Abstract. We study a new example of equation obtained as a result of a recent generalized symmetry classification of differential-difference equations defined on five points of one-dimensional lattice. We have established that in the continuous limit this new equation goes into the well-known Kaup–Kupershmidt equation. We have also proved its integrability by constructing an $L – A$ pair and conservation laws. Moreover, we present a possibly new scheme for deriving conservation laws from $L – A$ pairs.

Keywords: differential-difference equation, integrability, Lax pair, conservation law

Mathematics Subject Classification: 37K10, 35G50, 39A10

1. Introduction

We consider the differential-difference equation

$$u_{n,t} = \left(u_n^2 - 1\right) \left(u_{n+2}^2 u_{n+1}^2 - 1 - u_{n-2}^2 u_{n-1}^2 - 1\right),$$

where $n \in \mathbb{Z}$, while $u_n(t)$ is the unknown function of one discrete variable $n$ and one continuous variable $t$, and the index $t$ denotes time derivative. Equation (1) is obtained as a result of generalized symmetry classification of five-point differential-difference equations

$$u_{n,t} = F(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}),$$

carried out in [8]. Equation (1) coincides with the equation [8, (E17)] up to $u_n$ and $t$ scaling.

Equations (2) play an important role in the study of four-point discrete equations on the square lattice, which are very relevant for today, see e.g. [1,5,6,15]. No relation is known between (1) and any other known equation of the form (2). More precisely, we mean relations in the form of the transformations

$$u_n = \varphi(u_{n+k}, u_{n+k-1}, \ldots, u_{n+m}), \quad k > m,$$

and their compositions, see a detailed discussion of such transformations in [7]. The only information we have at the moment on (1) is that it possesses a nine-point generalized symmetry of the form:

$$u_{n,\theta} = G(u_{n+4}, u_{n+3}, \ldots, u_{n-4}).$$

In this article we explore equation (1) in details. We have found in Section 2 its continuous limit, which is the well-known Kaup–Kupershmidt equation [4,10]:

$$U_\tau = U_{xxxx} + 5UU_{xx} + \frac{25}{2} U_x U_{xx} + 5U^2 U_x,$$

where the indices $\tau$ and $x$ denote $\tau$ and $x$ partial derivatives. In order to justify the integrability of (1), we construct an $L – A$ pair in Section 3 and show that it provides an infinity hierarchy of conservation laws in Section 4. In Section 5 we discuss possible generalizations of a construction.
scheme for the conservation laws, which has been formulated in Section 3 by example of equation (1).

2. Continuous limit

In a list of equations of the form (2), presented in [8], most of equations go in continuous limit into the Korteweg-de Vries equation. The exceptions are (1) and the following two equations:

\[ u_{n,t} = u^2_n (u_{n+2}u_{n+1} - u_{n-1}u_{n-2}) - u_n (u_{n+1} - u_{n-1}), \]

\[ u_{n,t} = (u_n + 1) \left( \frac{u_{n+2}u_{n+1} + 1}{u_{n+1}} - \frac{u_{n-2}u_{n-1} + 1}{u_{n-1}} \right) + (1 + 2u_n)(u_{n+1} - u_{n-1}), \]

which correspond to equations (E15) and (E16) of [8]. Equation (5) has been known for a long time [17]. Equation (6) has been found recently in [2] and it is related to (5) by a composition of transformations of the form (3). These three equations in the continuous limit correspond to the fifth order equations of the form:

\[ U_\tau = U_{xxxxx} + F(U_{xxx}, U_{xx}, U_x, U). \]

There is a complete list of integrable equations of the form (7), see [3, 11, 14]. Two equations play the main role there, namely, (4) and the Sawada-Kotera equation [16]:

\[ U_\tau = U_{xxxxx} + 5U U_{xxx} + 5U_x U_{xx} + 5U^2 U_x. \]

All the other are transformed into these two by transformations of the form:

\[ \hat{U} = \Phi(U, U_x, U_{xx}, \ldots, U_{x...x}). \]

It has been known [1] that equation (5) goes in the continuous limit into the Sawada-Kotera equation (8). The other results below are new. Using the substitution

\[ u_n(t) = \frac{2\sqrt{2}}{3} + \sqrt{\frac{2}{16}} \varepsilon^2 U \left( \tau - \frac{9}{80} \varepsilon^5 t, x + \frac{2}{3} \varepsilon t \right), \]

in equation (1), we get at \( \varepsilon \to 0 \) the Kaup-Kupershmidt equation (4).

