Quicksort Is Optimal For Many Equal Keys

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I prove that the average number of comparisons for median-of-$k$ Quicksort (with fat-pivot a.k.a. three-way partitioning) is asymptotically only a constant $\alpha_k$ times worse than the lower bound for sorting random multisets with $\Omega(n^\varepsilon)$ duplicates of each value (for any $\varepsilon > 0$). The constant is $\alpha_k = \ln(2)/(H_{k+1} - H_{(k+1)/2})$, which converges to $1$ as $k \to \infty$, so Quicksort is asymptotically optimal for inputs with many duplicates. This resolves a conjecture by Sedgewick and Bentley [37, 38] and constitutes the first progress on the analysis of Quicksort with equal elements since Sedgewick’s 1977 article.

1. Introduction

Sorting is one of the basic algorithmic tasks that is used as a fundamental stepping stone for a multitude of other, more complex challenges. Quicksort is the method of choice for sorting in practice. Any undergraduate student learns as a justification for its quality that the average number of comparisons is $\Theta(n \log n)$ for a random permutation of $n$ distinct elements, putting it into the same complexity class as, e.g., Mergesort. By choosing random pivots we can make this average the expected behavior regardless of the order of the input. Moreover, the hidden constants in $\Theta(n \log n)$ are actually small: Quicksort needs $2 \ln(2)n \log(n) + O(n)$ comparisons in expectation, i.e., asymptotically only a factor $2 \ln(2) \approx 1.39$ more than the information-theoretic lower bound for any comparison-based sorting method. By choosing the pivot as median of $k$ sample elements in each step, for $k$ a fixed, odd integer, this constant can be reduced to $\alpha_k = \ln(2) / (H_{k+1} - H_{(k+1)/2})$ [35], where $H_k = \sum_{i=1}^{k} 1/i$. Note that $\alpha_k$ converges to $1$, so Quicksort’s expected behavior is asymptotically optimal on random permutations.

It is also well-known that Quicksort rarely deviates much from this expected behavior. Folklore calculations show that Quicksort needs, e.g., at most $7$ times the expected number of comparisons with probability at least $1 - 1/n^2$ (cf. Appendix E). This result can be strengthened [30] to guarantee at most $\alpha$ times the expected costs for any $\alpha > 1$ with probability $1 - O(n^{-c})$ for all constants $c$ (independent of $\alpha$). Median-of-$k$ Quicksort is hence optimal not only in expectation, but almost always!

These guarantees certainly justify the wide-spread use of Quicksort, but they only apply to inputs with $n$ distinct elements. In many applications, duplicates (i.e., elements that are equal w.r.t. the order relation) appear naturally. The SQL clause GROUP BY C, for example, is typically implemented by first sorting rows w.r.t. column C to speed up the computation of

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1 I write $\log_2$ for $\logarithmus\ d\ uit\ a\ is$, and $\ln$ for $\logarithmus\ n\ ur\ a\ is$. 

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aggregating functions like \texttt{COUNT} or \texttt{SUM}. Many rows with equal \texttt{C}-entries are expected in such applications.

At the KnuthFest celebrating Donald Knuth’s 1000000th birthday, Robert Sedgewick gave a talk titled “Quicksort is optimal” \footnote{Sedgewick presented the conjecture in a less widely-known talk already in 1999 \cite{Sedgewick99}. Since they never published their results in an article, I briefly reproduce their arguments below.} presenting a result that is at least very nearly so: Quicksort with fat-pivot (a.k.a. three-way) partitioning uses $2 \ln 2 \approx 1.39$ times the number of comparisons needed by any comparison-based sorting method for any randomly ordered input, with or without equal elements. He closed with the conjecture that this factor can be made arbitrarily close to 1 by choosing the pivot as the median of $k$ elements for sufficiently large $k$. This statement is referred to as the Sedgewick-Bentley conjecture.

The Sedgewick-Bentley conjecture is a natural generalization of the distinct-keys case (for which it is known to hold true), and sorting experts will not find the claim surprising, since it is the theoretical justification for a long-established best practice of general-purpose sorting. As observed many times in practice, Quicksort with the fat-pivot partitioning method by Bentley and McIlroy \cite{Bentley93} and ninther (a.k.a. pseudomedian of nine) pivot sampling comes to within ten percent of the optimal expected comparison count for inputs with or without equal keys.

In this paper, I prove the Sedgewick-Bentley conjecture for inputs with “many duplicates”, i.e., where every key value occurs $\Omega(n^\varepsilon)$ times for $\varepsilon > 0$ an arbitrary constant. I also derive the constant of proportionality explicitly: it turns out to be the same $\alpha_k$ mentioned above. (Sedgewick and Bentley did not include a guess about the constant in their conjecture). While Quicksort can certainly be outperformed for particular types of inputs, the combination of simple, efficient code and almost universal proven optimality is unsurpassed; it is good news that the latter also includes inputs with equal keys.

The confirmation of the Sedgewick-Bentley conjecture is not a surprising outcome, but it has been surprisingly resistant to all attempts to formally prove it: no progress had been made in the nearly two decades since it was posed (see Section 1.3 for some presumable reasons for that), so restricting the problem might be sensible. I would like to remark that this restriction is a technical requirement of my analysis, but we have no reason to believe that Quicksort performs much different when it is violated. More importantly, it is not the purpose of this paper to suggest sorting inputs with many duplicates as a natural problem per se, for which tailored methods shall be developed. (One is tempted to choose a hashing-based approach when we know that the number $u$ of different values is small, but our results show that precisely for such instances, Quicksort will be competitive to tailored algorithms!)

Indeed, the strength of Quicksort is that it smoothly adapts to the actual amount of duplication without requiring explicit precautions for that case. The concept is analogous (but orthogonal) to adaptive sorting methods \cite{Sen90} that take advantage of existing (partial) order in the input. The purpose of this paper is thus to finally deliver a mathematical proof that median-of-$k$ Quicksort is an optimal “entropy-adaptive” \footnote{Sen and Gupta \cite{Sen90} call such algorithms distribution sensitive.} sorting method, at least for a large class of inputs.

\textbf{Methods.} The restriction to many duplicates allows us to work in an alternative input model without affecting overall costs by much (cf. Fact 4.2): instead of fixing the exact multiplicities of a multiset, we fix a discrete probability distribution over $[1..u]$ and draw $n$ elements independently.
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and identically distributed (i.i.d.) according to that distribution. We choose this universe distribution so as to obtain the same multiplicities in expectation, but all values in the input are now mutually independent unlike for the multiset model. This is the what makes the analysis feasible.

The analysis of Quicksort on discrete i.i.d. inputs proceeds in two steps: First, I use that costs of (median-of-$k$) Quicksort correspond to costs of searching in a ($k$-fringe-balanced) binary search tree (BST). A concentration argument shows that for inputs with many duplicates, the search tree reaches a stationary state with high probability (w.h.p.) after a short prefix of the input, and is thus independent of the (larger) rest of the input.

The second step is to determine the expected search costs in a $k$-fringe-balanced BST built by repeatedly inserting i.i.d. elements until all $u$ values appear in the tree; (I call the tree saturated then). I show that the search costs are asymptotic to $\alpha_k$ times the entropy of the universe distribution. To do so I derive lower and upper bounds from the recurrence of costs (with the vector of probability weights as argument) using the aggregation property of the entropy function. To the best of my knowledge, the analysis of search costs in fringe-balanced trees with equal keys is novel, as well.

Although most used techniques are elementary, keeping the error terms under control so that only $\Omega(n^\epsilon)$ duplicates are required made several refinements of classical methods necessary that might be of independent interest. All methods and proofs are therefore developed in detail in the appendix.

Outline. After introducing a bit of notation (Section 1.1), the remainder of this section is devoted to related work (Section 1.2), and a formal statement of my main result (Section 1.3). I also briefly sketch techniques of previous attempts to prove the Sedgewick-Bentley conjecture and why they do not succeed. Section 2 recapitulates how fat-pivot Quicksort works. Sections 4–7 study the cost of fat-pivot Quicksort on discrete i.i.d. inputs as outlined above. The results are summarized in Section 8. Detailed proofs are given in the appendix.

1.1. Preliminaries and Input Models

This section introduces the most important notation; a comprehensive list is given in Appendix A for reference. I write vectors in bold font $\mathbf{x} = (x_1, \ldots, x_n)$ and always understand them as column vectors (even when written in a row in the text). By default, all operations on vectors are meant component-wise. By $\sum x$, I denote $x_1 + \cdots + x_n$, the total of $\mathbf{x}$. $H_n = \sum_{i=1}^n \frac{1}{i}$ is the $n$th harmonic number.

I use Landau-symbols ($O$-classes etc.) in the formal sense of Flajolet and Sedgewick [14, Section A.2]. For a random variable $X$, $\mathbb{E}[X]$ is its expectation and $\mathbb{P}[X = x]$ denotes the probability for the event $X = x$. For $q \in [0,1]^u$ with $\Sigma q = 1$, I write $U \overset{\text{d}}{=} D(q)$ to say that $\mathbb{P}[U = i] = q_i$ for $i \in [u]$. By $H$ or $H_{ld}$ I denote the binary Shannon entropy $H(q) = \sum_{i=1}^u q_i \log(1/q_i)$; likewise $H_{\log}$ is the base $e$ entropy. Appendix G collects some technical properties of this function.

1.2. Previous Work

The problem of multiset sorting attracted considerable attention in the literature. Unless stated otherwise, all results concern the number of ternary comparisons, i.e., one comparison has as result $<, = \text{ or } >$. 

Multiset Sorting. The input is a random permutation of a fixed multiset, where value \( v \in [1..u] \) appears \( x_v \in \mathbb{N} \) times. I write \( n = x_1 + \cdots + x_u = \Sigma x \). Munro and Raman \[33\] prove a lower bound of \( n \log n - \sum_{i=1}^u x_i \log x_i - n \log e \pm O((\log n)) \) (ternary) comparisons, which is probably more intuitively written in terms of the entropy as \( H(x/n)n - n \log e \pm O((\log n)) \). We reproduce their main argument in Appendix J and extend it to i.i.d. inputs.

The conceptually simplest algorithm coming close to this bound is to insert elements into a splay tree, collecting all duplicates in linear lists inside the nodes. By static optimality (Theorem 2 of Sleator and Tarjan \[42\]), this needs \( O(\text{median selection}) \), and thus at least \( \log n \); however the constant of proportionality is one plus the constant for deterministic median selection, and so is optimal up to a constant factor. That factor is at least 2 (using semisplaying), and we need linear extra space. Already in 1976, Munro and Spira \[32\] described simple variants of Mergesort and Heapsort that collapse duplicate elements whenever discovered. They are optimal up to an \( O(n) \) error term w.r.t. comparisons, but do not work in place. (Their Heapsort requires a non-standard extract-min variant that does not work in place.) The first in-place method was the adapted Heapsort of Munro and Raman \[33\]; it does not use the element-collapsing technique, but rather removes all duplicates from the heap in one bulk extract-min operation. None of these methods made it into practical library implementations since they incur significant overhead w.r.t. existing sorting methods when there are not many equal keys in the input.

Quicksort on Multiset Permutations. Building on Burge’s analysis of BSTs \[5\], Sedgewick analyzed several Quicksort variants on random permutations of multisets in his 1977 article \[36\]. For fat-pivot Quicksort without sampling, he found the exact result: \( 2H_Q(x) + n - u \) ternary comparisons on average, where \( H_Q(x) = \sum_{1 \leq i < j \leq u} x_i x_j / (x_i + \cdots + x_j) \). Interestingly, Sedgewick found fat-pivot partitioning not advisable for practical use at that time; this only changed with the success of the implementation of Bentley and McIlroy \[4\].

Two decades later, Sedgewick and Bentley \[37,38\] combined this exact bound, but somewhat unwieldy result with the bound \( H_Q(q) \leq H(q) \ln 2 \) and concluded that with at most \( (2 \ln(2) H(x/n) + 1)n \) ternary comparisons on average, fat-pivot Quicksort is asymptotically optimal for sorting a random permutation of any fixed multiset—up to the constant factor \( 2 \ln 2 \). The bound \( H_Q(q) \leq H(q) \ln 2 \) was proven in a seemingly unrelated context by Allen and Munro \[1\] Theorem 3.2] that appeared just one year after Sedgewick’s Quicksort paper \[36\]. Allen and Munro studied the move-to-root heuristic for self-organizing BSTs, which they found to have the same search costs in the long run as a BST built by inserting elements drawn i.i.d. according to the access distribution until saturation. We will consider this connection between Quicksort and search trees in detail in Section 4. Parameters of BSTs under the discrete i.i.d. model were studied in more detail later \[25,22\].

Katajainen and Pasanen considered Quicksort-based approaches for multiset sorting. They argued (indirectly) that a fat-pivot Quicksort uses on average \( 2n \ln(n) - \sum_{i=1}^u x_i \ln(x_i) \pm O(n) \) comparisons (their Theorem 3), since “Due to the three-way partitions, all redundant comparisons between a pivot and elements equal to the pivot are avoided” \[23\]. Note however that this only shows that we use at most \( H(x/n)n + (2 \pi^2 - 1)n \ln n \pm O(n) \) comparisons, which is not entropy-optimal. In a companion paper they described a stable Quicksort version with exact median selection and show that it needs \( O(\tilde{H}(x/n)n) \) comparisons even in the worst case \[24\]; however the constant of proportionality is one plus the constant for deterministic median selection, and thus at least 3 \[9\].

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\[9\] Sedgewick and Bentley \[37,38\] compare this number against \( \ln((x_1,\ldots,x_u)) = \ln(p!/(x_1!\cdots x_u!)) \), i.e., the logarithm of the number of different input orderings (given by the multinomial coefficient). This information-theoretic argument lower bounds the number of needed yes/no questions (i.e., binary comparisons), but more elaboration is necessary for ternary comparisons. The lower bound of Munro and Raman \[33\] (cf. Appendix J) uses a reduction to distinct elements and yields the desired bound for ternary comparisons.
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1.3. New Results

We now state the main claim of this paper. Note that the error terms of asymptotic approximations in the results mentioned above only involved $n$, so there was no need to specify an explicit relation between the profile of the multiset $\mathbf{x} = (x_1, \ldots, x_n)$, the universe size $u$, and the number of elements $n$; here we are not so fortunate. We hence include $n$ as sub- or superscript whenever the dependence is important.

**Theorem 1.1 (Main Result):** Let $(\mathbf{x}^{(n)})_{n \in \mathbb{N}}$ with $\mathbf{x}^{(n)} \in \mathbb{N}^n$ and $n = x_1 + \cdots + x_n$, for all $n$ be a sequence of profiles with “many duplicates”, i.e., there is a constant $\varepsilon > 0$ so that $\min_n x_1^{(n)} = \Omega(n^\varepsilon)$. Abbreviate the corresponding (binary) entropy by $\mathcal{H}_n = \mathcal{H}(\mathbf{x}^{(n)}/n)$.

The expected number of (ternary) comparisons used by median-of-$k$ Quicksort with fat-pivot partitioning to sort a random permutation of any multiset with profile $\mathbf{x}^{(n)}$ fulfills

$$\mathbb{E}[C_{\mathbf{x}^{(n)}}] = \alpha_k \mathcal{H}_n n \pm O\left((\mathcal{H}_n^{1-\delta} + 1)n\right), \quad (n \to \infty),$$

for any constant $\delta < \frac{2}{k+5}$. $\mathbb{E}[C_{\mathbf{x}^{(n)}}]$ is asymptotically optimal up to the factor $\alpha_k$ in the leading term.

