Stabilization of the Nonconforming Virtual Element Method

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Index

1 Introduction

2 A Duality Technique for Constructing Semi-Inner Products

3 Stabilization in the Nonconforming Virtual Element Method

4 Numerical Experiments
Index

1 Introduction

2 A Duality Technique for Constructing Semi-Inner Products

3 Stabilization in the Nonconforming Virtual Element Method

4 Numerical Experiments
The Nonconforming Virtual Element Method

- We are in the framework of the Poisson problem.
- Recall the nonconforming virtual element space on an element $P$ of a polygonal grid $\Omega_h$

$$V^h_k(P) = \left\{ v \in H^1(P) : \frac{\partial v}{\partial n} \in P_{k-1}(e) \forall e \subset E_P, \Delta v \in P_{k-2}(P) \right\}$$

- Degrees of freedom on $V^h_k(P)$:

$$\frac{1}{h_e} \int_e v \ m_\alpha \ \forall \ m_\alpha \in M_{k-1}(e), \forall e \in E_P$$

$$\frac{1}{|P|} \int_P v \ m_\alpha \ \forall \ m_\alpha \in M_{k-2}(P).$$
Virtual Element Discretization

- The global discrete bilinear form is given by the sum of the elemental contributions $a_h^P$

$$a_h^P(u, v) = a^P(\Pi_k^\nabla,^P u, \Pi_k^\nabla,^P v) + \sigma^P((I - \Pi_k^\nabla,^P)u, (I - \Pi_k^\nabla,^P)v).$$

- $\sigma^P(\cdot, \cdot)$ a computable, symmetric and positive semidefinite bilinear form

$${C_1}a^P(v, v) \leq \sigma^P(v, v) \leq {C_2}a^P(v, v) \quad \forall v \in V_h^k(P) \cap \ker(\Pi_k^\nabla,^P)$$

- Optimal error estimates hold if

$$\frac{1 + \min\{1, C_2\}}{\min\{1, C_1\}} \text{ is uniformly bounded in } h$$

- Wish to use as weak assumptions on the mesh as possible.
Index

1 Introduction

2 A Duality Technique for Constructing Semi-Inner Products

3 Stabilization in the Nonconforming Virtual Element Method

4 Numerical Experiments
Our Idea

- Trace on $\partial P$ is not accessible.
- Degrees of freedom span a (known) space $V_k^*(P)$ in uniformly stable duality relation with $V_k^h$. 

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Stabilization of the NC-VEM  
Polygonal Methods for PDEs
Our Idea

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Idea

To build a semi-inner product in $V_k^*(P)$ and, by duality, get a semi-inner product in $V_k^h(P)$. 
Ingredients

- $V$ Hilbert space and its dual $V'$.
- A seminorm $|\cdot|: V \to \mathbb{R}^+$ with kernel $\widehat{W} \subset V$ and projector $\widehat{\Pi}: V \to \widehat{W}$.
- A seminorm $|\cdot|_*: V' \to \mathbb{R}^+$.
- A projector $\widehat{\Pi}^*: V' \to V'$.
- Two finite dimensional subspaces $W \subset V$, $W^* \subset V'$.

Assumptions:

- $\widehat{W}$ is finite dimensional.
- $||v|| \leq C|v|$ holds on ker($\widehat{\Pi}$).
- $\widehat{W} \subset W$, $\widehat{W}^* \subset W^*$.
- A pair of inf-sup conditions on $W$ and $W^*$

\[
\inf_{w \in W} \sup_{\eta \in W^*} \frac{\langle \eta, w \rangle}{||\eta||_* ||w||} \geq \beta, \\
\quad \quad \inf_{\eta \in W^*} \sup_{w \in W} \frac{\langle \eta, w \rangle}{||\eta||_* ||w||} \geq \beta.
\]
Algebraic Construction of Semi-inner Products

It holds:

