Coordinate Finite Type Rotational Surfaces in Euclidean Spaces

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Abstract

Submanifolds of coordinate finite-type were introduced in [11]. A submanifold of a Euclidean space is called a coordinate finite-type submanifold if its coordinate functions are eigenfunctions of $\Delta$. In the present study we consider coordinate finite-type surfaces in $\mathbb{E}^4$. We give necessary and sufficient conditions for generalized rotation surfaces in $\mathbb{E}^4$ to become coordinate finite-type. We also give some special examples.

1 Introduction

Let $M$ be a connected $n-$dimensional submanifold of a Euclidean space $\mathbb{E}^m$ equipped with the induced metric. Denote $\Delta$ by the Laplacian of $M$ acting on smooth functions on $M$. This Laplacian can be extended in a natural way to $\mathbb{E}^m$ valued smooth functions on $M$. Whenever the position vector $x$ of $M$ in $\mathbb{E}^m$ can be decomposed as a finite sum of $\mathbb{E}^m$-valued non-constant functions of $\Delta$, one can say that $M$ is of finite type. More precisely the position vector $x$ of $M$ can be expressed in the form $x = x_0 + \sum_{i=1}^{k} x_i$, where $x_0$ is a constant map $x_1, x_2, ..., x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then $M$ is said to be of $k$-type. Similarly, a smooth map $\phi$ of an $n$-dimensional Riemannian manifold $M$ of $\mathbb{E}^m$ is said to be of finite type if $\phi$ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of $\Delta$ ([2], [3]). For the position vector field $\vec{H}$ of $M$ it is well known (see eg. [3]) that $\Delta x = -n \vec{H}$, which shows in particular that $M$ is a minimal submanifold in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic. In [14] Takahashi proved that an $n$-dimensional submanifold of $\mathbb{E}^m$ is of 1-type (i.e., $\Delta x = \lambda x$) if and only if it is either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of some hypersphere of $\mathbb{E}^m$. As a generalization of T. Takahashi’s condition, O. Garay considered in [9], submanifolds of Euclidean space whose position vector field $x$ satisfies the differential equation $\Delta x = Ax$, for some $m \times m$ diagonal matrix $A$. Garay called such submanifolds coordinate finite type submanifolds. Actually coordinate finite type submanifolds are finite type submanifolds whose
type numbers are at most \( m \). Each coordinate function of a coordinate finite type submanifold \( m \) is of 1-type, since it is an eigenfunction of the Laplacian \([11]\).

In [8] by G. Ganchev and V. Milousheva considered the surface \( M \) generated by a \( W \)-curve \( \gamma \) in \( E^4 \). They have shown that these generated surfaces are a special type of rotation surfaces which are introduced first by C. Moore in 1919 (see [13]). Vranceanu surfaces in \( E^4 \) are the special type of these surfaces [15].

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in \( E^4 \). Section 3 tells about the generalized surfaces in \( E^4 \). Further this section provides some basic properties of surfaces in \( E^4 \) and the structure of their curvatures. In the final section we consider coordinate finite type surfaces in euclidean spaces. We give necessary and sufficient conditions for generalized rotation surfaces in \( E^4 \) to become coordinate finite type.

## 2 Basic Concepts

Let \( M \) be a smooth surface in \( E^n \) given with the patch \( X(u, v): (u, v) \in D \subset E^2 \).

The tangent space to \( M \) at an arbitrary point \( p = X(u, v) \) of \( M \) span \( \{X_u, X_v\} \).

In the chart \( (u, v) \) the coefficients of the first fundamental form of \( M \) are given by

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \tag{1}
\]

where \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product. We assume that \( W^2 = EG - F^2 \neq 0 \), i.e. the surface patch \( X(u, v) \) is regular. For each \( p \in M \), consider the decomposition \( T_p E^n = T_p M \oplus T^\perp_p M \) where \( T^\perp_p M \) is the orthogonal component of \( T_p M \) in \( E^n \). Let \( \tilde{\nabla} \) be the Riemannian connection of \( E^4 \). Given unit local vector fields \( X_1, X_2 \) tangent to \( M \).

