Research Article

Blow-Up Analysis for a Reaction-Diffusion Model with Nonlocal and Gradient Terms

Xuhui Shen\textsuperscript{1} and Lun Lan\textsuperscript{2}

\textsuperscript{1}School of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China
\textsuperscript{2}Yuncheng Branch of Chengdu Dayun Automobile Co., Ltd., Yuncheng 044000, China

Correspondence should be addressed to Xuhui Shen; xhuishen@sxufe.edu.cn

Received 7 December 2021; Accepted 12 January 2022; Published 28 March 2022

Academic Editor: Julien Bruchon

Copyright © 2022 Xuhui Shen and Lun Lan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we investigate the blow-up phenomena for the following reaction-diffusion model with nonlocal and gradient terms:

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + a u^p \int_{\Omega} u^\alpha \, dx - |\nabla u|^q \quad \text{in} \; \Omega \times (0,t^*), \\
\partial \Omega &\quad h(u) \quad \text{on} \; \partial \Omega \times (0,t^*), \\
\end{aligned}
\end{equation}

Here $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded and convex domain with smooth boundary, and constants $m$, $p$, $q$, $\alpha$ are supposed to be positive. Utilizing the Sobolev inequality and the differential inequality technique, lower bound for blow-up time is derived when blow-up occurs. In addition, we give an example as application to illustrate the abstract results obtained in this paper.

1. Introduction

As we all know, reaction-diffusion models can be used to illustrate many natural phenomena such as heat flow, combustion, and gravitational potentials, so they have received extensive attention from many scholars [1, 2]. Since early sixties, lots of papers concerning the problem of blow-up or global existence of solutions to reaction-diffusion models have been published. After that, qualitative properties of reaction-diffusion models were investigated, such as the blow-up set, blow-up rate, blow-up profile, and boundedness of global solutions (see the papers [3–6] and books [7, 8]).

Especially, when the solution of reaction-diffusion models blows up, one would like to know in what time the solution blows up. Weissler in [9] firstly studied the blow-up time of reaction-diffusion models, and then much attention has been paid to finding the blow-up time of solutions. However, much work has been done in deriving the upper bound for the blow-up time [10]. In practical situations, lower bound for the blow-up time is more useful than upper bound for the blow-up time in predicting the critical state of the systems. This makes the study of the lower bound for the blow-up time more meaningful. In 2006, Payne and Schaefer introduced the first differential inequality technique to give the lower bound for the blow-up time (see [11]). Based on Payne’s methods, lots of works are devoted to giving the lower bound for the blow-up time when blow-up occurs (refer to [12–17]).

In this paper, we investigate the blow-up time of the following reaction-diffusion model with nonlocal and gradient terms:

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + a u^p \left( \int_{\Omega} u^\alpha \, dx \right)^m - |\nabla u|^q \quad \text{in} \; (0,t^*), \\
\frac{\partial u}{\partial y} &= h(u) \quad \text{on} \; \partial \Omega \times (0,t^*), \\
u(x,0) &= u_0(x) \geq 0 \quad \text{in} \; \Omega,
\end{aligned}
\end{equation}
where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded and convex domain with smooth boundary, \( m, p, q, \alpha \) are positive constants, and \( t^* \) is the maximal existence time of the solution \( u \). Also, we assume that \( h \) is nonnegative \( C^1 (\mathbb{R}_+) \) function, where \( \mathbb{R}_+ = (0, +\infty) \). \( u_0 (x) \) is assumed to be nonnegative \( C^1 (\mathbb{R}_+) \) function, which is compatible with the boundary conditions. Problem (1) can describe many physical phenomena and biological species theories. For example, in the density of some biological species for population dynamics, nonlocal source term represents the births of the species, and gradient terms can illustrate the natural or the accidental deaths.

