UNIFIED PRODUCTS FOR JORDAN ALGEBRAS. APPLICATIONS

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ABSTRACT. Given a Jordan algebra $A$ and a vector space $V$, we describe and classify all Jordan algebras containing $A$ as a subalgebra of codimension $\dim_k(V)$ in terms of a non-abelian cohomological type object $J_A(V, A)$. Any such algebra is isomorphic to a newly introduced object called unified product $A \ ♮ V$. The crossed/twisted product of two Jordan algebras are introduced as special cases of the unified product and the role of the subsequent problem corresponding to each such product is discussed. The non-abelian cohomology $H_nab^2(V, A)$ associated to two Jordan algebras $A$ and $V$ which classifies all extensions of $V$ by $A$ is also constructed. Several applications and examples are given: we prove that $H_nab^2(k, k^n)$ is identified with the set of all matrices $D \in M_n(k)$ satisfying $2D^3 - 3D^2 + D = 0$.

INTRODUCTION

Jordan algebras have been first introduced in mathematical physics by P. Jordan [16] in order to achieve an axiomatization for the algebra of observables in quantum mechanics. Since then, Jordan algebra theory has developed into a rich and fruitful one, as evidenced by the various contexts where it makes an appearance such as supersymmetry, the theory of superstrings, projective geometry, Lie algebras and algebraic groups, representation theory or functional analysis (see [9, 11, 12, 13, 23, 25] and the references therein). One of the papers which generated a lot of interest and determined a turning point for Jordan algebra theory is [17] where all simple finite-dimensional formally real algebras are classified. Another important contribution is the paper of A. Albert [8] where an example of an exceptional Jordan algebra is constructed: the 27-dimensional algebra of self-adjoint $3 \times 3$ matrices over the octonions. The study of Jordan algebras experienced a major change in the second half of the last century due mainly to the algebra school in Novosibirsk. In this context, major contributions which lead to a better understanding of the field have been achieved by E. Zelmanov which extended results from the classical theory of finite dimensional Jordan algebras to the infinite dimensional case. In particular, all simple Jordan algebras, including infinite-dimensional ones, have been classified by what is now called Zelmanov’s exceptional theorem ([25, Chapter 8]): any simple exceptional Jordan algebra is an Albert algebra of dimension 27 over its center. The close relation between Jordan algebras and Lie algebras is also highlighted by the famous Kantor-Koecher-Tits construction [23].
The present paper is devoted to the study of the extending structures problem (ES-problem) introduced in [3] for arbitrary categories and the extension problem listed below. In the context of Jordan algebras it comes down to the following problem:

**Extending structures problem.** Let $A$ be a Jordan algebra and $E$ a vector space containing $A$ as a subspace. Describe and classify up to an isomorphism that stabilizes $A$ (i.e. acts as the identity on $A$) the set of all Jordan algebra structures that can be defined on $E$ such that $A$ becomes a Jordan subalgebra of $E$.

Equivalently, if we fix $V$ a complement of $A$ in the vector space $E$, the ES-problem can be restated by asking for the description and classification of all Jordan algebras which contain and stabilize $A$ as a subalgebra of codimension equal to $\dim(V)$. As explained in [4], the ES problem generalizes and unifies two famous and intensively studied open problems: the extension problem (introduced at the level of groups by Hölder) and the factorization problem (stemmed in group theory as well, in the work of Ore). The extension problem, formulated for Jordan algebras, can be stated as follows:

**Extension problem.** Let $A$ and $V$ be two given Jordan algebras. Describe and classify all extensions of $V$ by $A$, i.e. all Jordan algebra $E$ that fit into an exact sequence of Jordan algebras:

$$0 \rightarrow A \overset{i}{\rightarrow} E \overset{p}{\rightarrow} V \rightarrow 0.$$

The precise meaning of classification in the context of the extension problem appears in Section 1. The first result in this direction was proved by Jacobson [15, Theorem 12] who gave a partial answer to the extension problem in the case of null extensions (i.e. such that $a \cdot b := 0$, for all $a, b \in A$): in this case, similar to the group (or Lie algebra) case, the extensions of $V$ by $A$ are parameterized by the second cohomology group $H^2(V, A)$.

The paper is organized as follows: in Section 1 we recall some basic concepts in the context of Jordan algebras. Section 2 is devoted to the study of the extending structures problem following the strategy we previously developed in [4, 6]. Theorem 2.10, the main result of this section, provides the theoretical answer to the ES-problem by constructing a non-abelian cohomological type object $\mathcal{J}_A(V, A)$ which classifies all Jordan algebras containing and stabilizing $A$ as a subalgebra of codimension equal to $\dim(V)$. Any such algebra is isomorphic to a unified product $A \natural V$, a product associated to a Jordan algebra $A$ and a vector space $V$ connected through two actions and a generalized cocycle.

It is worth mentioning that the unified product generalizes spin factor Jordan algebras (Example 2.3) as first constructed in [17]. Theorem 2.10 offers the theoretical answer to the ES-problem. However, computing the classifying object $\mathcal{J}_A(V, A)$, for a given Jordan algebra $A$ and a vector space $V$, is a highly non-trivial task. In Theorem 2.17 we explicitly compute $\mathcal{J}_A(k, A)$ which is the key step in classifying finite dimensional flag Jordan algebras introduced in Definition 2.11 as a generalization of supersolvable Jordan algebras. Another application is given in Theorem 2.19 which proves an Artin type theorem for Jordan algebras. More precisely, we show that a Jordan algebra on which a finite group acts can be written as a twisted product between its subalgebra of invariants and a given complement. This is a first step for possible future developments in the invariant theory of Jordan algebras.
One of the main special cases of the unified product is the crossed products of Jordan algebras, will be treated separately in Section 3. As in the group (or Lie algebra) case, the crossed product of two Jordan algebras plays a key role in the study of the extension problem. The theoretical answer to the extension problem in its full generality is given in Theorem 3.5: all extensions of a given Jordan algebra $V$ by a Jordan algebra $A$ are classified by the non-abelian cohomology $H^2_{nab}(V, A)$ which is explicitly constructed. This result is the non-abelian generalization of [15, Theorem 12]. The classification of finite dimensional Jordan algebras was only recently initiated: in [18] all three-dimensional Jordan algebras over $\mathbb{R}$ are classified while in [24] all four-dimensional Jordan algebras over an algebraically closed field of characteristic different than 2 are classified. The classification of nilpotent Jordan algebras was completed up to dimension 5 ([14, 20]) and partial results are available in dimension 6 [2]. In order to obtain classification results in higher dimensions or for other classes of Jordan algebras, certain constructions such as the crossed or the bicrossed product might be useful. In fact, Corollary 3.2 proves the crucial role played by crossed products for the classification problem: any finite dimensional Jordan algebra is isomorphic to an iterated crossed product of the form $(\cdot \cdot \cdot ((S_1 \# S_2) \# S_3) \# \cdots \# S_t)$, where $S_i$ is a finite dimensional simple Jordan algebra for all $i = 1, \cdots, t$ (recall that simple Jordan algebras have already been classified).

Computing explicitly the non-abelian cohomology $H^2_{nab}(V, A)$ of two given Jordan algebras $A$ and $V$ is a really challenging problem. In Example 3.7 we compute $H^2_{nab}(k, A)$ and in particular we show that

$$H^2_{nab}(k, k^n) \cong \{ D \in M_n(k) \mid 2D^3 - 3D^2 + D = 0 \}$$

where $M_n(k)$ denotes the usual space of $n \times n$-matrices over $k$.

1. Preliminaries

Throughout, we work over a field $k$ of characteristic $\neq 2$. A bilinear map $f : W \times W \to V$ will be called symmetric if $f(x, y) = f(y, x)$, for all $x, y \in W$. If $V$ is a vector space, $V^*$ denotes its dual. $\text{End}_k(V)$ stands for the associative and unital endomorphisms algebra of $V$. The associator of an arbitrary algebra $(A, \cdot)$ is defined by $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$, for all $x, y, z \in A$.

