CONVERGENCE PROPERTIES OF INEXACT LEVENBERG-MARQUARDT METHOD UNDER HÖLDERIAN LOCAL ERROR BOUND

Haiyan Wang
School of Mathematical Sciences, Shanghai Jiao Tong University
Shanghai 200240, China

Jinyan Fan∗
School of Mathematical Sciences, Shanghai Jiao Tong University
Key Lab of Scientific and Engineering Computing (Ministry of Education)
Shanghai 200240, China

(Communicated by Liqun Qi)

Abstract. In this paper, we study convergence properties of the inexact Levenberg-Marquardt method under the Hölderian local error bound condition and the Hölderian continuity of the Jacobian. The formula of the convergence rates are given, which are functions with respect to the Levenberg-Marquardt parameter, the perturbation vector, as well as the orders of the Hölderian local error bound and Hölderian continuity of the Jacobian.

1. Introduction. Consider the system of nonlinear equations

\[ F(x) = 0, \]  

where \( F(x) : \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable. In some cases, (1) may have no solutions. Throughout the paper, we assume that the solution set of (1), denoted by \( X^* \), is nonempty.

The Levenberg-Marquardt (LM) method is one of the most important methods for nonlinear equations (cf. [13, 14]). At each iteration, it solves the linear system

\[ (J_k^T J_k + \lambda_k I)d = -J_k^T F_k \]  

to obtain the LM step \( \bar{d}_k \), where \( F_k = F(x_k) \), \( J_k = F'(x_k) \) is the Jacobian, and \( \lambda_k > 0 \) is the LM parameter introduced to overcome the singularity or near singularity of \( J_k \). The LM method is a regularization of the Gauss-Newton method.

It is usually expensive to solve (2) exactly especially for large-scale problems, while it is often efficient to use inexact methods to find an approximate solution that satisfies some appropriate conditions. In the inexact LM method, the inexact LM step \( d_k \) satisfies the system

\[ (J_k^T J_k + \lambda_k I)d = -J_k^T F_k + p_k, \]  

2010 Mathematics Subject Classification. 65K05, 90C30.

Key words and phrases. Nonlinear equations, inexact Levenberg-Marquardt method, Hölderian local error bound, Hölderian continuity, convergence rate.

The authors are supported by Chinese NSF grants 11971309.

∗ Corresponding author: Jinyan Fan.
where \( p_k \in \mathbb{R}^n \) is a perturbation vector and measures how exactly the linear system (2) is solved. When \( p_k = 0 \), the inexact LM method becomes the LM method.

Facchinei and Kanzow [3] showed that, if \( \lambda_k \to 0 \) and \( \| p_k \| \leq o(\| J_k^T F_k \|) \), then the inexact LM method converges superlinearly under the nonsingularity and Lipschitz continuity of the Jacobian; moreover, if \( \lambda_k = O(\| J_k^T F_k \|) \) and \( \| p_k \| = O(\| J_k^T F_k \|^2) \), then it converges quadratically.

However, the assumption of the nonsingularity of the Jacobian is too strong. The local error bound condition is a weaker condition than the nonsingularity. It requires that

\[
\text{dist}(x, X^*) \leq \frac{1}{c} \| F(x) \|, \quad \forall x \in N(x^*)
\]

holds for some constant \( c > 0 \), where \( \text{dist}(x, X^*) \) is the distance from \( x \) to \( X^* \), \( \| \cdot \| \) refers to the 2-norm, and \( N(x^*) \) is some neighbourhood of \( x^* \in X^* \).

Suppose \( \lambda_k = \| F_k \|^\alpha \) and \( \| p_k \| = \| F_k \|^\alpha + \theta \), where \( \alpha \) and \( \theta \) are positive constants. Dan et al. [2], Fan et al. [7] and Fischer et al. [11] gave the convergence rates of the inexact LM method for some special \( \alpha \) and \( \theta \), respectively, under the local error bound condition and Lipschitz continuity of the Jacobian. Later in [9], Fan et al. obtained the formula of the convergence rate of the inexact LM method under the same conditions. It is a continuous function with respect to both \( \alpha \) and \( \theta \). Interested readers are referred to [4, 10, 5, 6, 19] for related work.

In real applications, some nonlinear equations may not satisfy the local error bound condition, but satisfy the Hölderian local error bound condition.

**Definition 1.1.** We say \( F(x) \) provides a Hölderian local error bound of order \( \gamma \in (0, 1] \) in some neighbourhood of \( x^* \in X^* \), if there exists a constant \( c > 0 \) such that

\[
\text{dist}(x, X^*) \leq \frac{1}{c} \| F(x) \|^\gamma, \quad \forall x \in N(x^*).
\]

By (4) and (5), we can see that the Hölderian local error bound condition is more generalized; it includes the local error bound condition as a special case when \( \gamma = 1 \). Thus, the local error bound condition is stronger. For example, the Powell singular function

\[
h(x_1, x_2, x_3, x_4) = (x_1 + 10x_2, \sqrt{5}(x_3 - x_4), (x_2 - 2x_3)^2, \sqrt{10}(x_1 - x_4)^2)^T
\]

satisfies the Hölderian local error bound condition of order \( \frac{1}{2} \) around the zero point, but does not satisfy the local error bound condition (cf. [1]).

