Spectral portraits and the resolvent growth of a model problem associated with the Orr–Sommerfeld equation

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The Orr–Sommerfeld equation is obtained by linearization of the Navier–Stokes system in the infinite spatial layer \((x, \xi, \eta) \in \mathbb{R}^3\), where \(|x| \leq 1\) and \((\xi, \eta) \in \mathbb{R}^2\). This equation has the form

\[
(D^2 - \alpha^2)y - i\alpha R [q(x)(D^2 - \alpha^2) - q''(x)] y = i\alpha R \lambda (D^2 - \alpha^2)y.
\]

Here \(D = d/dx\), \(\alpha\) is the wave number \((\alpha \neq 0)\) appearing after the separation of the variables \((\xi, \eta) \in \mathbb{R}^2\), \(R\) is the Reynolds number, \(q(x)\) is the stationary unperturbed velocity profile and \(\lambda\) is the spectral parameter. Usually the boundary conditions

\[
y(\pm 1) = y'(\pm 1) = 0
\]

are associated with equation (1).

Our main goal is to understand the spectrum behaviour of problem (1), (2) as the Reynolds number \(R\) tends to the infinity (this means that the liquid becomes to be ideal). For these purposes we associate with (1), (2) a problem of the form

\[
-i\varepsilon^2 z'' + q(x)z = \lambda z
\]

assuming that \(q(x)\) is the same as in (1) and \(\varepsilon^2 = (\alpha R)^{-1}\). It turns out that this problem serves as a good model for the study of the original Orr–Sommerfeld problem (1)–(2). Now the question is: for which function \(q(x)\) we can describe the spectral portraits of problem (3), (4) as \(\varepsilon \to 0\)? The answer is not simple even for particular functions \(q(x)\), for example \(q(x) = x^2\) or \(q(x) = (x - \alpha)^2\), \(\alpha \in \mathbb{R}\). Only in the case \(q(x) = x\) the problem can be solved explicitly [1], [2]. At the moment we have general result only for analytic monotonous functions.

We shall say that a function \(q(x)\) belongs to the class \(AM\) if the following conditions hold: \(q(x)\) is analytic on the segment \([-1, 1]\) and admits an analytic continuation in a domain \(G\) such that the range \(q(G)\) contains the semistrip \(\Pi = \{\lambda \mid a < \text{Re} \lambda < b, \text{Im} \lambda < 0\}\) where \(a = q(-1)\), \(b = q(1)\). Moreover, if \(D = q^{-1}(\Pi) \subset G\), then \(q(z)\) is continuous on the closure \(\overline{D}\) and the map \(q(z) : \overline{D} \to \Pi\) is one to one.

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Assuming that \( q(x) \in AM \) we introduce the functions

\[
Q^\pm(\lambda) = \int_{\xi_\lambda}^{\pm 1} \sqrt{i(q(\xi) - \lambda)} \, d\xi, \quad \lambda \in \Pi,
\]

where \( \xi_\lambda \in \mathcal{D} \) is the root of the equation \( q(\xi) - \lambda = 0 \), and the function

\[
Q(\lambda) = \int_{-1}^{1} \sqrt{i(q(x) - \lambda)} \, dx, \quad \lambda \in \Pi.
\]

These functions are analytic in \( \Pi \) and continuous in \( \mathbb{C} \), provided \( q(x) \in AM \).

The following theorem is a generalization of the previous author’s result \cite{3}.

**Theorem 1.** Let \( q(x) \in AM \). Then the curves \( \tilde{\gamma}_\pm \) determined in the semistrip \( \Pi \) by the equations \( \text{Re} \, Q^\pm(\lambda) = 0 \) have the only intersection point in \( \Pi \), say \( \lambda_0 \). The curve \( \tilde{\gamma}_\infty \) determined in \( \Pi \) by the equation \( \text{Re} \, Q(\lambda) = 0 \), is a function with respect to the imaginary axis and intersect the curves \( \tilde{\gamma}_\pm \) in the only point \( \lambda_0 \). Denote by \( \gamma_+ \), \( \gamma_- \) and \( \gamma_\infty \) the parts of \( \tilde{\gamma}_+ \), \( \tilde{\gamma}_- \) and \( \tilde{\gamma}_\infty \) connecting the intersection point \( \lambda_0 \) with the points \( a \), \( b \) and \( -i\infty \), respectively. Denote also

\[
\Gamma = \gamma_+ \cup \gamma_- \cup \gamma_\infty.
\]

