Semimodules over commutative semirings and modules over unitary commutative rings

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**ABSTRACT**

It is well-known that the lattice of all submodules of a module is modular. However, this is not the case for the lattice of subsemimodules of a semimodule. We show examples and describe these lattices for a given semimodule. We study closed and splitting subsemimodules and submodules of a given semimodule or module $M$, respectively. We derive a sufficient condition under which the lattice $L_c(M)$ of closed subsemimodules is a homomorphic image of the lattice $L(M)$ of all subsemimodules. We describe the ordered set of splitting submodules of a module and show a natural bijective correspondence between this poset and the poset of all projections of this module. We show that this poset is orthomodular. This result extends the case known for the poset of closed subspaces of a Hilbert space which is used in the logic of quantum mechanics.

**1. Introduction**

It is well-known that any physical theory determines a class of event-state systems. To avoid details, in the case of quantum mechanics this event-state system is considered within the framework of a Hilbert space $\mathbf{H}$ whose projection operators are identified with the closed subspaces of $\mathbf{H}$.

It was recognized in 1936 by Birkhoff and von Neumann [1] and 1937 by Husimi [2], see also [3] or [4], that if the Hilbert space $\mathbf{H}$ is of infinite dimension then the lattice of its closed subspaces need not be modular, contrary to the case of the lattice of all subspaces. However, a later inspection showed that a supremum need not exist when the subspaces are not orthogonal (cf. [5]). This was the reason why orthomodular posets were introduced (see e.g. [5]) and intensively studied during the last decades.

The natural question arises whether the property that the closed subspaces of $\mathbf{H}$ form an orthomodular lattice or an orthomodular poset is a privilege of a Hilbert space. It was...
already answered in the negative [6,7]: there are vector spaces that are not Hilbert spaces for which the splitting subspaces form orthomodular posets.

Since the tools for determining the orthomodular poset of splitting subspaces of a given vector space can be used also for modules and, more generally, for semimodules [8,9], we extend our study to closed subsemimodules and submodules. We define splitting subsemimodules and prove that for a given semimodule \( M \), the set of its splitting subsemimodules forms a bounded poset with an antitone involution which, in the case when \( M \) is a module, turns out to be even an orthomodular poset. Similarly as for a Hilbert space, we use the method of projections and the bijective correspondence between the poset of projections and the poset of splitting submodules.

The concepts used for posets (i.e. ordered sets) and lattices are taken from monographs [5,10]. We hope that the study of closed and splitting subsemimodules and submodules and their lattices and posets can illuminate some properties of these concepts also in vector spaces, in particular in Hilbert spaces. Moreover, it may show that some physical theories need not be developed by using Hilbert spaces, but can be considered in a more general setting.

\[ \text{2. Semimodules over semirings} \]

There are various definitions of a semiring in literature. We use that taken from the monograph [11].

Recall that a \textit{commutative semiring} is an algebra \((S, \oplus, \cdot, 0, 1)\) of type \((2, 2, 0, 0)\) satisfying the following conditions:

- \((S, \oplus, 0)\) and \((S, \cdot, 1)\) are commutative monoids,
- \((x \oplus y)z \approx xz \oplus yz,\)
- \(x0 \approx 0.\)

Of course, every unitary commutative ring and every bounded distributive lattice is a commutative semiring.

Semimodules and semirings were studied by several authors, let us mention at least the papers [8,9,12,13]. Since these concepts are defined differently by the different authors, for the reader’s convenience we provide the following definition.

\textbf{Definition 2.1:} A \textit{semimodule} over a commutative semiring \((S, \oplus, \cdot, 0, 1)\) is an ordered quadruple \((M, +, \cdot, \vec{0})\) such that \(\cdot\) is a mapping from \(S \times M\) to \(M\) and the following conditions are satisfied for \(\vec{x}, \vec{y} \in M\) and \(a, b \in S\):

- \((M, +, \vec{0})\) is a commutative monoid,
- \(a(\vec{x} + \vec{y}) \approx a\vec{x} + a\vec{y},\)
- \((a \oplus b)\vec{x} \approx a\vec{x} + b\vec{x},\)
- \((ab)\vec{x} \approx a(b\vec{x}),\)
- \(1\vec{x} \approx \vec{x},\)
- \(0\vec{x} \approx a\vec{0} = \vec{0}.\)

Recall that a subset \(U\) of a semimodule \(M = (M, +, \cdot, \vec{0})\) over a commutative semiring \((S, \oplus, \cdot, 0, 1)\) (or the corresponding ordered quadruple \((U, +, \cdot, \vec{0})\)) is called a
subsemimodule of M if \( \vec{x} + \vec{y}, a\vec{x} \in U \) for all \( \vec{x}, \vec{y} \in U \) and \( a \in S \). Let \( L(M) \) denote the set of all subsemimodules of M.

Contrary to the case of vector spaces, not every semimodule has a basis. We define the notion of a basis for semimodules as follows.

