Gradient potential estimates in elliptic obstacle problems with Orlicz growth

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Abstract
In this paper, we consider the solutions of the non-homogeneous elliptic obstacle problems with Orlicz growth involving measure data. We first establish the pointwise estimates of the approximable solutions to these problems via fractional maximal operators. Furthermore, we obtain pointwise and oscillation estimates for the gradients of solutions by the non-linear Wolff potentials, and these yield results on $C^{1,\alpha}$-regularity of solutions.

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1 Introduction and main results
In this paper, we consider the non-homogeneous elliptic obstacle problems with Orlicz growth and they are related to measure data problems of the type

\[ \text{div} (a(x, Du)) = \mu \quad \text{in} \quad \Omega \] (1.1)

where $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ is a bounded open set and $\mu$ is a bounded Radon measure on $\Omega$. Moreover we assume that $\mu(\mathbb{R}^n \setminus \Omega) = 0$ and $a = a(x, \eta) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable for each $x \in \Omega$ and differentiable for almost every $\eta \in \mathbb{R}^n$ and there exist constants $0 < v \leq \lambda$.
$1 \leq L < +\infty$ such that for all $x \in \Omega, \eta, \lambda \in \mathbb{R}^n$,
\begin{equation}
\begin{aligned}
D_\eta a(x, \eta) \lambda \cdot \lambda & \geq v \frac{g(|\eta|)}{|\eta|} |\lambda|^2, \\
|a(x, \eta)| + |\eta||D_\eta a(x, \eta)| & \leq Lg(|\eta|),
\end{aligned}
\end{equation}
where $D_\eta$ denotes the differentiation in $\eta$ and $g(t) : [0, +\infty) \to [0, +\infty)$ satisfies
\begin{equation}
\begin{aligned}
& g(t) = 0 \iff t = 0, \\
& g(\cdot) \in C^1(\mathbb{R}^+), \\
& 1 \leq i_g =: \inf_{t > 0} \frac{tg'(t)}{g(t)} \leq \sup_{t > 0} \frac{tg'(t)}{g(t)} =: s_g < \infty.
\end{aligned}
\end{equation}
We define
\begin{equation}
G(t) := \int_0^t g(\tau) \, d\tau \quad \text{for } t \geq 0.
\end{equation}
It is straightforward to see that $G(t)$ is convex and strictly increasing. The standard example for $G(\cdot)$ is
\begin{equation}
G(t) = \int_0^t (\mu + s^2)^{\frac{p-2}{2}} \, ds
\end{equation}
with $\mu \geq 0, p \geq 2$, then (1.2) is reduced to $p$-growth condition.

The obstacle condition that we impose on the solution is of the form $u \geq \psi$ a.e. on $\Omega$, where $\psi \in W^{1,G}(\Omega) \cap W^{2,1}(\Omega)$ is a given function and $G$ is defined as (1.4). In the classical setting, we consider an inhomogeneity $f \in L^1(\Omega) \cap (W^{1,G}(\Omega))'$, where $(W^{1,G}(\Omega))'$ is the dual of $W^{1,G}(\Omega)$, the obstacle problem can be formulated by the variational inequality
\begin{equation}
\int_{\Omega} a(x, Du) \cdot D(v - u) \, dx \geq \int_{\Omega} f(v - u) \, dx
\end{equation}
for all functions $v \in u + W^{1,G}_0(\Omega)$ that satisfy $v \geq \psi$ a.e. on $\Omega$. However, we are more interested in solutions to obstacle problems with measure data in the sense that we want to replace the inhomogeneity $f$ by a bounded Radon measure $\mu$. And the solutions to the obstacle problems can be obtained by approximation with solutions to variational inequalities (1.5). The definition of approximable solutions is described precisely in Definition 1.3.

In fact, the nonlinear operator $a(\cdot, \cdot)$ is built upon the model case
\begin{equation}
\text{div} \left( \kappa(x) \frac{g(|Du|)}{|Du|} Du \right) = \mu \quad \text{in } \Omega
\end{equation}
where $\kappa : \Omega \to [c, +\infty)$ is bounded measurable and separated from zero function, and $g$ satisfies (1.3). According to Hwang and Lieberman [15], the case $i_g \geq 1$ is known as the degenerate case, and the case $0 < i_g \leq s_g \leq 1$ is known as the singular case. This type of elliptic equations were first introduced by Lieberman [22] and moreover he proved $C^{\alpha}$- and $C^{1,\alpha}$-regularity of the solutions for these elliptic equations in his paper. Since then there has been significant advances in regularity theory for this class of equations, we refer to these article [3, 6–8, 23, 28].

In this work, we are interested in the connections between regularity properties of the solutions and Wolff potentials of data $\mu$ and $\psi$. The Wolff potential was introduced by Maz’ya and Havin [25] and the relevant fundamental contributions were attributed to Hedberg and Wolff [14]. The last years have seen important developments in nonlinear potential theory, with a deeper analysis of the interactions between fine properties of Sobolev functions, regularity theory of nonlinear elliptic equations and nonlinear potentials. The fundamental
results due to Kilpeläinen and Malý [16, 17] are the pointwise estimates of solutions to the nonlinear equations of $p$-Laplace type via the Wolff potentials. Later these results have been extended to a general setting by Trudinger and Wang [31, 32] by means of a different approach. Further results for the gradient of solutions have been achieved by Duzaar, Kuusi and Mingione [10, 11, 18, 24]. Moreover, Scheven [29, 30] extended the above-mentioned results to elliptic obstacle problems with $p$-growth. For more results, please see [19–21, 26].

As for the elliptic equations with Orlicz growth, Baroni [2] obtained pointwise gradient estimates for solutions of equations with constant coefficients by the nonlinear potentials. Later, these results were upgraded by Xiong and Zhang in [33] to elliptic obstacle problems with measure data. Our goal in this paper is to obtain the pointwise and oscillation estimates for the gradient of solutions to obstacle problems with Dini-$BMO$ coefficients. The idea of the proof goes back to Kuusi and Mingione [18, 24]. We first derive excess decay estimates for solutions of obstacle problems by using some comparison estimates. Then iterating the resulting estimates, we give the pointwise estimates of fractional maximal operators. Finally, these estimates allow to draw conclusions about pointwise and oscillation estimates for the gradients of solutions.

Next, we summarize our main results. We begin by presenting some definitions, notations and assumptions.

**Definition 1.1** A function $B : [0, +\infty) \to [0, +\infty)$ is called a Young function if it is convex and $B(0) = 0$.

**Definition 1.2** Assume that $B$ is a Young function, the Orlicz class $K^B(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ satisfying
\[
\int_{\Omega} B(|u|) \, d\xi < \infty.
\]

The Orlicz space $L^B(\Omega)$ is the linear hull of the Orlicz class $K^B(\Omega)$ with the Luxemburg norm
\[
\|u\|_{L^B(\Omega)} := \inf \left\{ \alpha > 0 : \int_{\Omega} B\left(\frac{|u|}{\alpha}\right) \, d\xi \leq 1 \right\}.
\]

Furthermore, the Orlicz-Sobolev space $W^{1,B}(\Omega)$ is defined as
\[
W^{1,B}(\Omega) = \left\{ u \in L^B(\Omega) \cap W^{1,1}(\Omega) \mid Du \in L^B(\Omega) \right\}.
\]

Here, $D$ stands for gradient. The space $W^{1,B}(\Omega)$, equipped with the norm $\|u\|_{W^{1,B}(\Omega)} := \|u\|_{L^B(\Omega)} + \|Du\|_{L^B(\Omega)}$, is a Banach space. Clearly, $W^{1,B}(\Omega) = W^{1,p}(\Omega)$, the standard Sobolev space, if $B(t) = t^p$ with $p \geq 1$.

Note that for the Luxemburg norm there holds the inequality
\[
\|u\|_{L^B(\Omega)} \leq \int_{\Omega} B(|u|) \, d\xi + 1.
\]

The subspace $W^{1,B}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,B}(\Omega)$. The above properties about Orlicz space can be found in [27].

For every $k > 0$ we let
\[
T_k(s) := \begin{cases} 
  s & \text{if } |s| \leq k, \\
  k \, sgn(s) & \text{if } |s| > k.
\end{cases}
\]
Moreover, for given Dirichlet boundary data \( h \in W^{1,G}(\Omega) \), we define

\[
T^1_{h}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : T_k(u - h) \in W^{1,G}_0(\Omega) \text{ for all } k > 0 \right\}.
\]

We now give the definition of approximable solutions.

**Definition 1.3** Suppose that an obstacle function \( \psi \in W^{1,G}(\Omega) \), measure data \( \mu \in M_b(\Omega) \) and boundary data \( h \in W^{1,G}(\Omega) \) with \( h \geq \psi \) a.e. are given. We say that \( u \in T^1_{h}(\Omega) \) with \( u \geq \psi \) a.e. on \( \Omega \) is a limit of approximating solutions of the obstacle problem \( OP(\psi; \mu) \) if there exist functions \( f_i \in (W^{1,G}(\Omega))^\prime \cap L^1(\Omega) \) with \( f_i \rightharpoonup \mu \) in \( M_b(\Omega) \) as \( i \to +\infty \) satisfies

\[
\limsup_{i \to +\infty} \int_{B_R(x_0)} |f_i| dx \leq |\mu|(B_R(x_0))
\]

and solutions \( u_i \in W^{1,G}(\Omega) \) with \( u_i \geq \psi \) of the variational inequalities

\[
\int_{\Omega} a(x, Du_i) \cdot D(v - u_i) dx \geq \int_{\Omega} f_i(v - u_i) dx
\]

for \( \forall v \in u_i + W^{1,G}_0(\Omega) \) with \( v \geq \psi \) a.e. on \( \Omega \), such that for \( i \to +\infty \),

\[
u_i \to u \text{ a.e. on } \Omega
\]

and

\[
u_i \to u \text{ in } W^{1,1}(\Omega).
\]

For the inequalities (1.6) with constant coefficients, the existence of approximating solutions converging in the sense of the above definition has been proved in our preceding work [33]. Then the existence in this paper can be obtained by minor adjustments.

Let us next turn our attention to the classical non-linear Wolff potential which is defined by

\[
W^\mu_{\beta,p}(x, R) := \int_0^R \left( \frac{|\mu|(B_\rho(x))}{\rho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\rho}{\rho}
\]

for parameters \( \beta \in (0, n] \) and \( p > 1 \). We also abbreviate

\[
W_{\beta,\psi}(x, R) := \int_0^R \left( \frac{|\mu|(B_\rho(x))}{\rho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\rho}{\rho}
\]

with \( D\Psi(B_\rho(x)) := \int_{B_\rho(x)} \left( \frac{g(D\psi)}{|D\psi|^2} \right) (D^2\psi + 1) d\xi. \)

Now we recall the definitions of the centered maximal operators as follows.

**Definition 1.4** Let \( \beta \in [0, n] \), \( x \in \Omega \) and \( R < \text{dist}(x, \partial\Omega) \), and let \( u \) be an \( L^1(\Omega) \)-function or a measure with finite mass; the restricted fractional \( \beta \) maximal function of \( u \) is defined by

\[
M_{\beta, R}(u)(x) := \sup_{0 < r \leq R} r^\beta \frac{|u|(B_r(x))}{|B_r(x)|} = \sup_{0 < r \leq R} r^\beta \int_{B_r(x)} |u| d\xi.
\]
Note that when $\beta = 0$ the one defined above is the classical Hardy-Littlewood maximal operator.

Moreover, we define

$$\overline{M}_{\beta,R}(\psi)(x) := \sup_{0<r \leq R} r^\beta \frac{D\psi(B_r(x))}{|B_r(x)|} = \sup_{0<r \leq R} r^\beta \int_{B_r(x)} \left( \frac{g(|D\psi|)}{|D\psi|} |D^2\psi| + 1 \right) \, d\xi.$$  

**Definition 1.5** Let $\beta \in [0, n]$, $x \in \Omega$ and $R < \text{dist}(x, \partial \Omega)$, and let $u$ be an $L^1(\Omega)$-function or a measure with finite mass; the restricted sharp fractional $\beta$ maximal function of $u$ is defined by

$$M^\#_{\beta,R}(u)(x) := \sup_{0<r \leq R} r^{-\beta} \int_{B_r(x)} |u - (u)_{B_r(x)}| \, d\xi.$$  

When $\beta = 0$ the one defined above is the Fefferman-Stein sharp maximal operator.

Throughout this paper we write

$$\theta(a, B_r(x_0))(x) := \sup_{\eta \in \mathbb{R}^n \setminus \{0\}} \frac{|a(x, \eta) - \overline{a}_{B_r(x_0)}(\eta)|}{g(|\eta|)},$$

where

$$\overline{a}_{B_r(x_0)}(\eta) := \int_{B_r(x_0)} a(x, \eta) \, dx.$$  

Then we can easily check from (1.2) that $|\theta(a, B_r(x_0))| \leq 2L$. In addition, we assume that $a(x, \eta)$ satisfies the Dini-$BMO$ regularity. More precisely,

**Definition 1.6** We say that $a(x, \eta)$ is $(\delta, R)$-vanishing for some $\delta, R > 0$, if

$$\omega(R) := \sup_{x_0 \in \Omega} \int_{0<r \leq R} \theta(a, B_r(x_0)) \, dx \leq \delta.$$  

(1.7)

Throughout this paper, we always assume that $\delta$ is a small positive constant.

Finally we state our main results of this paper. The first result is the following Theorem that shows some pointwise estimates of the approximable solutions to of the non-homogeneous quasilinear elliptic obstacle problems involving measure data via fractional maximal operators.

