The Spectral Density of the Dirac Operator above $T_c$ Rep

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Abstract

The importance of the spectral density of the Dirac operator in studying spontaneous chiral symmetry breaking and anomalous $U(1)_A$ symmetry breaking are reviewed. It is shown that both types of symmetry breaking can be traced to effects of modes near zero virtuality. Above $T_c$, where chiral symmetry is restored, it is shown on general grounds that (in the $m_q \to 0$ limit), the density of states vanishes at zero virtuality faster than $\lambda$, where $\lambda$ is the virtuality—i.e. $\rho(\lambda) \sim |\lambda|^{\alpha}$ is not possible for $\alpha \leq 1$. Isospin invariance is used to show that $\rho(\lambda) \sim m_q^{1-\alpha}|\lambda|^\alpha$ is also not possible for $\alpha \leq 1$.

State-of-the-art lattice calculations are reviewed in light of these constraints. In particular, it is argued that violations of these constraints by lattice calculations indicate possible large systematic errors; this raises questions about $U(1)_A$ violating effects seen on the lattice. It is also shown that above $T_c$, the Dirac spectrum has a gap near zero (in the $m_q \to 0$ limit) unless contributions from quark-line-connected and disconnected contributions conspire to cancel.

I. INTRODUCTION

It is a great pleasure to give this talk at a workshop in honor of Mannque Rho’s 60th birthday. Mannque has been a central figure in our field for quite a long time. When I began working in nuclear physics about 15 years ago, it became quite clear to me that Mannque was quite special. I soon realized that our field could be roughly divided into two classes— theoretical physicists with powerful and elegant mathematical tools at their disposal, and those with a phenomenological bent who had the intuition to make the kinds of simplifications needed to deal with the complications of realistic strongly interacting systems and make experimentally relevant predictions. Mannque is almost unique in belonging to both groups.

Having divided the field this way, let me begin this talk with an apology. Virtually nothing which I will deal with here has immediate phenomenological consequences. The problem
with which I am dealing concerns the QCD vacuum in thermal equilibrium above the chiral phase transition. This raises the obvious question of just how does one heat up vacuum? An idealized experimental setup—as designed by a typical American student—consists of an empty test tube held above a large candle. I have been told that this method will not work in practice, and that the only practical way to explore this phase in the laboratory is through ultrarelativistic heavy ion collisions. Unfortunately, the way the physics of the phase of interest gets translated into experimental signatures which can be observed in heavy ion reactions is by no means clear. I will barely touch on this question in this talk, and what I do suggest will turn out to be highly sensitive to assumptions about the dynamics of the phase transition and will be highly speculative.

A theorist might wish to attack a problem so far removed from experiment for reasons other than the sheer delight in being unable to be shown wrong by our grungy experimental friends. The fact is that we believe we know the underlying theory (QCD) and we can cleanly formulate well-posed questions about QCD in thermal equilibrium in terms of the degrees of freedom of QCD. Moreover, at least in principle, these questions can be answered to arbitrary precision via sufficiently good lattice simulations. Of course in practice, present day lattice simulations have some serious shortcomings for technical reasons associated with a lack of sufficient computer power, but there is every reason to believe that as computers get better, the lattice community will slowly converge on the right answers.

Since we are ultimately concerned with issues of nonperturbative QCD, but we will be working with the QCD degrees of freedom—quarks and gluon—it is probably useful to introduce a few tools. One essential tool is the composite operator or interpolating field. These beasts are constructed from the quark and gluon operators. Typically one is interested in local gauge-invariant operators with fixed transformation properties under Lorentz transformations, parity, and isospin rotations. I will generically note such operators as $J$ and examples include:

$$J \sim \bar{q} i\gamma_{5} \tau q$$
$$J \sim F_{\mu\nu}F^{\mu\nu}$$

which have the quantum numbers of a pseudoscalar isovector and scalar isoscalar, respectively. One can construct an infinite number of such operators. Essentially all of the information one can extract from QCD is encoded in the various n-point correlation functions of these currents. A typical example is the two-point function $\langle J(x, t)J(0, 0) \rangle$ where one can choose the appropriate boundary conditions of interest (e.g., time-ordered, advanced, retarded, etc.) These correlators tell us about the response of the QCD vacuum given a disturbance associated with $J$.

