A NOTE ON CENTRAL MOMENTS IN C∗-ALGEBRAS

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Abstract. We present sharp estimates of the maximum of $k^{th}$ central moments of normal elements in $C^*$-algebras. We shall obtain an estimate for the upper bound of weaken moments of general elements as well.

1. Introduction

Variance and higher-order central moments are well-studied concepts in probability theory and in (quantum) statistics. Our starting point is a lemma that originates in Murthy and Sethi's paper [5] and provides an estimate for the standard variance of real random variables in terms of their largest and smallest values. It says that if $X$ is a discrete real random variable which attains the reals $x_i$ with probability $p_i$-s then

$$\text{Var}(X) = \sum_i p_i x_i^2 - \left( \sum_i p_i x_i \right)^2 \leq \frac{1}{4} (M - m)^2$$

for any $m \leq X \leq M$ constants. Quite recently K. Audenaert gave a sharp extension of the result for complex variables, and even for matrices in terms of different types of quantum variances, see [1]. Actually, R. Bhatia and R. Sharma obtained Audenaert’s result with a different method in [2]. Moreover, in the setting of $C^*$-algebras the full extension of these variance estimates appeared in M. Rieffel’s paper [9]. His approach is based on the proof of a variant of Arveson’s distance formula for $C^*$-algebras [10]. Later T. Batthacharyya and P. Grover [3] presented a new proof of these inequalities exploring the Birkhoff–James orthogonality.

Our aim here is to provide sharp inequalities on the higher moments of normal elements in $C^*$-algebras. We shall prove that our results are sharp for the $k^{th}$ moments when $k$ is even. The approach we are using is different from the above mentioned ones and rather elementary as well. In the first part of the paper we shall study the moments of Hermitian elements. Then, after these preliminary results, we can give the estimates for normal elements. In the last section of the paper we shall present an inequality for weaken central moments of general $C^*$-algebra elements.

2. Central moments of Hermitian elements

Let $\mathcal{A}$ denotes a unital $C^*$-algebra, and let $\mathcal{S}(\mathcal{A})$ stands for the set of states of $\mathcal{A}$, i.e. the set of positive linear functionals (of norm 1). Then $\mathcal{S}(\mathcal{A})$ is compact in the weak-* topology. The $k^{th}$ root of the maximal $k^{th}$ central moment of $a \in \mathcal{A}$ is defined by

$$\Delta_k(a) = \sup_{\omega \in \mathcal{S}(\mathcal{A})} \omega(\left| a - \omega(a) \right|^k)^{1/k} = \max_{\omega \in \mathcal{S}(\mathcal{A})} \omega(\left| a - \omega(a) \right|^k)^{1/k}.$$
where we used the substitution $m$ and the proof is completed. □

In [9] Theorem 3.10 Rieffel proved for any $a \in A$ that
\[
\Delta_2(a) = \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\|.
\]
Furthermore, he also observed that the factor norm obeys the strong Leibniz inequality which leads him to prove that the standard deviation is actually a strongly Leibniz seminorm (for the details see [10]). We note that L. Molnár [6] proved for self-adjoint $a$ that $\Delta_2(a)$ is the same as the factor norm of $a$ in $A/\mathbb{C}$. The proof there used the simple geometrical fact that
\[
\inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\| = \frac{1}{2} \text{diam } \sigma(a),
\]
where $\sigma(a)$ is the spectrum of $a$.

Now we can give a geometric meaning to the higher-order moments of Hermitian elements as well. From here on we shall denote the maximum of any polynomial $p$ on the interval $[0, 1]$ by $\|p\|_\infty$. The $C^*$-algebra generated by the single element $a$ is denoted by $C^*(a)$.

Here is our first statement.

**Proposition 1.** Let $a$ be a Hermitian element of a unital $C^*$-algebra $A$. Then
\[
\|p_k\|_\infty^{1/k} \text{diam } \sigma(a) \leq \Delta_k(a),
\]
where $p_k(x) = x(1 - x)^k + (-1)^k x^k (1 - x)$.

