C*-ALGEBRAS ASSOCIATED WITH ENDOmorphisms AND POLyMORphisms OF COMPACT ABELIAN GROUPS

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Abstract. A surjective endomorphism or, more generally, a polymorphism in the sense of [23], of a compact abelian group $H$ induces a transformation of $L^2(H)$. We study the C*-algebra generated by this operator together with the algebra of continuous functions $C(H)$ which acts as multiplication operators on $L^2(H)$. Under a natural condition on the endo- or polymorphism, this algebra is simple and can be described by generators and relations. In the case of an endomorphism it is always purely infinite, while for a polymorphism in the class we consider, it is either purely infinite or has a unique trace. We prove a formula allowing to determine the $K$-theory of these algebras and use it to compute the $K$-groups in a number of interesting examples.

1. Introduction

Let $H$ be a compact abelian group. Let $\alpha$ be an automorphism, a surjective endomorphism or an algebraic polymorphism (see below) of $H$. For Haar measure on $H$, the transformation $\alpha$ will define an operator $s_\alpha$ on the Hilbert space $L^2H$. This operator will be unitary, isometric or a Markov operator, respectively. We can form the C*-algebra $C^*(s_\alpha,C(H))$ generated in $\mathcal{L}(L^2H)$ by $s_\alpha$ together with $C(H)$ acting as multiplication operators on $L^2H$. The case where $\alpha$ is an automorphism is classical. In this case $s_\alpha$ is unitary and $C^*(s_\alpha,C(H))$ is essentially the crossed product by $\alpha$. The study of the case where $\alpha$ is an endomorphism was started in the 70s and by now there is quite some literature. The best known example is the case where $\alpha$ is the shift on $H = \prod_{k \in \mathbb{N}} \mathbb{Z}/n$. The corresponding C*-algebra is $O_n$, [4]. This case was also considered in [1] from the point of view of W*-algebras and factor representations.

The notion of a general measure preserving polymorphism was suggested in [26],[24]. The case of algebraic polymorphisms was discussed in [23]. The question of studying the C*-algebra corresponding to a polymorphism has hardly been touched upon so far, cf. however [11] where C*-algebras associated with such polymorphisms have been discussed in the setting of an associated groupoid.

In the present paper we analyze the structure of such C*-algebras and study their $K$-theory. We start with the case of an algebraic endomorphism $\alpha$ of the compact...
abelian group $H$. We assume that $\alpha$ is surjective with finite kernel and exact (i.e. the union of the kernels of the $\alpha^n$ is dense, see section 2).

In fact, to study $C^*(s_\alpha, C(H))$, most of the time it is more useful to work, rather than with $\alpha$, with the dual endomorphism $\phi = \hat{\alpha}$ of the dual group $G = \hat{H}$. Fourier transform transforms $C^*(s_\alpha, C(H))$ isomorphically into the C*-algebra $\mathfrak{A}[\phi]$ acting on $\ell^2G$. As in [16] and [8], but still somewhat surprisingly, this C*-algebra $\mathfrak{A}[\phi]$ which is originally defined by a concrete representation, can also be characterized as a universal algebra given by generators and relations. The structure of $\mathfrak{A}[\phi]$ (and thus of $C^*(s_\alpha, C(H))$) is governed by two “complementary” maximal abelian subalgebras, one being the algebra of continuous functions on $H$, the other one the algebra of continuous functions on a compactification (with respect to $\phi$) of $G$. The algebras $\mathfrak{A}[\phi]$ all have a similar structure, in particular they are simple, nuclear and purely infinite. They therefore belong to a very well understood class of C*-algebras. In particular by the Kirchberg-Phillips classification [18, 20], they are completely determined by their $K$-theory.

In section 3 we derive a Pimsner-Voiculescu type formula that can be used to determine the $K$-theory of $\mathfrak{A}[\phi]$. We prove that there is an exact sequence of the form

\[ K_*(C(H)) \xrightarrow{1-b(\phi)} K_*(C(H)) \xrightarrow{\phi} K_*\mathfrak{A}[\phi] \]

We should point out that $b(\phi)$ is not simply the map induced by $\phi$, even though it is related to this map by a simple equation. The determination of $b(\phi)$ in specific examples sometimes requires extra work. Since $K_*(C(H))$ is always torsion-free, the exact sequence (1) is particularly useful for computations. We use it to explicitly determine the $K$-theory of $\mathfrak{A}[\phi]$ for several examples, including the case of endomorphisms of $\mathbb{T}^n$, of $\prod_k \mathbb{Z}/p$ and of a solenoid group.

Algebras such as those in section 2 have been studied by quite a few authors. The simplicity of the algebra $\mathfrak{A}[\phi]$ and its description as a universal algebra has been established already in [16] even in a more general setting. Constructions along the same lines are considered in the thesis of F.Vieira, [27]. As pointed out to us by R.Exel, the simplicity of $\mathfrak{A}[\phi]$ could also be established using an approach as in [13] and [12]. One virtue of our approach here is its simplicity together with the fact that it reveals interesting structural properties of $\mathfrak{A}[\phi]$ and its canonical subalgebras.

Special cases of the algebra $\mathfrak{A}[\phi]$ for $H = \mathbb{T}^n$ or for $H = \prod_k \mathbb{Z}/p$ had also occurred before in [8, 9, 10], where again it was shown that in these examples $\mathfrak{A}[\phi]$ is purely infinite simple and its $K$-theory was partially computed. Our proof here that $\mathfrak{A}[\phi]$ is purely infinite simple is very similar to that in [8]. For the case of an endomorphism of $\mathbb{T}^n$, an algebra which is easily seen to be isomorphic to $\mathfrak{A}[\phi]$ has been described as a Cuntz-Pimsner algebra in [14], using Exel’s concept of a transfer operator [13]. In this paper, it was also proved that, for an expansive endomorphism of $\mathbb{T}^n$, the algebra is simple purely infinite and its $K$-theory was determined (using Pimsner’s extension which leads to a sequence similar to (1)). Our computation of the $K$-theory is somewhat simpler and more general.
Let us now turn to the case of an algebraic polymorphism of the compact abelian group $H$. This is a multivalued map determined typically by a pair of endomorphisms. We will restrict ourselves to the case of what we call rational polymorphisms. We introduce a concept of independence (motivated by the notion of relatively prime principal ideals in number theory) for two commuting surjective endomorphisms of a compact abelian group. This concept is interesting in itself. It leads to hyperfiniteness of the orbit equivalence relation for the semigroup generated by the two endomorphisms (and to nuclearity of the $C^*$-algebra). We discuss this point in section 6. The concept of independence could also be generalized to more general commuting pairs of not necessarily algebraic endomorphisms of a suitable measure space.

A rational polymorphism is then roughly speaking a product of $\alpha$ by the “inverse” of $\beta$ for two commuting and independent endomorphisms $\alpha$ and $\beta$ of the compact abelian group $H$. In [23] a Markov operator on $L^2(H)$ was associated with an algebraic polymorphism. This operator fits very well with the construction mentioned above. For a rational polymorphism it is a partial isometry. We can then generalize the construction and analysis of the algebra $A[\varphi]$ to a similar construction of an algebra $A[\varphi/\psi]$ associated with a rational polymorphism induced by a pair of commuting and independent endomorphisms (actually below $\varphi$ and $\psi$ will denote the dual endomorphisms of the dual group). Again, these algebras can be characterized by generators and relations and are simple and nuclear. They are purely infinite if the kernels of the given endomorphisms of the compact group $H$ do not have the same number of elements. If these kernels have the same size, then $A[\varphi/\psi]$ has a unique trace.

We also study the $K$-theory of $A[\varphi/\psi]$. By an argument similar to the case of a single endomorphism, we show that the $K$-groups satisfy an exact sequence of Pimsner-Voiculescu type but with somewhat more complicated ingredients. In particular the map $1 - b(\varphi)$ in formula (1) is replaced by $b(\psi) - b(\varphi)$ where $b(\psi)$ is the map corresponding to the second endomorphism $\psi$. Again, this sequence often suffices to explicitly compute the $K$-theory of $A[\varphi/\psi]$.

In all our arguments concerning the structure and the $K$-theory of $A[\varphi/\psi]$, the independence of the pair $(\varphi, \psi)$ plays a crucial role.

2. THE ALGEBRA ASSOCIATED WITH AN ENDOMORPHISM OF A COMPACT ABELIAN GROUP

Let $H$ be a compact abelian group and $G = \hat{H}$ its dual discrete group. We usually denote the group operation on $H$ by multiplication with neutral element 1 and on $G$ by addition with neutral element 0. We also assume that $G$ is countable. Let $\alpha$ be a surjective endomorphism of $H$ with finite kernel. We denote by $\varphi$ the dual endomorphism $\chi \mapsto \chi \circ \varphi$ of $G$ (i.e. $\varphi = \hat{\alpha}$). By duality, $\varphi$ is injective and has finite cokernel, i.e. the quotient $G/\varphi G$ will be finite. Both $\alpha$ and $\varphi$ induce isometric endomorphisms $s_\alpha$ and $s_\varphi$ of the Hilbert spaces $L^2H$ and $\ell^2G$, respectively.
We will also assume that
\[ \bigcap_{n \in \mathbb{N}} \varphi^n G = \{0\} \]
which, by duality, means that
\[ \bigcup_{n \in \mathbb{N}} \text{Ker} \alpha^n \]
is dense in \( H \) (this implies in particular that \( H, G \) can not be finite).

**Remark 2.1.** In ergodic theory a measure preserving endomorphism \( T : X \to X \) is called *exact* if it has the property that \( \bigcap_{n \in \mathbb{N}} T^{-n}(\mathcal{M}) = \mathcal{N} \) where \( \mathcal{M} \) denotes the sigma-algebra of all measurable sets while \( \mathcal{N} \) is the trivial sigma-algebra of sets of measure 0 or 1. For an algebraic endomorphism of the compact group \( H \) this condition means exactly that the subgroup \( \bigcup_{n \in \mathbb{N}} \text{Ker} \alpha^n \) is dense in \( H \). Exact endomorphisms of \( H \) can be characterized as those endomorphisms that have no quotient factor automorphisms. Among all endomorphisms the exact ones are generic and are the most interesting ones.

The standing assumption for the rest of the paper will be that \( \alpha \) is a surjective endomorphism of the compact abelian group \( H \) with non-trivial finite kernel, satisfying the exactness condition of Remark 2.1 (i.e. the union of the kernels of \( \alpha^n \) is dense), or equivalently, that the dual endomorphism \( \varphi \) of the dual group \( G \) is injective with \( 1 < |G/\varphi(G)| < \infty \) and \( \bigcap_{n \in \mathbb{N}} \varphi^n(G) = \{0\} \).