To complete our research, we focus our attention on the following blow-up phenomena of the reaction-diffusion models (see [18–21]). Marras, et al. in [20] studied

\[
\begin{cases}
    u_t = \Delta u^m + a \int_\Omega u^p \, dx - bu^q - c |\nabla u|^2 & \text{in } \Omega \times (0, t^*), \\
    \frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial \Omega \times (0, t^*), \\
    u(x, 0) = u_0 (x) \geq 0 & \text{in } \overline{\Omega},
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 1) \) is a bounded and convex domain with smooth boundary. When \( p > q \), they determined lower bounds for \( t^* \) when blow-up occurs in \( \Omega \subset \mathbb{R}^2 \) and \( \Omega \subset \mathbb{R}^3 \).

Ding and Shen in [21] investigated the following nonlocal reaction-diffusion model:

\[
\begin{cases}
    (h(u))_t = \nabla \cdot \left( \rho (|\nabla u|^2) \nabla u \right) + a (x) f (u) & \text{in } \Omega \times (0, t^*), \\
    \frac{\partial u}{\partial \nu} + \gamma u = 0 & \text{on } \partial \Omega \times (0, t^*), \\
    u(x, 0) = u_0 (x) \geq 0 & \text{in } \overline{\Omega},
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is a bounded and convex domain whose boundary is sufficiently smooth. Assuming that

\[
a (x) f (u) \leq a_1 + a_2 u^\beta \left( \int_\Omega u^\beta \, dx \right)^m,
\]

they obtained a lower bound for the blow-up time when \( \Omega \subset \mathbb{R}^3 \). Moreover, an upper bound for the blow-up time and global solution were also discussed.

Inspired by the research studies mentioned above, we deal with the blow-up phenomena of problem (1). The highlight of this paper is to investigate the model with both nonlocal terms and gradient terms, making the research closer to reality. In addition, there is little research on the blow-up phenomenon of the solution of problem (1) and even less research on the lower bound for the blow-up time. The key to achieving our work is to build suitable auxiliary functions. Since auxiliary functions given in problems (2) and (3) are no longer applicable, we need to establish new auxiliary functions and use the Sobolev inequality to accomplish our research.

The paper is organized as follows. In Section 2, when \( \Omega \subset \mathbb{R}^N (N \geq 3) \), we derive a lower bound for the blow-up time when blow-up occurs. In Section 3, an example is given to illustrate the application of the abstract results obtained in this paper.

## 2. Lower Bound for Blow-Up Time

### When \( \Omega \subset \mathbb{R}^N (N \geq 3) \)

In this section, we seek the lower bound for the blow-up time when \( \Omega \subset \mathbb{R}^N (N \geq 3) \). For this aim, we firstly assume

\[
0 < h(u) \leq bu^\beta,
\]

with constants \( b > 0, \beta > 1 \). Moreover, we suppose constants \( m > 0, p > 1, q > 2 \). Let the auxiliary function be defined as follows:

\[
A (t) = \int_\Omega u^\alpha \, dx,
\]

where

\[
a > \max \left\{ 2, N (\beta - 1), \frac{N (p - 1)}{2}, \frac{q}{q - 2} + 1 \right\}.
\]

Owing to Corollary 9.14 in [22], we know that

\[
W^{1,2} (\Omega) \rightarrow L^{(2N/N-2)} (\Omega), \quad N \geq 3,
\]
which implies

\[ \left( \int_{\Omega} w^{2N/(N-2)} \, dx \right)^{(N-2)/2N} \leq C \left( \int_{\Omega} w^2 \, dx + \int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2}. \]  

(9)

Here \( w \in W^{1,2}(\Omega) \) and \( C = C(N, \Omega) \) is a Sobolev embedding constant depending on \( N \) and \( \Omega \). Inequality (9) will be used in our proof. The main results are stated next.

