Post-Newtonian non-equilibrium kinetic theory

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The post-Newtonian hydrodynamic equations for a non-perfect fluid are developed within the framework of a post-Newtonian Boltzmann equation. The post-Newtonian components of the energy-momentum tensor are determined by considering the relativistic Eckart decomposition for a viscous and heat conducting fluid. From the relativistic Grad distribution function its post-Newtonian expression is derived. The hydrodynamic equations for the mass density, mass-energy density and momentum density are determined from a post-Newtonian transfer equation and Grad’s distribution function. In the non-relativistic limit the Newtonian hydrodynamic equations for mass, momentum and energy densities are recovered.

I. INTRODUCTION

In 1938 Einstein, Infeld and Hoffmann [1] proposed a method of successive approximations in powers of $1/c^2$ for the solution of Einstein’s field equations which goes beyond the Newtonian gravitational theory. In the post-Newtonian theory the components of the metric tensor in the order $1/c^n$ are determined from Einstein’s field equations once the energy-momentum tensor in the order $1/c^{n-2}$ is known. For an energy-momentum tensor described by a constitutive equation of a perfect fluid the hydrodynamic equations in the first post-Newtonian approximation were determined by Chandrasekhar [2] and Weinberg [3] while the corresponding ones in the second post-Newtonian approximation by Chandrasekhar and Nutku [4] (see also [5, 6]).

A subject which is interesting to investigate is the inclusion of non-equilibrium quantities related with the viscous stress and the heat flux in the post-Newtonian energy-momentum tensor of a non-perfect fluid. The role of the viscous stress in the post-Newtonian theory was first analysed in [7].

The aim of this work is to derive within the framework of kinetic theory of gases the post-Newtonian hydrodynamic equations for a viscous and heat conducting fluid. We call attention to the fact that the introduction of non-equilibrium quantities related with the viscous stress and heat flux does not change the determination of the components of the metric tensor from Einstein’s field equations in the first post-Newtonian approximation.

We start by determining the post-Newtonian expressions for the components of the energy-momentum tensor which for a non-perfect fluid is described by the Eckart decomposition [8].

At equilibrium the one-particle distribution function for a relativistic gas is characterized by the Maxwell-Jüttner distribution function (see e.g. [9]) and its post-Newtonian version was determined in [10]. For non-equilibrium processes the one-particle distribution function may be characterized by Grad’s distribution function [11]. In the relativistic case Grad’s distribution function is a function of the fourteen fields of particle number density, four-velocity, absolute temperature, dynamic pressure, pressure deviator and heat flux (see e.g. [9]). In this work the post-Newtonian approximation of the relativistic Grad distribution function is derived which is used to determine the contributions of the viscous stress and heat flux to the components of the energy-momentum tensor.

The Boltzmann equation in the first post-Newtonian approximation was derived in [12, 13] and its version in the second post-Newtonian approximation in [5, 6]. Here from the first post-Newtonian Boltzmann equation the so-called Maxwell-Enskog transfer equation [14, 15] is derived for arbitrary macroscopic quantities which are associated with mean values of microscopic quantities.

The hydrodynamic equations for the mass density, mass-energy density and momentum density are obtained from the transfer equation by considering the rest mass and the post-Newtonian expressions for the components of the particle momentum four-vector together with the post-Newtonian Grad distribution function. These hydrodynamic equations correspond to the ones that follow from the conservation equations of the particle four-flow and energy-momentum tensor in the post-Newtonian approximation. Without the relativistic corrections the hydrodynamic equations recover the Newtonian hydrodynamic equations for the mass, momentum and energy densities of a non-perfect fluid.
The paper is outlined as follows: in Section II we determine the components of the post-Newtonian non-equilibrium energy momentum tensor in the Eckart decomposition. The post-Newtonian Boltzmann equation, the transfer equation and Grad’s distribution function are determined Sections III and IV, respectively. In Section V the post-Newtonian hydrodynamic equations are derived and the conclusions of the work are stated in Section VI. Here Latin indices take the values 1, 2, 3 and the Greek indices take the values 0, 1, 2, 3. Furthermore, the indices of Cartesian tensors will be written as subscripts, the summation convention over repeated indices will be assumed, the partial differentiation will be denoted by $\partial / \partial x^i$ and the covariant differentiation is denoted by a semicolon.

