Are Maxwell knots integrable?

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Abstract

We review properties of the null-field solutions of source-free Maxwell equations. We focus on the electric and magnetic field lines, which actually can be knotted and/or linked at every given moment. We analyse the fact that the Poynting vector induces self-consistent time evolution of these lines and demonstrate that the Abelian link invariant is integral of motion. The same is expected to be true also for the non-Abelian invariants (like Jones and HOMFLY-PT polynomials or Vassiliev invariants), and many integrals of motion can imply that the Poynting evolution is actually integrable. We also consider particular examples of the field lines for the particular family of finite energy source-free "knot" solutions, attempting to understand when the field lines are closed – and can be discussed in terms of knots and links. Based on computer simulations we conjecture that Ranada’s solution, where every pair of lines forms a Hopf link, is rather exceptional. In general, only particular lines (a set of measure zero) are closed and form knots/links, while all the rest are twisting around them and remain unclosed. Still, conservation laws of Poynting evolution and associated integrable structure should persist.

1 Introduction

Knots are playing an increasingly important role in mathematical physics, but in phenomenological physics they remain rather exotic. This is despite knots served as an inspiration to many great physicists, starting at least from Lord Kelvin [1]. In fact the reason can be simple: we are not looking around with enough attention to notice unconventional things. From this perspective, it was a great breakthrough when in 1989 A.F.Ranada demonstrated [2, 3] that Hopf link appears naturally in a solution to the ordinary Maxwell equations, i.e. is no less natural than the conventional plane wave solution. In this paper we are going to discuss some surprising properties of such knot-revealing solutions and mention some open problems – without going into important but relatively sophisticated mathematics, like in [4] or [5, 6]. We begin in s.2 from consideration of the field lines of electric and magnetic fields. When they are closed, they can be knotted or linked. A given field line evolves in time – this evolution appears to proceed along the Poynting vector field, what, however imposes a non-trivial and non-expected condition on the parametrization – we discuss these issues in s.3 and s.3.3. Moreover, closed and linked field lines remain linked, and this is guaranteed by the conservation laws provided by knot invariants – which in this context serve as equations of motion and can even make the evolution of field lines integrable – see s.4 and s.5. We illustrate this general consideration by analysis of the Ranada’s Hopf solution in s.6.1 and by a brief discussion of more general solutions in s.6.2. In Hopf solution all field lines are closed but unknotted, and every pair of field lines forms a Hopf link with no non-trivial links of higher order. However, in other solutions generic field lines are not closed, only some are – but they can form quite non-trivial knots and links. There is still a question, which knots and links can appear and how abundant they are – see [5] for a possible approach to this problem. We end with a brief summary in s.7.

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2 Field lines at a given time moment

In this paper we review properties of solutions of Maxwell equations:

\[
\begin{align*}
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad \nabla \cdot B = 0; \\
\frac{\partial E}{\partial t} - \nabla \times B &= 0, \quad \nabla \cdot E = 0.
\end{align*}
\] (1)

In particular, we are interested in the topological structure of the field lines so let us define the field line itself. To be more specific, let us concentrate only on the electric field lines, however our analysis can be carried out for the magnetic field lines analogously. Let \( E \) be an electric field at a given moment of time and \( x(s) \) – a smooth curve in some parametrization \( s \). Then \( x(s) \) is called an electric field line if the following condition holds for any point on the curve:

\[
\frac{dx(s)}{ds} \parallel E(x(s))
\] (2)

In other words, the tangent vector of the curve should be collinear to the electric field at each point on the curve. We note that the parametrization of the field line is ambiguously defined. To proceed with practical calculation, we will fix a parametrization that is suitable for our purposes. For example, one may consider a field line in different parametrizations:

\[
\begin{align*}
\frac{dx(s_1)}{ds_1} &= E(x(s_1)), \\
\frac{dx(s_2)}{ds_2} &= \frac{E(x(s_2))}{|E(x(s_2))|}, \\
\frac{dx(s_3)}{ds_3} &= \frac{E(x(s_3))}{|E(x(s_3))|^2}.
\end{align*}
\] (3)

All these equations define the same field line. The second parametrization \( s_2 \) corresponds to the length of the field line, but we mainly use the third one \( s_3 \) in our analysis of the knot invariants.