**Theorem 1.7** Under the assumptions (1.2), (1.3) and (1.7), let $\psi \in W^{1,G}(\Omega) \cap W^{2,1}(\Omega)$, $g(|D\psi|)/|D^2\psi| \in L^1_{\text{loc}}(\Omega)$. Assume that $u \in W^{1,1}(\Omega)$ with $u \geq \psi$ a.e. is a limit of approximating solutions to $OP(\psi; u)$ with measure data $\mu \in \mathcal{M}_b(\Omega)$ (in the sense of Definition 1.3), and

$$\sup_{r > 0} \int_0^r \omega(\rho) \frac{1}{1 + \overline{\omega}} \frac{d\rho}{\rho} < +\infty,$$  

(1.8)

then there exists a constant $c = c(n, i_g, s_g, v, L)$ and a radius $R_0 > 0$, depending on $n, i_g, s_g, v, L, \omega(\cdot)$, such that

$$M^\#_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x)$$

$$\leq c \left[ R^{1-\alpha} \int_{B_R(x)} |Du| \, d\xi + W^\mu_{1-\alpha + \frac{a}{s_g}, \frac{a}{g} + 1}(x, 2R) + W^{\frac{a}{g} + 1}(x, 2R) \right]$$

$$+ c \int_0^{2R} \{\omega(\rho)\}^{1 + \overline{\omega}} G^{-1} \left[ \int_{B_{\rho \lambda}(x)} \|G(|D\psi|) + G(|\psi|)\| \, d\xi \right] \frac{d\rho}{\rho^\overline{\omega}}.$$  

(1.9)
Further assume that
\[
\sup_{r>0} \frac{[\omega(r)]^{\frac{1}{1+\alpha}}}{r^\alpha} \leq c_0, \tag{1.10}
\]
for some \( \hat{\alpha} \in [0, \beta) \), then
\[
M_{\alpha,R}(Du)(x) \leq c \left\{ R^{-\alpha} \int_{B_R(x)} |Du| \, d\xi + \left[ M_{1-\alpha_2,R}(\mu)(x) \right]^{\frac{1}{\alpha_2}} + \left[ M_{1-\alpha_2,R}(\psi)(x) \right]^{\frac{1}{\alpha_2}} \right\}
+ c \left[ W_{\mu}^{\hat{\alpha}} \frac{1}{1+\alpha} (x, 2R) + W_{\psi}^{\hat{\alpha}} \frac{1}{1+\alpha} (x, 2R) \right] + c \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\alpha}} G^{-\frac{1}{\alpha}} \left[ \int_{B_\rho(x)} [G(|D\psi|) + G(|\psi|)] \, d\xi \right] \frac{d\rho}{\rho^{1+\alpha}} \tag{1.11}
\]
holds uniformly in \( \alpha \in [0, \hat{\alpha}] \), where \( c = c(n, i_g, s_g, v, L, \hat{\alpha}, c_0, \omega(\cdot), diam(\Omega)) \), \( 0 < R \leq \min \{ R_0, dist(x_0, \partial \Omega) \} \) and \( \beta \) is as in Lemma 3.11.

Thanks to Theorem 1.7, we derive pointwise and oscillation estimates for the gradients of solutions to obstacle problems.

**Theorem 1.8** In the same hypotheses of Theorem 1.7, let \( B_{4R}(x_0) \subseteq \Omega, x, y \in B_{\frac{R}{4}}(x_0), 0 < R \leq \frac{1}{2} \), and for some \( \hat{\alpha} \in [0, \beta) \), assume that
\[
c_0 := \sup_{r>0} \int_0^r \frac{[\omega(\rho)]^{\frac{1}{1+\alpha}}}{\rho^\alpha} \, d\rho < +\infty. \tag{1.12}
\]
Then there exists a constant \( c = c(n, i_g, s_g, v, L, \hat{\alpha}, \omega(\cdot), diam(\Omega)) \) such that
\[
|Du(x_0)| \leq c \left[ \int_{B_R(x_0)} |Du| \, d\xi + W_{\mu}^{\hat{\alpha}} \frac{1}{1+\alpha} (x_0, 2R) + W_{\psi}^{\hat{\alpha}} \frac{1}{1+\alpha} (x_0, 2R) \right] + c \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\alpha}} G^{-\frac{1}{\alpha}} \left[ \int_{B_\rho(x_0)} [G(|D\psi|) + G(|\psi|)] \, d\xi \right] \frac{d\rho}{\rho^{1+\alpha}}. \tag{1.13}
\]
\[
|Du(x) - Du(y)| \leq c \int_{B_R(x_0)} |Du| \, d\xi \left( \frac{|x - y|}{R} \right)^\alpha
+ c \left[ W_{\mu}^{\hat{\alpha}} \frac{1}{1+\alpha} i_g, i_g+1 (x, 2R) + W_{\psi}^{\hat{\alpha}} \frac{1}{1+\alpha} i_g, i_g+1 (x, 2R) \right] |x - y|^\alpha
+ c \left[ W_{\mu}^{\hat{\alpha}} \frac{1}{1+\alpha} i_g, i_g+1 (y, 2R) + W_{\psi}^{\hat{\alpha}} \frac{1}{1+\alpha} i_g, i_g+1 (y, 2R) \right] |x - y|^\alpha
+ c \left[ \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\alpha}} G^{-\frac{1}{\alpha}} \left[ \int_{B_\rho(x)} [G(|D\psi|) + G(|\psi|)] \, d\xi \right] \frac{d\rho}{\rho^{1+\alpha}} \right] |x - y|^\alpha
+ c \left[ \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\alpha}} G^{-\frac{1}{\alpha}} \left[ \int_{B_\rho(y)} [G(|D\psi|) + G(|\psi|)] \, d\xi \right] \frac{d\rho}{\rho^{1+\alpha}} \right] |x - y|^\alpha, \tag{1.14}
\]
holds uniformly in \( \alpha \in [0, \hat{\alpha}] \), where \( \beta \) is as in Lemma 3.11 and \( x, y \) is the Lebesgue’s point of \( Du \).
Remark 1.9  Our results extend the results of Scheven [29] (case $p \geq 2$) to obstacle problems with Orlicz growth. Besides, we merely require a much weaker condition that $a(\cdot, \cdot)$ satisfies the Dini-$BMO$ regularity compared with [29].

The remainder of this paper is organized as follows. Section 2 contains some notions and preliminary results. In Sect. 3, we derive the excess decay estimate for solutions to these problems by using some comparison estimates. In Sect. 4, we obtain pointwise and oscillation estimates for the gradients of solutions by the Wolff potentials.

2 Preliminaries

In this section, we introduce some notions and results which will be used in this paper. Firstly, we denote by $m$ any number in the natural number set $\mathbb{N}$, it is easily verified that

$$\| f - (f)_\Omega \|_{L^2(\Omega)} = \min_{c \in \mathbb{R}^m} \| f - c \|_{L^2(\Omega)}$$

for any measurable set $\Omega \subseteq \mathbb{R}^n$ and every functions $f : \Omega \to \mathbb{R}^m$ such that $f \in L^2(\Omega)$. If $q \in [1, \infty)$, we have

$$\| f - (f)_\Omega \|_{L^q(\Omega)} \leq 2 \min_{c \in \mathbb{R}^m} \| f - c \|_{L^q(\Omega)}.$$  \hspace{1cm} (2.1)

Definition 2.1  A Young function $B$ is called an $N$-function if

$$0 < B(t) < +\infty \text{ for } t > 0$$

and

$$\lim_{t \to +\infty} \frac{B(t)}{t} = \lim_{t \to 0} \frac{t}{B(t)} = +\infty.$$  \hspace{1cm} (2.2)

It’s obvious that $G(t)$ is an $N$-function.

The Young conjugate of a Young function $B$ will be denoted by $B^*$ and defined as

$$B^*(t) = \sup_{s \geq 0} \{ st - B(s) \} \text{ for } t \geq 0.$$  \hspace{1cm} (2.3)

In particular, if $B$ is an $N$-function, then $B^*$ is an $N$-function as well.

Definition 2.2  A Young function $B$ is said to satisfy the global $\vartriangle_2$ condition, denoted by $B \in \vartriangle_2$, if there exists a positive constant $C$ such that for every $t > 0$,

$$B(2t) \leq CB(t).$$

Similarly, a Young function $B$ is said to satisfy the global $\nabla_2$ condition, denoted by $B \in \nabla_2$, if there exists a constant $\theta > 1$ such that for every $t > 0$,

$$B(t) \leq \frac{B(\theta t)}{2\theta}.$$  \hspace{1cm} (2.4)

Remark 2.3  For an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\vartriangle_2$ condition $f(2t) \lesssim f(t)$ for $t \geq 0$, it is easy to prove that $f(t + s) \leq c[f(t) + f(s)]$ holds for every $t, s \geq 0$.

Next let us recall a basic property of $N$-function, which will be used in the sequel.
Lemma 2.4 [34] If $B$ is an $N$-function, then $B$ satisfies the following Young’s inequality

$$ st \leq B^*(s) + B(t), \text{ for } \forall s, t \geq 0. $$

Furthermore, if $B \in \triangle_2 \cap \nabla_2$ is an $N$-function, then $B$ satisfies the following Young’s inequality with $\forall \varepsilon > 0$,

$$ st \leq \varepsilon B^*(s) + c(\varepsilon) B(t), \text{ for } \forall s, t \geq 0. $$

Note that $G(t)$ satisfies the Young’s inequality.

Another important property of Young’s conjugate function is the following inequality, which can be found in [1]:

$$ B^*\left(\frac{B(t)}{t}\right) \leq B(t). \quad (2.3) $$

Lemma 2.5 [7, 34] Under the assumption (1.3), $G(t)$ is defined in (1.4). Then we have

1. $G(t)$ is strictly convex $N$-function and

$$ G^*(g(t)) \leq cG(t) \text{ for } t \geq 0 \text{ and some } c > 0; $$

2. $G(t) \in \nabla_2$.

Note that $G(t)$ satisfies $\Delta_2$ and $\nabla_2$ conditions, then $W^{1,G}(\Omega)$ is a reflexive Banach space, see [13], Theorem 6.1.4.

Lemma 2.6 [4, 9] Under the assumptions (1.2) and (1.3), $G(t)$ is defined in (1.4). Then there exists $c = c(n, i_g, s_g, v, L) > 0$ such that

$$ [a(x, \eta) - a(x, \xi)] \cdot (\eta - \xi) \geq cG(|\eta - \xi|), \text{ for every } x \in \Omega, \eta, \xi, \in \mathbb{R}^n. \quad (2.4) $$

Especially, we have

$$ a(x, \eta) \cdot \eta \geq cG(|\eta|), \text{ for every } x \in \Omega, \eta \in \mathbb{R}^n. \quad (2.5) $$

The following iteration lemma turns out to be very useful in the sequel.

Lemma 2.7 [12] Let $f(t)$ be a nonnegative function defined on the interval $[a, b]$ with $a \geq 0$. Suppose that for $s, t \in [a, b]$ with $t < s$,

$$ f(t) \leq \frac{A}{(s-t)^\alpha} + \frac{B}{(s-t)^\beta} + C + \theta f(s) $$

holds, where $A, B, C \geq 0, \alpha, \beta > 0$ and $0 \leq \theta < 1$. Then there exists a constant $c = c(\alpha, \theta)$ such that

$$ f(\rho) \leq c\left(\frac{A}{(R-\rho)^\alpha} + \frac{B}{(R-\rho)^\beta} + C\right) $$

for any $\rho, R \in [a, b]$ with $\rho < R$.

The proof of the following lemma can be found in [33].

Lemma 2.8 Let $\Omega \subset \mathbb{R}$ be a bounded domain. Assume that $1 + i_g \leq n, h \in T_0^{1,G}(\Omega)$ satisfies

$$ \int_{\Omega \cap \{|x| \leq \delta\}} |Dh|^{1+i_g} dx \leq Mk + M^{1+i_g} $$

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for \( \forall \ k > 0 \), and fixed constants \( M > 0 \). Then we have
\[
\int_{\Omega} |h|^{1+\alpha} \, dx \leq c_1 M^{\frac{1+\alpha}{s_g}},
\]
\[
\int_{\Omega} |Dh|^{1+\beta} \, dx \leq c_2 M^{\frac{1+\beta}{s_g}},
\]
where \( 0 < \alpha < \min \left\{ 1, \frac{n(i_g-1)+1}{n-1} \right\} \), \( 0 < \beta < \min \left\{ 1, \frac{n(i_g-1)+1}{n-1} \right\} \), \( c_1 = c_1(\Omega, n, i_g, \alpha) \), \( c_2 = c_2(\Omega, n, i_g, \beta) \).

### 3 Comparison estimates

Our goal in this section is a suitable comparison estimate between the solutions of obstacle problems and the solutions of elliptic equations. Therefore we can obtain the excess decay estimate for solutions of obstacle problems with measure data. Let’s start with the following lemma, which will be useful in the sequel.