## II. ENERGY-MOMENTUM TENSOR

The hydrodynamic equations of a relativistic fluid are determined by the conservation laws of the particle four-flow $N^\mu$ and energy-momentum tensor $T^{\mu\nu}$, namely

$$N^\mu_{;\mu} = 0, \quad T^{\mu\nu}_{;\mu} = 0. \quad (1)$$

The identification of the relativistic non-equilibrium quantities with the non-relativistic ones is attained by introducing the decompositions of the particle four-flow $N^\mu$ and energy-momentum tensor $T^{\mu\nu}$ with respect to the four-velocity $U^\mu$ (such that $U^\mu U_\mu = c^2$). In the literature, there exist two representations for the non-equilibrium particle four-flow and energy-momentum tensor known as the Eckart [8] and the Landau and Lifshitz [16] decompositions. Both representations make use of the projector

$$\Delta^{\mu\nu} = g^{\mu\nu} - \frac{1}{c^2} U^\mu U^\nu, \quad (2)$$

which projects an arbitrary four-vector into another four-vector perpendicular to the four-velocity. The projector has the following properties

$$\Delta^{\mu\nu} U_\nu = 0, \quad \Delta^{\mu\nu} \Delta_{\nu\sigma} = \Delta^{\mu\sigma}, \quad \Delta^{\mu\nu} \Delta_{\nu\sigma} = \Delta^{\mu\sigma}, \quad \Delta^{\mu\nu} = 3,$$

and in a local Minkowski rest frame where $U^\mu = (c, \vec{0})$ it reads $\Delta^{\mu\nu} = \text{diag}(0, -1, -1, -1)$.

By using the projector one can introduce for an arbitrary four vector $A^\mu$ and tensor $A^{\mu\nu}$ the following representations

$$A^{(n)} = \Delta^{\mu\nu} A_\nu, \quad A^{(\mu\nu)} = \frac{1}{2} (\Delta^{\mu\nu} \Delta_{\nu\tau} + \Delta^{\nu\tau} \Delta_{\mu\nu}) A^{\tau}, \quad (4)$$

$$A^{[\mu\nu]} = \frac{1}{2} (\Delta^{\mu\nu} \Delta_{\nu\tau} - \Delta^{\nu\tau} \Delta_{\mu\nu}) A^{\tau}, \quad A^{(\mu\nu)} = A^{(\mu\nu)} - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\tau\sigma} A^{(\tau\sigma)}, \quad (5)$$

which are associated with a four-vector, a symmetric tensor, an antisymmetric tensor and a symmetric traceless tensor that have only the spatial components in a local Minkowski rest frame, respectively. Note that $\Delta_{\mu\nu} A^{(\mu\nu)} = 0$ hold.

Here we shall use the Eckart decomposition where the particle four-flow $N^\mu$ and the energy-momentum tensor $T^{\mu\nu}$ for a viscous heat conducting fluid are represented as:

$$N^\mu = n U^\mu, \quad T^{\mu\nu} = p^{(\mu\nu)} - (p + \varpi) \Delta^{\mu\nu} + \frac{c}{c^2} U^\mu U^\nu + \frac{1}{c^2} \left( U^\mu q^{(\nu)} + U^\nu q^{(\mu)} \right). \quad (6)$$

The quantities $n$, $p^{(\mu\nu)}$, $p + \varpi$, $q^{(\mu)}$ and $\epsilon$ introduced by the above decompositions are identified as follows:

$$n = \frac{1}{c^2} N^\mu U_\mu \quad \text{particle number density}, \quad (7)$$

$$p^{(\mu\nu)} = \left( \Delta^{\mu\nu} \Delta_{\nu\tau} - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\tau\sigma} \right) T^{\tau\sigma} \quad \text{pressure deviator}, \quad (8)$$

$$(p + \varpi) = -\frac{1}{3} \Delta^{\mu\nu} T^{\mu\nu} \quad \text{hydrostatic + dynamic pressures}, \quad (9)$$

$$q^{(\mu)} = \Delta^{\mu\nu} U_\nu T^{\nu\sigma} \quad \text{heat flux}, \quad (10)$$

$$\epsilon = \frac{1}{c^2} U_\mu T^{\mu\nu} U_\nu \quad \text{energy density}. \quad (11)$$
The hydrostatic pressure \( p \) and the energy density \( \epsilon \) refer to equilibrium quantities of the energymomentum tensor and the dynamic pressure \( \varpi \) is the non-equilibrium part of its trace. Furthermore, the energy density \( \epsilon = \rho \varepsilon (1 + \varepsilon/c^2) \) is a sum of two terms one related with the mass density \( \rho = m/n \) where \( m \) is the particle rest mass – and the other with the internal energy density \( \rho \varepsilon \).

The post-Newtonian approximations refer to the solutions of Einstein’s field equations in successive powers of \( 1/c^2 \) \cite{1–3}. Here we shall follow the work of Chandrasekhar \cite{2} and write the first post-Newtonian approximation for the components of the metric tensor as

\[
g_{00} = 1 - \frac{2U}{c^2} + \frac{2}{c^4} (U^2 - 2\Phi), \quad g_{0i} = \frac{\Pi_i}{c^2}, \quad g_{ij} = - \left( 1 + \frac{2U}{c^2} \right) \delta_{ij}, \tag{12}\]

where the scalar gravitational potentials \( U, \Phi \) and the vector gravitational potential \( \Pi_i \) in the above equations satisfy the Poisson equations

\[
\nabla^2 U = -4\pi G\rho, \quad \nabla^2 \Phi = -4\pi G\rho \left( V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho} \right), \quad \nabla^2 \Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial \partial x^i}. \tag{13}\]

Here \( G \) is the universal gravitational constant and \( V \) the three hydrodynamic velocity.