**Definition 2.2**: Let \( M = (M, +, \cdot, 0) \) be a semimodule over a commutative semiring \( (S, \oplus, \cdot, 0, 1) \) and I a non-empty set. Let

\[
A := \{ f \in S^I \mid f = 0 \text{ almost everywhere} \}
\]

and let \( \vec{b}_i \in M \) for all \( i \in I \). Then \( B := \{ \vec{b}_i \mid i \in I \} \) is called a basis of M if for every \( \vec{x} \in M \) there exists exactly one \( f \in A \) with

\[
\sum_{i \in I} f(i) \vec{b}_i = \vec{x}.
\]

In the following we will assume that M has a basis B. Then M is isomorphic to the subsemimodule \( (A, +, \cdot, 0) \) of \( (S, \oplus, \cdot, 0)^I \). Hence we may identify M with this subsemimodule. In the sequel we denote the coordinates of the element \( \vec{x} \) of M with respect to the basis

\[
B := \{ \vec{b}_i \mid i \in I \}
\]

by \( x_i, \; i \in I \). The situation is analogous for an arbitrary non-empty set I.

The concept of an inner product on semimodules was investigated in [9]. For the reader’s convenience we recall the definition of the inner product as well as the concept of orthogonality for subsemimodules.

**Definition 2.3**: On M we define an inner product as follows: If \( \vec{x}, \vec{y} \in M \) then

\[
\vec{x} \cdot \vec{y} := \sum_{i \in I} x_i y_i.
\]

We write \( \vec{x} \perp \vec{y} \) if \( \vec{x} \cdot \vec{y} = 0 \). Moreover, for \( C \subseteq M \) we put

\[
C^\perp := \{ \vec{x} \in M \mid \vec{x} \perp \vec{y} \text{ for all } \vec{y} \in C \}.
\]

**Lemma 2.4**: Let \( \vec{a}, \vec{b} \in M \). Then (i) and (ii) hold:

(i) If \( a\vec{x} = \vec{b} \) for all \( \vec{x} \in M \) then \( \vec{a} = \vec{b} \),
(ii) if \( \vec{a} \perp \vec{x} \) for all \( \vec{x} \in M \) then \( \vec{a} = 0 \).

**Proof**: In case \( a\vec{x} = \vec{b} \) for all \( \vec{x} \in M \) we have \( a_i = \vec{a} \vec{b}_i = \vec{b} \vec{b}_i = b_i \) for all \( i \in I \). Assertion (ii) is a special case of (i).
Proposition 2.5: If $U, W \in L(M)$ then

- $U^\perp \in L(M)$,
- $U \subseteq W$ implies $W^\perp \subseteq U^\perp$,
- $U \subseteq U^\perp^\perp$,
- $U^{\perp\perp\perp} = U^\perp$,
- $U \subseteq W^\perp$ if and only if $W \subseteq U^\perp$,
- $\{\vec{0}\}^\perp = M$ and $M^\perp = \{\vec{0}\}$.

(The last assertion follows from Lemma 2.4.) Thus $\perp\perp$ is a closure operator on $(L(M), \subseteq)$.

Definition 2.6: A subsemimodule $U$ of $M$ is called closed if $U^{\perp\perp} = U$. Let $L_c(M)$ denote the set of all closed subsemimodules of $M$. Obviously, $L_c(M) = \{U^\perp \mid U \in L(M)\}$.

Let $U, W, U_j \in L(M)$ for all $j \in J$. Put

$$U + W := \{\vec{x} + \vec{y} \mid \vec{x} \in U, \vec{y} \in W\},$$

$$\sum_{j \in J} U_j := \left\{ \text{sums of finitely many elements of } \bigcup_{j \in J} U_j \right\},$$

$$U \vee W := (U + W)^{\perp\perp},$$

$$\bigvee_{j \in J} U_j := \left( \sum_{j \in J} U_j \right)^{\perp\perp},$$

$$L(M) := (L(M), +, \cap, ^\perp, \{\vec{0}\}, M),$$

$$L_c(M) := (L_c(M), \vee, \cap, ^\perp, \{\vec{0}\}, M).$$

We can describe the properties of the just defined concepts as follows.

Lemma 2.7:

(i) If $U_j \in L(M)$ for all $j \in J$ then

$$\left( \sum_{j \in J} U_j \right)^{\perp} = \bigcap_{j \in J} U_j^\perp,$$

$$\left( \bigcap_{j \in J} U_j \right)^{\perp} \supseteq \sum_{j \in J} U_j^\perp.$$

(ii) If $U_j \in L_c(M)$ for all $j \in J$ then

$$\left( \bigvee_{j \in J} U_j \right)^{\perp} = \bigcap_{j \in J} U_j^\perp.$$
\[
\left( \bigcap_{j \in J} U_j \right) ^\perp = \bigvee_{j \in J} U_j ^\perp.
\]

**Proof:**

(i) The first assertion is clear and the second easily follows by applying Proposition 2.5.
(ii) This follows from the fact that by Proposition 2.5, \( \perp \) is an antitone involution of \((L_c(M), \subseteq)\).

Using Lemma 2.7 we obtain immediately

**Theorem 2.8:** \( L(M) \) is a complete lattice with an antitone unary operation \( \perp \) and \( L_c(M) \) a complete lattice with an antitone involution \( \perp \).

