THE FREIDLIN-WENTZELL LDP WITH RAPIDLY GROWING COEFFICIENTS

P. CHIGANSKY AND R. LIPTSER

Abstract. The Large Deviations Principle (LDP) is verified for a homogeneous diffusion process with respect to a Brownian motion \( B_t \),

\[
X^\varepsilon_t = x_0 + \int_0^t b(X^\varepsilon_s) ds + \varepsilon \int_0^t \sigma(X^\varepsilon_s) dB_s,
\]

where \( b(x) \) and \( \sigma(x) \) are locally Lipschitz functions with super linear growth. We assume that the drift is directed towards the origin and the growth rates of the drift and diffusion terms are properly balanced. Nonsingularity of \( a = \sigma \sigma^* (x) \) is not required.

1. Introduction

In this paper we extend the set of conditions, under which Freidlin-Wentzell’s Large Deviation Principle (LDP) for a homogeneous diffusion process remains valid. We consider a family \( \{(X^\varepsilon_t)_{t \geq 0}\}_{\varepsilon \rightarrow 0} \) of diffusions, where \( X^\varepsilon_t \in \mathbb{R}^d, d \geq 1 \) is defined by the Itô equation

\[
X^\varepsilon_t = x_0 + \int_0^t b(X^\varepsilon_s) ds + \varepsilon \int_0^t \sigma(X^\varepsilon_s) dB_s,
\]

relative to a standard Brownian motion \( B_t \), where \( b(x) \) and \( \sigma(x) \) are vector and matrix valued continuous functions of dimensions \( d \) and \( d \times d \) respectively, guaranteeing existence of the unique weak solution.

The classical Freidlin-Wentzell setting [8] (see e.g. Dembo and Zeitouni, [4]) covers the model (1.1) with bounded \( b(x) \) and \( \sigma(x) \) and uniformly positive definite diffusion matrix \( a(x) = \sigma \sigma^* (x) \). Various LDP versions can be found in Dupuis and Ellis [5], Feng [6], Feng and Kurtz [7], Friedman [9], Liptser and Pukhalskii, [12], Mikami [15], Narita [16], Stroock [23], Ren and Zhang [22]. In the recent paper [19], Puhalskii extends LDP to (1.1) with continuous and unbounded coefficients and singular \( a(x) \), assuming \( b(x) \) and \( a(x) \) are Lipschitz continuous functions (concerning singular \( \sigma(x) \) see also Liptser et al, [14]). Being Lipschitz continuous, the entries of \( b, \sigma \) grow not faster than linearly and, thereby, \textit{automatically} guarantee one of the necessary conditions for LDP (\( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^d \))

\[
\lim_{C \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P} \left( \sup_{t \leq T} \| X^\varepsilon_t \| > C \right) = -\infty, \quad \forall \ T > 0.
\]

Relinquishing the linear growth condition for \( b, \sigma \) would require additional assumptions providing (1.2).

This paper is inspired by Puhalskii’s remark in [19]:

\[\text{Research supported by a grant from the Israel Science Foundation.}\]
If the drift is directed towards the origin, then no restrictions are needed on the growth rate of the drift coefficient.

In particular, in this case the LDP holds, regardless of the growth rate of \( b(x) \), for a constant diffusion matrix (not necessarily nonsingular).

In this paper, we show that in fact LDP remains valid for (1.1) with non-constant diffusion term, if its growth rate is properly balanced relatively to the drift (see (H-3) of Theorem 2.1 below). Our result is formulated in terms of Khasminskii-Veretennikov’s condition (H-2) (see [11] and [17], [18])

The rest of the paper is organized as follows. In Sections 2 and 3, the main result, notations and preliminary facts on the LDP are given. Sections 4 - 6 contain the proof of the main result. Auxiliary technical details are gathered in Appendices A - C.

2. Notations and the main result

The following notations and conventions are used throughout the paper.

- * denotes the transposition symbol
- all vectors are columns (unless explicitly stated otherwise)
- \( |x| \) and \( \|x\| \) denote the \( \ell_1 \) and \( \ell_2 \) (Euclidian) norms of \( x \in \mathbb{R}^d \)
- \( \langle \langle x, y \rangle \rangle \) denotes the scalar product of \( x, y \in \mathbb{R}^d \)
- \( \|x\|_2^2 = \langle \langle x, \Gamma x \rangle \rangle \) with an nonnegative definite matrix \( \Gamma \)
- \( a(x) = \sigma(x)\sigma^*(x) \)
- \( a^\perp(x) \) denotes the Moore-Penrose pseudoinverse matrix of \( a(x) \) (see [1])
- \( \nabla V(x) \) is the gradient (row) vector of \( V(x) \):
  \[ \nabla V(x) := \left( \frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_d} \right) \]
- \( \langle M, N \rangle_t \) is the joint quadratic variation process of continuous martingales \( M_t \) and \( N_t \); for brevity \( \langle M, M \rangle_t = \langle M \rangle_t \)
- a.s. abbreviates “almost surely”; when the corresponding measure is not specified the Lebesgue measure on \( \mathbb{R}_+ \) is understood
- \( \rho \) is the locally uniform metric on \( \mathcal{C}_{(0, \infty)}(\mathbb{R}^d) \)
- \( I \) denotes \( d \times d \) identity matrix
- the convention \( 0/0 = 0 \) is kept throughout
- \( X^\varepsilon = (X^\varepsilon_t)_{t \geq 0} \)
- \( \inf \{ \emptyset \} = \infty \).

We study the LDP for the family \( \{X^\varepsilon\}_{\varepsilon \to 0} \) in the metric space \( (\mathcal{C}_{(0, \infty)}(\mathbb{R}^d), \rho) \) with \( \rho(x, y) = \sum_{k=1}^\infty 2^{-k}(1^{\sup_{t \leq k} \|x_t - y_t\|}) \), \( x, y \in \mathcal{C}_{(0, \infty)}(\mathbb{R}^d) \). Recall that \( \{X^\varepsilon\}_{\varepsilon \to 0} \) satisfies the LDP with the good rate function \( J(u) : \mathcal{C}_{(0, \infty)}(\mathbb{R}^d) \to [0, \infty] \) and the rate \( \varepsilon^2 \), if the level sets of \( J(u) \) are compacts and for any closed set \( F \) and open set \( G \) in \( \mathcal{C}_{(0, \infty)}(\mathbb{R}^d) \),

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \log P(X^\varepsilon \in F) \leq - \inf_{u \in F} J(u), \\
\lim_{\varepsilon \to 0} \varepsilon^2 \log P(X^\varepsilon \in G) \geq - \inf_{u \in G} J(u).
\]

Our main result is
Theorem 2.1. Assume:

(H-1) entries of $b(x)$ and $\sigma(x)$ are locally Lipschitz continuous functions,

(H-2) $\lim_{\|x\| \to \infty} \langle x, b(x) \rangle / \|x\| = -\infty$,

(H-3) for some positive constants $K$ and $L$, $\frac{\langle x, a(x)x \rangle}{\|x\| |\langle x, b(x) \rangle|} \leq K, \forall \|x\| > L$.

Then $\{X^\varepsilon_t\}_{\varepsilon \to 0}$ obeys the LDP in the metric space $(C_{[0,\infty)}(\mathbb{R}^d), \varrho)$ with the rate $\varepsilon^2$ and the rate function

$$J(u) = \begin{cases} \frac{1}{2} \int_0^\infty \|\dot{u}_t - b(u_t)\|^2_{a^\varepsilon(u_t)} dt, & u \in \Gamma \\ \infty, & u \notin \Gamma, \end{cases}$$

where

$$\Gamma = \left\{ u \in C_{[0,\infty)} : u_0 = x_0, du_t \ll dt, \int_0^\infty \|\dot{u}_t\|^2 dt < \infty, \ a(u_t) a^\varepsilon(u_t) [\dot{u}_t - b(u_t)] = [\dot{u}_t - b(u_t)] \ a.s. \right\}.$$ 

Remark 2.1. In the scalar case (recall $0/0=0$)

$$J(u) = \begin{cases} \frac{1}{2} \int_0^\infty \frac{\dot{u}_t - b(u_t)^2}{\sigma^2(u_t)} dt, & du_t = \dot{u}_t dt, u_0 = x_0, \int_0^\infty \dot{u}_t^2 dt < \infty \\ \infty, & \text{otherwise}. \end{cases}$$

Example 2.1. A typical example within the scope of Theorem 2.1 is

$$X^\varepsilon_t = x_0 - \int_0^t (X^\varepsilon_s)^3 ds + \varepsilon \int_0^t |X^\varepsilon_s|^{3/2} dB_s.$$

3. Preliminaries

We follow the framework, set up by A.Puhalskii (see [20], [21]):

Exponential tightness \hspace{1cm} Local LDP \hspace{1cm} $\iff$ LDP

The exponential tightness in the metric space $(C_{[0,\infty)}, \varrho)$ is convenient to verify in terms of, so called, $C$-exponential tightness conditions introduced by A.Puhalskii (see e.g. [12]), which are based on D. Aldous’s “stopping time and tightness” concept (see [2], [3]). To this end, let us assume that the diffusion processes are defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}^\varepsilon = (\mathcal{F}^\varepsilon_t)_{t \geq 0}, \mathbb{P})$, satisfying the usual conditions, where the filtration $\mathbb{F}^\varepsilon$ may depend on $\varepsilon$.

