SPECTRAL EXCISION AND DESCENT FOR ALMOST PERFECT COMPLEXES

CHANG-YEON CHOUGH

Abstract. We show that almost perfect complexes of commutative ring spectra satisfy excision and $v$-descent. These results generalize Milnor excision for perfect complexes of ordinary commutative rings and $v$-descent for almost perfect complexes of locally noetherian derived stacks by Halpern-Leistner and Preygel, respectively.

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1. Introduction

1.1. Suppose we are given a square of associative rings $\sigma$:

\[
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow & & \downarrow \\
B & \rightarrow & B'
\end{array}
\]

for which the map $B \rightarrow B'$ is surjective. According to Milnor (see [11, §2]), the image of $\sigma$ under the functor which assigns to each associative ring $R$ the category of finitely generated projective $R$-modules is a pullback diagram of categories. To work in the more general context of structured ring spectra, recall that if $R$ is a commutative ring, then a chain complex of $R$-modules $M$ (viewed as an object of the derived category of $R$-modules) is perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules (see, for example, [13, Tag 0657]). More generally, we say that $M$ is pseudo-coherent if it is quasi-isomorphic to a bounded above complex of finitely generated free $R$-modules (see, for example, [11, p.79]). Suppose now that $R$ is an $\mathbb{E}_\infty$-ring in the sense of [8, 7.1.0.1] and let $\text{Mod}_R$ denote the $\infty$-category of $R$-modules (see [8, 7.1.1.2]). Then the notions of perfect and pseudo-coherent modules over commutative rings can be generalized to the setting of $\mathbb{E}_\infty$-rings, and

\[
\begin{array}{ccc}
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\end{array}
\]
we obtain the notions of *perfect* and *almost perfect* modules over \( R \), respectively (see [8, 7.2.4.1] and [8, 7.2.4.10]). We will denote by \( \text{Perf}(R) \) and \( \text{APerf}(R) \) the full subcategories of \( \text{Mod}_R \) spanned by the perfect and almost perfect \( R \)-modules, respectively. The constructions \( R \mapsto \text{Perf}(R) \), \( \text{APerf}(R) \) determine functors \( \text{Perf} : \text{CAlg}^{cn} \to \text{Cat}_\infty \), where \( \text{CAlg}^{cn} \) and \( \text{Cat}_\infty \) denote the \( \infty \)-category of connective \( \mathbb{E}_\infty \)-rings and the \( \infty \)-category of \( \infty \)-categories, respectively (see [8, p.1201] and [6, 3.0.0.1]).

1.2. One of the main results in this paper is the following analogue of Milnor’s result in the setting of \( \mathbb{E}_\infty \)-rings:

**Theorem 1.3.** Suppose we are given a pullback square of connective \( \mathbb{E}_\infty \)-rings \( \sigma : \)

\[
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow & & \downarrow \\
B & \rightarrow & B'.
\end{array}
\]

If the induced map \( \pi_0 B \rightarrow \pi_0 B' \) is a surjection of commutative rings, then the diagram of \( \infty \)-categories \( \text{APerf}(\sigma) : \)

\[
\begin{array}{ccc}
\text{APerf}(A) & \rightarrow & \text{APerf}(A') \\
\downarrow & & \downarrow \\
\text{APerf}(B) & \rightarrow & \text{APerf}(B').
\end{array}
\]

determined by the extension of scalars functors is a pullback square in the \( \infty \)-category \( \text{Cat}_\infty \).

**Remark 1.4.** In our proof of 1.3 we will make use of the notion of a *universal descent morphism* of \( \mathbb{E}_\infty \)-rings in the sense of [7, D.3.1.1], which was introduced originally by Akhil Mathew in [9, 3.18]. We note that the notion of universal descent morphisms makes sense more generally for \( \mathbb{E}_2 \)-rings. However, it is in the commutative setting that the class of universal descent morphisms has the descent property of [7, D.3.5.8], which will play an important role in our discussion of almost perfect complexes. For this reason, we focus our attention to the case of \( \mathbb{E}_\infty \)-rings (unlike Milnor’s excision, which works for associative rings).

**Remark 1.5.** Under the additional assumption that the underlying map of commutative rings \( \pi_0 A' \rightarrow \pi_0 B' \) is surjective, \( \text{APerf}(\sigma) \) is already known to be a pullback diagram by virtue of [7, 16.2.0.2] and [7, 16.2.3.1].