**Proof:** This follows from Proposition 2.5 and Lemma 2.7.

The next theorem shows a sufficient condition under which the lattice \( L_c(M) \) is a homomorphic image of \( L(M) \). This condition is not too strong because it does not use suprema but only intersections and orthogonality.

**Theorem 2.9:**

(i) Assume \((U \cap W) ^{\perp \perp} = U ^{\perp \perp} \cap W ^{\perp \perp}\) for all \( U, W \in L(M) \). Then \( \perp \perp \) is a surjective homomorphism from \( L(M) \) to \( L_c(M) \).
(ii) Assume

\[
\left( \bigcap_{j \in J} U_j \right) ^{\perp \perp} = \bigcap_{j \in J} U_j ^{\perp \perp}
\]

for every family \((U_j; j \in J)\) of subsemimodules of \( M \). Then \( \perp \perp \) is a complete surjective homomorphism from \( L(M) \) to \( L_c(M) \).

**Proof:** Let \( U, W, U_j \in L(M) \) for all \( j \in J \).

(i) We have

\[
(U + W) ^{\perp \perp} = (U ^{\perp} \cap W ^{\perp}) ^{\perp} = (U ^{\perp \perp} \cap W ^{\perp \perp}) ^{\perp} = (U ^{\perp \perp} + W ^{\perp \perp}) ^{\perp} = U ^{\perp \perp} \lor W ^{\perp \perp},
\]

\[
(U \cap W) ^{\perp \perp} = U ^{\perp \perp} \cap W ^{\perp \perp},
\]

\[
(U ^{\perp}) ^{\perp \perp} = (U ^{\perp \perp}) ^{\perp},
\]

\[
\{ \vec{0} \} ^{\perp \perp} = \{ \vec{0} \},
\]
(ii) We have
\[
\left( \sum_{j \in J} U_j \right)^\perp = \left( \bigcap_{j \in J} U_j \right)^\perp = \left( \bigcup_{j \in J} U_j \right)^\perp = \left( \bigcap_{j \in J} U_j \right)^\perp = \bigvee_{j \in J} U_j^\perp,
\]
\[
\left( \bigcap_{j \in J} U_j \right)^\perp = \bigcap_{j \in J} U_j^\perp,
\]
\[
(U^\perp)^\perp = (U^\perp)^\perp,
\]
\[
\{0\}^\perp = \{0\},
\]
\[
M^\perp = M. \quad \blacksquare
\]

The conditions in Theorem 2.9 are sufficient, but need not necessarily hold as the following example shows.

**Example 2.10**: Consider the semiring \((S, \oplus, \cdot, 0, 1)\) where \(S = \{0, 1\}\) and the operations \(\oplus\) and \(\cdot\) are determined by the tables

\[
\begin{array}{c|cc}
\oplus & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 0 & 1
\end{array}
\quad
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
\]

Put \(M := (S, \oplus, \cdot, 0)^2\). Then \(M\) has the following subspaces:

\[
U_1 = \{(0, 0)\},
\]
\[
U_2 = \{(0, 0), (0, 1)\},
\]
\[
U_3 = \{(0, 0), (1, 1)\},
\]
\[
U_4 = \{(0, 0), (1, 0)\},
\]
\[
U_5 = \{(0, 0), (0, 1), (1, 1)\},
\]
\[
U_6 = \{(0, 0), (1, 0), (1, 1)\},
\]
\[
M = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.
\]

The Hasse diagram of \((L(M), \subseteq)\) is presented in Figure 1.

The lattice \(L(M)\) is not modular because it contains sublattices isomorphic to \(N_5\), e.g. the sublattice \(\{U_1, U_2, U_4, U_6, M\}\). The unary operation \(\perp\) looks as follows:

\[
\begin{array}{c|ccccccc}
U & U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & M \\
\hline
U^\perp & M & U_4 & U_3 & U_2 & U_1 & U_1 & U_1
\end{array}
\]

Hence, \(L_c(M) = \{U_1, U_2, U_4, M\}\). The Hasse diagram of \((L_c(M), \subseteq)\) is depicted in Figure 2.

It should be remarked that \(M\) does not satisfy the assumptions of Theorem 2.9 since
\[
(U_2 \cap U_6)^\perp = U_1^\perp = U_1 \neq U_2 = U_2 \cap M = U_2^\perp \cap U_6^\perp.
\]
**Example 2.11:** Consider the semiring \((S, \oplus, \cdot, 0, 1)\) where \(S = \{0, 1\}\) and the operations \(\oplus\) and \(\cdot\) are determined by the tables

\[
\begin{array}{c|cc}
\oplus & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

\((S, \oplus, \cdot, 0, 1)\) is in fact the two-element field. Put \(M := (S, \oplus, \cdot, 0)^2\). Hence \(M\) is the two-dimensional vector space over the two-element field. Then \(M\) has the following subspaces:

\[
\begin{align*}
U_1 &= \{(0, 0)\}, \\
U_2 &= \{(0, 0), (0, 1)\}, \\
U_3 &= \{(0, 0), (1, 1)\}, \\
U_4 &= \{(0, 0), (1, 0)\}, \\
M &= \{(0, 0), (0, 1), (1, 0), (1, 1)\}.
\end{align*}
\]

The Hasse diagram of \((L(M), \subseteq)\) is presented in Figure 3.

