UNIFORM EXPONENTIAL MIXING FOR CONGRUENCE
COVERS OF CONVEX COCOMPACT HYPERBOLIC
MANIFOLDS

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Abstract. Let $\Gamma$ be a Zariski dense convex cocompact subgroup contained
in an arithmetic lattice of $\text{SO}(n,1)^\circ$. We prove uniform exponential mixing
of the geodesic flow for congruence covers of the hyperbolic manifold $\Gamma\setminus\mathbb{H}^n$
avoiding finitely many prime ideals. This extends the work of Oh-Winter
who proved the result for the $n = 2$ case. Following their approach, we use
Dolgopyat’s method for the proof of exponential mixing of the geodesic flow.
We do this uniformly over congruence covers by establishing uniform spectral
bounds for the congruence transfer operators associated to the geodesic flow.
This requires another key ingredient which is the expander machinery due to
Bourgain-Gamburd-Sarnak extended by Golsefidy-Varjú.

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1. Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space for $n \geq 2$. Let $K \subset \mathbb{R}$
be a totally real number field, $\mathcal{O}_K$ be the corresponding ring of integers, and $Q$
be a quadratic form of signature $(n,1)$ defined over $K$. Let $G = \text{SO}_Q(\mathbb{C}) < \text{GL}_{n+1}(\mathbb{C})$
be an algebraic group defined over $K$ such that $G(\mathbb{R}) \cong \text{SO}(n,1)$ and $G^\sigma(\mathbb{R}) \cong \text{SO}(n+1)$,
which is compact, for all nontrivial embeddings $\sigma : K \hookrightarrow \mathbb{R}$. Let $G = G(\mathbb{R})^\circ$
which we recognize as the group of orientation preserving isometries of $\mathbb{H}^n$. We identify
$\mathbb{H}^n$ and its unit tangent bundle $T^1(\mathbb{H}^n)$ with $G/K$ and $G/M$ respectively where
$M \subset K$ are compact subgroups of $G$. Let $A = \{a_t : t \in \mathbb{R}\} < G$ be a one-
parameter subgroup of semisimple elements such that its right translation action
on $G/M$ corresponds to the geodesic flow.

Let $\Gamma < G$ be a Zariski dense convex cocompact subgroup. By the work of Stoy-
anov [Sto11], exponential mixing of the geodesic flow on the hyperbolic manifold
$T^1(\Gamma\setminus\mathbb{H}^n) = \Gamma\setminus G/M$ is known for the Bowen-Margulis-Sullivan measure. In this

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paper, when $\Gamma$ is defined arithmetically and satisfies the strong approximation property, we establish the uniform exponential mixing for all congruence covers of $\Gamma\backslash \mathbb{H}^n$ corresponding to ideals $q \subseteq \mathcal{O}_K$ avoiding finitely many prime ideals, extending the work of Oh-Winter [OW16] for $n = 2$ to arbitrary dimensions $n \geq 2$.

Let $\tilde{\pi} : \hat{G} \to G$ be a simply connected cover defined over $\mathbb{K}$. For all ideals $q \subseteq \mathcal{O}_K$, let $\pi_q : G(\mathcal{O}_K) \to G(\mathcal{O}_K/q)$ be the canonical quotient map. Let $\Gamma < G(\mathbb{K})$ be a Zariski dense torsion-free convex cocompact subgroup such that $\tilde{\pi}^{-1}(\Gamma)$ is contained in $G(\mathcal{O}_K)$ and $\{\text{tr}(\text{Ad}(g)) : g \in \tilde{\pi}^{-1}(\Gamma)\}$ generates the ring $\mathcal{O}_K$. We impose these conditions so that $\Gamma$ satisfies the strong approximation property. For all nontrivial ideals $q \subseteq \mathcal{O}_K$, let $\Gamma_q < \Gamma$ be a congruence subgroup of level $q$, meaning that $\tilde{\pi}^{-1}(\Gamma_q) = \tilde{\pi}^{-1}(\Gamma) \cap \langle \ker(\pi_q), \{e, -e\} \rangle$. For all nontrivial ideals $q \subseteq \mathcal{O}_K$, let $N_K(q)$ be the ideal norm and $m_q^{\text{BMS}}$ be the Bowen-Margulis-Sullivan measure on $\Gamma_q \backslash G$ induced from the one on $\Gamma \backslash G$.

**Theorem 1.1.** There exist $\eta > 0$, $C > 0$ and a nontrivial proper ideal $q_0 \subseteq \mathcal{O}_K$ such that for all square free ideals $q \subseteq \mathcal{O}_K$ coprime to $q_0$, and for all $M$-invariant functions $\phi, \psi \in C^1(\Gamma_q \backslash G, \mathbb{R})$, we have

$$\left| \int_{\Gamma_q \backslash G} \phi(xa_t)\psi(x) \, dm_q^{\text{BMS}}(x) - \frac{1}{m_q^{\text{BMS}}(\Gamma_q \backslash G)} m_q^{\text{BMS}}(\phi) \cdot m_q^{\text{BMS}}(\psi) \right| \leq C N_K(q)^C \epsilon^{-\eta t} \|\phi\|_{C^1} \|\psi\|_{C^1}. $$

**Remark.**

1. Theorem 1.1 can be used to show the existence of a uniform resonance free strip for the resolvent of the Laplacian as well as a uniform zero-free strip of the Selberg zeta functions on the family of hyperbolic manifolds $\Gamma_q \backslash \mathbb{H}^n$ for nontrivial ideals $q \subseteq \mathcal{O}_K$ (cf. [OW16]).

2. When the critical exponent of $\Gamma$ satisfies $\delta_\Gamma > \frac{n-1}{2}$, Theorem 1.1 has been established by Mohammadi-Oh in [MO15].

3. In a forthcoming version, we extend Theorem 1.1 to arbitrary functions which are not necessarily $M$-invariant.

Fix a Haar measure on $G$. This induces a left $G$-invariant measure on $\Gamma_q \backslash G$ for all ideals $q \subseteq \mathcal{O}_K$. For all ideals $q \subseteq \mathcal{O}_K$ and for all $\phi, \psi \in L^2(\Gamma_q \backslash G, \mathbb{C})$, we define the matrix coefficient by the usual inner product

$$\langle a_t \phi, \psi \rangle = \int_{\Gamma_q \backslash G} \phi(xa_t)\overline{\psi(x)} \, dx $$

where the use of the $G$-invariant measure is implicit. We denote by $m_q^{\text{BR}}$ and $m_q^{\text{BR}+}$ the unstable Burger-Roblin measure and the stable Burger-Roblin measure on $\Gamma_q \backslash G$ compatible with the choice of the Haar measure for all ideals $q \subseteq \mathcal{O}_K$.

**Corollary 1.1.1.** There exist $\eta > 0$, $C > 0$ and a nontrivial proper ideal $q_0 \subseteq \mathcal{O}_K$ such that for all square free ideals $q \subseteq \mathcal{O}_K$ coprime to $q_0$, and for all $M$-invariant functions $\phi, \psi \in C^1(\Gamma_q \backslash G, \mathbb{R})$, there exists $C_{\phi, \psi} > 0$ such that we have

$$\left| e^{(1-\delta)t} \langle a_t \phi, \psi \rangle - \frac{1}{m_q^{\text{BMS}}(\Gamma_q \backslash G)} m_q^{\text{BR}}(\phi) \cdot m_q^{\text{BR}}(\psi) \right| \leq C_{\phi, \psi} N_K(q)^C \epsilon^{-\eta t} \|\phi\|_{C^1} \|\psi\|_{C^1},$$

where $C_{\phi, \psi}$ can be taken to depend only the supports of the functions $\phi$ and $\psi$. 

1.1. Outline of the proof of Theorem 1.1. First we recount the proof of exponential mixing of the geodesic flow on the single manifold $T^1(\Gamma\backslash\mathbb{H}^n)$. From the works of Bowen and Ratner [Bow70, Rat73], it is well known that there are Markov sections for the geodesic flow on $T^1(\Gamma\backslash\mathbb{H}^n)$. This allows us to model the geodesic flow as a suspension space of subshift of finite type on bi-infinite sequences $\Sigma$. For any function $\phi \in C^1(\Gamma\backslash G/M, \mathbb{R})$, we can integrate out the strong stable direction of the Markov sections and take the Laplace transform in the flow direction to be left to deal with functions on one sided infinite sequences $\Sigma^+$ instead. Moreover, by Pollicott’s observation which was used by many others [AGY06, Dol98, Sto11], the Ruelle-Perron-Frobenius theorem can be used to cleanly write the Laplace transform of the correlation function as an infinite sum involving the transfer operators $L_\xi : C(\Sigma^+, \mathbb{R}) \to C(\Sigma^+, \mathbb{R})$ for $\xi \in \mathbb{C}$ defined by

$$L_\xi(h)(x) = \sum_{x' \in \sigma^{-1}(x)} e^{-(a+\delta_1-ib)\tau(x')} h(x').$$

Now, by a Paley-Wiener type of analysis, we can extract the desired exponential decay using the inverse Laplace transform formula given that the Laplace transform has a holomorphic extension to the left of the imaginary axis. To obtain the holomorphic extension, we are lead to find spectral bounds for the transfer operators. A major advancement regarding this study is the work of Dolgopyat [Dol98]. For transfer operators with large frequencies $|\Im(\xi)| \gg 1$, he was able to obtain spectral bounds by working explicitly on the strong unstable leaves of the Markov sections rather than the purely symbolic space in order to use the geometry of the manifold to obtain sufficiently strong bounds. The geometry provides a crucial local non-integrability condition which implies highly oscillating summands in the transfer operator and hence large cancellations which provide the required bounds. Moreover, he provided the right technical framework for the whole process to work in harmony. What is left are the transfer operators with small frequencies $|\Im(\xi)| \ll 1$ but they do not cause any problems due to the availability of the complex Ruelle-Perron-Frobenius theorem and a compactness argument which completes the proof.

For the proof of uniform exponential mixing, we can proceed as above, but we must use instead the congruence transfer operators $M_{\xi,q} : C(\Sigma^+, L^2(\Gamma_q \backslash \Gamma, \mathbb{C})) \to C(\Sigma^+, L^2(\Gamma_q \backslash \Gamma, \mathbb{C}))$ for $\xi \in \mathbb{C}$ and ideal $q \subset O_K$ defined by

$$M_{\xi,q}(H)(x) = \sum_{x' \in \sigma^{-1}(x)} e^{-(a+\delta_1-ib)\tau(x')} c_q(x')^{-1}H(x').$$

Here the cocycle $c_q$ “keeps track of the coordinate in the fibers” of the congruence cover $T^1(\Gamma_q\backslash\mathbb{H}^n) \to T^1(\Gamma\backslash\mathbb{H}^n)$. With this formulation, we now require spectral bounds for the congruence transfer operators uniform in the ideals $q \subset O_K$. A simple but crucial observation of Oh-Winter [OW16] is that the cocycle is locally constant. Consequently, there is no interference with Dolgopyat’s method for large frequencies $|\Im(\xi)| \gg 1$ and following [OW16, Sto11], it can be carried through uniformly in the ideals $q \subset O_K$. But now, since there are countably many ideals $q \subset O_K$, the compactness argument for small frequencies $|\Im(\xi)| \ll 1$ fails to hold uniformly in the ideals $q \subset O_K$.

Nevertheless, Bourgain-Gamburd-Sarnak demonstrated in their breakthrough work [BGS11] that the required bounds are attainable using entirely new methods of expander graphs. Built on their work is the expander machinery of Golsefidy-Varjú [GV12] which is what we use. The expander machinery captures the spectral
properties of a certain family of Cayley graphs coming from the congruence setting. This time, we obtain large cancellations of the summands in the congruence transfer operator due to the cocyle. We follow [OW16, MOW17] to use the expander machinery along with other techniques to take care of the spectral bounds for small frequencies $|\Im(\xi)| \ll 1$. In particular, Bourgain-Kontorovich-Magee has shown in the appendix of [MOW17] a method to use the expander machinery more directly which we adapt in this paper. In their work however, Zariski density of certain subgroups of $\Gamma$ which appears in the argument are crucial for successfully using the expander machinery. This was a trivial point in their work in the Schottky semigroups and continued fractions settings but in the setting of this paper, this is not at all a triviality, although it is quite natural to expect it to be true. The proof of this fact is the main new ingredient in this paper and completes the necessary tools to obtain spectral bounds for small frequencies $|\Im(\xi)| \ll 1$ uniform in the ideals $q \subset \mathcal{O}_K$ which completes the proof.

1.2. Organization of the paper. We start with reviewing the necessary background in Section 2 and also go through important constructions for the rest of the paper. Then we first go through the expander machinery part of the argument in Section 3 to obtain uniform spectral bounds for small frequencies. In Section 4, we use Dolgopyat’s method to obtain uniform spectral bounds for large frequencies. Finally in Section 5, we use the obtained uniform spectral bounds to go through arguments by Policott along with Paley-Wiener theory to prove uniform exponential mixing.

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2. Background and notations

Fix an integer $n \geq 2$ and let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space, i.e., the unique complete simply connected $n$-dimensional Riemannian manifold with constant negative sectional curvature. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm respectively on any tangent space of $\mathbb{H}^n$ induced by the hyperbolic metric. Similarly, we denote by $d$ the distance function on $\mathbb{H}^n$ induced by the hyperbolic metric. Let $K \subset \mathbb{R}$ be a totally real number field, $\mathcal{O}_K$ be the corresponding ring of integers, and $Q$ be a quadratic form of signature $(n, 1)$ defined over $K$. Let $G = \text{SO}_Q(\mathbb{C}) < \text{GL}_{n+1}(\mathbb{C})$ be an algebraic group defined over $K$ such that $G(\mathbb{R}) \cong \text{SO}(n, 1)$ and $G(\mathbb{R}) \cong \text{SO}(n + 1)$, which is compact, for all nontrivial embeddings $\sigma : K \hookrightarrow \mathbb{R}$. Let $G = G(\mathbb{R})^o$ which we recognize as the group of orientation preserving isometries of $\mathbb{H}^n$ by recalling that $\text{Isom}_+(\mathbb{H}^n) \cong \text{SO}(n, 1)^o$. Let $\Gamma < G$ be a torsion-free convex cocompact subgroup and Zariski dense in $G$. Let $o \in \mathbb{H}^n$ be a reference point and $v_o \in T^1_1(\mathbb{H}^n)$ be a reference tangent vector at $o$. Then we have the stabilizers $K = \text{Stab}_G(o)$ and $M = \text{Stab}_G(v_o) < K$. Note that $K \cong \text{SO}(n)$ and it is a maximal compact subgroup of $G$ and $M \cong \text{SO}(n - 1)$. Our base hyperbolic manifold is $X = \Gamma \backslash \mathbb{H}^n \cong \Gamma \backslash G / K$, its unit tangent bundle is $T^1(\mathbb{H}^n) \cong \Gamma \backslash G / M$ and its oriented orthonormal frame bundle is $F_{\text{SO}}(X) \cong \Gamma \backslash G$ which is a principal $\text{SO}(n)$-bundle over $X$ and a principal $\text{SO}(n - 1)$-bundle over $T^1(\mathbb{H}^n)$.
one parameter subgroup of semisimple elements $A = \{ a_t : t \in \mathbb{R} \} < G$, where $C_G(A) = AM$, parametrized such that its canonical right action on $G/M$ and $G$ corresponds to the geodesic flow and frame flow respectively. We choose any Riemannian metric on $G$ such that it is left $G$-invariant and right $K$-invariant [Sas58, Mok78] and again use the notations $\langle \cdot, \cdot \rangle, \| \cdot \|$ and $d$ on $G$ and any of its quotient spaces. In particular, the metric descends down to $\mathbb{H}^n \cong G/K$ and coincides with the previous hyperbolic metric.

To make use of the strong approximation theorem of Weisfeiler [Wei84] later on, we need to work on the simply connected cover $\hat{G}$ endowed with the covering map $\pi : \hat{G} \to G$ defined over $K$. Let $\hat{G} = G(\mathbb{R})$ which is connected and projects down to $\pi(\hat{G}) = G$. Let $\hat{G} < \hat{G}$ be a convex cocompact subgroup containing $\ker(\pi) = \{ c, -e \}$ as the only torsion elements and Zariski dense in $\hat{G}$. To be able to discuss the notion of congruence subgroups, let us suppose that there is a totally real number field $K$ with ring of integers $\mathcal{O}_K$ such that $\hat{G} < G(\mathcal{O}_K)$ which is indeed discrete in $\hat{G}$ by our assumption that $G^p(\mathbb{R})$ is compact for all nontrivial embeddings $\sigma : K \to \mathbb{R}$. Another assumption we will make to use the strong approximation theorem is that the subring of $K$ generated by $\{ \text{tr}(\text{Ad}(g)) : g \in \hat{G} \}$ is $\mathcal{O}_K$. Then we take $\hat{G}$ introduced previously to be $\hat{G} = \pi(\hat{G})$.

Let $\partial_\infty(\mathbb{H}^n)$ denote the boundary at infinity and $\Lambda(\Gamma) \subset \partial_\infty(\mathbb{H}^n)$ denote the limit set of $\Gamma$. We denote $\{ \mu_\Gamma^{PS} : x \in \mathbb{H}^n \}$ to be the Patterson-Sullivan density of $\Gamma$ [Pat76, Sul79], i.e., the set of finite Borel measures on $\partial_\infty(\mathbb{H}^n)$ supported on $\Lambda(\Gamma)$ such that

1. $g_\ast \mu_\Gamma^{PS} = \mu_\Gamma^{PS}$ for all $g \in \Gamma$, for all $x \in \mathbb{H}^n$
2. $\frac{d\mu_\Gamma^{PS}}{d\mu_\Gamma^{PS}}(\xi) = e^{\delta_\Gamma \beta_\xi(y,x)}$ for all $\xi \in \partial_\infty(\mathbb{H}^n)$, for all $x, y \in \mathbb{H}^n$

where $\beta_\xi$ denotes the Busemann function at $\xi \in \partial_\infty(\mathbb{H}^n)$ defined by $\beta_\xi(y, x) = \lim_{t \to \infty} (d(\xi(t), y) - d(\xi(t), x))$, where $\xi : \mathbb{R} \to \mathbb{H}^n$ is any geodesic such that $\lim_{t \to \infty} \xi(t) = \xi$. We allow tangent vector arguments for the Busemann function as well in which case we will use their basepoints in the definition. Since $\Gamma$ is convex cocompact, for all $x \in \mathbb{H}^n$, the measure $\mu_\Gamma^{PS}$ is the $\delta_\Gamma$-dimensional Hausdorff measure on $\partial_\infty(\mathbb{H}^n)$ supported on $\Lambda(\Gamma)$ corresponding to the spherical metric on $\partial_\infty(\mathbb{H}^n)$ with respect to $x$, up to scalar multiples. Now, using the Hopf parametrization via the homeomorphism $G/M \cong T^1(\mathbb{H}^n) \to \{(u^+, u^-) \in \partial_\infty(\mathbb{H}^n) \times \partial_\infty(\mathbb{H}^n) : u^+ \neq u^- \} \times \mathbb{R}$ defined by $u \mapsto (u^+, u^-, t = \beta_{u^+} - (o, u))$, we define the Bowen-Margulis-Sullivan (BMS) measure $m^{BMS}$ on $G/M$ [Mar04, Bow71, Kai90] by

$$dm^{BMS}(u) = e^{\delta_\Gamma \beta_{u^+}(o, u)} e^{\delta_\Gamma \beta_{u^-}(o, u)} d\mu_\Gamma^{PS}(u^+) d\mu_\Gamma^{PS}(u^-) dt$$

Note that this definition only depends on $\Gamma$ and not on the choice of reference point $o \in \mathbb{H}^n$ and moreover $m^{BMS}$ is left $\Gamma$-invariant. We now define induced measures on other spaces all of which we call the BMS measures and denote by $m^{BMS}$ by abuse of notation. Since $M$ is compact, we can use any Haar measure on $M$ to lift $m^{BMS}$ to a right $M$-invariant measure on $G$. By left $\Gamma$-invariance, $m^{BMS}$ now descends to a measure on $\Gamma \backslash G$. By right $M$-invariance, $m^{BMS}$ descends once more to a measure on $\Gamma \backslash G/M$. It can be checked that the BMS measures are invariant with respect to the geodesic flow or the frame flow as appropriate, i.e., right $A$-invariant. We denote $\Omega = \text{supp}(m^{BMS}) \subset \Gamma \backslash G/M$ which is compact since $\Gamma$ is convex cocompact, and it is also invariant with respect to the geodesic flow.
2.1. Markov sections. We use Markov sections on $\Omega \subset T^1(X) \cong \Gamma \backslash G/M$, as developed by Bowen and Ratner [Bow70, Rat73], to obtain a symbolic coding of the dynamical system at hand. Let $W^{wu}(w) \subset T^1(X)$ and $W^{ws}(w) \subset T^1(X)$ denote the leaves through $w \in T^1(X)$ of the strong unstable and strong stable foliations, and $W^{ss}(w) \subset W^{wu}(w)$ and $W^{wu}(w) \subset W^{uw}(w)$ denote the open balls of radius $\epsilon > 0$ with respect to the induced distance functions $d_{su}$ and $d_{ws}$, respectively. We use similar notations for the weak unstable and weak stable foliations by replacing ‘su’ with ‘wu’ and ‘ss’ with ‘ws’ respectively. Define $N^+ < G$ and $N^- < G$ to be the expanding and contracting horospherical subgroups, i.e.,

$$N^+ = \left\{ n^+ \in G : \lim_{t \to \infty} a_t n^+ a_{-t} = e \right\}$$

$$N^- = \left\{ n^- \in G : \lim_{t \to -\infty} a_t n^- a_{-t} = e \right\}$$

respectively. Then we can more explicitly write $W^{wu}(w) = wN^+, W^{ws}(w) = wN^-$, $W^{wu}(w) = wAN^+$ and $W^{ws}(w) = wAN^-$ for all $w \in T^1(X)$. We recall that for all $w \in T^1(X)$, for all $u \in W^{wu}(w)$, for all $s \in W^{wu}(w)$, there is a unique intersection denoted by $[u, s] = W^{wu}(u) \cap W^{wu}(s)$ and moreover $[\cdot, \cdot]$ defines a homeomorphism from $W^{wu}(w) \times W^{ws}(w)$ onto its image, where $c_0$ and $\gamma$ are constants from [Rat73] which we use in this subsection. Subsets $U \subset W^{wu}(w) \cap \Omega$, $S \subset W^{ws}(w) \cap \Omega$ and $R \subset [W^{wu}(w), W^{ws}(w)] \cap \Omega$ are called proper if $U = \text{int}^{su}(U)$, $S = \text{int}^{ss}(S)$ and $R = \text{int}^{su,ss}(R)$ respectively, where the superscripts signify that the interiors and closures are taken in the topology of the indicated containing set. We will often drop the superscripts henceforth and include it whenever further clarity in notations is required. For any proper sets $U \subset W^{wu}(w) \cap \Omega$ and $S \subset W^{ws}(w) \cap \Omega$ for some $w \in \Omega$, we call the proper set $R = [U, S] = \{ [u, s] \in \Omega : u \in U, s \in S \} \subset \Omega$ a rectangle of size $\delta$ if $\text{diam}_{wu}(U_j), \text{diam}_{ws}(S_j) \leq \delta$ for some $\delta > 0$, and we call $w$ the center of $R$. For any rectangle $R = [U, S]$, we generalize the notation and define $[v_1, v_2] = [u_1, s_2]$ for all $v_1 = [u_1, s_1] \in R$, for all $v_2 = [u_2, s_2] \in R$.

**Definition 2.1.** A set $\mathcal{R} = \{R_1, R_2, \ldots, R_N\} = \{[U_1, S_1], [U_2, S_2], \ldots, [U_N, S_N]\}$ for some $N \in \mathbb{Z}_{>0}$ consisting of rectangles is called a complete set of rectangles of size $\delta$ in $\Omega$ if

1. $R_j \cap R_k = \emptyset$ for all integers $1 \leq j, k \leq N$ with $j \neq k$
2. $\text{diam}_{wu}(U_j), \text{diam}_{ws}(S_j) \leq \delta$ for all integers $1 \leq j \leq N$
3. $\Omega = \bigcup_{j=1}^{N} \bigcup_{t \in [0, \delta]} R_j a_t$.

We set $R = \bigcup_{j=1}^{N} R_j$ and $U = \bigcup_{j=1}^{N} U_j$. Let $\mathcal{P} : R \to R$ be the Poincaré first return map and $\sigma = (\text{proj}_1 \circ \mathcal{P}) | U : U \to U$ be its projection where $\text{proj}_1 : R \to R$ is the projection defined by $\text{proj}_1([u, s]) = u$ for all $[u, s] \in R$. Let $\tau : R \to \mathbb{R}_{>0}$ be the first return time map defined by $\tau(u) = \inf \{ t \in \mathbb{R}_{>0} : u a_t \in R \}$ so that $\mathcal{P}(u) = u a_{\tau(u)}$ for all $u \in R$. Note that $\tau$ is constant on $[u, S_j]$ for all $u \in U_j$, for all integers $1 \leq j \leq N$. For future convenience we define $\overline{\tau} = \sup_{u \in R} \tau(u)$ and $
abla = \inf_{u \in R} \tau(u)$. Define the cores $\tilde{R} = \{u \in R : \mathcal{P}^k(u) \in \text{int}(R) \}$ for all $k \in \mathbb{Z}$ and $\tilde{U} = \{u \in U : \sigma^k(u) \in \text{int}(U) \}$ for all $k \in \mathbb{Z}_{>0}$. We note that the cores are both residual subsets (complements of meager sets) of $R$ and $U$ respectively.

**Definition 2.2.** We call the complete set $\mathcal{R}$ a Markov section if in addition to Properties 1–3, the following property
Henceforth, we fix a positive geodesic flows) of arbitrarily small size was proved by Bowen and Ratner \cite{Bow70, Rat73}. Henceforth, we fix a positive geodesic flows) of arbitrarily small size was proved by Bowen and Ratner \cite{Bow70, Rat73}. Henceforth, we fix a positive geodesic flows) of arbitrarily small size was proved by Bowen and Ratner \cite{Bow70, Rat73}. Henceforth, we fix a positive.

\[ \hat{\delta} < \min \left( 1, \epsilon_0, \gamma, \frac{1}{4} \text{Inj}(T^1(X)) \right) \]

where \( \text{Inj}(T^1(X)) \) denotes the injectivity radius of \( T^1(X) \). We also fix \( \mathcal{R} = \{ R_1, R_2, \ldots, R_N \} = \{ [U_1, S_1], [U_2, S_2], \ldots, [U_N, S_N] \} \) to be a Markov section of size \( \delta \) in \( \Omega \). We introduce the distance function \( d \) on \( U \) defined by

\[
d(u, u') = \begin{cases} 
  d_{su}(u, u'), & u, u' \in U_j \text{ for some integer } 1 \leq j \leq N \\
  1, & \text{otherwise.}
\end{cases}
\]

**Remark.** This definition makes sense since \( \text{diam}_{d_{su}}(U_j) \leq \hat{\delta} < 1 \) for all integers \( 1 \leq j \leq N \). This notation should not cause any confusion but we will use \( d_{su} \) whenever further clarity is required.

### 2.2. Symbolic dynamics

Let \( \mathcal{A} = \{ 1, 2, \ldots, N \} \) be the alphabet for the symbolic coding corresponding to the Markov sections. Define the \( N \times N \) transition matrix \( T \) by

\[
T_{j,k} = \begin{cases} 
  1, & \text{int}(R_j) \cap \mathcal{P}^{-1}(\text{int}(R_k)) \neq \emptyset \\
  0, & \text{otherwise}
\end{cases}
\]

for all integers \( 1 \leq j, k \leq N \). The transition matrix \( T \) is topologically mixing \cite{Rat73}*{Theorem 4.3}, i.e., there is a fixed \( N_T \in \mathbb{Z}_{>0} \) such that all the entries of \( T^{N_T} \) are positive. This definition is equivalent to the one in \cite{Rat73} in the setting of Markov sections. Define the spaces of bi-infinite and infinite admissible sequences respectively by

\[
\Sigma = \{ (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathcal{A}^{\mathbb{Z}} : T_{x_j, x_{j+1}} = 1 \text{ for all } j \in \mathbb{Z} \}
\]

\[
\Sigma^+ = \{ (x_0, x_1, \ldots) \in \mathcal{A}^{\mathbb{Z}_{\geq 0}} : T_{x_j, x_{j+1}} = 1 \text{ for all } j \in \mathbb{Z}_{\geq 0} \}.
\]

We will use the term admissible sequences for finite sequences as well in the natural way. For any \( \theta \in (0, 1) \), we can endow \( \Sigma \) with the distance function \( d_{\theta}(x, y) = \theta^{\inf\{|j| \in \mathbb{Z}_{\geq 0} : x_j \neq y_j \text{ for } j \in \mathbb{Z} \}} \) for all \( x, y \in \Sigma \). We can similarly endow \( \Sigma^+ \) with a distance function which we also denote by \( d_{\theta} \).

**Definition 2.3.** For all \( k \in \mathbb{Z}_{>0} \), for all admissible sequences \( x = (x_0, x_1, \ldots, x_k) \), we define the corresponding cylinder to be \( C[x] = \{ u \in U : \sigma^j(u) \in \text{int}(U_{x_j}), 0 \leq j \leq k \} \) with length \( \text{len}(C[x]) = k \). We will denote cylinders simply by \( C \) (or other typewriter style letters) when we do not need to specify the corresponding admissible sequence. A closed cylinder \( C \) will be the closure of some cylinder \( C' \), i.e., \( C = \overline{C'} \).

**Remark.** For all admissible pairs \( (j, k) \), the restricted maps \( \sigma|_{C[j,k]} : C[j,k] \rightarrow \text{int}(U_k) \) and \( \tau|_{C[j,k]} : C[j,k] \rightarrow \mathbb{R}_{>0} \) are Lipschitz.
By a slight abuse of notation, let $\sigma$ also denote the shift map on $\Sigma$ or its subspaces. There are natural continuous surjections $\zeta: \Sigma \to R$ and $\zeta^+: \Sigma^+ \to U$ defined by $\zeta(x) = \int_{\Sigma}^{\infty} \mathcal{P}^{-j}(\text{int}(R_x)) \, dx$ for all $x \in \Sigma$ and $\zeta^+(x) = \int_{\Sigma}^{\infty} \sigma^{-j}(\text{int}(U_x)) \, dx$ for all $x \in \Sigma^+$. Define $\hat{\Sigma} = \zeta^{-1}(R)$ and $\hat{\Sigma}^+ = (\zeta^+)^{-1}(U)$. Then the restrictions $\zeta|_{\hat{\Sigma}}: \hat{\Sigma} \to R$ and $\zeta^+|_{\hat{\Sigma}^+}: \hat{\Sigma}^+ \to \hat{\Sigma}$ are bijective and satisfy $\zeta|_{\hat{\Sigma}} \circ \sigma|_{\hat{\Sigma}} = \mathcal{P}|_R \circ \zeta|_{\hat{\Sigma}}$ and $\zeta^+|_{\hat{\Sigma}^+} \circ \sigma|_{\hat{\Sigma}^+} = \mathcal{P}|_U \circ \zeta^+|_{\hat{\Sigma}^+}$.

For $\theta \in (0, 1)$ sufficiently close to 1, the maps $\zeta$ and $\zeta^+$ are Lipschitz [Bow75, Lemma 2.2]. We fix $\theta$ to be any such constant. We now introduce some function spaces corresponding to $\Sigma, \Sigma^+$ and $U$. Let $B(\Sigma, \mathbb{R}) \supset C(\Sigma, \mathbb{R}) \supset C^{\text{Lip}(d)}(\Sigma, \mathbb{R})$ denote the spaces of functions $f: \Sigma \to \mathbb{R}$ which are bounded, continuous and Lipschitz respectively. The first two are Banach spaces with the $L^\infty$ norm $\|f\|_\infty$, and $C^{\text{Lip}(d)}(\Sigma, \mathbb{R})$ is the Banach space with the norm $\|f\|_{\text{Lip}(d)} = \|f\|_\infty + \text{Lip}_d(f)$ where

$$\text{Lip}_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d_\theta(x, y)} : x, y \in \Sigma \text{ such that } x \neq y \right\}$$

is the Lipschitz seminorm, for all $f \in C^{\text{Lip}(d)}(\Sigma, \mathbb{R})$. We use similar notations for function spaces with domain space $\Sigma^+$ or $U$. We use similar notations again for target space $\mathbb{C}$.

Since $(\tau \circ \zeta)|_{\hat{\Sigma}^+}$ and $(\tau \circ \zeta^+)|_{\hat{\Sigma}^+}$ are Lipschitz, there are unique Lipschitz extensions of $\zeta$ and $\zeta^+$ which we denote by $\tau_\Sigma: \Sigma \to \mathbb{R}_{\geq 0}$ and $\tau_{\Sigma^+}: \Sigma^+ \to \mathbb{R}_{\geq 0}$ respectively. Note that the resulting maps are distinct from $\tau \circ \zeta$ and $\tau \circ \zeta^+$ because they may differ precisely on $x \in \Sigma$ for which $\zeta(x) \in \partial(\mathbb{C})$ and $x \in \Sigma^+$ for which $\zeta^+(x) \in \partial(\mathbb{C})$ respectively, for some cylinder $\mathbb{C} \subset U$ with $\text{len}(\mathbb{C}) = 1$. Then the previous properties extend to $\zeta(\sigma(x)) = \zeta(x) a_{\tau_\Sigma}(x)$ for all $x \in \Sigma$ and $\zeta^+(\sigma(x)) = \text{proj}_1(\zeta^+(x) a_{\tau_{\Sigma^+}}(x))$ for all $x \in \Sigma^+$.

### 2.3. Thermodynamics.

**Definition 2.4.** For all $f \in C^{\text{Lip}(d)}(\Sigma, \mathbb{R})$, called the potential, the pressure is

$$\text{Pr}_\sigma(f) = \sup_\nu \int_\Sigma f \, d\nu + h_\nu(\sigma)$$

where the supremum is taken over all $\sigma$-invariant Borel probability measures $\nu$ on $\Sigma$ and $h_\nu(\sigma)$ is the measure theoretic entropy of $\nu$ with respect to $\sigma$.

For all $f \in C^{\text{Lip}(d)}(\Sigma, \mathbb{R})$, there is in fact a unique $\sigma$-invariant Borel probability measure $\nu_f$ on $\Sigma$ which attains the supremum in Definition 2.4 called the $f$-equilibrium state [Bow08, Theorems 2.17 and 2.20] and it satisfies $\nu_f(\Sigma) = 1$ [Che02, Corollary 3.2].

In particular, we will consider the probability measure $\nu_{(-\delta_\tau \tau_\Sigma)}$ on $\Sigma$ which we will denote simply by $\nu_\Sigma$ and has corresponding pressure $\text{Pr}_{\sigma}(-\delta_\tau \tau_\Sigma) = 0$. According to above, $\nu_{\hat{\Sigma}}(\hat{\Sigma}) = 1$. Define the corresponding probability measure $\nu_R = \zeta_* (\nu_\Sigma)$ on $R$ and note that $\nu_R(\hat{\Sigma}) = 1$. Now consider the suspension space $R^\tau = (R \times \mathbb{R}_{\geq 0})/\sim$ where $\sim$ is the equivalence relation on $R \times \mathbb{R}_{\geq 0}$ defined by $(u, t + \tau(u)) \sim (\mathcal{P}(u), t)$. Then we have a bijection $R^\tau \to \Omega$ defined by $(u, t) \mapsto u\alpha_t$. We can define the measure $\nu^\tau$ on $R^\tau$ as the product measure $\nu_R \times m_{\text{Leb}}$ on $\{(u, t) \in R \times \mathbb{R}_{\geq 0} : 0 \leq t < \tau(u)\}$. Then using the aforementioned bijection we have the pushforward measure which, by abuse of notation, we also denote by $\nu^\tau$ on $T^1(X)$ supported on $\Omega$. 

Theorem 2.5. We have $\nu^\tau = \frac{m_{\text{BMS}}^{\tau}}{\max(1, |U/M|)}$.

Proof. Sullivan [Sul84] proved that $m_{\text{BMS}}^{\tau}$ is the unique measure of maximal entropy for the geodesic flow on $T^1(X)$. By [Che02, Theorem 4.4], $\nu^\tau$ is also the unique measure of maximal entropy on $T^1(X)$. Thus they are equal after normalization. \hfill \blacksquare

Finally, we define the probability measure $\nu_U = (\text{proj}_U)_* (\nu_R)$ and note that $\nu_U(U) = 1$ and $\nu_U(\tau) = \nu_R(\tau)$. As a consequence of the RPF theorem, $\nu_U$ is in fact a Gibbs measure [Bow08, PP90], i.e., there are $c_1^U, c_2^U > 0$ such that

$$c_1^U e^{-\delta_U \tau_U(x)} \leq \nu_U(C) \leq c_2^U e^{-\delta_U \tau_U(x)}$$

(1)

for all $x \in C$, for all cylinders $C$ with $\text{len}(C) = k \in \mathbb{Z}_{\geq 0}$, since $\lambda_0 = 1$. See the next subsection for terms and notations.

2.4. Transfer operators. First we recall the classic transfer operator and associated theorems.

Definition 2.6. For all $f \in B(\Sigma^+, C)$, we define the transfer operator $\mathcal{L}_f : B(\Sigma^+, C) \to B(\Sigma^+, C)$ by

$$\mathcal{L}_f(h)(x) = \sum_{x' \in \sigma^{-1}(x)} e^{f(x')} h(x')$$

for all $x \in \Sigma^+$, for all $h \in B(\Sigma^+, C)$.

Remark. In fact, if $f \in C(\Sigma^+, C)$ or $f \in C^{\text{Lip}(d_\sigma)}(\Sigma^+, C)$, then the transfer operator $\mathcal{L}_f$ preserves the subspace $C(\Sigma^+, C)$ or $C^{\text{Lip}(d_\sigma)}(\Sigma^+, C)$ respectively.

The following theorem is a consequence of the Ruelle-Perron-Frobenius (RPF) theorem and the theory of Gibbs measures [Bow08, PP90].