It is interesting that equation (6) has two different continuous limits. The substitution

\[ u_n(t) = -\frac{4}{3} - \varepsilon^2 U \left( \tau - \frac{18}{5} \varepsilon^5 t, x + \frac{4}{3} \varepsilon t \right), \]

in (6) leads to equation (4), while the substitution

\[ u_n(t) = -\frac{2}{3} + \varepsilon^2 U \left( \tau - \frac{18}{5} \varepsilon^5 t, x + \frac{4}{3} \varepsilon t \right), \]

leads to equation (8). As well as (1), equation (6) deserves further study.

In conclusion, let us present a picture that shows the link between discrete and continuous equations:

\[ \begin{array}{ccc}
\text{(1)} & \text{(6)} & \text{(5)} \\
\text{(9)} & \text{(10)} & \text{(11)} \\
\text{(4)} & \text{(10)} & \text{(11)} & \text{(8)}
\end{array} \]
3. L – A Pair

As the continuous limit shows, equation (1) should be close to equation (5) in its integrability properties. Following the L – A pair \([1, (15,17)]\), we look for an L – A pair of the form:

\[
L_n \psi_n = 0, \quad \psi_{n,t} = A_n \psi_n
\]  

(12)

with the operator \(L_n\) of the form:

\[
L_n = l_n^{(2)}T^2 + l_n^{(1)}T + l_n^{(0)} + l_n^{(-1)}T^{-1},
\]

where \(l_n^{(k)}, \ k = -1, 0, 1, 2,\) depend on the finite number of functions \(u_{n+j}\). Here \(T\) is the shift operator: \(Th = h_{n+1}\). In this case the operator \(A_n\) can be chosen in the form

\[
A_n = a_n^{(1)}T + a_n^{(0)} + a_n^{(-1)}T^{-1}.
\]

The compatibility condition for the system (12) has the form:

\[
\frac{d(L_n \psi_n)}{dt} = (L_{n,t} + L_n A_n) \psi_n = 0
\]  

(13)

and it must be satisfied on virtue of equations (1) and \(L_n \psi_n = 0\).

If we suppose that the coefficients \(l_n^{(k)}\) depend on \(u_n\) only, as in [1], then we can see that \(a_n^{(k)}\) depend on \(u_{n-1}, u_n\) only. However, in this case the problem has no solution. Therefore we pass to the case when the functions \(l_n^{(k)}\) depend on \(u_n, u_{n+1}\). Then the coefficients \(a_n^{(k)}\) must depend on \(u_{n-1}, u_n, u_{n+1}\). In this case we have managed to find the operators \(L_n\) and \(A_n\) with one irremovable arbitrary constant \(\lambda\), which plays here the role of spectral parameter:

\[
L_n = u_n \sqrt{u_{n+1}^2 - 1}T^2 + u_{n+1}T + \lambda \left( u_n - u_{n+1} \sqrt{u_{n+1}^2 - 1}T^{-1} \right),
\]

(14)

\[
A_n = \frac{\sqrt{u_{n+1}^2 - 1}}{u_n} \left( \sqrt{u_n^2 - 1}(u_{n+1}T + u_{n-1}T^{-1}) - \lambda^{-1}u_{n-1}T - \lambda u_{n+1}T^{-1} \right).
\]

(15)

The \(L – A\) pair \([12,14,15]\) can be rewritten in the standard form with \(3 \times 3\) matrices \(\hat{L}_n, \hat{A}_n\):

\[
\Psi_{n+1} = \hat{L}_n \Psi_n, \quad \Psi_{n,t} = \hat{A}_n \Psi_n.
\]

Here a new spectral function is given by

\[
\Psi_n = 2^{-n} \begin{pmatrix}
\frac{\sqrt{u_{n+1}^2 - 1}}{u_n} \psi_{n+1} \\
\psi_n \\
\psi_{n-1}
\end{pmatrix},
\]

and the matrices \(\hat{L}_n, \hat{A}_n\) read:

\[
\hat{L}_n = \begin{pmatrix}
-\frac{1}{\sqrt{u_n^2 - 1}} & -\frac{\lambda}{u_{n+1}} & \frac{\lambda\sqrt{u_{n+1}^2 - 1}}{u_n} \\
\frac{u_n}{\sqrt{u_{n+1}^2 - 1}} & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\]

