An EOQ model for weibull distribution deterioration with time-dependent cubic demand and backlogging

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Abstract. In this article we introduce an economic order quantity model with weibull deterioration and time dependent cubic demand rate where holding costs as a linear function of time. Shortages are allowed in the inventory system are partially and fully backlogging. The objective of this model is to minimize the total inventory cost by using the optimal order quantity and the cycle length. The proposed model is illustrated by numerical examples and the sensitivity analysis is performed to study the effect of changes in parameters on the optimum solutions.

1. Introduction
It is important to maintain and control the inventories of deteriorating items for the modern industries. Deterioration is defined as decay, dryness, spoilage, obsolescence, damage, pilferage, evaporation. Ghare and Schrader [5] initiated an EOQ model for deterioration taking fixed rate. Covert and Philip [2] derived an inventory system under conditions of constant demand and instantaneous delivery, the distribution of the time to deteriorating items is represented by weibull distribution. Jalan et al. [7] studied an inventory model for deteriorating items with linearly increasing demand rate and shortages. Wu et al. [15] derived an inventory of items that deteriorate at a weibull distribution and demand rate with a continuous function of time. Giri et al. [6] generalized this model with the ramp-type demand and weibull deterioration by taking single-period EOQ model. Dye [3] derived an economic order quantity model for deteriorating items which follows the weibull distribution. Mukhopadhyay et al. [9] formulated the replenishment a policy with demand rate is used price-dependent. Srichandan et al. [11] obtained an inventory system for perishable items under the inflationary conditions and two parameter weibull distributions for deteriorating items.

These models consider that, when there is a shortage, all the customers will be waiting for the arrival of the next replenishment (complete backlogging case) or all customers who are not served leave the system (lost sales case). Commonly, in many real-life situations, there are customers willing to wait for the next replenishment until they satisfy their demands, while others do not want to or cannot wait and leave the system. In such situations, the partial backlogging was used to formulate the model. In this line, there are many researchers contributing numerous papers. Abad [1] studied the optimal pricing problem of the perishability and shortages. Wee [14] developed an inventory model with a quantity discount, pricing and partial backordering. Wu [16] generalized an EOQ model with Weibull distribution deterioration and shortages are allowed partial backlogging. Skouri and Papachristos [10] formulated an inventory model for deteriorating items, time dependent demand,
backlogging rate is an exponentially decreasing. Wu [17] derived this model by taking Weibull distribution deterioration and shortages. Dye and Ouyang [4] presented an EOQ model for perishable with goods stock-level dependent selling rate and shortages are considered partial backlogging rate. Yan [18] established an inventory model for perishable goods with freshness-dependent demand rate and partial backlogging. Valliathal and Uthayakumar [13] described an EOQ model with general ramp type demand, deteriorating of the items and backlogging of unfulfilled demand. Karthikeyan and Santhi [8] presented an inventory model for deteriorating items in which shortages are not allowed and demand rate is a cubic function of time and salvage cost.

Recently, Umakanta [12] introduced an EOQ model for weibull deterioration with demand rate as quadratic and partially backlogging. They have not considered the effect of cubic demand, complete backlogging and without shortage. In this paper, we developed an economic order quantity model for weibull deteriorating items and partial backlogging. In addition, the effect of cubic demand, complete backlogging and without shortage are considered. The two parameter weibull distribution is used to represent the time to deterioration. Here in this model, the total inventory cost is minimized. Numerical examples are provided for with and without shortage model and sensitivity analysis is performed to show the effect of changes in the parameters of the optimum solution.

2. Assumptions and notation of the model

2.1 Assumptions

1. The replenishment rate is infinite and the lead time is zero.
2. The demand rate is time dependent cubic function, i.e., \( D(t) = a + bt + ct^2 + dt^3 \) \( a \geq 0, b \neq 0, c \neq 0, d \neq 0 \).
3. \( T \) is the length of the replenishment cycle, \( Q \) is the order quantity per cycle. \( t_1, t_2 \) and \( Q \) are decision variables.
4. \( I_1(t) \): Inventory level at time \( 0 \leq t \leq t_1 \) in which the product has demand and deterioration.
5. \( I_2(t) \): Inventory level at time \( t_1 \leq t \leq t_1 + t_2 \) in which the product has shortage.
6. The rate of deterioration \( \theta(t) = \alpha \beta t^{-\beta-1} \), follows two parameter Weibull distribution, where \( \alpha(0 < \alpha << 1) \) is the scale parameter, \( \beta > 1 \) is the shape parameter. The assumed that the deterioration increases with time \( t > 0 \).
7. The unsatisfied demand is backlogged at a rate \( \exp(-\delta(t_1 + t_2 - t)) \), where \( t_1 + t_2 - t \) is the time up to the next replenishment and \( \delta \) is the backlogging parameter \( 0 \leq \delta \leq 1 \).

