PRODUCT OF FUNCTIONS IN $BMO$ AND $\mathcal{H}^1$ IN NON-HOMOGENEOUS SPACES

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Abstract

Under the assumption that the underlying measure is a non-negative Radon measure which only satisfies some growth condition and may not be doubling, we define the product of functions in the regular $BMO$ and the atomic block $\mathcal{H}^1$ in the sense of distribution, and show that this product may be split into two parts, one in $L^1$ and the other in some Hardy-Orlicz space.

AMS Subject Classification: 42B25; 42B30; 42B35

Keywords: local Hardy space, local BMO space, atomic block, block, non doubling measure, Hardy-Orlicz space.

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1 Introduction

In their paper [1], Bonami, Iwaniec, Jones and Zinsmeister defined the product of functions \( f \in BMO(\mathbb{R}^n) \) and \( h \in \mathcal{H}^1(\mathbb{R}^n) \) as a distribution operating on a test function \( \varphi \in \mathcal{D}(\mathbb{R}^n) \) by the rule
\[
\langle f \times h, \varphi \rangle := \langle f \varphi, h \rangle.
\]  
(1.1)

They proved that such distribution can be written as the sum of a function in \( L^1(\mathbb{R}^n) \) and a distribution in a Hardy-Orlicz space \( \mathcal{H}^{\psi}(\mathbb{R}^n, \nu) \) where
\[
\nu(t) = \frac{t}{\log(e + t)} \quad \text{and} \quad d\nu(x) = \frac{dx}{\log(e + |x|)}.
\]  
(1.2)

Bonami and Feuto in [2] considered the case where \( BMO(\mathbb{R}^n) \) is replaced by its local version \( \text{bmo}(\mathbb{R}^n) \) introduced by Golberg in [3], and proved that in this case, the weighted Hardy-Orlicz space is replaced by a space of amalgam type in the sense of Wiener [4]. Following the idea in [1] and [2], the author in [5] generalized this result in the setting of space of homogeneous type \((\phi, d, \mu)\). We recall that a space of homogeneous type is a non-empty set \( X \) equipped with a quasi metric \( d \) and a positive Radon measure \( \mu \) such that
\[
\mu(B(x, r)) \leq C\mu(B(x, r)), \quad x \in X, \ r > 0
\]  
(1.3)

where \( B(x, r) = \{y \in X: d(x, y) < r\} \) is the ball centered at \( x \) and having radius \( r \).

This doubling condition is an essential assumption for most results in classical function spaces, Calderón-Zygmund theory and operators theory. However, it has been shown recently (see [6], [7], [8], [9] and [10], and the reference therein) that one can drop the doubling condition and still obtain interesting results in the classical Calderón-Zygmund theory and on the classical Hardy and \( BMO \) spaces. In particular, Tolsa in [7] introduced, when the measure satisfies only the growth condition (1.4), the regular bounded mean oscillation space \( \text{RBMO}(\mu) \) and its predual space \( \mathcal{H}_{ab}^{1,\infty}(\mu) \). He showed that these spaces have similar properties to those of the classical \( BMO \) and \( \mathcal{H}^1 \) defined for doubling measures.

The purpose of this paper is to define the product of function in \( \text{RBMO}(\mu) \) and \( \mathcal{H}_{ab}^{1,\infty}(\mu) \) in the sense of distribution, and to prove that some results obtained in [2], [5] and [1] are valid in this context. To make our idea clear, let us give some notations and definitions.

Let \( n,d \) be some fixed integers with \( 0 < n \leq d \). We consider \((\mathbb{R}^d, |\cdot|, \mu)\), where \(|\cdot|\) is the Euclidean metric and \( \mu \) a positive Radon measure that only satisfies the following growth condition
\[
\mu(B(x, r)) \leq C_0 r^n, \quad \text{for all} \ x \in \mathbb{R}^d \text{ and } r > 0,
\]  
(1.4)

where \( C_0 > 0 \) is an absolute constant. Throughout the paper, by a cube \( Q \subset \mathbb{R}^d \), we mean a closed cube with sides parallel to the axis and centered at some point \( x_Q \) of \( \text{supp}(\mu) \), and if \( ||\mu|| < \infty \), we allow \( Q = \mathbb{R}^d \) too.

If \( Q \) is a cube, we denote by \( \ell(Q) \) the side length of \( Q \) and for \( \alpha > 0 \), we denote \( \alpha Q \) the cube with same center as \( Q \), but side length \( \alpha \) times as long. We will always choose the constant \( C_0 \) in (1.4) such that for all cubes \( Q \), we have \( \mu(Q) \leq C_0 \ell(Q)^n \).

For two fixed cubes \( Q \subset R \) in \( \mathbb{R}^d \), set
\[
S_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{\ell^n(2^k Q)}
\]  
(1.5)
where $N_{Q,R}$ is the smallest positive integer $k$ such that $\ell(2^k Q) \geq \ell(R)$ (in the case $R = \mathbb{R}^d \neq Q$ we set $N_{Q,R} = \infty$).

