A SIMPLE DELAYED NEURAL NETWORK WITH LARGE CAPACITY FOR ASSOCIATIVE MEMORY

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Abstract. We consider periodic solutions of a system of difference equations with delay arising from a discrete neural network. We show that such a small network possesses a huge amount of stable periodic orbits with large domains of attraction if the delay is large, and thus the network has the potential large capacity for associative memory and for temporally periodic pattern recognition.

1. Introduction. Multistability in a dynamical system referees to the coexistence of multiple stable patterns such as equilibria and periodic orbits. It has been shown that the coexistence of multiple equilibria/fixed points in a neural network lies at the basis of the mechanism for associative content-addressable memory storage and retrieval [15, 21, 22, 28, 32]. It has also been known that stable periodic orbits and limiting cycle attractors are important for memory storage and other neural activities as some form of memories are encoded as temporally patterned spike trains [8, 15, 16] (see also limiting cycle attractors in excitable cells [24] and in neural circuits constructed from invertebrate neurons [23]). According to Milton and Black [30], there are over 30 diseases of the nervous system in which recurrence of symptoms or the appearance of oscillatory signs are a defining feature. It was also noted in [29] that more than 25 years of experimental and theoretical work indicates that the onset of oscillations in neurons and in neuron populations is characterized by multistability.

Time delays are intrinsic properties of the nervous systems and unavoidable in electronic implementations due to axonal conduction times, distances of interneurons and the finite switching speeds of amplifiers. See, for example, [2, 3, 4, 5, 15, 16, 17, 18, 19, 20, 26, 27, 42]. Periodic orbits for delay differential equations and systems have been extensively studied in the literature. In particular, for a system of two coupled delay differential equations describing the information processing of two identical neurons with delayed feedback, the series of papers [9, 10, 11, 12, 13, 14] established the coexistence of multiple periodic orbits and gave detailed description of their domains of attraction and the structure of the global attractor.

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as the Morse decomposition of these periodic orbits and their connecting orbits. Most of these periodic orbits, except one, are unstable but with large domains of attractions in high dimensional submanifolds. These should be useful for secured encoding and communication, but less useful for associative memory due to the lack of their stability. See also [34, 35, 36, 37, 38, 39] for work on the transient behaviors and on domains of attraction for a network of two neurons with delayed feedback.

Our purpose here is to show that a very simple and small network of neurons with delayed excitation/inhibition exhibits multistability and possesses a very large number of stable attractive periodic orbits with prescribed periods if the delay is sufficiently large and the updating is discrete. This is in sharp contrast with the aforementioned dynamical behaviors of a continuously updating network of neurons with delay for which at most one periodic solution can be stable. This phenomenon of multistability due to the coupling of discretely updating and time delay was observed in the series of papers [15, 16, 17] where it was shown that time delayed recurrent loops have a potentially large capacity to encode information in the form of temporally patterned spike trains. The model used in the aforementioned papers is a hybrid delay differential equation where the membrane potential of one neuron increases linearly until it reaches the firing threshold, when the neuron fires and is then reset to its resting membrane potential, the firing of the neuron excites another inhibitory interneuron such that at a time \( \tau > 0 \) later the membrane potential of the excitatory neuron is decreased by a fixed amount \( \Delta \). Under general conditions on \( \Delta \) and \( \tau \), it was shown numerically that multiple periodic orbits exist and the presence of noise can cause switches between basins of attraction. Similar results were obtained for some simple scalar second-order delay differential equations with negative feedback, see [1, 6, 7, 25].

The model considered in this paper is simple in its mathematical formulation and also for its hardware implementation. This is a system of two difference equations coupled through an excitatory feedback with an integer delay given below

\[
\begin{align*}
  x(n) &= \beta x(n-1) + \alpha f(y(n-k)), \\
  y(n) &= \beta y(n-1) + \alpha f(x(n-k)),
\end{align*}
\] (1.1)

where \( n \in \mathbb{N} \) (the set of all nonnegative integers), \( \alpha > 0 \) (the case where \( \alpha < 0 \) can be transformed into the case of \( \alpha > 0 \) via a simple change of variables), \( \beta \in (0, 1) \), \( k \geq 1 \) is a fixed integer, \( f : \mathbb{R} \to \mathbb{R} \) is a nonlinear function satisfying standard conditions of the usual signal function such as McCulloch-Pitts step-function and sigmoid function with large gain \( f'(0) \). Such a system describes the dynamics, updating discretely, of a network of two identical neurons where the information processing of a neuron involves the internal decay and feedback from another with a delay. It will be shown that for each positive prime integer \( p|2k \) such a system has \((2^p - 2)/p\) distinct periodic orbits with the minimal period \( p \) and some prescribed signs (a similar but a little more complicated formula for the number of the distinct periodic orbits will also be given when \( p \) is not a prime number). Our approach is based on the elegant method recently developed by Walther in [40, 41], that allows us to construct a closed bounded convex subset in a \( 2k \)-dimensional Euclidean space (based on some simple analysis of the periodic orbits of (1.1) with the simple step function \( f \)) and a contractive self-mapping defined on this subset such that a periodic point of the mapping gives a stable and attractive \( p \)-periodic orbit.
The fact that in a continuous system of two coupled delay differential equations most of periodic orbits, except one, are unstable but with large domains of attractions in high dimensional submanifolds and the fact that a discrete analogue is capable of generating a large number of stable periodic orbits seem to suggest that a combination of discrete and continuous signal processing maybe the most effective way for neural information processing in order to achieve the optimal quality, large capacity, secured encoding and easy retrieval. We must emphasize that while the network of two neurons, when updated discretely, allows the coexistence of an amazingly large number of stable periodic orbits, it is still a remaining open problem how this can be used as a device for memory storage. The problem is that the model involves very limited number of parameters ($\beta$: the internal decay rate; $\alpha$: the synaptic weight; and $k$ the delay) and thus it is difficult to train such a network so that a large number of memories can be stored as stable periodic orbits. How our results can be extended to large networks with more complicated connection topology remains to be an interesting open problem.

