STEENROD COALGEBRAS

JUSTIN R. SMITH

ABSTRACT. This paper shows that a functorial version of the “higher diagonal” of a space used to compute Steenrod squares actually contains far more topological information — including (in some cases) the space’s integral homotopy type.

1. INTRODUCTION

It is well-known that the Alexander-Whitney coproduct is functorial with respect to simplicial maps. If $X$ is a simplicial set, $C(X)$ is the unnormalized chain-complex and $R S_2$ is the bar-resolution of $\mathbb{Z}_2$ (see [12]), it is also well-known that there is a unique homotopy class of $\mathbb{Z}_2$-equivariant maps (where $\mathbb{Z}_2$ transposes the factors of the target)

$$\xi_X: R S_2 \otimes C(X) \to C(X) \otimes C(X)$$

cohomology, and that this extends the Alexander-Whitney diagonal. We will call such structures, Steenrod coalgebras and the map $\xi_X$ the Steenrod diagonal.

With some care (see appendix A), one can construct $\xi_X$ in a manner that makes it functorial with respect to simplicial maps although this is seldom done since the homotopy class of this map is what is generally studied. Essentially, [18, 20, 21] show that $C(X)$ possesses the structure of a functorial coalgebra over an operad $\mathcal{S}$ (see example 2.15) and that the arity-2 portion of this operad-action is a functorial version of $\xi_X$. Throughout this paper, we will assume this functorial version of $\xi_X$.

It is natural to ask whether $\xi_X$ encapsulates more information about a topological space than its cup-product and Steenrod operations. The present paper answers this question affirmatively for degeneracy-free simplicial sets. Roughly
speaking, these are simplicial sets whose degeneracies do not satisfy any relations other than the minimal set of identities all face- and degeneracy-operators must satisfy — see definition \[2.4\] and proposition \[2.5\]. Every simplicial set is canonically homotopy equivalent to a degeneracy-free one (see proposition \[2.6\]). The only place degeneracy-freeness is used in this paper is lemma \[5.3\].

**Corollary.** \[5.9\] Let \( X \) and \( Y \) be pointed, reduced degeneracy-free simplicial sets with normalized chain-complexes \( N(X) \) and \( N(Y) \), let \( \mathcal{R} = \mathbb{Z}_p \) for some prime \( p \) or a subring of \( \mathbb{Q} \), and let
\[
f : N(X) \otimes \mathcal{R} \rightarrow N(Y) \otimes \mathcal{R}
\]
be a (purely algebraic) chain map that makes the diagram
\[
\begin{array}{ccc}
R S_2 \otimes N(X) \otimes \mathcal{R} & \xrightarrow{1 \otimes f} & R S_2 \otimes N(Y) \otimes \mathcal{R} \\
\xi_X \otimes 1 & & \xi_Y \otimes 1 \\
N(X) \otimes \mathcal{R} \otimes N(X) \otimes \mathcal{R} & \xrightarrow{f \otimes f} & N(Y) \otimes \mathcal{R} \otimes N(Y) \otimes \mathcal{R}
\end{array}
\]
commute exactly (i.e., not merely up to a chain-homotopy). Then \( f \) induces a simplicial map
\[
f_\infty : \mathcal{R}_\infty X \rightarrow \mathcal{R}_\infty Y
\]
between \( \mathcal{R} \)-completions that makes the diagram
\[
\begin{array}{c}
\phi_X \\
\mathcal{R}_\infty X \xrightarrow{f_\infty} \mathcal{R}_\infty Y \\
\phi_Y
\end{array}
\]
commute. If \( f \) is surjective, then \( f_\infty \) is a fibration, and if \( f \) is also a homology equivalence, then \( f_\infty \) is a trivial fibration.

If \( f \) is a surjective homology equivalence, \( \mathcal{R} = \mathbb{Z} \), and \( X \) and \( Y \) are nilpotent then there exists a homotopy equivalence
\[
\tilde{f} : |X| \rightarrow |Y|
\]
of topological realizations. Corollary \[5.12\] states that nilpotent, degeneracy-free spaces are homotopy equivalent if and only if there exists a homology equivalence of their chain-complexes that make diagram \[1.1\] commute.
Here, \( \tilde{R} \) is a pointed version of the \( R \)-free simplicial abelian group functor — see definitions 4.2 and 4.5.

Because of the canonical homotopy equivalence between all simplicial sets and degeneracy-free ones, the result above implies:

**Corollary.** \([5.13] \) If \( X \) and \( Y \) are pointed reduced simplicial sets and \( f: C(X) \to C(Y) \) is a morphism of Steenrod coalgebras — over unnormalized chain-complexes — then \( f \) induces a commutative diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow \phi_{(\partial f(X))} & & \downarrow \phi_{(\partial f(Y))} \\
\tilde{R}(\partial \circ f(X)) & \xrightarrow{f_{\infty}} & \tilde{R}(\partial \circ f(Y)) \\
\end{array}
\]

where \( g_X \) and \( g_Y \) are homotopy equivalences if \( X \) and \( Y \) are Kan complexes — and homotopy equivalences of their topological realizations otherwise. In particular, if \( X \) and \( Y \) are nilpotent, \( \mathbb{R} = \mathbb{Z} \), and \( f \) is an integral homology equivalence, then the topological realizations \(|X|\) and \(|Y|\) are homotopy equivalent.

Here, \( f \) and \( \partial \) are functors defined in definition 2.2. Singular simplicial sets are always Kan complexes.

The reader might wonder how the Steenrod diagonal can contain any information beyond the structure of a space at the prime 2. The answer is that it forms part of an operad structure that contains information about all primes — and the only part of this complex operad structure needed to compute, for instance, Steenrod \( p \)-th powers is the Steenrod diagonal.

For example, let \( X \) be a simplicial set with functorial higher diagonal

\[
h: R S_2 \otimes C(X) \to C(X) \otimes C(X)
\]

let \( \Delta = h([\cdot] \otimes \cdot): C(X) \to C(X) \otimes C(X) \) — the Alexander-Whitney diagonal — and let \( \Delta_2 = h([1, 2] \otimes \cdot): C(X) \to C(X) \otimes C(X) \). \( \Delta \)
straightforward calculation shows that
\[(1 \otimes \Delta) \circ \Delta_2 : C(X) \to C(X)^{\otimes 3}\]
has the property that
\[
\partial \{(1 \otimes \Delta) \circ \Delta_2 \} = (1 \otimes \Delta) \circ \partial \Delta_2
\]
\[
= (1 \otimes \Delta) \circ \{(1,2) - 1\} \Delta
\]
\[
= (1,2,3)(\Delta \otimes 1) \circ \Delta - (1 \otimes \Delta) \circ \Delta
\]
\[
= \{(1,2,3) - 1\}(1 \otimes \Delta) \circ \Delta
\]
where \((1,2,3)\) is a cyclic permutation of the factors. It follows that \(\Delta\) and \(\Delta_2\) incorporate information about \(X\) at the prime 3.

This paper’s general approach to homotopy theory is the end result of a lengthy research program involving some of the 20th century’s leading mathematicians. In [15], Daniel Quillen proved that the category of simply-connected rational simplicial sets is equivalent to that of commutative coalgebras over \(\mathbb{Q}\). In [24], Sullivan analyzed the algebraic and analytic properties of these coalgebras, developing the concept of minimal models and relating them to de Rham cohomology. That work was dual to Quillen’s and had the advantage of being far more direct.

Since then, a major goal has been to develop a similar theory for integral homotopy types.

In [17], Smirnov asserted that the integral homotopy type of a space is determined by a coalgebra-structure on its singular chain-complex over an \(E_\infty\)-operad. Smirnov’s proof was somewhat opaque and several people known to the author even doubted the result’s validity. In any case, the \(E_\infty\)-operad involved was complex, being uncountably generated in all dimensions.

In [21], the author showed that the chain-complex of a space was naturally a coalgebra over an \(E_\infty\)-operad \(\mathcal{S}\) and that this could be used to iterate the cobar construction. The paper [19] applied those results to show that this \(\mathcal{S}\)-coalgebra determined the integral homotopy type of a simply-connected space.

In [13]¹, Mandell showed that the mod-\(p\) cochain complex of a \(p\)-nilpotent space had a algebra structure over an operad that determined the space’s \(p\)-type. In [14], Mandell showed

¹Based on Mandell’s 1997 thesis.
that the cochains of a nilpotent space whose homotopy groups are all finite have an algebra structure over an operad that determined its integral homotopy type.

The paper [18] showed that the $\mathcal{S}$-coalgebra structure of a chain-complex had a “transcendental” structure that determines a nilpotent space’s homotopy type (without the finiteness conditions of [14]). It essentially reprised the main result of [19], using a very different proof-method. The present paper shows that this transcendental structure even manifests in the sub-operad of $\mathcal{S}$ generated by its arity-2 component, $R_{S_2}$.

I am indebted to Dennis Sullivan for several interesting discussions.

2. Definitions

Given a simplicial set, $X$, $C(X)$ will always denote its unnormalized chain-complex and $N(X)$ its normalized one (with degeneracies divided out).

We consider variations on the concept of simplicial set.

**Definition 2.1.** Let $\Delta_+$ be the ordinal number category whose morphisms are order-preserving monomorphisms between them. The objects of $\Delta_+$ are elements $n = \{0 \to 1 \to \cdots \to n\}$ and a morphism

$$\theta: m \to n$$

is a strict order-preserving map ($i < k \implies \theta(i) < \theta(j)$). Then the category of *delta-complexes*, $D$, has objects that are contravariant functors

$$\Delta_+ \to \text{Set}$$

to the category of sets. The chain complex of a delta-complex, $X$, will be denoted $N(X)$.

**Remark.** In other words, delta-complexes are just simplicial sets without degeneracies.

A simplicial set gives rise to a delta-complex by “forgetting” its degeneracies — “promoting” its degenerate simplices to nondegenerate status. Conversely, a delta-complex can be converted into a simplicial set by equipping it with degenerate simplices in a mechanical fashion. These operations define functors:
**Definition 2.2.** The functor
\[ \mathfrak{f} : S \to D \]
is defined to simply drop degeneracy operators (degenerate simplices become nondegenerate). The functor
\[ \mathfrak{d} : D \to S \]
equips a delta complex, \( X \), with degenerate simplicies and operators via
\[ (2.1) \quad \mathfrak{d}(X)_m = \bigsqcup_{m \geq n} X_n \]
for all \( m > n \geq 0 \).

**Remark.** The functors \( \mathfrak{f} \) and \( \mathfrak{d} \) were denoted \( F \) and \( G \), respectively, in [16]. Equation 2.1 simply states that we add all possible degeneracies of simplices in \( X \) subject only to the basic identities that face- and degeneracy-operators must satisfy.

Although \( \mathfrak{f} \) promotes degenerate simplicies to nondegenerate ones, these new nondegenerate simplices can be collapsed without changing the homotopy type of the complex: although the degeneracy operators are no longer built in to the delta-complex, they still define contracting homotopies.

