Infinite Component Relativistic Wave Equations

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Abstract

We construct an infinite component relativistic wave equation which is a linear first order differential equation identical in form to a Dirac like equation, describing composite fields possessing multiple spin and energy states. The main motivation for such a construction is to give a description of Hadronic fields moving along their so called Regge Trajectories, however this may be generalized to other composite fields. In order to construct the equation so that it may accommodate physical states the concept of Spin Frames is introduced, and it is found that such an equation may propagate physical fields whose spin states differ by two units of angular momentum namely $\Delta J = 2$. The solution for the free field case is given by boosting a rest frame spinor with the infinite dimensional Lorentz Boost which are constructed as well. Finally we discuss the relevance of the groups $GL(4R)$, and $GL(3,1,R)$ and their appearance with regards to the wave equation at hand.

1 Introduction

The construction of relativistic wave equations has been of great interest to physicists, in particular in the last century where relativistic quantum mechanics has evolved to the now celebrated quantum field theory. Besides the complete relativistic description of a free filed, the latter has invoked a need for such equations, so that mechanisms such as free field perturbation theory will have some degree of success. In this context much work has been invested on studying free field equations describing particles possessing a particular spin [1, 4, 12](those have been mostly of fields carrying spin 0, 1, 2, $\frac{1}{2}$, and even $\frac{3}{2}$), however not as much has focused on relativistic wave equations describing fields possessing multiple states of spin$^1$ and rest mass (bound states), or so called composite fields. A particular case in point which we will utilize repeatedly as

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$^1$Some of these constructions were initially made by Majorana [8], and Dirac [5].
a fitting case for such a construction of relativistic wave equations, and in fact will provide a major motivation for our work is an Hadronic field. A Hadron is a composite entity which can be excited along a particular Regge trajectory and therefore may poses different states both in energy, and in angular momentum. It follows that the physical motivation for constructing such an equation is to have a collective description of interactions among such fields taking into account the various energy and angular momentum states such fields may acquire. This description would be an effective treatment of Hadronic fields whose fundamental interactions are that of constituent quarks and gluons, those however would not be specified in such a treatment, but would be inferred from the fact that excited states of the composite fields correspond to various interactions of constituent fields. It is therefore appropriate that we limit our discussion to kinematic considerations with respect to such an infinite component wave equation.

To demonstrate further the physical motivation for such an equation in the context of Hadron physics it would be instructive to examine the decomposition of physical currents of Hadronic fields. As an example, a spin $\frac{1}{2}$ field may be used to demonstrate one such current decomposition.

According to the Gordon identity [11] the current for a spin $\frac{1}{2}$ field is given (in momentum space) by:

$$ j^\mu(p,p') = \langle p' | \frac{E^\mu + p^\mu}{2m} + \frac{i\sigma^\mu\nu q_\nu}{2m} | p \rangle $$

(1)

where $q_\nu = p'_\nu - p_\nu$, and $\sigma^{\mu\nu}$ comprise the six Lorentz generators.

This decomposition is of course based on the existence of a linear wave equation, or what is know as the Dirac Equation. What is revelling about this decomposition is that the current is split into two parts; one describing the linear motion of the current given by the first term on the right hand side of (1), the second describing the spinning part of the current given by the second term on the right. The latter also gives us the magnetic moment of the electron.

It is evident that such a decomposition will describe a particle of a definite spin (in this case it is $\frac{1}{2}$), and indeed by the use of the Bargmann Wigner equations [1] such a decomposition should be attained for a field of any spin. However, for a particle acquiring different spin states as it interacts such a decomposition would have to yield another term to its current, namely that which has an expectation value between different angular momentum states, indicating that such a particle may acquire additional higher multipole moments. Indeed it was shown in [6, 10] that Hadrons may interact through their quadrupole moments produced by a digluon intermediate state which leads to Hadron structure deformation. This would be a hint that an additional term to (1) would have to be added, and would be proportional to:

$$ \Delta j^\mu = \langle p' | j | c_1 \tau_1^{\mu\nu} l_{1\nu} + c_2 \tau_2^{\mu\nu\rho} l_{2\nu} l_{3\rho} + ... + c_n \tau_n^{\mu\nu...\rho\sigma} l_{2\nu} l_{3\rho} ... l_{n\rho} | p \rangle $$

(2)
where the terms $\tau^{\mu\nu...\rho}_n$ are tensors associated with multipole moments that a particle may possess, and the vectors $l_{n\rho}$ are four vectors comprised of linear combinations of the vectors $p$ and $p'$.

The first tensor in (2) is a symmetric ‘shear’ tensor describing quadrupolar pulsations [9] providing excitations between angular momentum states that differ by $\Delta J = 2$ along a given Regge trajectory. Higher terms in (2) will produce transitions of $\Delta J = 3, 4,...n$.

In a sense this formalism would be an attempt to give a description of an entire Regge trajectory, and therefore its consistency is dependent on the presence of strong interactions. After all a Regge (Hadronic) field is considered a bound state very much like a Hydrogen atom whose excitations are dependent on electromagnetic interactions. For the present wave equation (which carries multiple states of angular momentum) to meet its ultimate goal in describing such excitations, one has to assume as a priori that any interactions of interest must require that Hadrons appear as asymptotic states.

2 A First Construction of the Equation

2.1 Preliminary considerations

In this section we will attempt to construct the infinite component wave equation more or less in a ‘naive’ approach which will initially render the field in question un-physical since it will fail to satisfy a causal condition. This initial effort however will not be in vain since it does point the way on how to overcome such a problem by utilizing the same formalism with some modifications.

The current (2) suggests that an infinite component wave equation describing a field possessing multiple spin states could be a first order linear differential equation similar to that of the Dirac equation. In fact the main motivation for such a construction is derived from Dirac’s inspiration for searching relativistic first order linear equations to describe interaction of particles which possess spin. We therefore conjecture on the form of this equation namely:

$$ (i\partial^{\mu}_x X_{\mu} - M_{\alpha}) \psi(x_{\alpha}) = 0. \quad (3) $$

The terms $X_{\mu}$ whose construction will be given below, are infinite dimensional matrices describing transitions between different spin states, and are analogous to the gamma matrices appearing in the Dirac equation. The index $\mu$ is a Lorentz index, while $\alpha$ is an infinite (spin) frame index for which we reserve the discussion on its meaning for later.\(^2\) The term $M_{\alpha}$ is a real diagonal matrix whose entrees correspond to the masses of each (spin) frame.

Because we insist that $(M_{\alpha})_{jj}$ correspond to the masses of the Regge field as it acquires different spin states we are actually saying that the mass shell condition $p^2\alpha_j = m^2_{\alpha_j}$ is satisfied for each spin component of the field $\psi$. In fact one of the major driving conditions for building the representations of $X_{\mu}$

\(^2\)The first letters in the Greek alphabet will be reserved for frame indices.
will be this mass shell condition, however it should be noted that for the wave equation (3) to be Lorentz invariant such a condition is not necessary.

A famous case in point is the infinite component wave equation constructed by Majorana [8] from which two important features are worth while examining. First, in his treatment the matrix $M$ is considered to be a constant (and thus does not carry any spin frame index), which leads to a particle mass spectrum of the form $m = \frac{M}{\sqrt{j}}$. This does not correspond to the observed spectrum of Hadrons, never the less it is a unique and a remarkable result. This result also implies that in the rest frame the operator $p^0$ does not correspond to $m$, and therefore it follows that Majorana does not enforce the relation $p^2 - m^2 = 0$. The second feature of this equation is that the matrix $X^0$ is positive definite, meaning that in the Fourier transform of the field $\psi$, negative frequencies are excluded, implying that quantization of the field $\psi$ would be a bit awkward since this field would not correspond to neither Fermi nor Bose Einstein statistics. Such features will be avoided from the following treatment by first introducing an infinite spin frame. This means that for each spin state $p^0_\alpha$ is different, and if the latter is to be interpreted as the mass of the particle in the rest frame then it also must follow that $p^2_\alpha = m^2_\alpha$. This means that each spin state has a different mass forcing $M_\alpha$ not to be a constant.

