Abstract

We consider a family of non-commutative 4d Minkowski spaces with the signature (1,3) and two types of spaces with the signature (2,2). The Minkowski spaces are defined by the common reflection equation and differ in anti-involutions. There exist two Casimir elements and the fixing of one of them leads to the non-commutative "homogeneous" spaces $H_3$, $dS_3$, $AdS_3$ and light-cones. We present the quasi-classical description of the Minkowski spaces. There are three compatible Poisson structures - quadratic, linear and canonical. The quantization of the former leads to the considered Minkowski spaces. We introduce the horospheric generators of the Minkowski spaces. They lead to the horospheric description of $H_3$, $dS_3$ and $AdS_3$. The irreducible representations of Minkowski spaces $H_3$ and $dS_3$ are constructed. We find the eigen-functions of the Klein-Gordon equation in the terms of the horospheric generators of the Minkowski spaces. They give rise to eigen-functions on the $H_3$, $dS_3$, $AdS_3$ and light-cones.

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1 Introduction

The non-commutative Minkowski spaces (NCMS) naturally arise in the string theory. The origin of the non-commutativity is the B-field background of the open string that ends on a D-brane (for reviews, see [1, 2]). On the other hand a string propagation on curved space-time manifolds such as AdS$_3$, its Euclidean version H$_3$ and dS$_3$ was investigated in detail over the last few years. These investigations include, in particular, AdS-CFT correspondence and solutions of 2+1 gravity like BTZ black holes. Non-commutative deformation of AdS(dS)/CFT correspondence was analyzed in [3, 4, 5].

We consider a particular family of NCMS. Their distinguishing feature is a natural action of the quantum Lorentz group $U_q(SL_2)$. NCMS of the signature (1, 3) drew an active attention over the last fifteen years starting with the first works [6, 7]. At present their complete classification is known [8, 9, 10, 11]. In fact, the spaces we consider here are included in the list of Ref. [10]. As in Ref. [11] we fit the commutation relation on NCMS in the form of the reflection equation.
We construct irreducible representations of associative algebra generated by non-commutative coordinates of NCMS with the signature \((1,3)\) (see also [12]). By fixing one of Casimir elements we define the non-commutative Lobachevsky spaces (NCLS) \((H_3)\) and the non-commutative imaginary Lobachevsky spaces (NCILS) \((dS_3)\). These algebras have a natural description in terms of the non-commutative analog of the horospheric coordinates. The Casimir element that determines the homogeneous spaces is one of the horospheric generators. It allows us to define corresponding homogeneous spaces in terms of the rest horospheric generators.

The final goal of this paper is the solutions of the Klein-Gordon equations on NCMS in terms of the horospheric coordinates. By analogy with the classical case, the solutions are products of \(q\)-cylindric functions. The reduction of these solutions to NCLS, NCILS and the non-commutative cone is straightforward.

To include in the consideration the non-commutative \(AdS_3\) we consider non-commutative deformations of the Minkowski spaces with the signature \((+, -, +, -)\) and \((+, -, -, +)\). While in the classical case these spaces are isomorphic since they are governed by the isomorphic groups \(SL(2, \mathbb{R}) \oplus SL(2, \mathbb{R})\) and \(SU(1, 1) \oplus SU(1, 1)\) in the non-commutative case the situation is different. Though the commutation relations in these algebras are the same, the algebras are distinguished in different anti-involutions. Instead of the quantum Lorentz group we have the action of non-isomorphic Hopf algebras. It is \(\mathcal{U}_q(SL_2(\mathbb{R})) \oplus \mathcal{U}_q(SL_2(\mathbb{R}))\) with \(|q| = 1\) in the former case and \(\mathcal{U}_qSU(1,1) \oplus \mathcal{U}_qSU(1,1)\) and \(q \in \mathbb{R}\) in the latter. The horospheric description and the solutions of the Klein-Gordon equations in the case of the space of type \((+, -, +, -)\) are presented as in the case of the signature \((1,3)\).

The properties of the non-commutative "homogeneous spaces" can be arranged in the following table.

| Notations |\(q\) | Hopf algebra symmetry | NCMS |
|-----------|------|----------------------|------|
| \(H_3\)  | \(q \in \mathbb{C}\) | \(\mathcal{U}_q(SL(2, \mathbb{C}))\) | \(M_{1,3}^{1,3}\) |
| \(C_{1,3}^{1,3}\) | \(q \in \mathbb{C}\) | \(\mathcal{U}_q(SL(2, \mathbb{C}))\) | \(M_{1,3}^{1,3}\) |
| \(dS_3\) | \(q \in \mathbb{C}\) | \(\mathcal{U}_q(SL(2, \mathbb{C}))\) | \(M_{1,3}^{1,3}\) |
| \(AdS^\pm_3\) | \(|q| = 1\) | \(\mathcal{U}_q(SL(2, \mathbb{R})) \oplus \mathcal{U}_q(SL(2, \mathbb{R}))\) | \(M_{q}^{1,3}\) |
| \(C_{q}^{2,2}\) | \(|q| = 1\) | \(\mathcal{U}_q(SL(2, \mathbb{R})) \oplus \mathcal{U}_q(SL(2, \mathbb{R}))\) | \(M_{q}^{2,2}\) |
| \(AdS^\pm_3\) | \(q \in (0,1]\) | \(\mathcal{U}_q(SU(1,1)) \oplus \mathcal{U}_q(SU(1,1))\) | \(M_{q}^{2,2}\) |
| \(C_{q}^{2,2}\) | \(q \in (0,1]\) | \(\mathcal{U}_q(SU(1,1)) \oplus \mathcal{U}_q(SU(1,1))\) | \(M_{q}^{2,2}\) |

We also consider the quasi-classical approximations of NCMS. In this way we obtain quadratic Poisson algebras with anti-involutions. They are described by the classical reflection equation. The Poisson algebras have two classical Casimir functions. The symplectic leaves of these structures in the case of the signature \((1,3)\) are the classical horospheres or spheres of \(H_3\) and \(dS_3\). We construct the linear and the canonical Poisson structures that are compatible with the quadratic one.

**Notations.**
Classical variables are denoted by small letters, while their non-commutative deformation (quantization) with capital letters. We do not introduce a special notation for the non-commutative multiplication.
The coordinates of the Minkowski space $\mathbb{M}^4 = (x_1, x_2, x_3, x_4)$ or $(y_0, y_1, y_2, y_3)$ we identify with the matrix elements of the matrix $\mathbf{x}$

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = y_0 \text{Id} + \sum_{\alpha} \epsilon_{\alpha} y_{\alpha} \sigma_{\alpha}, \quad \epsilon_{\alpha} = 1, \text{ or } i. \quad (1.1)$$

The choice $1$ or $i$ in front of $y_{\alpha}$ defines the signature of the Minkowski space. The generators of the non-commutative Minkowski space we also arrange in the matrix form

$$\mathbf{X} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}. \quad (1.2)$$

The deformation parameter is $q = \exp \theta \in (0, 1]$, or $q = \exp i \theta$, $(|q| = 1)$.

2 Horospheric coordinates on the classical Minkowski spaces

There are two types of Minkowski spaces with the signature $(+,-,-,-)$ and $(+,+,\ldots,-)$. While the first one allows us to describe the Lobachevsky space $H_3$ and the Imaginary Lobachevsky space $dS_3$, the second leads to $AdS_3$. We will consider them separately.

2.1 Minkowski space $\mathbb{M}^{1,3}$ in the horospheric description

The Minkowski space $\mathbb{M}^{1,3}$ can be identified with the space of Hermitian matrices

$$\mathbb{M}^{1,3} = \{ \mathbf{x} \in \text{Mat}_{\mathbb{C}} | \mathbf{x}^\dagger = \mathbf{x} \}, \quad (\bar{x}_1 = x_1, \bar{x}_2 = x_3, \bar{x}_4 = x_4). \quad (2.1)$$

The metric is $ds^2 = \det(dx) = dx_1 dx_4 - dx_2 dx_3$. Another set of coordinates is $y_a$, $(a = 0, \ldots, 3)$ corresponds to the choice $\epsilon_{\alpha} = 1$ in (1.1)

$$\mathbf{x} = \sum_{a=0}^{3} y_a \sigma_a, \quad \sigma_0 = \text{Id}.$$

It leads to the metric

$$ds^2 = dy_0^2 - \sum_{j=1}^{3} dy_j^2. \quad (2.2)$$

The group $\text{SL}(2, \mathbb{C})$ is the double covering of the proper Lorentz group $\text{SO}(1,3)$ and acts on the Minkowski space $\mathbb{M}^{1,3} = \{ y_0, \ldots, y_3 \}$ as

$$\mathbf{x} \to g^\dagger \mathbf{x} g, \quad g \in \text{SL}(2, \mathbb{C}), \quad (2.3)$$

where $g^\dagger$ is the Hermitian conjugated matrix. The action preserves

$$\det \mathbf{x} = y_0^2 - y_1^2 - y_2^2 - y_3^2 = x_1 x_4 - x_2 x_3 \quad (2.4)$$

and thereby the metric (2.2) on $\mathbb{M}^{1,3}$.

The time-like part $\mathbb{M}^{1,3+}$ of $\mathbb{M}^{1,3}$ corresponds to the matrices with $\det \mathbf{x} > 0$, while $\det \mathbf{x} < 0$ corresponds to the space-like part $\mathbb{M}^{1,3-}$. The equation $\det \mathbf{x} = 0$ selects the light cone

$$\mathbb{C}^{1,3} = \{ \det \mathbf{x} = x_1 x_4 - x_2 x_3 = 0 \}. \quad (2.5)$$
We introduce the horospheric coordinates $x \sim (r, h, z, \bar{z})$. If $x_1 \neq 0$ then

$$x_1 = rh, \quad x_2 = rhz, \quad x_3 = rh\bar{z}, \quad x_4 = r(h|z|^2 + \varepsilon h^{-1}).$$

(2.6)

Here

$$z \in \mathbb{C}, \quad h \in \mathbb{R} \setminus 0, \quad \varepsilon = \pm 1, 0, \quad r^2\varepsilon = \det x.$$ 

and

$$z = x_2x_1^{-1}, \quad \bar{z} = x_3x_1^{-1}, \quad r = \sqrt{|\det x|}, \text{ for } \det x \neq 0,$$

$$h = \left\{ \begin{array}{ll} x_1(|\det x|)^{-\frac{1}{2}} & \text{for } \varepsilon = \pm 1, \\ x_1 & \text{for } \varepsilon = 0. \end{array} \right.$$ 

The case $\varepsilon = 1$ corresponds to the time-like part of $M^{1,3}$, $\varepsilon = -1$ corresponds to the space-like part, and $\varepsilon = 0$ to the light-cone $C^{1,3}$.

The horospheric coordinates on the light-cone $C^{1,3}$ are $(h, z, \bar{z})$

$$x_1 = h, \quad x_2 = hz, \quad x_3 = \bar{x}_2, \quad x_4 = h|z|^2.$$ 

(2.7)

To describe the case $x_1 = 0$ we put $\varepsilon = -1$, $h \to 0$, $r < \infty$ and $z \to \infty$ such that

$$\lim hz = \exp(it), \quad x_2 = r\exp(it) \quad \text{and} \quad x_4 \text{ takes an arbitrary real value. Thus, the horospheric description has the form}$$

$$(r, \exp(it), x_4), \quad x_2 = r\exp(it) \quad x_3 = r\exp(-it).$$

Consider the commutative algebra $S(M^{1,3})$ of the Schwartz functions on $M^{1,3}$. The invariant integral with respect to the SL$(2, \mathbb{C})$ action on $S(M^{1,3})$

$$I(f) = \int f(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4$$

takes the form in the horospheric coordinates

$$I(g) = \int g(z, \bar{z}, h, r)r^3hdrdhdzd\bar{z}, \quad g \in S(M^{1,3}).$$ 

(2.8)

### 2.2 Homogeneous spaces, embedded in $M^{1,3}$

The action of SL$(2, \mathbb{C})$ (2.3) leads to the foliation of $M^{1,3}$. The orbits are defined by fixing $\det x$.