The lattice \(L(M)\) is modular and the unary operation \(\perp\) looks as follows:

\[
\begin{array}{c|ccccccccc}
U & U_1 & U_2 & U_3 & U_4 & M \\
\hline
U_\perp & M & U_4 & U_3 & U_2 & U_1 \\
\end{array}
\]

Hence, \(L_c(M) = L(M)\). It is easy to see that \(M\) satisfies the assumptions of Theorem 2.9.
3. Splitting subsemimodules

It can be easily checked that for a subsemimodule $U$ of $M$, the semimodule $U^\bot$ need not be a complement of $U$ in the lattice $L(M)$ or $L_c(M)$, see e.g. Example 2.10. This is the motivation for introducing the following concept.

**Definition 3.1:** We call a subsemimodule $U$ of $M$ splitting if $U + U^\bot = M$ and $U \cap U^\bot = \{0\}$. Let $L_s(M)$ denote the set of all splitting subsemimodules of $M$.

Clearly, $\{0\}, M \in L_s(M)$.

**Example 3.2:** The splitting subsemimodules of the semimodule from Example 2.10 are exactly the closed ones. In Example 2.11 we have $L_s(M) = \{U_1, U_2, U_4, M\} \subseteq L(M) = L_c(M)$.

**Lemma 3.3:** Every splitting subsemimodule of $M$ is closed.

**Proof:** Assume $U \in L_s(M)$, $a \in U^{\bot\bot}$ and $d \in M$. Then there exist $b, c \in U$ and $e, f \in U^{\bot}$ with $b + c = a$ and $e + f = d$. Since $\tilde{a} \in U^{\bot\bot}$, $\tilde{b}, \tilde{c} \in U$ and $\tilde{e}, \tilde{f} \in U^{\bot}$, we have

$$\tilde{a} \in U^{\bot\bot}, \tilde{b}, \tilde{c} \in U \quad \text{and} \quad \tilde{e}, \tilde{f} \in U^{\bot},$$

we have

$$\tilde{a} \tilde{f} = \tilde{b} \tilde{e} = \tilde{b} f = 0$$

and hence

$$\tilde{a} d = \tilde{a} (e + f) = \tilde{a} e + \tilde{a} f = \tilde{a} e = (b + c) e = \tilde{b} e + \tilde{c} e = \tilde{b} e = \tilde{b} e + \tilde{b} f = \tilde{b} (e + f) = \tilde{b} d.$$

According to Lemma 2.4, $\tilde{a} = \tilde{b} \in U$. This shows $U^{\bot\bot} \subseteq U$. The converse inclusion follows from Proposition 2.5. ■

Recall that if $(P, \leq, 0, 1)$ is a bounded poset, then a unary operation $'$ on $P$ is called a complementation if $\sup(x, x') = 1$ and $\inf(x, x') = 0$ for all $x \in P$. If $'$ is, moreover, an antitone involution then $(P, \leq, 0, 1)$ is called an orthoposet. In the sequel, we will denote sup and inf by $\lor$ and $\land$, respectively, provided they exist.

**Corollary 3.4:** $L_s(M) := (L_s(M), \subseteq, \bot, \{0\}, M)$ is an orthoposet.
It is a question whether the poset \((L_c(M), \subseteq)\) of splitting subsemimodules of \(M\) is a lattice depending of the choice of the semiring \(S\). It turns out that in some particular cases this is true.

Assume that \(S = (S, \lor, \land, 0, 1)\) is a non-trivial bounded distributive lattice where 0 is meet-irreducible, i.e. \(x \land y = 0\) implies \(0 \in \{x, y\}\). Let \(I\) be a non-empty set, put
\[
M := \{ \vec{x} \in S^I \mid x_i = 0 \text{ for almost all } i \in I \}
\]
and consider the subsemimodule \(M = (M, \lor, \land, \vec{0})\) of \((S, \lor, \land, 0)^I\). For every subset \(J\) of \(I\) put \(U_J := \{ \vec{x} \in M \mid x_i = 0 \text{ for all } i \in J \}\).

A mapping \(f\) from a poset \((P, \leq)\) to a poset \((Q, \leq)\) is called an antiisomorphism if it is bijective and if for all \(x, y \in P, x \leq y\) is equivalent to \(f(y) \leq f(x)\).

In the following theorem we use the formulation ‘\(x_i = 0\) for almost all \(i \in I\)’ which is the same as ‘\(\{i \in I \mid x_i = 0\}\) is a cofinite subset of \(I\).

**Theorem 3.5:** Let \((S, \lor, \land, 0, 1)\) be a non-trivial bounded distributive lattice where 0 is meet-irreducible and put
\[
M := \{ \vec{x} \in S^I \mid x_i = 0 \text{ for almost all } i \in I \}
\]
for a non-empty set \(I\). Then \((L_c(M), \subseteq) = (L_c(M), \subseteq)\) is an atomic Boolean algebra and the mapping \(J \mapsto U_J\) an antiisomorphism between the posets \((2^I, \subseteq)\) and \((L_c(M), \subseteq)\).

