Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint

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Abstract

This is a survey of history, methods and developments in the theory of cycle spaces of flag domains, and new results on double fibration transforms and their applications.

Contents:

§0. Introduction

Part I: Background

§1. Flag Domains and Compact Subvarieties
  Measurability
  Compact subvarieties

§2. Basic Facts on the Cycle Space
  Hermitian trichotomy

§3. The Exhaustion Function and \((q + 1)\)-Completeness
  Cohomology vanishing theorems
  The Stein property for cycle spaces of measurable open orbits

§4. Early Problems and Results on the Double Fibration Transform
  Double fibration
  Pull–back
  Push–down
  Flag domain case

Part II: The Complex Geometric Approach

§5. Introduction to the Complex Geometric Approach

§6. The Equivalences \(\Omega_{\text{adpt}} = \Omega_{\text{AG}} = \Omega_I\)
  Adapted complex structures
  Basic properties of plurisubharmonic functions
  The adapted structure for Riemannian symmetric spaces
  Proper actions
  Incidence geometry and the domain \(\Omega_I\)
  Domains defined by invariant hypersurfaces
  The equality \(\Omega_I = \Omega_{\text{AG}}\)

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0 Introduction

Cycle space theory is a basic chapter in complex analysis. Since since the 1960’s its importance has been underlined by its role in the geometry of flag domains and applications, by means of double fibration transforms, to variation of Hodge structure and to the representation theory of semisimple Lie groups. This developed very slowly until a few of years ago when methods of complex analytic geometry, in particular the methods of Schubert slices, Schubert domains, Iwasawa domains and supporting hypersurfaces, were introduced. Early in 2002 those methods were used to settle a number of outstanding questions. This effectively enabled the use of double fibration transforms in all flag domain situations. This has very interesting consequences for geometric construction of representations of semisimple Lie groups, especially for the construction of singular representations. It also has many potential interesting consequences for automorphic cohomology and other aspects of variation of Hodge structure. In this article we survey the recent results, filling in the background as necessary, and present some new results that help to complete the picture.

Part I, “Background”, is an exposition of flag domains and their cycle spaces before the introduction of the new complex geometric methods. Section 1 recalls the very basic results on flag domains and compact subvarieties. Section 2 goes into the complex structure of these cycle spaces and describes the three basic possibilities. Section 3 describes a particular exhaustion function for measurable flag domains, and its consequences for cohomology vanishing theorems that are crucial to the double fibration transform and application to semisimple representation theory. The basic aspects of that double fibration transform are described in Section 4.

Part II, “The Complex Geometric Approach”, introduces the methods of Schubert slices, Schubert domains, Iwasawa domains and supporting hypersurfaces, and the use of Kobayashi hyperbolicity in this context. In order to orient the reader who is not working in complex analysis, Section 5 is an introduction to the methods and ideas, and a sketch of the remainder of Part II. Section 6 introduces several related domains, and proves certain equivalences among
them, specifically \( \Omega_{\text{adpe}} \cong \Omega_{AG} \cong \Omega_I \). Section 8 goes into the key notion of transversal Schubert varieties, explaining an enhanced duality theory and the argument that the cycle space \( \Omega_{W}(D) \) is equal to the Schubert domain \( \Omega_S(D) \) in all but certain exceptional cases related to hermitian symmetric spaces. The situation of hermitian symmetric spaces is completely described in Section 8. Another new element in this picture, the use of Kobayashi hyperbolicity, is described in Section 8. In Section 11 these ingredients are combined to describe the maximal domain of hyperbolicity; the result is \( \Omega_{AG} \cong \Omega_I \cong \Omega_{S}(D) \cong \Omega_{W}(D) \), again except in certain exceptional cases related to hermitian symmetric spaces which are described completely in Section 8.

Part III, “Applications and Open Problems”, applies the results of Part II to the mechanism of the double fibration transform, and discusses certain applications. The material on the double fibration transform, key parts of which are new, appears in Section 11. Consequences for representations of real reductive Lie groups are discussed in Section 12, and in Section 13 there is a discussion of variation of Hodge structure and automorphic cohomology.

Part I: Background.

In this Part we describe the early results on the cycle space and the double fibration transform. For the most part those results are based on Lie structure theory.

1 Flag Domains and Compact Subvarieties.

We begin by reviewing the basic setup for flag domains and their maximal compact subvarieties which was presented in Wolf [W2]. We review the part of [W2] that is relevant to the theory of cycle spaces of flag domains.

Let \( G \) be a complex semisimple Lie group and \( Q \) a parabolic subgroup. The compact algebraic homogeneous space \( Z = G/Q \) is called a complex flag manifold. Write \( g \) and \( q \) for the respective Lie algebras of \( G \) and \( Q \). Then \( Q \) is the \( G \)-normalizer of \( q \). Thus we may view \( Z \) as the set of \( G \)-conjugates of \( q \). The correspondence is \( z \leftrightarrow q_z \) where \( q_z \) is the Lie algebra of the isotropy subgroup \( Q_z \) of \( G \) at \( z \).

Let \( G_0 \) be a real form of \( G \) in the sense that there is a homomorphism \( \varphi : G_0 \to G \) such that \( \varphi(G_0) \) is closed in \( G \) and \( d\varphi : g_0 \to g \) is an isomorphism onto a real form of \( g \). In this paper we will only consider the situation where \( \varphi \) is an inclusion, \( \varphi : G_0 \to G \) and so we now assume \( G_0 \subset G \) and that \( G_0 \) is noncompact.

Write \( g \mapsto \overline{g} \) for complex conjugation of \( G \) over \( G_0 \) and of \( g \) over \( g_0 \). We recall some of the basic facts about \( G_0 \)-orbits on \( Z \).

If \( z \in Z \) then \( q_z \cap \overline{q_z} \) contains a Cartan subalgebra \( h \) of \( g \). We may assume that \( h = \overline{h} \), in other words that \( h \) is the complexification of a Cartan subalgebra \( h_0 = h \cap g_0 \) of \( g_0 \). There is a choice of positive root system \( \Delta^+ = \Delta^+(g, h) \) such that \( q_z \) is the standard parabolic subalgebra \( q_\Phi \) defined by some subset \( \Phi \subset \Psi \) where \( \Psi = \Psi(g, h, \Delta^+) \) is the corresponding simple root system. In other words, \( q_z = q_\Phi \), where

\[
\Phi^r = \{ \alpha \in \Delta \mid \alpha \text{ is a linear combination of elements of } \Phi \}, \\
\Phi^s = \{ \alpha \in \Sigma^+ \mid \alpha \notin \Phi^r \}, \text{ and} \\
q_\Phi = q_\Phi^r + q_\Phi^s \text{ with } q_\Phi^r = h + \sum_{\alpha \in \Phi^r} g_\alpha \text{ and } q_\Phi^s = \sum_{\alpha \in \Phi^s} g_{-\alpha}.
\]

(1.1)

It follows that \( G_0 \) acts on \( Z \) with only finitely many orbits; in particular there are open orbits. We refer to the open orbits as flag domains. As \( G_0 \)-invariant open subsets of \( Z \), the flag domains \( D \subset Z \) are \( G_0 \)-homogeneous complex manifolds.

Measurability.
A flag domain $D = G_0(z) \subset Z$ is called measurable if it carries a $G_0$–invariant volume element. This is the type of flag domain currently of most interest in representation theory. More precisely, the following conditions are equivalent:

1.2a) The orbit $G_0(z)$ is measurable.
1.2b) $G_0 \cap Q_z$ is the $G_0$–centralizer of a (compact) torus subgroup of $G_0$.
1.2c) $D$ has a $G_0$–invariant, possibly–indefinite, Kähler metric, thus a $G_0$–invariant measure obtained from the volume form of that metric.

1.2d) $\Phi^\flat = \Phi^\flat$, and $\Phi^n = -\Phi^n$ where $q_z = q_\alpha$.
1.2e) $q_z \cap q_\bar z$ is reductive, i.e. $q_z \cap q_\bar z = q_z^* \cap q_\bar z^*$.
1.2f) $q_z \cap q_\bar z = q_\alpha^2$.
1.2g) $\Phi$ is Ad $(G)$–conjugate to the parabolic subalgebra $q^* + q^n$ opposite to $q$.

In particular, since (1.2g) is independent of choice of $z$, if one open $G_0$–orbit on $Z$ is measurable then all open $G_0$–orbits are measurable.

Condition (1.2d) holds whenever the Cartan subalgebra $h_0 = h \cap g_0$ of $g_0$ corresponds to a compact Cartan subgroup $H_0 \subset G_0$. (Here $h = \mathfrak{t}$ is the Cartan subalgebra relative to which $q_z = q_\alpha$.) For in that case $\mathfrak{t} = -\alpha$ for every $\alpha \in \Delta(g, h)$. In particular, if $G_0$ has discrete series representations (so that by a result of Harish–Chandra it has a compact Cartan subgroup) then every open $G_0$–orbit on $Z$ is measurable. Condition (1.2d) is also automatic if $Q$ is a Borel subgroup of $G$, and more generally Condition (1.2g) provides a quick test for measurability.

Compact subvarieties.

We now fix $z \in Z$ such that $D = G_0(z)$ is open in $Z$. For convenience we suppose that $z$ is the base point in $Z = G/Q$, so $Q = Q_z$ and $q = q_z$. For notational consistency with many papers in this area, we write $L$ for the Levy component $Q^\flat$ of $Q$. So $D$ is measurable if and only if $Q \cap g_0$ is a real form $L_0$ of $L$, and in that case $D \cong G_0/L_0$.

Fix a Cartan involution $\theta$ of $g_0$ that stabilizes the Cartan subgroup $H_0 \subset G_0$, and denote its fixed point sets on $g_0$ and $G$ by $K_0 = G_0^\theta$ and $K = G^\theta$. Then $K_0$ is a maximal compact subgroup of $G_0$ and $K$ is its complexification. $L \cap K_0$ is a real form of $L \cap K$ and $K_0(z) \cong K_0/(L_0 \cap K_0)$.

As $D$ is open we may assume $h$ chosen so that $H_0 \cap K_0$ is a Cartan subgroup of $K_0$, in other words so that $H_0$ is a fundamental Cartan subgroup of $G_0$. Use $\mathfrak{h}$ for the standard Weyl basis construction of a $\theta$–stable compact real form $g_0 \subset g$. Then $G_0 \cap G_u = K_0$ and $\mathfrak{t} = (h \cap l) + (h \cap r_+) + (h \cap r_-)$. Thus $K(z) \cong K/(K \cap Q)$ is a complex flag submanifold of $Z$, and $K_0$ acts transitively on it. In summary

**Lemma 1.3** $K(z) = K_0(z)$; in particular it is a compact complex submanifold of $D$.

We write $C_0$ for the compact complex submanifold $K_0(z) \subset D$. It will be the base cycle in a certain cycle space discussed below. The discussion leading to Lemma 1.3 shows that $C_0$ is both the unique $K$–orbit in $D$ that is compact and the unique $K_0$–orbit in $D$ that is complex. This is the origin of what is known as “Matsuki duality.”

**Example 1.4** Let $Z$ be the complex projective space $\mathbb{C}^{\mathbb{P}n}$ and let $G_0 = SU(n, 1)$. Let $\{e_1, \ldots, e_{n+1}\}$ denote the standard basis of $\mathbb{C}^{n+1}$ absolute to which the hermitian form defining $G_0$ is $\langle u, v \rangle = \left( \sum_{1 \leq a \leq n} u_a \overline{v}_a - u_{n+1} \overline{v}_{n+1} \right)$. Then $G_0$ has three orbits on $Z$: the (open) unit ball $B$ in $\mathbb{C}^{n}$ inside $Z$, consisting of the negative definite lines, the $(2n - 1)$–sphere $S$ which is the boundary of $B$, consisting of the null lines, and the complement $D$ of $B \cup S$, consisting of the positive definite lines. $D$ is the non–convex open $G_0$–orbit on $Z$. Here $C_0$ is the hyperplane at infinity, complement to $\mathbb{C}^{n}$ in $Z$. In homogeneous coordinates $[z^1, \ldots, z^{n+1}]$, $B$ is given by $\sum_{1 \leq a \leq n} |z^a|^2 < |z^{n+1}|^2$, $S$ is given by $\sum_{1 \leq a \leq n} |z^a|^2 = |z^{n+1}|^2$, $D$ is given by $\sum_{1 \leq a \leq n} |z^a|^2 > |z^{n+1}|^2$, and $C_0$ is given by $|z^{n+1}|^2 = 0$.

Later we will see $C_0$ as the base cycle in $D$. In this case $C_0$ is maximal among the (complex) subvarieties of $Z$ contained in $D$. 

2 Basic Facts on the Cycle Space.

Basic facts about the cycle space are given in Wells & Wolf [WeW] and in Wolf [W7]. We review some of that material now, and briefly indicate some of the applications of cycle spaces to variation of Hodge structure, specifically to period matrix domains, and construction of automorphic cohomology classes by Poincaré $\vartheta$–series. Those applications, and several others, will be discussed in more detail in Part III below using the tools which are described in Part II.

Definition. Let $E = \{g \in G \mid g C_0 = C_0\}$. Then $E$ is a closed complex subgroup of $G$, so the quotient manifold $\Omega := \{g C_0 \mid g \in G\} \cong G/E$ has a natural structure of $G$–homogeneous complex manifold. Since $C_0$ is compact and $D$ is open, the subset $\{g C_0 \mid g \in G \text{ and } g C_0 \subset D\}$ is open in $\Omega$, and thus has a natural structure of complex manifold. The cycle space of $D$ is

\[ \Omega_W(D) : \text{topological component of } C_0 \text{ in } \{g C_0 \mid g \in G \text{ and } g C_0 \subset D\}. \]

Thus $\Omega_W(D)$ has a natural structure of complex manifold.

Hermitian trichotomy.

In order to understand the structure of $Z$, $D$ and $\Omega_W(D)$ we may assume that $G_0$ is simple, because $G_0$ is local direct product of simple groups, and $Z$, $D$ and $\Omega_W(D)$ break up as global direct products along the local direct product decomposition of $G_0$. From this point on $G_0$ is simple unless we say otherwise.

Since $g_0$ is simple and $\epsilon$ contains $t$, there are four possibilities, one trivial. The trivial one is the case $\epsilon = g$, in other words the case where $G_0$ acts transitively on $Z$, and $\Omega_W(D)$ is reduced to a single point. There are just a few possibilities for this (Wolf [W8]). From now on we ignore this trivial case and concentrate on the other three:

1. **Hermitian holomorphic case.** $G_0/K_0$ is a bounded symmetric domain $B$, we have the usual $g = p_+ + t + p_-$, and $\epsilon$ is one of $t + p_\pm$. In this case $D$ is measurable, say $D = G_0/L_0$, and there is a holomorphic double fibration

   \[ D \leftarrow G_0/(L_0 \cap K_0) \rightarrow B \]

   In other words, the two projections are simultaneously holomorphic for some choice between $B$ and the complex conjugate structure $\overline{B}$ and some choice of invariant complex structure on $G_0/(L_0 \cap K_0)$. In this case $G_0/(L_0 \cap K_0)$ is the incidence space $I(D)$ that we’ll meet later, and $\Omega_W(D)$ is $B$ or $\overline{B}$; see [W8] or Wolf–Zierau [WZ1].

2. **Hermitian non–holomorphic case.** $G_0/K_0$ is a bounded symmetric domain $B$ and $\epsilon = t$. In this case we cannot adjust invariant complex structures so that the double fibration indicated just above will be holomorphic. Here it was recently proved that $\Omega_W(D)$ is biholomorphic to $B \times \overline{B}$. See [W7] or [WZ1], and Huckleberry–Wolf [HW3] or [WZ2].

3. **Generic (or non–hermitian) case.** $G_0/K_0$ does not have a $G_0$–invariant complex structure. Then $t$ is a maximal subalgebra of $g$. In this case $K$ is the identity identity component of $E$, $\Omega$ is an affine homogeneous space, and we will describe the structure of $\Omega_W(D)$ in that context. The precise structure was worked out only very recently in [HW3] and Fels–Huckleberry [FH].

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1 By holomorphic fibration we mean a holomorphically locally trivial fiber space, essentially a holomorphic fiber bundle except perhaps lacking a complex structure group.
In the rest of this article, we describe complex geometric methods that lead to the developments indicated above, and to other applications and developments through the use of double fibration transforms. The new developments include aspects of the theory of holomorphic double fibration transforms themselves (Section 11). The areas of application include aspects of the representation theory of semisimple Lie groups (Section 12) and variation of Hodge structure (Section 13).

3 The Exhaustion Function and \((q + 1)\)-Completeness.

Measurable open orbits \(D = G_0(z) \subset Z\) carry an especially useful real analytic exhaustion function \(\varphi : D \to \mathbb{R}\) whose Levi form \(\mathcal{L}(\varphi)\) has at least \(n - q\) positive eigenvalues at every point of \(D\), where \(n = \dim C\) and \(q = \dim C_0\). Thus \(\varphi\) is strongly \(q\)-pseudoconvex and \(D\) is \((q + 1)\)-complete. In this section we review that development from [2], [WeW], and [SW], and then we indicate applications to cohomology over \(D\) and to the Stein property of \(\Omega_W(D)\).

The exhaustion function \(\varphi : D \to \mathbb{R}\) was first described in Schmid’s thesis [2] in the setting where \(G_0\) has a compact Cartan subgroup and \(Z = G/B\) where \(B\) is a Borel subgroup of \(G\). The \(\theta\)-stable real form \(G_u\) of \(G\) acts transitively on \(Z\). The canonical line bundle \(K_Z \to Z\), and the (dual) anticanonical line bundle \(K^*_Z \to Z\), are \(G_u\)-homogeneous and have \(G_u\)-invariant metrics. Let \(h_u\) denote the \(G_u\)-invariant hermitian metric on \(K^*_Z \to Z\). In this setting the isotropy subgroup \(L_0\) of \(G_0\) at a point \(z \in D\) of the open orbit is just a compact Cartan subgroup, so the anticanonical bundle \(K_0^* \to D\) has a \(G_0\)-invariant hermitian metric \(H_0\). Then one has the \(C^\infty\) (real analytic) positive function \(\varphi = \log h_0/h_u\) on \(D\). If \(g(z) \in \mathfrak{bd}(D)\) then \(\text{Ad}(g)((1 + q_-) + \text{Ad}(g)(1 + q_-)) \subsetneq \mathfrak{g}\), and it follows that \(\varphi\) goes to infinity as one approaches \(g(z)\) from the interior of \(D\). From this one sees that \(\varphi\) is an exhaustion function for \(D\). Root space considerations allow one to compute \(\sqrt{-1} \partial \overline{\partial} \log h_0\) and \(\sqrt{-1} \partial \overline{\partial} \log h_u\) and see the Levi form \(\mathcal{L}(\varphi)\) explicitly. It follows immediately that \(\mathcal{L}(\varphi)\) has at least \(n - q\) positive eigenvalues at every point of \(D\).

Somewhat later, Wells and Wolf [WeW] noted that Schmid’s argument could be adapted to the more general setting where the only requirement is that the isotropy subgroup \(L_0\) of \(G_0\) at a point \(z \in D\) is compact. Somewhat after that, Schmid and Wolf [SW] further adapted the argument to the (even more general) situation where \(D\) is a measurable open \(G_0\)-orbit in a complex flag manifold \(Z = G/Q\). Thus every measurable open orbit \(D\) is \((q + 1)\)-complete.