The components of the four-velocity up to the order \( 1/c^4 \) are

\[
U^0 = c \left[ 1 + \frac{1}{c^2} \left( \frac{V^2}{2} + U \right) \right] + \frac{1}{c^4} \left( \frac{3V^4}{8} + \frac{5U V^2}{2} + \frac{U^2}{2} + 2\Phi - \Pi_i V_i \right), \quad U^i = \frac{V_i U^0}{c}. \tag{14}\]

The components of the projector in the first post-Newtonian approximation follows from \cite{11} and \cite{14} and read

\[
\Delta^{00} = -V^2/c^2 + O(c^{-4}), \quad \Delta^{0i} = -V_i/c + O(c^{-3}), \quad \Delta^{ij} = - \left( 1 - \frac{2U}{c^2} \right) \delta_{ij} - \frac{V_i V_j}{c^2} + O(c^{-4}). \tag{15}\]

Here \( O(c^{-n}) \) denotes the order of the \( n \)th inverse power of the light speed.

Let us first analyse the components of the pressure deviator \( p^{(\mu\nu)} \). From the relationship \( U_\mu p^{(\mu\nu)} = g_{\nu\sigma} U^\sigma p^{(\mu\nu)} = 0 \) we have that

\[
\begin{align*}
(g_{00} U^0 + g_{0j} U^j) p^{(0j)} + (g_{0j} U^0 + g_{jk} U^k) p^{(ij)} &= 0, \tag{16} \\
(g_{00} U^0 + g_{0j} U^j) p^{(00)} + (g_{0j} U^0 + g_{jk} U^k) p^{(0j)} &= 0, \tag{17}
\end{align*}
\]

which by considering \cite{12} and \cite{14} imply the following relations for the time and space-time components of the pressure deviator

\[
\begin{align*}
p^{(00)} &= p^{(ij)} \frac{V_i V_j}{c^2} + O(c^{-4}), & p^{(0i)} &= p^{(ij)} \frac{V_i}{c} + O(c^{-3}). \tag{18}
\end{align*}
\]

In order to fulfill the traceless condition of the pressure deviator up to the order of the first post-Newtonian approximation we represent the spatial components of the pressure deviator as

\[
p^{(ij)} = p_{ij} + \frac{1}{2c^2} (p_{ik} V_k V_j + p_{jk} V_k V_i) + O(c^{-4}). \tag{19}\]

It is easy to verify that the above representation fulfills the traceless condition \( \Delta_{\mu\nu} p^{(\mu\nu)} = g_{\mu\nu} p^{(\mu\nu)} = 0 \) up to the first post-Newtonian approximation. Above we have introduced the non-relativistic pressure deviator

\[
p_{ij} = p_{ij} - \frac{p_{rr}}{3} \delta_{ij}, \quad \text{such that} \quad \delta_{ij} p_{ij} = 0. \tag{20}\]

For the components of the heat flux we make use of the relationship \( U_\mu q^{(\mu)} = g_{\mu\nu} U^\nu q^{(\mu)} = 0 \) and get that its time component becomes

\[
q^{(0)} = q_i \frac{V_i}{c} + O(c^{-3}), \tag{21}\]

where \( q^{(i)} = q_i \) is the non-relativistic heat flux vector.

The non-relativistic pressure deviator \( p_{ij} \) and the heat flux vector \( q_i \) vanish at equilibrium and in the non-relativistic limiting case we have \( p^{(ij)} = p_{ij}, p^{(00)} = 0, p^{(0i)} = 0, q^{(i)} = q_i \) and \( q^{(0)} = 0 \).
The energy-momentum tensor components can be obtained now from (6) together with (15), (18) – (21), yielding
\[ T^{00} = \rho c^2 \left[ 1 + \left( V^2 + \varepsilon + 2U \right) \right], \]  \[ T^{i0} = \rho c V_i \left[ 1 + \frac{1}{c^2} \left( V^2 + 2U + \varepsilon + \frac{\rho}{\mu} \right) \right] + \frac{p_{ij} V_j}{c} + \frac{q_i}{c}, \]  \[ T^{ij} = \rho V_i V_j + p \delta_{ij} + p_{ij} + \frac{\rho}{c^2} \left( V^2 + 2U + \varepsilon + \frac{\rho}{\mu} \right) V_i V_j - 2 \frac{p_{ij}}{c^2} U \delta_{ij} + \frac{1}{c^2} (q_i V_j + q_j V_i) \]
\[ + \frac{1}{2c^2} (p_{ik} V_k V_j + p_{jk} V_k V_i). \]  (24)

In the above equations the dynamic pressure \( \pi \) was not taken into account, since for rarefied monatomic gases its constitutive equation is proportional to the velocity divergent and the coefficient of proportionality – the bulk or volume viscosity – is of order of \( 1/c^4 \) (see [9]).

