Axiomatic $G_1$-vertex algebras

Haisheng Li
Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102
and
Department of Mathematics, Harbin Normal University, Harbin, China

Abstract

Inspired by Borcherds’ work on “$G$-vertex algebras,” we formulate and study an axiomatic counterpart of Borcherds’ notion of $G$-vertex algebra for the simplest nontrivial elementary vertex group, which we denote by $G_1$. Specifically, we formulate a notion of axiomatic $G_1$-vertex algebra, prove certain basic properties and give certain examples. The notion of axiomatic $G_1$-vertex algebra is a nonlocal generalization of the notion of vertex algebra. We also show how to construct axiomatic $G_1$-vertex algebras from a set of compatible $G_1$-vertex operators.

The results of this paper had been reported in June 2001, at the International Conference on Lie Algebras in the Morningside center, Beijing, China, and been reported on November 30, 2001, in the Quantum Mathematics Seminar, at Rutgers-New Brunswick. We noticed that a paper of Bakalov and Kac appeared today (math.QA/0204282) on noncommutative generalizations of vertex algebras, which has certain overlaps with the current paper. On the other hand, most of their results are orthogonal to the results of this paper.

1 Introduction

It has been well known that vertex (operator) algebras introduced in [B1] and [FLM] are mathematical counterparts of chiral algebras in 2-dimensional quantum conformal field theory (cf. [BPZ]). Later, higher dimensional analogues of vertex algebras (or chiral algebras), which are expected to play the same role in higher dimensional quantum field theory as vertex algebras play in 2-dimensional quantum field theory, were also established by Borcherds in [B2] by introducing a notion of $G$-vertex algebra. In this notion, $G$ is what Borcherds called an elementary vertex group and a $G$-vertex algebra is an “associative algebra” in a certain “relaxed multilinear category” associated to $G$. For the simplest nontrivial elementary vertex group $G$, which is denoted here by $G_1$, as it was proved in [Sn] (cf. [B2]), the notion of commutative $G_1$-vertex algebra is equivalent to the notion of ordinary vertex algebra. On the other hand, it had been known (earlier) that vertex (operator) algebras are analogous to commutative associative algebras, as the Jacobi identity for ordinary vertex algebras amounts to certain commutativity and associativity properties (see [FLM], [FHL], [DL], [Li1]). (Ordinary vertex algebras are also analogous to Lie algebras in many aspects.) In view of this analogy, a natural exercise is to establish the corresponding analogues of noncommutative associative algebras, or more or less, to establish the axiomatic analogues of Borcherds’ $G_1$-vertex algebras.

\[1\]Partially supported by NSF grant DMS-9970496 and a grant from Rutgers Research Council
In this paper we formulate and study a notion of what we call axiomatic $G_1$-vertex algebra, where axiomatic $G_1$-vertex algebras are the corresponding analogues of noncommutative associative algebras in contrast with the analogy between vertex algebras and commutative associative algebras. This notion is defined by using all the axioms in the definition of the notion of vertex algebra except for the Jacobi identity which is replaced by the \textit{weak associativity}: For any algebra elements $u, v, w$, there exists a nonnegative integer $l$ (depending only on $u$ and $w$) such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2) w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w.$$ \hspace{1cm} (1.1)

(See [DL] and [Li1] for this property.) It is expected that this notion of axiomatic $G_1$-vertex algebra is equivalent to the Borcherds’ notion of $G_1$-vertex algebra.

In terms of this notion, ordinary vertex algebras are exactly axiomatic $G_1$-vertex algebras that also satisfy a certain weak commutativity property, which was discovered to be an axiom in [DL] and [Li1]. Such a weak commutativity is also called \textit{locality} in the literature. Trivial examples of (nonlocal) axiomatic $G_1$-vertex algebras are noncommutative associative algebras (with identity element). We also give three constructions of (nonlocal) axiomatic $G_1$-vertex algebras from ordinary vertex algebras. All the three constructions are natural analogues of those in the classical associative algebra theory. In the classical theory, if $G$ is an abelian group, any (commutative) $G$-graded associative algebra $A$ can be made a (noncommutative) associative algebra by using a normalized 2-cocycle on $G$. Our first construction is an exact analogue of this for a $G$-graded ordinary vertex algebra $V$ and a normalized 2-cocycle $\epsilon$ on $G$. Also, in the classical theory, if $A$ is a (commutative) associative algebra and $n$ is a positive integer, we have the noncommutative matrix algebra $M(n, A)$ of all $n \times n$ matrices over $A$ for $n \geq 2$. As our second construction, we show that for any ordinary vertex algebra $V$, the vector space $M(n, V)$ has a natural axiomatic $G_1$-vertex algebra structure. In fact, this follows from a general result. Just as in [FHL] for ordinary vertex algebras, it can be easily shown that tensor products of axiomatic $G_1$-vertex algebras are also axiomatic $G_1$-vertex algebras, see also [B2]. In particular, the tensor product of an ordinary vertex algebra with an associative algebra is an axiomatic $G_1$-vertex algebra. The axiomatic $G_1$-vertex algebra $M(n, V)$ is naturally isomorphic to the tensor product axiomatic $G_1$-vertex algebra $V \otimes M(n, \mathbb{C})$. Now, let $A$ be an associative algebra acted by a group $G$ by automorphisms. Associated to $A$ and $G$ there is an associative algebra called the cross product of $A$ with $G$ (or the skew algebra), whose underlying vector space is $A \otimes \mathbb{C}[G]$. Our third construction is an analogous cross product (or skew product) construction of axiomatic $G_1$-vertex algebras from an ordinary vertex algebra acted by a group $G$. For cross product axiomatic $G_1$-vertex algebras, we also derive a Jacobi-like identity and motivated by this, we define a notion of restricted (weak) axiomatic $G_1$-vertex algebra by using a Jacobi-like identity as its main axiom. This notion turns out to unify all the examples mentioned above.

There is a viewpoint about vertex (operator) algebras which is that vertex (operator) algebras are “algebras” of vertex operators just as classical associative algebras are algebras of linear operators. From this point of view, vertex operators ought to give rise to vertex (operator) algebras just as linear operators naturally give rise to classical as-
sociative algebras. In [Li1], for any abstract vector space $W$, a notion of (weak) vertex operator on $W$ was defined and it was proved that any set of “pairwise mutually local” (weak) vertex operators on $W$ in a certain canonical way generates a vertex algebra with $W$ as a natural module. This is an analogue of the classical fact that any set of pairwise commuting linear operators on $W$ generates a commutative associative algebra with $W$ as a module. (See [Li2-3], [GL] for generalizations in certain directions.) In the context of $G$-vertex algebras, a theorem of Borcherds ([B2], Theorem 7.9) states that any compatible set of vertex operators in a certain sense on a vector space $W$ generates a $G$-vertex algebra acting on $W$. Borcherds’ theorem is also in the same spirit of the corresponding theorems of [Li1], [Li2] and [GL], and it can be viewed as a noncommutative version of those corresponding theorems. In this paper, we also study (weak) vertex operators as defined in [Li1], but they are renamed as (weak) $G_1$-vertex operators according to Borcherds’ notion of $G$-vertex algebra. We first define a notion of compatibility, where the notion of compatibility is weaker than the notion of locality and it asserts that the operator product expansion is of a certain form. We then prove an analogous theorem (Theorem 5.25) of Borcherds’. To prove this, we prove that any closed compatible space, in a certain sense, of (weak) $G_1$-vertex operators on a vector space $W$ has a natural (weak) axiomatic $G_1$-vertex algebra structure with $W$ as a natural module (Theorem 5.22), which is analogous to a result obtained in [Li1] (cf. [MN]) for ordinary vertex algebras.

In [EK], Etingof and Kazhdan established and studied a notion of quantum vertex operator algebra where it was proved that a certain $h$-adic (topological) version of the weak associativity holds. To a certain extent, quantum vertex operator algebras are $h$-adic (topological) axiomatic $G_1$-vertex algebras. Much of the current work can be carried on to quantum vertex operator algebras and details will appear in a coming paper.

Recently, there has been active research in physics on noncommutative field theory (field theory on noncommutative manifolds) (cf. [DN]). It seems that noncommutative field theories are related to (nonlocal) axiomatic $G_1$-vertex algebras. This is also part of our motivation systematically to study axiomatic $G_1$-vertex algebras.

This paper is organized in the following manner. In Section 2, we define a notion of axiomatic $G_1$-vertex algebra and present certain basic properties. In Section 3, we discuss various examples and we introduce a notions of restricted (weak) axiomatic $G_1$-vertex algebra to unify many examples. In Section 4, we define the notion of module and present certain basic properties. In Section 5, we show how to construct axiomatic $G_1$-vertex algebras from a set of compatible $G_1$-vertex operators on a vector space $W$ and prove the main results.

2 Axiomatic $G_1$-vertex algebras

In this section, we define the notion of (weak) axiomatic $G_1$-vertex algebra and we establish certain basic properties analogous to those (cf. [DL], [FHL], [LL], [Li1]) for ordinary vertex algebras.

Let $x, y, z, x_i, y_i, z_i, i = 0, 1, \ldots$ be mutually commuting independent formal variables.
throughout this paper. We shall use the standard formal variable notations and conventions as defined in [FLM] and [FHL]. In particular, for a vector space $U$,

$$U[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} a(n)x^n \mid a(n) \in U \right\},$$  

$$U((x)) = \left\{ \sum_{n \geq r} a(n)x^n \mid r \in \mathbb{Z}, a(n) \in U \right\} \subset U[[x, x^{-1}]],$$  

$$U[[x]] = \left\{ \sum_{n \geq 0} a(n)x^n \mid a(n) \in U \right\} \subset U((x)).$$

The spaces $U[[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]], U((x_1, \ldots, x_n))$ and $U[[x_1, \ldots, x_n]]$ are also defined in the obvious ways.

The formal delta function is the following formal series:

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n \in \mathbb{C}[[x, x^{-1}]].$$

In formal calculus the following binomial expansion convention is implemented:

$$(x_1 \pm x_2)^n = \sum_{i \geq 0} \binom{n}{i} x_1^{n-i} x_2^i \in \mathbb{C}[[x_1, x_2, x_1^{-1}]],$$

where $\binom{n}{i} = \frac{1}{i!}n(n-1)\cdots(n+1-i)$ for $n \in \mathbb{Z}$, $i \in \mathbb{N}$. Furthermore, by definition

$$\delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n \in \mathbb{Z}} x_0^{-n}(x_1 - x_2)^n = \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} \binom{n}{i} (-1)^i x_0^{-n} x_1^{-n} x_2^i.$$  

Then we have

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right).$$

As it was emphasized in [FLM], in formal calculus, associativity in general does not hold for products, and on the other hand, associativity does hold under the assumption that the products and their involved subproducts exist. For example, for three formal series $A, B$ and $C$, we have $A(BC) = (AB)C (= ABC)$ if $ABC, AB$ and $BC$ all exist.

The following is a reformulation of Proposition 3.4.2 of [LL] with a slightly different proof (cf. [Li1], part 3 of the proof of Proposition 2.2.4):

**Lemma 2.1** Let $U$ be a vector space and let

$$A(x_1, x_2) \in U((x_1))(x_2),$$  

$$B(x_1, x_2) \in U((x_2))(x_1),$$  

$$C(x_0, x_2) \in U((x_2))(x_0).$$
Then
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) A(x_1, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) B(x_1, x_2) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) C(x_0, x_2)
\] (2.11)

if and only if there exist nonnegative integers \(k\) and \(l\) such that
\[
(x_1 - x_2)^k A(x_1, x_2) = (x_1 - x_2)^k B(x_1, x_2),
\] (2.12)
\[
(x_0 + x_2)^l A(x_0 + x_2, x_2) = (x_0 + x_2)^l C(x_0, x_2).
\] (2.13)

**Proof.** Let \(k\) and \(l\) be nonnegative integers such that
\[
x_0^k C(x_0, x_2) \in U((x_2))[[x_0]], \quad x_1^l B(x_1, x_2) \in U((x_2))[[x_1]].
\]

Then (2.12) follows from (2.11) by applying \(\text{Res}_{x_0} x_0^k\) and (2.13) follows from (2.11) by applying \(\text{Res}_{x_1} x_1^l\). Now we shall show that (2.11) also follows from (2.12) and (2.13). Let \(l'\) be a nonnegative integer such that \(l' \geq l\) and \(x_1^l B(x_1, x_2) \in U((x_2))[[x_1]]\). Then (2.12) and (2.13) with \(l\) being replaced by \(l'\) still hold, and furthermore,
\[
(x_1 - x_2)^k x_1^{l'} A(x_1, x_2) = (x_1 - x_2)^k x_1^{l'} B(x_1, x_2) \in U((x_2))[[x_1]] \cap U((x_1))(x_2)).
\]

Thus
\[
\left[(x_1 - x_2)^k x_1^{l'} A(x_1, x_2) \right]_{x_1 = x_2 + x_0} = \left[(x_1 - x_2)^k x_1^{l'} A(x_1, x_2) \right]_{x_1 = x_0 + x_2}.
\] (2.14)

Using (2.12), (2.14) and (2.13) we get
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_0^k x_1^{l'} A(x_1, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) x_0^k x_1^{l'} B(x_1, x_2)
\]
\[
= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) (x_1 - x_2)^k x_1^{l'} A(x_1, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) (x_1 - x_2)^k x_1^{l'} B(x_1, x_2)
\]
\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \left[(x_1 - x_2)^k x_1^{l'} A(x_1, x_2) \right]_{x_1 = x_2 + x_0}
\]
\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \left[(x_1 - x_2)^k x_1^{l'} A(x_1, x_2) \right]_{x_1 = x_0 + x_2}
\]
\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \left[x_0^k (x_0 + x_2)^{l'} A(x_0 + x_2, x_2) \right]
\]
\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \left[x_0^k (x_0 + x_2)^{l'} C(x_0, x_2) \right]
\]
\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) x_0^k x_1^{l'} C(x_0, x_2).
\] (2.15)

Multiplying both sides by \(x_0^{-k} x_1^{-l'}\) we obtain (2.11). □

The notion of axiomatic \(G_1\)-vertex algebra is defined as follows:
Definition 2.2 An axiomatic $G_1$-vertex algebra is a vector space $V$ equipped with a linear map

$$Y(\cdot, x) : V \to (\text{End} V)[[x, x^{-1}]]$$

$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

and equipped with a distinguished vector $1 \in V$ such that all the following axioms hold:

For $u, v \in V$,

$$u_n v = 0 \text{ for } n \text{ sufficiently large};$$

$$Y(1, x) = 1;$$

(2.17) (2.18)

for $v \in V$,

$$Y(v, x)1 \in V[[x]] \text{ and } \lim_{x \to 0} Y(v, x)1(\equiv v_1 1) = v;$$

(2.19)

and for $u, w \in V$, there exists $l \in \mathbb{N}$ such that for all $v \in V$,

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w$$

(2.20)

(the weak associativity).

Note that the integer $l$ in (2.20) only depends on $u$ and $w$, but not $v$. A weak axiomatic $G_1$-vertex algebra satisfies all the axioms for an axiomatic $G_1$-vertex algebra except that the weak associativity axiom is replaced by the weaker one: For any $u, v, w \in V$ there exists a nonnegative integer $l$ such that (2.20) holds.

Of course, a finite-dimensional weak axiomatic $G_1$-vertex algebra is automatically an axiomatic $G_1$-vertex algebra.

Remark 2.3 In the notion of (weak) axiomatic $G_1$-vertex algebra, $G_1$ represents the simplest nontrivial elementary vertex group defined in [B2]; roughly speaking, it is the pair $(H_1, K_1)$, where $H_1 = \mathbb{C}[D]$, being considered as the universal enveloping algebra of the 1-dimensional Lie algebra $\mathbb{C}D$, is a cocommutative Hopf algebra and $K_1 = \mathbb{C}[[x]][x^{-1}] = \mathbb{C}((x))$, a commutative associative algebra and an $H_1$-module with $D$ acting as $d/dx$. It is expected that the notion of axiomatic $G_1$-vertex algebra is equivalent to Borcherds’ notion of $G_1$-vertex algebra defined in [B2]. Within this paper, when there is no confusion, we shall take the liberty simply to use the term “$G_1$-vertex algebra.”

Remark 2.4 Recall from [B1] and [FLM] (cf. [Li1]) that a vertex algebra is a vector space $V$ such that all the axioms for an axiomatic $G_1$-vertex algebra except for the weak associativity hold and such that for $u, v \in V$,

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_0)Y(v, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)$$

$$= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2)$$

(2.21)
(the Jacobi identity). In view of Lemma 2.1 (cf. [FHL], [DL], [Li1]), the Jacobi identity (2.21) is equivalent to the following weak commutativity and weak associativity: For \( u, v \in V \), there exists \( k \in \mathbb{N} \) such that

\[
(x_1 - x_2)^k Y(v, x_1)Y(u, x_2) = (x_1 - x_2)^k Y(v, x_2)Y(u, x_1);
\]

for \( u, w \in V \), there exists \( l \in \mathbb{N} \) such that for all \( v \in V \),

\[
(x_0 + x_2)^l Y(v, x_0 + x_2)Y(u, x_2)w = (x_0 + x_2)^l Y(v, x_0)v, x_2)w.
\]

In view of this, ordinary vertex algebras are axiomatic \( G_1 \)-vertex algebras. Furthermore, ordinary vertex algebras are analogous to commutative associative algebras while axiomatic \( G_1 \)-vertex algebras are analogous to associative algebras.