The physical motivation for introducing frames was explained by Ne’eman [2] when considering relativistic excitations of fields which carry multiple angular momentum states. The analogy is made with that of Einstein’s gravity in a local frame on a 4-manifold with a defined local metric using tetrads. In this formalism any general transformation under the covariant group (diffeomorphisms) can describe a transformation from one local frame to another in a gravitational field. In our case, excitations of Hadronic fields really correspond to structural deformations of these ‘extended’ objects. Since the problem of describing relativistic extended objects (in four dimensions) is yet to be formulated, spin is one of the local properties that may be used to describe such excitations. Similar to the situation in gravity, one can describe the transformation of a four manyfold of an extended object from one geometrical state (frame) to another by describing the spin states associated with each geometrical state with the use of some symmetry group. Since an Hadronic field in theory can be excited to an infinite amount of geometrical states (frames) through structural deformations, where each geometrical state corresponds to a particular spin state (frame); hence the term infinite spin frames.

It should be stressed that an infinite (spin) has no relation and should not to be confused with a Lorentz frame. The indices $\mu$, and $\alpha$ represent two distinct coordinates; one representing a space time coordinate, the other a spin coordinate. The relation between $\alpha$ the spin frame, and $j$ the spin of the particle will be given in section (4).

\[I would like to thank E.C.G Sudarshan for clarifying to me this point.\]
2.2 The representations of $X^\mu$

Since the matrices $X^\mu$ should comprise a four vector (a statement which has yet to be proven) it should follow that:

$$[S^\mu{}^\nu, X^\lambda] = i\eta^\mu{}^\lambda X^\nu - i\eta^\nu{}^\lambda X^\mu$$  \hspace{1cm} (4)

In light of this the following spherical basis can be defined:

$$X^+ = X^1 + iX^2$$
$$X^- = X^1 - iX^2$$
$$X^3 = X^3$$
$$X^0 = X^0.$$  \hspace{1cm} (5)

This together with (4), the general representations of $X^\mu$ can be written by the use of the Wigner Eckart theorem. One particular representation which will be convenient for the development of the subject has been given by Weyl [13] for the spherical basis (5), and is given by:

$$\langle \alpha, j, m | X^+ | \alpha, j - 1, m - 1 \rangle = -c^- (\alpha, j) \sqrt{(j + m)(j + m - 1)}$$
$$\langle \alpha, j, m - 1 | X^- | \alpha, j - 1, m \rangle = c^- (\alpha, j) \sqrt{(j - m + 1)(j - m)}$$
$$\langle \alpha, j, m | X^3 | \alpha, j - 1, m \rangle = c^- (\alpha, j) \sqrt{(j + m)(j - m)}$$
$$\langle \alpha, j, m | X^+ | \alpha, j, m - 1 \rangle = c^- (\alpha, j) \sqrt{(j + m + 1)(j - m + 1)}$$
$$\langle \alpha, j, m | X^- | \alpha, j, m - 1 \rangle = c^- (\alpha, j) \sqrt{(j - m + 2)(j - m + 1)}$$
$$\langle \alpha, j, m | X^3 | \alpha, j, m - 1 \rangle = c^- (\alpha, j) \sqrt{(j + m + 1)(j - m)}$$
$$\langle \alpha, j, m | X^0 | \alpha, j, m \rangle = c^0 (\alpha, j).$$  \hspace{1cm} (6)

The term general representations with regards to (5) is used since the coefficients $c^+ (\alpha, j)$, $c^- (\alpha, j)$, $c (\alpha, j)$, and $c^0 (\alpha, j)$ have yet to be determined. As will be shown these representations are proportional to the infinite dimensional Lorentz boosts [7], just as in the case of the Dirac equation where the gamma matrices are a result of boosts acting on a field.

The coefficient $c (\alpha, j)$ furnishes a finite representation of (3) and therefore is of no use to us in the current development. To obtain the coefficients $c^+ (\alpha, j)$, $c^- (\alpha, j)$, and $c^0 (\alpha, j)$ we enforce the following condition:

$$\left( \partial^2 + m^2 \right) \psi = 0.$$  \hspace{1cm} (7)

\(^4\text{It will be shown below that in fact an inclusion of such terms is inconsistent with certain conditions the wave equation must fulfill.}\)
To arrive at such a condition one can use Dirac’s trick by operating on (3) from the left with its conjugate for a particular frame and spin to give:

\[
\left[ (-i \partial_\nu^\mu X_\nu - M_\alpha) (i \partial_\mu^\nu X_\mu - M_\alpha) \right]_{j,j} \psi(x_\alpha) = \\
\left[ \frac{1}{2} \partial_\nu^\mu \partial_\mu^\nu \{ X_\nu X_\mu \} + m_\alpha^2 \right]_{j,j} \psi(x_\alpha) = 0,
\]

and it appears that in order to arrive at a mass shell condition the \(X^\mu\)'s have to fulfill the familiar statement for each spin frame namely,

\[
\langle \alpha, j, m \| \{ X_\nu X_\mu \} \| \alpha', j, m \rangle = 2 \eta_{\nu\mu} \delta_{\alpha\alpha'}.
\]  \hspace{1cm} (9)

It should be emphasized that unlike the gamma matrices appearing in the Dirac equation, the statement \(\{ X_\nu X_\mu \} = 2 \eta_{\nu\mu}\) is not true in general, but should only apply when this expression is taken between equal values of spin.

Keeping this in mind we note that when \(\mu \neq \nu\), it is easy to see that the condition (9) is satisfied. First, for the case \(\mu = 0, \nu = i\), or vice versa the term

\[
\langle \alpha, j, m \| X^0 X^i \| \alpha, j, m \rangle = 0.
\]

This is because according to (4), \(X^0\) is spherical tensor of rank zero, while \(X^i\) is a spherical tensor of rank one. This term therefore can only have expectation values between \(j\) and \(j \pm 1\). Second, for the case where both \(\mu\) and \(\nu\) are space indices one can decompose their bilinear as follows:

\[
X^i X^j = \frac{1}{3} \delta^{ij} D_3 + \frac{1}{2} [X^i X^j] + \frac{1}{2} \left( \left( X^i X^j \right) - \frac{2}{3} D_3 \right),
\]  \hspace{1cm} (10)

where \(D_3 = X^i X_i\).

The nine bilinears according to the decomposition of (10) have split into one spherical scalar, three spherical vectors, and five spherical tensors of rank two containing the symmetric combination in (9). When \(i \neq j\) the latter symmetric combination can only have expectation values between \(j\) and \(j \pm 2\). Second, for the case where both \(\mu\) and \(\nu\) are space indices one can decompose their bilinear as follows:

\[
X^i X^j = \frac{1}{3} \delta^{ij} D_3 + \frac{1}{2} [X^i X^j] + \frac{1}{2} \left( \left( X^i X^j \right) - \frac{2}{3} D_3 \right),
\]  \hspace{1cm} (10)

where \(D_3 = X^i X_i\).

For the case \(\mu = \nu = 0\) one gets from (9) that \(c^0(\alpha, j) = \pm 1\). For the case \(\mu = \nu = i\) the situation is more problematic for which the representations of \(X^\mu\) will have to be modified. To illustrate how naive our approach has been so far in dealing with this construction, and the problem that arises it is sufficient to evaluate the term \(X^i X_i\). Using the representations (6), the following is obtained

\[
-D_3 = -(j(j + 1) + m) c^2(\alpha, j) - c^-(\alpha, j) c^+(\alpha, j - 1)(j + m)(2j - 1) + c^+(\alpha, j) c^-(\alpha, j + 1)(j - m + 1)(2j + 3) - m
\]  \hspace{1cm} (11)

The terms proportional to \(c(\alpha, j)\) have been included in this step to justify the earlier assumption made on their irrelevance.

\[\text{5}\] There are more conditions related to Lorentz invariance (to be discussed) that the mass matrix must fulfill in order to be called a mass matrix, though (7) will suffice to obtain the coefficients \(c^+(\alpha, j), c^-(\alpha, j), \text{ and } c^0(\alpha, j)\).

\[\text{6}\] The terms proportional to \(c(\alpha, j)\) have been included in this step to justify the earlier assumption made on their irrelevance.
This expression should not depend neither on \( j \), nor on \( m \), in fact the expression above should equal \(-X^i X_i = 3\). Further, \( c^0(\alpha, j), c^+(\alpha, j), c^-(\alpha, j)\), and \( c(\alpha, j)\) are all independent of \( m \), therefore from (11) the following two conditions are obtained:

\[
1 + c^2(\alpha, j) = c^+(\alpha, j)c^-(\alpha, j + 1)(2j + 3) - c^-(\alpha, j)c^+(\alpha, j - 1)(2j - 1),
\]

