Community structure in directed networks

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We consider the problem of finding communities or modules in directed networks. The most common approach to this problem in the previous literature has been simply to ignore edge direction and apply methods developed for community discovery in undirected networks, but this approach discards potentially useful information contained in the edge directions. Here we show how the widely used benefit function known as modularity can be generalized in a principled fashion to incorporate the information contained in edge directions. This in turn allows us to find communities by maximizing the modularity over possible divisions of a network, which we do using an algorithm based on the eigenvectors of the corresponding modularity matrix. This method is shown to give demonstrably better results than previous methods on a variety of test networks, both real and computer-generated.

At the most fundamental level a network consists of a set of nodes or vertices connected in pairs by lines or edges, but many variations and extensions are possible, including networks with directed edges, weighted edges, labels on nodes or edges, and others. This flexible structure lends itself to the modeling of a wide array of complex systems and networks have, as a result, attracted considerable attention in the recent physics literature\textsuperscript{[1, 2, 3, 4]}. Many networks are found to display “community structure,” dividing naturally into communities or modules with dense connections within communities but sparser connections between them. Communities have proven to be of interest both in their own right as functional building blocks within networks and for the insights they offer into the dynamics or modes of formation of networks, and a large volume of research has been devoted to the development of algorithmic tools for discovering communities—see [5] for a review. Nearly all of these methods, however, have one thing in common: they are intended for the analysis of undirected network data. Many of the networks that we would like to study are directed, including the world wide web, food webs, many biological networks, and even some social networks. The commonest approach to detecting communities in directed networks has been simply to ignore the edge directions and apply algorithms designed for undirected networks. This works reasonably well in some cases, although in others it does not, as we will see in this paper. Even in the cases where it works, however, it is clear that in discarding the directions of edges we are throwing away a good deal of information about our network’s structure, information that, at least in principle, could allow us to make a more accurate determination of the communities.

Several previous studies, including our own, have touched on this problem in the context of other analyses of directed network data\textsuperscript{[6, 7, 8, 9]}, but they have typically not tackled the community structure problem directly. In this paper we propose a method for the discovery of communities in directed networks that makes explicit use of the information contained in edge directions. The method we propose is an extension of the well-established modularity optimization method for undirected networks\textsuperscript{[10]}, a method that has been shown to be both computationally efficient and highly effective in practical applications\textsuperscript{[3]}.

The premise of the modularity optimization method is that a good division of a network into communities will give high values of the benefit function $Q$, called the modularity, defined by \textsuperscript{[11]}

\begin{equation}
Q = \left( \text{fraction of edges within communities} \right) - \left( \text{expected fraction of such edges} \right)
\end{equation}

Large positive values of the modularity indicate when a statistically surprising fraction of the edges in a network fall within the chosen communities; it tells us when there are more edges within communities than we would expect on the basis of chance.

The expected fraction of edges is typically evaluated within the so-called configuration model, a random graph conditioned on the degree sequence of the original network, in which the probability of an edge between two vertices $i$ and $j$ is $k_i k_j / 2m$, where $k_i$ is the degree of vertex $i$ and $m$ is the total number of edges in the network. The modularity can then be written

\begin{equation}
Q = \frac{1}{2m} \sum_{ij} \left[ A_{ij} - \frac{k_i k_j}{2m} \right] \delta_{c_i, c_j},
\end{equation}

where $A_{ij}$ is an element of the adjacency matrix, $\delta_{ij}$ is the Kronecker delta symbol, and $c_i$ is the label of the community to which vertex $i$ is assigned. Then one maximizes $Q$ over possible divisions of the network into communities, the maximum being taken as the best estimate of the true communities in the network. Neither the size nor the number of communities need be fixed; both can be varied freely in our attempt to find the maximum.

In practice, the exhaustive optimization of modularity is computationally hard, known to be \textit{NP}-complete over the set of all graphs of a given size\textsuperscript{[12]}, so practical methods based on modularity optimization make use of approximate optimization schemes such as greedy
algorithms, simulated annealing, spectral methods, and others.

Now consider a directed network. In searching for communities in such a network we again look for divisions of the network in which there are more edges within communities than we expect on the basis of chance, but we now take edge direction into account. The crucial point to notice is that the expected positions of edges in the network depend on their direction. Consider two vertices, A and B. Vertex A has high out-degree but low in-degree while vertex B has the reverse situation. This means that a given edge is more likely to run from A to B than vice versa, simply because there are more ways it can fall in the first direction than in the second. Hence if we observe in our real network that there is an edge from A to B, it should be considered a bigger surprise than an edge from A to B and hence should make a bigger contribution to the modularity, since modularity should be high for statistically surprising configurations.