Let \( \chi(M) \) and \( \chi^\perp(M) \) be the space of the smooth vector fields tangent to \( M \) and the space of the smooth vector fields normal to \( M \), respectively. Consider the second fundamental map: \( h: \chi^\perp(M) \times \chi(M) \to \chi^\perp(M); \)

\[
h(X_i, X_j) = \tilde{\nabla} X_i X_j - \nabla X_i X_j \quad 1 \leq i, j \leq 2. \tag{2}
\]

where \( \tilde{\nabla} \) is the induced. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal normal frame field \( \{N_1, N_2, \ldots, N_{n-2}\} \) of \( M \), recall the shape operator \( A: \chi^\perp(M) \times \chi(M) \to \chi(M); \)

\[
A_{N_k} X_j = -\langle \tilde{\nabla} X_j N_k \rangle^T, \quad X_j \in \chi(M), \quad 1 \leq k \leq n - 2 \tag{3}
\]

This operator is bilinear, self-adjoint and satisfies the following equation:

\[
\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = h^k_{ij}, \quad 1 \leq i, j \leq 2. \tag{4}
\]

The equation (2) is called Gaussian formula, and

\[
h(X_i, X_j) = \sum_{k=1}^{n-2} h^k_{ij} N_k, \quad 1 \leq i, j \leq 2 \tag{5}
\]
where $c_{ij}^k$ are the coefficients of the second fundamental form.

Further, the Gaussian and mean curvature vector of a regular patch $X(u, v)$ are given by

$$K = \sum_{k=1}^{n-2} (h_{11}^k h_{22}^k - (h_{12}^k)^2),$$

and

$$H = \frac{1}{2} \sum_{k=1}^{n-2} (h_{11}^k + h_{22}^k) N_k,$$

respectively, where $h$ is the second fundamental form of $M$. Recall that a surface $M$ is said to be minimal if its mean curvature vector vanishes identically [2].

For any real function $f$ on $M$ the Laplacian of $f$ is defined by

$$\Delta f = -\sum_i (\nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f).$$

3 Generalized Rotation Surfaces in $\mathbb{E}^4$

Let $\gamma = \gamma(s) : I \rightarrow \mathbb{E}^4$ be a W-curve in Euclidean 4-space $\mathbb{E}^4$ parametrized as follows:

$$\gamma(v) = (a \cos cv, a \sin cv, b \cos dv, b \sin dv), \quad 0 \leq v \leq 2\pi,$$

where $a, b, c, d$ are constants ($c > 0, d > 0$). In G. Ganchev and V. Milousheva considered the surface $M$ generated by the curve $\gamma$ with the following surface patch:

$$X(u, v) = (f(u) \cos cv, f(u) \sin cv, g(u) \cos dv, g(u) \sin dv),$$

where $u \in J, 0 \leq u \leq 2\pi$, $f(u)$ and $g(u)$ are arbitrary smooth functions satisfying

$$c^2 f'^2 + d^2 g'^2 > 0 \quad \text{and} \quad (f')^2 + (g')^2 > 0.$$

These surfaces are first introduced by C. Moore in [13], called generalized rotation surfaces.

We choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that $e_1, e_2$ are tangent to $M$ and $e_3, e_4$ normal to $M$ in the following (see, [8]):

$$e_1 = \frac{X_u}{\|X_u\|}, \quad e_2 = \frac{X_v}{\|X_u\|},$$

$$e_3 = \frac{1}{\sqrt{(f')^2 + (g')^2}} (g' \cos cv, g' \sin cv, -f' \cos dv, -f' \sin dv),$$

$$e_4 = \frac{1}{\sqrt{c^2 f'^2 + d^2 g'^2}} (-dg \sin cv, dg \cos cv, cf \sin dv, -cf \cos dv).$$
Hence the coefficients of the first fundamental form of the surface are

\[
\begin{align*}
E &= \langle X_u, X_u \rangle = (f')^2 + (g')^2 \\
F &= \langle X_u, X_v \rangle = 0 \\
G &= \langle X_v, X_v \rangle = c^2 f'^2 + d^2 g'^2
\end{align*}
\]  

