Multi-scaling mix and non-universality between population and facility density

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(Dated: May 2, 2014)

The distribution of facilities is closely related to our social economic activities. Recent studies have reported a scaling relation between population and facility density with the exponent depending on the type of facility. In this paper, we show that generally this exponent is not universal for a specific type of facility. Instead by using Chinese data we find that it increases with Per Capital GDP. Thus our observed scaling law is actually a mixture of some multi-scaling relations. This result indicates that facilities may change their public or commercial attributes according to the outside environment. We argue that this phenomenon results from the unbalanced regional economic level and suggest a modification for previous model by introducing consuming capacity. The modified model reproduces most of our observed properties.

PACS numbers: 89.75.Hc, 89.75.Da, 89.40.Dd

I. INTRODUCTION

Facilities and infrastructures such as hospitals, schools, internet routers, origin from the development of human social civilization and in turn shape our modern daily life. This complex evolution process raises an interesting question that how these facilities are distributed and how they correlate with the whole social economic system. A better understanding on this issue could help to provide better public service and to save social opportunity cost. It is believed that population density and economic ingredient are crucial on deciding the locations of these facilities. This. But unevenly distributed population and economic level make the question very complicated. Although studies ranging from business economics, system engineering, computer science, geography to even biology have addressed on the issue, both theoretical basis and empirical demonstration are inadequate. This causes the arbitrary assumption of uniformly distributed nodes in many spatial network models even though it is far from the reality.

Intuitively, the number of facilities in an area increases with the corresponding population. The related studies can be traced back to a so-called p-median problem which aims to find the precise locations of facilities so that the mean distance that one reaches his nearest facility is minimized. Numerical and analytic treatments have been used to suggest a relation \( D \sim \rho^\alpha \) where \( D \) is the facility density, \( \rho \) is the population density and \( \alpha = 2/3 \) depending on types of facility, ranges from \( 0 \) to \( 1 \) rather than just fixed \( 2/3 \). More specifically, commercial facility and public one has \( \alpha = 1 \) and \( \alpha = 2/3 \), respectively, while for facility with both attributes, the exponent lies intermediate. These findings stimulated a recent study in which a general model based on the tradeoff between the profit (commercial concern) and the social opportunity cost (public service) was proposed. The value of \( \alpha \) in that model represents the type of facility, which is described by the relative weight of commercial and public attributes. In other words it assumes a universal scaling exponent for a specific type of facility. (Actually all the previous studies tacitly approve this assumption.) Thus if we choose a part of these facilities according to some other properties, the scaling exponent is expected to be unchanged in this sample.

As we will show in this paper, this is not always true. At least in Chinese cases, samples from a specific type of facility according to Per Capital GDP yield an increasing scaling exponent. Therefore the macroscopic power law relation between facility and population density is actually a mixture of different scaling functions. This multi-scaling property indicates a different picture that the attribute of a facility changes with the outside environment and consequently affects its real distribution. In the next section we will provide empirical evidences for the above arguments. And in section III, we will try to offer an explanation and present a modified model to reproduce the multi-scaling property.

II. EMPIRICAL STUDY

We have gathered 7-year empirical data (2001 – 2007) including the positions of 4 typical types of facility (hospital, post office, school and theater), the population, GDP and the area of every county. Despite of the temporal fluctuations, there are totally more than 287 counties and about 60500 hospitals, 56300 post offices, 309200 schools and 5000 theaters. All these data come from the CHINA CITY STATISTICAL YEARBOOK whose electronic versions can be found and downloaded at [http://ishare.iask.sina.com.cn](http://ishare.iask.sina.com.cn). These data allow us to calculate the population density, facility density, Per Capital GDP and their correlation at county level. Such coarse-grained treatment was also applied in the previous study due to the resolution limitation of the facility.
The class from 1 to 4 describes the increase of the Per Capital GDP level. For each class the data is logarithmic binned and is plotted on the log-log scale. Dotted lines are their corresponding fits, which are measured as $D \sim \rho^{0.7}$ (blue) and $D \sim \rho^{0.96}$ (red). (b) The yearly scaling exponents for all the four facilities. Dotted lines are their corresponding averages calculated as 0.71, 0.96, 0.83, 0.77 for post office, theater, school, and hospital, respectively.