Without the non-equilibrium pressure deviator \( p_{ij} \) and heat flux vector \( q_i \) the components of the energy-momentum tensor (22) – (24) reduce to the well known expressions in the literature [2, 3] for perfect fluids.

III. POST-NEWTONIAN KINETIC THEORY

In kinetic theory of gases the space-time evolution of the one-particle distribution function \( f(x, p, t) \) in the phase space spanned by the spatial coordinates \( x \) and momentum \( p \) of the particles is governed by the Boltzmann equation. The first post-Newtonian approximation of the Boltzmann equation read [5, 6, 12]
\[ \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} \left[ 1 + \frac{1}{c^2} \left( \frac{\v^2}{2} + U \right) \right] + \frac{\partial U}{\partial x^i} \frac{\partial f}{\partial v_i} - \frac{1}{c^2} \left[ 4v_i v_j \frac{\partial U}{\partial v_j} + 3v_i \frac{\partial U}{\partial t} \right] \\
- \left( \frac{3\v^2}{2} - 3U \right) \frac{\partial U}{\partial x^i} - 2 \frac{\partial \Phi}{\partial x^i} \frac{\partial \Pi}{\partial v_i} - \frac{\partial \Pi_i}{\partial v_i} \right] \frac{\partial f}{\partial v_i} = Q(f, f). \]  (25)

Above \( Q(f, f) \) is the collision operator of the Boltzmann equation, which takes into account the binary collisions of the particles and is represented by an integral of the product of two particle distribution functions at collision.

Furthermore, the energy-momentum tensor in kinetic theory of gases is defined in terms of the one-particle distribution function by [9]
\[ T^{\mu\nu} = m^4 c \int u^\mu u^\nu f \sqrt{-g} \frac{d^3 u}{u_0}. \]  (26)

Here \( m \) is the particle rest mass, \( u^\mu = p^\mu / m \) the particle four-velocity and \( \sqrt{-g} \frac{d^3 u}{u_0} \) the invariant integration element. The components of the particle four-velocity in the post-Newtonian approximation reads
\[ u^0 = c \left[ 1 + \frac{1}{c^2} \left( \frac{\v^2}{2} + U \right) \right] + \frac{1}{c^4} \left( \frac{3\v^4}{8} + \frac{5U\v^2}{2} + \frac{U^2}{2} + 2\Phi - \Pi v_i \right), \]
\[ u^i = v_i \frac{u^0}{c}, \]  (27)
which has the same structure as [14], the difference being that the hydrodynamic \( \mathbf{V} \) is substituted by the particle velocity \( \mathbf{v} \).

The invariant integration element of the energy-momentum tensor [20] in the first post-Newtonian approximation was determined in [10] and is given by
\[ \sqrt{-g} \frac{d^3 u}{u_0} = \left\{ 1 + \frac{1}{c^2} \left[ 2\v^2 + 6U \right] \right\} \frac{d^3 v}{c}. \]  (28)

At equilibrium the collision operator of the Boltzmann equation vanishes and the one-particle distribution function becomes the Maxwell-Jüttner distribution function (see e.g. [6])
\[ f^{(0)} = \frac{n}{4\pi m^2 c k T K_2(\zeta)} \exp \left( \frac{-p^\mu U_\mu}{kT} \right). \]  (29)
Above \( T \) is the absolute temperature, \( k \) the Boltzmann constant and \( \zeta = mc^2/kT \) a relativistic parameter which is given by the ratio of the rest energy of the gas particles \( mc^2 \) and the thermal energy of the gas \( kT \). In the ultra-relativistic limiting case \( \zeta \ll 1 \) and in the non-relativistic limiting case \( \zeta \gg 1 \). Furthermore, \( K_2(\zeta) \) denotes the modified Bessel function of the second kind.

The first post-Newtonian approximation of the Maxwell-Jüttner distribution function \( f^{(1)} \) was determined in \( [11] \) and its expression is

\[
f^{(0)} = \frac{n e^{-\frac{mV^2}{2kT}}}{(2\pi mkT)^2} \left[ 1 - \frac{1}{c^2} \left( \frac{15kT}{8m} + \frac{m(V_iV_j)}{2kT} + \frac{2mUV^2}{kT} + \frac{3mV^2V_j}{8kT} + \frac{m(V_iV_j)V^2}{kT} \right) \right]. \tag{30}
\]

Here \( V_i = v_i - V \) is the so-called peculiar velocity which is the particle velocity in the gas frame i.e., the difference of the particle velocity \( v_i \) and the gas velocity \( V \).