We put these insights to work as follows. Given the joint in/out-degree sequence of our directed network, we can create a directed equivalent of the configuration model, which will have an edge from vertex j to vertex i with probability \(k_{ij}^{in}/m\), where \(k_{ij}^{in}\) and \(k_{ij}^{out}\) are the in- and out-degrees of the vertices. (Note that there is no factor of 2 in the denominator now.) Then the equivalent of Eq. (2) is

\[
Q = \frac{1}{m} \sum_{ij} A_{ij} - \frac{k_{ij}^{in}k_{ij}^{out}}{m} \delta_{c_i,c_j}, \tag{3}
\]

where \(A_{ij}\) is defined in the conventional manner to be 1 if there is an edge from j to i and 0 otherwise. Note that indeed edges \(j \rightarrow i\) make larger contributions to this expression if \(k_{ij}^{in}\) and/or \(k_{ij}^{out}\) is small.

Now we search for the division of the network into communities \(\{c_i\}\) such that \(Q\) is maximized. One can in principle make use of any of the methods previously applied to modularity maximization, such as simulated annealing or greedy algorithms. Here we derive the appropriate generalization of the spectral optimization method of Newman [13], which is both computationally efficient and appears to give excellent results in practice.

We consider first the simplified problem of dividing a directed network into just two communities. We define \(s_i\) to be +1 if vertex \(i\) is assigned to community 1 and −1 if it is assigned to community 2. Note that this implies that \(\sum_i s_i^2 = n\). Then \(\delta_{c_i,c_j} = \frac{1}{2}(s_i s_j + 1)\) and

\[
Q = \frac{1}{2m} \sum_{ij} \left[ A_{ij} - \frac{k_{ij}^{in}k_{ij}^{out}}{m} \right] (s_i s_j + 1)
= \frac{1}{2m} \sum_{ij} s_i A_{ij} s_j = \frac{1}{2m} s^T B s, \tag{4}
\]

where \(s\) is the vector whose elements are the \(s_i\), \(B\) is the so-called modularity matrix with elements

\[
B_{ij} = A_{ij} - \frac{k_{ij}^{in}k_{ij}^{out}}{m}, \tag{5}
\]

and we have made use of \(\sum_{ij} A_{ij} = \sum_i k_{in} = \sum_j k_{out} = m\). Our goal is now to find the \(s\) that maximizes \(Q\) for a given \(B\).

In the undirected case the modularity matrix is symmetric but in the present case it is, in general, not, and the lack of symmetry will cause technical problems if we blindly attempt to duplicate the eigenvector-based machinery presented for undirected networks in [13]. Luckily, however, we can easily restore symmetry to our problem by adding (4) to its own transpose to give

\[
Q = \frac{1}{4m} s^T (B + B^T) s, \tag{6}
\]

The matrix \(B + B^T\) is now manifestly symmetric and it is on this symmetric matrix that we focus forthwith. Notice that \(B + B^T\) is not the same as the modularity matrix for a symmetrized version of the network in which direction is ignored and hence we expect methods based on the true directed modularity to give different results, in general, to methods based on the undirected version.

The leading constant \(1/4m\) in Eq. (6) is conventional, but makes no difference to the position of the maximum of \(Q\), so for the sake of clarity we neglect it in defining our optimization procedure.

Following [13], we now write \(s\) as a linear combination of the eigenvectors \(v_i\) of \(B + B^T\) thus: \(s = \sum_i a_i v_i\) with \(a_i = v_i^T \cdot s\). Then

\[
Q = \sum_i a_i v_i^T (B + B^T) \sum_j a_j v_j = \sum_i \beta_i (v_i^T \cdot s)^2, \tag{7}
\]

where \(\beta_i\) is the eigenvalue of \(B + B^T\) corresponding to eigenvector \(v_i\). Let us assume the eigenvalues to be labeled in decreasing order \(\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n\). Under the normalization constraint \(s^T \cdot s = n\) the maximum of \(Q\) is achieved when \(s\) is parallel to the leading eigenvector \(v_1\), but normally this solution is forbidden by the additional condition that \(s_i = \pm 1\). We do the best we can, however, and make \(s\) as close as possible to parallel with \(v_1\), meaning we choose the value of \(s\) that maximizes \(v_1^T \cdot s\). It is straightforward to show that this gives \(s_1 = +1\) if \(v_1^{(1)} > 0\) and \(s_1 = -1\) if \(v_1^{(1)} < 0\), where \(v_1^{(1)}\) is the ith element of \(v_1\). (If \(v_1^{(1)} = 0\) then \(s_1 = \pm 1\) are equally good solutions to the maximization problem.)