A Jordan algebra is a vector space $A$ together with a bilinear map $\cdot : A \times A \to A$, called multiplication, such that for any $a, b \in A$ we have:

$$a \cdot b = b \cdot a, \quad (a^2 \cdot b) \cdot a = a^2 \cdot (b \cdot a)$$

i.e. $\cdot$ is commutative and satisfies the Jordan identity $(a^2, b, a) = 0$, for all $a, b \in A$. For all unexplained notions pertaining to Jordan algebra theory we refer the reader to [15, 22, 25, 26]. A Jordan algebra $A$ is called abelian if it has trivial multiplication, i.e. $a \cdot b = 0$ for all $a, b \in A$. If $A$ is an associative algebra, then $A$ with the new product defined by $x \cdot y := 2^{-1}(xy + yx)$, for all $x, y \in A$ becomes a Jordan algebra and is denoted by $A^+$. In particular, $\text{End}_k(V)^+$ is a Jordan algebra for any vector space $V$. A Jordan algebra is said to be special if it is isomorphic to a subalgebra of $A^+$, for an associative algebra $A$. Non-special Jordan algebras are called exceptional. We denote by $\text{Aut}_J(A)$
the automorphism group of the Jordan algebra $A$. If $A$ is a Jordan algebra, then it is well known [21] that the following polarization relation holds for any $x, y, z \in A$:

$$(x^2, y, z) + 2(x \cdot z, y, x) = 0$$

(1)

This can be easily proved by taking $a := x + \lambda z$, $b := y$, for any $\lambda \in k^*$, in the Jordan identity. For an arbitrary non-associative algebra $(A, \cdot)$ we define the so-called polarization map of $A$ as follows:

$$P : A \times A \times A \to A, \quad P(x, y, z) := (x^2, y, z) + 2(x \cdot z, y, x)$$

(2)

for all $x, y, z \in A$.

**Remark 1.1.** Unlike the case of associative or Lie algebras, showing that the Jordan algebra identity holds is rather cumbersome. More precisely, the major drawback in the case of Jordan algebras is that one needs to check the Jordan identity on the entire underlying vector space not only on the elements of its $k$-basis.

The unusual situation in Remark 1.1 is explained by the following result that will be used later on in the proof of Theorem 2.4:

**Proposition 1.2.** Let $(W, \circ)$ be a commutative algebra such that $W = A + V$, for two subspaces $A, V \subseteq W$. Assume that the Jordan identity $(\alpha^2, \beta, \alpha) = 0$ holds for all $\alpha, \beta \in A \cup V$. Then, $(W, \circ)$ is a Jordan algebra if and only if for any $a, b \in A$, $x, y \in V$ we have:

$$P(a, b, x) + P(x, b, a) + P(a, y, x) + P(x, y, a) = 0$$

(3)

where $P : W^3 \to W$ is the polarization map of $W$.

**Proof.** For arbitrary elements $X, Y \in W$ we write $X = a + x$ and $Y = b + y$, for some $a, b \in A$ and $x, y \in V$. Then, it can be easily proved that:

$$(X^2 \cdot Y) \cdot X = (a^2 \cdot b) \circ a + (a^2 \cdot b) \circ x + (a^2 \cdot y) \circ a + (a^2 \cdot y) \circ x +$$

$$+ (x^2 \cdot b) \circ a + (x^2 \cdot b) \circ x + (x^2 \cdot y) \circ a + (x^2 \cdot y) \circ x +$$

$$+ 2((a \circ x) \cdot b) \circ a + 2((a \circ x) \cdot b) \circ x + 2((a \circ x) \cdot y) \circ a +$$

$$+ 2((a \circ x) \cdot y) \circ x$$

and

$$X^2 \cdot (Y \cdot X) = a^2 \circ (b \cdot a) + a^2 \circ (b \cdot x) + a^2 \circ (y \cdot a) + a^2 \circ (y \cdot x) +$$

$$+ x^2 \circ (b \cdot a) + x^2 \circ (b \cdot x) + x^2 \circ (y \cdot a) + x^2 \circ (y \cdot x) +$$

$$+ 2(a \circ x) \cdot (b \cdot a) + 2(a \circ x) \cdot (b \cdot x) + 2(a \circ x) \cdot (y \cdot a) +$$

$$+ 2(a \circ x) \cdot (y \cdot x)$$

Taking into account that the Jordan identity $(\alpha^2, \beta, \alpha) = 0$ holds for any pair of elements $\alpha, \beta \in A \cup V$, we obtain that $(X^2 \cdot Y) \cdot X = X^2 \cdot (Y \cdot X)$ if and only if

$$(a^2, b, x) + (a^2, y, x) + (x^2, b, a) + (x^2, y, a) +$$

$$+ 2(a \circ x, b, a) + 2(a \circ x, b, x) + 2(a \circ x, y, a) + 2(a \circ x, y, x) = 0$$

which is equivalent to (3), using the formula (2). The proof is now finished. □
Definition 1.3. A right action of a Jordan algebra $A$ on a vector space $V$ is a bilinear map $\triangleleft: V \times A \to V$ such that for any $a \in A$ and $x \in V$ we have:
\[(x \triangleleft a^2) \triangleleft a = (x \triangleleft a) \triangleleft a^2\]  
(4)

Similarly, a left action of $A$ on $V$ is a bilinear map $\triangleright: A \times V \to V$ such that for any $a \in A$ and $x \in V$ we have:
\[a \triangleright (a^2 \triangleright x) = a^2 \triangleright (a \triangleright x)\]  
(5)

The canonical maps $\triangleright: A \times A \to A$ and $\triangleright: A \times A^* \to A^*$ given for any $a, b \in A$ and $a^* \in A^*$ by:
\[a \triangleright b := a \cdot b, \quad (a \triangleright a^*)(b) := a^*(a \cdot b)\]  
(6)
are left actions of $A$ on $A$ and $A^* = \text{Hom}_k(A, k)$, respectively.

Remarks 1.4. 1. Since a Jordan algebra $A$ is commutative any right/left action is a left/right action too; nevertheless, we shall use both types of actions throughout the paper in order to indicate precisely the position of the acting algebra. The set of all right/left actions of $A$ on $V$ is in bijection with the set of all linear maps $\varphi : A \to \text{End}_k(V)$ such that $\varphi(a) \circ \varphi(a^2) = \varphi(a^2) \circ \varphi(a)$, for all $a \in A$ in the endomorphism algebra of $V$. In certain references, such a map $\varphi : A \to \text{End}_k(V)$ is called a Jacobson representation of $A$ on $V$. Moreover, the left/right action condition (5) appears as one of the compatibilities in the definition of a Jordan bimodule or a representation of a Jordan algebra [10, 19].

2. Let $\varphi : A \to \text{End}_k(V)^+$ be a morphism of Jordan algebras. If we denote $\varphi(a)(x) = a \triangleright x$, we have that $(a \cdot b) \triangleright x = 2^{-1}(a \triangleright (b \triangleright x) + b \triangleright (a \triangleright x))$, for all $a, b \in A$ and $x \in V$. In particular, for $b = a$ we obtain that $a^2 \triangleright x = a \triangleright (a \triangleright x)$. Using this relation we can easily see that (5) holds, i.e. any Jordan algebra map $\varphi : A \to \text{End}_k(V)^+$ induces a left action of $A$ on $V$, as expected.

