A NEW BASIS FUNCTION APPROACH TO 'T HOOFT EQUATION

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We present the new basis functions to investigate the 't Hooft equation, the lowest order mesonic Light-Front Tamm-Dancoff equation for SU(N C) gauge theories. We find the wave function can be well approximated by new basis functions and obtain an analytic formula for the mass of the lightest bound state. Its value is consistent with the precedent results.

1 Introduction

It is expected that light-front (LF) quantization provides an powerful tool for studying many-body relativistic field theories. The bare vacuum is equal to the physical vacuum in the LF coordinate, since all constituents must have non-negative longitudinal momentum. This simple structure of the true vacuum enables us to avoid the insuperable problems which appeared in the Tamm-Dancoff (TD) approximation in the equal time frame. Therefore, the TD approximation is commonly used in the context of the LF quantization.

Many authors have developed the effective techniques for solving LFTD equations in several models. Bergknoff first applied LFTD approximation to the massive Schwinger model model, which is the extension of the simplest (1+1)-dimensional QED. He obtained the so-called Bergknoff equation, which is the light front Einstein-Schrödinger equation truncated to one fermion-antifermion pair. Mo and Perry presented a prevailing method to handle the ground state and the excited state in the massive Schwinger model. They concluded that the Jacobi polynomials are appropriate basis functions to study the massive Schwinger model. Harada and his coworker investigated the massive Schwinger model with SU(2) flavor symmetry, including up to four fermion sectors. They applied of simpler basis functions, which are essentially equivalent to the Jacobi polynomials, to the model. Sugihara and collaborators studied 2-dimensional SU(N C) Quantum ChromoDynamics(QCD), including four fermion sectors, by means of the basis functions of Harada et al.

Although excellent papers exist concerning massless and massive Schwinger models and 2-dimensional QCD, including excited states, it is worth analyzing the "t Hooft" equation, the lowest order LFTD equation in SU(N C)
gauge theory. This is because there is a mathematical interest in the basis function method. There is no mathematical evidence that the wave function can be expanded in terms of the conventional basis function. Contrarily, there is evidence that the conventional method breaks down if we try to improve the approximation. We would therefore like to improve the basis functions so as to avoid such difficulties.

2 Conventional Basis function Method

The ’t Hooft equation for two dimensional SU(N_C) gauge theory is given in the form

\[ M^2 \Phi(x) = \int_{-1/2}^{1/2} dy H(x,y)\Phi(y) \]

\[ \equiv \frac{4(m^2 - 1)}{1 - 4x^2} \Phi(x) - \varphi \int_{-1/2}^{1/2} dy \frac{\Phi(y)}{(y-x)^2}, \quad -1/2 \leq x \leq 1/2, \quad (1) \]

where \( \Phi \) is a wave function of a bound state, \( M \) denotes the normalized meson mass, \( m \) stands for the normalized (anti-)quark mass and \( \varphi \) denotes the finite part integral. In Eq. (1), we shifted the variable \( x \) total amount of \(-1/2\) compared with the variable in Refs. [8], in order to show the symmetry of the wave function transparently.

According to Harada and collaborators [6], one can expect that the wave function could be expanded as follows:

\[ \Phi(x) = \lim_{N \to \infty} \sum_{j=0}^{N} a_j(1 - 4x^2)^{\beta + j}. \quad (2) \]

The exponent \( \beta \) and the normalized quark mass \( m \) are related to each other by the equation [8,11]

\[ (m^2 - 1) + \beta \pi \cot \beta \pi = 0. \quad (3) \]

The authors of references [6,8] adopted the positive smallest solution \( \beta_0(m) \) of Eq. (3) as \( \beta \) in Eq. (2).

Mo and Perry, and Harada and his collaborators presented effective way to determine the coefficients \( a_j \)'s. We will briefly reproduce their procedures. By the use of the expansion in Eq. (2) truncated to given finite number \( N \) for the wave function \( \Phi \), we multiply both sides of Eq. (1) by \((1 - 4x^2)^{\beta + i}\) and integrate them over \( x \), then we obtain

\[ M^2 \tilde{N} \tilde{a} = \tilde{H} \tilde{a}, \quad \tilde{a} = \{a_0, a_1, \cdots, a_{n-1}\}. \quad (4) \]
In order to obtain eigenvalues of the generalized eigenvalue equation, we have to solve the eigenvalue problem for norm matrix $\hat{N}$, first, i.e.,

$$\hat{N}\vec{v}_i = \lambda_i \vec{v}_i.$$  \hspace{1cm} (5)

Next, we introduce a transformation matrix $\hat{W}$ by

$$\hat{W} = \begin{bmatrix} \vec{v}_1 \sqrt{\lambda_1} & \cdots & \vec{v}_n \sqrt{\lambda_n} \end{bmatrix}.$$  \hspace{1cm} (6)

Then, we can transform Eq. (4) into a usual eigenvalue problem of the form

$$M^2\vec{b} = \hat{W}\hat{H}\hat{W}\vec{b}, \quad \vec{a} = \hat{W}\vec{b}.$$  \hspace{1cm} (7)

For $N = 3$ and $m = 0.01$, we find, for the ground state boson,

$$\beta = 0.00552328, \quad M^2 = 0.0366342, \quad a_0 = 1, \quad a_1 = 0.00203562, \quad a_2 = -0.000579369, \quad a_3 = 0.000165813.$$  \hspace{1cm} (8)

The values of the LHS and the RHS of Eq. (1) are shown in Fig. 1. In Fig. 1, the solid line represents the LHS and the dotted line stands for the RHS. The RHS with $N = 9$, which is indicated by the dashed line, is exhibited for comparison. The coincidence of the LHS and the RHS is inadmissible for small values of $x$. On the other hand, for $x \approx \pm 1/2$, the difference between the LHS and the RHS is not allowable. Only the RHS has sharp spikes at the end points. This behavior is not changed much even if we improve the order of approximation.