(16)

\[
\hat{A}_n = \begin{pmatrix}
\lambda^{-1} - \frac{u_{n-2}}{u_n} \sqrt{u_{n-1}^2 - 1} & u_{n+1} \sqrt{u_n^2 - 1} & \frac{(u_n^2 - 1)(u_{n+1}^2 - 1) - u_{n-1}^2}{u_n^2} \\
\frac{u_{n+1} \sqrt{u_n^2 - 1} - \lambda^{-1}u_{n-1}}{u_n + \lambda^{-1}u_{n-2} \sqrt{u_{n-1}^2 - 1}} & 0 & \lambda \frac{u_{n+1} \sqrt{u_n^2 - 1} - u_{n-1} (u_{n+1}^2 - 1)}{u_n \sqrt{u_{n-1}^2 - 1}} \\
u_n + \lambda^{-1}u_{n-2} \sqrt{u_{n-1}^2 - 1} - \lambda^{-1}u_{n-1} & u_n u_{n-1} & \lambda + \frac{u_{n-2}}{u_n} \sqrt{u_{n-1}^2 - 1}
\end{pmatrix}.
\]

(17)

In this case, unlike [13], the compatibility condition can be represented in matrix form:

\[
\hat{L}_{n,t} = \hat{A}_{n+1} \hat{L}_n - \hat{L}_n \hat{A}_n,
\]
Introducing Lax pair. The operator

without using the spectral function $\Psi_n$.

There are two methods to construct the conservation laws by using such matrix $L - A$ pairs [5,9,12]. However, we do not see how to apply those methods in case of the matrices [16] and [17]. In the next section, we will use a different scheme for deriving conservation laws from the $L - A$ pair [12], and that scheme seems to be new.

4. Conservation laws

The structure of operators [14,15] allows us to rewrite the $L - A$ pair [12] in form of the Lax pair. The operator $L_n$ has the linear dependence on $\lambda$:

$$L_n = P_n - \lambda Q_n,$$

where

$$P_n = u_n \sqrt{u_{n+1}^2 - 1} T^2 + u_{n+1} T, \quad Q_n = u_{n+1} \sqrt{u_n^2 - 1} T^{-1} - u_n.$$

Introducing $\hat{L}_n = Q_n^{-1} P_n$, we get an equation of the form:

$$\hat{L}_n \psi_n = \lambda \psi_n.$$  \hspace{1cm} (19)

The functions $\lambda \psi_n$ and $\lambda^{-1} \psi_n$ in the second equation of (12) can be expressed in terms of $\hat{L}_n$ and $\psi_n$, using (19) and its consequence $\lambda^{-1} \psi_n = \hat{L}_n^{-1} \psi_n$. As a result we have:

$$\psi_{n,t} = \hat{A}_n \psi_n,$$

where

$$\hat{A}_n = \frac{\sqrt{u_n^2 - 1}}{u_n} \left( \sqrt{u_{n+1}^2 - 1} (u_{n+1} T + u_{n-1}^{-1}) - u_{n-1} T P_n^{-1} Q_n + u_{n+1} T^{-1} Q_n^{-1} P_n \right).$$

It is important that new operators $\hat{L}_n$ and $\hat{A}_n$ in the $L - A$ pair [19,20] do not depend on the spectral parameter $\lambda$. For this reason, the compatibility condition can be written in the operator form, without using $\psi$-function:

$$\hat{L}_{n,t} = \hat{A}_n \hat{L}_n - \hat{L}_n \hat{A}_n = [\hat{A}_n, \hat{L}_n],$$  \hspace{1cm} (21)

i.e. it has now the form of the Lax equation. The difference between this $L - A$ pair and well-known Lax pairs for the Toda and Volterra equations is that now the operators $\hat{L}_n$ and $\hat{A}_n$ are nonlocal. Nevertheless, using the definition of inverse operators, which are linear:

$$P_n P_n^{-1} = P_n^{-1} P_n = 1, \quad Q_n Q_n^{-1} = Q_n^{-1} Q_n = 1,$$

we can check that (21) is true by direct calculation.

The conservation laws of equation (1), which are expressions of the form

$$\rho^{(k)}_{n,t} = (T - 1) \sigma^{(k)}_n, \quad k \geq 0,$$

can be derived from the Lax equation (21), notwithstanding nonlocal structure of the operators $\hat{L}_n, \hat{A}_n$, see [18]. For this we must, first of all, represent the operators $\hat{L}_n, \hat{A}_n$ as formal series in powers of $T^{-1}$:

$$H_n = \sum_{k \in \mathbb{N}} h^{(k)}_n T^k.$$  \hspace{1cm} (23)

Formal series of this kind can be multiplied according the rule: $(a_n T^k)(b_n T^j) = a_n b_{n+k} T^{k+j}$. The inverse series can be obtained by definition (22), for instance:

$$Q_n^{-1} = -(1 + q_n T^{-1} + (q_n T^{-1})^2 + \ldots + (q_n T^{-1})^k + \ldots) \frac{1}{u_n}, \quad q_n = \frac{u_{n+1}}{u_n} \sqrt{u_n^2 - 1}. $$
The series $\hat{L}_n$ has the second order:

$$\hat{L}_n = \sum_{k=2}^{n} l_n^{(k)} T^k = -\left(\sqrt{u_{n+1}^2 - 1} T^2 + u_{n+1} u_n T + u_{n+1} u_{n-1} \sqrt{u_n^2 - 1} + \ldots \right).$$