Our first observation consists with the previous studies. Indeed a scaling relation between population and facility density emerges in all the four types of facility (Fig. 1(a)). A detailed analysis on their scaling exponents indicates that despite of the yearly fluctuations, the exponents stay around their own averages which are measured as 0.71, 0.96, 0.83, 0.77 for post office, theater, school, and hospital respectively. According to Ref. [15], post office and theater whose scaling exponents are close to $2/3$ and 1 can be viewed as the representative of public and commercial facility. This is consistent with our experience that post office disregarding profit is necessary everywhere while theater behaves oppositely.

To examine the universality assumption of scaling exponent, we classify the counties into four different classes. Specifically, we calculate the Per Capital GDP for each county and divide their logarithmic values into four equal intervals. Those counties whose Per Capital GDP lie in a common interval belong to the same class. Consequently counties in class 1 have the lowest Per Capital GDP while those in class 4 have the highest one. This classification distributes about 40 counties in class 1 and over 70 ones in each of the other three. Although the number of counties in class 1 is almost half less, the number of the corresponding facilities is still large enough to apply statistical analysis. Another problem is whether the Per Capital GDP correlates obviously with the population density so that our method might cause serious statistical bias. This possibility is basically ruled out as the correlation turns out to be very weak (correlation coefficient $< 0.25$).

For each type of facility, we study their scaling relation in different classes. As illustrated in Fig. 2 in each class the relation between population and facility density is still a power law. But their scaling exponents are not equal but increase clearly with the Per Capital GDP level (class number). This result contrasts with the universality assumption of the scaling exponent in Ref. [15]. Instead it indicates that the observed scaling relation $D \sim \rho^\alpha$ is actually composed of some multi-scaling behaviors. If we accept the physical meaning of $\alpha$ interpreted in Ref. [15], the multi-scaling phenomenon reveals an interesting fact that facilities can change their attributes according to the outside economic environment. Particularly, they tend to be commercial in high Per Capital GDP level area but still provide necessary public services in poor developed places. Further detailed analysis suggests that this multi-scaling property as well as the positive correlation between the multi-scaling exponents and Per Capital GDP level occurs every year in all the types of facility regardless of their temporal fluctuations (Fig. 3). For school and hospital (Fig. 3(a) and Fig. 3(b)), the exponents vary from $2/3$ to 1, which almost covers the whole possible range. In contrast, the exponents of post office are much more stable. As shown in Fig. 3(c) they stay near 0.7 and rise up to no more than 0.85. Similarly in Fig. 3(d), the range of the exponents of theater narrows around 0.9. It seems that purely commercial or public facilities are not likely to behave diverse attributes, which is somewhat consistent with our intuition.

### III. Explanation and Model Modification

All these findings are not captured by the previous models and thus require a more complete theory. As will be presented later, a small modification by introducing the consuming capacity can reproduce most of these properties. Before that, we will first review the model of...
This means that the number of visitors to the $i$-th facility, which is denoted as $n_i$, is better off moved to other locations with higher population for higher profit. This strategy is applied by all the facilities during the relocating process. Finally the system will reach an equilibrium so that every facility has almost the same profit, i.e. $n_i \sim N_p/N_f$. Then by using the expression $\rho_i$ and $D_i$ calculated above, we arrive at $D \sim \rho$. On the other hand, public facility concerns prior the social opportunity cost caused by the distance between visitors and facilities, which is described by $n_i\langle r_i \rangle$ with $\langle r_i \rangle \sim \sqrt{s_i}$ representing the average distance to the $i$-th facility. To provide better public service, facilities at lower-cost places should be relocated to those with higher $n_i\langle r_i \rangle$. Then in steady state, $n_i\langle r_i \rangle$ becomes the same for all the facilities. Again by using the expression $\rho_i$ and $D_i$, we have $D \sim \rho^{2/3}$. For facilities with both attributes, Ref. \cite{15} defines a general quantity
\[ c_i = n_i\langle r_i \rangle^\beta, \quad (1) \]
where $\beta$ is tunable within the range $[0, 1]$. Analogous to the above analysis, one can derive the final scaling relation $D \sim n^{2/\beta}$, which gives $\alpha = 2/(\beta + 2)$.