(12)

and

\[
-D_3 = -j(j + 1)c^2(\alpha, j) - c^+(\alpha, j)c^-(\alpha, j + 1)(j + 1)(2j + 3) - c^-(\alpha, j)c^+(\alpha, j - 1)j(2j - 1).
\]

(13)

These relations enable to eliminate the term \( c(\alpha, j) \) completely, which does not have any bearing on the representation of \( X^i \), so it might as well be set equal to zero. Hence the only possible way to satisfy the condition \( D_3 = -3 \) would be to set

\[
c^+(\alpha, j)c^-(\alpha, j + 1) = -c^-(\alpha, j)c^+(\alpha, j - 1) = \frac{1}{2j + 1}
\]

(14)

This implies however that:

\[
c^+(\alpha, j + 1)c^-(\alpha, j + 2) = c^-(\alpha, j)c^+(\alpha, j - 1)
\]

(15)

which says that \( c^+(\alpha, j) \), and \( c^-(\alpha, j) \) are both independent of \( j \) contradicting (14). Although an impossible result, statements (14, 15) do suggest that the form \( c^+(\alpha, j) \sim c^-(\alpha, j - 1) \sim \frac{1}{\sqrt{2j + 1}} \) up to some phase is the right prescription if there was another set of coefficients requiring more freedom with respect to how one writes the representations for \( X^i \). Thus there should exist \( c_1^+(\alpha, j_1, j_2) \), \( c_2^+(\alpha, j_1, j_2) \), \( c_1^- (\alpha, j_1, j_2) \), \( c_2^- (\alpha, j_1, j_2) \), for each \( X^i \), and \( c_1^0(\alpha, j_1), c_2^0(\alpha, j_2) \) for each \( X^0 \) with \( (|j_1 - j_2| = 1) \). Thus expression (14) would read:

\[
\begin{align*}
c_1^+(\alpha, j_1)c_2^-(\alpha, j_2) &= -c_1^- (\alpha, j_1)c_2^+ (\alpha, j_2) = \frac{\eta}{2j_1 + 1} \\
c_1^+(\alpha, j_2)c_2^- (\alpha, j_1) &= -c_1^- (\alpha, j_2)c_2^+ (\alpha, j_1) = \frac{\eta}{2j_2 + 1},
\end{align*}
\]

(16)

where \( \eta \) is some phase, and the contradiction of the form of (15) is avoided. Although these additions are motivated by the above arguments they have not been fully justified, and it seems a bit blurry at this point how such conditions may arise from the representations given by (6). To put these on a more firm footing we proceed at this stage to the next section to better formulate the representations of \( X^\mu \).
3 A Second Construction of the Equation

3.1 Defining the odd and even representations

As indicated in the last section it seems that there should be two sets of matrices $X^\mu$ in order to fulfill (9). It will be shown that having two of these matrices corresponds to a kind of chirality that exist in the Dirac case, where the gamma matrices in the chiral representation mix left and right components of the spinor field. The correspondence between the current wave equation and the Dirac equation is that the two sets of $X^\mu$ will act as the sigma matrices appearing in the gamma matrices of the chiral representation, performing the mixing mentioned above.\(^7\)

We proceed with this construction by first defining the projection operator:

$$\langle \alpha, j, m' | P_n | \alpha', j', m' \rangle = \delta_{\alpha\alpha'} \delta_{j,j'} \delta_{m,m'}, \quad \text{(17)}$$

with $n = j - j_{\text{min}}$, and where $j_{\text{min}}$ is the lowest value of $j$ for a particular representation. This matrix operator has $(2j + 1) \times (2j + 1)$ dimensions with $m = j, j - 1, ... - j$ which forms a subspace of the infinite dimensional identity, hence

$$I = \sum_{n=0}^{\infty} P_n.$$

Using (3.1) it is possible to define the odd, and even projection operators respectively $P_{2n+1}$, and $P_{2n}$ which have the property of splitting the Hilbert space $|jm\rangle$ into two separate spaces which we will call ‘odd’ and ‘even’ respectively. These are given by:

$$|j_o, m\rangle = P_o |j, m\rangle = \sum_n P_{2n+1} |j, m\rangle$$

$$|j_e, m\rangle = P_e |j, m\rangle = \sum_n P_{2n} |j, m\rangle. \quad \text{(18)}$$

where $|j_o - j_e| = 1$.

Thus for example for a space with half integral spins

$$j_e = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ..., \quad j_o = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, ...$$

(19)

Now define the following operators:

\(^7\)There may well be other infinite representations not necessarily of this type that may fulfill conditions that have been mentioned (in particular (9)), though I am not aware of any.
\[ L^i = \sum_{n} P_{2n+1}X^iP_{2n} + P_{2n}X^iP_{2n+1} \] (20)

\[ N^i = \sum_{n} P_{2n+1}X^iP_{2n} + P_{2n}X^iP_{2n-1} \] (21)

\[ L^0 = \sum_{n} P_{2n+1}X^0P_{2n+1} \] (22)

\[ N^0 = \sum_{n} P_{2n}X^0P_{2n}, \] (23)

where it is understood that for the lowest term in (21) \( P_{-1} = 0 \).

From this splitting (18) it is apparent that with respect to rotations the operators \( L^i, N^i, \) and \( L^0, N^0 \) are spherical tensors of rank one and zero respectively, and by the methods of (6) their representations are given by:

\[ \begin{align*}
\langle \alpha, j_o, m \mid L^+_o \mid \alpha, j_e, m - 1 \rangle &= -l^-(\alpha, j_o)\sqrt{(j_o + m)(j_o + m - 1)} \\
\langle \alpha, j_o, m - 1 \mid L^-_o \mid \alpha, j_e, m \rangle &= l^-(\alpha, j_o)\sqrt{(j_o - m)(j_o - m - 1)} \\
\langle \alpha, j_o, m \mid L^3_o \mid \alpha, j_e, m \rangle &= l^-(\alpha, j_o)\sqrt{(j_o + m)(j_o - m)} \\
\langle \alpha, j_e, m - 1 \mid L^-_o \mid \alpha, j_o, m \rangle &= -l^+(\alpha, j_e)\sqrt{(j_e - m - 1)(j_e - m)} \\
\langle \alpha, j_e, m \mid L^3_o \mid \alpha, j_o, m \rangle &= l^+(\alpha, j_e)\sqrt{(j_e + m)(j_e - m)} \\
\langle \alpha, j_e, m \mid L^0_o \mid \alpha, j_o, m \rangle &= l^0(\alpha, j_e)
\end{align*} \] (24)

and

\[ \begin{align*}
\langle \alpha, j_e, m \mid N^+_o \mid \alpha, j_o, m - 1 \rangle &= -n^-(\alpha, j_e)\sqrt{(j_e + m)(j_e + m - 1)} \\
\langle \alpha, j_e, m - 1 \mid N^-_o \mid \alpha, j_o, m \rangle &= n^-(\alpha, j_e)\sqrt{(j_e - m)(j_e - m - 1)} \\
\langle \alpha, j_e, m \mid N^3_o \mid \alpha, j_o, m \rangle &= n^-(\alpha, j_e)\sqrt{(j_e + m)(j_e - m)} \\
\langle \alpha, j_o, m \mid N^+_o \mid \alpha, j_e, m - 1 \rangle &= n^+(\alpha, j_o)\sqrt{(j_o - m - 1)(j_o - m)} \\
\langle \alpha, j_o, m - 1 \mid N^-_o \mid \alpha, j_e, m \rangle &= -n^+(\alpha, j_o)\sqrt{(j_o + m)(j_o + m + 1)} \\
\langle \alpha, j_o, m \mid N^3_o \mid \alpha, j_e, m \rangle &= n^+(\alpha, j_o)\sqrt{(j_o + m)(j_o - m)} \\
\langle \alpha, j_o, m \mid N^0_o \mid \alpha, j_e, m \rangle &= n^0(\alpha, j_o)
\end{align*} \] (25)

The terms \( l(\alpha, j), n(\alpha, j) = 0 \) by the same arguments of the previous section.

