Wigner function approach to single electron coherence in quantum Hall edge channels

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Recent electron quantum optics experiments performed with on-demand single electron sources call for a mixed time/frequency approach to electronic quantum coherence. Here, we present a Wigner function representation of first order electronic coherence and show that it provides a natural visualization of the excitations emitted by recently demonstrated single electron sources. It also gives a unified perspective on single particle and two particle interferometry experiments. In particular, we introduce a non-classicality criterion for single electron coherence and discuss it in the context of Mach-Zehnder interferometry. Finally, the electronic Hanbury Brown and Twiss and the Hong Ou Mandel experiments are interpreted in terms of overlaps of Wigner function thus connecting them to signal processing.

PACS numbers: 73.23-b, 73.43.-f, 71.10.Pm, 73.43.Lp
Keywords: quantum transport, quantum Hall effect, quantum coherence

I. INTRODUCTION

Recent experiments have demonstrated the importance of single[12] and two particle[3-6] quantum coherence in the field of quantum coherent electronics. The advent of on demand single electron sources[7-13] has opened the way to a new generation of experiments dealing with excitations having a finite spatial extension and various shapes instead of a continuous stream of indistinguishable excitations. These experiments open the way to the controlled preparation, manipulation and characterization of single to few electron excitations in ballistic conductors, an emerging field called electron quantum optics[14]. Since these experiments access time scales comparable to the coherence time of electrons within conductors through finite frequency current[13,14] and noise measurements[17,18], a time resolved approach to electronic quantum coherence is required.

In this paper, we present a unified view of the various representations of single electron coherence in quantum Hall edge channels and we introduce in the present context a time/frequency representation based on the Wigner function introduced in quantum mechanics[19] and signal processing[20]. The Wigner function is commonly used in quantum optics[21] and has been recently been measured in cavity QED to demonstrate the decay of quantum superpositions of two quasi-classical states of the electromagnetic field[22].

Although all the representations of electronic coherence contain exactly the same information, each of them has its advantages and drawbacks. First, the time domain representation of single electron coherence is suitable for analyzing time-dependent aspects as well as to define the proper notions of coherence and dephasing times[23]. However, information on the electron or hole nature of excitations in hidden in the phase of this quantity. On the other hand, the frequency domain representation is perfect for discussing the nature of excitations and is the natural representation for the electronic analogue of homodyne tomography[24] but it is not well suited for describing real time aspects. The mixed time/frequency representation called the electronic Wigner function combines the advantages of both representations: it gives a direct access to both the time evolution and energy content of single electron coherence. Moreover, it provides a natural non-classicality criterion for single electron coherence and for example, enable us to discuss non-classicality in Mach-Zehnder interferometry.

Historically, the Wigner function has been introduced in the theory of quantum transport at the end of the 80s to understand the limits of a semi-classical treatment in semiconductors[25] to study phonon interaction effect[26] and also to model various quantum devices built from semiconductors[27,28]. Although we deal with the same
concept, we consider here low dimensional conductors in which many-body effects as well as interaction and decoherence in the presence of the Fermi sea are crucial. Moreover, our primary motivation is to discuss the various representation of single electron coherence and show the relevance of the electronic Wigner function for understanding electron quantum optics experiments.

The price to pay is that the electronic Wigner function, as of today, cannot be measured directly at a given time and frequency contrary to cavity QED experiments. Nevertheless, we will show that it provides a unified view of single electron interference experiments as well as of two particle interference experiments based on the Hanbury Brown and Twiss (HBT) effect. These include the Hong, Ou, Mandel (HOM) experiment which involve two electron interferences from two different sources in full generality. By doing so, we will greatly simplify the discussion of the various protocols for reconstructing single electron coherence. We show that the recently proposed tomography protocol does indeed reconstruct the electronic Wigner function. Moreover, this approach, by providing a natural visualization of single electron coherence, suggests that specifically designed alternative tomography protocols may lead to single electron reconstruction from less measurements than our generic homodyne protocol. We finally sketch a possible connection between the search for such optimized tomography protocols and the problematic of compressed sensing in signal processing.

This paper is structured as follows: in Sec. II, single electron coherence and its relation to Glauber’s coherence in quantum optics is revisited. We then discuss the time and frequency representations of this quantity and we introduce the electronic Wigner function. Section III is devoted to examples: we first consider single electron coherence in the presence of a classical voltage drive. The cases of a sinusoidal drive and of Lorentzian pulse are discussed. We then discuss the electronic Wigner function emitted by the mesoscopic capacitor used as an on demand single electron source. Section IV is devoted to interferometry experiments, starting with Mach-Zehnder interferometry and then discussing the HBT and HOM two particle interference experiments.

II. SINGLE ELECTRON COHERENCE

A. Definition and simple examples

Mach-Zehnder interferometry has shown the importance of first order electron coherence. It is usually defined as the non equilibrium Keldysh Green’s function already used to describe electronic coherence in the many body approaches to the decoherence problem in diffusive conductors:

\[ G^{(e)}_\rho(x, t; y, t') = \text{Tr}(\psi^\dagger(x, t) \rho \psi(y, t')). \]  

As noticed by Glauber et al. this correlator is also relevant for atom-counting experiment with fermionic cold atom system. A similar quantity can be defined for hole excitations:

\[ G^{(h)}_\rho(x, t; y, t') = \text{Tr}(\psi^\dagger(x, t) \rho \psi(y, t')). \]  

The electron and hole coherences at coinciding times are related using the canonical anti commutation relations

\[ G^{(e)}(x, t; y, t) = \delta(x - y) - G^{(h)}(y, t; x, t). \]  

The first important difference with photon quantum optics comes from the fact that, in a metallic conductor, the Fermi sea, which plays here the role of the vacuum state, has a non vanishing single particle coherence whereas coherence vanishes in the photon vacuum. At zero temperature, the single electron coherence within a single chiral channel at equal times is given by:

\[ G^{(e)}_\mu(x, t; y, t) = \frac{i}{2 \pi} \frac{e^{ik_F (x - y)} y - x + \text{i} \theta^+}{y - x + \text{i} \theta^+}, \]  

where \( k_F(\mu) \) denotes the Fermi momentum associated with the chemical potential \( \mu \) of the edge channel under consideration. At non-vanishing electronic temperature \( T_e \), these correlators decay over the thermal length scale \( l(T_e) = \hbar v_F/k_B T_e \) where \( v_F \) denotes the Fermi velocity:

\[ G^{(e)}_\mu,T_e(x, t; y, t) = -\frac{i}{2l(T_e)} e^{ik_F (x - y)} \text{sinh} \left( \frac{\pi(y - x + \text{i} \theta^+)}{l(T_e)} \right). \]  

This suggests to decompose the single electron coherence into a Fermi sea contribution due to the chemical potential \( \mu \) of the conductor and a contribution due to excitations above this ground state:

\[ G^{(e)}_\rho(x, t; y, t') = G^{(e)}_\mu,T_e(x, t; y, t') + \Delta G^{(e)}_\rho(x, t; y, t'). \]  

The case of an ideal single electron excitation helps clarifying the physical meaning of \( \Delta G^{(e)}_\rho \). Such a source generates a many body state obtained from a Fermi sea by adding one extra-particle in a normalized wave packet \( \varphi_c \):

\[ \psi^\dagger(x) |F\rangle = \int_{-\infty}^{+\infty} \varphi_c(x) \psi^\dagger(x) |F\rangle \, dx \]  

where \( |F\rangle \) denotes the Fermi sea at a fixed chemical potential. In momentum space, \( \varphi_c \) only has components on single particle states above the Fermi level. Then, Wick’s theorem leads to single electron coherence at initial time:

\[ \Delta G^{(e)}_{\psi^\dagger|\varphi_c|F}(x, 0; y, 0) = \varphi_c(x) \varphi_c^*(y). \]  

In the same way, the single electron coherence of the state obtained by adding a single hole excitation to the Fermi sea \( \psi^\dagger|\varphi_h|F\rangle \) (\( \varphi_h(k) = 0 \) above the Fermi level) is given by

\[ \Delta G^{(e)}_{\psi^\dagger|\varphi_h|F}(x, 0; y, 0) = -\varphi_h(x) \varphi_h^*(y). \]  

The – sign
reflects the fact that a hole is the absence of an electron in the Fermi sea.

The excess single electron coherence \( \Delta G_p^{(e)}(x, t; y, t) \) contains information on both the shape of the wave packet and its phase dependence. More precisely, \( \Delta G_p^{(e)}(x, t; x, t) \) encodes the average density and thus the shape of the wave packet whereas the \( x - y \) dependence encodes the phase dependence. The shape of the wave packet gives access to its length (or duration in the time domain) which we denote by \( l_2 \) (resp. \( T_1 \)).

But a realistic single electron source does not necessarily emit a perfectly coherent wave packet: it can also emit a statistical mixture of them or the electrons may have experienced some decoherence due to Coulomb interactions.

How can we measure the decay of single electron coherence? As recently noticed by G. Haack et al. the first degree of coherence originally introduced by Mandel for photons:

\[
g^{(1)}(x, l) = \frac{\Delta G_p^{(e)}(x + \frac{1}{2}, x - \frac{1}{2})}{\sqrt{\Delta G_p^{(e)}(x + \frac{1}{2}, x + \frac{1}{2}) \Delta G_p^{(e)}(x - \frac{1}{2}, x - \frac{1}{2})}} \tag{9}
\]

A very simple description of decoherence consists in introducing a phenomenological decoherence coefficient in front of the excess single electron coherence associated with a coherent wave packet:

\[
\Delta G_p^{(e)}(x, y) \approx D(x - y) \varphi_e(x) \varphi_e(y)^* \tag{10}
\]

In this case, the decaying length of this decoherence coefficient is precisely \( l_\phi \). However such an approach to decoherence is justified only when the electron under consideration can still be distinguished from electron/hole pairs generated in the Fermi sea by Coulomb interactions. Such a form of decoherence can then obtained in a single electron model under the influence of an harmonic environment. Otherwise, a more complete approach should be used. Nevertheless, using Eq. (10) into Eq. (9) shows that \( g^{(1)}(x, l) \) decays over the same length scale \( l_\phi \) as \( D(x - y) \) which is called the dephasing length (the dephasing time in the time domain).

It then follows from (9) that the length scale governing the decay of the single electron coherence \( \Delta G_p^{(e)}(x, y) \) combines the decaying length of the wave packet and \( l_\phi \):

\[
\frac{1}{l_2} = \frac{1}{2l_1} + \frac{1}{l_\phi} \tag{11}
\]

This formula mimics the famous expression of the total decoherence rate in NMR in terms of the relaxation and dephasing rates.