**Lemma 3.1** Assume that \( g(t) \) satisfies (1.3), \( G(t) \) is defined in (1.4). Then we have

(1) for any \( \beta \geq 1 \),
\[
\beta s_g \leq \frac{g(\beta t)}{g(t)} \leq \beta s_g \text{ and } \beta^{1+i_g} \leq \frac{G(\beta t)}{G(t)} \leq \beta^{1+i_g}, \text{ for every } t > 0,
\]
for any \( 0 < \beta < 1 \),
\[
\beta^{i_g} \leq \frac{g(\beta t)}{g(t)} \leq \beta^{i_g} \text{ and } \beta^{1+i_g} \leq \frac{G(\beta t)}{G(t)} \leq \beta^{1+i_g}, \text{ for every } t > 0.
\]

(2) for any \( \beta \geq 1 \),
\[
\beta^{1+i_g} (t) \leq \frac{G^{-1}(\beta t)}{G^{-1}(t)} \leq \beta^{1+i_g} \text{ and } \beta^{1+i_g} \leq \frac{G^{-1}(\beta t)}{G^{-1}(t)} \leq \beta^{1+i_g}, \text{ for every } t > 0,
\]
for any \( 0 < \beta < 1 \),
\[
\beta^{1+i_g} (t) \leq \frac{G^{-1}(\beta t)}{G^{-1}(t)} \leq \beta^{1+i_g} \text{ and } \beta^{1+i_g} \leq \frac{G^{-1}(\beta t)}{G^{-1}(t)} \leq \beta^{1+i_g}, \text{ for every } t > 0.
\]

**Proof** We first consider the case (1).

For \( \beta \geq 1 \), (1.3) allows us to estimate
\[
\frac{i_g}{\beta t} \leq \frac{g'(\beta t)}{g(\beta t)} \leq \frac{s_g}{\beta t}.
\]
By integrating the above inequality over \([1, \beta]\), we obtain
\[
i_g \log \beta \leq \log \frac{g(\beta t)}{g(t)} \leq s_g \log \beta.
\]
Hence
\[
\beta^{i_g} \leq \frac{g(\beta t)}{g(t)} \leq \beta^{s_g}.
\]
We make use of (1.3) again to get
\[ s g \int_0^t g(s) ds \geq \int_0^t s g'(s) ds = tg(t) - \int_0^t g(s) ds, \]
which implies that
\[ tg(t) \leq (s_g + 1) G(t), \]
then we have
\[ (\log G(t))' \leq (s_g + 1) (\log t)'.$\]
By integrating the above inequality over \([t, \beta t]\) for any \(t > 0\), we conclude that
\[ \log G(\beta t) = \hat{G}(\beta s_g + 1) G(t). \]
Thus
\[ G(\beta t) \leq \beta^{s_g + 1} G(t). \]
Likewise,
\[ G(\beta t) \geq \beta^{s_g + 1} G(t). \]
For \(0 < \beta < 1\), we can prove it in the same way.

As for the case (2), being \(g\) strictly increasing and with infinite limit, then \(g^{-1}\) exists, is defined for all \(t \in \mathbb{R}\) and it is strictly increasing. Because of (1), for \(\beta \geq 1\), we have
\[ g(\beta t) \leq \beta^{s_g} g(t). \]
Consequently,
\[ \beta t \leq g^{-1}(\beta^{s_g} g(t)). \]
Now we choose \(t = g^{-1}(s)\), then we get
\[ \beta g^{-1}(s) \leq g^{-1}(\beta^{s_g} s), \]
then
\[ \beta^{s_g - 1} g^{-1}(s) \leq g^{-1}(\beta s). \]
In other cases we can prove it similarly. ⊓⊔

It’s obvious that Lemma 3.1 implies that
\[ L^{1+s_g}(\Omega) \subset L^G(\Omega) \subset L^{1+s_g}(\Omega) \subset L^1(\Omega), \]
and \(g\) and \(G\) satisfy \(\triangle_2\) condition. Then from Remark 2.3, we know that \(g\) and \(G\) satisfy the subadditivity property: \(g(t + s) \leq c[g(t) + g(s)]\), \(G(t + s) \leq c[G(t) + G(s)]\), for every \(t, s \geq 0\).

The next lemma provides us with the comparison estimate between the solutions of elliptic obstacle problems with measure data and the solutions of the corresponding homogeneous obstacle problems.

**Lemma 3.2** Assume that conditions (1.2)–(1.3) are fulfilled, let \(B_{2R}(x_0) \subset \Omega, f \in L^1(B_R(x_0)) \cap (W^{1,G}(B_R(x_0)))'\) and the map \(u \in W^{1,G}(B_R(x_0))\) with \(u \geq \psi\) solves the variational inequality
\[ \int_{B_R(x_0)} a(x, Du) \cdot D(v - u) dx \geq \int_{B_R(x_0)} f(v - u) dx \] (3.2)
for any \( v \in u + W_0^{1,G}(B_R(x_0)) \) that satisfy \( v \geq \psi \) a.e. on \( B_R(x_0) \). Let \( w \in u + W_0^{1,G}(B_R(x_0)) \) with \( w \geq \psi \) be the weak solution of the homogeneous obstacle problem

\[
\int_{B_R(x_0)} a(x, Dw) \cdot D(v - w) dx \geq 0.
\]  

(3.3)

Then there exists \( c = c(n, i_g, s_g, \psi, L) \) such that

\[
\int_{B_R(x_0)} |Du - Dw| \, dx \leq c \left[ R \int_{B_R(x_0)} |f| \, dx \right]^\frac{1}{i_g}.
\]

(3.4)

**Proof** Without loss of generality we may assume that \( x_0 = 0, R = 1 \) by defining

\[
\hat{u}(x) = \frac{u(Rx + x_0)}{R}, \quad \hat{w}(x) = \frac{w(Rx + x_0)}{R}, \quad \hat{f}(x) = Rf(Rx + x_0).
\]

Case 1: \( \int_{B_1} |f| \, dx \leq 1 \). If \( 1 + i_g > n \), because of \( u - w \in W_0^{1,G}(B_1) \), \( u - w \in W_0^{1,i_g}(B_1) \), then we make use of Sobolev’s inequality to get \( u - w \in L^\infty(B_1) \). Now we take \( v = \frac{u + w}{2} \in u + W_0^{1,G}(B_1) \) as comparison functions in the variational inequalities (3.2) and (3.3), which implies that

\[
\int_{B_1} |Du - Dw|^{1 + i_g} \, dx \leq c \int_{B_1} [G(|Du - Dw|) + 1] \, dx \\
\leq c \int_{B_1} [a(x, Dw) - a(x, Du)] \cdot (Dw - Du) \, dx + c \\
\leq c \int_{B_1} f(w - u) \, dx + c \\
\leq c \| u - w \|_{L^\infty(B_1)} \int_{B_1} |f| \, dx + c \\
\leq c \| Du - Dw \|_{L^{1 + i_g}(B_1)} + c,
\]

where we used Lemma 3.1 and Lemma 2.6. Then we have

\[
\int_{B_1} |Du - Dw| \, dx \leq c.
\]

If \( 1 + i_g \leq n \), we define

\[
D_k := \{ x \in B_1 : |u(x) - w(x)| \leq k \}, \quad \forall k > 0,
\]

and let \( v_k := u + T_k(w - u) \) and \( \overline{v}_k := w - T_k(w - u) \in u + W_0^{1,G}(B_1) \) as comparison functions in the variational inequalities (3.2) and (3.3) respectively, then we have

\[
\int_{B_1} [a(x, Dw) - a(x, Du)] \cdot D[T_k(w - u)] \, dx \leq k.
\]

(3.5)

Then for every \( k \geq 1 \), we have

\[
\int_{D_k} |Du - Dw|^{1 + i_g} \, dx \leq c \int_{B_1} [a(x, Dw) - a(x, Du)] \cdot D[T_k(w - u)] \, dx + c \\
\leq ck.
\]

Now using Lemma 2.8, we obtain

\[
\int_{B_1} |Du - Dw| \, dx \leq c
\]
Case 2: $\int_{B_1} |f| dx > 1$. We let

$$
\bar{u}(x) = A^{-1} u(x), \quad \bar{v}(x) = A^{-1} v(x), \quad \bar{w}(x) = A^{-1} w(x),
$$

$$
\bar{\psi}(x) = A^{-1} \psi(x), \quad \bar{f}(x) = A^{-1} f(x), \quad \bar{a}(x, \eta) = A^{-1} a(x, \eta), \quad \bar{G}(t) = \int_0^t \bar{g}(\tau) d\tau,
$$

where $A = (\int_{B_1} |f| dx)^{\frac{1}{|s|}} > 1$. Then we can easily obtain $\int_{B_1} |\bar{f}| dx = 1$, $\bar{a}$ satisfies (1.2) and $\bar{G}$ satisfies (1.3).

Moreover, $\bar{u} \in W^{1,\bar{G}}(B_1)$ with $\bar{u} \geq \bar{\psi}$ solves the variational inequality

$$
\int_{B_1} \bar{a}(x, D\bar{u}) \cdot D(\bar{v} - \bar{u}) dx \geq \int_{B_1} (\bar{v} - \bar{u}) \bar{f} dx
$$

for any $\bar{v} \in \bar{u} + W^{1,\bar{G}}_0(B_1)$ that satisfy $\bar{v} \geq \bar{\psi}$ a.e. on $B_1$.

And $\bar{w} \in \bar{u} + W^{1,\bar{G}}_0(B_1)$ with $\bar{w} \geq \bar{\psi}$ solves the inequality

$$
\int_{B_1} \bar{a}(x, D\bar{w}) \cdot D(\bar{v} - \bar{u}) dx \geq 0.
$$

Similar to the case 1 we have

$$
\int_{B_1} |D\bar{u} - D\bar{w}| dx \leq c.
$$

In conclusion, we have

$$
\int_{B_1} |Du - Dw| dx \leq c(\int_{B_1} |f| dx)^{\frac{1}{|s|}},
$$

which finishes our proof. \qed

**Corollary 3.3** Assume that conditions (1.2)–(1.3) are fulfilled, let $w$ be as in Lemma 3.2 and $\mu \in \mathcal{M}_b(\Omega)$ and $u$ be a limit of approximating solutions for OP($\psi, \mu$), in the sense of Definition 1.3. Then there exists $c = c(n, i_g, s_g, \nu, L)$ such that

$$
\int_{B_R(x_0)} |Du - Dw| dx \leq c \left[ \frac{|\mu|(B_R(\chi_0))}{R^{n-1}} \right]^{|s|}.
$$

(3.6)

The proof of Corollary 3.3 is similar to the proof of Corollary 4.2 in [33], so we don’t repeat it here.

Next, we prove the Caccioppoli’s inequality for homogeneous obstacle problems.

**Lemma 3.4** Assume that conditions (1.2)–(1.3) are fulfilled, let $B_R(x_0) \subseteq \Omega, \psi \in W^{1,\bar{G}}(B_R(x_0))$ and $u \in W^{1,\bar{G}}(B_R(x_0))$ with $u \geq \psi$ solves the inequality

$$
\int_{B_R(x_0)} a(x, Du) \cdot D(v - u) dx \geq 0
$$

(3.7)

for any $v \in u + W^{1,\bar{G}}_0(B_R(x_0))$ with $v \geq \psi$ a.e. on $B_R(x_0)$. Then there exists $c = c(n, i_g, s_g, \nu, L) > 0$ such that

$$
\int_{B_R(x_0)} G(|Du|) dx \leq c \int_{B_R(x_0)} G \left( \frac{|u - \lambda|}{R} \right) dx + c \int_{B_R(x_0)} \left[ G \left( \frac{|\psi|}{R} \right) + G(|D\psi|) \right] dx.
$$

(3.8)
for every $\lambda \geq 0$.

**Proof** Without loss of generality we may assume that $x_0 = 0$, let $\eta \in C^\infty_0(B_t)$, $\eta = 1$ on $B_t$, $|D\eta| \leq \frac{c}{s-t}, 0 \leq \eta \leq 1,$ for any $0 < t < s \leq R$. We take $v = u - \eta(u - \lambda) + \eta \psi \geq \psi$ a.e. on $B_R$ as testing function for the inequality (3.7), then we have

$$\int_{B_t} a(x, Du) \cdot D(-\eta(u - \lambda) + \eta \psi)dx \geq 0,$$

which implies that

$$\int_{B_t} a(x, Du) \cdot (Du)\eta dx \leq \int_{B_t} a(x, Du) \cdot (D\eta)(u - \lambda)dx + \int_{B_t} a(x, Du) \cdot D(\eta \psi)dx.$$

Then combing Young’s inequality with Lemmas 3.1, 2.5, 2.6, we have

$$\int_{B_t} G(|Du|)dx \leq \int_{B_t} G(|Du|)\eta dx \leq c \int_{B_t} a(x, Du) \cdot (Du)\eta dx \leq c \int_{B_t} |a(x, Du)||D\eta||u - \lambda|dx + c \int_{B_t} |a(x, Du)||D(\eta \psi)|dx \leq c \epsilon \int_{B_t} G^\lambda(|Du|)dx + c(\epsilon) \int_{B_t} G(|D\psi|)dx \leq c \epsilon \int_{B_t} G(|Du|)dx + c(\epsilon) \int_{B_t} G(|D\psi|)dx \leq c \epsilon \int_{B_t} G(|Du|)dx + c(\epsilon) \int_{B_t} G(|D\psi|)dx + c(\epsilon) \int_{B_t} \left(\frac{R}{s-t}\right)^{c_1} G\left(\frac{|u - \lambda|}{R}\right) + \left(\frac{R}{s-t}\right)^{c_1} G\left(\frac{|\psi|}{R}\right)dx.$$

Now we take $\epsilon$ small enough to get $c \epsilon \leq \frac{1}{2}$, then we make use of Lemma 2.7, we obtain

$$\int_{B_{\frac{R}{2}}} G(|Du|)dx \leq c \int_{B_R} \left[ G\left(\frac{|u - \lambda|}{R}\right) + G\left(\frac{|\psi|}{R}\right) \right] dx + c \int_{B_R} G(|D\psi|)dx. \quad (3.9)$$

$\square$

To obtain Lemma 3.6, we need a new Sobolev type inequality that can be found in [5].

