On the initial value problem for the wave equation in Friedmann–Robertson–Walker space–times

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The propagator $W(t_0, t_1)(g, h)$ for the wave equation in a given space–time takes initial data $(g(x), h(x))$ on a Cauchy surface $\{(t, x) : t = t_0\}$ and evaluates the solution $(u(t_1, x), \partial_t u(t_1, x))$ at other times $t_1$. The Friedmann–Robertson–Walker space–times are defined for $t_0, t_1 > 0$, whereas for $t_0 \to 0$, there is a metric singularity. There is a spherical means representation for the general solution of the wave equation with the Friedmann–Robertson–Walker background metric in the three spatial dimensional cases of curvature $K = 0$ and $K = -1$ given by S. Klainerman and P. Sarnak. We derive from the expression of their representation three results about the wave propagator for the Cauchy problem in these space–times. First, we give an elementary proof of the sharp rate of time decay of solutions with compactly supported data. Second, we observe that the sharp Huygens principle is not satisfied by solutions, unlike in the case of three-dimensional Minkowski space–time (the usual Huygens principle of finite propagation speed is satisfied, of course). Third, we show that for $0 < t_0 < t$ the limit,

$$\lim_{t_0 \to 0^+} W(t_0, t)(g, h) = W(0, t)(g)$$

exists, it is independent of $h(x)$, and for all reasonable initial data $g(x)$, it gives rise to a well-defined solution for all $t > 0$ emanating from the space–time singularity at $t = 0$. Under reflection $t \to -t$, the Friedmann–Robertson–Walker metric gives a space–time metric for $t < 0$ with a singular future at $t = 0$, and the same solution formulae hold. We thus
have constructed solutions \( u(t, x) \) of the wave equation in Friedmann–Robertson–Walker space–times which exist for all \(-\infty < t < 0\) and \(0 < t < +\infty\), where in conformally regularized coordinates, these solutions are continuous through the singularity \( t = 0 \) of space–time, taking on specified data \( u(0, \cdot) = g(\cdot) \) at the singular time.

1. Introduction

The family of Friedmann–Robertson–Walker metrics, Lorentzian metrics for space–time which are spatially homogeneous, and which have a singularity at \( t = 0 \), have played a central role in general relativity and cosmology [1,2]. In particular, they provide the simplest case of space–times with a ‘big bang’ singularity, and thus are important in the current interpretation of our universe. Lorentzian metrics with a singularity (without loss of generality at \( t = 0 \)) have played a particular role in discussions of the origins of space–time, having the striking feature of exhibiting divergence of the curvature and the energy–momentum tensor, whether in the past (‘big bang’) or at a future collapse of (a region of) space–time. The linear wave equation in a background space–time metric describes the propagation of energy and matter fields in the linearized regime, such as for example electromagnetic fields [3]. It is also the first model to consider when investigating the propagation of linear fluctuations of the Einstein metric tensor around this background. This paper discusses a representation of the wave propagator, and several of its consequences, for the initial value problem for this wave equation, both for non-zero \( t \) and for the singular time \( t = 0 \).

In a short paper, Klainerman & Sarnak [4] derived an explicit spherical means representation of the propagator for the scalar wave equation in a Friedmann–Robertson–Walker space–time in the case that the pressure and cosmological constant vanish (‘dust’ models), following ideas in John [5,6]. The Lorentzian line element corresponding to the classical Friedmann–Robertson–Walker space–time metrics takes the form

\[
ds^2 = -dt^2 + S(t)^2 d\sigma^2, \tag{1.1}
\]

where \( d\sigma^2 \) is the line element for each underlying spatially homogeneous time slice \( \{(t, x) : t = t_0\} \). In the cases we consider these are space–like hypersurfaces corresponding to Euclidian space \( \mathbb{R}^3 \) in the case of curvature \( K = 0 \), and to hyperbolic space \( \mathbb{H}^3 \) in the case of curvature \( K = -1 \). Under the time change

\[
\frac{dt}{d\tau} = S(t), \quad t(0) = 0,
\]

the line element (1.1) is transformed to the form

\[
ds^2 = S(\tau)^2 (-d\tau^2 + d\sigma^2), \tag{1.2}
\]

which is conformal to the half space \( \mathbb{R}_+ \times \mathbb{R}^3 = \{(t, x) : \tau > 0\} \) endowed with the Minkowski metric. This conformal correspondence is of course singular at \( \tau = 0 \) (which is the image of \( t = 0 \)), at which time the metric interpretation is that all of space is contracted to a point. Expressed in the transformed time variables \( \tau \), the scale factor \( S(\tau) \) is given explicitly in [1]

\[
\begin{align*}
S(\tau) &= \tau^2, \quad \text{when } K = 0 \\
S(\tau) &= \cosh(\tau) - 1, \quad \text{when } K = -1.
\end{align*}
\tag{1.3}
\]

The wave propagator \( W(\tau_0, \tau)(g, h) \) is the solution operator for the wave equation

\[
\Box u = 0, \quad u(\tau_0, x) = g(x), \quad \partial_\tau u(\tau_0, x) = h(x), \tag{1.4}
\]

where the D’Alembertian operator in Friedmann–Robertson–Walker space–times is given by

\[
\Box u = -\frac{1}{S^2} \partial^2_\tau u - \frac{2S}{S^3} \partial_\tau u + \frac{1}{S^2} \Delta_\sigma u. \tag{1.5}
\]
Here, $\Delta_\sigma$ is the Laplace–Beltrami operator corresponding to the Riemannian metric on the time slices $\{t = \tau_0 > 0\}$, in particular for the metrics in the two cases $K = 0$ and $K = -1$ [7]. Namely given data $(g(x), h(x))$ in some appropriate class of functions or distributions, the wave propagator $W(\tau_0, \tau_1)(g, h)$ is defined to be the solution operator

$$W(\tau_0, \tau_1)(g, h) := (u(\tau_1, x), \partial_\tau u(\tau_1, x)), \quad \tau_0 > 0, \; \tau_1 > 0,$$

where of course $u(\tau, x)$ is the solution to (1.4). We do not take up the case of positive curvature $K = +1$ as this space–time has a recurrent singularity, and the solution formulae for the wave propagator are not as simply presented as in the above two cases.

