Rays to renormalizations
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Summary. Let $K_P$ be the filled Julia set of a polynomial $P$, and $K_f$ the filled Julia set of a renormalization $f$ of $P$. We show, loosely speaking, that there is a finite-to-one function $\lambda$ from the set of $P$-external rays having limit points in $K_f$ onto the set of $f$-external rays to $K_f$ such that $R$ and $\lambda(R)$ share the same limit set. In particular, if a point of the Julia set $J_f = \partial K_f$ of a renormalization is accessible from $\mathbb{C} \setminus K_f$ then it is accessible through an external ray of $P$ (the converse is obvious). Another interesting corollary is that a component of $K_P \setminus K_f$ can meet $K_f$ only in a single (pre-)periodic point. We also study a correspondence induced by $\lambda$ on arguments of rays. These results are generalizations to all polynomials (covering notably the case of connected Julia set $K_P$) of some results of Levin and Przytycki (1996), Blokh et al. (2016) and Petersen and Zakeri (2019) where it is assumed that $K_P$ is disconnected and $K_f$ is a periodic component of $K_P$.

1. Introduction

1.1. Polynomial external rays. Let $Q: \mathbb{C} \to \mathbb{C}$ be a non-linear polynomial considered as a dynamical system. Conjugating $Q$ if necessary by a linear transformation, one can assume without loss of generality that $Q$ is monic centered, i.e., $Q(z) = z^{\deg(Q)} + az^{\deg(Q)-2} + \cdots$.

We briefly recall the necessary definitions (see e.g. [DH1], [CG], [Mil0], [LS91] for details). The filled Julia set $K_Q$ of $Q$ is the complement $\mathbb{C} \setminus A_Q$ to the basin of infinity $A_Q = \{z : Q^n(z) \to \infty \text{ as } n \to \infty\}$, and $J_Q = \partial A_Q = \partial K_Q$ is the Julia set (here and below $Q^n(z)$ is the image of $z$ by the $n$-iterate $Q$ of $Q$ for $n$ non-negative and the full preimage of $z$ by $Q^{[n]}$ for $n$ negative).

Let $u_Q : A_Q \to \mathbb{R}_+$ be Green’s function in $A_Q$ such that $u_Q(z) \sim \log |z| + o(1)$ as $z \to \infty$. For all $z$ in some neighborhood $W$ of $\infty$, $u_Q(z) = \log |B_Q(z)|$.

2020 Mathematics Subject Classification: 37F10, 37F20, 37F25.
Key words and phrases: Julia set, renormalization, external rays.
Received 29 January 2021; revised 17 February 2021.
Published online 9 March 2021.

DOI: 10.4064/ba210129-19-2 [1] © Instytut Matematyczny PAN, 2021
where $B_Q$ is the Böttcher coordinate of $Q$ at $\infty$, i.e., a univalent function from $W$ onto $\{ w : |w| > R \}$, for some $R > 1$, such that $B_Q(Q(z)) = B_Q(z)^\text{deg } Q$ for $z \in W$ and $B_Q(z)/z \to 1$ as $z \to \infty$.

An equipotential of $Q$ of level $b > 0$ is the level set $\{ z : u_Q(z) = b \}$. Alternatively, the equipotential containing a point $z \in A_Q$ is the closure of the union $\bigcup_{n>0} Q^{-n}(Q^n(z))$ and $u_Q(z) = \lim_{n \to \infty} (\text{deg}(Q))^{-n} \log |Q^n(z)|$ is the level of this equipotential where $b = u_Q(z)$ is called the $Q$-level of $z \in A_Q$. Note that $u_Q(Q(z)) = (\text{deg } Q)u_Q(z)$ for all $z \in A_Q$.

The gradient flow for Green’s function (potential) $u_Q$ equipped with direction from $\infty$ to $J_Q$ defines $Q$-external rays. More specifically, the gradient flow has singularities precisely at the critical points of $u_Q$ which are preimages by $Q^n$, $n = 0, 1, \ldots$, of critical points of $Q$ that lie in the basin of infinity $A_Q$. If a trajectory $R$ of the flow that starts at $\infty$ does not meet a critical point of $u_Q$, it extends as a smooth (analytic) curve, external ray $R$, up to $J_Q$. If $R$ does meet a critical point of $u_Q$, one should consider instead two corresponding (non-smooth) left and right external rays as left and right limits of smooth external rays tending to $R$ (for a visualization of such rays, see e.g., Figures 1(a-b) of [LP96] or images in [PZ19–PZ20]; to get an impression about the geometry of the Julia set of renormalizable polynomials, see e.g. the computer images of [Pic]). Each external ray $R$ is parameterized by the level of equipotential $b \in (+\infty, 0)$.

The argument $\tau \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ of an external ray $R$ is the argument of the curve $R$ asymptotically at $\infty$. Informally, $\tau$ is the argument at which $R$ crosses the “circle at infinity”. The correspondence between external rays and their arguments is one-to-one on smooth rays and two-to-one on non-smooth ones. If $R$ is a $Q$-external ray of argument $\tau$ then $Q(R)$ is also a ray of argument $\sigma_{\text{deg}(Q)}(\tau)$ where $\sigma_k(t) = tk \mod(1)$. Note that, for any $b$ large enough, $B_Q$ maps the equipotential of level $b$ onto the round circle $\{|w| = e^b\}$ and arcs of external rays from this equipotential to $\infty$ onto standard rays that are orthogonal to this circle. Finally, $K_Q$ is connected if and only if $B_Q$ extends as a univalent function to the basin of infinity $A_Q$, if and only if all external rays of $Q$ are smooth.

Let $S = \{|z| = 1\}$ be the unit circle which we identify—when this is not confusing—with $\mathbb{T}$ via the exponential $\mathbb{T} \ni t \mapsto \exp(2\pi it) \in S$.

1.2. Polynomial-like maps and renormalization. Let us recall [DH2] that a triple $(W, W_1, f)$ is a polynomial-like map if $W, W_1$ are topological discs, $W_1 \subset W$ and $f : W_1 \to W$ is a proper holomorphic map of some degree $m \geq 2$. The set of non-escaping points $K_f = \bigcap_{n=1}^{\infty} f^{-n}(W)$ is called the filled Julia set of $(W, W_1, f)$. By the Straightening Theorem [DH2], there exists a monic centered polynomial $G$ of degree $m$ which is hybrid equivalent to $f$, i.e., there is a quasiconformal homeomorphism $h : \mathbb{C} \to \mathbb{C}$ which is
conformal a.e. on $K_f$, such that $G \circ h = h \circ f$ near $K_f$. The map $h$ is called a straightening. This implies in particular that $K_f$ is the set of limit points of $\bigcup_{n \geq 0} f^{-n}(z)$ for any $z \in W$ with, perhaps, at most one exception.

We say that another polynomial-like map $(\tilde{W}, \tilde{W}_1, \tilde{f})$ of the same degree $m$ is equivalent to $(W, W_1, f)$ if there is a component $E$ of $W \cap \tilde{W}$ such that $K_f \subset E$ and $f = \tilde{f}$ in a neighborhood of $K_f$. Taking a point $z$ as above close to $J_f = \partial K_f$, it follows (cf. [McM, Theorem 5.11]) that $K_f = \tilde{K}_f$ and that this is indeed an equivalence relation for polynomial-like maps. Denote by $f$ the equivalence class of the polynomial-like map $(W, W_1, f)$, by $K_f$, $J_f$ the corresponding filled Julia set and Julia set of (any representative of) $f$, and by $f$ the restriction to a neighborhood of $K_f$ of an $f$-representative (i.e., for any two representatives $(W^{(i)}, W_1^{(i)}, f_i)$, $i = 1, 2$, we have $f_1 = f_2 = f$ in a neighborhood of $K_f$).