The definition immediately implies that

**Proposition 2.3.** If \( X \) is a simplicial set and \( Y \) is a delta-complex, \( C(X) = \mathcal{N}(\mathfrak{f}(X)) \), \( \mathcal{N}(\mathfrak{d}(Y)) = \mathcal{N}(Y) \), and \( C(X) = \mathcal{N}(\mathfrak{d} \circ \mathfrak{f}(X)) \).

**Definition 2.4.** A simplicial set, \( X \), is defined to be **degeneracy-free** if
\[ X = \mathfrak{d}(Y) \]
for some delta-complex, \( Y \).

**Remark.** Compare definition 1.10 in chapter VII of [6]. In a manner of speaking, \( X \) is freely generated by the degeneracy operators acting on a basis consisting of the simplices of \( Y \). Lemma 1.2 in chapter VII of [6] describes other properties of degeneracy-free simplicial sets (hence of the functor \( \mathfrak{d} \)).

In [16], Rourke and Sanderson also showed that one could give a “somewhat more intrinsic” definition of degeneracy-freeness:

---

2 Their definition has a typo, stating that \( \Delta_+ \) consists of **surjections** rather than **injections**.
Proposition 2.5. If \( X \) is a simplicial set, let \( \text{Core}(X) \) consist of the nondegenerate simplices and their faces. This is a delta-complex and there exists a canonical map
\[
\gamma : \text{d}(\text{Core}(X)) \to X
\]
sending simplices of \( \text{Core}(X) \) to themselves in \( X \) and degeneracies to suitable degeneracies of them. Then \( X \) is degeneracy-free if and only if \( \gamma \) is an isomorphism.

Theorem 1.7 of [16] shows that there exists an adjunction:

\[
\text{d} : \mathcal{D} \leftrightarrow \mathcal{S} : \text{f}
\]

The composite (the counit of the adjunction)
\[
\text{f} \circ \text{d} : \mathcal{D} \to \mathcal{D}
\]
maps a delta complex into a much larger one — that has an infinite number of (degenerate) simplices added to it. There is a natural inclusion
\[
\iota : X \to \text{f} \circ \text{d}(X)
\]
and a natural map (the unit of the adjunction)
\[
g : \text{d} \circ \text{f}(X) \to X
\]
The functor \( g \) sends degenerate simplices of \( X \) that had been “promoted to nondegenerate status” by \( \text{f} \) to their degenerate originals — and the extra degenerates added by \( \text{d} \) to suitable degeneracies of the simplices of \( X \).

Rourke and Sanderson also prove:

Proposition 2.6. If \( X \) is a simplicial set and \( Y \) is a delta-complex then

1. \( |Y| \) and \( |\text{d}Y| \) are homeomorphic
2. the map \( |\gamma| : |\text{d} \circ \text{f}(X)| \to |X| \) is a homotopy equivalence.
3. \( \text{f} : \mathcal{H}S \to \mathcal{H}D \) defines an equivalence of categories, where \( \mathcal{H}S \) and \( \mathcal{H}D \) are the homotopy categories, respectively, of \( \mathcal{S} \) and \( \mathcal{D} \). The inverse is \( \text{d} : \mathcal{H}D \to \mathcal{H}S \). In particular, if \( X \) is a Kan complex, the natural map
\[
g : \text{d} \circ \text{f}(X) \to X
\]
is a homotopy equivalence.

Remark. Here, \( |*| \) denotes the topological realization functors for \( \mathcal{S} \) and \( \mathcal{D} \).
Proof. The first two statements are proposition 2.1 of [16] and statement 3 is theorem 6.9 of the same paper. The final statement follows from Whitehead’s theorem and the fact that Kan complexes are fibrant in the Quillen model structure of simplicial sets (see [6]). □

Definition 2.7. We will denote the category of \( R \)-free chain complex by \( Ch \) and ones that are bounded from below in dimension 0 by \( Ch_0 \).

We make extensive use of the Koszul Convention (see [8]) regarding signs in homological calculations:

**Definition 2.8.** If \( f: C_1 \to D_1, g: C_2 \to D_2 \) are maps, and \( a \otimes b \in C_1 \otimes C_2 \) (where \( a \) is a homogeneous element), then \((f \otimes g)(a \otimes b)\) is defined to be \((-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)\).

**Remark 2.9.** If \( f_i, g_i \) are maps, it isn’t hard to verify that the Koszul convention implies that \((f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2)\).

The set of morphisms of chain-complexes is itself a chain complex:

**Definition 2.10.** Given chain-complexes \( A, B \in Ch \) define

\[
\text{Hom}_Z(A, B)
\]

to be the chain-complex of graded \( R \)-morphisms where the degree of an element \( x \in \text{Hom}_Z(A, B) \) is its degree as a map and with differential

\[
\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f
\]

As a \( R \)-module \( \text{Hom}_Z(A, B)_k = \prod_j \text{Hom}_Z(A_j, B_{j+k}) \).

**Remark.** Given \( A, B \in Ch^{S_n} \), we can define \( \text{Hom}_{ZS_n}(A, B) \) in a corresponding way.

Now we define the concept of operad:

**Definition 2.11.** If \( G \) is a discrete group, let \( Ch^G_0 \) denote the category of chain-complexes equipped with a right \( G \)-action. This is again a closed symmetric monoidal category and the forgetful functor \( Ch^G_0 \to Ch_0 \) has a left adjoint, \((-)[G]\). This applies to the symmetric groups, \( S_n \), where we regard \( S_1 \) and \( S_0 \) as the trivial group. The category of collections is defined to be the product

\[
\text{Coll}(Ch_0) = \prod_{n \geq 0} Ch^{S_n}_0
\]
Its objects are written $\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$. Each collection induces an endofunctor (also denoted $\mathcal{V}$) $\mathcal{V} : \text{Ch}_0 \to \text{Ch}_0$

$$\mathcal{V}(X) = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{\mathbb{Z}S_n} X^\otimes$$

where $X^\otimes = X \otimes \cdots \otimes X$ and $S_n$ acts on $X^\otimes$ by permuting factors. This endofunctor is a monad if the defining collection has the structure of an operad, which means that $\mathcal{V}$ has a unit $\eta : R \to \mathcal{V}(1)$ and structure maps

$$\gamma_{k_1, \ldots, k_n} : \mathcal{V}(n) \otimes \mathcal{V}(k_1) \otimes \cdots \otimes \mathcal{V}(k_n) \to \mathcal{V}(k_1 + \cdots + k_n)$$

satisfying well-known equivariance, associativity, and unit conditions — see [22], [10].

We will call the operad $\mathcal{V} = \{\mathcal{V}(n)\} \Sigma$-cofibrant if $\mathcal{V}(n)$ is $\mathbb{Z}S_n$-projective for all $n \geq 0$.

Remark. The operads we consider here correspond to symmetric operads in [22].

The term “unital operad” is used in different ways by different authors. We use it in the sense of Kriz and May in [10], meaning the operad has a $0$-component that acts like an arity-lowering augmentation under compositions. Here $\mathcal{V}(0) = R$.

The term $\Sigma$-cofibrant first appeared in [3].

We can also define operads in terms of compositions:

**Definition 2.12.** If $\mathcal{V}$ is an operad with components $\mathcal{V}(n)$ and $\mathcal{V}(m)$, define the $i^{th}$ composition, with $1 \leq i \leq n$

$$\circ_i : \mathcal{V}(n) \otimes \mathcal{V}(m) \to \mathcal{V}(n + m - 1)$$

by

$$\mathcal{V}(n) \otimes \mathcal{V}(m) \xrightarrow{\mathcal{V}(n) \otimes Z^{i-1} \otimes \mathcal{V}(m) \otimes Z^{n-i}} \mathcal{V}(n) \otimes \mathcal{V}(1)^{i-1} \otimes \mathcal{V}(m) \otimes \mathcal{V}(1)^{n-i} \xrightarrow{1 \otimes \eta^{i-1} \otimes 1 \otimes \eta^{n-i}} \mathcal{V}(n + m - 1)$$

Here $\eta : \mathbb{Z} \to \mathcal{V}(1)$ is the unit.
Remark. Operads were originally called composition algebras and defined in terms of these operations — see [5].

It is well-known that the compositions and $\gamma$ determine each other — see definition 2.12 and proposition 2.13 of [22]. It is also well-known (see lemma 2.14 of [22]) that:

**Lemma 2.13.** Compositions obey the identities

$$
(a \circ_i b) \circ_j c = \begin{cases} 
(-1)^{\dim b \cdot \dim c} (a \circ_{j-n+1} c) \circ_i b & \text{if } i + n - 1 \leq j \\
\quad a \circ_i (b \circ_{j-i+1} c) & \text{if } i \leq j < i + n - 1 \\
(-1)^{\dim b \cdot \dim c} (a \circ_j c) \circ_{i+m-1} b & \text{if } 1 \leq j < i
\end{cases}
$$

where arity $c = m$, arity $a = n$, and

$$
(2.4) \quad a \circ_{\sigma(i)} (\sigma \cdot b) = T_{1,\ldots,\sigma(i),\ldots,1}(\sigma) \cdot (a \circ_i b)
$$

for $\sigma \in S_n$, where $T_{\alpha_1,\ldots,\alpha_n}(\sigma) \in S_{\sum \alpha_i}$ is a permutation that permutes the $n$ blocks

$$
\{1,\ldots,\alpha_1\}, \{\alpha_1 + 1, \alpha_1 + \alpha_2\}, \ldots, \\
\{\alpha_1 + \cdots + \alpha_n - 1 + 1, \alpha_1 + \cdots + \alpha_n\}
$$

via $\sigma$.

A simple example of an operad is:

**Example 2.14.** For each $n \geq 0$, $\mathcal{S}_0(n) = \mathbb{Z}S_n$, with structure-map a $\mathbb{Z}$-linear extension of

$\gamma_{\alpha_1,\ldots,\alpha_n} : S_n \times S_{\alpha_1} \times \cdots \times S_{\alpha_n} \to S_{\alpha_1+\cdots+\alpha_n}$

defined by

$\gamma_{\alpha_1,\ldots,\alpha_n}(\sigma \times \theta_1 \times \cdots \times \theta_n) = T_{\alpha_1,\ldots,\alpha_n}(\sigma) \circ (\theta_1 \oplus \cdots \oplus \theta_n)$

with $\sigma \in S_n$ and $\theta_i \in S_{\alpha_i}$ where $T_{\alpha_1,\ldots,\alpha_n}(\sigma) \in S_{\sum \alpha_i}$ is defined above, in lemma 2.8. See [21] for explicit formulas and computations.