**Previous Approaches And Why They Fail for $k > 1$.** Sedgewick’s analysis [36] is based on explicitly solving the recurrence for the expected number of comparisons. Since it has a vector, namely the profile $\mathbf{x}$, as parameter, tricky differencing operations are required to obtain a telescoping recurrence. They rely on symmetries that are only present for the most basic version of Quicksort: it has now been 40 years since Sedgewick’s article appeared, and not the slightest generalization of the analysis to, say, median-of-3 Quicksort, has been found.

Following our approach outlined above we can alternatively compute the expected costs to search each element of the multiset in a BST built by inserting the same elements in random order. The random number of comparisons can thus be written as the scalar product $\mathbf{I}^T \mathbf{x}$, where $\mathbf{I}$ is the node-depth vector of the BST (cf. Section 5). For an ordinary BST, once an element is present in the tree, any further insertions of the same value are without effect; so we obtain the same tree no matter how many duplicates of this element follow later. This means that the resulting tree has exactly the same shape as when we insert elements drawn i.i.d. according to $\mathcal{D}(\mathbf{q})$ with $\mathbf{q} = \mathbf{x}/n$ until saturation. The expected search costs in the latter case are found to be precisely $2\mathcal{H}_Q(\mathbf{q}) + 1$ by a comparatively simple argument[11]: multiplying by $n$ gives the Quicksort costs.

For median-of-$k$ Quicksort we obtain $k$-fringe-balanced trees, and now a certain number of duplicate insertions do affect the shape of the tree; after all, this is the way the balancing is achieved in first place (details are given in Section 4). As the multiset model corresponds to drawing elements without replacement, the probabilities for the values change after each insertion. Analyzing the search cost then essentially reduces to solving the vector-recurrence for Quicksort with pivot sampling that has resisted all attempts for 40 years. One might conclude that the case $k = 1$ can be explicitly solved precisely because we were effectively working in the i.i.d. model instead of the multiset model.

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Assume we search $\ell \in [u]$. We always have one final comparison with outcome $-\ell$; the remaining comparisons are on the search path and compare $\ell$ to some $j \neq \ell$. We compare $\ell$ to $j$ iff among the values between $\ell$ and $j$, $j$ was the first to be inserted into the tree, which happens with probability $q_j/(q_1 + \cdots + q_j)$. Adding up the indicator variables for these events and multiplying by the probability $q_{\ell}$ to search that value $\ell$, we obtain

$$1 + \sum_{\ell=1}^u q_{\ell} \sum_{j \neq \ell} \frac{q_j}{q_1 + \cdots + q_j} = 1 + 2 \sum_{1 \leq i < j \leq u} \frac{q_i q_j}{q_1 + \cdots + q_j} = 2\mathcal{H}_Q(\mathbf{x}/n) + 1.$$
My Assumption. The only hope I see to make progress for $k > 1$ is thus to cling to the i.i.d. model even though it is not equivalent to the multiset model anymore. We thereby retain independent insertions and the analysis of search costs in fringe-balanced trees becomes conceivable. However, we face two new problems:

1. How often we search each value $v$ is now a random variable $X_v$, and the random number of comparisons is $\Gamma X$, the \textit{product} of two random variables. Since $\Gamma$ is the node-depth vector of a tree built by inserting a multiset with profile $X$ (in random order), the two random variables \textit{not independent}.

2. To compare with previous results, we would like to at least approximate the multiset model. Due to the inherent variance in the random profile $X$, we can only do so for sufficiently “tame” profile vectors $x$.

My suggested assumption, $\Omega(n^\varepsilon)$ occurrences of each value, is a simple sufficient condition for addressing both of these problems. It is possible to slightly weaken it at the price of a more clumsy and artificial criterion: For the first issue, we can tolerate values with minuscule multiplicities in $O(n^\delta)$ as long as $\delta < \varepsilon$ (so that we have a nonempty \textit{separating range} $(\delta, \varepsilon)$), and the total number of these rare values is $O(n^{1-\varepsilon})$. The second issue only requires $u = O(n^{1-\varepsilon})$ with no restriction on the individual probabilities.

This concludes the introductory overview. We start the main part of the article by describing the algorithm under study.

2. Fat-Pivot Quicksort

By “fat-pivot” I mean a partitioning method that splits the input into three parts: elements (strictly) smaller than the pivot, elements equal to the pivot, and elements (strictly) larger than the pivot. Instead of only one pivot element in binary partitioning, we now obtain a “fat” pivot segment that separates the left and right subproblems. This idea is more commonly known as \textit{three-way partitioning}, but I prefer the vivid term fat pivot to make the distinction between fat-pivot partitioning and partitioning around \textit{two pivots} clear. The latter has become fashionable again in recent years [22, 45, 44]. Since all duplicates of the pivot are removed before recursive calls, the number of partitioning rounds is bounded by $u$, the number of different values in the input.

To simplify the presentation, I assume in this work that partitioning \textit{retains the relative order} of elements that go to the same segment; a corresponding reference implementation is given in Appendix B. I always refer to this Quicksort variant in this paper. It uses the median of $k$ sample elements as pivot, where the sample size $k = 2t + 1$, $t \in \mathbb{N}$, is a tuning parameter of the algorithm that we consider a fixed constant in the analysis. Whenever I speak of costs, I mean the number of ternary key comparisons. I would like to remark that all results derived in this paper carry over to other fat-pivot partitioning implementations that do not necessarily retain the relative order, as long as they use $n \pm O(n^{1-\varepsilon})$ comparisons; my Ph.D. thesis [44] gives details on that.

3. Input Models

This section formally defines the input models and some related notation.
3.1. Discrete I.I.D. Model

In the discrete i.i.d. model (a.k.a. probability model \[25\] or expected-profile model \[14\]) with parameters \(u \in \mathbb{N}\) and \(q \in (0,1)^u\) with \(\Sigma q = 1\), an input of size \(n\) consists of \(n\) i.i.d. (independent and identically distributed) random variables \(U_1, \ldots, U_n\) with \(U_i \sim D(q)\) for \(i = 1, \ldots, n\). The domain \([u]\) is called the universe, and \(q\) the (probability vector of the) universe distribution.

We denote by \(X_v\), for \(v \in [1..u]\), the number of elements \(U_j\) that have value \(v\); the vector \(X = (X_1, \ldots, X_u)\) of all these multiplicities is called the profile of the input \(U = (U_1, \ldots, U_n)\). Clearly, \(X\) has a multinomial distribution, \(X \sim \text{Mult}(n, q)\), with mean \(\mathbb{E}[X] = nq\).

The distribution function of the universe distribution is \(F_U(v) = \mathbb{P}[U \leq v] = \sum_{i=1}^{[v]} q_i\) for \(v \in [0, u + 1]\), and I denote its (generalized) inverse by \(F_U^{-1} : (0, 1) \rightarrow [1..u]\) with \(F_U(x) = \inf\{v \in [1..u] : F_U(v) \geq x\}\).

3.2. Multiset Model

In the random multiset permutation model (a.k.a. exact-profile model), we have parameters \(u \in \mathbb{N}\), the universe size, and \(x \in \mathbb{N}^u\), the fixed profile. An input under this model always has size \(n = \Sigma x\), and is formed as a uniformly chosen random permutation of the multiset with \(x_v\) copies of the number \(v\) for \(v = 1, \ldots, u\).

3.3. Relation of the Two Input Models

The two input models for \(x = qn = \mathbb{E}[X]\) are closely related. The random profile \(X\) is highly concentrated (see Devroye\’s Chernoff bound for multinomial variables, \[\text{Lemma D.2}\]), so the discrete i.i.d. model essentially adds a certain amount of “noise” or “smoothing” to a fixed profile \(x\)—not more, but also not less: we have \(X_v = x_v + O(n^{1-\delta})\) w.h.p. for all \(v \in [u]\) as long as \(u = O(n^{1-\varepsilon})\) as \(n \rightarrow \infty\) and \(\delta < \varepsilon/2\), (see \[\text{Lemma D.4}\] in \[\text{Appendix D}\].) We use this below to show that the asymptotic result of our analysis actually applies equally to both models \[\text{Fact 4.2}\].

4. Quicksort and Search Trees

It has become folklore that the comparison costs of (classic) Quicksort are the same as the internal path length of a BST, which equals the cost to construct the tree by successive insertions. Hibbard \[17\] first described this fact in 1962, right after Hoare’s publication of Quicksort itself \[18\] \[19\]. Formally we associate a recursion tree to each execution of Quicksort: Each partitioning step contributes a node labeled with the used pivot value. Its left and right children are the recursion trees of the left and right recursive calls, respectively. See \[\text{Figure 1}\] for an example.

![Figure 1: Execution trace of Quicksort without pivot sampling and its recursion tree. The recursion tree coincides with the BST obtained by successively inserting the original input. An animated version of this example (and its extension to finge-balancing and equal keys) is available online: youtu.be/yi6syj9nksk](https://www.example.com/figure1)

Recursion trees obviously fulfill the search tree property; in fact, the recursion tree is exactly the binary search tree that results from successively inserting the input elements into an initially
empty tree (in the order they appear in the original input) if Quicksort always uses the first element of the list as pivot and partitions so that the relative order of elements smaller resp. larger than the pivot is retained (as done in our reference Quicksort, see Appendix B). Even the same set of comparisons is used in both processes, albeit in a different order.

**Pivot Sampling and Fringe Balancing.** The correspondence extends to median-of-k Quicksort with an appropriate fringe-balancing rule for BSTs. This is a little less widely known, but also well researched [10]. Upon constructing a k-fringe-balanced search tree, we collect up to k − 1 elements in a leaf; this corresponds to using Insertionsort for subproblems with n ≤ k − 1 in Quicksort. Once a leaf has collected k = 2t+1 elements, it is split: Simulating the pivot selection process in Quicksort, we find the median from the k elements in the leaf and use it as the label of a new inner node. Its children are two new leaves with the elements that did not become pivots: the t smaller elements go to the left, the t larger to the right. Because of the dynamic process of splitting leaves, the correspondence is best shown in an animation; I prepared a simple one at [youtu.be/yi6syj9nksk](youtu.be/yi6syj9nksk). Appendix C gives the full insertion procedure and some further remarks to clarify details.

**Equal Keys.** The correspondence of costs in Quicksort and search trees also extends to inputs with equal keys, if we consider a weighted path length, where the weight of each node is the multiplicity $x_v$ of its key value $v$.

We have to precisely specify how duplicate insertions are dealt with in fringe-balanced trees. This depends on where the duplicate element $x$ is found: if $x$ appears as key of an inner node, the repeated insertion is without effect; but if $x$ is found in a leaf, another copy of $x$ is added to the leaf. This unequal treatment might seem peculiar at first sight, but does exactly what we need: duplicates do play a role for selecting pivots—likely values contribute more duplicates to a random sample and are thus more likely to be selected as pivot—but once a pivot has been selected, all its copies are removed in this single partitioning step no matter how many there are. A simple inductive argument then yields the following correspondence.

**Fact 4.1 (Recursion Trees):** Consider median-of-k Quicksort with fat-pivot partitioning as described in Appendix B. For any input $U = (U_1, \ldots, U_n)$ we have that sorting $U$ with Quicksort and inserting $U$ successively into an initially empty k-fringe-balanced tree executes the same set of (ternary) key comparisons.

The costs for inserting one element are bounded by the height of the tree. It is another folklore theorem that Quicksort recursion trees have logarithmic height w.h.p.; we give the precise statement and a formal proof in Appendix E. It follows that the costs of Quicksort react “smoothly” to small changes in the profile of an input: changing the value of one input element changes the overall costs by at most $O(\log n)$ with high probability. Using the concentration of random profiles (see Section 3.3), we obtain the following statement.

**Fact 4.2 (Multiset Costs From Discrete i.i.d. Costs):** Assume $u = O(n^{1-\epsilon})$ as $n \to \infty$. The expected costs for Quicksort in the random multiset model with $x \in \mathbb{N}^u$ differ at most by $O(n^{1-\delta})$ from the costs of the corresponding discrete i.i.d. model (where $n = \Sigma x$ and $q = x/n$) for any $\delta < \epsilon/2$.

The proof follows the well-known good-split-bad-split strategy using Chernoff’s bound to show that we have not too many bad nodes w.h.p. Interestingly, the usual definition of good splits does not yield independent events, a technical problem that seems not to have been addressed in the literature so far. A slight twist in the definition of “good” suffices to restore independence, though.
5. Saturated Fringe-Balanced Trees

We now consider a \( k \)-fringe-balanced tree \( T \) built by successively inserting elements drawn i.i.d. \( D(q) \) into an initially empty tree. Instead of fixing the number of insertions up front, we consider in this section the process of indefinitely inserting i.i.d. elements; we will see in the next section that this suffices for a good approximation of Quicksort’s costs. While it may happen that not all values in the universe appear in a given input, it becomes less and less likely not to have seen all possible outcomes if we keep drawing elements. Since the universe size is finite and insertions of values already present in an inner node are without effect, this means that \( T \) reaches one of finitely many stationary states almost surely. We call such trees saturated (w.r.t. the given, fixed universe).

Stochastic Model. Let \( P \) be the label of the root of \( T \); the distribution of \( P \) is a key ingredient to (recursively) describe saturated trees. (\( P \) is also the pivot chosen in the first partitioning step of median-of-\( k \) Quicksort.) When the first leaf overflows, \( P \) is chosen as the median of the first \( k = 2t + 1 \) inserted values, which are i.i.d. \( D(q) \) distributed, so it is given by

\[
P \overset{D}{=} f^{-1}_U(\Pi),
\]

where \( \Pi \) has a Beta\((t + 1, t + 1)\) distribution, i.e., the density \( f_\Pi(z) = \frac{z^t(1-z)^t}{B(t + 1, t + 1)} \) for \( B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) the beta function. This is the generalized inversion method of random sampling, see Devroye [7, Sec.V.3.4], illustrated in our Figure 2, which is based on the fact that Beta\((t + 1, t + 1)\) is the distribution of the median of \( 2t + 1 \) i.i.d. uniformly in \((0,1)\) distributed random variables (see, e.g., [7, Sec.I.4.3]). For convenient notation, we write \( D = (D_1, D_2) = (\Pi, 1 - \Pi) \) for the induced spacings; see Figure 3.

We further denote by \( V_1 \) and \( V_2 \) the probability that a random element \( U \overset{D}{=} D(q) \) belongs to the left resp. right subtree of the root, and by \( H = \mathbb{P}[U = P] \) the probability to “hit” the root’s value. These quantities are fully determined by \( P \) (see also Figure 3):

\[
V_1 = q_1 + \cdots + q_{P-1}, \quad V_2 = q_{P+1} + \cdots + q_u, \quad H = q_P.
\]

(In the boundary case \( P = 1 \), we have \( V_1 = 0 \), and similarly \( V_2 = 0 \) for \( P = u \).) Finally, we denote by \( Z_1 \) and \( Z_2 \) the “zoomed-in” universe distributions in the left resp. right subtree:

\[
Z_1 = \left( \frac{q_1}{V_1}, \ldots, \frac{q_{P-1}}{V_1} \right), \quad Z_2 = \left( \frac{q_{P+1}}{V_2}, \ldots, \frac{q_u}{V_2} \right).
\]
Z_1 is not well-defined for P = 1; we set it to the empty vector Z_1 = () in this case. Similarly Z_2 = () for P = u.