For a fixed $\rho > 1$ and $p \in (1, \infty]$, a function $b \in L^1_{loc}(\mu)$ is called a $p$-atomic block if

(i) there exists some cube $\ell$ such that $\text{supp } b \subset \ell$,

(ii) $\int_{\mathbb{R}^d} b \, d\mu = 0$,

(iii) there are functions $a_j$ supported on cubes $Q_j \subset \ell$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \sum_{j=1}^{\infty} \lambda_j a_j$ and

$$\|a_j\|_{L^p(\mu)} \leq (\mu(\ell_{Q_j}))^{\frac{1}{p} - 1} \left( S_{Q_j, R} \right)^{-1},$$

where we used the natural convention that $\frac{1}{\infty} = 0$. We put

$$|b|_{\mathcal{H}^{1,p}_{atb}(\mu)} := \sum_j |\lambda_j|.$$

**Definition 1.1.** ([7]) We say that $h \in \mathcal{H}^{1,p}_{atb}(\mu)$ if there are $p$-atomic blocks $b_j$ such that

$$h = \sum_{j=1}^{\infty} b_j \text{ with } \sum_{j=1}^{\infty} |b_j|_{\mathcal{H}^{1,p}_{atb}(\mu)} < \infty,$$

The atomic block Hardy space $\mathcal{H}^{1,p}_{atb}(\mu)$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{H}^{1,p}_{atb}(\mu)}$ defined by

$$\|h\|_{\mathcal{H}^{1,p}_{atb}(\mu)} = \inf_{\sum_{j=1}^{\infty} |b_j|_{\mathcal{H}^{1,p}_{atb}(\mu)}} \sum_{j=1}^{\infty} |b_j|_{\mathcal{H}^{1,p}_{atb}(\mu)}, \quad h \in \mathcal{H}^{1,p}_{atb}(\mu),$$

where the infimum is taken over all possible decomposition of $h$ into atomic blocks.

As it is proved in Proposition 5.1 and in Theorem 5.5 of [7], the definition of $\mathcal{H}^{1,p}_{atb}(\mu)$ does not depend on $p$ and we have that, for all $1 < p < \infty$, the spaces $\mathcal{H}^{1,p}_{atb}(\mu)$ are topologically equivalent to $\mathcal{H}^{1,\infty}_{atb}(\mu)$. So in the sequel, we shall use the notation $\mathcal{H}^{1}(\mu)$ instead of $\mathcal{H}^{1,\infty}_{atb}(\mu)$, and take $p = 2$.

When $b \in L^1_{loc}(\mu)$ satisfies only Condition (i) and (iii) of the definition of atomic blocks, we say that it is a $p$-block and put $|b|_{\mathcal{H}^{1}_{atb}(\mu)} = \sum_j |\lambda_j|$. Moreover, we say that $h$ belongs to the local Hardy space $\mathcal{H}^{1}_{atb}(\mu)$ (see [7]) if there are $p$-atomic blocks or $p$-blocks $b_j$ such that

$$h = \sum_{j=1}^{\infty} b_j,$$

where $\sum_{j=1}^{\infty} |b_j|_{\mathcal{H}^{1}_{atb}(\mu)} < \infty$, $b_j$ is an atomic block if $\text{supp } b_j \subset R_j$ and $\ell(R_j) \leq 1$, and $b_j$ is a block if $\text{supp } b_j \subset R_j$ and $\ell(R_j) > 1$. We define the $\mathcal{H}^{1}_{atb}(\mu)$ norm of $h$ by

$$\|h\|_{\mathcal{H}^{1}_{atb}(\mu)} = \inf_{\sum_{j=1}^{\infty} |b_j|_{\mathcal{H}^{1}_{atb}(\mu)}} \sum_{j=1}^{\infty} |b_j|_{\mathcal{H}^{1}_{atb}(\mu)}.$$
where the infimum is taken over all possible decompositions of $h$ into atomic blocks or blocks.

The definition of local Hardy space is independent of $\rho > 0$ and for $1 < p < \infty$, we have $h_{ab}^{1,p}(\mu) = h_{ab}^{1,\infty}(\mu)$ (see Proposition 3.4 and Theorem 3.8 of [9]). This allow us to just denote it by $h^1(\mu)$ and consider also $\rho = 2$.

In Theorem 5.5 of [7] and Theorem 3.8 of [9], it is proved that the dual space of $H^1(\mu)$ and $h^1(\mu)$ are respectively RBMO($\mu$) and its local version rbmo($\mu$) (see Section 2 for more explanations about these spaces).

Let $h = \sum b_j$ belonging to $H^1(\mu)$, where the atomic block $b_j$ is supported in the cube $R_j$ and satisfies $b_j = \sum_1^\infty \lambda_{ij} a_{ij}$ for $a_{ij}$'s and $\lambda_{ij}$'s as in the definition of atomic blocks. For $f \in$ RBMO($\mu$), we denote by $f_{\hat{R}}$ the mean value of $f$ over the cube $\hat{R}$, which is an appropriate dilation of the cube $R$ (see Section 2 for more explanation). We can see from the proof of Theorem 1.2 that the double series

$$\sum_{j=1}^{\infty} \left( f - f_{\hat{R}_j} \right) b_j = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} \lambda_{ij} \left( f - f_{\hat{R}_j} \right) a_{ij} \right)$$

converges normally in $L^1(\mu)$, while

$$\sum_{j=1}^{\infty} f_{\hat{R}_j} b_j = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} f_{\hat{R}_j} \lambda_{ij} a_{ij} \right)$$

converges in the Hardy-Orlicz space $H^{\varphi}(\nu)$, where $\varphi(t) = \frac{t}{\log(e+t)}$ and $d\nu(x) = \frac{d\mu(x)}{\log(e+|x|)}$. Since both convergence implies convergence in the sense of distribution, we define the product of $f$ and $h$ as the sum of both series by

$$f \times h = \sum_{j=1}^{\infty} \left( f - f_{\hat{R}_j} \right) b_j + \sum_{j=1}^{\infty} f_{\hat{R}_j} b_j.$$

It follows that

**Theorem 1.2.** For $f$ in RBMO($\mu$) and $h$ in $H^1(\mu)$, the product $f \times h$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion

$$f \times h \in L^1(\mu) + H^{\varphi}(\nu).$$

When we replaced RBMO($\mu$) by its local version rbmo($\mu$) as define in [9] (see also [11]) we obtain the analogous of the result in [2]. We also obtain interesting results by replacing both RBMO($\mu$) and $H^1(\mu)$ with their local version.

