THE LOCAL LIFTING PROBLEM FOR $A_4$

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Abstract. We solve the local lifting problem for the alternating group $A_4$, thus showing that it is a local Oort group. Specifically, if $k$ is an algebraically closed field of characteristic 2, we prove that every $A_4$-extension of $k[[s]]$ lifts to characteristic zero.

1. Introduction

This paper concerns the local lifting problem about lifting Galois extensions of power series rings from characteristic $p$ to characteristic zero:

Problem 1.1. (The local lifting problem) Let $k$ be an algebraically closed field of characteristic $p$ and $G$ a finite group. Let $k[[z]]/k[[s]]$ be a $G$-Galois extension (that is, $G$ acts on $k[[z]]$ by $k$-automorphisms with fixed ring $k[[s]]$). Does this extension lift to characteristic zero? That is, does there exist a DVR $R$ of characteristic zero with residue field $k$ and a $G$-Galois extension $R[[Z]]/R[[S]]$ that reduces to $k[[z]]/k[[s]]$?

We will refer to a $G$-Galois extension $k[[z]]/k[[s]]$ as a local $G$-extension. Basic ramification theory shows that any group $G$ that occurs as the Galois group of a local extension is of the form $P \rtimes \mathbb{Z}/m$, with $P$ a $p$-group and $p \nmid m$. In [CGH11], Chinburg, Guralnick, and Harbater ask, given a prime $p$, for which groups $G$ (of the form $P \rtimes \mathbb{Z}/m$) is it true that all local $G$-actions (over all algebraically closed fields of characteristic $p$) lift to characteristic zero? Such a group is called a local Oort group (for $p$). Due to various obstructions (The Bertin obstruction of [Ber98], the KGB obstruction of [CGH11], and the Hurwitz tree obstruction of [BW09]), the list of possible local Oort groups is quite limited. In particular, due to [CGH11, Theorem 1.2] and [BW09], if a group $G$ is a local Oort group for $p$, then $G$ is either cyclic, dihedral of order $2p^n$, or the alternating group $A_4$ ($p = 2$). Cyclic groups are known to be local Oort — this is the so-called Oort conjecture, proven by Obus-Wewers and Pop in [OW14], [Pop14]. Dihedral groups of order $2p$ are known to be local Oort for $p$ odd due to Bouw-Wewers ([BW06]) and for $p = 2$ due to Pagot ([Pag02]). The group $D_9$ is local Oort by [Obu15]. Our main theorem is:

Theorem 1.2. If $k$ is an algebraically closed field of characteristic 2, then every $A_4$-extension of $k[[s]]$ lifts to characteristic zero. That is, the group $A_4$ is a local Oort group for $p = 2$.

This result was announced by Bouw (see the beginning of [BW06]), but the proof has not been written down. Our proof uses a simple idea that avoids the “Hurwitz tree” machinery.
A deformation of a local $A_4$-extension by what we call their “break” (this is a jump in the higher ramification filtration). One then uses the following strategy of Pop ([Pop14]), sometimes known as the “Mumford method”: First, make an equicharacteristic is a jump in the higher ramification filtration). One then uses the following strategy of Pop communicated to the author after the first draft of this paper was written (see Remark 5.3). Namely, one first classifies local $A_4$-extensions with small breaks lift. An induction finishes the proof.

We remark that Florian Pop has his own similar proof of Theorem 1.2, which was communicated to the author after the first draft of this paper was written (see Remark 5.3).

The main motivation for the local lifting problem is the following global lifting problem, about deformation of curves with an action of a finite group (or equivalently, deformation of Galois branched covers of curves).

**Problem 1.3.** (The global lifting problem) Let $X/k$ be a smooth, connected, projective curve over an algebraically closed field of characteristic $p$. Suppose a finite group $\Gamma$ acts on $X$. Does $(X, \Gamma)$ lift to characteristic zero? That is, does there exist a DVR $R$ of characteristic zero with residue field $k$ and a relative projective curve $X_R/R$ with $\Gamma$-action such that $X_R$, along with its $\Gamma$-action, reduces to $X$?