How can one calculate these things? In principle, there is a straightforward way to calculate the expectation value of any observable given in terms of quark and gluon field operators. The method is functional integration. Here we will discuss the problem for imaginary time (Euclidean space) and “all” one needs to do to get real time results is analytically continue. The Euclidean functional integral for the expectation value of any operator is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [DA][D\bar{\tau}][Dq] e^{iS_{\text{QCD}}} \mathcal{O}$$
\[ Z = \int [DA] [D\mathbf{q}] [D\mathbf{q}] e^{iS_{QCD}} \]  

(2)

where \( S_{QCD} \) is the action.

Here we are interested in QCD at finite temperatures. One of the virtues of the Euclidean space functional integral is that one can calculate finite thermal expectation values exactly as one calculates zero temperature ones. Indeed the functional integral expression in eqs. (2) holds for thermal expectation values. The only difference from the free-space case concerns the class of configurations over which one integrates. To use functional integrals at finite temperatures one simply imposes the boundary condition \[ \] that all boson fields (gluons in this case) are periodic in time with a period given by \( \beta \), the inverse temperature, while fermion field configurations are anti-periodic:

\[ A_\mu(x, t) = A_\mu(x, t + \beta) \quad q(x, t) = -q(x, t + \beta) \]

The above formulation holds for any operator. The operator which I will stress in this talk is the density of states of the Dirac operator; the question I will address is how this relates to various correlation functions. The operator can be defined in the following way:

\[ \rho(\lambda)_A = 1/\text{Vol} \sum_j \delta(\lambda - \lambda_j) \]  

(3)

where \( i\lambda_j \) is an eigenvector of the (Euclidean) covariant Dirac operator acting on spinors,

\[ D_\mu \gamma^\mu q_j = i\lambda_j q_j, \]

and Vol is the volume of space-time with a finite temporal extent given by \( \beta \) and, ultimately, infinite spatial extent. Note that the Dirac operator \( D_\mu \gamma^\mu \) is anti-Hermitian in Euclidean space and hence has purely imaginary eigenvalues. Note also that \( \rho_A \) is implicitly a functional of the gauge configuration \( A \) through the covariant derivative.

A useful property of the Dirac operator is that it anti-commutes with \( \gamma_5 \) which, in turn, implies that for every nonzero eigenvector \( q_j \), \( \gamma_5 q_j \) is an eigenvector with opposite eigenvalue. This, in turn, implies that \( \rho_A(\lambda) \) is an even function of \( \lambda \). The parameter \( \lambda \) will be referred to as the virtuality. There is an extremely simple way to represent \( \rho_A \) as a trace which we will find useful later:

\[ \rho_A(\lambda) = \lim_{\epsilon \to 0} \pi/\text{Vol} \text{ Im} \left( \text{Tr} \left[ \frac{1}{\lambda + iD_\mu \gamma^\mu - i\epsilon} \right] \right) \]  

(4)

where \( \text{Tr} \) indicates a trace over Dirac indices and also a functional trace over configurations.

The thermal expectation of \( \rho_A \) will be of prime interest. I will define \( \rho(\lambda) \) to be expectation value:

\[ \rho(\lambda) = \langle \rho(\lambda)_A \rangle \]  

(5)

It is not easy to gain an immediate intuitive grasp of the meaning of \( \rho(\lambda) \). However, as will be stressed in this talk, \( \rho(\lambda) \) in the vicinity of \( \lambda = 0 \) is intimately related to the spontaneous breaking of chiral symmetry and anomalous breaking of \( U(1) \) axial symmetry.
Before proceeding, let me remind you of some well-known facts about chiral symmetry in QCD. The up and down quark masses in QCD (which I will generically call $m_q$), are much less than the typical hadronic scales. Neglecting the quark masses is a reasonable first approximation which we will take here and the effects of nonzero quark masses can be computed via chiral perturbation theory. Now if we neglect quark masses the QCD Lagrangian is invariant under
\[ q \rightarrow e^{i\gamma_5 \vec{\tau} \cdot \vec{\theta}/2} \]
where $\vec{\tau}$ are the Pauli matrices and $\vec{\theta}$ are arbitrary parameters. While the underlying theory has this symmetry, the ground state (i.e. the vacuum) does not, at least for $T < T_c$: the symmetry is spontaneously broken. One easy way to see this is to note that the chiral condensate $\langle \bar{q}q \rangle$ is nonzero. This indicates spontaneously chiral symmetry breaking since under a chiral rotation through $\pi$, in the $a$ direction, the operator $\bar{q}q$ transforms into $\bar{q}i\gamma_5 \tau_a q$ which has a zero expectation value. The consequence of this spontaneous breaking are profound: through Goldstone’s theorem we see that the symmetry breaking implies the existence of massless isovector pseudoscalars which we interpret as pions. (In nature, they are merely very light and not massless since the quark masses are not exactly zero). Similarly, the spontaneous symmetry breaking implies that the pions don’t interact at threshold (in the zero quark mass limit) which allows one to formulate a chiral perturbation theory for low energy interactions. [2]