Then the spectrum of model problem \cite{3}, \cite{4} is concentrated along \( \Gamma \) as \( \varepsilon \to 0 \), i.e. \( \forall \delta > 0 \) there are no eigenvalues outside the \( \delta \)-neighbourhood of \( \Gamma \), provided that \( \varepsilon < \varepsilon_0 = \varepsilon_0(\delta) \). Moreover, the eigenvalues of the problem near the curves \( \gamma_+ \), \( \gamma_- \) and \( \gamma_\infty \) are determined from the conditions

\[
i \int_{\xi_\lambda}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \varepsilon \pi (k - 1/4), \quad k \in \mathbb{Z},
\]

\[
i \int_{-1}^{\xi_\lambda} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \varepsilon \pi (k - 1/4), \quad k \in \mathbb{Z},
\]

\[
i \int_{-1}^{1} \sqrt{i(q(x) - \lambda)} \, dx = \varepsilon \pi k, \quad k \in \mathbb{Z}.
\]

More precisely, let \( \mu^+_k \), \( \mu^-_k \) and \( \mu_k \) are solutions of these equations lying on the curves \( \gamma_+ \), \( \gamma_- \) and \( \gamma_\infty \) outside \( \delta \)-neighbourhoods of the points \( a \), \( b \) and \( \lambda_0 \). Then there is a constant \( C = C(\delta) \) such that each circle of the radius \( C\varepsilon^2 \) centered at the points \( \mu^-_k \), \( \mu^+_k \) and \( \mu_k \) contains the only eigenvalue. The eigenvalue counting functions along the curves \( \gamma_+ \), \( \gamma_- \) and \( \gamma_\infty \) have representations

\[
N_\pm(\lambda) = \pm \frac{1}{i \pi \varepsilon} \int_{\xi_\lambda}^{\pm 1} \sqrt{i(q(\xi) - \lambda)} \, d\xi + O(1),
\]

\[
N(\lambda) = \frac{1}{i \pi \varepsilon} \int_{-1}^{1} \sqrt{i(q(\xi) - \lambda)} \, dx + O(1).
\]
where $O(1)$ is uniformly bounded as $\varepsilon \to 0$ provided that $\lambda$ lies outside some neighbourhoods of the points $a$, $b$ and $\lambda_0$.

Our next results clarify the connection of model problem \[3 \], \[4 \] with original Orr-Sommerfeld problem \[1 \]-\[2 \].

**Theorem 2.** Let $q(x) \in AM$. Denote by $\Delta_m(\lambda, \varepsilon)$ and $\Delta_{OS}(\lambda, \varepsilon)$ the characteristic determinants of the model and Orr-Sommerfeld problems, respectively. Then

$$\Delta_{OS}(\lambda, \varepsilon) = \Delta_m(\lambda, \varepsilon) R(\lambda)(1 + O(\varepsilon)),$$

where $R(\lambda) \neq 0$ for $\lambda \in \Pi$ and $|O(\varepsilon)| \leq C\varepsilon$ for $\lambda \in \Pi \setminus \Gamma_\delta$. Here $\Gamma_\delta$ is the $\delta$-neighbourhood of the limit spectral graph $\Gamma$ determined in Theorem 1, and the constant $C$ depends only on $\delta$.

The technique, developed in \[4 \] allows to prove the similarity of the zero counting functions for a pair of holomorphic functions, provided that they have similar growth near the curves where the zeros are located (in our case on the boundary of $\Gamma_\delta$). Hence, Theorem 1 leads to the following result.

