ON FACTORIZATION OF $q$-DIFFERENCE EQUATION FOR CONTINUOUS $q$-ULTRASPHERICAL POLYNOMIALS

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Abstract. We prove that a customary Sturm-Liouville form of second-order $q$-difference equation for the continuous $q$-ultraspherical polynomials $C_n(x; \beta|q)$ of Rogers can be written in a factorized form in terms of some explicitly defined $q$-difference operator $D_{x}^{\beta,q}$. This reveals the fact that the continuous $q$-ultraspherical polynomials $C_n(x; \beta|q)$ are actually governed by the $q$-difference equation $D_{x}^{\beta,q}C_n(x; \beta|q) = (q^{-n/2} + \beta q^{n/2}) C_n(x; \beta|q)$, which can be regarded as a square root of the equation, obtained from its original form.

1. Introduction

It is well known that for many purposes it proves practical, as in the case of linear second-order ordinary differential equations, to represent the difference equation of hypergeometric type for classical orthogonal polynomials in Sturm-Liouville (or self-adjoint) form [1]

$$\Delta \left[ \sigma(s) \rho(s) \frac{\nabla f(s)}{\nabla x(s)} \right] + \lambda \rho(s) f(s) = 0,$$

(1.1)

where $\Delta y(s) := y(s + 1) - y(s)$ and $\nabla y(s) := y(s) - y(s - 1)$ (we employ standard notations of the theory of special functions, see, for example, [2] or [3]).

The important feature of this form (1.1) is that it requires the introduction of a function $\rho(s)$ through the Pearson-type difference equation

$$\Delta \left[ \sigma(s) \rho(s) \right] = \tau(s) \rho(s),$$

(1.2)

with polynomials $\sigma(s)$ and $\tau(s)$ of respective degrees at most two and one, which characterize an original form of the difference equation (1.1). The full importance of the self-adjoint form (1.1) becomes apparent when one takes into account that the same function $\rho(s)$ enables one to formulate the orthogonality property of solutions of equation (1.1). Moreover, one can construct explicit representation ([1], p.66)

$$f_n(s) := \frac{B_n}{\rho(s)} \nabla x_1(s) \cdots \nabla x_{n-1}(s) \nabla x_n(s) \left[ \rho(s + n) \prod_{k=1}^{n} \sigma(s + k) \right]$$

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in terms of the function $\rho(s)$ for the polynomial solutions $f_n(s)$ of equation (1.1), which correspond to the values $\lambda_n := -n\tau' - n(n-1)\sigma'\sigma''/2$ of the parameter $\lambda$ (for a more detailed discussion of this topic, see [1]).

An example to illustrate this point is provided by the continuous $q$-Hermite polynomials of Rogers,

$$H_n(x|q) := \sum_{k=0}^{n} \binom{n}{k}_q e^{(n-2k)\theta}, \quad 0 < q < 1,$$  \hspace{1cm} (1.3)

which are orthogonal on the finite interval $-1 \leq x := \cos \theta \leq 1$ with respect to the weight function

$$\tilde{w}(x|q) := \frac{1}{\sin \theta} \left( e^{2i\theta}, e^{-2i\theta}; q \right)_\infty.$$ \hspace{1cm} (1.4)

These polynomials $H_n(x|q)$ satisfy the following $q$-difference equation

$$D_q \left[ \tilde{w}(x|q) D_q H_n(x|q) \right] = \frac{4q(1-q^{-n})}{(1-q)^2} H_n(x|q) \tilde{w}(x|q),$$ \hspace{1cm} (1.5)

written in self-adjoint form (1.1) (see [4], p.115). The symbol $D_q$ in (1.5) is the conventional notation for the Askey-Wilson divided-difference operator (see, for example, [3], p.529), defined as

$$D_q f(x) := \frac{\delta_q f(x)}{\delta_q x}, \quad \delta_q g(e^{i\theta}) := g(q^{1/2} e^{i\theta}) - g(q^{-1/2} e^{i\theta}), \quad x = \cos \theta.$$ \hspace{1cm} (1.6)