It is generally believed that in the high temperature phase of QCD that chiral symmetry is restored. That is to say that the symmetry of the thermal state is the same as the symmetry of the Lagrangian. One consequence of this is above $T_c$ the chiral condensate vanishes.

Let me now turn to the connection between the chiral condensate and the spectrum of the Dirac operator. Banks and Casher proved a remarkable result in 1980: [3]
\[ \lim_{m_q \to 0} \langle \bar{q}q \rangle = \pi \rho(0) \tag{6} \]
The chiral condensate in the zero quark mass limit is directly proportional to the density of states at zero virtuality. The proof of this is quite straightforward:
\[
\langle \bar{q}q(x = 0) \rangle = \langle \text{tr}[S_A(0,0)] \rangle \tag{7}
\]
where $S_A$ is the quark propagator in the presence of a background gluon field,
\[
(D_\mu \gamma^\mu + m_q)S_A(x,y) = \delta^4(x-y), \tag{8}
\]
and tr indicates a trace over the Dirac indices only. Since $S_A(x,y)$ is simply the inverse of the operator $D_\mu \gamma^\mu + m_q$, it is clear that spatially averaging over all $x$ will simply yield the functional trace of $(D_\mu \gamma^\mu + m_q)^{-1}$:
\[
\int d^4x \text{tr}[S_A(x,x)] = Tr \left[ \frac{1}{D_\mu \gamma^\mu + m} \right]
\]
One can exploit translational invariance to note the
\[
\langle \bar{q}q \rangle = \langle \frac{1}{\text{Vol}} \int d^4x \bar{q}q(x) \rangle = 1/\text{Vol} \langle Tr \left[ \frac{1}{D_\mu \gamma^\mu + m} \right] \rangle
\]
Taking the limit as $m \to 0$ and comparing with eq. (4) immediately yields the Banks-Casher relation of eq. (6).

One obvious and beautiful thing about the derivation is that it only depends on the Euclidean functional integral formulation of the problem and hence is automatically valid at finite temperature. Above the phase transition where chiral symmetry is restored and $\langle \bar{q}q \rangle$ goes to zero (or more strictly $O(m_q)$ ) we see that $\rho(0) \to 0$ (or more precisely $O(m_q)$). So we see that $\rho(0)$ serves as an order parameter for chiral symmetry and its restoration.

The spectral density of the Dirac operator also gives us useful insights into the $U(1)_A$ problem (i.e. the axial $U(1)$ problem). It is well known that the QCD Lagrangian with massless quarks has an additional axial symmetry:

$$q \to e^{i\gamma_5 \theta/2}q$$

This symmetry is anomalously broken by quantum effects—the current associated with it is not conserved. The reason for this is quite simple: divergences in the theory require regularization; there is no way to regularize which simultaneously respects $U(1)_A$ symmetry and gauge symmetry. Since the renormalization of the theory requires unbroken gauge symmetry, $U(1)_A$ must break. The consequences of this are far reaching. For example, this is an essential ingredient in the understanding of why there is no $U(1)_A$ Goldstone boson. 

Now, just as there is a connection between the spectral density at zero virtuality and the spontaneous breaking of ordinary chiral symmetry, there is an important connection between the states at zero virtuality and the spontaneous/anomalous breaking of the $U(1)_A$. The point is very simple—all effects associated with $U(1)_A$ breaking in massless QCD can be traced to modes of the Dirac operator in the neighborhood of zero virtuality. To make this concrete, consider what I will call as the $U(1)_A$ susceptibility, $\chi_{U(1)_A}$, which is simply the spatial integral of the correlator in the isovector pseudoscalar ($\pi$) channel minus correlator in the isoscalar pseudoscalar ("$\eta'$") channel:

$$\chi_{U(1)_A} = \int d^4x \langle j_{\pi}(x)j_{\pi}(0) - j_{\eta'}(x)j_{\eta'}(0) \rangle$$

with $j_{\pi} = \bar{q}\gamma_5 \tau q$ and $j_{\eta'} = \bar{q}\gamma_5 q$. An analysis very similar to that of the Banks-Casher relation immediately yields:

$$\chi_{U(1)_A} = 2 \int d\lambda \rho(\lambda) \frac{m_q^2}{\lambda^2 + m_q^2}$$