**Proof.** Assume that $\xi$ denote a point where the polynomial $p_k$ attahces its maximum on $[0, 1]$. Let $\varphi_1$ and $\varphi_2$ be multiplicative functionals of the commutative $C^*(a)$ generated by $a$ which satisfy $\varphi_1(a) = \max \{ \lambda : \lambda \in \sigma(a) \}$ and $\varphi_2(a) = \min \{ \lambda : \lambda \in \sigma(a) \}$. Set the state $\varphi = \xi \varphi_1 + (1 - \xi) \varphi_2$, and pick one of its extensions to the whole $A$ which is also a state. Then we obtain that
\[
\varphi((a - \varphi(a))^k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \varphi(a^{k-j}) \varphi(a)^j
\]
\[
= \sum_{j=0}^{k} \sum_{l=0}^{j} \binom{k}{j} \binom{j}{l} (-1)^l (1 - \xi)^{j-l} \xi^l \varphi_1(a)^{k-j+l} \varphi_2(a)^{j-l}
\]
\[
+ \sum_{j=0}^{k} \sum_{l=0}^{j} \binom{k}{j} \binom{j}{l} (-1)^l (1 - \xi)^{j-l+1} \xi^l \varphi_1(a)^{k-j+l} \varphi_2(a)^{j-l}
\]
\[
= \sum_{m=0}^{k} \xi(1 - \xi)^m \binom{k}{m} (-1)^m \varphi_1(a)^{k-m} \varphi_2(a)^m \sum_{l=0}^{k-m} \binom{k-m}{l} (-1)^l \xi^l
\]
\[
+ \sum_{l=0}^{k} \sum_{j=l}^{k} \binom{k}{l} \binom{k-l}{j-l} (-1)^l (1 - \xi)^{j-l+1} \xi^l \varphi_1(a)^{k-j+l} \varphi_2(a)^{j-l}
\]
\[
= \xi(1 - \xi)^k (\varphi_1(a) - \varphi_2(a))^k + (1 - \xi) \xi^k (\varphi_2(a) - \varphi_1(a))^k
\]
\[
= p_k(\xi)(\varphi_1(a) - \varphi_2(a))^k,
\]
where we used the substitution $m = j - l$. Therefore we get that
\[
\max_{\omega \in \mathcal{S}(A)} |\omega((a - \varphi(a))^k)|^{1/k} \geq \|p_k\|_\infty^{1/k} \text{diam } \sigma(a)
\]
and the proof is completed. □
The next statement shows that the converse inequality holds as well.

**Proposition 2.** Let $a$ denote a Hermitian element of a unital $C^*$-algebra $A$. Then

$$\hat{\Delta}_k(a) \leq \|p_k\|_\infty^{1/k} \text{diam } \sigma(a),$$

where $p_k(x) = x(1 - x)^k + (-1)^k x^k(1 - x)$. Moreover, the equality holds with an $\omega \in S(C^*(a))$, which is the convex sum of Dirac masses concentrated on the farthest spectrum points of $a$.

**Proof.** To prove the inequality we shall reduce the problem to the finite dimensional setting. The Gelfand–Naimark–Segal construction tells us that $A$ can be realized as a *-subalgebra of the full operator algebra of a Hilbert space $\mathcal{H}$. Then the Hermitian $a$ has the spectral decomposition

$$a = \int_{\sigma(a)} \lambda \, dp(\lambda),$$

where $dp(\lambda)$ denotes the spectral distribution of $a$. Let us choose a sequence of simple functions $s_n$ which uniformly approximates the identity on the spectrum $\sigma(A) \subseteq \mathbb{R}$. Now fix an $n$ and let us assume that $s_n = \sum_{i=1}^{m} \gamma_i \chi_{E_i}$, where $E_i$'s are disjoint Borel sets of $\mathbb{R}$ and their union is $\sigma(a)$. Set

$$a_n = \int_{\sigma(a)} s_n \, dp = \sum_{i} \gamma_i p_{E_i}.$$  

Then for any state $\omega$ on $\mathcal{B}(\mathcal{H})$, we have

$$\omega((a_n - \omega(a_n))^k) = \omega\left(\left(\sum_{i=1}^{m} (\gamma_i - \omega(a_n))p_{E_i}\right)^k\right)$$

