BIALGEBRAS, FROBENIUS ALGEBRAS AND ASSOCIATIVE YANG-BAXTER EQUATIONS FOR ROTA-BAXTER ALGEBRAS

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Abstract. Rota-Baxter operators and bialgebras go hand in hand in their applications, such as in the Connes-Kreimer approach to renormalization and the operator approach to the classical Yang-Baxter equation. We establish a bialgebra structure that is compatible with the Rota-Baxter operator, called the Rota-Baxter antisymmetric infinitesimal (ASI) bialgebra. This bialgebra is characterized by generalizations of matched pairs of algebras and double constructions of Frobenius algebras to the context of Rota-Baxter algebras. The study of the coboundary case leads to an enrichment of the associative Yang-Baxter equation (AYBE) to Rota-Baxter algebras. Antisymmetric solutions of the equation are used to construct Rota-Baxter ASI bialgebras. The notions of an \(\mathcal{O}\)-operator on a Rota-Baxter algebra and a Rota-Baxter dendriform algebra are also introduced to produce solutions of the AYBE in Rota-Baxter algebras and thus to provide Rota-Baxter ASI bialgebras. An unexpected byproduct is that a Rota-Baxter ASI bialgebra of weight zero gives rise to a quadri-bialgebra instead of bialgebra constructions for the dendriform algebra.

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1. **Introduction**

This paper develops a bialgebra theory for Rota-Baxter algebras by applying methods from double constructions of Frobenius algebras, associative Yang-Baxter equations, $\mathcal{O}$-operators and dendriform algebras.

1.1. **Hopf algebras and Lie bialgebras.** A bialgebra structure is the coupling of an algebra structure and a coalgebra structure by certain compatibility conditions, sometimes called distributive laws. The most well-known such structures are the Hopf algebra for associative algebras and the Lie bialgebra for Lie algebras. The central importance of bialgebras rests on their connection with other structures arising from mathematics and physics. Hopf algebra has its origin in topology, and serves as universal enveloping algebras of Lie algebras and quantization in terms of quantum groups and Yang-Baxter equation.

Lie bialgebras also arise naturally in the study of Yang-Baxter equations, and are closely related to quantum groups as deformations of universal enveloping algebras. Etingof and Kazhdan proved that every Lie bialgebra has a corresponding quantized universal enveloping algebra, that is, there exists a quantization for every Lie bialgebra.

Further, bialgebras naturally lead to other important algebraic structures of independent importance. Lie bialgebras are characterized by Manin triples of Lie algebras, while Manin triples of Lie algebras with respect to (symmetric) nondegenerate invariant bilinear form give constructions of self-dual (or quadratic) Lie algebras which are useful in conformal field theory and quantized universal algebras. In addition, the Manin triple approach of the Lie bialgebra is interpreted as cocycle conditions which provide a rich structure theory and effective constructions for the Lie bialgebra. In particular, the study of coboundary case leads to the introduction of the classical Yang-Baxter equation (CYBE). (antisymmetric) solutions of the CYBE give rise to Lie bialgebras, whereas $\mathcal{O}$-operators on Lie algebras introduced in [34], which are a natural generalization of Rota-Baxter operators, provide the needed solutions of the CYBE.

Similarly, Manin triples of Lie algebras with respect to a (skew-symmetric) nondegenerate 2-cocycle, which are called the parakähler structures in geometry or phase spaces in mathematical physics, correspond to left-symmetric bialgebras. The associative analog of the Lie bialgebra is the antisymmetric infinitesimal (ASI) bialgebra, for which the Manin triples of associative algebras with respect to (symmetric) nondegenerate invariant bilinear form are called double constructions of Frobenius algebras, the latter being widely applied to areas such as 2d topological quantum field theory and string theory [34]. Further details can be found in [3, 4] for example.

1.2. **Bialgebra structures of Rota-Baxter algebras.** The study of Rota-Baxter algebras originated from the work of the mathematician G. Baxter in probability and, in its early study, attracted the attention of well-known analysts and combinatorists including Atkinson, Cartier and Rota. In the Lie algebra context, the defining relation of Rota-Baxter operators is precisely the operator form of the classical Yang-Baxter equation, named after C.-N. Yang and R. J. Baxter. This area has expanded tremendously in recent years with its broad applications and connections, notably in the Connes-Kreimer approach to renormalization of quantum field theory, where Hopf algebras and Rota-Baxter algebras are the fundamental structures, especially in the algebraic Birkhoff factorization. See [7, 2, 3, 5, 24] for further details.

Given the joint roles played by bialgebras and Rota-Baxter algebras in their applications, it is desirable to study their composed structures. Some progresses have been made in this direction. There has been quite much study equipping free Rota-Baxter algebras with a bialgebra or Hopf
algebra structures [3, 18, 22], thus providing rich structures on Rota-Baxter algebras and leading to connections with combinatorics and number theory [26, 44]. There is also a notion of Rota-Baxter bialgebra [37], as a quintuple \((A, \cdot, \Delta, P, Q)\) where \((A, \cdot, \Delta)\) is a bialgebra, \((A, \cdot, P)\) is a Rota-Baxter algebra of certain weight and \((A, \Delta, Q)\) is a Rota-Baxter coalgebra of another weight. Yet, compatibility relations between the operators \(P\) and \(Q\) are still needed to have a bialgebra theory for Rota-Baxter algebras.

In this paper, we establish a bialgebra theory for Rota-Baxter algebras by extending the approach of double construction of Frobenius algebras to ASI bialgebras, after resolving challenges imposed by the extra restrictions of the Rota-Baxter operators. The method is quite general and might be adapted for the other operators such as the differential, Nijenhuis, average and Reynolds operators. Note that these structures are different from Hom-Lie algebras [25] which also have an extra linear operator, but the operator entails a twist of the existing Jacobian identity rather than adding a new relation to the structure.

To give a thorough and uniform treatment of the possible compatibility conditions among the multiplication, comultiplication and the linear operators, we introduce an admissibility condition between a linear operator and a Rota-Baxter algebra. Then we are able to apply the Rota-Baxter operator to the matched pairs of algebras, Frobenius algebras, associative Yang-Baxter equation, \(O\)-operators and dendriform algebras, so that the full theory of ASI bialgebras can be extended to Rota-Baxter algebras.

1.3. Outline of the paper. The main notions and constructions in this paper are summarized in the following diagram.

In Section 2, we introduce the notion of a Rota-Baxter antisymmetric infinitesimal (ASI) bialgebra together with some preliminary examples. This notion is built on Rota-Baxter operators on an ASI bialgebra, but with a special set of compatibility conditions between the unary and binary operations that is the key to a complete bialgebra theory comparable to those of Lie bialgebras and ASI bialgebras. To gain a good understanding of these compatibility conditions for later applications, we conceptualize these conditions in Section 2.2, to the notion of an admissible quadruple of a Rota-Baxter algebra, as the compatibility of a Rota-Baxter algebra and a linear operator in terms of dual representations.

With the preparation in Section 2.2, in Section 3, we give the general notion of a matched pair of Rota-Baxter algebras, before specializing to the case when the underlying linear spaces of the two Rota-Baxter algebras are dual to each other, providing an equivalent condition for Rota-Baxter ASI bialgebras (Theorem 3.5). Then this notion is tied in with the notion of a double construction of Rota-Baxter Frobenius algebra (Theorem 3.12) and thus provides another equivalent condition of Rota-Baxter ASI bialgebras (Theorem 3.13).
The remaining part of the paper studies and constructs Rota-Baxter ASI bialgebras through their relationship with Rota-Baxter operations on associative Yang-Baxter equations, O-operators and dendriform algebras.

In Section 4, we focus on coboundary Rota-Baxter ASI bialgebras. From their characterization equations (Corollary 4.4), we extract the notion of an admissible associative Yang-Baxter equation in a Rota-Baxter algebra, whose study further leads to the notion of an O-operator on a Rota-Baxter algebra. Thus such O-operators provide the needed solutions of the admissible associative Yang-Baxter equation to give rise to Rota-Baxter ASI bialgebras (Theorem 4.21). Several cases are considered where the conditions and computations can be made explicit, providing examples for the general results (Proposition 4.24). In particular, a Rota-Baxter operator on any Rota-Baxter algebra is naturally an O-operator on this Rota-Baxter algebra and hence produces a Rota-Baxter ASI bialgebra (Corollary 4.26).

In view of the role played by dendriform algebras [34] in the study of ASI bialgebras [3], especially in constructing O-operators, we introduce in Section 5 the notion of a Rota-Baxter dendriform algebra as a dendriform algebra with a Rota-Baxter operator, which gives a natural O-operator on the associated Rota-Baxter algebra (Proposition 5.9). Hence there is a construction of Rota-Baxter ASI bialgebras from Rota-Baxter dendriform algebras via such O-operators, applying the methods introduced in the previous section. Especially, any Rota-Baxter algebra of weight zero already carries a natural Rota-Baxter dendriform algebra structure and hence produces a Rota-Baxter ASI bialgebra (Corollary 5.12). This phenomenon, of obtaining a nontrivial bialgebra structure directly from the base algebraic structure as displayed here and in the above Corollary 4.26, is new in Rota-Baxter algebras, not found in previously considered algebra structures such as Lie algebras and associative algebras. We further found that although a Rota-Baxter algebra of weight zero gives a dendriform algebra [2], a Rota-Baxter ASI bialgebra of weight zero does not give a bialgebra construction for the dendriform algebra, but rather a quadri-bialgebra [38] (Remark 5.5 and Corollary 5.6).

Notations: Throughout this paper, we fix a field K. All vector spaces, tensor products, and linear homomorphisms are over K. After Section 2 all the vector spaces and algebras are finite dimensional unless otherwise specified. By an algebra, we mean an associative algebra not necessarily having a unit.

2. Rota-Baxter ASI Bialgebras and their Admissibility Conditions

In this section, we first introduce the notion of a Rota-Baxter antisymmetric infinitesimal (ASI) bialgebra as an enrichment of ASI bialgebras, with a key role played by representations of a Rota-Baxter dendriform algebra on a dual space. We then give a focused study of such representations for better understanding of Rota-Baxter ASI bialgebras and their applications.

2.1. Rota-Baxter ASI Bialgebras. We first recall the notion of antisymmetric infinitesimal bialgebras [3, 8, 23] as the associative analog of Lie bialgebras [13, 16].

Definition 2.1. An antisymmetric infinitesimal bialgebra or simply an ASI bialgebra is a triple \((A, \cdot, \Delta)\) consisting of a vector space \(A\) and linear maps \(\cdot : A \otimes A \to A\) and \(\Delta : A \to A \otimes A\) such that

(a) the pair \((A, \cdot)\) is an associative algebra,
(b) the pair \((A, \Delta)\) is a coassociative coalgebra, and
(c) with the flip map \(\sigma : A \otimes A \to A \otimes A\), the following equations hold.

\[
\Delta(a \cdot b) = (R_A(b) \otimes \text{id})\Delta(a) + (\text{id} \otimes L_A(a))\Delta(b),
\]
A Rota-Baxter coalgebra, then \((A, \Delta, Q)\) is an antisymmetric infinitesimal bialgebra, and a linear operator \(P\) such that
\[
\Delta(a) = \Delta(b) = \sigma(id \otimes R_A(b) - L_A(b) \otimes id)\Delta(a), \quad \forall a, b \in A. \tag{2}
\]

The terms infinitesimal and antisymmetric come from Eqs. (1) and (2) respectively. When \(A\) is finite dimensional, an ASI bialgebra can be equivalently formulated in terms of a matched pair of algebras and a double construction of Frobenius algebras.

We will extend the theory of ASI bialgebras to the context of Rota-Baxter algebras.

**Definition 2.2.** \([11]\) Let \(\lambda \in K\) be given. A **Rota-Baxter algebra of weight** \(\lambda\) is a pair \((A, P)\), consisting of an algebra \(A\) and a linear operator \(P : A \to A\) such that
\[
P(aP(b)) = P(aP(b)) + P(P(a)b) + \lambda P(ab), \quad \forall a, b \in A. \tag{3}
\]

Such a linear operator \(P\) is called a **Rota-Baxter operator of weight** \(\lambda\) on \(A\). We will suppress the weight if there is no danger of confusion.

Dualizing the notion of a Rota-Baxter algebra, we have

**Definition 2.3.** \([17, 27, 57]\) A **Rota-Baxter coalgebra** of weight \(\lambda \in K\) is a triple \((A, \Delta, Q)\) where \((A, \Delta)\) is a coalgebra and \(Q : A \to A\) is a linear operator such that
\[
(Q \otimes Q)\Delta(a) = (Q \otimes id)\Delta Q(a) + \Delta Q(a) + \lambda \Delta Q(a), \quad \forall a \in A. \tag{4}
\]

Any algebra or coalgebra naturally comes with Rota-Baxter operators given by the scalar multiplications. Thus Rota-Baxter algebras and coalgebras can be regarded as generalizations of algebras and coalgebras. Generalizing the well-known duality between a coalgebra and an algebra \([1]\), for a vector space \(A\) and linear maps \(Q : A \to A, \Delta : A \to A \otimes A\), if \((A, \Delta, Q)\) is a Rota-Baxter coalgebra, then \((A^*, \Delta^*, Q^*)\) is a Rota-Baxter algebra. When \(A\) is finite dimensional, the converse is also true \([27]\).

We now introduce the main notion of the paper as an enrichment and generalization of ASI bialgebras in the context of Rota-Baxter algebras.

**Definition 2.4.** A **Rota-Baxter antisymmetric infinitesimal (ASI) bialgebra** is a quintuple \((A, \cdot, \Delta, P, Q)\) or simply \(((A, P), \Delta, Q)\) consisting of a vector space \(A\) and linear maps
\[
\cdot : A \otimes A \to A, \quad \Delta : A \to A \otimes A, \quad P, Q : A \to A
\]
such that

(a) \((A, \cdot, \Delta)\) is an antisymmetric infinitesimal bialgebra,
(b) \((A, \cdot, P)\) is a Rota-Baxter algebra,
(c) \((A, \Delta, Q)\) is a Rota-Baxter coalgebra, and
(d) the following compatibility conditions hold.

\[
Q(aP(b)) = Q(a)P(b) + Q(Q(a)b) + \lambda Q(ab), \tag{5}
\]
\[
Q(P(a)b) = P(a)Q(b) + Q(aQ(b)) + \lambda aQ(b), \quad \forall a, b \in A, \tag{6}
\]
\[
(id \otimes Q)\Delta P = (P \otimes Q)\Delta + (P \otimes id)\Delta P + \lambda (P \otimes id)\Delta, \tag{7}
\]
\[
(Q \otimes id)\Delta P = (Q \otimes P)\Delta + (id \otimes P)\Delta P + \lambda (id \otimes P)\Delta. \tag{8}
\]

We give some preliminary examples. More substantial examples will be provided later after the needed tools are developed.