From the form of eq. (11) it is obvious that in the limit $m_q \to 0$ the susceptibility must vanish unless there is strength in the immediate neighborhood of zero. The important thing to realize is that as was pointed out a couple of years back this is generic: all $U(1)_A$ violating amplitudes get their strength entirely from the $\lambda \sim 0$ region.

A few years ago, motivated by an insight of the previous sort (in the context of the instanton liquid model), Shuryak asked the following provocative question: since we know above $T_c$, $\rho(0) = 0$ and we know that $U(1)_A$ violating amplitudes come from modes at $\lambda = 0$, is it possible that all effects of the anomalous $U(1)_A$ breaking vanish in the chirally restored phase? At first sight this seems completely nuts—the anomaly is an operator equation and the axial current is not conserved regardless of the state of the system. On the other hand,
it is well known that it is only due to subtle interplay between topology and the anomaly that any effects of $U(1)_A$ breaking are seen. In fact, a couple of years ago I showed on very general grounds based on the positivity of the measure in the QCD along the lines of the QCD inequalities of Weingarten, Witten and Vafa [7] that unless there are contributions to the functional integral which form a set of measure zero in the massless limit of the theory, $U(1)_A$ violating amplitudes must vanish above $T_c$. This raises the obvious question as to whether such a set of measure zero can contribute. As noted by Lee and Hatsuda [8] and by Evans, Hsu and Schwetz [9], such contributions are possible in a dilute instanton gas where one might expect $\rho(\lambda) \sim m_u m_d \delta(\lambda)$. Such a form clearly is a set of measure zero in the massless quark limit but would contribute to $U(1)_A$ violating amplitude. This leaves us with the question of whether QCD actually does yield $U(1)_A$ violating effects above $T_c$ or not.

Now the preceding analysis is highly formal and mathematical and it is useful to remind ourselves of what Bertrand Russell once said: “Mathematics may be defined as the subject in which we never know what we are talking about.”

Given the bleak prospects of using purely formal reasoning to answer this, it seems there are two possible avenues of attack. The first is through phenomenology and the second is via lattice studies.

As I noted at the outset it is very difficult to use phenomenology to pin things down since so much of the dynamics is unknown. There is at least one scenario where we can get a dramatic effect but it depends on many unproven assumptions. The first is that $U(1)_A$ violating effects do vanish above $T_c$; this is, of course, what we are trying to check. The second is that the transition is second order and happens sufficiently rapidly to get out of thermal equilibrium over large domains and develops an instability for pionic growth. This is the so-called disoriented chiral condensate scheme of Wilczek and Rajagopal [10]. If this happens, then along with collective enhancements of low $P_T$ pions we would get enhancements of low $P_T$ $\eta$'s. Clearly this scenario is highly speculative and we turn to the question of what the lattice can tell us.

II. LATTICE RESULTS

In principle, the lattice can answer our question about $U(1)_A$ violations above $T_c$. Of course, there may be the usual problems of interpretations given effects of finite masses, lattice spacings and lattice sizes and the like. While there have been a number of lattice calculations of quantities sensitive to $U(1)_A$ violations many of them depend on very numerically unstable quantities such as screening masses in various channels which we know are highly sensitive to quark mass effects and threshold effects. A more natural quantity to study is the $U(1)_A$ violating susceptibility defined in eq. (11). To my knowledge there are only two calculations in the literature of this quantity. The first by Chandrasekharan and Christ [11] and the second by Bernard et al. [12]. Both of these papers claim that that they see evidence that $\chi_{U(1)_A}$ does not go to zero above $T_c$ indicating that $U(1)_A$ violating effects do not vanish above $T_c$. In particular, Chandrasekharan and Christ claim that they see a small but nonvanishing $\chi_{U(1)_A}$ for modestly small quark masses. Bernard et al. did a more systematic study; they calculate for a number of modestly small quark masses and extrap-
olate back to zero and obtain a nonzero value. In contrast, the ordinary $SU(2) \times SU(2)$ chiral symmetry breaking susceptibility (the pion–sigma channels) goes nicely to zero.

