HOPF-HECKE ALGEBRAS, INFINITESIMAL CHEREDNIK ALGEBRAS AND DIRAC COHOMOLOGY

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Abstract. In [Fl], Hopf-Hecke algebras and Barbasch-Sahi algebras are defined in order to provide a general framework for the study of Dirac cohomology. The aim of this paper is to explore new examples of these definitions and to contribute to their classification.

Hopf-Hecke algebras, which generalize the Drinfeld Hecke algebras defined by Drinfeld in [Dr], are distinguished by an orthogonality condition and a PBW property. The PBW property for algebras as the ones considered has been of great interest in the literature and we extend this discussion by partial results on the classification of such deformations and by a class of examples.

We study infinitesimal Cherednik algebras of $GL_n$ as defined in [EGG] as new examples of Hopf-Hecke algebras with a generalized Dirac cohomology. We will see that they are in fact Barbasch-Sahi algebras, that is, a version of Vogan’s conjecture analogous to [HP1] is available for them. Finally, we compute the Dirac cohomology of all their finite-dimensional modules using the classification results of [DT].

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1. INTRODUCTION

1.1. Let $\mathbb{F}$ be a field of characteristic 0. All vector spaces and tensor products are over $\mathbb{F}$, all modules are finite-dimensional left modules.

We recall some definitions from [Fl]: The module $V$ of a cocommutative Hopf algebra $H$ is called orthogonal if it admits a non-degenerate $H$-invariant symmetric bilinear form. In this situation let $\kappa : V \wedge V \to H$ be a linear map and let $I_\kappa$ be the ideal in the semidirect/smash product algebra $T(V) \rtimes H$ generated by elements of the form $v \otimes w - w \otimes v - \kappa(v, w)$ for $v, w \in V$. Then the algebra

$$A = (T(V) \rtimes H)/I_\kappa$$
is called Hopf-Hecke algebra if it has the PBW property, i.e., if it is a flat deformation of $S(V) \rtimes H$. Now the PBW property for such algebras, usually independent of orthogonality of $V$ and often also for special choices of $H$, has been of great interest in the literature. Such special choices include Drinfeld Hecke algebras ([Dr, RS, SW1]) which include the Hecke algebra studied by Lusztig ([Lu]) and symplectic reflection algebras ([EG]). The discussion is summarized for instance in the survey [SW2] and the articles [WW, Kh2].

In our case, the PBW property is equivalent to $\kappa$ being $H$-linear and a Jacobi-identity which allows us to draw certain conclusions on the classification of maps $\kappa$ suitable for obtaining algebras $A$ with the PBW property (sec. 2.1 and 2.2). In particular, we construct a class of examples of such maps $\kappa$ (2.21).  

1.2 In [Fl], a theory of Dirac cohomology is developed for Hopf-Hecke algebras with the aim to generalize results on the Dirac cohomology of semisimple Lie algebras ([HP1]), graded affine Hecke algebras ([BCT]), symplectic reflection algebras and, more generally, Drinfeld Hecke algebras ([Ci]). For a Hopf-Hecke algebra $A$ as above, we have a Dirac element  

$$D := \sum_i v_i \otimes v^i$$

in $A \otimes C$, where $(v_i)_i$ and $(v^i)_i$ are dual bases of $V$ and where we identify elements in $V$ with elements in $C(V)$. Let $S$ be a spin module of $C$. If $M$ is an $A$-module, then $M \otimes S$ is an $A \otimes C$-module, so in particular, $D$ acts on $M \otimes S$. The Dirac cohomology of $M$ is defined as  

$$H^D(M) := \ker D/(\ker D \cap \text{im} D).$$

Let us assume $H$ is pointed, then $H$ is generated as algebra by group-like and primitive elements and it has a pin cover ([Fl, sec. 2]). In particular, there is a certain pointed cocommutative Hopf algebra $\tilde{H}$ which is a $\mathbb{Z}_2$-graded algebra and an algebra map $\Delta_C : \tilde{H} \rightarrow H \otimes C(V)$. The Hopf-Hecke algebra $A$ is called Barbasch-Sahi algebra, if  

$$D^2 \in Z(A \otimes C) + \Delta_C(\tilde{H}^{\text{even}}),$$

where $Z(A \otimes C)$ is the center of $A \otimes C$.

For a Barbasch-Sahi algebra $A$, it is proved in [Fl] that if $M$ is an $A$-module with central character and non-vanishing Dirac cohomology, then its central character is determined by its Dirac cohomology ([Fl, thm. 4.3]), a result which parallels a result called “Vogan’s conjecture” in [HP1].

1.3 We consider the infinitesimal Cherednik algebras of $GL_n$ for $n \geq 1$ which are parameterized by a polynomial $\xi$. They can be regarded as generalized deformations of $U(sl_{n+1})$ relative to $U(gl_n)$ (cp. [EGG], ex. 4.7). In our notation, they correspond to the choices $H = U(gl_n)$ and $V = h \oplus h^*$, where $h \simeq \mathbb{C}^n$ is the standard $gl_n$-module. We denote these algebras by $\mathcal{H}_\xi$, and we show:

**Proposition 1.1** (4.13). $\mathcal{H}_\xi$ is a Barbasch-Sahi algebra.

For fixed $\xi$, let $w(z) = \sum_{k \geq 0} w_k z^k$ be the polynomial uniquely defined up to a constant by  

$$e^{-\partial/2}(2 \sinh(\partial/2))^n z^n w(z) = \partial^n (z^n \xi(z)) .$$

Let $P(\mu) := \sum_{k \geq 0} w_k h_k(\mu + \rho)$, where $h_k$ are the complete symmetric homogeneous polynomials (defined in [20]) and $\rho$ is the Weyl vector of $gl_n$. In [DT] it is proved that the action of the Casimir element of $H_\xi$ is determined by $P$. 
Furthermore, it is shown there that every finite-dimensional irreducible $\mathcal{H}_\xi$-module is the irreducible quotient $L(\lambda)$ of a Verma module uniquely determined by a dominant $\mathfrak{gl}_n$-weight $\lambda$, and the dominant weights $\lambda$ making $L(\lambda)$ finite-dimensional are characterized depending on $\xi$.

For a fixed $H_\xi$-module, we write $V_\mu$ for a highest weight $\mathfrak{gl}_n$-submodule with highest weight $\mu$ and $n_\mu$ for its multiplicity. Let $S$ be the unique spin module of $C(V)$ (dim $V$ is even). We prove:

**Theorem 1.2** \([4.24, 4.32]\). The Dirac cohomology of the finite-dimensional irreducible $\mathcal{H}_\xi$-module $L(\lambda)$ is isomorphic to the submodule $\bigoplus_{\mu} n_\mu V_\mu$ of $L(\lambda) \otimes S$, where the sum ranges over those weights $\mu$ satisfying $P(\lambda) = P(\mu - (\tfrac{1}{2}, \ldots, \tfrac{1}{2}))$.

In [DT] thm. 4.1 it is also shown that for every finite-dimensional irreducible $L(\lambda)$, there is a unique vector of non-negative integers $\nu$ such that $L(\lambda) = \bigoplus_{0 \leq \nu' \leq \nu} V_{\lambda - \nu'}$. This allows us to compute the explicit structure of $L(\lambda) \otimes S$ as $\mathfrak{gl}_n$-module in \([4.27]\) which together with the previous result determines the Dirac cohomology of all finite-dimensional modules of $H_\xi$. As a consequence, we show:

**Corollary 1.3** \([4.29, 4.32]\). $V_{\lambda^0}$ for $\lambda^0 := \lambda + (\tfrac{1}{2}, \ldots, \tfrac{1}{2})$ and $V_{\lambda^n}$ for $\lambda^n := \lambda + (\tfrac{1}{2}, \ldots, -\nu_n - \tfrac{1}{2})$ appear with multiplicity one in the Dirac cohomology of $\mathcal{M}$.

For $1 \leq i < n$, if $\lambda^i := \lambda - (0, \ldots, 0, \nu_i + 1, 0, \ldots, 0)$ is a dominant weight, then $V_{\lambda^i}$ for $\lambda^i := \lambda + (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\nu_i - \tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})$ appears with multiplicity one in the Dirac cohomology of $L(\lambda)$.

1.4 The PBW property of Hopf-Hecke algebras and its relation with the map $\kappa$ is explored in section 2; some of the main results on the Dirac cohomology of Hopf-Hecke algebras from [Fl] are recalled in section 3, and the Dirac cohomology of infinitesimal Cherednik algebras of $\mathfrak{gl}_n$ is studied in section 4.

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## 2. Hopf-Hecke algebras

We start with a brief review of basic Hopf algebra theory, for more information on this we refer to [Mo].

When working with a coalgebra $C$ we will refer to its counit as $\varepsilon : C \to \mathbb{F}$ and to its coproduct as $\Delta : C \to C \otimes C$. When working with a Hopf algebra $H$, the same convention applies and the antipode is referred to as $S : H \to H$. For an element $c \in C$, we will use Sweedler’s notation $c^{(1)} \otimes c^{(2)}$ for the coproduct $\Delta(c) \in C \otimes C$ which does not necessarily represent a pure tensor, but implies a summation over several pure tensors in general. The $i$-fold coproduct for $c$ is written as $c^{(i)} \otimes \ldots \otimes c^{(i-i)}$ in $C^{\otimes i}$, where the notation is justified due to cocommutativity. The Hopf algebra $H$ is an $H$-module itself via the (left) adjoint action

$$h \cdot k = h^{(1)} k Sh^{(2)}$$

for $h, k \in H$.

**Definition 2.1.** A coalgebra $C$ is called **pointed** if every simple subcoalgebra is one-dimensional. An element $c \in C$ is called **group-like** if $\Delta c = c \otimes c$ and **primitive** if $\Delta c = 1 \otimes c + c \otimes 1$. The set of group-likes is denoted by $G(C)$, the set of primitives is denoted by $P(C)$.

Basic Hopf algebra theory tells us that for every Hopf algebra $H$, $G(H)$ is a group with multiplication in $H$ and $P(H)$ is a Lie subalgebra of $H$ with the commutator.
Definition 2.2. If $H$ is a Hopf algebra and $B$ is an $H$-module algebra, then the \textit{semidirect/smash product} $B \rtimes H$ is the algebra generated by $B$ and $H$ and the relation $hb = (h(1) \cdot h)h(2)$ for all $h \in H, b \in B$.

By a well-known structure theorem, any cocommutative pointed Hopf algebra $H$ over a field $\mathbb{F}$ of characteristic 0 has the form $H = U(P(H)) \rtimes \mathbb{F}[G(H)]$, where $U(P(H))$ is the universal enveloping algebra of the Lie algebra of primitive elements and $\mathbb{F}[G(H)]$ is the group algebra of the group of group-likes in $H$. This applies in particular to any cocommutative Hopf algebra over $\mathbb{F} = \mathbb{C}$, because every simple cocommutative coalgebra over an algebraically closed field is one-dimensional.

2.1. Hopf-Hecke algebras and the PBW property. We fix a cocommutative Hopf algebra $H$ with finite-dimensional module $V$.

We review the definition of Hopf-Hecke algebras ([Fl] def. 3.1) and we will study their structure. $H$ as a cocommutative Hopf algebra acts on $V$ and hence, the tensor algebra $T(V)$ is an $H$-module algebra. The semidirect/smash product $T(V) \rtimes H$ is the algebra generated by $T(V)$ and $H$ and the relation

$$hv = (h(1) \cdot v)h(2)$$

for all $h \in H, v \in V$.

Definition 2.3. A bilinear form $\langle \cdot, \cdot \rangle$ on $V$ is called $H$-\textit{invariant} if $\langle h(1) \cdot v, h(2) \cdot w \rangle = \langle v, w \rangle$, or equivalently if $\langle h \cdot v, w \rangle = \langle v, Sh \cdot w \rangle$, for all $h \in H, v, w \in V$ (cp. [Fl] lem. 2.3). $V$ is called an \textit{orthogonal} module if it admits a non-degenerate $H$-invariant symmetric bilinear form.

In [Fl] sec. 2, a \textit{pin cover} of $H$ with respect to $V$ is constructed for any pointed cocommutative Hopf algebra $H$ over $\mathbb{F}$ with an orthogonal module $V$. Note also that for such Hopf algebras, $V$ is orthogonal if and only if every group-like element acts as an orthogonal operator and every primitive element acts as skew-symmetric operator.

Definition 2.4. Let $\kappa : V \wedge V \to H$ be a linear map. We denote by $I_\kappa$ the ideal of $T(V) \rtimes H$ generated by elements of the form $vw - wk - \kappa(v \wedge w)$ for $v, w \in V$. The $H$-module algebra

$$A = A_\kappa := (T(V) \rtimes H)/I_\kappa$$

(2.1)

is called \textit{Hopf-Hecke algebra} of $(H,V,\kappa)$ if $V$ is an orthogonal module and if it satisfies the \textit{PBW property}, that is, if it is a flat deformation of $S(V) \rtimes H$.

In other words, let $\overline{A}$ be the associated graded algebra of $A$ with respect to the filtration of the tensor factor $T(V)$. Now $A$ satisfies the PBW property if the natural surjection from $S(V) \rtimes H$ to $A$ is an isomorphism.