The quadric

$$L = \{ \det x = r_0^2 > 0, \quad x_1 > 0 \}$$

is the upper sheet of the two-sheeted hyperboloid. It is a model of the Lobachevsky space, or $H_3$. The metric on $H_3$ is the restriction of the invariant metric $dx_1dx_4 - dx_2dx_3$ on $r = \text{const}$. In what follows we assume $r_0 = 1$. The horospheric coordinates on $L$ have the restrictions $h > 0$. Since SU$(2)$ leaves the point $y_0 = 1, y_\alpha = 0$ the Lobachevsky space is the coset $L \sim \text{SL}(2, \mathbb{C})/\text{SU}(2) \sim \text{SO}(1, 3)/\text{SO}(3)$.

Consider the commutative algebra $S(L)$ of Schwartz functions on $L$. Functions from $S(L)$ are infinitely differentiable with all derivatives tending to zero when $|z| \to \infty, h \to \infty, h \to 0$ faster than any power. Let $I_3$ be the ideal in $S(M^{1,3})$ generated by $f(\det x - r_0^2) = 0$. The algebra $S(L)$ can be described as the factor-algebra $S(M^{1,3})/I_1$ with the additional condition $x_1 > 0$. 

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In the similar way we describe the upper sheet of the light-cone \( C^{1,3} \) as \( S(M^{1,3})/I_0 \). The horospheric coordinates (2.6) being restricted on \( C^{1,3+} \) satisfy the condition \( (r = 1, \ h > 0, \ \varepsilon = 0) \). \( C^{1,3+} \) is the quotient \( SL(2, \mathbb{C})/B_C \), where \( B_C \) is the subgroup of the form

\[
\tilde{B}_C = \left\{ \begin{pmatrix} \exp(i\phi) & w \\ 0 & \exp(-i\phi) \end{pmatrix}, \ w \in \mathbb{C} \right\}.
\]

The space \( \text{IL} = \{ \det x = -1 \} \) is called the \textit{Imaginary Lobachevsky space}. The corresponding quadric is \( y_0^2 - \sum \alpha y_\alpha^2 = -1 \). It is the de Sitter space \( \text{IL} \sim dS = SL(2, \mathbb{C})/SU(1, 1) \sim SO(1, 3)/SO(1, 2) \), since

\[
g^\dagger \sigma_3 g = \sigma_3, \quad \text{for } g \in SU(1, 1).
\]

As before, \( S(\text{IL}) \sim S(M^{1,3})/I_{-1} \), but in contrast with the \( \text{L} \) and \( \text{C}^+ \) the horospheric radius \( h \) of \( \text{IL} \) can take an arbitrary value \( h \in \mathbb{R} \setminus \{0\} \).

We partially compactify \( \text{IL} \) with respect to the coordinate \( h \). Two "limiting" space \( \Xi^\pm = h \rightarrow \pm \infty \) are called \textit{absolutes}. It follows from (2.5) and (2.9) that \( \Xi^\pm \) can be considered as the projectivization of the cone \( C^{1,3} \). The both absolutes are homeomorphic to \( \mathbb{C} \) and therefore can be compactify to \( \Xi^\pm \sim \mathbb{CP}^1 \). Note, that while \( \Xi^\pm \) are two components of the boundary of the \( \text{IL} \), \( \Xi^+ \) is the boundary of \( C^{1,3+} \) and the \( \text{L} \).

### 2.3 Minkowski space \( M^{2,2} \) in the horospheric description

We identify \( M^{2,2} \) with the space \( \text{Mat}_{\mathbb{R}}(2) \). It is obtained from \( M^{1,3} \) by the Wick rotation \( y_2 \rightarrow -iy_2 \)

\[
M^{2,2} = \{ x \in \text{Mat}_{\mathbb{R}}(2) \}, \quad x = y_0 \text{Id} + y_1 \sigma_1 + iy_2 \sigma_2 + y_3 \sigma_3.
\]

The metric on \( M^{2,2} \) assumes the form

\[
d s^2 = dx_1 dx_4 - dx_2 dx_3 = dy_0^2 - dy_1^2 + dy_2^2 - dy_3^2.
\]

Thus, \( M^{2,2} \) has the signature \((+ - + -)\). The group \( G = SL(2, \mathbb{R}) \oplus SL(2, \mathbb{R}) \) acts on \( M^{2,2} \) as

\[
x \rightarrow g_2^{-1} x g_1, \quad g_k \in SL(2, \mathbb{R}).
\]

The transformed matrix belongs to \( \text{Mat}_{\mathbb{R}}(2) \), since \( g_1, g_2 \in SL(2, \mathbb{R}) \). Moreover, (2.13) preserves \( \det x \) and therefore the metric.

The horospheric coordinates take the similar form as for \( M^{1,3} \) after replacing in (2.6), (2.7)

\[
z \rightarrow z_1, \quad \bar{z} \rightarrow z_2, \quad (z_1, z_2 \in \mathbb{R})
\]

\[
x_1 = rh, \quad x_2 = rhz_1, \quad x_3 = rhz_2, \quad x_4 = r(hz_1 z_2 + \varepsilon h^{-1}).
\]

The invariant integral on \( M^{2,2} \) is the same as on \( M^{1,3} \) (2.10).
### 2.4 Homogeneous subspaces embedded in $\mathbb{M}^{2,2}$

The action of $G = \text{SL}(2,\mathbb{R}) \oplus \text{SL}(2,\mathbb{R})$ on $\mathbb{M}^{2,2}$ preserves the determinant $\det x$. We have the two types of nontrivial $G$-orbits.

The first type is $\text{AdS}_3$ spaces

$$\text{AdS}_3^\pm = \{ x \mid \det x = \varepsilon r_0^2 , \varepsilon = \pm 1 \}. \quad (2.16)$$

In the commutative case the spaces $\text{AdS}_3^+$ and $\text{AdS}_3^-$ are isomorphic. The spaces $\text{AdS}_3^\pm$ is identified with the quotient $(\text{SL}(2,\mathbb{R}) \oplus \text{SL}(2,\mathbb{R}))/\langle \text{SL}(2,\mathbb{R}) \rangle$. Then $\text{AdS}_3 \sim \text{SL}(2,\mathbb{R})$.

The second type is cone

$$C_2^2 = \{ x \mid \det x = 0 \}.$$

It is the quotient $C_2^2 \sim (\text{SL}(2,\mathbb{R}) \oplus \text{SL}(2,\mathbb{R}))/\langle \tilde{B} \rangle$, where $\tilde{B}$ is the subgroup of $\text{SL}(2,\mathbb{R}) \oplus \text{SL}(2,\mathbb{R})$ preserving a point of $C_2^2$. For example, it can be defined as follows

$$\tilde{B} = \left\{ (g_2,g_1) \mid \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) g_1 = g_2 \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) \right\}.$$

### 2.5 Minkowski space $\tilde{\mathbb{M}}^{2,2}$ in the horospheric description

Consider the Minkowski space $\tilde{\mathbb{M}}^{2,2}$ with the signature $(+−−+)$. Though $\tilde{\mathbb{M}}^{2,2}$ is isomorphic to $\mathbb{M}^{2,2}$, we consider it separately, since they become non-isomorphic in the non-commutative case.

We identify $\tilde{\mathbb{M}}^{2,2}$ with the space of matrix

$$\tilde{\mathbb{M}}^{2,2} = \{ x \in \text{Mat}_\mathbb{C}(2) \mid \bar{x}_1 = x_4 , \bar{x}_2 = x_3 \}. \quad (2.17)$$

It is obtained from $\mathbb{M}^{1,3}$ by the Wick rotations $y_3 \rightarrow iy_3$.

$$x = y_0 \text{Id} + y_1 \sigma_1 + y_2 \sigma_2 + iy_3 \sigma_3.$$

The definition (2.17) is equivalent to

$$\tilde{\mathbb{M}}^{2,2} = \{ x \in \text{Mat}_\mathbb{C} \mid x^\sigma_3 x^\dagger = \sigma_3 \}. \quad (2.18)$$

The isomorphism between $\mathbb{M}^{2,2}$ (2.11) and $\tilde{\mathbb{M}}^{2,2}$ is defined by the conjugation

$$\tilde{x} = gxg^{-1} , \quad \tilde{x} \in \tilde{\mathbb{M}}^{2,2} , \quad x \in \mathbb{M}^{2,2} , \quad g = \left( \begin{array}{cc} 1 & −i \\ i & 1 \end{array} \right).$$

It follows from (2.18) that the group $G = \text{SU}(1,1) \oplus \text{SU}(1,1)$ acts on $\tilde{\mathbb{M}}^{2,2}$

$$x \rightarrow g_2^{-1} x g_1 , \quad g_k \in \text{SU}(1,1). \quad (2.19)$$

It means that $\tilde{\mathbb{M}}^{2,2} \sim \text{SU}(1,1) \oplus \text{SU}(1,1)/\text{SU}(1,1)$. The action (2.19) preserves $\det x$ and therefore the metric.

The horospheric coordinates on $\tilde{\mathbb{M}}^{2,2}$ take the form

$$x_1 = \frac{r}{2}(h(1 + z_1 z_2) + \varepsilon h^{-1} + ih(z_1 - z_2)) , \quad x_4 = \bar{x}_1 , \quad (2.20)$$

$$x_2 = \frac{r}{2}(h(1 - z_1 z_2) - \varepsilon h^{-1} - ih(z_1 + z_2)) , \quad x_3 = \bar{x}_2 , \quad (2.21)$$

where $z_1, z_2 \in \mathbb{R}$.

We again can determine the homogeneous spaces embedded in $\tilde{\mathbb{M}}^{2,2}$ by fixing $\det x = \varepsilon r^2$. If $\varepsilon = \pm 1$ and $r = r_0$ we come to $\text{AdS}_3^\pm$ isomorphic to the defined above. The spaces $\text{AdS}_3^\pm$ can be identified with $\text{SU}(1,1)$.
2.6 Short summary

For completeness we consider the Euclidean space $\mathbb{R}^4$ and the embedded sphere $S_3$

$$x = y_0 I + i \sum_{\alpha} y_\alpha \sigma_\alpha, \quad \det x = y_0^2 + \sum_{\alpha} y_\alpha^2,$$

$$S_3 = \{ x \mid \det x = r_0 \} \sim SU(2) \oplus SU(2)/SU(2).$$

The horospherical description of $S_3$ does not exists. We will not generalize $S_3$ to the non-commutative case. We summarize the structure of the homogeneous spaces in the following table, that will be generalized to the non-commutative situation.

| Notations | Notations | Coset construction | Ambient space |
|-----------|-----------|--------------------|---------------|
| $L$       | $H_3$     | $SL(2, \mathbb{C})/SU(2)$ | $M^{1,3}$     |
| $C^{1,3}$ | $dS_3$    | $SL(2, \mathbb{C})/B\mathbb{C}$ | $M^{1,3}$     |
| $IL$      | $dS_3$    | $SL(2, \mathbb{C})/SU(1, 1)$ | $M^{1,3}$     |
| $C^{2,2}$ | $AdS_3$   | $SL(2, \mathbb{R}) \oplus SL(2, \mathbb{R})/SL(2, \mathbb{R})$ | $M^{2,2}$     |
| $S_3$     | $SU(1, 1) \oplus SU(1, 1)/SU(1, 1)$ | $M^{2,2}$     |
| $C^{2,2}$ | $AdS_3$   | $SU(2) \oplus SU(2)/SU(2)$ | $\mathbb{R}^4$ |

3 Laplace operator and its eigen-functions

In this paper we generalize to the noncommutative case the following facts concerning the eigen-functions of the Laplace operator.

The solutions of the Klein-Gordon equation on $M^4$

$$\Delta f_\nu(x_1, x_2, x_3, x_4) = \nu^2 f_\nu(x_1, x_2, x_3, x_4), \quad \Delta = \frac{\partial^2}{\partial x_1 \partial x_4} - \frac{\partial^2}{\partial x_3 \partial x_2}. \quad (3.1)$$

are the exponents

$$f_\nu(x_1, x_2, x_3, x_4) = \exp(\xi x), \quad (\xi x) = \sum \xi_i x_i \quad \nu^2 = \xi_1 \xi_4 - \xi_2 \xi_3. \quad (3.2)$$

We will consider $\Delta$ and its eigen-functions in the horospheric coordinates.