Theorem 2.7. For all $f \in C^{\text{Lip}(d_\sigma)}(\Sigma^+, \mathbb{R})$, the operator $\mathcal{L}_f : C(\Sigma^+, C) \to C(\Sigma^+, C)$ and its dual $\mathcal{L}_f^* : C(\Sigma^+, C)^* \to C(\Sigma^+, C)^*$ has eigenvectors with the following properties. There exist a unique positive function $h \in C^{\text{Lip}(d_\sigma)}(\Sigma^+, \mathbb{R})$ and a unique Borel probability measure $\nu$ on $\Sigma^+$ such that

1. $\mathcal{L}_f(h) = e^{\text{Pr}_\sigma(f)} h$
2. $\mathcal{L}_f^*(\nu) = e^{\text{Pr}_\sigma(f)} \nu$
3. the eigenvalue $e^{\text{Pr}_\sigma(f)}$ is simple and the rest of the spectrum of $\mathcal{L}_f$ is contained in a disk of radius $r < e^{\text{Pr}_\sigma(f)}$
4. $\nu(h) = 1$ and the Borel probability measure $\mu$ defined by $d\mu = h d\nu$ is $\sigma$-invariant and is the projection of the $f$-equilibrium state to $\Sigma^+$, i.e., $\mu$ is the pushforward of $\nu_f$ via the map $\Sigma \to \Sigma^+$ defined by $(\ldots, x_{-1}, x_0, x_1, \ldots) \mapsto (x_0, x_1, \ldots)$.

Similarly, we would like to define transfer operators on $B(U, C)$ specifically corresponding to the function $\xi \tau$ for any $\xi \in C$, but we need to deal with some technicalities due to the boundaries of the rectangles. For all admissible pairs $(j, k)$, it is not hard to see from the Markov section construction that there are unique natural Lipschitz maps $\sigma^{(1)}_{j,k} : U_k \to \overline{C[j,k]}$ which extends $(\sigma_{C[j,k]})^{-1} : \text{int}(U_k) \to C[j,k]$ and $\tau^{(1)}_{j,k} : \overline{C[j,k]} \to \mathbb{R}_{>0}$ which extends $\tau_{C[j,k]} : C[j,k] \to \mathbb{R}_{>0}$. Note that $\tau_{(x_0,x_1)}(\xi^+(x)) = \tau_{\Sigma^+}(x)$ for all $x \in \Sigma^+$. 
Definition 2.8. For all $\xi \in \mathbb{C}$, we define the transfer operator $L_{\xi} : B(U, \mathbb{C}) \to B(U, \mathbb{C})$ by

$$L_{\xi}(h)(u) = \sum_{(j,k) \in \sigma^{-1}(u)} e^{\xi \tau_{j,k}(u')} h(u')$$

for all $u \in U_k$, for all $k \in A$, for all $h \in B(U, \mathbb{C})$, where it is understood that the sum is only over admissible pairs.

Remark. Due to the careful treatment of the boundaries of the rectangles, it is clear that for all $\xi \in \mathbb{C}$, the transfer operator $L_{\xi}$ preserves the subspaces $C(U, \mathbb{C})$ and $C^{\text{Lip}(d)}(U, \mathbb{C})$. Let $\xi \in \mathbb{C}$ and $h \in B(U, \mathbb{C})$. Then for all $u \in \text{int}(U)$, we can in fact write $L_{\xi}(h)(u) = \sum_{u' \in \sigma^{-1}(u)} e^{\xi \tau(u')} h(u')$. With these observations, if $h \in C(U, \mathbb{C})$, then we can equivalently define $L_{\xi}(h)$ by the previous expression on $\text{int}(U)$ and then extend continuously to a function on $U$ (where the unique existence is now known by Definition 2.8). This will help us avoid the more cumbersome Definition 2.8. Finally, note that $L_{\xi} \circ (\zeta^+)^* = (\zeta^+) \circ L_{\xi}$.

The following is the RPF theorem in this setting.

Theorem 2.9. For all $a \in \mathbb{R}$, the operator $L_{a\tau} : C(U, \mathbb{C}) \to C(U, \mathbb{C})$ and its dual $L_{a\tau}^* : C(U, \mathbb{C})^* \to C(U, \mathbb{C})^*$ has eigenvectors with the following properties. There exist a unique positive function $h \in C^{\text{Lip}(d)}(U, \mathbb{R})$ and a unique Borel probability measure $\nu$ on $U$ such that

1. $L_{a\tau}(h) = e^{Pr_{a\tau}(a\tau)} h$
2. $L_{a\tau}^*(\nu) = e^{Pr_{a\tau}(a\tau)} \nu$
3. the eigenvalue $e^{Pr_{a\tau}(a\tau)}$ is simple and the rest of the spectrum of $L_f$ is contained in a disk of radius $r < e^{Pr_{a\tau}(a\tau)}$
4. $\nu(h) = 1$ and the Borel probability measure $\mu$ defined by $d\mu = h \, dv$ is $\sigma$-invariant and is the projection of the $a\tau$-equilibrium state to $U$, i.e.,

$\mu = (\text{proj}_1 \circ \zeta^*)^+ (\nu_{a\tau})$

Remark. Recall $L_{\xi \Sigma^+} \circ (\zeta^+) = (\zeta^+) \circ L_{\xi}$ for all $\xi \in \mathbb{C}$. Using this, it is clear from the proofs that the $h$ in Theorem 2.9 pulls back to the corresponding one in Theorem 2.7 and the $\nu$ in Theorem 2.9 is the pushforward of the corresponding one in Theorem 2.7, both via the map $\zeta^+$. By the same property, the eigenvalues in Theorem 2.9 are the same as the corresponding ones in Theorem 2.7.

Now we normalize the transfer operators for convenience. Let $a \in \mathbb{R}$. Define $\lambda_a = e^{Pr_{a\tau}(a\tau + 0)}$ which is the largest eigenvalue of $L_{-(a\tau + 0)}$ and recall that $\lambda_0 = 1$. Define the eigenvectors, the unique positive function $h_a \in C^{\text{Lip}(d)}(U, \mathbb{R})$ and the unique probability measure $\nu_a$ on $U$ with $\nu_a(h_a) = 1$ such that $L_{-(a\tau + 0)}(h_a) = \lambda_a h_a$ and $L_{-(a\tau + 0)}^*(\nu_a) = \lambda_a \nu_a$, provided by Theorem 2.9. Note that $d\nu_U = h_0 \, dv$. By abuse of notation, we will denote the pullback $(\zeta^+)^*(h_a)$, which is the corresponding eigenvector for $L_{-(a\tau + 0)}^*$, by $h_a$ as well. Moreover, by perturbation theory for operators as in [Kat95, Chapter 8] and [PP90, Proposition 4.6], we can fix $a_0' > 0$ such that the map $[-a_0', a_0'] \to \mathbb{R}$ defined by $a \mapsto \lambda_a$ and the map $[-a_0', a_0'] \to C(U, \mathbb{R})$ defined by $a \mapsto h_a$ are Lipschitz. In particular, there is a $C > 0$ such that $|\lambda_a - 1| \leq C|a|$ and $|h_a(u) - h_0(u)| \leq C|a|$ for all $u \in U$, for all
\[ |a| \leq a_0'. \] For all \( a \in \mathbb{R} \), we define
\[
f^{(a)} = -(a + \delta_1)\tau + \log \circ h_0 - \log \circ (h_0 \circ \sigma) - \log(\lambda_a).
\] (2)

Then we can fix a \( A_f > 0 \) such that \( |f^{(a)}(u) - f^{(0)}(u)| \leq A_f |a| \) for all \( u \in U \), for all \( |a| \leq a_0' \). For all \( k \in \mathbb{Z}_{>0} \), we use the notation
\[
f_k^{(a)}(u) = \sum_{j=0}^{k-1} f^{(a)}(\sigma^j(u)) \quad \quad \quad \tau_k(u) = \sum_{j=0}^{k-1} \tau(\sigma^j(u))
\]
for all \( u \in U \), and when \( k = 0 \) we mean the empty sum which is 0. By a slight abuse of notation, for all \( \xi = a + ib \in \mathbb{C} \), we define \( \mathcal{L}_\xi = \mathcal{L}_{f^{(a)} + ib\tau} : C(U, \mathbb{C}) \to C(U, \mathbb{C}) \), i.e.,
\[
\mathcal{L}_\xi(h)(u) = \sum_{u' \in \sigma^{-1}(u)} e^{(f^{(a)} + ib\tau)(u')} h(u')
\]
\[
= \frac{1}{\lambda_a h_0(u)} \sum_{u' \in \sigma^{-1}(u)} e^{-(a+\delta_1-ib)\tau(u')(h_0h)(u')}
\]
and for all \( k \in \mathbb{Z}_{>0} \), its \( k \)th iteration is
\[
\mathcal{L}^{(k)}_\xi(h)(u) = \sum_{u' \in \sigma^{-k}(u)} e^{(f^{(a)} + ib\tau_k)(u')} h(u')
\]
for all \( u \in \text{int}(U) \), and then extend continuously to a function on \( U \), for all \( h \in C(U, \mathbb{C}) \). Then the transfer operators are normalized such that the largest eigenvalue of \( \mathcal{L}_a \) is 1 with normalized eigenvector \( \chi_U \) and \( \mathcal{L}_0^{(\nu_U)} = \nu_U \).

We leave it to the reader to make the appropriate adjustments to define the operators on \( B(U, \mathbb{C}) \) as in Definition 2.8.

**Remark.** We can also define \( f_{\xi}^{(a)} \) by replacing \( \sigma|_U \) by \( \sigma|_{\Sigma^+} \), \( \tau \) by \( \tau_{\Sigma^+} \) and using \( h_0 \in C^{\text{Lip}(d_{\Sigma^+})}(\Sigma^+, \mathbb{R}) \) in Eq. (2), which would serve to define, by abuse of notation, a corresponding normalized operator \( \mathcal{L}_\xi : C(\Sigma^+, \mathbb{C}) \to C(\Sigma^+, \mathbb{C}) \) for all \( \xi \in \mathbb{C} \). Then \( \mathcal{L}_\xi \circ (\xi^*)^* = (\xi^*)^* \circ \mathcal{L}_\xi \) for all \( \xi \in \mathbb{C} \) similar to the final remark after Definition 2.8.

Now fix a
\[
T_0 > \max \left( \|\tau\|_\infty, \text{Lip}_d(\tau), \text{Lip}_{d_{\Sigma^+}}(\tau_{\Sigma^+}) \right),
\]
\[
\sup_{|a| \leq a_0'} \|f^{(a)}\|_\infty, \sup_{|a| \leq a_0'} \text{Lip}_d(f^{(a)}), \sup_{|a| \leq a_0'} \text{Lip}_{d_{\Sigma^+}}(f^{(a)}_\Sigma^+)
\]
where we define the essentially Lipschitz seminorm for any function \( \varphi : U \to \mathbb{R} \) to be
\[
\text{Lip}_d^e(\varphi) = \sup \left\{ \frac{|\varphi(u) - \varphi(u')|}{d(u, u')} : u, u' \in U, u, u' \in C, C \subset U \text{ is a cylinder with } \text{len}(C) = 1 \right\}
\]
and call \( \varphi \) essentially Lipschitz if \( \text{Lip}_d^e(\varphi) < \infty \). This is possible by [PS16, Lemma 4.1].
2.5. Cocycles and congruence transfer operators. We recall some definitions from [OW16] regarding the congruence setting. Noting that \( T^1(\mathbb{H}^n) \) is a locally isometric cover of \( T^1(X) \), for all \( j \in \mathcal{A} \), choose homeomorphic lifts \( \overline{R}_j = [\overline{U}_j, \overline{S}_j] \subset T^1(\mathbb{H}^n) \cong G/M \) of \( R_j \). Define \( \overline{R} = \bigsqcup_{j=1}^N \overline{R}_j \) and \( \overline{U} = \bigsqcup_{j=1}^N \overline{U}_j \). For all \( u \in R \), let \( \overline{u} \in \overline{R} \) denote the unique lift in \( \overline{R} \).

**Definition 2.10.** The cocycle \( c : R \to \Gamma \) is a map such that for all \( u \in R \), we have \( \overline{u}a_{\tau(u)} \in c(u)\overline{R} \).

**Lemma 2.11.** The cocycle \( c \) is locally constant, i.e., if \( u_1, u_2 \in R_x \cap \mathcal{P}^{-1}(R_y) \) for some \( x, y \in \mathcal{A} \), then \( c(u_1) = c(u_2) \).

**Proof.** Let \( u_1, u_2 \in R_x \cap \mathcal{P}^{-1}(R_y) \) for some \( x, y \in \mathcal{A} \) but suppose \( c(u_1) \neq c(u_2) \). Then \( \overline{u}_j \in \overline{R}_x \) and \( u_j = \Gamma \overline{u}_j \) for all \( j \in \{1, 2\} \). By definition, \( \overline{u}_j = c(u_j)^{-1}\overline{u}_j a_{\tau(u_j)} \in \overline{R}_x \) for all \( j \in \{1, 2\} \) and by local isometry, we have

\[
0 \neq d(c(u_1)\overline{u}_1, c(u_2)\overline{u}_1) \leq d(\overline{u}_1, c(u_2)\overline{u}_1) + \delta \leq d(\overline{u}_2, c(u_2)\overline{u}_1) + 2\delta \leq d(c(u_2)\overline{u}_2, c(u_2)\overline{u}_1) + 3\delta \leq 4\delta \leq \text{Inj}(T^1(X))
\]

and so there is a geodesic from \( c(u_1)\overline{u}_1 \) to \( c(u_2)\overline{u}_1 \) of positive length less than the injectivity radius. But such a geodesic would project isometrically to a closed geodesic on \( T^1(X) \) since \( c(u_j)\overline{u}_1 \) projects to \( \Gamma c(u_j)\overline{u}_1 = \Gamma \overline{u}_1 \) for all \( j \in \{1, 2\} \) which is a contradiction. \( \square \)

Justified by the lemma above, for all \( k \in \mathbb{Z}_{>0} \), we use the notation

\[
c^k(u) = \prod_{j=0}^{k-1} c(\sigma^j(u)) = c(u)c(\sigma(u))\cdots c(\sigma^{k-1}(u))
\]

for all \( u \in U \), and when \( k = 0 \) we mean the empty product which is \( e \in \Gamma \). Note that the order in the product is important in the definition above.

**Corollary 2.11.1.** If \( u_1, u_2 \in \mathcal{C} \) for some cylinder \( \mathcal{C} \subset U \) with \( \text{len}(\mathcal{C}) = k \in \mathbb{Z}_{>0} \), then \( c^k(u_1) = c^k(u_2) \).

For all ideals \( q \subset \mathcal{O}_K \), we have the canonical quotient map \( \pi_q : \hat{G}(\mathcal{O}_K) \to \hat{G}(\mathcal{O}_K/q) \) and we define the principal congruence subgroup of level \( q \) to be \( \ker(\pi_q) \). We would like to define the congruence subgroup of \( \hat{\Gamma} \) of level \( q \) to be the normal subgroup \( \hat{\Gamma}_q = \ker(\pi_q|_{\hat{\Gamma}}) \subset \hat{\Gamma} \). However, we make a minor modification and assume as before that \( \hat{\Gamma}_q \subset \hat{\Gamma} \) contains \( \ker(\hat{\pi}) = \{e, -e\} \) as the only torsion elements, i.e., we define \( \hat{\Gamma}_q = \langle \ker(\pi_q|_{\hat{\Gamma}}), \ker(\hat{\pi}) \rangle \subset \hat{\Gamma} \). Again we define \( \Gamma_q = \hat{\pi}(\hat{\Gamma}_q) \). For all nontrivial \( q \subset \mathcal{O}_K \), also define the finite group \( F_q = \Gamma_q \backslash \Gamma \cong \hat{\Gamma}_q \backslash \hat{\Gamma} \). By the strong approximation theorem of Weisfeiler [Wei84], there is a nontrivial proper ideal \( q_0 \subset \mathcal{O}_K \) such that for all ideals \( q \subset \mathcal{O}_K \) coprime to \( q_0 \), the map \( \pi_q|_{\hat{\Gamma}} \) is in fact surjective and hence induces the isomorphism \( \pi_q|_{\hat{\Gamma}} : F_q \to \hat{G}_q \) where we define the finite group \( \hat{G}_q = \{e, -e\}\backslash \hat{G}(\mathcal{O}_K/q) \), by a slight abuse of notation. Without loss of generality, we assume that for all \( (y, z) \in \mathcal{A}^2 \), the same ideal \( q_0 \subset \mathcal{O}_K \) is sufficient to apply both the strong approximation theorem and [GV12, Corollary 6] for the subgroups \( \hat{H}^p(y, z) \subset \hat{\Gamma} \) which is to be introduced later. This condition will appear throughout the paper in order for the strong approximation theorem to apply. For all nontrivial ideals \( q \subset \mathcal{O}_K \), define the congruence cocycle \( c_q : R \to F_q \) by \( c_q = \pi_{\Gamma_q} \circ c \), i.e., \( c_q(x) = \Gamma_q c(x) \) for all \( x \in R \).
Let \( q \subset \mathcal{O}_K \) be a nontrivial ideal. Let \( X_q = \Gamma_q \backslash \mathbb{H}^2 \) be the congruence cover of \( X \) of level \( q \). Note that we have the isometries \( T^1(X_q) \cong \Gamma_q \backslash G/M \) and \( F_{SO}(X_q) \cong \Gamma_q \backslash G \). Recall that \( m_{BMS}^{\Gamma} \) is left \( \Gamma \)-invariant, so in particular it is left \( \Gamma_q \)-invariant. Thus it descends to the measure \( m_{q}^{BMS} \) on \( \Gamma_q \backslash G \) which, by right \( M \)-invariance, descends once more to the measure \( m_{\mathfrak{m}}^{BMS} \) on \( \Gamma_q \backslash G/M \) by abuse of notation. We call both of these the congruence BMS measures. Note that \( m_{\mathfrak{m}}^{BMS}(\Gamma_q \backslash G/M) = \# F_q \cdot m_{\mathfrak{m}}^{BMS}(\Gamma \backslash G/M) \). Let \( p_q : T^1(X_q) \to T^1(X) \) be the locally isometric covering map and \( \Omega_q = p_q^{-1}(\Omega) = \text{supp}(m_{\mathfrak{m}}^{BMS}) \subset \Gamma_q \backslash G/M \). Now we need a Markov section on \( \Omega_q \) compatible with the Markov section on \( \Omega \). For all \( x \in \mathcal{A} \), for all \( g \in F_q \), we define \( R_{x,g}^q = gR_x \subset T^1(X_q) \) and similarly define \( U_{x,g}^q \) and \( S_{x,g}^q \). We also define \( R^q = \bigcup_{x \in \mathcal{A}, g \in F_q} R_{x,g}^q \) and similarly define \( U^q \) and their corresponding cores. To facilitate notation, we make the identification \( R \times F_q \cong R^q \) via the isometry \( (u, g) \mapsto g\pi \) and also similar identifications for \( U^q \) and their corresponding cores. Using this identification, define the measure \( \nu_{R^q} \) on \( R^q \) to be the pushforward of \( \nu_R \times m_{F_q} \) where \( m_{F_q} \) is simply the counting measure on \( F_q \). Note that \( \nu_{R^q}(R^q) = \# F_q \). It can be checked that 

\[
\mathcal{R}^q = \{ R_{x,g}^q : x \in \mathcal{A}, g \in F_q \}
\]

is a Markov section of size \( \hat{\delta} \). We have the natural first return map \( \mathcal{P}_q : \mathcal{R}^q \to \mathcal{R}^q \) defined by \( \mathcal{P}_q(u, g) = (\mathcal{P}(u), gc_q(u)) \) for all \( (u, g) \in \mathcal{R}^q \). It is easy to check that the corresponding first return time map is \( \tau_q = \rho_q^2(\tau) = \tau \circ p_q \). Note that \( \tau_q \) is independent of the \( F_q \) component. Like \( R^q \) and the measure \( \nu^q \), we define \( \mathcal{R}^{q, \tau} \) and the measure \( \nu^{q, \tau} \) in a similar fashion. Again by abuse of notation, we have the measure \( \nu^{q, \tau} \) on \( T^1(X_q) \) supported on \( \Omega_q \). Note that \( \nu^{q, \tau}(T^1(X_q)) = \nu_{R^q}(\tau_q) = \# F_q \cdot \nu_{R^q}(\tau) \) and so together with Theorem 2.5, we have 

\[
\frac{\nu^{q, \tau}}{\# F_q \cdot \nu_{R^q}(\tau)} = \frac{m_{BMS}^{\Gamma_q}}{\# F_q \cdot \nu_{R^q}(\tau)}.
\]

Finally, \( \mathcal{P}_q \) induces the shift map \( \sigma_q : \mathcal{U}^q \to \mathcal{U}^q \) defined by \( \sigma_q(u, g) = (\sigma(u), gc_q(u)) \) for all \( (u, g) \in \mathcal{U}^q \).

**Definition 2.12.** For all \( \xi \in \mathbb{C} \), for all nontrivial ideals \( q \subset \mathcal{O}_K \), we define the congruence transfer operator \( \mathcal{M}_{\xi \tau_q, q} : C(\mathcal{U}^q, \mathbb{C}) \to C(\mathcal{U}^q, \mathbb{C}) \) by

\[
\mathcal{M}_{\xi \tau_q, q}(h)(u, g) = \sum_{(u', g') \in \sigma_{q, \xi}^{-1}(u, g)} e^{\xi \tau_q(u', g')} h(u', g')
\]

\[
= \sum_{u' \in \sigma_{q, \xi}^{-1}(u)} e^{\xi \tau_q(u', gc_q(u')^{-1})} h(u', gc_q(u')^{-1})
\]

for all \( (u, g) \in \text{int}(\mathcal{U}^q) \), and then extend continuously to a function on \( \mathcal{U}^q \), for all \( h \in C(\mathcal{U}^q, \mathbb{C}) \).

**Remark.** As before, for all \( \xi \in \mathbb{C} \), for all nontrivial ideals \( q \subset \mathcal{O}_K \), the congruence transfer operator \( \mathcal{M}_{\xi \tau_q, q} \) preserves \( \mathcal{C}^{\text{Lip}(\delta)}(\mathcal{U}^q, \mathbb{C}) \).

Let \( q \subset \mathcal{O}_K \) be a nontrivial ideal. We make the identification \( C(\mathcal{U}^q, \mathbb{C}) \cong C(U, L^2(F_q, \mathbb{C})) \). Note that for all \( u \in U \), we have the induced left \( \mathbb{C}[F_q] \)-module automorphism on \( L^2(F_q, \mathbb{C}) \) given by \( (c_q(u)^{-1})\phi)(g) = \phi(gc_q(u)^{-1}) \) for all \( g \in F_q \), for all \( \phi \in L^2(F_q, \mathbb{C}) \) (where \( L^2(F_q, \mathbb{C}) \) is viewed as a module using the left regular representation). Again for normalization, for all \( \xi = a + ib \in \mathbb{C} \), we define \( \mathcal{M}_{\xi, q} : \)
\[ C(U, L^2(F_q, \mathbb{C})) \to C(U, L^2(F_q, \mathbb{C})) \]

\[
\mathcal{M}_{\xi,q}(H)(u) = \sum_{u' \in \sigma^{-1}(u)} e^{(f(a)^{+} + ib\tau)(u')} c_q(u')^{-1} H(u') \\
= \frac{1}{\lambda_a h_0(u)} \sum_{u' \in \sigma^{-1}(u)} e^{-(a + \delta - ib)\tau(u')} c_q(u')^{-1}(h_0H)(u') \tag{3}
\]

and for all \( k \in \mathbb{Z}_{>0} \), its \( k \)-th iteration is

\[
\mathcal{M}_{\xi,q}^k(H)(u) = \sum_{u' \in \sigma^{-k}(u)} e^{(f(a)^{+} + ib\tau)(u')} c_q(u')^{-1} H(u')
\]

for all \( u \in \text{int}(U) \), and then extend continuously to a function on \( U \), for all \( H \in C(U, L^2(F_q, \mathbb{C})) \).

**Remark.** As before, since \((c_q \circ \zeta^+)|_{\Sigma^+}\) is Lipschitz, there is a unique Lipschitz extension \( c_{q,\Sigma^+} : \Sigma^+ \to F_q \) using the discrete metric on \( F_q \) which we again note is distinct from \( c_q \circ \zeta^+ \). Then replacing \( \sigma_U \) by \( \sigma_{\Sigma^+}, \tau \) by \( \tau_{\Sigma^+}, f(a) \) by \( f_{\Sigma^+} \), \( c_q \) by \( c_{q,\Sigma^+} \), and using \( h_0 \in C^{\text{Lip}(d_A)}(\Sigma^+, \mathbb{R}) \) in Eq. (3), we can similarly define, by abuse of notation, a corresponding normalized operator \( \mathcal{M}_{\xi,q} : C(\Sigma^+, L^2(F_q, \mathbb{C})) \to C(\Sigma^+, L^2(F_q, \mathbb{C})) \) for all \( \xi \in \mathbb{C} \). Then \( \mathcal{M}_{\xi,q} \circ (\zeta^+)^* = (\zeta^+)^* \circ \mathcal{M}_{\xi,q} \) for all \( \xi \in \mathbb{C} \) similar to the final remark after Definition 2.8. Again we leave it to the reader to make the appropriate adjustments to define the operators on \( B(U, L^2(F_q, \mathbb{C})) \) as in Definition 2.8.

**Remark.** As above, for all \( u \in R \), the cocycle \( c(u)^{-1} \) acts on \( L^2(F_q, \mathbb{C}) \) by a left \( \mathbb{C}[F_q] \)-module automorphism and following definitions we see that the action is the same as that of \( c_q(u)^{-1} \). This justifies dropping the subscript of the cocycle in the definition of the transfer operator above whenever required. We prefer to keep the subscript in the Section 3 so that the cocycle takes values in a finite group and we prefer to drop the subscript in Section 4 where it is unnecessary.

### 2.6 Main technical theorem.

We introduce some inner products, norms and seminorms. Let \( q \subset O_q \) be a nontrivial ideal and \( H, H_1, H_2 \in C(U, L^2(F_q, \mathbb{C})) \). Let \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote the usual \( L^2 \) inner product and norm on any appropriate space respectively. Similarly, let \( \| \cdot \|_{\infty} \) denote the usual \( L^\infty \) norm (i.e., essential supremum norm) on any appropriate space. In particular, we have

\[
\langle H_1, H_2 \rangle = \int_U (H_1(u), H_2(u)) \, d\nu_U(u) = \int_U \sum_{g \in F_q} H_1(u)(g) \cdot \overline{H_2(u)(g)} \, d\nu_U(u)
\]

\[
\| H \|_2 = \left( \int_U \| H(u) \|^2 \, d\nu_U(u) \right)^{\frac{1}{2}}
\]

and \( \| H \|_{\infty} = \sup_{u \in U} \| H(u) \|_2 \) since \( H \in C(U, L^2(F_q, \mathbb{C})) \subset L^2(U, L^2(F_q, \mathbb{C})) \). For convenience we will also denote \( \| H \| \in C(U, L^2(F_q, \mathbb{R})) \) and \( \| H \| \in C(U, \mathbb{R}) \) to be the functions defined by \( \| H \|(u) = \| H(u) \| \in L^2(F_q, \mathbb{R}) \) and \( \| H \|(u) = \| H(u) \|_2 \) for all \( u \in U \), respectively. We use the notations \( \text{Lip}_{d_A}(H) = \text{Lip}_{d_A}(H \circ \zeta^+) \) for the Lipschitz seminorm and \( \| H \|_{\text{Lip}(d_A)} = \| H \circ \zeta^+ \|_{\text{Lip}(d_A)} = \| H \|_{\infty} + \text{Lip}_{d_A}(H) \) for the Lipschitz norm. Similarly, we use the notations

\[
\text{Lip}_d(H) = \sup \left\{ \frac{\| H(u) - H(u') \|_2}{d(u, u')} : u, u' \in U \text{ such that } u \neq u' \right\}
\]
for the Lipschitz seminorm and \( \|H\|_{\text{Lip}(d)} = \|H\|_{\infty} + \text{Lip}_d(H) \) for the Lipschitz norm. We generalize this to another useful norm denoted by

\[
\|H\|_{1,b} = \|H\|_{\infty} + \frac{1}{\max(1, |b|)} \text{Lip}_d(H).
\]

We denote any operator norms simply by \( \| \cdot \|_{\text{op}} \).

**Remark.** We have \( \|H\|_2 \leq \|H\|_{\infty} \) and \( \|H\|_{\text{Lip}(d)} \leq C_\theta \|H\|_{\text{Lip}(d)} \) for some fixed \( C_\theta > 0 \).

For all nontrivial ideals \( q \subset \mathcal{O}_q \), we define

\[ \mathcal{V}_q = C^{\text{Lip}(d)}(U, L^2(F_q, \mathbb{C})) = \{ H \in C(U, L^2(F_q, \mathbb{C})) : \|H\|_{\text{Lip}(d)} < \infty \} \]

and when \( q \) is coprime to \( q_0 \), we can similarly define

\[ \mathcal{W}_q = C^{\text{Lip}(d)}(U, L^2(\tilde{G}_q, \mathbb{C})) \subset \mathcal{V}_q \]

where \( L^2_0(\tilde{G}_q, \mathbb{C}) = \{ \phi \in L^2(\tilde{G}_q, \mathbb{C}) : \sum_{g \in \mathcal{G}_q} \phi(g) = 0 \} \).

For all ideals \( q \subset \mathcal{O}_q \), we denote the ideal norm by \( N_\mathcal{G}(q) = \#(\mathcal{O}_q/q) \) and we say \( q \) is square free if it is a nontrivial proper ideal without any square prime ideal factors. Now we can state Theorem 2.13 which is the main technical theorem regarding spectral bounds. The theorem is proved in Sections 3 and 4 as a simple consequence of Theorems 3.1 and 4.1.

**Theorem 2.13.** There exist \( \eta > 0, C \geq 1, a_0 > 0 \) and a nontrivial proper ideal \( q'_0 \subset \mathcal{O}_q \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( b \leq b_0 \), for all square free ideals \( q \subset \mathcal{O}_q \) coprime to \( q_0 q'_0 \), for all integers \( k \in \mathbb{Z}_{\geq 0} \), for all \( H \in \mathcal{W}_q \), we have

\[
\left\| \mathcal{M}^{k}\xi,q(H) \right\|_2 \leq C N_\mathcal{G}(q)^{C} e^{-nk} \|H\|_{1,b}.
\]

### 3. Spectral bounds for small \( |b| \) using expander machinery

In this section, we prove Theorem 3.1. We fix \( b_0 > 0 \) to be the one from Theorem 4.1 where if we examine the proof of Theorem 4.2 it is clear that we can assume \( b_0 = 1 \).

**Theorem 3.1.** There exist \( \eta > 0, C \geq 1, a_0 > 0 \) and a nontrivial proper ideal \( q'_0 \subset \mathcal{O}_q \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( b \leq b_0 \), for all square free ideals \( q \subset \mathcal{O}_q \) coprime to \( q_0 q'_0 \), for all integers \( k \in \mathbb{Z}_{\geq 0} \), for all \( H \in \mathcal{W}_q \), we have

\[
\left\| \mathcal{M}^{k}\xi,q(H) \right\|_2 \leq C N_\mathcal{G}(q)^{C} e^{-nk} \|H\|_{\text{Lip}(d)}.
\]

To prove this, we first make some reductions as in [OW16].

#### 3.1. Reductions.

Let \( q, q' \subset \mathcal{O}_q \) be ideals such that \( q \subset q' \). We have the canonical quotient map \( \pi_{q,q'} : \tilde{G}(\mathcal{O}_q/q) \to \tilde{G}(\mathcal{O}_q/q') \) which induces the pull back \( \pi_{q,q'}^* : L^2(\tilde{G}(\mathcal{O}_q/q'), \mathbb{C}) \to L^2(\tilde{G}(\mathcal{O}_q/q), \mathbb{C}) \). Define \( \tilde{E}_q^a = \pi_{q,q'}^*(L^2(\tilde{G}(\mathcal{O}_q/q'), \mathbb{C})) \subset L^2(\tilde{G}(\mathcal{O}_q/q), \mathbb{C}) \). Define \( \tilde{E}_q^a = \tilde{E}_q^a \cap \left( \bigoplus_{q' \subset q''} \tilde{E}_{q''}^a \right) \). Then, for all ideals \( q \subset \mathcal{O}_q \), we have the orthogonal decomposition

\[
L^2(\tilde{G}(\mathcal{O}_q/q), \mathbb{C}) = \bigoplus_{q \subset q' \subset \mathcal{O}_q} \tilde{E}_q^a.
\]
Remark. We exclude $q' = \mathcal{O}_K$ above because the subspace $\dot{E}^q_{\mathcal{O}_K} \subset L^2(\tilde{\mathcal{G}}(\mathcal{O}_K/q), \mathbb{C})$ consists of constant functions.

Similarly, using the same procedure with the induced quotient map $\pi_{q,q'} : \tilde{G}_q \to \tilde{G}_{q'}$ for ideals $q, q' \subset \mathcal{O}_K$ such that $q \subset q'$, we can obtain a similar orthogonal decomposition

$$L^2_0(\tilde{G}_q, \mathbb{C}) = \bigoplus_{q \subset q' \subset \mathcal{O}_K} E^q_{q'}$$

for all ideals $q \subset \mathcal{O}_K$. Again let $q, q' \subset \mathcal{O}_K$ be ideals such that $q \subset q'$. For all $\phi \in L^2(\tilde{G}_q, \mathbb{C})$, define $\phi \in L^2(\tilde{G}(\mathcal{O}_K/q), \mathbb{C})$ by $\phi(g) = \phi(\{e, -e\}g)$ for all $g \in \tilde{G}(\mathcal{O}_K/q)$. Then we can describe $E^q_{q'}$ by

$$E^q_{q'} = \{ \phi \in L^2(\tilde{G}_q, \mathbb{C}) : \phi \in \dot{E}^q_{q'} \}.$$ 

In the relevant case when $q$ is coprime to $q_0$, the subspace $\mathbb{E}^q_{q'} \subset L^2(\tilde{G}_q, \mathbb{C})$ can also be thought of as consisting of “new functions” invariant under $G_q$ but not invariant under $G_{q'}$ for any $q' \subsetneq q'$, using the isomorphism $\pi_{q,q'} : F_q \to \tilde{G}_q$. Continuing the case when $q$ is coprime to $q_0$, we define $\mathbb{W}^q_{q'} = \{ H \in \mathbb{W}_q : H(u) \in \mathbb{E}^q_{q'} \text{ for all } u \in U \}$, so that we have the orthogonal decomposition

$$\mathbb{W}_q = \bigoplus_{q \subset q' \subset \mathcal{O}_K} \mathbb{W}^q_{q'}.$$ 

Let $q, q' \subset \mathcal{O}_K$ be ideals coprime to $q_0$ such that $q \subset q'$. We have the canonical projection operator $e_{q,q'} : \mathbb{W}_q \to \mathbb{W}^q_{q'}$ and we can define the canonical projection map $\text{proj}_{q,q'} = (\pi_{q,q'})^{-1} |_{E^q_{q'}} : E^q_{q'} \to E^q_{q'}$ since $\pi_{q,q'}$ is injective and $E^q_{q'} \subset \pi_{q,q'}^*(L^2(\tilde{G}_q', \mathbb{C}))$. This is equivalent to defining $(\text{proj}_{q,q'}(F))(g) = F(\tilde{g})$ for all $g \in \tilde{G}_q'$, for all $F \in \mathbb{E}^q_{q'}$, where $\tilde{g}$ is any lift of $g$ with respect to quotient map $\pi_{q,q'} : \tilde{G}_q \to \tilde{G}_{q'}$. We use the same notation $\text{proj}_{q,q'} : \mathbb{W}^q_{q'} \to \mathbb{W}^q_{q'}$ for the induced projection map defined pointwise. The congruence transfer operator $\mathcal{M}_{\xi,q}$ preserves $\mathbb{W}^q_{q'}$ for all $\xi \subset \mathbb{C}$. The projection operator commutes with the congruence transfer operator, i.e., $e_{q,q'} \circ \mathcal{M}_{\xi,q} = \mathcal{M}_{\xi,q} \circ e_{q,q'}$ for all $\xi \subset \mathbb{C}$, and the projection map is equivariant with respect to the congruence transfer operator, i.e., $\text{proj}_{q,q'} \circ \mathcal{M}_{\xi,q} = \mathcal{M}_{\xi,q} \circ \text{proj}_{q,q'}$ for all $\xi \subset \mathbb{C}$. By surjectivity of $\pi_{q,q'}$, we can denote $\blacktriangle_{q,q'} = \#(\ker(\pi_{q,q'})) = \#(\dot{G}_q \setminus \pi_{q,q'} \dot{G}_{q'})$ and by direct calculation it can be checked that $\|H\|_2 = \sqrt{\dot{\triangle}_{q,q'} \|\text{proj}_{q,q'}(H)\|_2}$ and $\|H\|_{\text{Lip}(d)} = \sqrt{\dot{\triangle}_{q,q'} \|\text{proj}_{q,q'}(H)\|_{\text{Lip}(d)}}$.