Remark 2.5. We review [Fl] rem. 3.2, since it will make the following structure theory more transparent:

First, we note that the definition of Hopf-Hecke algebra is closely related to that of continuous Hecke algebras in [EGG]: if $G$ is a reductive algebraic group and $\mathfrak{g}$ its Lie algebra, then the Hopf algebra $H = U(\mathfrak{g}) \rtimes \mathbb{F}[G]$ can be viewed as subalgebra of the algebra of algebraic distributions $\mathcal{O}(G)^*$ on $G$. If we replace $H$ with $\mathcal{O}(G)^*$ in the definition above and drop the orthogonality condition on $V$, we have the definition of continuous Hecke algebras in the sense of [EGG].

Second, we observe that a special case of our definition is the situation of $H$ being the group algebra of a finite group $G$. In this context, every module $V$ is orthogonal, and the class of algebras $A_\kappa$ has been studied in [Dr, RS, SW1].
Definition 2.6. We say that a linear map $\kappa : V \wedge V \to H$ has the Jacobi property if the following Jacobi identity holds in $A_\kappa$ for all $x, y, z \in V$:

$$[\kappa(x, y), z] + [\kappa(y, z), x] + [\kappa(z, x), y] = 0.$$  

The following fact is well-known ([BG], [EGG, thm. 2.4], [WW, thm. 3.1], [Kh2, thm. 2.5]):

Proposition 2.7. $A_\kappa$ has the PBW property if and only if $\kappa$ is $H$-linear and $\kappa$ has the Jacobi property.

In order to study the image of maps with the Jacobi property, we make some definitions.

Definition 2.8 (Notation). We define the notation

$$h \triangleright v := h \cdot v - \varepsilon(h)v$$

for all $h \in H, v \in V$ and as in the proof of [EGG] prop 2.8:

$$(v_1, \ldots, v_k, x, y) := (\kappa(x, y)_{(1)} \triangleright v_1) \wedge \cdots \wedge (\kappa(x, y)_{(k)} \triangleright v_k) \otimes \kappa(x, y)_{(k+1)} \in \wedge^k V \otimes H$$

for all $v_1, \ldots, v_k, x, y \in V$.

Note that (in contrast to the dot “.”) the triangle “$\triangleright$” does not denote an action of the algebra $H$.

Definition 2.9. For any $i \geq 0$, we define $K'_i \subset H$ to be the subspace of those $h \in H$ satisfying

$$\sum_k r^k \otimes h^k = \Delta h \in H \otimes H$$

for all $v_1, \ldots, v_{i+1} \in V$. Let $K_i := \Delta^{-1}(K'_i \otimes H)$.

Remark 2.10. Note that the left-hand side of (2.2) can be expanded. For instance, for $i = 1$, the expansion reads

$$h \cdot (v_1 \wedge v_2) - (h \cdot v_1) \wedge v_2 - v_1 \wedge (h \cdot v_2) + \varepsilon(h)v_1 \wedge v_2,$$

where the dot denotes the action of $H$ on $\wedge(V)$.

Note also that $K'_0 = \{ h \in H : h \cdot v = \varepsilon(h) v \}$, and that due to cocommutativity, $\Delta^{-1}(K'_i \otimes H) = \Delta^{-1}(K'_i \otimes K'_i)$ for all $i \geq 0$.

Lemma 2.11. $(K_i)_{i \geq 0}$ is an algebra filtration of $H$.

Proof. First, we want to show $K_i \subset K_{i+1}$ for any $i \geq 0$. We consider $h \in K_i$ and we write

$$\Delta h = \sum_k r^k \otimes h^k$$

with $(h^k)_k$ in $H$ and $(r^k)_k$ in $K'_i$. Then

$$\Delta h = \sum_k \left( (h^k_{(1)} \triangleright v_{(1)}) \wedge \cdots \wedge (h^k_{(i+1)} \triangleright v_{(i+1)}) \otimes h^k_{(I+2)} \right) = 0,$$

for any $v_1, \ldots, v_{i+2} \in V$, so $h \in K_{i+1}$, as desired.

To see that we obtain an algebra filtration, consider $i, j \geq 0$ and let $m := i + j$. If $a, b \in H$, then

$$(ab) \triangleright v = a \cdot (b \triangleright v) + \varepsilon(b)(a \triangleright v)$$

for all $v \in V$. Let us use the shorthand notations $S(a, b, v) := a \cdot (b \triangleright v)$ and $T(a, b, v) := \varepsilon(b)(a \triangleright v)$. Then for all $v_1, \ldots, v_{m+1} \in V$,

$$((ab)_{(1)} \triangleright v_1) \wedge \cdots \wedge ((ab)_{(m+1)} \triangleright v_{m+1}) \otimes (ab)_{m+2}$$

$$= \left( (a_{(1)}b_{(1)} \triangleright v_1) \wedge \cdots \wedge (a_{(m+1)}b_{(m+1)} \triangleright v_{m+1}) \right) \otimes (ab)_{m+2}$$

$$= (S(a_1, b_1, v_1) + T(a_1, b_1, v_1)) \wedge \cdots \wedge (S(a_{m+1}, b_{m+1}, v_{m+1}) + T(a_{m+1}, b_{m+1}, v_{m+1})) \otimes (ab)_{m+2}.$$
Now we can simplify the wedge product using the distributive law and after swapping wedge factors and relabeling \(v_1, \ldots, v_{m+1}\) as \(v_1', \ldots, v_{m+1}'\) when necessary, every summand will contain a factor
\[
S(a_1, b_1, v'_1) \wedge \cdots \wedge S(a_{i+1}, b_{i+1}, v'_{i+1})
\]
or a factor
\[
T(a_1, b_1, v_1) \wedge \cdots \wedge T(a_{j+1}, b_{j+1}, v'_{j+1}),
\]
so every summand vanishes if \(a \in K'_i\) and \(b \in K'_j\). Hence for such \(a\) and \(b\), the product \(ab\) lies in \(K'_{m+1}\).

**Lemma 2.12.** \(K_i\) is a subcoalgebra of \(H\) and a submodule of \(H\) under the adjoint action for all \(i \geq 0\).

**Proof.** Consider \(h \in K_i\). We can write \(\Delta h = \sum_{k} r_k \otimes h^k\) for linearly independent \((h^k)_k\) in \(H\) and suitable elements \((r^k)_k\) in \(K'_i\). For any index \(j\), let \(p\) be a projection of \(H\) onto \(F h^j\) along \(h^k\) for all \(k \neq j\). Then
\[
(r^j(1) \triangleright v_1) \wedge \cdots \wedge (r^j(i+1) \triangleright v_{i+1}) \otimes r^j(i+2) \otimes h^j
\]

\[
= (\text{id} \otimes \text{id} \otimes p)(\sum_k (r^k(1) \triangleright v_1) \wedge \cdots \wedge (r^k(i+1) \triangleright v_{i+1}) \otimes r^k(i+2) \otimes h^k)
\]

\[
= (\text{id} \otimes \text{id} \otimes p)(\sum_k (r^k(1) \triangleright v_1) \wedge \cdots \wedge (r^k(i+1) \triangleright v_{i+1}) \otimes h^k(i+1) \otimes h^k) = 0,
\]
so \(\Delta r^j \in K'_i \otimes H\), so \(r^j \in K_i\), and \(K_i\) is a subcoalgebra, as desired.

To see that \(K_i\) is a submodule of \(H\), we first note that for all \(h, k \in H\) and all \(x \in V\),
\[
(k \cdot h) \triangleright x = (k(1)hSk(2)) \triangleright x = k(1) \cdot (h \triangleright (Sk(2) \cdot x)).
\]

So assume \(h \in K_i\). Then
\[
(((k \cdot h)(1) \triangleright v_1) \wedge \cdots \wedge ((k \cdot h)(i+1) \triangleright v_{i+1}) \otimes (k \cdot h)(i+2)
\]

\[
= (k(1) \cdot ((h(1) \triangleright (Sk(2)v_1)) \wedge \cdots \wedge (h(i+1) \triangleright (Sk(2)v_{i+1}))) \otimes k(i+2) \cdot h(i+2)
\]

\[
= k(1) \cdot ((h(1) \triangleright (Sk(2)v_1)) \wedge \cdots \wedge (h(i+1) \triangleright (Sk(2)v_{i+1}))) \otimes k(i+3) \cdot h(i+2)
\]

\[
= 0
\]
and indeed, \(k \cdot h \in K_i\).

Using the introduced and studied notations, we have a counterpart to [EGG] prop. 2.8 on the “support” of \(\kappa\):

**Proposition 2.13.** Assume \(\kappa : V \wedge V \to H\) has the Jacobi property. Then \(\text{im } \kappa \subset K_2\).

**Proof.** This is a word-for-word translation of [EGG] prop. 2.8 and the associated lemmas:

Note that in \(A\),
\[
[h, v] = (h(1) \cdot v)h(2) - e(h(1))vh(2) = (h(1) \triangleright v)h(2),
\]
so using our new notation, the Jacobi identity reads
\[
(v \cdot x, y) + (x \cdot y, v) + (y \cdot v, x) = 0
\]
for all \(v, x, y \in V\). Now as in [EGG] lem. 2.10, this implies
\[
(z, u|x, y) = (x, y|z, u)
\]
for all \( z, u, x, y \in V \). Now as in [EGG, lem. 2.11], this implies
\[
(z, u, v \mid x, y) = 0
\]
for all \( z, u, v, x, y \in V \).

Hence if \( h = \kappa(x, y) \in H \) for elements \( x, y \in V \), then
\[
(h_{(1)} \triangleright z) \wedge (h_{(2)} \triangleright u) \wedge (h_{(3)} \triangleright v) \otimes h_{(4)} = 0
\]
for all \( z, u, v \in V \), so \( \Delta h \in K'_2 \otimes H \) and hence \( h \in K_2 \).

**Remark 2.14.** Compare this with [Kh2, prop. 4.3] which is formulated for a cocommutative bialgebra and with an additional deformation parameter \( \lambda \) (and note that the above proof works for a cocommutative bialgebra, as well).

We can reformulate the recent findings:

**Definition 2.15.** For each \( i \geq 1 \), we define the linear map
\[
T_i : H \rightarrow \text{End}(\wedge^i V) \otimes H, h \mapsto (h_{(1)} \triangleright \cdot) \wedge \cdots \wedge (h_{(i)} \triangleright \cdot) \otimes h_{(i+1)} .
\]

By the definition, \( \ker T_i = K_{i-1} \), so there is an inverse map \( T_i^{-1} : T_i(H) \rightarrow H/K_{i-1} \).

For a linear map \( \kappa : V \wedge V \rightarrow H \), let us use the notation \( [\kappa]_i \) for its image in \( \text{Hom}(V \wedge V, H/K_i) \) for any \( i \geq 0 \).

**Corollary 2.16.** If \( \kappa : V \wedge V \rightarrow H \) has the Jacobi property, then for all \( x, y \in V \),
\[
[k]_1(x, y) = T_1^{-1}(T_1 \circ \kappa(x, \cdot)(y) + T_1 \circ \kappa(\cdot, y)(x)) ,
\]
\[
[k]_2(x, y) = T_2^{-1}(T_2 \circ \kappa(\cdot, \cdot)(x, y)) ,
\]
and \( [\kappa]_3 = 0 \).

**Proof.** The three statements are reformulations of the Jacobi identity (2.3), (2.4) and (2.5), respectively.

Extending [EGG, 2.3], we have the following class of examples:

**Definition 2.17.** Consider elements \( \tau \in (V \wedge V)^* \otimes K_0, \)
\[
\sigma = \sum_m \sigma_m \otimes h^m \in (V \wedge V)^* \otimes K_1 , \quad \theta = \sum_i \theta_i \otimes k^i \in (V \wedge V)^* \otimes K_2 ,
\]
which can be viewed as linear maps from \( V \wedge V \) to \( K_0, K_1 \) and \( K_2 \), respectively. Using those we define new linear maps from \( V \wedge V \) to \( H \): \( \kappa_\tau(x, y) := \tau(x, y), \)
\[
\kappa_\sigma(x, y) := \sum_m \sigma_m(h^m_{(1)} \triangleright x, y)h^m_{(2)} + \sigma_m(x, h^m_{(1)} \triangleright y)h^m_{(2)} ,
\]
\[
\kappa_\theta(x, y) := \sum_i \theta_i(k^i_{(1)} \triangleright x, k^i_{(2)} \triangleright y)k^i_{(3)}
\]
for all \( x, y \in V \), and
\[
(2.6) \quad \kappa := \kappa_\tau + \kappa_\sigma + \kappa_\theta .
\]
Lemma 2.19. Each of \( \kappa_\sigma \) and \( \kappa_\theta \) actually only depend on \( [\sigma] \) and \( [\theta] \) in \( K_1/K_0 \) and \( K_2/K_1 \), respectively. This is, because if \( h \in K_0 \) and \( k \in K_1 \), then
\[
h_{(1)} \triangleright x \otimes h_{(2)} = h_{(1)} \triangleright y \otimes h_{(2)} = 0
\]
and
\[
(k_{(1)} \triangleright x) \wedge (k_{(2)} \triangleright y) \otimes k_{(3)} = 0.
\]

Remark 2.18. \( \kappa_\sigma \) and \( \kappa_\theta \) actually only depend on \( [\sigma] \) and \( [\theta] \) in \( K_1/K_0 \) and \( K_2/K_1 \), respectively. This is, because if \( h \in K_0 \) and \( k \in K_1 \), then
\[
h_{(1)} \triangleright x \otimes h_{(2)} = h_{(1)} \triangleright y \otimes h_{(2)} = 0
\]
and
\[
(k_{(1)} \triangleright x) \wedge (k_{(2)} \triangleright y) \otimes k_{(3)} = 0.
\]

Lemma 2.19. Each of \( \kappa_\tau, \kappa_\sigma \) or \( \kappa_\theta \) as in the definition is \( H \)-linear if and only if the corresponding map \( \tau, \sigma \) or \( \theta \) is \( H \)-linear, respectively. In particular, \( \kappa \) is \( H \)-linear if \( \tau, \sigma \) and \( \theta \) are \( H \)-linear.

