BERRY-ESSEEN BOUNDS AND MODERATE DEVIATIONS FOR THE NORM, ENTRIES AND SPECTRAL RADIUS OF PRODUCTS OF POSITIVE RANDOM MATRICES

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Abstract. Let \((g_n)_{n \geq 1}\) be a sequence of independent and identically distributed positive random \(d \times d\) matrices and consider the matrix product \(G_n = g_n \ldots g_1\). Under suitable conditions, we establish the Berry-Esseen bounds on the rate of convergence in the central limit theorem and moderate deviation expansions of Cramér type, for the matrix norm \(\|G_n\|\) of \(G_n\), for its \((i,j)\)-th entry \(G_n^{i,j}\) and for its spectral radius \(\rho(G_n)\).

1. Introduction

Fix an integer \(d \geq 2\). Let \((g_n)_{n \geq 1}\) be a sequence of independent and identically distributed (i.i.d.) positive random \(d \times d\) matrices of the same probability law \(\mu\). Set \(G_n = g_n \ldots g_1\) and denote by \(\|G_n\|\) any matrix norm of the product \(G_n\). It has been of great interest in recent years to investigate the asymptotic behaviors of the random matrix product \(G_n\) since the pioneering work of Furstenberg and Kesten [13]. In [13] the strong law of large numbers (SLLN) for the matrix norm \(\|G_n\|\) was established: if \(\mathbb{E}(\max\{0, \log \|g_1\|\}) < \infty\), then

\[
\lim_{n \to \infty} \frac{1}{n} \log \|G_n\| = \lambda, \quad a.s.,
\]

where \(\lambda\) is a constant called the upper Lyapunov exponent of the product \(G_n\). This result can be seen as a direct consequence of Kingman’s subadditive ergodic theorem [21]. The central limit theorem (CLT) for \(\|G_n\|\) was also proved in [13]: for any \(y \in \mathbb{R}\),

\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{\log \|G_n\| - n\lambda}{\sigma \sqrt{n}} \leq y \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} dt =: \Phi(y),
\]

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where \( \sigma^2 > 0 \) is the asymptotic variance corresponding to the product \( G_n \). The conditions used in [13] for the proof of (1.2) have been relaxed later by Hennion [17] to the second moment condition together with the allowability and positivity condition that we will present in the next section. We mention that in the case of invertible random matrices, the CLT (1.2) was established by Le Page [22], and has been extended by Goldsheid and Guivarc’h [14] to a multidimensional version, and by Benoist and Quint [3] to the general framework of reductive groups under optimal moment conditions.

In [25] the authors proved a Berry-Esseen bound and a moderate deviation expansion for the norm cocycle \( \log |G_n|^x \) jointly with the Markov chain \( X^x_n = G_n^x/|G_n|^x \), where \( x \) is any starting point on the unit sphere and \(| \cdot |\) is the euclidean norm in \( \mathbb{R}^d \). For related results about the vector norm \( |G_n|^x \) we refer to [22, 5, 1, 15, 8, 4, 10, 11, 23, 24]. However, this type of results for other important quantities like the matrix norm \( \|G_n\| \), the entries \( G^{i,j}_n \) and the spectral radius \( \rho(G_n) \) of \( G_n \) are absent in the literature. The goal of the present paper is to fill this gap by extending the results of [25] to the matrix norm, to the entries and to the spectral radius for the product \( G_n \) of positive random matrices, jointly with the Markov chain \( (X^x_n)_{n \geq 0} \).

Let us explain briefly the main results that we obtain for the matrix norm. We would like to quantify the error in the normal approximation (1.2). We do this in two ways. The first way is to estimate the absolute error. In this spirit, under suitable conditions we prove the following Berry-Esseen bound: there exists a constant \( C > 0 \) such that for all \( n \geq 1 \),

\[
\sup_{y \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\log \|G_n\| - n\lambda}{\sigma \sqrt{n}} \leq y \right) - \Phi(y) \right| \leq \frac{C}{\sqrt{n}}.
\] (1.3)

Our result (1.3) is clearly a refinement of (1.2) by giving the rate of convergence. In fact, a more general version of the Berry-Esseen bound for the couple \((X^x_n, \log \|G_n\|)\) with a target function \( \phi \) on \( X^x_n \) is given in Theorem 2.1.

The second way is to study the relative error in (1.2). Along this line we prove the following Cramér type moderate deviation expansion: as \( n \to \infty \), uniformly in \( y \in [0, o(\sqrt{n})] \),

\[
\mathbb{P} \left( \frac{\log \|G_n\| - n\lambda}{\sigma \sqrt{n}} \geq y \right) = e^{\frac{y^3}{3} \zeta(\frac{y}{\sqrt{n}})} \left[ 1 + O \left( \frac{y + 1}{\sqrt{n}} \right) \right],
\] (1.4)

where \( \zeta \) is the Cramér series (see (2.10)). Note that the expansion (1.4) clearly implies the moderate deviation principle for the matrix norm \( \|G_n\| \), see Corollary 2.5, which to the best of our knowledge was not known before.

The results (1.3) and (1.4) concern the matrix norm \( \|G_n\| \), but we also prove that they remain valid (under stronger conditions) when the matrix norm \( \|G_n\| \) is replaced by the entries \( G^{i,j}_n \) or the spectral radius \( \rho(G_n) \): see
Theorems 2.2 and 2.6. The corresponding strong law of large numbers and the central limit theorem were established in [13, 12, 17] for the entries $G^{i,j}_n$, and in [17] for the spectral radius $\rho(G_n)$. However, our Theorems 2.2 and 2.6 on Berry-Esseen bounds and Cramér type moderate deviation expansions for the entries $G^{i,j}_n$ and the spectral radius $\rho(G_n)$ are new.

The proofs of (1.3) and (1.4) are based on the recent results established in [25] about the Berry-Esseen bound and the Cramér type moderate deviation expansion for the norm cocycle $\log |G_n x|$ and on a comparison between $\|G_n\|$ and $|G_n x|$ (Lemma 3.1), where $x$ is a vector in $\mathbb{R}^d$ with strictly positive components.

To prove (1.3) and (1.4) when the matrix norm $\|G_n\|$ is replaced by the entries $G^{i,j}_n$, in addition to the use of the aforementioned results established in [25], we do a careful quantitative analysis of the comparison between $\log G^{i,j}_n := \log \langle e_i, G_n e_j \rangle$ and $\log |G_n e_j|$, where $(e_i)_{1 \leq k \leq d}$ is the canonical orthonormal basis in $\mathbb{R}^d$. This comparison is possible due to a regularity condition which ensures that all the entries in the same column of the matrix $g \in \text{supp } \mu$ (the support of $\mu$) are comparable: see condition A3. Note that this condition is weaker than the Furstenberg-Kesten condition (2.1) used in [13], which says that all the entries of the matrix $g \in \text{supp } \mu$ are comparable.

Using the results mentioned above for the matrix norm $\|G_n\|$ and for the vector norm $|G_n x|$ established in [25], we then prove the corresponding results for the spectral radius $\rho(G_n)$ based on the Collatz-Wielandt formula: see Theorems 2.2 and 2.6.