**Example 2.5.** As noted above, for a given scalar \(\lambda\) in the base field, the scalar product by \(-\lambda\) is a Rota-Baxter operator of weight \(\lambda\). Thus for any algebra \(R\), \((R, -\lambda id)\) is a Rota-Baxter algebra of weight \(\lambda\). Similarly, for any coalgebra \(C\), the pair \((C, -\lambda id)\) is a Rota-Baxter coalgebra of weight \(\lambda\).
\( \lambda \). It then follows that for any ASI bialgebra \((R, \cdot, \Delta)\), together with the scalar product operators \( P = Q = -\lambda \text{id} \), the quintuple \((A, \cdot, \Delta, P, Q)\) is a Rota-Baxter ASI bialgebra of weight \( \lambda \).

**Example 2.6.** A basic example of Rota-Baxter algebras of weight \(-1\) is the direct sum \( A \oplus B \) of two algebras with the projection operator to either of the two summands. The same construction gives Rota-Baxter coalgebras of weight \(-1\). Now let \((A, \cdot_A, \Delta_A)\) and \((B, \cdot_B, \Delta_B)\) be ASI bialgebras. It is direct to check that the direct sum \( A \oplus B \) with the component operations \( \cdot_A + \cdot_B \) and \( \Delta_A + \Delta_B \) is again an ASI bialgebra. Let \( P_A, P_B : A \oplus B \to A \oplus B \) be the projections to \( A \) and \( B \) respectively. Then the quintuple \((A \oplus B, \cdot_A + \cdot_B, \Delta_A + \Delta_B, P_A, P_B)\) is a Rota-Baxter ASI bialgebra. For this it remains to check the compatibility conditions in Eqs. (5)-(8). For \( x \in A, a \in B \), we have

\[
(\text{id}_{A \oplus B} \otimes P_B)(\Delta_A + \Delta_B)P_A(x + a) = (P_A \otimes P_B)(\Delta_A + \Delta_B)(x + a) = 0,
\]

\[
(P_A \otimes \text{id}_{A \oplus B})(\Delta_A + \Delta_B)P_A(x + a) = (P_A \otimes \text{id}_{A \oplus B})(\Delta_A + \Delta_B)(x + a) = \Delta_A(x).
\]

Hence

\[
(\text{id}_{A \oplus B} \otimes P_B)(\Delta_A + \Delta_B)P_A = (P_A \otimes P_B)(\Delta_A + \Delta_B) + (P_A \otimes \text{id}_{A \oplus B})(\Delta_A + \Delta_B)P_A
\]

\[
- (P_A \otimes \text{id}_{A \oplus B})(\Delta_A + \Delta_B),
\]

that is, Eq. (7) holds. The other equations can be verified in the same way.

**Remark 2.7.** As we can see in Definition 2.4, the key to the notion of a Rota-Baxter ASI algebra is a suitable choice of compatibility conditions among the multiplication, comultiplication and the linear operators. Our choice of the conditions in Eqs. (5)-(8) was motivated by our goal of developing a full theory expanding those of Lie bialgebras (resp. ASI bialgebras), including their close connections with matched pairs, Manin triples (resp. double constructions of Frobenius algebras) and furthermore with classical (resp. associative) Yang-Baxter equations, 0-operators and pre-Lie (resp. dendriform) algebras. These connections and applications of Rota-Baxter ASI algebras will be presented in the later sections of the paper. For this purpose, we first need a conceptual understanding of these compatibility conditions which we will explore in the next subsection.

### 2.2. Rephrasing Rota-Baxter ASI bialgebras in terms of admissible quadruples

We will work with the representation theory of an algebra in the sense of bimodules, instead of one-sided modules.

**Definition 2.8.** Let \( A \) be an algebra. A **representation** of \( A \) or an \( A \)-**bimodule** is a triple \((V, \ell, r)\), abbreviated as \( V \), consisting of a vector space \( V \) and linear maps

\[
\ell, r : A \to \text{End}_K(V)
\]

such that

\[
\ell(a)(\ell(b)v) = \ell(ab)v, \quad (r(ab))r(b) = vr(ab), \quad (\ell(a)v)r(b) = \ell(a)(vr(b)), \quad \forall a, b \in A, v \in V.
\]

With \( L = L_A \) and \( R = R_A \) denoting the left and right multiplications by elements of \( A \) respectively, the triple \((A, L, R)\) is a representation of \( A \), called the **adjoint representation** of \( A \).

Let \( \ell, r : A \to \text{End}_K(V) \) be linear maps. Define a multiplication on \( A \oplus V \) by

\[
(a + u) \cdot (b + v) := ab + (\ell(a)v + ur(b)), \quad \forall a, b \in A, u, v \in V.
\]

Then as is well known, \( A \oplus V \) is an algebra, denoted by \( A \rtimes_{\ell, r} V \) and called the **semi-direct product** of \( A \) by \( V \), if and only if \((V, \ell, r)\) is a representation of \( A \).

Extending the concept of a representation of an algebra to the context of Rota-Baxter algebras, we give
Definition 2.9. A representation or a bimodule of a Rota-Baxter algebra \((A, P)\) of weight \(\lambda\) is a quadruple \((V, \ell, r, \alpha)\) where \((V, \ell, r)\) is an \(A\)-bimodule and \(\alpha\) is a linear operator on \(V\) such that
\[
\ell (P(a)) \alpha(v) = \alpha(\ell (P(a)) v) + \alpha(\ell (a) \alpha(v)) + \lambda \alpha(\ell (a)v)
\] (10)
and
\[
\alpha(v)r(P(a)) = \alpha(\alpha(v) r(a)) + \alpha(vr(P(a))) + \lambda \alpha(vr(a)), \quad \forall a \in A, v \in V.
\] (11)

Two representations \((V_1, \ell_1, r_1, \alpha_1)\) and \((V_2, \ell_2, r_2, \alpha_2)\) of a Rota-Baxter algebra \((A, P)\) are called equivalent if there exists a linear isomorphism \(\varphi : V_1 \rightarrow V_2\) such that
\[
\varphi(\ell_1(a)(v)) = \ell_2(a) \varphi(v), \quad \varphi(vr_1(a)) = \varphi(v)r_2(a), \quad \varphi(\alpha_1(v)) = \alpha_2 \varphi(v), \quad \forall a \in A, v \in V_1.
\] (12)

It follows immediately from the definition that the quadruple \((A, L, R, P)\) is a representation of the Rota-Baxter algebra \((A, P)\), called the adjoint representation of \((A, P)\).

For representations of Rota-Baxter algebras in the sense of one-sided modules, we refer the reader to [4, 5]. Semi-direct products of Rota-Baxter algebras can also be characterized by their representations.

Proposition 2.10. Let \((A, P)\) be a Rota-Baxter algebra of weight \(\lambda\). Let \((V, \ell, r)\) be a representation of the algebra \(A\) and let \(\alpha\) be a linear operator on \(V\). Define a linear map
\[
P_{A \oplus V} : A \oplus V \rightarrow A \oplus V, \quad P_{A \oplus V}(a + u) := P(a) + \alpha(u),
\] (13)
which is simply denoted by \(P_{A \oplus V} := P + \alpha\). Then together with the multiplication defined in Eq. (9), \((A \oplus V, P_{A \oplus V})\) is a Rota-Baxter algebra of weight \(\lambda\) if and only if \((V, \ell, r, \alpha)\) is a representation of \((A, P)\). The resulting Rota-Baxter algebra is denoted by \((A \kappa_{\ell, r}, V, P + \alpha)\) and is called the semi-direct product of \((A, P)\) by its representation \((V, \ell, r, \alpha)\).

The proof will be omitted since this result is a special case of the matched pairs of Rota-Baxter algebras in Theorem [3, 4] when \(B = V\) is equipped with the zero multiplication.

We fix more notations. Denote the usual pairing between the dual space \(V^*\) and \(V\) by
\[
\langle \cdot, \cdot \rangle : V^* \times V \rightarrow K, \quad \langle v^*, v \rangle := v^*(v), \quad \forall v \in V, v^* \in V^*.
\]
For a linear map \(\varphi : V \rightarrow W\), we denote the transpose map by \(\varphi^* : W^* \rightarrow V^*\) given by
\[
\langle \varphi^*(w^*), v \rangle = \langle w^*, \varphi(v) \rangle, \quad \forall v \in V, w^* \in W^*.
\]

For a representation \((V, \ell, r)\) of an algebra \(A\), define the triple \((V^*, r^*, \ell^*)\) where the linear maps \(\ell^*, r^* : A \rightarrow \text{End}_K(V^*)\) are defined by
\[
\langle v^\ell(a), v \rangle = \langle v^*, \ell(a)v \rangle, \quad \langle r^*(a)v^*, v \rangle = \langle v^*, vr(a) \rangle, \quad \forall a \in A, v^* \in V^*, v \in V.
\] (14)

Then the triple \((V^*, r^*, \ell^*)\) is again a representation of \(A\) [4, Lemma 2.1.2], called the dual representation of \((V, \ell, r)\). However, this property does not hold for representations of Rota-Baxter algebras, that is, the linear dual of a representation of a Rota-Baxter algebra is not necessarily a Rota-Baxter representation. In fact, the following extra conditions are needed.

Lemma 2.11. Let \((A, P)\) be a Rota-Baxter algebra of weight \(\lambda\). Let \((V, \ell, r)\) be a representation of the algebra \(A\) and let \(\beta : V \rightarrow V\) be a linear map. The quadruple \((V^*, r^*, \ell^*, \beta^*)\) is a representation of \((A, P)\) if and only if the linear operator \(\beta\) satisfies
\[
\beta(vr(P(a))) - \beta(v)r(P(a)) - \beta(\beta(v)r(a)) - \lambda \beta(v)r(a) = 0, \quad (15)
\]
\[
\beta(\ell(P(a))v) - \ell(P(a))\beta(v) - \beta(\ell(a)\beta(v)) - \lambda \ell(a)\beta(v) = 0, \quad \forall a \in A, v \in V. \quad (16)
\]
Proof. Since \((V^*, r^*, ℓ^*)\) is an \(A\)-bimodule, we just need to determine when \(β^*\) satisfies Eqs. (11) and (13) where \(α\) is replaced by \(β^*\). By Eq. (14), we have
\[
\langle ℓ(P(a))β^*(v^*) − β^*(ℓ(P(a))v^*) − β^*(ℓ(α)v^*) − λβ^*(ℓ(α)v^*), v \rangle = \langle v^*, β(vr(P(a))) − β(v)r(P(a)) − β(β(v)r(a)) − λβ(v)r(a), \forall a ∈ A, v ∈ V, v^* ∈ V^* \rangle.
\]
Thus Eq. (11) is equivalent to Eq. (15). The same argument proves Eq. (16).

We introduce a notion to conceptualize this key property.

**Definition 2.12.** Use the same data \((A, P, (V, ℓ, r))\) and \(β\) as in Lemma 2.11. If any (and hence both) of the equivalent conditions is satisfied, we say that \(β\) is **admissible to the Rota-Baxter algebra** \((A, P)\) on \((V, ℓ, r)\). To allow more flexibility in applying this notion, we also say that \((A, P)\) is **β-admissible** on \((V, ℓ, r)\) or that the quadruple \((V, ℓ, r, β)\) is **admissible**. When \((V, ℓ, r)\) is taken to be the adjoint representation \((A, L, R)\) of the algebra \(A\), we say that \(β\) is **admissible to** \((A, P)\) or simply \((A, P)\) is **β-admissible**.

Then we have

**Corollary 2.13.** Let \((A, P)\) be a Rota-Baxter algebra of weight \(λ\). A linear operator \(Q\) on \(A\) is admissible to \((A, P)\) if and only Eqs. (5)- (8) hold.

Further the definition of a Rota-Baxter ASI bialgebra can be rephrased as

**Proposition 2.14.** A quintuple \((A, ·, Δ, P, Q)\) with notations in Definition 2.4 is a Rota-Baxter ASI bialgebra if and only if it satisfies conditions (6)-(8) in Definition 2.4 and that \(P^*\) and \(P^*\) are admissible to the Rota-Baxter algebras \((A, P)\) and \((A^*, Q^*)\) respectively.

**Proof.** Of the compatibility conditions in Eqs. (6)-(8) for a Rota-Baxter ASI bialgebras, Corollary 2.13 already equals Eqs. (5) and (6) to the admissibility of \(Q\) to the Rota-Baxter algebra \((A, P)\). Further, Eqs. (6) and (7) can be rewritten as
\[
P^*(uQ^*(v)) = P^*(u)Q^*(v) + P^*(P^*(u)v) + λP^*(u)v, \quad (17)
P^*(Q^*(u)v) = Q^*(u)P^*(v) + P^*(uP^*(v)) + λuP^*(v), \quad ∀u, v ∈ A^*.
\]
This means that \(P^*\) is admissible to the Rota-Baxter algebra \((A^*, Q^*)\).

3. Matched pairs of Rota-Baxter algebras, double constructions of Rota-Baxter Frobenius algebras and Rota-Baxter ASI bialgebras

In this section, we introduce the notions of a matched pair of Rota-Baxter algebras and a double construction of Rota-Baxter Frobenius algebra. Generalizing the characterizations of Lie bialgebras and ASI bialgebras in terms of Manin triples for Lie algebras or double constructions of Frobenius algebras for associative algebras \([6, 7, 14]\), we prove that these new notions give equivalent conditions for a Rota-Baxter ASI bialgebra.

3.1. Matched pairs of Rota-Baxter algebras. We first recall the concept of a matched pair of algebras \([6]\).

**Definition 3.1.** A matched pair of algebras consists of algebras \((A, ·_A)\) and \((B, ·_B)\), together with linear maps \(ℓ_A, r_A : A → \text{End}_K(B)\) and \(ℓ_B, r_B : B → \text{End}_K(A)\) such that
(a) \((A, ℓ_B, r_B)\) is a representation of \((B, ·_B)\),
(b) \((B, ℓ_A, r_A)\) is a representation of \((A, ·_A)\) and
(c) the following compatibility conditions hold: for \( a, a' \in A \) and \( b, b' \in B \),
\[
\begin{align*}
\ell_A(a)(b \cdot b') &= \ell_A(ar_B(b))b' + (\ell_A(a)b) \cdot_B b' ; \\
(b \cdot b')r_A(a) &= b r_A(\ell_B(b'))a + b \cdot_B (b'r_A(a)) ; \\
\ell_B(b)(a \cdot a') &= \ell_B(b r_A(a))a' + (\ell_B(b)a) \cdot_B a' ; \\
(a \cdot a')r_B(b) &= a r_B(\ell_A(a')b) + a \cdot_B (a' r_B(b)) ; \\
\ell_A(\ell_B(b)ab')(b r_A(a)) &= b r_A(a b'r_B(b')) + b \cdot_B (\ell_A(a)b') ; \\
\ell_B(\ell_A(a)b)a' + (a r_B(b')) \cdot_B a' &= a r_B(b r_A(a')) + a \cdot_B (\ell_A(b)a').
\end{align*}
\] (19) (20) (21) (22) (23) (24)

There is a characterization of ASI bialgebras by matched pairs of algebras.