Recall (see [12]) that the family of diffusion processes is $C$-exponentially tight if for any $T > 0, \eta > 0$ and any $\mathbb{F}^\varepsilon$-stopping time $\theta$,

$$\lim_{C \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left( \sup_{t \leq T} \|X^\varepsilon_t\| > C \right) = -\infty, \hspace{1cm} (3.1)$$

$$\lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \mathbb{P} \left( \sup_{t \in [\Delta]} \|X^\varepsilon_{\theta+t} - X^\varepsilon_{\theta}\| > \eta \right) = -\infty. \hspace{1cm} (3.2)$$
The family of diffusion processes obeys the local LDP in \((C_{[0,\infty)}(\mathbb{R}^d), \vartheta)\) if for any \(T > 0\) there exists a local rate function \(J_T(u)\) such that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left( \sup_{t \leq T} \left\| X_t^\varepsilon - u_t \right\| \leq \delta \right) \leq -J_T(u) \quad (3.3)
\]
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left( \sup_{t \leq T} \left\| X_t^\varepsilon - u_t \right\| \leq \delta \right) \geq -J_T(u). \quad (3.4)
\]

Under the conditions (3.1)-(3.4), the family of diffusion processes obeys the LDP with the rate \(\varepsilon^2\) and the good rate function
\[
J(u) = \sup_T J_T(u), \quad u \in C_{[0,\infty)}(\mathbb{R}^d),
\]
where
\[
J_T(u) = \begin{cases} 
\frac{1}{2} \int_0^T \| \dot{u}_t - b(u_t) \|^2_{a^{\otimes}(u_t)} dt, & u \in \Gamma_T \\
\infty, & u \notin \Gamma_T,
\end{cases}
\]
with
\[
\Gamma_T = \left\{ u \in C_{[0,T]} : \quad u_0 = x_0, \quad du_t \ll dt, \quad \int_0^T \| \dot{u}_t \|^2 dt < \infty \right. \\
\left. a(u_t) a^{\otimes}(u_t) [\dot{u}_t - b(u_t)] = [\dot{u}_t - b(u_t)] \text{ a.s.} \right\}.
\]
Thus the proof of Theorem 2.1 is reduced to establishing (3.1) - (3.4).

4. The proof of \(C\)-exponential tightness

4.1. Auxiliary lemma. Let \(\mathcal{D}\) be a nonlinear operator acting on continuously differentiable functions \(V(x) : \mathbb{R}^d \to \mathbb{R}\) as follows:
\[
\mathcal{D}V(x) = \langle \langle \nabla V(x), b(x) \rangle \rangle + \frac{1}{2} \langle \langle \nabla V(x), a(x) \nabla V(x) \rangle \rangle.
\]

**Lemma 4.1.** Assume there exists twice continuously differentiable nonnegative function \(V(x)\) such that
(a-1) \(\lim_{C \to \infty} \inf_{\|x\| \geq C} V(x) = \infty\)
(a-2) for some \(L > 0\), \(\mathcal{D}V(x) \leq 0, \forall \|x\| > L\).
Then (3.1) holds.

**Proof.** Notice that (3.1) is equivalent to
\[
\lim_{C \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log P(\Theta_C \leq T) = -\infty, \quad (4.1)
\]
where
\[
\Theta_C = \inf \{ t : \| X_t^\varepsilon \| \geq C \}, \quad C > 0 \quad (4.2)
\]
are stopping times relative to \(\mathbb{F}^\varepsilon\).

We use (a) of Proposition A.1 to estimate \(\log P(\Theta_C \leq T)\). An appropriate martingale \(M^\varepsilon_t\) is constructed with the help of function \(V(x)\). Let
\[
\Psi(x) = \begin{pmatrix} V_{11}(x) & V_{12}(x) & \cdots & V_{1d}(x) \\
V_{21}(x) & V_{22}(x) & \cdots & V_{2d}(x) \\
\vdots & \vdots & \ddots & \vdots \\
V_{d1}(x) & V_{d2}(x) & \cdots & V_{dd}(x) \end{pmatrix}.
\]
By applying the Itô formula we find that
\[ \varepsilon^{-2} V(x_\Theta^\varepsilon) = \varepsilon^{-2} V(x_0) + \int_0^{\Theta \wedge T} \varepsilon^{-2} \langle \nabla V(X_s^\varepsilon), b(X_s^\varepsilon) \rangle \, ds \]
\[ + \int_0^{\Theta \wedge T} \varepsilon^{-1} \langle \nabla V(X_s^\varepsilon), \sigma(X_s^\varepsilon) dB_s \rangle + \int_0^{\Theta \wedge T} \frac{1}{2} \text{trace} \left( \Psi(X_s^\varepsilon) a(X_s^\varepsilon) \right) \, ds. \]
We choose \( M_t^\varepsilon = \int_0^t \varepsilon^{-1} \langle \nabla V(X_s^\varepsilon), \sigma(X_s^\varepsilon) dB_s \rangle \), which has the variation process \( \langle M \rangle_t = \int_0^t \varepsilon^{-2} \langle \nabla V(X_s^\varepsilon), a(X_s^\varepsilon) \nabla V(X_s^\varepsilon) \rangle \, ds \). Clearly
\[ M_0^\varepsilon = \varepsilon^{-2} V(X_0^\varepsilon) - \varepsilon^{-2} V(x_0) \]
\[ - \int_0^{\Theta \wedge T} \varepsilon^{-2} \langle \nabla V(X_s^\varepsilon), b(X_s^\varepsilon) \rangle \, ds \]
\[ - \int_0^{\Theta \wedge T} \frac{1}{2} \text{trace} \left( \Psi(X_s^\varepsilon) a(X_s^\varepsilon) \right) \, ds. \]
Hence, by the definition of \( \mathcal{D} \), one gets
\[ M_0^\varepsilon = \varepsilon^{-2} V(X_0^\varepsilon) - \varepsilon^{-2} V(x_0) \]
\[ - \int_0^{\Theta \wedge T} \frac{1}{2} \text{trace} \left( \Psi(X_s^\varepsilon) a(X_s^\varepsilon) \right) \, ds - \int_0^{\Theta \wedge T} \varepsilon^{-2} \mathcal{D} V(X_s^\varepsilon) \, ds. \] (4.3)
On the set \( \{ \Theta \leq T \} \), we have
\[ \varepsilon^{-2} V(X_\Theta^\varepsilon) - \varepsilon^{-2} V(x_0) \geq \varepsilon^{-2} \inf_{||x|| \leq C} V(x) - \varepsilon^{-2} V(x_0), \]
and
\[ \left| \int_0^{\Theta \wedge T} \frac{1}{2} \text{trace} \left( \Psi(X_s^\varepsilon) a(X_s^\varepsilon) \right) \, ds \right| \leq \frac{T}{2} \sup_{||x|| \leq C} \left| \text{trace} \left( \Psi(x) a(x) \right) \right|, \]
and, by (a-2),
\[ - \int_0^{\Theta \wedge T} \varepsilon^{-2} \mathcal{D} V(X_s^\varepsilon) \, ds \]
\[ \geq - \int_0^{\Theta \wedge T} \varepsilon^{-2} I_{\{||X_s|| \leq L\}} \mathcal{D} V(X_s^\varepsilon) \, ds \geq -\varepsilon^{-2} T \sup_{||x|| \leq L} |\mathcal{D} V(x)|. \]
These inequalities and (4.3) imply
\[ M_0^\varepsilon - \frac{1}{2} \langle M \rangle_0 \geq \varepsilon^{-2} \inf_{||x|| \geq C} V(x) - \varepsilon^{-2} V(x_0) \]
\[ - \frac{T}{2} \sup_{||x|| \leq C} \left| \text{trace} \left( \Psi(x) a(x) \right) \right| - \varepsilon^{-2} T \sup_{||x|| \leq L} |\mathcal{D} V(x)| \]
on the set \( \{ \Theta \leq T \} \). Hence, due to (a) of Proposition A.1
\[ \varepsilon^2 \log P(\Theta \leq T) \leq \]
\[ - \inf_{||x|| \geq C} V(x) + V(x_0) + \frac{T \varepsilon^2}{2} \sup_{||x|| \leq C} \left| \text{trace} \left( \Psi(x) a(x) \right) \right| + T \sup_{||x|| \leq L} |\mathcal{D} V(x)| \]
\[ \xrightarrow[\varepsilon \to 0]{} - \inf_{||x|| \geq C} V(x) + V(x_0) + T \sup_{||x|| \leq L} |\mathcal{D} V(x)| \]
and it is left to recall that by (a-1) \( \lim_{C \to \infty} \inf_{||x|| \geq C} V(x) = \infty. \) \( \square \)
4.2. The proof of (3.1). We apply Lemma 4.1 to

$$V(x) = \frac{c\|x\|^2}{1 + \|x\|}$$

with a positive parameter $c \leq \frac{1}{K}$ for $K$ from (H-3) of Theorem 2.1. The function $V(x)$ is twice continuously differentiable and satisfies (a-1). It is left to show that $V(x)$ satisfies (a-2) as well.