It is a major result of Grothendieck ([SGA03, XIII, Corollaire 2.12]) that the global lifting problem can be solved whenever $\Gamma$ acts with tame (prime-to-$p$) inertia groups, and $R$ can be taken to be the Witt ring $W(k)$. More generally, the local-global principle states that $(X, \Gamma)$ lifts to characteristic zero over a complete DVR $R$ if and only if the local lifting problem holds (over $R$) for each point of $X$ with nontrivial stabilizer in $\Gamma$. Specifically, if $x$ is such a point, then its complete local ring is isomorphic to $k[[z]]$. The stabilizer $I_x \subseteq \Gamma$ acts on $k[[z]]$ by $k$-automorphisms, and we check the local lifting problem for the local $I_x$-extension $k[[z]]/k[[z]]^{I_x}$. Thus, the global lifting problem is reduced to the local lifting problem. Proofs of the local-global principle have been given by Bertin-Mézard ([BM00]), Green-Matignon ([GM98]), and Garuti ([Gar96]).

One consequence of the local-global principle and Theorem 1.2 is the following:

**Corollary 1.4.** The groups $A_4$ and $A_5$ are so-called Oort groups for every prime. That is, if $\Gamma \in \{A_4, A_5\}$ acts on a smooth projective curve $X$ over an algebraically closed field of positive characteristic $p$, then $(X, \Gamma)$ lifts to characteristic zero.

**Proof.** By the local-global principle (see also [CGH08, Theorem 2.4]), it suffices to show that every cyclic-by-$p$ subgroup of $A_4$ or $A_5$ is a local Oort group for $p$. The only subgroups of $A_4$ of this form for any $p$ are isomorphic to the trivial group, $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/3$, or $A_4$. The subgroups of $A_5$ of this form are isomorphic to the trivial group, $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/5$, $D_3$, $A_4$, and $D_5$. All these are local Oort groups for the relevant primes, as has been noted above.

1.1. **Conventions/Notation.** Throughout, $k$ is an algebraically closed field of characteristic 2. The ring $R$ is a large enough complete discrete valuation ring of characteristic zero with residue field $k$, maximal ideal $m$, and uniformizer $\pi$. We normalize the valuation $v$ on $R$ so that $v(2) = 1$. In any polynomial or power series ring with coefficients in $R$, the expression $\alpha(x)$ for $x \in R$ means a polynomial or power series with coefficients in $xk$.

The ring $k[[t]]$ is always a $\mathbb{Z}/3$-extension of $k[[s]]$ with $t^3 = s$. Likewise, $R[[T]]$ is always a $\mathbb{Z}/3$-extension of $R[[S]]$ with $T^3 = S$. If $k[[z]]/k[[s]]$ is an extension, it is always assumed to
contain \( k[[t]] \). Our convention for variables is that lowercase letters represent the reduction of capital letters from characteristic 0 to characteristic 2 (e.g., \( t \) is the reduction of \( T \)).

We write \( \zeta_3 \) for a primitive 3rd root of unity in any ring.

2. \( A_4 \)-extensions

We start with the basic structure theory of \( A_4 \)-extensions.

2.1. \( A_4 \)-extensions in characteristic 2.

Lemma 2.1. Let \( K \subseteq L \subseteq M \) be a tower of field extensions of characteristic 2 such that \( L/K \) is \( \mathbb{Z}/3 \)-Galois and \( \text{Gal}(M/L) \) is \( \mathbb{Z}/2 \)-Galois. Let \( \sigma \) be a generator of \( \text{Gal}(L/K) \). For \( \ell \in L \), let \( \overline{\ell} \) denote the image of \( \ell \) in \( L/(F-1)L \), where \( F \) is Frobenius. Suppose \( M \cong L[x]/(x^2-x-a) \), and let \( d \) be the dimension of the \( \mathbb{F}_2 \)-vector space generated by \( \overline{a}, \overline{\sigma(a)}, \) and \( \overline{\sigma^2(a)} \). If \( N \) is the Galois closure of \( M \) over \( L \), then \( \text{Gal}(N/K) \) can be expressed as a semi-direct product \( \cong (\mathbb{Z}/2)^d \rtimes \mathbb{Z}/3 \).

Proof. By the Schur-Zassenhaus theorem, it is enough to prove that \( \text{Gal}(N/L) \cong (\mathbb{Z}/2)^d \).

But \( N/L \) is clearly generated by Artin-Schreier roots of \( a, \sigma(a), \) and \( \sigma^2(a) \). Thus the result follows from Artin-Schreier theory.

Corollary 2.2. If \( d = 2 \) in Lemma 2.1 then \( \text{Gal}(N/K) \cong A_4 \).

Proof. The group \( \text{Gal}(N/K) \) must be a semi-direct product \( (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3 \) that is nonabelian (as there exists a non-Galois subextension). The only such group is \( A_4 \).