**Theorem 3.2.** [8] Let \((A, \cdot)\) be an algebra. Suppose that there is an algebra \((A^*, \circ)\) on the linear dual \(A^*\). Let \(\Delta : A \rightarrow A \otimes A\) be the linear dual of \(\circ : A^* \otimes A^* \rightarrow A^*\). Then \((A, \cdot, \Delta)\) is an ASI bialgebra if and only if \((A, A^*, R^*, L^*, R^*_o, L^*_o)\) is a matched pair of algebras.

By [8], for algebras \((A, \cdot_A), (B, \cdot_B)\) and linear maps \(\ell_A, r_A : A \rightarrow \text{End}_K(B), \ell_B, r_B : B \rightarrow \text{End}_K(A)\), define a multiplication on the direct sum \(A \oplus B\) by
\[
(a + b) \star (a' + b') := (a \cdot_A a' + ar_B(b') + \ell_B(b)a') + (b \cdot_B b' + \ell_A(a)ba' + br_A(a')),
\] (25)

for \(a, a' \in A\) and \(b, b' \in B\). Then \((A \oplus B, \star)\) is an algebra if and only if \((A, \cdot_A), (B, \cdot_B), \ell_A, r_A, \ell_B, r_B)\) is a matched pair of \((A, \cdot_A)\) and \((B, \cdot_B)\). We denote the resulting algebra \((A \oplus B, \star)\) by \(A \succ B\) or simply \(A \bowtie B\). Further, for any algebra \(C\) whose underlying vector space is a linear direct sum of two subalgebras \(A\) and \(B\), there is a matched pair \((A, B, \ell_A, r_A, \ell_B, r_B)\) such that there is an isomorphism from the resulting algebra \((A \bowtie B, \star)\) via Eq. (25) to the algebra \(C\) and the restrictions of the isomorphism to \(A\) and \(B\) are the identity maps.

We extend this property to Rota-Baxter algebras.

**Definition 3.3.** A **matched pair of Rota-Baxter algebras** is a sextuple \(((A, P_A), (B, P_B), \ell_A, r_A, \ell_B, r_B)\) where \((A, P_A)\), \((B, P_B)\) are Rota-Baxter algebras, \((B, \ell_A, r_A, P_B)\) is a representation of \((A, P_A)\), \((A, \ell_B, r_B, P_A)\) is a representation of \((B, P_B)\), and \((A, B, \ell_A, r_A, \ell_B, r_B)\) is a matched pair of algebras.

**Theorem 3.4.** Let \((A, P_A)\) and \((B, P_B)\) be Rota-Baxter algebras of weight \(\lambda\) and let \((A, B, \ell_A, r_A, \ell_B, r_B)\) be a matched pair of the algebras \(A\) and \(B\). On the resulting algebra \(A \bowtie B\) from Eq. (25), define the linear map
\[
P_{A\bowtie B} : A \bowtie B \rightarrow A \bowtie B, \quad P_{A\bowtie B}(a + b) = P_A(a) + P_B(b), \quad \forall a \in A, b \in B.
\] (26)

Then the pair \((A \bowtie B, P_{A\bowtie B})\) is a Rota-Baxter algebra of weight \(\lambda\) if and only if \(((A, P_A), (B, P_B), \ell_A, r_A, \ell_B, r_B)\) is a matched pair of the Rota-Baxter algebras \((A, P_A)\) and \((B, P_B)\).

**Proof.** Computing two sides of the Rota-Baxter equation (3) for the operator \(P_{A\bowtie B}\), for \(a, a' \in A\) and \(b, b' \in B\), on the left hand side of the equation we have
\[
\begin{align*}
P_{A\bowtie B}(a + b) \star (a' + b') &= (P_A(a) + P_B(b)) \star (P_A(a') + P_B(b')) \\
&= P_A(a) \cdot_A P_A(a') + P_A(a)r_B(P_B(b')) + \ell_B(P_B(b'))P_A(a') \\
&\quad + P_B(b) \cdot_B P_B(b') + \ell_B(P_B(b'))P_B(b') + P_B(b)r_A(P_A(a')),
\end{align*}
\] and on the right hand side of the equation, by a similar computation we obtain
\[
\begin{align*}
P_{A\bowtie B}(a + b) \star (a' + b') &= (a + b) \star (a' + b') + \lambda(a + b) \star (a' + b') \\
&= P_A(a) \cdot_A a' + P_A(a)r_B(P_B(b')) + \lambda P_A(a)r_B(P_B(b')) + ar_B(P_B(b')) \\
&\quad + P_A(\ell_B(P_B(b))a' + \ell_B(b)P_A(a')) + \lambda \ell_B(b)a' + P_B(b) \cdot_B b' + b \cdot_B P_B(b') + \lambda b \cdot_B b'.
\end{align*}
\]
+P_B(\ell_A(Aa)b^\prime) + \ell_A(A)P_B(b^\prime) + \ell_A(b)r_A(a^\prime) + b\lambda r_A(a^\prime)).

Now if \((A, P_A), (B, P_B), \ell_A, r_A, \ell_B, r_B)\) is a matched pair of the Rota-Baxter algebras \((A, P_A)\) and \((B, P_B)\), then each term in the six-term sum of the left hand side equals the corresponding term of the right hand side. Therefore \(P_{A\oplus B}\) is a Rota-Baxter operator of weight \(\lambda\).

Conversely, suppose that \(P_{A\oplus B}\) satisfies the Rota-Baxter equation \((3.2)\). Taking \(a^\prime = b = 0\) in the equation and comparing, we obtain two of the four equalities in order for \(((A, P_A), (B, P_B), \ell_A, r_A, \ell_B, r_B)\) to be a matched pair of Rota-Baxter algebras. Likewise, taking \(a = b^\prime = 0\), we obtain another two of the four equalities in order to have a matched pair of Rota-Baxter algebras. \(\square\)

**Theorem 3.5.** Let \((A, \cdot, P)\) be a Rota-Baxter algebra. Suppose that there is a Rota-Baxter algebra \((A^*, \circ, Q^*)\) on the linear dual \(A^*\) of \(A\). Let \(\Delta : A \to A \otimes A\) denote the linear dual of the multiplication \(\circ : A^* \otimes A^* \to A^*\) on \(A^*\), that is, \((A, \Delta, Q)\) is a Rota-Baxter coalgebra. Then the quintuple \((A, \cdot, \Delta, P, Q)\) is a Rota-Baxter ASI bialgebra if and only if the sextuple \(((A, P), (A^*, Q^*), R^*, L^*, R_o^*, L_o^*)\) is a matched pair of Rota-Baxter algebras.

**Proof.** \((\Longrightarrow)\) If \((A, \cdot, \Delta, P, Q)\) is a Rota-Baxter ASI bialgebra, then \((A, \cdot, \Delta)\) is an ASI bialgebra and the linear operators \(P\) and \(P^*\) are admissible to \((A, \cdot, P)\) and \((A^*, \circ, Q^*)\) respectively. The former means that \((A, A^*, R^*, L^*, R_o^*, L_o^*)\) is a matched pair of algebras by Theorem [3.2] and the latter means that \((A^*, R^*, L^*, Q^*)\) is a representation of \((A, \cdot, P)\) and \((A, R_o^*, L_o^*, P)\) is a representation of \((A^*, \circ, Q^*)\). Hence \(((A, P), (A^*, Q^*), R^*, L^*, R_o^*, L_o^*)\) is a matched pair of Rota-Baxter algebras.

\((\Longleftarrow)\) By definition, if \(((A, P), (A^*, Q^*), R^*, L^*, R_o^*, L_o^*)\) is a matched pair of Rota-Baxter algebras, then \((A, A^*, R^*, L^*, R_o^*, L_o^*)\) is a matched pair of algebras and the linear operators \(Q\) and \(P^*\) are admissible to \((A, P)\) and \((A^*, Q^*)\) respectively. Hence \((A, \cdot, \Delta, P, Q)\) is a Rota-Baxter ASI bialgebra. \(\square\)

### 3.2. Double constructions of Rota-Baxter Frobenius algebras

We recall the concept of a double construction of Frobenius algebra. See [7] for details.

**Definition 3.6.** A bilinear form \(\mathcal{B}(,\, )\) on an algebra \(A\) is called **invariant** if

\[
\mathcal{B}(ab, c) = \mathcal{B}(a, bc), \quad \forall \, a, b, c \in A.
\]

A Frobenius algebra \((A, \mathcal{B})\) is an algebra \(A\) with a nondegenerate invariant bilinear form \(\mathcal{B}(,\, )\). A Frobenius algebra \((A, \mathcal{B})\) is called **symmetric** if \(\mathcal{B}(,\, )\) is symmetric.

Let \((A, \cdot)\) be an algebra. Suppose that there is an algebra structure \(\circ\) on its dual space \(A^*\), and an algebra structure on the direct sum \(A \oplus A^*\) of the underlying vector spaces of \(A\) and \(A^*\) which contains both \((A, \cdot)\) and \((A^*, \circ)\) as subalgebras. Define a bilinear form on \(A \oplus A^*\) by

\[
\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall \, a^*, b^* \in A^*, x, y \in A.
\]

If \(\mathcal{B}_d\) is invariant, so that \((A \oplus A^*, \mathcal{B}_d)\) is a symmetric Frobenius algebra, then the Frobenius algebra is called a **double construction of Frobenius algebra** associated to \((A, \cdot)\) and \((A^*, \circ)\), which is denoted by \((A \boxtimes A^*, \mathcal{B}_d)\). The notation \(A \boxtimes A^*\) is justified since the algebra on \(A \oplus A^*\) comes from a matched pair from \(A\) and \(A^*\) in Eq. \((25)\). Indeed, we have

**Theorem 3.7.** [3.1, Theorem 2.2.1] Let \((A, \cdot)\) and \((A^*, \circ)\) be algebras. Then there is a double construction of Frobenius algebra associated to \((A, \cdot)\) and \((A^*, \circ)\) if and only if \((A, A^*, R^*, L^*, R_o^*, L_o^*)\) is a matched pair of algebras.

We now extend these notions and properties to Rota-Baxter algebras.
Definition 3.8. A Rota-Baxter Frobenius algebra is a triple \((A, P, \mathfrak{B})\) where \((A, P)\) is a Rota-Baxter algebra and \((A, \mathfrak{B})\) is a Frobenius algebra. Let \(\hat{P} : A \rightarrow A\) denote the adjoint linear transformation of \(P\) under the nondegenerate bilinear form \(\mathfrak{B}:\)

\[
\mathfrak{B}(P(a), b) = \mathfrak{B}(a, \hat{P}(b)), \quad \forall a, b \in A. \tag{29}
\]

It is remarkable that the symmetric Frobenius property of a Rota-Baxter algebra \((A, P)\) naturally guarantees a representation on the dual space \(A^*\).

Proposition 3.9. Let \((A, P, \mathfrak{B})\) be a Rota-Baxter symmetric Frobenius algebra. Then for the adjoint operator \(\hat{P}\) in Eq. (29), the quadruple \((A^*, R^*, L^*, \hat{P}^*)\) is a representation of the Rota-Baxter algebra \((A, P)\) that is equivalent to \((A, L, R, P)\).

Conversely, let \((A, P)\) be a Rota-Baxter algebra and \(Q : A \rightarrow A\) be a linear map that is admissible to \((A, P)\). If the resulting representation \((A^*, R^*, L^*, Q^*)\) of \((A, P)\) is equivalent to \((A, L, R, P)\), then there exists a nondegenerate bilinear form \(\mathfrak{B}(, )\) such that \((A, P, \mathfrak{B})\) is a Rota-Baxter Frobenius algebra for which \(\hat{P} = Q\).

Proof. For \(a, b, c \in A\), by the Rota-Baxter relation in Eq. (2), we obtain

\[
0 = \mathfrak{B}(P(a)P(b), c) - \mathfrak{B}(P(aP(b)), c) - \mathfrak{B}(P(P(a)b), c) - \mathfrak{B}(\lambda P(ab), c)
\]

\[
= \mathfrak{B}(P(a), P(b)c) - \mathfrak{B}(aP(b), \hat{P}(c)) - \mathfrak{B}(P(ab), \hat{P}(c)) - \mathfrak{B}(\lambda ab, \hat{P}(c))
\]

\[
= \mathfrak{B}(a, \hat{P}(P(b)c) - P(b)\hat{P}(c) - \hat{P}(b\hat{P}(c)) - \lambda b\hat{P}(c))
\]

yielding \(\hat{P}(P(b)c) - P(b)\hat{P}(c) - \hat{P}(b\hat{P}(c)) - \lambda b\hat{P}(c) = 0\). This gives Eq. (3). Applying the symmetry of \(\mathfrak{B}\), a similar argument gives Eq. (4). Hence \((A^*, R^*, L^*, \hat{P}^*)\) is a representation of \((A, P)\). Define a linear map \(\phi : A \rightarrow A^*\) by

\[
\phi(a)(b) := \langle \phi(a), b \rangle = \mathfrak{B}(a, b), \quad \forall a, b \in A.
\]

The nondegeneracy of \(\mathfrak{B}\) gives the bijectivity of \(\phi\). Also for \(a, b, c \in A\), we have

\[
\phi(L(a)b)c = \mathfrak{B}(ab, c) = \mathfrak{B}(c, ab) = \mathfrak{B}(ca, b) = \langle \phi(b), ca \rangle = \langle R^*(a)\phi(b), c \rangle = R^*(a)\phi(b)c
\]

and similarly \(\phi(aR(b))c = \phi(a)L^*(b)c\) and \(\phi(P(a))b = \hat{P}^*(\phi(a))b\). Hence \((A, L, R, P)\) is equivalent to \((A^*, R^*, L^*, \hat{P}^*)\) as representations of \((A, P)\).

Conversely, suppose that \(\phi : A \rightarrow A^*\) is the linear isomorphism giving the equivalence between \((A, L, R, P)\) and \((A^*, R^*, L^*, Q^*)\). Define a bilinear form \(\mathfrak{B}(, )\) on \(A\) by

\[
\mathfrak{B}(a, b) := \langle \phi(a), b \rangle, \quad \forall a, b \in A.
\]

Then a similar argument gives the Rota-Baxter Frobenius algebra \((A, P, \mathfrak{B})\) and \(\hat{P} = Q\). \(\square\)

We now extend the notion of double constructions to Rota-Baxter Frobenius algebras.