As was observed in [5], one may eliminate the weight function $\tilde{w}(x|q)$ from (1.5) by utilizing its readily verified property that

$$\exp \left( \pm i \ln q^{1/2} \partial_\theta \right) \tilde{w}(x|q) = -\frac{e^{\pm 2i\theta}}{\sqrt{q}} \tilde{w}(x|q).$$ \hspace{1cm} (1.7)

It should be noted that following [5] we find it more convenient to write (1.7) (and subsequent $q$-difference equations) in terms of the shift operators (or the operators of the finite displacement, [6]) $e^{\pm a \partial_\theta} g(\theta) := g(\theta \pm a)$ with respect to the variable $\theta$.

This elimination of the weight function $\tilde{w}(x|q)$ from (1.5) yields the following $q$-difference equation

$$\frac{1}{2i \sin \theta} \left[ \frac{e^{i\theta}}{1 - q e^{-2i\theta}} (e^{i\ln q \partial_\theta} - 1) + \frac{e^{-i\theta}}{1 - q e^{2i\theta}} (1 - e^{-i\ln q \partial_\theta}) \right] H_n(x|q)$$

$$= \left( q^{-n} - 1 \right) H_n(x|q)$$ \hspace{1cm} (1.8)

for the continuous $q$-Hermite polynomials $H_n(x|q)$. The resultant $q$-difference equation (1.8) then admits factorization of the form

$$(D_q^x)^2 H_n(x|q) = q^{-n} H_n(x|q),$$ \hspace{1cm} (1.9)

where the $q$-difference operator $D_q^x$ is equal to

$$D_q^x := \frac{1}{1 - e^{-2i\theta}} e^{i \ln q^{1/2} \partial_\theta} + \frac{1}{1 - e^{2i\theta}} e^{-i \ln q^{1/2} \partial_\theta}$$

$$\equiv \frac{1}{2i \sin \theta} \left( e^{i\theta} e^{i \ln q^{1/2} \partial_\theta} - e^{-i\theta} e^{-i \ln q^{1/2} \partial_\theta} \right), \quad x = \cos \theta.$$ \hspace{1cm} (1.10)
This means that the continuous \( q \)-Hermite polynomials are in fact governed by a simpler \( q \)-difference equation,
\[
\mathcal{D}_q H_n(x|q) = q^{-n/2} H_n(x|q),
\]
which represents a "square root" of (1.8) or (1.9).

This curious interrelation between two \( q \)-difference equations (1.5) and (1.11), studied in detail in [5], leads to the natural question whether the continuous \( q \)-Hermite polynomials \( H_n(x|q) \) represent the exceptional case or there exist other instances of orthogonal polynomials from the Askey \( q \)-scheme [4], which admit the same type of factorization in corresponding \( q \)-difference equations for them.

The present paper is aimed at proving that the continuous \( q \)-ultraspherical (Rogers) polynomials \( C_n(x;\beta|q) \) exhibit the same property of factorization as the continuous \( q \)-Hermite polynomials \( H_n(x|q) \). The next section collects those known facts about the \( q \)-ultraspherical polynomials \( C_n(x;\beta|q) \) and their \( q \to 1 \) limit counterpart, the Gegenbauer (ultraspherical) polynomials \( C_n(\gamma) \), which are needed in section 3 for proving that a \( q \)-difference equation for the \( C_n(x;\beta|q) \), derived from its appropriate self-adjoint form like (1.1), does admit a factorization of the type (1.9). In the concluding section 4 we briefly discuss some special and limit cases of the parameter \( \beta \), which are related with other well-known families of \( q \)-polynomials.

