Eluding the No-Hair Conjecture for Black Holes *

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ABSTRACT

I discuss a recent analytic proof of bypassing the no-hair conjecture for two interesting (and quite generic) cases of four-dimensional black holes: (i) black holes in Einstein-Yang-Mills-Higgs (EYMH) systems and (ii) black holes in higher-curvature (Gauss-Bonnet (GB) type) string-inspired gravity. Both systems are known to possess black-hole solutions with non-trivial scalar hair outside the horizon. The ‘spirit’ of the no-hair conjecture, however, seems to be maintained either because the black holes are unstable (EYMH), or because the hair is of secondary type (GB), i.e. it does not lead to new conserved quantum numbers.

1 Introduction

The ‘no-hair conjecture’ of Black holes may be best attributed to Wheeler [1], who, motivated by earlier uniqueness theorems on Black-hole solutions of Einstein-Maxwell theory [2], stated that all exteriors of stationary black hole solutions are uniquely characterized by at most three conserved ‘charges’: mass, angular momentum and electric charge, i.e there are no non-trivial exterior fields, outside the horizon of a black-hole, other than those associated with long range Abelian gauge forces, obeying a Gauss law constraint.

The surprising discovery of the Bartnik-McKinnon (BM) non-trivial particle-like structure [3] in the Einstein-Yang-Mills system opened many possibilities for the existence of non-trivial solutions to Einstein-non-Abelian-gauge systems [4]. Indeed, soon after its discovery, many other self-gravitating structures with non-Abelian gauge fields have been discovered [5]. These include black holes with non-trivial non-Abelian hair, thereby leading to the possibility of evading the no-hair conjecture [6]. The physical reason for the existence of these classical solutions is the ‘balance’ between the non-Abelian gauge-field repulsion and the gravitational attraction. As I shall argue in this talk [7], such a balance allows for dressing black hole solutions by non-trivial configurations (outside the horizon) not only of non-Abelian gauge fields, whose presence should not have come as a great surprise since they obey the Gauss-law constraint in a rather similar spirit with the electromagnetic-field case, but also of (scalar) fields that are not associated with a Gauss-law. It is the presence of the latter that leads to an ‘apparent’ evasion of the no-hair conjecture [8].

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[1] In this article I shall concentrate on classical hair. Quantum hair is not covered by the no-hair conjecture, and it is a totally different, but equally important, issue of black-hole physics, which, however, will not be touched upon here.
In this talk I will explain in some detail the reasons for such a bypassing of the no-hair conjecture \[7\], by concentrating on two physically interesting examples, which capture the essential generic features of the problem. The first example is that of a spontaneously broken Yang-Mills theory in a non-trivial black-hole space-time (EYMH) \[8\]. This system has been recently examined from a stability point of view, and found to possess instabilities \[9, 7\], thereby making the physical importance of the solution rather marginal, but also indicating another dimension of the no-hair conjecture, not considered in the original analysis, that of stability. The fact that the hairy black hole is unstable implies a violation of the ‘letter’ but not of the ‘spirit’ of the no-hair conjecture. The second example is that of higher-curvature string-inspired gravity, and in particular a dilaton-graviton (super)string-effective action containing curvature-square terms of Gauss-Bonnet (GB) type \[10\]. The dilaton hair in that case is of secondary type \[11\], i.e. it does not lead to new conserved quantum numbers as it is expressed in terms of the ADM mass \[10\]. Again, the ‘spirit’ of the no-hair conjecture seems to be maintained.

I will start my talk by briefly presenting a method, developed in collaboration with E. Winstanley \[7\], for proving the possibility of an evasion of the no-hair conjecture for these systems. Our approach was inspired by a recent elegant proof of the no-hair theorem for black holes by Bekenstein \[12\] in a variety of cases with scalar fields \[12\]. The theorem is formulated in such a way so as to rule out a multicomponent scalar field dressing an asymptotically flat, static, spherically-symmetric black hole. The basic assumption of the theorem is that the scalar field is minimally coupled to gravity and bears a non-negative energy density as seen by any observer, and the proof relies on very general principles, such as energy-momentum conservation and the Einstein equations. From the positivity assumption and the conservation equations for the energy momentum tensor \( T_{MN} \) of the theory, \( \nabla^M T_{MN} = 0 \), one obtains for a spherically-symmetric space-time background the condition that near the horizon the radial component of the energy-momentum tensor and its first derivative are negative

\[
T_r^r < 0, \quad (T_r^r)' < 0
\]  

with the prime denoting differentiation with respect to \( r \). This implies that in such systems there must be regions in space outside the horizon where both quantities in \( (1) \) change sign. This contradicts the results following from Einstein’s equations though \[12\], and this contradiction constitutes the proof of the no-hair theorem, since the only allowed non-trivial configurations are Schwarzschild black holes. We note, in passing, that there are known exceptions to the original version of the no-hair theorem \[4\], such as conformal scalar fields coupled to gravity, which come from the fact that in such theories the scalar fields diverge at the horizon of the black hole \[13\], and therefore the original assumptions of the theorem are violated.

The interest for our case is that the theorem, if true for the EYMH system, would rule out the existence of non-trivial hair due to a Higgs field with a double (or multiple) well potential, as is the case for spontaneous symmetry breaking. Given that stability issues are not involved in the proof, it would be of interest to reconcile the results of the theorem with the situation in the case of EYMH or GB systems, where at least we know that explicit black-hole solutions with non-trivial hair exist \[8, 10\]. As we shall show below, the formal reason for bypassing the modern version of the no-hair theorem \[12\] lies in the violation of the key relation among the components of the stress tensor, \( T^t_t = T^\theta_\theta \), shown to hold in the case of ref. \[12\]. The physical reason for the ‘elusion’ of the above no-hair conjecture lies in the fact that the presence of the repulsive non-Abelian gauge interactions, or the higher-curvature gravitational terms, balances the gravitational attraction, by producing terms that make the result \( (1) \) inapplicable in this case. For the GB system there is an additional simpler reason to expect that the no-scalar-hair theorem can be bypassed. In the presence of curvature-squared terms, the modified Einstein’s equation leads to an effective stress...
tensor that involves the gravitational field. This implies that the assumption of positive definiteness
of the time-component of this tensor, which in the Einstein case is the local energy density of the
field, breaks down, leading to a relaxation of one of the key constraints/assumptions of the no-hair
theorem of ref. [12].