Direct computations give $\nabla V(x) = \frac{(2 + \|x\|)\|x\|}{(1 + \|x\|)^2}$. Denote

$$r(x) := \frac{(2 + \|x\|)\|x\|}{(1 + \|x\|)^2}$$

and notice that $r(x) \leq 1$. By assumption (H-2) of Theorem 2.1, one can choose $L > 0$ sufficiently large so that $\langle \langle x, b(x) \rangle \rangle < 0$ for any $\|x\| \geq L$. On the other hand, by assumption (H-3) of Theorem 2.1, $-1 + \frac{c}{2} \|x\| \|\langle \langle x, a(x) \rangle \rangle\| \leq -\frac{1}{2}$ for $\|x\| \geq L$ and

$$\mathcal{D} V(x) = \left( \frac{r(x)}{\|x\|} \langle \langle x, b(x) \rangle \rangle + \frac{c^2 r^2(x)}{2} \frac{\langle \langle x, a(x) \rangle \rangle}{\|x\|^2} \right)$$

$$= \left( -\frac{r(x)}{\|x\|} \|\langle \langle x, b(x) \rangle \rangle\| + \frac{c^2 r^2(x)}{2} \frac{\langle \langle x, a(x) \rangle \rangle}{\|x\|^2} \right)$$

$$= cr(x) \|\langle \langle x, b(x) \rangle \rangle\| \left( -1 + \frac{c}{2} r(x) \frac{\langle \langle x, a(x) \rangle \rangle}{\|x\| \|\langle \langle x, b(x) \rangle \rangle\|} \right)$$

$$\leq cr(x) \|\langle \langle x, b(x) \rangle \rangle\| \left( -1 + \frac{c}{2} \frac{\langle \langle x, a(x) \rangle \rangle}{\|x\| \|\langle \langle x, b(x) \rangle \rangle\|} \right)$$

$$\leq -\frac{1}{2} cr(x) \|\langle \langle x, b(x) \rangle \rangle\|$$

and (a-2) follows. \qed

4.3. The proof of (3.2). The obvious inclusion

$$\left\{ \sup_{t \leq \Delta} \|X_{\theta+t}^\varepsilon - X_\theta^\varepsilon\| > \eta \right\}$$

$$\subseteq \left\{ \sup_{t \leq \Delta} \|X_{\theta+t}^\varepsilon - X_\theta^\varepsilon\| > \eta, \ \Theta_C = \infty \right\} \cup \left\{ \Theta_C \leq T \right\}$$

reduces the proof to verifying

$$\lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \left( \sup_{t \leq \Delta} \|X_{\theta+t}^\varepsilon - X_\theta^\varepsilon\| > \eta, \ \Theta_C = \infty \right) = -\infty$$

(4.4)

for any fixed $C$. Indeed if (4.4) holds, then

$$\lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \left( \sup_{t \leq \Delta} \|X_{\theta+t}^\varepsilon - X_\theta^\varepsilon\| > \eta \right)$$

$$\leq \lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \left( \sup_{t \leq \Delta} \|X_{\theta+t}^\varepsilon - X_\theta^\varepsilon\| > \eta, \ \Theta_C = \infty \right)$$

$$\vee \lim_{C \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\Theta_C \leq T)$$
and, thus, (3.2) is implied by (4.4) and (4.1). So, it is left to check (4.4) for any entry \( x^\varepsilon_t \) of \( X^\varepsilon_t \):

\[
\lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \mathbb{P} \left( \sup_{t \leq \Delta} |x^\varepsilon_{\theta t} - x^\varepsilon_{\theta t}| > \eta, \ \Theta_C = \infty \right) = -\infty.
\]

A generic entry of \( X^\varepsilon_t \) satisfies

\[
x^\varepsilon_t = x^\varepsilon_0 + \int_0^t \gamma^\varepsilon_s ds + \varepsilon m^\varepsilon_t,
\]

where \( \gamma^\varepsilon_t \) is \( \mathbb{P}^\varepsilon \)-adapted continuous random process and \( m_t \) is \( \mathbb{P}^\varepsilon \)-continuous martingale with \( \langle m^\varepsilon \rangle_t = \int_0^t \mu^\varepsilon_s ds \). Since \( b \) and \( \sigma \) are locally Lipschitz continuous functions, there is a constant \( l_C \), such that \( |\gamma^\varepsilon_{C \wedge t}| \leq l_C \) and \( \mu^\varepsilon_{C \wedge t} \leq l_C \).

Taking into account that

\[
\sup_{t \leq \Delta} \int_0^t \gamma^\varepsilon_s ds \geq \eta, \ \Theta_C = \infty \subseteq \{ l_C \Delta \geq \eta \} = \emptyset, \text{ for } \Delta < \eta/l_C,
\]

it is left to verify

\[
\lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \mathbb{P} \left( \sup_{t \leq \Delta} |\varepsilon m^\varepsilon_{\theta t} - \varepsilon m^\varepsilon_{\theta t}| > \eta, \ \Theta_C = \infty \right) = -\infty.
\]

Due to the obvious inclusion

\[
\left\{ \sup_{t \leq \Delta} |\varepsilon m^\varepsilon_{\theta t} - \varepsilon m^\varepsilon_{\theta t}| > \eta, \ \Theta_C = \infty \right\} =
\left\{ \sup_{t \leq \Delta} |\varepsilon m^\varepsilon_{\Theta C \wedge (\theta+t)} - \varepsilon m^\varepsilon_{\Theta C \wedge \theta}| > \eta, \ \Theta_C = \infty \right\}
\subseteq \left\{ \sup_{t \leq \Delta} |\varepsilon m^\varepsilon_{\Theta C \wedge (\theta+t)} - \varepsilon m^\varepsilon_{\Theta C \wedge \theta}| > \eta \right\},
\]

we shall verify

\[
\lim_{\Delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\theta \leq T} \mathbb{P} \left( \sup_{t \leq \Delta} |\varepsilon m^\varepsilon_{\Theta C \wedge (\theta+t)} - \varepsilon m^\varepsilon_{\Theta C \wedge \theta}| > \eta \right) = -\infty.
\]

Notice that \( n^\varepsilon_t := \varepsilon m^\varepsilon_{\Theta C \wedge (\theta+t)} - \varepsilon m^\varepsilon_{\Theta C \wedge \theta} \) is a continuous martingale relative to \( \mathbb{F}^\varepsilon_{\Theta C \wedge \theta+t} \geq 0 \) (see e.g. Ch. 4, §7 in [13]) with \( \langle n^\varepsilon \rangle_t = \varepsilon^2 \int_{\Theta C \wedge \theta}^{\Theta C \wedge (\theta+t)} \mu^\varepsilon_s ds \leq \varepsilon^2 l_C t \). By the statement (d) of Proposition A.1, \( \mathbb{P} \left( \sup_{t \leq \Delta} |n^\varepsilon_t| \geq \eta \right) \leq 2e^{-\eta^2/(2l_C \varepsilon^2 \Delta)} \), so that, \( \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left( \sup_{t \leq \Delta} |n^\varepsilon_t| \geq \eta \right) \leq -\frac{\eta^2}{2l_C \Delta} \to -\infty \).

### 5. Local LDP upper bound

We start with the observation that (3.3) holds if for any \( T > 0 \)

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P} \left( \sup_{t \leq T} \|X^\varepsilon_t - u_t\| \leq \delta, \ \Theta_C = \infty \right) \leq -J_T(u), \tag{5.1}
\]

since by the inclusion

\[
\left\{ \sup_{t \leq T} \|X^\varepsilon_t - u_t\| \leq \delta \right\} \subseteq \left\{ \sup_{t \leq T} \|X^\varepsilon_t - u_t\| \leq \delta, \ \Theta_C = \infty \right\} \cup \left\{ \Theta_C \leq T \right\}
\]
we have
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P \left( \sup_{t \leq T} \| X_t^\varepsilon - u_t \| \leq \delta \right)
\]
\[
\leq \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P \left( \sup_{t \leq T} \| X_t^\varepsilon - u_t \| \leq \delta, \Theta_C = \infty \right) \vee \lim_{\varepsilon \to 0} \varepsilon^2 \log P (\Theta_C \leq T),
\]
and, by (4.1), the last term goes to \(-\infty\) as \(C \to \infty\).

We omit the standard proof for \(u_0 \neq x_0\) or \(du_t \not\in dt\) (see, e.g. [4]). The rest of the proof is split into several steps.

5.1. \(u_0 = x_0\), \(d u_t \ll dt\), \(\int_0^T \| \dot{u}_s \|^2 ds < \infty\). Define the set
\[
\mathcal{A} = \left\{ \sup_{t \leq T} \| X_t^\varepsilon - u_t \| \leq \delta, \Theta_C = \infty \right\}.
\]

With a continuously differentiable vector-valued function \(\lambda(s)\) of the size \(d\), let us introduce a continuous local martingale \(U_t = \int_0^t \langle \langle \lambda(s), \varepsilon \sigma(X_s^\varepsilon) dB_s \rangle \rangle ds\) and its martingale exponential \(3_t = e^{U_t - 0.5(U)_t}\), where
\[
\langle (U)_t \rangle = \int_0^t \varepsilon^2 \langle \langle \lambda(s), a(X_s^\varepsilon) \lambda(s) \rangle \rangle ds.
\]

It is well known that \(3_t\) is a continuous positive local martingale, as well as a supermartingale. Consequently, \(E_{3T} \leq 1\) and, therefore,
\[
1 \geq E_{\mathcal{A}} 3_T. \tag{5.2}
\]

The required upper bound for \(P(\mathcal{A})\) is obtained by estimating \(3_T\) from below on \(\mathcal{A}\). Since \(U_t = \int_0^t \langle \langle \lambda(s), dX_s^\varepsilon - b(X_s^\varepsilon) ds \rangle \rangle\),
\[
U_T - 0.5 \langle (U)_T \rangle = \int_0^T \left[ \langle \langle \lambda(s), dX_s^\varepsilon - b(X_s^\varepsilon) ds \rangle \right]
\]
\[
- \frac{\varepsilon^2}{2} \langle \langle \lambda(s), a(X_s^\varepsilon) \lambda(s) \rangle \rangle ds\right]
\]
\[
= \int_0^T \left[ \langle \langle \lambda(s), \dot{u}_s - b(u_s) \rangle \rangle - \frac{\varepsilon^2}{2} \langle \langle \lambda(s), a(u_s) \lambda(s) \rangle \rangle \right] ds
\]
\[
+ \int_0^T \langle \langle \lambda(s), dX_s^\varepsilon - \dot{u}_s ds \rangle \rangle
\]
\[
+ \int_0^T \langle \langle \lambda(s), b(u_s) - b(X_s^\varepsilon) \rangle \rangle ds
\]
\[
+ \int_0^T \frac{\varepsilon^2}{2} \langle \langle \lambda(s), [a(u_s) - a(X_s^\varepsilon)] \lambda(s) \rangle \rangle ds.
\]

We derive lower bounds on the set \(\mathcal{A}\) for each term in the right hand side of (5.3). Applying the Itô formula to \(\langle \langle \lambda(t), X_t^\varepsilon - u_t \rangle \rangle\), and taking into account that \(X_0^\varepsilon = u_0\), we find that
\[
\langle \langle \lambda(T), X_T^\varepsilon - u_T \rangle \rangle = \int_0^T \langle \langle \lambda(s), dX_s^\varepsilon - \dot{u}_s ds \rangle \rangle + \int_0^T \langle \langle \dot{\lambda}(s), X_s^\varepsilon - u_s \rangle \rangle ds.
\]
Therefore,
\[ \int_0^T \langle \lambda(s), dX_s^\varepsilon - \dot{u}_s ds \rangle \geq -\left| \langle \lambda(T), X_T^\varepsilon - u_T \rangle \right| - \int_0^T \langle \dot{\lambda}(s), X_s^\varepsilon - u_s \rangle ds \geq -r_1 \delta, \]
with \( r_1 := r_1(\lambda, T, C) \geq 0 \), independent of \( \varepsilon \).