**Theorem 1.** Let \( u \) be a nonnegative classical solution of the problem (1). Suppose (5)–(7) hold. If the solution \( u \) blows up in the measure \( A(t) \) in some finite time \( t^* \), then \( t^* \) is bounded by

\[ t^* \geq \int_{A(0)}^{\infty} \frac{d\eta}{D + D_1 \eta^{\alpha-\beta/2} + D_2 \eta^{\alpha-\beta} + D_3 \eta^{\alpha+p-1/\alpha+\eta} + D_4 \eta^{2\alpha(m+1)-(p+1)(N-2)/2aN(p-1)}}. \]

(10)

where

\[ D = B_1, D_1 = B_2 \alpha - N(\beta - 1), D_2 = B_2 \alpha - N(\beta - 1)/\alpha, D_3 = a \alpha C(\beta - 1), D_4 = a \alpha C(\beta - 1)/\alpha, \]

(11)

\[ B_1 = a \alpha C(\beta - 1)/\alpha, D_4 = a \alpha C(\beta - 1)/\alpha, \]

(12)

\[ B_2 = a \alpha C(\beta - 1)/\alpha, D_4 = a \alpha C(\beta - 1)/\alpha. \]

(13)

\[ \varepsilon_1 = \frac{\alpha^2 L_0}{(\alpha + \beta - 1) \sigma}, \]

\[ \varepsilon_2 = \frac{\alpha^2 L_0}{(\alpha + \beta - 1) \sigma}, \]

\[ \varepsilon_3 = \frac{\alpha^2 L_0}{(\alpha + \beta - 1) \sigma}. \]

(15)

where \( |\Omega| \) is the measure of the bounded and convex domain \( \Omega, L_0 = \min_{\Omega} (x \cdot y) > 0 \), and \( d = \max_{\Omega} |x| \).

Proof. Using (5)–(7) and the divergence theorem, we have

\[ A'(t) = a \int_{\Omega} u^{\alpha-1} u t \, dx = a \int_{\Omega} u^{\alpha-1} \left[ \Delta u + au^p \left( \int_{\Omega} u^m \, dx \right)^m - |\nabla u|^2 \right] \, dx \]

\[ = -\alpha (\alpha - 1) \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + a \int_{\Omega} u^{\alpha-1} h(u) \, dx + a \int_{\Omega} u^{\alpha+p-1} \left( \int_{\Omega} u^m \, dx \right)^m \]

\[ \leq -\alpha (\alpha - 1) \int_{\Omega} u^{\alpha-1} |\nabla u|^2 \, dx \]

(16)

It follows from the Hölder inequality that

\[ A'(t) \leq -\alpha (\alpha - 1) \int_{\Omega} u^{\alpha-1} |\nabla u|^2 \, dx. \]
\[
\int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx \\
\leq \left( \int_{\Omega} u^{\alpha-1} |\nabla u|^q \, dx \right)^{2/q} \left( \int_{\Omega} u^{\alpha-1 - (qq^{-2})} \, dx \right)^{q^{-2q}} \\
\leq \int_{\Omega} u^{\alpha-1} |\nabla u|^q \, dx + \frac{q-2}{q} \left( \frac{q}{2} \right)^{-2q-2} \int_{\Omega} u^{\alpha-1 - qq^{-2}} \, dx,
\]
which is equivalent to

\[
-\alpha \int_{\Omega} u^{\alpha-1} |\nabla u|^q \, dx \leq -\alpha \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + \frac{(q-2)\alpha}{q} \left( \frac{q}{2} \right)^{-2q-2} \int_{\Omega} u^{\alpha-1 - qq^{-2}} \, dx.
\]

(18)

Inserting (18) into (16), we derive

\[
A'(t) \leq -\alpha^2 \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + b\alpha \int_{\partial\Omega} u^{\alpha+\beta-1} \, dS + a\alpha \int_{\Omega} u^{\alpha+\beta-1} \left( \int_{\Omega} u^\alpha \, dx \right)^m \\
+ \frac{(q-2)\alpha}{q} \left( \frac{q}{2} \right)^{-2q-2} \int_{\Omega} u^{\alpha-1 - qq^{-2}} \, dx.
\]