2.2 Notations

To develop this mathematical model, the following notations are used throughout this paper:

- \( A \) : Order cost per unit,
- \( C \) : Purchase cost per unit,
- \( HC \) : Holding cost per unit time, \( H(t) = g + h(t), g > 0, h > 0 \)
- \( C_s \) : Shortage cost per unit per unit time,
- \( C_d \) : Lost sales per unit,
- \( C_{d} \) : Deterioration cost per unit of deteriorated item,
- \( \delta \) : Backlogging parameter is \( \delta \) which lies in \( 0 \leq \delta \leq 1 \)
- \( t_1 \) : Length of time interval with positive or zero inventory level in a replenishment cycle, where \( t_1 \geq 0 \)
- \( t_2 \) : Length of time interval with negative inventory (or shortages) in a cycle, where \( t_2 \geq 0 \)
$Q$: Order quantity per cycle, i.e. $Q = Q_0 + BI$

$T$: Length of the inventory cycle,

$I(t)$: Inventory level at time $t$,

$I_1(t)$: Inventory level at time $t$, $0 \leq t \leq t_1$

$I_2(t)$: Inventory level at time $t$, $t_1 \leq t \leq t_1 + t_2$

$Q_0$: The maximum inventory level,

$BI$: The maximum inventory level during the stockout period,

$TC$: Total relevant inventory cost,

### 3. Model formulation

#### 3.1 The model with partial backlogging

![Figure 1](image.png)

The instantaneous inventory level $I(t)$ at time $t$ ($0 \leq t \leq t_1 + t_2$) can be modelled by the following differential equations. During the interval $[0, t_1]$, the inventory is depleted due to the combined effects of demand as well as the deterioration and Shortages occur during the period $[t_1, t_1 + t_2]$. The behavior of inventory in a cycle is depicted in Figure 1. Hence, the differential equation below represents the inventory status is given by

$$
\frac{dI(t)}{dt} + \alpha \beta t^{\beta-1} I(t) = -(a + bt + ct^2 + dt^3), \ 0 \leq t \leq t_1
$$

(1)

With boundary conditions $I(t_0) = Q_0$ and $I(t_1) = 0$.

The solution of the equation (1) is

$$
I_1(t) = \left[ a(t_1-t) + \frac{b}{2}(t_1^2-t^2) + \frac{c}{3}(t_1^3-t^3) + \frac{d}{4}(t_1^4-t^4) + \frac{aa t_1^{\beta+1}}{\beta+1} - \frac{aa t_1^{\beta+1}}{\beta+1} \right] , \ 0 \leq t \leq t_1
$$

(2)
We get the maximum inventory level is
$$Q_0 = \left[ a t_1 + \frac{b t_1^2}{2} + \frac{c t_1^3}{3} + \frac{d t_1^4}{4} + \frac{a a t_1^{\beta+1}}{\beta + 1} + \frac{b a t_1^{\beta+2}}{\beta + 2} + \frac{c a t_1^{\beta+3}}{\beta + 3} + \frac{d a t_1^{\beta+4}}{\beta + 4} \right]$$  (3)

Due to shortage during \([t_1, t_1 + t_2]\), the demand at time \('t'\) is partially backlogged at rate \(e^{-\delta(t_1 + t_2 - t)}\). The differential equation governing the amount of demand is backlogged:
$$\frac{dI_2(t)}{dt} = -(a + bt + ct^2 + dt^3) e^{-\delta(t_1 + t_2 - t)} \quad t_1 \leq t \leq t_1 + t_2$$  (4)

with boundary condition \(I_2(t_1) = 0\), the solution of Equation (4) is
$$I_2(t) = \left[ a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{c}{3}(t_1^3 - t^3) + \frac{d}{4}(t_1^4 - t^4) \right] - a\delta\left\{t_2(t_1 - t) + t_2(t_1 - t) - \frac{1}{2}(t_1^2 - t^2)\right\}$$

$$I_2(t) = -b\delta\left\{\frac{1}{2}t_2(t_1^2 - t^2) + \frac{1}{2}t_2(t_1^2 - t^2) - \frac{1}{3}(t_1^3 - t^3)\right\} - c\delta\left\{\frac{1}{3}t_1(t_1^3 - t^3) + \frac{1}{3}t_2(t_1^3 - t^3) - \frac{1}{4}(t_1^4 - t^4)\right\}$$

$$I_2(t) = -d\delta\left\{\frac{1}{4}t_1(t_1^4 - t^4) + \frac{1}{4}t_2(t_1^4 - t^4) - \frac{1}{5}(t_1^5 - t^5)\right\}$$

The maximum backlogged inventory \(BI\) is obtained at \(t = t_1 + t_2\) then from equation (5)
$$BI = -I_2(t_1 + t_2) = +d\left\{t_1^3 t_2 + t_1 t_2^3 + \frac{3t_1^2 t_2^2}{2} + \frac{t_2^4}{4}\right\} - a\delta\left\{\frac{t_2^2}{2}\right\} - b\delta\left\{\frac{t_1 t_2^2 + t_2^3}{6}\right\}$$

$$BI = -I_2(t_1 + t_2) = -c\delta\left\{\frac{t_1^2 t_2}{2} + t_1 t_2^3 + \frac{t_2^4}{12}\right\} - d\delta\left\{\frac{t_1^2 t_2^3}{2} + \frac{t_1^3 t_2^2}{3} + \frac{t_1^4}{4} + \frac{t_2^5}{20}\right\}$$  (6)