Further, with \( r_i := r_i(\lambda, T, C) \geq 0 \), \( i = 2, 3 \), due to the local Lipschitz continuity of \( \sigma \) and \( a \), we find that
\[ \int_0^T \langle \lambda(s), b(u_s) - b(X_s^\varepsilon) \rangle ds \geq -r_2(\lambda, C, T) \delta \]
\[ \int_0^T \frac{\varepsilon^2}{2} \langle \lambda(s), [a(u_s) - a(X_s^\varepsilon)]\lambda(s) \rangle ds \geq -\varepsilon^2 r_3(\lambda, C, T) \delta. \]

Hence with \( r := r_1 + r_2 + \varepsilon^2 r_3 \),
\[ \log 3T \geq \int_0^T \left[ \langle \nu(s), \dot{u}_s - b(u_s) \rangle - \frac{1}{2} \langle \nu(s), a(u_s) \nu(s) \rangle \right] ds - r(\lambda, T, C) \delta. \]
Set \( \nu(s) = \varepsilon^2 \lambda(s) \) and rewrite the above inequality as:
\[ \log 3T \geq \frac{1}{\varepsilon^2} \int_0^T \left[ \langle \nu(s), \dot{u}_s - b(u_s) \rangle - \frac{1}{2} \langle \nu(s), a(u_s) \nu(s) \rangle \right] ds \]
\[ - r \left( \frac{\nu}{\varepsilon^2}, T, C \right) \delta. \]
This lower bound, along with (5.2), provides the following upper bound
\[ \varepsilon^2 \log P(\mathfrak{A}) \leq - \int_0^T \left[ \langle \nu(s), \dot{u}_s - b(u_s) \rangle - \frac{1}{2} \langle \nu(s), a(u_s) \nu(s) \rangle \right] ds \]
\[ + \varepsilon^2 r \left( \frac{\nu}{\varepsilon^2}, T, C \right) \delta. \]

Clearly \( \lim_{\varepsilon \to 0} \varepsilon^2 r \left( \frac{\nu}{\varepsilon^2}, T, C \right) < \infty \) and, hence,
\[ \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P(\mathfrak{A}) \leq - \int_0^T \left[ \langle \nu(s), \dot{u}_s - b(u_s) \rangle - \frac{1}{2} \langle \nu(s), a(u_s) \nu(s) \rangle \right] ds. \quad (5.4) \]

Since the left hand side of (5.4) is independent of \( \nu(s) \), (5.1) is derived by minimizing the right hand side of (5.4) with respect to \( \nu(s) \). Two difficulties arise on the way to direct minimization:
- the matrix \( a(u_s) \) may be singular
- the entries of \( \nu(s) \) should be continuously differentiable functions.
Assume first \( a(u_s) \) is a positive definite matrix, uniformly in \( s \), and write
\[ \langle \nu(s), \dot{u}_s - b(u_s) \rangle - \frac{1}{2} \langle \nu(s), a(u_s) \nu(s) \rangle = \frac{1}{2} \| \dot{u}_s - b(u_s) \|_{a^{-1}(u_s)}^2 \]
\[ - \frac{1}{2} \left\| a^{1/2}(u_s)(\nu(s) - a^{-1}(u_s)[\dot{u}_s - b(u_s)]) \right\|^2. \]
If the entries of \( a^{-1}(u_s)[\dot{u}_s - b(u_s)] \) are continuously differentiable functions, then, by taking \( \nu(s) \equiv -a^{-1}(u_s)[\dot{u}_s - b(u_s)] \) we find that

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P(\mathcal{A}) \leq -\frac{1}{2} \int_0^T \|\dot{u}_s - b(u_s)\|_{a^{-1}(u_s)}^2 ds.
\]  

(5.5)

In the general case, due to \( \int_0^T \|\dot{u}_s\|^2 ds < \infty \), the entries of \( a^{-1}(u_s)[\dot{u}_s - b(u_s)] \) are square integrable with respect to the Lebesgue measure on \([0, T]\). Choose a maximizing sequence \( \nu_n(s) \), \( n \geq 1 \), of continuously differentiable functions such that \( \lim_{n \to \infty} \int_0^T \|\nu_n(s) - a^{-1}(u_s)[\dot{u}_s - b(u_s)]\|^2 ds = 0 \). Since all the entries of \( a(u_s) \) are uniformly bounded on \([0, T]\)

\[
\lim_{n \to \infty} \int_0^T \|a^{1/2}(u_s)(\nu_n(s) - a^{-1}(u_s)[\dot{u}_s - b(u_s)])\|^2 ds = 0
\]

and (5.5) holds too.

Now we drop the uniform nonsingularity assumption of \( a(u_s) \). The upper bound in (5.5) remains valid with \( a(u_s) \) replaced by \( a\beta(u_s) \equiv a(u_s) + \beta I \), where \( \beta \) is a positive number and \( I \) is \((d \times d)\)-unit matrix:

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P(\mathcal{A}) \leq -\frac{1}{2} \int_0^T \|\dot{u}_s - b(u_s)\|_{[a\beta(u_s) + \beta I]^{-1}}^2 ds.
\]

For any fixed \( s \), the function \( \|\dot{u}_s - b(u_s)\|_{[a\beta(u_s) + \beta I]^{-1}} \) increases with \( \beta \downarrow 0 \) and by Lemma B.1 possesses the limit

\[
\lim_{\beta \to 0} \|\dot{u}_s - b(u_s)\|_{[a\beta(u_s) + \beta I]^{-1}}^2 = \begin{cases} \|\dot{u}_s - b(u_s)\|_{a\beta(u_s)}^2, & a(u_s) a\beta(u_s)[\dot{u}_s - b(u_s)] = [\dot{u}_s - b(u_s)] \\ \infty, & \text{otherwise.} \end{cases}
\]

Thus the required upper bound

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P(\mathcal{A}) \leq \begin{cases} -\int_0^T \frac{1}{2} \|\dot{u}_s - b(u_s)\|_{a\beta(u_s)}^2 ds, & a(u_s) a\beta(u_s)[\dot{u}_s - b(u_s)] = [\dot{u}_s - b(u_s)], \text{ a.s.} \\ \infty, & \text{otherwise} \end{cases}
\]

follows by the monotone convergence theorem.

5.2. \( u_0 = x_0 \), \( du \ll dt \), \( \int_0^T \|\dot{u}_s\|^2 ds = \infty \). We emphasize that \( du \ll dt \) on \([0, T]\) implies \( \int_0^T \|\dot{u}_s\| ds < \infty \) and return to the upper bound from (5.4). Since \( b \) and \( \sigma \) are locally Lipschitz, one can choose a constant \( L \) (depending on \( u(s) \)), so that, \( \|\langle \nu(s), b(u_s) \rangle\| \leq \|b(u_s)\| \|\nu(s)\| \leq L \|\nu(s)\| \) and \( \|\langle \nu(s), a(u_s) \nu(s) \rangle\| \leq L \|\nu(s)\|^2 \). Then, (5.4) implies

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P(\mathcal{A}) \leq -\int_0^T \left[ \langle \nu(s), \dot{u}_s \rangle - L \|\nu(s)\| - \frac{L}{2} \|\nu(s)\|^2 \right] ds.
\]

Let \( \nu_n(s) \) be a sequence of continuously differentiable functions, approximating the bounded (for each fixed \( p > 0 \)) function \( L^{-1} \langle \dot{u}_s, I \{\|u_s\| \leq p\} \rangle \) in the
sense that \( \lim_{n \to \infty} \int_0^T \| \frac{1}{L} \dot{u}_s I_{\| \dot{u}_s \| \leq \beta} - \nu_n(s) \|^2 ds = 0 \). Thus,

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P(\mathfrak{A}) \leq -\frac{1}{2L} \int_0^T \| \dot{u}_s \|^2 I_{\| \dot{u}_s \| \leq \beta} ds + \int_0^T \| \dot{u}_s \| ds \quad \text{as } p \to \infty \to -\infty
\]

6. Local LDP lower bound.

If \( \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P( \sup_{t \leq \Theta_C \wedge T} |X^\varepsilon_t - u_t| \leq \delta ) \leq -J_T(u) = -\infty \), then the corresponding lower bound in the local LDP is equal \(-\infty\) too.

So in this section we examine the local LDP lower bound for \( J_T(u) \) from (5.1) when \( J_T(u) < \infty \). The latter means that we may restrict ourselves to analyzing test functions with the properties:

1. \( u_0 = x_0 \)
2. \( du_t \ll dt \)
3. \( a(u_t) [\dot{u}_t - b(u_t)] = [\dot{u}_t - b(u_t)] \) a.s.
4. \( \int_0^T \| \dot{u}_t - b(u_t) \|_{a^\beta(u_t)}^2 dt < \infty, \forall T > 0 \)
5. \( \int_0^T \| \dot{u}_t \|^2 dt < \infty \).