When the boundedness condition A3 of Furstenberg-Kesten type is relaxed to a moment condition A4, we are also able to establish Berry-Esseen type bounds and moderate deviation principles for the entries $G^{i,j}_n$ and the spectral radius $\rho(G_n)$: see Theorems 2.3 and 2.7. Note that under condition A4, the Markov chain $(X^i_n)_{n \geq 0}$ is no longer separated from the coordinates $e_i$ and an important step to prove Theorems 2.3 and 2.7 is to establish the Hölder regularity of the stationary measure $\nu$ shown in Proposition 3.3, which is also of independent interest. The proof of Proposition 3.3 is based on the large deviation bounds for the norm cocycle $\log |G_n x|$ stated in Theorem 3.4.

In closing this section, we mention that Berry-Esseen bounds and moderate deviations for random matrices on different aspects have been considered in the literature, see e.g. Chen, Gao and Wang [9] for eigenvalues of a single random matrix when the dimension goes to $\infty$.

2. Main results

2.1. Notation and conditions. For any integer $d \geq 2$, denote by $\mathcal{M}_+$ the multiplicative semigroup of $d \times d$ matrices with non-negative entries in $\mathbb{R}$. A non-negative matrix $g \in \mathcal{M}_+$ is said to be allowable, if every row
and every column of $g$ contains a strictly positive entry. We write $\mathcal{M}_+^d$ for the subsemigroup of $\mathcal{M}_+$ with strictly positive entries. Equip the space $\mathbb{R}^d$ with the standard scalar product $\langle \cdot , \cdot \rangle$ and the Euclidean norm $| \cdot |$. For a vector $x$, we write $x \geq 0$ (resp. $x > 0$) if all its components are non-negative (resp. strictly positive). Denote by $S_+^{d-1} = \{ x \geq 0 : |x| = 1 \}$ the intersection of the unit sphere with the positive quadrant. The space $S_+^{d-1}$ is endowed with the Hilbert cross-ratio metric $d$, i.e., for any $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $S_+^{d-1}$,
\[
  d(x,y) = \frac{1 - m(x,y)m(y,x)}{1 + m(x,y)m(y,x)},
\]
where
\[
  m(x,y) = \sup \{ \alpha > 0 : \alpha y_i \leq x_i, \forall i = 1, \ldots, d \}.
\]
It is shown in [17] that there exists a constant $C > 0$ such that $|x - y| \leq C d(x,y)$ for any $x, y \in S_+^{d-1}$. We refer to [17] for more properties of the metric $d$.

Let $\mathcal{C}(S_+^{d-1})$ be the space of continuous complex-valued functions on $S_+^{d-1}$ and $\mathbf{1}$ be the constant function with value $1$. Throughout the paper we always assume that $\gamma > 0$ is a fixed small enough constant. For any $\varphi \in \mathcal{C}(S_+^{d-1})$, set
\[
  \| \varphi \|_\gamma := \| \varphi \|_\infty + [\varphi]_\gamma, \quad \| \varphi \|_\infty := \sup_{x \in S_+^{d-1}} |\varphi(x)|, \quad [\varphi]_\gamma := \sup_{x,y \in S_+^{d-1}} \frac{|\varphi(x) - \varphi(y)|}{d^\gamma(x,y)}.
\]
We introduce the Banach space
\[
  \mathcal{B}_\gamma := \left\{ \varphi \in \mathcal{C}(S_+^{d-1}) : \| \varphi \|_\gamma < +\infty \right\}.
\]
Let $(g_n)_{n \geq 1}$ be a sequence of i.i.d. positive random matrices of the same probability law $\mu$ on $\mathcal{M}_+$. Denote by $\text{supp} \mu$ the support of the measure $\mu$. Consider the matrix product $G_n = g_n \ldots g_1$ and denote by $G_n^{i,j}$ the $(i,j)$-th entry of $G_n$, where $1 \leq i, j \leq d$. It holds that
\[
  G_n^{i,j} = \langle e_i, G_ne_j \rangle,
\]
where $(e_k)_{1 \leq k \leq d}$ is the canonical orthonormal basis of $\mathbb{R}^d$. For any $g \in \mathcal{M}_+$, denote by $\rho(g)$ the spectral radius of $g$, and by $\|g\|$ its operator norm, i.e., $\|g\| = \sup_{x \in S_+^{d-1}} |gx|$. By Gelfand’s formula, it holds that $\rho(g) = \lim_{k \to \infty} \|g^k\|^{1/k}$. In this paper, we are interested in Berry-Esseen bounds and moderate deviation asymptotics for the matrix norm $\|G_n\|$, the entries $G_n^{i,j}$ and the spectral radius $\rho(G_n)$.

Let $\iota(g) = \inf_{x \in S_+^{d-1}} |gx|$ and $N(g) = \max\{\|g\|, \iota(g)^{-1}\}$. We shall need the following exponential moment condition:
A1. There exists a constant \( \eta \in (0,1) \) such that \( E[N(g_1)^\eta] < +\infty \).

Let \( \Gamma_\mu \) be the smallest closed subsemigroup of \( M_+ \) generated by \( \text{supp}\mu \). We will use the allowability and positivity conditions:

A2. (i) (Allowability) Every \( g \in \Gamma_\mu \) is allowable.

(ii) (Positivity) \( \Gamma_\mu \) contains at least one matrix belonging to \( M_+^c \).

It follows from the Perron-Frobenius theorem that every \( g \in M_+^c \) has a dominant eigenvalue which coincides with its spectral radius \( \rho(g) \). The corresponding eigenvector is denoted by \( v_g \). It is easy to see that \( v_g \in S_{d-1}^d \).

The following condition ensures that all the entries in each column of the matrix \( g \in \text{supp}\mu \) are comparable.

A3. For any \( 1 \leq j \leq d \), there exists a constant \( C > 1 \) such that for any \( g = (g_{i,j})_{1 \leq i,j \leq d} \in \text{supp}\mu \),

\[
1 \leq \frac{\max_{1 \leq i,j \leq d} g_{i,j}}{\min_{1 \leq i,j \leq d} g_{i,j}} \leq C.
\]

Note that the set of such type of matrices forms a subsemigroup of \( M_+ \), because if two positive matrices \( g_1 \) and \( g_2 \) satisfy condition A3, then so does the product \( g_2 g_1 \), as will be seen from Lemma 3.2 where an equivalent description of condition A3 will be provided.

It is easy to see that condition A3 implies condition A2. However, our condition A3 is clearly weaker than the Furstenberg-Kesten condition used in [13]: there exists a constant \( C > 1 \) such that for any \( g = (g_{i,j})_{1 \leq i,j \leq d} \in \text{supp}\mu \),

\[
1 \leq \frac{\max_{1 \leq i,j \leq d} g_{i,j}}{\min_{1 \leq i,j \leq d} g_{i,j}} \leq C. \tag{2.1}
\]

This condition plays an essential role in [13] for the proofs of the strong law of large numbers and the central limit theorem for entries \( G_{n,j} \).