It follows from the representations (24), and (25) that the bilinears of the operators \( L^\mu, N^\mu \) have expectation values only between the following states:

\[ \langle j, m \mid L^\mu L^\nu \mid j' = j, m' = m, m \pm 2 \rangle, \quad \langle j \mid N^\mu N^\nu \mid j' = j, m' = m, m \pm 2 \rangle, \quad \langle j \mid L^1 N^3 \mid j' = j \pm 2, m' = m, m \pm 2 \rangle, \quad \langle j \mid N^1 L^1 \mid j' = j \pm 2, m' = m, m \pm 2 \rangle. \]
With these representations the matrices $X^\mu$ can be constructed by adding a two dimensional index to their labels denoted by $a, b = 1, 2$ to give $X^\mu_{ab}$, which are defined as follows:

\begin{align*}
X_{12}^i &= L^i + N^i \\
X_{21}^i &= L^i - N^i \\
X_{11}^i &= X_{22}^i = 0. \quad (26)
\end{align*}

and

\begin{align*}
X_{11}^0 &= X_{22}^0 = \frac{1}{2}(L^0 + N^0) \\
X_{12}^0 &= X_{21}^0 = 0. \quad (27)
\end{align*}

The above representations could also be written more compactly as:

\begin{align*}
\langle \alpha, j, m | X^\mu | \alpha', j', m' \rangle &= \langle \alpha, j, m \mid (X_{11}^0 X_{12}^1 X_{21}^i X_{22}^i) \mid \alpha', j', m' \rangle \quad (28)
\end{align*}

In defining the multiplication with regards to the indices $a, b$ we introduce a raising and lowering metric $\epsilon_{ab} = \epsilon^{ab}$, where $\epsilon_{11} = \epsilon_{22} = \epsilon_{12} = \epsilon_{21} = 0$, and whose representation is given by

\begin{align*}
\epsilon_{ab} &= \epsilon_{\alpha, \alpha' j, j', m, m'} \delta_{ab}, \quad (29)
\end{align*}

where

\begin{equation}
\epsilon_{\alpha, \alpha' j, j', m, m'} = \begin{cases} 
\delta_{\alpha, \alpha'} \delta_{j, j'} \delta_{m, m'} & j = j_0 \\
-\delta_{\alpha, \alpha'} \delta_{j, j'} \delta_{m, m'} & j = j_e
\end{cases} \quad (30)
\end{equation}

The metric $\epsilon_{ab}$ has the following properties:

\begin{align*}
\{ \epsilon_{ab}, X^i \} &= 0 \\
[\epsilon_{ab}, X^0] &= 0, \quad (31)
\end{align*}

and behaves like the $4 \times 4$ matrix $\gamma^0$ that appears in the Dirac theory.

Using the metric (30) the following bilinear can be defined:

\begin{align*}
Q_{\mu \nu}^{ab} &= (X^\mu X^\nu)_{ab} = X^\mu_{ac} \epsilon^{cde} X^\nu_{db}. \quad (32)
\end{align*}

The condition (9) can now be implemented, and due to (26) only terms proportional to $(L^i)^2$ and $(N^i)^2$ need to be considered when evaluating the following:

\begin{align*}
(D_4)_{ab} &= \langle \alpha, j, m \mid (X^\mu X^\mu)_{ab} \mid \alpha, j, m \rangle = \langle \alpha, j, m \mid X_1^\mu X_1^\mu \mid \alpha, j, m \rangle \\
&= \langle \alpha, j, m \mid X_2^\mu X_2^\mu \mid \alpha, j, m \rangle \quad (33)
\end{align*}
In order to determine the coefficients \( l^\pm(\alpha, j), n^\pm(\alpha, j), l^0(\alpha, j), \) and \( n^0(\alpha, j) \) it would be sufficient to evaluate \((X^1)^2\), (any other of the space components will do as well), and \((X^0)^2\). Using the representation (24), and (25) together with (74) it follows that

\[
\langle \alpha, j, m | (L^1 + N^1)(L^1 - N^1) | \alpha, j, m \rangle = \frac{1}{2} \left[ l^-(\alpha, j)l^+(\alpha, j - 1)(j^2 - j + m^2) - n^+(\alpha, j)n^-(\alpha, j + 1)(j^2 + 3j + m^2 + 2) \right].
\]  
(35)

Because \( l^\pm(\alpha, j), \) and \( n^\pm(\alpha, j), \) are independent of \( m \) one immediate consequence of (35) is

\[
n^+(\alpha, j)n^-(\alpha, j + 1) = l^-(\alpha, j)l^+(\alpha, j - 1). \tag{36}
\]

Now (35) reads

\[
\langle \alpha, j, m | (L^1 + N^1)(L^1 - N^1) | \alpha, j, m \rangle = -l^-(\alpha, j)l^+(\alpha, j - 1)(2j + 1). \tag{37}
\]

By choosing

\[
l^-(\alpha, j) = l^+(\alpha, j - 1) = \frac{\eta}{\sqrt{2j + 1}},
\]

\[
n^+(\alpha, j) = n^-(\alpha, j + 1) = \frac{\eta}{\sqrt{2j + 1}} \tag{38}
\]

with \( \eta \) being either a pure complex number, or a pure real number with a modulus of unity, the condition (9) can be satisfied for all the space components of \( X^\mu \) up to a sign. The choice \( \eta = i \) will make \( \langle j, m | (X^1)\rangle^2 | j, m \rangle = 1 \) (up to a sign), but more importantly would mean that the matrices \( X^\nu \) are anti-Hermitian which would be consistent if \( \psi \) in (3) is to be interpreted as a field.

For the time component \( X^0 \), the condition (9), and the condition that in the rest frame \( p_\alpha^0 = m^\alpha \) implies that

\[
n^0(\alpha, j) = l^0(\alpha, j) = 1 \tag{39}
\]

With this condition (9) reads as follows:

\[
\langle \alpha, j, m | \{ X_\nu X_\mu \} | \alpha, j, m \rangle = 2\Delta_{\mu\nu}, \tag{40}
\]

where

\[
\Delta_{\mu\nu} = \begin{cases} 
\eta_{\mu\nu} & j = j_0 \\
-\delta_{\mu\nu} & j = j_e
\end{cases}
\]  
(41)
This peculiarity has a physical consequence and in fact is necessary for any physical interpretation as will be discussed in the next section. Although the choice of sign in (39) is arbitrary, other conventions will not change the outcome (41), namely that the metric changes from a Euclidian metric to a Minkowskian metric for states that differ by $\Delta J = 1$ in (41).

### 3.2 Lorentz Generators

At this stage it would be useful to look at commutation relations of the type $[X^a, X^b]_{ab}$, where one contraction with respect to the indices $a, b$ is obtained, so as to project a $(1,1)$ component with respect to these indices. To get a feel what such terms correspond to it would be instructive to evaluate as an example the following commutator:

$$
[X^1, X^2]_{11} = X^1_{12} \epsilon^{22} X^2_{21} - X^2_{12} \epsilon^{22} X^1_{21}
$$

Implementing the definitions (26), and a bit of algebra we get:

$$
[X^1, X^2]_{11} = -i \epsilon^{22} \left( N^+ N^- - N^- N^+ L^- L^- - L^+ L^- + N^+ L^- + N^- L^+ \right)
$$

$$
= -i \epsilon^{22} \left( [N^+, N^-] + [L^-, L^+] + \{L^+, N^-\} - \{L^-, N^+\} \right).
$$

Using the representations (25), (24), and the properties (26) the four expressions in (43) can be evaluated for their non-vanishing matrix elements which are given by (we drop the labels ‘odd’, and ‘even’ from $j$):

$$
\langle j, m | [N^+, N^-] | j, m \rangle = 2m \left( \frac{2j + 3}{2j + 1} \right)
$$

$$
\langle j, m | [L^-, L^+] | j, m \rangle = -2m \left( \frac{-2j + 1}{2j + 1} \right)
$$

$$
\langle j, m | L^+, N^- | j - 2, m \rangle = \langle j, m | L^- N^+ | j - 2, m \rangle = \sqrt{\frac{(j + m)(j + m - 1)(j - m)(j - m - 1)}{(2j + 1)(2j - 3)}}
$$

$$
\langle j, m | N^+ L^- | j + 2, m \rangle = \langle j, m | N^- L^+ | j + 2, m \rangle = \sqrt{\frac{(j + m + 1)(j + m + 2)(j - m + 1)(j - m + 2)}{(2j + 1)(2j + 3)}}.
$$

(44)
The first two terms in (43) add, while the last two exactly cancel, and we are left with

\[ \frac{i}{2} [X^1, X^2]_{11} = \frac{i}{2} [X^1, X^2]_{22} = \langle j, m | J_3 | j, m \rangle = m, \]  

(45)

which is nothing more than the representation of the 3-generator of angular momentum. Upon evaluating

\[ \frac{i}{2} \left( [X^2, X^3]_{ab} + i [X^1, X^3]_{ab} \right) = \langle j, m | J^+ | j, m - 1 \rangle = \sqrt{(j + m)(j - m + 1)} \]

\[ \frac{i}{2} \left( [X^2, X^3]_{ab} - i [X^1, X^3]_{ab} \right) = \langle j, m | J^- | j, m + 1 \rangle = \sqrt{(j + m + 1)(j - m)} \]  

(46)

it is evident that the terms \( \frac{i}{2} [X^i, X^j]_{ab} \) furnish the finite spinorial representations of the group of rotation \( SU(2) \).