If \( K = k((s)) \) in Lemma 2.1 above, then after a change of variable, we may assume that \( L = k((t)) \) with \( t^3 = s \). Then, it is easy to see that an Artin-Schreier representative \( a \) of \( M/L \) may be chosen uniquely such that \( a \in t^{-1}k[t^{-1}] \) and \( a \) has only odd-degree terms. We say that such an \( a \) is in standard form. In this case, a standard exercise shows that the break in the higher ramification filtration of \( M/L \) (i.e., the largest \( i \) such that the higher ramification group \( G_i \) is nontrivial) occurs at \( \text{deg}(a) \), thought of as a polynomial in \( t^{-1} \).

Corollary 2.3. Suppose \( K = k((s)) \) and \( L = k((t)) \). Suppose \( a \in t^{-1}k[t^{-1}] \subseteq L \) is in standard form. Using the notation of Lemma 2.1 we have \( \text{Gal}(N/K) \cong A_4 \) if and only if \( a \) has no nonzero terms of degree divisible by 3.

Proof. Since linear combinations of elements of \( L \) in standard form are also in standard form, Lemma 2.1 and Corollary 2.2 imply that \( \text{Gal}(N/K) \) is \( A_4 \) if and only if if the \( \mathbb{F}_2 \)-subspace \( V \) of \( L \) generated by \( a, \sigma(a), \) and \( \sigma^2(a) \) has dimension 2. If \( a \) has no nonzero terms of degree divisible by 3, then \( a + \sigma(a) + \sigma^2(a) = 0 \) is the only \( \mathbb{F}_2 \)-linear relation that holds among the conjugates of \( a \), so \( \text{dim} V = 2 \) (note that \( a \neq 0 \) since it is an Artin-Schreier representative of \( M/L \)). Conversely, if \( a \) has a nonzero term of degree divisible by 3, then either no \( \mathbb{F}_2 \)-linear relation holds, or \( a \in k((s)) \) (in which case \( a = \sigma(a) = \sigma^2(a) \)). In either case, \( \text{dim} V \neq 2 \).

If \( d = 2 \) in the context of Lemma 2.1, we say that \( a \in L \) gives rise to the \( A_4 \)-extension \( N/K \). By abuse of notation, if \( K \cong k((s)) \), we say that the break of \( N/K \) is the ramification break of \( M/L \). This is the same as the unique ramification break of \( N/L \) in either the upper or lower numbering. Furthermore, if \( K = k((s)) \) and \( N = k((z)) \), we also say that \( a \) gives rise to the extension \( k[[z]]/k[[s]] \).

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Proposition 2.4. If $K = k((s))$ and $N/K$ is an $A_4$-extension with break $\nu$, then $\nu \equiv 1$ or 5 (mod 6).

Proof. If $a$ gives rise to $N/K$ and is in standard form, we know that $\nu$ is the degree of $a$ in $t^{-1}$. This must be odd, and by Corollary 2.3, it cannot be divisible by 3. \hfill \qedsymbol

2.2. $A_4$-extensions in characteristic zero. The story in characteristic zero (or odd characteristic) is completely analogous. We state the result for reference and omit the proof, which is the same as in \S 2.1 with Kummer theory substituted for Artin-Schreier theory.

Proposition 2.5. Let $K \subseteq L \subseteq M$ be a tower of separable field extensions of characteristic $\neq 2$ such that $L/K$ is $\mathbb{Z}/3$-Galois and $\text{Gal}(M/L)$ is $\mathbb{Z}/2$-Galois. Let $\sigma$ be a generator of $\text{Gal}(L/K)$. For $\ell \in L^\times$, let $\ell$ denote the image of $\ell$ in $L^\times/(L^\times)^2$. Suppose $M \cong L[x]/(x^2-a)$, and let $d$ be the dimension of the $\mathbb{F}_2$-subspace of $L^\times/(L^\times)^2$ generated by $\overline{\pi}, \sigma(a)$, and $\sigma^2(a)$. If $N$ is the Galois closure of $M$ over $L$, then $\text{Gal}(N/K)$ can be expressed as a semi-direct product $\cong (\mathbb{Z}/2)^d \times \mathbb{Z}/3$. In particular, if $d = 2$, then $\text{Gal}(N/K) \cong A_4$.

In the context of Proposition 2.5 we again say that $a \in L$ gives rise to $N/K$.

3. Characteristic 2 deformations

For this section, let $K$, $L$, $M$, $N$ be as in \S 2.1 with $K = k((s))$ and $L = k((t))$. Let $N = k((z))$. Suppose $\text{Gal}(N/K) \cong A_4$, and $N/K$ is given rise to by $a \in t^{-1}k[t^{-1}]$. Our goal is to prove the following proposition.