**Proof:** It is clear that for \(\vec{a}, \vec{b} \in M\) we have \(\vec{a} \perp \vec{b}\) if and only if for all \(i \in I\) either \(a_i = 0\) or \(b_i = 0\) (or both). Hence, for \(U \in L(M)\) we have \(U^\perp = U_K\) where
\[
K = \{ i \in I \mid \text{there exists some } \vec{x} \in U \text{ with } x_i \neq 0 \}.
\]
Obviously, \(U_J^\perp = U_{J^\perp}\) for all \(J \subseteq I\). This shows \(L_c(M) = \{ U_J \mid J \subseteq I \}\) and by calculation, \(L_c(M) \subseteq L_S(M)\) and thus \(L_c(M) = L_S(M)\). Now let \(S, T \subseteq I\). If \(S \subseteq T\) then \(U_T \subseteq U_S\). Conversely, assume \(U_T \subseteq U_S\). Suppose \(S \not\subseteq T\). Then there exists some \(j \in S \setminus T\). Let \(\vec{a}\) denote the element of \(M\) with \(a_j = 1\) and \(a_i = 0\) otherwise. Then \(\vec{a} \in U_T \setminus U_S\) contradicting \(U_T \subseteq U_S\). Hence \(S \subseteq T\). This shows that \(S \subseteq T\) is equivalent to \(U_T \subseteq U_S\) completing the proof of the theorem.

It should be remarked that in any non-trivial bounded chain the smallest element is meet-irreducible.

### 4. The poset of projections

The next concept plays a crucial role in our study.

**Definition 4.1:** A projection of \(M\) is a linear mapping \(P\) from \(M\) to \(M\) satisfying \(P(P(\vec{x})) = P(\vec{x})\) and \((P\vec{x}) \cdot \vec{y} = \vec{x} \cdot (P\vec{y})\) for all \(\vec{x}, \vec{y} \in M\). We write \(P\vec{x}\) instead of \(P(\vec{x})\). Let \(Pr(M)\) denote the set of all projections of \(M\) and \(P, Q \in Pr(M)\). We define \(P \leq Q\) if \(P(M) \subseteq Q(M)\), and, moreover, \((P + Q)(\vec{x}) := P\vec{x} + Q\vec{x}\) and \(PQ\vec{x} := P(Q(\vec{x}))\) for all \(\vec{x} \in M\). Note that \(PQ \in Pr(M)\) if \(PQ = QP\) since then we have
\[
(PQ\vec{x})\vec{y} = (Q\vec{x})(P\vec{y}) = \vec{x}(QP\vec{y}) = \vec{x}(PQ\vec{y})
\]
for all \(\vec{x}, \vec{y} \in M\). Let \(0\) denote the constant mapping from \(M\) to \(M\) with value \(\vec{0}\) and \(I\) the identical mapping from \(M\) to \(M\).
Clearly, $0, I \in \text{Pr}(M)$.

**Lemma 4.2:** Let $P, Q \in \text{Pr}(M)$.

(i) The following are equivalent:
(a) $P \leq Q$,
(b) $PQ = P$,
(c) $QP = P$.

(ii) Assume $PQ = QP$. Then the infimum $P \wedge Q$ exists and $P \wedge Q = PQ$.

**Proof:** Let $\vec{a}, \vec{b} \in M$.

(i) (a) $\Rightarrow$ (b): Since $P\vec{b} \in P(M) \subseteq Q(M)$, there exists some $\vec{c} \in M$ with $P\vec{b} = Q\vec{c}$. Now for all $\vec{a}, \vec{b} \in M$:

$$
(PQ\vec{a})\vec{b} = (Q\vec{a})(P\vec{b}) = (Q\vec{a})(Q\vec{c}) = \vec{a}(Q^2\vec{c}) = \vec{a}(Q\vec{c}) = \vec{a}(P\vec{b}) = (P\vec{a})\vec{b}
$$

showing $PQ = P$ by Lemma 2.4. (b) $\Rightarrow$ (c): We have for all $\vec{a}, \vec{b} \in M$:

$$
(QP\vec{a})\vec{b} = (P\vec{a})(Q\vec{b}) = \vec{a}(QP\vec{b}) = \vec{a}(P\vec{b}) = (P\vec{a})\vec{b}
$$

showing $QP = P$. (c) $\Rightarrow$ (a): We have $P(M) = QP(M) \subseteq Q(M)$.

(ii) Obviously, $PQ$ is a linear mapping from $M$ to itself. Moreover,

$$
(PQ)^2 = PQPQ = P^2Q^2 = PQ,
(PQ\vec{a})\vec{b} = (Q\vec{a})(P\vec{b}) = \vec{a}(QP\vec{b}) = \vec{a}(P\vec{b}) = (P\vec{a})\vec{b}
$$

showing $PQ \in \text{Pr}(M)$. Now $(PQ)P = PQ$, i.e. $PQ \leq P$, and $(PQ)Q = PQ$, i.e. $PQ \leq Q$. Now let $R \in \text{Pr}(M)$. If $R \leq P$, then $R(PQ) = (RP)Q = RQ = R$ and hence $R \leq PQ$. This shows $PQ = P \wedge Q$. ■

Moreover, we can prove the following.