The main results will be described and proved in Section 2, followed by short discussions in Section 3 related to the issue of training and distributed delays.

2. Introduction. We consider the following nonlinear discrete-time system

$$\begin{align*}
    x(n) &= \beta x(n - 1) + \alpha f(y(n - k)), \\
    y(n) &= \beta y(n - 1) + \alpha f(x(n - k)),
\end{align*}$$

(2.1)

where $n \in \mathbb{N}$ (the set of all nonnegative integers), $\alpha > 0$, $k \geq 1$ is a fixed integer, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function satisfying the following conditions:

(H1) $$\begin{align*}
    |f(x) - 1| &\leq \epsilon \quad \text{if } x \in (r, R], \\
    |f(x) + 1| &\leq \epsilon \quad \text{if } x \in [-R, -r),
\end{align*}$$

for some constants $\epsilon > 0$, $R > r > 0$;

(H2) $$|f(x) - f(y)| \leq L|x - y|$$

if $x, y \in [-R, -r]$ or $x, y \in [r, R]$, where $L > 0$ is a constant.

Note that if $\epsilon = 0$, $r = 0$ and $R = \infty$, then $f$ must be the widely used McCulloch-Pitts nonlinearity given by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases}$$

(2.2)

and in this case, $L = 0$.

System (2.1) describes the evolution of a discrete-time network of two identical neurons with excitatory interactions, where $\beta \in (0, 1)$ is the internal decay rate, $f$ is the signal transmission function, and $k$ is the signal transmission delay. See [42] for general backgrounds on delayed neural networks, and [43, 44] for results about the existence of $k$-periodic orbits and $2k$-periodic orbits. In this section, we consider the existence, stability, multiplicity and domain of attraction of $p$-periodic solutions for every positive integer $p$ with $p|2k$.

Recall that by a solution of (2.1), we mean a sequence $\{(x(n), y(n))\}$ of points in $\mathbb{R}^2$ that is defined for every integer $n \geq -k$ and satisfying (2.1) for $n \in \mathbb{N}$. In what follows, in a statement involving a "$p$-periodic solution", we always mean the $p$ is the minimum period of the solution.
Letting \( w_j(n) = x(n+j-k-1) \) and \( w_{k+j}(n) = y(n+j-k-1) \) for \( j = 1, \ldots, k \), we can rewrite (2.1) as

\[
w(n+1) = F(w(n)),
\]

where \( F : \mathbb{R}^{2k} \to \mathbb{R}^{2k} \) is given by, for any \( w = (w_1, \ldots, w_{2k}) \in \mathbb{R}^{2k} \), the following formula

\[
F_j(w) = \begin{cases} 
  w_{j+1} & \text{for } j \in \{1, \ldots, k-1\}, \\
  \beta w_k + \alpha f(w_k+1) & \text{for } j = k, \\
  \beta w_{k+j+1} & \text{for } j \in \{k+1, \ldots, 2k-1\}, \\
  \beta w_{2k} + \alpha f(w_1) & \text{for } j = 2k.
\end{cases}
\]

We shall denote by \( \{w(n, w^0)\}_{w \in \mathbb{N}} \) the solution of (2.4) with initial value \( w(0) = w^0 \in \mathbb{R}^{2k} \).

For \( w = (w_1, \ldots, w_{2k}) \in \mathbb{R}^{2k} \), let

\[
|w| = \max\{|w_j|, j = 1, \ldots, 2k\}.
\]

Let

\[
\Sigma = \{\sigma = (\sigma_1, \ldots, \sigma_{2k}) \in \mathbb{R}^{2k} : \sigma_j \in \{-1, 1\}, j = 1, \ldots, 2k\}.
\]

For any \( m \in \mathbb{N} \), let \([m] \in \{1, \ldots, 2k\} \) so that \( m \equiv [m] \pmod{2k} \).

Define a mapping \( \pi : \Sigma \to \Sigma \) by

\[
(\pi \sigma)_j = \begin{cases} 
  \sigma_{j+1} & \text{for } j \in \{1, \ldots, 2k-1\}, \\
  \sigma_1 & \text{for } j = 2k,
\end{cases}
\]

for any \( \sigma \in \Sigma \). As usual, the mapping \( \pi^p : \Sigma \to \Sigma \) with \( p \geq 2 \) is given by \( \pi^p \sigma = \pi(\pi^{p-1} \sigma) \) for \( \sigma \in \Sigma \) inductively. Clearly,

\[
\pi^{2k} \sigma = \sigma \text{ for every } \sigma \in \Sigma.
\]

Moreover, for any integer \( k \geq 2 \) and \( p \in \{2, \ldots, 2k-1\} \) and for any \( \sigma \in \Sigma \), we have

\[
(\pi^p \sigma)_j = \begin{cases} 
  \sigma_{j+p} & \text{for } j = 1, \ldots, 2k-p, \\
  \sigma_{p-s} & \text{for } j = 2k-s \text{ with } s = 0, \ldots, p-1.
\end{cases}
\]

Let

\[
\Sigma_1 = \{\sigma \in \Sigma : \pi \sigma = \sigma\}
\]

denote the set of all fixed points of \( \pi \). For any \( p \in \{2, \ldots, 2k\} \), let

\[
\Sigma_p = \{\sigma \in \Sigma : \pi^p \sigma = \sigma \text{ and } \pi^q \sigma \neq \sigma \text{ for any } q \in \{1, \ldots, p-1\}\}
\]

denote the set of all \( p \)-periodic points of \( \pi \).