We discussed the convergence results of the LM method under the Hölderian local error bound condition and the Hölderian continuity of the Jacobian [21, 1, 18]. In this paper, we aim to investigate the local convergence of the inexact LM method under the Hölderian local error bound condition and Hölderian continuity of the Jacobian. We first study the inexact LM method with \( \lambda_k = \| F_k \|^\alpha \) and \( p_k = \| F_k \|^\alpha + \theta \), then consider \( \lambda_k = \| J_k^T F_k \|^\alpha \) and \( p_k = \| J_k^T F_k \|^\alpha + \theta \). The formulas of the convergence rates of the inexact LM method are given. They are functions with respect to \( \alpha \), \( \theta \) and the orders of the Hölderian local error bound and Hölderian continuity of the Jacobian as well.

The paper is organized as follows. In Section 2, we study the convergence rate of the inexact LM method with \( \lambda_k = \| F_k \|^\alpha \) and \( p_k = \| F_k \|^\alpha + \theta \). The convergence rate of the inexact LM method with \( \lambda_k = \| J_k^T F_k \|^\alpha \) and \( p_k = \| J_k^T F_k \|^\alpha + \theta \) is given in Section 3. Finally we conclude the paper in Section 4.
2. **Convergence rate of inexact LM method with** \( \lambda_k = \| F_k \|^\alpha \). In this section, we discuss the convergence rate of the inexact LM method. The next iterate \( x_{k+1} \) is computed by

\[
x_{k+1} = x_k + d_k,
\]

where \( d_k \) is given by (3) with

\[
\lambda_k = \| F_k \|^\alpha \quad \text{and} \quad \| p_k \| = \| F_k \|^\alpha + \theta.
\]

Here \( \alpha, \theta > 0 \).

We make the following assumption.

**Assumption 2.1.** (a) \( F(x) \) provides a Hölderian local error bound of order \( \gamma \in (0, 1] \) in some neighbourhood of \( x^* \in X^* \), i.e., there exist constants \( c > 0 \) and \( 0 < b < 1 \) such that

\[
dist(x, X^*) \leq \frac{1}{c} \| F(x) \|^{\gamma}, \quad \forall x \in N(x^*, b),
\]

where \( N(x^*, b) = \{ x \in \mathbb{R}^n : \| x - x^* \| \leq b \} \).

(b) \( J(x) \) is Hölderian continuous of order \( \nu \in (0, 1] \), i.e., there exists a constant \( \kappa_{bh} > 0 \) such that

\[
\| J(x) - J(y) \| \leq \kappa_{bh} \| x - y \|^\nu, \quad \forall x, y \in N(x^*, b). \tag{10}
\]

Note that, by (10), we have

\[
\| F(y) - F(x) - J(x)(y - x) \| = \left\| \int_0^1 J(x + t(y - x))(y - x)dt - J(x)(y - x) \right\|
\leq \| y - x \| \int_0^1 \| J(x + t(y - x)) - J(x) \| dt
\leq \kappa_{bh} \| y - x \|^{1+\nu} \int_0^1 t^\nu dt
= \frac{\kappa_{bh}}{1+\nu} \| y - x \|^{1+\nu}. \tag{11}
\]

Thus, there exists a constant \( \kappa_{bf} > 0 \) such that

\[
\| F(y) - F(x) \| \leq \kappa_{bf} \| y - x \|, \quad \forall x, y \in N(x^*, \frac{b}{2}). \tag{12}
\]

In the following, we denote by \( \bar{x}_k \) the vector in \( X^* \) that is closest to \( x_k \), i.e.,

\[
\| \bar{x}_k - x_k \| = dist(x_k, X^*). \tag{13}
\]

Without loss of generality, we assume that the sequence \( \{ x_k \} \) generated by (7) converges to \( X^* \) and lies in \( N(x^*, \frac{b}{4}) \).

**Lemma 2.2.** Under Assumption 2.1, if \( 0 < \alpha < 2\gamma(1 + \nu) \), then there exists a constant \( \bar{c} > 0 \) such that

\[
\| d_k \| \leq \bar{c}\| \bar{x}_k - x_k \|^{\min\{1, 1+\nu, \frac{\gamma}{\nu}\}}. \tag{14}
\]

**Proof.** Since \( x_k \in N(x^*, \frac{b}{2}) \), we have

\[
\| \bar{x}_k - x^* \| \leq \| \bar{x}_k - x_k \| + \| x_k - x^* \| \leq \frac{b}{2}. \tag{15}
\]

