Variations of the \(q\)-Garnier system

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Abstract
We study several variations of the \(q\)-Garnier system corresponding to various directions of discrete time evolutions. We also investigate a relation between the \(q\)-Garnier system and Suzuki’s higher order \(q\)-Painlevé system using a duality of the \(q\)-KP system.

Keywords: \(q\)-Garnier system, \(q\)-Painlevé equation, Lax pair, \(q\)-KP system, Painlevé method

1. Introduction

The \(q\)-Garnier system is a multivariable extension of the \(q\)-Painlevé VI equation \([7]\). It was first formulated in \([26]\) and has been studied in \([20, 23, 27]\) from different points of view.\(^4\) For the differential Garnier system, two kinds of deformation have been considered: the continuous shift of the position of singular points and the discrete shift of exponents (Schlesinger transformations), and these two deformations are usually considered in different manners. For difference Garnier system, since both kinds of deformation are discrete, it is natural to expect that those deformations can be treated in a unified way.

The aim of this paper is to consider such variations of \(q\)-Garnier system corresponding to various deformation directions. The main problems and results are as follows.

- In order to give various \(q\)-Garnier systems a simple form, we need to choose dependent variables for each direction appropriately. We will give a method to find such variables using contiguity type scalar Lax pairs.
- In \([20]\) a connection between Suzuki’s higher order \(q\)-Painlevé system \([28, 29]\) and the \(q\)-Garnier system is suggested by comparing their special solutions. However, the relation is not so obvious since their Lax forms are apparently very different. We will clarify the connection using a duality of the \(q\)-KP system \([8, 9]\).

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\(^4\) Other discrete analogues such as elliptic and additive Garnier systems have been studied in \([21, 24, 37]\) and \([2, 3]\).
This paper is organized as follows. In section 2, based on our previous work [20], we rewrite Sakai’s $2 \times 2$ matrix Lax form into a scalar Lax form. In section 3, we exemplify the scalar Lax pairs and the evolution equations for the various $q$-Garnier systems by choosing the appropriate dependent variables. In section 4, as an application of the results in section 3, we give two reductions from case $N=3$ of the various $q$-Garnier systems to $q$-Painlevé systems of type $E^{(1)}_7$. In section 5, extending the duality for reduced $q$-KP system [8, 9], we show the equivalence between Suzuki’s system and a certain variation of the $q$-Garnier system.

2. $q$-Garnier system

In this section we recall Sakai’s $2 \times 2$ matrix Lax form [26] for the $q$-Garnier system and its corresponding scalar Lax form [20].

2.1. Sakai’s matrix Lax form

In [26], the $q$-Garnier system has been formulated by Sakai. For an unknown function $Y(z) = \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}$, the $2 \times 2$ matrix Lax form for the $q$-Garnier system is given by

$$Y(qz) = A(z)Y(z), \quad (2.1)$$

$$Y(z) = B(z)Y(z). \quad (2.2)$$

Here $q$ is a (complex) parameter, and $A$ and $B$ are $2 \times 2$ matrices depending on $z$ and other parameters. The matrix $A$ is a polynomial in $z$ (see properties (2.4) below); on the other hand, the matrix $B$ is rational in $z$. We denote the time evolution by $T(\ast) = \ast$. The compatibility condition for the systems (2.1) and (2.2) leads

$$A(z)B(z) = B(qz)A(z), \quad (2.3)$$

which gives the $q$-Garnier system. Let $\alpha_1, \ldots, \alpha_{2N+2}, \kappa_1, \kappa_2, \theta_1$ and $\theta_2$ be (complex) parameters. Then the matrix $A$ is characterized by the following properties:

(i) $A(z) = A_0 + A_1z + \cdots + A_{N+1}z^{N+1}$,

(ii) $A_{N+1} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$, $A_0$ has eigenvalues $\theta_1$ and $\theta_2$,

(iii) $|A(z)| = \kappa_1 \kappa_2 \prod_{i=1}^{2N+2} (z - \alpha_i)$, $\prod_{i=1}^{2N+2} \alpha_i = \theta_1 \theta_2). \quad (2.4)$

The matrix $A$ satisfying these properties (i)–(iii) has $2N+1$ arbitrary parameters except parameters $\{\alpha_i, \kappa_i, \theta_i\}$. The $2N$ parameters among them can be interpreted as variables for the $q$-Garnier system and the remaining parameter is a gauge freedom. The deformation direction in [26] is given by $\overline{\pi}_i = q\alpha_i, \pi_j = \alpha_j, \overline{\pi}_i = \kappa_i$ and $\overline{\theta}_i = q\theta_i (i \in \{r,s\}, j \in \{1, \ldots, 2N+2\} \setminus \{r,s\})$.

We must choose appropriate coordinates of variables according to the deformation directions in order to obtain simple forms of the $q$-Garnier systems. This content will be shown in section 3.

5 The matrix $B$ of the equation (2.1) can be chosen according to various deformation directions.
2.2. Scalar Lax form

We recall the scalar Lax form and derive the simple form of the time evolution equation from its compatibility condition.

**Lemma 2.1.** The systems (2.1) and (2.2) can be rewritten into the following linear $q$-difference equations:

\[
L_1(z) := F\left(\frac{z}{q}\right)y(qz) - \left\{ F\left(\frac{z}{q}\right)A_{11}(z) + F(z)A_{22}\left(\frac{z}{q}\right) \right\} y(z) + F(z)|A\left(\frac{z}{q}\right)|y\left(\frac{z}{q}\right) = 0,
\]

\[
L_2(z) := F(z)\overline{y}(z) - H(z)y(qz) + G(z)y(z) = 0,
\]

\[
L_3(z) := \overline{T}(z)|B(z)|y(qz) - H(qz)|A(z)|\overline{y}(z) + G(z)\overline{y}(qz) = 0.
\]

Here,

\[
y(z) = y_1(z), \quad F(z) = A_{12}(z), \quad G(z) = A_{11}(z)B_{12}(z) - A_{12}(z)B_{11}(z), \quad H(z) = B_{12}(z).
\]

**Proof.** We write the system (2.1) as:

\[
\begin{aligned}
&y_1(qz) = A_{11}(z)y_1(z) + A_{12}(z)y_2(z), \\
y_2(qz) = A_{21}(z)y_1(z) + A_{22}(z)y_2(z).
\end{aligned}
\]

Eliminating the component $y_2(z)$ from the equation (2.7), we obtain the linear equation $L_1 = 0$ in (2.5). The system (2.2) is also rewritten as:

\[
\begin{aligned}
&\overline{y}_1(z) = B_{11}(z)y_1(z) + B_{12}(z)y_2(z), \\
&\overline{y}_2(z) = B_{21}(z)y_1(z) + B_{22}(z)y_2(z).
\end{aligned}
\]

Eliminating the component $y_2(z)$ from the first equations in (2.7) and (2.8), we obtain the deformation equation $L_2 = 0$ in (2.5).

On the other hand, rewriting (2.1) and (2.2) into:

\[
\overline{T}(qz) = \overline{A}(z)\overline{y}(z), \quad \overline{y}(qz) = B(qz)\overline{y}(qz),
\]

and eliminating $y_2(qz)$, $\overline{y}_1(z)$ and $\overline{y}_2(qz)$ from the system (2.9), then we have:

\[
\overline{A}_{12}(z)|B(qz)|y(qz) - H(qz)|A(z)|\overline{y}(x) - \{ A_{12}(z)B_{22}(qz) - \overline{A}_{22}(z)B_{12}(z) \} y(qz) = 0.
\]

Moreover applying the compatibility condition (2.3) to (2.10), we can derive the deformation equation $L_3 = 0$ in (2.5).

