Weingarten map of the hypersurface in 4-dimensional Euclidean space and its applications

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Abstract
In this paper, by taking into account the beginning of the hypersurface theory in Euclidean space $E^4$, a practical method for the matrix of the Weingarten map (or the shape operator) of an oriented hypersurface $M^3$ in $E^4$ is obtained. By taking this efficient method, it is possible to study of the hypersurface theory in $E^4$ which is analog the surface theory in $E^3$. Furthermore, the Gaussian curvature, mean curvature, fundamental forms and Dupin indicatrix of $M^3$ is introduced.

1 Introduction
Let $x = \sum_{i=1}^{4} x_i e_i$, $y = \sum_{i=1}^{4} y_i e_i$, $z = \sum_{i=1}^{4} z_i e_i$ be three vectors in $\mathbb{R}^4$, equipped with the standard inner product given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4,$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. The norm of a vector $x \in \mathbb{R}^4$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$. The vector product (or the ternary product or cross product) of the vectors $x, y, z \in \mathbb{R}^4$ is defined by

$$x \otimes y \otimes z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$
Some properties of the vector product are given as follows: (for the vector product in \( \mathbb{R}^4 \), see \[1, 2, 5\])

\[
\begin{align*}
e_1 \otimes e_2 \otimes e_3 &= -e_4 \\
e_2 \otimes e_3 \otimes e_4 &= e_1 \\
e_3 \otimes e_4 \otimes e_1 &= e_2 \\
e_4 \otimes e_1 \otimes e_2 &= -e_3 \\
e_3 \otimes e_2 \otimes e_1 &= e_4
\end{align*}
\]

\( e_1 \otimes e_2 \otimes e_3 \) and \( e_2 \otimes e_3 \otimes e_4 \) are skew-symmetric.

ii. \( \|x \otimes y \otimes z\|^2 = \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle & \langle x, z \rangle \\ \langle y, x \rangle & \langle y, y \rangle & \langle y, z \rangle \\ \langle z, x \rangle & \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} \) (2)

iii. \( \langle x \otimes y \otimes z, t \rangle = \det (x, y, z, t) \).

Let \( M^3 \) be an oriented 3-dimensional hypersurface in 4-dimensional Euclidean space \( E^4 \). Let examine the implicit and parametric equations of \( M^3 \).

Firstly; the implicit equation of \( M^3 \) can be defined by

\[
M^3 = \left\{ X \in E^4 | f : U \subset E^4 \xrightarrow{df} \mathbb{R}, f(X) = \text{const.}, \nabla f|_P \neq 0, P \in M^3 \right\}
\]

where \( \nabla f|_P \) is the gradient vector of \( M^3 \). The unit normal vector field of \( M^3 \) is defined by

\[
N = \frac{\nabla f}{\| \nabla f \|}.
\]

The Weingarten map (or the shape operator) of \( M^3 \) is defined by

\[
S : \chi(M^3) \rightarrow \chi(M^3), \quad S(X) = D_N X
\]

where \( D \) is the connection of \( E^4 \) and \( \chi(M^3) \) is the space of vector fields of \( M^3 \). Then the Gauss curvature \( K \) and mean curvature \( H \) of \( M^3 \) are given by \( K = \det S \) and \( H = \frac{1}{2} \text{Tr} S \), respectively. Also, the \( q-th \) fundamental forms of \( M^3 \) are given by \( \mathbf{I}^q \),

\[
\mathbf{I}^q(X, Y) = \langle S^{q-1}(X), Y \rangle, \quad \forall X, Y \in \chi(M^3).
\]

Secondly, to examine parametric form of the hypersurface \( M^3 \) given by the implicit equation in the eq (3), let consider

\[
\phi : U \subset \mathbb{R}^3 \rightarrow E^4 \\
(u, v, w) \rightarrow \phi(u, v, w) = (\varphi_1(u, v, w), \varphi_2(u, v, w), \varphi_3(u, v, w), \varphi_4(u, v, w))
\]