This implies that \( \bar{x}_k \in N(x^*, \frac{b}{2}) \). Hence, it follows from (9) and (12) that

\[
c^{\alpha} \| \bar{x}_k - x_k \|^\gamma \leq \lambda_k = \| F_k \|^\alpha \leq \kappa_{bf}^\alpha \| \bar{x}_k - x_k \|^\alpha \tag{16}
\]
Define
\[ \phi_k(d) = \|F_k + J_k d\|_2^2 + \lambda_k \|d\|_2^2. \] (17)
Then, the LM step \( \bar{d}_k \) given by (2) is the minimizer of \( \phi_k(d) \). By (11) and (16),
\[ \|\bar{d}_k\| \leq \frac{\phi_k(\bar{x}_k - x_k)}{\lambda_k} \leq \frac{\|F_k + J_k (\bar{x}_k - x_k)\|_2^2 + \lambda_k \|\bar{x}_k - x_k\|_2^2}{\lambda_k} \]
\[ \leq \frac{\kappa_{bf}^2 c^{-\frac{\alpha}{2}}}{(1 + v)^2} \|\bar{x}_k - x_k\|_2^{2 + 2v - \frac{2}{\phi}} + \|\bar{x}_k - x_k\|_2^2 \]
\[ \leq \left( \frac{\kappa_{bf}^2 c^{-\frac{\alpha}{2}}}{(1 + v)^2} + 1 \right) \|\bar{x}_k - x_k\|_2^2 \min\{1, 1 + v - \frac{2}{\phi}\}. \] (18)
Thus, by (8) and (12),
\[ \|d_k\| \leq \|\bar{d}_k\| + \|d_k - \bar{d}_k\| \]
\[ = \|d_k\| + \|(J_k^T J_k + \lambda_k I)^{-1} p_k\| \]
\[ \leq \|\bar{d}_k\| + \|p_k\| = \|\bar{d}_k\| + \|F_k\|^{\theta} \]
\[ \leq \sqrt{\frac{\kappa_{bf}^2 c^{-\frac{\alpha}{2}}}{(1 + v)^2} + 1} \|\bar{x}_k - x_k\|_2^{\min\{1, 1 + v - \frac{2}{\phi}\}} + \kappa_{bf}^\theta \|\bar{x}_k - x_k\|_2^\theta \]
\[ \leq \tilde{c} \|\bar{x}_k - x_k\|_2^{\min\{1, 1 + v - \frac{2}{\phi}, \theta\}}, \] (19)
where \( \tilde{c} = \sqrt{\frac{\kappa_{bf}^2 c^{-\frac{\alpha}{2}}}{(1 + v)^2} + 1} + \kappa_{bf}^\theta \). The proof is completed. \( \square \)

Next we use the singular value decomposition (SVD) technique to derive the convergence rate of the inexact LM method (7). Suppose the SVD of \( J(\bar{x}_k) \) is
\[ J(\bar{x}_k) = \bar{U}_k \Sigma_k \bar{V}_k^T = (\bar{U}_{k,1}, \bar{U}_{k,2}) \left( \begin{array}{c} \Sigma_{k,1} \ 0 \\ \bar{V}_{k,1}^T \\ \bar{V}_{k,2}^T \end{array} \right) = \bar{U}_{k,1} \Sigma_{k,1} \bar{V}_{k,1}^T, \] (20)
where \( \Sigma_{k,1} = \text{diag}(\sigma_{k,1}, \ldots, \sigma_{k,r}) > 0 \). Suppose the SVD of \( J_k \) is
\[ J_k = U_k \Sigma_k V_k^T = (U_{k,1}, U_{k,2}) \left( \begin{array}{c} \Sigma_{k,1} \\ \Sigma_{k,2} \end{array} \right) \left( \begin{array}{c} V_{k,1}^T \\ V_{k,2}^T \end{array} \right) \]
\[ = U_{k,1} \Sigma_{k,1} V_{k,1}^T \]
\[ + U_{k,2} \Sigma_{k,2} V_{k,2}^T, \] (21)
where \( \Sigma_{k,1} = \text{diag}(\sigma_{k,1}, \ldots, \sigma_{k,r}) > 0 \) and \( \Sigma_{k,2} = \text{diag}(\sigma_{k,r+1}, \ldots, \sigma_{k,n}) \geq 0 \). In the following, if the context is clear, we suppress the subscription \( k \) in \( U_{k,i}, \Sigma_{k,i} \) and \( V_{k,i} \), and write
\[ J_k = U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T \] (22)
for convenience.

By some computations, we have
\[ d_k = -(J_k^T J_k + \lambda_k I)^{-1} J_k^T F_k + (J_k^T J_k + \lambda_k I)^{-1} p_k \]
\[ = -V_1 (\Sigma_1^2 + \lambda_k I)^{-1} \Sigma_1 U_1^T F_k - V_2 (\Sigma_2^2 + \lambda_k I)^{-1} \Sigma_2 U_2^T F_k \]
\[ + V_1 (\Sigma_1^2 + \lambda_k I)^{-1} V_1^T p_k + V_2 (\Sigma_2^2 + \lambda_k I)^{-1} V_2^T p_k, \] (23)
and
\[ F_k + J_k d_k = F_k - J_k (J_k^T J_k + \lambda_k I)^{-1} J_k^T F_k + J_k (J_k^T J_k + \lambda_k I)^{-1} p_k \]
\[ = \lambda_k U_1 (\Sigma_1^2 + \lambda_k I)^{-1} U_1^T F_k + \lambda_k U_2 (\Sigma_2^2 + \lambda_k I)^{-1} U_2^T F_k \]  
(24)

The estimations of \(\|U_1 U_1^T F_k\|\) and \(\|U_2 U_2^T F_k\|\) are given as follows.