The paper is organized as follows, in Section 2 we recall the definition of the space RBMO($\mu$), its local version and some properties involved.

Section 3 is devoted to auxiliary results and prerequisites in Orlicz spaces while in Section 4 we give the proof of the main results and their extensions.

Throughout the paper, the letter $C$ is used for non-negative constants that may change from one occurrence to another. Constants with subscript, such as $C_0$, do not change in different occurrences. The notation $A \approx B$ stands for $C^{-1} A \leq B \leq C A$, $C$ being a constant not depending on the main parameters involved.
2 Prerequisite about $RBMO(\mu)$, $\mathfrak{rbmo}(\mu)$, $\mathcal{H}^1(\mu)$ and $\mathfrak{h}^1(\mu)$ spaces

Definition 2.1. Let $\alpha > 1$ and $\beta > \alpha^n$, we say that a cube $Q$ is an $(\alpha, \beta)$-doubling cube if $\mu(\alpha Q) \leq \beta \mu(Q)$.

It is proved in [7] that there are a lot of "big" doubling cubes and also a lot of "small" doubling cubes, this due to the facts that $\mu$ satisfies the growth Condition (1.4) and $\beta > \alpha^n$. More precisely, given any point $x \in \text{supp}(\mu)$ and $c > 0$, there exists some $(\alpha, \beta)$-doubling cube $Q$ centered at $x$ with $\ell(Q) \geq c$.

On the other hand, if $\beta > \alpha^n$ then, for $\mu$-a.e. $x \in \mathbb{R}^d$, there exists a sequence of $(\alpha, \beta)$-doubling cubes $\{Q_k\}_{k \in \mathbb{N}}$ centered at $x$ with $\ell(Q_k) \to 0$ as $k \to \infty$.

In the following, for any $\alpha > 1$, we denote by $\beta_\alpha$ one of these big constants $\beta$. For definiteness, one can assume that $\beta_\alpha$ is twice the infimum of these $\beta$'s.

Given $\rho > 1$, we let $N$ be the smallest non-negative integer such that $2^N Q$ is $(\rho, \beta_\rho)$-doubling and we denote this cube by $\tilde{Q}$.

Definition 2.2. (9) Let $\rho > 1$ be some fixed constant.

(a) Let $1 < \eta < \infty$. We say that $f \in L^1_{\text{loc}}(\mu)$ is in $RBMO(\mu)$ if there exists a non-negative constant $C_2$ such that for any cube $Q$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(x) - f_Q| d\mu(x) \leq C_2,$$

(2.1)

and for any two $(\rho, \beta_\rho)$-doubling cubes $Q \subset R$

$$|f_Q - f_R| \leq C_2 S_{Q,R}.$$  

(2.2)

Let us put

$$\|f\|_{RBMO(\mu)} = \inf \{C_2 : (2.1) \text{ and } (2.2) \text{ hold} \}.$$  

(2.3)

(b) Let $1 < \eta \leq \rho < \infty$. We say that $f \in L^1_{\text{loc}}(\mu)$ belongs to $\mathfrak{rbmo}(\mu)$ if there exists some constant $C_3$ such that (2.7) holds for any cube $Q$ with $\ell(Q) \leq 1$ and $C_3$ instead of $C_2$, (2.2) holds for any two $(\rho, \beta_\rho)$-doubling cubes $Q \subset R$ with $\ell(Q) \leq 1$ and $C_3$ instead of $C_2$, and

$$\frac{1}{\mu(\eta Q)} \int_Q |f(x)| d\mu(x) \leq C_3$$

(2.4)

for any cube $Q$ with $\ell(Q) > 1$. We set

$$\|f\|_{\mathfrak{rbmo}(\mu)} = \inf \{C_3 : (2.7), (2.2) \text{ and } (2.4) \text{ hold} \}.$$  

(2.5)

We should have referred to the choice of constants $\eta, \rho$ and $\beta$ in the terminology, but it is proved in [7] and [9] that $RBMO(\mu)$ and $\mathfrak{rbmo}(\mu)$ are independent of their choice. We also have (see Proposition 2.5 of [7] and Proposition 2.2 of [9]) that $(RBMO(\mu), \|\cdot\|_{RBMO(\mu)})$ and $(\mathfrak{rbmo}(\mu), \|\cdot\|_{\mathfrak{rbmo}(\mu)})$ are Banach spaces of functions (modulo additive constants).

We have that $S_{Q,R} \approx 1 + \delta(Q,R)$ (see [3]), where

$$\delta(Q,R) = \max \left( \int_{Q_R \setminus Q} \frac{d\mu(x)}{|x - x_Q|^\gamma}, \int_{R_Q \setminus R} \frac{d\mu(x)}{|x - x_R|^\gamma} \right),$$

(2.6)
and there exists a constant $\kappa > 0$ such that for all cubes $Q \subset R$ we have
\[
\delta(Q,R) \leq \kappa \left(1 + \log \left(\frac{\ell(R)}{\ell(Q)}\right)\right). \tag{2.7}
\]

**Lemma 2.3.** Let $f \in RBMO(\mu)$ and $\varphi \in D(\mathbb{R}^d)$. Then the pointwise product $f \varphi \in RBMO(\mu)$. Moreover, if $f \in \text{rbmo}(\mu)$ then $f \varphi \in \text{rbmo}(\mu)$.