From equations (3) and (6), we have
$$Q = Q_0 + BI$$
$$Q = \left[ a t_1 + \frac{b t_1^2}{2} + \frac{c t_1^3}{3} + \frac{d t_1^4}{4} + \frac{a a t_1^{\beta+1}}{\beta + 1} + \frac{b a t_1^{\beta+2}}{\beta + 2} + \frac{c a t_1^{\beta+3}}{\beta + 3} + \frac{d a t_1^{\beta+4}}{\beta + 4} + a t_2 + \right]$$

$$Q = \left[ b \left( t_1 t_2 + \frac{t_2^2}{2} \right) + c \left( t_1^2 t_2 + t_1 t_2^2 + \frac{t_2^3}{3} \right) + d \left( t_1^3 t_2 + t_1 t_2^3 + \frac{3t_1^2 t_2^2}{2} + \frac{t_2^4}{4}\right) - a\delta\left\{\frac{t_2^2}{2}\right\}$$

$$-b\delta\left\{\frac{t_1 t_2^2 + t_2^3}{6}\right\} - c\delta\left\{\frac{t_1^2 t_2^3}{2} + \frac{t_1^3 t_2^2}{3} + \frac{t_1^4}{4} + \frac{t_2^5}{20}\right\}$$  (7)

Cost components:
The cost components per cycle of the total inventory cost are follows
Ordering cost \((OC) = A\)
Holding cost \((HC) = \int_{0}^{t_1} (g + ht) I_1(t) \, dt\)

\[
HC = \begin{bmatrix}
ag_1t_1^{\beta+2} + bg_1t_1^{\beta+3} + cgt_1^{\beta+4} + dgt_1^{\beta+5} \\
\frac{2}{\beta+2} + \frac{3}{\beta+3} + \frac{4}{\beta+4} + \frac{5}{\beta+5}
\end{bmatrix}
\]

The cost of deterioration per cycle \((DC)\) is

\[
DC = C_d \left[ Q_0 - \int_{0}^{t_1} (a + bt + ct^2 + dt^3) \, dt \right]
\]

\[
DC = C_d \left[ \frac{aatt_1^{\beta+1}}{\beta+1} + \frac{bat_1^{\beta+2}}{\beta+2} + \frac{cat_1^{\beta+3}}{\beta+3} + \frac{dat_1^{\beta+4}}{\beta+4} \right]
\]

The cost of shortage per cycle due to backlogged \((SC)\) is

\[
SC = -c_1 \int_{t_1}^{t_1+t_2} I_2(t) \, dt, \quad t_1 \leq t_1 + t_2
\]

\[
= C_1 \left[ \frac{at_1^{2}}{2} + b \left( \frac{tt_1^{2}}{2} + t_1^{3} \right) + c \left( \frac{tt_1^{2}}{2} + t_1^{3} \right) \right]
\]

\[
= C_1 \left[ \frac{at_1^{2}}{2} + b \left( \frac{tt_1^{2}}{2} + t_1^{3} \right) + c \left( \frac{tt_1^{2}}{2} + t_1^{3} \right) \right]
\]

The cost due to lost sales \((LS)\) is given by

\[
LS = C_2 \int_{t_1}^{t_1+t_2} \left[ a + bt + ct^2 + dt^3 \right] \left[ 1 - e^{-\delta(t_1+t_2-t)} \right] \, dt, \quad t_1 \leq t_1 + t_2
\]

\[
LS = C_2 \left[ a\delta t_1^{2} + \frac{a\delta t_1^{2}}{2} + \frac{b\delta t_1^{2}}{2} + \frac{b\delta t_1^{2}}{2} + \frac{c\delta t_1^{2}}{6} + \frac{c\delta t_1^{2}}{3} + \frac{c\delta t_1^{2}}{12} + \frac{d\delta t_1^{2}}{4} + \frac{d\delta t_1^{2}}{20} \right]
\]

The purchase cost per cycle \((PC)\) is given by \(PC = CQ\)

\[
PC = C \left[ \frac{at_1^{2}}{2} + \frac{bt_1^{2}}{3} + \frac{ct_1^{2}}{4} + \frac{dt_1^{2}}{5} + \frac{aatt_1^{\beta+1}}{\beta+1} + \frac{bat_1^{\beta+2}}{\beta+2} + \frac{cat_1^{\beta+3}}{\beta+3} + \frac{dat_1^{\beta+4}}{\beta+4} \right]
\]

\[
PC = C \left[ \frac{at_1^{2}}{2} + \frac{bt_1^{2}}{3} + \frac{ct_1^{2}}{4} + \frac{dt_1^{2}}{5} + \frac{aatt_1^{\beta+1}}{\beta+1} + \frac{bat_1^{\beta+2}}{\beta+2} + \frac{cat_1^{\beta+3}}{\beta+3} + \frac{dat_1^{\beta+4}}{\beta+4} \right]
\]
The total annual cost per unit time is given by
\[ TC = \frac{1}{T} [OC + HC + DC + SC + LS + PC] \]