Proof. For \( \kappa_\tau \), the assertion is tautological. For \( \kappa_\sigma, \kappa_\theta \) let us first note that for any \( h, k \in H \) and any \( x \in V \),
\[
h \triangleright (Sk \cdot x) = Sk_{(1)} \cdot ((k_{(2)}hSk_{(3)}) \triangleright x) = Sk_{(1)} \cdot ((k_{(2)}) \cdot h) \triangleright x
\]
using the adjoint action in \( H \). Now a linear map from \( V \wedge V \) to \( H \) is \( H \)-linear, if the corresponding element in \( (V \wedge V)^* \otimes H \) is \( H \)-invariant. So we can verify for any \( h \in H, x, y, z \in V \):
\[
(h \cdot \kappa_\theta)(x, y) = \sum_i \theta_i(k_{(1)}^i \triangleright (Sh_{(1)} \cdot x), k_{(2)}^i \triangleright (Sh_{(2)} \cdot y))h_{(3)} \cdot k_{(3)}^i
\]
\[
= \sum_i \theta_i(Sh_{(1)} \cdot (h_{(2)} \cdot k_{(1)}^i) \triangleright x, Sh_{(3)} \cdot (h_{(4)} \cdot k_{(2)}^i) \triangleright y)h_{(5)} \cdot k_{(3)}^i
\]
\[
= \sum_i (h_{(1)} \cdot \theta_i)((h_{(2)} \cdot k_{(1)}^i)_{(1)} \triangleright x, (h_{(2)} \cdot k_{(2)}^i)_{(2)} \triangleright y)h_{(2)} \cdot k_{(3)}^i
\]
\[
= \kappa_{h, \theta}(x, y),
\]
and analogously for \( \kappa_\sigma \).

Remark 2.20. Obviously, one way of obtaining \( H \)-linear \( \tau, \sigma, \theta \) is by choosing \( H \)-linear elements in \( (V \wedge V)^* \) and \( H \)-invariant (that is, \( H \)-central) elements in \( K_0, K_1 \) and \( K_2 \). The map \( \kappa \) generated according to \( \eqref{2.19} \) will be \( H \)-linear and will have the Jacobi property, so \( A_k \) will be a PBW deformation. If additionally \( V \) is an orthogonal \( H \)-module, \( A_k \) will be a Hopf-Hecke algebra.

Proposition 2.21. Let \( \kappa \) be as in \( \eqref{2.19} \). Then it has the Jacobi property.

In particular, if additionally \( \tau, \sigma, \theta \) are \( H \)-linear, then \( A = A_k \) has the PBW property.

Proof. As in \( \cite[\text{EGG}]{thm.2.13} \): By \( \cite{2.19} \) the PBW property is equivalent to the Jacobi identity if \( \kappa \) is \( H \)-linear.

To verify the Jacobi property, we consider elements \( x, y, z \in V \). Recall that the Jacobi identity reads
\[
0 = (\kappa(x, y)_{(1)} \triangleright z)\kappa(x, y)_{(2)} + (\kappa(y, z)_{(1)} \triangleright x)\kappa(y, z)_{(2)} + (\kappa(z, x)_{(1)} \triangleright y)\kappa(z, x)_{(2)}.
\]
Now for all \( h \in K_0 \) and all \( v \in V \),
\[
0 = (h_{(1)} \triangleright v) \otimes h_{(2)},
\]
which verifies the Jacobi identity for \( \kappa_\tau \).

Also, for every index \( m \) and all \( x, y, z \in V \),
\[
0 = (h_{(1)}^m \triangleright x) \wedge (h_{(2)}^m \triangleright y) \wedge z \otimes h_{(3)}^m + (h_{(1)}^m \triangleright x) \wedge y \otimes (h_{(2)}^m \triangleright z) \otimes h_{(3)}^m + x \wedge (h_{(1)}^m \triangleright y) \wedge (h_{(2)}^m \triangleright z) \otimes h_{(3)}^m,
\]
because $h^m \in K_1$, so

$$0 = \sigma_m(h^m_{(1)} \triangleright x, y)(h^m_{(2)} \triangleright z)h^m_{(3)} + \sigma_m(x, h^m_{(1)} \triangleright y)(h^m_{(2)} \triangleright z)h^m_{(3)}$$

$$+ \sigma_m(h^m_{(1)} \triangleright z, x)(h^m_{(2)} \triangleright y)h^m_{(3)} + \sigma_m(z, h^m_{(1)} \triangleright x)(h^m_{(2)} \triangleright y)h^m_{(3)}$$

$$+ \sigma_m(h^m_{(1)} \triangleright y, z)(h^m_{(2)} \triangleright x)h^m_{(3)} + \sigma_m(y, h^m_{(1)} \triangleright z)(h^m_{(2)} \triangleright y)h^m_{(3)},$$

which verifies the Jacobi identity for $\kappa_\sigma$.

Finally for every index $i$ and all $x, y, z \in V$,

$$0 = (k^i_{(1)} \triangleright x) \wedge (k^i_{(2)} \triangleright y) \wedge (k^i_{(3)} \triangleright z) \otimes k^i_{(4)},$$

because $k^i \in K_2$, so

$$0 = (\theta_i(k^i_{(1)} \triangleright x, k^i_{(2)} \triangleright y)(k^i_{(3)} \triangleright z) + \theta_i(k^i_{(1)} \triangleright z, k^i_{(2)} \triangleright x)(k^i_{(3)} \triangleright y) + \theta_i(k^i_{(1)} \triangleright y, k^i_{(2)} \triangleright z)(k^i_{(3)} \triangleright x))k^i_{(4)},$$

which verifies the Jacobi identity for $\kappa_\sigma$. $\square$

2.2. Maps with the Jacobi property for pointed cocommutative Hopf algebras. In the following we consider the case of a pointed cocommutative Hopf algebra $H$ over $\mathbb{F}$ (a field of characteristic 0). We recall that this includes all cocommutative Hopf algebras over $\mathbb{C}$.

Let $H$ be a cocommutative pointed Hopf algebra. Recall that by the structure theorem for cocommutative pointed Hopf algebras over a field of characteristic 0, $H = H^1 \times \mathbb{F}[G(H)]$, where $H^1$ is the universal enveloping algebra of the Lie algebra of primitive elements in $H$ and $\mathbb{F}[G(H)]$ is the group algebra of the group of group-like $G(H)$ in $H$. For each group-like element $g \in G(H)$, $H^1 g$ is a subcoalgebra of $H$ and $H = \bigoplus_{g \in G(H)} H^1 g$ as coalgebras.

Still, $V$ is a finite-dimensional $H$-module.

**Definition 2.22.** Let $\kappa$ be a linear map $V \wedge V \to H$. Then we write

$$\kappa = \sum_{g \in G(H)} \kappa_g$$

with component maps $\kappa_g : V \wedge V \to H^1 g$ for all $g \in G(H)$. Note that the sum has only finitely many non-zero contributions, because the image of $\kappa$ is finite-dimensional.

**Lemma 2.23.** A linear map $\kappa : V \wedge V \to H$ has the Jacobi property if and only if $\kappa_g$ has the Jacobi property for all $g \in G(H)$.

**Proof.** For every $g \in G(H)$, let $p_g : H \to H^1 g$ be the linear projection along $H^1 g'$ for all $g' \neq g$. Then we can apply $\text{id}_V \otimes p_g$ to the Jacobi identity in $V \otimes H$ to obtain the Jacobi identity for $\kappa_g$. $\square$

**Definition 2.24.** Let $C$ be a coalgebra. A filtration $(C_k)_k$ of $C$ as vector space is called coalgebra filtration if

$$\Delta C_k \subset \sum_{0 \leq i \leq k} C_i \otimes C_{k-i}.$$ 

Let $C_0$ be the coradical of $C$, i.e. the sum of simple subcoalgebras of $C$. The coradical filtration of $C$ is defined inductively by $C_{k+1} := \Delta^{-1}(C_0 \otimes C + C \otimes C_k)$.

We recall well-known facts from the theory of coalgebras: The coradical filtration is a coalgebra filtration such that $C = \bigcup_{k \geq 0} C_k$ for every coalgebra $C$. If $C$ is a pointed coalgebra, for instance any coalgebra over $\mathbb{C}$, then $C_0 = \bigoplus_{g \in G(C)} \mathbb{F}g$ for the set of group-like elements $G(C)$ in $C$.

We have the following information on the group-like elements $g$ which are necessary to determine $\kappa$ and the corresponding maps $\kappa_g$ (cp. [RS sec. 1], [EGG sec. 2.3]):
Proposition 2.25. Let $\kappa : V \wedge V \to H$ be an linear map with the Jacobi property. Then the following holds for every $g \in G(H)$, where $(g - 1)$ denotes the corresponding operator on $V$:

- $\kappa_g = 0$ if $\text{rank}(g - 1) \not\in \{0, 1, 2\}$.
- If $\text{rank}(g - 1) = 1$, then $\kappa_g(x, y) = 0$ for all $x, y \in \ker(g - 1)$.
- If $\text{rank}(g - 1) = 1$ and $g$ acts diagonalizable, then $\kappa_g(x, y) = 0$ for all $x, y \in V$ satisfying $(g - 1) \cdot x \wedge y + x \wedge (g - 1) \cdot y = 0$.
- If $\text{rank}(g - 1) = 2$, then $\kappa_g(x, y) = 0$ for all $x, y \in V$ satisfying $(g - 1) \cdot x \wedge (g - 1) \cdot y = 0$.

Proof. We fix $g \in G(H)$. Then by Proposition 2.25, $\kappa_g$ has the Jacobi property, so it is enough to consider the case $\kappa = \kappa_g$.

It is a basic statement on coalgebras that every finite-dimensional subspace is contained in a finite-dimensional subcoalgebra. Let $C$ be such a finite-dimensional subcoalgebra of $H^3(g)$ (which is a subcoalgebra of $H$) containing $(\text{im} \, \kappa_g)$. Let $(C_k)_{k \geq 0}$ be the coradical filtration of $C$ and let $k$ be minimal such that $\text{im} \, \kappa \subset C_k$. Note that $C_0 = Fg$ now, because $g$ is the unique group-like element in $C$.

Then we can write $\kappa_g = \sum_i \theta_i h_i$ with suitable non-zero $(\theta_i)_i$ in $(V \wedge V)^*$ and $(h_i)_i$ in $C_k$. Let $J$ be the set of indices $j$ such that $h_j \in C_k \setminus C_{k-1}$ (where we set $C_{-1} = 0$). Since $k$ was chosen minimally, $J \neq \emptyset$. For every $j \in J$, let $p_j$ be a projection of $C_k$ onto $F h_j$ along $C_{k-1}$ and along $h_i$ for all $i \neq j$. Then

$$(\text{id} \otimes p_j) \circ \Delta(h^j) = \delta_{ij} g \otimes h_j$$

for all $i$.

Thus if we apply $(\text{id} \otimes p_j)$ to (2.25), this yields

$$0 = (g - 1) \cdot z \wedge (g - 1) \cdot u \wedge (g - 1) \cdot v \otimes \theta_j(x, y) h^j$$

for all $z, u, v, x, y \in V$, so the operator $(g - 1)$ has rank at most 2.

If we apply $(\text{id} \otimes p_j)$ to the Jacobi identity $0 = (x | y, z) + (y | z, x) + (z | x, y)$ in $V \otimes H$ for any $x, y, z \in V$, we obtain

$$0 = ((g - 1) \cdot x) \theta_j(y, z) + ((g - 1) \cdot y) \theta_j(z, x) + ((g - 1) \cdot z) \theta_j(x, y) \otimes h^j.$$ 

Let us assume that $(g - 1)$ has rank 1, then this shows that $\theta_j(x, y) = 0$ for all $x, y \in \ker(g - 1)$.

Let us additionally assume that $g$ acts diagonalizably, then $(g - 1)$ acts diagonalizably, hence there is vector $z \in V$ and an invertible scalar $r \in F \setminus \{0\}$ such that $(g - 1) \cdot z = rz$ and $V = F z \oplus \ker(g - 1)$. We pick a linear functional $\alpha \in V^*$ such that $\alpha(z) = r^{-1}$, i.e. $\alpha((g - 1) \cdot z) = 1$ and $(g - 1) \cdot x = \alpha((g - 1) \cdot x)(g - 1) \cdot z$ for all $x \in V$. Then applying $\alpha \otimes \text{id}$ to the last equation we get

$$\theta_j(x, y) = r^{-1} \theta_j(\alpha((g - 1) \cdot x) z, y) + \theta_j(\alpha((g - 1) \cdot y) z)$$

$$= r^{-1} \theta_j(\alpha((g - 1) \cdot x)(g - 1) \cdot z \wedge y + x \wedge \alpha((g - 1) \cdot y)(g - 1) \cdot z)$$

$$= r^{-1} \theta_j((g - 1) \cdot x \wedge y + x \wedge (g - 1) \cdot y).$$

This proves that $\theta_j(x, y) = 0$ if $(g - 1) \cdot x \wedge y + x \wedge (g - 1) \cdot y = 0$.