**Proof.** Let $f \in RBMO(\mu)$ and $\varphi \in D(\mathbb{R}^d)$ with support in the cube $Q_0$. We assume without loss of generality that $f_Q = 0$. The pointwise product $f \varphi$ belongs to $RBMO(\mu)$ if and only if for some real number $\rho > 1$, there exists $C > 0$ and a collection of numbers $\{C_Q(f \varphi)\}_Q$ (i.e. for each cube $Q$, there exists $C_Q(f \varphi) \in \mathbb{R}$) such that
\[
\int_Q |(f \varphi)(x) - C_Q(f \varphi)| \, d\mu(x) \leq C \tag{2.8}
\]
and
\[
|C_Q(f \varphi) - C_R(f \varphi)| \leq CS_{Q,R} \text{ for any two cubes } Q \subset R. \tag{2.9}
\]

**A-The choice of the numbers $C_Q(f \varphi)$ satisfying (2.8)**

Let $Q$ be a cube in $\mathbb{R}^d$. If

1. $\mu(Q \cap Q_0) = 0$, or
2. $\mu(Q \cap Q_0) > 0$ and $Q \not\subset 2Q_0$

then we take $C_Q(f \varphi) = 0$. In the case 1 we have $\int_Q |f \varphi| \, d\mu = 0$ while in the case 2 we have $Q_0 \subset 5Q$ so that
\[
\int_Q |f \varphi| \, d\mu = \int_{Q \setminus Q_0} |f \varphi| \, d\mu \leq \int_{Q_0} |f \varphi| \, d\mu \leq C \|\varphi\|_{L^\infty} \|f\|_{RBMO(\mu)} \mu(\rho Q).
\]
for any $\rho > 5$. We suppose now that $\mu(Q \cap Q_0) > 0$ and $Q \subset 2Q_0$.

We put $C_Q(f \varphi) = f_Q \varphi_Q$. It follows that
\[
\int_Q |f \varphi - f_Q \varphi_Q| \, d\mu = \int_Q |(f - f_Q) \varphi + f_Q (\varphi - \varphi_Q)| \, d\mu \leq \|\varphi\|_{L^\infty} \|f\|_{RBMO(\mu)} \mu(\rho Q) + |f_Q| \int_Q |\varphi - \varphi_Q| \, d\mu.
\]
But
\[
|f_Q| = |f_Q - f_{2Q_0}| \leq S_{Q,2Q_0} \|f\|_{RBMO(\mu)}
\]
\[
\leq C(1 + \delta_{Q,2Q_0}) \|f\|_{RBMO(\mu)} \leq C(1 + \log \left(\frac{2\ell(Q_0)}{\ell(Q)}\right)) \|f\|_{RBMO(\mu)},
\]
according to Lemma 2.4 of [8]. So that taking into consideration the following classical result
\[
\int_Q |\varphi - \varphi_Q| \, d\mu \leq C \|\nabla \varphi\|_{L^\infty} \ell(Q) \mu(Q)
\]

and the fact that $2\ell(Q_0) \geq \ell(Q)$, we obtain

$$
|f_Q| \int_Q |(\varphi - \varphi_Q)| d\mu \leq C(1 + \log(2\ell(Q_0)/\ell(Q)))\ell(Q)\|\varphi\|_{RBMO(\mu)}
\leq C\mu(Q)\|f\|_{RBMO(\mu)}.
$$

**B- Prove that the collection satisfy (2.9)**

Let $Q \subset R$ be two cubes. If $R \cap Q_0 = \emptyset$ or $Q \not\subset 2Q_0$, then $C_Q(f\varphi) = C_R(f\varphi) = 0$. Thus there is nothing to prove.

We suppose that $R \cap Q_0 \neq \emptyset$ and $Q \subset 2Q_0$.

If $R \not\subset 2Q_0$ then $C_R(f\varphi) = 0$ and $Q_0 \subset 5R$, so that

$$
C_R(f\varphi) = |f_R| |\varphi_R - \varphi_Q| \leq |f_R| |R - Q| + |f_R| |\varphi_R - \varphi_Q|.
$$

Let us estimate the second term.

$$
|f_R| |\varphi_R - \varphi_Q| \leq C|f_R| (\ell(R) + \ell(Q) + \text{dist}(x_Q, x_R))
\leq C(1 + |f_R| \text{dist}(x_Q, x_R)),
$$

where $x_Q$ and $x_R$ denote the centers of the cubes $Q$ and $R$ respectively. But $\text{dist}(x_Q, x_R) \leq C\ell(Q_R)$ and $|f_R| \leq |f_{Q_R}| + |f_{Q_R} - f_R|$, which leads to

$$
|f_R| \text{dist}(Q, R) \leq C\ell(Q_R) (|f_{Q_R}| + |f_{Q_R} - f_R|)
\leq C \|f\|_{RBMO(\mu)} + S_{Q, Q_R} \|f\|_{RBMO(\mu)} \leq C \|f\|_{RBMO(\mu)}.
$$

The result follow.

Let us consider know the particular case where $f$ belongs to $\text{rbmo}(\mu)$. For any cube $Q$ such that $\ell(Q) > 1$, we have

$$
|C_Q(f\varphi)| \leq |\varphi_Q| |\varphi| \leq |\varphi| L^- \|f\|_{\text{rbmo}(\mu)} \mu(\tilde{Q}) / \mu(\hat{Q}) \leq C \|\varphi\| L^- \|f\|_{\text{rbmo}(\mu)}
$$

for some positive constant $C$ and fixed $1 < \eta \leq \rho$, since $\ell(\hat{Q}) \geq \ell(Q)$. It follows that $f\varphi \in \text{rbmo}(\mu)$.