**Theorem 3.2.** There exist $\eta > 0, C \geq 1, \eta > 0, a_0 > 0$ and $q_0 \in \mathbb{Z}_{>0}$ such that for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| < b_0$, for all square free ideals $q \subset \mathcal{O}_K$ coprime to $q_0$ with $N_K(q) > q_1$, for all integers $k \in \mathbb{Z}_{>0}$, for all $H \in \mathbb{W}^q_{q'}$, we have

$$\|\mathcal{M}_{\xi,q}^k(H)\|_2 \leq C N_K(q) e^{-\eta k} \|H\|_{\text{Lip}(d)}.$$ 

**Proof that Theorem 3.2 implies Theorem 3.1.** Fix $\eta > 0, C \geq 1, a_0 > 0$ and $q_1 \in \mathbb{Z}_{>0}$ to be the $\eta, C, a_0$ and $q_1$ from Theorem 3.2 respectively. Fix $C = C_1 + 1/2$. Set $q_0' \subset \mathcal{O}_K$ to be the product of all nontrivial prime ideals $p \subset \mathcal{O}_K$ with $N_K(p) \leq q_1$ so that if $q \subset \mathcal{O}_K$ is an ideal coprime to $q_0q_0'$ and $q \subset q' \subset \mathcal{O}_K$ is a proper ideal, then $N_K(q') > q_1$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| < b_0$. Let $q \subset \mathcal{O}_K$ be an
ideal coprime to \(q_0q_k\), \(k \in \mathbb{Z}_{>0}\) and \(H \in W_q\). We have

\[
\|M_{\xi,q}^k(H)\|_2^2 = \sum_{q < q' \subseteq \mathcal{O}_k} \|\epsilon_{q,q'}(M_{\xi,q}^k(H))\|_2^2
\]

\[
= \sum_{q < q' \subseteq \mathcal{O}_k} \|\epsilon_{q,q'}(\mathcal{M}_{q,q'}(M_{\xi,q}^k(H)))\|_2^2
\]

\[
= \sum_{q < q' \subseteq \mathcal{O}_k} \|\epsilon_{q,q'}(M_{\xi,q}^k(H))\|_2^2.
\]

Since \(\text{proj}_{q,q'}(\epsilon_{q,q'}(H)) \in W_q^q\), we can now apply Theorem 3.2 to get

\[
\|M_{\xi,q}^k(H)\|_2^2 \leq \sum_{q < q' \subseteq \mathcal{O}_k} C_1^2 N\,(q')^{2C_1} e^{-2\eta k} \|\epsilon_{q,q'}(H)\|_{\text{Lip}(\delta)}^2
\]

\[
= \sum_{q < q' \subseteq \mathcal{O}_k} C_1^2 N\,(q')^{2C_1} e^{-2\eta k} \|\epsilon_{q,q'}(H)\|_{\text{Lip}(\delta)}^2
\]

\[
\leq C_1^2 N\,(q)\,e^{-2\eta k} \|H\|_{\text{Lip}(\delta)}^2 \sum_{q < q' \subseteq \mathcal{O}_k} 1
\]

\[
\leq C^2 N\,(q)\,e^{-2\eta k} \|H\|_{\text{Lip}(\delta)}^2.
\]

\[\square\]

**Theorem 3.3.** There exist \(C_s > 0\), \(a_0 > 0\) and \(\kappa \in (0,1)\) and \(q_1 \in \mathbb{Z}_{>0}\) such that for all \(\xi = a + ib \in \mathbb{C}\) with \(|a| < a_0\) and \(|b| \leq b_0\), for all square free ideals \(q \subset \mathcal{O}_k\) coprime to \(q_0\) with \(N\,(q) > q_1\), there exists an integer \(s \in (0, C_s \log(N\,(q)))\) such that for all integers \(j \in \mathbb{Z}_{>0}\), for all \(H \in W_q^q\), we have

\[
\|M_{\xi,q}^j(H)\|_2 \leq N\,(q)^{-j}s \|H\|_{\text{Lip}(\delta)}.
\]

**Proof that Theorem 3.3 implies Theorem 3.2.** Fix \(C_s, a_0 > 0\) and \(\kappa \in (0,1)\) and \(q_1 \in \mathbb{Z}_{>0}\) to be the ones from Theorem 3.3. Fix

\[
C_M = \max \left(0, \sup_{|\Re(z)| \leq a_0, (0) \subseteq q \subseteq \mathcal{O}_k} \log \|M_{\xi,q}\|_{\text{op}} \right)
\]

\[
\leq \max \left(0, \sup_{|\Re(z)| \leq a_0} \log \|\mathcal{C}_z\|_{\text{op}} \right) \leq \max(0, \log(N\,(q_0)^s))
\]

where we use operator norms for operators on \(L^2(U, L^2(F_q, \mathcal{C}))\) and \(L^2(U, \mathbb{R})\) respectively. Fix \(C = \max(C_s, C_0) + 1\), \(C_0 \geq 1\) and \(\eta = \frac{C}{C_0} > 0\). Let \(\xi = a + ib \in \mathbb{C}\) with \(|a| < a_0\) and \(|b| \leq b_0\). Let \(q \subset \mathcal{O}_k\) be an ideal coprime to \(q_0\) with \(N\,(q) > q_1\) and \(s \in (0, C_s \log(N\,(q)))\) be the corresponding integer provided by Theorem 3.3. Then \(\|M_{\xi,q}^m\| \leq N\,(q)^{m}\,C_M^{s}\) for all integers \(0 \leq m < s\). Let \(k \in \mathbb{Z}_{>0}\) and \(H \in W_q^q\). Writing \(k = js + m\) for some integers \(j \in \mathbb{Z}_{>0}\) and \(0 \leq m < s\), and using
Theorem 3.3, we have
\[ \left\| M_{\xi,q}^t(H) \right\|_2 \leq \left\| M_{\xi,q}^m \cdot M_{s,\xi,q}^s(H) \right\|_2 \]
\[ \leq N_\xi(q)^{CM}C_sN_\xi(q)^{-j_0\kappa}\left\| H \right\|_{\text{Lip}(d_\theta)} \]
\[ \leq N_\xi(q)^{CM}C_s\frac{\log(N_\xi(q))}{\epsilon} e^{-(j_0+s+m)\theta} \left\| H \right\|_{\text{Lip}(d_\theta)} \]
\[ \leq N_\xi(q)^{CM}C_s^{1+1}e^{-\epsilon\theta} \left\| H \right\|_{\text{Lip}(d_\theta)} \]
\[ = CN_\xi(q)^C e^{-nk} \left\| H \right\|_{\text{Lip}(d)} \].

The rest of the section is devoted to obtaining strong bounds which are crucial for the proof of Theorem 3.3 in Subsection 3.5.

3.2. Approximating the transfer operator. The aim of this subsection is to approximate the transfer operator by a convolution with a measure as in [OW16].

Let \( q \in O _{\xi} \) be an ideal coprime to \( q_0 \). Recall the property \( M_{\xi,q} \circ (\zeta ^+)^* = (\zeta ^+)^* \circ M_{\xi,q} \) for all \( \xi \in \mathbb{C} \) from a remark in Subsection 2.5, and also recall that \( \zeta ^+ \) is surjective. In light of these observations, without losing any information we can abuse notation slightly to interpret \( M_{\xi,q}(H) \in C(\Sigma ^+, L^2(\tilde{G}_q, \mathbb{C})) \) for any \( H \in C(U, L^2(\tilde{G}_q, \mathbb{C})) \) as the operator \( M_{\xi,q} : C(\Sigma ^+, L^2(\tilde{G}_q, \mathbb{C})) \to C(\Sigma ^+, L^2(\tilde{G}_q, \mathbb{C})) \) applied to the pullback \( H \in C(\Sigma ^+, L^2(\tilde{G}_q, \mathbb{C})) \) with the map \((\zeta ^+)^*\) suppressed. We will do this throughout the rest of this section. From the same remark, recall the map \( c_{q,\Sigma ^+} : \Sigma ^+ \to \tilde{G}_q \). By Corollary 2.11.1, the map can be described simply by \( c_{q,\Sigma ^+}(x) = c_q(u) \) for all \( x \in \Sigma ^+ \), for any choice of \( u \in \mathbb{C}[x_0, x_1] \) and hence we see that there is dependence only on the first two entries. Thus it makes sense to introduce the notation \( c_{q,\Sigma ^+}(j, k) \) for all admissible pairs \((j, k)\) in the natural way. In particular, with this notation we have
\[ c_{q,\Sigma ^+}^k(x) = c_{q,\Sigma ^+}(x_0, x_1)c_{q,\Sigma ^+}(x_1, x_2)\cdots c_{q,\Sigma ^+}(x_{k-1}, x_k) \]
for all \( x \in \Sigma ^+ \). For the rest of this section we will henceforth drop the subscript \( \Sigma ^+ \) in \( \tau_{\Sigma ^+}, f_{\Sigma ^+}^{(a)} \) and \( c_{q,\Sigma ^+} \) but we reiterate that it should not be confused with \( \tau \circ \zeta ^+ \), \( f_{\Sigma ^+}^{(a)} \circ \zeta ^+ \) and \( c_q \circ \zeta ^+ \) respectively.

We introduce some notations for admissible sequences to save space. We define \( \alpha ^j = (\alpha_1, \alpha_{j-1}, \ldots, \alpha_1) \) for all admissible sequences \((\alpha_j, \alpha_{j-1}, \ldots, \alpha_1)\), for all \( j \in \mathbb{Z}_{>0} \). Also, when sequences are themselves written in a sequence, they are to be concatenated. For all \( y \in A \), denote \( \omega(y) \in \Sigma ^+ \) to be any sequence such that \((y, \omega(y))\) is admissible. We extend the notation naturally to admissible sequences as well so that \( \omega(\alpha^j) = \omega(\alpha_1) \) for all admissible sequences \( \alpha^j \), for all \( j \in \mathbb{Z}_{>0} \). For the rest of the section, any written sums over sequences will be understood to mean sums only over admissible sequences.

Let \( q \in O _{\xi} \) be an ideal coprime to \( q_0 \). For any measure \( \mu \) on \( \tilde{G}_q \) and \( \phi \in L^2(\tilde{G}_q, \mathbb{C}) \) the convolution \( \mu \ast \phi \in L^2(\tilde{G}_q, \mathbb{C}) \) is defined by
\[ (\mu \ast \phi)(g) = \sum_{h \in \tilde{G}_q} \mu(h)\phi(gh^{-1}) \]
for all \( g \in \tilde{G}_q \).

Now we present a bound which will be used often.
Lemma 3.4. There exists $C > 1$ such that for all $|a| < a'_0$, for all $x \in \Sigma^+$, for all $k \in \mathbb{Z}_{>0}$, we have
\[ \sum_{\alpha^k} e^{f(a) (\alpha^k, x)} \leq C. \]

Proof. Fix $C = e^{A_f a'_0} > 1$. Let $|a| < a'_0$. Then $|f(a) - f(0)| \leq A_f |a| < A_f a'_0$ and so we have
\[ \sum_{\alpha^k} e^{f(a) (\alpha^k, x)} \leq e^{A_f a'_0} \sum_{\alpha^k} e^{f(0) (\alpha^k, x)} = e^{A_f a'_0} L^k(\chi_U) = e^{A_f a'_0} = C. \]

We will denote the constant $C$ provided by Lemma 3.4 by $C_f$.

For all $\xi = a + ib \in \mathbb{C}$, for all ideals $q \subset \mathcal{O}_K$ coprime to $q_0$, for all $x \in \Sigma^+$, for all integers $0 < r < s$, for all admissible sequences $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$, we define the measures
\[
\mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{\xi, x} = \sum_{\alpha'} e^{f(a) + ibr_x}(\alpha', x) \delta_{q_{r+1}}^{r+1}(\alpha_{r+1}, \alpha', x) \\
\nu_0^{a, q, x, r} = \sum_{\alpha'} e^{f(a) + b \alpha'} \delta_{q_{r+1}}^{r+1}(\alpha_{r+1}, \alpha', x) \\
\mu^{a, q, x} = \left( \sum_{\alpha'} e^{f(a) + b \alpha'} \delta_{q_{r+1}}^{r+1}(\alpha_{r+1}, \alpha', x) \right) \nu^{a, q, x, r}_0 \\
\nu^{a, q, x} = e^{f(a)}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \nu_0^{a, q, x, r}
\]
on $\tilde{G}_q$ and also for all $H \in C(U, L^2(\tilde{G}_q, \mathbb{C}))$, define the function
\[
\varphi_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{b, H} = e^{-r} \delta_{q_{r+1}}(\alpha_{r+1}, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))) \ast H(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))
\]
in $L^2(\tilde{G}_q, \mathbb{C})$ where we note that $\|\varphi_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{b, H}\| \leq \|H\|_{\infty}$. Before we proceed, we record an easy lemma which relate the measures defined above.

Lemma 3.5. There exists $C > 0$ such that for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a'_0$, for all ideals $q \subset \mathcal{O}_K$ coprime to $q_0$, for all $x \in \Sigma^+$, for all integers $0 < r < s$, for all admissible sequences $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$, we have $|\mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{\xi, x} \leq \mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x}$, and moreover we have both $\mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x} \leq C \mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x}$ and $\nu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x} \leq C \nu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x}$.

Proof. Fix $C = e^{\frac{\varphi_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x}}{e^{A_f a'_0}}}$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a'_0$, $q \subset \mathcal{O}_K$ be an ideal coprime to $q_0$, $x \in \Sigma^+$, $0 < r < s$ be integers and $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$ be an admissible sequence. Denote $\mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{\xi, x}$ by $\mu$, $\mu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x}$ by $\bar{\mu}$, and $\nu_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}^{a, q, x}$ by $\nu$. It is an easy computation to check $f_{x-r}^{a}(\alpha^s, x) = f_{x-r}^{a}(\alpha^s, x) + f_r^{a}(\alpha^r, x)$ and the first inequality of the lemma $|\mu| \leq \bar{\mu}$ from definitions. We also
have
\[ \left| f_s^{(a)}(\alpha^s, x) - f_{s-r}^{(a)}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) + f_r^{(a)}(\alpha^r, x) \right| \]
\[ = \left| f_s^{(a)}(\alpha^s, x) - f_{s-r}^{(a)}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \right| \]
\[ \leq \sum_{k=0}^{s-r-1} \left| f(a)(\sigma^k(\alpha^s, x)) - f(a)(\sigma^k(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \right| \]
\[ \leq \sum_{k=0}^{s-r-1} \text{Lip}_{d_\theta}(f(a)) \cdot d_\sigma(\sigma^k(\alpha^s, x), \sigma^k(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \]
\[ \leq \text{Lip}_{d_\theta}(f(a)) \sum_{k=0}^{s-r-1} \theta^{s-r-k} \]
\[ \leq \frac{T_0 \theta}{1 - \theta}. \]

Hence the rest of the lemma follows by comparing \( \hat{\mu} \) and \( \nu \) and using the above computed estimate. \( \blacksquare \)

Now we return to our goal of approximating the transfer operator.

**Lemma 3.6.** For all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0' \), for all ideals \( q \subset \mathcal{O}_K \) coprime to \( q_0 \), for all \( x \in \Sigma^+ \), for all integers \( 0 < r < s \), for all \( H \in \mathcal{V}_q \), we have

\[ \left\| M^s_{\xi, q}(H)(x) - \sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_s} \mu^{(a, q, x)}_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})} \ast \phi^{q, H}_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})} \right\|_2 \]
\[ \leq C_f \text{Lip}_{d_{\theta}(H)} \theta^{s-r}. \]

**Proof.** Let \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0' \), \( q \subset \mathcal{O}_K \) be an ideal coprime to \( q_0 \), \( x \in \Sigma^+ \), \( 0 < r < s \) be integers and \( H \in \mathcal{V}_q \). We calculate that

\[ \sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_s} \mu^{(a, q, x)}_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})} \ast \phi^{q, H}_{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})} \]
\[ = \sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_s} \left( \sum_{\alpha'} e^{(f(\alpha^s)+ib\tau_r)(\alpha^r, x) (\delta_{q}^{s-r-1}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))} \right) \]
\[ \ast (\delta_{q}^{s-r-1}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \ast H(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \]
\[ = \sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_s} \left( \sum_{\alpha'} e^{(f(\alpha^s)+ib\tau_r)(\alpha^r, x) (\delta_{q}^{s-r-1}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))} \right) \]
\[ \ast (\delta_{q}^{s-r-1}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \ast H(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) \]
\[ = \sum_{\alpha'} e^{(f(\alpha^s)+ib\tau_r)(\alpha^r, x) (\delta_{q}^{s-r-1}(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))} \ast H(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})). \]
Thus
\[
\left\| M_{x,q}^{c}(H)(x) - \sum_{\alpha_{r+1},\alpha_{r+2},\ldots,\alpha_{s}} \mu_{(\alpha_{r},\alpha_{r+1},\ldots,\alpha_{r+1})} \ast \phi_{(\alpha_{s},\alpha_{s-1},\ldots,\alpha_{r+1})} \right\|_{2}
\]
\[
= \left\| \sum_{\alpha} e^{(f_{\alpha}^{*}+ib\tau_{\alpha})(\alpha,x)} \delta_{\alpha}^{*}(\alpha,x) \ast H(\alpha^{*},x) \right\|_{2}
\]
\[
= \left\| \sum_{\alpha} e^{(f_{\alpha}^{*}+ib\tau_{\alpha})(\alpha,x)} \delta_{\alpha}^{*}(\alpha,x) \ast H(\alpha^{*},x) - H(\alpha^{*},x) \right\|_{2}
\]
\[
\leq \sum_{\alpha} \left\| e^{(f_{\alpha}^{*}+ib\tau_{\alpha})(\alpha,x)} \delta_{\alpha}^{*}(\alpha,x) \ast (H(\alpha^{*},x) - H(\alpha^{*},x)) \right\|_{2}
\]
\[
\leq \sum_{\alpha} e^{(f_{\alpha}^{*})(\alpha,x)} \text{Lip}_{d_{p}}(H) \cdot d_{\phi}(\alpha^{*},x,\alpha^{*},x,\alpha^{*},x,\alpha^{*},x,\alpha^{*},x,\alpha^{*},x,\alpha^{*},x,\alpha^{*})
\]
\[
\leq C_{f} \text{Lip}_{d_{p}}(H)\theta^{s-r}
\]
where we used the fact that convolutions with $\delta_{\alpha}^{*}(\alpha,x)$ preserves the norm and also Lemma 3.4. \(\Box\)

We will use this approximation to study the convolution, rather than dealing with the transfer operator directly, and obtain strong bounds. This is the objective of Subsection 3.4 but first we need to establish an important fact in Subsection 3.3 which will be used in Subsection 3.4.

### 3.3. Zariski density of $\bar{H}^{p}(y,z)$ in $\tilde{G}$

In this subsection, we prove Lemma 3.7 which will be required to use the expander machinery of Golsefidy-Varjú [GV12].

For all $(y,z) \in A^{2}$, define $H^{p}(y,z) = (S^{p}(y,z)) < \Gamma < G$ where
\[
S^{p}(y,z) = \left\{ \prod_{j=0}^{p} c(\alpha_{p+1-j},\alpha_{p-j}) \prod_{j=0}^{p} c(\tilde{\alpha}_{1+j},\tilde{\alpha}_{j})^{-1} : \alpha_{p+1} = \tilde{\alpha}_{p+1} = y, \alpha_{0} = \tilde{\alpha}_{0} = z \right\}.
\]

Define $\bar{S}^{p}(y,z) = \tilde{\pi}^{-1}(S^{p}(y,z)) \subset \tilde{\Gamma} < \tilde{G}$ and $\bar{H}^{p}(y,z) = (\bar{S}^{p}(y,z)) < \tilde{\Gamma} < \tilde{G}$. Since ker($\tilde{\pi}$) = \{e, $\epsilon$\}, we note that $\bar{H}^{p}(y,z) = \tilde{\pi}^{-1}(H^{p}(y,z))$.

**Lemma 3.7.** There exists $p_{0} \in \mathbb{Z}_{>0}$ such that for all integers $p > p_{0}$, for all $(y,z) \in A^{2}$, the subgroup $\bar{H}^{p}(y,z)$ is Zariski dense in $\tilde{G}$.

To prove this, we first make some reductions.

**Lemma 3.8.** For all $(y,z) \in A^{2}$, there exists $p_{0} \in \mathbb{Z}_{>0}$ such that for all integers $p > p_{0}$, the subgroup $H^{p}(y,z)$ is Zariski dense in $G$.

**Proof that Lemma 3.8 implies Lemma 3.7.** Let $(y,z) \in A^{2}$. By Lemma 3.8, there is a $p_{0}(y,z) \in \mathbb{Z}_{>0}$ such that for all integers $p > p_{0}(y,z)$, the subgroup $H^{p}(y,z) < G$ is Zariski dense in $G$. If $\bar{H}^{p}(y,z)$ is not Zariski dense in $G$, then consider $\bar{\pi}(\bar{H}^{p}(y,z)) \subset G$ which is the image of a proper subvariety of $\tilde{G}$ and hence must be a Zariski constructible set. But it cannot contain any Zariski open subset of $G$.
for dimensional reasons. Thus it is contained in a finite union of proper subvarieties of $G$ which contradicts the Zariski density of $H^p(y,z)$ in $G$. Now we can simply choose $p_0 = \max_{(y,z) \in A^2} p(y,z)$. □

Through an isometry, we will now view the hyperbolic space in the upper half space model $\mathbb{H}^n \cong \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ with boundary at infinity $\partial_{\infty}(\mathbb{H}^n) \cong \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\} \cup \{\infty\} \cong \mathbb{R}^{n-1} \cup \{\infty\}$ and we also use the isometry $T(\mathbb{H}^n) \cong \mathbb{H}^n \times \mathbb{R}^n$. By a slight abuse of notation, sometimes we will make the distinction and sometimes we will make the identification which will always be clear from context. Let $(e_1, e_2, \ldots, e_n)$ be the standard basis on $\mathbb{R}^n$. Let $B_{\mathbb{E}}^\mathbb{E}(u) \subset \mathbb{R}^{n-1}$ denote the open Euclidean ball of radius $\epsilon > 0$ centered at $u \in \mathbb{R}^{n-1}$. Let $d_{\mathbb{E}}$ denote the Euclidean distance. By a sphere $\hat{V} \subset \partial_{\infty}(\mathbb{H}^n)$ we mean that $\hat{V} \subset \mathbb{H}^n \cup \{\infty\}$ is a $(n-2)$-dimensional sphere or an affine hyperplane when we use the upper half space model. Let $\pi_1 : \mathbb{H}^n \times \mathbb{R}^n \rightarrow \mathbb{H}^n$ be the tangent bundle projection map and $\pi_2 : \mathbb{H}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection onto the directional component. For convenience, we denote $\overline{\Omega} \subset T^1(\mathbb{H}^n)$ to be the full preimage of $\Omega$ under the covering map $T^1(\mathbb{H}^n) \rightarrow T^1(X)$. For all $j \in A$, denote by the $\overline{R}_j, \overline{U}_j$ and $\mathcal{S}_j$ the cores of $\overline{R}_j, \overline{U}_j$ and $\mathcal{S}_j$ respectively.

**Lemma 3.9.** Let $(y, z) \in A^2$. Applying any appropriate isometry, we assume that the vectors in $\overline{U}_y$ have direction $\pi_2(\overline{U}_y) = -e_n$ and their basepoints lie on the hyperplane $\langle \pi_1(\overline{U}_y), e_n \rangle = 1$. Then for all $\epsilon > 0$, there exists $p_0 \in \mathbb{Z}_{>0}$ such that for all integers $p > p_0$, if $H^p(y,z)$ is not Zariski dense in $G$, then for all $u \in \overline{U}_y$, we have that $\Lambda(H^p(y,z))$ is contained in a 0-dimensional or $(n-2)$-dimensional sphere $\hat{V} \subset \partial_{\infty}(\mathbb{H}^n)$ such that $\hat{V} \cap B_{\mathbb{E}}^\mathbb{E}(u^+) \neq \emptyset$.

**Proof that Lemma 3.9 implies Lemma 3.8.** Suppose Lemma 3.8 is false. Then the set

$$P_{NZ}(y,z) = \{p \in \mathbb{Z}_{>0} : H^p(y,z) \text{ is not Zariski dense in } G\}$$

is unbounded for some $(y, z) \in A^2$. Apply any appropriate isometry to assume $\pi_2(\overline{U}_y) = -e_n$ and $\langle \pi_1(\overline{U}_y), e_n \rangle = 1$. By properness of $\overline{U}_y$, there are $w_0 \in \overline{\Omega}$ and $\delta_1 > 0$ such that $W_0^{\mathbb{E}}(w_0) \cap \overline{\Omega} \subset \overline{U}_y$. Consider the diffeomorphism $\Phi : W_0^{\mathbb{E}}(w_0) \rightarrow B_{\mathbb{E}}^\mathbb{E}(w^+_1)$ defined by $\Phi(w) = w^+$ for all $w \in W_0^{\mathbb{E}}(w_0)$. Since $G$ is Zariski dense in $G$, recalling that the set of attracting fixed points of hyperbolic elements of $G$ is dense in $\Lambda(G)$, we use the contrapositive of [Win15, Proposition 3.12] to conclude that $\Lambda(G) \cap B_{\mathbb{E}}^\mathbb{E}(w^+_1) \subset \mathbb{R}^{n-1}$ is not contained in any $(n-2)$-dimensional sphere in $\partial_{\infty}(\mathbb{H}^n)$. Thus there is a set of distinct points $\{u^+_1, u^+_2, \ldots, u^+_{n+1}\} \subset \Lambda(G) \cap B_{\mathbb{E}}^\mathbb{E}(w^+_1)$ such that it is not contained in any $(n-2)$-dimensional sphere in $\partial_{\infty}(\mathbb{H}^n)$. But $\{(w_1, w_2, \ldots, w_{n+1}) \in (\mathbb{R}^{n-1})^{n+1} : \{w_1, w_2, \ldots, w_{n+1}\} \text{ is not contained in any } (n-2)\text{-dimensional sphere in } \partial_{\infty}(\mathbb{H}^n)\}$ is an open subset. Hence there is an $\epsilon > 0$ such that if $w_j \subset \Lambda(G) \cap B_{\mathbb{E}}^\mathbb{E}(w^+_1) \neq \emptyset$ for all integers $1 \leq j \leq n + 1$, then $\{w_1, w_2, \ldots, w_{n+1}\}$ is also not contained in any $(n-2)$-dimensional sphere in $\partial_{\infty}(\mathbb{H}^n)$. Now there are corresponding vectors $u_j = \Phi^{-1}(w^+_j) \in \overline{U}_y$ for all integers $1 \leq j \leq n + 1$. Lemma 3.9 provides a corresponding $p_0 \in \mathbb{Z}_{>0}$ for this $\epsilon$. Since $P_{NZ}(y,z)$ is unbounded, we can fix some $p \in P_{NZ}(y,z)$ with $p > p_0$ so that $H^p(y,z)$ is not
Zariski dense in $G$. Then Lemma 3.9 implies that $\Lambda(H^p(y,z))$ is contained in a 0-dimensional or $(n-2)$-dimensional sphere $\hat{V} \subset \partial_{\infty}(\mathbb{H}^n)$ such that $\hat{V} \cap B^y(u_2^+) \neq \emptyset$ for all integers $1 \leq j \leq n + 1$ which is a contradiction. 

Lemma 3.9 is a consequence of the following lemmas put together.

**Lemma 3.10.** Let $(y, z) \in \mathbb{A}^2$. If $u_0 \in \overline{R}_y$ such that $\mathcal{P}^{p+1}(u_0) \in g_z\overline{R}_z$ for some $g_z \in \Gamma$ and $v_0 \in g_z\overline{R}_z$ such that $\mathcal{P}^{-(p+1)}(v_0) \in h\overline{R}_y$ for some $h \in \Gamma$, then in fact $h \in S^p(y, z) \subset H^p(y, z)$.

**Proof.** Let $(y, z) \in \mathbb{A}^2$. Suppose $u_0 \in \overline{R}_y$ such that $\mathcal{P}^{p+1}(u_0) \in g_z\overline{R}_z$ for some $g_z \in \Gamma$ and $v_0 \in g_z\overline{R}_z$ such that $\mathcal{P}^{-(p+1)}(v_0) \in h\overline{R}_y$ for some $h \in \Gamma$. Let $u'_0 = h^{-1}\mathcal{P}^{-(p+1)}(v_0) \in \overline{R}_y$. Then $\mathcal{P}^{p+1}(u'_0) \in g'_z\overline{R}_z$ where $g'_z = h^{-1}g_z$. Now by definitions, there are admissible sequences $(\alpha_{p+1}, \alpha_p, \ldots, \alpha_0)$ and $(\hat{\alpha}_{p+1}, \hat{\alpha}_p, \ldots, \hat{\alpha}_0)$ with $\alpha_{p+1} = \hat{\alpha}_{p+1} = y$ and $\alpha_0 = \hat{\alpha}_0 = z$ where $g_z = c^{p+1}(\alpha_{p+1}, \alpha_p, \ldots, \alpha_0) = \prod_{j=0}^{p} c(\alpha_{p+1-j}, \alpha_{p-j})$ and $g'_z = c^{p+1}(\hat{\alpha}_{p+1}, \hat{\alpha}_p, \ldots, \hat{\alpha}_0) = \prod_{j=0}^{p} c(\hat{\alpha}_{p+1-j}, \hat{\alpha}_{p-j})$. Thus $h = g_zg'_z^{-1} = \prod_{j=0}^{p} c(\alpha_{p+1-j}, \alpha_{p-j}) \prod_{j=0}^{p} c(\hat{\alpha}_{p+1-j}, \hat{\alpha}_{p-j})^{-1}$ where $\alpha_{p+1} = \hat{\alpha}_{p+1} = y$ and $\alpha_0 = \hat{\alpha}_0 = z$ which shows that $h \in S^p(y, z) \subset H^p(y, z)$ by definition.

We will use the above procedure to produce elements of $S^p(y, z) \subset H^p(y, z)$.

**Lemma 3.11.** Let $(y, z) \in \mathbb{A}^2$. For all $\epsilon > 0$, there exists $p_1 \in \mathbb{Z}_{> 0}$ such that for all integers $p > p_1$, for all $u \in \overline{U}_y$, there exists $u_0 \in \overline{U}_y$ with $d_{su}(u, u_0) < \epsilon$ such that $\mathcal{P}^{p+1}(u_0) \in \Gamma\overline{R}_z$.

**Proof.** Fix $c_0, \lambda > 0$, $\epsilon_0$ to be the constants $c, \log(\lambda), q$ respectively from the Anosov property in [Rat73]. Recall the trajectory isomorphism $\psi$ from [Rat73]. Also recall $\tau = \inf_{u' \in R} \tau(u') = \inf_{u' \in \mathbb{R}} \tau(u')$. Let $(y, z) \in \mathbb{A}^2$. Let $\epsilon > 0$. Fix an integer $p_1 \geq N_T$ such that $c_02\delta e^{-\lambda_0 z(p+1-N_T)} < \frac{\epsilon}{2}$ and let $p > p_1$ be an integer. Let $u \in \overline{U}_y$. Since $\overline{U}_y \subset \mathbb{U}_y$ is a residual set, there is a $u_1 \in \mathbb{U}_y$ with $d_{su}(u_1, u) < \frac{\epsilon}{2}$. Consider $u_2 = \mathcal{P}^{p+1-N_T}(u_1) \in g_y\overline{R}_y'$ for some $g_y' \in \Gamma$. Since $T$ is topologically mixing and $g_y\overline{R}_y' \subset g_y\overline{R}_y$ is a residual set, there is a $u_3 \in g_y\overline{R}_y'$ such that $v = \mathcal{P}^{N_T}(u_3) \in g_z\overline{R}_z$ for some $g_z \in \Gamma$. Let $u_0 = \mathcal{P}^{-(p+1-N_T)}((u_3, u_2)) \in \mathcal{P}^{-(p+1-N_T)}([g_y\overline{R}_y', u_2]) \subset [\overline{U}_y, u_1] = \overline{U}_y$.

Denote $w = [u_0, u_2] \in \mathbb{S}_y'$ where $w_0 \in \overline{R}_y'$ is the center. Then $\mathcal{P}^{p+1}(u_0) = (\mathcal{P}^{N_T} \circ \mathcal{P}^{p+1-N_T})(\mathcal{P}^{-(p+1-N_T)}((u_3, u_2))) = \mathcal{P}^{N_T}((u_3, u_2)) \in \mathcal{P}^{N_T}([u_3, g_y'\mathbb{S}_y']) \subset [v, g_z\mathbb{S}_z] \subset g_z\overline{R}_z$

as desired. Also

\[ d_{su}(u_1, u_0) < c_0e^{-\lambda_0 z(p+1-N_T)}d_{su}(u_2, \psi_{u_2}^{-1}([u_3, u_2])) \]

\[ < c_02\delta e^{-\lambda_0 z(p+1-N_T)}d_{su}(\psi_{w}^{-1}(u_2), \psi_{w}^{-1}([u_3, u_2])) \]

\[ < c_02\delta e^{-\lambda_0 z(p+1-N_T)} < \frac{\epsilon}{2} \]

since $p > p_1$. This implies $d_{su}(u, u_0) < \epsilon$. See the construction of Markov sections in [Rat73] for more details for the bounds. 

\[ \Box \]
Lemma 3.12. Let \( (y, z) \in \mathcal{A}^2 \). There exist \( B > 0 \) and \( p_2 \in \mathbb{Z}_{>0} \) such that for all \( g \in \Gamma \), for all \( v \in g \mathcal{R}_z \), for all integers \( p > p_2 \), the following holds. Applying any appropriate isometry, we assume that \( v = (e_n, -e_n) \). Then there exists \( v_0 \in [v, g \tilde{Z}_z] \) such that

1. \( v_0^r \in B_{\mathcal{B}}^E(0) \)
2. \( \mathcal{P}^{-(p+1)}(v_0) \in \Gamma \mathcal{R}_y \).

Proof. Let \( (y, z) \in \mathcal{A}^2 \). Let \( w_0 \in \mathcal{R}_z \) be the center. Fix some distinct \( w_1, w_2 \in \mathcal{Y}_z \neq \emptyset \) which is possible using the definition of \( \mathcal{Y}_z \) and the fact that \( \Lambda(\Gamma) \) is a nonempty perfect set (since \( \Gamma \) is nonelementary). Let \( \delta_1 = \frac{1}{2} d_{\mathcal{B}}(w_1, w_2) > 0 \). Now we make a modification to ensure that Property 2 of the lemma holds. By a similar argument as in Lemma 3.11 with minor modifications, we conclude that there is a \( p_2 \in \mathbb{Z}_{>0} \) such that for any chosen \( p > p_2 \), for all \( j \in \{1, 2\} \), there is a \( \tilde{w}_j \in \mathcal{Y}_z \) with

\[
d_{\mathcal{B}}(\tilde{w}_j, w_j) < \delta_1 \quad \text{such that} \quad \mathcal{P}^{-(p+1)}(\tilde{w}_j) \in \Gamma \mathcal{R}_y.
\]

Now suppose \( v \in g \mathcal{R}_z \) for some \( g \in \Gamma \). If \( [w_0, g^{-1} v] \in W_{\mathcal{B}}^E(\tilde{v}_2) \), then set \( \tilde{v}_0 = \tilde{w}_2 \) and otherwise set \( \tilde{v}_0 = \tilde{w}_1 \). Then \( d_{\mathcal{B}}([w_0, g^{-1} v], \tilde{v}_0) > \delta_1 \) and also \( \mathcal{P}^{-(p+1)}(\tilde{v}_0) \in \Gamma \mathcal{R}_y \). Set \( v_0 = [v, g \tilde{v}_0] \in [v, g \mathcal{R}_z] \). Now by left \( G \)-invariance we have \( d_{\mathcal{B}}([g w_0, v], g \tilde{v}_0) > \delta_1 \) and we need to extend the bound to obtain a uniform lower bound for \( d_{\mathcal{B}}(v, v_0) \). Using the homeomorphism \( \mathcal{U}_z \times \mathcal{Y}_z \to \mathcal{R}_z \) defined by \( (v', w') \to (v', w') \) and the compactness of the sets \( \mathcal{U}_z \) and \( \mathcal{Y}_z \), there is a \( \delta \in (0, \delta_1) \) such that

\[
\inf \{ d_{\mathcal{B}}([v', u], [v', u']) : v' \in \mathcal{R}_z \text{ and } u, u' \in \mathcal{Y}_z \text{ with } d_{\mathcal{B}}(u, u') \geq \delta_1 \} > \delta
\]

where \( \delta \) only depends on the rectangle \( \mathcal{R}_z \). This implies that \( d_{\mathcal{B}}(v, v_0) > \delta \). We can also check that

\[
\mathcal{P}^{-(p+1)}(v_0) = \mathcal{P}^{-(p+1)}([v, g \tilde{v}_0]) \in \mathcal{P}^{-(p+1)}([g \mathcal{U}_z, g \tilde{v}_0]) \subset \Gamma \mathcal{R}_y
\]

by the Markov property since \( \mathcal{P}^{-(p+1)}(g \tilde{v}_0) \in \Gamma \mathcal{R}_y \). So Property 2 still holds.

Apply any appropriate isometry to assume \( v = (e_n, -e_n) \). Then \( v_0 \in \mathcal{A}_{\delta, \delta}^\mathcal{B}(v) = W_{\mathcal{B}}^E(v) \setminus \mathcal{W}_{\mathcal{B}}^E(v) \). Note that by homogeneity of \( \mathcal{X}^\mathcal{B} \) and compactness of \( \mathcal{A}_{\delta, \delta}^\mathcal{B}(v) \subset \mathcal{X}^\mathcal{B} \times \mathbb{R}^n \), the distance function \( d_{\mathcal{B}} |_{\mathcal{A}_{\delta, \delta}^\mathcal{B}(v)} \) and the Euclidean distance function \( d_{\mathcal{B}} |_{\mathcal{A}_{\delta, \delta}^\mathcal{B}(v)} \) (measuring the distance between basepoints of tangent vectors along the horosphere) are equivalent, where the implied constant is uniform in \( g \in \Gamma \) and \( v \in g \mathcal{R}_z \) (since we are assuming \( v = (e_n, -e_n) \)). Since \( v^- = \infty \), there is a diffeomorphism \( \Phi : \mathcal{A}_{\delta, \delta}^\mathcal{B}(v) \to \mathcal{A}_{\delta, \delta}^B(0) \) for some \( 0 < \delta_2 < B \) defined by \( \Phi(w) = w^- \) for all \( w \in \mathcal{A}_{\delta, \delta}^\mathcal{B}(v) \), where \( \mathcal{A}_{\delta, \delta}^B(0) = \mathcal{B}_{\mathcal{B}}^E(0) \setminus \mathcal{B}_{\mathcal{B}}^E(0) \subset \mathbb{R}^{-1} \) and \( B \) only depends on \( \delta \). Thus \( v_0^r \in \mathcal{B}_{\mathcal{B}}^E(0) \) which is Property 1.

In the proof of Lemma 3.13, we will also denote by \( \mathcal{B}_{\mathcal{B}}^E(u) \subset \mathbb{R}^n \) the open Euclidean ball of radius \( \epsilon > 0 \) centered at \( u \in \mathbb{R}^n \) where we will explicitly show the containment to be clear and otherwise the notation will mean the open Euclidean ball in \( \mathbb{R}^{-1} \) as defined before.