### Expected Node Depths

Let \( T \) be a random k-fringe-balanced tree resulting from inserting i.i.d. \( D(q) \) elements until saturation. Each value \( v \in [u] \) then appears as the key of one inner node of \( T \); let \( \Gamma_v \) denote its depth, i.e., the (random) number of nodes on the path (including endpoints) from the root to the node containing \( v \) in the (random) tree \( T \). The vector \( \Gamma = (\Gamma_1, \ldots, \Gamma_u) \) is called the (random) node-depths vector of \( T \); cf. Figure 4. Finally, we write \( A_q = \Gamma^T q \). This is the average depth of a node drawn according to \( D(q) \) in the (random) tree \( T \); note that we average the costs over the searched key, but consider the tree fixed; so \( A_q \) is a random variable since \( T \) remains random: the (weighted) average node depth in a random saturated k-fringe-balanced tree.

The expected node depth, or equivalently, the expected search cost in the tree, can be described recursively: The root contributes one comparison to any searched element, and with probability \( H \) the search stops there. Otherwise, the sought element is in the left resp. right subtree with probability \( V_1 \) resp. \( V_2 \), and the expected search costs in the subtrees are given recursively by \( A_{Z_1} \) and \( A_{Z_2} \). With the notation from above, this yields a distributional recurrence for \( A_q \):

\[
A_q \overset{D}{=} 1 + V_1 A_{Z_1}^{(1)} + V_2 A_{Z_2}^{(2)}, \quad (u \geq 1), \tag{5.1}
\]

\[
A() = 0, \quad \tag{5.2}
\]

where \((A_q^{(1)})\) and \((A_q^{(2)})\) are independent copies of \( (A_q) \), which are also independent of \( (V_1, V_2, Z_1, Z_2) \).

### 6. Quicksort With Many Duplicates

Let \( C_{n,q} \) denote the number of comparisons used by our reference Quicksort to sort \( n \) i.i.d. \( D(q) \) numbers using median-of-\( k \) sampling. The dependence on \( k \) will be left implicit to keep notation readable. In this section, I derive an asymptotic approximation of \( \mathbb{E}[C_{n,q}] \) that separates the influence of \( n \) from that of \( q \): the costs of Quicksort are approximately the costs of searching \( n \) random elements from the universe distribution in a saturated tree whose shape is independent of the searched elements. Before stating this separation theorem (Theorem 6.3) formally, let us first consider an example.

**Example.** Let the universe size be \( u = 5 \). Assume that \( k \) is small, say, \( k = 3 \), and \( n \) is very large. Then it is very likely that each of the five values occurs at least \( k \) times in the input; let us assume this for the rest of the example. Each of the five values is then used as a pivot in exactly one partitioning step; all duplicates of the element are removed in this step, and only there. The leaves will thus always be empty, i.e., Insertionsort is always called for empty subranges. One possible outcome for the recursion tree is shown in Figure 4.

![Figure 4](exemplary_recursion_tree.png)

**Figure 4:** Exemplary recursion tree for \( u = 5 \). Each node represents a partitioning step, with the given pivot value. Child links correspond to child recursive calls. Empty leaves are not shown. The node-depths vector for this tree is \( \Gamma = (3, 2, 3, 1, 2) \).
The cost of a single partitioning step is one comparison per element, plus the comparisons for selecting the pivot. For example, the partitioning step corresponding to pivot 2 gets as input all elements smaller than 4, i.e., $X_1 + X_2 + X_3$ many. The number of comparisons used in this step is hence $X_1 + X_2 + X_3 \pm c \cdot k$ for some global constant $c$ depending on the median-selection algorithm. The expected number of comparisons under a discrete i.i.d. model with many duplicates. The expected number of comparisons for sorting, conditional on the recursion tree from Figure 4 are then

$$C_n = \begin{cases} 
X_1 + X_2 + X_3 + X_4 + X_5 & \pm c \cdot k \\
X_1 + X_2 + X_3 & \pm c \cdot k \\
X_3 & \pm c \cdot k \\
X_5 & \pm c \cdot k 
\end{cases}$$

Each line corresponds to one partitioning step and each column corresponds to one value of the universe. Reading the sum column-wise, we find that, up to the given error terms, sorting costs are the cost of searching each input element in the (final) recursion tree:

$$\sum_{X} (\pm 3 \cdot n \pm 3 \cdot c \cdot k)$$

For non-degenerate inputs, the recursion tree will depend only on the first $\nu$ many duplicates of each kind, we impose the following restriction on how many duplicates, we expect few enough to have non-degenerate input w.h.p., but small enough to not make a large error in the profile of the remaining $\nu - \nu_T$ elements. To deal with degenerate cases, where not all elements occur at least $k$ times. To deal with these, we impose the following restriction on how $q$ (and thus $u$) may evolve with growing $n$.

**Definition 6.1 (Many Duplicates):** Let $(q^{(n)})_{n \in \mathbb{N}}$ be a sequence of stochastic vectors, where $q^{(n)}$ has $u_{n}$ entries, i.e., $q^{(n)} \in (0,1)^{u_{n}}$ and $\sum q^{(n)} = 1$, for all $n \in \mathbb{N}$. An input of size $n \in \mathbb{N}$ under the i.i.d. model for $(q^{(n)})$ consists of the $n$ i.i.d. $\mathcal{D}(q^{(n)})$ distributed random variables. The i.i.d. model is said to have many duplicates if there is a constant $\varepsilon > 0$ so that $\mu_n = \Omega(n^{-1+\varepsilon})$ as $n \to \infty$ where $\mu_n := \min_{q^{(n)}} q^{(n)}$ is the smallest probability.

This condition ensures that every value occurs $\Omega(n^\varepsilon)$ times in expectation. (It might hence be more appropriate to say many duplicates of each kind, but I refrain from doing so for conciseness.) With many duplicates, we expect few degenerate inputs in the following sense.

**Definition 6.2 (Profile-Degenerate Inputs):** Let $\nu \in [0,1]$ and $k \in \mathbb{N}$. An input vector $U = (U_1, \ldots, U_n) \in [u]^n$ of size $n$ is called $(\nu,k)$-profile-degenerate if not all $u$ elements of the universe appear at least $k$ times in the first $n_T = \lceil n^\nu \rceil$ elements $U_1, \ldots, U_{n_T}$ of $U$. If the parameters are clear from the context or are not important, we call $U$ simply profile-degenerate.

For non-degenerate inputs, the recursion tree will depend only on the first $n_T$ elements, and the profile of the remaining $n - n_T$ elements is independent of this tree. By choosing $n_T$ large enough to have non-degenerate inputs w.h.p., but small enough to not make a large error in ignoring the first $n_T$ elements for the search costs, we obtain the following theorem; the detailed proof is given in Appendix F.

**Theorem 6.3 (Separation Theorem):** Consider median-of-$k$ Quicksort with fat-pivot partitioning under a discrete i.i.d. model with many duplicates. The expected number of comparisons fulfills

$$E[C_{n,q^{(n)}}] = E[A_{q^{(n)}}] \cdot n \pm O(n^{1-\varepsilon}), \quad (n \to \infty), \quad (7)$$
for a constant $\varepsilon > 0$; more precisely, we need $\varepsilon \in (0, \tilde{\varepsilon})$ when $\mu_n = \min_r q_r^{(n)} = \Omega(n^{-1+\varepsilon})$.

Recall that $A^{(n)}_q$ is the average node depth in a saturated $k$-fringe-balanced tree built from $D(q^{(n)})$. It depends only on the universe distribution $q = q^{(n)}$ (and $k$), but not on $n$ itself. We have therefore separated the influence of $n$ and $q$, and can investigate $\mathbb{E}[A_q]$ in isolation.

7. The Expected Node Depth

In this section we are concerned with computing $\mathbb{E}[A_q]$, the expected node depth of saturated $k$-fringe-balanced trees built from i.i.d. $D(q)$ elements. As before $k = 2t+1$. Previous work only covers the BST case ($k = 1$), where the result is known exactly: $\mathbb{E}[A_q] = 2H_q(q)+1 \leq 2H_{\ln}(q)+1$ where $H_q(q) = \sum_{1 \leq i \leq u} q_i / (q_i + \cdots + q_u)$ is the "Quicksort entropy" [1, Theorems 3.1 and 3.4]. For general fringe-balanced trees, no closed form seems in reach, but we can give an asymptotic approximation in terms of the entropy of $q$.

Theorem 7.1 (Expected Node Depth): Assume a sequence of universe distributions $(q^{(i)})_{i \in \mathbb{N}}$ for which $H_{q^{(i)}} \to_{i \to \infty} \infty$. The expected node depth of a saturated $k$-fringe-balanced tree built from i.i.d. $D(q^{(i)})$ keys is given by

$$\mathbb{E}[A_{q^{(i)}}] = \frac{H_{\ln}(q^{(i)})}{H_{k+1} - H_{t+1}} + O\left(H_{t+1} \log(H_{t+1})\right), \quad (i \to \infty). \quad (8)$$

I prove Theorem 7.1 in Appendix H. An inductive argument shows that $\mathbb{E}[A_q]$ is bounded from above and below by functions of the form $c \cdot H_{\ln}(q) + d$ for constants $c$ and $d$. I show the bounds for a whole family of pairs $(c, d)$, so that we get the same $c$ for upper and lower bound in the limit.

8. Conclusion

Combining Theorems 6.3 and 7.1 we obtain an asymptotic approximation of the comparison count in Quicksort in the discrete i.i.d. model with many duplicates. By Fact 4.2 we can translate these back to the multiset model, which proves our main result, Theorem 1.1 (page 5).

A detailed computation tracing the error terms is given in Appendix I. A particularly simple example of distributions covered by our analysis is the uniform distribution, $\mathbf{q} = \left(\frac{1}{u}, \ldots, \frac{1}{u}\right)$, where $u = u_n = O(n^{1-\varepsilon})$. This model coincides with the random $u$-ary files studied by Sedgewick [36] in his 1977 article. Since the entropy of the discrete uniform distribution is simply $H(q) = \log(u)$, we obtain $\mathbb{E}[C_{n,q}] \sim \alpha_k \log(u)$ — the same form as for random permutation only with $\log n$ replaced by $\log u$. For the special case of the uniform distribution, we can also strengthen the error bound of Theorem 7.1 for $\mathbb{E}[A_q]$ to $O(1)$ using completely different techniques; see my Ph.D. thesis [34, Sec. 8.7.7] for details.

It is remarkable that we obtain the very same constant $\alpha_k$ in the analysis of median-of-$k$ Quicksort with and without equal keys: the relative benefit of pivot sampling remains constant. All technicalities aside, we can interpret this result as follows: The average node depth in search trees on continuous i.i.d. (or randomly permuted distinct) data is of order $\log n$, whereas for discrete i.i.d. data (with many duplicates), it is of order $H(q)$; in fact, the depth is of this order with high probability. Fringe-balancing allows to control the constant of proportionality, $\alpha_k$, and has the same relative effect in both cases. Table I shows the first values of $\alpha_k$; most savings happen for small $k$.

Since $H_{k+1} - H_{(k+1)/2} \sim \ln(k+1) - \ln((k+1)/2) = \ln(2)$ as $k \to \infty$, we have $\alpha_k \to 1$, and we have confirmed the conjecture of Sedgewick and Bentley for inputs with many duplicates:
fat-pivot Quicksort with pivot sampling is optimal on randomly ordered inputs, with (many) or without equal elements.

**Extensions and Future Directions.** The methods used in this paper can readily be generalized to analyze skewed sampling, i.e., when we pick an order statistic other than the median of the sample. It is known that the number of comparisons is minimal for the median \[28\], so this might not be very interesting in its own right, but it can be used to analyze ninther sampling and similar schemes \[44, \text{Sec. 6.4}\].

I conjecture that \(\alpha_k \mathcal{H}(x/n)n\) is the correct leading term for any profile \(x \in \mathbb{N}^n\). If we have, however, a discrete i.i.d. model with \(q_v \ll n^{-1}\) so that \(v\) is unlikely to be present in the input at all, a modified “entropy” must appear instead of \(\mathcal{H}(q)\). (This case cannot occur in the multiset model since \(x_v \geq 1\).) In the limit when all individual \(q_i\) are small, this modified entropy would have to equal \(\log(n)\). Finding this unified expression is an open problem.

Another line of future research is to extend the present analysis to multiway Quicksort, e.g., the Yaroslavskiy-Bentley-Bloch dual-pivot Quicksort used in Java. The uniform case is known \[44\], but the techniques of the present paper do not carry over: the partitioning costs for such methods also depend on \(q\), which means that we do not get matching lower and upper bounds for \(E[A_q]\) using the method of Appendix H any more.

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Appendix

A. Index of Notation

In this appendix, I collect the notations used in this work. Some might be standard, but I prefer including a few more to misunderstandings caused by omissions.

A.1. Generic Mathematical Notation

\[N, N_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \ldots\text{natural numbers } N = \{1, 2, 3, \ldots\}, N_0 = N \cup \{0\}, \text{integers}\]

\[\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}, \text{rational numbers } \mathbb{Q} = \{p/q : p \in \mathbb{Z} \wedge q \in \mathbb{N}\}, \text{real numbers } \mathbb{R}, \text{and complex numbers } \mathbb{C}.\]

\[\mathbb{R}_{>1}, \mathbb{N}_{\geq 3} \text{ etc.} \ldots\text{restricted sets } X_{\text{pred}} = \{x \in X : x \text{ fulfills pred}\}.\]

\[\ln(n), \log_2(n) \ldots \text{natural and binary logarithm; } \ln(n) = \log_\text{e}(n), \log_2(n).\]

\[x \ldots \text{to emphasize that } x \text{ is a vector, it is written in \textbf{bold};}\]

\[\text{components of the vector are not written in bold: } x = (x_1, \ldots, x_d);\]

\[\text{unless stated otherwise, all vectors are column vectors.}\]

\[X \ldots \text{to emphasize that } X \text{ is a random variable it is Capitalized.}\]

\[[a, b) \ldots \text{real intervals, the end points with round parentheses are excluded, those with square brackets are included.}\]

\[[m..n], [n] \ldots \text{integer intervals, } [m..n] = \{m, m+1, \ldots, n\}; [n] = \{1..n\}.\]

\[[\text{stmt}], [x = y] \ldots \text{Iverson bracket, } [\text{stmt}] = 1 \text{ if stmt is true, } [\text{stmt}] = 0 \text{ otherwise.}\]

\[\|x\|_\infty \ldots \text{∞-norm or maximum-norm; for } x \in \mathbb{R}^d \text{ we have } \|x\|_\infty = \max_{r=1,\ldots,d} |x_r|.\]

\[x + 1, 2^x, f(x) \ldots \text{element-wise application on vectors; } (x_1, \ldots, x_d) + 1 = (x_1 + 1, \ldots, x_d + 1) \text{ and}\]

\[2^x = (2^{x_1}, \ldots, 2^{x_d}); \text{for any function } f : \mathbb{C} \to \mathbb{C} \text{ write}\]

\[f(x) = (f(x_1), \ldots, f(x_d)) \text{ etc.}\]

\[\sum x \ldots \text{“total” of a vector; for } x = (x_1, \ldots, x_d), \text{ we have } \sum x = \sum_{i=1}^d x_i.\]

\[x^T, x^T y \ldots \text{“transpose” of vector/matrix } x; \text{ for } x, y \in \mathbb{R}^n, \text{ I write } x^T y = \sum_{i=1}^n x_i y_i.\]

\[H_n, \ldots \text{n-th harmonic number; } H_n = \sum_{i=1}^n 1/i.\]

\[O(f(n)), \pm O(f(n)), \Omega, \Theta, \sim \ldots \text{asymptotic notation as defined, e.g., by Flajolet and Sedgewick [14 Section A.2]; } f = g \pm O(h) \text{ is equivalent to } |f - g| \in O(|h|).\]

\[x \pm y \ldots \text{with absolute error } |y|; \text{ formally the interval } x \pm y = [x - |y|, x + |y|]; \text{ as with}\]

\[O\text{-terms, I use “one-way equalities”: } z = x \pm y \text{ instead of } z \in x \pm y.\]

\[\Gamma(z) \ldots \text{the gamma function, } \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.\]

\[\psi(z) \ldots \text{the digamma function, } \psi(z) = \frac{d}{dz} \ln(\Gamma(z)).\]

\[B(\alpha, \beta) \ldots \text{beta function; } B(\alpha, \beta) = \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} \, dz = \Gamma(\alpha) \Gamma(\beta)/\Gamma(\alpha + \beta).\]

\[I_{\alpha, \beta}(x) \ldots \text{incomplete regularized beta function; } I_{\alpha, \beta}(x) = \int_x^\infty z^{\alpha-1} (1-z)^{\beta-1} \, dz / B(\alpha, \beta).\]
A.2. Stochastics-related Notation

\[ \mathbb{P}[E], \mathbb{P}[X = x] \] \ldots probability of an event \( E \) resp. probability for random variable \( X \) to attain value \( x \).

\[ \mathbb{E}[X] \] \ldots expected value of \( X \); I write \( \mathbb{E}[X \mid Y] \) for the conditional expectation of \( X \) given \( Y \), and \( \mathbb{E}_X[f(X)] \) to emphasize that expectation is taken w.r.t. random variable \( X \).

