ON THE BI-LIPSCHITZ CONTACT EQUIVALENCE OF PLANE COMPLEX FUNCTION-GERMS

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Abstract. In this note, we consider the problem of bi-Lipschitz contact equivalence of complex analytic function-germs of two variables. Basically, it is inquiring about the infinitesimal sizes of such function-germs up to bi-Lipschitz changes of coordinates. We show that this problem is equivalent to right topological classification of such function-germs.

1. Contact equivalence

Two $\mathbb{K}$-analytic function-germs $f, g: (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, at the origin $0$ of $\mathbb{K}^n$, are (\(\mathbb{K}\)-analytically) contact equivalent if the ideals (in $\mathcal{O}_{\mathbb{K}^n, 0}$) generated by $f$ and, respectively, generated by $g$ are $\mathbb{K}$-analytically isomorphic. As is well known, this classical (\(\mathbb{K}\)-analytic) contact equivalence admits moduli. For a complete description and answer to Zariski problème des modules pour les branches planes in the uni-branch case, see [5], (see also [6] for an answer towards the general case). Over the years several generalizations of the notion of (\(\mathbb{K}\)-analytic) contact equivalence appeared, and for some rough ones moduli do not appear.

More precisely, we will say that two function-germs $f, g: (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ at the origin $0$ of $\mathbb{K}^n$ are bi-Lipschitz contact equivalent if there exists $H: (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ a bi-Lipschitz homeomorphism and there exist positive constants $A$ and $B$, and $\sigma \in \{-1, +1\}$ such that

$$A |f(p)| \leq |g \circ H(p)| \leq B |f(p)| \text{ when } \mathbb{K} = \mathbb{C},$$

$$Af(p) \leq \sigma \cdot (g \circ H(p)) \leq Bf(p) \text{ when } \mathbb{K} = \mathbb{R},$$

for any point $p \in \mathbb{K}^n$ close to $0$.

When the bi-Lipschitz homeomorphism $H$ is also subanalytic, we will say that the functions $f$ and $g$ are subanalytically bi-Lipschitz contact equivalent.

A consequence of the main result of [1] on bi-Lipschitz contact equivalence of Lipschitz function-germs is the following finiteness theorem

**Theorem (1).** For any given pair $n$ and $k$ of positive integers, the subspace of polynomial function-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ of degree smaller than or equal to $k$ has finitely many bi-Lipschitz contact equivalence classes.

Later on, Ruas and Valette (see [7]) obtained for real mappings a result more general than that of [1], and which again ensures the finiteness of the bi-Lipschitz contact equivalent classes for polynomial functions germ $(\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ with given bounded degree. However, we observe that in the aforementioned papers [1, 7], the proofs of the finiteness theorems for bi-Lipschitz contact equivalence do not say anything about the corresponding recognition problem.
The preprint [2] completely solves the recognition problem of subanalytic contact bi-Lipschitz equivalence for continuous subanalytic function-germs \((\mathbb{R}^2,0) \to (\mathbb{R},0)\) by providing an explicit combinatorial object which completely characterizes the corresponding orbit.

In the present note, we solve the recognition problem for the subanalytic bi-Lipschitz contact equivalence of complex analytic function-germs \((\mathbb{C}^2,0) \to (\mathbb{C},0)\).

Our main result, Theorem 4.2, states that the subanalytic bi-Lipschitz contact equivalence class of a plane complex analytic function-germ \(f: (\mathbb{C}^2,0) \to (\mathbb{C},0)\) determines and is determined by purely numerical data, namely: the Puiseux pairs of each branch of its zero locus, the multiplicities of its irreducible factors and the intersection numbers of pairs of branches of its zero locus. It is a consequence of Theorem 3.6 which explicits the order of an irreducible function-germ \(g\) along real analytic half-branches at \(0\) as an affine function of the contact of the half-branch and the zero locus of \(g\) the function-germ.

Last, combining the main result of [9] and our main result, we eventually get that two complex analytic function germs \(f,g: (\mathbb{C}^2,0) \to (\mathbb{C},0)\) are subanalytically bi-Lipschitz contact equivalent if, and only if, they are right topologically equivalent, i.e. there exists a homeomorphism \(\Phi: (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)\) such that \(f = g \circ \Phi\).

2. Preliminaries

We present below some well known material about complex analytic plane curve-germs. It will be used in the description and the proof of our main result.