Definition 3.10. Let \((A, \cdot, P)\) be a Rota-Baxter algebra. Suppose that \((A^*, \circ, Q^*)\) is a Rota-Baxter algebra. A double construction of Rota-Baxter Frobenius algebra associated to \((A, \cdot, P)\) and \((A^*, \circ, Q^*)\) is a double construction \((A \bowtie A^*, \mathfrak{B}, \mathfrak{D})\) of Frobenius algebra associated to \((A, \cdot)\) and \((A^*, \circ)\) such that \((A \bowtie A^*, P + Q^*, \mathfrak{B}, \mathfrak{D})\) is a Rota-Baxter Frobenius algebra, that is, \(P + Q^*\) is a Rota-Baxter operator on \(A \bowtie A^*\).

By definition, both \((A, \cdot, P)\) and \((A^*, \circ, Q^*)\) are Rota-Baxter subalgebras of \((A \bowtie A^*, P + Q^*)\). Thus the Rota-Baxter algebra \((A \bowtie A^*, P + Q^*)\) comes from a matched pair of the Rota-Baxter algebras \((A, P)\) and \((A^*, Q^*)\) in Theorem 3.4.
Lemma 3.11. Let \((A \bowtie A^\circ, P + Q^\circ, \mathfrak{B}_d)\) be a double construction of Rota-Baxter Frobenius algebra associated to \((A, P)\) and \((A^\circ, Q^\circ)\).

(a) The adjoint \(\widehat{P + Q^\circ}\) of \(P + Q^\circ\) with respect to \(\mathfrak{B}_d\) is \(Q + P^\circ\). Further \(Q + P^\circ\) is admissible to \((A \bowtie A^\circ, P + Q^\circ)\).

(b) \(Q\) is admissible to \((A, P)\).

(c) \(P^\circ\) is admissible to \((A^\circ, Q^\circ)\).

Proof. (\(\square\)) For \(a, b \in A, a^\circ, b^\circ \in A^\circ\), by Eq. (\(\mathfrak{F}\)), we have
\[
\mathfrak{B}_d((P + Q^\circ)(a + a^\circ), b + b^\circ) = \mathfrak{B}(P(a) + Q^\circ(a^\circ), b + b^\circ) = \langle P(a), b^\circ \rangle + \langle Q^\circ(a^\circ), b \rangle = \mathfrak{B}_d(a + a^\circ, (Q + P^\circ)(b + b^\circ)).
\]

Hence the adjoint \(\widehat{P + Q^\circ}\) of \(P + Q^\circ\) with respect to \(\mathfrak{B}_d\) is \(Q + P^\circ\). By Proposition 3.9, \(P + Q^\circ = Q + P^\circ\) is admissible to \((A \bowtie A^\circ, P + Q^\circ)\).

(\(\square\)) By Item (\(\mathfrak{F}\)), \(Q + P^\circ\) is admissible to \((A \bowtie A^\circ, P + Q^\circ)\). By Eqs. (\(\mathfrak{I}\)) and (\(\mathfrak{G}\)), this is true if and only if, for \(a, b \in A, a^\circ, b^\circ \in A^\circ\),
\[
(Q + P^\circ)((a + a^\circ)(P(b) + Q^\circ(b^\circ))) = (Q(a) + P^\circ(a))(P(b) + Q^\circ(b^\circ))
\]
\[
+ (Q + P^\circ)((Q(a) + P^\circ(a^\circ))(b + b^\circ)) + \lambda(Q(a) + P^\circ(a^\circ))(b + b^\circ)
\]
and
\[
(Q + P^\circ)((P(a) + Q^\circ(a^\circ))(b + b^\circ)) = (P(a) + Q^\circ(a^\circ))(Q(b) + P^\circ(b^\circ))
\]
\[
+ (Q + P^\circ)((a + a^\circ)(Q(b) + P^\circ(b^\circ))) + \lambda(a + a^\circ)(Q(b) + P^\circ(b^\circ)).
\]

Now taking \(a^\circ = b^\circ = 0\) in the above equations gives the admissibility of \(Q\) to \((A, P)\).

(\(\square\)) Likewise, taking \(a = b = 0\) in the above equations yields the desired admissibility:
\[
P^\circ(a^\circ Q^\circ(b^\circ)) = P^\circ(a^\circ)Q^\circ(b^\circ) + P^\circ(P^\circ(a^\circ)b^\circ) + \lambda P^\circ(a^\circ)b^\circ,
\]
\[
P^\circ(Q^\circ(a^\circ)b^\circ) = Q^\circ(a^\circ)P^\circ(b^\circ) + P^\circ(a^\circ P^\circ(b^\circ)) + \lambda a^\circ P^\circ(b^\circ).
\]

Extending Theorem 3.7 to Rota-Baxter Frobenius algebras, we obtain

Theorem 3.12. Let \((A, \cdot, P)\) be a Rota-Baxter algebra. Suppose that there is a Rota-Baxter algebra structure \((A^\circ, \circ, Q^\circ)\) on its dual space \(A^\circ\). Then there is a double construction of Rota-Baxter Frobenius algebra \((A \oplus A^\circ, P_{A\oplus A^\circ}, \mathfrak{B}_d)\) associated to \((A, \cdot, P)\) and \((A^\circ, \circ, Q^\circ)\) if and only if 
\((A, P), (A^\circ, Q^\circ), R^\circ, L^\circ, R_0^\circ, L_0^\circ)\) is a matched pair of Rota-Baxter algebras.

Proof. (\(\Leftarrow\)) To the given double construction \((A \oplus A^\circ, P_{A\oplus A^\circ}, \mathfrak{B}_d)\) associated to the Rota-Baxter algebras \((A, P)\) and \((A^\circ, Q^\circ)\) implies that \((A \oplus A^\circ, \mathfrak{B}_d)\) is a double construction of Frobenius algebra associated to \(A\) and \(A^\circ\). Hence by Theorem 3.7, \((A, A^\circ, R^\circ, L^\circ, R_0^\circ, L_0^\circ)\) is a matched pair of algebras for which the algebra on \(A \oplus A^\circ\) is the algebra \(A \bowtie A^\circ\). Since the Rota-Baxter operator \(P_{A\oplus A^\circ}\) is \(P + Q^\circ\), \((A, P), (A^\circ, Q^\circ), R^\circ, L^\circ, R_0^\circ, L_0^\circ)\) is a matched pair of Rota-Baxter algebras.

(\(\Rightarrow\)) If \((A, P), (A^\circ, Q^\circ), R^\circ, L^\circ, R_0^\circ, L_0^\circ)\) is a matched pair of Rota-Baxter algebras, then \((A, A^\circ, R^\circ, L^\circ, R_0^\circ, L_0^\circ)\) is a matched pair of algebras. Hence by Theorem 3.7 again, \((A \bowtie A^\circ, \mathfrak{B}_d)\) is a Frobenius algebra. By Theorem 3.7, the matched pair of Rota-Baxter algebras also equips the algebra \(A \bowtie A^\circ\) with the Rota-Baxter operator \(P + Q^\circ\), giving us a Rota-Baxter Frobenius algebra. This is what we need.

Combining Theorems 3.5 and 3.12, we have
Theorem 3.13. Let \((A, \cdot, P)\) be a Rota-Baxter algebra. Suppose that there is a Rota-Baxter algebra \((A^*, \circ, Q^*)\) on the linear dual \(A^*\) of \(A\). Let \(\Delta : A \to A \otimes A\) denote the linear dual of the multiplication \(\circ : A^* \otimes A^* \to A^*\) on \(A^*\). Then the following conditions are equivalent.

(a) The sextuple \((A, P), (A^*, Q^*), R^*, L^*, R^*_\circ, L^*_\circ\) is a matched pair of Rota-Baxter algebras.

(b) There is a double construction of Rota-Baxter Frobenius algebra associated to \((A, \cdot, P)\) and \((A^*, \circ, Q^*)\).

(c) The quintuple \((A, \cdot, \Delta, P, Q)\) is a Rota-Baxter ASI bialgebra.

4. Coboundary Rota-Baxter ASI bialgebras, admissible associative Yang-Baxter equations and \(\mathcal{O}\)-operators

In this section, we study the coboundary Rota-Baxter ASI bialgebras and show that they can be given by antisymmetric solutions of the \(Q\)-admissible associative Yang-Baxter equation in a Rota-Baxter algebra. We also give the notion of \(\mathcal{O}\)-operators on a Rota-Baxter algebra and show that \(\mathcal{O}\)-operators provide antisymmetric solutions of \(Q\)-admissible associative Yang-Baxter equation in suitable Rota-Baxter algebras.

4.1. Coboundary Rota-Baxter ASI bialgebras. For given \(r \in A \otimes A\), define

\[
\Delta(a) := \Delta_r(a) := (\text{id} \otimes L(a) - R(a) \otimes \text{id})(r), \quad \forall a \in A.
\]  

(30)

Definition 4.1. A Rota-Baxter ASI bialgebra \(((A, P), \Delta, Q)\) is called coboundary if \(\Delta\) is defined by Eq. (30) for some \(r \in A \otimes A\).

Remark 4.2. Let \((A, P)\) be a \(Q\)-admissible Rota-Baxter algebra and \(r \in A \otimes A\). If \(\Delta : A \to A \otimes A\) is given by Eq. (30), then \(\Delta\) satisfies Eq. (1). Moreover, by [3, Proposition 2.3.4], \(\Delta\) satisfies Eq. (2) if and only if \(r\) satisfies

\[
(L(a) \otimes \text{id} - \text{id} \otimes R(a))(\text{id} \otimes L(b) - R(b) \otimes \text{id})(r + \sigma(r)) = 0, \quad \forall a, b \in A.
\]  

(31)

By [3, Proposition 2.3.3], we know that \(\Delta^*\) defines an algebra on \(A^*\) if and only if

\[
(\text{id} \otimes \text{id} \otimes L(a) - R(a) \otimes \text{id} \otimes \text{id})(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = 0, \quad \forall a \in A,
\]  

(32)

where for \(r = \sum_i a_i \otimes b_i\),

\[
r_{12}r_{13} = \sum_{i,j} a_i a_j \otimes b_i \otimes b_j, \quad r_{13}r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i b_j, \quad r_{23}r_{12} = \sum_{i,j} a_j \otimes a_i \otimes b_i b_j.
\]

Hence in order for \(((A, P), \Delta, Q)\) to be a Rota-Baxter ASI bialgebra, we just need to further require that \((A^*, \Delta^*, Q^*)\) is a \(P^*\)-admissible Rota-Baxter algebra, that is, \((A, \Delta, Q)\) is a Rota-Baxter coalgebra and Eqs. (7) and (8) hold.

Theorem 4.3. Let \((A, P)\) be a \(Q\)-admissible Rota-Baxter algebra and \(r \in A \otimes A\). Define a linear map \(\Delta : A \to A \otimes A\) by Eq. (30). Suppose that \(\Delta^*\) defines an associative multiplication on \(A^*\). Then the following conclusions hold.

(a) Eq. (4) holds if and only if for \(a \in A\),

\[
(id \otimes Q(L(a)) - id \otimes L(Q(a)))(Q \otimes id - id \otimes P)(r) + (Q(R(a)) \otimes id - R(Q(a)) \otimes id)(P \otimes id - id \otimes Q)(r) = 0.
\]  

(33)
Then (and (that))

\begin{align*}
\text{\textit{Proof.}} & \text{ Set } r = \sum_i a_i \otimes b_i. \text{ By Eq. (31), we have} \\
& (\text{id} \otimes L(P(a)) - R(P(a)) \otimes \text{id}) + \lambda \text{id} \otimes L(a)) (P \otimes \text{id} - \lambda \text{id} \otimes P(r) = 0. \\
\end{align*}

(c) Eq. (33) holds if and only if for \( a \in A \),

\begin{align*}
(\text{id} \otimes L(P(a)) - R(P(a)) \otimes \text{id} - \lambda \text{id} \otimes P(L(a)) \\
- Q(R(a)) \otimes \text{id} = 0.
\end{align*}

Hence Eq. (3) holds if and only if Eq. (31) holds. Items (3) and (4) can be proved by the same argument. \( \square \)

By Remark 4.2 and Theorem 4.3, we have

\textbf{Corollary 4.4.} Let \((A, P)\) be a \(Q\)-admissible Rota-Baxter algebra and \( r \in A \otimes A \). Then the linear map \( \Delta \) defined by Eq. (37) induces a \( P\)-admissible Rota-Baxter algebra \((A', \Delta', Q')\) such that \(((A, P), \Delta, Q)\) is a Rota-Baxter ASI bialgebra if and only if Eqs. (37)-(35) are satisfied.

\textbf{Example 4.5.} We continue with the notations in Example 2.6. Suppose in addition that \((A, \cdot_A, \Delta_A)\) and \((B, \cdot_B, \Delta_B)\) are coboundary ASI bialgebras, that is, there exist \( r_1 \in A \otimes A \) and \( r_2 \in B \otimes B \) such that

\begin{align*}
\Delta_A(a) &= (\text{id} \otimes L(a) - R(a) \otimes \text{id})(r_1), \forall a \in A, \\
\Delta_B(b) &= (\text{id} \otimes L(b) - R(b) \otimes \text{id})(r_2), \forall b \in B.
\end{align*}

Then \((A \otimes B, \cdot_A + \cdot_B, \Delta_A + \Delta_B)\) is a coboundary ASI bialgebra with \( r = r_1 + r_2 \):

\begin{align*}
(\Delta_A + \Delta_B)(a + b) &= (\text{id}_{A \otimes B} \otimes L(a + b) - R(a + b) \otimes \text{id}_{A \otimes B})(r_1), \forall a \in A, b \in B.
\end{align*}

It is straightforward to check that Eqs. (33)-(35) hold for \( P = P_A, Q = P_B \) and \( r = r_1 + r_2 \). Hence the Rota-Baxter ASI bialgebra \((A \otimes B, \cdot_A + \cdot_B, \Delta_A + \Delta_B, P_A, P_B)\) is coboundary.

We now prove a self-duality of a Rota-Baxter ASI bialgebra and give its construction on the double space.
**Theorem 4.6.** Let \((A, P), \Delta, Q)\) be a Rota-Baxter ASI bialgebra. Let \(\delta : A^* \rightarrow A^* \otimes A^*\) be the linear dual of the multiplication on \(A\). Then \((A^*, Q^*), -\delta, P^*)\) is also a Rota-Baxter ASI bialgebra. Further there is a Rota-Baxter ASI bialgebra structure on the direct sum \(A \oplus A^*\) of the underlying vector spaces of \(A\) and \(A^*\) which contains the two Rota-Baxter ASI bialgebras as Rota-Baxter ASI sub-bialgebras.

**Proof.** Denote the product on the algebra \(A^*\) by \(\circ\). By [1, Remark 2.2.4], \((A^*, \circ, -\delta)\) is an ASI bialgebra. Moreover, \(Q\) is admissible to the Rota-Baxter algebra \((A, P)\) whose algebra structure is given by \(-\delta^*\) if and only if \(Q\) is admissible to the Rota-Baxter algebra \((A, P)\) whose algebra structure is given by \(\delta^*\). Therefore with the fact that \(P^*\) is admissible to \((A^*, Q^*)\), we show that \((A^*, Q^*), -\delta, P^*)\) is a Rota-Baxter ASI bialgebra.

