Infrared Modified Gravity with Dynamical Torsion

V. NIKIFOROVA$^{a,c}$, S. RANDJBAR-DAEMI$^b$, V. RUBAKOV$^{c}$

$^a$Physics Department, Moscow State University, Moscow, Russia
$^b$The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy
$^c$Institute for Nuclear Research of the Russian Academy of Sciences, Moscow, Russia

We continue the recent study of the possibility of constructing a consistent infrared modification of gravity by treating the vierbein and connection as independent dynamical fields. We present the generalized Fierz–Pauli equation that governs the propagation of a massive spin-2 mode in a model of this sort in the backgrounds of arbitrary torsionless Einstein manifolds. We show explicitly that the number of propagating degrees of freedom in these backgrounds remains the same as in flat space-time. This generalizes the recent result that the Boulware–Deser phenomenon does not occur in de Sitter and anti-de Sitter backgrounds. We find that, at least for weakly curved backgrounds, there are no ghosts in the model. We also briefly discuss the interaction of sources in flat background.

\footnote{E-mail: seif@ictp.trieste.it, rubakov@ms2.inr.ac.ru}
1 Introduction and summary

The possibility that gravity may get modified at large distances attracts considerable interest, which is motivated, in particular, by the accelerated expansion of the Universe. Deforming General Relativity in the infrared is, however, not at all easy. The problems one has to worry about are well understood in the context of the Fierz–Pauli theory \cite{1} of massive graviton. Already at the linearized level about Minkowski background, the Fierz–Pauli gravity exhibits the van Dam–Veltman–Zakharov discontinuity \cite{2,3}, which, taken at face value, implies that the theory deviates from General Relativity even at short distances. Below the Vainshtein radius, however, non-linear effects become important \cite{4} and, in fact, these effects may cure the van Dam–Veltman–Zakharov problem \cite{5}. The disaster occurs in curved backgrounds, where an extra, Boulware–Deser propagating mode shows up \cite{6,7,8}, over and beyond the five modes of massive graviton. This extra mode has ghost kinetic term and makes the theory unacceptable.

There are various theories that pretend to be free of at least some of the above problems. These include theories which deform General Relativity in backgrounds other than Minkowski \cite{9,10,11}, theories with extra dimensions \cite{12,13,14,15,16} (for a review see Ref. \cite{17}) and theories with broken Lorentz-invariance \cite{18,19,20,21,22} (for a review see Ref. \cite{23}). Yet another option is to consider theories whose independent fields are both vierbein and connection and whose Lagrangians contain terms quadratic in curvature as well as mass terms for the torsion field. Some theories of the latter type have both massless and massive spin-2 modes in the spectrum about Minkowski background and nevertheless are free of pathologies (ghosts or tachyons) at the linearized level in this background \cite{24,25,26,27}.

It is worth noting that theories with dynamical vierbein and connection may be viewed as gauge theories with spontaneously broken local Lorentz invariance \cite{30,31}. This viewpoint opens up a possibility of unification of gravity with the Yang–Mills theory \cite{32,33,34}. We will not pursue this approach and concentrate on degrees of freedom describing dynamical space-time geometry itself.

Once a geometrical theory has a massive spin-2 mode at the linearized level in Minkowski background, a question arises of whether or not this theory has the Boulware–Deser mode in curved backgrounds. Recently, this question has been addressed \cite{35} in the context of

\footnote{Vierbein and connection as gauge fields of the Poincare group have been introduced by Kibble \cite{28} and studied extensively by Hehl and collaborators. For a historical review and references to earlier work see Ref. \cite{29}. Our point here is to study the possibility of a consistent infrared modification of gravity.}
one of the models of Refs. [24, 25, 26, 27]. The analysis in Ref. [35] has been restricted
to de Sitter and anti-de Sitter backgrounds. The result has been encouraging, as it has
been found that there are no Boulware–Deser modes in these backgrounds. In this paper
we extend the analysis to arbitrary Einstein backgrounds with vanishing torsion and show
that the Boulware–Deser mode does not appear in this more general case as well. If the
curvature of the background is sufficiently small, propagating modes are neither ghosts nor
tachyons. Hence, our analysis supports the conjecture of Ref. [35] that the model is healthy
at least for sufficiently weak fields.

As a by-product, we identify the field that reduces to the massive spin-2 field as the
curvature of the background is switched off, and obtain the equation for this field that serves
as a generalization of the Fierz–Pauli equation to arbitrary Einstein backgrounds.

In the end of this paper we consider the interaction of sources in our model at the
linearized level in Minkowski background. We rederive the result of Refs. [25] that within
the appropriate range of parameters, there are no ghosts or tachyons. The interaction
between symmetric and conserved energy-momentum tensors is mediated by both massless
and massive spin-2 fields, the relative strength depending on the parameters of the model.
In this sense our model is indeed an infrared-modified gravity. The interaction due to
massive spin-2 field does exhibit the van Dam–Veltman–Zakharov discontinuity. We leave
for future study the questions of whether the Vainshtein phenomenon occurs in our model,
and if so, whether it cures the van Dam–Veltman–Zakharov problem.

The paper is organized as follows. We introduce the model, its field equations and
the Einstein manifold solutions in section 2. In section 3 we discuss the behaviour of
vector components of the torsion field in the Einstein backgrounds. The results of section 3
have been already obtained in Ref. [35]; we present them for completeness and later use.
Section 4 contains the main results of this paper. In section 4.1 we derive the equation that
the massive tensor field perturbations obey in the Einstein backgrounds. This equation
may be viewed as a generalization of the Fierz–Pauli equation. We then proceed to show
that in arbitrary Einstein backgrounds, this equation describes five propagating modes, the
right number for massive spin-2 field. We begin in section 4.2 by counting the number
of constraints — equations that contain time derivatives of at most first order — among
the ten equations constituting the generalized Fierz–Pauli system. We find that there are
five such constraints, which suggests that there are indeed five propagating modes. In
section 4.3 we make use of the St"uckelberg formalism to confirm this result and show that
at least in weakly curved backgrounds, neither of the propagating modes is a ghost. We
note, however, that the ghost problem may reappear in backgrounds of sufficiently high
curvature. Finally, in section 5 we study the linearized theory with sources in Minkowski background and obtain both the fields induced by the sources and the action that describes the interaction of the sources. Some of the results of this section are contained in Ref. 25; however, our emphasis will be different. We conclude in section 6 by discussing directions for future studies.