Let us assume that $(g - 1)$ has rank 2. We apply $(\text{id} \otimes p_j)$ to (2.25) to obtain

$$(g - 1) \cdot z \wedge (g - 1) \cdot u \otimes \theta_j(x, y) = (g - 1) \cdot x \wedge (g - 1) \cdot y \otimes \theta_j(z, u)$$

for all $z, u, x, y \in V$. Since $(g - 1)$ has rank 2, we can pick $z, u$ such that $(g - 1) \cdot z \wedge (g - 1) \cdot u$ is non-zero. So $\theta_j(x, y)$ has to be zero if $(g - 1) \cdot x \wedge (g - 1) y = 0$. 


Hence \( \theta_j \) has to vanish on the subspaces as stated for every \( j \in J \). Hence
\[
\kappa(x, y) = \sum_{i \notin J} \theta_i(x, y) h^i =: \kappa'(x, y)
\]
on these subspaces, but \( \text{im} \kappa' \subset C_{k-1} \). We repeat the argument inductively replacing \( \kappa \) by \( \kappa' \) each time until \( \text{im} \kappa' \subset C_{-1} = 0 \). \( \square \)

To compare this with the classical situation of \( H \) being the group-algebra of a finite group, we note:

**Corollary 2.26.** Let \( \kappa : V \wedge V \to H \) be a linear map such that \( A = A_\kappa \) is a Hopf-Hecke algebra (i.e. \( V \) is an orthogonal \( H \)-module, \( \kappa \) is \( H \)-linear and has the Jacobi property). Fix \( g \in G(H) \) such that \( \text{rk}(g - 1) = 1 \). Then
\[
\text{im} \kappa_g \subset \{ x \in H^1 g : gxg^{-1} = -x \}.
\]
In particular, if \( H \) is the group algebra of a finite group, then \( \kappa_g = 0 \) for all \( g \) with \( \text{rk}(g - 1) = 1 \).

**Proof.** Since \( V \) is an orthogonal \( H \)-module, \( g \) acts diagonally and orthogonally. Hence there is a non-zero vector \( v \in V \) such that \( g \cdot v = -v \) and \( V = \mathbb{F}v \oplus \ker(g - 1) \). Now \( \kappa_g(x, y) = 0 \) for all \( x, y \in \ker(g - 1) \) and \( \kappa_g(x, y) = 0 \) for all \( x, y \in \mathbb{F}v \), because this space is one-dimensional. Assume \( x = v \) and \( y \in \ker(g - 1) \). Then due to \( H \)-linearity,
\[
g\kappa_g(x, y) g^{-1} = \kappa_g(g \cdot x, g \cdot y) = -\kappa_g(x, y),
\]
so indeed \( \text{im} \kappa_g \) lies in the subspace of \( H^1 g \) on which \( g \) acts by \(-1\).

If \( H \) is the group algebra of a finite group, then \( H^1 g = \mathbb{F}g \), so any \( g \) acts trivially on \( H^1 g \). \( \square \)

**Definition 2.27.** For every \( p \geq 0 \) and a linear map \( \kappa : V \wedge V \to H \), we define
\[
\kappa_p := \sum_{g \in G(H), \text{rk}(g - 1) = p, \text{im} \kappa_g \subset K_p} \kappa_g.
\]
We observe that if \( \kappa \) has the Jacobi property, by \[2.13\] and \[2.25\] \( \kappa_p = 0 \) for \( p > 2 \) and the condition \( \text{im} \kappa_g \subset K_2 \) in the definition of \( \kappa_2 \) is obsolete.

**Lemma 2.28.** For every \( \kappa : V \wedge V \to H \) with the Jacobi property, \( \kappa_0 \) is of the form \[2.6\].

**Proof.** This is true by definition of \( \kappa_0 \). \( \square \)

**Proposition 2.29.** Assume that every group-like \( g \in H \) with \( \text{rk}(g - 1) = 1 \) acts diagonally on \( V \) (for instance, assume \( V \) is an orthogonal module). Then for every \( \kappa : V \wedge V \to H \) with the Jacobi property, \( \kappa_1 \) is of the form
\[
\kappa_1(x, y) = \sum_m \sigma_m(h^m_{(1)} \triangleright x, y) h^m_{(2)} + \sigma_m(x, h^m_{(1)} \triangleright y) h^m_{(2)}
\]
with \( h^m \) in \( K_1 \) and \( \sigma_m \in (V \wedge V)^* \) for every \( m \). In particular, it is of the form \[2.6\].

**Proof.** By \[2.28\] it is enough to show the assertion for \( \kappa = \kappa_g \) for a fixed \( g \in G(H) \) acting diagonally on \( V \) with \( \text{rk}(g - 1) = 1 \) and such that \( \kappa_g \subset K_1 \).

We can write \( \kappa = \sum_i \sigma_i h^i \) with linearly independent \( h^i \) in \( H^1 g \cap K_1 \) and suitable \( \sigma_i \) in \((V \wedge V)^*\). Let \( J \) be the set of indices \( j \) such that \( h^j \) lies in maximal degree \( d \) of the coradical filtration. Since \( \text{rk}(g - 1) = 1 \) and \( g \) acts diagonally, by \[2.28\] we know that
\[
\sigma_j(x, y) = \tilde{\sigma}_j((g - 1) \cdot x \wedge y + x \wedge (g - 1) \cdot y)
\]
for some $\tilde{\sigma}_j$ in $(V \wedge V)^*$. We define

$$\kappa'(x, y) := \sum_{j \in J} \tilde{\sigma}_j(h^j_{(1)} \triangleright x, y)h^j_{(2)} + \tilde{\sigma}_j(x, h^j_{(1)} \triangleright y)h^j_{(2)},$$

then by 2.24 $\kappa'$ has the Jacobi property, so $\kappa'' = \kappa - \kappa'$ has the Jacobi property, but the image of $\kappa''$ lies in degree $\leq d - 1$ of the coradical filtration, because the highest degree terms of $\kappa$ and $\kappa'$ cancel. We can replace $\kappa$ by $\kappa''$ and proceed inductively until the image of $\kappa''$ lies in degree $-1$, so $\kappa'' = 0$.

**Proposition 2.30.** For every $g \in G(H)$ with $\text{rnk}(g - 1) = 2$, let $\theta_g$ be a form in $(V \wedge V)^*$ which does not vanish on $(g - 1)V \wedge (g - 1)V$. Then for every $\kappa : V \wedge V \to H$ with the Jacobi property, $\kappa_2$ is of the form

$$\kappa_2(x, y) = \sum_{g \in G(H), \text{rnk}(g - 1) = 2} \theta_g(h^\theta_{(1)} \triangleright x, h^\theta_{(2)} \triangleright y)h^\theta_{(3)}$$

with $h^\theta$ in $H^1 g \cap K_2$ for every $g$. In particular, it is of the form (2.6).

**Proof.** By 2.23 it is enough to show this for $\kappa = \kappa_g$ for a fixed $g \in G(H)$ with $\text{rnk}(g - 1) = 2$.

Consider $\theta_g$, a skew-symmetric bilinear form on $V$ which does not vanish on $(g - 1)V \wedge (g - 1)V$. Since $\text{rnk}(g - 1) = 2$, the restriction of any other skew-symmetric bilinear form on $V$ to $(g - 1)V \wedge (g - 1)V$ is just a scalar multiple of the restriction of $\theta_g$.

We can write $\kappa = \sum_i \theta_i k^i$ with linearly independent $k^i$ in $H^1 g \cap K_2$ and suitable $\theta_i$ in $(V \wedge V)^*$. Let $J$ be the set of indices $j$ such that $k^j$ lies in maximal degree $d$ of the coradical filtration. Since $\text{rnk}(g - 1) = 2$, by 2.25 we know that

$$\theta_j(x, y) = \tilde{\theta}_j((g - 1) \cdot x, (g - 1) \cdot y) = r_j \theta_g((g - 1) \cdot x, (g - 1) \cdot y)$$

for some $\tilde{\theta}_j$ in $(V \wedge V)^*$ and for some $r_j \in \mathbb{F}$. We define $h^j := r_j k^j$ and

$$\kappa'(x, y) := \sum_{j \in J} \theta_g(h^j_{(1)} \triangleright x, h^j_{(2)} \triangleright y)h^j_{(3)},$$

then by 2.24 $\kappa'$ has the Jacobi property, so $\kappa'' = \kappa - \kappa'$ has the Jacobi property, but the image of $\kappa''$ lies in degree $\leq d - 1$ of the coradical filtration, because the highest degree terms of $\kappa$ and $\kappa'$ cancel. We can replace $\kappa$ by $\kappa''$ and proceed inductively until the image of $\kappa''$ lies in degree $-1$, so $\kappa'' = 0$. This way we see that

$$\kappa(x, y) = \sum_p \theta_g(h^p_{(1)} \triangleright x, h^p_{(2)} \triangleright y)h^p_{(3)}$$

for some $(h^p)_p$ in $H^1 g \cap K_2$, but now we can define $h^\theta := \sum_p h^p$ and the assertion follows. \hfill $\square$

**Remark 2.31.** It might be an interesting question which maps $\kappa$ have the Jacobi property other than the ones characterized in 2.24, 2.28, 2.29 and 2.30, all being of the form (2.6).

Note that by 2.23 and 2.30 it is enough to consider the case $\kappa = \kappa_g$ for a fixed group-like $g$ with $\text{rnk}(g - 1) \in \{0, 1\}$, and if $V$ is an orthogonal module, then by the results in 2.28, and 2.29 an example extending our partial characterization would necessarily satisfy $\kappa_g \subset K_{\text{rnk}(g - 1)}$.

If $H$ is the group-algebra of a finite group, there can be no such maps, because every module $V$ is orthogonal and $\text{im} \kappa_g \subset \mathbb{F}g \subset K_{\text{rnk}(g - 1)}$ for every $g \in G(H)$. 

HOPF-HECKE ALGEBRAS, INFINITESIMAL CHEREDNIK ALGEBRAS AND DIRAC COHOMOLOGY 12
3. Dirac cohomology for Hopf-Hecke algebras

For the convenience of the reader we would like to recall some central notions and results from [Fl] which will be used in the course of this paper.

We fix a cocommutative Hopf algebra $H$, an orthogonal (in particular, finite-dimensional) $H$-module $V$ and an $H$-linear map $\kappa : V \wedge V \to H$ with the Jacobi property, so $A = A_\kappa$ is a Hopf-Hecke algebra. Since $V$ is fixed, we use the shorthand $C$ for the Clifford algebra $C(V)$.

For a general Hopf-Hecke algebra $A$, we have the following definitions and results ([Fl, 3.2]):

**Definition 3.1.** Let $(v^k)_k$, $(u^k)_k$ be a pair of orthogonal bases of $V$. Then the Casimir element $\Omega$ and the Dirac element $D$ are defined to be

$$\Omega := \sum_k v^k v^k \in A, \quad D := \sum_k v^k \otimes v^k \in A \otimes C.$$  

**Lemma 3.2.** The Casimir and the Dirac element are independent of the choice of dual bases, and $D^2 = \Omega \otimes 1 + \frac{1}{2} \sum_{k<l} \kappa(v^k, u^l) \otimes [v^k, v^l]$ in $A \otimes C$, where the commutator is taken in $C$.

Let $S$ be a fixed spin module of $C$ and let $M$ be an $A$-module. Then $M \otimes S$ is an $A \otimes C$-module, and in particular the Dirac operator acts on $M \otimes S$.

**Definition 3.3.** The Dirac cohomology of $M$ is defined as

$$H^D(M) := \ker D / (\text{im} D \cap \ker D).$$

**Lemma 3.4.** If $D$ acts diagonalizably on $M \otimes S$ (e.g., as a normal operator), then $H^D(M) \cong \ker D^2$.

In preparation of the study of infinitesimal Cherednik algebras, we assume now $H$ is the universal enveloping algebra of a finite-dimensional Lie algebra over a field of characteristic 0, so in particular $H$ is pointed, since $1 \in H$ is the unique group-like element.

Let us recall the definition of $\text{Biv}(V)$, the Lie subalgebra of the Clifford algebra $C(V)$ generated by elements $w_1 w_2$ for vectors $w_1, w_2 \in V$, and the fact that there is a Lie algebra isomorphism $\phi : \text{Biv}(V) \to \mathfrak{so}(V)$ which sends a bivector $w_1 w_2$ to the skew-symmetric endomorphism $v \mapsto [w_1, w_2]$, where the commutator is taken in $C$ leaving $V$ invariant. For more information on Clifford algebras we refer to [HP2, MC].

Now we have the following concrete description of the pin cover of $H$ as constructed in [Fl, prop. 2.10]:

**Definition 3.5.** Let $\tilde{H} := H \oplus H$, let $\pi : \tilde{H} \to H$ be the natural projection onto the first copy of $H$, and let $\gamma : H \to C$ be the map $\tilde{h} \mapsto \phi^{-1}(\pi(\tilde{h}) \cdot)$, where the element $\pi(\tilde{h})$ of $H$ can be viewed as a skew-symmetric endomorphism of $V$, since $V$ is an orthogonal module.

