TWO CLASSES OF GENERALIZED FUNCTIONS USED IN NONLOCAL FIELD THEORY

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Abstract. We elucidate the relation between the two ways of formulating causality in nonlocal quantum field theory: using analytic test functions belonging to the space $S^0$ (which is the Fourier transform of the Schwartz space $\mathcal{D}$) and using test functions in the Gelfand-Shilov spaces $S^0_\alpha$. We prove that every functional defined on $S^0$ has the same carrier cones as its restrictions to the smaller spaces $S^0_\alpha$. As an application of this result, we derive a Paley-Wiener-Schwartz-type theorem for arbitrarily singular generalized functions of tempered growth and obtain the corresponding extension of Vladimirov’s algebra of functions holomorphic on a tubular domain.

1. Introduction

One of the most advanced branches of nonlocal quantum field theory (QFT) is based on treating nonlocal fields as highly singular operator-valued generalized functions that must be averaged with analytic test functions. The Haag-Ruelle scattering theory and the PCT and spin-statistics theorems have thus been extended to nonlocal interactions; this generalization proved feasible for an arbitrarily singular ultraviolet behavior of the vacuum expectation values (see [1]). Such an approach also turned out to be effective for describing strong interactions phenomenologically [2]. Gelfand and Shilov [3] introduced test function spaces of type $S^0$, which are convenient for these applications. The specific choice of test functions should be adapted to the model under study, but the spaces $S^0_\alpha$ play a dominant role in deriving the general theorems because the Fourier transforms of their elements have compact supports and these spaces are hence suitable for fields with arbitrarily singular high-energy behavior. The subscript $\alpha$ controls the decrease of elements of $S^0_\alpha$ at infinity. Namely, by definition [3], they decrease no slower than $\exp\{-|x/A|^{1/\alpha}\}$ with some $A > 0$; this space proves to be nontrivial only if $\alpha > 1$. The mathematical foundation for extending the results of the axiomatic approach [4], [5] to fields defined on $S^0_\alpha$ is their angular localizability [6]. Specifically, for each (closed) cone of directions $K \subset \mathbb{R}^d$ we can define the space $S^0_\alpha(K)$ in such a way that $S^0_\alpha(\mathbb{R}^d) = S^0_\alpha$ and the correspondence $K \rightarrow S^0_\alpha(K)$ is a one-to-one mapping satisfying the structural relations

$$S^0_\alpha(K_1 \cap K_2) = S^0_\alpha(K_1) + S^0_\alpha(K_2), \quad S^0_\alpha(K_1 \cup K_2) = S^0_\alpha(K_1) \cap S^0_\alpha(K_2).$$

(1)

Relations (1) imply dual relations for the generalized functions composing the dual space $S^0_\alpha'$; as a consequence of these dual relations, each element of $S^0_\alpha'$ has a unique minimal closed carrier cone. In the treatment of nonlocal fields of the class $S^0_\alpha$ with a dense invariant domain $D$ in the Hilbert space of states, it is natural to replace the microcausality axiom with the condition that for any field components $\phi_\iota$ and $\phi_{\iota'}$, the matrix elements of the commutator or anticommutator

$$\langle \Phi, [\phi_\iota(x), \phi_{\iota'}(x')] \rangle \psi = \langle \Phi, \psi \rangle$$

(+) $(\Psi, \Psi \in D)$

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are continuous in the topology of $S^0_\alpha(\mathbb{K})$, where $\mathbb{K} = \{(x, x') \in \mathbb{R}^8 : (x - x')^2 \geq 0\}$. As shown in [1], this condition ensures the normal spin-statistics connection and the PCT invariance of nonlocal field theory.

A number of works (see, e.g., [7], [8] and the references therein) on nonlocal theory use another test function space $S^0_\alpha$, which is just the Fourier transform of the Schwartz space $\mathcal{D}$ of smooth functions of compact support. An argument in favor of this choice is that the vacuum expectation values of any field theory (both local and nonlocal) whose state space has a positive metric are bounded in any difference variable if the other variables are fixed (this is discussed in great detail in [9].) For this reason, the space $S^0_\alpha$, which is a formal limit of $S^0_{\alpha}$ as $\alpha \to \infty$, is apparently suitable for realizing the majority of models, although it gives a slightly narrower framework. As shown in [10], here again, there is a natural definition of the spaces $S^0_\alpha(K)$ associated with cones, which leads to an alternative formulation of causality as the continuity property of matrix elements under the topology of $S^0(\mathbb{K})$. The connection between this formulation and those proposed by other authors is also discussed there.

But a thorough elucidation of the interplay between the two generalizations of local commutativity stated above has been an open issue up to now. The point is that the topological structure of the spaces $S^0(K)$, which are constructed of Banach spaces using two (projective and inductive) limits, is quite complicated. In contrast, the spaces $S^0_\alpha(K)$ enter the well-studied class of DFS spaces and have properties that are most convenient for applications. In case of difficulties with functionals and fields defined on $S^0$, we may work with their restrictions to $S^0_\alpha$. But then the question arises whether the carrier cones of the restrictions defined using test functions in $S^0_\alpha$ are the same as the carrier cones of the initial functionals. As shown in Sec. 3 below, the answer is affirmative. When coupled with formulas [1], this immediately implies that every functional in $S^0_\alpha$ has a smallest carrier cone. Another consequence of this result is established in Sec. 5, where the Paley-Wiener-Schwartz theorem, which plays an important part in QFT, is extended to the functional class $S^0_\alpha$. It should be noted that the direct derivation of this extension by analogy to what was done in [12] for $S^0_{\alpha}$ is vastly more sophisticated. Section 2 is devoted to necessary preliminaries. We do not dwell on the motivation for the definitions used because it has been detailed in [6], [10]. The main tool used below to derive the theorem on carriers of restrictions is Hörmander’s $L^2$-estimates [13] for solutions of nonhomogeneous Cauchy-Riemann equations. In Sec. 4, this technique is adapted to the problem under study, which is required because the weight functions that define the norms in Hörmander’s estimates are logarithmically plurisubharmonic, whereas the indicator functions defining the spaces $S^0_\alpha(K)$ and $S^0_{\alpha}(K)$ are not of such form. We refer to Vladimirov’s treatise [14] for the facts about plurisubharmonic functions and the duality of convex cones. In Sec. 6, the central theorem of the paper is extended to the more general class of spaces $S^0_\alpha$. Section 7 contains concluding remarks.