### B. Analogy with Glauber’s coherences

In Glauber’s approach to photo-detection, a photon detector is a quantum device designed to detect a single photon. In such a detector, a single photon causes the photoionization of an atom and the emitted electron is then amplified to give a macroscopic signal. Old photo-multipliers work exactly this way: the incoming photon is absorbed, leading to the emission of a single electron which is then amplified by the secondary emission of electrons in dynodes. The initial photo-emission stage can be described using elementary time dependent perturbation theory. The resulting photo-detection signal is then obtained as

\[
I_D(t) = \int_0^t K_D(\tau, \tau') G_p^{(1)}(x_D, \tau; x_D, \tau') d\tau d\tau' \tag{12}
\]

where \( x_D \) denotes the position of the detector. The quantity \( G_p^{(1)}(x_D, \tau | x_D, \tau') \) only depends on the quantum state of the electromagnetic field and is indeed Glauber’s single photon coherence function:

\[
G_p^{(1)}(x, \tau; x', \tau') = \text{Tr} \left( E^{(\tau)}(x, \tau) \rho \cdot E^{(-)}(x', \tau') \right) \tag{13}
\]

where \( E^{(\pm)} \) denote the positive (resp. negative) frequency part of the electric field operator and \( \rho \) is the electromagnetic field initial density operator. The function \( K_D(\tau, \tau') \) characterizes the response of the detector to the absorption of a single photon. Broadband detectors have a local response in time \( K_D(\tau - \tau') \approx \delta(\tau - \tau') \) and therefore measure the (integrated) photo-count:

\[
I_D(t) \approx \int_0^t G_p^{(1)}(x_D, \tau', x_D, \tau) d\tau'. \tag{14}
\]

On the other hand, narrow band detectors select a single frequency and therefore measure the Fourier transform of Glauber’s single photon coherence with respect to time.

The analogy between Glauber’s single photon coherence in Eq. (13) and the single electron coherence function given by Eq. (11) is then obvious. But what would be the analogous of photo-detection for electrons? The idea is simply to extract an electron from the conductor we want to probe and to amplify the corresponding charge deposited into the detector. Naturally, the stage corresponding to photo-ionization is simply tunneling of electrons from the conductor into the detector which could for example be an STM tip or a nearby dot. Of course this approach does not take into account the electrostatic coupling between the conductor and the detector. Assuming a pointlike detector located at position \( x \), the average tunneling current from the conductor to the detector contains two contributions arising from electron transmitted from the conductor to the reservoir and vice versa:

\[
I_D(t) = \int_0^t \left( G_p^{(e)}(x, \tau; x, \tau') K_a(\tau, \tau') - G_p^{(h)}(x, \tau; x, \tau') K_e(\tau, \tau') \right) d\tau d\tau' \tag{14}
\]

In this expression, \( K_a \) and \( K_e \) characterize the detector and respectively account for available single electron and hole states within the reservoir and for the eventual energy filtering of the detector. Such a detection scheme.
has been recently implemented experimentally to study electron relaxation in quantum Hall edge channels in these experiments, a quantum dot is used to filter energies. This corresponds to a narrow-band detection in the quantum optics language.

C. Representations of single electron coherence

1. The time and frequency domains

Since measurements are usually performed locally, let us consider the single electron coherence function at a given position $x$: $G^{(e)}_{\rho} (t,t') = g^{(e)}_{\rho}(x,t; x,t')$. In the time domain, the diagonal $G^{(e)}_{\rho} (t,t)$ is nothing but the average electronic density at time $t$ and position $x$. Consequently, in chiral edge channels with Fermi velocity $v_F$, the excess current with respect to the chemical potential $\mu$ is

$$\langle i(x,t) \rangle = -e v_F \Delta G^{(e)}_{\rho}(t,t).$$

The off diagonal ($t \neq t'$) excess single electron coherence $\Delta G^{(e)}_{\rho}(t,t')$ is complex. Introducing $\tilde{t} = (t + t')/2$ and $\tau = t - t'$, the decay of $|\Delta G^{(e)}_{\rho}(t,t')|$ with increasing $\tau$ defines the coherence time of the source at time $t$. So this representation is indeed appropriate to discuss the coherence time as well as to discuss time dependence. But it is not well suited for understanding the nature (electron or hole) of the excitations emitted by the source since it is encoded in the $t - t'$ dependence of the phase of $\Delta G^{(e)}_{\rho}(t,t')$.

Going to the frequency domain gives access to the nature of excitations. The single electron coherence in the frequency domain is defined as the double Fourier transform:

$$\tilde{G}^{(e)}_{\rho}(\omega_+,\omega_-) = \int G^{(e)}_{\rho}(t,t') e^{i(\omega_+ t - \omega_- t')} dt dt'.$$

As shown on Fig. 1, the Fourier plane can then be divided into four quadrants. The (e) or electron quadrant defined by $\omega_+ > 0$ and $\omega_- > 0$ contains the contribution of excitations that correspond to single particle levels having positive energies with respect to the chemical potential $\mu = 0$. The (h) or hole quadrant defined by $\omega_+ < 0$ and $\omega_- < 0$ contains the contribution of excitations that correspond to single particle levels with negative energies with respect to the chemical potential $\mu = 0$. Finally, the (e/h) or electron/hole quadrants are defined by $\omega_+ \omega_- \leq 0$ correspond to the coherence between electronic and hole excitations. These coherences are produced in the presence of a superposition of electron and hole excitations. At this point, it is worth mentioning that the electronic or hole nature of excitations is defined with respect to a given chemical potential, here chosen as $\mu = 0$. With respect to another chemical potential, excitations would be categorized differently.

Using the Fourier decomposition of the single electron creation and destruction operators, the single electron coherence in the frequency domain satisfies

$$\frac{v_F}{2\pi} \tilde{G}^{(e)}_{\rho}(\omega_+,\omega_-) = \langle c^\dagger(\omega_-) c(\omega_+) \rangle_{\rho}$$

where $c(\omega)$ and $c^\dagger(\omega)$ respectively denote the fermion destruction and creation operators at position $x$ (see Appendix A). The electron occupation number can thus be recovered from the diagonal in the frequency domain. In particular, an equilibrium state corresponds to a singular single electron coherence in the frequency domain:

$$\tilde{G}^{(e)}_{\rho}(\omega_+ - \omega_-) = \frac{2\pi}{v_F} \delta(\omega_+ - \omega_-) f_{\mu,T} \left( \frac{\omega_+ + \omega_-}{2} \right).$$

A convenient way to visualize the $(\omega_+ - \omega_-)$ plane uses $\omega = (\omega_+ + \omega_-)/2$ and $\Omega = \omega_+ - \omega_-$. The fact that, at zero temperature, the Fermi sea coherence is localized in the frequency domain leads to Cauchy-Schwarz inequalities discussed in Appendix A. They imply that, at zero temperature, a source that does not emit any excess hole excitation (resp. electronic excitation) has non vanishing single electron coherence only within the (e) (resp. (h)) quadrant.

To summarize, the frequency domain representation of the single electron coherence is clearly well suited to visualizing the nature of excitations with respect to a given chemical potential. But recovering time dependence aspect is more difficult since it is encoded in the phase dependence of the off diagonal terms $\tilde{G}^{(e)}_{\rho}(\omega_+ - \omega_-)$ for $\omega_+ \neq \omega_-$. Since it is difficult to detect a single electron in one shot on a sub nanosecond time scale, it may be easier to access single electron coherence in the frequency domain than in the time domain although a protocol has recently been proposed using Mach-Zehnder interferometry. We shall come back to this question in section IV.

However, working with sources that are able to inject a single to few electrons and holes strongly rises the need for a representation of the single electron coherence giving access to real time phenomenon as well as to the nature of excitations. Such a time/frequency representation has been known for a long time in quantum statistical physics: it is the Wigner function which we will now discuss for single electron coherence.

2. The Wigner function

In order to capture both the $\tilde{t} = (t + t')/2$ dependence of single electron coherence and the nature of excitations, we define the Wigner distribution function as:

$$W^{(e)}_{\rho,x} (\tilde{t},\omega) = \int v_F \tilde{G}^{(e)}_{\rho,x}(\tilde{t} + \tau/2, \tilde{t} - \tau/2) e^{i\omega \tau} d\tau$$

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which is dimensionless due to the presence of the velocity $v_F$. In a similar way, one would define the Wigner function for the hole excitations by substituting $g^{(h)}_{\rho,z}$ into (19). Note that due to the hermiticity properties (A1) of the single electron and single hole coherences, these Wigner functions are real.

Originally, the Wigner function was introduced to provide a bridge between classical and quantum mechanics for a single particle (20). In classical mechanics, a particle has a well defined position and momentum and an ensemble of such particles is represented by a probability distribution over phase space. For a quantum particle, the uncertainty principle prevents the particle to be perfectly localized. Still, the Wigner distribution function is the proper generalization of the probability distribution: it is a real distribution over phase space whose integration over position (resp. momentum) gives the probability distribution of the momentum (resp. position). The Wigner function is thus normalized but contrarily to the classical probability distribution, it is not always positive.

Significant differences are expected for the Wigner function associated with single electron coherence. In the stationary case, the Wigner function defined by Eq. (19) is nothing but the time independent electronic distribution function at position $x$:

$$W^{(e)}_{\rho,z}(\bar{t}, \omega) = f_e(\omega, x).$$  \hfill (20)

For a Wigner function to be interpreted as a time dependent electronic distribution function, it has to satisfy $0 \leq W^{(e)}_{\rho,z}(\bar{t}, \omega) \leq 1$. The positivity condition is needed in order to have an interpretation as a probability density. In a chiral system, the electronic and hole Wigner distributions are related by

$$W^{(h)}_{\rho,z}(\bar{t}, \omega) = 1 - W^{(c)}_{\rho,z}(\bar{t}, -\omega)$$  \hfill (21)

where the minus sign in front of $\omega$ in the r.h.s reflects the fact that $W^{(h)}_{\rho,z}(\bar{t}, \omega)$ is the Wigner function for hole excitations at time $\bar{t}$ and energy $\hbar \omega$. Consequently, the upper bound on $W^{(c)}_{\rho,z}(\bar{t}, \omega)$ is also required for a probabilistic interpretation of the hole Wigner distribution function. For a probability distribution in the $(\bar{t}, \omega)$ plane, this upper bound expresses the Pauli principle.

In full generality, the Wigner function for conduction electrons within a metal contains the Fermi sea: $W^{(e)}_{\rho,z}(\bar{t}, \omega) \to 1$ for sufficiently negative $\omega$ and $W^{(e)}_{\rho,z}(\bar{t}, \omega) \to 0$ at high enough energy. In between, the Wigner function can get various values, sometimes negative or strictly larger than one. In such cases, it cannot be interpreted as a probability distribution. We shall come back on this issue while discussing explicit examples in the next section.

