Stabilization of Quantum Information: A Unified Dynamical-Algebraic Approach

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The notion of symmetry is shown to be at the heart of all error correction/avoidance strategies for preserving quantum coherence of an open quantum system S e.g., a quantum computer. The existence of a non-trivial group of symmetries of the dynamical algebra of S provides state-space sectors immune to decoherence. Such noiseless sectors, that can be viewed as a noncommutative version of the pointer basis, are shown to support universal quantum computation and to be robust against perturbations. When the required symmetry is not present one can generate it artificially resorting to active symmetrization procedures.

I. INTRODUCTION

Stabilizing quantum-information processing against the environmental interactions as well as operation imperfections is a vital goal for any practical application of the protocols of Quantum Information and Quantum Computation theory. To the date three kind of strategies have been devised in order to satisfy such a crucial requirement: a) Error Correcting Codes which, in analogy with classical information theory, stabilize actively quantum information by using redundant encoding and measurements; b) Error Avoiding Codes pursue a passive stabilization by exploiting symmetry properties of the environment-induced noise for suitable redundant encoding; c) Noise suppression schemes in which, exploiting symmetry properties and with no redundant encoding, the unwanted interactions are averaged away by frequently iterated external pulses. We shall show how all these schemes derive conceptually from a unified dynamical-algebraic framework.

Such an underlying dynamical-algebraic structure is provided by the reducibility of the operator algebra describing the faulty interactions of the coding quantum system. This property in turns amounts to the existence of a non-trivial group of symmetries for the global dynamics. We describe a unified framework which allows us to build systematically new classes of error correcting codes and noiseless subsystems. Moreover we shall argue how by using symmetrization strategies one can artificially produce noiseless subsystems and to perform universal quantum computation within these decoherence-free sectors.

II. NOISELESS SUBSYSTEMS

Suppose that S is quantum system coupled to an environment. Without further assumptions the decoherence induced by this coupling is likely to affect all the states of S. Suppose now that S turns out to be bi-partite, say $S = S_1 + S_2$, and moreover that the environment is actually coupled just with $S_1$. If this is the case one can encode information in the quantum state of $S_2$ in an obvious noiseless way.

The idea underlying the algebraic constructions that will follow is conceptually nothing but an extension of the extremely simple example above: the symmetry of the system plus environment dynamical algebra provides S with an hidden multi-partite structure such that the environment is not able to extract information out of some of these “virtual” subsystems.

Let S be an open quantum system, with (finite-dimensional) state-space $H$, and self-Hamiltonian $H_S$, coupled to its environment through the hamiltonian $H_I = \sum_{\alpha} S_{\alpha} \otimes B_{\alpha} \neq 0$, where the $S_{\alpha}$’s ($B_{\alpha}$’s) are system (environment) operators. The unital associative algebra $A$ closed under hermitian conjugation $S \rightarrow S^\dagger$, generated by the $S_{\alpha}$’s and $H_S$ will be referred to as the interaction algebra. In general $A$ is a reducible $^\dagger$-closed subalgebra of the algebra $\text{End}(H)$ of all the linear operators over $H$; it can be written as a direct sum of $d_J \times d_J$ (complex) matrix algebras each one of which appears with a multiplicity $n_J$.

\[
A \cong \bigoplus_{J \in \mathcal{J}} \text{Id}_{n_J} \otimes M(d_J, \mathbb{C}).
\] (1)

where $\mathcal{J}$ is suitable finite set labelling the irreducible components of $A$. The associated state-space decomposition reads

\[
H \cong \bigoplus_{J \in \mathcal{J}} \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}.
\] (2)

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These decompositions encode all information about the possible quantum stabilization strategies.

Knill et al noticed that (1) implies that each factor \( \mathbb{C}^{p_j} \) in eq. (2) corresponds to a sort of effective subsystem of \( S \) coupled to the environment in a state independent way. Such subsystems are then referred to as noiseless. In particular one gets a noiseless code i.e., a decoherence-free subspace, \( C \subset \mathcal{H} \) when in equation (2) there appear one-dimensional irreps \( J_0 \) with multiplicity greater than one \( C \cong \mathbb{C}^{n_{J_0}} \otimes \mathbb{C}^\lambda \).

The commutant of \( A \) is defined as \( A' \) in \( \text{End}(\mathcal{H}) \) of \( A \) by \( A' := \{ X \mid [X, A] = 0 \} \).

The existence of a NS is equivalent to
\[
A' \cong \oplus_{J \in J} \mathcal{M}(n_J, \mathbb{C}) \otimes \text{Id}_{d_J} \neq \mathbb{C} \text{Id}
\] (3)
The condition \( A' \neq \mathbb{C} \text{Id} \) is amounts to the existence of a non-trivial group of symmetries \( G \subset U A' \). One has that the more symmetric a dynamics, the more likely it supports NSs.

