Stability of non-constant equilibrium solutions for two-fluid non-isentropic Euler-Maxwell systems arising in plasmas

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Abstract. We consider the periodic problem for two-fluid non-isentropic Euler-Maxwell systems in plasmas. By means of suitable choices of symmetrizers and an induction argument on the order of the time-space derivatives of solutions in energy estimates, the global smooth solution with small amplitude is established near a non-constant equilibrium solution with asymptotic stability properties. This improves the results obtained in [15] for models with temperature diffusion terms by using the pressure functions \( p^\nu \) in place of the unknown variables densities \( n^\nu \).

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1 Introduction and main results

Recently there are many mathematical researches on the Euler-Maxwell systems which are used for modeling the motion of fluid plasmas (see [1, 19, 26] and references therein). In this article, we consider the period problem for the two-fluid non-isentropic compressible Euler-Maxwell system

\[
\begin{align*}
\partial_t n^\nu + \nabla \cdot (n^\nu u^\nu) &= 0, \\
\partial_t (n^\nu u^\nu) + \nabla \cdot (n^\nu u^\nu \otimes u^\nu) + \nabla p^\nu &= q^\nu n^\nu (E + u^\nu \times B) - n^\nu u^\nu, \\
\partial_t E^\nu + \nabla \cdot ((E^\nu + p^\nu) u^\nu) &= q^\nu n^\nu u^\nu \cdot E - n^\nu |u^\nu|^2 - (E^\nu \cdot n^\nu e_1), \\
\partial_t E - \nabla \times B &= n^e u^e - n^i u^i, \\
\partial_t B + \nabla \times E &= 0, \\
\nabla \cdot B &= 0, \\
E &= n^i - n^e + b(x),
\end{align*}
\]  

(1.1)

where \( \mathbb{T} = (\mathbb{R}/\mathbb{Z})^3 \) denotes a three-dimensional torus and \( q^e = -1 \ (q^i = 1) \) is the charge of electrons (ions). Here, \((n^\nu, u^\nu, \theta^\nu, E, B) : \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R}^1 \times \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3 \times \mathbb{R}^3 \) and \( \nabla \) is the usual gradient. Physically, \( n^\nu > 0 \) is the density of fluid, \( u^\nu \) is the velocity, \( \theta^\nu > 0 \) is the absolute temperature, \( E \) is the electric field, and \( B \) is the magnetic field. Functions \( p^\nu = n^\nu \theta^\nu, \ E^\nu = n^\nu e^\nu + \frac{1}{2} n^\nu |u^\nu|^2, \ e^\nu = c_s \theta^\nu \) and \( b(x) > 0 \) denote respectively pressure, total energy, internal energy and a doping term. The constants \( c_s > 0, \theta_l > 0 \) and \( e_l = c_s \theta_l \) stand for the coefficient of heat conduction, the back ground temperature and the background internal energy, respectively.

Next we set \( c_s = \theta_l = e_l = 1 \) for the sake of simplicity. This is not an essential restriction in
the study of global existence of smooth solutions. Then for \(n^\nu > 0\), system (1.1) is equivalent to

\[
\begin{aligned}
\frac{\partial}{\partial t} n^\nu + \nabla \cdot (n^\nu u^\nu) &= 0, \\
\frac{\partial}{\partial t} u^\nu + (u^\nu \cdot \nabla)u^\nu + \frac{1}{n^\nu} \nabla p^\nu &= q^\nu (E + u^\nu \times B) - u^\nu, \\
\frac{\partial}{\partial t} \theta^\nu + u^\nu \cdot \nabla \theta^\nu + \theta^\nu \nabla \cdot u^\nu &= \frac{1}{2} |u^\nu|^2 - (\theta^\nu - 1), \\
\frac{\partial}{\partial t} E - \nabla \times B &= n^e u^e - n^i u^i, \\
\nabla \cdot E &= n^i - n^e + b(x), \\
\frac{\partial}{\partial t} B + \nabla \times E &= 0, \\
\nabla \cdot B &= 0, \\
\nu &= e, i, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T},
\end{aligned}
\]  

(1.2)

with the initial condition

\[
(n^\nu, u^\nu, \theta^\nu, E, B)|_{t=0} = (n_0^\nu, u_0^\nu, \theta_0^\nu, E_0, B_0), \quad x \in \mathbb{T}, \quad \nu = e, i,
\]  

(1.3)

which satisfies the compatibility condition

\[
\nabla \cdot E_0 = n_0^i - n_0^e + b(x), \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{T}.
\]  

(1.4)

Now we consider equilibrium solution \((n^\nu, u^\nu, \theta^\nu, E, B) = (\bar{n}^\nu(x), 0, \bar{\theta}^\nu, \bar{E}(x), \bar{B}(x))\) to be a steady-state solution of (1.2). Then we get

\[
\begin{aligned}
\frac{1}{\bar{n}^e} \nabla \bar{p}^e &= \bar{E} = \frac{1}{\bar{n}^i} \nabla \bar{p}^i, \\
\bar{p}^\nu &= \bar{n}^\nu \bar{\theta}^\nu, \\
\bar{\theta}^\nu &= 1, \\
\nabla \times \bar{E} &= 0, \\
\nabla \cdot \bar{E} &= \bar{n}^i - \bar{n}^e + b(x), \\
\nabla \times \bar{B} &= 0, \\
\nabla \cdot \bar{B} &= 0, \quad x \in \mathbb{T},
\end{aligned}
\]  

(1.5)

which implies that \(\bar{B}\) is a constant vector in \(\mathbb{R}^3\). Moreover, the same analysis as that for Proposition 1.1 in [15] gives the existence and uniqueness of a smooth periodic solution for (1.5).

**Lemma 1.1** Suppose \(b = b(x)\) to be a smooth periodic function such that \(b \geq \text{const.} > 0\) in \(\mathbb{T}\). Then the periodic problem (1.5) has a unique smooth solution \((\bar{n}^\nu, 0, \bar{\theta}^\nu, \bar{E}, \bar{B})\) with \(\bar{\theta}^\nu = 1\) and \(\bar{n}^\nu \geq \text{const.} > 0\) in \(\mathbb{T}\), \(\nu = e, i\).

For \(n^\nu, \theta^\nu > 0\), (1.2) is a nonlinear and symmetrizable hyperbolic-parabolic system in the sense of Friedrichs. Then following the result of Kato [13] and the famous work of Majda [18], the periodic problem (1.2)-(1.3) admits a unique local smooth solution when initial values are smooth.

**Proposition 1.1** *(Local existence of smooth solutions, see [13,18])* Assume integer \(s \geq 3\) and (1.4) holds. Let \(\bar{B} \in \mathbb{R}^3\) be any given constant vector and \((\bar{n}^\nu, 0, \bar{\theta}^\nu, \bar{E}, \bar{B})\) be an equilibrium solution of (1.2) in the sense of Lemma 1.1. Suppose \((n_0^\nu - \bar{n}^\nu, u_0^\nu, \theta_0^\nu - 1, E_0 - \bar{E}, B_0 - \bar{B}) \in H^s(\mathbb{T})\) with \(n_0^\nu, \theta_0^\nu \geq 2\kappa\) for some given constant \(\kappa > 0\). Then there exists \(T > 0\) such that periodic problem (1.2)-(1.3) admits a unique smooth solution which satisfies \(n^\nu, \theta^\nu \geq \kappa\) in \([0, T] \times \mathbb{T}\) and

\[
(n^\nu - \bar{n}^\nu, u^\nu, \theta^\nu - 1, E - \bar{E}, B - \bar{B}) \in C^1([0, T]; H^{s-1}(\mathbb{T})) \cap C([0, T]; H^s(\mathbb{T})), \quad \nu = e, i.
\]
There are mathematical investigations in numerical simulations [3], the asymptotic limits with small parameters [22], the existence of solutions for Euler-Maxwell systems. Particularly, some of them are concerned with the global existence and asymptotic stability of small-amplitude smooth solutions near constant equilibrium states. For one-dimensional simplified isentropic Euler-Maxwell system in which the energy equation is not contained, the global existence of weak solutions is proved by the compensated compactness method [2]. For the three-dimensional isentropic Euler-Maxwell systems, the existence of global smooth small solutions to the Cauchy problem in $\mathbb{R}^3$ is established for $s \geq 3$ and the asymptotic behaviors of solutions when $s \geq 4$ [28]. By using suitable choices of symmetrizers and energy estimates, the global existence and the long time behaviors of smooth solutions to the periodic problem in $\mathbb{T}$ and to the initial value problem in $\mathbb{R}^3$ for $s \geq 3$ are established [23, 24]. By high- and low-frequency decomposition methods, uniform (global) classical solutions to the initial value problem in Chemin-Lerner’s spaces with critical regularity is constructed [32]. For $s \geq 4$, by the tools of Fourier analysis, the decay rates of global smooth solutions in $L^q$ with $2 \leq q \leq \infty$ when the time goes to infinity are presented [4, 5]. And for $s \geq 6$, the long-time decay rates of global smooth solutions in $H^{s-2k}(\mathbb{R}^3)$ with $0 \leq k \leq \lfloor s/2 \rfloor$ are also established [27]. For the three-dimensional non-isentropic Euler-Maxwell systems, the existence of global smooth small solutions to the Cauchy problem in $\mathbb{R}^3$ is established [8, 29]. For the Euler-Maxwell systems without damping, an additional relation was made to establish such a global existence result for the one-fluid model [10]. And for the two-fluid case without damping, the global stability of a constant neutral background is proved [11], in the sense that irrotational, smooth and localized perturbations of a constant background with small amplitude lead to global smooth solutions in $\mathbb{R}^3$.

All these results above hold when the solutions are close to a constant equilibrium solution of the Euler-Maxwell systems where $b(x)$ is a positive constant (for example $b(x) = 1$). When $b(x)$ is a small perturbation of a constant, the Cauchy problem for compressible Euler-Maxwell systems are considered [17], and the time decay rates of smooth solutions are established. When $b(x)$ is large, such a stability problem is much more complicated than before. Recently, motivated by the Guo-Strauss’s work on the study of the damped Euler-Poisson system on the general bounded domain [12], by employing an induction argument on the order of the derivatives of solutions, the stabilities of non-constant equilibrium solutions for the isentropic Euler-Maxwell systems [7, 25] and non-isentropic Euler-Maxwell systems with temperature diffusion terms [9, 15], respectively. Very recently, with the help of choosing a new symmetrizer matrix, the stability of the one-fluid non-isentropic Euler-Maxwell systems is considered [16]. However, there is no result on the stability of non-constant equilibrium solutions for the two-fluid non-isentropic Euler-Maxwell systems without temperature diffusion effects so far. The goal here is to consider this problem.

