DIRAC VERSUS REDUCED PHASE SPACE QUANTIZATION

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Abstract

The relationship between the Dirac and reduced phase space quantizations is investigated for spin models belonging to the class of Hamiltonian systems having no gauge conditions. It is traced out that the two quantization methods may give similar, or essentially different physical results, and, moreover, it is shown that there is a class of constrained systems, which can be quantized only by the Dirac method. A possible interpretation of the gauge degrees of freedom is given.

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1 Introduction

There are two main methods to quantize the Hamiltonian systems with first class constraints: the Dirac quantization\cite{1} and the reduced phase space quantization\cite{2}, whereas two other methods, the path integral method\cite{3,2} and the BRST quantization\cite{4} being the most popular method for the covariant quantization of gauge-invariant systems, are based on and proceed from them\cite{2,5}. The basic idea of the Dirac method consists in imposing quantum mechanically the first class constraints as operator conditions on the states for singling out the physical ones\cite{1}. The reduced phase space quantization first identifies the physical degrees of freedom at the classical level by the factorization of the constraint surface with respect to the action of the gauge group, generated by the constraints. Then the resulting Hamiltonian system is quantized as a usual unconstrained system\cite{2}. Naturally, the problem of the relationship of these two methods arises. It was discussed in different contexts in literature\cite{6}, and there is an opinion that the differences between the two quantization methods can be traced out to a choice of factor ordering in the construction of various physical operators.

We investigate the relationship of the two methods of quantization for the special class of Hamiltonian systems with first class constraints corresponding to different physical models of spinning particles. The specific general property of the examples of constrained systems considered here is the following: their constraints generate SO(2) transformations and, hence, corresponding gauge orbits topologically are one-spheres $S^1$. This fact implies that these systems do not admit gauge conditions, and, therefore, for the construction of their reduced phase spaces we shall use a general geometrical approach to the Dirac–Bergmann theory of the constrained systems\cite{4,8}.

2 Plane Spin Model

The first model we are going to consider is the plane spin model, which is a subsystem of the $(3+1)$–dimensional models of massless particles with arbitrary helicity\cite{9}, and of the $(2+1)$–dimensional relativistic models of fractional spin particles\cite{10}. The initial phase space of the model is a cotangent bundle $T^*S^1$ of the one–dimensional sphere $S^1$, that is a cylinder $S^1 \times \mathbb{R}$. It can be described locally by an angular variable $0 \leq \varphi < 2\pi$ and the conjugate momentum $S \in \mathbb{R}$. The symplectic two–form $\omega$ in terms of the local variables $\varphi$, $S$ has the form $\omega = dS \wedge d\varphi$, and, thus, we have locally $\{\varphi, S\} = 1$. Actually, any $2\pi$–periodical function of the variable $\varphi$ that is considered as a variable, taking values in $\mathbb{R}$, can be considered as a function on the phase space, i.e., as an observable, and any observable is connected with the corresponding $2\pi$–periodical function. Therefore, we can introduce the functions $q_1 = \cos \varphi$, $q_2 = \sin \varphi$, $q_1^2 + q_2^2 = 1$, as the dependent functions on the phase space of the system. For these functions we have $\{q_1, q_2\} = 0$, $\{q_1, S\} = -q_2$, $\{q_2, S\} = q_1$. Any function on the phase space can be considered as a function of dependent coordinates $q_1$, $q_2$ and $S$, which will be taken below as the quantities, forming a restricted set of observables whose quantum analogs have the commutators which are in the direct correspondence with their Poisson brackets.