Example 2.5 Just as ordinary vertex algebras can be constructed from commutative associative algebras with identity element equipped with a derivation (cf. [B1]), axiomatic \( G_1 \)-vertex algebras can be constructed from associative algebras with identity element equipped with a derivation. Let \( A \) be an associative algebra with identity element 1 equipped with a derivation \( d \) (possibly zero). Define a linear map

\[
Y(\cdot, x) : A \to (\text{End}A)[[x]] \subset (\text{End}A)[[x, x^{-1}]]
\]

by

\[
Y(a, x)b = (e^{xd}a)b = \sum_{i \geq 0} \frac{1}{i!}(d^i a)bx^i.
\]

All the axioms except for the weak associativity clearly hold. Since \( d \) is a derivation of \( A \), \( e^{xd} \) is an automorphism of the associative algebra \( A[[x]] \) (by considering \( d \) as a derivation of \( A[[x]] \) with \( d(x) = 0 \), so that for \( a, b, c \in A \),

\[
Y(a, x_0 + x_2)Y(b, x_2)c = (e^{(x_0 + x_2)d}a)(e^{x_2d}b)c
= (e^{x_2d}(e^{x_0d}a)b)c
= Y(Y(a, x_0)b, x_2)c.
\]

This proves the weak associativity, so \( A \) equipped with the distinguished vector 1 and the linear map \( Y \) defined in (2.23) is an axiomatic \( G_1 \)-vertex algebra. In particular, by taking \( d = 0 \) we see that any associative algebra with identity is an axiomatic \( G_1 \)-vertex algebra.

Next, we give some consequences of the definition. First, as in [Li1] and [LL] for vertex algebras we have the following \( D \)-bracket-derivative formula (which in fact follows from the same proof of [LL]):
Proposition 2.6 Let $V$ be a weak axiomatic $G_1$-vertex algebra. Define a linear operator $D$ on $V$ by
\[
D(v) = v_{-2}1 = \left( \frac{d}{dx}Y(v, x)1 \right) |_{x=0} \text{ for } v \in V.
\] (2.27)

Then
\[
[D, Y(v, x)] = Y(D(v), x) = \frac{d}{dx}Y(v, x) \text{ for } v \in V.
\] (2.28)

Proof. Let $u, v \in V$. Then there exists $l \in \mathbb{N}$ such that
\[
(x_2 + x_0)^l Y(Y(u, x_0)1, x_2)v = (x_2 + x_0)^l Y(u, x_0 + x_2)Y(1, x_2)v.
\] (2.29)

With $Y(1, x) = 1$ we have
\[
(x_2 + x_0)^l Y(Y(u, x_0)1, x_2)v = (x_0 + x_2)^l Y(u, x_0 + x_2)v.
\] (2.30)

We may assume that $x^l Y(u, x)v \in V[[x]]$ by replacing $l$ with a bigger integer if necessary, so that
\[
(x_0 + x_2)^l Y(u, x_0 + x_2)v = (x_2 + x_0)^l Y(u, x_2 + x_0)v.
\] (2.31)

Then (2.30) can be also written as
\[
(x_2 + x_0)^l Y(Y(u, x_0)1, x_2)v = (x_2 + x_0)^l Y(u, x_2 + x_0)v.
\] (2.32)

Multiplying both sides by $(x_2 + x_0)^{-l}$ we get
\[
Y(Y(u, x_0)v, x_2) = Y(u, x_2 + x_0)v = e^{x_0 \frac{d}{dx_2}} Y(u, x_2)v.
\] (2.33)

Extracting the coefficient of $x_0$ we obtain
\[
Y(D(u), x_2)v = \frac{d}{dx_2}Y(u, x_2)v.
\] (2.34)

This proves the second equality of (2.28).

For the first equality, let $u, v \in V$ and let $l \in \mathbb{N}$ be such that
\[
(x_2 + x_0)^l Y(Y(u, x_0)v, x_2)1 = (x_2 + x_0)^l Y(u, x_0 + x_2)Y(v, x_2)1.
\] (2.35)

In view of the creation property, $Y(Y(u, x_0)v, x_2)1$ involves only nonnegative powers of $x_2$, so that we may multiply both sides by $(x_0 + x_2)^{-l}$ to get
\[
Y(Y(u, x_0)v, x_2)1 = Y(u, x_0 + x_2)Y(v, x_2)1.
\] (2.36)

In view of the Taylor theorem we have
\[
Y(Y(u, x_0)v, x_2)1 = e^{x_2 \frac{d}{dx_2}} Y(u, x_0)Y(v, x_2)1.
\] (2.37)
Extracting the coefficient of $x^2$ from both sides and using the creation property we get

$$D(Y(u, x_0)v) = Y(u, x_0)D(v) + \frac{d}{dx_0} Y(u, x_0)v.$$  \hspace{1cm} (2.38)

That is,

$$[D, Y(u, x_0)]v = \frac{d}{dx_0} Y(u, x_0)v.$$ \hspace{1cm} (2.39)

This proves the first equality of (2.28), completing the proof. \hfill $\Box$

Combining Proposition 2.6 with the Taylor theorem we immediately have the first part of the following Proposition:

**Corollary 2.7** Let $V$ be a weak axiomatic $G_1$-vertex algebra and let $D \in \text{End} V$ be defined as in Proposition 2.6. Then for $v \in V$,

$$e^{xD}Y(v, x_1)e^{-xD} = Y(e^{xD}v, x_1) = Y(v, x_1 + x),$$  \hspace{1cm} (2.40)

$$Y(v, x)1 = e^{xD}v.$$ \hspace{1cm} (2.41)

**Proof.** Applying the second equality of (2.40) to 1, and then setting $x_1 = 0$ and using the creation property we obtain (2.41). \hfill $\Box$

**Remark 2.8** Recall from [LL] that a weak vertex algebra is a vector space $V$ equipped with a linear map $Y$ from $V$ to $(\text{End} V)[[x, x^{-1}]]$ and equipped with a distinguished vector 1 such that $Y(1, x) = 1$ and such that (2.19) and (2.28) hold. In view of Proposition 2.6, any weak axiomatic $G_1$-vertex algebra is a weak vertex algebra.

Let $V$ be a weak axiomatic $G_1$-vertex algebra. A subalgebra of $V$ is a subspace $U$ such that

$$1 \in U,$$ \hspace{1cm} (2.42)

$$u_n u' \in U \quad \text{for} \quad u, u' \in U, \ n \in \mathbb{Z}.$$ \hspace{1cm} (2.43)

Then $U$ itself equipped with the linear map $Y$ restricted to $U$ is a weak axiomatic $G_1$-vertex algebra.

Let $U$ be a subspace of $V$. We define the stabilizer $\text{Stab}(U)$ of $U$ in $V$ as

$$\text{Stab}(U) = \{ v \in V \mid v_n U \subset U \quad \text{for all} \ n \in \mathbb{Z} \}.$$ \hspace{1cm} (2.44)

Then $U$ is a subalgebra if and only if $1 \in U$ and $U \subset \text{Stab}(U)$.

**Lemma 2.9** The stabilizer $\text{Stab}(U)$ of $U$ in $V$ is a subalgebra.
Proof. Clearly, $1 \in \text{Stab}(U)$. Let $a, b \in \text{Stab}(U)$ and $u \in U$. Then there exists a nonnegative integer $l$ such that

$$(x_2 + x_0)^l Y(Y(a, x_0)b, x_2)u = (x_2 + x_0)^l Y(a, x_0 + x_2)Y(b, x_2)u.$$ \hspace{1cm} (2.45)

With $a, b \in \text{Stab}(U)$, we have

$$(x_2 + x_0)^l Y(Y(a, x_0)b, x_2)u \in U[[x_0, x_0^{-1}, x_2, x_2^{-1}]],$$

so

$$(x_2 + x_0)^l Y(Y(a, x_0)b, x_2)u \in U[[x_0, x_0^{-1}, x_2, x_2^{-1}]].$$ \hspace{1cm} (2.46)

Then

$$(x_2 + x_0)^l Y(Y(a, x_0)b, x_2)u \in V((x_2)) \cap U[[x_0, x_0^{-1}, x_2, x_2^{-1}]] = U((x_2))((x_0)).$$ \hspace{1cm} (2.47)

Consequently, $Y(Y(a, x_0)b, x_2)u \in U((x_2))((x_0))$, since

$$(x_2 + x_0)^{-l} F(x_0, x_2) \in U((x_2))((x_0)) \text{ for any } F(x_0, x_2) \in U((x_2))((x_0)).$$

Then $(a_m b)_n u \in U$ for all $m, n \in \mathbb{Z}$. Thus $a_m b \in \text{Stab}(U)$ for all $m \in \mathbb{Z}$. Therefore, $\text{Stab}(U)$ is a subalgebra. \qquad \square

Let $S$ be a subset of a weak axiomatic $G_1$-vertex algebra $V$. Denote by $\langle S \rangle$ the subalgebra of $V$ generated by $S$, which is by definition the smallest subalgebra of $V$ containing $S$.

**Proposition 2.10** For any subset $S$ of $V$, the subalgebra $\langle S \rangle$ generated by $S$ is linearly spanned by vectors

$$u_{n_1}^{(1)} \cdots u_{n_r}^{(r)} 1$$ \hspace{1cm} (2.48)

for $r \geq 0$, $u^{(i)} \in S$, $n_1, \ldots, n_r \in \mathbb{Z}$.

**Proof.** Let $U$ be the subspace linearly spanned by vectors in (2.48). Since, any subalgebra that contains $S$ must contain $U$, we have $U \subseteq \langle S \rangle$. To prove $\langle S \rangle \subseteq U$, since $S \subseteq U$, it suffices to prove that $U$ is a subalgebra. Since $S \subseteq \text{Stab}(U)$ and $\text{Stab}(U)$ is a subalgebra (Lemma 2.9), we have $\langle S \rangle \subseteq \text{Stab}(U)$. Consequently, $U \subseteq \langle S \rangle \subseteq \text{Stab}(U)$. Then $U$ is a subalgebra (clearly, $1 \in U$), so that $\langle S \rangle \subseteq U$. This proves $U = \langle S \rangle$, completing the proof. \qquad \square

For ordinary vertex algebras, due to the Borcherds’ commutator formula, we know (cf. [FHL]) that vertex operators $Y(u, x_1)$ and $Y(v, x_2)$ commute if and only if $u_i v = 0$ for $i \geq 0$. For (weak) axiomatic $G_1$-vertex algebras, we in general do not have Borcherds’ commutator formula. Nevertheless, here we have:
Proposition 2.11 Let $V$ be a weak axiomatic $G_1$-vertex algebra and let $u,v \in V$, $k \in \mathbb{N}$, $q \in \mathbb{C}^*$. Then
\[(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = q(x_1 - x_2)^k Y(v, x_2) Y(u, x_1) \] (2.49)
if and only if
\[x^k Y(u, x)v \in V[[x]], \quad Y(u, x)v = q e^{xD} Y(v, -x) u. \] (2.50)
(2.51)
In particular, $[Y(u, x_1), Y(v, x_2)] = 0$ if and only if $Y(u, x)v \in V[[x]]$ and $Y(u, x)v = e^{xD} Y(v, -x) u$.

Proof. Assume (2.49) holds. For any $w \in V$, in view of Lemma 2.1, (2.49), together with the weak associativity relation
\[(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2) w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w \]
for some nonnegative integer $l$, gives
\[
x_0^{-l} \delta \left( \frac{x_1 - x_0}{x_0} \right) Y(u, x_1) Y(v, x_2) w - x_0^{-l} \delta \left( \frac{x_2 - x_1}{-x_0} \right) q Y(v, x_2) Y(u, x_1) w
= x_2^{-l} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) w. \] (2.52)
With (2.49), after multiplied by $(x_1 - x_2)^k$ the left-hand side of (2.52) involves only nonnegative powers of $x_0$, so is the right-hand side. Then (by taking $\text{Res}_{x_1}$)
\[x_0^k Y(Y(u, x_0)v, x_2) w \in V((x_2))[[x_0]] \quad \text{for } w \in V. \] (2.53)
Since the vertex operator map $Y$ is injective (from the creation property (2.19)), we obtain (2.50). Similarly, (2.49), together with the weak associativity relation
\[(-x_0 + x_1)^l q Y(v, -x_0 + x_1) Y(u, x_1) w = (-x_0 + x_1)^l q Y(Y(v, -x_0)u, x_1) w \]
for some nonnegative integer $l$, gives
\[
x_0^{-l} \delta \left( \frac{x_1 - x_0}{x_0} \right) Y(u, x_1) Y(v, x_2) w - x_0^{-l} \delta \left( \frac{x_2 - x_1}{-x_0} \right) q Y(v, x_2) Y(u, x_1) w
= x_1^{-l} \delta \left( \frac{x_2 + x_0}{x_1} \right) q Y(Y(v, -x_0)u, x_1) w. \] (2.54)
Using (2.40) we get
\[
x_1^{-l} \delta \left( \frac{x_2 + x_0}{x_1} \right) q Y(Y(v, -x_0)u, x_1) w
= x_1^{-l} \delta \left( \frac{x_2 + x_0}{x_1} \right) q Y(Y(v, -x_0)u, x_2 + x_0) w
= x_1^{-l} \delta \left( \frac{x_2 + x_0}{x_1} \right) q Y(e^{x_0 D} Y(v, -x_0)u, x_2) w. \] (2.55)
Thus

\[ x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)w = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) qY(e^{x_0D}Y(v, -x_0)u, x_2)w, \] (2.56)

which (by taking Res_{x_1}) gives

\[ Y(Y(u, x_0)v, x_2)w = qY(e^{x_0D}Y(v, -x_0)u, x_2)w. \] (2.57)

Again, with \( Y \) being injective, the skew-symmetry relation (2.51) follows immediately.

On the other hand, assume that (2.51), (2.52) and (2.54) hold. Let \( w \in V \). There exists a nonnegative integer \( l \) such that \( x^lY(v, x)w \in V[[x]] \) and

\[
(x_0 + x_2)^lY(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^lY(Y(u, x_0)v, x_2)w,
\] (2.58)

\[
(-x_0 + x_1)^lY(v, -x_0 + x_1)Y(u, x_1)w = (-x_0 + x_1)^lY(Y(v, -x_0)u, x_1)w. \] (2.59)

Then

\[ x_2^l(x_0 + x_2)^lY(Y(u, x_0)v, x_2)w = x_2^l(x_0 + x_2)^lY(u, x_0 + x_2)Y(v, x_2)w \in V((x_0))[[x_2]], \]

hence

\[
\left[ x_2^l(x_0 + x_2)^lY(Y(u, x_0)v, x_2)w \right]_{x_2 = -x_0 + x_1} = \left[ x_2^l(x_0 + x_2)^lY(Y(u, x_0)v, x_2)w \right]_{x_2 = x_1 - x_0}. \]

(2.60)

From (2.51) and (2.52) we have

\[ qY(Y(v, -x_0)u, x_1) = Y(e^{-x_0D}Y(u, x_0)v, x_1) = Y(Y(u, x_0)v, x_1 - x_0). \] (2.61)

Using (2.58), (2.59), (2.61) and (2.60) we get

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_0^k x_1 x_2^l Y(u, x_1)Y(v, x_2)w
\]

\[
- x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) x_0^k x_1 x_2^l qY(v, x_2)Y(u, x_1)w
\]

\[
= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_0^k x_1 x_2^l Y(u, x_0 + x_2)^lY(v, x_2)w
\]

\[
- x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) x_0^k x_1 (-x_0 + x_1)^l qY(v, -x_0 + x_1)Y(u, x_1)w
\]

\[
= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \left[ x_0^k x_1 x_2^l Y(Y(u, x_0)v, x_2)w \right]
\]

\[
- x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \left[ x_0^k x_1 (-x_0 + x_1)^l qY(Y(v, -x_0)u, x_1)w \right]
\]

\[
= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \left[ x_0^k x_1 x_2^l Y(Y(u, x_0)v, x_2)w \right]
\]

\[
- x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \left[ x_0^k x_1 (-x_0 + x_1)^l Y(Y(u, x_0)v, x_1 - x_0)w \right]
\]
Remark 2.12 As we have seen in the proof of Proposition 2.11, for weak axiomatic $G_1$-vertex algebras, the weak commutativity relation (2.49) amounts to the following Jacobi identity

\[ x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) \ Y(u, x_1)Y(v, x_2)w - x^{-1}_0 \delta \left( \frac{x_2 - x_1}{-x_0} \right) qY(v, x_2)Y(u, x_1)w \]