2 The model

2.1 Action

We follow the notations of Refs. 24, 25, 26, 35 and denote the vierbein by \( e^i_\mu \) and the connection by \( A_{ij\mu} = -A_{ji\mu} \), where \( \mu = 0, 1, 2, 3 \) and \( i, j = 0, 1, 2, 3 \) are the space-time and tangent space indices, respectively. We often use the tangent space basis, in which the indices are raised and lowered by the Minkowski metric \( \eta_{ij} \). The connection can be viewed as an \( O(1, 3) \) gauge field. It is conveniently decomposed as follows,

\[
A_{ijk} \equiv A_{ij\mu}e^\mu_k = \frac{1}{2}(T_{ijk} - T_{jik} - T_{kij} + C_{ijk} - C_{jik} - C_{kij}),
\]

where

\[
C_{ijk} = e^\mu_j e^\nu_k (\partial_\mu e_{i\nu} - \partial_\nu e_{i\mu}) = -C_{ikj}
\]
is constructed from vierbein and \( T_{ijk} = -T_{ikj} \) is the torsion tensor. The latter can in turn be decomposed into its irreducible components under the local \( O(1, 3) \) group,

\[
T_{ijk} = \frac{2}{3}(t_{ijk} - t_{ikj}) + \frac{1}{3}(\eta_{ij}v_k - \eta_{ik}v_j) + \varepsilon_{ijkl}a^l
\]

where the field \( t_{ijk} \) is symmetric with respect to the interchange of \( i \) and \( j \) and satisfies the cyclic and trace identities,

\[
t_{ijk} + t_{jki} + t_{kij} = 0, \quad \eta^{ij}t_{ijk} = 0, \quad \eta^{ik}t_{ijk} = 0
\]
The 24 independent components of \( T_{ijk} \) break up into 4 components of \( v_i \), 4 components of \( a_i \) and 16 independent components of \( t_{ijk} \).

The curvature, as usual in gauge theories, is defined by

\[
F_{ijmn} = e^\mu_m e^\nu_n (\partial_\mu A_{ij\nu} - \partial_\nu A_{ij\mu} + A_{ik\mu}A^k_{j\nu} - A_{ik\nu}A^k_{j\mu})
\]
The model studied in this paper, as well as in Ref. 35, is defined by the action

\[
S = \int d^4x \; e \; (L_F + L_T),
\]
where \( e = \det e^i_\mu \),

\[
L_F = c_1 F + c_2 + c_3 F_{ij} F^{ij} + c_4 F_{ij} F^{ji} + c_5 F^2 + c_6 (\varepsilon_{ijkl} F^{ijkl})^2
\]

and

\[
L_T = \alpha \left( t_{ijkl} t^{ijkl} - v_i v^i + \frac{9}{4} a_i a^i \right),
\]

with

\[
F_{ij} = \eta^{ik} F_{kjl}, \quad F = \eta^{ik} F_{jk}, \quad \varepsilon \cdot F = \varepsilon_{ijkl} F^{ijkl}.
\] (3)

Here \( c_1, \ldots, c_6 \) are “coupling constants” obeying, apart from sign restrictions, the only condition

\[
c_3 + c_4 + 3c_5 = 0.
\]

In what follows, three combinations of these parameters will be used,

\[
\tilde{\alpha} &= \alpha + \frac{2}{3} c_1 \quad (4) \\
\Lambda &= -\frac{c_2}{6c_1} \quad (5) \\
\varkappa &= 2\Lambda + \frac{\tilde{\alpha}}{2\kappa} \quad (6)
\]

By appearance, the term \( L_F \) has the form of kinetic term for the connection (plus the cosmological constant term \( c_2 \)), while \( L_T \) is torsion mass term.

We note in passing that the Lagrangian \( L_T \) does not explicitly break local \( O(1,3) \) invariance, so the entire action is invariant under both local frame rotations and general coordinate transformations. For \( c_2 = 0 \), the model admits Minkowski space-time as a solution of the field equations. In that case, the local \( O(1,3) \) invariance is spontaneously broken by the background value of the vierbein field, cf. Refs. [30, 31, 32, 33, 34].

The model with \( c_2 = 0 \) is free of ghosts and tachyons in Minkowski background provided the parameters satisfy certain inequalities [24, 25, 26, 27], which in our notations read

\[
c_1 > 0, \quad c_5 < 0, \quad c_6 > 0, \quad \alpha < 0, \quad \tilde{\alpha} > 0.
\] (7)

Non-vanishing value of \( c_2 \) enables one to have de Sitter or anti-de Sitter solution with vanishing torsion in this model, with cosmological constant equal to \( \Lambda \). In the latter case, the requirement of the absence of tachyons imposes one more condition [35], \( c_5 \varkappa > 0 \), i.e.,

\[
\tilde{\alpha} > -4\Lambda c_5.
\] (8)

Since \( c_5 < 0 \), the latter condition is non-trivial for positive \( \Lambda \). Once the above conditions are satisfied, the theory is healthy in de Sitter and anti-de Sitter backgrounds [35].
There are two sets of field equations in our model. One consists of the gravitational field equations obtained by varying the action with respect to vierbein, $v^a$, 

$$c_1 F_{ji} + c_3 (F^m_{i} F_{mj} - F_{jm}^{mn} F_{mn}) + c_4 (F^m_{i} F_{jm} - F_{jm}^{mn} F_{mn}) + 2c_5 F_{ji} F + 2c_6 \varepsilon_{kmn} F^{kmn} i (\varepsilon_{rpq} F^{rqs}) + (D^k + v^k) F_{ijk} + H_{ij} - \frac{1}{2} \eta_{ij} (L_F + L_T) = 0 \quad (9)$$

where

$$F_{ijk} = \alpha \left[ (t_{ijk} - t_{ikj}) - (\eta_{ij} v_k - \eta_{ik} v_j) - \frac{3}{4} \varepsilon_{ijkl} a^l \right]$$

$$H_{ij} = T_{nm} F^{mn} j - \frac{1}{2} T_{jmn} F^{mn}$$

and $D_i$ is the covariant derivative with respect to the connection $A_{ijk}$. Note that these equations have both symmetric and antisymmetric parts.