2. Basic definitions and notation

**Definition 1.** Let $U$ be an open cone in $\mathbb{R}^d$. The space $S^{0,b}(U)$ is the intersection (projective limit) of the Hilbert spaces $H^{0,B}_N(U)$ ($B > b$, $N = 0, 1, 2, \ldots$) consisting of entire functions on $\mathbb{C}^d$ and endowed with the scalar products

$$\langle f, g \rangle_{U,B,N} = \int f(z)g(z) \left(1 + |x|\right)^{2N} e^{-2B|x|} \frac{dxdy}{(1+|x|)^N} \quad (z = x + iy),$$

(3)
where $\delta_U(x)$ is the distance of $x$ from $U$. It can also be defined as the intersection of the Banach spaces $E_{N^0}^0(U)$ of entire functions with the norms

$$\|f\|_{U,B,N} = \sup_{z \in \mathbb{C}^d} |f(z)| (1 + |x|)^N e^{-B\delta_U(x) - B|y|}. \quad (4)$$

The equivalence of the system of norms (4) to that defined by scalar products (3) is easily proved from Cauchy’s integral formula in the same way as for the spaces $S_\alpha^0(U)$ in [13]. The space $S^0(U)$ is the union (inductive limit) of the spaces $S_{0,b}^0(U)$ ($b \to \infty$). If $K$ is a closed cone, then $S^0(K)$ is defined as the inductive limit of the spaces $S^0(U)$, where $U$ ranges those open cones that contain $K$ as a compact subcone, which is denoted by $K \Subset U$. All these spaces are continuously embedded in the space $S^0(\{0\})$ of analytic functions of exponential type. It corresponds to the degenerate closed cone consisting of one point, namely, the origin. (It is valid and convenient to say that the same space corresponds to the empty open cone.) The spaces $S_{0,b}^0(U)$ belong to the class FN of nuclear Fréchet spaces. This fact, which is essential for applications to QFT, can be established in the same way as in deriving Theorem 2 of Ref. [12]. As a consequence, they also belong to the Fréchet-Schwartz class FS and are Montel spaces. In particular, they are reflexive. The spaces $S^0(U)$ and $S^0(K)$, being countable inductive limits of such spaces, inherit nuclearity (see Sec. III.7.4 in [15]) and are obviously Hausdorff spaces. The spaces $S^0(U)$ are complete, as is proved in [16] using the acyclicity of the injective sequence $S^{0,\nu}(U)$, $\nu = 1, 2, \ldots$. Together with nuclearity and barrelledness, this implies that they are Montel spaces (see Exer. 19 in Chap. IV in [15]) and hence are reflexive. Whether $S^0(K)$ has such properties is still an open question.

**Definition 2.** The space $S^0_\alpha(U)$, where $\alpha > 1$ and $U$ is an open cone in $\mathbb{R}^d$, is the inductive limit of the Hilbert spaces $H_{\alpha,A}^{0,0}(U)$, $A > 0$, $B > 0$, consisting of entire functions on $\mathbb{C}^d$ and endowed with the scalar products

$$\langle f, g \rangle_{U,A,B} = \int f(z)\overline{g(z)} e^{2(|x/A|^{1/\alpha} - B\delta_U(x) - B|y|)} \, dx \, dy. \quad (5)$$

This inductive limit coincides (see [3]) with that of the Banach spaces $E_{\alpha,A}^{0,0}(U)$ of entire functions with the norms

$$\|f\|_{U,B,A} = \sup_{z \in \mathbb{C}^d} |f(z)| e^{(|x/A|^{1/\alpha} - B\delta_U(x) - B|y|)}. \quad (6)$$

The space $S^0_\alpha(K)$, where $K$ is a closed cone, is defined as the inductive limit of the spaces $S^0_\alpha(U)$, $U \supseteq K$. All the spaces $S^0_\alpha(U)$ and $S^0_\alpha(K)$ are continuously embedded into $S^0_\alpha(\{0\})$ which is obviously the same as $S^0(\{0\})$. As shown in [12], they belong to the class DFS and even to the class DFN. (These abbreviations denote the respective strong dual spaces of Fréchet-Schwartz spaces and of nuclear Fréchet spaces.) Therefore, they are Montel spaces and reflexive.

**Definition 3.** A closed cone $K \subset \mathbb{R}^d$ is said to be a carrier cone of a functional $v \in S^0(\mathbb{R}^d)$ (or $S^0_\alpha(\mathbb{R}^d)$) if $v$ has a continuous extension to the space $S^0(K)$ (or $S^0_\alpha(K)$).

Such an extension, if it exists, is unique because $S^0_\alpha$ is dense in $S^0_\alpha(K)$, as shown in [12], and this also implies that $S^0$ is dense in $S^0(K)$ (see [10] for the details). Hence, the subspace of functionals carried by $K$ is algebraically identified with $S^0(K)$ or $S^0_\alpha(K)$.

In what follows, we use the following elementary lemma.

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1. For arbitrary cones $V_1$, $V_2$, the notation $V_1 \Subset V_2$ means that $\overline{V_1} \setminus \{0\} \subset V_2$, where $\overline{V_1}$ is the closure of $V_1$. 
Lemma 1. Let $E$ be a linear space and let $L_0$, $L_1$, and $L_2$ be its subspaces endowed with locally convex topologies and such that $L_0 \subset L_1 \cap L_2$. We assume that $L_1 + L_2$ and $L_1 \cap L_2$ are equipped with the respective inductive and projective topologies. If $L_0$ is dense in each of $L_1$, $L_2$, and $L_1 \cap L_2$ and the injections $L_0 \to L_1$, $L_0 \to L_2$ are continuous, then

$$(L_1 + L_2)' = L_1' \cap L_2',$$

where the dual spaces are regarded as linear subspaces of $L_0'$.