Proposition 3.1. Suppose that $\nu > 6$ and all $A_4$-extensions $N'/K$ with break $\leq \nu - 6$ lift to characteristic zero. Then $N'/K$ lifts to characteristic zero.

Our proof follows an idea of Pop ([Pop14]). As in [Pop14] and [Obu15], we make a deformation in characteristic 2 so that the generic fiber has “milder” ramification than the special fiber.

Proposition 3.2. Let $A = k[[\varpi, s]] \supseteq k[[s]]$, and let $\mathcal{K} = \text{Frac}(A)$. There exists an $A_4$-extension $\mathcal{N}/\mathcal{K}$, with $\mathcal{N} \supseteq N$, having the following properties:

(1) The unique $\mathbb{Z}/3$-subextension $\mathcal{L}/\mathcal{K}$ of $\mathcal{N}/\mathcal{K}$ is given by $\mathcal{L} = \mathcal{K}[t] \subseteq \mathcal{N}$.

(2) If $\mathcal{C}$ is the integral closure of $A$ in $\mathcal{N}$, we have $\mathcal{C} \cong k[[\varpi, s]]$. In particular, $(\mathcal{C}/(\varpi))/(A/(\varpi))$ is $A_4$-isomorphic to the original extension $k[[t]]/k[[s]]$.

(3) Let $\mathcal{B} = A[t] \subseteq L$. Let $\mathcal{R} = A[\varpi^{-1}]$, let $\mathcal{S} = \mathcal{B}[\varpi^{-1}]$, and let $\mathcal{T} = \mathcal{C}[\varpi^{-1}]$. Then $\mathcal{T}/\mathcal{R}$ is an $A_4$-extension of Dedekind rings, branched at 2 maximal ideals. Above the ideal $(s)$, the inertia group is $A_4$, and the break is $\nu - 6$. The other branched ideal has inertia group $\mathbb{Z}/2 \times \mathbb{Z}/2$, unique ramification break 1, and can be chosen to be of the form $(s - \mu^3)$, where $\mu \in \varpi^2 k[[\varpi]] \setminus \{0\}$ is arbitrary.

Proof. Define $\mathcal{L}$ by adjoining $t$ to $\mathcal{K}$. We proceed by deforming $a$ to an element of $\mathcal{L}$. Let $\mu \in \varpi^2 k[[\varpi]] \setminus \{0\}$. Let $a' = a/t^{-6} = a/s^{-2}$, and deform $a$ to the element $\tilde{a} := a'(s - \mu^3)^{-2} = a' \prod_{a=1}^3 (\zeta_3^a t - \mu)^{-2} \in \mathcal{B}(\varpi) \subseteq \mathcal{L}$. Note that $\tilde{a}$ reduces to $a$ (mod $\varpi$). Observe also that $\text{Gal}(\mathcal{L}/\mathcal{K}) \cong \mathbb{Z}/3$, and the $\mathbb{F}_2$-vector space generated by the images of $a'$ (and thus $\tilde{a}$) under this Galois action has dimension 2. By Corollary 2.2, $\tilde{a} \in \mathcal{L}$ gives rise to an $A_4$-extension $\mathcal{N}/\mathcal{K}$. We claim that this is the extension we seek.
Property (1) is obvious. To show property (3), first note that $S/R$ is branched exactly above the ideal $(s)$. The $\mathbb{Z}/2$-subextensions of $N/L$ are the Artin-Schreier extensions corresponding to $\tilde{a}$, $\sigma(\tilde{a})$, and $\sigma^2(\tilde{a})$, where $\sigma$ generates $\text{Gal}(L/K)$. Each of these is ramified at most above the ideals $(t)$ and $(\zeta_3^\alpha t - \mu)$, for $\alpha \in \{1, 2, 3\}$. We will see in the next paragraph that all of these ideals ramify in each $\mathbb{Z}/2$-subextension. Thus the ramification groups of $T/S$ above these ideals are all $\mathbb{Z}/2 \times \mathbb{Z}/2$. Since the three $\mathbb{Z}/2$-subextensions are Galois conjugate over $K$, there can only be one higher ramification jump for each ideal, and it is determined, say, by the Artin-Schreier subextension corresponding to $\tilde{a}$.

