Unitary representation of the Poincaré group for classical relativistic dynamics

A. D. Bermúdez Manjarres

Universidad Distrital Francisco José de Caldas
Cra 7 No. 40B-53, Bogotá, Colombia
ad.bermudez168@uniandes.edu.co

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Abstract

We give a unitary irreducible representation of the proper Poincaré group that leads to an operational version of the classical relativistic dynamics of a massive spinless particle. Unlike quantum mechanics, in this operational theory there is no uncertainty principle between position and momentum. It will be shown that the theory contains the Koopman-von Neumann formalism as a particular case, and a explicit connection with relativistic Hamiltonian mechanics will be given.

1 Introduction

The structure of the Poincaré group and algebra plays a fundamental role in the formulation of any relativistic theory. Quantum mechanics and quantum field theory, being theories of operators acting on a Hilbert space, lead to the study of unitary representations of the Poincaré group. There is a vast literature on the subject, we refer to the books [1, 2] and references therein. On the other hand, the theory of Poisson brackets representation of Lie groups is the most natural way to study the Poincaré group in the context of classical Hamiltonian mechanics [3, 4].

Classical mechanics can be recast in the same mathematical language of quantum mechanics. This old approach is due to Koopman [5] and von Neumann [6]. Whether for derivation of purely classical results or for comparison between quantum and classical mechanics, the Koopman-von Neumann formalism (hereafter abbreviated as KvN) has received increasing attention in the past two decades (see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]), the possibility of formulating quantum-classical hybrid theories has also increased the interest in this formalism [22, 23, 24, 25]. The existence of the KvN theory raises the question of the classification of the unitary representations of the groups of
space-time symmetries in the context of classical mechanics. An irreducible unitary representation of the Galilei group that leads to classical mechanics was given recently [26], and it was used to study the limitation imposed by Galilean covariance on quantum-classical hybrids systems [27].

In this paper we will give a new kind of unitary representation of the proper Poincaré group. We do not attempt to give a complete classification of this kind of representations, we prefer to focus on a single relevant case. This classical representation differs from the quantum one in the use of non-observable operators, what Sudarshan refers to as hidden dynamical variables [28]. The representation given here results in the correct classical relativistic dynamics of a massive spinless particle.

The KvN theory starts with the Liouville equation, and from it a Hilbert space and a set of relevant operators are built. We could construct a unitary representation of the Poincaré group starting directly from the KvN theory, but we will take a different approach as we want to avoid the use of previous results from analytical mechanics. Instead, from the beginning we will postulate the existence of a Hilbert space $\mathcal{H}_C$, and then we will look for a realization of the proper Poincaré group where the space-time transformations are represented by unitary transformations acting on $\mathcal{H}_C$. We will demand the states belonging to $\mathcal{H}_C$ to contain all the information that can classically be known about a particle, this leads to an absence of uncertainty principles between the observables of the theory. We will show that the theory constructed this way contains the KvN theory as a particular case.

The structure and methods of this paper are very similar to the one in [26], we just adapt them to the case of the Poincaré group. This work is organized as follows: in the following section we give a brief summary of the proper Poincaré algebra, and we present the conventions that will be followed throughout this work. For reasons of necessity, our notation for the elements of the Poincaré algebra is not the standard one used in quantum mechanics.

In section 3 we introduce the Hilbert space $\mathcal{H}_C$ where the operators from the Poincaré group act upon, and we state the action of these operators over the vectors of $\mathcal{H}_C$. Here, we also define the position and velocity operators, as they will be necessary to give a physical interpretation to the Poincaré algebra. We then proceed to find an irreducible unitary representation of the Poincaré group that is equivalent to the dynamics of a free particle. The extension to the case of a particle interacting with an external force is also given.

In section 4 we give the momentum-based alternative representation of the Poincaré group. We will show that the change to phase space variables entails a quantum canonical transformation, just as in the non-relativistic case. The developments of this section leads directly to the KvN theory for a relativistic (spinless) particle.

Section 5 is devoted to show the close relation of the operational theory constructed in the previous sections with relativistic Hamiltonian mechanics.

Finally, in section 6 we investigate the transformation properties, under Lorentz boosts, of the operators used in the theory. It will be found that the momentum and velocity operators transform as expected from relativistic physics.
However, the position operator will not transform as the coordinates do in special relativity, instead it will transform as a Newton-Wigner operator.

The Einstein summation convention is used throughout this paper. We work in units where \( c = 1 \).

## 2 The proper Poincaré group and algebra

The proper Poincaré group is a ten parameter group that consists of space and time translations, rotations, and Lorentz transformation (boosts). The generators of these space-time transformations will be associated with Hermitian operators as follows: \( \hat{J}_i \) will be the generator of rotations around \( i \)-axis; \( \hat{\lambda}_{x_i} \) stands for the space displacement generator in the \( i \)-direction; \( \hat{K}_i \) denotes the Lorentz boost along the \( i \)-axis; \( \hat{L} \) stands for the time displacement generator. All of these operators will act on a Hilbert space to be described in the next section. The space-time transformations of the Poincaré group, not including the Lorentz boosts, will be realized by unitary operators with the following convention:

| Space – Time Transformations | Unitary Operator |
|-----------------------------|------------------|
| Rotation around the \( i \) axis | \( e^{-i\theta \hat{J}_i} \) |
| Spatial displacement         | \( e^{-ia \hat{\lambda}_{x_i}} \) |
| Time displacement            | \( e^{i\tau \hat{L}} \) |

The operator that generates a finite Lorentz transformation to a moving frame with speed \( v \) along the \( i \)-axis will be denoted by

\[
e^{is\hat{K}_i},
\]

where \( s = \tanh^{-1} v \) is the rapidity.