Let \(r \in A \otimes A^* \subset (A \otimes A^*) \otimes (A \otimes A^*)\) correspond to the identity map \(\text{id}\) on \(A\). Let \(\{e_1, e_2, \cdots , e_n\}\) be a basis of \(A\) and \(\{e^1, e^2, \cdots , e^n\}\) its dual basis. Then \(r = \sum_{i=1}^n e_i \otimes e^i\). Let \((A \leadsto A^*, \star)\) denote the algebra structure on \(A \oplus A^*\) induced by the matched pair \((A, A^*, R^*, L^*, R^*_o, L^*_o)\) of algebras. Define

\[
\Delta_{A \oplus A^*}(u) = (\text{id} \otimes L_{A \oplus A^*}(u) - R_{A \oplus A^*}(u) \otimes \text{id})(r), \quad \forall u \in A \leadsto A^*.
\]

Moreover, \((A \leadsto A^*, P + Q^*)\) is a \((Q + P^*)\)-admissible Rota-Baxter algebra by Lemma 3.11. Hence Eqs. (33) and (34) hold. Since

\[
((P + Q^*) \otimes \text{id} - \text{id} \otimes (Q + P^*)) (r) = \sum_{i=1}^n (P(e_i) \otimes e^i - e_i \otimes P^*(e^i)) = 0,
\]

\[
((Q + P^*) \otimes \text{id} - \text{id} \otimes (P + Q^*)) (r) = \sum_{i=1}^n (Q(e_i) \otimes e^i - e_i \otimes Q^*(e^i)) = 0,
\]

Eqs. (33)–(35) hold. By [2, Theorem 2.3.6], we know that \(r\) satisfies Eqs. (31) and (32), and \((A \leadsto A^*, \star, \Delta_{A \oplus A^*})\) is an ASI bialgebra containing \((A, \cdot, \Delta)\) and \((A^*, \circ, -\delta)\) as ASI sub-bialgebras. Therefore \((A \leadsto A^*, P + Q^*), \star, \Delta_{A \oplus A^*}, Q + P^*)\) is a Rota-Baxter ASI bialgebra. It is obvious that it contains \((A, P), \Delta, Q)\) and \((A^*, Q^*), -\delta, P^*)\) as Rota-Baxter ASI sub-bialgebras. This completes the proof.

\[\square\]

### 4.2. Admissible associative Yang-Baxter equation in a Rota-Baxter algebra

As a consequence of Corollary 4.4, we obtain

**Corollary 4.7.** Let \((A, P)\) be a \(Q\)-admissible Rota-Baxter algebra and \(r \in A \otimes A\). Then the linear map \(\Delta\) defined by Eq. (31) induces a \(P^*\)-admissible Rota-Baxter algebra \((A^*, \Delta^*, Q^*)\) such that \((A, P), \Delta, Q)\) is a Rota-Baxter ASI bialgebra if Eq. (17) and the following equations hold:

\[
r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0, \quad (37)
\]

\[
(P \otimes \text{id} - \text{id} \otimes \sigma)(r) = 0, \quad (38)
\]

\[
(Q \otimes \text{id} - \text{id} \otimes P)(r) = 0. \quad (39)
\]

This leads us to the following variation of the associative Yang-Baxter equation.

**Definition 4.8.** Let \((A, P)\) be a Rota-Baxter algebra. Suppose that \(r \in A \otimes A\) and \(Q : A \rightarrow A\) is a linear map. Then Eq. (37) with conditions given by Eqs. (38) and (39) is called the \(Q\)-admissible associative Yang-Baxter equation (AYBE) in \((A, P)\) or simply the \(Q\)-admissible AYBE.

**Remark 4.9.** Eq. (37) is simply the associative Yang-Baxter equation (AYBE) in an associative algebra \([3, 19]\), as an analogue of the classical Yang-Baxter equation in a Lie algebra \([10, 11]\). Also if \(r\) is antisymmetric (that is, \(r = -\sigma(r)\)), then Eq. (38) holds if and only if Eq. (39) holds.
Remark 4.10. Continuing with the notations in Example 4.5, we make an observation that distinguishes the admissible AYBE from the AYBE. Suppose that $r_1$ and $r_2$ are distinct skew-symmetric solutions of the AYBE in the algebras $A$ and $B$ respectively. Then $r = r_1 + r_2$ is a skew-symmetric solution of the AYBE in $A \oplus B$. However, $r$ is not a solution of the $P_B$-admissible AYBE in the Rota-Baxter algebra $(A \oplus B, P_A)$ since in this case $(P_A \otimes \text{id}_{A \otimes B} - \text{id}_{A \otimes B} \otimes P_B)(r) = r_1 - r_2$ is nonzero.

By Corollary 4.11, we have the following conclusion.

Corollary 4.11. Let $(A, P)$ be a $Q$-admissible Rota-Baxter algebra and $r \in A \otimes A$ an antisymmetric solution of the $Q$-admissible AYBE in $(A, P)$. Then $((A, P), \Delta, Q)$ is a Rota-Baxter ASI bialgebra, where the linear map $\Delta = \Delta_r$ is defined by Eq. (43).

We now study solutions of the $Q$-admissible AYBE. For a vectors space $A$, the isomorphism $A \otimes A \cong \text{Hom}(A^*, K) \otimes A \cong \text{Hom}(A^*, A)$ identifies an $r \in A \otimes A$ with a map from $A^*$ to $A$ which we still denote by $r$. Explicitly, writing $r = \sum_i a_i \otimes b_i$, then

$$r : A^* \to A, \quad r(a^*) = \sum \langle a^*, a_i \rangle b_i, \quad \forall a^* \in A^*.$$  

(40)

We called $r \in A \otimes A$ nondegenerate if the map $r^\dagger : A^* \to A$ defined by Eq. (41) is bijective.

Theorem 4.12. Let $(A, \cdot, P)$ be a Rota-Baxter algebra and $r \in A \otimes A$ antisymmetric. Let $Q : A \to A$ be a linear map. Then $r$ is a solution of the $Q$-admissible AYBE in $(A, \cdot, P)$ if and only if $r$ satisfies

$$r(a^*) \cdot r(b^*) = R_A^*(r(a^*)) b^* + a^* L_A^* (r(b^*))), \quad \forall a^*, b^* \in A^*,$$

(41)

$$Pr = rQ^*.$$  

(42)

Proof. By [1, Proposition 2.4.7], $r$ is a solution of the AYBE in $(A, \cdot)$ if and only if Eq. (41) holds. Moreover, let $r = \sum_i a_i \otimes b_i$ and for $a^* \in A^*$, we have

$$r(Q^*(a^*)) = \sum_i \langle Q^*(a^*), a_i \rangle b_i = \sum_i \langle a^*, Q(a_i) \rangle b_i, \quad P(r(a^*)) = \sum_i \langle a^*, a_i \rangle P(b_i).$$

So $Pr = rQ^*$ if and only if Eq. (42) holds. This completes the proof. □

We next relate the admissible AYBE to Connes cocycles and Frobenius algebras.

Definition 4.13. [3] An antisymmetric bilinear form $\omega : A \otimes A \to K$ on an algebra $A$ is a cyclic 1-cocycle in the sense of Connes [23], or simply a Connes cocycle, if

$$\omega(ab, c) + \omega(bc, a) + \omega(ca, b) = 0, \quad \forall a, b, c \in A.$$  

(43)

Let $(A, P)$ be a Rota-Baxter algebra and $\omega : A \otimes A \to K$ be a bilinear form. Suppose that $\omega$ is a nondegenerate Connes cocycle on $A$. Define $\hat{P} : A \to A$ to be the (right) adjoint linear transformation of $P$ with respect to $\omega$: $\omega(P(a), b) = \omega(a, \hat{P}(b)), \quad \forall a, b \in A.$

(44)

Proposition 4.14. Let $(A, P)$ be a Rota-Baxter algebra. Let $r \in A \otimes A$ and $Q : A \to A$ be a linear map. Suppose that $r$ is antisymmetric and nondegenerate. Let $\omega$ be the bilinear form defined by the inverse $r^{-1} : A \to A^*$ of the linear bijection corresponding to $r$:

$$\omega(a, b) := \langle r^{-1}(a), b \rangle, \quad \forall a, b \in A.$$  

Then $r$ is a solution of the $Q$-admissible AYBE in $(A, P)$ if and only if $\omega$ gives a nondegenerate Connes cocycle $(A, \omega)$ with respect to which $Q$ is the adjoint $\hat{P}$ of $P$. 

\textbf{Proof.} By \cite[Proposition 2.1]{1}, we only need to check that Eq. (\ref{eq:45}) holds if and only if the adjoint \(\hat{P}\) of \(P\) with respect to \(\omega\) is \(Q\). But the latter holds if and only if for all \(a, b \in A\),
\[
0 = \omega(P(a), b) - \omega(a, Q(b)) = \langle r^{-1}P(a), b \rangle - \langle r^{-1}(a), Q(b) \rangle = \langle r^{-1}P(a), b \rangle - \langle Q^*r^{-1}(a), b \rangle,
\]
which holds if and only if \(r^{-1}P = Q^*r^{-1}\), which means \(Pr = rQ^*\), that is, \((P \otimes \text{id} - \text{id} \otimes Q)(r) = 0\). This completes the proof. \(\square\)

Now let \((A, P, \mathcal{B})\) be a Rota-Baxter symmetric Frobenius algebra of weight zero. Then under the natural bijection \(\text{Hom}(A \otimes A, K) \cong \text{Hom}(A, A^*)\), the bilinear form \(\mathcal{B}\) corresponds to the linear map (see also the proof of Proposition \ref{prop:3.9})
\[
\phi : A \to A^*, \quad \langle \phi(a), b \rangle := \mathcal{B}(a, b), \quad \forall a, b \in A.
\]
By pre-composing, we obtain a bijection
\[
\text{Hom}(A^*, A) \to \text{Hom}(A, A), \quad r \mapsto P_r := r\phi, \quad \forall r \in \text{Hom}(A^*, A). \tag{45}
\]

\textbf{Theorem 4.15.} Let \((A, P, \mathcal{B})\) be a Rota-Baxter symmetric Frobenius algebra of weight zero. Then an antisymmetric \(r \in A \otimes A\) is a solution of the \(\hat{P}\)-admissible AYBE in \((A, P)\) if and only if the corresponding \(P_r\) from Eq. (\ref{eq:45}) is a Rota-Baxter operator of weight zero on \(A\) such that \(PP_r = P_rP\).

\textbf{Proof.} By \cite[Corollary 3.17]{1}, \(r\) is a solution of AYBE in the algebra \(A\), that is, Eq. (\ref{eq:45}) holds, if and only if \(P_r\) is a Rota-Baxter operator of weight zero on \(A\).

Moreover, set \(r = \sum_i a_i \otimes b_i\). For \(a \in A\), we have
\[
PP_r(a) = Pr(\phi(a)), \quad P_rP(a) = \langle \phi(\phi(a)), b_i \rangle = \sum_i \mathcal{B}(a, P_r(\phi(a)))b_i \]
\[
= \sum_i \langle \phi(\phi(a)), P_r(\phi(a)) \rangle b_i = r\hat{P}^*(\phi(a)).
\]
Thus \(Pr = r\hat{P}^*\) if and only if \(PP_r = P_rP\). \(\square\)

\textbf{Corollary 4.16.} Let \((A, P, \mathcal{B})\) be a Rota-Baxter symmetric Frobenius algebra of weight zero and \(r \in A \otimes A\) antisymmetric. If \((A, P_r)\) is a Rota-Baxter algebra of weight zero and \(PP_r = P_rP\), where \(P_r\) is defined by Eq. (\ref{eq:45}), then \((A, P, \Delta, \hat{P})\) is a Rota-Baxter ASI bialgebra, where \(\Delta = \Delta_r\), is given by Eq. (\ref{eq:45}). In particular, if \(\hat{P} = -P\), then for the inverse image \(r = r_P\) of \(P\) under the bijection in Eq. (\ref{eq:45}), the triple \((A, P, \Delta_r, -P)\) is a Rota-Baxter ASI bialgebra.

\textbf{Proof.} Note that \((A, P)\) is a \(\hat{P}\)-admissible Rota-Baxter algebra due to Proposition \ref{prop:4.15}. Hence the first conclusion follows from Corollary \ref{cor:3.17} and Theorem \ref{thm:4.15}.

Now suppose \(\hat{P} = -P\). Note that the element \(r_P \in \text{Hom}(A^*, A)\) corresponding to \(P\) under the bijection in Eq. (\ref{eq:45}) is defined by
\[
r_P(a^*) := P(\phi^{-1}(a^*)), \quad \forall a^* \in A^*. \tag{46}
\]
Then, for \(a^*, b^* \in A^*\),
\[
\langle r_P(a^*), b^* \rangle + \langle a^*, r_P(b^*) \rangle = \langle P(\phi^{-1}(a^*)), b^* \rangle + \langle a^*, P(\phi^{-1}(b^*)) \rangle
\]
\[
= \mathcal{B}(P(\phi^{-1}(a^*)), P(\phi^{-1}(b^*))) + \mathcal{B}(P(\phi^{-1}(b^*)), P(\phi^{-1}(a^*))
\]
\[
= \mathcal{B}(\phi^{-1}(a^*), \hat{P}(\phi^{-1}(b^*))) + \mathcal{B}(\phi^{-1}(b^*), \hat{P}(\phi^{-1}(a^*))) = 0.
\]
Hence \(r_P\) (or its corresponding 2-tensor) is antisymmetric. Since \(PP_r = P\) under the bijection in Eq. (\ref{eq:45}), the second conclusion holds from the first conclusion. \(\square\)
4.3. $\mathcal{O}$-operators on Rota-Baxter algebras. The importance of Theorem 4.12 leads us to the next notion.

**Definition 4.17.** Let $(A, P)$ be a Rota-Baxter algebra of weight $\lambda$. Let $(V, \ell, r)$ be a representation of the algebra $A$ and $\alpha : V \to V$ be a linear map. A linear map $T : V \to A$ is called a **weak $\mathcal{O}$-operator associated to $(V, \ell, r)$ and $\alpha$** if $T$ satisfies

\[
T(u)t(v) = T(\ell(T(u))v + ur(T(v))), \quad \forall u, v \in V,
\]

\[
PT = T\alpha.
\]

If in addition, $(V, \ell, r, \alpha)$ is a representation of $(A, P)$, then $T$ is called an **$\mathcal{O}$-operator associated to $(V, \ell, r, \alpha)$**.

**Example 4.18.** (a) Let $(A, P)$ be a Rota-Baxter algebra of weight $\lambda$. Then the identity map $id$ on $A$ is an $\mathcal{O}$-operator associated to $(A, L, 0, P)$ or $(A, 0, R, P)$.