3.1 Scalar fields on $M^{1,3}$ in the horospheric coordinates

The metric on $M^{1,3}$ in the horospheric coordinates takes the form

$$ds^2 = g_{jk} dx_j dx_k = \varepsilon dr^2 - \varepsilon r^2 h^{-2} dh^2 - r^2 h^2 dz d\bar{z}.$$ 

Then one can rewrite $\Delta = \frac{1}{(\det g)^{\frac{1}{2}}} \frac{\partial}{\partial j} g^{jk} (\det g)^{\frac{1}{2}} \partial_k$ as

$$\Delta = r^{-2} \left[ h^2 \frac{\partial^2}{\partial h^2} + 3 \frac{\partial}{\partial h} + 4 \varepsilon h^{-2} \frac{\partial^2}{\partial z \partial \bar{z}} - r^2 \frac{\partial^2}{\partial r^2} - 3 r \frac{\partial}{\partial r} \right] \quad (3.3)$$

and we come the eigenvalue problem

$$r^{-2} \left[ h^2 \frac{\partial^2}{\partial h^2} + 3 \frac{\partial}{\partial h} + 4 \varepsilon h^{-2} \frac{\partial^2}{\partial z \partial \bar{z}} - r^2 \frac{\partial^2}{\partial r^2} - 3 r \frac{\partial}{\partial r} \right] f_\nu(\bar{z}, h, z; r) = \nu^2 f_\nu(\bar{z}, h, z; r). \quad (3.4)$$
Let $Z_\nu(x)$ be a cylindric function. It means that $Z_\nu(x)$ is a solution of the equation

$$\frac{\partial^2 Z_\nu}{\partial x^2} + \frac{1}{x} \frac{\partial Z_\nu}{\partial x} + \left(1 - \frac{\nu^2}{x^2}\right) Z_\nu = 0.$$ 

We will prove the non-commutative analog of the following statement

**Proposition 3.1** The basic harmonics of the eigen-value problem (3.4) are

$$f_\nu(\bar{z}, h, z; r) = r^{-1} h^{-1} \exp(i \mu z + i \bar{\mu} \bar{z}) Z_{\alpha}(r \nu) Z_{\alpha}(2i \epsilon \frac{1}{2} |\mu| h^{-1}), \epsilon = \pm 1,$$

and

$$f_\nu(\bar{z}, h, z; r) = h^{\alpha - 1} \exp(i \mu z + i \bar{\mu} \bar{z}), \quad \epsilon = 0, \quad \alpha^2 = \nu^2 + 1,$$

where $\mu, \alpha \in \mathbb{C}$.

**Proof**

The variables of the equation (3.4) can be separated

$$f_\nu(\bar{z}, h, z; r) = \exp(i \mu z + i \bar{\mu} \bar{z}) v_{\alpha,h}(h) \chi_{\nu,\alpha}(r),$$

where $\alpha^2 - 1$ is the separation constant

$$\left[ h^2 \frac{\partial^2}{\partial h^2} + 3h \frac{\partial}{\partial h} + (-4 \epsilon h^{-2} |\mu|^2 - \alpha^2 + 1) \right] v_{\alpha,h} = 0,$$

and

$$\left[ r^2 \frac{\partial^2}{\partial r^2} + 3r \frac{\partial}{\partial r} + r^2 (\nu^2 - 1) - \alpha^2 + 1 \right] \chi_{\nu,\alpha}(r) = 0.$$

The solutions of (3.8) and (3.9) have the following form

$$\chi_{\nu,\alpha}(r) = r^{-1} Z_{\alpha}(r \sqrt{\nu^2 - 1}),$$

$$v_{\alpha,h} = \begin{cases} h^{-1} Z_{\alpha}(2i \epsilon \frac{1}{2} |\mu| h^{-1}), & \epsilon = \pm 1, \\ h^{\alpha - 1}, & \epsilon = 0. \end{cases}$$

In this way we come to (3.5), (3.6). \(\square\)

It follows from (3.3) that the restrictions of the Klein-Gordon equation to the homogeneous spaces assume the form

$$\left( h^2 \frac{\partial^2}{\partial h^2} + 3h \frac{\partial}{\partial h} + 4\epsilon h^{-2} \frac{\partial^2}{\partial z \partial \bar{z}} \right) f_\nu(h, z, \bar{z}) = (\nu^2 - 1) f_\nu(h, z, \bar{z}),$$

$$H_3 \rightarrow \epsilon = 1, \quad dS_3 \rightarrow \epsilon = -1.$$

Thus, we come to the following statement

**Corollary 3.1** The basic harmonics on $H_3(L), dS_3(IL)$ and the light-cone $C^{1,3}$ are

$$f_\nu(\bar{z}, h, z) = h^{-1} \exp(i \mu z + i \bar{\mu} \bar{z}) Z_{\nu,2^{-1}}(2i \epsilon \frac{1}{2} |\mu| h^{-1}), \epsilon = \pm 1,$$

and

$$f_\nu(\bar{z}, h, z) = h^{\alpha - 1} \exp(i \mu z + i \bar{\mu} \bar{z}), \quad \epsilon = 0, \quad \alpha^2 = \nu^2 + 1.$$
3.2 Scalar fields on $\mathbb{M}^{2,2}$ in the horospheric coordinates

It is easy to pass from the eigen-functions of the Klein-Gordon equation on $\mathbb{M}^{1,3}$ to the eigen-functions of the Klein-Gordon equation on $\mathbb{M}^{2,2}$. We come to the equation

$$r^{-2} \left[ h^{2} \frac{\partial^{2}}{\partial h^{2}} + 3 \frac{\partial}{\partial h} + 4\varepsilon h^{-2} \frac{\partial^{2}}{\partial z_{1} \partial z_{2}} - r^{2} \frac{\partial^{2}}{\partial r^{2}} - 3r \frac{\partial}{\partial r} \right] f_{\nu}(z_{2}, h, z_{1}; r) = \nu^{2} f_{\nu}(z_{2}, h, z_{1}; r). \quad (3.13)$$

The analog of Proposition 3.1 has the form

**Proposition 3.2** The basic harmonics of the eigen-value problem (3.13) are

$$f_{\nu}(\bar{z}, h, z; r) = r^{-1} h^{-1} \exp(\mu_{1} z_{1} + i \mu_{2} z_{2}) Z_{\nu}(2i \varepsilon \frac{1}{2} \sqrt{\mu_{1} \mu_{2}} h^{-1}), \varepsilon = \pm 1, \quad (3.14)$$

and

$$f_{\nu}(\bar{z}, h, z; r) = h^{\nu - 2} \exp(i \mu_{1} z_{1} + i \mu_{2} z_{2}), \varepsilon = 0, \quad \alpha^{2} = \nu^{2} + 1, \quad (3.15)$$

where $\mu_{1} \mu_{2}, \alpha \in \mathbb{R}$.

Thus, we come to the scalar field on AdS$_{3}$ and on $\mathbb{C}^{2,2}$.

**Corollary 3.2** The basic harmonics on AdS$_{3}$ and the light-cone $\mathbb{C}^{2,2}$ are

$$f_{\nu}(\bar{z}, h, z) = h^{-1} \exp(i \mu_{1} z_{1} + i \mu_{2} z_{2}) Z_{\nu}(2i \varepsilon \frac{1}{2} \sqrt{\mu_{1} \mu_{2}} h^{-1}), \varepsilon = \pm 1, \quad (3.16)$$

and

$$f_{\nu}(\bar{z}, h, z) = h^{\alpha - 1} \exp(i \mu_{1} z_{1} + i \mu_{2} z_{2}), \quad \alpha^{2} = \nu^{2} + 1, \quad \varepsilon = 0. \quad (3.17)$$

4 Non-commutative 4d Minkowski space $\mathbb{M}^{1,3}_{\delta,q}$.

4.1 Definition

We define an algebra generated by matrix elements of (1.2).

**Definition 4.1** The non-commutative 4d Minkowski space $\mathbb{M}^{1,3}_{\delta,q}$, $0 < q \leq 1$, $\delta \in \mathbb{N}$ is the unital associative algebra with the anti-involution $\ast$ and four generators $X_{j}$, $j = 1, \ldots, 4$ with the quadratic relations

$$X_{1}X_{3} = q^{-\delta} X_{3}X_{1}, \quad X_{1}X_{2} = q^{\delta} X_{2}X_{1}, \quad (4.1)$$

$$[X_{2}, X_{3}] = q^{\delta - 2}(1 - q^{2})X_{1}X_{4}, \quad (4.2)$$

$$X_{2}X_{4} = q^{\delta - 2}X_{4}X_{2}, \quad (4.3)$$

$$X_{3}X_{4} = q^{-\delta + 2}X_{4}X_{3}, \quad (4.4)$$

$$[X_{1}, X_{4}] = 0, \quad (4.5)$$

such that

$$X_{1}^{\ast} = X_{1}, \quad X_{2}^{\ast} = X_{3}, \quad X_{4}^{\ast} = X_{4}. \quad (4.6)$$
This space was described in [10]. Following this approach we cast the relations in $M_{\delta,q}^{1,3}$ in the form of the reflection equation. Consider the basis in Mat(2)

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Define two R-matrices

$$R(q) = q^{-1}(E_1 \otimes E_1 + E_4 \otimes E_4) + (E_1 \otimes E_4 + E_4 \otimes E_1) + q^{-1}(1 - q^2)E_3 \otimes E_2, \quad (4.7)$$

$$R^{(2)}(q) = (E_1 \otimes E_1 + E_4 \otimes E_4) + q^{\delta-1}(E_1 \otimes E_4 + E_4 \otimes E_1). \quad (4.8)$$

The R-matrices satisfy the Yang-Baxter type equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad (4.9)$$

$$R_{12}R_{13}^{(2)}R_{23}^{(2)} = R_{23}^{(2)}R_{13}^{(2)}R_{12}. \quad (4.10)$$

It can be checked straightforwardly that the relations (4.1)-(4.5) are equivalent to the reflection equation

$$R(q)X^{(1)}R^{(2)}(q)X^{(2)} = X^{(2)}R^{(2)}(q)X^{(1)}R^{(2)}(q), \quad (4.11)$$

where $X^{(1)} = X \otimes I_d$ and $X^{(2)} = I_d \otimes X$ and

$$R^{\dagger}(q) = q^{-1}(E_1 \otimes E_1 + E_4 \otimes E_4) + (E_1 \otimes E_4 + E_4 \otimes E_1) + q^{-1}(1 - q^2)E_2 \otimes E_3.$$

**Lemma 4.1** The algebra $M_{\delta,q}^{1,3}$ has two independent Casimir elements

$$K_1 = X_1^{\delta-2}X_4^{\delta}, \quad (4.12)$$

$$K_2 = X_1X_4 - q^{-\delta}X_3X_2. \quad (4.13)$$

**Proof.**

It can checked in straightforward way that the both expressions commute with the generators of $M_{\delta,q}^{1,3}$.$\square$

The Casimir operator $K_2$ (4.12) is the quantum determinant $K_2 = \det_q X$. In an irreducible module over $M_{\delta,q}^{1,3}$ $K_2 = \varepsilon r^2 \in \mathbb{R}$ is a scalar. It allows us to define the time-like part $M_{\delta,q}^{1,3+}$, $(\varepsilon = 1)$, the space-like part $M_{\delta,q}^{1,3-}$, $(\varepsilon = -1)$, and the light cone $C_{\delta,q}$, $(\varepsilon = 0)$.

**4.2 Standard basis**

Represent the matrix $X$ in the basis of the sigma-matrices

$$X = Y_0I_d + \sum_{\alpha=1}^{3} Y_\alpha \sigma_\alpha.$$ 

The advantage of this basis is that its generators are self-conjugate with respect to the anti-involution $Y_\alpha^* = Y_\alpha$. Let $\theta = \ln q$, $0 < q \leq 1$ and

$$c_1(\theta, \delta) = \frac{1}{2} (\cosh \theta \delta + \cosh \theta (2 - \delta)), \quad c_2(\theta, \delta) = \frac{1}{2} (\cosh \theta \delta - \cosh \theta (2 - \delta)), \quad$$

$$c_3(\theta \delta) = \frac{1}{2} (\sinh \theta \delta + \sinh \theta (2 - \delta)), \quad c_4(\theta, \delta) = \frac{1}{2} (\sinh \theta \delta - \sinh \theta (2 - \delta)).$$
In terms of $Y_a$ the commutation relations in the algebra $\mathbb{M}^{1,3}_{\delta,\gamma}$ assume the form

\begin{align}
Y_1 Y_0 &= c_1 Y_0 Y_1 + c_2 Y_3 Y_1 - i c_3 Y_0 Y_2 - i c_4 Y_3 Y_2, \\
Y_1 Y_3 &= c_2 Y_0 Y_1 + c_1 Y_3 Y_1 - i c_4 Y_0 Y_2 - i c_3 Y_3 Y_2, \\
Y_2 Y_0 &= i c_3 Y_0 Y_1 + i c_4 Y_3 Y_1 + c_1 Y_0 Y_2 + c_2 Y_3 Y_2, \\
Y_2 Y_3 &= i c_4 Y_0 Y_1 + i c_3 Y_3 Y_1 + c_2 Y_0 Y_2 + c_1 Y_3 Y_2, \\
[Y_1, Y_2] &= \frac{i}{2} q^{\delta - 2} (1 - q^2) (Y_0^2 - Y_3^2), \\
[Y_0, Y_3] &= 0.
\end{align}

Note that for $q \to 1$ $c_1(\theta, \delta) \to 1$ while $c_j \to 0$, $j = 2, 3, 4$ and we come to the commutative space $\mathbb{M}^{1,3} = (y_0, y_1, y_2, y_3)$.