We come to the plane spin model by introducing the ‘spin’ constraint

$$\psi = S - \theta = 0,$$ \hfill (2.1)
where $\theta$ is an arbitrary real constant. Let us consider the Dirac quantization of the system. To this end we take as the Hilbert space the space of complex $2\pi$–periodical functions of the variable $\varphi$ with the scalar product $(\Phi_1, \Phi_2) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_1(\varphi) \Phi_2(\varphi) d\varphi$. The operators $\hat{q}_1$ and $\hat{q}_2$, corresponding to the functions $q_1$ and $q_2$, are the operators of multiplication by the functions $\cos \varphi$ and $\sin \varphi$, respectively, whereas the operator $\hat{S}$ is defined by $\hat{S}\Phi = (-id/d\varphi + c)\Phi$, where $c$ is an arbitrary real constant. The operators $\hat{q}_1$, $\hat{q}_2$ and $\hat{S}$ are Hermitian operators with respect to the introduced scalar product, and they satisfy the relation $[\hat{A}, \hat{B}] = i\{A, B\}$, $A, B = q_1, q_2, S$. The quantum analog of the constraint (2.3) gives the equation for the physical state wave functions: $(\hat{S} - \theta)\Phi_{phys} = 0$. Decomposing the function $\Phi_{phys}(\varphi)$ over the orthonormal basis, formed by the functions $e^{ik\varphi}$, $k \in \mathbb{Z}$, we find this equation has a nontrivial solution only when $c = \theta + n$, where $n$ is some fixed integer, $n \in \mathbb{Z}$. In this case the corresponding physical normalized wave function is $\Phi_{phys}(\varphi) = e^{in\varphi}$. The only physical operator $\hat{S}$, i.e., an operator commuting with the quantum constraint $\hat{\psi}$ here is $\hat{S}$, which is reduced to the constant $\theta$ on the physical subspace.

Now we come back to the classical theory in order to construct the reduced phase space of the model. Let us show that for the surface, defined by Eq. (2.3), there is no 'good' gauge condition, but, nevertheless, the reduced phase space of the system can be constructed. Indeed, it is clear that the one–parameter group of transformations, generated by the constraint $\psi$, consists of the rotations of the phase space. This group acts transitively on the constraint surface, and we have only one gauge orbit, which is the constraint surface itself. The gauge conditions must single out one point of an orbit. In our case we have to define only one gauge condition, let us denote it by $\chi$. The function $\chi$ must be such that the pair of equations $\psi = 0$, $\chi = 0$ would determine a set, consisting of only one point, and in this point we should have $\{\psi, \chi\} \neq 0$. Recall that any function on the phase space of the system under consideration can be considered as a function of the variables $\varphi$ and $S$, which is $2\pi$–periodical with respect to $\varphi$. Thus, we require the $2\pi$–periodical function $\chi(\varphi, S)$ turn into zero at only one point $\varphi = \varphi_0$ from the interval $0 \leq \varphi < 2\pi$ when $S = \theta$. Moreover, we should have $\{\psi, \chi\}(\varphi_0, \theta) = -\partial \chi(\varphi, \theta)/\partial \varphi|_{\varphi=\varphi_0} \neq 0$. It is clear that such a function does not exist. Nevertheless, here we have the reduced phase space that consists of only one point. Therefore, the reduced space quantization is trivial: physical operator $\hat{S}$ takes here constant value $\theta$ in correspondence with the results obtained by the Dirac quantization method. When the described plane spin model is a subsystem of some other system, the reduction means simply that the cylinder $T^*S^1$ is factorized into a point, where $S = \theta$, and that wave functions do not depend on the variable $\varphi$.

Let us point out one interesting analogy in interpretation of the situation with nonexistence of a global gauge condition. Here the condition of $2\pi$–periodicity can be considered as a 'boundary' condition. If for a moment we forget about it, we can take as a gauge function any monotonic function $\chi(\varphi, S)$, $\chi \in \mathbb{R}$, such that $\chi(\varphi_0, \theta) = 0$ at some point $\varphi = \varphi_0$, and, in particular, we can choose the function $\chi(\varphi, S) = \varphi$. The 'boundary' condition excludes all such global gauge conditions. In this sense the situation is similar to the situation in the non–Abelian gauge theories where without taking into account the boundary conditions for the fields it is also possible to find global gauge conditions, whereas the account of those leads, in the end, to the nonexistence of global gauge conditions [11].
3 Rotator Spin Model