(19)

Recalling inequality (2) in [20], we have

\[
\int_{\partial\Omega} u^{\alpha+\beta-1} \, dS \leq \frac{N}{L_0} \int_{\Omega} u^{\alpha+\beta-1} \, dx + \frac{(\alpha + \beta - 1)d}{L_0} \int_{\Omega} u^{\alpha+\beta-2} \, dx.
\]

(20)

We apply the Hölder inequality and the Young inequality to the term \( \int_{\Omega} u^{\alpha+\beta-2} |\nabla u| \, dx \) to deduce

\[
\int_{\Omega} u^{\alpha+\beta-2} |\nabla u| \, dx \leq \left( \epsilon_1 \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx \right)^{1/2} \left( \frac{1}{\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx \right)^{1/2} \\
\leq \frac{\epsilon_1}{2} \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + \frac{1}{2\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx,
\]

(21)

where \( \epsilon_1 \) is given in (15). The substitution of (21) into (20) yields

\[
\int_{\partial\Omega} u^{\alpha+\beta-1} \, dS \leq \frac{N}{L_0} \int_{\Omega} u^{\alpha+\beta-1} \, dx + \frac{(\alpha + \beta - 1)d\epsilon_1}{2L_0} \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx \\
+ \frac{(\alpha + \beta - 1)d}{2L_0\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx.
\]

(22)

We insert (22) into (19) to get
Applying (7) and Sobolev inequality (9), we have

\[ A'(t) \leq -\frac{1}{2}\int_\Omega u^{\alpha-2} |\nabla u| dx + \frac{abN}{L_0} \int_\Omega u^{\alpha+\beta-1} dx \]

\[ + \frac{b}{2L_0\epsilon_1} \int_\Omega u^{\alpha+2\beta-2} dx + aa \int_\Omega u^{\alpha+p-1} dx \left( \int_\Omega u^\alpha dx \right)^m \]

\[ + \frac{(q-2)\alpha}{q} \left( \frac{q}{2} \right)^{-2q-2} \int_\Omega u^{\alpha-1-q/2} dx \]

\[ = -2\int_\Omega |\nabla u|^{2\beta-2} dx + \frac{abN}{L_0} \int_\Omega u^{\alpha+\beta-1} dx \]

\[ + \frac{b}{2L_0\epsilon_1} \int_\Omega u^{\alpha+2\beta-2} dx + aa \int_\Omega u^{\alpha+p-1} dx \left( \int_\Omega u^\alpha dx \right)^m \]

\[ + \frac{(q-2)\alpha}{q} \left( \frac{q}{2} \right)^{-2q-2} \int_\Omega u^{\alpha-1-q/2} dx. \]

(23)

From (7), the Hölder inequality, and the Young inequality, we can deduce that

\[ \int_\Omega u^{\alpha+\beta-1} dx \leq \left( \int_\Omega u^{\alpha+2\beta-2} dx \right)^{\beta-1/\alpha+2\beta-2} \leq \frac{\alpha + \beta - 1}{\alpha + 2\beta - 2} \int_\Omega u^{\alpha+2\beta-2} dx + \frac{\beta - 1}{\alpha + 2\beta - 2} |\Omega|, \]

(24)

\[ \int_\Omega u^{\alpha-1-q/2} dx \leq \left( \int_\Omega u^{\alpha+2\beta-2} dx \right)^{(\alpha-1)(q-2) - q' (\alpha+2\beta-2)(q-2)} |\Omega|^{(2\beta-1)(q-2) + q' (\alpha+2\beta-2)(q-2)} \]

\[ \leq \frac{\alpha - 1}{\alpha + 2\beta - 2} + q \int_\Omega u^{\alpha+2\beta-2} dx + \frac{(2\beta - 1)(q-2) + q}{(\alpha + 2\beta - 2)(q-2)} |\Omega|. \]