Thus we arrive at a simple algorithm for splitting a network: we calculate the eigenvector corresponding to the largest positive eigenvalue of the symmetric matrix \(B + B^T\) and then assign communities based on the signs of the elements of the eigenvector.

As in the undirected case, the spectral method typically provides an excellent guide to the broad outlines of the optimal partition, but may err in the case of individual vertices, a situation that can be remedied by adding a “fine-tuning” stage to the algorithm in which vertices are moved back and forth between communities in an effort to increase the modularity, until no further improvements can be made [13]. We have incorporated such a fine-tuning in all the calculations presented here.
So far we have discussed the division of a network into two communities. There are a variety of ways of generalizing the approach to more than two communities but the simplest, which we adopt here, is repeated bisection. That is, we first divide the network into two groups using the algorithm above and then divide those groups and so forth. The process stops when we reach a point at which further division does not increase the total modularity of the network.

The subdivision of a community contained within a larger network requires a slight generalization of the method above. Consider the change in modularity $\Delta Q$ of an entire network when a community $g$ within it is subdivided and, defining $s_i$ as before for vertices in $g$, we find

$$\Delta Q = \frac{1}{2m} \left[ \sum_{i,j \in g} (B_{ij} + B_{ji}) \frac{s_is_j + 1}{2} - \sum_{i,j \in g} (B_{ij} + B_{ji}) \right]$$

$$= \frac{1}{4m} \sum_{i,j \in g} \left[ (B_{ij} + B_{ji}) - \delta_{ij} \sum_{k \in g} (B_{ik} + B_{ki}) \right] s_is_j$$

$$= \frac{1}{4m} s_i^T (B^{(g)} + B^{(g)T}) s$$  \hspace{1cm} (8)

where we have made use of $s_i^2 = 1$ for all $i$ and

$$B^{(g)}_{ij} = B_{ij} - \delta_{ij} \sum_{k \in g} B_{ik}. \hspace{1cm} (9)$$

In other words, $B^{(g)}$ is the submatrix of $B$ for the subgraph $g$ with the sum of each row subtracted from the corresponding diagonal element. Although $B^{(g)}$, like $B$, is in general asymmetric, the sum $B^{(g)} + B^{(g)T}$ is symmetric and hence Eq. (8) has the same functional form as Eq. (6) and we can apply the same method to maximize $\Delta Q$.

Our complete algorithm for discovering communities or groups in a directed network is thus as follows. We construct the modularity matrix, Eq. (4), for the network and find the most positive eigenvalue of the symmetric matrix $B + B^T$ and the corresponding eigenvector. Each vertex is assigned to one of two groups depending on the sign of the corresponding element of the eigenvector and then we fine-tune the assignments as described above to maximize the modularity. We then further subdivide the communities using the same method, but with the generalized modularity matrix, Eq. (9), fine tuning after each division. If the algorithm finds no division giving a positive value of $\Delta Q$ for a particular community then we can increase the modularity no further by subdividing this community and we leave it alone. When all communities reach this state the algorithm ends.

We now give a number of examples of the application of our method. We consider four different directed networks of varying degrees of complexity, starting with a relatively simple but important example: the world wide web.

Weblogs or “blogs” are personal web sites on which their proprietors record brief thoughts on topics of their choosing, often with links to other blogs with related discussions. In a recent study, Adamic and Glance looked at a network of 1225 blogs focusing on US politics. In this network the vertices represent the blogs and there is a directed link between vertices if one blog links to another. Adamic and Glance also characterized the political persuasion of each blog as conservative or liberal based on textual content.

When fed into our community finding algorithm, the blog network divides into two clear communities, with one being composed almost entirely of conservative blogs and the other of liberal blogs. (The algorithm places 97% of the blogs characterized by Adamic and Glance as conservative in the first community and 93% of those characterized as liberal in the second.) The algorithm finds no subdivision of either community that gives any increase in the modularity, indicating that the network consists of only two tightly knit communities corresponding closely to the traditional left-right division of US politics. This serves as a particularly clear demonstration of the algorithm’s ability to find meaningful structure in network data. But on the other hand this particular network gives very similar results when analyzed using the undirected form of the spectral modularity algorithm, in which edge direction is entirely ignored. The principal interest in our algorithm derives from its ability to find structure in networks where the simpler undirected version fails, so let us turn to examples of this kind.