Note that a $T$-periodic system generates a single electron distribution invariant in $(t, t') \mapsto (t+T, t'+T)$ which can thus be decomposed as a Fourier transform with respect to $\tau = t - t'$ and a Fourier series with respect to $\bar{t} = (t + t')/2$:

$$G^{(e)}_{\rho}(t, t') = \sum_{n = -\infty}^{+\infty} e^{-2\pi i n f \bar{t}} \int G^{(e)}_{\rho,n}(\omega) e^{-i \omega \tau} \frac{d\omega}{2\pi}$$  \hfill (22)

where $f = 1/T$ is the driving frequency. This single electron coherence leads to a $T$-periodic Wigner function:

$$W^{(e)}_{\rho}(t+T, \omega) = W^{(e)}_{\rho}(t, \omega)$$

whose expression as a Fourier series reads:

$$W^{(e)}_{\rho}(\bar{t}, \omega) = \sum_{n = -\infty}^{+\infty} v_p G^{(e)}_{\rho,n}(\omega) e^{-2\pi i n f \bar{t}}$$  \hfill (23)

in which the position $x$ has been dropped out for simplicity. As we shall see, this expression is of great use in numerical evaluations of the Wigner function in the framework of Floquet scattering theory.

Integrating the Wigner function of a quantum particle over position or momentum gives the probability distribution of the conjugated variable. Here, partial integrals of the Wigner function give access to physically relevant quantities such as the average excess current with respect to a chemical potential $\mu$:

$$\langle i(x, \bar{t}) \rangle_{\rho} = -e \int \Delta W^{(e)}_{\rho,z}(\bar{t}, \omega) \frac{d\omega}{2\pi}$$  \hfill (24)

where $\Delta W^{(e)}_{\rho,z}(\bar{t}, \omega)$ denotes the excess Wigner function with respect to the Wigner function $\Theta(\mu/\hbar - \omega)$ of the
Fermi sea at chemical potential $\mu$. Note that measuring this quantity requires broadband high frequency measurements\footnote{[14,15]}. In the same way, averaging over time gives access to the electronic distribution function at position $x$:

$$f_e(\omega, x) = \frac{1}{T} \int_{-T/2}^{T/2} W^{(e)}(t, \omega) dt.$$  \hspace{1cm} (25)

This quantity can be measured using dc current measurements through an adjustable energy filter\footnote{[23]}. Note that the Wigner function for electrons in a conductor does not satisfy the normalization condition satisfied by the Wigner function of a single quantum particle: integrating the excess Wigner function $\Delta W^{(e)}(t, \omega)$ with respect to a given chemical potential in the $(t, \omega)$ plane gives the total excess charge in $-e$ units.

Finally, let us point out that as of today, there is no way to directly access the value of the Wigner function at a given point $(t, \omega)$ in a quantum conductor. By contrast, such direct measurements of the Wigner function in cavity QED are possible and have indeed been performed\footnote{[24]}. They rely on the measurement of the parity of the photon number, the equivalent of which is, as far as we know, not accessible in electron quantum optics. The problem of reconstructing the Wigner function for electrons in quantum Hall edge channels through interferometry experiments will be discussed in section \hspace{1cm} IV.

III. EXAMPLES

Let us now discuss several important examples of single electron coherences emitted by various electronic sources. We shall first consider the single electron coherence emitted by a driven Ohmic contact.

A. Voltage drives

1. General properties

The single electron coherence emitted by an ideal Ohmic contact driven by a time dependent voltage $V(t)$ is:

$$G_V^{(e)} \left( t + \frac{\tau}{2}, -t - \frac{\tau}{2} \right) = \exp \left( \frac{i e}{\hbar} \int_{-t - \frac{\tau}{2}}^{t + \frac{\tau}{2}} V(\tau') d\tau' \right) G_0^{(e)}(\tau).$$  \hspace{1cm} (26)

In the case of a $T$-periodic potential, the single electron coherence can be obtained in terms of the photo assisted transition amplitudes $p_n[V_{ac}]$ associated with the a.c. component of the drive\footnote{[23]}. The resulting Wigner function is then expressed as

$$W^{(e)}(t, \omega) = \sum_{n_+, n_- \in \mathbb{Z}} p_{n_+}[V_{ac}]p_{n_-}[V_{ac}] e^{2\pi i (n_- - n_+) \omega f t} f_{\bar{\mu}}(\omega - \pi(n_+ + n_-) f)$$  \hspace{1cm} (27)

where $\bar{\mu} = \mu - eV_{dc}$ denotes the chemical potential shifted by the d.c. component of the drive. The electronic occupation number is obtained by averaging Eq. \hspace{1cm} (27) over the time $t$. This selects terms with $n_+ = n_-$, thus leading to

$$f_e(\omega) = \sum_{n \in \mathbb{Z}} |p_n[V_{ac}]|^2 f_{\bar{\mu}}(\omega - 2\pi n f).$$  \hspace{1cm} (28)

This expression clearly show the interpretation of $|p_n[V_{ac}]|^2$ as the probability of photo assisted transition. However, the Wigner function contains more information than the photo assisted probabilities. These are the terms with $n_+ \neq n_-$ in Eq. \hspace{1cm} (27) which are sensitive to the phases of the photo assisted transition amplitudes. They play a crucial role in ensuring that the current noise of the driven channel is equal to the current noise at thermal equilibrium as is expected for a coherent states of the edge magnetoplasmon mode\footnote{[26]}.

In the case of a sinusoidal drive $V_{ac}(t) = V_0 \cos(2\pi f t)$ at frequency $f$, $p_n[V_{ac}] = J_n(\omega V_0/\hbar)$ where $J_n$ denotes the Bessel function of order $n$. The Wigner function is then obtained as:

$$W^{(e)}(t, \omega) = \sum_{n \in \mathbb{Z}} J_n \left( \frac{2V_0}{\pi f} \cos(2\pi f t) \right) e^{\beta_{\text{el}}(\hbar \omega + \pi n f)} + 1$$  \hspace{1cm} (29)

where $\beta_{\text{el}} = 1/k_B T_{\text{el}}$. At zero temperatures, this Wigner function exhibits singularities in the variable $\omega$ each time $\hbar \omega$ is a multiple of $\hbar f/2$. Quantum effects are expected to be dominant in the regime of low temperature and low photon number. On the contrary, for large photon number and high temperature, quantum features are expected to be small. Let us now turn to these two limiting regimes of small $eV_0 \ll \hbar f$ and large amplitudes $eV_0 \gg \hbar f$.

2. Small amplitudes

The regime of small amplitudes is most suitably discussed in the sinusoidal case. Then, $\hbar f$ represents the energy of photons absorbed or emitted by the electron gas and the condition $eV_0 \ll \hbar f$ expresses that the physics is dominated by single photon processes. In this regime, the first order contribution in $eV_0/\hbar f$ to the Wigner function is:

$$\frac{\partial W^{(e)}(t, \omega)}{\partial (eV_0/\hbar f)} \bigg|_{V_0 = 0} = F_{\mu,T_{\text{el}}} (\omega) \cos(2\pi f t)$$  \hspace{1cm} (30)

where $F_{\mu,T_{\text{el}}} (\omega) = f_{\mu,T_{\text{el}}} (\omega - \pi f) - f_{\mu,T_{\text{el}}} (\omega + \pi f)$ is, at zero temperature, the characteristic function of the interval $[\hbar^{-1} \mu - \pi f, \hbar^{-1} \mu + \pi f]$. Note that this contribution does not affect the electronic occupation number $f_e(\omega)$ but contributes to the average current by $(e^2/\hbar)V(t)$ as expected from the linear response of a chiral edge channel. However, as we shall see in section \hspace{1cm} IV, it is instrumental to the recently proposed single electron tomography protocol\footnote{[23]}.
The first non trivial contribution to the electron distribution function arises at second order in $eV_0$ and corresponds to processes in which a single photon is absorbed to promote one electron from the Fermi sea above the Fermi level:

$$\frac{\partial^2 W^{(e)}(t, \omega)}{\partial^2 (eV_0/\hbar f)} \bigg|_{V_0=0} = g_{\mu, T}(\omega) \cos^2 (2\pi ft)$$  \hspace{1cm} (31)

where $g_{\mu, T}(\omega) = f_{\mu, T_0}(\omega + 2\pi f) + f_{\mu, T_0}(\omega - 2\pi f) - 2 f_{\mu, T_0}(\omega)$ is, at zero temperature, equal to 1 for $\mu < \hbar \omega \leq \mu + hf$, to to $-1$ when $\mu - hf \leq \hbar \omega < \mu$ and vanishes everywhere else.

At higher amplitudes, multiphotonic processes contribute. At zero temperature, the Wigner function exhibits singularities for $\omega$ multiple of $\pi f$ but only the even multiples contribute to singularities in the occupation number as expected from the theory of photon-assisted noise.

3. Large amplitudes

Let us now turn to the opposite regime of large voltage amplitudes where the physics is dominated by multiphotonic processes. In this case, let us make the discussion slightly more general by considering a smooth time dependent periodic voltage drive that varies on a scale $\Delta V = \max(V(t)) - \min(V(t))$ over a time scale $T = 1/f$ where $f$ denotes the driving frequency.

To discuss the features of the Wigner function on energy scales of the order $e\Delta V$, we have to consider $\mathcal{G}^{(e)}(t, t+\tau)$ over time scales such that $|\tau| \ll 1$. We can then assume that $V(t')$ is constant and equal to $V(t)$ between $t-\tau/2$ and $t+\tau/2$. This immediately leads to an adiabatic expression for the Wigner function as a time-dependent Fermi distribution:

$$W^{(e)}(t, \omega) \approx f_{\mu, T}(\omega + eV(t)/\hbar)$$  \hspace{1cm} (32)

corresponding to a time dependent chemical potential $\mu(t) = \mu - eV(t)$. However, at zero temperature, quantum interference effects lead to quantum corrections to this expression. They arise from the time dependence of the voltage drive over the interval $[t-\tau/2, t+\tau/2]$. Considering a time $t$ such that $V''(t) \neq 0$, the $\omega$ dependence of $W^{(e)}(t, \omega)$ exhibits a Fresnel-like diffraction pattern. At zero temperature, assuming that $V''(t) > 0$, we find

$$W^{(e)}(t, \omega) \simeq \int_{\omega+eV(t)/\hbar}^{\omega+eV(t)/\hbar} \text{Ai}(x) \, dx$$  \hspace{1cm} (33)

where $\text{Ai}(x)$ denotes the Airy function and

$$\delta \omega(t) = \frac{1}{2} \left( \frac{e|V''(t)|}{\hbar} \right)^{1/3}.$$  \hspace{1cm} (34)

The Wigner function thus exhibits quantum ripples on the Fermi plateau around the value one and an exponential decay at high energy. These ripples are clearly visible on Fig. 2(a) presenting the Wigner function generated by a sinusoidal drive for $eV_0/\hbar f = 20$ at zero temperature. On this figure, they appear as oscillations on the top part of the waves of the driven Fermi sea (the semi classical potential is $\mu(t) = \mu - eV(t)$ and therefore $V''(t) > 0$ corresponds to $\mu''(t) < 0$). Due to them, the Wigner function can be greater than one. For $V''(t) < 0$, a similar computation shows that

$$W^{(e)}(t, \omega) \simeq \int_{-\infty}^{-\omega+eV(t)/\hbar} \text{Ai}(x) \, dx.$$  \hspace{1cm} (35)

On Fig. 2(a), the corresponding ripples correspond to the oscillations in the bottom of the wave part of the driven Fermi sea. They lead to negative values of the Wigner function.