By the same method, and upon the use of representation (24, 25) the following operators are evaluated

\[ K^i = S^{00}_{ab} = \frac{i}{2} [X^i, X^0]_{ab}. \]  

(47)

Evaluating the commutation relations

\[ [K^i, K^j] = -i \epsilon_{ijk} J^k \]

(48)

it follows that the operators \( K^i \) furnish the representations of the boost generators of the Lorentz group. It should be noted that the representation of \( \epsilon_{ab} \) (29) establishes the consistency of these commutation relations, and provides the motivation for its definition.

4 Physical States

4.1 ‘Even’ and ‘odd’ trajectories and spin frames

In this section we would like to investigate the physical consequences of equation (3) in light of the representations and properties of \( X^\mu \) discussed in the previous sections.

Equation (3) can be written in component form which manifestly resembles a chiral component of the Dirac equation in its chiral representation, (while providing an entirely different physical description) and is given in the form:

\[ \begin{pmatrix} \partial_0^0 X^0 - M_{aa} \alpha \vspace{1mm} & -\nabla_\alpha \cdot X_{ab} \\ -\nabla_\alpha \cdot X_{ba} & \partial_0^0 X^0 - M_{bb} \alpha \end{pmatrix}_{jj'} \begin{pmatrix} \psi^a(x_\alpha) \\ \psi^b(x_\alpha) \end{pmatrix}_{j'} = 0. \]  

(49)
where it is understood that $a, b = 1, 2$, and that upper and lower indices are connected via the metric (29), with the term $M_{ab} = \delta_{ab} \delta_{j', j} m_\alpha$.

As stated in section (2) the mass shell condition should be satisfied for each frame, meaning each state of a definite spin should satisfy $p^2_\alpha = m^2_\alpha$. This is done by effecting the operation (8) which gives the second order equation

$$
\left( \partial^2_\alpha I_{aa} + \epsilon_{aa} M^2_\alpha \right) \left( \partial^2_\alpha - \nabla^2_\alpha I_{aa} + \epsilon_{aa} M^2_\alpha \right) \psi_j = 0,
$$

(50)

where $I_{aa}$ is just the identity matrix.

We have attempted to achieve a Klein-Gordon equation however the result in (50) is not quite such an equation if the term $M_\alpha$ is to be interpreted as the mass of the particle in a particular spin state. The source of what seems to be a problem is the alternating signs of the terms $(\partial^2_\alpha I_{aa} + \epsilon_{aa} M^2_\alpha)$ as one goes from $j_o$ to $j_e$. By the definition of $\epsilon_{ab}$ (29) it is obvious that for $j = j_o$ one has the equation

$$
(\partial^2_\alpha + M^2_\alpha) \psi_{j_o} = 0,
$$

(51)

while for $j = j_e$ Eq. (50) reads

$$
(\partial^2_\alpha + M^2_\alpha) \psi_{j_e} = 0.
$$

(52)

So it is evident that states that carry $j = j_o$ do satisfy the Klein-Gordon equation while those that carry $j = j_e$ do not. It should be stressed that had a different convention been chosen for the metric, $j_e$ would satisfy (51), while $j_o$ would satisfy (52). As an example, in the former case $\psi_{j_o}$ would be a trajectory of states carrying half integral spin $\frac{3}{2}, \frac{5}{2}, \frac{7}{2} ....$ and does satisfy the Klein-Gordon equation. On the other hand $\psi_{j_e}$ is a trajectory of states carrying the half integral spins $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} ....$, and which does not satisfy the Klein-Gordon equation. The natural question to ask now is why aren’t these two trajectories on the same physical footing?

To resolve this quandary one has to look more carefully at the Lorentz transformations furnished by the representations of $X^\mu$, particularly the boosts. Under an infinitesimal Lorentz boost $\psi$ transforms as

$$
\psi'(x^\alpha) = 1 + i S^0_{ab} \phi_{i0} \psi(x^\alpha)
$$

$$
= 1 - \frac{1}{2} [X^i, X^0]_{ab} \phi_{i0} \psi(x^\alpha),
$$

(53)

and it is evident from the representations (24, 25) that the above transformation is non-unitary since the Lorentz boost are anti-Hermitian. Furthermore,
according to these representations a Lorentz boost takes the field from a state \( j \) to a state \( j \pm 1 \). From a physical stand point there is absolutely no reason why one observer may measure one value of spin in one frame, and a second observer may measure a different value in another (boosted) frame. The boost being represented non-unitarily is exactly the right prescription since these correspond to non-physical transitions. What seems as a conflict in (52) is simply a statement that states that correspond to \( j = j_e \) (keeping in mind our initial convention of \( \epsilon_{ab} \)) are not physical states, and therefore do not have a rest frame description nor a mass. As a result equation (49) applies to fields whose states differ in two units of angular momentum.

The concept of infinite spin frames which has been used repeatedly so far becomes more clear. It is phenomenologically well known that a Hadron can be excited to different spin states when interacting with an external field, due to its structure (comprised of gluons and quarks). In calculations of physical processes such as amplitudes and cross sections, one should take into account all possible states that the particle can be found in, which provides the main motivation for the wave equation at hand. A Regge trajectory can be defined as a sum on an infinite ensemble of free states, or frames (in \( J/m^2 \) space). In other words if a particle is free, it is said to be in a specific spin frame as far as its spin and rest mass are concerned. Applying a boost to a particle similar to the one in (53) alters its Lorentzian (kinetic) frame, however it does not alter its spin frame. The fact that the boost in (53) may change the field’s angular momentum component by \( \Delta J = \pm 1 \) has no physical consequence what so ever since the transition to such a state is done non-unitarily, and therefore it not observed. Thus a particular spin frame which is described by a Regge field will span angular momentum values \( J-1 < J < J+1 \). If a Regge field is measured to be in a physical state with some spin \( J \) then by the argument presented above its \( J-1, J+1 \) components cannot correspond to any physical observables, and therefore don’t have a rest frame description in the equation (3).

The field describing an entire Regge trajectory as a sum of spin frames thus may be written as:

\[
\Psi^a(x) = \sum_{\alpha=0}^{\infty} \sum_j \psi_j^a(x_\alpha) \tag{54}
\]

where \( j \) is either \( j_o \), or \( j_e \), and the sum on their values is given by:

\[
2\alpha + (j_o)_{\text{min}} - 1 \leq j_o \leq 2\alpha + (j_o)_{\text{min}} + 1, \tag{55}
\]

for and odd trajectory, while for an even trajectory

\[
2\alpha - (j_e)_{\text{min}} \leq j_o \leq 2\alpha - (j_e)_{\text{min}} + 2. \tag{56}
\]

In the latter case when \( \alpha = 0 \) the left hand side of (56) should be taken as zero.
4.2 Left and right boosts and free particle solutions

We now proceed to obtain free particle solutions to the wave equation (3) by boosting a rest frame spinor. Before performing such a boost it would be worthwhile to further investigate the properties of the Lorentz generators given by (81). One of these is given by:

$$S_{12}^{0} = \frac{i}{2}(X_{12}^{2}X_{22} - X_{11}^{0}X_{12}).$$  \hspace{1cm} (57)