Another important operad is:

**Example 2.15.** The operad, $\mathcal{S}$, defined in [21] is given by $\mathcal{S}(n) = R\mathbb{Z}S_n$ — the normalized bar-resolution of $\mathbb{Z}$ over $\mathbb{Z}S_n$. This is well-known (like the closely-related Barrett-Eccles operad defined in [2]) to be a Hopf-operad, i.e. equipped with an operad morphism

$$\delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$$

and is important in topological applications. See [21] for formulas for the structure maps.
For the purposes of this paper, the main example of an operad is

**Definition 2.16.** Given any $C \in \text{Ch}$, the associated coendomorphism operad, $\text{CoEnd}(C)$ is defined by

$$\text{CoEnd}(C)(n) = \text{Hom}_\mathbb{Z}(C, C \otimes n)$$

Its structure map

$$\gamma_{\alpha_1, \ldots, \alpha_n} : \text{Hom}_\mathbb{Z}(C, C \otimes n) \otimes \text{Hom}_\mathbb{Z}(C, C \otimes \alpha_1) \otimes \cdots \otimes \text{Hom}_\mathbb{Z}(C, C \otimes \alpha_n) \to \text{Hom}_\mathbb{Z}(C, C \otimes \alpha_1 + \cdots + \alpha_n)$$

simply composes a map in $\text{Hom}_\mathbb{Z}(C, C \otimes n)$ with maps of each of the $n$ factors of $C$.

This is a non-unital operad, but if $C \in \text{Ch}$ has an augmentation map $\varepsilon : C \to R$ then we can regard $\varepsilon$ as the generator of $\text{CoEnd}(C)(0) = R \cdot \varepsilon \subset \text{Hom}_\mathbb{Z}(C, C \otimes 0) = \text{Hom}_\mathbb{Z}(C, \mathcal{R})$.

We use the coendomorphism operad to define the main object of this paper:

**Definition 2.17.** A coalgebra over an operad $\mathcal{V}$ is a chain-complex $C \in \text{Ch}$ with an operad morphism $\alpha : \mathcal{V} \to \text{CoEnd}(C)$, called its structure map. We will sometimes want to define coalgebras using the adjoint structure map,

$$\alpha : C \to \prod_{n \geq 0} \text{Hom}_\mathbb{Z}_{S_n}(\mathcal{V}(n), C \otimes n)$$

(2.5)

where $S_n$ acts on $C \otimes n$ by permuting factors or the set of chain-maps

$$\alpha_n : C \to \text{Hom}_\mathbb{Z}_{S_n}(\mathcal{V}(n), C \otimes n)$$

for all $n \geq 0$ or even

$$\beta_n : \mathcal{V}(n) \otimes C \to C \otimes n$$

It is not hard to see how compositions (in definition 2.12) relate to coalgebras

**Proposition 2.18.** Let $\beta_n : \mathcal{V}(n) \otimes C \to C \otimes n$ for all $n \geq 0$ define a coalgebra over an operad $\mathcal{V}$ and, for any $x \in \mathcal{V}(n)$ and any $n \geq 0$ define

$$\Delta_x = \beta_n(x \otimes \ast) : C \to C \otimes n$$

If $x \in \mathcal{V}(n)$ and $y \in \mathcal{V}(m)$, then

$$\Delta_{y \circ i, x} = 1 \otimes \cdots \otimes 1 \otimes \Delta_y \otimes 1 \otimes \cdots \otimes \circ \Delta_x$$
Proof. Immediate, from definitions 2.12 and 2.16. □

2.1. Types of coalgebras.

**Example 2.19.** Coassociative coalgebras are precisely the coalgebras over $\mathcal{S}_0$ (see 2.14).

**Definition 2.20.** Commute is an operad defined to have one basis element $\{b_i\}$ for each integer $i \geq 0$. Here the arity of $b_i$ is $i$ and the degree is 0 and these elements satisfy the composition-law: $\gamma(b_n \otimes b_{k_1} \otimes \cdots \otimes b_{k_n}) = b_K$, where $K = \sum_{i=1}^n k_i$. The differential of this operad is identically zero. The symmetric-group actions are trivial.

**Example 2.21.** Coassociative, commutative coalgebras are the coalgebras over Commute.

We can define a concept dual to that of a free algebra generated by a set:

**Definition 2.22.** Let $D$ be a coalgebra over an operad $\mathcal{V}$, equipped with a Ch-morphism $\varepsilon : \lceil D \rceil \rightarrow E$, where $E \in \text{Ch}$. Then $D$ is called the cofree coalgebra over $\mathcal{V}$ cogenerated by $\varepsilon$ if any morphism in Ch

$$f : \lceil C \rceil \rightarrow E$$

where $C \in \mathcal{S}_0$, induces a unique morphism in $\mathcal{S}_0$

$$\alpha_f : C \rightarrow D$$

that makes the diagram

$$\begin{array}{ccc}
\lceil C \rceil & \xrightarrow{[\alpha_f]} & \lceil D \rceil \\
\downarrow f & & \downarrow \varepsilon \\
\downarrow E & & \end{array}$$

Here $\alpha_f$ is called the classifying map of $f$. If $C \in \mathcal{S}_0$ then

$$\alpha_f : C \rightarrow L_{\mathcal{V}}[C]$$

will be called the classifying map of $C$.

**Remark 2.23.** This universal property of cofree coalgebras implies that they are unique up to isomorphism if they exist.

The paper [22] explicitly constructs cofree coalgebras for many operads:
$L V C$ is the general cofree coalgebra over the operad $V$ — here, $C$, is a chain-complex that is not necessarily concentrated in nonnegative dimensions. Then [22] constructs $D = L V E$ as the maximal submodule of

$$\prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), E \otimes n)$$

on which the dual of the structure-maps of $V$ define a coalgebra-structure: let $\iota: D \to \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), E \otimes n)$ be the inclusion of chain-complexes. In the notation of definition [2.22], an $V$-coalgebra, $C$, is defined by its structure map (see equation [2.5])

$$s: C \to \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), C \otimes n)$$

and its classifying map $\alpha_f: D \to L V C$ is the coalgebra morphism defined by the diagram

$$\begin{array}{ccc}
C & \xrightarrow{s} & \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), C \otimes n) \\
\downarrow{\alpha_f} & & \downarrow{\prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(1, f \otimes n)} \\
D & \xrightarrow{\iota} & \prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(V(n), E \otimes n)
\end{array}$$

An inductive argument shows that this is the unique coalgebra morphism compatible with $f$.

In all cases, definition [2.22] implies the existence of an adjunction

$$[\ast]: \text{Ch}_0 \leftrightarrows L V [\ast]$$

where $[\ast]: \mathcal{C} \to \text{Ch}_0$ is the forgetful functor from coalgebras to chain-complexes.

3. STEENROD COALGEBRAS

We begin with:

**Definition 3.1.** A Steenrod coalgebra, $(C, \delta)$ is a chain-complex $C \in \text{Ch}$ equipped with a $\mathbb{Z}_2$-equivariant chain-map

$$\delta: R S_2 \otimes C \to C \otimes C$$
where \( \mathbb{Z}_2 \) acts on \( C \otimes C \) by swapping factors and \( R_S 2 \) is the bar-resolution of \( \mathbb{Z} \) over \( \mathbb{Z} S_2 \). A morphism \( f : (C, \delta_C) \to (D, \delta_D) \) is a chain-map \( f : C \to D \) that makes the diagram commute.

Steenrod coalgebras are very general — the underlying coalgebra need not even be coassociative. The category of Steenrod coalgebras is denoted \( S \).

**Definition 3.2.** Let, \( \mathcal{F} \), denote the free operad generated by \( R_S 2 \).

**Remark.** See sections 5.2 and 5.5 of [11] or section 5.8 of [3] for an explicit construction of \( \mathcal{F} \). For instance

\[
\mathcal{F}(3) = R_S 2 \otimes_{\mathbb{Z} S_2} \left( \mathbb{Z} S_3 \otimes_{\mathbb{Z} S_2} R_S 2 \oplus \mathbb{Z} S_3 \otimes_{\mathbb{Z} S_2} R_S 2 \right)
\]

where \( S_2 = \mathbb{Z}_2 \) swaps the summands and \( \mathbb{Z} S_3 \) acts on \( \mathcal{F}(3) \) by acting on the factors \( \mathbb{Z} S_3 \) inside the parentheses.

**Proposition 3.3.** The identity map of \( R_S 2 \) uniquely extends to an operad-morphism

\[
\xi : \mathcal{F} \to \mathcal{G}
\]

and the kernel is an operadic ideal (see section 5.2.16 of [11]) denoted \( \mathcal{K} \).

**Remark.** The image, \( \xi(\mathcal{F}) \subset \mathcal{G} \), is the suboperad generated by \( \mathcal{G}(2) = R_S 2 \).

**Proof.** All statements follow immediately from the defining property of free operads. \( \square \)

Although the construction of \( \mathcal{F} \) is fairly complex, it is easy to describe coalgebras over \( \mathcal{F} \):

**Proposition 3.4.** The category of coalgebras over \( \mathcal{F} \) is identical to that of Steenrod coalgebras.
Proof. If $C$ is an $\mathcal{F}$-coalgebra then there exists a $\mathbb{Z}S_2$-morphism $F(2) \otimes C = R\mathbb{Z}S_2 \otimes C \to C \otimes C$ so $C$ is a Steenrod coalgebra. If $C$ is a Steenrod coalgebra, it has an adjoint structure map $R\mathbb{Z}S_2 \to \text{Hom}_{\mathbb{Z}}(C, C \otimes C) = \text{CoEnd}(C)(2)$ that uniquely extends to an operad-morphism $F \to \text{CoEnd}(C)$ It is also clear that this correspondence respects morphisms. □

This has a number of interesting consequences:

**Theorem 3.5.** If $C$ is a chain-complex, there exists a universal Steenrod coalgebra $L_\mathcal{F}C$ — the cofree coalgebra over $\mathcal{F}$ cogenerated by $C$ — equipped with a chain-map $\varepsilon: L_\mathcal{F}C \to C$ with the property that, given any Steenrod coalgebra $D$ and any chain-map $f: D \to C$, there exists a unique morphism of Steenrod coalgebras $\bar{f}: D \to L_\mathcal{F}C$ that makes the diagram commute.

Proof. The conclusions are nothing but the defining properties of a cofree coalgebra over $\mathcal{F}$. So the result follows immediately from proposition 3.4. □

4. THE DOLD-KAN FUNCTOR AND VARIANTS

Throughout this section, $\mathcal{R}$ is a ring defined by

$$\mathcal{R} = \begin{cases} \mathbb{Z}_p & \text{for some prime } p \\ \mathcal{R} \subset \mathbb{Q} & \end{cases}$$

We recall classic results regard simplicial abelian groups:
Definition 4.1. Let $s\text{AB}$ denote the category of simplicial abelian groups and $s\text{AB}_0 \subset s\text{AB}$ the full subcategory of $\mathbb{Z}$-free pointed, reduced simplicial abelian groups. If $A \in s\text{AB}$, let

1. $\{A\}$ denote the Moore complex of $A$ — a (not necessarily $\mathbb{Z}$-free chain complex made up of the simplices of $A$)
2. $NA \subset \{A\}$ denote the normalized chain-complex of $A$ defined by

$$NA_n = \bigcap_{i=0}^{n-1} \ker d_i \subset A_n$$

where $d_i : A_n \to A_{n-1}$ are the face-operators. The boundary is defined by $\partial_n = (-1)^n d_n : NA_n \to NA_{n-1}$.

The Dold-Kan functor from the category of arbitrary chain complexes (not necessarily $\mathbb{Z}$-free) to $s\text{AB}$ is denoted $\Gamma$.