**Remark 1.6.** In the special case where \( \sigma \) is a pullback square of connective \( \mathbb{E}_\infty \)-rings for which the map \( \pi_0 B \rightarrow \pi_0 B' \) is a surjection whose kernel is a nilpotent ideal of \( \pi_0 B \), 1.3 can be deduced from [7, 16.2.0.2] and [7, 16.2.3.1].

**Remark 1.7.** By restricting the equivalence of 1.3 to the full subcategories spanned by the dualizable objects (see the proof of 2.15), we immediately deduce that the canonical map \( \text{Perf}(A) \to \text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B) \) is an equivalence of \( \infty \)-categories. We remark that this equivalence is proven in [9, 2.23] under the additional assumption that the map \( \pi_0 A' \rightarrow \pi_0 B' \) is a surjection.

**Remark 1.8.** According to [4, 1.17] of Markus Land and Georg Tamme, \( \text{Perf}(\sigma) \) is a pullback diagram of \( \infty \)-categories if \( \sigma \) is a pullback square of \( \mathbb{E}_1 \)-rings for which the functor \( \text{Perf}(B \odot A') \rightarrow \text{Perf}(B') \) is conservative (see [4, 1.3] for more details). In the case of connective \( \mathbb{E}_\infty \)-rings, the condition appearing in [4, 1.17] is satisfied if the map \( \pi_0 B \rightarrow \pi_0 B' \) is surjective.
commutative rings
for which the induced map $cn\text{Grothendieck topology on (CAlg}\subseteq\text{descent. In particular, the functor Perf : CAlg}$

Remark 1.14. see 3.8.

universal) descent for the map $f\subseteq A$ remove the noetherian assumption on

3.3.6] of Halpern-Leistner and Preygel shows that the functor $\text{Mod}^\text{acn}$

Remark 1.13.

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Remark 1.12.

$\pi\subseteq A$ the induced map $\pi\subseteq A$ for the morphism

Theorem 1.11. Let $f : A \subseteq B$ be a $v$-cover of connective $\mathbb{E}_\infty$-rings for which the underlying

map of commutative rings $\pi_0f : \pi_0A \subseteq \pi_0B$ is of finite presentation. Then $f$ is of universal

A$Perf-descent: that is, for every morphism $A \subseteq A'$ in the $\infty$-category $\text{CAlg}^\text{acn}$ of connective

$\mathbb{E}_\infty$-rings, the induced map $\text{APerf}(A') \subseteq \text{lim } \text{APerf}(B'^*)$ is an equivalence of $\infty$-categories,

where $B'^*$ denotes the Čech nerve of the map $A' \subseteq A' \otimes_A B$ (formed in the opposite of the

$\infty$-category $\text{CAlg}^\text{acn}$).

Remark 1.12. If $f : A \subseteq B$ is a morphism of noetherian simplicial commutative rings for

which the induced map $\pi_0f$ is a finitely presented $v$-cover of ordinary commutative rings, it then follows from [3, 3.3.1] of Halpern-Leistner and Preygel that almost perfect complexes satisfy (not necessarily universal) descent for the morphism $f$; see 2.13. Our descent result

1.11 can be regarded as a generalization of their work (in the affine case): it holds without the locally noetherian assumption appearing in [3, 3.3.1] and is valid more generally for connective

$\mathbb{E}_\infty$-rings, rather than merely for simplicial commutative rings. Moreover, the A$Perf$-descent

for the morphism $f$ is universal (that is, A$Perf$ satisfies descent for arbitrary base change of

the map $f$).

Remark 1.13. Let $f : A \subseteq B$ be a morphism of noetherian connective $\mathbb{E}_\infty$-rings for which the induced map $\pi_0f$ is a finitely presented $v$-cover of ordinary commutative rings. Then [3 3.3.6] of Halpern-Leistner and Preygel shows that the functor $\text{Mod}^\text{acn}$ satisfies (not necessarily universal) descent for the map $f$ (see 2.14). Using a slight variant of the proof of 1.11 we can remove the noetherian assumption on $A$ and $B$ to show that $f$ is of universal $\text{Mod}^\text{acn}$-descent; see 3.8.