(11)

where \( \langle \cdot, \cdot \rangle \) is the standard scalar product in \( \mathbb{E}^4 \). Since

\[ EG - F^2 = ((f')^2 + (g')^2) (c^2 f'^2 + d^2 g'^2) \]

does not vanish, the surface patch \( X(u, v) \) is regular. Then with respect to the frame field \( \{e_1, e_2, e_3, e_4\} \), the Gaussian and Weingarten formulas (2)-(3) of \( M \) look like (see, [7]):

\[
\begin{align*}
\tilde{\nabla}_{e_1} e_1 &= -A(u) e_2 + h_{11}^1 e_3, \\
\tilde{\nabla}_{e_1} e_2 &= A(u) e_1 + h_{22}^1 e_4, \\
\tilde{\nabla}_{e_2} e_2 &= h_{12}^1 e_3, \\
\tilde{\nabla}_{e_2} e_1 &= h_{12}^2 e_4,
\end{align*}
\]

(12)

and

\[
\begin{align*}
\tilde{\nabla}_{e_1} e_3 &= -h_{11}^1 e_1 + B(u) e_4, \\
\tilde{\nabla}_{e_1} e_4 &= -h_{12}^2 e_2 - B(u) e_3, \\
\tilde{\nabla}_{e_2} e_3 &= -h_{12}^2 e_2, \\
\tilde{\nabla}_{e_2} e_4 &= -h_{12}^1 e_1,
\end{align*}
\]

(13)

where

\[
\begin{align*}
A(u) &= \frac{c^2 f f' + d^2 g g'}{\sqrt{(f')^2 + (g')^2 (c^2 f'^2 + d^2 g'^2)}}, \\
B(u) &= \frac{cd(f f' + g g')}{\sqrt{(f')^2 + (g')^2 (c^2 f'^2 + d^2 g'^2)}}, \\
h_{11}^1 &= \frac{d^2 f' g - c^2 f g'}{\sqrt{(f')^2 + (g')^2 (c^2 f'^2 + d^2 g'^2)}}, \\
h_{22}^1 &= \frac{g f'' - f g'''}{((f')^2 + (g')^2)^{3/2}}, \\
h_{12}^2 &= \frac{cd(f g' - f' g)}{\sqrt{(f')^2 + (g')^2 (c^2 f'^2 + d^2 g'^2)}}, \\
h_{11}^2 &= h_{22}^1 = h_{12}^1 = 0.
\end{align*}
\]

(14)

are the differentiable functions. Using (6)-(7) with (14) one can get the following results:
Proposition 1 \[1\] Let $M$ be a generalized rotation surface given by the parametrization (14), then the Gaussian curvature of $M$ is

$$K = \frac{(c^2 f'^2 + d^2 g^2)(g'f'' - f'g'')(d^2 g f' - c^2 fg')(c^2 f' - f g')(c^2(f')^2 + (g')^2)}{((f')^2 + (g')^2)^2(c^2 f'^2 + d^2 g^2)^2}.$$

An easy consequence of Proposition 1 is the following.

Corollary 2 \[1\] The generalized rotation surface given by the parametrization (14) has vanishing Gaussian curvature if and only if the following equation

$$(c^2 f'^2 + d^2 g^2)(g'f'' - f'g'')(d^2 g f' - c^2 fg')(c^2 f' - f g')(c^2(f')^2 + (g')^2) = 0,$$

holds.

The following results are well-known;

Proposition 3 \[1\] Let $M$ be a generalized rotation surface given by the parametrization (14), then the mean curvature vector of $M$ is

$$\overrightarrow{H} = \frac{1}{2}(h_{11} + h_{22})e_3 = \left(\frac{(c^2 f'^2 + d^2 g^2)(g'f'' - f'g'')(d^2 g f' - c^2 fg')(c^2(f')^2 + (g')^2)}{2((f')^2 + (g')^2)^{3/2}(c^2 f'^2 + d^2 g^2)^2}\right)e_3.$$

An easy consequence of Proposition 3 is the following.

Corollary 4 \[1\] The generalized rotation surface given by the parametrization (14) is minimal surface in $\mathbb{E}^4$ if and only if the equation

$$(c^2 f'^2 + d^2 g^2)(g'f'' - f'g'')(d^2 g f' - c^2 fg')(c^2(f')^2 + (g')^2) = 0,$$

holds.