By varying the action with respect to the connection $A_{ij\mu}$ one finds another set of equations,

$$c_3 \left\{ \eta^{jk} (D_m + \frac{2}{3} v_m) F^{jm} - \eta^{jk} (D_m + \frac{2}{3} v_m) F^{jm} - (D^i + \frac{2}{3} v^i) F^{ik} + (D^j + \frac{2}{3} v^j) F^{jk} \right\}$$

$$+ c_4 \left\{ \eta^{jk} (D_m + \frac{2}{3} v_m) F^{jm} - \eta^{jk} (D_m + \frac{2}{3} v_m) F^{jm} - (D^i + \frac{2}{3} v^i) F^{ki} + (D^j + \frac{2}{3} v^j) F^{kj} \right\}$$

$$+ c_5 \left\{ \eta^{jk} (D^j + \frac{2}{3} v^j) F - \eta^{jk} (D^i + \frac{2}{3} v^i) F \right\} + 4c_6 \left\{ \varepsilon^{ijk} (D_m + \frac{2}{3} v_m) (\varepsilon \cdot F) \right\}$$

$$- \left( \frac{4}{3} \right)^k [mnp] + \varepsilon^{k mnp} a^p \right\} \left\{ c_3 (\eta^{jm} F^{jn} - \eta^{jm} F^{jn}) + c_4 (\eta^{im} F^{mj} - \eta^{mj} F^{im}) \right\}$$

$$+ 2c_5 \eta^{im} \eta^{jn} F + 2c_6 \varepsilon^{ijk} (\varepsilon \cdot F) \right\} + H_{ijk} = 0 \quad (10)$$

where

$$H_{ijk} = -\tilde{\alpha} (t_{kij} - t_{kji}) + \tilde{\alpha} (\eta_{ki} v_j - \eta_{kj} v_i) - \frac{3\tilde{\alpha}}{2} \varepsilon_{ijkl} a^l$$

Equations (9) and (10) are not completely independent because of the Bianchi identity. In a theory with torsion, this identity reads

$$D_k F_{ijlm} + T^n_{kl} F_{ijmn} + \text{cyclic (klm)} = 0 \quad (11)$$

Contracting this identity one obtains

$$D^i F_{ij} - \frac{1}{2} D_j F = T^i_{kj} F^{k i} + \frac{1}{2} T^i_{kl} F^{k l j} \quad (12)$$

We will see that these identities provide useful constraints in the linearized theory.
2.3 Einstein backgrounds

Let us consider torsion-free backgrounds. For vanishing torsion, the curvature tensor and its contractions reduce to the Riemann tensor, the Ricci tensor and the Ricci scalar, $F_{ijkl} = R_{ijkl}, F_{ij} = R_{ij}$ and $F = R$, respectively. By inspecting eq. (9) one finds that it is satisfied for the Einstein manifolds. The Riemann tensor for these manifolds has the following form

$$R_{ijkl} = \Lambda (\eta_{ik} \eta_{jl} - \eta_{il} \eta_{jk}) + W_{ijkl},$$

where the Weyl tensor $W_{ijkl}$ has all symmetries of the Riemann tensor and is traceless in all pairs of indices. One then has

$$R_{ij} = 3\Lambda \eta_{ij}, \quad R = 12 \Lambda.$$

Using these properties one finds that eq. (9) reduces to the relationship (5) between $\Lambda$ and coupling constants. Equation (10) is satisfied for the Einstein manifolds automatically.

An important property of the Weyl tensor of the Einstein manifolds follows from the Bianchi identity, namely

$$\nabla^i W_{ijkl} = 0$$

We will repeatedly make use of this property in what follows.

3 Linearized theory in Einstein backgrounds: pseudovector $a_i$ and vector $v_i$

One of the main purposes of this paper is to study field perturbations about general torsion-free Einstein backgrounds. The analysis of pseudovector $a_i$ and vector $v_i$ has been performed in Ref. [35] where it has been shown that the vector field $v_i$ does not have its own propagating modes, while the pseudovector field $a_i$ is a gradient, and its longitudinal part obeys the massive Klein–Gordon equation. These properties are exactly the same as in the theory about Minkowski background. For the sake of completeness and presentation of useful formulas, we recapitulate the analysis here.

We do not use special notation for the background objects, unless there is a risk of an ambiguity; the subscript (1) refers to linearized perturbations. The torsion components vanish for our backgrounds, and we do not label their perturbations by the subscript (1).

To study the fields $v_i$ and $a_i$, it suffices to consider certain combinations of the full equations (9) and (10). We begin with the antisymmetric part of the gravitational equation.
where \( c_1 \) whose complete form is
\[
c_1 F_{[ij]} + \frac{c_3}{2}(F^{m_i}_j F_{jm} - F^{m_j}_i F_{im}) - \frac{1}{2}(F^{mn}_{ij} + F^{mn}_{ji}) (c_3 F_{mn} + c_4 F_{mn}) \\
+ 2c_5 F_{[ijkl]} - c_6 (\varepsilon_{kmn} F^{k}{m{n}} - \varepsilon_{kmn} F^{k}{m{n}}) (\varepsilon_{rpsq} F^{rpsq}) \\
+ (D^k + v^k) F_{[ij]k} + H_{[ij]} = 0. \tag{13}
\]
By linearizing this equation about the Einstein background, one obtains
\[
(c_1 - 4\Lambda c_3) F_{(1)[ij]} - \nabla^k F_{(1)[ij]k} - (c_3 - c_4) W_j^{[mn]} F_{(1)[mn]} = 0. \tag{14}
\]
where \( \nabla \) denotes the covariant derivative with respect to the background metric. By linearizing \( F_{ij} \) defined in (3) one finds the following explicit expression for the antisymmetric part
\[
F_{(1)[ij]} = -\frac{2}{3\alpha} \nabla^k F_{[ij]k} \tag{15}
\]
Hence, eq. (14) is an algebraic equation for \( F_{(1)[ij]} \) whose only solution is
\[
F_{(1)[ij]} = 0 \tag{16}
\]
This result simplifies all other equations.

Now, let us write the curl of the torsion equation (10). Namely, we contract eq. (10) with \( \varepsilon_{ijkl} \) to find the complete curl equation,
\[
(c_3 - c_4) \varepsilon_{ijkl} D^l F^{jk} - 12c_6 D_l (\varepsilon \cdot F) - \frac{2}{3} \varepsilon_{ijkl} t^i n^k (c_3 F^{jn} + c_4 F^{nj}) - 8c_6 v_l (\varepsilon \cdot F) \\
- \frac{2}{3} (c_3 - c_4) \varepsilon_{ijkl} v^i F^{jk} - 2(c_3 F_{jl} + c_4 F_{lj}) a^j + \frac{2}{9} \alpha a_l = 0 \tag{17}
\]
Substituting (16) into eq. (17) and using the fact that \( \varepsilon \cdot F = 6\nabla^i a_i \) we obtain the following linearized equation for \( a_l \),
\[
8c_6 \nabla l (\nabla \cdot a) - \left( 2\Lambda c_5 + \frac{\bar{\alpha}}{2} \right) a_l = 0 \tag{18}
\]
This equation shows that the pseudovector field is a gradient, \( a_l = \nabla l \sigma \). Its longitudinal part obeys the Klein-Gordon equation
\[
\left( \nabla^2 - \frac{c_5 \bar{\alpha}}{8c_6} \right) \sigma = 0 \tag{19}
\]
where \( \bar{\alpha} \) is defined in (6). The spin zero field \( \sigma \) has healthy kinetic term and is not a tachyon provided that the inequality (8) is satisfied. The mass of this field coincides with the flat space result when \( \Lambda = 0 \).
It is worth noting that using the explicit form of the right hand side of eq. (15) and the fact that $a_l$ is a gradient, one obtains from eq. (16) the following constraint

$$\nabla^k t_{k[mn]} = \nabla_{[m} v_{n]}$$

(20)

This constraint already suggests that at least some components of the field $v_i$ are not independent.