We remark that unknown variables of $q$-Garnier systems appear as the coefficients in $z$ of the polynomials $F$, $G$ and $H$. The polynomial $F$ is of degree $N$ in $z$ and the $N$ points such that $F(z) = 0$ are not singularities but ‘apparent singularities’ (i.e. the solutions are regular there). This means that the two relations:
\[ \frac{y(qz)}{y(z)} = A_{11}(z), \quad \text{for} \quad F(z) = 0, \]
\[ \frac{y(z)}{y(z)} = \frac{|A(\frac{z}{q})|}{A_{22}(\frac{z}{q})}, \quad \text{for} \quad F(\frac{z}{q}) = 0, \quad (2.11) \]
derived from \( L_1 = 0 \) in (2.5) are consistent with each other.

A pair of the deformation equations \( L_2 = L_3 = 0 \) in (2.5) is equivalent to the scalar Lax pair \( L_1 = 0 \), so we call the pair \( L_2 = L_3 = 0 \) the contiguity type scalar Lax pair. We conveniently deal with the Lax pair \( L_2 = L_3 = 0 \) [22], since the explicit form of the equation \( L_1 = 0 \) is rather complicated.

**Lemma 2.2.** The compatibility of the scalar Lax pair \( L_2 = L_3 = 0 \) in (2.5) gives the following relation:
\[ G(z)G(z) = H(z)H(qz)|A(z)|, \quad \text{for} \quad F(z) = 0, \]
\[ |B(z)|F(z)\overline{F}(z) = H(z)H(qz)|A(z)|, \quad \text{for} \quad G(z) = 0. \quad (2.12) \]

**Proof.** Under the condition \( F(z) = 0 \), eliminating \( y(z) \) and \( y(qz) \) from \( L_2(z) = L_3(qz) = 0 \), we obtain the first relation of (2.12). Similarly, for \( G(z) = 0 \), eliminating \( y(qz) \) and \( \overline{y}(z) \) from \( L_2(z) = L_3(qz) = 0 \), we have the second relation of (2.12).

The relation (2.12) may be insufficient for the compatibility of the Lax pair \( L_2 = L_3 = 0 \). Additional conditions for the sufficiency will be considered for each case in section 3.

Though we have derived the scalar Lax pair from the matrix Lax form, the scalar Lax pair \( L_2 = L_3 = 0 \) can also be obtained from a certain method using Padé interpolation (see for example [20]). In the following sections, we will discuss based on the Lax pair obtained by the Padé method

### 3. Variations of the \( q \)-Garnier system

In this section, we consider Lax pairs and evolution equations for several directions of \( q \)-Garnier systems.

#### 3.1. Notation

Fix a positive integer \( N \) and a parameter \( q \). Let \( a_1, \ldots, a_N, b_1, \ldots, b_N, c_1, c_2, d_1, d_2 \) be parameters with a constraint \( \prod_{i=1}^{N+1} \frac{a_i}{b_i} = q \prod_{i=1}^{2} \frac{c_i}{d_i} \) and \( T_a : a \mapsto qa \) be the \( q \)-shift operator of parameter \( a \). In this section, we use the following notations:
\[ A(z) = \prod_{i=1}^{N+1} (z - a_i), \quad B(z) = \prod_{i=1}^{N+1} (z - b_i), \quad A_i(z) = \frac{A(z)}{z - a_i}, \quad B_i(z) = \frac{B(z)}{z - b_i}, \]
\[ A_{ij}(z) = \frac{A(z)}{(z - a_i)(z - a_j)}, \quad B_{ij}(z) = \frac{B(z)}{(z - b_i)(z - b_j)}, \quad F(z) = \sum_{i=0}^{N} f_i z^i, \quad (3.1) \]

\(^6\)For the proof of the correspondence between (2.5) and (3.11), see appendix A.1 as an example.
where $f_0, \ldots, f_N$ are variables depending on parameters $a_i, b_i, c_i$ and $d_i$.

We consider the following four directions $T_1, \ldots, T_4$ defined by:

$$T_1 = T_{a_1}^{-1} T_{b_1}^{-1}, \quad T_2 = T_{a_1}^{-1} T_{b_1}^{-1} T_{b_2}^{-1}, \quad T_3 = T_{b_1} T_{d_1}, \quad T_4 = T_{a_{e+1}}^{-1} T_{c_1}^{-1},$$

and the corresponding shifts are denoted as $\mathbf{X} := T_i(X)$ and $\mathbf{X} := T_i^{-1}(X)$. The operators $T_i$ play the role of time evolutions of the $q$-Garnier system. The directions $T_1$, $T_3$, $T_4$ (and $T_{a_i}^{-1} T_{b_i}^{-1}$, $T_1 T_{d_1}$, $T_{a_i}^{-1} T_{c_1}^{-1}$ obtained by obvious symmetry from $T_1, T_3$ and $T_4$) are fundamental ones, and others (e.g. $T_2$) are given by compositions of them (and their inverses).

The following data will be described by the polynomials $F(z), G(z)$ and $H(z) = 1 + b_1 z$. The polynomial $F$ is common for all cases as given above, while one should take the polynomials $G$ and $H$ differently in sections 3.2.1–3.2.4 as given below (see section 3.2.5).

3.2. Results for each direction

In the following subsections, we show the three items: (a) scalar Lax pair, (b) time evolution equation, and (c) the $L_1$ equation.

In item (b), we give the time evolution equation as necessary and sufficient condition for compatibility of the scalar Lax pair (for the proof of the sufficiency, see appendix A.2). In item (c), the $L_1$ equations in each subsection are expressed in different forms, however, we will show that they are equivalent with each other (theorem 3.1).

3.2.1. Direction $T_1 = T_{a_1}^{-1} T_{b_1}^{-1}$. This case is considered in [20] and the direction $T_1$ is an inverse of the original direction in [26].

(a) Scalar Lax pair:

$$L_2(z) = F(z)\gamma(z) - A_1(z)\gamma + (z - b_1)G(z)\gamma(z) = 0,$$

$$L_3(z) = T_1^{-1}(z)\gamma(z) + (z - a_1)G(z)\gamma(z) = q^{-1} c_2 B_1(z) \gamma(z) = 0,$$

where $A_1(z), B_1(z), F(z)$ are as in (3.1) and $G(z) = \sum_{i=0}^{N-1} g_i z^i$ and $H(z) = 1$.

(b) Time evolution equation:

$$G(z) F(z) = c_1 c_2 \frac{A_1(z) B_1(z)}{(z - a_1)(z - b_1)} \quad \text{for} \quad F(z) = 0,$$

$$F(z) T_1(z) = q c_1 c_2 A_1(z) B_1(z) \quad \text{for} \quad G(z) = 0,$$

$$f_N q = q (g_{N-1} - c_1)(g_{N-1} - c_2),$$

$$f_0 q = a_1 b_1 (g_0 - \frac{d_1}{a_1 b_1} A(0)) \left( g_0 - \frac{d_2}{a_1 b_1} A(0) \right),$$

where $2N$ variables $\frac{f_0}{f_N}, \ldots, \frac{f_1}{f_N}, g_0, \ldots, g_{N-1}$ are the dependent variables.

The data for each subsection is obtained by considering the same Padé problem (see appendix B).
(c) The $L_1$ equation:

$$L_1(z) = A(z)F\left(\frac{z^2}{q}\right)y(qz) + qc_1c_2B\left(\frac{z^2}{q}\right)F(z)y\left(\frac{z}{q}\right)$$

$$- \left\{ (z - a_1)(z - b_1)F\left(\frac{z^2}{q}\right)G(z) + \frac{F(z)}{G\left(\frac{z}{q}\right)} V\left(\frac{z}{q}\right) \right\} y(z) = 0,$$

where $V(z) = qc_1c_2A(z)B(z) - F(z)\bar{F}(z)$.