The following condition concerns the existence of the harmonic moments of the entries of \( g_1 \):

A4. For any \( 1 \leq i, j \leq d \), there exists a constant \( \delta > 0 \) such that

\[
E\left[ (g_{i,j}^{\delta})^{-\delta} \right] < \infty.
\]

Condition A4 is used to establish Berry-Esseen type bounds and moderate deviation principles for the entries \( G_{n,j}^{i,j} \) and the spectral radius \( \rho(G_n) \), see Theorems 2.3 and 2.7, where condition A3 is not assumed. Note that the conditions A3 and A4 do not imply each other. However, under the moment assumption A1, condition A3 (and therefore also (2.1)) implies condition A4. The converse is not true.
For any \( x \in S_{d+1}^d \) and allowable matrix \( g \in M_+ \), we write \( g \cdot x := \frac{gx}{|gx|} \) for the projective action of the matrix \( g \) on the projective space \( S_{d+1}^d \). For any starting point \( x \in S_{d+1}^d \), set \( X_0^x = x \) and 
\[
X_n^x = G_n \cdot x, \quad n \geq 1.
\]
Then \( (X_n^x)_{n \geq 0} \) forms a Markov chain on \( S_{d+1}^d \) with the transfer operator \( P \) given as follows: for any \( \varphi \in C(S_{d+1}^d) \),
\[
P \varphi(x) = \int_{\Gamma_\mu} \varphi(g \cdot x) \mu(dg), \quad x \in S_{d+1}^d.
\]
Under conditions \( A_1 \) and \( A_2 \), the Markov chain \( (X_n^x)_{n \geq 0} \) possesses a unique stationary probability measure \( \nu \) on \( S_{d+1}^d \) such that for any \( \varphi \in C(S_{d+1}^d) \),
\[
\int_{S_{d+1}^d} \int_{\Gamma_\mu} \varphi(g \cdot x) \mu(dg) \nu(dx) = \int_{S_{d+1}^d} \varphi(x) \nu(dx).
\]
Moreover, the support of \( \nu \) is given by \( \text{supp} \nu = \{ v_g \in S_{d+1}^d : g \in \Gamma_\mu \cap \mathcal{M}_\mu^d \} \).

Under conditions \( A_1 \) and \( A_2 \), it is shown in [25] that uniformly in \( x \in S_{d+1}^d \),
\[
\sigma^2 := \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ (\log |G_n x| - n \lambda)^2 \right] \in [0, \infty),
\]
where \( \lambda \) is the upper Lyapunov exponent defined by (1.1). Equivalent formulations of \( \sigma^2 \) will be given in Proposition 2.8. We shall need the following conditions.

**A5.** The asymptotic variance \( \sigma^2 \) satisfies \( \sigma^2 > 0 \).

**A6.** (Non-arithmeticity) For \( t > 0, \theta \in [0, 2\pi) \) and a function \( \varphi : S_{d+1}^d \to \mathbb{R} \), the equation
\[
|gx|^t \varphi(g \cdot x) = e^{i\theta} \varphi(x), \quad \forall g \in \Gamma_\mu, \forall x \in \text{supp} \nu,
\]
has no trivial solution except that \( t = 0, \theta = 0 \) and \( \varphi \) is a constant.

Note that condition \( A_6 \) implies \( A_5 \). If the additive subgroup of \( \mathbb{R} \) generated by the set \( \{ \log \rho(g) : g \in \Gamma_\mu \cap \mathcal{M}_\mu^d \} \) is dense in \( \mathbb{R} \), then both conditions \( A_5 \) and \( A_6 \) are fulfilled (see [8]). This sufficient condition was introduced by Kesten [20] and is usually easier to verify in practice.

2.2. **Berry-Esseen bounds.** The goal of this section is to present our results on the Berry-Esseen bounds for the matrix norm \( \| G_n \| \), the entries \( G_{n,i}^j \) and the spectral radius \( \rho(G_n) \). Let us first state the result for the operator norm \( \| G_n \| \). Denote \( (S_{d+1}^d)^c = \{ x > 0 : |x| = 1 \} \), which is the interior of the projective space \( S_{d+1}^d \).
Theorem 2.1. Assume conditions \( A1, A2 \) and \( A5 \). Then, for any compact set \( K \subset (S^d_{+})^\circ \), there exists a constant \( C > 0 \) such that for all \( n \geq 1 \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\sup_{y \in \mathbb{R}} \sup_{x \in K} \left| E \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \|G_n\| - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma .
\] (2.4)

Since all matrix norms are equivalent, it can be easily checked that in Theorem 2.1, the operator norm \( \| \cdot \| \) can be replaced by any matrix norm.

It would be interesting to show that (2.4) holds uniformly in \( x \in S^d_{+} \) instead of \( x \in K \). Note that Theorem 2.1 is proved under the exponential moment condition \( A1 \). It is not clear how to establish Theorem 2.1 under the polynomial moment condition on the matrix law \( \mu \).

If the stronger condition \( A3 \) holds instead of condition \( A2 \), then we are able to prove the following Berry-Esseen bounds for the scalar product \( \langle f, G_n x \rangle \) and for the spectral radius \( \rho(G_n) \).

Theorem 2.2. Assume conditions \( A1, A3 \) and \( A5 \).

(1) There exists a constant \( C > 0 \) such that for all \( n \geq 1 \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\sup_{y \in \mathbb{R}} \sup_{f, x \in S^d_{+}} \left| E \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma .
\] (2.5)

(2) For any compact set \( K \subset (S^d_{+})^\circ \), there exists a constant \( C > 0 \) such that for all \( n \geq 1 \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\sup_{y \in \mathbb{R}} \sup_{x \in K} \left| E \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma .
\] (2.6)

In particular, taking \( \varphi = 1 \), \( f = e_i \) and \( x = e_j \) in (2.5), we get the Berry-Esseen bound for the entries \( G_n^{i,j} \). The Berry-Esseen bounds (2.5) and (2.6) are new.

If condition \( A3 \) is replaced by the weaker one \( A4 \) (under \( A1 \), condition \( A4 \) is weaker than \( A3 \)), then we are able to establish the following result.

Theorem 2.3. Assume conditions \( A1, A4 \) and \( A6 \).

(1) There exists a constant \( C > 0 \) such that for all \( n \geq 1 \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\sup_{y \in \mathbb{R}} \sup_{f, x \in S^d_{+}} \left| E \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \rho(G_n) - n\lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}} \|\varphi\|_\gamma .
\] (2.7)
For any compact set $K \subset (S^d_{+})^{\circ}$, there exists a constant $C > 0$ such that for all $n \geq 1$ and $\varphi \in \mathcal{B}_{\gamma}$,

$$\sup_{y \in \mathbb{R}} \sup_{x \in K} \left| \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \log \rho(G_n) - n\lambda - n\sigma \sqrt{n} \leq y \right\} } \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C \log n}{\sqrt{n}} \| \varphi \|_{\gamma}.$$  

(2.8)

The proof of (2.7) and (2.8) relies on the Hölder regularity of the stationary measure $\nu$ established in Proposition 3.3. To prove Proposition 3.3, the large deviation bounds for the norm cocycle $\log |G_n|$ (see Theorem 3.4) is required. This explains why the non-arithmeticity condition A6 is assumed in Theorem 2.3.

It seems to be a challenging problem to improve (2.7) and (2.8) by replacing $\log n \sqrt{n}$ with $1 \sqrt{n}$.