In this section we are going to describe the C*-algebra \( C^*(s_\alpha, C(K)) \) generated in \( L(L^2H) \) by \( C(K) \), acting by multiplication operators, and by the isometry \( s_\alpha \). Via Fourier transform it is isomorphic to the C*-algebra \( C^*(s_\varphi, C^*G) \) generated in \( L(\ell^2G) \) by \( C^*G \), acting via the left regular representation, and by the isometry \( s_\varphi \). These two unitarily equivalent representations are useful for different purposes.

**Remark 2.2.** It seems to be interesting to replace in this construction Haar measure on \( H \) by another \( \alpha \)-invariant measure \( \mu \) and to consider then the C*-algebra of operators on \( L^2(H, \mu) \) generated by \( s_\alpha \) and \( C(K) \). In this way one would obtain for instance algebras analogous to the algebras \( O_A \) introduced in [7].

\( C^*(s_\varphi, C^*G) \) is generated by an isometry \( s = s_\varphi \) together with unitary operators \( u_g, g \in G \), satisfying the relations

\[ u_g u_h = u_{gh} \quad su_g = u_{\varphi(g)} s \quad \sum_{g \in G/\varphi G} u_g s s^* u_g^* = 1 \]

Even though the analysis of the structure of \( C^*(s_\varphi, C^*G) \) is essentially a straightforward generalization of constructions in [6] and [8], it is interesting enough to merit a separate discussion.

**Definition 2.3.** Let \( H, G \) and \( \alpha, \varphi \) be as above. We denote by \( \mathfrak{A}[\varphi] \) the universal C*-algebra generated by an isometry \( s \) and unitary operators \( u_g, g \in G \) satisfying the relations (2).

We will show that \( \mathfrak{A}[\varphi] \cong C^*(s_\alpha, C(H)) \cong C^*(s_\varphi, C^*G) \).
Lemma 2.4. The C*-subalgebra $D$ of $\mathfrak{A}[\varphi]$ generated by all projections of the form $u_g s^n s^m u_g^*$, $g \in G$, $n, m \in \mathbb{N}$ is commutative. Its spectrum is the “$\varphi$-adic completion”

$$G_\varphi = \lim_{\leftarrow n} G/\varphi^n G$$

It is an inverse limit of the finite spaces $G/\varphi^n G$ and becomes a Cantor space with the natural topology.

$G$ acts on $D$ via $d \mapsto u_g d u_g^*$, $g \in G$, $d \in D$. This action corresponds to the natural action of the dense subgroup $G$ on its completion $G_\varphi$ via translation. The map $D \to D$ given by $x \mapsto sx^n s^m$ corresponds to the map induced by $\varphi$ on $G_\varphi$.

Proof. It is easily checked that the maps $C^*\{u_g s^n s^m u_g^* : g \in G\} \to C(G/\varphi^n(G))$, $n \in \mathbb{N}$ that map $u_g s^n s^m u_g^*$ to the characteristic function of the one point set $\{\varphi^n(G) + g\}$ extend to an isomorphism of the inductive limits with the asserted properties (this is basically the same situation as in [6] or in [8]).

From now on we will denote the compact abelian group $G_\varphi$ by $K$. By construction, $G$ is a dense subgroup of $K$. The dual group of $K$ is the discrete abelian group

$$L = \lim_{\to\varphi_n} \ker (\alpha^n : H \to H)$$

Because of the condition that we impose on $\alpha$, $L$ can be considered as a dense subgroup of $H$.

The groups $K$ and $L$ will play an important role in the analysis of $\mathfrak{A}[\varphi]$. They are in a sense complementary to $H$ and $G$. By Lemma 2.4, the C*-algebra $D$ is isomorphic to $C(K)$ and to $C^*(L)$.

Lemma 2.5. The C*-subalgebra $B_\varphi$ of $\mathfrak{A}[\varphi]$ generated by $C(H)$ together with $C(K)$ (or equivalently by $C^*G$ together with $C^*L$) is isomorphic to the crossed product $C(K) \rtimes G$. It is simple and has a unique trace.

Proof. The action of the dense subgroup $G$ by translation on $K$ is obviously minimal (every orbit is dense). Therefore the crossed product $C(K) \rtimes G$ is simple. It also has a unique trace, the Haar measure on $K$ being the only invariant measure. The fact that an invariant measure on $K$ extends uniquely to a trace on the crossed product, if all the stabilizer groups are trivial, is well known, but not easy to pin down in the literature. Here is a very simple argument in the present case: Let $E : C(K) \rtimes G \to C(K)$ be the canonical conditional expectation and let $e_1^{(n)}, e_2^{(n)}, \ldots, e_{N(\varphi^n)}^{(n)}$, with $N(\varphi^n) = |G/\varphi^n(G)|$, be the minimal projections in $C(G/\varphi^n(G)) \subseteq C(K)$. Then, for any $x$ in the crossed product, $\sum_{i=1}^{N(\varphi^n)} e_i^{(n)} x e_i^{(n)}$ converges to $E(x)$ for $n \to \infty$. For any trace $\tau$ on the crossed product, we have

$$\tau(x) = \tau\left(\sum_{i=1}^{N(\varphi^n)} e_i^{(n)} x e_i^{(n)}\right) = \tau\left(\sum_{i=1}^{N(\varphi^n)} e_i^{(n)} x e_i^{(n)}\right)$$

and therefore $\tau(x) = \tau(E(x))$.

Finally, by Lemma 2.4, $B_\varphi$ is generated by a covariant representation of the system $(C(K), G)$. The induced surjective map $C(K) \rtimes G \to B_\varphi$ has to be injective, thus an isomorphism.
The simple algebra $B_{\varphi}$ has a very interesting structure. In particular it can be represented as a crossed product in different ways. To see this, note first that $C(K) \rtimes G$ can also be written as $C^*(L) \rtimes G$ ($L$ being the dual group of $K$). The action of $G$ on $C^*(L)$, dual to the action of $G$ on $C(K)$ by translation, is determined by the commutation relation

\begin{equation}
    u_g w_l = \langle g | l \rangle w_l u_g \quad g \in G, \; l \in L
\end{equation}

Here we denote the unitary generators of $C^*(G)$ and $C^*(L)$ by $u_g, \; w_l$, respectively and use the embedding of $L$ into the dual group $H$ of $G$ to obtain the pairing between $G$ and $L$. Therefore $B_{\varphi}$ can also be described as the universal C*-algebra generated by unitary representations $g \mapsto u_g$ and $l \mapsto w_l$ of $G$ and $L$ satisfying the commutation relation (3). In particular, the relation (3) can also be interpreted as an action of $L$ on $C^*(G)$ and we see that $B_{\varphi}$ is also isomorphic to the crossed product $C^*(G) \rtimes L$.

Now similarly the action of $L$ on $C^*(G)$ determined by the commutation relation (3) corresponds to the action of $L$ on $C(H)$ by translation, under duality. Therefore we get the following intriguing chain of isomorphisms

\[
    B_{\varphi} \cong C(K) \rtimes G \cong C^* L \rtimes G \cong C^* G \rtimes L \cong C(H) \rtimes L
\]

This chain of isomorphisms is similar - though easier - to the duality result in [9].

The map $x \mapsto sx s^*$ defines a natural endomorphism $\gamma_{\varphi}$ of $B_{\varphi}$.

**Theorem 2.6.** The algebra $A[\varphi]$ is simple, nuclear and purely infinite. Moreover, it is isomorphic to the semigroup crossed product $B_{\varphi} \rtimes_{\gamma_{\varphi}} \mathbb{N}$ (i.e. to the universal unital C*-algebra generated by $B_{\varphi}$ together with an isometry $t$ such that $t x t^* = \gamma_{\varphi}(x), \; x \in B_{\varphi}$).

**Proof.** $A[\varphi]$ contains $B_{\varphi}$ as a unital subalgebra. The condition $\alpha_{\lambda}(s) = \lambda s, \; \alpha_{\lambda}(b) = b, \; b \in B_{\varphi}$ defines for each $\lambda \in \mathbb{T}$ an automorphism of $A$ and integration of $\alpha_{\lambda}(x)$ over $\mathbb{T}$ determines a faithful conditional expectation $A \to B_{\varphi}$. The proof now is very similar to the corresponding proof in [8]. The representation as a crossed product $C(K) \rtimes G$ of $B_{\varphi}$ gives a natural faithful conditional expectation $B_{\varphi} \to D \cong C(K)$. The composition of these expectations gives a faithful conditional expectation $E : A[\varphi] \to D$.

Now, this expectation can be represented in a different way using only the internal structure of $A[\varphi]$. The relations (2) immediately show that the linear combinations of elements of the form $z = s^n d u_g s^m, \; n, \; m \in \mathbb{N}, \; g \in G, \; d \in D$ are dense in $A[\varphi]$. For such an element $z$ we have $E(z) = s^n d s^n$ if $n = m, \; g = 0$, and $E(z) = 0$ otherwise. The subalgebra $D$ of $A[\varphi]$ is the inductive limit of the finite-dimensional subalgebras $D_n \cong C(G/\varphi^n G)$. Note that the minimal projections in $D_n$ are all of the form $u_g s^o s^n u_g^*$. Let

\[
    z = d + \sum_{i=1}^{m} s^{k_i} d_i u_{g_i} s^{l_i}
\]
be an element of $\mathfrak{A}[\varphi]$ such that for each $i$, $k_i \neq l_i$ or $g_i \neq e$ and such that $d, d_i \in \mathcal{D}_n$ for some large $n$ (such elements are dense in $\mathfrak{A}[\varphi]$).

Let also $n$ be large enough so that the projections $u_d, cu_d^*$, $i = 1, \ldots, m$, are pairwise orthogonal for each minimal projection $e$ in $\mathcal{D}_n$ (this means that the $g_i$ are pairwise distinct mod $\varphi^n(G)$).

We have $E(z) = d$ and there is a minimal projection $e$ in $\mathcal{D}_n$ such that $E(z)/e = \lambda e$ with $|\lambda| = |E(z)||$. Since $E$ is faithful, $\lambda > 0$ if $z$ is positive $\neq 0$.

Let $h' \in \mathcal{G}$ such that $e = u_h s^n s^m u_h^*$ and the product $e s^{k_i} d_i u_d, s^i e$ is non-zero only if $g_i \varphi^i(h) = \varphi^i(\lambda)$ or $g_i = \varphi^i(h) \varphi^i(h)^{-1} \mod \varphi^n(G)$.

Let $f \in \varphi^n G$ such that $\varphi^i(f) \neq \varphi^i(f)$ for all $i$ for which $k_i \neq l_i$ (such an $f$ obviously exists) and let $k \geq 0$ such that $\varphi^i(f) \neq \varphi^i(f) \mod \varphi^n G$ for those $i$.

Then, setting $h' = hf$, we obtain

$$g_i \varphi^i(h') \neq \varphi^i(h') \mod \varphi^n G, \quad i = 1, \ldots, m$$

If we now set $e' = u_h s^n s^m u_h^*$, then $e'$ is a minimal projection in $\mathcal{D}_{n+k}$, $e' \leq e$ and $e' s^{k_i} d_i u_d, s^i e' = 0$ for $i = 1, \ldots, m$.