The above operators are postulated to obey the commutation relations from the Lie algebra of the proper Poincaré group. The derivation of the Poincaré algebra can be found in, for example, [1]. The commutator relations not involving
the time evolution operator are

\[
\begin{align*}
\{\hat{\lambda}_x, \hat{\lambda}_x\} & = 0, \quad (2a) \\
\{\hat{J}_i, \hat{J}_j\} & = i\varepsilon_{ijk}\hat{J}_k, \quad (2b) \\
\{\hat{J}_i, \hat{\lambda}_x\} & = i\varepsilon_{ijk}\hat{\lambda}_k, \quad (2c) \\
\{\hat{J}_i, \hat{K}_j\} & = i\varepsilon_{ijk}\hat{K}_k, \quad (2d) \\
\{\hat{K}_i, \hat{K}_j\} & = -i\varepsilon_{ijk}\hat{J}_k. \quad (2e)
\end{align*}
\]

The commutators that do involve \(\hat{\mathcal{L}}\) are

\[
\begin{align*}
\{\hat{\mathcal{J}}_\alpha, \hat{\mathcal{L}}\} & = 0, \quad (3a) \\
\{\hat{\mathcal{K}}_i, \hat{\mathcal{L}}\} & = i\hat{\lambda}_x, \quad (3b) \\
\{\hat{\lambda}_x, \hat{\mathcal{L}}\} & = 0, \quad (3c) \\
\{\hat{\mathcal{K}}_i, \hat{\lambda}_x\} & = i\delta_{ij}\hat{\mathcal{L}}. \quad (3d)
\end{align*}
\]

In the operational version of classical dynamics (relativistic or not), the generator of the space-time transformations do not have the same physical interpretation as in quantum mechanics. For example, the generator of rotation \(\hat{\mathcal{J}}\) is not an angular momentum operator, and \(\hat{\mathcal{L}}\) is not an energy operator. For this reason, we are using the symbols \((\hat{\mathcal{J}}, \hat{\lambda}_x, \hat{\mathcal{K}}, \hat{\mathcal{L}})\) instead of the more familiar \((\hat{\mathcal{J}}, \hat{\mathcal{P}}, \hat{\mathcal{K}}, \hat{\mathcal{H}})\).

We will see in section 5 the relation between the time evolution operator \(\hat{\mathcal{L}}\) and the Liouville equation of classical statistical mechanics; for this reason, we will call \(\hat{\mathcal{L}}\) as the Liouvillian operator.

3 Classical representation of the Poincaré Algebra

In classical relativistic dynamics there is no uncertainty principle between the position \(\mathbf{r}\) and the velocity \(\mathbf{v}\) of a particle, both quantities are postulated to be measurable as precisely as desired. This lack of uncertainty principle will be formulated with a commutation relation in Eq. (11a). The state of the classical particle will be described by a vector in a suitable Hilbert space \(\mathcal{H}_C\), and these states have to contain information of the position and velocity of the particle. The situation is almost the same as in the non-relativistic case, with the only exception that, here, the speed should not be allowed to be greater than 1 for
physically acceptable states. The vectors $|\psi\rangle \in \mathcal{H}_C$ are postulated to be of the form

$$|\psi\rangle = \int \langle r, v | \psi \rangle |r, v\rangle \, dr \, dv, \quad (4)$$

such that $\psi(r, v) = \langle r, v | \psi \rangle$ is a square integrable function, and the kets $|r, v\rangle$ obey the orthonormality condition

$$\langle r', v' | r, v \rangle = \delta(r - r') \delta(v - v'). \quad (5)$$

For physically acceptable states, the limits of integration in (4) are $(-\infty, \infty)$ for each coordinate and $(-1, 1)$ for each component of the velocity. However, the formalism to be developed below requires that we accept in $\mathcal{H}_C$ states with values of the speed greater than 1. Nevertheless, we will see that the unitary time evolution is such that if an state is physically acceptable at $t = 0$, it remains that way for all times.

The theory will be statistical in nature (though it can describe particle trajectories if non-normalizable states are allowed, see section 6). The probability $P(r, v)$ of finding the particle with position $r$ and velocity $v$ is given by the Born rule

$$P(r, v) = |\langle r, v | \Psi \rangle|^2. \quad (6)$$

The position $\hat{R} = \left(\hat{X}_1, \hat{X}_2, \hat{X}_3\right)$ and velocity $\hat{V} = \left(\hat{V}_1, \hat{V}_2, \hat{V}_3\right)$ operators are defined by their action on the base kets

$$\hat{X}_i |r, v\rangle = x_i |r, v\rangle, \quad (7)$$
$$\hat{V}_j |r, v\rangle = v_j |r, v\rangle. \quad (8)$$

In order to form an irreducible set of operators in the Hilbert space we are considering, we define the velocity-translation operator $\hat{\lambda}_v = \left(\hat{\lambda}_{v_1}, \hat{\lambda}_{v_2}, \hat{\lambda}_{v_3}\right)$ by

$$e^{-i b \cdot \hat{\lambda}_v} |r, v\rangle = |r, v + \mathbf{b}\rangle. \quad (9)$$

The translation produced by $\hat{\lambda}_v$ can increase the speed above 1, but we will find that this have no negative effect on the rest of the theory.

The translation operators $\hat{\lambda}_r$ and $\hat{\lambda}_v$ are conjugated in the quantum sense to $\hat{R}$ and $\hat{V}$, respectively. The following commutation relations are postulated

\[1\]The interpretation of $\hat{\lambda}_r$ and $\hat{\lambda}_v$ is not entirely clear. Usually, they are understood to be non-observables or hidden variables [28]. However, it is tempting to consider them as observables because (1) they are related to ergodic properties [29], (2) to avoid superselection rules [12], and (3) they are useful to construct a classical measurement theory [16]. Moreover, these operator are related to the Bopp operators in the Wigner phase space representation of quantum mechanics [30].
to be satisfied

\[
\begin{align*}
[\hat{X}_\alpha, \hat{X}_\beta] &= [\hat{V}_\alpha, \hat{V}_\beta] = 0, \\
[\hat{X}_\alpha, \hat{\lambda}_{\nu_\beta}] &= [\hat{V}_\alpha, \hat{\lambda}_{\nu_\beta}] = 0, \\
[\hat{X}_\alpha, \hat{\lambda}_{x_\beta}] &= [\hat{V}_\alpha, \hat{\lambda}_{x_\beta}] = i\delta_{\alpha\beta}.
\end{align*}
\] (10a)

(10b)

(10c)

Furthermore, \( \hat{R}, \hat{V}, \) and \( \hat{\lambda}_\nu \) are postulated to rotate as vector operators

\[
\begin{align*}
[\hat{J}_i, \hat{X}_j] &= i\varepsilon_{ijk}\hat{X}_k, \\
[\hat{J}_i, \hat{V}_j] &= i\varepsilon_{ijk}\hat{V}_k, \\
[\hat{J}_i, \hat{\lambda}_{\nu_i}] &= i\varepsilon_{ijk}\hat{\lambda}_{\nu_k}.
\end{align*}
\] (11)

(12)

(13)

The effect of rotation and space displacement on the base kets will be

\[
e^{-i\hat{\lambda}_r \hat{\lambda}_r} |r, v\rangle = |r + a, v\rangle, \] (14)

\[
e^{-i\hat{\lambda}_n \hat{\lambda}_n} |r, v\rangle = |r + \theta \hat{n} \times r + \theta \hat{n} \times v\rangle. \] (15)