2 Eluding the no-hair conjecture for the EYMH system

We start our discussion from the EYMH system. Consider the EYMH theory with a (gauge-fixed)
Lagrangian
\[ L_{EYMH} = -\frac{1}{4\pi} \left\{ \frac{1}{4} |F_{MN}|^2 + \frac{1}{8} \phi^2 |A_M|^2 + \frac{1}{2} \partial_M \phi|^2 + V(\phi) \right\} \] (2)
where \( A_M \) denotes the Yang-Mills field, \( F_{MN} \) its field strength, \( \phi \) is the Higgs field and \( V(\phi) \) its potential. All the indices are contracted with the help of the back ground gravitational tensor \( g_{MN} \). In the spirit of Bekenstein’s modern version of the no-hair theorem [12], we now examine
the energy-momentum tensor of the model (2). It can be written in the form
\[ 8\pi T_{MN} = -\mathcal{E} g_{MN} + \frac{1}{4\pi} \left\{ F_{MP} F^P_N + \frac{\phi^2}{4} A_M A_N + \partial_M \phi \partial_N \phi \right\} \] (3)
with \( \mathcal{E} \equiv -L_{EYMH} \). We consider Yang-Mills fields of the form [8]
\[ A = (1 + \omega(r)) [\hat{\tau}_r \phi d\theta + \hat{\tau}_\theta \sin \theta d\varphi] \] (4)
where \( \tau_i, i = r, \theta, \varphi \) are the generators of the \( SU(2) \) group in spherical-polar coordinates.

Consider, now, an observer moving with a four-velocity \( u^M \). The observer sees a local energy
density
\[ \rho = \mathcal{E} + \frac{1}{4\pi} \left\{ u^M F_{MP} F^P_N u^N + \frac{\phi^2}{4} (u^M A_M)^2 + (u^M \partial_M \phi)^2 \right\}, \quad u^M u_M = -1. \] (5)

To simplify the situation let us consider a space-time with a time-like killing vector, and
suppose that the observer moves along this killing vector. Then \( u^M \partial_M \phi = 0 \) and by an appropriate
gauge choice \( u^M A_M = 0 = u^M F_{MN} \). This gauge choice is compatible with the spherically-
symmetric ansatz for \( A_M \) of ref. [8]. Under these assumptions,
\[ \rho = \mathcal{E} \] (6)
and the requirement that the local energy density as seen by any observer is non-negative implies
\[ \mathcal{E} \geq 0. \] (7)

We are now in position to proceed with the announced proof of the bypassing of the no-hair theorem
of ref. [12] for the EYMH black hole of ref. [8]. To this end we consider a spherically-symmetric
ansatz for the space-time metric \( g_{MN} \), with an invariant line element of the form
\[ ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad \Gamma = \Gamma(r), \ \Lambda = \Lambda(r). \] (8)
To make the connection with the black hole case we further assume that the space-time is asymptotically flat.
From the conservation of the energy-momentum tensor, following from the invariance of the effective action under general co-ordinate transformations, one has for the $r$-component of the conservation equation

$$
[(−g)\gamma^r]′ − \frac{1}{2} (−g)^{\frac{3}{2}} \left( \frac{∂}{∂r} g_{MN} \right) T^{MN} = 0 \tag{9}
$$

with the prime denoting differentiation with respect to $r$. The spherical symmetry of the space time implies that $T^{θθ} = T^{ϕϕ}$. Hence, (9) can be written as

$$(e^{\frac{−Ω}{4}r^2}T^r_r)' - \frac{1}{2}e^{\frac{−Ω}{4}r^2}r^2 \left[ Γ' T^t_t + Λ' T^r_r + \frac{4}{r} T^θ_θ \right] = 0. \tag{10}$$

Observing that the terms containing $Λ$ cancel, and integrating over the radial coordinate $r$ from the horizon $r_h$ to a generic distance $r$, one obtains

$$T^r_r(r) = e^{-\frac{Ω}{4}r^2} \int_{r_h}^{r} dr e^{\frac{Ω}{4}r^2} \left[ Γ' T^t_t + \frac{4}{r} T^θ_θ \right]. \tag{11}$$

Note that the assumption that scalar invariants, such as $T_{MN}T^{MN}$ are finite on the horizon (in order that the latter is regular), implies that the boundary terms on the horizon vanish in (11).

It is then straightforward to obtain

$$(T^r_r)' = \frac{1}{2} \left[ Γ' T^t_t + \frac{4}{r} T^θ_θ \right] - \frac{e^{-Ω/4}r^2}{r^2} (e^{Ω/4}r^2)' T^r_r. \tag{12}$$

Ansatz (4) for the gauge field yields

$$T^t_t = -E \tag{13a}$$
$$T^r_r = -E + F \tag{13b}$$
$$T^θ_θ = -E + J \tag{13c}$$

with

$$\mathcal{F} \equiv \frac{e^{-Λ}}{4π} \left[ \frac{2ω'τ^2}{r^2} + φ^2 \right] \tag{14a}$$
$$\mathcal{J} \equiv \frac{1}{4π} \left[ \frac{ω'}{r} e^{-Λ} + \frac{(1 - ω^2)^2}{4r^2} \right]. \tag{14b}$$

Substituting (14) in (13) yields

$$T^r_r(r) = \frac{e^{-Ω/4}r^2}{r^2} \int_{r_h}^{r} \left\{ -(e^{Ω/4}r^2)' E + \frac{2}{r^2} \mathcal{J} \right\} dr \tag{15}$$

$$(T^r_r)'(r) = -\frac{e^{-Ω/4}r^2}{r^2} (e^{Ω/4}r^2)' F + \frac{2}{r^2} \mathcal{J} \tag{16}$$

where $E$ is expressed as

$$E = \frac{1}{4π} \left[ \frac{(ω')^2}{r^2} e^{-Λ} + \frac{(1 - ω^2)^2}{2r^4} + \frac{φ^2(1 + ω)^2}{4r^2} + \frac{1}{2} (φ')^2 e^{-Λ} + \frac{λ}{4} (φ^2 - v^2)^2 \right]. \tag{17}$$
We now turn to the Einstein equations for the first time, following the analysis of ref. [12]. Our aim is to examine whether there is a contradiction with the requirement of the non-negative energy density. These equations read for our system

\[ e^{-\Lambda}(r^{-2} - r^{-1}\Lambda') - r^{-2} = 8\pi T^t_t = -8\pi E \]
\[ e^{-\Lambda}(r^{-1}\Gamma' + r^{-2}) - r^{-2} = 8\pi T^r_r. \]  
(18)

Integrating out the first of these yields

\[ e^{-\Lambda} = 1 - \frac{8\pi}{r} \int_{r_h}^r E r^2 dr - \frac{2M'}{r} \]  
(19)

where \( M' \) is a constant of integration.