**Lemma 3.5** Let $G(t)$ is defined in (1.4). Set

$$S(t) := G(t) \left[ \frac{G(t)}{t} \right]^{-\frac{1}{n}} \text{ for } t > 0.$$

Then there exists a constant $c$ depending only on $n$ such that

$$G^{-1}\left( \int_{B_R(x)} G\left(\frac{|u - m(u)|}{R}\right) d\xi \right) \leq c S^{-1}\left( \int_{B_R(x)} S(c|Du|)d\xi \right).$$
Proof: Without loss of generality we may assume that $x_0 = 0$, $R = 1$ by defining
\[ \tilde{u}(x) := \frac{u(x_0 + Rx)}{R}, \quad \tilde{\psi}(x) := \frac{\psi(x_0 + Rx)}{R}. \]

Now we let
\[ S(t) := G(t) \left[ \frac{G(t)}{t} \right]^{-\frac{1}{n}} \quad \text{for} \quad t > 0. \]

From Remark 2.3, Lemmas 3.4 and 3.5, we obtain
\begin{align*}
\int_{B_{\frac{1}{2}}(y)} G(|Du|)dx & \leq c(G \circ S^{-1}) \left( \int_{B_{\rho}(y)} S(|Du|)dx \right) \\
& + c \int_{B_{\rho}(y)} G(|D\psi|)dx + c \int_{B_{\rho}(y)} G \left( \frac{|\psi|}{\rho} \right) dx. \tag{3.11}
\end{align*}

Now we let $r \leq 1$, $\alpha \in (0, 1)$ and a point $y \in B_{\alpha r} := B_{\alpha r}(0)$. We take $\rho = (1 - \alpha)r$, note that $B_{(1-\alpha)r}(y) \subset B_r$, from (3.11) we get
\begin{align*}
\int_{B_{\frac{1}{2}(1-\alpha)r}(y)} G(|Du|)dx & \leq c(G \circ S^{-1}) \left( \int_{B_{(1-\alpha)r}(y)} S(|Du|)dx \right) + c \int_{B_{(1-\alpha)r}(y)} G(|D\psi|)dx \\
& + \frac{c}{[(1-\alpha)r]^{1+\alpha}} \int_{B_{(1-\alpha)r}(y)} G(|\psi|)dx.
\end{align*}

On the other hand, thanks to Hölder’s inequality, we have
\begin{align*}
\int_{B_{(1-\alpha)r}(y)} S(|Du|)dx &= \int_{B_{(1-\alpha)r}(y)} [G(|Du|)]^{\frac{n-1}{n}} |Du|^\frac{1}{n} dx \\
& \leq \left( \int_{B_{(1-\alpha)r}(y)} G(|Du|)dx \right)^{\frac{n-1}{n}} \left( \int_{B_{(1-\alpha)r}(y)} |Du|dx \right)^{\frac{1}{n}}. \tag{3.12}
\end{align*}
Now we consider a Young function \( E(t) := S(t^n) \) and its Young conjugate function \( E^* \). Firstly, Obviously \( E(t) \) is increasing and satisfies (2.2) and \( E(0) = 0 \). Then by calculating, we know that
\[
2n - 1 \leq \frac{tE'(t)}{E(t)} \leq (n - 1)(s_g + 1) + 1,
\]
whence
\[
\tilde{E}(t) := \int_0^t \frac{E(s)}{s} ds \simeq E(t)
\]
and \( \tilde{E} \) is convex, so we can suppose \( E \) convex. And it is easy to see that \( E(t) \) satisfies \( \Delta_2 \) and \( \nabla_2 \) conditions. In conclusion, \( E(t) \) satisfies Young’s inequality. We change variable \( (s = \sigma^{\frac{1}{n}}) \) in the definition of the Young’s conjugate function, for \( \alpha > 0, \alpha^{\frac{n-1}{n}} s \geq S(s^n) \),
\[
E^*(\alpha^{\frac{n-1}{n}}) = \sup_{s>0} \alpha^{\frac{n-1}{n}} s - S(s^n) \leq c(n) \left[ \sup_{\sigma>0} \alpha^{n-1} \sigma - [S(\sigma)]^n \right]^{\frac{1}{n}} := [F^*(\alpha^{n-1})]^{\frac{1}{n}}, \quad (3.13)
\]
where \( F(t) := [S(t)]^n \). From (2.3), we deduce that
\[
F^*(\{G(\tau)\}^{n-1}) = F^* \left( \frac{[S(\tau)]^n}{\tau} \right) = F^* \left( \frac{F(\tau)}{\tau} \right) \leq F(\tau) = [S(\tau)]^n.
\]
Next, we take \( \tau = G^{-1}(\alpha) \), then we have
\[
F^*(\alpha^{n-1}) \leq [S(G^{-1}(\alpha))]^n. \quad (3.14)
\]
Coupling (3.13) with (3.14) tells us that for \( \alpha = \int_{B(1-\alpha)r(y)} G(|Du|) dx \),
\[
E^* \left( \int_{B(1-\alpha)r(y)} G(|Du|) dx \right)^{\frac{n-1}{n}} \leq c(n)(S \circ G^{-1}) \left( \int_{B(1-\alpha)r(y)} G(|Du|) dx \right). \quad (3.15)
\]
Note that \( t \mapsto (G \circ S^{-1})(t) \) is increasing. On the other hand, using Lemma 3.1 provides us
\[
S(\beta t) \geq S(t) \beta^{1 + i_g - \frac{ig}{n}} \quad \beta \geq 1,
\]
which implies that
\[
\beta S^{-1}(t) \geq S^{-1} \left[ t^{1 + i_g - \frac{ig}{n}} \right].
\]
So that \( G \circ S^{-1} \) satisfies \( \Delta_2 \) condition. Then by Remark 2.3, we have
\[
(G \circ S^{-1})(t + s) \leq c((G \circ S^{-1})(s) + (G \circ S^{-1})(t)) \quad \text{for} \quad s, t \geq 0.
\]
Thanks to Lemma 3.1, (3.12), (3.15) and Young’s inequality, we get
\[
\int_{B(1-\alpha)r(y)} G(|Du|) dx \leq c(G \circ S^{-1}) \left( \int_{B(1-\alpha)r(y)} S(|Du|) dx \right) + c \int_{B(1-\alpha)r(y)} G(|D\psi|) dx
\]
\[
+c[(1 - \alpha)r]^{-1 - ig} \int_{B(1-\alpha)r(y)} G(|\psi|) dx
\]
\[
\leq c(G \circ S^{-1}) \left[ \left( \int_{B(1-\alpha)r(y)} G(|Du|) dx \right)^{\frac{n-1}{n}} \left( \int_{B(1-\alpha)r(y)} |Du| dx \right)^{\frac{1}{n}} \right]
\]
In order to prove Lemma 3.8, we introduce the following Lemma 3.7. Since we have shown which finishes our proof.

\[\text{\(c \in G \circ S^{-1}\left\{ \varepsilon \mathbf{E}^* \left[ \left( \int_{B_{1-\alpha r}(y)} G(|Du|)dx \right)^{\frac{1}{s\gamma}} \right] \right\} \)}\]

\[+ c \int_{B_{1-\alpha r}(y)} G(|Du|)dx + c[(1 - \alpha)r]^{-1 - s\gamma} \int_{B_{1-\alpha r}(y)} G(|\psi|)dx\]

\[\leq c^{\frac{1}{s\gamma}} \int_{B_{1-\alpha r}(y)} G(|Du|)dx + cG \left( \int_{B_{1-\alpha r}(y)} |Du|dx \right)\]

\[+ c \int_{B_{1-\alpha r}(y)} G(|Du|)dx + c[(1 - \alpha)r]^{-1 - s\gamma} \int_{B_{1-\alpha r}(y)} G(|\psi|)dx\]

\[\leq c^{\frac{1}{s\gamma}} \int_{B_{1-\alpha r}(y)} G(|Du|)dx + c[(1 - \alpha)r]^{-1 - s\gamma} \int_{B_{1-\alpha r}(y)} G(|\psi|)dx.\]

Since \(y \in B_{ar}\), the ball \(B_{ar}\) can be covered by some balls included in \(B_r\) such that only a finite and independent of \(\alpha\) number of balls of double radius intersect, then we have

\[\int_{B_{ar}} G(|Du|)dx \leq c^{\frac{1}{s\gamma}} \int_{B_r} G(|Du|)dx + c[(1 - \alpha)r]^{-ns\gamma} G \left( \int_{B_1} |Du|dx \right)\]

\[+ c \int_{B_1} G(|Du|)dx + c[(1 - \alpha)r]^{-1 - s\gamma} \int_{B_1} G(|\psi|)dx.\]

We make use of Lemma 2.7 to get

\[\int_{B_{\frac{1}{4}}} G(|Du|)dx \leq cG \left( \int_{B_1} |Du|dx \right) + c \int_{B_1} [G(|Du|) + G(|\psi|)]dx,\]

which finishes our proof. \(\square\)

In order to prove Lemma 3.8, we introduce the following Lemma 3.7. Since we have shown the Caccioppoli’s inequality (that is, Lemma 3.4), then we can get Lemma 3.7 by combining [9], Theorem 7 and [9], Proposition 6, for more comments see [9], Theorem 9.

**Lemma 3.7** We assume that \(u \in W^{1,G}(B_R(x_0))\) solves the inequality (3.16), then we have

\[\left( \int_{B_R(x_0)} G(|Du|)^\gamma dx \right)^{\frac{1}{\gamma}} \leq c \int_{B_{3R}(x_0)} G(|Du|)dx\]

for some \(c > 0\) and \(\gamma > 1\), depending only on \(n, i_R, s_\gamma, v\) and \(L\).

Now we prove the comparison estimate between the solutions of a homogeneous obstacle problem and the solutions of a desired obstacle problem, and it will be crucial for dealing with Dini-BMO vector fields \(a\).
Lemma 3.8 Under the conditions (1.2)–(1.3), we assume that $B_{2R}(x_0) \subseteq \Omega$, $u \geq 0$, $\psi \in W^{1,G}(B_{2R}(x_0))$, and $u \in W^{1,G}(B_{2R}(x_0))$ with $u \geq \psi$ solves the inequality

$$
\int_{B_R(x_0)} a(x, Du) \cdot D(v-u)\,dx \geq 0
$$

(3.16)

for any $v \in u + W^{1,G}_{0}(B_{R}(x_0))$ with $v \geq \psi$ a.e. on $B_{R}(x_0)$. Assume that $w \in W^{1,G}(B_{2R}(x_0))$ with $w \geq \psi$ solves the inequality

$$
\int_{B_{R}(x_0)} \overline{a}_{B_{R}(x_0)}(Dw) \cdot D(v-w)\,dx \geq 0
$$

(3.17)

and $w = u$ on $\partial B_{R}(x_0)$. Then we have

$$
\int_{B_{R}(x_0)} |Du - Dw|\,dx \leq c\omega(R)^{1\over 1+\sigma}
$$

where $c = c(n, i_g, s_g, v, L)$.

**Proof** Without loss of generality we may assume that $x_0 = 0$. We take $v = {u + w \over 2} \in u + W^{1,G}_{0}(B_{R})$ as comparison functions in the variational inequalities (3.16) and (3.17), which implies that

$$
\int_{B_{R}} [a(x, Du) - \overline{a}_{B_{R}(x_0)}(Dw)] \cdot D(u-w)\,dx \leq 0.
$$

Then from (1.2), (1.7), Lemmas 2.6 and 3.7, we have

$$
\int_{B_{R}} G(|Du - Dw|)\,dx \leq \int_{B_{R}} [\overline{a}_{B_{R}}(Du) - \overline{a}_{B_{R}}(Dw)] \cdot D(u-w)\,dx
$$

$$
\leq \int_{B_{R}} \overline{a}_{B_{R}}(Du) - a(x, Du) \cdot D(u-w)\,dx
$$

$$
\leq \int_{B_{R}} \theta(a, B_{R})g(|Du|)|Du - Dw|\,dx
$$

$$
\leq c\varepsilon \int_{B_{R}} G(|Du - Dw|)\,dx + c \int_{B_{R}} \theta(a, B_{R})G^{\ast}(g(|Du|))\,dx
$$

$$
\leq c\varepsilon \int_{B_{R}} G(|Du - Dw|)\,dx
$$

$$
+ c \left( \int_{B_{R}} \theta(a, B_{R})\gamma\,dx \right)^{\frac{1}{\gamma}} \left( \int_{B_{R}} G(|Du|)^{\gamma}\,dx \right)^{\frac{1}{\gamma}}
$$

$$
\leq c\varepsilon \int_{B_{R}} G(|Du - Dw|)\,dx + c\omega(R) \int_{\Omega} G(|Du|)\,dx,
$$

where we used the fact $\theta \leq 2L$. Now we choose $\varepsilon$ small enough to get

$$
\int_{B_{R}} G(|Du - Dw|)\,dx \leq c\omega(R) \int_{B_{R}^{2R}} G(|Du|)\,dx.
$$

Because $G$ is convex, we obtain

$$
G \left( \int_{B_{R}} |Du - Dw|\,dx \right) \leq \int_{B_{R}} G(|Du - Dw|)\,dx.
$$
Using Remark 2.3, Lemmas 3.1 and 3.6, we conclude
\[
\int_{B_R} |Du - Dw|dx \leq G^{-1} \left( c \omega(R) \int_{B_{3R}^G} G(|Du|)dx \right) \\
\leq c \omega(R)^{\frac{1}{1+\beta}} G^{-1} \left( \int_{B_{3R}^G} G(|Du|)dx \right) \\
\leq c \omega(R)^{\frac{1}{1+\beta}} \left\{ \int_{B_{2R}} |Du|dx + G^{-1} \left[ \int_{B_{2R}} [G(|D\psi|) + G(|\psi|)]dx \right] \right\},
\]
and the proof is complete. \hfill \Box

The following two lemmas show some comparison estimates. Since Lemma 4.4 and Lemma 4.5 in [33] give similar results, the two lemmas in our paper can be obtained by modifying their proving process a little.

