Correlation functions of the spin-1 analog of the XXZ model

Makoto Idzumi†

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606

Exact integral representations of spin one-point functions (ground state expectation values) are reported for the spin-1 analog of the XXZ model in the region $-1 < q < 0$. The method enables one to calculate arbitrary $n$-point functions in principle. We also report a construction of level 2 irreducible highest weight representations of $U_q(\hat{sl}_2)$ in terms of boson and fermion operators, and explicit forms of related vertex operators.

†On leave from Department of Applied Physics, Faculty of Engineering, The University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113
§1. Introduction

Recent developments of a representation theoretical approach to the XXZ quantum spin chain for $\Delta < -1$ ($-1 < q < 0$) enable one not only to diagonalize the Hamiltonian but also to calculate arbitrary spin correlation functions exactly in the form of integral representations\textsuperscript{1,2}. Higher spin analogs of the XXZ model could be attacked as well and excitation spectra were obtained\textsuperscript{3}. The method fully uses the representation theory of quantum affine algebras: the XXZ model and its higher spin analogs has an exact symmetry $U_q(\widehat{sl}_2)$ if the spin chain is infinite in both directions. A space of states is identified with a tensor product of an irreducible highest weight module over $U_q(\widehat{sl}_2)$ of level $k$ (if the spin is $k/2$) and its dual. Vertex operators play a key role in this identification. The vacuum (the ground state), which should be unique if a boundary condition is fixed, is identified with the unique one-dimensional submodule in the space of states. In terms of the vacuum and the vertex operators we can write down an exact expression of an arbitrary spin $n$-point correlation function (cf. eq.(4.2)), where the correlation function means the expectation value with respect to the vacuum.

In ref.2, correlations of the XXZ model were calculated through a concrete realization of the level 1 irreducible highest weight modules over $U_q(\widehat{sl}_2)$ in terms of boson\textsuperscript{4} (the procedure is called bosonization). A success in obtaining arbitrary correlations in ref.2 partly relies on the simplicity of the boson calculus. If one tries to apply the same method to the higher spin problem, he would realize that a way to the goal is not so easy to go ahead: first he must know a concrete realization of level $k$ irreducible highest weight representations which is known very complicated for $k > 1$, he must find the explicit forms of vertex operators related to the representations, and finally using these vertex operators he must perform calculations to get some simpler and more explicit expression of correlation functions.

The present paper treats with the spin 1 analog of the XXZ model in the region $-1 < q < 0$. We construct the level 2 irreducible highest weight representations explicitly
in terms of boson, and Neveu-Schwarz and Ramond fermions, and give explicit forms of vertex operators (§3). After giving exact expressions of arbitrary correlations for spin \( k/2 \), we specialize ourselves to calculations of spin one-point functions for spin 1 (§4). Since we shall follow the previous works completely, we omit to expose details of the approach (cf. ref.2; and refs.1,3). The method is only sketched quickly at the beginning of §4. We prepare necessary notations in the next section and report novel results in the subsequent sections.

§2. Notations

Notations for \( U_q(\hat{sl}_2) \) follow the refs.3,2. We set \( F = \mathbb{Q}(q) \) for a field where a parameter \( q \) is an indeterminate here (but later in §4 we shall regard it as a complex number in a range \(-1 < q < 0\)). Set \( P^* = \mathbb{Z}h_0 + \mathbb{Z}h_1 + \mathbb{Z}d \), \( P = \mathbb{Z}\Lambda_0 + \mathbb{Z}\Lambda_1 + \mathbb{Z}\delta \), \( \hat{h} = Fh_0 \oplus Fh_1 \oplus Fd \) (the Cartan sub-algebra) and \( \hat{h}^* = F\Lambda_0 \oplus F\Lambda_1 \oplus F\delta = F\alpha_0 \oplus F\alpha_1 \oplus F\Lambda_0 \) (The basis \{\( \Lambda_0, \Lambda_1, \delta \)\} of \( \hat{h}^* \) is dual to \{\( h_0, h_1, d \)\} of \( \hat{h} \), and \( \alpha_1 = 2\Lambda_1 - 2\Lambda_0, \alpha_0 = \delta - \alpha_1 \)). Define a symmetric bilinear form on \( \hat{h} \) and the one induced on \( \hat{h}^* \) by

\[
(h_i, h_j) = \begin{cases} 2 & \text{for } i = j, \\ -2 & \text{for } i \neq j \end{cases}, \quad (h_i, d) = \delta_{i,0}, \quad (d, d) = 0;
\]

\[
(\Lambda_i, \Lambda_j) = \frac{1}{2} \delta_{i,j}, \quad \Lambda_i, \delta = 1, \quad (\delta, \delta) = 0,
\]

\[
(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{for } i = j, \\ -2 & \text{for } i \neq j \end{cases}, \quad (\alpha_{i}, \Lambda_{0}) = \delta_{i,0}, \quad (\Lambda_{0}, \Lambda_{0}) = 0.
\]

A quantum affine algebra \( U_q(\hat{sl}_2) \) is an associative algebra with unit 1 over a field \( F = \mathbb{Q}(q) \) generated by \( e_i, f_i \ (i = 0, 1), \ q^h \ (h \in P^*) \) with relations

\[
q^hq^{h'} = q^{h+h'}, \quad q^0 = 1, \quad q^he_iq^{-h} = q^{(\alpha_i,h)}e_i, \quad q^hf_if_i^{q^{-h}} = q^{-(\alpha_i,h)}f_i,
\]

\[
[e_i, f_j] = \delta_{ij}t_i - t_i^{-1} \quad (t_i = q^{h_i}),
\]

where \( h, h' \in P^*, \ i, j = 0, 1, \) and

\[
e_i^3 - [3]e_i^2e_j + [3]e_i^2e_j - e_j^3 = 0 \quad (i \neq j),
\]

\[
f_i^3f_j - [3]f_i^2f_jf_i + [3]f_i^2f_jf_i - f_j^3f_i = 0 \quad (i \neq j).
\]
We have used and will use notations such as

\[ [x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [n]! = [n][n-1] \cdots [2][1], \quad \binom{n}{k} = \frac{[n]!}{[k]![n-k]!}. \]

Let us define the coproduct and the antipode on the generators by

\[ \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h \quad (h \in P^*); \]

\[ a(e_i) = -t_i^{-1}e_i, \quad a(f_i) = -f_i t_i, \quad a(q^h) = q^{-h} \quad (h \in P^*). \]

The Drinfeld’s realization of \( U_q' (\hat{sl}_2) \), which is a subalgebra of \( U_q (\hat{sl}_2) \) generated by \( \{ e_i, f_i, t_i | i = 0, 1 \} \) only, is defined as follows: generators are \( x_m^\pm (m \in \mathbb{Z}) \), \( a_m (m \in \mathbb{Z}_{\neq 0}) \), \( \gamma \), and \( K \), and relations are

\[ \gamma \in \text{center}, \]

\[ [a_m, a_n] = \delta_{m+n,0} \frac{2m}{m} \frac{\gamma^m - \gamma^{-m}}{q - q^{-1}}, \quad [a_m, K] = 0, \]

\[ K x_m^\pm K^{-1} = q^{\pm 2} x_m^\pm, \quad [a_m, x_n^\pm] = \pm \frac{2m}{m} \gamma^{\pm m/2} x_{m+n}^\pm, \]

\[ x_{m+1}^\pm x_m^\mp - q^{\pm 2} x_m^\pm x_{m+1}^\pm = q^{\pm 2} x_m^\pm x_{n+1}^\pm - x_{m+1}^\pm x_m^\pm, \]

\[ [x_m^+, x_n^-] = \frac{1}{q - q^{-1}} (\gamma^{\frac{1}{2}(m-n)} \psi_{m+n} - \gamma^{-\frac{1}{2}(m-n)} \varphi_{m+n}), \quad (2.1) \]

where

\[ \sum_{m=0}^{\infty} \psi_m z^{-m} = K \exp \left( (q - q^{-1}) \sum_{m=1}^{\infty} a_m z^{-m} \right), \]

\[ \sum_{m=0}^{\infty} \varphi_m z^m = K^{-1} \exp \left( -(q - q^{-1}) \sum_{m=1}^{\infty} a_m z^m \right), \]

and \( \psi_m = \varphi_m = 0 \) for \( m > 0 \). The bracket \( [X, Y] \) means \( XY - YX \). We note that \( a_m \ (m \in \mathbb{Z}_{\neq 0}) \) and the center generated by \( \gamma \) form a Heisenberg subalgebra with respect to the bracket \( \{ \ , \ \} \). We therefore regard this \( a_m \) as boson. The Chevalley generators \( \{ e_i, f_i, t_i \} \) of \( U_q' (\hat{sl}_2) \) are given by the identification

\[ t_0 = \gamma K^{-1}, \quad t_1 = K, \quad e_1 = x_0^+, \quad f_1 = x_0^-, \quad e_0 t_1 = x_1^+, \quad t_1^{-1} f_0 = x_{-1}^+, \quad (2.2) \]
The coproduct of the Drinfeld generators is known partially: for $k \geq 0$ and $l > 0$ we have

$$
\Delta(x^+_k) = x^+_k \otimes \gamma^k + \gamma^{2k}K \otimes x^+_k + \sum_{i=0}^{k-1} \gamma^{(k+3i)/2} \psi_{k-i} \otimes \gamma^{k-i}x^+_i \mod N_- \otimes N^2_+,
$$

$$
\Delta(x^-_l) = x^-_l \otimes \gamma^{-l} + K^{-1} \otimes x^-_l + \sum_{i=1}^{l-1} \gamma^{(l-i)/2} \varphi_{-l+i} \otimes \gamma^{-l+i}x^-_i \mod N_- \otimes N^2_+,
$$

$$
\Delta(a_l) = a_l \otimes \gamma^{l/2} + \gamma^{3l/2} \otimes a_l \mod N_- \otimes N_+,
$$

$$
\Delta(a_{-l}) = a_{-l} \otimes \gamma^{-3l/2} + \gamma^{-l/2} \otimes a_{-l} \mod N_- \otimes N_+.
$$

(2.3)

Here $N_+$ and $N^2_+$ are left $F[\gamma, \psi, \varphi, r, -s \in \mathbb{Z}_{\geq 0}]$-modules generated by $\{x^\pm_m | m \in \mathbb{Z}\}$ and $\{x^\pm_m x^n_n | m, n \in \mathbb{Z}\}$ respectively. It gives sufficient information for our calculation.