For illustrative purposes, we first consider an artificial computer-generated network, designed specifically to test the performance of the algorithm. In this network of 32 vertices, vertex pairs are connected by edges independently and uniformly at random with some probability $p$. The edges are initially undirected. The network is then divided into two groups of 16 vertices each and edges that fall within groups are assigned directions at random but edges between groups are biased so that they are more likely to point from group 1 to group 2 than vice versa.

FIG. 1: Community assignments for the two-community random network described in the text from (a) a standard undirected modularity maximization which ignores edge direction and (b) the algorithm of this paper. The shaded regions represent the communities discovered by the algorithms. The true community assignments are denoted by vertex shape.
By construction, there is no community structure to be found in this network if we ignore edge directions—the positions of the edges are entirely random—and this is confirmed in Fig. 1b, which shows the results of the application of the undirected modularity maximization algorithm. If we take the directions into account, however, using the algorithm presented in this paper, the two communities are detected almost perfectly: just one vertex out of 32 is misclassified—see Fig. 1b.

Even in networks where there is clear community structure contained in the positions of the edges it is still possible for the directions to contain additional useful information. As an example of this type of behavior, consider the network shown in Fig. 2 which has 32 vertices and three communities. For two of the communities, containing 14 vertices each, there is a high probability of connection between pairs of vertices that fall in the same community but a lower probability if one of the vertices is in a different community. Structure of this kind, in which edge direction does not play a role, can in principle be found by algorithms designed for undirected networks. The third community, however, is different. It has four vertices, each of which has a high probability of connection to every other vertex. The only feature that distinguishes this third community as separate is the direction of its edges—two of the four vertices have high probability of ingoing edges, the other two have high probability of outgoing edges, and there are also a small number of additional edges running from the former to the latter. These last edges are statistically surprising in the sense considered here and hence tend to bind the third community together.

Applied to this network, the standard undirected community detection algorithm finds the two large communities with ease, but the remaining community is not found and its vertices are dispersed by the algorithm among the other communities (Fig. 2a). Our directed algorithm, on the other hand, finds all three communities without difficulty (Fig. 2b). Again the algorithm has made use of information contained in the edge directions to identify community structures not accessible to previous methods.

Returning now to real-world networks, we show in Fig. 3 an example of the performance of our algorithm on, in this case, a word network. The network represents connections between a set of technical terms, such as “vertex” and “edge,” contained in a glossary of network jargon derived from recent review papers by Newman [3] and Boccaletti et al. [4]. Vertices in this network represent terms and there is a directed edge from one vertex to another if the first glossary term was used in the definition of the second. Because circular definitions are unhelpful and normally avoided, most edges in the network are not reciprocated.

Figure 3 shows the communities found in this network by our directed modularity algorithm. The algorithm finds six communities in this case that appear to correspond to groupings of terms clustered around a few basic concepts. For instance, one group deals with words describing basic network structure, such as “edge” and “graph,” while another deals with terms describing directed networks. A third group contains the terms “vertex” and “degree” and related concepts and the remaining groups are associated with clustering, communities, and paths respectively. Thus, the algorithm again appears to find meaningful structure in the network, of the type that could be useful in understanding the broader shape of otherwise poorly understood systems.

We have also applied the undirected modularity maximization algorithm to this same network, which results in four groups. Two of these are closely similar to ones found by the directed algorithm—the groups dealing with edges and with directed networks. The other groups, however, contain a mix of terms that do not correspond closely to any obvious network concepts, with words like “vertex,” “diameter,” “cycle,” and “motif” grouped together. As discussed above, the undirected algorithm

FIG. 2: Community assignments for the three-community random network described in the text as generated by (a) standard undirected modularity maximization and (b) the algorithm of this paper.

FIG. 3: The network of technical terms described in the text along with the community assignments determined by a standard undirected modularity maximization (boxes) and the algorithm of this paper (shaded groups).
has less information at its disposal, the directions of the edges having been discarded, so it is natural that it is unable to detect some of the structure found by its directed counterpart.

In summary, we have presented a method for detecting community structure in directed networks that makes explicit use of information contained in edge directions, information that most other algorithms discard. Our method is an extension of the established modularity maximization method widely used to determine community structure in undirected networks. We have applied the method to a variety of networks, both real and simulated, showing that it is able to recover known community structure in previously studied networks and extract additional and revealing structural information not available to algorithms that ignore edge direction. The computational efficiency of the algorithm is essentially identical to that of the corresponding algorithm for undirected networks and hence we see no reason to continue to use the undirected algorithm on directed graphs; we recommend the use of the full directed algorithm in all cases where researchers wish to analyze both edge placement and edge direction.

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