- \( N = \dim(W) = \dim(W^*) \), \( M = \dim(\hat{W}) = \dim(\hat{W}^*) \)
- There exist a basis \( B = \{e_m\}_{m=1,...,N} \) for \( W \) and a basis \( B^* = \{\eta_n\}_{n=1,...,N} \) for \( W^* \) such that
  \[
  \langle \eta_n, e_m \rangle = \delta_{n,m} \quad m, n = 1, \ldots, N.
  \]
- For \( v \in W, \xi \in W^* \)
  \[
  v = \sum_{m=1}^{N} v_m e_m \iff v = (v_m)_{m=1}^{N}, \quad \xi = \sum_{n=1}^{N} \xi_n \eta_n \iff \xi = (\xi_n)_{n=1}^{N}.
  \]
Algebraic Construction of Semi-inner Products

- \( S \in \mathbb{R}^{N \times N} \) symmetric positive definite matrix and associated bilinear form \( s(\cdot, \cdot): W \times W \to \mathbb{R} \), with
  \[
  s(v, w) = w^T Sv, \quad s(v, w) \leq A|v||w|, \quad \alpha |v|^2 \leq s(v, v)
  \]

- \( S^\dagger \in \mathbb{R}^{N \times N} \) reflexive generalized inverse of \( S \): given \( \eta \in \mathbb{R}^N \), the saddle point problem
  \[
  \begin{cases}
  Sw + P^T \lambda = \eta, \\
  Pw = 0
  \end{cases}
  \]
  \((P\text{ matrix realization of } \hat{\Pi})\)

  has a unique solution \((w, \lambda) \in \mathbb{R}^N \times \mathbb{R}^M\). We set
  
  \[
  S^\dagger \eta = w, \quad S^\dagger = (I \ 0) \begin{pmatrix} S & P^T \\ P & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}
  \]

- Semi-inner product inducing a semi-norm equivalent to \(|\cdot|_*
  \[
  s^*: W \times W \to \mathbb{R}, \quad s^*(\eta, \xi) = \eta^T S^\dagger \xi.
  \]
Mesh Assumptions

1. \( \exists \gamma_0 > 0 \) such that \( \forall \Omega_h, \) each \( P \in \Omega_h \) is star-shaped with respect to a ball of radius \( > \gamma_0 h_P. \)

2. \( \exists \gamma_1 > 0 \) such that \( \forall \Omega_h, \) the distance between any two vertices of each \( P \in \Omega_h \) is \( > \gamma_1 h_P. \)

Possible alternatives to (2):

3. \( \exists \gamma_2 > 0 \) such that \( \forall \Omega_h \) and each pair of adjacent edges \( e, e' \in \mathcal{E}_P, P \in \Omega_h \)

\[
1/\gamma_2 \leq h_e/h_{e'} \leq \gamma_2
\]

4. \( \exists \) integer \( N^* > 0 \) such that \( \forall \Omega_h, \) every \( P \in \Omega_h \) has at most \( N^* \) edges.

Our choice:

5. We require that \( \forall \Omega_h, \) each \( P \in \Omega_h \) a part of \( \partial P \) satisfies 3 and the remaining part satisfies 4.
Step 1: Degrees of Freedom and Stable Duality

The degrees of freedom stem from the action of a basis of

$$V_k^*(P) = \gamma_P^* N_{k-1}(\partial P) \oplus P_{k-2}(P) \subset (H^1(P))',$$

where:

- $N_{k-1}(\partial P) = \{ \lambda \in L^2(\partial P) \mid \lambda|_e \in P_{k-1}(e), \forall e \in \mathcal{E}_P \} \subset H^{-1/2}(\partial P)$.
- $\gamma_P^*$ is the adjoint of the trace operator.

Lemma

Let $f_{\partial P} v = \frac{1}{|\partial P|} \int_{\partial P} v$. For all $v \in V_k^h(P)$

$$\sup_{\eta \in V_k^*(P)} \frac{\langle \eta, v \rangle}{\sqrt{|\langle \eta, 1 \rangle|^2 + |\eta|_{-1,P}^2}} \geq \sqrt{\int_{\partial P} v^2} + |v|_{1,P}^2.$$
Step 2: Reduction to the Boundary