Every positive element $x \neq 0$ of $\mathfrak{A}[\varphi]$ can be approximated up to an arbitrary $\varepsilon$ by a positive element $z$ as above. Thus, if $\varepsilon$ is small enough, $e' x e'$ is close to $\lambda e'$ and therefore invertible in $e' \mathfrak{A}[\varphi] e'$. Thus the product $s^{-k_i} d_i u_d, s^i e'$ is invertible in $\mathfrak{A}[\varphi]$. This shows, at the same time, that $\mathfrak{A}[\varphi]$ is purely infinite and simple. Moreover, it follows that the natural map from $\mathfrak{A}[\varphi]$ to the semigroup crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$, is an isomorphism. The fact that this crossed product is nuclear $(B_\varphi)$ is nuclear and, using a standard dilation, $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$ is Morita equivalent to a crossed product $B_\varphi^\infty \rtimes_{\gamma_\varphi} \mathbb{Z}$, where $B_\varphi^\infty$ is nuclear) then shows that $\mathfrak{A}[\varphi]$ is nuclear.

Corollary 2.7. The natural map induces an isomorphism $\mathfrak{A}[\varphi] \cong C^*(s_\varphi, C^*G) \cong C^*(s, C(K))$.

Proof. Since $C^*(s_\varphi, C^*G)$ is generated by elements satisfying the relations \cite{2}, there is a natural surjective map $\mathfrak{A}[\varphi] \rightarrow C^*(s_\varphi, C^*G) \cong C^*(s, C(K))$. By simplicity of $\mathfrak{A}[\varphi]$ this map has to be injective. \qed

2.1. Examples. Let us now look at a number of examples.

2.1.1. Let $H = \prod_{k \in \mathbb{N}} \mathbb{Z}/n$, $G = \bigoplus_{k \in \mathbb{N}} \mathbb{Z}/n$ and $\alpha$ the one-sided shift on $H$ defined by $\alpha((a_k)) = (a_{k+1})$. We obtain $K = \prod_{k \in \mathbb{N}} \mathbb{Z}/n = H$ and $L = \bigoplus_{k \in \mathbb{N}} \mathbb{Z}/n \cong G$. The algebra $B_\varphi$ is a UHF-algebra of type $n^\infty$ and $\mathfrak{A}[\varphi]$ is isomorphic to $O_n$. It is interesting to note that the UHF-algebra $B_\varphi$ is generated by two maximal abelian subalgebras both isomorphic to $C(K)$.

2.1.2. Let $H = \mathbb{T}$, $G = \mathbb{Z}$ and $\alpha$ the endomorphism of $H$ defined by $\alpha(z) = z^n$. The algebra $B_\varphi$ is a Bunce-Deddens-algebra of type $n^\infty$ and $\mathfrak{A}[\varphi]$ is isomorphic to a natural subalgebra of the algebra $Q_\mathbb{N}$ considered in \cite{5}. In this case, we also get for $B_\varphi$ the interesting isomorphism $C(\mathbb{Z}_n) \rtimes \mathbb{Z} \cong C(\mathbb{T}) \rtimes L$ where $\mathbb{Z}$ acts on the Cantor space $\mathbb{Z}_n$ by the odometer action (addition of 1) and $L$ denotes the subgroup of $\mathbb{T}$ given by all $n^k$-th roots of unity, acting on $\mathbb{T}$ by translation.
2.1.3. Let $H = \mathbb{T}^n$, $G = \mathbb{Z}^n$ and $\alpha$ an endomorphism of $H$ determined by an integral matrix $T$ with non-zero determinant. We assume that the condition

$$\bigcap_{n \in \mathbb{N}} \varphi^n G = \{0\}$$

is satisfied (this is in fact not very restrictive).

The algebra $B_\varphi$ is a higher-dimensional analogue of a Bunce-Deddens-algebra. In the case where $H$ is the additive group of the ring $R$ of algebraic integers in a number field of degree $n$ and the matrix $T$ corresponds to an element of $R$, the algebra $\mathfrak{A}[\varphi]$ is isomorphic to a natural subalgebra of the algebra $\mathfrak{A}[R]$ considered in [8]. It is also isomorphic to the algebra studied in [14].

2.1.4. Let $H = \mathbb{F}_p[[t]]$, $G = \mathbb{F}_p[t]$ and $\varphi$ an endomorphism of $G$ determined by multiplication by a non-zero element $P$ in the ring $\mathbb{F}_p[t]$. In the simplest case where $P = t$, we are back in the situation of 2.1.1. Then the algebra $B_\varphi$ is a UHF-algebra of type $p^\infty$ and $A[\varphi]$ is isomorphic to $C_p$. For endomorphisms $\alpha, \varphi$ induced by an arbitrary $P$, the algebra $A[\varphi]$ is naturally a subalgebra of the ring $\mathfrak{C}$-algebra considered in [10].

More generally, instead of multiplication by $P$, we could also consider an arbitrary $\mathbb{F}_p$-linear injective endomorphism $\varphi$ of $\mathbb{F}_p[t]$ with finite-dimensional cokernel and such that the intersection of the images of all powers of $\varphi$ is 0.

2.1.5. Let $p$ and $q$ be natural numbers that are relatively prime and $\gamma$ the endomorphism of $\mathbb{T}$ defined by $z \mapsto z^p$. We take

$$H = \lim_\leftarrow \mathbb{T} \quad G = \mathbb{Z}[\frac{1}{p}]$$

$\alpha_q$ the endomorphism of $H$ induced by $z \mapsto z^q$ and $\varphi_q$ the endomorphism of $G$ defined by $\varphi_q(x) = qx$. These endomorphisms satisfy our hypotheses ($G/\varphi^n G = \mathbb{Z}/q^n$ and $\bigcap \varphi^n G = \{0\}$). We find that $K = \mathbb{Z}_q$ (the $q$-adic completion of $\mathbb{Z}$).

3. Computation of the $K$-theory for $\mathfrak{A}[\varphi]$

In section 2 we had considered the natural subalgebra $B_\varphi = C(K) \rtimes G$ of $\mathfrak{A}[\varphi]$. Since, by definition,

$$K = \lim_\leftarrow G/\varphi^n G$$

we can represent $B_\varphi$ as an inductive limit $B = \lim \leftarrow B_n$ with $B_n = C(G/\varphi^n G) \rtimes G$. It is well known ("imprimitivity") that

$$C(G/\varphi^n G) \rtimes G \cong M_{N(\varphi)}(C(\varphi^n G))$$

where $N(\varphi) = |G/\varphi^n G|$. Consider the natural inclusion

$$C^* G \cong C^*(\varphi^n G) \longrightarrow M_{N(\varphi)}(C^*(\varphi^n G)) \cong B_n$$

into the upper left corner of $M_{N(\varphi)}$, considered as a map $C^* G \to B_n$. This map induces an isomorphism $\kappa_n : K_*(C^* G) \to K_*(B_n)$ in $K$-theory.
We let moreover \( \iota_n \) denote the map \( K_*(B_n) \to K_*(B_{n+1}) \) induced by the inclusion \( B_n \hookrightarrow B_{n+1} \) and define

\[
b(\varphi)_n : K_*(C^*(G)) \to K_*(C^*(G))
\]

by \( b(\varphi)_n = \kappa_{n+1}^{-1} \iota_n \kappa_n \).

Now, the commutative diagram

\[
\begin{array}{c}
C^*(G) \xrightarrow{\kappa_0} B_0 \xrightarrow{\cong} B_1 \xrightarrow{\cong} C^*(G) \\
\downarrow \kappa_n \downarrow \cong \downarrow \kappa_{n+1} \\
C^*(\varphi^n G) \xrightarrow{\cong} C(X_n) \ltimes \varphi^n G \\
\downarrow \kappa_n \downarrow \cong \\
B_n \xleftarrow{\cong} B_{n+1}
\end{array}
\]

with \( X_n = \varphi^n G/\varphi^{n+1} G \), shows that \( b(\varphi)_n = b(\varphi)_0 \) for all \( n \). We write \( b(\varphi) \) for this common map.

We obtain the following commutative diagram

\[
\begin{array}{c}
K_*(C^*(G)) \xrightarrow{b(\varphi)} K_*(C^*(G)) \xrightarrow{b(\varphi)} K_*(C^*(G)) \\
\downarrow \kappa_1 \downarrow \kappa_2 \\
K_*(B_0) \xrightarrow{\iota_0} K_*(B_1) \xrightarrow{\iota_1} K_*(B_2)
\end{array}
\]

One immediate consequence is the following formula for the \( K \)-theory of \( B_\varphi \)

\[
(4) \quad K_*(B_\varphi) = \lim_{b(\varphi)} K_*(C^*(G))
\]

We note however, that the problem remains to determine a suitable formula for the map \( b(\varphi) \), given a specific endomorphism \( \varphi \). We will consider a few examples below.

Since, by Theorem 2.6, \( \mathfrak{A}[\varphi] \) can be represented as a crossed product \( B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N} \), we are now in a position to derive a formula for the \( K \)-theory of \( \mathfrak{A}[\varphi] \).

**Theorem 3.1.** The \( K \)-groups of \( \mathfrak{A}[\varphi] \) fit into an exact sequence as follows

\[
\begin{array}{c}
K_0 C^*(G) \xrightarrow{1-b(\varphi)} K_0 C^*(G) \\
K_1 \mathfrak{A}[\varphi] \xleftarrow{1-b(\varphi)} K_1 C^*(G)
\end{array}
\]
Proof. From Theorem 2.6 we know that $\mathfrak{A}[\varphi]$ is isomorphic to the semigroup crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$. Using the Pimsner-Voiculescu sequence [21] in combination with a simple dilation argument as in [5] (or directly appealing to the results in [22] or in [17]) we see that there is an exact sequence

\begin{equation}
K_*B_\varphi \xrightarrow{1-\gamma_\varphi} K_*B_\varphi \rightarrow K_*\mathfrak{A}[\varphi]
\end{equation}

In order to determine the kernel and cokernel of the map $K_*B_\varphi \xrightarrow{1-\gamma_\varphi} K_*B_\varphi$, consider the commutative diagram

\[
\begin{array}{c}
K_*C^*(G) \xrightarrow{b(\varphi)} K_*C^*(G) \xrightarrow{b(\varphi)} K_*C^*(G) \\
\kappa_0 \downarrow \hspace{1cm} \kappa_1 \downarrow \hspace{1cm} \kappa_2 \\
K_*B_0 \xrightarrow{\iota_0} K_*B_1 \xrightarrow{\iota_1} K_*B_2 \\
\hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
K_*B_\varphi \xrightarrow{=} K_*B_\varphi \xrightarrow{=} K_*B_\varphi
\end{array}
\]

By construction, it is clear that $\gamma_{\varphi^*}^n = \kappa_n + 1$ (where we still denote the composition $K_*C^*(G) \xrightarrow{\kappa_n} K_*B_n \rightarrow K_*B_\varphi$ by $\kappa_n$). Let $\kappa$ denote the map (isomorphism)