From the properties of the metric $\epsilon_{ab}$ given by (31) it is evident that (57) furnishes two representations of a boost which could be obtained by anti-commuting the metric $\epsilon_{ab}$ either to the left, or to the right; thus yielding two representations that differ by an overall minus sign (the same goes for $S_{01}^{0}$). Due to this ambiguity it is appropriate to define left and right boosts given by:

$$\begin{align*}
(S_{ab}^{0})_{LL} &= \epsilon^{cc'}X_{ac}^{i}X_{c'b}^{0} \\
(S_{ab}^{0})_{RR} &= X_{ac}^{i}X_{c'b}^{0}\epsilon^{cc'}.
\end{align*}$$  \hspace{1cm} (58)

The representations (58) suggests that the wave equation (3) supports left and right fields which transform differently under Lorentz boosts according to

$$\begin{align*}
\psi(x^{\alpha})_{aR} &= 1 + i(S_{ab}^{0})_{RR}^{\alpha}_{\alpha'}\xi \\
\psi(x^{\alpha})_{aL} &= 1 + i(S_{ab}^{0})_{LL}^{\alpha}_{\alpha'}\xi,
\end{align*}$$  \hspace{1cm} (59)

where $\xi$ is a rest frame spinor with $(2j + 1) \times 1$ dimensions. Splitting the above two components we find the following:

$$\begin{align*}
(\psi_1)_{R} &= 1 - \frac{1}{2}X_{12}^{i}\phi_{i}\xi \\
(\psi_2)_{R} &= 1 - \frac{1}{2}X_{21}^{i}\phi_{i}\xi,
\end{align*}$$  \hspace{1cm} (60)

and

$$\begin{align*}
(\psi_2)_{L} &= 1 + \frac{1}{2}X_{12}^{i}\phi_{i}\xi \\
(\psi_2)_{L} &= 1 + \frac{1}{2}X_{21}^{i}\phi_{i}\xi,
\end{align*}$$  \hspace{1cm} (61)

Because of the left/right splitting the wave equation (3) doubles in dimension with respect to indices $ab$ which is analogous to the Dirac case in the chiral representation. The matrices $X^{\mu}$ now take on the form\textsuperscript{8}:

$$\begin{align*}
X_{abLR}^{i} &= -X_{abLR}^{i} \\
X_{aRL}^{0} &= X_{aRL}^{0},
\end{align*}$$  \hspace{1cm} (62)

or in a left/right tensor form

$$\begin{align*}
X^{i} &= \begin{pmatrix} 0 & X_{RL}^{i} \\ -X_{LR}^{i} & 0 \end{pmatrix} \\
X^{0} &= \begin{pmatrix} 0 & X_{RL}^{0} \\ X_{LR}^{0} & 0 \end{pmatrix}
\end{align*}$$  \hspace{1cm} (63)

\textsuperscript{8}With this definition $X_{21}^{i}$ is different from its initial definition by a minus sign.
with
\[ X^i_{RL} = \begin{pmatrix} 0 & X^i_{12} \\ X^{i*}_{21} & 0 \end{pmatrix} \]
\[ X^0_{RL} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \] (64)

The representation (63) assume that multiplication of any two \( X^\mu_{ab} \) is done between left and right components which induces a left/right metric \( g_{LR} \). Thus the four dimensional space \( abLR \) is very similar in nature to that of the four dimensional space that exist in Dirac case of the chiral representation.

Equation (3) in the rest frame with both left and right components takes a finite dimensional form and can be written as the following (in momentum space):
\[
\begin{pmatrix} -M_\alpha & p^0_\alpha \\ p^0_\alpha & -M_\alpha \end{pmatrix}_{jj} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}_j = 0,
\] (65)

where \( \psi \) has \( 2j + 1 \) components, and the matrix above \( 2(2j + 1) \times 2(2j + 1) \) dimensions (for a spin \( \frac{1}{2} \) the above is identical to the rest frame Dirac equation). The vanishing determinant of this matrix produces \( 2(2j + 1) \) eigenvalues; \( 2j + 1 \) with a value of \( -M_\alpha \), and \( 2j + 1 \) with a value of \( M_\alpha \), implying that the equation (3) admits negative particle solutions.

To obtain a solution to the free particle wave equation in a moving frame we can apply a Lorentz boosts to the rest frame spinor, keeping in mind that such boosts must not take the field out of its spin frame. Effectively what needs to be done is to decompose the full Lorentz transformation with respect to each spin frame, and to project those spherical tensors which don’t take the particle out of its spin frame. To understand this better we write the Lorentz transformations as the following:
\[
U_{\text{Lorentz}} = e^{iS^{\mu\nu}\omega_{\mu\nu}} = 1 + iS^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}(iS^{\mu\nu}\omega_{\mu\nu})^2 + \frac{1}{6}(iS^{\mu\nu}\omega_{\mu\nu})^3 + \ldots
\] (66)

With respect to the different powers of \( S^{\mu\nu} \) it is evident that each power can be decomposed into its irreducible spherical tensors. For example the first term in (66) is a scalar, the second is a vector, the third is comprised of rank two tensor and a scalar, the fourth is comprised of a vector and a rank three tensor, and so on. For each term in the series above that contains a spherical tensor of rank \( n \), a field with a spin component of \( J \) will be taken by the action of each of these terms to a state of \( J \pm n \), each being independent from the other. If the wave equation (3) is to describes a free particle in a specific state, and this equation at most connects field components that differ in angular momentum by \( \Delta J = 0, \pm 1 \), with a physical spin frame spanning components of \( J - 1 < J < J + 1 \), it follows that the only relevant tensors (as far as a physical boost is concerned) are those that could carry the field components from \( J \)
to \( J \pm 0.1 \), namely those tensors which are scalars or vectors. If higher order terms are to be considered for each frame, for example tensors of rank two from the third term in (66), then the free wave equation would describe transitions between physical frames. Physically this wouldn’t be consistent since a wave equation of a non-interacting particle under a Lorentz transformation must retain its description of the same particle of the same spin. In effect this means that one can choose a subset of solutions to the wave equation which correspond to the physical solutions while neglecting those that correspond to non-physical solutions. From this picture it stems that a particular Lorentz transformation can describe a particular boost for an infinite amount of spin frames, however each one will be disjoint from the other as required by physical constraints.

In order to extract from the boosts (66) the appropriate components fitting each spin, the Lorentz boosts generators (in our chiral representation (58)) can be written as the following:

\[
\mathbf{X}_{abRL} \rightarrow (P^\alpha \mathbf{X} P^\alpha)_{abRL},
\]

where \( P^\alpha \) is the projection operator which is defined in a similar way to that in (17), and \( \alpha \) determines the range of \( J \) given by (55). This does not alter the representation of the boosts whatsoever since the upper and lower values of the representations (24, 25) for any particular \( J \) are within the limits of (55).

Now apply this boost as in (66) to a rest frame spinor:

\[
\psi(p^\alpha)_{a j} = (e^{iS^{\mu\nu}\omega_{\mu\nu}})_{\text{physical}} \xi
\]

\[
= \sum_b \sum_j \left( 1 - (P^\alpha \mathbf{X} P^\alpha)_{ab jj'} \cdot \left( \frac{\phi}{2} \right) \right)^2 + \sum_b \sum_j \left( (P^\alpha \mathbf{X} P^\alpha)_{ab jj'} \cdot \left( \frac{\phi}{2} \right) \right)^2 ..... \xi_{j o}.
\]

(68)

The second term just gives the original boost back while the third term gives:

\[
\frac{1}{2} \sum_{jm} \sum_{j'} P^\alpha_{\omega jj''mm'} \phi_{j''} \phi_{j'''} = \delta_{ab} \delta_{jj'} \delta_{m'm''} \frac{1}{2} \left( \frac{\phi}{2} \right)^2
\]

(69)

where it is assumed that \( j' \) belongs to a physical state, meaning the one which actually propagates and for which \((\epsilon_{ab})_{jj'} m'm = 1\). Because of (69) the fourth term in the series gives:

\[
\frac{1}{3!} (P^\alpha \mathbf{X} P^\alpha)^3_{ab jj'} = \frac{1}{3!} \mathbf{X},
\]

(70)
and when summing on all terms in the series the right handed boosted field is
given by:

\[ \psi(p^\alpha)_{aR j} = \left( e^{iS^{\alpha\mu}\omega_{\mu}} \right)_{\text{physical}} \xi = (\cosh(\phi) - X_{RL} \cdot n \sinh(\phi)) \xi. \]  