**Theorem 4.3:** Let $M$ be a semimodule. Then $(\text{Pr}(M), \leq, 0, I)$ is a bounded poset.

**Proof:** We apply Lemma 4.2. Let $P, Q, R \in \text{Pr}(M)$. Since $P^2 = P$ we have $P \leq P$, if $P \leq Q \leq P$ then $P = PQ = Q$, and if $P \leq Q \leq R$ then

$$
PR = (PQ)R = P(QR) = PQ = P,
$$

i.e. $P \leq R$. Thus, $(\text{Pr}(M), \leq)$ is a poset. Clearly, $0 \leq P \leq I$. ■

It is elementary to check the following Proposition.

**Proposition 4.4:** The mapping $P \mapsto P(M)$ is an order homomorphism from the bounded poset $(\text{Pr}(M), \leq, 0, I)$ to the bounded poset $(L(M), \subseteq, \{\vec{0}\}, M)$. 

5. Modules over rings

In this section we will investigate modules over unitary commutative rings instead of semimodules over commutative semirings. Of course, every module \( M \) over a unitary commutative ring \( S \) is a semimodule but now \((M,+)\) is a commutative group. It means that on \( M \) there is also a binary operation — of subtraction. This enables us to reach stronger results than those above for semimodules.

In the sequel we assume that the semimodule \( M \) over the commutative semiring \( S \) is a module over the unitary commutative ring \( S \), i.e. \((M,+)\) is a commutative group.

In this section let \( L(M), L_c(M) \) and \( L_s(M) \) denote the set of all submodules, closed submodules and splitting submodules of \( M \), respectively.

It is well known \([14]\) that for a module \( M \), the lattice \( L(M) \) is modular, unlike the case for semimodules (see Example 2.10).

**Definition 5.1:** Let \( P, Q \in Pr(M) \). We define \((P - Q)\tilde{x} := P\tilde{x} - Q\tilde{x}\) for all \( \tilde{x} \in M \). Further, \( P' := I - P \) and \( P \perp Q \) if \( P \leq Q \).

**Lemma 5.2:** Let \( P, Q \in Pr(M) \). Then \( P' \in Pr(M) \), and \( P \perp Q \iff PQ = 0 \iff QP = 0 \).

**Proof:** Let \( \tilde{a}, \tilde{b} \in M \). Clearly, \( P' \) is a linear mapping from \( M \) to itself,

\[
(P')^2 = (I - P)(I - P) = I - P - P + P^2 = I - P = P',
\]

\[
(P' \tilde{a}) \tilde{b} = ((I - P) \tilde{a}) \tilde{b} = (\tilde{a} - P \tilde{a}) \tilde{b} = \tilde{a} \tilde{b} - (P \tilde{a}) \tilde{b} = \tilde{a} \tilde{b} - \tilde{a}(P \tilde{b}) = \tilde{a}(\tilde{b} - P \tilde{b}) = \tilde{a}(\tilde{b} - P \tilde{b}) = \tilde{a}(\tilde{b} - P \tilde{b})
\]

showing \( P' \in Pr(M) \). Finally,

\[
P \perp Q \iff P \leq Q' \iff PQ' = P \iff QP = 0 \iff Q'P = P.
\]

Now \( PQ' = P \iff P(I - Q) = P \iff P - PQ = P \iff PQ = 0 \) and \( Q'P = P \iff (I - Q)P = P \iff P - QP = P \iff QP = 0 \). \( \blacksquare \)

By Theorem 4.3, \((Pr(M), \leq, 0, 1)\) is a bounded poset. Now we can prove for modules a bit more.

**Lemma 5.3:** Let \( M \) be a module. Then \( Pr(M) := (Pr(M), \leq', 0, 1) \) is a bounded poset with an antitone involution.

**Proof:** Let \( P, Q \in Pr(M) \). If \( P \leq Q \) then

\[
Q'P' = (I - Q)(I - P) = I - P - Q + QP = I - Q = Q'
\]

according to Lemma 4.2, i.e. \( Q' \leq P' \). Finally, \( P''' = I - (I - P) = P \). \( \blacksquare \)

For a splitting submodule \( U \) of a module \( M \) we can show now that every element of \( M \) can be uniquely decomposed into a sum of two elements, one belonging to \( U \) and the other to \( U^\perp \).
Lemma 5.4: Let $U \in L_s(M)$ and $\vec{a} \in M$. Then there exist unique $b \in U$ and $c \in U^\perp$ with $\vec{b} + \vec{c} = \vec{a}$.

Proof: Because $M = U + U^\perp$ there exist $\vec{b} \in U$ and $\vec{c} \in U^\perp$ with $\vec{b} + \vec{c} = \vec{a}$. If $\vec{d} \in U, \vec{e} \in U^\perp$ and $\vec{d} + \vec{e} = \vec{a}$ then $\vec{b} - \vec{d} = \vec{e} - \vec{c} \in U \cap U^\perp = \{0\}$ and hence $\vec{b} = \vec{d}$ and $\vec{c} = \vec{e}$. 