$\kappa : \lim_{\rightarrow b(\varphi)} K_*C^*(G) \rightarrow K_*(B_\varphi)$

induced by the commutative diagram. For an element of the form $[x_0, x_1, \ldots]$ in the inductive limit we then obtain

$\gamma_{\varphi^*} \circ \kappa([x_0, x_1, x_2, \ldots]) = \kappa([a, x_0, x_1, \ldots])$

(where $a$ is arbitrary). Therefore the exact sequence (5) becomes isomorphic to

\begin{equation}
\lim_{\rightarrow b(\varphi)} K_*C^*(G) \xrightarrow{1-\sigma} \lim_{\rightarrow b(\varphi)} K_*C^*(G) \rightarrow K_*\mathfrak{A}[\varphi]
\end{equation}

where $\sigma$ is the shift defined by

$\sigma([x_0, x_1, x_2, \ldots]) = [a, x_0, x_1, \ldots]$

Consider the natural map $j : K_*C^*G \rightarrow \lim_{\rightarrow b(\varphi)} K_*C^*(G)$ defined by

$j(x) = [x, b(\varphi)(x), b(\varphi)^2(x), \ldots]$

If $(1-\sigma)[x_0, x_1, \ldots] = 0$, then there is $n$ such that $x_n = x_{n+1} = b(\varphi)(x_n)$ and thus $[x_0, x_1, \ldots] = [x_n, x_n, \ldots]$. This shows that $\text{Ker}(1-\sigma) = j(\text{Ker}(1-b(\varphi))) \cong \text{Ker}(1-b(\varphi))$. 

If we divide $\lim_{\to \ker(1 - \sigma)} K^*(G)$ by $\ker(1 - \sigma)$, then $[x_0, x_1, \ldots]$ becomes identified with $[x_1, x_2, \ldots]$ and thus to an element of the form $[x, b(\varphi)(x), \ldots]$ which is in the image of $j$. Also $j$ maps $\ker(1 - b(\varphi))$ to $\ker(1 - \sigma)$ and thus induces an isomorphism from the cokernel of $1 - b(\varphi)$ to the cokernel of $1 - \sigma$. This shows that $j$ induces a transformation from the sequence

$$K^*(G) \xrightarrow{1 - b(\varphi)} K^*(G) \xrightarrow{\varphi} \mathfrak{A}[\varphi]$$

into the exact sequence (2), which is an isomorphism on kernels and cokernels (in fact $j$ transforms $1 - b(\varphi)$ not into $1 - \sigma$ but into $1 - \sigma^{-1}$ - this however does not affect exactness).

There is a connection between $b(\varphi)$ and $\varphi_\ast$ which is described in the following Lemma.

**Lemma 3.2.** On $K^*(G)$ we have the identity $b(\varphi)\varphi_\ast = N(\varphi)\text{id}$ where $N(\varphi) = |G/\varphi(G)|$.

**Proof.** Under the identification $B_1 \cong M_{N(\varphi)}(C^*(G))$, the map $\iota_0 \kappa_0 \varphi_\ast$ is induced by the embedding of $C^*(G) \cong C^*(\varphi G)$ along the diagonal of $M_{N(\varphi)}(C^*(G))$. Therefore $\iota_0 \kappa_0 \varphi_\ast = N(\varphi)\kappa_1$ ($\kappa_1$ is induced by the embedding in the upper left corner). The assertion now follows from the definition of $b(\varphi)$ as $\kappa_1^{-1} \iota_0 \kappa_0$.

### 3.1. Examples.

**3.1.1.** Let $H = \prod_{k \in \mathbb{N}} \mathbb{Z}/n$, $G = \bigoplus_{k \in \mathbb{N}} \mathbb{Z}/n$ and $\alpha$ the one-sided shift on $H$ defined by $\alpha((a_k)) = (a_{k+1})$. As we explained in 2.1.1 in this case $\mathfrak{A}[\varphi] \cong \mathcal{O}_n$.

It is well known (and easy to see) that $K_0 C(H) \cong C(H, \mathbb{Z})$, $K_1 C(H) = 0$. If we describe the elements of $H$ by sequences $(x_0, x_1, \ldots)$ with $x_1 \in \mathbb{Z}/n$, then on $f \in C(H, \mathbb{Z}) \cong K_0 C(H)$, the map $b(\varphi)$ is given by

$$b(\varphi)(f)(x_0, x_1, \ldots) = \sum_{k=0}^{n-1} f(k, x_1, x_2, \ldots)$$

while $\varphi_\ast$ is described by $\varphi_\ast f(x_0, x_1, \ldots) = f(x_1, x_2, \ldots)$. The application of Theorem 3.1 and the inductive limit description of $K^*_\varphi B_\varphi$ of course leads to the well known formulas for the $K$-theory of $B_\varphi$ and $\mathcal{O}_n$, i.e $K_0(B_\varphi) = \mathbb{Z}[x_1^n]$, $K_1(B_\varphi) = 0$ and $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)$, $K_1(\mathcal{O}_n) = 0$.

**3.1.2.** Let $\alpha$ be an endomorphism of $H = \mathbb{T}^n$ with finite kernel and $\varphi$ the dual endomorphism of $G = \mathbb{Z}^n$. We assume that the intersection of all $\varphi^k(\mathbb{Z}^n)$ is $\{0\}$.

We know that there is a grading (and exterior product) preserving isomorphism of $K_\varphi(C(\mathbb{T}^n))$ with the exterior algebra $\Lambda^* \mathbb{Z}^n = \bigoplus_{p=0}^{n} \Lambda^p \mathbb{Z}$. The endomorphism $\varphi_\ast$ of $K_\varphi(C(\mathbb{T}^n))$ induced by $\varphi$ corresponds to the endomorphism $\Lambda\varphi$ of $\Lambda^* \mathbb{Z}^n$.

The associated endomorphism $b(\varphi)$ of $\Lambda^* \mathbb{Z}^n$ is determined by the formula $b(\varphi)\varphi_\ast = N(\varphi)\text{id}$ from Lemma 3.2. In the present case we have $N(\varphi) = |\det \varphi|$. 

Consider the Poincaré isomorphism $D : \Lambda G \cong \Lambda G'$ (here we write $G'$ for the algebraic dual $\text{Hom}(G, \mathbb{Z})$) and denote by $\varphi'$ the endomorphism of $G'$ which is dual to $\varphi$ under the natural pairing $G \times G' \to \mathbb{Z}$.

By [15], (6.62), one has

$$\Lambda \varphi (D \Lambda \varphi' D^{-1}) = \det \varphi \text{id}$$

Therefore the unique solution $b(\varphi)$ (in endomorphisms of $\Lambda \mathbb{Z}^n$) for the equation $b(\varphi) \Lambda \varphi = |\det \varphi| \text{id}$ corresponds under the Poincaré isomorphism to $\varepsilon \Lambda \varphi'$ with $\varepsilon = \text{sgn}(\det \varphi)$. The restriction of $b(\varphi)$ to $\Lambda^1 \mathbb{Z}^n \cong \mathbb{Z}^n$ for instance is the complementary matrix to $\varphi$ determined by Cramer’s rule. Thus we obtain

$$K_* \mathcal{A}[\varphi] \cong \Lambda G'/ (1 - \varepsilon \Lambda \varphi') \Lambda G' \oplus \text{Ker}(1 - \varepsilon \Lambda \varphi')$$

where the first term has the natural even/odd grading. The second term $\text{Ker}(1 - \varepsilon \Lambda \varphi')$ is $\Lambda^n \mathbb{Z}^n \cong \mathbb{Z}$ if $\det \varphi > 0$ and $\{0\}$ if $\det \varphi < 0$. It contributes to $K_0$ if $n$ is odd and to $K_1$ if $n$ is even.

For instance, if $\varphi = k \text{id}$, then $\det \varphi = k^n$, and $\Lambda \varphi = k^p \text{id}$ on $\Lambda^p \mathbb{Z}^n$. Therefore $b(\varphi)|_{\Lambda^p \mathbb{Z}^n} = k^{n-p} \text{id}$. We thus obtain

$$K_* \mathcal{A}[\varphi] = \bigoplus_{0 \leq p \leq n} \Lambda^p \mathbb{Z}^n / (1 - k^{n-p}) \Lambda^p \mathbb{Z}^n \oplus \mathbb{Z}$$

(understood with the natural even/odd grading for the first term on the right hand side and $\mathbb{Z}$ contributing to $K_0$ or $K_1$ depending on the parity of $n$). The same type of formula has been obtained in [14] using an exact sequence similar to the one in Theorem 3.1, which itself however has been obtained in a rather different way.

3.1.3. Consider the solenoid group $H$ of Example 2.1.5, i.e.

$$H = \lim_{\leftarrow} \mathbb{T} \quad G = \mathbb{Z} \left[\frac{1}{p}\right]$$

with the endomorphism $\varphi_q$ determined on $G$ by $\varphi_q(x) = qx$ ($q$ prime to $p$). The description of $G$ as an inductive limit of groups of the form $\mathbb{Z}$ immediately leads to the formulas

$$K_0(C^*G) = \mathbb{Z} \quad K_1(C^*G) = \mathbb{Z} \left[\frac{1}{p}\right]$$

Now $\varphi_q$ acts as id on $K_0(C^*G)$ and by multiplication by $q$ on $K_1(C^*G)$. Since $N(\varphi_q) = q$, we infer from Lemma 3.2 that $b(\varphi) = q \text{id}$ on $K_0(C^*G)$ and $b(\varphi) = \text{id}$ on $K_1(C^*G)$. Thus the exact sequence of Theorem 3.1 shows that

$$K_0(\mathcal{A}[\varphi]) = \mathbb{Z}/(q - 1) + \mathbb{Z} \left[\frac{1}{p}\right] \quad K_1(\mathcal{A}[\varphi]) = \mathbb{Z} \left[\frac{1}{p}\right]$$
4. Algebraic polymorphisms

Let \((X, \mu)\) be a Lebesgue space with continuous measure \(\mu\) (a measure space isomorphic to the unit interval with Lebesgue measure).

**Definition 4.1.** A measure preserving polymorphism \(\Pi\) of the Lebesgue space \((X, \mu)\) to itself is a diagram between three Lebesgue spaces:

\[
\Pi : (X, \mu) \leftarrow (X \times X, \nu) \rightarrow (X, \mu)
\]

where the left and right arrows are the projections \(\pi_1, \pi_2\) onto the first and second component of the product space \((X \times X, \nu)\), and where we assume that \(\pi_i \ast \nu = \mu, i = 1, 2\).

Because of the condition \(\pi_i \ast \nu = \mu\), the operators \(v_1, v_2: L^2(X, \mu) \rightarrow L^2(X \times X, \nu), i = 1, 2\), induced by \(\pi_i\), are isometries.

