Comparisons of \(\delta\)-Two-Stream and \(\delta\)-Four-Stream Radiative Transfer Schemes in RRTMG for Solar Spectra

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Abstract

Five \(\delta\)-two-stream and \(\delta\)-four-stream schemes are compared in solar spectra using the Rapid Radiative Transfer Model for General Circulation Models Applications (RRTMG). By calculating the flux and heating rate in various atmosphere, it is found that, in accuracy, the \(\delta\)-four-stream schemes overwhelmingly outperform the \(\delta\)-two-stream schemes. The precision of adding algorithm of the \(\delta\)-four-stream spherical harmonic expansion (\(\delta\)-4SDA) is comparable to that of adding algorithm of the \(\delta\)-four-stream discrete ordinates method (\(\delta\)-4DDA). Furthermore, the accuracy of the adding algorithm of \(\delta\)-Eddington approximation (\(\delta\)-2SDA) is close to that of \(\delta\)-two-stream approximation with Practical Improved Flux Method (\(\delta\)-PIFM), while adding algorithm of \(\delta\)-two-stream discrete ordinates method (\(\delta\)-2DDA) produces the poorest results among the five approximate schemes. For the RRTMG model with radiative transfer calculation, the computational time of \(\delta\)-4DDA is about 1.5 times that of \(\delta\)-two-stream schemes, and the computational time of \(\delta\)-4SDA is about 88% that of \(\delta\)-4DDA.

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1. Introduction

Atmospheric radiative transfer is a key process for physical processes in the atmosphere. Mathematically, the radiative transfer process is expressed as an integral-differential equation which generally has no exact solution for scattering media. Therefore, lots of approximate methods have been developed (Liu 1973; Joseph et al. 1976; Wiscombe 1977; Karp et al. 1980; Liu et al. 1988; Meador and Weaver 1988; Stamnes et al. 1988; Stephens et al. 1990; Shibata and Uchiyama 1992; Kylberg et al. 1995; Li and Ramaswamy 1996; Fu et al. 1997; Li and Dobbie 1998; Thomas and Stamnes 1999; Ritter and Geleyn 2008; Zhang et al. 2010; Doicu et al. 2013; Efremenko et al. 2013; Zhang and Li 2013; Zhang et al. 2013; Wang 2017a). Among these approximate methods, adding algorithm of \(\delta\)-Eddington approximation (referred as \(\delta\)-2SDA) and adding algorithm of \(\delta\)-two-stream discrete ordinates method (DOM), referred as \(\delta\)-2DDA, are widely used in numerical models because of their high computational efficiency.

In recent two decades, the \(\delta\)-two-stream approximation methods have been compared under the inclusion of gaseous absorption and cloud in radiative transfer models like BCC_RAD, Fu-Liou model and CanAM4.5 RAD (King and Harshvardhan 1986; Meador and Weaver 1988; Kay et al. 2001; Rüßmann 2002; Lu et al. 2009; Zhang and Li 2013; Zhang et al. 2013; Yang et al. 2019; Wang 2017b). It is found that the \(\delta\)-two-stream approximation method can produce accurate results for the upward/downward radiative flux and heating rate under the clear-sky condition, however, it leads to large errors under cloudy sky conditions. The cloud heating calculated by the \(\delta\)-two-stream approximation might have been underestimated by as much as 10% and the errors in flux can be over 10 Wm\(^{-2}\) (Kay et al. 2001; Lu et al. 2009; Zhang and Li 2013; Zhang et al. 2013; Yang et al. 2019). Zdunkowski et al. (1980) provided another \(\delta\)-two-stream approximation method, Practical Improved Flux Method (referred as \(\delta\)-PIFM), and suggested that it can produce a more accurate result in most cases compared to other \(\delta\)-two-stream approximations. However, there are still large errors in the flux and heating rate produced by \(\delta\)-PIFM in cloudy conditions. It indicates that the \(\delta\)-two-stream approximation method is not sufficient to characterize strong anisotropic scattering in solar spectra.