2. Rogers and Gegenbauer polynomials

To proceed further we need to recall in this section some standard facts about continuous \( q \)-ultraspherical (Rogers) polynomials and their \( q \to 1 \) limit counterpart, Gegenbauer (ultraspherical) polynomials. The continuous \( q \)-ultraspherical polynomials
\[
C_n(x;\beta|q) := \sum_{k=0}^{n} \frac{(\beta; q)_{n-k}}{(q; q)_{n-k}} e^{i(n-2k)\theta}, \quad 0 < q < 1,
\]
are known to be orthogonal on the finite interval \(-1 \leq x := \cos \theta \leq 1,\)
\[
\frac{1}{2\pi} \int_{-1}^{1} C_m(x;\beta|q) C_n(x;\beta|q) \tilde{w}(x;\beta|q) \, dx = d_n^{-1}(\beta; q) \delta_{mn},
\]
where
\[
d_n(\beta; q) := \frac{(1-\beta q^n)}{(1-\beta)} \frac{(q; q)_n}{(\beta^2; q)_n} \frac{(\beta, \beta^2 q; q)_\infty}{(\beta q, \beta^2; q)_\infty}, \quad |\beta| < 1, \quad (2.2)
\]
with respect to the weight function (see, for example, [4], p.86)
\[
\tilde{w}(x;\beta|q) := \frac{1}{\sin \theta} \left( \frac{e^{2i\theta} e^{-2i\theta}}{\beta e^{2i\theta}; q}_\infty \right).
\]
They satisfy the Sturm-Liouville type \( q \)-difference equation
\[
D_q [\tilde{w}(x;\beta|q) D_q C_n(x;\beta|q)] = \lambda_n(\beta) C_n(x;\beta|q) \tilde{w}(x;\beta|q) \quad (2.4)
\]
with eigenvalues $\lambda_n(\beta) := 4q (1 - q^{-n}) (1 - \beta^2 q^n)/(1 - q)^2$ (see, for example, [4], p. 86). Note that the $D_q$ in (2.4) is the same Askey–Wilson divided-difference operator, defined above in (1.6), namely,

$$D_q = \frac{\sqrt{q}}{i(1-q)} \frac{1}{\sin \theta} \left( e^{i \ln q^{1/2} \partial_\theta} - e^{-i \ln q^{1/2} \partial_\theta} \right), \quad \partial_\theta \equiv \frac{d}{d\theta}. \tag{2.5}$$

Observe also that one readily derives from definition (2.3) the relation

$$\tilde{w}(x; \beta q \| q) = [(1 + \beta)^2 - 4\beta x^2] \tilde{w}(x; \beta \| q) \tag{2.6}$$

between the weight functions $\tilde{w}(x; \beta \| q)$ with the two distinct parameters $\beta$ and $\beta q$. Therefore a $q$-analogue of the factor $\sigma(s)$ from the self-adjoint equation (1.1) in the case of the $q$-difference equation (2.4) is just

$$\sigma_q(x; \beta) := (1 + \beta)^2 - 4\beta x^2.$$

If one sets $\beta = q^\gamma$ in (2.1) and then evaluates its limit as $q \to 1$, this results in

$$\lim_{q \to 1} C_n(x; q^{\gamma} \| q) = C_n^{(\gamma)}(x),$$

where $C_n^{(\gamma)}(x)$ are the Gegenbauer polynomials:

$$C_n^{(\gamma)}(x) := \sum_{k=0}^{n} \frac{(\gamma)_k (\gamma)_{n-k}}{k! (n-k)!} e^{i(n-2k)\theta}, \quad x = \cos \theta. \tag{2.7}$$

The self-adjoint form of the second-order differential equation for the Gegenbauer polynomials (2.7) is known to be of the form

$$\frac{d}{dx} \left[ (1 - x^2) w(x) \frac{dC_n^{(\gamma)}(x)}{dx} \right] + n(n + 2\gamma) w(x) C_n^{(\gamma)}(x) = 0, \tag{2.8}$$

where $w(x) := (1 - x^2)^{\gamma - 1/2}$, $\gamma > -1/2$, is the orthogonality weight function for the $C_n^{(\gamma)}(x)$ on the finite interval $-1 \leq x \leq 1$. After eliminating the weight function $w(x)$ from (2.8), one can rewrite it as

$$\left[ (1 - x^2) \frac{d^2}{dx^2} - (2\gamma + 1) x \frac{d}{dx} + n(n + 2\gamma) \right] C_n^{(\gamma)}(x) = 0. \tag{2.9}$$