The Casimirs assume the forms

\begin{align}
K_1 &= (Y_0 + Y_3)^{\delta - 2} (Y_0 - Y_3)^\delta, \\
K_2 &= \left(1 - \frac{1 - q^{-2}}{2}\right) Y_0^2 - q^{-\delta} Y_1^2 - q^{-\delta} Y_2^2 - \left(1 - \frac{1 - q^{-2}}{2}\right) Y_3^2.
\end{align}

### 4.3 Quantum Lorentz group action on $\mathbb{M}^{1,3}_{\delta,\gamma}$

We start with a pair of the standard $\mathcal{U}_q(\text{SL}_2)$ Hopf algebra [13, 14]. The first one is generated by $A, B, C, D$ and the unit 1 with relations

\begin{align}
AD &= DA = 1, \quad AB = qBA, \quad BD = qDB, \\
AC &= q^{-1}CA, \quad CD = q^{-1}DC, \\
[B, C] &= \frac{1}{q - q^{-1}} (A^2 - D^2).
\end{align}

There is a copy of this algebra $\mathcal{U}_q^s(\text{SL}_2)$ generated by $A^*, B^*, C^*, D^*$ with the relations coming from (4.20) $U^* V^* = (VU)^*$. They commute with $A, B, C, D$.

The pair $\mathcal{U}_q(\text{SL}_2), \mathcal{U}_q^s(\text{SL}_2)$ forms a Hopf algebra $\mathcal{U}_q^{(s)}(\text{SL}_2)$, where the coproduct and the antipode are twisted in the consistent way [17]

\begin{align}
\Delta(A) &= \Delta(A) = A \otimes A, \\
\Delta(B) &= A \otimes B + B \otimes D (A^*)^s, \\
\Delta(C) &= A \otimes C + C \otimes D (A^*)^{-s}, \\
S \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} D & -q^{-1}(A^*)^{-s}B \\ -q(A^*)^sC & A \end{pmatrix}, \\
\Delta(A^*) &= \Delta(A^*) = A^* \otimes A^*, \\
\Delta(B^*) &= A^* \otimes B^* + B^* \otimes D^* A^*, \\
\Delta(C^*) &= A^* \otimes C^* + C^* \otimes D^* A^{-s}, \\
S \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} &= \begin{pmatrix} D^* & -q A^{-s} B^* \\ -q^{-1} A^s C^* & A^* \end{pmatrix}.
\end{align}
The counit on $U_q^{(s)}(\text{SL}(2, \mathbb{C}))$ assumes the form

$$
\varepsilon(A) = 1, \quad \varepsilon(B, C) = 0.
$$

(4.24)

There are two Casimir elements in $U_q^{(s)}(\text{SL}(2, \mathbb{C}))$ which commute with any $u \in U_q(\text{SL}_2(\mathbb{C}))$.

$$
\Omega_q := \frac{(q^{-1} + q)(A^2 + A^{-2}) - 4}{2(q^{-1} - q)^2} + \frac{1}{2}(BC + CB)
$$

(4.25)

$$
\bar{\Omega}_q := \frac{(q^{-1} + q)(A^{*2} + A^{*-2}) - 4}{2(q^{-1} - q)^2} + \frac{1}{2}(B^{*}C^{*} + C^{*}B^{*})
$$

(4.26)

**Proposition 4.1** $M_{\delta,q}^{1,3}$ is a right module over the Hopf algebra $U_q^{(s)}(\text{SL}(2, \mathbb{C}))$.

**Proof.**  
We define the action of the quantum group $U_q^{(s)}(\text{SL}(2, \mathbb{C}))$ on $M_{\delta,q}^{1,3}$

$$
\begin{pmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{pmatrix}
.A = 
\begin{pmatrix}
q^\frac{1}{2}X_1 & q^{-\frac{1}{2}}X_2 \\
q^{-\frac{1}{2}}X_3 & q^{\frac{1}{2}}X_4
\end{pmatrix},
$$

(4.27)

$$
\begin{pmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{pmatrix}
.B = 
\begin{pmatrix}
0 & X_1 \\
0 & X_3
\end{pmatrix},
$$

(4.28)

$$
\begin{pmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{pmatrix}
.C = 
\begin{pmatrix}
X_2 & 0 \\
X_4 & 0
\end{pmatrix},
$$

(4.29)

$$
\begin{pmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{pmatrix}
.A^{*} = 
\begin{pmatrix}
q^{\delta}X_1 & q^{\frac{1}{2}}X_2 \\
q^{-\frac{1}{2}}X_3 & q^{-\delta}X_4
\end{pmatrix}. 
$$

(4.30)

The direct calculations show that the commutation relations in $M_{\delta,q}^{1,3}$ are compatible with the coproduct in $U_q^{(s)}(\text{SL}(2, \mathbb{C}))$. Moreover,

$$(X_j.a)^* = a^*.X_j^*.$$ 

The Schwartz space $S(M_{\delta,q}^{1,3})$ as the series with the rapidly decreasing coefficients

$$S(M_{\delta,q}^{1,3}) = \{ \sum_{m,k,l,n} a_{m,k,l,n} w(m, k, l, n), \quad a_{m,k,l,n} \in \mathbb{C} \},$$

(4.31)

where $|a_{m,k,l,n}| < (1 + m^2 + k^2 + l^2 + n^2)^j$, for any $j \in \mathbb{N}$, when $|m|, |k|, |l|, |n| \to \infty$.
Proposition 4.2 The Jackson integral

\[
\langle f \rangle = \int d_{q^2} X_3 d_{q^2} X_1 d_{q^2} X_4 f(X_1, X_2, X_3, X_4) \tag{4.32}
\]

is invariant functional on \( S(\mathcal{M}^{1,3}_{\delta, q}) \) with respect to the action of \( \mathcal{U}^{(s)}_q(\text{SL}(2, \mathbb{C})) \)

\[
\langle f.u \rangle = \varepsilon(u) \langle f \rangle,
\]

where \( \varepsilon(u) \) is the counit (4.24).

The proof will follow from the actions of the generators \( A, A^*, B, C \) on the ordered monomials \( w(m, k, l, n) \) presented in Section 9.

5 Module over \( \mathcal{M}^{1,3}_{\delta, q} \)

Here we construct representations of \( \mathcal{M}^{1,3}_{\delta, q} \) in the infinite-dimensional spaces \( E^+, E^-, E^0 \) (right modules over \( \mathcal{M}^{1,3}_{\delta, q} \)). The representations of algebras satisfying the more general reflection equations were constructed in Ref. [12]. Three types of the spaces correspond to the time-like part \( K_2 > 0 \), the space-like part \( K_2 < 0 \), and the light cone \( K_2 = 0 \). Here we assume that \( \delta \in \mathbb{Z} \).

5.1 Time-like part \( (K_2 > 0) \).

We will define the module \( E^+_{\alpha, \rho} \), depending on \( \alpha, \rho \in \mathbb{R} \). Consider \( \mathbb{C} \)-valued functions on the \( \mathbb{Z} \) lattice with the vertices \( q^j \). Using the Jackson integral we introduce the Hermitian metric

\[
\langle f|g \rangle = \int d_{q^2} u d_{q^2} \overline{u} e_{q^2}(u\overline{u}) f(u) g(u),
\]

where \( e_{q^2}(u) = e_{q^2}((1 - q^2)u) \) (A.1) and the Jackson integral (A.5) is defined by the double series

\[
\langle f|g \rangle = (1 - q^2)^2 \sum_{m,n \in \mathbb{Z}} q^{2(m+n)} \overline{f(q^{2m})g(q^{2n})} + e_{q^2}(-q^{2(m+n)}) \overline{f(-q^{2m})g(-q^{2n})}.
\]

The module \( E^+_{\alpha, \rho} \) is the space of functions on the lattice with finite norm \( \langle f|f \rangle^{\frac{1}{2}} < \infty \). Let \( \lambda = \lambda(\delta, q) = q^\frac{\delta}{2} - 1 (1 - q^2)^\frac{1}{2} \).

Proposition 5.1 The space \( E^+_{\alpha, \rho} \) \( (\rho \in \mathbb{R}^+, \alpha \in \mathbb{R}^+) \) is the right module of the algebra \( \mathcal{M}^{1,3}_{\delta, q} \)

\[
f(u)X_1 = \alpha^{-1} \rho f(uq^\delta), \quad f(u)X_2 = \lambda \rho D_u f(u),
\]

\[
f(u)X_3 = -\lambda \rho u f(u), \quad f(u)X_4 = \alpha \rho f(uq^{2-\delta}).
\]

The Casimirs on \( E^+_{\alpha, \rho} \) have the values \( K_2 = \rho^2 q^-2 = R^2, \ K_1 = \alpha^2 \rho^{2\delta - 2} \). The anti-involution corresponds to the Hermitian conjugation

\[
\langle fX_a|g \rangle = \langle f|X^*_a g \rangle.
\]

\( D_u \) is defined by (A.2).
Proof
We have the following relations for $T_u: f(u) \to f(uq)$, $u: f(u) \to uf(u)$ and the for difference operator (A.2)

\[ f(u)(u \cdot T_q) = qf(u)(T_q \cdot u), \quad f(u)(D_u \cdot T_q) = q^{-1}f(u)(T_q \cdot D_u), \quad (5.5) \]

\[ f(u)[D_u, u] = -f(uq^2). \]

They imply the commutation relations (4.1) - (4.5) for (5.3). The conjugation formula $X_2^* = X_3$ comes from the relation

\[ D_q e_{q^2}(au) = ae_{q^2}(au). \]

The shift operators $X_1$ and $X_4$ are self-conjugate due to (5.2). □

Note that the operator $X_1 + X_4$ acts as

\[ f(u) \to \rho(\alpha^{-1}f(\delta) + \alpha(\rho q^{2-\delta})). \]

Since $\alpha$ and $\rho$ are positive this operator is positive definite

\[ \langle (X_1 + X_4)f | f \rangle > 0, \quad \text{if} \ f \neq 0. \]

This fact is in agreement with the classical limit $\frac{1}{2}(X_1 + X_4) \to y_0$, and $y_0 > 0$.

5.2 Space-like part ($K_2 < 0$).

Replace the measure in the integral (5.1)

\[ \langle f | g \rangle = \int d_q u d_{q^2} \bar{ue}_{q^2}(-w\bar{u})f(\bar{u})g(u). \quad (5.6) \]

It defines the Hermitian pairing in the module $E_{\alpha, \rho}$.

**Proposition 5.2** The space $E_{\alpha, \rho}$, $(\alpha, \rho \in \mathbb{R}^+)$ is the right module of the algebra $\mathbb{M}_{\delta, q}^{1, 3}$

\[ f(u)X_1 = \alpha^{-1}\rho f(\delta), \quad f(u)X_2 = \lambda\rho D_u f(u), \quad (5.7) \]

\[ f(u)X_3 = \lambda\rho u f(u), \quad f(u)X_4 = -\alpha\rho f(q^{2-\delta}). \]

with $K_2 = -R^2 = -\rho^2 q^{-2}$ and $K_1 = (-1)^{\delta}\alpha^2\rho^{2\delta-2}$.

The proof of this Proposition is the same as the previous one. We just change consistently the sign in the measure and in the definition of $X_3, X_4$.