\[
A = \begin{array}{c}
\frac{agt_1^2}{2} + \frac{bgt_1^3}{3} + \frac{cgt_1^4}{4} + \frac{dgt_1^5}{5}
+ \frac{agat_1^2}{(\beta + 2)} + \frac{bgat_1^2}{(\beta + 3)} + \frac{cgat_1^2}{(\beta + 4)}
+ \frac{agt_1^2}{(\beta + 5)} + \frac{bgat_1^2}{(\beta + 1)(\beta + 2)} + \frac{cgat_1^2}{(\beta + 1)(\beta + 4)}
+ \frac{bht_1^3}{6} + \frac{cht_1^5}{8} + \frac{dht_1^6}{10} + \frac{aht_1^7}{12} + \frac{bht_1^7}{2(\beta + 3)} + \frac{cht_1^7}{2(\beta + 4)}
+ \frac{dht_1^7}{2(\beta + 5)}
+ \frac{aht_1^7}{2(\beta + 6)} + \frac{bht_1^7}{(\beta + 2)(\beta + 3)} + \frac{cht_1^7}{(\beta + 2)(\beta + 4)}
+ \frac{dht_1^7}{(\beta + 2)(\beta + 5)} + \frac{bht_1^7}{(\beta + 2)(\beta + 6)}
\end{array}
\]

\[
C = \frac{1}{t_1 + t_2} + C_1 + C_2
\]

\[
TC = \frac{1}{t_1 + t_2} + C_1 + C_2
\]

\[
\frac{\partial TC}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial TC}{\partial t_2} = 0
\]

Providing that the optimal values for \( t_1 \) and \( t_2 \) obtained from equation (14) satisfies the sufficient condition

\[
\frac{\partial^2 TC}{\partial t_1^2} \times \frac{\partial^2 TC}{\partial t_2^2} - \left( \frac{\partial^2 TC}{\partial t_1 \partial t_2} \right)^2 > 0
\]

3.2. The special case of completely backlogging
Due to shortage during \([t_1, t_1 + t_2]\), the demand at time \(t \) is completely backlogged. Hence, the inventory level can be represented by the following differential equation:

\[
\frac{dI_2(t)}{dt} = -(a + bt + ct^2 + dt^3), \quad t_1 \leq t \leq t_1 + t_2
\]

with the boundary condition \(I_2(t_1) = 0\)

\[
I_2(t) = \left[ a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{c}{3}(t_1^3 - t^3) + \frac{d}{4}(t_1^4 - t^4) \right]
\]

The maximum backlogged inventory \(BI \) is obtained at \( t = t_1 + t_2 \) then from equation (15)

\[
BI = -I_2(t_1 + t_2) = \left[ at_2 + b \left( t_2 t_2^2 - \frac{t_2^2}{2} \right) + c \left( t_2^2 t_2^2 + t_2^3 + \frac{t_2^3}{3} \right) + d \left( t_2^3 t_2^3 + \frac{3t_2^2 t_2^2}{2} + \frac{t_2^4}{4} \right) \right]
\]

Hence the order quantity during \([0, t_1 + t_2]\)

\[
Q = Q_0 + BI
\]

\[
Q = \left[ at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} + \frac{dt_1^4}{4} + \frac{a t_1^{\beta+1}}{\beta+1} + \frac{b t_1^{\beta+2}}{\beta+2} + \frac{c t_1^{\beta+3}}{\beta+3} + \frac{d t_1^{\beta+4}}{\beta+4} \right]
\]

The total shortage cost during interval \([t_1, t_1 + t_2]\) is

\[
SC = C_1 \int_{t_1}^{t_1 + t_2} I_2(t) \, dt, \quad t_1 \leq t \leq t_1 + t_2
\]

\[
= C_1 \left[ \frac{t_2^3}{2} + b \left( \frac{t_2^2}{2} + \frac{t_2^3}{6} \right) + c \left( \frac{t_2^2 t_2^3}{2} + \frac{t_2^3}{3} + \frac{t_2^4}{4} \right) + d \left( \frac{t_2^3 t_2^3}{2} + \frac{t_2^4}{4} + \frac{t_2^5}{20} \right) \right]
\]

Purchase cost per cycle \((PC) = CQ\)

\[
PC = CQ = \left[ a(t_2 + t_1) + \frac{b}{2}(t_2^2 + t_1^2) + \frac{c}{3}(t_2^3 + t_1^3) + \frac{d}{4}(t_2^4 + t_1^4) + \frac{a t_1^{\beta+1}}{\beta+1} \right]
\]

\[
- \frac{a t_1^{\beta+1}}{\beta+1} - \frac{b t_1^{\beta+2}}{\beta+2} - \frac{c t_1^{\beta+3}}{\beta+3} - \frac{d t_1^{\beta+4}}{\beta+4}
\]

\[
+ \frac{b a t_1^{\beta+2}}{\beta+2} + \frac{c a t_1^{\beta+3}}{\beta+3} + \frac{d a t_1^{\beta+4}}{\beta+4}
\]

\[
+ \frac{b a t^d_1^{\beta+2}}{\beta+2} + \frac{c a t^d_1^{\beta+3}}{\beta+3} - \frac{d a t^d_1^{\beta+4}}{\beta+4}
\]

\[
+ \frac{b a t^d_2^{\beta+2}}{\beta+2} + \frac{c a t^d_2^{\beta+3}}{\beta+3} + \frac{d a t^d_2^{\beta+4}}{\beta+4}
\]