In contrast to (2.8), this differential equation is evidently not self-adjoint; but to transform it into self-adjoint equation (2.8) one needs only to multiply it by $w(x)$ from the left and employ the readily verified identity

$$w(x) \left[ (1 - x^2) \frac{d^2}{dx^2} - (2\gamma + 1) x \frac{d}{dx} \right] = \frac{d}{dx} \left[ (1 - x^2) w(x) \frac{d}{dx} \right]. \tag{2.10}$$
3. Factorization for Rogers polynomials

To eliminate the weight function \( \tilde{w}(x; \beta | q) \) from \( q \)-difference equation (2.4), we employ first the relations

\[
\exp \left( \mp i \ln q^{1/2} \frac{\partial}{\partial \theta} \right) \tilde{w}(x; \beta | q) = \frac{1}{\sqrt{q}} \left( 1 - \beta e^{\mp 2i\theta} \right) \left( \beta q - e^{\pm 2i\theta} \right) \tilde{w}(x; \beta | q),
\]

which are straightforward to derive upon using the explicit expression (2.3) for \( \tilde{w}(x; \beta | q) \) and relation (2.6). Substituting (3.1) into (2.4), one obtains the \( q \)-difference equation

\[
\frac{1}{\sin \theta} \left[ e^{i\theta} \left( 1 - \beta e^{-2i\theta} \right) \left( 1 - \beta q e^{-2i\theta} \right) \left( 1 - q e^{-2i\theta} \right) \left( 1 - e^{1\ln q \partial_\theta} \right) - 1 \right]
+ e^{-i\theta} \left( 1 - \beta e^{2i\theta} \right) \left( 1 - \beta q e^{2i\theta} \right) \left( 1 - q e^{2i\theta} \right) \left( 1 - e^{-1\ln q \partial_\theta} \right) C_n(x; \beta | q)
= 2 \left( q^{-n} - 1 \right) \left( 1 - \beta^2 q^n \right) C_n(x; \beta | q)
\]

for the \( q \)-ultraspherical polynomials \( C_n(x; \beta | q) \), which does not contain the weight function \( \tilde{w}(x; \beta | q) \). This equation is a \( q \)-extension of the second-order differential equation (2.9) for the Gegenbauer polynomials \( C_n^{(\gamma)}(x) \).

The next step is to use two simple trigonometric identities

\[
e^{\pm i\theta} = \pm \frac{2}{1 - e^{\mp 2i\theta}}
\]

in order to write a \( q \)-difference operator on the left side of equation (3.2) as

\[
2 \left[ \left( 1 - \beta e^{-2i\theta} \right) \left( 1 - \beta q e^{-2i\theta} \right) \left( 1 - q e^{-2i\theta} \right) \left( 1 - e^{1\ln q \partial_\theta} \right) - 1 \right]
+ e^{-i\theta} \left( 1 - \beta e^{2i\theta} \right) \left( 1 - \beta q e^{2i\theta} \right) \left( 1 - q e^{2i\theta} \right) \left( 1 - e^{-1\ln q \partial_\theta} \right) C_n(x; \beta | q)
= 2 \left[ \left( 1 - \beta e^{-2i\theta} \right) \left( 1 - \beta q e^{-2i\theta} \right) \left( 1 - q e^{-2i\theta} \right) \left( 1 - e^{1\ln q \partial_\theta} \right) - 1 \right]
+ e^{-i\theta} \left( 1 - \beta e^{2i\theta} \right) \left( 1 - \beta q e^{2i\theta} \right) \left( 1 - q e^{2i\theta} \right) \left( 1 - e^{-1\ln q \partial_\theta} \right)
- \left( 1 - \beta e^{-2i\theta} \right) \left( 1 - \beta q e^{-2i\theta} \right) \left( 1 - q e^{-2i\theta} \right) \left( 1 - e^{1\ln q \partial_\theta} \right) - 1 \right] C_n(x; \beta | q)\]