**Lemma 2.3.** Under Assumption 2.1, we have

(i) \(\|U_1 U_1^T F_k\| \leq \kappa_{hf} \|\bar{x}_k - x_k\|\);

(ii) \(\|U_2 U_2^T F_k\| \leq 2\kappa_{hj} \|\bar{x}_k - x_k\|^{1+v}\).

**Proof.** Result (i) follows immediately from (12). By Assumption 2.1 (b) and the theory of matrix perturbation [17], we have

\[ \|\text{diag}(\Sigma_1 - \Sigma_{k,1}, \Sigma_2)\| \leq \|J_k - J(\bar{x}_k)\| \leq \kappa_{hf} \|\bar{x}_k - x_k\|^v. \]  
(25)

Thus,

\[ \|\Sigma_1 - \Sigma_{k,1}\| \leq \kappa_{hf} \|\bar{x}_k - x_k\|^v, \quad \|\Sigma_2\| \leq \kappa_{hf} \|\bar{x}_k - x_k\|^v. \]  
(26)

Let \(\bar{J}_k = U_1 \Sigma_1 V_1^T\) and \(\delta_k = -\bar{J}_k^T s_k\), where \(\bar{J}_k^T\) is the pseudo-inverse of \(J_k\). Then \(\bar{s}_k\) is the least squares solution of \(\min_{x \in \mathbb{R}^n} \|F_k + \bar{J}_k \bar{s}_k\|\). By (11) and (26),

\[ \|U_2 U_2^T F_k\| = \|F_k + \bar{J}_k \bar{s}_k\| \]
\[ \leq \|F_k + \bar{J}_k (\bar{x}_k - x_k)\| \]
\[ \leq \|F_k + J_k (\bar{x}_k - x_k)\| + \|\bar{J}_k - J_k\| (\bar{x}_k - x_k)\| \]
\[ \leq \frac{\kappa_{hj}}{1+v} \|\bar{x}_k - x_k\|^{1+v} + \|U_2 \Sigma_2 V_2^T (\bar{x}_k - x_k)\| \]
\[ \leq \frac{\kappa_{hj}}{1+v} \|\bar{x}_k - x_k\|^{1+v} + \kappa_{hj} \|\bar{x}_k - x_k\|^{1+v} \]
\[ \leq 2\kappa_{hj} \|\bar{x}_k - x_k\|^{1+v}. \]  
(27)

The proof is completed. \(\square\)

**Theorem 2.4.** Under Assumption 2.1, if \(0 < \alpha < 2\gamma(1 + v)\) and \(\max\{\alpha + \theta - \frac{\alpha}{\gamma}, \alpha + \theta + v - \frac{\alpha}{\gamma}\} > 0\), then the sequence \(\{x_k\}\) generated by the inexact LM method (7) satisfies

\[ \|\bar{x}_{k+1} - x_k\| \leq \hat{c}\|\bar{x}_k - x_k\|^r(\gamma, v, \alpha, \theta), \]  
(28)

where \(\hat{c} > 0\) is a constant and

\[ r(\gamma, v, \alpha, \theta) = \begin{cases} \gamma \min \left\{ \alpha + \theta, (1 + v)\theta, (1 + v)(1 + v - \frac{\alpha}{\gamma}) \right\}, & \text{if } 0 < \theta < 1, \\ \max \left\{ \alpha + \theta - \frac{\alpha}{\gamma}, \alpha + \theta + v - \frac{\alpha}{\gamma} \right\}, & \text{if } 0 < \theta < 1, \\ \gamma \min \left\{ 1 + \alpha, 1 + v, (1 + v)(1 + v - \frac{\alpha}{\gamma}) \right\}, & \text{if } \theta \geq 1. \end{cases} \]  
(29)

**Proof.** Since \(\{x_k\}\) converges to \(X^*\), we assume that \(\kappa_{hj} \|\bar{x}_k - x_k\|^{v} \leq \frac{\alpha}{\gamma}\) holds for all sufficiently large \(k\) without loss of generality. By (26),