2.3. Moderate deviation expansions. In this section we formulate the moderate deviation results for the matrix norm $\|G_n\|$, the entries $G_{ij}^n$ and the spectral radius $\rho(G_n)$. We need some additional notation. For any $s \in (-\eta, \eta)$, define the transfer operator $P_s$ as follows: for any $\varphi \in \mathcal{C}(S^d_{+})$,

$$P_s \varphi(x) = \int_{\Gamma_s} |gx|^s \varphi(g \cdot x) \mu(dg), \quad x \in S^d_{+}.$$  

(2.9)

We see that $P_0$ coincides with the transfer operator $P$ defined by (2.2). Based on the perturbation theory for linear operators [18], it was shown in [25] that under conditions A1 and A2, the transfer operator $P_s$ has spectral gap properties on the Banach space $\mathcal{B}_{\gamma}$ and possesses a dominating eigenvalue $\kappa(s)$. Moreover, the function $\kappa$ is analytic, real-valued and strictly convex in a small neighborhood of 0 under the additional condition A5. Denote $\Lambda = \log \kappa$ and $\gamma_k = \Lambda^{(k)}(0)$, $k \geq 1$, then it holds that $\gamma_1 = \lambda$ and $\gamma_2 = \sigma^2$. Throughout this paper, we write $\zeta$ for the Cramér series of $\Lambda$:

$$\zeta(t) = \frac{\gamma_3}{6\gamma_2^3} + \frac{\gamma_4 \gamma_2 - 3 \gamma_3^2}{24 \gamma_2^2} t + \frac{\gamma_5 \gamma_2^2 - 10 \gamma_4 \gamma_3 \gamma_2 + 15 \gamma_3^3}{120 \gamma_2^9} t^2 + \cdots,$$  

(2.10)

which converges for $|t|$ small enough.

The following result concerns the Cramér type moderate deviations for the operator norm $\|G_n\|$. Recall that $(S^d_{+})^{\circ} = \{x > 0 : |x| = 1\}$.

**Theorem 2.4.** Assume conditions A1, A2 and A5. Then, for any compact set $K \subset (S^d_{+})^{\circ}$, we have, as $n \to \infty$, uniformly in $x \in K$, $y \in [0, o(\sqrt{n})]$. 


and \( \varphi \in B_\gamma, \)

\[
\mathbb{E} \left[ \frac{\varphi(X_n^x) 1_{\{ \log \|G_n\| - n\lambda \geq \sqrt{n}\sigma y \}}}{1 - \Phi(y)} \right] = e^{\frac{3}{\sqrt{n}} \zeta \left( \frac{y}{\sqrt{n}} \right)} \left[ \nu(\varphi) + \|\varphi\|_\gamma O \left( \frac{y + 1}{\sqrt{n}} \right) \right],
\]

(2.11)

\[
\mathbb{E} \left[ \frac{\varphi(X_n^x) 1_{\{ \log \|G_n\| - n\lambda \leq -\sqrt{n}\sigma y \}}}{\Phi(-y)} \right] = e^{-\frac{3}{\sqrt{n}} \zeta \left( -\frac{y}{\sqrt{n}} \right)} \left[ \nu(\varphi) + \|\varphi\|_\gamma O \left( \frac{y + 1}{\sqrt{n}} \right) \right].
\]

(2.12)

Like in Theorem 2.1, it can also be checked that in Theorem 2.4 the operator norm \( \| \cdot \| \) can be replaced by any matrix norm.

Note that condition A3 is not required in Theorem 2.4. Theorem 2.4 is new even for \( \varphi = 1 \) and the expansions (2.11) and (2.12) remain valid even when \( \nu(\varphi) = 0 \). As a particular case, Theorem 2.4 implies the following moderate deviation principle for \( \log \|G_n\| \) with a target function \( \varphi \) on the Markov chain \( X_n^x \).

**Corollary 2.5.** Assume conditions A1, A2 and A5. Then, for any real-valued function \( \varphi \in B_\gamma \) satisfying \( \nu(\varphi) > 0 \), for any Borel set \( B \subseteq \mathbb{R} \) and any positive sequence \( (b_n)_{n \geq 1} \) satisfying \( \frac{b_n}{n^2} \to 0 \) and \( \frac{b_n}{\sqrt{n}} \to \infty \), we have, uniformly in \( x \in K \),

\[
-\inf_{y \in B^0} \frac{y^2}{2\sigma^2} \leq \liminf_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) 1_{\{ \frac{\log \|G_n\| - n\lambda}{b_n} \in B \}} \right] \leq \limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) 1_{\{ \frac{\log \|G_n\| - n\lambda}{b_n} \in B \}} \right] \leq -\inf_{y \in \bar{B}} \frac{y^2}{2\sigma^2},
\]

(2.13)

where \( B^0 \) and \( \bar{B} \) are respectively the interior and the closure of \( B \).

Note that the target function \( \varphi \) in (2.13) is not necessarily positive and it can vanish on some part of the projective space \( S^{d-1}_+ \). The moderate deviation principle (2.13) is new, even for \( \varphi = 1 \).

As in Theorem 2.1, it would be interesting to prove that Theorem 2.4 holds uniformly in \( x \in S^{d-1}_+ \) instead of \( x \in K \).

Now we formulate Cramér type moderate deviation expansions for the scalar product \( \langle f, G_n x \rangle \) as well as for the spectral radius \( \rho(G_n) \).
Theorem 2.6. Assume conditions A1, A3 and A5. Then, we have:

(1) as \( n \to \infty \), uniformly in \( f, x \in \mathbb{S}_+^{d-1} \), \( y \in [0, o(\sqrt{n})] \) and \( \varphi \in B_{\gamma} \),

\[
\frac{\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log(f, G_n x) - n \lambda \geq \sqrt{n} \rho(y)\}} \right]}{1 - \Phi(y)} = e^{\frac{y^2}{\sqrt{n}}} \zeta\left(\frac{y}{\sqrt{n}}\right) \left[ \nu(\varphi) + \| \varphi \|_{\gamma} O\left(\frac{y + 1}{\sqrt{n}}\right) \right],
\]

(2.14)

\[
\frac{\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log(f, G_n x) - n \lambda \leq -\sqrt{n} \rho(y)\}} \right]}{\Phi(-y)} = e^{-\frac{y^2}{\sqrt{n}}} \zeta\left(-\frac{y}{\sqrt{n}}\right) \left[ \nu(\varphi) + \| \varphi \|_{\gamma} O\left(\frac{y + 1}{\sqrt{n}}\right) \right];
\]

(2.15)

(2) for any compact set \( K \subset (\mathbb{S}_+^{d-1})^0 \), as \( n \to \infty \), uniformly in \( x \in K \), \( y \in [0, o(\sqrt{n})] \) and \( \varphi \in B_{\gamma} \),

\[
\frac{\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log(\rho(G_n x) - n \lambda \geq \sqrt{n} \rho(y)\}} \right]}{1 - \Phi(y)} = e^{\frac{y^2}{\sqrt{n}}} \zeta\left(\frac{y}{\sqrt{n}}\right) \left[ \nu(\varphi) + \| \varphi \|_{\gamma} O\left(\frac{y + 1}{\sqrt{n}}\right) \right],
\]

(2.16)

\[
\frac{\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log(\rho(G_n x) - n \lambda \leq -\sqrt{n} \rho(y)\}} \right]}{\Phi(-y)} = e^{-\frac{y^2}{\sqrt{n}}} \zeta\left(-\frac{y}{\sqrt{n}}\right) \left[ \nu(\varphi) + \| \varphi \|_{\gamma} O\left(\frac{y + 1}{\sqrt{n}}\right) \right].
\]

(2.17)

As a particular case of (2.14) and (2.15) with \( f = e_i \) and \( x = e_j \), we get

the Cramér type moderate deviation expansions for the entries \( G_n^{ij} \). The

expansions (2.14)-(2.17) are all new even for \( \varphi = 1 \).

We end this subsection by giving moderate deviation principles for the

scalar product \( (f, G_n x) \) and for the spectral radius \( \rho(G_n) \). Recall that for a

Borel set \( B \), we write respectively \( B^{\circ} \) and \( \overline{B} \) for its interior and closure.