where \( (u, v, w) \in R \subset \mathbb{R}^3 \) and \( \varphi_i, 1 \leq i \leq 4 \) are the real functions defined on \( R \). \( M^3 = \phi(R) \subset E^4 \) is a hypersurface if only if the frame field \( \{ \phi_u, \phi_v, \phi_w \} \) of \( M^3 \) is linearly independent system. It can be also seen by taking the Jacobian matrix \( \phi_* = [\phi_u \phi_v \phi_w] \) of the differential map of \( \phi \). It is clear that if \( \text{rank} \phi_* = 3 \), then the vector system \( \{ \phi_u, \phi_v, \phi_w \} \) is linearly independent. Furthermore, \( \phi_u, \phi_v, \phi_w \) are the tangent vectors of the parameter curves \( \alpha(u) = \ldots \)
Then by using the Weingarten operator the below equalities can be written

$$M$$ the unit normal vector field of $$M$$ map of hypersurface

In this original section, a practical method for the matrix of the Weingarten map of hypersurface $$M^3$$ is defined by

$$N = \frac{\phi_u \otimes \phi_v \otimes \phi_w}{\|\phi_u \otimes \phi_v \otimes \phi_w\|}$$  \hspace{1cm} (4)

and it has the following properties:

$$\langle N, \phi_u \rangle = \langle N, \phi_v \rangle = \langle N, \phi_w \rangle = 0.$$  \hspace{1cm} (5)

By using the Weingarten operator the below equalities can be written

$$S(\phi_u) = D_{\phi_u} N = \frac{\partial N}{\partial u}$$

$$S(\phi_v) = D_{\phi_v} N = \frac{\partial N}{\partial v}$$

$$S(\phi_w) = D_{\phi_w} N = \frac{\partial N}{\partial w}.$$  \hspace{1cm} (6)

2 The matrix of the Weingarten map of hypersurface $$M^3$$ in $$E^4$$

In this original section, a practical method for the matrix of the Weingarten map of hypersurface $$M^3$$ in $$E^4$$ is introduced.

Let $$M^3$$ be an oriented hypersurface with the parametric equation $$\phi(u, v, w)$$. Then $$\{\phi_u, \phi_v, \phi_w\}$$ is linearly independent and we also can write

$$S(\phi_u) = a_{11} \phi_u + a_{21} \phi_v + a_{31} \phi_w$$

$$S(\phi_v) = a_{12} \phi_u + a_{22} \phi_v + a_{32} \phi_w$$

$$S(\phi_w) = a_{13} \phi_u + a_{23} \phi_v + a_{33} \phi_w$$  \hspace{1cm} (7)

and the Weingarten matrix is given by

$$S = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix},$$

where $$a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3$$. Using the equation (7), we have the following systems of linear equations:

$$\begin{cases}
\langle S(\phi_u), \phi_u \rangle = a_{11} \phi_{11} + a_{21} \phi_{12} + a_{31} \phi_{13} \\
\langle S(\phi_u), \phi_v \rangle = a_{11} \phi_{12} + a_{21} \phi_{22} + a_{31} \phi_{23} \\
\langle S(\phi_u), \phi_w \rangle = a_{11} \phi_{13} + a_{21} \phi_{23} + a_{31} \phi_{33},
\end{cases}$$

$$\begin{cases}
\langle S(\phi_v), \phi_u \rangle = a_{12} \phi_{11} + a_{22} \phi_{12} + a_{32} \phi_{13} \\
\langle S(\phi_v), \phi_v \rangle = a_{12} \phi_{12} + a_{22} \phi_{22} + a_{32} \phi_{23} \\
\langle S(\phi_v), \phi_w \rangle = a_{12} \phi_{13} + a_{22} \phi_{23} + a_{32} \phi_{33},
\end{cases}$$

$$\begin{cases}
\langle S(\phi_w), \phi_u \rangle = a_{13} \phi_{11} + a_{23} \phi_{12} + a_{33} \phi_{13} \\
\langle S(\phi_w), \phi_v \rangle = a_{13} \phi_{12} + a_{23} \phi_{22} + a_{33} \phi_{23} \\
\langle S(\phi_w), \phi_w \rangle = a_{13} \phi_{13} + a_{23} \phi_{23} + a_{33} \phi_{33},
\end{cases}$$  \hspace{1cm} (8)
where
\[
\langle \phi_u, \phi_u \rangle = \phi_{11}, \quad \langle \phi_u, \phi_v \rangle = \phi_{12}, \quad \langle \phi_u, \phi_w \rangle = \phi_{13}, \\
\langle \phi_v, \phi_v \rangle = \phi_{22}, \quad \langle \phi_v, \phi_w \rangle = \phi_{23}, \quad \langle \phi_w, \phi_w \rangle = \phi_{33}.
\] (9)