In the non-relativistic kinetic theory of gases Grad’s distribution function plays an important role to describe the non-equilibrium behavior of gases not too far from a local equilibrium. Grad’s moment method \( [11] \) was proposed in 1949 for a non-relativistic gas and takes into account the thirteen moments of the one-particle distribution function: particle number density, hydrodynamic velocity, pressure tensor and heat flux vector. For the case of a relativistic gas one has to include the field of the dynamic pressure which is the trace of the pressure tensor in non-equilibrium and in this case Grad’s distribution function is described by fourteen basic fields. The relativistic Grad’s distribution function in terms of the fourteen fields of particle number density \( n \), four-velocity \( \bar{U} \), absolute temperature \( T \), dynamic pressure \( \varpi \), pressure deviator \( p^{(\mu\nu)} \) and heat flux \( q^{(\mu)} \) is (see e.g. \( [9] \))

\[
f = f^{(0)} \left[ 1 + q^{(\mu)} \frac{\zeta}{p} \left( \frac{G}{mc^2} \right) + \varpi \left( \frac{U_{\nu}p^{\nu}}{kT} \right) \right.
\]

\[
\left. + \frac{\varpi}{p} \left( \frac{1 - 5G\zeta - \zeta^2 + 2G^2\zeta^2}{20G + 3\zeta - 13G\zeta^2 - 2G^2\zeta^2 + 2G^3\zeta^2} \right) \left( \frac{U_{\mu}p^{\mu} - \frac{3\zeta}{mc^2} \left( 6G\zeta + \zeta - G^2\zeta \right)}{2G^2\zeta^3} \right) \right]
\]

\[
\left. + \frac{\varpi}{p} \left( \frac{15G + 2\zeta - 6G^2\zeta + 5G^2 + \zeta^3 - G^2\zeta^3}{1 - 5G\zeta - \zeta^2 + 2G^2\zeta^2} \right) \right] \}, \tag{31}
\]

where \( f^{(0)} \) is the Maxwell-Jüttner distribution function \( [29] \) and \( G \) denotes the ratio of the Bessel functions \( G = K_3(\zeta)/K_2(\zeta) \).

The first post-Newtonian approximation of Grad’s distribution function \( f^{(1)} \) is obtained by taking into account the expressions for the hydrodynamic four-velocity \( \bar{U} \), particle four-velocity \( \V \), the asymptotic expression for the modified Bessel functions of second kind \( K_n(\zeta) \) for large values of \( \zeta \gg 1 \)

\[
K_n(\zeta) = \sqrt{\frac{\pi}{2\zeta \csc \zeta}} \left[ 1 + \frac{4n^2 - 1}{8\zeta} + \frac{(4n^2-1)(4n^2-9)}{2!(8\zeta)^2} + \ldots \right], \tag{32}
\]

and the relations for the components of the pressure deviator and heat flux, namely

\[
p^{(0)} = p^{(\mu\nu)} \frac{V_iV_j}{c^2} + \frac{2}{c^2} p^{(\mu\nu)} \frac{V_j}{c} (4UV_i - \Pi_i) + \mathcal{O}(c^{-6}), \tag{33}
\]

\[
p^{(1)} = \frac{p^{(\mu\nu)} V_j}{c} + p^{(\mu)} V_j \frac{c}{c^3} (4UV_i - \Pi_i) + \mathcal{O}(c^{-5}). \tag{34}
\]

\[
q = q^{(\mu)} \frac{V_i}{c} + q^{(\mu)} \frac{c}{c^3} (4UV_i - \Pi_i) + \mathcal{O}(c^{-5}). \tag{35}
\]

After a long calculation one can obtain the first post-Newtonian approximation of Grad’s distribution function

\[
f = f^{(0)} \left[ 1 + \left( \frac{m}{kT} \right)^2 \frac{V_iV_j}{2\rho} \left\{ p_{ij} \left[ 1 + \frac{c^2}{c^2} \frac{V^2 + V^2 + 2V_iV_j}{2} + 6U - \frac{5kT}{2m} \right] + \frac{p_{ik}V_kV_i + p_{jk}V_kV_j}{2c^2} \right\} 
\]

\[
+ \frac{\varpi}{p} \left( \frac{mc^2}{kT} - \frac{3m^2V^4}{2kT} \right) \left[ \frac{V_iV_j}{\rho} \frac{m}{kT} \right] \left\{ 1 - \frac{m}{5kT} \right\} + \frac{c^2}{c^2} \left[ \frac{V^2 + V^2 + 3U}{2} \right]
\]

\[
+ \frac{V_iV_j}{4m} - \frac{m}{20kT} (5V^4 + 4(V_jV_j)^2 + 6V^2V^2 + 28V^2U + 12(V_jV_j)V^2) \right\} \right]. \tag{36}
\]
In the above equations $f^{(0)}$ is the first post-Newtonian approximation for the Maxwell-Jüttner distribution function \[ f^{(0)} \]

By neglecting the $1/c^2$ terms \[ f^{(1)} \] reduces to well-known non-relativistic Grad’s distribution function \[ f^{(2)} \] and perform the integrations. Hence it

If we introduce in \[ f^{(3)} \] the post-Newtonian mass density \[ f^{(4)} \] and the invariant element of integration \[ f^{(5)} \] into the definition of the energy-momentum tensor \[ f^{(6)} \] and integrate the resulting equation by making use of the integrals of the appendix, we get the components of the energy-momentum tensor \[ f^{(7)} \]. For a complete identification one has to consider the equation of state $p = \rho kT/m$ and the equation for specific internal energy $\varepsilon = 3kT/2m$ of a monatomic gas.