2.1. Embedded topology of complex plane curves.

Let \(f: (\mathbb{C}^2,0) \to (\mathbb{C},0)\) be the germ at \(0\) of an irreducible analytic function. It admits a Puiseux parameterization of the following kind:

\[
x \to (x^m, \Psi(x)) \text{ with } \Psi(x) = x^{\beta_1} \varphi_1(x^{e_1}) + \ldots + x^{\beta_s} \varphi_s(x^{e_s}),
\]

where each function \(\varphi_i\) is a holomorphic unit at \(x = 0\), the integer number \(m\) is the multiplicity of the function \(f\) at the origin and \((\beta_1, e_1), \ldots, (\beta_s, e_s)\) are the Puiseux pairs of \(f\). Then we can write down,

\[
f(x^m, y) = U(x,y)\Pi_{i=1}^{m}(y - \Psi(\omega^i x)),
\]

where \(\omega\) is a primitive \(m\)-th root of unity, the function \(U\) is a holomorphic unit at the origin, and \(\Psi\) is a function like in Equation (1).

The following relations determines the Puiseux pairs of \(f\). Let us write \(\Psi(x) = \sum_{j \geq m} a_j x^j\) and \(e_0 := m\) and \(\beta_{s+1} := +\infty\). We recall that

\[
\beta_{i+1} = \min\{j : a_j \neq 0 \text{ and } e_i | j\} \text{ and } e_{i+1} := \gcd(e_i, \beta_{i+1})
\]

for \(i = 0, \ldots, s - 1\). We deduce there exists positive integers \(m_1, \ldots, m_s\), such that for each \(k = 1, \ldots, s\), we find

\[
m = e_1 m_1 = e_2 m_2 m_1 = \ldots = e_k (m_k \cdots m_1)
\]

We recall that the irreducibility of the function \(f\) implies that \(e_s = 1\).
Remark 1. Let $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be an irreducible analytic function-germ and let $X$ be its zero locus. The ideal $I_X$ of $\mathbb{C}[x, y]$ consisting of all the functions vanishing on $X$ is generated by $f$. If $g = \lambda f$ is any other generator of $I_X$, then the functions $f$ and $g$ have the same Puiseux pairs. Thus we will speak of the Puiseux pairs of the branch $X$.

Let $f_1, f_2: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be irreducible analytic function-germs, and let $X_1$ and $X_2$ be the respective zero sets of $f_1$ and $f_2$.

The intersection number at $0$ of the branches $X_1$ and $X_2$ is defined as:

$$(X_1, X_2)_0 = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]}{(f_1, f_2)}$$

where $(f_1, f_2)$ denotes the ideal generated by $f_1$ and $f_2$.

Notation: Let $\Phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a homeomorphism and let $X$ be a subset germ of $(\mathbb{C}^2, 0)$. We will write

$$\Phi: (\mathbb{C}^2, X, 0) \rightarrow (\mathbb{C}^2, Y, 0)$$

to mean that the subset germ $Y$ is the germ of the image $\Phi(X)$ of $X$.

The following classical result completely described the classification of embedded complex plane curve germs:

**Theorem 2.1** \(\text{(II)}\). Let $f, g: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be reduced analytic function-germs and let $X$ and $Y$ be the respective zero sets of $f$ and $g$. Let $X = \bigcup_{i=1}^r X_i$ and $Y = \bigcup_{i=1}^s Y_i$ be the irreducible components of $X$ and $Y$ respectively. There exists a homeomorphism $\Phi: (\mathbb{C}^2, X, 0) \rightarrow (\mathbb{C}^2, Y, 0)$ if and only if, up to a re-indexation of the branches of $Y$, the components $X_i$ and $Y_i$ have the same Puiseux pairs, and each pair of branches $X_i$ and $X_j$ have the same intersection numbers as the pair $Y_i$ and $Y_j$.

We end-up this subsection in recalling a recent result of Parusiński \[9\]. It is as much a generalization of Theorem 2.1 to the non reduced case, as it is an improvement in the sense that it provides a more rigid statement.

**Theorem 2.2.** Let $f, g: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be complex analytic function-germs (thus not necessarily reduced). There exists a germ of homeomorphism $\Phi: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $g \circ \Phi = f$ (the function-germs $f$ and $g$ are then said topologically right-equivalent) if, and only if, there exists a one-to-one correspondence between the irreducible factors of $f$ and $g$ which preserves the multiplicities of these factors, their Puiseux pairs and the intersection numbers of any pairs of distinct irreducible components of the respective zero loci of $f$ and $g$.

### 2.2. Lipschitz geometry of complex plane curve singularities.

The Lipschitz geometry of complex plane curve singularities we are interested in is the Lipschitz geometry which comes from being embedded in the plane. It is described in a collection of three articles over 40 years, initiated with the seminal paper \[10\], followed then by \[4\] and concluding for now with the recent preprint \[8\]. Those papers state that the Lipschitz geometry of complex plane curve singularities determines and is determined by the embedded topology of such singularities. The version of this result which we are going to use is the following one:

**Theorem 2.3.** Let $X$ and $Y$ be germs of complex analytic plane curves at $0 \in \mathbb{C}^2$. Then, there exists a homeomorphism $\Phi: (\mathbb{C}^2, X, 0) \rightarrow (\mathbb{C}^2, Y, 0)$ if, and only if, there exists a (subanalytic) bi-Lipschitz homeomorphism $H: (\mathbb{C}^2, X, 0) \rightarrow (\mathbb{C}^2, Y, 0)$. 
The version stated above is almost Theorem 1.1 of [8]. The exact statement of Theorem 1.1 of [8] does not require the subanalyticity of the homeomorphism $H$. However, we observe that the proof presented there actually guarantees the subanalyticity of the mapping $H$.