When \( \{ S_\alpha \} \) is a commuting set of hermitian operators, \( A \) is an abelian algebra and Eq. (3) [with \( d_J = 1 \)] is the decomposition of the state-space according the joint eigenspaces of the \( S_\alpha \)'s. The pointer basis discussed in relation to the so-called environment-induced superselection is nothing but an orthonormal basis associated to the resolution \( \mathbb{I} \). Thus the NS's provide in a sense the natural noncommutative extension of the pointer basis.

Now we discuss the relation between NSs and error correction. The interaction algebra \( A \) has to do be thought of generated by error operators and it is assumed to satisfy Eq. (3). Let \( \{ J_{\lambda \mu} \} (J \in J, \lambda = 1, \ldots, n_J; \mu = 1, \ldots, d_J) \) be an orthonormal basis associated to the decomposition (1). Let \( \mathcal{H}_{\mu} = \text{span}\{ |J_{\lambda \mu}\rangle | \lambda = 1, \ldots, n_J \} \), and let \( \mathcal{H}_\mu' \) be defined analogously. The next proposition shows that to each NS there is associated a family of ECCs called \( A \)-codes.

The \( \mathcal{H}_\mu' \)'s (\( \mathcal{H}_\mu \)) are \( A \)-codes (\( A' \)-codes) for any subset \( E \) of error operator such that \( \forall e_i, e_j \in E \Rightarrow e_i^\dagger e_j \in A \).

The standard stabilizer codes are recovered when one considers a \( N \)-partite qubit system, and an abelian subgroup \( G \) of the Pauli group \( P \). Let us consider the state-space decomposition (2) associated to \( G \). If \( G \) has \( k < N \) generators then \( |G| = 2^k \), whereas from commutativity it follows \( d_J = 1 \) and \( |J| = |G| \).

Moreover one finds \( n_J = 2^{N - k} \) : each of the \( 2^k \) joint eigenspaces of \( G \) (stabilizer code) encode \( N - k \) logical qubits. It follows that
\[
\mathcal{H} = \bigoplus_{j=1}^{2^k} \mathbb{C}^{2^{N-k}} \otimes \mathbb{C} \cong \mathbb{C}^{2^{N-k}} \otimes \mathbb{C}^{2^k}.
\] (4)
The allowed errors belong to the algebra \( A = \text{Id}_{2^{N-k}} \otimes M(2^k, \mathbb{C}) \). In particular errors \( e_i, e_j \in A \cap P \) are such that \( e_i^\dagger e_j \) either belong to \( G \) or anticommute with (at least) one element \( G \). In this latter case one has a non-trivial action on the \( \mathbb{C}^{2^k} \) factor.

Let us finally stress out that the NSs approach described so far works even in the case in which the interaction operators \( S_\alpha \) represent, rather than coupling with external degrees of freedom, internal unwanted internal interactions i.e., \( H_S' = H_S + \sum_\alpha S_\alpha \).

III. COLLECTIVE DECOHERENCE.

Collective decoherence arises when a multi-partite quantum system, is coupled symmetrically with a common environment. This is the paradigmatic case for the emergence of noiseless subspaces and NS’s as well. More specifically one has a \( N \)-qubit system \( \mathcal{H}_N := (\mathbb{C}^2)^\otimes N \) and the relevant interaction algebra \( A_N \) coincides with the algebra of completely symmetric operators over \( \mathcal{H}_N \). The commutant \( A_N' \) is the group algebra \( \nu(\mathfrak{S}_N) \), where \( \nu \) is the natural representation of the symmetric group \( S_N \) over \( \mathcal{H}_N : \nu(\pi) \otimes_{j=1}^{N} |j\rangle = \otimes_{j=1}^{N} |\pi(j)\rangle, (\pi \in S_N) \). Using elementary \( su(2) \) representation theory one finds:

\( A_N \) supports NS with dimensions
\[
n_J = \frac{(2J + 1)! N!}{(N/2 + J + 1)! (N/2 - J)!}
\] (5)
where \( J \) runs from 0 (1/2) for \( N \) even (odd). If in the above \( A_N \) is replaced by its commutant, the above result holds with \( n_J = 2J + 1 \).