The object of this paper was to draw several conclusions from the spherical means expression for the solution operator for (1.4) given in reference [4]. Using an explicit form of the propagator, a modification of that given in [4], we make three observations about solutions to the wave equation in a Friedmann–Robertson–Walker background metrics. First, we derive decay estimates for solutions in transformed time $\tau \gg 0$ (and by consequence in physical time $t \gg 0$), given compactly supported initial data $(g(x), h(x))$ posed on the Cauchy surface $\{\tau = \tau_0 > 0\}$, using a remark of John [5]. In short, the result is the following: for the case $K = 0$, solutions of the wave equation satisfy the estimate

$$|(u(t, \cdot), \partial_\tau u(t, \cdot)|_{L^\infty(\mathbb{R}^3)} \leq O(t^{-1})$$

for large time $t$, a rate identical to that of four-dimensional Minkowski space–time. In the case of negative curvature $K = -1$, solutions obey a faster decay rate, namely

$$|(u(t, \cdot), \partial_\tau u(t, \cdot)|_{L^\infty(\mathbb{R}^3)} \leq O(t^{-2}).$$

Our second main result is that, whereas solutions satisfy the general Huygen’s principle of finite propagation speed, the sharp Huygen’s principle is not valid, either for cases $K = 0$ or $K = -1$. This is in contrast to the case of the Minkowski metric on $\mathbb{R}^{1+3}$. The consequence is that signals do not propagate sharply on the light cone, but instead, after the passage of the wavefront, they leave a fading residual within the interior of the light cone. More precisely, the support of the kernel of the wave propagator is the full interior of the light cone, with the light cone itself being its singular support. It is reminiscent of wave propagation phenomena in the Minkowski metric for even space dimensions. This phenomenon was recently observed in [8] for the Klein–Gordon wave propagator in the de Sitter space–time metrics.

Third, a principal feature of Friedmann–Robertson–Walker space–times is that they are singular at times $t = \tau = 0$, a fact that is central in our current picture of cosmology. However, it should not be assumed from this that the initial value problem for the wave equation is without meaning for $t_0 = 0$. Indeed, we show that for fixed $t_1 > 0$ the limit of the wave propagator

$$\lim_{t_0 \to 0^{t_0 > 0}} W(t_0, t_1)(g, h) := W(0, t_1)(g)$$

exists, and gives rise to well-defined solutions with admissible initial data being precisely half of the Hilbert space of standard Cauchy data for the wave equation. Under time reflection $t \mapsto -t$, the Friedmann–Robertson–Walker space–times are also solutions of Einstein’s equations with the same properties of spatial homogeneity, and the above initial value problem can just as well be run backwards in time from $t = 0$, thus providing a full class of solutions of the wave equation which are global in space–time, both for positive and negative time $t \in \mathbb{R}$, with specified initial data $u(0, x) = g(x)$ at $t = 0$. In conformally regularized coordinates about $t = 0$, these solution are continuous, indeed smooth, through the time slice $t = 0$, leading to the interpretation that they pass information continuously through the singularity in space–time from the past to the future.

This last phenomenon is analogous to that of the conformally invariant wave equation

$$\partial_\tau^2 u - \Delta u + \frac{1}{6} Ru = 0,$$  \hspace{1cm} (1.6)

where $R$ is the scalar curvature of the background metric. This is a more special case, as the equation is invariant under conformal transformations. Using an appropriate conformal
transformation that regularizes the singularity at \( t = 0 \), both initial data \((g(x), h(x))\) can be specified at the singular time \( t_0 = 0 \). In this case, the sharp Huygens principle holds as well both initial data \((g(x), h(x))\) can be specified at the singular time \( t_0 = 0 \). It is interesting to us that in the case of the usual wave equation (1.4) in a cosmologically relevant space–time such as with the Friedmann–Robertson–Walker metric, the initial value problem posed at a singular time has a solution for even one specified initial datum \( g(x) \).

2. Wave equation for \( K = 0 \)

(a) The solution operator in Minkowski space–time

For comparison purposes, we recall the spherical means representation of solutions \( u(t, x) \) of the initial value problem for the wave equation in Minkowski space–time \( \mathbb{R}^{1+3} \), namely

\[
\partial_t^2 u - \Delta u = 0, \quad u(t_0, x) = g(x), \quad \partial_t u(t_0, x) = h(x).
\]

Define the spherical means operator to be

\[
M_f(r, x) := \frac{1}{4\pi r^2} \int_{S_r(x)} f(y) \, dS_r(y).
\] (2.1)

Then, \( rM_u(t, \cdot)(r, \cdot) \) satisfies the wave equation in one space dimension. From this we deduce the spherical means representation of the solution (also known as Kirchhoff’s formula), given by

\[
u(t, x) = \partial_t ((t - t_0)\mathcal{M}_g(t - t_0, x)) + (t - t_0)\mathcal{M}_h(t - t_0, x)
= \frac{1}{4\pi(t - t_0)} \int_{S_{t-t_0}(x)} \nabla g(y) \cdot \frac{y - x}{|y - x|} \, dS_{t-t_0}(y) + \frac{1}{4\pi(t - t_0)^2} \int_{S_{t-t_0}(x)} g(y) \, dS_{t-t_0}(y)
+ \frac{1}{4\pi(t - t_0)} \int_{S_{t-t_0}(x)} h(y) \, dS_{t-t_0}(y).
\] (2.2)

For \( g \in C^1(\mathbb{R}^3) \) and \( h \in C^0(\mathbb{R}^3) \), this expression is a solution to the wave equation (1.4) in the sense of distributions. When \( g \) and \( h \) are more regular, the result is a classical solution to (1.4).

The finite propagation speed of solutions (Huygen’s principle), the sharp Huygens principle and the time decay of solutions with compactly supported initial data follow immediately from this expression. Indeed, the solution at point \((t, x)\) is influenced only by initial data \((g(x), h(x))\) at time \( t = t_0 \) at points on the sphere \( S_{t-t_0}(x) \), and in particular, the integral kernel of the solution operator at \((t, x)\) given by the spherical means formula (2.2) is zero off of the light cone \(|(t', x') : |t - t'| = |x - x'|| \). A derivation and discussion of this expression of the solution to the wave equation can be found in [5,6].