Taking $\text{Res}_{x_0}$, then using (2.50) we obtain (2.49). \qed

Remark 2.13 As we have seen in the proof of Proposition 2.11, for weak axiomatic $G_1$-vertex algebras, the weak commutativity relation (2.49) amounts to the following Jacobi identity

\[ x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2)w - x^{-1}_0 \delta \left( \frac{x_2 - x_1}{-x_0} \right) qY(v, x_2)Y(u, x_1)w \]

Taking $\text{Res}_{x_0}$, then using (2.50) we obtain (2.49). \qed

Remark 2.14 It was known (cf. [FHL], [Li1], [LL]) that in the theory of ordinary vertex algebras, under the skew-symmetry, the weak commutativity is equivalent to the weak associativity. Proposition 2.11 and Corollary 2.13 are in the same spirit.

\[ x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2)w - x^{-1}_0 \delta \left( \frac{x_2 - x_1}{-x_0} \right) qY(v, x_2)Y(u, x_1)w \]
Let $S$ be a subset of a weak axiomatic $G_1$-vertex algebra $V$. We define the localizer $L_V(S)$ of $S$ in $V$ to consist of $v \in V$ such that for every $w \in S$ there exists a nonnegative integer $k$ such that
\[(x_1 - x_2)^k[Y(v, x_1), Y(w, x_2)] = 0.\]
In view of Corollary 2.13 we have
\[L_V(S) = \{v \in V \mid Y(v, x)w = e^{xD}Y(w, -x)v \text{ for every } w \in S\}.\] (2.66)

**Proposition 2.15** For any subset $S$ of $V$, the localizer $L_V(S)$ is a subalgebra.

**Proof.** Clearly $1 \in L_V(S)$, so we must prove that $u_n v \in L_V(S)$ for $u, v \in L_V(S), n \in \mathbb{Z}$. In view of Proposition 2.11, we must show
\[Y(u_n v, x)w = e^{xD}Y(w, -x)u_n v \text{ for every } w \in S.\] (2.67)

Let $u, v \in L_V(S)$ and $w \in S$ and let $l$ be a nonnegative integer such that
\[(x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w = (x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w \text{ (2.68)}\]
\[(x_0 + x_2)^l Y(u, x_0)Y(w, -x_2)v = (x_0 + x_2)^l Y(w, -x_2)Y(u, x_0)v. \text{ (2.69)}\]

With $v \in L_V(S)$, $w \in S$, we also have
\[Y(v, x_2)w = e^{x_2 D}Y(w, -x_2)v. \text{ (2.70)}\]

Using (2.68)-(2.70) and the conjugation formula (2.40) we get
\[(x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w = (x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w \]
\[= (x_0 + x_2)^l Y(u, x_0 + x_2)e^{x_2 D}Y(w, -x_2)v \]
\[= e^{x_2 D}(x_0 + x_2)^l Y(u, x_0)Y(w, -x_2)v \]
\[= e^{x_2 D}(x_0 + x_2)^l Y(w, -x_2)Y(u, x_0)v. \text{ (2.71)}\]

Multiplying both sides by $(x_2 + x_0)^{-l}$ we obtain
\[Y(Y(u, x_0)v, x_2)w = e^{x_2 D}Y(w, -x_2)Y(u, x_0)v, \text{ (2.72)}\]
as desired. \(\square\)

A subset $S$ of a weak axiomatic $G_1$-vertex algebra $V$ is said to be local if $S \subset L_V(S)$, that is, for any $u, v \in S$, there exists a nonnegative integer $k$ such that
\[(x_1 - x_2)^k[Y(u, x_1), Y(v, x_2)] = 0.\] (2.73)

In view of Remark 2.12, any local subalgebra of $V$ is an ordinary vertex algebra. Furthermore, ordinary vertex algebras are exactly local weak axiomatic $G_1$-vertex algebras.
Lemma 2.16  Any maximal local subspace \( A \) of a weak axiomatic \( G_1 \)-vertex algebra \( V \) is an ordinary vertex algebra.

Proof. It suffices to prove that \( A \) is a subalgebra. Clearly, \( A + C_1 \) is a local subspace of \( V \). With \( A \) being maximal, we must have \( A = A + C_1 \), hence \( 1 \in A \). Let \( a, b \in A \), \( n \in \mathbb{Z} \). Since \( A \) is local, we have \( A \subset L_V(A) \). In view of Proposition 2.13, we get \( a_n b \in L_V(A) \). This also implies that \( a_n b \in A \). By Proposition 2.13 again we have \( a_n b \in L_V(\{a_n b\}) \). This shows that \( A + C_n b \) is local. Again, since \( A \) is maximal, we must have \( a_n b \in A \). This proves that \( A \) is a subalgebra of \( V \), as we need.

Proposition 2.17  Let \( V \) be a weak axiomatic \( G_1 \)-vertex algebra and let \( S \) be a local subset of \( V \). Then the subalgebra \( \langle S \rangle \) of \( V \) generated by \( S \) is an ordinary vertex algebra.

Proof. It follows from Zorn’s lemma that \( S \) is contained in some maximal local subspace \( A \) of \( V \). By Lemma 2.16, \( A \) is an ordinary vertex algebra, so is \( \langle S \rangle \) as a subalgebra of \( A \). □

The following Proposition follows immediately from the proof of the corresponding proposition in [FHL] for ordinary vertex algebras:

Proposition 2.18  Let \( V_1, \ldots, V_n \) be (weak) axiomatic \( G_1 \)-vertex algebras. Then the tensor product space

\[
V_1 \otimes \cdots \otimes V_n,
\]

equipped with the vertex operator map \( Y \) defined by

\[
Y(v_1 \otimes \cdots \otimes v_n, x) = Y(v_1, x) \otimes \cdots \otimes Y(v_n, x)
\]

(2.74)

and equipped with the vacuum vector

\[
1 = 1 \otimes \cdots \otimes 1,
\]

(2.75)

is a (weak) axiomatic \( G_1 \)-vertex algebra. □

Remark 2.19  The notions of left ideal and right ideal for a weak axiomatic \( G_1 \)-vertex algebra are defined in the obvious ways.

3 Constructing \( G_1 \)-vertex algebras from ordinary vertex algebras

In this section we shall give several ways to construct (weak) axiomatic \( G_1 \)-vertex algebras from ordinary vertex algebras. Specifically, we consider a certain twisting of abelian group-graded vertex algebras by a normalized 2-cocycle and the cross product of a vertex algebra \( V \) with a group \( G \) which acts on \( V \) by automorphisms. We also consider the axiomatic
$G_1$-vertex algebra $M(n,V)$ of all $n \times n$ matrices over a vertex algebra $V$. We introduce the notions of restricted weak axiomatic $G_1$-vertex algebra to unify the given examples.

Let $G$ be an abelian group. The group algebra $\mathbb{C}[G]$ is a commutative associative algebra. Let $\epsilon(\cdot, \cdot)$ be a normalized 2-cocycle of $G$ in the sense that $\epsilon(\cdot, \cdot)$ is a $\mathbb{C}^*$-valued function on $G \times G$ such that

$$
\epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma),
$$

(3.1)

$$
\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1.
$$

(3.2)

Then $\mathbb{C}[G]$ becomes a noncommutative associative algebra (with the same identity element) by defining

$$
e^{\alpha} \circ e^{\beta} = \epsilon(\alpha, \beta)e^{\alpha + \beta} \quad \text{for} \quad \alpha, \beta \in G.
$$

(3.3)

More generally, let $A$ be a $G$-graded associative algebra. Then, in the same way we can make $A$ an associative algebra by using a normalized 2-cocycle $\epsilon$ of $G$. We next discuss an exact analogue for axiomatic $G_1$-vertex algebras of this fact.

**Example 3.1** Let $G$ be an abelian group and let $V$ be a $G$-graded axiomatic $G_1$-vertex algebra in the sense that $V$ is an axiomatic $G_1$-vertex algebra equipped with a $G$-grading $V = \bigoplus_{g \in G} V^g$ such that

$$
u_n v \in V^{g+h} \quad \text{for} \quad u \in V^g, \quad v \in V^h, \quad n \in \mathbb{Z}.
$$

(3.4)

Assume that $1 \in V^0$. Let $\epsilon$ be a normalized 2-cocycle of $G$. Define a linear map

$$
Y_\epsilon(\cdot, x) : V \rightarrow (\text{End}V)[[x, x^{-1}]]
$$

(3.5)

$$
v \mapsto Y_\epsilon(v, x)
$$

by

$$
Y_\epsilon(u, x)v = \epsilon(g, h)Y(u, x)v \quad \text{for} \quad u \in V^g, \quad v \in V^h, \quad g, h \in G.
$$

(3.6)

With $\epsilon$ being normalized, it is clear that all the axioms except for the weak associativity hold. Let $u \in V^{g_1}, \quad v \in V^{g_2}, \quad w \in V^{g_3}$. Then

$$
Y_\epsilon(u, x_1)Y_\epsilon(v, x_2)w = \epsilon(g_1, g_2 + g_3)\epsilon(g_2, g_3)Y(u, x_1)Y(v, x_2)w
$$

(3.7)

and

$$
Y_\epsilon(Y_\epsilon(u, x_0)v, x_2)w = \epsilon(g_1, g_2)\epsilon(g_1 + g_2, g_3)Y(Y(u, x_0)v, x_2)w.
$$

(3.8)

With the property (3.1), we easily see that the weak associativity holds for $Y_\epsilon$. Thus, $(V, Y_\epsilon, 1)$ is an axiomatic $G_1$-vertex algebra.

Furthermore, we assume that $V$ is an ordinary vertex algebra. From (3.7) (using the obvious symmetry) we have

$$
Y_\epsilon(v, x_2)Y_\epsilon(u, x_1)w = \epsilon(g_2, g_1 + g_3)\epsilon(g_1, g_3)Y(v, x_2)Y(u, x_1)w.
$$

(3.9)
Then it follows immediately from (3.7)-(3.9), (3.1) and the Jacobi identity (2.21) that

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - c(g, h) x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) Y(v, x_2) Y(u, x_1)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)$$

(3.10)

for $u \in V^g$, $v \in V^h$, $g, h \in G$, where

$$c(g, h) = \epsilon(g, h) \epsilon(h, g)^{-1}. \quad (3.11)$$

It is easy to see that $c(\cdot, \cdot)$ is a $\mathbb{C}^*$-valued bilinear form on $G$.

**Remark 3.2** Note that for an ordinary vertex algebra $V$, the axiomatic $G_1$-vertex algebra $(V, Y, \Theta, G, c, (\cdot, \cdot))$ obtained in Example 3.1 in fact has a natural generalized vertex algebra structure $(V, Y, \mathbb{C}, G, c, (\cdot, \cdot))$ with $(\cdot, \cdot) = 0$ in the sense of [DL]. The notion of generalized vertex algebra naturally generalizes the notions of ordinary vertex algebra and vertex superalgebra. In the notion of generalized vertex algebra, if the form $(\cdot, \cdot)$ is (zero) trivial, then the linear map $Y$ maps $V$ to $\text{Hom}(V, V((x)))$ and the generalized Jacobi identity for the generalized vertex algebra reduces to (3.10). For convenience, we call a generalized vertex algebra with trivial form $(\cdot, \cdot)$ a **restricted generalized vertex algebra**. It is easy to see that any restricted generalized vertex algebra is an axiomatic $G_1$-vertex algebra.

**Remark 3.3** Let $L$ be an integral lattice. For any normalized 2-cocycle $\epsilon(\cdot, \cdot)$ of $L$, a restricted generalized vertex algebra $V_L$ was constructed in [DL]. It was proved in [DL] that for a certain $\epsilon(\cdot, \cdot)$, $V_L$ is a vertex superalgebra.

**Example 3.4** In view of Proposition 2.18 and Example 2.5, the tensor product of any axiomatic $G_1$-vertex algebra $V$ with any associative algebra $A$ is an axiomatic $G_1$-vertex algebra. In particular, the tensor product $V \otimes M(n, \mathbb{C})$ is an axiomatic $G_1$-vertex algebra. If we naturally identify the space $V \otimes M(n, \mathbb{C})$ with the vector space $M(n, V)$ of $n \times n$ matrices with entries in $V$, then for $A = (a_{ij})$, $B \in M(n, V)$,

$$Y(A, x)B = (Y(a_{ij}, x))B \quad \text{(the formal matrix product).} \quad (3.12)$$

Furthermore, the general linear group $GL(n, \mathbb{C})$ naturally acts on $M(n, V)$, and so does the unitary group $U(n)$.

In the classical associative algebra theory, for an algebra $A$ acted by a group $G$, there is a notion of cross product (or skew group algebra) where the underlying vector space is $A \otimes \mathbb{C}[G]$ and the multiplication is given by

$$(ag)(bh) = ag(b)gh \quad \text{for } a, b \in A, \ g, h \in G. \quad (3.13)$$
When $G$ acts on $A$ trivially, the cross product algebra becomes the usual product of $A$ with the group algebra $\mathbb{C}[G]$. In the following we give an analogue for axiomatic $G_1$-vertex algebras. To do so, first we define the notion of automorphism of a (weak) axiomatic $G_1$-vertex algebra in the obvious way: An automorphism of $V$ is an invertible linear endomorphism $\psi$ of $V$ such that $\psi(1) = 1$ and such that $\psi(u_n v) = \psi(u)_n \psi(v)$ for $u, v \in V$, $n \in \mathbb{Z}$.

**Example 3.5** Let $V$ be an axiomatic $G_1$-vertex algebra and let $G$ be a group acting on $V$ as automorphisms. Define a vertex operator map $Y$ on $V[G] = V \otimes \mathbb{C}[G]$ by

$$Y(ug, x)(vh) = Y(u, x)g(v)gh \quad \text{for } u, v \in V, \ g, h \in G. \quad (3.14)$$

Taking $1e$ to be the vacuum vector, where $e$ denotes the identity element of $G$, we easily see that all the axioms except for the weak associativity hold. Furthermore, let $u, v, w \in V, g_1, g_2, g_3 \in G$. Since $G$ acts as automorphisms of $V$, we have

$$Y(ug_1, x_1)Y(vg_2, x_2)wg_3 = Y(ug_1, x_1)Y(v, x_2)g_2(w)(g_2g_3)$$

$$= Y(u, x_1)g_1(Y(v, x_2)g_2(w))g_1(g_2g_3)$$

$$= Y(u, x_1)Y(g_1(v), x_2)(g_1g_2)(w)g_1(g_2g_3) \quad (3.15)$$

and

$$Y(Y(ug_1, x_0)vg_2, x_2)wg_3 = Y(Y(u, x_0)g_1(v)g_1g_2, x_2)wg_3$$

$$= Y(Y(u, x_0)g_1(v), x_2)(g_1g_2)(w)(g_1g_2)g_3. \quad (3.16)$$

Now, the weaker version of the weak associativity for $Y$ on $V[G]$ follows immediately. Therefore, $V[G]$ equipped with this vertex operator map $Y$ is a weak axiomatic $G_1$-vertex algebra. It is clear that if $\dim \mathbb{C}[G]w < \infty$ for every $w \in V$, $V[G]$ is an axiomatic $G_1$-vertex algebra. In particular, this is true if $V$ is an ordinary vertex operator algebra (with the two grading restrictions), since any automorphism group preserves every homogeneous subspace of $V$.