In fact, the entire field $v_i$ is not dynamical by itself, as it can be expressed through the tensor $t_{ijk}$. To see this, one makes use of the trace of the torsion equation. For obtaining its general form, one contracts eq. (10) with $\eta_{jk}$ and finds

$$-3c_5 \left( D_j F^{(ij)} - \frac{1}{2} D^i F \right) + (c_3 - c_4) D_j F^{[ij]} - 2c_5 \left( v_j F^{(ij)} - \frac{1}{2} v^i F \right) + (c_3 - c_4) D_j F^{[ij]}$$

$$+ \frac{2}{3} (c_3 - c_4) v_i F^{[ij]} - 3c_5 t^{(ij)}_{jl} F_{[jn]} + \frac{1}{3} (c_3 - c_4) t^{[ij]}_{jn} F_{[jn]}$$

$$- \frac{1}{2} (c_3 - c_4) \varepsilon^{ijn} a_l F_{[jn]} + 6c_6 a^i (\varepsilon \cdot F) + \frac{3}{2} \tilde{\alpha} v^i = 0$$

(21)

Now, one substitutes $F^{[ij]} = 0$ in the linearized version of this equation and finds

$$\left( D_j F^{ij} - \frac{1}{2} D^i F \right) = \kappa v^i$$

(22)

The left hand side of eq. (22) vanishes for vanishing torsion, so it is proportional to torsion at the linearized level. To obtain the explicit expression one makes use of the Bianchi identity (12). Linearizing this identity in the Einstein background, one finds

$$\left( D^i F_{ij} - \frac{1}{2} D_j F \right) = 2\Lambda v^i + \frac{1}{2} T^{i}_{kl} W^{kl}$$

so that eq. (22) becomes

$$v_i = \frac{4c_5}{3\tilde{\alpha}} W_{ijkl} t^{[kl]}$$

Hence, the vector field $v_i$ is not an independently propagating field, exactly as in the flat space.

4 Spin-2 mode in Einstein backgrounds

4.1 Generalized Fierz–Pauli equation

Let us now derive the equations for propagating tensor perturbations. To this end, we make use of the results of section 3 and write the linearized gravitational equation (9) in the
following form,
\[ c_1 \left( F^{(1)}_{1ij} - \frac{1}{2} \eta_{ij} F^{(1)} \right) + \nabla^k F^{(1)}_{ijk} + 3c_5 W_{jmn} F^{mn}_{(1)} = 0 \] (24)

Hereafter we treat the field \( F^{(1)}_{ij} \) as well as the combination \( \nabla^k F^{(1)}_{ijk} \) as symmetric with respect to the interchange of indices \( i \) and \( j \), see eqs. (15) and (16). The expression for \( \nabla^k F^{(1)}_{ijk} \) is, explicitly,
\[ \nabla^k F^{(1)}_{ijk} = -\alpha \left[ 3 \nabla^k t_{k(ij)} - \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) + \eta_{ij} \nabla \cdot v \right] \] (25)

The remaining equation is the torsion equation (10) whose linearized version is
\[ \nabla_i F^{(1)}_{jk} - \nabla_j F^{(1)}_{ik} + \frac{1}{6} (\eta_{ik} \nabla_j F^{(1)} - \eta_{jk} \nabla_i F^{(1)}) - \frac{1}{3} \left( 2\Lambda + \frac{\tilde{\alpha}}{2c_5} \right) \left[ (\eta_{ik} v_j - \eta_{jk} v_i) + 4t_{k[ij]} \right] = 0 \] (26)

Here \( v_i \) should be expressed in terms of \( t_{ijkl} \) according to (23).

Equation (26) together with eq. (22) may be used to express \( t_{ijkl} \) in terms of \( F^{(1)}_{1ij} \),
\[ t_{ijkl} = -\frac{1}{4\varkappa} \left[ (3\nabla_i F^{(1)}_{jk}) - 3\nabla_j F^{(1)}_{ik}) + \left( \eta_{ik} \nabla^l F^{(1)}_{(1)kl} - \eta_{jk} \nabla^l F^{(1)}_{(1)li} \right) - \left( \eta_{ik} \nabla_j F^{(1)} - \eta_{jk} \nabla_i F^{(1)} \right) \right], \] (27)

where \( \varkappa \) is defined in (3). It is straightforward to check that eq. (20), with \( t_{ijkl} \) and \( v_i \) expressed through \( F^{(1)}_{1ij} \), is identically satisfied.

Equation (27) determines, in fact, the full tensor \( t_{ijkl} \) in terms of \( F^{(1)}_{1ij} \). Indeed, due to the identities (3), one has
\[ t_{ijkl} = \frac{2}{3} \left( t_{i[jk]} + t_{j[ik]} \right) \] (28)

Substituting this into eq. (25) one obtains
\[ \nabla^l F_{ijkl} = -\frac{\alpha}{4\varkappa} \left[ 6\nabla^2 F^{(1)}_{ijkl} - 3 \left( \nabla_j \nabla^l F^{(1)}_{(1)kl} + \nabla^l \nabla_j F^{(1)}_{(1)kl} \right) - 3 \left( \nabla_k \nabla^l F^{(1)}_{(1)jl} + \nabla^l \nabla_k F^{(1)}_{(1)jl} \right) + 6\eta_{jk} \nabla^m \nabla^m F^{(1)}_{(1)mn} - 4\eta_{jk} \nabla^2 F^{(1)} + 4 \nabla_j \nabla_k F^{(1)} \right] \] (29)

Thus, equation (24) becomes the equation for the field \( F^{(1)}_{1ij} \),
\[ \nabla^k F^{(1)}_{ijk} + c_1 \left( F^{(1)}_{1ij} - \frac{1}{2} \eta_{ij} F^{(1)} \right) + 3c_5 W_{jmn} F^{mn}_{(1)} = 0 \] (30)

where the first term is given by the right hand side of (29).
Equation (30) is a closed equation for the field \( F_{(1)ij} \), while the fields \( t_{ijk} \) and \( v_i \) are expressed through \( F_{(1)ij} \) according to eqs. (27), (28) and (23). One can check that all equations of the linearized theory are satisfied provided that the field \( F_{(1)ij} \) obeys eq. (30).