The following simple observation is useful for our existence result.

Lemma 2.1. \(^1\)

(i): For any \( p \in \{1, \ldots, 2k\} \), if \( \Sigma_p \neq \emptyset \) then \( p|2k \);
(ii): \( \Sigma = \bigcup_{p|2k} \Sigma_p \);

\(^1\)We wish to thank Cornelius Greither and a referee for bringing our attention to the connection of (2.10) with the well-known M"{o}bius inversion formula described below: Let \( p = \prod_{i=1}^{l} p_i^{m_i} \), where \( p_i, i = 1, 2, \ldots, l \) are primes. For every subset \( I \) of the set \( \{1, \ldots, l\} \), let \( p_I = \prod_{i \in I} p_i \). Then by the M"{o}bius inversion theorem (see pp. 234-236 of G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 5th edition, Oxford University Press, 1979) \( N(p) = \sum_{I \subseteq \{1, \ldots, l\}} (-1)^{|I|} 2^{|I|} / p_I \). We also want to thank Xingfu Zou for his help in this formulation.
(iii): The number of elements of the set $\Sigma_p$ is given by

$$N(p) = \begin{cases} 2 & \text{if } p = 1, \\ 2^p - 2 & \text{if } p \text{ is a prime integer}, \\ 2^p - \sum_{q|p,q<p}N(q) & \text{elsewhere}. \end{cases} \quad (2.10)$$

**Proof.** (i) Consider the case where $k \geq 2$, $p \in \{2, \cdots, 2k - 1\}$ and $\Sigma_p \neq \emptyset$. For a fixed $\sigma \in \Sigma_p$, define a sequence $\{s_n\}_{n \in \mathbb{N}}$ by

$$s_n = \begin{cases} \sigma_n & \text{for } n = 1, \cdots, 2k, \\ \sigma_{[n]} & \text{elsewhere.} \end{cases} \quad (2.11)$$

Therefore, $s_{n+2k} = s_n$ for $n \in \mathbb{N}$. So $2k$ is a period of $\{s_n\}_{n \in \mathbb{N}}$. By (2.8) and $\pi^p\sigma = \sigma$, we have

$$\sigma_j = \sigma_{j+p} \quad \text{for } j = 1, \cdots, 2k - p, \quad (2.12)$$

Note that for $j = 2k - s$ with $s \in \{0, \cdots, p - 1\}$, we have $j + p = 2k + (p - s)$ with $p - s \in \{1, \cdots, p\}$. So, $[j + p] = p - s$. This, together with (2.11) and (2.12), yields

$$s_n = s_{n+p} \text{ for } n = 1, \cdots, 2k - p,$$

and

$$s_n = \sigma_{[n+p]} = s_{n+p} \text{ for } n = 2k - (p - 1), \cdots, 2k.$$ 

For $n \geq 2k + 1$, we have $n = l(2k) + q$ with some $l \geq 1$ and $q \in \{1, \cdots, 2k - 1\}$. Therefore,

$$s_{n+p} = \sigma_{(2k)\cdot q + p} = s_{q + p} = s_q = s_n.$$

Thus, $p$ is the minimum period of $\{s_n\}_{n \in \mathbb{N}}$. As $p < 2k$ and as $2k$ is a period of $\{s_n\}_{n \in \mathbb{N}}$, we conclude that $p|2k$. This proves (i). (ii) follows easily as $\Sigma = \cup_{p=1}^{2k} \Sigma_p$. (iii) is simple.

We can now state our existence result.

**Theorem 2.2.** Let (H1) be satisfied with $\beta \in (0, \frac{1}{2})$, $\epsilon \in (0, \kappa \beta)$ with $\kappa \beta = 1 - 2\beta$, $R > b^*$ with

$$a^* = \alpha(1 - \epsilon) - \beta b^*, \quad b^* = \frac{\alpha}{1 - \beta}(1 + \epsilon). \quad (2.13)$$

Then, for any positive integer $p$ with $p|2k$ and for every $\sigma \in \Sigma_p$, (2.3) has a $p$-periodic solution $\{w(n, w^\sigma)\}_{n \in \mathbb{N}}$ with $\text{sign } w^\sigma_j = \sigma_j$ and $r < |w^\sigma_j| < R$ for $1 \leq j \leq 2k$.

**Proof.** Let $r_* = \min\{a^* - r, R - b^*\}$. Fix $c \in [0, r_*)$ and define $a_c = a^* - c$ and $b_c = b^* + c$. Let

$$\Omega(\sigma, c) = \{w \in \mathbb{R}^{2k} : a_c \leq |w_j| \leq b_c, \text{sign}\{w_j\} = \sigma_j, \ j = 1, \cdots, 2k\}. \quad (2.14)$$

We will prove that (2.3) has a $p$-periodic solution $\{w(n, w^\sigma)\}$ with $w^\sigma \in \Omega(\sigma, c)$.