For $U \in L_s(M)$ let $P_U$ denote the unique mapping from $M$ to $M$ with $P_U(\vec{x}) \in U$ and $\vec{x} - P_U(\vec{x}) \in U^\perp$ for all $\vec{x} \in M$. In the notation of Lemma 5.4, $P_U(\vec{a}) = \vec{b}$ and $\vec{a} - P_U(\vec{a}) = \vec{c}$.

Now we can show that the poset of splitting submodules of $M$ is isomorphic to the poset of its projections.

Theorem 5.5: The mappings $U \mapsto P_U$ and $P \mapsto P(M)$ are mutually inverse isomorphisms between $L_s(M)$ and $Pr(M)$.

Proof: Let $U, W \in L_s(M), P, Q \in Pr(M)$ and $\vec{a}, \vec{b} \in M$.

Obviously, $P_U$ is a linear mapping from $M$ to itself and $(P_U)^2 = P_U$. Moreover,

$$(P_U \vec{a})\vec{b} = (P_U \vec{a})(\vec{b} - P_U \vec{b}) + P_U \vec{b} = (P_U \vec{a})(\vec{b} - P_U \vec{b}) + (P_U \vec{a})(P_U \vec{b})$$

$$= \vec{a}(P_U \vec{b} - P_U \vec{b}) + (P_U \vec{a})(P_U \vec{b}) = \vec{a}(P_U \vec{b} - P_U \vec{b}) + (P_U \vec{a})(P_U \vec{b}) = (P_U \vec{a})(P_U \vec{b})$$

$$= (P_U \vec{a} - P_U \vec{a})\vec{b} + (P_U \vec{a})(P_U \vec{b}) = (P_U \vec{a} - P_U \vec{a})\vec{b} + (P_U \vec{a})(P_U \vec{b})$$

$$= (\vec{a} - P_U \vec{a})(P_U \vec{b}) + (P_U \vec{a})(P_U \vec{b}) = ((\vec{a} - P_U \vec{a}) + P_U \vec{a})(P_U \vec{b}) = \vec{a}(P_U \vec{b})$$

showing $P_U \in Pr(M)$.

Now $(\vec{a} - P\vec{a})(\vec{b}) = (P(\vec{a} - P\vec{a}))\vec{b} = (P\vec{a} - P\vec{a})\vec{b} = 0$ and hence $\vec{a} - P\vec{a} \in (P(M))^\perp$.

This shows $P(M) + (P(M))^\perp = M$. If $\vec{a} \in (P(M))^\perp$ then $(P\vec{a})(\vec{b}) = (P^2\vec{a})(\vec{b}) = (P\vec{a})(P\vec{b}) = 0 = \vec{0}^\perp$ and hence $P\vec{a} = \vec{0}^\perp$ showing $P(M) \cap (P(M))^\perp = \{0\}$. Hence $P(M) \in L_s(M)$.

Obviously, $P_U(M) = U$.

Since $\vec{a} - P\vec{a} \in (P(M))^\perp$, we have $P = P_{P(M)}$.

If $U \subseteq W$ then $P_U(M) = U \subseteq W = P_W(M)$, i.e. $P_U \leq P_W$.

If, conversely, $P \leq Q$ then $P(M) \subseteq Q(M)$.

Of course, $P_{\{0\}} = 0$, $P_M = I$ and $P_{U^\perp} = I - P_U = (P_U)'$.

The next lemma shows that the supremum of two commuting projections always exists.

Lemma 5.6: Let $P, Q \in Pr(M)$ and assume $PQ = QP$. Then $P \vee Q = P + Q - PQ$.

Proof: We have

$$P'Q' = (I - P)(I - Q) = I - Q - P + PQ = I - P - Q + PQ = (I - Q)(I - P) = Q'P'.$$

Since $'$ is an antitone involution on $Pr(M)$ we have $P \vee Q = (P' \wedge Q')'$ and hence

$$P \vee Q = (P' \wedge Q')' = (P'Q')' = I - (I - P - Q + PQ) = P + Q - PQ$$

according to Lemma 4.2.
Corollary 5.7: If \( P, Q \in \text{Pr}(M) \) and \( P \perp Q \) then \( P \wedge Q = 0 \) and \( P \vee Q = P + Q \).

**Proof:** This follows from Lemmas 4.2, 5.2 and 5.6.

Recall from [5] that an orthomodular poset is a bounded poset \((P, \leq, \prime, 0, 1)\) with an antitone involution such that for all \( x, y \in P \):

if \( x \leq y' \) then \( x \vee y \) exists, and if \( x \leq y \) then \( y = x \vee (y \wedge x') \).

The notion of an orthomodular poset is well-defined: If \( x \leq y \) then \( x \vee y' \) exists and hence \( x' \wedge y \) exists, too. Moreover, \( x' \wedge y \leq x' \) and hence \( (x' \wedge y) \vee x \) exists.

Our final result shows that the splitting submodules of \( M \) form an orthomodular poset. This was already shown for vector spaces over fields (cf. [6]).

**Theorem 5.8:** Let \( M \) be a module. Then \( \text{Pr}(M) \) is an orthomodular poset, \( L_s(M) \) is isomorphic to \( \text{Pr}(M) \), and \( U \vee W = U + W \) in \( L_s(M) \) for every \( U, W \in L_s(M) \) with \( U \perp W \) (i.e. \( U \subseteq W' \)).