Now we state the main results of this article.

**Theorem 1.1** Let $s \geq 3$ and (1.4) hold. Let $B \in \mathbb{R}^3$ be any given constant vector and $(\vec{n}^\nu, 0, 1, \vec{E}, B)$ be an equilibrium solution of (1.2) satisfying $\vec{n}^\nu \geq $ const. > 0 in the sense of Lemma 1.1. Then
there exist constants $\delta_0 > 0$ and $C > 0$, independent of any given time $t > 0$, such that if
\[
\|(n_0^\nu - \bar{n}^\nu, u_0^\nu, \theta_0^\nu - 1, E_0 - \bar{E}, B_0 - \bar{B})\|_s \leq \delta_0, \quad \nu = e, i,
\]
where $\| \cdot \|_m$ is the norm of usual Sobolev spaces $H^m(\mathbb{T})$, periodic problem (1.2)-(1.3) has a unique global smooth solution $(n^\nu, u^\nu, \theta^\nu, E, B)$ satisfying
\[
\sum_{\nu = e, i} \left\| (n^\nu(t, \cdot) - \bar{n}^\nu, u^\nu(t, \cdot), \theta^\nu(t, \cdot) - 1) \right\|^2_s + \left\| (E(t, \cdot) - \bar{E}, B(t, \cdot) - \bar{B}) \right\|^2_s \leq C \sum_{\nu = e, i} \left\| (n_0^\nu - \bar{n}^\nu, u_0^\nu, \theta_0^\nu - 1, E_0 - \bar{E}, B_0 - \bar{B}) \right\|^2_s, \quad \forall t \geq 0.
\]
Furthermore,
\[
\lim_{t \to \infty} \left\| (n^\nu(t) - \bar{n}^\nu, u^\nu(t), \theta^\nu(t) - 1) \right\|_{s-1} = 0, \quad \nu = e, i,
\]
\[
\lim_{t \to \infty} \left\| (E(t) - \bar{E}) \right\|_{s-1} = 0,
\]
and
\[
\lim_{t \to +\infty} \left( \left\| \partial_t B(t) \right\|_{s-2} + \left\| \nabla B(t) \right\|_{s-2} \right) = 0,
\]
where $\| \cdot \|_m$ is defined in the next section.

**Remark 1.1** Obviously, the result above still holds for system (1.2) in case the temperature equation contains temperature diffusion terms (see [15]).

**Remark 1.2** The result in Theorem 1.1 for two-fluid non-isentropic Euler-Maxwell systems still holds for two-fluid non-isentropic Euler-Poisson systems which can be regarded as a special case of the former systems by setting $B = 0$ and $E = -\nabla \Psi$ (see [7, 25]).

**Remark 1.3** Different from the proof process in [15], we choose a new symmetrizer like (2.25) here. The effect of temperature diffusion in non-isentropic Euler-Maxwell equations has been released successfully by this suitable choices of symmetrizers and the techniques of using the pressure functions $p^\nu$ in place of the unknown variables densities $n^\nu$ (see [16, 18]).

**Remark 1.4** It should be emphasized that the velocity relaxation and temperature terms of the considered two-fluid non-isentropic Euler-Maxwell system here play a key role in the proof of Theorem 1.1. We shall study in the other forthcoming work the case of non-relaxation for which the proof is much more complicated to carry out.
The proof of Theorem 1.1 is mainly based on the suitable choices of symmetrizers and an induction argument on the order of the time-space derivatives of solutions. These techniques, firstly employed by Peng [25] in the one-fluid isentropic case and then extended by Feng-Peng-Wang [7] to the two-fluid isentropic case, can release the difficulty due to the appearance of non-constant equilibrium solutions. Besides of these techniques, by using diffusion effects of temperature, Feng-Wang-Li [9, 15] proved the stability of non-constant equilibrium solution of the periodic problems to non-isentropic models for $s \geq 6$. Very recently, Liu-Peng [16] considered the stability of the one-fluid non-isentropic models for $s \geq 3$. It should be pointed out that the techniques of choosing a non-diagonal symmetrizers and making a change of unknown variables in [16] can replace the help of diffusion effects of temperature used in [9]. We remark that the two-fluid non-isentropic Euler-Maxwell systems are much more complex than both the two-fluid isentropic and the one-fluid non-isentropic Euler-Maxwell systems because they contain two different charged fluids energy equations besides the density and velocity equations. Different from the one-fluid non-isentropic Euler-Maxwell systems in [16], we shall overcome the difficulties caused by the coupling of two fluids when we establish the energy estimates. This can be done by employing new multiplier functions. Indeed, firstly we introduce a new potential function $\psi$ such that $\nabla \psi = E - E$, and then another function $\eta'' = Q'' + q\nu \psi$ to establish the estimate of $\partial_t^k Q''$ indirectly (see Lemma 3.8), where $Q'' = (\ln p'') - \ln \bar{p}'$. This yields a recurrence relation in Lemma 3.9 which allows us to obtain the estimates by induction on $(k, m)$ with $k$ decreasing and $m$ increasing. Then Theorem 1.1 follows in the way by combining these previous estimates with Proposition 1.1 and the standard continuity argument.

We conclude this section by stating the arrangement of the rest of this article. In Section 2, we recall some useful preliminary Lemmas and reformulate the periodic problem under consideration. In section 3, detailed energy estimates are established. In section 4, we complete the proof of Theorems 1.1 by using an induction argument and combining the estimates above.

2 Preliminaries and Reformulation of periodic problem (1.2)-(1.3)

2.1 Preliminaries

In this subsection, we want to make preliminary works for proving Theorem 1.1. Firstly, let us introduce some notations for the use throughout this paper. The expression $f \sim g$ means $\gamma g \leq f \leq \frac{1}{\gamma} g$ for a constant $0 < \gamma < 1$. We denote by $\| \cdot \|_s$ the norm of the usual Sobolev space $H^s(\mathbb{T})$, and by $\| \cdot \|$ and $\| \cdot \|_{L^p}$ the norms of $L^2(\mathbb{T})$ and $L^p(\mathbb{T})$, respectively, where $2 < p \leq +\infty$. We also denote by $\langle \cdot, \cdot \rangle$ the inner product over $L^2(\mathbb{T})$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$
For \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3 \), \( \beta \leq \alpha \) stands for \( \beta_j \leq \alpha_j \) for \( j = 1, 2, 3 \), and \( \beta < \alpha \) stands for \( \beta \leq \alpha \) and \( \beta \neq \alpha \). For \( T > 0 \) and \( m \geq 1 \), we define the Banach space 
\[
B_{m,T}(\mathbb{T}) = \bigcap_{k=0}^{m} C^k \left([0,T], H^{m-k}(\mathbb{T})\right),
\]
with the norm 
\[
|||f|||_m = \sqrt{\sum_{k+|\alpha|\leq m} \|\partial_t^k \partial^\alpha f\|_2^2}, \quad \forall f \in B_{m,T}(\mathbb{T}).
\]

Obviously, \( \| \cdot \|_m \leq ||| \cdot |||_m \).

Next, we introduce three lemmas which will be used in the proof of Theorem 1.1.

**Lemma 2.2** (Poincaré inequality, see [6].) Let \( 1 \leq p < \infty \) and \( \Omega \subset \mathbb{R}^3 \) be a bounded connected open domain with Lipschitz boundary. Then there exists a constant \( C > 0 \) depending only on \( p \) and \( \Omega \) such that
\[
\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega),
\]
where
\[
u_\Omega = \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx
\]
is the average value of \( u \) over \( \Omega \).

**Lemma 2.3** (see [25]) Let \( s \geq 3 \) and \( u, v \in B_{s,T}(\mathbb{T}) \). It holds
\[
|||uv|||_s \leq C |||u|||_s |||v|||_s.
\]

**Lemma 2.4** see [25] Let \( s \geq 3 \) and \( v \in B_{s,T}(\mathbb{T}) \) satisfying \( \partial_t v = f(x,v,\partial_x v) \), with \( f \) being a smooth function such that \( f(x,0,0) = 0 \). Then for all \( t \in [0,T] \), we have
\[
\|\partial_t^k \partial^\alpha v(t, \cdot)\|_s \leq C \|v(t, \cdot)\|_s, \quad \forall k + |\alpha| \leq s,
\]
where the positive constant \( C \) may depend continuously on \( \|v\|_s \).

### 2.2 Reformulation of periodic problem (1.2)-(1.3)

Firstly, we introduce new unknown variables. Noticing that \( p^\nu = n^\nu \theta^\nu \), then the summation of the first equation multiplying \( \theta^\nu \) and the third equation in (1.2) multiplying \( n^\nu \) gives
\[
\partial_t p^\nu + u^\nu \cdot \nabla p^\nu + 2p^\nu \nabla \cdot u^\nu = \frac{p^\nu}{2\theta^\nu} |u^\nu|^2 - \frac{p^\nu}{\theta^\nu} (\theta^\nu - 1), \quad \nu = e, i.
\]

Let
\[
q^\nu = \ln p^\nu, \quad \bar{q}^\nu = \ln \bar{p}^\nu, \quad \nu = e, i.
\]

Then for \( p^\nu > 0 \), it follows that
\[
\partial_t q^\nu + u^\nu \cdot \nabla q^\nu + 2\nabla \cdot u^\nu = \frac{1}{2\theta^\nu} |u^\nu|^2 - \frac{1}{\theta^\nu} (\theta^\nu - 1), \quad \nu = e, i.
\]
Suppose \((n^\nu, u^\nu, \theta^\nu, E, B)\) to be a local smooth solution to the periodic problem (1.2)-(1.3). Now, for \(\nu = e, i\), we introduce the perturbed variables

\[
Q^\nu = q^\nu - \bar{q}^\nu, \quad \Theta^\nu = \theta^\nu - 1, \quad F = E - \bar{E}, \quad G = B - \bar{B}, \quad V^\nu = \begin{pmatrix} Q^\nu \\ u^\nu \\ \Theta^\nu \end{pmatrix}, \quad Z = \begin{pmatrix} V^e \\ V^i \\ F \\ G \end{pmatrix}. \tag{2.15}
\]