Remark 1.14. As a consequence of 1.11 we will see in 3.9 that $f$ is also of universal $\text{Perf}$-

descent. In particular, the functor $\text{Perf} : \text{CAlg}^\text{acn} \subseteq \text{Cat}_\infty$ satisfies descent with respect to the

Grothendieck topology on $(\text{CAlg}^\text{acn})^\text{op}$ which is characterized by the following property: a sieve $\mathcal{E} \subseteq (\text{CAlg}^\text{acn})^\text{op}$ is a covering if and only if it contains a finite collection of maps $\{A \subseteq A_i\}_{1 \leq i \leq n}$ for which the induced map $f : A \subseteq \prod A_i$ is a $v$-cover such that the underlying map of commutative rings $\pi_0f$ exhibits $\pi_0(\prod A_i)$ as a finitely presented $\pi_0A$-algebra. In contrast
with [2, 11.2] for ordinary perfect schemes, the sheaf Perf is not hypercomplete (that is, it is not true in general that if $U_\bullet : \Delta^{op} \to (\text{CAlg}^{cn})^{op}$ is a hypercovering with respect to the above Grothendieck topology in the sense of [7, A.5.7.1], then the composition $\Delta \xrightarrow{U_\bullet} \text{CAlg}^{cn} \xrightarrow{\text{Perf}} \text{Cat}_\infty$ is a limit diagram). To see this, we note that for every connective $\mathbb{E}_\infty$-ring $R$, the truncation map $R \to \pi_0 R$ is a $v$-cover, so that the constant cosimplicial $\mathbb{E}_\infty$-ring with value $\pi_0 R$ is a hypercovering of $R$ with respect to the above Grothendieck topology. Consequently, if the functor Perf is hypercomplete, then the natural map $\text{Perf}(R) \to \text{Perf}(\pi_0 R)$ is an equivalence of $\infty$-categories, which is false in general.

Remark 1.15. Fix integers $a \leq b$. Since the functor Perf is a sheaf for the Grothendieck topology appearing in 1.14 by virtue of 3.9, so is the subfunctor Perf$^{[a,b]}$ which assigns to each connective $\mathbb{E}_\infty$-ring $R$ the full subcategory Perf$^{[a,b]}(R) \subseteq \text{Perf}(R)$ spanned by those perfect complexes whose Tor-amplitude is contained in $[a, b]$; see, for example, [8, 7.2.4.21]. As Akhil Mathew pointed out, the functor Perf$^{[a,b]} : \text{CAlg}^{cn} \to \text{Cat}_\infty$ does not take values in the $\infty$-category of $(b - a + 1)$-categories in the sense of [6, 2.3.4.1], unlike in the case of ordinary perfect schemes of [2, 11.2]. Consequently, the fact that Perf$^{[a,b]}$ is a sheaf does not guarantee that it is a hypercomplete sheaf (see [6, 6.5.2.9]). In fact, for a connective $\mathbb{E}_\infty$-ring $R$, the canonical map Perf$^{[a,b]}(R) \to \text{Perf}^{[a,b]}(\pi_0 R)$ is not an equivalence in general, so that Perf$^{[a,b]}$ is not hypercomplete.

1.16. Conventions. We will make use of the theory of $\infty$-categories and the theory of spectral algebraic geometry developed in [6], [8], and [7].

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2. The $v$-covers and universal descent morphisms of connective $\mathbb{E}_\infty$-rings

2.1. Let $f : X \to Y$ be a morphism of quasi-compact and quasi-separated schemes. Recall from [2, 2.1] (see also [12, 2.9]) that $f$ is said to be a $v$-cover if, for every valuation ring $V$ and every morphism of schemes $\text{Spec } V \to Y$, there exist an extension of valuation rings $V \to W$ (that is, an injective local homomorphism) and a morphism of schemes $\text{Spec } W \to X$, which fit into a commutative diagram

$$
\begin{array}{ccc}
\text{Spec } W & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\text{Spec } V & \xrightarrow{f} & Y.
\end{array}
$$

In the spectral setting, we define the $v$-cover of connective $\mathbb{E}_\infty$-rings as follows:

Definition 2.2. Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings (see [8, 7.1.0.1]). We will say that $f$ is a $v$-cover if the induced morphism $\text{Spec}(\pi_0 f) : \text{Spec}(\pi_0 B) \to \text{Spec}(\pi_0 A)$ is a $v$-cover of ordinary schemes in the sense of 2.1.