At a given moment of time there are different types of field lines. The force lines with endpoints are prohibited by the zero divergence condition. In other words, endpoints of the force lines are electric/magnetic charges, which are absent in our case. Generally, several types are available: closed loops of finite length and endless curves of infinite length. The curves with an infinite length may go to infinity or be located in the finite area. It would be interesting to find sufficient conditions of closedness of the field lines. We discuss particular examples of field lines in s.6.1, s.6.2.

3 Time evolution of force lines

In our paper we consider only null-field solutions of Maxwell equations. This means that at every moment of time and at every point in space the following conditions hold:

\[
E \cdot B = 0, \quad E^2 = B^2.
\] (4)

Generally, Maxwell equations do not preserve these conditions, which makes them non-trivial. However, such solutions exist and a procedure of constructing them was described in detail in s.7 of [6]. The simplest solution of this kind is the Ranada’s Hopf solution and we discuss it in s.6.1.

We mainly focus on the following property of null-field solutions. It turns out that one can define self-consistent time evolution of the field lines [7, 8]. Generally, at different moments of time the space is filled with different field lines, due to the electric field evolution according to Maxwell equations. To give a prescription of how a field line deforms and moves in space, one needs to define a velocity vector \( v \) for each point of a field line. This vector \( v \) is normal to the field line at each point. This structure will match the field lines at different moments of time.

However, to define the time evolution of the field lines, a velocity vector field should obey a very
special, yet natural condition. The condition requires that any two points of a field line become points of another field line in the further moment of time when moved along the velocity vector field (see Fig.1). In other words, points of a field lines can not become points of different field lines. Alternatively, one could require that the following curve

$$x'(s) = x(s) + v(x(s))dt$$  \hspace{1cm} (5)$$

is a field line that is defined at the moment \( t + dt \).

Figure 1: The picture demonstrates the condition that the vector field \( v \) should obey to define self-consistent time evolution of the field lines. For any two points on a field line at the moment \( t \) the ends of the vectors \( vdt \) at the corresponding points lie on a field line that is defined at the moment \( t + dt \).

In case of null-field solutions, the velocity vector \( v \) can be chosen as the normalized Poynting vector:

$$p = \frac{E \times B}{|E||B|}$$  \hspace{1cm} (6)$$

3.1 Poynting time evolution

In this section we demonstrate that the Poynting vector in a null-field solution induces self-consistent time evolution of the force lines. To do this, we explicitly check the self-consistency of the condition from the previous section.

We consider an electric field line \( x(s) \) of a null-field solution at the moment \( t \). In this section we fix the parametrization \( s \) by the following equation:

$$\frac{dx(s)}{ds} = E(x(s), t)$$  \hspace{1cm} (7)$$

Then we consider an auxiliary line \( x'(s) \) that is obtained from \( x(s) \) by a slight shift along the Poynting vector \( p \):

$$x'(s) = x(s) + p(x(s), t)dt$$  \hspace{1cm} (8)$$

Note that we shift along the Poynting vector by \( dt \). The main claim is that \( x'(s) \) coincides as geometrical objects with some field line at the moment \( t + dt \). Namely, a tangent vector of the line \( x'(s) \) is collinear to the electric field at the moment \( t + dt \), it means that they define the same field line:

$$\frac{dx'(s)}{ds} \parallel E(x'(s), t + dt)$$  \hspace{1cm} (9)$$
We should note that this collinearity holds only in the first order in $dt$. To explicitly see this, we compute the corresponding cross product:

$$\frac{dx'(s)}{ds} \times \mathbf{E}(x'(s), t + dt) = \left( \mathbf{E} + (\mathbf{E} \cdot \nabla) p \right) \times \left( \mathbf{E} + (p \cdot \nabla) \mathbf{E} dt + \frac{\partial \mathbf{E}}{\partial t} dt + o(dt) \right) =$$

$$= dt \mathbf{E} \times \left( (p \cdot \nabla) \mathbf{E} - (\mathbf{E} \cdot \nabla) p + \frac{\partial \mathbf{E}}{\partial t} \right) + o(dt) = o(dt) \quad (10)$$