Introduce a suitable subspace $\hat{V}_k^h(P) \subset V_k^h(P)$

\[
\mathcal{A}_k(P) = \{ q \in \mathbb{P}_k(P) \mid \Delta q = 0 \} \subset \mathbb{P}_k(P)
\]

\[
\hat{\mathbb{P}}_k(P) = \left\{ p \in \mathbb{P}_k(P) \mid \int_P pq = 0 \ \forall q \in \mathcal{A}_k(P) \right\} \subset \mathbb{P}_k(P)
\]

\[
\hat{V}_k^h(P) = \left\{ v \in V_k^h(P) \mid \int_P \nabla v \cdot \nabla q = 0, \ \forall q \in \hat{\mathbb{P}}_k(P) \right\}
\]

Lemma

\[
\sup_{\eta \in \mathbb{N}_{k-1}(\partial P)} \frac{\langle \gamma^*_P \eta, v \rangle}{\sqrt{\langle \gamma^*_P \eta, 1 \rangle^2 + |\gamma^*_P \eta|_{-1,P}^2}} \geq \sqrt{\int_{\partial P} v^2} + |v|_{1,P}^2 \quad \forall v \in \hat{V}_k^h(P)
\]

Remarks: $\ker(\nabla_P^k) \subset \hat{V}_k^h(P)$, $\dim(\hat{V}_k^h(P)) = \dim(\mathbb{N}_{k-1}(\partial P))$. 
Step 3: Transfer to the Dual

- Set:
  - $V = (H^1(P))'$ and $V' = H^1(P)$;
  - $W = \gamma_P^*(N_{k-1}(\partial P))$ and $W^* = \mathring{V}_k^h(P)$;
  - $\widehat{W} = \gamma_P^*(P_0(\partial P))$ and $\widehat{W}^* = P_0(P)$.

- Build a bilinear form $\sigma_P^*$ on $N_{k-1}(\partial P)$ (whose elements are explicitly known) such that
  \[ \sigma_P^*(\eta, \eta) \approx |\gamma_P^*\eta|_{-1, P}^2, \quad \sigma_P^*(\eta, \mu) \lesssim |\gamma_P^*\eta|_{-1, P} |\gamma_P^*\mu|_{-1, P} \]

- Let $S$ be the matrix associated with $\sigma_P^*$, and $S^\dagger$ its reflexive generalized inverse. Then a stabilizing bilinear form on $W^* = \mathring{V}_k^h(P)$ is given by
  \[ \sigma^P(v, w) = w^T S^\dagger v, \quad \forall v, w \in \mathring{V}_k^h(P) \]
Step 4: Factoring out Higher Order Polynomials

- \(|\gamma_P^* \eta|_{-1,P} \simeq |\eta|_{-1/2,P}\) \quad \forall \eta \in H^{-1/2}(\partial P).
- \(\sigma_P^*(\eta, \eta) \simeq |\eta|^2_{-1/2,\partial P}, \quad \sigma_P^*(\eta, \mu) \lesssim |\eta|_{-1/2,\partial P}|\mu|_{-1/2,\partial P}.
- \(N_{k-1}(\partial P) = N_0(\partial P) \oplus N_0^\perp(\partial P).
- \(N_0^\perp(\partial P) = \{ \eta \in N_{k-1}(\partial P) \mid \int_e \eta = 0, \forall e \in \mathcal{E}_P \}.

Corollary

If mesh assumption (5) holds, for \(\eta = \eta^0 + \eta^\perp \in N_{k-1}(\partial P)\), we have

\[|\eta|^2_{-1/2,\partial P} \simeq |\eta^0|^2_{-1/2,\partial P} + |\eta^\perp|^2_{-1/2,\partial P} \simeq |\eta^0|^2_{-1/2,\partial P} + \sum_{e \in \mathcal{E}_P} h_e \int_e |\eta - \eta^0|^2\]

We get

\[\sigma_P^*(\eta, \xi) = s^0(\eta^0, \xi^0) + \sum_{e \in \mathcal{E}_P} h_e \int_e (\eta - \eta^0)(\xi - \xi^0),\]

\[s^0(\eta^0, \eta^0) \gtrsim |\eta^0|^2_{-1/2,\partial P}, \quad s^0(\eta^0, \mu^0) \lesssim |\eta^0|_{-1/2,\partial P}|\mu^0|_{-1/2,\partial P}\]
Step 5: Stabilization for the Lowest Order NC-VEM