Let us consider now the rotator spin model \[12\]. The initial phase space of the system is described by a spin three–vector \(S\) and a unit vector \(q\), \(q^2 = 1\), being orthogonal one to the other, \(qS = 0\). The variables \(q_i\) and \(S_i, i = 1, 2, 3\), can be considered as dependent coordinates in the phase space of the system. The Poisson brackets for these coordinates are \(\{q_i, q_j\} = 0, \{S_i, S_j\} = \epsilon_{ijk}S_k, \{S_i, q_j\} = \epsilon_{ijk}q_k\). Using these Poisson brackets, we find the following expression for the symplectic two–form: \(\omega = dp_i \wedge dq_i = d(\epsilon_{ijk}S_jq_k) \wedge dq_i\). Introducing the spherical angles \(\varphi, \vartheta\) \((0 \leq \varphi < 2\pi, 0 \leq \vartheta \leq \pi)\) and the corresponding momenta \(p_\varphi, p_\vartheta \in \mathbb{R}\), we can write the parameterization for the vector \(q, q = (\cos \varphi \sin \vartheta, \sin \varphi \sin \vartheta, \cos \vartheta)\) and corresponding parameterization for the vector \(S = S(\vartheta, \varphi, p_\varphi, p_\vartheta)\), whose explicit form we do not write down here (see Ref. \[8\]). Then for the symplectic two–form we get the expression \(\omega = dp_\varphi \wedge d\vartheta + dp_\vartheta \wedge d\varphi\). From this relation we conclude that the initial phase space of the system is symplectomorphic to the cotangent bundle \(T^*S^2\) of the two–dimensional sphere \(S^2\), furnished with the canonical symplectic structure.

The rotator spin model is obtained from the initial phase space by imposing the constraint \[
\psi = \frac{1}{2}(S^2 - \rho^2) = 0, \quad \rho > 0,
\] (3.1) fixing the spin of the system. Using the Dirac method, we quantize the model in the following way. The state space is a space of the square integrable functions on the two–dimensional sphere. The scalar product is \((\Phi_1, \Phi_2) = \int_{S^2} \Phi_1(\varphi, \vartheta)\Phi_2(\varphi, \vartheta)\sin \vartheta d\vartheta d\varphi\). The above mentioned parameterization allows us to use as the operator \(\hat{S}\) the usual orbital angular momentum operator expressed via spherical angles. The wave functions as the functions on a sphere are decomposable over the complete set of the spherical harmonics: \(\Phi(\varphi, \vartheta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm}Y_{lm}(\varphi, \vartheta)\), and, therefore, the quantum analog of the first class constraint (3.1),
\[
(S^2 - \rho^2)\Phi_{phys} = 0,
\] (3.2) leads to the quantization condition for the constant \(\rho\):
\[
\rho^2 = n(n + 1),
\] (3.3) where \(n > 0\) is an integer. Only in this case equation (3.2) has a nontrivial solution of the form \(\Phi_{phys}^{n}(\vartheta, \varphi) = \sum_{m=-n}^{n} \Phi_{nm}Y_{nm}(\varphi, \vartheta)\), i.e., with the choice of (3.3) we get the states with spin equal to \(n\): \(\hat{S}^2\Phi_{phys}^{n} = n(n + 1)\Phi_{phys}^{n}\). Thus, we conclude that the Dirac quantization leads to the quantization (3.3) of the parameter \(\rho\) and, as a result, the quantum system describes the states with integer spin \(n\).

Let us turn now to the construction of the reduced phase space of the system. The constraint surface of the model can be considered as a set composed of the points specified by two orthonormal three–vectors. Each pair of such vectors can be supplemented by a unique third three–vector, defined in such a way that we get an oriented orthonormal basis in three dimensional vector space. It is well known that the set of all oriented orthonormal bases in three dimensional space can be smoothly parameterized by the elements of the Lie group \(SO(3)\). Thus, the constraint surface in our case is diffeomorphic to the group manifold of the Lie group \(SO(3)\).