To some this may seem to settle the question; but to quote Mark Twain, “It’s differences of opinion which make horse races.” There are several reasons to doubt the lattice studies. The first is the obvious one; the calculation involves a small number from an extrapolation of quantities with unknown and potentially large systematic errors due to finite masses, lattice spacings sizes etc. Moreover at least one class of these systematic errors (finite quark masses compounded by finite lattice spacing) can easily account for a spurious $U(1)_A$ violating amplitude without affecting ordinary chiral symmetry. The reason is quite simple: due to the fermion doubling problem, viable lattice schemes in general violate axial symmetries in a spurious way. There are different schemes for dealing with fermion doubling which all have the general feature that the spurious axial symmetry violating effects vanish in the continuum limit of $a \to 0$. The calculations by Chandrasekharan and Christ and Bernard et al. used staggered fermions. These have the property that even at finite lattice spacing one axial symmetry is conserved (for $m_q = 0$). This conserved axial current is the one which is conventionally associated with the pion channel. On the other hand, the $U(1)_A$ symmetry is explicitly broken by the lattice formulation and is only restored in the continuum limit. Thus, for example, even in the case of noninteracting massless fermions in this lattice formulation one would see $U(1)_A$ violations until $a$ is sent to zero.

Accordingly it would be very useful to study the anatomy of these calculations to see whether the results are coming from lattice artifacts. As was discussed in the introduction, $\chi_{U(1)_A}$ is completely determined by $\rho(\lambda)$ near zero. Thus if one could measure $\rho(\lambda)$ for the lattice configurations contributing, one could see from where the $U(1)_A$ violating amplitudes get their strength. In fact, as we will see, the $U(1)_A$ violations above $T_c$ in ref. 11 get their strength in a manner which appears to be inconsistent with chiral restoration suggesting the result may well be due to lattice artifacts.

While a direct lattice calculation of $\rho(\lambda)$ is what we want, there are presently no available such calculations. This is not surprising as it would be a numerical nightmare to calculate. Fortunately an integral transform of $\rho(\lambda)$ has been calculated by Chandrasekharan and Christ. In particular they calculate

$$f(m_\xi) = \int d\lambda \rho(\lambda) \frac{1}{i\lambda + m_\xi}$$

This is actually a relatively easy calculation to do on the lattice since it amounts to calculating $\langle \bar{q}q \rangle$ with a quark mass of $m_\xi$ in the propagator instead of $m_q$. Hence standard codes can be used. Since the quark mass in the propagator can be made much less than that in the functional determinant (which is implicitly contained in $\rho$) one can call such a calculation partially quenched. Note that by construction, $f(m_q) = \langle \bar{q}q \rangle$.

Chandrasekharan and Christ calculated $f(m_\xi)$ for various choices of coupling constants which correspond to different values of the temperature. (Let me remind you that by changing the coupling constant with fixed physical observables one is effectively changing the lattice spacing. Since the number of steps in the temporal direction is fixed, this in turn changes the periodicity of the lattice in time and hence the temperature.) The important thing for our purposes is that for temperatures above $T_c$ and small $m_\xi$, the function is
apparently linear on a log-log plot over several decades implying that \( f(m_\xi) \sim m_\xi^\alpha \) for sufficiently small \( \lambda \). Moreover it is found that in all cases \( \alpha < 1 \).

Now it is easy to see that this behavior implies

\[
\rho(\lambda) = c|\lambda|^\alpha,
\]

where \( c \) is a constant, and this form applies at sufficient small \( \lambda \). The simplest way to see this is to put this form into eq. (12) and evaluate the integral. One finds that \( \rho \sim |\lambda|^\alpha \) implying \( f \sim m_\xi^\alpha \). Indeed this is seen easily by dimensional analysis. In the following section, I will show that there is a constraint imposed by chiral restoration which shows that a form \( \rho \sim |\lambda|^\alpha \) with \( \alpha < 1 \) cannot occur. This in turn raises questions as to whether the lattice results are dominated by artifacts.