In order to improve the accuracy of radiative transfer, the adding of \(\delta\)-four-stream DOM (referred as \(\delta\)-4DDA) and adding of \(\delta\)-four-stream spherical harmonic expansion (referred as \(\delta\)-4SDA) were developed (Zhang and Li 2013; Zhang et al. 2013). The accuracy of \(\delta\)-4DDA and \(\delta\)-4SDA was assessed in BCC_RAD and Fu-Liou model respectively, and it shows that the relative errors of them are generally less than 1% in both flux and heating rate (Zhang and Li 2013; Zhang et al. 2013). In addition, Wang (2017a, b) presented a unified formulation of radiative transfer and found that hemispherical harmonics method with four components (\(\delta\)-HSHM4) is nearly as accurate as \(\delta\)-4DDA.

Although many works have been done for comparison, there are still limitations in previous studies in which, with respect to \(\delta\)-4DDA and \(\delta\)-4SDA, little attention has been paid to the requirement of a large sample size to represent the accuracy of schemes in complex atmosphere situations. Besides, Rapid Radiative Transfer Model for General Circulation Models Applications (RRTMG) developed by Clough et al. (2005) is widely used in Weather Research and Forecasting Model (WRF) and General Circulation Models (GCMs). How is the performance of various radiative transfer schemes in RRTMG for solar spectra? As we know, there is no investigation before. Our motivation for this work is to provide a convincing reference to show the realistic effect of the adding of \(\delta\)-two- and \(\delta\)-four-stream schemes in RRTMG for solar spectra.

2. Comparisons of accuracy

2.1 Ideal atmosphere

The radiative transfer model used in this work, known as RRTMG SW, is a shortwave part of RRTMG. RRTMG SW adopts the correlated-k approach with 13 bands ranges from 50000 cm\(^{-1}\) to 2600 cm\(^{-1}\) including most greenhouse gases. In this subsection, the accuracy of three \(\delta\)-two-stream approximation schemes (\(\delta\)-2SDA, \(\delta\)-2DDA and \(\delta\)-PIFM) and two \(\delta\)-four-stream approximation schemes (\(\delta\)-4SDA and \(\delta\)-4DDA) are investigated in an ideal atmosphere.

The Mid-Latitude Summer atmospheric profile (Anderson et al. 1986) is used and vertically divided into 200 layers with a resolution of 0.25 km. The benchmark results are calculated from
the DOM with $\delta$-128-stream scheme ($\delta$-128S). Five different sky conditions are performed: clear sky; sky containing low cloud (positioned from 1.0 to 2.0 km with liquid water content LWC = 0.22 g/m$^3$, and effective radius $r_e = 5.89 \mu$m); sky containing middle cloud (positioned from 4.0 to 5.0 km with LWC = 0.28 g/m$^3$, and $r_e = 6.2 \mu$m); sky containing high cloud (positioned from 10.0 to 12.0 km with ice water content LWC = 0.0048 g/m$^3$, and mean effective size $D_e = 41.5 \mu$m); sky containing all three clouds. The solar zenith angle is set to $0^\circ$ and surface albedo is set to 0.2.

Figure 1 displays the absolute errors in the heating rate of all five approximation schemes against benchmark results. Under the clear-sky condition, all five approximation schemes produce accurate results (Figs. 1b and 1c). The absolute values of relative errors are 0.60%, 1.79% and 1.22% for $\delta$-2DDA, $\delta$-PIFM and $\delta$-2SDA while those of $\delta$-4DDA and $\delta$-4SDA are 0.46% and 0.73%, respectively. In the cloudy sky (Figs. 1d-o), $\delta$-2DDA produces the poorest results compared to other schemes, as the maximum error can reach −2.24 K/day at the cloud top under the sky containing middle cloud. Considering that the maximum error of $\delta$-2SDA can reduce by about half compared with that of $\delta$-2DDA under the sky containing water cloud (Figs. 1e and 1h), the results of $\delta$-2SDA are more accurate than those of $\delta$-2DDA. The accuracy of $\delta$-PIFM is close to that of $\delta$-2SDA. In contrast, the errors in $\delta$-four-stream schemes reduce sharply and are bounded to 0.2 K/day for $\delta$-4SDA and 0.1 K/day for $\delta$-4DDA (Figs. 1f, 1i, 1l and 1o). It is notable that experiments for different solar zenith angles are also conducted (Figures not shown). Similar results apart from the different scales of overall errors are achieved.