We recall ([Fl def. 2.5]) that for a pointed cocommutative Hopf algebra $H$ with an orthogonal module $V$ we have an algebra $\mathbb{Z}_2$-gradation which assigns each group-like element the determinant of the corresponding operator on $V$ and each primitive element even degree. Obviously in our setting, $H = H^{even}$ and $\tilde{H} = \tilde{H}^{even}$ irrespective of the module $V$.

From the discussion in [Fl sec. 2] and in particular the construction in the proof of [Fl prop. 2.10], we have the following result:
Proposition 3.6. \((\tilde{H},\pi,\gamma)\) is the pin cover of \(H\) with respect to \(V\) in the sense of [Fl] def. 2.11, and it splits in the sense of [Fl] def. 2.13.

We recall the definition ([Fl] def. 2.11) of the diagonal map,
\[
\Delta_C : \tilde{H} \to H \otimes C, \tilde{h} \mapsto \pi(\hat{h}(1)) \otimes \gamma(\hat{h}(2)),
\]
and of \(H' := \tilde{H} / \ker \Delta_C\). Since the pin cover splits by our construction, we have \(H' \cong H\), we can consider \(H\) as a Hopf subalgebra of \(\tilde{H}\) and we have algebra maps \(\gamma|_H : H \to C, h \mapsto \phi^{-1}(h)\), and \(\Delta_C|_H : H \to H \otimes C, h \mapsto h(1) \otimes \gamma|_H(h(2))\), which we denote by \(\gamma, \Delta_C\), as well (abusing notation).

Finally, we recall that the Hopf-Hecke algebra \(A\) defined by \((H,V,\kappa)\) is called Barbasch-Sahi algebra if \(D\) satisfies the Parthasarathy condition,
\[
D^2 \in Z(A \otimes C) + \Delta_C(H^{\text{even}}).
\]
Now with the Hopf algebra \(H\) as above, this is equivalent to
\[
D^2 \in Z(A \otimes C) + \Delta_C(H).
\]

4. Infinitesimal Cherednik algebras of \(\text{GL}_n\)

4.1. Motivation. We fix a cocommutative Hopf algebra \(H\) over \(\mathbb{C}\) and a completely reducible \(H\)-module \(V\).

Proposition 4.1. If \(V\) admits both a symmetric and a skew-symmetric non-degenerate \(H\)-invariant bilinear form, then \(V\) is of the form \(V \cong W \oplus W^*\) for an \(H\)-module \(W\).

Proof. Since \(V\) is completely reducible, we can decompose \(V\) as a direct sum of simple submodules, and we can group these simple submodules such that
\[
V = \bigoplus_{i=1}^k V_i^{a_i} \oplus \bigoplus_{j=1}^m W_j^{b_j} \oplus (W_j^*)^{c_j}
\]
with positive integers \((a_i)_i, (b_j)_j, (c_j)_j\) and self-dual modules \((V_i)_i\) and such that \((V_i), (W_j), (W_j^*)_j\) are all pairwise non-isomorphic simple \(H\)-modules. As \(V\) admits a non-degenerate \(H\)-invariant bilinear form, it is self-dual, so \(b_j = c_j\) for each \(j\). Hence, it is enough to show that \(a_i\) is even for each \(i\).

Consider two simple submodules \(V'\) and \(V''\) of \(V\) and let \(\alpha\) be a non-degenerate \(H\)-invariant bilinear form on \(V\). Then \(v \mapsto \alpha(v, v')\) is an \(H\)-linear map from \(V'\) to \((V'')^*\), but since \(V'\) and \(V''\) are simple, the map has to be an isomorphism or 0. Hence the restriction of \(\alpha\) to \(V_i^{a_i}\) has to be non-degenerate for each \(i\). This means that \(V_i^{a_i}\) admits both a symmetric and a skew-symmetric non-degenerate \(H\)-invariant bilinear form for each \(i\).

We consider a fixed index \(i\). Since \(V_i\) is self-dual, there is an \(H\)-linear isomorphism \(V_i \to V_i^*\) or, equivalently, a non-degenerate \(H\)-invariant bilinear form \(\alpha\) on \(V_i\). We can viewed \(\alpha\) as the sum of a symmetric and a skew-symmetric bilinear form, and since \(\alpha\) is \(H\)-invariant, both summands have to be \(H\)-invariant, as well. Since \(V_i\) is simple, the space of \(H\)-linear endomorphisms, equivalently, \(H\)-invariant bilinear forms is one-dimensional. Hence \(\alpha\) has to be symmetric (case a) or skew-symmetric (case b).

We write \(V_i^{a_i} = V_i \otimes C^{a_i}\) and we pick a basis \((e_p)_{1 \leq p \leq a_i}\) of \(C^{a_i}\). Let \(\beta\) be a non-degenerate \(H\)-invariant skew-symmetric (case a) or symmetric (case b) bilinear form on \(V_i^{a_i}\). Now for every \(1 \leq p, q \leq a_i\), the map \((v, v') \mapsto \beta(v \otimes e_p, v' \otimes e_q)\) is an \(H\)-invariant bilinear form on \(V_i\), so it has to be a multiple of \(\alpha\). Hence \(\beta(v \otimes e_p, v' \otimes e_q) = \gamma(e_p, e_q)\alpha(v, v')\) for scalars \((\gamma(e_p, e_q))_{p,q}\), which
defines a bilinear form \( \gamma \) on \( \mathbb{C}^n \). For \( \beta \) to be skew-symmetric (case a) or symmetric (case b), \( \gamma \) has to be skew-symmetric. Now if \( a_i \) is odd, \( \gamma \) cannot be non-degenerate, so there is a vector \( e \in \mathbb{C}^n \) such that \( \gamma(e,e') = 0 \) for all \( e' \in \mathbb{C}^n \), and consequently, \( \beta(v \otimes e, v' \otimes e') = 0 \) for all \( v, v' \in V_i \) and \( e' \in \mathbb{C}^n \). This is a contradiction, since \( \beta \) was assumed to be non-degenerate. Hence \( a_i \) has to be even, which was to be shown. \( \square \)

**Proposition 4.2.** The finite-dimensional \( H \)-modules \( V \) which admit both a symmetric and a skew-symmetric non-degenerate \( H \)-invariant bilinear form are exactly the \( H \)-modules of the form \( V \cong W \oplus W^* \) for finite-dimensional \( H \)-modules \( W \).

**Proof.** It remains to show that modules of the form \( W \oplus W^* \) admit forms as required. Let \((\cdot, \cdot) : W^* \otimes W \to \mathbb{C} \) be the natural pairing. By definition of the contragredient action of \( H \) on \( W^* \), the pairing is \( H \)-invariant. We define the forms \( \alpha, \beta \) by

\[
\alpha(y + x, y' + x') := (y, x') + (y', x), \quad \beta(y + x, y' + x') := (y, x') - (y', x).
\]

Then since \((\cdot, \cdot)\) is \( H \)-invariant, \( \alpha \) and \( \beta \) are \( H \)-invariant. By definition, they are non-degenerate, bilinear and also symmetric and skew-symmetric, respectively. \( \square \)

**Remark 4.3.** One might want to look for Hopf-Hecke algebras constructed from completely reducible orthogonal \( H \)-modules \( V \) with a non-degenerate \( H \)-invariant skew-symmetric bilinear form. Then \([4,2]\) tells us that these modules are exactly the ones of the form \( W \oplus W^* \).

Now if we take \( H \) to be the universal enveloping algebra of the Lie algebra of a reductive algebraic group, a class of such Hopf-Hecke algebras called *infinitesimal Cherednik algebras* is defined in [EGG].

**Remark 4.4.** The infinitesimal Hecke algebras of \( \text{Sp}_{2n} \) with the standard module \( V = \mathbb{C}^{2n} \) classified in [EGG] 4.1.2 and studied in [Kh1] [TK] are not Hopf-Hecke algebras, since the module does not have a non-degenerate invariant symmetric form; this follows from the above discussion, for instance, because we saw that a simple module cannot have a symmetric and a skew-symmetric non-degenerate invariant form at the same time.

We will study the Dirac cohomology of those algebras for the group \( \text{GL}_n \).

### 4.2. Infinitesimal Cherednik algebras of \( \text{GL}_n \) as Hopf-Hecke algebras

For a fixed \( n \geq 1 \) we consider the general linear group \( G = \text{GL}_n(\mathbb{C}) \), its Lie algebra \( \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \) and its universal enveloping algebra \( H = U(\mathfrak{g}) \). We consider the standard Lie algebra (and hence \( H \)-)module \( \mathfrak{h} = \mathbb{C}^n \). We define \( V := \mathfrak{h} \oplus \mathfrak{h}^* \) as module, where \( \mathfrak{h}^* \) is the usual contragredient module, and we denote the pairing of \( \mathfrak{h}^* \) and \( \mathfrak{h} \) by \((\cdot, \cdot)\).

The following definitions are from [EGG]:

**Definition 4.5.** For all \( m \geq 0, x \in \mathfrak{h}^* \) and \( y \in \mathfrak{h} \), let \( r_m(x,y) \) be the coefficient of \( \tau^m \) in the expansion of the polynomial function \( A \mapsto (x,(1-\tau A)^{-1} \cdot y) \det(1-\tau A)^{-1} \) in \( S(\mathfrak{gl}_n^* ) \) viewed as element in \( S(\mathfrak{gl}_n^* ) \cong S(\mathfrak{gl}_n) \cong U(\mathfrak{gl}_n) \), where the first identification is via the trace pairing \( \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathbb{C}, (A,B) \mapsto \text{Tr}(AB) \) and the second identification is via the symmetrization map.

Let \( \xi(z) = \sum_{m \geq 0} \xi_m z^m \) be a polynomial. We define a map \( \kappa = \kappa_\xi : V \wedge V \to H \) by \( \kappa(x,x') = \kappa(y,y') = 0 \) and

\[
\kappa(y,x) = \sum_{m \geq 0} \xi_m r_m(x,y) \tag{4.1}
\]
for all \( x, x' \in \mathfrak{h}^* \) and all \( y, y' \in \mathfrak{h} \). Let \( I_\kappa \) be the ideal of \( T(V) \rtimes H \) generated by elements of the form \( vw - wv - \kappa(v, w) \) for \( v, w \in V \). The algebra

\[
\mathcal{H}_\kappa = (T(V) \rtimes H)/I_\kappa
\]

is called infinitesimal Cherednik algebra.

There is an alternative definition of \( \kappa \) in terms of \( \xi \) as explained in [EGG 4.2] (see also [DT, 3.1]):

**Definition 4.6.** Let \( \hat{\xi} \) be the polynomial

\[
\hat{\xi}(z) = \frac{1}{2\pi^n} \theta^n(z^n(\xi(z))) = \sum_{m \geq 0} \frac{1}{2\pi^n} \frac{(m + n)!}{m!} \xi_m z^m.
\]

Also, we define the notations \( \langle v, w \rangle_H := v^T \cdot w \), which is an Hermitian inner product on \( \mathfrak{h} \), and \( |v| := (\sum_i |v_i|^2)^{1/2} \) for all \( v \in \mathfrak{h} \), the Euclidean norm.

For every non-zero \( v \in \mathfrak{h} \), let \( v \otimes \mathfrak{m} \) denote the rank-one endomorphism \( v(\cdot, v)_H \) of \( \mathfrak{h} \) viewed as element in \( \mathfrak{gl}_n \), so \( \xi(v \otimes \mathfrak{m}) \) can be viewed as element in \( S(\mathfrak{gl}_n) \) or \( \mathcal{U}(\mathfrak{gl}_n) \).

**Lemma 4.7 ([EGG 4.2]).** With the definitions as above,

\[
\kappa(x, y) = \int_{|v|=1} \langle x, (v \otimes \mathfrak{m}) \cdot y \rangle \hat{\xi}(v \otimes \mathfrak{m}) dv,
\]

for all \( x \in \mathfrak{h}^* \), \( y \in \mathfrak{h} \).

**Proof.** We recall results from [EGG 4.2]: Let \( F_m \in S(\mathfrak{g}^*) \) be defined by

\[
F_m(A) := \int_{|v|=1} \langle A \cdot v, v \rangle^m_H dv
\]

for all \( A \in \mathfrak{gl}_n \). According to the computations in [EGG 4.2], \( F_m(A) \) equals the coefficient of \( \tau^m + 1 \) in

\[
2\pi^n \frac{(m + 1)!}{(m + n)!} \det(1 - \tau A)^{-1}.
\]

As explained in [EGG 4.2], under the identification \( S(\mathfrak{g}) \simeq S(\mathfrak{g}^*) \),

\[
\int_{|v|=1} \langle x, (v \otimes \mathfrak{m}) \cdot y \rangle |(v \otimes \mathfrak{m})|^m_H dv = \int_{|v|=1} \langle x, (v \otimes \mathfrak{m}) \cdot y \rangle \langle A \cdot v, v \rangle^m_H dv
\]

\[
= \frac{1}{m + 1} dF_m(A(y \otimes x)) = 2\pi^n \frac{m!}{(m + n)!} r_m
\]

where \( A \in \mathfrak{g} \) symbolizes the argument of a polynomial function in \( S(\mathfrak{g}^*) \), and where \( r_m \) is the coefficient of \( \tau^m \) in \( (x, (1 - \tau A)^{-1} \cdot y) \det(1 - \tau A)^{-1} \).