**Example 3.6** We continue with Example 3.5 studying the cross product weak axiomatic $G_1$-vertex algebra. Let us assume that $V$ is an ordinary vertex algebra. From (3.15) (using the obvious symmetry) we have

$$Y(vg_2, x_2)Y(ug_1, x_1)wg_3 = Y(v, x_2)Y(g_2(u), x_1)(g_2g_1)(w)g_2(g_1g_3). \quad (3.17)$$

We are not able to see weak commutativity. On the other hand, notice that

$$Y(g_1(v)g_2, x_2)Y(g_2^{-1}(u)g_1, x_1)(g_1^{-1}g_2^{-1}g_1g_2)(w)(g_1^{-1}g_2^{-1}g_1g_2g_3)$$

$$= Y(g_1(v), x_2)Y(u, x_1)(g_1g_2)(w)(g_1g_2g_3). \quad (3.18)$$
Then we have the following Jacobi-like identity:

\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(ug_1, x_1)Y(vg_2, x_2)wg_3 \]
\[ -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) Y(g_1(v)g_2, x_2)Y(g_2^{-1}(u)g_1, x_1)(g_1^{-1}g_2^{-1}g_1g_2)(w)(g_1^{-1}g_2^{-1}g_1g_23) \]
\[ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(ug_1, x_0)vg_2, x_2)wg_3. \] (3.19)

Let \( R \) be the linear endomorphism of \( V[G] \otimes V[G] \otimes V[G] \) uniquely determined by

\[ R(vg_2 \otimes ug_1 \otimes wg_3) = g_1(v)g_2 \otimes g_2^{-1}(u)g_1 \otimes (g_1^{-1}g_2^{-1}g_1g_2)(w)(g_1^{-1}g_2^{-1}g_1g_23). \] (3.20)

Then

\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(ug_1, x_1)Y(vg_2, x_2)wg_3 \]
\[ -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) (Y \otimes Y)(x_2, x_1)R(vg_2 \otimes ug_1 \otimes wg_3) \]
\[ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(ug_1, x_0)vg_2, x_2)wg_3, \] (3.21)

where \( (Y \otimes Y)(x_1, x_2) \) is the linear map from \( V[G] \otimes V[G] \otimes V[G] \) to \( V[G](\langle x_1 \rangle)(\langle x_2 \rangle) \) defined by

\[ (Y \otimes Y)(x_1, x_2)(a \otimes b \otimes c) = Y(a, x_1)Y(b, x_2)c. \] (3.22)

If \( G \) is abelian, the Jacobi-like identity reduces to

\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(ug_1, x_1)Y(vg_2, x_2)wg_3 \]
\[ -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) Y(g_1(v)g_2, x_2)Y(g_2^{-1}(u)g_1, x_1)wg_3 \]
\[ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(ug_1, x_0)vg_2, x_2)wg_3. \] (3.23)

**Remark 3.7** The classical cross product algebra or skew group algebra was employed by Yamksulna [Y] in a certain study of twisted modules. It would be interesting if one can apply the cross product axiomatic \( G_1 \)-vertex algebra in that type of study.

Motivated by Example 3.6, we introduce the following notion of restricted weak axiomatic \( G_1 \)-vertex algebra:

**Definition 3.8** A **restricted weak axiomatic \( G_1 \)-vertex algebra** is a vector space \( V \) equipped with a linear map \( Y \) from \( V \) to \( \text{Hom}(V, V(\langle x \rangle)) \), a distinguished vector \( 1 \in V \) and a linear
map $R \in \text{End}(V \otimes V \otimes V)$ such that $Y(1, x) = 1$, for $v \in V$, $Y(v, x)1 \in V[[x]]$ and
\[ \lim_{x \to 0} Y(v, x)1 = v \] and such that the following Jacobi-like identity holds for $u, v, w \in V$:
\[ x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2)w \]
\[ -x_0^{-1}\delta \left( \frac{x_2 - x_1}{-x_0} \right) (Y \otimes Y)(x_2, x_1)R(v \otimes u \otimes w) \]
\[ = x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)w, \] (3.24)
where $(Y \otimes Y)(x_1, x_2)$ is the linear map from $V \otimes V \otimes V$ to $V((x_1))(x_2))$ defined by
\[ (Y \otimes Y)(x_1, x_2)(u, v, w) = Y(u, x_1)Y(v, x_2)w. \] (3.25)

**Remark 3.9** Let $u, v, w \in V$ and let $l$ be a nonnegative integer such that
\[ x_1^l(Y \otimes Y)(x_2, x_1)R(v \otimes u \otimes w) \in V[[x_1, x_2, x_2^{-1}]]. \]
By taking Res$_{x_1} x_1^l$ of the Jacobi-like identity (3.24) we get
\[ (x_0 + x_2)^lY(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^lY(Y(u, x_0)v, x_2)w. \] (3.26)

Then, a restricted weak axiomatic $G_1$-vertex algebra is indeed a weak axiomatic $G_1$-vertex algebra. Furthermore, let $k$ be a nonnegative integer such that $x^kY(u, x)v \in V[[x]]$. Then we get
\[ (x_1 - x_2)^kY(u, x_1)Y(v, x_2)w = (x_1 - x_2)^k(Y \otimes Y)(x_2, x_1)R(v \otimes u \otimes w). \] (3.27)

**Remark 3.10** We explain that many examples of weak axiomatic $G_1$-vertex algebras, discussed previously, are restricted weak axiomatic $G_1$-vertex algebras. First, from Example 3.6 the cross product of an ordinary vertex algebra with a group is a restricted weak axiomatic $G_1$-vertex algebra.

Second, any restricted generalized vertex algebra defined in Remark 3.2 is a restricted weak axiomatic $G_1$-vertex algebra, where
\[ R(v \otimes u \otimes w) = c(g, h)(v \otimes u \otimes w) \quad \text{for } u \in V^g, \ v \in V^h, \ w \in V, \ g, h \in G. \] (3.28)

Third, for any ordinary vertex algebra $V$ and any associative algebra $A$, the tensor product axiomatic $G_1$-vertex algebra $V \otimes A$ is a restricted weak axiomatic $G_1$-vertex algebra, where
\[ R(uv \otimes vb \otimes wc) = ub \otimes va \otimes wc \quad \text{for } u, v, w \in V, \ a, b, c \in A. \] (3.29)
Indeed, the Jacobi-like identity holds because of the Jacobi identity of $V$ and the relations

\begin{align}
Y(ua,x_{1})Y(vb,x_{2})wc &= Y(u,x_{1})Y(v,x_{2})wa(bc), \\
(Y \otimes Y)(x_{2},x_{1})R(vb \otimes ua \otimes wc) &= Y(va,x_{2})Y(ub,x_{1})wc = Y(v,x_{2})Y(u,x_{1})wa(bc), \\
Y(Y(ua,x_{0})vb,x_{2})wc &= Y(Y(u,x_{0})v,x_{2})wa(bc).
\end{align}

(3.30)

(3.31)

Then, for any ordinary vertex algebra $V$, $M(n,V)$ is a restricted weak axiomatic $G_{1}$-vertex algebra. On the other hand, by taking $V = \mathbb{C}$, we see that any associative algebra $A$ is a restricted weak axiomatic $G_{1}$-vertex algebra, where the classical associativity axiom is expressed in terms of delta functions as

\begin{align}
x_{0}^{-1}\delta \left( \frac{x_{1} - x_{2}}{x_{0}} \right) a(bc) - x_{0}^{-1}\delta \left( \frac{x_{2} - x_{1}}{-x_{0}} \right) a(bc) &= x_{2}^{-1}\delta \left( \frac{x_{1} - x_{0}}{x_{2}} \right) (ab)c.
\end{align}

(3.33)

Remark 3.11 Since restricted weak axiomatic $G_{1}$-vertex algebras are still too general to study, we may consider a special class of restricted weak axiomatic $G_{1}$-vertex algebras with $R = R' \otimes 1$, where $R' \in \text{End}(V \otimes V)$. Restricted generalized vertex algebras, matrix axiomatic $G_{1}$-vertex algebras $M(n,V)$ and the cross product axiomatic $G_{1}$-vertex algebras $V[G]$ with $G$ abelian are of this type.

Remark 3.12 We here define a notion of axiomatic $G_{1}$-vertex operator algebra. An \textit{axiomatic $G_{1}$-vertex operator algebra} is an axiomatic $G_{1}$-vertex algebra $V$ equipped with a distinguished vector $\omega \in V$, called the \textit{conformal vector}, such that

\begin{align}
[L(m),L(n)] &= (m-n)L(m+n) + \frac{1}{12}(m^{3} - m)\text{rank}V \delta_{m+n,0} \\
Y(\omega,x) &= \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}
\end{align}

(3.34)

(3.35)

for $m,n \in \mathbb{Z}$, where

\begin{align}
Y(L(-1)v,x) &= \frac{d}{dx}Y(v,x), \\
[L(m),Y(v,x)] &= \sum_{i \geq 0} \binom{m+1}{i}x^{m+1-i}Y(L(i-1)v,x)
\end{align}

(3.36)

(3.37)

for $v \in V$ and such that

\begin{align}
V &= \coprod_{n \in \mathbb{Z}} V_{(n)},
\end{align}

(3.38)

\begin{align}
\dim V_{(n)} < \infty \text{ for all } n \in \mathbb{Z} \text{ and } V_{(n)} = 0 \text{ for } n \text{ sufficiently negative, where for } n \in \mathbb{Z},
\end{align}

(3.39)

\begin{align}
V_{(n)} &= \{ v \in V \mid L(0)v = nv \}.
\end{align}


4 Modules for axiomatic $G_1$-vertex algebras

In this section we shall define the notion of module for a (weak) axiomatic $G_1$-vertex algebra and we obtain certain analogous results of ordinary vertex algebras for axiomatic $G_1$-vertex algebras.

Let $V$ be a weak axiomatic $G_1$-vertex algebra, fixed throughout this section. We first define the notion of $V$-module.

**Definition 4.1** A $V$-module is a vector space $W$ equipped with a linear map

$$Y_W(\cdot, x) : V \to (\text{End}W)[[x, x^{-1}]]$$

$$v \mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (v_n \in \text{End}W) \quad (4.1)$$

such that all the following axioms hold: For every $v \in V$, $w \in W$,

$$v_n w = 0 \quad \text{for } n \text{ sufficiently large} ; \quad (4.2)$$

$$Y_W(1, x) = 1_W \quad (\text{where } 1_W \text{ is the identity operator on } W) ; \quad (4.3)$$

for any $u, v \in V$ and $w \in W$, there exists $l \in \mathbb{N}$ such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2) w . \quad (4.4)$$

If $V$ is an axiomatic $G_1$-vertex algebra, for the notion of $V$-module, we use the stronger weak associativity: For any $u, v \in V$ and $w \in W$, there exists $l \in \mathbb{N}$ such that for all $v \in V$, (4.4) holds

We next discuss some consequences of the definition. First, by carefully examining the first half of the proof of Proposition 2.6 we find that the same argument (with $v$ being replaced by $w \in W$) gives:

**Proposition 4.2** Let $(W, Y_W)$ be a $V$-module. Then

$$Y_W(Dv, x) = \frac{d}{dx} Y_W(v, x) \quad \text{for } v \in V$$

(recall the linear operator $D$ on $V$). \hfill $\Box$

The notions of submodule, irreducible module and module homomorphism are defined in the obvious ways.

The following result tells us how a certain commutativity relation of vertex operators on $V$ is related to that of the vertex operators on other modules:
Proposition 4.3 Let \( u, v \in V \), \( k \in \mathbb{N} \) and \( q \in \mathbb{C}^* \). If
\[
(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = q(x_1 - x_2)^k Y(v, x_2) Y(u, x_1)
\] (4.6)
on \( V \), then for any \( V \)-module \((W, Y_W)\),
\[
(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = q(x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1).
\] (4.7)
In particular, if \([Y(u, x_1), Y(v, x_2)] = 0\) on \( V \), then \([Y_W(u, x_1), Y_W(v, x_2)] = 0\) on \( W \). On the other hand, if \( W \) is a faithful module and if (4.7) holds, then (4.4) holds.

Proof. From Proposition 2.11 we have
\[
x^k Y(u, x) v \in V[[x]] \quad \text{and} \quad Y(u, x)v = q e^{xD} Y(v, -x) u.
\] (4.8)
Then (4.7) follows from the same proof of the “if” part of Proposition 2.11. On the other hand, if \( W \) is a faithful module and if (4.7) holds, the same proof of the “only if” part of Proposition 2.11 shows that (4.8) holds. Then (4.6) follows from (the “if” part of) Proposition 2.11. \( \square \)

Proposition 4.4 Let \( V \) be a restricted generalized vertex algebra and let \((W, Y_W)\) be a module for \( V \) viewed as an axiomatic \( G_1 \)-vertex algebra. Then
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - c(g, h) x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1)
\] 
\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0) v, x_2)
\] (4.9)
for \( u \in V^g, v \in V^h \) with \( g, h \in G \).

Proof. For \( u \in V^g, v \in V^h \), by Proposition 4.3, we have
\[
(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = c(g, h)(x_1 - x_2)^k Y_W(v, x_2) Y_W(u, x_1).
\] (4.10)
Then it follows immediately from Lemma 2.1. \( \square \)

The following Proposition follows from the proof of the corresponding result in [FHL] for ordinary vertex algebras.

Proposition 4.5 Let \( V_1, \ldots, V_r \) be (weak) axiomatic \( G_1 \)-vertex algebras and let \( W_i \) be a \( V_i \)-module for \( i = 1, \ldots, r \). Then \( W_1 \otimes \cdots \otimes W_r \) is a \( V_1 \otimes \cdots \otimes V_r \)-module with the vertex operator map \( Y \) defined by
\[
Y(v^{(1)} \otimes \cdots \otimes v^{(r)}, x) = Y(v^{(1)}, x) \otimes \cdots \otimes Y(v^{(r)}, x). \quad \square \tag{4.11}
\]

Example 4.6 Let \( V \) be a (weak) axiomatic \( G_1 \)-vertex algebra, \( W \) a \( V \)-module and let \( n \) be a positive integer. Then \( W^n \) is an \( M(n, V) \)-module.
As an immediate consequence of Proposition 4.3 we have:

**Corollary 4.7** Let $U$ and $V$ be (weak) axiomatic $G_1$-vertex algebras and let $W$ be a module for the tensor product $U \otimes V$. Then $W$ is a natural $U$-module and $V$-module and furthermore, the actions of $U$ and $V$ on $W$ commute. $\square$

Now, we have:

**Proposition 4.8** Let $V$ be an axiomatic $G_1$-vertex algebra and let $n$ be a positive integer. Then any irreducible $M(n,V)$-module is of the form $W^n$, where $W$ is an irreducible $V$-module. On the other hand, for any irreducible $V$-module $W$, $W^n$ is an irreducible $M(n,V)$-module.

**Proof.** Note that $\mathbb{C}^n$ is the only irreducible module up to equivalence for the matrix algebra $M(n,\mathbb{C})$ and any $M(n,\mathbb{C})$-module is completely reducible. Since any $M(n,V)$-module $W$ is naturally an $M(n,\mathbb{C})$-module, we have the canonical decomposition

$$M = \text{Hom}_{M(n,\mathbb{C})}(\mathbb{C}^n, M) \otimes \mathbb{C}^n, \tag{4.12}$$

where $\text{Hom}_{M(n,\mathbb{C})}(\mathbb{C}^n, M)$ is naturally a $V$-module. If $M$ is an irreducible $M(n,V)$-module, $\text{Hom}_{M(n,\mathbb{C})}(\mathbb{C}^n, M)$ is necessarily an irreducible $V$-module.

On the other hand, let $W$ be an irreducible $V$-module. For $1 \leq i,j \leq n$, denote by $E_{ij}$ be the matrix whose entry is 1 at $ij$-position and is zero elsewhere. Also, for $1 \leq i \leq n$, denote by $e_i$ the element of $\mathbb{C}^n$ whose $i$-th entry is 1 and others are zero. Then

$$W^n = W \otimes \mathbb{C}^n = \sum_{i=1}^n W \otimes e_i$$

and $E_{ii}W^n = W \otimes e_i$ for $i = 1, \ldots, n$. Since $W$ is an irreducible $V$-module, for $1 \leq i \leq n$, any nonzero element of $W \otimes e_i$ generates $W^n$ as an $M(n,V)$-module. For any nonzero $w \in W^n$, since $w = E_{11}w + \cdots + E_{nn}w$, $E_{ii}w \neq 0$ for some $i$. Then it follows that any nonzero element $w$ of $W^n$ generates $W^n$ as an $M(n,V)$-module. That is, $W^n$ is an irreducible $M(n,V)$-module. $\square$

Next, we shall derive a certain compatibility of vertex operators $Y_W(v, x)$ for $v \in V$ and for a $V$-module $(W, Y_W)$. First we have:

**Lemma 4.9** Let $(W, Y_W)$ be a $V$-module and let $u, v \in V$. Then there exists a nonnegative integer $k$ such that

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))). \tag{4.13}$$

**Proof.** Let $w \in W$. Then there exists a nonnegative integer $l$ such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w = (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2)w. \tag{4.14}$$
Just as in [FHL], [DL], or [LL], we notice that the expression on the left-hand side of (4.14) involves only finitely many negative powers of $x_2$ and the expression on the right-hand side involves only finitely many negative powers of $x_0$. Consequently, the common quantity lies in $W((x_0, x_2))$. Let $k$ be a nonnegative integer such that $x^k Y(u, x) v \in V[[x]]$. (Of course, $k$ depends only on $u$ and $v$.) Set

$$p(x_0, x_2) = x_0^k (x_2 + x_0)^l Y_W(Y(u, x_0)v, x_2)w.$$  

(4.15)

Then $p(x_0, x_2) \in W[[x_0, x_2]][x_2^{-1}]$ and

$$x_0^k (x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = p(x_0, x_2).$$  

(4.16)

Applying $e^{-x_2 \frac{x_2}{x_0}}$ to both sides and then using the Taylor theorem we have

$$(x_0 - x_2)^k x_0^l Y_W(u, x_0) Y_W(v, x_2)w = e^{-x_2 \frac{p(x_0, x_2)}{x_0}} p(x_0, x_2) = p(x_0 - x_2, x_2).$$  

(4.17)

Hence

$$(x_0 - x_2)^k Y_W(u, x_0) Y_W(v, x_2)w = x_0^l p(x_0 - x_2, x_2) \in W((x_0, x_2)).$$  

(4.18)

Since $k$ is independent of $w$, (4.13) follows. 

