Deformation of quark stars under strong internal magnetic fields

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Abstract. The pulsar detector has observed the existence of a strong magnetic field on compact stars. So much we can learn from this observation, one of them is the deformation of a compact star under strong magnetic fields. In this note, we study the deformation of quark stars due to a magnetic field. Note that quark stars and neutron stars have a fundamental difference in their equation of states such as the composition and crusts. Here, we used two equation of states of quark matter, i.e., CIDDM model with the scalar and with vector Coulomb terms included [1]. In both equation of states, we vary the magnetic field value in the center of the star to see how the internal magnetic field affects the shape of the quark star.

1. Introduction
After Anomalous Detectors X-ray pulsars (AXPs) and Soft-Gamma Repeaters (SGR) detect strong magnetic fields present in a pulsar where reported as remnants of supernovae are classified as young neutron stars [2, 3], many researchers are interested in studying effect of magnetic fields on relativistic stars. The observations from AXPs detect the magnetic field value at the star surface of about $10^{14} - 10^{15}$ G. The results of SGR and AXPs observations are believed to be magnetars i.e. neutron stars with very strong magnetic fields [4–7]. But as we know that neutron stars are not the only product of supernova explosions, but there are types of stars that have enormous mass densities that can break down neutrons into their quark-quark to form quark stars [8]. Quark stars are hypothesis stars that contain three quarks, namely up, down, and strange. The up and down quarks themselves are formed because of the very strong gravitational pressure and are able to break down neutrons in neutron stars into their constituents, up and down quarks. Strange quarks are formed from converted of up and down quarks through weak interactions [8, 9].

There are two ways to give the magnetic field effect to the relativistic star, first on the microscopic aspect, that is, the presence of a strong magnetic field is found in the energy levels of charged particles caused by Landau quantization, thus modifying the equation of state (EoS) for matters in stars, researchers have used this method a lot and apply it to EoS relativistic stars i.e. hyperon stars [10, 12], quark stars [11] and hybrid stars [13]. The second is by macroscopic means where the magnetic field effect is added into tensor energy-momentum as perturbative term, as which is done by Ref. [16, 17] (see also the references therein).

We can apply the deformations that occur in relativistic stars using Hartle and Thorne approximation [14, 15], which is a multipole expansion of the schwarzschild metrics, and pair it
with the sum of the magnetic field at the energy-momentum tensor in such a way as to cause the deformation [16, 17].

2. Formalism Detail
We have solved the equations that Hartle and Thorne have derived from their paper, which are used to describe the deformation of slow-motion rotation on relativistic stars [14, 15]. In this study, we assume that the magnetic field that causes deformation in which our formulation is exactly the same as that done on Ref. [16, 17], where Hartle and Thorne’s perturbation metrics are expressed as follows

\[
ds^2 = - e^{\nu} \left[ 1 + 2 \left( h_0 + h_2 P_2(\cos \theta) \right) \right] dt^2 + e^{\lambda} \left[ 1 + \frac{\epsilon^\lambda}{r} \left( m_0 + m_2 P_2(\cos \theta) \right) \right] dr^2 + r^2 \left[ 1 + 2k^2 P_2(\cos \theta) \right] \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

where \( m_0, m_2, h_0, h_2 \) and \( k \) is a function of \( r \) and \( P_2(\cos \theta) \) is Legendre function. The perturbation metrics have three summations, the first part is the Schwarzschild metric, the second part with the subscript '0' is the monopole correction and the last with subscript '2' is the quadrupole correction. The perturbation metrics need to be solved separately by connecting them to the energy-momentum tensor which also undergoes expansion in the magnetic field, which is formulated as follows

\[
\epsilon = \epsilon_m + \frac{B^2}{8\pi} \tag{2}
\]

\[
P_\perp = P_m + \frac{B^2}{8\pi} \tag{3}
\]

\[
P_\parallel = P_m - \frac{B^2}{8\pi}, \tag{4}
\]

where \( \epsilon_m \) and \( P_m \) is the energy density and thermodynamics pressure of matter, \( B^2/8\pi \) is the magnetic stress along z-axis, Total pressure can be written in Legendre polynomial form

\[
P = P_m + p_0 + p_2 P_2(\cos \theta) \tag{5}
\]

\[
\epsilon = \epsilon_m + 3p_0, \tag{6}
\]

where \( p_0 = B^2/24\pi \) is the monopole factor and \( p_2 = -4B^2/24\pi \) is the quadrupole factor.
We find the following equations after solving Einstein’s field equations:

\[
\frac{dP}{dr} = - \frac{G (\varepsilon_m + P_m)(m + 4\pi r^3 P_m)}{r^2 (1 - 2Gm/r)} \quad (7)
\]