From now on, let us fix a monic centered polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d > 1$.

We say that $f$ is a renormalization of $P$ (cf. [McM], [Inou]) if $f$ is an equivalence class of polynomial-like maps such that $K_f$ is a connected proper subset of $K_P$ and, for some $r \geq 1$, $f = P^r$ in a neighborhood of $K_f$.

1.3. Assumptions. Suppose that

(p1) $f$ is a renormalization of $P$.

To avoid a situation when an external ray of $P$ can have a limit point in $J_f$ as well as a limit point off $J_f$, we introduce another condition:

(p2) There exists a representative $(W^*, W_1^*, f)$ of the renormalization $f$ of $P$ and some $b_*>0$ as follows. If $z \in \partial W_1^*$ belongs to an external ray of $P$ which has a limit point in $K_f$ then the $P$-level of $z$ is at least $b_*$, i.e., $u_P(z) \geq b_*$.

Let us stress that external rays of $P$ as in (p2) can cross the boundaries of $W^*$, $W_1^*$ many times (or e.g. have joint arcs with the boundaries).

This condition holds if $W^*$ is obtained by the following frequently used construction that we only indicate here; see [Mil], [McM], [Inou] for details. In the first step, a simply connected domain $W_0$ is built using an appropriate Yoccoz puzzle so that $\partial W_0 = L_{\text{hor}} \cup L_{\text{vert}} \cup F$ where $L_{\text{hor}}$ is a union of finitely many arcs of a fixed equipotential of $P$, $L_{\text{vert}}$ is a union of finitely many arcs of external rays of $P$ between ends of arcs of $L_{\text{hor}}$, and $F$ is a finite set of repelling periodic points of $J_P$ or/and their preimages such that $K_f \subset W_0 \cup F$ and $f : f^{-1}(W_0) \to W_0$ is a branched covering. By the construction, every external ray of $P$ to $J_f \setminus F$ must cross the “horizontal” part $L_{\text{hor}}$ so that (p2) is obviously satisfied for the set of those rays. If either $L_{\text{vert}} = F = \emptyset$ (as in Example [1] that follows) or $F \cap K_f = \emptyset$, one can take $W^* = W_0$ so that
(p2) holds for $W_1^* = f^{-1}(W^*)$. If $F \subset J_f$, then $W_0 \setminus f^{-1}(W_0)$ is a degenerate annulus. Then, in the second step, $W^*$ is modified from $W_0$ by “thickening” \[\text{[Mil1, p. 12]}\] around points of the set $F$, which adds only finitely many rays (tending to $F$). Then (p2) holds for $W_1^* = f^{-1}(W^*)$ as well.

**Example 1.** Assume that the Julia set of the polynomial $P$ is disconnected and $K$ is a component of $K_P$ different from a point. In this case $K = K_f$ for some renormalization $f$ of $P$ and conditions (p1)–(p2) are fulfilled. The boundary of $W^*$ (hence of $W_1^*$, too) can be chosen to be merely a component of an equipotential that encloses $K$. With such a choice, each intersection point of an external ray of $P$ with $\partial W^*$ has a fixed level so every external ray can cross the boundaries of $W^*$ and $W_1^*$ at most once.

Our goal is to study a correspondence between external rays of $P$ that have limit points in $J_f$, on the one hand, and external, or polynomial-like rays of the renormalization $f$, on the other (up to a change of straightening, see below). In the case of disconnected Julia set $J_P$ and the renormalization $f$ as in Example 1 this has been done in \([LP96], [ABC16, Sect. 6]\), and \([PZ19]\).

### 1.4. Polynomial-like rays.

For a curve $\alpha : [0, 1) \to \hat{\mathbb{C}}$, the limit (or principal, or accumulation) set of $\alpha$ is $\Pr(\alpha) = \overline{\alpha} \setminus \alpha$.

Let us define external rays of the renormalization $f$. By \([DH2]\), since $K_f$ is connected, the monic centered polynomial $G$ of degree $m$ which is hybrid equivalent to any representative of $f$ is uniquely defined by $f$. Let $h$ be a straightening of $f$. By this we mean a quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ which is conformal a.e. on $K_f$ and satisfies $G \circ h = h \circ f$ on some neighborhood of $K_f$. One can also assume that $h$ is conformal at $\infty$ such that $h'(\infty) \neq 0$.

As the filled Julia set $K_G$ is connected, given $t \in \mathbb{T}$ there is a unique external ray of $G$ of argument $t$, denoted by $R_{t,G}$. Its $h^{-1}$-image $l^h_t := h^{-1}(R_{t,G})$ is called the polynomial-like ray to $K_f$ of argument $t$. As $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, $\Pr(l^h_t) = h^{-1}(\Pr(R_{t,G}))$. Note that the straightening $h$ is not unique. However, the polynomial $G$ is unique, and if $\tilde{h}$ is another straightening, although $\tilde{h}$ defines another system of polynomial-like rays, the homeomorphism $\tilde{h}^{-1} \circ h : \mathbb{C} \to \mathbb{C}$ maps $l^h_t$ onto $\tilde{l}^h_t$ and $\Pr(l^h_t)$ onto $\Pr(\tilde{l}^h_t)$.

In what follows we fix a straightening map $h : \mathbb{C} \to \mathbb{C}$ (see Theorem 3(e) and its proof though). Then the set $\{l_t\}$ of polynomial-like rays is fixed, too (where we omit $h$ in $l^h_t$ as $h$ is fixed). For brevity, $P$-external rays are called $P$-rays, or just rays, and polynomial-like rays to $K_f$ are $f$-rays, or polynomial-like rays.

### 1.5. Main results.

Given a connected compact set $K \subset \mathbb{C}$ which is different from a point, we say that a curve $\gamma : [0, 1) \to \Omega := \mathbb{C} \setminus K$ converges to a prime end $\hat{P}$ of $K$ if, for a conformal homeomorphism $\psi : \mathbb{C} \setminus K \to \{|z| > 1\}$,
the curve $\psi \circ \gamma : [0,1) \to \{|z| > 1\}$ converges to a single point $P \in \mathbb{S}$; we say that $\gamma$ converges to the prime end $\hat{P}$ non-tangentially if moreover $\psi \circ \gamma$ converges to the point $P$ non-tangentially, i.e., the set $\psi \circ \gamma((1-\epsilon,1))$ lies inside a sector (Stolz angle) $\{z : |\arg(z-P)-\arg P| \leq \alpha\}$ for some $\epsilon > 0$ and $\alpha \in (0,\pi/2)$. Furthermore, we say that two curves $\gamma_1, \gamma_2 : [0,1) \to \Omega$ are $K$-equivalent if they both converge to the same prime end and moreover have the same limit sets $\Pr(\gamma_1) = \Pr(\gamma_2)$ in $\partial K$.

By Lindelöf’s theorem (see e.g. [Pom, Theorem 2.16]), if two curves converge to the same prime end non-tangentially, they share the same limit set. Therefore, if $\gamma_1, \gamma_2$ converge to the same prime end of $K$ non-tangentially, then $\gamma_1, \gamma_2$ are also $K$-equivalent.

The following statement was proved in [ABC16 (2)] in the set up of Example 1.