Remark. Note that the simplices of a simplicial abelian group form a chain-complex: this is the Moore complex, which “forgets” the extra structure of a simplicial abelian group. It is well-known (see [6, chapter III]) that

$$\pi_i(A) = H_i(\{A\})$$

for $i \geq 0$.

Given a simplicial set, we can construct a simplicial abelian group from it:

Definition 4.2. If $X$ is a simplicial set, $\mathcal{R}X$ denotes the $\mathcal{R}$-free simplicial abelian group generated by $X$. The Hurewicz map

$$h_X : X \to \mathcal{R}X$$

sends a simplex $x \in X$ to $1 \cdot x \in \mathcal{R}X$.

Remark. It is well-known (see [1], chapter I, § 2), that

$$\pi_i(\mathcal{R}X) \cong H_i(X, \mathcal{R})$$

and that the Hurewicz map induces

$$\pi_i(h_X) : \pi_i(X) \to \pi_i(\mathcal{R}X) = H_i(X, \mathcal{R})$$

— the Hurewicz homomorphism from homotopy groups to homology groups (hence the name “Hurewicz map”).

We have the classic Dold-Kan results (see [9] and [6, chapter III], corollary 2.3, theorem 2.5, and corollary 2.12):
Proposition 4.3. For any chain-complex, $C$, concentrated in nonnegative dimensions

$$N \Gamma C \cong C$$

and for any simplicial abelian group $A$

$$\Gamma N A \cong A$$

These correspondences define an equivalence of categories between the category of (not necessarily $\mathbb{Z}$-free) chain-complexes concentrated in nonnegative dimensions and simplicial abelian groups.

In addition, there is an adjunction

$$N(*) : \text{Ch}_0 \rightleftarrows \text{S} : \Gamma *$$

where $N(*)$ is the (normalized) integral chain-complex functor (see [6, chapter III]). The functor $\Gamma *$ is defined by

$$\Gamma(C)_m = \bigoplus_{m-n} C_n$$

for all $m > n \geq 0$, where $m$ and $n$ are objects of the ordinal number category $\Delta$.

Remark. Equation 4.2 simply says that one forms all possible “formal” degeneracies of $C$ and defines face and degeneracy operators to via the defining identities that they satisfy.

We define a pointed variant:

Definition 4.4. If $C \in \text{Ch}_0$, then $C = C^+ \oplus Z_0$, where $Z_0$ is concentrated in dimension 0 and equal to $\mathbb{Z}$ there. We define

$$\tilde{\Gamma} C = \Gamma C / \Gamma Z_0 \cong \Gamma C^+$$

— where / denotes a quotient-group.

Remark. Since $C = C^+ \oplus Z_0$, the equivalence of categories in proposition [4.3] implies that

$$\Gamma C = \tilde{\Gamma} C \times \Gamma Z_0$$

If $C \in \text{Ch}_+$, note that $\tilde{\Gamma} C = \Gamma C$, since $\{\Gamma C\}_0 = 0$ — the trivial abelian group.

The reason we need a pointed variant is that the Bousfield-Kan cosimplicial resolution (see [1]) of a space requires it to be pointed.

We also have the free abelian group functor (denoted $FA*$ in [9]):
Definition 4.5. If $X \in S$, $\mathcal{R}X$ is the $\mathcal{R}$-free simplicial abelian group generated by the simplices of $X$. If $X \in S_0$ — i.e., if $X$ is pointed and reduced — then we have the Bousfield-Kan pointed version of the free abelian group functor (see [1]), the quotient

$$\tilde{\mathcal{R}}X = \mathcal{R}X / \mathcal{R}^*$$

where $*$ is the sub-simplicial set generated by the basepoint of $X$.

Proposition 4.6. If $X$ is a simplicial set and $N(X)$ is the normalized integral chain-complex of $X$ then

$$\mathcal{R}X = \Gamma(N(X) \otimes \mathcal{R})$$

If $X$ is pointed and reduced then

$$\tilde{\mathcal{R}}X = \tilde{\Gamma}(N(X) \otimes \mathcal{R}) \times \Gamma \mathcal{R}_0$$

If $X$ and $Y$ are pointed reduced simplicial sets and

$$f: N(X) \otimes \mathcal{R} \to N(Y) \otimes \mathcal{R}$$

is a chain-map of normalized chain-complexes, then $f$ induces

$$\tilde{\mathcal{R}}^{n-1} \tilde{\Gamma} f: \tilde{\mathcal{R}}^{n-1} \tilde{\Gamma} (N(X) \otimes \mathcal{R}) = \tilde{\mathcal{R}}^n X \to \tilde{\mathcal{R}}^{n-1} \tilde{\Gamma} (N(Y) \otimes \mathcal{R}) = \tilde{\mathcal{R}}^n Y$$

for all $n > 0$.

Proof. This follows immediately from the Dold-Kan results in proposition 4.3.

We also have:

Proposition 4.7. If $A$ is a simplicial abelian group, then $A$ is degeneracy-free.

Remark. Also see section 3 in chapter VIII of [6]. This implies that $\mathcal{R}X$ is degeneracy-free for any simplicial set, $X$.

Proof. If $NA$ is the normalized Moore complex $\{A\}$, then the Dold-Kan correspondence implies that

$$A \cong \Gamma NA$$

The conclusion follows by comparing equations 4.2 and 2.1.

18
5. The Main Result

We begin with a definition:

**Definition 5.1.** Let $S$ denote the category of simplicial sets and $S_0$ that of pointed, reduced simplicial sets.

By following the procedure in appendix A, we get:

**Proposition 5.2.** If $X \in S$ is a simplicial set, then the unnor-
malized chain complex of $X$, $C(X)$ has a natural Steenrod coal-
gebra structure and there exists a functor

$$C(\ast): S \to \mathcal{S}$$

from the category of simplicial sets to that of Steenrod coalgebras concentrated in nonnegative dimensions. This projects to a Steenrod coalgebra structure on the normalized chain-complex, $N(X)$.

**Proof.** See appendix A and proposition A.4 for the details. □

The main (only?) reason we are interested in degeneracy-
free simplicial sets is:

**Lemma 5.3.** If $X$ is a degeneracy-free simplicial set, its non-
degenerate simplices form a delta-complex, $\bar{X}$, and there is a
natural inclusion

$$\bar{X} \to f(X)$$

inducing an inclusion of Steenrod coalgebras

$$\iota: N(X) = N(\bar{X}) \to N(f(X)) = C(X)$$

**Remark.** Although all simplicial sets have an inclusion of chain-complexes

$$N(X) \to C(X)$$

the Steenrod coalgebra structure of $N(X)$ is defined as a *quotient* of that of $C(X)$ by the degenerate simplices. It follows that this inclusion of chain-complexes does not necessarily imply one of Steenrod coalgebras.

**Proof.** This follows from proposition 2.3 □

We also define

**Definition 5.4.** If $X$ is a pointed, reduced simplicial set with unnormalized chain-complex $C(X)$ and $\mathcal{R}$ is a ring defined by

$$\mathcal{R} = \begin{cases} \mathbb{Z}_p & \text{for some prime } p \\ \mathcal{R} \subset \mathbb{Q} & \end{cases}$$
then

\[ C(X) \otimes \mathcal{R} = \{ \mathcal{R}X \} \]

— the Moore complex of \( \mathcal{R}X \) — and we can define a chain map

\[ \gamma_X : C(\mathcal{R}X) \otimes \mathcal{R} \to C(X) \otimes \mathcal{R} \]

by \( \mathcal{R} \)-linear extension. Since \( \mathcal{R}X \) is a simplicial abelian group, proposition \[4.7\] implies that it is degeneracy-free and lemma \[5.3\] implies that the inclusion of chain-complexes

\[ N(\mathcal{R}X) \otimes \mathcal{R} \to C(\mathcal{R}X) \otimes \mathcal{R} \]

is a morphism of Steenrod coalgebras inducing a morphism (see definition \[2.22\]):

\[ F_X : N(\mathcal{R}X) \otimes \mathcal{R} \to L_\mathcal{R}(C(X) \otimes \mathcal{R}) \]

(via the adjunction in equation \[2.7\] — also see diagram \[2.6\] where \( L_\mathcal{R}(C(X) \otimes \mathcal{R}) \) is the cofree coalgebra constructed in \[22\].

**Proposition 5.5.** If \( X \) is a pointed, reduced simplicial set with normalized chain-complex \( N(X) \), \( \mathcal{R} \) is a ring defined by

\[ \mathcal{R} = \begin{cases} 
\mathbb{Z}_p & \text{for some prime } p \text{ or} \\
\mathcal{R} \subset \mathbb{Q} & 
\end{cases} \]

and

\[ h_X : X \to \mathcal{R}X \]

is the Hurewicz map, (see definition \[4.2\]) then the composite chain map

\[ N(X) \otimes \mathcal{R} \xrightarrow{N(h_X)} N(\mathcal{R}X) \otimes \mathcal{R} \xrightarrow{\gamma_X} N(X) \otimes \mathcal{R} \]

is the identity map of \( N(X) \otimes \mathcal{R} \), where \( \gamma_X \) is defined in definition \[5.4\].

**Proof.** Just verify this on each simplex: if \( x \in X \), then \( h_X(x) = 1 \cdot x \in \mathcal{R}X \) and \( f(1 \cdot x) = 1 \cdot x = x \in N(X) \otimes \mathcal{R} \).

**Corollary 5.6.** If \( X \) is a pointed, reduced degeneracy-free simplicial set, with normalized chain-complex \( N(X) \), \( \mathcal{R} \) is a ring defined by

\[ \mathcal{R} = \begin{cases} 
\mathbb{Z}_p & \text{for some prime } p \\
\mathcal{R} \subset \mathbb{Q} & 
\end{cases} \]
then the diagram

\[
\begin{array}{ccc}
N(X) \otimes R & \xrightarrow{\alpha_X} & N(\tilde{R}X) \otimes R \\
N(h_X) \otimes 1 & \downarrow & F_X \\
N(\tilde{R}X) \otimes R & \underset{L_\Sigma(C(X) \otimes R)}{\longrightarrow} \n\end{array}
\]

commutes, where

1. \(h_X: X \to \tilde{R}X\) is the Hurewicz map (see definition 4.2),
2. \(\alpha_X: N(X) \otimes R \to L_\Sigma(C(X) \otimes R)\) is the unique morphism of Steenrod coalgebras induced by the chain-map
   \(\iota \otimes 1: N(X) \otimes R \to C(X) \otimes R\)
3. \(F_X: N(\tilde{R}X) \otimes R \to L_\Sigma(C(X) \otimes R)\) is the unique morphism of Steenrod coalgebras induced by \(\gamma_X\) in definition 5.4.

Proof. Since \(h_X: X \to \tilde{R}X\) is simplicial, \(N(h_X) \otimes 1\) is a morphism of Steenrod coalgebras. Proposition 5.5 implies that the morphisms \(\alpha_X\) and \(F_X \circ (N(h_X) \otimes 1)\) are both induced by \(\iota \otimes 1\). Since induced maps to cofree coalgebras are unique, the triangle must commute (see theorem 3.5). \(\square\)

One of the main results in this paper is:

**Theorem 5.7** (Injectivity Theorem). Under the hypotheses of definition 5.4 the map

\[F_X: N(\tilde{R}X) \otimes R \to L_\Sigma(C(X) \otimes R)\]

is injective.