The energy scale $\hbar \delta \omega(t)$ associated with these ripples compares to $\hbar f$ through

$$\frac{\hbar \delta \omega(t)}{\hbar f} = 1 \left( \frac{2\pi e T^2 |V''(t)|}{\hbar f} \right)^{1/3}.$$  \hspace{1cm} (36)

For a moderately varying voltage such as a sinusoidal drive, $T^2 V''(t)$ is of the order of the total drive amplitude $\Delta V$. Therefore in the case of a large amplitude $e\Delta V \gg \hbar f$, the scale $\hbar \delta \omega(t)$ is significantly larger than $\hbar f$ as can already be seen on Fig. 2(a).

When $\hbar \delta \omega(t) \gg \hbar f$, this energy scale gives the temperature above which the quantum ripples disappear. A convenient way to understand finite temperature effects is to remember that thermal fluctuations will smooth the Wigner function over an energy scale equal to $k_B T_{\text{el}}$. The quantum ripples are thus expected to disappear at finite temperature, when $k_B T_{\text{el}} \gtrsim \hbar \delta \omega(t)$. In this regime, the adiabatic result given by Eq. (32) is recovered as can be seen from Fig. 2(b).

Finally, one might consider a voltage drive that is strongly peaked around specific times. In this case, the local energy scale $\hbar \delta \omega(t)$ might become of the order of $eV(t)$ itself and the Wigner function is locally dominated by these interferences effect. In such a situation, one would not be able to see the overall picture of the Fermi step at chemical potential $\tilde{\mu}(t) = \mu - eV(t)$. As we shall see in section 3.1B.2 Lorentzian pulses realize such a situation.

B. Single electron sources

1. The mesoscopic capacitor

An on demand single electron source can be realized using a mesoscopic capacitor operated in the non linear regime. This source has been demonstrated in 2007 by G. Fève et al. Properly operated, it emits a single electron and a single hole excitation per period at GHz repetition rate. One of its main advantages is that these excitations are energy resolved and that their average energies and width can be tuned to some extent.
A description of this source can be achieved within a non interacting electron approximation using the framework of the Floquet scattering theory. This approach has been developed by Moskalets and Böttiker to describe quantum mechanical pumping in mesoscopic conductors\textsuperscript{29} and since then has been applied to a variety of systems among which the mesoscopic capacitor\textsuperscript{52}. In particular it has been used to predict the low and finite frequency noise emitted by a periodically driven mesoscopic conductor\textsuperscript{21,25,53}. These theoretical results have been compared to experimental results on finite frequency noise of the source\textsuperscript{53}. The Floquet scattering amplitude for electrons propagating through a driven quantum conductor is:

\[ S_{W}(t, t') = \exp \left( \frac{ie}{\hbar} \int_{t'}^{t} V_{d}(\tau) d\tau \right) S_{0}(t - t') \]  \hspace{1cm} (37)

where \( V_{d}(\tau) \) is the periodic a.c. driving voltage applied to the dot and \( S_{0}(t - t') \) is the scattering amplitude across the undriven conductor, expressed in real time. Knowing the Floquet scattering amplitude (37) leads to the real time single electron coherence emitted by the driven mesoscopic conductor\textsuperscript{21} in terms of the Floquet scattering amplitudes \( S_{n}(\omega) \) defined as:

\[ c_{\text{out}}(\omega) = \sum_{n=-\infty}^{+\infty} S_{n}(\omega) c_{\text{in}}(\omega + 2\pi nf) \]  \hspace{1cm} (38)

A Fourier transform then leads to the general expression for the Wigner function emitted by a source described within the framework of Floquet scattering theory:

\[ W^{(e)}(t, \omega) = \sum_{n, k=-\infty}^{+\infty} S_{k}(\omega + \pi nf) S_{n+k}(\omega - \pi nf)^{*} \times f_{\mu}(\omega + 2\pi f(k + n/2)) e^{-2\pi i nf} \]  \hspace{1cm} (39)

This expression can then be used to compute either analytically or numerically the Wigner function within Floquet scattering theory.

Let us now discuss the numerical results for the mesoscopic capacitor driven by a square voltage: \( V(t) = V_{d} \) for \( 0 < t \leq T/2 \) and \( V(t) = -V_{d} \) for \( T/2 < t \leq T/2 \). The results presented here have been obtained for realistic values of the parameters of the mesoscopic capacitor. We consider \( \hbar f/\Delta = 0.06, k_{B}T_{d}/\Delta = 0.01 \) and \( eV_{d} = \Delta/2 \) so that the voltage step corresponds to the level spacing of the dot. These results have been obtained by evaluating the single electron coherence numerically using a specific form for the Floquet scattering theory already used to interpret the experimental data\textsuperscript{53}.

\[ S_{0}(\omega) = \frac{\sqrt{1 - D - e^{2\pi i h(\omega - \omega_{0})}/\Delta}}{1 - \sqrt{1 - D} e^{2\pi i h(\omega - \omega_{0})}/\Delta}. \]  \hspace{1cm} (40)

Here \( \Delta \) denotes the level spacing within the dot and \( D \) the dot to lead transmission controlling the tunneling
between the dot and the chiral edge channel. Another control is the position $\hbar \omega_0$ of the energy levels of the dot which can be controlled by applying a d.c. voltage to its top gate. Note that electron/hole symmetry is realized when $\hbar \omega_0$ is an integer multiple of $\Delta$. Depending on $D$, various behaviors are expected.

At $D = 1$, electrons go around the dot only once and feel the effect of the voltage drive during a very short time $\tau_0$ which is the time of flight around the dot. As shown on Fig. 3 excitation emission is concentrated at the times where the voltage drive changes. This is expected since it is precisely when the drive changes that the electrons going through the dot feel a sudden change of the electrical potential. Between two changes, the dot acts as a purely elastic scatterer and therefore we expect to see the emission of electrons as if they were coming straight out of the reservoir. Consequently, the average current should be a succession of current pulses of duration $\tau_0$ corresponding to the sudden changes of the voltage drive.

From an edge magnetoplasmon perspective, the state generated by the mesoscopic capacitor at $D = 1$ is a coherent state. It is therefore a coherent superposition of many electron/hole pairs and therefore we expect the single electron coherence to have an important contribution in the $(e/h)$ quadrants. We then expect that excitations are created close to the Fermi level which is confirmed by Fig. 3. We then expect the excitations to have an important contribution to the sudden changes of the voltage drive.

When $D$ decreases, the density of states within the dot becomes more and more textured. Consequently we expect the source to emit electron and hole excitations that are better and better defined and time shifted by a half period. Figure 4 confirms this physical picture: it clearly shows the succession of electronic and hole excitations emitted by the mesoscopic capacitor near its optimal point.

The shape of these pulses can indeed be understood very simply by considering Lorentzian wave packets in energy, truncated to energies above the Fermi level:

$$\bar{\varphi}_e(\omega) = \frac{N_e \Theta(\omega)}{\omega - \omega_e - i\gamma_e/2}$$

where $N_e$ ensures normalization and $\gamma_e$ denotes the electron escape rate from the quantum dot. For $|\hbar \omega_0| < \Delta$, we expect the electron to be emitted by the mesoscopic capacitor at energy $\hbar \omega_e = \Delta/2 - \hbar \omega_0$ whereas the hole is expected at energy $\hbar \omega_h = \hbar \omega_0 - \Delta/2$. The electronic escape rate is then given by $\gamma_e = 2D\Delta/\hbar(2 - D)$.

To understand the limit $\gamma_e/\omega_e \ll 1$, let us first neglect the truncation of the wave packet. Within this approximation, the associated excess Wigner function is given by:

$$\Delta W^{(e)}(t, \omega) \sim 2\gamma_e \Theta(t) \frac{\sin(2(\omega - \omega_e)t)}{\omega - \omega_e} e^{-\gamma_e t}.$$  \hspace{1cm} (42)

This expression leads to a triangular shape in the $(\omega - \omega_e)/\gamma_e, \gamma_e t)$ variables which reflects the Heisenberg time/energy uncertainty principle: right after its emission, the spreading in energy is large and then becomes...
sharper. As expected the duration of the excitation is governed by $\tau_e = 1/\gamma_e$. This Wigner function also exhibits some negative values appearing as dark quantum ripples on Fig. 3(b) but apart from these, it is mainly a positive bump. Taking into account the truncation of the Lorentzian in Eq. (11) alters this image at low energies: it leads to the vanishing of the single electron coherence outside the $(e)$ quadrant as shown on Fig. 3(a). Consequently, for $\omega \lesssim \omega_e^2/2$, the Fourier transform of $\Delta W^{(e)}(t, \omega)$ with respect to $t$ is much smaller for $\omega \lesssim \omega_e^2/2$ as can be seen on Fig. 3(b). The residual interference pattern shows broader and fainter fringes as $\omega$ goes to zero. These time oscillations at fixed $\omega$ arise from the residual coherence in the $(e)$ quadrant ($\omega_{e} > 0$) of the Fourier plane and $\omega_{+} = \omega_{-} = 2\omega$.

This explains the band seen for $|\omega| \lesssim \Delta/2$ on Fig. 4 for $|\omega| < |\omega_e|/2$: the positive bump gives way to a fainter pattern of interferences fringes. This truncation effect, which can be interpreted as an expression of the Pauli principle, is of course sharper when $\gamma_{e} \ll \omega_{e}$. Its consequences in the time domain are discussed in Appendix [1]. The same remarks apply to hole excitations which appear as dips in the Fermi sea.

With $D$ going down closer to zero, the escape times of the electron and hole increase as one enters the shot noise regime of the source. When these escape times become comparable to the half period $T/2$, the electron and the hole do not have the time to escape before the voltage drive changes again. In this case, electron/hole coherences are expected to reappear since the capacitor generates a superposition of various electron/hole pair excitations of the form $|F\rangle + \psi^\dagger [\varphi_{e}]\psi[\varphi_{h}]|F\rangle$. The $(e/h)$ coherences for such a superposition are of the form $\varphi_{e}(x)\varphi_{h}^*(y)$ and $\varphi_{h}(x)\varphi_{e}^*(y)$. In the Wigner distribution function, they appear at $\omega \simeq (\omega_{e} + \omega_{h})/2$ and oscillate in time at angular velocity $\omega_{e} - \omega_{h}$ (remember that $\omega_{e} > 0$ for an electronic excitation and $\omega_{h} < 0$ for a hole excitation). We thus expect fast oscillations in time at mid position between the electron and hole energies.