**Proof:** According to Lemma 5.3, \( \text{Pr}(M) \) is a bounded poset with an antitone involution. Now let \( P, Q \in \text{Pr}(M) \). If \( P \perp Q \) then \( P \vee Q = P + Q \). If \( P \leq Q \) then \( P \perp Q' \), \( P \vee Q' = P + Q', P' \wedge Q = (P \vee Q')' \), \( P \perp (P' \wedge Q) \) and

\[
P \vee (P' \wedge Q) = P + (P + Q')' = P + \mathbb{I} - (P + \mathbb{I} - Q) = Q.
\]

The second part of the theorem follows from Theorem 5.5 and from

\[
U + W \subseteq U \vee W = (P_U + P_W)(M) \subseteq U + W
\]

for every \( U, W \in L_s(M) \) with \( U \perp W \).

**Example 5.9:** Consider the ring \((\mathbb{Z}_4, +, \cdot, 0, 1)\) of residue classes of the integers modulo 4 and put \( M := (\mathbb{Z}_4, +, \cdot, 0)^2 \) and \( A := \{0, 2\} \). Then \( M \) has the following submodules:

\[
\begin{align*}
U_1 &= \{0\}^2, \\
U_2 &= \{0\} \times A, \\
U_3 &= \{(0,0), (2,2)\}, \\
U_4 &= A \times \{0\}, \\
U_5 &= \{0\} \times \mathbb{Z}_4, \\
U_6 &= \{(0,0), (0,2), (2,1), (2,3)\}, \\
U_7 &= \{(0,0), (1,1), (2,2), (3,3)\}, \\
U_8 &= A^2, \\
U_9 &= \{(0,0), (1,3), (2,2), (3,1)\}, \\
U_{10} &= \{(0,0), (1,2), (2,0), (3,2)\}, \\
U_{11} &= \mathbb{Z}_4 \times \{0\},
\end{align*}
\]
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Figure 4. Lattice of submodules of a module $M$ over $\mathbb{Z}_4$.

Figure 5. Orthomodular poset of splitting submodules of $M$.

\[ U_{12} = A \times \mathbb{Z}_4, \]
\[ U_{13} = \{(0, 0), (0, 2), (1, 1), (1, 3), (2, 0), (2, 2), (3, 1), (3, 3)\}, \]
\[ U_{14} = \mathbb{Z}_4 \times A, \]

\[ M \]

The Hasse diagram of $(L(M), \subseteq)$ is presented in Figure 4.

The unary operation $\perp$ looks as follows:

| $U$ | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ | $U_6$ | $U_7$ | $U_8$ | $U_9$ | $U_{10}$ | $U_{11}$ | $U_{12}$ | $U_{13}$ | $U_{14}$ | $M$ |
|-----|------|------|------|------|------|------|------|------|------|---------|---------|---------|---------|-------|
| $U^\perp$ | $M$ | $U_{14}$ | $U_{13}$ | $U_{12}$ | $U_{11}$ | $U_9$ | $U_8$ | $U_7$ | $U_6$ | $U_5$ | $U_4$ | $U_3$ | $U_2$ | $U_1$ |

Hence, $L_c(M) = L(M)$ and $L_s(M) = \{U_1, U_5, U_6, U_{10}, U_{11}, M\}$. The Hasse diagram of $(L_s(M), \subseteq)$ is depicted in Figure 5.

One can easily see that $(L_s(M), \subseteq, \perp, \{(0, 0), M\})$ is the orthomodular lattice $MO_2$ and hence an orthomodular poset.

In our examples, the poset of splitting subsemimodules or splitting submodules is a lattice. In general, this need not hold. G. Birkhoff and J. von Neumann proved [1] that in the case of an infinite-dimensional Hilbert space over the field of complex numbers this poset is not a lattice but only an orthomodular poset. However, this need not hold only for Hilbert spaces. Posets of splitting subspaces which need not form lattices are intensively
studied by Vetterlein (cf. [15]). However, up to now, we do not have an example of finite dimension.

6. Conclusion

We described the lattice of closed subsemimodules $L_c(M)$ of a semimodule $M$ over a commutative semiring and the lattice $L(M)$ of all subsemimodules of $M$. As shown, $L_c(M)$ need not be a sublattice of $L(M)$. A sufficient condition under which $L_c(M)$ is a homomorphic image of the lattice $L(M)$ of all subsemimodules was given. We introduced the concept of a splitting subsemimodule being a subsemimodule for which orthogonality induces a complementation in the lattice $L(M)$. It was shown that every splitting subsemimodule is closed and that for a module $M$, the poset $L_c(M)$ of splitting submodules of $M$ forms an orthomodular poset. In the case when $M$ is a module, this poset turns out to be isomorphic to the poset of projections on $M$. Hence, the situation is analogous to the situation of orthomodular lattices of closed subspaces of a Hilbert space which rises the question about possible application in the logic of quantum mechanics.

Open problems:

1. Find a module of finite dimension over a commutative ring whose poset of splitting submodules does not form a lattice.

2. Given an orthomodular poset $P$, find a module whose poset of splitting submodules is isomorphic to $P$.

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