3.2.2. Direction $T_2 = T_{a_1}^{-1} T_{b_1}^{-1} T_{d_1}^{-1} T_{b_2}^{-1}$.

(a) Scalar Lax pair:

$$L_2(z) = F(z)y(z) - A_{1,2}(z)(1 + hz)y(qz) + (z - b_1)(z - b_2)G(z)y(z) = 0,$$

$$L_3(z) = \bar{F}\left(\frac{z^2}{q}\right)y(z) + (z - a_1)(z - a_2)G\left(\frac{z^2}{q}\right)y(z) - qc_1c_2B_{1,2}\left(\frac{z^2}{q}\right)(1 + hz)y\left(\frac{z}{q}\right) = 0,$$

where $A_{1,2}(z), B_{1,2}(z), F(z)$ are as in (3.1) and $G(z) = \sum_{i=0}^{N-2} g_i z^i$ and $H(z) = 1 + hz$.

(b) Time evolution equation:

$$G(z)G\left(\frac{z}{q}\right) = \frac{c_1c_2A_{1,2}(z)B_{1,2}(z)}{(1 + hz)(1 + qh^2)}, \quad \text{for} \quad F(z) = 0,$$

$$F(z)\bar{F}(z) = qc_1c_2A_{1,2}(z)B_{1,2}(z)(1 + hz)(1 + qh^2), \quad \text{for} \quad G(z) = 0,$$

$$F(z)\bar{F}(z) = (z - a_1)(z - a_2)(z - b_1)(z - b_2)G(z)G\left(\frac{z}{q}\right), \quad \text{for} \quad 1 + hz = 0,$$

$$f_0 f_{\bar{N}_2} = q^2 g_{N-2} - c_1 h)(g_{N-2} - c_2 h),$$

$$f_0 f_{\bar{N}_2} = \frac{g_0}{g_{0}} - (g_0 - \frac{d_1 A(z)}{a_1 a_2 b_1 b_2}) \left( g_0 - \frac{d_1 A(z)}{a_1 a_2 b_1 b_2} \right),$$

where $2N$ variables $\frac{a_1}{a_2}, \ldots, \frac{a_1}{a_2}, g_0, \ldots, g_{N-2}$ and $h$ are the dependent variables.

(c) The $L_1$ equation:

$$L_1(z) = A(z)F\left(\frac{z^2}{q}\right)y(qz) + qc_1c_2B\left(\frac{z^2}{q}\right)F(z)y\left(\frac{z}{q}\right)$$

$$- \frac{1}{1 + hz} \left\{ (z - a_1)(z - b_1)(z - a_2)(z - b_2)F\left(\frac{z^2}{q}\right)G(z) + \frac{F(z)}{G\left(\frac{z}{q}\right)} V\left(\frac{z}{q}\right) \right\} y(z) = 0,$$

where $V(z) = qc_1c_2(1 + hz)(1 + qh^2)A_{1,2}(z)B_{1,2}(z) - F(z)\bar{F}(z)$.

3.2.3. Direction $T_3 = T_{c_1} T_{d_1}$. The deformation direction $T_3$ is the same as that of Suzuki’s system [28] (see section 5.3).

(a) Scalar Lax pair:

$$L_2(z) = F(z)y(z) - A(z)y(qz) + G(z)y(z) = 0,$$

$$L_3(z) = \bar{F}\left(\frac{z^2}{q}\right)y(z) + G\left(\frac{z^2}{q}\right)y(qz) - qc_1c_2B\left(\frac{z^2}{q}\right)y\left(\frac{z}{q}\right) = 0,$$
where $A(z), B(z), F(z)$ are as in (3.1) and $G(z) = \sum_{i=0}^{N+1} g_i z^i, g_{N+1} = \frac{c_{\text{eq}}}{A_0(0)}$ and $H(z) = 1$.

(b) Time evolution equation:

\[
G(z)\frac{dG}{dz} = c_1c_2 A_N(z) B(z), \quad \text{for} \quad F(z) = 0,
\]

\[
zF(z)\frac{dF}{dz} = q_c c_2 A_N(z) B(z), \quad \text{for} \quad G(z) = 0,
\]

where $2N$ variables $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_N}$ are the dependent variables.

(c) The $L_1$ equation:

\[
L_1(z) = A(z) F(\frac{z}{q}) y(z) + q_c c_2 B(\frac{z}{q}) F(z) y(\frac{z}{q}) - \left\{ F(\frac{z}{q}) G(z) + \frac{F(z)}{G(\frac{z}{q})} V(\frac{z}{q}) \right\} y(z) = 0,
\]

where $V(z) = q_c c_2 A_N(z) B(z) - z F(z) F(z)$.

3.2.4. Direction $T_4 = T_{a_{N+1}}^{-1} T_{c_1}^{-1}$.

(a) Scalar Lax pair:

\[
L_2(z) = F(z) y(z) - A_{N+1}(z) y(qz) + G(z) y(z) = 0,
\]

\[
L_3(z) = T_4(\frac{z}{q}) y(z) + (z - a_{N+1}) G(\frac{z}{q}) y(z) + q_c c_2 B(\frac{z}{q}) y(\frac{z}{q}) = 0,
\]

where $A_{N+1}(z), B(z), F(z)$ are as in (3.1) and $G(z) = \sum_{i=0}^{N} g_i z^i, g_N = c_1$ and $H(z) = 1$.

(b) Time evolution equation:

\[
G(z)\frac{dG}{dz} = \frac{q_c c_2 A_{N+1}(z) B(z)}{z - a_{N+1}}, \quad \text{for} \quad F(z) = 0,
\]

\[
F(z)\frac{dF}{dz} = q_c c_2 A_{N+1}(z) B(z), \quad \text{for} \quad G(z) = 0,
\]

\[
f_{\alpha \beta} = \left( g_0 + \frac{d_A(0)}{a_{N+1}} \right) \left( g_0 + \frac{d_A(0)}{a_{N+1}} \right),
\]

where $2N$ variables $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}, g_0, \ldots, g_{N-1}$ are the dependent variables.

(c) The $L_1$ equation:

\[
L_1(z) = A(z) F(\frac{z}{q}) y(z) + q_c c_2 B(\frac{z}{q}) F(z) y(\frac{z}{q}) - \left\{ (z - a_{N+1}) F(\frac{z}{q}) G(z) + \frac{F(z)}{G(\frac{z}{q})} V(\frac{z}{q}) \right\} y(z) = 0,
\]

where $V(z) = q_c c_2 A_{N+1}(z) B(z) - F(z) F(z)$.

3.2.5. Relations among variables $g_i$ and $h$ for each direction.

**Theorem 3.1.** The $L_1$ equations for each direction $T_1, \ldots, T_4$: (3.5), (3.8), (3.11) and (3.14) are equivalent with each other if the coefficients $g_i$ in $G(z)$ and $h$ are related as:

\[7]
\[
\frac{(z - b_1)G(z) \frac{\Delta z}{A(z)}}{A_1(z)} = \frac{(z - b_1)(z - b_2) G(z) \frac{\Delta z}{A_1(z)}}{H(z) \frac{\Delta z}{A_{N+1}(z)}}
\]
\[
= \frac{G(z) \frac{\Delta z}{A(z)}}{A(z)} = \frac{G(z) \frac{\Delta z}{A_{N+1}(z)}}{A(z)}, \quad \text{for } F(z) = 0.
\]

Here \( G(z) \frac{\Delta z}{A(z)} \) means \( G(z) \) used in section 3.2.∗ and \( H(z) \frac{\Delta z}{A_{N+1}(z)} = 1 + hz. \)

\textbf{Proof.} The relation (3.15) is obtained by comparing ratios \( y(qz) \) in \( L_1 \) under the condition \( F(z) = 0 \). Then the equivalence of the \( L_1 \) equations can be understood in various ways. The equivalence is obvious if the \( L_1 \) equations are obtained as in section 2. Alternatively, if starting from a certain Padé problem, one can check the equivalence directly using characteristic properties as in the appendix A.1.