Inequalities of John-Nirenberg type are valid in both spaces. More precisely we have

**Theorem 2.4.** [7] Let $f \in RBMO(\mu)$. For any cube $Q$ and any $\lambda > 0$, we have

$$
\mu\left(\{x \in Q : |f(x) - f_Q| > \lambda\}\right) \leq C_4\mu(\rho Q) \exp\left(-\frac{C_5\lambda}{\|f\|_{RBMO(\mu)}}\right),
$$

where the constants $C_4 > 0$ and $C_5 > 0$ depend only on $\rho > 1$.
As we can see in Theorem 2.6 of [9], one can replace in the previous theorem the space $RBMO(\mu)$ by its local version $r\text{tmo}(\mu)$ provided the cube $Q$ satisfies $\ell(Q) \leq 1$, while for cubes $Q$ such that $\ell(Q) > 1$ we have $\mu(\{x \in Q : |f(x)| > \lambda\}) \leq C_\mu(\rho Q) \exp\left(-\frac{C_\lambda}{\|f\|_{\text{tmo}(\mu)}}\right)$.

An immediate consequence of this result is that there exists a non-negative constant $C_6$, which can be chosen as big as we like, such that for all cube $Q$ and constant $\not\equiv f \in RBMO(\mu)$,

$$\frac{1}{\mu(\rho Q)} \int_Q \exp \left( \frac{|f - f_Q|}{C_6 \|f\|_{RBMO(\mu)}} \right) d\mu \leq 1. \quad (2.11)$$

We also have the following:

**Lemma 2.5.** Let $\not\equiv f \in RBMO(\mu)$ and $Q$ the unit cube. We have

$$\int_{\mathbb{R}^d} \left( \exp \left( \frac{|f(x) - f_Q|}{k} \right) - 1 \right) d\mu(x) \leq 1 \quad (2.12)$$

where $k = C_7 \|f\|_{RBMO(\mu)}$.

**Proof.** Let $f \in RBMO(\mu)$ with $\|f\|_{RBMO(\mu)} \neq 0$. We have

$$\int_{\mathbb{R}^d} \frac{|f(x) - f_Q|}{(1 + |x|)^{2n+\kappa}} d\mu(x) = \int_Q \frac{|f(x) - f_Q|}{(1 + |x|)^{2n+\kappa}} d\mu(x) + \int_{Q^c} \frac{|f(x) - f_Q|}{(1 + |x|)^{2n+\kappa}} d\mu(x),$$

where $Q^c = \mathbb{R}^d \setminus Q$. The first term in the right hand side is less that $\mu(\rho Q)$. For the second term, we have

$$\int_{Q^c} \frac{e^{\left|f(x) - \mu\right|_{\text{tmo}(\mu)}} - 1}{(1 + |x|)^{2n+\kappa}} d\mu(x) = \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \frac{e^{\left|f(x) - \mu\right|_{\text{tmo}(\mu)}} - 1}{(1 + |x|)^{2n+\kappa}} d\mu(x) \leq C \sum_{k=0}^{\infty} 2^{-(2n+\kappa)k} \int_{2^{k+1}Q} \left( e^{\left|f(x) - \mu\right|_{\text{tmo}(\mu)}} - 1 \right) d\mu(x).$$

Furthermore, there exists a non-negative constant $K$ such that

$$|f_Q - f_{2^{k+1}Q}| \leq KS_{Q,R} \|f\|_{RBMO(\mu)}$$

for two cubes $Q \subset R$,

as we can see in the proof of Lemma 2.8 in [7]. We also have $S_{Q,2^{k+1}Q} \leq (k + 2)$, which leads to $|f_Q - f_{2^{k+1}Q}| \leq \log(2\log(k+2)) \|f\|_{RBMO(\mu)}$.

Hence

$$\sum_{k=0}^{\infty} 2^{-(2n+\kappa)k} \int_{2^{k+1}Q} \left( e^{\left|f(x) - \mu\right|_{\text{tmo}(\mu)}} - 1 \right) d\mu(x) \leq C \sum_{k=0}^{\infty} 2^{-(2n+\kappa)k} \frac{e^{\left|f(x) - f_{2^{k+1}Q}\right|_{\text{tmo}(\mu)}} - 1}{(1 + \rho_{2^{k+1}Q})^{2n+\kappa}}.$$

If we choose \( C_6 > \frac{K}{(n+\kappa)\log 2} \) then the above series converges. Finally we have

\[
\int_{\mathbb{R}^d} \frac{|f(x) - f_Q|}{(1 + |x|)^{2n+\kappa}} d\mu(x) \leq K_1,
\]

where \( K_1 \) is a non-negative constant not depending on \( f \).

Thus the result follows from taking \( C_7 = \max(C_6, K_1)C_6 \).

\( \square \)

### 3 Some properties of Orlicz and Hardy-Orlicz space

For the definition of Hardy-Orlicz space, we need the maximal characterization of \( \mathcal{H}^1(\mu) \) given in [8].

Let \( f \in L^1_{\text{loc}}(\mu) \), we set

\[
\mathcal{M} f(x) = \sup_{\phi \in F(x)} \left| \int_{\mathbb{R}^d} f\phi d\mu \right|,
\]

where for \( x \in \mathbb{R}^d \), \( F(x) \) is the set of \( \phi \in L^1(\mu) \cap C^1(\mathbb{R}^d) \) satisfying the following conditions:

\[
\|\phi\|_{L^1(\mu)} \leq 1,
\]

\[
0 \leq \phi(y) \leq \frac{1}{|y-x|^n} \quad \text{for all } y \in \mathbb{R}^d
\]

and

\[
|\nabla \phi(y)| \leq \frac{1}{|y-x|^{n+1}} \quad \text{for all } y \in \mathbb{R}^d.
\]