The one–parameter group of canonical transformations, generated by the constraint \(\psi\), acts in the following way: \(q(\tau) = q \cos(S\tau) + (S \times q)S^{-1}\sin(S\tau), S(\tau) = S\), where
\[ S = \sqrt{S^2}. \]
Hence, we see that the gauge transformations are the rotations about the direction, given by the spin vector. Thus, in the case of a general position the orbits of the one–parameter group of transformations under consideration are one dimensional spheres. Note, that only the orbits, belonging to the constraint surface where \( S = \rho \neq 0 \), are interesting to us. It is clear that an orbit is uniquely specified by the direction of the spin three–vector \( S \) whose length is fixed by the constraint \( \psi \). As a result of our consideration, we conclude that the reduced phase space of the rotator spin model is the coset space \( \text{SO}(3)/\text{SO}(2) \), which is diffeomorphic to the two–dimensional sphere \( S^2 \). Due to the reasons discussed for the preceding model there is no gauge condition in this case either. In fact, since \( \text{SO}(3) \) is a nontrivial fiber bundle over \( S^2 \), we can neither find a mapping from \( S^2 \) to \( \text{SO}(3) \) whose image would be diffeomorphic to the reduced phase space. In other words, in this case the reduced phase space cannot be considered as a submanifold of the constraint surface.

Our next goal is to write an expression for the symplectic two–form on the reduced phase space. We can consider the variables \( S_i \) as dependent coordinates in the reduced phase space, and the symplectic two–form on it may be expressed in terms of them. With the help of an orthonormal basis formed by the vectors \( q, s = S/S \) and \( q \times s \), we get for the symplectic two-form on the reduced phase space the following expression [8]:

\[ \omega = -\frac{1}{2\rho^2} (S \times dS) \wedge dS. \] (3.4)

Thus, we see that the dependent coordinates \( S_i \) in the reduced phase space of the system provide a realization of the basis of the Lie algebra \( \text{so}(3) \):

\[ \{S_i, S_j\} = \epsilon_{ijk}S_k. \] (3.5)

The quantization on the reduced phase space can be performed with the help of the geometric quantization method proceeding from the classical relations (3.4), (3.5) and \( S^2 = \rho^2 \). This was done in detail, e.g., in Ref. [17], and we write here the final results of this procedure. The constant \( \rho \) is quantized:

\[ \rho = j, \quad 0 < 2j \in \mathbb{Z}, \] (3.6)
i.e., it can take only integer or half–integer value, and the Hermitian operators, corresponding to the components of the spin vector, are realized in the form:

\[ \hat{S}_1 = \frac{1}{2}(1-z^2)d/dz + jz, \quad \hat{S}_2 = \frac{1}{2}(1+z^2)d/dz - jz, \quad \hat{S}_3 = zd/dz - j, \]

where \( z = e^{-i\vartheta}/2 \), or, in terms of the dependent coordinates, \( z = (S_1 - iS_2)/(\rho + S_3) \). Operators \( \hat{S}_i \) act in the space of holomorphic functions \( f(z) \) with the scalar product \( \langle f_1, f_2 \rangle = \frac{2j+1}{\pi} \int \overline{f_1(z)}f_2(z)(1 + |z|^2)^{-(2j+2)}d^2z \), in which the functions \( \psi^m_j \propto z^{j+m} \), \( m = -j, -j+1, ..., j \), form the set of eigenfunctions of the operator \( \hat{S}_3 \) with the eigenvalues \( s_3 = m \). These operators satisfy the relation \( \hat{S}_2 = j(j+1) \), and, therefore, we have the \( (2j+1) \)–dimensional irreducible representation \( D_j \) of the Lie group SU(2).