Remark 2.3. Alternatively, we can define a $v$-cover of affine spectral Deligne-Mumford stacks $\text{Spec}(f) : \text{Spec } B \to \text{Spec } A$ as in 2.1 by virtue of the universal property of 0-truncations of spectral Deligne-Mumford stacks (see [7, 1.4.6.3]).
2.4. We now summarize some of the formal properties of \[2.2\] (which follow immediately from the case of ordinary schemes and \[8\] 7.2.1.23):

**Lemma 2.5.**

(i) The collection of \( v \)-covers of connective \( E_\infty \)-rings contains all equivalences and is stable under composition.

(ii) Suppose we are given a pushout diagram of connective \( E_\infty \)-rings

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{f'} & B'.
\end{array}
\]

If \( f \) is a \( v \)-cover, then so is \( f' \).

2.6. The notion of \( v \)-covers appearing in \[2.2\] is closely related to the notion of a *universal descent morphism* of \( E_\infty \)-rings (which makes sense more generally for \( E_2 \)-rings): recall from \[7\] D.3.1.1 that a morphism of \( E_\infty \)-rings \( f : A \to B \) is said to be a *universal descent morphism* if the smallest stable subcategory of \( \text{LMod}_A \) which is closed under retracts and contains all \( A \)-modules of the form \( N \otimes_A B \), where \( N \) is a left \( A \)-module, coincides with \( \text{LMod}_A \). We note that for a finitely presented map \( f : A \to B \) of noetherian commutative rings, the map \( f \) (regarded as a morphism of discrete \( E_\infty \)-rings) is a universal descent morphism if and only if \( \text{Spec}(f) : \text{Spec} B \to \text{Spec} A \) is a \( v \)-cover of ordinary schemes; see \[2\] 11.26.

**Remark 2.7.** The class of universal descent morphisms, which will play a central role in our proofs of \[1.3\] and \[1.11\] allows us to do descent theory in the spectral setting. In particular, if \( f : A \to B \) is a universal descent morphism of \( E_\infty \)-rings, it then follows from \[7\] D.3.5.8 that the canonical map \( \text{Mod}_A \to \lim \text{Mod}_B \) is an equivalence of symmetric monoidal \( \infty \)-categories, where \( B^* \) denotes the Čech nerve of \( f \) formed in the \( \infty \)-category \( (\text{CAlg}^{cn})^{\text{op}} \).

**Remark 2.8.** Let \( f : A \to B \) be a morphism of \( E_\infty \)-rings. Let \( B^* \) denote the Čech nerve of \( f \) (so that \( B^n \) is given by the \((n+1)\)-fold tensor product \( B \otimes_A \cdots \otimes_A B \) for each \( n \geq 0 \)). We denote by \( \text{Tot}^*(B/A) \) the Tot-tower of \( B^* \): that is, the sequence of \( A \)-modules

\[
\cdots \text{Tot}^2(B/A) \to \text{Tot}^1(B/A) \to \text{Tot}^0(B/A).
\]

Here \( \text{Tot}^n(B/A) \) denotes the limit of the diagram \( \{B^n\} \) taken over the full subcategory \( \Delta_{\leq n} \subseteq \Delta \) spanned by those objects \([m] \in \Delta \) such that \( m \leq n \). We note that if \( f : A \to B \) is a universal descent morphism, then \( A \) is a retract of \( \text{Tot}^n(B/A) \) for some integer \( n \geq 0 \) (see the proof of \[7\] D.3.2.1). We will use this fact to deduce some descending properties of universal descent morphisms; see \[2.11\].

2.9. Let \( R \) be a connective \( E_\infty \)-ring. Recall that an \( R \)-module \( M \) is said to be *almost connective* if it is \((-m)\)-connective for some integer \( m \gg 0 \) (see \[8\] p.1201). We will let \( \text{Mod}^{\text{cn}}_R \subseteq \text{Mod}_R \) denote the full subcategory spanned by the almost connective \( R \)-modules. We note that the construction \( R \mapsto \text{Mod}^{\text{cn}}_R \) determines a functor \( \text{Mod}^{\text{cn}} : \text{CAlg}^{\text{cn}} \to \text{Cat}_{\infty} \). According to \[8\] 7.2.4.10, an \( R \)-module \( M \) is *almost perfect* if it is \( m \)-connective for some integer \( m \) and is an almost compact object of the \( \infty \)-category \( \text{Mod}^{\text{cn}}_R \) of \( m \)-connective \( R \)-modules (that is, for every integer \( n \geq 0 \), \( \tau_{\leq n}M \) is compact as an object of \( \tau_{\leq n} \text{Mod}^{\text{cn}}_R \)). We will say that an \( R \)-module \( M \) is *perfect to order \( n \)* if, for every filtered diagram \( \{N_\alpha\} \)
of 0-truncated $R$-modules, the canonical map $\colim \text{Ext}^i_A(M, N_\alpha) \to \text{Ext}^i_A(M, \colim N_\alpha)$ is bijective for $i < n$ and injective for $i = n$; see [7 2.7.0.1]. We remark that an $R$-module $M$ is almost perfect if and only if it is perfect to order $n$ for each integer $n$ (see [7 2.7.0.2]).