Substituting these expressions into (1.2), and taking into account (1.5), we obtain

\[
\begin{aligned}
\partial_t Q^\nu + u^\nu \cdot \nabla Q^\nu + 2\nabla \cdot u^\nu + u^\nu \cdot \nabla q^\nu = & \frac{1}{2\theta^\nu}u^\nu|^2 - \frac{1}{\theta^\nu}\Theta^\nu, \\
\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \theta^\nu \nabla Q^\nu + \Theta^\nu \nabla q^\nu = & q_\nu (F + u^\nu \times (\bar{B} + G)) - u^\nu, \\
\partial_t \Theta^\nu + u^\nu \cdot \nabla \Theta^\nu + \theta^\nu \nabla \cdot u^\nu = & \frac{1}{\theta^\nu}u^\nu|^2 - \Theta^\nu, \\
\partial_t F - \nabla \times G = & n^e u^e - n^i u^i, \quad \nabla \cdot F = N^i - N^e, \\
\partial_t G + \nabla \times F = & 0, \quad \nabla \cdot G = 0, \quad \nu = e, i, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T},
\end{aligned} \tag{2.16}
\]

with the initial condition :

\[
Z|_{t=0} = Z_0 = \left( Q_0^\nu, u_0^\nu, \Theta_0^\nu, Q_0^i, u_0^i, \Theta_0^i, F_0, G_0 \right), \quad x \in \mathbb{T}, \tag{2.17}
\]

which satisfies the compatibility condition :

\[
\nabla \cdot F_0 = N_0^i - N_0^e, \quad \nabla \cdot G_0 = 0, \quad x \in \mathbb{T}. \tag{2.18}
\]

Here \(N_0^\nu = n_0^\nu - \bar{n}^\nu\), \(Q_0^\nu = \ln (n_0^\nu \theta_0^\nu) - \ln (\bar{n}^\nu)\), \(\Theta_0^\nu = \theta_0^\nu - 1\), \(F_0 = E_0 - \bar{E}\) and \(G_0 = B_0 - \bar{B}\).

A direct computation gives

\[
N^\nu = n^\nu - \bar{n}^\nu = \frac{p^\nu}{\theta^\nu} - \frac{\bar{p}^\nu}{\theta^\nu} = \frac{n^e}{\theta^\nu} - \frac{\bar{n}^e}{\theta^\nu} \sim Q^\nu + \Theta^\nu, \quad \nu = e, i, \tag{2.19}
\]

which implies that \(N^\nu\) can be regarded as a function of \(Q^\nu\) and \(\Theta^\nu\) with order one.

Next, we can also rewrite the non-isentropic Euler equations of (2.16) in the matrix form :

\[
\partial_t V^\nu + \sum_{j=1}^{3} A_j^\nu (u^\nu, \theta^\nu) \partial_j V^\nu + L^\nu(x) V^\nu = K^\nu (u^\nu, \theta^\nu, F, G, x), \quad \nu = e, i, \tag{2.20}
\]

Supplemented by the Maxwell equations

\[
\begin{aligned}
\partial_t F - \nabla \times G = & n^e u^e - n^i u^i, \quad \nabla \cdot F = N^i - N^e, \\
\partial_t G + \nabla \times F = & 0, \quad \nabla \cdot G = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}, \tag{2.21}
\end{aligned}
\]

with

\[
A_j^\nu (u^\nu, \theta^\nu) = \begin{pmatrix} u_j^\nu & 2e_j^T & 0 \\ 0 & \theta^\nu e_j^T & u_j^\nu \\ 0 & 0 & \theta^\nu e_j^T \end{pmatrix}, \quad j = 1, 2, 3, \quad \nu = e, i. \tag{2.22}
\]
\[
L^\nu(x) = \begin{pmatrix}
0 & (\nabla \bar{q}^\nu)^T & 0 \\
0 & 0 & \nabla \bar{q}^\nu \\
0 & 0 & 0
\end{pmatrix}, \quad \nu = e, i, \quad (2.23)
\]

\[
K^\nu(u^\nu, \theta^\nu, F, G, x) = \begin{pmatrix}
\frac{1}{2\theta^\nu}|u^\nu|^2 - \frac{1}{\theta^\nu}\Omega^\nu \\
q^\nu \left( F + u^\nu \times (\bar{B} + G) \right) - u^\nu \\
\frac{1}{2}|u^\nu|^2 - \Theta^\nu
\end{pmatrix}, \quad \nu = e, i, \quad (2.24)
\]

where \((e_1, e_2, e_3)\) denotes the canonical basis of \(\mathbb{R}^3\), \(I_3\) denotes the \(3 \times 3\) unit matrix and we use \([\cdot]^T\) to denote the transpose of a vector.

Obviously, system (2.20) for \(V^\nu\) is symmetrizable hyperbolic when both \(\theta^\nu\) and \(p^\nu = n^\nu\theta^\nu\) are positive. In fact, we can choose a symmetric and positive definite matrix as

\[
A^\nu_0(p^\nu, \theta^\nu) = \begin{pmatrix}
p^\nu & 0 & -p^\nu \theta^\nu \\
0 & p^\nu I_3 & 0 \\
-p^\nu \theta^\nu & 0 & \frac{2p^\nu}{|\theta^\nu|^2}
\end{pmatrix}, \quad \nu = e, i, \quad (2.25)
\]

which implies that

\[
\tilde{A}_j^\nu(p^\nu, u^\nu, \theta^\nu) = A^\nu_0(p^\nu, \theta^\nu) A^\nu_j(u^\nu, \theta^\nu) = \begin{pmatrix}
p^\nu u^\nu_j & p^\nu e^T_j - \frac{p^\nu}{\theta^\nu} u^\nu_j \\
p^\nu e_j & \frac{p^\nu}{\theta^\nu} u^\nu_j I_3 & 0 \\
-p^\nu \theta^\nu u^\nu_j & 0 & \frac{2p^\nu}{|\theta^\nu|^2} u^\nu_j
\end{pmatrix}, \quad \nu = e, i, \quad (2.26)
\]

is symmetric, and then system (2.20) is symmetrizable hyperbolic when \(n^\nu > 0\) and \(\theta^\nu > 0\). Moreover, we have

\[
A^\nu_0(p^\nu, \theta^\nu)L^\nu(x) = \begin{pmatrix}
0 & p^\nu(\nabla \bar{q}^\nu)^T & 0 \\
0 & 0 & \frac{p^\nu}{\theta^\nu} \nabla \bar{q}^\nu \\
0 & \frac{p^\nu}{\theta^\nu}(\nabla \bar{q}^\nu)^T & 0
\end{pmatrix}, \quad \nu = e, i. \quad (2.27)
\]
Then it follows from $\nabla \tilde{q}^{\nu} = \frac{1}{p^\nu} (\nabla \tilde{p}^\nu)$ that the following matrix

$$B^\nu(p^\nu, u^\nu, \theta^\nu, x) = \sum_{j=1}^{3} \partial_j \tilde{A}^\nu_j(p^\nu, u^\nu, \theta^\nu) - 2A^\nu_0(p^\nu, \theta^\nu)L^\nu(x)$$

$$= \begin{pmatrix}
\nabla \cdot (p^\nu u^\nu) & (\nabla p^\nu)^T - 2 \frac{p^\nu}{\theta^\nu} (\nabla p^\nu)^T - \nabla \cdot \left( \frac{p^\nu u^\nu}{\theta^\nu} \right) \\
\nabla p^\nu & \nabla \cdot \left( \frac{p^\nu u^\nu}{\theta^\nu} \right) I_3 - 2 \frac{p^\nu}{\theta^\nu \theta^\nu} (\nabla \tilde{p}^\nu) \\
-\nabla \cdot \left( \frac{p^\nu u^\nu}{\theta^\nu} \right) & 2 \frac{p^\nu}{\theta^\nu \theta^\nu} (\nabla \tilde{p}^\nu)^T - 2 \nabla \cdot \left( \frac{p^\nu u^\nu}{|\theta^\nu|^2} \right)
\end{pmatrix}, \quad \nu = e, i,$$

(2.28)

is antisymmetric at the point $(p^\nu, u^\nu, \theta^\nu) = (\tilde{p}^\nu, 0, \tilde{\theta}^\nu) = (\bar{p}^\nu, 0, 1)$.

From now on, let $T > 0$ and $Z$ be a smooth solution of (2.20)-(2.21) with the initial condition (2.17) defined on time interval $[0, T]$. We set

$$\omega_T = \sup_{0 \leq t \leq T} |||Z(t)|||_s.$$

(2.29)

From the continuous embedding $H^{s-1}(T) \hookrightarrow L^\infty(T)$ for $s \geq 3$, there is a constant $C > 0$ such that

$$||f||_{L^\infty} \leq C||f||_{s-1}, \quad \forall f \in H^{s-1}(T).$$

In the following, we always suppose that integer $s \geq 3$ and $\omega_T$ is small enough, as a consequence, it is easy to see that

$$0 < \text{const.} \leq \frac{3}{4} \bar{n}^\nu \leq n^\nu = \bar{n}^\nu + N^\nu \leq \frac{4}{3} \bar{n}^\nu, \quad \frac{3}{4} \leq \theta^\nu = 1 + \Theta^\nu \leq \frac{4}{3}, \quad \nu = e, i,$$

(2.30)

and then

$$0 < \text{const.} \leq \frac{9}{16} \bar{n}^\nu \leq p^\nu = n^\nu \theta^\nu \leq \frac{16}{9} \bar{n}^\nu, \quad \nu = e, i,$$

(2.31)

3 Energy estimates

In this section we begin to use the normal energy method to obtain some uniform-in-time a priori estimates for smooth solutions to the periodic problem (2.20)-(2.21) with the initial condition (2.17). In the first subsection, we establish energy estimates with dissipation estimates of velocity $u^\nu$ and temperature $\Theta^\nu$. In the second subsection, we establish a recurrence relationship for proving Theorem 1.1 by combining the dissipation estimates for density $N^\nu$, pressure variables $Q^\nu$ and electric field $F$. 