Thus, we see that for the rotator spin model the reduced phase space quantization method leads to the states with integer or half–integer spin, depending on the choice of the quantized parameter \( \rho \), and gives in general the results physically different from the results obtained with the help of the Dirac quantization method. Let us stress once again here that within the Dirac quantization method in this model the spin operator \( \hat{S} \) has a nature of the orbital angular momentum operator, and it is this nature that does not allow spin to take half-integer values [18].
4 Top Spin Model

Let us consider now the top spin model \[^{13}\]. The initial phase space of the model is described by the spin three–vector \(\mathbf{S}\), and by three vectors \(\mathbf{e}_i\) such that \(\mathbf{e}_i \times \mathbf{e}_j = \delta_{ij}\). Denote the components of the vectors \(\mathbf{e}_i\) by \(E_{ij}\). The components \(S_i\) of the vector \(\mathbf{S}\) and the quantities \(E_{ij}\) form a set of dependent coordinates in the phase space of the system. The corresponding Poisson brackets are

\[
\{E_{ij}, E_{kl}\} = 0, \quad \{S_i, E_{jk}\} = \epsilon_{ikl}E_{jl}, \quad \{S_i, S_j\} = \epsilon_{ijk}S_k.
\]

(4.1)

The vectors \(\mathbf{e}_i\) form a right orthonormal basis in \(\mathbb{R}^3\). The set of all such bases can be identified with the three–dimensional rotation group. Taking into account Eqs. (4.1) we conclude that the initial phase space is actually the cotangent bundle \(T^*\text{SO}(3)\), represented as the manifold \(\mathbb{R}^3 \times \text{SO}(3)\). Using Eqs. (4.1), one can get the following expression for the symplectic two–form \(\omega\) on the initial phase space: \(\omega = \frac{1}{2}d(\mathbf{S} \times \mathbf{e}_l) \wedge d\mathbf{e}_l = \frac{1}{2}d(\epsilon_{ijk}S_jE_{lk}) \wedge dE_{li}\).

It is useful to introduce the variables \(J_i = \mathbf{e}_i \mathbf{S} = E_{ij}S_j\). For these variables we have the following Poisson brackets: \(\{J_i, E_{jk}\} = -\epsilon_{ijl}E_{lk}\), \(\{J_i, J_j\} = -\epsilon_{ijk}J_k\). Note, that we have the equality \(S_iS_i = J_iJ_i\).

The phase space of the top spin model is obtained from the phase space, described above, by introducing two first class constraints

\[
\psi = \frac{1}{2}(\mathbf{S}^2 - \rho^2) = 0, \quad \chi = \mathbf{S}\mathbf{e}_3 - \kappa = 0,
\]

(4.2)

where \(\rho > 0, |\kappa| < \rho\). Consider now the Dirac quantization of the model. Let us parameterize the matrix \(E\), which can be identified with the corresponding rotation matrix, by the Euler angles, \(E = \hat{E}(\alpha, \beta, \gamma)\), and use the representation where the operators, corresponding to these angles are diagonal. In this representation state vectors are functions of the Euler angles, and the operators \(\hat{S}_i\) and \(\hat{J}_i\) are realized as linear differential operators, acting on such functions \[^{19}\]. The quantum analogs of the constraints \(\psi\) and \(\chi\) turn into the equations for the physical states of the system:

\[
(\hat{S}_i^2 - \rho^2)\Phi_{\text{phys}} = 0, \quad (\hat{J}_i^2 - \kappa)\Phi_{\text{phys}} = 0.
\]

(4.3)

An arbitrary state vector can be decomposed over the set of the Wigner functions, corresponding to either integer or half–integer spins \[^{19}\]: \(\Phi(\alpha, \beta, \gamma) = \phi_{jmk}\hat{D}_m^j(\alpha, \beta, \gamma)\), where \(j = 0, 1, \ldots\), or \(j = 1/2, 3/2, \ldots\), and \(k, m = -j, -j+1, \ldots, j\). The Wigner functions \(\hat{D}_m^j\) have the properties: \(\hat{S}_i^2\hat{D}_m^j = j(j+1)\hat{D}_m^j\), \(\hat{S}_3\hat{D}_m^j = m\hat{D}_m^j\), \(\hat{J}_i\hat{D}_m^j = k\hat{D}_m^j\). Using the decomposition of the state vector, we see that Eqs. (4.3) have nontrivial solutions only when \(\rho^2 = j(j+1)\), and \(\kappa = k\), for some integer or half–integer numbers \(j\) and \(k\), such that \(-j \leq k \leq j\). In other words we get the following quantization condition for the parameters of the model:

\[
\rho^2 = j(j+1), \quad \kappa = k, \quad -j \leq k \leq j, \quad 0 < 2j \in \mathbb{Z}.
\]