**Remark.** This is essentially the only place we need lemma 5.3, which is the only reason we are interested in degeneracy-free simplicial sets.

The commutative diagram in corollary 5.6 and this result imply that

\[N(h_X) \otimes 1 = F_X^{-1} \circ \alpha_X: N(X) \otimes R \to N(\tilde{R}X) \otimes R\]

so that the geometrically-relevant Hurewicz map is uniquely determined by the Steenrod coalgebra structure of \(N(X) \otimes R\).

Proof. See appendix B. \(\square\)
**Proposition 5.8.** If $X$ and $Y$ are pointed reduced degeneracy-free simplicial sets, $\mathcal{R}$ is a ring defined by

\[
\mathcal{R} = \begin{cases} 
\mathbb{Z}_p & \text{for some prime } p \\
\mathcal{R} \subset \mathbb{Q}
\end{cases}
\]

and

\[
f: N(X) \otimes \mathcal{R} \to N(Y) \otimes \mathcal{R}
\]

is a morphism of Steenrod coalgebras, then the diagram

\[
\begin{array}{ccc}
N(h_X) \otimes 1 & \xrightarrow{f} & N(h_Y) \otimes 1 \\
\downarrow \gamma_X & & \downarrow \gamma_Y \\
N(\tilde{R}X) \otimes \mathcal{R} & \xrightarrow{N(\tilde{\Gamma}f) \otimes 1} & N(\tilde{R}Y) \otimes \mathcal{R}
\end{array}
\]

commutes, which $\tilde{\Gamma}f: \tilde{R}X \to \tilde{R}Y$ is defined in proposition 4.6.

**Remark.** These two data-points (i.e., the chain-complex and the chain-map induced by the Hurewicz map) suffice to define $\mathcal{R}^\ast X$ — the cosimplicial space used to construct Bousfield and Kan's $\mathcal{R}$-completion, $\mathcal{R}_\infty X$ (see [1]).

**Proof.** The fact that $\tilde{\Gamma}f$ (see definition 4.6) maps each simplex of $\tilde{R}X$ (i.e., generator of $N(\tilde{R}X)$) via $f$ implies that the diagram of chain-maps

\[
\begin{array}{ccc}
N(\tilde{R}X) \otimes \mathcal{R} & \xrightarrow{N(\tilde{\Gamma}f) \otimes 1} & N(\tilde{R}Y) \otimes \mathcal{R} \\
\downarrow \gamma_X & & \downarrow \gamma_Y \\
N(X) \otimes \mathcal{R} & \xrightarrow{f} & N(Y) \otimes \mathcal{R}
\end{array}
\]

commutes, where $\gamma_X$ and $\gamma_Y$ are given in definition 5.4. The uniqueness of induced maps to cofree coalgebras (see definition 2.22 and theorem 3.5) and the fact that the target, $L_C (C(Y) \otimes \mathcal{R})$, is cofree implies that the induced diagram of Steenrod coalgebras

\[
\begin{array}{ccc}
N(\tilde{R}X) \otimes \mathcal{R} & \xrightarrow{N(\tilde{\Gamma}f) \otimes 1} & N(\tilde{R}Y) \otimes \mathcal{R} \\
\downarrow F_X & & \downarrow F_Y \\
L_C (C(X) \otimes \mathcal{R}) & \xrightarrow{L_C f} & L_C (C(Y) \otimes \mathcal{R})
\end{array}
\]
commutes. The conclusion follows from the commutativity of the diagram

\[
\begin{array}{ccc}
N(\tilde{\mathcal{R}}X \otimes \mathcal{R}) & \xrightarrow{N(\tilde{f}) \otimes 1} & N(\tilde{\mathcal{R}}Y \otimes \mathcal{R}) \\
\downarrow F_X & & \downarrow F_Y \\
L_{\mathcal{R}}(C(X) \otimes \mathcal{R}) & \xrightarrow{L_{\mathcal{R}}(\tilde{f}) \otimes 1} & L_{\mathcal{R}}(C(Y) \otimes \mathcal{R}) \\
\uparrow \alpha_X & & \uparrow \alpha_Y \\
N(X) \otimes \mathcal{R} & \xrightarrow{f} & N(Y) \otimes \mathcal{R}
\end{array}
\]

where \( \alpha_X : N(X) \otimes \mathcal{R} \to L_{\mathcal{R}}(C(X) \otimes \mathcal{R}) \) and \( \alpha_Y : N(Y) \otimes \mathcal{R} \to L_{\mathcal{R}}(C(Y) \otimes \mathcal{R}) \) are induced by the inclusions of Steenrod coalgebras

\[
\iota_X : N(X) \to C(X) \\
\iota_Y : N(Y) \to C(Y)
\]

respectively (compare corollary \[5.9\]). \(\square\)

Our main topological result is

**Corollary 5.9.** Let \( X \) and \( Y \) be pointed, reduced degeneracy-free simplicial sets with normalized chain-complexes \( N(X) \) and \( N(Y) \), respectively, with their functorial Steenrod diagonals. If \( \mathcal{R} \) is a ring defined by

\[
\mathcal{R} = \begin{cases} 
\mathbb{Z}_p & \text{for some prime } p \\
\mathcal{R} \subset \mathbb{Q}
\end{cases}
\]

and

\[
f : N(X) \otimes \mathcal{R} \to N(Y) \otimes \mathcal{R}
\]

is a morphism of Steenrod coalgebras, then \( f \) induces

\[
f_\infty : \mathcal{R}_\infty X \to \mathcal{R}_\infty Y
\]
of \( \mathbb{Z} \)-completions that makes the diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\phi_X \downarrow & f \downarrow & \phi_Y \\
\mathbb{R}_\infty X & \rightarrow & \mathbb{R}_\infty Y \\
q_X \downarrow & \downarrow & q_Y \\
\mathbb{R}X & \rightarrow & \mathbb{R}Y
\end{array}
\]

commute. Here

\[
\phi_X : X \rightarrow \mathbb{R}_\infty X \\
\phi_Y : Y \rightarrow \mathbb{R}_\infty Y
\]

are the canonical maps (see 4.2 in \([1, \text{chapter I}]\)) and \(q_X\) and \(q_Y\) are maps to the 0-coskeleta.

Remark. The vertical composites are just the respective Hurewicz maps.

Proof. Proposition 4.6 implies that the chain-map, \( f \) induces morphisms of simplicial abelian groups

\[
\tilde{\mathbb{R}}^{-i-1} \tilde{\Gamma} f : \tilde{\mathbb{R}}^i X \rightarrow \tilde{\mathbb{R}}^i Y
\]

for all \( i > 0 \). The fact that \( f \) preserves Steenrod diagonals and proposition 5.8 implies that the diagram

\[
\begin{array}{ccc}
N(X) \otimes \mathbb{R} & \rightarrow & N(Y) \otimes \mathbb{R} \\
N(h_X) \downarrow & f \downarrow & N(h_Y) \otimes 1 \\
N(\mathbb{R}X) \otimes \mathbb{R} & \rightarrow & N(\mathbb{R}Y) \otimes \mathbb{R} \\
N((\tilde{\Gamma}f)) \downarrow & \downarrow \\
N(\tilde{\mathbb{R}}X) \otimes \mathbb{R} & \rightarrow & N(\tilde{\mathbb{R}}X) \otimes \mathbb{R}
\end{array}
\]

commutes, where \( h_X \) and \( h_Y \) are Hurewicz maps. If we take \( \tilde{\Gamma} \) of this diagram (5.3), proposition 4.3 implies that we get a commutative diagram of simplicial abelian groups

\[
\begin{array}{ccc}
\tilde{\mathbb{R}}X & \rightarrow & \tilde{\mathbb{R}}Y \\
\tilde{\mathbb{R}}h_X \downarrow & \downarrow & \tilde{\mathbb{R}}h_Y \\
\tilde{\mathbb{R}}^2 X & \rightarrow & \tilde{\mathbb{R}}^2 Y \\
\tilde{\mathbb{R}}\tilde{f} \downarrow & \downarrow \\
\tilde{\mathbb{R}}f & \rightarrow & \tilde{\mathbb{R}}f
\end{array}
\]
Now recall the cosimplicial resolutions $R^\bullet X$ and $R^\bullet Y$ defined in example 4.1 of [3, chapter VII, section 4]. They have levels

$$(R^\bullet X)^n = \tilde{R}^{n+1} X$$

$n \geq 0$, with coface maps

$$\delta_X^i = \tilde{R}^i h \tilde{R}^{n-i+1} \tilde{X} \to \tilde{R}^{n+1} X$$
$$\delta_Y^i = \tilde{R}^i h \tilde{R}^{n-i+1} \tilde{Y} \to \tilde{R}^{n+1} Y$$

for $i = 0, \ldots, n+1$, where $h: * \to \tilde{R}*$ is the Hurewicz map of the space to its right. In addition, they have codegeneracy maps

$$s_X^i = \tilde{R}^i \gamma \tilde{R}^{m-i} \tilde{X} \to \tilde{R}^{m+1} X$$
$$s_Y^i = \tilde{R}^i \gamma \tilde{R}^{m-i} \tilde{Y} \to \tilde{R}^{m+1} Y$$

for $i = 0, \ldots, n$, where $\gamma(\alpha \cdot \beta) = \alpha \beta$ for $\alpha, \beta \in R$. If $0 \leq i < n+1$ the diagram

$\begin{array}{ccc}
\tilde{R}^{m-i+1} X & \xrightarrow{\tilde{R}^{i+1} f} & \tilde{R}^{m-i+1} Y \\
\downarrow h_X & & \downarrow h_Y \\
\tilde{R}^{m-i+2} X & \xrightarrow{\tilde{R}^{i+2} f} & \tilde{R}^{m-i+2} Y
\end{array}$

commutes by the naturality of Hurewicz maps. Composing this with $\tilde{R}^n$ shows that the maps $\tilde{R}^n \tilde{f}$ preserve cofaces $\delta_i$ for $i < n+1$. The only coface that uses the topology of $X$ and $Y$ — beyond their bare chain-complexes — is (remarkably!) $\delta_{n+1}$. Applying $\tilde{R}^n$ to diagram 5.4 implies that this is also preserved.

It follows that the maps in equation 5.2 commute with all cofaces and codegeneracies so that they define a morphism of cosimplicial spaces

$$R^\bullet f: R^\bullet X \to R^\bullet Y$$

that induces a morphism $f_\infty$ of total spaces that makes diagram 5.1 commute. $\square$

**Proposition 5.10.** Under the hypotheses of corollary 5.9, if $f$ is surjective, $R_\infty f$ is a fibration. If $f$ is a surjective homology equivalence, then $R_\infty f$ is a trivial fibration.

**Proof.** All of the coface maps except for the 0th in $Z^\bullet X$ are morphisms of simplicial abelian groups. It follows that $R^\bullet X$ is “group-like” in the sense of section 4 in chapter X of [1]. The
conclusion follows from proposition 4.9 section 4 in chapter X of [1].