However in this nicely compact model, the assumption that profit equals to population is too simple. After all, the profit is yet closely related to the commodity prices and people’s consuming capacity. Facilities providing luxury commodity are not opened at poor places even though they are densely populated because no one can bear such high level consumption. On the other hand, facilities in the regions with lower population but higher consuming capacity can still benefit from high prices. Therefore facilities with both attributes, due to their ingredient of commercial part, take chance to locate in the regions with lower population and gain at developed area even though the service there provided by their public ingredient is adequate enough. This causes the number of facility in a well developed area is larger than those in poor places even if they are equally populated, which indicates a more rapid increase of facility in high economic-level places. If we assume the consuming capacity has a positive correlation with Per Capital GDP, which seems plausible, the above explanation for multi-scaling does make sense.

To make the explanation more convincing, it is useful to introduce an alternative expression of Eq.(1) to characterize the transition from public to commercial facility, expressed as
\[ c_i = \lambda n_i + (1 - \lambda)n_i\langle r_i \rangle, \quad (2) \]
where $\lambda \in [0, 1]$ is a tunable parameter controlling the relative weight of commercial or public attribute and consequently determining the final exponent $\alpha$ as $\alpha = (2/(\beta + 2))\lambda$, just as the role of $\beta$ in Eq.(1). Eq.(2) has the same physical meaning and similar effect to Eq.(1) but turns out to be more difficult to apply analytical treatment. However, one can still prove its scaling property by the method used in Ref. \cite{16}. To introduce the consuming capacity to the model, we denote $m_i$ as the average expense consumed by every person living in Voronoi cell $V_i$ so that the profit equals to $m_i n_i$. Then Eq.(2) is rewritten as
\[ c_i = \lambda m_i n_i + (1 - \lambda)n_i\langle r_i \rangle. \quad (3) \]

The only modification compared to Eq.(2) lies in the first term of the righthand of Eq.(3), i.e. $\lambda \to \lambda m_i$. This modification does not affect the macroscopic scaling law between $D$ and $\rho$ qualitatively, but changes the relative weight of commercial attributes on microscopic level. Particularly, for large $m_i$ the first term $\lambda m_i n_i$ is enhanced by the effective weight $\lambda m_i$, which causes the system to concern more about profit. This leads the facility to be more commercial and the scaling exponent to be close to 1. On the other hand if $m_i$ is small, the second term $(1 - \lambda)n_i\langle r_i \rangle$ takes over, then the system concerns more about social opportunity cost just like public...
FIG. 4: Multi-scaling relation between $D$ and $\rho$ simulated by the modified model with $\mu = 1$. The data is plotted on the log-log scale. The class from 1 to 4 describes the increase of the level of $m$. The relation in each class is power law but the exponent increases with the class number just as in Fig. 4. Dotted lines are their corresponding fits plotted here for guiding eyes. Inset: The simulated multi-scaling exponent $\mu$ (red) and $\beta$ (purple), which represent the pure commercial and public facility, the exponents stable at 1 and 2/3, respectively. For intermediate $\mu$, such as $\mu = 1$ (blue), the exponent increases with the classification. The result is averaged over 50 simulations.

FIG. 5: The relation between the scaling exponent $\alpha$ and the parameter $\mu$. The green points are the simulation result which is averaged over 50 realizations. The red dotted lines are the analytical prediction. Inset: The scaling relation between population and facility density simulated by our modified model with $\mu = 1$. The simulation data is logarithmic binned and is plotted on the log-log scale. Red dotted line is the fit measured as $D \sim \rho^{0.8}$.

Facility and the exponent becomes close to $2/3$. Therefore even for the same $\lambda$ (i.e. the same type of facility), different $m_i$ leads to different system behavior. This is the reason why multi-scaling emerges and why their exponents increase with the economic level. Moreover, if $\lambda = 0$ (purely public facility), Eq.(3) becomes $m_i$ independent. So the system degenerates to the classical model in Ref.[13] and thus displays only a single scaling relation with the exponent $\alpha = 2/3$. On the other hand, if $\lambda = 1$ (purely commercial facility), Eq.(3) depends totally on the first term. But $m_i$ in this case has no effect on the exponent but only changes the coefficient of the scaling relation, leaving the only exponent $\alpha = 1$ which is also independent of $m_i$. Therefore multi-scaling property can be less pronounced in a very commercial or public facility, which explains the narrow range of the multi-scaling exponents observed in post office and theater.