We have to note here that in order to solve the generalized eigenvalue problem, the norm matrix should be positive definite. We cannot advance the above procedure beyond $N \approx 12$, because some of the eigenvalues of the norm matrix $\hat{N}$ become almost zero or negative. These facts strongly suggest that the conventional basis function is not appropriate basis function for 2 dimensional field theories.

3  New Basis Function

In order to introduce new basis function, we first notice that there are infinite solutions $\beta_1 < \beta_2 < \cdots$ of Eq. (8) in addition to solution $\beta_0$. We are easily led
to

\[ 0 \ll \beta_n - \beta_0 - n < 1/2, \]
\[ 0 < \beta_n - \beta_k - (n - k) \ll 1, \quad n > k. \]  \tag{9} \]

The above relations imply that Eq. 2 never incorporates terms like \((1 - 4x^2)^{\beta_n + j}\) with positive integer \(n\) and non-negative integer \(j\).

We posit that the wave function is given by an infinite series

\[ \Phi(x) = \lim_{N \to \infty} \sum_{n=0}^{N} \sum_{j=0}^{N-n} c_n^j (1 - 4x^2)^{\beta_n(m) + j}. \]  \tag{10} \]

Substituting Eq. 10 into Eq. 1, we have

\[ 0 = M^2 \Phi(x) - \left. \int_{-1/2}^{1/2} dy H(x, y) \Phi(y) \right|_{1-4x^2=\epsilon} \]
\[ = -4 \sum_{n=0}^{\infty} c_n^0 (m^2 - 1 + \pi \beta_n \cot \pi \beta_n) \epsilon^{\beta_n - 1} \]
\[ + \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} F_{nj}(m; \beta_n; M; c \cdots) \epsilon^{\beta_n + j} \]
\[ + \sum_{k=0}^{\infty} G_k(m; \beta_0, \beta_1; \cdots; M; c \cdots) \epsilon^k. \]  \tag{11} \]

Of course, the first line in Eq. 11 cancels automatically because of the definition of \(\beta_n\)’s. Suppose that we truncate series in Eq. 10 to \(O(\epsilon^{N})\). That is, we set \(c_n^j = 0\) for \(n + j > N\). We demand Eq. 11 to hold up to \(O(\epsilon^{N-1})\). Then we have \(N(N+3)/2\) non-trivial equations. On the other hand, there are \((N+1)(N+2)/2\) unknown parameters. The number of parameters is larger than that of non-trivial equations by 1. Thus, we can solve the equations for \(c_n^j\) in terms of \(M^2\). Another equation of use to us is obtained by multiplying both sides of Eq. 1 by \(\Phi(x)\) and integrating them over \(x\),

\[ M^2 \int_{-1/2}^{1/2} dx |\Phi(x)|^2 = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} dx dy \Phi(x) H(x, y) \Phi(y). \]  \tag{12} \]

For a given \(m\), we put \(M^2 = M_i^2\). We can then solve Eq. 11 for \(c_n^j\) in terms of \(M_i\). We thus obtain the \(M_i\) dependent truncated wave function, say,
\[ \Phi(x; M_i). \] We can calculate a new mass eigenvalue \( M_{i+1} \) using this wave function as

\[ M_{i+1}^2 = \frac{\langle \Phi(M_i)|H|\Phi(M_i) \rangle}{\langle \Phi(M_i)|\Phi(M_i) \rangle}. \quad (13) \]

For \( N \leq 15 \) and \( m = 0.01 \), mass \( M^2 \) converges in 5 iterations. For \( 0 < m < 0.5 \), we obtain \( M^2 \)'s and fit them by polynomials, as follows:

\[ M^2(m) = 3.6276m + 3.5803m^2 + 0.0636m^3 + O(m^4). \quad (14) \]

It should be noted here that the coefficients of \( m \) are consistent with Bergknoff’s result. In order to see the efficacy of this new basis function expansion, we show the wave functions in Fig. 2. There, the thin solid line represents the LHS in Eq. (1), provided that the wave function was approximated by Eq. (10) with \( N = 15 \). The dotted line denotes the RHS with \( N = 2 \), the dot-dashed line exhibits the RHS with \( N = 3 \), the dot-dot-dashed line represents the RHS with \( N = 4 \), the dot-dash-dashed line stands for the RHS with \( N = 5 \), and the dashed line exhibits the RHS with \( N = 10 \). The thick solid line indicates the RHS in Eq. (1) with wave function given in Eq. (10) with \( N = 15 \).

4 Summary and Discussion

In the previous section we have introduced the new basis function and calculated the mass eigenvalue of the bound state using the new basis function. We have found that (1) the new basis function gives an effective approximation of the wave function, and (2) the mass eigenvalues are consistent with the results of the precursors.

It should be noted that Eq. (14) is, mathematically, the most general expansion. This means that there is no room to introduce any other additional terms like \( d(1 - 4x^2)\gamma \) for \( \gamma \neq \beta_n + j \) with non-negative integers \( n \) and \( j \). If we introduce such terms, then the following equality should hold

\[ 0 = 4d \left( m^2 - 1 + \pi \gamma \cot(\pi \gamma) \right) (1 - 4x^2)^{-1}. \quad (15) \]

This demands that \( d \equiv 0 \).
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