On the Adaptive Numerical Solution to the Darcy–Forchheimer Model †

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Abstract: We considered a primal-mixed method for the Darcy–Forchheimer boundary value problem. This model arises in fluid mechanics through porous media at high velocities. We developed an a posteriori error analysis of residual type and derived a simple a posteriori error indicator. We proved that this indicator is reliable and locally efficient. We show a numerical experiment that confirms the theoretical results.

Keywords: Darcy–Forchheimer; mixed finite element; a posteriori error estimates

1. Introduction

The Darcy–Forchheimer model constitutes an improvement of the Darcy model which can be used when the velocity is high [1]. It is useful for simulating several physical phenomena, remarkably including fluid motion through porous media, as in petroleum reservoirs, water aquifers, blood in tissues or graphene nanoparticles through permeable materials. Let \( \Omega \) be a bounded, simply connected domain in \( \mathbb{R}^2 \) with a Lipschitz-continuous boundary \( \partial \Omega \). The problem reads as follows: given known functions \( g \) and \( f \), find the velocity \( u \) and the pressure \( p \) such that

\[
\begin{align*}
\frac{1}{\rho} \left( \frac{1}{K} - 1 \right) u + \frac{\beta}{\rho} |u| u + \nabla p &= g \quad \text{in } \Omega, \\
\nabla \cdot u &= f \quad \text{in } \Omega, \\
u \cdot n &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \mu \) is the dynamic viscosity, \( \rho \) denotes the fluid density, \( \beta \) is the Forchheimer number \( K \) denotes the permeability tensor, \( g \) represents gravity, \( f \) is compressibility, and \( n \) is the unit outward normal vector to \( \partial \Omega \).

We make use of the finite element method to approximate the solution of problem (1). We present the approach by Girault and Wheeler [1], who introduced the primal formulation, in which the term \( \nabla \cdot u \) undergoes weakening by integration by parts. It is shown in [1] that problem (1) has a unique solution in the space \( X \times M \), where \( X := [L^3(\Omega)]^2 \) and \( M := W^{1,3/2}(\Omega) \cap L_0^2(\Omega) \) (we use the standard notations for Lebesgue and Sobolev spaces).

2. Discrete Problem

To pose a discrete problem, we can use a family \( \{ \mathcal{T}_h \} \) of conforming triangulations to divide the domain \( \Omega \) such that \( \Omega = \bigcup_{T \in \mathcal{T}_h} T \), \( \forall h > 0 \) represents the mesh...
size. Here we follow [2] and choose the following conforming discrete subspaces of $X$ and $M$, respectively:

$$X_h := \left\{ v_h \in [L^2(\Omega)]^2; \forall T \in \mathcal{T}_h, v_h|_T \in [\mathcal{P}_0(T)]^2 \right\} \subset X,$$

$$M_h := Q_h^1 \cap L_0^2(\Omega) \subset M,$$

where $Q_h^1 := \left\{ q_h \in C^0(\Omega); \forall T \in \mathcal{T}_h, q_h|_T \in \mathcal{P}_1(T) \right\}$.

Then, the discrete problem consists in finding $(u_h, p_h) \in X_h \times M_h$ such that

$$\begin{cases}
\int_{\Omega} \left( \frac{K}{\rho} K^{-1} u_h + \frac{P}{\rho} |u_h| u_h \right) \cdot v_h \, dx + \int_{\Omega} \nabla p_h \cdot v_h \, dx = - \int_{\Omega} g \cdot v_h \, dx, & \forall v_h \in X_h, \\
\int_{\Omega} \nabla q_h \cdot u_h \, dx = - \int_{\Omega} q_h f \, dx, & \forall q_h \in M_h.
\end{cases} \tag{2}$$

It is shown in [2] that problem (2) has a unique solution and that the sequence $\{(u_h, p_h)\}_h$ converges to the exact solution of problem (1) in $X \times M$. Furthermore, under additional regularity assumptions on the exact solution, some error estimates were derived in [2].

3. Novel Error Estimator and Adaptive Algorithm

We denote by $E_{\Omega}, E_{\partial \Omega}$ and $E_T$, respectively, the sets of edges $e$ belonging to the interior domain, the boundary and the element $T$; $h_e$ denotes the length of a particular edge $e$; and $h_T$ is the diameter of a given element $T$. We denote by $\mathbb{J}_e(v)$ the jump of $v$ across the edge $e$ in the direction of $n_e$, a fixed normal vector to side $e$. Finally, we use the operator $A(u_h, p_h) : = \frac{K}{\rho} K^{-1} u_h + \frac{P}{\rho} |u_h| u_h + \nabla p_h - g$.

On every triangle $T \in \mathcal{T}_h$, we propose the following a posteriori error indicator:

$$\theta_T = \left( h_T^2 \| A(u_h, p_h) \|_{L^2(T)}^2 + \| \nabla \cdot u_h - f \|_{L^2(T)}^2 + \frac{1}{2} \sum_{e \in E^{\Omega} \cap \partial T} h_T^{-1} \| J_e(u_h \cdot n) \|_{L^2(e)}^2 + \sum_{e \in E_{\partial \Omega} \cap \partial T} h_T^{-1} \| u_h \cdot n \|_{L^2(e)}^2 \right)^{1/2}$$

We also define the global a posteriori error indicator $\theta := \left( \sum_{T \in \mathcal{T}_h} \theta_T^2 \right)^{1/2}$.

Theorem 1. For the primal-mixed method (2), there exists a positive constant $C_1$, independent of $h$, and a positive constant $C_2$, independent of $h$ and $T$, such that

$$\| (u - u_h, p - p_h) \|_{X \times M} \leq C_1 \theta,$$

$$\theta_T \leq C_2 \| (u - u_h, p - p_h) \|_{[L^2(W_T)]^2 \times W^{1,3/2}(W_T)}, \quad \forall T \in \mathcal{T}_h,$$

where $W_T = \bigcup_{e \in E_T \cap \partial T \neq \emptyset} T'$.

We propose an adaptive algorithm based on the a posteriori error indicator $\theta$. Given an initial mesh, we follow the iterative procedure described in Figure 1. Each new mesh is generated as suggested in [3].
4. Numerical Experiment

We performed several simulations in FreeFem++ [4], validating the theoretical results. Here we select an example on an L-shaped domain, $\Omega = (-1,1)^2 \setminus [0,1]^2$, and focus on the data $f$ and $g$ so that the exact solution is

$$p(x,y) = \frac{1}{x - 1.1}, \quad u(x,y) = \left( \exp(x) \sin(y), \exp(x) \cos(y) \right). \quad (3)$$

Thus the solution has a singularity in pressure close to the line $x = 1$. Figure 2 shows the mesh refinement by the adaptive algorithm. Figure 3, bottom, represents the evolution with respect to degrees of freedom (DOF) of error and indicator; on the right, we can observe the evolution of the efficiency index with DOF.
5. Discussion

The adaptive algorithm was tested on an example with a singularity. From Figure 2 we can observe that the algorithm refined the mesh near the singularity, as expected. Since it is an academic example with a known solution, we could compute the exact error. The graphs in Figure 3 confirm that the error was lower for the adaptive refinement. Additionally, since the exact error and estimator followed close to parallel lines, we confirm that the indicator gives a consistent measure of the error. This could also be checked by the efficiency index, which is the ratio of indicator to exact total error.

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