A quasi optimal stabilization term:

\[ s^0(\eta, \mu) = s_{L2}^0(\eta, \mu) = \sum_{e \in \mathcal{E}_P} h_e \int_e \left( \eta - \int_{\partial P} \eta \right) \left( \mu - \int_{\partial P} \mu \right). \]

If assumption (4) holds, \( \forall \, v, w \in V^h_k(P) \cap \ker(\Pi^{\nabla, P}_k) \)

\[ a^P(v, v) \lesssim \sigma^P(v, v) \lesssim (1 + \log(h_P/\hat{h}_P)) a^P(v, v) \quad (\hat{h}_P = \min_{e \in \partial P} h_e). \]
Step 5: Stabilization for the Lowest Order NC-VEM

1. A quasi optimal stabilization term:

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If assumption (4) holds, \( \forall v, w \in V_h^k(P) \cap \text{ker}(\Pi^{\nabla, P}_k) \)

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2. An optimal stabilization based on a wavelet decomposition of \( N_0(\partial P) \): if assumption (5) holds, \( \forall v, w \in V_h^k(P) \cap \text{ker}(\Pi^{\nabla, P}_k) \)

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1. A quasi optimal stabilization term:

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If assumption (4) holds, \( \forall v, w \in V^h_k(P) \cap \text{ker}(\Pi_k^{\nabla}, P) \)

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\[ a^P(v, v) \lesssim \sigma^P(v, v) \lesssim a^P(v, v). \]

3. Construct a second, explicitly known, discrete space, in a stable duality relation with \( N_0(\partial P) \) and apply the duality strategy twice. If assumption 3 holds, the resulting \( s^0 \) ensures \( \forall v, w \in V^h_k(P) \cap \text{ker}(\Pi_k^{\nabla}, P) \)

\[ a^P(v, v) \lesssim \sigma^P(v, v) \lesssim a^P(v, v). \]
Index

1 Introduction

2 A Duality Technique for Constructing Semi-Inner Products

3 Stabilization in the Nonconforming Virtual Element Method

4 Numerical Experiments
**Goal**: to test the robustness of five different stabilizations when using sequences of meshes challenging mesh regularity assumptions (2)–(5).

(3) fails, (5) ok

(4) fails, (5) ok

(4) fails, (5) ok.
Hexagonal Meshes

$k = 1$

$k = 2$

$k = 3$

$k = 4$

Relative error $H^1$

Mesh size $h$

Relative error $H^1$

Mesh size $h$

Relative error $H^1$

Mesh size $h$

Relative error $H^1$

Mesh size $h$

σ1a ←→ σ2a ←→ σ3a ←→ σ4a

σ1b ←→ σ2b ←→ σ3b ←→ σ4b
Polygonal Meshes with Increasing Number of Edges

$k = 1$

$k = 2$

$k = 3$

$k = 4$
### Numerical Experiments

#### Dyadic Meshes

The graphs illustrate the relative error $L^2$ for different mesh sizes $h$ and polynomial orders $k$. The error decreases as the mesh size decreases and the polynomial order increases. The diagrams show logarithmic scales for the $x$- and $y$-axes, highlighting the exponential decrease in error.

- **$k = 1$**: The error decreases rapidly with decreasing mesh size, indicating a high order of approximation.
- **$k = 2$**: The error continues to decrease at a similar rate, with a slight increase in the slope compared to $k = 1$.
- **$k = 3$**: The error decreases even more rapidly, indicating a further increase in the order of approximation.
- **$k = 4$**: The error decreases at the highest rate, with the steepest slope among the graphs, indicating the highest order of approximation.

The diagrams are labeled with different symbols for each polynomial order, allowing for easy comparison of the error trends. The legends at the bottom of the page indicate the symbols used for each polynomial order.
Conclusions

- Some stabilizations appear more robust than others.
- Details of this work can be found in

S. Bertoluzza, G. Manzini, M. Pennacchio, and DP. Stabilization of the nonconforming Virtual Element Method, arxiv: 2103.03742, 2021.
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Thank you very much for your attention!