Note that we have taken into account the underlined term in \[ f^{(8)} \] which refers to the dynamic pressure, but the contribution of this term to the energy-momentum tensor vanishes.

**IV. HYDRODYNAMIC EQUATIONS**

The determination of the hydrodynamic equations from a transfer equation derived from the Boltzmann equation is an old task in the literature of the kinetic theory of gases which goes back to the works of Maxwell \[ f^{(9)} \] and Enskog \[ f^{(10)} \]. Here we shall determine the so-called Maxwell-Enskog transfer equation in the first post-Newtonian approximation.

To this end we multiply the Boltzmann equation \[ f^{(11)} \] by an arbitrary function $\Psi(x, v, t)$ and integrate the resulting equation by taking into account the invariant element of integration \[ f^{(12)} \]. The post-Newtonian version of the Maxwell-Enskog transfer equation, reads

\[
\begin{align*}
\frac{\partial}{\partial t} \int \Psi \left[ 1 + \frac{1}{c^2} \left( \frac{5v^2}{2} + 7U \right) \right] f \, dv + \frac{\partial}{\partial x^i} \int \Psi v_i \left[ 1 + \frac{1}{c^2} \left( \frac{5v^2}{2} + 7U \right) \right] f \, dv \\
+ \frac{2}{c^2} \int \Psi \left[ \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x^i} v_i \right] f \, dv \\
- \frac{\partial U}{\partial x^i} \int \Psi v_i \left[ 1 + \frac{1}{c^2} \left( \frac{7}{2} v^2 + 3U \right) \right] f \, dv + \frac{1}{c^2} \int \Psi v_i \frac{\partial U}{\partial v^i} + 3v_i \frac{\partial U}{\partial t} \\
- v_j \left( \frac{\partial \Pi_j}{\partial x^i} - \frac{\partial \Pi_i}{\partial x^j} \right) - 2 \frac{\partial \Phi}{\partial x^i} \frac{\partial \Pi_i}{\partial t} f \, dv = \int \Psi \left[ 1 + \frac{1}{c^2} \left( 2v^2 + 6U \right) \right] \mathcal{Q}(f, f) \, dv. \\
\end{align*}
\]

Now the hydrodynamic equations can be obtained from the transfer equation \[ f^{(13)} \] by choosing values of the arbitrary function $\Psi(x, v, t)$ and integration of the resulting equations.

We begin by determining the mass density balance equation and for that end we choose $\Psi = m^4$ in \[ f^{(14)} \], take into account Grad’s distribution function \[ f^{(15)} \] and perform the integrations. Hence it follows

\[
\begin{align*}
\frac{\partial}{\partial t} \rho \left[ 1 + \frac{1}{c^2} \left( \frac{V^2}{2} + U \right) \right] + \frac{\partial}{\partial x^i} \rho v_i \left[ 1 + \frac{1}{c^2} \left( \frac{V^2}{2} + U \right) \right] = -2 \frac{\rho}{c^2} \left( \frac{\partial U}{\partial t} + v_i \frac{\partial U}{\partial x^i} \right) \\
= -2 \frac{\rho}{c^2} \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho V_i}{\partial x^i} \right) + \frac{2U}{c^2} \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho V_i}{\partial x^i} \right),
\end{align*}
\]

by rearranging the last term. For the underlined term we can use the Newtonian approximation of the continuity equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x^i} = 0.
\]

If we introduce in \[ f^{(16)} \] the post-Newtonian mass density \[ f^{(17)} \]

\[
\rho^* = \rho \left[ 1 + \frac{1}{c^2} \left( \frac{V^2}{2} + 3U \right) \right],
\]

we get the continuity equation for the mass density $\rho^*$ in the post-Newtonian approximation

\[
\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* v_i}{\partial x^i} = 0.
\]
As it should be, the above equation corresponds to the post-Newtonian balance equation of the mass density for a perfect fluid \[^2,13\].