The requirement for asymptotic flatness of space-time implies the following asymptotic behaviour for the energy-density functional \( E \sim O(r^{-3}) \) as \( r \to \infty \), so that \( \Lambda \sim O(r^{-1}) \). In order that \( e^\Lambda \to \infty \) at the horizon, \( r \to r_h \), \( M' \) is fixed by

\[ M' = \frac{r_h}{2} \]  
(20)

The second of the equations (18) can be rewritten in the form

\[ e^{-\frac{\Gamma}{2}} r^{-2} (r^2 e^{\frac{\Gamma}{2}})' = \left[ 4\pi r T^r_r + \frac{1}{2r} \right] e^\Lambda + \frac{3}{2r}. \]  
(21)

Consider, first, the behaviour of \( T^r_r \) as \( r \to \infty \). Asymptotically, \( e^{\frac{\Gamma}{2}} \to 1 \), and so the leading behaviour of \( (T^r_r)' \) is

\[ (T^r_r)' = \frac{2}{r}[\mathcal{J} - \mathcal{F}]. \]  
(22)

We, now, note that the fields \( \omega \) and \( \phi \) have masses \( \frac{v}{2} \) and \( \mu = \sqrt{\lambda} \) respectively. From the field equations and the requirement of finite energy density their behaviour at infinity must then be

\[ \omega(r) \sim -1 + ce^{-\frac{\Gamma}{2}r} \]
\[ \phi(r) \sim v + ae^{-\sqrt{2}\mu r} \]  
(23)

for some constants \( c \) and \( a \). Hence, the leading asymptotic behaviour of \( \mathcal{J} \) and \( \mathcal{F} \) is

\[ \mathcal{J} \sim \frac{1}{4\pi} \left[ \frac{c^2 \mu^2}{4r^2} e^{-\nu r} + \frac{2c^2}{r^4} e^{-\nu r} + \frac{v^2 c^2}{4r^2} e^{-\nu r} \right] \]
\[ \mathcal{F} \sim \frac{1}{4\pi} \left[ \frac{c^2 \nu^2}{2r^2} e^{-\nu r} + 2a^2 \mu^2 e^{-2\sqrt{2}\mu r} \right] \]  
(24)

since \( e^{-\Lambda} \to 1 \) asymptotically.

The leading behaviour of \( (T^r_r)' \), therefore, is

\[ (T^r_r)' \sim \frac{1}{4\pi} \left[ \frac{2c^2}{r^4} e^{-\nu r} - 2a^2 \mu^2 e^{-2\sqrt{2}\mu r} \right]. \]  
(25)

There are two possible cases: (i) \( 2\sqrt{2}\mu > v \) (corresponding to \( \lambda > 1/8 \)); in this case \( (T^r_r)' > 0 \) asymptotically, (ii) \( 2\sqrt{2}\mu \leq v \) (corresponding to \( \lambda \leq 1/8 \) ); then, \( (T^r_r)' < 0 \) asymptotically.
Since $\mathcal{J}$ vanishes exponentially at infinity, and $\mathcal{E} \sim O[r^{-3}]$ as $r \to \infty$, the integral defining $T^r_r(r)$ converges as $r \to \infty$ and $|T^r_r|$ decreases as $r^{-2}$.

Thus, in case (i) above, $T^r_r$ is negative and increasing as $r \to \infty$, and in case (ii) $T^r_r$ is positive and decreasing.

Now turn to the behaviour of $T^r_r$ at the horizon. When $r \approx r_h$, $\mathcal{E}$ and $\mathcal{J}$ are both finite, and $\Gamma'$ diverges as $1/(r - r_h)$. Thus the main contribution to $T^r_r$ as $r \approx r_h$ is

$$T^r_r(t) = \int_{r_h}^r \left(-e^{\frac{\Gamma'}{2}r^2}\right)\frac{\Gamma'}{2} \mathcal{E} dr$$

which is finite.

At the horizon, $e^\Gamma = 0$; outside the horizon, $e^\Gamma > 0$. Hence $\Gamma' > 0$ sufficiently close to the horizon, and, since $\mathcal{E} \geq 0, T^r_r < 0$ for $r$ sufficiently close to the horizon.

Since $\mathcal{F} \sim O[r - r_h]$ at $r \approx r_h$, $(T^r_r)'$ is finite at the horizon and the leading contribution is

$$(T^r_r)'(r_h) \approx -\frac{\Gamma'}{2} \mathcal{F} + \frac{2}{r} \mathcal{J}.$$  \hspace{1cm} (27)

From ref. 8 we record the relation

$$re^{-\frac{\Lambda}{2}} = e^{-\Lambda} \left[ e^{\Lambda} (1 + \omega)^2 + \frac{m}{r} e^{\Lambda} \frac{\phi^2}{4} (1 - \omega^2)^2 r^2 \right]$$  \hspace{1cm} (28)

where $e^{-\Lambda} = 1 - \frac{2m(r)}{r}$. Hence,

$$(T^r_r)' = -\frac{e^{-\Lambda}}{4\pi r} \left[ \frac{(\omega')^2}{r^2} + (\phi')^2 \right] \left\{ (\omega')^2 + \frac{1}{2} r^2 (\phi')^2 - \frac{1}{2} e^{\Lambda} \frac{(1 - \omega^2)^2}{r^2} \right\}$$

$$-\frac{\phi^2}{4} e^{\Lambda} (1 + \omega)^2 + \frac{m}{r} e^{\Lambda} \frac{\phi^2}{4} (1 - \omega^2)^2 r^2 + \frac{\phi^2}{4} \frac{1}{r^2} (1 + \omega)^2 \right\}.$$  \hspace{1cm} (29)