**Theorem 1** (cf. [ABC16, Theorem 6.9]). Assume (p1)–(p2) hold. For each $P$-ray $R$ that has an accumulation point in $K_f$ we have $\Pr(R) \subset J_f$ and there is a unique polynomial-like ray $l = \lambda(R)$ such that the curves $l, R$ are $K_f$-equivalent. Moreover, $l, R$ converge to a single prime end of $K_f$ non-tangentially. Furthermore, $\lambda : R \mapsto l$ maps the set of $P$-rays to $K_f$ onto the set of polynomial-like rays, and is “almost injective”: $\lambda$ is one-to-one except when one and only one of the following (i)–(ii) holds. Suppose that $\lambda^{-1}(\ell) = \{R_1, \ldots, R_k\}$ with $k > 1$.

(i) $k = 2$ and both rays $R_1, R_2$ are non-smooth and share a common arc starting at a critical point of Green’s function of $P$, or

(ii) there is $z \in J_f$ such that $\Pr(R_i) = \{z\}$, $i = 1, \ldots, k$, at least two of the rays $R_1, \ldots, R_k$ are disjoint, and, for some $n \geq 0$, $P^n(z) \in Y$ where $Y \subset J_f$ is a finite collection of repelling or parabolic periodic points of $P$ that depends merely on $K_f$.

If $K_P$ is connected then (i) is not possible.

Note that in case (ii) any two disjoint $P$-rays completed by the joint limit point $z$ split the plane into two domains such that one of them contains $K_f \setminus \{z\}$, and the other one, points from $K_P \setminus K_f$. In particular, if $K_P$ is connected, the second domain must contain a component of $K_P \setminus K_f$ that goes all the way to a pre-periodic point $z \in J_f$. In fact, this is “if and only if”: see Theorem 2(b) below.

For an illustration, see e.g. pictures in [McM, p. 116, explained in Example IV, p. 115] of a “dragon” filled Julia set of a quadratic polynomial $P$ admitting three renormalizations; the maps $\lambda$ corresponding to these renormalizations are one-to-one except at countably many polynomial-like rays.

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(1) One can show that if $\gamma_1$ converges to a single point $a \in \partial K$, then $\gamma_2$ is $K$-equivalent to $\gamma_1$ if and only if $\gamma_1, \gamma_2$ are homotopic through a family of curves in $\Omega$ converging to $a$.

(2) In [ABC16], a different terminology is used.
where \( \lambda \) is 6-to-1 in the top picture, 2-to-1 in the left bottom and 3-to-1 in the right bottom. In all three cases, the landing points of rays where \( \lambda \) is not one-to-one are (pre-)periodic to a fixed point of \( P \) where six \( P \)-rays land.

The next two theorems are consequences of the proof of Theorem 1.

**Theorem 2.** Assume (p1)–(p2).

(a) If a point \( a \in J_f \) is accessible along a curve \( s \in \mathbb{C} \setminus K_f \), then \( a \) is the landing point of a \( P \)-ray \( R \); moreover the curves \( s, R \) are \( K_f \)-equivalent.

(b) There exists a finite set \( Y \subset J_f \) of repelling or parabolic periodic points of \( f \), as follows. Let \( S \) be a component of \( K_P \setminus K_f \) such that \( (S \setminus S) \cap J_f \neq \emptyset \). Then \( S \setminus S \) is a single point \( b \in J_f \), and moreover \( f^n(b) \in Y \) for some \( n \geq 0 \).

Note that part (a) is in fact an easy corollary of Lemma 2.1 similar to a result of [LP96]. Part (b) is void if (and only if) \( K_f \) is itself a component of \( K_P \).

For the next statement, we introduce the following notations. Let \( \Lambda \subset \mathbb{T} \) be the set of arguments of all \( P \)-rays that have their limit points in \( J_f \). Observe that by Theorem 1 the whole limit sets of such rays are in \( J_f \) and, given \( \tau \in \Lambda \), there is a unique \( P \)-ray, denoted by \( R_{\tau,P} \), which has its limit set in \( J_f \). Indeed, this is obvious if the \( P \)-ray of argument \( \tau \) is smooth. On the other hand, if there are two \( P \)-rays, left and right, of argument \( \tau \), only one of them can have its limit point in \( J_f \) because the other one must go to another component of \( K_P \). Now, the map \( \lambda \) of Theorem 1 induces a map \( p: \Lambda \rightarrow \mathbb{T} \) such that for all \( \tau \in \Lambda \),

\[
\lambda(R_{\tau,P}) = l_{p(\tau)}.
\]

By Theorem 1, \( \text{Pr}(l_{p(\tau)}) = \text{Pr}(R_{\tau,P}) \), and moreover \( R_{\tau,P}, l_{p(\tau)} \) are \( K_f \)-equivalent.

Given a positive integer \( k \), let \( \sigma_k: \mathbb{T} \rightarrow \mathbb{T}, \sigma_k(t) = kt \mod 1 \). Recall that \( \deg(f) = m \). Let \( D := \deg(P^r) = d^r \).

**Theorem 3** (cf. [PZ19]).

(a) \( \Lambda \) is a compact nowhere dense subset of \( \mathbb{T} \) which is invariant under \( \sigma_D \).

(b) \( \sigma_m \circ p = p \circ \sigma_D \) on \( \Lambda \).

(c) The map \( p: \Lambda \rightarrow \mathbb{T} \) is surjective and finite-to-one, and moreover “almost injective” as defined in Theorem 1.

(d) \( p: \Lambda \rightarrow \mathbb{T} \) extends to a continuous monotone degree one map \( \tilde{p}: \mathbb{T} \rightarrow \mathbb{T} \).

(e) The map \( p \) is unique in the following sense: if \( \tilde{p}: \Lambda \rightarrow \mathbb{T} \) corresponds to another straightening \( \tilde{h} \), then \( \tilde{p}(t) = p(t) + k/(m - 1) \mod 1 \) for some \( k = 0, 1, \ldots, m - 1 \).

In the set up of Example 1, i.e., when \( K_P \) is disconnected and \( K_f \) is a periodic component of \( K_P \), Theorem 3 was proved in [PZ19] (by a different
method), with part (c) replaced by an explicit bound for the cardinality of fibers of the map \( p \) as well as with an extra statement about the Hausdorff dimension of the set \( \Lambda \).

A detailed proof of the main Theorem 1 is given in Sect. 2 and the proofs of Theorems 2, 3 are in Sect. 3. The proof of Theorem 1 follows rather closely the proofs of [LP96, Lemma 2.1] and [ABC16, Theorems 6.8–6.9]. An essential difference is that we have to adapt the proofs to the situation that external rays of \( P \) can cross the boundary of \( W_1^* \) as in (p2) many times.

2. Proof of Theorem 1. Let \( f : W_1^* \to W^* \) be a representative of \( f \) as in (p2). As \( K_f \) is connected, all the critical points of \( f \) are in \( K_f \). Hence, for each \( k \), \( f^k : f^{-k}(W_1^* \setminus K_f) \to W_1^* \setminus K_f \) is an unbranched (degree \( m^k \)) map. Therefore, \( L_k := f^{-k}(\partial W_1^*) \) is the boundary of a simply connected domain \( f^{-k}(W_1^*) \).