Making use of the coproduct and the antipode, we can define canonically the tensor product and the dual representations of $U_q(\widehat{sl}_2)$: (i) given representations $(\pi_V, V)$, $(\pi_W, W)$, the tensor product representation $(\pi_{V \otimes W}, V \otimes W)$ is defined to be $\pi_{V \otimes W} = (\pi_V \otimes \pi_W) \circ \Delta$; (ii) given a representation $(\pi, V)$, the dual representation $(\pi^{*a^\pm 1}, V^*)$ is defined to be $\pi^{*a^\pm 1} = t \pi \circ a^\pm 1$. We also denote them $V^{*a^\pm 1}$ as modules over $U_q(\widehat{sl}_2)$ with the action given by $\pi^{*a^\pm 1}$. Note that they are left $U_q(\widehat{sl}_2)$-modules.

Set $V^{(k)} = Fu_0 \oplus Fu_1 \oplus \cdots \oplus Fu_k$ and $V^{(k)}_2 = V^{(k)} \otimes F[z, z^{-1}]$. Then, equations

$$
\pi(t_1)u_j = q^{k-2j}u_j, \quad \pi(e_1)u_j = [j]u_{j-1}, \quad \pi(f_1)u_j = [k-j]u_{j+1},
$$

$$
\pi(t_0) = \pi(t_1)^{-1}, \quad \pi(e_0) = \pi(f_1), \quad \pi(f_0) = \pi(e_1)
$$

or

$$
\pi(\gamma)u_j = u_j, \quad \pi(K)u_j = q^{k-2j}u_j, \quad \pi(a_m)u_j = a^j_m u_j,
$$

$$
\pi(x^+_m)u_j = q^{m(k-2j)}[j]u_{j-1}, \quad \pi(x^-_m)u_j = q^{m(k-2j)}[k-j]u_{j+1},
$$

where

$$
a^j_m = -\frac{[2m][jm]}{m[m]}q^{m(k-j)+1} + \frac{[km]}{m},
$$

5
define a \((k + 1)\)-dimensional \((\text{spin-}k/2)\) representation \((\pi, V^{(k)})\) of \(U'_{q}(\widehat{sl}_2)\). Equations

\[
\pi_z(x) = \pi(x) \otimes \text{id} \quad \text{for} \quad x = e_1, f_1, t_1, t_0,
\]

\[
\pi_z(e_0) = \pi(f_1) \otimes z, \quad \pi_z(f_0) = \pi(e_1) \otimes z^{-1},
\]

\[
\pi_z(q^d)(u_j \otimes z^n) = q^n u_j \otimes z^n
\]

define a representation \((\pi_z, V_z^{(k)})\) of \(U_{q}(\widehat{sl}_2)\). In terms of the Drinfeld generators, the action of \(U_{q}(\widehat{sl}_2)\) on \(V_z^{(k)}\) is written as

\[
\pi_z(\gamma) = \pi(\gamma) \otimes \text{id}, \quad \pi_z(K) = \pi(K) \otimes \text{id}, \quad \pi_z(a_m) = \pi(a_m) \otimes z^m,
\]

\[
\pi_z(x_m^+) = \pi(x_m^+) \otimes z^m, \quad \pi_z(x_m^-) = \pi(x_m^-) \otimes z^m.
\]

\section*{§3. Level 2 irreducible highest weight representations of \(U_{q}(\widehat{sl}_2)\)}

and vertex operators

Set \(P_+ = \mathbb{Z}_{\geq 0}\Lambda_0 \otimes \mathbb{Z}_{\geq 0}\Lambda_1\). For \(\lambda \in P_+\), a \(U_{q}(\widehat{sl}_2)\)-module \(V(\lambda)\) is called an irreducible highest weight module with highest weight \(\lambda\) if the following conditions are satisfied: there is a vector \(|\lambda\rangle \in V(\lambda)\), called the highest weight vector, and \(q^h|\lambda\rangle = q^{(\lambda,h)}|\lambda\rangle \quad (h \in P^*)\), \(e_i|\lambda\rangle = 0, f_i^{(\lambda,h_i)+1}|\lambda\rangle = 0 \quad (i = 0, 1)\), and \(V(\lambda) = U_{q}(\widehat{sl}_2)|\lambda\rangle\). We say that \(V(\lambda)\) has level \(k\) if \(\langle \lambda, c \rangle = k \in \mathbb{Z}_{\geq 0}\). The \(V(\lambda)\) has a weight-space decomposition \(V(\lambda) = \bigoplus_{\xi \in P} V(\lambda)_{\xi}\). We sometimes write a weight of a weight vector \(v\) as \(\text{wt} (v)\).

To construct the level 2 irreducible highest weight modules, we may need to introduce some fermions, since we know that we really have to introduce fermions to construct the corresponding level 2 modules over an affine Lie algebra \(\widehat{sl}_2\) (see, for instance, ref.6). Incidentally, there exists a work by Bernard\(^7\) on level 1 representations of \(U_{q}(B_r^{(1)})\). It is helpful to us since \(\mathfrak{o}_3 \simeq sl_2\) and \(B_r^{(1)} \simeq \mathfrak{o}_{2r+1}\). So, let us construct our modules refering to these references as a guide.
Let $P = \mathbb{Z}^2$, $Q = \mathbb{Z} \alpha$ be the weight/root lattice of $sl_2$, and let $F[P], F[Q]$ be their group algebras (Do not confuse this $P$ with $P = \mathbb{Z} \Lambda_0 + \mathbb{Z} \Lambda_1 + \mathbb{Z} \delta$, and this $Q$ with the field of rational numbers). The basis elements of $F[P]$ are written multiplicatively as $e^{n\alpha}$ ($n \in \mathbb{Z}$). Let $F^a = F[a_{-1}, a_{-2}, \cdots]$ be the boson Fock space.

In contrast to the level 1 case, we need furthermore two species of fermions to construct the boson vacuum space in $V(\lambda)$. Define $$\phi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \phi_n z^{-n}, \quad \phi^R(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n}$$
by the anticommutation relations $$[\phi_m, \phi_n]_+ = \delta_{m+n,0} \frac{q^{2m} + q^{-2m}}{q + q^{-1}}.$$
We call $\{\phi_n|n \in \mathbb{Z} + \frac{1}{2}\}$ NS fermions (after Neveu-Schwarz), and $\{\phi_n|n \in \mathbb{Z}\}$ R fermions (after Ramond). Let $F^{\phi^{NS}} = F[\phi_{-1/2}, \phi_{-3/2}, \cdots]$ and $F^{\phi^R} = F^{\psi} \otimes \mathbb{C}^2 = F[\psi_{-1}, \psi_{-2}, \cdots] \otimes \mathbb{C}^2$ be the fermion Fock spaces. Here we have set for R fermion $\phi_n = \psi_n \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $n \neq 0$ and $\phi_0 = \psi_0 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $\psi_n$ ($n \neq 0$) satisfy the same anti-commutation relations as $\phi_n$ and $\psi_0 = 1/(q + q^{-1})^{1/2}$ is an ordinary number (i.e., $\psi_0$ commutes with all $\psi_n$, $n \neq 0$), because the total Fock space of R fermion $\{\phi_n\}$ decomposes into two mutually isomorphic sectors.

Now we can define the action of $U'_q(\hat{sl}_2)$ on spaces $W = F^{a} \otimes F^{\phi^i} \otimes F[P]$ ($i = NS, R$). The latter two components form the boson vacuum space. The bosons $\{a_m\}$ act on the first component of $W$ as

$$a_m(f \otimes g \otimes e^\beta) = a_m f \otimes g \otimes e^\beta \quad \text{for} \quad m < 0,$$

$$a_m(f \otimes g \otimes e^\beta) = [a_m, f] \otimes g \otimes e^\beta \quad \text{for} \quad m > 0.$$

Define operators $e^\beta$ and $\partial_\alpha$, $\beta \in \mathbb{C} \alpha$, by

$$e^{\beta_1}(f \otimes g \otimes e^{\beta_2}) = f \otimes g \otimes e^{\beta_1 + \beta_2},$$