**Definition 4.2.** The Markov operator \(s_\Pi\) corresponding to the polymorphism \(\Pi\) is the operator on \(L^2(X)\) defined by the formula:

\[
s_\Pi = v_1^* v_2,
\]

The operator \(s_\Pi\) is a Markov operator in the sense that it is a contraction (\(\|M\| \leq 1\)), it is positive (\(Mf \geq 0\) if \(f \geq 0\)) and it preserves the constant function 1 (\(M(1) = M^*(1) = 1\)).

**Remark 4.3.** A product of the form \(v_1^* v_2\) where \(v_1, v_2\) are isometries is a partial isometry, if and only if the range projection of \(v_2\) commutes with the range projection of \(v_1\). Therefore \(s_\Pi\) is a partial isometry only under additional assumptions.

In this article we will consider only algebraic polymorphisms of compact groups as discussed in [23].

**Definition 4.4** (cf. [23], 1.1). Let \(H\) be a compact group which is not finite. A closed subgroup \(C \subset H \times H\) is an algebraic correspondence of \(H\) if \(\pi_1(C) = \pi_2(C) = H\) for the two coordinate projections \(\pi_i: H \times H \rightarrow H, i = 1, 2\).

Every algebraic correspondence gives rise to a polymorphism

\[
\Pi_C : (H, \lambda) \leftarrow (H \times H, \nu) \rightarrow (H, \lambda)
\]

where \(\nu\) is the extension of the Haar measure on \(C\) to \(H \times H\) such that \(\nu((H \times H) \setminus C) = 0\). Clearly, the \(\pi_i: C \rightarrow H\) are homomorphisms and send the Haar measure on \(C\) to the Haar measure on \(H\). The associated polymorphism therefore is measure preserving.

**Definition 4.5.** Given two surjective endomorphisms \(\alpha\) and \(\beta\) of \(H\) with finite kernel we define a correspondence \(C_{\alpha, \beta}\) and an associated polymorphism \(\Pi_{C_{\alpha, \beta}}\) by setting

\[
C_{\alpha, \beta} = \{ (\alpha(h), \beta(h)) : h \in H\}
\]
If \( \text{Ker} \alpha \cap \text{Ker} \beta = \{1\} \), we can identify \( H \) with \( C \) via the map \( h \mapsto (\alpha(h), \beta(h)) \). Under this identification, the isometries \( \nu_1, \nu_2 \) correspond to the isometric operators \( s_\alpha \) and \( s_\beta \) on \( L^2(H) \) as in section 2. Their range projections commute if \( H \) is abelian (under Fourier transform they simply correspond to the orthogonal projections onto \( \ell^2(\text{Im} \alpha) \) and \( \ell^2(\text{Im} \beta) \)). Therefore the associated Markov operator \( s_\Pi \) is a partial isometry if \( H \) is abelian and \( \text{Ker} \alpha \cap \text{Ker} \beta = \{1\} \).

5. Independence

Let \( \varphi \) and \( \psi \) be two injective endomorphisms of a discrete abelian group \( G \) satisfying the conditions of section 2.

We will assume that \( \varphi \) and \( \psi \) commute. For much of our discussion (in particular concerning polymorphisms defined by a pair of endomorphisms) we need a stronger condition which is described in the following lemma. This condition is well known in number theory for the case where \( \varphi, \psi \) are given by multiplication by an algebraic number on the ring of algebraic integers.

**Lemma 5.1.** Let \( \varphi, \psi \) be injective commuting endomorphisms of the abelian group \( G \) such that \( G/\varphi(G) \) and \( G/\psi(G) \) are both finite. Then the following conditions are equivalent:

(a) \( \varphi(G) + \psi(G) = G \)

(b) \( \varphi(G) \mapsto G \) induces an isomorphism \( \varphi(G)/(\varphi(G) \cap \psi(G)) \cong G/\psi(G) \).

(c) \( \varphi(G) \cap \psi(G) = \varphi \psi(G) \)

**Proof.** The induced map in (b) is injective. It is also surjective if we assume (a). Thus (a) \( \Rightarrow \) (b). The fact that \( \varphi \psi(G) \subset \varphi(G) \cap \psi(G) \), together with the isomorphism \( G/\psi(G) \cong \varphi(G)/\varphi \psi(G) \) shows that (b) \( \Rightarrow \) (c).

Finally, (c) implies that \( (\varphi(G) + \psi(G))/(\varphi(G) \cap \psi(G)) \cong G/\varphi(G) \oplus G/\psi(G) \cong G/(\varphi(G) \cap \psi(G)) \) (using the fact that \( G/\varphi(G) \cong \psi(G)/\varphi \psi(G) \)). This implies (a), since both sides of the equation are finite. \( \square \)

**Definition 5.2.** We say that \( \varphi, \psi \) are independent (or relatively prime) if the equivalent conditions in 5.1 are satisfied.

Applying condition (a) in 5.1 inductively, we see that then also each power of \( \varphi \) is prime to each power of \( \psi \) (Proof: \( \varphi(G) + \psi(G) = \varphi(\varphi(G) + \psi(G)) + \psi(G) = \varphi^2(G) + \varphi \psi(G) + \psi(G) = \varphi^2(G) + \psi(G) \)).

**Remark 5.3.** For \( \varphi, \psi \) independent we have the following version of the Chinese remainder theorem

\[ G/\psi \varphi(G) = G/(\psi(G) \cap \varphi(G)) \cong G/\psi(G) \oplus G/\varphi(G) \]

The second isomorphism is a consequence of condition (a) above.

The following lemma is just a reformulation of Lemma 5.1 for the dual group.

**Lemma 5.4.** Let \( \alpha \) and \( \beta \) be two commuting surjective endomorphisms of the compact abelian group \( H \) with finite kernel and let \( \varphi = \hat{\alpha}, \psi = \hat{\beta} \) be the dual endomorphisms of the dual group \( G = H \). The following are equivalent
(a) $\varphi$ and $\psi$ are independent.
(b) $\text{Ker} \alpha \cap \text{Ker} \beta = \{1\}$.
(c) $\alpha(\text{Ker} \beta) = \text{Ker} \beta$.
(c$'$) $\beta(\text{Ker} \alpha) = \text{Ker} \alpha$.
(d) The subgroup $\text{Ker} \alpha \text{Ker} \beta$ generated by $\text{Ker} \alpha$ and $\text{Ker} \beta$ equals $\text{Ker} \alpha \beta$.

**Definition 5.5.** We say that $\alpha$ and $\beta$ are independent if they satisfy the equivalent conditions in Lemma 5.4.

**Remark 5.6.** Condition (b) in Lemma 5.4 says that the partitions of $H$ into cosets with respect to $\text{Ker} \varphi$ and $\text{Ker} \psi$ are independent for the Haar measure. This means that two functions invariant under $\text{Ker} \varphi$ and $\text{Ker} \psi$, respectively, are independent in $L^2(H)$ in the probabilistic sense. Based on this observation one could also define independence for more general (non-algebraic) commuting endomorphisms of a measure space.

**Lemma 5.7.** Let $v$ and $w$ be two isometries in a unital $C^*$-algebra. Then the identity $vw^* = w^*v$ implies that $v$ and $w$ commute (but not conversely).

**Proof.** The identity $vw^* = w^*v$ implies that $v^*w^*vw = 1$. On the other hand, if $a,b$ are isometries (here $b=vw$, $a=wv$) such that $a^*b = 1$, then $a = b$. □

**Lemma 5.8.** Let $\varphi$ and $\psi$ be injective endomorphisms of the abelian group $G$. Let $s_\varphi$ and $s_\psi$ be the isometries in $\mathcal{L}(\ell^2 G)$ defined by $s_\varphi(\xi_g) = \xi_{\varphi(g)}$ and $s_\psi(\xi_g) = \xi_{\psi(g)}$. Let further $e_\varphi = s_\varphi s_\varphi^*$ and $e_\psi = s_\psi s_\psi^*$ be the range projections. Then

(a) Assume that $\varphi$ and $\psi$ commute and are independent. Then $e_\psi e_\varphi = e_\psi e_\varphi$ and $s_\varphi^* s_\psi = s_\psi s_\varphi$.
(b) If $s_\varphi^* s_\psi = s_\psi s_\varphi$, then $\psi$ and $\varphi$ commute and are independent.

**Proof.** (a) Since $e_\varphi$ is exactly the orthogonal projection onto $\ell^2(\varphi G)$, condition (c) in Lemma 5.1 translates to $e_\varphi e_\varphi = e_\varphi$ or $s_\varphi s_\psi s_\psi s_\psi^* = s_\varphi s_\psi s_\psi^* s_\varphi^* = s_\psi s_\varphi^* s_\psi^* s_\varphi^*$ (for the last equality we use the fact that $\varphi \psi = \psi \varphi$). This is true if and only if $s_\varphi^* s_\psi = s_\psi s_\varphi^*$.
(b) Lemma 5.1 shows that the hypothesis implies that $s_\varphi$ and $s_\psi$ and thus also $\varphi$ and $\psi$, commute. Now we use again that, for commuting $s_\varphi$, $s_\psi$, the identity $s_\varphi^* s_\psi = s_\psi s_\varphi^*$ holds if and only if $s_\varphi^* s_\psi s_\psi s_\psi^* = s_\varphi s_\psi s_\psi^* s_\varphi^*$, thus iff $e_\psi e_\varphi = e_\psi e_\varphi$ which is equivalent to condition (c) in Lemma 5.1. □

6. **Orbit partitions for endomorphisms and for pairs of independent commuting endomorphisms**

We briefly discuss here the orbit partition corresponding to endomorphisms and semigroups of endomorphisms of a compact abelian group $H$. As in the previous sections we consider only surjective endomorphisms $\alpha$ with finite kernel for which the subgroup $L_\alpha = \bigcup_n \text{Ker} \alpha^n$ is dense in $H$.

Let $\alpha$ and $\beta$ be two such endomorphisms which commute and are independent in the sense of Definition 5.5. From the remark after Definition 5.2 we see that then every power of $\alpha$ is independent from every power of $\beta$. Therefore, from Lemma 5.4 we obtain the following identities
\( L_\alpha \cap L_\beta = \{1\} \quad L_\alpha L_\beta = L_{\alpha \beta} \quad \alpha(L_\beta) = L_\beta \quad \beta(L_\alpha) = L_\alpha \)

**Definition 6.1.** Suppose that \( S \) is a commutative semigroup of endomorphisms of the compact abelian group \( H \). The orbit of the point \( x \in H \) with respect to the semigroup \( S \) is the set:

\[
\text{orb}_S(x) = \{y \in H : \exists \sigma_1, \sigma_2 \in S, \sigma_1(x) = \sigma_2(y)\}
\]

In particular if \( S = \{\alpha^n, n \in \mathbb{N}\} \) then we obtain the orbit of the endomorphism \( \alpha \)

\[
\text{orb}_\alpha(x) = \{y : \exists n, m \in \mathbb{N}, \alpha^n(y) = \alpha^m(x)\}
\]

If \( S \) is generated by two commuting endomorphisms \( \alpha \) and \( \beta \) then the orbit of a point \( x \) with respect to the semigroup \( S \) is

\[
\text{orb}_S(x) = \text{orb}_{\alpha,\beta}(x) = \{y : \exists n, m, s, t \in \mathbb{N}, \alpha^n \beta^s(y) = \alpha^m \beta^t(x)\}
\]

It is clear that this definition defines a partition of the group \( H \) into orbits. This partition is called the orbit partition of the semigroup \( S \). Sometimes it is better to speak about the orbit equivalence relation as a Borel subset of the product \( H \times H \) (i.e. the set of all pairs of points which belong to the same orbit).