Proof. We note that $L_0$ is dense in $L_1 + L_2$ if it is dense in $L_1$ and $L_2$ and the natural mapping $(L_1 + L_2)' \to L_0'$ is hence injective along with $L_1' \to L_0'$ and $L_2' \to L_0'$. Clearly, $(L_1 + L_2)' \subset L_1' \cap L_2'$ and we need only show the converse inclusion. Let $v \in L_1' \cap L_2'$ and let $v_1$ and $v_2$ be its continuous extensions to $L_1$ and $L_2$. Because the projective topology on $L_1 \cap L_2$ is the upper bound of the topologies induced by those of $L_1$ and $L_2$, the functionals $v_1$ and $v_2$ are continuous on $L_1 \cap L_2$ and hence coincide on this subspace by the denseness condition. Therefore, the formula $\hat{v}(f_1 + f_2) = v_1(f_1) + v_2(f_2)$ defines a linear extension of $v$ to $L_1 + L_2$ which is continuous by the definition of inductive topology. □

3. Theorem on the restriction of functionals

Theorem 1. Let $v \in S^0$ and let $\alpha > 1$. If the restriction $v|_{S^0_\alpha}$ is carried by a closed cone $K$, then so is $v$.

Proof. This statement, combined with the obvious converse implication, can be expressed by the relation

$$S^0 \cap S^0_\alpha(K) = S^0_\alpha(K),$$

where all the spaces are regarded as vector subspaces of $S^0_\alpha$, which is permissible because $S^0 \alpha$ is obviously dense in $S^0$ and $S^0_\alpha$ is dense in $S^0_\alpha(K)$ and in $S^0(K)$ as pointed out above. For $K = \{0\}$, equality (7) is valid by definition, and the cone $K$ is henceforth assumed nontrivial. We begin by deriving the dual formula

$$S^0(K) = S^0 + S^0_\alpha(K)$$

and then apply Lemma 1. Let $f \in S^0(U), U \supseteq K$. Using the dilation invariance of the spaces involved, we assume that $b < 1$ without loss of generality. Here and in the next two sections, we use the Euclidean norm in $\mathbb{R}^d$, denoted by $| \cdot |$. We choose a nonnegative function $\chi_0 \in C_0^\infty$ with support in the ball $B_\epsilon = \{x : |x| < \epsilon\}$ and such that $\int U \chi_0(x - \xi) \, dx = 1$ and set

$$\chi(x) = \int_U \chi_0(x - \xi) \, d\xi.$$

We decompose $f$ as

$$f = f_1 + f_2, \quad f_1(z) = f(z)\chi(x), \quad f_2(z) = f(z)(1 - \chi(x)).$$

The functions $f_1$ and $f_2$ are not analytic but respectively behave at infinity as elements of $S^0$ and $S^0_\alpha(K)$. Indeed, we have

$$|f_1(z)| \leq \|f\|_{U,1,N} (1 + |x|)^{-N} e^{\epsilon+|y|}$$

for any $N$ because $\delta_U(x) \leq \epsilon$ for $x \in \text{supp } \chi$. Further, let $V$ be an open cone chosen so that $K \subseteq V \subseteq U$. Then there is a constant $\gamma > 0$ such that $\delta_V(x) \geq \gamma|x|$ for all points of $\text{supp}(1 - \chi)$ except a compact subset. At these points, we have $\delta_U(x) \leq \delta_V(x) \leq 2\delta_V(x) - \gamma|x|$. Therefore,

$$|f_2(z)| \leq C \|f\|_{U,1,N} e^{-\gamma|x|+2\delta_V(x)+|y|}.$$  \hspace{1cm} (10)

To obtain an analytic decomposition, we write

$$f = f'_1 + f'_2, \quad f'_1 = f_1 - \psi, \quad f'_2 = f_2 + \psi$$
and subject \(\psi\) to the equations

\[
\frac{\partial \psi}{\partial \bar{z}_j} = \eta_j,
\]

where

\[
\eta_j = \frac{1}{2} \frac{\partial \chi}{\partial \bar{z}_j}, \quad j = 1, \ldots, d.
\]

The functions \(\eta_j(z)\) are nonzero only for \(x \in \partial U + B_\epsilon\), where \(\partial U\) is the boundary of \(U\), and satisfy the estimate

\[
|\eta_j(z)| \leq C_{j,N} (1 + |x|)^{-N} e^{\gamma \|y\|}
\]

for each \(N\). It remains to verify that there exists a solution of Eqs. (11) with the required behavior at infinity. We characterize this behavior by a plurisubharmonic function, which allows applying an existence theorem due to Hörmander [13].

**Lemma 2.** Let \(U'\) and \(U\) be open cones in \(\mathbb{R}^d\) such that \(U' \subset U\) and let \(\alpha > 1\). For each system of functions \(\eta_j, j = 1, \ldots, d\), satisfying (12) and having support in an \(\epsilon\)-neighborhood of the boundary of \(U\), there is a plurisubharmonic function \(\rho(z)\) with values in \((-\infty, +\infty)\) such that for any \(B > \sqrt{d}\) and \(N = 0, 1, 2, \ldots\), the following inequalities hold:

\[
\rho(z) \geq \max_j \log |\eta_j(z)| \quad \text{for} \quad x \in \partial U + B_\epsilon,
\]

\[
\rho(z) \leq -N \log(1 + |x|) + B|y| + C_{B,N}
\]

everywhere,

\[
\rho(z) \leq -|x|^{1/\alpha} + B|y| + C_B
\]

for \(x \in U'\),

where \(C_{B,N}\) and \(C_B\) are constants.