(b) Let $(A, P)$ be a Rota-Baxter algebra of weight zero. Then the Rota-Baxter operator $P : A \to A$ is an $\mathcal{O}$-operator associated to the representation $(A, L, R, P)$.

Theorem 4.12 is rewritten in terms of $\mathcal{O}$-operators as follows.

**Corollary 4.19.** Let $(A, P)$ be a Rota-Baxter algebra and $r \in A \otimes A$ antisymmetric. Let $Q : A \to A$ be a linear map. Then $r$ is a solution of the $Q$-admissible AYBE in $(A, P)$ if and only if $r$ is a weak $\mathcal{O}$-operator associated to $(A^*, R^*, L^*)$ and $Q^*$. If in addition, $(A, P)$ is a $Q$-admissible Rota-Baxter algebra, then $r$ is a solution of the $Q$-admissible AYBE in $(A, P)$ if and only if $r$ is an $\mathcal{O}$-operator associated to the representation $(A^*, R^*, L^*, Q^*)$.

We next show that $\mathcal{O}$-operators give numerous solutions of the admissible AYBE in semi-direct product Rota-Baxter algebras and give rise to Rota-Baxter ASI bialgebras.

We first consider admissible quadruples for semi-direct products of Rota-Baxter algebras.

**Theorem 4.20.** Let $(A, P)$ be a Rota-Baxter algebra of weight $\lambda$ and let $(V, \ell, r)$ be a representation of the algebra $A$. Let $Q : A \to A$ and $\alpha, \beta : V \to V$ be linear maps. Then the following conditions are equivalent.

(a) There is a Rota-Baxter algebra $(A \ltimes_{\ell, r} V, P + \alpha)$ such that the linear operator $Q + \beta$ on $A \oplus V$ is admissible to $(A \ltimes_{\ell, r} V, P + \alpha)$.

(b) There is a Rota-Baxter algebra $(A \ltimes_{r, \ell} V^*, P + \beta^*)$ such that the linear operator $Q + \alpha^*$ on $A \oplus V^*$ is admissible to $(A \ltimes_{r, \ell} V^*, P + \beta^*)$.

(c) The following conditions are satisfied:

(i) $(V, \ell, r, \alpha)$ is a representation of $(A, P)$, that is, Eqs. (10) and (11) hold;

(ii) $Q$ is admissible to $(A, P)$, that is, Eqs. (3) and (4) hold;

(iii) $(V, l, r, \beta)$ is an admissible quadruple of $(A, P)$, that is, Eqs. (7) and (8) hold;

(iv) For $a \in A, u \in V$, we have

\[
\beta(\ell(\alpha(u))a) = \beta(\ell(Q(a))u) + \ell(Q(a))\alpha(u) + \lambda\ell(Q(a))u,
\]

\[
\beta(\alpha(u)r(a)) = \beta(uQ(a)) + \alpha(u)r(Q(a)) + \lambda u r(Q(a)).
\]

**Proof.** $(\text{i}) \iff (\text{ii})$. By Proposition 2.11, $(A \ltimes_{\ell, r} V, P + \alpha)$ is a Rota-Baxter algebra if and only if $(V, \ell, r, \alpha)$ is a representation of the Rota-Baxter algebra $(A, P)$. Let the product on $(A \ltimes_{\ell, r} V)$ be denoted by $\ast$. Let $a, b \in A$ and $u, v \in V$. Then we have

\[
(Q + \beta)(((P + \alpha)(a + u)) \ast (b + v)) = Q(P(a) \cdot b) + \beta(\alpha(u)r(b)) + \beta(\ell(P(a))v),
\]
((P + α)(a + u)) * ((Q + β)(b + v)) = P(a) \cdot Q(b) + α(u)r(Q(b)) + β(V(b)\cdot β(v)),
(Q + β)((a + u) * ((Q + β)(b + v))) = Q(a \cdot Q(b)) + β(α(Q(b))) + β(β(α(Q(b))),
λ(a + u) * ((Q + β)(b + v)) = λa \cdot Q(b) + λur(Q(b)) + λ(β(α(Q(b)));

Therefore Eq. (3) holds (where Q is replaced by Q + β, P by P + α, a by a + u, and b by b + v) if and only if Eq. (4) (corresponding to u = v = 0), Eq. (16) (corresponding to b = u = 0) and Eq. (50), where a is replaced by b, (corresponding to a = v = 0) hold. Similarly, Eq. (3) holds (where Q is replaced by Q + β, P by P + α, a by a + u, and b by b + v) if and only if Eq. (4), Eq. (15) and Eq. (15) hold. Hence Condition (3) holds if and only if Condition (4) holds.

(F) ←→ (G) In Item (4), take

\[ V = V', \ell = r', r = r', \beta = \beta', \alpha = \alpha'. \]

Then from the above equivalence between Condition (4) and Condition (5), we have Condition (5) holds if and only if the conditions (3)-(11) in Condition (6) as well as the following two equations hold (for all \( a \in A, u^* \in V^* \)):

\[
\alpha^r(r^*(Q(a))u^*) + r^*(Q(a))u^* + \lambda r^*(Q(a))u^*, \quad (51)
\]

\[
\alpha^r(u^*r^*(Q(a))) + \beta^r(u^*)r^*(Q(a)) + \lambda u^*r^*(Q(a)), \quad (52)
\]

For \( a \in A, u \in V \) and \( u^* \in V^* \), we have

\[
\langle \alpha^r(u^*r^*(Q(a))), u \rangle = \langle \beta^r(u^*)r^*(Q(a)), u \rangle = \langle \alpha^r(u^*)r^*(Q(a)), u \rangle = \langle \beta^r(u^*)r^*(Q(a)), u \rangle = \langle \alpha^r(u^*)r^*(Q(a)), u \rangle = \langle \beta^r(u^*)r^*(Q(a)), u \rangle = \langle \alpha^r(u^*)r^*(Q(a)), u \rangle = \langle \beta^r(u^*)r^*(Q(a)), u \rangle.
\]

Hence Eq. (52) holds if and only if Eq. (45) holds. Similarly, Eq. (51) holds if and only if Eq. (51) holds. Therefore Condition (5) holds if and only if Condition (6) holds.

Here is our main result on antisymmetric solutions of the admissible AYBE and the constructions of Rota-Baxter ASI bialgebras.

**Theorem 4.21.** Let \((V, \ell, r, β)\) be an admissible quadruple of a Rota-Baxter algebra \((A, P)\) of weight \(λ\) and let \((V^*, r^*, , β^* )\) be the representation of \((A, P)\) defined in Lemma 2.11. Let \(Q : A \rightarrow A\) and \(α : V \rightarrow V\) be linear maps. Let \(T : V \rightarrow A\) be a linear map which is identified as an element in \((A \otimes A) \otimes A \subseteq (A \otimes A) \otimes A \otimes A\).

(a) The element \(r = T - σ(T)\) is an antisymmetric solution of the \((Q + α^*)\)-admissible AYBE in the Rota-Baxter algebra \((A \otimes A) \otimes A \otimes A\) if and only if \(T\) is a weak \(O\)-operator associated to \((V, \ell, r, α)\).

(b) Assume that \((V, \ell, r, α)\) is a representation of \((A, P)\). If \(T\) is an \(O\)-operator associated to \((V, \ell, r, α)\) and \(Tβ = QT\), then \(r = T - σ(T)\) is an antisymmetric solution of the \((Q + α^*)\)-admissible AYBE in the Rota-Baxter algebra \((A \otimes A) \otimes A \otimes A\).

Proof. (3) Corollary 3.10] shows that \(r\) satisfies Eq. (57) if and only if Eq. (57) holds.
Let \( \{e_1, e_2, \cdots, e_n\} \) be a basis of \( V \) and \( \{e^1, e^2, \cdots, e^n\} \) be its dual basis. Then \( T = \sum_{i=1}^n T(e_i) \otimes e^i \in (A \bowtie_{\gamma, \cdot} V^*) \otimes (A \bowtie_{\gamma, \cdot} V^*) \). Hence

\[
    r = T - \sigma(T) = \sum_{i=1}^n (T(e_i) \otimes e^i - e^i \otimes T(e_i)).
\]

Note that

\[
    ((P + \beta^*) \otimes \text{id})(r) = \sum_{i=1}^n (PT(e_i) \otimes e^i - \beta^*(e^i) \otimes T(e_i)),
\]

\[
    (\text{id} \otimes (Q + \alpha^*))(r) = \sum_{i=1}^n (T(e_i) \otimes \alpha^*(e^i) - e^i \otimes QT(e_i)).
\]

Further,

\[
    \sum_{i=1}^n \beta^*(e^i) \otimes T(e_i) = \sum_{i=1}^n \sum_{j=1}^n \langle \beta^*(e^i), e_j \rangle e^j \otimes T(e_i) = \sum_{j=1}^n e^j \otimes \sum_{i=1}^n \langle e^i, \beta(e_j) \rangle T(e_i)
\]

\[
    = \sum_{i=1}^n e^i \otimes T(\sum_{j=1}^n \langle \beta(e_i), e_j \rangle e^j) = \sum_{i=1}^n e^i \otimes T\beta(e_i),
\]

and similarly, \( \sum_{i=1}^n T(e_i) \otimes \alpha^*(e^i) = \sum_{i=1}^n T\alpha(e_i) \otimes e^i \). Therefore \((P + \beta^* \otimes \text{id})(r) = (\text{id} \otimes (Q + \alpha^*))(r)\) if and only if \( PT = T\alpha \) and \( QT = T\beta \). Hence the conclusion follows.

(\[\]) It follows from Item (\[\]) and Theorem 4.20. \[\]

4.4. Some cases and examples. According to Theorem 4.21, from an \( \Theta \)-operator \( T \) of a Rota-Baxter algebra \((A, P)\) associated to a representation \((V, l, r, \alpha)\) and an admissible quadruple \((V, l, r, \beta)\) satisfying \( T\beta = QT \), one can get a skew-symmetric solution of the admissible AYBE in a semi-direct product Rota-Baxter algebra. If in addition, \((A, P)\) is \( Q \)-admissible and Eqs. (49)–(50) hold, then one can get a Rota-Baxter ASI bialgebra in the semi-direct product Rota-Baxter algebra.

In general, \( \beta \) is not related to \( \alpha \). When \( \beta \) does depend on \( \alpha \) in certain way: \( \beta = \Pi(\alpha) \), say for a Laurent series \( \Pi \in K[x, x^{-1}] \), then it is natural to expect that the double dual of a representation is the representation itself: \( \Pi(\beta) = \Pi^2(\alpha) = \alpha \). This happens when \( \Pi(x) \) is either \( \pm x \), or \( -x + \theta \), or \( \theta x^{-1} \) when \( x \) is invertible and \( 0 \neq \theta \in K \), that is, when \( \beta = \pm \alpha \) or \( -\alpha + \theta \text{id} \) or \( \beta = \theta \alpha^{-1} \).

We will investigate these instances more carefully since they provide interesting examples and applications of our general construction of Rota-Baxter ASI bialgebras.

To emphasize, for all \( \Pi \) in the set

\[
    \{\pm x\} \cup (-x + K^x) \cup K^x x^{-1}, \quad K^x := K \setminus \{0\},
\]

we have \( \Pi^2(\alpha) = \alpha \) and \( \Pi(\alpha^*) = \Pi(\alpha)^* \). Moreover, for any linear map \( T : V \to A \), it is obvious that \( T\Pi(\alpha) = \Pi(P)T \) when \( T\alpha = PT \).

Applying Theorem 4.20, we conclude

**Proposition 4.22.** Let \((A, P)\) be a Rota-Baxter algebra of weight \( \lambda \). Let \((V, \ell, r)\) be a representation of the algebra \( A \) and \( \alpha : V \to V \) be a linear map. For \( \Pi \in \{\pm x\} \cup (-x + K^x) \cup K^x x^{-1} \), there is a Rota-Baxter algebra \((A \bowtie_{\gamma, \cdot} V^*, P + \Pi(\alpha^*))\) that is \( \Pi(P + \Pi(\alpha^*)) \)-admissible (that is, \( (\Pi(P) + \alpha^*) \)-admissible) if and only if the \( \Pi \)-admissible equations (associated to the quadruple \((V, \ell, r, \alpha)\)) hold. Here
(a) when \( \Pi = \theta x \) with \( \theta = \pm 1 \), the \( \Pi \)-admissible equations are
\[
\begin{align*}
(\theta P + P + \lambda \text{id})(aP(b)) + \lambda P(ab) &= 0, \quad (53) \\
(\theta P + P + \lambda \text{id})(P(a)b) + \lambda P(ab) &= 0, \quad (54) \\
\ell(P(a))\alpha(v) &= \alpha(\ell(P(a))v) + \alpha(\ell(a)\alpha(v)) + \lambda \alpha(\ell(a)v), \quad (55) \\
\alpha(\nu r(P(a)) &= \alpha(\alpha(\nu r(P(a))) + \lambda \alpha(\nu r(a)), \quad (56) \\
(\theta \alpha + \alpha + \lambda \text{id})(\ell(a)\alpha(v)) + \lambda \alpha(\ell(a)v) &= 0, \quad (57) \\
(\theta \alpha + \alpha + \lambda \text{id})(\alpha(\nu r(a)) + \lambda \alpha(\nu r(a)) &= 0, \quad (58) \\
(\theta \alpha + \alpha + \lambda \text{id})(\ell(P(a))\nu) + \lambda \alpha(\ell(a)v) &= 0, \quad (59) \\
(\theta \alpha + \alpha + \lambda \text{id})(\nu r(P(a))) + \lambda \alpha(\nu r(a)) &= 0, \quad \forall a, b \in A, \nu \in V; \quad (60)
\end{align*}
\]

(b) when \( \Pi = -x + \theta \) with \( \theta \neq 0 \), the \( \Pi \)-admissible equations are Eqs. (55)-(57) and
\[
\begin{align*}
(\lambda + \theta)(P(ab) + aP(b) - \theta ab) &= 0, \quad (61) \\
(\lambda + \theta)(P(ab) + P(a)b - \theta ab) &= 0, \quad (62) \\
(\lambda + \theta)(l(a)\alpha(v) + \alpha(l(a)v) - \theta l(a)v) &= 0, \quad (63) \\
(\lambda + \theta)(\alpha(\nu r(a)) + \lambda \alpha(\nu r(a)) - \theta \nu r(a)) &= 0, \quad (64) \\
(\lambda + \theta)(l(P(a))\nu + \alpha(l(a)v) - \theta l(a)v) &= 0, \quad (65) \\
(\lambda + \theta)(\nu r(P(a)) + \alpha(\nu r(a)) - \theta \nu r(a)) &= 0, \quad \forall a, b \in A, \nu \in V; \quad (66)
\end{align*}
\]

(c) when \( \Pi = \theta x^{-1}, \theta \neq 0 \) (in which case assume that \( P \) and \( \alpha \) are invertible), the \( \Pi \)-admissible equations are
\[
\begin{align*}
P(aP(b)) &= P(P(a)b) = \theta ab, \quad (67) \\
\alpha(\ell(a)\alpha(v)) &= \alpha(\ell(P(a)v)) = \theta \ell(a)v, \quad (68) \\
\ell(P(a))\alpha(v) &= (\lambda \alpha + 2\theta \text{id})(\ell(a)v), \quad \alpha(\nu r(P(a))) = (\lambda \alpha + 2\theta \text{id})(\nu r(a)), \quad \forall a, b \in A, \nu \in V. \quad (69)
\end{align*}
\]

Proof. The statement is a consequence of Theorem 4.20 in the case that \( \beta = \Pi(\alpha) \) and \( Q = \Pi(P) \). We provide some details for Case (i). Cases (i) and (ii) are similarly verified.