**Lemma 3.9** Assume that conditions (1.2)–(1.3) are fulfilled, let \( u \in W^{1,G}(B_R(x_0)) \) with \( u \geq \psi \) solves the inequality
\[
\int_{B_R(x_0)} a(x, Du) \cdot D(v - u)dx \geq 0
\]
for any \( v \in u + W^{1,G}_0(B_R(x_0)) \) with \( v \geq \psi \) a.e. on \( B_R(x_0) \). Let \( w \in u + W^{1,G}_0(B_R(x_0)) \) be a weak solution of the equation
\[
- \text{div} (a(x, Dw)) = - \text{div} (a(x, D\psi)) \quad \text{on} \quad B_R(x_0)
\]
and \( \psi \in W^{1,G}(B_R(x_0)) \cap W^{2,1}(B_R(x_0)) \), \( \text{div} (a(x, D\psi)) \in L^1(B_R(x_0)) \). Then there exists
\( c = c(n, i_g, s_g, v, L) > 0 \) such that
\[
\int_{B_R(x_0)} |Du - Dw|dx \leq c \left( R \int_{B_R(x_0)} (|\text{div} (a(x, D\psi))| + 1)dx \right)^{\frac{1}{\beta}}. \tag{3.18}
\]

**Lemma 3.10** Assume that conditions (1.2)–(1.3) are fulfilled, let \( f, g \in L^1(B_R(x_0)) \cap (W^{1,G}(B_R(x_0)))' \) and \( u, w \in W^{1,G}(B_R(x_0)) \) with \( u - w \in W^{1,G}_0(B_R(x_0)) \) be weak solutions of
\[
\begin{align*}
- \text{div} (a(x, Du)) &= f \quad \text{on} \quad B_R(x_0), \\
- \text{div} (a(x, Dw)) &= g \quad \text{on} \quad B_R(x_0).
\end{align*} \tag{3.19}
\]
Then the following comparison estimates hold:
\[
\int_{B_R(x_0)} |Du - Dw|dx \leq c \left( R \int_{B_R(x_0)} (|f| + |g| + 1)dx \right)^{\frac{1}{\beta}}, \tag{3.20}
\]
where \( c = c(n, i_g, s_g, v, L) > 0 \).

Next we state an excess decay estimate for a homogeneous comparison problem.

**Lemma 3.11** (see [2], lemma 4.1) If \( v \in W^{1,G}_{loc}(\Omega) \) is a local weak solution of
\[
\text{div} (a(Dv)) = 0 \quad \text{in} \quad \Omega
\]
\[
(3.21)
\]
under the assumptions (1.2) and (1.3), then there exist constants $\beta \in (0, 1)$ and $C = C(n, i_g, s_g, v, L)$ such that
\[
\int_{B_\rho(x_0)} |Dv - (Du)_{B_\rho(x_0)}| \, dx \leq C \left( \frac{\rho}{R} \right)^\beta \int_{B_R(x_0)} |Dv - (Du)_{B_R(x_0)}| \, dx, \tag{3.22}
\]
where $0 < \rho \leq R$, $B_{2R}(x_0) \subset \Omega$.

We want to obtain a similar excess decay estimate, but with error terms, for solutions to (3.2). And our approach is to transfer the excess decay estimate from Lemma 3.11 to solutions of an obstacle problem by employing a multistep comparison argument.

**Lemma 3.12** Assume that conditions (1.2)–(1.3) are fulfilled, let $B_{2R}(x_0) \subset \Omega, \psi \in W^{1,G}(B_R(x_0)) \cap W^{2,1}(B_R(x_0)), \frac{g(|D\psi|)}{|D\psi|} |D^2\psi| \in L^1(B_R(x_0))$. Let $u \in W^{1,1}(B_R(x_0))$ with $u \geq \psi$ a.e. be a limit of approximating solutions for $OP(\psi; u)$ with measure data $\mu \in M(R(B_R(x_0))$ (in the sense of Definition 1.3). Then there exists $\beta \in (0, 1)$ such that
\[
\int_{B_\rho(x_0)} |Du - (Du)_{B_\rho(x_0)}| \, dx \\
\leq c \left( \frac{\rho}{R} \right)^\beta \int_{B_R(x_0)} |Du - (Du)_{B_R(x_0)}| \, dx \\
+ c \left( \frac{R}{\rho} \right)^{n} \left[ \left( \frac{|\mu|(B_R(x_0))}{R^{n-1}} \right) \right]^{\frac{1}{\gamma}} + \left( R \int_{B_R(x_0)} \left( \frac{g(|D\psi|)}{|D\psi|} |D^2\psi| + 1 \right) \, dx \right)^{\frac{1}{2}} \\
+ c \left( \frac{R}{\rho} \right)^n \omega(R)^{\frac{1}{2}}\left\{ \int_{B_{R^2}(x_0)} |Du| \, dx + \frac{1}{G^{-1}} \left[ \int_{B_{R^2}(x_0)} G(|D\psi|) + G(|\psi|) \, dx \right] \right\},
\]
where $0 < \rho \leq R$, $c = c(n, i_g, s_g, v, L)$ and $\beta$ is as in Lemma 3.11.

**Remark 3.13** Since we obtained all comparison estimates for regularized problems on $L^1$–level, our results also hold in measure data problems.

**Proof** Without loss of generality we may assume that $x_0 = 0$ and that $w_1, w_2, w_3 \in W^{1,G}(B_R)$ satisfy separately
\[
\begin{cases}
\int_{B_R} a(x, Dw_1) \cdot D(v - w_1) \, dx \geq 0, \text{ for } \forall v \in w_1 + W^{1,G}_0(B_R) \text{ with } v \geq \psi \text{ a.e. on } B_R, \\
w_1 \geq \psi, w_1 \geq 0 \text{ a.e. on } B_R, \\
w_1 = u \text{ on } \partial B_R,
\end{cases}
\]
\[
\begin{cases}
\int_{B_R} \bar{a}_{B_R} \frac{1}{2} (Dw_2) \cdot D(v - w_2) \, dx \geq 0, \text{ for } \forall v \in w_1 + W^{1,G}_0(B_R) \text{ with } v \geq \psi \text{ a.e. on } B_R, \\
w_2 \geq \psi \text{ a.e. on } B_R, \\
w_2 = w_1 \text{ on } \partial B_R,
\end{cases}
\]
\[
\begin{cases}
- \text{div} \left( \bar{a}_{B_R} \frac{1}{2} (Dw_3) \right) = - \text{div} \left( \bar{a}_{B_R} \frac{1}{2} (D\psi) \right) \text{ on } B_R, \\
w_3 = w_1 \text{ on } \partial B_R,
\end{cases}
\]
\[
\begin{cases}
- \text{div} \left( \bar{a}_{B_R} \frac{1}{2} (Dw_4) \right) = 0 \text{ on } B_R, \\
w_4 = w_1 \text{ on } \partial B_R.
\end{cases}
\]
In order to remove the inhomogeneity, we make use of Corollary 3.3 to get the comparison estimate
\[
\int_{B_R} |Du - Dw_1| \, dx \leq c \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right]^{\frac{1}{16}}.
\] (3.23)

Then we use Lemma 3.8 to obtain
\[
\int_{B_R^2} |Dw_1 - Dw_2| \, dx \leq c \omega(R)^{\frac{1}{16}} \left\{ \int_{B_R} |Dw_1| \, dx + G^{-1} \left[ \int_{B_R} [G(|D\psi|) + G(|\psi|)] \, dx \right] \right\}
\leq c \omega(R)^{\frac{1}{16}} \left\{ \int_{B_R} |Du| \, dx + \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right]^{\frac{1}{16}} \right\}
+ c \omega(R)^{\frac{1}{16}} \left\{ G^{-1} \left[ \int_{B_R} [G(|D\psi|) + G(|\psi|)] \, dx \right] \right\}.
\] (3.24)

Next, in order to transition to an obstacle-free problem, we apply Lemma 3.9 to get
\[
\int_{B_R^2} |Dw_2 - Dw_3| \, dx \leq c \left( R \int_{B_R^2} \left( |\text{div} (\overline{a}B_R(D\psi))| + 1 \right) \, dx \right)^{\frac{1}{16}}.
\] (3.25)

Now we reduce to a homogeneous equation. An application of Lemma 3.10 with \( f = - \text{div} \left( \overline{a}B_R(D\psi) \right) \) and \( g = 0 \) implies that
\[
\int_{B_R^2} |Dw_3 - Dw_4| \, dx \leq c \left( R \int_{B_R^2} \left( |\text{div} (\overline{a}B_R(D\psi))| + 1 \right) \, dx \right)^{\frac{1}{16}}.
\] (3.26)

Thanks to Lemma 3.11, there exists \( \beta \in (0, 1) \) such that
\[
\int_{B_\rho^2} |Dw_4 - (Dw_4)_{B_\rho^2}| \, dx \leq c \left( \frac{\rho}{R} \right)^\beta \int_{B_\rho^2} |Dw_4 - (Dw_4)_{B_\rho^2}| \, dx
\leq c \left( \frac{\rho}{R} \right)^\beta \int_{B_\rho^2} |Dw_4 - Du| + |Du - (Du)_{B_R^2}| \, dx.
\] (3.27)

Finally we together with (3.23) (3.24) (3.25) (3.26) and (3.27), we infer
\[
\int_{B_\rho} |Du - (Du)_{B_\rho}| \, dx \leq \int_{B_\rho} |Du - (Dw_4)_{B_\rho}| \, dx
\leq \int_{B_\rho} \left( |Du - Dw_1| + |Dw_1 - Dw_2| + |Dw_2 - Dw_3| + |Dw_3 - Dw_4| + |Dw_4 - (Dw_4)_{B_\rho}| \right) \, dx
\leq c \left( \frac{R}{\rho} \right)^n \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right]^{\frac{1}{16}} + \left( R \int_{B_\rho} \left( |\text{div} (\overline{a}B_R(D\psi))| + 1 \right) \, dx \right)^{\frac{1}{16}}
+ c \left( \frac{R}{\rho} \right)^n \omega(R)^{\frac{1}{16}} \left\{ \int_{B_R} |Du| \, dx + \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right]^{\frac{1}{16}} \right\}
\]
\[ + c \left( \frac{R}{\rho} \right)^n \omega(R)^{\frac{1}{1+g}} G^{-1} \left[ \int_{B_R} |G(|D\psi|) + G(|\psi|)|dx \right] \]
\[ + c \left( \frac{\rho}{R} \right)^\beta \left[ \int_{B_{\frac{R}{2}}} (|Dw_4 - Du| + |Du - (Du)_{B_R}|)dx \right] \]
\[ \leq c \left( \frac{\rho}{R} \right)^\beta \int_{B_R} |Du - (Du)_{B_R}|dx \]
\[ + c \left( \frac{R}{\rho} \right)^n \left[ \frac{|\mu(B_{R})|}{R^{n-1}} \right]^{\frac{1}{\nu}} + \left( R \int_{B_R} \left( \frac{g(|D\psi|)}{|D\psi|} |D^2\psi| + 1 \right) dx \right)^{\frac{1}{\nu}} \]
\[ + c \left( \frac{R}{\rho} \right)^n \omega(R)^{\frac{1}{1+g}} \left[ \int_{B_{R_{i}}} |Du|dx + G^{-1} \left[ \int_{B_{R}} [G(|D\psi|) + G(|\psi|)]dx \right] \right] , \]
where we used the fact \( \omega \leq 2L \) and the condition (1.2) in the last step. \qed

4 The proof of gradient estimates

This section is devoted to obtain pointwise and oscillation estimates for the gradients of solutions by the sharp maximal function estimates. We start with a pointwise estimate of fractional maximal operator by precise iteration methods.