\section{4. Reduction to q-\( E_7^{(1)} \) system}

In this section we consider some \( q \)-Painlevé equations of type \( E_7^{(1)} \) as a reduction of \( q \)-Garnier systems.

\subsection{4.1. Notation}

Throughout this section, we consider the case \( N = 3 \) and specialize parameters as:
\[
c_1 = c_2, \quad d_1 = d_2.
\]

(4.1)

Among the directions \( T_i \) considered in previous section, the directions consistent with this specialization are the following two:
\[
T_1 = T_{a_1}^{-1}T_{b_1}^{-1}, \quad T_2 = T_{a_1}^{-1}T_{a_2}^{-1}T_{b_1}^{-1}T_{b_2}^{-1},
\]

(4.2)

and inconsistent ones \( T_3 \) and \( T_4 \) will be omitted. Under the specialization (4.1), we can and will impose constraints on dependent variables as:
\[
f_0 = f_1 = 0, \quad f_1 = w, \quad f_2 = -fw,
\]

(4.3)

where \( f \) is one of the unknown variables for the reduced system and \( w \) is a gauge freedom.

In the following, we also impose additional constraints (4.4) or (4.10) in order to reduce the variables \( g_i \) and \( h \) of \( q \)-Garnier systems to a variable \( g \) and a parameter \( e \). Note that the meaning of the parameter \( e \) and the reduced variable \( g \) are different depending on directions in sections 4.2.1 and 4.2.2 (see remark 4.1). The results of section 4.2.1 are known in [18, 20] and those of section 4.2.2 are new.

\subsection{4.2. Results for each direction}

\subsubsection{4.2.1. Reduction from q-Garnier system in section 3.2.1}

The direction is \( T_1 = T_{a_1}^{-1}T_{b_1}^{-1}. \) In this case we can and will impose an additional condition:
\[
g_0 = e, \quad g_1 = eg, \quad g_2 = c_1, \quad (4.4)
\]

where \( e = d_1a_2a_3a_4b_1^{-1}. \)

(a) Scalar Lax pair

Under the conditions (4.1), (4.3) and (4.4), we can reduce the Lax pair (3.3) to the following Lax pair:
\[ L_2(z) = wz(1 - fz)y(z) - \prod_{i=2}^{4} (z - a_i)y(qz) + (z - b_1)(e + egz + c_1z^2)y(z) = 0, \]

\[ L_3(z) = \frac{w^2}{q^2}(1 - \frac{fz}{q})y(z) + (z - a_1)(e + egz + c_1(z^2))y(z) - qe^2 \prod_{i=2}^{4} (\frac{z}{q} - b_i)y(\frac{z}{q}) = 0. \]

(4.5)

The Lax pair (4.5) has been given in [18].

(b) Time evolution equation and equation for \( w \)

\[ (\frac{e}{c_1}f^2 + \frac{e}{c_1}gf + 1)(\frac{e}{qc_1}f^2 + \frac{e}{qc_1}gf + 1) = \prod_{i=2}^{4} (1 - af)(1 - bf), \]

\[ \frac{z_i^2(1 - fz_1)(1 - \frac{fz_2}{q})}{z_i^2(1 - fz_2)(1 - \frac{fz_1}{q})} = \prod_{i=2}^{4} (z_i - a_i)(z_i - b_i) \]

\[ \text{and} \]

\[ w = \prod_{i=2}^{4}(z_i - a_i)(z_i - b_i) \]

(4.6)

where \( z = z_1, z_2 \) are solutions of the equation \( e + egz + c_1z^2 = 0 \). The bi-rational equation (4.6) is equivalent to the variation of \( q \)-Painlevé equation of type \( E_7^{(1)} \) in [18, 20].

We remark that the configuration of eight singular points\(^8\) is given by two points on a line \( g = \infty \) and six points on a parabolic curve \( ef^2 + egf + c_1 = 0 \) as follows:

\[ (f, g) = (\frac{1}{a_1}, \infty), (\frac{1}{b_1}, \infty), (\frac{1}{a_i} - \frac{1}{a_i} - \frac{a_c}{e}), (\frac{1}{b_i} - \frac{1}{b_i} - \frac{b_c}{e}), \quad (i = 2, 3, 4). \]

(4.8)

(c) The \( L_1 \) equation

\[ L_1(z) = \prod_{i=1}^{4} (z - a_i)(1 - \frac{fz}{q})y(qz) + q^2c_1^2 \prod_{i=1}^{4} (\frac{z}{q} - b_i)(1 - fz)y(\frac{z}{q}) \]

\[ - \{ (z - a_i)(z - b_i)(1 - \frac{fz}{q})\varphi(z) + q(1 - fz)\frac{\varphi(\frac{z}{q})}{\varphi(z)}V(z) \} y(z) = 0. \]

(4.9)

where \( V(z) = qe^2 \prod_{i=1}^{4}(z - a_i)(z - b_i) - w^2(1 - fz)(1 - \frac{fz}{q}) \) and \( \varphi(z) = e + egz + c_1z^2 \).

The \( L_1 \) equation (4.9) has been given in [18].

4.2.2. Reduction from \( q \)-Garnier system in section 3.2.2. The direction is \( T_2 = T_{a_1}^{-1}T_{a_2}^{-1}T_{b_1}^{-1}T_{b_1}^{-1} \).

We impose an additional condition:

\[ g_0 = e, \; \; \; g_1 = -\frac{e}{g}, \; \; \; h = -\frac{e}{c_1g}. \]

(4.10)

where \( e = d_1a_1a_2(b_1b_2)^{-1} \).

(a) Scalar Lax pair

\(^8\)For the theory of the configuration of eight singular points, see [10, 25].
Under the conditions (4.1), (4.3) and (4.10), one can reduce the Lax pair (3.6) to the following Lax pair:

\[ L_2(z) = wz(y(z) - 4 \prod_{i=3}^{4}(z - a_i)(1 - \frac{ez}{c_1g})y(qz) + e \prod_{i=1}^{2}(z - b_i)(1 - \frac{z}{qg})y(z) = 0, \]

\[ L_3(z) = \frac{w^2}{q}(1 - \frac{f_z}{q})y(z) + e \prod_{i=1}^{2}(z - a_i)(1 - \frac{z}{qg})y(z) - qe^2 \prod_{i=3}^{4}(z - b_i)(1 - \frac{ez}{c_1g})y(\frac{z}{q}) = 0. \]

The scalar Lax pair (4.11) is equivalent to the one in [10, 35].

(b) Time evolution equation and equation for w

\[ \frac{(ezfg - 1)(ezfg - 1)}{(fg - 1)(fg - 1)} = \prod_{i=1}^{2}(1 - a_ifg)(1 - b_ifg), \]

\[ \frac{(fg - \frac{e}{z})(fg - \frac{e}{q})}{(fg - 1)(fg - 1)} = \prod_{i=1}^{2}(g - a_i)(g - b_i). \]

\[ w\bar{w} = \frac{\prod_{i=1}^{4}(g - a_i)(g - b_i)}{qg^2(1 - e)(g - qe)(fg - 1)(fg - 1)}. \]

The bi-rational equation (4.12) is equivalent to the \(q\)-Painlevé equation of type \(E_7^{(1)}\) in [4, 10, 25, 35], and the configuration of 8 singular points is the well-known standard one.