Let \( \mathcal{R} \) denote the set of all \( P \)-rays \( R \) such that \( R \) has a limit point in \( J_f \). First, we show that all limit points of \( R \in \mathcal{R} \) are in \( J_f \), introducing some notations along the way. Let

\[
b_{*,k} = \inf\{u_P(z) : z \in R \cap L_k, R \in \mathcal{R}\}.
\]

By (p2), \( b_{*,0} > 0 \). As \( R \in \mathcal{R} \) implies \( P^r(R) \in \mathcal{R} \), we have \( b_{*,k} \geq b_{*,0}/D^k \), hence \( b_{*,k} > 0 \), for all \( k \). Let \( R \in \mathcal{R} \) and \( k \geq 0 \). Since \( R \cap L_k \) is a closed set and \( b_{*,k} > 0 \), there exists a unique point \( z_k(R) \in R \cap L_k \) such that \( u_P(z_k(R)) = \inf\{u_P(z) : z \in R \cap L_k\} \). Observe that the arc \( \Gamma_{k,R} \) of \( R \) from \( z_k(R) \) down to \( J_f \) lies entirely in \( f^{-k}(W_1^*) \). As \( \bigcap_{k \geq 0} f^{-k}(W_1^*) = K_f \), we see immediately that the limit set of \( R \), which is \( \bigcap_{k \geq 0} \overline{T_{K,R}} \), is a subset of \( J_f \).

Before proceeding with more notations and the main lemma, let us note that \( b_{*,k} = b_{*,0}/D^k \), \( k = 1,2,\ldots \). Indeed, as \( f^k : f^{-k}(W_1^* \setminus K_f) \to W_1^* \setminus K_f \) is an unbranched covering, each component of \( f^{-k}(R) \) is an arc of some ray from \( \mathcal{R} \). This implies that \( b_{*,k} \leq b_{*,0}/D^k \). The opposite inequality was seen before.

Now, choose a conformal isomorphism \( \psi \) from \( C \setminus K_f \) onto \( D^* = \{ |z| > 1 \} \) such that \( \psi(z)/z \to e \) as \( z \to \infty \), for some \( e > 0 \). A curve \( \hat{R} \) in \( D^* \) with limit set in \( S = \{ |z| = 1 \} \) is called a \( K \)-related ray if its preimage \( \psi^{-1}(\hat{R}) \) is a \( P \)-ray \( R \in \mathcal{R} \), i.e., \( R \) has its limit set in \( K_f \). The argument of \( \hat{R} \) is said to be the argument of the ray \( \psi^{-1}(\hat{R}) \). Let \( A_K = \psi(W^* \setminus K_f) \) be an “annulus” with boundary curves \( \psi(\partial W^*) \) and \( S \). Denote \( \tilde{z}_k(R) = \psi(z_k(R)) \). Note that \( \tilde{z}_k(R) \in \psi(L_k) \cap \hat{R} \) and the arc of the \( R \)-related ray \( \hat{R} \) from \( \tilde{z}_k(R) \) to \( S \) is contained in the “annulus” between \( \psi(L_k) \) and \( S \). An arc of a \( K \)-related ray \( \tilde{R} = \psi(R) \) from the point \( \tilde{z}_0(R) = \psi(z_0(R)) \) \( \in \psi(L_0) \) to \( S \) is called a \( K \)-related arc. Its argument is the argument of the corresponding ray. The following main lemma and its proof are minor adaptations of the ones of [LP96, Lemma 2.1].
Lemma 2.1.

1° Every K-related arc has a finite length, and hence converges to a unique point of $\mathbb{S}$.

2° For every closed arc $I \subset \mathbb{S}$ (in particular a point), the set $K(I)$ of arguments of all K-related arcs converging to a point of $I$ is a non-empty compact set.

3° The set of all K-related arcs in $\{z : 1 < |z| < 1 + \epsilon\}$ converging to a point $z_0$ lies in a Stolz angle

$$\{z : |\arg(z - z_0) - \arg z_0| \leq \alpha\},$$

where $\alpha \in (0, \pi/2)$ and $\epsilon$ do not depend on $z_0 \in \mathbb{S}$.

Proof. 1° Let $B_{*,k} = \sup \{u_P(z) : z \in L_k\}$. For every $k \geq 0$ there is a number $C_k$ such that, for every ray $R \in \mathcal{R}$, the length of the arc $R_k$ of $R$ between the points $z_k(R)$ and $z_{k+1}(R)$ is bounded by $C_k$. This is because the latter arc is an arc of a $P$-ray that joins two equipotentials of positive levels $B_{*,k}$, $b_{*,k}$. Denote $\tilde{L}_k = \psi(L_k)$. Then $\tilde{L}_k$ is a compact subset of $\mathcal{A}_K$ which surrounds $\mathbb{S}$. By the above, every $K$-related arc $\tilde{R}$ splits into arcs $\tilde{R}_k = \psi(R_k)$, $k \geq 0$, i.e., $\tilde{R}_k$ is the arc of $\tilde{R}$ joining $\tilde{z}_k(\tilde{R})$ and $\tilde{z}_{k+1}(\tilde{R})$. For every $k$, the supremum of the lengths over all arcs $\tilde{R}_k$ of the $K$-related rays $\tilde{R}$ is bounded by

$$\tilde{C}_k = C_k \sup \{|\psi'(z)| : z \in \overline{W}_1^* \setminus f^{-k-2}(W_1^*)\}.$$ 

Let $A_{1,K} = \psi(W_1^* \setminus K_f)$ and $g = \psi \circ f \circ \psi^{-1} : A_{1,K} \to \mathcal{A}_K$. Then $z$ tends to $\mathbb{S}$ if and only if $g(z)$ tends to $\mathbb{S}$. It is well-known (see e.g. [P86]) that $g$ extends to an expanding holomorphic map in an annulus $U_0 = \{z : 1 - \rho_0 < |z| < 1 + \rho_0\}$ for some $\rho_0 > 0$. This means that after passing if necessary to an iterate of $g$ (which we also denote $g$) we have

$$|(g^{-1})'(z)| < c < 1$$

for every $z \in U_0$ and for every branch $g^{-1}$ such that $g^{-1}(z) \in U_0$.

Fix a set $\tilde{L}_m \subset U = A_K \cap U_0$ for some $m$ large enough. Then, for each $n = 1, 2, \ldots, \tilde{L}_{n+m} = \{z \in U : g^n(z) \in \tilde{L}_{n+m}\}$. Denote by $l_n$ the supremum of the lengths of $\tilde{R}_{n+m}$ over all $R \in \mathcal{R}$. Note that each $l_n$ is finite, because $l_n \leq \tilde{C}_{m+n}$. In fact, much more is true: as $g^n(\tilde{R}_{n+m})$ is $\tilde{S}_m$ for some ray $S \in \mathcal{R}$, [4] yields $l_n < c^n l_0$. Given a K-related ray $\tilde{R}$, the length of its arc from the point $\tilde{z}_m(\tilde{R})$ to $\mathbb{S}$, which is in the component of $\mathcal{C} \setminus \gamma_0$ containing $\mathbb{S}$, is bounded from above by $\sum_{n=0}^{\infty} c^n l_0 < \infty$. Moreover, the same argument shows the following

Claim 1. The lengths of the arcs of K-related rays $\tilde{R}$ between $\tilde{z}_k(\tilde{R})$ and $\mathbb{S}$ tend uniformly to zero (exponentially in $k$).
2° Fix a closed non-degenerate arc \( I \subset \mathbb{S} \). There exists a \( K \)-related ray converging to a point of \( I \). Indeed, otherwise no \( K \)-related ray ends in the arc \( g^n(I) \), for any \( n \). This is impossible because \( g^n(I) = \mathbb{S} \) for large \( n \) and the set of \( K \)-related rays is non-empty (for example, it contains images by \( \psi \) of \( P \)-rays landing at repelling periodic points of the polynomial-like map \( f : W_1^* \to W^* \); for the existence of such \( P \)-rays, see [Mil0], [EL89], [LP96]). We need to show that the set \( K(I) \) of arguments of all \( K \)-related rays ending in \( I \) is closed.