The effect of \( \hat{L} \) is temporal displacement in the state vectors, namely

\[
e^{-it\hat{L}} |\Psi(0)\rangle = |\Psi(t)\rangle. \] (16)

Equation (16) implies a Schrödinger-like equation

\[
\frac{d}{dt} |\Psi(t)\rangle = -i\hat{L} |\Psi(t)\rangle. \] (17)

As in the non-relativistic quantum [31] and classical cases [26], \( \hat{R} \) and \( \hat{V} \) will be related by

\[
\frac{d}{dt} \langle \hat{R} \rangle = \langle \hat{V} \rangle, \] (18)

or, equivalently, by

\[
\hat{V} = i \left[ \hat{L}, \hat{R} \right]. \] (19)

Unfortunately, under the unitary transformation generated by \( \hat{K} \), the position operator \( \hat{R} \) cannot be made to transform as expected from a Lorentz transformation (see section 6). However, \( \hat{V} \) can be made to obey the velocity

\footnote{We will not postulate the effect of Lorentz transformation on \( \hat{R} \) or on \( |r, v\rangle \). The transformation equation will be found a posteriori once the form of the generators is known.}
addition formula. For example, a boost with rapidity \( s = \tanh^{-1} v \) in the \( z \)-axis will give

\[
e^{is\hat{K}_z} \hat{V}_x e^{-is\hat{K}_z} = \frac{(1 - v^2)^{1/2} \hat{V}_x}{1 - v \hat{V}_z},
\]

\[
e^{is\hat{K}_z} \hat{V}_y e^{-is\hat{K}_z} = \frac{(1 - v^2)^{1/2} \hat{V}_y}{1 - v \hat{V}_z},
\]

\[
e^{is\hat{K}_z} \hat{V}_z e^{-is\hat{K}_z} = \frac{\hat{V}_z - v}{1 - v \hat{V}_z}.
\]

To end this section we point out that the operators \( \{ \hat{R}, \hat{V} \} \) form a complete set of commuting observables in \( \mathcal{H}_C \). This is, any operator \( \hat{A} \) that commutes with all the elements of \( \{ \hat{R}, \hat{V} \} \) is of the form \( \hat{A} = \hat{A}(\hat{R}, \hat{V}) \). Physically, this represents the fact that the particle we are considering has no internal degrees of freedom. Now, due to the commutation relations (10a) to (10c), no operator \( \hat{A}(\hat{R}, \hat{V}) \) can commute with all the elements of \( \{ \hat{\lambda}_r, \hat{\lambda}_v \} \). Hence, only a multiple of the identity can commute with all the elements of the set \( \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \} \).

Let us remind here the Schur’s lemma [32]: *A set of self-adjoint operators is irreducible if and only if any operator that commutes with all members of the set is a multiple of the identity.* Hence, we conclude that \( \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \} \) is an irreducible set in \( \mathcal{H}_C \). In the next subsection we will give a realization of the Poincaré algebra in terms of \( \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \} \), this means that we will have, via the exponential map, an irreducible unitary representation of the Poincaré group.

### 3.1 Free particle

We can easily find a suitable realization for \( \hat{J} \) and \( \hat{L} \). The operator

\[
\hat{J}_k = \varepsilon_{ijk} \left( \hat{X}_j \hat{\lambda}_{xk} + \hat{V}_j \hat{\lambda}_{vk} \right),
\]

satisfy Eqs. (23), (24), (11), (10), (13) and (15). On the other hand, we expect the wave functions of free particles to evolve in time according to \( \psi(\mathbf{r}, \mathbf{v}) \rightarrow \psi(\mathbf{r} - \mathbf{v}t, \mathbf{v}) \), and that is accomplished if the effect of the Liouvillian on the base kets is given by

\[
e^{-it\hat{L}} |\mathbf{r}, \mathbf{v}\rangle = |\mathbf{r} + \mathbf{v}t, \mathbf{v}\rangle.
\]

A viable solution to Eq. (24) is

\[
\hat{L} = \hat{V}_z \hat{\lambda}_v.
\]
The operator (25) is invariant under translations and rotations, as it should be from the Poincaré algebra. \( \hat{J} \) and \( \hat{L} \) have the same form as the generator of the Galilei algebra for the non-relativistic case.

The only non-trivial task is to find the generators of Lorentz boosts \( \hat{K} \), however, trial and error give the suitable operator

\[
\hat{K}_i = \left( \hat{X}_i \hat{L} - \left\{ \delta_{ij} - \hat{V}_i \hat{V}_j \right\} \hat{\lambda}_{v_j} \right)_S - t \hat{\lambda}_{x_i},
\]

where we used the notation \( (\hat{A} \hat{B})_S = \frac{1}{2} (\hat{A} \hat{B} + \hat{B} \hat{A}) \). There is, actually, some freedom in the choice of \( \hat{K} \). We can remove the term \( t \hat{\lambda}_x \) without affecting the commutation relations of the Poincaré algebra. However, it is preferable to keep this term as it enhances the physical interpretation of the generator of the Lorentz transformations, as we will see in section 5.

Under the Schrödinger evolution

\[
i \frac{\partial}{\partial t} \psi(r, \mathbf{v}, t) = \hat{L} \psi(r, \mathbf{v}, t),
\]

a wave function evolve as expected from a free particle \( \psi(r, \mathbf{v}) \rightarrow \psi(r - \mathbf{v}t, \mathbf{v}) \).

Moreover, the Heisenberg evolution of \( \hat{R} \) and \( \hat{V} \) matches the behavior of a free particle

\[
\begin{align*}
\frac{d}{dt} \hat{R}(t) &= i \left[ \hat{L}, \hat{R} \right] = \hat{V}, \\
\frac{d}{dt} \hat{V}(t) &= i \left[ \hat{L}, \hat{V} \right] = 0.
\end{align*}
\]

To summarize, in view of the relations (10a) to (10c), the operators \( \hat{K}, \hat{J} \) and \( \hat{L} \) given by (23), (25), and (26), respectively, satisfy all the commutation relations of the Poincaré algebra. They generate a representation of the Poincaré group that is irreducible in the Hilbert space we are considering, and this representation gives the correct dynamics of a free particle.

### 3.2 Interaction with an external field

The generators \( \hat{J} \) and \( \hat{K} \) are understood to be geometrical in nature, they do not change in the presence of interactions. Hence, the relations (24) to (27) remain valid.