Equation (30) does not look similar to the Fierz–Pauli equation yet. To write it in a more familiar way, let us introduce the field

\[
 u_{ij} = F_{(1)ij} - \frac{1}{6} \eta_{ij} F_{(1)}
\]

Then eq. (30) becomes

\[
 \nabla^2 u_{ij} - \nabla^k \nabla_i u_{kj} - \nabla^k \nabla_j u_{ki} + \nabla_i \nabla_j u + \eta_{ij} \left( \nabla^k \nabla^l u_{kl} - \nabla^2 u \right) + 6 \Lambda \left( u_{ij} - \frac{1}{2} \eta_{ij} u \right) \\
- \left( 2 \Lambda + \frac{2 \kappa c_1}{3 \alpha} \right) \left( u_{ij} - \eta_{ij} u \right) + \left( 1 - \frac{2 \kappa c_5}{\alpha} \right) W_{ilkj} u^{lk} = 0 \tag{31}
\]

This equation reduces to the Fierz–Pauli equation in Minkowski background, with the mass of the spin-2 field given by

\[
 m^2 = \frac{\tilde{\alpha} c_1}{3 \alpha c_5} .
\]

So, eq. (31) may be viewed as the generalization of the Fierz–Pauli equation to the Einstein backgrounds.

Equation (31) has particularly simple form in de Sitter and anti-de Sitter backgrounds for which \( W_{ilkj} = 0 \). In that case the trace and divergence of this equation together with the equation itself imply that \( u_{ij} \) has to be traceless and divergence free. We then obtain the Klein–Gordon equation with the mass given by

\[
 M^2 = 4 \Lambda \left( 1 + \frac{c_1}{3 \alpha} \right) + \frac{\tilde{\alpha} c_1}{3 \alpha c_5}
\]

in accord with Ref. [35]. This mass obeys the Higuchi bound [36] provided the inequality (8) is satisfied.

A remark is in order. Besides the tensor mode propagating according to eq. (31), there is of course a massless tensor mode. The latter corresponds to perturbations of the vierbein field with vanishing torsion, and propagates according to the linearized Einstein equations (written relative to an orthonormal basis of the background geometry), \( R_{(1)ij} = 0 \). For this mode \( F_{(1)ij} = 0 \), so eq. (31) is trivially satisfied.

### 4.2 Counting the number of constraints

Let us show that out of the ten equations satisfied by ten components of \( u_{ij} \), five are constraints which involve at most first order time derivatives. The remaining five involve at
most second order time derivatives. This suggests that the field $u_{ij}$ describes five propagating modes, the right number for massive spin-2 field.

Obtaining four out of five constraints is straightforward. Indeed, the divergence of eq. (31) gives first-order equations,

$$\tilde{M}^2 \nabla^i (u_{ij} - \eta_{ij} u) = \left( 1 - \frac{2 \zeta_5}{\alpha} \right) W_{jikl} \nabla^k u^{il} \quad (33)$$

where

$$\tilde{M}^2 = M^2 - 2 \Lambda$$

These are obviously the four constraints.

To see that there are actually five constraints, let us choose the coordinates in the background manifold such that the background metric components are $g_{0a} = 0 = g^{0a}$, where $a = 1, 2, 3$ and $g^{00} = g_{00}^{-1}$. Working in coordinate basis rather than orthonormal one which we have used so far, we perform the $(3 + 1)$-decomposition of the field components $u_{\mu \nu}$ as well as the field equations. It is straightforward to see that $(00)$- as well as $(a0)$-components of eq. (31) involve at most first order time derivatives. These are the four constraints corresponding to eq. (33). The $(ab)$-components of eq. (31) — six equations — are superficially second order in time. However, one combination of the latter is in fact a constraint. To see this, let us write the trace of eq. (31),

$$u = \frac{2 \zeta_5 - \alpha}{M^2 \zeta c_1} W_{imkn} \nabla^i \nabla^n u^{mk} \quad (34)$$

To show that no time derivatives higher than first order enter this equation, we write the right hand side of this equation in an expanded form,

$$u = \frac{2 \zeta_5 - \alpha}{M^2 \zeta c_1} W_{\mu \sigma \nu} \nabla^\mu \nabla^\nu u^{\lambda \sigma} = \frac{2 \zeta_5 - \alpha}{M^2 \zeta c_1} W^{\nu \nu \nu \nu} \nabla^2 u_{ab} + \ldots \quad (35)$$

where dots denote the terms which have at most one time derivative. Now, we substitute $\nabla^2 u_{ab}$ from eq. (31) into eq. (35). One can show that the expression for $\nabla^2 u_{ab}$ does not involve any term with more than one $\nabla_0$. Thus, the substitution shows that eq. (35) is indeed another constraint reducing the number of propagating modes from ten to five.

### 4.3 St"{u}ckelberg treatment

A simple way to isolate the dangerous degrees of freedom is to make use of the St"{u}ckelberg trick. As an example, this trick enables one to see in rather straightforward manner how the van Dam–Veltman–Zakharov phenomenon and related effects emerge in the Fierz–Pauli
theory in Minkowski background, and how the Boulware–Deser ghost mode appears in that theory in curved backgrounds. Let us make use of the Stückelberg trick to see that no Boulware–Deser mode is present in the theory with the field equation (31).

The quadratic action which corresponds to eq. (31) is, up to an overall factor,

\[ S = S_{\text{inv}} + S_m + S_W \]

where

\[ S_{\text{inv}} = \int d^4x \sqrt{-g} \left\{ -\frac{1}{2} \nabla^k u_{ij} \nabla_k u^{ij} + \nabla^k u_{ik} \nabla_i u^{k} - \nabla^k u_{lk} \nabla^l u^{i} + \frac{1}{2} \nabla_i u \nabla^i u \right. \]

\[ \left. - \Lambda \left( u_{ij} u^{ij} + \frac{1}{2} u^2 \right) - W_{iklj} u^{ik} u^{lj} \right\} \] (36)

\[ S_m = -\frac{\tilde{M}^2}{2} \int d^4x \sqrt{-g} \left( u_{ij} u^{ij} - u^2 \right) \] (37)

\[ S_W = s \int d^4x \sqrt{-g} W_{iklj} u^{ik} u^{lj} \] (38)

Here

\[ s = 1 - \frac{2 \kappa c_5}{\alpha} \]

The part (36) of the action is invariant under the gauge transformation

\[ u_{ij} = \bar{u}_{ij} + \nabla_i \zeta_j + \nabla_j \zeta_i \] (39)

while the terms (37) and (38) are not invariant. To implement the Stückelberg procedure, one introduces new fields \( \bar{u}_{ij}, \xi_i \) and \( \phi \) and writes

\[ u_{ij} = \bar{u}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \phi \]

Altogether, there are now 15 fields. The theory is invariant under two gauge symmetries,

\[ \bar{u}_{ij} \rightarrow \bar{u}_{ij} + \nabla_i \zeta_j + \nabla_j \zeta_i , \quad \xi_i \rightarrow \xi_i - \zeta_i \]

and

\[ \xi_i \rightarrow \xi_i + \nabla_i \psi , \quad \phi \rightarrow \phi - 2 \psi \] (40)

Provided that all fields have kinetic terms quadratic in derivatives, these symmetries eliminate 10 degrees of freedom, and there remain 5 propagating modes. It is not at all guaranteed, however, that the change of variables (39) does not introduce higher-derivative terms in the action. A counterexample is given by the Fierz–Pauli gravity in curved backgrounds: the kinetic term for the field \( \phi \) has four derivatives (there are terms like \( (\Box \phi)^2 \)), and the sixth propagating mode, a ghost, appears in the spectrum [7, 8, 23]; this is the
way the Boulware–Deser mode is seen in the Stückelberg formalism. Let us show that this phenomenon does not occur in our model.