In order to illustrate the general ideas let us consider \( N = 3 \). One has \((\mathbb{C}^2)^\otimes 3 \cong \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \). The last term can be written as \( \text{span}\{ |\psi_{\beta}\rangle \}_{\alpha, \beta = 1}^2 \) where
\[
|\psi_1^1\rangle = 2^{-1/2}(|010\rangle - |100\rangle),
|\psi_1^2\rangle = 2^{-1/2}(|011\rangle - |101\rangle),
|\psi_2^1\rangle = 2/\sqrt{6} [1/2(|010\rangle + |100\rangle) - |001\rangle],
|\psi_2^2\rangle = 2/\sqrt{6} [|110\rangle - 1/2(|011\rangle + |101\rangle)].
\] (6)
The vectors $|\psi_1^3\rangle$ and $|\psi_2^3\rangle$ ( $|\psi_0^1\rangle$ and $|\psi_2^3\rangle$) span a two-dimensional $A_3$-code ($A_3'$-code). Taking the trace with respect to the index $\alpha$ ($\beta$) one gets the $A_3'$ ($A_3$) NS’s. Moreover the first term corresponds to a trivial four-dimensional $A_3'$ code.

IV. NS SYNTHESIS BY SYMMETRIZATION

A typical situation in which NSs could arise is when the interaction algebra is contained in some reducible group representation $\rho$ of a finite order (or compact) group $G$. Suppose that the irrep decomposition of $H$ associated to $\rho$ has the form of Eq. (4) in which the $J$ labels a set of $G$-irreps $\rho_J$ (dim $\rho_J = d_J$). $\rho$ by extends linearly to the group algebra $\mathcal{G} := \oplus_{g \in G} \mathcal{U}(g)$ giving rise to a decomposition as in Eq. (5). It follows

If $A \subset \rho(\mathcal{G})$ then the dynamics supports (at least) $|J|$ NS’s with dimensions $\{n_J(\rho)\}_{J \in \mathcal{J}}$.

The non-trivial assumption in the above statement is the reducibility of $\rho$ in that any subalgebra of operators belongs to a group-algebra. As already stressed this is equivalent to a symmetry assumption. When this required symmetry is lacking one can artificially generate it by resorting the so-called bang-bang techniques. These are physical procedures, involving iterated external ultra-fast “pulses” $\{\rho_g\}_{g \in G}$, whereby one can synthesize, from a dynamics generated by the $S_\alpha$’s, to a dynamics generated by $\pi_\rho(S_\alpha)$’s where

$$\pi_\rho: X \rightarrow \pi_\rho(X) := \frac{1}{|G|} \sum_{g \in G} \rho_g X \rho_g^\dagger \in \rho(\mathcal{G})' .$$

projects any operator $X$ over the commutant of the operator $\rho(\mathcal{G})$ generated by the bang-bang operations. If one will, preserving the system self-dynamics, to get rid of unwanted interactions $\bar{g}_\alpha$ with the environment he has to look for a group $G \subset U(H)$, such that i) $H_S \in \mathcal{G}'$, ii) the $S_\alpha$’s transform according to non-trivial irreps under the (adjoint) action of $G$. Then it can be shown that $\pi_G(S_\alpha) = 0$: the effective dynamics of $S$ is unitary.

To understand how this strategies might be useful for artificial NSs synthesis is sufficient to notice that Prop. 2 holds even for the commutant by replacing the $n_J$’s with the $d_J$’s. Since the $G$-symmetrization of an operator belongs to $\rho(\mathcal{G})'$, one immediately finds that:

$G$-symmetrization of $A$ supports (at least) $|J|$ NS’s with dimensions $\{d_J(\rho)\}_{J \in \mathcal{J}}$.

It is remarkable that NSs do not allow just for safe encoding of quantum information but even for its manipulation. Form the mathematical point of view this result stems once again quite easily from the basic Eq. (6) which shows that the elements of $A'$ have non-trivial action over the $C^{n_J}$ factors. Therefore:

If an experimenter has at disposal unitaries in $U A'$ universal QC is realizable within the NSs. When such gates are not available from the outset they can be obtained through a $G$-symmetrization of a couple of generic hamiltonians, where $\mathcal{G} := U A$.

V. NS: ROBUSTENESS

In this section we prove a robustness result for NSs extending analogous ones obtained for decoherence-free subspaces. Let $H_S (H_B)$ denote the system state (environment) state-space. Here $S$ represents the NS and the environment includes both the coupled factor in Eq. (4) and the external degrees of freedom.

The evolution of the subsystem $S$ is given by $E_\varepsilon(t) := \text{tr}_B [e^{t \mathcal{L}_S (\rho \otimes \sigma)}]$, where $\rho \in S(H_S)$, $\sigma \in S(H_B)$, and the Liouvillian operator is given by $\mathcal{L}_S := \mathcal{L}_0 + \varepsilon \mathcal{L}_1$, where $e^{t \mathcal{L}_0}$ acts trivially over $S$ i.e., $e^{t \mathcal{L}_0 (\rho \otimes \sigma)} = \rho \otimes \sigma^t$. In particular $\rho \otimes I_B$ is a fixed point.