Hence, the total relevant cost per unit time is given by

\[
TC = \frac{1}{T} \left[ OC + HC + DC + SC + PC \right]
\]
The conditions for minimization of the total cost \( TC \) per unit time are

\[
\frac{\partial TC}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial TC}{\partial t_2} = 0 \tag{21}
\]

Providing that the optimal values for \( t_1 \) and \( t_2 \) obtained from equation (21) satisfies the sufficient condition

\[
\frac{\partial^2 TC}{\partial t_1^2} \times \frac{\partial^2 TC}{\partial t_2^2} - \left( \frac{\partial^2 TC}{\partial t_1 \partial t_2} \right)^2 > 0
\]

### 3.3. Without shortage model

![Graphical presentation of inventory level](image-url)
The decreases of the inventory level occurs due to demand and deterioration during the time \([0, T]\). Hence, the differential equation below represents the inventory status is given by

\[
\frac{dI(t)}{dt} + \theta I(t) = -D(t), \quad 0 \leq t \leq T
\]

Now putting the value of \(\theta\) in above equation we get,

\[
\frac{dI(t)}{dt} + \alpha t^{\beta-1} I(t) = -(a + bt + ct^2 + dt^3), \quad 0 \leq t \leq T
\]

with boundary conditions \(I(0) = Q\) and \(I(T) = 0\).

The solution of the equation (22) is

\[
I(t) = \begin{bmatrix}
a(T-t) + \frac{b}{2} (T^2-t^2) + \frac{c}{3} (T^3-t^3) + \frac{d}{4} (T^4-t^4) + a\alpha T^{\beta+1} - a\alpha t^{\beta+1} \\
+ b\alpha T^{\beta+2} \frac{\beta}{2} + b\alpha t^{\beta+2} \frac{\beta}{2} + c\alpha T^{\beta+3} \frac{\beta}{3} + c\alpha t^{\beta+3} \frac{\beta}{3} + d\alpha T^{\beta+4} \frac{\beta}{4} + d\alpha t^{\beta+4} \frac{\beta}{4} \\
+ a\alpha t^{\beta+1}
\end{bmatrix}, \quad 0 \leq t \leq T
\]

We get the optimum order quantity is given by

\[
Q = \begin{bmatrix}
aT + \frac{b^2}{2} + \frac{cT^3}{3} + \frac{dT^4}{4} + a\alpha T^{\beta+1} + b\alpha T^{\beta+2} + c\alpha T^{\beta+3} + d\alpha T^{\beta+4}
\end{bmatrix}
\]

**Cost components**

The cost components per cycle of the total inventory cost are follows

**Ordering cost** \((OC) = A\)

**Holding cost per cycle** \((HC) = \int_0^T (g + ht) I(t) \, dt\)

\[
HC = \begin{bmatrix}
\frac{agT^2}{2} + \frac{bgT^3}{3} + \frac{cgT^4}{4} + \frac{dgT^5}{5} + \frac{ag\alpha T^{\beta+2}}{(\beta+2)} + \frac{bg\alpha T^{\beta+3}}{(\beta+3)} + \frac{cg\alpha T^{\beta+4}}{(\beta+4)} \\
+ \frac{dgaT^{\beta+5}}{(\beta+5)} - \frac{ag\alpha T^{\beta+2}}{(\beta+2)} - \frac{bg\alpha T^{\beta+3}}{(\beta+3)} - \frac{cg\alpha T^{\beta+4}}{(\beta+4)} - \frac{dgaT^{\beta+5}}{(\beta+5)} \\
+ \frac{ahT^3}{6} + \frac{bhT^4}{8} + \frac{chT^5}{10} + \frac{dhT^6}{12} + \frac{aahT^{\beta+3}}{2(\beta+3)} + \frac{bahT^{\beta+4}}{2(\beta+4)} + \frac{cahT^{\beta+5}}{2(\beta+5)} \\
+ \frac{dahT^{\beta+6}}{2(\beta+6)} - \frac{aahT^{\beta+3}}{2(\beta+3)} - \frac{bahT^{\beta+4}}{2(\beta+4)} - \frac{cahT^{\beta+5}}{2(\beta+5)} - \frac{dahT^{\beta+6}}{2(\beta+6)}
\end{bmatrix}
\]

The cost of deterioration per cycle \((DC)\) is

\[
DC = C_d \left[ Q - \int_0^T (a + bt + ct^2 + dt^3) \, dt \right]
\]

\[
DC = C_d \left[ \frac{aa\alpha T^{\beta+1}}{\beta+1} + \frac{baT^{\beta+2}}{\beta+2} + \frac{caT^{\beta+3}}{\beta+3} + \frac{daT^{\beta+4}}{\beta+4} \right]
\]