$$= \sum_{i=1}^{m} (\gamma_i - \omega(a_n))^k \omega(p_{E_i})$$

$$= \mathbb{E}((X - \mathbb{E}(X))^k) =: \mu_k^X,$$

where $X$ is a discrete random variable on $\{1, \ldots, m\}$ which takes the value $\gamma_i$ with probability $\omega(p_{E_i})$ ($1 \leq i \leq m$). We claim that the $k$th central momentum of $X$ is maximal if its distribution is concentrated at the farthest points of $\sigma(a_n)$. In fact, from the shift invariance property of $\mu_k$, one can assume that $\mathbb{E}(X) = 0$. Hence $\mu_k^X$ is at most the maximum of the linear function of $s_i$'s

$$\max \sum_{i} \gamma_i^k s_i \text{ under the linear constraints }$$

$$\sum_{i} \gamma_i s_i = 0, \quad \sum_{i} s_i = 1 \text{ and } s_1, \ldots, s_m \geq 0.$$

It is simple that we can optimize the above problem at the corners of the convex polytope $\{(s_1, \ldots, s_m) \in \Delta_m : \sum_{i} s_i \gamma_i = 0\} \subseteq \mathbb{R}^m$, where $\Delta_m$ denotes the convex hull of the standard basis vectors of $\mathbb{R}^m$ (i.e. $\Delta_m$ is the standard $m$-1-simplex). One can readily see that these corners, which lie at the intersection of $\Delta_m$ and an affine hyperplane of $\mathbb{R}^m$, must be on the edges of $\Delta_m$. That is, if $\mu_k$ is maximal then the distribution $(\omega(p_{E_i}), \ldots, \omega(p_{E_m}))$ should be concentrated on at most two atoms.

Now let us say that $\omega(p_{E_i})$ and $\omega(p_{E_j}) = 1 - \omega(p_{E_i})$ are positive (or at least one of them is positive, and $\omega(p_{E_i}) = 0$ if $k \neq i, j$). Since $c\mu_k(X) = \mu_k(c(X - \lambda))$ for any reals $c$ and $\lambda$, a simple computation gives that

$$\mu_k^X = (\gamma_i - \gamma_j)^k \omega(p_{E_i})(1 - \omega(p_{E_i}))^k + (-1)^k \omega(p_{E_j})^k(1 - \omega(p_{E_j})))$$


whenever $\gamma_i \geq \gamma_j$ holds. Actually, we get

$$\mu_k^X(X) \leq |\gamma_i - \gamma_j|^k \max_{x \in [0,1]} |p_k(x)|$$

which implies that

$$\omega((a_n - \omega(a_n))^k)^{1/k} \leq \|p_k\|^{1/k}_{\infty} \text{diam } \sigma(a_n).$$

Since $a_n \to a$ in norm, $\text{diam } \sigma(a_n) = \text{diam ran } s_n \to \text{diam } \sigma(a)$ ($n \to \infty$), and $\omega$ is continuous, we conclude that

$$\Delta_k(a) \leq \|p_k\|^{1/k}_{\infty} \text{diam } \sigma(a).$$

From the construction in the proof of Proposition 1, we get the existence of a convex combination of two Dirac masses concentrated on the largest and the smallest spectrum points of $a$ such that the equality holds. \qed

Remark. We note that a minor modification of the last proof immediately gives an upper bound on the $k$th central moment of any Hermitian (for odd $k$). In fact, $E(|X - E(X)|^k)$ is maximal for any real discrete $X$ if its distribution is concentrated on the largest and the smallest values of $X$. As a corollary one can readily prove the inequality

$$\Delta_k(a) \leq \|q_k\|^{1/k}_{\infty} \text{diam } \sigma(a) = 2\|q_k\|^{1/k}_{\infty} \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\|,$$

where $q_k(x) = x(1 - x)^k + x^k(1 - x)$. We leave the details to the interested reader.

Combining the above propositions we get the following result for self-adjoints.

Corollary 1. Let $a$ be a Hermitian element of a unital $C^*$-algebra. For any $k$,

$$\hat{\Delta}_k(a) = \|p_k\|_{\infty}^{1/k} \text{diam } \sigma(a) = 2\|p_k\|^{1/k}_{\infty} \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\|.$$

3. Higher-order moments of normal elements

Relying on the results of the previous section we can prove some estimates on the central moments of normal elements as well. Instead of the equality for the weaken moments $\Delta_k(a)$, we shall get sharp estimates for $\Delta_k(a)$, at least when $k$ is even. For the upper bound, we have the following.