The action \( S_{\text{inv}} = S_{\text{inv}}(\bar{u}) \) contains only the field \( \bar{u}_{ij} \) and is obviously second order in derivatives. The change of variables (39) gives rise to the derivative terms in the mass part of the action,

\[
S_m(\bar{u}, \xi, \phi) = \frac{\tilde{M}^2}{2} \int d^4x \sqrt{-g} \left\{ (\nabla_i \xi_j - \nabla_j \xi_i)(\nabla^i \xi^j - \nabla^j \xi^i) - 2u \nabla^2 \phi - 3\Lambda \nabla_i \phi \nabla^i \phi + \ldots \right\} \tag{41}
\]

where the contribution proportional to \( \Lambda \) appears in the process of integration by parts, and dots denote terms with less than two derivatives. The term (38) also contains derivatives,

\[
S_W(\bar{u}, \xi, \phi) = s \int d^4x \sqrt{-g} W^{ijkl} \left\{ 2\bar{u}_{ij} \nabla_k \nabla_l \phi + 4 [\nabla_i (\xi_j + \nabla_j \phi)] [\nabla_k (\xi_l + \nabla_l \phi)] + \ldots \right\} \tag{42}
\]

The part (41) of the action gives healthy kinetic term for the (gauge fixed) vector field \( \xi^i \) and provides kinetic mixing between the fields \( \phi \) and \( u_{ij} \). This property is precisely the same as in the Fierz–Pauli theory in Minkowski background [37]. The sum \( (S_{\text{inv}} + S_m) \) can be diagonalized by the shift of the field (cf. Ref. [37]),

\[
\tilde{u}_{ij} \rightarrow \bar{u}_{ij} + \frac{\tilde{M}^2}{2} \eta_{ij} \phi
\]

As a result, the field \( \phi \) obtains healthy kinetic term

\[
S_{\phi} = -\frac{3}{4} \tilde{M}^2 \left( \tilde{M}^2 - 2\Lambda \right) \int d^4x \sqrt{-g} \nabla_i \phi \nabla^i \phi
\]

The overall sign here is normal (non-tachyonic), provided that inequality \([8]\) is satisfied.

For the de Sitter or anti-de Sitter background one has \( W_{ijkl} = 0 \), so the Stückelberg action is clearly second order in derivatives in that case and the field \( u_{ij} \) describes five propagating modes, in agreement with Ref. [35].

In the general Einstein background, the Weyl tensor does not vanish, and the term (42) looks dangerous, as it appears to contain four derivatives. However, integrating by parts and using the properties of the Weyl tensor, we write this term in the following form,

\[
S_W(\bar{u}, \xi, \phi) = s \int d^4x \sqrt{-g} W^{ijkl} \left\{ 2\bar{u}_{ij} \nabla_k \nabla_l \phi - W_{iklm}(\nabla^m \phi \nabla^j \phi + 4 \nabla^m \phi \xi^j + 2 \xi_j \xi^j) + \ldots \right\}
\]

Hence, this term contains in fact at most two derivatives, so the number of propagating modes remains equal to five. The Boulware–Deser phenomenon is absent in our model, at least in the Einstein backgrounds.
In sufficiently weak background fields, when $|W_{ijkl}| \ll \tilde{M}^2$, the mass term $S_m$ dominates over $S_W$, so the propagating modes are not ghosts. In stronger background fields, the term $S_W$ induces explicit two-derivative term in the action for the field $\phi$, and also extra kinetic mixing between this field and $\bar{u}_{ij}$, so there may appear ghost modes. We will comment on this point in section 6.

5 Interaction between sources in Minkowski background

Let us now consider the linearized theory in Minkowski background and study the interaction between sources. Our purpose here is twofold. First, we will confirm that all modes in this theory linearized about flat space-time are neither ghosts nor tachyons. Second, we will see that the interaction between conserved energy-momentum tensors which couple to the vierbein and do not directly couple to connection is mediated by both massless and massive spin-2 fields, so our model is indeed an infrared-modified gravity.

Let us denote the sources coupled to the vierbein and connection by $J_i^\mu$ and $S^{ij\mu}$, respectively, and introduce the source term in the action,

$$S_{\text{source}} = \int d^4x \left( 2h_i^\mu J_i^\mu - \frac{1}{2}A_{ij\mu}S^{ij\mu} \right)$$

where $h_i^\mu$ is defined by

$$e_i^\mu = \delta_i^\mu + h_i^\mu$$

We will still work with objects like $J^{ij}$, $S^{ijk}$, defined by $J^{ij} = J_i^\mu \delta_j^\mu$, $S^{ijk} = S^{ij\mu} \delta_k^\mu$. Note that the source $J^{ij}$ in general is not symmetric.

The theory is invariant under linearized local frame rotations and linearized general coordinate transformations. The requirement that the source term respects these gauge symmetries gives two conservation laws,

$$\partial_j J_{ij} = 0$$

and

$$\partial_j S^{ij} = 4J[ij]$$

Note that eq. implies

$$\partial_j J_{(ij)} = -\partial_j J_{[ij]}$$

The local symmetries also enable one to set the antisymmetric part of $h_{ij}$ equal to zero. Thus, in what follows we use the gauge $h_{ij} = h_{ji}$.
Let us write down the linearized field equations with the sources. We omit the subscript $(1)$ in this section. The linearized gravitational equation is

$$c_1 \left( F_{ji} - \frac{1}{2} \eta_{ij} F \right) + \partial^k F_{ijk} = J_{ij}$$

(47)

Note that the antisymmetric part of this equation gives

$$F_{[ij]} = -\frac{2}{3\alpha} J_{[ij]}$$

(48)

which is a constraint. The propagation equation is thus the symmetric part of (47), namely,

$$c_1 \left( F_{(ij)} - \frac{1}{2} \eta_{ij} F \right) + \partial^k F_{(ij)k} = J_{(ij)}$$

(49)