\( X \equiv Y \) \ldots equality in distribution; \( X \) and \( Y \) have the same distribution.

\( \mathds{1}_E, \mathds{1}_{\{X \leq 5\}} \) \ldots indicator variable for event \( E \), i.e., \( \mathds{1}_E \) is 1 if \( E \) occurs and 0 otherwise;
\( \{X \leq 5\} \) denotes the event induced by the expression \( X \leq 5 \).

\( \text{B}(p) \) \ldots Bernoulli distributed random variable; \( p \in [0, 1] \).

\( \mathcal{U}(a, b) \) \ldots discrete uniformly in \( (a, b) \subset \mathbb{R} \) distributed random variable.

\( \mathcal{U}[a, b] \) \ldots discrete uniformly in \([a..b] \subset \mathbb{Z} \) distributed random variable.

\( \mathcal{D}(p) \) \ldots discrete random variable with weights \( p \); for \( p \in [0, 1]^d \), for \( I \equiv \mathcal{D}(p) \), we have \( I \in [1..d] \) and \( \mathbb{P}[I = i] = p_i \) for \( i \in [d] \) and 0 otherwise.

\( \text{Beta}(\alpha, \beta) \) \ldots Beta distributed random variable with shape parameters \( \alpha \in \mathbb{R}_{>0} \) and \( \beta \in \mathbb{R}_{>0} \).

\( \text{Bin}(n, p) \) \ldots binomial distributed random variable with \( n \in \mathbb{N}_0 \) trials and success probability \( p \in [0, 1] \); \( X \equiv \text{Bin}(n, p) \) is equivalent to \( (X, n - X) \equiv \text{Mult}(n, p, 1 - p) \).

\( \text{Mult}(n, p) \) \ldots multinomially distributed random variable; \( n \in \mathbb{N}_0 \) and \( p \in [0, 1]^d \) with \( \Sigma p = 1 \).

\( \mathcal{H}(p), \mathcal{H}_{id}(p), \mathcal{H}_{ln}(p) \) Shannon entropy of information theory; \( \mathcal{H}_{id}(p_1, \ldots, p_d) = \sum_{r=1}^{d} p_r \log(1/p_r) \); similarly \( \mathcal{H}_{ln} \) is the base-\( e \) entropy. \( \mathcal{H}_{ln}(p_1, \ldots, p_d) = \sum_{r=1}^{d} p_r \ln(1/p_r) \);
\( I \) write \( \mathcal{H} \) for \( \mathcal{H}_{id} \).

stochastic vector \ldots \( A \) vector \( p \) is called stochastic if \( 0 \leq p \leq 1 \) and \( \Sigma p = 1 \).

A.3. Notation for the Algorithm

\( n \) \ldots length of the input array, i.e., the input size.

\( u \) \ldots universe size \( u \in \mathbb{N} \).

\( q \) \ldots probability weights of the discrete i.i.d. model; \( q \in (0, 1)^u \), \( \Sigma q = 1 \).

\( U_i \) \ldots \( i \)th element of the input; \( U_i \equiv \mathcal{D}(q) \), for all \( i \) and \( (U_1, \ldots, U_n) \) are (mutually) independent.

\( k, t \) \ldots sample size \( k = 2t + 1 \) for \( t \in \mathbb{N}_0 \);

ternary comparison \ldots operation to compare two elements, outcome is either \(<, = \) or \( >\).

\( X \) \ldots profile of the input; \( X_v \) is the number of occurrences of value \( v \) in \( U_1, \ldots, U_n \);
\( X \equiv \text{Mult}(n, q) \).
A.4. Notation for the Analysis

\( C_{n,q} \) (random) number of (ternary) comparisons needed by Algorithm B.1 to sort \( n \) i.i.d. \( D(q) \) numbers; the dependence on \( k \) is left implicit.

\( \Pi \) continuous pivot value; \( \Pi \sim \text{Beta}(t + 1, t + 1) \)

\( D = (D_1, D_2) \) continuous spacings of the unit interval (0, 1) induced by \( \Pi \), i.e., \( D = (\Pi, 1 - \Pi) \).

\( P \) (random) value of chosen pivot in the first partitioning step; \( P = F_U^{-1}(\Pi) \), for \( F_U \) the cumulative distribution function of \( D(q) \).

\( V = (V_1, V_2) \) non-pivot class probabilities, see Equation (3).

\( H \) hitting probability, Equation (3).

\( Z = (Z_1, Z_2) \) zoomed-in distributions; see Equation (4).

\( \Gamma \) node-depths vector in search tree; \( \Gamma_v \) is the random cost of searching value \( v \in [u] \).

\( A_q \) random \( q \)-weighted average node depth of a \( k \)-fringe-balanced tree built from inserting \( D(q) \) elements till saturation, \( A_q = \Gamma^T q \); see Equation (5).

B. Reference Quicksort

Algorithm B.1 is our reference fat-pivot Quicksort implementation with median-of-\( k \) sampling. It uses a simple order-preserving partitioning method that operates on linked lists.

```plaintext
QUICKSORT_k(L)
1 if (L.length) \leq k - 1
2 return INSERTIONSORT(L)
3 end if
4 P := MEDIAN(L[1 .. k])
5 L_1, L_2, L_3 := new List
6 while ¬ L.empty
7 U := L.removeFirstElement()
8 case distinction on cmp(U, P)
9 in case < do L_1.append(U)
10 in case = do L_2.append(U)
11 in case > do L_3.append(U)
12 end cases
13 end while
14 L_1 := QUICKSORT_k(L_1)
15 L_3 := QUICKSORT_k(L_3)
16 return L_1.append(L_2).append(L_3)
```

Apart from selecting the median, one partitioning step in Algorithm B.1 uses exactly \( n \) ternary comparisons (calls to cmp). As we always use the first \( k \) elements as sample elements, and add the \( t \) small sampled-out elements as the first elements to \( L_1 \) resp. the \( t \) large sampled-out elements as the first elements to \( L_3 \), we exactly mimic the behavior of a \( k \)-fringe-balanced tree (see below); Fact 4.1 is obviously fulfilled.

Our average-case results apply to typical methods like the array-based implementation by Bentley and McIlroy [4] or the simpler variant of Sedgewick and Wayne [39], as well. More
C. Fringe-Balanced Trees

precisely, for these methods Fact 4.1 holds in distribution for any i.i.d. input, be it discrete or continuous, and this is all we need.

C. Fringe-Balanced Trees

In this appendix, we give some details on fringe-balanced trees. Unfortunately, the concept appears under a handful of different names in the literature: locally balanced search trees [33], fringe-balanced trees [34], diminished trees [16], and iR / SR trees [20, 21]. I use the term fringe-balanced trees, since it is the most vivid term and has been widely adopted in the analysis-of-algorithms community, see, e.g., the relatively recent monograph [10] by Drmota.

Along with the different names come slight variations in the definitions; we remark that our definition (deliberatively) differs a bit in the base cases from usual definitions to precisely mimic Quicksort recursion trees.

A $k$-fringe-balanced tree, for $k = 2t + 1$ an odd integer, is a binary search tree whose leaves can store between 0 and $k - 1$ elements. An empty fringe-balanced tree is represented as Leaf(), a single empty leaf. The $k$-fringe-balanced tree $T$ corresponding to a sequence of elements $x_1, \ldots, x_n$ is obtained by successively inserting the elements into an initially empty tree using Algorithm C.1 more explicitly, with $T_0 = \text{Leaf}()$ and $T_i = \text{Insert}_k(T_{i-1}, x_i)$ for $1 \leq i \leq n$, we have $T = T_n$.

Algorithm C.1. Insert into $k$-fringe-balanced tree.

\begin{algorithm}
\begin{algorithmic}
\State \textbf{if} $T$ is $\text{Leaf}(U)$
\State \hspace{1em} Append $x$ to $U$
\State \hspace{1em} \textbf{if} $|U| \leq k - 1$ \textbf{then} \text{return} \hspace{1em} \text{Leaf}(U) \hspace{1em} \textbf{end if}
\hspace{1em} \text{// Else: Split the leaf}
\State $P := \text{Median}(U_1, \ldots, U_k)$
\State $C_1, C_2 := \text{new empty list}$
\State \hspace{1em} \textbf{for} each $U$ in $U$
\State \hspace{2em} \textbf{case distinction} on $\text{cmp}(U, P)$
\State \hspace{3em} \textbf{in case} $U < P$ \textbf{do} append $U$ to $C_1$
\State \hspace{3em} \textbf{in case} $U > P$ \textbf{do} append $U$ to $C_2$
\hspace{2em} \text{// In case $U = P$, we drop $U$.}
\State \hspace{1em} \textbf{end cases}
\State \hspace{1em} \textbf{end for}
\State \textbf{return} $\text{Inner}(P, \text{Leaf}(C_1), \text{Leaf}(C_2))$
\State \textbf{else} $T$ is $\text{Inner}(P, T_1, T_2)$
\State \hspace{1em} \textbf{case distinction} on $\text{cmp}(x, P)$
\State \hspace{2em} \textbf{in case} $x = P$ \textbf{do} \text{return} $T$ \hspace{1em} \text{// tree unchanged}
\State \hspace{2em} \textbf{in case} $x < P$ \textbf{do} \text{return} $\text{Inner}(P, \text{Insert}_k(T_1, x), T_2)$
\State \hspace{2em} \textbf{in case} $x > P$ \textbf{do} \text{return} $\text{Inner}(P, T_1, \text{Insert}_k(T_2, x))$
\State \hspace{1em} \textbf{end cases}
\State \textbf{end if}
\end{algorithmic}
\end{algorithm}

Growing trees with Algorithm C.1 enforces certain shapes upon the lowest subtrees, i.e., at the fringe of the tree, hence the name fringe-balanced. Searching an element in a fringe-balanced tree works as in an ordinary BST, except for the leaves, where we sequentially search through the buffer; Algorithm C.2 shows pseudocode for completeness. (Note that elements collected in leaves are not sorted.)
Algorithm C.2. Search in fringe-balanced trees.

\begin{algorithm}
\begin{algorithmic}[1]
\Function{Search}{$T, x$}
\If{$T$ is Leaf($U$)}
\State \Return \Call{SequentialSearch}{$U, x$}
\ElsIf{$T$ is Inner($P, T_1, T_2$)}
\State \textbf{case distinction} on $\text{cmp}(U, P)$
\State \textbf{in case} $U = P$ \textbf{do} \Return "Found"
\State \textbf{in case} $U < P$ \textbf{do} \Return \Call{Search}{$T_1, x$}
\State \textbf{in case} $U > P$ \textbf{do} \Return \Call{Search}{$T_2, x$}
\EndIf
\EndFunction
\end{algorithmic}
\end{algorithm}

Many parameters like path length, height and profiles of fringe-balanced trees have been studied when the trees are built from a random permutation of $n$ distinct elements, see, e.g., Drmota [10]. The case of equal elements has not been considered except for the trivial case $k = 1$, i.e., ordinary BSTs; see Kemp [25], Archibald and Clément [2].

D. Stochastic Preliminaries

For the reader’s convenience, we collect a few basic tail bounds here: a classical and less known Chernoff concentration bound, and a bound for the far end of the lower tail.

**Lemma D.1 (Chernoff Bound):** Let $X \overset{\text{d}}{=} \text{Bin}(n, p)$ for $n \in \mathbb{N}$ and $p \in (0, 1)$ and let $\delta \geq 0$. Then

$$\Pr\left[\left|\frac{X}{n} - p\right| \geq \delta\right] \leq 2 \exp(-2\delta^2 n). \tag{9}$$

This bound appears, e.g., as Theorem 2.1 of McDiarmid [29].

The following lemma is a handy, but less well-known bound for the multinomial distribution that appears—indeed rather hidden—as Lemma 3 in a paper by Devroye [6] from 1983. (Its proof is also discussed on math stack exchange: [math.stackexchange.com/q/861058](https://math.stackexchange.com/q/861058)).

**Lemma D.2 (Chernoff Bound for Multinomial):** Let $X \overset{\text{d}}{=} \text{Bin}(n, p)$ for $n \in \mathbb{N}$ and $p \in (0, 1)^u$ with $\Sigma q = 1$. Further, let $\delta \in (0, 1)$ with $\delta \geq \sqrt{20u/n}$ be given. Then

$$\Pr\left[\sum_{i=1}^{u} \left|\frac{X_i}{n} - p_i\right| \geq \delta\right] \leq 3 \exp(-\delta^2 n/25) \tag{10}$$

**Lemma D.3 (Far-End Left-Tail Bound):** Let $X^{(n)} \overset{\text{d}}{=} \text{Bin}(n, p^{(n)})$ be a sequence of random variables and $k \in \mathbb{N}$ a constant, where $p^{(n)}$ satisfies $p^{(n)} = \omega\left(\frac{\log n}{n}\right)$ as $n \to \infty$ and is bounded away from 1, i.e., there is a constant $\varepsilon > 0$ so that $p^{(n)} \leq 1 - \varepsilon$ for all $n$. Then $\Pr[X^{(n)} \leq k] = o(n^{-c})$ as $n \to \infty$ for any constant $c$.