Another helpful observation is that (3.4) holds if for any \( C > 0 \)

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P \left( \sup_{t \leq \Theta_C \wedge T} \| X^\varepsilon_t - u_t \| \leq \delta \right) \geq -J_T(u) \quad (6.2)
\]

due to

\[
\left\{ \sup_{t \leq \Theta_C \wedge T} \| X^\varepsilon_t - u_t \| \leq \delta \right\} \subseteq \left\{ \sup_{t \leq T} \| X^\varepsilon_t - u_t \| \leq \delta \right\} \bigcup \{ \Theta_C \leq T \}
\]

and (4.1).

6.1. Nonsingular \( a(x) \). In this section, the matrix \( a(x) \) is assumed to be uniformly nonsingular in \( x \in \mathbb{R} \), in the sense that \( a(x) \geq \beta I \) for a positive number \( \beta \). Let \( \lambda(s) := \sigma^{-1}(X^\varepsilon_s) [\dot{u}_s - b(X^\varepsilon_s)] \) and introduce a martingale

\[
U_t = \int_0^{\Theta_C \wedge T} \frac{1}{\varepsilon} \langle \lambda(s), dB_s \rangle \quad \text{and its martingale exponential } \exp_{\Theta_C \wedge T} = e^{U_t - 0.5(U)_t},
\]

\( t \leq T \), where \( (U)_t = \int_0^{\Theta_C \wedge T} \frac{1}{\varepsilon} \| \lambda(s) \|^2 ds \).

By (iv) and (v) of (6.1), \( (U)_T \leq \text{const.} \) and, hence, \( E_{\Theta_C \wedge T} = 1 \). We use this fact in order to define a new probability measure \( Q^\varepsilon \) by \( dQ^\varepsilon = \exp_{\Theta_C \wedge T} dP \). Since \( \exp_{\Theta_C \wedge T} \) is positive \( P \)-a.s., \( P \ll Q^\varepsilon \) as well and \( dP = \exp_{\Theta_C \wedge T} dQ^\varepsilon \).

We proceed with the proof of (6.2) by applying

\[
P(\mathfrak{A}) = \int_{\mathfrak{A}} \exp_{\Theta_C \wedge T}^{-1} dQ^\varepsilon \quad (6.3)
\]

to the set \( \mathfrak{A} = \left\{ \sup_{t \leq \Theta_C \wedge T} \| X^\varepsilon_t - u_t \| \leq \delta \right\} \), and estimating from below the right hand side in (6.3). In order to realize this program, it is convenient to have a semimartingale description of the process \( X^\varepsilon_{\Theta_C \wedge T} \) under \( Q^\varepsilon \). Recall that the random process \( B_{\Theta_C \wedge T} \) is a martingale under \( P \) with the variation process \( \langle B \rangle_{\Theta_C \wedge T} = (\Theta_C \wedge t) I \). It is well known (see e.g. Theorem 2, Ch.
4.5 in [13]) that $B_{\Theta C \wedge t}$ is a continuous semimartingale under $Q^\varepsilon$ with the decomposition $B_{\Theta C \wedge t} = \tilde{B}_t + A_t^B$, where $\tilde{B}_t$ is a martingale (under $Q^\varepsilon$) with $\langle \tilde{B} \rangle_t \equiv \langle B \rangle_{\Theta C \wedge t}$ and, by the Girsanov theorem,

$$A_t^B = \int_0^{\Theta C \wedge t} \frac{1}{\varepsilon} \sigma^{-1}(X_s^\varepsilon)[\dot{u}_s - b(X_s^\varepsilon)] ds.$$

In particular,

$$X_s^\Theta_{\Theta C \wedge t} = u_{\Theta C \wedge t} + \varepsilon \int_0^{\Theta C \wedge t} \sigma(X_s^\varepsilon) d\tilde{B}_s, \quad t \leq T, \quad Q^\varepsilon\text{-a.s.}$$

As the next preparatory step we derive the semimartingale decomposition of $U_t$ under $Q^\varepsilon$. As before, the continuous martingale $U_t$ under $P$ is transformed to a semimartingale under $Q^\varepsilon$:

$$U_t = \tilde{U}_t + A_t^U$$

with continuous $Q^\varepsilon$-martingale $\tilde{U}_t$, having the variation process $\langle \tilde{U} \rangle_t \equiv \langle U \rangle_t$, $P$- and $Q^\varepsilon$-a.s., and a continuous drift $A_t^U \equiv \langle U \rangle_t$.

Thus, $U_t = \tilde{U}_t + \langle U \rangle_t, \quad t \leq T, \quad Q^\varepsilon$-a.s. and, thereby, $\tilde{\xi}_T = e^{\tilde{\xi}_T - \frac{1}{2} \langle U \rangle_T}$. Consequently, (6.3) is transformed to

$$P(\tilde{\mathfrak{A}}) = \int_{\tilde{\mathfrak{A}}} \exp \left( - \tilde{U}_T - \frac{1}{2} \langle U \rangle_T \right) dQ^\varepsilon$$

$$= \int_{\tilde{\mathfrak{A}}} \exp \left( - \tilde{U}_T - \frac{1}{2\varepsilon^2} \int_0^{\Theta C \wedge T} \|\dot{u}_s - b(X_s^\varepsilon)\|_{a^{-1}(X_s^\varepsilon)}^2 ds \right) dQ^\varepsilon.$$

We are now in the position to derive a lower bound for the right hand side. Replacing $\tilde{\mathfrak{A}}$ with a smaller set $\tilde{\mathfrak{A}} \cap \mathfrak{B}$, where $\mathfrak{B} = \{ |\varepsilon^2 \tilde{U}_T| < \eta \}$, write

$$P(\tilde{\mathfrak{A}}) \geq \int_{\tilde{\mathfrak{A}} \cap \mathfrak{B}} \exp \left( - \frac{\eta}{\varepsilon^2} - \frac{1}{2\varepsilon^2} \int_0^{\Theta C \wedge T} \|\dot{u}_s - b(X_s^\varepsilon)\|_{a^{-1}(X_s^\varepsilon)}^2 ds \right) dQ^\varepsilon.$$

By the local Lipschitz continuity of $b, \sigma$ and the uniform nonsingularity of $a(x)$,

$$\|\dot{u}_s - b(X_s^\varepsilon)\|_{a^{-1}(X_s^\varepsilon)}^2 - \|\dot{u}_s - b(u_s)\|_{a^{-1}(u_s)}^2 \leq l_C(\|\dot{u}_s\| + 1)^2 \delta, \quad \delta \leq 1,$$

on the set $\tilde{\mathfrak{A}} \cap \mathfrak{B}$ for any $s \leq \Theta C \wedge T$. Then,

$$P(\tilde{\mathfrak{A}}) \geq \int_{\tilde{\mathfrak{A}} \cap \mathfrak{B}} \exp \left( - \frac{\eta}{\varepsilon^2} - \frac{\delta l_C}{\varepsilon^2} \int_0^T (\|\dot{u}_s\| + 1)^2 ds - \frac{1}{2\varepsilon^2} \int_0^{\Theta C \wedge T} \|\dot{u}_s - b(u_s)\|_{a^{-1}(u_s)}^2 ds \right) dQ^\varepsilon$$

$$\geq \int_{\tilde{\mathfrak{A}} \cap \mathfrak{B}} \exp \left( - \frac{\eta}{\varepsilon^2} - \frac{\delta l_C}{\varepsilon^2} \int_0^T (\|\dot{u}_s\| + 1)^2 ds - \frac{1}{2\varepsilon^2} \int_0^T \|\dot{u}_s - b(u_s)\|_{a^{-1}(u_s)}^2 ds \right) dQ^\varepsilon.$$

Consequently,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log P(\tilde{\mathfrak{A}}) \geq -\eta - \delta l_C \int_0^T (\|\dot{u}_s\| + 1)^2 ds - J_T(u) + \lim_{\varepsilon \to 0} \varepsilon^2 \log Q^\varepsilon(\tilde{\mathfrak{A}} \cap \mathfrak{B}).$$
We prove now that \( \lim_{\epsilon \to 0} \epsilon^2 \log Q^\epsilon(\tilde{\mathcal{A}} \cap \mathcal{B}) = 0 \) by showing
\[
\lim_{\epsilon \to 0} Q^\epsilon(\Omega \setminus \tilde{\mathcal{A}}) = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} Q^\epsilon(\Omega \setminus \mathcal{B}) = 0.
\]

To this end, recall that
\[
\Omega \setminus \tilde{\mathcal{A}} = \left\{ \epsilon \sup_{t \leq T} \left\| \int_0^{\Theta_{C \wedge t}} \sigma(X_s^\epsilon)dB_s \right\| > \delta \right\}
\]
and
\[
\Omega \setminus \mathcal{B} = \left\{ \epsilon \left\| \int_0^{\Theta_{C \wedge T}} \sigma^{-1}(X_s^\epsilon)[\dot{u}_s - b(X_s^\epsilon)]dB_s \right\| > \eta \right\}.
\]

We verify (6.4) componentwise. Let \( L^\epsilon_t \) denote any entry of \( \int_0^{\Theta_{C \wedge t}} \sigma(X_s^\epsilon)dB_s \) or \( \int_0^{\Theta_{C \wedge T}} \sigma^{-1}(X_s^\epsilon)[\dot{u}_s - b(X_s^\epsilon)]dB_s \). We show that
\[
\lim_{\epsilon \to 0} Q^\epsilon(\epsilon \sup_{t \leq T} |L^\epsilon_t| > \delta) = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} Q^\epsilon(\epsilon |L^\epsilon_T| > \delta) = 0.
\]

In both cases, \( L^\epsilon_t \) is a continuous \( Q^\epsilon \)-martingale with \( \langle L^\epsilon \rangle_t = \int_0^t g(s)ds \) and \( \int_\Omega \int_0^T g(s)dsdQ^\epsilon < \infty \). Then (6.5) holds by Doob’s inequality:
\[
\lim_{\epsilon \to 0} Q^\epsilon(\epsilon \sup_{t \leq T} |L^\epsilon_t| > \delta) \leq \frac{4\epsilon^2}{\delta^2} \int_\Omega \int_0^T g(s)dsdQ^\epsilon \xrightarrow{\epsilon \to 0} 0.
\]

Now, for any fixed \( \delta \) and \( \eta \),
\[
\lim_{\epsilon \to 0} \epsilon^2 \log P(\tilde{\mathcal{A}}) \geq -\eta - \delta l_C \int_0^T (\|u_s\| + 1)^2 ds - J_T(u).
\]

The required lower bound
\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P(\tilde{\mathcal{A}}) \geq -J_T(u)
\]
follows by taking \( \lim_{\eta \to 0} \lim_{\epsilon \to 0} \).