### III. CONSTRAINTS

Recall that we are studying the symmetry restored phase. In a chiral symmetric phase one knows that the log of the partition function is an even analytic function of the quark mass near zero. The reason for this is quite easy to understand. Look at the \( n^{th} \) derivative of \( \log(Z) \) with respect to the quark masses:

\[
\frac{\partial^n \log(Z)}{\partial m_q^n} \bigg|_{m_q=0} = \frac{1}{V} \langle (d^4 x \overline{q}q)^n \rangle.
\]

In the chiral restored phase this quantity should not change under a chiral rotation. Under a chiral rotation through \( \pi \), \( \overline{q}q \rightarrow -\overline{q}q \), thus the preceding expression must vanish for all odd \( n \); in general, for even \( n \) we expect nonzero finite values implying that \( \log(Z) \) is an even analytic function of \( m_q \).

We should recall that the quark condensate is given by

\[
\langle \overline{q}q \rangle = \frac{1}{Vol} \frac{\partial \log(Z)}{\partial m_q},
\]

and that from the Banks-Casher analysis,

\[
\langle \overline{q}q \rangle = \int d\lambda \rho(\lambda) \frac{m_q}{\lambda^2 + m_q^2}.
\]

Combining these we see that

\[
\lim_{m_q \rightarrow 0} \frac{1}{Vol} \frac{\partial^n \log(Z)}{\partial m_q^n} = \frac{\partial^{n-1}}{\partial m_q^{n-1}} \int d\lambda \rho(\lambda) \frac{m_q}{\lambda^2 + m_q^2}.
\]

Recall that this must be zero for all odd \( n \).

Before proceeding with a careful analysis, let me begin with a small swindle. For your convenience, I will identify the swindle at the outset: assume that in eq. (13) the quark mass derivative acts only on \( m_q/\lambda^2 + m_q^2 \) and not on \( \rho(\lambda) \). (I should remind you that \( \rho \) depends implicitly on \( m_q \) through the functional determinant.) Diagrammatically, this corresponds
to including only quark-line–connected diagrams. I will label any quantity calculated at this level with the superscript $qlc$. From eq. (16) we see that

$$\lim_{m_q \to 0} \left. \frac{\partial^n \log(Z)}{\partial m_q^n} \right|_{qlc}^\Delta = \lim_{m_q \to 0} \int d\lambda \rho(\lambda) \frac{\partial^{n-1} m_q}{\partial \lambda^{n-1}} \frac{m_q}{\lambda^2 + m_q^2} = - \int d\lambda \rho(\lambda) \frac{i}{\lambda^n}. \quad (17)$$

Now if $\rho(\lambda) \sim |\lambda|^\alpha$ for small $\lambda$, then from eq. (17) one sees that the integral diverges for any $n > \alpha$. On the other hand if $\log(Z)$ is an even analytic function this is impossible. We conclude from this analysis the following: if only quark-line–connected diagrams contribute, then $\rho$ (for $m_q = 0$) is infinitely flat in $\lambda$—it goes to zero faster than any power law. This is what you would expect if $\rho$ had a gap at zero. By a gap at zero, I simply mean that $\rho(\lambda) = 0$ unless $\lambda$ is greater than some minimum. Alternatively, one could imagine a situation in which $\rho$ had an essential singularity at $\lambda = 0$, such as $\rho \sim e^{-1/\lambda}$. In any case, $\rho$ is infinitely flat at the origin.

My guess is that the preceding description is correct—that above $T_c$, $\rho$ does, in fact, have a gap. I cannot prove it however. The problem is that the preceding analysis only included the quark-line–connected diagrams. It is possible in principle that the quark-line–disconnected parts, coming from derivatives of $m_q$ acting on $\rho(\lambda)$, conspire to cancel the divergent quark-line–connected part. One can prove, however, in the chiral restored phase, that for $m_q = 0$, $\rho$ must go to zero faster than linearly: $\left. \frac{\partial \rho}{\partial \lambda} \right|_{\lambda=0} = 0$.