3. On the irreducible functions case

This section is devoted to the relation between the order of a given irreducible plane complex function-germ $f$ along any real analytic half-branch germ at the origin $0$ of $\mathbb{C}^2$, and the contact (at the origin) between the half-branch and the zero locus $X$ of $f$. (Both notions of order and contact will be recalled below.) Theorem 3.6 is the main result of the section and the key new ingredient to complete the subanalytic bi-Lipschitz contact classification. It states that the contact and the order satisfies an affine relation whose coefficients can be explicitly computed by means of the Puiseux data of $X$ presented in sub-Section 2.1.

We suppose given some local coordinates $(w,y)$ centered at the origin of $\mathbb{C}^2$.

Let $\Gamma$ be a real-analytic half branch germ at the origin of $\mathbb{C}^2$, that is the image of (the restriction of) a real analytic map-germ $\gamma : (\mathbb{R}^+,0) \to (\mathbb{C}^2,0)$ defined as $s \to \gamma(s) = (w(s),y(s))$. When $\Gamma$ is not contained in the $y$-axis, we can assume that $\gamma(s) = (s^e u(s),s^e v(s))$ for positive integers $e,e'$ with $u(z),v(z) \in \mathcal{O}_1 := \mathbb{C}\{z\}$ and $u(0),v(0) \neq 0$.

When $\Gamma$ is not contained in the $y$-axis, we want to find a holomorphic change of coordinates $w \to x(w)$ so that

$$x(z^e u(z)) = z^e \iff u(z) \cdot x(z^e u(z)) = 1 \tag{4}$$

writing $x$ as $x(w) := w \cdot x(w)$ for a local holomorphic unit $x$. Thus Equation (4) admits a holomorphic solution. The mapping $\Theta : (w,y) \to (x(w),y) = (x,y)$ is bi-holomorphic in a neighbourhood of the origin. In the new coordinates $(x,y)$, the mapping $\gamma$ now writes as $s \to (s^e, s^e v(s))$.

**Vocabulary.** A map-germ $\phi : (\mathbb{R}^+,0) \to (\mathbb{C}^2,0)$ is *ramified analytic* if there exists a function germ $\tilde{\phi} \in \mathcal{O}_1$ and (co-prime) positive integers $p,q$ such that $\phi(t) = \tilde{\phi}(t^p/q)$. We will further say that $\phi$ is a *ramified analytic unit* if $\tilde{\phi}$ is a holomorphic unit.

When $\Gamma$ is not contained in the $y$-axis, we re-parameterize $\gamma$ with $s(t) := t^{e/m}$ for $t \in \mathbb{R}^+$, so that $\gamma(t) := \gamma(s(t)) = (t^m, y(t))$ where $y$ is ramified analytic with $y(0) = 0$ and $m$ is the multiplicity of the function $f$ at the origin.

If $\Gamma$ is contained in the $y$-axis then we take $s = t$ and $\Theta$ is just the identity mapping.

We recall that the Puiseux pairs introduced in sub-Section 2.1 are bi-holomorphic invariant. We denote again $f = f(x,y)$ for $f \circ \Theta^{-1}$ and use the Puiseux decomposition for $f(x^m,y)$ given in Equation (2) to define for each $k = 0, \ldots, s$, the function germ $\Psi_k \in \mathcal{O}_1$ as

$$\Psi_0(x) \ := \ 0,$$

$$\Psi_k(x) \ := \ x^{\beta_1} \varphi_1(x^{e_1}) + \ldots + x^{\beta_k} \varphi_k(x^{e_k}) \text{ when } k \geq 1.$$ 

Note that $\Psi_k(x) = \theta_k(x^{e_k})$ for some function germ $\theta_k \in \mathcal{O}_1$.

For each $l = 1, \ldots, m$, we can write

$$y(t) = \Psi(\omega^l t) + t^\lambda u_l(t)$$
where \( \lambda_l \in \mathbb{Q}_{>0} \cup \{+\infty\} \) for \( u_l \) is a ramified analytic unit, and with the convention that we write the null function 0 as \( 0 = t^+u_l(t) \). Thus the half-branch \( \Gamma \) is contained in \( X \) if and only if there exists \( l \) such that \( \lambda_l = +\infty \).