Cohomology vanishing theorems.

The theorem of Andreotti and Grauert [AnG] says that if a complex manifold \(D = \{ z \in D \mid \varphi(z) < 0 \}\) is \((q + 1)\)-complete, and if \(S \to D\) is a coherent analytic sheaf, then the cohomologies \(H^r(D; S) = 0\) for for all \(r > q\).

Since our measurable open \(G_0\)-orbit \(D\) is \((q + 1)\)-complete, we have the vanishing \(H^r(D; \mathcal{O}(E)) = 0\) for \(r > q\), for every holomorphic vector bundle \(E \to D\). On the other hand, if \(E \to D\) is a (sufficiently) negative bundle [GrS] the methods based on the Bott–Borel–Weil Theorem show that \(H^r(D; \mathcal{O}(E)) = 0\) for \(r < q\). Thus, finally,

\[
(3.1) \quad \text{if } E \to D \text{ is (sufficiently) negative, then } H^r(D; \mathcal{O}(E)) = 0 \text{ for } r \neq q.
\]

This will be very important when we discuss double fibration transforms.

The Stein property for cycle spaces of measurable open orbits.

When \(D\) is a measurable open orbit, Wolf [W7] combined his extension of boundary component theory of bounded symmetric domains [W2], or see [W4] with the exhaustion function \(\varphi : D \to \mathbb{R}\), to prove that \(\Omega_W(D)\) is a Stein manifold. We review the argument.
Case: $D$ is of hermitian holomorphic type. Here we may assume $D = G_0(z)$ and $E = KP$, so $\Omega = G/E$ is the compact hermitian symmetric space dual to the bounded symmetric domain $\mathcal{B}$. Thus $\mathcal{B} \subset \Omega_W(D) \subset \Omega$ and $\Omega_W(D)$ is invariant by the action of $G_0$ on $\Omega$. The $G_0$–orbit structure of $\Omega$, and the closure relations among the $G_0$–orbits, are known precisely in terms of partial Cayley transforms ([W2], [W4]). If $\mathcal{O}$ is a $G_0$–orbit in $\Omega_W(D)$, then it contains every open $G_0$–orbit whose closure contains $\mathcal{O}$, because $\Omega_W(D)$ is open in $\Omega$. Some operator norm arguments show that $\Omega_W(D)$ cannot contain an open orbit different from $\mathcal{B}$. It follows that $\Omega_W(D) = \mathcal{B}$, and in particular $\Omega_W(D)$ is Stein.

Case: $D$ is not of hermitian holomorphic type. Then $E$ has identity component $K$, so $E$ is reductive and $\Omega = G/E$ is affine. Define $\beta : \Omega_W(D) \to \mathbb{R}^+$ by $\beta(gC_0) = \sup_{y \in C_0} \varphi(g(y))$. Since $\varphi$ is an exhaustion function and the $gC_0$ are compact, one sees that $\beta : \Omega_W(D) \to \mathbb{R}^+$ blows up at every boundary point of $\Omega_W(D)$. From the specific construction of $\varphi$, and a close look at the real analytic variety given by $d\varphi = 0$, one sees that $\beta$ is continuous, piecewise $C^\omega$ and plurisubharmonic. Now a modification suggested by results of Docquier and Grauert ([DG]) gives a $C^\omega$ strictly plurisubharmonic exhaustion function $\psi = \varphi + \nu$ constructed as follows. Since $\Omega$ is Stein there is a proper holomorphic embedding $f : \Omega \to \mathbb{C}^{2n+1}$ with closed image, by Remmert's theorem. Define $\nu(C) := ||f(C)||^2$ for $C \in \Omega_W(D)$. Since $\Omega_W(D)$ carries a strictly plurisubharmonic exhaustion function, it is Stein.

4 Early Problems and Results on the Double Fibration Transform.

We start this section with a review of some basic facts on the double fibration transform from [WZ3]. We then specialize (initially as in [WZ2]) to the case of an open orbit $D = G_0(z) \subset Z$, and present some new results that clear up several open problems in that flag domain case. Finally we give a quick indication of the consequences for variation of Hodge structure and for semisimple representation theory. Now we start with the general setup, indicate its technical requirements, and specialize it to our flag domain situation. Several of the problems that come up here are settled later in Section 11 using methods developed in Part II below.

Double fibration.

Let $D$ be a complex manifold (later it will be an open orbit of a real reductive group $G_0$ on a complex flag manifold $Z = G/Q$ of its complexification). We suppose that $D$ fits into a holomorphic double fibration, in other words that there are complex manifolds $M$ and $\mathcal{I}(D)$ with simultaneously holomorphic fibrations:

\begin{equation}
\begin{array}{ccc}
D & \xleftarrow{\mu} & \mathcal{I}(D) \\
& \nu & \\
& M
\end{array}
\end{equation}

(Later $M$ will be a cycle space and $\mathcal{I}(D)$ will be an incidence space for points and cycles.) Given a coherent analytic sheaf $\mathcal{E} \to D$ we construct a coherent sheaf $\mathcal{E}' \to M$ and a transform

\begin{equation}
P : H^0(D; \mathcal{E}) \to H^0(M; \mathcal{E}')
\end{equation}

under mild conditions on (4.1). In fact we give several variations on the construction. This construction is fairly standard (see, for example, [BE], [PR1], and [M]), but we need several results specific to the case of flag domains.
Pull–back.

The first step is to pull cohomology back from \( D \) to \( \mathcal{I}(D) \). Let \( \mu^{-1}(\mathcal{E}) \rightarrow \mathcal{I}(D) \) denote the inverse image sheaf. For every integer \( r \geq 0 \) there is a natural map

\[
\mu^{(r)} : H^r(D; \mathcal{E}) \rightarrow H^r(\mathcal{I}(D); \mu^{-1}(\mathcal{E}))
\]

given on the Čech cocycle level by \( \mu^{(r)}(c)(\sigma) = c(\mu(\sigma)) \) where \( c \in Z^r(D; \mathcal{E}) \) and where \( \sigma = (w_0, \ldots, w_r) \) is a simplex. For \( q \geq 0 \) we consider the Buchdahl \( q \)-condition

\[
\text{the fiber } F \text{ of } \mu : \mathcal{I}(D) \rightarrow D \text{ is connected and } H^r(F; \mathbb{C}) = 0 \text{ for } 1 \leq r \leq q - 1.
\]

**Proposition 4.5** (See [Bu.]) Fix \( q \geq 0 \). If \( (4.4) \) holds, then \( (4.3) \) is an isomorphism for \( r \leq q - 1 \) and is injective for \( r = q \). If the fibers of \( \mu \) are cohomologically acyclic then \( (4.2) \) is an isomorphism for all \( r \).

As usual, \( \mathcal{O}_X \rightarrow X \) denotes the structure sheaf of a complex manifold \( X \) and \( \mathcal{O}(\mathbb{E}) \rightarrow X \) denotes the sheaf of germs of holomorphic sections of a holomorphic vector bundle \( \mathbb{E} \rightarrow X \). Let \( \mu^*(\mathcal{E}) := \mu^{-1}(\mathcal{E}) \otimes_{\mathcal{O}(\mathcal{I}(D))} \mathcal{O}(\mathcal{I}(D)) \rightarrow \mathcal{I}(D) \) denote the pull–back sheaf. It is a coherent analytic sheaf of \( \mathcal{O}(\mathcal{I}(D)) \)-modules. If \( \mathbb{E} = \mathcal{O}(\mathbb{E}) \) for some holomorphic vector bundle \( \mathbb{E} \rightarrow D \), then \( \mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E})) \), where \( \mu^*(\mathbb{E}) \) is the pull–back bundle. In any case, \( [\sigma] \mapsto [\sigma] \otimes 1 \) defines a map \( \iota : \mu^{-1}(\mathcal{E}) \rightarrow \mu^*(\mathcal{E}) \) which in turn specifies maps in cohomology, the coefficient morphisms

\[
i_p : H^p(\mathcal{I}(D); \mu^{-1}(\mathcal{E})) \rightarrow H^p(\mathcal{I}(D); \mu^*(\mathcal{E})) \quad \text{for } p \geq 0.
\]

Our natural pull–back maps are the compositions \( j^{(p)} = \iota_p \cdot \mu^{(p)} \) of \( (4.3) \) and \( (4.4) \):

\[
j^{(p)} : H^p(D; \mathcal{E}) \rightarrow H^p(\mathcal{I}(D); \mu^*(\mathcal{E})) \quad \text{for } p \geq 0.
\]

If \( \mathcal{E} = \mathcal{O}(\mathbb{E}) \) for some holomorphic vector bundle \( \mathbb{E} \rightarrow D \), then \( \mu^*(\mathcal{E}) = \mathcal{O}(\mu^*(\mathbb{E})) \), we realize these sheaf cohomologies as Dolbeault cohomologies, and the pull–back maps \( (4.7) \) are given by pulling back \([\omega] \mapsto [\mu^*(\omega)] \) on the level of differential forms.

Push–down.

In order to push the \( H^q(\mathcal{I}(D); \mu^*(\mathcal{E})) \) down to \( M \) we assume that

\[
\nu : \mathcal{I}(D) \rightarrow M \quad \text{is a proper map and } M \text{ is a Stein manifold}.
\]

The Leray direct image sheaves \( \mathcal{R}^p(\mu^*(\mathcal{E})) \rightarrow M \) are coherent \([GrR] \). As \( M \) is Stein

\[
H^q(M; \mathcal{R}^p(\mathcal{E})) = 0 \quad \text{for } p \geq 0 \text{ and } q > 0.
\]

Thus the Leray spectral sequence collapses and gives

\[
H^p(\mathcal{I}(D); \mu^*(\mathcal{E})) \cong H^0(M; \mathcal{R}^p(\mu^*(\mathcal{E}))).
\]

**Definition 4.11** The *double fibration transform* for the holomorphic double fibration \( (4.1) \) is the composition

\[
P : H^p(D; \mathcal{E}) \rightarrow H^0(M; \mathcal{R}^p(\mu^*(\mathcal{E})))
\]

of the maps \( (4.7) \) and \( (4.10) \).
In order that the double fibration transform \((4.12)\) be useful, one wants two conditions to be satisfied. They are

\begin{align}
(4.13) & \quad P : H^p(D; \mathcal{E}) \to H^0(M; \mathcal{R}^q(\mu^*\mathcal{E}))) \text{ should be injective, and} \\
(4.14) & \quad \text{there should be an explicit description of the image of } P.
\end{align}

Assuming \((4.8)\), injectivity of \(P\) is equivalent to injectivity of \(j^p\) in \((4.17)\). The most general way to approach this is the combination of vanishing and negativity in Theorem 4.15 below, based on the Buchdahl conditions \((4.4)\).

The general (assuming \((4.8)\)) injectivity question uses a spectral sequence argument for the the relative Dolbeault complex of the holomorphic fibration \(\mu : \mathcal{I}(D) \to D\). See [WZ2] for the details. The end result is

**Theorem 4.15** Fix \(q \geq 0\). Suppose that the fiber \(F\) of \(\mu : \mathcal{I}(D) \to D\) is connected satisfies \((4.4)\). Assume \((4.8)\) that \(v : \mathcal{I}(D) \to M\) is proper and \(M\) is Stein, say with fiber \(C\). Let \(\Omega^*_v(E) \to \mathcal{I}(D)\) denote the sheaf of relative \(\mu^*E\)-valued holomorphic \(r\)-forms on \(\mathcal{I}(D)\) with respect to \(\mu : \mathcal{I}(D) \to D\). Suppose that \(H^q(C; \Omega^*_\mu(E)|_C) = 0\) for \(p < q\), and \(r \geq 1\). Then \(P : H^q(D; \mathcal{E}) \to H^0(M; \mathcal{R}^q(\mu^*\mathcal{E}))\) is injective.

**Remark 4.16** In the cases of interest to us, \(\mathcal{E} = \mathcal{O}(\mathcal{E})\) for some holomorphic vector bundle \(\mathcal{E} \to D\), and \(P\) has an explicit formula. The Leray derived sheaf is given by

\begin{equation}
(4.17) \quad \mathcal{R}^q(\mu^*(\mathcal{O}(\mathcal{E}))) = \mathcal{O}(\mathcal{E}^\dagger) \text{ where } \mathcal{E}^\dagger \to M \text{ has fiber } H^q(\nu^{-1}(C); \mathcal{O}(\mu^*(\mathcal{E})|_{\nu^{-1}(C)})) \text{ at } C.
\end{equation}

Let \(\omega\) be an \(\mathcal{E}\)-valued \((0, q)\)-form on \(D\) and \([\omega] \in H^q_D(D, \mathcal{E})\) its Dolbeault class. Then

\[P([\omega])\]

is the section of \(\mathcal{E}^\dagger \to M\) whose value \(P([\omega])(C)\) at \(C \in M\) is \([\mu^*(\omega)]_{\nu^{-1}(C)}\).

In other words,

\begin{equation}
(4.18) \quad P([\omega])(C) = [\mu^*(\omega)]_{\nu^{-1}(C)} \in H^0(M; \mathcal{E}^\dagger).
\end{equation}

This is most conveniently interpreted by viewing \(P([\omega])(C)\) as the Dolbeault class of \(\omega|_C\), and by viewing \(C \mapsto [\omega]|_C\) as a holomorphic section of the holomorphic vector bundle \(\mathcal{E}^\dagger \to M\).

**Flag domain case.**

Now let \(D = G_0(z_0)\) be an open orbit in the complex flag manifold \(Z = G/Q\), and \(M\) is replaced by the cycle space \(\Omega_W(D)\). Our double fibration \((4.1)\) is replaced by

\begin{equation}
(4.19) \quad \begin{array}{ccc}
\mu & \mathcal{I}(D) & \nu \\
\downarrow & \downarrow & \downarrow \\
D & \Omega_W(D) & \text{ where } \mathcal{I}(D) := \{(z, C) \in D \times \Omega_W(D) \mid z \in C\} \text{ is the incidence space. Given a homogeneous holomorphic vector bundle } \mathcal{E} \to D, \text{ and the number } q = \dim_{\mathbb{C}} C_0, \text{ the Leray derived sheaf involved in the double fibration transform satisfies } (4.17). \text{ Here that takes the form}
\end{array}
\end{equation}

\begin{equation}
(4.20) \quad \mathcal{R}^q(\mu^*(\mathcal{O}(\mathcal{E}))) = \mathcal{O}(\mathcal{E}^\dagger) \text{ where } \mathcal{E}^\dagger \to \Omega_W(D) \text{ has fiber } H^q(C; \mathcal{O}(\mathcal{E}|_C)) \text{ at } C \in \Omega_W(D).
\end{equation}

Evidently, \(\mathcal{E}^\dagger \to \Omega_W(D)\) is globally \(G_0\)-homogeneous and infinitesimally \(g\)-homogeneous, and \(H^q(C; \mathcal{O}(\mathcal{E}|_C))\) can be calculated in any reasonable case from the Bott–Borel–Weil Theorem,
especially when $E \to D$ is negative. Thus $R^q(\mu^*(O(E)))$ is given explicitly by (4.20) in the flag domain case.

Using methods of complex geometry to be described in Part II, we will see in Part III that $\Omega_W(D)$ is a contractible Stein manifold in general, so $\mathbb{E}^i \to \Omega_W(D)$ is holomorphically trivial, and that $F$ satisfies (4.4) for all $q$, so the double fibration transforms are injective. Thus, in the flag domain case, we will have a complete answer to (4.13) and some sharp progress toward (4.14). See Section 11.

Now let us take a quick historical look back at the hermitian trichotomy of Section 2. In the hermitian holomorphic case $\Omega_W(D)$ is $B$ or $\mathcal{B}$, and one knows [WZ2, Section 4] that $F$ and $\Omega_W(D)$ are contractible Stein manifolds. In the hermitian nonholomorphic case, where $\Omega_W(D) = B \times \mathcal{B}$ ([W7] or [WZ1], and [HW3] or [WZ3]), there had only been partial information (see [WZ2, Theorem 6.6]) on contractibility of $F$. There had been essentially no information in the nonhermitian case.

One more remark. In some cases one knows that $H^q(D; E)$ is an irreducible representation space for a group under which all our constructions are equivariant, and one sees directly that $P$ is an intertwining operator, thus zero or injective. In practice, however, we usually look for implications in the other directions. See Section 11.

**Part II: The Complex Geometric Approach.**

In this Part we describe the methods and results in complex geometry that lead to a structure theory for the cycle space, various associated domains, and the double fibration transform.

5 Introduction to the Complex Geometric Approach.

Our goal here is to explain recent results which have led to the characterization of $\Omega_W(D)$ as being equivalent to a certain universal domain $\Omega_{AG}$ in all but the well-understood trivial and the hermitian holomorphic cases as discussed in Section 2. Without further reference we exclude those cases in the sequel.

In the present section we outline the relevant results and methods in a nontechnical way. In the following sections we give enough details so that the reader should have no difficulty working through the literature. See [HW3] and [FH] for complete details.

Building on experience with transversal varieties gained in [HW1], [W9] and [HS], the Schubert domain $\Omega_S(D)$ was introduced in [4] as a tool for understanding complex analytic properties of $\Omega_W(D)$. The motivation for this is quite transparent in [HS], although there only the case of $G_0 = SL_n(\mathbb{R})$ was considered.

We discuss [HS] with the benefit of hindsight and a more up to date notation. Let $G_0 = K_0A_0N_0$ be an Iwasawa decomposition of the real form $G_0$. The corresponding set $KAN$ corresponds to the open $AN$–orbit in the spherical affine homogeneous space $\Omega := G/K$ and is a proper, Zariski open subset of $G$. We refer to a Borel subgroup $B \subset G$ as an Iwasawa–Borel subgroup of $G$ if it contains an Iwasawa factor $A_0N_0$. Of course these are just the Borel subgroups which occur as the isotropy groups at points of the closed $G_0$–orbit in $G/B$. Given an Iwasawa–Borel subgroup $B$ in $G$ and a point $z \in Z = G/Q$, the closure $S = c(\mathcal{O})$ of the orbit $\mathcal{O} = B.z$ is the associated Iwasawa–Schubert variety or Schubert cycle. Let $Y$ denote the complement of the Schubert cell $\mathcal{O} \in S$, i.e., $Y := S \setminus \mathcal{O}$.

An Iwasawa–Schubert variety $T = c(\mathcal{O})$ is transversal (relative to an open $G_0$–orbit $D$) if (1) $T \cap D$ is nonempty and contained in the open $B$–orbit $\mathcal{O}$, and (2) $\text{codim} T = q = \dim C_0$. 

10
and the intersection $T \cap C_0$ is transversal at each of its points. In this dimension $n - q$, the Iwasawa decomposition of $G_0$ implies that $Y \subset Z \setminus D$ (Theorem 5.7).

It was shown in [15] that for an open $SL_n(\mathbb{R})$-orbit $D$ in a flag manifold $Z$ of $SL_n(\mathbb{C})$, and $C \in \text{bd}(\Omega_W(D))$, there exists a transversal Schubert variety $T$ such that $Y \cap C \neq \emptyset$. The method of incidence varieties ([16], [19]) then shows that the algebraic variety $A_Y := \{C \in \Omega : C \cap Y\}$ contains the polar set $H_Y$ of a nonconstant meromorphic function which is produced by the method of trace transform. In this context we refer to $H_Y$ as an incidence hypersurface. It follows immediately that $\Omega_W(D)$ is a Stein domain in $\Omega$, because each of its boundary points is contained in an analytic hypersurface which is entirely contained in its complement.