Theorem 1. Let $a$ be a normal element of a unital $C^*$-algebra $A$. Then

$$\max_{\omega \in \mathcal{S}(A)} \omega([a - \omega(a)]^k)^{1/k} \leq 2\|q_k\|^{1/k}_{\infty} \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\|,$$

where $q_k(x) = x(1 - x)^k + x^k(1 - x)$.

Proof. First, denote $\omega_0$ a state where $\Delta_k(a)$ takes its values. Let $(\mathcal{H}, \pi, \varphi)$ be the GNS representation of $\omega_0$ restricted to $C^*(a)$. We recall that $\pi$ is contractive and $\mathcal{H}$ is separable now. Relying on the Weyl–von Neumann–Berg theorem [3] Theorem 39.4], for any $\varepsilon > 0$ we can find a diagonal operator $d$ (i.e. its eigenvectors of unit length form a complete orthonormal system in $\mathcal{H}$) and a compact perturbation $k$ such that $\pi(a) = d + k$ and $\|k\| \leq \varepsilon$ hold. Let $r_0(a)$ denote the radius of the smallest circle that contains $\sigma(a)$. Then $r_0(a) = \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\|$ follows because $a$ is normal. Since the spectrum as a set function is upper semicontinuous (in the usual Hausdorff distance) (see e.g. [4]) and $r_0(\pi(a)) \leq r_0(a)$, we can assume by translation of $a$ with const-1 that $d = \sum \lambda_i p_i$, where $p_i$-s are orthogonal projections and $\lambda_i \in (r_0(a) + \varepsilon)\mathbb{D}$. Set the diagonal elements

$$\hat{d} := \sum_{i}(\lambda_i p_i \oplus -\bar{\lambda}_i p_i) \quad \text{and} \quad \hat{d} := \sum_{i} |\lambda_i|(p_i \oplus -p_i)$$
in $B(\mathcal{H} \oplus \mathcal{H})$. Obviously, $r_0(d) = r_0(\tilde{d}) = r_0(d)$ and $\sigma(d) \subseteq \sigma(\tilde{d})$. Since any positive linear functional of $C^*(d)$ (and $C^*(\tilde{d})$) is in the weak-$*$ closure of the convex hull of the Dirac masses (evaluation functionals) on $\sigma(d)$, see e.g. [10 Proposition 2.5.7], (and on $\sigma(\tilde{d})$, respectively), we have by means of the Gelfand transform that

$$\Delta_k(d) \leq \Delta_k(\tilde{d}).$$

Thus, with a positive $\varepsilon'$, the equality (1) in our Remark shows that

$$\max_{\omega \in \mathcal{S}(A)} \omega(|a - \omega(a)|^k)^{1/k} \leq (1 + \varepsilon') \max_{\omega \in \mathcal{S}(B(\mathcal{H}))} \omega(|d - \omega(d)|^k)^{1/k}$$

$$= (1 + \varepsilon') \max_{\omega} \omega(\sum_i |\lambda_i|^k |(p_i + - p_i) - \omega(p_i + - p_i)|^k)^{1/k}$$

$$= (1 + \varepsilon') \max_{\omega \in \mathcal{S}(B(\mathcal{H}))} \omega(|d - \omega(d)|^k)^{1/k}$$

$$\leq 2(1 + \varepsilon')\|q_k\|^{1/k} \inf_{\lambda \in \mathbb{C}} |d - \lambda 1|$$

$$= 2(1 + \varepsilon')\|q_k\|^{1/k} \inf_{\lambda \in \mathbb{C}} |d - \lambda 1|$$

Note that $\varepsilon' \to 0$ if $\varepsilon \to 0$, hence the proof is completed. \hfill \square

We recall that the direct sum $A \oplus A$ is an ordinary Banach $*$-algebra which is a $C^*$-algebra with norm

$$\|a \oplus b\| = \max\{\|a\|, \|b\|\}.$$ To prove the main theorem of this section, we need a preliminary lemma.