6.2. General \( a(x) \). This part of the proof requires perturbation arguments. The idea is to use the already obtained local LDP lower bound for the uniformly nonsingular \( a(x) \). Let \( W_t \) be a standard \( d \) dimensional Brownian motion, independent of \( B_t \), defined on the same stochastic basis. Since \( b \) and \( \sigma \) are assumed to be locally Lipschitz continuous, one can introduce the perturbed diffusion process controlled by a free parameter \( \beta \in (0, 1) \):
\[
X_{t}^{\epsilon, \beta} = x_0 + \int_0^t b(X_s^\epsilon, \beta)ds + \epsilon \int_0^t [\sigma(X_s^\epsilon, \beta)dB_s + \sqrt{\beta}dW_s].
\]

The process \( X_{t}^{\epsilon, \beta} \), defined in (6.6), solves the Itô equation \( X_{t}^{\epsilon, \beta} = x_0 + \int_0^t b(X_s^\epsilon, \beta)ds + \epsilon \int_0^t [\sigma(X_s^\epsilon, \beta) + \beta I]^{1/2}dB_s \). with respect to a standard Brownian motion \( B^\beta_t = \int_0^t [a(X_s^\epsilon, \beta) + \beta I]^{-1/2}[\sigma(X_s^\epsilon, \beta)dB_s + \sqrt{\beta}dW_s] \). Then the family \( \{(X_{t}^{\epsilon, \beta})_{t \leq T}\}_{\epsilon \geq 0} \) satisfies the local LDP lower bound. Indeed, the matrix \( a_\beta(x) \) is uniformly nonsingular, its entries are locally bounded and satisfy the assumption (H-3) of Theorem 2.1 since
\[
\frac{\langle x, a_\beta(x)x \rangle}{\|x\|} \geq \frac{\langle x, a(x)x \rangle}{\|x\|} + \beta \frac{\|x\|}{\|\langle x, b(x) \rangle\|}.
\]
and $\frac{\|x\|}{\|x, b(x)\|}$ converges to zero as $\|x\| \to \infty$ by (H-2). In particular, with $\Theta_C^\beta = \inf \{t : \|X_t^\varepsilon, \beta\| \geq C \}$ and $u_0 = x_0$, $du_t \ll dt$, $\int_0^T \|\dot{u}_t\|^2 dt < \infty$, we have

$$\lim\lim_{\delta \to 0} \varepsilon^2 \log P \left( \sup_{t \leq \Theta_C^\beta} \|X_t^\varepsilon - u_t\| \leq \delta \right) \geq -\frac{1}{2} \int_0^T \|\dot{u}_s - b(u_s)\|^2 (a(u_s) + \beta I)^{-1} ds. \quad (6.7)$$

Further, we will use (6.7) to establish

$$\lim\lim_{\delta \to 0} \varepsilon^2 \log P \left( \sup_{t \leq T} \|X_t^\varepsilon - u_t\| \leq \delta \right) \geq -\frac{1}{2} \int_0^T \|\dot{u}_s - b(u_s)\|^2 (a(u_s) + \beta I)^{-1} ds. \quad (6.8)$$

To this end, we introduce the filtration $G^\varepsilon = \{G^\varepsilon_t\}_{t \geq 0}$, with the general conditions, generated by $(X^\varepsilon_t, X^\varepsilon, \beta)_t \geq 0$ and notice that both $\Theta^\beta C$ (see (4.2)) and $\Theta^\beta C$ are stopping times relative to $G^\varepsilon$. Hence,

$$\tau^\beta C = \Theta^\beta \wedge \Theta^\beta C \quad (6.9)$$

is a stopping time as well relative to $G^\varepsilon$. Obviously,

$$\lim\lim_{C \to \infty} \varepsilon^2 \log P (\tau^\beta C \leq T) = -\infty.$$

However, the proof of (6.8) requires a stronger property:

$$\lim\lim_{C \to \infty} \varepsilon^2 \log \sup_{\beta \in [0,1]} P (\tau^\beta_C \leq T) = -\infty. \quad (6.10)$$

It is clear, that (6.10) is valid if it is valid with $\tau^\beta_C$ replaced by $\Theta^\beta C$. The latter is verified along the lines of Lemma 4.1 proof:

$$\varepsilon^2 \log \sup_{\beta \in [0,1]} P (\Theta^\beta C \leq T) \leq -\inf_{\|x\| \geq C} V(x) + V(x_0)$$

$$+ \frac{T \varepsilon^2}{2} \sup_{\beta \in [0,1]} \sup_{\|x\| \leq C} \left| \text{trace} \left( \Psi(x)[a(x) + \beta I] \right) \right| + T \sup_{\beta \in [0,1]} \sup_{\|x\| \leq L} \left| \mathcal{D}_\beta V(x) \right|$$

$$\xrightarrow{\varepsilon \to 0} -\inf_{\|x\| \geq C} V(x) + V(x_0) + T \sup_{\beta \in [0,1]} \sup_{\|x\| \leq L} \left| \mathcal{D}_\beta V(x) \right| \xrightarrow{C \to \infty} -\infty,$$
where $\mathcal{D}_\beta V(x) = \langle \nabla V(x), b(x) \rangle + \frac{1}{2} \langle \nabla V(x), a_\beta(x) \nabla V(x) \rangle$. We are now in the position to prove (6.8). With $\delta \leq \beta^{1/4}$, write

$$
\left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - u_t \| \leq \delta \right\} \\
= \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - u_t \| \leq \delta \right\} \bigcap \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| \leq \beta^{1/4} \right\} \\
\bigcup \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - u_t \| \leq \delta \right\} \bigcap \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| > \beta^{1/4} \right\} \\
\subseteq \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - u_t \| \leq \beta^{1/4} \right\} \bigcap \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| \leq \beta^{1/4} \right\} \\
\bigcup \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| > \beta^{1/4} \right\} \\
\subseteq \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - u_t \| \leq \beta^{1/4} \right\} \bigcup \left\{ \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| > \beta^{1/4} \right\}
$$

Hence,

$$
P\left( \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - u_t \| \leq \delta \right) \leq 3 \left\{ P\left( \sup_{t \leq T} \| X^\varepsilon - u_t \| \leq 2 \beta^{1/4} \right) \\
\bigvee P\left( \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| > \beta^{1/4} \right) \bigvee P\left( \tau_C^\beta \leq T \right) \right\}.
$$

Clearly, $\Theta^\beta_C$ can be replaced by $\tau_C^\beta$, and so

$$
- \frac{1}{2} \int_0^T \| \dot{u}_s - b(u_s) \|_{(a(u_s) + \beta I)^{-1}}^2 ds \leq \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left( \sup_{t \leq T} \| X^\varepsilon - u_t \| \leq 2 \beta^{1/4} \right) \\
\bigvee \left\{ \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left( \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| > \beta^{1/4} \right) \bigvee \lim_{\varepsilon \to 0} \varepsilon^2 \log \sup_{\beta \in [0,1]} P\left( \tau_C^\beta \leq T \right) \right\}.
$$

Recall the following facts:

1) by Lemma B.1 and (6.1),

$$
\lim_{\beta \to 0} \int_0^T \| \dot{u}_s - b(u_s) \|_{(a(u_s) + \beta I)^{-1}}^2 ds = \int_0^T \| \dot{u}_s - b(u_s) \|^2_{a_\beta(u_s)} ds;
$$

2) by Lemma C.1,

$$
\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left( \sup_{t \leq \tau_C^\beta \wedge T} \| X^\varepsilon - X^\varepsilon \| > \beta^{1/4} \right) = -\infty;
$$
3) by (6.10), \( \lim_{\beta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log \sup_{\beta \in (0,1]} P(\tau^\beta_C \leq T) = -\infty. \)

Hence, passing to the limit \( \beta \to 0 \) and then \( C \to \infty \) in (6.11) and taking into account 1) - 3), one gets the required lower bound

\[
\lim_{\beta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P(\sup_{t \leq T} \|X^\beta_t - u_t\| \leq 2\beta^{1/4}) \geq -\frac{1}{2} \int_0^T \|\dot{u}_s - b(u_s)\|_{\alpha\oplus(u_s)}^2 ds.
\]

\[\square\]

**APPENDIX A. Exponential estimates for martingales**

**Proposition A.1.** (Lemma A.1 in [10]) Let \( M = (M_t)_{t \geq 0} \), \( M_t \in \mathbb{R} \), be a continuous local martingale with \( M_0 = 0 \) and the predictable variation process \( \langle M \rangle_t \) defined on some stochastic basis with general conditions. Let \( \tau \) be a stopping time, \( \alpha \) and \( B \) positive constants and \( \mathfrak{A} \) some measurable set.