For $r \approx r_h$, this expression simplifies to

$$(T^r_r)'(r_h) \approx \mathcal{J}(r_h) \left[ \frac{2}{r_h} + \frac{4\pi}{r_h} \mathcal{F}(r_h) \right]$$

$$-\mathcal{F}(r_h) \left[ \frac{1}{2} + \frac{\lambda}{2} \frac{2r^2}{r_h^2} \right]$$

$$= \mathcal{F}(r_h) \left[ \frac{(1 - \omega^2)^2}{2r^2} + \frac{\phi^2}{4r^2} (1 + \omega)^2 + \frac{\lambda}{4} r_h (\phi^2 - \nu^2)^2 - \frac{1}{2} r_h \right]$$

$$+ \frac{2}{r_h} \mathcal{J}.$$  \hspace{1cm} (31)

where $\mathcal{F} = e^{\Lambda} \mathcal{F}(r) = \frac{4\pi}{4\pi} \frac{(2\omega')^2}{r^2} + (\phi')^2].$
Consider for simplicity the case $r_h = 1$. Then, from the field equations (3)

$$
\omega_h' = \frac{1}{D} \left[ \frac{1}{4} \phi_h^2 (1 + \omega_h) - \omega_h (1 - \omega_h^2) \right] = \frac{A}{D} \tag{32}
$$

$$
\phi_h' = \frac{1}{D} \left[ \frac{1}{2} \phi_h (1 + \omega_h)^2 + \lambda \phi_h (\phi_h^2 - v^2) \right] = \frac{B}{D} \tag{33}
$$

where

$$
D = 1 - (1 - \omega_h^2)^2 - \frac{1}{2} \phi_h^2 (1 + \omega_h)^2 - \frac{1}{2} \lambda (\phi_h^2 - v^2)^2. \tag{34}
$$

Then the expression (31) becomes

$$
(T_T')' (r_h) = \frac{1}{4\pi D} \left[ 8\pi DJ - A^2 - \frac{1}{2} B^2 \right] = \frac{C}{4\pi D}. \tag{35}
$$

From the field equations (8)

$$
D = 1 - 2m_h' \tag{36}
$$

which is always positive because the black holes are non-extremal. Thus the sign of $(T_T')'(r_h)$ is the same as that of $C$. The EYMH system possesses two branches of solutions (8), labelled by the number of nodes $k$ of the gauge field. A detailed analysis (7) for the case of black hole solutions possessing at most one node, examined in refs. (8), (7), shows that for both branches $(k = 1, 0)$ $C$ is non-negative. This implies that $(T_T')'(r_h)$ is positive for all the black hole solutions having one node in $\omega$, regardless of the value of the Higgs mass $v$.

Let us now check on possible contradictions with Einstein’s equations. Consider first the case $\lambda > 1/8$. Then, as $r \to \infty$, $T_T^r < 0$ and $(T_T^r)' > 0$. As $r \to r_h$, $T_T^r < 0$ and $(T_T^r)' > 0$. Hence there is no contradiction with Einstein’s equations in this case. Next, consider the case $\lambda \leq 1/8$. In this case, as $r \to \infty$, $T_T^r > 0$ and $(T_T^r)' < 0$, whilst as $r \to r_h$, $T_T^r < 0$ and $(T_T^r)' > 0$. Hence, there is an interval $[r_a, r_h]$ in which $(T_T^r)'$ is positive and there exists a ‘critical’ distance $r_c \in (r_a, r_h)$ at which $T_T^r$ changes sign. However, unlike the case when the gauge fields are absent (12), here there is no contradiction with the result following from Einstein equations, because $(T_T^r)' > 0$ in some open interval close to the horizon, as we have seen above.

In conclusion the method of ref. (12) cannot be used to prove a ‘no-scalar-hair’ theorem for the EYMH system, as expected from the existence of the explicit solution of ref. (8). The key difference is the presence of the positive term $\frac{2}{4\pi} J$ in the expression (11) for $(T_T^r)'$. This term is dependent on the Yang-Mills field and vanishes if this field is absent, or if the field is Abelian. Thus, there is a sort of ‘balancing’ between the gravitational attraction and the non-Abelian gauge field repulsion, which is responsible for the existence of the classical non-trivial black-hole solution of ref. (8). However, as shown in ref. (8), this solution is not stable against (linear) perturbations of the various field configurations. Thus, although the ‘letter’ of the ‘no-scalar-hair’ theorem of ref. (12), based on non-negative scalar-field-energy density, is violated, its ‘spirit’ is maintained in the sense that there exist instabilities which imply that the solution cannot be formed as a result of collapse of stable matter. However, stability is a new dimension in the no-hair conjecture, not included in the original formulation. Therefore, it seems fair to say that the above analysis constitutes an analytic proof of bypassing or, better, ‘eluding’ the no-hair conjecture.
3 Dilatonic Hair in Higher-Curvature Gravity

As a second example of a physical theory of black holes not covered by the no-hair theorem of ref. [12] I shall describe black hole solutions [10] of the Einstein-dilaton system in the presence of the (higher-derivative) curvature-squared terms of Gauss-Bonnet (GB) type [10]. These solutions were found numerically by P. Kanti, J. Rizos and K. Tamvakis, and are endowed with a non-trivial dilaton field outside the horizon, thus possessing dilaton hair. The treatment of the curvature-squared terms in ref. [10] is non-perturbative and the solutions are present for any value of $\alpha'/g^2$, where $\alpha'$ is the string Regge slope, and $g$ is the gauge coupling constant of the low-energy theory. What I shall argue in this section, in connection with a bypassing of the no-hair theorem of ref. [12], is that the presence of the higher-derivative GB terms provides the necessary ‘repulsion’ in the effective theory that balances the gravitational attraction, thereby leading to black holes dressed with non-trivial classical dilaton hair. This is an analogous phenomenon to that occurring in the case of Einstein-Yang-Mills systems discussed in the previous section.

The action I shall use will be the effective low-energy action obtained from (super)strings. I shall concentrate on the bosonic part of the gravitational multiplet which consists of the dilaton, graviton, and antisymmetric tensor fields. I shall ignore the antisymmetric tensor for simplicity. As is well known in low-energy effective field theory, there are ambiguities in the coefficients of such terms, due to the possibility of local field redefinitions which leave the $S$-matrix amplitudes of the effective field theory invariant, according to the equivalence theorem. To $\mathcal{O}(\alpha')$ the freedom of such redefinitions is restricted to two generic structures, which cannot be removed by further redefinitions [15]. One is a curvature-squared combination, and the other is a four-derivative dilaton term. Thus, a generic form of the string-inspired $\mathcal{O}(\alpha')$ corrections to Einstein’s gravitation have the form

$$L = -\frac{1}{2}R - 4\left(\partial_\mu \phi\right)^2 + \frac{\alpha'}{8g^2}\epsilon^\phi (c_1 R^2 + c_2 \left(\partial_\rho \phi\right)^4)$$

(37)

where $\alpha'$ is the Regge slope, $g^2$ is some gauge coupling constant (in the case of the heterotic string that we concentrate for physical reasons), and $R^2$ is a generic curvature-dependent quadratic structure, which can always be fixed to correspond to the Gauss-Bonnet (GB) invariant

$$R^2_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

(38)

The coefficients $c_1$, $c_2$ are fixed by comparison with string scattering amplitude computations, or $\sigma$-model $\beta$-function analysis. It is known that in the three types of string theories, Bosonic, Closed-Type II Superstring, and Heterotic Strings, the ratio of the $c_1$ coefficients is 2:0:1 respectively [15]. The case of superstring II effective theory, then, is characterized by the absence of curvature-squared terms. In such theories the fourth-order dilaton terms can still be, and in fact they are, present. In such a case, it is straightforward to see from the modern proof of the no-scalar hair theorem of ref. [12] that such theories, cannot sustain to order $\mathcal{O}(\alpha')$, any non-trivial dilaton hair. On the other hand, the presence of curvature-squared terms can drastically change the situation [10], as I will now describe.