**Lemma 1.** For any normal $a$ in $A$,

$$\Delta_k(a) = \Delta_k(a \oplus a^*).$$

**Proof.** Obviously, for an $\omega \in \mathcal{S}(A)$, the functionals $\omega \oplus 0$ and $0 \oplus \omega$ are also states on $A \oplus A$, hence $\Delta_k(a) \leq \Delta_k(a \oplus a^*)$. Conversely, let $(\mathcal{H}, \pi)$ be the GNS representation of the $C^*$-algebra $A \oplus A$. Fix an $\omega \in \mathcal{S}(A \oplus A)$ where $\Delta_k(a \oplus a^*) = \omega(|a \oplus a^* - \omega(a \oplus a^*)|^k)^{1/k}$. Since $\Delta_k(a \oplus a^*) = \Delta_k(a \oplus a^*)$, for any complex $c$ with modulus 1, we can assume without loss of generality that $\omega(a \oplus a^*) \in \mathbb{R}$. Let $v \in \mathcal{H}$ be the unit vector for which $\omega(\cdot) = \langle \pi(\cdot)v, v \rangle$. The ranges of the projections $\pi(1 \oplus 0)$ and $\pi(0 \oplus 1)$ are orthogonal, hence one can write $v$ as the direct sum $x \oplus y$, where $x = \pi(1 \oplus 0)v$ and $y = \pi(0 \oplus 1)v$. Furthermore,

$$\omega(a \oplus a^*) = \langle \pi(a \oplus a^*)v, v \rangle$$

$$= \langle (\pi(a \oplus 0) + \pi(0 \oplus a^*))x \oplus y, x \oplus y \rangle$$

$$= \|x\|^2 \langle \pi(a \oplus 0)x, x \rangle + \|y\|^2 \langle \pi(0 \oplus a^*)y, y \rangle + \langle y, \pi(0 \oplus a^*)y \rangle$$

$$= \|x\|^2 \omega_1(a) + (1 - \|x\|^2) \omega_2(a)$$

holds with $\omega_1, \omega_2$ and $\omega_3 \in \mathcal{S}(A)$ because $\mathcal{S}(A)$ is convex and closed under conjugation. Then

$$\omega(|a \oplus a^* - (a \oplus a^*)|^k) = \omega(|a - \omega_3(a)|^k \oplus |a^* - \omega_3(a)|^k)$$

$$= \omega(|a - \omega_3(a)|^k \oplus |(a - \omega_3(a))^*|^k)$$

$$= \omega_2(|a - \omega_3(a)|^k),$$

which is exactly what we need to get $\Delta_k(a \oplus a^*) \leq \Delta_k(a)$. \hfill \square
Theorem 2. Let $a$ be a normal element of a unital $C^*$-algebra $A$. Then

$$2\|p_k\|^{1/k}_\infty \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\| \leq \Delta_k(a).$$

Proof. Pick a $\lambda_0$ such that $\inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\| = \|a - \lambda_0 1\|$ and let $\tilde{a} := (a - \lambda_0 1)/\|a - \lambda_0 1\|$. Clearly, $\inf_{\lambda \in \mathbb{C}} \|\tilde{a} - \lambda 1\| = \|\tilde{a}\| = 1$. By rotation around 0, we can assume that $i \in \sigma(\tilde{a})$. Since $\sigma(\tilde{a} \oplus \tilde{a}^*) = \sigma(\tilde{a}) \cup \sigma(\tilde{a})$, we get from the spectral mapping theorem that

$$\inf_{\lambda \in \mathbb{C}} \|i(\tilde{a} \oplus \tilde{a}^*) + 1| - \lambda 1\| = \frac{1}{2} \text{diam} \sigma(\tilde{a} \oplus \tilde{a}^*) = 1.$$

On the other hand, from Proposition 2, $\hat{\Delta}_k([i(\tilde{a} \oplus \tilde{a}^*) + 1])$ attains its maximum on the convex sums of the Dirac masses $\delta_0$ and $\delta_2$ on $C^*(\tilde{a} \oplus \tilde{a}^*)$ (and their extensions to $A$). However, for any state $\omega_0 := \lambda \delta_0 + (1 - \lambda) \delta_2$, we clearly have via the Gelfand transform that

$$\omega_0([i(\tilde{a} \oplus \tilde{a}^*) + 1]) = \omega_0(i(\tilde{a} \oplus \tilde{a}^*) + 1) = 1 - \lambda.$$