2.10. One of the main ingredients in our proofs of [13.3 and 11.1] is the following descent result:

**Proposition 2.11.** Let $f : A \to B$ be a universal descent morphism of connective $\mathbb{E}_\infty$-rings, let $M$ be an $A$-module, and let $n$ be an integer. Then:

(i) The $A$-module $M$ is $n$-connective if and only if the $B$-module $M \otimes_A B$ is $n$-connective.

(ii) The $A$-module $M$ is almost connective if and only if the $B$-module $M \otimes_A B$ is almost connective.

(iii) The $A$-module $M$ is perfect to order $n$ if and only if the $B$-module $M \otimes_A B$ is perfect to order $n$.

(iv) The $A$-module $M$ is almost perfect if and only if the $B$-module $M \otimes_A B$ is almost perfect.

**Proof.** Assertions (ii) and (iv) follow immediately from (i) and (iii), respectively. The “only if” directions of assertions (i) and (iii) follow from [8 7.2.1.23] and [7 2.7.3.1], respectively. To prove “if” directions, we note that since $f$ is a universal descent morphism, it follows from [7 D.3.2.1] that $A$ is a retract of $\text{Tot}^m(B/A)$ for some $m \geq 0$. To prove the “if” direction of (i), assume that $M \otimes_A B$ is $n$-connective. In particular, $M \otimes_A \text{Tot}^m(B/A)$ is $n$-connective when viewed as a $B$-module. Since $M$ is a retract of the connective $A$-module $M \otimes_A \text{Tot}^m(B/A)$, $M$ is $n$-connective as desired. To complete the proof of (iii), let $\{N_\alpha\}$ be a filtered diagram of 0-truncated $A$-modules. We wish to show that the canonical map $\phi_i : \colim \text{Ext}^i_A(M, N_\alpha) \to \text{Ext}^i_A(M, \colim N_\alpha)$ is bijective for $i < n$ and injective for $i = n$. For this, it will suffice to show that $\phi_i$ is a retract of a bijection for $i < n$ and an injection for $i = n$. Since $A$ is a retract of $\text{Tot}^m(B/A)$, we deduce that $N_\alpha$ is a retract of $\tau_{\leq 0}(N_\alpha \otimes_A \text{Tot}^m(B/A))$. We are therefore reduced to proving that the horizontal map in the diagram

$$
\begin{array}{ccc}
\colim \text{Ext}^i_A(M, \tau_{\leq 0}(N_\alpha \otimes_A \text{Tot}^m(B/A))) & \longrightarrow & \text{Ext}^i_A(M, \colim \tau_{\leq 0}(N_\alpha \otimes_A \text{Tot}^m(B/A))) \\
\downarrow & & \downarrow \\
\colim \text{Ext}^i_B(M \otimes_A B, \tau_{\leq 0}(N_\alpha \otimes_A \text{Tot}^m(B/A))) & \longrightarrow & \text{Ext}^i_B(M \otimes_A B, \colim \tau_{\leq 0}(N_\alpha \otimes_A \text{Tot}^m(B/A)))
\end{array}
$$

is bijective for $i < n$ and injective for $i = n$, where we regard $N_\alpha \otimes_A \text{Tot}^m(B/A)$ as a $B$-module. Since the forgetful functor $\text{Mod}_B \to \text{Mod}_A$ commutes with the formation of 0-truncations and preserves small colimits (see [8 4.2.3.7]), the vertical maps are equivalences. Consequently, the desired result follows from our assumption that $M \otimes_A B$ is perfect to order $n$ as a module over $B$. \square

2.12. Before stating our next result which will be needed in our proof of [11.1], we recall a bit of terminology (see [5 3.1.1]):