\( \hat{L} \), on the other hand, is now to be understood as the generator of a dynamical evolution in time, and its form change accordingly. Thus, relations (28) to (30) could get modified. The equation of motion is still postulated to be

\[
\frac{d}{dt} |\Psi(t)\rangle = -i\hat{L} |\Psi(t)\rangle.
\]
In the non-relativistic case, $\hat{L}$ is identified by demanding that the acceleration is independent of the unobservable operators $\hat{\lambda}_r$ and $\hat{\lambda}_v$, this is, the acceleration obeys the Newton equation
\[
\frac{d}{dt} \hat{V}(t) = \frac{1}{m} \hat{F}(\hat{R}, \hat{V}),
\] (30)
for some $\hat{F}$. The mass $m$ and the force operator $\hat{F}$ are to be found from experiments.

We can make a similar demand to find $\hat{L}$ in the relativistic case. It can be checked that
\[
\hat{L} = \hat{V}_i \hat{\lambda}_{xi} + \left( \frac{1}{m_0} \hat{\gamma}^{-1} \left\{ F_i - (\hat{V} \cdot \hat{F}) \hat{V}_i \right\} \right) \hat{\lambda}_{vi},
\] (31)
where $\hat{\gamma} = \left( 1 - \hat{V}^2 \right)^{-1}$, reproduces the relativistic force equation
\[
m_0 \frac{d}{dt} \left( \hat{\gamma} \hat{V}(t) \right) = \hat{F}(\hat{R}, \hat{V}).
\] (32)

**Example: Constant force.**

We can take the case of a constant force as an example. For simplicity, we can consider the one-dimensional case of a particle restricted to move in the $x$-axis. In this case, the Liouvillian reduces to
\[
\hat{L} = \hat{V} \hat{\lambda}_x + \frac{F}{m_0} \left( \left\{ 1 - \hat{V}^2 \right\} \frac{3}{2} \hat{\lambda}_v \right).
\] (33)

The Heisenberg equation of motion for $\hat{V}$ can be computed to give
\[
\frac{d}{dt} \hat{V} = i \left[ \hat{L}, \hat{V} \right] = \frac{F}{m_0} \left\{ 1 - \hat{V}^2 \right\}^{3/2}.
\] (34)

Equation (34) can be inverted and integrated to obtain $\hat{V}(t)$ and $\hat{X}(t)$. Assuming $\hat{V}(0) = 0$ and $\hat{X}(0) = 0$, we have
\[
\hat{V}(t) = \frac{\frac{1}{m_0} Ft \sqrt{1 + \left( \frac{Ft}{m_0} \right)^2}},
\]
and
\[
\hat{X}(t) = \frac{F}{m_0} \left( \sqrt{1 + \left( \frac{Ft}{m_0} \right)^2} - 1 \right).
\]

We can see that the operators $\hat{V}(t)$ and $\hat{X}(t)$ are as expected compared to the known behavior of this system in relativistic dynamics. Moreover, knowing
the solution to the Heisenberg equation, we can pass to the Schrödinger picture and write the evolution of basis kets as

\[ e^{-it\hat{L}} |x, v\rangle = |x(t), v(t)\rangle, \quad (35) \]

where \( x(t) = \frac{F}{m_0} \sqrt{1 + \left( \frac{Ft}{m_0} \right)^2} \) and \( v(t) = \frac{Ft}{m_0} \left[ 1 + \left( \frac{Ft}{m_0} \right)^2 \right]^{-1/2} \). For basis kets with \( v < 1 \), the time evolution guarantees that \( v(t) < 1 \) for all times. Hence, a physically acceptable state will remain so for all times. This result has been obtained for the particular case of a constant force in one dimension, however, it is a general result since, by construction, the Heisenberg evolution of the operators \( \hat{R}(t) \) and \( \hat{V}(t) \) mimic the behavior of \( r(t) \) and \( v(t) \) obtained from the relativistic force equation.

## 4 Lagrangian Operator

It is possible to write the elements of the Poincaré algebra in terms of a canonical momentum operator instead of the velocity, obtaining, then, the relativistic KvN theory. To define the momentum operator we first have to write the equation of motion in the Euler-Lagrange form. Equation (32) can be rewritten as

\[ -\left[ \hat{L}, \left[ \hat{\lambda}_v, \hat{T}^* \right] \right] = \hat{F}_i, \quad (36) \]

where the kinetic co-energy [34] is given by \( \hat{T}^* = m_0[1 - (1 - \hat{V}^2)^{1/2}] \). So far, we were not worried about the nature of the \( \hat{F}_i \). Relativistic principles restrict the possible 3-forces depending on the tensor decomposition of the 4-force [34]. We will focus our attention to Lorentz-like forces. This kind of forces can be obtained from a generalized potential [26]

\[ \hat{U}(\hat{R}, \hat{V}) = \hat{\phi}(\hat{R}) - \hat{V}_i \hat{A}_{i}(\hat{R}), \quad (37) \]

according to

\[ \hat{F}_k = -i \left[ \hat{\lambda}_{xk}, \hat{U} \right] - \left[ \hat{L}, \left[ \hat{\lambda}_{uk}, \hat{U} \right] \right] = \hat{E}_k + \left( \hat{V} \times \hat{B} \right)_k, \quad (38) \]

where

\[ \hat{E}_k = -\frac{\partial \hat{\phi}}{\partial \hat{x}_k} - \frac{\partial \hat{A}_k}{\partial t}, \quad (39) \]

\[ \hat{B}_k = \left( \nabla \times \hat{A} \right)_k. \quad (40) \]

The relativistic force equation can then be rewritten as

\[ \Phi[\hat{L}] = 0, \quad (41) \]
where the Lagrangian operator is
\[ \hat{\mathcal{L}} = \hat{T}^* - \hat{U}, \] (42)
and \( \Phi \) is the superoperator defined by
\[ \Phi = -\left[ \hat{\mathcal{L}}, \hat{\lambda}_v, \right] - i \left[ \hat{\lambda}_x, \right]. \] (43)

### 4.1 Momentum Representation

As in the nonrelativistic case [26], the Lagrangian operator is used to define the canonical momentum operator by
\[ \hat{\mathcal{P}} = i \left[ \hat{\mathcal{L}}, \hat{\lambda}_v \right]. \] Using Eq. (42), we obtain
\[ \hat{\mathcal{P}} = m_0 \hat{\gamma} \hat{\mathcal{V}} + \hat{\mathcal{A}}. \] (44)
The following commutation relations for \( \hat{\mathcal{P}} \) are satisfied
\[ \left[ \hat{X}_i, \hat{P}_j \right] = 0, \] (45)
\[ \left[ \hat{P}_i, \hat{\lambda}_x \right] = \frac{\partial \hat{A}_i}{\partial \hat{X}_j}, \] (46)
\[ \left[ \hat{P}_i, \hat{\lambda}_v \right] = im_0 \hat{\gamma} (\delta_{ij} + V_i V_j \hat{\gamma}). \] (47)