To determine the ramification, we consider the Artin-Schreier extension of the complete discrete valuation field $k((\omega))(t)$ (resp. $k((\omega))((\zeta_3^\alpha t - \mu)$) for $\alpha \in \{1, 2, 3\}$) given by $\tilde{a}$. Since $t$ is a unit in $k((\omega))[[\zeta_3^\alpha t - \mu]]$ for any $\alpha$ and $\zeta_3^3 t - \mu$ is a unit in $k((\omega))[t]$ and in $k((\omega))[[\zeta_3^\alpha t - \mu]]$ for any $\alpha' \neq \alpha$ in $\{1, 2, 3\}$, the degree of the pole of $\tilde{a}$ with respect to $t$ (resp. $\zeta_3^3 t - \mu$) is $\nu - 6$ (resp. 2). Since $\nu - 6$ is odd, we have that the Artin-Schreier extension of $k((\omega))(t)$ given by $\tilde{a}$ ramifies and has ramification break $\nu - 6$. To calculate the ramification break for the corresponding extension of $k((\omega))((\zeta_3^\alpha t - \mu))$, we assume $\alpha = 3$ for simplicity and we write $\tilde{a}$ as a Laurent series in $(t - \mu)$. Note that $\tilde{a} = t^{-1}(t - \mu)^{-2}x^2$ for some $x \in k((\omega))[[t - \mu]]$, and that

$$t^{-1} = \mu^{-1} + \mu^{-2}(t - \mu) + \text{higher order terms in } (t - \mu).$$

So

$$\tilde{a} = c\mu^{-1}(t - \mu)^{-2} + c\mu^{-2}(t - \mu)^{-1} + \theta,$$

where $\theta \in k((\omega))[[t - \mu]]$ and $c \in k((\omega))$ is the “constant” term of $x^2$ (in fact, it is easy to see that $c \in k((\mu^2)) = k((\omega^4)))$. Let $b = \sqrt{c\mu^{-1}(t - \mu)^{-1}}$. After replacing $a$ with $\tilde{a} + b^2 - b$, which does not change the Artin-Schreier extension, we see that $\tilde{a}$ has a simple pole (since $c \neq \mu^3$, the principal part does not vanish). So this extension ramifies with ramification break 1. This shows property (3).

For property (2), it suffices by [GM98, I, Theorem 3.4] to show that the total degree of the different of $T/S$ is equal to the degree of the different of $N/K$. Clearly, we may replace $R$ by $S$ and $K$ by $L$. Call these degrees $\delta_{T/S}$ and $\delta_{N/L}$, respectively.

Since the ramification break of $M/L$ is $\nu$, and $N/L$ is the compositum of Galois conjugates of $M/L$, we have that $N/L$ has $\nu$ as its single ramification break in the upper numbering, and all nontrivial higher ramification groups of $N/L$ have order 4. Using Serre’s different formula ([Ser68, IV, Proposition 4]), we obtain $\delta_{N/L} = 3(\nu + 1)$.

For $\delta_{T/S}$, we add up the contributions from the different branched ideals separately. For the ideal $(t)$, arguing as in the previous paragraph, we have a $\mathbb{Z}/2 \times \mathbb{Z}/2$-extension with single ramification break $\nu - 6$. This gives a contribution of $3(\nu - 5)$ to $\delta_{T/S}$. For each of the branched ideals $(\zeta^\alpha t - \mu)$ ($\alpha \in \{1, 2, 3\}$), we have ramification group $\mathbb{Z}/2 \times \mathbb{Z}/2$ with ramification break 1. Using Serre’s different formula again, we get a contribution of $3 \cdot 3 \cdot 2 = 18$ to $\delta_{T/S}$. Thus $\delta_{T/S} = 3(\nu - 5) + 18 = \delta_{N/L}$ and we are done.

We omit the proof of the following proposition, which follows from Proposition 3.2 exactly as [Pop14, Theorem 3.6] follows from [Pop14, Key Lemma 3.2].

**Proposition 3.3.** Let $Y \to W$ be a branched $A_1$-cover of projective smooth $k$-curves. Suppose that the local inertia at each totally ramified point is an extension $k[[z]]/k[[s]]$ having break $\leq \nu$ and given rise to by an Artin-Schreier generator in standard form divisible by $t^6$.
in $k[t^{-1}]$. Set $W = W \times_k k[[\varpi]]$. Then there is an $A_4$-cover of projective smooth $k[[\varpi]]$-curves $\mathcal{Y} \to W$ with special fiber $Y \to W$ such that the totally ramified points on the generic fiber $\mathcal{Y}_\eta \to W_\eta$ have breaks $\leq \nu - 6$.