The conserved densities $\rho_n^{(k)}$ of equation (1) can be found as:

$$\rho_n^{(0)} = \log l_n^{(2)}, \quad \rho_n^{(k)} = \text{res} \, \hat{L}_n, \quad k \geq 1,$$

where the residue of formal series (23) is defined by the rule: $\text{res} \, H_n = h_n^{(0)}$, see [18]. Corresponding functions $\sigma_n^{(k)}$ can easily be found by direct calculation.

Conserved densities $\rho_n^{(k)}$ below have been found in this way and then simplified in accordance with the rule:

$$\rho_n^{(k)} = c_k \rho_n^{(k)} + (T - 1) g_n^{(k)},$$

where $c_k$ are constant. First three densities of equation (1) read:

$$\rho_n^{(0)} = \log(u_n^2 - 1),$$

$$\rho_n^{(1)} = u_{n+1} u_{n-1} \sqrt{u_n^2 - 1},$$

$$\rho_n^{(2)} = (u_n^2 - 1)(2u_{n+2} u_{n-2} \sqrt{u_{n+1}^2 - 1} \sqrt{u_{n-1}^2 - 1} + u_{n+1}^2 u_{n-1} u_{n-1} u_{n-1} \sqrt{u_{n-1}^2 - 1}) + u_{n+1} u_{n-1} u_n \sqrt{u_n^2 - 1}(u_{n+2} \sqrt{u_{n+1}^2 - 1} + u_{n-2} \sqrt{u_{n-1}^2 - 1}).$$

5. Discussion of the construction scheme

In previous section we have outlined a construction scheme for the conservation laws by example of equation (1). It can easily be generalized to equations of an arbitrarily high order:

$$u_{n,t} = F(u_{n+M}, u_{n+M-1}, \ldots, u_{n-M}).$$

Let such equation have an $L - A$ pair of the form (12) with a linear in $\lambda$ operator $L_n$, and let the operators $P_n, Q_n$ of (18) have the form:

$$R_n = \sum_{k=k_1}^{k_2} r_n^{(k)} T^k, \quad k_1 \leq k_2 \in \mathbb{Z},$$

with the coefficients $r_n^{(k)}$ depending on the finite number of functions $u_{n+j}$. We require that

$$\hat{L}_n = Q_n^{-1} P_n = \sum_{k \leq N} l_n^{(k)} T^k$$

has a positive order $N \geq 1$. If $N \leq -1$, then we change $\lambda \to \lambda^{-1}$ and introduce $\hat{L}_n = P_n^{-1} Q_n$ of a positive order. In case $N = 0$ the scheme does not work.

As $\lambda^k \psi_n = \hat{L}_n^k \psi_n$ for any integer $k$, we can consider operators $A_n$ of the form:

$$A_n = \sum_{k=m_1}^{m_2} a_n^{(k)} [T] \lambda^k, \quad m_1 \leq m_2 \in \mathbb{Z},$$

where $a_n^{(k)} [T]$ are operators of the form (25). Then we can rewrite $A_n$ as:

$$\hat{A}_n = \sum_{k=m_1}^{m_2} a_n^{(k)} [T] \hat{L}_n^k = \sum_{k \leq N} \hat{a}_n^{(k)} T^k.$$
We are led to the Lax equation (21) with \( \hat{L}_n, \hat{A}_n \) of the form (23) and, therefore, we can construct the conserved densities as written above, namely, in accordance with (24) with the only difference: \( \rho_n^{(0)} = \log l_n^{(N)} \).

It should be remarked that the scheme can easily be applied to equation (5) with the \( L - A \) pair \([1, (15, 17)]\).

This scheme can also be applied in a quite similar way in the continuous case, namely, to PDEs of the form

\[ u_t = F(u, u_x, u_{xx}, \ldots, u_{x...x}) \]

We consider the operators (25) with \( D_x \) instead of \( T \), which become the differential operators, where \( D_x \) is the operator of total \( x \)-derivative. Besides, \( k_2 \geq k_1 \geq 0 \) and the coefficients \( r_n^{(k)} \) depend on a finite number of functions \( u, u_x, u_{xx}, \ldots \). Instead of (23) we consider the formal series in powers of \( D_x^{-1} \). A theory of such formal series and, in particular, a definition of the residue are discussed in [13].

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