\[
\frac{dm}{dr} = 4\pi r^2 \varepsilon_m \quad (8)
\]

\[
\frac{dm_0}{dr} = 12\pi r^2 p_0 \quad (9)
\]

\[
\frac{dh_0}{dr} = 4\pi G r e^{\lambda} p_0 + \frac{1}{r} G P e^{\lambda} m_0 + \frac{1}{r^2} G e^{\lambda} m_0 \quad (10)
\]

\[
\frac{d}{dr}(h_2 + k_2) = h_2 \left( \frac{1}{r} - \frac{1}{2} \frac{d
u}{dr} \right) + \frac{e^\lambda}{r} G m_2 \left( \frac{1}{r} + \frac{1}{2} \frac{d
u}{dr} \right) \quad (11)
\]

\[
h_2 + \frac{e^\lambda}{r} G m_2 = 0 \quad (12)
\]

\[
\frac{dh_2}{dr} + \frac{dk_2}{dr} \left( 1 + \frac{r}{2} \frac{d
u}{dr} \right) = 4\pi G r e^{\lambda} p_2 + \frac{e^\lambda}{r} G m_2 + \frac{e^\lambda}{r} G \frac{d
u}{dr} m_2 + \frac{3}{r} e^\lambda h_2 + \frac{2}{r} e^\lambda k_2, \quad (13)
\]

we also get the following equations from Bianchi’s contracted identity:

\[
\frac{d\nu}{dr} = - \frac{2}{(\varepsilon_m + P_m)} \frac{dP}{dr} \quad (14)
\]

\[
\frac{dp_0}{dr} = -2 \frac{d\nu}{dr} p_0 - (\varepsilon_m + P_m) \frac{dh_0}{dr} \quad (15)
\]

\[
p_2 = - (\varepsilon_m + P_m) h_2 \quad (16)
\]

\[
\frac{dp_2}{dr} = - \frac{1}{2} \frac{d\nu}{dr} p_2 - (\varepsilon_m + P_m) \frac{dh_2}{dr} \quad (17)
\]

Equation (7) - (17) selected and simplified in such a way as to be solved using numerical methods, provided that EoS of a relativistic star is known, in this case it is a quark star. The energy density equations of the materials we use are based on the author’s research of Ref. [1], where in that paper the material density energy is applied with CIDDM model with Lagrangian as follows

\[
\mathcal{L}^{\text{Int}(CIDDM)} = -\kappa_1 n_B^{1/3} n_B^{(s)} - \kappa_3 \delta n_B n_B e^{-bn_B n_B^{(s)}}, \quad (18)
\]

where \(\kappa_1\) is the independent isospin quark parameter, \(\kappa_3\) is a quark dependent interaction parameter, \(a\) and \(b\) are isospin parameters in the CIDDM model, \(\delta = 3 (n_d - n_u) / (n_d + n_u)\) is an up-down quark asymmetry parameter. Then \(n_0\), \(n_B^{(s)}\) and \(n_B^{(ve)}\) are vector-isoscalar, scalar-isoscalar and vector-isovector baryon number density respectively. From Ref. [1], they focus on investigating Lagrangian density interactions

\[
\mathcal{L} = \mathcal{L}^{\text{Int}(CIDDM)} + \Delta \mathcal{L}^{(i)\text{Int}}, \quad (19)
\]

where for \(i = 1\) using Coulomb scalar density model,

\[
\Delta \mathcal{L}^{(1)\text{Int}} = \kappa_1^{(1)} n_B^{1/3} n_B^{(s)}, \quad (20)
\]

and for \(i = 2\) using Coulomb vector density model,

\[
\Delta \mathcal{L}^{(2)\text{Int}} = \kappa_2^{(2)} n_B^{4/3}, \quad (21)
\]
research results from Ref. [1] found a stable condition on $\kappa_2^{(1)} = 0.3$ for range value $0.45 \leq \kappa_1 \leq 0.54$, and $\kappa_2^{(2)} = -0.8$ for range value $0.63 \leq \kappa_1 \leq 0.83$. In this paper we use EoS with degrees of freedom $\kappa_1$ and $\kappa_2$.

The magnetic field term $B$ obtained using ansatz from Ref. [10] sebagai berikut

$$B(n_B/n_0) = B_{surf} + B_0[1 - \exp\{-\beta (n_B/n_0)\gamma\}], \quad (22)$$

with assumption that the condition of magnetic field in the center of the star that must be higher than the surface. we set the value of saturation density $n_0 = 0.15 fm^{-3}$. Here we set at $\beta = 0.01$ and $\gamma = 2$ according to Ref. [10]. the magnetic field at the surface is $B_{surf} = 10^{15} G$, which is according to the observations data and $B_0 = 5 \times 10^{18}$ also obtain from Ref. [10].