The proof of Lemma 2 is given in the next section, and we now derive formula (17). We choose a cone \(U'\) such that \(V \subset U' \subset U\) and set \(g(z) = 2 \rho(z) + (d + 1) \log(1 + |z|^2)\), where \(\rho\) is defined in Lemma 2. In view of (13) the functions \(\eta_j\) belong to \(L^2(\mathbb{C}^d, e^{-\rho}d\lambda)\), where \(d\lambda = dx dy\) is the Lebesgue measure on \(\mathbb{C}^d\). By the definition of \(\eta_j\), the consistency conditions

\[
\frac{\partial \eta_j}{\partial \bar{z}_k} = \frac{\partial \eta_k}{\partial \bar{z}_j}
\]

are satisfied. Theorem 15.1.2 in [13] shows that the system of equations (11) has a solution \(\psi\) such that

\[
2 \int |\psi|^2 e^{-\rho}(1 + |z|^2)^{-2}d\lambda \leq \int |\eta|^2 e^{-\rho}d\lambda.
\]

It follows that

\[
\psi \in L^2(\mathbb{C}^d, e^{2(1 + |x|)^{2(N-d-3)} e^{-2B|y|}d\lambda}), \quad \psi \in L^2(\mathbb{C}^d, e^{2|z/A|^{1/\alpha} - \delta_V(x) - B'|y|}d\lambda)
\]

for all \(N\) and for arbitrary \(A > 1\) and \(B' > B\). The first membership relation is ensured by (14), and the second follows from (15) because for \(x \not\in U'\), we have \(\delta_V(x) \geq \gamma' |x|\) with some \(\gamma' > 0\) and obviously

\[
2(|z/A|^{1/\alpha} - \delta_V(x) - B'|y|) \leq -g(z) - 2 \log(1 + |z|^2)
\]

by (14) again. Referring to (9) and (10) and keeping definitions (3) and (5) in mind, we conclude that the analytic functions \(f_1'\) and \(f_2'\) belong to the respective spaces \(S^0\) and \(S^0_\alpha(V)\). Relation (8) is thus proved.

**Lemma 3.** The space \(S^0_\alpha\) is dense in the intersection \(S^0 \cap S^0_\alpha(K)\) endowed with the projective topology.

In fact, an approximating sequence for \(f \in S^0 \cap S^0_\alpha(K)\) is easy to construct by setting \(f_\nu = \sigma_\nu f\), where \(\sigma_\nu(z)\) is a sequence of Riemann sums for the integral \(\int \sigma_0(z - \xi) \, d\xi\), with \(\sigma_0\) being a function in \(S^0_\alpha\) whose integral is unity. The sequence \(f_\nu \in S^0_\alpha\) is bounded in both the spaces \(S^0\) and \(S^0_\alpha(K)\) and converges to \(f\) uniformly on compact subsets of \(\mathbb{C}^d\) by the Vitali–Montel theorem. Therefore, by the standard argument in Sec. II.2.1 in [3], \(f_\nu \to f\) in the topology of either of these spaces.
We now use Lemma 1, taking into account that the inductive topology $\mathcal{T}$ on $S^0(K)$ defined by the mappings $S^0 \to S^0(K)$ and $S^0(K) \to S^0(K)$ coincides with the original topology $\tau$ of this space by Grothendieck’s version of the open mapping theorem (also see Raikov’s Supp. 1 to the Russian edition of [18]). Indeed, $\tau$ is not stronger than $\mathcal{T}$ by the definition of the latter, and Grothendieck’s theorem is applicable here because the space $(S^0(K), \tau)$, being an inductive limit of Fréchet spaces and a Hausdorff space, belongs to the class (b) (the spaces of this class are also called ultrabornological) and the space $(S^0(K), \mathcal{T})$ belongs to the class $\mathcal{UF}$, i.e., can be covered by a countable family of its Fréchet subspaces.\footnote{This name is used for a vector subspace of a Hausdorff, locally convex space $E$ if it can be made into a complete metrizable locally convex space by giving a topology that is stronger than the one induced on it by the topology of $E$.} This is the case because both the spaces $S^0$ and $S^0(K)$ are in the class $\mathcal{UF}$ and $\mathcal{T}$ is representable as the quotient topology of the sum $S^0 \oplus S^0(K)$ modulo a closed subspace (see Prop. 28 in Chap. V in [18] and Lemma 6 in Supp. 1 to the Russian edition of this book). Theorem 1 is thus proved.

\[ \square \]

**Corollary 1.** For any pair of closed cones $K_1, K_2$ in $\mathbb{R}^d$, the relation

\[ S^0(K_1 \cap K_2) = S^0(K_1) \cap S^0(K_2) \]  

holds. Each element of $S^0$ has a unique minimal carrier cone.

In view of Theorem 1, formula (18) immediately follows from an analogous relation obtained in [6] for the spaces $S^0_\alpha(K)$, which is in turn a consequence of (13); we emphasize that its derivation essentially uses the fact that these are FS spaces. The existence of the smallest carrier cone of $v \in S^0$ also immediately follows from the existence of such a carrier for the restriction $v|S^0_\alpha$ or, alternatively, from relation (13) by the usual compactness considerations.

### 4. Approximation of the Indicator Functions

by Plurisubharmonic Functions

The possibility of using plurisubharmonic functions to describe the topology of spaces occurring in the theory of Fourier-Laplace transformation was discussed in Sec. 15.2 in [13]. In our case, where another class of spaces comes into play, this issue calls for further examination. The method presented below and based on using an analytic function with special properties and on systematically constructing the upper envelopes of families of plurisubharmonic functions seems quite general and can also be applied to other problems.