Definition 1.5. Let $A$ and $V$ be two Jordan algebras. An extension of $V$ by $A$ is a triple $(E, i, p)$ consisting of a Jordan algebra $E$ and two morphisms of Jordan algebras that fit into an exact sequence of Jordan algebras
\[0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} V \longrightarrow 0\]

Two extensions $(E, i, p)$ and $(E', i', p')$ of $V$ by $A$ are called cohomologous and we denote this by $(E, i, p) \approx (E', i', p')$ if there exists a Jordan algebra map $\varphi : E \to E'$ such that the following diagram is commutative:
\[\begin{array}{ccc}
A & \xrightarrow{i} & E \\
\downarrow{\text{Id}} & & \downarrow{\varphi} \\
A & \xrightarrow{i'} & E' \\
\end{array} \quad \begin{array}{ccc}
E & \xrightarrow{p} & V \\
\downarrow{\text{Id}} & & \downarrow{\text{Id}} \\
E' & \xrightarrow{p'} & V \\
\end{array}\]  
(7)

We denote by $\text{Ext}(V, A)$ the set of all equivalence classes of extensions of $V$ by $A$ via $\approx$. We say that $\varphi : E \to E'$ stabilizes $A$ (resp. co-stabilizes $V$) if the left square (resp. the right square) of diagram (7) is commutative.
Any Jordan algebra map \( \varphi : E \to E' \) which makes diagram (7) commutative is an isomorphism and thus \( \approx \) is indeed an equivalence relation on the class of all extensions of \( V \) by \( A \). Hilbert’s extension problem, formulated for Jordan algebras, is the following difficult open question: *For two given Jordan algebras \( A \) and \( V \) compute explicitly the classifying object \( \text{Ext}(V, A) \).*

2. Extending structures problem

This section aims at providing a theoretical answer to the extending structures problem for Jordan algebras. To this end, we introduce the following:

**Definition 2.1.** Let \( A \) be a Jordan algebra and \( E \) a vector space containing \( A \) as a subspace. Two Jordan algebra structures on \( E \) denoted by \( \cdot_E \) and \( \cdot_{E'} \), both containing \( A \) as a subalgebra, are called equivalent, and we denote this by \((E, \cdot_E) \equiv (E, \cdot_{E'})\), if there exists a Jordan algebra isomorphism \( \varphi : (E, \cdot_E) \to (E, \cdot_{E'}) \) which stabilizes \( A \), that is \( \varphi(a) = a \), for all \( a \in A \). We denote by \( \text{Jext}(E, A) \) the set of all equivalence classes on the set of all Jordan algebras structures on \( E \) containing \( A \) as a subalgebra via the equivalence relation \( \equiv \).

The set \( \text{Jext}(E, A) \) defined above is the object responsible for the classification part of the extending structures problem. We will show that \( \text{Jext}(E, A) \) is parameterized by a cohomological type object which we will be able to construct explicitly.

**Definition 2.2.** Let \( A = (A, \cdot) \) be a Jordan algebra and \( V \) a vector space. An extending datum of \( A \) through \( V \) is a system \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V) \) consisting of four bilinear maps

\[
\triangleleft : V \times A \to V, \quad \triangleright : V \times A \to A, \quad f : V \times V \to A, \quad \cdot_V : V \times V \to V
\]

Let \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V) \) be an extending datum. We denote by \( A \uplus \Omega(A, V) = A \uplus V \) the vector space \( A \times V \) together with the bilinear map \( \circ : (A \times V) \times (A \times V) \to A \times V \) defined by:

\[
(a, x) \circ (b, y) := (ab + x \triangleright b + y \triangleright a + f(x, y), \ x \triangleleft b + y \triangleleft a + x \cdot_V y)
\]

for all \( a, b \in A \) and \( x, y \in V \). The object \( A \uplus V \) is called the unified product of \( A \) and \( V \) if it is a Jordan algebra with the multiplication given by (8). In this case the extending datum \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V) \) is called a Jordan extending structure of \( A \) through \( V \). The maps \( \triangleleft \) and \( \triangleright \) are called the actions of \( \Omega(A, V) \) and \( f \) is called the (non-abelian) cocycle of \( \Omega(A, V) \).

In the sequel we use the following convention: if any of the maps of an extending datum \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V) \) is trivial then we will write the quadruple \((\triangleleft, \triangleright, f, \cdot_V)\) without it. The first natural example of a unified product was first constructed in [17] related to the classification of simple finite-dimensional formally real Jordan algebras.

**Example 2.3.** Let \( V \) be a vector space equipped with a symmetric bilinear form \( f : V \times V \to k \) and let \( J(V, f) := k \times V \) with the multiplication

\[
(a, x) \circ (b, y) := (ab + f(x, y), \ bx + ay)
\]
for all \(a, b \in k\) and \(x, y \in V\). Then \(J(V, f)\) is a Jordan algebra and \(J(V, f) = k \otimes V\) is a unified product of the Jordan algebras \(k\) (with the usual multiplication) and \(V\): the maps \(\triangleright\) and \(\cdot_V\) are both trivial and the (right) action \(\triangleleft\) of \(k\) on \(V\) is just the \(k\)-vector space structure on \(V\). The Jordan algebra \(J(V, f)\) is called spin factor Jordan algebra, the Clifford Jordan algebra or the Jordan algebra associated to a symmetric bilinear form. Moreover, \(J(V, f)\) is special and if \(f\) is nondegenerate and \(\dim_k(V) > 1\) then \(J(V, f)\) is simple \([26]\).

Let \(\Omega(A, V)\) be an extending datum of \(A\) through \(V\). It will be worth the while to write down the following relations which hold in \(A \otimes V\):

\[
(a, 0) \circ (b, y) = (a \cdot b + y \triangleright a, y \triangleleft a) \quad (9)
\]

\[
(0, x) \circ (b, y) = (x \triangleright b + f(x, y), x \triangleleft b + x \cdot_V y) \quad (10)
\]

for all \(a, b \in A\) and \(x, y \in V\).

**Theorem 2.4.** Let \(\Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V)\) be an extending datum of a Jordan algebra \(A = (A, \cdot)\) through a vector space \(V\). The following statements are equivalent:

1. \(A \otimes V\) is a unified product;
2. The following compatibilities hold for any \(a, b \in A, x, y \in V\):
   
   (E1) \(f : V \times V \rightarrow A\) and \(\cdot_V : V \times V \rightarrow V\) are symmetric maps;

   (E2) \(\triangleleft : V \times A \rightarrow V\) is a right action of \(A\) on \(V\), i.e. \((x \triangleleft a^2) \triangleleft a = (x \triangleleft a) \triangleleft a^2\);

   (E3) \(a \cdot (x \triangleright a^2) + (x \triangleleft a^2) \triangleright a = a^2 \cdot (x \triangleright a) + (x \triangleleft a) \triangleright a^2\);

   (E4) \(x \triangleright (f(x, x) \cdot a) + x \triangleright (x^2 \triangleright a) + f(x^2 \triangleleft a, x) = f(x, x) \cdot f(x, x) + x^2 \triangleright f(x, x) + (x \triangleleft a) \triangleright f(x, x) + f(x^2, x \triangleleft a)\);

   (E5) \(x \triangleleft (f(x, x) \cdot a) + x \triangleleft (x^2 \triangleright a) + (x^2 \triangleleft a) \cdot_V x = x^2 \triangleleft (x \triangleright a) + (x \triangleleft a) \triangleleft f(x, x) + x^2 \cdot_V (x \triangleleft a)\);

   (E6) \(x \triangleright (y \triangleright f(x, x)) + x \triangleright f(x^2, y) + f(y \triangleleft f(x, x) + x^2 \cdot_V y, x) = f(x, x) \cdot f(x, x) + x^2 \triangleright f(x, x) + f(x, x) \cdot_V y) + f(x, x) + f(x^2, x \cdot_V y)\);

   (E7) \(x \triangleleft (y \triangleright f(x, x)) + x \triangleleft f(x^2, y) + (y \triangleleft f(x, x)) \cdot_V x + (x^2 \cdot_V y) \cdot_V x = x^2 \triangleleft f(x, x) + (x \cdot_V y) \triangleleft f(x, x) + x^2 \cdot_V (x \cdot_V y)\);

   (E8) The following missing relations hold in the algebra \((A \otimes V, \circ)\):

   \[
P(a, b, x) + P(x, b, a) + P(a, y, x) + P(x, y, a) = 0 \quad (11)
   \]

   where \(P : (A \otimes V)^3 \rightarrow A \otimes V\) is the polarization map of \(A \otimes V\) and we identify \(\alpha = (\alpha, 0), z = (0, z)\), for any \(\alpha \in A\) and \(z \in V\).
Theorem (JES). Let \( \Omega(A, V) = \langle \preceq, \succeq, f, \cdot_V \rangle \) be an extending datum of a Jordan algebra \( A \) through a vector space \( V \) such that \( f := 0 \) is the trivial map. Then, using Theorem 2.4, we obtain that \( \Omega(A, V) = \langle \preceq, \succeq, f := 0, \cdot_V \rangle \) is a Jordan extending structure of \( A \) through \( V \) if and only if \((V, \cdot_V)\) is a Jordan algebra (by (E1) and (E7)) and \((A, V, \preceq, \succeq, f, \cdot_V)\) is a matched pair of Jordan algebras in the sense of [7, Definition 2.1].