Since the system \{\phi_u, \phi_v, \phi_w\} is linearly independent, using the equations (2) and (9), we have
\[
\|\phi_u \otimes \phi_v \otimes \phi_w\|^2 = \begin{vmatrix}
\phi_{11} & \phi_{12} & \phi_{13} \\
\phi_{12} & \phi_{22} & \phi_{23} \\
\phi_{13} & \phi_{23} & \phi_{33}
\end{vmatrix} \neq 0.
\]

Also, 3-linear equation systems given by the equation 8 have the determinant
\[
\begin{vmatrix}
\phi_{11} & \phi_{12} & \phi_{13} \\
\phi_{12} & \phi_{22} & \phi_{23} \\
\phi_{13} & \phi_{23} & \phi_{33}
\end{vmatrix} = \Delta.
\]

Because of the property \(\|\phi_u \otimes \phi_v \otimes \phi_w\|^2 = \Delta \neq 0\), these 3-linear equations systems can be solved by Cramer method. Then using the equations (6), (8) and (9) the matrix \(S\) of the Weingarten map in \(M^3\) can be found. Although \(S\) is a symmetric linear operator, the matrix presentation \((a_{ij})\) of \(S\) with respect to \{\phi_u, \phi_v, \phi_w\} is not necessary to be symmetric because the system \{\phi_u, \phi_v, \phi_w\} is not orthonormal.

2.1 Special Case

If we take the orthogonal frame field \{\phi_u, \phi_v, \phi_w\} of the hypersurface \(M^3\), then we have \(\phi_{12} = \phi_{13} = \phi_{23} = 0\) from the equation (9). Then, the system \(\{U = \frac{\phi_u}{\|\phi_u\|}, \quad V = \frac{\phi_v}{\|\phi_v\|}, \quad W = \frac{\phi_w}{\|\phi_w\|}\}\) is an orthonormal frame field. Furthermore, we can write the following equations
\[
\begin{align*}
S(U) &= c_1 U + c_2 V + c_3 W \\
S(V) &= c_2 U + c_4 V + c_5 W \\
S(W) &= c_3 U + c_5 V + c_6 W,
\end{align*}
\] (10)

then, the matrix of the Weingarten map can be calculated as follows:
\[
S = \begin{pmatrix}
c_1 & c_2 & c_3 \\
c_2 & c_4 & c_5 \\
c_3 & c_5 & c_6
\end{pmatrix}.
\]
By using the equations (4), (6) and (10), the coefficients $c_i \in \mathbb{R}, 1 \leq i \leq 6$ can be calculated as follows:

$$
c_1 = \langle S(U), U \rangle = \frac{1}{||\phi_u||} \langle \frac{\partial N}{\partial u}, \phi_u \rangle,
$$

$$
c_2 = \langle S(U), V \rangle = \frac{1}{||\phi_u||} \frac{1}{||\phi_v||} \langle \frac{\partial N}{\partial u}, \phi_v \rangle,
$$

$$
c_3 = \langle S(U), W \rangle = \frac{1}{||\phi_u||} \frac{1}{||\phi_w||} \langle \frac{\partial N}{\partial u}, \phi_w \rangle,
$$

$$
c_4 = \langle S(V), V \rangle = \frac{1}{||\phi_v||} \langle \frac{\partial N}{\partial v}, \phi_v \rangle,
$$

$$
c_5 = \langle S(V), W \rangle = \frac{1}{||\phi_v||} \frac{1}{||\phi_w||} \langle \frac{\partial N}{\partial v}, \phi_w \rangle,
$$

$$
c_6 = \langle S(W), W \rangle = \frac{1}{||\phi_w||} \langle \frac{\partial N}{\partial w}, \phi_w \rangle.
$$

(11)

By using the equation (5), we can also write six equations as below:

$$
\langle \frac{\partial N}{\partial u}, \phi_u \rangle + \langle N, \phi_{uu} \rangle = 0,
$$

$$
\langle \frac{\partial N}{\partial u}, \phi_v \rangle + \langle N, \phi_{uv} \rangle = 0,
$$

$$
\langle \frac{\partial N}{\partial u}, \phi_w \rangle + \langle N, \phi_{uw} \rangle = 0,
$$