$$\partial_\alpha(f \otimes g \otimes e^\beta) = (\alpha, \beta) f \otimes g \otimes e^\beta.$$
The fermions $\{\phi_m\}$ act on $W$ as
\[
\phi_m(f \otimes g \otimes e^\beta) = f \otimes (\phi_m g) \otimes e^\beta
\]
with $\phi_m(f \otimes 1 \otimes e^\beta) = 0$ for $m > 0$. We let also $K = 1 \otimes 1 \otimes q^{\partial_\alpha}$ and
\[
\gamma = q^2 \otimes 1 \otimes 1.
\]
(Note that the element $c \in P^*$ act as scalar 2 on the level 2 module and that the $\gamma$ is regarded as $q^c$.) Then, from the Drinfeld’s relations (2.1), forms of generating functions (often called the currents) $x^\pm(z) = \sum_{n \in \mathbb{Z}} x^\pm_n z^{-n}$ acting on $W$ are determined as
\[
x^+(z) = \eta \exp\left(\sum_{m=1}^{\infty} \frac{a_m}{2m} q^{-m} z^m \right) \exp\left(-\sum_{m=1}^{\infty} \frac{a_m}{2m} q^{-m} z^{-m} \right) \phi(z) e^{\alpha z^{1/2} + \frac{1}{2} \phi_\alpha},
\]
\[
x^-(z) = \eta \exp\left(-\sum_{m=1}^{\infty} \frac{a_m}{2m} q^m z^m \right) \exp\left(\sum_{m=1}^{\infty} \frac{a_m}{2m} q^m z^{-m} \right) \phi(z) e^{-\alpha z^{1/2} - \frac{1}{2} \phi_\alpha},
\]
where $\eta = (q + q^{-1})^{1/2}$ (Our definition of currents is different from the one in ref.2). With these actions we can obtain the irreducible modules. Let $F_{even/odd}$ be subspaces of the Fock space $F$ consisting of even/odd particle states. Then the followings are $U_q(\widehat{sl}_2)$-modules:
\[
V(2\Lambda_0) \simeq \left( F^a \otimes F^{\phi NS}_{even} \otimes F[2Q] \right) \oplus \left( F^a \otimes F^{\phi NS}_{odd} \otimes e^\alpha F[2Q] \right),
\]
\[
V(2\Lambda_1) \simeq \left( F^a \otimes F^{\phi NS}_{even} \otimes e^\alpha F[2Q] \right) \oplus \left( F^a \otimes F^{\phi NS}_{odd} \otimes F[2Q] \right),
\]
\[
V(\Lambda_0 + \Lambda_1)
\]
\[
\simeq \left( F^a \otimes F^{\psi}_{even} \otimes \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right) \otimes e^{\frac{\alpha}{2}} F[Q] \oplus \left( F^a \otimes F^{\psi}_{odd} \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right) \otimes e^{\frac{\alpha}{2}} F[Q]
\]
\[
\overset{\text{or}}{=} \left( F^a \otimes F^{\psi}_{odd} \otimes \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right) \otimes e^{\frac{\alpha}{2}} F[Q] \oplus \left( F^a \otimes F^{\psi}_{even} \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right) \otimes e^{\frac{\alpha}{2}} F[Q].
\]
The highest weight vectors are $1 \otimes 1 \otimes 1$ for $V(2\Lambda_0)$, $1 \otimes 1 \otimes e^\alpha$ for $V(2\Lambda_1)$, and $1 \otimes (1 \otimes \left( \begin{array}{c} 1 \\ 1 \end{array} \right)) \otimes e^{\frac{\alpha}{2}}$ or $1 \otimes (1 \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right)) \otimes e^{\frac{\alpha}{2}}$ for $V(\Lambda_0 + \Lambda_1)$ according to the two realizations, respectively.
We can define the grading operator $d$ of $U_q(\hat{sl}_2)$ by

\[
d(a_{-m_1} \cdots a_{-m_r} \otimes \phi_{-k_1} \cdots \phi_{-k_s} v \otimes e^\beta) = (-\sum_{j=1}^r m_j n_j - \sum_{j=1}^s k_j - \frac{(\beta, \beta)}{4} + \frac{(\lambda, \lambda)}{4})(a_{-m_1} \cdots a_{-m_r} \otimes \phi_{-k_1} \cdots \phi_{-k_s} v \otimes e^\beta)
\]

\[
= (-\sum_{m=1}^\infty mN_m^a - \sum_{k>0} kN_{-m}^\phi - \frac{1}{8} \partial_\alpha^2 + \frac{(\lambda, \lambda)}{4})(a_{-m_1} \cdots a_{-m_r} \otimes \phi_{-k_1} \cdots \phi_{-k_s} v \otimes e^\beta)
\]

on $V(\lambda)$, where the boson and the fermion number operators are defined by (for $m > 0$)

\[
N_m^a = \frac{1}{[a_m, a_{-m}]} a_{-m}a_m = \frac{m}{[2m]^2} a_{-m}a_m,
\]

\[
N_m^\phi = \frac{1}{[\phi_m, \phi_{-m}]^+} \phi_{-m}\phi_m = \frac{q + q^{-1}}{q^{2m} + q^{-2m}} \phi_{-m}\phi_m.
\]

With this definition of the action of $d$, the $V(\lambda)$ become irreducible highest weight $U_q(\hat{sl}_2)$-modules.

We checked the validity of the expressions for currents $x^\pm(z)$ for ‘higher’ 26 weight vectors (i.e., vectors $f_{i_r} \cdots f_{i_1}|\lambda\rangle$ with $r$ small, $|\lambda\rangle$ the highest weight vector) of $V(2\Lambda_i)$, $i = 0, 1$, and 35 weight vectors of $V(\Lambda_0 + \Lambda_1)$; we checked the defining relations for the Chevalley generators on these vectors.

At this point, let us define the normal-ordered product of two fermion fields. We want to define it by

\[
N[\phi_m\phi_n] = \phi_m\phi_n \quad \text{for } m, n \geq 0 \text{ or } m, n \leq 0;
\]

\[
N[\phi_m\phi_{-n}] = -\phi_{-n}\phi_m \quad \text{for } m, n > 0;
\]

\[
N[\phi_{-m}\phi_n] = \phi_{-m}\phi_n \quad \text{for } m, n > 0.
\]

For the fermion fields $\phi(z)$, we set $N[\phi(z)\phi(w)] = \sum \sum N[\phi_m\phi_n]z^{-m}w^{-n}$. This definition yields $N[\phi(z)\phi(z)] = 0$ for NS fermion; but for R fermion $N[\phi(z)\phi(z)] = (\phi_0)^2$. So we re-define the normal order product of the fermion fields as

\[
: \phi(z)\phi(w) := N[\phi(z)\phi(w)] - \varepsilon(\phi_0)^2,
\]

9
\[ \varepsilon = \begin{cases} 0 & \text{for NS} \\ 1 & \text{for R.} \end{cases} \]

The propagator has the following form:

\[ \langle \phi(w_1) \phi(w_2) \rangle \overset{\text{def}}{=} \phi(w_1) \phi(w_2) - : \phi(w_1) \phi(w_2) : = \frac{\sigma(w_2/w_1)(1 - w_2/w_1)}{(1 - q^2 w_2/w_1)(1 - q^{-2} w_2/w_1)}, \]

\[ \sigma(w_2/w_1) = \begin{cases} \frac{(w_2/w_1)^{1/2}}{q+q^{-1}}(1 + w_2/w_1) & \text{for NS} \\ \frac{1}{q+q^{-1}}(1+w_2/w_1) & \text{for R}. \end{cases} \]

It is defined in a region \(|q^2 w_2/w_1|, |q^{-2} w_2/w_1| < 1\). We note a relation

\[ \frac{\langle \phi(w_1) \phi(w_2) \rangle}{w_1(1 - w_2/w_1)} - \frac{\langle \phi(w_2) \phi(w_1) \rangle}{w_2(1 - w_1/w_2)} = \frac{\delta(q^2 w_2/w_1)}{q^{-1}(q^2 - q^{-2})w_1} - \frac{\delta(q^{-2} w_2/w_1)}{q(q^2 - q^{-2})w_1}. \]

Now let us find expressions for vertex operators acting on them. They are the \(q\)-deformation of the vertex operators of Tsuchiya-Kanie type\(^8\), and are first introduced by Frenkel-Reshetikhin\(^9\). We write level \(k\) highest weights as \(\lambda_m = (k - m)\Lambda_0 + m\Lambda_1\), \(m = 0, 1, \ldots, k\). For two level \(k\) highest weight modules \(V(\lambda_m), V(\lambda_{k-m})\) and the \((k+1)\)-dimensional representation \(V\), consider a map

\[ \Phi_{\lambda_m}^{\lambda_{k-m}} V(z) : V(\lambda_m) \longrightarrow V(\lambda_{k-m}) \otimes V_z \]

acting as an intertwiner

\[ \Delta(x) \circ \Phi(z) = \Phi(z) \circ x \quad \text{for} \quad x \in U_q(\hat{sl}_2). \quad (3.1) \]

This is a vertex operator. Precisely speaking, it is defined as an intertwiner \(\Phi(z) : V(\lambda_m) \longrightarrow V(\lambda_{k-m}) \otimes V_z\) where \(M \otimes N = \bigoplus_\xi \prod_\nu M_\nu \otimes N_{\xi-\nu}\). It exists uniquely up to normalization for each \(m = 0, 1, \ldots, k\) (The existence is proven in ref.10 for more general setting). We normalize it as

\[ \tilde{\Phi}_{\lambda_m}^{\lambda_{k-m}} V(z)(|\lambda_m\rangle) = |\lambda_{k-m}\rangle \otimes u_{k-m} + \cdots. \quad (3.2) \]
Here and in the following, we will use the notation \( \tilde{\Phi} \) for normalized vertex operators (such as (3.2)), and \( \Phi \) for the others (unnormalized ones). Note that for level \( k \) we have \((k + 1)\) vertex operators (for \( m = 0, 1, \ldots, k \)) of this type.

Let us determine the explicit form of \( \Phi(z) \) which satisfies the intertwining relations (3.1). We set

\[
\Phi(z) = \sum_{j=0}^{k} \Phi_j(z) \otimes u_j
\]

and call \( \Phi_j(z) \) the \( j \)-th component of \( \Phi(z) \).

From the intertwining relation with \( f_1 \), we get relations among \( \Phi_j \) such as

\[
\begin{align*}
\Phi_0(z)f_1 &= q^{-k}f_1\Phi_0(z), \\
\Phi_j(z)f_1 &= q^{2j-k}f_1\Phi_j(z) + [k - j + 1]\Phi_{j-1}(z), \quad j = 1, \ldots, k.
\end{align*}
\]

(3.3)

The second equations yield

\[
\Phi_{j-1}(z) = \frac{1}{[k - j + 1]} \left( \Phi_j(z)f_1 - q^{2j-k}f_1\Phi_j(z) \right) = \frac{1}{[k - j + 1]} \oint_{w=0} \frac{dw}{2\pi i w} \left( \Phi_j(z)x^{-}(w) - q^{2j-k}x^{-}(w)\Phi_j(z) \right)
\]

for \( j = 1, \ldots, k \), since \( f_1 = x_0 = \oint_{w=0} \frac{dw}{2\pi i w} (x^{-}(w)/w) \). The contour around \( w = 0 \) is chosen such that the integrand is convergent wherein.

