6} \right) + \frac{k c^2}{a^2} \left( \frac{A}{2} - 2 \right), \quad \dot{a} = Ha
\]

where \( H = \dot{a}/a \) is the Hubble parameter, \( A(a) \equiv d\ln D/d\ln a \) and \( V \) is the potential, assumed constant. The equilibrium points of this system, obtained by setting the left hand sides of both equations to zero, correspond to static solutions defined by \( \dot{a} = 0 = \ddot{a} \). It can be shown that depending upon the value of the potential \( V \) there are up to two equilibrium points for this system which correspond to saddle and centre equilibrium points respectively [33]. It is instructive to contrast this with the classical general relativistic case where there is only one equilibrium point – the Einstein static universe – corresponding to a saddle point which is unstable.

A further important finding in this connection is that the center equilibrium point always occurs in the effective domain \( (a < a_s) \) while in contrast the saddle equilibrium point always occurs in the classical domain \( (a > a_s) \). Thus the crucial consequence of the LQC corrections is to induce an extra centre equilibrium point which is capable of supporting oscillations in its neighbourhood and which importantly always occurs in the effective domain. An illustration of the role of the center equilibrium point in determining the dynamics is depicted in Fig. 2. The circular orbits correspond to oscillations around the centre equilibrium point, which eventually get disrupted when the system enters an inflationary phase.

### 3.2 Bounces from higher order corrections

As was mentioned, if the matter source is a free scalar with a zero potential and the universe is spatially flat \( (k = 0) \), then there are no higher derivative correction terms present and in that case the sum of all higher order terms can be trusted to provide a consistent effective equation [24]. In that case, including all higher order correction terms for the example [3],
Figure 2: A stroboscopic phase diagram, showing the density of discretized trajectories in the plane spanned by the scale factor and its time derivative. The circular feature illustrates oscillations around a stable equilibrium point near $a \sim 4$, $da/dt = 0$ around which the universe performs a number (possibly infinite) of small initial oscillations. At some point, it comes close to a saddle point and escapes into an inflationary phase, as indicated by the protruding linear feature.
but ignoring for now the inverse volume effects, the Friedman equation becomes

\[
\frac{8\pi G p_\phi^2}{3c^4 a^5} = \frac{a^3}{\ell_p^2} \left( 1 - \sqrt{1 - 4\frac{\ell_p^2 \dot{a}^2}{c^2 a^2}} \right) \tag{5}
\]

where \( p_\phi \) is constant in the absence of a potential. Defining the constant \( C = 4\pi G \ell_p^2 p_\phi^2 / 3c^4 \) of the dimension length\(^6\), this equation can readily be solved to give \( \dot{a} = (c/\ell_p) a^{-5} C \sqrt{a^6 / C - 1} \).

In contrast to the classical case, there exists a solution for \( \dot{a} = 0 \) at a non-zero value of the scale factor \( a = C^{1/6} \). Moreover, we can take a further time derivative of \( \dot{a} \) and, after some simplifications, compute \( \ddot{a} = -5\dot{a}^2/a + 3c^2 C / a^5 \ell_p^2 \). This demonstrates that at the point where \( \dot{a} = 0 \) we have \( \ddot{a} > 0 \), implying a minimum – or a bounce rather than a collapse. Thus a contracting universe within this framework reaches a non-zero size before re-expanding, never hitting a singularity. This has been discussed in [34] in an oscillatory context in parallel to what is mentioned in Sec. 3.1.

One can even compute precisely how a wave function of the universe behaves, i.e. not only find solutions for expectation values such as \( a \) but also for fluctuations and other wave function parameters as well [24, 25]. It turns out that they evolve through the bounce unscathed, in such a way that the wave function extends to a regime before the classical singularity. Detailed pictures have been developed numerically for this model [26, 27], focusing on unsqueezed Gaussian states only and large values of \( p_\phi \). The results remain true if inverse volume corrections are included, although details of the solutions clearly change. This is an example of a model where different types of quantum corrections have been included. In fact, the set of corrections employed in this case is the complete set, since the neglected higher derivative correction terms are exactly zero for this model.