In this case the operator $X_1 + X_4$ is not positive-definite in accordance with the classical description of the space-like part.

5.3 The light-cone $K_2 = 0$.

Consider the space $E^0$ of holomorphic functions on $\mathbb{C}^*$ equipped with the Hermitian form

\[ \langle f | g \rangle = \frac{1}{2\pi i} \int \frac{dw}{w} \overline{f(w)}g(w). \quad (5.8) \]

The following Proposition can be established in the direct way.

**Proposition 5.3** $E^0$ is the right module over the light-cone $\mathbb{C}_{\delta, q}^{1, 3}$

\[ f(w)X_1 = f(wq\delta), \quad f(w)X_2 = \frac{\delta-1}{2} w^{-1} f(wq), \quad (5.9) \]

\[ f(w)X_3 = \frac{\delta-1}{2} w f(wq), \quad f(w)X_4 = f(wq^{2-\delta}), \]

with $K_2 = 0$ and $K_1 = 1$. 

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6 Horospheric description.

6.1 Horospheric generators

We introduce another set of generators - the non-commutative analog of the horospheric coordinates \((Z^*, H, Z, R)\), \((H^* = H, R^* = R, (Z^*)^* = Z)\)

\[
\begin{align*}
X_1 &= RH, \quad X_2 = RHZ, \quad X_3 = RZ^*H, \quad (6.1) \\
X_4 &= R(Z^*HZ + \varepsilon H^{-1}), \quad \varepsilon = \pm 1, \ 0.
\end{align*}
\]

The defining relations

\[
ZH = q^{-\delta}HZ, \quad Z^*H = q^\delta HZ^*, \quad [R, H] = [R, Z] = [R, Z^*] = 0,
\]

\[
ZZ^* = q^{2\delta - 2}Z^*Z - \varepsilon q^{\delta - 2}(1 - q^2)H^{-2}.
\]

yield the relations (4.1)–(4.5). The Casimir elements are

\[
K_2 = \varepsilon R^2 \quad K_1 = R^{2\delta - 2}H^{\delta - 2}(Z^*HZ + \varepsilon H^{-1})^\delta.
\]

The inverse relations assume the form

\[
H = R^{-1}X_1, \quad Z = X_1^{-1}X_2, \quad Z^* = X_3X_1^{-1}, \quad R = \varepsilon K_1.
\]

In terms of the horospheric generators the action of \(U_q^{(s)}(SL(2, \mathbb{C}))\) takes the form

\[
\begin{align*}
Z^*A &= z^*, & H.A &= q^{\frac{1}{2}}H, & Z.A &= q^{-1}z, & R.A &= R, \\
Z^*.A^* &= q^{\frac{\delta - 2}{2}}Z^*, & H.A^* &= q^{\frac{\delta - 1}{2}}H, & Z.A^* &= Z, & R.A^* &= R, \\
Z^*.B &= 0, & H.B &= 0, & Z.B &= q^{-\frac{1}{2}}, & R.B &= 0, \\
Z^*.C &= q^{\frac{3}{2} - \delta}H^{-2}, & H.C &= HZ, & Z.C &= -q^{\frac{1}{2}}Z^2, & R.C &= 0.
\end{align*}
\]

(6.6)

It follows from these relations that \(R\) is invariant with respect to the \(U_q^{(s)}(SL(2, \mathbb{C}))\) action \(R.u = \varepsilon(u)R\).

Define the analog of the Schwartz space \(\mathcal{S}(M_{\delta,q}^{1,3})\) (4.31) in terms of the ordered monomial \(\hat{w}(m, k, n) = Z^m H^k Z^n\). Since \(R\) is a center element its position is irrelevant. Let

\[
\begin{align*}
\hat{f}(Z^*, H, Z, R)|^\dagger &= \sum_{m, k, n, l} a_{m, k, n, l} \hat{w}(m, k, n)R^l, \quad a_{m, k, n, l} \in \mathbb{C}.
\end{align*}
\]

(6.7)

For \(\mathcal{S}(M_{\delta,q}^{1,3})\), the coefficients satisfy the condition

\[
|a_{m, k, n, l}| < (1 + m^2 + k^2 + l^2 + n^2)^j, \text{ for any } j \in \mathbb{N}, \text{ when } |m|, |k|, |l|, |n| \to \infty.
\]

The invariant integral (4.32) is well defined functional on (6.7). It assumes the form

\[
I_{q^2}(f) = \int d_qZ^* d_qH d_qZ d_qR |\dagger f(Z^*, H, Z, R)H|^\dagger.
\]

(6.8)
6.2 Homogeneous spaces

Consider an irreducible representation of algebra (6.3). Then one can fix the Casimir operator (6.4) \( K_2 = \varepsilon R^2, \ R^2 = r^2 \in \mathbb{R}^+ \). It allows us to define the non-commutative analog of Lobachevsky spaces and the cone. Let us fix the ideal \( I_\varepsilon = \{ K_2 - \varepsilon r^2 = 0 \} \).

\[
S(\mathbb{M}^{1,3}_{\delta,q})/I_\varepsilon \sim H_3 (\varepsilon = 1), \ dS_3 (\varepsilon = -1), \ C^{1,3}_{\delta,q} (\varepsilon = 0).
\]

As we observed above the action of the quantum Lorentz group preserves these spaces. It justifies the notion of homogeneous spaces in the noncommutative situation.

We can directly define their generators using the horospheric description of \( \mathbb{M}^{1,3}_{\delta,q} \).

**Definition 6.1** The non-commutative Lobachevsky space \( \mathbb{I}_{\delta,q} (H_3) \), the non-commutative Imaginary Lobachevsky space \( \mathbb{II}_{\delta,q} (dS_3) \) and the non-commutative cone \( C^{1,3}_{q,\delta} \) are the associative unital algebras with an anti-involution and the defining relations

\[
ZH = q^{-\delta}HZ, \ Z^*H = q^\delta HZ^*,
\]

\[
ZZ^* = q^{2\delta-2}Z^*Z - \varepsilon q^{\delta-2}(1-q^2)H^{-2}.
\]

\[
(Z)^* = Z^*, \ H^* = H,
\]

\[
H_3 \sim \varepsilon = 1, \ dS_3 \sim \varepsilon = -1, \ C^{1,3}_{\delta,q} \sim \varepsilon = 0.
\]

In addition we define the non-commutative absolute.

**Definition 6.2** The non-commutative absolute \( \Xi_{\delta,q} \) is the associative algebra with two generators and the commutation relation

\[
ZZ^* = q^{-2+2\delta}Z^*Z.
\] (6.9)

6.3 Representations of the horospheric generators

For the time-like part \( \varepsilon = 1 \) we considered the space \( E^+_{\alpha,\rho} \). We have from (5.3), (6.1), (6.2)

\[
f(u)H = \rho f(uq^\delta), \quad (6.10)
\]

\[
f(u)Z = \alpha^{-1}\lambda f(uq^{-\delta}) - \frac{f(uq^{-2\delta})}{(1-q^2)}u^{-1}, \quad f(u)Z^* = -\alpha^{-1}q^{-\delta}\lambda uf(uq^{-\delta}),
\]

and \( K_2 = \rho^2q^{-2} \).

Similarly, the representation of the space-like part \( (\varepsilon = -1) \) of \( \mathbb{M}^{1,3}_{\delta,q} \) in the space \( E^-_{\alpha,\rho} \) assumes the the following form. The horospheric generators \( H, Z \) are represented as before (6.10), while

\[
f(u)Z^* = q^{-\delta}\lambda uf(uq^{-\delta}).
\]

The horospheric generators corresponding to the light cone \( C^{1,3}_{\delta,q} \) act in the space \( E_0 (5.8) \) as

\[
f(w)H = f(wq^\delta), \quad (6.11)
\]

\[
f(w)Z = q^{\delta-1}w f(wq^{-\delta}), \quad f(w)Z^* = q^{\delta-1}w^{-1}f(wq^{-1-\delta}). \quad (6.12)
\]

The module \( E^0 \) serves simultaneously as the module of the non-commutative absolute \( \Xi_{\delta,q} \). It is easy to derive from (6.12) that the generators \( Z, Z^* \) that satisfy (6.9) are represented as follows

\[
f(w)Z = w f(wq^{-1-\delta}), \quad f(w)Z^* = w^{-1}f(wq^{-1-\delta}). \quad (6.13)
\]
7 Non-commutative Minkowski space $\mathbb{M}_q^{2,2}$.

7.1 Definition

The space $\mathbb{M}_q^{2,2}$ is related to the Hopf algebra $\mathcal{U}_q(\text{SL}_2(\mathbb{R}))$. It implies that $|q| = 1$, $(q = \exp i\theta)$. In this case the quadratic relations (4.1) – (4.5) are valid for $\delta = 1$ only.

**Definition 7.1** The non-commutative 4d Minkowski space $\mathbb{M}_q^{2,2}$, $|q| = 1$ is the unital associative $\ast$-algebra with four generators $X_j$, $j = 1, \ldots, 4$ that satisfies the quadratic relations

\begin{align*}
X_1X_3 &= q^{-1}X_3X_1, \quad X_1X_2 = qX_2X_1, \quad (7.1) \\
[X_2, X_3] &= (q^{-1} - q)X_1X_4, \quad (7.2) \\
X_2X_4 &= qX_4X_2, \quad (7.3) \\
X_3X_4 &= q^{-1}X_4X_3, \quad (7.4) \\
[X_1, X_4] &= 0, \quad (7.5)
\end{align*}

and with the anti-involution

\begin{align*}
X_1^* &= X_1, \quad X_2^* = q^{-1}X_2, \quad X_3^* = qX_3, \quad X_4^* = X_4. \quad (7.6)
\end{align*}

The reflection equation describing these relations is reduced to the standard RTT form for the Hopf algebra $\mathcal{A}_q(\text{GL}(2, \mathbb{R}))$

\begin{equation}
R(q)X^{(1)}X^{(2)} = X^{(2)}X^{(1)}R^\dagger(q), \quad (7.7)
\end{equation}

since $R^{(2)} = \text{Id} \otimes \text{Id}$ for $\delta = 1$ and

\begin{equation}
R(q) = q^{-1}(E_1 \otimes E_1 + E_4 \otimes E_4) + (E_1 \otimes E_4 + E_4 \otimes E_1) + q^{-1}(1 - q^2)E_3 \otimes E_2.
\end{equation}

In the classical limit $\mathcal{A}_q(\text{GL}(2, \mathbb{R}))$ passes to the algebra $A(\text{GL}(2, \mathbb{R}))$. It allows to identify $\mathbb{M}_q^{2,2}$ with a non-commutative deformation of the Minkowski space $\mathbb{M}_q^{2,2}$.

From Lemma 4.1 we have the form of the Casimir elements of $\mathbb{M}_q^{2,2}$

\begin{align*}
K_1 &= X_1^{-1}X_4, \quad (7.8) \\
K_2 &= X_1X_4 - q^{-1}X_3X_2. \quad (7.9)
\end{align*}

Note that $K_2$ is not self adjoint. The self adjoint element is $\tilde{K}_2 = (1 + q^2)K_2$.

7.2 Quantum group $\mathcal{U}_q(\text{SL}_2(\mathbb{R})) \oplus \mathcal{U}_q(\text{SL}_2(\mathbb{R}))$ action on $\mathbb{M}_q^{2,2}$.

Consider the Hopf algebra $\mathcal{U}_q(\text{SL}_2)$ (4.20). The following conditions pick up the $\ast$-Hopf algebra $\mathcal{U}_q(\text{SL}_2(\mathbb{R}))$

\begin{align*}
A^* &= A, \quad B^* = -B, \quad C^* = -C
\end{align*}

with the untwisted coproduct (4.21)

\begin{align*}
\Delta(A) &= \Delta(A) = A \otimes A, \\
\Delta(B) &= A \otimes B + B \otimes D, \\
\Delta(C) &= A \otimes C + C \otimes D,
\end{align*}

\begin{equation}
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\end{equation}
and the untwisted antipode (4.22)

\[ S \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} D & -q^{-1}B \\ -qC & A \end{array} \right), \quad (7.10) \]

The Casimir element of \( \mathcal{U}_q(\text{SL}_2(\mathbb{R})) \) has the form

\[ \Omega_q := \frac{(q^{-1} + q)(A^2 + A^{-2}) - 4}{2(q^{-1} - q)^2} + \frac{1}{2}(BC + CB), \quad (7.11) \]

**Proposition 7.1** \( M_{2,2}^{q,2} \) is the module over the *-Hopf algebra \( \mathcal{U}_q(\text{SL}_2(\mathbb{R})) \).