Before we prove Proposition 3.1 we recall Harbater-Katz-Gabber covers (or HKG-covers) from [Kat86]. Let $G \cong P \times \mathbb{Z}/m$, with $P$ a $p$-group and $p \nmid m$. If $k[[z]]/k[[s]]$ is a local $G$-extension, then the associated HKG-cover is the unique branched $G$-cover $X \to \mathbb{P}^1_k$ tamely ramified of index $m$ above $s = \infty$ and totally ramified above $s = 0$ ($s$ being a coordinate on $\mathbb{P}^1_k$), such that the formal completion of $X \to \mathbb{P}^1_k$ above 0 yields $k[[z]]/k[[s]]$.

**Proof of Proposition 3.1**: The proof is essentially the same as the proof of [Obr15 Proposition 1.11], which itself is adapted from [Pop14]. We include it here for completeness.

Let $Y \to W = \mathbb{P}^1$ be the Harbater-Katz-Gabber cover associated to $k[[z]]/k[[s]]$, let $\mathcal{Y} \to W$ be the $A_4$-cover over $k[[\varpi]]$ guaranteed by Proposition 3.3. Let $\mathcal{Y}_\eta \to W_\eta$ be the generic fiber of $\mathcal{Y} \to W$. Recall that we assume that every local $A_4$-extension with break $\leq \nu - 6$ lifts to characteristic zero. Furthermore, by [Pag02] and the theory of tame ramification, every abelian extension of $k[[s]]$ (and thus of $k((\varpi))/((s))$) with Galois group a proper subgroup of $A_4$ lifts to characteristic zero. So the local-global principle tells us that $\mathcal{Y}_\eta \to W_\eta$ lifts to a cover $\mathcal{Y}_{\mathcal{O}_1} \to W_{\mathcal{O}_1}$ over some characteristic zero complete discrete valuation ring $\mathcal{O}_1$ with residue field $k((\varpi))$. Then, [Pop14 Lemma 4.3] shows that we can “glue” the covers $\mathcal{Y} \to W$ and $\mathcal{Y}_{\mathcal{O}_1} \to W_{\mathcal{O}_1}$ along the generic fiber of the former and the special fiber of the latter, in order to get a cover defined over a rank two characteristic zero valuation ring $\mathcal{O}$ with residue field $k$ lifting $Y \to W$ (cf. [Pop14 p. 319, second paragraph]). Note that this process works starting with any $A_4$-extension of $k[[s]]$ with break $\nu$, and that such extensions can be parameterized by some affine space $\mathbb{A}^N$ (with one coordinate corresponding to each possible coefficient in an entry of an Artin-Schreier generator in standard form).

To conclude, we remark that [Pop14 Proposition 4.7] and its setup carry through exactly in our situation, with our $\mathbb{A}^N$ playing the role of $\mathbb{A}^d$ in [Pop14]. Indeed, we have that the analog of $\Sigma_4$ in that proposition contains all closed points, by the paragraph above. Thus we can in fact lift $Y \to W$ over a discrete characteristic zero valuation ring. Applying the easy direction of the local-global principle, we obtain a lift of $k[[z]]/k[[s]]$. This concludes the proof of Proposition 3.1.

4. The Form of a Lift

We start by reviewing lifts of $\mathbb{Z}/2$-extensions of $k[[t]]$. The following lemma is well-known, but difficult to cite directly from the literature. We provide a proof.

**Lemma 4.1.** Let $k((u))/k((t))$ be a $\mathbb{Z}/2$-extension with Artin-Schreier generator $a \in t^{-1}k[t^{-1}]$ in standard form and ramification break $\nu$. Let $A$ be a lift of $a$ to $T^{-1}R[T^{-1}]$ of degree $\nu$. If $\Phi \in 1 + T^{-1}m[T^{-1}]$ has degree $\nu$ or $\nu + 1$ and satisfies

$$\Phi = H^2 + 4A + o(4)$$

for some $H \in 1 + T^{-1}m[T^{-1}]$, then the normalization of $R[[T]]$ in $M := \text{Frac}(R[[T]])[\sqrt{\Phi}]$ is a lift of $k[[u]]/k[[t]]$ to characteristic zero. Furthermore, $\Phi$ has simple roots.
Proof. The extension \( k((u))/k((t)) \) is given by adjoining an element \( y \) such that \( y^2 - y = a \). Making a substitution \( \sqrt{\Phi} = H + 2Y \), the expression for \( \Phi \) given in the lemma yields

\[
H^2 + 4HY + 4Y^2 = H^2 + 4A + o(4),
\]