Now if we write \( \hat{\xi}(z) = \sum_{m \geq 0} \xi_m z^m \), then by definition, \( \hat{\xi}_m = \frac{1}{2\pi^n} \frac{(m + n)!}{m!} \xi_m \) for all \( m \geq 0 \), so

\[
\int_{|v|=1} \langle x, (v \otimes \mathfrak{m}) \cdot y \rangle \hat{\xi}(v \otimes \mathfrak{m}) dv = \sum_{m \geq 0} \frac{2\pi^n}{(m + n)!} \frac{m!}{\xi_m r_m(x, y) \sum_{m \geq 0} \xi_m r_m(x, y)}.
\]

\( \square \)
Remark 4.8. In fact, [EGG, thm. 4.2] says that \((T(V) \times H)/I_\kappa\) has the PBW property if and only if \(\kappa\) is of this form. Since we will see that \(V\) is an orthogonal \(U(\mathfrak{gl}_n)\)-module, the Hopf-Hecke algebras \((H, V, \kappa)\) with \(H = U(\mathfrak{gl}_n)\) and \(V = \mathfrak{h} \oplus \mathfrak{h}^*\) are exactly the infinitesimal Cherednik algebras.

We also note that the presentation of infinitesimal Cherednik algebras is in “reverse order” here: In [EGG], infinitesimal Cherednik algebras for a reductive algebraic group \(G\) over \(\mathbb{C}\) are parametrized by \(G\)-invariant distributions on the closed subscheme “of complex reflections” \(\Phi\). In particular, these distributions are parametrized by polynomials. The relation between polynomial and resulting deformation is computed to be (4.3). After evaluating the integral, the equivalent formulation (4.1) is given.

The center of these algebras has been shown to be a polynomial algebra in \(n\) variables in [TT]. Their representation theory has been studied and in particular their finite-dimensional irreducible modules have been classified in [TT]. Universal infinitesimal Cherednik algebras, which are the analogs of infinitesimal Cherednik algebras, but with \(\xi_1, \ldots, \xi_n\) viewed as formal parameters, have been identified with \(W\)-algebras of the same type and a 1-block nilpotent element in [TT].

We want to see that \(H_\xi\) is a Hopf-Hecke algebra in our notation and we want to find a description of \(D^2\).

Definition 4.9. Let \(\langle \cdot, \cdot \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{C}\) be the natural pairing, which is \(\mathfrak{g}\)-invariant. We define a form \(\langle \cdot, \cdot \rangle\) on \(V\) by
\[
\langle x + y, x' + y' \rangle := \langle x, y' \rangle + \langle x', y \rangle
\]
for all \(x, x' \in \mathfrak{h}^*\) and \(y, y' \in \mathfrak{h}\).

We pick dual bases \((x_i)\), \((y_i)\) of \(\mathfrak{h}^*\) and \(\mathfrak{h}\), respectively, and we define
\[
(v_k) = (x_1, \ldots, x_n, y_1, \ldots, y_n), \quad (v^k) = (y_1, \ldots, y_n, x_1, \ldots, x_n).
\]

Lemma 4.10. In the situation as in the definition, \(\langle \cdot, \cdot \rangle\) is a symmetric \(g\)-invariant bilinear form on \(V\), i.e., \(V\) is an orthogonal \(H\)-module and \(H_\xi\) is a Hopf-Hecke algebra, and \((v_k)\), \((v^k)\) is a pair of dual bases for \(V\) with respect to \(\langle \cdot, \cdot \rangle\).

Proof. \(\langle \cdot, \cdot \rangle\) makes \(V\) an orthogonal \(H\)-module with the described pair of dual bases, because the natural pairing \(\langle \cdot, \cdot \rangle\) is \(\mathfrak{g}\)-invariant, as we have seen in 4.2 already.

By construction, \(H_\xi\) has the PBW property, so it is a Hopf-Hecke algebra. \(\square\)

Proposition 4.11. The pin cover \(\tilde{H}\) of \(H\) splits, so \(\tilde{H} \cong H\) as algebras, we can identify \(H\) with a Hopf subalgebra of \(\tilde{H}\) and \(\gamma : \tilde{H} \to C\), \(\Delta_C : \tilde{H} \to H \otimes C\) restrict to algebra maps \(\gamma : H \to C\), \(\Delta_C : H \to H \otimes C\) (abusing notation). Furthermore, \(\gamma(E_{ij}) = \frac{1}{2}(y_i x_j - x_j y_i) \in C\) for the elementary matrix \(E_{ij}\) in \(\mathfrak{gl}(\mathfrak{h}) \cong \mathfrak{gl}_n\) which sends \(y_j\) to \(y_i\).

Proof. As discussed in sec. 3 the pin cover splits, because \(H\) is the universal enveloping algebra of a Lie algebra. Consequently, \(H\) can be identified with a Hopf subalgebra of \(\tilde{H}\), \(\tilde{H} \cong H\), and we have the restricted algebra maps as asserted.

To verify \(\gamma(E_{ij}) = \frac{1}{2}(y_i x_j - x_j y_i)\), we can compute explicitly in \(C\):
\[
[y_i x_j, y_k] = y_i x_j y_k + y_j y_k x_j = 2y_i(x_j, y_k),
\]
\[
- [x_j y_i, y_k] = x_j y_k y_i + y_k x_j y_i = 2y_i(x_j, y_k),
\]
and similarly \([y_i x_j, x_k] = -2x_j(x_k, y_i) = [-x_j y_i, x_k]\). \(\square\)

We recall the definitions of the Casimir element \(\Omega = \sum_k v_k v^k\) in \(A = H_\xi\) and of the Dirac element \(D = \sum_k v_k \otimes v^k\) in \(A \otimes C\) (3.1) for any pair of dual bases \((v_k)\) and \((v^k)\), so in particular for the choice made in 4.9.
Lemma 4.12. Let $D \in A \otimes C$ be the Dirac element for $A = \mathcal{H}_\xi$. Then
\begin{equation}
D^2 = \Omega \otimes 1 - 2 \int_{|v| = 1} \xi(v \otimes \mathfrak{v}) \otimes \gamma(v \otimes \mathfrak{v}) \, dv ,
\end{equation}
where $v \otimes \mathfrak{v}$ stands for a rank-one matrix in $\mathfrak{gl}_n$ for any non-zero $v \in \mathfrak{h}$ and where we identify $H = \mathcal{U}(\mathfrak{gl}_n)$ with a Hopf subalgebra of $H$.

Proof. We invoke \cite{22} to obtain
\begin{align*}
D^2 &= \Omega \otimes 1 + \frac{1}{2} \sum_{k<l} \kappa(v_k, v_l) \otimes [v^k, v^l] = \Omega \otimes 1 + \frac{1}{2} \sum_{i,j} \kappa(y_j, x_i) \otimes [x_j, y_i] \\
&= \Omega \otimes 1 - 2 \sum_{i,j} \kappa(y_j, x_i) \otimes \gamma(E_{ij}) ,
\end{align*}
where $E_{ij} = y_i \otimes x_j$ as above is an element in $\mathfrak{gl}_n$ for all $i, j$. Using the integral formula \cite{13} for $\kappa$, we obtain
\begin{align*}
\sum_{i,j} \kappa(y_j, x_i) \otimes \gamma(E_{ij}) &= \sum_{i,j} \int_{|v| = 1} \xi(v \otimes \mathfrak{v}) \otimes (x_i, (v \otimes \mathfrak{v})y_j) \gamma(E_{ij}) \, dv \\
&= \int_{|v| = 1} \xi(v \otimes \mathfrak{v}) \otimes \gamma(v \otimes \mathfrak{v}) \, dv ,
\end{align*}
as desired. \hfill \Box

In the following we want to find an even more explicit expression for $D^2$ in terms of polynomials derived from $\xi$, which will allow us to prove that $D$ satisfies the Parthasarathy condition and hence $\mathcal{H}_\xi$ is a Barbasch-Sahi algebra. We need some auxiliary lemmas.

Definition 4.13. For any $\epsilon \in \mathbb{C}$, we define $\nabla_\epsilon$, an operator on polynomials, by
\[ \nabla_\epsilon f(z) := f(z + \epsilon) - f(z + \epsilon - 1) . \]
For $k \geq 0$, let $B_k(z)$ be the $k$-th Bernoulli polynomial defined by the generating series
\[ \sum_{k \geq 0} B_k(z) \frac{t^k}{k!} = \frac{te^{tz}}{e^t - 1} \]
\cite{23} \cite{23}. We recall that $B_k$ satisfies $\nabla_1 B_k(z) = B_k(z + 1) - B_k(z) = kz^{k-1}$ for every $k \geq 0$ \cite{23} \cite{23}.

Lemma 4.14. Let $p$ be a polynomial. For any $\epsilon \in \mathbb{C}$, there is a polynomial $f$ satisfying $\nabla_\epsilon f(z) = p(z)$ and $f$ is characterized by this relation uniquely up to the constant term.

Proof. To construct $f$, we write $p(z) = \sum_{i \geq 0} p_i z^i$ and we let $B_n(z)$ be the $n$-th Bernoulli polynomial. Then
\[ \nabla_\epsilon f(z) = p(z) \iff \nabla_1 f(z) = p(z + 1 - \epsilon) = \sum_{i \geq 0} \frac{p_i}{i + 1} (i + 1)(z + 1 - \epsilon)^i , \]
hence $f(z) := \sum_{i \geq 0} \frac{p_i}{i + 1} B_{i+1}(z + 1 - \epsilon) + f_0$ satisfies this recurrence relation for any scalar $f_0$.

For uniqueness, let $f_2$ be another polynomial satisfying the same recurrence relation. Then $f_d = f - f_2$ is a polynomial satisfying $\nabla_\epsilon f_d(z) = f_d(z + \epsilon) - f_d(z + \epsilon - 1) = 0$. Hence $f_d$ attains the same value at, say, all integer numbers, so it has to be a constant polynomial. \hfill \Box
Lemma 4.15. For every polynomial $p$, let $f$ be a polynomial satisfying $\nabla_{1/2} f(z) = p(z)$. Then
\begin{equation}
(4.5) \quad p(z)\gamma = f(z + \gamma) + \frac{1}{2}p(z) - f(z + \frac{1}{2}) \quad \text{in } \mathbb{C}[z, \gamma] \mod (\gamma^2 - \frac{1}{4}).
\end{equation}

Proof. We claim that for every polynomial $p$, there are polynomials $f, q$ such that
\begin{equation}
(4.6) \quad p(z)\gamma = f(z + \gamma) + q(z) \quad \text{in } \mathbb{C}[z, \gamma] \mod (\gamma^2 - \frac{1}{4}).
\end{equation}
First we note that it is enough to show this for polynomials $p$ of the form $p(z) = (k + 1)z^k$, because those form a basis. Consider $k = 0$. Then $p(z)\gamma = \gamma = (z + \gamma) - z$, which verifies the claim. Assume the claim is true for all non-negative integers $0 \leq k < K$ for some $K \geq 1$, and hence for all polynomials $p$ of degree at most $K - 1$. We consider $p(z) = (K + 1)z^K$, $f(z) = z^{K+1}$ and $q(z) = -z^{K+1}$, then
\[ p(z)\gamma = f(z + \gamma) + q(z)
\]
for polynomials $p', q'$ (because $\gamma^2 = \frac{1}{4}$), and $\deg p' \leq K - 1$. This proves the claim by induction.

We assume now $f, q$ are as in (4.6). Then we can substitute $\gamma = \pm \frac{1}{2}$ to get
\[ q(z) = \pm \frac{1}{2}p(z) - f(z \pm \frac{1}{2}). \]
However, the two choices of substitution should yield the same result, so
\[ \frac{1}{2}p(z) - f(z + \frac{1}{2}) = -\frac{1}{2}p(z) - f(z - \frac{1}{2}) \quad \Leftrightarrow \quad f(z + \frac{1}{2}) - f(z - \frac{1}{2}) = p(z) \]
and using the choice $\gamma = \frac{1}{2}$ to obtain the above expression of $q$ in terms $p, f$, we have
\[ p(z)\gamma = f(z + \gamma) + \frac{1}{2}p(z) - f(z + \frac{1}{2}), \]
as desired. \hfill \Box

Lemma 4.16. Let $v$ be a vector in $\mathfrak{h}$ with $|v| = 1$, and let $v \otimes \mathfrak{v}$ be the corresponding rank-one matrix in $\mathfrak{gl}_n$. Then $\gamma(v \otimes \mathfrak{v})^2 = \frac{1}{4}$ in $C(V)$.

Proof. We write $v = \sum_i v_i x_i$, then by linearity of $\gamma$,
\[ \gamma(v \otimes \mathfrak{v}) = \sum_{i,j} v_i \mathfrak{v}_j \gamma(E_{ij}) = \frac{1}{4} \sum_{i,j} v_i \mathfrak{v}_j [x_i, y_j] = \frac{1}{4} \left[ \sum_i v_i x_i, \sum_i \mathfrak{v}_i y_i \right] = \frac{1}{4} [v, v^*], \]
where $v$ and $v^* := \sum_i \mathfrak{v}_i x_i$ can be regarded as elements in $V$ or in $C(V)$, and where we used the value of $\gamma(E_{ij})$ as discussed in (4.11).

Now in $C(V)$, $v^2 = \langle v, v \rangle = 0, (v^*)^2 = \langle v^*, v^* \rangle = 0$ and $vv^* + v^* v = 2\langle v, v^* \rangle = 2$. Hence,
\[ \gamma(v \otimes \mathfrak{v})^2 = \frac{1}{16}(vv^* vv^* +vv^* v - v v^* v^* - v v^* v^*) = \frac{1}{16}(v(2 - vv^*)v^* + v^*(2 - v^* v)v)
\]
\[ = \frac{1}{8}(vv^* + v^* v) = \frac{1}{4}, \]
as desired. \hfill \Box

We are ready to give a refined formula for $D^2$.