We can now make the definitions
\[ \hat{\lambda}_{p_i} = \frac{1}{m_0} \left[ \hat{\gamma}^{-1} \left( \hat{\lambda}_{v_i} - V_i V_k \hat{\lambda}_{v_k} \right) \right] S, \] (48)
\[ \hat{\lambda}'_{x_j} = \hat{\lambda}_{x_j} - \frac{\partial \hat{A}_i}{\partial \hat{X}_j} \hat{\lambda}_{p_i}, \] (49)

and, then, write the commutation relations for the set \( \{ \hat{\mathcal{R}}, \hat{\mathcal{P}}, \hat{\lambda}'_{x}, \hat{\lambda}_{p} \} \) as
\[ \left[ \hat{X}_i, \hat{X}_j \right] = \left[ \hat{X}_i, \hat{P}_j \right] = \left[ \hat{P}_i, \hat{P}_j \right] = 0, \] (50a)
\[ \left[ \hat{X}_i, \hat{\lambda}_{p} \right] = [\hat{P}_\alpha, \hat{\lambda}'_{x_\beta}] = [\hat{\lambda}'_{x_j}, \hat{\lambda}_{p_i}] = 0, \] (50b)
\[ \left[ \hat{X}_i, \hat{\lambda}'_{x_j} \right] = \left[ \hat{P}_i, \hat{\lambda}_{p_j} \right] = i\delta_{ij}. \] (50c)

The set of operators \( \{ \hat{\mathcal{R}}, \hat{\mathcal{P}}, \hat{\lambda}'_{x}, \hat{\lambda}_{p} \} \) is irreducible in the Hilbert space we are considering in view of the preceding set of equations (50a) to (50c). We can write the elements of the Poincaré algebra \( (\hat{J}, \hat{K}, \hat{L}) \) in terms of \( \{ \hat{\mathcal{R}}, \hat{\mathcal{P}}, \hat{\lambda}'_{x}, \hat{\lambda}_{p} \} \), obtaining, then, a momentum-based irreducible representation of the Poincaré
This alternative representation is gauge dependent due to the presence of the vector potential $\boldsymbol{A}$. We can give an example for the free particle, defining

$$\mathcal{H} = \sqrt{\mathbf{p}^2 + m_0^2}, \quad (51)$$

the elements of the algebra becomes

$$\mathcal{J}_k = \varepsilon_{ijk} \left( \hat{X}_j \hat{\lambda}_{x_k} + \hat{P}_j \hat{\lambda}_{p_k} \right), \quad (52)$$

$$\mathcal{L}' = \hat{P}_i \hat{\mathcal{H}}^{-1} \hat{X}_{x_i}, \quad (53)$$

$$\mathcal{K}_i = \left( \hat{X}_i \hat{\mathcal{L}} - \hat{\mathcal{H}} \hat{\lambda}_{p_i} \right)_S - t \hat{\lambda}_{x_i}. \quad (54)$$

In presence of a general Lorentz interactions, the general form of the Liouvillian is given by the rather complicated expresion

$$\mathcal{L}' = \left[ \hat{\mathcal{H}}^{-1} \left( \hat{P}_i - \hat{\lambda}_{x_i} \right) \left( \hat{\lambda}_{x_i}' + \frac{\partial \hat{\lambda}_{p_i}}{\partial \hat{X}_j} \right) + \hat{\mathbf{F}} \cdot \hat{\lambda}_{p} \right] - \hat{\mathcal{H}}^{-2} \left\{ \hat{\lambda}_i \left( \hat{\mathbf{F}} \cdot \hat{\mathbf{P}} \right) + \left( \hat{P}_i - \hat{\lambda}_i \right) \left( \hat{\mathbf{F}} \cdot \hat{\mathbf{A}} \right) \right\}_S \times \left( \delta_{ij} - \hat{P}_i \hat{\lambda}_j - \hat{P}_j \hat{\lambda}_i + \hat{\lambda}_i \hat{\lambda}_j \right) \hat{\lambda}_{p_j}. \quad (55)$$

where $\hat{\mathbf{F}}$ is the Lorentz force operator given in (38) and now $\hat{\mathcal{H}} = m_0 \hat{\gamma} = \sqrt{\left( \mathbf{p} - \mathbf{A} \right)^2 + m_0^2}$. Equation (55) is the relativistic generalization of the minimal-coupling rule for the KvN theory given in [33]. For purely electric fields ($\mathbf{A} = 0$), the Liouvillian (55) considerably simplifies to

$$\mathcal{L}' = \hat{\mathcal{H}}^{-1} \hat{\mathbf{P}} \cdot \hat{\lambda}'_{e} + \hat{\mathbf{F}} \cdot \hat{\lambda}_{p}. \quad (56)$$

The change from $\{ \hat{\mathbf{R}}, \hat{\mathbf{V}}, \hat{\lambda}_r, \hat{\lambda}_v \}$ to $\{ \hat{\mathbf{R}}, \hat{\mathbf{P}}, \hat{\lambda}'_{r}, \hat{\lambda}_p \}$ is a quantum canonical transformation, i.e., a transformation that preserve the commutation relations. Unitary transformations are canonical, but the converse is not necessarily true [35]. In the nonrelativistic case, the pass from velocity representation to momentum representation is performed by a composition of a scale and a unitary transformation [26], we will now show that the same is true in the relativistic case. The transformation $\{ \hat{\mathbf{R}}, \hat{\mathbf{V}}, \hat{\lambda}_r, \hat{\lambda}_v \} \rightarrow \{ \hat{\mathbf{R}}, \hat{\mathbf{P}}, \hat{\lambda}'_{r}, \hat{\lambda}_p \}$ can be given in two steps. First, we make the change

$$\hat{\mathbf{V}} \rightarrow m_0 \hat{\mathbf{V}}, \quad (57a)$$

$$\hat{\lambda}_v \rightarrow \frac{1}{m_0} \hat{\lambda}_v. \quad (57b)$$
The second step consists in a similarity transformation by the unitary operator

\[ \hat{C} = \hat{C}_2 \hat{C}_1, \]  

(58a)

\[ \hat{C}_1 = \exp \left[ \frac{i}{2} \left( \hat{V}^2 \cdot \hat{\lambda}_v \right) s \right], \]  