The fidelity is defined as

$$F_\varepsilon(t) := \text{tr}_S [\rho E_\varepsilon(t) \rho] = \langle \rho \otimes I_B, e^{t \mathcal{L}} (\rho \otimes \sigma) \rangle = \sum_{n=0}^{\infty} \varepsilon^n f_n(t).$$

One has $e^{t \mathcal{L}} = e^{t \mathcal{L}_0} E_\varepsilon e^{-t \mathcal{L}_0}$, in which, by defining $\mathcal{L}_1(\tau) := e^{-\tau \mathcal{L}_0} \mathcal{L}_1 e^{\tau \mathcal{L}_0}$

$$E_\varepsilon := e^{-t \mathcal{L}_0} e^{t \mathcal{L}} e^{t \mathcal{L}_0} = T \exp \left( \int_0^t \mathcal{L}_1(\tau) d\tau \right)$$

$$= \sum_{n=0}^{\infty} \varepsilon^n \int_0^t d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \mathcal{L}_1^{\tau_1} \cdots \mathcal{L}_1^{\tau_n} .$$


that holds for Markovian evolutions

We assume that the infinitesimal generator $L$ of the dynamical semi-group has the following standard (Lindblad) form

$$
< \rho \otimes I_B, \mathcal{L}_1(\rho \otimes \sigma^r) > = 0.
$$

(10)

We assume that the infinitesimal generator $\mathcal{L}$ of the dynamical semi-group has the following standard (Lindblad) form that holds for Markovian evolutions

$$
\mathcal{L}(\rho) := \frac{1}{2} \sum_{\mu} ([L_{\mu} \rho, L_{\mu}^\dagger] + [L_{\mu}, \rho L_{\mu}^\dagger]).
$$

(11)

Perturbing the Lindblad operators $L_{\mu} \mapsto L_{\mu} + \varepsilon \delta L_{\mu}$, one gets $\mathcal{L} \mapsto \mathcal{L} + \varepsilon [\mathcal{L}_1 + \varepsilon \mathcal{L}_2]$, where

$$
\mathcal{L}_1(\omega) := \frac{1}{2} \sum_{\mu} ((\delta L_{\mu} \omega, L_{\mu}^\dagger) + [\delta L_{\mu}, \omega L_{\mu}^\dagger] + [L_{\mu} \omega, \delta L_{\mu}^\dagger] + [L_{\mu}, \omega \delta L_{\mu}^\dagger]),
$$

(12)

$$
\mathcal{L}_2(\omega) := \frac{1}{2} \sum_{\mu} ((\delta L_{\mu} \omega, \delta L_{\mu}^\dagger) + [\delta L_{\mu}, \omega \delta L_{\mu}^\dagger]).
$$

(13)

Moreover $L_{\mu} := I_S \otimes B_{\mu}$, and $\delta L_{\mu} := X_{\mu} \otimes A_{\mu}$. Let us consider the first two terms of Eq. (12) for a given $\mu$ and with $\omega = \rho \otimes \sigma$

$$
2 \delta L_{\mu} (\rho \otimes \sigma) L_{\mu}^\dagger - L_{\mu}^\dagger \delta L_{\mu} (\rho \otimes \sigma) - (\rho \otimes \sigma) L_{\mu}^\dagger \delta L_{\mu} \\
= 2 X_{\mu} \rho \otimes A_{\mu} \sigma L_{\mu}^\dagger - X_{\mu} \rho \otimes L_{\mu}^\dagger A_{\mu} \sigma - \rho X_{\mu} \otimes \sigma L_{\mu}^\dagger A_{\mu}
$$

(14)

multiplying by $\rho \otimes I_B$ and taking the trace

$$
\text{tr}_S(\rho X_{\mu} \rho) \text{tr}_B(2 A_{\mu} \sigma L_{\mu}^\dagger - L_{\mu}^\dagger A_{\mu} \sigma - \sigma L_{\mu}^\dagger A_{\mu}) = 0.
$$

(15)

Reasoning in the very same way, even the last terms of Eq. (12) give a vanishing contribution. This show that relation (10) is fulfilled by $\mathcal{L}_1$.

VI. CONCLUSIONS.

The possibility of noiseless encoding and processing of quantum information is traced back to the existence of an underlying multi-partite structure. The origin of such hidden structure is purely algebraic and it is dictated by the interactions between the systems and the environment: When the latter admits non trivial symmetry group then noiseless subsystems allowing for universal quantum computation exist. This NSs approach, introduced by Knill et al as a generalization of decoherence-free subspaces, is robust against perturbations and provides an analog of the pointer basis in the noncommutative realm. The notion of NS has been shown to be crucial for a unified understanding and designing of error correction/avoidance strategies. In particular we argued how one can use decoupling/symmetrization techniques for artificial synthesis of systems supporting NSs, and how to perform on such NSs non-trivial computations.
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