The proof of this constraint is actually quite simple. Consider the susceptibility associated with the pion channel,

$$\chi_\pi = \int d^4x \langle j_\pi(x)j_\pi(0) \rangle, \quad (18)$$

with $j_\pi = \bar{q}\gamma_5\tau q$. By standard analysis of the propagator analogous to the derivation of the Banks-Casher result discussed in the introduction, one finds:

$$\chi_\pi = \int d\lambda \frac{\rho(\lambda)}{\lambda^2 + m_q^2} = \frac{\langle \bar{q}q \rangle}{m_q}. \quad (19)$$

This last form is just a functional integral derivation of the Ward identity implicit in the Gell-Mann–Oakes-Renner relation. Now suppose that for small $\lambda$, $\rho = c|\lambda|^\alpha$ with $\alpha \leq 1$, and $c$ is a constant of proportionality. It is a trivial exercise to evaluate the integral in definition of $\chi_\pi$ using this form, and one finds that

$$\chi_\pi \sim cm_q^{\alpha-1}. \quad (20)$$

Let us suppose for the moment that the form $\rho = c|\lambda|^\alpha$ survives in the chiral limit of $m_q \to 0$ so that $c$ is finite in this limit. Evaluating the integral in eq. (20) for this case gives for small $m_q$

$$\chi_\pi = c \frac{\pi}{m_q^{1-\alpha} \cos(\alpha \pi/2)} \quad (21)$$

which diverges in the chiral limit $m_q \to 0$, provided $\alpha \leq 1$. (For $\alpha = 1$ it diverges as $\log(m_q)$.) Moreover, if as hypothesized, one were studying the chiral restored phase then as $m_q \to 0$ the susceptibilities in the pion and $\sigma$ channels must be equal. On the other hand
\[
\chi_\sigma = \frac{1}{\text{Vol}} \frac{\partial^n \log(Z)}{\partial m_q^n}.
\]

(22)

Thus, if \(\chi_\pi\) (and hence \(\chi_\sigma\)) diverge in the chiral limit we see that \(\log(Z)\) is not an analytic function of \(m_q\) in contradiction to the system being in a chiral restored phase.

¿From the preceding analysis one can deduce that in the chiral limit of \(m_q \to 0\) and the chirally restored phase, \(\rho(\lambda)\) cannot go as \(c|\lambda|^\alpha\) for \(\alpha \leq 1\). This appears to be inconsistent with the lattice results of Chandrasekharan and Christ. Of course, the preceding analysis was done for the chiral limit of \(m_q \to 0\), while the lattice studies were done with a fixed quark mass. This raises the following possibility. Suppose the constant \(c\) used above were not a constant but depended on the quark mass in such a way that it went to zero as \(m_q \to 0\). Would such a scenario allow a \(U(1)_A\) violating susceptibility while still being constant with the observation that \(f(m_\xi) \sim m_\xi^\beta\)? At first sight it appears to be easy to do this: \(f(\xi) \sim m_q^{1-\alpha} m_\xi^\beta\), i.e. the constant \(c\) could be proportional to \(m_q\). This possibility is consistent with the lattice data in Chandrasekharan and Christ. Since they do not do the calculation for multiple values of \(m_q\) at fixed \(m_\xi\), it is impossible to tell if this behavior is present. If \(f(m_\xi)\) behaves this way it is clear that for small \(\lambda\) and \(m_q\),

\[
\rho(\lambda) = b m_q^{1-\alpha} |\lambda|^\alpha
\]

(23)

with \(b\) a constant. If \(\rho\) is given by eq. (23) it is a simple matter to verify that \(\chi_\pi\) and \(\chi_\sigma\) are finite and equal and that \(\chi_{U(1)_A} = (1 - \alpha)\chi_\pi\).

The preceding scenario appears to show a way to reconcile the lattice data with chiral restoration and \(U(1)_A\) violation. In fact, this scenario is inconsistent, unless there is spontaneous symmetry breaking of isospin and is therefore not physically realized. This leaves an inconsistency which suggests that the lattice results may be dominated by unphysical lattice artifacts. The issue of isospin has up until now been ignored. Implicitly I have taken the mass of the two light quarks to be degenerate. Assuming isospin is not spontaneously broken, this is not an issue since at the end of the problem both the up and down quark masses are taken to zero. If isospin is spontaneously broken, the order in which the limit is taken can matter, but if it is not spontaneously broken, then the ordering is irrelevant and we might just as well take the masses equal. The problem with the form in eq. (23) is that it depends on the quantity \(m_q^{1-\alpha}\). If one generalizes to \(m_u \neq m_d\) one must generalize \(m_q^{1-\alpha}\) into some function of \(m_u\) and \(m_d\) which goes to \(m_q^{1-\alpha}\) when \(m_u = m_d = m_q\). Consider the following quantity:

\[
\chi_{ud} = \int d^4 x \langle \bar{u}(x) u(0) \rangle
\]