**Proof**

Define the right action of \( \mathcal{U}_q(\text{SL}_2(\mathbb{R})) \) on \( M_{2,2}^{q,2} \)

\[ \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) . A = \left( \begin{array}{cc} q^{\frac{1}{2}}X_1 & q^{-\frac{1}{2}}X_2 \\ q^{\frac{1}{2}}X_2 & q^{-\frac{1}{2}}X_4 \end{array} \right), \quad (7.12) \]

\[ \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) . B = \left( \begin{array}{cc} 0 & X_1 \\ 0 & X_3 \end{array} \right), \quad (7.13) \]

\[ \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) . C = \left( \begin{array}{cc} X_2 & 0 \\ X_4 & 0 \end{array} \right). \quad (7.14) \]

The consistency of multiplication in \( M_{2,2}^{q,2} \) and the comultiplication in \( \mathcal{U}_q(\text{SL}_2(\mathbb{R})) \) follows from Proposition 4.1, since (7.12) – (7.14) comes from (4.27) – (4.29).

We also define the left action of \( \mathcal{U}_q(\text{SL}_2(\mathbb{R})) \) on \( M_{2,2}^{q,2} \)

\[ A. \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) = \left( \begin{array}{cc} q^{-\frac{1}{2}}X_1 & q^{-\frac{1}{2}}X_2 \\ q^{\frac{1}{2}}X_2 & q^{\frac{1}{2}}X_4 \end{array} \right), \]

\[ B. \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) = \left( \begin{array}{cc} -X_3 & -X_4 \\ 0 & 0 \end{array} \right), \]

\[ C. \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & X_4 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ -X_1 & -X_2 \end{array} \right). \]

It is also consistent with the multiplication in \( M_{2,2}^{q,2} \).

In addition from (7.6) and (7.10) one has

\[ S(u)^* . X_j^* = (X_j . u)^* \quad u = A, B, C. \]

\[ \square \]

### 7.3 Horospheric generators and homogeneous spaces

The non-commutative analog of the horospheric generators \((Z_2, H, Z_1, R)\) on \( M_{2,2}^{q,2} \) assumes the form

\[ X_1 = RH, \quad X_2 = RHZ_1, \quad X_3 = RZ_2H, \quad X_4 = R(Z_2HZ_1 + \varepsilon H^{-1}), \quad \varepsilon = \pm 1, \, 0. \quad (7.15) \]

\[ (7.16) \]
It follows from (7.6) that

\[ Z_1^* = Z_2, \quad H^* = H, \quad Z_1^* = Z_1, \quad R^* = R. \]

The defining relations

\[ Z_1 H = q^{-1} H Z_1, \quad Z_2 H = q H Z_2, \quad [R, H] = [R, Z_1] = [R, Z_2] = 0, \quad (7.17) \]
\[ [Z_1, Z_2] = \varepsilon(q - q^{-1}) H^{-2}. \]

yield the relations (7.1)–(7.5). The Casimir elements are

\[ K_2 = \varepsilon R^2, \quad K_1 = H^{-1}(Z_2 H Z_1 + \varepsilon H^{-1}). \quad (7.18) \]

The horospheric coordinates can be expressed in terms of \( X_j \) as

\[ H = R^{-1} X_1, \quad Z_1 = X_1^{-1} X_2, \quad Z_2 = X_3 X_1^{-1}, \quad R^2 = \varepsilon K_2. \quad (7.19) \]

In terms of the horospheric generators the right action of \( \mathcal{U}_q(SL_2(\mathbb{R})) \) takes the form

\[
\begin{align*}
Z_2.A &= Z_2, & H.A &= q^{\frac{1}{2}} H, & Z_1.A &= q^{-1} Z_1, & R.A &= R, \\
Z_2.B &= 0, & H.B &= 0, & Z_1.B &= q^{-\frac{1}{2}}, & R.B &= 0, & (7.20) \\
Z_2.C &= q^{\frac{1}{2}} H^{-2}, & H.C &= H Z_1, & Z_1.C &= -q^{\frac{1}{2}} Z_1^2, & R.C &= 0.
\end{align*}
\]

It follows from these relations that \( R \) is invariant with respect to the \( \mathcal{U}_q(SL_2(\mathbb{R})) \) action.

Let \( \mathcal{S}(\mathbb{M}^{2,2}) \) be the algebra of the ordered Schwartz functions on \( \mathbb{M}^{2,2}_q \). Consider an irreducible representation of the associative algebra \( \mathbb{M}^{2,2}_q \). We fix the value of the Casimir operator (7.18) \( K_2 = \varepsilon R^2, \quad R^2 = r^2 \in \mathbb{R}^+ \) and the ideal \( I_\varepsilon = \{ K_2 - \varepsilon R^2 = 0 \} \). It allows us to define the non-commutative analog of AdS\(^\pm\)\(3\) and the cone \( C^{2,2}_q \) as

\[ \mathcal{S}(\mathbb{M}^{2,2})/I_\varepsilon \sim \text{AdS}_3^{\pm}(\varepsilon = \pm 1), \quad C^{2,2}_q(\varepsilon = 0). \]

The direct description of these algebras in terms of the horospheric generators has the form

**Definition 7.2** The non-commutative \( \text{AdS}_3^{\pm} \) spaces and the non-commutative cone \( C^{2,2}_q \) are the associative unital algebras with the anti-involution and the defining relations

\[ Z_1 H = q^{-1} H Z_1, \quad Z_2 H = q H Z_2, \]
\[ [Z_1, Z_2] = \varepsilon(q - q^{-1}) H^{-2}, \]

where \( \text{AdS}_3^{\pm} \rightarrow (\varepsilon = \pm 1) \) and \( C^{2,2}_q \rightarrow (\varepsilon = 0) \).

### 7.4 The non-commutative Minkowski space \( \tilde{\mathbb{M}}^{2,2}_q \)

**Definition 7.3** The non-commutative 4d Minkowski space \( \tilde{\mathbb{M}}^{2,2}_q, \quad q \in (0, 1] \) is the unital associative \(*\)-algebra with four generators \( X_j, \ j = 1, \ldots, 4 \) that satisfies the quadratic relations (7.1)–(7.5) with the anti-involution

\[ X_1^* = X_4, \quad X_2^* = q X_3. \quad (7.21) \]
This algebra was considered in Ref. [16].

The commutation relations are the same as for \( M^{2,2}_q \) and the only difference is the anti-involution (compare (7.6) and (7.21) and the reality of \( q \). Therefore, we have the same Casimirs \( K_1 \) (7.8) and \( K_2 \) (7.9). It allows us to define the associative algebras

\[
\widetilde{\text{AdS}}^\pm_3 \rightarrow K_2 = \pm r^2 \neq 0, \\
\widetilde{C}^{2,2}_q \rightarrow K_2 = 0.
\]

Define the Hopf algebra \( U_q(SU(1,1)) \) by the anti-involution

\[
A^* = A, \quad B^* = -C, \quad C^* = -B. \tag{7.22}
\]

The Hopf algebra \( U_q(SU(1,1)) \oplus U_q(SU(1,1)) \) acts on \( \widetilde{M}^{2,2}_q \) in the similar way as \( U_q(SL_2(\mathbb{R})) \oplus U_q(SL_2(\mathbb{R})) \) acts on \( M^{2,2}_q \) (Proposition 7.1). The anti-involutions in \( U_q(SU(1,1)) \) and in \( \widetilde{M}^{2,2}_q \) are consistent.

8 Quasi-classical description of the non-commutative Minkowski spaces.

8.1 Quadratic Poisson algebras

Consider first the Minkowski space \( M^{1,3}_{\delta,q} \). In the limit \( q \rightarrow 1 \) we come in the first order of \( \theta = \log q \) to the Poisson algebra \( M^{1,3} \). It is a commutative algebras of functions on \( M^{1,3} \) equipped with the quadratic Poisson brackets

\[
\{x_j, x_k\}_2 = \lim_{\theta \rightarrow 0} \frac{X_jX_k - X_kX_j}{\theta}.
\]

Explicitly the brackets take the form

\[
\{x_1, x_2\}_2 = \delta x_1 x_2, \quad \{x_1, x_3\}_2 = -\delta x_1 x_3, \tag{8.1}
\]
\[
\{x_4, x_2\}_2 = (2 - \delta)x_2 x_4, \quad \{x_4, x_3\}_2 = -(2 - \delta)x_3 x_4, \tag{8.2}
\]
\[
\{x_2, x_3\}_2 = -2x_1 x_4, \quad \{x_1, x_4\}_2 = 0. \tag{8.3}
\]

In addition, there is the anti-involution on \( M^{1,3} \) that comes from the anti-involution on the algebra \( M^{1,3}_{\delta,q} \)

\[
\{x_j, x_k\}_2 = -\{x_j^*, x_k^*\}_2. \tag{8.4}
\]

It takes the form

\[
x_1^* = x_1, \quad x_4^* = x_4, \quad x_2^* = x_3.
\]

There are two Casimir functions

\[
k_1 = x_1 x_4 - x_2 x_3, \quad k_2 = x_1^\delta x_4^\delta. \tag{8.5}
\]
The Poisson algebra $\mathcal{M}^{1,3}$ can be represented in the form of the classical reflection equation. Let $\theta = \ln q$. The quantum $R$ matrices (4.8) have the expansion $R = Id \otimes Id + \theta r$, $R^{(2)} = Id \otimes Id + \theta r^{(2)}$, where

$$r = -(E_1 \otimes E_1 + E_4 \otimes E_4) - 2E_3 \otimes E_2,$$  \hspace{1cm} (8.6)

and

$$r^{(2)} = (\delta - 1)(E_1 \otimes E_4 + E_4 \otimes E_1)$$ \hspace{1cm} (8.7)

Then the reflection equation in the quasi-classical limit gives

$$\{x_1, x_2\}_2 = r(x_1 \otimes x_2) - (x_2 \otimes x_1)r^\dagger + x_2 r^{(2)} x_1 - x_1 r^{(2)} x_2.$$ \hspace{1cm} (8.8)

It can be checked that the Poisson structure (8.1)–(8.3) can be extracted from this equation.

We rewrite the Poisson algebra $\mathcal{M}^{1,3}$ in terms of the horospheric coordinates (2.6), (2.7)

$$\{h, z\}_2 = \delta h z, \quad \{h, \tilde{z}\}_2 = -\delta h \tilde{z}, \quad \{r, h\}_2 = \{r, z\}_2 = \{r, \tilde{z}\}_2 = 0,$$ \hspace{1cm} (8.9)

$$\{z, \tilde{z}\}_2 = 2(\delta - 1)\tilde{z} \tilde{z} - 2\varepsilon h^{-2}.$$ \hspace{1cm} (8.10)

The Casimir functions are

$$k_1 = \varepsilon r^2, \quad k_2 = h^{-2}(h^2|z|^2 + \varepsilon)\delta.$$ 

The symplectic leaves of this structure are two-dimensional surfaces

$$\varepsilon r^2 = c_1, \quad h^{-2}(h^2|z|^2 + \varepsilon)\delta = c_2.$$ 

If $\delta = 0, 2$ the symplectic leaves are the horospheres, i.e. the orbits of unipotent subgroups. For $\delta = 1$ the leaves are the orbits of groups conjugated to SU(2) ($\varepsilon = 1$) or SU(1,1) ($\varepsilon = -1$).

The quadratic Poisson algebra $\widetilde{\mathcal{M}}^{2,2}$ that arises in the case $\tilde{\mathcal{M}}^{2,2}_q$ coincides with (8.1) – (8.3) for $\delta = 1$. It differs from $\mathcal{M}^{1,3}$ by the anti-involution

$$x_1^* = x_4, \quad x_2^* = x_3.$$ 

In the case of $\tilde{\mathcal{M}}^{2,2}_q$ one should put $q = \exp i\theta$. Therefore we come to the Poisson algebra $\mathcal{M}^{2,2}$

$$\{x_1, x_2\}_2 = ix_1 x_2, \quad \{x_1, x_3\}_2 = -ix_1 x_3,$$

$$\{x_4, x_2\}_2 = ix_2 x_4, \quad \{x_4, x_3\}_2 = -ix_3 x_4,$$

$$\{x_2, x_3\}_2 = -2ix_1 x_4, \quad \{x_1, x_4\}_2 = 0.$$ 

with the anti-involution $x_j^* = x_j$. In the last two cases the classical reflection equations are simplified since $r^{(2)} = Id \otimes Id$.