Lemma 3.13. Let \( (y, z) \in \mathcal{A}^2 \). For all \( \epsilon > 0 \), there exists \( p_3 \in \mathbb{Z}_{>0} \) such that the following holds. Suppose that \( H^p(y, z) \) is not Zariski dense in \( \mathcal{G} \) for some integer
Suppose further that there exists \( u_0 \in \hat{\mathbb{R}}_y \) such that \( v = \mathcal{P}^{p+1}(u_0) \in \Gamma \hat{\mathbb{R}}_z \). Applying any appropriate isometry, we assume that \( u_0 = (e_n, -e_n) \). Then \( \Lambda(H^p(y, z)) \) is contained in a 0-dimensional or \((n - 2)\)-dimensional sphere \( \hat{V} \subset \partial_{\infty}(\mathbb{H}^n) \) such that \( \hat{V} \cap B^E_{\epsilon}(0) \neq \emptyset \).

**Proof.** First we fix some notations and constants. Let \((y, z) \in A^2 \). Without loss of generality, let \( \epsilon \in (0, 1) \). We will fix a \( \epsilon_{1,1} > 0 \) which will be required for the case when \( n > 2 \) and \( H^p(y, z) \) is elementary and a \( \epsilon_{1,2} > 0 \) for the other cases as follows. Let \( C = \partial_{\infty}(\mathbb{H}^n) \setminus B^E_{\epsilon}(0) \). Define Banana\((c_1, c_2) \subset \mathbb{H}^n \) to be the unique banana neighborhood of the geodesic with limit points \( c_1, c_2 \in \partial_{\infty}(\mathbb{H}^n) \) such that \( e_n \in \partial(\text{Banana}(c_1, c_2)) \). Consider the function \( \varphi : C \times C \to \mathbb{R} \) defined by

\[
\varphi(c_1, c_2) = \inf \left\{ u_n \in \mathbb{R} : (u_1, u_2, \ldots, u_n) \in \partial(\text{Banana}(c_1, c_2)) \cap \left( B^E_{\epsilon}(0) \times \mathbb{R} \right) \right\}
\]

for all \((c_1, c_2) \in C \times C \). Fix \( \epsilon_{1,1}' = \inf_{(c_1, c_2) \in C \times C} \varphi(c_1, c_2) \). Then \( \epsilon_{1,1}' > 0 \) because \( C \times C \) is compact and \( \varphi \) is positive continuous. Fix \( \epsilon_{1,1} = \min(\frac{1}{2}, \epsilon_{1,1}') \). Now fix

\[
\epsilon_{1,2} = \min \left( \frac{1}{2}, \frac{\epsilon}{3}, \frac{\epsilon}{12}, \frac{\epsilon}{\sqrt{6}} \right)
\]

where the bounds will become clear later. Fix \( \epsilon_1 = \min(\epsilon_{1,1}, \epsilon_{1,2}) \). Fix \( B > 0 \) and \( p_2 \in \mathbb{Z} > 0 \) to be the constants provided by Lemma 3.12. Fix \( L_0 > 0 \) such that \( \frac{L_0}{m} < \frac{B}{2} \) and a \( \epsilon_2 \in (0, B) \).

Suppose \( H^p(y, z) \) is not Zariski dense in \( G \) for some \( p \in \mathbb{Z} > 0 \). Suppose \( u_0 \in \hat{\mathbb{R}}_y \) such that \( v = \mathcal{P}^{p+1}(u_0) \in g_z\hat{\mathbb{R}}_z \) for some \( g_z \in \Gamma \) and apply any appropriate isometry to assume \( u_0 = (e_n, -e_n) \). First consider the case that \( H^p(y, z) \) is elementary with the limit set \( \Lambda(H^p(y, z)) = \hat{V} = \{V_1, V_2\} \). For all \( h' \in H^p(y, z) \), since \( h' \) must be hyperbolic (\( \Gamma \) is torsion-free convex cocompact), it preserves the hypersurface \( V = \partial(\text{Banana}(h'^\perp, h'^+)) = \partial(\text{Banana}(V_1, V_2)) \subset \mathbb{H}^n \) with boundary at infinity \( \partial_{\infty}(V) = \hat{V} \subset \partial_{\infty}(\mathbb{H}^n) \) which is a 0-dimensional sphere. Hence the orbit \( H^p(y, z)\pi_1(u_0) \) is contained in \( \hat{V} \). Now consider the case that \( H^p(y, z) \) is nonelementary, and hence \( \#\Lambda(H^p(y, z)) = \infty \), if \( n > 2 \) (recall that if \( n = 2 \), then being nonelementary is equivalent to being Zariski dense in \( G \)). Since \( \frac{H^p(y, z)}{\mathbb{Z}} < G \) is a nontrivial proper connected subgroup, the Karpelevič-Mostow theorem [DSO09, Kar53, Mos55] implies that \( \frac{H^p(y, z)}{\mathbb{Z}} \), and in particular \( H^p(y, z) \), preserves a proper totally geodesic submanifold which in this case is a \((n - 1)\)-dimensional Euclidean half sphere or a half affine hyperplane in \( \mathbb{H}^n \) perpendicular to \( \mathbb{R}^{n-1} \). Consequently, \( \Lambda(H^p(y, z)) \) is contained in a unique \((n - 2)\)-dimensional sphere \( \hat{V} \subset \partial_{\infty}(\mathbb{H}^n) \). This further implies that \( H^p(y, z) \) preserves any spherical hypersurface or half affine hyperplane in \( \mathbb{H}^n \) with boundary at infinity \( \hat{V} \). In particular, the orbit \( H^p(y, z)\pi_1(u_0) \) is contained in \( V \subset \mathbb{H}^n \) which is the unique spherical hypersurface or half affine hyperplane with the corresponding spherical boundary at infinity \( \partial_{\infty}(V) = \hat{V} \subset \partial_{\infty}(\mathbb{H}^n) \).

For any of the cases, let \( V \subset \mathbb{H}^n \) be the hypersurface containing the orbit \( H^p(y, z)\pi_1(u_0) \) as described above. We will use Lemma 3.12 to find a point of the orbit close to 0 \( \in \mathbb{R}^{n-1} \). In order to apply Lemma 3.12, first suppose \( p > p_2 \) and define the isometry \( \iota \) which simply scales by some factor \( L > 0 \) such that
\( \iota(v) = (e_n, -e_n) \). It is clear that there is a \( p_{3,1} \in \mathbb{Z}_{>0} \) satisfying \( e^{-\frac{1}{L_n}} < \frac{1}{L_n} \), so that for all \( p > p_{3,1} \) we have \( L > L_0 \). Since \( v \in g_2 \tilde{R}_E \) with \( \iota(v) = (e_n, -e_n) \), we can apply Lemma 3.12 to obtain a vector \( v_0 \in g_2 \tilde{S}_E \) which satisfies the properties listed in the lemma. Since \( \pi_1(\tilde{R}_g) \) is contained in some hyperbolic ball, so there is a \( \delta_1 \in (0, \varepsilon_0) \) such that if \( d_E(g_0 \pi_1(\tilde{R}_g), \partial_\infty(H^n)) < \delta_1 \) for some \( g \in G \), then \( g \pi_1(\tilde{R}_g) \subset B_{\varepsilon_0/3}(c_g) \subset \mathbb{R}^n \) for some \( c_g \in \mathbb{R}^n \). Note that \( \tilde{A}_{\delta, \tilde{B}}(\iota(v)) \) is compact and the geodesic flow is smooth and \( \Phi \left( \tilde{A}_{\delta, \tilde{B}}(\iota(v)) \right) = \tilde{A}_{\delta, \tilde{B}}(0) \), where we use the notations from the proof of Lemma 3.12. Hence there is a \( p_{3,2} \in \mathbb{Z}_{>0} \) such that if \( p > p_{3,2} \), then \( d_E(\iota(V), \iota(v_0)) < \delta_1 \). Fix \( p_3 = \max(p_2, p_{3,1}, p_{3,2}) \) and assume \( p > p_3 \) henceforth. But recalling that \( \iota(V) \in h_1(\tilde{R}_g) \) for some \( h \in H^p(y, z) \) by Property 2 of Lemma 3.12 and Lemma 3.10, we have \( h_1(\tilde{R}_g) \subset B_{\varepsilon_0/3}(c_{h_1}) \subset B_{\varepsilon_0/3}(\iota(v_0)) \subset \mathbb{R}^n \) for some \( c_{h_1} \in \mathbb{R}^n \). In particular \( h_1(\iota(u_0)) \in B_{\varepsilon_0/3}(\iota(v_0)) \cap \iota(V) \subset \mathbb{R}^n \) with nonempty intersection. By Property 1 of Lemma 3.12, we have \( v_0 = \iota(V) \), so rescaling \( \iota^{-1} \) gives \( v_0 \in B_{\varepsilon_0/3}(0) \) and \( h_1(\iota(u_0)) \in B_{\varepsilon_0/3}(v_0) \cap V \subset \mathbb{R}^n \) which implies \( h_1(\iota(u_0)) \in B_{\varepsilon_0/3}(0) \cap V \subset \mathbb{R}^n \). For ease of notation, let \( w = (\hat{w}, \hat{v}_n) \) and write \( w = (\hat{w}, \hat{v}_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \).

Now consider the case that \( H^p(y, z) \) is elementary so that \( V \) is the hypersurface \( \partial(\text{Banana}(V_1, V_2)) \). If \( V_1, V_2 \notin B_{\varepsilon_0/3}(0) \), then the definition of \( c_{1,1} \) gives a contradiction to \( w \in B_{\varepsilon_0/3}(0) \cap V \subset \mathbb{R}^n \). Hence either \( V_1 \in B_{\varepsilon_0/3}(0) \) or \( V_2 \in B_{\varepsilon_0/3}(0) \) which proves the lemma in this case.

Now consider the case that \( H^p(y, z) \) is nonelementary if \( n > 2 \), and \( V = S_{E_1}^{-1}(c, r_{u_0}) \cap \mathbb{H}^n \) is a spherical hypersurface with center \( c = (\tilde{c}, c_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \) and radius \( r_{u_0} > 0 \) and with corresponding spherical boundary at infinity \( \partial_\infty(V) = \tilde{V} = S_{E_1}^{-1}(\tilde{c}, r_{\tilde{V}}) \) with center \( \tilde{c} \) and radius \( r_{\tilde{V}} > 0 \). Recalling that \( \iota_1(u_0) = e_n = (0, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \), we have \( e_n, w \in S_{E_1}^{-1}(c, r_{u_0}) \) with \( \|w\| < c_1 \) and \( S_{E_1}^{-1}(c, r_{u_0}) \cap \mathbb{R}^{n-1} = \tilde{V} = S_{E_1}^{-1}(\tilde{c}, r_{\tilde{V}}) \). Now, there is a point \( k \in \tilde{V} \) closest to \( 0 \in \mathbb{R}^{n-1} \), i.e., satisfying \( d_E(k, 0) = \inf_{k \in \tilde{V}} d_E(k', 0) \) and such a point must be on the ray emanating from the center \( \tilde{c} \) passing through \( 0 \in \mathbb{R}^{n-1} \). Thus, using \( r_{u_0} = d_E(c, e_n)^2 = \|\tilde{c}\|^2 + (1 - c_n)^2 \) gives

\[
\begin{align*}
d_E(k, 0) &= d_E(\tilde{c}, k) - d_E(\tilde{c}, 0) = \|\tilde{V} - \|\tilde{c}\| = \sqrt{r_{u_0}^2 - c_n^2} - \|\tilde{c}\| \quad (8) \\
&= \sqrt{\|\tilde{c}\|^2 + (1 - c_n)^2 - c_n^2 - \|\tilde{c}\|} \quad (9) \\
&= \sqrt{\|\tilde{c}\|^2 + 1 - 2c_n - \|\tilde{c}\|} \quad (10).
\end{align*}
\]

Now, to bound the right hand side, we will use the constraint on the center \( c \) that it must lie on the hyperplane which is the perpendicular bisector of the line segment from \( w \) to \( e_n \). This forces the center to be of the form

\[
c = (\tilde{c}, c_n) = \left( \frac{\tilde{w} + w_n}{2} + \frac{\tilde{w}, \tilde{c}}{1 - w_n} - \frac{\|\tilde{w}\|^2}{2(1 - w_n)} \right).
\]
If $\|\tilde{c}\| \geq 1$, then
\[
d_{E}(k, 0) = \frac{|1 - 2c_n|}{\sqrt{\|\tilde{c}\|^2 + 1 - 2c_n + \|\tilde{c}\|}} \leq \frac{|1 - 2c_n|}{\|\tilde{c}\|},
\]
\[
eq \frac{|w_n| + 2\langle \tilde{w}, \tilde{c} \rangle}{\|\tilde{c}\|} - \frac{||\tilde{w}||^2}{\|\tilde{c}\|(1 - w_n)} \leq |w_n| + \frac{2\|\tilde{w}\|}{1 - w_n} + \frac{\|\tilde{w}\|^2}{1 - w_n} < \epsilon_1 + \frac{2\epsilon_1}{1 - \epsilon_1} + \frac{\epsilon_1^2}{1 - \epsilon_1} \leq \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} = \frac{\epsilon^2}{3} < \epsilon
\]
using $\|\tilde{w}\| < \epsilon_1, w_n \in (0, \epsilon_1)$ and the definition of $\epsilon_1$. Now suppose $\|\tilde{c}\| < 1$. Going back to Eq. (10), if either $\|\tilde{c}\| = 0$ or $\|\tilde{c}\|^2 + 1 - 2c_n = 0$, then $d_{E}(k, 0) = \sqrt{1 - 2c_n}$.

Now assuming both $\|\tilde{c}\|^2 + 1 - 2c_n > 0$ and $\|\tilde{c}\| > 0$, we have
\[
d_{E}(k, 0) = \frac{|1 - 2c_n|}{\sqrt{\|\tilde{c}\|^2 + 1 - 2c_n + \|\tilde{c}\|}}.
\]

If $1 - 2c_n = 0$, then trivially $d_{E}(k, 0) = 0$. If $\|\tilde{c}\|^2 > 2c_n - 1 > 0$, then
\[
d_{E}(k, 0) = \frac{|1 - 2c_n|}{\sqrt{\|\tilde{c}\|^2 + 1 - 2c_n + \|\tilde{c}\|}} < \frac{|1 - 2c_n|}{\|\tilde{c}\|} < \frac{|1 - 2c_n|}{\sqrt{2c_n - 1}} = \sqrt{1 - 2c_n}.
\]

If $1 - 2c_n > 0$, then
\[
d_{E}(k, 0) = \frac{|1 - 2c_n|}{\sqrt{\|\tilde{c}\|^2 + 1 - 2c_n + \|\tilde{c}\|}} < \sqrt{\|\tilde{c}\|^2 + 1 - 2c_n} < \sqrt{1 - 2c_n} = \sqrt{1 - 2c_n}.
\]

So in any case,
\[
d_{E}(k, 0)^2 \leq |1 - 2c_n| = |w_n + 2\langle \tilde{w}, \tilde{c} \rangle - \frac{||\tilde{w}||^2}{1 - w_n}| \leq |w_n| + \frac{2\|\tilde{w}\| \cdot ||\tilde{c}||}{1 - w_n} + \frac{\|\tilde{w}\|^2}{1 - w_n}.
\]

Finally, using $\|\tilde{c}\| < 1, \|\tilde{w}\| < \epsilon_1$ and $w_n \in (0, \epsilon_1)$ gives us the bound
\[
d_{E}(k, 0) < \sqrt{\epsilon_1 + \frac{2\epsilon_1}{1 - \epsilon_1} + \frac{\epsilon_1^2}{1 - \epsilon_1} \leq \sqrt{\frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} = \epsilon}
\]
which proves the lemma in this case.

Now consider the case that $H^p(y, z)$ is nonelementary if $n > 2$, and $V \subset \mathbb{H}^n$ is a half affine hyperplane containing both $\pi_1(u_0) = e_n$ and $w = h\pi_1(u_0)$, with corresponding boundary at infinity $\partial_\infty(V) = \hat{V}$ where $\hat{V} \cap \mathbb{R}^{n-1} \subset \mathbb{R}^{n-1}$ is also an affine hyperplane. Any affine hyperplane containing $e_n$ and $w$ must also contain the straight line through $e_n$ and $w$. Hence $\hat{V}$ contains the intersection of that straight line with $\mathbb{R}^{n-1}$ which we call $k$. It is easily to calculate that the point is $k = \frac{1}{1-w_n} \tilde{w}$ and hence
\[
d_{E}(k, 0) = \|k\| = \frac{||\tilde{w}||}{1 - w_n} < \frac{\epsilon_1}{1 - \epsilon_1} < \frac{\epsilon^2}{3} < \epsilon
\]
which proves the lemma in this case.

\textbf{Proof of Lemma 3.9.} Let $(y, z) \in \mathcal{A}^2$. Applying any appropriate isometry, we assume that the vectors in $\mathcal{U}_y$ have direction $\pi_2(\mathcal{U}_y) = -e_n$ and their basepoints lie on the hyperplane $\langle \pi_1(\mathcal{U}_y), e_n \rangle = 1$. Let $\epsilon > 0$. There are $C, \delta > 0$ such that for all $u_1, u_2 \in \mathcal{U}_y$ with $d_{su}(u_1, u_2) < \delta$, we have $d_{su}(u_1, u_2) \leq d_{E}(u_1^*, u_2^*) \leq C d_{su}(u_1, u_2)$. Let $\epsilon' = \min(\frac{\epsilon}{2\delta}, \epsilon)$. Fix $p_1, p_3 \in \mathbb{Z}_{>0}$ to be the constants provided by Lemmas 3.11 and 3.13 corresponding to $\epsilon'$ and $\frac{\epsilon}{2}$ respectively. Fix $p_0 = \max(p_1, p_3)$
and let $p > p_0$ be an integer such that $H^p(y, z)$ is not Zariski dense in $G$. Let $u \in \overline{U}_y$. Using Lemma 3.11, we obtain the vector $u_0 \in \overline{U}_y$ such that $d_{uv}(u_0, u) < \epsilon'$, which implies $d_{E}(u_0^+, u^+) < \frac{\epsilon'}{2}$, and $v = P^{v+1}(u_0) \in \Gamma R_z$. We remark that translations parallel to $\mathbb{R}^{n-1}$ are both hyperbolic and Euclidean isometries and so applying such a translation preserves all hyperbolic and Euclidean properties that we are dealing with. Hence, without loss of generality, we assume that $u_0 = (\epsilon_n, -\epsilon_n)$.

Now the hypotheses of Lemma 3.13 are satisfied and we may apply it to conclude that $\Lambda(H^p(y, z))$ is contained in a 0-dimensional or $(n - 2)$-dimensional sphere $\hat{V} \subset \partial_\infty(\mathbb{H}^n)$ such that $\hat{V} \cap B^{\mathbb{E}}_{\|z\|/2}(u_0^+) \neq \emptyset$ and hence $\hat{V} \cap B^{\mathbb{E}}(u^+) \neq \emptyset$. ■

### 3.4. $L^2$-flattening lemma

In this subsection we fix any $p > p_0$ from Lemma 3.7 so that the lemma applies when it is needed in Lemma 3.18. The aim of this subsection is to prove the following $L^2$-flattening type lemma. The arguments here are expanded on [MOW17, Appendix].

**Lemma 3.14.** There exist $C > 0, C_0 > 0$ and $l \in \mathbb{Z}_{>0}$ such that for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$, for all square free ideals $q \in \mathcal{O}_K$ coprime to $q_0$, for all $x \in \Sigma^n$, for all integers $C_0 \log(N_K(q)) \leq r < s$ with $r \in \mathbb{Z}$, for all admissible sequences $(\alpha_x, \alpha_{x-1}, \ldots, \alpha_{r+1})$, for all $\phi \in E_{q}^\mathbb{R}$ with $\|\phi\|_2 = 1$, we have both

$$\left\|\mu_{(\alpha_x, \alpha_{x-1}, \ldots, \alpha_{r+1})} \ast \phi\right\|_2 \leq CN_K(q)^{-\frac{1}{2}} \left\|\nu_{(\alpha_x, \alpha_{x-1}, \ldots, \alpha_{r+1})}\right\|_1$$

and

$$\left\|\nu_{(\alpha_x, \alpha_{x-1}, \ldots, \alpha_{r+1})} \ast \phi\right\|_2 \leq CN_K(q)^{-\frac{1}{2}} \left\|\mu_{(\alpha_x, \alpha_{x-1}, \ldots, \alpha_{r+1})}\right\|_1.$$

The proof uses two tools. The first is Lemma 3.21 derived from lower bounds of nontrivial irreducible representations of Chevalley groups. The second is the expander machinery of Golsefidy-Varjú [GV12] which we cannot use directly in its raw form but culminates in Lemma 3.20. Due to Lemma 3.5, we focus on $\nu_0^{a, q, x, r}$ and our goal is to use [GV12] to obtain bounds on the operator norm but this requires the measure to be what we call “nearly flat”. Although $\nu_0^{a, q, x, r}$ is not nearly flat, it suffices to estimate $\nu_0^{a, q, x, r}$ by $\nu_1^{a, q, x, r}$ which breaks up into convolutions of nearly flat measures. The following is the procedure to do exactly that.

Let $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ for some $r' \in \mathbb{Z}_{>0}$ and some integer $l > 1$, and $\alpha^r$ be an admissible sequence. To better facilitate manipulations of sequences, we introduce the following additional notations. Define

$$\alpha_j^l = (\alpha_{jl}, \alpha_{jl-1}, \ldots, \alpha_{j(l-1)+1})$$

$$\alpha_j^{(k)_1} = (\alpha_{jl}, \alpha_{jl-1}, \ldots, \alpha_{j(l-k)+1})$$

$$\alpha_j^{(k)_2} = (\alpha_{jl-(l-k)}, \alpha_{jl-(l-k)-1}, \ldots, \alpha_{(j-1)(l+1)}$$

for all integers $1 \leq j \leq r'$ and $1 \leq k \leq l - 1$. For example, with these notations and conventions we have $\alpha^r = (\alpha_j^{r'}, \alpha_j^{r'-1}, \ldots, \alpha_j^1) = (\alpha, \alpha_{r-1}, \ldots, \alpha_1)$ and $\alpha_j^l = (\alpha_j^{(k)_2}, \alpha_j^{(l-k)_2})$ for all integers $1 \leq j \leq r'$. We also have $\sigma^k(\alpha^j) = \alpha^{j-k}$ for all integers $1 \leq j \leq r$ and $0 \leq k \leq j - 1$. 

For all $a \in \mathbb{R}$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, for all admissible sequences $\alpha^r$, we compute that

$$f_r^{(a)}(\alpha^r, x) = \sum_{k=0}^{r-1} f^{(a)}(\alpha^r_k, x) = \sum_{k=0}^{r-1} f^{(a)}(\alpha^{r-k}, x)$$

$$= \sum_{k=0}^{p-1} f^{(a)}(\alpha^{r-j}, x) + \sum_{j=0}^{r'-3} \sum_{l=1}^{l-1} f^{(a)}(\alpha^{r-p-(jl+k)}, x)$$

$$+ \sum_{k=0}^{2l-p-1} f^{(a)}(\alpha^{2l-p-k}, x)$$

$$= \sum_{k=0}^{p-1} f^{(a)}(\alpha^{r-j}, x) + \sum_{j=0}^{r'-3} \sum_{l=0}^{l-1} f^{(a)}(\alpha^{r-p-jl}, x)$$

$$+ \sum_{k=0}^{2l-p-1} f^{(a)}(\alpha^{2l-p}, x)$$

$$= f_p^{(a)}(\alpha^r, x) + \sum_{j=0}^{r'-3} f_l^{(a)}(\alpha^{r-p-jl}, x) + f_{2l-p}^{(a)}(\alpha^{2l-p}, x)$$

$$= f_{2l-p}^{(a)}(\alpha^{2l-p}, x) + \sum_{j=2}^{r'-1} f_l^{(a)}(\alpha^{j+1)(l-p)}, x) + f_p^{(a)}(\alpha^r, x).$$

We can estimate each term in the sum above so that in the $j$th term, we remove dependence on $\alpha_{k_l}^{(p)}$ for all distinct integers $1 \leq j, k \leq r'$.

Remark. Such an estimate is not required for $j = 1$ since the first term does not have any dependence on $\alpha_{k_l}^{(p)}$ for all $2 \leq k \leq r'$.

Lemma 3.15. There exists $C > 0$ such that for all $|a| < a'_0$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, for all admissible sequences $\alpha^r$, we have

$$\left| f_l^{(a)}(\alpha^{(j+1)(l-p)}, x) - f_l^{(a)}(\alpha_j^{(l-p)}, \alpha_j^l, \omega(\alpha_j^l)) \right| \leq C \theta^j$$

for all integers $2 \leq j \leq r' - 1$ and

$$\left| f_p^{(a)}(\alpha^r, x) - f_p^{(a)}(\alpha_r^l, \omega(\alpha_r^l)) \right| \leq C \theta^j.$$
Proof. Fix $C = T_0 \theta^{l-p}$. Let $|a| < a_0'$, $x \in \Sigma^+$, $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, and $\alpha'$ be an admissible sequence. We make the estimate

$$\left| f_{l_1}^{(a)}(\alpha^{(j+1)(l-p)}, x) - f_{l_1}^{(a)}(\alpha_{j+1}^{(l-p)}, \omega(\alpha_j^{l})) \right|$$

$$\leq \left| f_{l_1}^{(a)}(\alpha_{j+1}^{(l-p)}, \alpha_j^{l}, x) - f_{l_1}^{(a)}(\alpha_{j+1}^{(l-p)}, \omega(\alpha_j^{l})) \right|$$

$$\leq \sum_{k=0}^{l-1} \left| f^{(a)}(\sigma^k(\alpha_{j+1}^{(l-p)}, \alpha_j^{l}, \omega(\alpha_j^{l}))) - f^{(a)}(\sigma^k(\alpha_{j+1}^{(l-p)}, \alpha_j^{l}, \omega(\alpha_j^{l}))) \right|$$

$$\leq \sum_{k=0}^{l-1} \text{Lip}_{d_\sigma}(f^{(a)}) \cdot d_\sigma(\sigma^k(\alpha_{j+1}^{(l-p)}, \alpha_j^{l}, \omega(\alpha_j^{l})))$$

$$\leq \text{Lip}_{d_\sigma}(f^{(a)}) \sum_{k=0}^{l-1} \theta^{2l-p-k}$$

$$\leq \frac{T_0 \theta^{l-p+1}}{1 - \theta} = C \theta^l$$

for all integers $2 \leq j \leq r' - 1$. Similarly

$$\left| f_{p_1}^{(a)}(\alpha^{r}, x) - f_{p_1}^{(a)}(\alpha_{r'}^{l}, \omega(\alpha_r^{l})) \right|$$

$$\leq \left| f_{p_1}^{(a)}(\alpha_{r'}, \alpha^{(r'-1)l}, x) - f_{p_1}^{(a)}(\alpha_{r'}, \omega(\alpha_r^{l})) \right|$$

$$\leq \sum_{k=0}^{p-1} \left| f^{(a)}(\sigma^k(\alpha_{r'}, \alpha^{(r'-1)l}, x)) - f^{(a)}(\sigma^k(\alpha_{r'}, \omega(\alpha_r^{l}))) \right|$$

$$\leq \sum_{k=0}^{p-1} \text{Lip}_{d_\sigma}(f^{(a)}) \cdot d_\sigma(\sigma^k(\alpha_{r'}, \alpha^{(r'-1)l}, \omega(\alpha_r^{l})))$$

$$\leq \text{Lip}_{d_\sigma}(f^{(a)}) \sum_{k=0}^{p-1} \theta^{l-k}$$

$$\leq \frac{T_0 \theta^{l-p+1}}{1 - \theta} = C \theta^l.$$

To make sense of the notations in what follows, we make the convention that $\alpha_j^{(l-p)2}$ is the empty sequence for all admissible sequences $\alpha'$, for all $j \in \{0, r' + 1\}$. In light of the calculations and Lemma 3.15 above, for all $a \in \mathbb{R}$, for all ideals $q \subset \mathcal{O}_K$ coprime to $q_0$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, for all integers $0 \leq j \leq r'$, for all admissible sequences $\alpha'$, define the coefficients

$$F_{x,r',j}(\alpha_{j+1}^{(l-p)}, \alpha_j^{l}) = \begin{cases} 
 f_{2l-p}^{(a)}(\alpha^{2l-p}, x), & j = 1 \\
 f_{l_1}^{(a)}(\alpha_{j+1}^{(l-p)}, \alpha_j^{l}, \omega(\alpha_j^{l})), & 2 \leq j \leq r' - 1 \\
 f_{r}^{(a)}(\alpha_j^{l}, \omega(\alpha_j^{l})), & j = r'.
\end{cases}$$
where we show the dependence of the admissible choices of $\alpha_j^{(p)1}$ on $\alpha_j^{(l-p)2}$ and $\alpha_j^{(l-p)2}$ (or more precisely only on the last entry of $\alpha_j^{(l-p)2}$ and the first entry of $\alpha_j^{(l-p)2}$). The measures above satisfy a property as shown in Lemma 3.16 which we call nearly flat.

**Lemma 3.16.** There exists $C > 1$ such that for all $|a| < a_0$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, for all integers $0 \leq j \leq r'$, for all pairs of admissible sequences $(\alpha_j^{(l-p)2}, \alpha_j^{(l-p)2})$ with $(\alpha_j^{(l-p)2}, \alpha_j^{(l-p)2}) = (\alpha_j^{(l-p)2}, \alpha_j^{(l-p)2})$, we have

$$E_{j,(\alpha_j^{(l-p)2}, \alpha_j^{(l-p)2})} \leq C.$$

**Proof.** Fix $C = e^{T_0\left(\frac{\theta}{1 - \theta} + p\right)} > 1$. Let $|a| < a_0$, $x \in \Sigma^+$, $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$. Let $0 \leq j \leq r'$ be an integer and $(\alpha_j^{(l-p)2}, \alpha_j^{(l-p)2})$ and $(\tilde{\alpha}_j^{(l-p)2}, \tilde{\alpha}_j^{(l-p)2})$ be two pairs of admissible sequences with $(\alpha_j^{(l-p)2}, \alpha_j^{(l-p)2}) = (\tilde{\alpha}_j^{(l-p)2}, \tilde{\alpha}_j^{(l-p)2})$. We have the elementary calculation

$$\begin{align*}
\left| f^{(a)}_{2l-p}(\alpha^{2l-p}, x) - f^{(a)}_{2l-p}(\tilde{\alpha}^{2l-p}, x) \right| \\
\leq \sum_{k=0}^{2l-p-1} \left| f^{(a)}(\sigma^k(\alpha^{2l-p}, x)) - f^{(a)}(\sigma^k(\tilde{\alpha}^{2l-p}, x)) \right| \\
\leq \sum_{k=0}^{l-p-1} \text{Lip}_{d_\theta}(f^{(a)}) \cdot d_\theta(\sigma^k(\alpha^{2l-p}, x), \sigma^k(\tilde{\alpha}^{2l-p}, x)) \\
+ \sum_{k=l-p}^{l-1} \text{Lip}_{d_\theta}(f^{(a)}) \cdot d_\theta(\sigma^k(\alpha^{2l-p}, x), \sigma^k(\tilde{\alpha}^{2l-p}, x)) \\
+ \sum_{k=l}^{2l-p-1} \text{Lip}_{d_\theta}(f^{(a)}) \cdot d_\theta(\sigma^k(\alpha^{2l-p}, x), \sigma^k(\tilde{\alpha}^{2l-p}, x)) \\
\leq \text{Lip}_{d_\theta}(f^{(a)}) \left( \sum_{k=0}^{l-p-1} \theta^{l-p-k} + \sum_{k=l-p}^{l-1} 1 + 0 \right) \\
\leq T_0 \left( \frac{\theta}{1 - \theta} + p \right) \\
= \log(C).
\end{align*}$$
Similarly

\[
\left| f_l^{(a)}(\alpha_{j+1}^{(l-p)2}, \alpha_j^l, \omega(\alpha_j^l)) - f_l^{(a)}(\tilde{\alpha}_{j+1}^{(l-p)2}, \tilde{\alpha}_j^l, \omega(\tilde{\alpha}_j^l)) \right|
\]

\[
\leq \sum_{k=0}^{l-1} \left| f_l^{(a)}(\sigma_k(\alpha_{j+1}^{(l-p)2}, \alpha_j^l, \omega(\alpha_j^l))) - f_l^{(a)}(\sigma_k(\tilde{\alpha}_{j+1}^{(l-p)2}, \tilde{\alpha}_j^l, \omega(\tilde{\alpha}_j^l))) \right|
\]

\[
\leq \sum_{k=0}^{l-p-1} \text{Lip}_{d_k}(f_l^{(a)}) \cdot d_0(\sigma_k(\alpha_{j+1}^{(l-p)2}, \alpha_j^l, \omega(\alpha_j^l)), \sigma_k(\tilde{\alpha}_{j+1}^{(l-p)2}, \tilde{\alpha}_j^l, \omega(\tilde{\alpha}_j^l)))
\]

\[
+ \sum_{k=l-p}^{l-1} \text{Lip}_{d_k}(f_l^{(a)}) \cdot d_0(\sigma_k(\alpha_{j+1}^{(l-p)2}, \alpha_j^l, \omega(\alpha_j^l)), \sigma_k(\tilde{\alpha}_{j+1}^{(l-p)2}, \tilde{\alpha}_j^l, \omega(\tilde{\alpha}_j^l)))
\]

\[
= \text{Lip}_{d_k}(f_l^{(a)}) \left( \sum_{k=0}^{l-p-1} \theta^{l-p-k} + \sum_{k=l-p}^{l-1} 1 \right)
\]

\[
\leq T_0 \left( \frac{\theta}{1 - \theta} + p \right)
\]

\[= \log(C).\]

Similarly again

\[
\left| f_p^{(a)}(\alpha_{r'}, \omega(\alpha_{r'})) - f_p^{(a)}(\tilde{\alpha}_{r'}, \omega(\tilde{\alpha}_{r'})) \right|
\]

\[
\leq \sum_{k=0}^{p-1} \left| f_p^{(a)}(\sigma_k(\alpha_{r'}, \omega(\alpha_{r'}))) - f_p^{(a)}(\sigma_k(\tilde{\alpha}_{r'}, \omega(\tilde{\alpha}_{r'}))) \right|
\]

\[
\leq \text{Lip}_{d_k}(f_p^{(a)}) \sum_{k=0}^{p-1} d_0(\sigma_k(\alpha_{r'}, \omega(\alpha_{r'})), \sigma_k(\tilde{\alpha}_{r'}, \omega(\tilde{\alpha}_{r'})))
\]

\[
\leq \text{Lip}_{d_k}(f_p^{(a)}) \sum_{k=0}^{p-1} 1
\]

\[= T_0 p
\]

\[< \log(C).\]

The lemma now follows from definitions. \[\square\]

For all \(a \in \mathbb{R}\), for all ideals \(q \subset \mathcal{O}_K\) coprime to \(q_0\), for all \(x \in \Sigma^+\), for all \(r \in \mathbb{Z}_{>0}\) with factorization \(r = r'l\) with \(l > p\), we also define the measure

\[
\nu^{a,q,x,r,r'}_1 = \sum_{\alpha_1^{(l-p)2}, \alpha_2^{(l-p)2}, \ldots, \alpha_i^{(l-p)2}} \eta^{a,q,x,r,r'}_j \cdot \sigma_{j_1}^{(l-p)2} \cdots \sigma_{j_i}^{(l-p)2}
\]

which in particular consists of convolutions of nearly flat measures. Lemma 3.17 shows that we can estimate \(\nu^{a,q,x,r}_0\) with \(\nu^{a,q,x,r,r'}_1\) up to a multiplicative constant depending on the factorization \(r = r'l\) and vice versa.

**Lemma 3.17.** There exists \(C > 0\) such that for all \(|a| < a_0\), for all ideals \(q \subset \mathcal{O}_K\) coprime to \(q_0\), for all \(x \in \Sigma^+\), for all \(r \in \mathbb{Z}_{>0}\) with factorization \(r = r'l\) with \(l > p\), we have \(\nu^{a,q,x,r}_0 \leq e^{r'C\theta} \nu^{a,q,x,r,r'}_1\) and \(\nu^{a,q,x,r,r'}_1 \leq e^{r'C\theta} \nu^{a,q,x,r}_0\).
Proof. Fix $C > 0$ to be the one from Lemma 3.15. Let $|a| < a_0'$, $q \in \mathcal{O}_K$ be an ideal coprime to $q_0$, $x \in \Sigma^+$, and $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$. Denote $\nu_0^{a,q,r}$ by $\nu_0$, $\nu_1^{a,q,r,r'}$ by $\nu_1$, $\eta_{j,(\alpha^{l-2}_j),\alpha_j^{l-2}(p)}$ by $\eta_{j,(\alpha^{l-2}_j+1),\alpha_j^{l-2}(p)}$, and $E_{j,(\alpha^{l-2}_j+1),\alpha_j^{l-2}(p)}$ by $E_{j,(\alpha^{l-2}_j),\alpha_j^{l-2}(p)}$. First we have

$$\nu_0 = \sum_{\alpha^r} e^{f(a)}(\alpha^r) \delta_{c_{q+1}}(\alpha+1,\alpha^r, x)$$

$$= \sum_{\alpha^r} e^{f(a)}(\alpha^r) + \sum_{j=2}^{r'} f(a)(\alpha^{j-1}l-p, x) + f_p(a)(\alpha^r, x) \delta_{c_{q+1}}(\alpha+1,\alpha^r, x).$$

Now by construction, for fixed sequences $\alpha_1^{l-2}, \alpha_2^{l-2}, \ldots, \alpha_r^{l-2}$, the terms $E_{j,(\alpha^{l-2}_j+1),\alpha_j^{l-2}(p)}$ and

$$c_q^j(\alpha_j, x) = c_q((\alpha_j)^{l-2}, \alpha_j^{l-2}(p), \omega(\alpha_j^{l-2}))$$

depend only on the choice of $\alpha_j^{l-2}$ and not on $\alpha_j^{l-2}$ for all integers $1 \leq j, k \leq r'$. We also note that $c_q(\alpha_1, x)$ depends only on the choice of $\alpha_1^{l-2}$ since $l = p \geq 1$. So we can do the manipulations

$$\nu_1 = \sum_{\alpha_1^{l-2}, \alpha_2^{l-2}, \ldots, \alpha_r^{l-2}} \eta_{j,(\alpha_j^{l-2}+1),\alpha_j^{l-2}(p)} = \sum_{\alpha_1^{l-2}, \alpha_2^{l-2}, \ldots, \alpha_r^{l-2}} \delta_{c_q}(\alpha, x) * \left( \sum_{j=1}^{r'} \prod_{j=1}^{r'} E_{j,(\alpha_j^{l-2}+1),\alpha_j^{l-2}(p)} \delta_{c_q}(\alpha_{j+1}, \alpha^l, x) \right)$$

$$= \sum_{\alpha_1^{l-2}, \alpha_2^{l-2}, \ldots, \alpha_r^{l-2}} \left( \prod_{j=1}^{r'} E_{j,(\alpha_j^{l-2}+1),\alpha_j^{l-2}(p)} \right) \delta_{c_q}(\alpha, x) * \left( \prod_{j=1}^{r'} \delta_{c_q}(\alpha_{j+1}, \alpha^l, x) \right)$$

$$= \sum_{\alpha_1^{l-2}, \alpha_2^{l-2}, \ldots, \alpha_r^{l-2}} \left( \prod_{j=1}^{r'} E_{j,(\alpha_j^{l-2}+1),\alpha_j^{l-2}(p)} \right) \delta_{c_q}(\alpha, x) * \delta_{c_q}(\alpha_{j+1}, \alpha^l, x) * \delta_{c_q}(\alpha_{2j+1}, \alpha_{2l}, x) * \cdots * \delta_{c_q}(\alpha_{r+1}, \alpha^{r'l}, x)$$

Hence the lemma follows by comparing the above two expressions for $\nu_0$ and $\nu_1$ and using Lemma 3.15. 