If \( f \) is also a homology equivalence, then \( R^\bullet f: R^\bullet X \to R^\bullet Y \) is a pointwise trivial fibration. The final statement follows from theorem 4.13 in chapter VIII of [1], and the fact that \( \Delta^\bullet \) is cofibrant in coS.

□

If \( R = \mathbb{Z} \) and spaces are nilpotent, we can say a bit more:

**Corollary 5.11.** Under the hypotheses of corollary 5.9, if \( Y \) is also nilpotent and a Kan complex then \( \phi_Y: Y \to Z_\infty Y \) is a weak equivalence with a homotopy-inverse, \( \phi': Z_\infty Y \to Y \), that fits into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_X} & Z_\infty X \\
\downarrow \phi_X & & \downarrow Z_\infty f \\
Y & \xrightarrow{\phi_Y} & Z_\infty Y
\end{array}
\]

where

1. \( \phi_Y: Y \to Z_\infty X \) is a weak equivalence
2. a morphism of cellular coalgebras, \( f: C(X) \to C(Y) \), induces a map of simplicial sets

\[
X \xrightarrow{\phi_X} Z_\infty X \xrightarrow{Z_\infty f} ZY \xrightarrow{\phi'} Y
\]

If \( Y \) is not a Kan complex, then the diagram that results from applying the topological realization functor, \(| \ast |\), to all terms of diagram 5.5 commutes.

**Remark.** For instance, singular simplicial sets are always Kan complexes.

**Proof.** The main statement (that \( \phi_Y \) is a weak equivalence) follows from proposition 3.5 in chapter V of [1]. □

Our final result is:

**Corollary 5.12.** If \( X \) and \( Y \) are pointed, reduced, nilpotent degeneracy-free simplicial sets that are Kan complexes, with normalized chain-complexes \( N(X) \) and \( N(Y) \), respectively, then \( X \) and \( Y \) are homotopy equivalent if and only if there exists a morphism of Steenrod coalgebras

\[
f: N(X) \to N(Y)
\]
Proof. Any homotopy equivalence \( g: X \to Y \) induces a map like \( f \) in the statement. Conversely, given \( f \) as above, we get

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\phi_X & \downarrow & \phi_Y \\
\mathbb{Z}_\infty X & \xrightarrow{f_\infty} & \mathbb{Z}_\infty Y \\
q_X & \downarrow & q_Y \\
\tilde{\mathbb{Z}}X & \xrightarrow{\gamma f} & \tilde{\mathbb{Z}}Y \\
\end{array}
\]

where \( \phi_X \) and \( \phi_Y \) are weak equivalences and the map at the bottom is a weak equivalence. Since \( \phi_X \) and \( \phi_Y \) are weak equivalences, it follows that the maps \( q_X \) and \( q_Y \) are homotopic to the Hurewicz maps of \( \mathbb{Z}_\infty X \) and \( \mathbb{Z}_\infty Y \), respectively. Since \( f \) is a weak equivalence, it follows that \( f_\infty \) induces isomorphisms in homology (recall that \( \pi_n(\tilde{\mathbb{Z}}X) \cong H_n(X) = H_n(\mathbb{Z}_\infty X) \) for all \( n \geq 0 \), and that a corresponding statement holds for \( Y \)). Whitehead’s theorem implies the existence of a homotopy inverse for \( \phi_Y \) and

\[
\phi_Y^{-1} \circ f_\infty \circ \phi_X: X \to Y
\]
is a homotopy equivalence. \( \square \)

Since arbitrary simplicial sets are homotopy equivalent to degeneracy-free ones, we also get

**Corollary 5.13.** If \( X \) and \( Y \) are pointed reduced simplicial sets, \( R \) is a ring defined by

\[
R = \begin{cases} 
\mathbb{Z}_p & \text{for some prime } p \text{ or } \\
R \subset \mathbb{Q} & 
\end{cases}
\]

then any morphism of Steenrod coalgebras (over unnormalized chain-complexes)

\[
f: C(X) \otimes R \to C(Y) \otimes R
\]
induces a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\partial \circ f(X)} & \tilde{R}(\partial \circ f(X)) \\
g_X \downarrow & & \downarrow \tilde{f}_f \\
\partial \circ f(Y) & \xrightarrow{\phi(\partial \circ f(Y))} & \tilde{R}(\partial \circ f(Y)) \\
g_Y \downarrow & & \downarrow \tilde{g} \\
\end{array}
\]

where:

1. the functors $f$ and $\partial$ are defined in definition 2.2
2. the maps $g_X$ and $g_Y$ are defined in equation 2.3 and are homotopy equivalences, by proposition 2.6.

In particular, if $X$ and $Y$ are nilpotent, $\mathcal{R} = \mathbb{Z}$, and $f$ is a homotopy equivalence, then the topological realizations, $|X|$ and $|Y|$, are homotopy equivalent.

Proof. This follows immediately from corollary 5.9 and proposition 2.3 which implies that $N(\partial \circ f(X)) = C(X)$. □

APPENDIX A. Functorial Steenrod Diagonals

In this section, we construct a functorial Steenrod coalgebra structure described in proposition 5.2 — the basic construction first appeared in [21]. Also see [7] for an alternative functorial form of Steenrod coalgebra.

We begin with a contracting cochain on the normalized chain-complex of a standard simplex:

**Definition A.1.** Let $\Delta^k$ be a standard $k$-simplex with vertices $\{[0], \ldots, [k]\}$ and $j$-faces $\{[i_0, \ldots, i_j]\}$ with $i_0 < \cdots < i_j$ and let $s^k$ denote its normalized chain-complex with boundary map $\partial$. This is equipped with an augmentation

\[ \epsilon : s^k \to \mathbb{Z} \]

that maps all vertices to $1 \in \mathbb{Z}$ and all other simplices to $0$. Let

\[ \iota_k : \mathbb{Z} \to s^k \]
denote the map sending $1 \in \mathbb{Z}$ to the image of the vertex $[k]$. Then we have a contracting cochain

\[ \varphi_k([i_0, \ldots, i_t]) = \begin{cases} (-1)^{t+1} [i_0, \ldots, i_t, k] & \text{if } i_t \neq k \\ 0 & \text{if } i_t = k \end{cases} \]

and $1 - \iota_k \circ \epsilon = \partial \circ \varphi_k + \varphi_k \circ \partial$.

**Theorem A.2.** The unnormalized chain-complex, $U^k$, of $[i_0, \ldots, i_k] = \Delta^k$ has a Steenrod coalgebra structure

\[ \xi: R S_2 \otimes U^k \to U^k \otimes U^k \]

that is natural with respect to order-preserving mappings of vertex-sets $[i_0, \ldots, i_k] \to [j_0, \ldots, j_t]$ with $j_0 \leq \cdots \leq j_t$ and $t \geq k$. If $c \in U^k$ is degenerate then one of the two factors in each term of $h(* \otimes c)$ is degenerate so that $h$ induces a well-defined Steenrod coalgebra structure on the normalized chain-complex of $\Delta^k$, denoted $N^k$.

**Remark.** The author has a Common LISP program for computing $\xi(x \otimes C(\Delta^k))$ — the number of terms is exponential in the dimension of $x$.

**Proof.** If $C = s^k = C(\Delta^k)$ — the normalized chain complex — we can define a corresponding contracting homotopy on $C \otimes C$ via

\[ \Phi = 1 \otimes \varphi_k + \varphi_k \otimes \iota_k \circ \epsilon \]

where $\varphi_k$, $\iota_k$, and $\epsilon$ are as in definition A.1. Above dimension 0, $\Phi$ is effectively equal to $1 \otimes \varphi_k$. Now set $M_2 = C \otimes C$ and $N_2 = \text{im}(\Phi)$. In dimension 0, we define $f_2$ for all $n$ via:

\[ \xi(A \otimes [0]) = \begin{cases} [0] \otimes [0] & \text{if } A = [] \\ 0 & \text{if dim } A > 0 \end{cases} \]

This clearly makes $s^0$ a Steenrod coalgebra.

Suppose that the $\xi$ are defined below dimension $k$. Then the Steenrod coalgebra structure of $C(\partial \Delta^k)$ is well-defined and satisfies the conclusions of this theorem. We define $\xi(a[a_1| \ldots |a_j] \otimes [0, \ldots, k])$ by induction on $j$,

\[ \xi(A \otimes s^k) = \Phi \circ \xi(\partial A \otimes s^k) + (-1)^{\text{dim } A} \Phi \circ \xi(A \otimes s^k) \]

where $A \in A(\mathbb{Z}_2, 1) \subset RS_n$ and the term $\xi(A \otimes s^k)$ refers to the Steenrod coalgebra structure of $C(\partial \Delta^k)$. 

29
The terms \( \xi(A \otimes \partial s^k) \) and \( \xi(\partial A \otimes s^k) \) are defined by induction on the dimension of \( A \) and we ultimately get an expression for \( \xi(x \otimes [0, \ldots, k]) \) as a sum of tensor-products of sub-simplices of \([0, \ldots, k] \) — given as ordered lists of vertices.

We claim that this Steenrod coalgebra structure is natural with respect to ordered mappings of vertices. This follows from the fact that the only significant property that the vertex \( k \) has in equations A.1 and A.2 is that it is the highest numbered vertex.

The final statement (regarding degenerate simplices) follows from three facts:

1. It is true for the Alexander-Whitney coproduct (the starting point of our induction),
2. The boundary of a degenerate simplex is a linear combination of degenerate simplices (nondegenerate faces cancel out), and
3. \( \Phi \) of a term with a degenerate factor has a degenerate factor.

Here is an example of some higher coproducts:

We conclude this section some computations of higher coproducts:

**Example A.3.** If \([0, 1, 2] = \Delta^2\) is a 2-simplex, then

\[
(\text{A.3}) \quad \xi(\, \otimes \Delta^2) = \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2
\]

— the standard (Alexander-Whitney) coproduct — and

\[
\xi\left(\ ((1, 2) \otimes \Delta^2 \right) = -[1, 2] \otimes [0, 1, 2] + [0, 1, 2] \otimes [0, 2] \\
- [0, 1] \otimes [0, 1, 2]
\]

or, in face-operations

\[
(\text{A.4}) \quad \xi\left(\ ((1, 2) \otimes \Delta^2 \right) = -F_0 \Delta^2 \otimes \Delta^2 + \Delta^2 \otimes F_1 \Delta^2 \\
- F_2 \Delta^2 \otimes \Delta^2
\]

*Proof.* If we write \( \Delta^2 = [0, 1, 2] \), we get

\[
\xi(\, \otimes \Delta^2) = [0, 1, 2] \otimes [2] + [0, 1] \otimes [1, 2] + [0] \otimes [0, 1, 2]
\]
To compute $\xi([1,2] \otimes \Delta^2)$ we have a version of equation A.2:

$$
\xi(e_1 \otimes \Delta^2) = \Phi_2(\xi(\partial e_1 \otimes \Delta^2) - \Phi_2(\xi(e_1 \otimes \partial \Delta^2)
$$

Now

$$
\Phi_2(\xi(\partial e_1 \otimes \Delta^2) = (1 \otimes \varphi_2)([2] \otimes [0, 1, 2] - [1, 2] \otimes [0, 1] + [0, 1, 2] \otimes [0])
$$

where the + sign on the term $[1, 2] \otimes [0, 1, 2]$ is due to the Koszul convention and definition. We also get

$$
\Phi_2(\xi(\partial e_1 \otimes \Delta^2) = (1 \otimes \varphi_2)([0, 1, 2] \otimes [2] + [0, 1, 2] \otimes [1, 2] + [0] \otimes [0, 1, 2]) = 0
$$

In addition, proposition A.5 implies that

$$
\xi(e_1 \otimes \partial \Delta^2) = -[1, 2] \otimes [1, 2] + [0, 2] \otimes [0, 2] - [0, 1] \otimes [0, 1]
$$

so that

$$
\Phi_2(\xi(e_1 \otimes \partial \Delta^2) = -[0, 1] \otimes [0, 1, 2]
$$

We conclude that

$$
\xi([1,2] \otimes \Delta^2) = -[1, 2] \otimes [0, 1, 2] - [0, 1, 2] \otimes [0, 2] + [0, 1] \otimes [0, 1, 2]
$$

which implies equation A.4.