**Theorem 3.** Let $q(x) \in AM$. Then the limit spectral graph of the Orr-Sommerfeld problem coincides with the graph $\Gamma$ of the corresponding model problem, i.e. \forall $\delta > 0$ there are no eigenvalues of problem \[1 \], \[2 \] outside the $\delta$-neighbourhood of $\Gamma$, provided that $R > R_0 = R_0(\delta)$. Moreover, the eigenvalue counting functions $N_\pm(\lambda)$ and $N_\infty(\lambda)$ along the curves $\gamma_\pm$ and $\gamma_\infty$ for problem \[1 \], \[2 \] have representations

$$N_\pm(\lambda) = \pm \frac{1}{i\pi\varepsilon} \int_{\xi_\lambda}^{+1} \sqrt{i(q(\xi) - \lambda)} \, d\xi \, (1 + o(1)),$$

$$N(\lambda) = \frac{1}{i\pi\varepsilon} \int_{-1}^{-1} \sqrt{i(q(\xi) - \lambda)} \, d\xi \, (1 + o(1))$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly for all $\lambda$ lying outside some neighbourhoods of the points $a$, $b$ and $\lambda_0$.

Next our goal is to understand the resolvent behaviour of the operators corresponding to problems in question. Denote by $z_k = z_k(x, \varepsilon)$ the normed eigenfunctions of the model problems (for simplicity we assume that there is no associated functions, although they do appear for some discrete values of $\varepsilon_j \to 0$). It is known that the system \{$z_n$\}$_{n=1}^\infty$ forms a Riesz basis in $L_2(-1, 1)$, therefore, the Gramm matrix \{$(z_k, z_j)$\}$_{k,j=1}^\infty$ generates a bounded and boundedly invertible operator $T$ in $l_2$. The number

$$C = C(\varepsilon) = \|T\| \cdot \|T^{-1}\|$$

characterizes "the quality" of the basis \{$z_k$\}$_{1}^\infty$. It is called the Riesz constant.

Let $L = L(\varepsilon)$ be the operator in $L_2(-1, 1)$ associated with model problem \[3 \], \[4 \]. It can be easily proved that

$$\|(L - \lambda)^{-1}\| \leq \frac{C(\varepsilon)}{\text{dist}(\lambda, \sigma(L))}$$

where $C(\varepsilon)$ is defined by (5). The question is: how rapidly grows the constant $C(\varepsilon)$?
Theorem 4. Denote by $\Omega$ the domain bounded by the curves $\gamma_+$, $\gamma_-$ and the segment $[a, b]$. Then, given compact $K$ in $\Omega$ there are numbers $\tau > 0$ and $\varepsilon_0 > 0$ such that $\| (L - \lambda)^{-1} \| \geq e^{\tau/\varepsilon}$ for all $\lambda \in K$ and $\varepsilon < \varepsilon_0$. In particular, the constant $C(\varepsilon)$ defined in [5] grows exponentially, i.e. there is a number $\eta > 0$ such that $C(\varepsilon) \geq e^{\eta/\varepsilon}$.

It was proved in [5] that there is no operator in $L_2(-1, 1)$ associated with Orr-Sommerfeld problem. However, there is such an operator in the Sobolev space $W_2^1[-1, 1]$. Denote this operator by $S$ and consider the operator $T$ in $l_2$ generated by the Gramm matrix $\{(y_k, y_j)\}_{k,j=1}^\infty$ assuming that $y_k$ are the normed eigenfunctions of the operator $S$ in the space $W_2^1[-1, 1]$ and the scalar product is taken in the same space.

Theorem 5. The resolvent $(S - \lambda)^{-1}$ and the constant $C(\varepsilon)$ defined by [5] admit the same estimates from below as in Theorem 4.

The estimates from below for the resolvents $(L - \lambda)^{-1}$ and $(S - \lambda)^{-1}$ confirm the following fact: for sufficiently small $\varepsilon > 0$ the pseudospectra of these operators occupies ”almost whole” domain $\Omega$. So, the last theorems prove some observations of Trefethen [6] and Reddy, Schmidt and Henningson [7] on the pseudospectra of the Orr–Sommerfeld operator. We remark also that a weaker estimate $C(\varepsilon) \geq \tau \varepsilon^{1/3}$ for the Riesz constant was obtained by Stepin [8].

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