For an exact endomorphism the orbit of each point is a dense countable set; so this partition is not measurable in the usual sense. This means that there is no measurable structure on the space of orbits (no natural quotient under that partition). This fact is usually expressed as the ergodicity of the orbit equivalence relation.

The structure of an orbit for one exact endomorphism \( \alpha \) is easy to describe: One obviously has \( \text{orb}_\alpha(1) = L_\alpha \); for a generic point \( x \in H \), using surjectivity of \( \alpha \) one can choose a two sided sequence \( z_k, k \in \mathbb{Z} \) such that \( z_0 = x \), and \( z_{k+1} = \alpha(z_k) \) for all \( k \in \mathbb{Z} \). Then the \( \alpha \)-orbit is given by

\[
\text{orb}_\alpha(x) = \bigcup_{k \in \mathbb{Z}} z_k L_\alpha
\]

**Definition 6.2.** A partition with countable blocks (or an equivalence relation with countable classes) is called hyperfinite if it is a union of a sequence of decreasing measurable partitions (equivalently a limit of a sequence of monotonously increasing equivalence relations with finite classes).

**Theorem 6.3.** ([2], [25]) The orbit partition of an arbitrary measure preserving or nonsingular endomorphism of a standard measure space is hyperfinite.

This is a generalisation of Dye’s theorem which says that the orbit partition for a single nonsingular automorphisms is hyperfinite. Connes-Feldman-Weiss [3] proved that a nonsingular action of any amenable group is hyperfinite. They also gave an alternative proof of Theorem 6.3, [3, Corollary 13].

The situation with semigroups is much more complicated even if we consider algebraic endomorphisms. Here we claim only the following fact about algebraic endomorphisms.

\(^1\)Recall that the expression "decreasing sequence of partitions" in measure theory means that the sequence of the elements of the partitions which contain a given point, monotonously increases.
Theorem 6.4. The orbit partition for the semigroup generated by two independent commuting endomorphisms of a compact abelian group is hyperfinite.

Sketch of the argument. From the definition of the orbit partition for a commutative semigroup and Lemma 5.4 resp. formulas (7) and (8), we can conclude that the orbit partition of the semigroup generated by two commuting independent endomorphisms \( \alpha \) and \( \beta \) has the form:

\[
\text{orb}_{\alpha,\beta}(x) = \bigcup_{y \in \text{orb}_\alpha(x)} \text{orb}_\beta(y) = \bigcup_{y \in \text{orb}_\beta(x)} \text{orb}_\alpha(y)
\]

This is a partition of product type. Since \( L_\alpha \cap L_\beta = \{1\} \), the intersection of an \( \alpha \)-orbit with a \( \beta \)-orbit consists of at most one point. Since we also know that the orbit partition for one endomorphism is hyperfinite (see Theorem 6.3), we can now apply a general theorem to the effect that the product of hyperfinite partitions is hyperfinite. It is possible to apply the same argument as in [3], Corollary 12 but for the group \( \mathbb{Z}^2 \) instead of \( \mathbb{Z} \).

The claim of Theorem 6.4 is not true for an arbitrary pair of endomorphisms. Perhaps it is not true even if the endomorphisms \( \alpha \) and \( \beta \) commute but are not independent. However the theorem is valid for a pair of arbitrary (non algebraic) commuting measure preserving (or non singular) endomorphisms of the measure space which are independent in the sense of 5.6.

7. The C*-algebra of a rational polymorphism

We consider now a pair \( \varphi, \psi \) of injective endomorphisms of the (countable) abelian group \( G \) and define the operators \( s_\varphi, s_\psi \) on \( \ell^2G \) by \( s_\varphi(\xi_g) = \xi_{\varphi(g)} \) and \( s_\psi(\xi_g) = \xi_{\psi(g)} \).

The fact that \( s_\varphi, s_\psi \) are isometries satisfying \( s_\varphi s_\psi^* = s_\psi^* s_\varphi \) encodes the hypothesis that \( \varphi, \psi \) are injective, commute and are independent, see Lemma 5.8. This will be our assumption from now on. Moreover we assume that \( \varphi \) and \( \psi \) are exact, i.e. that

\[
\bigcap \varphi^n(G) = \bigcap \psi^n(G) = \{0\}
\]

and that \( \varphi, \psi \) have finite cokernel. We call this a standard pair of endomorphisms from now on.

Definition 7.1. We say that \( \Pi \) is a rational polymorphism of the compact abelian group \( H \), if it is determined as in Definition 4.5 by a correspondence \( C_{\varphi,\psi} \), where \( \varphi, \psi \) is a standard pair of endomorphisms of the dual group \( \hat{G} = \hat{H} \). We then denote \( \Pi \) by \( \varphi/\psi \).

Each element of \( G \) induces a unitary operator \( u_g \) on \( \ell^2G \) defined by \( u_g(\xi_t) = \xi_{g+t} \). For a rational polymorphism as in Definition 7.1, \( s_\varphi \) commutes with \( s_\psi^* \) and the partial isometry \( s = s_\varphi s_\psi^* \) describes the Markov operator. We propose to describe the C*-algebra generated by \( s \) and the \( u_g, g \in G \).

The operators \( u_g : g \in G \) and \( s = s_\varphi s_\psi^* = s_\psi^* s_\varphi \) in \( \mathcal{L}(\ell^2G) \) satisfy the following relations:

P1 \( s^n \) is a partial isometry for each \( n = 1, 2, \ldots \) (i.e. the \( s^n s^n \) are projections).
Lemma 7.3. (a) For each $n$, we have

$$\sum_{g \in G/\varphi(G)} u_g s^n s^n u_{-g} = 1$$

(b) Any two projections of the form $u_g s^n s^n u_g^*$ and $u_h s^m s^m u_h^*$, $g, h \in G$, $n, m \in \mathbb{N}$, commute.

Proof. (a) We observe first that a product $ab$ of two partial isometries is a partial isometry if and only if the range projection of $b$ commutes with the support projection of $a$. Moreover $s^n s^n \leq s^m s^m$ if $n \geq m$. Therefore we conclude from property P1 that all projections of the form $s^n s^n$ and $s^m s^m$ commute.

Since $\varphi(G) + \psi(G) = G$ by Definition 5.1 (a), and thus $G/\varphi G = \psi G/\varphi G$, the identity $\sum_{g \in G/\varphi(G)} u_g s^n s^n u_g^* = 1$ from P4 can be rewritten as

$$\sum_{h \in G/\varphi(G)} u_{\psi(h)} s s^* u_{\psi(h)}^* = 1$$

Conjugating this by $s \cdot s^*$ and using property P3 one finds that $s s^* = \sum_{h \in G/\varphi(G)} u_{\varphi(h)} s^2 s^2 u_{\varphi(h)}^*$ and thus, applying P4 again

$$\sum_{g, h \in G/\varphi(G)} u_{g + \varphi(h)} s s^* u_{g + \varphi(h)}^* = \sum_{g \in G/\varphi^2(G)} u_g s^2 s^2 u_g^* = 1$$

Iterating this procedure we find the first identity in (a). The proof of the second identity is of course the same, replacing $\varphi$ by $\psi$ and $s^n s^n$ by $s^m s^m$.

(b) Using the comment after Definition 5.2 for any $n, m \in \mathbb{N}$ we have $\varphi^m(G) + \psi^m(G) = G$. Therefore we can rewrite any two projections of the form $u_g s^m s^n u_g^*$ and $u_h s^m s^m u_h^*$ as $u_{\varphi^m(g')} s^m s^n u_{\varphi^m(g')}^*$ and $u_{\psi^m(h')} s^m s^m u_{\psi^m(h')}^*$. However, by P3, $u_{\varphi^m(g')}$ commutes with $s^m s^m$ and $u_{\psi^m(h')}$ commutes with $s^m s^n$. \qed

By point (b) of this Lemma, the projections of the form $u_g s^n s^n u_{-g}$, $u_h s^m s^m u_{-h}$, $g, h \in G$, $n, m \in \mathbb{N}$, generate a commutative subalgebra $\mathcal{D}$ of $\mathfrak{A}[\varphi/\psi]$. The following Lemma is then basically a special case of Lemma 2.4.
Lemma 7.4. The $C^*$-subalgebra $\mathcal{D}$ of $\mathcal{A}[\phi/\psi]$ generated by all projections of the form $u_g s^n s^m u_g^*$, $u_g s^n s^m u_g^*$, $g \in G$, $n, m \in \mathbb{N}$ is commutative. Its spectrum is the completion

$$G_{\psi\phi} = \lim_{\rightarrow n} G/(\psi\phi)^n G$$

$G$ acts on $\mathcal{D}$ via $d \mapsto u_g d u_g^*$, $g \in G$, $d \in \mathcal{D}$. This action corresponds to the natural action of the dense subgroup $G$ on its completion $G_{\psi\phi}$ via translation.

We have $G_{\psi\phi} \cong G_\phi \times G_\psi$ and $\mathcal{D} \cong \mathcal{D}_\phi \otimes \mathcal{D}_\psi$ with $\mathcal{D}_\phi \cong C(G_\phi)$ and $\mathcal{D}_\psi \cong C(G_\psi)$. The map $Ds^* s \to Dss^*$ given by $x \mapsto sxs^*$ corresponds to the map induced by $\phi\psi^{-1}$ on $(\psi(G))_{\psi\phi} \cong G_\phi \times (\psi(G))_\psi$.

Proof. The first part of the assertion is just a special case of Lemma 2.4. According to Remark 5.3 we have $G_{\psi\phi} \cong G_\phi \times G_\psi$. □

We will denote the compact abelian group $G_{\psi\phi} \cong G_\psi \times G_\phi$ by $K$. Its dual group is the discrete abelian group

$$L = \lim_{\rightarrow n} \ker (\hat{\psi}\hat{\phi}^n : H \to H)$$

Because of the condition that we impose on $\psi, \phi, L$ can be considered as a dense subgroup of $H$.

The torus $T$ acts on $\mathcal{A}[\phi/\psi]$ by automorphisms $\alpha_t$, $t \in \mathbb{R}$ defined by

$$\alpha_t(s) = e^{it} s, \quad \alpha_t(u_g) = u_g$$

The fixed point algebra $B_{\psi\phi}$ is the subalgebra of $\mathcal{A}[\phi/\psi]$ generated by all $u_g$, $g \in G$ and by the $s^n s^m$, $s^n s^m$. Integration over $T$ gives a faithful conditional expectation $\mathcal{A}[\phi/\psi] \to B_{\psi\phi}$.