\[ \|J \Sigma_1^2 + \lambda_k I\|^{-1} \leq \|\Sigma_1^{-2}\| \leq \frac{1}{(\sigma - \kappa_{hj} \|\bar{x}_k - x_k\|^v)^2} \leq \frac{4}{\sigma^2}. \]  
(30)
Note that, on the one hand, by (16),
\[\|\Sigma_2 (\Sigma_2^2 + \lambda_k I)^{-1}\| \leq \frac{1}{2\sqrt{\lambda_k}} \leq \frac{1}{2e^{\frac{1}{\lambda_k}}} \|\bar{x}_k - x_k\|^{-\frac{\alpha}{2}};\] (31)
on the other hand, by (16) and (26),
\[\|\Sigma_2 (\Sigma_2^2 + \lambda_k I)^{-1}\| \leq \frac{\|\Sigma_2\|}{\lambda_k} \leq \kappa_h j e^{-\frac{\alpha}{2}} \|\bar{x}_k - x_k\|^{-\frac{\alpha}{2}}.\] (32)
Hence, it follows from (16), (24), (26), (30), (31), (32), \(\|\Sigma_2^2 + \lambda_k I\|^{-1}\| \leq \lambda_k^{-1}\) and Lemma 2.3 that
\[
\|F_k + J_k d_k\| \leq \frac{4k^{1+\alpha}}{\sigma^2} \|\bar{x}_k - x_k\|^{1+\alpha} + 2\kappa_{hj} \|\bar{x}_k - x_k\|^{1+\alpha} + \frac{2k^{1+\alpha}}{\sigma^2} \|\bar{x}_k - x_k\|^{1+\alpha} + \bar{L} \|\bar{x}_k - x_k\|^{\max\{1+\alpha, 1+\alpha + \theta, \max\{1+\alpha, \theta - \frac{\alpha}{2}, \theta + \alpha + v - \frac{\alpha}{2}\}\}},
\] (33)
where \(\bar{L} = \max\{\kappa_h j^{1+\alpha}, \frac{r_{b}^{1+\alpha}}{c^2} \} \) and \(c_1 = \frac{4k^{1+\alpha}}{\sigma^2} + 2\kappa_{hj} + \frac{2k^{1+\alpha}}{\sigma^2} + \bar{L}.
\)
Therefore, by (9), (11) and Lemma 2.2, we have
\[
\|\bar{x}_{k+1} - x_{k+1}\| \leq \frac{1}{c} \|F_{k+1}\|^{\gamma} \leq \frac{1}{c} \left(\|F_k + J_k d_k\| + \kappa_{hj} \|d_k\|^{1+\alpha}\right)^{\gamma} \leq \frac{1}{c} \left(c_1 \|\bar{x}_k - x_k\|^{\min\{1+\alpha, 1+\alpha, \max\{1+\alpha, \theta - \frac{\alpha}{2}, \theta + \alpha + v - \frac{\alpha}{2}\}\}} + \kappa_{hj} \|\bar{x}_k - x_k\|^{\min\{1+\alpha, 1+\alpha + \theta, (1+\alpha)(1+\alpha + \theta - \frac{\alpha}{2})\}}\right)^{\gamma} \leq \tilde{c} \|\bar{x}_k - x_k\|^{r(\gamma, v, \alpha, \theta)},
\] (34)
where \(\tilde{c} = \frac{(c_1 + \kappa_j \bar{c}^{\gamma})^{\gamma}}{c}\) and
\[
r(\gamma, v, \alpha, \theta) = \gamma \min\left\{1 + \alpha, (1+\alpha + \theta, (1+\alpha)\theta, (1+\alpha)(1+\alpha + \theta - \frac{\alpha}{2})\right\} \max\{1+\alpha, 1+\alpha, \max\{1+\alpha, \theta - \frac{\alpha}{2}, \theta + \alpha + v - \frac{\alpha}{2}\}\}.\] (35)
Thus, we obtain (29). The proof is completed. \(\square\)

As we know, the Hölderian local error bound condition becomes the local error bound condition when \(v = 1\), and the Hölderian continuity becomes the Lipschitz continuity when \(v = 1\). So we have:

**Corollary 1.** Under Assumption 2.1 and \(v = 1\), if \(0 < \alpha < 4\), then the sequence \(\{x_k\}\) generated by the inexact LM method (7) satisfies
\[
\|\bar{x}_{k+1} - x_{k+1}\| \leq \tilde{c} \|\bar{x}_k - x_k\|^{r(\alpha, \theta)},
\] (36)
where
\[
r(\alpha, \theta) = \left\{\begin{array}{ll}
\min\{1 + \alpha, 2\alpha, 4 - \alpha, \max\{\frac{\alpha}{2} + \theta, \theta + 1\}\}, & \text{if } 0 < \theta < 1, \\
\min\{1 + \alpha, 2, 4 - \alpha, \max\{\frac{\alpha}{2} + \theta, \theta + 1\}\}, & \text{if } \theta \geq 1.
\end{array}\right.
\] (37)
Corollary 1 indicates that, if $0 < \theta < 1$, then
\[
   r(\alpha, \theta) = \begin{cases} \alpha + \theta, & \text{if } \alpha \in (0, \theta], \\ 2\theta, & \text{if } \alpha \in (\theta, 4 - 2\theta], \\ 4 - \alpha, & \text{if } \alpha \in (4 - 2\theta, 4); \end{cases} \tag{38}
\]
and if $\theta \geq 1$, then
\[
   r(\alpha, \theta) = \begin{cases} 1 + \alpha, & \text{if } \alpha \in (0, 1), \\ 2, & \text{if } \alpha \in [1, 2], \\ 4 - \alpha, & \text{if } \alpha \in (2, 4). \end{cases} \tag{39}
\]
Both (38) and (39) coincide with the results given in [7].

3. Convergence rate of inexact LM method with $\lambda_k = \|J_k^T F_k\|^\alpha$. In this section, we discuss the convergence rate of the inexact LM method
\[
x_{k+1} = x_k + d_k, \tag{40}
\]
where $d_k$ is given by (3) with
\[
   \lambda_k = \|J_k^T F_k\|^\alpha \text{ and } \|p_k\| = \|J_k^T F_k\|^{\alpha + \theta}. \tag{41}
\]
Without loss of generality, we also assume that the sequence \{x_k\} generated by (40) converges to $X^*$ and lies in $N(x^*, \frac{b}{4})$.