**Definition 2.13.** Let $\mathcal{C}$ be an $\infty$-category which admits fiber products, and let $\mathcal{D}$ be an $\infty$-category. Let $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ be a functor. We will say that a morphism $f : C_0 \to C$ in $\mathcal{C}$ is of $F$-descent if the composition $\Delta_+ \overset{C_0}{\longrightarrow} \mathcal{C}^{\text{op}} \overset{F}{\longrightarrow} \mathcal{D}$ is a limit diagram, where $C_\bullet$ denotes the Čech nerve of $f$, regarded as an augmented simplicial object of $\mathcal{C}$ with $C_{-1} \simeq C$ (see [6 p.543]). That is, the canonical map $F(C) \to \lim_{[n] \in \Delta^{\text{op}}} F(C_n)$ is an equivalence in $\mathcal{D}$. In this case, we will also say that $F$ satisfies descent for $f$. If this condition holds for every base
change $C'' \times_C C_0 \to C'$ of $f$ along a morphism $C' \to C$, we will say that $f$ is of universal $F$-descent, or that $F$ satisfies universal descent for $f$.

**Theorem 2.14.** Let $f : A \to B$ be a universal descent morphism of connective $\mathbb{E}_\infty$-rings. Then the map $f$ is of universal $\text{Mod}^{\text{acn}}$-descent (see §2.9) and of universal $\text{APerf}$-descent.

*Proof.* The property of being a universal descent morphism of $\mathbb{E}_\infty$-rings is stable under pushouts (see [7, D.3.1.6]), so it will suffice to show that $f$ is of $\text{Mod}^{\text{acn}}$-descent and of $\text{APerf}$-descent. Let $B^\bullet$ denote the Čech nerve of $f$, so that the canonical map $\text{Mod}_A \to \lim \text{Mod}_{B^\bullet}$ is an equivalence of symmetric monoidal $\infty$-categories (see §2.7). Using (2.13) we deduce the desired results by restricting to the full subcategories spanned by the almost connective and almost perfect modules, respectively.

**Corollary 2.15.** Let $f : A \to B$ be a universal descent morphism of connective $\mathbb{E}_\infty$-rings. Then $f$ is of universal $\text{Perf}$-descent.

*Proof.* As in the proof of (2.14) it will suffice to show that $f$ is of $\text{Perf}$-descent. Using (2.14) we obtain an equivalence of symmetric monoidal $\infty$-categories $\text{APerf}(A) \to \lim \text{APerf}(B^\bullet)$. Then the desired result follows by restricting to the full subcategories spanned by the dualizable objects (see [8, 4.6.1.7]), because the full subcategory of $\lim \text{APerf}(B^\bullet)$ spanned by the dualizable objects can be identified with $\lim \text{Perf}(B^\bullet)$ by virtue of [8, 4.6.1.11].

**Remark 2.16.** In the situation of (2.11) it follows from (2.15) that the $A$-module $M$ is perfect if and only if the $B$-module $M \otimes_A B$ is perfect (see [9, 3.28]).

**Remark 2.17.** The proof of (2.15) shows that every morphism which is either of universal $\text{Mod}^{\text{acn}}$-descent or of universal $\text{APerf}$-descent is of universal $\text{Perf}$-descent.

**Remark 2.18.** The converse of (2.15) is not necessarily true. For example, if $R = \text{Sym}^*(\Sigma^2 \mathbb{Q})$ denotes the free $\mathbb{E}_\infty$-algebra over $\mathbb{Q}$ on a single generator $t$ of degree 2, then we will see from (3.9) that the truncation map $R \to \pi_0 R$ is of universal $\text{Perf}$-descent. However, the extension of scalars functor $\text{Mod}_R \to \text{Mod}_{\pi_0 R}$ is not conservative (since the image of the localization $R[t^{-1}]$ vanishes), so that the truncation map is not a universal descent morphism (see §2.7).

### 3. Proof of Theorems 1.3 and 1.11

#### 3.1. Using the notion of a universal descent morphism of [7, D.3.1.1] (see also §2.6), we now provide a proof of 1.3