(58b)

\[ \hat{C}_2 = \exp [i A \cdot \hat{\lambda}_p]. \]  

(58c)

The proof that \( \hat{C} \) leads to the correct canonical transformation \( \{ \hat{R}, m_0 \hat{V}, \hat{\lambda}_r, \hat{\lambda}_p \} \rightarrow \{ \hat{R}, \hat{P}, \hat{\lambda}_r, \hat{\lambda}_p \} \) is given in the appendix. We point out that the role of \( \hat{C}_1 \) is to pass from the nonrelativistic kinematic momentum to the relativistic one

\[ \hat{C}_1 \left( m_0 \hat{V} \right) \hat{C}_1^{-1} = m_0 \gamma \hat{V}, \]  

(59)

and the role of \( \hat{C}_2 \) is to go into the canonical momentum by inclusion of the vector potential

\[ \hat{C}_2 \left( m_0 \gamma \hat{V} \right) \hat{C}_2^{-1} = m_0 \gamma \hat{V} + \hat{A}. \]  

(60)

There are two nonequivalent ways to define phase space kets. We could define them by the unitary transformation \( |r, p \rangle = \hat{C} |r, v \rangle \), as it was done in [26]. However, this would result in \( \hat{P} \) having the undesirable eigenvalue equation \( \hat{P} |r, p \rangle = m_0 v |r, p \rangle \). In the next section we will relate the operational theory we have just developed with the results of Hamiltonian mechanics, the following simple identification is then preferable

\[ |r, p \rangle \equiv |r, v \rangle. \]  

(61)

On \( |r, p \rangle \), \( \hat{R} \) and \( \hat{P} \) act as multiplicative operators

\[ \hat{R} |r, p \rangle = r |r, p \rangle, \]  

(62a)

\[ \hat{P} |r, p \rangle = p |r, p \rangle. \]  

(62b)

That \( \hat{\lambda}_r \) and \( \hat{\lambda}_p \) act as translation operator on \( |r, p \rangle \) follow from the commutation relations \( \hat{C} \) to \( \hat{C}^{-1} \).

The Hilbert space spanned by the kets \( |r, p \rangle \), the set of operators \( \{ \hat{R}, \hat{P}, \hat{\lambda}_r, \hat{\lambda}_p \} \), the Liouvillian \( \hat{L} \) or \( \hat{L}^* \), together with the Born rule \( P(r, p) = |\langle r, v | \Psi \rangle|^2 \) give the relativistic generalization of the KvN formulation of classical mechanic.

## 5 Relation with Hamiltonian mechanics

In this section, we will link the theory developed above with the usual relativistic Hamiltonian mechanics. It is enough to show the derivation of Hamiltonian
mechanics from the KvN formalism. Consider the wave function

$$\psi(r, p) = \langle r, p | \psi \rangle.$$  \hfill (63)

The position and momentum operators act as multiplication operators when acting on $\psi(r, p)$

$$\hat{X}_i \psi(r, p) = x_i \psi(r, p),$$  \hfill (64)

$$\hat{P}_j \psi(r, p) = p_j \psi(r, p).$$  \hfill (65)

On the other hand, the operators $\hat{\lambda}_p$ and $\hat{\lambda}_r$ act as derivatives

$$\hat{\lambda}_r \psi(r, p) = -i \nabla_r \psi(r, p),$$  \hfill (66)

$$\hat{\lambda}_p \psi(r, p) = -i \nabla_p \psi(r, p).$$  \hfill (67)

With the help of the Poisson bracket $\{a, b\} = \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial p_j} - \frac{\partial a}{\partial p_j} \frac{\partial b}{\partial x_i}$, we can write the components of $\hat{\lambda}_r$ and $\hat{\lambda}_p$ as

$$\hat{\lambda}_{r_i} = -i \{, p_i \},$$  \hfill (68)

$$\hat{\lambda}_{p_j} = i \{, x_j \}.$$  \hfill (69)

The elements of the Poincaré algebra (52) to (54) can be written as

$$\hat{\mathcal{L}}' = -i \{ , H \},$$  \hfill (70)

$$\hat{\mathcal{J}} = -i \{ , r \times p \},$$  \hfill (71)

$$\hat{\mathcal{K}} = -i \{ , rH - pt \},$$  \hfill (72)

where the Hamiltonian function is $H = \sqrt{p^2 + m_0^2}$. We can see that the elements of the Poincaré algebra are related, but are not identified with, the energy $H$, the angular momentum $r \times p$, and the center of energy $rH - pt$.

For a dynamical evolution, the Liouvillian retains the form (70), but the Hamiltonian function changes to

$$H = \sqrt{(p - A)^2 + m_0^2} + \phi(r).$$  \hfill (73)

The Schrödinger-like equation (17) becomes

$$\frac{\partial \psi}{\partial t} + \{\psi, H\} = 0,$$  \hfill (74)

or written for the complex conjugate $\psi^*$

$$\frac{\partial \psi^*}{\partial t} + \{\psi^*, H\} = 0.$$  \hfill (75)
Defining the probability density in phase-space \( \rho = |\psi|^2 \), equations (74) and (75) can be combined to give

\[
\frac{\partial \rho}{\partial t} + \{ \rho, H \} = 0.
\]  

(76)

Equation (76) is the classical Liouville equation. Hence, the operational formulation of classical mechanics is equivalent to classical statistical mechanics.

Particle dynamics is recovered when we considerer classical pure states, this is, when the probability density is allowed to be a point-like distribution

\[
\rho(t) = \delta(r - r(t))\delta(p - p(t)).
\]  

(77)

6 Transformation properties

We can find the transformation properties of an operator under a finite Lorentz transformation just by computing its commutator with \( \hat{K} \). For example, computing the commutators between \( \hat{K} \) and \( \hat{V} \) gives

\[
[\hat{K}_j, \hat{V}_k] = i (\delta_{jk} - \hat{V}_j \hat{V}_k).
\]  

(78)

The commutation relation (78) is also found in the quantum case [36]. Finite Lorentz transformation are obtained by a similarity transformation

\[
e^{is\hat{K}_j} \hat{V}_k e^{-is\hat{K}_j}.
\]  

(79)

The computation of (79) reduces to a computation of nested commutator via the Baker-Campbell-Hausdorff expansion. The similarity transformation (79) has the same outcome as in the quantum case since the commutator algebras coincide. The result is that the velocity operator obeys the velocity addition formula. For example, a boost in the z-axis results in the Eqs. (20) to (22).