**Proof of Lemma 2.** Let $e$ be a unit vector in $\mathbb{R}^d$ and $\theta > 0$. We let $R_e$ denote the ray $\{\lambda e: \lambda \geq 0\}$ and $K_{e, \theta}$ denote the circular cone $\{\lambda x: |x-e| \leq \theta, \lambda \geq 0\}$. We assume that $\theta$ is less than the angular separation between the cones $U$ and $U'$. It suffices to prove that for every $e$, there exists a plurisubharmonic function $\rho_e(z)$ bounded by (13) (with a constant independent of $e$) and satisfying estimates of forms (13) and (15) but respectively for $x \in R_e + B_e$ and $x \notin K_{e, \theta}$. Then the upper envelope

\[ \rho(z) = \lim_{z' \to z} \sup \{\rho_e(z'): e \in \partial U, |e| = 1\} \]  

satisfies all the required conditions because $U' \subset \bar{C}K_{e, \theta}$ for every $e \in \partial U$. We emphasize that according to Sec. 10.3 in [13], function (19) is plurisubharmonic because the family $\{\rho_e\}$ is locally uniformly bounded from above. The space $S^0_\alpha(\mathbb{R})$, which is the Fourier transform of $S^0_\alpha(\mathbb{R})$, contains a nonnegative even function $\omega$ such that supp $\omega \subset [-\delta, \delta]$, \( \int \omega(t) \, dt = 1 \), and \( |\omega^{(n)}(t)| \leq C_0 A_0^n t^{n\alpha} \), where $A_0$ and $\delta$ can be taken arbitrarily small (see Sec. IV.8.3 in [3]). Let $\Omega$ be the convolution of $\omega$ by the characteristic function of the segment $|t| \leq 1 + \delta$ and let $1 + 2\delta < \pi/3$. Then $\cos(\xi t) > 1/2$ for $|\xi| < 1$ and...
The right-hand sides of (14) and (15) are invariant under rotations, as is the function \( \rho \). We consider the function
\[
\rho(\zeta) = -|\text{Re}\zeta/A|^{1/\alpha} + (1 + 2\delta)|\text{Im}\zeta| + \text{Const.}
\]
We can now construct the functions \( \rho_0 \). We first assume that \( e \) is the first basis vector and define \( \rho_1 \) as the upper envelope of the family
\[
\rho_0(z_1 - r) + \sum_{j > 1} \rho_0(z_j) + H(r), \quad r \geq 0.
\]
Because \( H(r) \neq -\infty \), we have \( \rho(z) > -\infty \) everywhere. If \( x = \text{Re}z \) lies in the \( e \)-neighborhood of \( \lambda e \), then \(|x_1 - |x| < |x_1 - \lambda| + |\lambda - |x|| < 2\epsilon \) and \( |x_j| < \epsilon \) for all \( j > 1 \). Hereafter, we assume that \( \epsilon < 1/2 \). Then
\[
\rho_1(z) \geq \sum_j |y_j| + H(|y|) \geq |y| + H(|x|) \geq \max \log |\eta_j(x, y)|
\]
by our construction, and the required lower bound is satisfied. Further, using (21), (23), and the elementary inequalities
\[
|x_j/A|^{1/\alpha} \geq N \log(1 + |x_j|) - C_{N,A}, \quad |x_1| \leq |x_1 - r| + r,
\]
and \( \sum |y_j| \leq \sqrt{d}|y| \), we conclude that \( \rho_1 \) satisfies (24) if \( \delta \) is sufficiently small. Finally, if \( x \notin \pm K_{e,\theta} \), then \( \sum_{j > 1} |x_j|^{1/\alpha} \geq |\theta' x|^{1/\alpha} \) with some \( \theta' > 0 \), and if \( x \in -K_{e,\theta} \), we have \( |x_1 - r| \geq |x_1| \). Therefore, the last desired bound on \( \rho_1 \) is also satisfied if \( \delta \) and \( A_0 \) are sufficiently small. Now let \( e \) be an arbitrary unit vector on the boundary of \( U \), and let \( T_e \) be an orthogonal transformation taking it to the first basis vector. The function \( \rho_e(z) = \rho_1(T_e z) \) is plurisubharmonic and also satisfies all the required constraints because the right-hand sides of (14) and (15) are invariant under rotations, as is the function \( |y| + H(|x|) \) majorizing \( \max j \log |\eta_j(x, y)| \). This completes the proof of Lemma 2. \( \square \)

5. Fourier-Laplace transforms of functionals of the class \( S^{10} \)

When coupled with the Paley-Wiener-Schwartz-type theorem established in [12] for functionals of the class \( S^{10} \), Theorem 1 readily implies the following result.

**Corollary 2.** Let \( V \) be an open connected cone in \( \mathbb{R}^d \), and let \( \alpha > 1 \). Suppose a function \( u(\zeta) \) is analytic on the tubular domain \( T^V = \mathbb{R}^d + iv \) and satisfies the estimate
\[
|u(\zeta)| \leq C_{e,R}(W) \exp \left\{ \varepsilon |\text{Im}\zeta|^{-1/(\alpha - 1)} \right\} \quad (\text{Im}\zeta \in W, \ |\zeta| \leq R)
\]
for arbitrary \( \varepsilon, R > 0 \) and each cone \( W \subset V \). If the boundary value of \( u \) is a Schwartz distribution (i.e., belongs to \( \mathcal{D}' \)), then this function satisfies the stronger inequality
\[
|u(\zeta)| \leq C_R(W) |\text{Im}\zeta|^{-N_{R,W}} \quad (\text{Im}\zeta \in W, \ |\zeta| \leq R)
\]
In fact, as shown in [12], every function analytic in $T^V$ and having property $24$ is the Laplace transform of a functional $v \in S_0^0(V^*)$, where $V^* = \{ x : x\eta \geq 0, \forall \eta \in V \}$ is the dual cone$^3$ of $V$, and its boundary value is the Fourier transform of $v$. The Fourier transformation isomorphically maps $S_0$ onto $D$ (see Sec. III.2 in [8]), and hence Theorem 1 immediately gives $v \in S_0^0(V^*)$. Now $25$ results from the elementary estimate