Returning to the general case, we note that if $T$ is Poincaré dual to $C_0$, then it is indeed a transversal Schubert variety. However, at least initially there is no reason to believe that, given $C \in \text{bd}(\Omega_W(D))$, there exists $T$ with $C \cap Y \neq \emptyset$.

Domains with supporting analytic hypersurfaces at each of their boundary points have optimal character from the complex analytic viewpoint. Thus the following Schubert domain was introduced in [1]. Let $\Omega_S(D)$ be the connected component containing $C_0$ of the complement of the union of all Iwasawa–Borel invariant intersection hypersurfaces $H_Y$ which are defined by transversal Schubert varieties. Since $K_0$ acts transitively on such Borel groups, the set which is removed is a compact family of hypersurfaces and clearly $\Omega_S(D)$ is a proper, open, Stein domain in $\Omega$ which contains $\Omega_W(D)$. It should be emphasized that the sets $Y$ are possibly very far away from the boundary of the cycle space and that the inclusion $\Omega_W(D) \subset \Omega_S(D)$ could theoretically be proper.

In principle there could be a plethora of domains $\Omega_S(D)$, but experience with real forms of $SL_n(\mathbb{C})$ (see [17] for the remaining cases) and classical hermitian symmetric spaces ([11], [19]) suggests that they might all be the same, agreeing with a domain which is defined by removing all Iwasawa–Borel invariant hypersurfaces from $\Omega$.

More precisely, let $\Omega_I$ be defined as the connected component containing $C_0$ of the complement of the union of all $B$–invariant algebraic hypersurfaces in $\Omega$, where $B$ runs over all Iwasawa–Borel subgroups of $G$. Note that, without further information, $\Omega_I$ could theoretically be empty. But in any case, using the same argument as above, it is a Stein domain and of course $\Omega_I \subset \Omega_S(D)$ for every open orbit $D$ in every $G$–flag manifold $Z = G/Q$. In order to understand the relation of the cycle spaces to the universal domain $\Omega_{AG}$, it is natural to compare $\Omega_I$ and $\Omega_{AG}$. In fact, using the identification of $\Omega_{AG}$ with the maximal domain of definition $\Omega_{adpt}$ of the adapted complex structure in the tangent bundle of the Riemannian symmetric space $G_0/K_0$ ([20], [19]; see Section 3 below), an elementary argument involving the transported norm function shows that $\Omega_{AG} \subset \Omega_I$ [1].

In another guise $\Omega_I$ had been considered and, from a completely different viewpoint the above inclusion had been shown for classical groups [8]. Recently, a purely algebraic proof was given in [21].

Regarding $\Omega_I$ as the polar $\tilde{X}_0$ (the two definitions are easily seen to be equivalent (see [1]), but nevertheless reflect two very different aspects of the subject), Barchini proved the opposite inclusion [3].

As a consequence of considerations of cycle spaces associated to non–open $G_0$–orbits, it is implicitly shown by case by case methods in [19] that $\Omega_{AG} \subset \Omega_W(D)$ for classical groups and exceptional hermitian groups. Thus $\Omega_{AG} = \Omega_I \subset \Omega_S(D)$ was known at this point.

We indicate our contributions; they will be sketched in more detail below. In [17] we carried out the general program which was indicated by the naive flag arguments of [8]. For every $C \in \text{bd}(\Omega_W(D))$ there exists a transversal Schubert variety $T$ so that $Y \cap C \neq \emptyset$; in
particular, \( \Omega_W(D) = \Omega_S(D) \) and thus \( \Omega_W(D) \) is a Stein domain with respect to functions in the image of the trace transform.

The construction of \( T \) results from the construction of a Schubert variety \( Y_p \) containing an arbitrarily given boundary point \( p \in \text{bd}(D) \), with codim \( Y_p = q + 1 \), and of course \( Y_p \subset Z \setminus D \). This is a consequence of a sort of triality that initiates with Matsuki duality. It is of interest that the moment map, Morse–theoretic method for realizing this duality (\cite{MUV}, \cite{BL}) plays a direct role in our considerations (see Section 7). Furthermore, it is shown that, given \( C \) in the boundary of the cycle domain, there exits \( p \in C \) so that \( Y_p \) is indeed the \( Y \) of a transversal Schubert variety \( T \) (see Theorem 7.6). So the incidence variety method along with the inclusion \( \Omega_{AG} \subset \Omega_I \) yields \( \Omega_{AG} \subset \Omega_W(D) = \Omega_S(D) \).

Satisfied with identifying \( \Omega_{AG} \) with \( \Omega_{adpt} \) or \( \Omega_I \), we have actually not given its original definition [\cite{AkG}]. We do that now because we will need its computable nature.

The symmetric space \( \Omega_0 := G_0/K_0 \) is embedded as a totally real submanifold of half–dimension of \( \Omega = G/K \) as the \( G_0 \)–orbit of the neutral point. General principles imply that there are \( G_0 \)–invariant neighborhoods of \( \Omega_0 \) in \( \Omega \) on which the \( G_0 \)–action is proper. One looks for a canonically defined domain for which this is the case.

Let \( g_0 = t_0 \oplus p_0 \) be a Cartan decomposition set up in the usual way with respect to a compact real form \( g_0 \) of \( g \). Let \( a_0 \subset p_0 \) be maximal abelian as above. For \( \alpha \) a root of \( a_0 \), let \( H_\alpha := \{ A \in a_0 : \alpha(A) = 2 \} \) and define \( \omega_0 \) as the connected component containing 0 of the complement in \( a_0 \) of the union of the \( H_\alpha \). Then \( \Omega_{AG} := G_0.\exp(i\omega_0)z_0 \), where \( z_0 \) is a base point corresponding to \( C_0 \), is an open neighborhood of \( \Omega_0 \) and is maximal with respect to the property that every \( G_0 \)–isotropy group is compact. The \( G_0 \)–action on \( \Omega_{AG} \) is proper (see Section 7).

The fact that the \( G_0 \)–action is proper indicates that an invariant metric (perhaps of canonical nature) is playing a role. Thus, in [\cite{H}] methods were introduced to study the hyperbolicity of such domains. Let us recall the basic facts which are relevant for such considerations.

The Kobayashi pseudo–metric on a complex manifold \( X \) can be defined as follows. First, define a disk in \( X \) as the biholomorphic image of the unit disk in the complex plane. A chain of disks is the union of finitely many such disks which overlap (on open subsets) to form a connected set. Given \( p,q \in X \) consider a chain \( \kappa = \Delta_1 \cup \ldots \cup \Delta_m \) with \( p \in \Delta_1 \), \( q \in \Delta_m \) and \( \Delta_i \cap \Delta_{i+1} \neq \emptyset \). Let \( p_i \in \Delta_i \cap \Delta_{i+1} \) and \( d_i \) be the distance from \( p_{i-1} \) to \( p_i \) computed in the Poincaré metric of \( \Delta_i \). Adding up these distances we obtain a number \( d(\kappa) \) which also depends on the choices of the \( p_i \) which are regarded as part of the data of the chain. Finally, define the Kobayashi pseudo–distance between \( p \) and \( q \) as the minimum of all such \( d(\kappa) \) as \( \kappa \) runs over all such chains. This defines a pseudo–distance function on \( X \times X \) which, if it is nonzero for all \( p,q \in X \), is the Kobayashi metric [\cite{R}]. In this case \( X \) is called Kobayashi hyperbolic.

The methods introduced in [\cite{E}] could be regarded as leading to the group–theoretic version of the fact that the complement in \( \mathbb{CP}^m \) of \( 2m + 1 \) hyperplanes in general position is Kobayashi hyperbolic (a result of classical geometry). It is shown in [\cite{E}] that, e.g., \( \Omega_I \) is hyperbolic. These methods, refined in [\cite{EH}], lead to the following result (see Theorem 5.1) in the non–hermitian case. Let \( H \) be any Iwasawa–Borel invariant hypersurface in \( \Omega \) and define \( \Omega_H \) to be the connected component containing \( C_0 \) in \( \Omega \) of the complement of \( \bigcup_{k \in K_0} k(H) \). In other words, the definition of \( \Omega_H \) is analogous to \( \Omega_S(D) \) and \( \Omega_I \) except that one initially has only a single hypersurface.

**Theorem 5.1** If \( G_0 \) is not of hermitian type, then \( \Omega_H \) is Kobayashi hyperbolic.

Summarizing the above, if \( G_0 \) is not of hermitian type then, for any Iwasawa–Borel invariant hypersurface \( H \) in the complement of \( \Omega_W(D) \)

\[
\Omega_{AG} = \Omega_I \subset \Omega_W(D) = \Omega_S(D) \subset \Omega_H
\]
and each of these domains is Stein and Kobayashi hyperbolic.

The main new development in [FH] is summarized as follows.

**Theorem 5.3** Suppose that $G_0$ is not of hermitian type. If $\hat{\Omega}$ is a $G_0$–invariant, Kobayashi hyperbolic Stein domain that contains $\Omega_{AG}$, then $\hat{\Omega} = \Omega_{AG}$.

It follows that the inclusions in (5.2) are equalities. With some additional remarks to handle the hermitian case (which in fact is much simpler), it follows that $\Omega_W(D) = \Omega_{AG}$ in all but the well understood trivial and hermitian holomorphic cases discussed in Section 2.

We complete these introductory remarks by outlining the basic ideas of [FH]. Roughly speaking, the goal of that work is to reduce to the case of $SL_2$. This requires a rather detailed analysis of the $G_0$–action on $bd(\Omega_{AG})$, in particular a detailed description of closed orbits and orbit closures (see the subsection on closed orbits in Section 10 below). The boundary $bd(\Omega_{AG})$ is not smooth, but for the purposes of [FH] it is sufficient to consider points in a tractable open, dense stratum $bd_{gen}(\Omega_{AG})$ (see the subsection on genericity in Section 10 below). For $z \in bd_{gen}(\Omega_{AG})$ one determines a 3–dimensional simple subgroup $S \subset G$ which is defined over $\mathbb{R}$ such that $S.z \cap \Omega_{AG}$ is the associated domain $\Omega_{AG}(S)$ for the group $S$ with respect to its noncompact real form $S_0$.

Since no difficulties are introduced by going to finite covers, it may be assumed that the associated affine variety $\Omega(S)$ is the complement of the diagonal in $\mathbb{C}P_1 \times \mathbb{C}P_1$, i.e., the 2–dimensional affine quadric. Regarding it as a closed $S$–orbit in $\Omega$, we refer to $\Omega(S)$ as a $Q_2$–slice.

The $S$–action on $\mathbb{C}P_1 \times \mathbb{C}P_1$ is the standard diagonal action. The intersection $\Omega(S) \cap \Omega_{AG} = \Omega_{AG}(S)$ is a 2–dimensional polydisk $\Delta$ which can be regarded as being the product $B^+ \times B^−$ of an $S_0$–invariant (1–dimensional) disk with its exterior in $\mathbb{C}P_1$. Due to the genericity assumption, the point $z$ can be chosen to lie in $B^+ \times bd(\mathbb{B}^−)$.

It is shown in [FH] that if $\hat{\Omega}(S)$ is a $S_0$–invariant Stein domain in $\Omega(S)$ which contains $\Omega_{AG}(S)$ and the boundary point $z$, then it contains the open set $\hat{\Omega} \cap (B^+ \times \mathbb{C}P_1) \cong B^+ \times \mathbb{C}$. It is straightforward to check that if a complex manifold $X$ contains a (biholomorphic) copy of $Y$ of $\mathbb{C}$, then the Kobayashi distance between any two points in $Y$ is zero. Thus the domain $\hat{\Omega}(S)$ is certainly not Kobayashi hyperbolic.

The proof of Theorem 5.3 is then immediate, because, if $\hat{\Omega}$ is a $G_0$–invariant, Stein domain which properly contains $\Omega_{AG}$, then there is a generic boundary point $z$ such that $\hat{\Omega}(S) \cap \hat{\Omega}$ contains both $\Omega_{AG}(S)$ and $z$. Since $\Omega_{AG}(S)$ is Stein and $S_0$–invariant, it contains the copies of $\mathbb{C}$ as above and therefore $\hat{\Omega}$ contains these as well. As a consequence $\hat{\Omega}$ is not hyperbolic and Theorem 5.3 follows.

6 The Equivalences $\Omega_{adpt} = \Omega_{AG} = \Omega_I$.

Here three $G_0$ domains are introduced from three different viewpoints. The domain $\Omega_{adpt}$ in the tangent bundle of the Riemannian symmetric space $\Omega_0 = G_0/K_0$ can be defined by either metric or symplectic properties. The equivalence of these two ways of viewing $\Omega_{adpt}$ are important for complex analytic considerations.

The domains $\Omega_{AG}$ and $\Omega_I$ are defined as neighborhoods of $\Omega_0$ in the affine homogeneous space $\Omega = G/K$, where a base point $x_0$ has been chosen so that $\Omega_0 = G_0.x_0$. The domain $\Omega_{AG}$ is defined from the point of view of group actions. The domain $\Omega_I$ can be seen from several viewpoints. Ours is that of incidence divisors which are defined by Schubert varieties in $Z$ of Iwasawa–Borel subgroups $B \subset G$. 

13
Adapted complex structures.

Beginning with the metric standpoint, let \((M,g)\) be a real analytic Riemannian manifold which for simplicity is assumed to be complete. The differential \(\gamma_\ast\) of a geodesic is a map \(\gamma_\ast : T\mathbb{R} \to TM\) of tangent bundles and can be viewed as an orbit of the \(\mathbb{R}\)–action defined by geodesic flow. Let \(\mathbb{R}^*\) act by scalar multiplication in the fibers of \(TM\). This action extends in the usual way to a map \(\mathbb{R} \times TM \to TM\). Identify \(T\mathbb{R}\) with the complex plane \(\mathbb{C}\) by \((t, s \frac{d}{dt}) \mapsto t + is\) and define \(\gamma^C : \mathbb{C} \cong T\mathbb{R} \to TM\) by \(z = s + it \mapsto s \cdot \gamma_\ast(t)\), where the multiplication comes from the \(\mathbb{R}^*\)–action.

**Definition 6.1** An integrable complex structure \(J\) on a starlike neighborhood of \(A\) of the \(0\)–section in \(TM\) is **adapted** if for every geodesic \(\gamma\) there is a disk \(\Delta = \Delta(0) \subset \mathbb{C}\) such that \(\gamma^C|\Delta : \Delta \to A\) is holomorphic.

The existence and uniqueness of adapted structures are proved in \([\text{LS}];\) also see \([\text{Ha}]\). The uniqueness statement says: If \(J_1\) and \(J_2\) are adapted structures on \(A\), then \(J_1 = J_2\).

The symplectic side of the picture was developed in \([\text{Gus}];\) also see \([\text{Fr}]\). Let \(\lambda_{std}\) (resp. \(\omega_{std}\)) be the standard \(1\)–form (resp. \(2\)–form) on the cotangent bundle \(T^*M\). Let \(\rho_g = TM \to \mathbb{R}\) be the norm–function defined by the metric, i.e., \(\rho_g(\cdot) := \| \cdot \|^2_g\). If \(\psi_g : TM \to T^*M\) denotes the diffeomorphism defined by the metric, then we have the forms \(\theta_g := \psi_g^*(\lambda_{std})\) and \(\omega_g := \psi_g^*(\omega_{std})\).

Given \(A\) as above, \(dd^c \rho_g = \theta_g\) is regarded as a differential equation for a complex structure \(J\) on \(A\). The local existence of integrable such structures is shown in \([\text{Gus}]\) and the same strong uniqueness theorem as that stated above is proved; in particular, the locally defined \(J\)'s automatically glue together. Furthermore, \(dd^c \rho_g = d\theta_g = \omega_g\) is Kählerian, i.e., \(\rho_g\) is a strictly plurisubharmonic function on every adapted neighborhood \(A\).

The connection between these two notions of adapted structure is given by the following result.

**Theorem 6.2** \([\text{LS}]\) The \(1\)–form \(\theta_g\) on a domain \(A\) equipped with the adapted complex structure in the Riemannian sense satisfies \(dd^c \rho_g = \theta_g\).

As a consequence of the uniqueness theorem, the Riemannian and symplectic notions of adapted structure are equivalent. This allows the use of properties of plurisubharmonic functions, which we now briefly summarize, in the Riemannian setting.

**Basic properties of plurisubharmonic functions.**

A (smooth) strictly plurisubharmonic function \(\rho : X \to \mathbb{R}\) on a complex manifold is by definition a potential of a Kähler form \(dd^c \rho = \omega\). In other words, in holomorphic coordinates the complex Hessian \(H(\rho) := \left(\frac{\partial^2 \rho}{\partial z \partial \bar{z}}\right)\) is positive definite. Equivalently, the restriction \(\rho|_C\) to every (local) complex curve \(C\) is strictly subharmonic in the sense that the Laplacian of \(\rho|_C\) is negative. On disks in \(C\) such functions have the (strong) mean value property.

Strictly plurisubharmonic functions have strong convexity properties. In fact, holomorphic coordinates can be chosen so that \(\rho\) is strictly convex on the underlying real domain. Thus the maximum principle holds: A strictly plurisubharmonic function never takes on a (local) maximum value. The following also reflects this strong convexity.

**Proposition 6.3** Let \(\rho : X \to \mathbb{R}\) be strictly plurisubharmonic. Let \(M \subset X\) be a connected local real submanifold such that (1) \(\rho\) has some constant value \(c\) on \(M\) and (2) \(c\) is a minimal value of \(\rho\) in a neighborhood of \(M\). Then \(\dim_{\mathbb{R}} M \leq \dim_{\mathbb{C}} X\)

This follows immediately from the fact that the complex Hessian of \(\rho\) is positive definite, and therefore if \(V\) is a real subspace of the tangent space that is isotropic with respect to the real Hessian of \(\rho\), then \(\dim_{\mathbb{R}} V \leq \dim_{\mathbb{C}} X\).
The adapted structure for Riemannian symmetric spaces.

Here only irreducible Riemannian symmetric spaces $\Omega_0 = G_0/K_0$ of negative curvature are considered. In other words $G_0$ is noncompact and simple. If $g_0 = p_0 \oplus p_0$ is a Cartan decomposition, then $T\Omega_0 = G_0 \times_{K_0} p_0$. Now we recall the Riemannian notion of adapted complex structure on domains $\mathcal{A}$ in $T\Omega_0$.

Define the polar coordinate map $\Psi : T\Omega_0 \to \Omega = G/K$ by $[(g_0, \xi)] \mapsto g_0 \exp(i\xi).x_0$. It is well defined and $G_0$–equivariant. Furthermore, $\Psi_*(p) : T_p T\Omega_0 \to T_{\Psi(p)} \Omega$ is an isomorphism at points $p$ of the $0$–section.

Let $\Omega_{\text{adpt}}$ be the connected component containing the $0$–section of the set where $\Psi$ has maximal rank, which is $\{p \in T\Omega_0 | \Psi_*(p) \text{ is an isomorphism}\}$. Let $J$ be the (integrable) complex structure on $\Omega_{\text{adpt}}$ which is defined by pulling back the complex structure of $\Omega$ by $\Psi$.