The set \( (\hat{H}, \hat{P}) \) transform as the elements of a 4-vector. This follows from the brackets

\[
[\hat{K}_i, \hat{P}_j] = \delta_{ij} \hat{H},
\]  

(80)

\[
[\hat{K}_k, \hat{H}] = i \hat{P}_k.
\]  

(81)

For example, a boost in the z-axis gives

\[
e^{is\hat{K}_z} \hat{H} e^{-is\hat{K}_z} = \hat{H} \cosh b - \hat{P}_z \sinh b,
\]  

(82)

\[
e^{is\hat{K}_z} \hat{P}_z e^{-is\hat{K}_z} = \hat{P}_z \cosh b - \hat{H} \sinh b,
\]  

(83)

\[
e^{is\hat{K}_z} \hat{P}_i e^{-is\hat{K}_z} = \hat{P}_i \quad (i = 1, 2).
\]  

(84)
The sets \((\hat{L}, \hat{\lambda}_v)\) and \((\hat{L}', \hat{\lambda}_p)\) also transform as a 4-vector in view of the Poincaré algebra relations (3b) and (3d).

The position operator \(\hat{R}\) does not transform as the spatial components of a 4-vector. It is the exact same situation encountered in relativistic quantum mechanics for the Newton-Wigner operator \([37, 38]\) and for the very same reasons. As in quantum theory, the formalism developed in this paper does not consider space and time on equal grounds. An Hermitian operator is assigned to position, whereas time is considered to be just a parameter. It can be shown that \(\hat{R}\) transforms, up to a symmetrization, as a Newton-Wigner operator. For example, a Lorentz transformation in the z-axis gives

\[
e^{is\hat{K}_z} e^{-is\hat{K}_z} = \gamma^{-1} \hat{z} + v\gamma^{-1} \left( \frac{\hat{z}\hat{V}_z}{1 - v\hat{V}_z} \right),
\]

\[e^{is\hat{K}_z} \hat{X}_i e^{-is\hat{K}_z} = \hat{X}_i + v \left( \frac{\hat{z}\hat{V}_i}{1 - v\hat{V}_z} \right) \quad (i = 1, 2).
\]

To prove the above statement, is sufficient to show that the right and left hand sides of the previous equations agree up to the firsts lowest orders in a series expansion. The equality of the entire expressions follows since both sides are a one parameter group of transformation. The left-hand side of Eq. (85) can be expanded using the Baker-Campbell-Hausdorf formula

\[
e^{X}Y e^{-X} = Y + [X,Y] + \frac{1}{2} [X,[X,Y]] + \ldots,
\]

where

\[
\left[ \hat{K}_z, \hat{z} \right] = i \left( \hat{z}\hat{V}_z \right),
\]

\[
\left[ \hat{K}_z, \left[ \hat{K}_z, \hat{z} \right] \right] = \hat{z} - 2\hat{z}\hat{V}_z.
\]

The right hand side of (85) can be expanded using \(\gamma^{-1} = 1 - \frac{1}{2}v^2 - \frac{1}{8}v^4\ldots\) and \(\frac{1}{1-v\hat{V}_z} = 1 + v\hat{V}_z + \ldots\). It follows that both sides agree up to order \(v^2\), hence, Eq. (85) is an identity. A similar reasoning shows the validity of Eq. (86).

### 7 Concluding Remarks

We obtained an operational formulation of classical relativistic dynamics from a unitary irreducible representation of the Poincaré group, extending the results given in [26] for the Galilei group and non-relativistic mechanics. Moreover, we have given, from first principles, a relativistic generalization of the minimal coupling rule in the KvN theory given in [13]. These results are completely independent from quantum mechanics as no classical limit was employed anywhere.

As in the nonrelativistic case [26], the theory can be equivalently formulated in terms of the velocity or in terms of the canonical momentum. The passing from a theory formulated in the tangent bundle of configuration space to a theory in phase space is done by a quantum canonical transformation that is
a composition of a scale and a unitary transformation. However, contrary to the nonrelativistic case, it was found that it is preferable to not define the kets \(|r, p\rangle\) with the same unitary operator used to pass from \(\{\hat{R}, m_0\hat{V}, \lambda_r, \frac{1}{m_0} \lambda_v\}\) to \(\{\hat{R}, \hat{P}, \lambda'_r, \lambda'_p\}\). The ad hoc identification \(|r, p\rangle\) given in Eq. (61) was chosen in order for us to get the relativistic KvN formalism and, later, relativistic Hamiltonian mechanics.

The majority of the operators we used in this paper transform as expected from relativistic physics. The sole exception is the position operator as it transforms not as the spatial components of a 4-vector but as a Newton-Wigner operator. The relation between our position operator and the Newton-Wigner function in Hamiltonian mechanics [39] remains to be explored.

The classification of all the classical unitary irreducible representation of the proper Poincaré group remains unsolved as we have only given one possible realization. In this regard, the physically most relevant open problem is to find representations that leads to the dynamics of massless particles and the inclusion of the (classical) intrinsic angular momentum.

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A Appendix

The purpose of this appendix is to prove that the quantum canonical transformation

\[ \hat{R} \rightarrow \hat{\lambda}, \]

\[ m_0 \hat{V} \rightarrow \hat{P} = m_0 \hat{V} + \hat{A}, \]

\[ \hat{\lambda}_x \rightarrow \hat{\lambda}_x' = \hat{\lambda}_x - \frac{\partial \hat{A}}{\partial \hat{X}^i} \hat{\lambda}_x, \]

\[ \frac{1}{m_0} \hat{\lambda}_v \rightarrow \hat{\lambda}_p = \frac{1}{m_0} \left[ \frac{\gamma}{\gamma^2} \left( \hat{\lambda}_v - V \hat{V}_k \hat{\lambda}_v \right) \right]_S, \]

can be obtained via the unitary operator

\[ \hat{C} = \hat{C}_2 \hat{C}_1, \]

\[ \hat{C}_1 = \exp \left[ \frac{i}{2} \left( \hat{V}^2 \hat{V} \cdot \hat{A} \right)_S \right], \]

\[ \hat{C}_2 = \exp \left[ i A \cdot \hat{A} \right]. \]

First, we have the immediate result \( e^{i \hat{C} \hat{R} e^{-i \hat{C}}} = \hat{R} \) since \( \hat{C} \) does not depend on \( \hat{\lambda}_x \). We can proceed for \( \hat{\lambda}_x \) as follows: since \( \hat{C}_1 \) is not function of \( \hat{R} \), we have

\[ \hat{C} \hat{\lambda}_x \hat{C}^{-1} = \hat{C}_2 \hat{\lambda}_x \hat{C}_2^{-1}. \]