Besides, corresponding results of downward flux at the surface and upward flux at the top of the atmosphere (TOA) are given in Table 1. Large errors are produced by $\delta$-2DDA, as the error can be over 10 W/m$^2$ for the upward flux at TOA in the cloudy sky. For $\delta$-PIFM and $\delta$-2SDA, the errors decrease sharply in water cloudy sky cases while they are still large in ice cloudy sky case. The results of $\delta$-4DDA and $\delta$-4SDA are more accurate compared to those of $\delta$-two-stream schemes, as the errors of $\delta$-4DDA and $\delta$-4SDA are bounded to 1.5 W/m$^2$ in clear and cloudy sky cases. And $\delta$-4SDA produces slightly more accurate results than those of $\delta$-4DDA for upward flux at TOA, in contrast, $\delta$-4DDA produces slightly more accurate results than those of $\delta$-4SDA for downward flux at the surface in most cases.

Generally speaking, the accuracy of $\delta$-four-stream schemes overwhelmingly outperforms that of $\delta$-two-stream schemes. For $\delta$-four-stream schemes, the accuracy of $\delta$-4DDA scheme is slightly more precise than that of $\delta$-4SDA. For $\delta$-two-stream schemes, the accuracy of $\delta$-2DDA is the poorest while $\delta$-2SDA and $\delta$-PIFM produce similar accuracy.

### 2.2 Realistic atmosphere

It is worth noting that Table 1 shows several counterexamples to the general conclusions mentioned above, which may result from the contingencies caused by the minimum of sample size and is due to be checked out in this subsection.

All results in Section 2.1 are based on ideal conditions. There are remarkable limitations due to using only one reference atmospheric profile, fixed surface albedo and inflexible cloud distri-

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![Figure 1](image-url)
bution to describe such a complex process as solar-atmosphere interaction. Considering these limitations, further exploration is considered in more realistic atmosphere by using satellite observation data combined with reanalysis data. The 1° × 1° resolution cloud properties used in this work is collected from the Terra platform Level-3 Moderate Resolution Imaging Spectroradiometer (MODIS) Atmosphere Monthly Global Product (MOD08_M3; Platnick et al. 2015). The surface albedo and profiles of the atmosphere with a spatial resolution of 1° × 1° and 37 vertical levels from 1000 hPa to 1 hPa come from ERA-Interim reanalysis (Dee et al. 2011). All profiles are interpolated to a thickness of 0.25 km for each layer. The data of the lowest several layers may be useless because of ground-clutter contamination. Only those 1° × 1° cells with solar zenith angle less than 85° are used. To get fresher and more representative samples, data of January 2018 and July 2018 are chosen not only to represent winter and summer in the northern hemisphere but also to cover all earth latitudes. There are 54096 domains covered all the latitudes from 90°S to 65°N in January 2018 and 52397 domains covered all the latitudes from 62°S to 90°N in July 2018.

For each 1° × 1° domain, radiative transfer simulation was conducted. Upward flux at TOA and downward flux at the surface are calculated by RRTMG_SW using δ-128S and five approximation schemes. Figure 2a shows the downward flux at the surface of the benchmark (δ-128S) in January 2018. Figures 2b, 2c, 2d, 2e, and 2f give the statistical view of deviations of five approximation schemes against the benchmark. The data of each 20-degree latitude range are combined into a bar. This kind of statistical graph called violin plot contains two parts in each bar; the outer shape represents all possible errors with thickness indicating how

| Atmospheric conditions | δ-128S | δ-2DDA | δ-2SDA | δ-PIFM | δ-4DDA | δ-4SDA |
|------------------------|--------|--------|--------|--------|--------|--------|
| Clear sky              | 236.60 | 237.67 | 238.51 | 237.97 | 236.13 | 237.29 |
| Low cloud              | 870.51 | 881.06 | 869.87 | 868.22 | 871.86 | 869.51 |
| Middle cloud           | 949.52 | 960.84 | 949.81 | 948.81 | 950.71 | 948.88 |
| High cloud             | 306.44 | 289.75 | 299.22 | 301.75 | 306.28 | 306.70 |
| Full cloud             | 991.92 | 1001.41 | 991.26 | 992.68 | 991.56 | 991.35 |