(b) The solution operator in Friedmann–Robertson–Walker space–time

For \( K = 0 \), the scale factor (1.3) takes the form \( S(\tau) = \tau^2 \), thus the wave equation (1.5) in the Friedmann–Robertson–Walker space–time takes the form

\[
\partial_{\tau}^2 u + \frac{4}{\tau} \partial_{\tau} u - \Delta u = 0 \quad 0 < \tau, \tau_0; \ x \in \mathbb{R}^3
\]

\[
\begin{aligned}
u(\tau_0, x) &= g(x) \\
\partial_{\tau} v(\tau_0, x) &= h(x)
\end{aligned}
\] (2.3)

Assume that \( g \in C^2(\mathbb{R}^3) \) and \( h \in C^1(\mathbb{R}^3) \), and impose the condition that \( \text{supp}(g, h) \subseteq B_R(0) \) for some \( R > 0 \), where \( B_R(0) = \{ x \in \mathbb{R}^3 : |x| \leq R \} \), \(|x|\) being the usual Euclidean distance. Equation (2.3)
has an explicit expression for the general solution, indeed, define a transformed function
\[ v(\tau, x) = \frac{1}{\tau} \partial_\tau (r^3 u), \tag{2.4} \]
which has an inverse expression for \( u(\tau, x) \)
\[ \tau^3 u(\tau, x) = \int_0^{\tau - \tau_0} (r + \tau_0) v(r + \tau_0, x) \, dr + \tau_0^3 g(x). \tag{2.5} \]
The new function \( v(\tau, x) \) satisfies the following wave equation
\[ \partial_\tau^2 v = \Delta v \quad \tau, \tau_0 > 0, \quad x \in \mathbb{R}^3 \tag{2.6} \]
as can be seen by a comparison with the ultrahyperbolic wave equation with five-dimensional time variable, with solutions that are radial in time. The initial data for (2.6) are defined through the transformation (2.4)
\[ v(\tau_0, x) = 3\tau_0^2 g(x) + \tau_0^2 h(x) := \phi(x) \quad \text{and} \quad \partial_\tau v(\tau_0, x) = 3g(x) + \tau_0^2 \Delta g(x) + \tau_0 h(x) := \psi(x). \tag{2.7} \]
Equation (2.6) is the wave equation in the Minkowski metric on the domain \( \{(\tau, x) : \tau > 0, x \in \mathbb{R}^3\} \), whose solution operator is expressed by the spherical means formula (2.2)
\[ v(\tau, x) = \partial_\tau ((r - \tau_0) M_\phi(r - \tau_0, x)) + (r - \tau_0) M_\psi (r - \tau_0, x), \tag{2.8} \]
where \( M_\phi(r, x) \) is the spherical mean of the function \( f(x) \), (respectively, \( \phi \) and \( \psi \)) (2.1). Substituting the expression for \( v(\tau, x) \) back into the expression for \( u(\tau, x) \) yields the following solution formula
\[ \tau^3 u(\tau, x) = \int_0^{\tau - \tau_0} (r + \tau_0) \partial_\tau (r^3 \phi(r, x)) \, dr + \int_0^{\tau - \tau_0} (r + \tau_0) r M_\psi (r, x) \, dr + \tau_0^3 g(x) \]
\[ = \tau (r - \tau_0) M_\phi(r - \tau_0, x) - \int_0^{\tau - \tau_0} r M_\phi (r, x) \, dr \]
\[ + \int_0^{\tau - \tau_0} (r + \tau_0) r M_\psi (r, x) \, dr + \tau_0^3 g(x). \tag{2.9} \]

To present the formula in more useful form, consider the initial data in separate cases: (i) \( (g(x), h(x) = 0) \), and (ii) \( g(x) = 0, h(x) \).

In case (i), using the definitions for \( (\phi, \psi) \) given in (2.7), the expression (2.9) gives the formula
\[ u(\tau, x) = \frac{1}{\tau^3} \left( \int_0^{\tau - \tau_0} (r + \tau_0) \partial_\tau \left( \frac{1}{4\pi r} \int_{S_{r}(\chi)} 3\tau_0 g(y) \, dS_r(y) \right) \, dr \right. \]
\[ + \left. \int_0^{\tau - \tau_0} (r + \tau_0) r \left( \frac{1}{4\pi r^2} \int_{S_{r}(\chi)} 3g(y) + \tau_0^2 \Delta g(y) \, dS_r(y) \right) \, dr + \tau_0^3 g(x) \right) \]
\[ = \frac{1}{\tau^3} \left( \int_0^{\tau - \tau_0} \partial_\tau \left( (r + \tau_0) \frac{1}{4\pi r} \int_{S_{r}(\chi)} 3\tau_0 g(y) \, dS_r(y) \right) \, dr + \int_0^{\tau - \tau_0} \frac{1}{4\pi} \int_{S_{r}(\chi)} 3g(y) \, dS_r(y) \, dr \right. \]
\[ + \left. \left( \int_0^{\tau - \tau_0} \frac{1}{4\pi} \int_{S_{r}(\chi)} \frac{r^2}{\tau_0^3} \Delta g(y) \, dS_r(y) \, dr + \tau_0^3 g(x) \right) \right). \]

Applying the divergence theorem to the expression containing the Laplacian, and using the fact that the fundamental solution for the Laplacian is given by \((-1/4\pi)(1/r)\), this gives the following expression for the wave propagator:
\[ u(\tau, x) = \frac{1}{\tau^3} \left( \frac{1}{4\pi} \int_{S_{r}(\chi)} \frac{r^3}{\tau_0^3} (r - \tau_0)^3 g(y) \, dS_{r-\tau_0}(y) + \frac{1}{4\pi} (r - \tau_0) \int_{S_{r-\tau_0}} \nabla g(y) \cdot \frac{y - x}{|y - x|} \, dS_{r-\tau_0}(y) \right. \]
\[ + \frac{3}{4\pi} \int_0^{\tau - \tau_0} g(y) \, dS_r(y) \, dr \bigg). \tag{2.10} \]
In case (ii), using (2.7), the expression (2.9) gives rise to
\[
 u(\tau, x) = \frac{1}{\tau^3} \left( \int_0^{\tau - \tau_0} (r + \tau_0) \partial_\tau \left( \frac{1}{4\pi r} \int_{S_r(x)} r_0^2 h(y) dS_r(y) \right) dr 
+ \int_0^{\tau - \tau_0} (r + \tau_0) \frac{1}{4\pi r} \int_{S_r(x)} \tau_0 h(y) dS_r(y) dr \right)
= \frac{1}{\tau^3} \left( \frac{\tau}{4\pi (\tau - \tau_0)} \int_{S_{\tau - \tau_0}(x)} h(y) dS_{\tau - \tau_0}(y) + \frac{\tau_0}{4\pi} \int_0^{\tau - \tau_0} \int_{S_r(x)} h(y) dS_r(y) dr \right).
\] (2.11)

The sum of the expressions (2.10) and (2.11) gives the solution formula for the wave propagator in the case of general initial data. For \( g \in C^1(\mathbb{R}^3) \) and \( h \in C^0(\mathbb{R}^3) \), these formulae express a distributional (or weak) solution. With more regularity, the expressions give a classical \( C^2 \) solution. In these expressions, the integral densities for surfaces and volumes are not given with respect to the background Lorentzian metric \( g \) restricted to the time slice \( \{(\tau, x) : \tau = \tau_0\} \), which we denote \( g_{\tau_0} \). This is implemented using the scale factor \( S(\tau_0) = \tau_0^2 \) and the substitutions \( dS_{\tau - \tau_0} = S^{-2}(\tau_0) dS_{\tau - \tau_0}(g_{\tau_0}) := \tau_0^{-4} dS_{\tau - \tau_0}(g_{\tau_0}) \), and \( dS_r dr = S^{-3}(\tau_0) (S^3(\tau_0) dS_r dr) := \tau_0^{-6} dV(g_{\tau_0}) \). Additionally, the time variable \( \tau \) is not the same as \( t \) of the Friedmann–Robertson–Walker metric; it may be recovered through the time change \( t = \tau^3/3 \).