The corresponding physical state vectors have the form

\[
\Phi_{\text{phys}}(\alpha, \beta, \gamma) = \sum_{m=-j}^{j} \varphi_m\hat{D}_m^j(\alpha, \beta, \gamma).
\]
Thus, we see that the Dirac quantization of the top spin model leads to an integer or half–integer spin system.

Proceed now to the construction of the reduced phase space of the system. As the constraints ψ and χ have zero Poisson bracket, we can consider them consecutively. Let us start with the constraint ψ. From the expressions for the Poisson brackets (4.1) it follows that the group of gauge transformations, generated by the constraint ψ, acts in the initial phase space variables as follows:

\[ e_i(\tau) = e_i \cos(S\tau) + (S \times e_i)S^{-1}\sin(S\tau) + S(Se_i)S^{-2}(1 - \cos(S\tau)), \quad S(\tau) = S, \]

where \( S = \sqrt{S^2} \). We see that the transformation under consideration have the sense of the rotation by the angle \( S\tau \) about the direction of the spin vector. Let us consider the initial phase space of the system being diffeomorphic to \( R^3 \times SO(3) \) as a trivial fibre bundle over \( R^3 \) with the fibre \( SO(3) \). The gauge transformations act in fibres of this bundle. It is clear that the constraint surface, defined by the constraint ψ, is a trivial fibre subbundle \( S^2 \times SO(3) \). As \( SO(3)/SO(2) = S^2 \), then after the reduction over the action of the gauge group we come to the fibre bundle over \( S^2 \) with the fibre \( S^2 \). As it follows from general theory of fibre bundles [24], this fibre bundle is again trivial. Thus the reduced phase space, obtained using only the constraint ψ, is the direct product \( S^2 \times S^2 \). The symplectic two–form on this reduced space can be written in the form \( \omega = -(2\rho^2)^{-1}(\epsilon_{ijk}S_idS_j \wedge dS_k - \epsilon_{ijk}J_idJ_j \wedge dJ_k) \). Here the quantities \( S_i \) and \( J_i \) form a set of dependent coordinates in the reduced phase space under consideration: \( S_iS_i = J_iJ_i = \rho^2 \).

Let us turn our attention to the constraint χ. It is easy to get convinced that the transformations of the gauge group, generated by this constraint act in the initial phase space in the following way: \( e_i(\tau) = e_i \cos(\tau) + (e_3 \times e_i) \sin(\tau), \quad i = 1, 2, e_3(\tau) = e_3, \quad S(\tau) = S \).

So, we see that the gauge group, generated by the constraint χ, acts only in one factor of the product \( S^2 \times S^2 \), which is a reduced phase space obtained by us after reduction with the help of the constraint ψ. Thus we can consider only that factor, which is evidently described by the quantities \( J_i \). From such point of view, the constraint surface, defined by the constraint χ, is a one dimensional sphere \( S^1 \), where the group of gauge transformations acts transitively. Hence, after reduction we get only one point. Thus, the final reduced phase space is a two–dimensional sphere \( S^2 \), and the symplectic two–form on the reduced phase space has the form given by Eq. (3.4). Therefore, the reduced phase space we have obtained, coincides with the reduced phase space for the rotator spin model. Hence the geometric quantization method gives again the quantization condition (3.6) for the parameter \( \rho \), while the parameter \( \kappa \) remains unquantized here. Therefore, while for this model unlike the previous one, two methods of quantization lead to the quantum system, describing either integer or half–integer spin states, nevertheless, the corresponding quantum systems are different: the Dirac method gives discrete values for the observable \( \hat{J}_3 \), whereas the reduced phase space quantization allows it to take any value \( \kappa \), such that \( \kappa^2 < j^2 \) for a system with spin \( j \).