The requirement that $p^{(n)}$ be bounded away from 1 can be lifted, but it simplifies the proof. For the application in the present paper the version is sufficient as is.
Proof of Lemma D.3: For better readability, we drop the superscript from \( p^{(n)} \) when \( n \) is clear from the context. Let \( c \) be an arbitrary constant.

\[
P[X^{(n)} \leq k] \cdot n^c = n^c \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \quad (11)
\]

\[
\leq n^c \sum_{i=0}^{k} \frac{1}{i!} \left( \frac{p}{1-p} \right)^i n^i (1-p)^n \quad (12)
\]

\[
\leq n^{c+k} (1-p)^n \cdot O(1) \quad (13)
\]

\[
= \exp \left( n \ln(1-p) + (c+k) \ln(n) \right) \cdot O(1) \quad (14)
\]

using \( \ln(x) \leq x - 1 \), this is

\[
\leq \exp \left( -np \pm O(\log n) \right) \cdot O(1) \quad (15)
\]

\[
\rightarrow 0 \quad (16)
\]

since \( p = \omega \left( \frac{\log n}{n} \right) \). This proves the claim. \( \square \)

We give an immediate consequence of the above bounds on the relation of multiset models and discrete i.i.d. models: when a discrete i.i.d. model has expected profile \( x \), the random profile \( X \) is actually very close to \( x \) w.h.p..

Lemma D.4 (Concentration for Profiles): Let \( x \in \mathbb{N}_n^0 \) be given and set \( X \overset{\text{D}}{=} \text{Mult}(n, q) \) with \( n = \Sigma x \) and \( q = x/n \). Then we have that

\[
P[X = x \pm \delta n] \leq \begin{cases} 2u \exp(-2\delta^2 n), & \text{for all } \delta \geq 0; \\ 3 \exp(-\delta^2 n/25), & \text{for all } \delta \geq \sqrt{20u/n}. \end{cases} \quad (17)
\]

Asymptotically when \( u \in O(n^\nu) \) for \( \nu \in [0, 1) \) we have

\[
X = x \pm c n^{(1+\nu)/2+\varepsilon} \quad \text{with probability at least } 1 \pm o(n^{-d}) \quad (18)
\]

for any combination of constants \( c > 0 \) and \( \varepsilon > 0 \) and \( d \).

Proof: We start noting that for any \( \delta \in (0, 1) \) it holds that

\[
P \left[ \left\| \frac{X}{n} - q \right\|_\infty \geq \delta \right] = P \left[ \bigvee_{i=1}^u \left| \frac{X_i}{n} - q_i \right| \geq \delta \right]. \quad (19)
\]

We can now either continue with the union bound for any \( \delta \geq 0 \) to find

\[
P \left[ \left\| \frac{X}{n} - q \right\|_\infty \geq \delta \right] \leq \sum_{i=1}^u P \left[ \left| \frac{X_i}{n} - q_i \right| \geq \delta \right] \quad (20)
\]

\[
\leq u \cdot 2 \exp(-2\delta^2 n) \quad (21)
\]

[Lemma D.1]
using the classical Chernoff bound, or we apply the multinomial Chernoff bound to
\[
\Pr \left[ \left\| X/n - q \right\|_\infty \geq \delta \right] \leq \Pr \left[ \sum_{i=1}^u \left\| X/n - q_i \right\|_\infty \geq \delta \right] \leq 3 \exp(-\delta^2 n/25). \tag{22}
\]
which requires \( \delta \geq \sqrt{20u/n} \). Combining both paths yields the first part of the claim.

For the second part, let the constants \( \nu \in [0,1) \), \( \varepsilon > 0 \), \( c > 0 \) and \( d \) be given. Assume \( u = O(n^\nu) \), i.e., \( u \leq \mu n^\nu \) for a constant \( \mu > 0 \) and large enough \( n \). We obtain an asymptotically valid choice of \( \delta \) for Equation (23) whenever \( \delta = \omega(n^{(\nu-1)/2}) \); then for large enough \( n \) we will have \( \delta > \sqrt{20dn^{(\nu-1)/2}} \geq \sqrt{20u/n} \). Setting \( \delta = cn^{(\nu-1)/2+\varepsilon} = \omega(n^{(\nu-1)/2}) \) is thus valid and we obtain for large \( n \) that
\[
n^d \cdot \Pr \left[ X \notin x \pm cn^{(\nu+1)/2+\varepsilon} \right] = n^d \cdot \Pr \left[ \left\| X - x \right\|_\infty \geq n\delta \right] = n^d \cdot \Pr \left[ \left\| X/n - q \right\|_\infty \geq \delta \right] \leq 3n^d \exp(-\delta^2 n/25) \tag{26}
= 3 \exp(-n^{\nu+2\varepsilon}/25 + d \ln n) \tag{27}
\rightarrow 0 \quad (n \rightarrow \infty), \tag{28}
\]
which implies the claim. \( \square \)

E. Height of Recursion Trees

In this appendix, we recapitulate the well-known folklore result that Quicksort recursion trees—or equivalently (fringe-balanced) binary search trees—have logarithmic height with high probability. To be specific, we cite here a result of Mahmoud \cite{mahmoud} (page 101) on the height of (ordinary) binary search trees:

**Lemma E.1 (BSTs Have Log-Height with High Probability):**
For any \( \varepsilon > 0 \) it holds: The probability that a binary search tree built from a random permutation of \( n \) distinct elements has height \( \geq (\alpha + \varepsilon) \ln n \) is at most \( Kc_x n^{-\eta_x} = O(n^{-\eta_x}) \) where \( K > 0 \) is some constant (independent of \( \varepsilon \)), \( c_x = 1/(\Gamma(x)(1-2/x)) \), \( \eta_x = -(x \ln(2/x) + x - 1) \) and \( \alpha \approx 4.31107 \) is the unique root of \( \eta_x \) for \( x \in (4,5) \).

By Fact 4.1 this result translates immediately to the height of Quicksort recursion trees.

Intuitively, the probability for a certain height in a \( k \)-fringe-balanced search tree is at most that for an ordinary binary search tree: Let \( \hat{J} \) and \( \hat{J}^{(k)} \) be maximum of the two subproblem sizes of the root node in an ordinary BST and a \( k \)-fringe-balanced tree, respectively. Since subproblem sizes are more balanced in fringe-balanced trees, \( \hat{J} \) is typically larger than \( \hat{J}^{(k)} \), so that inductively, the same holds for the heights. Similarly, considering a discrete i.i.d. input instead of a random permutation makes the tree no higher since the potentially removed duplicates of the pivot are subtracted from subproblem sizes.

Mahmoud’s derivation is based on extensive knowledge on random BSTs (also given in \cite{mahmoud}), in particular he uses the exact expected number of leaves at any given level. Generalizing this for fringe-balanced trees with duplicates seems a daunting task.
Variants of Lemma E.1 are sometimes covered in (advanced) algorithms courses and textbooks, see, e.g., Exercise 4.20 of Mitzenmacher and Upfal [31] Section 2.4 of Dubhashi and Panconesi [11], and Section 4.7 of Erickson [12]. There, a more intuitive argument is given by bounding the probability of many “bad” splits at nodes using Chernoff bounds, however neither of these resources gives a detailed formal proof. The argument is appealingly simple, and fairly easy to extend, but it indeed requires some care to really guarantee the needed independence. We therefore give a detailed proof below and obtain the following result.

Lemma E.2 (Logarithmic Height of Recursion Trees With High Probability): Let $T$ be a $k$-fringe-balanced search tree built by inserting $n$ i.i.d. $D(q)$ numbers into an initially empty tree. Then $T$ has height $O(\log n)$ w.h.p.; more precisely $T$ has height $\geq c \ln n$ with probability $\leq 2n^\eta$ for $n \geq n_0 = e^{30000}$ (i.e., with probability $O(n^\eta)$), where

$$\eta = 1 - 2c\delta^2$$

$$\delta = p - \frac{1}{c} \cdot \left( \frac{1}{\ln(1/\alpha)} + 1 \right)$$

$$p = 0.99 - 2I_{\alpha=0.01.1}(t+1, t+1) = 0.99 - 2 \frac{k!}{t!t!} \int_\alpha^{1} x^t(1-x)^{t} \, dx$$

where $\alpha \in (\frac{1}{7}, 1)$ is a parameter that can be chosen arbitrarily as long as $\delta > 0$. This result is independent of $q$, and holds in particular when $q$ depends on $n$.

| $k$ | $c$ | $\eta$ | $k$ | $c$ | $\eta$ |
|-----|-----|--------|-----|-----|--------|
| 1   | 12  | -1.72  | 5   | 7   | -0.70  |
| 1   | 13  | -2.86  | 5   | 8   | -2.24  |
| 1   | 20  | -11.53 | 5   | 20  | -22.45 |
| 3   | 9   | -1.94  | $\infty$ | 4  | -2.03  |
| 3   | 10  | -3.37  | $\infty$ | 5  | -4.01  |
| 3   | 20  | -19.02 | $\infty$ | 20 | -33.71 |

Table E.1: A few exemplary values for the constants in Lemma E.2

The given constants are certainly not best possible; they could be improved by a stronger version of the Chernoff bound; also at the price of making $n_0$ even more ridiculously large, some numbers can be tweaked further. It is not my intention to do so here.

We note that for Quicksort, an alternative route is to analyze a modified version of the algorithm that makes some technicalities vanish and performs no better than the original Quicksort; see, e.g., Seidel [40]. Moreover, much stronger concentration results are known for the overall number of comparisons in Quicksort, see McDiarmid and Hayward [30] or the streamlined description in Section 7.6 of Dubhashi and Panconesi [11]. There the exponent of the probability bound is arbitrarily large for one fixed bound $c \ln n$. It seems not possible to obtain such a stronger bound for the height of the tree, though.

Concentration results are typically framed in terms of randomized Quicksort. To emphasize the relation between Quicksort recursion trees and search trees, our reference Quicksort (Algorithm B.1) is not randomized, but deterministically chooses the first elements for the sample. Causing an exceptionally high recursion tree can hence be attributed to a specific input in our scenario; the following definition expresses that idea.

Definition E.3 (Height-degenerate Inputs): An input of length $n$ is called $h$-height-degenerate (w.r.t. our reference fat-pivot median-of-$k$ Quicksort) if the recursion tree of QUICKSORT$_k$ on this input has a height $> h \ln(n)$.
From Lemma E.2 we immediately obtain the following fact.

**Corollary E.4 (Probability of Height-Degeneracy):** For any $k$, the probability that an input of $n$ elements is $13$-height-degenerate is in $O(1/n^2)$ as $n \to \infty$.

We will in the following simply use “height-degenerate” to mean $13$-height-degenerate.

---

**Proof of Lemma E.2** Let $k = 2t + 1$ be given. We will follow the folklore proof that the height of randomly grown BSTs is typically logarithmic: we determine a constant probability $p > 0$ (independent of $n$) so that a single partitioning step yields a reasonably balanced split; since long paths in the recursion tree cannot contain more than a certain number of such balanced nodes, we can bound the probability of seeing such a long path in a recursion tree.

**Outline.** A few technicalities need to be addressed in the formal proof:

1. To tweak constants, we will introduce a parameter $\alpha \in (1/2, 1)$ and require subproblems at a node $v$ to contain at most $\alpha n(v)$ elements, where $n(v)$ is the size of the sublist that $v$ corresponds to.

2. The probability to lose a given fraction of elements depends on the subproblem size (a discretization effect) and in our case of inputs with duplicates also on $q$. We can therefore only lower bound the probability for a balanced node by a constant $p$. To obtain a term for $p$ that we can actually evaluate, we resort to asymptotic approximations, which are only a valid lower bound for sublist sizes larger than a threshold $n_0$.

3. Since the precise probability to lose a fraction of elements depends on the sublist size, also the events that a certain node be balanced are dependent.

The last point is a problem since the standard Chernoff bounds requires mutual independence; nevertheless this issue is not addressed in the sources cited above. Since the pivot choices themselves are done independently, we only need a little trick to obtain independent events: A node is only considered good when it is balanced and additionally a biased coin flip yields heads, where the bias is chosen so that the overall probability for a good node is the same for all nodes. This gives us a sequence of independent indicator variables to which we can apply our machinery as usual.

**Balanced nodes.** We use the following notation in this proof. $v$ denotes a node of the recursion tree. By $n(v)$ we mean the sublist size at $v$, i.e., the number of elements in the subproblem of the recursive call that $v$ corresponds to. $d(v)$ denotes the depth of $v$, i.e., the number of nodes on the path from the root to $v$, including endpoints. Finally, if $n(v) > k$, we use $J_r(v)$, $r = 1, 2$, to denote the size of the $r$th subproblem at $v$; this is the subproblem size of $v$’s left resp. right child.

We are now in the position to formalize the notion of balanced nodes: Let $\alpha \in (1/2, 1)$ be a fixed number and $n_0 \geq k$ a constant (to be chosen later). We call an inner node $\alpha$-balanced if $n(v) \leq n_0$ or if $J_r(v)/n(v) \leq \alpha$ for both $r = 1, 2$. ($(\alpha, n_0)$)-balanced would be more appropriate; the dependence on $n_0$ is understood implicitly). An $\alpha$-balanced node hence has no subproblem with more than an $\alpha$-fraction of the elements, or has a negligibly small sublist anyway.

The key idea is now that any path in a recursion tree for $n$ elements can contain at most

$$\log_\alpha(n_0/n) = \log_{1/\alpha}(n/n_0) = \frac{1}{\ln(1/\alpha)} \ln(n/n_0) \leq \frac{1}{\ln(1/\alpha)} \ln(n)$$  (32)
\(\alpha\)-balanced nodes before reaching a node \(v\) with \(n(v) \leq n_0\) since considering only the size reduction at these \(\alpha\)-balanced nodes already reduces the \(n\) initial elements to \(\leq \alpha^{\log_\alpha(n_0/n)} n = n_0\) elements. From there on, at most \(n_0\) additional \(\alpha\)-balanced nodes can follow since each reduces the subproblem size by at least one.

**Balanced is not good enough.** We are now formalizing the idea sketched above to obtain independent indicator variables. For a node with \(n(v) \geq n_0\), we define \(p_b(v) = \mathbb{P}[v \text{ \(\alpha\)-balanced } | n(v)]\). Note that \(p_b(v)\) only depends on \(n(v)\) and \(k\), but since the number of its possible subproblems sizes is finite, \(p_b(v)\) will necessarily differ for different values of \(n(v)\), even without pivot sampling \((k = 1)\) and without our threshold \((n_0 = 0)\).

However, we will show below that we can find choices for \(n_0\) and \(\alpha\) so that at least \(p_b(v) \geq p\) for a given constant \(p = p(\alpha)\) in all possible trees.

For such a \(p\), we call a node \(v\) \((\alpha, p)\)-good if it is \(\alpha\)-balanced and additionally \(B(v) = 1\), where \(B(v) \equiv B\left(\frac{p}{p_b(v)}\right)\) which is independent of all other random variables. The distribution of \(B(v)\) is conditional on the given tree via \(n(v)\), so one might imagine first drawing a random recursion tree, and then assigning its nodes labels good or bad.

Since every good node is also balanced, we cannot have more than \(\frac{1}{\ln(1/\alpha)} \ln(n) + n_0\) good nodes on any path in a recursion tree for input size \(n\).