From the above expression for \( u \), which contains integrals over the interior of the backward light cone, one can see that for any given \( (\tau, x), \tau > \tau_0 \), the domain of dependence is \( B_{\tau - \tau_0}(x) \). It then follows that the solution is identically zero outside of the union of interiors of all the future light cones emanating from \( B(0) \), namely \( UR := \{(\tau, x) : |x| \leq R + (\tau - \tau_0), \tau > \tau_0\} \), that is, \( \text{supp}(u) \subseteq UR \), the statement of finite propagation speed. In addition, consider the union of all future light cones
\[
V_R = \bigcup_{y \in B_R(0)} \{(x, y) : |x - y| = \tau - \tau_0, \tau > \tau_0\}.
\]
Based on the representation (2.10) and (2.11) of the wave propagator, the solution \( u(\tau, x) \) is generally non-zero in \( UR \setminus V_R \), that is, inside the envelope of light cones over the support of the initial data, and it is given there by the expression
\[
 u(\tau, x) = \frac{1}{\tau^3} \left( \frac{3}{4\pi} \int_0^{\tau - \tau_0} \int_{S_r(x)} g(y) dS_{\tau - \tau_0}(y) dr + \frac{\tau_0}{4\pi} \int_0^{\tau - \tau_0} \int_{S_r(x)} h(y) dS_{\tau - \tau_0}(y) dr \right),
\]
valid for \( \tau > |x| + \tau_0 + R \). One can see from these observations that for any given \( x \), after a given time (and in particular for \( \tau > |x| + \tau_0 + R \)) the solution at that point will generally persist indefinitely, spatially constant with value related to the average value of the initial data, however with asymptotically diminishing magnitude in time. Thus, the sharp Huygen's principle does not hold for solutions of the wave equation in this space–time, a fact that is in contrast to the ordinary three-dimensional wave equation, as discussed in pages 130–131 of [6]. Of course, there is a similar statement in the case \( 0 < \tau < \tau_0 \).

(c) Rate of decay

The explicit expression (2.9) for the solution operator is useful for estimates on the rate of decay of solutions as \( \tau \to \infty \), which we quantify in the following statement.

**Theorem 2.1.** Suppose \( g \in C^1(\mathbb{R}^3), h \in C^0(\mathbb{R}^3) \), \( \text{supp}(g, h) \subseteq B_R(0) \). Then, the solution to (2.3) decays to zero at rate of \( O(\tau^{-3}) \) uniformly throughout \( UR \) as \( \tau \) tends to infinity.

Similar estimates of the decay rate hold for \( \partial_\tau u(\tau, x) \), and therefore for the wave propagator \( W(\tau_0, \tau)(g, h) \).
Proof. We consider initial data \( g \in C^1(\mathbb{R}^3) \), \( h \in C^0(\mathbb{R}^3) \), with \( \text{supp}(g,h) \subseteq B_R(0) \). Define constants
\[
C_g := \sup_{x \in \mathbb{R}^3} |g(x), \nabla g(x)| = |g(x)|_{C^1(B_R)} \quad \text{and} \quad C_h := \sup_{x \in \mathbb{R}^3} |h(x)| = |h(x)|_{C^0(B_R)},
\]
we show that
\[
|u(\tau, x)| \leq \frac{C_R}{\tau^3} (C_g + C_h).
\]
To prove this, taking as a sample calculation, we examine the decay rate of the second term in the second equality of (2.10). The calculations for the remaining terms follow similarly.
\[
\frac{1}{\tau^3} \left| 1 \left[ \frac{\tau^2}{4\pi} \left( \tau - \tau_0 \right) \right] \int_{S_{\tau \tau_0}} \nabla g(y) \cdot \frac{y - x}{|y - x|} \, dS_{\tau \tau_0}(y) \right| \leq \frac{1}{\tau^3} \left( \frac{\tau^2}{4\pi} \right) \frac{C_g}{S_{\tau \tau_0}(\cap B_R(0))} \int_{S_{\tau \tau_0}(\cap B_R(0))} \, dS_{\tau \tau_0}(y) \leq \frac{C_g}{\tau^3} R^2 = O(\tau^{-3}).
\]
Note that the term \( |\tau^2(\tau - \tau_0)| \sim \tau^2 \) for \( \tau \rightarrow +\infty \), and that \( \int_{S_{\tau \tau_0}(\cap B_R(0))} \, dS_{\tau \tau_0}(y) \) is bounded by \( 4\pi \, \text{min}\{
\}
\]
\[
4\pi \, \text{min}\{(\tau - \tau_0)^2, R^2\} \]
for geometrical reasons. In the second line, we have used that the support of the initial data is compact, indeed \( \text{supp}(g) \subseteq B_R(0) \).

A similar analysis handles the remaining terms, yielding
\[
\frac{1}{\tau^3} \left| 1 \left[ \frac{\tau^3 - (\tau - \tau_0)^3}{(\tau - \tau_0)^2} \right] \int_{S_{\tau \tau_0}} g(y) \, dS_{\tau \tau_0}(y) \right| = O(\tau^{-3})
\]
\[
\frac{1}{\tau^3} \left| 3 \left[ \frac{\tau^2}{4\pi} \left( \tau - \tau_0 \right) \right] \int_{S_{\tau \tau_0}} g(y) \, dS_{\tau \tau_0}(y) \, dr \right| = O(\tau^{-3})
\]
\[
\frac{1}{\tau^3} \left| 4\pi \left[ \frac{\tau^2}{4\pi} \right] \frac{h(y)}{S_{\tau \tau_0}(y)} \, dS_{\tau \tau_0}(y) \right| = O(\tau^{-3})
\]
for large \( \tau \rightarrow +\infty \). The result is that for \( (\tau, x) \in U \), for \( \tau \) large, \( u(\tau, x) = O(\tau^{-3}) \), and uniformly so throughout \( U \).