(a) if \( M_\tau - \frac{1}{2} \langle M \rangle_\tau \geq \alpha \) on \( \mathfrak{A} \), then \( P(\mathfrak{A}) \leq e^{-\alpha} \);
(b) if \( M_\tau \geq \alpha \) and \( \langle M \rangle_\tau \leq B \) on \( \mathfrak{A} \), then \( P(\mathfrak{A}) \leq e^{-\frac{\alpha^2}{2B}} \);
(c) \( P(\sup_{t \leq \tau} |M_t| \geq \alpha, \langle M \rangle_\tau \leq B) \leq 2e^{-\frac{\alpha^2}{2B}} \);
(d) \( P(\sup_{t \leq \tau} |M_t| \geq \alpha) \leq 2e^{-\frac{\alpha^2}{2B}} \vee P(\langle M \rangle_T > B) \).

**APPENDIX B. Pseudoinverse of nonnegative definite matrices**

Let \( A^\oplus \) be the Moore-Penrose pseudoinverse matrix of \( A \) (see [1]).

**Lemma B.1.** For \( d \times d \) nonnegative definite matrix \( A \) and \( x \in \mathbb{R}^d \),

\[
\lim_{\beta \to 0} \langle x, (A + \beta \mathbf{I})^{-1}x \rangle = \begin{cases} \|x\|^2_{A^\oplus}, & AA^\oplus x = x \\ \infty, & \text{otherwise.} \end{cases}
\]

**Proof.** Let \( S \) be an orthogonal matrix, \( S^*S = I \), such that \( D := S^*AS \) is a diagonal matrix. Then, due to \( S^*(A + \beta \mathbf{I})S = D + \beta \mathbf{I} \), we have \( S^*(A + \beta \mathbf{I})^{-1}S = (D + \beta \mathbf{I})^{-1} \) and \( S(D + \beta \mathbf{I})^{-1}S^* = (A + \beta \mathbf{I})^{-1} \). Write \( (y := S^*x) \)

\[
\langle x, (A + \beta \mathbf{I})^{-1}x \rangle = \langle x, S(D + \beta \mathbf{I})^{-1}S^*x \rangle = \langle S^*x, (D + \beta \mathbf{I})^{-1}S^*x \rangle \\
= \langle y, (D + \beta \mathbf{I})^{-1}y \rangle = \langle y, (D + \beta \mathbf{I})^{-1}DD^\oplus y \rangle \\
+ \langle y, (D + \beta \mathbf{I})^{-1}(I - DD^\oplus)y \rangle.
\]

Since \( \lim_{\beta \to 0}(D + \beta \mathbf{I})^{-1}DD^\oplus = D^\oplus \), one gets

\[
\lim_{\beta \to 0} \langle y, (D + \beta \mathbf{I})^{-1}DD^\oplus y \rangle = \|y\|^2_{D^\oplus} = \|x\|^2_{A^\oplus}
\]

while \( \lim_{\beta \to 0} \langle y, (D + \beta \mathbf{I})^{-1}(I - DD^\oplus)y \rangle \neq \infty \) only if \( (I - DD^\oplus)y = 0 \). Since the latter condition is nothing but \( (I - AA^\oplus)x = 0 \), the desired statement holds.

\[\square\]

**APPENDIX C. Exponential negligibility of \( X^\varepsilon,\beta - X^\varepsilon \)**

We start with the auxiliary result.
Proposition C.1. Let $Y_t$ be a nonnegative continuous semimartingale defined on a stochastic basis (with general conditions):

$$Y_t = \int_0^t h_1(s)Y_sds + \varepsilon \int_0^t h_2(s)Y_sdM'_s + \varepsilon \sqrt{\beta} \int_0^t h_3(s)\sqrt{Y_s}dM''_s + \varepsilon^2 \beta \int_0^t h_4(s)ds,$$

(C.1)

where $h_i(s), i = 1, \ldots, 4,$ are bounded predictable processes and $M'_i, M''_i$ are continuous martingales, $d(M'_i)_t = m'_i(t)dt$, $d(M''_i)_t = m''_i(t)dt$, $(M'_i, M''_i)_t \equiv 0$ with bounded $m'_i(t)$ and $m''_i(t)$. Assume that for any $T > 0$ and $\beta > 0$,

$$\lim_{L \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left(\sup_{t \leq T} \sqrt{Y_t} > L\right) = -\infty.$$  

Then, for any $T > 0$,

$$\lim_{\beta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left(\sup_{t \leq T} |Y_t| > \beta^{1/4}\right) = -\infty.$$  

Proof. Obviously $Y_t$ solves an integral equation

$$Y_t = \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} \left[\varepsilon \sqrt{\beta} h_3(s)\sqrt{Y_s}dM''_s + \varepsilon^2 \beta h_4(s)ds\right],$$

where $\mathcal{E}_t = \exp \left(\int_0^t [h_1(s) - \varepsilon^2 0.5 h_2^2(s)]ds + \int_0^t \varepsilon h_2(s)dM'_s\right)$. Let for definiteness $|h_i| \leq r$, where $r$ is a constant. Then, with $\varepsilon \leq 1$,

$$\sup_{t \leq T} |\log \mathcal{E}_t| \leq T(r + 0.5 r^2) + \sup_{t \leq T} |\varepsilon \int_0^t h_2(s)dM'_s|.$$  

Hence the random variable $\sup_{t \leq T} |\log \mathcal{E}_t|$ is bounded on the set

$$\{\sup_{t \leq T} |\varepsilon \int_0^t h_2(s)dM'_s| \leq C\}.$$  

Moreover, it is exponentially tight in the sense that

$$\lim_{C \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left(\sup_{t \leq T} |\log \mathcal{E}_t| > C\right) = -\infty.$$  

(C.3)

The latter is implied by

$$\lim_{C \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log P\left(\sup_{t \leq T} |\varepsilon \int_0^t h_2(s)dM'_s| > C\right) = -\infty$$  

(C.4)

since the martingale $N_t = \varepsilon \int_0^t h_2(s)dM'_s$ has $\langle N \rangle_t = \varepsilon^2 \int_0^t h_2^2(s)m'_i(s)ds$ and, with some positive number $r_1$, we have $\varepsilon^2 h_2^2(s)m'_i(s) \leq \varepsilon^2 r_1$. Then, by taking into account that $P\left(\langle N \rangle_T > \varepsilon^2 r_1 T\right) = 0$ and applying the statement (d) of Proposition A.1, we obtain $P\left(\sup_{t \leq T} |N_t| > C\right) \leq 2e^{-C^2/(2\varepsilon^2 r_1 T)}$ providing (C.4).

Now we estimate $\sup_{t \leq T} |Y_t|$ on the set $\{\sup_{t \leq T} |\log \mathcal{E}_t| \leq C\}$. Write

$$\sup_{t \leq T} |Y_t| \leq e^{C} T \varepsilon^2 \beta + e^{C} \sup_{t \leq T} \left|\int_0^t \mathcal{E}_s^{-1} \varepsilon \sqrt{\beta} h_3(s)\sqrt{Y_s}dM''_s\right|. $$
This upper bound and (C.2), (C.3) reduces the proof of Proposition C.1 to:

\[
\lim_{\beta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P \left( \sup_{t \leq T} \left| \int_0^t \mathcal{E}_s^{-1} \epsilon \sqrt{\beta h_3(s)} \sqrt{Y_s^\nu} dM_s^\nu \right| > \beta^{1/4}, \sup_{t \leq T} \sqrt{Y_t} \leq L, \sup_{t \leq T} |\log \mathcal{E}_t| \leq C \right) = \infty
\]

for any \( C > 0 \) and \( L > 0 \). Introduce the martingale

\[
N_t^n = \int_0^t \mathcal{E}_s^{-1} \epsilon \sqrt{\beta h_3(s)} \sqrt{Y_s} dM_s^\nu \quad \text{with} \quad (N_t^n)_t = \int_0^t \mathcal{E}_s^{-2} \epsilon^2 \beta h_3^2(s) Y_s m^\nu(s) ds
\]

and denote \( \mathcal{E} = \{ \sup_{t \leq T} \sqrt{Y_t} \leq L, \sup_{t \leq T} |\log \mathcal{E}_t| \leq C \} \). With \( r_2 \geq h_3^2(s) L m^\nu(s) \), we find that

\[
(N_T^n)_T \leq e^{2C} r_2 T \epsilon^2 \beta.
\]

Hence,

\[
P \left( \sup_{t \leq T} |N_t^n| > \beta^{1/4}, \mathcal{E} \right) = P \left( \sup_{t \leq T} |N_t^n| > \beta^{1/4}, (N_T^n)_T \leq e^{2C} r_2 T \epsilon^2 \beta, \mathcal{E} \right)
\]

\[
\leq P \left( \sup_{t \leq T} |N_t^n| > \beta^{1/4}, (N_T^n)_T \leq e^{2C} r_2 T \epsilon^2 \beta \right).
\]

By (c) of Proposition A.1 the latter term is bounded from above by

\[
2 \exp \left( \frac{\beta^{1/2}}{2 e^{2C} r_2 T \epsilon^2 \beta} \right).
\]

Then we obtain

\[
\lim_{\epsilon \to \infty} \epsilon^2 \log P \left( \sup_{t \leq T} |N_t^n| > \beta^{1/4}, \mathcal{E} \right) \leq -\frac{1}{2 e^{2C} r_2 T \beta^{1/2}} \xrightarrow{\beta \to 0} -\infty.
\]

\( \square \)

We apply Proposition C.1 in order to prove

**Lemma C.1.** For any \( T > 0 \) and \( C > 0 \),

\[
\lim_{\beta \to 0} \lim_{\epsilon \to 0} \epsilon^2 \log P \left( \sup_{t \leq \tau_C^\beta T} \|X_{t,\beta}^\epsilon - X_t^\epsilon\| > \beta^{1/4} \right) = -\infty.
\]