(12)

$$
\langle \frac{\partial N}{\partial v}, \phi_v \rangle + \langle N, \phi_{vv} \rangle = 0,
$$

$$
\langle \frac{\partial N}{\partial v}, \phi_w \rangle + \langle N, \phi_{vw} \rangle = 0,
$$

$$
\langle \frac{\partial N}{\partial w}, \phi_w \rangle + \langle N, \phi_{ww} \rangle = 0.
$$

(13)

Also, by using the equations (2) and (9), we find

$$
\|\phi_u \otimes \phi_v \otimes \phi_w\|^2 = \begin{vmatrix}
\phi_{22} & 0 & 0 \\
0 & \phi_{11} & 0 \\
0 & 0 & \phi_{33}
\end{vmatrix} = \|\phi_u\|^2 \|\phi_v\|^2 \|\phi_w\|^2.
$$

(13)
Hence we find the coefficients $c_1, c_2, c_3, c_4, c_5, c_6$ of the Weingarten matrix in the equation (10) as follows:

\[ c_1 = -\frac{1}{\|\phi_u\| \|\phi_v\| \|\phi_w\|} \det (\phi_{uu}, \phi_u, \phi_v, \phi_w), \]
\[ c_2 = -\frac{1}{\|\phi_u\|^2 \|\phi_v\| \|\phi_w\|} \det (\phi_{uv}, \phi_u, \phi_v, \phi_w), \]
\[ c_3 = -\frac{1}{\|\phi_u\| \|\phi_v\|^2 \|\phi_w\|} \det (\phi_{uw}, \phi_u, \phi_v, \phi_w), \]
\[ c_4 = -\frac{1}{\|\phi_u\| \|\phi_v\|^2 \|\phi_w\|^2} \det (\phi_{vw}, \phi_u, \phi_v, \phi_w), \]
\[ c_5 = -\frac{1}{\|\phi_u\|^2 \|\phi_v\|^2 \|\phi_w\|} \det (\phi_{ww}, \phi_u, \phi_v, \phi_w). \]  

(14)

So, by taking into account the equations (4), (13) and (14) we have the symmetric Weingarten matrix

\[
S = \begin{pmatrix}
\varphi_{11} & \varphi_{12} & \varphi_{13} \\
\sqrt{\varphi_{11}\varphi_{22}} & \sqrt{\varphi_{11}\varphi_{33}} & \\
\sqrt{\varphi_{12}} & \sqrt{\varphi_{12}} & \sqrt{\varphi_{22}\varphi_{33}} \\
\sqrt{\varphi_{13}} & \sqrt{\varphi_{23}} & \sqrt{\varphi_{33}} \\
\end{pmatrix}.
\]  

(15)

where

\[ \varphi_{11} = -\langle \phi_{uu}, N \rangle, \quad \varphi_{12} = -\langle \phi_{uv}, N \rangle, \quad \varphi_{13} = -\langle \phi_{uw}, N \rangle, \]
\[ \varphi_{22} = -\langle \phi_{vv}, N \rangle, \quad \varphi_{23} = -\langle \phi_{vw}, N \rangle, \quad \varphi_{33} = -\langle \phi_{ww}, N \rangle. \]

Finally the following theorem can be given for hypersurface $M^3$ in $E^4$:

**Theorem 1** Let $M^3$ be an oriented hypersurface in $E^4$. Then the Gaussian curvature and the mean curvature of $M^3$ can be given by:

\[
K = \frac{\varphi_{11}\varphi_{22}\varphi_{33} + 2\varphi_{12}\varphi_{13}\varphi_{23} - \varphi_{12}^2\varphi_{33} - \varphi_{13}^2\varphi_{22} - \varphi_{23}^2\varphi_{11}}{\phi_{11}\phi_{22}\phi_{33}},
\]

and

\[
H = \frac{1}{3} \left( \frac{\varphi_{11}}{\phi_{11}} + \frac{\varphi_{22}}{\phi_{22}} + \frac{\varphi_{33}}{\phi_{33}} \right),
\]

respectively.