(25)

Combining (24) and (25) with (23), we obtain

\[ A'(t) \leq -2\int_\Omega |\nabla u|^{2\beta-2} dx + \frac{abN (\beta - 1)}{L_0 (\alpha + 2\beta - 2)} |\Omega| + \frac{a[(2\beta - 1)(q-2) + q]}{q(\alpha + 2\beta - 2)} \left( \frac{q}{2} \right)^{-2q-2} |\Omega| \]

\[ + \frac{abN (\alpha + \beta - 1)}{L_0 (\alpha + 2\beta - 2)} + \frac{a[(\alpha - 1)(q-2) - q]}{q(\alpha + 2\beta - 2)} \left( \frac{q}{2} \right)^{-2q-2} + \frac{b}{2L_0\epsilon_1} \int_\Omega u^{\alpha+2\beta-2} dx \]

\[ + aa \int_\Omega u^{\alpha+p-1} dx \left( \int_\Omega u^\alpha dx \right)^m \]

\[ \leq -2\int_\Omega |\nabla u|^{2\beta-2} dx + B_1 + B_2 \int_\Omega u^{\alpha+2\beta-2} dx + aa \int_\Omega u^{\alpha+p-1} dx \left( \int_\Omega u^\alpha dx \right)^m, \]

(26)

where \( B_1 \) and \( B_2 \) are defined in (13) and (14), respectively.

Applying (7) and Sobolev inequality (9), we have
\[
\int_\Omega u^{\alpha+2\beta-2} \, dx \\
\leq \left( \int_\Omega u^\alpha \, dx \right)^{1-\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega u^{\alpha/2} \, 2^{N/N-2} \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \\
\leq \left( \int_\Omega u^\alpha \, dx \right)^{1-\left(\beta-1\right)\left(N-2\right)/\alpha} \left[ C^{2N/N-2} \left( \int_\Omega u^\alpha \, dx + \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \right] \\
= C^{2N\left(\beta-1\right)/\alpha} \left( \int_\Omega u^\alpha \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha},
\]

\[
\int_\Omega u^{\alpha+p-1} \, dx \left( \int_\Omega u^\alpha \, dx \right)^m \\
\leq \left( \int_\Omega u^\alpha \, dx \right)^{2a-\left(p-1\right)\left(N-2\right)/2a} \left( \int_\Omega u^{\alpha/2} \, 2^{N/N-2} \, dx \right)^{\left(p-1\right)\left(N-2\right)/2a} \left( \int_\Omega u^\alpha \, dx \right)^m \\
\leq \left( \int_\Omega u^\alpha \, dx \right)^{2a-\left(p-1\right)\left(N-2\right)/2a+m} \left[ C^{\left(2N/N-2\right)\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega u^\alpha \, dx + \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} \right] \\
= C^N\left(p-1\right)/\alpha \left( \int_\Omega u^\alpha \, dx \right)^{2a-\left(p-1\right)\left(N-2\right)/2a+m} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(p-1\right)/2a\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}}.
\]

Thanks to the basic inequality, we rewrite (27) and (28) as

\[
\left(k_1 + k_2\right)^l \leq k_1^l + k_2^l, \quad k_1, k_2 > 0, \quad 0 \leq l < 1,
\]

\[
\int_\Omega u^{\alpha+2\beta-2} \, dx \\
\leq C^{\left(2N\left(\beta-1\right)/\alpha\right)\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega u^\alpha \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}}.
\]