### Table 1. Comparisons of upward flux at TOA and downward flux at the surface (W/m²) for the five approximation schemes. The cosine of solar zenith μ₀ = 1. The values in parentheses display errors (W/m²) of the five schemes against δ-128S.

| Atmospheric conditions | F↑(TOA) | F↓(surface) |
|------------------------|---------|-------------|
| Clear sky              | 1075.39 | 1075.39     |
| Low cloud              | 175.67  | 175.67      |
| Middle cloud           | 150.50  | 150.50      |
| High cloud             | 984.95  | 984.95      |
| Full cloud             | 76.47   | 76.47       |

Fig. 2. The statistical results of downward flux (W/m²) at the surface in January 2018 arranged by: (a) the benchmark computed from δ-128S, and the deviations of (b) δ-PIFM, (c) δ-2DDA, (d) δ-2SDA, (e) δ-4DDA and (f) δ-4SDA.
common and another part inside shows a few statistical values. For example, the first bar in Fig. 2b displays statistical information of all deviations of $1^\circ \times 1^\circ$ domains between 90°S to 70°S; the width of blue bar indicates the frequency of samples and the red line, lower black cross, middle yellow cross and upper black cross present the mean, 10% quantiles, median and 90% quantiles, respectively.

Except for the information of different latitudes shown in Fig. 2, some statistical values are also calculated for global samples (similarly hereinafter). For downward flux errors at the surface in January 2018, $\delta$-PIFM creates the largest mean error with the value of 1.03 W/m$^2$ compared to other schemes, while it has the smallest standard deviation with the value of 1.14 W/m$^2$ among three $\delta$-two-stream schemes. $\delta$-2DDA produces the smallest mean error with the value of $-0.7$ W/m$^2$ among three $\delta$-two-stream schemes, while the largest standard deviation with the value of 2.44 W/m$^2$ compared to other schemes. And the maximum error of $\delta$-2DDA can reach 15.72 W/m$^2$ in tropical areas. For $\delta$-four-stream schemes, the mean errors decrease significantly, with the value of $-0.52$ W/m$^2$ for $\delta$-PIFM and $-0.7$ W/m$^2$ for $\delta$-2DDA. The standard deviations also obviously reduce compared to those of $\delta$-two-stream schemes, with the value of $0.27$ W/m$^2$ for $\delta$-4SDA and the value of $0.18$ W/m$^2$ for $\delta$-4DDA. And the errors of $\delta$-4SDA and $\delta$-4DDA are bounded to 2 W/m$^2$.

Figure 3 is the same configured as Fig. 2 except for upward flux at TOA. $\delta$-2DDA produces the largest mean error and standard deviation, with the value of 6.97 W/m$^2$ and 5.27 W/m$^2$, respectively, among three $\delta$-two-stream schemes. And the maximum error of $\delta$-2DDA can reach 30.85 W/m$^2$ in tropical areas, it is consistent with the conclusion that large errors occur in areas with large cloud fraction for $\delta$-2DDA from Fig. 1. $\delta$-PIFM and $\delta$-2SDA have similar accuracy, as the mean errors (standard deviations) are 1.64 W/m$^2$ (3.53 W/m$^2$) for $\delta$-PIFM and 1.98 W/m$^2$ (3.53 W/m$^2$) for $\delta$-2SDA. For $\delta$-four-stream schemes, the errors of $\delta$-4SDA and $\delta$-4DDA are bounded to 1.5 W/m$^2$. And the mean errors (standard deviations) are $-0.3$ W/m$^2$ (0.38 W/m$^2$) for $\delta$-4DDA and 0.28 W/m$^2$ (0.22 W/m$^2$) for $\delta$-4SDA.