To recover a decay estimate in terms of our original time variable \( t \), use the fact that \( t = (\tau^3 / 3) \), implying that \( \tau = (3t)^{1/3} \). Therefore, the decay of the solution of the wave equation in flat Friedmann–Robertson–Walker space–time is \( O(t^{-1}) \), which is identical to the rate of decay of solutions for the wave equation for the Minkowski metric in three-dimensional space.

3. Wave equation for \( K = -1 \)

(a) The solution

We now solve the wave equation in the Robertson–Walker space–time for constant curvature \( K = -1 \). We know in this case the scaling factor takes the form \( S(\tau) = \cosh(\tau) - 1 \), giving rise to the following system
\[
\partial_\tau^2 u + 2 \coth \left( \frac{\tau}{2} \right) \partial_\tau u - \Delta_g u = 0, \quad \tau > 0, \ x \in \mathbb{H}^3 \quad (3.1)
\]
and
\[
u(\tau_0, x) = g(x), \quad \partial_\tau u(\tau_0, x) = h(x), \ \tau_0 > 0.
\]
Assume similar support constraints to our initial data as in the case for \( K = 0 \), namely that \( (g(x), h(x)) \) are supported in the geodesic ball \( B_R(0) \). Then, equation (3.1) has an explicit geodesic spherical means expression for solutions, similar to the one given in (2.9). Define a
transformed function
\[ v(\tau, x) = \frac{4}{\sinh(\tau/2)} \partial_\tau \left( \sinh^3 \left( \frac{\tau}{2} \right) u(\tau, x) \right), \] (3.2)
in analogy with (2.4). The inverse of this transformation is given by
\[ \sinh^3 \left( \frac{\tau}{2} \right) u(\tau, x) = \int_0^{\tau - \tau_0} \frac{1}{4} \sinh \left( \frac{r + \tau_0}{2} \right) v(r + \tau_0, x) \, dr + \sinh^3 \left( \frac{\tau_0}{2} \right) u(\tau_0, x). \] (3.3)

Then, \( v(\tau, x) \) satisfies the equation
\[ \partial_\tau^2 v = L v \] (3.4)
with \( L = \Delta_\sigma + 1 \) and with Cauchy data given on the hypersurface \( \tau = \tau_0 > 0 \)
\[ v(\tau_0, x) = 3 \sinh(\tau_0) g(x) + 4 \sinh^2 \left( \frac{\tau_0}{2} \right) h(x) := \phi(x) \]
and
\[ \partial_\tau v(\tau_0, x) = 3 \cosh(\tau_0) g(x) + 4 \sinh^2 \left( \frac{\tau_0}{2} \right) \Delta_\sigma g + \sinh(\tau_0) h(x) := \psi(x). \] (3.5)

For (3.4), there is an explicit spherical means formula for the solution, given in [9], which is the hyperbolic analogue of (2.2), namely
\[ v(\tau, x) = \sinh(\tau - \tau_0) M_\phi(\tau - \tau_0, x) + \partial_\tau (\sinh(\tau - \tau_0) M_\phi(\tau - \tau_0, x)), \] (3.6)
where, in the case \( K = -1 \), the geodesic spherical mean of a function \( f(x) \) is given by an integral over the geodesic sphere \( S_r(x) \) of radius \( r \) about \( x \):
\[ M_f(r, x) := \frac{1}{4\pi (\sinh(r))^2} \int_{S_r(x)} f(y) \, dS_r(y), \]
and where \( dS_r(x) \) is the element of spherical surface area. Using (3.6) and the inversion formula (3.3), the solution operator can be expressed as
\[ \sinh^3 \left( \frac{\tau}{2} \right) u(\tau, x) = \int_0^{\tau - \tau_0} \frac{1}{4} \sinh \left( \frac{r + \tau_0}{2} \right) \partial_\tau (\sinh(r)M_\phi(r, x)) \, dr \]
\[ + \int_0^{\tau - \tau_0} \frac{1}{4} \sinh \left( \frac{r + \tau_0}{2} \right) \sinh(r) \sinh^2 \left( \frac{\tau_0}{2} \right) M_{\Delta_\sigma g}(r, x) \, dr + \sinh^3 \left( \frac{\tau_0}{2} \right) g(x). \] (3.7)

Using the definitions for \( (\phi, \psi) \) given in (3.5), this expresses the solution of the wave propagator for the hyperbolic case in terms of spherical means over geodesic spheres in hyperbolic geometry. A more transparent expression is obtained from substituting the actual initial data. As in the Euclidian case, this is separated into the two cases: (i) \( (g(x), h(x) = 0) \), and (ii) \( (g(x) = 0, h(x)) \).

In case (i), the formula (3.7) gives the expression
\[ u(\tau, x) = \frac{1}{\sinh^3(\tau/2)} \left( \int_0^{\tau - \tau_0} \frac{3}{4} \sinh \left( \frac{r + \tau_0}{2} \right) \partial_\tau (\sinh(r)M_\phi(r, x)) \, dr \right) \]
\[ + \int_0^{\tau - \tau_0} \frac{3}{4} \sinh \left( \frac{r + \tau_0}{2} \right) \partial_\tau (\sinh(r) \sinh(\tau_0) M_\phi(r, x)) \, dr \]
\[ + \int_0^{\tau - \tau_0} \sinh \left( \frac{r + \tau_0}{2} \right) \sinh(r) \sinh^2 \left( \frac{\tau_0}{2} \right) M_{\Delta_\sigma g}(r, x) \, dr + \sinh^3 \left( \frac{\tau_0}{2} \right) g(x) \]
\[ = \frac{1}{\sinh^3(\tau/2)} \left( \frac{3}{4} \sinh \left( \frac{\tau - \tau_0}{2} \right) \sinh(\tau_0) M_\phi(\tau - \tau_0, x) \right) \]
\[ + \int_0^{\tau - \tau_0} \frac{3}{8} \left( \sinh \left( \frac{r + \tau_0}{2} \right) \cosh(\tau_0) + \sinh \left( \frac{r - \tau_0}{2} \right) \right) \sinh(r) M_\phi(r, x) \, dr \]
\[ + \int_0^{\tau - \tau_0} \sinh \left( \frac{r + \tau_0}{2} \right) \sinh(r) \sinh^2 \left( \frac{\tau_0}{2} \right) M_{\Delta_\sigma g}(r, x) \, dr + \sinh^3 \left( \frac{\tau_0}{2} \right) g(x) \]. (3.8)