(71)

Putting the expressions for the rapidity and the explicit expression for \( X_{RL} \) together with a summation on \( \alpha \), (71) can be written as:

\[
\begin{pmatrix}
\Psi_1(p) \\
\Psi_2(p)
\end{pmatrix}
_R = \sum_{\alpha} \frac{1}{\sqrt{2m^\alpha(E^\alpha + m^\alpha)}} \begin{pmatrix}
E^\alpha + m^\alpha & -p \cdot X_{12} \\
-p \cdot X_{21} & E^\alpha + m^\alpha
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix},
\]

(72)

while the left handed field is given by

\[
\begin{pmatrix}
\Psi_1(p) \\
\Psi_2(p)
\end{pmatrix}
_L = \sum_{\alpha} \frac{1}{\sqrt{2m^\alpha(E^\alpha + m^\alpha)}} \begin{pmatrix}
E^\alpha + m^\alpha & p \cdot X_{12} \\
p \cdot X_{21} & E^\alpha + m^\alpha
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix},
\]

(73)

The above representation establishes the fact that under a physical Lorentz transformation the following is true

\[
(U^{-1}_{\text{Lorentz}} X_{ab}^{\mu} U^{\mu}_{\text{Lorentz}})_{\text{physical}} = \Lambda_{\mu}^{\nu} X_{ab}^{\nu},
\]

(74)

where \( \Lambda_{\mu}^{\nu} \) are the familiar four dimensional Lorentz transformation. This establishes that the wave equation (3) is Lorentz invariant.

To complete the construction of the wave equation (3) we need to address the question of what values should the mass matrix \( M_{ab}^{\alpha} \) take for those entries which correspond to non-physical states. For example let’s assume that particles corresponding to the trajectory \( j_o \) are the physical states then according to (71, 72) it is clear that

\[
(M_{11}^{\alpha})_{j_o j_o} = (M_{22}^{\alpha})_{j_o j_o} = m_{j_o}^{\alpha}.
\]

(75)

Inserting (71, 72) into the wave equation (3) we also find

\[
(M_{11}^{\alpha})_{j_o \pm 1 j_o \pm 1} = (M_{22}^{\alpha})_{j_o \pm 1 j_o \pm 1} = m_{j_o}^{\alpha}.
\]

(76)

These relations are also consistent with (74), meaning the mass matrix in equation (3) transforms as a scalar under a Lorentz transformation.

5 The GL(3,1R) and GL(4R) Groups

5.1 GL(3,1)/GL(4R) splitting

In the introduction it was proposed that the non-locality of Hadrons may manifest itself in local parameters such as spin. If equation (3) describes a collection
of states acquired by a Regge field then it is reasonable to assume that this
equation could give rise to currents discussed in (2) when interactions are con-
cerned (for which no treatment will be given in the present work). Due to this
and the fact that the representation of the Lorentz generators are proportional
to the commutators of $X^\mu X^\nu$ suggest that these bi-linears may have further
properties worth exploring.

In section (2) the tensorial decomposition of the bi-linear $X^i X^j$ was per-
formed, and was given by: \(^9\):

$$X^i X^j = \frac{1}{3} \delta^{ij} D_3 + \frac{1}{2} [X^i X^j] + \frac{1}{2} \left( \{ X^i X^j \} - \frac{2}{3} D_3 \right), \quad (77)$$

where $D_3 = X^i X^i$.

The decomposition (77) splits the bilinear $iX^i X^j$ into three irreducible
spherical tensors. The first is a symmetric tensor $D_3 \eta^{\mu\nu}$ which has one in-
dependent term, the second an anti-symmetric traceless tensor $S^{ij}$ which has
three terms, and the last is a traceless symmetric tensor $T^{ij}$ which has five
terms. Given this, and the representations (24, 25), it follows that the first
term transforms as a spherical tensor of rank zero, the second term transforms
as a spherical tensor of rank one, and the third as a spherical tensor of rank
two, all under rotations, making nine terms all together.

Now define the following irreducible spherical tensors:

$$T_0^0 = D \quad J^k = \epsilon_{kij} S^{ij} \quad J^+ = J^1 + i J^2 \quad J^- = J^1 - i J^2$$

$$T_0^2 = \frac{1}{\sqrt{6}} (T_{11} + T_{22} - 2 T_{33})$$

$$T_{+1}^2 = T_{13} + iT_{23} \quad T_{-1}^2 = T_{13} - iT_{23}$$

$$T_{+2}^2 = T_{11} - T_{22} + 2iT_{22} \quad T_{-2}^2 = T_{11} - T_{22} - 2iT_{22} \quad (78)$$

\(^9\)Due to the left/right representations of the Lorentz transformation it will be convenient
in what follows to contract such bi-linears with the metric $\epsilon_{ab}$ from either the left or the
right. Also since $\epsilon_{ab}$ commutes with such terms the labels $a, b$ are suppressed, and it is
understood that all multiplications are done with this metric between left/right, and right/left
components.
By the Wigner Eckart theorem the following commutation relations follow:

\[
\begin{align*}
[J^3, T^0] &= 0 \\
[J^3, T^j_m] &= m T^j_m \\
[J^\pm, T^1_m] &= \sqrt{(1 \mp m)(1 \pm m + 1)} T^1_{m \pm 1} \\
[J^\pm, T^2_m] &= \sqrt{2(1 \mp m)(2 \pm m + 1)} T^2_{m \pm 1}.
\end{align*}
\]  

(79)

The last three of these could also be put in the form:

\[
[J^i, T^{jk}] = i \epsilon^{ijk} T^{lk} + i \epsilon^{ikl} T^{jl}.
\]

(80)

Since the angular momentum operator is given by (46), then (80) can also be written as:

\[
\begin{align*}
[S^{ij}, S^{kl}] &= -i \eta^{jk} S^{il} + i \eta^{il} S^{jk} \\
&\quad + i \eta^{ik} S^{lj} - i \eta^{lj} S^{ik} \\
[S^{ij}, T^{kl}] &= -i \eta^{jk} T^{il} - i \eta^{il} T^{jk} \\
&\quad + i \eta^{ik} T^{lj} - i \eta^{lj} T^{ik} \\
[S^{ij}, D_3] &= 0 \\
[T^{ij}, D_3] &= 0
\end{align*}
\]

(81)

These commutation relations are satisfied if the operators:

\[
Q^{ij} = \frac{i}{2} X^i X^j
\]

(82)

(note that this definition of \(Q^{ij}\) is different from that of (29) by a factor of \(1/2\)) satisfy the commutation relations:

\[
[Q^{ij}, Q^{kl}] = i \eta^{jk} Q^{il} - i \eta^{il} Q^{kj},
\]

(83)

which are the commutation relations of the group \(GL(3R)\), and which make \(Q^{ij}\) a \(GL(3R)\) group element. Omitting the operator \(D_3\), the irreducible spherical tensors (78) furnish the representation of the group \(SL(3R)\). This group has been shown by Ne’eman [9] to give the correct Regge excitations of Hadrons with respect to angular momentum, and its emergence should not come as a big surprise. Extended objects unlike point like particles posses properties that are indicative of their structure; one such example is the particle’s spin altered by some deformation of its structure. The operators \(T^{ij}\) are the shear tensors that describe such deformations, and lead to excitations of angular momentum with \(\Delta J = 2\). Indeed such excitations characterize a Regge trajectory [3], where
resonances of Hadrons are observed to have these selection rules.

