Stability of solutions for controlled nonlinear systems under perturbation of state constraints

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Problem statement

We consider a nonlinear system with unbounded control and state constraints

\[ x'(t) = f(t, x(t), u(t)), \quad \text{for a.e. } t \in [0, T], \quad (1) \]

\[ x(t) \in A_{0,t} := \{ x \mid h(t, x) \leq 0 \}, \quad \text{for all } t \in [0, T], \quad (2) \]

where \( f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N \) and \( h : [0, T] \times \mathbb{R}^N \to \mathbb{R}^P \). We call \( f \)-trajectories the solutions of (1) for measurable controls \( u(\cdot) \).
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• We are given a reference trajectory \( x(\cdot) \), such that \( h(0, x(0)) < 0 \), with control \( u(\cdot) \in L^\infty(0, T) \) satisfying (1)-(2). Our goal is to design a neighboring feasible trajectory \( x^\epsilon(\cdot) \) satisfying (1) and

\[ x^\epsilon(0) = x(0), \quad \| x(\cdot) - x^\epsilon(\cdot) \|_{L^\infty(0, T)} \leq \lambda, \quad \| u(\cdot) \|_{L^2(0, T)}^2 - \| u^\epsilon(\cdot) \|_{L^2(0, T)}^2 \leq \lambda, \quad \lambda \ll 1 \]
\[ x^\epsilon(t) \in A_{\epsilon, t} := \{ x \mid \epsilon + h(t, x) \leq 0 \} \text{ for all } t \in [0, T]. \quad (3) \]

Important for interior point methods and perturbations!
Problem statement (illustrated)
Theorem

Under assumptions (H-1)-(H-6), for any \( \lambda > 0 \), there exists \( \varepsilon > 0 \) and a \( f \)-trajectory \( x^\varepsilon(\cdot) \) on \([0, T]\) such that \( x^\varepsilon(0) = \bar{x}(0) \), \( x^\varepsilon(t) \in \text{Int } A_{\varepsilon, t} \) for all \( t \in [0, T] \), and

\[
\| \bar{x}(\cdot) - x^\varepsilon(\cdot) \|_{L^\infty(0, T)} \leq \lambda.
\]

Moreover if (H-7) is satisfied, then, for any mapping \( R(\cdot) \in C^0([0, T], \mathbb{R}^{M,M}) \) with positive semidefinite matrix values, one can choose \( \varepsilon > 0 \) and \( x^\varepsilon(\cdot) \) such that the controls \( u^\varepsilon(\cdot) \) satisfy

\[
\left| \| R(\cdot)^{1/2} \bar{u}(\cdot) \|_{L^2(0, T)}^2 - \| R(\cdot)^{1/2} u^\varepsilon(\cdot) \|_{L^2(0, T)}^2 \right| \leq \lambda.
\]
Some prior literature

\[ F(t, x) = \{ f(t, x, u) \mid u \in U(t, x) \} \] and \( x(t) \in K(t) = A_{0, t} := \{ x \mid h(t, x) \leq 0 \} \)

bounded \( F \): \( \exists c \in \mathbb{R}, \ F(t, x) \subset c(1 + \| x \|)B_N \) i.e. \( \| f(t, x, u) \| \leq c(1 + \| x \|) \)

- [Rampazzo, 1999]: bounded \( F \) and real-valued \( h(\cdot) \in C^2 \)
- [Bettiol et al., 2010]: bounded \( F \) and real-valued \( h(\cdot) \)
- [Bressan and Facchi, 2011]: bounded \( F \) and compact convex \( K \) both time-independent + relaxation
- [Bettiol and Vinter, 2011]: bounded \( F \) and \( K \) time-independent + inf-relaxation

Control-affine systems \( x'(t) = a(t, x) + b(t, x)u \) with \( u \in \mathbb{R}^m \) are unbounded!!
**Assumptions**

\[ A_\epsilon := \{(t, x) | t \in [0, T], x \in A_{\epsilon,t}\}. \]

**(H-1)** (Regular perturbation of \( A \))

\[
\forall \lambda > 0, \ \exists \epsilon > 0, \ \forall (t, x) \in A_0 \cap ([0, T] \times \|\bar{x}(\cdot)\|_{L^\infty(0,T)\mathcal{B}N}),
\]

\[
d_{A_{\epsilon,t}}(x) \leq \lambda.
\]

**(H-2)** (Uniform continuity from the right of \( d_{\partial A_{\epsilon,t}} \) w.r.t. \( \epsilon \) and \( t \)) There exist \( \epsilon_0 > 0, \Delta_0 > 0, \) and \( \omega_A(\cdot) \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) such that \( \omega_A(0) = 0 \) and, for all \( \epsilon \leq \epsilon_0, \) and all \( (t, x) \in A_0 \cap ([0, T] \times 2\|\bar{x}(\cdot)\|_{L^\infty(0,T)\mathcal{B}N}), \)

\[
\forall \delta \in [0, \min(\Delta_0, T - t)], \ \|d_{\partial A_{\epsilon,t+\delta}}(x) - d_{\partial A_{\epsilon,t}}(x)\| \leq \omega_A(\delta).
\]