Theorem 2.7. Assume either conditions A1, A3, A5, or conditions A1, A4, A6. Then, for any real-valued function \( \varphi \in B_{\gamma} \) satisfying \( \nu(\varphi) > 0 \), for any Borel set \( B \subset \mathbb{R} \) and any positive sequence \( (b_n)_{n \geq 1} \) satisfying \( \frac{b_n}{n} \to 0 \) and \( \frac{b_n}{\sqrt{n}} \to \infty \), we have

(1) uniformly in \( f, x \in \mathbb{S}_+^{d-1} \),

\[
- \frac{\inf_{y \in B^{\circ}} \frac{y^2}{2\sigma^2}}{\sqrt{2 \pi}} \leq \liminf_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log(f, G_n x) - n \lambda \leq \frac{b_n}{\sqrt{n}} \}} \right] - \limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log(f, G_n x) - n \lambda \geq \frac{b_n}{\sqrt{n}} \}} \right] \leq - \frac{\inf_{y \in B} \frac{y^2}{2\sigma^2}}{\sqrt{2 \pi}}.
\]

(2.18)
(2) for any compact set $K \subset (S^d_{-1})^\circ$, uniformly in $x \in K$,
\[
- \inf_{y \in B^\circ} \frac{y^2}{2\sigma^2} \leq \liminf_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{I} \left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \in B \right\} \right] \\
\leq \limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{I} \left\{ \frac{\log \rho(G_n) - n\lambda}{b_n} \in B \right\} \right] \leq - \inf_{y \in B} \frac{y^2}{2\sigma^2},
\]
(2.19)

Under conditions $A_1$, $A_3$ and $A_5$, the moderate deviation principles (2.18) and (2.19) follow directly from Theorem 2.6, just as we obtained (2.13) from Theorem 2.4. Under conditions $A_1$, $A_4$, $A_6$, (2.18) and (2.19) cannot be deduced from Theorem 2.6. In fact, the proof turns out to be delicate and is carried out using the Hölder regularity of the stationary measure $\nu$, see Proposition 3.3.

2.4. Formulas for the asymptotic variance. In this section, we give alternative expressions for the asymptotic variance $\sigma^2$ defined by (2.3). These expressions can be useful while applying the theorems and the corollaries stated before, where $\sigma$ appears.

**Proposition 2.8.** (1) Under conditions $A_1$ and $A_2$, we have
\[
\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \left( \log \|G_n\| - n\lambda \right)^2 \right].
\]
(2.20)

(2) Under conditions $A_1$ and $A_3$, we have
\[
\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \left( \log \langle f, G_n x \rangle - n\lambda \right)^2 \right] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \left( \log \rho(G_n) - n\lambda \right)^2 \right],
\]
(2.21)

where the convergence in the first equality holds uniformly in $f, x \in S^d_{-1}$.

We mention that for invertible matrices, the expression (2.20) has been established in [4, Proposition 14.7]. For positive matrices, both (2.20) and (2.21) are new.

3. Proofs of Berry-Esseen bounds

The goal of this section is to establish Theorems 2.1, 2.2 and 2.3.

3.1. Proof of Theorems 2.1 and 2.2. In order to prove Theorem 2.1, we shall use the following result which was shown in [7, Lemma 4.5].

**Lemma 3.1.** Under condition $A_2 (i)$, for any $x \in (S^d_{-1})^\circ$, we have
\[
\tau(x) := \inf_{g \in P_\mu} \frac{|gx|}{\|g\|} > 0.
\]
Moreover, for any compact set $K \subset (S^d_{-1})^\circ$, it holds that $\inf_{x \in K} \tau(x) > 0$. 
We now proceed to prove Theorem 2.1 based on Lemma 3.1 and the Berry-Esseen bound for the norm cocycle \(|G_n x|\) established in [25].

**Proof of Theorem 2.1.** Without loss of generality, we assume that the target function \(\varphi\) is non-negative. Under conditions of Theorem 2.1, the following Berry-Esseen bound for the norm cocycle \(|G_n x|\) with a target function \(\varphi\) on the Markov chain \(X_n^x\) has been recently established in [25]: there exists a constant \(C > 0\) such that for all \(n \geq 1\) and \(\varphi \in \mathcal{B}_\gamma\),

\[
\sup_{y \in \mathbb{R}} \sup_{x \in S^d_{+}^{-1}} \left| \mathbb{E} \left[ \varphi(X_n^x) 1_{\left\{ \frac{\log |G_n x| - n \lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] - \nu(\varphi) \Phi(y) \right| \leq \frac{C}{\sqrt{n}} \|\varphi\|_\gamma. \tag{3.1}
\]

On the one hand, using the fact that \(\log |G_n x| \leq \log \|G_n\|\), we deduce from (3.1) that there exists a constant \(C > 0\) such that for all \(y \in \mathbb{R}, x \in S^d_{+}^{-1}, n \geq 1\) and \(\varphi \in \mathcal{B}_\gamma\),

\[
\mathbb{E} \left[ \varphi(X_n^x) 1_{\left\{ \frac{\log \|G_n\| - n \lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] \leq \nu(\varphi) \Phi(y) + \frac{C}{\sqrt{n}} \|\varphi\|_\gamma.
\]

On the other hand, by Lemma 3.1, we see that for any compact set \(K \subset (S^d_{+}^{-1})^\circ\), there exists a constant \(C_1 > 0\) such that for all \(n \geq 1\) and \(x \in K\),

\[
\log \|G_n\| \leq \log |G_n x| + C_1. \tag{3.2}
\]

Combining this inequality with (3.1), we obtain that, with \(y_1 = y - \frac{C_1}{\sigma \sqrt{n}}\), uniformly in \(y \in \mathbb{R}, x \in K, n \geq 1\) and \(\varphi \in \mathcal{B}_\gamma\),

\[
\mathbb{E} \left[ \varphi(X_n^x) 1_{\left\{ \frac{\log \|G_n\| - n \lambda}{\sigma \sqrt{n}} \leq y \right\}} \right] \geq \nu(\varphi) \Phi(y_1) - \frac{C}{\sqrt{n}} \|\varphi\|_\gamma. \tag{3.3}
\]

By elementary calculations, we find that there exists a constant \(C_2 > 0\) such that for all \(y \in \mathbb{R}\) and \(n \geq 1\),

\[
\Phi(y_1) - \Phi(y) = -\frac{1}{\sqrt{2\pi}} \int_{y - \frac{C_1}{\sigma \sqrt{n}}}^{y} e^{-\frac{t^2}{2}} dt \geq -\frac{C_2}{\sqrt{n}}. \tag{3.4}
\]

This, together with (3.3), yields the desired lower bound. The proof of Theorem 2.1 is complete. \(\square\)

Now we proceed to prove Theorem 2.2. For any \(0 < \epsilon < 1\), set

\[
S^d_{+,-\epsilon} = \left\{ x \in S^d_{+} : \langle x, e_j \rangle \geq \epsilon \text{ for all } 1 \leq j \leq d \right\}.
\]

The following result provides an equivalent formulation of condition \(A_3\), which will be used to prove Theorems 2.2 and 2.6. For any matrix \(g \in \text{supp} \mu\), we denote \(g \cdot S^d_{+}^{-1} = \{ g \cdot x : x \in S^d_{+}^{-1} \}\).
Lemma 3.2. Condition A3 is equivalent to the following statement: there exists a constant \( \epsilon \in (0, \frac{\sqrt{2}}{2}) \) such that

\[
g \cdot S_{+}^{d-1} \subset S_{+, \epsilon}^{d-1}, \quad \text{for any } g \in \text{supp} \mu.
\]  