**Probability of long paths.** We can now bound the probability of having a tree of height \(\geq c \ln n\). First note that the overall number of nodes is bounded by \(n\) (at least one pivot is removed in each step), so the number of leaves in the recursion tree is trivially bounded by \(n\).

By the union bound, the probability that any of these leaves has depth \(\geq h\) is at most \(n\) times the probability that one leaf has depth \(\geq h\). Let hence \(v\) be one of the leaves in the recursion tree and let \(v_1, \ldots, v_{d(v)} = v\) be the nodes on the path from the root \(v_1\) down to \(v\); recall that \(d(v)\) is the depth of leaf \(v\).

The sublist corresponding to \(v\) is the result of \(d(v) - 1\) successive partitioning steps, each of which is either \((\alpha, p)\)-good or not. Let \(G_1, \ldots, G_{d(v) - 1}\) be the corresponding indicator random variables where \(G_i\) is 1 if and only if \(v_i\) is good. By construction, we have \(\mathbb{P}[G_i = 1] = p_b(v) \cdot \frac{p}{p_b(v)} = p\) independently of the tree. We now extend \(G_1, \ldots, G_{d(v) - 1}\) to an infinite sequence of random variables by i.i.d. \(B(p)\) variables for all \(i \geq d(v)\); the \(G_i\) then form are all i.i.d. random variables.

Now recall that the number of \(\alpha\)-balanced nodes on any path is at most \(n_0 + \frac{1}{\ln(1/\alpha)} \ln n\). For any \(h \in \mathbb{N}\), we thus have

\[
\mathbb{P}[d(v) \geq h] \leq \mathbb{P}\left[G_1 + \cdots + G_h \leq n_0 + \frac{1}{\ln(1/\alpha)} \ln n\right] \quad (33)
\]

\[
= \mathbb{P}\left[X_h \leq \hat{\gamma} \ln n\right], \quad (34)
\]

\[
\leq \mathbb{P}\left[X_h \leq \gamma \ln n\right], \quad (35)
\]

where \(X_h \overset{D}{=} \text{Bin}(h, p)\)

\[
\hat{\gamma} = \frac{1}{\ln(1/\alpha)} + \frac{n_0}{\ln n}, \quad (36)
\]

\[
\gamma = \frac{1}{\ln(1/\alpha)} + 1 \geq \hat{\gamma}, \quad (n \geq e^{n_0}). \quad (37)
\]
For \( n \geq e^{n_0} \) and \( h \) so that \( \delta := p - \gamma \ln(n)/h > 0 \) we then have
\[
\begin{align*}
P[d(v) \geq h] & \leq P[X_h \leq \gamma \ln n] \quad \text{(39)} \\
& = P[p - \frac{X_h}{h} \geq p - \gamma \frac{\ln(n)}{h}]; \quad \text{(40)} \\
& \leq P \left[ \left| \frac{X_h}{h} - p \right| \geq \delta \right] \quad \text{(41)} \\
& \leq 2 \exp(-2\delta^2 h). \quad \text{(42)}
\end{align*}
\]

With \( h = c \ln(n) \), we have \( \delta = p - \frac{\gamma}{c} \), which is independent of \( n \), and positive for any \( c > \gamma/p \).

With this bound we finally find
\[
\begin{align*}
P[T \text{ has height } \geq c \ln n] & \leq n P[d(v) \geq c \ln n] \quad \text{(43)} \\
& \leq 2n \exp(-2c\delta^2 \ln n) \quad \text{(44)} \\
& = 2n^{1-2c\delta^2}, \quad \text{(45)}
\end{align*}
\]

which implies the claim.

**A lower bound for balanced nodes: how to choose the constants.** It remains to show that we can actually find values \( n_0 \) and \( p = p(\alpha) \) (at least for some choices of \( \alpha \)) so that so that \( p_b(v) \geq p \) in all nodes \( v \) in all possible trees \( T \).

To this end, we derive an upper bound for the probability \( 1 - p_b(v) \) that the root \( v \) of \( T \) is not \( (\alpha, n_0) \)-balanced. For \( n \leq n_0 \), we are done since \( p_b(v) = 1 \) and any \( p > 0 \) will do. So assume \( n > n_0 \). By the union bound we have
\[
P[v \text{ not } \alpha\text{-balanced}] = P[\exists r : J_r^{(n)} \geq \alpha n] \leq \sum_r P[J_r^{(n)} \geq \alpha n], \quad \text{(46)}
\]

so it suffices to consider the subproblems \( r = 1, 2 \) in isolation.

Recall that after a pivot value \( P \) is chosen according to Equation (2) on page 9, the probabilities \( V_r \) for any other element to belong to the \( r \)th subproblem are fully determined. \( P \) is turn is fully determined by the choice of \( D \). Hence, conditionally on \( D \), we have Bin\((n - k, V_r)\) elements that go to the \( r \)th subproblem, plus up to \( t \) from the sample. Moreover we always have \( V_r \leq D_r \) (cf. Figure 3). Conditional on \( D \), \( J_r^{(n)} \) is hence smaller than \( \tilde{J}_r \equiv \text{Bin}(n, D_r) + t \) in stochastic order, i.e., for all \( j \) we have that \( P[J_r \geq j \mid D] \leq P[\tilde{J}_r \geq j \mid D] \). (By averaging over all choices for \( D \) the same relation holds also unconditionally.)

This is nothing but the precise formulation of the fact that (in stochastic order) subproblem sizes for inputs with duplicates are no larger than for random-permutation inputs, since we potentially exclude duplicates of pivots from recursive calls.

The good thing about \( D_r \) is that — unlike \( J_r \) — it does not depend on \( n \): \( D_r \equiv \text{Beta}(t+1, t+1) \) in every node. Also — unlike \( V_r \) — it does not depend on \( q \). Since \( J_r \) is concentrated around \( nD_r \), \( J_r \) is likely to be \( \geq \alpha n \) only for \( D_r \gtrsim \alpha \). Precisely for \( \delta > 0 \) a constant we have
\[
P[\tilde{J}_r \gtrsim (D_r + \delta)n \mid D_r] \leq P\left[\frac{\tilde{J}_r}{n} - D_r \gtrsim \delta \mid D_r\right] \leq 2 \exp(-2\delta^2 n) \quad \text{(47)}
\]

\[\leq 2 \exp(-2\tilde{\delta}^2 n) \quad \text{for any } \tilde{\delta} < \delta. \quad \text{(49)}\]
Using this and separately considering $D_r$ larger resp. smaller $\alpha - \delta$ yields

\[
\mathbb{P}[J_r \geq \alpha n] \leq \mathbb{P}[\tilde{J}_r \geq \alpha n] = E_D[I_{\{D_r < \alpha - \delta\}} \cdot \mathbb{P}[J_r \geq \alpha n \mid D_r = d, d < \alpha - \delta]] + E_D[I_{\{D_r > \alpha - \delta\}} \cdot \mathbb{P}[J_r \geq \alpha n \mid D_r = d, d > \alpha - \delta]]
\]

(50)

\[
\leq \mathbb{P}[D_r < \alpha - \delta] \cdot 2 \cdot \exp(-2\delta^2 n) + \mathbb{P}[D_r > \alpha - \delta] \cdot 1
\]

(51)

if we choose, say $\delta = 0.01$, we have $2 \cdot \exp(-2\delta^2 n) \leq 0.005$ for $n \geq n_0 = 30000$

\[
\leq \mathbb{P}[D_r > \alpha - 0.01] + 0.005 \quad (n \geq n_0).
\]

(52)

Plugging in above, we find that with $n_0 = 30000$, we can choose

\[
p = 0.99 - 2 \cdot I_{\alpha-0.01,1}(t+1, t+1) \leq \mathbb{P}[v \text{ $\alpha$-balanced}] \quad (n \geq n_0).
\]

(53)

Since $p = p(\alpha)$ is continuous and $\geq 0.97$ for $\alpha = 1$ there is always a valid choice $\alpha < 1$ with $p > 0$. We are free to choose any such $\alpha$; the resulting constant $c$ for the achieved height bound then has to satisfy $c > (1 + 1/\ln(1/\mu))/p$. It is not clear in general which choice yields the best bounds, so we keep it as a parameter in the analysis. \qed

Remarks. Two remarks are in order about Lemma E.2.

- **Height-Bound for any input.**
  The attentive reader might have noticed that we do not make use of the assumption that the input consists of $D(q)$ elements. In fact, the above proof works for any randomly permuted input, since we actually compute the subproblem sizes in the most unfavorable case: when all elements are distinct.

  For randomized Quicksort, the random-order assumption is also vacuous; we thus have proved the much more general statement that randomized Quicksort has $O(\log n)$ recursion depth w.h.p. for any input.

- **Height-Bound in Terms of $q$.**
  For saturated trees, our bound on the height in terms of $n$ is meaningless. By similar arguments as above we can show that the height is in $O(\log(1/\mu))$ with high probability as $1/\mu \to \infty$. Here $\mu$ is the smallest probability $q_v$: Intuitively, after $\ln(1/\mu)$ balanced subdivisions of the unit interval, we are left with segments of size less than $\mu$, so after so many partitioning rounds, we have reduced the subuniverse sizes to 1. The subproblems are then solved in one further partitioning step.

  This bound is intuitively more appealing, but for our use case in the proof of the separation theorem in Appendix F we are dealing with non-saturated trees and the $\log n$ bound turns out more convenient. (There we only require that $1/\mu$ does not grow too fast with $n$, but we do not have any guarantee that it grows at all. The height-bound $c \log(1/\mu)$ only holds with high probability as $1/\mu$ goes to infinity; we would then need a case distinction on the growth rate of $1/\mu \ldots$)

F. Proof of the Separation Theorem

We basically have to show that degenerate inputs are so unlikely that we can ignore them, and use this to separate the distribution of recursion trees and search costs. We then address the costs of sampling pivots, and put the results together.
Probability of Degenerate Profiles. We start noting the following basic fact.

Fact F.1: In any discrete i.i.d. model, \( u_n \leq \frac{1}{1+\varepsilon} \) for all \( n \). In particular, an i.i.d. model with many duplicates has \( u_n = O(n^{1-\varepsilon}) \).

Proof: Since \( \mu_n \) is the smallest entry of \( q^{(n)} \) we have \( 1 = \sum q^{(n)} \geq u_n \mu_n \), so \( u_n \leq 1/\mu_n \). The second part follows directly from the definition.

We can bound the probability of degenerate inputs using Chernoff bounds. The elementary far-end lower-tail bound [Lemma D.3](#) actually yields the following slightly stronger asymptotic result.

Lemma F.2 (Non-Degenerate w.h.p.): Assume a discrete i.i.d. model with \( \mu_n = \min_v q_v^{(n)} = \Omega(n^{-\rho}) \) for \( \rho \in (0,1) \) and let \( k \in \mathbb{N} \) and \( \rho < \nu < 1 \). Then the probability of an input of size \( n \) to be \((\nu,k)\)-profile-degenerate is in \( o(n^{-c}) \) for any constant \( c \).

Proof: Let \( \rho \in [0,1), k \) and \( \nu \in (\rho,1) \) be given. Set \( \varepsilon = \nu - \rho > 0 \) and denote by \( Y = Y^{(n)} \) the profile of the first \( nT \) elements of the input. Clearly \( Y^{(n)} \) is \( \Omega(nT; q^{(n)}) \). Assume w.l.o.g. that the minimal probability is always \( q_1^{(n)} = \mu_n \). A standard application of the union bound yields

\[
\Pr[-Y^{(n)} \geq k] = \Pr \left[ \bigvee_{v=1}^{u_n} Y_v^{(n)} < k \right] \leq \sum_{v=1}^{u_n} \Pr[Y_v^{(n)} < k] \leq u_n \cdot \Pr[Y_1^{(n)} < k].
\]

Now \( Y_1^{(n)} \overset{D}{=} \text{Bin}(nT,\mu_n) \) with \( \mu_n = \Omega(n^{-\rho}) = \Omega(n^{-\rho/[\nu T]} = \omega(\log \text{num}) \), and we always have \( \mu_n \leq \frac{1}{2} < 1 \), so we can apply [Lemma D.3](#) for any given constant \( c \), we have \( \Pr[Y_1^{(n)} < k] = o(nT^{-c+1}/\nu) \). By Fact F.1, \( u_n = o(n \log n) = o(n) \), so we find

\[
n^c \cdot \Pr[-Y^{(n)} \geq k] \leq n^c u_n \Pr[Y_1^{(n)} < k] \overset{(55)}{=} o(n^{c+1}) \cdot o \left( \frac{n^{-c+1}}{\nu T} \right) \overset{(56)}{=} o(1),
\]

since \( nT \sim n^\nu \). So the input is \((\nu,k)\)-degenerate with high probability.

Proof of [Theorem 6.3](#) In [Theorem 6.3](#) we assume a discrete i.i.d. model with many duplicates, i.e., \( \mu_n = \Omega(n^{1+\varepsilon}) \) with \( \varepsilon \in (0,1) \), and an \( \varepsilon \in (0,\varepsilon) \) is given. We set

\[
\nu := \frac{(1-\varepsilon) + (1-\varepsilon)}{2} = 1 - \frac{\varepsilon + \varepsilon}{2} \in (1-\varepsilon,1-\varepsilon).
\]

Then, by [Lemma F.2](#), an input is \((\nu,k)\)-degenerate with probability in \( o(n^{-c}) \) for all \( c \). This also means that the overall cost contribution of degenerate inputs to expected costs is in \( o(n^{-c}) \) for all \( c \), and hence covered by the error term in [Equation 7](#) since costs for any input are at most quadratic in \( n \).

We will thus, for the remainder of this proof, assume that the input is not \((\nu,k)\)-degenerate, i.e., each of the values of the universe appears at least \( k \) times among the first \( nT = \lceil n^\nu \rceil \) elements.
Independence of Recursion Trees. We now turn to the distribution of the recursion trees. The shape of the recursion tree is determined by at most $u \cdot k$ elements: we have at most $u$ partitioning rounds since each of the $u$ elements of the universe becomes a pivot in at most one partitioning step, and each partitioning step inspects $k$ elements for choosing its pivot.

Also, for each of the $u$ values in the universe, at most the first $k$ occurrences in the input, reading from left to right, can influence the tree: if a value $v \in [u]$ is already contained in an inner node, all further duplicates of $v$ are ignored. Otherwise, all occurrences of $v$ must appear in a single leaf, which can hold up to $k - 1$ values, so there are never more than $k - 1$ copies of $v$ in the tree. The leaf will overflow at the latest upon inserting the $k$th occurrence of $v$, and then a new internal node with pivot $v$ is created.

In a non-degenerate input, the first $k$ duplicates appear among the first $n_T$ elements $U_1, \ldots, U_{n_T}$ of the input, so all pivots are chosen based on these elements only. Moreover, after these $n_T$ insertions, all $u$ values appear as labels of inner nodes. All leaves are empty then and remain so for good: the recursion tree has reached a saturated state.

We denote by $Γ = Γ(q)$ the node-depths vector for the final recursion tree and by $\tilde{X}$ the profile of $U_{n_T+1}, \ldots, U_n$. Since they are derived from disjoint ranges of the i.i.d. input, $Γ$ and $\tilde{X}$ are stochastically independent.