Remark 4.10 Lemma 4.9 in a slightly different form has been obtained in [LL] (Proposition 3.3.12).

To generalize Lemma 4.9 for the products of more than two vertex operators we shall need to assume that $V$ is an axiomatic $G_1$-vertex algebra and assume that the stronger weak associativity holds on $W$.

**Proposition 4.11** Let $V$ be an axiomatic $G_1$-vertex algebra and let $(W, Y_W)$ be a $V$-module (with the stronger weak associativity). Then for any $v^{(1)}, \ldots, v^{(r)} \in V$, there exists a nonnegative integer $k$ such that

$$\left( \prod_{1 \leq i < j \leq r} (x_i - x_j)^k \right) Y_W(v^{(1)}, x_1) \cdots Y_W(v^{(r)}, x_r) \in \text{Hom}(W, W((x_1, \ldots, x_r))).$$  

(4.19)

**Proof.** First, we prove the special case with $W = V$ by induction on $r$. For $r = 2$, it has been proved by Lemma 4.9. Assume that the assertion holds for a certain $r \geq 2$. Let $v^{(1)}, \ldots, v^{(r+1)} \in V$ and $w \in W$. From the (stronger) weak associativity, there exists $l \in \mathbb{N}$ such that

$$(x_{0i} + x_{r+1})^l Y(v^{(i)}, x_{0i} + x_{r+1}) Y(v, x_{r+1})w = (x_{0i} + x_{r+1})^l Y(Y(v^{(i)}, x_{0i})v, x_{r+1})w$$  

(4.20)
for all \( v \in V \) and for \( i = 1, \ldots, r \). Then
\[
\left( \prod_{i=1}^{r} (x_{0i} + x_{r+1})^l \right) Y(v^{(1)}, x_{01} + x_{r+1}) \cdots Y(v^{(r)}, x_{0r} + x_{r+1}) Y(v^{(r+1)}, x_{r+1}) w
\]
\[=
\left( \prod_{i=1}^{r} (x_{0i} + x_{r+1})^l \right) Y(v^{(1)}, x_{01} + x_{r+1}) \cdots Y(v^{(r-1)}, x_{0r-1} + x_{r+1})
\cdot Y(Y(v^{(r)}, x_{0r}) v^{(r+1)}, x_{r+1}) w
\]
\[=
\left( \prod_{i=1}^{r} (x_{0i} + x_{r+1})^l \right) Y(Y(v^{(1)}, x_{01}) \cdots Y(v^{(r)}, x_{0r}) v^{(r+1)}, x_{r+1}) w. \quad (4.21)
\]

Notice that for the second equality we are using the stronger version of the weak associativity. From the inductive hypothesis there exists a nonnegative integer \( k' \) such that
\[
\left( \prod_{1 \leq i < j \leq r} (x_{0i} - x_{0j})^{k'} \right) Y(v^{(1)}, x_{01}) \cdots Y(v^{(r)}, x_{0r}) v^{(r+1)} \in V((x_{01}, \ldots, x_{0r})) \quad (4.22)
\]
so that there exists a nonnegative integer \( k'' \) (only depending on \( v^{(i)}'s, \) not \( v \)) such that
\[
\left( \prod_{1 \leq i < j \leq r} (x_{0i} - x_{0j})^{k'} \right) x_{01}^{k''} \cdots x_{0r}^{k''} Y(v^{(1)}, x_{01}) \cdots Y(v^{(r)}, x_{0r}) v^{(r+1)} \in V[[x_{01}, \ldots, x_{0r}]] \quad (4.23)
\]
Combining (4.21) with (4.22) we get
\[
\left( \prod_{i=1}^{r} (x_{0i} + x_{r+1})^l \prod_{1 \leq i < j \leq r} (x_{0i} - x_{0j})^{k'} \right) x_{01}^{k''} \cdots x_{0r}^{k''} \cdot
\cdot Y(v^{(1)}, x_{01} + x_{r+1}) \cdots Y(v^{(r)}, x_{0r} + x_{r+1}) Y(v^{(r+1)}, x_{r+1}) w
\in W[[x_{01}, \ldots, x_{0r}]][x_{r+1})]. \quad (4.24)
\]
Therefore (by substituting \( x_{0i} = x_{i} - x_{r+1} \))
\[
\left( \prod_{i=1}^{r} x_{i}^l \prod_{1 \leq i < j \leq r} (x_{i} - x_{j})^{k'} \right) (x_{1} - x_{r+1})^{k''} \cdots (x_{r} - x_{r+1})^{k''} \cdot
\cdot Y(v^{(1)}, x_{1}) \cdots Y(v^{(r)}, x_{r}) Y(v^{(r+1)}, x_{r+1}) w
\in W[[x_{1}, \ldots, x_{r}][x_{r+1})]. \quad (4.25)
\]
That is,
\[
\left( \prod_{1 \leq i < j \leq r} (x_{i} - x_{j})^{k'} \right) (x_{1} - x_{r+1})^{k''} \cdots (x_{r} - x_{r+1})^{k''} \cdot
\cdot Y(v^{(1)}, x_{1}) \cdots Y(v^{(r)}, x_{r}) Y(v^{(r+1)}, x_{r+1}) w
\in W((x_{1}, \ldots, x_{r}, x_{r+1})]. \quad (4.26)
\]
Since $k'$ and $k''$ are independent of $w$, this finishes the induction, proving the special case.

For the general case, similar to the special case, (from the (stronger) weak associativity), there exists $l \in \mathbb{N}$ such that

$$
\left( \prod_{i=1}^{r} (x_{0i} + x_{r+1}) \right) Y_W(v^{(1)}, x_{01} + x_{r+1}) \cdots Y_W(v^{(r)}, x_{0r} + x_{r+1}) Y_W(v^{(r+1)}, x_{r+1}) w
$$

$$
= \left( \prod_{i=1}^{r} (x_{0i} + x_{r+1}) \right) Y_W(\cdots Y(W(v^{(1)}, x_{01}) \cdots Y(v^{(r)}, x_{0r}) v^{(r+1)}, x_{r+1}) w. \quad (4.27)
$$

The rest directly follows from the proof and the result of the special case.  

\[ \blacksquare \]

Remark 4.12 Let $V$ and $W$ be given as in Proposition 4.11. Let $w^* \in W^*$, $v^{(1)}, \ldots, v^{(r)} \in V$, $w \in W$. In view of Proposition 4.11, there exist nonnegative integers $k$ and $l$ such that

$$
\langle w^*, Y_W(v^{(1)}, x_1) \cdots Y_W(v^{(r)}, x_r) w \rangle = \left( \prod_{1 \leq i < j \leq r} (x_i - x_j)^{-k} \prod_{i=1}^{r} x_i^{-l} \right) p(x_1, \ldots, x_r) \quad (4.28)
$$

for some $p(x_1, \ldots, x_r) \in \mathbb{C}[[x_1, \ldots, x_r]]$, where we are using the binomial expansion convention.

Remark 4.13 For an axiomatic $G_1$-vertex algebra $V$, a left ideal of $V$ amounts to a $V$-submodule of $V$. Unlike in the case of ordinary vertex algebras, the notions of left and right ideals are in general different. Thus, the simplicity of $V$ as a $V$-module does not amount to the simplicity of $V$ as an axiomatic $G_1$-vertex algebra.

5 Axiomatic $G_1$-vertex algebras generated by compatible $G_1$-vertex operators

In this section we study (weak) $G_1$-vertex operators on an arbitrary vector space and we show how a suitable set of (weak) $G_1$-vertex operators gives rise to an axiomatic $G_1$-vertex algebra. We recover the corresponding result of [Li1].

Let $W$ be a vector space fixed throughout this section.

Definition 5.1 A weak $G_1$-vertex operator on $W$ is a formal series

$$
a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1} \in (\text{End}W)[[x, x^{-1}]] \quad (5.1)
$$

such that for every $w \in W$, $a_n w = 0$ for $n$ sufficiently large. Namely, a weak $G_1$-vertex operator on $W$ is an element of $\text{Hom}(W, W((x)))$.

All weak $G_1$-vertex operators on $W$ constitute the space $\text{Hom}(W, W((x)))$. We alternatively denote this space by $\mathcal{E}_{G_1}(W)$.
Remark 5.2 In [Li1] and [LL], an element of $\text{Hom}(W,W((x)))$ was simply called a weak vertex operator. We here chose to use the term “weak $G_1$-vertex operator” because of the expected connection with Borcherds’ notion of $G$-vertex algebra.

Set

$$D = \frac{d}{dx}. \quad (5.2)$$

Then $D$ is a natural endomorphism of $\mathcal{E}_{G_1}(W)$.

Motivated by Proposition 4.11 and by [B2] we define the following notion of compatibility.

Definition 5.3 An (ordered) sequence $(\psi^{(1)}, \ldots, \psi^{(r)})$ in $\mathcal{E}_{G_1}(W)$ is said to be compatible if there exists a nonnegative integer $k$ such that

$$\left( \prod_{1 \leq i < j \leq r} (x_i - x_j)^k \right) \psi^{(1)}(x_1) \cdots \psi^{(r)}(x_r) \in \text{Hom}(W,W((x_1, \ldots, x_r))). \quad (5.3)$$

A set or a space $S$ of weak $G_1$-vertex operators on $W$ is said to be compatible if any finite sequence in $S$ is compatible. A weak $G_1$-vertex operator $a(x)$ on $W$ is called a $G_1$-vertex operator if $\{a(x)\}$ is compatible. Then weak $G_1$-vertex operators in a compatible set are $G_1$-vertex operators. It is important to note that compatibility in general depends on the order. Clearly, $(\text{End}W)((x))$ is a compatible subspace of $\mathcal{E}_{G_1}(W)$.

Remark 5.4 It is easy to see that the linear span of any compatible set of weak $G_1$-vertex operators on $W$ is compatible.

Example 5.5 Let $V$ be an axiomatic $G_1$-vertex algebra and let $(W,Y_W)$ be a $V$-module. It follows immediately from Proposition 4.11 that the image of $V$ under $Y_W$ is a compatible space of weak $G_1$-vertex operators on $W$.

It is in general not a good idea to use the definition directly to check the compatibility of a set of weak $G_1$-vertex operators. In the following, we prove that certain pairwise relations imply compatibility.

Lemma 5.6 Let $a(x), b(x) \in \mathcal{E}_{G_1}(W)$. Assume that there exists a nonnegative integer $k$ such that

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} \psi^{(i)}(x_2)\phi^{(i)}(x_1) \quad (5.4)$$

for some $\psi^{(i)}(x), \phi^{(i)}(x) \in \mathcal{E}_{G_1}(W)$. Then the ordered sequence $(a(x), b(x))$ is compatible.
Proof. Let \( w \in W \). Since \( b(x_2) \in \text{Hom}(W,W((x_2))) \), \( (x_1 - x_2)^k a(x_1) b(x_2) w \) involves only finitely many negative powers of \( x_2 \). On the other hand, the expression on the right-hand side of (5.4), after applied to \( w \), involves only finitely many negative powers of \( x_1 \). Consequently, \( (x_1 - x_2)^k a(x_1) b(x_2) w \in W((x_1, x_2)). \) Thus \( (x_1 - x_2)^k a(x_1) b(x_2) \in \text{Hom}(W,W((x_1, x_2))). \) That is, \( (a(x), b(x)) \) is compatible. \( \square \)

Furthermore, we have:

**Proposition 5.7** Let \( S \) be a set of weak \( G_1 \)-vertex operators on \( W \) such that for any \( a(x), b(x) \in S \), there exists a nonnegative integer \( k \) such that

\[
(x_1 - x_2)^k a(x_1) b(x_2) = \sum_{i=1}^{r} \alpha_i (x_1 - x_2)^k b^{(i)}(x_2) a^{(i)}(x_1)
\]

(5.5)

for some \( \alpha_i \in \mathbb{C} \), \( a^{(i)}(x), b^{(i)}(x) \in S \). Then \( S \) is compatible.

**Proof.** We must prove that any sequence in \( S \) of finite length is compatible. We shall use induction on the length \( n \) of sequences. If \( n = 2 \), it has been proved by Lemma 5.4. Assume that any sequence in \( S \) of length \( n \) is compatible. Let \( \psi^{(1)}, \ldots, \psi^{(n+1)} \in S \). From the inductive hypothesis, there exists a nonnegative integer \( k_1 \) such that

\[
\left( \prod_{2 \leq i < j \leq n+1} (x_i - x_j)^{k_1} \right) \psi^{(2)}(x_2) \cdots \psi^{(n+1)}(x_{n+1}) \in \text{Hom}(W,W((x_2, \ldots, x_{n+2}))).
\]

(5.6)

From (5.3) there exists a nonnegative integer \( k_2 \) such that

\[
(x_1 - x_2)^{k_2} \psi^{(1)}(x_1) \psi^{(2)}(x_2) = \sum_{i=1}^{r} \alpha_i (x_1 - x_2)^{k_2} b^{(i)}(x_2) a^{(i)}(x_1)
\]

(5.7)

for some \( \alpha_i \in \mathbb{C} \), \( a^{(i)}(x), b^{(i)}(x) \in S \). From the inductive hypothesis again, there exists a nonnegative integer \( k_3 \) such that

\[
\left( \prod_{i=3}^{n+1} (x_1 - x_i)^{k_3} \prod_{3 \leq i < j \leq n+1} (x_i - x_j)^{k_3} \right) a^{(s)}(x_1) \psi^{(3)}(x_3) \cdots \psi^{(n+1)}(x_{n+1}) 
\]

\[
\in \text{Hom}(W,W((x_1, x_3, x_4, \ldots, x_{n+1})))
\]

(5.8)

for \( s = 1, \ldots, r \). Because of (5.7), we have

\[
(x_1 - x_2)^{k_2} \left( \prod_{i=3}^{n+1} (x_1 - x_i)^{k_3} \prod_{3 \leq i < j \leq n+1} (x_i - x_j)^{k_3} \right) \psi^{(1)}(x_1) \cdots \psi^{(n+1)}(x_{n+1})
\]

\[
= (x_1 - x_2)^{k_2} \left( \prod_{i=3}^{n+1} (x_1 - x_i)^{k_3} \prod_{3 \leq i < j \leq n+1} (x_i - x_j)^{k_3} \right) 
\]

\[
\cdot \sum_{s=1}^{r} \alpha_s a^{(s)}(x_2) a^{(s)}(x_1) \psi^{(3)}(x_3) \cdots \psi^{(n+1)}(x_{n+1}).
\]

(5.9)
From (5.8), the right-hand side of (5.9) lies in
\[ \text{Hom}(W, W((x_2))((x_1, x_3, x_4, \ldots, x_{n+1}))), \]
and so does the left-hand side of (5.9). Combining this with (5.6) we see that
\[ \left( \prod_{1 \leq i < j \leq n+1} (x_i - x_j)^{k_1+k_2+k_3} \right) \psi^{(1)}(x_1) \psi^{(2)}(x_2) \cdots \psi^{(n+1)}(x_{n+1}) \]
\[ \in \text{Hom}(W, W((x_1, x_2, \ldots, x_{n+1}))). \]  
(5.10)
This proves that the sequence \((\psi^{(1)}, \ldots, \psi^{(n+1)})\) is compatible, completing the induction. □

**Remark 5.8** Recall from [Li1] that weak \((G_1)\)-vertex operators \(a(x)\) and \(b(x)\) are said to be *mutually local* if there exists a nonnegative integer \(k\) such that
\[ (x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k b(x_2)a(x_1). \]  
(5.11)
A set \(S\) of weak vertex operators on \(W\) is said to be local if any two (maybe the same) weak vertex operators in \(S\) are mutually local.

As an immediate consequence of Proposition 5.7 we have:

**Corollary 5.9** Any local set of weak \(G_1\)-vertex operators on \(W\) is compatible. □

**Lemma 5.10** Let \(a(x), b(x) \in \mathcal{E}_{G_1}(W)\) be such that the sequence \((a(x), b(x))\) is compatible. Define
\[ T(a(x)b(y)) = (-y + x)^{-k} \left( (x - y)^k a(x)b(y) \right), \]
(5.12)
where \(k\) is any nonnegative integer such that
\[ (x - y)^k a(x)b(y) \in \text{Hom}(W, W((x, y))). \]
(5.13)
Then \(T(a(x)b(y))\) does not depend the choice of \(k\) and it lies in \(\text{Hom}(W, W((y))((x))).\)
Furthermore,
\[ (x - y)^k T(a(x)b(y)) = (x - y)^k a(x)b(y) \]
(5.14)
for any nonnegative integer \(k\) such that (5.13) holds.