To simplify the formulas, we omit the explicit coordinate and time dependence of the fields meaning $\mathbf{E} = \mathbf{E}(x(s), t)$ and $p = p(x(s), t)$. We compute the expression in the brackets using the following identity for null-field solutions:

$$E_\beta p_\alpha - E_\alpha p_\beta = \epsilon_{\alpha\beta\gamma} B_\gamma \quad (11)$$

Taking the derivative of (11) and using the Maxwell equations, we obtain the expression of the form:

$$(p \cdot \nabla) \mathbf{E} - (\mathbf{E} \cdot \nabla) p + \frac{\partial \mathbf{E}}{\partial t} = -(\nabla \cdot p) \mathbf{E} \quad (12)$$

The r.h.s is collinear to the electric field $\mathbf{E}$ that ensures (10). Finally, we conclude that the Poynting vector defines the self-consistent time evolution of the field line. Therefore, we can think of a field line as a strand where each point moves with the velocity vector $p$. From now we understand the time dependence of the field lines $x(s, t)$ as it is induced by the Poynting vector:

$$\frac{dx(s, t)}{dt} = p(x(s, t), t) \quad (13)$$

It is evident now why we consider the normalized Poynting vector. If one chooses another normalization for the Poynting vector it will spoil (12).

### 3.2 Examples

For demonstrative purposes let us discuss plane wave null-field solutions of Maxwell equations to analyse the structure of the field lines and its time evolution induced by the Poynting vector.

- **Plane wave with linear polarization**

  The plane wave with linear polarization has the form:

  $$(E \cos(z - t) \quad 0 \quad 0) \quad \begin{pmatrix} 0 \\ E \cos(z - t) \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (14)$$

  Note that at a given moment of time the the electric field changes along the z-axis. However, all the electric field lines are straight lines parallel to the z-axis. The field lines fill the whole space except "singular" xy-planes with coordinates $z = t + \pi/2 + \pi n, n \in \mathbb{Z}$. We can write an explicit formula for the electric field line that goes through the point $(y_0, z_0)$ in the yz-plane at the moment $t_0$:

  $$x(s, t) = \begin{pmatrix} s E \cos(z_0 - t_0) \\ y_0 \\ z_0 + t - t_0 \end{pmatrix} \quad (15)$$

  At each moment of time the tangent vector of the field line is the electric field:

  $$\frac{dx(s, t)}{ds} = \mathbf{E}(x(s, t), t) \begin{pmatrix} E \cos(z_0 - t_0) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E \cos(z_0 + t - t_0 - t) \\ 0 \\ 0 \end{pmatrix} \quad (16)$$
We note that the tangent vector of the particular field line does not change over time. This is due to the fact \( \nabla \cdot \mathbf{p} = 0 \) and we discuss this in s.3.3. The velocity of the points of the field line is the Poynting vector:

\[
\frac{d\mathbf{x}(s, t)}{dt} = \mathbf{p}(\mathbf{x}(s, t), t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

(17)

The structure of the field lines flows along the \( z \)-axis over time as a whole and do not change. This is evident from the coordinate independence of the Poynting vector.

- **Plane wave with elliptic polarization**

This solution has the form:

\[
\mathbf{E} = \begin{pmatrix} E_x \cos(z - t) \\ E_y \sin(z - t) \\ 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -E_y \sin(z - t) \\ E_x \cos(z - t) \\ 0 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

(18)

This solution of Maxwell equations has a slightly more complicated structure of the field lines, however the time evolution is the same as in the previous example – the field lines move along the \( z \)-axis, while the structure of the field lines remains the same. The explicit formula for the field line that goes through the point \((x_0, y_0, z_0)\) at the moment \(t_0\):

\[
\mathbf{x}(s, t) = \begin{pmatrix} x_0 + s E_x \cos(z_0 - t_0) \\ y_0 + s E_y \sin(z_0 - t_0) \\ z_0 + t - t_0 \end{pmatrix}
\]

(19)

The field lines are normal to the \( z \)-axis and all field lines on a particular \( xy \)-plane are parallel. However, the direction of the field lines at a fixed moment of time is continuously changing along the \( z \)-axis. Namely, the direction rotates clockwise or anticlockwise along the \( z \)-axis at a fixed moment of time depending on the mutual sign of \( E_x, E_y \).