Tolsa proved in Theorem 1.2 of [8] that a function \( f \in L^1(\mu) \) belongs to the Hardy space \( \mathcal{H}^1(\mu) \) if and only if \( \int_{\mathbb{R}^d} f d\mu = 0 \) and \( \mathcal{M} f \in L^1(\mu) \). Moreover, in this case we have

\[
\|f\|_{\mathcal{H}^1(\mu)} \approx \|f\|_{L^1(\mu)} + \|\mathcal{M} f\|_{L^1(\mu)}.
\]

Hardy-Orlicz spaces are defined via this maximal characterization. We recall that for a continuous function \( \mathcal{P} : [0, \infty) \to [0, \infty) \) increasing from zero to infinity (but not necessarily convex), the Orlicz space \( L^\mathcal{P}(\mu) \) consists of \( \mu \)-measurable functions \( f : \Omega \to \mathbb{R} \) such that

\[
\|f\|_{L^\mathcal{P}(\mu)} := \inf \left\{ k > 0 : \int_{\mathbb{R}^d} \mathcal{P} \left( k^{-1} |f| \right) d\mu \leq 1 \right\} < \infty.
\]

In general, the nonlinear functional \( \|\cdot\|_{L^\mathcal{P}(\mu)} \) need not satisfy the triangle inequality. It is well known that \( L^\mathcal{P}(\mu) \) is a complete linear metric space, see [12]. The \( L^\mathcal{P} \)-distance between \( f \) and \( g \) is given by

\[
\text{dist}_{\mathcal{P}}[f, g] := \inf \left\{ \rho > 0 : \int_{\mathbb{R}^d} \mathcal{P} \left( \rho^{-1} |f - g| \right) d\mu \leq \rho \right\} < \infty.
\]
The Hardy-Orlicz space $\mathcal{H}^p(\mu)$ consists of local integrable function $f$ such that $\mathcal{M} f \in L^p(\mu)$. We put
\[
\|f\|_{\mathcal{H}^p(\mu)} = \|\mathcal{M} f\|_{L^p(\mu)}.
\] (3.8)
It comes from what precede that $\mathcal{H}^p(\mu)$ is a complete linear metric space, a Banach space when $\mathcal{P}$ is convex. These spaces have previously been dealt with by many authors, see \cite{13,14,15} and further references given there. When we consider the Orlicz function $\varphi(t) = \frac{t}{\log(e+t)}$, we have the following results given in \cite{11}.

- If $\text{dist}_\varphi[f,g] \leq 1$ then $\|f-g\|_{L^p(\mu)} \leq \text{dist}_\varphi[f,g] \leq 1$.
- The sequence $(f_j)_{j>0}$ converges to $f$ in $L^p(\mu)$ if and only if $\|f_j-f\|_{L^p(\mu)} \to 0$.
- We have duality between the Orlicz space $L^\varphi(\mu)$ associated to the Orlicz function $\Xi(t) = e^t - 1$ and $L^\varphi(\mu)$ with $\varphi(x) = x \log(e+x)$ in the sense that for $f \in L^\varphi(\mu)$ and $g \in L^{\varphi'}(\mu)$ we have
\[
\|fg\|_{L^1(\mu)} \leq \|f\|_{L^\varphi(\mu)} \|g\|_{L^{\varphi'}(\mu)}.
\] (3.9)

- For $f,g \in L^p(\mu)$, we have the following substitute of the additivity
\[
\|f+g\|_{L^p(\mu)} \leq 4 \|f\|_{L^p(\mu)} + 4 \|g\|_{L^p(\mu)}.
\] (3.10)

- Let
\[
d\sigma = \frac{d\mu}{(1+|x|)^{2n+\kappa}} \quad \text{and} \quad d\nu = \frac{d\mu}{\log(e+|x|)},
\] (3.11)
for $f \in L^\varphi(\sigma)$ and $g \in L^1(\mu)$, we have $f g \in L^\varphi(\nu)$ and
\[
\|f g\|_{L^\varphi(\nu)} \leq C \|f\|_{L^\varphi(\sigma)} \|g\|_{L^1(\mu)}.
\] (3.12)
and for $f \in RBMO(\mu)$ and $g \in L^1(\mu)$,
\[
\|f g\|_{L^\varphi(\nu)} \leq C \|f\|_{RBMO(\mu)} + \|g\|_{L^1(\mu)},
\] (3.13)
where $\|f\|_{RBMO^+(\mu)} = \|f\|_{RBMO(\mu)} + \|f_Q\|_{L^1(\mu)}$.

4 Proof of the main results

Proof of Theorem \[1.2\] Let $f \in RBMO(\mu)$ and $h \in \mathcal{H}^1(\mu)$, $h$ having the $p$-atomic blocks decomposition given in (1.8), i.e.
\[
h = \sum_j b_j,
\] (4.1)
where $b_j = \sum_{i=1}^{n} \lambda_{ij} a_{ij}$ is the atomic-block supported in the cube $R_j$, $a_{ij}$ supported in the cube $Q_{ij} \subset R_j$ and $\|a_{ij}\|_{L^p(\mu)} \leq \mu(\partial Q_{ij})^{-1} (S_{Q_{ij},R_j})^{-1}.