Let us note here one interesting property of the system. We can use a combination of the Dirac and reduced phase space quantization methods. After the first reduction with the help of the constraint ψ, the system, described by the spin vector and the ‘isospin’ vector \( \tau \) with the components \( I_i = -J_i, S_iS_i = I_iI_i \), can be quantized according to Dirac by imposing the quantum analog of the constraint χ on the state vectors for singling out the physical states. In this case we have again the quantization of the parameter \( \kappa \) as in the pure Dirac quantization method, and, therefore, here the observable \( \hat{J}_3 \) can take
only integer or half-integer value. Hence, in this sense, such a combined method gives the results coinciding with the results of the Dirac quantization method.

5 Discussion and conclusions

The first considered model gives an example of the classical constrained system with finite number of the degrees of freedom for which there is no gauge condition, but nevertheless, the reduced phase space can be represented as a submanifold of the constraint surface. As we have seen, Dirac and reduced phase space quantization methods lead to the coinciding physical results for this plane spin model. Moreover, we have revealed an interesting analogy in interpretation of the situation with nonexistence of a global gauge condition for this simple constrained system with the situation taking place for the non-Abelian gauge theories [11].

The rotator and top spin models give examples of the classical systems, in which there is no global section of the space of gauge orbits. In spite of impossibility to impose gauge conditions such systems admit the construction of the reduced phase space. These two models demonstrate that the reduced phase space and the Dirac quantization methods can give essentially different physical results.

Thus, for Hamiltonian systems with first class constraints we encounter two related problems.

The first problem consists in the choice of a ‘correct’ quantization method for such systems. From the mathematical point of view any quantization leading to a quantum system, which has the initial system as its classical limit, should be considered as a correct one, but physical reasonings may distinguish different quantization methods. Consider, for example, the above mentioned systems. The rotator spin model, quantized according to the Dirac method, represents by itself the orbital angular momentum system with additional condition (3.3), singling out the states with a definite eigenvalue of angular momentum operator \( \hat{S}^2 \). This eigenvalue, in turn, is defined by the concrete value of the quantized parameter of the model: \( \rho^2 = n(n+1) > 0 \). On the other hand, the reduced phase space quantization of the model gives either integer or half-integer values for the spin of the system. If we suppose that the system under consideration is to describe orbital angular momentum, we must take only integer values for the parameter \( \rho \) in the reduced phase space quantization method. But in this case we must, nevertheless, conclude, that the reduced phase space quantization method of the rotator spin model describes a more general system than the quantum system obtained as a result of the Dirac quantization of that classical system.

The Dirac quantization of the top spin model, or its combination with the reduced phase space quantization gives us a possibility to interpret this system as a system having spin and isospin degrees of freedom (with equal spin and isospin: \( \hat{S}^2 = \hat{I}_i \hat{I}_i = j(j+1) \)), but in which the isospin degrees of freedom are ‘frozen’ by means of the condition \( \hat{I}_3 \Phi_{\text{phys}} = -k\Phi_{\text{phys}} \). On the other hand, as we have seen, the reduced space quantization method does not allow one to have such interpretation of the system since it allows the variable \( I_3 \) to take any (continuous) value \(-\kappa\) restricted only by the condition \( \kappa^2 < j^2 \), i.e., the operator \( \hat{I}_3 \) (taking here only one value) cannot be interpreted as a component of the isospin vector operator. From this point of view a ‘more correct’ method of quantization is the Dirac quantization method.
In this respect it is worth to point out that there is a class of physical models, for which it is impossible to get the reduced phase space description, and which, therefore, can be quantized only by the Dirac method.