### 8.2 Trihamiltonian structure

In addition to the quadratic brackets we introduce the linear and the canonical brackets in the space of functions on the Minkowski spaces. Consider for brevity the case $\mathcal{M}^{1,3}$. The linear brackets assume the form

$$\{x_1, x_2\}_1 = \delta x_2, \quad \{x_1, x_3\}_1 = -\delta x_3,$$ \hspace{1cm} (8.11)

$$\{x_4, x_2\}_1 = (\delta - 2)x_2, \quad \{x_4, x_3\}_1 = (2 - \delta)x_3,$$ \hspace{1cm} (8.12)
\{x_2, x_3\}_1 = -2(x_1 - x_4), \quad \{x_1, x_4\}_1 = 0. \quad (8.13)

For \(\delta = 1\) it is the Lie-Poisson brackets on the Lie algebra \(\text{gl}(2, \mathbb{R})\). Two Casimir functions of this structure are

\[
\begin{align*}
  k_1 &= (\delta - 2)x_1 - \delta x_4 \sim y_0 - (\delta - 1)y_3, \\
  k_2 &= x_2x_3 - (x_1 - x_4)^2 \sim y_1^2 - y_2^2 - 2y_3^2.
\end{align*}
\]

The linear brackets (8.11)-(8.13) can be reproduced in terms of the canonical variables \((v, u)\), \(\{v, u\} = 1\)

\[
\begin{align*}
  x_1 &= -\delta uv - \frac{1}{2}k_1, \quad x_2 = v, \quad x_3 = 2u^2v, \\
  x_4 &= (2 - \delta)uv - \frac{1}{2}k_1.
\end{align*}
\]

The quantization of the linear brackets is straightforward \(u \rightarrow U, \quad v \rightarrow V = \partial U\), where \(V\) is standing on the right.

The canonical brackets are

\[
\begin{align*}
  \{x_2, x_3\}_0 &= -2, \quad \{x_1, x_j\}_0 = \{x_4, x_j\}_0 = 0. \quad (8.14)
\end{align*}
\]

Note that the both Poisson structure are anti-invariant with respect to the involution.

**Lemma 8.1** The Poisson structures on \(\mathcal{M}^{1,3}\) are compatible.

It means that their linear combination

\[
\{ , \}_\lambda = \{ , \}_2 + \lambda \{ , \}_1 + \lambda^2 \{ , \}_0 \quad \lambda \in \mathbb{C}
\]

is again a Poisson brackets. To prove it we introduce the new algebra \(\mathcal{M}^{1,3}_\lambda\) with the shifted generators

\[
\begin{align*}
  x_1 &\rightarrow x_1 + \lambda, \quad x_4 \rightarrow x_4 - \lambda, \quad x_2 \rightarrow x_2, \quad x_3 \rightarrow x_3. \quad (8.16)
\end{align*}
\]

This shift does not spoil the Jacobi identity of the brackets and we come to (8.15). \(\square\)

## 9 The Laplace operator and its eigen-functions.

In this section we consider in details the case \(\mathcal{M}^{1,3}_{\delta,q}\) and shortly reproduce the similar construction for \(\mathcal{M}^{2,2}_q\). The case \(\widetilde{\mathcal{M}}^{2,2}_q\) will not be considered in this paper.

### 9.1 The Laplace operator on \(\mathcal{M}^{1,3}_{\delta,q}\)

Consider the action of \(\mathcal{U}^{(s)}_q(\text{SL}(2, \mathbb{C}))\) on the ordered monomials \(w(m, k, l, n) = X_2^m X_1^k X_4^l X_2^n\). It follows from (4.27) - (4.30) that

\[
\begin{align*}
  w(m, k, l, n).A &= q^\frac{m+k-l-n}{2} w(m, k, l, n), \quad w(m, k, l, n).A^* = q^\frac{(\delta-1)(m-k+l-n)}{2} w(m, k, l, n), \\
  w(m, k, l, n).B &= q^\frac{m+k-l-n+1}{2} -\delta(n-1) \frac{1-q^{2n}}{1-q^2} w(m, k + 1, l, n - 1) + \\
  &\quad -q^\frac{m+k-l-n+1}{2} \frac{1-q^{2l}}{1-q^2} w(m + 1, k, l - 1, n), \quad (9.1)
\end{align*}
\]
Proposition 9.1

The action of the Laplace operator on \( M_{b,q}^{1,3} \) assumes the form

\[
\Delta_q = \left\{ -q^{1-\delta} D_{X_1} D_{X_2} \right\}
\]

\[
f(X_3, X_1, X_4, X_2) \Delta_q = \left\{ \frac{1}{(q - q^{-1})^2} q^{-1} (T_{X_1}^{-1} T_{X_2}^{-1} T_{X_3}^{-1} T_{X_4}^{-1} - T_{X_1}^{-1} T_{X_2} T_{X_3} T_{X_4}^{-1}) + q^{1-\delta} \right\} f(X_3, X_1, X_4, X_2)
\]

Proof

One can define the action of the Casimir element on \( \mathcal{U}_q^{(s)}(SL(2, \mathbb{C})) \) (4.25) on the ordered monomials using (9.1). Since

\[
w(m, k, l, n) \Delta_q = \frac{m + k + l + n}{2} w(m, k - 1, l, n)
\]
we have from (9.4)
\[ w(m, k, l, n) \Delta_q = \]
\[ \frac{1}{(1 - q^2)^2} \left[ q^{m-k-l-n+1} + q^{m-k+l+n+3} + q^{m+k-l+n+3} + q^{-m-k-l+n+1} \right. \]
\[ - q^{-m-k-l-n+1} - q^{m+k+l+n+3} - q^{m-k-l+n+1} - q^{-m-k-l+n+1} \]
\[ + q^{-m-k+l+1+\delta(k-l+1)} \frac{(1 - q^{2m})(1 - q^{2n})}{(1 - q^2)^2} w(m - 1, k + 1, l + 1, n - 1) \]
\[ + q^{m-k-3l+n+5-\delta(k-l+1)} \frac{(1 - q^{2k})(1 - q^{2l})}{(1 - q^2)^2} w(m + 1, k - 1, l - 1, n + 1) . \]

This relation implies (9.5). \( \square \)

**Remark 9.1** *In the classical limit* \( \lim_{q \to 1} \Delta_q = \Delta \) (3.3).

### 9.2 The Laplace operator on \( \mathbb{M}^{1,3}_{\delta_q} \) in terms of the horospheric generators

Define the ordered monomial
\[ \dot{w}(m, k, n) = (Z^*)^m H^k Z^n , \]
and let
\[ F(Z^*, H, Z) = \sum_{m, k, n} a_{m, k, n} \dot{w}(m, k, n) . \]

Consider the action of the operator \( \Delta_q \) on the Schwartz space (4.31).

**Proposition 9.2** *The action of the Casimir operator \( \Delta_q \) in terms of horospheric generators takes the form*

\[ F(Z^*, H, Z, R) \Delta_q = \frac{1}{(1 - q^2)^2} \left[ q^{-1} T_H - q^2 + q T_H^{-1} \right] F(Z^*, H, Z, R) \]
\[ + q^{1-\delta} D_Z \cdot D_{Z^*} H^{-2} T_H^{\delta-1} F(Z^*, H, Z, R) \frac{1}{\delta} . \]

**Proof**

We have already define the action of \( \mathcal{U}_q^{(s)}(\text{SL}(2, \mathbb{C})) \) on the horospheric generators (6.6). Then the action of \( A, A^*, B, C \) on the ordered monomial \( \dot{w}(m, k, n) \) takes the form
\[ \dot{w}(m, k, n) A = q^{-n+\frac{1}{2}} \dot{w}(m, k, n) , \]
\[ \dot{w}(m, k, n) (A^*)^s = q^{(1-\delta)(-2m+k)} \dot{w}(m, k, n) , \]
\[ \dot{w}(m, k, n) B = q^{-n+\frac{k+1}{2}} \frac{1 - q^{2n}}{1 - q^2} \dot{w}(m, k, n - 1) , \]
\[ \dot{w}(m, k, n) C = q^{n - \frac{3(k-1)}{2}} q^{\delta(k-1)} \frac{1 - q^{2m}}{1 - q^2} \dot{w}(m - 1, k - 2, n) - q^{-n+\frac{k+3}{2}} \frac{1 - q^{2n-2k}}{1 - q^2} \dot{w}(m, k, n + 1) . \]

Thus, we have
\[ \dot{w}(m, k, n) \Omega_q = \frac{q^{-k+1}(1 - q^{k+1})^2}{(1 - q^2)^2} \dot{w}(m, k, n) + \]
\[ + q^{(\delta-1)(k-1)} \frac{(1 - q^{2m})(1 - q^{2n})}{(1 - q^2)^2} \dot{w}(m - 1, k - 2, n - 1) . \]
On the other hand
\[ \hat{w}(m, k, n) R^\alpha.M = q^{\alpha} \hat{w}(m, k, n) R^\alpha, \]
and therefore
\[ \hat{w}(m, k, n) R^\alpha.\Omega_q.M = \left[ \frac{\alpha}{2} \right]^2 q^2 \hat{w}(m, k, n) R^\alpha.\]
it follows from (9.4) that
\[ \hat{w}(m, k, n) R^\alpha.\Delta_q = q^{(\delta-1)(k-1)}(1 - q^{2m})(1 - q^{2n}) \hat{w}(m - 1, k - 2, n - 1) R^\alpha. \]
In this way we come to (9.6). \(\Box\)

**Remark 9.2** In the classical (9.6) takes the form of the Laplace operator in horospheric coordinates \(\lim_{q \to 1} \Delta_q = \Delta\) (3.3).

Our main goal is to find the eigen-functions of \(\Delta_q\)
\[ F_\nu(Z^*, H, Z, R).\Delta_q = \left[ \frac{\nu}{2} \right]^2 q^2 F_\nu(Z^*, H, Z, R). \] (9.10)
These functions are expressed through the \(q\)-exponents (A.1) and the three types of \(q\)-cylindric functions (A.4). For \(|q| \neq 1\) they can be defined by the expansion
\[ Z_\nu^{(j)}(z) = \frac{1}{(1 - q^2)^\alpha \Gamma_{q^2} (\alpha + 1)} \sum_{m=0}^{\infty} q^{(2-\delta)(m+\alpha)} z^{\alpha+2m} (q^2, q^2)_m (q^{2\alpha+2}, q^2)_m z^{2\alpha+2m}, \]
where \(\Gamma_{q^2} (\alpha + 1)\) is the \(q^2\)-\(\Gamma\)-function (A.3). We assume that \(\frac{|z|}{2(1-q^2)} < 1\) for \(\delta = 1\). It can be checked that \(Z_\nu^{(j)}\) satisfies the difference equation (A.4).