**Notation.** Let \( \lambda := \max_{l=1,\ldots,m} \lambda_l \).

Let \( l \in \{1,\ldots,m\} \) so that \( \lambda = \lambda_l \). When \( \Gamma \) is not contained in \( X \) (equivalently \( \lambda < +\infty \)) and convening further that \( \beta_0 = 0 \) and \( \beta_{s+1} = +\infty \), there exists a unique integer \( k \in \{0,\ldots,s\} \) such that

\[
\beta_k \leq \lambda < \beta_{k+1},
\]

and consequently we can write

\[
y(t) = \Psi_k(\omega^t) + t^\lambda u(t)
\]

for \( u \) a ramified analytic unit. (Note that \( \Psi = \Psi_k + R_k \) where \( R_k(x) = (\Psi - \Psi_k)(x) = O(x^{\beta_{k+1}}) \).)

Evaluating the function \( f \) along the parameterized arc \( t \rightarrow \gamma(t) \) using Equation (2) gives

\[
f(\gamma(t)) = f(t^m, y(t)) = f(t^m, \Psi_k(t) + t^\lambda u(t)) = U(t)^{\prod_{i=1}^m [\Psi_k(\omega^i t) + t^\lambda u(t) - \Psi(\omega^i t)]}
\]

where \( t \rightarrow U(t) \) is a ramified analytic unit. Since the function \( t \rightarrow f(\gamma(t)) \) is a ramified analytic function, there exist a ramified analytic unit \( V \) and a number \( \nu \in \mathbb{Q}_{>0} \cup \{+\infty\} \) such that

\[
f(\gamma(t)) = t^\nu V(t).
\]

The number \( \nu \) of Equation (5) is called the *order of the function \( f \) along the parameterized curve \( t \rightarrow \gamma(t) \).*

**Lemma 3.1.** 1) Assume \( \Gamma \) is contained in the \( y \)-axis. The order of the function \( f \) along the parameterized curve \( t \rightarrow \gamma(t) = (0, t^e \psi(t)) \) is \( \nu = m \cdot e' \).

2) Assume \( \Gamma \) is not contained in the \( y \)-axis. The order of \( \nu \) the function \( f \) along the parameterized curve \( t \rightarrow \gamma(t) \) is given by

\[
\nu = e_k \lambda + (e_0 - e_1)\beta_1 + \ldots + (e_{k-1} - e_k)\beta_k \in \mathbb{Q}_{>0} \cup \{+\infty\}
\]

**Proof.** If \( \Gamma \) is contained in the \( y \)-axis, then the order of \( f \) along \( t \rightarrow (0, t^e \psi(t)) \) is \( m \cdot e' \).

We can assume that \( \Gamma \) is parameterized as \( \mathbb{R}_+ \ni t \rightarrow \gamma(t) = (t^m, \psi_k(t) + t^\lambda u(t)) \).

For \( i \in \{1,\ldots,m\} \) such that \( l - i \) is not a multiple of \( m_1 \), the order of \( \Psi_k(\omega^i t) + t^\lambda u(t) - \Psi(\omega^i t) \) is \( \beta_1 \). There are \( m_1 - 1 = (e_1 - 1) = e_0 - e_1 \) such indices \( i \).

For any \( 0 < j < k \), when \( i \in \{1,\ldots,m_1 - 1\} \) is such that \( l - i \) is a multiple of \( m_1 \ldots m_j \) but not a multiple of \( m_1 \ldots m_{j+1} \), the order of \( \Psi_k(\omega^i t) + t^\lambda u(t) - \Psi(\omega^i t) \) is \( \beta_j \). There are \( e_j - e_{j+1} \) such indices.

When \( i \in \{1,\ldots,m\} \) is such that \( l - i \) a multiple of \( m_1 \ldots m_k \), the order of \( \Psi_k(\omega^i t) + t^\lambda u(t) - \Psi(\omega^i t) \) is \( \lambda \). There are \( e_k \) such indices.

We just add-up all these order to get the desired number \( \nu \), once we have checked that this sum does not depend on the index \( l \) such that \( \lambda = \lambda_l \). Let \( r \in \{1,\ldots,m\} \) be an index such that \( \lambda_r = \lambda \). Thus \( y(t) = \Psi_k(w^r t) + t^\lambda u_r(t) \). If \( l - r \) is not a multiple of \( m_1 \ldots m_k \), then we check again that \( 0 = y(t) - y(t) = t^\lambda (u(t) - u_r(t)) + t^\beta W \) for a ramified analytic unit \( W \) and \( \beta_j \leq \beta_{k-1} < \lambda \), which is impossible. Necessarily \( l - r \) is a multiple of \( m_1 \ldots m_k \) and thus \( \Psi_k(w^r t) = \Psi_k(\omega^i t) \), so that \( \nu \) is well defined. \( \square \)
Now we can introduce a sort of normalized parameterization of real analytic half-branch germs in order to do bi-Lipschitz geometry. More precisely,