**Proposition 6.4** The structure $J$ on $\Omega_{\text{adpt}}$ is adapted.

**Proof.** Let $x_0 = (x_0, 0)$, the neutral point in $T\Omega_0$. Identify $\Omega_0$ with $G_0.x_0$. Recall that the geodesics through $g(x_0)$ are given by $1$–parameter groups: $\gamma(t) = g \exp(t\xi).x_0$ for $\xi \in p_0$. Thus $s\gamma_\tau(t) = [(g, \exp(t\xi).e, s\xi)]$ and $\Psi \circ \gamma_\tau(t + is) = g \exp(t + is\xi).x_0$. Consequently, for an appropriately small disk, $\gamma_\tau : \Delta \to \Omega_{\text{adpt}}$ is holomorphic. $\square$

**Proper actions.**

An action $L \times M \to M$ of a topological group on a topological space is said to be proper if the induced map $L \times M \to M \times M$, $(g, x) \mapsto (g(x), x)$, is proper. Under minimal assumptions on the spaces at hand, this can be expressed as follows: For all sequences $\{g_n\} \subset L$ and $\{x_n\} \subset M$ such that $x_n \to x$ and $g_n(x_n) \to y$ there exists a convergent subsequence $g_{n_k} \to g \in L$. All isotropy groups of a proper action are compact.

The $G_0$–action on $\Omega_0 = G_0/K_0$ is proper, so the $G_0$–action on $M = T\Omega_0$ is as well. Although $\Psi : \Omega_{\text{adpt}} \to \Omega$ has finite fibers, it does not immediately follow that the $G_0$–action on its image is proper. Nevertheless, the $G_0$–isotropy groups in $\Psi(\Omega_{\text{adpt}})$ are at most finite extensions of the corresponding (compact) isotropy groups in $\Omega_{\text{adpt}}$ and therefore are themselves compact. It would therefore be natural to consider canonically defined neighborhoods of $\Omega_0$ in $\Omega$ in which the $G_0$–isotropy groups are compact.

From the point of view of the Riemannian conjugate locus there is a very natural candidate for such a domain. See [AkG]. In order to define it, assume as usual that the Cartan involution of $G_0$ is the restriction of that for $G$ which in turn defines its maximal compact subgroup $G_u$. Consider the restriction of the polar coordinate map $\Psi$ to the fiber $T_{x_0}\Omega_0 \cong p_0$ at the neutral point in $T\Omega_0$. It maps sufficiently small open neighborhoods of $0 \in p_0$ diffeomorphically onto neighborhoods of the neutral point $x_0$ in the $G_u$–orbit $G_u.x_0 = G_u/K_0$ in $\Omega$.

The maximal such set is determined as follows by the conjugate locus of the invariant metric. Let $a_0$ be a maximal abelian subalgebra of $g_0$ which is contained in $p_0$, $\Lambda_0$ be its set of (real) roots, and for $\alpha \in \Lambda_0$, let $H_\alpha$ be the affine hypersurface $\{\xi \in a_0 : \alpha(\xi) = \frac{2}{\pi}\}$. Define $\omega_\alpha$ to be the connected component containing $0 \in p_0 \setminus \bigcup_{\alpha \in \Lambda_0} H_\alpha$ and let $\Sigma_0 := K_0 \exp(\omega_\alpha)$. Then $G_0 \times_{K_0} \Sigma_0$ is naturally embedded in $T\Omega_0$ as an open neighborhood of the $0$–section by the action map $(g, \xi) \mapsto [(g_0, \xi)]$ and $\Psi|_{G_0 \times_{K_0} \Sigma_0} : G_0 \times_{K_0} \Sigma \to \Omega$ is a diffeomorphism onto its image (see e.g. [AkG]). This image $\Psi(G_0 \times_{K_0} \Sigma_0) = G_0.\exp(i\omega_\alpha).x_0$ was considered in [AkG], and we denote by $\Omega_{AG}$. The situation can now be summarized as follows.

**Proposition 6.5** The restriction of $\Psi$ to the open subset $G_0 \times_{K_0} \Sigma_0$ of $\Omega_{\text{adpt}}$ is a diffeomorphism onto its image $\Omega_{AG}$; in particular the $G_0$–action on $\Omega_{AG}$ is proper. Furthermore, $\Omega_{AG}$ comes equipped with the $\Psi$–induced, $G_0$–invariant strictly plurisubharmonic function $\rho = \rho_\Psi \circ \Psi^{-1}$ and its associated Kähler form $\omega := dd^c \rho$. 15
The following result sheds more light on the picture.

**Theorem 6.6** The domain $G_0 \times_{K_0} \Sigma_0$ is a maximal domain of definition for the adapted complex structure. In particular $\Omega_{adpt} = G_0 \times_{K_0} \Sigma_0$ and $\Psi : \Omega_{adpt} \to \Omega_{AG}$ is biholomorphic.

**Incidence geometry and the domain $\Omega_L$.**

Recall that an Iwasawa–Borel subgroup is a Borel subgroup $B \subset G$ that contains a factor $A_0N_0$ of an Iwasawa decomposition $G_0 = K_0A_0N_0$, where $K_0$ can be any maximal compact subgroup of $G_0$.

Consider our usual cycle space setup for an open $G_0$–orbit $D$ in $Z = G/Q$, where $\Omega_W(D)$ is regarded as an open neighborhood of $\Omega_0 = G_0C_0 = G_0z_0$ in $\Omega = G/K$. Recall that a $B$–Schubert variety $S$ in $Z$ is defined to be the closure of a $B$–orbit $O$ in $Z$. Write $S = O \cup Y$, where $Y$ is the (finite) union of $B$–orbits on the boundary of $O$. The following remark motivates a number of our considerations.

**Theorem 6.7** If $S \cap D \neq \emptyset$, then $\text{codim}_Z S \leq \dim_C C_0$.

**Proof.** If $S \cap D \neq \emptyset$, then $O \cap D \neq \emptyset$ as well. Without loss of generality we may assume that $G_0 = K_0A_0N_0$, $B \supset A_0N_0$ and $K_0z_0 = C_0$ is the base cycle. In particular, $A_0N_0C_0 = D$ and the real codimension of $A_0N_0z$ in $D$ is at most the real dimension of $C_0$ for every $z \in D$. Since $B \supset A_0N_0$, it follows that $O \cap C_0 = \emptyset$ and the desired dimension bound holds. □

**Invariant incidence varieties.** Let $B$ be an Iwasawa–Borel subgroup and $S$ a $B$–Schubert variety with $\text{codim}_Z S = \dim_C C_0 = q$ and $S \cap C_0 \neq \emptyset$. Using the same type of argument as above, we make the following observation [BH].

**Proposition 6.8** The intersection $S \cap D$ is contained in $O$, and the intersection $O \cap C_0$ with the base cycle is nonempty and transversal at each of its points. Furthermore, each of its components is an $A_0N_0$–orbit.

Note that the above Schubert varieties are completely determined topologically by the Poincaré dual $PD(C_0)$ in $H_*(Z; \mathbb{Z})$. In particular, there exist such varieties and given one we refer to the components of $S \cap O$ as Schubert slices. Using deeper considerations, we have the following improvement of the above proposition (see [HW3]).

**Proposition 6.9** Every Schubert slice $\Sigma$ intersects every cycle $C \in \Omega_W(D)$ in exactly one point, and that intersection is transversal.

Turning to incidence varieties, we consider $B$ as above and let $Y$ be any closed $B$–invariant subvariety of $Z$. Then the incidence variety $A_Y := \{C \in \Omega : C \cap Y \neq \emptyset\}$ is a closed, $B$–invariant algebraic subvariety of $\Omega$. It is a proper subvariety if and only if $C_0 \notin Y$. We only consider this case. Since $B$ has only finitely many orbits in $Z$, there are only finitely many candidates for $A_Y$.

Consider the special case $Y = S \setminus O$, where $S$ is a Schubert variety containing a Schubert slice $\Sigma$ as above. Here, due to Proposition [BH] we refer to $S$ as a transversal Schubert variety.

Recall that $Y$ has the structure of a very ample Cartier divisor. Let $\Gamma(S, \mathcal{O}(+Y))$ denote the space of meromorphic functions on $S$ with poles only on $Y$ and let $\Gamma(\Omega, \mathcal{O}(+A_Y))$ be the analogously defined space of functions on $\Omega$. The trace transform $T_\Sigma : \Gamma(S, \mathcal{O}(+Y)) \to \Gamma(\Omega, \mathcal{O}(+A_Y))$ is defined by $T_\Sigma(f)(C) = \sum_{p \in C \cap Y} f(p)$. (BM, HS, Appendix). Of course this is first defined at the cycles $C$ which intersect $S$ generically. This resulting function in $\Gamma(\Omega, \mathcal{O}(+A_Y))$ arises via analytic continuation. With care about cancellations in the defining sum, one proves
Theorem 6.10. Given $C \in A_Y$ there exists $f \in \Gamma(S, \mathcal{O}(\ast Y))$ such that the polar set $\pi(T_S(f))$ contains $C$. In particular, $A_Y$ is a $B$–invariant complex hypersurface.

Incidence varieties in $\Omega$ which are hypersurfaces are denoted by $H_Y$ and are called incidence hypersurfaces. If $S$ is a transversal Schubert variety and $Y = S \setminus \mathcal{O}$, then, by Proposition 6.4, the incidence hypersurface $H_Y$ is contained in the complement of $\Omega_W(D)$ in $\Omega$.

One of our original goals was to prove that $\Omega_W(D)$ is a domain of holomorphy in $\Omega$, i.e., that given a divergent sequence $\{C_n\} \subset \Omega_W(D)$ there exists a function $f \in \mathcal{O}(\Omega_W(D))$ with $\lim_{n \to \infty} |f(C_n)| = \infty$. The following criterion (see, for example, [GuR]), formulated in the setting where $\Omega$ is affine, is useful for this.

Theorem 6.11. If for every point $C$ in the boundary $\text{bd}(\Omega_W(D))$ in $\Omega$ there exist a complex hypersurface $H$ in $\Omega$ which is contained in $\Omega \setminus \Omega_W(D)$ with $C \in H$, then $\Omega_W(D)$ is a domain of holomorphy.

One consequence of our recent work is that for every $C \in \text{bd}(\Omega_W(D))$ there exists an incidence hypersurface $H_Y$ defined by a transversal Schubert variety ([HW3], also Section 7).

Domains defined by invariant hypersurfaces.

Let $B \subset G$ be an Iwasawa–Borel subgroup and $H$ a $B$–invariant complex hypersurface in $\Omega$. The family $\{g_0 H\}_{g_0 \in G_0}$ consists of all such hypersurfaces which are equivalent in the sense that the variation within the family only depends on the choice of $B$. Since $H$ is $A_0N_0$–invariant for some Iwasawa decomposition $G_0 = A_0N_0$, this family is the same as $\{k_0 H\}_{k_0 \in K_0}$. The connected component of $\bigcap_{k_0 \in K_0} \Omega \setminus k_0 H$ containing the neutral point $x_0 \in \Omega$ is a $G_0$–invariant domain defined by $H$. We denote it $\Omega_H$.

If $S$ is a transversal Schubert variety, we have the associated incidence hypersurface $H = H_Y$ where $Y = S \setminus \mathcal{O}$. The connected component $\Omega_S(D)$ containing $x_0$ of the intersection of all $\Omega_H$ where $H = H_Y$ is an incidence hypersurface associated to such a Schubert variety is referred to as the Schubert domain associated to $D$ in $\Omega$.

Finally, the Iwasawa domain is defined to be the connected component containing $x_0$ of the intersection of all the $\Omega_H$, i.e., as $H$ ranges over all the (finitely many) $B$–invariant hypersurfaces.

Proposition 6.12. $\Omega_H$, $\Omega_S(D)$ and $\Omega_I$ are $G_0$–invariant domains of holomorphy in $\Omega$.

Proof. By definition these domains are $G_0$–invariant, connected open subsets of $\Omega$. Every boundary point of such a domain is contained in a hypersurface $k_0 H$ which is contained in its complement in $\Omega$. The desired result then follows from Theorem 6.11. □

From the definitions, $\Omega_I \subset \Omega_S(D)$ and, if $H$ is a Schubert incidence hypersurface then $\Omega_S(D) \subset \Omega_H$.

The domain $\Omega_I$ can be viewed from several different perspectives. For example, it is the same as the polar $\tilde{X}_0$ which is defined to be $\{gx_0 \in \Omega : g \in G \text{ and } X_0 \subset gx_0\}$ where $X_0$ is the closed $G_0$–orbit in $\tilde{Z} = G/B$ and $k_0$ is the open $K$–orbit in $\tilde{Z}$ (see [Ba]). R. Zierau remarked that $\tilde{X}_0 = \Omega_I$ (see [HW3]). The polar can also be regarded as a type of cycle space (see [GM]).

The equality $\Omega_I = \Omega_{AG}$.

As noted above, the domain $\Omega_I$ has various guises. From the point of view of holomorphic extension of certain special functions on $\Omega_0$, it is shown in [KS] that $\Omega_{AG} \subset \Omega_I$ for the classical groups. From the polar viewpoint it is shown in [Ba] that $\Omega_I \subset \Omega_{AG}$. Equality of these domains comes down to the opposite inclusion, $\Omega_{AG} \subset \Omega_I$. That was proved in [H] using the plurisubharmonic function $\rho$ that is defined by the adapted complex structure; see Proposition 6.3. Now we put all this together.
Theorem 6.13 \( \Omega_I = \Omega_{AG} \).

**Proof.** For the inclusion \( \Omega_{AG} \subset \Omega_I \) let \( \rho \) be the \( G_0 \)-invariant strictly plurisubharmonic function of Proposition 5.5. Suppose to the contrary that some hypersurface \( H \) which is invariant under an Iwasawa–Borel subgroup \( B \) has nonempty intersection with \( \Omega_{AG} \) or, equivalently, for some Iwasawa decomposition \( G = K_0 A_0 N_0 \) and some \( x_1 \in \Omega_{AG} \) the orbit \( AN.x_1 \) is not open.

The orbit \( A_0 N_0.x_0 = \Omega_0 \) is totally real of half dimension in \( \Omega \). On the other hand, no \( A_0 N_0 \)-orbit in \( AN.x_1 \cap \Omega_{AG} \) is of this type. Now \( \rho(x_0) = 0 \) and \( \rho > 0 \) along all other \( G_0 \)-orbits. For \( x \in \Sigma_0 \) near \( x_0 \), i.e., for \( \rho(x) > 0 \) sufficiently small, \( A_0 N_0.x \) must still be totally real.

Let \( r \) be the smallest value of \( \rho|_{\Sigma_0} \) such that for some \( x_1 \in \{ \rho = r \} \) the orbit \( A_0 N_0.x_1 \) is not totally real. Such a value must exist, because the total reality of an \( A_0 N_0 \)-orbit is equivalent to the openness of the corresponding \( AN \)-orbit; in particular, we might as well let \( x_1 \) be the same point as that which was denoted by \( x_1 \) at the outset.

Apply Proposition 6.3 to the complex manifold \( X = AN.x_1 \cap \Omega_{AG} \); the strictly plurisubharmonic function \( \rho|_X \) and the real submanifold \( M := A_0 N_0.x_1 \). It follows that \( \dim M \leq \dim_{\mathbb{R}} X \). But this implies that \( \dim_{\mathbb{R}} M \leq \dim_{\mathbb{R}} \Omega_0 - 2 \), contrary to the fact that the \( A_0 N_0 \)-orbits in \( \Omega_{AG} \) have the same dimension as \( \Omega_0 \).

Our proof of the inclusion \( \Omega_I \subset \Omega_{AG} \) follows in [Ba]. In the flavor of the present paper we use the fact that the \( G_0 \)-action on \( \Omega_I \) is proper. That follows from the Kobayashi hyperbolicity of \( \Omega_I \) ([1]), see Section 2.3.

Suppose that there is a sequence \( \{ z_n \} \subset \Omega_{AG} \cap \Omega_I \) with \( z_n = z \in \partial(\Omega_{AG}) \cap \partial\Omega_I \). From the definition of \( \Omega_{AG} \), it follows that there exist \( \{ g_m \} \subset G \) and \( \{ x_m \} \subset \exp(\omega_0) \) such that \( g_m(x_m) = z_m \). Write \( g_m = k_m a_m n_m \) in a \( K_0 A_0 N_0 \) decomposition of \( G_0 \). Since \( \{ k_m \} \) is contained in the compact group \( K_0 \), it may be assumed that \( k_m \rightarrow k \); therefore that \( g_m = a_m n_m \). Since \( \omega_0 \) is relatively compact in \( a \), it may also be assumed that \( x_m = \exp(\omega_0) \). Thus \( x_m = T_m x_0 \), where \( \{ T_m \} \subset \exp(\omega_0) \) and \( T_m \rightarrow T \). Write \( a_m n_m (x_m) = a_m n_m T_m x_0 = \tilde{a}_m \tilde{n}_m x_0 \), where \( \tilde{a}_m = a_m A_m \) and \( \tilde{n}_m = T_m^{-1} n_m T_m \) are elements of \( A \) and \( N \), respectively. Now \( \{ z_m \} \) and the limit \( z \) are contained in \( \Omega_I \) which is in turn contained in \( AN \cdot x_0 \). Furthermore, \( AN \) acts freely on this orbit. Thus \( \tilde{a}_m \rightarrow a \in A \) and \( \tilde{n}_m \rightarrow n \in N \) with \( \tilde{a} \tilde{n} \cdot x_0 = z \). Since \( T_m \rightarrow T \), it follows that \( a_m \rightarrow a \in A_0 \) and \( n_m \rightarrow n \in N_0 \) with \( a n.x = z \). Since \( z \notin \Omega_{AG} \), it follows that \( x \in \partial(\exp(\omega_0)) \), and \( z \in \Omega_I \) implies that \( x \in \Omega_I \).

On the other hand, since \( x \in \partial(\exp(\omega_0)) \), the isotropy group \( G_x \) is noncompact. But \( \Omega_I \) is Kobayashi hyperbolic ([1]), see Section 2.3. Therefore the \( G \)-action on \( \Omega_I \) is proper (see e.g. [1]) and consequently \( x \notin \Omega_I \), which is a contradiction. That completes the proof. \( \square \)

## 7 Transversal Schubert Varieties.

In this Section we outline the methods introduced in [HW3] and their main applications, in particular the fact that \( \Omega_W(D) = \Omega_S(D) \). The intermediate results, which follow from a sort of triality, are of complex analytic interest. For example, at every boundary point \( p \in \partial(D) \) we construct a \((q + 1)\)-codimensional Schubert variety \( S \) with \( p \in S \) and \( S \subset Z \setminus D \). Due to the existence of the family \( \Omega_W(D) \) of \( q \)-dimensional cycles in \( D \), this exhibits the maximal possible degree of holomorphic convexity. It should be underlined that \( S \) arises as an extension of a complex analytic manifold in the the boundary orbit and therefore its relation to the signature of the Hessian of a boundary defining function is unclear.