At each moment of time the tangent vector is the electric field and the velocity is the Poynting vector:

\[
\frac{d\mathbf{x}(s, t)}{ds} = \mathbf{E}(\mathbf{x}(s, t), t) \begin{pmatrix} E_x \cos(z_0 - t_0) \\ E_y \sin(z_0 - t_0) \\ 0 \end{pmatrix} = \begin{pmatrix} E_x \cos(z_0 + t - t_0 - t) \\ E_y \sin(z_0 + t - t_0 - t) \\ 0 \end{pmatrix}
\]

(20)

\[
\frac{d\mathbf{x}(s, t)}{dt} = \mathbf{p}(\mathbf{x}(s, t), t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

(21)

### 3.3 Peculiarities of Poynting evolution

In this section we derive the parametrization of the field lines that is suitable for our further analysis of the link invariants. Also we argue that in this parametrization the commutativity of flows along \( s \) and \( t \) is evident.

As has been shown in the section 3.1, the tangent vector of the auxiliary line

\[
\mathbf{x}'(s) = \mathbf{x}(s) + \mathbf{p}(\mathbf{x}(s), t)dt
\]

(22)

is not equal to the electric field at the moment \( t + dt \) but only collinear. Graphically it can be shown in the following parallelogram of vectors 3.3. This parallelogram is the consequence of the
formula (12). As can be seen, the upper vector is a slightly rescaled electric field. This fact can be thought of as the change in the parametrization:

\[ ds' = ds(1 + (\nabla \cdot p)dt) \]  

We are free to multiply the electric field by a function \( f(E^2) \) and the corresponding field lines will remain the same. However, the function \( f \) can be chosen in such a way that the parametrization does not change:

\[ ds' = ds \]  

In other words, the tangent vector of the auxiliary line (the upper vector in the parallelogram) will be equal to the vector \( E f(E^2) \) on the shifted line at the moment \( t + dt \). Namely:

\[
\frac{dx(s)}{ds} = E(x, t) f(E^2(x, t)) \tag{25}
\]

\[
\frac{dx'(s)}{ds} = E(x', t + dt) f(E^2(x', t + dt)) \tag{26}
\]

From the last equation in the first order in \( dt \) we obtain the following constraint on the function \( f \):

\[
(\nabla \cdot p) \left( f + E^2 \frac{\partial f}{\partial E^2} \right) = 0 \tag{27}
\]

Note that if \( (\nabla \cdot p) = 0 \), the equation (26) will be satisfied automatically with any function \( f \). Otherwise, if \( (\nabla \cdot p) \neq 0 \), the function is determined uniquely:

\[ f(E^2) = \frac{1}{E^2} \tag{28} \]

To simplify the formulas, let us introduce the rescaled fields:

\[ e := \frac{E}{E^2} \tag{29} \]

Note that the rescaling of the fields does not spoil the null field conditions (4). In these new notations both the analog of the formula (12) and the parallelogram 3.3 look simpler and do not contain the term with \( (\nabla \cdot p) \):

\[
(p \cdot \nabla)e + \frac{\partial e}{\partial t} = (e \cdot \nabla)p \tag{30}
\]

This formula is crucial for our analysis of the topological structure of the field lines. The rescaling of the electric fields does not change the structure of the field lines, however the formulas and calculations in the new notations are much simpler.

As was discussed in the previous section 3.1, each point of the field line moves over time with the velocity \( p \). Therefore, the system of the equations on the field lines takes the following form:

\[
\frac{dx(s, t)}{ds} = e(x(s, t), t) \tag{31}
\]

\[
\frac{dx(s, t)}{dt} = p(x(s, t), t) \tag{32}
\]
Using the equation (30), we obtain the following property:

\[
\frac{de(x(s,t),t)}{dt} = \frac{dp(x(s,t),t)}{ds}
\]

This property ensures commutativity of flows along \( s \) and \( t \) in the chosen parametrization.