It follows from the Hölder inequality and the Young inequality that

\[
C^{2N\left(\beta-1\right)/\alpha} \left( \int_\Omega u^\alpha \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} \\
\leq \left[ \epsilon_2^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} \left( \int_\Omega u^\alpha \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \right]^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} \\
\leq \left( \frac{\alpha}{\alpha-N\left(\beta-1\right)} \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}} C^{2N\left(\beta-1\right)/\alpha-N\left(\beta-1\right)} \left( \int_\Omega u^\alpha \, dx \right)^{\left(\beta-1\right)\left(N-2\right)/\alpha} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^2 \, dx \right)^{\left(N\left(\beta-1\right)/\alpha\right)^{\left(\beta-1\right)\left(N-2\right)/\alpha}}.
\]
\( C^{(N(p-1)/\alpha)} \left( \int_{\Omega} u^p \, dx \right)^{\frac{2\alpha - (p-1)(N-2)/2\alpha}{N(p-1)/2\alpha}} \left( \int_{\Omega} |\nabla u^{a/2}|^2 \, dx \right)^{\frac{N}{N(p-1)/2\alpha}} \)

\[
\leq \left[ \epsilon_3^{-N(p-1)/2\alpha - N(p-1)/2\alpha} C^{2N(p-1)/2\alpha - N(p-1)/2\alpha} \right] \left( \int_{\Omega} u^p \, dx \right)^{2\alpha - (p-1)(N-2)/2\alpha} \left( \int_{\Omega} |\nabla u^{a/2}|^2 \, dx \right)^{2\alpha - N(p-1)/2\alpha} 
\]

\[
+ \left( \epsilon_3 \int_{\Omega} |\nabla u^{a/2}|^2 \, dx \right)^{N(p-1)/2\alpha} \leq \frac{2\alpha - N(p-1)}{2\alpha} \epsilon_3^{-N(p-1)/2\alpha - N(p-1)/2\alpha} C^{2N(p-1)/2\alpha - N(p-1)/2\alpha} \left( \int_{\Omega} u^p \, dx \right)^{2\alpha - (p-1)(N-2)/2\alpha} \left( \int_{\Omega} |\nabla u^{a/2}|^2 \, dx \right)^{2\alpha - N(p-1)/2\alpha} 
\]

\[
\left( \epsilon_3 \int_{\Omega} |\nabla u^{a/2}|^2 \, dx \right)^{N(p-1)/2\alpha} \leq \frac{2\alpha - N(p-1)}{2\alpha} \epsilon_3^{-N(p-1)/2\alpha - N(p-1)/2\alpha} C^{2N(p-1)/2\alpha - N(p-1)/2\alpha} \left( \int_{\Omega} u^p \, dx \right)^{2\alpha - (p-1)(N-2)/2\alpha} \left( \int_{\Omega} |\nabla u^{a/2}|^2 \, dx \right)^{2\alpha - N(p-1)/2\alpha} 
\]

where \( \epsilon_2, \epsilon_3 \) are given in (15). Moreover, combining (33) and (31) and (32) and (30), respectively, we have

\[
\int_{\Omega} u^{p+1} \, dx \left( \int_{\Omega} u^a \, dx \right)^m \leq C^{(N(p-1)/\alpha)} \left( \int_{\Omega} u^a \, dx \right)^{(a+1)/\alpha + m} 
\]

Inserting (34) and (35) into (26), we obtain

\[
A'(t) \leq B_1 + B_2 C^{2N(p-1)/\alpha} A^{a-\beta(p-1)(N-2)/\alpha} (t) + B_2 \frac{\alpha - N(p-1)}{\alpha} \epsilon_2^{-N(p-1)/2\alpha} C^{(N(p-1)/2\alpha - N(p-1)/2\alpha)} A^{a-\beta(p-1)(N-2)/\alpha - N(p-1)/2\alpha} (t) \]

\[
+ \frac{2\alpha - N(p-1)}{2\alpha} \epsilon_3^{-(N(p-1)/2\alpha - N(p-1)/2\alpha)} C^{2N(p-1)/2\alpha - N(p-1)/2\alpha} A^{a-\beta(p-1)(N-2)/\alpha - N(p-1)/2\alpha} (t) 
\]