Let \( A \) be a Jordan algebra \( A \) and \( V \) a vector space. In the sequel, we shall denote by \( \mathcal{JES}(A, V) \) the set of all Jordan extending structures of \( A \) through \( V \) (or, equivalently, all systems \( \Omega(A, V) = \langle \preceq, \succeq, f, \cdot_V \rangle \) satisfying the compatibility conditions (E1)-(E8) of Theorem 2.4). Note that the set \( \mathcal{JES}(A, V) \) contains at least one element, namely the extending structure \( \Omega(A, V) = \langle \preceq, \succeq, f, \cdot_V \rangle \) consisting only of trivial maps and whose associated unified product is the direct product between \( A \) and the abelian Jordan algebra \( V \).

Let \( \Omega(A, V) = \langle \preceq, \succeq, f, \cdot_V \rangle \in \mathcal{JES}(A, V) \) be a Jordan extending structure and \( A \natural V \) the associated unified product. We have an injective Jordan algebra map defined as follows:

\[
i_A : A \rightarrow A \natural V, \quad i_A(a) = (a, 0)
\]

which allows us to see \( A \) as a Jordan subalgebra of \( A \natural V \) via the identification \( A \cong i_A(A) = A \times \{0\} \). Furthermore, the converse also holds: any Jordan algebra structure on a vector space \( E \) containing \( A \) as a Jordan subalgebra is isomorphic to a unified product.

**Theorem 2.7.** Let \( A = (A, \cdot) \) be a Jordan algebra, \( E \) a vector space and \( \cdot : E \) a Jordan algebra structure on \( E \) containing \( A \) as a Jordan subalgebra. Then there exists a Jordan
Proof. Let \( \cdot_E \) be Jordan algebra structure on \( E \) containing \( A \) as a Jordan subalgebra, i.e. \( a \cdot_E b = a \cdot b \), for all \( a, b \in A \). Working over a field allows us to find a linear map \( p : E \to A \) such that \( p(a) = a \), for all \( a \in A \). Consequently, \( V := \ker(p) \) is a complement of \( A \) in \( E \), i.e. \( E = A \oplus V \). Using the retraction \( p \) we can now define the extending datum of \( A \) through \( V \) as follows:

\[
\begin{align*}
\triangleright &= \triangleright_p : V \times A \to A, & \triangleright a := p(x \cdot_E a) = p(a \cdot_E x) & (13) \\
\triangleleft &= \triangleleft_p : V \times A \to V, & \triangleleft a := x \cdot_E a - p(x \cdot_E a) & (14) \\
f = f_p : V \times V \to A, & f(x, y) := p(x \cdot_E y) & (15) \\
\cdot_V &= (\cdot_V)_p : V \times V \to V, & x \cdot_V y := x \cdot_E y - p(x \cdot_E y) & (16)
\end{align*}
\]

for any \( a \in A \) and \( x, y \in V \). First of all, it is straightforward to see that the above maps are well defined bilinear maps: \( x \triangleleft a \in V \) and \( x \cdot_V y \in V = \ker(p) \), for all \( x, y \in V \) and \( a \in A \). We will show that \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V) \) is a Jordan extending structure of \( A \) through \( V \) and \( \varphi : A \circ V \to E, \varphi(a, x) := a + x \) is an isomorphism of Jordan algebras that stabilizes \( A \). The strategy we use, relaying on Theorem 2.4, is the following: \( \varphi : A \times V \to E, \varphi(a, x) := a + x \) is a linear isomorphism between the Jordan algebra \( E = (E, \cdot_E) \) and the direct product of vector spaces \( A \times V \) with the inverse given by \( \varphi^{-1}(y) := (p(y), y - p(y)) \), for all \( y \in E \). Thus, there exists a unique Jordan algebra structure on \( A \times V \) such that \( \varphi \) is an isomorphism of Jordan algebras and this unique multiplication on \( A \times V \) is given for any \( a, b \in A \) and \( x, y \in V \) by:

\[
(a, x) \circ (b, y) := \varphi^{-1}\left( \varphi(a, x) \cdot_E \varphi(b, y) \right)
\]

We are now left to prove that the above multiplication coincides with the one associated to the system \( (\triangleleft_p, \triangleright_p, f_p, (\cdot_V)_p) \) as defined by (8). Indeed, using intensively the commutativity of \( \cdot_E \), for any \( a, b \in A \) and \( x, y \in V \) we have:

\[
\begin{align*}
(a, x) \circ (b, y) &= \varphi^{-1}\left( \varphi(a, x) \cdot_E \varphi(b, y) \right) = \varphi^{-1}\left( (a + x) \cdot_E (b + y) \right) \\
&= \varphi^{-1}\left( a \cdot_E (b + y) + a \cdot_E b + x \cdot_E b + x \cdot_E y \right) \\
&= \left( p(a \cdot_E b) + p(a \cdot_E y) + p(x \cdot_E b) + p(x \cdot_E y), a \cdot b + a \cdot_E y + x \cdot_E b + x \cdot_E y - p(a \cdot_E b) - p(a \cdot_E y) - p(x \cdot_E b) - p(x \cdot_E y) \right) \\
&= \left( a \cdot b + x \triangleright b + y \triangleright a + f(x, y), x \triangleleft b + y \triangleleft a + x \cdot_V y \right)
\end{align*}
\]

as desired. Moreover, the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & A \circ V \\
\downarrow{Id} & & \downarrow{\varphi} \\
A & \xrightarrow{i} & E
\end{array}
\]

is obviously commutative which shows that \( \varphi \) stabilizes \( A \) and this finishes the proof. \( \square \)
Theorem 2.7 shows that the classification of all Jordan algebra structures on $E$ that contain $A$ as a subalgebra comes down to the classification of the unified products $AζV$ for a given complement $V$ of $A$ in $E$. In order to describe the classifying sets $\text{Jext} (E, A)$ introduced in Definition 2.1, we need the following:

**Lemma 2.8.** Let $Ω(A, V) = (≤, ∨, f, ·, V)$ and $Ω′(A, V) = (≤′, ∨′, f′, ·′, V)$ be two Jordan algebra extending structures of a Jordan algebra $A = (A, ·)$ through $V$ and $AζV$, respectively $AζV′$, the associated unified products. Then there exists a bijection between the set of all morphisms of Jordan algebras $ψ : AζV → AζV′$ which stabilize $A$ and the set of pairs $(r, v)$, where $r : V → A$, $v : V → V$ are two linear maps satisfying the following compatibility conditions for any $a ∈ A$, $x, y ∈ V$:

1. $v(x) ≤′ a = v(x ≤ a)$;
2. $v(x) ·′ v′ a = r(x ≤ a) + x · a − a · r(x)$;
3. $v(x · v′ y) = v(x) ·′ v(y) + v(x) ≤′ r(y) + v(y) ≤′ r(x)$;
4. $r(x · v′ y) = r(x) · r(y) + v(x) ·′ r(y) + v(y) ·′ r(x) + f′(v(x), v(y)) − f(x, y)$

Under the above bijection the morphism of Jordan algebras $ψ = ψ_{(r,v)} : AζV → AζV′$ corresponding to $(r, v)$ is given for any $a ∈ A$ and $x ∈ V$ by:

$ψ(a, x) = (a + r(x), v(x))$

Moreover, $ψ = ψ_{(r,v)}$ is an isomorphism if and only if $v : V → V$ is bijective and $ψ_{(r,v)}$ co-stabilizes $V$ if and only if $v = \text{Id}_V$, the identity of $V$.