4 Knot invariants as invariants of motion

There are solutions of Maxwell equations that exhibit topologically nontrivial structure of the electric field lines that makes them worth studying. Firstly, some solutions have the property that all/some of the field lines are closed loops. Secondly, a particular closed field line can be topologically nontrivial i.e. it can be a knot. Thirdly, a pair of closed field lines can be linked.

A closed field line could be identified with some string or strand. Each element of the string moves with velocity equal to the Poynting vector at the corresponding point of the space. As soon as one identifies the field lines with moving strands, natural questions appear. One can ask if the topological structure of the strands is preserved over time. Namely, if a strand was knotted, can it become a different knot? If a pair of strands was linked together, can they become unlinked at a further moment of time? These questions are quite similar and refer to the situation that involves crossings of strands.

There are tools to address this type of questions. They are called knot and link invariants. These invariants are preserved under smooth deformations of knots and links that do not admit crossings of lines. In our work we use the Gauss linking integral [9] which is an example of a link invariant:

\[
\text{Link}_{x,y} = \oint dx_\alpha \oint dy_\beta \epsilon_{\alpha\beta\gamma} \frac{x_\gamma - y_\gamma}{|x - y|^3}
\]

where \( x_\alpha, y_\beta \) are parametrizations and their indices \( x, y \) reflect belonging to the particular line. The linking integral computes a numerical invariant widely known as the linking number. Therefore, we show that the linking integral computed for a pair of electric field lines does not change over time, i.e. it is an integral of the Poynting evolution. This fact gives the answer for the second question, while the answer for the first one involves non-Abelian generalization of the Gauss integral and we leave it for future studies.

4.1 Gauss linking integral

In this section we consider an application of the Gauss linking integral to the study of the structure of the field lines. We show that the linking integral applied for a pair of field lines is preserved under the time evolution of the field lines induced by the Poynting vector. The linking integral is a topological invariant and it is defined for a pair of closed lines \( x(s_x), y(s_y) \):

\[
\text{Link}_{x,y} = \int ds_x ds_y \epsilon_{\alpha\beta\gamma} \frac{dx_\alpha}{ds_x} \frac{dy_\beta}{ds_y} \frac{x_\gamma - y_\gamma}{|x - y|^3}
\]

where \( s_x, s_y \) are parametrizations and their indices \( x, y \) reflect belonging to the particular line. The linking integral computes a numerical invariant widely known as the linking number. Therefore,
up to the normalization factor this integral is integer-valued. The linking number shows how many times one line winds around another.

We consider a null-field solution of the Maxwell equations with the property that all its field lines are closed. Such solutions exist and we discuss the particular example in s.6.1. As was shown in s.3.3, time independent parametrization of the electric field lines is obtained by rescaling the fields as (29):

\[ \mathbf{e} = \mathbf{E}/\mathbf{E}^2 \]  

In this parametrization a field line obeys the following equations:

\[ \frac{dx(s, t)}{ds} = e(x(s, t), t) \]  
\[ \frac{dx(s, t)}{dt} = p(x(s, t), t) \]  

The crucial property of this parametrization is that the flows along \( s \) and \( t \) commute:

\[ \frac{d\mathbf{e}(x(s, t), t)}{dt} = \frac{d\mathbf{p}(x(s, t), t)}{ds} \]  

We pick up two distinct field lines \( x(s_x, t), y(s_y, t) \) and consider the corresponding linking integral. Taking into account the fact that the tangent vectors to the field lines are rescaled electric fields (37), we represent the linking integral in the form:

\[ \text{Link}_{x(t), y(t)} = - \int ds_x ds_y \epsilon_{\alpha \beta \gamma} e_{\alpha}^x e_{\beta}^y \frac{\partial}{\partial x_\gamma} \frac{1}{|x - y|} \]  

where we simplify the notation \( e_{\alpha}^x := e_\alpha(x(s_x, t), t) \) and \( x_\gamma := x_\gamma(s_x, t) \). The time dependence of the whole linking integral is encoded in the time dependence of the field lines. Time evolution of the field lines in turn is induced by the Poynting vector (38). We also used the fact that the ratio in the integrand (35) is a total derivative:

\[ \frac{x_\gamma - y_\gamma}{|x - y|^2} = -\frac{\partial}{\partial x_\gamma} \frac{1}{|x - y|} \]  