Applying Lemma 2.5 to the endomorphism $\psi\phi$ we obtain

Lemma 7.5. The $C^*$-subalgebra $B_{\psi\phi}$ of $\mathcal{A}[\phi/\psi]$ generated by $C(H)$ together with $C(K)$ (or equivalently by $C^*G$ together with $C^*L$) is isomorphic to the crossed product $C(K) \rtimes G$. It is simple and has a unique trace.

As in section 2 we also have the isomorphisms

$$B_{\psi\phi} \cong C(K) \rtimes G \cong C^* L \rtimes G \cong C^*G \rtimes L \cong C(H) \rtimes L$$

Now, again as in section 2, we compose the conditional expectation $\mathcal{A}[\phi/\psi] \to B_{\psi\phi}$ with the natural expectation $B_{\psi\phi} \to \mathcal{D}$, coming from the crossed product representation, to obtain a faithful conditional expectation $E : \mathcal{A}[\phi/\psi] \to \mathcal{D}$.

Lemma 7.6. (a) $s^* u_g s \neq 0 \Rightarrow g \in \phi(G)$ and $s u_g s^* \neq 0 \Rightarrow g \in \psi(G)$

(b) $s u_g s \neq 0 \Rightarrow g \in (\phi(G) \cap \psi(G))$

(c) Every element in $\mathcal{A}[\phi/\psi]$ can be approximated by linear combinations of elements of the form $u_g s^n d u_t s^m u_h$ or $u_g s^n d u_t s^m u_h$ with $d \in \mathcal{D}$, $n, m \geq 0$, $g, h, t \in G$. 

Proof. (a) follows immediately from condition P4. (b) follows from the identities
\[ f_1 = \sum_{g \in \varphi(G)/\varphi(G)} u_g f_1 f_2 u^{-1}_g \quad f_2 = \sum_{h \in \varphi(G)/\varphi(G)} u_h f_1 f_2 u^{-1}_h \]
where \( f_1 = ss^*, \ f_2 = s^*s \).
Finally, using (a) and (b) one sees that the set of elements described in (c) is invariant under multiplication by \( s, s^*, u_g \) on the right or on the left. \( \square \)

**Lemma 7.7.** (a) \( s \) and \( s^* \) normalize \( \mathcal{D} \).
(b) An element \( z \) of the form \( u_g s^n d u_t s^m u_h \) or \( u_g s^n d u_t s^m u_h \) with \( d \neq 0 \) is in \( \mathcal{D} \) if and only if \( n = m \) and \( g + h + t = 0 \).
(c) Let \( z \) be an element of the form \( u_g s^n d u_t s^m u_h \) or \( u_g s^n d u_t s^m u_h \). If \( n \neq m \) or \( g + h + t \neq 0 \), then \( E(z) = 0 \).

Proof. (a) follows from Lemma 7.6(a) and (b).
(b) If \( n = m \) and \( g + h + t = 0 \), then combining Lemma 7.6(a) and (b) with the fact that \( s, s^* \) normalize \( \mathcal{D} \), we see that \( z \in \mathcal{D} \). Conversely, if \( z \) is in \( \mathcal{D} \), then \( z \) has to fixed by \( \alpha_t, t \in \mathbb{R} \) whence \( n = m \) and by \( \beta_\chi, \chi \in \hat{G} \) whence \( g + h + t = 0 \). (c) is immediate from the definition of \( E \). \( \square \)

**Lemma 7.8.** Let
\[ z = d + \sum_{i=1}^{k} u_{n_i} s^{n_i} d_i u_{t_i} s^{m_i} u_{h_i} + \sum_{i=k+1}^{m} u_{g_i} s^{n_i} d_i u_{t_i} s^{m_i} u_{h_i} \]
be an element of \( \mathfrak{A}[\varphi/\psi] \) (cf. Lemma 7.6(c)) such that for each \( i \), either \( n_i \neq m_i \) or \( g_i + h_i + t_i \neq 0 \) with \( d, d_i \in \mathcal{D} \). Then
(a) There is a projection \( e \in \mathcal{D} \), such that for all \( i \) we have \( eu_{g_i} s^{n_i} d_i u_{t_i} s^{m_i} u_{h_i} e = 0 \) and \( eu_{g_i} s^{n_i} d_i u_{t_i} s^{m_i} u_{h_i} e = 0 \), and such that \( ede = \lambda e \) with \( |\lambda| = \|E(z)\| \).
(b) For the projection \( e \) in (a) one has \( eze = \lambda e \) with \( |\lambda| = \|E(z)\| \).

Proof. (a) \( \mathcal{D} \) is normalized by \( s, s^* \) and the \( u_g \). Arguing exactly as in the proof of Theorem 2.6 we see that there is a projection \( e \in \mathcal{D} \) which is transported to an orthogonal projection by each of the \( u_{g_i} s^{n_i} d_i u_{t_i} s^{m_i} u_{h_i} \) and \( u_{g_i} s^{n_i} d_i u_{t_i} s^{m_i} u_{h_i} \) and such that \( eze = \lambda e \) with \( |\lambda| = \|E(z)\| = \|d\| \). In fact for \( e \) we can choose \( e'_1 \otimes e'_2 \in \mathcal{D}_\varphi \otimes \mathcal{D}_\psi \cong \mathcal{D} \), where \( e'_1 \) and \( e'_2 \) are chosen for \( \varphi \) and \( \psi \) respectively as in the proof of theorem 2.6. One has \( E(z) = d, eze = de \) and (b) is then clear. \( \square \)

The algebra \( B_{\psi_\varphi} \) is the inductive limit of the algebras \( B_n \cong M_{N(\psi_\varphi)^n}(C^*(G)) \) (see the discussion at the beginning of section 3). It naturally contains the inductive limit \( U_{\psi_\varphi} \) of the subalgebras \( M_{N(\psi_\varphi)^n}(\mathbb{C}) \subset M_{N(\psi_\varphi)^n}(C^*(G)) \). This inductive limit is a \( UHF \)-algebra of type \( N(\psi_\varphi)^\infty \).

**Remark 7.9.** We have \( N(\psi_\varphi) = N(\psi)N(\varphi) \) (this is a slight generalization of the well known fact that the absolute norm in number theory is multiplicative). The proof follows from the facts that \( G/\psi(G) \cong G/\psi_\varphi(G)/(\psi(G)/\psi_\varphi(G)) \) and that \( \psi(G)/\psi_\varphi(G) \cong G/\varphi(G) \).
Lemma 7.10. Assume that $N(\varphi) \geq N(\psi)$. There is a unitary $u \in M_{N(\psi)}(\mathbb{C}) \subset U_{\psi}$ such that the partial isometry $t = us$ satisfies $tt^* = t^*t$.

Let $\tau$ be the unique trace state on $B_{\psi\varphi}$ and $x \in B_{\psi\varphi}$ such that $x = xt^*t$. Then $\tau(txt^*) = N(\varphi)^{-1}N(\psi)\tau(x)$.

Proof. The projections $ss^*$ and $s^*s$ are both contained in $U_{\psi\varphi}$ (in fact even in $M_{N(\psi\varphi)}(\mathbb{C}) \subset U_{\psi\varphi}$) and satisfy $N(\varphi)^{-1} = \tau(ss^*) < \tau(s^*s) = N(\psi)^{-1}$ for the unique trace state $\tau$ on $U_{\psi\varphi}$. Therefore we can choose a unitary $u \in M_{N(\psi\varphi)}(\mathbb{C})$ as required. The unique trace state $\tau$ on $B_{\psi\varphi}$ is obviously given as the pointwise limit of the natural trace states $\tau_n$ on $B_n \cong M_{N(\psi\varphi)n}(C^*(G))$ (obtained from the composition of the natural map $M_{N(\psi\varphi)n}(C^*(G)) \to M_{N(\psi\varphi)n}(\mathbb{C})$ given by the trivial representation of $G$ and the normalized trace on $M_{N(\psi\varphi)n}(\mathbb{C})$). These states clearly satisfy $\tau_n(sxs^*) = N(\varphi)^{-1}N(\psi)\tau_n(s^*ss)$ for $x \in B_n$ (under the isomorphism $M_{N(\psi\varphi)}(\mathbb{C}) \cong M_{N(\psi)}(\mathbb{C}) \otimes M_{N(\varphi)}(\mathbb{C})$ we have $s(e_{11} \otimes 1)s^* = 1 \otimes e_{11}$). Therefore $\tau(sxs^*) = N(\varphi)^{-1}N(\psi)\tau(s^*ss)$.

Theorem 7.11. Let $\Pi = \varphi/\psi$ be a rational polymorphism of the compact abelian group $H$. The associated universal $C^*$-algebra $\mathfrak{A}[\varphi/\psi]$ is simple. It is purely infinite if $N(\varphi) \neq N(\psi)$.

Proof. Let $I$ be a non-zero closed ideal in $\mathfrak{A}[\varphi/\psi]$ and $h$ a non-zero positive element in $I$. Then $E(h) \neq 0$, and by Lemma 7.8 in combination with Lemma 7.6 (c), there is a projection $e$ in $\mathcal{D}$ (thus also in the subalgebra $B_{\psi\varphi}$) such that $che - \|E(h)\|e < \|E(h)\|/2$. It follows that $e \in I$. Since $e \in B_{\psi\varphi}$ and $B_{\psi\varphi}$ is simple unital, also $1 \in I$.

This shows that $\mathfrak{A}[\varphi/\psi]$ is simple. Possibly replacing $s$ by $s^*$ we can always assume that $N(\varphi) \geq N(\psi)$.

If now $N(\varphi) > N(\psi)$, then choosing $t$ as in Lemma 7.10 we have by 7.10 that $\tau(t^kx^k) = (N(\varphi)N(\psi)^{-1})^{-k}$. Take $k$ large enough so that $\tau(t^kx^k) < \tau(e)$. In the UHF-algebra $U_{\psi\varphi}$ there is then a unitary $u$ such that $ut^kx^ku^* \leq e$. Thus $t^kx^k\psi\psi^k$ is invertible in the unital $C^*$-algebra $t^k\mathfrak{A}[\varphi/\psi]t^k$. This shows that $t^k\mathfrak{A}[\varphi/\psi]t^k\psi\psi$ is purely infinite which immediately implies that $\mathfrak{A}[\varphi/\psi]$ is purely infinite too.

Corollary 7.12. Let $\Pi = \varphi/\psi$ be a rational polymorphism of the compact abelian group $H$. The natural map from $\mathfrak{A}[\varphi/\psi]$ to the $C^*$-algebra of operators on $L^2(H)$ generated by $\Phi$ and $C(H)$ is an isomorphism.