Despite this seeming robustness, however, small changes to the details of the model, such as for example using an alternative choice of factor ordering of the quantum operator determining the dynamics, could render the higher derivative correction terms non-zero and hence the model incomplete. During the long evolution times for the universe, such effects can easily add up and change the behaviour considerably. Thus, the exact bounce solutions using only higher order terms cannot easily be extended beyond the simplest model. For example, the addition of a matter potential or the inclusion of anisotropies and inhomogeneities, which constitute crucial ingredients for a realistic cosmological model, could lead to strong quantum back-reaction effects caused by the spread and deformation of the quantum wave packet on its mean position. This highlights the need to bear in mind the robustness of the models employed. The occurrence of a bounce in this case could be prevented by quantum back-reaction effects if they turned out to be substantial. While it still remains to be seen whether or not this is the case, some qualitative results are already known as described in the next subsection. Prior to knowing the outcome of such studies, therefore, scenarios of this type while providing suggestive insights, cannot be guaranteed to be robust.
3.3 Role of the quantum gravity state

The crucial question to be understand before one can draw conclusions about the robustness of scenarios in quantum cosmology is the quantum back-reaction of an evolving state on its expectation values. Since cosmology requires long time evolution, such quantum corrections which are typically small but almost always present can add up to yield significant contributions. In general, the presence of such back-reaction effects implies that one is dealing with a highly coupled system of equations which describes the time evolutions of the infinitely many state parameters. This system of equations can be solved exactly only in very rare cases, such as the free scalar system available for cosmology \[24\], i.e. a free, massless scalar in a spatially flat universe \((k = 0)\). As a free system, in analogy with the harmonic oscillator in quantum mechanics or free quantum field theories, this model can be used as the basis of a perturbation expansion in interacting systems. In our context, this can in particular be applied to scalar field models with non-vanishing potentials, thus providing them with mass or self-interactions, or to models with anisotropies and inhomogeneities. So far, the case of a non-vanishing potential has been studied in some detail.

Thus, there are new correction terms which change the free solutions and which can be derived and analyzed. It is particularly important to reanalyze the bouncing scenarios in presence of these new correction terms in order to make sure that earlier conclusions concerning the existence of bounces are robust. It is not just the presence of the potential, which already changes the classical dynamics, but as seen in the quantum mechanical example of Sec. 2.3 also deviations from the free model which result in the spreading and other features of a state to back-react on the effective evolution of expectation values. Thus in addition to inverse volume and higher order corrections, corrections of the third type discussed above become important.

Since the spreading itself is time dependent, there are new equations in addition to (1) for the dynamics of quantum variables which do not have classical counterparts. Already in the free system, this provides crucial information on the behavior of dynamical coherent states showing how a state generically spreads as it evolves. With such solutions, one can find relations between, e.g., fluctuations of a generic state before and after the bounce. The ratio of these quantum variables turns out to be rather weakly restricted \[35, 36\], such that even in the highly controlled free system the pre-bounce state is not strongly determined by conditions one may pose after the bounce. The degree of asymmetry of fluctuations before and after the bounce is determined by a parameter \(\eta\) which defines the squeezing of the state of the universe.

For models with a self-interacting scalar field, the same spreading effects happen but now they back-react on the behavior of expectation values. Thus, Eq. (1) is replaced with an effective Friedmann equation capturing this quantum back-reaction. This has the form \[46, 47\]

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \left( \rho \left( 1 - \frac{\rho_Q}{\rho_{\text{crit}}} \right) + \frac{1}{2} \sqrt{1 - \frac{\rho_Q}{\rho_{\text{crit}}} \eta V(\phi) + \frac{a^6 V(\phi)^2}{2\rho_{\phi}^2} \eta^2} \right)
\]  

(6)
where $\rho_Q$ is an expression for energy density carrying quantum corrections and $\rho_{\text{crit}} = \frac{3c^4}{8\pi G \mu^2}$ is a critical density defined in terms of a fundamental length parameter $\mu$ which arises, as in the case of $a_*$, due to quantum gravity. (The previous example (5) for a free scalar with $V(\phi) = 0$, when solved for $(\dot{a}/a)^2$, corresponds to $\mu = \ell_P$. In general, $\mu$ can differ from the Planck length by numerical factors but, more importantly, can also depend on the scale factor $a$. The precise behavior reflects the underlying class of quantum gravity states whose evolution is described by these effective equations [14, 48].)

Moreover, $\eta$ in (6) is a parameter determining the dynamical squeezing of the state. Thus, the precise squeezing, or the value of quantum correlations summarized by the parameter $\eta$, are important for the fate of the bounce: when $\rho_Q \sim \rho_{\text{crit}}$, such that a free model would bounce at this point, with the dominant term on the right hand side depends on $\eta$. Now since this is not a constant but a dynamical variable which changes in time; it is generically non-zero at the would-be bounce. Consequently, the condition $\rho_Q \sim \rho_{\text{crit}}$ would not result in a bounce because in this case $\dot{a} \neq 0$. Whether or not there will always exist a time time at which $\dot{a}$ vanishes is yet to be determined from an analysis of the coupled equations of motion which is still in progress. Only if this is ensured can one conclude the robustness of the bounce.