To confirm this picture, Fig. 5 depicts the Wigner function emitted by the mesoscopic capacitor in the low $D$ regime with a shift in the energy levels of the dot ($\omega_{0} \neq 0$). As expected, it exhibits fast oscillations precisely at the mid position between the electron and hole peaks. Note also the way the triangular shape of the hole excitation dips are truncated close to the Fermi surface due to the Pauli exclusion principle. Note that such oscillations were also visible on Fig. 4 but they are exactly at the Fermi energy since for this figure, the capacitor is assumed to be operated at an electron/hole symmetric point.

2. Lorentzian pulses

More recently, another source of single to few electronic excitations in a 2DEG at zero magnetic field has been developed[13,14] based on ideas by Levitov, Ivanov, Lee and Lesovik[15,16]. The idea is to use Lorentzian voltage pulses carrying a quantized charge in units of $e$ to generate purely electronic excitations. Experimentally, minimizing the partition noise of these excitations is used to minimize the production of spurious electron/hole pairs[13,15]. Contrary to the excitations studied in the previous paragraph, these electron pulses are time resolved instead of being energy resolved. Moreover, excitations carrying more than an elementary charge under the form of a coherent wave packet of $n$ electrons can be generated. More precisely, such an excitation is a Slater determinant built from the $1 \leq k \leq n$ mutually orthogonal electronic wavefunctions given by

$$\varphi_{k}^{(\tau_{0})}(\omega) = \sqrt{2\tau_{0}} \Theta(\omega)e^{-\omega\tau_{0}}L_{k-1}(2\omega\tau_{0}) \quad (43)$$

where $\tau_{0}$ denotes the duration of the Lorentzian pulse and $L_{k}$ is the $k$th Laguerre polynomial[15]. Their single electron coherence can be computed analytically in the case of a single pulse and also from the Floquet scattering theory in the case of a periodic train of pulses[10].

For a single Levitov excitation of charge $-ne$ ($n \geq 1$) and width $\tau_{0}$, the associated excess Wigner function is given by:

$$\Delta W^{(e)}(t, \omega) = \sqrt{4\pi} \Theta(\omega)e^{-2\omega\tau_{0}} \sum_{k=0}^{n-1} \sum_{l=0}^{k} \left(\frac{2e\tau_{0}}{\sqrt{\omega\tau}}\right)^{2l+1} L_{k-1}^{(2l)}(4\omega\tau_{0}) J_{l+\frac{1}{2}}(2\omega t) \quad (44)$$

where $J_{l+\frac{1}{2}}$ denotes the Bessel function of order $l + 1/2$ and $L_{n}^{(m)}$ is the generalized Laguerre polynomial[15] of order $n$. Note that for $n = 1$, the wave function $\varphi_{0}^{(\tau_{0})}$ being exponential in energy, one expects the Wigner distribution function to have a similar form as Eq. (42) up to the exchange of $t$ and $\omega$.

However, in order to discuss the physics of these Lorentzian pulses, it is more convenient to consider a periodic train of excitations since, in this case, non integer values of the charge can be considered[13,14]. Using Eq. (40), we have plotted the Wigner function of a train of Lorentzian pulses $f_{\tau_{0}} = 0.05$ at zero temperature and for increasing values of $\alpha$ which denotes the average charge per pulse in units of $-e$.

Figure 7 show trains of time-resolved excitations that are not separated in energy from the Fermi level (compare with Fig. 4) as expected. Time-energy uncertainty leads to the spreading of the excitation in time close to the Fermi surface. Increasing the amplitude of the drive or equivalently $\alpha$, we see that these excitations grow in the energy direction. We also see the appearance of maxima that are indeed the quantum ripples discussed in section 11 A 3 But in the present case, they are more prominent since we are not in an adiabatic limit. More interesting, we see that the Fermi sea is left pristine each time $\alpha$ is an integer confirming that for positive integer $\alpha$, the source is emitting a strain of purely electronic excitations.
On the contrary, when $\alpha > 0$ is not an integer, hole excitations are expected since in this case, each pulse should be understood as a collective excitation. Comparing $\alpha = 1/2, 3/2$ and $5/2$, we see that the hole contribution diminishes: this is not surprising in view of the Pauli principle since one adds a half electronic excitation on top of a Slater determinant in which more and more electronic states close to the Fermi level are populated.

IV. INTERFEROMETRY

Let us now discuss interferometry experiments, starting first with single particle interferences (Mach-Zehnder interferometry) and then discussing two particle interferometry experiments based on the Hanbury Brown Twiss
arms of the MZI are $\tau$ each arm, thus leading to respective times of flights $\tau_1$ and $\tau_2$ enclosing a magnetic flux $\Phi_B$ and recombine at a beam splitter $B$. An electronic source (S) is located onto the incoming channel $1$. It then gives rise to an excess Wigner function in the outcoming channel $1$. This is the Wigner function $W^{(e)}_{1,\text{out}}(t,\omega)$ given by:
\[
W^{(e)}_{1,\text{out}}(t,\omega) = \sum_{j=1,2} M_{j,j} W^{(e)}_{1,\text{in}}(t-\tau_j) + 2 |M_{1,2}| \cos(\omega \tau_{12} + \phi) \Delta W^{(e)}_{1,\text{in}}(t-\tau,\omega)
\]
where $\tau_{12} = \tau_1 - \tau_2$ and $\tau = (\tau_1 + \tau_2)/2$ respectively denote the difference and the average of the two times of flights and $\phi = \text{Arg}(M_{12}) + 2\pi \phi_B$ accounts for the magnetic phase. Note that this is the Wigner function version of the discussion of the ideal MZI interferometer by Haack et al.\cite{23}. The coefficients $M_{i,j}$ are associated with the beam splitters and, in the present case, are given by:
\[
M_{11} = (1 - T_A)(1 - T_B), \quad M_{22} = T_A T_B, \quad M_{12} = \sqrt{T_A (1 - T_A)} \sqrt{T_B (1 - T_B)}.
\]
In Eq. (46), the first two terms represent the contribution of electrons traveling classically along the two arms of the interferometer whereas the last term represents quantum interference effects associated with propagation along both arms. This quantum term presents oscillations in $\omega$ taking place over a scale $2\pi/|\tau_{12}|$. Changing the magnetic flux through the MZI interferometer sweeps the phase of these quantum oscillations. These quantum fringes in the Wigner function are indeed characteristic of quantum superpositions. They have been recently observed for Schrödinger cat states in cavity QED\cite{22} and they are now routinely seen in circuit QED experiments\cite{59–61}.

Figure 9(a) and 9(b) present the Wigner functions obtained by sending through an ideal MZI an energy resolved single electron wavepacket given by Eq. (11). The two time shifted wave packets are clearly visible. We have chosen the time of flight difference $\tau_{12}$ larger than the duration of each wave packet so that the quantum fringes are clearly visible. In this regime, the two classical components of $\Delta W^{(e)}_{1,\text{out}}(t,\omega)$ are clearly separated from the quantum interference features. Due to the quantum interference contribution, the Wigner function exhibits pronounced regions with negative values thus preventing it from any quasi-classical interpretation.

Although they could be observed in a full quantum tomography of the single electron coherence\cite{21}, these fringes could also be observed through their impact on the marginal distributions of the Wigner function, that is to say the average current (see Eq. (24)) and the electron distribution function (see Eq. (25)).

In order to observe the effect on the average current, the typical scale $2\pi/|\tau_{12}|$ of quantum oscillations in the Wigner function must be comparable or larger than the energy spread of the excess coherence of the source so that these oscillations are not averaged to zero when integrating over $\omega$. This condition expresses that quantum interferences can be observed in the time domain only when the difference of times of flights $\tau_{12}$ is comparable or smaller than the coherence time of the source\cite{21}. This is still slightly the case on Fig. 9(b) since we have considered $\tau_{12} = 2 \tau_0$. The average current then shows two overlapping peaks whereas the electron distribution exhibit mainly one peak with some light structure. Changing the magnetic flux by half a quantum decreases the size of the second peak of the average current and also decreases the size of the central peak of the electron distribution function.

On the other hand, in the case of a strongly unbalanced MZI, such as the one depicted on Fig. 9(b), interferences cannot be observed using a time resolved measurement.
The average current depicted on Fig. 9(b)-(iii) exhibits two peaks that correspond to the two classical contributions to the excess Wigner function. These two peaks do not change when the magnetic flow is varied. In this case, a frequency resolved detector able to restore the overlap between electronic wavepackets is more appropriate to reveal the interferences. As shown on Fig. 9(b), the quantum fringes of the Wigner function can be revealed by measuring the electron distribution function. This is the single electron version of the channeled spectrum observed in optical interferometers using white light.

In practice, observing these effects might be quite challenging due to decoherence within the interferometer itself; although a proper design of the sample leads to partial protection against decoherence.

Moreover, observing a channeled spectrum requires an unbalanced interferometer in which $|\tau_{12}|$ is greater than the inverse of the typical energy spreading of the electrons sent into the interferometer. Therefore, the best way to observe a channeled spectrum would be to use a non equilibrium distribution spreading over a broad energy range such as a double step generated by a QPC or the one generated by an ac sinusoidal voltage. A typical energy range of 100 to 500 µeV corresponds to frequencies from 24 to 120 GHz so that an imbalance of a few µm would be sufficient to observe a channeled spectrum. However, such an experiment would rely on a continuous stream of indistinguishable electrons. In order to observe a channeled spectrum with a single electron source, a larger imbalance is required and excitations that are spread in energy such as Lorentzian pulses with short duration (typically 10 ps) may be appropriate. An alternative approach is to use energy resolved excitations and change their energies to probe the interferences at the single electron level. But discussing the observability of these effects in a realistic situation would require computing the Wigner distribution function at the output of a Mach-Zehnder interferometer in the presence of interactions, which would go beyond the scope of the present paper.