**Lemma 3.1.** Under Assumption 2.1 and $v > \frac{2}{\gamma} - 2$, there exists a constant $\bar{c} > 0$ such that
\[
   \bar{c} \|\bar{x}_k - x_k\|^{\frac{2}{\gamma} - 1} \leq \|J_k^T F_k\| \leq \kappa_{bf}^2 \|\bar{x}_k - x_k\|. \tag{42}
\]
**Proof.** By (12),
\[
   \|J_k^T F_k\| \leq \|J_k\| \|F_k - F(\bar{x}_k)\| \leq \kappa_{bf}^2 \|\bar{x}_k - x_k\|. \tag{43}
\]
Let $V_k = F_k - F(\bar{x}_k) - J_k(x_k - \bar{x}_k)$. Then,
\[
   F_k^T J_k(x_k - \bar{x}_k) = \|F_k\|^2 - F_k^T V_k. \tag{44}
\]
By (9), (11) and (12),
\[
   \|F_k^T J_k\| \|\bar{x}_k - x_k\| \geq c_2 \|\bar{x}_k - x_k\|^{\frac{2}{\gamma} - 1} - \frac{\kappa_{hf} \kappa_{bf}}{1 + v} \|\bar{x}_k - x_k\|^{1 + v} \geq \bar{c} \|\bar{x}_k - x_k\|^{\frac{2}{\gamma} - 1}. \tag{45}
\]
Since $v > \frac{2}{\gamma} - 2$, we know that
\[
   \|F_k^T J_k\| \geq c_2 \|\bar{x}_k - x_k\|^{\frac{2}{\gamma} - 1} - \frac{\kappa_{hf} \kappa_{bf}}{1 + v} \|\bar{x}_k - x_k\|^{1 + v} \geq \bar{c} \|\bar{x}_k - x_k\|^{\frac{2}{\gamma} - 1} \tag{46}
\]
holds for some $\bar{c} > 0$. The proof is completed. \hfill \Box

**Lemma 3.2.** Under Assumption 2.1 and $v > \frac{2}{\gamma} - 2$, if $0 < \alpha < \frac{2\gamma(1+v)}{2 - \gamma}$, then there exists a constant $\bar{c} > 0$ such that
\[
   \|d_k\| \leq \bar{c} \|\bar{x}_k - x_k\|^{\min\{1,1+v-(\frac{2-\gamma}{2\gamma})\alpha, \beta\}}. \tag{47}
\]
**Proof.** Similarly to (15), we have $\bar{x}_k \in N(x^*, \frac{b}{4})$. It follows from (42) that
\[
   \bar{c}^{\alpha} \|\bar{x}_k - x_k\|^{\frac{(2-\gamma)\alpha}{\gamma}} \leq \lambda_k = \|J_k^T F_k\|^\alpha \leq \kappa_{bf}^{2\alpha} \|\bar{x}_k - x_k\|^\alpha. \tag{48}
\]
Since $\tilde{d}_k$ is the minimizer of $\phi_k(d)$ defined in (17), by (11) and (48),
\[ \|\tilde{d}_k\|^2 \leq \frac{\phi_k(\tilde{d}_k)}{\lambda_k} \leq \frac{\phi_k(x_k - \tilde{x}_k)}{\lambda_k} = \frac{\|F_k + J_k(\tilde{x}_k - x_k)\|^2 + \lambda_k\|\tilde{x}_k - x_k\|^2}{\lambda_k} \leq \frac{\kappa_{h_j}^2}{c^\alpha(1 + v)^2}\|\tilde{x}_k - x_k\|^2 + \min\{1, 1 + v - \frac{(2 - \gamma)\alpha}{\gamma}\} + \frac{\kappa_{h_j}^2}{c^\alpha(1 + v)^2} + 1\|\tilde{x}_k - x_k\|^2 \min\{1, 1 + v - \frac{(2 - \gamma)\alpha}{\gamma}\}. \] (49)

It then follows from (41) and (42) that
\[ \|d_k\| \leq \|\tilde{d}_k\| + \|d_k - \tilde{d}_k\| = \|\tilde{d}_k\| + \|J_k^T J_k + \lambda_k I\|^{-1}p_k \leq \|\tilde{d}_k\| + \frac{\|p_k\|}{\lambda_k} = \|\tilde{d}_k\| + \|J_k^T F_k\|^{\theta} \leq \sqrt{\frac{\kappa_{h_j}^2}{c^\alpha(1 + v)^2} + 1\|\tilde{x}_k - x_k\|^2 + \kappa_{h_j}^2\|\tilde{x}_k - x_k\|^{\theta}} \leq \hat{c}\|\tilde{x}_k - x_k\|^\min\{1, 1 + v - \frac{(2 - \gamma)\alpha}{\gamma}\}, \] (50)
where $\hat{c} = \sqrt{\frac{\kappa_{h_j}^2}{c^\alpha(1 + v)^2} + 1 + \kappa_{h_j}^2\theta}$. The proof is completed.