(c) The \(L_4\) equation

\[ L_4(z) = \prod_{i=1}^{4}(z - a_i)(1 - \frac{f_z}{q})y(qz) + q^2e^2 \prod_{i=1}^{4}(z - b_i)(1 - \frac{z}{qg})y(\frac{z}{q}) \]

\[ - \frac{1}{1 + \frac{e}{c_1g}} \left( e^2 \prod_{i=1}^{2}(z - a_i)(z - b_i)(1 - \frac{f_z}{q}) (1 - \frac{z}{g}) + \frac{q(1 - f_z)}{e(1 - \frac{z}{qg})} V(\frac{z}{q}) \right)y(z) = 0, \]

where \(V(z) = qe^2(1 - \frac{ez}{c_1g})(1 - \frac{ez}{c_1g}) \prod_{i=3}^{4}(z - a_i)(z - b_i) - w\bar{w}^2(1 - \frac{f_z}{q} - 1 - \frac{f_z}{q})z. \)

**Remark 4.1.** The \(L_4\) equations for each direction \(T_1\) and \(T_2\) are equivalent under the relation [18]

\[ \frac{(1 - a_2f)(1 - b_2f)}{(e^{\frac{1}{84.2.2}} g + e^{\frac{1}{84.2.2}} f + 1)} \frac{1 - f}{e^{\frac{1}{84.2.2}} g^2 + e^{\frac{1}{84.2.2}} f^2} = 1, \]

where \(g^{84.2.2}\) and \(e^{84.2.2}\) means \(g\) and \(e\) used in section 4.2. This result follows from a reduction of theorem 3.1. A direct proof is given in [18].

**Remark 4.2.** If we include more directions such as \(T_3^{-1}T_6^{-1}T_8^{-1}\) and \(T_3^{-1}T_6^{-1}T_8\), in addition to \(T_1\) as fundamental ones, then all the other directions can be obtained by compositions of them. Thanks to the symmetry of the parameters, the equations for such additional fundamental directions are similar to that of \(T_1\). Therefore the results of this section are enough to obtain all the directions of \(q\)-Painlevé equation of type \(E_7^{(1)}\).
5. Relation to Suzuki’s system

5.1. q-KP and its duality

A Lax form of a certain isomonodromic system has been formulated as a reduction of the q-KP hierarchy [9] as follows. Let $x_{ij}, r_i, t_j \in \mathbb{C}$ and $m, n \in \mathbb{N}$ be parameters. For an unknown $Y(z) \in \mathbb{C}^n$, we consider a system of q-difference equations

$$Y(qz) = A(z)Y(z), \quad (5.1)$$

and its deformation system

$$\tilde{Y}(z) = B(z)Y(z). \quad (5.2)$$

Here, $A(z) = dX_m \cdots X_2X_1$, $X_i(z) = \begin{bmatrix} x_{i,1} & 1 \\ x_{i,2} & 1 \\ \vdots & \vdots \\ x_{i,m-1} & 1 \\ r_j^{-1}z & x_{i,n} \end{bmatrix}$, $d = \text{diag}[t_1, \ldots, t_n]$. \quad (5.3)

In this section, we focus consideration on the linear equation (5.1).

Extending the result of [8, 9], one obtains the following:

**Theorem 5.1.** Two cases $(m, n)$ and $(n, m)$ of the linear equation (5.1) are equivalent under the following transformation:

$$x_{ij} \leftrightarrow -x_{ji}, \quad r_i \leftrightarrow t_i, \quad z \leftrightarrow Tz. \quad (5.4)$$

**Proof.** Setting $Y_i = Y, Y_{i+1} = X_iY_i$ ($1 \leq i \leq m$) and defining components of $Y$ by $Y_{ij} = (Y_i)_j$, then we have relations:

$$Y_{i+1,j} = x_{ij}Y_{ij} + Y_{i+1,j}, \quad Y_{m+1,j} = t_j^{-1}TzY_{1,j}, \quad Y_{i,n+1} = r_i^{-1}zY_{i,1}. \quad (5.5)$$

Applying a transformation (5.4) and

$$m \leftrightarrow n, \quad Y_{ij} \leftrightarrow Y_{ji}, \quad (5.6)$$

into the relations (5.5), then the cases $(m, n)$ and $(n, m)$ of the linear equation (5.1) can be transformed with each other. \quad \blacksquare

We remark that the transformation $z \leftrightarrow Tz$ in (5.4) is a kind of q-Laplace transformation, and it transforms $Tz = qTz$ into $Tz = qTz$ i.e. $Tz = qTz^{-1}$.

9 For the related works on q-KP, see [8] (affine Weyl group symmetry), [14, 30] (the case $(m, n) = (2, 3), (2, 4)$), [11, 33] (Yang–Baxter maps), [31] (q-UC hierarchy), [1, 5] (non-commutative version).

10 For convenience, extra parameters $r_i$ and $t_j$ are introduced though they are redundant.
5.2. q-Garnier system as the case \((m, n) = (2N + 2, 2)\)

Consider the case \((m, n) = (2N + 2, 2)\) of the matrix \(A\) given by:

\[
A(z) = dX_{2N+2} \cdots X_2X_1,
\]

\[
X_i = \begin{bmatrix} x_{i,1} & 1 \\ r_{i-1}z & x_{i,2} \end{bmatrix}, \quad d = \text{diag}[t_1, t_2].
\]  

(5.7)

**Proposition 5.2.** The matrix \(A\) (5.7) is equivalent to the Sakai’s matrix \(A\) (2.4).

**Proof.** The matrix \(A\) (5.7) takes the form:

(i) \(A(z) = \sum_{i=0}^{N+1} A_i z^i,\)

(ii) \(A_{N+1} = \begin{bmatrix} t_1 \prod_{i=1}^{N+1} r_{2i-1}^{-1} & 0 \\ t_2 \prod_{i=1}^{N+1} r_{2i}^{-1} & \end{bmatrix}, \quad A_0 = \begin{bmatrix} \theta_1 & * \\ 0 & \theta_2 \end{bmatrix},\)

(iii) \(\det A(z) = t_1 t_2 \prod_{i=1}^{2N+2} r_i^{-1}(z - \alpha_i),\)

(5.8)

where \(\theta_j = t_j \prod_{i=1}^{2N+2} x_{ij}, \alpha_i = r_{i,1}x_{i,2}.\) These properties (5.8) corresponds exactly to the properties (2.4) for Sakai’s matrix \(A\), by a gauge transformation with lower triangular matrix

\[
\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}
\]

Conversely, any matrix \(A\) with properties (5.8) can be factorized into the form \(A\) (5.7). Note that the variables \(x_{ij}\) can be obtained by considering the Ker\(A\) at \(z = \alpha_i\) due to the property (iii).

**Remark 5.3.** In the case of differential Garnier system, its relation to the KP hierarchy is not so clear. However, it is known that the Garnier system can been derived through a similarity reduction of the UC hierarchy [32].

5.3. Suzuki’s system as the case \((m, n) = (2, 2N + 2)\)

Consider the case \((m, n) = (2, 2N + 2)\) of the matrix \(A\) (5.3):

\[
A(z) = dX_2X_1,
\]

\[
X_i = \begin{bmatrix} x_{i,1} & 1 \\ r_{i-1}z & x_{i,2} \\ & \ddots \\ && \ddots \\ &&& x_{i,2N+1} & 1 \\ &&&& r_{i}^{-1}z & x_{i,2N+2} \end{bmatrix}, \quad d = \text{diag}[t_1, \ldots, t_{2N+2}].
\]  

(5.9)
Proposition 5.4. The matrix $A$ (5.9) has the following form:

$$A(z) = \begin{cases}
    \alpha_1 \varphi_1 & t_1 \\
    \alpha_2 \varphi_2 & t_2 \\
    \vdots & \vdots \\
    \alpha_{2N} \varphi_{2N} & t_{2N}
\end{cases}, \quad (5.10)
$$

Conversely, any matrix $A$ of the form (5.10) can be factorized as (5.9).