**Proof.** By using the equation (15) and the definitions of the Gaussian curvature $K$ and the mean curvature $H$, the theorem can be easily proved. ■
Example 2 Let $M^3$ be an oriented hypersurface with the implicit equation $xy = 1$ in $E^4$. The parametric equation of $M^3$ can be given by

$$\phi (u, v, w) = \left( u, \frac{1}{u}, v, w \right).$$

Then, we obtain $\phi_u \otimes \phi_u \otimes \phi_w = \left(-\frac{1}{u^2}, -1, 0, 0\right)$ and the unit normal field

$$N = \frac{1}{\sqrt{1+u^4}} \left(-1, -u^2, 0, 0\right).$$

By using the orthonormal basis $\left\{ \frac{\phi_u}{\|\phi_u\|}, \frac{\phi_v}{\|\phi_v\|}, \frac{\phi_w}{\|\phi_w\|} \right\}$, we have

$$S \left( \frac{\phi_u}{\|\phi_u\|} \right) = \frac{2u^3}{(1+u^4)^{3/2}} \phi_u,$$

$$S \left( \frac{\phi_v}{\|\phi_v\|} \right) = 0,$$

$$S \left( \frac{\phi_w}{\|\phi_w\|} \right) = 0.$$

So, we find the Weingarten matrix $S$ as:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 3 Let $S^3$ be a hypersphere with the implicit equation $x^2 + y^2 + z^2 + t^2 = 1$ in $E^4$. The parametric equation of $S^3$ can be given by

$$\phi (u, v, w) = (\sin u \cos v \sin w, \sin u \sin v \sin w, \cos u \sin w, \cos w).$$

Then, $\{\phi_u, \phi_v, \phi_w\}$ is an orthogonal system. Also we have the orthonormal basis $\{U, V, W\}$ of $S^3$ such that

$$U = \frac{\phi_u}{\|\phi_u\|} = (\cos u \cos v, \cos u \sin v, - \sin u, 0),$$

$$V = \frac{\phi_v}{\|\phi_v\|} = (-\sin v, \cos v, 0, 0),$$

$$W = \frac{\phi_w}{\|\phi_w\|} = (\sin u \cos v \cos w, \sin u \sin v \cos w, \cos u \cos w, - \sin w).$$

Furthermore, the unit normal vector field $N$ can be found:

$$N = (-\sin u \cos v \sin w, -\sin u \sin v \sin w, -\cos u \sin w, -\cos w).$$

Then using the equation (15), we obtain $S = I_3$.

Theorem 4 Let $M^3$ be an oriented hypersurface in $E^4$ and let $\{X_P, Y_P, Z_P\}$ be a linearly independent vector system of the tangent space $T_{M^3}(P)$. Then, we have

i. $S (X_P) \otimes S (Y_P) \otimes S (Z_P) = K (P) (X_P \otimes Y_P \otimes Z_P)$

ii. $(S (X_P) \otimes Y_P \otimes Z_P) + (X_P \otimes S (Y_P) \otimes Z_P) + (X_P \otimes Y_P \otimes S (Z_P)) = 3H (P) (X_P \otimes Y_P \otimes Z_P),$

where $K$ and $H$ are the Gaussian curvature and the mean curvature of $M^3$, respectively.
Proof. By using (i), (ii) parts of the equation (2) and considering the definitions of the Gaussian curvature $K$ and the mean curvature $H$ the theorem can be easily proved. \[4\], it is proved that these equations are also provided for closed hypersurfaces.

Theorem 5 Let $M^3$ be an oriented hypersurface in $E^4$ and let $I^q$, $K$, $H$ be the $q$-th fundamental forms, the Gaussian curvature and the mean curvature, respectively. Then we have

$$I^4 - 3H I^3 + \frac{3K}{h} I^2 - K I = 0 \quad (16)$$

where $h$ is the harmonic mean of the non-zero principal curvatures of $M^3$.

Proof. Let $k_1, k_2, k_3$ be the characteristic values of the Weingarten map $S$ (or the principal curvatures of $M^3$). Then we obtain the characteristic polynomial $P_S(\lambda)$ of the Weingarten map $S$ of $M^3$ as

$$P_S(\lambda) = \det (\lambda I^3 - S) = \lambda^3 - (k_1 + k_2 + k_3) \lambda^2 + (k_1 k_2 + k_1 k_3 + k_2 k_3) \lambda - (k_1 k_2 k_3).$$

By using the Cayley-Hamilton theorem, we obtain

$$S^3 - (k_1 + k_2 + k_3) S^2 + (k_1 k_2 + k_1 k_3 + k_2 k_3) S - (k_1 k_2 k_3) I_3 = 0.$$ 

By using the definitions of the $q$-th fundamental forms, the Gaussian curvature, the mean curvature and the harmonic mean

$$h = \frac{3}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}}$$

of the principal curvature $k_1, k_2, k_3$,

we obtain the equation (16). \[5\]