Proof of Theorem 1.7 Proof of (1.9) We define
\[ B_i := B(x, \frac{R}{H^i}) = B(x, R_i), \text{ for } i = 0, 1, 2, \ldots, \]
\[ A_i := \int_{B_i} |Du - (Du)_{B_i}|dx, \ k_i := |(Du)_{B_i} - S|, \ S \in \mathbb{R}^n. \]
We take \( H = H(n, i_g, s_g, v, L) > 1 \) large enough to have
\[ c \left( \frac{1}{H} \right)^\beta \leq \frac{1}{4}, \]
with \( \beta \) as in Lemma 3.11 and we apply Lemma 3.12 to obtain
\[ \int_{B_{i+1}} |Du - (Du)_{B_{i+1}}|d\xi \]
\[ \leq \frac{1}{4} \int_{B_i} |Du - (Du)_{B_i}|d\xi + cH^n \left\{ \left[ \frac{\mu(B_{i})}{R_i^{n-1}} \right]^{\frac{1}{\nu}} + \left[ \frac{D\Psi(B_{i})}{R_i^{n-1}} \right]^{\frac{1}{\nu}} \right\} \]
\[ + cH^n \omega(R_i)^{\frac{1}{1+g}} \left\{ \int_{B_i} |Du|d\xi + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\} . \]
Now we reduce the value of \( R_0 \)-in a way depending only on \( n, i_g, s_g, v, L \) and \( \omega(\cdot) \)- to get
\[ cH^n \omega(R_i)^{\frac{1}{1+g}} \leq cH^n \omega(R_0)^{\frac{1}{1+g}} \leq \frac{1}{4} , \]
which together with the following estimate
\[ \int_{B_i} |Du|d\xi \leq \int_{B_i} |Du - (Du)_{B_i}|d\xi + k_i + |S|, \]
\( \square \)
we reduce that
\[
A_{i+1} \leq \frac{1}{2} A_i + c \left\{ \left[ \frac{|\mu|(\mathcal{B}_i)}{R_{i}^{n-1}} \right]^{1/\gamma} + \left[ \frac{D\Psi(B_i)}{R_{i}^{n-1}} \right]^{1/\gamma} \right\}^{+} + c \left[ \omega(R_i) \right]^{1 + \varepsilon} \left\{ k_i + |S| + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}
\]
whenever \( i \geq 0 \). On the other hand, we calculate
\[
|k_{i+1} - k_i| \leq |(Du)_{B_{i+1}} - (Du)_{B_i}|
\leq \int_{B_{i+1}} |Du - (Du)_{B_i}| d\xi
\leq H^n \int_{B_i} |Du - (Du)_{B_i}| d\xi = H^n A_i,
\]
from which we see that for \( m \in \mathbb{N} \),
\[
k_{m+1} = \sum_{i=0}^{m} (k_{i+1} - k_i) + k_0 \leq H^n \sum_{i=0}^{m} A_i + k_0.
\]
At this stage we sum up (4.1) over \( i \in \{0, \ldots, m - 1\} \), which allows us to infer the inequality
\[
\sum_{i=1}^{m} A_i \leq \frac{1}{2} \sum_{i=0}^{m-1} A_i + c \sum_{i=0}^{m-1} \left\{ \left[ \frac{|\mu|(\mathcal{B}_i)}{R_{i}^{n-1}} \right]^{1/\gamma} + \left[ \frac{D\Psi(B_i)}{R_{i}^{n-1}} \right]^{1/\gamma} \right\}^{+} + c \sum_{i=0}^{m-1} \left[ \omega(R_i) \right]^{1 + \varepsilon} \left\{ k_i + |S| + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.
\]
Consequently,
\[
\sum_{i=1}^{m} A_i \leq A_0 + 2c \sum_{i=0}^{m-1} \left\{ \left[ \frac{|\mu|(\mathcal{B}_i)}{R_{i}^{n-1}} \right]^{1/\gamma} + \left[ \frac{D\Psi(B_i)}{R_{i}^{n-1}} \right]^{1/\gamma} \right\}^{+} + c \sum_{i=0}^{m-1} \left[ \omega(R_i) \right]^{1 + \varepsilon} \left\{ k_i + |S| + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.
\]
For every integer \( m \geq 1 \) we employ (4.2) to gain
\[
k_{m+1} \leq c A_0 + c k_0 + c \sum_{i=0}^{m-1} \left\{ \left[ \frac{|\mu|(\mathcal{B}_i)}{R_{i}^{n-1}} \right]^{1/\gamma} + \left[ \frac{D\Psi(B_i)}{R_{i}^{n-1}} \right]^{1/\gamma} \right\}^{+} + c \sum_{i=0}^{m-1} \left[ \omega(R_i) \right]^{1 + \varepsilon} \left\{ k_i + |S| + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.
\]
we take into account the definition of \( A_0 \) to get
\[
k_{m+1} \leq c \int_{B_R} |Du - (Du)_{B_R}| + |Du - S|d\xi + c \sum_{i=0}^{m-1} \left\{ \left[ \frac{|\mu|(\mathcal{B}_i)}{R_{i}^{n-1}} \right]^{1/\gamma} + \left[ \frac{D\Psi(B_i)}{R_{i}^{n-1}} \right]^{1/\gamma} \right\}^{+} + c \sum_{i=0}^{m-1} \left[ \omega(R_i) \right]^{1 + \varepsilon} \left\{ k_i + |S| + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.
\]
+ c \sum_{i=0}^{m-1} \left[ \omega (R_i) \right]^{1 \over 1+q} \left\{ k_i + |S| + G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)] d\xi \right] \right\} \quad (4.4)

for every \( m \geq 0 \). In the previous inequality we choose 0 = S and multiply both sides by \( R_{m+1}^{1-\alpha} \), taking into account that \( \alpha \in [0, 1] \) and \( R_m + 1 \leq R_i \) for \( 0 \leq i \leq m + 1 \), we get

\[
R_{m+1}^{1-\alpha} k_{m+1} \leq c R^{1-\alpha} \int_{\mathbb{R}^n} |Du| d\xi + c \sum_{i=0}^{m-1} R_i^{1-\alpha} \left\{ \left[ \frac{|\mu| (B_i)}{R_i^{n-1}} \right]^{1 \over 1+q} + \left[ \frac{D\Phi (B_i)}{R_i^{n-1}} \right]^{1 \over 1+q} \right\} + c \sum_{i=0}^{m-1} R_i^{1-\alpha} \left[ \omega (R_i) \right]^{1 \over 1+q} \left\{ k_i + G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)] d\xi \right] \right\}
\]

and therefore

\[
R_{m+1}^{1-\alpha} k_{m+1} \leq c R^{1-\alpha} \int_{\mathbb{R}^n} |Du| d\xi + c \sum_{i=0}^{m} \left\{ \left[ \frac{|\mu| (B_i)}{R_i^{n-1-i(1-\alpha)}} \right]^{1 \over 1+q} + \left[ \frac{D\Phi (B_i)}{R_i^{n-1-i(1-\alpha)}} \right]^{1 \over 1+q} \right\} + c \sum_{i=0}^{m} R_i^{1-\alpha} \left[ \omega (R_i) \right]^{1 \over 1+q} \left\{ k_i + G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)] d\xi \right] \right\}. \quad (4.5)
\]

Now we employ the definition of Wolff potential to estimate the last term on the right-hand side in (4.5) and we find

\[
\sum_{i=0}^{\infty} \left[ \frac{|\mu| (B_i)}{R_i^{n-1-i(1-\alpha)}} \right]^{1 \over 1+q} \leq \frac{c}{\log 2} \int_{R}^{2R} \left[ \frac{|\mu| (B_\rho)}{\rho^{n-1-i(1-\alpha)}} \right]^{1 \over 1+q} d\rho + \sum_{i=0}^{\infty} \frac{c}{\log H} \int_{R_{i+1}}^{R_i} \left[ \frac{|\mu| (B_\rho)}{\rho^{n-1-i(1-\alpha)}} \right]^{1 \over 1+q} d\rho \leq c W_1^{\mu} + c \int \frac{1}{\rho^{1-\alpha} + \frac{\alpha}{1-q} \xi + 1} (x, 2R), \quad (4.6)
\]

in the last step, we apply \( |\mu| (B_{2R}) < +\infty \) by assumption, then \( |\mu| (\partial B_\rho) > 0 \) can hold at most for countably many radii \( \rho \in (R, 2R) \). Similarly, we have

\[
\sum_{i=0}^{\infty} \left[ \frac{D\Phi (B_i)}{R_i^{n-1-i(1-\alpha)}} \right]^{1 \over 1+q} \leq c W_1^{\mu} + c \int \frac{1}{\rho^{1-\alpha} + \frac{\alpha}{1-q} \xi + 1} (x, 2R),
\]

\[
\sum_{i=0}^{\infty} \left[ \omega (R_i) \right]^{1 \over 1+q} \leq c \int_{0}^{2R} \left[ \omega (\rho) \right]^{1 \over 1+q} d\rho ,
\]

\[
\sum_{i=0}^{\infty} R_i^{1-\alpha} \left[ \omega (R_i) \right]^{1 \over 1+q} G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)] d\xi \right] \leq c \int_{0}^{2R} \left[ \omega (\rho) \right]^{1 \over 1+q} G^{-1} \left[ \int_{B_\rho} [G(|D\psi|) + G(|\psi|)] d\xi \right] d\rho. \quad (4.7)
\]

Consequently,

\[
R_{m+1}^{1-\alpha} k_{m+1} \leq c M + c \sum_{i=0}^{m} R_i^{1-\alpha} \left[ \omega (R_i) \right]^{1 \over 1+q} , \quad (4.7)
\]
where
\[ M := c \left[ R^{1-\alpha} \int_{B_R} |Du|d\xi + W^\mu_{1-\alpha+\frac{q}{p+1}j_x+1}(x, 2R) + W^{|\psi|}_{1-\alpha+\frac{q}{p+1}j_x+1}(x, 2R) \right] + c \int_0^{2R} \left[ \omega(\rho) \right]^\frac{1}{\gamma+\rho} G^{-1} \left[ \int_{B_\rho} [G(|D\psi|) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^\alpha}. \]

We now prove by induction that
\[ R^{1-\alpha}_m k_{m+1} \leq (c + c^*) M, \quad (4.8) \]
holds for every \( m \geq 0 \), some positive constants \( c, c^* > 1 \).

Firstly, the case \( m = 0 \) of (4.7) is trivial. Then we assume that \( R^{1-\alpha}_i k_i \leq (c + c^*) M \) for \( i \leq m \) and prove it for \( m + 1 \). Because of (1.8), we further reduce the value of \( R_0 \) to get
\[ \int_0^{2R} \left[ \omega(\rho) \right]^\frac{1}{\gamma+\rho} \frac{d\rho}{\rho} \leq \int_0^{2R_0} \left[ \omega(\rho) \right]^\frac{1}{\gamma+\rho} \frac{d\rho}{\rho} \leq \frac{1}{2(c + c^*)c}. \]

Then taking (4.7) into account we obtain
\[ R^{1-\alpha}_{m+1} k_{m+1} \leq c M + c \sum_{i=0}^m (c + c^*) M \left[ \omega(R_i) \right]^\frac{1}{\gamma+\rho} \leq (c + c^*) M, \]
therefore (4.8) follows for every integer \( m \geq 0 \). Now we define
\[ C_m := R^{1-\alpha}_m A_m = R^{1-\alpha}_m \int_{B_m} |Du - (Du)_{B_m}|d\xi \]
\[ h_m := \int_{B_m} |Du|d\xi. \]

Then it’s easy to obtain
\[ R^{1-\alpha}_m h_m = R^{1-\alpha}_m \int_{B_m} |Du|d\xi \]
\[ \leq R^{1-\alpha}_m \int_{B_m} |Du - (Du)_{B_m}| + |(Du)_{B_m}|d\xi \]
\[ = R^{1-\alpha}_m k_m + C_m \]
\[ \leq C M + C_m \]
with \( M \) as in (4.7), and so we just need to look for a bound on \( C_m \). Applying (4.6) and keeping in mind the definition of \( M \) in (4.7), we gain
\[ \left[ \frac{|\mu|(|B_i|)}{R^{n-1}_i} \right]^\frac{1}{\gamma+\rho} \leq C R^{\alpha-1}_i W^\mu_{1-\alpha+\frac{q}{p+1}j_x+1}(x, 2R) \]
\[ \leq C R^{\alpha-1}_i M. \]
Similarly, we have
\[
\left[ D\Psi(B_i) \right]^{\frac{1}{\gamma_i}} \leq c R_i^{\alpha-1} M,
\]
\[
[\omega(R_i)]^{\frac{1}{\gamma_i}} G^{-1} \left[ \int_{B_i} |G(|D\psi|) + G(|\psi|)| d\xi \right] \leq c R_i^{\alpha-1} M.
\]
Therefore, referring to (4.1) we have
\[
A_{m+1} \leq \frac{1}{2} A_m + c R_m^{\alpha-1} M + c k_m.
\]
In turn, we make use of (4.8) to obtain
\[
A_{m+1} \leq \frac{1}{2} A_m + c R_m^{\alpha-1} M.
\]
then multiply both sides by $R_{m+1}^{1-\alpha}$, we get
\[
C_{m+1} \leq \frac{1}{2} \left( \frac{R_{m+1}}{R_m} \right)^{1-\alpha} C_m + C \left( \frac{R_{m+1}}{R_m} \right)^{1-\alpha} M
\]
\[
\leq \frac{1}{2} C_m + C_1 M.
\]
(4.9)

Now we shall prove by induction that
\[
C_m \leq 2 C_1 M \quad (4.10)
\]
holds whenever $m \geq 0$. When $k = 0$, we have
\[
C_0 = R^{1-\alpha} \int_{B_R} |Du - (Du)_{B_R}| d\xi
\]
\[
\leq 2 R^{1-\alpha} \int_{B_R} |Du| d\xi
\]
\[
\leq 2 M.
\]
Then we assume (4.10) holds for $k = m - 1$, then using (4.9) we gain
\[
C_m \leq 2 C_1 M \quad \text{for} \quad m \geq 0.
\]
Now we consider $r \leq R$ and determine the integer $i \geq 0$ such that $R_{i+1} \leq r \leq R_i$, then we have
\[
r^{1-\alpha} \int_{B_r} |Du| d\xi \leq \left( \frac{R_i}{R_{i+1}} \right)^n R_i^{1-\alpha} \int_{B_i} |Du| d\xi
\]
\[
\leq C h_i^{1-\alpha} R_i^{1-\alpha} h_i
\]
\[
\leq C M.
\]
Recalling the definition of $M$ and the restricted maximal operator we in turn obtain

$$M_{1-\alpha, R}(\{|Du|\})(x)$$

$$\leq c \left[ R^{1-\alpha} \int_{B_R} |Du|d\xi + W^\mu_{1-\alpha, 1+\frac{\alpha}{i\theta+1}, 1+1}(x, 2R) + W^{\psi}_{1-\alpha, 1+\frac{\alpha}{i\theta+1}, 1+1}(x, 2R) \right]$$

$$+c \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+i\theta}} G^{-1} \left[ f_{B_\rho} [G(\{|D\psi|\}) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^\alpha}.$$ 

Moreover, because of

$$M_{\alpha, R}^\#(u)(x) \leq M_{1-\alpha, R}(\{|Du|\})(x),$$

then we have

$$M_{\alpha, R}^\#(u)(x) + M_{1-\alpha, R}(Du)(x)$$

$$\leq c \left[ R^{1-\alpha} \int_{B_R} |Du|d\xi + W^\mu_{1-\alpha, 1+\frac{\alpha}{i\theta+1}, 1+1}(x, 2R) + W^{\psi}_{1-\alpha, 1+\frac{\alpha}{i\theta+1}, 1+1}(x, 2R) \right]$$

$$+c \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+i\theta}} G^{-1} \left[ f_{B_\rho} [G(\{|D\psi|\}) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^\alpha}.$$ 