(3.5)

Proof. We first show that the assertion (3.5) implies condition A3. For any matrix \( g = (g^{i,j})_{1 \leq i, j \leq d} \in \text{supp} \mu \), we see that for any \( 1 \leq i, j \leq d \),

\[
\langle e_i, g \cdot e_j \rangle = \frac{g^{i,j}}{\sqrt{\sum_{i=1}^{d}(g^{i,j})^2}}.
\]  

(3.6)

Using (3.5) and the definition of \( S_{+, \epsilon}^{d-1} \), we get that there exists \( \epsilon \in (0, \frac{\sqrt{2}}{2}) \) such that \( \langle e_i, g \cdot e_j \rangle \geq \epsilon \) for all \( 1 \leq i, j \leq d \). This implies condition A3 with \( C = \sqrt{\frac{1}{d-1} \left( \frac{1}{\epsilon^2} - 1 \right)} \) by taking maxima and minima by rows in (3.6).

We next prove that condition A3 implies the assertion (3.5). For any \( x \in S_{+}^{d-1} \), we write \( x = \sum_{j=1}^{d} x_j e_j \), where \( x_j \geq 0 \) satisfies \( \sum_{j=1}^{d} x_j^2 = 1 \). It is easy to see that \( \sum_{j=1}^{d} x_j \geq 1 \). For any \( 1 \leq i \leq d \), it holds that

\[
\langle e_i, g \cdot x \rangle = \frac{1}{|g_x|} \sum_{j=1}^{d} x_j \langle e_i, g e_j \rangle = \frac{\sum_{j=1}^{d} x_j g^{i,j}}{\sqrt{\sum_{i=1}^{d}(\sum_{j=1}^{d} g^{i,j} x_j)^2}}.
\]

Since \( \sum_{j=1}^{d} x_j^2 = 1 \), we get \( (\sum_{j=1}^{d} g^{i,j} x_j)^2 \leq (\sum_{j=1}^{d} (g^{i,j})^2)^2 \) using the Cauchy-Schwarz inequality. Combining this with condition A3 and the fact that \( \sum_{j=1}^{d} x_j \geq 1 \), we obtain \( \langle e_i, g \cdot x \rangle \geq \sum_{j=1}^{d} \frac{x_j g^{i,j}}{\sqrt{C^2 x_j^2}} \geq \frac{1}{C d} \), so that the assertion (3.5) holds with \( \epsilon = \frac{1}{C d} \). \( \square \)

Using Lemma 3.2, Theorem 2.1 and the Berry-Esseen bound (3.1), we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality, we assume that the target function \( \varphi \) is non-negative.

We first prove the Berry-Esseen bound (2.5) for the scalar product \( \langle f, G_n x \rangle \).

On the one hand, using the fact that \( \log \langle f, G_n x \rangle \leq \log |G_n x| \), we deduce from the Berry-Esseen bound (3.1) that there exists a constant \( C > 0 \) such that for all \( y \in \mathbb{R}, f, x \in S_+^{d-1}, n \geq 1 \) and \( \varphi \in B_\gamma \),

\[
\mathbb{E} \left[ \varphi(X_n^x) 1 \left\{ \log \frac{\log(f,G_n x) - n \lambda_{\gamma}}{\varphi_{\gamma}} \right\} \right] \geq \nu(\varphi) \Phi(y) - \frac{C}{\sqrt{n}} \| \varphi \|_\gamma.
\]  

(3.7)

On the other hand, note that \( \log \langle f, G_n x \rangle = \log |G_n x| + \log \langle f, X_n^x \rangle \). By Lemma 3.2, we see that there exists a constant \( C_1 > 0 \) such that for all \( f, x \in S_+^{d-1} \) and \( n \geq 1 \),

\[
\log |G_n x| \leq \log \langle f, G_n x \rangle + C_1.
\]  

(3.8)
Using this inequality and again the Berry-Esseen bound (3.1), we obtain that, with \( y_1 = y + \frac{C}{\sqrt{n}} \), uniformly in \( y \in \mathbb{R}, f, x \in S^{d-1}_+, n \geq 1 \) and \( \varphi \in B_\gamma 
exists C > 0 \) such that for all \( y \in \mathbb{R}, x \in K, n \geq 1 \) and \( \varphi \in B_\gamma 
exists C > 0 \),

\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \left( f(G_n^x) \right) \cdot n}{\sigma \sqrt{n}} \leq y \right\}} \right] \leq \nu(\varphi) \Phi(y_1) + \frac{C}{\sqrt{n}} \|\varphi\|_\gamma.
\]

It is easy to show that \( \Phi(y_1) - \Phi(y) \leq \frac{C}{\sqrt{n}} \), uniformly in \( y \in \mathbb{R} \). Together with the above inequality, this leads to the desired upper bound and ends the proof of the Berry-Esseen bound (2.5).

We next prove the bound (2.6) for the spectral radius \( \rho(G_n) \). Since \( \rho(G_n) \leq \|G_n\| \), by Theorem 2.1, we get the following lower bound: there exists a constant \( C > 0 \) such that for all \( y \in \mathbb{R}, x \in K, n \geq 1 \) and \( \varphi \in B_\gamma 
exists C > 0 \),

\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \left( f(G_n^x) \right) \cdot n}{\sigma \sqrt{n}} \leq y \right\}} \right] \geq \nu(\varphi) \Phi(y) - \frac{C}{\sqrt{n}} \|\varphi\|_\gamma.
\]

The upper bound is carried out by using the Collatz-Wielandt formula in conjunction with the Berry-Esseen bound (2.5) for the entries \( G_n^{i,j} \). Denote by \( C_+ = \{x \in \mathbb{R}^d : x \geq 0\} \setminus \{0\} \) the positive quadrant in \( \mathbb{R}^d \) except the origin. According to the Collatz-Wielandt formula, the spectral radius of the positive matrix \( G_n \) can be represented as follows:

\[
\rho(G_n) = \sup_{x \in C_+} \min_{1 \leq i \leq d, \langle e_i, x \rangle > 0} \frac{\langle e_i, G_n x \rangle}{\langle e_i, x \rangle}, \tag{3.9}
\]

It follows that there exists a constant \( \epsilon \in (0, \sqrt{\frac{2}{d}}) \) such that for all \( x \in S^{d-1}_+, \rho(G_n) \geq \min_{1 \leq i \leq d} \langle e_i, G_n x \rangle \geq \min_{1 \leq i \leq d} \langle e_i, X_n^x \rangle |G_n x| \geq \epsilon |G_n x|, \tag{3.10}
\]

where in the last inequality we used Lemma 3.2. Using the bound (3.1) and the inequality (3.10), we deduce that there exists a constant \( C > 0 \) such that for all \( x \in S^{d-1}_+, y \in \mathbb{R}, n \geq 1 \) and \( \varphi \in B_\gamma 
exists C > 0 \),

\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \left( f(G_n^x) \right) \cdot n}{\sigma \sqrt{n}} \leq y \right\}} \right] \leq \nu(\varphi) \Phi(y) + \frac{C}{\sqrt{n}} \|\varphi\|_\gamma.
\]