The last important step is to employ a readily verified identity

\[
\frac{1 - \beta q e^{2i\theta}}{1 - q e^{\mp 2i\theta}} e^{\pm i \ln q^{1/2} \partial_\theta} = e^{\pm i \ln q^{1/2} \partial_\theta} \frac{1 - \beta e^{\mp 2i\theta}}{1 - e^{\pm 2i\theta}}
\]

for the shift operators \( \exp \left( \pm i \ln q^{1/2} \partial_\theta \right) \), which enter into first two terms in (3.3). With the aid of (3.4) one can thus cast (3.3) into the form

\[
2 \left[ \frac{1 - \beta e^{-2i\theta}}{1 - e^{-2i\theta}} e^{1\ln q^{1/2} \partial_\theta} \frac{1 - \beta e^{-2i\theta}}{1 - e^{-2i\theta}} e^{1\ln q^{1/2} \partial_\theta} - \frac{1 - \beta e^{2i\theta}}{1 - e^{2i\theta}} e^{-1\ln q^{1/2} \partial_\theta} \frac{1 - \beta e^{2i\theta}}{1 - e^{2i\theta}} e^{-1\ln q^{1/2} \partial_\theta}
- \left( 1 - \beta e^{-2i\theta} \right) \left( 1 - \beta q e^{-2i\theta} \right) \left( 1 - \beta q e^{-2i\theta} \right) \left( 1 - q e^{-2i\theta} \right) \left( 1 - e^{1\ln q \partial_\theta} \right) - 1 \right] C_n(x; \beta | q)
\]
can be written as diagonal with respect to the orthonormal basis \( L \) in operator on the Hilbert space \( L \) of equation (3.2):

\[
H \text{ is defined on the linear span } \langle x \rangle \text{ of } S \text{ with respect to the scalar product (3.8). Since (2.3) is defined by (2.3), the polynomials } p_n(x) := d_n^{1/2}(\beta; q) C_n(x; \beta | q), \text{ which is everywhere dense in } L^2(S^1). \text{ We close } (D_x^\beta, q)^2 \text{ with respect to the scalar product (3.8). Since } (D_x^\beta, q)^2 \text{ is diagonal with respect to the orthonormal basis } p_n(x), \text{ its closure } (\overline{D_x^\beta, q})^2 \text{ is a self-adjoint operator, which coincides on } H \text{ with } (D_x^\beta, q)^2. \text{ According to the theory of self-adjoint operators (see [7], Chapter 6), we can take a square root of the operator } (\overline{D_x^\beta, q})^2 \text{ as the operator } (\overline{D_x^\beta, q})^{1/2} \text{ does. We denote this operator by } \overline{D_x^\beta, q}. \text{ It is evident that on the subspace } H \text{ the operator } \overline{D_x^\beta, q} \text{ coincides with the } D_x^\beta, q. \text{ That is, the } D_x^\beta, q \text{ is a well-defined operator on the Hilbert space } L^2(S^1) \text{ with everywhere dense subspace of definition. Moreover, according to the definition of a function of a self-adjoint operator (see [7], Chapter 6), we have } D_x^\beta, q p_n(x) = (q^{-n/2} + \beta q^{n/2}) p_n(x). \text{ This means that the} \]

\[
2 \left[ (D_x^\beta, q)^2 - (1 + \beta)^2 \right] = 2 \left( D_x^\beta, q + 1 + \beta \right) \left( D_x^\beta, q - 1 - \beta \right),
\]

where \( D_x^\beta, q \) is equal to (cf. (1.10))

\[
D_x^\beta, q := \frac{1 - \beta e^{-2\theta}}{1 - e^{-2\theta}} e^{i \ln q^{1/2} \partial_\theta} + \frac{1 - \beta e^{2\theta}}{1 - e^{2\theta}} e^{-i \ln q^{1/2} \partial_\theta} \equiv D_x^q + \beta D_x^{1/q}
\]

\[
\equiv \frac{1}{2i \sin \theta} \left[ (e^{i \theta} - \beta e^{-i \theta}) e^{i \ln q^{1/2} \partial_\theta} - (e^{-i \theta} - \beta e^{i \theta}) e^{-i \ln q^{1/2} \partial_\theta} \right].
\]