**Proof.** To start with, it can be easily seen that if a linear map $ψ : AζV → AζV′$ makes the following diagram commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & AζV \\
\downarrow{Id_A} & & \downarrow{ψ} \\
A & \xrightarrow{i_A} & AζV'
\end{array}
\]

then there exists two linear maps $r : V → A$, $v : V → V$ such that $ψ(a, x) = (a + r(x), v(x))$, for all $a ∈ A$, and $x ∈ V$.

Let $ψ = ψ_{(r,v)}$ be such a linear map, i.e. $ψ(a, x) = (a + r(x), v(x))$, for some linear maps $r : V → A$, $v : V → V$. We will prove that $ψ$ is a morphism of Jordan algebras if and only if the compatibility conditions (M1)-(M4) hold. To this end, it is enough to prove that the compatibility

\[ψ((a, x) ∘ (b, y)) = ψ(a, x) ∘′ ψ(b, y)\]  \hspace{1cm} (17)

holds on all generators of $AζV$. We leave out the detailed computations and only indicate the main steps of this verification. First, it is easy to see that (17) holds for the pair $(a, 0)$, $(b, 0)$, for all $a, b ∈ A$. Secondly, we can prove that (17) holds for the pair $(a, 0)$, $(0, x)$ if and only if (M1) and (M2) hold. Finally, (17) holds for the pair $(0, x)$, $(0, y)$ if and only if (M3) and (M4) hold. The last statement is elementary: we just note that if
v : V → V is bijective, then ψ(r,v) is an isomorphism of Jordan algebras with the inverse given for any b ∈ A and y ∈ V by:

$$\psi^{-1}_{(r,v)}(b, y) = (b - r(v^{-1}(y)), v^{-1}(y))$$

This finishes the proof. □

Arising from Lemma 2.8 the following concept will be used for the classification of unified products:

**Definition 2.9.** Let A be a Jordan algebra and V a vector space. Two Jordan algebra extending structures of A by V, Ω(A, V) = (≺, ⊲, f, ·V) and Ω′(A, V) = (≺′, ⊲′, f′, ·′V) are called equivalent, and we denote this by Ω(A, V) ≡ Ω′(A, V), if there exists a pair of linear maps (r, v), where r : V → A and v ∈ Autk(V) such that (≺′, ⊲′, f′, ·′V) is defined via (≺, ⊲, f, ·V) using (r, v) as follows:

- $$x ≺ a = v(v^{-1}(x) ≪ a)$$
- $$x ⊲′ a = r(v^{-1}(x) ≪ a) + v^{-1}(x) ⊲ a - a \cdot r(v^{-1}(x))$$
- $$f′(x, y) = f(v^{-1}(x), v^{-1}(y)) + r(v^{-1}(x) \cdot v^{-1}(y)) + r(v^{-1}(x)) \cdot r(v^{-1}(y))$$
- $$-v^{-1}(y) ⊲ r(v^{-1}(x))$$
- $$x \cdot′_V y = v(v^{-1}(x) \cdot v^{-1}(y)) - v(v^{-1}(x) ≪ r(v^{-1}(y))) - v(v^{-1}(y) ≪ r(v^{-1}(x)))$$

for all a ∈ A, x, y ∈ V.

By putting together all the results proved in this section we obtain the following theoretical answer to the extending structures problem for Jordan algebras:

**Theorem 2.10.** Let A be a Jordan algebra, E a vector space that contains A as a subspace and V a complement of A in E. Then:

1. ≡ from Definition 2.9 is an equivalence relation on the set JES(A, V) of all Jordan extending structures of A through V. We denote by J_{A}(V, A) := JES(A, V)/≡, the quotient set.

2. The map

$$J_{A}(V, A) \to Jext(E, A), \quad (≺, ⊲, f, ·V) \mapsto (A \cdot V, ◦)$$

is bijective, where (≺, ⊲, f, ·V) is the equivalence class of (≺, ⊲, f, ·V) via ≡.

**Proof.** We observe that Ω(A, V) ≡ Ω′(A, V) in the sense of Definition 2.9 if and only if there exists an isomorphism of Jordan algebras $\psi : A \cdot V \to A \cdot V'$ which stabilizes A. Therefore, ≡ is an equivalence relation on the set JES(A, V) of all Jordan algebra extending structures Ω(A, V). Now, the conclusion follows from Theorem 2.4, Theorem 2.7 and Lemma 2.8. □

The main application of the results proven in this section will be given in the next section. Here we consider only two of them: the first one refers to the classification of a special
class of finite dimensional Jordan algebras while the second one to a possible further development of the invariant theory for Jordan algebras.

**Application: flag Jordan algebras.** This section deals with the following special case of extending structures:

**Definition 2.11.** Let \( A \) be a Jordan algebra and \( E \) a vector space containing \( A \) as a subspace. A Jordan algebra structure on \( E \) such that \( A \) is a subalgebra is called a flag extending structure of \( A \) to \( E \) if there exists a finite chain of subalgebras of \( E \)

\[
E_0 := A \subset E_1 \subset \cdots \subset E_m = E
\]

such that \( E_i \) has codimension 1 in \( E_{i+1} \), for all \( i = 0, \ldots , m - 1 \). A finite dimensional Jordan algebra \( E \) is called flag if it is a flag extending structure of \( \{0\} \).

The flag extending structures of \( A \) to \( E \) can be obtained by following the method described below. We start by describing and classifying all unified products \( A \sharp V_1 \), for a 1-dimensional vector space \( V_1 \). The next natural step consists in describing and classifying all unified products between the unified products previously described and a 1-dimensional vector space. Continuing this process will render all flag extending structures of \( A \) to \( E \) after \( \dim_k(V) \) steps.

The following concept will be useful for our approach:

**Definition 2.12.** Let \( A \) be a Jordan algebra and \( V \) a vector space of dimension 1 with basis \( \{x\} \). A flag datum of \( A \) is a system \((D, \lambda, a_0, \alpha_0) \in \text{End}_k(A) \times A^* \times A \times k \) such that \( A_{(D, \lambda, a_0, \alpha_0)} := A \sharp_k x \) with the multiplication given for all \( a, b \in A \) by:

\[
(a, x) \circ (b, x) = (a \cdot b + D(a + b) + a_0, (\lambda(a + b) + \alpha_0) x)
\]

is a Jordan algebra. The set of all flag data of \( A \) will be denoted by \( \mathcal{F}(A) \).

Explicitly, if \( \{e_i \mid i \in I\} \) is a \( k \)-basis of a Jordan algebra \( A \) then \( A_{(D, \lambda, a_0, \alpha_0)} \) is the commutative algebra having \( \{x, e_i \mid i \in I\} \) as a \( k \)-basis and the multiplication given for all \( i, j \in I \) by:

\[
e_i \circ e_j := e_i \cdot e_j, \quad e_i \circ x := x \circ e_i := D(e_i) + \lambda(e_i) x, \quad x \circ x := a_0 + \alpha_0 x
\]

The set of all Jordan extending structures \( JES(A, V) \) of a Jordan algebra \( A \) through a 1-dimensional vector space \( V \) is parameterized by \( \mathcal{F}(A) \).