*Proof of 1.3.* We wish to show that the canonical map $\text{APerf}(A) \to \text{APerf}(A') \times_{\text{APerf}(B')} \text{APerf}(B)$ is an equivalence of $\infty$-categories. The assertion that this map is fully faithful follows immediately from the first half of [7, 16.2.0.2], which supplies a fully faithful functor $\text{Mod}_A \to \text{Mod}_{A'} \times_{\text{Mod}_{B'}} \text{Mod}_B$. To prove the essential surjectivity, note that we can identify the objects of $\text{APerf}(A') \times_{\text{APerf}(B')} \text{APerf}(B)$ with triples $(M', N, \alpha)$, where $M'$ is an almost perfect $A'$-module, $N$ is an almost perfect $B$-module, and $\alpha : M' \otimes_{A'} B' \to N \otimes_B B'$ is an equivalence of $B'$-modules. Since almost perfect modules are almost connective (that is, $m$-connective for some integer $m$), we may assume that $M'$ and $N$ are both $n$-connective for some integer $n$. Using the second part of [7, 16.2.0.2], we can choose an $n$-connective $A$-module $M$ such that $M \otimes_A A' \simeq M'$ and $M \otimes_A B \simeq N$ in $\text{Mod}_{A'}$ and $\text{Mod}_B$, respectively.
To complete the proof, it will suffice to show that $M$ is almost perfect. Since the diagram $\sigma$ is a pullback square, the map $A \to A' \times B$ is a universal descent morphism in the sense of \cite[D.3.1.1]{7}, so that the desired result follows from \cite[2.11]{2}.

### 3.2

The proof of 1.11 will require some preliminary results.

**Lemma 3.3.** Let $f : A \to B$ be morphism of connective $E_\infty$-rings. Then the canonical map $\text{Mod}^\text{cn}_B \to \varprojlim_{n \geq 0} \text{Mod}^\text{cn}_{B \otimes_{A} \tau \leq n A}$ is an equivalence of $\infty$-categories. Moreover, it restricts to equivalences of $\infty$-categories $\text{APerf}(B) \to \varprojlim_{n \geq 0} \text{APerf}(B \otimes_{A} \tau \leq n A)$ and $\text{Perf}(B) \to \varprojlim_{n \geq 0} \text{Perf}(B \otimes_{A} \tau \leq n A)$.

**Proof.** We first note that there is an equivalence of $\infty$-categories $\text{Mod}^\text{cn}_A \to \varprojlim \text{Mod}^\text{cn}_{\tau \leq n A}$ (see \cite[2.5.9.3]{7}). Then \cite[10.2.2.3]{7} guarantees that the natural map

$$\text{Mod}^\text{cn}_B \cong \text{Mod}^\text{cn}_A \otimes_{\text{Mod}^\text{cn}_A} \text{Mod}^\text{cn}_{\tau \leq n A} \to \varprojlim \text{Mod}^\text{cn}_{\tau \leq n A}$$

is an equivalence of $\infty$-categories. Since the tensor product $\text{Mod}^\text{cn}_B \otimes_{\text{Mod}^\text{cn}_A} \text{Mod}^\text{cn}_{\tau \leq n A}$ can be identified with $\text{Mod}^\text{cn}_{\tau \leq n A}$ (see \cite[10.2.1.7]{7}), we can identify $\text{Mod}^\text{cn}_B$ with the limit of the diagram $\{\text{Mod}^\text{cn}_{\tau \leq n A}\}$ in the $\infty$-category of (not necessarily small) $\infty$-categories $\text{Cat}_\infty$ of \cite[3.0.0.5]{6}. The restrictions to almost perfect and perfect modules follow immediately from \cite[2.7.3.2]{7}.

### 3.4

Since an almost connective module is $m$-connective for some integer $m$ (see \cite[2.9]{2}), we immediately obtain the following:

**Corollary 3.5.** Let $f : A \to B$ be morphism of connective $E_\infty$-rings. Then the canonical map $\text{Mod}^\text{acn}_B \to \varprojlim_{n \geq 0} \text{Mod}^\text{acn}_{\tau \leq n A}$ is an equivalence of $\infty$-categories.

**Lemma 3.6.** Let $f : A \to B$ be a map of $E_\infty$-rings which exhibits $A$ as a square-zero extension of $B$ by a $B$-module $M$ in the sense of \cite[7.4.1.6]{8}. Then $f$ is a universal descent morphism.

**Proof.** We have a pullback diagram of $E_\infty$-rings

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
B & \to & B \oplus \Sigma M,
\end{array}
$$

where $\Sigma M$ denotes the suspension of $M$ (see \cite[7.4.1.7]{8}), so that the desired result follows from the definition of a universal descent morphism (see \cite[D.3.1.1]{7}); alternatively, the desired assertion can be deduced from \cite[11.20]{2}.