This is an immediate consequence of the next claim which follows, basically, from Claim 1 and will also be useful later on. Given a \( K \)-related ray \( \tilde{R}_t \) of argument \( t \) (i.e., \( t \in \Lambda \)) consider its arc \( \tilde{r}_t \) between \( \hat{L}_0 \) and \( \mathbb{S} \), parameterized as a curve \( \tilde{r}_t : [b^*,0,0] \to A_K \cup \mathbb{S} \) as follows. For any \( b \in [b^*,0,0] \), define the point \( r_t(x) \in A_K \) to be such that \( \psi^{-1}(r_t(x)) \) is a point of a \( P \)-ray of argument \( t \) and equipotential level \( b \). Finally, let \( \tilde{r}_t(0) = \lim_{b \to 0} \tilde{r}_t(x) \in \mathbb{S} \) where the limit exists by 1°.

**Claim 2.** The family \( \mathcal{R} = \{\tilde{r}_t\}_{t \in \Lambda} \) is a compact subset of \( C[b^*,0,0] \).

Let us first show that this family is equicontinuous. In view of Claim 1, this will follow from the equicontinuity of the restricted family \( \mathcal{R}_m = \{\tilde{r}_t : [b^*,0,b^*,0/D^m] \to A_K \} \) for each integer \( m > 1 \). Fix \( m \) and consider two objects: a compact set \( E_m \subset \mathbb{C} \) bounded by the equipotential of levels \( b^*,0 \) and \( b^*,0/D^m \) of \( P \) and a family \( \mathcal{R}_m \) of (closed) arcs in \( E_m \) of all \( P \)-rays that join the equipotential levels \( b^*,0, b^*,0/D^m \) and are parameterized by the equipotential level \( b \in [b^*,0,b^*,0/D^m] \). It is easy to see that this is a compact subset of \( C[b^*,0,b^*,0/D^m] \) (indeed, map this family by a fixed high iterate of \( P \) to a family of smooth arcs of \( P \)-rays which are preimages of segments of standard rays by the Böttcher coordinate \( B_P \) at infinity; hence, this new family is compact; then pull it back). As \( \mathcal{R}_m \subset C[b^*,0,b^*,0/D^m] \) is compact, it is equicontinuous. In turn, since \( \psi^{-1} \) is a homeomorphism on \( E_m \) (onto its image) and each \( \psi^{-1}(\tilde{r}_t) \) is in \( \mathcal{R}_m \), the family \( \mathcal{R}_m \) is equicontinuous too. Thus \( \mathcal{R} \) is an equicontinuous family.

It remains to prove that it is closed. So suppose a sequence \( \tilde{r}_{t_n} \) converges uniformly in \( [b^*,0,0] \). In particular, \( \tilde{r}_{t_n} \) crosses \( \hat{L}_k \) for each \( k \) large enough. One can assume that \( t_n \) tends to some \( t \). Then the sequence of arcs of \( P \)-rays \( \psi^{-1} \circ \tilde{r}_{t_n} \), on the one hand, tends, uniformly on each interval \( [b^*,0,b^*,0/D^m] \), to an arc \( r \) of a \( P \)-ray of argument \( t \), on the other hand, crosses each \( L_k \) with \( k \) large. Hence, \( r \) has a limit point in \( K_f \). Applying \( \psi \) we find that the limit of \( \tilde{r}_{t_n} \) is a \( K \)-related arc, which ends the proof of the claim.

This proves 2° when \( I \) is not a single point. By the intersection of compacta, 2° also holds if \( I \) is a point.

3° Every branch of \( g^{-n} \) is a well defined univalent function in every disc contained in \( U_0 \). Hence, by the Koebe distortion theorem (see e.g. [Gol]), one
can choose $0 < \rho' < \rho_0$ such that for every $z \in U' = \{z : 1 - \rho' < |z| < 1 + \rho'\}$, every $n = 1, 2, \ldots$ and every branch $g^{-n}$,

$$\frac{|(g^{-n})'(x)|}{|g^{-n})'(y)|} < 2$$

whenever $|z - x| < \rho'$ and $|z - y| < \rho'$.

We reduce $U'$ further as follows. By Claim 1, fix $m_0 > m$ such that the length of the arc of any $K$-related ray $R$ between $\tilde{z}_{m_0}(R)$ and $S$ is less than $\rho'$. On the other hand, if $z$ lies in an unbounded component of $R \setminus z_{m_0}(R)$, i.e., in the arc of $R$ between $z_{m_0}(R)$ and $\infty$, then $u_{J}(z) \geq b_{*,m_0}$, in particular, there is $r > 0$ independent of $z$ and $R$ as above such that the distance between $z$ and $J_{P}$ is at least $r$. Therefore, there exists some $\rho_1 \in (0, \rho')$ such that for every $z \in \{z : 1 < |z| < 1 + \rho_1\}$; if $z$ belongs to a $K$-related ray $R$ then $z$ lies in an arc of $R$ between $\tilde{z}_{m_0}(R)$ and $S$. Let

$$U_1 = \{z : 1 - \rho_1 < |z| < 1 + \rho_1\}.$$  

We introduce the following notations:

Given $x \in U_1$, denote by $l_x$ the part of the $K$-related ray passing through $x$ between $x$ and $S$ (if such a ray exists). This notation is correct: as already noted before, if another $K$-related ray passes through $x$ and next ramifications from $l_x$, it goes to a component of $\psi(J(f))$, not to $S$. So it is not $K$-related.

Denote by $h_x$ the interval which joins $x$ and $S$, orthogonal to $S$. Denote by $l(x)$ and $h(x)$ the corresponding Euclidean lengths. Find a large enough $N$ such that $\tilde{\gamma}_0 := \tilde{L}_N$ in $U_1$. By the choice of $U_1$,

$$l(x) < \rho' \quad \text{for all } x \text{ between } \tilde{\gamma}_0 \text{ and } S.$$ 

Let $\tilde{\gamma}_1 = g^{-1}(\tilde{\gamma}_0)$. There exists a positive $\beta_0$ less than 1 such that

$$\frac{h(x)}{l(x)} > \beta_0$$

for all points $x$ in the annulus $V$ between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.

Fix the maximal $\epsilon_0 > 0$ such that

$$U_2 = \{z : 1 - \epsilon_0 < |z| < 1 + \epsilon_0\}$$

does not intersect $\tilde{\gamma}_1$. We intend to prove assertion $3^o$ of our lemma with

$$\alpha = \arccos\left(\frac{\beta_0}{8L}\right)$$

where $L = \sup\{|g'(z)| : z \in U_0\}$ and with $\epsilon$ between 0 and $\epsilon_0$ so small that $1 < |z| < 1 + \epsilon$ and $h(z)/|z - z_0| \geq 2 \cos \alpha$ implies $|\arg(z - z_0) - \arg z_0| \leq \alpha$. 
It is enough to prove that
\[
\frac{h(x)}{l(x)} > \beta = \frac{\beta_0}{4L}
\]
for all \( x \in U_2 \). Assume the contrary: there exists \( x_* \in U_2 \) that belongs to some \( K \)-related ray \( \tilde{R} \) with
\[
h(x_*) / l(x_*) \leq \beta.
\]
Choose the minimal \( n \geq 1 \) such that \( g^n(x_*) \in V \).