The purchase cost per cycle \((PC)\) is

\[
PC = CQ
\]
\[ PC = C \left[ aT + \frac{bT^2}{2} + \frac{cT^3}{3} + \frac{dT^4}{4} + \frac{a\alpha T^{\beta+1}}{\beta + 1} + \frac{b\alpha T^{\beta+2}}{\beta + 2} + \frac{c\alpha T^{\beta+3}}{\beta + 3} + \frac{d\alpha T^{\beta+4}}{\beta + 4} \right] \]  

\[ TC = \frac{1}{T} [OC + HC + DC + PC] \]

\[ T C = \frac{1}{T} \left[ \begin{array}{l} \frac{a g T^2}{2} + \frac{b g T^3}{3} + \frac{c g T^4}{4} + \frac{d g T^5}{5} + \frac{a g \alpha T^{\beta+2}}{\beta + 2} + \frac{b g \alpha T^{\beta+3}}{\beta + 3} + \frac{c g \alpha T^{\beta+4}}{\beta + 4} \\ + \frac{a h T^3}{6} + \frac{b h T^4}{8} + \frac{c h T^5}{10} + \frac{d h T^6}{12} + \frac{a a h T^{\beta+3}}{\beta + 3} + \frac{b a h T^{\beta+4}}{\beta + 4} + \frac{c a h T^{\beta+5}}{\beta + 5} \\ + \frac{a c h T^{\beta+6}}{2(\beta + 6)} - \frac{a a h T^{\beta+5}}{(\beta + 2)(\beta + 3)} + \frac{b a h T^{\beta+4}}{(\beta + 2)(\beta + 4)} - \frac{c a h T^{\beta+5}}{(\beta + 2)(\beta + 5)} + \frac{d a h T^{\beta+6}}{(\beta + 2)(\beta + 6)} \\ + C_{\delta} \frac{a a T^{\beta+1}}{\beta + 1} + \frac{b a T^{\beta+2}}{\beta + 2} + \frac{c a T^{\beta+3}}{\beta + 3} + \frac{d a T^{\beta+4}}{\beta + 4} \\ + C \left[ aT + \frac{bT^2}{2} + \frac{cT^3}{3} + \frac{dT^4}{4} + \frac{a\alpha T^{\beta+1}}{\beta + 1} + \frac{b\alpha T^{\beta+2}}{\beta + 2} + \frac{c\alpha T^{\beta+3}}{\beta + 3} + \frac{d\alpha T^{\beta+4}}{\beta + 4} \right] \right] \]

The condition for minimization of the total cost (TC) per unit time is

\[ \frac{dTC}{dT} = 0 \]  

The condition is also satisfied for the value \( T \) from equation (29)

\[ \frac{d^2TC}{dT^2} > 0 \text{ for all } T > 0 \]

4. **Numerical illustrations**

**Example 1. Partially backlogging model**

Let \( A = 500, \ a = 5, \ b = 9, \ c = 15, \ d = 20, \ g = 0.9, \ h = 0.4, \ \alpha = 0.5, \ \beta = 2, \ \ C_1 = 0.8, \ \ C_2 = 0.5, \ \ C = 0.6, \ \ C_{\delta} = 0.2, \ \delta = 0.5. \) We get the optimal values are \( t_1 = 0.0532, \ t_2 = 1.4538, \ Q = 63.4161 \) and \( TC = 385.6440. \)

**Example 2. Completely backlogging model**

Let \( A = 500, \ a = 5, \ b = 9, \ c = 15, \ d = 20, \ g = 0.9, \ h = 0.4, \ \alpha = 0.5, \ \beta = 2, \ \ C_1 = 0.8, \ \ C_2 = 0.5, \ \ C = 0.6, \ \ C_{\delta} = 0.2. \) We get the optimal values are \( t_1 = 0.3283, \ t_2 = 1.8413, \ Q = 211.5212 \) and \( TC = 337.0816. \)

**Example 3. Without shortage model**
Let $A = 500$, $a = 5$, $b = 9$, $c = 15$, $d = 20$, $g = 0.9$, $h = 0.4$, $\alpha = 0.5$, $\beta = 2$, $C = 0.6$, $C_d = 0.2$.
The optimal solutions are $T = 1.6213$, $Q = 155.1932$ and $TC = 469.3064$.

5. Sensitivity analysis
This section, we now study the effect of changes in the major parameters such as $A$, $a$, $b$, $c$, $d$, $\alpha$, $\beta$, $h$, $g$, $C$, $C_1$, $C_2$, $C_d$, $c$ and $\delta$ on $T$, $Q$, and $TC$ in the EOQ model of the Examples 1, 2 and 3. We change one parameter at a time and keep the other parameter remains unchanged to examine the sensitivity analysis. The results are summarized in Tables 1, 2 and 3 respectively.