The term involving the geodesic spherical mean of the Laplace–Beltrami operator is treated by integrations by parts. Using the fact that the fundamental solution is given by \(-1/4\pi \sinh(r)\) and
the definition of the geodesic spherical mean, we find

\[
\int_0^{r-t_0} \sinh \left( \frac{r + t_0}{2} \right) \sinh(r) \sinh^2 \left( \frac{t_0}{2} \right) M_{\Delta, g}(r, x) \, dr \\
= \int_{B_{r-t_0}(x)} \sinh \left( \frac{r + t_0}{2} \right) \sinh^2 \left( \frac{t_0}{2} \right) \frac{1}{4\pi \sinh(r)} \left( (\partial^2_2 + 2 \coth(r) \partial_r) g \right) \sinh^2(r) \, dS_t(\xi) \, dr \\
= \sinh^2(t_0/2) \sinh(\tau/2) \int_{S_{r-t_0}(x)} \partial_r g(y) \, dS_{r-t_0}(y) \\
+ \sinh^2(t_0/2) \cosh(\tau - t_0) - (1/2) \cosh(\tau/2) \sinh(t_0/2) \int_{S_{r-t_0}(x)} g(y) \, dS_{r-t_0}(y) \\
- \int_0^{r-t_0} \frac{3}{4} \sinh^2 \left( \frac{t_0}{2} \right) \sinh \left( \frac{r + t_0}{2} \right) \frac{1}{4\pi \sinh(r)} \sinh^2(x) \, dS_r(y) \, dr - \sinh^3 \left( \frac{t_0}{2} \right) \frac{3}{4} \sinh^2(x) \, dS_r(y) \, dr. \quad (3.9)
\]

Using this in the expression (3.8), one finds that

\[
u(\tau, x) = \frac{1}{\sinh^3(\tau/2)} \left( \sinh^2(t_0/2) \sinh(\tau/2) \int_{S_{r-t_0}(x)} \partial_r g(y) \, dS_{r-t_0}(y) \\
+ \frac{1}{2} \sinh \left( \frac{\tau}{2} \right) \sinh \left( \frac{t_0}{2} \right) \left( 3 \sinh(\tau - t_0) \cosh \left( \frac{t_0}{2} \right) + \cosh(\tau - t_0) \sinh \left( \frac{t_0}{2} \right) \right) \\
+ \frac{1}{2} \sinh^3 \left( \frac{t_0}{2} \right) \int_{S_{r-t_0}(x)} g(y) \, dS_{r-t_0}(y) \\
+ \int_0^{r-t_0} \int_{S_{r}(x)} \frac{3}{8} \left( \sinh \left( \frac{r + t_0}{2} \right) + \sinh \left( \frac{r - t_0}{2} \right) \right) \frac{1}{4\pi \sinh(r)} g(y) \, dS_r(y) \, dr \right). \quad (3.9)
\]

In the case (ii), the expression (3.7) is the following

\[
u(\tau, x) = \frac{1}{\sinh^3(\tau/2)} \left( \int_0^{r-t_0} \sinh \left( \frac{r + t_0}{2} \right) \partial_r \left( \sinh(r) \sinh^2 \left( \frac{t_0}{2} \right) M_b(r, x) \right) \, dr \\
+ \int_0^{r-t_0} \frac{1}{4} \sinh \left( \frac{r + t_0}{2} \right) \sinh(r) \sinh(t_0) M_b(r, x) \, dr \right) \\
= \frac{1}{\sinh^3(\tau/2)} \left( \int_{S_{r-t_0}(x)} h(y) \, dS_{r-t_0}(y) \\
+ \int_0^{r-t_0} \int_{S_{r}(x)} \frac{1}{4} \left( \cosh \left( \frac{r + t_0}{2} \right) - \cosh \left( \frac{r - t_0}{2} \right) \right) \frac{1}{4\pi \sinh(r)} h(y) \, dS_r(y) \, dr \right). \quad (3.10)
\]

Again, these expressions give a distributional solution to the wave equation when \((g(x), h(x)) \in C^1 \times C^0(\mathbb{R}^3)\). In the expressions (3.9) and (3.10), the integral densities for surfaces and volumes are not given with respect to the background Lorentzian metric \(g\) restricted to the time slice \(\{(r, x) : \tau = t_0\}\); this is implemented using the scale factor \(S(t_0) = \cosh(t_0) - 1 = 2 \sinh^2(t_0/2)\) and the substitutions \(dS_{r-t_0} = S^{-2}(t_0)(S^2(t_0) \, dS_{r-t_0}) := \frac{1}{4} \sinh^{-4}(t_0/2) \, dS_{r-t_0}(\xi_0)\), and \(dS_r \, dr = S^{-3}(t_0)(S^3(t_0) \, dS_r) \, dr := \frac{1}{8} \sinh^{-5}(t_0/2) \, dV_{g(x)}\). Additionally, the time variable \(\tau\) is not the same as \(t\) of the Friedmann–Robertson–Walker metric; it may be recovered by inverting the time change \(t = \sinh(\tau - \tau)\).

Similar to the case for \(K = 0\), the domain of dependence of the solution to the case \(K = -1\) is the geodesic ball \(B_{r-t_0}(0)\), so that the support of the solution is restricted to \(\text{supp}(u) = U_R := \{(r, x) : \text{dist}_g(x, 0) \leq R + \sinh(t - t_0), \tau > t_0\}\). Because of the final terms of the RHS in both of the expressions (3.9) and (3.10), it is evident that the support of the kernel of the wave propagator is not confined to the set \(V_R := \cup_{y \in B_{r}(0)} \{(r, x) : \text{dist}_g(x, y) = \tau - t_0\}\) of light cones eliminating from the initial data, rather it fills the interior of the future light cone. Namely the sharp Huygen’s principle does not hold in the negative curvature case, and in particular, solutions of the wave
equation in the hyperbolic case of Friedmann–Robertson–Walker space–times do not experience sharp propagation of signals. Furthermore, and in contrast to the case of \( K = 0 \), the solution in the interior of the light cone \( U_R \setminus V_R \) is not locally spatially constant, and is dictated by a kernel which is dependent upon the geodesic radius \( r \). Again, there is a similar statement for the case \( 0 < \tau < \tau_0 \).

(b) Rate of decay

The explicit expression via spherical means over geodesic spheres gives information about the decay of solutions for large times. This is quantified in the following statement.

**Theorem 3.1.** Suppose \( g \in C^1(\mathbb{H}^3) \), \( h \in C^0(\mathbb{H}^3) \), with support \( \text{supp}(g, h) \subset B_R(0) \). Then, the solution to (2.3) decays to zero at rate of \( O(\sinh^2(R) e^{-2t}) \) uniformly throughout \( U_R \) as \( \tau \) tends to infinity.