\[
= D + D_1 A^{a-\beta(p-1)(N-2)/\alpha} (t) + D_2 A^{a-\beta(p-1)(N-2)/\alpha - N(p-1)/2\alpha} (t) + D_3 A^{a+1/a+m} (t) \]

\[
+ D_4 A^{2\alpha(m+1)-(p-1)(N-2)/2\alpha - N(p-1)/2\alpha} (t) 
\]

where \( D, D_1, \ldots, D_4 \) are defined in (11) and (12). Integrating (36) between 0 and \( t' \), we arrive at
\[ t^* \geq \int_{A(0)}^{\infty} \frac{d\eta}{D + D_1 \eta^{1/2} + D_2 \eta^{1/3} + D_3 \eta^{2/3} + D_4 \eta^{3/2}} \]

(37)

3. Application

In what follows, an example is given as application to illustrate the abstract results in Theorem 1.

Example 1. Let \( u^0 \) be a nonnegative classical solution of the following problem:

\[
\begin{aligned}
&\frac{\partial \eta}{\partial t} = \Delta \eta + \frac{1}{100} \eta^4, & &\text{on } \partial \Omega \times (0, t^*), \\
&\eta(x, 0) = \frac{1999}{2000}, & &\text{in } \Omega,
\end{aligned}
\]

(38)

where \( \Omega = \{ \mathbf{r} = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1/100 \} \) is a spherical region in \( \mathbb{R}^3 \). Here we choose \( N = 3, m = 1, p = 2, q = 3, a = 2/3, b = 1/100, \alpha = 6, \beta = 2 \). It is easy to verify that conditions (5)–(7) are established. Moreover, form [23], we can calculate that the embedding constant \( C \approx 7 \cdot 5931 \). With simple calculations, we have \( L_0 = d = (1/10), |\Omega| = 4\pi/3000, B_1 \approx 2.8973 \times 10^{-3}, B_2 \approx 0.4381, \epsilon_1 = 36/7, \epsilon_2 = 4.5652, \epsilon_3 = 1, D_1 \approx 3.3265, D_2 \approx 2.7664, D_3 \approx 11.0223, D_4 \approx 11.5896, \) and

\[
A(0) = \int_{\Omega} \left( \frac{1999}{2000} + \frac{1}{20} |x|^2 \right)^6 \, dx \approx 41.838 \times 10^{-3}.
\]

(39)

It follows from Theorem 1 that

\[ t^* \geq \int_{A(0)}^{\infty} \frac{d\eta}{D + D_1 \eta^{5/6} + D_2 \eta^{2/3} + D_3 \eta^{5/6} + D_4 \eta^{23/9}} \]

(40)

\[ \approx \int_{4.183 \times 10^{-1}}^{\infty} \frac{d\eta}{2 \cdot 8973 \times 10^{-3} + 3 \cdot 3265 \eta^{2/3} + 2 \cdot 7664 \eta^{5/6} + 11 \cdot 0223 \eta^{1/2} + 11 \cdot 5896 \eta^{23/9}} \approx 0.4168. \]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (no. 2020L025), the National Natural Science Foundation of China (no. 61473180), and the Youth Natural Science Foundation of Shanxi Province (no. 20210302124533).