For all ideals $q \in \mathcal{O}_K$ coprime to $q_0$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, for all integers $0 \leq j \leq r'$, for all pairs of admissible
sequences \((\alpha_{j+1}^{(l-p)_2}, \alpha_j^{l})\) and \((\bar{\alpha}_{j+1}^{(l-p)_2}, \bar{\alpha}_j^{l})\) with \((\alpha_{j+1}^{(l-p)_2}, \alpha_j^{(l-p)_2}) = (\bar{\alpha}_{j+1}^{(l-p)_2}, \bar{\alpha}_j^{(l-p)_2})\), we calculate that
\[
c_q(\alpha_{jl+1}, \alpha^{jl}, x) = c_q(\alpha_{jl+1}, \alpha_{jl})c_q(\alpha_{jl}, \alpha_{jl-1}) \cdots c_q(\alpha_{(j-1)l+2}, \alpha_{(j-1)l+1})
\]
\[
= \prod_{k=0}^{l-1} c_q(\alpha_{jl+1-k}, \alpha_{jl-k})
\]
\[
= \prod_{k=0}^{p} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) \prod_{k=p+1}^{l-1} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}).
\]
We write the product in the final form because
\[
c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) = c_q(\alpha_{jl+1-k}, \alpha_{jl-k})
\]
for all \(p+1 \leq k \leq l-1\). We use this to obtain
\[
c_q(\alpha_{jl+1}, \alpha^{jl}, x)c_q(\bar{\alpha}_{jl+1}, \bar{\alpha}^{jl}, x)^{-1}
\]
\[
= \left( \prod_{k=0}^{p} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) \prod_{k=p+1}^{l-1} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) \right)^{-1}
\]
\[
\cdot \left( \prod_{k=0}^{p} c_q(\bar{\alpha}_{jl+1-k}, \bar{\alpha}_{jl-k}) \prod_{k=p+1}^{l-1} c_q(\bar{\alpha}_{jl+1-k}, \bar{\alpha}_{jl-k}) \right)^{-1}
\]
\[
= \prod_{k=0}^{p} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) \prod_{k=0}^{p} c_q(\bar{\alpha}_{jl+1-k}, \bar{\alpha}_{jl-k})
\]
\[
\cdot \left( \prod_{k=p+1}^{l-1} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) \right)^{-1} \left( \prod_{k=0}^{p} c_q(\bar{\alpha}_{jl+1-k}, \bar{\alpha}_{jl-k}) \right)^{-1}
\]
\[
= \prod_{k=0}^{p} c_q(\alpha_{jl+1-k}, \alpha_{jl-k}) \prod_{k=0}^{p} c_q(\bar{\alpha}_{jl-p+1+k}, \bar{\alpha}_{jl-p+k})^{-1}.
\]

Now by Lemma 3.7, we can use the result of Golsefidy-Varjú [GV12] to obtain the following lemma regarding spectral gap.

**Lemma 3.18.** There exists \(\epsilon \in (0, 1)\) such that for all \(r \in \mathbb{Z}_{>0}\) with factorization \(r = r'^l\) with \(l > p\), for all integers \(0 \leq j \leq r'\), for all pairs of admissible sequences \((\alpha_{j+1}^{(l-p)_2}, \alpha_j^{l})\), for all square free ideals \(\mathfrak{q} \subset \mathcal{O}_K\) coprime to \(q_0\), for all \(\phi \in L^2_0(\hat{\mathcal{G}}, \mathbb{C})\) with \(\|\phi\|_2 = 1\), there exist admissible sequences \((\beta_{jl+1}, \beta_j^{(p)_1}, \beta_{jl-p})\) and \((\tilde{\beta}_{jl+1}, \tilde{\beta}_j^{(p)_1}, \tilde{\beta}_{jl-p})\) with \(\beta_{jl+1} = \beta_{jl+1} = \alpha_{jl+1}\) and \(\beta_{jl-p} = \beta_{jl-p} = \alpha_{jl-p}\) such that
\[
\|\delta_g * \phi - \phi\|_2 \geq \epsilon
\]
where \(g = \prod_{k=0}^{p} c_q(\beta_{jl+1-k}, \beta_{jl-k}) \prod_{k=0}^{p} c_q(\tilde{\beta}_{jl-p+1+k}, \tilde{\beta}_{jl-p+k})^{-1} \).

**Proof.** Uniformity of \(\epsilon\) with respect to \(r \in \mathbb{Z}_{>0}\) with factorization \(r = r'^l\) with \(l > p\), integers \(0 \leq j \leq r'\), and pairs of admissible sequences \((\alpha_{j+1}^{(l-p)_2}, \alpha_j^{(l-p)_2})\) is trivial since it only depends on the first entry \(\alpha_{jl+1} \in \mathcal{A}\) of \(\alpha_{jl+1}^{(l-p)_2}\) and the last entry \(\alpha_{jl-p} \in \mathcal{A}\) of \(\alpha_j^{(l-p)_2}\) and there are only a finite number of such pairs. So
Lemma 3.7 implies that $\tilde{H} \subset O \subset \text{Cay}(\tilde{G})$. Although in the above proof we only know a priori that the graphs $\tilde{g}$ and $\tilde{h}$ are isomorphic, we can see that $\tilde{g}(\lambda r) \equiv \tilde{h}(\lambda r)$ for all $\phi \in O_K$, let $\tilde{S}_q^p = \{e, -e\} \pi_q(\tilde{H}^p)$ and $\tilde{H}^q = \{e, -e\} \pi_q(H^q) = \langle \tilde{S}_q^p \rangle$. Recalling the strong approximation theorem, Lemma 3.7 implies that $H^q = \tilde{G}_q$ for all ideals $q \subset O_K$ coprime to $q_0$. Hence again by Lemma 3.7, we can use [GV12, Corollary 6] to further conclude that the Cayley graphs $\text{Cay}(\tilde{G}_q, \tilde{S}_q^p) = \text{Cay}(\{e, -e\} \backslash G_k/q, \{e, -e\} \pi_q(\tilde{S}_q^p))$ form a family of expanders with respect to square free ideals $q \subset O_K$ coprime to $q_0$. For all square free ideals $q \subset O_K$ coprime to $q_0$, let $A_q : L^2(\tilde{G}_q, \mathbb{C}) \to L^2(\tilde{G}_q, \mathbb{C})$ be the adjacency operator defined by

$$A_q(\phi)(g) = \sum_{h \in \tilde{S}_q^p} \phi(gh) = \sum_{h \in \tilde{S}_q^p} \phi(gh^{-1}) = \sum_{h \in \tilde{S}_q^p} (\delta_h \ast \phi)(g)$$

for all $g \in \tilde{G}_q$ which we rewrite as $A_q(\phi) = \sum_{h \in \tilde{S}_q^p} \delta_h \ast \phi$ and note that it is self-adjoint. Its largest eigenvalue is $\lambda_1(A_q) = \|\tilde{S}_q^p\| = k$ with corresponding eigenvectors being the constant functions. We can choose $\epsilon \in (0, 1)$ coming from the expander property so that for all square free ideals $q \subset O_K$ coprime to $q_0$, the next largest eigenvalue is $\lambda_2(A_q) \leq (1 - \epsilon)k$. This corresponds to the graph Laplacian $\Delta_q = 1\text{d}_{L^2(\tilde{G}_q, \mathbb{C})} - \frac{1}{k} A_q$ having smallest eigenvalue $\lambda_1(\Delta_q) = 0$ with corresponding eigenvectors being the constant functions and having the next smallest eigenvalue $\lambda_2(\Delta_q) \geq \epsilon$ for all square free ideals $q \subset O_K$ coprime to $q_0$. Thus for all square free ideals $q \subset O_K$ coprime to $q_0$, for all $\phi \in L^2(\tilde{G}_q, \mathbb{C})$ with $\|\phi\|_2 = 1$, we have $\|\Delta_q(\phi)\|_2 = \frac{1}{k} A_q(\phi) - \phi \geq \epsilon$ which implies

$$\sum_{h \in \tilde{S}_q^p} \|\delta_h \ast \phi - \phi\|_2 \geq \sum_{h \in \tilde{S}_q^p} (\delta_h \ast \phi - \phi) \geq k\epsilon$$

and so there is a $g \in \tilde{S}_q^p$ such that $\|\delta_g \ast \phi - \phi\|_2 \geq \epsilon$. But $\tilde{S}_q^p \subset \tilde{H}^p < \tilde{G}$ and recall the induced isomorphisms $\pi_q : \tilde{G}_q \to \tilde{G}_q$ and $\pi : \tilde{G}_q \to \tilde{G}_q$. Following these isomorphisms, we can see that $g$ is in fact of the form

$$g = \prod_{k=0}^{p} \langle \beta^{j_l+1-k}, (\beta^{j_l-2k})^{-1} \rangle \prod_{k=0}^{p} \langle \beta^{j_l-p+1+k}, (\beta^{j_l-p+k})^{-1} \rangle.$$

Remark. Although in the above proof we only know a priori that the graphs $\text{Cay}(G(\tilde{G}_q, q), \pi_q(\tilde{S}_q^p))$ form a family of expanders with respect to square free ideals $q \subset O_K$ coprime to $q_0$, it is easy to see using the Cheeger constant formulation for a family of expanders that left quotients by $\{e, -e\}$ preserve this property.

Let $q \subset O_K$ be an ideal coprime to $q_0$. For all measures $\eta$ on $\tilde{G}_q$, we define $\tilde{\eta}$ to be the operator $\tilde{\eta} : L^2(\tilde{G}_q, \mathbb{C}) \to L^2(\tilde{G}_q, \mathbb{C})$ acting by the convolution $\tilde{\eta}(\phi) = \eta \ast \phi$ for all $\phi \in L^2(\tilde{G}_q, \mathbb{C})$ and $\tilde{\eta}^p$ will be its adjoint operator as usual. We also define $\eta^* \ast$ to be the measure on $\tilde{G}_q$ defined by $\eta^* \ast (g) = \eta(g^{-1})$ for all $g \in \tilde{G}_q$. It is then easy to see that these two operations commute, i.e., $\tilde{\eta}^* \ast = \tilde{\eta}^* \ast$ or $\tilde{\eta}^* \ast = \tilde{\eta}^* \ast$ for
Lemma 3.19. There exists $C \in (0, 1)$ such that for all $|a| < a'_0$, for all square free ideals $q \subset \mathcal{O}_k$ coprime to $q_0$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, for all integers $1 \leq j \leq l'$, for all pairs of admissible sequences $(\alpha_{j+1}, \alpha_j)$, for all $\phi \in L^2(\tilde{G}_q, \mathbb{C})$ with $\|\phi\|_2 = 1$, we have

$$
\eta^* \phi, \psi = \langle \phi, \tilde{\eta}(\psi) \rangle = \sum_{g \in \tilde{G}_q} \phi(g) \overline{\eta(\psi)}(g) = \sum_{g, h \in \tilde{G}_q} \phi(g) \overline{\eta(h^{-1})\psi(gh)}
$$

$$
= \sum_{g, h \in \tilde{G}_q} \eta(h^{-1}) \phi(gh^{-1}) \overline{\psi(g)} = \sum_{g \in \tilde{G}_q} (\eta^* \phi)(g) \overline{\psi(g)}
$$

$$
\leq \|\eta^* \phi, \psi\| = \langle \eta^* \phi, \eta^* \phi \rangle = \langle \eta^* \phi, \phi \rangle
$$

for all $\phi, \psi \in L^2(\tilde{G}_q, \mathbb{C})$.

Proof. Fix $\epsilon \in (0, 1)$ to be the one from Lemma 3.18 and $C_0 > 1$ to be the $C$ from Lemma 3.16. Fix $C = \sqrt{1 - \frac{\epsilon^2}{2}}$. Let $|a| < a'_0$, $q \subset \mathcal{O}_k$ be a square free ideal coprime to $q_0$, $x \in \Sigma^+$, $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$ with $l > p$, $1 \leq j \leq l'$ be an integer, $(\alpha_{j+1}, \alpha_j) \in (l(p), l^-p)$ be a pair of admissible sequences and $\phi \in L^2(\tilde{G}_q, \mathbb{C})$ with $\|\phi\|_2 = 1$. Denote $\eta_{a,q,x,r,r'}^j, (\alpha_{j+1}, \alpha_j) \in (l(p), l^-p)$ by $\eta$ and $E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}}^j, (\alpha_{j+1}, \alpha_j) \in (l(p), l^-p)$ by $E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}}$. Recalling that $E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}} \in \mathbb{R}$, we can define the measure

$$
A = \eta^* \eta = \left( \sum_{\alpha_{j+1}, \alpha_j} E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}} \delta_{\alpha_{j+1}, \alpha_j}^j(x) \right) \ast \left( \sum_{\alpha_{j+1}, \alpha_j} E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}} \delta_{\alpha_{j+1}, \alpha_j}^j(x) \right)
$$

$$
= \sum_{\alpha_{j+1}, \alpha_j} E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}} \delta_{\alpha_{j+1}, \alpha_j}^j(x) \ast \delta_{\alpha_{j+1}, \alpha_j}^j(x)
$$

where $(\alpha_{j+1}, \alpha_j) \in (l(p), l^-p)$, $\alpha_{j+1} \in (l(p), l^-p)$, $(\alpha_{j+1}, \alpha_j)$ henceforth. To begin estimating $\|\eta^* \phi\|_2$, we first use definitions and properties mentioned above to get

$$
0 \leq \|\eta^* \phi\|^2 = \langle \eta^* \phi, \eta^* \phi \rangle = \langle \eta^* (\eta^* \phi), \phi \rangle = \langle \eta^* \phi, \phi \rangle = \langle \eta^* \eta \phi, \phi \rangle = \langle \tilde{\eta}(\eta^* \phi), \phi \rangle.
$$

The calculations also show that $\tilde{A} = \tilde{\eta}^* \tilde{\eta}$ is a self-adjoint positive semidefinite operator. Moreover, it suffices to show $\langle \tilde{A}(\phi), \phi \rangle \leq C^2 \|\tilde{A}\|_1$, since

$$
\|\tilde{A}\|_1 = \sum_{\alpha_{j+1}, \alpha_j} E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}} = \left( \sum_{\alpha_{j+1}, \alpha_j} E_{\alpha_{j+1}, \alpha_j}^{\eta_{a,q,x,r,r'}} \right)^2 = \|\eta\|^2.
$$
Before we estimate \( \langle \tilde{A}(\phi), \phi \rangle \), we first use Lemma 3.18 to obtain a
\[
g = c_q^{(\beta_{j+1}, \beta_j, x)} c_q^{(\tilde{\beta}_{j+1}, \tilde{\beta}_j, x)}
= \prod_{k=0}^{p} c_q(\beta_{j+1-k}, \beta_j-k) \prod_{k=0}^{p} c_q(\tilde{\beta}_{j-p+1+k}, \tilde{\beta}_{j-p+k})^{-1}
\]
for some admissible sequences \((\beta_{j+1}, (\beta_j)_j), (\beta_{j+1}, (\beta_j)_j)\) and \((\tilde{\beta}_{j+1}, (\tilde{\beta}_j)_j), (\tilde{\beta}_{j+1}, (\tilde{\beta}_j)_j)\) with \(\beta_{j+1} = \tilde{\beta}_{j+1} = \alpha_{j+1}\) and \(\beta_{j-p} = \tilde{\beta}_{j-p} = \alpha_{j+p}\) such that \(\|\delta_g * \phi - \phi\|_2 \geq \epsilon\).

Expanding the norm, we have \(\|g * \phi\|_2^2 - 2\Re(\delta_g * \phi, \phi) + \|\phi\|_2^2 \geq \epsilon^2\). Thus
\[
\Re(\delta_g * \phi, \phi) \leq 1 - \frac{\epsilon^2}{2} \in (0, 1)
\]
using the fact that \(\|\delta_g * \phi\|_2 = \|\phi\|_2 = 1\). Now we use this inequality to begin estimating \(\langle \tilde{A}(\phi), \phi \rangle\). Since \(A\) is self-adjoint, we have
\[
\langle \tilde{A}(\phi), \phi \rangle = \Re(\tilde{A}(\phi), \phi)
= \Re \left( \sum_{(\alpha_j, \tilde{\alpha}_j)_j} E_{\alpha_j} E_{\tilde{\alpha}_j} \Re(\delta_{\alpha_j} \phi, \phi) \right)
\]
\[
\leq \left(1 - \frac{\epsilon^2}{2}\right) E_{\beta_j} E_{\tilde{\beta}_j} + \sum_{(\alpha_j, \tilde{\alpha}_j)_j \neq (\beta_j, \tilde{\beta}_j)_j} E_{\alpha_j} E_{\tilde{\alpha}_j} \Re(\delta_{\alpha_j} \phi, \phi)
\]
\[
\leq \left(1 - \frac{\epsilon^2}{2}\right) E_{\beta_j} E_{\tilde{\beta}_j} + \sum_{(\alpha_j, \tilde{\alpha}_j)_j \neq (\beta_j, \tilde{\beta}_j)_j} E_{\alpha_j} E_{\tilde{\alpha}_j} \Re(\delta_{\alpha_j} \phi, \phi)
\]
\[
= \left(1 - \frac{\epsilon^2}{2}\right) E_{\beta_j} E_{\tilde{\beta}_j} + \sum_{(\alpha_j, \tilde{\alpha}_j)_j \neq (\beta_j, \tilde{\beta}_j)_j} E_{\alpha_j} E_{\tilde{\alpha}_j}
\]
\[
= \left\| A \right\|_1 - \frac{\epsilon^2}{2} E_{\beta_j} E_{\tilde{\beta}_j}
\]
Finally, Lemma 3.16 gives
\[
\left\| A \right\|_1 \leq \sum_{(\alpha_j, \tilde{\alpha}_j)_j \neq (\beta_j, \tilde{\beta}_j)_j} E_{\alpha_j} E_{\tilde{\alpha}_j} \leq C_0^2 \sum_{(\alpha_j, \tilde{\alpha}_j)_j \neq (\beta_j, \tilde{\beta}_j)_j} E_{\beta_j} E_{\tilde{\beta}_j} \leq C_0^2 N^{2p} E_{\beta_j} E_{\tilde{\beta}_j}
\]
and thus
\[
\langle \tilde{A}(\phi), \phi \rangle \leq \left(1 - \frac{\epsilon^2}{2C_0^2 N^{2p}}\right) \left\| A \right\|_1 = C^2 \left\| A \right\|_1.
\]
Lemma 3.20. There exists $l_0 \in \mathbb{Z}_{>0}$ such that for all integers $l > l_0$, there exists $C \in (0, 1)$ such that for all $|a| < a_0$, for all square free ideals $q \subset \mathcal{O}_K$ coprime to $q_0$, for all $x \in \Sigma^+$, for all $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$, for all admissible sequences $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$, for all $\phi \in L^2_0(\mathbb{G}_q, \mathbb{C})$ with $\|\phi\|_2 = 1$, we have
\[
\left\| \nu_{a,x}^{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})} \ast \phi \right\|_2 \leq C r' \left\| \nu_{a,x}^{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})} \right\|_1.
\]

Proof. Fix $C_1 > 0$ to be the $C$ from Lemma 3.17 and $C_2 \in (0, 1)$ to be the $C$ from Lemma 3.19. Fix $C_3 = -\log(C_2) > 0$. There is an integer $l_0 \geq p$ such that $2C_1 \theta - C_3 < 0$ for all integers $l > l_0$. Fix an integer $l > l_0$ and $C = e^{\frac{1}{2} (2C_1 \theta - C_3)} \in (0, 1)$. Let $|a| < a'_0$, $q \subset \mathcal{O}_K$ be a square free ideal coprime to $q_0$, $x \in \Sigma^+$, $r \in \mathbb{Z}_{>0}$ with factorization $r = r'l$, $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$ be an admissible sequence and $\phi \in L^2_0(\mathbb{G}_q, \mathbb{C})$ with $\|\phi\|_2 = 1$. Denote $\nu_{a,x}^{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}$ by $\nu$, $\nu_{a,x}^{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}$ by $\nu_0$, $\nu_{a,x}^{(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})}$ by $\nu_0$, $\eta_{j, (\alpha_s^1, \ldots, \alpha_s^{r'})}$ by $\eta_j$, $\eta_{j, (\alpha_s^1, \ldots, \alpha_s^{r'})}$ by $\eta_j$, and $E_{j, (\alpha_s^1, \ldots, \alpha_s^{r'})}$ by $E_{j, (\alpha_s^1, \ldots, \alpha_s^{r'})}$. Using Lemma 3.17 and then Lemma 3.19 repeatedly $r'$ times, we have
\[
\|\nu_0 \ast \phi\|_2 
\leq e^{r'C_1 \theta} \|\nu_1 \ast \phi\|_2 
\leq e^{r'C_1 \theta} \sum_{\alpha_1, \ldots, \alpha_r} \left( \sum_{j=0}^{r'} \eta_j \right) \left( \sum_{j=1}^{r'} E_j \right) \|\phi\|_2 
\leq e^{r'C_1 \theta} C_2 \sum_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^{r'} \left( \sum_{j=1}^{r'} E_j \right) \|\phi\|_2 
= \left( e^{r'C_1 \theta} - C_3 \right)^{r'} \sum_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^{r'} E_j \|\phi\|_2 
= \left( e^{r'C_1 \theta} - C_3 \right)^{r'} \sum_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^{r'} E_j \|\phi\|_2.
\]

Note that in the above calculations $\eta_{0, (\alpha_s^1, \ldots, \alpha_s^{r'})} = \delta_{q}(\alpha_1, x)$ which preserves the norm when taking convolutions. As mentioned earlier, for fixed sequences $(\alpha_1, \ldots, \alpha_r)$, the term $E_{j, (\alpha_s^1, \ldots, \alpha_s^{r'})}$ depends only on the choice of $\alpha_j$ and not on $\alpha_k$ for all admissible $1 \leq j, k \leq r'$. Hence we can commute the inner sum and product to get
\[
\|\nu_0 \ast \phi\|_2 
\leq \left( e^{r'C_1 \theta} - C_3 \right)^{r'} \sum_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^{r'} E_j \|\phi\|_2 
= \left( e^{r'C_1 \theta} - C_3 \right)^{r'} \sum_{\alpha_1, \ldots, \alpha_r} \prod_{j=1}^{r'} E_j \|\phi\|_2.
\]

Recognizing this sum to be $\|\nu_1\|_1$, we use Lemma 3.17 once again to get
\[
\|\nu_0 \ast \phi\|_2 \leq \left( e^{r'C_1 \theta} - C_3 \right)^{r'} \|\nu_1\|_1 \leq \left( e^{r'C_1 \theta} - C_3 \right)^{r'} \|\nu_0\|_1 = C_r' \|\nu_0\|_1.
\]

Since $e^{r'C_1 \theta} (\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1})) > 0$, it follows that $\|\nu \ast \phi\|_2 \leq C_r' \|\nu\|_1$. \hfill □
Let \( q \subset \mathcal{O}_K \) be a square free ideal coprime to \( q_0 \) and \( \mu \) be a complex measure on \( \hat{G}_q \). We note that \( E_q^a \) is a \( \hat{\mu} \)-invariant submodule of the left \( \mathbb{C}[\hat{G}_q] \)-module \( L^2(\hat{G}_q, \mathbb{C}) \). Let \( \hat{\mu} \) denote the measure on \( \hat{G}(\mathcal{O}_K/q) \) defined by \( \hat{\mu}(g) = \mu(e^{-1}g) \) for all \( g \in \hat{G}(\mathcal{O}_K/q) \) and \( \hat{\mu} : L^2(\hat{G}(\mathcal{O}_K/q), \mathbb{C}) \to L^2(\hat{G}(\mathcal{O}_K/q), \mathbb{C}) \) denote the operator acting by convolution as before. Similar to above, \( E_q^a \) is a \( \hat{\mu} \)-invariant submodule of the left \( \mathbb{C}[\hat{G}(\mathcal{O}_K/q)] \)-module \( L^2(\hat{G}(\mathcal{O}_K/q), \mathbb{C}) \).

**Lemma 3.21.** There exists \( C > 0 \) such that for all square free ideals \( q \subset \mathcal{O}_K \) coprime to \( q_0 \), for all complex measures \( \mu \) on \( \hat{G}_q \), we have

\[
\|\hat{\mu}|_{E_q^a}\|_{\text{op}} \leq CN_K(q)^{-\frac{1}{2}}(\#\hat{G}_q)^{\frac{1}{2}}\|\mu\|_2.
\]

**Proof.** Let \( q \subset \mathcal{O}_K \) be a square free ideal coprime to \( q_0 \) and \( \mu \) be a complex measure on \( \hat{G}_q \). It will be fruitful to first work with \( \hat{\mu} \). One way to calculate the operator norm is using the equation

\[
\|\hat{\mu}|_{E_q^a}\|_{\text{op}} = \max_{\lambda \in \Lambda(\hat{\mu}|_{E_q^a})} \sqrt{\lambda}
\]

where \( \Lambda(\hat{\mu}|_{E_q^a}) \) is the set of eigenvalues of the self-adjoint positive semidefinite operator \( \hat{\mu} \langle \cdot, \cdot \rangle_{E_q^a} \) which is diagonalizable with nonnegative eigenvalues. Since \( \hat{\mu}|_{E_q^a} : \hat{E}_q^a \to \hat{E}_q^a \) is a \( \mathbb{C}[\hat{G}(\mathcal{O}_K/q)] \)-module homomorphism, its eigenspaces are submodules of \( \hat{E}_q^a \) and hence must contain at least one irreducible submodule \( V \) which corresponds to an irreducible representation \( \rho : \hat{G}(\mathcal{O}_K/q) \to \text{GL}(V) \). Now suppose we have the prime ideal factorization \( q = \prod_{j=1}^k p_j \) for some \( k \in \mathbb{Z}_{>0} \) and for some prime ideals \( p_1, p_2, \ldots, p_k \subset \mathcal{O}_K \). Then the Chinese remainder theorem gives \( \mathcal{O}_K/q \cong \prod_{j=1}^k \mathcal{O}_K/p_j \cong \prod_{j=1}^k \mathbb{F}_{N_K(p_j)} \) and hence \( \hat{G}(\mathcal{O}_K/q) \cong \prod_{j=1}^k \hat{G}(\mathcal{O}_K/p_j) \cong \prod_{j=1}^k \hat{G}(\mathbb{F}_{N_K(p_j)}) \). Thus we have \( \rho = \bigotimes_{j=1}^k \rho_j \) where \( \rho_j : \hat{G}(\mathbb{F}_{N_K(p_j)}) \to \text{GL}(V_j) \) for some complex vector space \( V_j \) is an irreducible representation for all integers \( 1 \leq j \leq k \). The significance of using \( \hat{E}_q^a \) is that its definition and \( V \subset \hat{E}_q^a \) forces \( \rho_j \) to be nontrivial for all integers \( 1 \leq j \leq k \). Now we use bounds directly from [KS13]. However, we note that the bounds originate from [Lan72, LS74, SZ93] which gives lower bounds on the degrees of nontrivial irreducible projective representations of Chevalley groups. By Schur’s lemma, projective irreducible representations correspond to irreducible representations of central group extensions as in our case. From the proof of [KS13, Proposition 4.2], there is a \( C_1 > 0 \) independent of anything such that \( \deg(\rho_j) = \dim(V_j) > C_1 N_K(p_j)^{n-2} > C_1 N_K(p_j) \) for all integers \( 1 \leq j \leq k \) if \( n \geq 6 \) and \( \deg(\rho_j) = \dim(V_j) > C_1 N_K(p_j) \) for all integers \( 1 \leq j \leq k \) if \( n < 6 \). In any case, we conclude that \( \deg(\rho) = \dim(V) > C_1 N_K(q) \). Thus for all \( \lambda \in \Lambda(\hat{\mu}|_{E_q^a}) \), we have

\[
C_1 N_K(q) \lambda \leq \text{tr}(\hat{\mu} \hat{\mu}^*) = \sum_{g \in \hat{G}(\mathcal{O}_K/q)} \langle \hat{\mu} \hat{\mu}(\delta_g), \delta_g \rangle = \sum_{g \in \hat{G}(\mathcal{O}_K/q)} \|\hat{\mu} \delta_g\|_2^2 = \# \hat{G}(\mathcal{O}_K/q) \cdot \|\hat{\mu}\|_2^2.
\]

Hence,

\[
\|\hat{\mu}|_{E_q^a}\|_{\text{op}}^2 = \max_{\lambda \in \Lambda(\hat{\mu}|_{E_q^a})} \lambda \leq C_1^{-1} N_K(q)^{-1} \# \hat{G}(\mathcal{O}_K/q) \cdot \|\hat{\mu}\|_2^2.
\]
Now we convert this to a bound for $\|\tilde{\mu}|_{E_q^3}|_{op}$. Let $\phi \in E_q^3$. Then $\tilde{\phi} \in \hat{E}_q^3$ and also $\tilde{\mu}(\phi) = 2\psi$ if $e (\mod q) \neq -e (\mod q)$ in $G(O_K/q)$ and $\tilde{\mu}(\phi) = \psi$ otherwise, where $\psi = \tilde{\mu}(\phi)$. In any case, $\|\tilde{\mu}(\phi)\|_2 = \|\psi\|_2 \leq \|\phi\|_2 \leq \|\tilde{\mu}(\phi)\|_2$. Being careful of similar cases, the above bound gives

$$
\|\tilde{\mu}(\phi)\|_2^2 \leq C_1^{-1}N_K(q)^{-1}\#\hat{G}(O_K/q) \cdot \|\tilde{\mu}\|_2^2 \|\phi\|_2^2
$$

which is independent of $q$ and $\mu$. Hence

$$\|\tilde{\mu}|_{E_q^3}|_{op} \leq C N_K(q)^{-1/2}(\#\hat{G}_q)^{1/2} \|\mu\|_2.
$$

Now we prove Lemma 3.14 by starting with Lemma 3.21 obtained from the lower bounds of nontrivial irreducible representations of Chevalley groups, and then using Lemma 3.20 obtained from the expansion machinery to continue to bound the right hand side by the $L^1$ norm and also remove the growth contributed by $\#\hat{G}_q$ essentially by fiat.

**Proof of Lemma 3.14.** Fix $C_1 > 0$ to be the $C$ from Lemma 3.21 and $C_2 > 0$ to be the $C$ from Lemma 3.5. Fix any integer $l > l_0$ where $l_0 \in \mathbb{Z}_{>0}$ is the constant provided by Lemma 3.20 and fix $C_3 \in (0,1)$ to be the corresponding $C$ from the same lemma. Fix $C = 2C_1C_2 > 0$ and $C_0 = -\frac{e}{2\log(C_3)} > 0$ where $e > 0$ depends on $n$ and is such that $\#\hat{G}_q \leq N_K(q)^{-c}$ for all nontrivial ideals $q \subset O_K$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$, $q \subset O_K$ be a square free ideal coprime to $q_0$, $x \in \Sigma^+$, $C_0 \log(N_K(q)) \leq r < s$ be integers with $r \in IZ$, $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$ be an admissible sequence and $\phi \in E_q^3$ with $\|\phi\|_2 = 1$. Let $\mu$ denote either $\mu^{\xi,q,x}_{(x,\alpha_{s-1},\ldots,\alpha_{r+1})}$ or $\mu^{\xi,q,x}_{(x,\alpha_{s-1},\ldots,\alpha_{r+1})}$ and $\nu$ denote $\nu^{\xi,q,x}_{(x,\alpha_{s-1},\ldots,\alpha_{r+1})}$. Applying Lemma 3.21 to $\mu$ and then using Lemma 3.5 gives

$$\|\mu \ast \phi\|_2 \leq C_1 C_2 N_K(q)^{-1/2}(\#\hat{G}_q)^{1/2} \|\nu\|_2.$$

By choice of $r$ and the function $\varphi = \delta_c - \frac{1}{\#\hat{G}_q} \chi_{\hat{G}_q} \in L^2_0(\hat{G}_q, \mathbb{C})$, which satisfies $\|\varphi\|_2 \leq 1$, we can use Lemma 3.20 to get

$$\|\nu\|_2 = \|\nu \ast \delta_c\|_2 \leq \left\|\nu \ast \frac{1}{\#G_q} \chi_{\hat{G}_q}\right\|_2 + \|\nu \ast \varphi\|_2 \leq \frac{\|\nu\|_1}{(\#G_q)^{1/2}} + C_4^r \|\nu\|_1 \leq 2 \frac{\|\nu\|_1}{(\#G_q)^{1/2}},$$

Using this bound in the previous inequality and recalling $C = 2C_1C_2$, we have

$$\|\mu \ast \phi\|_2 \leq C N_K(q)^{-1/2} \|\varphi\|_1.$$
3.5. $L^\infty$ and Lipschitz bounds and proof of Theorem 3.3. In this subsection we use Lemma 3.14 to prove Lemmas 3.22 and 3.24 which is then used to prove Theorem 3.3 by induction as in [OW16]. We start with fixing some notations and easy bounds.

Let $q \subset \mathcal{O}_k$ be a nontrivial proper ideal. Fix integers

$$r_q \in [C_0 \log(N_k(q)), C_0 \log(N_k(q)) + 1]$$

with $r_q \in l\mathbb{Z}$ and

$$s_q \leq \left( r_q - \frac{\log(N_k(q)) + \log(4C_1C_f)}{\log(\theta)}, C_4 \log(N_k(q)) \right)$$

where we fix $C_0$ and $l$ to be constants from Lemma 3.14, $C_1$ to be the constant from Lemma 3.23 and $C_4 = C_0 - \frac{1}{\log(\theta)} + \frac{l}{\log(\theta)} - \frac{\log(4C_1C_f)}{\log(\theta) \log(2)} + \frac{1}{\log(2)}$ so that there is enough room for the integer $s_q$ to exist. These definitions of constants ensure that $C_0 \log(N_k(q)) \leq r_q < s_q$ and $4C_1C_2s^{-r_q} \leq N_k(q)^{-1}$. For all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$, for all square free ideals $q \subset \mathcal{O}_k$ coprime to $q_0$, for all $x \in \Sigma^+$, for all integers $C_0 \log(N_k(q)) \leq r < s$ with $r \in l\mathbb{Z}$, for all admissible sequences $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_{r+1})$, we have

$$\|\mu^{q,x}_{(\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1})}\|_1 = e^{f^{(q)}_{s-r}(\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1},\omega(\alpha_{r+1}))} \left( \sum_{\alpha'_r} e^{f^{(q)}_{s-r}(\alpha'_r,\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1},\omega(\alpha_{r+1}))} \right) \leq C_f e^{f^{(q)}_{s-r}(\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1},\omega(\alpha_{r+1}))}$$

by Lemma 3.4 and hence Lemma 3.14 implies that for all $\phi \in E^q$ we have

$$\|\mu * \phi\|_2 \leq CC_f N_k(q)^{-\frac{1}{2}} e^{f^{(q)}_{s-r}(\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1},\omega(\alpha_{r+1}))} \|\phi\|_2$$

where $\mu$ denotes either $\mu^{q,x}_{(\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1})}$ or $\mu^{q,x}_{(\alpha_s,\alpha_{s-1},\ldots,\alpha_{r+1})}$ and $C$ is the constant from the same lemma. We will use this in Lemmas 3.22 and 3.24. We now start with the $L^\infty$ bound.