Following the above discussion we shall ignore, for simplicity, the fourth-derivative dilaton terms in (37), setting from now on $c_2 = 0$. However, we must always bear in mind that such terms are non-zero in realistic effective string cases, once the GB combination is fixed for the gravitational $\mathcal{O}(\alpha')$ parts. Then, the lagrangian for dilaton gravity with a Gauss Bonnet term reads

$$L = -\frac{1}{2}R - 4\left(\partial_\mu \phi\right)^2 + \frac{\alpha'}{8g^2}\epsilon^\phi R^2_{GB}$$

(39)

In four dimensions, the antisymmetric tensor field leads to the axion hair, already discussed in ref. [14]. Modulo unexpected surprises, we do not envisage problems associated with its presence as regards the results discussed in this work, and, hence, we ignore it for simplicity.
where $R^2_{GB}$ is the Gauss Bonnet (GB) term (38).

As I mentioned earlier, although we view (39) as a heterotic-string effective action, for simplicity, in this paper we shall ignore the modulus and axion fields, assuming reality of the dilaton ($S = e^\phi$ in the notation of ref. [14]). We commence our analysis by noting that the dilaton field and Einstein’s equations derived from (39), are

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \partial^\mu \phi] = -\frac{\alpha'}{4g^2} e^\phi R^2_{GB}$$  \hspace{1cm} (40)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} g_{\mu\nu} (\partial_\rho \phi)^2 - \alpha' K_{\mu\nu}$$  \hspace{1cm} (41)

where

$$K_{\mu\nu} = (g_{\mu\rho}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\rho}) \gamma^{\lambda\alpha\beta} D_{\gamma} [\hat{R}^{\rho\gamma}_{\alpha\beta} \partial_\kappa f]$$  \hspace{1cm} (42)

and

$$\gamma^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} (-g)^{-\frac{1}{2}}$$

$$\epsilon^{ijk} = -\epsilon_{ijk}$$

$$\hat{R}^{\mu\nu}_{\kappa\lambda} = \gamma^{\mu\nu\rho\sigma} R_{\rho\sigma\kappa\lambda}$$

$$f = \frac{e^\phi}{8g^2}$$

From the right-hand-side of the modified Einstein’s equation (41), one can construct a conserved “energy momentum tensor”, $D_\mu T^{\mu\nu} = 0$,

$$T_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} (\partial_\rho \phi)^2 + \alpha' K_{\mu\nu}$$  \hspace{1cm} (44)

It should be stressed that the time component of $-T_{\mu\nu}$, which in Einstein’s gravity would correspond to the local energy density $E$, may not be positive. Indeed, as we shall see later on, for spherically-symmetric space times, there are regions where this quantity is negative. The reason is that, as a result of the higher derivative GB terms, there are contributions of the gravitational field itself to $T_{\mu\nu}$. From a string theory point of view, this is reflected in the fact that the dilaton is part of the string gravitational multiplet. Thus, this is the first important indication on the possibility of evading the no-scalar-hair theorem of ref. [12] in this case. However, this by itself is not sufficient for a rigorous proof of an evasion of the no-hair conjecture. I shall come to this point later on.

At the moment, let me consider a spherically symmetric space-time having the metric

$$ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$  \hspace{1cm} (45)

where $\Gamma, \Lambda$ depend on $r$ solely. Using the above ansatz, the dilaton equation as well as the $(tt)$, $(rr)$ and $(\theta\theta)$ component of the Einstein’s equations take the form

$$\phi'' + \phi' \left( \frac{\Gamma'}{r} - \frac{\Lambda'}{2} + \frac{2}{r} \right) = \frac{\alpha' e^\phi}{2g^2 r^2} \left( \Gamma' \Lambda' e^{-\Lambda} + (1 - e^{-\Lambda})[\Gamma'' + \frac{\Gamma'}{2} (\Gamma' - \Lambda')] \right)$$  \hspace{1cm} (46)

$$\Lambda' \left( 1 + \frac{\alpha' e^\phi}{2g^2 r} \phi' (1 - 3e^{-\Lambda}) \right) = \frac{r \phi'^2}{4} + \frac{1 - e^\Lambda}{r} + \frac{\alpha' e^\phi}{g^2 r} (\phi'' + \phi'^2) (1 - e^{-\Lambda})$$  \hspace{1cm} (47)
\[ \Gamma'(1 + \frac{\alpha' e^{\phi}}{2g^2r}\phi'(1 - 3e^{-\Lambda})) = \frac{r\phi''}{4} + \frac{e^{\Lambda} - 1}{r} \] (48)

\[ \Gamma'' + \frac{\Gamma'}{2}(\Gamma' - \Lambda') + \frac{\Gamma' - \Lambda'}{r} = \frac{\phi''}{2} + \frac{\alpha' e^{\phi - \Lambda}}{g^2r} \left( \phi'\Gamma' + (\phi'' + \phi'^2)\Gamma' \right) \]

\[ + \frac{\phi'\Gamma'}{2}(\Gamma' - 3\Lambda') \] (49)

Before I proceed to study the above system it is useful to note that if I turn off the Gauss-Bonnet term, equation (10) can be integrated to give \( \phi' \sim \frac{1}{r^2}e^{(\Lambda - \Gamma)/2} \). A black hole solution should have at the horizon \( r_h \) the behaviour \( e^{-\Gamma}, e^\Lambda \to \infty \). Therefore the radial derivative of the dilaton would diverge on the horizon resulting into a divergent energy-momentum tensor

\[ T^t_t = -T^r_r = T^\theta_\theta = -\frac{e^{-\Lambda}}{4} \phi'^2 \to \infty \] (50)

Rejecting this solution we are left with the standard Schwarzschild solution and a trivial \((\phi = \text{constant})\) dilaton, in agreement with the no-hair theorem. This behaviour will be drastically modified by the Gauss-Bonnet term.