**Definition 3.2.** An analytic arc (at the origin of \( \mathbb{C}^2 \)) is the germ at 0 \( \in \mathbb{R}_+ \) of a mapping \( \alpha : [0, \varepsilon) \to \mathbb{C}^2 \) defined as \( t \to (x(t), y(t)) \) such that:

0) the mapping \( \alpha \) is not constant and \( \alpha(0) = 0 \),

1) there exist a positive integer \( e \) such that \( t \to \alpha(t^e) \) is (the restriction of) a real analytic mapping,

2) the arc is parameterized by the distance to the origin in the following sense: there exists positive constants \( a < b \) such that for \( 0 \leq t \ll 1 \) the following inequalities hold,

\[
 al \leq |\alpha(t)| \leq bt.
\]

We will denote any analytic arc by its defining mapping \( \alpha \). Note that the semi-analyticity of the image of an analytic arc \( \alpha \) implies a much better asymptotic than that proposed in the definition, namely we know that \( |\alpha(t)| = \alpha_1 t + \delta(t) \), with \( \alpha_1 > 0 \) and where \( \delta \) is ramified analytic such that \( \delta(0) = 0 \).

Let \( \alpha \) be a real analytic arc. The function \( t \to f \circ \alpha(t) \) is ramified analytic, thus as already seen in Equation [5] can be written as \( f \circ \alpha(t) = t^{\nu_f(\alpha)} V(t) \) for a ramified analytic function and \( \nu_f(\alpha) \in \mathbb{Q}_{>0} \cup \{+\infty\} \). The order of the function \( f \) along the real analytic arc \( \alpha \) is the well defined rational number \( \nu_f(\alpha) \).

Let \( C \) be a real-analytic half-branch germ at the origin of \( \mathbb{C}^2 \). Let \( \alpha \) and \( \beta \) be two real analytic arcs parameterizing \( C \). We check with an easy computation that \( \nu_f(\alpha) = \nu_f(\beta) \). Thus we introduce the following

**Definition 3.3.** The order of the function \( f \) along the real analytic half-branch \( C \) is the well defined number \( \nu_f(C) := \nu_f(\delta) \) for any arc \( \delta \) parameterizing \( C \).

Let us denote \( X(r) = \{ p \in X : |p| = r \} \) for \( r \) a positive real number.

Let \( \alpha \) be any analytic arc. The contact (at the origin) between the analytic arc \( \alpha \) and the complex curve-germ \( X \) is the rational number defined as

\[
 c(\alpha, X) = \lim_{t \to 0^+} \frac{\log(\text{dist}(\alpha(t), X(|\alpha(t)|)))}{\log(t)}.
\]

Let \( C \) be the image of the analytic arc \( \alpha \) above. Given any other analytic arc \( \beta \) parameterizing \( C \), it is a matter of elementary computations to check that \( c(\alpha, X) = c(\beta, X) \). Thus we present the following

**Definition 3.4.** The contact between the real-analytic half-branch \( C \) and the curve \( X \) is \( c(C, X) := c(\delta, X) \) for any analytic arc \( \delta \) parameterizing \( C \).

Let \( \Gamma \) be a real analytic half-branch at the origin of \( \mathbb{C}^2 \). Let \( \gamma \) be a parameterization of \( \Gamma \) of the form \( \mathbb{R}_+ \ni t \to (0, y(t)) \) when \( \Gamma \) is contained in the \( y \)-axis, where \( y \) is a ramified analytic function-germ. When \( \Gamma \) is not contained in the \( y \)-axis, possibly after a holomorphic change of coordinates at the origin of \( \mathbb{C}^2 \), we consider a parameterization of \( \Gamma \) of the form \( \mathbb{R}_+ \ni t \to (t^m, y(t)) \) for \( y \) ramified analytic.

When the half-branch \( \Gamma \) is not contained in \( X \) (and regardless of its position relatively to the \( y \)-axis), as already seen above, we can write \( y(t) \) as \( y(t) = \Psi_k(\omega^lt) + t^\lambda u(t) \) where \( \beta_k \leq \lambda < \beta_{k+1} \) for some integer \( k \in \{0, \ldots, s\} \), with \( u \) a ramified analytic unit and \( l \in \{1, \ldots, m\} \). Let \( \mu \) be the
order of $|\gamma(t)|$ at $t = 0$, that is the positive rational number $\mu$ such that $|\gamma(t)| = Mt^\mu + o(t^\mu)$ for a positive constant $M$. Thus we find

Lemma 3.5. The contact between $\Gamma$ and $X$ is $c(\Gamma, X) = \frac{\lambda}{m}$.