$$|u(\zeta)| = |(v, e^{i\zeta})| \leq \|v\|_{U,B,N} \|e^{i\zeta}\|_{U,B,N}$$

by norms $11$. Here, $B$ is arbitrarily large, $U$ is any open cone containing $V^*$ as a compact subcone, and $N$ depends on $B, U$ in general. The cone $U$ and another auxiliary cone $U'$ should be taken so that $V^* \in U \subseteq U' \subseteq \text{Int} W^*$ (here $\text{Int} W^*$ is the interior of $W^*$). This is possible because $W \subseteq V$ implies that $V^* \subseteq \text{Int} W^*$. Setting $\zeta = \xi + i\eta$, we have

$$\|e^{i\zeta}\|_{U,B,N} = \sup_{x,y} \exp \{ -x\eta - y\xi + N \log (1 + |x|) - B\delta_U(x) - B|y| \}. \tag{26}$$

Assuming that $|\xi| \leq R < B$, we can omit the terms dependent on $y$. If $x \notin U'$, then $\delta_U(x) / R > B \theta$, and for $|\eta| \leq R < B$, the exponent is dominated by a constant. Finally, let $x \in U'$. The inclusion $U' \subseteq \text{Int} W^*$ implies that there is a $\theta' > 0$ such that $x\eta \geq \theta' |x||\eta|$ for all $x \in U'$ and $\eta \in W$. Substituting this inequality in $26$, dropping the term $\delta_U(x)$, and locating the extremum, we obtain $25$ with some $C_R(W)$ proportional to $\|v\|_{U,B,N}$.

Another consequence of the same combination of Theorem 1 with Theorem 4 in [12] is an extension of the Paley-Wiener-Schwartz theorem to the generalized functions of the class $S_0^0$. We let $A_0(V)$ denote the space of functions analytic in $T^V$ and satisfying $25$, for each $W \subseteq V$ and every $R > 0$. Clearly, it is an algebra under pointwise multiplication.

**Theorem 2.** Let $V$ be an open connected cone in $\mathbb{R}^d$ and $V^*$ be its dual cone. The Laplace transformation $v \rightarrow (v, e^{i\zeta})$ is an isomorphism of the space $S_0^0(V^*)$ onto the algebra $A_0(V)$. Consequently, the elements of $S_0^0$ with a given closed carrier cone compose a convolution algebra. The function $u(\zeta)$ in $A_0(V)$ that is the Laplace transform of a functional $v \in S_0^0(K)$ tends to its Fourier transform $\hat{v}$ in the topology of $D'$ as $\text{Im} \zeta \rightarrow 0$ inside a fixed cone $W \subseteq V$.

**Proof.** We have just seen that every functional belonging to $S_0^0(V^*)$ has a Laplace transform defined on $T^V$ and satisfying $25$. Applied to its restriction to $S_0^0$, Theorem 4 in [12] shows that the Laplace transform is analytic in this tube. As noted above, by the same theorem, every function analytic in $T^V$ and having property $24$ (and particularly $u \in A_0(V)$) is the Laplace transform of a certain $v \in S_0^0(V^*)$, and $\int u(\xi + i\eta)f(\xi)d\xi \rightarrow (\hat{v}, f)$ as $W \ni \eta \rightarrow 0$ for each $f \in S_0^0$. On the other hand, it is well known that every function analytic on $T^V$ and satisfying $25$ has a boundary value belonging to $D'$, which is zero only if the function vanishes (see Theorem 3.1.15 in [19]). Therefore, $v$ belongs to $S_0^0$ and, by Theorem 1, to the space $S_0^0(V^*)$ as well. Hence, the Laplace transformation is a one-to-one mapping of $S_0^0(V^*)$ onto $A_0(V)$. The weak convergence $u(\xi + i\eta) \rightarrow \hat{v}$ on elements of $D$ implies the convergence in the strong topology of $D'$ because it is a Montel space. Furthermore, Theorem 4 of [12] shows that $S_0^0(V^*)$ is a convolution algebra and the convolution $v_1 * v_2$ of its two elements corresponds to the product $u_1 \cdot u_2$ of their Laplace transforms. If $v_1, v_2 \in S_0^0(V^*)$, then $u_1 \cdot u_2 \in A_0(V)$ and hence $v_1 * v_2 \in S_0^0(V^*)$. Therefore, $S_0^0(V^*)$ is a convolution algebra. It is worth noting that an arbitrary closed properly convex cone $K \subseteq \mathbb{R}^d$ is the dual cone of the interior of $K^*$. Theorem 2 is thus proved. We also note that Theorem 2 and the relation $V^* = (\text{ch} V)^*$, where $\text{ch}$ signifies the convex hull, imply that $A_0(V) = A_0(\text{ch} V)$. 

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$^3$We note that this cone is closed and convex and even properly convex (i.e., contains no entire line) because $V$ is open.
The algebra \( \mathcal{A}_0(V) \) is a generalization of Vladimirov’s algebra \([13, 20]\), formed by the Laplace transforms of tempered distributions that have support in a closed properly convex cone, to the case of analytic functionals with an arbitrary singularity, which means an arbitrary fast increase of the Laplace transforms at infinity. The algebra \( \mathcal{A}_0(V) \) can be made into a topological algebra by regarding it as the projective limit of the family of spaces \( \mathcal{A}_{0,r}(W) \) \((r > 0, W \subseteq V)\) defined in turn as the inductive limits of the Banach spaces \( \mathcal{A}_{0,R,N}(U) \) \((R > r, N = 0, 1, 2, \ldots, U \supseteq W)\) whose elements are analytic on \( T_R^U = \{ \zeta \in \mathbb{C}^d : |\zeta| < R, \text{Im} \zeta \in U \} \), with \( U \) an open cone, and have the finite norm
\[
\|u\|_{U,R,N} = \sup_{\zeta \in T_R^U} |\text{Im} \zeta|^N |u(\zeta)|.
\] (27)