From Proposition 1 and the inequality $|f| - |g|^k \leq |f - g|^k$ for continuous functions on the Gelfand space of $C^*(\tilde{a} \oplus \tilde{a}^*)$, we obtain

$$2\|p_k\|^{1/k}_\infty \inf_{\lambda \in \mathbb{C}} \|\tilde{a} - \lambda 1\| = 2\|p_k\|^{1/k}_\infty \inf_{\lambda \in \mathbb{C}} \|i(\tilde{a} \oplus \tilde{a}^*) + 1| - \lambda 1\|
\leq \max_{\omega_0 = \lambda \delta_0 + (1 - \lambda) \delta_2} \omega_0([i(\tilde{a} \oplus \tilde{a}^*) + 1] - \omega_0(i(\tilde{a} \oplus \tilde{a}^*) + 1))^{1/k}
\leq \max_{\omega \in \mathcal{S}(A \oplus A)} \omega([i(\tilde{a} \oplus \tilde{a}^*) + 1] - \omega(i(\tilde{a} \oplus \tilde{a}^*) + 1))^{1/k}
= \Delta_k(\tilde{a}),$$

where we used Lemma 1 in the last equality.

The estimate for the standard variance of normal matrices in the matrix algebra $M_n(\mathbb{C})$ first appeared in [1, Theorem 8]. From the above theorems we get the main result of the section.

Corollary 2. For any normal $a$ in a unital $C^*$-algebra $A$ and $k \in 2\mathbb{N}$,

$$\Delta_k(a) = 2\|p_k\|^{1/k}_\infty \inf_{\lambda \in \mathbb{C}} \|a - \lambda 1\|. $$

Example. Obviously, the equality $\hat{\Delta}_k(a) = \Delta_k(a)$ follows for self-adjoint $a$ and even $k$. However, one can construct an example which shows that $\Delta_k(a) < \Delta_k(a)$ can hold when $a$ is normal and $k$ is even. For instance, let $k = 2$ and define the normal matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\
0 & e^{2i\pi/3} & 0 \\
0 & 0 & e^{4i\pi/3} \end{pmatrix}. $$

From the previous corollary in $M_3(\mathbb{C})$, we get

$$\Delta_2(A) = 2\|p_2\|^{1/2}_\infty = 1.$$

On the other hand,

$$\hat{\Delta}_2(A) \leq \max_{X: \text{ran } X \subseteq \sigma(A)} |E(X^2) - E(X)^2|^{1/2}.$$

Some calculations give that the function $(p_1, p_2, p_3) \mapsto |E(X^2) - E(X)^2|$ defined on the probability simplex attains its maximum when $X$ is Bernoulli distributed with parameter $1/2$; that is,

$$\hat{\Delta}_2(A) \leq \frac{\sqrt{3}}{2}. $$
Example. If \( k \) is odd then it is straightforward to see that the Corollary 2 does not hold. In fact, let \( k = 3 \) and let
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Set the state \( \omega(X) = \frac{1}{4} \text{Tr } X \) on \( M_3(\mathbb{C}) \). Then
\[
\Delta_3(A) \geq |\omega((A - \omega(A))^3)|^{1/3} = \frac{\sqrt{50}}{3\sqrt{2}} \approx 0.868,
\]
while \( 2\|p_3(x)\|^{1/3}_{\infty} \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| = \frac{\sqrt{2}}{\sqrt{108}} \approx 0.648.\)

4. Notes on general elements

In this last section we have only few remarks and examples on the central moments of general elements. However, it would be interesting to know whether Rieffel’s theorem remains true for larger even moments; that is,
\[
\Delta_k(a) = 2\|q_k\|^{1/k}_{\infty} \inf_{\lambda \in \mathbb{C}} \|a - \lambda I\|
\]
follows for \( k \in 2\mathbb{N} \), where \( \|q_k\|_{\infty} \) denotes the largest \( k \)th central moments of the Bernoulli distribution? We do not even know the answer in the finite dimensional case, or when \( A = M_2(\mathbb{C}) \). In the noncommutative setting, we can prove the following weaker statement relying on the von Neumann inequality.

**Theorem 3.** For any \( k \) and \( a \) in a unital \( C^* \)-algebra \( A \), we have
\[
\max_{\omega \in S} |\omega((a - \omega(a))^k)|^{1/k} \leq 2\|q_k\|^{1/k}_{\infty} \inf_{\lambda \in \mathbb{C}} \|a - \lambda I\|,
\]
where \( q_k(x) = x(1-x)^k + x^k(1-x) \).