(24)

\[
\chi_{ud} = \frac{1}{\text{Vol}} \frac{\partial^2 \log(Z)}{\partial m_u \partial m_d}
\]

Plugging in \(\rho\) from eq. (23) into a generalization of eq. (16), for the case where \(m_u\) and \(m_d\) are varied independently, and evaluating the integral yields:

\[
\chi_{ud} = \frac{\partial m_q(m_u, m_d)^{1-\alpha}}{\partial m_d} \frac{b \pi m_u^\alpha}{\cos(\alpha \pi/2)}.
\]

(25)
If isospin and chiral symmetry are both unbroken then the result for $\chi_{ud}$ will be finite as $m_u, m_d \to 0$ and is independent of how the quark masses go to zero. Let us assume this to be true and show that it leads to a contradiction. On dimensional grounds we can always write

$$\frac{\partial m_q^{1-\alpha}}{\partial m_d} = (1 - \alpha)m_u^{-\alpha} g(m_u/m_d)$$

(26)

where $g$ is some presently unknown function. Inserting this form into eq.(25) gives

$$\chi_{ud} = g(m_u/m_d) \frac{b \pi m_u^{\alpha}}{\cos(\alpha \pi/2)}$$

(27)

By hypothesis, isospin and chiral symmetry are not spontaneously broken so that the value of $\chi_{ud}$ must be the same independently of how the chiral limit is approached. This in turn means that the value of $\chi_{u,d}$ has to be independent of the ratio $m_u/m_d$ since we can approach the chiral limit with this ratio fixed to any value we wish. From this we deduce that consistency requires $g = 1$ and $m_q^{1-\alpha} = m_d m_u^{-\alpha} \alpha$ which in turn implies

$$\rho(\lambda) = b m_d m_u^{-\alpha} |\lambda|^\alpha$$

(28)

At this point, however, we see a contradiction: the preceding expression is not invariant under $m_u \leftrightarrow m_d$; on the other hand since theory is isospin symmetric up to the values of the quark masses the spectral density must be invariant under the switching of the up and down quark masses. Thus we conclude that the scenario in which $\rho(\lambda) \sim m_q^{1-\alpha} |\lambda|^\alpha$ is not consistent.

**IV. SUMMARY**

Let me summarize the situation: I have discussed the concept of the spectral density near $\lambda = 0$, and argued that it was interesting and an important theoretical tool for understanding both spontaneous chiral symmetry breaking and the spontaneous/anomalous breaking of $U(1)_A$. One problem of physical interest that this tool may prove useful for is the question of whether the effects of $U(1)_A$ axial symmetry breaking are manifest above the chiral phase transition. Although present lattice simulations seem to indicate that $U(1)_A$ violating effects survive above the transition, it is possible that the calculations are dominated by lattice artifacts. One way to see if this is so, is to test whether the spectral densities implied by the lattice calculation are consistent with constraints imposed on the spectral density by chiral symmetry and isospin in the unbroken phase.

There are a number of such constraints: first, one sees that unless quark-line–disconnected graphs conspire to cancel the quark-line–connected ones, in the chiral limit, $\rho(\lambda)$ is infinitely flat in the sense of having all derivatives vanish as one would expect from a spectrum with a gap. Second, in the chiral limit, $\frac{\partial \rho}{\partial m_q} = 0$, indicating a behavior such as $\rho(\lambda, m_q = 0) \sim |\lambda|^\alpha$, is not possible for $\alpha \leq 1$. Finally it is shown that for small $\lambda$ and $m_q$ a behavior of the form $\rho(\lambda, m_q = 0) \sim m_q^{1-\alpha} |\lambda|^\alpha$ is not possible for $\alpha \leq 1$.

Given these constraints it is reasonable to ask whether the lattice calculation of Chandrasekharan and Christ, which have $\rho$ behaving like a power law with $\alpha < 1$ (and hence
in apparent violation of the constraint) are dominated by lattice artifacts. Since the lattice calculations of ref. 12 do not study the spectral density near zero it is impossible to know whether they are consistent with the constraints derived above. However, as these calculations were qualitatively similar to those in ref. 11 it is reasonable to question whether the spectral functions are consistent with being in a chirally restored phase. I conclude with a word of warning to potential consumers of lattice calculations for these problems: Let the buyer beware.

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