The Borel groups \( B \) considered here are Iwasawa–Borel subgroups, the ones that contain an Iwasawa component \( A_0 N_0 \). The Schubert varieties are always those which are closures \( S \) of orbits \( O \) in \( Z \) of Iwasawa–Borel subgroups, and \( Y := S \setminus O \). If \( \operatorname{codim} Y = \operatorname{dim} C_0 = q \) and \( S \cap C_0 \neq \emptyset \), then \( S \) is called a transversal Schubert variety (see Proposition 6.4).
As usual $D$ is an open $G_0$-orbit in $Z = G/Q$ and $C_0 = K_0.z_0$ is the base cycle. Here we view $\Omega_W(D) \subset \Omega$ with $\Omega := G.C_0$ in the cycle space $C^q(Z)$. In other words, in the case when $G_0^0 = K$ we do not replace $\Omega$ by the finite cover $G/K$ even if the $G$-stabilizer of $C_0$ properly contains $K$, and we do not exclude the hermitian holomorphic case where $\Omega = G.C_0$ is an associated compact hermitian symmetric space.

**Duality.**

Throughout this subsection $\gamma \in \text{Orb}_Z G_0$ (resp. $\kappa \in \text{Orb}_Z K$) denotes one of the finitely many $G_0$-orbits (resp. $K$-orbits) in $Z$. The first example of duality was proved at the level of open $G_0$-orbits [W2]: Every open $G_0$-orbit $\gamma$ contains a unique compact $K$-orbit $\kappa$.

Let us reformulate this in a way that makes sense for all $G_0$- and $K$-orbits. For this first observe that, since $G_0 \cap K = K_0$, the intersection $\gamma \cap \kappa$ of any two $G_0$- and $K$-orbits is $K_0$-invariant.

For $z \in \gamma \cap \kappa$ we refer to the orbit $K_0.z$ as isolated if it has a neighborhood $U$ in $\gamma$ so that $\kappa \cap U = K_0.z$. Finally, let us say that $(\gamma, \kappa)$ is a dual pair if $\gamma \cap \kappa$ contains an isolated $K_0$-orbit.

Matsuki’s duality theorem, which is an extension of the above statement for open orbits, states that there is a bijective map $\mu : \text{Orb}_Z G_0 \to \text{Orb}_Z K$ such that $(\gamma, \kappa)$ is a dual pair if and only if $\kappa = \mu(\gamma)$ [M].

In the original version, duality was defined by $\gamma \cap \kappa$ being nonempty and compact, but it was soon realized that this is equivalent to the condition that $\gamma \cap \kappa$ be a single $K_0$-orbit. The more recent proofs, which involve the Morse theory related to a certain moment map ([MUV], [BL]), use the above weaker notion. However, in the end, if an intersection $\gamma \cap \kappa$ contains an isolated $K_0$-orbit, then it is a $K_0$-orbit, and the intersection $\gamma \cap \kappa$ is transversal along that orbit.

In our work we use the following non-isolation property which is implicit in the proofs in [MUV] and [BL].

**Proposition 7.1** If $(\gamma, \kappa)$ is not a dual pair, then every $K_0$-orbit $K_0.z$ in $\gamma \cap \kappa$ is contained in a $K_0$-invariant locally closed submanifold $M \subset \gamma \cap \kappa$ with $\dim \mathbb{R} M = \dim \mathbb{R} K_0.z + 1$.

This symplectic approach also yields information about the topology of the $G_0$-orbits [HW3]:

**Proposition 7.2** If $(\gamma, \kappa)$ is a dual pair, but $\gamma \cap \kappa \neq \emptyset$, then the $K_0$-orbit $\gamma \cap \kappa$ is a $K_0$-equivariant strong deformation retract of $\gamma$.

**Triality.**

Here we describe a sort of triality where, in addition to the orbits $\gamma$ and $\kappa$ above, we incorporate Schubert varieties of Iwasawa Borel subgroups $B$. For this we fix an Iwasawa decomposition $G_0 = K_0 A_0 N_0$ and $B$ containing $A_0 N_0$. For $\kappa \in \text{Orb}_Z K$ let $\text{cf}(\kappa)$ denote its closure in $Z$ and define $S_\kappa$ to be the set of all $B$-Schubert varieties $S$ such that $\text{codim}_Z S = \dim_{\mathbb{R}} \kappa$ and $S \cap \text{cf}(\kappa) \neq \emptyset$.

The Schubert varieties of a fixed Borel subgroup generate the integral homology of $Z$ and consequently $S_\kappa$ is determined by the topological class of $\text{cf}(\kappa)$; in particular, it is non-empty.

The following can be regarded as a statement of triality.

**Theorem 7.3** If $(\gamma, \kappa)$ is a dual pair, then the following hold for every $S \in S_\kappa$.

1. $S \cap \text{cf}(\kappa)$ is contained in $\gamma \cap \kappa$ and is finite. If $x \in S \cap \kappa$, then $(AN)(x) = B(x) = \mathcal{O}$, where $S = \text{cf}(\mathcal{O})$, and $S$ is transversal to $\kappa$ at $x$ in the sense that the real tangent spaces satisfy $T_x(S) \oplus T_x(\kappa) = T_x(Z)$.
2. The set $\Sigma = \Sigma(\gamma, S, x) := A_0 N_0(x)$ is open in $S$ and closed in $\gamma$; in particular it is a locally closed complex submanifold of $Z$. 

19
3. Let $\text{cl}(\Sigma)$ and $\text{cl}(\gamma)$ denote closures in $Z$. Then the map $K_0 \times \text{cl}(\Sigma) \to \text{cl}(\gamma)$, given by $(k, z) \mapsto k(z)$, is surjective.

**Proof.** Let $x \in S \cap \text{cl}(\kappa)$. Since $g = t + a + n$ is the complexification of the Lie algebra version $\mathfrak{g}_0 = \mathfrak{t}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ of $G_0 = K_0 A_0 N_0$, we have $T_x(AN(x)) + T_x(K(x)) = T_x(Z)$. As $x \in S = \text{cl}(O)$ and $AN \subset B$, we have $\dim AN(x) \leq \dim B(x) \leq \dim O = \dim S$. Furthermore $x \in \text{cl}(\kappa)$. Thus $\dim K(x) \leq \dim \kappa$. If $x$ were not in $\kappa$, this inequality would be strict, in violation of the above additivity of the dimensions of the tangent spaces. Thus $x \in \kappa$ and $T_x(S) + T_x(\kappa) = T_x(Z)$. Since $\dim S + \dim \kappa = \dim Z$ this sum is direct, i.e., $T_x(S) \oplus T_x(\kappa) = T_x(Z)$. Now also $\dim AN(x) = \dim S$ and $\dim K(x) = \dim \kappa$. Thus $AN(x)$ is open in $S$, forcing $AN(x) = B(x) = O$. We have already seen that $K(x)$ is open in $\kappa$, forcing $K(x) = \kappa$. For assertion 1 it remains only to show that $S \cap \kappa$ is contained in $\gamma$ and is finite.

Denote $\tilde{\gamma} = G_0(x)$. If $\tilde{\gamma} \neq \gamma$, then $(\tilde{\gamma}, \kappa)$ is not dual, but $\tilde{\gamma} \cap \kappa$ is nonempty because it contains $x$. By the non–isolation property [7,1], we have a locally closed $K_0$–invariant manifold $M \subset \tilde{\gamma} \cap \kappa$ such that $\dim M = \dim K_0(x) + 1$. We know $T_x(S) \oplus T_x(\kappa) = T_x(Z)$, and $K(x) = \kappa$, so $T_x(A_0 N_0(x)) \cap T_x(M) = 0$. Thus $T_x(A_0 N_0(x)) + T_x(K_0(x))$ has codimension 1 in the subspace $T_x(A_0 N_0(x))$ of $T_x(\tilde{\gamma})$, which contradicts $G_0 = K_0 A_0 N_0$. We have proved that $(S \cap \text{cl}(\kappa)) \subset \gamma$. Since that intersection is transversal at $x$, it is finite. This completes the proof of assertion 1.

We have seen that $T_x(AN(x)) \oplus T_x(K(x)) = T_x(Z)$, so $T_x(A_0 N_0(x)) \oplus T_x(K_0(x)) = T_x(\gamma)$, and $G_0(x) = \gamma$. From the basic properties of a dual part, in particular the transversality of the intersection $\gamma \cap \kappa$, we have $\dim A_0 N_0(x) = \dim T_x(\gamma) - \dim T_x(\kappa \cap \gamma) = \dim T_x(Z) - \dim T_x(\kappa) = \dim AN(x) = \dim S$. Now $A_0 N_0(x)$ is open in $S$.

Every $A_0 N_0$–orbit in $\gamma$ meets $K_0(x)$ because $\gamma = G_0(x) = A_0 N_0 K_0(x)$. By the transversality of the intersection $\gamma \cap \kappa$, every such $A_0 N_0$–orbit has dimension at least that of $\Sigma = \Sigma N_0$. Since the orbits on the boundary of $S$ in $\gamma$ would necessarily be smaller, it follows that $\Sigma$ is closed in $\gamma$. This completes the proof of assertion 2.

The map $K_0 \times \Sigma \to \gamma$, by $(k, z) \mapsto k(z)$, is surjective because $K_0 A_0 N_0(x) = \gamma$. Since $K_0$ is compact and $\gamma$ is dense in $\text{cl}(\gamma)$, assertion 3 follows.

Let us now indicate the construction given in [HW3] of the supporting Schubert variety at each boundary point $p \in \text{bd}(D)$. For this it is convenient to refer to a $G_0$–orbit in $\text{bd}(D)$ as generic if it is open in $\text{bd}(D)$. The transversality property [HW3] also gives us

**Lemma 7.4** If $\gamma$ is a generic orbit in $\text{bd}(D)$, $\kappa$ is dual to $\gamma$ and $S = \text{cl}(O) \in S_\kappa$, then $\text{codim}_Z S > \dim C_\kappa$.

There are increasing sequences $\{O_k\}$ of $B$–orbits with $O_0 = O$, $\dim O_k = \dim O + k$ and $O_k \subset \text{cl}(O_{k+1})$, for $k \leq n - \dim C_\kappa$, so the above Schubert variety can be enlarged to obtain the following consequence of triality.

**Theorem 7.5** For every $p \in \text{bd}(D)$ there exists an Iwasawa-Borel subgroup $B \subset G$ and a $B$–Schubert variety $S$ with (1) $\text{codim}_Z S = \dim C_\kappa + 1$ and (2) $p \in S$ and $S \subset Z \setminus D$.

**Proof.** Given a point $p$ in a generic orbit $\gamma$, we may take an appropriate conjugate of the Iwasawa-Borel subgroup of Theorem [7,3] to obtain $S$ satisfying all of the required conditions except that it may be too small. In that case we enlarge it to be $(q + 1)$-codimensional by the above procedure. By Proposition [6,7] this Schubert variety is also contained in $Z \setminus D$.

If $p \in \text{bd}(D)$ is not generic, then nevertheless it is the limit $p_n \to p$ with $\gamma = G_0 p_n$ generic and with $B_n$–Schubert varieties $S_n$ which have the desired properties.
Given \( S \) is zero in homology, one proves that assumption on \( S \) contains \( C \). The equality \( \Omega \sim \). When \( \Omega \) is affine, this follows immediately from Theorem 6.11 and the equality \( \Omega = \Omega \). Note that \( \Omega \) is a Stein domain in \( \Omega \). Spaces of cycles in lower dimensional \( G_0 \)-orbits.

We recall the setting of [GM]. For \( Z = G/Q, \gamma \in \text{Orb}_Z(G_0) \) and \( \kappa \in \text{Orb}_Z(K) \) its dual, let \( G\{\gamma\} \) be the connected component of the identity of \( \{ g \in G : g(\kappa) \cap \gamma \) is non-empty and compact \}. Note that \( G\{\gamma\} \) is an open \( K \)-invariant subset of \( G \) that contains the identity. Define \( \mathcal{C}\{\gamma\} := G\{\gamma\}/K \). Finally, define \( \mathcal{C} \) as the intersection of all such cycle spaces \( \mathcal{C}\{\gamma\} \) as \( \gamma \) ranges over \( \text{Orb}_Z(G_0) \) and \( Q \) ranges over all parabolic subgroups of \( G \).

Proposition 7.10 \( \left( \bigcap_{D \subset G/B \text{ open}} \Omega_W(D) \right) \subset \mathcal{C} \).
Proof of Theorem. The polar $\hat{X}_0$ in $Z = G/B$ coincides with the cycle space $C_Z(\gamma_0)$, where $\gamma_0$ is the unique closed $G_0$-orbit in $Z$. As was shown above, this agrees with the Iwasawa domain $\Omega$ which in turn is contained in every Schubert domain $\Omega_S(D)$. Thus, for every open $G_0$-orbit $D_0$ in $Z = G/B$ we have the inclusions

$$\left(\bigcap_{D \subset G/B \text{ open}} \Omega_W(D)\right) \subset C \subset C_Z(\gamma_0) = \hat{X}_0 = \Omega_I \subset \Omega_S(D_0) = \Omega_W(D_0).$$

Intersecting over all open $G_0$-orbits $D$ in $G/B$, the equalities

$$\left(\bigcap_{D \subset G/B \text{ open}} \Omega_W(D)\right) = C = \Omega_I = \left(\bigcap_{D \subset G/B \text{ open}} \Omega_W(D)\right)$$

are forced, and $C = \Omega_{AG}$ is a consequence of $\Omega_I = \Omega_{AG}$. \hfill \Box

8 Cycle Domains in the Hermitian Case.

In this section $G_0$ is a group of hermitian type, in other words $\mathcal{B} = G_0/K_0$ is a bounded symmetric domain. We give a concrete description of $\Omega_W(D)$. As indicated in Section 2 either $\Omega_W(D) = \mathcal{B}$ or $\mathcal{B}$, or $G.C_0 = \Omega$ is affine. Thus it is enough to consider the latter case. At first we will replace $\Omega_W(D)$ by the connected component containing $x_0$ in its preimage in $G/K$ and denote the latter by $\Omega$.

Choosing a system of roots in the usual way, we regard the bounded symmetric domain $\mathcal{B}$ of $G_0$ as the $G_0$-orbit of the neutral point $x_0 \in X = G/P_\gamma$ and its complex conjugate as the orbit of the analogous point $\overline{x}_0 \in \overline{X} = G/P_\gamma$. We view $z_0 = (x_0, \overline{x}_0)$ as the base point for $\Omega = G.z_0 = G/K \hookrightarrow X \times \overline{X}$, i.e., the open $G$-orbit by its diagonal action. We identify $\mathcal{B} \times \overline{\mathcal{B}}$ with its image under the natural embedding $\mathcal{B} \times \overline{\mathcal{B}} \hookrightarrow X \times \overline{X}$ and note that this lies in $\Omega$.

The following is proved by a reduction to the polydisc case \cite{BHH}.

Proposition 8.1 $\Omega_{AG} = \mathcal{B} \times \overline{\mathcal{B}}$

In \cite{WZ} it was shown that $\Omega_W(D) \subset \mathcal{B} \times \overline{\mathcal{B}}$. Thus Corollary 7.7 together with Theorem 7.13 imply the following characterization.

Theorem 8.2 (\cite{HW3}, \cite{WZ3}) If $G_0$ is of hermitian type, then either (i) $\Omega$ is the compact dual symmetric space to $G_0/K_0$ and $\Omega_W(D) = \mathcal{B} \cup \overline{\mathcal{B}}$, or (ii) $\Omega = G/K$ is affine with $\hat{K}/\hat{K}$ finite and $\Omega_W(D) = \mathcal{B} \times \overline{\mathcal{B}}$. In case (ii) $\Omega_W(D)$ lifts bijectively to $G/K$.

Proof. It is enough to consider the case where $\Omega$ is affine. As we have seen above, after lifting to $\Omega = G/K$, $\Omega_W(D) \subset \mathcal{B} \times \overline{\mathcal{B}} = \Omega_{AG}$. But, by Theorem 7.13, $\Omega_{AG} = \Omega_I$, and $\Omega_W(D) = \Omega_S(D)$ by Corollary 7.7. Since by definition $\Omega_I \subset \Omega_S(D)$, the result follows at the level of $G/K$. Furthermore, $\mathcal{B} \times \overline{\mathcal{B}}$ is a cell. Thus it agrees with its image in $G/K$ by the canonical finite covering map (see Proposition 10.20). \hfill \Box

Remark 8.3 This theorem can also be proved using results from \cite{GM}.

9 Kobayashi Hyperbolicity.

The Kobayashi pseudometric $d_K$ is defined on any complex manifold $X$; see Section 3. If it is a metric, i.e., $d_K(x, y) > 0$ if $x \neq y$, then $X$ is said to be Kobayashi hyperbolic. One checks that it is a metric on bounded domains and vanishes identically in the case where $X$, e.g., is
normal crossing property. This means that the intersection is empty.

We wish to prove that a locally closed, irreducible real analytic subset \( P \subset \mathbb{C} \) contains \( (2k+1) \)-hyperplanes in general position, which is a proper, linear plane \( \mathbb{C} \). It is a classical result that the complement of the union of \((2k+1)\)-hyperplanes in general position in \( \mathbb{C} \) is Kobayashi hyperbolic \([\text{D}]\). Let us make the notion of general position precise in a context which is appropriate for our applications.

Since the complex manifolds which we consider are embedded in projective spaces by sections of line bundles, it is natural to regard a “point” as being in the projectivization \( \mathbb{P}(V^*) \) of the dual space of a complex vector space and a “hyperplane” as a point in \( \mathbb{P}(V) \). We regard a subset \( S \subset \mathbb{P}(V) \) as parameterizing a family of hyperplanes in \( \mathbb{P}(V^*) \). A non–empty subset \( S \subset \mathbb{P}(V) \) is said to have the normal crossing property if for every \( k \in \mathbb{N} \) there exist \( H_1, \ldots, H_k \in S \) so that for every subset \( I \subset \{1, \ldots, k\} \) the intersection \( \bigcap_{i \in I} H_i \) is \( |I| \)-codimensional. If \( |I| \geq \dim \mathbb{C} V \), this means that the intersection is empty.

In the sequel \( \langle S \rangle \) denotes the complex linear span of \( S \) in \( \mathbb{P}(V) \), i.e., the smallest complex subspace in \( \mathbb{P}(V) \) containing \( S \). If \( \langle S \rangle = \mathbb{P}(V) \) we say that \( S \) is a generating set.

**Proposition 9.2** A locally closed, irreducible real analytic subset \( S \) with \( \langle S \rangle = \mathbb{P}(V) \) has the normal crossing property.