Finally, taking into account that the factor \((q^{-n} - 1)(1 - \beta^2 q^n)\) on the right side of (3.2) can be written as \((q^{-n/2} + \beta q^{n/2})^2 - (1 + \beta)^2\), one arrives at the following factorized form of equation (3.2):

\[
(D_x^\beta, q)^2 C_n(x; \beta | q) = (q^{-n/2} + \beta q^{n/2})^2 C_n(x; \beta | q).
\]
continuous $q$-ultraspherical polynomials $C_n(x; \beta|q)$ are in fact governed by a simpler $q$-difference equation,
\begin{equation}
D_x^{\beta,q} C_n(x; \beta|q) = \left(q^{-n/2} + \beta q^{n/2}\right) C_n(x; \beta|q),
\end{equation}
which can be regarded as a "square root" of (3.7).

Observe that the $q$-difference operator $D_x^{\beta,q}$ in (3.9) may be expressed in terms of the
Askey-Wilson divided-difference operator $\bar{D}_q$, defined in (1.6), as
\begin{equation}
D_x^{\beta,q} = (1 + \beta) A_q + \frac{1 - q}{2\sqrt{q}} (1 - \beta) x D_q,
\end{equation}
where the $A_q$ is so-called averaging difference operator, that is (see, for example [8]),
\begin{equation}
(\mathcal{A}_q f)(x) = \frac{1}{2} \left(e^{i\ln q^{1/2} \partial_\theta} + e^{-i\ln q^{1/2} \partial_\theta}\right) f(x) \equiv \cos \left(\ln q^{1/2} \partial_\theta\right) f(x).
\end{equation}

We emphasize that $q$-difference equation (3.8) is consistent with the generating function
\begin{equation}
\sum_{n=0}^{\infty} t^n C_n(x; \beta|q) = \frac{\left(\beta t e^{i\theta}, \beta t e^{-i\theta} ; q\right)_{\infty}}{\left(t e^{i\theta}, t e^{-i\theta} ; q\right)_{\infty}}
\end{equation}
for the continuous $q$-ultraspherical polynomials $C_n(x; \beta|q)$ (see [2], p.169). Indeed, apply the $q$-difference operator $D_x^{\beta,q}$ to both sides of (3.12) to verify that
\begin{align*}
\sum_{n=0}^{\infty} t^n D_x^{\beta,q} C_n(x; \beta|q) &= D_x^{\beta,q} \sum_{n=0}^{\infty} t^n C_n(x; \beta|q) \\
&= \sum_{n=0}^{\infty} \left(q^{-n/2} + \beta q^{n/2}\right) t^n C_n(x; \beta|q).
\end{align*}
Equating coefficients of like powers of $t$ on the extremal sides of (3.13), one completes the another proof of equation (3.9).

As recalled in section 2, if $\beta = q^\gamma$, then the $q$-ultraspherical polynomials $C_n(x; q^\gamma|q)$ reduce to the Gegenbauer polynomials $C_n^{(\gamma)}(x)$ in the limit as $q \to 1$. This fact can be also expressed as the following limit property of the $q$-difference operator $D_x^{\beta,q}$ in (3.6):
\begin{equation}
\lim_{q \to 1} \left\{ \frac{1}{(\ln q)^2} \left[ (1 + q^\gamma) I - D_x^{q^\gamma,q} \right] \right\} = \frac{1}{4} \left[ (1 - x^2) \frac{d^2}{dx^2} - (2\gamma + 1)x \frac{d}{dx} \right],
\end{equation}
where $I$ is the identity operator.