We can extend the Steenrod coalgebra structure on simplices to one on degenerate simplices by regarding

$$
D_i \Delta^n = D_i[0,\ldots,n] = [0,\ldots,i,\ldots,n]
$$

and plugging these vertices into the formulas for the higher coproducts. For instance, example A.3 implies that

$$
\xi(([1,2] \otimes D_0[0,1]) = [0, 0, 1] \otimes [0, 1] - [0, 1, 2] \otimes [0, 0, 1]
$$
or
\[ \xi([(1, 2)] \otimes D_0 \Delta^1) = D_0 \Delta^1 \otimes \Delta^1 - \Delta^1 \otimes D_0 \Delta^1 \]
\[ - D_0 \Delta^1 \otimes D_0 F_1 \Delta^1 \]
and
\[ \xi([(1, 2)] \otimes D_1 [0, 1]) = [0, 1, 1] \otimes [0, 1] - [0, 1] \otimes [0, 1, 1] \]
\[ - [0, 1, 1] \otimes [1, 1] \]

or
\[ \xi([(1, 2)] \otimes D_1 \Delta^1) = D_1 \Delta^1 \otimes D_0 F_0 \Delta^1 - \Delta^1 \otimes D_1 \Delta^1 \]
\[ - D_1 \Delta^1 \otimes D_0 F_0 \Delta^1 \]

**Remark.** As in [7], the number of terms in \( \xi(e \otimes \Delta^n) \) grows exponentially with \( n \) and the dimension of \( e \in RS_2 \).

It follows that:

**Proposition A.4.** If \( X \) is a simplicial set, we can define a natural Steenrod coalgebra structure on the unnormalized chain-complex of \( X \):

\[ C(X) = \lim_{\to} N_k = \lim_{\to} C(\Delta^k) \]

for \( \Delta^n \in \Delta \downarrow X — the simplex category of X with Steenrod diagonal \)

\[ \xi: C(X) \to C(X) \otimes C(X) \]

This induces a natural Steenrod coalgebra structure on the normalized chain-complex

\[ \xi: N(X) \to N(X) \otimes N(X) \]

**Proof.** Theorem B.3 of [23] implies that this colimit of Steenrod coalgebras (i.e., coalgebras over \( \mathcal{F} \)) has a chain-complex that is the chain-complex of the colimit of chain-complexes — i.e. the unnormalized chain complex of \( X \).

The second statement follows from the fact that \( \xi \) of a degenerate simplex has a degenerate factor in \( C(X) \otimes C(X) \). \( \square \)

We conclude this section with a calculation that is crucial to this paper:

**Proposition A.5.** Let \( X \) be a simplicial set with \( C = N(X) \) and with coalgebra structure

\[ \xi: RS_2 \otimes N(X) \to N(X) \otimes N(X) \]
and suppose $RS_2$ is generated in dimension $n$ by $e_n = [(1,2)] \cdots [(1,2)]$. If $x \in C$ is the image of a $k$-simplex, then

\[ \xi(e_k \otimes x) = \eta_k \cdot x \otimes x \]

where $\eta_k = (-1)^{k(k+1)/2}$.

**Remark.** This is just a chain-level statement that the Steenrod operation $Sq^0$ acts trivially on mod-2 cohomology. A weaker form of this result appeared in [4].

It proves that Steenrod coalgebras of the form $C(X)$, for a simplicial set $X$ are not nilpotent: iterated coproducts of simplices never “peter out”. Although there are many natural ways to define the contracting homotopy, $\varphi_k$, and they give different versions of $\xi$, they all produce a result of the form

\[ \xi(e_k \otimes x) = \pm x \otimes x \]

when $x$ is a simplex since $\xi(e_i \otimes x)$ is a linear combination of tensor-products of sub-simplices of $x$. The conclusions of this paper are, therefore, valid for all of them.

**Proof.** Recall that $(RS_2)_n = \mathbb{Z}[\mathbb{Z}_2]$ generated by $e_n = [(1,2)] \cdots [(1,2)]$. Let $T$ be the generator of $\mathbb{Z}_2$ — acting on $C \otimes C$ by swapping the copies of $C$.

Since the normalized chain-complex, $N(\Delta^k)$, has the property that $N(\Delta^k)_j = 0$ for $j > k$

(A.5) \[ j > k \implies \xi(e_j \otimes N(\Delta^k)) = 0 \]

As in section 4 of [21], if $e_0 = [] \in RS_2$ is the 0-dimensional generator, we define

\[ \xi : RS_2 \otimes C \rightarrow C \otimes C \]

inductively by

(A.6) \[ \xi(e_0 \otimes [i]) = [i] \otimes [i] \]

\[ \xi(e_0 \otimes [0, \ldots, , k]) = \sum_{i=0}^{k} [0, \ldots, , i] \otimes [i, \ldots, , k] \]

Let $\sigma = \Delta^k$ and inductively define

(A.7) \[ \xi(e_k \otimes \sigma) = \Phi_k(\xi(\partial e_k \otimes \sigma)) + (-1)^k \Phi_k \xi(e_k \otimes \partial \sigma) \]

because of equation [A.5].
Expanding $\Phi_k$, we get
\[
\xi(e_k \otimes \sigma) = (1 \otimes \varphi_k)(\xi(\partial e_k \otimes \sigma)) + (\varphi_k \otimes \iota_k \circ \epsilon)\xi(\partial e_k \otimes \sigma)
\]
\[\text{(A.8)}\]
\[= (1 \otimes \varphi_k)(\xi(\partial e_k \otimes \sigma))\]

because $\varphi_k^2 = 0$ and $\varphi_k \circ \iota_k \circ \epsilon = 0$.

Noting that $\partial e_k = (1 + (-1)^k T)e_{k-1} \in RS_2$, we get
\[
\xi(e_k \otimes \sigma) = (1 \otimes \varphi_k)(\xi(e_{k-1} \otimes \sigma)) + (-1)^k(1 \otimes \varphi_k) \cdot T \cdot \xi(e_{k-1} \otimes \sigma)
\]
\[= (-1)^k(1 \otimes \varphi_k) \cdot T \cdot \xi(e_{k-1} \otimes \sigma)\]

again, because $\varphi_k^2 = 0$ and $\varphi_k \circ \iota_k \circ \epsilon = 0$. We continue, using equation [A.8] to compute $\xi(e_{k-1} \otimes \sigma)$:
\[
\xi(e_k \otimes \sigma) =(-1)^k(1 \otimes \varphi_k)) \cdot T \cdot \xi(e_{k-1} \otimes \sigma)
\]
\[= (-1)^k(1 \otimes \varphi_k) \cdot T \cdot (1 \otimes \varphi_k) \left(\xi(\partial e_{k-1} \otimes \sigma)\right)
\]
\[+ (-1)^{k-1}\xi(e_{k-1} \otimes \varphi e_k \times \partial \sigma)
\]
\[= (-1)^{k+1}\varphi_k \otimes \varphi_k \cdot T \cdot \left(\xi(\partial e_{k-1} \otimes \sigma)\right)
\]
\[+ (-1)^{k-1}\xi(e_{k-1} \otimes \varphi \partial \sigma)\]

— where the factor of $(-1)^{k+1}$ is the result of applying the Koszul Convention — $\varphi_k \otimes \varphi_k = \varphi_k \otimes \varphi_k$.

If $k-1 = 0$, then the left term vanishes. If $k-1 = 1$ so $\partial e_{k-1}$ is 0-dimensional then equation [A.8] gives $\xi(\varphi_k \otimes \varphi_k \circ \partial)$ and this vanishes when plugged into $\varphi_k \otimes \varphi_k$. If $k-1 > 1$, then $\xi(\partial e_{k-1} \otimes \varphi) = \varphi_k$ so it vanishes when plugged into $\varphi_k \otimes \varphi_k$.

In all cases, we can write
\[
\xi(e_k \otimes \sigma) = (-1)^{k+1}\varphi_k \otimes \varphi_k \cdot T \cdot (-1)^{k-1}\xi(e_{k-1} \otimes \varphi e_k \times \partial \sigma)
\]
\[= \varphi_k \otimes \varphi_k \cdot T \cdot \xi(e_{k-1} \otimes \varphi e_k \times \partial \sigma)
\]

If $\xi(e_{k-1} \otimes \Delta^{k-1}) = \eta_{k-1} \Delta^{k-1} \otimes \Delta^{k-1}$ (the inductive hypothesis), then
\[
\xi(e_{k-1} \otimes \varphi \partial \sigma) = \\
\sum_{i=0}^{k} \eta_{k-1} \cdot (-1)^i [0, \ldots, i - 1, i + 1, \ldots k] \otimes [0, \ldots, i - 1, i + 1, \ldots k]
\]
and the only term that does not get annihilated by $\varphi_k \otimes \varphi_k$ is

$(-1)^k[0, \ldots, k-1] \otimes [0, \ldots, k-1]$  

(see equation A.1). We get

$$\xi(e_k \otimes \sigma) = \eta_{k-1} \cdot \varphi_k \otimes \varphi_k \otimes (-1)^{k-1} \otimes [0, \ldots, k-1] \otimes [0, \ldots, k-1]$$

where the sign-changes are due to the Koszul Convention. We conclude that $\eta_k = (-1)^k \eta_{k-1}$. □

**Appendix B. Proof of Theorem 5.7**

If $R$ is a ring defined by

$$R = \begin{cases} \mathbb{Z}_p & \text{for some prime } p \text{ or } \\ \mathcal{R} \subset \mathbb{Q} & \end{cases}$$

let $F$ denote its field of fractions (either $\mathbb{Z}_p$ or $\mathbb{Q}$).