**Proof.** Choose a \( \lambda_0 \) such that \( \inf_{\lambda \in \mathbb{C}} \|a - \lambda I\| = \|a - \lambda_0 I\| \) holds and take \( \tilde{a} := (a - \lambda_0 I)/\|a - \lambda_0 I\| \). Then von Neumann’s inequality shows
\[
\|p(\tilde{a})\| \leq \sup_{z \in B} |p(z)| = \|p(u)\|,
\]
for any polynomial \( p \) where \( u \) stands for the bilateral shift on \( \ell^2(\mathbb{Z}) \). (Recall that \( \sigma(u) \) is the unit circle.) Thus any state \( \omega \) on \( A \) gives rise to a state \( \tilde{\omega} \) on \( B(\ell^2(\mathbb{Z})) \) such that \( \omega(p(\tilde{a})) = \tilde{\omega}(p(u)) \) holds for any polynomial \( p \) (see [4] Proposition 33.10). Hence from Theorem 1 we get
\[
1 = \|u\| = \inf_{\lambda \in \mathbb{C}} \|u - \lambda I\|
\leq \frac{1}{2\|q_k\|^{1/k}_{\infty}} \max_{\omega \in S(A)} |\omega((u - \omega(u))^k)|^{1/k}
\geq \frac{1}{2\|q_k\|^{1/k}_{\infty}} \max_{\omega \in S(A)} |\omega((\tilde{a} - \omega(\tilde{a}))|^k{1/k},
\]
which is what we intended to show. \( \square \)

Finally, our next example shows that the above inequality can be sharp for special elements, at least when \( k \) is even.

**Example.** Let us assume that \( S \) is a non-invertible isometry on a Hilbert space \( \mathcal{H} \). Then we can easily see that \( 2\|p_k\|^{1/k}_{\infty} \inf_{\lambda \in \mathbb{C}} \|S - \lambda I\| \leq \tilde{\Delta}(S) \), where \( p_k(x) = x(1-x)^k + (-1)^k x^k(1-x) \) as usual. In fact, set the Hermitian
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{C}).
\]
Obviously, \( \sigma(S) = \mathbb{D} \). The spectral mapping theorem gives that \( \sigma(p(A)) \subseteq \sigma(p(S)) \) hence \( \|p(A)\| \leq \|p(S)\| \) follows for any polynomial \( p \). This implies that any state \( \omega \) of \( M_2(\mathbb{C}) \) defines a state \( \tilde{\omega} \) on \( \mathcal{B}(\mathcal{H}) \) such that \( \omega(p(S)) = \tilde{\omega}(p(A)) \) holds for any \( p \). Therefore from Proposition 1

\[
2\|p_k\|^{1/k} \inf_{\lambda \in \mathbb{C}} \|S - \lambda I\| = 2\|p_k\|^{1/k} \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|
\leq \|p_k\|^{1/k} \diam(A)
\leq \max_{\omega} |\omega((A - \omega(A))^k)|^{1/k}
\leq \max_{\omega} |\omega((S - \omega(S))^k)|^{1/k}.
\]

Theorem 3 and the last inequality shows for any non-invertible isometry and \( k \in 2\mathbb{N} \) that

\[ \hat{\Delta}_k(S) = 2\|q_k\|^{1/k}. \]

Moreover, from Rieffel’s theorem ([10, Theorem 3.10], [3, Corollary 4.2]) we get \( \Delta_2(S) = \hat{\Delta}_2(S) \). Notice that \( S \) is Birkhoff–James orthogonal to \( I \); i.e. \( \|S\| = \inf_{\lambda \in \mathbb{C}} \|S - \lambda I\| \), hence there exists a state \( \varphi \) on \( \mathcal{B}(\mathcal{H}) \) such that \( \varphi(I) = 1 \) and \( \varphi(S) = 0 \) ([3 Corollary 3.3]). Actually with this state we have the equality

\[ \varphi((S - \varphi(S))^2)^{1/2} = \Delta_2(S) = 1. \]

However, for \( 4 \leq k \in 2\mathbb{N} \) the state \( \varphi \) determined by the Birkhoff–James orthogonality does not relate to the higher-order moments because \( \hat{\Delta}_k(S) > 1 \) while \( \varphi((S - \varphi(S))^k)^{1/k} = 1 \).

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