Observe also that the $q$-ultraspherical polynomials $C_n(x; \beta|q)$ are known to possess the simple transformation property
\begin{equation}
C_n(x; \beta | q^{-1}) = (\beta q)^n C_n(x; \beta^{-1}|q)
\end{equation}
with respect to the changes $q \to q^{-1}$ and $\beta \to \beta^{-1}$ (see [4], p.88). It is not hard to check that $q$-difference equation (3.9) agrees with this property (3.14) since by definition (3.6)
\begin{equation}
D_x^{\beta,q} \equiv \beta D_x^{\beta^{-1},q^{-1}}.
\end{equation}
We close this section with the following remark about equation (3.9). Koornwinder have recently examined raising and lowering relations for the Askey–Wilson polynomials $p_n(x; a, b, c, d| q)$ [9], which are known to reduce to the continuous $q$-ultraspherical polynomials $C_n(x; \beta| q)$, when one specializes the parameters $a, b, c, d$ as $a = -c = \sqrt{\beta}$ and $b = -d = \sqrt{q \beta}$. So equation (3.9) coincides with "the second order $q$-difference formula" (6.10) in Koornwinder’s paper [9], upon taking into account that variables $z$ and $t$ in (6.10) are equal to $e^{i\theta}$ and $\beta$, respectively, in our notations.

4. SPECIAL AND LIMIT CASES OF PARAMETER $\beta$

The $q$-difference equation (3.9) for the $q$-ultraspherical polynomials, derived in the previous section, does actually contain some special and limit cases of the parameter $\beta$, which correspond to other well-known families of $q$-polynomials. We recall (see, for example, [4], p.88) that in the case when $\beta = q^{\alpha+1/2}$ the $q$-ultraspherical polynomials $C_n(x; q^{\alpha+1/2}| q)$ reduce to (up to a normalization factor) the continuous $q$-Jacobi polynomials $P_n(\alpha, \alpha| q)$; when $\beta = q^{1/2}$ the $C_n(x; q^{1/2}| q)$ are related to the continuous $q$-Legendre polynomials $P_n(x| q)$; and when $\beta = q$ the $q$-ultraspherical polynomials $C_n(x; q| q)$ embrace the Chebyshev polynomials of the second kind $U_n(x)$.

There is also the limit case $\beta \to 1$, which leads to the Chebyshev polynomials of the first kind $T_n(x)$ in the following way:

$$\lim_{\beta \to 1} \frac{1-q^n}{2(1-\beta)} C_n(x; \beta| q) = T_n(x) \equiv \cos n\theta, \quad n = 1, 2, 3, \ldots.$$ 

But the point is that $q$-difference equation (3.9) in this limit reduces to the difference equation

$$\left[e^{i\ln q^{1/2}\partial_y} + e^{-i\ln q^{1/2}\partial_y}\right] T_n(x) = (q^{n/2} + q^{-n/2}) T_n(x), \quad (4.1)$$

although we all know well that the Chebyshev polynomials of the first kind $T_n(x)$ satisfy the second-order differential equation

$$\left(1 - x^2\right) \frac{d^2}{dx^2} - x \frac{d}{dx} + n^2 \right] T_n(x) = 0.$$

Nevertheless, there is no contradiction here since one readily verifies that the Chebyshev polynomials of the first kind $T_n(x) = \cos n\theta$, $n = 0, 1, 2, \ldots$, do satisfy difference equation (4.1) as well.

5. CONCLUDING REMARKS

To summarize, we have proved that the conventional $q$-difference equation (2.4) of Sturm-Liouville type for the continuous $q$-ultraspherical polynomials $C_n(x; \beta| q)$ of Rogers admits factorization of the form (3.9). The special case of the $C_n(x; \beta| q)$ with the vanishing parameter $\beta$ is known to correspond to the continuous $q$-Hermite polynomials $H_n(x| q)$. The above-presented formulas in this case when $\beta = 0$ are in accord with that obtained by M.Atakishiyev and A.Klimyk in [5]. So it would be of considerable interest to explore now whether the situation here described obtains for other families of
orthogonal polynomials on higher levels in the Askey $q$-scheme [4]. Work on clarifying this point is in progress.

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