Indeed, there are various pseudoclassical models containing first class nilpotent constraints of the form \[ \psi = \xi_{i_1} \cdots \xi_{i_n} G^{i_1 \cdots i_n} = 0, \] where \( \xi_{i_k} \), are real Grassmann variables with the Poisson brackets \( \{ \xi_k, \xi_l \} = -ig_{kl} \), \( g_{kl} \) being a real nondegenerate symmetric constant matrix. Here it is supposed that \( G^{i_1 \cdots i_n}, \ n \geq 2 \), are some functions of other variables, antisymmetric in their indices, and all the terms in a sum have simultaneously either even or odd Grassmann parity. For our considerations it is important that constraints (5.1) are the constraints, nonlinear in Grassmann variables, and that they have zero projection on the unit of Grassmann algebra. In the simplest example of relativistic massless vector particle in (3+1)-dimensional space–time \[ \xi_{a \mu} \] the odd part of the phase space is described by two Grassmann vectors \( \xi^a_{a \mu}, a = 1, 2, \) with brackets \( \{ \xi^a_{a \mu}, \xi^b_{b \nu} \} = -i\delta^{ab} g_{\mu\nu}, \) and the corresponding nilpotent first class constraint has the form:

\[ \psi = i\xi^1_{\mu} \xi^2_{\nu} g^{\mu\nu} = 0, \] where \( g_{\mu\nu} = \text{diag}(-1,1,1,1) \). This constraint is the generator of the SO(2)–rotations in the ‘internal isospin’ space: \( \xi^1_{\mu}(\tau) = \xi^1_{\mu} \cos \tau + \xi^2_{\mu} \sin \tau, \) \( \xi^2_{\mu}(\tau) = \xi^2_{\mu} \cos \tau - \xi^1_{\mu} \sin \tau. \) The specific property of this transformation is that having \( \xi^a_{a \mu}(\tau) \) and \( \xi^a_{a \mu} \), we cannot determine the rotation angle \( \tau \) because there is no notion of the inverse element for an odd Grassmann variable. Another specific feature of the nilpotent constraint (5.2) is the impossibility to introduce any, even local, gauge constraint for it. In fact, we cannot find a gauge constraint \( \chi \) such that the Poisson bracket \( \{ \psi, \chi \} \) would be invertible. Actually, it is impossible in principle to construct the corresponding reduced phase space for such a system. Obviously, the same situation arises for the constraint of general form (5.1). It is necessary to note here that in the case when the constraint \( \psi \) depends on even variables of the total phase space (see, e.g., ref. [16]), and, therefore, generates also transformations of some of them, we cannot fix the transformation parameter (choose a point in the orbit) from the transformation law of those even variables, because the corresponding parameter is present in them with a noninvertible factor, nonlinear in Grassmann variables. Therefore, the pseudoclassical systems containing the constraints of form (5.1) can be quantized only by the Dirac method, that was done in original papers [14]–[16].

Let us come back to the discussion of the revealed difference between two methods of quantization, and point out that the second related problem is clearing up the sense of gauge degrees of freedom. The difference appearing under the Dirac and reduced phase space quantization methods can be understood as the one proceeding from the quantum ‘vacuum’ fluctuations corresponding to the ‘frozen’ (gauge) degrees of freedom. Though these degrees of freedom are ‘frozen’ by the first class constraints, they reveal themselves through quantum fluctuations, and in the Dirac quantization method they cannot be completely ‘turned off’ due to the quantum uncertainty principle. Thus, we can suppose that the gauge degrees of freedom serve not simply for ‘covariant’ description of the system but have ‘hidden’ physical meaning, in some sense similar to the compactified degrees of freedom in the Kaluza–Klein theories. If we adopt such a point of view, we have to use only the Dirac quantization method. Further, the gauge principle cannot be considered then
as a pure technical principle. From here we arrive also at the conclusion that the Dirac separation of the constraints into first and second class constraints is not technical, and nature ‘distinguish’ these two cases as essentially different, since gauge degrees of freedom, corresponding to the first class constraints, may reveal themselves at the quantum level (compare with the point of view advocated in Ref. [21]).

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