In order to find an expression of \( \Phi_k(z) \), we use (3.1) for \( x = a_m, x_m^+ \) and \( K \) (the coproduct of these generators are given in eq.(2.3)). The result is

\[
\begin{align*}
& a_m\Phi_k(z) - \Phi_k(z)a_m = q^{(\frac{k}{2}+2)m}\frac{[km]}{m}z^m\Phi_k(z) \quad \text{for } m > 0, \\
& a_-m\Phi_k(z) - \Phi_k(z)a_-m = q^{-(\frac{k}{2}+2)m}\frac{[km]}{m}z^{-m}\Phi_k(z) \quad \text{for } m > 0, \\
& \Phi_k(z)x^+(w) - x^+(w)\Phi_k(z) = 0, \\
& K\Phi_k(z)K^{-1} = q^k\Phi_k(z).
\end{align*}
\]

(3.4a, 3.4b, 3.4c, 3.4d)
There is another equation
\[
\sum_{l=0}^{k+1} (-1)^l \left[ \begin{array}{l} k+1 \cr l \end{array} \right] f^l_k(z) f_1^{k+1-l} = 0
\]
obtained from (3.3), which, however, we shall not use.

From these relations, we propose the following explicit forms of vertex operators.

**Proposition 1.** An operator acting on the space \( F^a \otimes F^b \otimes F[P] \) with components

\[
\Phi_2(z) = \exp \left( \sum_{m=1}^{\infty} \frac{a_{m-1}}{[2m]} q^{m} z^{m} \right) \exp \left( -\sum_{m=1}^{\infty} \frac{a_{m}}{[2m]} q^{-3m} z^{-m} \right) e^{\alpha} (-q^4 z)^{2 \hat{\alpha}},
\]

\[
\Phi_1(z) = \int_{w_1=0} dw_1 \frac{1}{2\pi i} \frac{d w_1}{w_1} (\Phi_2(z) x^{-}(w_1) - q^2 x^{-}(w_1) \Phi_2(z))
\]

\[
= -\frac{1-q^4}{q^4 z} \int_{w_1=0} dw_1 \frac{1}{2\pi i} \frac{d w_1}{w_1 (1-q^{-2} w_1/z)(1-q^6 z/w_1)} : \Phi_2(z) x^{-}(w_1) :,
\]

\[
\Phi_0(z) = \frac{1}{2} \int_{w_2=0} dw_2 \frac{1}{2\pi i} \frac{d w_2}{w_2} (\Phi_1(z) x^{-}(w_2) - x^{-}(w_2) \Phi_1(z))
\]

\[
= \int_{w_2=0} dw_2 \frac{1}{2\pi i} \int_{w_1=0} dw_1 \left\{ \frac{1}{2} \frac{1-q^4}{q^4 z} (1-q^{-2} w_1/z)(1-q^6 z/w_1) w_2 (1-q^{-2} w_2/z)(1-q^6 z/w_2)
\times : \Phi_2(z) x^{-}(w_1) x^{-}(w_2) : 
\right. 
\left. + \frac{1}{2} \frac{1-q^4}{q^4 z} (1-q^{-2} w_2/w_1) (\phi(w_1) \phi(w_2)) (1-q^{-2} w_1/z)(1-q^6 z/w_1) w_2 (1-q^{-2} w_2/z)
\times : \Phi_2(z) x^{-}(w_1) x^{-}(w_2) : 
\right. 
\left. + \frac{1}{2} \frac{1-q^4}{q^4 z} (1-q^6 z/w_1) (\phi(w_2) \phi(w_1)) w_1 (1-q^{-2} w_1/z)(1-q^6 z/w_1) w_2 (1-q^6 z/w_2)
\times : \Phi_2(z) x^{-}(w_1) x^{-}(w_2) : 
\right. 
\left. \right\}
\]

\[
= \int_{w_2=0} dw_2 \frac{1}{2\pi i} \int_{w_1=0} dw_1 \left\{ \frac{1}{2} \frac{1-q^4}{q^4 z} \frac{w_1}{z} (1-q^{-2} w_1/z)(1-q^{-2} w_1/z)(1-q^6 z/w_1)(1-q^6 z/w_2)
\times \left[ (1-q^4) \frac{w_1}{z} (1-q^4 \frac{z}{w_1}) : \Phi_2(z) x^{-}(w_1) x^{-}(w_2) : 
\right. 
\right. 
\left. + \left[ \frac{w_1}{z} (1-q^6 \frac{z}{w_2})(1-q^{-2} \frac{w_1}{w_1}) (\phi(w_1) \phi(w_2)) 
\right. 
\right. 
\left. + q^4 (1-q^{-2} \frac{w_2}{z})(1-q^{-2} \frac{w_1}{w_2}) (\phi(w_2) \phi(w_1)) \right] : \Phi_2(z) x^{-}(w_1) x^{-}(w_2) : \right\}
\]
satisfies the relations (3.1), where \( \cdot \cdot \cdot \) denotes the normal ordering with respect to boson, fermion and lattice operators \( \dagger \), and the hat “\( \hat{\cdot} \)” denotes fermion contraction

\[
\hat{x}^{-}(w) = \eta \exp(-\sum_{m=1}^{\infty} \frac{a_{-m}}{2m} q^{m} w^{m}) \exp \left( \sum_{m=1}^{\infty} \frac{a_{m}}{2m} q^{m} w^{-m} \right) e^{-\alpha w} \dagger \frac{1}{2} \dagger \frac{1}{2} \partial_{z}.
\]

The normalized vertex operators (3.2) are given by

\[
\tilde{\Phi}_{\lambda_{2-m}}^{\lambda_{m}}(z) = (-q^{4}z)^{m/2} \Phi(z), \quad m = 0, 2;
\]
\[
\tilde{\Phi}_{\lambda_{1}}^{\lambda_{1}}(z) = \epsilon(-q^{4}z)^{1/2} \Phi(z)
\]

where the vertex operator \( \tilde{\Phi}_{\lambda_{1}}^{\lambda_{1}}(z) \) acts on the realization of \( V(\lambda_{1}) = V(\Lambda_{0} + \Lambda_{1}) \) with highest weight vector \( p_{\epsilon} \otimes \Phi_{\epsilon} \), where \( p_{\epsilon} = (1_{\epsilon}) \), \( \epsilon = \pm 1 \).

**Proof.** First of all, we note that we have made everything in the normal order in the above equations (cf. Appendix C). The proposed vertex operators satisfy all the equations (3.4a–d). In order to confirm that \( \Phi \) thus obtained are correct ones, we must check the intertwining relations (3.1) at least for the Chevalley generators (2.2). We checked the intertwining relations partially, but not all. So we compared matrix elements of \( \Phi_{2} \) to

\[\text{For bosons}\]

\[
\cdot_{m} a_{-n} := a_{-n} a_{m} + \delta_{mn} c_{m} \quad (m, n > 0);
\]
\[
\cdot_{-m} a_{n} := a_{-m} a_{n}; \quad (m, n > 0; \text{already normal-ordered});
\]
\[
\cdot_{m} a_{n} := a_{m} a_{n} = a_{n} a_{m} \quad (m, n > 0 \text{ or } m, n < 0).
\]

For lattice

\[
\cdot z^{\partial_{z}} e^{\beta} : = z^{(\alpha, \beta)} e^{\beta} z^{\partial_{z}} \quad (\beta = x\alpha);
\]
\[
\cdot e^{\beta} z^{\partial_{z}} : = e^{\beta} z^{\partial_{z}} \quad \text{(already normal-ordered)}.
\]
the correct ones, which are known for ‘higher’ weight vectors (i.e., vectors $f_i, \cdots f_i |\lambda\rangle$ with $r$ small, $|\lambda\rangle$ the highest weight vector) and obtained using the global crystal base of Kashiwara$^{11}$). Actually we checked ‘higher’ $26 \times 17$ matrix elements (including many 0 elements) for $\Phi_{2\Lambda_i-1}^\lambda V$ ($i = 0, 1$) and ‘higher’ $35 \times 15$ matrix elements for $\Phi_{\Lambda_0+\Lambda_1}^\lambda V$. We found that they are correct. Thus, though the proposition is not completely proven we think that it must be correct certainly.