**Proof.** Recall that \( X_t^\epsilon \) and \( X_{t,\beta}^\epsilon \) solve (1.1) and (6.6) respectively and \( \tau_C^\beta \) is given in (6.9). Set \( \Delta_t^{\epsilon,\beta} = X_{t,\beta}^\epsilon - X_t^\epsilon \). By (1.1) and (6.6),

\[
\Delta_t^{\epsilon,\beta} = \int_0^{\tau_C^\beta T} \left( b(X_{s,\beta}^\epsilon) - b(X_s^\epsilon) \right) ds + \epsilon \int_0^{\tau_C^\beta T} \left( \sigma(X_{s,\beta}^\epsilon) - \sigma(X_s^\epsilon) \right) dB_s + \epsilon \sqrt{\beta} W_{\tau_C^\beta T}^\beta.
\]

Due to the local Lipschitz continuity of \( b \) and \( \sigma \) and with \( 0/0 = 0 \), the vector-valued and matrix-valued functions:

\[
f(s) = \frac{b(X_{s,\beta}^\epsilon) - b(X_s^\epsilon)}{\|\Delta_s^{\epsilon,\beta}\|} \quad \text{and} \quad g(s) = \frac{\sigma(X_{s,\beta}^\epsilon) - \sigma(X_s^\epsilon)}{\|\Delta_s^{\epsilon,\beta}\|}
\]
are well defined and their entries are bounded by a constant depending on $C$. Hence
$$\Delta_{t}^{\varepsilon,\beta} = \int_{0}^{\tau_{C}^{\beta} \land t} \|\Delta_{s}^{\varepsilon,\beta}\|f(s)ds + \varepsilon \int_{0}^{\tau_{C}^{\beta} \land t} \|\Delta_{s}^{\varepsilon,\beta}\|g(s)dB_{s} + \varepsilon \sqrt{\beta W_{\tau_{C}^{\beta} \land t}}.$$ 

Since $\|\Delta_{t}^{\varepsilon,\beta}\|^2 = \langle \langle \Delta_{t}^{\varepsilon,\beta}, \Delta_{t}^{\varepsilon,\beta} \rangle \rangle$, by the Itô formula, we find that
$$\|\Delta_{t}^{\varepsilon,\beta}\|^2 = \int_{0}^{t} 2\|\Delta_{s}^{\varepsilon,\beta}\|\langle \langle \Delta_{s}^{\varepsilon,\beta}, f(s) \rangle \rangle ds$$
$$+ \varepsilon \int_{0}^{\tau_{C}^{\beta} \land t} 2\|\Delta_{s}^{\varepsilon,\beta}\|\langle \langle \Delta_{s}^{\varepsilon,\beta}, g(s)dB_{s} \rangle \rangle$$
$$+ \varepsilon \sqrt{\beta} \int_{0}^{\tau_{C}^{\beta} \land t} 2\langle \langle \Delta_{s}^{\varepsilon,\beta}, dW_{s} \rangle \rangle$$
$$+ \varepsilon^2 \int_{0}^{\tau_{C}^{\beta} \land t} \|\Delta_{s}^{\varepsilon,\beta}\|^2 \text{trace}[g(s)g^{\ast}(s)]ds$$
$$+ \varepsilon^2 \beta (\tau_{C}^{\beta} \land t)d. \tag{C.5}$$

Now, by letting $\phi(s) = \frac{2\langle \langle \Delta_{s}^{\varepsilon,\beta}, f(s) \rangle \rangle}{\|\Delta_{s}^{\varepsilon,\beta}\|}$ and $d\hat{B}_{s} = \frac{2\langle \langle \Delta_{s}^{\varepsilon,\beta}, g(s)dB_{s} \rangle \rangle}{\|\Delta_{s}^{\varepsilon,\beta}\|}$, we rewrite (C.5) as:
$$\|\Delta_{t}^{\varepsilon,\beta}\|^2 = \int_{0}^{\tau_{C}^{\beta} \land t} \|\Delta_{s}^{\varepsilon,\beta}\|^2 \{\phi(s)$$
$$+ \varepsilon^2 \text{trace}[g(s)g^{\ast}(s)]\}ds$$
$$+ \varepsilon \int_{0}^{\tau_{C}^{\beta} \land t} \|\Delta_{s}^{\varepsilon,\beta}\|^2 d\hat{B}_{s}$$
$$+ \varepsilon \sqrt{\beta} \int_{0}^{\tau_{C}^{\beta} \land t} \|\Delta_{s}^{\varepsilon,\beta}\| \frac{2\langle \langle \Delta_{s}^{\varepsilon,\beta}, dW_{s} \rangle \rangle}{\|\Delta_{s}^{\varepsilon,\beta}\|}$$
$$+ \varepsilon^2 \beta (\tau_{C}^{\beta} \land t)d. \tag{C.6}$$

With the notations
- $Y_{t} = \|\Delta_{t}^{\varepsilon,\beta}\|^2$
- $h_{1}(s) = I_{\{\tau_{C}^{\beta} \leq s\}}\{\phi(s) + \varepsilon^2 \text{trace}[g(s)g^{\ast}(s)]\}$
- $h_{2}(s) \equiv 1$
- $h_{4}(s) = I_{\{\tau_{C}^{\beta} \leq s\}}d$
- $M'_{t} = \hat{B}_{t}$, $m'(s) = \frac{4\langle \langle \Delta_{s}^{\varepsilon,\beta}, g(s)g^{\ast}(s)\Delta_{s}^{\varepsilon,\beta} \rangle \rangle}{\|\Delta_{s}^{\varepsilon,\beta}\|^2}$
- $M''_{t} = \int_{0}^{\tau_{C}^{\beta} \land t} \frac{2\langle \langle \Delta_{s}^{\varepsilon,\beta}, dW_{s} \rangle \rangle}{\|\Delta_{s}^{\varepsilon,\beta}\|}$, $m''(s) \equiv 4$,

the equation (C.6) is in the form of (C.1). Since $h_{i}(s), i = 1, \ldots, 4$ are bounded and $\sqrt{Y_{t}} \equiv \|X_{\tau_{C}^{\beta} \land t}^{\varepsilon,\beta} \rangle - X_{\tau_{C}^{\beta} \land t}^{\varepsilon,\beta} \| \leq \|X_{\tau_{C}^{\beta} \land t}^{\varepsilon,\beta} \rangle + \|X_{\tau_{C}^{\beta} \land t}^{\varepsilon,\beta} \| \leq 2C$, i.e., (C.2) holds too, the statement of the lemma follows from Proposition C.1. \qed
References

[1] Albert, A. (1972) Regression and the Moore-Penrose Pseudoinverse. Academic Press, New York and London.
[2] Aldous, G.J. (1976) Stopping time and tightness. Ann. Prob. 6, 2, p. 335-340.
[3] Aldous, G.J. (1981) Weak convergence and the general theory of processes. Incomplete draft of monograph. Department of Statistic. University of California, Berleley.
[4] Dembo, A., Zeitouni, O. (1998) Large Deviations Techniques and Applications. Springer, 2nd edition.
[5] Dupuis, P., Ellis, R. (1997) A Weak Convergence Approach to the Theory of Large Deviations. Wiley.
[6] Feng, J. (1999) Martingale problems for large deviations of Markov processes. Stoch. Proc. Appl. 81(2): p. 165-216.
[7] Feng, J., Kurtz, T.G. (2004) Large deviations for stochastic processes. (preliminary manuscript).
[8] Freidlin, M.I., Wentzell A.D. (1984) Random Perturbations of Dynamical Systems. N.Y. Springer.
[9] Friedman, A. (1976) Stochastic Differential Equations and Applications, volume 2. Academic Press.
[10] Guillin, A., R. Liptser, R. MDP for integral functionals of fast and slow processes with averaging, Stochastic Processes, Applications, (2005) 115, 7, 1187–1207.
[11] Khasminskii, R.Z. (1980). Stochastic stability of differential equations. Sijthoff & Noordhoff.
[12] Liptser, R.S., Pukhalskii, A.A. (1992) Limit theorems on large deviations for semimartingales. Stochastic and Stochastics Reports 38, 201–249.
[13] Liptser, R.Sh., Shiryaev, A.N. (1989) Theory of Martingales. Kluwer Acad. Publ.
[14] Liptser, R., Spokoiny, V., Veretennikov, A.Yu., Freidlin-Wentzell type large deviations for smooth processes. Markov Process and Relat. Fields. 8 (2002), pp. 611-636.
[15] Mikami, T. (1988) Some generalizations of Wentzell’s lower estimates on large deviations. Stochastics, 24(4), p. 269-284.
[16] Narita, K. (1988) Large deviation principle for diffusion processes. Tsukuba J. Math., 12(1), p. 211-229.
[17] Pardoux, É., Veretennikov, A.Yu. On Poisson equation and diffusion approximation 1, Ann. Probab. 29(3) pp. 1061–1085, (2001).
[18] Pardoux, É., Veretennikov, A.Yu. On Poisson equation and diffusion approximation. II. Ann. Probab. 31(3) pp. 1166–1192, (2003).
[19] Puhalskii, A. (2004) On some degenerate large deviation problems. Electronic J. Probab. 9:862-886.
[20] Puhalskii, A.A. On functional principle of large deviations New trends in Probability and Statistics. V.Sazonov and Sherwashidze (eds.), Vilnius, Lithuania, VSP/Moksas, pp. 198–218, (1991).
[21] A. Puhalskii, Large Deviations and Idempotent Probability, 2001, Chapman & Hall/CRC Press.
[22] Ren, J., Zhang, X. (2005) Freidlin-Wentzells large deviations for homeomorphism flows of non-Lipschitz SDEs. Bull. Sci. math. 129. p. 643655.
[23] Stroock, D.W. (1984) An Introduction to the Theory of Large Deviations Springer.

P. CHIGANSKY AND R. LIPTSER

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: pavel.chigansky@weizmann.ac.il

Department of Electrical Engineering Systems, Tel Aviv University, 69978 Tel Aviv, Israel
E-mail address: liptser@eng.tau.ac.il