**Proof.** Clearly, for any nonnegative integer \(k\) such that (5.13) holds, we have
\[ (-y + x)^{-k} \left( (x - y)^k a(x)b(y) \right) \in \text{Hom}(W, W((y))((x))). \]
(5.15)
We easily see that $(5.14)$ holds if $T(a(x)b(y))$ is well defined. It remains to prove that $T(a(x)b(y))$ is independent of $k$.

Let $k_1$ and $k_2$ be any two nonnegative integers such that

$$(x - y)^{k_1}a(x)b(y) \in \operatorname{Hom}(W,W((x,y))) \quad \text{for } i = 1, 2.$$  \hspace{1cm} (5.16)

Assume that $k_1 \geq k_2$. (For the case that $k_2 \geq k_1$, one simply exchanges $k_1$ with $k_2$ in the following argument.) Then

$$(-y + x)^{-k_1}((x - y)^{k_1}a(x)b(y)) = (-y + x)^{-k_1}((x - y)^{k_1-k_2}((x - y)^{k_2}a(x)b(y)))$$

$$= (-y + x)^{-k_1}(x - y)^{k_1-k_2}((x - y)^{k_2}a(x)b(y))$$

$$= (-y + x)^{-k_2}((x - y)^{k_2}a(x)b(y)).$$  \hspace{1cm} (5.17)

This proves the assertion. \quad \square

**Definition 5.11** Let $a(x), b(x) \in E_{G_1}(W)$ be such that the sequence $(a(x), b(x))$ is compatible. For $n \in \mathbb{Z}$, we define $a(x)_n b(x) \in (\text{End}W)[[x, x^{-1}]]$ by

$$a(x)_n b(x) = \text{Res}_{x_1} ((x_1 - x)^n a(x_1)b(x) - (-x + x_1)^n T(a(x_1)b(x))).$$  \hspace{1cm} (5.18)

Just as in [Li1] and [LL] with Lemma 5.10, we immediately have:

**Proposition 5.12** Let $a(x), b(x) \in E_{G_1}(W)$ be such that the sequence $(a(x), b(x))$ is compatible. We have

$$a(x)_n b(x) \in E_{G_1}(W) \quad \text{for } n \in \mathbb{Z}.$$  \hspace{1cm} (5.19)

Furthermore,

$$a(x)_n b(x) = 0 \quad \text{for } n \geq k,$$  \hspace{1cm} (5.20)

where $k$ is a nonnegative integer such that $(x - y)^k a(x)b(y) \in \operatorname{Hom}(W,W((x,y)))$. \quad \square

**Remark 5.13** Let $\psi, \phi, \psi^{(i)}, \phi^{(i)} \in E_{G_1}(W)$ for $i = 1, \ldots, r$ be such that

$$(x_1 - x_2)^k \psi(x_1)\phi(x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} \alpha_i \phi^{(i)}(x_2)\psi^{(i)}(x_1)$$  \hspace{1cm} (5.21)

for some nonnegative integer $k$ and some $\alpha_i \in \mathbb{C}$. (In view of Lemma 5.6, the sequence $(\psi, \phi)$ is compatible.) Then

$$T(\psi(x_1)\phi(x_2)) = (-x_2 + x_1)^{-k}((x_1 - x_2)^k \psi(x_1)\phi(x_2))$$

$$= (-x_2 + x_1)^{-k}((x_1 - x_2)^k \sum_{i=1}^{r} \alpha_i \phi^{(i)}(x_2)\psi^{(i)}(x_1))$$

$$= \sum_{i=1}^{r} \alpha_i \phi^{(i)}(x_2)\psi^{(i)}(x_1).$$  \hspace{1cm} (5.22)
Therefore,

\[ \psi(x)_n \phi(x) = \text{Res}_{x_1} \left( (x_1 - x)^n \psi(x_1) \phi(x) - \sum_{i=1}^{r} \alpha_i (-x + x_1)^i \phi^{(i)}(x) \psi^{(i)}(x_1) \right) \]  \hspace{0.5cm} (5.23)

In particular, if

\[ (x_1 - x_2)^k \psi(x_1) \phi(x_2) = \alpha(x_1 - x_2)^k \phi(x_2) \psi(x_1) \]

for some \( \alpha \in \mathbb{C} \), we have

\[ \psi(x)_n \phi(x) = \text{Res}_{x_1} \left( (x_1 - x)^n \psi(x_1) \phi(x) - \alpha(-x + x_1)^n \phi(x) \psi(x_1) \right). \]  \hspace{0.5cm} (5.24)

**Remark 5.14** In view of Remark 5.13, if \( a(x), b(x) \) are mutually local weak vertex operators on \( W \), then the current definition for \( a(x)_n b(x) \) coincides with the one given in [Li1] and [LL], where for any \( \alpha(x), \beta(x) \in \mathcal{E}_{G_1}(W) \) it was defined that

\[ \alpha(x)_n \beta(x) = \text{Res}_{x_1} \left( (x_1 - x)^n \alpha(x_1) \beta(x) - (-x + x_1)^n \beta(x) \alpha(x_1) \right). \]

If \( a(x), b(x) \) are weak vertex operators with the relation

\[ (x_1 - x_2)^k a(x_1) b(x_2) = -(x_1 - x_2)^k b(x_2) a(x_1), \]

then the current definition for \( a(x)_n b(x) \) is different from the one given in [Li1] and [LL]. The essential difference between the definitions is that the current definition only uses the product \( a(x_1) b(x_2) \), not the product \( b(x_2) a(x_1) \) while the definition given in [Li1] and [LL] uses both of the products. That is, one definition takes the associative algebra point of view and the other takes the Lie algebra point of view.

Writing \( a(x)_n b(x) \) for \( n \in \mathbb{Z} \) in terms of generating function as

\[ Y_{E}(a(x), x_0) b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) x_0^{-n-1}, \]  \hspace{0.5cm} (5.25)

we have

\[ Y_{E}(a(x), x_0) b(x) \in \mathcal{E}_{G_1}(W)((x_0)). \]  \hspace{0.5cm} (5.26)

Then

\[ Y_{E}(a(x_2), x_0) b(x_2) = \sum_{n \in \mathbb{Z}} (a(x_2)_n b(x_2)) x_0^{-n-1} \]

\[ = \text{Res}_{x_1} \left( x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) a(x_1) b(x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) T(a(x_1) b(x_2)) \right). \]  \hspace{0.5cm} (5.27)

**Remark 5.15** Since \( a(x)_n b(x) \) for \( n \in \mathbb{Z} \) are defined under the condition that the (ordered) sequence \( (a(x), b(x)) \) is compatible, \( Y_{E}(a(x), x_0) b(x) \) is a well defined element of \( \mathcal{E}_{G_1}(W)((x_0)) \) under the same condition. For any compatible space \( V \) of \( G_1 \)-vertex operators on \( W \), \( Y_{E} \) is a natural linear map from \( V \) to \( \text{Hom}(V, \mathcal{E}_{G_1}(W)((x_0))) \).
Proposition 5.16 Let $a(x), b(x) \in \mathcal{E}_g(W)$ be such that the sequence $(a(x), b(x))$ is compatible. Then for any $w \in W$, there exists a nonnegative integer $l$ such that

$$ (x_0 + x_2)^l a(x_0 + x_2) b(x_2) w \in W((x_0, x_2)). $$

(5.28)

Furthermore, if $l$ is a nonnegative integer such that (5.28) holds, then

$$ (x_2 + x_0)^l (Y_c(a(x_2), x_0) b(x_2)) w = (x_0 + x_2)^l a(x_0 + x_2) b(x_2) w. $$

(5.29)

Proof. Let $k$ be a nonnegative integer such that

$$ (x_1 - x_2)^k a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))). $$

In view of Lemma 5.10, we have

$$ (x_1 - x_2)^k T(a(x_1) b(x_2)) = (x_1 - x_2)^k a(x_1) b(x_2). $$

For any $w \in W$, since $(x_1 - x_2)^k a(x_1) b(x_2) w \in W((x_1, x_2))$, there exists a nonnegative integer $l$ such that

$$ x_1^l (x_1 - x_2)^k a(x_1) b(x_2) w \in W[[x_1, x_2]][x_2^{-1}]. $$

Then

$$ (x_0 + x_2)^l x_0^k a(x_0 + x_2) b(x_2) w \in W[[x_0, x_2]][x_2^{-1}], $$

(5.30)

which implies (5.28).

Now let $l$ be a nonnegative integer such that (5.28) holds. Let $k'$ be a nonnegative integer such that

$$ x_0^{k'} (x_0 + x_2)^l a(x_0 + x_2) b(x_2) w \in W[[x_0, x_2]][x_2^{-1}]. $$

Then

$$ x_1^l (x_1 - x_2)^{k'} a(x_1) b(x_2) w \in W[[x_1, x_2]][x_2^{-1}]. $$

(5.31)

Therefore,

$$ x_1^l (x_1 - x_2)^{k+k'} T(a(x_1) b(x_2)) w = x_1^l (x_1 - x_2)^{k+k'} a(x_1) b(x_2) w \in W[[x_1, x_2]][x_2^{-1}]. $$

(5.32)

Multiplying by $(-x_2 + x_1)^{-k-k'}$, which lies in $\mathbb{C}[x_2, x_2^{-1}][[x_1]]$, we get

$$ x_1^l T(a(x_1) b(x_2)) w \in W((x_2))[[x_1]]. $$

(5.33)

Just as in the ordinary vertex algebra theory (cf. [DL], [Li1] or [LL]), multiplying both sides of (5.27) by $(x_2 + x_0)^l$ we obtain (5.29). \qed
Remark 5.17 In view of (5.20), by multiplying both sides of (5.29) by \((x_2 + x_0)^{-1}\) we get
\[
(Y_\varepsilon(a(x_2), x_0)b(x_2))w = (x_2 + x_0)^{-1} [(x_0 + x_2)^{\dagger}a(x_0 + x_2)b(x_2)]w.
\]
(5.34)
On the other hand, notice that (5.31) implies that
\[
(x_0 + x_2)^{\dagger}a(x_0 + x_2)b(x_2)w \in W((x_0, x_2))
\]
(5.35)
so that the expression on the right-hand side of (5.34) is well defined. Then one can use (5.31) as an alternative definition for \(Y_\varepsilon(a(x_2), x_0)b(x_2)\).

Remark 5.18 Combining Lemma 5.10 and Proposition 5.16 with Lemma 2.1 we get
\[
x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y_\varepsilon(a(x_2), x_0)b(x_2)
= x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) a(x_1)b(x_2) - x_0^{-1}\delta \left( \frac{x_2 - x_1}{-x_0} \right) T(a(x_1)b(x_2)).
\]
(5.36)
Let \(1_W\) denote the identity operator on \(W\) and let \(a(x)\) be any weak \(G_1\)-vertex operator on \(W\). Since
\[
1_W(x_1)a(x_2) = a(x_2) \in \text{Hom}(W, W((x_2))) \subset \text{Hom}(W, W((x_1, x_2))),
\]
(5.37)
\[
a(x_1)1_W(x_2) = a(x_1) \in \text{Hom}(W, W((x_1))) \subset \text{Hom}(W, W((x_1, x_2))),
\]
(5.38)
the sequences \((1_W, a(x))\) and \((a(x), 1_W)\) are compatible, and \(T(a(x_1)1_W) = a(x_1)\). By (5.27) we have
\[
Y_\varepsilon(1_W(x), x_0)a(x) = \text{Res}_{x_1} \left( x_0^{-1}\delta \left( \frac{x_1 - x}{x_0} \right) 1_W(x_1)a(x) - x_0^{-1}\delta \left( \frac{x - x_1}{-x_0} \right) a(x) \right)
= \text{Res}_{x_1}x^{-1}\delta \left( \frac{x_1 - x_0}{x} \right) a(x)
= a(x).
\]
(5.39)
Similarly we have
\[
Y_\varepsilon(a(x), x_0)1_W(x) = \text{Res}_{x_1}x^{-1}\delta \left( \frac{x_1 - x_0}{x} \right) a(x_1)
= a(x + x_0)
= e^{x_0\frac{d}{dx}}a(x)
= e^{x_0D}a(x).
\]
(5.40)
Thus we have proved:
Lemma 5.19 For any $a(x) \in \mathcal{E}_{G_1}(W)$,

\[
Y_\mathcal{E}(1W, x_0)a(x) = a(x),
\]

\[
Y_\mathcal{E}(a(x), x_0)1W = e^{xaD}a(x). \quad \square
\] (5.42)

A compatible space $U$ of weak $G_1$-vertex operators on $W$ is said to be closed if

\[
a(x)_nb(x) \in U \quad \text{for } a(x), b(x) \in U, \quad n \in \mathbb{Z}.
\] (5.43)

Then for a closed compatible space $U$, we have a linear map $Y_\mathcal{E}$ from $U$ to $\text{Hom}(U, U((x_0)))$.

Remark 5.20 Let $V$ be an axiomatic $G_1$-vertex algebra and let $(W, Y_W)$ be a $V$-module. Then the image of $Y_W$ is a closed compatible subspace of $\mathcal{E}_{G_1}(W)$. Furthermore, for $u, v \in V$, $n \in \mathbb{Z}$,

\[
Y_W(u_n v, x) = Y_W(u, x)_n Y_W(v, x).
\] (5.44)

Indeed, for any $u, v \in V$, $w \in W$, in view of Definition 4.1 and Proposition 5.16, there exists a nonnegative integer $l$ such that

\[
(x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w = (x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w
\]

\[
(x_0 + x_2)^l Y_\mathcal{E}(Y_W(u, x_2), x_0)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w.
\] (5.45)

Then

\[
(x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w = (x_0 + x_2)^l Y_\mathcal{E}(Y_W(u, x_2), x_0)Y_W(v, x_2)w.
\] (5.46)

Noticing that both sides involve only finitely many negative powers of $x_0$, by multiplying by $(x_2 + x_0)^{-l}$ we get

\[
Y_W(Y(u, x_0)v, x_2)w = Y_\mathcal{E}(Y_W(u, x_2), x_0)Y_W(v, x_2)w.
\] (5.47)

which is (5.44) in terms of generating functions.

In view of Remark 5.20, if $(W, Y_W)$ is a faithful $V$-module, e.g., $W = V$ (the faithfulness follows from the creation property), $V$ can be naturally identified with a closed compatible subspace of $\mathcal{E}_{G_1}(W)$, containing $1_W$. Next, we shall show that for an abstract vector space $W$, any closed compatible subspace of $\mathcal{E}_{G_1}(W)$ that contains $1_W$ is a (weak) axiomatic $G_1$-vertex algebra with $W$ as a natural faithful module.