**Proof of (1.11)** We define

$$\hat{A}_i := R_i^{-\alpha} \int_{B_i} |Du - (Du)_{B_i}| d\xi$$

(4.11)

Using Lemma 3.12 (multiply both sides by $R_{i+1}^{-\alpha}$) we obtain

$$\hat{A}_{i+1} \leq c \left[ R_{i+1}^{1-\alpha} \hat{A}_i + c \left( \frac{R_i}{R_{i+1}} \right)^{n+\alpha} \left\{ \left[ \frac{|\mu|(|B_i|)}{R_i^{n-1+\alpha i\theta}} \right]^{\frac{1}{1+i\theta}} + \left[ \frac{D\Psi(B_i)}{R_i^{n-1+\alpha i\theta}} \right]^{\frac{1}{1+i\theta}} \right\} \right]$$

$$+c \left( \frac{R_i}{R_{i+1}} \right)^{n+\alpha} \frac{1}{R_i^\alpha} \omega(R_i) \left\{ \int_{B_i} |Du|d\xi + G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.$$ 

We choose $H = H(n, \alpha, \beta, \hat{A}, \beta) > 1$ large enough to gain

$$c \left( \frac{R_{i+1}}{R_i} \right)^{\beta-\alpha} = c \left( \frac{1}{H} \right)^{\beta-\alpha} \leq c \left( \frac{1}{H} \right) \leq \frac{1}{2}$$

and

$$\frac{|\mu|(|B_i|)}{R_i^{n-1+\alpha i\theta}} \leq H^{n-1+\alpha i\theta} \frac{|\mu|(B_{i-1})}{R_{i-1}^{n-1+\alpha i\theta}}.$$ 

From assumption (1.10), we have

$$\left[ \frac{\omega(R_i)}{R_i^\alpha} \right]^{\frac{1}{1+i\theta}} \leq \frac{1}{R_i^\alpha} \omega(R_i) \left[ \frac{1}{R_i^{n-1+\alpha i\theta}} \right] \leq C_0.$$ 

then because of the definition of restricted maximal operator, we conclude that

$$\hat{A}_{i+1} \leq \frac{1}{2} \hat{A}_i + c \left\{ \left[ M_{1-\alpha i\theta, R}(\mu)(x) \right]^{\frac{1}{1+i\theta}} + \left[ M_{1-\alpha i\theta, R}(\psi)(x) \right]^{\frac{1}{1+i\theta}} + \int_{B_i} |Du|d\xi \right\}$$

$$+c \frac{1}{R_i} \omega(R_i) \left\{ G^{-1} \left[ \int_{B_i} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.$$
for every $i \geq 0$. Moreover, from the case $\alpha = 1$ of inequality (1.9), we have
\[
\int_{B_i} |Du|d\xi \leq c \left[ \int_{B_R} |Du|d\xi + W^\mu_{\frac{1}{i+1},i+1}(x, 2R) + W^\psi_{\frac{1}{i+1},i+1}(x, 2R) \right]
\quad + c \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\epsilon}} G^{-1} \left[ \int_{B_\rho} [G(|D\psi|) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^\alpha}.
\]
\[
:= c M^*.
\]
(4.12)

So combined with the two previous inequalities, we obtain
\[
\hat{A}_{i+1} \leq \frac{1}{2} \hat{A}_i + c \left\{ \left[ M_{1-\alpha i, R} \mu(x) \right]^{\frac{1}{1+\epsilon}} + \left[ M_{1-\alpha i, R} \psi(x) \right]^{\frac{1}{1+\epsilon}} + M^* \right\}
\quad + c \frac{1}{R^\alpha_i} \omega(R_i)^{\frac{1}{1+\epsilon}} \left\{ G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}.
\]

Iterating the previous relation, we conclude
\[
\hat{A}_i \leq 2^{-j} \hat{A}_0 + c \sum_{j=0}^{i-1} 2^{-j} \left\{ \left[ M_{1-\alpha i_j, R} \mu(x) \right]^{\frac{1}{1+\epsilon}} + \left[ M_{1-\alpha i_j, R} \psi(x) \right]^{\frac{1}{1+\epsilon}} + M^* \right\}
\quad + c \sum_{j=0}^{i-1} 2^{-j} \frac{1}{R^\alpha_j} \omega(R_j)^{\frac{1}{1+\epsilon}} \left\{ G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}
\]
holds for every $i \geq 1$. Then similar to (4.6), we reduce
\[
\sum_{j=0}^{i-1} 2^{-j} \frac{1}{R^\alpha_j} \omega(R_j)^{\frac{1}{1+\epsilon}} \left\{ G^{-1} \left[ \int_{B_j} [G(|D\psi|) + G(|\psi|)]d\xi \right] \right\}
\quad \leq c \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\epsilon}} G^{-1} \left[ \int_{B_\rho} [G(|D\psi|) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^\alpha}.
\]

Then recalling (4.11) and (2.1) one easily deduces that
\[
\sup_{i \geq 0} \hat{A}_i \leq c \left\{ R^\alpha \int_{B_R} |Du|d\xi + \left[ M_{1-\alpha i_j, R} \mu(x) \right]^{\frac{1}{1+\epsilon}} + \left[ M_{1-\alpha i_j, R} \psi(x) \right]^{\frac{1}{1+\epsilon}} \right\}
\quad + c \left\{ M^* + \int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\epsilon}} G^{-1} \left[ \int_{B_\rho} [G(|D\psi|) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^\alpha} \right\}.
\]

For every $\rho \in (0, R]$, let $i \in \mathbb{N}$ be such that $R_{i+1} < \rho \leq R_i$, then we gain
\[
\rho^{-\alpha} \int_{B_\rho} |Du - (Du)_{B_\rho}|d\xi \leq c \frac{R^n_i}{\rho^n} R_{i+1}^{-\alpha} \int_{B_i} |Du - (Du)_{B_i}|d\xi \leq c \sup_{i \geq 0} \hat{A}_i,
\]
from which we obtain that

\[ M_{\alpha,R}(Du)(x) \leq c \left\{ R^{-\alpha} \int_{B_g} |Du| \, d\xi + \left[ M_{1-\alpha i_g,R}(\mu(x)) \right]^{\frac{1}{\tilde{\gamma}}} + \left[ M_{1-\alpha i_g,R}(\psi(x)) \right]^{\frac{1}{\tilde{\gamma}}} \right\} 
+ c \left\{ \int_{B_g} W^{\mu}_{\frac{1}{\tilde{\gamma}+1}}(x, 2R) + \int_{B_g} W^{\psi}_{\frac{1}{\tilde{\gamma}+1}}(x, 2R) \right\} 
+ c \int_{B_g}^{2R} \left[ \int_{B_g} |G(|D\psi|) + G(|\psi|)| \, d\xi \right] \frac{d\rho}{\rho^{1+\alpha}}. \]

This completes the proof of Theorem 1.7. \( \square \)

**Proof of Theorem 1.8** At first we give the proof of the estimate (1.13).

We choose \( S = 0 \) and make use of (4.4), we conclude

\[ k_{m+1} \leq c \int_{B_g(x_0)} |Du| \, dx + |Du| \, dx \]
\[ + c \sum_{i=0}^{m-1} \left[ \left[ \frac{|\mu|}{R_i^{n-1}} \right]^{\frac{1}{\tilde{\gamma}}} + \left[ \frac{D\psi(B_R_i(x_0))}{R_i^{n-1}} \right]^{\frac{1}{\tilde{\gamma}}} \right] \]
\[ + c \sum_{i=0}^{m-1} \omega(B_i) \left[ \int_{B_R_i(x_0)} |Du| \, dx + G^{-1} \left[ \int_{B_R_i(x_0)} |G(|D\psi|) + G(|\psi|)| \, dx \right] \right]. \]

On the other hand, we observe

\[ \sum_{i=0}^{+\infty} \left[ \frac{|\mu|}{R_i^{n-1}} \right]^{\frac{1}{\tilde{\gamma}}} \leq c W^{\mu}_{\frac{1}{\tilde{\gamma}+1}}(x_0, 2R), \]
\[ \sum_{i=0}^{+\infty} \left[ \frac{D\psi(B_R_i(x_0))}{R_i^{n-1}} \right]^{\frac{1}{\tilde{\gamma}}} \leq c W^{\psi}_{\frac{1}{\tilde{\gamma}+1}}(x_0, 2R), \]
\[ \sum_{i=0}^{+\infty} \omega(R_i) \left[ \int_{B_R_i(x_0)} |Du| \, dx + G^{-1} \left[ \int_{B_R_i(x_0)} |G(|D\psi|) + G(|\psi|)| \, dx \right] \right] \]
\[ \leq c \int_{B_R(x_0)}^{2R} \omega(\rho) \left[ \int_{B_R(x_0)} |G(|D\psi|) + G(|\psi|)| \, dx \right] \frac{d\rho}{\rho}. \]

Coupling (4.12) with above estimates, it follows that

\[ |Du(x_0)| = \lim_{m \to \infty} k_{m+1} \leq c \left\{ \int_{B_g(x_0)} |Du| \, dx + W^{\mu}_{\frac{1}{\tilde{\gamma}+1}}(x_0, 2R) + W^{\psi}_{\frac{1}{\tilde{\gamma}+1}}(x_0, 2R) \right\} 
+ c \int_{B_g(x_0)}^{2R} \left[ \int_{B_R(x_0)} |G(|D\psi|) + G(|\psi|)| \, dx \right] \frac{d\rho}{\rho}. \]

Next we prove the estimate (1.14). For every \( x, y \in B_g(x_0) \), we define

\[ r_i := \frac{r}{H^i}, \quad r \leq \frac{R}{2}, \quad k_i = |(Du)_{B_{r_i}(x)} - S|, \quad \bar{k}_i = |(Du)_{B_{r_i}(y)} - S|. \]
Taking advantage of (4.4) again, we obtain

\[ k_{m+1} \leq c \int_{B_r(x)} \left( |Du - (Du)_{B_r(x)}| + |Du - S| \right) d\xi \]

\[ + cr^\alpha \sum_{i=0}^{m-1} \left\{ \left[ \frac{|\mu|(|B_{r_i}(x))|}{r_i} \right]^{\frac{1}{\sigma}} + \left[ \frac{D\Psi(|B_{r_i}(x))|}{r_i} \right]^{\frac{1}{\sigma}} \right\} \]

\[ + cr^\alpha \sum_{i=0}^{m-1} \int_{B_{r_i}(x)} |Du|d\xi + G^{-1} \left[ \int_{B_{r_i}(x)} [G(|D\psi|) + G(|\psi|)]d\xi \right] \].

Moreover,

\[ \sum_{i=0}^{+\infty} \left[ \frac{|\mu|(|B_{r_i}(x))|}{r_i} \right]^{\frac{1}{\sigma}} \leq c W_{\frac{1}{1+\alpha}}^{\mu} \left[ 1, t \right] \left( x, 2r, \right) \]

\[ \sum_{i=0}^{+\infty} \left[ \frac{D\Psi(|B_{r_i}(x))|}{r_i} \right]^{\frac{1}{\sigma}} \leq c W_{\frac{1}{1+\alpha}}^{\mu} \left[ 1, t \right] \left( x, 2r, \right) \]

\[ \sum_{i=0}^{+\infty} \frac{1}{r_i} \left[ \omega(r_i) \right]^{\frac{1}{1+\sigma}} \leq c \int_0^{2r} \left[ \omega(\rho) \right]^{\frac{1}{1+\sigma}} \frac{d\rho}{\rho^{1+\alpha}} \leq c, \]

\[ \sum_{i=0}^{+\infty} \frac{1}{r_i} \left[ \omega(r_i) \right]^{\frac{1}{1+\sigma}} \leq c \int_0^{2r} \left[ \omega(\rho) \right]^{\frac{1}{1+\sigma}} \frac{d\rho}{\rho^{1+\alpha}} \cdot \]

Combining (4.12) with the previous estimates, we have

\[ k_{m+1} \leq c \int_{B_r(x)} \left( |Du - (Du)_{B_r(x)}| + |Du - S| \right) d\xi \]

\[ + cr^\alpha \left[ W_{\frac{1}{1+\alpha}}^{\mu} \left[ 1, R \right] + W_{\frac{1}{1+\alpha}}^{\mu} \left[ 1, t \right] \left( x, R, \right) \right] \]

\[ + cr^\alpha \left\{ \int_{B_r(x)} |Du|d\xi + \int_0^{R} \left[ \omega(\rho) \right]^{\frac{1}{1+\sigma}} G^{-1} \left[ \int_{B_{\rho}(x)} [G(|D\psi|) + G(|\psi|)]d\xi \right] \frac{d\rho}{\rho^{1+\alpha}} \right\}. \]

If \( x \) is a Lebesgue’s point of \( Du \), then let \( m \to \infty \), we derive

\[ |Du(x) - S| = \lim_{m \to \infty} k_{m+1} \]

\[ \leq c \int_{B_r(x)} \left( |Du - (Du)_{B_r(x)}| + |Du - S| \right) d\xi \]

\[ + cr^\alpha \left[ W_{\frac{1}{1+\alpha}}^{\mu} \left[ 1, R \right] + W_{\frac{1}{1+\alpha}}^{\mu} \left[ 1, t \right] \left( x, R, \right) \right] \]
\[ +cr^\alpha \left\{ \int_{B_r(x)} |Du|d\xi + \int_0^R [\omega(\rho)]^{1+\alpha \beta} G^{-1} \left\{ \int_{B_{\rho^\beta}(x)} [G(|D\psi|) + G(|\psi|)]d\xi \right\} \frac{d\rho}{\rho^{1+\alpha}} \right\} \]