(i) Let \( \beta = \theta \alpha \) and \( Q = \theta P \). Then by Theorem 4.21, \( (A \kappa_{r, \ell}^\alpha, V^*, P + \theta \alpha^*) \) is a Rota-Baxter algebra that is \( (\theta P + \alpha^*) \)-admissible if and only if the following conditions hold.

\[
\begin{align*}
&\text{(i) } (V, \ell, r, \theta \alpha) \text{ is an admissible quadruple of } (A, P). \text{ By Lemma 2.11, } (V^*, r^*, \ell^*, \theta \alpha^*) \text{ is a representation of } (A, P) \text{ if and only if} \\
&\quad \alpha(\nu r(P(a))) = (\alpha(\nu r(P(a))) - \theta \alpha(\nu r(a)) = 0, \quad (70) \\
&\quad \alpha(\ell(P(a))) = (\ell(P(a)))\alpha(v) - \theta \alpha(\ell(a)v) = 0, \forall \nu \in V, a \in A. \quad (71)
\end{align*}
\]

(ii) \( \theta P \) is admissible to \( (A, P) \). Applying Item (i) to the adjoint representation \( (V, L, R, P) \), this holds if and only if Eqs. (53) and (54) hold.

(iii) \( (V, \ell, r, \alpha) \) is a representation of \( (A, P) \). It holds if and only if Eqs. (55) and (56) hold.

(iv) Eqs. (59) and (60) hold, where \( \beta = \theta \alpha \) and \( Q = \theta P \).

A straightforward computation shows that Eqs. (70), (71), (55), (56), (59) with \( \beta = \theta \alpha \) and \( Q = \theta P \) and Eq. (54) with \( \beta = \theta \alpha \) and \( Q = \theta P \) hold if and only if Eqs. (55)-(60) hold.

Taking \( \lambda = 0 \) and \( \Pi = x \) in Proposition 4.22 (i), and \( \theta = -\lambda \) with \( \lambda \neq 0 \) and hence \( \Pi = x - \lambda \) in Proposition 4.22 (ii), we obtain an important case where the admissibility does not impose any restraints.
Corollary 4.23. Let \((A, P)\) be a Rota-Baxter algebra of weight \(\lambda\) and \((V, \ell, r, \alpha)\) be a representation of \((A, P)\). Then \((V^+, r^+, \ell^+, -\alpha^* - \lambda\text{id}_{V^+})\) is a representation of \((A, P)\) and hence gives rise to a Rota-Baxter algebra \((A \rhd_{r^+, \ell^+} V^+, P - \alpha^* - \lambda\text{id}_{V^+})\). Moreover, the linear operator \(-P - \lambda\text{id}_A + \alpha^*\) on \(A \oplus V^+\) is admissible to \((A \rhd_{r^+, \ell^+} V^+, P - \alpha^* - \lambda\text{id}_{V^+})\).

For antisymmetric solutions of the admissible AYBE and the resulting Rota-Baxter ASI bialgebras, we have

Proposition 4.24. Let \((A, P)\) be a Rota-Baxter algebra of weight \(\lambda\). Let \((V, \ell, r)\) be a representation of the algebra \(A\). Let \(\alpha : V \to V\) and \(T : V \to A\) be linear maps. Let \(\Pi \in \{\pm x\} \cup (-x + K^x) \cup K^x x^{-1}\).

(a) Let \((V, \ell, r, \Pi(\alpha))\) be an admissible quadruple of \((A, P)\). Then \(r = T - \sigma(T)\) is an antisymmetric solution of the \((\Pi(P) + \alpha^*)\)-admissible AYBE in \((A \rhd_{r^+, \ell^+} V^+, P + \Pi(\alpha^*))\) if and only if \(T\) is a weak \(\circ\)-operator associated to \((V, \ell, r)\) and \(\alpha\).

(b) Assume the validity of the \(\Pi\)-admissible equations, given respectively by Eqs. (53)-(60) for \(\Pi = \pm x\), by Eqs. (53)-(54), (51)-(60) for \(\Pi \in -x + K^x\), and by Eqs. (51)-(55) for \(\Pi \in K^x x^{-1}\). Then \((V, \ell, r, \alpha)\) is a representation of \((A, P)\) and there is a \((\Pi(P) + \alpha^*)\)-admissible Rota-Baxter algebra \((A \rhd_{r^+, \ell^+} V^+, P + \Pi(\alpha^*))\). If \(T\) is an \(\circ\)-operator associated to \((V, \ell, r, \alpha)\), then \(r = T - \sigma(T)\) is an antisymmetric solution of the \((\Pi(P) + \alpha^*)\)-admissible AYBE in the Rota-Baxter algebra \((A \rhd_{r^+, \ell^+} V^+, P + \Pi(\alpha^*))\). Further there is a Rota-Baxter ASI bialgebra \((A \rhd_{r^+, \ell^+} V^+, P + \Pi(\alpha^*))\), \(\Delta, \Pi(P) + \alpha^*\), where the linear map \(\Delta = \Delta_r\) is defined by Eq. (75) with \(r = T - \sigma(T)\).

Proof. (\(\Box\)) follows from Theorem 4.21 (a). (\(\square\)) follows from Proposition 4.22 and Theorem 4.21 (b).

We next focus on the case when \(Q = -P - \lambda\text{id}\) for a Rota-Baxter algebra \((A, P)\) of weight \(\lambda\). In this case, by Corollary 4.23, the \(\Pi\)-admissible equations associated to \((V, \ell, r, \alpha)\) hold automatically. Hence Proposition 4.24 (b) gives

Corollary 4.25. Let \((A, P)\) be a Rota-Baxter algebra of weight \(\lambda\). Let \(T : V \to A\) be an \(\circ\)-operator associated to a representation \((V, \ell, r, \alpha)\) of \((A, P)\). Then \(r = T - \sigma(T)\) is an antisymmetric solution of the \((-P - \lambda\text{id}_A + \alpha^*)\)-admissible AYBE in the Rota-Baxter algebra \((A \rhd_{r^+, \ell^+} V^+, P - \alpha^* - \lambda\text{id}_{V^+})\). Further there is a Rota-Baxter ASI bialgebra \((A \rhd_{r^+, \ell^+} V^+, P - \alpha^* - \lambda\text{id}_{V^+})\), \(\Delta, -P - \lambda\text{id}_A + \alpha^*\), where the linear map \(\Delta = \Delta_r\) is defined by Eq. (76) with \(r = T - \sigma(T)\).

We then display explicitly solutions of the admissible AYBE that give Rota-Baxter ASI bialgebras obtained from the \(\circ\)-operators in Example 4.18.

Corollary 4.26. Let \((A, P)\) be a Rota-Baxter algebra of weight \(\lambda\).

(a) Denote \(r_1 := \sum_{i=1}^n (e_i \otimes e^i - e^i \otimes e_i)\), where \(\{e_1, \cdots, e_n\}\) is a basis of \(A\) and \(\{e^1, \cdots, e^n\}\) is its dual basis. Then \(r_1\) is a solution of the \((-P - \lambda\text{id}_A + P^*)\)-admissible AYBE in the Rota-Baxter algebras \((A \rhd_{r^+, \ell^+} A^*, P - P^* - \lambda\text{id}_{A^*})\) and \((A \rhd_{r_{0, \lambda}^+, \ell^+} A^*, P - P^* - \lambda\text{id}_{A^*})\). Moreover, the two Rota-Baxter algebras are \((-P - \lambda\text{id}_A + P^*)\)-admissible and hence there are Rota-Baxter ASI bialgebras \((A \rhd_{r^+, \ell^+} A^*, P - P^* - \lambda\text{id}_{A^*})\), \(\Delta, -P - \lambda\text{id}_A + P^*\) and \((A \rhd_{r_{0, \lambda}^+, \ell^+} A^*, P - P^* - \lambda\text{id}_{A^*})\), \(\Delta, -P - \lambda\text{id}_A + P^*\), where the linear map \(\Delta = \Delta_{r_1}\) is defined by Eq. (77) with the above \(r_1\).

(b) Suppose that \(\lambda = 0\). The element \(r_2 := P - \sigma(P)\) is a solution of the \((-P + P^*)\)-admissible AYBE in the Rota-Baxter algebra \((A \rhd_{r^+, \ell^+} A^*, P - P^*)\). Moreover, there is a Rota-Baxter ASI bialgebra \((A \rhd_{r^+, \ell^+} A^*, P - P^*)\), \(\Delta, -P + P^*\), where the linear map \(\Delta = \Delta_{r_2}\) is defined by Eq. (77) with the above \(r_2\).
5. Dendriform algebras and Rota-Baxter dendriform algebras

We first recall the well-known fact that a Rota-Baxter algebra induces a dendriform algebra. Then we show that a Rota-Baxter ASI bialgebra gives a quadri-bialgebra introduced in [32]. We also prove that a Rota-Baxter dendriform algebra gives an $\mathcal{O}$-operator on the associated Rota-Baxter algebra and hence induces a Rota-Baxter ASI bialgebra.

5.1. Dendriform algebras and quadri-bialgebras.

Definition 5.1. [34] Let $A$ be a vector space with multiplications $<$ and $>$. Then $(A, <, >)$ is called a dendriform algebra if for all $a, b, c \in A$,

$$(a < b) < z = a < (b < c + b > c), (a > b) < c = a > (b < c), (a < b + a > b) > c = a > (b > c).$$

Let $(A, <, >)$ be a dendriform algebra. For $a \in A$, let $L_<(a), R_<(a)$ and $L_>(a), R_>(a)$ denote the left and right multiplication operators on $(A, <)$ and $(A, >)$, respectively. Furthermore, define linear maps

$$R_<, L_>: A \rightarrow \text{End}_K(A), \quad a \mapsto R_<(a), \quad a \mapsto L_>(a), \quad \forall a \in A.$$

As is well known, for a dendriform algebra $(A, <, >)$, the multiplication

$$a \ast b = a < b + a > b, \quad \forall a, b \in A$$

(72)
defines an algebra $(A, \ast)$, called the associated algebra of the dendriform algebra. Moreover, $(A, L_>, R_<)$ is a representation of the algebra $(A, \ast)$ [3, 34]. Further, a Rota-Baxter operator $P$ of weight zero on an algebra $(A, \ast)$ induces a dendriform algebra [3]:

$$a > b = P(a) \cdot b, \quad a < b = a \cdot P(b), \quad \forall a, b \in A.$$  

(73)

It is natural to ask what algebraic structures can be induced from the pair of Rota-Baxter operators on the algebra and the coalgebra in a Rota-Baxter ASI bialgebra of weight zero. To address this question, we first recall the following notions.

Definition 5.2. ([3, 38]) Let $(A, <, >)$ be a dendriform algebra and let $(A, \ast)$ be the associated algebra. A symmetric bilinear form $\mathcal{B} : A \otimes A \rightarrow K$ is called a 2-cocycle of $(A, <, >)$ if $\mathcal{B}$ satisfies

$$\mathcal{B}(a \ast b, c) = \mathcal{B}(b, c < a) + \mathcal{B}(a, b > c), \quad \forall a, b, c \in A.$$  

(74)

A Manin triple of dendriform algebras with respect to a nondegenerate 2-cocycle is a triple of dendriform algebras $(A, A^+, A^-)$ together with a nondegenerate 2-cocycle $\mathcal{B}$ on $A$, such that

(a) $A^+$ and $A^-$ are dendriform subalgebras of $A$;
(b) $A = A^+ \oplus A^-$ as vector spaces;
(c) $A^+$ and $A^-$ are isotropic with respect to $\mathcal{B}$.

Lemma 5.3. Let $(A, P, \mathcal{B})$ be a Rota-Baxter symmetric Frobenius algebra of weight zero. Then $\mathcal{B}$ satisfies

$$\mathcal{B}(a \ast b, c) = \mathcal{B}(a, b > c), \quad \forall a, b, c \in A,$$

(75)

where $(A, <, >)$ is the dendriform algebra defined in Eq. (73). In particular, $\mathcal{B}$ is a 2-cocycle of the dendriform algebra $(A, <, >)$.
**Proof.** Let \( \cdot \) denote the associative product on \((A, P)\). Let \( a, b, c \in A \). Then we have
\[
\mathcal{B}(a < b, c) = \mathcal{B}(a \cdot P(b), c) = \mathcal{B}(a, P(b) \cdot c) = \mathcal{B}(a, b > c).
\]
Furthermore, we have
\[
\mathcal{B}(a \ast b, c) = \mathcal{B}(a < b + a > b, c) = \mathcal{B}(a, b > c) + \mathcal{B}(c, a > b) = \mathcal{B}(a, b > c) + \mathcal{B}(b, c < a).
\]
Hence \( \mathcal{B} \) is a 2-cocycle of the dendriform algebra \((A, <, \ast)\). \( \square \)

**Corollary 5.4.** Let \(((A, P), \Delta, Q)\) be a Rota-Baxter ASI bialgebra, where \((A, \cdot, P)\) is a \(Q\)-admissible Rota-Baxter algebra of weight zero and \((A^\ast, \circ, Q^\ast)\) is a \(P\)-admissible Rota-Baxter algebra of weight zero for which the product \(\circ\) is given by \(\Delta^\ast : A^\ast \otimes A^\ast \to A^\ast\). With the product \(\star\) on \(A \bowtie A^\ast\) in Eq. (25), define
\[
x > y := (P + Q^\ast) (x \star y), \quad x < y := x \cdot (P + Q^\ast) (y), \quad \forall x, y \in A \bowtie A^\ast.
\]
Then \((A \bowtie A^\ast, <, \ast)\) is a dendriform algebra and \((A, \vartriangleleft_{\lambda}, \triangleright_{\lambda})\), \((A^\ast, \vartriangleleft_{\lambda}, \triangleright_{\lambda})\) are dendriform subalgebras. Moreover, with these dendriform algebra structures, \((A \bowtie A^\ast, A, A^\ast)\) is a Manin triple of dendriform algebras with respect to the nondegenerate 2-cocycle \(\mathcal{B}_d\) defined by Eq. (28).