If we still adopt the idea in Ref.[13] and follow the expression of Eq.(1), the above explanation indicates that the exponent $\beta$ in Eq.(1) should be modified to be a function of parameter $m_i$, i.e. $c_i = n_i \langle r_i \rangle^{\beta(m_i)}$. Then the multi-scaling exponent can be calculated as $2/(\beta(m_i) + 2)$ and the macroscopic exponent $\alpha$ can be given by taking an average over all possible $m_i$. To reproduce the multi-scaling property by simulation, we apply the following simple formula of $\beta(m)$

$$\beta(m) = 1 - \left(\frac{m}{m_{\max}}\right)^\mu, \quad (4)$$

where $m_{\max}$ represents the possible maximum of $m$ and $\mu$ is a parameter controlling the sensitivity of $\beta$ with $m$. If $\mu = 0$, the facility becomes purely commercial while if $\mu \to \infty$, it becomes purely public. So $\mu$ also controls the exponent $\alpha$. The real-world $\beta(m)$ can be quite different (probably related to the type of facility). Finding its precise expression could be a complicated task and is beyond the scope of this paper. The aim of our simulation is only to reproduce the multi-scaling phenomenon qualitatively. Our simulation follows the similar process to that of Ref.[13] except for the above-proposed modification. Specifically, we first distribute the population density $\rho$ and the economic level $m$ randomly on a plane. And then we further put some facilities of a certain type. At each time step, every facility calculates its benefit according to $c_i = n_i \langle r_i \rangle^{\beta(m_i)}$. Facility with the lowest $c_i$ then moves to the region with the highest benefit. This process repeats until the system is steady. At steady state, we measure various quantities and their relations just as done for empirical data. Note that the classification here is carried out according to the value of $m$. Other details of the simulation such as parameter setting are described in APPENDIX A. In Fig. 4 we plot the simulated relation between population and facility density for different classes (i.e. different level of $m$ or say different interval of $m$). Clearly it displays multi-scaling property with the exponents increasing with class number. The inset of Fig. 4 presents the results of the simulated multi-scaling exponents. For purely commercial ($\mu = 0$) or purely public ($\mu \to \infty$) facility, the exponents become stable at 1 or 2/3. But for intermediate
μ = 1, the exponents increase. In the inset of Fig. 5 we demonstrate that the whole scaling relation between D and ρ maintains in our modified model. And the scaling exponent α in the simulation decreases with the parameter μ, as shown in Fig. 5. Note that all these results can be calculated analytically. We present an analytical solution of α(μ) in Fig. 5, which is in good agreement with the simulation. More details about the analytical calculations are reported in APPENDIX B.

IV. CONCLUSION

We have analyzed the scaling relation between population and Chinese facility density at different Per Capital GDP levels. Our study does not see the universality of the scaling exponent but instead suggests a multi-scaling picture. More interestingly, such multi-scaling exponents increase with the Per Capital GDP regardless of the type of facility. These results indicate that facilities can change their commercial or public attributes according to the outside environment, i.e. they take chance to gain more profit in developed area but still fulfill their public-service responsibility in poor region. We have also provided possible explanation and suggest a modification by considering consuming capacity. The modified model can reproduce most of our observed properties.

Our study stresses on existence rather than universality. Indeed the multi-scaling property is observed every year for all the four types of facility. Therefore their occurrence is certain rather than coincidental. On the other hand it is also appealing to explore whether this phenomenon occurs in other more developed country where the economic system is more balanced and stable. Due to the limitation of data, the present study cannot cover this area. If this point is evidenced, it indicates the multi-scaling could be a common property. Otherwise it either requires explanations from sociocultural, economic, politics, etc or leads us to a long-time dynamic evolution picture of facility allocation, both of which are significant on understanding our social economic system or even providing guidelines to urban development.

This work was partially supported by the National Nature Science Foundation of China under Grant Nos. 11075057, 11035009 and 10979074.