The \( r \) component of the energy-momentum conservation equations reads:

\[ (e^{\Gamma/2}r^2T^r_r)' = \frac{1}{2}e^{\Gamma/2}r^2[\Gamma'T^t_t + \frac{4}{r}T^\theta_\theta] \] (51)

where the prime denotes differentiation with respect to \( r \). The spherical symmetry of the space-time implies \( T^\theta_\theta = T^\varphi_\varphi \). Integrating over the radial coordinate \( r \) from the horizon \( r_h \) to generic \( r \) yields

\[ T^r_r(r) = \frac{e^{-\Gamma/2}}{2r^2} \int_{r_h}^{r} e^{\Gamma/2}r^2[\Gamma'T^t_t + \frac{4}{r}T^\theta_\theta]dr \] (52)

The boundary terms on the horizon vanish, since scalar invariants such as \( T^\alpha_\beta T^{\alpha\beta} \) are finite there. For the first derivative of \( T^r_r \) we have

\[ (T^r_r)'(r) = \frac{e^{-\Gamma/2}}{r^2} \left( e^{\Gamma/2}r^2 \right)' (T^t_t - T^r_r) + \frac{2}{r} (T^\theta_\theta - T^r_r) \] (53)

Taking into account (44) and (45), one easily obtains

\[ T^t_t = -e^{-\Lambda} \phi'^2/4 - \frac{\alpha' e^{\phi - \Lambda}(\phi'' + \phi'^2)(1 - e^{-\Lambda})}{2g^2r^2} + \frac{\alpha' e^{\phi - \Lambda} \phi' \Lambda'(1 - 3e^{-\Lambda})}{2} \]

\[ T^r_r = e^{-\Lambda} \phi'^2/4 - \frac{\alpha' e^{\phi - \Lambda} \phi' \Gamma'(1 - 3e^{-\Lambda})}{2} \] (54)

\[ T^\theta_\theta = -e^{-\Lambda} \phi'^2/4 + \frac{\alpha' e^{-2\Lambda} \phi' \Gamma' \phi'^2 + \Gamma' \phi'^2 + \frac{3}{2}(\Gamma' - 3\Lambda'))}{2} \]

In the relations (44) there lies the second reason for a possibility of an evasion of the no-hair conjecture. Due to the presence of the higher curvature contributions, the relation \( T^t_t = T^\theta_\theta \) assumed in ref. 12, is no longer valid. The alert reader must have noticed, then, the similarity of the rôle played by the Gauss-Bonnet \( O(\alpha') \) terms in the lagrangian (39) with the case of the non-Abelian gauge black holes studied in ref. 6, and described in the previous section. We stress once, again, however, that in the GB case both the non-positivity of the “energy-density” \( T^t_t \), and the modified
relation \( T^t_\ell \neq T^\theta_\theta \), play equally important roles in leading to a possibility of having non-trivial classical scalar (dilaton) hair in GB black holes systems. Below I shall demonstrate rigorously this, by showing that there is no contradiction between the results following from the conservation equation of the “energy-momentum tensor” \( T_{\mu\nu} \) and the field equations, in the presence of non-trivial dilaton hair.

First, let me define what one means by ‘dilaton hair’. Far away from the origin the unknown functions \( \phi(r) \), \( e^{\Lambda(r)} \), and \( e^{\Gamma(r)} \) can be expanded in a power series in \( 1/r \). These expansions, substituted back into the equations, are finally expressed in terms of three parameters only, chosen to be \( \phi_\infty \), the asymptotic value of the dilaton, the ADM mass \( M \), and the dilaton charge \( D \) defined as

\[
D = -\frac{1}{4\pi} \int d^2\Sigma \mu D_\mu \phi \tag{55}
\]

where the integral is over a two-sphere at spatial infinity. The asymptotic solutions are

\[
e^{\Lambda(r)} = 1 + \frac{2M}{r} + \frac{16M^2 - D^2}{4r^2} + O(1/r^3) \tag{56}
\]

\[
e^{\Gamma(r)} = 1 - \frac{2M}{r} + O(1/r^3) \tag{57}
\]

\[
\phi(r) = \phi_\infty + \frac{D}{r} + \frac{MD}{r^2} + O(1/r^3) \tag{58}
\]

Note, now, that the solution near the horizon is characterized by the parameter \( \phi_h \). However, the parameters that characterize the solution near infinity \( (56)-(58) \) are \( M \) and \( D \). From this, we can infer that a relation must hold between the above parameters in order to be able to classify the solution as a one parameter family of black hole solutions. After some manipulation, the set of equations \( (46)-(49) \) can be rearranged to yield the identity

\[
\frac{d}{dr}\left( r^2 e^{\Gamma - \Lambda}/2(\Gamma' - \phi') - \frac{\alpha'}{g^2} e^{\Lambda/2} [(1 - e^{-\Lambda})(\phi' - \Gamma') + e^{-\Lambda} r \phi' \Gamma'] \right) = 0 \tag{59}
\]

Integrating this relation over the interval \( (r_h, r) \) we obtain the expression

\[
2M - D = \sqrt{\gamma_1 \lambda_1} (r_h^2 + \frac{\alpha' e^{\phi_h}}{g^2}) \tag{60}
\]

This equation is simply a connection between the set of parameters describing the solution near the horizon and the set \( M \) and \( D \). The rhs of this relation clearly indicates that the existing dependence of the dilaton charge on the mass does not take the simple form of an equality encountered in EYMD regular solutions of ref.\[17\]. In order to find the relation between \( M \) and \( D \) we follow refs.\[14\] and take into account the \( O(\alpha'^2) \) expression of the dilaton charge in the limit \( r \to \infty \)

\[
\phi(r) = \phi_\infty + \frac{D}{r} + ... = \phi_\infty + \left( \frac{e^{\phi_\infty} \alpha'}{2M g^2} + \frac{736\alpha'^2 \phi_\infty}{60(2M)^3 g^4} \right) \frac{1}{r} + ... \tag{61}
\]