**Step 1.** For any $c$ with $0 \leq c < r_*$, define

$$\Omega_2(c) = \{(x, y) \in \mathbb{R}^2 : a_c \leq |x| \leq b_c, \ a_c \leq |y| \leq b_c\}. \quad (2.15)$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = \beta x + \alpha f(y)$. We claim that for any $c \in [0, r_*)$ and $x, y \in \Omega_2(c)$, we have

$$\text{sign } g(x, y) = \text{sign } y \text{ and } a_c \leq |g(x, y)| \leq b_c. \quad (2.16)$$

We first note from (2.13) that

$$\beta b^* + \alpha(1 + \epsilon) = b^*. \quad (2.17)$$
We now distinguish four cases.

(a) \( x > 0, \ y > 0 \): In this case, we have
\[
 g(x, y) \leq \beta b_c + \alpha (1 + \epsilon) = b^* + \beta c < b_c,
\]
\[
 g(x, y) \geq \beta a_c + \alpha (1 - \epsilon) > \alpha (1 - \epsilon) > a^* \geq a_c.
\]

(b) \( x < 0, \ y > 0 \): In this case, we have
\[
 g(x, y) \leq -\beta a_c + \alpha (1 + \epsilon) < \alpha (1 + \epsilon) < b^* \leq b_c
\]
\[
 g(x, y) \geq -\beta b_c + \alpha (1 - \epsilon) = a^* - \beta c > a_c.
\]

(c) \( x < 0, \ y < 0 \): In this case, we have
\[
 g(x, y) \leq -\beta a_c + \alpha (1 - \epsilon)
\]
\[
 g(x, y) \geq -\beta b_c + \alpha (1 - \epsilon).
\]
Using the argument in (a), we get \(-b_c \leq g(x, y) \leq -a_c\).

(d) \( x > 0, \ y < 0 \): In this case, we have
\[
 g(x, y) \leq \beta b_c + \alpha (-1 + \epsilon),
\]
\[
 g(x, y) \geq \beta a_c + \alpha (-1 - \epsilon).
\]
Using the argument in (b), we get \(-b_c \leq g(x, y) \leq -a_c\). This proves the claim.

**Step 2.** Next, we show that for any \( \sigma \in \Sigma \) and \( c \in [0, r_*) \), \( F(\Omega(\sigma, c)) \subset \Omega(\pi(\sigma), c) \).

Let \( \sigma = (\sigma_1, \ldots, \sigma_{2k}) \) and \( w = (w_1, \ldots, w_{2k}) \in \Omega(\sigma, c) \). Then, we have from (2.6) that \( \pi \sigma = (\sigma_2, \ldots, \sigma_{2k}, \sigma_1) \), \( a_c \leq |w_j| \leq b_c \) and \( \text{sign}(w_j) = \sigma_j \). We have by (2.4) that \( F_j(w) = w_{j+1} \) for \( j \neq k, 2k \). So,
\[
a_c \leq |F_j(w)| \leq b_c \text{ and sign } (F_j(w)) = \text{sign} (w_{j+1}) = \sigma_{j+1} \text{ for } j \neq k, 2k.
\]

By (2.4), we have \( F_k(w) = g(w_k, w_{k+1}) \) and \( F_{2k}(w) = g(w_{2k}, w_1) \). From the above claim we have that \((w_k, w_{k+1}, w_{2k}, w_1) \in \Omega_2(c) \) and
\[
\text{sign}(F_k(w)) = \text{sign}(w_{k+1}) = \sigma_{k+1}, \text{ sign}(F_{2k}(w)) = \text{sign}(w_1) = \sigma_1.
\]
This shows that \( F(w) \in \Omega(\pi(\sigma), c) \).

Note that \( \Omega(\sigma, c) \) is convex and closed. If \( p|2k \) and \( \sigma \in \Sigma_p \), then we have \( F^{p}(\Omega(\sigma, c)) \subset \Omega(\pi^p(\sigma), c) = \Omega(\sigma, c) \). Therefore, the continuous mapping \( F^p \) has a fixed point which gives a \( p \)-periodic solution of (2.3), \( \{w(n, w^*)\}_{n \in \mathbb{N}} \) with \( w^* \in \Omega(\sigma, c) \). This completes the proof.

We now describe the coexistence and the domains of attraction of periodic solutions.

**Theorem 2.3.** Assume all conditions in Theorem 2.2 hold. Moreover, assume that (H2) hold with \( 0 \leq L < \frac{1 - \beta}{\alpha} \). Then

(i): For every integer \( p \) with \( p|2k \) and each \( \sigma \in \Sigma_p \), (2.3) has a unique \( p \)-periodic \( w(n, w^*) \) with
\[
w^* \in \Omega(\sigma, 0) = \{w \in \mathbb{R}^{2k}; a^* \leq |w_j| \leq b^*, \text{sign } w_j = \sigma_j, j = 1, \ldots, 2k\}.
\]
This solution is uniformly asymptotically stable. More precisely, let
\[
r^* = \min\{a^* - r, R - b^*\}
\]
and
\[
r(\sigma) = \min\{|w_j^*| - (a^* - r), (b^* + r_*) - |w_j^*|; j = 1, \ldots, 2k\}.
\]
Then for any \( w^0 \) with \( |w^0 - w^\sigma| < r(\sigma) \), we have
\[
|w(n, w^0) - w(n, w^\sigma)| \leq C_\xi^n |w^0 - w^\sigma| \quad \text{for } n = 0, 1, \cdots,
\]
where \( \xi = (\beta + \alpha L)^{1/k} \) and \( C = \xi^{1-k} \);