The mass-energy density hydrodynamic equation is obtained by choosing \(\Psi = m^4 u^i\) in the transfer equation \[^35\] by considering Grad’s distribution function \[^36\] and by integrating the resulting equation, yielding

\[
\frac{\partial}{\partial t} \left( \frac{1}{c^2} V^2 + 2U + \frac{3kT}{2m} \right) \rho V_i \left[ 1 + \frac{1}{c^2} \left( V^2 + 2U + \frac{5kT}{2m} \right) \right]
\]

\[
+ \rho \frac{\partial U}{\partial t} + \frac{1}{c^2} \left( \frac{\partial p_{ij} V_i}{\partial x^i} + \frac{\partial q_i}{\partial x^i} \right) = 0.
\]

(43)

Here we follow Chandrasekhar \[^2\] and introduce the expression for the post-Newtonian mass-energy density

\[
\sigma = \rho \left[ 1 + \frac{1}{c^2} \left( V^2 + 2U + \frac{5kT}{2m} \right) \right] = \rho \left[ 1 + \frac{1}{c^2} \left( V^2 + 2U + \varepsilon + \frac{\rho}{\rho} \right) \right],
\]

so that (43) can be rewritten as

\[
\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma V_i}{\partial x^i} + \frac{1}{c^2} \left( \frac{\rho U}{\partial t} - \frac{\partial p}{\partial t} \right) + \frac{1}{c^2} \left( \frac{\partial p_{ij} V_i}{\partial x^i} + \frac{\partial q_i}{\partial x^i} \right) = 0.
\]

(45)

Without the new contributions of the pressure deviator \(p_{ij}\) and heat flux \(q_i\), (45) corresponds to the balance equation for the mass-energy density of a perfect fluid in the post-Newtonian approximation \[^2\].

For the hydrodynamic equation of the momentum density we choose \(\Psi = m^4 u^i\) in the transfer equation \[^35\] use Grad’s distribution function \[^36\], integrate the resulting equation and get

\[
\frac{\partial \sigma V_i}{\partial t} + \frac{\partial \sigma V_i}{\partial x^i} + \frac{\partial}{\partial x^i} \left[ \rho \left( 1 - 2U \right) \right] + \frac{\partial p_{ij}}{\partial x^i} \left( 1 + \frac{1}{c^2} \left( 2V^2 + \varepsilon - 2U - \frac{\rho}{\rho} \right) \right)
\]

\[
+ 4 \frac{p_{ij}}{c^2} \frac{\partial U}{\partial x^i} - \frac{\partial \rho}{\partial x^i} \frac{\partial U}{\partial x^i} + \frac{4}{c^2} \frac{\partial V_i}{\partial x^i} + \frac{\partial \rho}{\partial x^i} \frac{\partial U}{\partial x^i} - \frac{\rho}{c^2} \frac{\partial \Phi}{\partial t} \left( \frac{\partial \Pi_i}{\partial x^i} + \frac{\partial \Pi_j}{\partial x^i} \right)
\]

\[
- \frac{1}{c^2} \frac{\partial (p_{ij} V_j + q_i)}{\partial t} + \frac{1}{c^2} \frac{\partial q_i}{\partial x^i} \left[ q_i V_j + q_j V_i \right] + \frac{(p_{ik} V_j + p_{jk} V_i) V_k}{2} = 0.
\]

(46)

Without the dissipative terms \(p_{ik}\) and \(q_i\), the above equation corresponds to eq. (68) of \[^2\].

The momentum density hydrodynamic equation \[^10\] can be rewritten by taking into account the mass-energy hydrodynamic equation \[^45\] as

\[
\rho \frac{dV_i}{dt} + \frac{\partial (p_{ij} + \rho q_{ij})}{\partial x^i} \left[ 1 - \frac{1}{c^2} \left( V^2 + 4U + \varepsilon + \frac{\rho}{\rho} \right) \right] - \rho \frac{\partial U}{\partial x^i} \left[ 1 + \frac{1}{c^2} \left( V^2 - 4U \right) \right]
\]

\[
- \rho \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x^i} + 4 \frac{\partial U}{\partial x^i} - \frac{2}{c^2} \frac{\partial \Phi}{\partial x^i} + \frac{\partial \Pi_i}{\partial x^i} \left[ 1 + \frac{1}{c^2} \left( p_{ij} V_i + q_i \right) \right] + \frac{1}{c^2} \frac{\partial (q_i V_j + q_j V_i)}{\partial x^i} - \frac{U V_i}{c^2} \left( \frac{\partial p_{jk} V_k}{\partial x^i} + \frac{\partial q_j}{\partial x^i} \right) = 0.
\]

(47)

Here we have introduced the material time derivative \(d/dt = \partial/\partial t + V_i \partial/\partial x^i\).