B. HOM and HBT interferometry

1. Noise and correlations from coherences

Two particle interferences are at the heart of the Hanbury Brown Twiss (HBT) and Hong Ou Mandel (HOM) experiments recently demonstrated with single electron excitations emitted by on demand single electron source. They enable us to imprint information about the single electron coherences into the current noise and correlations issued from an electronic beam splitter of respective reflection and transmission probabilities $R$ and $T$ (see Figs. 10 and 11). This remark underlies the recently proposed single electron tomography protocol aimed at reconstructing an unknown single electron coherence from noise measurements. Predictions have been made for HOM noise signals within the

FIG. 9: Excess single electron coherence at the output of an ideal Mach-Zehnder interferometer for energy resolved wave-packets emitted at energy $\hbar \omega_e$ and with duration $\tau_e = \gamma_e^{-1}$ such that $\omega_e \tau_e = 10$ and two differences of time of flights $\tau_{12}$:

(a) Partly overlapping wavepackets: $\tau_{12} = 2 \tau_e$

(b) Non-overlapping wavepackets: $\tau_{12} = 9 \tau_e$

(i) $\Delta W_{1,\text{out}}^{(c)}(t, \omega)$ given by Eq. [16] for $\phi = 0$ as a function of $t/\tau_e$ and $\omega \tau_e$

(ii) $\delta f_e(\omega)$

(iii) $\langle i(t) \rangle$ in units of $-e/\tau_e$. 

The average current depicted on Fig. 9(b)-(iii) exhibits two peaks that correspond to the two classical contributions to the excess Wigner function. These two peaks do not change when the magnetic flow is varied. In this case, a frequency resolved detector able to restore the overlap between electronic wavepackets is more appropriate to reveal the interferences. As shown on Fig. 9(b), the quantum fringes of the Wigner function can be revealed by measuring the electron distribution function. This is the single electron version of the channeled spectrum observed in optical interferometers using white light.

In practice, observing these effects might be quite challenging due to decoherence within the interferometer itself; although a proper design of the sample leads to partial protection against decoherence. Moreover, observing a channeled spectrum requires an unbalanced interferometer in which $|\tau_{12}|$ is greater than the inverse of the typical energy spreading of the electrons sent into the interferometer. Therefore, the best way to observe a channeled spectrum would be to use a non equilibrium distribution spreading over a broad energy range such as a double step generated by a QPC or the one generated by an ac sinusoidal voltage. A typical energy range of 100 to 500 µeV corresponds to frequencies from 24 to 120 GHz so that an imbalance of a few µm would be sufficient to observe a channeled spectrum. However, such an experiment would rely on a continuous stream of indistinguishable electrons. In order to observe a channeled spectrum with a single electron source, a larger imbalance is required and excitations that are spread in energy such as Lorentzian pulses with short duration (typically 10 ps) may be appropriate. An alternative approach is to use energy resolved excitations and change their energies to probe the interferences at the single electron level. But discussing the observability of these effects in a realistic situation would require computing the Wigner distribution function at the output of a Mach-Zehnder interferometer in the presence of interactions, which would go beyond the scope of the present paper.

1. Noise and correlations from coherences

Two particle interferences are at the heart of the Hanbury Brown Twiss (HBT) and Hong Ou Mandel (HOM) experiments recently demonstrated with single electron excitations emitted by on demand single electron source. They enable us to imprint information about the single electron coherences into the current noise and correlations issued from an electronic beam splitter of respective reflection and transmission probabilities $R$ and $T$ (see Figs. 10 and 11). This remark underlies the recently proposed single electron tomography protocol aimed at reconstructing an unknown single electron coherence from noise measurements. Predictions have been made for HOM noise signals within the

FIG. 9: Excess single electron coherence at the output of an ideal Mach-Zehnder interferometer for energy resolved wave-packets emitted at energy $\hbar \omega_e$ and with duration $\tau_e = \gamma_e^{-1}$ such that $\omega_e \tau_e = 10$ and two differences of time of flights $\tau_{12}$:

(a) Partly overlapping wavepackets: $\tau_{12} = 2 \tau_e$

(b) Non-overlapping wavepackets: $\tau_{12} = 9 \tau_e$

(i) $\Delta W_{1,\text{out}}^{(c)}(t, \omega)$ given by Eq. [16] for $\phi = 0$ as a function of $t/\tau_e$ and $\omega \tau_e$

(ii) $\delta f_e(\omega)$

(iii) $\langle i(t) \rangle$ in units of $-e/\tau_e$. 

The average current depicted on Fig. 9(b)-(iii) exhibits two peaks that correspond to the two classical contributions to the excess Wigner function. These two peaks do not change when the magnetic flow is varied. In this case, a frequency resolved detector able to restore the overlap between electronic wavepackets is more appropriate to reveal the interferences. As shown on Fig. 9(b), the quantum fringes of the Wigner function can be revealed by measuring the electron distribution function. This is the single electron version of the channeled spectrum observed in optical interferometers using white light.

In practice, observing these effects might be quite challenging due to decoherence within the interferometer itself; although a proper design of the sample leads to partial protection against decoherence. Moreover, observing a channeled spectrum requires an unbalanced interferometer in which $|\tau_{12}|$ is greater than the inverse of the typical energy spreading of the electrons sent into the interferometer. Therefore, the best way to observe a channeled spectrum would be to use a non equilibrium distribution spreading over a broad energy range such as a double step generated by a QPC or the one generated by an ac sinusoidal voltage. A typical energy range of 100 to 500 µeV corresponds to frequencies from 24 to 120 GHz so that an imbalance of a few µm would be sufficient to observe a channeled spectrum. However, such an experiment would rely on a continuous stream of indistinguishable electrons. In order to observe a channeled spectrum with a single electron source, a larger imbalance is required and excitations that are spread in energy such as Lorentzian pulses with short duration (typically 10 ps) may be appropriate. An alternative approach is to use energy resolved excitations and change their energies to probe the interferences at the single electron level. But discussing the observability of these effects in a realistic situation would require computing the Wigner distribution function at the output of a Mach-Zehnder interferometer in the presence of interactions, which would go beyond the scope of the present paper.

1. Noise and correlations from coherences

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Floquet scattering theory due to electron and hole coherence and also for the \( \nu = 2 \) edge channel system with short range interactions.

The aim of this paragraph is to revisit all these HBT interferometry experiments in terms of the Wigner function. We show that the Wigner function provides a simple and unified view of all these experiments and that it is the quantity of interest for their interpretation.

In the HBT interferometry setup depicted on Figs. 10 and 11, the outcoming current correlations between edge channels \( \alpha \) and \( \beta \) is defined as

\[
S_{\alpha\beta}^{\text{out}}(t, t') = \langle \rho^\alpha(t) \rho^\beta(t') \rangle - \langle \rho^\alpha(t) \rangle \langle \rho^\beta(t') \rangle.
\]

It can be expressed in terms of the incoming ones \( S_{\alpha\beta}^{\text{in}}(t, t') \) and of a contribution \( Q(t, t') \) coming from two particle interferences. This contribution \( Q(t, t') \) involves the incoming single particle coherences right upstream the QPC:

\[
Q(t, t') = (e\nu_F)^2 \left( g_1^{(e)} g_2^{(h)} + g_2^{(e)} g_1^{(h)} \right) (t', t) \tag{51}
\]

where, for HBT interferometry, \( g_2^{(e)} \) and \( g_1^{(h)} \) are the coherences emitted by a reservoir at chemical potential \( \mu_2 \) and electronic temperature \( T_{el} \). The final expressions for outcoming current correlations are

\[
S_{11}^{\text{out}} = R^2 S_{11}^{\text{in}} + T^2 S_{22}^{\text{in}} + RT Q \tag{52}
\]

\[
S_{22}^{\text{out}} = T^2 S_{11}^{\text{in}} + R^2 S_{22}^{\text{in}} + RT Q \tag{53}
\]

\[
S_{12}^{\text{out}} = S_{21}^{\text{out}} = RT (S_{11}^{\text{in}} + S_{22}^{\text{in}} - Q) \tag{54}
\]

where for simplicity we have omitted the \((t, t')\) arguments.

As discussed in Sec. II A, electron quantum optics differs from photon quantum optics on the nature of the vacuum. Even when sources are switched off, a non vanishing single electron coherence is present and, at non zero temperature, leads to a non zero current noise. We shall thus consider situations in which no source is switched on (off/off), one of the two is switched on (cases on/off and off/on) and both are switched on (on/on). Denoting by \( \Delta X \) the difference between \( X \) in the situation under consideration and \( X \) when the two sources are off, we obtain

\[
\Delta S_{11}^{\text{on/off}} = R^2 \Delta S_{11}^{\text{in}} + RT (\Delta Q)_{\text{on/off}} \tag{55a}
\]

\[
\Delta S_{11}^{\text{off/on}} = T^2 \Delta S_{22}^{\text{in}} + RT (\Delta Q)_{\text{off/on}} \tag{55b}
\]

where the HBT excess contribution in Eq. (55a) is given by:

\[
(\Delta Q)_{\text{on/off}}(t, t') = (e\nu_F)^2 \left( \Delta g_1^{(e)} g_2^{(h)} + \Delta g_2^{(e)} g_1^{(h)} \right) (t', t) \tag{56}
\]

with a similar expression for \((\Delta Q)_{\text{off/on}}(t', t)\). These expressions are relevant when one of the incoming channel is populated by an equilibrium distribution. Since an equilibrium state is the analogous of the vacuum for photons in the electron quantum optics channel, such a situation is analogous to the historical tabletop HBT experiment. It has been recently demonstrated with an on demand single electron source.

In the case of an HOM experiment (see Fig. 11), the two sources are switched on. In this case, the excess outcoming noise \( \Delta S_{11}^{\text{on/on}} \) is the sum of the excess HBT noises and of an HOM contribution corresponding to two particle interferences between excitations emitted by both sources:

\[
\Delta S_{11}^{\text{on/on}} = \Delta S_{11}^{\text{on/off}} + \Delta S_{11}^{\text{off/on}} + RT (\Delta Q)_{\text{HOM}} \tag{57}
\]

This HOM contribution (with respect to the situation where the two sources are switched off) is defined as:

\[
(\Delta Q)_{\text{HOM}}(t, t') = (e\nu_F)^2 \left( \Delta g_1^{(e)} \Delta g_2^{(h)} + (1 \leftrightarrow 2) \right) (t', t) \tag{58}
\]

In electronic transport, we can only access to the time averaged component of current correlations at a given
frequency $\Omega$:

$$S_{\alpha\beta}^{\text{exp}}(\Omega) = \int S_{\alpha\beta}^{\text{out}}(t + \tau/2, t - \tau/2) e^{i\Omega \tau} d\tau. \quad (59)$$

Although the experiments are often performed at low frequency ($\Omega \approx 0$), recent progresses in finite frequency noise measurements motivate considering the case of finite frequency noise. As a consequence, the HBT and HOM excess contributions to the experimental signals are given by overlaps in time of two excess coherences. Thanks to Plancherel’s theorem, they can also be expressed as overlaps of the corresponding Wigner functions. As we shall see, this makes the physical interpretation of the HBT and HOM two particle interferometry experiments extremely transparent.