**Proof.** By Theorem 3.13, \((A \bowtie A^\ast, \star, P + Q^\ast)\) is a Rota-Baxter algebra of weight zero. Then by Eq. (73), \((A \bowtie A^\ast, <, \ast)\) is a dendriform algebra and \((A, \vartriangleleft_{\lambda}, \triangleright_{\lambda})\), \((A^\ast, \vartriangleleft_{\lambda}, \triangleright_{\lambda})\) are dendriform subalgebras. Note that in fact \((A, \vartriangleleft_{\lambda}, \triangleright_{\lambda})\), \((A^\ast, \vartriangleleft_{\lambda}, \triangleright_{\lambda})\) are the dendriform algebras induced by the Rota-Baxter operators \(P\) and \(Q^\ast\) respectively by Eq. (73). Moreover, by Lemma 5.3, \(\mathcal{B}_d\) is a symmetric nondegenerate 2-cocycle of \((A \bowtie A^\ast, <, \ast)\). Hence the conclusion holds. \( \square \)

**Remark 5.5.** There are several bialgebra theories for dendriform algebras, such as dendriform bialgebra \([4, 15, 26, 42]\), bidendriform bialgebra \([20]\) and dendriform \(D\)-bialgebra \([5]\). It came as a surprise that, by \([38, \text{Theorem 5.3}]\), a Manin triple of dendriform algebras with respect to a nondegenerate 2-cocycle is none of them, but a **quadi-bialgebra** consisting of a quadri-algebra \([5]\) and a quadri-coalgebra satisfying certain compatibility conditions. See \([38, \text{Definition 5.2}]\).

Then as a direct consequence of Corollary 5.4, we obtain

**Corollary 5.6.** For a Rota-Baxter ASI bialgebra \(((A, P), \Delta, Q)\) of weight zero, the Rota-Baxter operators \(P\) and \(Q^\ast\) induce a quadi-bialgebra.

### 5.2. Rota-Baxter Dendriform Algebras

We introduce the notion of a Rota-Baxter dendriform algebra and study its relationships with \(\emptyset\)-operators on Rota-Baxter algebras.

**Definition 5.7.** Let \((A, <, \ast)\) be a dendriform algebra. A linear operator \(P\) on \(A\) is called a **Rota-Baxter operator of weight** \(\lambda\) if \(P\) satisfies
\[
P(a \circ b) = P(P(a \circ b) + P(a \circ b)) + \lambda P(a \circ b), \quad \forall a, b \in A, \circ \in \{<, \ast\}.
\]
Then \((A, <, \ast, P)\) is called a **Rota-Baxter dendriform algebra of weight** \(\lambda\).

By a simple verification, we obtain

**Proposition 5.8.** Let \((A, <, \ast, P)\) be a Rota-Baxter dendriform algebra of weight \(\lambda\). Then with the product \(\ast\) given by Eq. (72), \((A, \ast, P)\) is a Rota-Baxter algebra of weight \(\lambda\), called the **associated Rota-Baxter algebra** of \((A, <, \ast, P)\). On the other hand, let \((A, \cdot, P)\) be a Rota-Baxter algebra of weight zero, then \((A, <, \ast, P)\) is a Rota-Baxter dendriform algebra of weight zero, where \(\triangleright_{\ast}, \vartriangleleft_{\ast}\) is given by Eq. (73).
From Eqs. (10), (11) and Definition 5.7, we directly have

**Proposition 5.9.** Let \((A, <, >, P)\) be a Rota-Baxter dendriform algebra of weight \(\lambda\). Then \((A, L_>, R_<, P)\) is a representation of the Rota-Baxter algebra \((A, *, P)\). Furthermore, the identity map \(\text{id}\) on \(A\) is an \(\emptyset\)-operator on the associated Rota-Baxter algebra \((A, *, P)\) associated to the representation \((A, L_>, R_<, P)\).

Therefore, by Proposition 4.24, any Rota-Baxter dendriform algebra \((A, <, >, P)\) of weight \(\lambda\) that satisfies the corresponding II-admissible equations associated to \((A, L_>, R_<, P)\) for \(\Pi \in \{\pm x\} \cup (-x + K^\times) \cup K^x x^{-1}\) can give a solution of the admissible AYBE induced from the identity map \(\text{id}\) and hence gives a Rota-Baxter ASI bialgebra. We illustrate the construction explicitly by considering the case when \(Q = -P - \lambda \text{id}\), that is, the construction of Rota-Baxter ASI bialgebras by Rota-Baxter dendriform algebras of weight \(\lambda\).

**Proposition 5.10.** Let \((A, <, >, P)\) be a Rota-Baxter dendriform algebra of weight \(\lambda\). Let \((A, *, P)\) be the associated Rota-Baxter algebra of weight \(\lambda\) where \(*\) is given by Eq. (72). Let \(\{e_1, \cdots, e_n\}\) be a basis of \(A\) and \(\{e^1, \cdots, e^n\}\) be its dual basis. Denote

\[ r := \sum_{i=1}^{n} (e_i \otimes e^i - e^i \otimes e_i) \]

and define the linear map \(\Delta = \Delta_r\) by Eq. (30). Then \(r\) is a solution of the \((-P - \lambda \text{id}_A + P^*)\)-admissible AYBE in the Rota-Baxter algebra \((A \ltimes_{L^\times, R^\times} A^*, P - P^* - \lambda \text{id}_A)\). Further the Rota-Baxter algebra \((A \ltimes_{L^\times, R^\times} A^*, P - P^* - \lambda \text{id}_A^*)\) is \((-P - \lambda \text{id}_A + P^*)\)-admissible and hence there is a Rota-Baxter ASI bialgebra \(((A \ltimes_{L^\times, R^\times} A^*, P - P^* - \lambda \text{id}_A), \Delta, -P - \lambda \text{id}_A + P^*)\).

**Proof.** By Proposition 5.9, \(\text{id}\) is an \(\emptyset\)-operator on the associated Rota-Baxter algebra \((A, *, P)\) associated to the representation \((A, L_>, R_<, P)\). Note that \(\text{id} = \sum_{i=1}^{n} (e_i \otimes e^i)\). Then the conclusion follows from Corollary 4.25. \(\square\)

**Remark 5.11.** Corollary 4.26 is a special case of Proposition 5.10, since the former corresponds to the trivial Rota-Baxter dendriform algebra structure on a Rota-Baxter algebra \((A, \cdot, P)\) given by \(> = \cdot, <= 0\) or \(>= 0, <= \cdot\).

By Proposition 5.11 we have

**Corollary 5.12.** Let \((A, \cdot, P)\) be a Rota-Baxter algebra of weight zero. Let \((A, <, >, P)\) be the Rota-Baxter dendriform algebra of weight zero given in Proposition 5.8. Then we have

\[ L_> = L P, \quad R_< = R P. \]

Let \((A, *, P)\) be the associated Rota-Baxter algebra of \((A, <, >, P)\). Then \((A, L P, R P, P)\) is a representation of the Rota-Baxter algebra \((A, *, P)\). Let \(r \text{ and } \Delta = \Delta_r\) be as defined in Proposition 5.10 and Eq. (30) respectively. Then \(r\) is a solution of the \((-P + P^*)\)-admissible AYBE in the Rota-Baxter algebra \(((A \ltimes_{(R, P), (L, P)} A^*, P - P^*), \Delta, -P + P^*)\). Moreover, this Rota-Baxter algebra is \((-P + P^*)\)-admissible and hence there is a Rota-Baxter ASI bialgebra \(((A \ltimes_{(R, P), (L, P)} A^*, P - P^*), \Delta, -P + P^*)\). \(\square\)

**Remark 5.13.** Combining Corollary 4.26 and Corollary 5.12, from a Rota-Baxter algebra \((A, \cdot, P)\) of weight zero, there are four Rota-Baxter ASI bialgebras on the direct sum \(A \oplus A^\prime\) of the underlying vector spaces of \(A\) and \(A^\prime\) \((A \ltimes_{L^\times, R^\times} A^*, P - P^*), \Delta, -P + P^*)\), \((A \ltimes_{L, R} A^*, P - P^*), \Delta, -P + P^*)\), \((A \ltimes_{L, R} A^*, P - P^*), \Delta, -P + P^*)\), and \((A \ltimes_{(R, P), (L, P)} A^*, P - P^*), \Delta, -P + P^*)\), where the linear map \(\Delta\) is defined by Eq. (30) with \(r = P - \sigma(P)\) for the first case and with \(r = \sum_{i=1}^{n} (e_i \otimes e^i - e^i \otimes e_i)\) for the other three cases. Note that the Rota-Baxter algebra structure on \(A\) is \((A, \cdot, P)\) itself for the first three cases and is \((A, *, P)\) for the fourth case.
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References

[1] E. Abe, Hopf Algebras, Cambridge University Press, 1980.
[2] M. Aguiar, Pre-Poisson algebras, *Lett. Math. Phys.* 54 (2000), 263-277.
[3] M. Aguiar, On the associative analog of Lie bialgebras, *J. Algebra* 244 (2001), 492-532.
[4] M. Aguiar, Infinitesimal bialgebras, pre-Lie algebras and dendriform algebras, in “Hopf Algebras”, *Lect. Notes in Pure and Appl. Math.* 237 (2004), 1-33.
[5] M. Aguiar, J.-L. Loday, Quadri-algebras, *J. Pure Appl. Algebra* 191 (2004), 205-221.
[6] G. E. Andrews, L. Guo, W. Keigher and K. Ono, Baxter algebras and Hopf algebras, *Trans. Amer. Math. Soc.* 355 (2003), 4639-4656.
[7] C. Bai, A unified algebraic approach to the classical Yang-Baxter equation, *J. Phys. A: Math. Gen.* 40 (2007), 11073-11082.
[8] C. Bai, Left-symmetric bialgebras and an analogue of the classical Yang-Baxter equation, *Comm. Contemp. Math.* 10 (2008), 221-260.
[9] C. Bai, Double constructions of Frobenius algebras, Connes cocycles and their duality, *J. Noncommut. Geom.* 4 (2010), 475-530.
[10] C. Bai, L. Guo and X. Ni, Ω-operators on associative algebras and associative Yang-Baxter equations, *Pacific J. Math.* 256 (2012), 257-289.
[11] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* 10 (1960), 731-742.
[12] T. Brzeziński, Rota-Baxter systems, dendriform algebras and covariant bialgebras, *J. Algebra* 460 (2016), 1-25.
[13] V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.
[14] A. Connes, Non-commutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* 62 (1985), 41-144.
[15] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.* 210 (2000), 249-273.
[16] V. Drinfeld, Hamiltonian structure on the Lie groups, Lie bialgebras and the geometric sense of the classical Yang-Baxter equations, *Soviet Math. Dokl.* 27 (1983), 68-71.
[17] K. Ebrahimi-Fard, Rota-Baxter algebras and the Hopf algebra of renormalization, Ph.D. Dissertation, University of Bonn, 2006.
[18] K. Ebrahimi-Fard and L. Guo, Mixable shuffles, quasi-shuffles and Hopf algebras, *J. Alg. Comb.* 24 (2006), 83-101.
[19] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras I, *Selecta Math.* 2 (1996), 1-41.
[20] L. Foissy, Bidendriform bialgebras, trees, and free quasi-symmetric functions, *J. Pure Appl. Algebra* 209 (2007), 439-459.
[21] J. Fuchs, Affine Lie Algebras and Quantum Groups, Cambridge University Press, Cambridge, 1995.
[22] X. Gao, L. Guo and T. Zhang, Hopf algebras of rooted forests, cocycles and free Rota-Baxter algebras, *J. Math. Phys.* 57 (2016), 101701, 16 pp.
[23] L. Guo, An Introduction to Rota-Baxter Algebra, *Surveys of Modern Mathematics* 4, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.
[24] L. Guo, S. Pycha and B. Zhang, Renormalization of conical zeta values and the Euler-Maclaurin formula, *Duke Math J.* 166 (2017) 537 - 571.
[25] J. Hartwig, D. Larsson and S. Silvestrov, Deformations of Lie algebras using σ-derivations, *J. Algebra* 295 (2006), 314-361.
[26] M. Hoffman, Quasi-shuffle products, *J. Alg. Comb.* 11 (2000), 49-68.
[27] R.-Q. Jian and J. Zhang, Rota-Baxter coalgebras, arXiv:1409.3052.
[28] S. A. Joni and G.-C. Rota, Coalgebras and bialgebras in combinatorics, *Studies in Appl. Math.* 61 (1979), 93-139.
[29] S. Kaneyuki, Homogeneous symplectic manifolds and dipolarizations in Lie algebras, *Tokyo J. Math.* 15 (1992), 313-325.
[30] J. Kock, Frobenius Algebras and 2d Topological Quantum Field Theories, Cambridge University Press, Cambridge, 2004.

[31] B. A. Kupershmidt, What a classical r-matrix really is, J. Nonlinear Math. Phys. 6 (1999), 448-488.

[32] B.A. Kupershmidt, Non-abelian phase spaces, J. Phys. A: Math. Gen. 27 (1994), 2801-2810.

[33] A. Lauda and H. Pfeiffer, Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras, Topology Appl. 155 (2005), 623-666.

[34] J.-L. Loday, Dialgebras. In “Dialgebras and Related Operads”, Lecture Notes in Math. 1763, Springer, Berlin 2001, 7-66.

[35] J.-L. Loday and M. Ronco, Hopf algebra of the planar binary trees, Adv. Math. 139 (1998) 293-309.

[36] J.-L. Loday and M. Ronco, Order structure on the algebra of permutations and of planar binary trees, J. Alg. Comb. 15 (2002) 253-270.

[37] T. Ma and L. L. Liu, Rota-Baxter coalgebras and Rota-Baxter bialgebras, Linear Multilinear Algebra 64 (2016), 968-979.

[38] X. Ni and C. Bai, On quadri-bialgebras, arXiv:1704.04781v1.

[39] O. Pelc, A new family of solvable self-dual Lie algebras, J. Math. Phys. 38 (1997), 3832-3840.

[40] L. Qiao, X. Gao and L. Guo, Rota-Baxter modules toward derived functors, Algebra Represent. Theory 22 (2019), 321-343.

[41] L. Qiao and J. Pei, Representations of polynomial Rota-Baxter algebras, J. Pure Appl. Algebra 222 (2018), 1738-1757.

[42] M. Ronco, Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras, J. Algebra 254 (2002), 151-172.

[43] G. C. Rota, Baxter algebras and combinatorial identities. I, II, Bull. Amer. Math. Soc. 75 (1969), 325-329; 330-334.

[44] H. Yu, L. Guo and J.-Y. Thibon, Weak composition quasi-symmetric functions, Rota-Baxter algebras and Hopf algebras, Adv. Math. 344 (2019), 1-34.

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