4 Complete universe scenarios: Combining conceptual and observational issues

As was mentioned above, despite its many recent successes, one of the major shortcomings of general relativistic standard cosmology is the presence of an initial singularity at which the classical laws of physics break down and with it the classical notion of space-time, causality, and in effect scientific predictability. Thus the question of the origin of the universe cannot be addressed within the classical framework. In the above sections we have briefly discussed the general outlines of how quantum gravity effects may change the classical picture and resolve this dilemma.

In addition to the question of origin, the other central question in standard cosmology is how to situate the early inflationary phase of the universe within a fundamental theory of quantum gravity. According to current cosmological understanding such a phase is crucial for structure formation. Thus clearly a viable cosmological scenario needs to deal with both questions: providing mechanisms to remove classical singularities as well as initiating a successful phase of inflation (or an alternative to provide structure formation).

In the above discussions we have already alluded to this possibility, particularly since the presence of an inflationary phase is likely to be closely related to the presence of bounces, given that both require the condition $\ddot{a} > 0$ for their occurrence. A model successfully answering both these questions may provide the fascinating prospect of indirectly testing the bounce phase through precise observations of background radiations.

Clearly much work remains to be done, both theoretically and observationally, in order to be in a position to test such a possibility. Nevertheless, the recent developments make
it at least possible to study whether the existence of cosmologically viable scenarios of this type is plausible within the LQC framework.

4.1 An emergent non-singular universe setting the initial conditions for inflation

Attempts to construct oscillating cosmological models go back a number of decades [37]. These clearly predate the recent attempts to employ specific quantum gravity effects in order to avoid a singular origin for the universe. An interesting example of such an attempt is the so called emergent scenario according to which the universe is past eternal by being asymptotic to an Einstein static universe [38]. This provides an example of a non-singular cosmological model within the general relativistic framework. Despite its novelty, the main shortcoming of this scenario is that, as we noted above, within the general relativistic framework, the classical Einstein static universe corresponds to a saddle equilibrium point which is unstable to all homogeneous-isotropic perturbations, including quantum fluctuations. (Interestingly though this solution has been shown to be stable with respect to inhomogeneous perturbations [39].) This therefore prevents such a model from being past eternal, unless the universe is initially at the equilibrium point, which amounts to extreme fine tuning.

As we saw above the situation is radically different when LQC inverse volume corrections are included. In the semi-classical domain the appropriate equilibrium point turns out to be a centre. An important consequence of this change in the nature of the equilibrium point is that small perturbations, which are unavoidably present, do not lead to a radical departure from the equilibrium state, but result in oscillations about it. This allows the construction of a plausible emergent scenario with a non-singular origin for the universe [33, 40].

As an example of such a scenario, consider a positively curved FLRW universe sourced by a scalar field with a potential which qualitatively has the form given in Fig. 3. Such potentials have been shown to arise in a number of settings of interest, including in attempts at renormalization of theories of quantum gravity [41], as well as in low-energy limits of superstring theories [42]. Now let the field be initially placed far away to the left along the $\phi$-axis (i.e. $\phi \rightarrow -\infty$) with $\dot{\phi}_\text{in} > 0$ and let the universe be close to the centre equilibrium point and hence oscillating about it. A crucial consequence of the oscillations of the universe is to push the field uni-directionally to the right along the $\phi$-axis and eventually up the potential to the point where $\phi$ is such that $V(\phi) \approx \dot{\phi}_\text{in}^2/2$ at which point the oscillations stop (see [17, 33] for the details). The field then turns round and the slow-roll inflation begins, finally leading to oscillations about a minimum of the potential and subsequent reheating at the start of the standard hot big bang phase. Importantly, it has been shown that such oscillations can push the field high enough up the potential to successfully set the initial conditions for the onset of inflation [33] (see also [43] for an analysis in the presence of other matter sources). This allows the universe to subsequently undergo a sufficiently large number of $e$-foldings required by observations.
4.2 Higher order corrections and oscillations

The above oscillatory scenario employs inverse volume corrections. Such models have also been discussed for higher order corrections to construct oscillating universe models \[44, 34\]. These models are based on subseries of higher order terms in $\dot{a}$ with arbitrarily high powers. Still, these are not the complete equations which would be more difficult to obtain. This is because such oscillating models necessarily deviate from the exactly solvable one of a free scalar in a spatially flat universe, requiring non-zero potentials. As a result the inclusion of such an entire subseries of higher order terms is not consistent with ignoring the higher derivative correction terms. For example, in presence of a scalar potential, as in the case of the model considered in \[44\], a back-reaction occurs between the fluctuations of a quantum state and its expectation value, implying that for the model to be consistent further corrections need to be taken into account as in Sec. \[3.3\]. Almost all the higher order terms are exceedingly small compared to other corrections, and yet they are all added up. If one were to treat higher order terms as an approximation to the full equations in the presence of quantum back-reaction, self-consistency of the perturbative analysis would require the inclusion of only a finite number of higher order correction terms, rather than the full set. (In the variables introduced in \[24\], on the other hand, one can essentially perform a resummation to ensure this consistency.) This gives rise to equations which are noticeably different from those that have so far been used in such models.