3.1 Dissipation energy estimates for velocity $u^\nu$ and temperature $\Theta^\nu$.

Assume $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $1 \leq k + |\alpha| \leq s$. Applying $\partial_t^k \partial^\alpha$ to (2.20), we have

$$\partial_t V_{k,\alpha}^\nu + \sum_{j=1}^3 A_j^\nu(u^\nu, \theta^\nu) \partial_j V_{k,\alpha}^\nu + L^\nu(x) V_{k,\alpha}^\nu = K_{k,\alpha}^\nu + g_{k,\alpha}^\nu, \quad \nu = e, i,$$

where

$$V_{k,\alpha}^\nu = \partial_t^k \partial^\alpha V^\nu, \quad K_{k,\alpha}^\nu = \partial_t^k \partial^\alpha K^\nu,$$

and

$$g_{k,\alpha}^\nu = \sum_{j=1}^3 \left( A_j^\nu(u^\nu, \theta^\nu) \partial_j V_{k,\alpha}^\nu - \partial_t^k \partial^\alpha \left( A_j^\nu(u^\nu, \theta^\nu) \partial_j V^\nu \right) \right) + L^\nu(x) V_{k,\alpha}^\nu - \partial_t^k \partial^\alpha (L^\nu(x)V^\nu), \quad \nu = e, i.$$

For the Maxwell equations, we also have

$$\begin{cases}
\partial_t F_{k,\alpha} - \nabla \times G_{k,\alpha} = \partial_t^k \partial^\alpha (n^e u^e - n^i u^i), \quad \nabla \cdot F_{k,\alpha} = N_{k,\alpha}^i - N_{k,\alpha}^e, \\
\partial_t G_{k,\alpha} + \nabla \times F_{k,\alpha} = 0, \quad \nabla \cdot G_{k,\alpha} = 0,
\end{cases} \quad (3.34)$$

with $F_{k,\alpha} = \partial_t^k \partial^\alpha F, G_{k,\alpha} = \partial_t^k \partial^\alpha G, N_{k,\alpha}^\nu = \partial_t^k \partial^\alpha N^\nu, \forall \nu = e, i$, and etc.

**Lemma 3.5** Suppose that the conditions of Theorem 1.1 hold and $\omega_T$ is small enough independent of $T$, then there exists a positive constant $C_0$ such that, for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 1$ and $k + |\alpha| \leq s$, we have

$$\frac{d}{dt} \left( \sum_{\nu=e,i} \left( A_0^\nu(p^\nu, \theta^\nu) V_{k,\alpha}^\nu, V_{k,\alpha}^\nu \right) + \left\| F_{k,\alpha} \right\|^2 + \left\| G_{k,\alpha} \right\|^2 \right) + C_0 \sum_{\nu=e,i} \left( \left\| u_{k,\alpha}^\nu \right\|^2 + \left\| \Theta_{k,\alpha}^\nu \right\|^2 \right) \leq C \sum_{\nu=e,i} \left( \left\| \partial_t^k \left( u^\nu, \Theta^\nu \right) \right\|_{|\alpha|-1} + \left\| \partial_t^k Q^\nu \right\|^2_{|\alpha|} \right) + C \left\| \partial_t^k F \right\|^2_{|\alpha|-1} + C \sum_{\nu=e,i} \left\| V^\nu \right\|^2_{|\alpha|} \left\| Z \right\|_{|\alpha|}. \quad (3.35)$$

**Proof.** For all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $1 \leq k + |\alpha| \leq s$, multiplying (3.32) by $A_0^\nu(p^\nu, \theta^\nu) V_{k,\alpha}^\nu$ and taking integrations in $x$ over $T$, we obtain

$$\frac{d}{dt} \left( A_0^\nu(p^\nu, \theta^\nu) V_{k,\alpha}^\nu, V_{k,\alpha}^\nu \right) = I_1^\nu + I_2^\nu + I_3^\nu + I_4^\nu, \quad \nu = e, i, \quad (3.36)$$

with

$$I_1^\nu = \left( \partial_t A_0^\nu(p^\nu, \theta^\nu) V_{k,\alpha}^\nu, V_{k,\alpha}^\nu \right), \quad I_2^\nu = \left( B^\nu(p^\nu, u^\nu, \theta^\nu, x) V_{k,\alpha}^\nu, V_{k,\alpha}^\nu \right), \quad \nu = e, i,$n

$$I_3^\nu = 2 \left( A_0^\nu(p^\nu, \theta^\nu) \partial_t^k \partial^\alpha K^\nu, V_{k,\alpha}^\nu \right), \quad I_4^\nu = 2 \left( A_0^\nu(p^\nu, \theta^\nu) g_{k,\alpha}^\nu, V_{k,\alpha}^\nu \right), \quad \nu = e, i.$$

In the following, we estimate every term on the right-hand side of (3.36).

For the first term $I_1^\nu$, by (2.12), the third equation in (1.2) and the Sobolev embedding theorem [6], we obtain

$$\left\| \partial_t(p^\nu, \theta^\nu) \right\|_{L^\infty} \leq C \left\| (u^\nu, \nabla u^\nu, \Theta^\nu) \right\|_{L^\infty} \leq C \left\| V^\nu \right\|_{s}, \quad \nu = e, i.$$
and then

$$|I_1^\nu| = |\langle \partial_t A_0^\nu(p^\nu, \theta^\nu) V_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \rangle| \leq C \|\partial_t (p^\nu, \theta^\nu)\|_{L^\infty} \|V_{k,\alpha}^{\nu}\|^2 \leq C \|V^\nu\|^3_s, \quad \nu = e, i. \quad (3.37)$$

For the second term $I_2^\nu$, with the help of (2.28), (2.30)-(2.31) and the fact that the matrix $B^\nu(p^\nu, u^\nu, \theta^\nu, x)$ is antisymmetric at the point $(p^\nu, u^\nu, \theta^\nu) = (\bar{p}^\nu, 0, 1)$, we have

$$|I_2^\nu| = |\langle B^\nu(p^\nu, u^\nu, \theta^\nu, x) V_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \rangle| \leq C \|V^\nu\|^3_s, \quad \nu = e, i. \quad (3.38)$$

For the third term $I_3^\nu$, we split it as

$$I_3^\nu = 2 \left\langle A_0^\nu(p^\nu, \theta^\nu) \partial^k \partial^\alpha K_1^\nu, V_{k,\alpha}^{\nu} \right\rangle$$

$$= 2 \left\langle A_0^\nu(p^\nu, \theta^\nu) \partial^k \partial^\alpha K_1^\nu, V_{k,\alpha}^{\nu} \right\rangle + 2 \left\langle A_0^\nu(p^\nu, \theta^\nu) \partial^k \partial^\alpha K_2^\nu, V_{k,\alpha}^{\nu} \right\rangle, \quad \nu = e, i. \quad (3.39)$$

with

$$K_1^\nu = K_1^\nu (u^\nu, \theta^\nu, F, G, x) = \left( \begin{array}{c} -\frac{1}{\theta^\nu} \Theta^\nu \\ q^\nu (F + u^\nu \times (B + G)) - u^\nu \\ -\Theta^\nu \end{array} \right), \quad \nu = e, i, \quad (3.40)$$

$$K_2^\nu = K_2^\nu (u^\nu, \theta^\nu) = \left( \begin{array}{c} 1 \\ \frac{1}{2 \theta^\nu} \left( u^\nu \right)^2 \\ 0 \end{array} \right), \quad \nu = e, i. \quad (3.41)$$

Due to the fact that $K_2^\nu$ is a quadratic term of $u^\nu$, we get easily

$$\left\langle A_0^\nu(p^\nu, \theta^\nu) \partial^k \partial^\alpha K_2^\nu, V_{k,\alpha}^{\nu} \right\rangle \leq C \|A_0^\nu(p^\nu, \theta^\nu)\|_{L^\infty} \left\|\partial^k \partial^\alpha K_2^\nu\right\| \|V_{k,\alpha}^{\nu}\| \leq C \|V^\nu\|^3_s, \quad \nu = e, i. \quad (3.42)$$

For the first term on the right-hand side of (3.39), by (2.25) and (3.40), we have

$$2 \left\langle A_0^\nu(p^\nu, \theta^\nu) \partial^k \partial^\alpha K_1^\nu, V_{k,\alpha}^{\nu} \right\rangle - 2q^\nu \langle F_{k,\alpha}, n^\nu u_{k,\alpha}^\nu \rangle$$

$$= -2 \langle n^\nu u_{k,\alpha}^\nu, u_{k,\alpha}^\nu \rangle + 2q^\nu \langle (u^\nu \times G)_{k,\alpha}, n^\nu u_{k,\alpha}^\nu \rangle + 2 \left\langle p^\nu \left( \frac{\Theta^\nu}{\theta^\nu} - \frac{\Theta^\nu}{\theta^\nu}_{k,\alpha} \right), Q_{k,\alpha}^\nu \right\rangle$$

$$+ 2 \left\langle \frac{p^\nu}{\theta^\nu} \left( \frac{\Theta^\nu}{\theta^\nu}_{k,\alpha} - \frac{\Theta^\nu}{\theta^\nu} \right), \Theta^\nu_{k,\alpha} \right\rangle, \quad \nu = e, i. \quad (3.43)$$

For the second term on the right-hand side of (3.43), by (2.30) and the Leibniz formula, we obtain

$$\left| \left\langle (u^\nu \times G)_{k,\alpha}, n^\nu u_{k,\alpha}^\nu \right\rangle \right| \leq \left| (u^\nu \times G_{k,\alpha}, n^\nu u_{k,\alpha}^\nu) \right| + \sum_{l < k} C_l^k \left| \left\langle \partial_l^{k-l} u^\nu \times G_{l,\alpha}, n^\nu u_{k,\alpha}^\nu \right\rangle \right|$$

$$+ \sum_{\beta < \alpha} C^\beta_{\alpha} \left| \left\langle \partial^{\alpha-\beta} u^\nu \times G_{k,\beta}, n^\nu u_{k,\alpha}^\nu \right\rangle \right| + \sum_{\beta < \alpha, l < k} C^\beta_{\alpha} C_l^k \left| \left\langle u_{k-l,\alpha-\beta} \times G_{l,\beta}, n^\nu u_{k,\alpha}^\nu \right\rangle \right|$$

$$\leq C \|V^\nu\|^3_s \|Z\|_s, \quad \nu = e, i. \quad (3.44)$$
We only give the estimate for the most complicate term \( \sum_{\beta<\alpha,l<k} C_\alpha^\beta C_k^l \langle u_{k-l,\alpha-\beta} \times G_{l,\beta}, n^\nu u_{k,\alpha}^\nu \rangle \) on the right-hand side of (3.44), the estimates for other terms are omitted here for the sake of simplicity

\[
\left| \langle u_{k-l,\alpha-\beta} \times G_{l,\beta}, n^\nu u_{k,\alpha}^\nu \rangle \right| \leq \|u_{k-l,\alpha-\beta}\|_{L^\infty} \|G_{l,\beta}\| \|n^\nu u_{k,\alpha}^\nu\|
\leq C \|u_{k-l,\alpha-\beta}\|_{L^2} \|G_{l,\beta}\| \|n^\nu u_{k,\alpha}^\nu\|
\leq C \|V^\nu\|_s^2 \|Z\|_s, \quad \nu = e, i, \quad \text{when } l + |\beta| \leq 1,
\]

and

\[
\left| \langle u_{k-l,\alpha-\beta} \times G_{l,\beta}, n^\nu u_{k,\alpha}^\nu \rangle \right| \leq \|u_{k-l,\alpha-\beta}\|_{L^\infty} \|G_{l,\beta}\| \|n^\nu u_{k,\alpha}^\nu\|
\leq C \|u_{k-l,\alpha-\beta}\|_{L^2} \|G_{l,\beta}\| \|n^\nu u_{k,\alpha}^\nu\|
\leq C \|V^\nu\|_s^2 \|Z\|_s, \quad \nu = e, i, \quad \text{when } l + |\beta| \geq 2.
\]