or \( Y^2 - Y = A + o(1) \). Thus we see that the normalization of \( R[[T]](\pi) \) in \( M \) gives \( k((u))/k((t)) \) upon reduction modulo \( \pi \). By Serre’s different formula ([Ser68, IV, Proposition 4]), the degree of the different of \( k[[u]]/k[[t]] \) is \( \nu + 1 \). On the other hand, the normalization of \( R[[T]] \otimes_R \text{Frac}(R) \) in \( M \) is branched at at most \( \nu + 1 \) maximal ideals, corresponding to the roots of \( \Phi \) and also 0 if \( \Phi \) has degree \( \nu \). Since this is a tamely ramified \( \mathbb{Z}/2 \)-extension, the degree of its different is at most \( \nu + 1 \). By [GM98, 1, 3.4], the degree of the different is exactly \( \nu + 1 \) and the normalization of \( R[[T]] \) in \( M \) is a lift of \( k[[u]]/k[[t]] \). This also shows that the roots of \( \Phi \) are all simple. \( \Box \)

For Proposition 4.2 below, recall that \( s = t^3 \) and \( S = T^3 \).

**Proposition 4.2.** Let \( k[[z]]/k[[s]] \) be a local \( A_4 \)-extension with break \( \nu \) given rise to by \( a \in t^{-1}k[t^{-1}] \) in standard form. If \( F(T^{-1}) \) and \( H(T^{-1}) \) are in \( 1 + T^{-1}m[T^{-1}] \) such that \( F \) has degree \( (\nu + 1)/2 \) and

\[
F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1}) = H^2 + 4A + o(4),
\]

where \( A \) is a lift of \( a \) to \( T^{-1}R[T^{-1}] \) of degree \( \nu \), then the normalization of \( R[[S]] \) in the \( A_4 \)-extension of \( \text{Frac}(R[[S]]) \) given rise to by \( F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1}) \) is a lift of \( k[[z]]/k[[s]] \) to characteristic zero.

**Proof.** Let the local \( \mathbb{Z}/2 \)-extension \( k[[u]]/k[[t]] \) be given by normalizing \( k[[t]] \) in the Artin-Schreier \( \mathbb{Z}/2 \)-extension of \( k((t)) \) given by \( a \). Let \( L = \text{Frac}(R[[T]]) \). By Lemma 4.2, normalizing \( R[[T]] \) in the degree 2 Kummer extension \( M/L \) given by some polynomial \( \Phi \in 1 + T^{-1}m[T^{-1}] \) of degree \( \nu + 1 \) in \( T^{-1} \) such that \( \Phi = H^2 + 4A + o(4) \) with \( A \) as in the proposition gives a lift of \( k[[u]]/k[[t]] \) to characteristic zero, and such a \( \Phi \) has simple roots.

Let \( \sigma \) generate \( \text{Gal}(L/\text{Frac}(R[[S]]) \) (and also, by abuse of notation, \( \text{Gal}(k((t))/k((s))) \)). Write \( \Phi = F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1}) \) for some polynomial \( F \in 1 + T^{-1}m[T^{-1}] \) of degree \( (\nu + 1)/2 \) as in the proposition. Then \( \Phi \) has simple roots, and thus \( F(T^{-1}) \), \( F(\zeta_3 T^{-1}) \), and \( F(\zeta_3^2 T^{-1}) \) have pairwise disjoint simple roots. Consequently, the \( \mathbb{F}_2 \)-subspace of \( L^*/(L^*)^2 \) generated by \( \Phi, \sigma(\Phi), \) and \( \sigma^2(\Phi) \) has dimension 2. By Proposition 2.3, this is equivalent to the Galois closure \( N \) (over \( \text{Frac}(R[[S]]) \)) of \( M \) having Galois group \( A_4 \).

Let \( k((u))/k((t)) \) be the Artin-Schreier extension given by \( \sigma(a) \). Clearly, the normalization of \( R[[T]] \) in \( \text{Frac}(R[[T]]) \) with \( \sigma(\Phi) \) is a lift of \( k[[u]]/k[[t]] \). Note that \( k[[z]] \) is the normalization of \( k[[t]] \) in the compositum of \( k((u)) \) and \( k((u')) \). Analogously, \( N := \text{Frac}(R[[T]])(\sqrt{\Phi}, \sqrt{\sigma(\Phi)}) \) is the \( A_4 \)-extension given rise to by \( \Phi \). Note that \( \Phi \) and \( \sigma(\Phi) \) have exactly \( (\nu + 1)/2 \) zeroes in common. Thus [GM98, 1, Theorem 5.1] shows that the normalization of \( R[[T]] \) in \( N \) is a lift of the Klein four extension \( k[[z]]/k[[t]] \) (and is isomorphic to \( R[[Z]]/R[[T]] \) for \( Z \) reducing to \( z \)). We conclude that \( R[[Z]]/R[[S]] \) is a lift of \( k[[z]]/k[[s]] \). \( \Box \)
5. Proof of Theorem 1.2