Proof. The tangent cone at the origin of the curve $X$ is just the $x$-axis. Writing $\gamma(t) = (x(t), y(t))$, the half-branch is tangent to the $x$-axis if and only if $\lim_{t \to 0} x(t)^{-1}y(t) = 0$. When $\Gamma$ is transverse to the $x$-axis, we have $k = 0$ in the writing of $y(t)$ above, so that $\mu = \lambda$ and thus $c(\Gamma, X) = 1$.

Suppose that the half-branch $\Gamma$ is tangent to the $x$-axis, we deduce $\mu = m$ since the tangency hypothesis implies that $y(t) = o(t^m)$. Thus the mapping $t \to \gamma(t^{\frac{1}{m}}) = (t, y(t^{\frac{1}{m}}))$ is an analytic arc parameterizing $\Gamma$. In particular we must have $\lambda > m$.

Notation. Up to the end of this proof will use the notation $Const$ to mean a positive constant we do not want to precise further.

Let $\rho : (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$ be the function defined as $\rho(t) := \text{dist}(\gamma(t^{\frac{1}{m}}), X)$. First, since $\gamma$ is tangent to $X$ and the function $\rho$ is continuous and subanalytic, there exists a positive rational number $c$ such that

$$
\rho(t) = Const \cdot t^c + o(t^c).
$$

Second, we obviously have for $t$ positive and small enough $\rho(t) \leq t^{\frac{1}{m}}|u(t)|$ so that we deduce from Equation (4) that $c \geq \frac{\lambda}{m}$.

Let $r(t) := |\gamma(t^{\frac{1}{m}})|$, so that we find $r(t) = t + o(t)$. Let $t \to \phi(t)$ be any analytic arc on $X$ such that $\rho(t) = |\phi(t) - \gamma(t^{\frac{1}{m}})|$. From Equation (6) we get

$$
||\phi(t)| - r(t)| \leq Const \cdot t^c.
$$

Writing $\phi = (x_\phi, y_\phi)$, we see from Equation (7) that $x_\phi(t) = t + O(t^c)$. Let $\xi : (\mathbb{R}_+, 0) \to (\mathbb{C}, 0)$ be the ramified analytic function of the form $t \to \xi(t) := t^{\frac{1}{m}}[1 + O(t^{-1})]$ and such that $\xi(t)$ is a $m$-th root of $x_\phi(t)$. Thus $y_\phi(t) = \Psi(\omega^i \xi(t))$ for some $i \in \{1, \ldots, m\}$ and we observe that $y_\phi(t) = \Psi_k(\omega^i t^{\frac{1}{m}}) + o(t^{\frac{1}{m}})$. Since $y(t) = \Psi_k(\omega^i t^{\frac{1}{m}}) + t^{\frac{1}{m}}u(t^{\frac{1}{m}})$, with $u$ a ramified analytic function, and $|y_\phi(t) - y(t^{\frac{1}{m}})| \leq Const \cdot t^c$, we deduce that $\Psi_k(\omega^i T) = \Psi_k(\omega^j T)$. But this implies that $c \leq \frac{\lambda}{m}$, and thus $c = \frac{\lambda}{m}$.

From Equation (7) we deduce that

$$
\rho(t) \leq \text{dist}(\gamma(t^{\frac{1}{m}}), X(r(t)) \leq Const \cdot t^c.
$$

Combining Equation (6) and Equation (8) we get the result.

The next result will be key for Theorem 4.2, the main result of this note, is indeed the new ingredient to the range of questions we are dealing with here. We recall that the Puiseux data notation convenes that $e_{-1} = \beta_0 = 0, e_0 = m$ and $\beta_{s+1} = +\infty$.

Theorem 3.6. Let $\Gamma$ be a real analytic half-branch at the origin of $\mathbb{C}^2$ as above. The order of the function $f$ along $\Gamma$ is given by

$$
\nu_f(\Gamma) = e_k \cdot c(\Gamma, X) + (e_0 - e_1)\frac{\beta_1}{m} + \ldots + (e_{k-1} - e_k)\frac{\beta_k}{m}
$$

(9) \hspace{1cm} = e_k \left( c(\Gamma, X) - \frac{\beta_k}{m} \right) + \sum_{i=\min(k-1,0)}^{k-1} e_i \left( \frac{\beta_{i+1}}{m} - \frac{\beta_i}{m} \right),

(10)
where the integer number \( k \in \{0, \ldots, s\} \) in Equations (9) and (10) is uniquely determined when \( c(\Gamma, X) < +\infty \) by the following condition:

\[
\beta_k \leq m \cdot c < \beta_{k+1}.
\]

Proof. It is just a rewriting of Lemma 3.1 in term of the size \( t \) of any arc parameterizing \( \Gamma \) and uses Lemma 3.5.

A direct consequence of the above result is the following result about bi-Lipschitz contact equivalence.