Definition 4.17. Let $f_\xi(z)$ be the polynomial defined up to a constant by
\[ \nabla_0 f_\xi(z) = f_\xi(z) - f_\xi(z - 1) = \xi(z) = \frac{1}{2\pi n} \partial^n (z^n \xi(z)) \]
(the first and the last equality being the definitions of $\Delta_0$ and $\xi$, respectively). Furthermore, we define $\alpha, \beta \in U(\mathfrak{gl}_n)$ by
\[ \alpha := \int_{|v| = 1} -\xi(v \otimes \mathfrak{v}) + 2f_\xi(v \otimes \mathfrak{v}) \, dv, \quad \beta := \int_{|v| = 1} 2f_\xi(v \otimes \mathfrak{v} - \frac{1}{2}) \, dv, \]
and $C' := \int_{|v|=1} f_\xi(v \otimes \mathfrak{m}) \, dv$.

**Proposition 4.18.** Let $f := f_\xi$, $\alpha, \beta$ as in the definition. Then we have the following formula for $D^2$:

\begin{equation}
D^2 = (\Omega + \alpha) \otimes 1 - \Delta_C(\beta) .
\end{equation}

Furthermore, $(\Omega + \alpha) = 2(\sum_i x_i y_i + C')$ and this element is central in $A$, and $\beta, C'$ are central in $H$. In particular, $D$ satisfies the Parthasarathy condition and $H_\xi$ is a Barbasch-Sahi algebra.

**Proof.** We fix $v \in \mathfrak{h}$ with $|v| = 1$ and define elements $z := (v \otimes \mathfrak{m}) \otimes 1$, $\gamma := 1 \otimes \gamma(v \otimes \mathfrak{m})$ in $A \otimes C$. Then $z + \gamma = \Delta_C(v \otimes \mathfrak{m})$ and $\gamma^2 = \frac{1}{2}$ by \[\text{[4.18]}\]. We observe that $\nabla_{1/2} f(z - \frac{1}{2}) = \nabla_0 f(z) = \tilde{\xi}(z)$ by definition of $f = f_\xi$. So we can apply \[\text{[4.15]}\] to obtain

\[\tilde{\xi}(v \otimes \mathfrak{m}) \otimes \gamma(v \otimes \mathfrak{m}) = f(\Delta_C(v \otimes \mathfrak{m}) - \frac{1}{2}) + \left(\frac{1}{2}\tilde{\xi}(v \otimes \mathfrak{m}) - f(v \otimes \mathfrak{m})\right) \otimes 1 ,\]

which implies the new formula for $D^2$ from \[\text{[4.14]}\].

We define the shorthand $M_v := v \otimes \mathfrak{m} \in \mathfrak{gl}_n$ for any $v \in \mathfrak{h}$. To obtain the alternative formula for $(\Omega + \alpha)$ we note that

\[\Omega = \sum_i x_i y_i + y_i x_i = \sum_i 2x_i y_i + [y_i, x_i] = \sum_i 2x_i y_i + \int_{|v|=1} (x_i, M_v \cdot y_i) \tilde{\xi}(M_v) \, dv \]

\[= 2 \sum_i x_i y_i + \int_{|v|=1} \tilde{\xi}(M_v) \, dv ,\]

where we use that $\sum_i (x_i, (v \otimes \mathfrak{m}) \cdot y_i) = \sum_i |v_i|^2 = 1$.

We want to show now that $\Omega + \alpha$ is central. We will prove that $\Omega + \alpha$ commutes with a set of algebra generators of $A \otimes C$. Obviously, it commutes with elements of $C$, so it is enough to consider elements of $\mathfrak{h}, \mathfrak{b}^*$ and $\mathfrak{gl}_n$.

Let us first fix $y, v \in \mathfrak{h}$ such that $|v| = 1$ and $M := M_v$. We regard $M$ as an element in a universal enveloping algebra, so $M^k$ denotes a tensor power of $M$ for all $k \geq 0$. If $\mu : \mathfrak{gl}_n \to \mathfrak{gl}_n$ is the matrix multiplication, we have $\mu(M^k) = M$ for all $k \geq 1$, so we can compute in $A = T(V) \rtimes U(\mathfrak{gl}_n)$:

\[M^k y = \sum_{i=0}^k \binom{k}{i} (\mu(M^{k-i}) \cdot y) M^i = y M^k + \sum_{i=0}^{k-1} \binom{k}{i} (M \cdot y) M^i = y M^k + (M \cdot y)(M+1)^k - (M \cdot y) M^k ,\]

because $M$ is a primitive element $M$, so the coproduct of $M^k$ is just $\sum_{i=0}^k \binom{k}{i} M^{k-i} \otimes M^i$. Hence,

\[[M^k, y] = (M \cdot y)((M+1)^k - M^k)\]

for all $k \geq 0$ and hence for any polynomial $q$,

\[[q(M), y] = (M \cdot y) \nabla_1 q(M) .\]

In particular,

\[\int_{|v|=1} f(M_v) \, dv, y = \int_{|v|=1} (M_v \cdot y) \nabla_1 f(M_v) \, dv = \int_{|v|=1} (M_v \cdot y) \tilde{\xi}(M_v + 1) \, dv .\]
On the other hand,
\[
\left[ \sum_{i} x_i y_i, y \right] = \sum_{i} [x_i, y] y_i = - \int_{|v|=1} (x_i, M_v \cdot y) \tilde{\xi}(M_v) y_i \, dv
\]
\[
= - \int_{|v|=1} [\tilde{\xi}(M_v), M_v \cdot y] + (M_v \cdot y) \tilde{\xi}(M_v) \, dv = - \int_{|v|=1} (M_v \cdot y) \tilde{\xi}(M_v + 1) \, dv ,
\]
where we have used that \( M_v \cdot (M_v \cdot y) = M_v \cdot y \). So indeed, \( \Omega + \alpha \) commutes with any \( y \in \mathfrak{h} \). An exactly parallel argument shows that \( \Omega + \alpha \) commutes with any \( x \in \mathfrak{h}^* \). (Alternatively, this follows from the existence of an anti-involution of \( H_{\xi} \) sending \( y_i \leftrightarrow x_i \) and \( E_{ij} \leftrightarrow E_{ji} \) as described in [DT], sec. 2.1.)

Furthermore, we have seen already that \( \Omega \) commutes with elements from \( \mathfrak{gl}_n \), so it remains to show that \( \alpha, \beta \) and \( C' \) are central in \( \mathcal{U}(\mathfrak{gl}_n) \), too. Let \( g \) be any polynomial and consider the element \( h_g = \int_{|v|=1} g(M_v) \, dv \) in \( \mathcal{U}(\mathfrak{gl}_n) \). We note that \( h_g \) is invariant under the adjoint action of \( U(\mathfrak{h}) \), the unitary group of \( \mathfrak{h} \) with \( \langle \cdot, \cdot \rangle_H \), because \( MQ_{v}Q^* = M_{Qv} \) for all \( Q \in U(\mathfrak{h}) \), \( v \in \mathfrak{h} \) and the integral is invariant under the transformation \( v \mapsto Qv \). Now \( \mathfrak{gl}_n \) is just the complexified Lie algebra of \( U(\mathfrak{h}) \), so the center of \( \mathcal{U}(\mathfrak{gl}_n) \) is just the space of \( U(\mathfrak{h}) \)-invariants in \( \mathcal{U}(\mathfrak{gl}_n) \). Hence \( h_g \) is central in \( \mathcal{U}(\mathfrak{gl}_n) \), and in particular, \( \alpha \) and \( \beta \) are central in \( H = \mathcal{U}(\mathfrak{gl}_n) \).

Now \( D \) satisfies the Parthasarathy condition, because \( (\Omega + \alpha) \otimes 1 \) is central in \( A \otimes C \) and \( \beta \) is in \( H^{even} \), since \( H = H^{even} \) as discussed in [8].

**Remark 4.19.** We compare this with the results in [DT]: The polynomial \( f_{\xi} \) corresponds to the polynomial called “2\( \pi^n \int \)" there, the central element \( (\Omega + \alpha) \) is just “2\( t' \)" in the notation of the reference, where \( t' \) is the Casimir element studied there, and \( C' \) is what is denoted by \( C' \) there, as well.

4.3. **Dirac cohomology for \( H_{\xi} \).** Having seen that \( H_{\xi} \) is what we call a Barbasch-Sahi algebra, we can explore the Dirac cohomology of its modules. Here we will focus on the finite-dimensional modules, which have been studied in [DT].

**Definition 4.20.** Denote by \( C'(\lambda) \) the scalar by which \( C' \) acts on an irreducible highest weight \( \mathfrak{gl}_n \)-module with highest weight \( \lambda \).

We recall the definition of the complete symmetric homogeneous polynomial \( h_k \) of degree \( k \): it is the polynomial in \( n \) variables defined for \( x = (x_1, \ldots, x_n) \) by
\[
h_k(x) = \sum_{l_1 + \cdots + l_n = k, l_i \geq 0} x_1^{l_1} \cdots x_n^{l_n} .
\]

Let \( w = \sum_{k \geq 0} w_k z^k \) be a polynomial of degree \( \deg(\xi + 1) \) satisfying
\[
2\pi^n f_{\xi} = (2 \sinh(\partial/2))^{n-1} z^{n-1} w(z) .
\]
(We observe that \( 2 \sinh(\partial/2) = (e^{\partial/2} - e^{-\partial/2}) \) is just the operator \( \nabla_{1/2} \) which sends a polynomial \( p(z) \) to \( p(z + \frac{1}{2}) - p(z - \frac{1}{2}) \)).

Finally, we define a polynomial in \( n \) variables \( P(\mu) := \sum_{k \geq 0} w_k h_k(\mu + \rho) \), where \( \rho \) is the Weyl vector of \( \mathfrak{gl}_n \).

We cite the following result from [DT]:

**Lemma 4.21** ([DT] thm. 3.2), \( w \) exists and is uniquely defined up to a constant, and \( C'(\lambda) = P(\lambda) \).

We compute the relations between \( w, \tilde{\xi} \) and \( \xi \):
Lemma 4.22. \( w \) is the polynomial uniquely defined up to a constant by
\[
(2 \sinh(\partial/2))^n z^{n-1} w(z) = 2\pi^n \xi(z + \frac{1}{2}) .
\]
Equivalently, \( w \) is the polynomial uniquely defined up to a constant by
\[
e^{-\partial/2} (2 \sinh(\partial/2))^n z^{n-1} w(z) = \partial^n (z^n \xi(z)) .
\]

Proof. We verify that by definition of \( w \) and \( f_\xi \),
\[
(2 \sinh(\partial/2))^n z^{n-1} w(z) = 2\pi^n (2 \sinh(\partial/2)) f_\xi = 2\pi^n (f_\xi(z + \frac{1}{2}) - f_\xi(z - \frac{1}{2})) = 2\pi^n \xi(z + \frac{1}{2}) .
\]
Now the polynomial \( \tilde{w} \) satisfying
\[
(2 \sinh(\partial/2))^n z^{n-1} \tilde{w}(z) = \tilde{\xi}(z + \frac{1}{2})
\]
is uniquely defined, so \( w \) is uniquely defined up to a constant by 4.14.

We can apply the bijective translation operator \( e^{-\partial/2} \) on both sides and use the definition of \( \tilde{\xi} \) to obtain the second assertion. \( \square \)

From here we can go on to compute the action of \( D^2 \) and finally the Dirac cohomology for the finite-dimensional irreducible \( H_\xi \)-modules studied in [DT].

Definition 4.23. In the following, we identify \( (a_1, \ldots, a_n) \in \mathbb{C}^n \) with the weight \( a_1 E_{11}^* + \cdots + a_n E_{nn}^* \) of \( \mathfrak{gl}_n \). For every dominant \( \mathfrak{gl}_n \)-weight \( \lambda \), let \( V_\lambda \) be the irreducible highest weight \( \mathfrak{gl}_n \)-module with highest weight \( \lambda \). We also define the set
\[
\tilde{\Lambda} := \{ \lambda \text{ dominant } \mathfrak{gl}_n \text{-weight} : \exists \nu_n \in \mathbb{N}_0 : P(\lambda) = P(\lambda - (0, \ldots, 0, \nu_n + 1)) \} .
\]

Proposition 4.24. [DT, thm. 4.1] For any \( \lambda \in \tilde{\Lambda} \), there exists a unique irreducible finite-dimensional \( H_\xi \)-module \( L(\lambda) \). More precisely,
\[
L(\lambda) = \bigoplus_{0 \leq \nu' \leq \nu} V_{\lambda - \nu'} ,
\]
as \( \mathfrak{gl}_n \)-module, where

\[
\nu_i \text{ is the minimal non-negative integer such that } \nu' := \lambda - (0, \ldots, 0, \nu_i + 1, 0, \ldots, 0) \text{ is either not a dominant weight or } P(\lambda) = P(\lambda') \text{ for every } 1 \leq i \leq n .
\]

The irreducible finite-dimensional \( H_\xi \)-modules are exactly the modules \( L(\lambda) \) for \( \lambda \in \tilde{\Lambda} \).