Similar estimates of the decay rate hold for \( \partial_t \), \( u(\tau, x) \), and therefore for the wave propagator \( W(t_0, \tau)(g, h) \). Because of the relation (1.3) between conformal time \( \tau \) and physical time \( t \) in the hyperbolic case, the result is that solutions have the decay rate for large \( t \)

\[
|W(t_0, t)(g, h)|_{L^\infty} \leq O(t^{-2}).
\]

**Proof.** We are assuming that \( g \in C^1(\mathbb{H}^3) \), \( h \in C^0(\mathbb{H}^3) \) and \( \text{supp}(g, h) \subset B_R(0) \). Define constants

\[
C_g := \sup_{x \in \mathbb{H}^3} |(g(x), \nabla g(x))| \quad C_h := \sup_{x \in \mathbb{H}^3} |h(x)|.
\]

The result follows from estimates of the terms of the spherical means expression (3.9) and (3.10). As a sample calculation, examine the decay rate of the first term of the r.h.s. of (3.9). The estimates for the remaining terms will follow similarly. Using the compact support of \( g \), the domain of integration is \( B_{\tau - \tau_0}(\tau) \cap B_R(0) \).

\[
\frac{\sinh^2(\tau_0/2) \sinh(\tau/2)}{4\pi \sinh^3(\tau/2) \sinh(\tau - \tau_0)} \int_{S_{\tau - \tau_0}(\tau)} \partial_y g(y) \, dS_{\tau - \tau_0}(y) \leq \frac{\sinh^2(\tau_0/2)}{4\pi \sinh^2(\tau/2) \sinh(\tau - \tau_0)} \int_{S_{\tau - \tau_0}(\tau) \cap \partial B_{\tau}(0)} \nabla g \cdot N \, dS_{\tau - \tau_0}(y),
\]

where \( N \) is the exterior unit normal to the geodesic sphere \( S_{\tau}(0) \). By hypotheses \( |\nabla \cdot g(y)| \leq C_g \).

Then, for geometrical reasons, the integral term is bounded by \( C_g 4\pi \min(\sinh^2(\tau - \tau_0), \sinh^2(R)) \).

Finally, the first factor is bounded for large \( \tau \) by \( e^{-2t} \). A similar analysis applies to the other four terms of (3.9) and (3.10). For \( (\tau, x) \in U_R \) and for \( \tau \) large, the conclusion is that \( |u(\tau, x)| \leq O(e^{-2t}) \). Of course for \( (\tau, x) \notin U_R \) the solution is \( u(\tau, x) = 0 \).

Returning to physical variables, \( t = \sinh(\tau) - \tau \) however, unlike the case for \( K = 0 \), there is not a clean expression for the conformal time \( \tau \) in terms of physical time \( t \). The asymptotics of this expression are such that, for any \( a > 0 \) there is \( t_* > 0 \) such that for \( t > t_* \), \( \ln(t) - a < \tau(t) < \ln(t) + a \). Thus, \( e^{-2t} \sim e^{-2\ln(t)} = t^{-2} \), giving the decay of the solution of the wave equation in Friedmann–Robertson–Walker space–time, expressed in the physical time variables \( t \), as being \( O(t^{-2}) \).

### 4. The initial value problem at the singular time \( \tau_0 = 0 \)

Throughout the discussion above, we have retained the condition that \( \tau, \tau_0 > 0 \), owing to the space–time metric singularity at \( \tau = 0 \) (equivalently at \( t = 0 \)). However, given the explicit nature of the wave propagator \( W(t_0, \tau)(g, h) \), we are able to consider the limit

\[
\lim_{\tau_0 \to 0^+} W(t_0, \tau_1)(g, h)
\]
where the initial time $\tau_0$ is taken to zero while leaving time $\tau_1$ fixed. Because, in both cases $K = 0$ and $K = -1$, the transformed time variable behaves asymptotically as $t \sim \tau^3/3$ for $\tau \to 0$ in a neighbourhood of $t = 0 = \tau$, it suffices to work in the transformed time $\tau$.

(a) Case $K = 0$

For zero curvature $K = 0$ and $\tau_0 > 0$, the solution to the Cauchy problem takes the form

$$u(\tau, x) = \frac{1}{\tau^3} \int_0^{\tau-\tau_0} (r + \tau_0) \partial_r (rM_\phi(r, x)) \, dr + \frac{1}{\tau^3} \int_0^{\tau-\tau_0} (r + \tau_0) rM_\psi(r, x) \, dr + \frac{1}{\tau^3} \tau^3 g(x).$$

Using the definition in (2.6) for $(\phi(x), \psi(x))$, recalling that they themselves depend upon $\tau_0$, and taking the limit $\tau_0 \to 0$ of the resulting formula yields the following limiting expression for the solution:

$$u(\tau, x) = \frac{1}{\tau^3} \int_0^\tau 3\tau^2 M_g(r, x) \, dr$$

(4.1)

Remark that the limit as $\tau_0$ tends to zero of this solution has no dependence on the initial data $h(x)$.

**Theorem 4.1.** For $(g, h) \in C^1(\mathbb{R}^3) \times C^0(\mathbb{R}^3)$, the limit of the wave propagator exists,

$$\lim_{\tau_0 \to 0^+} W(0, \tau)(g, h) = (u(\tau, x), \partial_\tau u(\tau, x)),$$

it is independent of $h(x)$, and satisfies

$$\lim_{\tau \to 0^+} u(\tau, x) = g(x), \quad \lim_{\tau \to 0^+} \partial_\tau u(\tau, x) = 0.$$

Thus, the expression (4.1) gives a solution to the wave equation over the full half-line $\tau \in (0, +\infty)$, with initial data $(u(0, x), \partial_\tau u(0, x)) = (g(x), 0)$ given at the singular time $\tau = 0$.

**Proof.** From the expression (4.1) and l’Hôpital’s rule, one finds that

$$\lim_{\tau \to 0^+} u(\tau, x) = \lim_{\tau \to 0^+} M_g(\tau, x) = g(x)$$

for continuous initial data $g(x)$. Furthermore, one verifies that expression (4.1) satisfies

$$\lim_{\tau \to 0^+} \partial_\tau u(\tau, x) = 0.$$

Namely after differentiation, the expression for $\partial_\tau u(\tau, x)$ is explicitly

$$\partial_\tau u(\tau, x) = \frac{1}{\tau^3} \left( \frac{-3}{\tau} \int_0^\tau 3\tau^2 M_g(r, x) \, dr + 3\tau^2 M_g(\tau, x) \right),$$

for which, using l’Hôpital’s rule, one has

$$\lim_{\tau \to 0^+} \partial_\tau u(\tau, x) = \lim_{\tau \to 0^+} \frac{3}{4} \partial_\tau M_g(\tau, x),$$

and this vanishes as $\tau$ tends to zero since $M_g(\tau, x)$ is even in $\tau$.