Contribution of Pivot Selection. With the search costs $Γ^T X$, we get the actual costs up to an error of $\pm ck$ per partitioning step, for some constant $c$ depending on the median-selection algorithm. As argued above, we have at most $u$ partitioning steps, so overall cost are $C_{n,q} = Γ^T X \pm O(k \cdot u)$.

Overall Result. We now have all ingredients to compute the overall costs of Quicksort. Recall that $u_n = O(n^{1-ε})$ and $n_T \sim n^v$ with $1 - ε < v < 1 - ε$. Since a recursion tree cannot have a path longer than $u$, we always have $Γ \leq cu$ for a fixed constant $c$ that depends only on the cost measure, i.e., $Γ_v = O(n^{1-ε})$ for all $v \in [u]$. However, this estimate is very pessimistic; [Corollary E.4] shows that for most recursion trees actually $Γ_v = O(\log n)$ for all $v$.

We therefore further split the set of non-profile-degenerate inputs into height-degenerate ones where the height of the resulting recursion tree is $> 13 \ln n$ and all other ones (cf. [Definition E.3]). This gives the following the stochastic representation conditional on a not $(ν, k)$-profile-degenerate input.

$$C_{n,q} \overset{D}{=} Γ^T \tilde{X} \pm O(n^v),$$

where $\tilde{X}$ is independent of $Γ$ and $\tilde{X} \overset{D}{=} \text{Mult}(n, q(n))$. We thus find that

$$C_{n,q} \overset{D}{=} Γ^T \tilde{X} \pm O(n^{1-ε}),$$

(input neither profile- nor height-degenerate).
Taking expectations over all non-degenerate inputs in Equation [67], exploiting independence, and inserting $P[\text{height-deg.}] = O(1/n^2)$ (Corollary E.4) yields
\begin{align*}
E[C_{n,q}] &= E[I^T \hat{X}] + O(n^{1-\varepsilon}) \\
&= E[I^T \cdot E[\hat{X}]] + O(n^{1-\varepsilon}) \\
&= (E[I^T \cdot q^{(n)}) \cdot n + O(n^{1-\varepsilon}),
\end{align*}
(68)

(69)

(70)

with $A_q$ as given in Equation (5) on page 10. As argued above, the contribution of profile-degenerate inputs is in $o(n^{-c})$ for any $c$ and thus covered by $O(n^{1-\varepsilon})$, so Equation (70) holds also for the unconditional expectation. This concludes the proof of Theorem 6.3.

G. Properties of the Entropy Function

This section collects a few useful properties of the entropy function used in the proof of Theorem 7.1. We start with some observations that follow from well-known results of real analysis. Detailed proofs are given in my Ph.D. thesis [44].

Lemma G.1 (Elementary Properties of the Entropy Function):

Let $H_{ln} : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ with $H_{ln}(x) = \sum_{i=1}^{u} x_i \ln(1/x_i)$ be the base $e$ entropy function.

(a) $H_{ln}(x) = \ln(2) H_{ld}(x)$.

(b) For all $x \in [0, 1]^u$ with $\sum x = 1$ we have that $0 \leq H_{ln}(x) \leq \ln(u)$.

(c) $H_{ln}$ is Hölder-continuous in $[0, 1]^u$ for any exponent $h \in (0, 1)$, i.e., there is a constant $C = C_h$ such that $|f(y) - f(x)| \leq C_h u \cdot \|y - x\|_\infty^h$ for all $x, y \in [0, 1]^u$.

A possible choice for $C_h$ is given by
\[ C_h = \left( \int_0^1 |\ln(t) + 1|^{1-h} \right)^{1-h} \] (71)

For example, $h = 0.99$ yields $C_h \approx 37.61$.

$D = (\Pi, 1 - \Pi)$ can be seen as a random binary probability distribution, so it makes sense to ask for the expected entropy of this distribution.

 Lemma G.2 (Expected Entropy of Beta Variables): For $\Pi \equiv Beta(t + 1, t + 1)$ and $k = 2t + 1$ we have

\[ E[H_{ln}(D)] = E[H_{ln}(\Pi, 1 - \Pi)] = H_{k+1} - H_{t+1} \] (72)

Proof: We need the following integral, which is a special case of Equation (4.253-1), p. 540, of Gradshteyn and Ryzhik [15] with $r = 1$:
\[ \int_0^1 z^{a-1}(1 - z)^{b-1} \ln(z) \, dz = B(a, b)(\psi(a) - \psi(a + b)), \quad (a, b > 0). \] (73)

Here $\psi(z) = \frac{d}{dz} \ln(\Gamma(z))$ is the digamma function.
The proof is now simply by computing. By symmetry we have \( \mathbb{E}[\mathcal{H}_{\ln}(D)] = -2\mathbb{E}[II \ln(II)]; \) using the above integral and the relation \( \psi(n+1) = H_n - \gamma \) (Equation (5.4.14) of the DLMF [8]) we find that

\[
\mathbb{E}[II \ln(II)] = \int_0^1 x \ln(x) \frac{x^t(1-x)^t}{B(t+1,t+1)} \, dx
= \frac{B(t+2,t+1)}{B(t+1,t+1)} \int_0^1 \ln(x) \frac{x^{t+1}(1-x)^t}{B(t+2,t+1)} \, dx
= \frac{t+1}{k+1} (\psi(t+2) - \psi(k+2))
= \frac{t+1}{2t+2} (H_{t+1} - H_{k+1})
= \frac{1}{2} (H_{t+1} - H_{k+1}).
\]

Inserting yields the claim.

The special case of Lemma G.2 when \( t = 0 \) appears Section 5.0 of Bayer [3] and as Exercise 6.2.2–37 of Knuth [26].

* * *

Using the Chernoff bound for the multinomial distribution (Lemma D.2), we obtain the following concentration property of the entropy of a normalized multinomial variable.

**Lemma G.3 (Concentration of Entropy of Multinomials):** Let \( u \in \mathbb{N} \) and \( p \in (0,1)^u \) with \( \sum p = 1 \), and \( X \sim \text{Mult}(n,p) \). Then it holds that

\[
\mathbb{E} \left[ \mathcal{H}_{\ln} \left( \frac{X}{n} \right) \right] = \mathcal{H}_{\ln}(p) \pm \rho,
\]

where we have for any \( \delta \in (0,1) \) with \( \delta \geq \sqrt{20u/n} \), \( h \in (0,1) \) and \( C_h \) as in Equation (71) that

\[
\rho \leq C_h \delta^h (1 - 3e^{-\delta^2 n/25}) + 3u \ln(u) e^{-\delta^2 n/25}.
\]

If \( u = O(n^\nu) \) as \( n \to \infty \) for a constant \( \nu \in [0,1) \), then Equation (79) holds with an error of \( \rho = o(n^{-(1-\nu)/2+\varepsilon}) \) for any fixed \( \varepsilon > 0 \).

**Proof:** We start with Equation (23) to use it on \( \mathbb{E}[[\mathcal{H}_{\ln}(X/n) - \mathcal{H}_{\ln}(p)]], \) we divide the domain \( \Delta_u \) of \( \frac{X}{n} \) into the region of values with \( \| \cdot \|_\infty \)-distance at most \( \delta \) from \( p \), and all others. By Lemma G.1, \( \mathcal{H}_{\ln} \) is Hölder-continuous for any exponent \( h \in (0,1) \) with Hölder-constant \( C_h \). Using this and the boundedness of \( \mathcal{H}_{\ln} \) (Lemma G.1–(b)) yields

\[
\mathbb{E} \left[ \mathcal{H}_{\ln} \left( \frac{X}{n} \right) - \mathcal{H}_{\ln}(p) \right] \leq \sup_{\|\xi\|_\infty < \delta} \left\| \mathcal{H}_{\ln}(p + \xi) - \mathcal{H}_{\ln}(p) \right\| (1 - 3e^{-\delta^2 n/25}) + \sup_x \left| \mathcal{H}_{\ln}(x) - \mathcal{H}_{\ln}(p) \right| \cdot 3e^{-\delta^2 n/25}
\leq C_h \delta^h \cdot (1 - 3e^{-\delta^2 n/25}) + 3 \ln(u) e^{-\delta^2 n/25}.
\]

This proves the first part of the claim.

For the second part, we assume \( u = O(n^\nu) \), thus \( u \leq dn^\nu \) for a constant \( d \) and large enough \( n \). We obtain an asymptotically valid choice of \( \delta \) when \( \delta = \omega(n^{(\nu-1)/2}) \); then for large enough \( n \) we will have \( \delta > \sqrt{20dn^{(\nu-1)/2}} \geq \sqrt{20u/n} \).
Let now an $\varepsilon > 0$ be given and set $\tilde{\varepsilon} = \varepsilon + \nu/2$. We may further assume that $\tilde{\varepsilon} < \frac{1}{2}$ since the claim is vacuous for larger $\tilde{\varepsilon}$. We choose a Hölder exponent $h \in (0, 1)$ so that $h > \frac{1-2\tilde{\varepsilon}}{1-\nu}$ (this is possible since $\frac{1-2\tilde{\varepsilon}}{1-\nu} < 1$ for $\tilde{\varepsilon} > \nu/2$); this is equivalent to the relation

$$\frac{\nu - 1}{2} < -\frac{1 - \tilde{\varepsilon}}{h}. \quad (83)$$

We can thus pick $c$ between these two values, e.g., $c = \left(\frac{\nu - 1}{2} - \frac{1 - 2\tilde{\varepsilon}}{h}\right)/2$. Since $c > -(\nu - 1)/2$, the choice $\delta = n^c$ guarantees $\delta \geq \sqrt{20u/n}$ for large enough $n$ and we can apply Equation (79).

As we now show, these choices are sufficient to prove the claim $\rho = o(n^{-1/2+\tilde{\varepsilon}})$. To streamline the computations, we note that (by its definition) we can write $h$ as

$$h = \frac{1 - 2\tilde{\varepsilon}}{1 - \nu} + \frac{4}{1 - \nu} \lambda$$

for some constant $\lambda > 0$ and

$$h = \frac{1 - 2\tilde{\varepsilon}}{1 - \nu - 2\lambda''} = \frac{1 - 2\tilde{\varepsilon}}{1 + \nu - 2\lambda'}$$

for constants $\lambda'' > 0$ resp. $\lambda' > \nu$, (85) which implies $h \cdot c + (\frac{1}{2} - \tilde{\varepsilon}) = -\lambda < 0$ and $2c + 1 = \lambda' > 0$. With these preparations we find for large $n$ that

$$\rho \cdot n^{1/2 - \tilde{\varepsilon}} \leq \frac{C_h}{n^{1/2 - \tilde{\varepsilon}} (1 - 3 \exp(-\delta^2 n/25))} + 3 n^{1/2 - \tilde{\varepsilon}} \ln(n) \exp(-\delta^2 n/25) \quad (86)$$

$$\leq \frac{C_h}{n^{1/2 - \tilde{\varepsilon}} (1 - 3 \exp(-n^{\lambda'}/25))} + 3 \nu \ln(n) \exp(-3 n^{\lambda'} + \frac{1}{2} - \tilde{\varepsilon}) \ln(n) \quad (87)$$

$$\to 0 \quad (88)$$

for $n \to \infty$, which implies the claim. \hfill \square

**H. Derivation of Expected Node Depth**

**Proof of Theorem 7.1** We first show upper and lower bounds, and afterwards combine them to obtain the claimed asymptotic approximation.

Before we start let us state for reference the following aggregation property for the entropy function; it follows directly form Lemma 6.2.2E of Knuth [26, page 444]: Let $q \in [0, 1]^u$ with $\Sigma q = 1$ and consider a fixed pivot value $P \in [u]$. Recall the definition of $Z$, $V$, $H$ from Section 5. We have that

$$\mathcal{H}(q) = \mathcal{H}(V_1, H, V_2) + \sum_{r=1}^{2} V_r \mathcal{H}(Z_r). \quad (89)$$

($\mathcal{H}$ can be w.r.t. any base; we will use it with $\mathcal{H}_{\ln}$.)

**H.1. Upper Bound**

We now show a class of upper bounds characterized by a parameter $\varepsilon$. 

Proposition H.1 (Upper Bound): Let $A_q$ satisfy Equation (5) on page 10, and let $\varepsilon \in (0, 1)$ be given. Define

$$c = c_\varepsilon = \frac{1}{H - 4\varepsilon h},$$

$$d = d_\varepsilon = \frac{(t + 1) B(t + 1, t + 1)}{\varepsilon^{t+2}(1 - \varepsilon)^t},$$

where $\bar{H} = H_{k+1} - H_{t+1}$

and $\bar{h} = H_k - H_t$.

If $c \geq 0$, we have that $\mathbb{E}[A_q] \leq c \cdot \mathcal{H}_{\text{ln}}(q) + d$ for all stochastic vectors $q$.

Proof: Let $\varepsilon$ with $c = c_\varepsilon \geq 0$ be given. Note that $d = d_\varepsilon \geq 0$ holds for all $\varepsilon$. The proof is by induction on $u$, the size of the universe. If $u = 0$, i.e., $q = ()$, we have $\mathbb{E}[A_q] = 0$, see Equation (5.2). Since $d \geq 0$ and here $\mathcal{H}_{\text{ln}}(q) = 0$, the claim holds.

Now assume that $u \geq 1$ and the claim holds for all (strictly) smaller universe sizes. We start by taking expectations in Equation (5) and conditioning on the pivot value $P$:

$$\mathbb{E}[A_q] = 1 + \mathbb{E}_P \left[ \sum_{r=1}^{2} \mathbb{E}[V_r A_{Z_r} | P] \right]$$

using the inductive hypothesis

$$\leq 1 + \mathbb{E}_P \left[ \sum_{r=1}^{2} V_r (c \mathcal{H}_{\text{ln}}(Z_r) + d) \right]$$

$$= 1 + c \cdot \mathbb{E} \left[ \sum_{r=1}^{2} V_r \mathcal{H}_{\text{ln}}(Z_r) \right] + d \cdot \mathbb{E}[\Sigma V]$$

$$\leq c \cdot \mathcal{H}_{\text{ln}}(q) + 1 - c \cdot \mathbb{E}[\mathcal{H}_{\text{ln}}(V, H)] + d \cdot \mathbb{E}[\Sigma V]$$

It remains to show that $\varpi \leq d$. We consider two cases depending on the maximal probability $\mu = \max_{1 \leq v \leq u} q_v$.

1. Case $\mu < \varepsilon$:

In this case, all individual probabilities are smaller than $\varepsilon$, so it is plausible that we can bound the expected entropy of partitioning $\mathbb{E}[\mathcal{H}_{\text{ln}}(V, H)]$ from below. The subuniverse probabilities $V_r$ are quite close to the continuous spacings $D_r$: by definition (see also Figure 3 on page 9) we have in interval-arithmetic notation

$$V_r = D_r + (-2\varepsilon, 0), \quad (r = 1, 2).$$

For the expected partitioning entropy, this means

$$\mathbb{E}[\mathcal{H}_{\text{ln}}(V, H)] \geq \sum_{r=1}^{2} \mathbb{E}[V_r \ln(1/V_r)]$$
using Equation (99) and \( x \log(1/x) \geq (x - \varepsilon) \log(1/(x + \varepsilon')) \) for \( \varepsilon, \varepsilon' \geq 0 \) and \( x \in [0, 1] \), this is

\[
\geq \sum_{r=1}^{2} \mathbb{E}[(D_r - 2\varepsilon) \log(1/D_r)]
\geq \mathbb{E}[\mathcal{H}_\text{in}(D)] + 2\varepsilon \sum_{r=1}^{2} \mathbb{E}[\ln(D_r)]
\geq \tilde{H} + 2\varepsilon \sum_{r=1}^{2} (\psi(t + 1) - \psi(k + 1))
\geq \tilde{H} - 4\varepsilon (H_t - \varepsilon \tilde{h})
\geq 1/c.
\]

Hence \( c \) satisfies \( c \geq 1/\mathbb{E}[\mathcal{H}_\text{in}(V, H)] \geq 0 \), which implies

\[
\varpi \leq 1 - \frac{1}{\mathbb{E}[\mathcal{H}_\text{in}(V, H)]} \cdot \mathbb{E}[\mathcal{H}_\text{in}(V, H)] + d \cdot \mathbb{E}[\sum V] \leq 1
\]

The inductive step is proven in this case.