**Lemma 3.22.** There exist $\kappa_1 \in (0, 1)$ and $q_{1,1} \in \mathbb{Z}_{>0}$ such that for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$, for all square free ideals $q \subset \mathcal{O}_k$ coprime to $q_0$ with $N_k(q) > q_{1,1}$, for all $H \in \mathcal{W}_q^a$, we have

$$\|\mathcal{M}_{\xi,q}^H\|_\infty \leq \frac{1}{2} N_k(q)^{-\kappa_1} (\|H\|_\infty + \text{Lip}_{d_0}(H)).$$

**Proof.** There are $q_{1,1} \in \mathbb{Z}_{>0}$ and $\varepsilon \in (0, 1)$ such that $\frac{1}{2} - \frac{\log(2CC_f^2)}{\log(q)} > \varepsilon$ for all integers $q > q_{1,1}$. Fix any $\kappa_1 \in (0, \varepsilon)$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$, $q \subset \mathcal{O}_k$ be a square free ideal coprime to $q_0$ with $N_k(q) > q_{1,1}$, $H \in \mathcal{W}_q^a$ and $x \in \Sigma^+$. Denote $r_q$ by $r$ and $s_q$ by $s$. Now using the approximation from Lemma 3.6 and
then Lemma 3.14, we have
\[
\|\mathcal{M}_{\xi,a}^s(H)(x)\|_2 \\
\leq \left| \sum_{\alpha_r+1,\alpha_r+2,\ldots,\alpha_s} \mu_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^\xi \right| \phi_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^H \frown H \|_\infty \\
\leq \left| \sum_{\alpha_r+1,\alpha_r+2,\ldots,\alpha_s} \mu_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^\xi \right| \phi_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^H \frown H \|_2 \\
\leq \left| \sum_{\alpha_r+1,\alpha_r+2,\ldots,\alpha_s} \mu_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^\xi \right| \phi_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^H \frown H \|_\infty \\
\leq \left| \sum_{\alpha_r+1,\alpha_r+2,\ldots,\alpha_s} \mu_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^\xi \right| \phi_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^H \frown H \|_2 \\
\leq \sum_{\alpha_r+1,\alpha_r+2,\ldots,\alpha_s} CC_f N_k(q)^{-\frac{1}{2}} \left| \sum_{\alpha_r+1,\alpha_r+2,\ldots,\alpha_s} \phi_{\alpha_r,\alpha_r+1,\ldots,\alpha_s}^H \frown H \|_\infty \right. \\
\left. + C_f \text{Lip}_{d_\theta}(H) \theta^{s-r} \right) \\
\leq CC_f^2 N_k(q)^{-\frac{1}{2}} \left| H \right|_\infty + C_f \text{Lip}_{d_\theta}(H) \theta^{s-r} \\
\leq \frac{1}{2} N_k(q)^{-\kappa} \left( \left| H \right|_\infty + \text{Lip}_{d_\theta}(H) \right) \\
\end{align*}

since \( CC_f^2 N_k(q)^{-\frac{1}{2}} \leq \frac{1}{4} N_k(q)^{-\kappa} \) and
\[
C_f \theta^{s-r} \leq C_1 C_f \theta^{s-r} \leq \frac{1}{2} N_k(q)^{-\kappa} \\
\]
by definitions of the constants.

Recalling that we already fixed \( b_0 = 1 \), we now record an estimate.

**Lemma 3.23.** There exists \( C > 1 \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a'_0 \) and \( |b| \leq b_0 \), for all \( x, y \in \Sigma^+ \), for all \( s \in \mathbb{Z}_{>0} \), for all admissible sequences \( \alpha^s \), we have
\[
\left| 1 - e^{f(a)x + ib \tau_x}(\alpha^s, y) - f(a)x + ib \tau_x)(\alpha^s, x) \right| \leq C d_\theta(x, y). \\
\]

**Proof.** Fix \( C > \max \left( 1, \left( 1 + b_0 \right) \frac{T_0 \theta}{1 - \theta} \right) \). Let \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a'_0 \) and \( |b| \leq b_0 \). Let \( x, y \in \Sigma^+ \), \( s \in \mathbb{Z}_{>0} \) and \( \alpha^s \) be an admissible sequence. We calculate that
\[
\left| f(a)(\alpha^s, y) - f(a)(\alpha^s, x) \right| \leq \sum_{j=0}^{s-1} \left| f(a)(\sigma^j(\alpha^s, y)) - f(a)(\sigma^j(\alpha^s, x)) \right| \\
\leq \sum_{j=0}^{s-1} \text{Lip}_{d_\theta}(f(a)) \cdot d_0(\sigma^j(\alpha^s, y), \sigma^j(\alpha^s, x)) \\
\leq \text{Lip}_{d_\theta}(f(a)) \sum_{j=0}^{s-1} \theta^{-j} d_\theta(x, y) \\
\leq \frac{T_0 \theta}{1 - \theta} d_\theta(x, y). \\
\]
In the same way, we have a similar bound $|\tau_s(\alpha^s, y) - \tau_s(\alpha^s, x)| \leq \frac{T_0\theta}{1 - \theta}d_\theta(x, y)$. Thus, using $d_\theta(x, y) \leq 1$, we have

$$
\left| 1 - e^{(f_s^{(\alpha)} + ib\tau_s)(\alpha^s, y) - (f_s^{(\alpha)} + ib\tau_s)(\alpha^s, x)} \right|
\leq e^{(f_s^{(\alpha)}(\alpha^s, y) - f_s^{(\alpha)}(\alpha^s, x)) + b\tau_s(x, y)}
\leq e^{\frac{T_0\theta}{1 - \theta}d_\theta(x, y) + \frac{b_0T_0\theta}{1 - \theta}d_\theta(x, y)}
\leq Cd_\theta(x, y).
$$

Remark. This is the reason the approach of Bourgain-Gamburd-Sarnak is restricted to small $|b|$.

Now we can take care of the Lipschitz bound and although Lemma 3.6 cannot be used directly, we can use similar albeit more intricate estimates.

**Lemma 3.24.** There exist $\kappa_2 \in (0, 1)$ and $q_{1,2} \in \mathbb{Z}_{>0}$ such that for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| \leq b_0$, for all square free ideals $q \subset \mathcal{O}_K$ coprime to $q_0$ with $N_K(q) > q_{1,2}$, for all $H \in \mathcal{W}_q^\xi$, we have

$$\text{Lip}_{d_\theta}(\mathcal{M}_{\xi, q}^s(H)) \leq \frac{1}{2}N_K(q)^{-\kappa_2}(\|H\|_\infty + \text{Lip}_{d_\theta}(H)).$$

**Proof.** There are $q_{1,2} \in \mathbb{Z}_{>0}$ and $\epsilon \in (0, 1)$ such that $\frac{1}{2} - \frac{\log(4CC'Cr)}{\log(q)} > \epsilon$ for all integers $q > q_{1,2}$. Fix any $\kappa_2 \in (0, \epsilon)$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0'$ and $|b| \leq b_0$. Let $q \subset \mathcal{O}_K$ be a square free ideal coprime to $q_0$ with $N_K(q) > q_{1,2}$, $H \in \mathcal{W}_q^\xi$ and $x, y \in \Sigma^+$. Denote $r_q$ by $r$ and $s_q$ by $s$. First suppose that $d_\theta(x, y) = 1$. Then from the proof of Lemma 3.22, we can simply estimate as

$$\left\| \mathcal{M}_{\xi, q}^s(H)(x) - \mathcal{M}_{\xi, q}^s(H)(y) \right\|_2
\leq \left( 2CC_f^2N_K(q)^{-\frac{1}{2}}\|H\|_\infty + 2C_f\text{Lip}_{d_\theta}(H)\theta^{s-1} \right)d_\theta(x, y).$$

Now suppose $d_\theta(x, y) < 1$. Then of course $x_0 = y_0$ and hence all the sums which will appear are over the same set of admissible sequences and moreover $\delta_{\xi, q}^s(\alpha^s, x) =$
\[ \delta_{\xi}(\alpha^*, y). \] Thus we have

\[
\begin{align*}
&\| M_{\xi;q}^c(H)(x) - M_{\xi;q}^c(H)(y) \|_2 \\
\leq & \left\| \sum_{\alpha^*} e^{f_s^{(a)}(\alpha^*, x)} \delta_{\xi}^c(\alpha^*, x) * H(\alpha^*, x) \\
& - \sum_{\alpha^*} e^{f_s^{(a)}(\alpha^*, y)} \delta_{\xi}^c(\alpha^*, y) * H(\alpha^*, y) \right\|_2 \\
\leq & \left\| \sum_{\alpha^*} e^{f_s^{(a)}(\alpha^*, x)} \delta_{\xi}^c(\alpha^*, x) * (H(\alpha^*, x) - H(\alpha^*, y)) \right\|_2 \\
& + \left\| \sum_{\alpha^*} \left( e^{f_s^{(a)}(\alpha^*, x)} - e^{f_s^{(a)}(\alpha^*, y)} \right) \right\|_2 \\
& \cdot \| H(\alpha^*, x) - H(\alpha^*, y) \|_2 \\
= & K_1 + K_2 + K_3.
\end{align*}
\]

We easily estimate the first term \( K_1 \) as

\[
K_1 \leq \sum_{\alpha^*} e^{f_s^{(a)}(\alpha^*, x)} \| H(\alpha^*, x) - H(\alpha^*, y) \|_2 \\
\leq \text{Lip}_{d_\theta}(H) \theta^* d_\theta(x, y) \sum_{\alpha^*} e^{f_s^{(a)}(\alpha^*, x)} \leq C_f \text{Lip}_{d_\theta}(H) \theta^* d_\theta(x, y).
\]

Next we estimate the second term \( K_2 \) as

\[
K_2 \leq \sum_{\alpha^*} \left| e^{f_s^{(a)}(\alpha^*, x)} \right| \cdot \left| 1 - e^{f_s^{(a)}(\alpha^*, y) - (f_s^{(a)} + ib\tau_s)(\alpha^*, x)} \right| \\
\cdot \| H(\alpha^*, x) - H(\alpha^*, y) - (f_s^{(a)} + ib\tau_s)(\alpha^*, x) \|_2 \\
\leq \text{C}_1 \text{Lip}_{d_\theta}(H) \theta^* - \text{Lip}_{d_\theta}(H) \theta^* d_\theta(x, y) \sum_{\alpha^*} e^{f_s^{(a)}(\alpha^*, x)} \\
\leq \text{C}_1 \text{C}_f \text{Lip}_{d_\theta}(H) \theta^* - \text{Lip}_{d_\theta}(H) \theta^* d_\theta(x, y).
\]
Finally, using Lemma 3.14, we estimate the third and last term $K_3$ as

$$K_3 \leq \left\| \sum_{\alpha^r} \left( e^{(f_{j,\alpha_a} + i \beta \tau_a) (\alpha^r, x)} - e^{(f_{j,\alpha_a} + i \beta \tau_a) (\alpha^r, y)} \right) \right\|_2$$

$$\cdot \delta_{\xi, \alpha_a} (\alpha^r, x) * H(\alpha_0, \alpha_1, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))$$

$$\leq \left\| \sum_{\alpha^r} \left( e^{(f_{j,\alpha_a} + i \beta \tau_a) (\alpha^r, x)} \right) \cdot \left| 1 - e^{(f_{j,\alpha_a} + i \beta \tau_a) (\alpha^r, y) - (f_{j,\alpha_a} + i \beta \tau_a) (\alpha^r, x)} \right| \right\|_2$$

$$\cdot \delta_{\xi, \alpha_a} (\alpha^r, x) * |H| (\alpha_0, \alpha_1, \ldots, \alpha_{r+1}, \omega(\alpha_{r+1}))$$

$$\leq C_1 d_\theta (x, y) \sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_r} \left\| \sum_{\alpha^r} e^{(f_{j,\alpha_a} (\alpha^r, x)) \delta_{\xi, \alpha_a} (\alpha_{r+1}, \alpha^r, x) * \phi_{\frac{q}{H}} (\alpha_0, \alpha_1, \ldots, \alpha_{r+1})} \right\|_2$$

$$\leq C_1 d_\theta (x, y) \sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_r} \left( CC_f N_{\xi}(q)^{-\frac{1}{2}} e^{f_{j,\alpha_a} (\alpha_{r+1}, \alpha^r, x) \frac{1}{2}} \right)$$

$$\sum_{\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_r} \left( CC_f N_{\xi}(q)^{-\frac{1}{2}} \right)$$

$$\leq CC_1 C_f N_{\xi}(q)^{-\frac{1}{2}} \|H\|_\infty d_\theta (x, y).$$

So combining all three estimates, we have

$$\left\| \mathcal{M}_{\xi, \alpha_a} (H)(x) - \mathcal{M}_{\xi, \alpha_a} (H)(y) \right\|_2$$

$$\leq \left( CC_1 C_f^2 N_{\xi}(q)^{-\frac{1}{2}} \|H\|_\infty + (C_f \theta^s + C_1 C_f \theta^{s-r}) \operatorname{Lip}_{d_\theta} (H) \right) d_\theta (x, y).$$

Thus, for both cases $d_\theta (x, y) = 1$ and $d_\theta (x, y) < 1$, we have

$$\left\| \mathcal{M}_{\xi, \alpha_a} (H)(x) - \mathcal{M}_{\xi, \alpha_a} (H)(y) \right\|_2$$

$$\leq \left( 2CC_1 C_f^2 N_{\xi}(q)^{-\frac{1}{2}} \|H\|_\infty + 2C_1 C_f \operatorname{Lip}_{d_\theta} (H) \theta^{s-r} \right) d_\theta (x, y).$$

Since the above holds for all $x, y \in \Sigma^+$, we have

$$\operatorname{Lip}_{d_\theta} (\mathcal{M}_{\xi, \alpha_a} (H)) \leq 2CC_1 C_f^2 N_{\xi}(q)^{-\frac{1}{2}} \|H\|_\infty + 2C_1 C_f \operatorname{Lip}_{d_\theta} (H) \theta^{s-r}$$

$$\leq \frac{1}{2} N_{\xi}(q)^{-\kappa_2} (\|H\|_\infty + \operatorname{Lip}_{d_\theta} (H))$$

since $2CC_1 C_f^2 N_{\xi}(q)^{-\frac{1}{2}} \leq \frac{1}{2} N_{\xi}(q)^{-\kappa_2}$ and $2C_1 C_f \theta^{s-r} \leq \frac{1}{2} N_{\xi}(q)^{-1} \leq \frac{1}{2} N_{\xi}(q)^{-\kappa_2}$ by definitions of the constants.

\textbf{Proof of Theorem 3.3.} Fix $\kappa_1, \kappa_2 \in (0, 1)$ and $q_{1,1}, q_{1,2} \in \mathbb{Z}_{>0}$ to be the constants from Lemmas 3.22 and 3.24. Recall the constant $C_f$ and that we already fixed $b_0 = 1$. Fix $a_0 = a'_0$, $\kappa = \min(\kappa_1, \kappa_2) \in (0, 1)$ and $q_1 = \max(q_{1,1}, q_{1,2}) \in \mathbb{Z}_{>0}$. Let
\[ \xi = a + ib \in \mathbb{C} \text{ with } |a| < a_0 \text{ and } |b| \leq b_0. \] Let \( q \subset \mathcal{O}_K \) be a square free ideal coprime to \( q_0 \) with \( N_\mathcal{O}_K(q) > q_1 \). Denote \( s_q \) by \( s \). Note that Lemmas 3.22 and 3.24 together give
\[
\left\| \mathcal{M}_{\xi,q}^b(U) \right\|_\infty + \text{Lip}_{d_\theta}(\mathcal{M}_{\xi,a}^b(U)) \leq N_\mathcal{O}_K(q)^{-\kappa} (\left\| H \right\|_\infty + \text{Lip}_{d_\theta}(H))
\]
for all \( H \in \mathcal{W}_q^a \). Now let \( j \in \mathbb{Z}_{\geq 0} \) and \( H \in \mathcal{W}_q^a \). Then by induction we have
\[
\left\| \mathcal{M}_{\xi,a}^b(U) \right\|_2 \leq \left\| \mathcal{M}_{\xi,q}^b(U) \right\|_\infty \leq N_\mathcal{O}_K(q)^{-\kappa} (\left\| H \right\|_\infty + \text{Lip}_{d_\theta}(H))
\]
\[ = N_\mathcal{O}_K(q)^{-\kappa} \| H \|_{\text{Lip}(d_\theta)}. \]

4. Spectral bounds for large \(|b|\) using Dolgopyat’s method

In this section we will use Dolgopyat’s method [Dol98] to prove the following Theorem 4.1. We will closely follow [OW16, Sto11].

**Theorem 4.1.** There exist \( \eta > 0, C > 0, a_0 > 0 \) and \( b_0 > 0 \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \(|a| < a_0 \) and \(|b| > b_0 \), for all nontrivial ideals \( q \subset \mathcal{O}_K \), for all \( k \in \mathbb{Z}_{\geq 0} \), for all \( H \in \mathcal{V}_q \), we have
\[
\left\| \mathcal{M}_{\xi,q}^b(H) \right\|_2 \leq C e^{-\eta k} \| H \|_{1,b}.
\]

4.1. Dolgopyat’s method, notations, constants and preliminary lemmas.

As in [Sto11], we define a new metric which is crucial for the argument. Let \( D \) be a new metric on \( U \) defined by
\[
D(u, u') = \begin{cases} 
\inf\{\text{diam}_d(\mathfrak{C}) \in \mathbb{R}_{>0} : u, u' \in \mathfrak{C} \text{ where } \mathfrak{C} \text{ is a cylinder}\}, & u, u' \in U_j \\
1, & \text{for some } j \in \mathcal{A} \\
\end{cases} \\
\text{otherwise}
\]
for all \( u, u' \in U \).

**Remark.** The above definition makes sense since \( \text{diam}_d(U_j) \leq \hat{\delta} < 1 \) for all \( j \in \mathcal{A} \). Note that \( D(u, u') \leq D(u, u') \) for all \( u, u' \in U \). Finally, it is important to observe that \( C^{\text{Lip}(D)}(U, \mathbb{R}) \subset B(U, \mathbb{R}) \) but \( C^{\text{Lip}(D)}(U, \mathbb{R}) \not\subset C(U, \mathbb{R}) \). Moreover, \( h \) is measurable with respect to \( \nu_U \) for all \( h \in C^{\text{Lip}(D)}(U, \mathbb{R}) \).

We define the cone
\[
K_B(U) = \{ h \in C^{\text{Lip}(D)}(U, \mathbb{R}) : h > 0, |h(u) - h(u')| \leq Bh(u) D(u, u') \}
\]
for all \( u, u' \in U_j \), for all \( j \in \mathcal{A} \).

The following theorem captures the mechanism of Dolgopyat’s method [Dol98].

**Theorem 4.2.** There exist \( m \in \mathbb{Z}_{>0}, \eta \in (0, 1), a_0 > 0, b_0 > 0, E > \frac{1}{b_0} \) and a set of operators \( \{ \mathcal{N}_{a,j} : C^{\text{Lip}(D)}(U, \mathbb{R}) \to C^{\text{Lip}(D)}(U, \mathbb{R}) : |a| < a_0, j \in \mathcal{J}(b), |b| > b_0 \} \), where \( \mathcal{J}(b) \) is some finite index set for all \(|b| > b_0\), such that
\begin{enumerate}
\item \( \mathcal{N}_{a,j}(K_{E|b|}(U)) \subset K_{E|b|}(U) \) for all \(|a| < a_0, \) for all \( j \in \mathcal{J}(b), \) for all \(|b| > b_0\)
\item \( \|\mathcal{N}_{a,j}(h)\|_2 \leq \eta \|h\|_2 \) for all \( h \in K_{E|b|}(U), \) for all \(|a| < a_0, \) for all \( j \in \mathcal{J}(b), \)
\item for all \( \xi = a + ib \in \mathbb{C} \) with \(|a| < a_0 \) and \(|b| > b_0, \) for all nontrivial ideals \( q \subset \mathcal{O}_K, \) if \( H \in C(U, L^2(F_q, \mathbb{C})) \) and \( h \in K_{E|b|}(U) \) satisfy
\end{enumerate}
constant function defined by $H(u)$ for all $u \in U$

(1b) $\|H(u) - H(u')\|_2 \leq E[b|h(u)D(u, u')|$ for all $u, u' \in U_j$, for all $j \in \mathcal{A}$, then there exists $j \in \mathcal{J}(b)$ such that

(2a) $\|M_{\xi,q}^m(H)(u)\|_2 \leq N_{a,j}(h(u)$ for all $u \in U$

(2b) $\|M_{\xi,q}^m(H)(u) - M_{\xi,q}^m(H)(u')\|_2 \leq E[b]N_{a,j}(h(u)D(u, u')$ for all $u, u' \in U_j$, for all $j \in \mathcal{A}$.

Proof that Theorem 4.2 implies Theorem 4.1. Fix $m \in \mathbb{Z}_{>0}, a_0 > b_0 > 0, E > \frac{1}{a_0}$ to be the ones from Theorem 4.2. Let $\tilde{\eta} \in (0, 1)$ be the $\eta$ in Theorem 4.2. Fix $\eta = \frac{-\log(\tilde{\eta})}{m}$ > 0. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| > b_0$. Let $q \subset O_{\xi}$ be a nontrivial ideal, $k \in \mathbb{Z}_{>0}$, and $H \in V_{\xi}$. First set $h_0 \in K_{E[b]}(U)$ to be the constant function defined by $h_0(u) = \|H\|_{1,b}$ for all $u \in U$. Then $H$ and $h_0$ satisfy Properties 3.(1a) and 3.(1b) in Theorem 4.2. Thus, given $h_j \in K_{E[b]}(U)$ for any $j \in \mathbb{Z}_{>0}$, Theorem 4.2 provides a $J_j \in \mathcal{J}(b)$ and we inductively obtain $h_{j+1} = N_{a,j}(h_j) \in K_{E[b]}(U)$. Then $\|M_{\xi,q}^m(H)(u)\|_2 \leq h_j(u)$ for all $u \in U$ and hence $\|M_{\xi,q}^m(H)\|_2 \leq \sup_{\|\xi\| \leq a_0} \|\xi\|_{\text{op}} \leq \max(1, Ne^{T_0})$

where we use operator norms for operators on $L^2(U, L^2(F_q, \mathbb{C}))$ and $L^2(U, \mathbb{R})$ respectively. Fix $C = C_M \tilde{\eta}^{-1}$. Then writing $k = jm + l$ for some integers $j \in \mathbb{Z}_{>0}$ and $0 \leq l < m$, we have

$\|M_{\xi,q}^k(H)\|_2 = \|M_{\xi,q}^{m+l}(H)\|_2$

$\leq C_M \|M_{\xi,q}^m(H)\|_2$

$\leq C_M \|\tilde{\eta}^l \|H\|_{1,b}$

$\leq C \|\tilde{\eta}^{-1} \cdot e^{-\eta(j+1)m} \|H\|_{1,b}$

$\leq Ce^{-\eta k} \|H\|_{1,b}$.

The subsequent subsections are devoted to the proof of Theorem 4.2. We continue with the preliminaries. The following Lemma 4.3 is derived from the hyperbolicity of the geodesic flow akin to Eq. (5).

Lemma 4.3. There exist $c_0 \in (0, 1)$ and $\kappa_1 > \kappa_2 > 1$ such that for all $j \in \mathbb{Z}_{>0}$, we have both

$c_0 \kappa_2 j d(u, u') \leq d(\sigma^j(u), \sigma^j(u')) \leq c_0^{-1} \kappa_1 j d(u, u')$

$c_0 \kappa_2 j D(u, u') \leq D(\sigma^j(u), \sigma^j(u')) \leq c_0^{-1} \kappa_1 j D(u, u')$

for all $u, u' \in \mathcal{C}$, for all cylinders $\mathcal{C} \subset U$ with $\text{len}(\mathcal{C}) = j$.

Remark. The second line of inequalities follow from the first. We fix constants $c_0 \in (0, 1)$ and $\kappa_1 > \kappa_2 > 1$ as in Lemma 4.3 for the rest of the section and use these inequalities without further comments.
Lemma 4.4. There exist $p_0 \in \mathbb{Z}_{>0}, \rho \in (0, 1)$ such that for all $l \in \mathbb{Z}_{>0}$, for all cylinders $C$ with $\text{len}(C) = l$, for all subcylinders $C', C'' \subset C$ with $\text{len}(C') = l + 1$ and $\text{len}(C'') = l + p_0$ respectively, we have
\[
\text{diam}_d(C'') \leq \rho \text{diam}_d(C') \leq \text{diam}_d(C')
\]

Fix $\theta_0 \in (0, 1)$ from the local non-integrability condition in [Sto11, Section 2], fix any $\theta_1 \in (\theta_0, 1)$ and finally fix $p_1 \in \mathbb{Z}_{>0}$ such that $\theta_0 < \theta_1 - 32 \rho^m$. We record here the main lemma of Stoyanov [Sto11, Lemma 4.2] which is the output of the local non-integrability condition in a usable form. We refer the reader to that paper for the proof and [Sto11, Definition 4.1] for notations not defined here. We do not included them here because we do not need to use them directly.

Lemma 4.5. There exist integers $1 \leq m_1 \leq m_0$, vectors $\eta_1, \eta_2, \ldots, \eta_0 \in E^n(w_0)$ and $U_0 \subset U_1$ which is a finite nonempty union of cylinders of length $m_1$ such that $U = \sigma^{m_1}(U_0) = \text{int}(U)$ is dense in $U$ and we have

1. for all integers $m \geq m_0$, for all $j \in \{1, 2\}$, for all integers $1 \leq \ell \leq \ell_0$, there exist Lipschitz sections $v^j_\ell : U \to U$ of $\sigma^m$ (i.e., $\sigma^m(v^j_\ell(u)) = u$ for all $u \in U$) and $v^j_\ell(U)$ is a finite union of cylinders of length $m$

2. $v^j_\ell(U) \cap v^{j'}_{\ell'}(U) = \emptyset$ for all $(j, \ell), (j', \ell') \in \{1, 2\} \times \{1, 2, \ldots, \ell_0\}$ with $(j, \ell) \neq (j', \ell')$

3. there exists $\delta_0 > 0$ such that for all integers $1 \leq \ell \leq \ell_0$, for all $s \in r^{-1}(U_0)$, for all $0 < |r| \leq \delta_0$, for all $\eta \in B_\ell$ with $s + \eta \in r^{-1}(U_0)$, we have
\[
\frac{1}{\ell} \left( (\tau_m \circ v^2_\ell \circ \sigma^{m_1} - \tau_m \circ v^1_\ell \circ \sigma^{m_1})(r(s + \eta)) \right)
\]
\[
\quad - (\tau_m \circ v^1_\ell \circ \sigma^{m_1} - \tau_m \circ v^1_\ell \circ \sigma^{m_1})(r(s)) \geq \frac{\delta_0}{2}
\]

4. for all cylinders $C \subset U_0$, there exists $\delta' > 0$ such that
\[
C \subset \bigcup_{\ell=1}^{\ell_0} M^{\delta}_{\eta}(C)
\]
for all $\delta \in (0, \delta')$.

Fix $m_0, m_1, \ell_0, U, \delta_0$ as in Lemma 4.5. Since $U_0$ is a finite union of cylinders, by Property 4 in Lemma 4.5, we can fix a $\delta_1 > 0$ such that
\[
U_0 \subset \bigcup_{\ell=1}^{\ell_0} M^{\delta}_{\eta}(U_0)
\]
for all $\delta \in (0, \delta_1]$. Fix some $A_0 > 2c_0^{-1} e^{-\frac{\tau_0}{2m} - \ell \max(1, \frac{\tau_0}{\kappa^2 - 1})}$.

Lemma 4.6. For all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0' \text{ and } |b| > 1$, for all nontrivial ideals $q \subset O_K$, for all $k \in \mathbb{Z}_{>0}$, we have

1. if $h \in K_B(U)$ for some $B > 0$, then for all $j \in A$, for all $u, u' \in U_j$, we have
\[
|L^k_u(h)(u) - L^k_u(h)(u')| \leq A_0 \left( \frac{B}{\kappa^2} + 1 \right) L^k_u(h)(u) D(u, u')
\]
(2) if $H \in C(U, L^2(F_q, \mathbb{C}))$ and $h \in B(U, \mathbb{R})$ satisfy

$$
\|H(u) - H(u')\|_2 \leq B h(u) D(u, u')
$$

for all $u, u' \in U_j$, for all $j \in A$, for some $B > 0$, then for all $j \in A$, for all $u, u' \in U_j$, we have

$$
\|\mathcal{M}_{\xi,q}^k(H)(u) - \mathcal{M}_{\xi,q}^k(H)(u')\|_2 \leq A_0 \left( \frac{B}{\kappa_2} \mathcal{L}_{\sigma}^k(h)(u) + |b| \mathcal{L}_{\sigma}^k\|H\|(u) \right) D(u, u').
$$

Proof. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0'$ and $|b| > 1$, $q \in \mathcal{O}_R$ be a nontrivial ideal, and $k \in \mathbb{Z}_{>0}$. To prove Property 1, let $h \in K_B(U)$ for some $B > 0$. We consider $u, u' \in \text{int}(U_j)$ for some $j \in A$ and refer the reader to Definition 2.8 for the case when $u \in \partial(U_j)$ or $u' \in \partial(U_j)$. Let $v \in \sigma^{-k}(u)$ and $C$ be the cylinder with $\text{len}(C) = k$ containing $v$. Since $\sigma^k_C : C \to \text{int}(U_j)$ is a homeomorphism, there is a unique element which we denote by $\varphi(v) = (\sigma^k_C)^{-1}(u') \in C$ with $\sigma^k(\varphi(v)) = u'$. Since $v \in \sigma^{-k}(u)$ was arbitrary, we see that $\varphi$ defines a bijective map $\varphi : \sigma^{-k}(u) \to \sigma^{-k}(u')$. Using Lemma 4.3, for all integers $0 \leq l \leq k$, we have $d(\sigma^l(v), \sigma^l(\varphi(v))) \leq \frac{1}{c_0 \kappa_2^{-l}} d(u, u')$ and so

$$
\left| f_k^{(a)}(v) - f_k^{(a)}(\varphi(v)) \right| \leq \sum_{l=0}^{k-1} \left| f_k^{(a)}(\sigma^l(v)) - f_k^{(a)}(\sigma^l(\varphi(v))) \right|
$$

$$
\leq \sum_{l=0}^{k-1} \text{Lip}_d^e(f_k^{(a)}) \cdot d(\sigma^l(v), \sigma^l(\varphi(v)))
$$

$$
\leq \sum_{l=0}^{k-1} \frac{T_0}{c_0 \kappa_2^{k-l}} d(u, u')
$$

$$
\leq \frac{T_0}{c_0 (\kappa_2 - 1)} D(u, u').
$$

In the same way, we have a similar bound $|\tau_k(v) - \tau_k(\varphi(v))| \leq \frac{T_0}{c_0 (\kappa_2 - 1)} D(u, u')$. Using $\text{diam}_d(U_j) \leq \delta < 1$, we note that

$$
\left| 1 - e^{f_k^{(a)}(\varphi(v)) - f_k^{(a)}(v)} \right| \leq e^{\left| f_k^{(a)}(v) - f_k^{(a)}(\varphi(v)) \right|} \left| f_k^{(a)}(v) - f_k^{(a)}(\varphi(v)) \right| \leq A_0 D(u, u')
$$

(12)

and similarly

$$
\left| 1 - e^{(f_k^{(a)} + ib\tau_k)(\varphi(v)) - (f_k^{(a)} + ib\tau_k)(v)} \right|
$$

$$
\leq e^{\left| f_k^{(a)}(v) - f_k^{(a)}(\varphi(v)) \right|} \left| (f_k^{(a)} + ib\tau_k)(v) - (f_k^{(a)} + ib\tau_k)(\varphi(v)) \right| \leq |b| A_0 D(u, u').
$$

(13)
Thus, again using \( \text{diam}_d(U_j) \leq \hat{\delta} < 1 \), we have
\[
\left| L^k_a(h)(u) - L^k_a(h)(u') \right| \\
= \left| \sum_{v \in \sigma^{-k}(u)} e_{f^k(v)} h(v) - \sum_{v' \in \sigma^{-k}(u')} e_{f^k(v')}(v') h(v') \right| \\
\leq \sum_{v \in \sigma^{-k}(u)} \left| e_{f^k(v)} h(v) - e_{f^k(v)}(\varphi(v)) h(\varphi(v)) \right| \\
\leq \sum_{v \in \sigma^{-k}(u)} e_{f^k(\varphi(v))} |h(v) - h(\varphi(v))| + \sum_{v \in \sigma^{-k}(u)} \left| e_{f^k(v)}(v') - e_{f^k(\varphi(v))}(v') \right| h(v) \\
\leq \sum_{v \in \sigma^{-k}(u)} c_0 A_0 e_{f^k(v)} Bh(v) D(v, \varphi(v)) \\
+ \sum_{v \in \sigma^{-k}(u)} \left| 1 - e_{f^k(\varphi(v))}(v') - e_{f^k(v)}(v') \right| e_{f^k(v)} h(v).
\]

Now we use Eq. (12) and continue the bound as
\[
\left| L^k_a(h)(u) - L^k_a(h)(u') \right| \\
\leq c_0 A_0 B \frac{1}{\kappa\kappa} D(u, u') \sum_{v \in \sigma^{-k}(u)} e_{f^k(v)} h(v) + A_0 D(u, u') \sum_{v \in \sigma^{-k}(u)} e_{f^k(v)} h(v) \\
= \frac{A_0 B}{\kappa\kappa^2} L^k_a(h)(u) D(u, u') + A_0 L^k_a(h)(u) D(u, u') \\
= A_0 \left( \frac{B}{\kappa\kappa^2} + 1 \right) L^k_a(h)(u) D(u, u').
\]

Now to prove Property 2, suppose \( H \in C(U, L^2(F_q, \mathbb{C})) \) and \( h \in B(U, \mathbb{R}) \) satisfy \( \|H(u) - H(u')\|_2 \leq B h(u) D(u, u') \). Define the bijective map \( \varphi : \sigma^{-k}(u) \rightarrow \sigma^{-k}(u') \) as before. Similar to above, noticing that \( c^k(v) = c^k(\varphi(v)) \), we can use the same estimates to get
\[
\left\| M^k_{\xi,q}(H)(u) - M^k_{\xi,q}(H)(u') \right\|_2 \\
\leq \left\| \sum_{v \in \sigma^{-k}(u)} e_{f^k(v) + ib\tau_k(v)} c^k(v) H(v) - \sum_{v' \in \sigma^{-k}(u')} e_{f^k(v') + ib\tau_k(v')} c^k(v') H(v') \right\|_2 \\
\leq \sum_{v \in \sigma^{-k}(u)} \left\| e_{f^k(v) + ib\tau_k(v)} c^k(v) H(v) - e_{f^k(v) + ib\tau_k(v)} c^k(\varphi(v)) H(\varphi(v)) \right\|_2 \\
+ \sum_{v \in \sigma^{-k}(u)} \left\| e_{f^k(v) + ib\tau_k(v)} - e_{f^k(\varphi(v)) + ib\tau_k(v)} \right\| \|H(v)\|_2 \\
\leq \sum_{v \in \sigma^{-k}(u)} c_0 A_0 e_{f^k(v)} Bh(v) D(v, \varphi(v)) \\
+ \sum_{v \in \sigma^{-k}(u)} \left| 1 - e_{f^k(\varphi(v))}(v') - e_{f^k(v)}(v') \right| e_{f^k(v)} \|H\|(v).
Using Eq. (13), we can continue the bound as

\[ \left\| \mathcal{M}_{\xi,q}(H)(u) - \mathcal{M}_{\xi,q}(H)(u') \right\|_2 \leq c_0 A_0 B \cdot \frac{1}{c_0 \kappa_2^{m}} D(u, u') \sum_{v \in \sigma^{-k}(u)} e^{f_k^{(v)}(\nu)h(v)} \]

\[ + |b| A_0 D(u, u') \sum_{v \in \sigma^{-k}(u)} e^{f_k^{(v)}(\nu)} \|H\|(v) \]

\[ \leq \frac{A_0 B}{\kappa_2^{m}} C_k(h)(u) D(u, u') + |b| A_0 C_k \|H\|(u) D(u, u') \]

\[ = A_0 \left( \frac{B}{\kappa_2^{m}} C_k(h)(u) + |b| C_k \|H\|(u) \right) D(u, u'). \]

Now we fix some positive constants

\[ E > \max(1, 2A_0) \]  \hspace{1cm} (14)

\[ \epsilon_1 < \min \left( \frac{c_0 E_0}{\kappa_1^{m_1}}, \frac{\pi c_0^2 (\kappa_2 - 1)}{2T_0 \kappa_1^{m_1}} \right) \]  \hspace{1cm} (15)

\[ m > m_0 \text{ such that } \kappa_2^m > \max \left( 8A_0, \frac{4E^p \kappa_1^{m_1} \epsilon_1}{c_0^2}, \frac{4 \cdot 128E \kappa_1^{m_1}}{c_0^2 \delta_0 \rho_1} \right) \]  \hspace{1cm} (16)

\[ \mu < \min \left( \frac{E_1 \epsilon_1 c_0^2 \rho_0^p \kappa_2^{m_1}}{\kappa_1^{m}}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16 \delta_0 \rho_1}, \left( \frac{\delta_0 \rho_1}{64} \right)^2 \right) \]  \hspace{1cm} (17)

and also fix \( b_0 = 1 \). For all \( j \in \{1, 2\} \), for all integers \( 1 \leq l \leq \ell_0 \), fix \( v_j \) to be the corresponding sections of \( \sigma^m \) provided by Lemma 4.5.

After constructing the Dolgopyat operators in Subsection 4.2, Theorem 4.2 is a simple consequence of Lemmas 4.9, 4.10, 4.13 and 4.18.

4.2. Construction of Dolgopyat operators. For all \( |b| > b_0 \), we define the set \( \{ C_1(b), C_2(b), \ldots, C_{b_0}(b) \} \) of maximal cylinders \( C \subset U_0 \) with \( \text{diam}_d(C) \leq \frac{c_0}{|b|} \) so that \( U_0 = \bigcup_{j=1}^{b_0} C_j(b) \).

Remark. As a consequence of Lemma 4.3 and Eq. (15), we have \( \text{len}(C_j(b)) \geq m_1 + 1 \) for all integers \( 1 \leq j \leq b_0 \), for all \( |b| > b_0 \).

Now, Corollary 2.11.1 implies the following lemma.

**Lemma 4.7.** For all \( |b| > b_0 \), for all integers \( 1 \leq l \leq c_0 \), if \( u, u' \in C_l(b) \), then \( e^m(v_j^l(\sigma^{m_1}(u))) = e^m(v_j^l(\sigma^{m_1}(u'))) \) for all \( j \in \{1, 2\} \), for all integers \( 1 \leq l \leq \ell_0 \).

Let \( |b| > b_0 \). By Lemma 4.7, we can define \( C_{l,j,k}(b) = e^m(v_j^l(\sigma^{m_1}(u))) \) for any choice of \( u \in C_l(b) \), for all integers \( 1 \leq l \leq c_0 \), for all \( j \in \{1, 2\} \), for all integers \( 1 \leq \ell \leq \ell_0 \). We define \( \{ D_1(b), D_2(b), \ldots, D_{p_0}(b) \} \) to be the set of subcylinders \( D \subset C_l(b) \) for some integer \( 1 \leq l \leq c_0 \) with \( \text{len}(D) = \text{len}(C_l(b)) + p_0 \rho_1 \). We say \( D_1(b) \) and \( D_2(b) \) are adjacent if \( D_1(b), D_2(b) \subset C_l(b) \) for some integer \( 1 \leq l \leq c_0 \). We define \( \Xi(b) = \{ 1, 2, \ldots, p_0 \} \times \{ 1, 2 \} \times \{ 1, 2, \ldots, \ell_0 \} \) and we also define \( \Xi_0(b) = \sigma^{m_1}(D_k(b)) \) and \( \chi_j^l(b) = v_j^l(\Xi_0(b)) \) for all \( k, j, \ell \in \Xi(b) \). Note that \( \chi_j^l(b) \cap \chi_{j'}^{l'}(b) = \emptyset \) for all \( (k, j, \ell), (k', j', \ell') \in \Xi(b) \) with \( (k, j, \ell) \neq (k', j', \ell') \). For all \( J \subset \Xi(b) \), we define
the function $\beta_J = \chi_U - \mu \sum_{(k,j,\ell) \in J} X_{\ell,k}(b)$, and it can be checked that in fact $\beta_J \in C^{\text{Lip}(D)}(U, \mathbb{R})$.