(ii): Let
\[
\Omega = \{ w \in \mathbb{R}^{2k} : r < |w_j| < R, \ j = 1, \cdots, 2k \}. \quad (2.18)
\]
If \( \{w(n)\}_{n \in \mathbb{N}} \) is a \( p \)-periodic solution of (2.3) with \( w(n) \in \Omega \) for \( n = 1, \cdots, p \), then \( p|2k \) and there exists a unique \( \sigma \in \Sigma_p \) such that \( w(n) = w(n, w^\sigma) \) for \( n = 1, 2, \cdots \);

(iii): For any solution \( \{w(n, w^0)\}_{n \in \mathbb{N}} \) of (2.3) with \( a^* - r_* < |w^0_j| < b^* + r_* \), for \( 1 \leq j \leq 2k \), there exists a unique integer \( p \in \mathbb{N} \) with \( p|2k \) and a unique \( \sigma \in \Sigma_p \) such that
\[
|w(n, w^0) - w(n, w^\sigma)| \leq C_\xi^n |w^0 - w^\sigma| \quad \text{for } n = 0, 1, \cdots;
\]

(iv): For each integer \( p \in \mathbb{N} \) with \( p \not| 2k \), (2.3) has \( N(p) \) \( p \)-periodic solutions in \( \Omega \), and these solutions are uniformly asymptotically stable. For any integer \( p \in \mathbb{N} \) with \( p \not| 2k \), (2.3) has no \( p \)-periodic solution in \( \Omega \).

**Proof.** We will divide the long proof into five steps.

**Step 1.** Let \( \Omega(\sigma, c) \) be given in (2.14). We first show that if \( c \in [0, r_*) \), \( \sigma \in \Sigma_p \) and \( p|2k \), then for any integer \( l \) with \( 1 \leq l \leq k \) and for any \( w^l, w'^l \in \Omega(\sigma, c) \)
\[
|F^l_{k-j}(w^l) - F^l_{k-j}(w'^l)| \leq (\beta + \alpha L)|w^l - w'^l|, \quad 0 \leq j \leq l - 1,
\]
\[
|F^l_{2k-j}(w^l) - F^l_{2k-j}(w'^l)| \leq (\beta + \alpha L)|w^l - w'^l|, \quad 0 \leq j \leq l - 1,
\]
\[
|F^l_i(w^l) - F^l_i(w'^l)| \leq |w^l - w'^l|, \quad 1 \leq i \leq 2k, i \neq k-j, 2k-j \text{ for } 0 \leq j \leq l - 1. \quad (2.19)_l
\]
Using the same argument as that in Step 1 of the proof for Theorem 2.2, we have
\[
w^l_j w'^l_j > 0, \quad F^l_j(w^l)F^l_j(w'^l) > 0 \quad \text{for } j = 1, \cdots, 2k \text{ and } 1 \leq l \leq k.
\]
(2.19)_l holds when \( l = 1 \), because by (H2) we have
\[
|F_1(w^l) - F_1(w'^l)| = |\beta(w^l_1 - w'^l_1) + \alpha[f(w^l_{k+1}) - f(w'^l_{k+1})]| \leq (\beta + \alpha L)|w^l - w'^l|,
\]
\[
|F_{2k}(w^l) - F_{2k}(w'^l)| = |\beta(w^l_{2k} - w'^l_{2k}) + \alpha[f(w^l_1) - f(w'^l_1)]| \leq (\beta + \alpha L)|w^l - w'^l|
\]
and
\[
|F_i(w^l) - F_i(w'^l)| = |w^l_{i+1} - w'^l_{i+1}| \leq |w^l - w'^l| \quad \text{if } 1 \leq i \leq k - 1,
\]
\[
|F_i(w^l) - F_i(w'^l)| = |w^l_{k+1+i} - w'^l_{k+1+i}| \leq |w^l - w'^l| \quad \text{if } k + 1 \leq i \leq 2k - 1.
\]
Assuming now that (2.19)_l holds for some \( l \) with \( 1 \leq l \leq k-1 \), then for \( 1 \leq j \leq l \) we have
\[
|F^{l+1}_{k-j}(w^l) - F^{l+1}_{k-j}(w'^l)| = |F^l_{k-(j-1)}(w^l) - F^l_{k-(j-1)}(w'^l)| \leq (\beta + \alpha L)|w^l - w'^l|
\]
since \( 0 \leq j \leq l < 1 \), and
\[
|F^{l+1}_{k-j}(w^l) - F^{l+1}_{k-j}(w'^l)| \leq |\beta[F^l_k(w^l) - F^l_k(w'^l)] + \alpha[f(F^l_{k+1}(w^l)) - f(F^l_{k+1}(w'^l))]| \leq (\beta + \alpha L) \max\{|F^l_k(w^l) - F^l_k(w'^l)|, |F^l_{k+1}(w^l) - F^l_{k+1}(w'^l)|\} \leq (\beta + \alpha L)|w^l - w'^l|.
\]
Similarly, we have
\[
|F^{l+1}_{2k-j}(w^l) - F^{l+1}_{2k-j}(w'^l)| \leq (\beta + \alpha L)|w^l - w'^l| \quad \text{for } 0 \leq j \leq l.
\]
If \( i \neq k - j, 2k - j \) for \( 0 \leq j \leq l \) then \( i + 1 \neq k - j, 2k - j \) for \( 0 \leq j \leq l - 1 \) and hence, (2.19) implies that
\[
|F_i^{l+1}(w') - F_i^{l+1}(w'')| = |F_i^{l+1}(w') - F_i^{l+1}(w'')| \leq |w' - w''|.
\]
Then (2.19) holds. This proves the claim, and from (2.19) we get
\[
|F_k(w') - F_k(w'')| \leq (\beta + \alpha L)|w' - w''|.
\]