Proof. It is easy to check that the matrix $A$ (5.9) takes the form (5.10) under substitutions $\alpha_i = t_1 x_{1,i} x_{2,i}$, $\theta_i = r_i \prod_{j=1}^{2N+2} x_{i,j}$, $\varphi_k = l_k (x_{1,1+k} + x_{2,k})$ ($k \neq 2N+2$) and $\varphi_{2N+2} = t_{2N+2} (r_1^{-1} x_{1,1} + r_2^{-1} x_{2,2N+2})$.

Conversely, the matrix $A$ (5.10) can be factorized as $A$ (5.9); consider the $\text{Ker} A$ at $z = \theta_1$ using the property (ii).

Remark 5.5. In [29] Suzuki considered the matrix $A$ (5.10), with redundant parameters fixed as $r_1 = \frac{1}{q}$, $r_2 = 1$, $t_1 = \ldots = t_{2N+2} = 1$. Note that two variables among $\varphi_1$, $\varphi_2$, $\varphi_{2N+2}$ can be reduced by a gauge transformation and the constraint coming from the property (ii). He also gave the matrix $B$ corresponding with a time evolution $\overline{t_1} = q r_1$ (i.e. $\overline{t} = \frac{1}{q}$, $\overline{\theta_1} = q \theta_1$).

Thanks to theorem 5.1 and proposition 5.2, proposition 5.4, we have the conclusion that the Lax pair of the $q$-Garnier system and the one of Suzuki’s higher order $q$-Painlevé system are mutually transformed into each other through a $q$-Laplace transformation.

By the correspondence of parameters ($n = N + 1$)

| Matrix | Parameters | Specialization |
|--------|------------|---------------|
| $a_1, \ldots, a_n$ | $b_1, \ldots, b_n$ | $c_1 c_2$ | $d_1$ | $d_2$ |
| $a_1, \ldots, a_n$ | $\alpha_{n+1}, \ldots, \alpha_{2n}$ | $t_1$ | $\frac{b_1}{q}$ | $\prod_{i=1}^{n} (-a_i) \frac{\theta_1}{\theta_2}$ |
| $a_1, \ldots, a_n$ | $\alpha_{n+1}, \ldots, \alpha_{2n}$ | $t_1$ | $\frac{b_1}{q}$ | $\prod_{i=1}^{n} (-a_i) \frac{\theta_1}{\theta_2}$ |

(5.11)

Suzuki’s direction $\overline{r_i} = q r_i$, $\overline{\theta_1} = q \theta_1$ for (5.10) corresponds to $T_3 = T_{\overline{r}} T_{\overline{\theta}}$ in section 3.2.3.

6. Summary

In this paper, extending the previous work [20], we obtained the following two main results.

• Choosing the appropriate variables for each deformation direction, we obtained simple expression of the scalar Lax pairs and the evolution equations for the variations of the
\textit{q}-Garnier system. Consequently, we gave relations among these \textit{q}-Garnier systems. As a byproduct, we obtained the standard \textit{q}-Painlevé equations of type $E_7^{(1)}$ as a reduction of a certain \textit{q}-Garnier system.

- We clarified the relation between the \textit{q}-Garnier system \cite{20, 26} and Suzuki’s system \cite{29} by formulating the duality of the \textit{q}-KP system.

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\section*{Appendix A. Lax equations}

\subsection*{A.1. Correspondence to Sakai’s Lax form}

The $L_1$ equations in (2.5) and those in section 3.2 are all equivalent with each other. In this appendix, we give the correspondence\textsuperscript{11} in case of (2.5) and (3.11) as an example.

We call $q^{\rho_0}$ (resp. $q^{\rho_\infty}$) characteristic exponents at $x = 0$ (resp. $x = \infty$) when solutions $y(x)$ have the form:

\begin{align}
  y(x) &= k_{\rho_0} x^{\rho_0} (1 + O(x)), \quad \text{at } x = 0, \quad (A.1) \\
  y(x) &= k_{\rho_\infty} x^{\rho_\infty} (1 + O\left(\frac{1}{x}\right)), \quad \text{at } x = \infty. \quad (A.2)
\end{align}

Eliminating $y(z)$ and $y(qz)$ from $L_2(z) = L_2(qz) = L_3(z) = 0$ (3.9), we obtain $L_1(z) = 0$ (3.11). Then we have:

\textbf{Lemma A.1.} The $L_1$ equation (3.11) takes the following properties:

(i) It is a linear three term equation between $y(qz)$, $y(z)$ and $y(qz)$, and the coefficients of $y(qz)$, $y(z)$ and $y(qz)$ are polynomials of degree $2N + 1$ in $z$.

(ii) The coefficient of $y(qz)$ (resp. $y(qz)$) has zeros at $z = a_1, \ldots, a_{N+1}$ (resp. $z = qb_1, \ldots, qb_{N+1}$).

(iii) The characteristic exponents of solutions $y(z)$ are $d_1, d_2$ (at $z = 0$) and $c_1, c_2$ (at $z = \infty$) generically\textsuperscript{12}.

(iv) The $N$ points $z$ such that $F(z) = 0$ are apparent singularities (i.e. the solutions are regular there), where:

\begin{equation}
  \frac{y(qz)}{y(z)} = \frac{G(z)}{A(z)}, \quad \text{for } F(z) = 0, \quad (A.3)
\end{equation}

holds (see (2.11)).

\textsuperscript{11} Other cases are similar. The case of (2.5) and (3.5) is given in \cite{20, section 2.2}.

\textsuperscript{12} Generic case means $f_{00} \neq 0$. We remark that the reduced cases to $q$-$E_7^{(1)}$ system are nongeneric, where their $L_1$ equations (4.9) and (4.14) have exponents $d_1, qd_1$ (at $z = 0$) and $c_1, qc_1$ (at $z = \infty$).
Conversely, the $L_1$ equation (3.11) is uniquely characterized by these properties (i)–(iv).

On the other hand, thanks to lemma 2.1, Sakai’s Lax equation (2.1) is rewritten into the $L_1$ equation (2.5). Furthermore we have:

**Lemma A.2.** The $L_1$ equation (2.5) can be expressed as a form:

\[
L_1 : \prod_{i=1}^{N+1} (z - \alpha_i) F(\frac{z}{q}) y(qz) - \left\{ F(\frac{z}{q}) A_{11}(z) + F(z) A_{22}(z) \right\} y(z)
\]

\[
+ \kappa_1 \kappa_2 \prod_{i=N+2}^{2N+2} (\frac{x}{q} - \alpha_i) F(z) y(\frac{x}{q}) = 0.
\]

**Proof.** The proof is given using a gauge transformation:

\[
y(z) \rightarrow H(z) y(z), \quad \text{with} \quad \frac{H(qz)}{H(z)} = \prod_{i=1}^{N+1} (z - \alpha_i).
\]

Similarly to lemma A.1, we have:

**Lemma A.3.** Then the $L_1$ equation (A.4) has the following properties:

(i) it is a linear three term equation between $y(qz)$, $y(z)$ and $y_1(\frac{z}{q})$, and the coefficients of $y(qz)$, $y(z)$ and $y(\frac{z}{q})$ are polynomials of degree $2N + 1$ in $z$.