Definition 5 The generalized rotation surface given by the parametrization

$$f(u) = r(u) \cos u, \quad g(u) = r(u) \sin u, \quad c = 1, d = 1. \quad (15)$$

is called Vranceanu rotation surface in Euclidean 4-space $\mathbb{E}^4$.\[15\]

Remark 6 Substituting (15) into the equation given in Corollary 2 we obtain the condition for Vranceanu rotation surface which has vanishing Gaussian curvature;

$$r(u)r''(u) - (r'(u))^2 = 0. \quad (16)$$

Further, and easy calculation shows that $r(u) = \lambda e^{\mu u}, \quad (\lambda, \mu \in \mathbb{R})$ is the solution is this second degree equation. So, we get the following result.

Corollary 7 \[16\] Let $M$ is a Vranceanu rotation surface in Euclidean 4-space. If $M$ has vanishing Gaussian curvature, then $r(u) = \lambda e^{\mu u}$, where $\lambda$ and $\mu$ are real constants. For the case, $\lambda = 1, \mu = 0, r(u) = 1$, the surface $M$ is a Clifford torus, that is it is the product of two plane circles with same radius.
Corollary 8 \[1\] Let $M$ is a Vranceanu rotation surface in Euclidean 4-space. If $M$ is minimal then
\[ r(u) r''(u) - 3(r'(u))^2 - 2r(u)^2 = 0. \]
holds.

Corollary 9 \[1\] Let $M$ is a Vranceanu rotation surface in Euclidean 4-space. If $M$ is minimal then
\[ r(u) = \frac{\pm 1}{\sqrt{a \sin 2u - b \cos 2u}}, \] (17)
where, $a$ and $b$ are real constants.

Definition 10 The surface patch $X(u, v)$ is called pseudo-umbilical if the shape operator with respect to $H$ is proportional to the identity (see, \[2\]). An equivalent condition is the following:
\[ \langle h(X_i, X_j), H \rangle = \lambda^2 \langle X_i, X_j \rangle, \] (18)
where, $\lambda = \|H\|$. It is easy to see that each minimal surface is pseudo-umbilical.

The following results are well-known;

Theorem 11 \[1\] Let $M$ be a generalized rotation surface given by the parametrization (9) is pseudo-umbilical then
\[ (c^2 f^2 + d^2 g^2)(g' f'' - f' g'') - (d^2 g f' - c^2 f g')(f'')^2 + (g'')^2 = 0. \] (19)
The converse statement of Theorem 11 is also valid.

Corollary 12 \[1\] Let $M$ be a Vranceanu rotation surface in Euclidean 4-space. If $M$ pseudo-umbilical then $r(u) = \lambda e^{\mu u}$, where $\lambda$ and $\mu$ are real constants.

3.1 Coordinate Finite Type Surfaces in Euclidean Spaces
In the present section we consider coordinate finite type surfaces in Euclidean spaces $E^{n+2}$. A surface $M$ in Euclidean $m$-space is called coordinate finite type if the position vector field $X$ satisfies the differential equation
\[ \Delta X = AX, \] (20)
for some $m \times m$ diagonal matrix $A$. Using the Beltrami formula’s $\Delta X = -2\vec{H}$, with \[1\] one can get
\[ \Delta X = - \sum_{k=1}^{n} (h_{11}^k + h_{22}^k) N_k. \] (21)
So, using \[20\] with \[21\] the coordinate finite type condition reduces to
\[ AX = - \sum_{k=1}^{n} (h_{11}^k + h_{22}^k) N_k \] (22)
For a non-compact surface in $E^4$ O.J.Garay obtained the following:
Theorem 13 [10] The only coordinate finite type surfaces in Euclidean 4-space $E^4$ with constant mean curvature are the open parts of the following surfaces:

i) a minimal surface in $E^4$,

ii) a minimal surface in some hypersphere $S^3(r)$,

iii) a helical cylinder,

iv) a flat torus $S^1(a) \times S^1(b)$ in some hypersphere $S^3(r)$.

In [5] Chen-Dillen-Verstraelen-Vrancken proved the following theorem;

Theorem 14 [5] Assume $M$ is a surface in $E^4$ that is immersed in $S^3(r)$ and has constant mean curvature. Then $M$ is of restricted type if and only if $M$ is one of the following:

i) an open part of a minimal surface of $S^3(r)$,

ii) an open part of $S^2(r')$ for $0 < r' \leq r$,

iii) an open part of the product of two circles $S^1(a) \times S^1(b)$, where $a, b > 0$ and $a^2 + b^2 = r^2$.