We begin with a general result:

**Lemma B.1.** Let $C$ be a free abelian group, let

$$\hat{C} = \mathbb{Z} \oplus \prod_{i=1}^{\infty} C^{\otimes i}$$

Let $e: C \to \hat{C}$ be the function that sends $c \in C$ to

$$(1, c, c \otimes c, c \otimes c \otimes c, \ldots) \in \hat{C}$$

For any integer $t > 1$ and any set $\{c_1, \ldots, c_t\} \in C$ of distinct, nonzero elements, the elements

$$\{e(c_1), \ldots, e(c_t)\} \in F \otimes_{\mathbb{Z}} \hat{C}$$

are linearly independent over $F$. It follows that $e$ defines an injective function

$$\bar{e}: R[C] \to \hat{C}$$

**Proof.** We will construct a vector-space morphism

$$f: \mathbb{Q} \otimes_{\mathbb{Z}} \hat{C} \to V$$
such that the images, \( \{ f(e(c_i)) \} \), are linearly independent. We begin with the “truncation morphism”

\[
r_t: \hat{C} \to \mathbb{Z} \oplus \bigoplus_{i=1}^{t-1} C^\otimes i = \hat{C}_{t-1}
\]

which maps \( C^\otimes 1 \) isomorphically. If \( \{ b_i \} \) is a \( \mathbb{Z} \)-basis for \( C \), we define a vector-space morphism

\[
g: \hat{C}_{t-1} \otimes \mathbb{Z} F \to F[X_1, X_2, \ldots]
\]

by setting \( g(b_i) = X_i \) and extending \( F \)-linearly to \( \hat{C}_1 \otimes \mathbb{Z} F \). We extend this (as a morphism of \( F \)-modules only) via

\[
g(c_1 \otimes \cdots \otimes c_j) = g(c_1) \cdots g(c_j) \in F[X_1, X_2, \ldots]
\]

The map in equation B.1 is just the composite

\[
\hat{C} \otimes \mathbb{Z} F \xrightarrow{r_{t-1} \otimes 1} \hat{C}_{t-1} \otimes \mathbb{Z} F \xrightarrow{g} F[X_1, X_2, \ldots]
\]

It is not hard to see that

\[
p_i = f(e(c_i)) = 1 + f(c_i) + \cdots + f(c_i)^{t-1} \in F[X_1, X_2, \ldots]
\]

for \( i = 1, \ldots, t \). Since the \( f(c_i) \) are linear in the indeterminates \( X_i \), the degree-\( j \) component (in the indeterminates) of \( f(e(c_i)) \) is precisely \( f(c_i)^j \). It follows that a linear dependence-relation

\[
\sum_{i=1}^{t} \alpha_i \cdot p_i = 0
\]

with \( \alpha_i \in \mathbb{Q} \), holds if and only if

\[
\sum_{i=1}^{t} \alpha_i \cdot f(c_i)^j = 0
\]

for all \( j = 0, \ldots, t-1 \). This is equivalent to \( \det M = 0 \), where

\[
M = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
f(c_1) & f(c_2) & \cdots & f(c_t) \\
\vdots & \vdots & \ddots & \vdots \\
f(c_1)^{t-1} & f(c_2)^{t-1} & \cdots & f(c_t)^{t-1}
\end{bmatrix}
\]

Since \( M \) is the transpose of the Vandermonde matrix, we get

\[
det M = \prod_{1 \leq i < j \leq t} (f(c_i) - f(c_j))
\]
Since \( f|C \otimes \mathbb{F} \subset \hat{C} \otimes \mathbb{F} \) is injective, it follows that this only vanishes if there exist \( i \) and \( j \) with \( i \neq j \) and \( c_i = c_j \). The second conclusion follows. \( \square \)

This leads to the proof:

**Corollary B.2.** If \( X \) is a pointed, reduced degeneracy-free simplicial set, then then the morphism of Steenrod coalgebras

\[
F_X: N(\tilde{R}X) \otimes \mathcal{R} \to L_\mathcal{F}(C(X) \otimes \mathcal{R})
\]

in definition 5.4 is injective.

**Proof.** If \( E = N(\tilde{R}X) \otimes \mathcal{R} \) and \( C = C(X) \otimes \mathcal{R} \), the results of [22] imply that

\[
L_\mathcal{F}E \subset E \oplus \bigoplus_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{F}(n), E^\otimes n)
\]

and the map \( \gamma_X \) in definition 5.4 induces a commutative diagram

(B.2)

\[
\begin{array}{ccc}
C(\tilde{R}X) & \overset{\alpha}{\longrightarrow} & L_\mathcal{F}E \\
\downarrow F_X & & \downarrow L_\mathcal{F}\gamma_X \\
L_\mathcal{F}C & \overset{\prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(1, \gamma_X^{\otimes n})}{\longrightarrow} & \bigoplus_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{F}(n), C^\otimes n)
\end{array}
\]

where we follow the convention that \( \text{Hom}_{\mathbb{Z}S_0}(\mathcal{F}(0), E^0) = \mathcal{R} \), \( \text{Hom}_{\mathbb{Z}S_n}(\mathcal{F}(1), E) = E \). Here

1. \( \alpha \) is induced by the identity map of \( E \) (regarded as a chain-complex),
2. \( \gamma_X \) and \( F_X \) are defined in definition 5.4.

Let \( p_n \) be projection to a factor

\[
p_n: \bigoplus_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{F}(n), E^\otimes n) \to \text{Hom}_{\mathbb{Z}S_n}(\mathcal{F}(n), E^\otimes n)
\]

If \( \sigma \in \mathcal{F} \) is an \( m \)-simplex defining an element \( [\sigma] \in E_m \), proposition A.5 implies that

\[
p_2 \circ \alpha([\sigma]) = \eta_m \cdot (e_m \mapsto [\sigma] \otimes [\sigma]) \in \text{Hom}_{\mathbb{Z}S_0}(R_2, E \otimes E)
\]

where \( \eta_m = (-1)^{m(m-1)/2} \) (see proposition A.5) and \( \mathcal{F}(2) = R_2 \).

Let \( Z_2 = e_m \) and \( Z_k = e_m \circ_1 \cdots \circ_1 e_m \in \mathcal{F}(k) \) be the operad-composite (see definition 2.12 and proposition 2.17 of [21]).
The fact that operad-composites map to composites of coproducts (see proposition [2.18]) in a coalgebra implies that 

\[ p_k \circ \alpha([\sigma]) = \eta_m^{k-1} \cdot (Z_k \mapsto [\sigma] \otimes \cdots \otimes [\sigma]) \in \text{Hom}_{ZS_k}(F(k), E^{\otimes k}) \]

If \( \{\sigma_1, \ldots, \sigma_t\} \in \tilde{\mathcal{R}}X \) are distinct \( m \)-simplices then \( \{\gamma_X[\sigma_1], \ldots, \gamma_X[\sigma_t]\} \in C = C(X) \otimes \mathcal{R} \) are also distinct (although no longer generators).

Their images in \( \prod_{n=0}^\infty \text{Hom}_{ZS_n}(F(n), C^{\otimes n}) \) will have the property that 

\[ p_k \circ F_X([\sigma_i]) = \eta_m^{k-1} \cdot (Z_k \mapsto \gamma_X[\sigma_i] \otimes \cdots \otimes \gamma_X[\sigma_i]) \]

\[ \otimes k \text{ factors} \]

\[ \in \text{Hom}_{ZS_k}(F(k), C(X)^{\otimes k}) \]

Evaluation of elements of \( \prod_{n=1}^\infty \text{Hom}_{ZS_n}(F(n), C^{\otimes n}) \) on the sequence \( (\eta_m \cdot Z_2, \eta_m \cdot Z_3, \eta_m \cdot Z_4, \ldots) \) gives a homomorphism of \( \mathbb{Z} \)-modules 

\[ j : \prod_{n=0}^\infty \text{Hom}_{ZS_n}(F(n), C^{\otimes n}) \to \prod_{n=0}^\infty C^{\otimes n} \]

and \( j \circ \gamma_X(\sigma_i) \) is \( e(\gamma_X[\sigma_i]) \), as defined in lemma [B.1]. The conclusion follows from lemma [B.1]. \( \square \)

**References**

1. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, New York, 1972.
2. M. Barratt and P. Eccles, *On \( \Gamma_+ \)-structures. I. A free group functor for stable homotopy theory*, Topology (1974), 25–45.
3. Clemens Berger and Ieke Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. 78 (2003), no. 4, 681–721.
4. James F. Davis, *Higher diagonal approximations and skeletons of \( K(\pi, 1) \)'s*, Lecture Notes in Mathematics, vol. 1126, Springer-Verlag, 1983, pp. 51–61.
5. G. Gerstenhaber, *The cohomology structure of an associative ring*, Annals of Math. 78 (1962), no. 2, 268–288.
6. Paul G. Goerss and John F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, vol. 174, Birkhäuser, Boston, 1999.
7. Rocío González-Díaz and Pedro Real, *A combinatorial method for computing Steenrod squares*, Journal of Pure and Applied Algebra 139 (1999), 89–108.
8. V. K. A. M. Gugenheim, *On a theorem of E. H. Brown*, Illinois J. of Math. 4 (1960), 292–311.
9. Daniel M. Kan, *Functors involving C.S.S. complexes*, Trans. Amer. Math. Soc. **87** (1958), 330–346.

10. I. Kriz and J. P. May, *Operads, algebras, modules and motives*, Astérisque, vol. 233, Société Mathématique de France, 1995.

11. Jean-Louis Loday and Bruno Vallette, *Algebraic Operads*, Available online http://math.unice.fr/~brunov/Operads.pdf.

12. S. MacLane, *Homology*, Springer-Verlag, 1995.

13. M. Mandell, *$E_{\infty}$ algebras and $p$-adic homotopy theory*, Topology **40** (2001), no. 1, 43–94.

14. Michael Mandell, *Cochains and homotopy type*, Publications Mathématiques de l’Institut des Hautes Études Scientifiques **103** (2007), no. 1, 213–246.

15. D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295.

16. C. P. Rourke and B. J. Sanderson, *$\Delta$-sets. I. Homotopy theory*, Quart. J. Math. Oxford **22** (1971), 321–338.

17. V. A. Smirnov, *Homotopy theory of coalgebras*, Izv. Mat. Nauk SSR Ser. Mat. **49** (1985), no. 6, 575–592.

18. Justin R. Smith, *Cellular coalgebras over the operad $S_I$*, arXiv:1304.6328 [math.AT].

19. _______, *m-structures determine integral homotopy type*, arXiv designation math.AT/9809151.

20. _______, *$S$-coalgebras determine fundamental groups*, arXiv:1307.7098 [math.AT].

21. _______, *Iterating the cobar construction*, vol. 109, Memoirs of the A. M. S., no. 524, American Mathematical Society, Providence, Rhode Island, May 1994.

22. _______, *Cofree coalgebras over operads*, Topology and its Applications **133** (2003), 105–138.

23. _______, *Model-categories of coalgebras over operads*, Theory and Applications of Categories **25** (2011), 189–246.

24. Dennis Sullivan, *Infinitesimal calculations in topology*, Publications Mathématiques d’Institut des Hautes Études Scientifiques, vol. 47, 1977, pp. 269–331.

**DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA 19104**

_E-mail address:_ jsmith@drexel.edu

**URL:** http://vorpal.math.drexel.edu