The spaces \( \mathcal{A}_{0,r}(W) \) belong to the class DFS because the natural injections \( \mathcal{A}_{0,R',N'}(U') \rightarrow \mathcal{A}_{0,R,N}(U) \) \((R' > R, N' < N, U' \supseteq U)\) are compact mappings, i.e., the image of the unit ball of \( \mathcal{A}_{0,R',N'}(U') \) is relatively compact in \( \mathcal{A}_{0,R,N}(U) \). In fact, by the Montel theorem, we can choose a sequence \( u_\nu \) from any infinite subset of this image such that it converges to an analytic function \( u \) uniformly on each compact set in \( T_{R'}^U \). In particular, for every \( \epsilon > 0 \), we have
\[
\sup_{\zeta \in T_{R'}^U, |\text{Im} \zeta| \geq \epsilon} |\text{Im} \zeta|^N |u - u_\nu| \leq \epsilon
\]
if \( \nu \) is sufficiently large. On the other hand, \( \|u\|_{U',R',N'} \leq 1 \) and hence
\[
\sup_{\zeta \in T_{R'}^U, |\text{Im} \zeta| \leq \epsilon} |\text{Im} \zeta|^N |u - u_\nu| \leq \epsilon \|u - u_\nu\|_{U',R',N'} \leq 2\epsilon.
\]
Therefore, \( u_\nu \rightarrow u \) in the norm \( \| \cdot \|_{U,R,N} \). The Laplace transformation \( S^0(V^*) \rightarrow \mathcal{A}_0(V) \) is continuous in the strong topology of \( S^0(V^*) \). Moreover, it is continuous even if this space is given the projective limit topology by the natural embeddings in the DFS spaces \( S^{0,b}(U), b > 0, U \supseteq V^* \). This is the case because the constant \( C_R(W) \) in \([25]\) can be chosen proportional to \( \|v\|_{U,B,N} \), as shown above. We note that according to Sec. IV.4.5 in \([13]\), the projective limit topology is consistent with the duality of \( S^0(V^*) \) and \( S^0(V^*) \).

In the simplest, but important, case where \( V \) is the semi-axis \( \mathbb{R}_+ \) and \( V^* = \mathbb{R}_+ \), the space \( S^0(\mathbb{R}_+) \) coincides with \( S^0(\mathbb{R}_+^d) \), and we can be assured that its strong topology is identical to the projective limit topology because of the regularity of the injective sequence of spaces \( S^{0,\nu}(\mathbb{R}_+) \). This property, which means that every bounded set in \( S^0(\mathbb{R}_+) \) is contained and bounded in some \( S^{0,\nu}(\mathbb{R}_+) \), is in turn ensured by the acyclicity of the sequence (see \([21]\)). We can also assert that \( S^0(\mathbb{R}_+) \) endowed with the strong topology is an ultrabornological space and belongs to the class \( PUF \). The last conclusion can be deduced in the same manner as an analogous statement for \( D' \) in Supp. 2 to the Russian edition of \([13]\) because \( S^0(\mathbb{R}_+) \) is complete and belongs to Grothendieck’s class \( (S) \). Applying Raikov’s generalization of the open mapping theorem, which is proved in the same place, we then infer that the strong topology of \( S^0(\mathbb{R}_+) \) coincides with the bornological topology associated with the inverse image of the topology of \( \mathcal{A}_0(\mathbb{R}_+) \). Hence, the inverse image of every bounded set of \( \mathcal{A}_0(\mathbb{R}_+) \) is strongly bounded in \( S^0(\mathbb{R}_+) \). In other words, in this case the Laplace transformation is not only an algebraic but also a bornological isomorphism. This consideration demonstrates the subtleties arising in dealing with analytic functionals of the class \( S^0 \). In contrast, in the case of DFN spaces \( S^0_\alpha \), the ordinary open mapping theorem immediately shows that the Laplace transformation \( S^0_\alpha(V^*) \rightarrow \mathcal{A}_0^0(V) \) is a topological isomorphism (see \([12]\)).

6. Generalization to the spaces \( S^0_a \)

In some instances, there is a need to use more general spaces \( S^0_a \) instead of \( S^0_\alpha \). We recall \([22]\) that \( S^0_a \) is specified by a sequence of positive numbers \( \{a_\nu\} \) and consists of the
smooth functions on $\mathbb{R}^d$ that satisfy the inequalities
\begin{equation}
|x^k \partial^q f(x)| \leq CA^{|k|}B^{|q|}a_{|k|},
\end{equation}
where $C$, $A$, and $B$ are constants dependent on $f$ and the usual multi-index notation is used. The behavior of $f \in S^0_a$ as $|x| \to \infty$ is described by the function $1/a(|x/A|)$, where $|x| = \max_j |x_j|$ and
\begin{equation}
a(r) \overset{\text{def}}{=} \sup_{\nu} \frac{r^\nu}{a_\nu}.
\end{equation}
The function $a(r)$ is convex and monotone increasing faster than any power of its argument.

We can assume, without loss of generality, that the sequence $\{a_\nu\}$ also satisfies some regularity conditions, namely,
\begin{equation}
a_0 = 1, \quad a_{\nu+1} \geq a_\nu, \quad a_\nu^2 \leq a_{\nu-1}a_{\nu+1}.
\end{equation}
The first is a normalization and the others are features of the regularized sequence $a^*_\nu = \sup_{r \geq 1} r^\nu/a(r)$ which determines the same indicator function (for $r \geq 1$) because $a^*_\nu \leq a_\nu$ and $a^*_\nu \geq r^\nu/a(r)$.