It is clear from this calculation that the full Cauchy problem is not well posed for $\tau_0 = 0$. In general, solutions of the Cauchy problem posed at $\tau_0 > 0$ and propagated to times $0 < \tau < \tau_0$ will become singular as $\tau \to 0^+$. But there remains a full function space of initial data, depending upon one scalar function $g(x)$, for which the solution exists for the initial value problem consisting of data $(g(x), 0)$ posed at $\tau_0 = 0$, and propagated to arbitrary future (or past) times $\tau$. 
(b) $K = -1$

There is a similar calculation of the limit as $\tau_0 \to 0$ for the case of constant curvature $K = -1$. For $\tau_0 > 0$, the solution to the Cauchy problem takes the form

$$u(x, \tau) = \frac{1}{4 \sinh^3(\tau/2)} \int_0^{\tau - \tau_0} \sinh \left( \frac{r + \tau_0}{2} \right) \partial_r (\sinh(r) M_\phi(r, x)) \, dr$$

$$+ \frac{1}{4 \sinh^3(\tau/2)} \int_0^{\tau - \tau_0} \sinh \left( \frac{r + \tau_0}{2} \right) \sinh(r) M_\psi(r, x) \, dr + \frac{\sinh^3(\tau_0/2)}{\sinh^3(\tau/2)} g(x)$$

Again, recall that the Cauchy data $(\phi(x), \psi(x))$ given in (3.5) are defined in terms of $\tau_0$. Taking the limit $\tau_0 \to 0$ of this expression, after a similar short calculation, yields the following expression for a solution:

$$u(\tau, x) = \frac{1}{4 \sinh^3(\tau/2)} \int_0^{\tau} 3 \sinh(r/2) \sinh(r) M_\phi(r, x) \, dr. \quad (4.2)$$

Thus, similar to the case $K = 0$, the Cauchy problem for $\tau_0 = 0$ is ill-posed for arbitrary initial data. However, for the particular initial data $(g(x), 0)$, there is a well-defined solution starting from $\tau_0 = 0$, given by the expression (4.2). Its time derivative is well behaved in the limit, indeed

$$\lim_{\tau \to 0} u(\tau, x) = g(x), \quad \lim_{\tau \to 0} \partial_\tau u(\tau, x) = 0,$$

as is shown by short calculations similar to those of the previous section. This gives a meaning to the wave propagator when applied to data $(g(x), 0)$ posed at $\tau = 0$, resulting in a well-defined solution for $\tau > 0$ emanating from the singularity of space–time at $\tau_0 = \tau_0 = 0$. Under reflection $\tau \to -\tau$, this is also a solution to the wave equations (2.3), respectively (3.1) for $\tau < 0$, for which the evolution is continuous across the space–time singularity at $\tau = 0$.

5. Perspectives

The analysis in this paper of the wave propagator in a Friedmann–Robertson–Walker background space–time metric raises a number of basic questions, having to do with the propagation of signals in an expanding universe as well as having to do with the passage of information through a space–time singularity from the past to the future. The most basic questions are as follows.

The result of §4 is that certain information propagated by solutions of the wave equation can be transmitted continuously from the past to the future of a space–time singularity of the character of the singularity in a Friedmann–Robertson–Walker space–time [10,11]. As the wave equation (1.4) represents the linearized theory of the evolution of many physical quantities, this class of solutions is analogous to the theory of a linear stable manifold for the wave equation at the singularity of the Friedmann–Robertson–Walker space–times. It would be a fundamental question whether non-trivial families of solutions of nonlinear equations exist which pass in a similar way through a Friedmann–Robertson–Walker-like space–time singularity, in a way which transmits information from past to future. In particular, one should ask whether there exist families of solutions to the full Einstein equations which behave similarly, and which are regularized under an appropriate singular conformal transformation, these would carry a non-trivial past space–time into a non-trivial future. Following the analogy with stable manifold theory, the indication is that a large class of such solutions could exist.

A second question has to do with the measurement of large-scale distances based on decay properties of the intensity of supernovae. In large sky surveys using type Ia supernovae as standard candles, two basic pieces of information are compared with high precision: one is redshift, giving a precise measurement of relative velocity of the supernova away from the observer. The second is a measurement of the intensity of the supernova, from which we deduce distance from the observer. Given data from the second, observations show that the red shift, and therefore the relative velocity of distant objects, is not consistent with the Hubble hypothesis of a uniformly expanding universe, as this velocity is slightly and unexpectedly higher for more
distant objects. This is the phenomenon of apparent acceleration of the universe, which gives rise to the idea of the possibility of a non-zero cosmological constant, among other theories. With the precision of the description of solutions to the wave equation given in this article, and the decay rates that follow from it, it would merit another look at this study. Namely a reassessment in particular of the transformation between the measurement of supernova intensity and the distance from the observer. To apply our detailed results on the wave propagator, we would make the assumption that on a sufficiently large scale the space–time metric is close to the Friedmann–Robertson–Walker metric. The decay rates in fact involve the parameter of the curvature $K$, where in this paper we have described the two cases $K = 0$ and $K = -1$. Of course an expression from the same derivation for the wave propagator for any $0 \geq K > -\infty$ is available after a scaling. It is conceivable, assuming a Friedmann–Robertson–Walker background and therefore a fixed Hubble expansion rate, that the result could be an improved fit of the data without the apparent acceleration, and where the best fit for the distance data would also give an observed value for the space–time curvature $K$.

A third question has to do with the lack of the sharp Huygens principle with a Friedmann–Robertson–Walker background metric, and whether it can be seen in observations. This paper has addressed only the initial value problem for the wave equation, but by the Duhamel principle the inhomogeneously forced problem can also be expressed in terms of the wave propagator. This would be what is used to describe light from a steady distant source impinging over a long period of time on our location on Earth. Given the lack of a sharp Huygens principle, a wavefront which has passed a location will leave a small residual in the form of a decaying trace. We propose that an observer could conceivably see this in an effect on the red shift of spectral lines; the trace left by the passage in the past of a wavefront, if still detectable, would be of a slightly smaller red shift, in the form of a slight blue downshifted trace or shadow, owing to the slightly smaller velocity of recession of the source in the distant past when the wavefront was emitted.

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