2. Case \( \mu \geq \varepsilon \):

In the second case, there is a likely value \( v \in [u] \) with \( q_v \geq \varepsilon \). We will show a lower bound for having this value as label of the root.

We have \( II \overset{\text{d}}{=} \text{Beta}(t + 1, t + 1) \) and hence \( f_{II}(z) = \frac{z^{t(1-\varepsilon)}}{B(t+1,t+1)} \). Recall that \( P = f_U^{-1}(II) \) (Figure 2), so we can bound the probability to draw \( v \) from below by the smallest value of the integral over any \( \varepsilon \)-wide strip of the density:

\[
\mathbb{P}[P = v] \geq \min_{0 \leq \zeta \leq 1/\varepsilon} \int_{\zeta}^{\zeta + \varepsilon} f_{II}(z) \, dz
\geq \int_{0}^{\varepsilon} \frac{z^{t}(1-z)^{t}}{B(t+1,t+1)} \, dz
\geq \int_{0}^{\varepsilon} \frac{\varepsilon^{t+1}(1-\varepsilon)^{t}}{(t+1)B(t+1,t+1)} \, dz
\geq \frac{\varepsilon^{t+2}(1-\varepsilon)^{t}}{(t+1)B(t+1,t+1)}
\geq \frac{\varepsilon^{t+2}(1-\varepsilon)^{t}}{(t+1)B(t+1,t+1)}
\]

For the expected hitting probability, we thus have for any \( q \) with a \( q_v \geq \varepsilon \) that

\[
\mathbb{E}[H] \geq q_v \cdot \mathbb{P}[P = v] \geq \frac{\varepsilon^{t+2}(1-\varepsilon)^{t}}{(t+1)B(t+1,t+1)} = 1/d,
\]

so \( d \geq 1/\mathbb{E}[H] \). This implies

\[
\varpi - d \leq 1 - c \cdot \mathbb{E}[\mathcal{H}_\text{in}(V, H)] + d \cdot \mathbb{E}[\sum V] - d
\leq 1 - (1 - \mathbb{E}[\sum V]) \cdot \frac{1}{\mathbb{E}[H]}
\leq 0.
\]
This concludes the inductive step also in the second case. The inductive step is thus possible in both cases, and so the claim holds for all stochastic vectors \( q \) by induction.

\( \square \)

H.2. Lower Bound

The lower bound on \( \mathbb{E}[A_q] \) uses basically the same techniques; only a few details differ.

**Proposition H.2:** Let \( A_q \) satisfy Equation 5 on page 10 and let \( \varepsilon \in (0, 1/e) \) be given. Define

\[
\begin{align*}
    c &= c_\varepsilon = \frac{1}{H + 4\varepsilon + \varepsilon \ln(1/\varepsilon)}, \\
    d &= d_\varepsilon = (c_\varepsilon \ln(3) - 1) \frac{(t + 1) B(t + 1, t + 1)}{\varepsilon^{t+2}(1-\varepsilon) t}.
\end{align*}
\]

If \( d \geq 0 \), we have that \( \mathbb{E}[A_q] \geq c \cdot H \ln(q) - d \) all stochastic vectors \( q \).

**Proof:** Let \( \varepsilon \in (0, 1/e) \) with \( d = d_\varepsilon \geq 0 \) be given; \( c = c_\varepsilon \geq 0 \) holds for any \( \varepsilon \). The proof is similar to that of Proposition H.1, so we emphasize the differences and skip identical parts. If \( u = 0 \), i.e., \( q = () \), the claim holds since \( d \geq 0 \).

Now assume \( u \geq 1 \) and that the claim holds for all (strictly) smaller universe sizes. As for the upper bound, we find from the recurrence using the inductive hypothesis

\[
\mathbb{E}[A_q] \geq 1 + \mathbb{E}_P \left[ \sum_{r=1}^{2} V_r (c H \ln(Z_r) - d) \right]
\]

\[
\mathbb{E}[H \ln(V, H)] - d \cdot \mathbb{E}[\Sigma V],
\]

and it remains to show that \( \varpi \geq -d \). We consider the same two cases for \( \mu = \max_{1 \leq v \leq u} q_v \).

1. Case \( \mu < \varepsilon \):

   In this case, we bounded the expected entropy of partitioning, \( \mathbb{E}[H \ln(V, H)] \), from above. Similar to the computation for the upper bound, we find

   \[
   \mathbb{E}[H \ln(V, H)] = \sum_{r=1}^{2} \mathbb{E}[V_r \ln(1/V_r)] + \mathbb{E}[H \ln(1/H)]
   \]

   using Equation 59 and \( (x-\varepsilon) \log(1/(x-\varepsilon)) \leq x \ln(1/x) + \varepsilon' \) for \( 0 \leq \varepsilon \leq \varepsilon' \) and \( x \in [0, 1] \), and that \( x \ln(1/x) \) is increasing for \( x \in [0, 1/e] \), this is

   \[
   \leq \sum_{r=1}^{2} \mathbb{E}[D_r \ln(1/D_r) + 2\varepsilon] + \varepsilon \ln(1/\varepsilon)
   \]

   \[
   \geq \tilde{H} + 4\varepsilon + \varepsilon \ln(1/\varepsilon)
   \]

   So \( c \) satisfies \( c \leq \frac{1}{\mathbb{E}[H \ln(V, H)]} \), which implies

   \[
   \varpi \geq 1 - \frac{1}{\mathbb{E}[H \ln(V, H)]} \cdot \mathbb{E}[H \ln(V, H)] - d \cdot \mathbb{E}[\Sigma V] \]

   \[
   \geq -d.
   \]

   The inductive step is proven in this case.
2. Case \( \mu \geq \varepsilon \):

In the second case, there is a likely value \( v \in [u] \) with \( q_v \geq \varepsilon \). By the same arguments as in the proof of Proposition H.1, we find

\[
\mathbb{E}[H] \geq \frac{\varepsilon t^2 (1 - \varepsilon)^t}{t+1} B(t+1,t+1),
\]

so that \( d \geq (c \ln(3) - 1)/\mathbb{E}[H] \). This implies

\[
\varpi + d = 1 - c \cdot \mathbb{E}[\mathcal{H}_{\ln}(V,H)] + d(1 - \mathbb{E}[\Sigma V]) \geq 1 - c \cdot \ln(3) + \frac{c \ln(3) - 1}{\mathbb{E}[H]} \cdot \mathbb{E}[H] \geq 0.
\]

This concludes the inductive step also in the second case, so the claim holds for all stochastic vectors \( q \) by induction. \( \square \)

H.3. Asymptotic Approximation

We observe that \( c_\varepsilon \) converges to \( 1/\mathcal{H} \) for both bounds as \( \varepsilon \to 0 \), so there is hope to show \( \mathbb{E}[A_q] \sim \mathcal{H}_{\ln}(q)/\mathcal{H} \). We have to be precise about the limiting process and error terms, though. So let \( (q^{(i)})_{i \in \mathbb{N}} \) be a sequence of universe distributions for which \( \mathcal{H}_i := \mathcal{H}_{\ln}(q^{(i)}) \to \infty \) as \( i \to \infty \).

Now, consider \( c_\varepsilon \) and \( d_\varepsilon \) from Proposition H.1. As functions in \( \varepsilon \), they satisfy for \( \varepsilon \to 0 \)

\[
c_\varepsilon = \frac{1}{\mathcal{H}} \pm O(\varepsilon), \quad d_\varepsilon = O(\varepsilon^{-t-2})
\]

Since the bounds above hold simultaneously for all feasible values of \( \varepsilon \), we can let \( \varepsilon \) depend on \( \mathcal{H}_i \). If we set

\[
\varepsilon = \varepsilon_i = \mathcal{H}_i^{-\frac{1}{t+3}}
\]

we have \( \varepsilon_i \to 0 \) as \( i \to \infty \) and so \( c_{\varepsilon_i} > 0 \) for large enough \( i \). Then we have by Proposition H.1

\[
\mathbb{E}[A_{q^{(i)}}] \leq c_{\varepsilon_i} \mathcal{H}_i + d_{\varepsilon_i} = \mathcal{H}_i \mathcal{H}_i^{-\frac{1}{t+3}} + O\left(\mathcal{H}_i^{\frac{1}{t+3}}\right), \quad (i \to \infty).
\]

Now consider the lower bound. For \( c_\varepsilon \) and \( d_\varepsilon \) from Proposition H.2 we similarly find as \( \varepsilon \to 0 \) that

\[
c_\varepsilon = \frac{1}{\mathcal{H}} \pm O(\varepsilon \log \varepsilon), \quad d_\varepsilon = O(\varepsilon^{-t-2}).
\]

With the same \( \varepsilon_i \) as above (Equation (132)) we have \( d_{\varepsilon_i} \geq 0 \) for large enough \( i \), so by Proposition H.2 it holds that

\[
\mathbb{E}[A_{q^{(i)}}] \geq \frac{\mathcal{H}_i}{\mathcal{H}} \pm O\left(\mathcal{H}_i^{\frac{1}{t+3}} \log \mathcal{H}_i\right), \quad (i \to \infty).
\]

Together with Equation (133), this proves Theorem 7.1. \( \square \)
I. Proof of Main Result

With all these preparations done, the proof of our main result reduces to properly combining the ingredients developed above.

Proof of Theorem 1.1: Let the sequence of profiles be given and consider the corresponding model with \( n \) i.i.d. \( q^{(n)} = x^{(n)}/n \) distributed elements. By assumption \( \mu_n = \Omega(n^{-1+\epsilon}) \), so the model has many duplicates. We thus obtain from Theorem 6.3 that

\[
\mathbb{E}[C_{n,q^{(n)}}] = \mathbb{E}[A_{q^{(n)}}] \cdot n + O(n^{1-\delta'}), \quad (n \to \infty),
\]

for any \( \delta' \in (0, \varepsilon) \). If \( H_n = H(q^{(n)}) \to \infty \) as \( n \to \infty \), we can continue with Theorem 7.1 right away. To cover the case that \( (H_n) \) contains an infinite subsequence that is bounded, we add an error bound of \( O(n) \); this dominates \( \mathbb{E}[A_{q^{(n)}}] \cdot n \) (making the claim is essentially vacuous) for such inputs. So in any case we have that

\[
\mathbb{E}[C_{n,q^{(n)}}] = \alpha_k H_n \cdot n + O(H_n^{1-\delta} n + n^{1-\delta'}) \quad (n \to \infty),
\]

for any \( \delta \in (0, \frac{1}{1+\zeta}) = (0, \frac{2}{k+3}) \).

The many-duplicates assumption implicitly bounds the universe size to \( u_n = O(n^{1-\epsilon}) \) (Fact F.1), so can use Fact 4.2 to translate costs in the i.i.d. model back to those in the multiset model: for any \( \delta'' < \varepsilon/2 \), we have that

\[
\mathbb{E}[C_{x^{(n)}}] = \mathbb{E}[C_{n,q^{(n)}}] \pm O(n^{1-\delta''})
\]

\[
= \alpha_k H_n \cdot n + O(H_n^{1-\delta} n + n), \quad (n \to \infty).
\]

The optimality claim follows directly by comparing with the lower bound of Munro and Raman [33, Theorem 4].

J. Lower Bound For I.I.D. Sorting

We follow the elegant argument of Munro and Raman [33] for multiset sorting to obtain a lower bound for the discrete i.i.d. model. By averaging over the profiles, we obtain essentially the result of their Theorem 4, but with a weaker error term.

Theorem J.1 (Lower Bound): Let \( u = O(n^\nu) \) for a constant \( \nu \in [0,1) \) and \( q \in (0,1)^n \) with \( \Sigma q = 1 \). For any constant \( \varepsilon > 0 \), \( H_{ld}(q)n - n/\ln(2) \pm o(n^{1+\nu}/2^\nu) \) ternary comparisons on average are necessary to sort \( n \) i.i.d. \( D(q) \) elements as \( n \to \infty \).

We remark that one might expect to require at least \( H_{ld}(q)n \) comparisons since this is the entropy of a vector of \( n \) i.i.d. \( D(q) \) elements; but such entropy arguments must be taken with care: we might have \( H(q) \gg \log n \) for \( u \gg n \); then \( nH(q) \gg n \log n \) is certainly not a lower bound for sorting. (Theorem J.1 does not make a statement for such a case since the error bound dominates then.)

Proof of Theorem J.1: Let \( U_1, \ldots, U_n \) be \( n \) i.i.d. \( D(q) \) numbers and \( V_1, \ldots, V_n \) be a random permutation of \([n]\). The \( n \) vectors \( (U_i, V_i) \) are then all distinct, and all \( n! \) relative rankings w.r.t. lexicographic order are equally likely.

Assume we can sort \( U_1, \ldots, U_n \) with \( \mathbb{E}[C_{n,q}] \) ternary comparisons on average. We use this method to partially sort the \( n \) vectors \( (U_1, V_1), \ldots, (U_n, V_n) \) according to the first component
References only. We can then complete the sorting using Mergesort (separately) on each of the \( u \) classes of elements with same first component. (Any sorting method must already have determined the borders between these classes while sorting according to \( U_1, \ldots, U_n \).) The total number of comparisons we use is then no more than
\[
E[C_{n,q}] + \sum_{v=1}^{u} E[X_v \ld X_v] = E[C_{n,q}] + n E \left[ \sum_{v=1}^{u} \frac{X_v}{n} \ld X_v \right] + n \ld(n) \tag{141}
\]
with \( \rho \) as in [Lemma G.3]
\[
= n \ld(n) + E[C_{n,q}] + n H_{ld}(q) \pm n \frac{\rho}{\ln(2)} \tag{142}
\]
\[
\ge n \ld(n) - n/\ln(2) \pm O(\log n) \tag{143}
\]
since the latter is the well-known lower bound on the average number of (ternary or binary) comparisons for sorting a random permutation of \( n \) distinct elements, see, e.g., Equation 5.3.1–(37) of Knuth [26]. It follows that
\[
E[C_{n,q}] \ge H_{ld}(q)n - \frac{n}{\ln(2)} \pm n \frac{\rho}{\ln(2)}. \tag{144}
\]
For \( u = O(n^\nu) \) with \( \nu \in [0, 1) \) we get asymptotically for any \( \varepsilon > 0 \)
\[
E[C_{n,q}] \ge H_{ld}(q)n - \frac{n}{\ln(2)} \pm o(n^{1+\varepsilon/2}). \tag{145}
\]
\[\blacksquare\]

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