According to information given by both Figures 2-5 and statistical values mentioned above, $\delta$-four-stream schemes are overwhelmingly accurate than $\delta$-two-stream schemes. In $\delta$-two-stream schemes, the accuracy of $\delta$-2DDA is the most unstable, judged by no matter the extreme values or the standard deviations. $\delta$-2SDA and $\delta$-PIFM, however, are more stable and have little difference between each other in precision. In $\delta$-four-stream schemes, although $\delta$-4DDA has slightly lower mean errors and standard

![Image](57x43 to 397x374)
Fig. 4. As in Fig. 2, but for the results of downward flux (W/m²) at the surface in July 2018.

Fig. 5. As in Fig. 2, but for the results of upward flux (W/m²) at TOA in July 2018.
deviations than δ-4SDA for downward flux at the surface, the situation is contrary for upward flux at TOA.

3. Computational efficiency

The high computational efficiency is a crucial requirement for GCMs or remote sensing. Table 2 presents the comparison results of computer time consumed for different radiative transfer schemes. It is worth noting that the computational efficiency of δ-four-stream DOM solution derived by Fu (1991), referred as Fu-Liou-δ-4DOM, is also compared in Table 2. In Fu-Liou-δ-4DOM scheme, a method of the inverse matrix is used to solve the connection between layers. As proposed by Zhang and Li (2013) and Zhang et al. (2013), the accuracy of δ-4DDA are the same as that of Fu-Liou-δ-4DOM. However, due to some code improvements in δ-4DDA and δ-4SDA, both of them are much more efficient in this work. Thus, Fu-Liou-δ-4DOM scheme is transplanted from the Fu-Liou radiation model (Fu and Liou 1992) to RRTMG to behave as a reference for computational efficiency.

From Table 2, there is no obvious difference in computational time between three δ-two-stream schemes. For pure radiative transfer algorithm, the computational time of δ-4SDA is about 27% less than that of δ-4DDA. This is because the single-layer solution of δ-four-stream SHM is much simpler than that of δ-four-stream DOM, as discussed in Zhang and Li (2013). Because of several code improvements, the computational time of δ-4DDA (δ-4SDA) is about 54% (39%) that of Fu-Liou-δ-4DOM. For the RRTMG model with radiative transfer calculation including gaseous transmission and cloud absorption, the computational time of δ-4DDA (δ-4SDA) is about 70% (62%) that of Fu-Liou-δ-4DOM.

4. Summary and discussion

In this study, several approximate methods of radiative transfer solution are compared in both accuracy and efficiency. In the case of the ideal atmosphere, the accuracy of δ-2SDA is close to that of δ-PIFM while δ-2DDA produces the poorest results among the five approximate schemes, as the errors of δ-2DDA can be over 10 W/m² for the upward flux at TOA in the cloudy sky. The accuracy of δ-four-stream schemes overwhelmingly outperforms that of δ-two-stream schemes in the heating rate and flux, as the flux errors of δ-4DDA and δ-4SDA are bounded to 1.5 W/m² in clear and cloudy sky cases. And δ-4SDA is slightly more precise than δ-4SDA.

To eliminate the errors introduced by the limitations of sample size, further exploration with realistic atmosphere profiles derived from satellite observation data and reanalysis data is conducted. It indicates that with respect to upward flux at TOA and downward flux at the surface, the δ-four-stream schemes are superior to the δ-two-stream schemes by about an order magnitude less in errors. For δ-two-stream schemes, δ-PIFM seems to be as accurate as δ-2SDA, and δ-2DDA shows the poorest accuracy. For δ-four-stream schemes, the accuracy of δ-4SDA is comparable to that of δ-4DDA.

The computational time involved by δ-two-stream schemes and δ-four-stream schemes are minimal because all of them are analytical methods. Three δ-two-stream schemes cost almost the same computational time. For the RRTMG model with radiative transfer calculation, The computational time of δ-4SDA is about 1.5 times that of the δ-two-stream schemes. And the computational time of δ-4DDA (δ-4SDA) is about 70% (62%) that of Fu-Liou-δ-4DOM.

With the development of computer, considering overall accuracy and efficiency, both δ-4DDA and δ-4SDA seem to be excellent choices in most cases.

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