The non-commutative analog of the horospheric elementary harmonics (3.5) has the following form

**Proposition 9.3** The basic solutions of (9.10) are defined as
\[ F_\nu(Z^*, H, Z, R) = e(\bar{\mu}Z^*) V_\alpha(H) e(\mu Z) \Xi_{\nu,\alpha}(R), \quad (\varepsilon \neq 0), \] (9.12)
where \(\mu, \alpha \in \mathbb{C},\)
\[ V_\alpha(H) = H^{-1} Z_\nu^{(j)}(2(-\varepsilon)^{\frac{1}{2}} |\mu| q^{-\frac{\delta}{2}} H^{-1}), \]
\[ \Xi_{\nu,\alpha}(R) = \frac{1}{R} Z_\nu^{(3)}(2q^{1-\frac{\nu}{2}} - 1 - q^{\nu} R). \]

**Proof**
Represent the solutions in the form
\[ F_\nu(Z^*, H, Z, R) = V_\alpha(Z^*, H, Z) \Xi_{\nu,\alpha}(R). \] (9.13)
Substituting it in (9.10) and using the comultiplication relations (4.21) we find
\[ (V_\alpha(Z^*, H, Z) \Omega_q) (\Xi_{\nu,\alpha}(R) \Omega_q) R^{-2} - (V_\alpha(Z^*, H, Z) \Omega_q M) (\Xi_{\nu,\alpha}(R) \Omega_q M) R^{-2} \]
It follows from (9.7) - (9.9) that it can be rewritten as

\[(V_\alpha(Z^*, H, Z), \Omega_q) \Xi_{\nu,\alpha}(R) R^{-2} - V_\alpha(Z^*, H, Z) (\Xi_{\nu,\alpha}(R), \Omega_q, M R^{-2}) - \left[ \frac{\nu^2}{2} \right] V_\alpha(Z^*, H, Z) \Xi_{\nu,\alpha}(R) = 0.\]

In this way we come to the equations

\[V_\alpha(Z^*, H, Z), \Omega_q - \frac{q^{-\alpha+2} - 2q^2 + q^{\alpha+2}}{(1 - q^2)^2} V_\alpha(Z^*, H, Z) = 0, \quad (9.14)\]

and

\[\Xi_{\nu,\alpha}(R), \Omega_q, M + \left( \frac{q^{-\nu+2} - 2q^2 + q^{\nu+2}}{(1 - q^2)^2} R^2 - \frac{q^{-\alpha+2} - 2q^2 + q^{\alpha+2}}{(1 - q^2)^2} \right) \Xi_{\nu,\alpha}(R) = 0. \quad (9.15)\]

From (9.2) and (9.3) one rewrites the equation (9.15) as

\[q^2 \Xi_{\nu,\alpha}(q^2 R) - (q^2 + q^{-2}) \Xi_{\nu,\alpha}(R) + q^2 \Xi_{\nu,\alpha}(q R) = q^2(1 - q^2)^2 \Xi_{\nu,\alpha}(R).\]

Put \(z = 2q^2 \Xi(1 - q^2)(1 - q^2)^{-1} R\). Then we come to (A.4) with \(\delta = 1\) and \(Z^{(3)}(z) = \Xi_{\nu,\alpha}(R).\)

Consider now (9.14) and put

\[\tilde{V}_\alpha(Z^*, H, Z) = e(\tilde{\mu} Z^*) V_\alpha(H) e(\mu Z). \quad (9.16)\]

Assume that \(\varepsilon = \pm 1\) and

\[V_\alpha(H) = \sum_{k=0}^{\infty} c_k \frac{(1 - q^2)^{2k-2}}{(q^2, q^2)^k (q^{2\alpha+2}, q^2)^k} H^{-\alpha-2k-1}.\]

Substituting this expression and (A.1) in (9.16) we express \(\tilde{V}_\alpha\) in terms of monomials \(\tilde{\mu}(m, k, n)\).

Using the action of \(\Omega_q\) on monomials (9.8) we obtain

\[e(\tilde{\mu} Z^*) \sum_{k=0}^{\infty} c_k \frac{(1 - q^2)^{2k-2}}{(q^2, q^2)^k (q^{2\alpha+2}, q^2)^k} q^{-\alpha-2k+2} (1 - q^2) (1 - q^{2\alpha+2k}) H^{-\alpha-2k-1} e(\mu Z)\]

\[-\varepsilon |\mu|^2 e(\tilde{\mu} Z^*) \sum_{k=0}^{\infty} c_k \frac{(1 - q^2)^{2k}}{(q^2, q^2)^k (q^{2\alpha+2}, q^2)^k} q^{-(\alpha+2k+2)(\delta-1)} H^{-\alpha-2k-3} e(\mu Z) = 0.\]

Then the coefficients \(c_k\) satisfy the recurrence relation

\[c_{k+1} = -\varepsilon \tilde{\mu} \mu c_k q^{2\alpha+4k+2-\delta(\alpha+2k+2)}.\]

Taking \(c_0 = 1\) we find

\[c_k = (-\varepsilon)^k |\mu|^{2k} q^{-(\delta-k)(k+\alpha)-\delta k}.\]

Then

\[H_\alpha(H) = q^{\frac{4\beta}{\alpha}} |\mu|^{-\alpha} \sum_{k=0}^{\infty} (-\varepsilon)^k q^{(2-\delta)(k+\alpha)} (1 - q^2)^{2k} (q^2, q^2)^k (q^{2\alpha+2}, q^2)^k q^{-\frac{4}{\alpha+2k} (\tilde{\mu} \mu)^\frac{1}{2} + k} H^{-\alpha-2k-1}.\]

These series coincide with (9.11) up to a constant multiplier after the replacement \(2z = (-\varepsilon)^{\frac{3}{2}} q^{-\frac{3}{2}} H^{-1}.\) 

\(\square\)
Remark 9.3 In the classical limit we come to Proposition 3.1
\[ \lim_{q \to 1} e(\mu Z^\alpha) V_\alpha(H) e(\mu Z) \Xi_{\nu, \alpha}(R) = \exp(i\mu z + i\bar{\mu} \bar{z}) \nu_\alpha(h) \chi_{\nu, \alpha}(r). \]

As in the classical situation one can restrict the operator $\Delta_q$ on the non-commutative homogeneous spaces.

Corollary 9.1 The restrictions of $\Delta_q$ assume the form
\[ L_{q, \delta}, (H_3) : \Delta_q = \frac{1}{(1 - q^2)^2} [q^3 T_H - 2q^2 + qT_H^{-1}] + q^{1-\delta} D_Z \cdot D_Z H^{-2} T_H^{\delta-1}, \]
\[ IL_{q, \delta}, (dS_3) : \Delta_q = \frac{1}{(1 - q^2)^2} [q^3 T_H - 2q^2 + qT_H^{-1}] - q^{1-\delta} D_Z \cdot D_Z H^{-2} T_H^{\delta-1}. \]

Then we obtain the non-commutative deformations of the classical formulas (3.11), (3.12).

Corollary 9.2 The basic harmonics on the non-commutative $H_3$, $dS_3$ and the light-cone $C_{q, \delta}^{1,3}$ are
\[ F_\nu(\bar{z}, h, z) = e(\mu Z^\alpha) H^{-1} Z^{(j)}_{\alpha} (2i\varepsilon^2 |\mu|H^{-1}) e(\mu Z), \quad \varepsilon = \pm 1, \tag{9.17} \]
and
\[ F_\nu(\bar{z}, h, z) = e(\mu Z^\alpha) H^{\alpha-1} e(\mu Z), \quad \varepsilon = 0. \tag{9.18} \]
Here $\nu^2 = \alpha^2 - 1$.

9.3 The case $\mathbb{M}_q^{2,2}$

The action of $U_q(SL(2, \mathbb{R}))$ on $\mathbb{M}_q^{2,2}$ coincides with with the action of $U_q(SL(2, \mathbb{C}))$ on $\mathbb{M}_{q, \delta}^{1,3}$. The only difference is that $|q| = 1$ in the former case. Then it is straightforward to repeat the calculations of the last two subsections.

Proposition 9.4 The Laplace operator $\Delta_q$ on $\mathbb{M}_q^{2,2}$ coincides with $\Delta_q$, defined in Proposition 9.1 with $\delta = 1$.

Consider the horospheric description of $\mathbb{M}_q^{2,2}$ from 8.3. Let
\[ F(Z_2, H, Z_1) = \sum_{m,k,n} a_{m,k,n} \hat{w}(m, k, n). \]
where $\hat{w}(m, k, n) = Z_2^m H^k Z_1^n$ and $F(Z_2, H, Z_1)$ belongs to the Schwartz space. Define the action of the operator $\Delta_q$.

Proposition 9.5 The action of the Casimir operator $\Delta_q$ on $\mathbb{M}_q^{2,2}$ in terms of horospheric generators takes the form
\[ F(Z_2, H, Z_1, R).\Delta_q = \frac{1}{(1 - q^2)^2} [q^3 T_H - 2q^2 + qT_H^{-1}] F(Z_2, H, Z_1, R) \tag{9.19} \]
\[ + \varepsilon D_{Z_2} D_{Z_1} \frac{1}{H^{-2}} F(Z_2, H, Z_1, R) \]
\[ - \frac{1}{(1 - q^2)^2} [q^3 T_R - 2q^2 + qT_R^{-1}] F(Z_2, H, Z_1, R). \]
Now we investigate the basic eigen-functions

\[ F_\nu(Z_2, H, Z_1, R) \Delta_q = \left[ \frac{\nu^2}{2} \right] F_\nu(Z_2, H, Z_1, R) . \]

Again we come to the equation (A.4) for \( q \)-cylindrical functions but with \(|q| = 1\). The series (9.11) are ill-defined in this case. There exists another approach to the theory of \( q \)-cylindrical functions based on integral representations for an arbitrary \( j \) and generic \( q \in \mathbb{C} \) [15]. As above we denote these functions \( Z_{\alpha}^{(j)}(z) \).

**Proposition 9.6** The basic solutions on \( M_4^{2,2} \) are

\[ F_\nu(Z^*, H, Z, R) = e(\mu_2 Z_2) V_{\alpha}(H) e(\mu_1 Z_1) \Xi_{\nu,\alpha}(R), \ (\varepsilon \neq 0), \quad (9.20) \]

where \( \mu_1, 2 \in \mathbb{R}, \alpha \in \mathbb{C}, \)

\[ V_{\alpha}(H) = H^{-1} Z_{\alpha}^{(j)}(2(-\varepsilon)^{\frac{1}{2}} \sqrt{\mu_1 \mu_2} q^{-\frac{1}{2}} H^{-1}) \]

\[ \Xi_{\nu,\alpha}(R) = \frac{1}{R} Z_{\alpha}^{(3)}(2q^{1-\nu} - \frac{1 - q^\nu}{1 - q^2} R) \]

Restricting these solutions to \( R = r = \text{const} \) we define the Klein-Gordon operator on non-commutative AdS\( ^+_3 \)

\[ \Delta_q = \frac{1}{(1 - q^2)^2} [q^3 T_H - 2q^2 + q^{-1} H^{-1}] + \varepsilon D_{Z_2}D_{Z_1} H^{-2}. \]

The basic functions on the homogeneous spaces have the form

\[ F_\nu(Z_2, H, Z_1) = e(\mu_2 Z_2) H^{-1} Z_{\alpha}^{(j)}(2i\varepsilon^{\frac{1}{2}} \sqrt{\mu_1 \mu_2} H^{-1}) e(\mu_1 Z_1), \ \varepsilon = \pm 1, \]

\[ F_\nu(Z_2, H, Z_1) = e(\mu_2 Z_2) H^{\alpha - 1} e(\mu_1 Z_1), \ \varepsilon = 0, \ (\nu^2 = \alpha^2 - 1). \]

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11 Appendix

Here we reproduce some results from [18].

The \( q^2 \)-exponent \( e_{q^2}(x) \)

is

\[ e_{q^2}(x) = \frac{1}{(x, q^2)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q^2, q^2)_n}, \ |x| < 1. \quad (A.1) \]

it satisfies the difference equation

\[ D_x e_{q^2}(\mu x) = \mu e_{q^2}(\mu x), \]
where
\[ D_x f(x) = \frac{f(x) - f(q^2 x)}{1 - q^2} x^{-1}. \quad (A.2) \]

The \( q^2 \) Gamma function
\[ \Gamma_{q^2}(x) = \frac{(q^2; q^2)_{\infty}}{(q^{2x}; q^2)_{\infty}} (1 - q^2)^{1-x}. \quad (A.3) \]

The \( q \)-cylindric functions \( Z_{\alpha}^{(j)} \) are solutions of the difference equation
\[ Z_{\alpha}^{(j)}(q^{-1} z) - (q^{-\alpha} + q^\alpha) Z_{\alpha}^{(j)}(z) + Z_{\alpha}^{(j)}(qz) = q^{-\delta} (1 - q^2)^2 z^2 Z_{\alpha}^{(j)}(q^{1-\delta} z). \quad (A.4) \]

In [18] the \( q \)-cylindric functions are defined for \( j = 1, 2 \). This equation is the second order difference equation for \( j = 1, 2, 3 \). There exist the analytic continuation of \( Z_{\alpha}^{(j)} \) on the complex plane \( j(\delta) \) [15].

The Jackson integral
is the series
\[ \langle f \rangle = \int d_{q^2} u f(u) = (1 - q^2)^2 \sum_{m \in \mathbb{Z}} q^{2m}[f(q^{2m}) + f(-q^{2m})]. \quad (A.5) \]

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