2. A Wigner view on the HBT and HOM experiments

To begin with, let us first consider the HBT excess contribution given by Eq. (56). Rewriting excess hole coherences in terms of electronic coherences and performing the time averaging and Fourier transform as in Eq. (59) leads to the excess HBT contribution of the outgoing noise under the form:

$$\langle \Delta Q \rangle_{\text{on/off}}(\Omega) = -e\langle i_1 \rangle - e^2 \int \delta f_1(\omega) g_2(\omega, \Omega) \frac{d\omega}{2\pi} \quad (60)$$

where $\langle i_1 \rangle$ denotes the average dc current of source 1, $\delta f_1(\omega) = \Delta W_1^{(e)}(t, \omega)$ is the excess electron distribution function in the incoming channel 1 and

$$g_2(\omega, \Omega) = f_2(\omega - \Omega) + f_2(\omega + \Omega) \quad (61)$$

is a double step going from zero for $\omega \gg |\Omega|$ to 2 for $\omega \ll |\Omega|$ through an intermediate plateau at 1 for $|\omega| \lesssim |\Omega|$. The steps are thermally broadened over an energy scale $k_B T_1$. Let us first remark that the result depends on the excess electron distribution function $\delta f_1(\omega)$ which is precisely the time average part of single electron coherence in the incoming channel 1. This is expected since we are measuring a time averaged quantity and the incoming channel 2 is populated with a stationary (equilibrium) state.

The first contribution in the r.h.s. of Eq. (60) corresponds to classical (Poissonian) partitioning of electrons and hole excitations sent in the incoming channel 1. This noise has a white spectrum the corresponding term is independent from $\Omega$.

The second term in the r.h.s. of Eq. (60) is an even function of $\Omega$. Its minus sign arises from the fermionic statistics of electrons and expresses antibunching. In fact, Eqs (51) to (54) are also valid for bosons but the relation between particle and anti-particle coherences would lead to a plus sign in the case of bosons in Eq. (60). In an experience with bosons, $g_2(\omega, \Omega)$ would arise the bosonic occupations numbers in a stationary reference beam arriving from input arm 2. To understand the frequency dependence, we rewrite the average dc current in terms of the excess electron distribution function. Eq. (60) then becomes:

$$\langle \Delta Q \rangle_{\text{on/off}}(\Omega) = e^2 \int \delta f_1(\omega) (1 - g_2(\omega, \Omega)) \frac{d\omega}{2\pi} \quad (62)$$

This shows that at low zero temperature, the excess HBT contribution is given by the sum of electronic excitations and of hole excitations whose energies exceed $\hbar|\Omega|$. This expression generalizes the formula known for the outgoing current noise at zero frequency.

Let us now discuss the case of an HOM experiment. The same line of reasoning shows that the HOM contribution given by Eq. (58) is the overlap of the excess Wigner functions arriving from the two incoming channels:

$$\langle \Delta Q \rangle_{\text{HOM}}(\Omega) = -e^2 \int \left[ \Delta W_1^{(e)}(t, \omega) \Delta W_2^{(e)}(t, \omega + \Omega) + \Delta W_2^{(e)}(t, \omega) \Delta W_1^{(e)}(t, \omega + \Omega) \right] \frac{d\omega}{2\pi} \quad (63)$$

This simple expression immediately suggests what happens physically: this term will contribute only when the excitations emitted by the two sources overlap up to the frequency shift at which the outgoing current noise or correlations are measured. Consequently, the noise and correlation in the HOM experiment contain an interesting information about the single electron coherences injected in the HBT interferometer. Whereas the usual HOM experiment gives access to the overlap of the excess Wigner functions of the two sources modulo an experimentally controlled time shift, finite frequency measurements enable us to probe a shift in energy. As a consequence, such measurements, although notably difficult, could be used to obtain qualitative information on relaxation mechanisms.

Finally, in the case of two ideal single electron sources, each one emitting a single electron wave packet per period characterized by its wave function $\varphi_{1,2}$, the HOM contribution at low frequency reduces to

$$\langle \Delta Q \rangle_{\text{on/on}}(\Omega = 0) = -2e^2f |\langle \varphi_1 | \varphi_2 \rangle|^2. \quad (64)$$

However, note that this interpretation of the HOM contribution as a wave packet overlap is specific to ideal sources emitting well defined and coherent single electron wave packets. But this is not true in general. In full generality, the proper way to understand the outcome of an HOM experiment is in terms of overlaps of single electron coherences. The Wigner representation provides a transparent way to visualize what contributes to the experimental HOM signals.
3. Single electron tomography revisited

Expressions (60) and (63) enable us to understand the recently proposed single electron tomography protocol\(^\text{23}\) in a very simple way. As in the case of optical homodyne tomography\(^\text{25,27,28}\), the key is to find a controlled source which, by varying its parameters will enable us to reconstruct the single electron coherence emitted by the source to be characterized or, equivalently, \(\Delta W_1^{(e)}(t, \omega)\).

A natural source that spans the whole space of single electron state is an equilibrium reservoir: depending on its chemical potential, some states will be filled and others will be empty. Its Wigner function is constant in time and equal to the equilibrium Fermi distribution of the reservoir. Let us assume zero temperature for simplicity: by varying the chemical potential, the r.h.s of Eq. (60) will select a small slice in frequency of the unknown Wigner function. Eq. (60) expresses that the change in current noises when increasing the chemical potential from \(\mu_2\) to \(\mu_2 + d\mu_2\) is proportional to the population of single electron states within the incoming channel 1 whose energies are between \(\mu_2\) and \(\mu_2 + d\mu_2\). If such a state is populated, the incoming electrons from both sides anti bunch and the noise does not change. If such a state is empty, the electron emitted by the battery into this interval will be partitioned at the beam splitter adding a contribution to the noise. This is the idea of shot noise spectroscopy originally discussed in the context of photo assisted shot noise\(^\text{24,76}\) and more recently in the context of electron quantum optics\(^\text{25,28}\).

To capture the time dependence of the unknown excess Wigner function \(\Delta W_1^{(e)}(t, \omega)\), a time dependent source is required. As seen in section III A 2 in the limit of small drive \((eV_0 \ll hf)\), the Wigner distribution issued by a reservoir driven by a small sinusoidal voltage \(V_d(t) = V_0 \cos(2\pi n ft + \phi_0)\) is given at first order by:

\[
\frac{\partial \Delta W_1^{(e)}(t, \omega)}{\partial(eV_0/hf)} = -2\cos(2\pi n ft + \phi_0) F_{\mu, T_d}(\omega)
\]

where \(F_{\mu, T_d}(\omega)\) is the convolution of the characteristic function of the interval \([h^{-1}\mu - \pi f, h^{-1}\mu + \pi f]\) with \(-\frac{1}{4k_B T_d}\cos^2(\hbar\omega/2k_B T_d)^{-1}\). Equation (65) combined to Eq. (63) at zero frequency shows that in the Wigner distribution point of view, the response of the low frequency noise to a small a.c. drive applied on the incoming channel 2 contains the information about the Fourier transform in time of the excess Wigner distribution of the source to be characterized. In practice, the finite electronic temperature introduces a blurring of the Wigner function along the energy direction over a scale \(k_B T_d\).

Approaching single electron tomography from the Wigner function point of view immediately rises the question of alternative tomography protocols based on other reference signals. This question is strongly motivated by the problem of efficiency: single electron tomography based on HOM-like experiments rely on ultra-high sensitivity current noise measurements which can be quite costly in terms of acquisition time\(^\text{16}\).

Our generic tomography protocol\(^\text{23}\) reconstructs the unknown excess Wigner function \(\Delta W_1^{(e)}(t, \omega)\) by extracting from noise measurements its \(t\) dependence at fixed \(\omega\) as a Fourier series. The various examples discussed in the present paper suggest that getting an accurate view of these coherences may require heavy sampling in \(\omega\) as well as in the number of harmonics. It would thus be highly desirable to design alternative protocols based on a different controlled source so that a much lighter sampling is required to obtain an accurate picture of the Wigner function. This problem is reminiscent of the problematic of compressive sensing\(^\text{33,77-79}\) but here we deal with single electron coherence which is a “quantum signal”. The HOM experiment automatically generates the overlap between two such “quantum signals”. The practical problem is then to find which controlled source could lead to an accurate approximation of the unknown Wigner function through a minimal number of noise measurements in an HOM experiment.

This problem may lead to some interesting theoretical developments but in practice, very few well controlled sources are available. The ac and dc drive lead to our generic tomography protocol. The energy resolved excitations discussed in Sec. III B 1 cannot be considered as controlled due to decoherence between the mesoscopic capacitor and the QPC. Indeed, characterizing decoherence effects on single electron excitations is precisely one of the main motivation for single electron tomography. On the contrary, voltage pulses can be considered as controlled to some extend: high frequency current measurement\(^\text{10}\) can be used to control the shape of the pulse arriving onto the QPC, although in a limited bandwidth. Moreover, in some cases, noise measurement can help characterizing purely electronic pulses\(^\text{111,112}\). Specific voltage pulses may thus provide interesting families of functions for a reconstructing single electron coherence at minimal cost in some cases. However, finding out the appropriate family of excitations requires an a priori knowledge of the signal to be measured. Forthcoming and foreseeable experiments will for sure involve decoherence effects. Therefore, designing alternative tomography protocols based on HOM experiments requiring less measurements calls for an in depth modeling of decoherence in the Wigner function formalism. This would go beyond the scope of the present paper but is of clear interest from this signal processing perspective.

V. CONCLUSION

In this paper, we have introduced the time-frequency representation of the single electron coherence in quantum Hall edge channels generalizing the quasi-probability
distribution function introduced in quantum mechanics\cite{12} and commonly used in quantum optics. We have shown that this real valued function of time and frequency provides a very convenient way to image single electron coherence in a quantum conductor. It gives access to both the energy content and real time development of excitations emitted by electron sources, shedding new light on electron coherence and discussing it with a signal processing language. It also provides a natural framework to deal with the quantum nature of electron excitations, in which quantum interferences effects have a clear signature as pronounced non-classical regions where the Wigner function violates the bounds required by the Pauli principle and for interpreting it as a probability distribution. Along the same lines, it leads straightforwardly to the emergent semiclassical picture of the quasiparticle dynamics.

Moreover, the celebrated HBT and HOM interferometry experiments have a very simple interpretation in terms of overlaps of Wigner function. Consequently, Wigner function computations are directly useful for interpreting the results of these experiments at both the qualitative and quantitative levels. In fact, our main message is that the Wigner function is a very relevant representation of single electron coherence because it provides a simple and unified view of many single and two particle interference effects that have been experimentally demonstrated in electron quantum optics and quantum nano-electronics over the last fifteen years\cite{18,19,20}.