### 3.7

We are now ready to give the proof of 1.11. We will follow the strategy of Halpern-Leistner and Preygel in the proof of \cite[3.1.1]{3}, where square-zero extensions allow us to reduce to the case where $A$ is discrete.

**Proof of 1.11.** Since the collection of $v$-covers $f : A \to B$ between connective $E_\infty$-rings for which the underlying map of commutative rings $\pi_0 f : \pi_0 A \to \pi_0 B$ is of finite presentation is closed under pushouts (see \cite[2.5]{2} and \cite[7.2.1.23]{8}), it will suffice to show that the map $f$
Remark 3.12. In the special case where $X$ is representable by an affine spectral Deligne-Mumford stack $\text{Spec} A$, the full subcategory $\text{Perf}(X) \subseteq \text{QCoh}(X)$ corresponds to the full subcategory $\text{Perf}(A) \subseteq \text{Mod}_A$ under the equivalence of $\infty$-categories $\text{QCoh}(X) \simeq \text{Mod}_A$.

Remark 3.13. We note that $\text{QCoh}(X)$ admits a symmetric monoidal structure of $\text{Mod}_A^{\text{acn}}$. According to [7, 6.2.6.2], an object $F \in \text{QCoh}(X)$ is dualizable as an object of the symmetric monoidal $\infty$-category $\text{QCoh}(X)$ if and only if it belongs to $\text{Perf}(X)$.

3.14. The $\nu$-topology on schemes is not subcanonical, but it is when restricted to quasi-compact and quasi-separated perfect schemes; see [2, 4.2]. In the spectral setting, we have the following consequence of \[3.11\]
Corollary 3.15. Let $X$ be a quasi-compact and quasi-separated spectral algebraic space of \cite[1.6.8.1]{spectral}, and let $X : \mathrm{CAlg}^{cn} \to \mathbb{S}$ denote the functor represented by $X$ in the sense of \cite[1.6.4.1]{spectral}. Let $f : A \to B$ be a $v$-cover of connective $\mathbb{E}_\infty$-rings for which the underlying map of commutative rings $\pi_0 f : \pi_0 A \to \pi_0 B$ exhibits $\pi_0 B$ as a finitely presented algebra over $\pi_0 A$. Then $X$ satisfies universal descent for $f$.

Proof. As in the proof of \cite[1.11]{spectral}, we are reduced to proving that the functor $X$ satisfies descent for the map $f$. Let $B^\bullet$ denote Čech nerve of $f$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathbb{S})}(\mathrm{Spec} A, X) & \longrightarrow & \lim_m \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathbb{S})}(\mathrm{Spec} B^m, X) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^\otimes_{\mathrm{ex}}(\mathrm{Perf}(X), \mathrm{Perf}(A)) & \longrightarrow & \lim_m \mathrm{Fun}^\otimes_{\mathrm{ex}}(\mathrm{Perf}(X), \mathrm{Perf}(B^m)),
\end{array}
$$

where $\mathrm{Fun}^\otimes_{\mathrm{ex}}(\mathrm{Perf}(X), \mathrm{Perf}(A))$ denotes the $\infty$-category of exact symmetric monoidal functors from $\mathrm{Perf}(X)$ to $\mathrm{Perf}(A)$ and $\mathrm{Fun}^\otimes_{\mathrm{ex}}(\mathrm{Perf}(X), \mathrm{Perf}(B^\bullet))$ is defined similarly (see \cite[2.1.3.7]{higher}). We wish to show that the upper horizontal map is an equivalence. It follows from \cite[9.6.4.2]{spectral} that the vertical maps are equivalences. The desired result now follows from \cite[3.9]{spectral}, which guarantees that the bottom horizontal map is an equivalence. \hfill $\square$

3.16. Arguing as in the proof of 3.15 (using $A\mathrm{Perf}$ and \cite[9.5.5.1]{spectral} in place of $\mathrm{Perf}$ and \cite[9.6.4.2]{spectral}, respectively), we obtain another consequence of \cite[1.11]{spectral}.

Corollary 3.17. Let $X$ be a locally noetherian geometric stack of \cite[9.3.0.1]{spectral} and \cite[9.5.1.1]{spectral}. Let $f : A \to B$ be a $v$-cover of connective $\mathbb{E}_\infty$-rings for which the underlying map of commutative rings $\pi_0 f : \pi_0 A \to \pi_0 B$ exhibits $\pi_0 B$ as a finitely presented algebra over $\pi_0 A$. Then $X$ satisfies universal descent for the morphism $f$.

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