**Step 2.** We now show that if \( \{w(n, w^0)\}_{n \in \mathbb{N}} \) is a \( p \)-periodic solution of (2.3) with \( w(j, w^0) \in \Omega \) for \( j = 1, \ldots, p \) then \( |w(j, w^0)| \leq b^* \) for \( j = 1, \ldots, p \).

To prove this, we first obtain a \( p \)-periodic solution \( \{x(n), y(n)\} \) of (2.1) from \( \{w(n, w^0)\}_{n \in \mathbb{N}} \). Assume, by way of contradiction, that there exists \( n_0 > p \) such that \( |x(n_0)| > b^* \). We first consider the case where \( x(n_0) > b^* \). Then \( b^* < x(n_0) < R \), and we can write \( x(n_0) = b^* + \delta_0 \) with some \( \delta_0 > 0 \). We have from (2.1), (H1) and (2.17) that
\[
x(n_0 - 1) = \frac{1}{\beta} [x(n_0) - \alpha f(y(n_0 - k))] \geq \frac{1}{\beta} [\delta_0 + b^* - \alpha(1 + \epsilon)] = \frac{1}{\beta} \delta_0 + b^*.
\]
Repeating the above argument, we get
\[
x(n_0 - m) \geq \frac{1}{\beta} [x(n_0 - m + 1) - b^*] + b^*, \quad m = 1, 2, \ldots, k.
\]
In particular,
\[
x(n_0 - p) \geq \frac{1}{\beta^p} \delta_0 + b^* > b^* + \delta_0 = x(n_0),
\]
a contradiction to the \( p \)-periodicity. Similarly, we exclude the case \( x(n_0) < -b^* \).

**Step 3.** Using the same argument as that in Step 1 of the proof for Theorem 2.2, we obtain that
\[
\text{sign}(x, y) = \text{sign} y \quad \text{and} \quad r < |g(x, y)| < b^*
\]
if \((x, y)\) is from
\[
\Omega^*_2 = \{(x, y) \in \mathbb{R}^2 : r < |x| \leq b^*, \quad r < |y| \leq b^*\}.
\]
Therefore, \( F^k(\Omega^*(\sigma)) \subset \Omega^*(x^k \sigma) \), where
\[
\Omega^*(\sigma) = \{w \in \mathbb{R}^{2k} : r < |w_j| \leq b^* \text{ and } \text{sign}(w_j) = \sigma_j, \quad j = 1, \ldots, 2k\}.
\]
Furthermore,
\[
F_2^k(w')F_k^j(w'') > 0 \quad \text{for} \quad j = 1, 2, \ldots, 2k \quad \text{and} \quad w', \quad w'' \in \Omega^*(\sigma).
\]
Therefore, utilizing the same argument as that in Step 1, we get
\[
|F^k(w') - F^k(w'')| \leq (\beta + \alpha L)|w' - w''| \quad \text{for} \quad w', \quad w'' \in \Omega^*(\sigma). \tag{2.20}
\]

**Step 4.** We next show that if \( \{w(n, w^0)\}_{n \in \mathbb{N}} \) is a \( p \)-periodic solution of (2.3) with \( w(j, w^0) \in \Omega \) for \( j = 1, \ldots, p \), then \( p|2k \) and \( w(n, w^0) = w(n, w^*) \) for some \( \sigma \in \Sigma_p \).

By Lemma 2.1, we have \( \Omega = \cup_{p|2k} \cup_{\sigma \in \Sigma_p} \Omega(\sigma) \). Therefore, there exist \( q|2k \) and \( \sigma \in \Sigma_q \) such that \( w^q \in \Omega(\sigma) \). By the result in Step 2, we have \( w(j, w^q) \in \Omega^*(\sigma) \). By Theorem 2.2, there exists \( w^\sigma \in \Omega(\sigma, c) \subset \Omega(\sigma) \) such that \( \{w(n, w^\sigma)\}_{n \in \mathbb{N}} \) is a
\( p \)-periodic solution of (2.3). By the result in Step 2, we also have \( w(j, w^0) \in \Omega^* (\sigma) \) for \( j = 1, 2, \cdots \). Then, for any \( j \in \mathbb{N} \), we have
\[
|w(j, w^0) - w(j, w^\sigma)| = |w(pqk + j, w^{qk}) - w(pqk + j, w^{\sigma})| \\
= |(F_k)^{pq}(w(j, w^0)) - (F_k)^{pq}(w(j, w^{\sigma}))| \\
\leq (\beta + \alpha L)^{pq}|w(j, w^0) - w(j, w^{\sigma})|.
\]
Therefore, we must have \( w(j, w^0) = w(j, w^{\sigma}) \) for \( j \in \mathbb{N} \).