**Discussion:** (H-1) and (H-2) are implied by \( C^{1,1} \)-regularity of \( h \) with a surjective Jacobian \( \frac{\partial h(t,x)}{\partial x} \) at all \( (t, x) \in \partial A_0. \)
Assumptions (cont’d)

\[ A_\epsilon := \{(t, x) \mid t \in [0, T], x \in A_{\epsilon, t}\}. \]

(H-3) (Sublinear growth of f w.r.t. x and u)

\[ \exists \theta(\cdot) \in L_+^2(0, T), \forall t \in [0, T], \forall x \in \mathbb{R}^N, \forall u \in \mathbb{R}^M, \]

\[ \|f(t, x, u)\| \leq \theta(t)(1 + \|x\| + \|u\|). \]

(H-4) (Inward-pointing condition) There exist \( \epsilon_0 > 0, M_u > 0, M_v > 0, \xi > 0, \) and \( \eta > 0 \) such that for all \( \epsilon \leq \epsilon_0 \) and all \( (t, x) \in (\partial A_\epsilon + (0, \eta \mathcal{B}_N)) \cap A_\epsilon \cap ([0, T] \times (1 + 2\|\bar{x}(\cdot)\|_{L^\infty(0, T)} \mathcal{B}_N)), \)

we can find \( u \in M_u \mathcal{B}_M \) such that \( v := f(t, x, u) \) belongs to \( M_v \mathcal{B}_N \) and

\[ y + \delta(v + \xi \mathcal{B}_N) \subset A_{\epsilon, t+\delta} \quad (4) \]

for all \( \delta \in [0, \xi] \) and all \( y \in (x + \xi \mathcal{B}_N) \cap A_{\epsilon, t} \)

\[ \hookrightarrow \text{i.e. } F(t, x) \cap \text{Int Tan}(A_{\epsilon, t}(x)) \neq \emptyset \text{ on the boundary (\text{Tan} = Clark tangent cone).} \]

Discussion: (H-3) prevents finite-time explosion of trajectories. (H-4) is paramount to the construction (existence of control to “correct” the trajectory).
We further ask the inward-pointing vector to be of bounded norm and bounded control, uniformly over $\partial A_\epsilon$. 

Inward pointing condition illustrated

Yes!

No!
Assumptions (cont’d)

\[ R := e^{\|\theta(\cdot)\|_{L^1(0, T)}} \left[ 1 + \|\bar{x}(\cdot)\|_{L^\infty(0, T)} + (1 + M_u)\|\theta(\cdot)\|_{L^1(0, T)} \right. \]
\[ + \left. \|\theta(\cdot)\|_{L^2(0, T)}(\|\bar{u}(\cdot)\|_{L^2(0, T)} + \|\beta_u(\cdot)\|_{L^2(0, T)}) \right]. \quad (5) \]

(H-6) (Local Lipschitz continuity of \( f \) w.r.t. \( x \))

\[ \exists k_f(\cdot) \in L^2_+(0, T), \quad \forall t \in [0, T], \quad \forall x, y \in R^B_N, \quad \forall u \in (M_u + \|\bar{u}(\cdot)\|_{L^\infty(0, T)})B_M, \]
\[ \|f(t, x, u) - f(t, y, u)\| \leq k_f(t)\|x - y\|. \]

Discussion: (H-6) guarantees uniqueness and encompasses control-affine systems of the form
\[ x'(t) = a(t, x) + b(t, x)u \] with \( \tilde{k}_f(t) \)-Lipschitz functions \( a(t, \cdot) \) and \( b(t, \cdot) \), for some \( \tilde{k}_f(\cdot) \in L^2(0, T) \).
Assumptions (cont’d)

(H-5) (Left local absolute continuity of $f$ w.r.t. $t$)

$$\exists \gamma(\cdot) \in L^1_+(0, T), \exists \beta_u(\cdot) \in L^2_+(0, T), \forall 0 \leq s < t \leq T, \forall x \in (1 + 2\|\overline{x}(\cdot)\|_{L^\infty(0, T)})B_N,$$

$$\forall u_s \in (M_u + \|\overline{u}(s)\|)B_M, \exists u_t \in u_s + \beta_u(s)B_M,$$

$$\|f(t, x, u_t) - f(s, x, u_s)\| \leq \int_s^t \gamma(\sigma) d\sigma.$$ 

(H-7) (Hölderian selection of the controls in (H-5))

$$\exists \gamma(\cdot) \in L^1_+(0, T), \exists \alpha \in ]0, 1], \exists k_u(\cdot) \in L^2_+(0, T),$$

$$\forall 0 \leq s < t \leq T, \forall x \in (1 + 2\|\overline{x}(\cdot)\|_{L^\infty(0, T)})B_N,$$

$$\forall u_s \in (M_u + \|\overline{u}(s)\|)B_M, \exists u_t \in u_s + (t - s)^\alpha k_u(s)B_M,$$

$$\|f(t, x, u_t) - f(s, x, u_s)\| \leq \int_s^t \gamma(\sigma) d\sigma.$$ 

Discussion: (H-5) was introduced to tackle discontinuities in the dynamics, and showcased on a civil engineering example [Bettiol et al., 2012, Section 4]. We adapt it to control systems and refine it in (H-7).
Idea of the proof

The overall strategy to construct a neighboring $A_\epsilon$-feasible trajectory can be related to that of [Bettiol et al., 2012]. Modifying it to unbounded controls and time-varying constraints is however not straightforward.