The inverse vertex operator

$$\Phi_{\lambda_{k-m}}^\lambda \mu \nu V(z) : V(\lambda_{k-m}) \otimes V_z \longrightarrow V(\lambda_m)$$

which is an intertwiner and normalized as

$$\Phi_{\lambda_{k-m}}^\lambda \mu \nu V(z)(|\lambda_{k-m}\rangle \otimes u_{k-m}) = |\lambda_m\rangle + \cdots$$

(3.6)

can be obtained from the previous one as follows. First let us identify a vertex operator $V(\mu) \otimes V_z \rightarrow V(\lambda)$ with $V(\mu) \rightarrow V(\lambda) \otimes V^*_{\alpha}$

$$\Phi_\mu V_{\mu,j}(z) = \Phi_\mu V^*_{\alpha, j}(z), \quad j = 0, 1, \ldots, k,$$

where we write

$$\Phi_\mu V^*_{\alpha, j}(z) = k \sum_{j=0}^{k} \Phi_\mu V^*_{\alpha, j}(z) \otimes u_j^*,$$

$$\Phi_\mu V_{\mu,j}(z)(v) = \Phi_\mu V(z)(v \otimes u_j).$$

Second, we must note an isomorphism between left $U_q(\hat{sl}_2)$-modules (the charge conjugation)

$$V_{zq^{-2}} \otimes u_j \sim u_j \otimes \Phi_\mu V(z) = c_j u_{k-j},$$

$j = 0, 1, \ldots, k$, where dim$V = k + 1$ and

$$c_j = (-1)^j q^{j^2 + (1-k)j} \frac{1}{\binom{k}{j}}$$

14
(for level 2, \(c_0 = 1, c_1 = -1/2, c_2 = q^2\)). This gives

\[
\tilde{\Phi}_{\mu,j}^{\lambda V}(z) = c_{k-j} \tilde{\Phi}_{\mu,k-j}^{\lambda V}(z q^{-2}).
\]

These two equations determine the (unnormalized) vertex operators. The normalized vertex operators are

\[
\tilde{\Phi}_{\lambda_{k-m}V,j}^{\lambda_m}(z) = c_{k-j} \tilde{\Phi}_{\lambda_{k-m},k-j}^{\lambda_m}(z q^{-2}), \quad m = 0, 1, \ldots, k.
\]  

(3.7)

We said that the \(\tilde{\Phi}_{\lambda_{k-m}V}(z)\) is the inverse of \(\tilde{\Phi}_{\lambda_m}(z)\), since the following relations hold:

\[
\tilde{\Phi}_{\lambda_{k-m}V}(z) \circ \tilde{\Phi}_{\lambda_m}(z) = g_{\lambda_m} \times \text{id}_{V(\lambda_m)},
\]

\[
\tilde{\Phi}_{\lambda_m}(z) \circ \tilde{\Phi}_{\lambda_{k-m}V}(z) = g_{\lambda_m} \times \text{id}_{V(\lambda_{k-m}) \otimes V},
\]

where

\[
g_{\lambda_m} = q^{(k-m)m} \binom{k}{m} \left( \frac{q^{2(k+1)}; q^4}{q^2; q^4} \right)_{\infty},
\]

\[
(x; p)_m = \prod_{j=1}^{m} (1 - xp^{j-1}), \quad (x; p)_{\infty} = \prod_{j=1}^{\infty} (1 - xp^{j-1}).
\]

For level 2 it is

\[
g_{\lambda_m} = \begin{cases} 
\frac{1}{1-q^2} & \text{for } m = 0, 2 \\
\frac{1+q^2}{1-q^2} & \text{for } m = 1.
\end{cases}
\]

For general \(k\), they are proved in ref.3 using a solution to the \(q\)-KZ equation. In the present case \((k = 2)\), since we have explicit expressions for the vertex operators, they can be proved by direct calculation (Since the intertwining relations hold, it is sufficient to prove the relations on the highest weight vector).
§4. Integral representations of spin correlation functions

First we describe a general formulation for general \( k \) quickly (cf. refs.1,3). The spin \( k/2 \) Hamiltonian is defined as (cf. Appendix A)

\[
H = \sum_{l \in \mathbb{Z}} h_{l+1,l},
\]

\[
h_{l+1,l} = \cdots \otimes 1 \otimes h^l \otimes 1 \otimes \cdots,
\]

\[
h = (-1)^k (q^k - q^{-k}) \left[ \frac{d}{dz} \tilde{R}(z, 1) \right]_{z=1}
\]  

(4.1)

where

\[
\tilde{R}(z_1/z_2) : V^{(k)}_{z_1} \otimes V^{(k)}_{z_2} \rightarrow V^{(k)}_{z_2} \otimes V^{(k)}_{z_1}
\]

is the \( R \)-matrix which intertwines two \( U_q(\hat{sl}_2) \)-modules. The Hamiltonian formally acts on an infinite tensor product

\[
V^{\otimes \infty} = \cdots \otimes V^2 \otimes V^1 \otimes V^0 \otimes V^{-1} \otimes \cdots
\]

of the \((k + 1)\)-dimensional space \( V = V^{(k)} \), and has a symmetry

\[
[H_{XXZ}, U'_q(\hat{sl}_2)] = 0.
\]

In the following we assume \(-1 < q < 0\), and then \( \Delta < -1 \) \((\Delta = (q + q^{-1})/2)\). The basic idea in refs.1,3 is an identification of the space of states with some fixed boundary condition, which is a subspace of \( V^{\otimes \infty} \), with a level-zero \( U_q(\hat{sl}_2) \)-module \( V(\lambda) \otimes V(\mu)^* \).

This space is embedded into the infinite product space via vertex operators

\[
V(\lambda) \otimes V(\mu)^* \hookrightarrow V^{\otimes \infty}.
\]

Specializing the level \( k \) highest weight \( \lambda \) and the level \(-k\) lowest weight \( \mu \) corresponds to fixing a boundary condition. This boundary condition depends on the parameter \( q \). In the
limit $q \to 0$ (which is an Ising limit; cf. Appendix A) it is explicitly described as follows: any vector in $V(\lambda_m) \otimes V(\lambda_m')^*$ tends to a pure tensor of the form $\cdots \otimes u_{p(2)} \otimes u_{p(1)} \otimes u_{p(0)} \otimes u_{p(-1)} \otimes \cdots$ (when embedded into $V^{\otimes \infty}$) with a boundary condition

$$p(l) = \begin{cases} m & \text{if } l \equiv 0 \text{ mod } 2 \\ k - m & \text{if } l \equiv 1 \text{ mod } 2 \end{cases}$$

for $l \gg 1$,

$$= \begin{cases} m' & \text{if } l \equiv 0 \text{ mod } 2 \\ k - m' & \text{if } l \equiv 1 \text{ mod } 2 \end{cases}$$

for $-1 \gg l$.

Below we consider the cases $m' = m$ only. The vacuum vector in $V(\lambda) \otimes V(\lambda)^*$ is

$$\bar{v} = \sum_i v_i \otimes v_i^\ast (= |\text{vac}\rangle)$$

where $\{v_i\}$ is a base of $V(\lambda)$ and $\{v_i^\ast\}$ its dual base. This $\bar{v}$ forms a one-dimensional submodule. The subject of the present paper is to calculate the vacuum expectation value of an arbitrary local operator. A local operator $L$ is a linear operator acting on a finite tensor product of $V$:

$$L \in \text{End}(V \otimes \cdots \otimes V)$$

where $V \otimes \cdots \otimes V$ is understood as $n$th to 1st components in $V^{\otimes \infty}$. This acts on $V(\lambda) \otimes V(\lambda)^*$ via the vertex operators. The vacuum expectation value of $L$, or the correlation function, is derived, in the same manner as in ref. 2 for spin $1/2$, as

$$\langle L \rangle^{(\lambda)}_{z_n, \ldots, z_1} = \frac{\langle \text{vac} | g_{z_n, \ldots, z_1}^{(\lambda)} (L) \otimes \text{id}_{V(\mu)}^* | \text{vac} \rangle}{\langle \text{vac} | \text{vac} \rangle} = \frac{\text{tr}_{V(\lambda)}(q^{-2\rho} g_{z_n, \ldots, z_1}^{(\lambda)} (L))}{\text{tr}_{V(\lambda)}(q^{-2\rho})}$$

(4.2)

where $\rho = \Lambda_0 + \Lambda_1$ and

$$g_{z_n, \ldots, z_1}^{(\lambda)} (L) = (\Phi_\lambda^{(n)}(z_n, \ldots, z_1)^{-1} \circ (\text{id}_{V(\lambda)^{n-1}} \otimes L) \circ \Phi_\lambda^{(n)}(z_n, \ldots, z_1)),$$

$$\Phi_\lambda^{(n)}(z_n, \ldots, z_1)$$

$$= (\bar{\Phi}_\lambda^{(n-1)}(z_n) \otimes \text{id}_{V \otimes \cdots \otimes V}) \circ \cdots \circ (\bar{\Phi}_\lambda^{(2)}(z_2) \otimes \text{id}_V) \circ \bar{\Phi}_\lambda^{(1)}(z_1),$$

$$\Phi_\lambda^{(n)}(z_n, \ldots, z_1)^{-1}$$

$$= \bar{\Phi}_\lambda^{(1)}(z_1) \circ (\bar{\Phi}_\lambda^{(2)}(z_2) \otimes \text{id}_V) \circ \cdots \circ \bar{\Phi}_\lambda^{(n-1)}(z_n) \otimes \text{id}_{V \otimes \cdots \otimes V})/(g_\lambda g_{\lambda(1)} \cdots g_{\lambda(n-1)}).$$

17
Let us set 

\[ P_{j_n \ldots j_1} (z_n, \ldots, z_1 | \lambda) = \langle E_{i_n j_n} \otimes \cdots \otimes E_{i_1 j_1} \rangle_{z_n \ldots, z_1}^{(\lambda)} \]

where \( E_{ij} u_{j'} = \delta_{jj'} u_i \).

In the following we concentrate on calculations of one-point functions

\[ P_i^j (z|\lambda) = \frac{1}{g_\lambda} \frac{\text{tr}_{V(\lambda)} (q^{-2\rho} \tilde{\Phi}_V^\lambda(z) \tilde{\Phi}_{\bar{\lambda},j}^\mu (z))}{\text{tr}_{V(\lambda)} (q^{-2\rho})}, \]

\( \lambda = (k - m) \Lambda_0 + m \Lambda_1, \mu = m \Lambda_0 + (k - m) \Lambda_1, \; 0 \leq m \leq k \)

for level \( k = 2 \) (spin 1). Note that \( \rho = 2d + \frac{1}{2} \partial_\alpha \) in the present representations.