This ends the proof of the bound (2.6) for the spectral radius \( \rho(G_n) \). \( \square \)

### 3.2. Hölder regularity of stationary measures

In this section we present our results on the Hölder regularity of the stationary measure \( \pi_s \) and of the eigenmeasure \( \nu_s \). The regularity of \( \pi_s \) and \( \nu_s \) is central to establishing Berry-Esseen type bounds and moderate deviation asymptotics for the entries \( G_n^{i,j} \) and is also of independent interest. Hereafter, we denote

\[
I_\mu = \{ s \geq 0 : \mathbb{E}(\|g_1\|^s) < \infty \}.
\]

By Hölder’s inequality, it is easy to see that \( I_\mu \) is an interval on \( \mathbb{R} \). The interior of \( I_\mu \) is denoted by \( I_\mu^\circ \). For any \( s \in I_\mu \), define the transfer operator
$P_s$ as in (2.9): for any $\varphi \in B_\gamma$,
\[
P_s \varphi(x) = \int_{\Gamma_\nu} |gx|^s \varphi(g \cdot x) \mu(dg), \quad x \in S_{d-1}^+. \tag{3.11}
\]

It is proved in [7] that the operator $P_s$ has unique continuous strictly positive eigenfunction $r_s$ on $\mathbb{P}^{d-1}$ and unique probability eigenmeasure $\nu_s$ satisfying
\[
P_s r_s = \kappa(s) r_s \quad \text{and} \quad P_s \nu_s = \kappa(s) \nu_s.
\]

The family of probability kernels $q_n^s(x, g) = \frac{|gx|^s r_s(g \cdot x)}{\kappa(s) r_s(x)}$, $n \geq 1$, satisfies the cocycle property. Hence the probability measures $q_n^s(x, g_0 \ldots g_n) \mu(dg_1) \ldots \mu(dg_n)$ form a projective system on $\mathcal{M}_+^{N^r}$, so that there exists a unique probability measure $Q_s^\nu$ on $\mathcal{M}_+^{N^r}$, by the Kolmogorov extension theorem. The corresponding expectation is denote by $E_{Q_s^\nu}$. For any measurable function $\varphi$ on $(S_{d-1}^+ \times \mathbb{R})^n$, it holds that
\[
\frac{1}{\kappa^n(s)} E\left[r_s(X_1^s) | G_n x|^s \varphi(X_1^s, \log |G_1 x|, \ldots, X_n^s, \log |G_n x|)\right] = E_{Q_s^\nu}[\varphi(X_1^s, \log |G_1 x|, \ldots, X_n^s, \log |G_n x|)]. \tag{3.12}
\]

Under the changed measure $Q_s^\nu$, the Markov chain $(X_n^x)_{n \geq 0}$ has a unique stationary measure $\pi_s$ given by $\pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)}$, for any function $\varphi \in \mathcal{C}(S_{d-1}^+)$. 

**Proposition 3.3.** Assume either condition A3 or conditions A1, A4, A6. Then, for any $s \in \{0\} \cup I_\mu^c$, there exists a constant $\alpha > 0$ such that
\[
\sup_{f \in S_{d-1}^+} \int_{S_{d-1}^+} \frac{1}{|\langle f, x \rangle|^\alpha} \nu_s(dx) < \infty. \tag{3.13}
\]

In particular, for any $s \in \{0\} \cup I_\mu^c$, there exist constants $\alpha, C > 0$ such that for any $0 < t \leq 1$,
\[
\sup_{f \in S_{d-1}^+} \nu_s \left(\{ x \in S_{d-1}^+ : |\langle f, x \rangle| \leq t \} \right) \leq Ct^\alpha. \tag{3.14}
\]

Moreover, the assertions (3.13) and (3.14) remain valid when the eigenmeasure $\nu_s$ is replaced by the stationary measure $\pi_s$.

Under condition A3, the proof of the assertion (3.13) relies on the fact that $\sup \nu = \sup \nu_s$ $(s > 0)$ established in [7] and essentially on condition A3 which ensures that the Markov chain $(X_n^x)_{n \geq 0}$ stays forever in the interior of the projective space $S_{d-1}^+$: see Lemma 3.2. If condition A3 is replaced by A4, the main difficulty to prove (3.13) is that the Markov chain $(X_n^x)_{n \geq 0}$ is no longer separated from the coordinates $(e_k)_{1 \leq k \leq d}$; hence the proof can not follow directly from the fact that $\sup \nu = \sup \nu_s$. Instead, the main ingredient in our proof consists in the large deviation asymptotic
for the norm cocycle \( \log |G_n x| \) under the changed measure \( Q_s^x \) established in Theorem 3.4.

It is worth mentioning that in the case of invertible matrices, the corresponding result with \( s = 0 \) (in this case also \( \pi_0 = \nu_0 = \nu \)) has been obtained in [16]; we also refer to [5] for the detailed description of the method used in [16] and to [6, 4] for a different approach of the proof.

Before proving Proposition 3.3, let us give the precise large deviation result for the norm cocycle \( \log |G_n x| \) under the changed measure \( Q_s^x \). It is deduced from [24, Theorem 2.2] and will be used in the proof of regularity of the stationary measure \( \pi_x \) (see Proposition 3.3). As in (2.10), we denote \( \Lambda = \log \kappa \) and by \( \Lambda^* \) the Legendre transform of \( \Lambda \). In particular, we have \( \Lambda^*(q_s) = s q_s - \Lambda(s) \) if \( q_s = \Lambda'(s) \).

**Theorem 3.4.** Assume conditions A1, A2 and A6. Let \( s \in I_\mu, t \in I_\mu^0 \) be such that \( s < t \) and set \( q_s = \Lambda'(s) \) and \( q_t = \Lambda'(t) \). Then, for any positive sequence \((\ell_n)_{n \geq 1}\) satisfying \( \lim_{n \to \infty} \ell_n = 0 \), we have, as \( n \to \infty \), uniformly in \( |l| \leq \ell_n \) and \( x \in S_{d-1}^+ \),

\[
Q_s^x(\log |G_n x| \geq n(q_t + l)) = \frac{\nu_t(r_t) r_t(x) \exp\left(-n(\Lambda^*(q_t + l) - \Lambda^*(q_s) - s(q_t - q_s + l))\right)}{\nu_t(r_t) r_s(x) (t - s) \sigma_t \sqrt{2\pi n}} [1 + o(1)].
\]

**Proof.** By (3.12), we get

\[
Q_s^x(\log |G_n x| \geq n(q_t + l)) = \frac{1}{\kappa_s(n) r_s(x)} \mathbb{E}\left[r_s(X_n^x)|G_n x|^s \mathbb{1}_{\{|\log |G_n x|\geq n(q_t + l)\}}\right]
\]

\[
= \frac{1}{\kappa_s(n) r_s(x)} e^{s n(q_t + l)} \mathbb{E}\left[r_s(X_n^x) \psi_s(\log |G_n x| - n(q_t + l))\right],
\]

where \( \psi_s(u) = e^{sq_s} \mathbb{1}_{\{u > 0\}}, u \in \mathbb{R} \). From Theorem 2.2 in [24] it follows that for any \( t \in I_\mu^0, q_t = \Lambda'(t) \), \( \psi \in \mathcal{B}_s \) and measurable function \( \psi \) on \( \mathbb{R} \) such that \( u \mapsto e^{-s u} \psi(u) \) is directly Riemann integrable for some \( s' \in (0, s) \), we have, as \( n \to \infty \), uniformly in \( |l| \leq \ell_n \) and \( x \in S_{d-1}^+ \),

\[
\mathbb{E}\left[\varphi(X_n^x) \psi(\log |G_n x| - n(q_t + l))\right] = \frac{r_t(x) \exp(-n \Lambda^*(q_t + l))}{\nu_t(r_t) \sigma_t \sqrt{2\pi n}} \left[\nu_t(\varphi) \int_{\mathbb{R}} e^{-ty} \psi(y) dy + o(1)\right].
\]  