References

[1] J. W. Bebernes and D. H. Eberly, Mathematical Problems from Combustion Theory, Springer, New York, NY, USA, 1989.
[2] M. Weiss, “Crowding, diffusion, and biochemical reactions,” International Review of Cell and Molecular Biology, vol. 307, pp. 383–417, 2014.
[3] M. Chipot and F. B. Weissler, “Some blowup results for a nonlinear parabolic equation with a gradient term,” SIAM Journal on Mathematical Analysis, vol. 20, no. 4, pp. 886–907, 1989.
[4] K. Deng and H. A. Levine, “The role of critical exponents in blow-up theorems: the sequel,” Journal of Mathematical Analysis and Applications, vol. 243, no. 1, pp. 85–126, 2000.
[5] V. A. Galaktionov and J. L. Vázquez, “The problem of blow-up for nonlinear parabolic equations,” Discrete & Continuous Dynamical Systems, vol. 8, pp. 399–433, 2002.
[6] S. Frassu, C. Van Der Mee, and G. Viglialoro, “Boundedness in nonlinear attraction-repulsion Keller–Segel systems,” Mathematical Problems in Engineering, vol. 2014, Article ID 65428, 2021.
[7] P. Quittner and P. Souplet, Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States, Birkhäuser Advanced Texts, Basel, Switzerland, 2007.
[8] A. A. Samarskii, V. A. Kurdyumov, S. P. Mikhailov, and A. P. Galaktionov, Blow-up in Problems for Quasilinear Parabolic Equations, Walter de Gruyter, Berlin, Germany, 1995.
[9] F. B. Weissler, “Local existence and nonexistence for semilinear parabolic equations in L^p,” Indiana University Mathematics Journal, vol. 29, pp. 79–102, 1980.
[10] H. A. Levine, “The role of critical exponents in blow-up theorems,” SIAM Review, vol. 32, pp. 268–288, 1990.
[11] L. E. Payne and P. W. Schaefer, “Lower bounds for blow-up time in parabolic problems under Neumann conditions,” Applicable Analysis, vol. 85, pp. 1301–1311, 2006.
[12] J. T. Ding and B. Z. Guo, “Global and blow-up solutions for nonlinear parabolic equations with a gradient term,” Houston Journal of Mathematics, vol. 37, pp. 1265–1277, 2011.
[13] L. L. Zhang, N. Zhang, and L. X. Li, "Blow-up solutions and global existence for a kind of quasilinear reaction-diffusion equations," Zeitschrift für Analysis und ihre Anwendungen, vol. 33, pp. 247–258, 2014.

[14] L. E. Payne, G. A. Philippin, and S. Vernier Piro, "Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition," Zeitschrift für angewandte Mathematik und Physik, vol. 61, pp. 999–1007, 2010.

[15] H. M. Tian, L. L. Zhang, and X. Wang, "Blow-up phenomena in a class of coupled reaction-diffusion system with nonlocal boundary conditions," Applied Mathematics and Computation, vol. 414, Article ID 126667, 2022.

[16] J. Z. Zhang and F. S. Li, "Global existence and blow-up phenomena for divergence form parabolic equation with time-dependent coefficient in multidimensional space," Zeitschrift für Angewandte Mathematik und Physik, vol. 70, p. 16, 2019.

[17] Ph. Souplet, "Finite time blow-up for a nonlinear parabolic equation with a gradient term and applications," Mathematical Methods in the Applied Sciences, vol. 19, pp. 1317–1333, 1996.

[18] B. Abdelhedi and H. Zaag, "Construction of a blow-up solution for a perturbed nonlinear heat equation with a gradient and a non-local term," Journal of Differential Equations, vol. 272, pp. 1–45, 2021.

[19] C. Li, P. Kevin, S. Christina, and Z. G. Anna, "Mathematical models for cell migration: a non-local perspective," Philosophical Transactions of the Royal Society B, vol. 375, 2020.

[20] M. Marras, N. Pintus, and G. Viglialoro, "On the lifespan of classical solutions to a non-local porous medium problem with nonlinear boundary conditions," Discrete and Continuous Dynamical Systems-Series S, vol. 13, pp. 2033–2045, 2020.

[21] J. T. Ding and X. H. Shen, "Blow-up analysis for a class of nonlinear reaction diffusion equations with Robin boundary conditions," Mathematical Methods in the Applied Sciences, vol. 41, pp. 1683–1696, 2018.

[22] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, NY, USA, 2011.

[23] M. Mizuguchi, K. Tanaka, K. Sekine, and S. Oishi, "Estimation of Sobolev embedding constant on a domain dividable into bounded convex domains," Journal of Inequalities and Applications, vol. 17, pp. 1–18, 2017.