Oscillations also arise if one keeps the scalar field massless and non-interacting but allows positive spatial curvature, as was numerically studied in \[49\] for initial states which are unsqueezed. Although back-reaction still occurs, it is only active briefly at each recollapse since the bounce phases are still well described by the solvable model. There is
thus not much spread of the quantum state unless one considers many cycles of bounces and recollapses. In the presence of a scalar potential, however, back-reaction especially of correlations can be so strong that it might even prevent a single bounce from occurring. This example demonstrates that at present also scenarios based on higher order need to be treated with great care. Methods for a complete treatment including all quantum corrections in isotropic models are now available and being applied to oscillating scenarios [45, 46].

5 Conclusions

We have seen that loop quantum gravity gives rise to three different types of corrections in the effective equations: inverse volume corrections, higher order terms in $\dot{a}$ and quantum back-reaction, related to higher time derivative terms. The first two are typical for quantum geometry effects as they arise from loop quantum gravity, while the latter are of genuine quantum origin. At present the complete set of corrections is known precisely only for the case of a free massless scalar in a flat isotropic geometry [24, 25]. More general settings are then treated by perturbations around the solvable model, much like interacting quantum field theories are dealt with by perturbations around a free theory, in more general settings it still remains to be determined. However, as these developments are quite recent, and still ongoing, most models considered so far include only a subset of corrections that are expected to be present. The overlooked corrections, however, specially the higher derivative corrections, can potentially change the behaviour of the models dramatically. Although each correction is typically small, long evolution times are by definition involved in cosmological scenarios, where even small terms can have lasting effects. In view of these limitations, it is therefore important to pay particular attention to the questions of consistency and robustness of models employed, if their predictions are to be treated as wholly reliable.

Despite these shortcomings, however, recent developments in LQC have provided a framework which in principle allows the systematic derivation, presently in progress, and testing of all relevant quantum corrections. In yet more complicated models quantum dynamics has been analyzed in regimes where effective equations break down and are replaced by difference equations for the evolution of the quantum states which again have been shown to remain non-singular [50, 51, 52, 53]. These difference equations are capable of allowing the evolution of a state through strong quantum regimes where no effective description is valid. In fact, for general solutions to a difference equation it is easily possible for the quantum fluctuations to remain strong even after evolving through a classical singularity. Thus, although the quantum state would stay well-defined, it may not easily allow a semiclassical geometry after a “bounce.” Even recent studies of the effective equations indicate that fluctuations generically differ on both sides of the bounce [35, 36]. An emergence of a classical universe after a bounce could then only be understood in more complicated terms, based e.g. on decoherence [54]. The difficulty with employing difference equations for a quantum state rather than an effective geometry, however, is that they are
less intuitive and not easily conducive to the development of cosmological scenarios.

Despite the shortcomings mentioned above, the development of preliminary scenarios, using partial inclusion of corrections, have nevertheless been valuable. In particular, the emergent universe scenario sketched here, based on effective equations, provides an example of how LQC effects can potentially give rise to physically reasonable non-singular behaviour. The important features of this scenario are that it is non-singular, past-eternal, and oscillating. Importantly, the oscillations have a crucial function, providing a mechanism for setting the initial conditions for successful inflation. Furthermore, this scenario is in principle observationally testable. It sets the initial conditions for the standard hot big bang expansion which is broadly supported by current astronomical observations, but requires a positively curved universe. Thus although this scenario is at present compatible with observations, it can in principle be ruled out, especially given the planned future observations with far greater resolutions. Also, the turn over of the scalar field can potentially leave signatures in the CMB which may be observable [55]. Finally, inhomogeneous perturbations in effective equations [56, 57, 58, 59] may leave additional observable imprints [60, 61, 62, 63]. It should also be mentioned here that a number of other recent early universe models postulate a bounce (e.g. [64]), but none in a fully satisfactory way within the context of a theory of quantum gravity and without introducing other singularities.

The above discussion has demonstrated the potential of LQC to provide a unified framework to deal with the questions of origin and the early inflationary phase of the universe. Despite these preliminary successes, however, a great deal of work remains to be done. In particular, the construction of consistent and robust realistic scenarios needs to await the evaluation of the complete set of corrections which are expected to be present.

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