(ii) the coefficient of $y(qz)$ (resp. $y(\frac{z}{q})$) has zeros at $z = \alpha_1, \ldots, \alpha_{N+1}$ (resp. $z = q\alpha_{N+2}, \ldots, q\alpha_{2N+2}$),

(iii) the characteristic exponents of the solutions $y(z)$ are \( \theta_1 \prod_{i=1}^{N+1} (-\alpha_i), \theta_2 \prod_{i=1}^{N+1} (-\alpha_i) \) (at $z = 0$) and $\kappa_1, q^{-1}\kappa_2$ (at $z = \infty$),

(iv) $N$ points $z = \lambda_i$ such that $F(z) = 0$ are apparent singularities, where:

\[
\frac{y(qz)}{y(z)} = \frac{A_{11}(z)}{\prod_{i=1}^{N+1} (z - \alpha_i)}, \quad \text{for} \quad F(z) = 0.
\]

Conversely, the $L_1$ equation (A.4) is uniquely characterized by these properties (i)–(iv).

Hence, from lemmas A.1–A.3, we obtain:

**Proposition A.4.** The $L_1$ equations (3.11) and (2.5) are equivalent up to the gauge transformation (A.5) and changes of variables and parameters.
A.2. Sufficiency for the compatibility

In section 3.2, we stated that the equations in item (b) are sufficient for the compatibility of the equations in item (a). Here we will prove this fact in the case of section 3.2.3 as an example.

Eliminating \( y(z) \) and \( y(qz) \) from \( L_2(z) = L_3(z) = L_3(qz) = 0 \) (3.9), we obtain the following expression:

\[
L_4(z) = \bar{A}(z) F(z) G(z) + q^2 c_1 c_2 B(z) F(z) G(z) - \left\{ q F(z) G(z) + \frac{F(z)}{G(z)} V(z) \right\} y(z),
\]

(A.7)

where \( V \) is given in (3.11).

**Lemma A.5.** The \( L_4 \) equation (A.7) has the following properties:

(i) it is a linear three term equation between \( y(qz) \), \( y(z) \) and \( y(qz) \), and the coefficients of \( y(qz) \), \( y(z) \) and \( y(qz) \) are polynomials of degree \( 2N + 1 \) in \( z \),
(ii) the coefficient of \( y(qz) \) (resp. \( y(qz) \)) has zeros at \( z = a_1, a_2, \ldots, a_{N+1} \) (resp. \( z = b_1, b_2, \ldots, b_{N+1} \)),
(iii) the characteristic exponents of solutions \( y(z) \) are \( q(d_1, d_2) \) (at \( z = 0 \)) and \( q(c_1, c_2) \) (at \( z = \infty \)),
(iv) the \( N \) points \( z \) such that \( F(z) = 0 \) are apparent singularities, where:

\[
\frac{y(qz)}{y(z)} = \frac{q c_1 c_2 B(z)}{G(z)}, \quad \text{for} \quad F(z) = 0,
\]

(A.8)

Conversely, the equation \( L_4 = 0 \) (A.7) is uniquely characterized by these properties (i)–(iv).

**Proposition A.6.** The linear \( q \)-difference equations \( L_1 \) (3.11) and \( L_2 \) (3.9) are compatible if and only if the bi-rational equation (3.10) is satisfied.

**Proof.** The compatibility of \( L_1 \) and \( L_2 \) means that \( T(L_1) = L_4 \). This can be checked by the characterizations of the equations \( L_1 \) (resp. \( L_4 \)) in lemma A.1 (resp. A.5).

**Appendix B. From Padé interpolation to the \( q \)-Garnier system**

There is a convenient method to approach the continuous/discrete Painlevé equations using certain problem of Padé approximation [17, 34] (see also [12, 13]) or interpolation [6, 15, 16, 18, 19, 22, 36, 37]. In [20], this method is applied to the \( q \)-Garnier system in the case of section 3.2.1. Here, we illustrate the derivation of the Lax pair (3.9) through the Padé method in case of \( q \)-Garnier system in section 3.2.3 as an example.

**B.1. Derivation of scalar Lax pair**

Fix a positive integer \( N \in \mathbb{N} \) and a parameter \( q \) \((0 < |q| < 1)\). For parameters \( a_1, \ldots, a_N, b_1, \ldots, b_N \) and \( c \), we consider a function:

\[13\] Other cases can be proved in a similar way and the case of section 3.2.1 has been done in [20, section 2.3].
Let \( P(z) \) and \( Q(z) \) be polynomials of degree \( m \) and \( n \in \mathbb{Z}_{\geq 0} \) in \( z \) determined by the following Padé interpolation condition:

\[
\psi(z) = e^{\log z \sum_{i=1}^{N} \frac{(a_i z, b_i)}{(a_i, b_i)}_{\infty}}.
\]  \( \text{(B.1)} \)

among \( y \) and \( \psi \) determined by the following relations:

\[
\begin{align*}
\psi(z) &= \frac{P(z)}{Q(z)} \quad (z_i = q^s, s = 0, 1, \ldots, m + n). \quad \text{(B.2)}
\end{align*}
\]

The common normalizations of the polynomials \( P \) and \( Q \) in \( z \) are fixed as \( P(0) = 1 \). We consider the shift operation: \( \bar{z} = T(x) \) where \( T = T_z \).

We construct two linear \( q \)-difference relations: \( L_2 = 0 \) among \( y(z), y(qz), \bar{y}(z) \) and \( L_3 = 0 \) among \( y(z), \bar{y}(z), \bar{y}(\bar{z}) \) satisfied by the functions \( y = P \) and \( y = \psi Q \).

**Proposition B.1.** The linear relations \( L_2 \) and \( L_3 \) can be expressed as:

\[
\begin{align*}
L_2(z) &= F(z)\bar{y}(z) + A(z)(\frac{z}{q^{m+n}})y(qz) + \frac{c}{g_0}G(z)y(z) = 0, \\
L_3(z) &= \frac{z}{q}F(\bar{z})y(x) + \frac{1}{q}G(\frac{z}{q})\bar{y}(z) - (z)B(\bar{z})\bar{y}(\frac{z}{q}) = 0,
\end{align*}
\]  \( \text{(B.3)} \)

where \( A(z) = \prod_{i=1}^{m}(a_i z, b_i), B(z) = \prod_{i=1}^{m}(b_i z, 1) \) and \( F(z), G(z) \) are as in (3.9) and \( f_0, \ldots, f_N, g_0, \ldots, g_N \) depend only on parameters \( a_i, b_j, c \) and \( m, n \in \mathbb{Z}_{\geq 0}. \)

**Proof.** By the definition above, the linear relations \( L_2 \) and \( L_3 \) can be written as:

\[
\begin{align*}
L_2(z) \propto |y(z) y(qz) \bar{y}(z) \bar{y}(qz)| &= D_1(z)\bar{y}(z) - D_2(z)y(qz) + D_3(z)y(z) = 0, \\
L_3(z) \propto |y(z) \bar{y}(z) \bar{y}(\bar{z}) \bar{y}(\bar{z})| &= D_1(\frac{z}{q})y(z) + D_2(\frac{z}{q})\bar{y}(z) - D_2(\frac{z}{q})\bar{y}(\frac{z}{q}) = 0,
\end{align*}
\]  \( \text{(B.4)} \)

where \( y(z) = \left[ \begin{array}{c} P(z) \\ \psi(z)Q(z) \end{array} \right] \) and Casorati determinants

\[
D_1(z) = |y(z), y(qz)|, \quad D_2(z) = |y(z), \bar{y}(z)|, \quad D_3(z) = |y(qz), \bar{y}(z)|. \quad \text{(B.5)}
\]

