Application of stochastic calculus for certain classes of quantum models

A Pavelev¹, V Semin¹

¹Samara National Research University, Moskovskoe Shosse 34A, Samara, Russia, 443086

Abstract. In this work, stochastic calculus is applied for describing open quantum systems. The suggested approach is based on a modified stochastic Schroedinger equation. We apply the approach to investigate a qubit relaxation in Markovian and non-Markovian regimes.

1. Introduction
Stochastic calculus is a theory that describes stochastic (or random) processes in various areas such as systems theory, electronic engineering, financial mathematics and others [1]. Application of stochastic calculus to some models of quantum physics has the great interest to the scientific society, because this approach has many advantages [2].

The quantum theory has a probabilistic nature, and it makes stochastic calculus more appropriate for its investigation rather than deterministic differential equations [3]. Moreover, realistic quantum systems always interact with some environment, which has a very large number of degrees of freedom. The environment often can be described as a stochastic process with some statistical properties. The statistical properties of the environment allow us to cover both Markovian and non-Markovian regimes of dynamics [2]. The first regime completely neglects any memory effects of the environment. In other words, the system-environment correlation time is vanishingly small. Recent experiments have revealed that many systems show strong non-Markovian behavior [4]. It means, that for some classes of quantum models stochastic calculus is suitable and productive.

In this work we investigate the application of stochastic calculus for describing Markovian and non-Markovian relaxation of a qubit in a bosonic reservoir.

2. Stochastic Schroedinger equation
One of the useful approaches to quantum systems is based on a stochastic Schrodinger equation (SSE) [5]. The diffusive SSE is a stochastic differential equation for a state vector and has the following general structure:

$$d|\psi\rangle = A|\psi\rangle dt + \sum_i B_i |\psi\rangle dW_i,$$

(1)

where $A$, $B_i$ are some operators acting in the Hilbert space of the open quantum system, and $dW_i$ are independent Wiener processes.

Each solution of this equation represents an independent quantum trajectory. An averaging over a large number of quantum trajectories is needed to cover some physical. For example, the density matrix can be expressed through the following relation:

$$\rho = E(|\psi\rangle\langle\psi|),$$

(2)

where $E$ is the stochastic averaging.
3. Model
Let’s consider the master equation for one qubit in the bosonic thermostat [6].
\[
\frac{d\rho}{dt} = \gamma (\sigma_+\rho - 2\sigma_\rho + \rho\sigma_+),
\]
(3)
where \(\rho\) is the density matrix, \(\sigma_+\), \(\sigma_-\) raising and lowering operators, \(\gamma\) is the damping constant.

The stochastic Schrodinger equation corresponding to Eq. (3) is [3]:
\[
d|\psi\rangle = A|\psi\rangle dt + B|\psi\rangle dW,
\]
(4)
where operators \(A\) and \(B\) have the following form:
\[
A = \frac{\gamma}{2}\sigma_+, B = \sqrt{\gamma}\sigma_-.
\]
(5)
To generalize SSE to describe the non-Markovian regime, by analogy to the paper [7] and our recent work [8], one replaces the Wiener Markovian process with the non-Markovian Ornstein-Uhlenbeck process, which is well suited for the consideration of memory effects of the environment, and is defined by the equation:
\[
dX = -kX dt + dW,
\]
(6)
where \(k\) characterizes correlation time with the thermostat. The dynamics is obviously Markovian when \(k\) goes to 0.

By substituting the Ornstein-Uhlenbeck process instead of the Wiener process and require the martingale property, one can derive the equation for the normalized state vector:
\[
|\tilde{\psi}\rangle = |\psi\rangle\|\psi\|^{-1},
\]
(7)
which has the following form:
\[
d|\tilde{\psi}\rangle = \left(A - kX(B^\dagger + B) + \frac{1}{2}B(\tilde{\psi}|B^\dagger + B|\tilde{\psi}) - \frac{1}{8}B^2(\tilde{\psi}|B^\dagger + B|\tilde{\psi})^2\right)|\tilde{\psi}\rangle dt +
\]
\[
+ \left(B - \frac{1}{2}B(\tilde{\psi}|B^\dagger + B|\tilde{\psi})\right) d\tilde{W},
\]
(8)
where \(d\tilde{W}\) is the new Wiener process noise, that relates to the old one as [8]
\[
d\tilde{W} = dW - \langle \tilde{\psi}|B^\dagger + B|\tilde{\psi}\rangle dt
\]
(9)
Note, that this equation is nonlinear, but preserve the norm of the state vector. The last allows one efficiently simulate the solution.

4. Results of simulation
Results of numerical simulation of equation (8) are presented in the figures 1 (Markovian case) and 2 (non-Markovian case). On the X-axis is the dimensionless time \(\gamma t\), on the Y-axis is probability \(P\) to find the system either in the ground or in the excited energy states. The initial condition of the state vector is chosen to be \(|\tilde{\psi}(\gamma t_0)\rangle = |0; 1\rangle^T\). To obtain a stochastic trajectory we solve the equation (8) with the explicit Euler method [9]. The results are averaged over 2500 trajectories. The standard deviation is smaller than the thickness of the curves.

![Figure 1](image-url)
Figure 2. The dynamics of excited excited (solid line) and ground (dotted line) states probabilities in the non-Markovian environment, $k=0.8$.

The figures demonstrate that the non-Markovianity strongly affects the dynamics and shifted the stationary values of probabilities to find the quantum system in one of the energy states.

5. Conclusion
In this work we investigated the application of stochastic calculus for the describing of quantum systems on the example of Markovian and non-Markovian relaxation of a qubit. We numerically simulated the solution of the stochastic Schrodinger equation. We have shown, that non-Markovian dynamics differed in comparison with Markovian case. It means, that stochastic calculus gives a powerful framework for the analysis of open quantum systems.

6. Acknowledgments
This work was supported by Russian Foundation for Basic Research, grant N 18-32-00249.

7. References
[1] Cohen S 2015 *Stochastic Calculus and Applications* (New York: Birkhauser).
[2] Breuer H P 2002 *The Theory of Open Quantum Systems* (Oxford University Press).
[3] Barchielli A 2009 *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case* (Springer).
[4] Liu B H, Huang Y F, Li C F, Guo G C, Laine E M, Breuer P H and Piilo J 2011 Experimental control of the transition from Markovian to non-Markovian dynamics of open quantum systems *Nature Physics* 7 931-934.
[5] Gisin N 1989 Stochastic quantum dynamics and relativity *Helv. Phys. Acta.* 62(4) 363-371.
[6] Scully O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press).
[7] Barchielli A, Pellegrini C and Petruccione F 2010 Stochastic Schrödinger equations with coloured noise *Europhysics Letters* 91(2) 24001.
[8] Pavelev A V and Semin V V 2019 Investigation of non-markovian dynamics of two dipole-dipole interacting qubits based on numerical solution of the non-linear stochastic Schrödinger equation *Computer Optics* 43(2) 168-173 DOI: 10.18287/2412-6179-2019-43-2-168-173.
[9] Øksendal B 2003 *Stochastic Differential Equations* (Berlin: Springer).