We here introduce a notation for convenience. Let $U$ be a vector space and let

\[
a(x) = \sum_{n \in \mathbb{Z}} a_n x^{n-1} \in U[[x, x^{-1}]]
\]

be any formal series, e.g., a weak $G_1$-vertex operator on $W$. For $m \in \mathbb{Z}$, we set

\[
a(x)_{\geq m} = \sum_{n \geq m} a_n x^{n-1}.
\] (5.48)

Then for any polynomial $p(x)$ we have

\[
\text{Res}_x x^m p(x) a(x) = \text{Res}_x x^m p(x) a(x)_{\geq m}.
\] (5.49)

First, we have the following result:
Lemma 5.21 Let $\psi_1, \ldots, \psi_r, a, b, \phi_1, \ldots, \phi_s \in E_G(W)$. Assume that the ordered sequences $(a(x), b(x))$ and $(\psi_1(x), \ldots, \psi_r(x), a(x), b(x), \phi_1(x), \ldots, \phi_s(x))$ are compatible. Let $k$ be a nonnegative integer such that

$$
(x_1 - x_2)^k \left( \prod_{1 \leq p < q \leq r} (y_p - y_q)^k \right) \left( \prod_{i=1}^r (x_1 - y_i)^k (x_2 - y_i)^k \right) \left( \prod_{j=1}^s (x_1 - z_j)^k (x_2 - z_j)^k \right)
\cdot \left( \prod_{1 \leq i \leq r, 1 \leq j \leq s} (y_i - z_j)^k \right) \left( \prod_{1 \leq p < q \leq s} (z_p - z_q)^k \right)
\cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_1) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w 
\in \text{Hom}(W, W((y_1, \ldots, y_r, x_1, x_2, z_1, \ldots, z_s))).
$$

(5.50)

Let $w \in W$ and let $l$ be a nonnegative integer such that

$$
x_1^l (x_1 - x_2)^k \left( \prod_{1 \leq p < q \leq r} (y_p - y_q)^k \right) \left( \prod_{i=1}^r (x_1 - y_i)^k (x_2 - y_i)^k \right) \left( \prod_{j=1}^s (x_1 - z_j)^k (x_2 - z_j)^k \right)
\cdot \left( \prod_{1 \leq i \leq r, 1 \leq j \leq s} (y_i - z_j)^k \right) \left( \prod_{1 \leq p < q \leq s} (z_p - z_q)^k \right)
\cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_1) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w
\in W[[x_1]]((y_1, \ldots, y_r, x_1, x_2, z_1, \ldots, z_s)).
$$

(5.51)

Then

$$
(x_0 + x_2)^l \left( \prod_{j=1}^s (x_0 + x_2 - z_j)^k \right) \psi_1(y_1) \cdots \psi_r(y_r) (Y_{E}(a, x_0) b)(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w
= (x_0 + x_2)^l \left( \prod_{j=1}^s (x_0 + x_2 - z_j)^k \right) \psi_1(y_1) \cdots \psi_r(y_r) a(x_0 + x_2) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w.
$$

Proof. Set

$$
P = \prod_{1 \leq i < j \leq r} (y_i - y_j)^k, \quad Q = \prod_{1 \leq i < j \leq s} (z_i - z_j)^k, \quad R = \prod_{1 \leq i \leq r, 1 \leq j \leq s} (y_i - z_j)^k
$$
and

$$
S = \prod_{i=1}^r (x_0 + x_2 - y_i)^k (x_2 - y_i)^k.
$$

Let $m_1, \ldots, m_s$ be arbitrarily fixed integers. Since $\phi_1(z_1)_{\geq m_1} \cdots \phi_s(z_s)_{\geq m_s} w$ is a finite sum, from Proposition 5.16 there exists a nonnegative integer $l'$ (depending on $m_1, \ldots, m_s$) such that

$$
(x_0 + x_2)^{l'} (Y_{E}(a, x_0) b)(x_2) \phi_1(z_1)_{\geq m_1} \cdots \phi_s(z_s)_{\geq m_s} w
= (x_0 + x_2)^{l'} a(x_0 + x_2) b(x_2) \phi_1(z_1)_{\geq m_1} \cdots \phi_s(z_s)_{\geq m_s} w.
$$

(5.53)
Then using (5.49) we get
\[
\left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) PQRS \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) (x_0 + x_2)^{t'} \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(Y_\mathcal{E}(a, x_0)b)(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w
\]
\[
= \left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) PQRS \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot (x_0 + x_2)^{t'} \psi_1(y_1) \cdots \psi_r(y_r)(Y_\mathcal{E}(a, x_0)b)(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w
\]
\[
= \left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) PQRS \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot (x_0 + x_2)^{t'} \psi_1(y_1) \cdots \psi_r(y_r)a(x_0 + x_2)b(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w. \tag{5.54}
\]

Multiplying both sides by \((x_0 + x_2)^{t'}\) we get
\[
(x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) PQRS x_0^{t}(x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(Y_\mathcal{E}(a, x_0)b)(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w
\]
\[
= (x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) [PQRS x_0^{t}(x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)a(x_0 + x_2)b(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w]. \tag{5.55}
\]

Noticing that from (5.51), the expression in the bracket on the right-hand side of (5.55) involves only nonnegative powers of \(x_0\), then multiplying (5.55) by \(x_0^{-k}(x_2 + x_0)^{-t'}\) we get
\[
\left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) PQRS (x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(Y_\mathcal{E}(a, x_0)b)(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w
\]
\[
= \left(\prod_{i=1}^{s} \text{Res}_{z_i}z_i^{m_i}\right) PQRS (x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)a(x_0 + x_2)b(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w. \tag{5.56}
\]

Since \(l\) and \(k\) are independent of \(m_i\)’s and \(m_i\)’s are arbitrary, we have
\[
PQRS(x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)(Y_\mathcal{E}(a, x_0)b)(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w
\]
\[
= PQRS(x_0 + x_2)^{t'} \left(\prod_{i=1}^{s} (x_0 + x_2 - z_i)^k(x_2 - z_i)^k \right) \\
\cdot \psi_1(y_1) \cdots \psi_r(y_r)a(x_0 + x_2)b(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w. \tag{5.57}
\]
Noticing that we are allowed to multiply both sides by

\[
\prod_{1 \leq p < q \leq r} (y_p - y_q)^{-k} \prod_{1 \leq i \leq s, 1 \leq j \leq s} (y_i - z_j)^{-k} \prod_{j=1}^{s}(x_2 - z_j)^{-k} \prod_{1 \leq p < q \leq s} (z_p - z_q)^{-k}
\]

and by \( \prod_{i=1}^{s}(-y_i + x_0 + x_2)^k(-y_i + x_2)^k \) (but we are not allowed to multiply both sides by \( \prod_{i=1}^{s}(x_0 + x_2 - z_i)^{-k} \)), we get the desired result. \( \square \)

Now we are in a position to prove our first key result:

**Theorem 5.22** Let \( V \) be a subspace of \( \mathcal{E}_{G_1}(W) \) such that any sequence in \( V \) of length 2 or 3 is compatible and such that

\[
1_W \in V, \quad (5.58)
\]

\[
\psi(x)_n \phi(x) \in V \quad \text{for } \psi(x), \phi(x) \in V, \ n \in \mathbb{Z}. \quad (5.59)
\]

Then \( (V, Y_\mathcal{E}, 1_W) \) carries the structure of a weak axiomatic \( G_1 \)-vertex algebra with \( W \) as a natural faithful module where the vertex operator map \( Y_W \) is given by \( Y_W(\alpha(x), x_0) = \alpha(x_0) \). Furthermore, assume that for any \( \psi(x), \theta(x) \in V \), there exists a nonnegative integer \( k \) such that for every \( \phi(x) \in V \) there exists a nonnegative integer \( k' \) such that

\[
(x - y)^k(y - z)^k(x - z)^k \psi(x) \phi(y) \theta(z) \in \text{Hom}(W, W((x, y, z))). \quad (5.60)
\]

Then \( (V, Y_\mathcal{E}, 1_W) \) carries the structure of an axiomatic \( G_1 \)-vertex algebra.

**Proof.** For the assertion on the axiomatic \( G_1 \)-vertex algebra structure, with Proposition 5.12 and Lemma 5.19 we must prove the weak associativity, i.e., for \( \psi, \phi, \theta \in V \), there exists a nonnegative integer \( k \) such that

\[
(x_0 + x_2)^k Y_\mathcal{E}(\psi, x_0 + x_2) Y_\mathcal{E}(\phi, x_2) \theta = (x_0 + x_2)^k Y_\mathcal{E}(Y_\mathcal{E}(\psi, x_0) \phi, x_2) \theta. \quad (5.61)
\]

Let \( k \) and \( k' \) be nonnegative integers such that

\[
(x - y)^k(x - z)^k(y - z)^k' \psi(x) \phi(y) \theta(z) \in \text{Hom}(W, W((x, y, z))). \quad (5.62)
\]

For the first assertion, both \( k \) and \( k' \) depend on all \( \psi, \phi, \theta \) and for the second assertion, \( k \) depends only on \( \psi \) and \( \theta \).

Let \( w \in W \) be arbitrary and fixed. There exists a nonnegative integer \( l \) such that

\[
x^l y^l z^l (x - y)^k(x - z)^k(y - z)^k \psi(x) \phi(y) \theta(z) w \in W[[x, y, z]]. \quad (5.63)
\]

In view of Proposition 5.16, by replacing \( l \) with a larger integer if necessary we may assume that

\[
(x_2 + x)^l \phi(x_2 + x) \theta(x) w = (x_2 + x)^l (Y_\mathcal{E}(\phi, x_2) \theta(x)) w. \quad (5.64)
\]
Then

\[
x^l x_0^k (x_0 + x)^l x_0^k (x_0 - x_2)^l (x_2 + x)^l \psi(x_0 + x) \phi(x_2 + x) \theta(x) w
= x^l x_0^k (x_0 + x)^l x_0^k (x_0 - x_2)^l (x_2 + x)^l \psi(x_0 + x) (Y_\phi, \phi, x_2) \theta(x) (x) w.
\] (5.65)

Noticing that the expression on the left-hand side lies in \(W[[x, x_0, x_2]]\) by (5.63), we have

\[
x^l x_0^k (x_0 + x)^l x_0^k (x_0 - x_2)^l (x_2 + x)^l \psi(x_0 + x) (Y_\phi, \phi, x_2) \theta(x) (x) w \in W[[x, x_0, x_2]].
\] (5.66)

By multiplying by \(x^{-l} x_0^{-k} x_2^{-l} (x_0 - x_2)^{-l} (x_2 + x)^{-l}\), which lies in \(\mathbb{C}((x, x_0))((x_2))\), we have

\[
(x_0 + x)^l \psi(x_0 + x) (Y_\phi, \phi, x_2) \theta(x) (x) w \in W((x, x_0))((x_2)).
\] (5.67)

In view of Proposition 5.14, by considering components of \(Y_\phi, \phi, x_2, \theta\), we have

\[
(x_0 + x)^l \psi(x_0 + x) (Y_\phi, \phi, x_2, \theta) (x) w = (x_0 + x)^l (Y_\phi, \psi, x_0) Y_\phi, \phi, x_2, \theta) (x) w.
\] (5.68)

Then

\[
(x_0 + x_2)^l (x_0 + x)^l (x_0 + x_2 + x)^l \psi(x_0 + x_2 + x) \phi(x_2 + x) \theta(x) w
= (x_0 + x_2)^l (x_0 + x)^l (x_0 + x_2 + x)^l \psi(x_0 + x_2 + x) (Y_\phi, \phi, x_2) \theta(x) (x) w
= e^{x_2 \sigma_{x_0}} x_0^k (x_0 + x)^l (x_0 + x)^l \psi(x_0 + x) (Y_\phi, \phi, x_2) \theta(x) (x) w
= e^{x_2 \sigma_{x_0}} x_0^k (x_0 + x)^l (x_0 + x)^l (Y_\phi, \psi, x_0) Y_\phi, \phi, x_2, \theta) (x) w
= (x_0 + x_2)^l (x_0 + x)^l (x_0 + x_2 + x)^l (Y_\phi, \psi, x_0 + x_2) Y_\phi, \phi, x_2, \theta) (x) w.
\] (5.69)

On the other hand, let \(n \in \mathbb{Z}\) be arbitrarily fixed. Since \(\psi(x)_m \phi(x) = 0\) for \(m\) sufficiently large, there exists a nonnegative integer \(l' \in \mathbb{Z}\) such that

\[
(x_2 + x)^l (Y_\phi, \psi, x_0)_m \phi(x) (x_2 + x) \theta(x) w = (x_2 + x)^l (Y_\phi, \psi, x_0) \phi(x) (x_2 + x) \theta(x) w
\] (5.70)

for all \(m \geq n\). With (5.68), in view of Lemma 5.21, we have

\[
(x_0 + x_2)^l (x_0 + x_2 - x)^k \psi(x_0 + x_2) \phi(x_2) \theta(x) w
= (x_0 + x_2)^l (x_0 + x_2 - x)^k (Y_\phi, \psi, x_0) \phi(x_2) \theta(x) w.
\] (5.71)

Using (5.45), (5.70) and (5.71) we get

\[
\text{Res}_{x_0} x_0^k (x_0 + x + x)^l (x_0 + x_2)^l (x_2 + x)^l (Y_\phi, \psi, x_0, \phi, x_2) \theta(x) w
= \text{Res}_{x_0} x_0^k (x_0 + x + x)^l (x_0 + x_2)^l (x_2 + x)^l (Y_\phi, \psi, x_0, \phi, x_2) \theta(x) w
= \text{Res}_{x_0} x_0^k (x_0 + x + x)^l (x_0 + x_2)^l (x_2 + x)^l (Y_\phi, \psi, x_0, \phi, x_2) \theta(x) w
= \text{Res}_{x_0} x_0^k (x_0 + x + x)^l (x_0 + x_2)^l (x_2 + x)^l (Y_\phi, \psi, x_0, \phi, x_2) \theta(x) w
= \text{Res}_{x_0} x_0^k (x_0 + x + x)^l (x_0 + x_2)^l (x_2 + x)^l (Y_\phi, \psi, x_0, \phi, x_2) \theta(x) w
= \text{Res}_{x_0} x_0^k (x_0 + x + x)^l (x_0 + x_2)^l (x_2 + x)^l (Y_\phi, \psi, x_0, \phi, x_2) \theta(x) w.
\] (5.72)
Combining (5.72) with (5.69) we get
\[
\text{Res}_{x_0} x_0^n (x_0 + x_2 + x)^l (x_0 + x_2 + x)^{l+\ell'} (Y_\ell(\psi, x_0 + x_2) Y_\ell(\phi, x_2) \theta)(x) w
\]
\[
= \text{Res}_{x_0} x_0^n (x_0 + x_2 + x)^l (x_0 + x_2 + x)^{l+\ell'} (Y_\ell(\psi, x_0) \phi, x_2) \theta)(x) w.
\] (5.73)

Notice that both sides of (5.73) involve only finitely many negative powers of $x_2$. Then multiplying both sides by $(x + x_2)^{-l-\ell'}$ we get
\[
\text{Res}_{x_0} x_0^n (x_0 + x_2 + x)^l (x_0 + x_2 + x)^{l+\ell'} (Y_\ell(\psi, x_0 + x_2) Y_\ell(\phi, x_2) \theta)(x) w
\]
\[
= \text{Res}_{x_0} x_0^n (x_0 + x_2 + x)^l (x_0 + x_2 + x)^{l+\ell'} (Y_\ell(\psi, x_0) \phi, x_2) \theta)(x) w.
\] (5.74)

Since $n$ is arbitrary and $l$ and $k$ do not depend on $n$, we must have
\[
(x_0 + x_2 + x)^l (x_0 + x_2 + x)^k (Y_\ell(\psi, x_0 + x_2) Y_\ell(\phi, x_2) \theta)(x) w
\]
\[
= (x_0 + x_2 + x)^l (x_0 + x_2 + x)^k (Y_\ell(\psi, x_0) \phi, x_2) \theta)(x) w.
\] (5.75)

Notice that $(x + x_0 + x_2)^{-l} = (x + x_2 + x_0)^{-l}$ and that we are allowed to multiply the left-hand side of (5.75) by $(x + x_0 + x_2)^{-l}$ and to multiply the right-hand side by $(x + x_2 + x_0)^{-l}$. Then multiplying both sides by $(x + x_0 + x_2)^{-l}$ we obtain
\[
(x_0 + x_2)^l (Y_\ell(\psi, x_0 + x_2) Y_\ell(\phi, x_2) \theta)(x) w
\]
\[
= (x_0 + x_2)^l (Y_\ell(\psi, x_0) \phi, x_2) \theta)(x) w.
\] (5.76)

Since $k$ does not depend on $w$, we immediately have (5.61), as desired.