We have
\[
\|\lambda_{ij} (f-f_{R_j}) a_{ij}\|_{L^1(\mu)} \leq |\lambda_{ij}| \int_{Q_{ij}} |f-f_{R_j}| |a_{ij}| d\mu
\leq |\lambda_{ij}| \left( \int_{Q_{ij}} |f-f_{\partial Q_{ij}}| |a_{ij}| d\mu + \int_{Q_{ij}} |f_{R_j} - f_{\partial Q_{ij}}| |a_{ij}| d\mu \right)
\leq C |\lambda_{ij}| \|f\|_{RBMO(\mu)},
\]
according to Inequalities (2.13) and (1.6), which proves that the first series \( \sum_{j=1}^{\infty} \left( f - f_{R_j} \right) b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda_{ij} \left( f - f_{R_j} \right) a_{ij} \) converges normally in \( L^1(\mu) \), since the atomic decomposition theorem asserts that the double series \( \sum_{i,j} |\lambda_{ij}| \) converges. It remains to prove the convergence of

\[
S = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} \lambda_{ij} f_{R_j} a_{ij} \right) = \sum_{j=1}^{\infty} f_{R_j} b_j
\]

in \( M^p(\nu) \). For this purpose, we have to prove that the sequence \( S_N = M \left( \sum_{j=1}^{N} f_{R_j} b_j \right) \) is Cauchy in \( L^p(\nu) \). This is equivalent to prove that \( \lim_{l \to \infty} \| M \left( \tilde{S}_l^k \right) \|_{L^p(\nu)} = 0 \), where

\[
\tilde{S}_l^k = \sum_{j=l}^{k} f_{R_j} b_j \text{ with } l \leq k.
\]

Since

\[
M \left( f_{R_j} b_j \right) \leq |f - f_{R_j}| M(b_j) + |f| M(b_j),
\]

we have that

\[
\| M \left( \tilde{S}_l^k \right) \|_{L^p(\nu)} \leq 4 \left( \sum_{j=1}^{k} |f - f_{R_j}| M(b_j) \right)_{L^1(\mu)} + 4 \left( \sum_{j=1}^{k} |f| M(b_j) \right)_{L^p(\nu)},
\]

according to (3.10) and the fact that \( \|f\|_{L^p(\mu)} \leq \|f\|_{L^1(\mu)} \) for all measurable functions \( f \). Let us consider the first term in the second member of (4.5). We have

\[
\left\| \sum_{j=1}^{k} |f - f_{R_j}| M(b_j) \right\|_{L^1(\mu)} \leq \sum_{j=1}^{k} \left\| \sum_{i=1}^{\infty} |\lambda_{ij}| \left( |f - f_{Q_{ij}}| + |f_{R_j} - f_{Q_{ij}}| \right) M(a_{ij}) \right\|_{L^1(\mu)},
\]

since \( M(b_j) \leq \sum_{i=1}^{\infty} |\lambda_{ij}| M(a_{ij}) \). From the definition of \( M(a_{ij}) \), we have

\[
M(a_{ij})(x) \leq \mu(pQ_{ij})^{-1} (S_{Q_{ij},R_j})^{-1},
\]

so that taking into consideration relation (2.13), we obtain

\[
\left\| \left( |f - f_{Q_{ij}}| + |f_{R_j} - f_{Q_{ij}}| \right) M(a_{ij}) \right\|_{L^1(\mu)} \leq C \| f \|_{RBMO(\mu)}.
\]

Thus

\[
\lim_{l \to \infty} \left\| \sum_{j=1}^{k} |f - f_{R_j}| M(b_j) \right\|_{L^1(\mu)} = 0,
\]

since the double series \( \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} |\lambda_{ij}| \right) \) converges. Let us consider now the series

\[
\left\| \sum_{j=1}^{k} |f| M(b_j) \right\|_{L^p(\nu)} = \left\| |f| \sum_{j=1}^{k} M(b_j) \right\|_{L^p(\nu)}.
\]
We have
\[ \left\| \sum_{j=l}^{k} \mathcal{M} (b_j) \right\|_{L^1(\mu)} \leq C \sum_{j=l}^{k} \left( \sum_{i=1}^{n} |\lambda_{ij}| \right), \] (4.10)
according to Lemma 3.1 of [8]. Furthermore, we have
\[ \left\| f \right\|_{L^1(\mu)} \leq \| f \|_{RBMOC^* (\mu)} \left( \sum_{j=l}^{k} \mathcal{M} (b_j) \right), \] (4.11)
according to (3.13).

\[ |f| \sum_{j=l}^{k} \mathcal{M} (b_j) \leq \| f \|_{RBMOC^* (\mu)} \left( \sum_{j=l}^{k} \mathcal{M} (b_j) \right), \] (4.13)

We accordingly define \( \mathcal{N}_P^p \). Using the concavity described above, we have \( \varphi(st) \leq Cs\varphi(t) \) for \( s > 1 \). It follows that \( L^p \) is contained in \( L^p \) as a consequence of the fact that
\[ \| f \|_{L^p(\mu)} \leq \int_Q \| f \|_{L^p(\mu)} d\mu(x). \] The converse inclusion is not true.

**Theorem 4.2.** For \( h \in \mathcal{H}^1 (\mu) \) and \( f \in rbmo (\mu) \), the product \( f \times h \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion
\[ f \times h \in L^1 (\mu) + \mathcal{H}^*_p (\mu). \] (4.12)

**Proof.** The proof is inspired by the one given in [2] in the case of Lebesgue measure. Let \( f \in rbmo(\mu) \) and \( h \in \mathcal{H}^1 (\mu) \) being as in the proof of Theorem 1.2. The series
\[ \sum_j \left( \sum_i \lambda_{ij} (f - f_{R}) a_{ij} \right) \cdot \sum_j (f - f_{R}) \mathcal{M} (b_j) \] and \( \sum_j \mathcal{M} (b_j) \) converge normally in \( L^1 (\mu) \) and
\[ \mathcal{M} \left( \sum_j b_j f_{R} \right) \leq \sum_j \| f - f_{R} \| \mathcal{M} (b_j) + \| f \| \sum_j \mathcal{M} (b_j). \] (4.14)