In this section, let $k[[z]]/k[[s]]$ be a local $A_4$-extension given rise to by $a \in t^{-1}k[t^{-1}]$ in standard form. Recall that $\deg(a) = \nu$, where $\nu$ is the break in $k[[z]]/k[[s]]$. We will prove that $k[[z]]/k[[s]]$ lifts to characteristic zero by strong induction on $\nu$.

**Proposition 5.1.** If $\nu = 1$, then $k[[z]]/k[[s]]$ lifts to characteristic zero.

**Proof.** Since $\nu = 1$, we have $a = \overline{c}_1t^{-1}$, with $\overline{c}_1 \in k$. By Proposition 4.2 it suffices to find $F(T^{-1})$ and $H(T^{-1})$ in $1 + T^{-1}R[T^{-1}]$ such that $F$ has degree 1 and

$$F(\zeta_3T^{-1})F(\zeta_3^2T^{-1}) = H^2 + 4c_1T^{-1} + o(4),$$

where $c_1$ is a lift of $\overline{c}_1$ to $R$. This is accomplished by taking $H = 1$ and $F = 1 - 4c_1T^{-1}$. \qed

**Proposition 5.2.** If $\nu = 5$, then $k[[z]]/k[[s]]$ lifts to characteristic zero.

**Proof.** Since $\nu = 5$, we have $a = \overline{c}_1t^{-1} + \overline{c}_5t^{-5}$, with $\overline{c}_1, \overline{c}_5 \in k$. By Proposition 4.2 it suffices to find $F(T^{-1})$ and $H(T^{-1})$ in $1 + T^{-1}R[T^{-1}]$ such that $F$ has degree 3 and

$$F(\zeta_3T^{-1})F(\zeta_3^2T^{-1}) = H^2 + 4c_1T^{-1} + 4c_5T^{-5} + o(4),$$

where each $c_i$ is a lift of $\overline{c}_i$ to $R$.

Let $b \in R$ be any element such that $v(b) = 2/5$. Write

$$F(T^{-1}) = 1 + a_1T^{-1} + a_2T^{-2} + a_3T^{-3},$$

where

$$a_1 = -2b - 4c_1, \quad a_2 = b^2, \quad a_3 = -4c_5/b^2.$$  

Note that $v(a_1) = 7/5$, $v(a_2) = 4/5$, and $v(a_3) = 6/5$. Then

$$F(\zeta_3T^{-1})F(\zeta_3^2T^{-1}) = 1 - a_1T^{-1} - a_2T^{-2} + a_2^2T^{-4} - a_2a_3T^{-5} + o(4)$$

$$= 1 + (4c_1 + 2b)T^{-1} - b^2T^{-2} + b^4T^{-4} + 4c_5T^{-5} + o(4)$$

$$= (1 + bT^{-1} + b^2T^{-2})^2 + 4c_1T^{-1} + 4c_5T^{-5} + o(4).$$

We conclude by taking $H = 1 + bT^{-1} + b^2T^{-2}$. \qed

**Proof of Theorem 1.2:** We use strong induction on the break $\nu$ of $k[[z]]/k[[s]]$, which only takes values congruent to 1 or 5 modulo 6 (Proposition 2.4). The base cases $\nu = 1$ and $\nu = 5$ are Propositions 5.1 and 5.2 respectively. The induction step is Proposition 3.1. \qed

**Remark 5.3.** Florian Pop has informed the author of his own proof, which uses much the same method. In place of the deformation in Proposition 3.2 he uses one for which it is slightly more difficult to verify that it yields an $A_4$-extension, but which immediately reduces Theorem 1.2 to the case $\nu = 1$ (eliminating the need for Proposition 5.2).

**Question 5.4.** Given $k$, does there exist a particular DVR $R$ in characteristic zero such that all local $A_4$-extensions over $k$ lift over $R$? This is known for local $G$-extensions in characteristic $p$ where $G$ is cyclic with $v_p(|G|) \leq 2$ (see [GM98], where it is shown that $W(k)[\zeta_p]$ works). Since our proof is rather inexplicit, this question remains open for $A_4$. 

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