The linearized torsion equations in the presence of source terms have the following form,

$$c_3 \left( \eta^{ik} \partial_m F^{jm} - \eta^{jk} \partial_m F^{im} - \partial^i F^{jk} + \partial^j F^{ik} \right)$$

$$+ c_4 \left( \eta^{ik} \partial_m F^{mj} - \eta^{jk} \partial_m F^{mi} - \partial^i F^{kj} + \partial^j F^{ki} \right)$$

$$+ 2c_5 (\eta^{ik} \partial^j F - \eta^{jk} \partial^i F) + 4c_6 \epsilon^{ikm} \partial_m (\epsilon \cdot F) + H^{ijk} = \frac{1}{2} S^{ijk}$$

(50)

Making use of (11) and (2) one writes the source term (43) in terms of $h_{ij}$ and the components of the torsion. The result is

$$S_{source} = \int d^4 x \left( 2h_{ij}\tau^{ij} + \frac{2}{3} \eta_{ij} S^{ij} + \frac{1}{3} \tau_j S^{ij} + \frac{1}{4} \epsilon_{ijkm} a^m S^{ijk} \right)$$

(51)

where $\tau^{ij}$ is defined by

$$\tau^{ij} = J^{(ij)} - \frac{1}{2} \partial_m S^{m(ij)}$$

(52)

In view of eqs. (45) and (46) the source $\tau_{ij}$ is conserved, as it should.

Let us also define the following combinations of the sources,

$$S = \epsilon_{ijkl} \partial_i S^{jk}$$

and

$$\sigma_{ij} = J^{(ij)} - \frac{\alpha}{2} \partial^m S^{m(ij)}$$

(53)

It is now a matter of straightforward but tedious calculation to find the solution to eqs. (47).
The result is

\[ h_{ij} = \frac{1}{c_1} \frac{1}{k^2} \left( \tau_{ij} - \frac{1}{2} \eta_{ij} \tau \right) - \frac{\tilde{\alpha}}{c_1 \alpha} \frac{1}{k^2 + m^2} \left( \sigma_{ij} - \frac{1}{3} \eta_{ij} \sigma \right) \]  

(54)

\[ v^i = - \frac{1}{6\alpha} \left( S^{ij} + 8 \frac{c_3}{c_1} i k_m \sigma^m \right) \]  

(55)

\[ a_t = - \frac{1}{288m_0^2 c_0} \frac{k_i S}{k^2 + m^2} + \frac{1}{18\tilde{\alpha}} \varepsilon_{ijkl} \left\{ S^{ijk} + \frac{2(c_3 - c_4)}{3\tilde{\alpha}} k^i k_m S^{jkm} \right\} \]  

(56)

\[ t_{k[ij]} = \frac{i}{2\alpha} \frac{1}{k^2 + m^2} \left\{ k_i \left( \sigma_{jk} - \frac{1}{3} \eta_{jk} \sigma \right) - k_j \left( \sigma_{ik} - \frac{1}{3} \eta_{ik} \sigma \right) + \frac{k^m k^m}{m^2} (k_i \sigma_{mj} - k_j \sigma_{mi}) \right\} \]  

- \frac{c_3 - c_4}{36\alpha^2} k^m (k_i S_{jkm} - k_j S_{ikm} - 2k_k S_{ijm}) - \frac{1}{12\tilde{\alpha}} (\eta_{ik} S_{jkm} - \eta_{jk} S_{ikm}) \]  

- \frac{1}{6\alpha} \left\{ S_{ijk} + \frac{1}{2} \left( S_{ikj} - S_{jki} \right) + \frac{i c_3}{3c_1 \alpha} k^m (\eta_{ik} \sigma_{mj} - \eta_{jk} \sigma_{mi}) \right\} \]  

(57)

where \( m_0 \) is the flat limit of the mass of the pseudoscalar field (see eq. (19)),

\[ m^2_0 = \frac{\tilde{\alpha}}{16c_0} \]

and \( m \) is given by (32). Plugging these expressions back into the action (this amounts to calculating (1/2) of the source term (51)) we find the action that describes the interaction of the sources. We write its expression omitting the terms which are ultra-local in the sources,

\[ S_{\text{int}} = \int d^4k \left\{ \frac{1}{144\tilde{\alpha}} \frac{\ddot{S}}{k^2 + m^2} + \frac{1}{c_1} \frac{1}{k^2} \ddot{\tau}_{ij} \left( \tau_{ij} - \frac{1}{2} \eta_{ij} \tau \right) \right. 

- \frac{\tilde{\alpha}}{c_1 \alpha} \frac{1}{k^2 + m^2} \left[ \dot{\sigma}_{ij} \left( \sigma^{ij} - \frac{1}{3} \eta^{ij} \sigma \right) + 2\frac{k^m k^m}{m^2} \dot{\sigma}_{ij} \left( \sigma^{jm} - \frac{1}{3} \eta^{jm} \sigma \right) \right] \right\} \]  

(58)

where bar denotes complex conjugation. The three terms here correspond to exchange by massive spin-0 particle, massless spin-2 particle and massive spin-2 particle, respectively. The latter exhibits the van Dam–Veltman–Zakharov discontinuity, as is generally the case. It is clear that with the restrictions on parameters summarized in (7), neither of the modes is ghost or tachyon.

Of particular interest is the metric perturbation generated by the symmetric energy-momentum tensor \( \tau_{ij} = \tau_{(ij)} \) coupled to metric and not directly coupled to torsion. In this case we have \( J_{ij} = \sigma_{ij} = \tau_{ij} \), \( S_{ijk} = 0 \), \( \partial_t \tau^{ij} = 0 \) and the expression (54) becomes

\[ h_{ij} = \frac{1}{c_1} \frac{1}{k^2} \left( \tau_{ij} - \frac{1}{2} \eta_{ij} \tau \right) - \frac{\tilde{\alpha}}{c_1 \alpha} \frac{1}{k^2 + m^2} \left( \tau_{ij} - \frac{1}{3} \eta_{ij} \tau \right) \]  

Thus, in this case too, the interaction is mediated by both massless and massive spin-2 fields, with relative strength being a free parameter in our model. For completeness, let us
write down the linearized connection in this case,

\[ A_{ijk} = -\frac{i}{c_1 k^2} \left\{ k_i \left( \tau_{jk} - \frac{1}{2} \eta_{jk} \tau \right) - k_j \left( \tau_{ik} - \frac{1}{2} \eta_{ik} \tau \right) \right\} + \frac{i}{c_1 k^2 + m^2} \left\{ k_i \left( \tau_{jk} - \frac{1}{3} \eta_{jk} \tau \right) - k_j \left( \tau_{ik} - \frac{1}{3} \eta_{ik} \tau \right) \right\} \]

It is different from the Riemannian connection corresponding to the vierbein perturbation (59) (see also the first line in (57)), which clearly shows mixing between vierbein and torsion fields in our model.

6 Conclusions

The model discussed in this paper belongs to the class of modified gravities in the sense that the gravitational force is mediated by both massless and massive tensor fields. Yet the model successfully passes a non-trivial consistency check: in torsionless Einstein space backgrounds it has no pathologies in the spectrum, at least for small enough background curvature. Clearly, this model deserves further study.