Let $|b| > b_0$. Here we record a number of basic facts derived from Lemmas 4.3 and 4.4 regarding the above definitions which will be required later.

(1) We have the diameter bounds

$$\frac{\epsilon_1 \rho^p}{|b|} \leq \text{diam}_d(C(b)) \leq \frac{\epsilon_1}{|b|}$$

(18)

$$\rho^{p \rho_{p1} + 1} \frac{\epsilon_1}{|b|} \leq \text{diam}_d(D_k(b)) \leq \rho^{p_1} \frac{\epsilon_1}{|b|}$$

(19)

$$\frac{\rho^{p \rho_{p1} + 1} \epsilon_1}{|b|} \leq \text{diam}_d(Z_k(b)) \leq \rho^{p_1} \frac{\epsilon_1}{|b|}$$

(20)

$$\frac{\epsilon_1 c_0^2 \rho^{p \rho_{p1} + 1} \kappa_2^{m_1}}{|b| \kappa_1^m} \leq \text{diam}_d(X_{\ell,k}(b)) \leq \frac{\epsilon_1 \rho^{p_1} \kappa_1^{m_1}}{|b| c_0^2 \kappa_2^m}$$

(21)

(22)

for all integers $1 \leq l \leq c_b$, for all $(k, j, \ell) \in \Xi(b)$.

(2) Let $J \subset \Xi(b)$. The function $\beta_J$ is $D$-Lipschitz with Lipschitz constant

$$\text{Lip}_D(\beta_J) \leq \frac{\mu}{\min_{(k,j,\ell) \in J} \text{diam}_d(X_{\ell,k}^I(b))} \leq \frac{\mu |b| \kappa_1^m}{\epsilon_1 c_0^2 \rho^{p \rho_{p1} + 1} \kappa_2^{m_1}}.$$  

(23)

(3) Let $1 \leq l \leq c_b$ be an integer. If $u, u' \in \sigma^{m_1}(C(b))$, then

$$D(v_J^I(u), v_J^I(u')) \leq \frac{\epsilon_1 \kappa_1^{m_1}}{|b| c_0^2 \kappa_2^m}$$

(24)

and

$$|b| \cdot |(\tau_m (v_J^I(u)) - \tau_m (v_J^I(u'))) - (\tau_m (v_J^I(u')) - \tau_m (v_J^I(u)))| \leq \pi$$

(25)

for all $j \in \{1, 2\}$, for all integers $1 \leq \ell \leq \ell_0$, by using Eq. (15).

For all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0'$ and $|b| > b_0$, for all $J \subset \Xi(b)$, we define the Dolgopyat operators $N_{a,J} : C^{\text{Lip}(D)}(U, \mathbb{R}) \to C^{\text{Lip}(D)}(U, \mathbb{R})$ by $N_{a,J}(h) = \mathcal{L}^m_a(\beta_J h)$ for all $h \in C^{\text{Lip}(D)}(U, \mathbb{R})$.

**Definition 4.8.** For all $|b| > b_0$, a subset $J \subset \Xi(b)$ is said to be dense if for all integers $1 \leq l \leq c_b$, there exists $(k, j, \ell) \in J$ such that $D_k(b) \subset C_l(b)$. Denote $\mathcal{J}(b)$ to be the set of all dense subsets of $\Xi(b)$.

**4.3. Proof of Properties 1 and 3.(2b) in Theorem 4.2.**

**Lemma 4.9.** There exits $a_0 > 0$ such that for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| > b_0$, for all $J \subset \mathcal{J}(b)$, we have $N_{a,J}(K_{E|b|}(U)) \subset K_{E|b|}(U)$.

**Proof.** Fix $a_0 = a_0'$ and recall that we already fixed $b_0 = 1$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| > b_0$ and $J \in \mathcal{J}(b)$. Let $h \in K_{E|b|}(U)$ and $u, u' \in U_j$ for some
Lemma 4.10. There exists \( a_0 > 0 \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( |b| > b_0 \), for all nontrivial ideals \( q \subset O_K \), if \( h \in B(U, \mathbb{R}) \) and \( H \in C(U, L^2(F_q, \mathbb{C})) \) satisfy Properties 3.(1a) and 3.(1b) in Theorem 4.2, then for all \( J \in \mathcal{J}(b) \) we have

\[
\| \mathcal{M}_{\xi,q}^m(H)(u) - \mathcal{M}_{\xi,q}^m(H)(u') \|_2 \leq E|b|N_{a,J}(h)(u) D(u, u')
\]

for all \( u, u' \in U_J, \) for all \( j \in A \).

Proof. Fix \( a_0 = a_0' \) and recall that we already fixed \( b_0 = 1 \). Let \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( |b| > b_0 \). Let \( q \subset O_K \) be a nontrivial ideal and suppose \( H \in C(U, L^2(F_q, \mathbb{C})) \) and \( h \in B(U, \mathbb{R}) \) satisfy Properties 3.(1a) and 3.(1b) in Theorem 4.2. Let \( J \in \mathcal{J}(b) \) and \( u, u' \in U_J \) for some \( j \in A \). Applying Lemma 4.6 and Eqs. (14), (16) and (17), we have

\[
\| \mathcal{M}_{\xi,q}^m(H)(u) - \mathcal{M}_{\xi,q}^m(H)(u') \|_2 \leq A_0 \left( E|b| + \frac{E|b|}{8A_0} + \frac{E|b|}{2A_0} \right) L_a^m(h)(u) D(u, u')
\]

\[
\leq A_0 \left( \frac{E|b|}{8(1-\mu)} + \frac{E|b|}{2(1-\mu)} \right) L_a^m(h)(u) D(u, u')
\]

\[
\leq \left( \frac{E|b|}{6} + \frac{2E|b|}{3} \right) N_{a,J}(h)(u) D(u, u')
\]

\[
\leq E|b|N_{a,J}(h)(u) D(u, u').
\]
4.4. Proof of Property 2 in Theorem 4.2.

**Definition 4.11.** We say that a subset \( W \subset U \) is \((t, S)\)-dense if there exists a set of mutually disjoint cylinders \( \{B_1, B_2, \ldots, B_k\} \) for some \( k \in \mathbb{Z}_{>0} \), with \( \bigcup_{j=1}^{k} B_j = U \) such that for all integers \( 1 \leq j \leq k \), we have

1. \( \text{diam}_d(B_j) \leq tS \)
2. there exists a subcylinder \( B'_j \subset W \cap B_j \) with \( \text{diam}_d(B'_j) \geq t \).

**Lemma 4.12.** Let \( B > 0, C \geq 1 \). There exists \( \eta \in (0, 1) \) such that for all \( t > 0 \), for all \((t, C)\)-dense subset \( W \subset U \), for all \( h \in K_{B/t}(U) \), we have \( \int_{U} h^2 \, d\nu_U \geq \eta \int_{U} h^2 \, d\nu_U \).

**Proof.** Let \( B > 0, C \geq 1 \). Fix \( \eta = e^{-2BC} \cdot \frac{c_U}{c'_2} e^{-p_0 \delta_B \tau(1 - \frac{\log(C)}{\log(p)})} \in (0, 1) \). Let \( t > 0 \), \( W \subset U \) be a \((t, C)\)-dense subset and \( \{B_1, B_2, \ldots, B_k\} \) be the set of mutually disjoint cylinders provided by Definition 4.11. Then \( \bigcup_{j=1}^{k} B_j = U \) and hence \( \sum_{j=1}^{k} \nu_U(B_j) = 1 \) and \( \text{diam}_d(B_j) \leq tC \) for all integers \( 1 \leq j \leq k \). Let \( h \in K_{B/t}(U) \). Setting \( l_j = \inf_{u \in B_j} h(u) \) and \( L_j = \sup_{u \in B_j} h(u) \), and using the properties \( h \in K_{B/t}(U) \) and convexity of \( -\log \), we can derive that \( h(u) \geq l_j \geq L_j e^{-BC} \). Let \( 1 \leq j \leq k \) be an integer. Now, since \( W \) is \((t, C)\)-dense, there is a subcylinder \( B'_j \subset W \cap B_j \) such that \( \text{diam}_d(B'_j) \geq t \). Write \( \text{len}(B'_j) = \text{len}(B_j) + r_j p_0 + s_j \) for some \( r_j \in \mathbb{Z}_{\geq 0} \) and some integer \( 0 \leq s_j < p_0 \). By Lemma 4.4 we have \( t \leq \text{diam}_d(B'_j) \leq r_j \text{diam}(B_j) \leq r_j C t \) which implies \( r_j \leq -\frac{\log(C)}{\log(p)} \). Hence by the property of Gibbs measures in Eq. (1), we have \( \frac{\nu_U(B'_j)}{\nu_U(B_j)} \geq \frac{c'_U}{c_2} e^{-(r_j p_0 + s_j) \delta_B \tau} \geq \frac{c'_U}{c_2} e^{-p_0 \delta_B \tau(1 - \frac{\log(C)}{\log(p)})} \). Thus for all integers \( 1 \leq j \leq k \), we have \( \nu_U(W \cap B_j) \geq \nu_U(B'_j) \geq \frac{c'_U}{c_2} e^{-p_0 \delta_B \tau(1 - \frac{\log(C)}{\log(p)})} \nu_U(B_j) \). Thus we calculate that

\[
\int_{W} h(u)^2 \, d\nu_U(u) = \sum_{j=1}^{k} \int_{W \cap B_j} h(u)^2 \, d\nu_U(u) \geq \sum_{j=1}^{k} l_j^2 \nu_U(W \cap B_j) \\
\geq \sum_{j=1}^{k} L_j^2 e^{-2BC} \nu_U(W \cap B_j) \\
\geq e^{-2BC} \cdot \frac{c'_U}{c_2} e^{-p_0 \delta_B \tau(1 - \frac{\log(C)}{\log(p)})} \sum_{j=1}^{k} L_j^2 \nu_U(B_j) \\
\geq \eta \sum_{j=1}^{k} \int_{B_j} h(u)^2 \, d\nu_U(u) \\
= \eta \int_{U} h(u)^2 \, d\nu_U(u).
\]

**Lemma 4.13.** There exist \( a_0 > 0 \) and \( \eta \in (0, 1) \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( |b| > b_0 \), for all \( J \in \mathcal{J}(b) \), for all \( h \in K_{|E|b}(U) \), we have \( \|\mathcal{N}_{a,J}(h)\|_2 \leq \eta \|h\|_2 \).
Proof. Fix $B = E_1C_0 \rho^{p_0p_1+1} \kappa_2^{m_1} > 0$ and $C = \frac{\kappa_1 \kappa_2^{m_1}}{e_0^2 \rho^{p_0p_1+1} \kappa_2^{m_1}} \geq 1$. Fix $\eta' \in (0, 1) \in (0, 1)$. Recall that we already fixed $b_0 = 1$. Let $\xi = a + ib \in \mathbb{C}$ with $|a| < a_0$ and $|b| > b_0$. Let $J \in \mathcal{J}(b)$ and $h \in K_E(b)(U)$. We have the estimate $\mathcal{N}_{a,J}(h)^2 \leq e^{mA_{J}a_0}N_{0,J}(h)^2$ since $|f(\alpha) - f(0)| \leq A_f |\alpha| < A_J a_0$ and by the Cauchy-Schwarz inequality, we have

$$\mathcal{N}_{0,J}(h)^2 = \mathcal{L}_0^m(\beta_J h)^2 \leq \mathcal{L}_0^m(\beta_J L^2)(h^2).$$

We would like to apply Lemma 4.12 on $h$ but first we need to ensure all the hypotheses hold. Let $t = \frac{\varepsilon \rho^{p_0p_1+1} \kappa_2^{m_1}}{|b|}$ and note that $\frac{B}{t} = E|b|$. Let $W = \bigcup_{k=1}^{p_0} Z_k(b) \subset U$. We will show that $W$ is $(t, C)$-dense. Let $B_l = \sigma^{m_l}(C_l(b))$ for all integers $1 \leq l \leq c_b$. Then $B_l \cap B_l' = \emptyset$ and diam$_d(B_l) \leq lC$ by Lemma 4.3 for all integers $1 \leq l, l' \leq c_b$ with $l \neq l'$. Since $J$ is dense, for all integers $1 \leq l \leq c_b$, there is a $(k, j, \ell) \in J$ such that $D_k(b) \subset C_l(b)$ and so $Z_k(b) = \sigma^{m_l}(D_k(b)) \subset W \cap B_l$ with diam$_d(Z_k(b)) \geq t$ by Lemma 4.3. Hence $W$ is $(t, C)$-dense. Since $h \in K_{E}(b)(U)$, we can apply Lemma 4.12 to get $\int_W h^2 d\nu_U \geq \eta' \int_U h^2 d\nu_U$. Now using $\mathcal{L}_0^m(\nu_U) = \nu_U$ we conclude that $\int_W \mathcal{L}_0^m(\nu_U) \geq \eta' \int_U \mathcal{L}_0^m(h^2) d\nu_U$. Note that $\mathcal{L}_0^m(\beta_J)^2(u) \leq 1 - \mu e^{-mt} \geq \mu e^{-mt}$ for all $u \in W$ by choosing any one section whose image is $X_{j,k}^*$ for some $(k, j, \ell) \in J$. So putting everything together, we have

$$\int_U N_{0,J}(h^2) d\nu_U \leq \int_U e^{mA_{J}a_0}N_{0,J}(h)^2 d\nu_U \leq e^{mA_{J}a_0} \left( \int_W \mathcal{L}_0^m(\beta_J)^2 \mathcal{L}_0^m(h^2) d\nu_U + \int_{U \setminus W} \mathcal{L}_0^m(\beta_J)^2 \mathcal{L}_0^m(h^2) d\nu_U \right) \leq e^{mA_{J}a_0} \left( \int_W \mathcal{L}_0^m(h^2) d\nu_U \right) \leq e^{mA_{J}a_0} \left( 1 - \mu e^{-mt} \right) \int_W \mathcal{L}_0^m(h^2) d\nu_U = \eta^2 \int_U h^2 d\nu_U.$$

4.5. Proof of Property 3.2a in Theorem 4.2. We start with a lemma which is derived from Lemma 4.5. This is exactly [Sto11, Lemma 5.9] and we refer the reader to that paper for the proof.

Lemma 4.14. Suppose $D_k(b), D_{k'}(b) \subset C_l(b)$ for some integers $1 \leq k, k' \leq p_0$ and $1 \leq l \leq c_b$ such that $d(u_0, u_0') \geq \frac{1}{2} \text{diam}_d(C_l(b))$ and $\left\| \frac{r^{-1}(u_0) - r^{-1}(u_0')}{\|r^{-1}(u_0) - r^{-1}(u_0')\|}, \eta \right\| \geq \theta_1$
for some $u_0 \in D_k(b), u'_0 \in D_k(b)$ and some integer $1 \leq \ell \leq \ell_0$. Then we have
\[
|b| \cdot |(\tau_m(v^\ell_2(u)) - \tau_m(v^\ell_1(u))) - (\tau_m(v^\ell_2(u')) - \tau_m(v^\ell_1(u')))| \geq \frac{\delta_0 \rho \varepsilon_1}{16}
\]
for all $u \in Z_k(b)$, for all $u' \in Z_k(b)$.

Now, for all $\xi = a + ib \in \mathbb{C}$ with $|a| < a'_0$ and $|b| > b_0$, for all integers $1 \leq \ell \leq \ell_0$, for all $H \in C(U, L^2(F_q, \mathbb{C}))$ for some nontrivial ideal $q \subset \mathcal{O}_K$, for all $h \in K_{E|b}(U)$, we define the functions $\chi^f_1[\xi, H, h], \chi^f_2[\xi, H, h] : U \to \mathbb{R}$ by
\[
\chi^f_1[\xi, H, h](u) = \left\| e^{(f^m_{a_0} + ib\tau_m)(v^\ell_2(u))} c_{1, 1, \ell}(b)^{-1} H(v^\ell_2(u)) + e^{(f^m_{a_0} + ib\tau_m)(v^\ell_1(u))} c_{1, 2, \ell}(b)^{-1} H(v^\ell_1(u)) \right\|_2

(1 - \mu) e^{f^m_{a_0}(v^\ell_2(u))} h(v^\ell_2(u)) + e^{f^m_{a_0}(v^\ell_1(u))} h(v^\ell_1(u))
\]
and
\[
\chi^f_2[\xi, H, h](u) = \left\| e^{(f^m_{a_0} + ib\tau_m)(v^\ell_2(u))} c_{1, 1, \ell}(b)^{-1} H(v^\ell_2(u)) + e^{(f^m_{a_0} + ib\tau_m)(v^\ell_1(u))} c_{1, 2, \ell}(b)^{-1} H(v^\ell_1(u)) \right\|_2

(1 - \mu) e^{f^m_{a_0}(v^\ell_2(u))} h(v^\ell_2(u)) + e^{f^m_{a_0}(v^\ell_1(u))} h(v^\ell_1(u))
\]
for all $u \in \sigma^{m_1}(C_l(b))$, for all integers $1 \leq l \leq c_k$. We need another lemma before proving Property 3.2(a) in Theorem 4.2.

**Lemma 4.15.** Let $|b| > b_0$ and $q \subset \mathcal{O}_K$ be a nontrivial ideal. Suppose $H \in C(U, L^2(F_q, \mathbb{C}))$ and $h \in K_{E|b}(U)$ satisfy Properties 3.1(a) and 3.1(b) in Theorem 4.2. Then for all $(k, j, \ell) \in \Xi(b)$, we have
\[
\frac{1}{2} \leq \frac{h(v^\ell_j(u))}{h(v^\ell_j(u'))} \leq 2
\]
for all $u, u' \in Z_k(b)$ and also either of the alternatives
1. $\|H(v^\ell_j(u))\|_2 \leq \frac{2}{3} h(v^\ell_j(u))$ for all $u \in Z_k(b)$
2. $\|H(v^\ell_j(u'))\|_2 \geq \frac{4}{3} h(v^\ell_j(u))$ for all $u \in Z_k(b)$.

**Proof.** Recall that we already fixed $b_0 = 1$. Let $|b| > b_0$ and $q \subset \mathcal{O}_K$ be a nontrivial ideal. Suppose $H \in C(U, L^2(F_q, \mathbb{C}))$ and $h \in K_{E|b}(U)$ satisfy Properties 3.1(a) and 3.1(b) in Theorem 4.2. Let $(k, j, \ell) \in \Xi(b)$. We first note that using Eqs. (16) and (21), for all $u, u' \in Z_k(b)$, we have
\[
h(v^\ell_j(u')) \leq h(v^\ell_j(u)) + E|b|h(v^\ell_j(u))D(v^\ell_j(u), v^\ell_j(u'))
\]
\[
\leq h(v^\ell_j(u))(1 + E|b|\operatorname{diam}(X^f_{j,k}))
\]
\[
\leq h(v^\ell_j(u)) \left(1 + E|b| \frac{c_1p_{K^{m_1}}}{|b|c_0^2c_2^m}\right)
\]
\[
\leq 2h(v^\ell_j(u))
\]
which proves the first sequence of inequalities. If $\|H(v^\ell_j(u))\|_2 \geq \frac{1}{4} h(v^\ell_j(u))$ for all $u \in Z_k(b)$, then we are done. Otherwise there is a $u_0 \in Z_k(b)$ such that $\|H(v^\ell_j(u_0))\|_2 \leq \frac{1}{4} h(v^\ell_j(u_0))$. Then for all $u \in Z_k(b)$, using the above gives
\[
\|H(v^\ell_j(u_0))\|_2 \leq \frac{1}{4} h(v^\ell_j(u_0)) \leq \frac{1}{2} h(v^\ell_j(u)).
\]
Similarly proceeding as above, using Eqs. (16) and (21), for all \( u \in \mathbb{Z}_k(b) \), we have
\[
\|H(v_j^0(u))\|_2 \leq \|H(v_j^0(u_0))\|_2 + E|b|h(v_j^0(u))D(v_j^0(u), v_j^0(u_0))
\leq \frac{1}{2}h(v_j^0(u)) + E|b|h(v_j^0(u))\text{diam}_d(X_{j,k}^0)
\leq h(v_j^0(u)) \left( \frac{1}{2} + E|b| \cdot \frac{\epsilon_1 \rho^{\alpha_k} \kappa_1 \kappa_2}{|b| c_0^2 \kappa_2^m} \right)
\leq \frac{3}{4}h(v_j^0(u)).
\]

For any integer \( k \geq 2 \), let \( \Theta : (\mathbb{R}^k \setminus \{0\}) \times (\mathbb{R}^k \setminus \{0\}) \rightarrow [0, \pi] \) be the map which gives the angle defined by \( \Theta(w_1, w_2) = \arccos \left( \frac{\langle w_1, w_2 \rangle}{\| w_1 \| \| w_2 \|} \right) \) for all \( w_1, w_2 \in \mathbb{R}^k \setminus \{0\} \), where we use the standard inner product and norm.

**Lemma 4.16.** Let \( k \geq 2 \) be an integer. Suppose \( w_1, w_2 \in \mathbb{R}^k \setminus \{0\} \) such that \( \Theta(w_1, w_2) \geq \alpha \) and \( \frac{\| w_1 \|}{\| w_2 \|} \leq L \) for some \( \alpha \in [0, \pi] \) and \( L \geq 1 \). Then we have a stronger version of the triangle inequality
\[
\| w_1 + w_2 \| \leq \left( 1 - \frac{\alpha^2}{16L} \right) \| w_1 \| + \| w_2 \|.
\]

**Proof.** Let \( k \geq 2 \) be an integer. Let \( w_1, w_2 \in \mathbb{R}^k \setminus \{0\} \) such that \( \Theta(w_1, w_2) \geq \alpha \) and \( \frac{\| w_1 \|}{\| w_2 \|} \leq L \) for some \( \alpha \in [0, \pi] \) and \( L \geq 1 \). Let \( \epsilon = \frac{\alpha^2}{16L} \in [0, 1] \). Using the cosine law, we have
\[
\| w_1 + w_2 \|^2 = \| w_1 \|^2 + 2\| w_1 \| \cdot \| w_2 \| \cos(\Theta(w_1, w_2)) + \| w_2 \|^2
\leq \| w_1 \|^2 + 2\| w_1 \| \cdot \| w_2 \| \cos(\alpha) + \| w_2 \|^2.
\]

Hence it suffices to show that \( \| w_1 \|^2 + 2\| w_1 \| \cdot \| w_2 \| \cos(\alpha) + \| w_2 \|^2 \leq ((1 - \epsilon)\| w_1 \| + \| w_2 \|)^2 \) since \( (1 - \epsilon)\| w_1 \| + \| w_2 \| \geq 0 \). Using the double angle formula and some simplification, we see that this is equivalent to showing
\[
2\epsilon \left( \frac{\| w_1 \|}{\| w_2 \|} + 1 \right) - \epsilon^2 \frac{\| w_1 \|}{\| w_2 \|} \leq 4 \sin^2 \left( \frac{\alpha}{2} \right).
\]

Indeed, by our choice of \( \epsilon \) and the simple inequality \( \frac{\theta}{2} \leq \sin(\theta) \) for all \( \theta \in [0, \pi] \), we have
\[
2\epsilon \left( \frac{\| w_1 \|}{\| w_2 \|} + 1 \right) - \epsilon^2 \frac{\| w_1 \|}{\| w_2 \|} \leq 2\epsilon \left( \frac{\| w_1 \|}{\| w_2 \|} + 1 \right) \leq 4\epsilon L = 4 \cdot \frac{\alpha^2}{16} \leq 4 \sin^2 \left( \frac{\alpha}{2} \right).
\]

**Lemma 4.17.** Let \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0' \) and \( |b| > b_0 \). Let \( q \subset \mathcal{O}_k \) be a nontrivial ideal and suppose \( H \in C(U, L^2(F_q, \mathbb{C})) \) and \( h \in K_{E|b|}(U) \) satisfy Properties 3.1(a) and 3.1(b) in Theorem 4.2. For all integers \( 1 \leq l \leq c_0 \), there exists \( (k, j, \ell) \in \Xi(b) \) such that \( \mathcal{D}_k(b) \subset \mathcal{G}_l(b) \) and such that \( \chi_{\ell}^k[\xi, H, h](u) \leq 1 \) for all \( u \in \mathbb{Z}_k(b) \).

**Proof.** Recall that we already fixed \( b_0 = 1 \). Let \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0' \) and \( |b| > b_0 \). Let \( q \subset \mathcal{O}_k \) be a nontrivial ideal and suppose \( H \in C(U, L^2(F_q, \mathbb{C})) \) and \( h \in K_{E|b|}(U) \) satisfy Properties 3.1(a) and 3.1(b) in Theorem 4.2. Let \( 1 \leq l \leq c_0 \) be an integer. Since \( \frac{c_0}{|b|} \in (0, \delta_1] \), so it follows from Eq. (11) that \( \mathcal{G}_l(b) \subset M_{n/b}^{1/|b|}(U_0) \)
for some integer \(1 \leq \ell \leq \ell_0\). Hence there are \(u_0, u'_0 \in \mathbb{C}_l(b)\) such that \(d(u_0, u'_0) \geq \frac{1}{2} \text{diam}_2(\mathbb{C}_l(b))\) and \(\left\langle \frac{r^{-1}(u_0)-r^{-1}(u'_0)}{r^{-1}(u_0)-r^{-1}(u'_0)}\right\rangle \eta_k \geq \theta_1\). Take integers \(1 \leq k, k' \leq p_b\) such that \(u_k \in \mathbb{D}_k(b)\) and \(u'_0 \in \mathbb{D}_{k'}(b)\) so that the hypotheses of Lemma 4.14 are satisfied. Now, suppose Alternative 1 in Lemma 4.15 holds for one of \((k, j, \ell), (k', j, \ell) \in \mathbb{Z}(b)\) for some \(j \in \{1, 2\}\). Without loss of generality, we can assume it holds for \((k, j, \ell) \in \mathbb{Z}(b)\) and then it is a straightforward calculation to check that \(\chi^*_k[\xi, H, h](u) \leq 1\) for all \(u \in \mathbb{Z}(b)\), using Eq. (17). Otherwise, Alternative 2 in Lemma 4.15 holds for all of \((k_1, 1, \ell), (k_2, 2, \ell), (k', 1, \ell), (k', 2, \ell) \in \mathbb{Z}(b)\). Let \(u \in \mathbb{Z}_k(b)\) and \(u' \in \mathbb{Z}_{k'}(b)\). Note that \(\|H(v_j'(u))\|_2, \|H(v_j'(u'))\|_2 > 0\) for all \(j \in \{1, 2\}\). We would like to apply Lemma 4.16 but first we need to establish bounds on relative angle and relative size. We start with the former. For all \(j \in \{1, 2\}\), let \(u_j \in \{u, u'\}\) such that \(\|H(v_j'(u_j))\|_2 = \min(\|H(v_j'(u))\|_2, \|H(v_j'(u'))\|_2)\). Then recalling Eqs. (16) and (24), for all \(j \in \{1, 2\}\), we have

\[
\frac{\|H(v_j'(u)) - H(v_j'(u'))\|_2}{\min(\|H(v_j'(u))\|_2, \|H(v_j'(u'))\|_2)} \leq \frac{E|b|H(v_j'(u))D(v_j'(u), v_j'(u'))}{\|H(v_j'(u))\|_2} \leq 4E|b| \cdot \frac{e^1\kappa_1^{m_1}}{|b|c_0^2\kappa_2^{-m}} \leq \frac{\delta_0\rho\kappa_1}{128}.
\]

Using the isomorphism \(L^2(F_q, \mathbb{C}) \cong \mathbb{R}^{2#F_n}\) of real vector spaces and some elementary geometry, the above shows that \(\sin(\Theta(H(v_j'(u)), H(v_j'(u')))) \leq \frac{\delta_0\rho\kappa_1}{128}\) with \(\Theta(H(v_j'(u)), H(v_j'(u'))) \in [0, \frac{\pi}{2}]\), for all \(j \in \{1, 2\}\). A simple inequality \(\frac{\theta}{2} \leq \sin(\theta)\) for all \(\theta \in [0, \frac{\pi}{2}]\) gives the angular bound \(\Theta(H(v_j'(u)), H(v_j'(u'))) \leq 2\sin(\Theta(H(v_j'(u)), H(v_j'(u')))) \leq \frac{\delta_0\rho\kappa_1}{64}\) for all \(j \in \{1, 2\}\). For notational convenience, we define \(\varphi : U \rightarrow \mathbb{R}\) by \(\varphi(w) = b(\tau_m(v_j'(u)) - \tau_m(v_j'(w)))\) for all \(w \in U\). By Lemma 4.14 and Eq. (25), we have \(\frac{\delta_0\rho\kappa_1}{64} \leq |\varphi(u) - \varphi(u')| \leq \pi\). The second inequality is to ensure that we take the correct branch of angle in the following calculations. We will use these bounds to obtain a lower bound for \(\Theta(V_1(u), V_2(u))\) or \(\Theta(V_1(u'), V_2(u'))\) where we define

\[
V_j(w) = e^{(f_m^{(a)} + ib\tau_m)(v_j'(w))c_{1,j}\ell(b)^{-1}H(v_j'(w))}
\]

for all \(w \in U\), for all \(j \in \{1, 2\}\). Also note that the triangle inequality on spheres implies that for all \(w_1, w_2, w_3 \in \mathbb{R}^{2#F_n} \setminus \{0\}\) we have \(\Theta(w_1, w_3) \leq \Theta(w_1, w_2) + \Theta(w_2, w_3)\). Using the triangle inequality, we have

\[
\Theta(e^{(f_m^{(a)} + ib\tau_m)(v_j'(u))c_{1,j}\ell(b)^{-1}H(v_j'(u))}, e^{(f_m^{(a)} + ib\tau_m)(v_j'(u))c_{1,j}\ell(b)^{-1}H(v_j'(u))})
\]

\[
= \Theta(e^{i\varphi(u)c_{1,j}\ell(b)^{-1}H(v_j'(u))}, c_{1,2}\ell(b)^{-1}H(v_j'(u))}
\]

\[
\geq \Theta(e^{i\varphi(u)c_{1,j}\ell(b)^{-1}H(v_j'(u))}, e^{i\varphi(u')c_{1,j}\ell(b)^{-1}H(v_j'(u))})
\]

\[
- \Theta(e^{i\varphi(u')c_{1,j}\ell(b)^{-1}H(v_j'(u))}, c_{1,2}\ell(b)^{-1}H(v_j'(u')))
\]

\[
- \Theta(c_{1,2}\ell(b)^{-1}H(v_j'(u)), c_{1,2}\ell(b)^{-1}H(v_j'(u')))
\]

\[
- \Theta(e^{i\varphi(u')c_{1,j}\ell(b)^{-1}H(v_j'(u'))}, c_{1,2}\ell(b)^{-1}H(v_j'(u'))).
\]
Since cocycles act unitarily, we can continue the bound as

\[ \Theta \left( e^{(f_m^{(a)} + i \beta \tau_m)}(v_j(u)) c_{l, \ell}(b)^{-1} H(v_j(u)) \right) \]

\[ \geq \Theta \left( e^{[(\nu(u) - \nu(u'))}] c_{l, \ell}(b)^{-1} H(v_j(u)) \right) \]

\[ - \Theta \left( H(v_j(u)), H(v_j(u')) \right) - \Theta \left( H(v_j(u)), H(v_j(u')) \right) \]

\[ - \Theta \left( e^{i \nu(u')} c_{l, \ell}(b)^{-1} H(v_j(u')) \right) \]

\[ = |\nu(u) - \nu(u')| - \Theta \left( H(v_j(u)), H(v_j(u')) \right) - \Theta \left( H(v_j(u)), H(v_j(u')) \right) \]

Using the previously calculated angular bounds, we see that

\[ \Theta \left( e^{(f_m^{(a)} + i \beta \tau_m)}(v_j(u)) c_{l, \ell}(b)^{-1} H(v_j(u)) \right) \]

\[ \geq \frac{\delta_0 \rho c_1}{16} - \frac{\delta_0 \rho c_1}{64} \Theta \left( e^{(f_m^{(a)} + i \beta \tau_m)}(v_j(u)) c_{l, \ell}(b)^{-1} H(v_j(u)) \right) \]

\[ = \frac{\delta_0 \rho c_1}{32} - \Theta \left( e^{(f_m^{(a)} + i \beta \tau_m)}(v_j(u)) c_{l, \ell}(b)^{-1} H(v_j(u)) \right) \]

Hence \( \Theta(V_1(u), V_2(u)) + \Theta(V_1(u'), V_2(u')) \geq \frac{\delta_0 \rho c_1}{16} \) for all \( u \in \mathbb{Z}_k(b) \), for all \( u' \in \mathbb{Z}_k(b) \). Thus, without loss of generality, we can assume that \( \Theta(V_1(u), V_2(u)) \geq \frac{\delta_0 \rho c_1}{64} \) for all \( u \in \mathbb{Z}_k(b) \), which establishes the required bound on relative angle. For the bound on relative size, let \( (j, j') \in \{(1, 2), (2, 1)\} \) such that \( h(v_j'(u_0)) \leq h(v_j'(u_0)) \) for some \( u_0 \in \mathbb{Z}_k(b) \). Then by Lemma 4.15, we have

\[ \frac{\|V_2(u)\|_2}{\|V_j(u)\|_2} = \frac{\|e^{(f_m^{(a)} + i \beta \tau_m)}(v_j'(u)) c_{l, \ell}(b)^{-1} H(v_j'(u))\|_2}{\|e^{(f_m^{(a)} + i \beta \tau_m)}(v_j'(u)) c_{l, \ell}(b)^{-1} H(v_j'(u))\|_2} \]

\[ \leq \frac{4e^{(f_m^{(a)} + i \beta \tau_m)}(v_j'(u)) c_{l, \ell}(b)^{-1} H(v_j'(u))\|_2}{\|h(v_j'(u))\|_2} \]

for all \( u \in \mathbb{Z}_k(b) \), which establishes the required bound on relative size. Now applying Lemma 4.16 and Eq. (17) and \( \|H\| \leq h \) on \( \|V_j(u) + V_j'(u)\|_2 \) gives \( \chi_{\xi, H, h}(u) \leq 1 \) for all \( u \in \mathbb{Z}_k(b) \).}

**Lemma 4.18.** There exists \( a_0 > 0 \) such that for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( |b| > b_0 \), for all nontrivial ideals \( q \subset O_K \), if \( H \in C(U, L^2(F_q, \mathbb{C})) \) and \( h \in K_{E|b|}(U) \) satisfy Properties 3.1(a) and 3.1(b) in Theorem 4.2, then there exists \( J \in J(b) \) such that

\[ \|M_{\xi,q}^m(H)(u)\|_2 \leq N_{a,J}(h)(u) \]

for all \( u \in U \).

**Proof.** Fix \( a_0 = a_0' \) and recall that we already fixed \( b_0 = 1 \). Let \( \xi = a + ib \in \mathbb{C} \) with \( |a| < a_0 \) and \( |b| > b_0 \), and \( q \subset O_K \) be a nontrivial ideal. Suppose \( H \in \mathbb{C} \).
$C(U, L^2(F_q, \mathbb{C}))$ and $h \in K_{\mathcal{E}[\phi]}(U)$ satisfy Properties 3.(1a) and 3.(1b) in Theorem 4.2. For all integers $1 \leq l \leq c_b$, we can choose a $(k_l, j_l, \ell_l) \in \Xi(b)$ as guaranteed by Lemma 4.17. Let $J = \{(k_l, j_l, \ell_l) \in \Xi(b) : 1 \leq l \leq c_b\} \subset \Xi(b)$ which is then dense by construction and so $J \in \mathcal{J}(b)$. Now we show that $\|\mathcal{M}_{\Xi,q}^m(H)\| \leq N_{a,J}(h)$ for this choice of $J \in \mathcal{J}(b)$. We consider $u \in \text{int}(U)$ and refer the reader to Definition 2.8 for the case when $u \in \partial(U)$. If $u \notin \mathcal{Z}_b(b)$ for all $(k_l, j_l, \ell_l) \in J$, then $\beta_j(v) = 1$ for all branch $v \in \sigma^{-m}(u)$ and hence the bound follows trivially by definitions. Otherwise there is an integer $1 \leq l \leq c_b$ such that $u \in \mathcal{Z}_b(b)$ corresponding to $(k_l, j_l, \ell_l) \in J$. Note that by definition of $J$, we have $(k_l, j_l, \ell_l) \notin J$ for all $(k_l, j_l, \ell_l) \in \Xi(b)$ with $(j_l, \ell_l) \neq (j_l, \ell_l)$. Let $(j_l, \ell_l) \in \{(1, 2), (2, 1)\}$. Then by construction of $J$, we have $\chi_j^l(\xi, h, b)(u) \leq 1$ and $\beta_j(v_j^l(u)) = 1 - \mu$ and $\beta_j(v_j^l(u)) = 1$ for all $(k_l, j_l, \ell_l) \in \Xi(b)$ with $(j_l, \ell_l) \neq (j_l, \ell_l)$. Hence we compute that

$$
\begin{align*}
\|\mathcal{M}_{\Xi,q}^m(H)(u)\| & = \|\sum_{v \in \sigma^{-m}(u)} e^{f_m(v)+ib\tau_m(v)}c_m(v)^{-1}H(v)\|_2 \\
& \leq \sum_{v \in \sigma^{-m}(u), v \neq v_1^l(u), v \neq v_2^l(u)} \|e^{f_m(v)+ib\tau_m(v)}c_m(v)^{-1}H(v)\|_2 \\
& \quad + \|e^{f_m(v)+ib\tau_m(v)}v_1^l(u)c_{l,j_l,\ell_l}(b)H(v_1^l(u))\|_2 \\
& \quad + \|e^{f_m(v)+ib\tau_m(v)}v_2^l(u)c_{l,j_l,\ell_l}(b)H(v_2^l(u))\|_2 \\
& \leq \sum_{v \in \sigma^{-m}(u), v \neq v_1^l(u), v \neq v_2^l(u)} e^{f_m(v)}h(v) \\
& \quad + \left((1 - \mu)e^{f_m(v_1^l(u))}h(v_1^l(u)) + e^{f_m(v_2^l(u))}h(v_2^l(u))\right) \\
& \leq N_{a,J}(h)(u).
\end{align*}
$$

5. Uniform Exponential Mixing of the Geodesic Flow

The aim of this section is to prove Theorem 1.1 using the proven spectral bound Theorem 2.13.