**Proposition 2.13.** Let \( A \) be a Jordan algebra and \( V \) a vector space of dimension 1 with basis \( \{x\} \). Then there exists a bijection between the set \( JES(A, V) \) of all Jordan extending structures of \( A \) through \( V \) and the set \( \mathcal{F}(A) \) of all flag data of \( A \). Through the above bijection, the Jordan extending structure \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot_V) \) corresponding to \((D, \lambda, a_0, \alpha_0) \in \mathcal{F}(A) \) is given for all \( a \in A \) by:

\[
x \triangleleft a = \lambda(a)x, \quad x \triangleright a = D(a), \quad f(x, x) = a_0, \quad x \cdot_V x = \alpha_0 x
\]

**Proof.** Since \( V := kx \) has dimension 1, the set of all bilinear maps \( \triangleleft : V \times A \to V \), \( \triangleright : V \times A \to A \), \( f : V \times V \to A \) and \( \cdot_V : V \times V \to V \) is in bijection with the set of all systems \((D, \lambda, a_0, \alpha_0) \in \text{End}_k(A) \times A^* \times A \times k \) and the bijection is given such...
that (21) holds. The proof is now finished using Theorem 2.4 once we observe that the multiplication given by (19) coincides with the one of the unified product (8) associated to the extending datum defined by (21).

\[ \square \]

**Remark 2.14.** The compatibility conditions of a flag datum \((D, \lambda, a_0, \alpha_0) \in \mathcal{F}(A)\) of a Jordan algebra \(A\) as defined in Definition 2.12 can be written down explicitly using Theorem 2.4 and Proposition 2.13 by a straightforward computation. For instance, the axioms (E1)-(E7) from Theorem 2.4 are equivalent to the following five compatibility conditions:

\[
\begin{align*}
    a \cdot D(a^2) + \lambda(a^2) D(a) &= a^2 \cdot D(a) + \lambda(a) D(a^2), \\
    D(a_0 \cdot a) &= a_0 \cdot D(a) + \lambda(a) D(a_0), \\
    D^2(a_0) &= D(a_0) a_0 = a_0^2 + \alpha_0 D(a_0), \\
    \lambda(a_0 \cdot a) &= \lambda(a_0) \lambda(a), \\
    \lambda(D(a_0)) &= 0
\end{align*}
\]

for all \(a \in A\). Two more complicated compatibilities which we do not write down arise from the missing relations (11).

Using Theorem 2.7 and Proposition 2.13 we obtain:

**Corollary 2.15.** Let \(A\) be a Jordan algebra. Then a Jordan algebra \(E\) contains \(A\) as a subalgebra of codimension 1 if and only if there exists \((D, \lambda, a_0, \alpha_0) \in \mathcal{F}(A)\) a flag datum of \(A\) such that \(E \cong A_{(D, \lambda, a_0, \alpha_0)}\).

We can now provide the classification of all Jordan algebras \(A_{(D, \lambda, a_0, \alpha_0)}\). This will be done by computing \(\mathcal{J}_A(V, A)\) for a 1-dimensional vector space \(V\) and constitutes the main step in classifying flag Jordan algebras.

The following can be obtained as a consequence of all the above together:

**Theorem 2.17.** Let \(A\) be a Jordan algebra of codimension 1 in the vector space \(E\) and \(V\) a complement of \(A\) in \(E\). Then, \(\equiv\) from Definition 2.16 is an equivalence relation on the set \(\mathcal{F}(A)\) of all flag data of \(A\) and

\[
\text{Jext} (E, A) \cong \mathcal{J}_A(V, A) \cong \mathcal{F}(A) / \equiv
\]

The bijection between \(\mathcal{F}(A) / \equiv\) and \(\text{Jext} (E, A)\) is given by:

\[
(D, \lambda, a_0, \alpha_0) \mapsto A_{(D, \lambda, a_0, \alpha_0)}
\]

where \((D, \lambda, a_0, \alpha_0)\) is the equivalence class of \((D, \lambda, a_0, \alpha_0)\) via \(\equiv\) and \(A_{(D, \lambda, a_0, \alpha_0)}\) is the Jordan algebra constructed in (19).
The next example computes \( \mathcal{J}_A(k, A) \) and then describes all Jordan algebra structures which extend the Jordan algebra structure from \( A \) to a vector space of dimension \( 1 + \dim_k(A) \). The detailed computations are rather long but straightforward and will be omitted.

**Example 2.18.** Let \( A := k^n \) be the abelian Jordan algebra of dimension \( n \), i.e. \( a \cdot b = 0 \), for all \( a, b \in k^n \). We define the following sets:

\[
\mathcal{F}_1(k^n) := \{(D, a_0, \alpha_0) \in \text{End}_k(k^n) \times k^n \times k | D^2(a_0) = \alpha_0 D(a_0) \}
\]

\[
\mathcal{F}_2(k^n) := \{(D, \lambda, a_0, \alpha_0) \in \text{End}_k(k^n) \times (k^n)^* \times k^n \times k | D(a_0) = \lambda(a_0) = 0, \lambda \neq 0 \}
\]

Then the classifying object \( \mathcal{J}_{k^n}(k, k^n) \) is the coproduct of the following sets:

\[
\mathcal{J}_{k^n}(k, k^n) \cong (\mathcal{F}_1(k^n)/\equiv_1) \sqcup (\mathcal{F}_1(k^n)/\equiv_2)
\]

where \( \equiv_1 \) and \( \equiv_2 \) are the following equivalence relations: \((D, a_0, \alpha_0) \equiv_1 (D', a_0', \alpha_0') \) if and only if there exists a pair \((r, u) \in k^n \times k^* \) such that for all \( a \in k^n \) we have:

\[
D(a) = u D'(a), \quad \alpha_0 = u \alpha_0', \quad a_0 = u^2 a_0' + 2 u D'(r) - u \alpha_0' r
\]

and \((D, \lambda, a_0, \alpha_0) \equiv_2 (D', \lambda', a_0', \alpha_0') \) if and only if \( \lambda = \lambda' \) and there exists a pair \((r, u) \in k^n \times k^* \) such that for all \( a \in k^n \) we have:

\[
D(a) = u D'(a) - \lambda'(a) r, \quad \alpha_0 = u \alpha_0' + 2 \lambda'(r) r
\]

\[
a_0 = u^2 a_0' + 2 u D'(r) - u \alpha_0' r - 2 \lambda'(r) r
\]

Any \((n + 1)\)-dimensional Jordan algebra having a \( n \)-dimensional abelian subalgebra is isomorphic to \( k^n_{(D, a_0, \alpha_0)} \), for some \((D, a_0, \alpha_0) \in \mathcal{F}_1(k^n) \) or to \( k^n_{(D, \lambda, a_0, \alpha_0)} \), for some \((D, \lambda, a_0, \alpha_0) \in \mathcal{F}_2(k^n) \).

**Application: an Artin type theorem for Jordan algebras.** Let \( A \) be a Jordan algebra, \( V \) a vector space and \( \Omega(A, V) = \{\langle \cdot, \cdot \rangle, f, \cdot \cdot \} \) a Jordan extending structure of \( A \) through \( V \) such that \( \triangleright \) is trivial, i.e. \( x \triangleright a = 0 \), for all \( x \in V \) and \( a \in A \). Then, in this case the compatibility conditions (E1)-(E8) of Theorem 2.4 take a simpler form and the associated unified product, called the **twisted product** of \( A \) and \( V \), will be denoted by \( A \# \cdot \cdot \cdot V \). Thus \( A \# \cdot \cdot \cdot V = A \times V \) with the multiplication given by:

\[
(a, x) \circ (b, y) := (a \cdot b + f(x, y), x \cdot \cdot \cdot b + y \cdot \cdot \cdot a + x \cdot \cdot \cdot y)
\]

for all \( a, b \in A \) and \( x, y \in V \). Twisted products are generalizations of spin factor Jordan algebras from Example 2.3 which can be recovered by setting \( A := k \) and \( \cdot \cdot \cdot V := 0 \). These products will be the main ingredient in proving the following Artin type theorem. As in the classical case, the Jordan algebra version of the Artin theorem will provide a way of reconstructing a Jordan algebra out of its subalgebra of invariants.