The specialized characters \( Z^{(\lambda)} = \text{tr}_{V(\lambda)} (q^{-2\rho}) \), \( \lambda = 2\Lambda_0, 2\Lambda_1, \Lambda_0 + \Lambda_1 \), which appeared as denominators, are calculated straightforwardly. The result is

\[ Z^{(2\Lambda_i)} = q^{-(2\Lambda_i, 2\Lambda_i)} \cdot (-q^2; q^2)_\infty (-q^4; q^4)_\infty, \quad (4.3) \]

\[ Z^{(\Lambda_0 + \Lambda_1)} = q^{-1} \cdot (-q^2; q^2)_\infty (-q^4; q^4)_\infty. \quad (4.4) \]

Numerators of spin one-point functions are expressed by vertex-operator two-point functions

\[ F_{j_k}^{(\lambda)} (z_1/z_2) = \text{tr}_{V(\lambda)} (q^{-2\rho} \tilde{\Phi}_{\sigma(\lambda),j}^{\Lambda V} (z_1) \tilde{\Phi}_{\bar{\lambda},k}^{\sigma(\lambda)V} (z_2)). \]

Let us first calculate these functions. In ref.2, they introduced some auxiliary boson to simplify computations of traces slightly. It is, however, not necessary to do so. We perform direct calculations of traces. Write

\[ d = \bar{d}a + \bar{d} \phi - \frac{1}{8} \partial_\alpha^2 + \frac{(\lambda, \lambda)}{4} \]

on \( V(\lambda) \) where

\[ \bar{d}a = - \sum_{m=1}^{\infty} mN_m^a, \quad \bar{d} \phi = - \sum_{m=\frac{1}{2}, \frac{3}{2}, \ldots} mN_m^\phi; \]
\[ V_<(z_1, z_2, w_1, w_2) = \exp\left(-\sum_{m=1}^{\infty} a_{-m} \frac{q^m}{2m} \xi_m(z_1, z_2, w_1, w_2)\right), \]
\[ V_>(z_1, z_2, w_1, w_2) = \exp\left(\sum_{m=1}^{\infty} a_m \frac{q^m}{2m} \xi_{-m}(z_1, z_2, w_1, w_2)\right), \]
\[ \xi_m(z_1, z_2, w_1, w_2) = w_1^m + w_2^m - q^{4m} z_1^m - q^{4m} z_2^m. \]

When calculating traces directly, we use the following formulas: a trace on the boson Fock space

\[
\text{tr}_{\mathcal{F}^a}(q^{-4d^a} V_<(z_1, z_2, w_1, w_2) V_>(z_1, z_2, w_1, w_2)) = \left(\frac{q^6}{q^4}\right)_\infty \left(\frac{q^6 z_1}{z_2}\right)_\infty \left(\frac{q^6 z_2}{z_1}\right)_\infty \prod_{i,j=1,2} \left(\frac{q^2 w_i z_j}{w_j z_i}\right)_\infty \}
\]

where \((\cdot)_\infty = (\cdot; q^4)_\infty\) (see Appendix C for the normal ordering of boson exponentials); a trace on the total NS fermion Fock space

\[
\text{tr}_{\mathcal{F}^\phi}(\xi^{-2d^\phi} : \phi(w_1) \phi(w_2) ) = \left(\frac{-\xi; \xi^2}{q + q^{-1}}\right)_\infty \left(\frac{w_1}{w_2}\right)^{1/2} \sum_{N=0}^{\infty} \left(\frac{w_1}{w_2}\right) N \frac{q^{2N+1} + q^{-2N-1}}{1 + \xi^{-2N-1}} \\
- \sum_{N=-\infty}^{-1} \left(\frac{w_1}{w_2}\right) N \frac{q^{2N+1} + q^{-2N-1}}{1 + \xi^{2N+1}}
\]

where \(f_N(\xi) = \frac{q^{2N+1} + q^{-2N-1}}{1 + \xi^{-2N-1}}\) for \(N \geq 0\),
\[= -\frac{q^{2N+1} + q^{-2N-1}}{1 + \xi^{2N+1}}\] for \(N < 0;\)

a trace on any one of the two sectors of the Ramond fermion Fock space

\[
\text{tr}(\xi^{-2d^\phi} : \phi(w_1) \phi(w_2) ) = \left(\frac{-q^4; q^4}{q + q^{-1}}\right)_\infty \left[\frac{q^{2 w_1 w_2}}{1 - q^{2 w_1 w_2}} - \frac{q^{2 w_2 w_1}}{1 - q^{2 w_2 w_1}} \right];
\]

19
and a trace on the total root lattice

\[ \text{tr}_{F[Q]} \left( \xi^4 \partial^2 \alpha - \partial_y \alpha \frac{1}{2} \partial_\alpha \right) = \sum_{n \in \mathbb{Z}} \xi_n^2 (q^{-2} y)^n = \Theta_1 (q^{-1} y \frac{1}{2} | \xi) \]

where \( y = \frac{q^2 z_1 z_2}{w_1 w_2}. \)

For a technical reason, when we treat with NS fermions, we introduce a parameter \( \xi \) and calculate

\[ F_{jk}^{(\lambda)}(z_1, z_2 | \xi) = \text{tr}_{\nu(\lambda)} \left( q^{-2\rho'} \xi^{-2\tilde{d}^\alpha + \frac{1}{4} \partial^2 \alpha \tilde{\Phi}_\sigma^{\lambda V}} (z_1) \tilde{\Phi}_{\sigma,k}^{\sigma(\lambda)V} (z_2) \right) \]

where \( \rho' = 2d' + \frac{1}{2} \partial_\alpha \) and \( d' = \tilde{d}^a + \frac{(L, \lambda)}{4} \). Setting \( \xi = q^2 \) yields the desired function:

\[ F_{jk}^{(\lambda)}(z_1, z_2 | q^2) = F_{jk}^{(\lambda)}(z_1, z_2). \]

Define a symmetrization/antisymmetrization operation with respect to the parameter \( \xi \) by

\[ f^S(\xi) = \frac{1}{2} (f(\xi) + f(-\xi)), \]
\[ f^A(\xi) = \frac{1}{2} (f(\xi) - f(-\xi)) \]

for an arbitrary function of \( \xi \). We sometimes denote \( \sigma_0 = S, \sigma_1 = A \).

The results are listed below:

\[ F_{jk}^{(2\Lambda_i)}(z | \xi) = q^{-4(2-k)} \cdot (q^{-2} z^{-1})^i \cdot b(z) \cdot \int_{w_1=0}^{dw_1} \frac{dw_1}{2\pi i} \int_{w_2=0}^{dw_2} \frac{dw_2}{2\pi i} \left\{ (w_1 w_2)^{-\frac{1}{2}} \cdot A_k(w_1, w_2, z) \left[ N(w_1, w_2, z | \xi) \right]_{\sigma_i} \right\}, \]

(4.5a)

\[ F_{jk}^{(\Lambda_0 + \Lambda_1)}(z) = -q^{4(k-1)} z^{\frac{1}{2}} \cdot b(z) \cdot \int_{w_1=0}^{dw_1} \frac{dw_1}{2\pi i} \int_{w_2=0}^{dw_2} \frac{dw_2}{2\pi i} \left\{ (w_1 w_2)^{-1 - \frac{k}{2}} \cdot A_k(w_1, w_2, z) \times (1 - q^2)^2 (-q^4; q^4)_{\infty} \Pi'(z^{-\frac{k}{2}} w_1, z^{-\frac{k}{2}} w_2) \Theta_1 \left( (\frac{q^8 z}{w_1 w_2})^\frac{1}{2} | q^2 \right) \right\}, \]

(4.6a)
for \((j, k) = (2, 0), (0, 2)\):

\[
F_{11}^{(2\Lambda_i)}(z|\xi) = -q^{-4} \cdot (q^{-2}z - 1)^i \cdot b(z) \cdot \oint_{w_1 = 0} \frac{dw_1}{2\pi i} \oint_{w_2 = 0} \frac{dw_2}{2\pi i} \frac{1}{w_1} B(w_1, w_2, z) \left[ M(w_1, w_2, z|\xi) \right]^{\sigma_i} \right \},
\]

\(4.5b\)

\[
F_{11}^{(\Lambda_0+\Lambda_1)}(z) = q^{-1}(1 - q^4)^2(-q^4; q^4)_\infty b(z) \times \oint_{w_2 = 0} \frac{dw_2}{2\pi i} \left \{ \frac{1}{w_1} B(q^{-2}w_2, w_2, z) \Theta_1((q^{10}z)^{1/2}|q^2) \right \}.
\]

\(4.6b\)

Here are notations:

\[
A_0(w_1, w_2, z) = \frac{(q^6 \frac{w_1}{w_2})_\infty (q^6 \frac{w_2}{w_1})_\infty}{(q^{-2}w_1)_\infty (q^{-2}w_2)_\infty (q^{-2}z^{-1}w_1)_\infty (q^{-2}z^{-1}w_2)_\infty} \frac{1}{(q^6w_1^{-1})_\infty (q^6w_2^{-1})_\infty (q^{10}z w_1^{-1})_\infty (q^{10}z w_2^{-1})_\infty},
\]

\(4.7a\)

\[
A_2(w_1, w_2, z) = \frac{(q^6 \frac{w_1}{w_2})_\infty (q^6 \frac{w_2}{w_1})_\infty}{(q^2w_1)_\infty (q^2w_2)_\infty (q^{-2}z^{-1}w_1)_\infty (q^{-2}z^{-1}w_2)_\infty} \frac{1}{(q^6w_1^{-1})_\infty (q^6w_2^{-1})_\infty (q^{6}z w_1^{-1})_\infty (q^{6}z w_2^{-1})_\infty},
\]

\(4.7b\)

\[
B(w_1, w_2, z) = \frac{(q^6 \frac{w_1}{w_2})_\infty (q^6 \frac{w_2}{w_1})_\infty}{(q^2w_1)_\infty (q^{-2}w_2)_\infty (q^{-2}z^{-1}w_1)_\infty (q^{-2}z^{-1}w_2)_\infty} \frac{1}{(q^6w_1^{-1})_\infty (q^6w_2^{-1})_\infty (q^{6}z w_1^{-1})_\infty (q^{6}z w_2^{-1})_\infty};
\]

\(4.7c\)

\[
N(w_1, w_2, z|\xi) = (1 - q^2) (-\xi; \xi^2)_\infty \Pi(\frac{w_1}{w_2}|\xi) \Theta_1((q^{6}z)^{1/2}|\xi),
\]