Proposition 3.7. Let \((\mathbb{C}^2, X, 0)\) and \((\mathbb{C}^2, Y, 0)\) be two germs of irreducible complex plane curves defined by reduced function-germs \( f \) and \( g \) respectively. If there exists a subanalytic bi-Lipschitz homeomorphism \( H : (\mathbb{C}^2, X, 0) \rightarrow (\mathbb{C}^2, Y, 0) \) then there exist positive constants \( 0 < A < B < +\infty \) such that in a neighbourhood of the origin we find

\[
A |f| \leq |g \circ H| \leq B |f|.
\]

Proof. If it is not true, it happens along a real-analytic half-branch \( C \). Necessarily such a half-branch \( C \) must be tangent to the curve \( X \). Taking a parameterization of \( C \) by an arc \( \alpha \), we can for instance assume that \((f \circ \alpha(t))^{-1}(g \circ H \circ \alpha(t))\) goes to 0 as \( t \) goes to 0. Let \( \nu \) be the order of \( f(\alpha(t)) \) and \( \nu' \) the order of \( g(H(\alpha(t))) \). Theorem 3.6 provides

\[
\nu = (e_0 - e_1)\frac{\beta_1}{m} + \ldots + (e_{k-1} - e_k)\frac{\beta_k}{m} + e_k \cdot c(C, X)
\]

\[
\nu' = (e_0 - e_1)\frac{\beta'_1}{m} + \ldots + (e_{k'-1} - e_{k'})\frac{\beta'_{k'}}{m} + e_{k'} \cdot c(H^{-1}(C), Y).
\]

From the proofs of Lemma 3.1 and Lemma 3.5 we know that

\[
\beta_{k'} \leq m \cdot c(H^{-1}(C), Y) < \beta_{k'+1} \quad \text{and} \quad \beta_k \leq m \cdot c(C, X) < \beta_{k+1}.
\]

Since the contact is a bi-Lipschitz invariant we get \( c(C, X) = c(H^{-1}(C), Y) \). Besides \( \nu' > \nu \), thus we deduce \( k' > k \). This latter inequality implies

\[
m \cdot c(H^{-1}(C), Y) \geq \beta_{k'} \geq \beta_{k+1} > m \cdot c(C, X),
\]

which is impossible.

\[\square\]

4. Main Result

Let \( f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0) \) be a germ of analytic function. Let \( f = f_1^{m_1} \cdots f_r^{m_r} \) be the irreducible decomposition of the function, where \( f_1, \ldots, f_r \) are irreducible function-germs and \( m_1, \ldots, m_r \), the corresponding respective multiplicities, are positive integer numbers.

Let \( X_i \) be the zero locus of \( f_i \), let \( m_i \) be the multiplicity of \( f_i \) at \( 0 \) and let \( (\beta_j^{(i)}, e_j^{(i)})_{j=1}^{s_i} \) be its Puiseux pairs. Let \( \Gamma \) be a real analytic half-branch at the origin. Let \( c_i := c(\gamma, X_i) \) be the contact of \( \Gamma \) with \( X_i \) and let \( \nu_i = \nu_{f_i}(\Gamma) \) be the order of \( f_i \) along \( \Gamma \).

Since we have defined in Section 3 the order of an irreducible function-germ along \( \Gamma \), the order of \( f \) along \( \Gamma \) is defined as the sum of the order of each of its irreducible component weighted by the corresponding multiplicity (as a factor of the irreducible decomposition of \( f \)). From Theorem 4.2 we deduce straightforwardly the next
Lemma 4.1. The order $\nu$ of the function $f$ along $\Gamma$ is

$$\nu := m_1 \cdot \nu_1 + \ldots + m_r \cdot \nu_r$$

$$= \sum_{i=1}^{r} m_i \left[ e_{k_i}^{(i)} \left( c_i - \frac{\beta_{k_i}^{(i)}}{m} \right) + \sum_{j=\min(k_i-1,0)}^{k_i-1} e_{k_j}^{(i)} \left( \frac{\beta_{j+1}^{(i)}}{m} - \frac{\beta_{j}^{(i)}}{m} \right) \right]$$

where each of the integer $k_i \in \{0, \ldots, s_i\}$ is uniquely determined when $c_1 \cdot c_r < +\infty$ by the condition

$$\beta_{k_i}^{(i)} \leq m_i \cdot c_i < \beta_{k_i+1}^{(i)}.$$

The main result of this note is the following:

Theorem 4.2. Let $f$ and $g$ be two analytic function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Let $f = f_1^{m_1} \cdots f_r^{m_r}$ and $g = g_1^{n_1} \cdots g_s^{n_s}$ be respectively the irreducible decompositions of the functions $f$ and $g$. Let $X_i$ be the zero locus of $f_i$ and $Y_j$ be the zero locus of $g_j$.