**Step 5.** We now complete the proof.

(i). We obtain the existence and uniqueness of \( p \)-periodic solution \( \{w(n, w^\sigma)\}_{n \in \mathbb{N}} \) with \( w^0 \in \Omega(\sigma, 0) \) by using Theorem 2.2 and the result in Step 4. For any \( w^0 \) with \( |w^0 - w^\sigma| < r(\sigma) \), by the definition of \( r(\sigma) \), we can find \( c \in [0, r_*) \) such that \( w^\delta \in \Omega(\sigma, c) \). Note that by (2.19) and for any \( l \in \{1, \cdots, k - 1\} \), we have
\[
|F^l(w') - F^l(w^\sigma)| \leq |w' - w^\sigma| \text{ for } w', w^\sigma \in \Omega(\sigma, c).
\]
Let \( n = sk + q \) with some \( q \in \{1, \cdots, k - 1\} \) and \( s \geq 1 \), then
\[
|w(n, w^0) - w(n, w^\sigma)| = |F^{sk+q}(w^0) - F^{sk+q}(w^\sigma)| \leq |F^{sk}(w^0) - F^{sk}(w^\sigma)|.
\]
By (2.19), we have
\[
|F^{sk}(w^0) - F^{sk}(w^\sigma)| \leq (\beta + \alpha L)^s|w^0 - w^\sigma| \\
= C\xi^{sk+k-1}|w^0 - w^\sigma| \\
\leq C\xi^n|w^0 - w^\sigma|.
\]

(ii) follows from the result in Step 4.

(iii). We can find \( c \in [0, r_*) \) such that \( a^* - c \leq |w^0| \leq b^* + c \). Let \( \sigma \in \Sigma \) so that \( \sigma_j = \text{sign}(w^0_j) \), then \( w^0 \in \Omega(\sigma, c) \). Now, by (i) and the result in Step 1 we obtain (iii).

(iv). We have from the result in Step 4 that the period \( p \) of any given periodic solution of (2.3) must satisfy that \( p|2k \). This, together with (i) and the definition of \( N(p) \), completes the proof.

Note that two different periodic solutions may give rise to a single periodic orbit if they are the time-translation from each other. To determine exactly the number of periodic orbits, we introduce the following

**Definition 2.4.** Two periodic solutions \( w(n, w') \) and \( w(n, w'') \) of (2.3) are equivalent to each other, if there exists \( q \in \mathbb{N} \) such that
\[
w(n, w') = w(n + q, w'') \quad \text{for} \quad n = 0, 1, \cdots.
\]
Clearly, two equivalent periodic solutions \( w(\cdot, w') \) and \( w(\cdot, w'') \) of (2.3) give the same orbit
\[
O(w') = \{w(n, w'); \quad n = 0, 1, \cdots \} = O(w'') = \{w(n, w''); \quad n = 0, 1, \cdots \}.
\]

**Lemma 2.5.** For any fixed \( p \in \mathbb{N} \) with \( p|2k \), and any given \( \sigma \in \Sigma_p \), and \( \pi \in \Sigma_p \) with \( \sigma \neq \pi \), \( w^\sigma \) and \( w^\pi \) are equivalent to each other if and only if there exists \( q \in \{1, \cdots, p - 1\} \) such that \( \pi = \sigma^q \neq \sigma \).

**Proof.** Let \( w^0 = w(0, \sigma) \) and \( \pi^0 = w(0, \pi) \). Define \( \{\sigma^{(n)}\}_{n \geq 0} \) and \( \{\pi^{(n)}\}_{n \geq 0} \) by \( \sigma^{(n)} = (\sigma^{(n)}_1, \cdots, \sigma^{(n)}_{2k}) \) and \( \pi^{(n)} = (\pi^{(n)}_1, \cdots, \pi^{(n)}_{2k}) \), where
\[
\sigma^{(0)} = \sigma, \quad \sigma_j^{(n)} = \text{sign} \ F_j^n(w^0) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad j = 1, \cdots, 2k,
\]
\[
\pi^{(0)} = \pi, \quad \pi_j^{(n)} = \text{sign} \ F_j^n(w^0) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad j = 1, \cdots, 2k.
\]
Therefore, for any \( w^\sigma \) and \( w^{\overline{\sigma}} \) are equivalent to each other, then we can find \( q \in \{1, \cdots, p-1\} \) such that \( w_{\overline{\sigma}}(n) = w^\sigma(n+q) \) for \( n \geq 0 \). Note that \( w^\sigma(n) = F^n(w^0) \) and \( w_{\overline{\sigma}}(n) = F^n(w^0) \).

Therefore, for any \( j \in \{1, \cdots, 2k\} \) and \( n \geq 0 \), we have

\[
\sigma_j = \text{sign } w_j(n, w^0) = \text{sign } w_j(n + q, \sigma) = \sigma_j(n + q) = \pi^q \sigma_j^{(n)}.
\]

So, we get \( \sigma = \pi(0) = \pi^q \sigma(0) = \pi^q \sigma \).