**Theorem 3.3.** Under Assumption 2.1 and $v > \frac{2}{\gamma} - 2$, if $0 < \alpha < \frac{2\gamma(1 + v)}{2 - \gamma}$ and $\max\{\alpha + \theta - \frac{(2 - \gamma)\alpha}{2\gamma}, \alpha + \theta + v - \frac{(2 - \gamma)\alpha}{\gamma}\} > 0$, then the sequence $\{x_k\}$ generated by the inexact LM method (40) satisfies
\[ \|x_{k+1} - x_k\| \leq \hat{c}\|x_k - \tilde{x}_k\|^{r(\gamma, \alpha, \theta)}, \] (51)
where $\hat{c} > 0$ is a constant and
\[ r(\gamma, \alpha, \theta) = \begin{cases} \gamma \min\{\alpha + \theta, (1 + v)\theta, (1 + v)\theta, (1 + v)\theta, (1 + v)\theta, (1 + v)\theta, (1 + v)\theta, (1 + v)\theta, (1 + v)\theta\}, & \text{if } 0 < \theta < 1, \\ \max\{\alpha + \theta - \frac{(2 - \gamma)\alpha}{2\gamma}, \alpha + \theta + v - \frac{(2 - \gamma)\alpha}{\gamma}\}, & \text{if } \theta \geq 1. \end{cases} \] (52)

**Proof.** Similarly to (31) and (32), we have
\[ \|\Sigma_2(\Sigma_2 + \lambda_k I)^{-1}\| \leq \frac{1}{2\sqrt{\lambda_k}} \leq \frac{1}{2\sqrt{c}}\|\tilde{x}_k - x_k\|^{\frac{(2 - \gamma)\alpha}{\gamma}}; \] (53)
due to (48) and (26),
\[ \|\Sigma_2(\Sigma_2 + \lambda_k I)^{-1}\| \leq \frac{\|\Sigma_2\|}{\lambda_k} \leq \kappa_{h_j}^2\|\tilde{x}_k - x_k\|^{\frac{(2 - \gamma)\alpha}{\gamma}}. \] (54)
Then, it follows from (24), (26), (30), (42), (48), (53), (54), \(\|\Sigma_2 + \lambda_k I\|^{-1} \leq \lambda_k^{-1}\) and Lemma 2.3 that

\[
\|F_k + J_k d_k\| \leq \frac{4\kappa_{bf}^{1+2\alpha}}{\sigma^2} \|\bar{x}_k - x_k\|^{1+\alpha} + 2\kappa_{hj} \|\bar{x}_k - x_k\|^{1+v} + \frac{2\kappa_{bf}^{2(\alpha+\theta)}}{\sigma} \|\bar{x}_k - x_k\|^{\alpha+\theta} + L \|\bar{x}_k - x_k\| \max\{\alpha+\theta-\frac{(2-\gamma)\alpha}{2\gamma},\alpha+\theta+v-\frac{(2-\gamma)\alpha}{2\gamma}\}
\]

\[
\leq c_2 \|\bar{x}_k - x_k\| \min\{1,1+v,\alpha+\theta,\max\{\alpha+\theta-\frac{(2-\gamma)\alpha}{2\gamma},\alpha+\theta+v-\frac{(2-\gamma)\alpha}{2\gamma}\}\}
\]

\[
\leq \hat{c} \|\bar{x}_k - x_k\|^{r(\gamma,\nu)}
\]

where \(\hat{c} = \frac{c_2 + \kappa_{hj} c^{1+v}}{c}\) and

\[
r(\gamma,\nu) = \gamma \min\left\{1+\alpha,1+v,\alpha+\theta,(1+v) \theta,(1+v) \theta,1+v\right\}
\]

\[
\min\{\alpha+\theta-\frac{(2-\gamma)\alpha}{2\gamma},\alpha+\theta+v-\frac{(2-\gamma)\alpha}{2\gamma}\}\}
\]

Thus, we obtain (52). The proof is completed.

**Corollary 2.** Under Assumption 2.1 and \(\gamma = \nu = 1\), if \(0 < \alpha < 4\), then the sequence \(\{x_k\}\) generated by the inexact LM method (40) satisfies

\[
\|\bar{x}_{k+1} - x_{k+1}\| \leq \hat{c} \|\bar{x}_k - x_k\|^{r(\alpha,\theta)},
\]

where

\[
r(\alpha,\theta) = \begin{cases}
\min\left\{\alpha+\theta,2\theta,4-\alpha,\max\left\{\frac{\alpha}{2}+\theta,\theta+1\right\}\right\}, & \text{if } 0 < \theta < 1, \\
\min\left\{1+\alpha,2,4-\alpha,\max\left\{\frac{\alpha}{2}+\theta,\theta+1\right\}\right\}, & \text{if } \theta \geq 1.
\end{cases}
\]

From the above, we can see that Corollary 2 is the same as Corollary 1. That is because, under the local error bound condition and Lipschitz continuity of the Jacobian, \(\|F_k\|\) has the same order as \(\|J_k F_k\|\).