One issue to be understood is whether the model is consistent in more general backgrounds, including those with non-vanishing torsion. Another is whether the Vainshtein mechanism cures the van Dam–Veltman–Zakharov problem; this issue can probably be understood in an appropriate decoupling limit, in analogy to Refs. [37, 38, 7, 8, 5]. There is one more property of our model that may be related to the Vainshtein non-linearity. We have seen in section 4.3 that at the linearized level in the Einstein backgrounds, the longitudinal mode \( \phi \) may become a ghost for \( |W_{ijkl}| > m^2 \), where \( m \) is the mass of the spin-2 field. For spherical source, the Weyl tensor is of order \( |W_{ijkl}| \sim R S / r^3 \), where \( r \) is the distance to the source and \( R S \) is the Schwarzschild radius. Hence, the danger of a ghost occurs at

\[ r \lesssim r_3 = \left( \frac{R S}{m^2} \right)^{1/3} \]

Note that \( r_3 \) is the smallest of all Vainshtein radii in the Fierz–Pauli theories [37], the generic value being

\[ r_V = r_5 = \left( \frac{R S}{m^4} \right)^{1/5} \]

By analogy to other known infrared modified gravities we expect that the longitudinal sector of our model goes non-linear at least at \( r \lesssim r_3 \), so the Vainshtein mechanism may cure the potential ghost problem as well.
The model we studied in this paper is geometrical, and its full non-linear action is well defined. At the linearized level virtually every source produces perturbations of both metric and torsion. Thus, it would be interesting to understand whether or not black holes have similar property. If they do, the existence of torsionless Ricci flat solution (for zero cosmological constant) — Schwarzschild black hole — would mean that black holes in this model have torsion hair, and that the Birkhoff’s theorem is not valid.

**Acknowledgements**

We are indebted to V.P Nair for stimulating discussions and to Stanley Deser for helpful correspondence. S.R.D. is grateful to the participants of the first ΨG Workshop on the Consistent Modifications of Gravity, especially to Diego Blas, Cedric Deffayet, Gia Dvali and Arkady Vainshtein for their criticism and useful comments. V.R. is indebted to R.P. Woodard for useful discussion. V.R. has been supported in part by Russian Foundation for Basic Research grant 08-02-00473.
References

[1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173, 211 (1939).

[2] H. van Dam and M. J. G. Veltman, Nucl. Phys. B22, 397 (1970).

[3] V. I. Zakharov, JETP Lett. 12, 312 (1970).

[4] A. I. Vainshtein, Phys. Lett. B39, 393 (1972).

[5] E. Babichev, C. Deffayet and R. Ziour, “The Vainshtein mechanism in the Decoupling Limit of massive gravity,” arXiv:0901.0393 [hep-th]; E. Babichev, C. Deffayet and R. Ziour, to appear.

[6] D. G. Boulware and S. Deser, Phys. Rev. D 6, 3368 (1972).

[7] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, JHEP 09, 003 (2005), hep-th/0505147.

[8] C. Deffayet and J.-W. Rombouts, Phys. Rev. D72, 044003 (2005), gr-qc/0505134.

[9] S. Deser and A. Waldron, Phys. Rev. Lett. 87, 031601 (2001), hep-th/0102166.

[10] S. Deser and A. Waldron, Phys. Lett. B508, 347 (2001), hep-th/0103255.

[11] M. Porrati, JHEP 04, 058 (2002), hep-th/0112166.

[12] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B485, 208 (2000), hep-th/0005016.

[13] C. de Rham, G. Dvali, S. Hofmann, J. Khoury, O. Pujolas, M. Redi and A. J. Tolley, Phys. Rev. Lett. 100, 251603 (2008) arXiv:0711.2072 [hep-th].

[14] N. Kaloper and D. Kiley, JHEP 0705, 045 (2007) [arXiv:hep-th/0703190].

[15] N. Kaloper, Mod. Phys. Lett. A 23, 781 (2008) arXiv:0711.3210 [hep-th].

[16] T. Kobayashi, Phys. Rev. D 78, 084018 (2008) arXiv:0806.0924 [hep-th].

[17] G. Gabadadze, “ICTP lectures on large extra dimensions,” hep-ph/0308112.

[18] V. A. Rubakov, “Lorentz-violating graviton masses: Getting around ghosts, low strong coupling scale and VDVZ discontinuity,” arXiv:hep-th/0407104.

[19] S. L. Dubovsky, JHEP 0410, 076 (2004) arXiv:hep-th/0409124.
[20] S. L. Dubovsky, P. G. Tinyakov and I. I. Tkachev, Phys. Rev. Lett. 94, 181102 (2005) [arXiv:hep-th/0411158].

[21] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. Lett. 99, 131101 (2007) [arXiv:hep-th/0703264].

[22] D. Blas, D. Comelli, F. Nesti and L. Pilo, “Lorentz Breaking Massive Gravity in Curved Space,” arXiv:0905.1699 [hep-th].

[23] V. A. Rubakov and P. G. Tinyakov, Phys. Usp. 51, 759 (2008) [arXiv:0802.4379 [hep-th]].

[24] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. 64, 866 (1980) [Erratum-ibid. 65, 2079 (1981)].

[25] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. 64, 1435 (1980) [Erratum-ibid. 66, 741 (1981)].

[26] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. 64, 2222 (1980).

[27] E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. D 21, 3269 (1980).

[28] T. W. B. Kibble, J. Math. Phys. 2, 212 (1961).

[29] F. W. Hehl and Y. N. Obukhov, “Elie Cartan’s torsion in geometry and in field theory, an essay,” arXiv:0711.1535 [gr-qc].

[30] R. Percacci, Phys. Lett. B 144, 37 (1984).

[31] R. Percacci, Nucl. Phys. B 353, 271 (1991) [arXiv:0712.3545 [hep-th]].

[32] J. Dell, J. L. deLyra and L. Smolin, Phys. Rev. D 34, 3012 (1986).

[33] F. Nesti and R. Percacci, J. Phys. A 41, 075405 (2008) [arXiv:0706.3307 [hep-th]].

[34] S. H. S. Alexander, “Isogravity: Toward an Electroweak and Gravitational Unification,” arXiv:0706.4481 [hep-th].

[35] V. P. Nair, S. Randjbar-Daemi and V. Rubakov, “Massive Spin-2 fields of Geometric Origin in Curved Spacetimes,” arXiv:0811.3781 [hep-th].

[36] A. Higuchi, Nucl. Phys. B 325, 745 (1989).

[37] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Ann. Phys. 305, 96 (2003), hep-th/0210184.
[38] A. Nicolis and R. Rattazzi, JHEP 0406, 059 (2004) arXiv:hep-th/0404159.