**Theorem 2.19.** Let \( G \) be a finite group whose order is invertible in \( k \) and \( A \) a Jordan algebra. Assume the group morphism \( \varphi : G \to \text{Aut}_3(A) \) is an action of \( G \) on \( A \). If \( A^G := \{a \in A | \varphi(g)(a) = a, \forall g \in G \} \subseteq A \) is the corresponding subalgebra of invariants and \( V \) a complement of \( A^G \) in \( A \), then there exists a Jordan extending structure of \( A^G \) through \( V \) and an isomorphism of Jordan algebras \( A \cong A^G \# \cdot \cdot \cdot V \).
Proof. We denote $\varphi(g)(a) = g \rightarrow a$, for all $g \in G$ and $a \in A$. First, note that $A^G$ is a subalgebra of $A$ and $g \rightarrow (a \cdot b) = (g \rightarrow a) \cdot (g \rightarrow b)$, for all $g \in G$, $a$, $b \in A$. We define the trace map as follows for all $x \in A$:

$$t : A \rightarrow A^G, \quad t(x) := |G|^{-1} \sum_{g \in G} g \rightarrow x$$ (25)

Observe that $t(x) \in A^G$ and therefore for all $a \in A^G$ we have:

$$t(x \cdot a) = |G|^{-1} \sum_{g \in G} g \rightarrow (x \cdot a) = |G|^{-1} \sum_{g \in G} (g \rightarrow x) \cdot (g \rightarrow a) = t(x) \cdot a$$

Secondly, we note that the trace map $t : A \rightarrow A^G$ is a linear retraction of the canonical inclusion $A^G \hookrightarrow A$, i.e. $t(a) = a$, for all $a \in A^G$. Now, if we compute the canonical extending structure of $A^G$ through $V := \ker(t)$ associated to the trace map $t$, using the formulas (13)-(16) from the proof of Theorem 2.7, we obtain that for all $x \in V$ and $a \in A^G$ we have:

$$x \triangleright_t a = t(x \cdot a) = t(x) \cdot a = 0,$$

i.e. the action $\triangleright_t$ is trivial. Applying once again Theorem 2.7 we obtain that the map defined for all $a \in A^G$ and $x \in V$ by:

$$\vartheta : A^G \# V \rightarrow A, \quad \vartheta(a, x) := a + x$$

is an isomorphism of Jordan algebras. This finishes the proof. \qed

Remark 2.20. If the group $G$ in Theorem 2.19 is a finite cyclic group generated by an element $g$ then it can be easily seen that $\ker(t) = \{ a - (g \rightarrow a) \mid a \in A \}$.

An invariant theory, similar to the classical theory of groups acting on associative algebras, can be developed for Jordan algebras by studying the Jordan algebra extension $A^G \subseteq A$. For instance, the following problems arise naturally in this context:

Let $\varphi : G \rightarrow \text{Aut}_J(A)$ be an action of a group $G$ on a Jordan algebra $A$. If the subalgebra of invariants $A^G$ is special (resp. (semi)simple, solvable, nilpotent, etc.), is $A$ also special (resp. (semi)simple, solvable, nilpotent, etc.)?

3. CROSSED PRODUCTS AND THE EXTENSION PROBLEM FOR JORDAN ALGEBRAS

In this section we deal with crossed products of Jordan algebras. As mentioned before, this important construction can be obtained as a special case of the unified product. Indeed, let $\Omega(A, V) = (\triangleright, \triangleleft, f, \{-, -\})$ be an extending datum of the Jordan algebra $A = (A, \cdot)$ through a vector space $V$ such that $\triangleleft$ is trivial, i.e. $x \triangleleft a = 0$, for all $x \in V$ and $a \in A$. Applying Theorem 2.4 we obtain that $\Omega(A, V) = (\triangleright, f, \triangleright)$ is a Jordan extending structure of $A$ through $V$ if and only if $(V, \triangleright)$ is a Jordan algebra and the following compatibilities hold for all $a \in A$ and $x, y, z \in V$:

(CP1) $f : V \times V \rightarrow A$ is a symmetric map;

(CP2) $a \cdot (x \triangleright a^2) = a^2 \cdot (x \triangleright a)$;
we obtain the following result which shows that crossed products
that is trivial since for any 2.7
Firstly, if \( A \) is simple there is nothing to prove. On the contrary, if \( A \) has a proper ideal \( \{0\} \neq I \neq A \), it follows from Corollary 3.1 that \( A \cong I \# (A/I) \), a crossed product of \( I \) and \( A/I \). If \( I \) and \( A/I \) are simple the proof is finished; Otherwise, since \( \dim_k(I) \), \( \dim_k(A/I) < n \), the conclusion follows by induction.
**Application: the extension problem.** We deal with the extension problem for Jordan algebras in the classical way, by using crossed products. Indeed, note first any crossed product \( A \#_f V \) of two Jordan algebras \( A \) and \( V \) is an extension of \( V \) by \( A \) via the following canonical exact sequence:

\[
0 \longrightarrow A \longrightarrow A \#_f V \longrightarrow V \longrightarrow 0
\]  

where \( i_A : A \rightarrow A \#_f V, i_A(a) := (a,0) \) and \( \pi_V : A \#_f V \rightarrow V, \pi_V(a,x) := x \), for all \( a \in A \) and \( x \in V \). Conversely, we have:

**Theorem 3.3.** Let \( A \) and \( V \) be two Jordan algebras and \((E, i, \pi)\) an extension of \( V \) by \( A \), that is there exists an exact sequence of Jordan algebras

\[
0 \longrightarrow A \longrightarrow E \longrightarrow V \longrightarrow 0
\]

Then \((E, i, \pi)\) is cohomologous, in the sense of Definition 1.5, to a crossed product extension of the form \((27)\).

**Proof.** We identify \( A \cong \text{Im}(i) = \text{Ker}(\pi) \subseteq E \) which allows us to assume that \( A \) is an ideal of \( E \). Since we are working over a field, we can find a \( k \)-linear section \( s : V \rightarrow E \) of \( \pi \), i.e. \( \pi(s(x)) = x \), for all \( x \in V \). Then \( \psi : A \times V \rightarrow E, \psi(a,x) := a + s(x) \) is an isomorphism of vector spaces with the inverse \( \psi^{-1}(y) = (y - s(\pi(y)), \pi(y)) \), for all \( y \in E \). Using the section \( s \) we can define the following two bilinear maps:

\[
\triangleright = \triangleright_s : V \times A \rightarrow A, \quad x \triangleright a := s(x) \cdot_E a
\]

\[
f = f_s : V \times V \rightarrow A, \quad f(x,y) := s(x) \cdot_E s(y) - s(x \cdot_V y)
\]

for all \( x, y \in V \) and \( a \in A = \text{Ker}(\pi) \). Since \( \pi \) is a Jordan algebra map and \( s \) is a section of it we can easily see that these are well-defined maps. We shall prove that \((A, V, \triangleright_s, f_s)\) is a crossed system and \( \psi : A \#_{f_s} V \rightarrow E \) is an isomorphism of Jordan algebras that stabilizes \( A \) and co-stabilizes \( V \) and this will finish the proof. Instead of proving the compatibility conditions \((\text{CP}1)\)-\((\text{CP}5)\) for the pair \((\triangleright_s, f_s)\) by a rather long computation we will use the same trick as in Theorem 2.7 and Theorem 2.4. First, the map \( \psi : A \times V \rightarrow E, \psi(a,x) = a + s(x) \) is a linear isomorphism between the Jordan algebra \( E = (E, \cdot_E) \) and the direct product of vector spaces \( A \times V \). Thus, there exists a unique Jordan algebra structure on \( A \times V \) such that \( \psi \) is an isomorphism of Jordan algebras. This unique multiplication \( \bullet \) on \( A \times V \) is given for all \( a, b \in A \) and \( x, y \in V \) by:

\[
(a,x) \bullet (b,y) = \psi^{-1}(\psi(a,x) \cdot_E \psi(b,y)) = \psi^{-1}((a + s(x)) \cdot_E (b + s(y)))
\]

\[
= \psi^{-1}((a \cdot_E b + a \cdot_E s(y) + s(x) \cdot_E b + s(x) \cdot_E s(y)))
\]

\[
= (a \cdot_E b + a \cdot_E s(y) + s(x) \cdot_E b + s(x) \cdot_E s(y) - s(x \cdot_V y), x \cdot_V y)
\]

\[
= (a \cdot b + x \triangleright b + y \triangleright a + f(x,y), x \cdot_V y)
\]

This shows that the multiplication \( \bullet \) on \( A \times V \) coincides with the one given in \((26)\). Applying now Theorem 2.4 for \( a := 0 \) we obtain that \((A, V, \triangleright_s, f_s)\) is a crossed system,
\( \psi : A \#_{\psi} V \to E \) is an isomorphism of Jordan algebras and the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & A \#_{\psi} V \\
\downarrow \text{Id} & & \downarrow \psi \\
A & \xrightarrow{i} & E
\end{array}
\]

\[
\begin{array}{ccc}
& & V \\
\pi & \xrightarrow{} & V \\
\downarrow \text{Id} & & \downarrow \text{Id}
\end{array}
\]

is commutative since \( \pi(\psi(a, x)) = \pi(a) + \pi(s(x)) = x = \pi_V(a, x) \), for all \( a \in A \) and \( x \in V \). The proof is now finished. \( \square \)

Theorem 3.3 shows that computing the classifying object \( \text{Ext}(V, A) \) reduces to the classification of all crossed product extensions of the form (27).