Similarly, for the third term on the right-hand side of (3.43), we have

\[
\left| \langle p^\nu \left( \frac{\Theta_{k,\alpha}^\nu}{\theta^\nu} - \frac{\Theta_{k,\alpha}^\nu}{\theta^\nu} \right), Q_{k,\alpha}^\nu \rangle \right| \leq C \|\Theta^\nu\|_s^3 \|Q^\nu\|_s, \quad \nu = e, i.
\]

For the last term on the right-hand side of (3.43), it follows from the Leibniz formula that

\[
\left\langle p^\nu \left( \frac{\Theta_{k,\alpha}^\nu}{\theta^\nu} - \frac{2 \Theta_{k,\alpha}^\nu}{\theta^\nu} \right), \Theta_{k,\alpha}^\nu \right\rangle
= \left\langle n^\nu \left( \frac{\Theta_{k,\alpha}^\nu}{\theta^\nu} - \frac{2 \Theta_{k,\alpha}^\nu}{\theta^\nu} \right), \Theta_{k,\alpha}^\nu \right\rangle
= - \left\langle n^\nu \frac{\Theta_{k,\alpha}^\nu}{\theta^\nu}, \Theta_{k,\alpha}^\nu \right\rangle + \sum_{l<k} C_k^l \left\langle n^\nu \partial_j^{k-l} \left( \frac{1}{\theta^\nu} \right), \Theta_{k,\alpha}^\nu \right\rangle
\leq \sum_{\beta<\alpha} C_\alpha^\beta \left\langle n^\nu \partial_j^{\alpha-\beta} \left( \frac{1}{\theta^\nu} \right), \Theta_{k,\alpha}^\nu \right\rangle
\leq \sum_{\beta<\alpha} C_\alpha^\beta \left\langle n^\nu \partial_j^{\alpha-\beta} \left( \frac{1}{\theta^\nu} \right), \Theta_{k,\alpha}^\nu \right\rangle
\leq C \|\Theta^\nu\|_s^3, \quad \nu = e, i.
\]

Thus from (3.39)-(3.48), for \( \nu = e, i \), we obtain

\[
I_3^\nu = 2 \left\langle A_{k,\alpha}^\nu (p^\nu, \theta^\nu) \partial_j^{k} \partial^\alpha K^\nu, V_{k,\alpha}^\nu \right\rangle
\leq 2 q_{i,\alpha} \langle F_{k,\alpha}, n^\nu u_{k,\alpha}^\nu \rangle - 2 \langle n^\nu u_{k,\alpha}^\nu, u_{k,\alpha}^\nu \rangle - 2 \left\langle n^\nu \frac{\Theta_{k,\alpha}^\nu}{\theta^\nu}, \Theta_{k,\alpha}^\nu \right\rangle + C \|V^\nu\|_s^2 \|Z\|_s.
\]

Next, we begin to estimate \( I_4^\nu = 2 \left\langle A_{k,\alpha}^\nu (p^\nu, \theta^\nu) g_{k,\alpha}^\nu, V_{k,\alpha}^\nu \right\rangle \). By (3.33), we split \( g_{k,\alpha}^\nu \) as

\[
g_{k,\alpha}^\nu = g_{k,\alpha}^{\nu 1} + g_{k,\alpha}^{\nu 2}, \quad \nu = e, i,
\]

with

\[
g_{k,\alpha}^{\nu 1} = \sum_{j=1}^3 \left( A_j^\nu (u^\nu, \theta^\nu) \partial_j V_{k,\alpha}^\nu - \partial_j^\nu (A_j^\nu (u^\nu, \theta^\nu) \partial_j V^\nu) \right), \quad \nu = e, i,
\]
We first establish the estimate for $g_{k,\alpha}^{\nu}$. Using (2.22) and the Leibniz formula, we have

$$
A_{j}^{\nu}(u^{\nu}, \theta^{\nu}) \partial_{j} V_{k,\alpha}^{\nu} - \partial_{k}^{\nu} \partial^{\alpha} \left( A_{j}^{\nu}(u^{\nu}, \theta^{\nu}) \partial_{j} V^{\nu} \right)
$$

which implies

$$
\|g_{k,\alpha}^{\nu}\| = \left\| \sum_{j=1}^{3} \left( A_{j}^{\nu}(u^{\nu}, \theta^{\nu}) \partial_{j} V_{k,\alpha}^{\nu} - \partial_{k}^{\nu} \partial^{\alpha} \left( A_{j}^{\nu}(u^{\nu}, \theta^{\nu}) \partial_{j} V^{\nu} \right) \right) \right\| \leq C \| \|V^{\nu}\|_{s}^{2},
$$

and then

$$
2 \left\langle A_{0}^{\nu}(p^{\nu}, \theta^{\nu}) g_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \right\rangle \leq C \|A_{0}^{\nu}(p^{\nu}, \theta^{\nu})\|_{L^{\infty}} \|g_{k,\alpha}^{\nu}\| \|V_{k,\alpha}^{\nu}\| \leq C \| \|V^{\nu}\|_{s}^{3}. \quad (3.51)
$$

On the other hand, it follows from (2.13), (2.23) and (2.30)-(2.31) that matrix function $L(x)$ is regular bounded in $T$. Then by using the Leibniz formula, (2.25) and (2.27), we get

$$
\left\langle A_{0}^{\nu}(p^{\nu}, \theta^{\nu}) g_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \right\rangle
$$

which implies

$$
2 \left\langle A_{0}^{\nu}(p^{\nu}, \theta^{\nu}) g_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \right\rangle \leq C \left\| \|u^{\nu}_{k,\alpha}, \Theta_{k,\alpha}^{\nu}\| \|Q^{\nu}_{k,\alpha}\| \right\|_{|\alpha|}^{2} + C \left\| \|Q^{\nu}_{k,\alpha}\| \right\|_{|\alpha|}^{2} + C \left\| \|Q^{\nu}_{k,\alpha}\| \right\|_{|\alpha|}^{2},
$$

and thus,

$$
|I_{\nu}^{2}| \leq 2 \left| \left\langle A_{0}^{\nu}(p^{\nu}, \theta^{\nu}) g_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \right\rangle \right| + 2 \left| \left\langle A_{0}^{\nu}(p^{\nu}, \theta^{\nu}) g_{k,\alpha}^{\nu}, V_{k,\alpha}^{\nu} \right\rangle \right| \leq C \left\| \|\partial_{k}^{\nu} u_{k}^{\nu}, \Theta_{k}^{\nu}\| \|Q_{k}^{\nu}\| \right\|_{|\alpha|}^{2} + C \left\| \|Q^{\nu}_{k,\alpha}\| \right\|_{|\alpha|}^{2} + C \left\| \|V^{\nu}\|_{s}^{3}, \quad \nu = e, i. \quad (3.54)
$$

Moreover, a normal energy estimate for (3.34) gives

$$
\frac{d}{dt} \left( \|F_{k,\alpha}\|^{2} + \|G_{k,\alpha}\|^{2} \right) + 2 \left( \|u^{\nu}_{k,\alpha}, \Theta_{k,\alpha}^{\nu}\|^{2} \right) = 0.
$$

(3.55)
Hence, combining (3.37)-(3.38), (3.49) and (3.54)-(3.55), we obtain

\[
\frac{d}{dt} \left( \sum_{\nu=e,i} \langle A^\nu_0(p^\nu, \theta^\nu)V^\nu_{k,0}, V^\nu_{0,k} \rangle + \|F_{k,0}\|^2 + \|G_{k,0}\|^2 \right) + 2 \sum_{\nu=e,i} \langle n^\nu k_{0,0,0}, u^\nu_{k,0} \rangle + 2 \sum_{\nu=e,i} \langle n^\nu k_{0,0,0}, \Theta^\nu_{k,0} \rangle \\
\leq \varepsilon \sum_{\nu=e,i} \left| \left\langle (n^\nu k_{0,0,0})^2 + C \sum_{\nu=e,i} \left| \left( \partial^0_t u^\nu, \partial^0_t \Theta^\nu \right) \right|^2 + C \sum_{\nu=e,i} \left| \partial^0_t Q^\nu \right|^2 \right| + 2 \left( n^\nu u^\nu - n^i u^i \right)_{k,0} - 2 \left( (n^\nu u^\nu - n^i u^i)_{k,0} \right) + C \sum_{\nu=e,i} \left| \right| \nu \right|^2 \left| \right| Z \right|_s.
\]

(3.56)

Next, let us estimate the term 2 \left( (n^\nu u^\nu - n^i u^i)_{k,0} - 2 \left( (n^\nu u^\nu - n^i u^i)_{k,0} \right) + C \sum_{\nu=e,i} \left| \nu \right|^2 \left| \right| Z \right|_s.

(3.57)

where \alpha_0 \in \mathbb{N}^3 with \left| \alpha_0 \right| = |\alpha| - 1 for |\alpha| \geq 1.

Then, taking \varepsilon > 0 sufficiently small (for example \varepsilon = \frac{1}{17} ), by (2.30)-(2.31), the combination of (3.56)-(3.57) yields (3.35). \hspace{1cm} \square

**Remark 3.5** Lemma 3.5 is valid for |\alpha| \geq 1. The following Lemma concerns the L^2 estimates for \partial^0_t V^\nu (i.e. \alpha = 0), which is a starting point for employing the argument of induction.