We can attempt to do the same kind of analysis above while incorporating the operators $Q^0_j Q^0_0$, however from (24, 25) it is apparent that these are already irreducible. Furthermore there seems to be a peculiarity in the way these are defined with respect to the metric $\epsilon_{ab}$. Unlike the operators $Q_{ij}$, multiplication for left or right by $\epsilon_{ab}$ doesn’t affect the representations of these operators since $\epsilon_{ab}$ anti-commutes with $X^i$. On the other hand for the operators $Q^0_{i0}$ we have:

\[
(Q^{00})_{abLL} = \frac{i}{2} \epsilon_{cc'} X^{0}_{ac} X^{0c}_{b} = -(Q^{00})_{abRR} = \frac{i}{2} X^{0}_{ac} X^{0c}_{b} \epsilon_{cc'}
\]

For the case where multiplication is done from the left one finds the commutation relations:

\[
\left[ Q^{\mu\nu}, Q^{\lambda\sigma} \right] = i \delta^{\nu\lambda} Q^{\mu\sigma} - i \delta^{\mu\sigma} Q^{\lambda\nu},
\]

where $-\delta^{\mu\nu} = (-1, -1, -1, -1)$ is the four dimensional (negative) Euclidean metric, which would indicate that $(Q^{\mu\nu})$ is a $GL(4R)$ group element. However for the right hand multiplication we get the same commutation relation as in (85) but with $-\delta^{\mu\nu}$ replaced with $\eta^{\mu\nu} = (+1, -1, -1, -1)$ the Minkowskian metric, indicting that $Q^{\mu\nu}$ is a $GL(3, 1)$ group element. The source of this splitting can be understood by observing that

\[
\langle \alpha, j, m | \{ X^\mu X^\nu \} | \alpha, j m \rangle = \Delta^{\mu \nu}
\]

where $\Delta^{\mu \nu}$ is $\eta^{\mu \nu}$ if $J$ corresponds to an angular momentum of physical state, or $\delta^{\mu \nu}$ if $J$ corresponds to angular momentum of a non-physical state. Again as an example choosing $J = j_0$ to be a physical state described by the wave equation (3) would mean it’s free particle description is governed by the subgroup of $GL(3, 1)$, namely $SO(3, 1)$ with $\eta^{\mu \nu}$ being its invariant metric. Upon the action of a boost whose representations are made from bi-linears of $X^\mu$, a physical state is carried to a non-physical state in a non-unitary fashion and its ‘free’ description is now described by the subgroup of $GL(4R)$, namely $SO(4)$ with a negative metric; hence this state cannot correspond to any physical observable. Finally if a Wick rotation of the form

\[
X^0 \to iX^0 \quad X^i \to X^i
\]

is performed then the left and right multiplications with respect to the metric $\epsilon_{ab}$ are interchanged, and so do the roles of $\eta^{\mu \nu}$, and $-\delta^{\mu \nu}$. This should be expected since the symmetry groups $GL(3, 1R)$, and $GL(4R)$ are separated by such a rotation.

5.2 Establishing $X^\mu$ as Lorentz vector

Finally we would like to show that the commutation relations (85) (with $\eta^{\mu \nu}$ being the metric) are consistent with the statement that $X^\mu$ is indeed a Lorentz vector, which simply means the following [11]:
Although this statement has already been shown for each spin frame in (74) by a decomposition of a full Lorentz transformation with respect to each frame, it would be more desirable to show it in a more general fashion.

The above relation could be verified by using the representations (24, 25), which could turn out to be a lengthy process due to the latter being of the infinite dimensional type, involving Clebsch-Gordan coefficients, never the less it can be done. Fortunately there is a more efficient way to establish (88) which utilizes (85).

We start by writing the left hand side of (85) as the following:

\[ [Q^{\mu\nu},Q^{\lambda\sigma}] = \frac{i}{2} [Q^{\mu\nu},X^{\lambda}]X^{\sigma} + \frac{i}{2} X^{\lambda} [Q^{\mu\nu}X^{\sigma}] \]

(89)

Without any loss of generality the commutators on the right hand side of (89) may be written in the following way:

\[ [Q^{\mu\nu},X^{\lambda}] = i\eta^{\nu\lambda}X^{\mu} - i\eta^{\mu\lambda}X^{\nu} + \Theta^{\mu\nu\lambda}, \]

(90)

with an analogous expression for \([Q^{\mu\nu}X^{\sigma}]\), and we note that the tensor \(\Theta^{\mu\nu\lambda}\) is a reducible rank three tensor with no apparent symmetries.

Putting (90) back into (89), and using (85) the following is obtained:

\[ \frac{i}{2} \Theta^{\mu\nu\lambda}X^{\sigma} + \frac{i}{2} X^{\lambda} \Theta^{\mu\nu\sigma} = \eta^{\mu\lambda}Q^{\nu\sigma} - \eta^{\nu\sigma}Q^{\lambda\mu} = -i [Q^{\mu\nu},Q^{\lambda\sigma}] \]

(91)

This suggests that the tensors \(\Theta^{\mu\nu\lambda}\), can be written as the following:

\[ \Theta^{\mu\nu\lambda} = \frac{1}{2} \left( X^{\nu}X^{\mu}X^{\lambda} - X^{\lambda}X^{\nu}X^{\mu} + \Pi^{\mu\nu\lambda} \right), \]

(92)

again with a similar expression for \(\Theta^{\mu\nu\sigma}\), and \(\Pi^{\mu\nu\lambda}\) being a general a tensor with no apparent special properties. Putting these expressions on the left hand
side of (91), and upon comparing to the right hand side while writing $Q^\mu\nu$ in terms of $X^\mu X^\nu$, it follows that:

$$\Pi^{\mu\nu\lambda}X^\sigma + X^\lambda\Pi^{\mu\nu\sigma} = 0. \quad (93)$$

Interchanging the indices $\lambda$ with $\sigma$, a similar equation to (93) is obtained from which it follows that:

$$[\Pi^{\mu\nu\lambda}, X^\sigma] = 0. \quad (94)$$

Setting $\lambda = \sigma$, relations (93, 94), and (92) produce:

$$\Pi^{\mu\nu\lambda}X^\lambda = 0. \quad (95)$$

From which it follows that

$$\Pi^{\mu\nu\lambda} = 0, \quad (96)$$

and

$$\Theta^{\mu\nu\lambda} = [X^\nu X^\mu, X^\lambda]. \quad (97)$$

Putting (97) in (92) one obtains:

$$[S^{\mu\nu}, X^\lambda] = i\eta^{\mu\lambda}X^\nu - i\eta^{\nu\lambda}X^\mu. \quad (98)$$

Hence $X^\mu$ is a Lorentz vector.

### 6 Concluding Remarks

In this chapter we have constructed a relativistic infinite component wave equation describing a field carrying multiple states of angular momentum. The physical motivation for such a construction is to give a collective description for Hadronic fields which are composite fields exhibiting excitations in both angular momentum, and rest mass $(J/m^2)$ known as excitations along Regge trajectories.

The treatment above has been purely kinematic, and a fundamental assumption has been made regarding the compositeness of the field in question enabling it to acquire different states of spin. For such a relativistic equation to be consistent it was necessary to introduce the concept of spin frames, a direct result of the varying mass matrix appearing in (3). The physical motivation for introducing spin frames comes about from the non-unitarity of the Lorentz transformation constructed from the infinite dimensional matrices $X^\mu$ appearing in (3). Under a boost, the field $\psi$ is taken to an angular momentum state which differs from its original (pre-boosted) state by $\Delta J = \pm 1$. Such a transition does not correspond to an actual physical process, meaning the field under a boost remains with the same value of spin, or in other words every observer along any constant moving frame measures the same value of spin for the field in question. Thus it was shown that a free field is characterized by a spin frame which spans spin values $J - 1 < J < J + 1$, with $J$ being the actual physical
spin of the free field. As a result of this it was shown that the relativistic wave equation can support fields whose physical spin states differ by $\Delta J = 2$.

Due to these kinematic considerations it is apparent that when interactions are included the field $\psi$ may be excited to states that differ by $\Delta J = 2$ through the action of the quadrupolar tensor $T^{\mu\nu}$ which is also constructed form the matrices $X^\mu$. It therefore seems that structural (non-local) deformations lead to excitations of spin as should be expected. In an interacting theory it is conceivable that incorporation of fields like $\psi$ with other fields (possibly gauge fields) may lead to structural information regarding Hadronic fields through such transitions. As mentioned in the introduction of this chapter, having a linear relativistic wave equation similar to that of the Dirac equation should give rise to a current decomposition similar to that of the Gordon identity (1). Since the minimal physical transition in spin states according to the wave equation (3) are in two units of angular momentum, one should expect that a current decomposition similar to the latter should yield terms proportional to the quadrupolar moments of the Hadron. Of course a full current decomposition should yield higher multipole moments.

Although we have been referring to Hadron physics throughout this paper, the construction of the infinite component wave equation is a general construction that may fit other fields with the same characteristics. It is conceivable that a particle such as a quark, or an electron at some energy (not presently attainable) would get excited to a higher state of spin and rest mass due to some internal structure yet to be discovered.

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