To show that the integral (40) is time independent, we calculate its time derivative:

\[ \frac{d}{dt} \text{Link}_{x(t), y(t)} = \int ds_x ds_y \epsilon_{\alpha \beta \gamma} \left[ \frac{d e_\alpha^x}{dt} e_\beta^y \frac{\partial}{\partial x_\gamma} \frac{1}{|x - y|} + e_\alpha^x \frac{d e_\beta^y}{dt} \frac{\partial}{\partial x_\gamma} \frac{1}{|x - y|} + e_\alpha^x e_\beta^y \frac{d}{dt} \frac{\partial}{\partial x_\gamma} \frac{1}{|x - y|} \right] \]  

For the first two terms in the integrand we change the derivatives of time to the derivatives along the field line using the property (39). Integrating these terms by parts, we obtain the following expression:

\[ \frac{d}{dt} \text{Link}_{x(t), y(t)} = \int ds_x ds_y \epsilon_{\alpha \beta \gamma} \left[ e_\beta^x (p_\alpha^x e_\alpha^x - p_\alpha^y e_\alpha^y) - e_\alpha^x (p_\beta^x e_\beta^x - p_\beta^y e_\beta^y) \right] \frac{\partial}{\partial x_\gamma} \frac{\partial}{\partial x_\lambda} \frac{1}{|x - y|} \]  

Using the properties of the null-field solutions (11), we have an expression of the form:

\[ \frac{d}{dt} \text{Link}_{x(t), y(t)} = \int ds_x ds_y \left[ \left( e_\alpha^x B_\alpha^x + e_\alpha^y B_\alpha^y \right) \frac{\partial}{\partial x_\lambda} \frac{\partial}{\partial x_\lambda} \frac{1}{|x - y|} - \left( e_\alpha^x B_\beta^x + e_\alpha^y B_\beta^y \right) \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \frac{1}{|x - y|} \right] \]  

In the first terms 3d Dirac delta function appears:

\[ \frac{\partial}{\partial x_\lambda} \frac{\partial}{\partial x_\lambda} \frac{1}{|x - y|} = -4\pi \delta^{(3)}(x - y) \]  

The last terms turn out to be total derivatives and finally we obtain the following:

\[
\frac{d}{dt} \text{Link}_{x(t), y(t)} = - \int ds_x ds_y \frac{4\pi \delta^{(3)}(x - y)}{4\pi |x - y|} \left( (e^z \cdot B^y) + (e^y \cdot B^z) \right) - \int ds_x ds_y \left[ \frac{d}{ds_y} \left( \frac{B^x}{4\pi |x - y|} \right) + \frac{d}{ds_x} \left( \frac{B^y}{4\pi |x - y|} \right) \right] \tag{46}
\]

The first integral may contribute because of the presence of the delta function, but it is multiplied by \( e^z \cdot B^y \) and \( e^y \cdot B^z \) vanishing when \( x - y = 0 \) due to the null-field condition (4), so the integrand is zero. The second integral is zero because the integrand is a sum of total derivatives and the integral is along the closed line. So we conclude that the linking integral of a pair of electric field lines is preserved under the time evolution:

\[
\frac{d}{dt} \text{Link}_{x(t), y(t)} = 0 \tag{47}
\]

We note that the conservation of the linking integral is a consequence of the null-field condition and closeness of the lines. Our reasoning does not involve a direct verification that the field lines do not cross.

4.2 Non-Abelian knot/link invariants

At the present moment the colored HOMFLY-PT polynomial is one of the most powerful knot and link invariants [10, 11, 12]. See [13] for recent results in this area. It is defined as the vacuum expectation value of the Wilson loop operators along \( n \) components of the link \( \mathcal{L} \) in the 3d Chern-Simons theory on \( S^3 \) with the gauge group \( SU(N) \):

\[
H_{\mathcal{L}_1, \ldots, \mathcal{L}_n}^{R_1, \ldots, R_n} = \left\langle \prod_{i=1}^n \text{tr}_{R_i} P \text{exp} \left( \oint_{\mathcal{L}_i} A \right) \right\rangle_{CS}, \tag{48}
\]

where Chern-Simons action is given by

\[
S_{CS}[A] = \frac{k}{4\pi} \int_{S^3} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{49}
\]

Here \( R_i \) are the representations of the gauge group and \( \mathcal{L}_i \) are link components.