Experimentally, measuring single electron coherence or equivalently the Wigner function is an important but difficult challenge for electron quantum optics. However, as discussed in the present paper, we think that a variety of techniques is now available to recover information on the Wigner function ranging from amplitude interferometry\cite{21,22,23} to the HOM experiment and its many possible variants among which the recently proposed single electron tomography protocol\cite{24}.

To conclude, an important and still open question in electron quantum optics is to compute the Wigner function taking into account the effects of interactions experienced by the excitations emitted by a single electron source. In the case of pure single electron excitations, the bosonization technique already used to discuss the problem of quasi-particle relaxation in quantum Hall edge channels\cite{25} can be adapted. This leads to a complete unraveling of decoherence scenarios of single electron excitations in quantum Hall edge channels which will be discussed in a future publication.

Acknowledgments

We thank B. Roussel from ENS Lyon and Q. Lavigne from Telecom St. Etienne for their precious help in data visualization. We also thank J.M. Berroir and B. PLAIGS from LPA as well as Th. Jonckheere, Th. Martin, J. Rech and C. Wahl from CPT Marseille for useful discussions. We finally thank P. Borgnat from ENS Lyon for discussions about time/frequency representation in signal processing and compressed sensing. This work is supported by the ANR grant ”Ishot” (ANR-2010-BLANC-0412).

Appendix A: Properties of single electron coherence

1. Basic properties

As in photon quantum optics, first order coherences satisfy the hermiticity property:

\[
G_p^{(c)}(x, t; y, t')^* = G_p^{(c)}(y, t'; x, t) \quad \text{(A1a)}
\]

\[
G_p^{(h)}(x, t; y, t')^* = G_p^{(h)}(y, t'; x, t). \quad \text{(A1b)}
\]

In a region of free propagation at velocity \(v_F\), the single electron coherence obeys:

\[
G_p^{(c)}(x + v_F t + \tau; t + \tau; x' + v_F t' + \tau) = G_p^{(c)}(x, t; x', t'). \quad \text{(A2)}
\]

When considering coherences at a position \(x\) where propagation can be assumed to be free, Eq. (3) translates into a relation between electron and hole coherences in the time domain:

\[
G_{p,x}^{(c)}(t, t') = v_F^{-1} \delta(t - t') - G_{p,x}^{(h)}(t', t). \quad \text{(A3)}
\]

Taking the Fourier transform of this equation with respect to \(t - t'\) immediately leads to the relation between electron and hole Wigner functions\cite{21}.

In order to connect single electron coherence to the fermionic occupation number, we decompose

\[
\psi(x, t) = \int e^{-i\omega t} c(\omega) \frac{d\omega}{\sqrt{2\pi v_F}} \quad \text{(A4)}
\]

where \(c(\omega)\) and \(c(\omega)\) respectively create and destroy an energy at energy \(h\omega\) with respect to the reference Fermi energy. These operators satisfy the canonical anti commutation relations: \(\{c(\omega), c(\omega')\} = 0\) and \(\{c(\omega), c(\omega')\} = \delta(\omega - \omega')\).

2. Cauchy-Schwarz inequalities

Finally, the single electron coherence satisfies a Cauchy-Schwarz inequality associated with the Hermitian product on the operator space: \(\langle A | B \rangle = \text{Tr}(B A^\dagger)\). Due to the non zero coherence of the Fermi sea, this inequality is most useful in the frequency domain at zero temperature. Using \(A = c(\omega_-)\) and \(B = c(\omega_+)\) and Eq. (17), the Cauchy-Schwarz inequality writes:

\[
\left| \bar{G}_{p,x}^{(c)}(\omega_+, \omega_-) \right|^2 \leq \bar{G}_{p,x}^{(c)}(\omega_+, \omega_+) \bar{G}_{p,x}^{(c)}(\omega_-, \omega_-). \quad \text{(A5)}
\]
Decomposing $\tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_-)$ into a Fermi sea contribution at chemical potential $\mu = 0$ and an excess contribution which we assume to be regular

$$\tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_-) = \frac{2\pi}{v_F} \delta (\omega_+ - \omega_-) \Theta (-\omega_+) + \Delta \tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_-) \tag{A6}$$

and taking $\omega_+$ and $\omega_-$ positive shows that the Fermi sea contribution vanishes and leads to the following inequality satisfied over the (e) quadrant:

$$\left| \Delta \tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_-) \right|^2 \leq \Delta \tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_+) \Delta \tilde{G}^{(e)}_{p,\omega} (\omega_-, \omega_-). \tag{A7}$$

Considering the hole coherence and $\omega_+ < 0$, the same line of reasoning shows that the inequality (A7) is also true on the (h) quadrant.

Finally, let us consider $\omega_- < 0$ and $\omega_+ > 0$ and assume that the excess single electron coherence does not contribute to the electron occupation level below the Fermi level $\Delta \tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_-) = 0$. Then, the general inequality (A5) implies that $\Delta \tilde{G}^{(e)}_{p,\omega} (\omega_+, \omega_-)$ also vanishes: there are no electron/hole coherences. The same reasoning can be done on hole coherences and shows that there are no electron/hole coherences if the source does not contribute to the excess occupation number of electrons. Finally, the inequality (A7) also shows that if a source does not contribute to the excess hole (resp. electron) occupation number, it has vanishing coherence in the (h) quadrant (resp. (e) quadrant). This proves that, at zero temperature, as soon as a source does not lead to an excess of electrons (resp. holes), it only contributes to the excess coherence in the (h) (resp. (e)) quadrant. In other words, it only generates hole (resp. electronic) excitations.

**Appendix B: Energy resolved electronic wave packet**

It is possible to obtain an analytic formula for the truncated Lorentzian wave packet (11) in the time domain. Depending on the sign of $t$, we build a closed contour that contains the positive real axis and either the positive or negative imaginary axis. The integral over the imaginary half-axis can then be expressed in terms of the exponential integral function $\text{Ei}(x)$. This leads to:

$$\varphi_e(t) = -\frac{i N_e}{\sqrt{v_F} e^{-\gamma_e t-i\omega_e t}} \left( \text{Ei}(\frac{i\gamma_e}{2} + i\omega_e t) \right)$$

where $N_e = \left( \frac{1}{2} + \frac{i}{2} \arctan \left( \frac{\omega_e}{\gamma_e} \right) \right)^{-1/2}$ is the normalization factor. The exponential integral part is responsible for the non vanishing of the wave packet for $t < 0$ and for the oscillations of its envelope at $t > 0$. These features can be observed on the numerical evaluation of excess single electron coherence $\varphi_e(t+\tau/2)\varphi_e(t-\tau/2)^*$ depicted on Fig. 5(c) as well as on the current density $v_F|\varphi_e(t)|^2$ computed from the Wigner function (see panels (iii) on Fig. 5). Note that Eq. (B1) predicts a log ($t/\tau_e$) singularity of $\varphi_e(t)$ at $t = 0$ which is not visible on the numerics due to the UV cutoff in numerical integration.

1. Y. Ji, Y. Chung, D. Sprinzak, M. Heiblum, D. Mahalu, and H. Shtrikman, Nature 422, 415 (2003).
2. P. Rouleau, F. Portier, D.C. Glattli, P. Roche, A. Cavanna, G. Faini, U. Gennser, and D. Mailly, Phys. Rev. B 76, 161309 (2007).
3. W. Oliver, J. Kim, R. Liu, and Y. Yamamoto, Science 284, 299 (1999).
4. M. Henny, S. Oberholzer, C. Strunk, T. Heinzel, K. Esslín, M. Holland, and C. Schönberger, Science 284, 296 (1999).
5. P. Samuelsson, E. Sukhorukov, and M. Büttiker, Phys. Rev. Lett. 92, 026805 (2004).
6. I. Neder, N. Ofek, Y. Chung, M. Heiblum, D. Mahalu, and V. Umansky, Nature 448, 333 (2007).
7. F.-J. Ahlers, O. Kieler, B. Sagol, K. Pierz, and U. Siegner, Journal of Applied Physics 100, 093702 (2006).
8. G. Féve, A. Mahé, J.-M. Berroir, T. Kontos, B. PLAÈIS, D.C. Glattli, A. Cavanna, B. Etienne, and Y. Jin, Science 316, 1169 (2007).
9. M. Blumenthal, B. Kaestner, L. Li, S. Giblin, T. Janssen, M. Pepper, D. Anderson, G. Jones, and D. Ritchie, Nature Physics 3, 343 (2007).
10. C. Leicht, P. Mirovsky, B. Kaestner, F. Hols, V. Kashačeyevs, E. Kurganova, O. Zeitler, T. Weimann, K. Pierz, and H. Schumacher, Semicond. Sci. Technol. 26, 055010 (2011).
11. F. Battista and P. Samuelsson, Phys. Rev. B 83, 125324 (2011).
12. S. Hermelin, S. Takada, M. Yamamoto, S. Tarucha, A. Wiek, L. Saminadayar, C. Bäerle, and T. Meunier, Nature 477, 435 (2011).
13. J. Dubois, T. Jullien, C. Grenier, P. Degiovanni, P. Rouleau, and D.C. Glattli, Phys. Rev. B 88, 085301 (2013).
14. C. Grenier, R. Hervé, G. Féve, and P. Degiovanni, Mod. Phys. Lett. B 25, 1053 (2011), proceedings of the Stat-phys 24 satellite meeting "International Conference on Frustrated Spin Systems, Cold Atoms and Nanomaterials" (Hanoi, 14-16 July 2010).
15. J. Gabelli, G. Féve, J.-M. Berroir, B. PLAÈIS, A. Cavanna, B. Etienne, Y. Jin, and D.C. Glattli, Science 313, 499 (2006).
16. E. Boquillon, V. Freulon, J.-M. Berroir, P. Degiovanni, B. PLAÈIS, A. Cavanna, Y. Jin, and G. Féve, Nature Communications 4, 1839 (2013).
17. A. Mahé, F.D. Parmentier, E. Boquillon, J.-M. Berroir, D.C. Glattli, T. Kontos, B. PLAÈIS, G. Féve, A. Cavanna, and Y. Jin, Phys. Rev. B 82, 201309 (2010).
18. F.D. Parmentier, A. Mahé, A. Denis, J.-M. Berroir, D.C. Glattli, B. PLAÈIS, and G. Féve, Rev. Sci. Instrum. 82, 013904 (2011).
19. E. Wigner, Phys. Rev. 40, 749 (1932).
20. J. Ville, Câbles et Transmission 2, 61 (1948).
source emitting only electrons, this is always the case. Working with equal time coherences, we drop the time
dependence of the coherence. We are left with the form
\[ g^{(1)}(x, l) \] is well defined only when the product \( \Delta G^{x} \) is positive. But for a

\[ G^{x} = \rho(e) \] and the density matrix \( \rho(e) \).

Mathematicians