Given two Jordan algebras \( A \) and \( V \), we denote by \( CP(A, V) \) the set of all pairs \( (\triangleright, f) \) of bilinear maps \( \triangleright : V \times A \to A, f : V \times V \to A \) satisfying axioms (CP1)-(CP5).

**Definition 3.4.** Let \( A \) and \( V \) be two Jordan algebras. Two pairs \( (\triangleright, f) \) and \( (\triangleright', f') \) \( \in \) \( CP(A, V) \) are called cohomologous and we denote this by \( (\triangleright, f) \approx (\triangleright', f') \) if there exists a linear map \( r : V \to A \) such that

\[
x \triangleright' a = x \triangleright a - a \cdot r(x) \tag{28}
\]

\[
f'(x, y) = f(x, y) + r(x \cdot y) + r(x) \cdot r(y) - x \triangleright r(y) - y \triangleright r(x) \tag{29}
\]

for all \( a \in A, x, y \in V \).

We can now provide a theoretical answer to the extension problem for Jordan algebras:

**Theorem 3.5.** Let \( A \) and \( V \) be two Jordan algebras. Then:

1. \( \approx \) as defined in Definition 3.4 is an equivalence relation on the set \( CP(A, V) \) of all crossed systems of \( A \) and \( V \). We denote the quotient set by \( H^2_{\text{ab}}(V, A) := CP(A, V)/\approx \) and we call it the non-abelian cohomology of \( A \) and \( V \).
2. There exists a bijection given by:

\[
H^2_{\text{ab}}(V, A) \to \text{Ext}(V, A), \quad (\triangleright, f) \mapsto (A \#^f V, i_A, \pi_V)
\]

where \( (\triangleright, f) \) is the equivalence class of \( (\triangleright, f) \) via \( \approx \), \( (A \#^f V, i_A, \pi_V) \) is the crossed product extension given by (27) and \( \text{Ext}(V, A) \) is the classifying object of all extensions of \( V \) by \( A \) as introduced in Definition 2.1.

**Proof.** The proof follows from Theorem 3.3 once we observe that \( (\triangleright, f) \approx (\triangleright', f') \) in the sense of Definition 3.4 if and only if there exists an isomorphism of Jordan algebras \( \psi : A \#^f V \to A \#^{f'} V \) which stabilizes \( A \) and co-stabilizes \( V \). We just mention that the conditions given by (28) and (29) are the only non-trivial ones among the compatibilities listed in Definition 2.9 for the trivial right actions \( \triangleleft \) and \( \triangleleft' \). We also use the last statement of Lemma 2.8. Therefore, \( \approx \) is an equivalence relation on the set \( CP(A, V) \) of all crossed system \( A \) and \( V \) and the conclusion follows. \( \square \)

**Remark 3.6.** As in the case of the extension problem from group theory or Lie theory, Theorem 3.5 takes a simplified form in the 'abelian' case (i.e. when \( a \cdot b := 0 \), for all \( a, b \in A \)). These are called 'null extensions' or 'central extensions' and were first considered...
in [15, Theorem 12]. Central extensions were also studied in [1, Section 3] in connection to the classification of nilpotent Jordan algebras.

The next example parameterizes all extensions of a 1-dimensional Jordan algebra through an arbitrary Jordan algebra \( A \).

**Example 3.7.** For a given Jordan algebra \( A \) we will compute the non-abelian cohomology \( \text{H}^2_{\text{nab}}(k, A) \). Let \( \{x\} \) be a basis in \( k \): then any Jordan algebra structure on \( k \) has the form \( x \mapsto x = \varepsilon x \). Up two an isomorphism there are two Jordan algebra structures on \( k \) corresponding to \( \varepsilon \in \{0, 1\} \). Fix \( \varepsilon \in \{0, 1\} \) and denote by \( \text{H}^2_{\text{nab}}(k_\varepsilon, A) \) the corresponding non-abelian cohomology. Then \( \text{CP}(A, k_\varepsilon) \) identifies with the set of all pairs \( (D, a_0) \in \text{End}_k(A) \times A \) satisfying the following six compatibilities:

\[
\begin{align*}
  & a \cdot D(a^2) = a^2 \cdot D(a), \quad D(a_0 \cdot a) = a_0 \cdot D(a), \quad D^2(a_0) = a_0^2 + \varepsilon D(a_0), \quad \varepsilon D(a_0) = D(a_0); \\
  & 2a \cdot D^2(a) + a \cdot D(a_0) + a_0 \cdot a + D(a) + D^2(a) + 2D^3(a) = 0; \\
  & D(a) \cdot b + (a_0 \cdot b) \cdot a + a \cdot D(b) + D(a^2 \cdot b) + 2D(D(a) \cdot b) = 0.
\end{align*}
\]

for all \( a, b \in A \) and

\[
\text{H}^2_{\text{nab}}(k_\varepsilon, A) \cong \text{CP}(A, k_\varepsilon)/\sim
\]

where \( \sim \) is the following equivalence relation: \( (D, a_0) \sim (D', a'_0) \) if and only if there exists \( r \in A \) such that

\[
D'(a) = D(a) - a \cdot r, \quad a'_0 = a_0 + r^2 - 2D(r) + \varepsilon r
\]

(30)

for all \( a \in A \). Indeed, the set of all pairs of bilinear maps \( (\triangleright, f) \), where \( \triangleright : k \times A \to A \), \( f : k \times k \to A \) are in bijective correspondence with the set of all pairs \( (D, a_0) \in \text{End}_k(A) \times A \) and the bijection is given by \( x \triangleright a := D(a) \), for all \( a \in A \) and \( f(x, x) := a_0 \).

Via this identification, through a laborious but straightforward computation, we can easily prove that \( (\triangleright, f) \) is a crossed system if and only if the pair \( (D, a_0) \) satisfies the above compatibility conditions: we just mention that the first three compatibilities are equivalent to the axioms (CP2)-(CP4) while the last three compatibilities are equivalent to (CP5) (i.e. the missing relation (11) written for the trivial action \( < \)). If \( (D, a_0) \) is a pair as above then any extension of \( k_\varepsilon \) by the Jordan algebra \( A \) is cohomologous with the crossed product \( A^{(D, a_0)} := A \#_{a_0} D k \), which is the vector space \( A \times k \) with the multiplication given for any \( a, b \in A \) by:

\[
(a, x) \circ (b, x) := (a \cdot b + D(a + b) + a_0, \varepsilon x)
\]

As a special case, for a positive integer \( n \), let us take \( A := k_0^n \), the abelian \( n \)-dimensional Jordan algebra, and \( \varepsilon := 0 \). Then, we can easily prove that

\[
\text{H}^2_{\text{nab}}(k_0, k_0^n) \cong \{ D \in M_n(k) \mid 2D^3 - 3D^2 + D = 0 \}
\]

where \( M_n(k) \) is the usual space of \( n \times n \)-matrices over \( k \). Indeed, from the third and the fourth relation above we obtain \( a_0 := 0 \) while from the other compatibilities the only
non-trivial one comes down to $D(a) + 2D^2(a) = 3D^2(a)$, for all $a \in A = k^n$. Moreover, in this case the equivalence relation (30) becomes $D' = D$.

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