Let $q \subset Q_b$ be an ideal coprime to $q_0$. Similar to $\text{R}^{3,\tau}$, consider the suspension space $U^{\varphi,\tau} = (U \times \hat{G}_q \times \mathbb{R}_{\geq 0})/\sim$ where $\sim$ is the equivalence relation on $U \times \hat{G}_q \times \mathbb{R}_{\geq 0}$ defined by $(u, g, t + \tau(u)) \sim (\varphi(u), gc_q(u), t)$ for all $(u, g, t) \in U \times \hat{G}_q \times \mathbb{R}_{\geq 0}$. Define the norm $\|\phi\|_{B_q} = \|\phi\|_\infty + \text{Lip}_{d,|t|}(\phi)$ where the second term is defined to be

$$
\text{sup}\left\{\frac{|\phi(u, g, t) - \phi(u', g, t')|}{d(u, u')} + |t - t'| : u, u' \in U, g \in \hat{G}_q, t, t' \in [0, \tau(u)), t \neq (u', t')\right\}
$$

for any function $\phi : U^{\varphi,\tau} \to \mathbb{R}$ and define the associated space $B_q = \{\phi : U^{\varphi,\tau} \to \mathbb{R} : \phi(u, g, t)|_{[0, \tau(u))} \in C^1([0, \tau(u)), \mathbb{R}), \|\phi\|_{B_q} < \infty\}$. For all $\phi \in B_q$, for all $\xi \in \mathbb{C}$,
define the measurable functions \( \hat{\phi}_\xi \in B(U, L^2(\tilde{G}_q, \mathbb{C})) \) by

\[
\hat{\phi}_\xi(u)(g) = \int_0^{\tau(u)} \phi(u, g, t)e^{-\xi t} \, dt
\]

for all \( g \in \tilde{G}_q \), for all \( u \in U \).

**Remark.** For all \( u \in U \), for all \( g \in \tilde{G}_q \), the map \( \xi \mapsto \hat{\phi}_\xi(u)(g) \) is entire.

**Remark.** Let \( q \subset O_K \) be an ideal coprime to \( q_0 \), \( \phi, \psi \in B_q \) and \( \xi \in \mathbb{C} \). Because of \( \tau \) involved in the definition of \( \hat{\phi}_\xi \), it is not necessarily Lipschitz but it is essentially Lipschitz. However we can see from the proof of Lemma 5.2 that in fact \( M_{\xi, q}(\hat{\phi}_\xi - \xi) \) \( \in V_q \). Moreover, we can then deduce that if \( \sum_{g \in \tilde{G}_q} \phi(u, g, t) = 0 \) for all \( (u, t) \in U^\tau \), we have \( M_{\xi, q}(\hat{\phi}_\xi - \xi) \) \( \in W_q \).

### 5.1. Correlation function and its Laplace transform.

Let \( q \subset O_K \) be an ideal coprime to \( q_0 \) and \( \phi, \psi \in B_q \). Define the continuous correlation function \( \Upsilon_{\phi, \psi} \in L^\infty(\mathbb{R}^\geq, \mathbb{R}) \) by

\[
\Upsilon_{\phi, \psi}(t) = \sum_{g \in \tilde{G}_q} \int_U \int_0^{\tau(u)} \phi(u, g, r + t)\psi(u, g, r) \, dr \, d\nu_U(u)
\]

for all \( t \in \mathbb{R}^\geq \). We can decompose this into continuous functions as \( \Upsilon_{\phi, \psi} = \Upsilon_{\phi, \psi}^0 + \Upsilon_{\phi, \psi}^1 \) where we define

\[
\Upsilon_{\phi, \psi}^0(t) = \sum_{g \in \tilde{G}_q} \int_U \int_0^{\tau(u)} \phi(u, g, r + t)\psi(u, g, r) \, dr \, d\nu_U(u)
\]

\[
\Upsilon_{\phi, \psi}^1(t) = \sum_{g \in \tilde{G}_q} \int_U \int_0^{\max(0, \tau(u) - t)} \phi(u, g, r + t)\psi(u, g, r) \, dr \, d\nu_U(u)
\]

for all \( t \in \mathbb{R}^\geq \).

**Remark.** This decomposition is useful because the Laplace transform of \( \Upsilon_{\phi, \psi}^0 \) has a clean expression in terms of the congruence transfer operator. Combining this with the fact that \( \Upsilon_{\phi, \psi}(t) = \Upsilon_{\phi, \psi}^0(t) \) for all \( t \geq \tau \), it is clear that it suffices to study decay properties of \( \Upsilon_{\phi, \psi}^0 \).

Consider the Laplace transform on the half plane \( \hat{\Upsilon}_{\phi, \psi}^0 : \{ \xi \in \mathbb{C} : \Re(\xi) > 0 \} \rightarrow \mathbb{C} \) defined by

\[
\hat{\Upsilon}_{\phi, \psi}^0(\xi) = \int_0^\infty \Upsilon_{\phi, \psi}^0(t)e^{-\xi t} \, dt
\]

for all \( \xi = a + ib \in \mathbb{C} \) with \( a > 0 \).

**Remark.** Clearly \( \hat{\Upsilon}_{\phi, \psi}^0 \) is holomorphic. However, it is not at all obvious from the above definition whether it can be defined holomorphically on a larger half plane. Nevertheless, using the congruence transfer operator bounds, we will show that \( \hat{\Upsilon}_{\phi, \psi}^0 \) has a holomorphic extension to a larger half plane \( \{ \xi \in \mathbb{C} : \Re(\xi) > -a_0 \} \) for some \( a_0 > 0 \). This will allow us to apply the inverse Laplace transform formula to extract an exponential decay for \( \Upsilon_{\phi, \psi}^0 \).
Lemma 5.1. For all ideals \( q \subset \mathcal{O}_K \) coprime to \( q_0 \), for all \( \phi, \psi \in \mathcal{B}_q \), for all \( \xi = a + ib \in \mathbb{C} \) with \( a > 0 \), we have

\[
\hat{\mathcal{T}}_\phi^0(\xi) = \sum_{k=1}^{\infty} \lambda_a^k \left\langle \hat{\phi}_\xi, \mathcal{M}_{\xi,q}^k(\hat{\psi}_-\xi) \rightangle.
\]

Proof. Let \( q \subset \mathcal{O}_K \) be an ideal coprime to \( q_0 \) and \( \phi, \psi \in \mathcal{B}_q \) and \( \xi = a + ib \in \mathbb{C} \) with \( a > 0 \). Observe that since \( \nu_U(U) = 1 \), we can compute some of the integrals over \( \text{int}(U) \) instead of \( U \) when convenient to avoid the boundary. Define the product measure \( \nu \) by \( d\nu = dt \, dr \, d\nu_U \) for convenience. We calculate that

\[
\hat{\mathcal{T}}_\phi^0(\xi)
= \int_0^\infty \left( \sum_{g \in G_q} \int_U \int_{\tau(u) - \tau(u)} \phi(u, g, t + r) \psi(u, g, r) \, dt \, d\nu_U(u) \right) e^{-\xi t} \, dt
= \sum_{g \in G_q} \int_U \int_0^{\tau(u)} \int_0^{\infty} e^{-\xi t} \phi(u, g, r + t) \psi(u, g, r) \, dt \, d\nu_U(u)
= \sum_{g \in G_q} \int_U \int_0^{\tau(u)} \int_0^{\infty} e^{-\xi (t-r)} \phi(u, g, t) \psi(u, g, r) \, dt \, d\nu_U(t, r, u)
= \sum_{g \in G_q} \int_U \int_0^{\tau(u)} \int_0^{\infty} \sum_{k=1}^{\tau_k(u)} e^{-\xi(t-r)} \phi(u, g, t) \psi(u, g, r) \, dt \, d\nu_U(t, r, u)
= \sum_{k=1}^{\infty} \int_U \int_0^{\tau(u)} \int_0^{\tau(u)} \sum_{g \in G_q} \int_U \int_0^{\tau(u)} \int_0^{\tau(u)} \int_0^{\tau(u)} \sum_{\phi} \phi(u, g, t) \psi(u, g, r) \, dt \, d\nu_U(t, r, u)
= \sum_{k=1}^{\infty} \int_U e^{-\xi \tau_k(u)} \left\langle c_k^k(u) \hat{\phi}_\xi(\sigma^k(u)), \hat{\psi}_-\xi(u) \right\rangle \, d\nu_U(u).
\]

Now using the fact that cocycles act unitarily and \( \mathcal{L}_0^*(\nu_U) = \nu_U \), we continue the calculation as

\[
\hat{\mathcal{T}}_\phi^0(\xi)
= \sum_{k=1}^{\infty} \int_U e^{-\xi \tau_k(u)} \left\langle \hat{\phi}_\xi(\sigma^k(u)), c_q^k(u)^{-1} \hat{\psi}_-\xi(u) \right\rangle \, d\nu_U(u)
= \sum_{k=1}^{\infty} \int_{\text{int}(U)} \sum_{u \in \sigma^{-1}(u)} e^{f_q(u)} e^{-\xi \tau_k(u)} \left\langle \hat{\phi}_\xi(\sigma^k(u)), c_q^k(u)^{-1} \hat{\psi}_-\xi(u) \right\rangle \, d\nu_U(u)
= \sum_{k=1}^{\infty} \lambda_a^k \int_{\text{int}(U)} \left\langle \hat{\phi}_\xi(u), \sum_{u \in \sigma^{-1}(u)} e^{(f_q(u) + ib \tau_k(u))} c_q^k(u)^{-1} \hat{\psi}_-\xi(u) \right\rangle \, d\nu_U(u)
= \sum_{k=1}^{\infty} \lambda_a^k \left\langle \hat{\phi}_\xi, \mathcal{M}_{\xi,q}^k(\hat{\psi}_-\xi) \right\rangle.
\]
5.2. Exponential decay of the correlation function.

**Lemma 5.2.** There exists $C > 0$ such that for all ideals $q \subset \mathcal{O}_K$ coprime to $q_0$, for all $\phi \in \mathcal{B}_q$, for all $\xi = a + ib \in \mathbb{C}$ with $|a| \leq a'_0$, we have

$$
\left\| \hat{\phi}_\xi \right\|_2 \leq \left\| \hat{\phi}_\xi \right\|_\infty \leq CN_K(q)^C \frac{||\phi||_{\mathcal{B}_q}}{\max(1, |b|)} \left\| \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z}) \right\|_{1,b} \leq CN_K(q)^C \frac{||\phi||_{\mathcal{B}_q}}{\max(1, |b|)}.
$$

**Proof.** Fix $c > 0$ (depending on $n$) such that $\# \hat{G}_q \leq N_K(q)^{2c}$ for all nontrivial ideals $q \subset \mathcal{O}_K$. Fix $C_1 = (2 + \tau)e^{a_0\tau}$, $C_2 = \max \left( 2N\tau T_0 C_1, \frac{2N\tau T_0 C_1}{c_0\kappa_2} + \frac{N\tau T_0 e^{a_0\tau} (\tau + T_0)}{c_0\kappa_2} \right)$, $C_3 = C_2 + N\tau T_0 C_1$ and $C = \max(c, C_1, C_3)$. Let $q \subset \mathcal{O}_K$ be an ideal coprime to $q_0$, $\phi \in \mathcal{B}_q$, $\xi = a + ib \in \mathbb{C}$ with $|a| \leq a'_0$. For the first inequality, clearly

$$
\left\| \hat{\phi}_\xi \right\|_2 \leq \left\| \hat{\phi}_\xi \right\|_\infty = \sup_{u \in U} \left\| \hat{\phi}_\xi(u) \right\|_2 \leq (\# \hat{G}_q)^{\frac{1}{2}} \sup_{u \in U, g \in \hat{G}_q} \left| \hat{\phi}_\xi(u)(g) \right|.
$$

(26)

If $|b| \leq 1$, from the definition of $\hat{\phi}_\xi$ we have $\left\| \hat{\phi}_\xi \right\|_\infty \leq N_K(q)^c \tau e^{a_0\tau} ||\phi||_\infty \leq C_1 N_K(q)^c ||\phi||_{\mathcal{B}_q}$. If $|b| \geq 1$, integrating by parts gives

$$
\hat{\phi}_\xi(u)(g) = \int_0^{\tau(u)} \phi(u, g, t)e^{-\xi t} dt
$$

$$
= \left[ -\frac{1}{\xi} \phi(u, g, t)e^{-\xi t} \right]_{t=0}^{\tau(u)} + \frac{1}{\xi} \int_0^{\tau(u)} \frac{d}{dt}\phi(u, g, t) \cdot e^{-\xi t} dt
$$

for all $g \in \hat{G}_q$, for all $u \in U$, and hence

$$
\left\| \hat{\phi}_\xi \right\|_\infty \leq N_K(q)^c \left( \frac{2}{|b|} ||\phi||_\infty e^{a_0\tau} + \frac{\tau}{|b|} ||\phi||_{\mathcal{B}_q} e^{a_0\tau} \right) \leq C_1 N_K(q)^c \frac{||\phi||_{\mathcal{B}_q}}{|b|}.
$$

For the second inequality, recall that

$$
\left\| \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z}) \right\|_{1,b} = \left\| \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z}) \right\|_\infty + \frac{1}{\max(1, |b|)} \text{Lip}_d \left( \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z}) \right).
$$

For the first term, we can use the above bound to get

$$
\left\| \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z}) \right\|_\infty \leq Ne^{T_0} \leq Ne^{T_0} C_1 N_K(q)^c \frac{||\phi||_{\mathcal{B}_q}}{\max(1, |b|)}.
$$

For the second term, if $u \in U_j$ and $u' \in U_k$ where $j, k \in \mathcal{A}$ with $j \neq k$, then

$$
\left\| \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z})(u) - \mathcal{M}_{\xi,q}(\hat{\phi} \cdot \mathbb{Z})(u') \right\|_2 \leq 2N\tau T_0 \left\| \hat{\phi} \cdot \mathbb{Z} \right\|_\infty
$$

$$
\leq 2N\tau T_0 C_1 N_K(q)^c \frac{||\phi||_{\mathcal{B}_q}}{\max(1, |b|)} d(u, u').
$$

Now let $u, u' \in \text{int}(U_j)$ for some $j \in \mathcal{A}$. Again we refer the reader to Definition 2.8 for the case when $u \in \partial(U_j)$ or $u' \in \partial(U_j)$. With a similar computation (using the
Lemma 5.3. Conclude that for all square free ideals $q$ and $\tau \leq \tau(\varphi(v))$. Then remembering that $v$ and $\varphi(v)$ are in the same cylinder of length 1, we have

$$\left\| M_{\varphi}(\hat{\varphi}(v)) - M_{\varphi}(\hat{\varphi}(v)) \right\|_2 \leq \sum_{u \in \sigma^{-1}(u)} \left( |1 - e^{f(a) + ibr}(\varphi(v)) - (f(a) + ibr)(v)| e^{f(a)(v)} \right) \left\| \hat{\varphi}(v) \right\|_2$$

Putting this in Eq. (34), gives $\text{Lip}_{\delta}(\varphi(v)) \leq \text{Lip}_{\delta}(\varphi(v)) \cdot \left\| \hat{\varphi}(v) \right\|_\infty$

$$\leq \sum_{u \in \sigma^{-1}(u)} \left( e^{f(a)(\varphi(v)) - f(a)(v)} \right) \left( (f(a) + ibr)(\varphi(v)) - (f(a) + ibr)(v) \right) \cdot \left\| \hat{\varphi}(v) \right\|_\infty$$

Let $v \in \sigma^{-1}(u)$. To bound the terms in the sum, recall Eq. (26). Assume $\tau(v) \leq \tau(\varphi(v))$. Then remembering that $v$ and $\varphi(v)$ are in the same cylinder of length 1, we have

$$\left\| \hat{\varphi}(v)(g) - \hat{\varphi}(\varphi(v))(g) \right\| \leq \int_0^{\tau(v)} |\hat{\varphi}(v, g, t) - \hat{\varphi}(\varphi(v), g, t)| e^{C|t|} dt + \int_{\tau(v)}^{\tau(\varphi(v))} |\hat{\varphi}(\varphi(v), g, t)| e^{C|t|} dt \leq \frac{1}{c_{0h^2}} \left( e^{g\varphi(\varphi(v))} \right) d(u, u').$$

The case $\tau(\varphi(v)) \leq \tau(v)$ is similar and so

$$\left\| \hat{\varphi}(v) - \hat{\varphi}(\varphi(v)) \right\|_2 \leq \frac{e^{g\varphi(v)}(\tau + T_0)}{c_{0h^2}} N_\xi(q) \| \phi \|_{B_4} d(u, u').$$

Putting this in Eq. (34), gives $\text{Lip}_{\delta}(M_{\varphi}(\hat{\varphi}(v))) \leq C_2 N_\xi(q) \| \phi \|_{B_4}$. Hence we conclude that $\left\| M_{\varphi}(\hat{\varphi}(v)) \right\|_1 \leq C_3 N_\xi(q) \| \phi \|_{B_4}$. ■

**Lemma 5.3.** There exist $C > 0, \eta > 0$ and a nontrivial proper ideal $q_0 \subset O_\xi$ such that for all square free ideals $q \subset O_\xi$ coprime to $\psi_0 q_0$, for all $\psi \in B_3$ for all $\psi \in B_3$, such that $\sum_{g \in G_3} \psi(g, u, t) = 0$ for all $(u, t) \in U^\tau$, we have

$$|\psi(\varphi(t))| \leq C N_\xi(q) e^{-\eta t} \| \phi \|_{B_4}.$$  

**Proof.** Fix $C_1 > 1, \eta > 0, a_0 > 0$ and the nontrivial proper ideal $q_0 \subset O_\xi$ be the $C, \eta, a_0, q_0$ from Theorem 2.13 and $C_2 > c$ be the $C$ from Lemma 5.2 where $c$ is the constant from its proof. Fix $\eta = a_0 \in (0, 1/2 \min(a_0'))$ such that
sup_{|a| \leq 2a_0} \log(\lambda_a) \leq \frac{\delta}{2}. Fix C_3 = \max(2e^{\delta}C_1C_2^2, C_1 + 2C_2) and

C = \max \left( C_3, C_3 \sum_{k=1}^{\infty} e^{-\frac{\delta}{2}k}, 2\pi e^{\eta \pi} \right).

Let \( q \subset \mathcal{O}_K \) be a square free ideal coprime to \( q_0q_0^\prime \), \( \phi \in \mathcal{B}_q \) and \( \psi \in \mathcal{B}_q \) such that

\[ \sum_{g \in G} \psi(u,g,t) = 0 \] for all \( (u,t) \in U^\tau \). Recall that Lemma 5.1 gives

\[ \hat{\Upsilon}_{\phi,\psi}^0(\xi) = \sum_{k=1}^{\infty} \lambda_a^k \left\langle \hat{\phi}_\xi, \mathcal{L}^k_{\xi,a}(\hat{\psi}_\zeta) \right\rangle \]

for all \( \xi = a + ib \in \mathbb{C} \) with \( a > 0 \). Also, recalling previous remarks, we can see that for all integers \( k \geq 1 \), the map \( \xi \mapsto \lambda_a^k \left\langle \hat{\phi}_\xi, \mathcal{L}^k_{\xi,a}(\hat{\psi}_\zeta) \right\rangle \) is defined on \( \mathbb{C} \) and is entire. Hence, to show that \( \hat{\Upsilon}_{\phi,\psi}^0 \) has a holomorphic extension to the half plane \( \{ \xi \in \mathbb{C} : \Re(\xi) > -2a_0 \} \), it suffices to show that the above sum is absolutely convergent for all \( \xi = a + ib \in \mathbb{C} \) with \( |a| < 2a_0 \). Recalling \( \mathcal{L}^k_{\xi,a}(\hat{\psi}_\zeta) \in \mathcal{W}_q \) and noting \( \frac{1}{\max(1,|b|)} \leq \frac{2}{\tau+\pi} \), we use Theorem 2.13 and Lemma 5.2 to calculate that

\[ \left| \lambda_a^k \left\langle \hat{\phi}_\xi, \mathcal{L}^k_{\xi,a}(\hat{\psi}_\zeta) \right\rangle \right| \leq \lambda_a^k \left\| \hat{\phi}_\xi \right\|_2 \left\| \mathcal{L}^k_{\xi,a}(\hat{\psi}_\zeta) \right\|_2 \leq \lambda_a^k \left\| \hat{\phi}_\xi \right\|_2 \cdot C_1 N_K(q)^C_1 e^{-\eta(k-1)} \left\| \mathcal{L}^k_{\xi,a}(\hat{\psi}_\zeta) \right\|_{1,b} \leq \lambda_a^k C_2 N_K(q)^C_2 e^{-\frac{\eta}{2}k} \left\| \hat{\phi}_\xi \right\|_{1,b} \left\| \psi \right\|_{B_q} \]

for all \( \xi = a + ib \in \mathbb{C} \) where \( |a| < 2a_0 \), whose sum over integers \( k \geq 1 \) converges as desired. The above calculation also gives the important bound \( \left| \hat{\Upsilon}_{\phi,\psi}^0(\xi) \right| \leq C N_K(q)^C \left\| \hat{\phi} \right\|_{B_q} \left\| \psi \right\|_{B_q} \) for all \( \xi = a + ib \in \mathbb{C} \) where \( |a| < 2a_0 \). Since \( \Upsilon_{\phi,\psi}^0 \) is continuous and in \( L^\infty(\mathbb{R}_0, \mathbb{R}) \), we use the holomorphic extension and the inverse Laplace transform formula along the line \( \{ \xi \in \mathbb{C} : \Re(\xi) = -a_0 \} \) to obtain

\[ \Upsilon_{\phi,\psi}^0(t) = \frac{1}{2\pi i} \lim_{A \to \infty} \lim_{B \to -\infty} \int_{-B - a_0}^{-B + a_0} \hat{\Upsilon}_{\phi,\psi}^0(\xi) e^{\xi t} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Upsilon_{\phi,\psi}^0(-a_0 + ib) e^{(-a_0 + ib)t} d\xi \]

for all \( t > 0 \) and we already know \( \Upsilon_{\phi,\psi}^0(0) = 0 \). Then using the above bound, we have

\[ \left| \Upsilon_{\phi,\psi}^0(t) \right| \leq \frac{1}{2\pi} e^{-a_0 t} \left| \Upsilon_{\phi,\psi}^0(-a_0 + ib) \right| \]

\[ \leq \frac{1}{2\pi} e^{-a_0 t} \int_{-\infty}^{\infty} C N_K(q)^C \left\| \phi \right\|_{B_q} \left\| \psi \right\|_{B_q} d\xi \]

\[ = C \frac{N_K(q)^C}{1 + b^2} e^{-\eta t} \left\| \phi \right\|_{B_q} \left\| \psi \right\|_{B_q} \]
for all \( t \geq 0 \). Now, \( \Upsilon_{\phi,\psi}(t) = \Upsilon_{\phi,\psi}^0(t) \) for all \( t \geq 7 \) while
\[
|\Upsilon_{\phi,\psi}^0(t)| \leq \tau N_\kappa(q)(\phi)_0 \|s\|_0 \leq \frac{C}{2} N_\kappa(q) e^{-\eta t} \|\phi\|_{s_0} \|s\|
\]
for all \( t \in [0,\tau] \) and hence
\[
|\Upsilon_{\phi,\psi}(t)| \leq C N_\kappa(q) e^{-\eta t} \|\phi\|_{s_0} \|s\|.
\]

5.3. Integrating out the strong stable direction and proof of Theorem 1.1.

Let \( q \in \mathcal{O}_G \) be an ideal coprime to \( q_0 \). Given a \( \phi \in C^1(\Gamma_\kappa \backslash G/M, \mathbb{R}) \), we can convert it to a function in \( B_q \). By Rokhlin’s disintegration theorem with respect to the projection \( \text{proj}_1 : R \to U \), the probability measure \( \nu_R \) disintegrates to give the set of conditional probability measures \( \{\nu_u : u \in U\} \). For all \( j \in \mathcal{A} \), for all \( u \in U_j \), the measure \( \nu_u \) is actually defined on the fiber \( \text{proj}_1^{-1}(u) = \{u, S_j\} \) but we can push forward via the diffeomorphism \( [u, S_j] \to S_j \) defined by \( [u, s] \mapsto s \) to think of \( \nu_u \) as a measure on \( S_j \). Thinking of the overline as an isometry \( R \to \tilde{R} \), for all \( t \geq 0 \), we define \( \phi_t \in B_q \) by
\[
\phi_t(u, g, r) = \int_{S_j} \phi(g[\tilde{u}, \tilde{s}](t+r)) d\nu_u(s)
\]
for all \((u, g, r) \in U_j \times \tilde{G}_q \times [0, \tau(u)) \), for all \( j \in \mathcal{A} \), and in order to ensure that indeed \( \phi_t \in B_q \), we must define \( \phi_t(u, g, r) = \phi_t(g, u, g, r, \nu_u) \) for all \( r = [\tau_k(u), \tau_{k+1}(u)) \), for all \( k \geq 0 \).

Remark. We need to deal with some technicalities. Let \( q \in \mathcal{O}_G \) be an ideal coprime to \( q_0 \) and \( j \in \mathcal{A} \). Firstly, by smoothness of the strong unstable and strong stable foliations and compactness of \( R_j \), there is a \( C_1 > 1 \) such that \( d([u, s], [u', s]) = d([\tilde{u}, \tilde{s}], [\tilde{u}', \tilde{s}']) \leq C_1 d(u, u') \leq C_1 d_{u_0}(u, u') \) for all \( u, u' \in U_j \), for all \( s \in S_j \), for all \( g \in \tilde{G}_q \). Now, for all \( u \in U_j \), the Patterson–Sullivan density induces the measure \( \mu_{u, S_j}^R([u, s]) = e^{\beta \gamma [\tilde{u}, \tilde{s}]} d\nu_u^R([\tilde{u}, \tilde{s}]) \) on \( [u, S_j] \). For all \( u \in U_j \), pushing forward via the diffeomorphism \([u, S_j] \to S_j\) mentioned above gives the measure \( \mu_u^R = e^{\beta \gamma [\tilde{u}, \tilde{s}]} d\nu_u^R([\tilde{u}, \tilde{s}]) \) on \( S_j \), using the important fact that \([\tilde{u}, \tilde{s}] = (\tilde{s})^\gamma\). In fact, for all \( u \in U_j \), comparing with the definition of the BMS measure, it can be checked that \( \mu_u^R \) after normalization is exactly the conditional measures \( \nu_u \).

The nontrivial consequence is that \( \frac{d\phi_t}{d\nu_u} \in C^\infty(S_j, \mathbb{R}) \) for all \( u, u' \in U_j \). Together with compactness of \( R_j \), there is a \( C_2 > 0 \) such that \( |1 - \frac{d\phi_t}{d\nu_u}(s)| \leq C_2 d(u, u') \leq C_2 d_{u_0}(u, u') \) for all \( u, u' \in U_j \), for all \( s \in S_j \). An easy computation using the two derived inequalities shows that \( \|\phi_t\|_{S_0} \leq C\|\phi\|_{C^1} \) where \( C = C_1 + C_2 \), for all \( \phi \in C^1(\Gamma_\kappa \backslash G/M, \mathbb{R}) \), for all \( t \geq 0 \).

Lemma 5.4. There exist \( C > 0 \) and \( \eta > 0 \) such that for all ideals \( q \in \mathcal{O}_G \) coprime to \( q_0 \), for all \( \phi \in C^1(\Gamma_\kappa \backslash G/M, \mathbb{R}) \), we have
\[
|\phi(g[\tilde{u}, \tilde{s}](t+r)) - \phi_t(u, g, t + r)| \leq C e^{-\eta |t|} \|\phi\|_{C^1}
\]
for all \([u, s] \in R\), for all \( g \in \tilde{G}_q \), for all \( r \geq 0 \).

Proof. Fix \( C, \eta > 0 \) to be the constants \( \epsilon, \log(\lambda) \) from the Anosov property in [Rat73]. Let \( q \in \mathcal{O}_G \) be an ideal coprime to \( q_0 \) and \( \phi \in C^1(\Gamma_\kappa \backslash G/M, \mathbb{R}) \). Let \( j \in \mathcal{A} \), \([u, s] \in R_j \), \( g \in \tilde{G}_q \) and \( r \geq 0 \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( t + r \in [\tau_m(u), \tau_{m+1}(u)) \)
and let \( u_1 = \sigma^m(u) \in U_k \) for some \( k \in A \). Let \( s_1 \in S_k \). Then \( \bar{u}, \bar{s} \alpha_{\tau_m(u)} = \rho_m(u)[u, s] \) and \( \rho_m(u)[\bar{u}, \bar{s}] \) are both in \( \rho_m(u)[\bar{u}, \bar{s}] \). Noting that \( \tau_m(u) \leq t + r \), we have
\[
d(g|\bar{u}, \bar{s}|a_{2t+r}, g\rho_m(u)|\bar{u}, \bar{s}|a_{2t+r-\tau_m(u)})
\leq d_\Sigma(g|\bar{u}, \bar{s}|a_{2t+r}, g\rho_m(u)|\bar{u}, \bar{s}|a_{2t+r-\tau_m(u)})
\leq C e^{-\eta(2t+r-\tau_m(u))} \leq C e^{-\eta t}.
\]
Thus, \(|\phi(g|\bar{u}, \bar{s}|a_{2t+r}) - \phi(g\rho_m(u)|\bar{u}, \bar{s}|a_{2t+r-\tau_m(u)})| \leq C e^{-\eta t} \| \phi \|_{C^1}. Integrating over \( s_1 \in S_k \) with respect to the probability measure \( \nu_{u_1} \) gives \(|\phi(g|\bar{u}, \bar{s}|a_{2t+r}) - \phi_1(u,g,t+r)| \leq C e^{-\eta t} \| \phi \|_{C^1}.

**Corollary 5.4.1.** There are \( C, \eta > 0 \) such that for all ideals \( \mathfrak{q} \subset \mathcal{O}_K \) coprime to \( \mathfrak{q}_0 \), for all \( \phi, \psi \in C^1(\Gamma \backslash G/M, \mathbb{R}) \), we have
\[
\left| \int_{\Gamma \backslash G/M} \phi(xa_{2t}) \psi(x) \, dm_\mathfrak{q}^{\text{BMS}}(x) - \frac{m^{\text{BMS}}(\Gamma \backslash G/M)}{\nu_U(\tau)} \Upsilon_{x_0, \psi_0}(t) \right| \leq CN_K(\mathfrak{q}) C e^{-\eta t} \| \phi \|_{C^1} \| \psi \|_{C^1}.
\]

**Proof.** Fix \( c > 0 \) (depending on \( n \)) such that \( \# G_\mathfrak{q} \leq N_\mathfrak{q}(\mathfrak{c}^n) \) for all nontrivial ideals \( \mathfrak{q} \subset \mathcal{O}_K \). Fix \( C', \eta > 0 \) to be the \( C \) and \( \eta \) from Lemma 5.4. Fix \( C = \max(C'm^{\text{BMS}}(\Gamma \backslash G/M), c) \). Let \( \mathfrak{q} \subset \mathcal{O}_K \) be an ideal coprime to \( \mathfrak{q}_0 \), and \( \phi, \psi \in C^1(\Gamma \backslash G/M, \mathbb{R}) \). We have
\[
\int_{\Gamma \backslash G/M} \phi(xa_{2t}) \psi(x) \, dm_\mathfrak{q}^{\text{BMS}}(x)
= \frac{m^{\text{BMS}}(\Gamma \backslash G/M)}{\# G_\mathfrak{q} \cdot \nu_R(\tau)} \int_{\Omega_\mathfrak{q}} \phi(xa_{2t}) \psi(x) \, d\nu^{\mathfrak{q}, \tau}(x)
= \frac{m^{\text{BMS}}(\Gamma \backslash G/M)}{\nu_R(\tau)} \int_{R_\mathfrak{q}} \int_0^{\tau(x)} \phi(xa_{r+2t}) \psi(xa_r) \, dr \, d\nu_R(x)
= \frac{m^{\text{BMS}}(\Gamma \backslash G/M)}{\nu_R(\tau)} \sum_{g \in G_\mathfrak{q}} \int_{R_\mathfrak{q}} \int_0^{\tau(x)} \phi(gx_a_{r+2t}) \psi(gx_a_r) \, dr \, d\nu_R(x)
= \frac{m^{\text{BMS}}(\Gamma \backslash G/M)}{\nu_U(\tau)} \sum_{g \in G_\mathfrak{q}} \sum_{j=1}^N \int_{U_j} \int_{S_j} \int_0^{\tau(u)} \phi(g|\bar{u}, \bar{s}|a_{r+2t}) \psi(g|\bar{u}, \bar{s}|a_r) \, dr \, d\nu_u(s) \, d\nu_U(u).
\]
Now recall the definition of \( \Upsilon_{x_0, \psi_0} \) and use Lemma 5.4 to finish the proof.

**Proof of Theorem 1.1.** A single ideal \( \mathfrak{q} = \mathcal{O}_K \), corresponds to the single manifold \( \Gamma \backslash G/M \) and hence exponential mixing of the geodesic flow has already been established by works of Dolgopyat and Stoyanov (essentially using Dolgopyat’s method from Section 4 for large \( |b| \) and the complex RPF theorem for small \( |b| \) in a similar proof as in Lemma 5.3). Now, recall the remark before Lemma 5.4 and fix \( C_4 > 1 \) to be the constant described there. Fix \( C_1, \eta_1, C_2, \eta_2 > 0 \) to be the \( C \) and \( \eta \) from Corollary 5.4.1 and Lemma 5.3 respectively. Fix \( C_3 = \frac{m^{\text{BMS}}(\Gamma \backslash G/M)}{\nu_U(\tau)} C_2, \eta = \frac{1}{2} \min(\eta_1, \eta_2) \)
and $C = \max(C_1, C_2, C_1 + C_3 C_4^2)$. Let $\phi, \psi \in C^1(\Gamma_q \backslash G/M, \mathbb{R})$. Fix the nontrivial proper ideal $q_0 \subset \mathcal{O}_K$ from Lemma 5.3. Let $q \subset \mathcal{O}_K$ be a square free ideal coprime to $q_0 q_0'$. Consider the decomposition $\psi = \psi \hat{G}_q + \psi_0$ where $\psi \hat{G}_q$, defined by $\psi \hat{G}_q(x) = \sum_{g \in \hat{G}_q} \psi(gx)$ for all $x \in \Gamma_q \backslash G/M$, is $\hat{G}_q$-invariant and consequently $\psi_0$ satisfies $\sum_{g \in \hat{G}_q} \psi_0(gx) = 0$ for all $x \in \Gamma_q \backslash G/M$. Then

$$\left| \int_{\Gamma_q \backslash G/M} \phi(x a_t) \psi(x) \ dm^\text{BMS}_q(x) \right| \leq \left| \int_{\Gamma_q \backslash G/M} \phi(x a_t) \psi \hat{G}_q(x) \ dm^\text{BMS}_q(x) \right| + \left| \int_{\Gamma_q \backslash G/M} \phi(x a_t) \psi_0(x) \ dm^\text{BMS}_q(x) \right|$$

where the first term can be written as

$$\int_{\Gamma_q \backslash G/M} \phi(x a_t) \psi \hat{G}_q(x) \ dm^\text{BMS}_q(x) = \sum_{g \in \hat{G}_q} \int_{\Gamma_q \backslash G/M} \phi(g x a_t) \psi \hat{G}_q(g x) \ dm^\text{BMS}_q(x)$$

$$= \int_{\Gamma_q \backslash G/M} \left( \sum_{g \in \hat{G}_q} \phi(g x a_t) \right) \psi \hat{G}_q(x) \ dm^\text{BMS}_q(x)$$

using the $\hat{G}_q$-invariance of $\psi \hat{G}_q$. Hence it again reduces to a question of exponential mixing of the geodesic flow on $\Gamma \backslash G/M$ which is known as explained above. Thus it suffices to assume that $\psi = \psi^0$, i.e., $\sum_{g \in \hat{G}_q} \psi(gx) = 0$ for all $x \in \Gamma_q \backslash G/M$. Thus we have corresponding functions $\phi_1, \psi_0 \in \mathcal{B}_q$ with $\sum_g \psi_0(u, g, t) = 0$ for all $(u, t) \in U^r$ and $\|\phi_1\|_{\mathcal{B}_q} \leq C_4^{-1} \|\psi\|_{C^1}$ for all $t \geq 0$ and $\|\psi_0\|_{\mathcal{B}_q} \leq C_4^{-1} \|\psi\|_{C^1}$. Hence by Corollary 5.4.1 and Lemma 5.3, for all $t \geq 0$, letting $t' = \frac{t}{2}$, we have

$$\left| \int_{\Gamma_q \backslash G/M} \phi(x a_t) \psi(x) \ dm^\text{BMS}_q(x) \right| \leq \frac{n_1^\text{BMS}(\Gamma \backslash G/M)}{\nu(\tau)} \left| \sum_{\tau} \psi_0(t') \right| + C_1 N_K(q)^{C_1} e^{-n t'} \|\phi\|_{C^1} \|\psi\|_{C^1}$$

$$\leq C_3 C_4^2 N_K(q)^{C_2} e^{-n t'} \|\phi\|_{C^1} \|\psi\|_{C^1} + C_1 N_K(q)^{C_1} e^{-n t'} \|\phi\|_{C^1} \|\psi\|_{C^1}$$

$$\leq C N_K(q)^{C_1} e^{-n t} \|\phi\|_{C^1} \|\psi\|_{C^1}.$$ 

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