**Lemma 3.6** Assume that the conditions of Theorem 1.1 holds and \omega_T is sufficiently small independent of T, then there exists a positive constant C_0 such that, for all 0 \leq k \leq s, it holds

\[
\frac{d}{dt} \left( \sum_{\nu=e,i} \langle A^\nu_0(p^\nu, \theta^\nu)V^\nu_{k,0}, V^\nu_{0,k} \rangle + \|F_{k,0}\|^2 + \|G_{k,0}\|^2 \right) + C_0 \sum_{\nu=e,i} \left( ||u^\nu_{k,0}||^2 + ||\Theta^\nu_{k,0}||^2 \right) \leq C \sum_{\nu=e,i} \left| \left| \nu \right|^2 \left| \right| Z \right|_s.
\]

(3.58)

\[
2 \left| \langle A^\nu_0(p^\nu, \theta^\nu)g^2_{k,0} V^\nu_{k,0} \rangle \right| \leq C \left| \left| \nu \right|^2 \left| \right| Z \right|_s,
\]

(3.59)

**Proof.** Carefully checking the procedures of the proof for Lemma 3.5, we shall prove
and
\[
\left| \left< (n^e u^e - n^i u^i)_{k,\alpha} - (n^e u^e_{k,\alpha} - n^i u^i_{k,\alpha}), F_{k,\alpha} \right> \right| \leq C \sum_{\nu=e,i} \|\| V^\nu \|\|^2 \| Z \|_s, \tag{3.60}
\]
which are correspondence to (3.53) and (3.57), respectively. Obviously, (3.59) can be easily obtained through the following calculation
\[
g^2_{k,0} = L^\nu(x) V^\nu_{k,0} - \partial^k_t (L^\nu(x) V^\nu) = 0, \quad \nu = e, i.
\]
Next, (3.60) follows from the following calculations
\[
\left| \left< (n^e u^e - n^i u^i)_{k,\alpha} - (n^e u^e_{k,\alpha} - n^i u^i_{k,\alpha}), F_{k,\alpha} \right> \right| = \left| \left< (N^e u^e - N^i u^i)_{k,\alpha} - (N^e u^e_{k,\alpha} - N^i u^i_{k,\alpha}), F_{k,\alpha} \right> \right| \leq C \sum_{\nu=e,i} \|\| V^\nu \|\|^2 \| Z \|_s.
\]

\[
\Box
\]

### 3.2 Recurrence relationship

In order to prove Theorem 1.1, we have to control the terms \|\| \partial^k_t F \|\|_{|\alpha| - 1} \|\| \partial^k Q^\nu \|\|_{|\alpha|} \) appearing on the right-hand side of (3.35). This will be achieved in the following Lemma.

**Lemma 3.7** Assume that the conditions of Theorem 1.1 holds and \( \omega_T \) is sufficiently small independent of \( T \), then for all \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^3 \) with \( |\alpha| \geq 1 \) and \( k + |\alpha| \leq s \), it holds
\[
\|\| \partial^k_t N^\nu \|\|_{|\alpha|} \leq C \left( \|\| \partial^k_t Q^\nu \|\|_{|\alpha| - 1} + \|\| \partial^k_t \Theta^\nu \|\|_{|\alpha| - 1} \right) + C \|\| V^\nu \|\|^2 \| Z \|_s, \quad \nu = e, i, \tag{3.61}
\]
\[
\sum_{\nu=e,i} \|\| \partial^k Q^\nu \|\|_{|\alpha|} \leq C \sum_{\nu=e,i} \left( \|\| \partial^k_t (Q^\nu, u^\nu, \Theta^\nu) \|\|_{|\alpha| - 1} + \|\| \partial^{k+1} u^\nu \|\|_{|\alpha| - 1} \right) + C \sum_{\nu=e,i} \|\| V^\nu \|\|^2 \| Z \|_s, \tag{3.62}
\]
and
\[
\sum_{\nu=e,i} \|\| \partial^k F \|\|_{|\alpha| - 1} \leq C \sum_{\nu=e,i} \left( \|\| \partial^k_t (Q^\nu, u^\nu, \Theta^\nu) \|\|_{|\alpha| - 1} + \|\| \partial^{k+1} u^\nu \|\|_{|\alpha| - 1} \right) + C \sum_{\nu=e,i} \|\| V^\nu \|\|^2 \| Z \|_s. \tag{3.63}
\]

**Proof.** The first estimate (3.61) follows from (2.19). Next we prove (3.62), rewriting the second equation in (2.16) as
\[
\nabla Q^\nu = q_\nu F - (\partial_t u^\nu + u^\nu) + q_\nu u^\nu \times \vec{B} - \nabla q^\nu \Theta^\nu + r^\nu, \tag{3.64}
\]
with
\[
r^\nu = -(u^\nu \cdot \nabla) u^\nu - \Theta^\nu \nabla Q^\nu + q_\nu u^\nu \times \vec{G}, \quad \nu = e, i.
\]

For \( k \in \mathbb{N} \) and \( \beta \in \mathbb{N}^3 \) with \( \beta < \alpha \) and \( k + |\alpha| \leq s \), applying \( \partial^k_t \partial^\beta \) to (3.64) and taking the inner product with \( (\nabla Q^\nu)_{k,\beta} \) in \( L^2(\mathbb{T}) \), we have
\[
\left\| (\nabla Q^\nu)_{k,\beta} \right\|^2 = q_\nu \left< F_{k,\beta}, (\nabla Q^\nu)_{k,\beta} \right> + \left< u^\nu_{k+1,\beta} + u^\nu_{k,\beta}, (\nabla Q^\nu)_{k,\beta} \right> + q_\nu \left< u^\nu_{k,\beta} \times \vec{B}, (\nabla Q^\nu)_{k,\beta} \right> - \left< (\nabla q^\nu \Theta^\nu)_{k,\beta}, (\nabla Q^\nu)_{k,\beta} \right> + \left< r^\nu_{k,\beta}, (\nabla Q^\nu)_{k,\beta} \right>. \tag{3.65}
\]

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By the compatibility condition $\nabla \cdot F = N^i - N^e$, we get

$$\sum_{\nu = e, i} q_{\nu} \left\langle F_{k, \beta}, (\nabla Q^\nu)_{k, \beta} \right\rangle = \left\langle F_{k, \beta}, (\nabla Q^i - \nabla Q^e)_{k, \beta} \right\rangle = - \left\langle \nabla \cdot F_{k, \beta}, (Q^i - Q^e)_{k, \beta} \right\rangle = \left\langle (N^e - N^i)_{k, \beta}, (Q^i - Q^e)_{k, \beta} \right\rangle.$$  

Then (3.61) together with the Young inequality implies that

$$\left| \sum_{\nu = e, i} q_{\nu} \left\langle F_{k, \beta}, (\nabla Q^\nu)_{k, \beta} \right\rangle \right| \leq C \sum_{\nu = e, i} \left\| \partial_t^k (Q^\nu, \Theta^\nu) \right\|_{[a]}^2 + C \sum_{\nu = e, i} \left\| V^\nu \right\|^2_{[a]} \| Z \| s.$$  

In a similar way, by the Young inequality, we obtain

$$\left| \left\langle u_{k+1, \beta}^\nu, \Theta^\nu_{k, \beta} \right\rangle \right| \leq C \left( \left\| \partial_t^k u^\nu_{k+1} \right\|_{[a]-1}^2 + \left\| \partial_t^{k+1} u^\nu_{k+1} \right\|_{[a]-1}^2 \right) + \left\| (\nabla Q^\nu)_{k, \beta} \right\|^2,$$  

and

$$\left| \left\langle \nabla q^\nu \Theta^\nu_{k, \beta}, (\nabla Q^\nu)_{k, \beta} \right\rangle \right| \leq C \left\| \partial_t^k \Theta^\nu \right\|_{[a]-1}^2 + \left\| (\nabla Q^\nu)_{k, \beta} \right\|^2.$$  

On the other hand, by the Leibniz formula, we easily get

$$\left| \left\langle r_{k, \beta}^\nu, (\nabla Q^\nu)_{k, \beta} \right\rangle \right| \leq C \left\| V^\nu \right\|^2_{[a]} \| Z \| s, \ \nu = e, i.$$

Taking $\varepsilon > 0$ sufficiently small (for instance, $\varepsilon = \frac{1}{6}$), and combining (3.65)-(3.69), we have

$$\sum_{\nu = e, i} \left\| (\nabla Q^\nu)_{k, \beta} \right\|^2 \leq C \sum_{\nu = e, i} \left( \left\| \partial_t^k (Q^\nu, u^\nu, \Theta^\nu) \right\|_{[a]-1}^2 + \left\| \partial_t^{k+1} u^\nu \right\|_{[a]-1}^2 \right) + C \sum_{\nu = e, i} \left\| V^\nu \right\|^2_{[a]} \| Z \| s.$$  

Then (3.62) follows from the summation of these inequalities for all indexes $\beta < \alpha$.

In the end, by (3.64), we have

$$q_{\nu} F = \nabla Q^\nu + (\partial_t u^\nu + u^\nu) - q_{\nu} u^\nu \times \bar{B} + \nabla q^\nu \Theta^\nu - r^\nu,$$

this equality together (3.62) imply (3.63). $\square$

Now we give a dissipation estimate for $\left\| \partial_t^k Q^\nu \right\|^2$ and a refined estimate of (3.62) for $\left\| \partial_t^k Q^\nu \right\|^2$, with $k \leq s - 1$. It should be pointed out that the two estimates are necessary, i.e., the process by induction in the proof of Theorem 1.1 may not be closed without of them (see (4.83)).