3.2 Surface of Revolution of Coordinate Finite Type

A surface in $E^3$ is called a surface of revolution if it is generated by a curve $C$ on a plane $\Pi$ when $\Pi$ is rotated around a straight line $L$ in $\Pi$. By choosing $\Pi$ to be the $xz$-plane and line $L$ to be the $x$-axis the surface of revolution can be parameterized by

$$X(u, v) = (f(u), g(u) \cos v, g(u) \sin v),$$

where $f(u)$ and $g(u)$ are arbitrary smooth functions. We choose an orthonormal frame $\{e_1, e_2, e_3\}$ such that $e_1, e_2$ are tangent to $M$ and $e_3$ normal to $M$ in the following:

$$e_1 = \frac{X_u}{\|X_u\|}, \quad e_2 = \frac{X_v}{\|X_v\|}, \quad e_3 = \frac{1}{\sqrt{(f')^2 + (g')^2}}(g', -f' \cos v, -f' \sin v),$$

By covariant differentiation with respect to $e_1, e_2$ a straightforward calculation gives

$$\begin{align*}
\nabla_{e_1} e_1 &= h_{11} e_3, \\
\nabla_{e_2} e_2 &= -A(u)e_1 + h_{22} e_3, \\
\nabla_{e_2} e_1 &= A(u)e_2, \\
\nabla_{e_1} e_2 &= 0,
\end{align*}$$

(25)
where

\[ A(u) = \frac{g'}{g' \sqrt{(f')^2 + (g')^2}}, \]
\[ h_{11}^1 = \frac{g' f'' - f' g''}{((f')^2 + (g')^2)^{\frac{3}{2}}}, \]
\[ h_{22}^1 = \frac{f'}{g' \sqrt{(f')^2 + (g')^2}}, \]
\[ h_{12}^1 = 0. \]  \hspace{1cm} (26)

are the differentiable functions. Using (6)-(7) with (26) one can get

\[ \overline{H} = \frac{1}{2} (h_{11}^1 + h_{22}^1) e_3 \] \hspace{1cm} (27)

where \( h_{11}^1 \) and \( h_{22}^1 \) are the coefficients of the second fundamental form given in (26).

A surface of revolution defined by (23) is said to be of polynomial kind if \( f(u) \) and \( g(u) \) are polynomial functions in \( u \) and it is said to be of rational kind if \( f \) is a rational function in \( g \), i.e., \( f \) is the quotient of two polynomial functions in \( g \). [4]

For finite type surfaces of revolution B.Y. Chen and S. Ishikawa obtained in [6] the following results:

**Theorem 15** [6] Let \( M \) be a surface of revolution of polynomial kind. Then \( M \) is a surface of finite type if and only if either it is an open portion of a plane or it is an open portion of a circular cylinder.

**Theorem 16** [6] Let \( M \) be a surface of revolution of rational kind. Then \( M \) is a surface of finite type if and only if \( M \) is an open portion of a plane.

T. Hasanis and T. Vlachos proved the following.

**Theorem 17** [11] Let \( M \) be a surface of revolution. If \( M \) has constant mean curvature and is of finite type then \( M \) is an open portion of a plane, of a sphere or of a circular cylinder.

We proved the following result:

**Lemma 18** Let \( M \) be a surface of revolution given with the parametrization (23). Then \( M \) is a surface of coordinate finite type if and only if diagonal matrix \( A \) is of the form

\[ A = \begin{bmatrix}
  a_{11} & 0 & 0 \\
  0 & a_{22} & 0 \\
  0 & 0 & a_{33}
\end{bmatrix} \] \hspace{1cm} (28)
where
\[
\begin{align*}
a_{11} &= -g'(g'f'' - f'g'') + f' ((f')^2 + (g')^2)) / fg ((f')^2 + (g')^2) \\
a_{22} &= a_{33} = f' (g(g'f'' - f'g'') + f' ((f')^2 + (g')^2)) / g^2 ((f')^2 + (g')^2) \\
\end{align*}
\] (29)
are differentiable functions.