If \( y \) is a Lebesgue’s point of \( Du \), we have a similar result. Then coupling with the previous two estimates tells us

\[ |Du(x) - Du(y)| \leq c \int_{B_r(x)} (|Du - (Du)_{B_r(x)}| + |Du - S|) d\xi \]

\[ +cr^\alpha \left\{ W^{\mu_{-\alpha+\frac{1+\alpha \beta}{1+\beta}}}_{i_{\rho^\alpha}} (x, R) + W^{\psi_{-\alpha+\frac{1+\alpha \beta}{1+\beta}}}_{i_{\rho^\alpha}} (y, R) \right\} \]

\[ +cr^\alpha \left\{ \int_{B_r(x)} |Du|d\xi + \int_0^R [\omega(\rho)]^{1+\alpha \beta} G^{-1} \left\{ \int_{B_{\rho^\beta}(x)} [G(|D\psi|) + G(|\psi|)]d\xi \right\} \frac{d\rho}{\rho^{1+\alpha}} \right\} + c \int_{B_r(y)} (|Du - (Du)_{B_r(y)}| + |Du - S|) d\xi \]

\[ +cr^\alpha \left\{ W^{\mu_{-\alpha+\frac{1+\alpha \beta}{1+\beta}}}_{i_{\rho^\alpha}} (y, R) + W^{\psi_{-\alpha+\frac{1+\alpha \beta}{1+\beta}}}_{i_{\rho^\alpha}} (y, R) \right\} \]

\[ +cr^\alpha \left\{ \int_{B_r(y)} |Du|d\xi + \int_0^R [\omega(\rho)]^{1+\alpha \beta} G^{-1} \left\{ \int_{B_{\rho^\beta}(y)} [G(|D\psi|) + G(|\psi|)]d\xi \right\} \frac{d\rho}{\rho^{1+\alpha}} \right\} . \]

We now choose

\[ S := (Du)_{B_3r(x)}, \quad r := \frac{|x - y|}{2}, \]

it’s easy to see that \( B_r(y) \subseteq B_{3r}(x) \) and therefore

\[ \int_{B_r(x)} (|Du - (Du)_{B_r(x)}| + |Du - S|) d\xi + \int_{B_r(y)} (|Du - (Du)_{B_r(y)}| + |Du - S|) d\xi \]

\[ \leq c(n) \int_{B_{3r}(x)} |Du - (Du)_{B_{3r}(x)}| d\xi. \]

Now notice that \( x, y \in B_{\frac{R}{4}}(x_0) \), so \( |x - y| \leq \frac{R}{2} \) and then \( B_{3r}(x) \subseteq B_{\frac{3R}{4}}(x) \subseteq B_R(x_0) \). Therefore apply (1.11) to obtain

\[ \int_{B_{3r}(x)} |Du - (Du)_{B_{3r}(x)}| d\xi \leq cr^\alpha M^{\beta_{\frac{3R}{4}}} (Du)(x) \]

\[ \leq c \left( \frac{r}{R} \right)^\alpha \int_{B_{\frac{3R}{4}}(x)} |Du| d\xi \]

\[ +cr^\alpha \left\{ M^{\beta_{\frac{3R}{4}}} \left( \mu \right)(x) \right\} + \left\{ M^{\beta_{\frac{3R}{4}}} \left( \psi \right)(x) \right\} \]

\[ +cr^\alpha \left\{ W^{\mu_{\frac{1}{1+\beta}}}_{i_{\rho^\alpha}} (x, 2R) + W^{\psi_{\frac{1}{1+\beta}}}_{i_{\rho^\alpha}} (x, 2R) \right\}. \]
Because of the above estimates, we derive

\[+\frac{c r^\alpha}{\int_0^{2R} [\omega(\rho)]^{\frac{1}{1+\frac{\alpha}{g}}} G^{-1}} \left( \int_{B_\rho(x)} [G(|D\psi|) + G(|\psi|)] d\xi \right) \frac{d\rho}{\rho^{1+\alpha}}.\]

Moreover, due to (4.12), we have

\[
\int_{B_{\rho}(x)} |Du| d\xi \leq c \left[ \int_{B_{\frac{3\rho}{4}}(x)} |Du| d\xi + \int_{B_{\frac{3\rho}{4}}(x)} |Du| d\xi + W_{\frac{1}{1+\frac{\alpha}{g}}, i} + \frac{1}{g} + \frac{1}{i} + \frac{1}{g}(x, \frac{3R}{2}) + W_{\frac{1}{1+\frac{\alpha}{g}}, i} + \frac{1}{g} + \frac{1}{i} + \frac{1}{g}(y, \frac{3R}{2}) \right] + c \int_0^{\frac{3\rho}{4}} [\omega(\rho)]^{\frac{1}{1+\frac{\alpha}{g}}} G^{-1} \left[ \int_{B_{\rho}(x)} [G(|D\psi|) + G(|\psi|)] d\xi \right] \frac{d\rho}{\rho}.
\]

Because of the above estimates, we derive

\[|Du(x) - Du(y)| \leq c \left( \frac{r}{R} \right)^\alpha \int_{B_{\rho}(x)} |Du| d\xi + c \left[ \left[ M_{1-\alpha\frac{3R}{4}} (\mu(x)) \right]^{\frac{1}{1+\frac{\alpha}{g}}} + \left[ M_{1-\alpha\frac{3R}{4}} (\psi(x)) \right]^{\frac{1}{1+\frac{\alpha}{g}}} \right] + c \int_0^{\frac{3\rho}{4}} [\omega(\rho)]^{\frac{1}{1+\frac{\alpha}{g}}} G^{-1} \left[ \int_{B_{\rho}(x)} [G(|D\psi|) + G(|\psi|)] d\xi \right] \frac{d\rho}{\rho^{1+\alpha}},\]

where we used the fact that \(W_{\frac{1}{1+\frac{\alpha}{g}}, i} + \frac{1}{g} \leq c W_{\frac{1}{1+\frac{\alpha}{g}}, i} + \frac{1}{g} + \frac{1}{i} \leq c W_{\frac{1}{1+\frac{\alpha}{g}}, i} + \frac{1}{g} + \frac{1}{i} + \frac{1}{g}(x, 2R)\) and for other case there are similar inequalities.

For every \(\varepsilon > 0\), we know that there exists \(0 < r \leq R\) such that

\[M_{1-\alpha\frac{3R}{4}} (\mu(x)) \leq |B_1|^{-1} \left( \frac{\varepsilon}{3R_{1+\alpha\frac{3R}{4}}} \right)^{-1} + \varepsilon.
\]
We keep in mind the definition of Wolff potential and we derive
\[
\frac{\|\mu\|_{L^p\left(\frac{B_{3r}}{x}\right)}}{\left(\frac{3r}{4}\right)^{n-1+\alpha i\ell}} \leq C \left[ \int_{3r/4}^{r} \left( \frac{\|\mu\|_{L^p\left(B_{\rho}(x)\right)}}{\rho^{n-1+\alpha i\ell}} \right)^{\frac{1}{\ell g}} \frac{d\rho}{\rho} \right]^{i\ell}.
\]
Likewise,
\[
\frac{D\Psi(B_{3r}(x))}{\left(\frac{3r}{4}\right)^{n-1+\alpha i\ell}} \leq C \left[ W^{\mu}_{\alpha+\frac{1+\alpha}{1+\tau i}\ell} + 1 + (x, R) \right]^{i\ell}.
\]
Finally, we take into account the definition of \(r\) to obtain
\[
|Du(x) - Du(y)| \leq c \int_{B_{R}(x_0)} |Du| \, d\xi \left( \frac{|x - y|}{R} \right)^\alpha
\]
\[
+ c \left[ W^{\mu}_{\alpha+\frac{1+\alpha}{1+\tau i}\ell} + 1 + (x, 2R) \right] |x - y|^\alpha
\]
\[
+ c \left[ W^{\mu}_{\alpha+\frac{1+\alpha}{1+\tau i}\ell} + 1 + (y, 2R) \right] |x - y|^\alpha
\]
\[
+ c \int_0^{2R} \left[ \omega(\rho) \right]^{\frac{1}{1+\tau i}} G^{-1} \left[ \int_{B_{\rho}(x)} G(|D\psi|) + G(|\psi|) d\xi \right] \frac{d\rho}{\rho^{1+\alpha}} \right] |x - y|^\alpha
\]
Then we finish the proof of Theorem 1.8. \(\square\)

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References
1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Baroni, P.: Riesz potential estimates for a general class of quasilinear equations. Calc. Var. Partial Differ. Equ. 53(3–4), 803–846 (2015)
3. Beck, L., Mingione, G.: Lipschitz bounds and nonuniform ellipticity. Commun. Pure Appl. Math. 73, 944–1034 (2020)
4. Cho, Y.: Global gradient estimates for divergence-type elliptic problems involving general nonlinear operators. J. Differ. Equ. 264, 6152–6190 (2018)
5. Cianchi, A., Fusco, N.: Gradient regularity for minimizers under general growth conditions. J. Reine Angew. Math. 507, 15–36 (1999)
6. Cianchi, A., Maz'ya, V.: Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems. J. Eur. Math. Soc. (JEMS) 16, 571–595 (2014)
7. Cianchi, A., Maz’ya, V.: Global Lipschitz regularity for a class of quasilinear elliptic equations. Commun. Partial Differ. Equ. 36, 100–133 (2011)
8. Cianchi, A., Maz’ya, V.: Global boundedness of the gradient for a class of nonlinear elliptic systems. Arch. Ration. Mech. Anal. 212, 129–177 (2014)
9. Diencing, L., Ettwein, F.: Fractional estimates for non-differentiable elliptic systems with general growth. Forum Math. 20(3), 523–556 (2008)
10. Duzaar, F., Mingione, G.: Gradient estimates via non-linear potentials. Am. J. Math. 133, 1093–1149 (2011)
11. Duzaar, F., Mingione, G.: Gradient estimates via linear and nonlinear potentials. J. Funct. Anal. 259, 2961–2998 (2010)
12. Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific Publishing Co., Inc, River Edge (2003)
13. Harjulehto, P., Hästö, P.: Orlicz spaces and generalized Orlicz spaces. Lecture Notes in Mathematics, vol. 2236. Springer, Cham (2019)
14. Hedberg, L., Wolff, Th.H.: Thin sets in nonlinear potential theory. Ann. Inst. Fourier (Grenoble) 33, 161–187 (1983)
15. Hwang, S., Lieberman, G.: Hölder continuity of bounded weak solutions to generalized parabolic $p$-Laplacian equations I: degenerate case. Electron. J. Differ. Equ. 287, 32 (2015)
16. Kilpeläinen, T., Malý, J.: The Wiener test and potential estimates for quasilinear elliptic equations. Acta Math. 172, 137–161 (1994)
17. Kilpeläinen, T., Malý, J.: Degenerate elliptic equations with measure data and nonlinear potentials. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19, 591–613 (1992)
18. Kuusi, T., Mingione, G.: Universal potential estimates. J. Funct. Anal. 262, 4205–4269 (2012)
19. Kuusi, T., Mingione, G.: Vectorial nonlinear potential theory. J. Eur. Math. Soc. 20, 929–1004 (2018)
20. Kuusi, T., Mingione, G.: Guide to nonlinear potential estimates (English summary). Bull. Math. Sci. 4, 1–82 (2014)
21. Kuusi, T., Mingione, G., Sire, Y.: Nonlocal equations with measure data. Commun. Math. Phys. 337, 1317–1368 (2015)
22. Lieberman, G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural’tseva for elliptic equations. Commun. Partial Differ. Equ. 16, 311–361 (1991)
23. Lieberman, G.M.: Regularity of solutions to some degenerate double obstacle problems. Indiana Univ. Math. J. 40, 1009–1028 (1991)
24. Mingione, G.: Gradient potential estimates. J. Eur. Math. Soc. (JEMS) 13, 459–486 (2011)
25. Maz’ja, V.G., Havin, V.P.: A nonlinear potential theory. Uspehi Mat. Nauk 27, 67–138 (1972)
26. Ma, L., Zhang, Z.: Wolff type potential estimates for stationary Stokes systems with Dini-BMO coefficients. Commun. Contemp. Math. 22, 67–138 (2019)
27. Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Marcel Dekker Inc, New York (1991)
28. Rodrigues, J.F., Teymurazyan, R.: On the two obstacles problem in Orlicz–Sobolev spaces and applications (English summary). Complex Var. Ellip. Equ. 56, 769–787 (2011)
29. Scheven, C.: Gradient potential estimates in non-linear elliptic obstacle problems with measure data. J. Funct. Anal. 262, 2777–2832 (2012)
30. Scheven, C.: Elliptic obstacle problems with measure data: potentials and low order regularity. Publ. Mat. 56, 327–374 (2012)
31. Trudinger, N., Wang, X.: On the weak continuity of elliptic operators and applications to potential theory. Am. J. Math. 124, 369–410 (2002)
32. Trudinger, N., Wang, X.: Quasilinear elliptic equations with signed measure data. Discrete Contin. Dyn. Syst. 23, 477–494 (2009)
33. Xiong, Q., Zhang, Z.: Gradient potential estimates for elliptic obstacle problems. J. Math. Anal. Appl. 495, 124698 (2021)
34. Yao, F.P., Zhou, S.: Calderón–Zygmund estimates for a class of quasilinear elliptic equations. J. Funct. Anal. 272, 1524–1552 (2017)

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