By neglecting all terms of \(\mathcal{O}(c^{-2})\) order it follows the Newtonian momentum density hydrodynamic equation

\[
\rho \frac{dV_i}{dt} + \frac{\partial (p_{ij} + \rho q_{ij})}{\partial x^i} - \rho \frac{\partial U}{\partial x^i} = 0.
\]

(48)

From the subtraction of the mass density hydrodynamic equation \[^42\] from the mass-energy hydrodynamic equation \[^45\] one can obtain the total energy density hydrodynamic equation, which is a sum of the internal \(\rho \varepsilon\) and kinetic \(\rho V^2/2\) energy densities, namely

\[
\frac{1}{c^2} \left( \frac{\partial}{\partial t} \left( \rho \left( \frac{V^2}{2} + \varepsilon \right) \right) + \frac{\partial}{\partial x^i} \left( \rho \left( \frac{V^2}{2} + \varepsilon \right) V_i \right) - \rho V_i \frac{\partial U}{\partial x^i} + \frac{\partial [p_{ij} V_j + q_i + p V_i]}{\partial x^i} \right)
\]

\[
+ U \left( \frac{\partial p}{\partial t} + \frac{\partial p V_i}{\partial x^i} \right) = 0.
\]

(49)
Here we note that this equation is of order $\mathcal{O}(e^{-2})$ and we may use the Newtonian continuity equation (40) for the underlined term so that (49) reduces to the Newtonian total energy density hydrodynamic equation for a viscous and heat conducting fluid

$$\frac{\partial}{\partial t} \left[ \rho \left( \frac{V^2}{2} + \varepsilon \right) \right] + \frac{\partial}{\partial x^i} \left[ \rho \left( \frac{V^2}{2} + \varepsilon \right) V_i \right] - \rho V_i \frac{\partial U}{\partial x^i} + \frac{\partial}{\partial x^i} \left[ p_{ij} V_j + q_i + p V_i \right] = 0. \quad (50)$$

We call attention to the fact that the post-Newtonian contributions to this equation do not show up. As was pointed out by Chandrasekhar [2, 19] the first post-Newtonian contributions to the total energy density are obtained from the knowledge of the second post-Newtonian contributions to the mass-energy hydrodynamic equation. In order to obtain these contributions here we have to determine Grad’s distribution function in the second post-Newtonian approximation which is a heavy task and will be subject of a future work. The second post-Newtonian approximation to the Boltzmann equation and for the Maxwell-Jüttner distribution function were determined in [2, 3].

The internal energy density hydrodynamic equation follows from the elimination of the time derivative of the hydrodynamic velocity from (50) by using the Newtonian momentum density hydrodynamic equation (45), yielding

$$\rho \frac{d\varepsilon}{dt} + \frac{\partial q_i}{\partial x^i} + [p_{ij} + p\delta_{ij}] \frac{\partial V_i}{\partial x^i} = 0. \quad (51)$$

This equation refers to the well-known Newtonian internal energy density hydrodynamic equation for a viscous and heat-conducting fluid.

The hydrodynamic equations derived from the transfer equation (58) for the mass, mass-energy and momentum densities are the same as those which follows from the conservation laws for the particle four-flow (21) and energy-momentum tensor (22) (see [6]).

Although the dynamic pressure appears in Grad’s distribution function (50) it does not participate in the Newtonian and first post-Newtonian hydrodynamic equations. Furthermore, the collision term in the transfer equation vanishes, since mass, momentum and energy of a particle are conservative quantities at collision.

V. CONCLUSIONS

In this work the post-Newtonian energy-momentum tensor for a non-perfect gas described by the Eckart decomposition was determined. From the post-Newtonian Boltzmann equation a transfer equation was derived as well as the post-Newtonian expression for the relativistic Grad’s distribution function. From the knowledge of the post-Newtonian transfer equation and Grad’s distribution function the hydrodynamic equations for the mass density, mass-energy density and momentum density were determined which show the contributions of the viscous stress and heat conduction. The non-relativistic limiting case of these equations lead to the well-known Newtonian hydrodynamic equations for the mass, momentum and energy densities.

APPENDIX

For the integration of the equations in the previous sections we have used the following well-known integrals from the kinetic theory of gases (see e.g. [21])

$$I_n = \int V^n e^{-\frac{mV^2}{2m}} dV = \frac{1}{2} \Gamma \left( \frac{n+1}{2} \right) \left( \frac{kT}{m} \right)^{\frac{n+1}{2}} \Gamma(n+1) = n\Gamma(n), \quad \Gamma(1) = 1, \quad \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi},$$

$$\int e^{-\frac{mV^2}{2m}} V_i V_j d^3V = \frac{I_{23}}{3} \delta_{ij}, \quad \int e^{-\frac{mV^2}{2m}} V_i V_j V_k V_l d^4V = \frac{I_4}{15} \left[ \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right],$$

$$\int e^{-\frac{mV^2}{2m}} V_i V_j V_k V_l V_m d^5V = \frac{I_5}{105} \left[ \delta_{ij} \left( \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm} + \delta_{kl} \delta_{mn} \right) + \delta_{ik} \left( \delta_{jl} \delta_{mn} + \delta_{jm} \delta_{ln} + \delta_{jn} \delta_{kl} \right) \right] + \delta_{il} \left( \delta_{jk} \delta_{mn} + \delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km} \right) + \delta_{jm} \left( \delta_{ik} \delta_{ln} + \delta_{il} \delta_{kn} + \delta_{jn} \delta_{kl} \right) + \delta_{kn} \left( \delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{jn} \delta_{kl} \right) \right].$$
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