| Parameters | Optimum values | $t_1$ | $t_2$ | $Q$ | $TC$ |
|------------|----------------|------|------|-----|-----|
| $A$        |                | 500  | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
|            |                | 600  | 0.0533 | 1.4823 | 69.7639 | 462.0844 |
|            |                | 700  | 0.0534 | 1.5030 | 73.5036 | 542.4440 |
|            |                | 800  | 0.0535 | 1.5356 | 76.5736 | 626.7559 |
| $a$        |                | 3    | 0.0645 | 1.4436 | 60.2932 | 384.1750 |
|            |                | 5    | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
|            |                | 7    | 0.0455 | 1.4638 | 66.7078 | 386.1143 |
|            |                | 9    | 0.0384 | 1.4739 | 70.2081 | 387.5401 |
| $b$        |                | 7    | 0.0540 | 1.4140 | 61.5606 | 381.7034 |
|            |                | 8    | 0.0536 | 1.4339 | 62.4875 | 383.1732 |
|            |                | 9    | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
|            |                | 10   | 0.0531 | 1.4735 | 64.3563 | 390.0970 |
| $c$        |                | 12   | 0.0540 | 1.4249 | 63.3397 | 377.8584 |
|            |                | 13   | 0.0538 | 1.4349 | 63.3937 | 380.3527 |
|            |                | 14   | 0.0535 | 1.4445 | 63.4095 | 382.9589 |
|            |                | 15   | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
| $d$        |                | 18   | 0.0596 | 1.4370 | 57.9141 | 377.8584 |
|            |                | 19   | 0.0562 | 1.4418 | 60.6876 | 378.7082 |
|            |                | 20   | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
|            |                | 21   | 0.0505 | 1.4634 | 66.0970 | 393.1456 |
| $g$        |                | 0.7  | 0.0530 | 1.5106 | 80.1677 | 352.8869 |
|            |                | 0.8  | 0.0531 | 1.5009 | 71.3675 | 362.9681 |
|            |                | 0.9  | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
|            |                | 1.0  | 0.0534 | 0.9642 | 56.5808 | 451.5986 |
| $h$        |                | 0.4  | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
|            |                | 0.6  | 0.0533 | 0.9863 | 33.2002 | 725.5964 |
|            |                | 0.8  | 0.0536 | 0.7489 | 25.1297 | 675.0084 |
|            |                | 1.0  | 0.0541 | 0.3724 | 18.3196 | 1255.4000 |
| $\alpha$   |                | 0.4  | 0.0537 | 1.5055 | 68.2256 | 376.3233 |
|            |                | 0.5  | 0.0532 | 1.4538 | 63.4161 | 385.6440 |
5.1 Observations:
The following observations are based on the results of Table 1

1) When parameters $A$ increases, $t_1$, $t_2$, $Q$ and $TC$ increases.

2) When parameters $a$, $b$, $c$ and $d$ are increases, $t_1$ is decreases while $t_2$, $Q$ and $TC$ increases.

3) When parameters $\alpha$ and $\beta$ are increases, $t_1$, $t_2$ and $Q$ decreases while $TC$ increases.

4) When the value of the model parameter $C_1$ and $C_2$ increases, $t_1$ and $TC$ increases while $t_2$ and $Q$ decreases.

5) When parameter $g$, $h$, $C$, $\delta$ and $C_d$ are increases, $t_1$ and $TC$ increases while $t_2$ and $Q$ decreases.

| Table 2. Sensitivity analysis of the completely backlogging model |
|---------------------------------------------------------------|
| Parameters | Optimum values |
|-------------|----------------|
|             | $t_1$  | $t_2$  | $Q$    | $TC$    |
|-------------|--------|--------|--------|---------|
| 500         | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   |   |   |   |   |
|---|---|---|---|---|
| A | 600 | 0.3609 | 1.9041 | 242.2979 | 382.1038 |
|   | 700 | 0.3938 | 1.9552 | 272.4117 | 425.4143 |
|   | 800 | 0.4273 | 1.9970 | 301.9924 | 467.3100 |
| a | 3   | 0.3300 | 1.7321 | 206.6395 | 334.2240 |
|   | 5   | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 7   | 0.3141 | 1.8833 | 216.3196 | 339.8388 |
|   | 9   | 0.3020 | 1.9430 | 220.8863 | 342.5250 |
| b | 7   | 0.3404 | 1.8275 | 209.0469 | 334.4820 |
|   | 8   | 0.3345 | 1.8393 | 210.3056 | 335.7837 |
|   | 9   | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 10  | 0.3121 | 1.8536 | 212.8180 | 338.3760 |
| c | 12  | 0.3510 | 1.8156 | 208.3298 | 332.0964 |
|   | 13  | 0.3433 | 1.8242 | 209.0882 | 333.7751 |
|   | 14  | 0.3357 | 1.8328 | 210.8176 | 335.4366 |
|   | 15  | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
| d | 18  | 0.3571 | 1.8208 | 209.7261 | 332.0695 |
|   | 19  | 0.3422 | 1.8361 | 210.6193 | 334.6118 |
|   | 20  | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 21  | 0.3154 | 1.8565 | 212.4847 | 339.4837 |
| g | 0.7 | 0.2439 | 1.9278 | 214.3377 | 248.3508 |
|   | 0.8 | 0.2763 | 1.8941 | 212.5894 | 260.1756 |
|   | 0.9 | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 1.0 | 0.4925 | 0.8767 | 210.8569 | 437.0286 |
| h | 0.5 | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 0.7 | 0.3398 | 1.8321 | 211.3563 | 357.0654 |
|   | 0.8 | 0.3421 | 1.8301 | 211.2370 | 365.0516 |
|   | 1.0 | 0.3549 | 1.8293 | 211.0983 | 371.0395 |
| α | 0.4 | 0.4470 | 1.9246 | 213.3963 | 335.1096 |
|   | 0.5 | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 0.6 | 0.2553 | 1.8035 | 210.5574 | 339.0647 |
|   | 0.7 | 0.2087 | 1.7896 | 209.9856 | 341.0621 |
| β | 2   | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 3   | 0.3036 | 1.6116 | 211.5204 | 338.8140 |
|   | 4   | 0.2942 | 1.5359 | 210.4937 | 339.3393 |
|   | 5   | 0.2798 | 1.4903 | 209.5872 | 340.3501 |
| Ci | 0.6 | 0.2195 | 2.0193 | 231.6483 | 327.0375 |
|   | 0.7 | 0.2697 | 1.9327 | 220.6760 | 332.1738 |
|   | 0.8 | 0.3283 | 1.8413 | 211.5212 | 337.0816 |
|   | 0.9 | 0.3976 | 1.7426 | 204.1045 | 341.8069 |
|   | 0.1 | 0.3022 | 1.8571 | 212.7540 | 336.5305 |
5.2 Observations:
The following observations are based on the results of Table-2