The functions $f$ and $g$ are subanalytically bi-Lipschitz contact equivalent if, and only if, there exists a bijection $\sigma : \{f_1, \ldots, f_r\} \to \{g_1, \ldots, g_r(=s)\}$ between the irreducible factors of $f$ and $g$ such that

1) The multiplicities of each corresponding factors are equal, that is $m_i = n_{\sigma(i)}$,

2) The Puiseux pairs of $f_i$ and $g_{\sigma(i)}$ are the same, and

3) for any pair $i, j$, the intersection numbers $(X_i, X_j)_0$ and $(Y_{\sigma(i)}, Y_{\sigma(j)})_0$ are equal.

In particular, $f$ and $g$ are subanalytically bi-Lipschitz contact equivalent if, and only if, they are right topologically equivalent.

Proof. First assume that, $r = s$,

- the intersection numbers $(X_i, X_j)_0$ and $(Y_i, Y_j)_0$ are equal for any $i \neq j$ and,

- the Puiseux pairs of the functions $f_i$ and $g_i$ are equal and,

- the multiplicities $m_i$ and $n_i$ are equal, for $i = 1, \ldots, r$.

From Theorem 2.3 we deduce there exists $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ a subanalytic bi-Lipschitz homeomorphism such that $H(X_i) = Y_i$ for any $i = 1, \ldots, r$. For each $i = 1, \ldots, r$, Proposition 3.7 implies there exist positive constants $0 < A_i < B_i < +\infty$ such that in a neighbourhood of the origin we find

$$A_i |f_i| \leq |g_i \circ H| \leq B_i |f_i|.$$

Thus the functions $f$ and $g$ are bi-Lipschitz contact equivalent (via $h$).

The general situation, by hypothesis, is easily deduced from the special one above, since it consists only in changing the indexation of one of the family of irreducible factors (of the corresponding zero loci and the corresponding multiplicity).

Conversely, we assume now that there exists $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ a subanalytic bi-Lipschitz homeomorphism such that there exist positive constants $A < B$ such that in a neighbourhood of the origin the following inequalities hold true:

$$A |f| \leq |g \circ H| \leq B |f|.$$ 

We immediately find $H(X) = Y$ and $r = s$. Up to re-indexation of the branches $Y_i$, we also have $H(X_i) = Y_i$ for $i = 1, \ldots, r$. Using Theorem 2.3 again we deduce that the intersection numbers $(X_i, X_j)_0$ and $(Y_i, Y_j)_0$ are equal for any $i \neq j$ (let us denote each such number by $I_{i,j}$), the Puiseux pairs of the function-germs $f_i$ and $g_i$ are equal. It remains to prove that the
multiplicities $m_i$ and $n_i$ are also equal, for $i = 1, \ldots, r$. In order to prove that $m_1 = n_1$, let $C$ be any real-analytic half-branch such that the contact $c = c(C, X_1)$ is sufficiently large (and finite). More explicitly it means that the others contacts $c(C, X_i)$, for $i = 2, \ldots, r$, are equal to the intersection number $I_{i,1} := (X_i, X_1)_0$. Since $H$ is a subanalytic bi-Lipschitz homeomorphism such that $H(X_i) = Y_i$ for any $i = 1, \ldots, r$, the pre-image $H^{-1}(C)$ is still a real analytic half-branch. Since bi-Lipschitz homeomorphisms preserve the contact, we deduce that $c = c(H^{-1}(C), Y_1)$ and each contact $c(H^{-1}(C), Y_i)$ is equal to the contact $(Y_i, Y_1)_0$, for $i = 2, \ldots, r$. In other words we see
\begin{equation}
\nu_y(H^{-1}(C)) = c \cdot n_1 + I_{2,1} \cdot n_2 + \ldots + I_{r,1} \cdot n_r
\end{equation}
and
\begin{equation}
\nu_f(C) = c \cdot m_1 + I_{2,1} \cdot m_2 + \ldots + I_{r,1} \cdot m_r.
\end{equation}
Combining Equation (11) from the hypothesis, with Equations (12) and (13) we conclude that $c n_1 + I_{2,1} n_2 + \ldots + I_{r,1} n_r = c m_1 + I_{2,1} m_2 + \ldots + I_{r,1} m_r$.

Since the half-branch $C$ can be chosen asymptotically arbitrarily close to $X_1$, its contact $c$ goes $+\infty$, and thus we find $m_1 = n_1$. The same procedure can be applied for each remaining $i = 2, \ldots, r$, substituting $i$ for 1, thus we conclude that that $m_i = n_i$ for $i = 1, \ldots, r$,

thus proving what we wanted. \hfill $\Box$

The first immediate consequence of our main result is the following:

**Corollary 4.3.** Let $f$ and $g$ be two analytic function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. They are bi-Lipschitz contact equivalent if, and only if, they are subanalytically bi-Lipschitz contact equivalent.

The second consequence is:

**Corollary 4.4.** The subanalytic bi-Lipschitz contact equivalence classification of complex analytic plane function-germs has countably many equivalence classes.

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