Thus we just have to prove that the second term in the right hand side of (4.14) is in \( L^p_\mu \). Let \( Q \) be a cube of side length 1. By John-Nirenberg inequality on \( rbmo(\mu) \), we have that there exists \( c_7 > 0 \) (we can choose any number greater than \( \frac{1}{c_3} + \frac{2\pi}{c_3} \)) such that
\[ \int_Q \left( e^{c_7 \| f \|_{\text{rbmo}(\mu)}} - 1 \right) d\mu(x) \leq 1. \] (4.15)

We claim that for \( \psi \in L^1 (\mu) \)
\[ \| f \psi \|_{L^p(Q)} \leq C \| f \|_{rbmo(\mu)} \int_Q |\psi|d\mu. \] (4.16)
In fact, by homogeneity, we can assume that \( c \| f \|_{rbmo(\mu)} = 1 \) and it is sufficient to find some constant \( c \) such that for \( \int_Q |\psi| \, d\mu = c \) we have

\[
\int_Q \frac{|\psi|}{\log(e + |\psi|)} \, d\mu \leq 1.
\]

We have

\[
\int_Q \frac{|f\psi|}{\log(e + |f\psi|)} \, d\mu = \int_{Q \cap \{|f| \leq 1\}} \frac{|f\psi|}{\log(e + |f\psi|)} \, d\mu + \int_{Q \cap \{|f| > 1\}} \frac{|f\psi|}{\log(e + |f\psi|)} \, d\mu.
\]  

(4.17)

The first term in the second member is bounded by \( \int_Q |\psi| \, d\mu \) and for the second term, we have

\[
\int_{Q \cap \{|f| > 1\}} \frac{|f\psi|}{\log(e + |f\psi|)} \, d\mu \leq \int_{Q \cap \{|f| > 1\}} \frac{|f|}{\log(e + |\psi|)} \, d\mu \leq \|f\|_{L^2(\mu)} \left\| \frac{|\psi|}{\log(e + |\psi|)} \right\|_{L^p(\mu)} \leq C \left\| \frac{|\psi|}{\log(e + |\psi|)} \right\|_{L^p(\mu)}.
\]

But

\[
\int_Q \frac{|\psi|}{\log(e + |\psi|)} \log \left( e + \frac{|\psi|}{\log(e + |\psi|)} \right) \, d\mu \leq \int_Q \frac{|\psi|}{\log(e + |\psi|)} \log(e + |\psi|) \, d\mu \leq \int_Q |\psi| \, d\mu
\]

Thus if \( c < \frac{1}{2} \) and \( \int_Q |\psi| \, d\mu = c \) the result follows. We have an estimate for each cube \( j + Q \), and sum up. This finishes the proof.

Since we do not have any maximal function characterization of the local Hardy spaces on non-homogeneous space in the literature, we are going to define the local space corresponding to \( H^1_{\mu} \) in the same manner as in [2]. For this purpose, we put

\[
M^{(1)} f(x) = \sup_{F_{\text{loc}}(x)} \left| \int f \phi \, d\mu \right|,
\]

(4.18)

where \( F_{\text{loc}}(x) \) denotes the set of elements belonging to \( F(x) \) as define in Section [3] but having their support in the cube \( Q(x, 1) \) centered at \( x \) with side length 1. A locally integrable function \( f \) belongs to the space \( H^1_{\mu} \) if \( M^{(1)} f \in L^p(\mu) \).

**Proposition 4.3.** For \( h \) a function in \( H^1(\mu) \) and \( b \) a function in \( rbmo(\mu) \), the product \( b \times h \) can be given a meaning in the sense of distributions. Moreover, we have the inclusion

\[
b \times h \in L^1(\mu) + H^1_{\mu}(\mu).
\]

(4.19)

**Proof.** Let \( f \in rbmo(\mu) \) and \( h \in H^1(\mu) \) with \( h = \sum_j b_j \) where \( b_j \)’s are atomic blocks or blocks. Since we do not use the cancellation property of \( b_j \)’s to prove that the \( \sum_j (f - f_{R_j}) b_j \) converge absolutely in \( L^1(\mu) \), it follows that the result remains true in this case. Thus we just
have to prove that the second term belongs to the amalgam space \( h^s \mu \). This immediate if we prove that for any bock \( j \), the quantity \( \| \mathcal{M}^{(1)} b_j \|_{L^1(\mu)} \) is bounded by a constant which is independent on \( b_j \). Let \( b_j = \sum_{i=1}^{\infty} \lambda_{ij} a_{ij} \), where \( a_{ij} \) is supported in the cube \( Q_{ij} \subset R_j \) and satisfy \( \| a_{ij} \|_{L^\infty(\mu)} \leq (\mu(2Q_{ij})S_{Q_{ij},R_j})^{-1} \). For every integer \( i \), we have

\[
\mathcal{M}^{(1)} a_{ij}(x) \leq (\mu(2Q_{ij})S_{Q_{ij},R_j})^{-1} \chi_{2R_j}(x),
\]

where \( \chi_{2R_j} \) denote the characteristic function of \( 2R_j \). In fact, if \( \varphi \in F_{loc}(x) \) then \( \int a_{ij} \varphi d\mu \neq 0 \) only if \( x \in 2R_j \), since \( \ell(R_j) > 1 \). Proceeding as in the prove of Proposition 2.6 in \([8]\), we have

\[
\int_{\mathbb{R}^d} \mathcal{M}^{(1)} a_{ij}(x) d\mu(x) = \int_{2R_j} \mathcal{M}^{(1)} a_{ij}(x)d\mu(x) \leq C,
\]

where \( C \) is independent of \( i \) and \( j \). Then we conclude as in the proof of Theorem \( 4.2 \) \( \square \)

Acknowledgments. I would like to thank the referee for his through revision of the paper and his useful comment. I would also like to thank professors Aline Bonami and Xavier Tolsa for some helpful discussions.

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