This relation can be checked numerically for the black hole solution of ref.\[14\]. Any deviations from this relation are due to higher order terms which turn out to be small. The above relation \( (61) \) implies that the dilaton hair of the black hole solution, if exists, is a kind of ‘secondary hair’, in the terminology of ref.\[11\]. This hair is generated because the basic fields (gravitons) of the theory associated with the primary hair (mass) act as sources for the non-trivial dilaton configurations outside the horizon of the black hole.
To check the possibility of the evasion of the no-hair conjecture we first consider the asymptotic behaviour of $T^r_r$ as $r \to \infty$. Since $\Gamma'$ and $\Lambda' \sim O(1/r)$ as $r \to \infty$, we have the following asymptotic behaviour

\[
T^r_r \sim \frac{1}{4}(\phi')^2 + O\left(\frac{1}{r^6}\right)
\]

\[
T^\theta_\theta \sim \frac{1}{4}(\phi')^2 + O\left(\frac{1}{r^6}\right)
\]

(62)

In this limit, $e^{\Gamma/2} \to 1$, and so the leading behaviour of $(T^r_r)'$ is

\[
(T^r_r)' \sim \frac{2}{r}(T^\theta_\theta - T^r_r) \sim -\frac{1}{r}(\phi')^2 < 0 \quad \text{as} \quad r \to \infty
\]

(63)

Thus, $T^r_r$ is positive and decreasing as $r \to \infty$.

We now turn to the behaviour of the unknown functions at the event horizon. When $r \sim r_h$, we make the ansatz

\[
e^{-\Lambda(r)} = \lambda_1(r - r_h) + \lambda_2(r - r_h)^2 + ...
\]

\[
e^{\Gamma(r)} = \gamma_1(r - r_h) + \gamma_2(r - r_h)^2 + ...
\]

(64)

\[
\phi(r) = \phi_h + \phi'_h(r - r_h) + \phi''_h(r - r_h)^2 + ...
\]

with the subscript $h$ denoting the value of the respective quantities at the horizon. It can be shown [10] that this is the most general asymptotic solution $\Gamma' \to \infty$, $\phi, \phi'$ finite. As we can see, $\phi(r_h) \sim \text{constant}$ while $\Gamma'$ and $\Lambda'$ diverge as $(r - r_h)^{-1}$ and $-(r - r_h)^{-1}$ respectively. Then, the behaviour of the components of the energy-momentum tensor near the horizon is

\[
T^r_r = -\frac{\alpha'}{2g^2r^2}e^{\phi - \Lambda} \phi' \Gamma' + O(r - r_h)
\]

\[
T^t_t = \frac{\alpha'}{2g^2r^2}e^{\phi - \Lambda} \phi' \Lambda' + O(r - r_h)
\]

\[
T^\theta_\theta = \frac{\alpha'}{2g^2r^2}e^{\phi - 2\Lambda}[\Gamma'' \phi' + \frac{\Gamma'}{2}(\Gamma' - 3\Lambda')] + O(r - r_h)
\]

(65)

Taking into account the above expressions the leading behaviour of $T^r_r$ near the horizon is

\[
T^r_r(r) \sim -\frac{e^{\Gamma/2}}{r^2} \int_{r_h}^r \frac{\alpha'}{4g^2}e^{\Gamma/2}(\Gamma')^2 e^{-\Lambda} e^{\phi} \phi' dr + O(r - r_h)
\]

(66)

Therefore one observes that for $r$ sufficiently close to the event horizon, $T^r_r$ has opposite sign to $\phi'$.

For $(T^r_r)'$ near the horizon, we have

\[
(T^r_r)'(r) = \frac{\alpha'}{2g^2r^2}e^{\phi} \left\{-\Gamma'(\phi'' + \phi'^2) + \phi'\left[\frac{\Gamma'}{2}(\Gamma' + \Lambda') + 2e^{-\Lambda}\Gamma'' - \frac{2}{r}\Lambda'\right]\right\}
\]

\[-\frac{1}{4} \Gamma' e^{-\Lambda} \phi'^2 + O(r - r_h)
\]

(67)
where $\Gamma' + \Lambda' \sim O(1)$ for $r \sim r_h$. Adding the $(tt)$ and $(rr)$ components of the Einstein’s equations we obtain at the event horizon

$$\Gamma' + \Lambda' = \frac{1}{\mathcal{F}} \left[ \frac{1}{2} T_{h} \phi_h^2 + \frac{\alpha'}{2 g^2} e^{\phi_h} (\phi''_h + \phi'^2_h) \right] + \mathcal{O}(r - r_h) \quad (68)$$

where

$$\mathcal{F} = 1 + \frac{\alpha'}{2 g^2} \frac{e^{\phi_h}}{r_h} \phi'_h \quad (69)$$

From the $(\theta\theta)$ component we obtain

$$e^{-2 \Gamma''} = \frac{1}{2} e^{-2 \Gamma'} (\Gamma')^2 + \frac{1}{2} e^{-2 \Gamma'} \Gamma' \Lambda' + \mathcal{O}(r - r_h)$$

$$= -\frac{1}{r_h^2 \mathcal{F}^2} + \mathcal{O}(r - r_h) \quad (70)$$

Substituting all the above formulae into (67) yields, near $r_h$

$$(T^r_r)'(r) \sim -\frac{1}{4} \frac{\phi_h^2}{r_h \mathcal{F}} - \frac{\alpha'}{2 g^2} \frac{e^{\phi_h}}{r_h^2 \mathcal{F}^2} (\phi''_h + \phi'^2_h) - \frac{\alpha'}{4 g^2} \frac{e^{2 \phi_h}}{r_h^2 \mathcal{F}^2} \phi_h^2 + \mathcal{O}(r - r_h) \quad (71)$$

Next, we turn to the dilaton equation (46). At $r \sim r_h$, it takes the form

$$\frac{\phi'_h}{r_h \mathcal{F}} = -\frac{3}{4 \mathcal{F}^2} \frac{\alpha'}{g^2} \frac{e^{\phi_h}}{r_h^3} + \mathcal{O}(r - r_h) \quad (72)$$

Substituting for $\mathcal{F}$ (63), the following equation for $\phi'_h$ is derived

$$\frac{\alpha'}{2 g^2} \frac{e^{\phi_h}}{r_h} \phi'_h + \phi'_h + \frac{3}{r_h^2} \frac{\alpha'}{g^2} e^{\phi_h} = 0 \quad (73)$$

which has as solutions

$$\phi'_h = \frac{\alpha'}{2 g^2} e^{-\phi_h} \left( -1 \pm \sqrt{\frac{6 (\alpha')^2}{g^2} \frac{e^{2 \phi_h}}{r_h^2}} \right) \quad (74)$$