21
\[ M(w_1, w_2, z|\xi) = (1 - q^4)(-\xi; \xi^2)_{\infty} \varpi(\frac{w_1}{w_2}|\xi)\Theta_1((\frac{q^6 z}{w_1 w_2})^{1/2}|\xi); \]

\[ \Pi(w|\xi) = 1 - q \cdot f_0(\xi) + \sum_{m=1}^{+\infty} (w^m + w^{-m}) \frac{1}{2} [(1 - q^2)q^{-2m} + q(f_{m-1}(\xi) - f_m(\xi))], \]

\[ \varpi(w|\xi) = (w - q^2) \sum_{n \in \mathbb{Z}} f_n(\xi)w^n + (q + q^{-1})[1 + (1 - q^2) \sum_{n=1}^{+\infty} q^{-2n}w^n], \]

\[ \Pi'(w_1, w_2) = \frac{1}{2} \sum_{n=0}^{+\infty} \left[ w_1 \left(\frac{w_1}{w_2}\right)^n + w_2 \left(\frac{w_2}{w_1}\right)^n \right] \cdot q^{-2n-2}(1 + q^{4n+2}); \]

and

\[ b(\frac{z_1}{z_2}) = \frac{(q^6; q^4)_{\infty}^4}{(q^8; q^4)_{\infty}^4} \frac{(q^6 \frac{z_1}{z_2}; q^4)_{\infty}}{(q^2 \frac{z_2}{z_1}; q^4)_{\infty}}. \]

We note that \( \varpi(w|q^2) = q^{-1}(1 - q^4)\delta(q^2w). \)

Spin one-point functions are related to these functions. Non zero ones are

\[ \langle E_{jj}^{(\lambda_m)} \rangle = P_j^j(z|\lambda_m) = \frac{c_{k-j}}{c_m} \cdot \frac{1}{g_{\lambda_m}Z(\lambda_m)} \times F_{k-j,j}^{(\lambda_m)}(q^{-2}|q^2). \]

where \( j, m = 0, 1, 2 \) (for \( k = 2 \)). Observe that they are independent of \( z \). From eq.(4.5) and eq.(4.6), we have for the numerators

\[ F_{jk}^{(2\Lambda_i)}(q^{-2}|\xi) = q^{-4(2-k)} \cdot b(q^{-2}) \cdot \oint_{w_1=0} \frac{dw_1}{2\pi i} \oint_{w_2=0} \frac{dw_2}{2\pi i}\left\{ (w_1 w_2)^{-\frac{k}{2}} \cdot A_k(w_1, w_2, q^{-2}) \times \left[ (1 - q^2)(-\xi; \xi^2)_{\infty} \Pi(\frac{w_1}{w_2}|\xi)\Theta_1((q^4 \frac{w_1}{w_2})^{1/2}|\xi) \right]^{\sigma_i} \right\} \]

\[ (4.8a) \]

and

\[ F_{jk}^{(\Lambda_0+\Lambda_1)}(q^{-2}) = -q^{3k-4} \cdot b(q^{-2}) \cdot \oint_{w_1=0} \frac{dw_1}{2\pi i} \oint_{w_2=0} \frac{dw_2}{2\pi i}\left\{ (w_1 w_2)^{-1-\frac{k}{2}} \cdot A_k(w_1, w_2, q^{-2}) \times (1 - q^2)(-q^4; q^4)_{\infty} \Pi'(q^k w_1, q^k w_2)\Theta_1((q^6 \frac{w_1}{w_2})^{1/2}|q^2) \right\} \]

\[ (4.9a) \]
for \((j, k) = (2, 0), (0, 2), \) and

\[
F_{11}^{(2\Lambda_1)}(q^{-2}) = -\frac{1}{2} q^{-5} (1 - q^4)^2 (-q^2; q^4)_\infty b(q^{-2}) \cdot (q^4; q^4)_\infty (q^8; q^4)_\infty \\
\times \oint_{w_2=0} \frac{dw_2}{2\pi i} \frac{1}{w_1} \frac{1}{((q^{-2}w_2; q^2)_\infty (q^6w_2^{-1}; q^2)_\infty)^2} \Theta_1(q^3w_2^{-1}|q^2),
\]

(4.8b)

\[
F_{11}^{(\Lambda_0+\Lambda_1)}(q^{-2}) = q^{-1} (1 - q^4)^2 (-q^4; q^4)_\infty b(q^{-2}) \cdot (q^4; q^4)_\infty (q^8; q^4)_\infty \\
\times \oint_{w_2=0} \frac{dw_2}{2\pi i} \frac{1}{w_1} \frac{1}{((q^{-2}w_2; q^2)_\infty (q^6w_2^{-1}; q^2)_\infty)^2} \Theta_1(q^4w_2^{-1}|q^2).
\]

(4.9b)

We note that

\[
A_0(w_1, w_2, q^{-2}) = \frac{(q^6w_1^2; q^4)_\infty (q^6w_2^2; q^4)_\infty}{\prod_{t=1,2} ((q^{-2}w_t; q^2)_\infty (q^6w_t^{-1}; q^2)_\infty)},
\]

\[
A_2(w_1, w_2, q^{-2}) = \frac{(q^6w_1^2; q^4)_\infty (q^6w_2^2; q^4)_\infty}{\prod_{t=1,2} ((w_t; q^2)_\infty (q^4w_t^{-1}; q^2)_\infty)},
\]

and that

\[
b(q^{-2}) = \frac{(q^6; q^4)_\infty}{(q^8; q^4)_\infty} (q^2; q^4)_\infty (q^2; q^4)_\infty = (1 - q^4)(q^4; q^4)_\infty (q^8; q^4)_\infty.
\]

These are our final expressions for spin one-point functions.

We can prove the following relations which show ‘homogeneity’ of the spin chain.

**Proposition 2.** We have relations

\[
P^j_j(z|2\Lambda_0) = P^{2-j}_{2-j}(z|2\Lambda_1), \quad j = 0, 1, 2; \quad P^0_0(z|\Lambda_0 + \Lambda_1) = P^2_2(z|\Lambda_0 + \Lambda_1).
\]

Or equivalently, in terms of the function \(F\) they are written as

\[
F_{2-j,j}^{(2\Lambda_0)}(q^{-2}|q^2) = q^{2j-2} \cdot F_{j,2-j}^{(2\Lambda_1)}(q^{-2}|q^2), \quad j = 0, 1, 2;
\]

\[
F_{20}^{(\Lambda_0+\Lambda_1)}(q^{-2}) = q^{-2} \cdot F_{02}^{(\Lambda_0+\Lambda_1)}(q^{-2}),
\]

23
respectively.

Proof. Noticing a relation $\Theta_1^{\sigma_i}(q^2 z|q^2) = q^{-2} z^{-2} \Theta_1^{\sigma_1-i}(z|q^2)$ ($i = 0, 1$, $\sigma_0 = S$, $\sigma_1 = A$), and

$$A_2(q^{-2} w_1, q^{-2} w_2, q^{-2}) = A_0(w_1, w_2, q^{-2}),$$

we can prove it easily (by changing integral variables, e.g., $q^2 w_j$ by $w_j$, etc.)

Here we give a list of expansions in $q$ of our integral representations.

\begin{align*}
P_2^2(2\Lambda_0) &= 1 - 2q^2 + 5q^6 - 2q^8 - 9q^{10} + 3q^{12} + 19q^{14} + O(q^{16}); \\
P_0^0(2\Lambda_0) &= 2q^4 + q^6 - 4q^8 - 9q^{10} + 7q^{12} + 19q^{14} + O(q^{16}); \\
P_1^1(2\Lambda_0) &= 2q^2 - 2q^4 - 6q^6 + 6q^8 + 18q^{10} - 10q^{12} - 38q^{14} + O(q^{16}); \\
P_0^0(\Lambda_0 + \Lambda_1) &= 2q^2 - 4q^4 + 2q^6 + 6q^{10} - 4q^{12} - 6q^{14} + O(q^{16}); \\
P_1^1(\Lambda_0 + \Lambda_1) &= 1 - 4q^2 + 8q^4 - 4q^6 - 12q^{10} + 8q^{12} + 12q^{14} + O(q^{16}).
\end{align*}

From this list we get

$$\langle s^z \rangle^{(2\Lambda_1)} = 1 - 2q^2 - 2q^4 + 4q^6 + 2q^8 - 4q^{12} + O(q^{16}) = \frac{(q^2; q^4)_{\infty}^2}{(-q^4; q^4)_{\infty}}$$

where $s^z = \text{diag}(1, 0, -1)$, which agrees with the known Bethe Ansatz result\(^{12}\) (see Appendix B). We also note that the expectation value with respect to the vacuum in $V(\Lambda_0 + \Lambda_1) \otimes V(\Lambda_0 + \Lambda_1)^*a$ is zero, since from Proposition 2

$$\langle s^z \rangle^{(\Lambda_0 + \Lambda_1)} = P_0^0(z|\Lambda_0 + \Lambda_1) - P_2^2(z|\Lambda_0 + \Lambda_1) = 0.$$ 

This again agrees with the result in ref.12.

Finally, as was mentioned in section 3 we have not justified rigorously the correctness of our vertex operators. The proof remains to be completed.
Acknowledgements

The author wishes to thank Professors Tetsuji Miwa, Michio Jimbo, and Tetsuji Tokihiro for many helpful discussions and suggestions.

Appendix A. The Hamiltonian

Let us define symmetric and anti-symmetric parts of $h$ (defined by eq.(4.1)) by

$$h_S = \frac{1}{2}(h + P h P),$$
$$h_A = \frac{1}{2}(h - P h P)$$

where $P$ is the permutation operator in $\text{End}(V^{(k)} \otimes V^{(k)})$ such that $P(a \otimes b) = b \otimes a$.