Conversely, assume \( \sigma, \overline{\sigma} \in \Sigma_p \) are given so that

\[
\sigma = \pi^q \sigma \text{ for some } q \in \{1, \cdots, p-1\}. \quad (2.22)
\]

Let \( w^\sigma \) and \( w_{\overline{\sigma}} \), with \( w^\sigma(0) = w^0 \) and \( w_{\overline{\sigma}}(0) = \overline{w}^0 \), be the \( p \)-periodic solutions of (2.3). Since \( F^q(\Omega(\sigma, 0)) \subset \Omega(\pi^q \sigma, 0) \), we have that \( w(n + q, w^0) = w(n, F^q(w^0)) \) is a \( p \)-periodic solution of (2.3) with \( F^q(\overline{w}^0) \in \Omega(\pi^q \sigma, 0) \). Note that (2.22) gives

\[
\sigma_j = \overline{\sigma}_j = (\pi^q \sigma)_j = \text{sign } F_j^q(\overline{w}^0).
\]

This, together with \( a^* \leq |\overline{w}^0| \leq b^* \) for \( j = 1, \cdots, 2k \), yields that \( \overline{w}^0 \in \Omega(\pi^q \sigma, 0) \).

We then have, from the uniqueness of a \( p \)-periodic solution of (2.3), that

\[
w^\sigma(n + q) = w(n + q, w^0) = w(n, F^q(w^0)) = w_{\overline{\sigma}}(n) = w^\sigma(n) \text{ for } n = 0, 1, \cdots.
\]

Therefore, \( w^\sigma \) and \( w_{\overline{\sigma}} \) are equivalent to each other. This completes the proof.

**Theorem 2.4.** Let \( N^*(p) \) be the number of \( p \)-periodic solutions of (2.1) which are not equivalent to each other. Then for each \( p \in \mathbb{N} \) with \( p/2k \), we have

\[
N^*(p) = \frac{N(p)}{p}.
\]

**Proof.** We have shown that \( N(p) \) is exactly the number of the elements of \( \Sigma_p \). For each \( p \in \Sigma_p \), the \( p \)-periodic solution \( w^\sigma \) is equivalent to each of the following \( p \)-periodic solutions \( w^{\pi^i \sigma}, \cdots, w^{\pi^{p-1} \sigma} \). As \( \pi^q \sigma \neq \sigma \) for any \( q \in \{1, \cdots, p-1\} \), we conclude that \( w^{\pi^i \sigma} \) and \( w^{\pi^{i+1} \sigma} \) are not equivalent to each other when \( i \neq j \) and \( i, j \in \{1, \cdots, p-1\} \). Therefore, for each \( w^\sigma \), there are exactly \( (p-1) \) \( p \)-periodic solutions that are equivalent to each other. This completes the proof.

We conclude this section by listing values of \( N(p) \) and \( N^*(p) \) for \( p = 1, \cdots, 20 \).

| \( p \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( N(p) \) | 2 | 2 | 6 | 12 | 30 | 54 | 126 | 324 | 504 | 990 | 2046 |
| \( N^*(p) \) | 2 | 1 | 2 | 3 | 6 | 9 | 18 | 30 | 56 | 99 | 186 |

**Table 1:** The numbers of periodic solutions \( N(p) \) and periodic orbits \( N^*(p) \) for a given integer \( p, 1 \leq r \leq 11 \)

More values are given in the following table:

| \( p \) | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| \( N(p) \) | 4020 | 8190 | 16254 | 32730 | 65280 | 131070 | 261576 | 524286 | 1047540 |
| \( N^*(p) \) | 335 | 630 | 1161 | 2182 | 4080 | 7710 | 14532 | 27594 | 52377 |

**Table 2:** The values of periodic solutions \( N(p) \) and periodic orbits \( N^*(p) \) for a given integer \( p, 12 \leq p \leq 20 \)
3. **Discussions.** We show that a very simple and small network of neurons with delayed coupling exhibits multistability with a very large number of stable attractive periodic orbits with prescribed periods if the delay is sufficiently large.

The model is a system of two coupled difference equations with delayed feedback, it is simple in its mathematical formulation and perhaps for hardware implementation.

The existence of multiple equilibria/fixed points and stable periodic orbits and limiting cycle attractors in a neural network is important for associative content-addressable memory, and many neural activities are encoded as temporally patterned spike trains. Periodic orbits for delay differential equations and systems have been extensively studied in the literature. Though a system of two coupled delay differential equations describing the information processing of two identical neurons with delayed feedback exhibits the coexistence of multiple periodic orbits, all of these periodic orbits, except possibly one, are unstable but with large domains of attractions in high dimensional submanifolds. These are useful for secured encoding and communication, but less useful for associative memory due to the lack of their stability.

Our work shows that the discrete model is capable of generating a large number of stable periodic orbits. This raises the issue how a network utilizes a combination of discrete and continuous signal processing for neural information processing in order to achieve the optimal quality, large capacity, secured encoding and easy retrieval.

While the network of two neurons, when updated discretely, allows the coexistence of an amazingly large number of stable periodic orbits, it is still a remaining open problem how this can be used as a device for memory storage. The problem is that the model involves very limited number of parameters ($\beta$: the internal decay rate; $\alpha$: the synaptic weight; and $k$ the delay) and thus it is difficult to train such a network so that a large number of memories can be stored as stable periodic orbits. One way seems to replace the coupling term $\alpha f(y(n - k))$ by a weighted sum involving multiple delays

$$\sum_{j=1}^{\infty} \alpha_j f(y(t - jk))$$

and then to determine the coefficients $\alpha_j$ from the training data sets. This is equivalent to the training the density of the distribution of delay in a model of difference equations with distributed delays, and the problem will be addressed in a future work.

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