One can show [10] that the relation (74) guarantees the finiteness of $\phi''_h$, and hence of the “local density” $T^r_r$ (64). Both these solutions for $\phi'_h$ are negative, and hence, since $T^r_r(r_h)$ has the opposite sign to $\phi'_h$, $T^r_r$ will be positive sufficiently close to the horizon. Since $T^r_r \geq 0$ also at infinity, we observe that there is no contradiction with Einstein’s equations, thereby allowing for the existence of black holes with scalar hair. We observe that near the horizon the quantity $\mathcal{E}$ ($-T^r_r$), which in Einstein’s gravitation would be the local energy density of the field $\phi$, is negative. As we mentioned earlier, this constitutes one of the reasons one should expect an evasion of the no-scalar-hair conjecture in this black hole space time. Crucial also for this result was the presence of additional terms in (54), leading to $T^r_r \neq T^r_r$. Both of these features, whose absence in the case of Einstein-scalar gravity was crucial for the modern proof of the no-hair theorem, owe their existence in the presence of the higher-order $\mathcal{O}(\alpha')$ corrections in (59).

The physical importance of the restriction (74) lies on the fact that according to this relation, black hole solutions of a given horizon radius can only exist if the coupling constant of the Gauss-Bonnet term in (59) is smaller than a critical value, set by the magnitude of the horizon scale. In fact from (74), reality of $\phi'_h$ is guaranteed if and only if

$$e^{\phi_h} < \frac{g^2}{\sqrt{6 \alpha'} r_h^2} \quad (75)$$
In this picture, $\beta \equiv \frac{1}{4}e^{\phi_h}$ can then be viewed as the (appropriately normalized with respect to the Einstein term) coupling constant of the GB term in the effective lagrangian (39). For a black hole of unit horizon radius $r_h = 1$, the critical value of $\beta$, above which black hole solutions cannot exist, is then $\beta_c = g^2/4\sqrt{6}\alpha'$. One is tempted to compare the situation with the case of $SU(2)$ sphaleron solutions in the presence of Gauss Bonnet terms [17]. Numerical analysis of sphaleron solutions in such systems reveals the existence of a critical value for the GB coefficient above which solutions do not exist. In the sphaleron case this number depends on the number of nodes of the Yang-Mills gauge field. In our case, if one fixes the position of the horizon, then it seems that in order to construct black hole solutions with this horizon size the GB coefficient has to satisfy (75). Thus, a way of interpreting (74) is to view it as providing a necessary condition for the absence of naked singularities in space-time [10]. To understand better this latter point we should consider the scalar curvature in the limit $r \to r_h$. It is given by

$$R = \frac{2}{r_h^2} \left( \frac{1 \mp \sqrt{1 - \frac{6(\alpha')^2 \frac{e^{2\phi_h}}{g^2}}{r_h^4} \frac{1}{r_h^4}}}{1 \pm \sqrt{1 - \frac{6(\alpha')^2 \frac{e^{2\phi_h}}{g^2}}{r_h^4} \frac{1}{r_h^4}}} \right)^2$$

(76)

This expression shows that the curvature is singular at $r_h \to 0$, i.e. when the horizon shrinks. The point $r_h = 0$ can be reached only when $\phi_h = -\infty$. Thus the inequality (36), in a sense, forbids $r_h$ to become zero and reveal the singularity.

Above, we have argued on the possibility of having black holes in the system (39) that admit non-trivial dilaton hair outside their horizon. The key is the bypassing of the no-hair theorem [12], as a result of the curvature-squared terms. However, the hair appears to be of secondary type, not yielding new quantum numbers for the black hole, but expressed in terms of its $ADM$ mass. In ref. [10] Kanti et al. found explicit black-hole solutions of the equations of motion originating from (39) and provided evidence for the existence of black hole solutions to all orders in $\alpha'$. Unfortunately a complete analytic treatment of these equations is not feasible, and one has to use numerical methods. This complicates certain things, in particular it does not allow for a clear view of what happens inside the horizon, thereby not giving good information on the curvature singularity structure. Nevertheless, the existence of an horizon in those solutions is demonstrated [10], and, thus, the GB-dilaton system may be considered as constituting a second example of the ‘elusion’ of the no-hair conjecture.

4 Conclusions

In this paper I have discussed two quite generic examples of bypassing the no-scalar-hair conjecture for black holes: the Einstein-Yang-Mills-Higgs $SU(2)$ system [8, 7], possessing black holes with non-trivial Higgs hair outside the horizon, and the Gauss-Bonnet (GB) higher-curvature graviton-dilaton system, possessing black holes with dilaton hair [10].

I have presented a method of proving analytically the existence of scalar hair for such systems [7]. I believe this is of value, not only because the black hole solutions in both systems are known only numerically at present, but also because the above examples constitute two rather generic categories of black hole systems that may evade the no-hair conjecture. The physical origin behind the existence of the hairy black holes in both systems can be traced back to the existence of non-Abelian field repulsion: the repulsion due to the Yang-Mills gauge field in the EYMH case, or the repulsion due to the presence of the higher-curvature (gravitational) terms in the GB system. Such repulsion balances the gravitational attraction from the Einstein terms, leading to non-trivial black-hole space-time structure.
Although the ‘letter’ of the no-scalar-hair theorem is violated in both cases, however its ‘spirit’ remains valid since the black hole solutions in the EYMH system are unstable \[7, 9\], and the dilaton hair in the GB is of ‘secondary’ type \[8\], in the sense that it is not accompanied by the presence of any new quantity that characterizes the black hole given that the dilaton charge can be expressed in terms of the ADM mass of the black hole. It should be stressed, however, that irrespectively of the precise type of hair the set of solutions examined in this talk bypasses the conditions of the no-scalar-hair theorem \[12\]. Thus, our work \[7, 10\] may be viewed as demonstrating that there is plenty of room in the gravitational structure of Non-Abelian Gauge and/or Superstring Theory to allow for physically sensible situations that are not covered by the no-hair theorem as stated.

The relevance of the above evasion of the no-hair conjecture to the information-loss and quantum decoherence issues of a black-hole space time remains to be seen. Although the analytic proof allows the existence of hair, it does not provide any information on the amount of hair carried by the black holes, or on whether such hair is capable of storing enough information so as to maintain unitarity of the ‘black-hole plus matter’ system during Hawking evaporation. It seems more likely that quantum hair, if exists, will be more relevant for this purpose. Such questions, especially from the point of view of string-inspired theories \[13\], are left for future investigations.

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