4. **Conclusions.** The Hölderian local error bound condition and Hölderian continuity of the Jacobian are more general than the local error bound condition and Lipschitz continuity of the Jacobian, respectively. We investigate the local convergence properties of the inexact LM method under the Hölderian local error bound condition and Hölderian continuity of the Jacobian. The results obtained in this paper include the prior existing results of the inexact LM method under the local error bound condition and the Lipschitz continuity of the Jacobian as special cases.
In fact, the LM parameter $\lambda_k$ and the vector $p_k$ could be replaced by the more general choices

$$\kappa_1 \|F_k\|^{\alpha} \leq \lambda_k \leq \kappa_2 \|F_k\|^{\alpha}, \quad \|p_k\| \leq \kappa_3 \|F_k\|^{\alpha+\theta},$$

or

$$\tilde{\kappa}_1 \|J_k^TF_k\|^{\alpha} \leq \lambda_k \leq \tilde{\kappa}_2 \|J_k^TF_k\|^{\alpha}, \quad \|p_k\| \leq \tilde{\kappa}_3 \|J_k^TF_k\|^{\alpha+\theta},$$

where $\kappa_i, \tilde{\kappa}_i (i = 1, 2, 3)$ are positive constants. The convergence orders will be the same, so we omit the discussions for brevity.

REFERENCES

[1] M. Ahookhosh, F. J. Aragón, R. M. T. Fleming and P. T. Vuong, Local convergence of Levenberg-Marquardt methods under Hölderian metric subregularity, Adv. Comput. Math., 45 (2019), 2771–2806, arXiv:1703.07461.
[2] H. Dan, N. Yamashita and M. Fukushima, Convergence properties of the inexact Levenberg-Marquardt method under local error bound, Optimization Methods and Software, 17 (2002), 605–626.
[3] F. Facchinei and C. Kanzow, A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems, Mathematical Programming, 76 (1997), 493–512.
[4] J. Y. Fan, A modified Levenberg-Marquardt algorithm for singular system of nonlinear equations, Journal of Computational Mathematics, 21 (2003), 625–636.
[5] J. Y. Fan, The modified Levenberg-Marquardt method for nonlinear equations with cubic convergence, Mathematics of Computation, 81 (2012), 447–466.
[6] J. Y. Fan, J. C. Huang and J. Y. Pan, An adaptive multi-step Levenberg-Marquardt method, Journal of Scientific Computing, 78 (2019), 531–548.
[7] J. Y. Fan and J. Y. Pan, Inexact Levenberg-Marquardt method for nonlinear equations, Discrete Continuous Dynamical System-Series B, 4 (2004), 1223–1232.
[8] J. Y. Fan and J. Y. Pan, A note on the Levenberg-Marquardt parameter, Applied Mathematics and Computation, 207 (2009), 351–359.
[9] J. Y. Fan and J. Y. Pan, On the convergence rate of the inexact Levenberg-Marquardt method, Industrial and Management Optimization, 7 (2011), 199–210.
[10] J. Y. Fan and Y. X. Yuan, On the quadratic convergence of the Levenberg-Marquardt method without nonsingularity assumption, Computing, 74 (2005), 23–39.
[11] A. Fischera, P. K. Shiokhia and M. Wang, On the inexactness level of robust Levenberg-Marquardt methods, Optimization, 59 (2010), 273–287.
[12] C. T. Kelley, Solving Nonlinear Equations with Newton's Method, Fundamentals of Algorithms, SIAM, Philadelphia, 2003.
[13] K. Levenberg, A method for the solution of certain nonlinear problems in least squares, Quart. Appl. Math., 2 (1944), 164–168.
[14] D. W. Marquardt, An algorithm for least-squares estimation of nonlinear inequalities, SIAM J. Appl. Math., 11 (1963), 431–441.
[15] J. J. Moré, The Levenberg-Marquardt algorithm: implementation and theory, In: G. A. Watson, ed., Lecture Notes in Mathematics 630: Numerical Analysis, Springer-Verlag, Berlin, 1978, 105–116.
[16] M. J. D. Powell, Convergence properties of a class of minimization algorithms, Nonlinear Programming, 2 (1974), 1–27.
[17] G. W. Stewart and J.-G. Sun, Matrix Perturbation Theory, (Computer Science and Scientific Computing), Academic Press Boston, 1990.
[18] H. Y. Wang and J. Y. Fan, Convergence rate of the Levenberg-Marquardt method under Hölderian local error bound, Optimization Methods and Software, 2019.
[19] N. Yamashita and M. Fukushima, On the rate of convergence of the Levenberg-Marquardt method, Computing, 15 (2001), 239–249.
[20] Y. X. Yuan, Recent advances in trust region algorithms, Math. Program., Ser. B, 151 (2015), 249–281.
[21] X. D. Zhu and G. H. Lin, Improved convergence results for a modified Levenberg-Marquardt method for nonlinear equations and applications in MPCC, Optimization Methods and Software, 31 (2016), 791–804.

Received August 2019; revised November 2019.

E-mail address: haiyanjady@sjtu.edu.cn
E-mail address: jyfan@sjtu.edu.cn