1) When the value of the model parameters \( A \) increases, \( t_1, t_2, Q \) and \( TC \) increases.

2) When the value of the model parameters \( a, b, c \) and \( d \) are increases, \( t_1 \) is decreases while \( t_2, Q \) and \( TC \) increases.

3) When parameters \( \alpha \) and \( \beta \) are increases, \( t_1, t_2 \) and \( Q \) decreases while \( TC \) increases.

4) When parameter \( C_i \) increases, \( t_i \) and \( TC \) increases while \( t_2 \) and \( Q \) decreases.

5) When parameter \( g, h, C \) and \( C_d \) are increases, \( t_i \) and \( TC \) increases while \( t_2 \) and \( Q \) decreases.

| Parameters | Optimum values |
|------------|----------------|
| \( T \) | \( Q \) | \( TC \) |
| 500 | 1.6213 | 155.1932 | 469.3064 |
| 600 | 1.6895 | 171.0763 | 529.6909 |
| 700 | 1.7491 | 186.0204 | 587.8372 |
| 800 | 1.8022 | 200.2129 | 644.1425 |
| 3 | 1.6262 | 150.3956 | 464.3530 |
| 5 | 1.6213 | 155.1932 | 469.3064 |
| 7 | 1.6164 | 159.9458 | 474.2475 |
| 9 | 1.6116 | 164.6766 | 479.1763 |
| 7 | 1.6301 | 151.8576 | 464.3536 |
| 8 | 1.6257 | 153.5371 | 466.8350 |
| 9 | 1.6213 | 155.1932 | 469.3064 |
| 10 | 1.6169 | 156.8258 | 471.7678 |
| 12 | 1.6434 | 151.7124 | 460.9469 |
| 13 | 1.6359 | 152.8927 | 463.7612 |
| 14 | 1.6285 | 154.0451 | 466.5474 |
| 15 | 1.6213 | 155.1932 | 469.3064 |
| 18 | 1.6456 | 154.3331 | 462.6698 |
| 19 | 1.6332 | 154.7617 | 466.0253 |
| 20 | 1.6213 | 155.1932 | 469.3064 |
| 21 | 1.6099 | 155.6354 | 472.5176 |
### Observations

The following observations are based on the results of Table-3:

1. When the value of the model parameters \( A \) increases, \( T, Q \) and \( TC \) increases.
2. When parameters \( a, b, c \) and \( d \) are increases, \( T \) is decreases while \( Q \) and \( TC \) increases.
3. When parameters \( \alpha \) and \( \beta \) are increases, \( T \) and \( Q \) decreases while \( TC \) increases.
4. When parameters \( g, h, C \) and \( C_d \) are increases, \( T \) and \( TC \) increases while \( Q \) decreases.

### Three dimensional graphs are shown in the following Figures 3, 4 and 5

Figure 3 shows that in partially backlogging model, while increasing the Backlogging rate \( (\delta) \) we obtain that the Order quantity \( (Q) \) decreases and the total cost increases.

Figure 4 shows that in completely backlogging model, while increasing Ordering cost \( (A) \) we obtain that the Order quantity \( (Q) \) and the total cost are increases.

Figure 5 shows that in without shortage model, while increasing Ordering cost \( (A) \) we obtain that the Order quantity \( (Q) \) and the total cost are increases.
Figure 3. Total cost versus $Q$ and $\delta$

Figure 4. Total cost versus $Q$ and $A$
6. Conclusion
In this article, we developed an economic order quantity (EOQ) model for deteriorating items with time-dependent demand and two-parameter Weibull distribution for deteriorating items. Shortages are allowed in the inventory system are complete and partial backlogging model. The numerical examples have been given to illustrate it. Finally, the sensitivity analysis shows the changes in the values of different parameters. Our research result implies that, total cost per unit time of partial backlogging is more than the complete backlogging which is clear from the numerical examples and sensitivity analysis. In future we extend the study further the proposed model can be incorporate into more realistic assumptions, such as stock-dependent demand, price-dependent demand, price discount, quantity discount.

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