The Gauss linking integral naturally appears in the Abelian version of Chern-Simons theory. In the non-Abelian case it also arises in the perturbative expansion of the HOMFLY-PT invariant for a link in the limit \( k \to \infty \) [14]. Actually, the non-Abelian generalization provides an infinite family of the knot/link invariants appearing in the perturbative expansion of the HOMFLY-PT invariant. These invariants are widely known as Vassiliev invariants [15, 16, 17] and can be represented as the complicated contour integrals along knots/links. For example, we provide an explicit form of the second Vassiliev invariant of a link component \( \mathcal{L}_i \):

\[
\rho(\mathcal{L}_i) = \frac{1}{8\pi^2} \oint_{\mathcal{L}_i} dx \int dy \int dz \int dw, \epsilon_{\sigma_{\alpha\beta\gamma}} \epsilon_{\rho_{\mu\nu}} \frac{(w - y)_\alpha(z - x)_\beta}{|w - y|^3 |z - x|^3} - \frac{1}{32\pi^3} \oint_{\mathcal{L}_i} dx \int dy \int dz \int dw, \epsilon_{\alpha\beta\gamma}, \epsilon_{\mu\alpha\sigma}, \epsilon_{\nu\beta\lambda}, \epsilon_{\tau\rho\sigma} \int_{\mathbb{R}^3} d^3 w, \frac{(w - x)_\alpha(w - y)_\lambda(w - z)_\tau}{|w - x|^3 |w - y|^3 |w - z|^3} \tag{50}
\]

The similar analysis as in the section 4.1 can be carried out for these integrals when the curve \( \mathcal{L}_i \) is the field line. The application of this invariant will give answers to the questions about knot structure of the evolving field lines. We leave this question for future studies.

These integral-type invariants are obtained in the Lorentz gauge. It would be interesting to make a connection with other gauges and lift-up the story to the level of the \( R \)-matrices. For possible approach see [18].
5 Integrability

It is a well known fact that the Gauss linking integral is connected to the electric/magnetic helicity [19]:

\[ h_e = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \, E(x) \cdot \frac{E(y) \times (x - y)}{|x - y|^3} \] (51)

\[ h_m = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \, B(x) \cdot \frac{B(y) \times (x - y)}{|x - y|^3} \] (52)

These formulas are similar to the Gauss linking integral for two closed field lines (40). In this terms the helicity can be obtained by "summing" the linking integrals over all pairs of field lines:

\[ h_e = \sum_{i \neq j} \oint ds_i \oint ds_j \, E(x_i) \cdot \frac{E(x_j) \times (x_i - x_j)}{|x_i - x_j|^3} \] (53)

Beside the fact that the sum over field lines is not rigorously defined, one can think of the helicity as the average linking number of field lines. It turns out that these quantities are integrals of motion in the null-field solutions of Maxwell equations:

\[ \frac{dh_e}{dt} = \frac{dh_m}{dt} = \int_{\mathbb{R}^3} dx \, E(x) \cdot B(x) = 0 \] (54)

If we believe in conservation on knot/link invariants for closed field lines, we could attempt to produce integrals of motion in the same way as helisity - "summing" over all field lines:

\[ \text{Integral of motion} = \sum_{\text{Field lines}} \text{knot/link invariant} \] (55)

We should note that we propose this algorithm only as a hint to search of another integrals of motion in null-field solutions. However, if this strategy is correct and invariants of knots/links are connected with nontrivial integrals of motion, it implies that the system of Maxwell equations coupled with null-field condition is actually integrable.

6 Electromagnetic knots

In this section we consider particular examples of solutions of the Maxwell equations that exhibit nontrivial behaviour of the field lines. In recent papers [5, 6] a family of source-free finite action solutions of Maxwell equations was constructed. The solutions are interesting because the field lines are similar to knots and links. The electric and magnetic fields as functions of the coordinates are the rational functions and therefore the solutions are called rational electromagnetic fields. We note that the zeros of the denominator are complex, so the solutions do not have singularities.