For $a(x), b(x) \in V, \ w \in W$, in view of Proposition 5.16 there exists a nonnegative integer $l$ such that
\[
(x_0 + x_2)^l (Y_\ell(a(x), x_0) b(x))(x_2) w = (x_0 + x_2)^l a(x_0 + x_2) b(x_2) w.
\]
That is,
\[
(x_0 + x_2)^l Y_W (Y_\ell(a(x), x_0) b(x), x_2) w = (x_0 + x_2)^l Y_W (a(x), x_0 + x_2) Y_W (b(x), x_2) w, (5.77)
\]
Therefore $W$ is a $V$-module with $Y_W (\alpha(x), x_0) = \alpha(x_0)$ for $\alpha(x) \in V$. □

Our next goal is to prove that any compatible set $\mathcal{S}$ of $G_1$-vertex operators on $W$ gives rise to an axiomatic $G_1$-vertex algebra. To achieve this goal, we first need to show that for $a, b, c \in S, \ n \in \mathbb{Z}$, the sequences $(a(x)_n b(x), c(x))$ and $(c(x), a(x)_n b(x))$ are compatible, so that $c(x)_m (a(x)_n b(x))$ and $(a(x)_n b(x))_m c(x)$ are defined for $m \in \mathbb{Z}$. The following is another key result:

**Proposition 5.23** Let $\psi_1(x), \ldots, \psi_r(x), a(x), b(x), \phi_1(x), \ldots, \phi_s(x) \in \mathcal{E}_{G_1}(W)$. Assume that the ordered sequences $(a(x), b(x))$ and
\[
(\psi_1(x), \ldots, \psi_r(x), a(x), b(x), \phi_1(x), \ldots, \phi_s(x))
\]
are compatible. Then for any $n \in \mathbb{Z}$, the ordered sequence
\[
(\psi_1(x), \ldots, \psi_r(x), a(x)_n b(x), \phi_1(x), \ldots, \phi_s(x))
\]
is compatible.
Proof. Let $n \in \mathbb{Z}$ be arbitrarily fixed. From Proposition 5.16 there exists a nonnegative integer $k'$ such that

$$x_0^{k'+n}Y_{\mathcal{E}}(a, x_0)b \in \mathcal{E}_{G_1}(W)[x_0].$$  \hfill (5.78)

Let $k$ be a nonnegative integer such that

$$\left( \prod_{1 \leq i < j \leq r} (y_i - y_j)^k \right) \left( \prod_{1 \leq i \leq r, 1 \leq j \leq s} (y_i - z_j)^k \right) \left( \prod_{1 \leq i < j \leq s} (z_i - z_j)^k \right) \cdot (x_1 - x_2)^k \left( \prod_{i=1}^{r} (x_1 - y_i)^k (x_2 - y_i)^k \right) \left( \prod_{j=1}^{s} (x_1 - z_j)^k (x_2 - z_j)^k \right) \cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_1) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) \in \text{Hom}(W, W((y_1, \ldots, y_r, x_1, x_2, z_1, \ldots, z_s))).$$  \hfill (5.79)

Set

$$P = \prod_{1 \leq i < j \leq r} (y_i - y_j)^k, \quad Q = \prod_{1 \leq i < j \leq s} (z_i - z_j)^k, \quad R = \prod_{1 \leq i \leq r, 1 \leq j \leq s} (y_i - z_j)^k.$$

Let $w \in W$. Then there exists a nonnegative integer $l$ such that

$$x_1^l PQR(x_1 - x_2)^k \left( \prod_{i=1}^{r} (x_1 - y_i)^k (x_2 - y_i)^k \right) \left( \prod_{j=1}^{s} (x_1 - z_j)^k (x_2 - z_j)^k \right) \cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_1) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w \in W[[x_1]]((x_2, y_1, \ldots, y_r, z_1, \ldots, z_s)).$$  \hfill (5.80)

(cf. Lemma 5.21). Hence

$$PQR(x_0 + x_2)^l x_0^k \left( \prod_{i=1}^{r} (x_0 + x_2 - y_i)^k (x_2 - y_i)^k \right) \left( \prod_{j=1}^{s} (x_0 + x_2 - z_j)^k (x_2 - z_j)^k \right) \cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_0 + x_2) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w \in W[[x_0]]((x_2, y_1, \ldots, y_r, z_1, \ldots, z_s)).$$  \hfill (5.81)

In the following we are going to use the binomial expansions for $x_2^{l+k'} = ((x_2 + x_0) - x_0)^{l+k'}$, $(x_2 - y_i)^{k+k'} = ((x_2 + x_0 - y_i) - x_0)^{k+k'}$ and $(x_2 - z_j)^{k+k'} = ((x_2 + x_0 - z_j) - x_0)^{k+k'}$. Using (5.78) and Lemma 5.21 we obtain

$$x_2^{l+k'} \prod_{i=1}^{r} (x_2 - y_i)^{2k+k'} \prod_{j=1}^{s} (x_2 - z_i)^{2k+k'} \cdot \psi_1(y_1) \cdots \psi_r(y_r) (a(x_0) b(x)) x_2 \phi_1(z_1) \cdots \phi_s(z_s) w \in \text{Res}_{x_0 \to x_2^l} \prod_{i=1}^{r} (x_2 - y_i)^{2k+k'} \prod_{j=1}^{s} (x_2 - z_i)^{2k+k'}$$

41
\[
\psi_1(y_1) \cdots \psi_r(y_r) (Y_E(a, x_0) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w
\]
\[
= \text{Res}_{x_0} x_0^n \left( \sum_{p \geq 0} \frac{l + k'}{p} \right) (x_0 + x_2)^{l+k'-p} (-x_0)^p
\]
\[
\cdot \prod_{i=1}^r \left( \sum_{t \geq 0} \frac{k + k'}{t} \right) (x_2 + x_0 - y_i)^{k+k'-t} (-x_0)^t \left( \prod_{i=1}^r (x_2 - y_i)^k \right)
\]
\[
\cdot \sum_{q \geq 0} \frac{k + k'}{q} (x_0 + x_2 - z_j)^{k+k'-q} (-x_0)^q \left( \prod_{j=1}^s (x_2 - z_j)^k \right)
\]
\[
\cdot \psi_1(y_1) \cdots \psi_r(y_r) (Y_E(a, x_0) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w)
\]
\[
= \text{Res}_{x_0} x_0^n \left( \sum_{0 \leq p \leq k'} \frac{l + k'}{p} \right) (x_0 + x_2)^{l+k'-p} (-x_0)^p
\]
\[
\cdot \prod_{i=1}^r \left( \sum_{0 \leq t \leq k'} \frac{k + k'}{t} \right) (x_2 + x_0 - y_i)^{k+k'-t} (-x_0)^t \left( \prod_{i=1}^r (x_2 - y_i)^k \right)
\]
\[
\cdot \sum_{0 \leq q \leq k'} \frac{k + k'}{q} (x_0 + x_2 - z_j)^{k+k'-q} (-x_0)^q \left( \prod_{j=1}^s (x_2 - z_j)^k \right)
\]
\[
\cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_0 + x_2) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w.
\] (5.82)

Noticing that
\[
PQR \left( \prod_{i=1}^r (x_2 - y_i)^k \right) \left( \prod_{j=1}^s (x_2 - z_j)^k \right) (x_0 + x_2)^{l+k'-p} \left( \prod_{i=1}^r (x_2 + x_0 - y_i)^{k+k'-t} \right)
\]
\[
\cdot \left( \prod_{j=1}^s (x_0 + x_2 - z_j)^{k+k'-q} \right) \psi_1(y_1) \cdots \psi_r(y_r) a(x_0 + x_2) b(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w
\]
\[
\in W((y_1, \ldots, y_r, x_0, x_2, z_1, \ldots, z_s))
\] (5.83)

for \(0 \leq p, t, q \leq k'\), from (5.82) we have that
\[
PQR x_2^{l+k'} \prod_{i=1}^r (x_2 - y_i)^{2k+k'} \prod_{j=1}^s (x_2 - z_j)^{2k+k'}
\]
\[
\cdot \psi_1(y_1) \cdots \psi_r(y_r) a(x_0, b(x))(x_2) \phi_1(z_1) \cdots \phi_s(z_s) w
\]
lies in $W((y_1, \ldots, y_r, x_2, z_1, \ldots, z_s))$, and so

$$PQR \prod_{i=1}^{r}(x_2 - y_i)^{2k + k'} \prod_{j=1}^{s}(x_2 - z_j)^{2k + k'}$$

$$ \cdot \psi_1(y_1) \cdots \psi_r(y_r)(a(x)_n b(x))(x_2) \phi_1(z_1) \cdots \phi_s(z_s)w$$

$$ \in W((y_1, \ldots, y_r, x_2, z_1, \ldots, z_s)). \quad (5.84)$$

This proves that the sequence $(\psi_1(x), \ldots, \psi_r(x), a(x)_n b(x), \phi_1(x), \ldots, \phi_s(x))$ is compatible, since $k$ and $k'$ are independent of $w$. □

It follows from Proposition 5.23 that if $S$ is a compatible set of weak $G_1$-vertex operators on $W$, then for any $a^{(1)}, \ldots, a^{(r+1)} \in S$, the expression

$$Y_\mathcal{E}(a^{(1)}(x), x_1) \cdots Y_\mathcal{E}(a^{(r)}(x), x_r)a^{(r+1)}(x)$$

is recursively well defined.

The following result states that any maximal compatible subspace of $\mathcal{E}_{G_1}(W)$ is automatically closed and it has a weak axiomatic $G_1$-vertex algebra structure.

**Proposition 5.24** Let $V$ be a maximal compatible subspace of $\mathcal{E}_{G_1}(W)$. Then $1_W \in V$ and

$$a(x)_n b(x) \in V \quad \text{for } a(x), b(x) \in V, \ n \in \mathbb{Z}. \quad (5.85)$$

Furthermore, $(V, Y_\mathcal{E}, 1_W)$ carries the structure of a weak axiomatic $G_1$-vertex algebra with $W$ as a natural module where the vertex operator map $Y_\mathcal{E}$ is given by $Y_\mathcal{E}(a(x), x_0) = a(x_0)$.

**Proof.** Clearly the space spanned by $V$ and $1_W$ is still compatible. With $V$ being maximal we must have $1_W \in V$. Now, let $a(x), b(x) \in V$ and $n \in \mathbb{Z}$. In view of Proposition 5.23, any (ordered) sequence in $V \cup \{a(x)_n b(x)\}$ with one appearance of $a(x)_n b(x)$ is compatible. It follows from induction on the number of appearance of $a(x)_n b(x)$ and from Proposition 5.23 that any (ordered) sequence in $V \cup \{a(x)_n b(x)\}$ with any (finite) number of appearance of $a(x)_n b(x)$ is compatible. So the space spanned by $V$ and $a(x)_n b(x)$ is compatible. Again, with $V$ being maximal we must have $a(x)_n b(x) \in V$. This proves that $V$ is closed, and hence by Theorem 5.22 $(V, Y_\mathcal{E}, 1_W)$ carries the structure of a weak axiomatic $G_1$-vertex algebra with $W$ as a natural module. □

Let $S$ be a compatible set of $G_1$-vertex operators on $W$. By Zorn’s lemma there exists a maximal compatible space $V$ of $\mathcal{E}_{G_1}(W)$, containing $S$ and $1_W$, and then by Proposition 5.24 $(V, Y_\mathcal{E}, 1_W)$ carries the structure of a weak axiomatic $G_1$-vertex algebra with $W$ as a natural module. Now, $S$ as a subset of $V$ generates a subalgebra $\langle S \rangle$ of $V$. Then in view of Proposition 2.22 we obtain our main result (cf. [B2], Theorem 7.9):

**Theorem 5.25** Let $S$ be any compatible set of $G_1$-vertex operators on $W$. Then for any $a^{(1)}(x), \ldots, a^{(r)}(x) \in S$, the expression

$$Y_\mathcal{E}(a^{(1)}(x), x_1) \cdots Y_\mathcal{E}(a^{(r)}(x), x_r)1_W$$

43
is recursively well defined. Furthermore, if we set
\[ U = \text{span}\{a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r}1_W \mid r \geq 0, \, a^{(i)}(x) \in S, \, n_i \in \mathbb{Z}\}, \quad (5.86) \]
then \((U, Y_E, 1_W)\) carries the structure of a weak axiomatic \(G_1\)-vertex algebra with \(W\) as a natural module where the vertex operator map \(Y_W\) is given by \(Y_W(\alpha(x), x_0) = \alpha(x_0)\). □

As an immediate consequence of Proposition 5.7 and Theorem 5.25 we have:

**Corollary 5.26** Let \(S\) be a set of weak \(G_1\)-vertex operators on \(W\) such that for any \(a, b \in S\), there exists a nonnegative integer \(k\) such that
\[ (x_1 - x_2)^k a(x_1)b(x_2) = \sum_{i=1}^{r} \alpha_i(x_1 - x_2)^k b^{(i)}(x_2)a^{(i)}(x_1) \quad (5.87) \]
for some \(\alpha_i \in \mathbb{C}, \, a^{(i)}, b^{(i)} \in S, \, r \geq 1\). Then all the assertions of Theorem 5.25 hold. □

Recall from Corollary 5.9 that any space of pairwise mutually local vertex operators on \(W\) is compatible. Then in view of Theorem 5.22, any closed space of pairwise mutually local vertex operators on \(W\) is a weak axiomatic \(G_1\)-vertex algebra with \(W\) as a module. In [Li1], it was proved (cf. [MN]) that any closed space of pairwise mutually local vertex operators on \(W\) is an (ordinary) vertex algebra with \(W\) as a module. Theorem 5.22 does not directly imply the corresponding result of [Li1], but Theorem 5.22 together with Propositions 2.17 and 4.3 does.

**Theorem 5.27** Let \(S\) be a set of pairwise mutually local \(G_1\)-vertex operators on \(W\). Then for \(r \geq 0, \, a^{(i)} \in S, \, i = 1, \ldots, r\), the expression
\[ Y_E(a^{(1)}(x), x_1) \cdots Y_E(a^{(r)}(x), x_r)1_W \]
is recursively well defined and
\[ Y_E(a^{(1)}(x), x_1) \cdots Y_E(a^{(r)}(x), x_r)1_W = Y_E(a^{(1)}(x), x_1) \cdots Y_E(a^{(r)}(x), x_r)1_W, \quad (5.88) \]
where
\[ Y_E(\alpha(x), x_0)\beta(x) = \text{Res}_{x_1}\left( x_0^{-1} \delta\left(\frac{x_1-x}{x_0}\right) \alpha(x_1)\beta(x) - x_0^{-1} \delta\left(\frac{x-x_1}{-x_0}\right) \beta(x)\alpha(x_1) \right) \quad (5.89) \]
for \(\alpha(x), \beta(x) \in \mathcal{E}_{G_1}(W)\). Furthermore, if we set
\[ U = \text{span}\{a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r}1_W \mid r \geq 0, \, a^{(i)} \in S, \, n_i \in \mathbb{Z}\}, \quad (5.90) \]
then \((U, Y_E, 1_W)\) carries the structure of an ordinary vertex algebra with \(W\) as a natural module where the vertex operator map \(Y_W\) is given by \(Y_W(\alpha(x), x_0) = \alpha(x_0)\).
Proof. Let $U'$ be the space defined in Theorem 5.25. In view of Corollary 5.26, $U'$ is a weak axiomatic $G_1$-vertex algebra with $W$ as a natural module. Since $W$ is a faithful module and for any $a(x), b(x) \in S$, $Y_W(a(x), x_1) (= a(x_1))$ and $Y_W(b(x), x_2) (= b(x_2))$ are mutually local, by Proposition 4.3, $Y(a(x), x_1)$ and $Y(b(x), x_2)$ acting on $U'$ are mutually local. By Proposition 2.17, $U'$ is an ordinary vertex algebra because $S$ generates $U'$ as a weak axiomatic $G_1$-vertex algebra. In view of Proposition 4.4, $W$ equipped with the linear map $Y_W$ given by $Y_W(\alpha(x), x_2) = \alpha(x_2)$ is a module for $U'$ viewed as a vertex algebra.

Then for $\alpha(x), \beta(x) \in U'$,

$$Y_E(\alpha(x_2), x_0) \beta(x_2) = Y_W(Y_E(\alpha, x_0) \beta, x_2)$$

$$= \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(\alpha(x), x_1) Y_W(\beta(x), x_2)$$

$$- \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(\beta(x), x_2) Y_W(\alpha(x), x_1)$$

$$= \text{Res}_{x_1} \left( x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \alpha(x_1) \beta(x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \beta(x_2) \alpha(x_1) \right)$$

$$= Y_E(\alpha(x_2), x_0) \beta(x_2). \quad (5.91)$$

Now, (5.88) follows immediately from induction and we have $U = U'$. This completes the proof. □

References

[B1] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA 83 (1986), 3068-3071.

[B2] R. E. Borcherds, Vertex algebras, in “Topological Field Theory, Primitive Forms and Related Topics” (Kyoto, 1996), edited by M. Kashiwara, A. Matsuo, K. Saito and I. Satake, Progress in Math., 160, Birkhäuser, Boston, 1998, 35-77; q-alg/9706008.

[BPZ] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, Nucl. Phys. B241 (1984), 333-380.

[DL] C. Dong and J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math. Vol. 112, Birkhäuser, Boston, 1993.

[DLM] C. Dong, H. Li and G. Mason, Regularity of rational vertex operator algebras, Adv. Math. 132 (1997), 148-166.

[DM] C. Dong and G. Mason, On quantum Galois theory, Duke Math. J. 86 (1997), 305-321.
[DN] M. R. Douglas and N. A. Nekrasov, Noncommutative field theory, arXiv: hep-th/0106048.

[EK] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras, V, arXiv: math.QA/9808121.

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Memoirs Amer. Math. Soc. 104, 1993; preprint, 1989.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Appl. Math. Vol. 134, Academic Press, Boston, 1988.

[GL] Y.-C. Gao and H.-S. Li, Generalized vertex algebras generated by parafermion-like operators, J. Algebra 240 (2001), 771-807.

[K] V. G. Kac, Vertex Algebras for Beginners, University Lecture Series, Vol. 10, 1996.

[LL] J. Lepowsky and H.-S. Li, Introduction to vertex operator algebras and their modules, Monograph, in preparation.

[Li1] H.-S. Li, Local systems of vertex operators, vertex superalgebras and modules, J. Pure Appl. Alg. 109 (1996), 143-195; hep-th/9406185.

[Li2] H.-S. Li, Local systems of twisted vertex operators, vertex superalgebras and twisted modules, Contemporary Math. 193 (1996), 203-236.

[Li3] H.-S. Li, On certain higher dimensional analogues of vertex algebras, arXiv:math.QA/0104225.

[LZ] B. Lian and G. Zuckerman, Commutative quantum operator algebras, J. Pure Appl. Alg. 100 (1995), 117-139.

[MN] A. Matsuo and K. Nagatomo, Axioms for a Vertex Algebra and the Locality of Quantum Fields, MSJ Memoir, Vol. 4, Mathematical Society of Japan, 1999.

[Sn] C. Snydal, Equivalence of Borcherds G-vertex algebras and axiomatic vertex algebras, math.QA/9904104.

[Y] G. Yamaskulna, The relationship between skew group algebras and orbifold theory, arXiv: math.QA/0107180.