The definition of $S^0_a$ also includes the requirement
\begin{equation}
a_{\nu+1} \leq Ch^\nu a_\nu,
\end{equation}
where $C$ and $h$ are constants, which ensures that the multiplication by $x_j$ does not take the functions out of the space. Under the same condition, the Fourier operator maps $S^0_a$ isomorphically onto the space $S^0_a$. That space coincides with the space $D\{a_\nu\}$ used by Roumieu [23] as the basic space for the theory of ultradistributions. (these are just the elements of the dual space $D'(a_\nu)$). The space $S^0_a$ is nontrivial if and only if it satisfies the nonquasianalyticity criterion (see Sec. 1.3 in [19]), which is
\begin{equation}
\sum_{\nu=1}^{\infty} a_\nu^{-1/\nu} < \infty
\end{equation}
in Mandelbrojt’s version. From [31], coupled with [24], it follows that
\begin{equation}
a(r) \leq Ce^{r\epsilon},
\end{equation}
where $\epsilon$ can be taken arbitrarily small. In fact, [29] means that $\log a_\nu$ is a convex function of the index and increases monotonically beginning with the value zero. Therefore, the terms of series [31] is monotone decreasing. In particular,
\begin{equation}
\sum_{\mu > \nu/2} a_{\mu}^{-1/\mu} \geq \frac{\nu}{2} a_\nu^{-1/\nu},
\end{equation}
and the convergence of the series implies that $\nu^\nu \leq \epsilon^\nu a_\nu$ for sufficiently large $\nu$. Clearly, each element of $S^0_a$ has an analytic continuation to $\mathbb{C}^d$ satisfying
\begin{equation}
|f(x + iy)| \leq Ce^{-\log a(|x/A|) + B\sum |y_j|},
\end{equation}
Applying Cauchy’s formula and using [32] and the inequality
\begin{equation}
a((1 - \lambda)r_1 + \lambda r_2) \leq a(r_1)a(r_1)
\end{equation}
which follows from the convexity and monotonicity of $a(r)$ (with the normalization $a(0) = 1$) and holds for each $\lambda \in [0,1]$, we readily see that, conversely, [33] implies estimate [28] for the derivatives of the restriction of the entire function $f$ to $\mathbb{R}^d$ with constants $A$ and $B$ which are generally different from those in [33] but can be taken arbitrarily close to them.

\footnote{Some additional restrictions were also imposed in [3], but they are unnecessary. A simple construction of the theory of Fourier transformation for spaces of the type $S$ using only [20] is given in [22].}
Therefore, we can use inequality (33) as the basis for the definition of $S^0_a$ and then define $H^{0,B}_{a,A}(U)$, $E^{0,B}_{a,A}(U)$ and also $S^0_a(U)$, $S^0_a(K)$ using the replacement

$$|x/A|^{1/\alpha} \mapsto \log a(|x/A|)$$

in formulas (5) and (11). The notion of a carrier cone thus naturally extends to these spaces. It is significant that $S^0_a(U)$ and $S^0_a(K)$ are DFN spaces if (31) is fulfilled. This can be shown in a way analogous to that used in [12] for $S^0_a(U)$, the only difference being that the canonical injection $H^{0,B}_{a,A}(U) \to H^{0,B'}_{a,A'}(U)$ is a nuclear mapping for arbitrary $A' > A$, $B' > B$, whereas an analogous statement for $H^{0,B}_{a,A}(U)$ is true if $A'$ and $B'$ are sufficiently large compared with $A$ and $B$, which has no effect on the ultimate conclusion. Thereafter, the proof of Theorem 1 extends to the spaces $S^0_a$ almost literally, with replacement (35) in (15) and further. By Theorem 1.3.5 in [19], condition (31) ensures that $S^0_a(\mathbb{R})$ contains a nonnegative even function $\omega$ supported in an arbitrarily small segment $[-\delta, \delta]$ and having the properties

$$\int \omega(t) \, dt = 1, \quad |\omega^{(n)}(t)| \leq C_0 A_0^\alpha a_\nu,$$

where $A_0$ is sufficiently large compared with $1/\delta$. But according to [23] (Lemma 1 in Chap. II), for each sequence $a_\nu$ satisfying (29) and (31), there is a sequence $a'_\nu$ satisfying the same conditions and such that $a'_\nu \leq C_\varepsilon \varepsilon'^\nu a_\nu$ for any $\varepsilon > 0$. Therefore, $A_0$ in (36) can be taken arbitrarily small (with a proper increase of $C_0$), which allows constructing an analogue of the function $\rho_0$ used to derive Lemma 2. To ensure that the corresponding analogue of $\rho_1$ has the desired properties, we can use the inequality

$$\sum_{j=1}^d \log a(|x_j|) \geq \log a(|x/d|),$$

which, as well as (31), follows from the regularity properties of $a$. Ultimately, we obtain the following generalization of Theorem 1.

**Theorem 3.** The restriction of $v \in S^{0'}_a$ to each nontrivial space $S^0_a$ specified by a sequence $a_\nu$ satisfying (29) and (31) has the same carrier cones as the functional $v$ itself.

7. Conclusion

In [7], which was devoted to constructing the scattering theory for nonlocal fields defined on the space $S^0$ (denoted by $Z$ there) and, in particular, to deriving an analogue of the Lehmann-Symanzik-Zimmermann reduction formulas, the locality axiom was replaced by some regularity properties of Green’s functions in momentum space. The results obtained here can be applied to clarify the relation between Steinmann’s conditions and the above-stated generalization of local commutativity which is closer to the intuitive idea of causality and, as already noted, ensures the normal spin-statistics relation and the existence of the PCT symmetry. A tentative conclusion is that they are actually equivalent. Because $S^0$ is a basic space of functional analysis, a deeper insight into the topological properties of the spaces $S^0(K)$ and $S^{0'}(K)$, including a closer examination of possibilities for extending the results concerning $S^0(\mathbb{R}_+)$ that are stated at the end of Sec. 5, is also mathematically interesting. But a major aim in this paper is to show how we can obviate the topological problems arising in concrete applications by using the restrictions of functionals to appropriate test function spaces with more convenient properties. We also note that as shown in [21], an analogue of Theorem 1 holds for the restrictions of elements of $S^0_a$ to smaller spaces of the same class.
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