**Proof.** We proceed by induction over \( k \). For \( k = 1 \) there is nothing to prove. Given a set \( \{H_{s_1}, \ldots, H_{s_k}\} \) of hyperplanes with the normal crossing property and a subset \( I \subset \{s_1, \ldots, s_k\} \), define

\[
\Delta_I := \bigcap_{s \in I} H_s, \quad \mathcal{H}(I) := \{s \in S : H_s \supset \Delta_I\} \quad \mathcal{C}l_k := \bigcup_{J \subset \{s_1, \ldots, s_k\}, \Delta_J \neq \emptyset} \mathcal{H}(J).
\]

We wish to prove that \( S \setminus \mathcal{C}l_k \neq \emptyset \). For this, note that each \( \mathcal{H}(I) \) is a real analytic subvariety of \( S \). Hence, if \( S = \mathcal{C}l_k \), then \( S = \mathcal{H}(J) \) for some \( J \) with \( \Delta_J \neq \emptyset \). However, \( \{H \in \mathbb{P}(V^*) : H \supset \Delta_J\} \) is a proper, linear plane \( \mathcal{L}(J) \) of \( \mathbb{P}(V) \). Consequently, \( S \subset \mathcal{L}(J) \), and this would contradict \( \langle S \rangle = \mathbb{P}(V) \). Therefore, there exists \( s \in S \setminus \mathcal{C}l_k \), or equivalently, \( \{H_{s_1}, \ldots, H_{s_k}, H_s\} \) has the normal crossing property. \( \square \)
In the case of finitely many hyperplanes in \( \mathbb{CP}_m \) the condition that \( H_1, \ldots, H_{2m+1} \) are in general position is equivalent to their having the normal crossing property, in which case \( \mathbb{CP}(V^*) \setminus \bigcup H_j \) is Kobayashi hyperbolic ([13], or see [8], p. 137].

**Corollary 9.3** If \( S \) is a locally closed, irreducible and generating real analytic subset of \( \mathbb{P}(V) \), then there exist hyperplanes \( H_1, \ldots, H_{2m+1} \in S \) such that the complement \( \mathbb{P}(V^*) \setminus \bigcup H_j \) is Kobayashi hyperbolic.

Our main application of this result arises in the case where \( S \) is an orbit of the real form at hand.

**Corollary 9.4** Let \( G \) be a reductive complex Lie group, \( G_0 \) a real form, \( V^* \) an irreducible \( G \)-representation space and \( S \) a \( G_0 \)-orbit in \( \mathbb{P}(V) \). Then there exist hyperplanes \( H_1, \ldots, H_{2m+1} \in S \) so that \( \mathbb{P}(V^*) \setminus \bigcup H_j \) is Kobayashi hyperbolic.

**Proof.** From the irreducibility of the representation \( V^* \), it follows that \( V \) is likewise irreducible and this, along with the identity principle, implies that \( \langle S \rangle = \mathbb{P}(V) \).

**Invariant hyperbolic domains in \( \Omega \).**

We begin by briefly discussing the formalism for equivariantly embedding \( \Omega = G/K \) in a complex projective space. In order to avoid discussions of infinite-dimensional spaces, we let \( X \) be a \( G \)-equivariant, smooth, projective algebraic compactification of \( \Omega \). Since \( X \) is rational and we may assume that \( G \) is simply connected, every line bundle \( L \to X \) is a \( G \)-bundle; in particular there is a canonically induced action on the space \( \Gamma(X, L) \) of sections.

Now let \( B \) be an Iwasawa-Borel subgroup of \( G \) and \( H \) be a \( B \)-invariant complex hypersurface in \( \Omega \). Since \( H \) is algebraic, it is Zariski open in its closure \( c\ell(H) \) in \( X \).

Let \( L \to X \) be the line bundle defined by \( c\ell(H) \) and \( s \in \Gamma(X, L) \) the defining section. Since \( c\ell(H) \) is \( B \)-invariant, \( s \) is a \( B \)-eigenvector.

Define \( V \) to be the irreducible \( G \)-representation subspace of \( \Gamma(X, L) \) that contains \( s \). Let \( \varphi : X \to \mathbb{P}(V^*) \) denote the canonically associated \( G \)-equivariant meromorphic map.

**Proposition 9.5** Assume that \( G_0 \) is not of hermitian holomorphic type. Then the restriction \( \varphi|\Omega : \Omega \to \mathbb{P}(V^*) \) is a \( G \)-equivariant, finite–fibered regular morphism with image a quasi-projective \( G \)-orbit in \( \mathbb{P}(V^*) \).

**Proof.** Since \( \varphi \) is \( G \)-equivariant and its set of indeterminacies is therefore \( G \)-invariant, the restriction to the open \( G \)-orbit is base point free. The fact that \( \varphi|\Omega \) is finite–fibered is a consequence of \( \varphi \) being non-constant, e.g., \( s \) is not \( G \)-fixed, and the fact that the \( G \)-isotropy group \( G_{x_0} = K \) is dimension–theoretically a maximal subgroup of \( G \). (It is here that we use the assumption that \( G_0 \) is not of hermitian holomorphic type.)

We are now in a position to prove the main theorem of this section ([14], [FH]).

**Theorem 9.6** If \( G_0 \) is not of hermitian holomorphic type, \( B \subset G \) is an Iwasawa–Borel subgroup, and \( H \) is a \( B \)-invariant hypersurface in \( \Omega = G/K \), then \( \Omega_H \) is Kobayashi hyperbolic.

**Proof.** We replace \( \varphi \) by its restriction to \( \Omega \) and only discuss that map. By definition every section \( \tau \in V \) is the pull-back \( \varphi^*(\tau) \) of a hyperplane section. Thus, there is a uniquely defined \( B \)-hypersurface \( \tilde{H} \) in \( \mathbb{P}(V^*) \) with \( \varphi^{-1}(\tilde{H}) = H \). Let \( \Omega_H = \mathbb{P}(V^*) \setminus \bigcup_{g \in G_0} g(\tilde{H}) \). Applying Corollary 9.3 to \( \mathbb{P}(V^*) \) and \( S := G_0 \cdot \tilde{H} \subset \mathbb{P}(V) \), it follows that the domain \( \Omega_H \) is Kobayashi hyperbolic. Furthermore, the connected component of \( \varphi^{-1}(\Omega_H) \) which contains the base point \( x_0 \) is just the original domain \( \Omega_H \). Since holomorphic
maps are distance decreasing and $\phi : \Omega_H \to \widetilde{\Omega}_H$ is locally holomorphic, it follows that $\Omega_H$ is also Kobayashi hyperbolic.

Let us summarize what we have presented up to this point.

Summary 9.7 If $G_0$ is of hermitian holomorphic type, then either $\Omega$ is the compact hermitian symmetric space dual to the bounded symmetric domain $B = G_0/K_0$, and either $\Omega_W(D) = \Omega_S(D) = \overline{B}$. If $G_0$ is of hermitian nonholomorphic type, then $\Omega$ is affine and $\Omega_{AG} = \Omega_I = \Omega_W(D) = \Omega_S(D) = B \times \overline{B}$.

If $G_0$ is not of hermitian type, then, with the usual convention that $\Omega_W(D) \subset \Omega = G/K$, for $H$ any complex hypersurface which is invariant by an Iwasawa-Borel subgroup. All of the domains are $G_0$-invariant, Stein and Kobayashi hyperbolic and therefore the $G_0$-actions are proper in every case.

10 The Maximal Domain of Hyperbolicity.

Recall the basic sequence of inclusions

\[(10.1) \quad \Omega_{AG} = \Omega_I \subset \Omega_W(D) = \Omega_S(D) \subset \Omega_H \]

for any complex hypersurface $H \subset \Omega = G/K$ which is invariant under an Iwasawa-Borel subgroup of $G$, when $D$ is not of hermitian holomorphic type. All of these domains are $G_0$-invariant, Stein and Kobayashi hyperbolic; see Summary 9.7.

Our goal here is to outline the proof of the following main theorem of [FH].

**Theorem 10.2** The only $G_0$-invariant, Stein, Kobayashi hyperbolic domain in $\Omega$ which contains $\Omega_{AG}$ is $\Omega_{AG}$ itself.

This implies that the sequence (10.1) of inclusions is a sequence of equalities in the non–hermitian case. With the classification in the hermitian case (Section 8), this yields the following.

**Theorem 10.3** In the hermitian holomorphic case, $\Omega_W(D)$ is the bounded domain $B$ or $\overline{B}$ associated to $G_0$. In all other cases, $\Omega_{AG} = \Omega_I = \Omega_W(D) = \Omega_S(D)$.

The proof of Theorem 10.2 involves three main steps: (1) Understanding the invariant theory of the $G_0$–action on $\Omega$; in particular, the orbit structure on $\text{bd}(\Omega_{AG})$. (2) For every generic boundary point $z$, determining an $\mathfrak{sl}_2$-triple defined over $\mathbb{R}$ so that the orbit $S_z = Q_2$ intersects $\Omega_{AG}$ in a 2-dimensional polydisk which is the $\Omega_{AG}$ for the group $S$. (3) Proving Theorem 10.2 in the case of $S = \text{SL}_2(\mathbb{C})$.

All details which are omitted in the following sketch can be found in [FH]. To be consistent with the notation of that paper, we let $x \in \Omega$ be a general point and $x_0 \in \Omega$ the chosen base point with $G_{x_0} = K$.

**The linear model.**

Using a classical linearizing map (see [M1]) we $G_0$-equivariantly embed $\Omega$ in a linear space where Jordan decomposition can be used in an optimal way.

**The basic map.** Let $\sigma$ denote complex conjugation of $g$ over $g_0$. Write $\tau$ for the complex-linear extension to $g$ of the Cartan involution $\theta$ of $g_0$, so $\tau$ is the holomorphic involution of $g$. 

25
with fixed point set \( \mathfrak{f} \). We extend \( \theta \) conjugate–linearly to \( \mathfrak{g} \), so \( \theta \) becomes the conjugate–linear involution of \( \mathfrak{g} \) whose fixed point set is the compact real form \( \mathfrak{g}_0 \), in other words \( \theta \) becomes a Cartan involution of \( \mathfrak{g} \).

Define \( \eta : G \to \text{Aut}_\mathbb{R}(\mathfrak{g}) \) by \( \eta(x) = \sigma \circ \text{Ad}(x) \circ \tau \circ \text{Ad}(x^{-1}) \). We regard \( \sigma \), \( \tau \) and \( G \) as operating on \( \text{Aut}_\mathbb{R}(\mathfrak{g}) \) by conjugation, the action of \( G \) being via \( \text{Ad} \). Let \( N \) denote the normalizer of \( K \) in \( G \). The basic properties of \( \eta \) are as follows.

- \( \eta \) is right \( N \)-invariant and defines an embedding \( G/N \hookrightarrow \text{Aut}_\mathbb{R}(\mathfrak{g}) \).
- \( \eta \) is equivariant with respect to the left action of \( G_0 \) on \( G \).
- The image \( \text{Im}(\eta) \) is closed, \( \sigma \)-invariant and \( \tau \)-invariant, \( \sigma(\eta(x)) = \eta(x)^{-1} \) and \( \tau(\eta(x)) = \eta(\tau(x)) \).

To see that \( \text{Im}(\eta) \) is closed we use a basic lemma of invariant theory (see [Hu], p. 117):

**Lemma 10.4** Let \( V \) be a finite dimensional real vector space, \( H \) a closed reductive algebraic subgroup of \( \text{GL}_\mathbb{R}(V) \) and \( s \in \text{GL}_\mathbb{R}(V) \) an element which normalizes \( H \). Regard \( H \) as acting on \( \text{GL}_\mathbb{R}(V) \) by conjugation.

Then, for a semisimple \( s \) the orbit \( H.s \) is closed.

To prove that \( \text{Im}(\eta) \) is closed it is enough to show that \( G.\tau = \{ \text{Ad}(g)\tau\text{Ad}(g)^{-1} : g \in G \} \) is closed in \( \text{Aut}_\mathbb{C}(\mathfrak{g}) \). Since \( \tau \) is semi-simple and normalizes \( G \) in this representation, this is then immediate.

From now on, unless otherwise stated, we view \( \eta \) as a map \( \eta : \Omega \to \text{Aut}_\mathbb{R}(\mathfrak{g}) \) which is is essentially a diffeomorphism onto its closed image. In this language Lemma 10.4 also leads to

**Proposition 10.5** If \( \eta(x) = s \) is semi-simple, then \( G_0.s \) is closed.

**Proof.** It is enough to show that \( G_0.s \) is closed. By Lemma 10.4 the complex orbit \( G.s \) is closed. Define \( \hat{\sigma} : \text{Aut}_\mathbb{R}(\mathfrak{g}) \to \text{Aut}_\mathbb{R}(\mathfrak{g}) \) by \( \hat{\sigma}(\varphi) = (\sigma(\varphi))^{-1} \). Here \( \sigma \) acts by conjugation as usual. Note that \( \text{Im}(\eta) \) belongs to the fixed point set \( \text{Fix}(\hat{\sigma}) \). Since \( G.s \cap \text{Fix}(\hat{\sigma}) \) consists of only finitely many \( G_0 \)-orbits [Br], it follows that \( G_0.s \) is closed.

**JORDAN DECOMPOSITION.** Let \( \eta(x) = u \cdot s \) denote the Jordan decomposition of an element in \( \text{Im}(\eta) \). Then the unipotent factor \( u \in \text{Aut}_\mathbb{R}(\mathfrak{g})^u \), and \( u = \exp(\text{ad}(\nu)) = \text{Ad}(\exp(\nu)) \) for some nilpotent \( \nu \in \mathfrak{g}_0 \). If \( \varphi \in \text{Aut}_\mathbb{R}(\mathfrak{g}) \) let \( \mathfrak{g}^\varphi \) denote its fixed point set. The fact \( su = us \) can be expressed \( \nu \in \mathfrak{g}^s \). Also, \( \sigma(\eta(x)) = \eta(x)^{-1} \) implies \( \eta \in i\mathfrak{g}_0 \). Now compute \( \eta(\exp(\nu).x) = s \). In summary we have the following result, describing a lifting of the Jordan decomposition.

**Proposition 10.6** For \( x \in \Omega \) with Jordan decomposition \( \eta(x) = u \cdot s \) there exists a nilpotent element \( \nu \in \mathfrak{g}^s \cap i\mathfrak{g}_0 \) such that \( u = \text{Ad}(\exp(\nu)) \) and \( \eta(\exp(\nu).x) = s \).

**Orbit structure.**

Here we outline some basic information on the orbit structure of the \( G_0 \)-action on \( \Omega \). The main objective is an understanding of the \( G_0 \)-action on \( \text{bd}(\Omega_{AG}) \).

**Closed orbits.** In any invariant theoretic situation it is of central importance to move in a systematic way from a point in a non-closed orbit to a closed orbit in its closure. Here we accomplish this by means of special \( \mathfrak{sl}_2 \)-triples.

Since \( \sigma(s) = s^{-1} \) and \( s \) is semisimple, \( \mathfrak{g}^s \) is a \( \sigma \)-invariant reductive subalgebra of \( \mathfrak{g} \). Let \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \) be its \( \sigma \)-decomposition. Now apply the Jacobson–Morozov Theorem.

**Lemma 10.7** Let \( e \in \mathfrak{g}^s \cap i\mathfrak{g} \) be nonzero and nilpotent. There exists an \( \mathfrak{sl}_2 \)-triple \( (e, h, f) \) in \( \mathfrak{g}^s \) (i.e., \( [e, f] = h, [h, e] = 2e, [h, f] = -2f \)) such that \( e, f \in \mathfrak{q} \) and \( h \in \mathfrak{h} \).
Using this $sl_2$-triple we are able to explicitly move to a closed orbit.

Proposition 10.8 If $\eta(x) = us$ is the Jordan decomposition, then the orbit $G_0.\eta(x) = G_0.(su)$ contains the closed orbit $G_0.s \subset \text{Im}(\eta)$ in its closure $\text{cl}(G_0.\eta(x))$. In particular, $G_0.\eta(x)$ is closed if and only if $\eta(x)$ is semi-simple.

Proof. Let $u = \text{Ad}(\exp(\nu))$ with $\nu$ as in Proposition [10.4]. Hence, by Lemma [10.7] there is a $sl_2$-triple $(\nu, h, f)$ ($e = \nu$) such that $[th, \nu] = 2t\nu$, i.e., $\text{Ad}(\exp(th))(\nu) = e^{2t\nu}$ for every $t \in \mathbb{R}$. Note also that $\exp(\mathbb{R}H) \subset G_0 \cap \exp(g^*)$ by construction of the $sl_2$-triple. It follows that

$$\eta(\exp(th) \cdot x) = \exp(th).\!(us) = \text{Ad}(\exp(th)) \cdot us \cdot \text{Ad}(\exp(-th)) =$$

$$= \text{Ad}(\exp(th)) \cdot u \cdot \text{Ad}(\exp(-tH)) \cdot s =$$

$$= \text{Ad}(\exp(th)) \text{Ad}(\exp(\nu)) \text{Ad}(\exp(-tH)) \cdot s =$$

$$= \text{Ad}(\exp(e^{2t\nu})) \cdot s.$$  

For $t \rightarrow -\infty$ it follows $\exp(tH).\!(us) = \text{Ad}(\exp(e^{2t\nu})) \cdot s \rightarrow s$. Hence, the closed orbit $G.s$ lies in the closure of $G.(us)$. In particular $G_0.(us)$ is not closed if $u \neq 1$, i.e., if $\eta(x)$ is not semi-simple. This, together with Proposition [10.9] implies that $G.\eta(x)$ is closed if and only if $\eta(x)$ is semi-simple. Recall that the image $\text{Im}(\eta)$ is closed. This forces $s \in \text{Im}(\eta)$ and the proof is now complete. □

Elliptic elements. An element $x \in \Omega$ is called elliptic if $\eta(x)$ is elliptic in the sense that $t \mapsto \eta(\exp(t\nu)\cdot x)$ is smooth for every $\nu \in \mathfrak{sl}_2$. If $x \in G_u$, so $\theta(x) = x$, then $\theta(\eta(x)) = \eta(x)\theta$, so $\eta(x)$ is elliptic. Therefore $G_u.x_0 \subset \Omega_{\text{ell}}$. Since $\Omega_{\text{ell}}$ is $G_0$-invariant now $G_0.\exp(i\alpha_0).x_0 \subset \Omega_{\text{ell}}$. The opposite inclusion follows via classical methods. We have proved

Proposition 10.9 $\Omega_{\text{ell}} = G_0.\exp(i\alpha_0.x_0)$

The following is a key ingredient for understanding the $G_0$-orbit structure in $\text{bd}(\Omega_{\text{AG}})$.

Proposition 10.10 $\exp(i\alpha_0).x_0 \cap \text{cl}(\Omega_{\text{AG}}) = \text{cl}(\exp(i\omega_0).x_0)$

Proof. If $x \in \text{cl}(\exp(i\omega_0).x_0)$, then it is elliptic and therefore its orbit $G_0.x$ is closed. In other words $\exp(i\alpha_0).x_0 \cap \text{cl}(\Omega_{\text{AG}}) \subset \text{cl}(\exp(i\omega_0).x_0)$.

For the opposite inclusion, observe that if $s, s' \in \exp(i\alpha_0).x_0$ and $s' \in G_0.s$, then $s' = k_0(s)$ for some element $k_0$ of the Weyl group. Thus, if $s \in \text{cl}(\exp(i\omega_0).x_0)$, then $s' \in \text{cl}(\exp(i\omega_0).x_0)$ as well. Therefore, in order to prove the opposite inclusion it is enough to show that, given $s' \in \exp(i\alpha_0).x_0 \cap \text{cl}(\Omega_{\text{AG}})$, there exists $s \in \text{cl}(\exp(i\omega_0).x_0)$ with $s' \in G_0.s$.