Appendix A: simulation details

We use a coarse-grained simulation for our modified model. We first divide a plane into many unit squares whose area all equal to 1 and assume that all the situations in one unit square are approximately identical. Then we distribute for each unit square u the population ρu and the consuming capacity mu according to the corresponding distribution p(ρ) and p(m). And we further distribute randomly the initial number of facility Du(t = 0). We use denotation Du(t) to emphasize that the facility number changes with time during the simulation while ρu and mu are always fixed as soon as they are distributed. Since the area of all u is 1, ρu and Du is exactly the population density and facility density respectively. The number of people visiting facility i in place u, denoted as ni,u, is calculated as ni,u = ρu/Du while the average distance for these people to travel to facility i, denoted as ⟨ri,u⟩, is calculated as ⟨ri,u⟩ ~ √ni,u = 1/√Du. Then we can determine the benefit ci,u = ni,u⟨ri,u⟩/Du ~ μ/Du+β(mu) for every facility i in the unit square u. Note that this quantity only depends on place u and is equal for any i within this unit square, so we can replace ci,u by the denotation cu. At each time step of our simulation, we calculate the cu for each places according to the current Du(t). Then we eliminate a facility in the place with the lowest cu and create one in the unit square with the highest cu. This procedure repeats until the system reaches its steady state, at which the relation between D and ρ as well as the exponents is stable.

We test this coarse-grained method by repeating the simulation in Ref. [15], i.e. setting β(m) = constant as Eq. (1). We find the method behaves the same as the simulation in Ref. [15] and reproduces all their results. In our own simulation, we have 200 different unit squares. And we set p(ρ) = 1/ρ with ρ ∈ [0, 5000] and p(m) = 1/300 with m ∈ [0, 300]. The initial distribution of facility is also uniform with D(t = 0) ∈ [0, 1000]. We choose power-law distribution of p because i). it coincides with the real population distribution which is observed to be heavy-tailed; ii). the power-law formula leads to a uniformly distributed data points on log-log plot, which gives a better visualization. Note that other distributions do not change the simulation results qualitatively. By these parameters, we obtain β(m) = 1 – (m/300)β.

When the simulation reaches its steady state, we measure various quantities and their relations just as done for empirical data. Note that the classification here is carried out according to the value of m. Specifically, we divide the value of m (not the logarithmic value of m) into four equal intervals. Those unit square u whose mu lies in a common interval belongs to the same class. Although there is a bit difference from what we have done for empirical data, it does not change our conclusion at all.

Appendix B: analytical calculations

Since α = d(ln(Du))/dmu = ⟨αi⟩mu, the macroscopic exponent is exactly an average of αi over all possible mi, i.e. α = ∫mmax p(m)α(m)dm, where α(m) = 2/(β(m) + 2). Substituting the corresponding parameters and equation, we have

\[ α(μ) = \int_0^1 \frac{2}{3 - x^μ} dx. \]  

(B1)
This function gives a good agreement with the simulation, as plotted in Fig. [5]. Particularly, for $\mu = 0, 0.2, 0.5, 1, 2, 5, 10$, Eq.(B1) gives $\alpha = 1, 0.926, 0.866, 0.81, 0.76, 0.712, 0.692$, which is consistent with the simulation data $\alpha = 1, 0.91, 0.85, 0.8, 0.755, 0.71, 0.68$ in Fig. [5].

We can also calculate the multi-scaling exponent for each class. The calculation is given by $\alpha(C) = \int_{m_{\min}}^{m_{\max}} p(m|C)\alpha(m)dm$, where $p(m|C)$ is the conditional probability density under class $C$ and equals to $1/75$ in our simulation. $m_{\min}$ and $m_{\max}$ here is related to the specific class $C$. We calculate the multi-scaling exponents for $\mu = 1$ and have the theoretical result $0.69, 0.76, 0.84, 0.94$ for class $1, 2, 3, 4$, which is also consistent with the simulation data $0.68, 0.74, 0.81, 0.91$ in the inset of Fig. [4].

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[18] Since we assume that all the situations in one unit square are identical, both the number of people and the area belonging to a facility are equal everywhere, leading $n_i$ and $s_i$ to equaling to their corresponding average, i.e. $\rho_u/D_u$ and $1/D_u$, respectively.