**Lemma 3.8** Assume that the conditions of Theorem 1.1 holds and $\omega_T$ is sufficiently small independent of $T$, then for all $k \in \mathbb{N}$ with $k \leq s - 1$, we have

$$\left\| \partial_t^k Q^\nu \right\|^2 \leq C \sum_{\nu = e, i} \left( \left\| \partial_t^k (u^\nu, \Theta^\nu) \right\|^2 + \left\| \partial_t^{k+1} u^\nu \right\|^2 \right) + C \sum_{\nu = e, i} \left\| V^\nu \right\|^2_{[a]} \| Z \| s, \ \nu = e, i,$$  

and

$$\left\| \partial_t^k Q^\nu \right\|^2 \leq C \left\| \partial_t^{e-1} (u^\nu, \Theta^\nu) \right\|^2_{[1]} + C \left\| V^\nu \right\|^2_{[a]} \| Z \| s, \ \nu = e, i.$$  

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Proof. For $k \in \mathbb{N}$ with $k \leq s - 1$, applying $\partial_t^k$ to (3.64), we get

$$\nabla \partial_t^k Q^\nu - q_\nu \partial_t^k F = q_\nu \partial_t^k u^\nu \times B - \partial_t^k u^\nu - \partial_t^{k+1} u^\nu - \nabla q^\nu \partial_t^k \Theta^\nu + \partial_t^k r^\nu, \quad \nu = e, i. \quad (3.72)$$

Now, we define a potential function $\nabla \psi$ as

$$\nabla \psi = \tilde{E} - E = -F, \quad \int_T \psi(t,x)dx = 0.$$

Then

$$\nabla \cdot \left( \partial_t^k F + \nabla \partial_t^k \psi \right) = 0, \quad \forall \quad 0 \leq k \leq s - 1.$$

From (3.72), we have

$$\nabla \eta^\nu_k - q_\nu \left( \partial_t^k F + \nabla \partial_t^k \psi \right) = q_\nu \partial_t^k u^\nu \times B - \partial_t^k u^\nu - \partial_t^{k+1} u^\nu - \nabla q^\nu \partial_t^k \Theta^\nu + \partial_t^k r^\nu, \quad (3.73)$$

where

$$\eta^\nu_k = \partial_t^k \eta^\nu - \partial_t^k Q^\nu + q_\nu \partial_t^k \psi, \quad \nu = e, i. \quad (3.74)$$

Due to the fact that

$$\left\langle \partial_t^k F + \nabla \partial_t^k \psi, \nabla \eta^\nu_k \right\rangle = - \left\langle \partial_t^k \nabla \cdot F + \Delta \partial_t^k \psi, \eta^\nu_k \right\rangle = 0,$$

we obtain

$$\|\nabla \eta^\nu_k\|^2 \leq \|\partial_t^k u^\nu\|^2 + \|\partial_t^{k+1} u^\nu\|^2 + C \|\partial_t^k \Theta^\nu\|^2 + \|\partial_t^k r^\nu\|^2. \quad (3.75)$$

From (3.74), we have

$$\partial_t^k Q^\nu = \eta^\nu_k - q_\nu \partial_t^k \psi.$$

For $k = 0$, we get

$$Q^\nu = \eta^\nu - q_\nu \psi, \quad \bar{q}^\nu = Q^\nu + \bar{q}^\nu.$$

Since

$$N^\nu = n^\nu - \bar{n}^\nu = \frac{\bar{p}^\nu}{\bar{\theta}^\nu} - \frac{\bar{p}^\nu}{\bar{\theta}^\nu} = \frac{e^{\nu}}{\theta^\nu} - \frac{e^{\nu}}{\theta^\nu} = \frac{e^{\nu}}{\theta^\nu} \left( e^{Q^\nu} - 1 - \Theta^\nu \right) = \frac{e^{\nu}}{\theta^\nu} \left( e^{\nu} e^{-\nu} - 1 \right),$$

we have

$$-\Delta \psi = N^i - N^e = \frac{e^{\nu}}{\theta^\nu} \left( e^{\nu} e^{-\nu} - 1 \right) - \frac{e^{\nu}}{\theta^e} \left( e^{\nu} e^{\nu} - 1 \right) - \frac{e^{\nu}}{\theta^i} \Theta^i + \frac{e^{\nu}}{\theta^e} \Theta^e.$$

Thus,

$$-\Delta \psi + \left( \frac{e^{\nu}}{\theta^e} e^{\nu} + \frac{e^{\nu}}{\theta^i} e^{\nu} \right) \psi = \frac{e^{\nu}}{\theta^e} \left( \psi + e^{-\nu} \right) + \frac{e^{\nu}}{\theta^i} \left( e^{\nu} \left( \psi + e^{-\nu} \right) - 1 \right) + \frac{e^{\nu}}{\theta^e} \Theta^e - \frac{e^{\nu}}{\theta^i} \Theta^i.$$
Since \( \left( \frac{e^{q_f}}{\vartheta} e^{r_f} - \frac{e^{q_i}}{\vartheta} e^{r_i} \right) \geq \text{const.} > 0 \), taking the inner product of the previous equality with \( \psi \) in \( L^2(\mathbb{T}) \) and using an integration by parts, we get

\[
\| \nabla \psi \|^2 + C_0 \| \psi \|^2 \leq \sum_{\nu = e, i} \left( \| \eta^{\nu}_k \|^2 + \| \Theta^{\nu}_k \|^2 \right) \leq \sum_{\nu = e, i} \left( \| \nabla \eta^{\nu}_k \|^2 + \| \Theta^{\nu}_k \|^2 \right),
\]

(3.76)

where we have used Lemma 2.2.

For \( k \geq 1 \), due to the fact that

\[
\partial_\psi^k N^\nu = -q_\nu \frac{e^{q_f}}{\vartheta} \psi e^{r_f} \partial_\psi^k \psi + \left( \partial_\psi^k \left( \frac{e^{q_f}}{\vartheta} (e^{r_f} e^{-q_\psi} - 1) \right) + q_\nu \frac{e^{q_f}}{\vartheta} e^{r_f} \partial_\psi^k \psi \right) - \partial_\psi^k \left( \frac{e^{q_f}}{\vartheta} \Theta^\nu \right),
\]

we have

\[
- \Delta \partial_\psi^k \psi = \partial_\psi^k N^i - \partial_\psi^k N^e
\]

\[
= - \frac{e^{q_i}}{\vartheta} e^{r_i} \partial_\psi^k \psi + \left( \partial_\psi^k \left( \frac{e^{q_i}}{\vartheta} (e^{r_i} e^{-q_\psi} - 1) \right) + \frac{e^{q_i}}{\vartheta} e^{r_i} \partial_\psi^k \psi \right) - \partial_\psi^k \left( \frac{e^{q_i}}{\vartheta} \Theta^i \right)
\]

\[
- \frac{e^{q_e}}{\vartheta} e^{r_e} \partial_\psi^k \psi - \left( \partial_\psi^k \left( \frac{e^{q_e}}{\vartheta} (e^{r_e} e^{q_\psi} - 1) \right) - \frac{e^{q_e}}{\vartheta} e^{r_e} \partial_\psi^k \psi \right) + \partial_\psi^k \left( \frac{e^{q_e}}{\vartheta} \Theta^e \right),
\]

which implies that

\[
- \Delta \partial_\psi^k \psi + \left( \frac{e^{q_i}}{\vartheta} e^{r_i} + \frac{e^{q_e}}{\vartheta} e^{r_e} \right) \partial_\psi^k \psi
\]

\[
= \left( \partial_\psi^k \left( \frac{e^{q_i}}{\vartheta} (e^{r_i} e^{-q_\psi} - 1) \right) + \frac{e^{q_i}}{\vartheta} e^{r_i} \partial_\psi^k \psi \right) - \partial_\psi^k \left( \frac{e^{q_i}}{\vartheta} \Theta^i \right)
\]

\[
- \left( \partial_\psi^k \left( \frac{e^{q_e}}{\vartheta} (e^{r_e} e^{q_\psi} - 1) \right) - \frac{e^{q_e}}{\vartheta} e^{r_e} \partial_\psi^k \psi \right) + \partial_\psi^k \left( \frac{e^{q_e}}{\vartheta} \Theta^e \right).
\]

Since \( \left( \frac{e^{q_f}}{\vartheta} e^{r_f} + \frac{e^{q_i}}{\vartheta} e^{r_i} \right) \geq \text{const.} > 0 \), taking the inner product of the previous equality with \( \partial_\psi^k \psi \) in \( L^2(\mathbb{T}) \), by the Leibniz formula and Lemma 2.2, using an integration by parts, we get

\[
\| \nabla \partial_\psi^k \psi \|^2 + C_0 \| \partial_\psi^k \psi \|^2 \leq \sum_{\nu = e, i} \left( \| \eta^{\nu}_k \|^2 + \| \partial_\psi^k \Theta^{\nu}_k \|^2 \right) \leq \sum_{\nu = e, i} \left( \| \nabla \eta^{\nu}_k \|^2 + \| \partial_\psi^k \Theta^{\nu}_k \|^2 \right).
\]

(3.77)

From (3.74)-(3.77), we have

\[
\| \partial_\psi^k Q^\nu \|^2 \leq \| \eta^{\nu}_k \|^2 + \| \partial_\psi^k \psi \|^2
\]

\[
\leq C \sum_{\nu = e, i} \left( \| \partial_\psi^k u^\nu \|^2 + \| \partial_\psi^k+1 u^\nu \|^2 + C \| \partial_\psi^k \Theta^{\nu}_k \|^2 + \| \partial_\psi^k \Theta^\nu \|^2 \right)
\]

\[
\leq C \sum_{\nu = e, i} \left( \| \partial_\psi^k u^\nu \|^2 + \| \partial_\psi^k+1 u^\nu \|^2 + C \| \partial_\psi^k \Theta^{\nu}_k \|^2 \right) + C \sum_{\nu = e, i} \| V^\nu \| \| \Omega \| \| Z \| \| s. 
\]

This proves (3.70).
Next, from the first equation of system (2.16), we get
\[ \partial_t^\nu Q' = \partial_t^\nu \left( \frac{1}{2\vartheta^\nu} |u^\nu|^2 - u^\nu \cdot \nabla Q' - \frac{1}{\vartheta^\nu} \Theta^\nu \right) - 2\nabla \cdot \partial_t^\nu u^\nu - \partial_t^\nu u^\nu \cdot \nabla q'. \]

Thus, (3.71) follows from the Leibniz formula and Lemma 2.3.

Lemma 3.9 (Relation of recurrence) Suppose that the conditions of Theorem 1.1 hold and \( \omega_T \) is small enough independent of \( T \), then there exists a positive constant \( C_0 \) such that, for all \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^3 \) with \( |\alpha| \geq 1 \) and \( k + |\alpha| \leq s \), we have
\[
\begin{align*}
\frac{d}{dt} & \left( \sum_{\nu=\epsilon,i} \sum_{\beta \leq \alpha} \langle A^\nu_0(p^\nu, \theta^\nu) V^\nu_{k,\beta}, V^\nu_{k,\beta} \rangle + ||F_{k,\beta}||^2 + ||G_{k,\beta}||^2 \right) + C_0 \sum_{\nu=\epsilon,i} ||\partial_t^k V^\nu||^2_{|\alpha|} \\
& \leq C \sum_{\nu=\epsilon,i} (||\partial_t^k V^\nu||^2_{|\alpha|-1} + ||\partial_t^{k+1} u^\nu||^2_{|\alpha|-1}) + C \sum_{\nu=\epsilon,i} ||V^\nu||^2_s ||Z||_s. \tag{3.78}
\end{align*}
\]

Proof. For all \( k \in \mathbb{N}, \beta \in \mathbb{N}^3 \) with \( |\beta| \geq 1 \) and \( k + |\beta| \leq s \), it follows from Lemma 3.5 and Lemma 3.7 that
\[
\begin{align*}
\frac{d}{dt} & \left( \sum_{\nu=\epsilon,i} \langle A^\nu_0(p^\nu, \theta^\nu) V^\nu_{k,\beta}, V^\nu_{k,\beta} \rangle + ||F_{k,\beta}||^2 + ||G_{k,\beta}||^2 \right) + C_0 \sum_{\nu=\epsilon,i} ||(u^\nu_{k,\beta}, Q^\nu_{k,\beta}, \Theta^\nu_{k,\beta})||^2 \\
& \leq C \sum_{\nu=\epsilon,i} (||\partial_t^k (Q^\nu, u^\nu, \Theta^\nu)||^2_{|\beta|-1} + ||\partial_t^{k+1} u^\nu||^2_{|\beta|-1}) + C \sum_{\nu=\epsilon,i} ||V^\nu||^2_s ||Z||_s,
\end{align*}
\]

The summation of these inequalities for all \( \beta \) up to \( |\beta| \leq |\alpha| \), together with Lemma 3.6, (3.78) follows.