Given $s'$ as above, there exist sequences $\{s_n\} \subset \exp(i\omega_0).x_0$ and $\{s'_n\} \subset \Omega_{\text{AG}}$ such that $s'_n \in G_0.s_n$, $s'_n \rightarrow s'$ and $s_n \rightarrow s \in \text{cl}(\exp(i\omega_0).x_0)$. Consider the (real) categorical quotient map $\pi : \text{Aut}_R(\mathfrak{g}) \rightarrow \text{Aut}_R(\mathfrak{g})//G_0$. It is continuous, the base is Hausdorff and in every fiber there is exactly one closed $G_0$-orbit. Since $\pi(s_n) = \pi(s'_n)$, it follows that $G_0.s = G_0.s'$. □

Corollary 10.11 Let $\Omega_{\text{cl}}$ denote $\{x \in \Omega : G_0.x$ is closed $\}$. Then $\Omega_{\text{cl}} \cap \text{cl}(\Omega_{\text{AG}}) = G_0.\text{cl}(\exp(i\omega_0).x_0) = \Omega_{\text{ell}} \cap \text{cl}(\Omega_{\text{AG}})$.

Proof. From Proposition [10.9], $\Omega_{\text{AG}} \subset \Omega_{\text{ell}}$. By continuity, the semi-simple part of $\eta(x)$ is elliptic for every $x \in \text{cl}(\Omega_{\text{AG}})$. Thus $\Omega_{\text{cl}} \cap \text{cl}(\Omega_{\text{AG}}) \subset \Omega_{\text{ell}} \cap \text{cl}(\Omega_{\text{AG}})$, because elements of closed orbits are semi-simple. Proposition [10.9] gives $\Omega_{\text{cl}} \cap \text{cl}(\Omega_{\text{AG}}) \subset G_0.\text{cl}(\exp(i\omega_0).x_0)$, and from Proposition [10.10] it follows that $G_0.\text{cl}(\exp(i\omega_0).x_0) \subset \Omega_{\text{ell}} \cap \text{cl}(\Omega_{\text{AG}})$. So we have $\Omega_{\text{cl}} \cap \text{cl}(\Omega_{\text{AG}}) \subset G_0.\text{cl}(\exp(i\omega_0).x_0) = \Omega_{\text{ell}} \cap \text{cl}(\Omega_{\text{AG}})$. Finally, if $x \in \Omega_{\text{ell}}$, then in particular it is semi-simple and $G.x$ is closed. This proves the remaining inclusion. □

27
Existence of a $Q_2$-slice.

**Hilbert Lemma.** We have fixed $a_0$. Let $G_{0,y} \subset \text{bd}(\Omega_{AG})$ be a non-closed orbit. We determine a base point $z \in G_{0,y}$ with $\eta(z) = u \cdot s$ such that the point $x_1$ corresponding to $s$ is in $\text{bd}(\exp(i\omega_0).x_0)$. That uses the nilpotent element $\nu$ associated to $u$, and then an $s\mathfrak{l}_2$-triple $(\nu = e, h, f)$ defined over $\mathbb{R}$ with $z$ in the orbit $S.x_1$ of the corresponding complex group. We regard this as an analogue of the Hilbert Lemma in the case of actions of reductive complex Lie groups.

**Lemma 10.12** Every non-closed $G_{0}$-orbit in $\text{bd}(\Omega_{AG})$ contains a point $z = \exp(\nu) \cdot \exp(i\xi) \cdot x_0$ where $\xi \in \text{bd}(\omega_0) \subset a_0$ and where $\nu \in g^{\eta(\exp(i\xi))} \cap i\mathfrak{g}_{0}$ is nonzero and nilpotent.

**Proof.** Let $\eta(y) = su$ be the Jordan decomposition and let $\nu \in g^s \cap i\mathfrak{g}_{0}$ as in Proposition 10.6. Then $\eta(y) = \eta(\exp(-\frac{i}{2}\nu) \exp(\frac{i}{2}\nu) \cdot y) = \text{Ad}(\exp\nu) \circ \eta(\exp(\frac{i}{2}\nu) \cdot y) = u \cdot s$. By Proposition 10.5 and Proposition 10.4 the semisimple element $\eta(\exp(\frac{i}{2}\nu) \cdot y)$ is elliptic. Hence, Proposition 10.9 implies the existence of $g \in G$ and $\xi \in \text{bd}(\omega_0)$ such that $\exp(\frac{i}{2}\nu) \cdot y = g^{-1} \exp(i\xi) \cdot x_0$. Define $e := \text{Ad}(g)(-\frac{i}{2}\nu)$. Then $g \cdot y = \exp(e) \exp(i\xi) \cdot y = \exp(e) \exp(i\xi) \cdot x_0$. Finally, $e \in g^{s,s} = g^{\eta(\exp(i\xi))}$, and Lemma 10.12 is proved. \qed

Let $e$ be as above and $S$ be the complex group determined by the above $s\mathfrak{l}_2$-triple. Direct calculation shows that $S_{x_1}^0 \cong \mathbb{C}^*$. Thus the orbit $S.x_1$ is equivariantly biholomorphic to either the 2-dimensional affine quadric or to the complement of a smooth quadric curve in $\mathbb{CP}^2$. The former is naturally realized as the complement of the diagonal in $\mathbb{CP}_1 \times \mathbb{CP}_1$ and the latter as its quotient by the $\mathbb{Z}_2$-action defined by reversing its factors. At the level of homogeneous spaces the former is $S/T$, where $T$ is the complexification of a (compact) maximal torus $T_0$, and the latter is $S/N$, where $N$ is the normalizer of $T$ in $S$.

In order to remind the reader of the connection to the quadric, the orbit $S.x_1$ is referred to as a $Q_2$-slice whenever its intersection with $\Omega_{AG}$ is an $\Omega_{AG}$ associated to the real form $S_0$ of $S$ which is defined by the restriction of $\sigma$, the complex conjugation of $G$ over $G_{0}$.

**Genericity.** Starting with an arbitrary $G_{0}$-orbit on $\text{bd}(\Omega_{AG})$ it is difficult to determine how the $S$-orbit $S.x_1$ intersects $\Omega_{AG}$. However, for $G_{0}$-orbits of generic boundary points this will be relatively straightforward. Here a point $z \in \text{bd}(\Omega_{AG})$ is called generic if $G_{0,z}$ is not closed and if the point $x_1 = \exp(i\xi)x_0$ constructed above by the Hilbert Lemma is a smooth point of $\text{bd}(\exp(i\omega_0).x_0)$.

Recall that $\text{bd}(\omega_0)$ is defined by hyperplanes. We refer to the points in $\text{bd}(\exp(i\omega_0).x_0)$ which correspond to points which are contained in two or more hyperplanes as corners. Let $\mathcal{E}$ denote the $G_{0}$-saturation of the set of such corners, i.e., points $z$ of $\text{bd}(\Omega_{AG})$ such that $G_{0,z}$ has a corner in its closure. Let $\mathcal{C}$ be the set of points $z \in \text{bd}(\Omega_{AG})$ such that $G_{0,z}$ is closed, so $\mathcal{C} = G_{0}.\text{bd}(\exp(i\omega_0).x_0)$. Finally, let $\text{bd}(\Omega_{AG})_{\text{gen}}$ be the closure of the complement of $\mathcal{E} \cup \mathcal{C}$ in $\text{bd}(\Omega_{AG})$. It follows that this is contained in the set of generic points in the above sense. A careful look at the invariant theory for the $G_{0}$-action, leads to the following density result.

**Proposition 10.13** The set $\text{bd}(\Omega_{AG})_{\text{gen}}$ is open and dense in $\text{bd}(\Omega_{AG})$.

Using detailed knowledge of the $G_{0}$-isotropy groups at the smooth points of $\text{bd}(\exp(i\omega_0).x_0)$, one proves the desired result on the existence of $Q_2$-slices.

**Proposition 10.14** At every generic boundary point there exists a $Q_2$-slice.

Given $y \in \text{bd}(\Omega_{AG})_{\text{gen}}$ we of course move it via $G_{0}$ to an optimal point $z$ in $G_{0}.y$ so that $S.z$ contains the smooth boundary point $x_1 \in \text{bd}(\exp(i\omega_0).x_0)$ as above.
Analysis of a $Q_2$-slice.

Let $S = SL_2(\mathbb{C})$ and let $S_0 = SL_2(\mathbb{R})$ be embedded in $S$ as the subgroup of real matrices. As usual $K_0 = SO_2(\mathbb{R})$ and $K = SO_2(\mathbb{C})$. $D_0$ and $D_\infty$ denote the open $S_0$-orbits in $CP_1$ containing the respective $K_0$-fixed points $0$ and $\infty$.

$S$ acts diagonally on $Z = CP^1 \times CP^1$ with one open orbit $\Omega$, the complement of the diagonal $\text{diag}(CP^1)$ in $Z$. $\Omega$ is the complex symmetric space $S/K$. Note that in $CP^1 \times CP^1$ there are 4 open $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$-orbits: the bi-disks $D_0 \times D_\beta$ for any pair $(\alpha, \beta)$ from $\{0, \infty\}$. As $S_0$-spaces, the domains $D_0 \times D_\infty$ and $D_\infty \times D_0$ are equivariantly biholomorphic; further, they are subsets of $\Omega$, and the Riemannian symmetric space $S_0/K_0$ sits in each of them as the totally real orbit $S_0(0, \infty)$ or $S_0(\infty, 0)$. Each can be considered as the $\Omega_{AG}$ associated to $S$ and the real form $S_0$:

$$\Omega_{AG} = D_0 \times D_\infty = S_0 \cdot \exp i\omega_0 \cdot (0, \infty) \quad \text{or} \quad D_\infty \times D_0 = S_0 \cdot \exp i\omega_0 \cdot (\infty, 0).$$

We choose the first: $\omega_0 = (\frac{-i}{4}, \frac{i}{4})h_\alpha$ where $h_\alpha \in a$ is the normalized coroot, i.e., $\alpha(h_\alpha) = 2$.

Our main point here is to understand $S_0$-invariant Stein domains in $\Omega$ which properly contain $\Omega_{AG}$. By symmetry we may assume that said domain has non-empty intersection with $D_0 \times D_\infty$. Observe that $(D_0 \times D_0) \cap \Omega = D_0 \times D_0 \setminus \text{diag}(D_0)$, and other than diag($D_0$) all $S_0$-orbits in $D_0 \times D_0$ are closed real hypersurfaces. If $p \in D_0 \times D_0 \setminus \text{diag}(D_0)$ let $\Omega(p)$ be the domain bounded by $S_0 \cdot p$ and diag($D_0$). We shall show that a function which is holomorphic in a neighborhood of $S_0 \cdot p$ extends holomorphically to $\Omega(p)$.

$$\Sigma := \{(-s, s) : 0 \leq s < 1\} \subset D_0 \times D_0$$ is a geometric slice for the action of $S_0$ on $D_0 \times D_0$. We say that a (1-dimensional) complex curve $C \subset C^2 \subset Z$ is a supporting curve for $bd(\Omega(p))$ at $p$ if $C \cap c(\Omega(p)) = \{p\}$. Here $c(\Omega(p))$ denotes the topological closure in $D_0 \times D_0$.

**Proposition 10.15** If $p \in D_0 \times D_0 \setminus \text{diag}(D_0)$ there is a supporting curve for $bd(\Omega(p))$ at $p$.

**Proof.** We consider $D_0 \subset \mathbb{C}$ as the unit disc. We need only construct a supporting curve $C \subset C^2$ at each point $p_s = (-s, s) \in \Sigma, s \neq 0$. Define $C_s := \{-s + z, s + z : z \in \mathbb{C}\}$. To prove $C_s \cap c(\Omega(p_s)) = \{p_s\}$ let $d : D_0 \times D_0 \to \mathbb{R}$ be the distance function of the Poincaré metric of $D_0$. It is an $S_0$-invariant, and its values parameterize the $S_0$-orbits on $D_0 \times D_0$.

We now claim that $d(-s + z, s + z) \geq d(-s, s) = d(p_s)$ for $z \in \mathbb{C}$ and $(-s + z, s + z) \in D_0 \times D_0$, with equality only for $z = 0$, i.e., $C_s$ touches $c(\Omega(p_s))$ only at $p_s$. To prove this, we compare the Poincaré length of the Euclidean segment $\text{seg}(z - s, s + z)$ in $D_0$ with that of $\text{seg}(-s, s)$. Writing the corresponding integral for the length, it is clear from a glance at the integrand that $d(-s + z, s + z) > d(-s, s)$ for $z \neq 0$. This completes the proof. \qed

From the above construction the boundary hypersurfaces $S(p)$ are strongly pseudoconvex from the viewpoint of diag($D_0$). The smallest Stein domain containing a $S_0$-invariant neighborhood of $S_0(p)$ is $\Omega(p) \setminus \text{diag}(D_0)$, so the following is immediate.

**Corollary 10.16** If $p \in D_0 \times D_0 \setminus \text{diag}(D_0)$ and $f$ is holomorphic on a neighborhood of $S_0 \cdot p$ then $f$ extends holomorphically to $\Omega(p) \setminus \text{diag}(D_0)$. If $p \in D_\infty \times D_\infty \setminus \text{diag}(D_\infty)$ the analogous statement holds.

The set of generic boundary points is the union of the two $S_0$-orbits, $bd_{\text{gen}}(D_0 \times D_\infty) = (bd(D_0) \times D_\infty) \cup (D_0 \times bd(D_\infty))$. Let $z \in bd(D_0) \times D_\infty$ (or $z \in D_0 \times bd(D_\infty)$, respectively).

**Corollary 10.17** Let $\hat{\Omega} \subset Q_2 \subset CP^1 \times CP^1$ be an $S_0$-invariant Stein domain that contains $D_0 \times D_\infty$ and its boundary point $z$. Then $\hat{\Omega}$ also contains $(D_0 \times CP^1) \setminus \text{diag}(CP^1)$ (or also contains $(CP^1 \times D_\infty) \setminus \text{diag}(CP^1)$, respectively).
Proof. Let $B$ be a ball around $z$ which is contained in $\widehat{\Omega}$. For $p \in B(z) \cap (D_0 \times D_0)$ sufficiently close to $z$, $S_0 \cdot q \subset \widehat{\Omega}$ for all $q \in B(z) \cap (D_0 \times D_0)$. The result follows from Corollary 10.16. □

If $\widehat{\Omega}$ is as in Corollary 10.17, then the fibers of the projection of $\widehat{\Omega} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ to the first factor $\mathbb{CP}^1$ can be regarded as non-constant holomorphic curves $f : \mathbb{C} \to \widehat{\Omega}$. In particular, Corollary 10.18.

Characterization of cycle domains.

In the previous sections, in order to avoid unnecessary notation, we have often replaced $\Omega$ by its finite cover $\Omega \subset G/\widetilde{K}$: component of $C_0$ in $\{gC_0 : g \in G \text{ and } gC_0 \subset D\} \subset G/\widetilde{K}$.

Proposition 10.20 below, shows that in fact the covering $G/\widetilde{K} \to G/K$ is bijective on $\Omega \subset G/\widetilde{K}$, inducing a holomorphic diffeomorphism $\pi : G/K \to \Omega \subset G/\widetilde{K}$ and justifying the above–mentioned replacement. For our main theorem, however, even though we have not yet come to the proof of Proposition 10.20, we still consider $\Omega \subset G/K$.

Theorem 10.19 Either we are in the hermitian holomorphic case and $\Omega \subset G/K$ is a cell or $\Omega = \Omega_N = \Omega cr = \Omega W(D) = \Omega S(D)$.

Proof. If $G_0$ is of hermitian type, then the result is contained in Theorem 8.2 or see [HW3] or [WZ]. Otherwise we have the inclusions and equalities (10.1) from Summary 9.7. By Theorem 10.2 above, $\Omega H = \Omega AG$, and consequently all of those inclusions are equalities. □

Finally we view the cycle space as it really is, and verify that the standard projection $\pi : G/K \to G/\widetilde{K}$ restricts to a holomorphic diffeomorphism of $\Omega \subset G/\widetilde{K}$ onto $\Omega \subset G/K$. The projection $\pi$ is given as identification under the right action of the finite group $\Gamma = K \setminus \widetilde{K}$ on $G/K$. $\Gamma$ permutes the components of $\{gK : g \in G \text{ and } gC_0 \subset D\} = \pi^{-1}(\{gC_0 : g \in G \text{ and } gC_0 \subset D\})$.

Thus $\Omega \subset G/K$ is the quotient of $\Omega \subset G/K$ by its stabilizer in $\Gamma$.

By Theorem 10.19, $\Omega \subset G/K = \Omega AG$. Since $\Omega AG$ is a cell, a finite group of diffeomorphisms can act freely on it only if it is trivial. Thus one may indeed regard $\Omega \subset G/K$ as being in $\Omega = G/K$.

We summarize this as follows.

Proposition 10.20 The restriction $\pi : \Omega \subset G/K \to \Omega \subset G/K$ of the projection $G/K \to G/\widetilde{K}$ is biholomorphic. In particular, $\Omega \subset G/K$ is a cell.

The following Corollary is contained in [HW3]. Also see [HW4].

Corollary 10.21 In all cases, $\Omega \subset G/K$ is a contractible Stein manifold.
Part III: Applications and Open Problems.

In this Part we go more closely into applications of the complex geometric methods described and developed in Part II.

11 Recent Results on the Double Fibration Transform.

We continue the discussion of double fibration transforms from Section 4, taking advantage of the material just described in Part II.

As explained above in Proposition 4.2, there is an Iwasawa decomposition \( G_0 = K_0 A_0 N_0 \) such that the Schubert slice \( \Sigma := A_0 N_0(\omega_0) \subset D \) meets every cycle \( C \in \Omega_W(D) \) transversally in a single point within \( D \). That gives a map

\[
(11.1) \quad \phi : \Omega_W(D) \to \Sigma := A_0 N_0(\omega_0) \text{ by } C \mapsto (\Sigma \cap C) \subset D.
\]

Note that \( \phi^{-1}(z) \) consists of all cycles \( C \in \Omega_W(D) \) that contain \( z \), so \( \phi^{-1}(z) = \mu^{-1}(z) =: F \).

Let \( J_0 \) denote the isotropy subgroup of \( A_0 N_0 \) at \( \omega_0 \), and let \( F_0 = \mu^{-1}(\omega_0) \). Note that \( J_0 \) acts on \( F_0 \). Realize (11.1) as the \( A_0 N_0 \)-homogeneous fiber bundle \( (A_0 N_0) \times_{J_0} F_0 \to A_0 N_0/J_0 \). The subsets \( \Omega_W(D) \) and \( F_0 := \{ C \in \Omega \mid \omega_0 \in C \} \) are semialgebraic in \( \Omega \), so their intersection \( F_0 \) has only finitely many topological components. As \( \Omega_W(D) \) is simply connected, it follows that \( \Sigma = A_0 N_0/J_0 \) is a solvmanifold with finite fundamental group. Thus \( \Sigma \) is \( C^1 \)-diffeomorphic to a cell. Hence the fibration (11.1) is trivial. By Proposition 10.20, its total space is a cell. So now the base and total space of (11.1) are cohomologically trivial, and thus the same holds for the fiber \( F \). We have proved

**Theorem 11.2** \[HW4\] Let \( F \) denote the fiber of the holomorphic fibration \( \mu : \mathcal{I}(D) \to D \). Then \( F \) is connected and \( H^r(F; \mathbb{C}) = 0 \) for all \( r > 0 \). In particular (4.4) is satisfied for every \( q \), and the double fibration transforms \( P : H^q(D; \mathcal{E}) \to H^p(M; \mathcal{R}^q(\mu^* \mathcal{E})) \) are injective for all sufficiently negative \( \mathbb{E} \to D \).