6.1 Ranada’s Hopf solution

Our main example is the celebrated Hopf-Ranada solution:

\[ E + iB = \frac{1}{((t - i)^2 - r^2)^2} \left( \begin{array}{c} (x - iy)^2 - (t - i - z)^2 \\ i(x - iy)^2 + i(t - i - z)^2 \\ -2(x - iy)(t - i - z) \end{array} \right) \] (56)

This solution is the first member of the family. Here we list the properties of the field lines that were observed on the computer simulations (see Fig.2):

- All field lines are closed.
- Every field line is topologically unknot.
• Any two field lines are linked.
• Links are preserved over time.

The last observation was a hint to consider the Gauss linking integral and demonstrate this property analytically. To make an attempt to explain the other properties, we consider the simplified version of the electric field, namely the electric field far from the origin. On the coordinate scales much larger that the time $t$ the electric field has the form:

$$E_\infty(x) = \frac{1}{r^6} \begin{pmatrix} -x^2 + y^2 + z^2 \\ -2xy \\ -2xz \end{pmatrix}$$  \hspace{1cm} (57)$$

This electric field defines an integrable system of differential equations on the field lines and the solution has the following form:

$$x^2 + (y \cos \theta + z \sin \theta - b)^2 = b^2$$  \hspace{1cm} (58)$$

where all values of the radius $b$ and the angle $\theta$ are possible. The circles are normal to the plane $yz$, pass through the origin and their centers lie on the plane $yz$. This fact is in agreement with the Fig.2 because one can see that a part of a field line is similar to a circle (58).

We note that the simplified version of the electric field at large scales (57) defines closed field lines (58). The fact that the simplified field at large scales defines an integrable system of differential equations and closed field lines simultaneously might not be by chance. The integrability is connected with existence of the function that is constant along field lines (58):

$$h(x) = \frac{\sqrt{y^2 + z^2}}{x^2 + y^2 + z^2}$$  \hspace{1cm} (59)$$

$$\frac{d}{ds} h(x(s)) = 0, \quad \text{where} \quad \frac{dx(s)}{ds} = E_\infty(x)$$  \hspace{1cm} (60)$$

The simplified version of the field (57) is effectively two dimensional and one "integral of motion" $h(x)$ is sufficient to determine the field lines. In case of the full Hopf-Ranada solution two "integrals" might explain the closedness of the field lines.
6.2 Other knots/links

The higher members of the family of knot solutions exhibit more complicated structure of the field lines. Almost all field lines are tightly wound and we conjecture that they have infinite length (see Fig.3). However, there are rare field lines in space that tend to be closed (see Fig.4). In computer simulations we observe that they tend to be closed and conjecture that exactly closed field lines exist. These field lines represent various knots. Another observation is that non-closed field lines tightly wind around the closed ones as in the Fig.3. It would be interesting to find an efficient description of this complicated structure and classify knots and links that appear in null-field solutions. For a possible approach to the problem see [6].

7 Conclusion

Long ago hidden integrability was predicted to be one of the governing principles for dynamics in stringy models [20, 21, 22, 23]. Since then this was proved to be the case in quite a number
of examples. In this paper we suggest to search for integrability in generic solutions of Maxwell equations. The point is to reformulate Maxwell dynamics in terms of behavior of the field lines. At a given time we can define a system of non-intersecting world lines, and Maxwell dynamics convert them onto a system of world surfaces. Accordingly there are two directions \( s \) and \( t \) and two kinds of dynamics and, potentially, integrability – for \( s \) and \( t \)-evolutions. We provided some evidence that both kinds of integrability are really present, and \( t \)-integrability is related to conservation of topological invariants, like knot polynomials and Vassiliev coefficients of their expansions. It would be very interesting to develop these arguments into a full-fledged theory – most probably this will include non-Abelian considerations. In particular, at the \( SU(2) \) level one can exploit peculiar properties of Lorentz and conformal groups in 4d [5]. In another direction, one can relate evolution of field lines with loop equations and integrability of eigenvalue matrix models [24, 25, 26]. We hope for a new and profound progress on these issues, which would enrich our understanding of integrability of effective theories with the help of a very concrete and down-to-earth story of Maxwell equations.

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