Let again \( S \) be a spin module of the Clifford algebra \( C(V) \) with spin module \( S \) and let \( n_\mu \) be the multiplicity of \( V_\mu \) in \( M \otimes S \) for every \( \mu \).

Proposition 4.25. Consider \( M = L(\lambda) \) for a \( \lambda \in \tilde{\Lambda} \) as in 4.24. Then the kernel of \( D^2 \) acting on \( M \otimes S \) is \( \bigoplus_\mu n_\mu V_\mu \), where the sum ranges over all \( \mu \) satisfying
\[
P(\lambda) = P(\mu - (\frac{1}{2}, \ldots, \frac{1}{2})) .
\]

Proof. First we recall that \( D^2 = (\Omega + \alpha) - \Delta_C(\beta) \) according to 4.18.

Let \( v_\lambda \in V_\lambda \subseteq M \) be a highest weight vector. From 4.18 we also know that \( \Omega + \alpha = 2(\sum_i x_i y_i + C') \), so \( \Omega + \alpha \) acts on \( v_\lambda \) by the scalar
\[
2C'(\lambda) = 2P(\lambda) .
\]
We want to find the scalar by which \( \Delta_C(\beta) \) acts on \( V_\mu \subseteq (M \otimes S) \). That is we want to find the scalar by which
\[
\beta = 2 \int_{|v| = 1} f_\xi(v \otimes \nabla - \frac{1}{2}) dv
\]
acts on $V_\mu$, noting that $\Delta_C$ is $\mathcal{U}(\mathfrak{gl}_n)$-linear. Let $\mathfrak{h}_g$ be the diagonal matrices in $\mathfrak{gl}_n$. We use the twisted Harish-Chandra map $Z(\mathcal{U}(\mathfrak{gl}_n)) \to S(\mathfrak{h}_g)$ to see that this scalar is

$$2\int_{|v|=1} f_\xi(\mu + \rho, (|v_1|^2, \ldots, |v_n|^2)) - \frac{1}{2}) dv = 2\int_{|v|=1} f_\xi((\mu + \rho - (\frac{1}{2}, \ldots, \frac{1}{2})), (|v_1|^2, \ldots, |v_n|^2)) dv$$

$$= 2C(\mu - (\frac{1}{2}, \ldots, \frac{1}{2})) = 2P(\mu - (\frac{1}{2}, \ldots, \frac{1}{2})) .$$

This yields the desired characterization of the highest weight submodules contained in $\ker D^2$. 

We have the following information on the structure of $S$ as $\mathfrak{gl}_n$-module via $\gamma$ (cp. [Ko] prop. 3.17):

**Lemma 4.26.** The weights of $S$ are exactly the weights $(s_1, \ldots, s_n)$ in $\{\pm \frac{1}{2}\}^n$, and all weight spaces are one-dimensional. Hence $S \cong \bigwedge(h) \otimes (-\frac{1}{2}\operatorname{Tr})$ as $\mathfrak{gl}_n$-module.

**Proof.** Since $\dim V$ is even, there is a unique spin module and we can take it to be the left ideal generated by $u = y_1 \ldots y_n$ in $C(V)$, which is irreducible (this is explained for instance in [Ko] sec. 3).

Hence, a basis of $S$ is given by the elements $x_1^{e_1} \ldots x_n^{e_n} u$ for exponents $e_1, \ldots, e_n \in \{0,1\}$. We can compute directly

$$\gamma(E_{ii})x_i = \frac{1}{2}(y_i x_i - x_i y_i)x_i = -\frac{1}{2}x_i x_i = -\frac{1}{2}x_i ,$$

$$\gamma(E_{ii})y_i = \frac{1}{2}(y_i x_i - x_i y_i)y_i = \frac{1}{2}y_i x_i y_i = \frac{1}{2}y_i ,$$

and $\gamma(E_{ii})$ commutes with $x_j$ or $y_j$ in $C(V)$ for all $j \neq i$, so

$$\gamma(E_{ii})x_1^{e_1} \ldots x_n^{e_n} u = \frac{1}{2}(-1)^{e_i} x_1^{e_1} \ldots x_n^{e_n} u$$

for all $1 \leq i \leq n$. 

We can use this result to obtain the structure of $L(\lambda) \otimes S$:

**Proposition 4.27.** For any $\lambda \in \hat{\mathcal{N}}$, $L(\lambda) \otimes S$ decomposes as

$$(4.8) \bigoplus_{0 \leq \mu' \leq \lambda} \bigoplus\{V_\mu : \mu \text{ dominant weight, } \mu_i - (\lambda_i - \nu_i') \in \{\pm \frac{1}{2}\} \forall 1 \leq i \leq n\} .$$

In particular, the irreducible modules $V_\mu$ occurring are those with dominant weight $\mu$ satisfying

$$\mu_i \in \{\lambda_i + \frac{1}{2}, \lambda_i - \frac{1}{2}, \lambda_i, \nu_i - \frac{1}{2}\}$$

for all $1 \leq i \leq n$.

**Proof.** Let $\lambda'$ be a highest $\mathfrak{gl}_n$-weight and $V_{\lambda'}$ the corresponding irreducible highest weight module. Then by Pieri’s rule, $V_{\lambda'} \otimes \bigwedge(h)$ decomposes as

$$\bigoplus\{V_\mu : \mu \text{ dominant weight, } \mu_i - \lambda_i' \in \{0,1\} \forall 1 \leq i \leq n\} .$$

Now since $L(\lambda) = \bigoplus_{0 \leq \mu' \leq \lambda'} V_{\lambda' - \mu'}$ and $S = \bigwedge(h) \otimes (-\frac{1}{2}\operatorname{Tr})$, $L(\lambda) \otimes S$ decomposes as asserted, and listing the weights occurring just gives the desired characterization. 

This allows us the following conclusions on the irreducible highest weight $\mathfrak{gl}_n$-submodules of $\ker D^2$:

**Corollary 4.28.** The kernel of $D^2$ acting on $L(\lambda) \otimes S$ is the sum of those irreducible submodules $V_\mu$ appearing in (4.8) for which $P(\lambda) = P(\mu - (\frac{1}{2}, \ldots, \frac{1}{2}))$. 
\textbf{Corollary 4.29.} For any $\lambda \in \hat{A}$, $V_{\lambda^0}$ for $\lambda^0 := \lambda + (\frac{1}{2}, \ldots, \frac{1}{2})$ and $V_{\lambda^n}$ for $\lambda^n := \lambda + (\frac{1}{2}, \ldots, -\nu_n - \frac{1}{2})$ appear with multiplicity one in the kernel of $D^2$ acting on $L(\lambda) \otimes S$.

For $1 \leq i < n$, if $\lambda^i := \lambda - (0, \ldots, 0, \nu_i + 1, 0, \ldots, 0)$ is a dominant weight, then $V_{\lambda_i}$ for $\lambda_i := \lambda + (\frac{1}{2}, \ldots, \frac{1}{2}, -\nu_i - \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ appears with multiplicity one in $\ker D^2$.

\textbf{Proof.} Examining (4.8), we see that for each $0 \leq i \leq n$, there is only one possible weight $\nu'$ in the first sum such that $\mu = \lambda^i$ can be obtained in the second sum, so all $\lambda^i$ appear with multiplicity one in $L(\lambda) \otimes S$.

Now
\[ P(\lambda) = P(\lambda^0 - (\frac{1}{2}, \ldots, \frac{1}{2})) = P(\lambda^n - (\frac{1}{2}, \ldots, \frac{1}{2})) , \]

hence, $V_{\lambda^0}$ and $V_{\lambda^n}$ lie in $\ker D^2$.

Also, if $\lambda^i$ is dominant, then by (4.23)
\[ P(\lambda) = P(\lambda^i) = P(\lambda^i - (\frac{1}{2}, \ldots, \frac{1}{2})) , \]

so $V_{\lambda_i}$ lies in $\ker D^2$ in this case, as well. \hfill \Box

\textbf{Remark 4.30.} Let us consider examples for $n = 1$ and $n = 2$ (cp. the examples in [DT sec. 4]).

For $n = 1$, $\lambda$ is just a complex number and $\nu$ is non-negative integer (minimal) such that $P(\lambda) = P(\lambda - \nu - 1)$. Then $L(\lambda) = V_{\lambda} \oplus \cdots \oplus V_{\lambda - \nu}$, hence,
\[ L(\lambda) \otimes S = V_{\lambda + 1} \oplus 2V_{\lambda - \frac{1}{2}} \oplus \cdots \oplus 2V_{\lambda - \nu + 1} \oplus V_{\lambda - \nu - \frac{1}{2}} . \]

Now the only weights $\mu$ occurring in $L(\lambda) \otimes S$ such that $P(\lambda) = P(\mu - \frac{1}{2})$ are obviously $\lambda^0 = \lambda + \frac{1}{2}$ and $\lambda^1 = \lambda - \nu - \frac{1}{2}$. So the kernel of $D^2$ and, as we will see shortly, the Dirac cohomology is just $V_{\lambda^0} \oplus V_{\lambda^1}$.

For $n = 2$, we consider the deformation with $P = 18h_1 - \frac{9}{2}h_2 - 2h_3 + \frac{3}{2}h_4$ (note that the deformation is determined by $\xi$ or by $P$ due to the correspondence $[4.22]$ and assume $\lambda = (3, 0) - \rho$.

The second table displayed below shows $P(\mu + \rho)$ for each of the weights $\mu + \rho$ shown in the first table. From the position of the zero entries in the first row and the first column of the second table, we see that $\nu = (2, 2)$, so $L(\lambda) = \bigoplus_{(3, 0) \geq \mu + \rho \geq (1, -2)} V_{\mu}$ for our choice of $\lambda$ and $P$. Consequently,
\[ L(\lambda) \otimes S = \bigoplus_{(3, 0) \geq \mu + \rho \geq (0, -3)} m_{\mu} V_{\mu + (1/2, 1/2)} \]

with multiplicities $m_{\mu}$ as shown in the third table.

| $\mu + \rho$ | $P(\mu + \rho)$ | $m_{\mu}$ |
|-------------|-----------------|----------|
| (3, 0)      | (2, 0)          | (1, 0)   | (0, 0) | 0 -5 -12 0 | 1 2 2 1 |
| (3, 1)      | (2, 1)          | (1, 1)   | (0, 1) | 10 0 -10 4 | 2 4 4 2 |
| (3, 2)      | (2, 2)          | (1, 2)   | (0, 2) | 12 -4 -16 3 | 2 4 4 2 |
| (3, -3)     | (2, -3)         | (1, -3)  | (0, -3) | 0 -20 -30 0 | 1 2 2 1 |

Now referring to the second table again, we observe that the kernel of $D^2$ (and as we will see, the Dirac cohomology) is
\[ V_{(3.5, 0.5) + \rho} \oplus V_{(0.5, 0.5) + \rho} \oplus 4V_{(2.5, -0.5) + \rho} \oplus V_{(3.5, -2.5) + \rho} \oplus V_{(0.5, -2.5) + \rho} . \]

To finally relate the kernel of $D^2$ to the Dirac cohomology for the discussed modules $L(\lambda)$, let us recall some more results of [DT] on the finite-dimensional irreducible modules $L(\lambda)$ of the infinitesimal Cherednik algebras of $GL_n$:

\textbf{Remark 4.31 ([DT sec. 2]).} For any $\lambda \in \hat{A}$, $L(\lambda)$ is the unique irreducible quotient of a Verma module $M(\lambda)$, and there is a Shapovalov-type symmetric bilinear form on $M(\lambda)$ which induces a non-degenerate symmetric bilinear form on $L(\lambda)$. Adoints with respect to this form are given by the anti-involution $\sigma : \mathcal{H}_{\xi} \to \mathcal{H}_{\xi}$ sending $x_i \leftrightarrow y_i$ and $E_{ij} \leftrightarrow E_{ji}$. 
Proposition 4.32. If $M = L(\lambda)$ is a finite-dimensional irreducible module of the infinitesimal Cherednik algebra $H_c(\mathfrak{g}_n)$, then the Dirac cohomology of $M$ is isomorphic to the kernel of $D^2$ as characterized in 4.25, 4.28 and 4.29.

Proof. In this situation we can apply 3.4.

Let $\langle \cdot, \cdot \rangle_M$ be the non-degenerate symmetric bilinear form on $M$ from [DT] mentioned in 4.31 and let $\langle \cdot, \cdot \rangle_S$ be the non-degenerate symmetric bilinear form on the spin module $S$ constructed as in [HP2, p. 62–63]: we identify $S$ with $\Lambda(h^*)$ as in the proof of 4.26 and extend the non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ defined by $\langle x_i, x_j \rangle_S = 2\delta_{ij}$ to a non-degenerate symmetric bilinear form on $\Lambda(h^*) \simeq S$ using the determinant.

Now it can be verified that adjoints for elements of $C(V)$ acting on $S$ with respect to $\langle \cdot, \cdot \rangle_S$ are given by the anti-involution sending $x_i \leftrightarrow y_i$. Then the elements $x_i \otimes y_i + y_i \otimes x_i$ in $A \otimes C$ act as self-adjoint operators with respect to the non-degenerate symmetric bilinear tensor product form $\langle \cdot, \cdot \rangle_{M \otimes S}$ on $M \otimes S$, so $D$ acts as self-adjoint operator on $M \otimes S$.

In particular, $D$ acts diagonally, so $\ker D = \ker D^2$, $\ker D \cap \im D = 0$ and thus $H^D(M) \cong \ker D^2$. 

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