An explicit form of the spin-1 analog of the XXZ Hamiltonian (thus, $k = 2$) is given as follows. We set $q = -e^{-\lambda}$ below $(0 < \lambda < +\infty)$.

$$h_S = s_1 \otimes s_1 + s_2 \otimes s_2 + \text{ch}(2\lambda) \cdot s_3 \otimes s_3 - \left( \sum_{j=1}^{3} s_j \otimes s_j \right)^2 + 2sh^2(\lambda) \cdot \left[ (s_3^2)^2 \otimes \text{id} + \text{id} \otimes (s_3^2)^2 - (s_3 \otimes s_3)^2 - 2 \cdot \text{id} \otimes \text{id} \right] + (2 + 4e^{-\lambda}\text{ch}^2(\lambda)) \cdot (s_1 \otimes s_1 + s_2 \otimes s_2)s_3 \otimes s_3 + (2 + e\lambda) \cdot s_3 \otimes s_3(s_1 \otimes s_1 + s_2 \otimes s_2),$$

$$h_A = -sh(2\lambda)(s_3 \otimes \text{id} - \text{id} \otimes s_3).$$

We see that the anti-symmetric part has the form $s_3 \otimes \text{id} - \text{id} \otimes s_3$. This implies that the summation of $h_A$ over all sites vanishes. Thus we conclude that the Hamiltonian has the form

$$H = \sum_{l \in \mathbb{Z}} \cdots \otimes 1^+ \otimes h_S 1^+ \otimes 1^+ \otimes \cdots.$$

Let us see the $q \to 0$ limit of the Hamiltonian. In the limit $q \to 0^-$ ($\lambda \to +\infty$) the Hamiltonian tends to an antiferromagnetic Ising Hamiltonian with appropriate renormalization; thus we say that the limit $q \to 0$ is an Ising limit. Define

$$h^{ising} = \lim_{\lambda \to +\infty} \frac{1}{2 \text{sh}k\lambda} h = -\lim_{q \to 0^-} \frac{(-1)^k}{q^k - q^{-k}} h,$$
or more directly
\[ h^{Ising} = - \lim_{q \to 0} \left[ \frac{d}{dz} \hat{R}(z, 1) \right]_{z=1}. \]

Then we find \( h^{Ising} \cdot u_i \otimes u_j = H(i, j)u_i \otimes u_j, \) where \( H(i, j) = -j \) for \( i + j \leq k, = i - k \) for \( i + j \geq k. \) Especially its symmetric part is \( h^{Ising}_S \cdot u_i \otimes u_j = H_S(i, j)u_i \otimes u_j \) where \( H_S(i, j) = -\frac{1}{2}(i + j) \) for \( i + j \leq k, = \frac{1}{2}(i + j - 2k) \) for \( i + j \geq k. \)

Explicit forms are as follows (for \( k = 2 \)):
\[
\begin{align*}
    h^{Ising}_S &= \frac{1}{2} \left[ s^3 \otimes s^3 + (s^3)^2 \otimes \text{id} + \text{id} \otimes (s^3)^2 - (s^3 \otimes s^3)^2 - 2 \cdot \text{id} \otimes \text{id} \right], \\
    h^{Ising}_A &= -\frac{1}{2} (s^3 \otimes \text{id} - \text{id} \otimes s^3).
\end{align*}
\]

**Appendix B. The result of Ref.12**

In ref.12, an exact expression for the one-point function \( \langle s^z \rangle^{(\lambda_m)} \) (in vertex model language, it is called the mean staggered polarization) is obtained for \( m = 0, 1, \ldots, k \) and for arbitrary spin \( k/2 \) using the Bethe Ansatz. Their result is (we change notations slightly)
\[
\langle s^z \rangle^{(\lambda_m)} = \prod_{n=1}^{\infty} \left( \frac{1 + q^{(k+2)n}}{1 - q^{(k+2)n}} \right)^2 \cdot \left[ m - \frac{k}{2} - 2 \sum_{l=0}^{+\infty} (-1)^l (f(l, k - m) - f(l, m)) \right]
\]

\( m = 0, 1, \ldots, k, \) where \( k/2 \) is the spin (\( k \) is the level in the present paper) and
\[
f(l, m) = [(k + 2)l + m + 1] \cdot \frac{q^{2[(k+2)l+m+1]}}{1 - q^{2[(k+2)l+m+1]}}.
\]

It gives series expansions of \( \langle s^z \rangle^{(k\Lambda_1)} \) in \( q: \)
\[
\begin{align*}
    k = 1: & 2q^2 - 2q^4 + 2q^8 - 4q^{10} + 2q^{16} - 2q^{18} + 4q^{20} + O(q^{22}); \\
    k = 2: & 1 - 2q^2 - 2q^4 + 4q^6 + 2q^8 - 4q^{12} + 2q^{16} - 6q^{18} + O(q^{22});
\end{align*}
\]

and so on.

We found their infinite-product representations. Here we report the results.
\[
\begin{align*}
    \langle s^z \rangle^{(\Lambda_1)} &= \frac{1}{2} \langle \sigma^z \rangle^{(\Lambda_1)} = \frac{1}{2} \cdot \frac{(q^2; q^2)^2_\infty}{(q^4; q^4)^2_\infty} \quad \text{for } k = 1; \\
    \langle s^z \rangle^{(2\Lambda_1)} &= \frac{(q^2; q^2)^2_\infty}{(-q^4; q^4)^2_\infty} = \frac{(q^4; q^4)^2_\infty}{(-q^2; q^2)^2_\infty(-q^4; q^4)^2_\infty} \quad \text{for } k = 2.
\end{align*}
\]

(We checked them up to \( q^{150} \).)
Appendix C. Normal ordering of exponentials of boson

We collect here formulas for the normal ordering of exponentials of boson operators. We recall that

\[ \Phi_2(z) = F_<(z)F_>(z)e^{\alpha(-q^4z)^{\frac{1}{2}}\partial_z}, \]
\[ x^+(z) = \eta E^+_<(z)E^+_>(z)\phi(z)e^{\alpha z^{\frac{1}{2}}+\frac{1}{2}\partial_z}, \]
\[ x^-(z) = \eta E^-_<(z)E^-_>(z)\phi(z)e^{-\alpha z^{\frac{1}{2}}-\frac{1}{2}\partial_z}, \]

where

\[ F_<(z) = \exp\left(\sum_{m=1}^{\infty} \frac{a_{-m}}{2m} q^{5m} z^m \right), \quad F_>(z) = \exp\left(-\sum_{m=1}^{\infty} \frac{a_{m}}{2m} q^{-3m} z^{-m} \right), \]
\[ E^+_<(z) = \exp\left(\sum_{m=1}^{\infty} \frac{a_{-m}}{2m} q^{-m} z^m \right), \quad E^+_>(z) = \exp\left(-\sum_{m=1}^{\infty} \frac{a_{m}}{2m} q^{-m} z^{-m} \right), \]
\[ E^-_<(z) = \exp\left(-\sum_{m=1}^{\infty} \frac{a_{m}}{2m} q^{m} z^m \right), \quad E^-_>(z) = \exp\left(\sum_{m=1}^{\infty} \frac{a_{-m}}{2m} q^{m} z^{-m} \right), \]

\[ \eta = (q + q^{-1})^{1/2}, \quad [a_m, a_{-n}] = \delta_{m,n} \frac{[2m]^2}{m}. \]

We have the following:

\[ F_>(z_1)F_<(z_2) = (1 - q^2 z_2/z_1)F_<(z_2)F_>(z_1), \]
\[ F_>(z_1)E^-_<(z_2) = \frac{1}{1 - q^{-2} z_2/z_1} E^-_<(z_2)F_>(z_1), \]
\[ E^-_<(z_1)F_<(z_2) = \frac{1}{1 - q^2 z_2/z_1} F_<(z_2)E^-_<(z_1), \]
\[ E^-_<(z_1)E^-_<(z_2) = (1 - q^2 z_2/z_1)E^-_<(z_2)E^-_<(z_1), \]
\[ F_>(z_1)E^+_<(z_2) = (1 - q^{-4} z_2/z_1)E^+_<(z_2)F_>(z_1), \]
\[ E^+_<(z_1)F_<(z_2) = (1 - q^4 z_2/z_1) F_<(z_2)E^+_<(z_1), \]
\[ E^+_<(z_1)E^+_<(z_2) = (1 - q^{-2} z_2/z_1) E^+_<(z_2)E^+_<(z_1), \]
\[ E^+_<(z_1)E^-_<(z_2) = \frac{1}{1 - z_2/z_1} E^-_<(z_2)E^+_<(z_1), \]
\[ E^-_<(z_1)E^+_<(z_2) = \frac{1}{1 - z_2/z_1} E^+_<(z_2)E^-_<(z_1). \]
References

1) B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki: Commun. Math. Phys. 151 (1993) 89.
2) M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki: Phys. Lett. A168 (1992) 256.
3) M. Idzumi, T. Tokihiro, K. Iohara, M. Jimbo, T. Miwa and T. Nakashima: Int. J. Mod. Phys. A 8 (1993) 1479.
4) I.B. Frenkel and N.H. Jing: Proc. Nat’l. Acad. Sci. USA 85 (1988) 9373.
5) V. Chari and A. Pressley: Commun. Math. Phys. 142 (1991) 261.
6) J. Lepowsky and M. Primc: Lecture Notes in Mathematics 1052 (1984) 194.
7) D. Bernard: Lett. Math. Phys. 17 (1989) 239.
8) A. Tsuchiya and Y. Kanie: Adv. Stud. Pure Math. 16 (1988) 297.
9) I.B. Frenkel and N.Yu. Reshetikhin: Commun. Math. Phys. 146 (1992) 1.
10) E. Date, M. Jimbo and M. Okado: Osaka Univ. Math. Sci. preprint 1 (1991).
11) M. Kashiwara: RIMS preprint 756 (1991).
12) E. Date, M. Jimbo, K. Miki and M. Okado: Int. J. Mod. Phys. A 7, Suppl. 1A (1992) 151.