3. Result and Discussion

The result of the numerical calculation we get the values of $P, m, r, m_0, h_0, h_2$ and $k_2$, in this study we focus more on the deformation that occurs due to the presence of magnetic field. First we can define the value of mass and the new radius due to the presence of correction factor monopole and quadrupole, that is for total mass

$$M = m + m_0, \quad (23)$$

where $m$ is the spherical mass and $m_0$ is mass for corrections factor to the magnetic field. We can write the radius equatorial and polar according to Ref. [14–16]

$$R_e = R + \xi_0(R) - \frac{1}{2}(\xi_2(R) + Rk_2(R)) \quad (24)$$

$$R_p = R + \xi_0(R) + (\xi_2(R) + Rk_2(R)), \quad (25)$$

where $R$ radius of spherical stars, $\xi_0$ and $\xi_2$ define by

$$\xi_0(r) = -h_0 \frac{r(r - 2Gm(r))}{G(4\pi r^3 P_m + m(r))}, \quad (26)$$

$$\xi_2(r) = -h_2 \frac{r(r - 2Gm(r))}{G(4\pi r^3 P_m + m(r))}. \quad (27)$$

We can define degrees of deformation by using the equation of ellipticity

$$e = \sqrt{1 - \chi^2}, \quad (28)$$

where to prolate shape we define $\chi = R_e/R_p$ and for oblate shape $\chi = R_p/R_e$.

Figure 1 illustrates the relation between mass and star radius using two different models. In panel (a) of Fig. 1, we use the Coulomb scalar EOS CIDDM model with two values $\kappa_1 = 0.54$ and $0.45$ which are the representation of the equation of states with minimum and maximum stiffness allowed by the stability of two flavors strange matter equation of state constraint. It can be seen in both models that after applying the magnetic field that except the region close to maximum mass, the changes due to magnetic fields are not too significant. It is evident that around the maximum mass the relation of $R_p > R_e$ is fulfilled, while outside this region, on the contrary, $R_e > R_p$ is fulfilled by both of $\kappa_1$ values. Similar situation but for Coulomb vector can be also seen in panel (b) of Fig.1. In that case, we can observe that the quark star is slightly deformed into the prolate form around the maximum mass region this happens due to the assumption that the magnetic field direction puts in the z-axis.
Figure 1. Relation between total mass and radius of quark stars according CIDDM model

Figure 2. Relation between ellipticity and Mass, (a) for $\chi = R_p/R_e$ and (b) for $\chi = R_e/R_p$.

Table 1. Profile of quark stars on the maximum mass in each CIDDM model.

| Model                   | $M/M_{solar}$ | $R$/km  | $Re$/km | $Rp$/km | $e$   |
|------------------------|---------------|---------|---------|---------|-------|
| Coulomb scalar ($\kappa_1 = 0.54$) | 1.746         | 10.427  | 10.375  | 10.455  | 0.123 |
| Coulomb scalar ($\kappa_1 = 0.45$) | 2.073         | 12.336  | 12.298  | 12.355  | 0.096 |
| Coulomb vector ($\kappa_1 = 0.83$) | 1.223         | 7.324   | 7.247   | 7.361   | 0.175 |
| Coulomb vector ($\kappa_1 = 0.63$) | 1.606         | 9.226   | 9.135   | 9.282   | 0.176 |

Figure 2 is the relation between the mass and the ratio of the ellipticity of quark stars where with the range of values $e \sim 0 - 1$, 0 indicating no deformation at all. From Figure 2 (a) is the degree of deformation of the oblate form, in which stars tend to be oblate-shaped in small
masses. In Coulomb scalar, in both variations the value $\kappa_1$ is not so different that the $e$ value is in the small value range i.e 0 – 0.2 which indicates the deformation is not very significant. The same thing happens to Coulomb vector but the difference of degrees of deformation is higher than Coulomb scalar, this can happen because in Lagrangian Coulomb vector the influence of its baryon number is more significant as shown in Equation (21), where magnetic field is depend on baryon number as shown in Equation (22). In Figure 2 (b) it describes for deformation to prolate form, where all models tend to be prolate shape on maximum mass. We can see that detail value in Table 1.

4. Conclusion
We can conclude that the magnetic field effect added to the momentum energy tensor with the value of the magnetic field constant at the surface of the quark star $B_0 = 10^{15}$ G for both CIDDf models has less significant effects (the radius difference exists in second order of magnitude) but make the star tend to be prolate in the maximum mass because the magnetic field in the neutron star is on the z-axis to form a repulsive force on the z-axis and the deformation is not so significant possible because of the gravitational force dominates on a high mass of quark stars.

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