Proof. The map exists and is surjective by universality of $\mathfrak{A}[\varphi/\psi]$. It is injective since $\mathfrak{A}[\varphi/\psi]$ is simple by Theorem 7.11.

Corollary 7.13. Let $h = t^k\psi\psi$ be the support projection of the partial isometry $t$ defined in Lemma 7.10 and $\gamma$ the endomorphism of $hB_{\psi\varphi}h$ defined by $\gamma(x) = txt^*$. Then $h\mathfrak{A}[\varphi/\psi]h$ is isomorphic to the semigroup crossed product $B_{\psi\varphi} \rtimes \gamma \mathbb{N}$ (and thus $\mathfrak{A}[\varphi/\psi]$ itself is Morita equivalent to this crossed product).

If $N(\varphi) = N(\psi)$, $\mathfrak{A}[\varphi/\psi]$ has a unique trace state.

Proof. By definition, clearly there is a surjective map from the universal algebra $h\mathfrak{A}[\varphi/\psi]h$ onto the crossed product. By simplicity of $\mathfrak{A}[\varphi/\psi]$, this map is an isomorphism.
By Lemma 7.10 if \( N(\psi) = N(\varphi) \), then \( \tau(txt^*) = t(x) \) for \( x \) in the hereditary subalgebra of \( B_{\psi\varphi} \) generated by \( t^*t \). Therefore \( \tau \) extends uniquely to a trace state on \( \mathcal{A}[\varphi/\psi] \). This follows for instance from [19], Corollary 1.2. In fact, \( \mathcal{A}[\varphi/\psi] \) can be seen as the \( C^* \)-algebra of a groupoid with object space \( K \) and with the set of points in \( K \) with non-trivial stabilizer group of zero measure for \( \tau \). In our case at hand one can also give a very easy direct proof (cf. the argument given in the proof of Lemma 2.5). In fact one can easily construct partitions of unity in \( C(K) \) consisting of pairwise orthogonal projections \( e_i \) with small trace such that for an element \( z \) such as in (9) one has \( e_i z e_i = d e_i \) for nearly all \( i \). We omit the details.

8. Computation of the \( K \)-theory of \( \mathcal{A}[\varphi/\psi] \)

From formula (4) in section 3 we conclude

\[
K_*(B_{\psi\varphi}) = \lim_{b(\psi\varphi)} K_*(C^*(G))
\]

where \( b(\psi\varphi) \) is the homomorphism playing the role of the \( b(\varphi) \) from section 3 for the endomorphism \( \psi\varphi \) (i.e. \( b(\psi\varphi) = \kappa^{-1}_1\psi\varphi\kappa_0 \)).

In the following we need however a somewhat finer analysis of the \( K \)-theory homomorphisms involved in this formula. For this we consider the \( C^* \)-subalgebras \( B_{mn} = C(G/\psi^m\varphi^nG) \times G \cong M_{N(\psi^m)N(\varphi^n)}(C^*(G)) \) of \( B_{\psi\varphi} \) and the corresponding natural maps \( \kappa_{mn} : K_*(C^*(G)) \rightarrow K_*(B_{mn}) \).

As in section 3 we denote by \( t_{mn}^\psi, t_{mn}^\varphi \) the maps \( K_0 B_{mn} \rightarrow K_0 B_{m+1,n} \) and \( K_1 B_{mn} \rightarrow K_1 B_{m,n+1} \) induced by the canonical inclusions and set \( b(\psi)_{mn} = \kappa_{m+1,n-1}^{-1} t_{mn}^\psi \kappa_{mn} \), \( b(\psi)_{mn} = \kappa_{m+1,n}^{-1} t_{mn}^\varphi \kappa_{mn} \).

**Lemma 8.1.** The endomorphisms \( b(\varphi)_{mn} \) and \( b(\psi)_{mn} \) of \( K_*(C^*(G)) \) do not depend on \( m, n \). Denoting these endomorphisms by \( b(\varphi) \) and \( b(\psi) \) we have \( b(\psi\varphi) = b(\psi)b(\varphi) = b(\varphi)b(\psi) \).

**Proof.** The independence of \( b(\varphi)_{mn} \) and \( b(\psi)_{mn} \) from \( m, n \) follows exactly as in section 3. The fact that \( b(\psi\varphi) \) is the product of \( b(\psi) \) and \( b(\varphi) \) (and that these maps commute) can be seen from the following commutative diagram

\[
\begin{array}{c}
\begin{CD}
K_*C^*(G) @>b(\varphi)>> K_*C^*(G) @>b(\psi)>> K_*C^*(G) \\
\downarrow{\kappa_{00}} @. \downarrow{\kappa_{01}} @. \downarrow{\kappa_{11}} \\
K_*B_{00} @>t_{00}^\varphi>> K_*B_{01} @>t_{01}^\psi>> K_*B_{11}
\end{CD}
\end{array}
\]

and the fact that \( t_{01}^\varphi t_{00}^\psi = t_{01}^\psi \) (and the analogous diagram with the order of \( b(\varphi) \) and \( b(\psi) \) inverted, which also commutes).

It will be convenient to represent \( K_*(B_{\psi\varphi}) \) rather than as in (10) as an inductive limit of the system \( C_{mn} \) where \( C_{mn} = K_*(C^*(G)) \) for all \( m, n \in \mathbb{N} \) and connecting maps \( b(\psi)b(\varphi)^t : C_{mn} \rightarrow C_{(m+k)(n+l)} \).
The $(m, n)$ form a directed set for the order $(m, n) \leq (m', n') \Leftrightarrow m \leq m'$ and $n \leq n'$.
Lemma 8.1 shows that $\lim_{b(\psi)} K_0(C^*(G)) \cong \lim_{\rightarrow mn} C_{mn}$.

**Theorem 8.2.** There is an exact sequence

$$
\lim_{\rightarrow b(\psi)} K_0(C^*(G)) \xrightarrow{\bar{b}(\psi)-\bar{b}(\varphi)} \lim_{\rightarrow b(\psi)} K_0(C^*(G)) \xrightarrow{\downarrow} K_0\mathcal{A}[\varphi/\psi] \xrightarrow{\uparrow} K_1\mathcal{A}[\varphi/\psi] = \lim_{\rightarrow b(\psi)} K_1(C^*(G)) \xrightarrow{\bar{b}(\psi)-\bar{b}(\varphi)} \lim_{\rightarrow b(\psi)} K_1(C^*(G))
$$

where $\bar{b}(\psi), \bar{b}(\varphi)$ are the endomorphisms of the inductive limit canonically induced by $b(\psi), b(\varphi)$ (note that $b(\psi), b(\varphi)$ commute with $b(\psi\varphi)$).

**Proof.** From the representation of $\mathcal{A}[\varphi/\psi]$ as a crossed product in Corollary 7.13 we get as in section 3 the Pimsner-Voiculescu sequence

$$K_*B_{\psi\varphi} \xrightarrow{1-\gamma_*} K_*B_{\varphi/\psi} \xrightarrow{\kappa} K_*\mathcal{A}[\varphi/\psi]$$

As in section 3 the maps $\kappa_{mn} : C_{mn} \rightarrow K_*(B_{mn})$ induce an isomorphism $\kappa : \lim_{\rightarrow mn} C_{mn} \rightarrow K_*(B_{\psi\varphi})$. Let $b(\varphi)$ and $b(\psi)$ denote the automorphisms of $\lim_{\rightarrow mn} C_{mn}$ induced by applying $b(\varphi)$ or $b(\psi)$, respectively to each $C_{mn}$ (thus $\bar{b}(\varphi)$ and $\bar{b}(\psi)$ correspond to the left shifts with respect to $m$ or $n$, respectively, on a sequence representing an element of the inductive limit). The identity $\gamma_*\kappa_{mn} = \kappa_m(\kappa_{m+1}(n-1))$ shows that $\gamma_*\kappa = \kappa^{1-\gamma_*}(\bar{b}(\psi)-\bar{b}(\varphi))$. Now $\bar{b}(\psi) - \bar{b}(\varphi)$ has the same kernel and cokernel as $1-\gamma_*$. Therefore we get the asserted exact sequence from (11). □

Because of exactness of the direct limit functor we obtain from Theorem 8.2 the following short exact sequence

$$0 \rightarrow \lim_{\rightarrow b(\psi)} \text{Coker} (b(\psi) - b(\varphi)) \rightarrow K_0\mathcal{A}[\varphi/\psi] \rightarrow \lim_{\rightarrow b(\psi)} \text{Ker} (b(\psi) - b(\varphi)) \rightarrow 0$$

where the cokernel $(K_0(C^*(G))/(b(\psi) - b(\varphi))K_0(C^*(G)))$ is taken on $K_0$, while the kernel of $b(\psi) - b(\varphi)$ is taken on $K_1(C^*(G))$. Of course, there also is the analogous exact sequence with the role of $K_0$ and $K_1$ interchanged.

### 8.1. Example.
Let $\varphi$ and $\psi$ be endomorphisms of $\mathbb{Z}$ determined by two relatively prime numbers $p$ and $q$. It is clear that this pair of endomorphisms defines a rational polymorphism of $\mathbb{T}$. The map $(\psi\varphi)_*$ induced by the product is the identity on $K_0(C(\mathbb{T})) = \mathbb{Z}$ and multiplication by $pq$ on $K_1(C(\mathbb{T})) = \mathbb{Z}$. The identity $b(\psi\varphi)(\psi\varphi)_* = pq$ shows that $b(\psi\varphi)$ is multiplication by $pq$ on $K_0(C(\mathbb{T})) = \mathbb{Z}$ and the identity on $K_1(C(\mathbb{T})) = \mathbb{Z}$. It follows that $K_0(B_{\psi\varphi}) \cong \mathbb{Z}[1/pq], K_1(B_{\psi\varphi}) \cong \mathbb{Z}$.

Moreover $\text{Ker} (b(\psi) - b(\varphi)) = 0$ on $K_0(C(\mathbb{T}))$ and $\text{Ker} (b(\psi) - b(\varphi)) = \mathbb{Z}$ on $K_1(C(\mathbb{T}))$. 


Coker \( b(\psi) - b(\varphi) = \mathbb{Z}/(p - q) \) on \( K_0(C(\mathbb{T})) \) and Coker \( b(\psi) - b(\varphi) = \mathbb{Z} \) on \( K_1(C(\mathbb{T})) \). From this and formula (12), \( K_0(\mathfrak{A}[\varphi/\psi]) = \mathbb{Z} \) on \( K_0(C(\mathbb{T})) \) and Coker \( b(\psi) - b(\varphi) \) can easily be computed. Thus, if \( p - q = 1 \), we get \( K_0(\mathfrak{A}[\varphi/\psi]) = \mathbb{Z} \). If for instance \( p = 5 \), \( q = 3 \), we get \( K_0(\mathfrak{A}[\varphi/\psi]) = \mathbb{Z} + \mathbb{Z}/2 \), \( K_1(\mathfrak{A}[\varphi/\psi]) = \mathbb{Z} \).

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