Next, using Lemma 3.6 and Lemma 3.8, we obtain the following result.

Lemma 3.10 Suppose that the conditions of Theorem 1.1 hold and \( \omega_T \) is small enough independent of \( T \), then there exists a positive constant \( C_0 \) such that, for all \( s \geq 3 \), we have
\[
\begin{align*}
\frac{d}{dt} & \left( \sum_{\nu=\epsilon,i} \langle A^\nu_0(p^\nu, \theta^\nu) \partial_t^\nu V^\nu, \partial_t^\nu V^\nu \rangle + ||\partial_t^k F||^2 + ||\partial_t^k G||^2 \right) + C_0 \sum_{\nu=\epsilon,i} ||\partial_t^k V^\nu||^2 \\
& \leq C \sum_{\nu=\epsilon,i} ||\partial_t^{k-1} V^\nu||^2_1 + C \sum_{\nu=\epsilon,i} ||V^\nu||^2_s ||Z||_s. \tag{3.79}
\end{align*}
\]

4 Proof of Theorem 1.1

The proof of Theorem 1.1 is mainly based on the following a priori estimates which is a consequence on the estimates obtained in Section 3.

Proposition 4.2 (A priori estimates.) Assume that the conditions of Theorem 1.1 holds and \( \omega_T \) is sufficiently small independent of \( T \), then for all \( t \in [0, T] \), there exist positive constants \( C_0 \) and
\[ C \text{ such that} \]
\[ \|Z(t, \cdot)\|_s^2 + C_0 \int_0^t \left( \sum_{e,i} \|V^{\nu}(\tau, \cdot)\|_s^2 + \|F(\tau, \cdot)\|_s^2 \right) d\tau \leq C \|Z_0\|_s^2. \]  
(4.80)

**Proof.** First, for any fixed index \( k \in \mathbb{N} \) with \( k \leq s - 1 \), we employ the induction on space derivatives \( |\alpha| \) with \( 1 \leq |\alpha| \leq s - k \) for (3.78). The step of the induction is increasing from \( |\alpha| = 1 \) to \( |\alpha| = s - k \). More precisely, for \( |\alpha| \geq 2 \), \( |\partial^k V^{\nu}|_{|\alpha|-1}^2 \) on the right-hand side of (3.78) can be controlled by \( \sum_{\nu=e,i} \|\partial^k V^{\nu}\|_{|\alpha|}^2 \) in the proceeding step on the left-hand side of (3.78) multiplying a small positive constant. Then we have
\[
\frac{d}{dt} \left( \sum_{|\alpha| \leq s-k} m_{k,\alpha} \left( \sum_{\nu=e,i} \langle A^\nu V^{\nu}_{s-k,\alpha}, V^{\nu}_{s-k,\alpha} \rangle + \|F_{s-k,\alpha}\|^2 + \|G_{s-k,\alpha}\|^2 \right) \right) + C_0 \sum_{\nu=e,i} \|\partial^k V^{\nu}\|_{s-k}^2 
\leq C \sum_{\nu=e,i} \left( \|\partial^k V^{\nu}\|^2 + \|\partial^k u^{\nu}\|_{s-k-1}^2 \right) + C \sum_{\nu=e,i} \|V^{\nu}\|_s^2 \|Z\|_s. 
\]  
(4.81)

where \( m_{k,\alpha} \) are positive constants.

Next, we continue to employ the induction on time derivatives \( k \) from \( k = s \) to \( k = 0 \). The corresponding estimate for \( k = s \) is given by (3.79). For \( k = s - 1 \), (4.81) yields
\[
\frac{d}{dt} \left( \sum_{|\alpha| \leq s-1} m_{s-1,\alpha} \left( \sum_{\nu=e,i} \langle A^\nu V^{\nu}_{s-1,\alpha}, V^{\nu}_{s-1,\alpha} \rangle + \|F_{s-1,\alpha}\|^2 + \|G_{s-1,\alpha}\|^2 \right) \right) + C_0 \sum_{\nu=e,i} \|\partial^{s-1} V^{\nu}\|_{s-k}^2 
\leq C \sum_{\nu=e,i} \left( \|\partial^{s-1} V^{\nu}\|^2 + \|\partial^s u^{\nu}\|_{s-k-1}^2 \right) + C \sum_{\nu=e,i} \|V^{\nu}\|_s^2 \|Z\|_s. 
\]  
(4.82)

Obviously, the term \( \sum_{\nu=e,i} \|\partial^1 V^{\nu}\|_{s-k}^2 \) on the right-hand side of (3.79) can be controlled by the same term on the left-hand side of (4.82) multiplying a small positive constant. In a similar way, the term \( \sum_{\nu=e,i} \|\partial^{s-1} u^{\nu}\|_{s-k-1}^2 \) can be controlled by \( \sum_{\nu=e,i} \|\partial^s V^{\nu}\|^2 \) in the proceeding step. Then by induction on \( k \), we obtain
\[
\frac{d}{dt} \left( \sum_{k+|\alpha| \leq s} m_{k,\alpha} \left( \sum_{\nu=e,i} \langle A^\nu V^{\nu}_{k,\alpha}, V^{\nu}_{k,\alpha} \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) \right) + C_0 \sum_{\nu=e,i} \sum_{k=0}^s \|\partial^k V^{\nu}\|_{s-k}^2 
\leq C \sum_{\nu=e,i} \sum_{k=0}^{s-1} \left( \|\partial^k V^{\nu}\|^2 + \|\partial^{k+1} u^{\nu}\|^2 \right) + C \sum_{\nu=e,i} \|V^{\nu}\|_s^2 \|Z\|_s. 
\]  
(4.83)

where the positive constants \( m_{k,\alpha} \) are possibly amended based on (4.81). Due to the following equivalence relation
\[ \sum_{k=0}^s \|\partial^k V^{\nu}\|_{s-k}^2 \sim \|V^{\nu}\|_s^2, \]
then from (3.58), (3.70) and (4.83), with a modification again the constants \( m_{k,\alpha} \), we have
\[
\frac{d}{dt} \left( \sum_{k+|\alpha| \leq s} m_{k,\alpha} \left( \sum_{\nu=e,i} \langle A^\nu V^{\nu}_{k,\alpha}, V^{\nu}_{k,\alpha} \rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) \right) + \frac{3}{2} \sum_{\nu=e,i} \|V^{\nu}\|_s^2 
\leq C \sum_{\nu=e,i} \|V^{\nu}\|_s^2 \|Z\|_s. \]
Since $\omega_T$ is sufficiently small, we further get
\[
\frac{d}{dt} \left( \sum_{k+|\alpha| \leq s} m_{k,\alpha} \left( \sum_{\nu = e,i} \langle A_{00}^{\nu} V_{k,\alpha}^\nu, V_{k,\alpha}^\nu \rangle + ||F_{k,\alpha}||^2 + ||G_{k,\alpha}||^2 \right) \right) + \sum_{\nu = e,i} |||V^\nu|||_s^2 \leq 0.
\]
Noting the following equivalence relation
\[
|||Z|||_s^2 \sim \sum_{k+|\alpha| \leq s} m_{k,\alpha} \left( \sum_{\nu = e,i} \langle A_{00}^{\nu} V_{k,\alpha}^\nu, V_{k,\alpha}^\nu \rangle + ||F_{k,\alpha}||^2 + ||G_{k,\alpha}||^2 \right),
\]
we obtain
\[
|||Z(t, \cdot)|||_s^2 + \sum_{\nu = e,i} \int_0^t |||V^\nu(\tau, \cdot)|||_s^2 d\tau \leq |||Z(0, \cdot)|||_s^2 \leq C |||Z_0|||_s^2, \quad t \in [0, T],
\]
where Lemma 2.4 is used.

From the second and the last two equations in (2.16), we obtain the estimates for electric-magnetic fields as
\[
|||F|||_{s-1}^2 \leq C \sum_{\nu = e,i} |||V^\nu|||_s^2 + C \sum_{\nu = e,i} |||V^\nu|||_s^2 |||Z|||_s,
\]
and
\[
|||\partial_t G|||_{s-2}^2 + |||\nabla_x G|||_{s-2}^2 \leq C \sum_{\nu = e,i} |||V^\nu|||_s^2 + C \sum_{\nu = e,i} |||V^\nu|||_s^2 |||Z|||_s.
\]
Due to the fact that $\omega_T$ is sufficiently small, combining (4.84)-(4.86) yields (4.80).

It is obvious that (4.80) gives (1.6) and the global existence of smooth solution $(n^\nu, u^\nu, \theta^\nu, E, B)$ to periodic problem (1.2)-(1.3). Finally, for all $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^3$ with $k + |\beta| \leq s - 1$, from (1.6), we get
\[
\partial_t^k \partial_\beta (n^\nu - \bar{n}^\nu, u^\nu, \theta^\nu - 1, E - \bar{E}) \in L^2 (\mathbb{R}^+; L^2(\mathbb{T})) \cap W^{1,\infty} (\mathbb{R}^+; L^2(\mathbb{T})), \quad \nu = e, i,
\]
which implies (1.7)-(1.8). Furthermore, if $k + |\beta| \geq 1$, noticing $\bar{B}$ is a constant vector, we have
\[
\partial_t^k \partial_\beta \bar{B} \in L^2 (\mathbb{R}^+; L^2(\mathbb{T})) \cap W^{1,\infty} (\mathbb{R}^+; L^2(\mathbb{T})),
\]
which gives (1.9).

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