**Remark 11.3** \[HW4\] The fiber space projection \( \phi : \Omega_W(D) \to \Sigma \) is the restriction to open subsets of a holomorphic bundle projection \( \tilde{\phi} : AN \times J \tilde{F}_0 \to \mathcal{O} \), as follows. Let \( \tilde{F}_0 := \{ C \in \Omega \mid \omega_0 \in C \} \). The complex submanifold \( \mathcal{O} = B(\omega_0) \subset Z \) where \( B \) is a Borel subgroup of \( G \) that contains \( A_0 N_0 \). Thus \( \Sigma = \mathcal{O} \cap D \) is open in \( \mathcal{O} \) by the discussion of Schubert cells and Schubert slices in Theorem 7.3 above. \( A \) and \( N \) are the respective complexifications of \( A_0 \) and \( N_0 \). \( \mathcal{I}(D) \) is open in \( AN(\tilde{F}_0) \), which is the total space of \( \tilde{\phi} : AN \times J \tilde{F}_0 \to \mathcal{O} \). The connection with \( \phi : \Omega_W(D) \to \Sigma \) is that \( \Omega_W(D) = A_0 N_0(\tilde{F}_0) \), which is open in \( AN(\tilde{F}_0) = AN \times J \tilde{F}_0 \), where \( J \) is the isotropy subgroup of \( AN \) at \( \omega_0 \). Since it is the restriction of \( \tilde{\phi} \), the map \( \phi : \Omega_W(D) \to \Sigma \) is holomorphic.

Now we have adequately addressed the injectivity requirement (1.13) for the double fibration transform of a flag domain, and we turn to the question (1.14) of its image. Since the Stein manifold \( \Omega_W(D) \) is contractible, every holomorphic vector bundle \( \mathbb{E} \to \Omega_W(D) \) is holomorphically trivial, and in particular the Leray derived bundles \( \mathbb{E}^\dagger = \mathbb{H}^q(\mathcal{I}(D); \mu^*(\mathcal{E}))_{|_{\mu^{-1}(\omega)}} \) over \( \Omega_W(D) \) are holomorphically trivial. Here we have two requirements for (1.14): we need

\[
(11.4) \quad \text{a canonical choice of holomorphic trivialization of } \mathbb{E}^\dagger \to \Omega_W(D), \quad \text{and}
\]

\[
(11.5) \quad \text{an explicit (in that trivialization) system of PDE that specifies the the image of } P.
\]

This is work in progress.
12 Unitary Representations of Real Reductive Lie Groups.

In this Section we look at some of the implications of the double fibration transform for representations of real reductive Lie groups.

Harish–Chandra’s analysis of the holomorphic discrete series can be viewed from the perspective of the double fibration transform as follows. Let \( G_0 \) be of hermitian type, \( B = G_0/K_0 \). In this case, of course, \( D = B = \mathcal{I}(D) = \Omega_W(D) \), the double fibration transform is the identity, (11.3) is completely standard, and the system (11.3) consists of the \( \mathcal{D} \) operator. Let \( E_\lambda \to B \) denote the homogeneous holomorphic hermitian vector bundle associated to the representation of \( G_0 \) having highest weight \( \lambda \), and the explicit holomorphic trivialization of \( \rho \) where \( \rho \) is half the sum of the positive roots and \( \beta \) is the maximal root.

Narasimhan and Okamoto \([NO]\) extended the Harish–Chandra construction to “almost all” discrete series representations of a real group \( G_0 \) of hermitian type, again always working over \( D = B = \mathcal{I}(D) = \Omega_W(D) \) where the double fibration transform is more or less invisible.

The double fibration transform first became visible, at least in degenerate form, in Schmid’s holomorphic construction of the discrete series \([S3], [S5]\). There \( \Omega = G/B \) for some Borel subgroup \( B \) and \( D = G_0/T_0 \) where \( T_0 \) is a compact Cartan subgroup, \( T_0 \subset K_0 \subset G_0 \). Only the “real form” \( \phi : D \to G_0/K_0 \) of the double fibration appears: there \( G_0/K_0 \) appears instead of the cycle \( \Omega_W(D) \); correspondingly \( D \) appears instead of the incidence space \( \mathcal{I}(D) \). Injectivity of this real double fibration transform \( \phi^L(D;E) : H^0(D;E) \to H^0(G_0/K_0;\mathcal{E}_0) \) is given by Schmid's “Identity Theorem”. That theorem says that, under appropriate restrictions, a Dolbeault class \( \omega \in H^0(D;\mathcal{E}) \) is zero if and only if every restriction \( \omega \) is cohomologous to zero on every fiber of \( \phi : D \to G_0/K_0 \). This was extended a bit by Wolf \([W3]\), for flag domains of the form \( D \cong G_0/L_0 \) with \( G_0 \) general reductive and \( L_0 \) compactly embedded in \( G_0 \).

The double fibration transform first appeared in modern form in the paper \([WeW]\) of Wells and Wolf on Poincaré series and automorphic cohomology. The only restriction there was that \( D \cong G_0/L_0 \) with \( L_0 \) compact, and a small extension of the Identity Theorem was used to, in effect, prove injectivity of the double fibration transform.

The Penrose transform applies to the case \( D = SU(2,2)/S(U(1) \times U(1,2)) \). There \( L_0 \) is noncompact, and perhaps that is the first such case to be studied carefully. See \([BE]\). Background work on interesting flag domains with noncompact isotropy includes, of course, parts of Berger’s classification \([B]\) of semisimple symmetric spaces, Wolf’s study \([W]\) of isotropic pseudo–riemannian manifolds, and of course \([W2]\). Important cases of construction of unitary representations using double fibration transforms on flag domains with noncompact isotropy were studied in Dunne–Zierau \([DZ]\) and Patton–Rossi \([PR2]\). This area was first studied systematically in Wolf–Zierau \([WZ2]\).

Finally, as noted in \([W7]\), there are indications of a strong relation between the double fibration transforms of \([WZ2]\) and the construction of unitary representations by indefinite harmonic theory of (Rawnsley, Schmid & Wolf \([RSW]\)).

13 Variation of Hodge Structure.

In this Section we indicate the connection between Griffiths’ theory of moduli spaces for compact Kähler manifolds (period matrix domains and linear deformation spaces), on the one hand, and flag domains, cycle spaces and double fibration transforms on the other hand. Along the way we will sketch some relevant aspects automorphic cohomology theory as developed by Wallach,
Wells, Williams and Wolf.

For Griffiths’ theory see [Gr1] and [Gr2]. There are expositions contained in [Gr3], [S1], [We1], and [We2]. The first Hodge–Riemann bilinear relation specifies a complex flag manifold $Z = G/Q$ and the second Hodge–Riemann bilinear relation specifies an open $G_{0}$–orbit $D \subset Z$.

Let $X$ denote a compact Kähler manifold, $H^{r}_{0}(X; \mathbb{C})$ and $H^{r}_{0}(X; \mathbb{R})$ the complex and real spaces of primitive cohomology classes in degree $r$, and $H^{r}_{0}(X; \mathbb{C}) = \sum_{p+q=r} H^{p,q}_{0}(X; \mathbb{C})$ the decomposition by bidegree. This specifies the Hodge filtration $(F^{0} \subset F^{1} \subset \cdots \subset F^{r})$ of $H^{r}_{0}(X; \mathbb{C})$, where $F^{s} = \sum_{i \leq s} H^{i,-i}_{0}(X; \mathbb{C})$, and thus the complex flag $\mathcal{F}(X) = (F^{0} \subset F^{1} \subset \cdots \subset F^{r})$ where $u$ is the integer part of $(r-1)/2$. We have a nondegenerate bilinear form $b$ on $H^{r}_{0}(X; \mathbb{C})$ given (on Dolbeault representative differential forms) by $b(\xi, \eta) = (-1)^{(r+1)/2} \int \omega^{n-r} \wedge \xi \wedge \eta$. Here $\omega$ is the Kähler form of $X$. Evidently $b(H^{p,q}_{0}(X; \mathbb{C}), H^{p',q'}_{0}(X; \mathbb{C})) = 0$ unless $p + p' = r = q + q'$. Define $w(\xi) = (\sqrt{-1})^{p-q} \xi$ for $\xi \in H^{p,q}_{0}(X; \mathbb{C})$. One can formulate the Hodge–Riemann bilinear relations as (1) $b$ pairs $H^{p,q}_{0}(X; \mathbb{C})$ with its complex conjugate $H^{p,q}_{0}(X; \mathbb{C})$ and (2) $b(\xi, \eta) := b(w(\xi), \eta)$ is positive definite on $H^{r}_{0}(X; \mathbb{C})$.

If $r$ is even, say $r = 2t$, then $b$ is symmetric. It is positive definite on $H^{r-2t}_{0}(X; \mathbb{C}) \oplus H^{r-t}_{0}(X; \mathbb{C})$ for $i < t$, negative definite on $H^{t}_{0}(X; \mathbb{C})$. The (identity component of the) isometry group of $(H^{r}_{0}(X; \mathbb{C}), b)$ is the complex special orthogonal group $G = SO(2h + k; \mathbb{C})$ where $k = \dim H^{2}_0(X; \mathbb{C})$ and $h = \sum_{i < t} h_{i}$ with $h_{i} = \dim H^{r-2i}_{0}(X; \mathbb{C})$. The dimension sequence of the flag $\mathcal{F}(X)$ specifies the complex flag manifold $Z = G/Q$ consisting of all the flags $\mathcal{E} = (E^{0} \subset E^{1} \subset \cdots \subset E^{t-1})$ in $H^{r}_{0}(X; \mathbb{C})$ with $b(E^{t-1}, E^{t-1}) = 0$. The (identity component of the) isometry group of $(H^{r}_{0}(X; \mathbb{R}, b)$ is the identity component $G_{0} = SO(2h, k)^{0}$ of the real special orthogonal group $SO(2h, k)$. The second bilinear relation above shows that the isotropy subgroup $L_{0}$ of $G_{0}$ at $\mathcal{F}(X)$ is compact. It follows that $L_{0}$ is of the form $(U(h_{0}) \times \cdots \times U(h_{t-1}) \times SO(k)$. The flag $\mathcal{F}(X)$ ranges (as $X$ varies) in the open $G_{0}$–orbit

$$D = \{ \mathcal{E} \mid b \gg 0 \text{ on } E^{t-1} + E^{t-1} \} \cong SO(2h, k)/(U(h_{0}) \times \cdots \times U(h_{t-1}) \times SO(k)).$$

Here $U(h_{i})$ preserves $(H^{r-2i}_{0}(X; \mathbb{C}) \oplus H^{r-t}_{0}(X; \mathbb{C})) \cap H^{r}_{0}(X; \mathbb{R})$, and $SO(k)$ preserves $H^{t}_{0}(X; \mathbb{R})$.

If $r$ is odd, say $r = 2t - 1$, then $b$ is antisymmetric, so $H^{r}_{0}(X; \mathbb{C})$ has even dimension $2m$ and the isometry group of $(H^{r}_{0}(X; \mathbb{C}), b)$ is the complex symplectic group $G = Sp(m; \mathbb{C})$. The dimension sequence of the flag $\mathcal{F}(X)$ specifies the complex flag manifold $Z = G/Q$ consisting of all the flags $\mathcal{E} = (E^{0} \subset E^{1} \subset \cdots \subset E^{t-1})$ in $H^{r}_{0}(X; \mathbb{C})$ with $b(E^{t-1}, E^{t-1}) = 0$. The isometry group of $(H^{r}_{0}(X; \mathbb{R}, b)$ is the real symplectic group $G_{0} = Sp(m; \mathbb{R})$. As above, $G_{0}$ has compact isotropy subgroup $L_{0}$ at $\mathcal{F}(X)$, necessarily of the form $U(h_{0}) \times \cdots \times U(h_{t})$. The flag $\mathcal{F}(X)$ ranges (as $X$ varies) in the open $G_{0}$–orbit $D = \{ \mathcal{E} \mid b \text{ nondegenerate on each } \mathcal{F}(H^{r-2i}_{0}(X; \mathbb{C}) \oplus H^{r-t}_{0}(X; \mathbb{C})) \}$, which is realized as $Sp(m; \mathbb{R})/(U(h_{0}) \times \cdots \times U(h_{t}))$ where $h_{i} = \dim H^{r-2i}_{0}(X; \mathbb{C})$ as before.

Since $G_{0}$ has compact isotropy subgroup $L_{0}$ on $D$, we have $L_{0} \subset K_{0}$, and the holomorphic double fibration (4.19) is supplemented by maps $D = G_{0}/L_{0} \to G_{0}/K_{0} \subset \Omega_{W}(D)$.

Choose a basis $\{ \gamma_{1}, \ldots, \gamma_{r} \}$ of the space $H^{r}_{0}(X; \mathbb{Z})/(\text{torsion})$ of $r$–cycles on $X$. Given $\mathcal{F}(X)$ we have a basis $\{ \omega^{1}, \ldots, \omega^{u} \}$ of $H^{r-1}_{0}(X; \mathbb{C})$, then $H^{r-1}_{0}(X; \mathbb{C})$, continuing through the $b$–isotropic space of the flag $\mathcal{F}(X)$. That defines a $u \times v$ period matrix

$$\Pi(X) := \left( \begin{array}{cccc}
\int_{\gamma_{1}} \omega^{1} & \cdots & \int_{\gamma_{e}} \omega^{1} \\
\vdots & & \vdots \\
\int_{\gamma_{1}} \omega^{u} & \cdots & \int_{\gamma_{e}} \omega^{u}
\end{array} \right)$$

which of course specifies $\mathcal{F}(X)$. As in the case of period matrices of Riemann surfaces, one can change the basis $\{ \gamma_{i} \}$ by any integral element of $G_{0}$ and change the basis $\{ \omega^{j} \}$ by any element of
$G_0$ that does not change $\mathcal{F}(X)$. Thus the moduli space for $r$-forms of compact Kähler manifolds $X$ with given Hodge numbers $h_{p,q}^0 := \dim H^{p,q}_0(X; \mathbb{C})$, $p + q = r$, is the arithmetic quotient

$$\Gamma \backslash D = G_\mathbb{Z} \backslash G_0 / L_0$$

(13.1)

$$= SO(2h, k; \mathbb{Z}) / SO(2h, k) / (U(h_0) \times \cdots \times U(h_{-1}) \times SO(k)) \text{ for } r \text{ even},$$

$$= Sp(m; \mathbb{Z}) / Sp(m; \mathbb{R}) / (U(h_0) \times \cdots \times U(h_{1})) \text{ for } r \text{ odd},$$

where $h_i = h_{0,-i}^0$ and $\Gamma = G_\mathbb{Z}$ is defined by the lattice $H_r(X; \mathbb{Z})$ in $H_r(X; \mathbb{R})$. A variation of Hodge structure of $X$ corresponds to a deformation of Kähler structure of $X$, that is a fiber space $\psi : U \to V$ and a distinguished point $v_0 \in V$ such that the $X_v = \psi^{-1}(v)$ are compact Kähler (or algebraic) manifolds, $X = X_{v_0}$, with $h_{0,q}^{0}(X_v) = h_{0,q}^{0}(X)$, and such that the $X_v$ vary holomorphically (or algebraically). That defines a holomorphic map of $V \to \Gamma \backslash D$.

Classically one constructs automorphic functions on $\Gamma \backslash D$ as quotients of $\Gamma$–invariant sections of holomorphic line bundles over $D$ (automorphic forms of a given weight). Also classically $D$ is a bounded symmetric domain $Sp(g; \mathbb{R}) / U(g)$ and one works in a fixed holomorphic trivialization of the line bundles over $D$, so the $\Gamma$–invariance condition is expressed by a transformation law. In this way one constructs the function field of the moduli space $\Gamma \backslash D$.

The classical theory of automorphic functions must be modified in our context because in general $D$ has no nonconstant holomorphic functions $\mathbb{W}$. And in general nontrivial homogeneous vector bundles over $D$ have no nonzero holomorphic sections. Instead one considers sufficiently negative homogeneous holomorphic vector bundles $\mathbb{E} \to D$. Roughly speaking, those are the bundles whose $L^2$ cohomology, and whose sheaf cohomology, viewed as $G_0$–modules, have the same underlying Harish–Chandra module. Their cohomology occurs in degree $\dim C_0$ where $C_0 \cong K_0 / L_0$. One looks for automorphic cohomology, meaning $\Gamma$–invariant classes in $H^q(D; \mathcal{O}(\mathbb{E}))$. That is a bit remote from the idea of a function field for $\Gamma \backslash D$.

In much of the literature one considers only the situation where $G_0$ is of hermitian type and the bounded symmetric domain $\mathcal{B} = G_0 / K_0$ is used instead of $\Omega_W(D)$. (Of course they are the same if $D$ is of hermitian holomorphic type.) When $G_0$ is not of hermitian type then again $G_0 / K_0$ is used instead of $\Omega_W(D)$, and it is considered somewhat of an obstacle that $G_0 / K_0$ is not a complex manifold. Our use of $\Gamma \backslash \Omega_W(D)$ addresses this point.

In connection with construction of automorphic cohomology, Wells [We1] showed by direct computation that $\Omega_W(D)$ is a Stein manifold in one particular case ($r = 2$). That result was extended in Wells–Wulf [WeW] to the more general situation of open $G_0$–orbits $D$ of the form $G_0 / L_0$ with $L_0$ compact, using a special case of the double fibration transform together with somewhat general methods of complex analysis (Andreotti–Grauert [AnG], Andreotti–Norguet [AN], Dociquer–Grauert [DG]) associated to questions of holomorphic convexity and the Levi problem. The goal of [WeW] was construction of automorphic cohomology as convergent Poincaré $\varphi$–series $\varphi \Gamma(c) := \sum_{\gamma \in \Gamma} \gamma^*(c)$ where $c \in H^q(D; \mathcal{O}(\mathbb{E}))$ is a $K_0$–finite cohomology class. The relevant estimates were derived from semisimple representation theory, specifically from Hecht–Schmid [HS] and Schmid [S4], and the passage between $D$ and $\Omega_W(D)$.

This theory of Poincaré $\varphi$–series and automorphic cohomology later was developed quite a bit. According to [WaW], if $\Gamma$ is any discrete subgroup of $G_0$, $\mathbb{E} \to D$ is sufficiently negative and $1 \leq p \leq \infty$ then every $\Gamma$–invariant $L^p(\Gamma \backslash D)$ class in $H^q(D; \mathcal{O}(\mathbb{E}))$ can be realized as a Poincaré series $\varphi \Gamma(c)$ where $c \in H^q(D; \mathcal{O}(\mathbb{E}))$ is $L^p(D)$. In particular this is close to the idea of catching all of the function field. The “sufficiently” part of the “sufficiently negative” condition on $\mathbb{E} \to D$ is relaxed in Wallach–Wulf [WaW] by construction of an appropriate reproducing
Finite dimensionality of automorphic cohomology was proved by Williams ([Wi1], [Wi2], [Wi3]), using index theory of Moscovici and Connes, for the case where $\Gamma \backslash D$ is compact. Despite this development, automorphic cohomology has not yet been effectively applied to variation of Hodge structure. We expect that the new information on the double fibration transform, presented above, will make a difference here.

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