Consider small subintervals $[0, T] = \bigcup_{i \in [0, N_0-1]} [t_i, t_{i+1}]$ and proceed iteratively.

- If the $i$th-trajectory stays in $A_\epsilon$ over $[t_i, t_{i+1}]$, move to the next time interval.

- Otherwise, (H-4) provides us with an inward-pointing control $u_i$ to stay in $A_\epsilon$ for a short time.
  
  - apply $u_i$ on $[t_i, t_i + t_\epsilon]$,
  - apply $\bar{u}(\cdot - t_\epsilon)$ on $[t_i + t_\epsilon, t_{i+1}]$,
  - apply $\bar{u}(\cdot)$ over $[t_{i+1}, T]$

By adequately choosing $t_\epsilon$, we prove that the resulting control after $N_0$ iterations is $L^2$-close from $\bar{u}(\cdot)$ and that the obtained trajectory is in $A_\epsilon$. 

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If the \(i\)th-trajectory stays in \(A_\epsilon\) over \([t_i, t_{i+1}]\), move to the next time interval. (H-4) provides us with an inward-pointing control \(u_i\) to stay in \(A_\epsilon\) for a short time.

- apply \(u_i\) on \([t_i, t_i + t_\epsilon]\),
- apply \(\bar{u}(\cdot - t_\epsilon)\) on \([t_i + t_\epsilon, t_{i+1}]\),
- apply \(\bar{u}(\cdot)\) over \([t_{i+1}, T]\)
Consider small subintervals 

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and proceed iteratively.

If the \( i \)-th trajectory stays in \( A_{\epsilon} \) over \([t_i, t_{i+1}]\), move to the next time interval. (H-4) provides us with an inward-pointing control \( u_i \) to stay in \( A_{\epsilon} \) for a short time.

- apply \( u_i \) on \([t_i, t_i + t_\epsilon]\),
- apply \( \bar{u}(\cdot - t_\epsilon) \) on \([t_i + t_\epsilon, t_{i+1}]\),
- apply \( \bar{u}(\cdot) \) over \([t_{i+1}, T]\)

By adequately choosing \( t_\epsilon \), we prove that the resulting control after \( N_0 \) iterations is \( L^2 \)-close from \( \bar{u}(\cdot) \) and that the obtained trajectory is in \( A_{\epsilon} \).
Example: Consider an electric motor

\[ x'(t) = a(t, x) + b(t, u), \]

with a bounded \( a \in C^{1,1}([0, 2] \times \mathbb{R}, \mathbb{R}) \) and constraints \( h(x) = 1 - |x| \), for controls \( u \in \mathbb{R} \). The motor suffers an incident at \( T = 1 \). If it is a power surge

\[
 b(t, u) = \tilde{b}(t)u = \begin{cases} 
 u & \text{if } t \in [0, 1] \\
 u/\sqrt{t-1} & \text{if } t \in ]1, 2] 
\end{cases},
\]

then (H-3) holds for \( \theta \equiv \tilde{b} + \|f\|_{\infty} \) and so does (H-7) after some computation. If the incident consists in a power decline

\[
 b(t, u) = \begin{cases} 
 \arctan(u) & \text{if } t \in [0, 1] \\
 (1 - \frac{\sqrt{t-1}}{2}) \arctan(u) & \text{if } t \in ]1, 2] 
\end{cases},
\]

then the system is bounded and (H-7) holds with \( u_t = u_s, \gamma(\sigma) = \frac{1}{4\sqrt{\sigma-1}} \) for \( \sigma \in ]1, 2] \) and \( \gamma(\sigma) = 0 \) otherwise. In both cases (H-4) is satisfied, so perturbing the constraints still allows for a trajectory and control close to the reference ones as per Theorem 1.
We have proven that one can approximate trajectories of systems with unbounded control (e.g. control-affine) under assumptions similar to those of bounded systems.

Systems $\tilde{f}$ with Lipschitz (or Hölderian) control constraints $t \sim U(t)$ can be considered by projecting over $U(t)$, i.e. $f(t, x, u) = \tilde{f}(t, x, \text{proj}_{U(t)}(u))$ if $f$ satisfies the above assumptions.

State constraints of order 2 (or more), e.g. $\ddot{x} = u$ with $x$ constrained, do not enter into the proposed framework (requires Lie brackets, see Franco Rampazzo’s recent work)
Bettiol, P., Bressan, A., and Vinter, R. (2010).
On trajectories satisfying a state constraint: $W^{1,1}$ estimates and counterexamples.
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