The lengths \( h^{(i)} \) and \( l^{(i)} \) of the curves \( g^i(h_{x_*}) \) and \( g^i(l_{x_*}) \) cannot exceed \( \rho' \) for all \( i = 0, 1, \ldots, n \). This holds for \( l^{(i)} \) by (3), because \( g^i(x_*) \) is between \( \tilde{\gamma}_0 \) and \( \mathbb{S} \). We cope with \( h^{(i)} \)'s by induction: Length \( h^{(0)} \) < \( \rho \) by the definition of \( U_1 \). If it holds for all \( i \leq j - 1 \) then by (2),
\[
\frac{h^{(j-1)}}{l^{(j-1)}} \leq 4\beta = \beta_0 / L.
\]
Then
\[
h^{(j)} \leq Lh^{(j-1)} \leq \beta_0 \cdot l^{(j-1)} < l^{(j-1)} < \rho'.
\]
Now we use the assumption (6) and again apply (2) to obtain, for \( z_* = g^n(x_*) \in \tilde{S}_N \),
\[
\frac{h(z_*)}{l(z_*)} \leq \frac{h^{(n)}}{l^{(n)}} \leq 4\beta = \beta_0 / L < \beta_0.
\]
This contradicts (4).

**Comment.** The key bound (5) can also be seen directly from (4) (with, for instance, \( \beta = \beta_0 / 10 \)) by applying, besides the Koebe distortion bound (2), another distortion bound: there is a function \( \epsilon : (0, 1) \to (0, +\infty) \), with \( \epsilon(r) \to 0 \) as \( r \to 0 \), such that for any univalent function \( \varphi \) on the unit disc, if \( \varphi(0) = 0 \) and \( \varphi'(0) = 1 \), then
\[
\left| \log \frac{\varphi(z)}{z} \right| < \epsilon(|z|)
\]
(see e.g. [Gol]). This bound is applied to the function
\[
\varphi(z) = \frac{g^{-n}(w + \rho_0 z) - g^{-n}(w)}{(g^{-n})'(w)\rho_0}
\]
where \( n \) is minimal such that \( g^n(x) \in V \), and \( w \in \mathbb{S} \) is the projection of \( g^n(x) \) to \( \mathbb{S} \) and reducing \( \rho' \). Note that \( (g^{-n})'(w) > 0 \) because \( g \) preserves \( \mathbb{S} \).

We continue as follows (cf. [ABC16, proof of Theorem 6.9]). Recall that a straightening \( h : \mathbb{C} \to \mathbb{C} \) is a quasiconformal homeomorphism which is holomorphic at \( \infty \) and \( h'(\infty) \neq 0 \). It conjugates the polynomial-like map \( f \) to the polynomial \( G \) near their filled Julia sets \( K_f \) and \( K_G \). Let \( B_G : A_G \to \mathbb{D}^* \) be the Böttcher coordinate of \( G \) such that \( B_G(z)/z \to 1 \) as \( z \to \infty \), which is well defined in the basin of infinity \( A_G = \mathbb{C} \setminus K_G \) as \( K_G \) is connected.
We have the following picture:

\[(7)\quad \mathbb{D}^* \xrightarrow{\psi^{-1}} \mathbb{C} \setminus K_f \xrightarrow{h} \mathbb{C} \setminus K_G \xrightarrow{B_G} \mathbb{D}^*.\]

Consider the map \(\Psi := \psi \circ h^{-1} \circ B_G^{-1} : \mathbb{D}^* \to \mathbb{D}^*\) from the uniformization plane of the polynomial \(G\) to the \(g\)-plane of \(K\)-related rays. It is a quasiconformal homeomorphism which is holomorphic at \(\infty\). For \(u \in \mathbb{S}\), let \(L_u = \Psi(r_u \cap \mathbb{D}^*)\) where \(r_u = \{tu : t > 0\}\) is a standard ray in the uniformization plane of \(G\)\(^{(3)}\).

**Lemma 2.2.** The curve \(L_u\) converges non-tangentially to a unique point \(z_0 = z_0(u)\) of the unit circle \(\mathbb{S}\). Moreover, there is \(\beta \in (0, \pi/2)\) such that, for any \(u \in \mathbb{S}\) and all \(z \in L_u\) close enough to \(\mathbb{S}\),

\[(8)\quad |\arg(z - z_0) - \arg z_0| \leq \beta.\]

Here \(\beta\) depends only on the quasiconformal deformation of the straightening map \(h\). Furthermore, for every \(z_0 \in \mathbb{S}\) there exists a unique \(u\) such that \(L_u\) lands at \(z_0\).

**Proof of Lemma 2.2** (cf. [ABC16, Section 6]). The map \(\Psi : \mathbb{D}^* \to \mathbb{D}^*\) extends to a homeomorphism of the closures and then to a quasiconformal homeomorphism \(\Psi^*\) of \(\mathbb{C}\) by \(\Psi^*(z) = 1/\Psi^*(1/\bar{z})\) (see [Ahl]). Note that the quasiconformal deformations of \(\Psi\) and \(\Psi^*\) are the same, equal to the quasiconformal deformation \(M\) of the straightening map \(h\). Consider the curve \(L_u^* = \Psi^*(r_u)\). It is an extension of the curve \(L_u\), which crosses \(\mathbb{S}\) at a point \(z_0 = \Psi^*(u)\). As a quasiconformal image of a straight line, the curve \(L_u^*\) has the following property [Ahl]: there exists \(C = C(M) > 0\) such that

\[|z - z_0|/|z - 1/\bar{z}| < C\]

for every \(z \in L_u^*\).

Therefore, \(L_u^*\) tends to \(z_0\) non-tangentially; moreover, \[(8)\] holds for some \(\beta = \beta(C(M))\). The last claim follows from the fact that \(\Psi^*\) is a homeomorphism. \(\blacksquare\)

Now, define the correspondence \(\lambda\) as follows (having in mind \[(7)\]). Let \(R\) be a \(P\)-ray to \(K_f\). By Lemma 2.1, the \(K\)-related ray \(\tilde{R} = \psi(R)\) tends to a point \(z_0 \in \mathbb{S}\). By Lemma 2.2, there exists a unique \(L_u\) which tends to \(z_0\). The curve \(\psi^{-1}(L_u) = h^{-1} \circ B_G^{-1}(\{tu : t > 1\})\) is a polynomial-like ray \(l_{\tau}\) where \(u = e^{2\pi i \tau}\). Let

\[\lambda(R) := \psi^{-1}(L_u).\]

The correspondence \(\lambda\) is “onto” by the first claim of Lemma 2.2 along with Lemma 2.1(2°).

Now, both curves \(\tilde{R}, L_u\) in \(\mathbb{D}^*\) tend to the point \(z_0 \in \mathbb{S}\) non-tangentially, by Lemmas 2.1 and 2.2 respectively. Then, by definition, the \(P\)-ray \(R\) and the

\[\text{(3) Note that the curve } L_u \text{ lies in the left-hand disc } \mathbb{D}^* \text{ of } (7) \text{ while the point } u \text{ is at the boundary of the right-hand disc there.}\]
polynomial-like ray \( \lambda(R) \) converge to a single prime end of \( \mathcal{K}_\mathfrak{f} \) non-tangentially, hence \( R \) and \( \lambda(R) \) are also \( \mathcal{K}_\mathfrak{f} \)-equivalent. Finally, the condition that \( R \) and \( \lambda(R) \) are \( \mathcal{K}_\mathfrak{f} \)-equivalent uniquely determines the polynomial-like ray \( \lambda(R) \).