Using the relations:

\[
\frac{\psi(qz)}{\psi(z)} = \frac{B(z)}{A(z)}, \quad \frac{\psi(z)}{\psi(z)} = z,
\]  \( \text{(B.6)} \)

we can rewrite the Casorati determinants (B.5) into the following determinants:

\[
\begin{align*}
D_1(z) &= \frac{\psi(z)}{A(z)} \left\{ cB(z)P(z)Q(qz) - A(z)P(qz)Q(z) \right\} =: \frac{\psi(z)}{A(z)} \prod_{i=0}^{m+n-1} (\frac{z}{q^i})_{1 \leq i \leq 0} F(z), \\
D_2(z) &= \frac{\psi(z)}{A(z)} \left\{ zP(z)Q(z) - P(z)Q(qz) \right\} =: \frac{\psi(z)}{A(z)} \prod_{i=0}^{m+n-1} (\frac{z}{q^i})_{1 \leq i \leq 0} G(z), \\
D_3(z) &= \frac{\psi(z)}{A(z)} \left\{ zA(z)P(qz)Q(z) - cB(z)P(z)Q(qz) \right\} =: \frac{\psi(z)}{A(z)} \prod_{i=0}^{m+n-1} (\frac{z}{q^i})_{1 \leq i \leq 0} G(z).
\end{align*}
\]  \( \text{(B.7)} \)
Then the constants $c_0 = \frac{g_0}{r}$, $g_{N+1} = -\frac{g_0}{\prod_{i=1}^{N}(-a_i)}$ are determined through the expansions around $x = 0$ and $x = \infty$. As a result, we obtain the desired equations.

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References

[1] Doliwa A 2014 Non-commutative rational Yang–Baxter maps *Lett. Math. Phys.* vol. 104 pp 299–309
[2] Dzhamay A, Sakai H and Takenawa T 2013 Discrete Hamiltonian structure of schlesinger transformations (arXiv:1302.2972 [math-ph])
[3] Dzhamay A and Takenawa T 2015 *Geometric Analysis of Reductions from Schlesinger Transformations to Difference Painlevé Equations* vol 651 (Providence, RI: American Mathematical Society) pp 87–124
[4] Grammaticos B and Ramani A 1999 On a novel $q$-discrete analogue of the Painlevé VI equation *Phys. Lett. A* vol. 257 pp 288–92
[5] Hasegawa K 2013 Quantizing the discrete Painlevé VI equation: the Lax formalism *Lett. Math. Phys.* vol. 103 pp 865–79
[6] Ikawa Y 2013 Hypergeometric solutions for the $q$-Painlevé equation of type $E_6^{(1)}$ by the Padé method *Lett. Math. Phys.* vol. 103 pp 743–63
[7] Jimbo M and Sakai H 1996 A $q$-analogue of the sixth Painlevé equation *Lett. Math. Phys.* vol. 38 pp 145–54
[8] Kajiwara K, Noumi M and Yamada Y 2002 Discrete dynamical systems with $W(A_{m+1}^{(1)} \times A_{n+1}^{(1)})$ symmetry *Lett. Math. Phys.* vol. 60 pp 211–9
[9] Kajiwara K, Noumi M and Yamada Y 2002 $q$-Painlevé systems arising from $q$-KP hierarchy *Lett. Math. Phys.* vol. 62 pp 259–68
[10] Kajiwara K, Noumi M and Yamada Y 2017 Geometric aspects of Painlevé equations *J. Phys. A: Math. Theor.* vol. 50 pp 073001
[11] Kuniba A, Okado M, Takagi T and Yamada Y 2003 Geometric crystal and tropical R for $D_{m}^{(1)}$ *Int. Math. Res. Not.* vol. 5 pp 2565–620
[12] Mano T 2012 Determinant formula for solutions of the Garnier system and Padé approximation *J. Phys. A: Math. Theor.* vol. 45 pp 135206–19
[13] Mano T and Tsuda T 2017 Hermite–Padé approximation, isomonodromic deformation and hypergeometric integral *Math. Z.* vol. 285 pp 397–431
[14] Masuda T 2003 On the rational solutions of $q$-Painlevé V equation *Nagoya Math. J.* vol. 169 pp 119–43
[15] Nagao H 2015 The Padé interpolation method applied to $q$-Painlevé equations *Lett. Math. Phys.* vol. 105 pp 503–21
[16] Nagao H 2016 Lax pairs for additive difference Painlevé equations (arXiv:1604.02530 [nlin.SI])
[17] Nagao H 2017 The Padé interpolation method applied to $q$-Painlevé equations II (differential grid version) *Lett. Math. Phys.* vol. 107 pp 107–27
[18] Nagao H 2017 A variation of the $q$-Painlevé system with affine Weyl group symmetry of type $E_7^{(1)}$ *SIGMA* vol. 13 pp 92–109
[19] Nagao H 2017 Hypergeometric special solutions for $d$-Painlevé equations (arXiv:1706.10101 [nlin.SI])
[20] Nagao H and Yamada Y 2018 Study of $q$-Garnier system by Padé method *Funkcialaj Ekvacioj* vol. 61 pp 109–33 accepted
[21] Nijhoff F and Delice N 2016 On elliptic Lax pairs and isomonodromic deformation systems for elliptic lattice equations (arXiv:1605.00829 [nlin.SI])
[22] Noumi M, Tsujimoto S and Yamada Y 2013 *Padé Interpolation for Elliptic Painlevé Equation* vol 40 (Berlin: Springer) pp 463–82
[23] Ormerod C M and Rains E M 2017 An elliptic Garnier systems *Commun. Math. Phys.* vol. 355 pp 741–66
[24] Ormerod C M and Rains E M 2016 Computation relations and discrete Garnier systems SIGMA 12 110–59
[25] Sakai H 2001 Rational surfaces with affine root systems and geometry of the Painlevé equations Commun. Math. Phys. 220 165–221
[26] Sakai H 2005 A $q$-analog of the Garnier system Funkcialaj Ekvacoj 48 273–97
[27] Sakai H 2005 Hypergeometric solution of $q$-Schlesinger system of rank two Lett. Math. Phys. 73 237–47
[28] Suzuki T 2015 A $q$-analogue of the Drinfeld–Sokolov hierarchy of type A and $q$-Painlevé system AMS Contemp. Math. 651 25–38
[29] Suzuki T 2017 A reformulation of the generalized $q$-Painlevé VI system with $W(A^{(1)}_{2n+1})$ symmetry J. Integrable Syst. 2 1–18
[30] Takenawa T 2003 Weyl group symmetry of type $D_1^{(1)}$ in the $q$-Painlevé V equation Funkcialaj Ekvacoj 46 173–86
[31] Tsuda T 2010 On an integrable system of $q$-difference equations satisfied by the universal characters: its Lax formalism and an application to $q$-Painlevé equations Comm. Math. Phys. 293 347–59
[32] Tsuda T 2014 UC hierarchy and monodromy preserving deformation J. Reine Angew. Math. 690 1–34
[33] Yamada Y 2001 A birational representation of Weyl group, combinatorial R-matrix and discrete Toda equation Physics and Combinatorics 2000 (Singapore: World Scientific) pp 305–19
[34] Yamada Y 2009 Padé method to Painlevé equations Funkcialaj Ekvacoj 52 83–92
[35] Yamada Y 2011 Lax formalism for $q$-Painlevé equations with affine Weyl group symmetry of type $E_6^{(1)}$ Int. Math. Res. Not. 17 3823–38
[36] Yamada Y 2014 A simple expression for discrete Painlevé equations RIMS Kokyuroku Bessatsu B 47 087–95
[37] Yamada Y 2017 An elliptic Garnier system from interpolation SIGMA 13 69–77