Then, by a Baker-Campbell-Hausdorff expansion \( e^{X}Ye^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} [X, Y]^{(n)} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots \), we have that

\[ e^{i \hat{A} \hat{\lambda}_x} e^{-i \hat{A} \hat{\lambda}_x} = \hat{\lambda}_x + i \hat{[A \cdot \hat{A}, \hat{\lambda}_x]} - \hat{[A \cdot \hat{A}, \hat{[A \cdot \hat{A}, \hat{\lambda}_x]}]} + \ldots \]

\[ = \hat{\lambda}_x - \frac{\partial \hat{A}}{\partial \hat{X}_i} \hat{\lambda}_x. \]
The procedure for $m_0 \hat{V}$ and $\frac{1}{m_0} \hat{\lambda}_V$ is more involved as it entails showing equality between power series. We start by considering the transformation of $m_0 \hat{V}$. Let us first define the kinematic momentum operator $\hat{\Pi} = m_0 \hat{\gamma} \hat{V}$. This operator obeys the following commutation relation

$$[\hat{P}_j, \hat{\lambda}_{p_i}] = [\hat{\Pi}_j, \hat{\lambda}_{p_i}] = i\delta_{ij}.$$ 

The transformation given by $\hat{C}_1$ on $m_0 \hat{V}$ is

$$\hat{C}_1 \left( m_0 \hat{V} \right) \hat{C}_1^{-1} = m_0 \hat{\gamma} \hat{V} = \hat{\Pi}_i.$$  

(A-7)

We prove this affirmation by expanding both sides of (A-7) in power series. The left hand side can be expanded via the Baker-Campbell-Hausdorff formula, whereas in the right hand side we can expand the Lorentz factor $\hat{\gamma}$ in a Maclaurin series as follows

$$\hat{C}_1 \left( m_0 \hat{V}_i \right) \hat{C}_1^{-1} = m_0 \hat{V}_i + \frac{im_0}{2} \left[ \hat{V}^2, \hat{V}_i \right] \ldots.$$  

(A-8)

We can see that $\hat{C}_1 \left( m_0 \hat{V}_i \right) \hat{C}_1^{-1}$ and $m_0 \hat{\gamma} \hat{V}_i$ coincide at order $\hat{V}^2$. We can prove that they coincide at all orders by mathematical induction. Let us assume the series agree up to the $n$th term, this is, we assume that

$$\frac{i^n}{2^n n!} \left[ \hat{V}^2, \hat{V}_i \right]^{(n)} = \hat{V}_i \hat{V}^{2n} \prod_{k=1}^{n} \left( \frac{2k - 1}{2k} \right).$$  

(A-10)

Then, the equality of the power series is proven by noting that they also agree in the $n+1$ term. We have that

$$\frac{i^{n+1}}{2^{n+1} (n+1)!} \left[ \hat{V}^2, \hat{V}_i \right]^{(n+1)} = \frac{i}{2(n+1)} \left[ \hat{V}^2, \hat{V}_i \right]^{(n+1)} \hat{V}_i \hat{V}^{2n} \prod_{k=1}^{n} \left( \frac{2k - 1}{2k} \right).$$  

(A-11)

thus, Eq. (A-7) has been proven.

We now proceed to show that

$$\hat{C}_2 \left( \hat{\Pi}_i \right) \hat{C}_2^{-1} = \hat{\Pi}_i + \hat{A}_i.$$  

(A-12)
This can be proven by noting that \( [i\mathbf{A} \cdot \lambda_p, \hat{H}_i] = \hat{A}_i \) and \( [i\mathbf{A} \cdot \lambda_p, [i\mathbf{A} \cdot \lambda_p, \hat{H}_i]] = 0 \). Equation (A-12) then follows by a Baker-Campbell-Hausdorff expansion. In summary, we have shown that

\[
\hat{C} \left( m_0 \hat{V}_i \right) \hat{C}^{-1} = \hat{C}_2 \hat{C}_1 \left( m_0 \hat{V}_i \right) \hat{C}_1^{-1} \hat{C}_2^{-1} = \hat{H}_i + \hat{A}_i = \hat{P}_i. \quad (A-13)
\]

The transformation of \( \hat{\lambda}_v \) involves, again, a power series comparison. The claim is that

\[
\hat{C}_1 \left( \frac{1}{m_0} \hat{\lambda}_{v_i} \right) \hat{C}_1^{-1} = \frac{1}{m_0} \left[ \hat{\gamma}^{-1} \left( \hat{\lambda}_{v_i} - \hat{V}_i \hat{V}_k \hat{\lambda}_{v_k} \right) \right]_S. \quad (A-14)
\]

With the help of

\[
\hat{\gamma}^{-1} = 1 - \frac{1}{2} \hat{V}^2 - \frac{1}{8} \hat{V}^4 - \frac{1}{16} \hat{V}^6 + \ldots, \quad (A-15)
\]

we can write both sides of (A-14) as

\[
\frac{1}{m_0} \left[ \hat{\gamma}^{-1} \left( \hat{\lambda}_{v_i} - \hat{V}_i \hat{V}_k \hat{\lambda}_{v_k} \right) \right]_S = \frac{1}{m_0} \hat{\lambda}_{v_i} - \frac{1}{m_0} \left( \hat{V}_i \hat{V}_k \hat{\lambda}_{v_k} \right)_S \quad \text{and} \quad -\frac{1}{2} \hat{V}^2 \left( \hat{\lambda}_{v_i} - \hat{V}_i \hat{V}_k \hat{\lambda}_{v_k} \right) + \ldots. \quad (A-16)
\]

This comparison is more elaborated than the previous one since the brackets \( \left( \hat{V}^2 \hat{V} \cdot \lambda_v \right)_S, \frac{1}{m_0} \hat{\lambda}_{v_i} \) produce terms of different order in \( \hat{V}^2 \). However, it can be checked that the first terms of both series agree, and the equality of the entire series can be established, again, by mathematical induction. Since \( \hat{C}_2 \hat{\lambda}_p \hat{C}_2^{-1} = \hat{\lambda}_p \), we finally obtain

\[
\hat{C} \left( \frac{1}{m_0} \hat{\lambda}_{v_i} \right) \hat{C}^{-1} = \frac{1}{m_0} \left[ \hat{\gamma}^{-1} \left( \hat{\lambda}_{v_i} - \hat{V}_i \hat{V}_k \hat{\lambda}_{v_k} \right) \right]_S = \hat{\lambda}_p. \quad (A-17)
\]