It remains to prove the “almost injectivity” of \( \lambda \). This is a direct consequence of the one-to-one correspondence between \( \mathcal{K} \)-related rays and curves \( \mathcal{L}_u \) established above and the following claim whose proof is identical to the one of \([\text{ABC16, Theorem 6.8}] \) (for completeness, we reproduce it below with obvious changes in notation). While passing from \( \mathcal{K} \)-related rays to \( \mathcal{P} \)-rays we use the fact that if a \( \mathcal{K} \)-related ray is periodic, the corresponding \( \mathcal{P} \)-ray converges to a periodic point of \( \mathcal{P} \) which is either repelling or parabolic (by the Snail Lemma \([\text{Mil0}] \), it cannot be irrationally indifferent).

**Lemma 2.3.** Any point \( w \in \mathcal{S} \) is the landing point of precisely one \( \mathcal{K} \)-related ray, except when one and only one of the following holds:

(i) \( w \) is the landing point of exactly two \( \mathcal{K} \)-related rays which are non-smooth and have a common smooth arc that goes to \( w \);

(ii) \( w \) is a landing point of at least two disjoint \( \mathcal{K} \)-related rays, in which case \( w \) is a (pre)periodic point of \( g \) and some iterate \( g^n(w) \) belongs to a finite set \( \hat{Y} \) (depending only on \( \mathcal{K} \)) of \( g|_{\mathcal{S}} \)-periodic points each of which is the landing point of finitely many, but at least two, \( \mathcal{K} \)-related rays, which are periodic of the same period depending merely on the landing point \( w \).

Moreover, if \( w \) is periodic then (i) cannot hold.

**Proof.** Assume that there are two \( \mathcal{K} \)-related rays landing at a point \( w \in \mathcal{S} \) and that (i) does not hold. We need to prove that then (ii) holds. Since (i) does not hold, there exist disjoint \( \mathcal{K} \)-related rays landing at \( w \). Let us study this case in detail.

Associate to any such pair of rays \( \hat{R}_t, \hat{R}_{t'} \) an open arc \( (\hat{R}_t, \hat{R}_{t'}) \) of \( \mathcal{S} \) as follows. Two points of \( \mathcal{S} \) with the arguments \( t, t' \) split \( \mathcal{S} \) into two arcs. Let \( (\hat{R}_t, \hat{R}_{t'}) \) be the one that contains no arguments of \( \mathcal{K} \)-related rays except possibly for those that land at \( w \). Geometrically, this means the following. The \( \mathcal{K} \)-related rays \( \hat{R}_t, \hat{R}_{t'} \) together with \( w \in \mathcal{S} \) split the plane into two domains. The arc \( (\hat{R}_t, \hat{R}_{t'}) \) corresponds to one of them, disjoint from \( \mathcal{S} \). Let \( L(\hat{R}_t, \hat{R}_{t'}) = \delta \) be the angular length of \( (\hat{R}_t, \hat{R}_{t'}) \). Clearly, \( 0 < \delta < 1 \). Now we make a few observations.

1. If \( \mathcal{K} \)-related disjoint rays of arguments \( t_1, t_1' \) land at a common point \( w_1 \), while \( \mathcal{K} \)-related disjoint rays of arguments \( t_2, t_2' \) land at a point \( w_2 \neq w_1 \), then the arcs \( (\hat{R}_{t_1}, \hat{R}_{t'_1}), (\hat{R}_{t_2}, \hat{R}_{t'_2}) \) are disjoint.

(\text{(4)}) In \([\text{ABC16, Theorem 6.8(ii)}] \), it is claimed erroneously that all \( \mathcal{K} \)-related rays to the point \( w \) are smooth (cf. \([\text{PZ20}] \)). Note that this claim is not relevant to the rest of \([\text{ABC16}] \).
The above follows from the definition of the arc \((\hat{R}_t, \hat{R}_t')\).

(2) If disjoint \(K\)-related rays \(\hat{R}_t, \hat{R}_t'\) of arguments \(t, t'\) land at a common point \(w\), then the \(K\)-related rays \(g(\hat{R}_t), g(\hat{R}_t')\) are also disjoint and land at the common point \(g(w)\). Moreover,

\[
L(g(\hat{R}_t), g(\hat{R}_t')) \geq \min\{D\delta \pmod{1}, 1 - D\delta \pmod{1}\} > 0.
\]

Indeed, the images \(g(\hat{R}_t), g(\hat{R}_t')\) are disjoint near \(g(w)\), because \(g\) is locally one-to-one. Hence, \(g(\hat{R}_t) \cap g(\hat{R}_t') = \emptyset\), because otherwise the corresponding \(P\)-rays would have their limit sets in different components of \(KP\), a contradiction since both rays \(g(\hat{R}_t), g(\hat{R}_t')\) are \(K\)-related. Since the argument of \(g(\hat{R}_t)\) is represented by the point \(Dt \pmod{1} \in (0, 1)\), we get the inequality of (2).

Let us consider the following set \(\hat{Z}(K)\) of points in \(\mathbb{S}\): \(w \in \hat{Z}(K)\) if and only if there is a pair of disjoint \(K\)-related rays \(\hat{R}, \hat{R}'\) which both land at \(w\) and satisfy \(L(\hat{R}, \hat{R}') \geq 1/(2D)\). Denote by \(\hat{Y}(K)\) the set of periodic points which are in forward images of the points of \(\hat{Z}(K)\).

(3) If the set \(\hat{Z}(K)\) is non-empty, then it is finite, and consists of (pre)-periodic points.

Indeed, \(\hat{Z}(K)\) is finite by (1). Assume \(w \in \hat{Z}(K)\). Then by (2) some iterate \(g^n(w)\) must hit \(\hat{Z}(K)\) again.

To complete the proof, choose disjoint \(K\)-related rays \(\hat{R}_t, \hat{R}_t'\) landing at \(w \in \mathbb{S}\) and use this to prove that all claims of (ii) hold.

We show that the orbit \(w, g(w), \ldots\) cannot be infinite. Indeed, otherwise by (1)–(2), we have a sequence of non-degenerate pairwise disjoint arcs \((g^n(\hat{R}_t), g^n(\hat{R}_t')) \subset \mathbb{S}, n = 0, 1, \ldots\) By (2), some iterates of \(w\) must hit the finite set \(\hat{Z}(K)\) and hence \(\hat{Y}(K)\) (which are therefore non-empty), a contradiction.

Hence for some \(0 \leq n < l\), \(g^n(w) = g^l(w)\); let us verify that the other claims of (ii) hold. Replacing \(w\) by \(g^n(w)\), we may assume that \(w\) is a (repelling) periodic point of \(g\) of period \(k = l - n\). By (2), \(w \in \hat{Y}(K)\). By [LP96, Theorem 1], the set of \(K\)-related rays landing at \(w\) is finite, and each \(K\)-related ray landing at \(w\) is periodic with the same period. Hence, (ii) holds. Finally, the last claim of the lemma follows because a periodic non-smooth ray must have infinitely many broken points, hence, no other ray can have a common arc with it that goes up to the Julia set; see [ABC16, Lemma 6.1] for details.

3. Proofs of Theorems 2–3

3.1. Theorem 2. Part (a) is an immediate corollary of Lemma 2.1 and Lindelöf’s theorem, as in [LP96]. Indeed, since a curve \(s \subset W \setminus K_f\) converges
to a point \( a \in K_f \), the curve \( \tilde{s} = \psi(s) \) converges to a point \( z_0 \in \hat{S} \), and the limit of the function \( \psi^{-1} \) along the curve \( \tilde{s} \) exists and equals \( a \). By Lemma 2.1 there is a \( K \)-related ray \( R \) that tends to \( z_0 \), and it tends non-tangentially. Then, by [Pom] Corollary 2.17, the \( P \)-ray \( R \) converges to the same point \( a \). By definition, the curves \( s, R \) are \( K_f \)-equivalent.