**Proof.** Assume that the surface of revolution $M$ given with the parametrization (23) is of coordinate finite type. Then, from the equality (22)
\[
\Delta X = -(h_{11}^1 + h_{22}^1)e_3.
\] (30)
Further, substituting (26) into (30) and using (24) we get the result. $\blacksquare$

**Remark 19** If the diagonal matrix $A$ is equivalent to a zero matrix then $M$ becomes minimal. So the surface of revolution $M$ is either an open portion of a plane or an open portion of a catenoid.

Minimal rotational surfaces are of coordinate finite type.

For the non-minimal case we obtain the following result;

**Proposition 20** Let $M$ be a non-minimal surface of revolution given with the parametrization (23). If $M$ is coordinate finite type surface then
\[
ff' + \lambda gg' = 0
\] (31)
holds, where $\lambda$ is a nonzero constant.

**Proof.** Suppose that the entries of the diagonal matrix $A$ are real constants. Then using (24) one can get the following differential equations
\[
\begin{align*}
-\frac{g'(g'f'' - f'g'') + f' ((f')^2 + (g')^2))}{fg ((f')^2 + (g')^2)} &= c_1 \\
\frac{f' (g(g'f'' - f'g'') + f' ((f')^2 + (g')^2))}{g^2 ((f')^2 + (g')^2)} &= c_2.
\end{align*}
\]
where $c_1, c_2$ are nonzero real constants. Further, substituting one into another we obtain the result. $\blacksquare$

**Example 21** The round sphere given with the parametrization $f(u) = r \cos u$, $g(u) = r \sin u$ satisfies the equality (31). So it is a coordinate finite type surface.

**Example 22** The cone $f(u) = g(u)$ satisfies the equality (31). So it is a coordinate finite type surface.
3.3 Generalized Rotation Surfaces of Coordinate Finite Type

In the present section we consider generalized rotation surfaces of coordinate finite type surfaces in Euclidean 4-spaces $\mathbb{E}^4$.

We proved the following result;

**Lemma 23** Let $M$ be a generalized rotation surface given with the parametrization (9). Then $M$ is a surface of coordinate finite type if and only if diagonal matrix $A$ is of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

where

$$a_{11} = a_{22} = -g'(u)((d^2 f' g - c^2 f' g')((f')^2 + (g')^2) + (g'' f' - f' g'')(c^2 f^2 + d^2 g^2)),$$

$$a_{33} = a_{44} = \frac{f'(u)((d^2 f' g - c^2 f' g')((f')^2 + (g')^2) + (g'' f' - f' g'')(c^2 f^2 + d^2 g^2))}{g(u)((f')^2 + (g')^2)(c^2 f^2 + d^2 g^2)},$$

are differentiable functions and $h_{11}^1, h_{22}^1$ the coefficients of the second fundamental form given in (14).

If the matrix $A$ is a zero matrix then $M$ becomes minimal. So minimal rotational surfaces are of coordinate finite type.

We prove the following result.

**Proposition 24** Let $M$ be a generalized rotation surface given by the parametrization (9). If $M$ is a coordinate finite type then

$$ff' = cgg'$$

holds, where, $c$ is a real constant.

**Proof.** Suppose that the entries of the diagonal matrix $A$ are real constants. Then, substituting the first equation in (33) into second one we get the result.

An easy consequence of Proposition 24 is the following.

**Corollary 25** Let $M$ be a Vranceanu rotation surface in Euclidean 4-space. If $M$ is a coordinate finite type, then

$$rr' \left( \cos^2 u - c \sin^2 u \right) = r^2 \cos u \sin u (1 + c)$$

holds, where, $c$ is a real constant.
We obtain the following result:

**Theorem 26** Let $M$ be a Vranceanu rotation surface in Euclidean 4-space. Then $M$ is of restricted type if and only if $M$ is one of the following:

i) an open part of a Clifford torus,

ii) a minimal surface given with the parametrization \[ r(u) = \frac{\pm \lambda}{\sqrt{(1 + c) \cos 2u + (1 - c)}} \quad c \neq 1 \] (34)

where, $\lambda$ and $c$ are real constants.

In [12] C. S. Houh investigated Vranceanu rotation surfaces of finite type and proved the following

**Theorem 27** [12] A flat Vranceanu rotation surface in $\mathbb{E}^4$ is of finite type if and only if it is the product of two circles with the same radius, i.e. it is a Clifford torus.

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