3 Dupin indicatrix of the hypersurface in $E^4$

Let $X, Y, Z$ be three principal vectors according to the principal curvatures $k_1, k_2, k_3$ of $M^3$. If we consider the orthonormal basis $\{X, Y, Z\}$ of $M^3$ then for any tangent vector $W_P \in T_{M^3}(P)$, we can write $W_P = x X_P + y Y_P + z Z_P$, where $x, y, z \in \mathbb{R}$, and

$$S(W_P) = x S(X_P) + y S(Y_P) + z S(Z_P)$$

$$= x k_1 X_P + y k_2 Y_P + z k_3 Z_P$$

Here, the Dupin indicatrix $\mathcal{D}$ of $M^3$ can be defined by

$$\mathcal{D} = \{ W_P = (x, y, z) \in T_{M^3}(P) | \langle S(W_P), W_P \rangle = k_1 x^2 + k_2 y^2 + k_3 z^2 = \pm 1 \}.$$
In another words, the Dupin indicatrix corresponds to a hypercylinder which has the equation
\[ k_1 x^2 + k_2 y^2 + k_3 z^2 = \pm 1. \]

Now, we will examine the Dupin indicatrix according to the Gaussian curvature \( K \):

1) Let \( K(P) > 0 \).
   - If \( k_1, k_2, k_3 > 0 \) then for equation of the Dupin indicatrix, we can write
     \[ k_1 x^2 + k_2 y^2 + k_3 z^2 = \pm 1. \]
     Hence, the Dupin indicatrix is the ellipsoidal class and this equation is called ellipsoidal cylinder in \( E^4 \). In this condition, \( P \in M^3 \) is called an ellipsoidal point.
   - If \( k_1 > 0, k_2 < 0 \) or \( k_3 < 0 \) or \( k_3 > 0 \), \( k_2 < 0 \) then for equation of the Dupin indicatrix, we can write
     \[ k_1 x^2 - k_2 y^2 - k_3 z^2 = \pm 1. \]
     Hence, the Dupin indicatrix is the hyperboloidal class and this equation is called hyperboloidal cylinder one or two sheets in \( E^4 \). In this condition, \( P \in M^3 \) is called a hyperboloidal point.

2) Let \( K(P) < 0 \).
   - If only one of \( k_i \)'s, \( i = 1, 2, 3 \) is negative, then for the equation of the Dupin indicatrix, we can write
     \[
     \begin{cases}
     k_1 x^2 + k_2 y^2 - k_3 z^2 = \pm 1, \\
     k_1 x^2 - k_2 y^2 + k_3 z^2 = \pm 1, \\
     -k_1 x^2 + k_2 y^2 + k_3 z^2 = \pm 1.
     \end{cases}
     \]
     The above equations are called one or two sheeted hyperboloidal cylinder in \( E^4 \). Then \( P \in M^3 \) is called a hyperboloidal point.
   - If \( k_1, k_2, k_3 < 0 \) then the Dupin indicatrix is the ellipsoidal class and this equation is called ellipsoidal cylinder in \( E^4 \). So \( P \in M^3 \) is called an ellipsoidal point.

3) Let \( K(P) = 0 \).
   - If \( k_1 = 0 \) or \( k_2 = 0 \) or \( k_3 = 0 \), then for the equation of the Dupin indicatrix for each case, we get
     i) If \( k_1 = 0 \), \( k_2, k_3 \) are the same or different signs then \( k_2 y^2 + k_3 z^2 = \pm 1 \).
     ii) If \( k_2 = 0 \), \( k_1, k_3 \) are the same or different signs then \( k_1 x^2 + k_3 z^2 = \pm 1 \).
     iii) If \( k_3 = 0 \), \( k_1, k_2 \) are the same or different signs then \( k_1 x^2 + k_2 y^2 = \pm 1 \).
     These equations are called elliptic cylinder or hyperbolic cylinder in \( E^4 \). In this condition, \( P \in M^3 \) is called an elliptic cylinder or hyperbolic cylinder point.
   - If \( k_1 = k_2 = k_3 = 0 \) then the point \( P \in M^3 \) is a flat point.
   - If any two of \( k_i \)'s, \( i = 1, 2, 3 \) are zero and other positive or negative then \( k_3 z^2 = \pm 1 \) or \( k_2 y^2 = \pm 1 \) or \( k_1 x^2 = \pm 1 \).
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