Let us prove part (b). The closed set \( S \cup K_f \) is connected and so too is its complement (by the Maximum Principle). Consider the set \( \tilde{S} = \psi(S) \subset \mathbb{D}^* \). Let \( I = \tilde{S} \setminus \tilde{S} \). Then \( I \) is a connected closed subset of the unit circle \( \hat{S} \).

Let us prove \( I \) is a single point. Otherwise there is an interior point \( x \in I \) which is \( g \)-periodic. Let \( \beta \) be a \( K \)-related ray that lands at \( x \). Notice that since \( x \) is an interior point of \( I \), \( \beta \) must cross \( \tilde{S} \). Now, since \( x \) is \( g \)-periodic, \( R = \psi^{-1}(\beta) \) is a periodic \( P \)-ray, hence it converges to a periodic point \( a \in \mathbb{S} \setminus S \) of \( P \) and crosses \( S \), a contradiction since \( S \subset K_P \). This proves that \( I \) is a single point; denote it by \( z_0 \).

Choose two sequences \( z_n', z_n'' \) of \( S \) tending to \( z_0 \) from the left and from the right respectively, and two sequences of \( K \)-related rays \( l_n', l_n'' \) so that \( l_n' \) lands at \( z_n' \) and \( l_n'' \) lands at \( z_n'' \). Then, passing perhaps to subsequences, by Claim 2 in the proof of Lemma 2.1 the sequence \( l_n' \) tends to a \( K \)-related ray \( l' \) and \( l_n'' \) tends to a \( K \)-related ray \( l'' \), where \( l' \) and \( l'' \) land at the same \( z_0 \). By the above, \( l', l'' \) are disjoint.

Now we apply Lemma 2.3 to conclude that \( z_0 \) is \( g \)-(pre-)periodic, and some iterate of \( z_0 \) lies in a finite set \( Y \subset \hat{S} \) of periodic points, which is independent of \( z_0 \). Hence, the point \( a \) is \( P \)-(pre-)periodic, and some iterate of \( a \) lies in a finite set \( Y \subset J_f \) of periodic points, which is independent of \( a \). As every point of \( Y \) is a landing point of a periodic ray, it can be either repelling or parabolic.

**3.2. Theorem** 3. Proof of (b), (c): It follows from the definition of \( \Lambda \) that \( \sigma_D(\Lambda) = \Lambda \) and \( \sigma_m \circ p = p \circ \sigma_D \) on \( \Lambda \). By invariance and since \( \Lambda \neq \mathbb{T} \), the set \( \Lambda \) contains no intervals; (c) is a reformulation of a part of the statement of Theorem 1.

Proof of (a), (d): Considering \( \Lambda \) as a subset of \( \mathbb{S} = \{ |z| = 1 \} \) define a new map \( p_K : \Lambda \to \mathbb{S} \) as follows: for \( \tau \in \Lambda \), let \( p_K(\tau) \in \mathbb{S} \) be the landing point of a \( K \)-related ray of argument \( \tau \). Recall the map \( \Psi = \psi \circ h^{-1} \circ B_{G}^{-1} : \mathbb{D}^* \to \mathbb{D}^* \) introduced in the proof of Theorem 1 and its quasi-conformal extension \( \Psi^* : \mathbb{C} \to \mathbb{C} \). By Lemma 2.2 and the definition of the maps \( \lambda \) and \( p \), we have

\[
p_K = \Psi^*|_{\mathbb{S}} \circ p.
\]

Since \( \Psi^* : \mathbb{S} \to \mathbb{S} \) is an orientation preserving homeomorphism, it is enough to prove (a), (d) with \( p \) replaced by \( p_K \). By Lemma 2.2, \( p_K^{-1}(I) \) is closed in \( \mathbb{S} \) for any closed arc \( I \subset \mathbb{S} \). Therefore, \( \Lambda = p_K^{-1}(\mathbb{S}) \) is closed and the map \( p_K : \Lambda \to \mathbb{S} \) is continuous. To show (d), define an extension \( \tilde{p}_K : \mathbb{S} \to \mathbb{S} \) of
$p_K : A \to \mathbb{S}$ in an obvious way as follows. Let $J := (t_1, t_2)$ be a component of $\mathbb{S} \setminus A$. Then $p_K(t_1) = p_K(t_2) = w_J$ because otherwise there would be a point of $\mathbb{S}$ with no $K$-related rays landing at it. Let $\tilde{p}_K(\tau) = w_J$ for all $\tau \in J$. Then $\tilde{p}_K : \mathbb{S} \to \mathbb{S}$ is continuous. Now, given $t \in \mathbb{S}$, the set $\tilde{p}_K^{-1}\{\{t\}\}$ is either a singleton or a non-trivial closed arc. This follows from the definition of $\tilde{p}_K$ and because $K$-related rays with different arguments do not intersect unless case (i) of Theorem 2.3 takes place. Therefore, $\tilde{p}_K : \mathbb{S} \to \mathbb{S}$ is monotone and of degree one.

Proof of (e): Let $\tilde{h}$ be another straightening, $\tilde{\Psi} : \mathbb{D}^* \to \mathbb{D}^*$ the corresponding quasiconformal map and $\tilde{\Psi}^* : \mathbb{C} \to \mathbb{C}$ its quasiconformal extension. As $p_K : \mathbb{S} \to \mathbb{S}$ is independent of the straightening, by [9] we have $\tilde{p} = T|_{\mathbb{S}} \circ p$ where $T = (\tilde{\Psi}^*)^{-1} \circ \tilde{\Psi}^*$. On the other hand, on $\mathbb{D}^*$, $T = (B_G \circ \tilde{h}) \circ (B_G \circ h)^{-1}$, hence $T$ commutes with $z \mapsto z^m$ for $|z| > 1$ near $\mathbb{S}$, by definitions of $h, B_G$. Therefore, the homeomorphism $\nu := T|_{\mathbb{S}} : \mathbb{S} \to \mathbb{S}$ commutes with $z \mapsto z^m$ on $\mathbb{S}$, too. It is then well known that $\nu(z) = vz$ for some $v \in \mathbb{C}$ with modulus 1 such that $v^m = v$ (proof: as $\nu(1)^m = \nu(1)$ let $v = \nu(1)$, so that a homeomorphism $\nu_0 = v^{-1} \nu : \mathbb{S} \to \mathbb{S}$ commutes with $z \mapsto z^m$ too and $\nu_0(1) = 1$; then there is a lift $\tilde{\nu}_0 : \mathbb{R} \to \mathbb{R}$ of $\nu_0$ such that $\tilde{\nu}_0(0) = 0$, $\tilde{\nu}_0 - 1$ is 1-periodic and $\tilde{\nu}_0(mx) = m\tilde{\nu}_0(x)$ for all $x \in \mathbb{R}$, which in turn implies $\tilde{\nu}_0(n/m^k) = n/m^k$ for all $n, k \in \mathbb{Z}_{>0}$; by continuity, $\tilde{\nu}_0(x) = x$ for all $x$).

Acknowledgments. In [Le12], we answered, under an extra assumption, a question posed by Alexander Blokh to the author whether an accessible point of the filled Julia set $K_f$ of a renormalization $f$ of $P$ by some curve outside of $K_f$ is always accessible by an external ray of $P$ (i.e., by a curve outside of the filled Julia set $K_P$). Theorem 2(a) strengthens this result of [Le12], under a weaker assumption (p2). Theorem 3 was added following a recent work [PZ19] which also served as an inspiration for writing up this paper. Finally, we would like to thank Feliks Przytycki for a helpful discussion and the referee for comments that helped to improve the exposition.

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