Using (3.15) with \( \varphi = r_s \) and \( \psi = \psi_s \), we obtain that, uniformly in \( |l| \leq \ell_n \) and \( x \in S_{d-1}^+ \),

\[
\mathbb{E}\left[r_s(X_n^x) \psi_s(\log |G_n x| - n(q_t + l))\right] = \frac{r_t(x)}{\nu_t(r_t)} \frac{e^{-n \Lambda^*(q_t + l)}}{(t - s) \sigma_t \sqrt{2\pi n}} [1 + o(1)].
\]
We conclude the proof of Theorem 3.4 by using the fact that \( \Lambda^*(q) = sq - \Lambda(s) \) and \( \Lambda(s) = \log \kappa(s) \).

**Proof of Proposition 3.3.** As mentioned before, we only need to establish (3.13) and (3.14) for the stationary measure \( \pi_s \) since \( r_s \) is bounded away from infinity and 0 uniformly on \( S_{d+\epsilon}^{d-1} \).

We first prove Proposition 3.3 under condition A3. By Lemma 3.2, the Markov chain \( (X_n^x)_{n\geq 0} \) stays in the space \( S_{d+\epsilon}^{d-1} \), and therefore the support of its stationary measure \( \nu \) is included in \( S_{d+\epsilon}^{d-1} \). Since \( \text{supp} \nu_s = \text{supp} \nu \) for \( s \in I_{\mu} \) (by [7, Proposition 3.1]), it holds that \( \text{supp} \nu_s \subset S_{d+\epsilon}^{d-1} \). As a consequence we also have \( \text{supp} \pi_s \subset S_{d+\epsilon}^{d-1} \). This implies that \( \langle f, x \rangle \geq \epsilon \) for all \( f \in S_{d+\epsilon}^{d-1} \), \( x \in \text{supp} \pi_s \), and so the bounds (3.13) and (3.14) hold under condition A3.

We next prove Proposition 3.3 under conditions A1, A4, A6. We divide the proof into two steps. It is worth mentioning that the assertions shown below remain valid when \( s = 0 \).

**Step 1.** We prove that there exist two constants \( C_1, C_2 > 0 \) and an integer \( n_0 \geq 1 \) satisfying \( C_1 > \Lambda'(s) \) such that, for any \( n \geq n_0 \), it holds uniformly in \( f, x \in S_{d+\epsilon}^{d-1} \) that

\[
I_n := \mathbb{Q}_s^x \left( \langle f, X_n^x \rangle \leq e^{-C_1 n} \right) \leq e^{-C_2 n}.
\]

(3.16)

Let \( s \in I_{\mu}, t \in I_{\mu}^0 \) be such that \( s < t \) and set \( q_s = \Lambda'(s) \) and \( q_t = \Lambda'(t) \) (we allow \( s \) to be 0). Substituting \( X_n^x = \frac{G_n x}{|G_n x|} \) into (3.16), we have

\[
I_n \leq \mathbb{Q}_s^x \left( \log |G_n x| > n q_t \right) + \mathbb{Q}_s^x \left( \log \langle f, G_n x \rangle \leq - (C_1 - q_t) n \right).
\]

(3.17)

Since \( s < t \), by Theorem 3.4 we get that there exists a constant \( c > 0 \) such that the first term on the right-hand side of (3.17) is bounded by \( e^{-cn} \), uniformly in \( x \in S_{d+\epsilon}^{d-1} \). For the second term on the right-hand side of (3.17), applying the Markov inequality and the change of measure formula (3.12), it follows that for a sufficiently small constant \( c_1 > 0 \), uniformly in \( f, x \in S_{d+\epsilon}^{d-1} \),

\[
\mathbb{Q}_s^x \left( \log \langle f, G_n x \rangle \leq - (C_1 - q_t) n \right) \\
\leq e^{-c_1 (C_1 - q_t) n} \mathbb{P}_{x} \left( \frac{1}{\langle f, G_n x \rangle c_1} \right) \\
= e^{-c_1 (C_1 - q_t) n} \mathbb{P}_{x} \left( \frac{|G_n x|^s r_s(X_n^x)}{\kappa^n(s) r_s(x) \langle f, G_n x \rangle c_1} \right) \\
\leq e^{-c_1 (C_1 - q_t) n} \mathbb{P}_{x} \left( \frac{|G_n x|^s r_s(X_n^x)}{\kappa^n(s) r_s(x) \min_{1 \leq i, j \leq d} \langle e_i, G_n e_j \rangle c_1} \right),
\]

(3.18)
where in the last line we used the fact that \( \min_{1 \leq i,j \leq d} \langle e_i, g e_j \rangle = \inf_{f,x \in \mathbb{S}^{d-1}} \langle f, g x \rangle \) for any \( g \in \Gamma_\mu \). Since \( |G_n x| \leq \|G_n\| \) and the function \( r_s \) is uniformly bounded and strictly positive on \( \mathbb{S}^{d-1} \), using the Hölder inequality leads to

\[
\mathbb{E} \left( \frac{|G_n x|^s r_s(X_n^x)}{\kappa^s(s) r_s(x)} \frac{1}{\langle e_i, G_n e_j \rangle^{c_1}} \right) \\
\leq \kappa^{-n}(s) \mathbb{E} \left( \frac{1}{\langle e_i, G_n e_j \rangle^{c_1 p'}} \right) \\
\leq \kappa^{-n}(s) \mathbb{E} \left( \frac{1}{\langle e_i, G_n e_j \rangle^{c_1 p'}} \frac{\mathbb{E} \left( \|G_n\|^{q p} \right)^{\frac{1}{q}}}{\min_{1 \leq i,j \leq d} \langle e_i, g e_j \rangle^{c_1 p'}} \right),
\]

(3.19)

where \( 1/p + 1/p' = 1 \) with \( p, p' > 1 \). Recall that \( c_1 > 0 \) can be taken sufficiently small. Taking \( p \) sufficiently close to 1 \( (p' \) sufficiently large) and using condition \( A_4 \), we get that the right-hand side of (3.19) is dominated by \( e^{-C_n} \) with some constant \( C > 0 \). Consequently, in view of (3.18), choosing the constant \( C_1 > 0 \) sufficiently large, we obtain that the right-hand side of (3.18) is bounded by \( e^{-C_2 n} \) with some constant \( C_2 > 0 \), uniformly in \( f, x \in \mathbb{S}^{d-1} \).

**Step 2.** From the construction of \( Q_s^x \) and the definition of \( \pi_s \), one can verify that for any \( x \in \mathbb{S}^{d-1} \) and \( n \geq 1 \), \( \pi_s = (Q_s^x)^n \ast \pi_s \), where \( \ast \) stands for the convolution of two measures. Combining this with (3.16), we get that for any \( s \in I_\mu \), uniformly in \( f \in \mathbb{S}^{d-1} \),

\[
\pi_s(\{ x : \langle f, x \rangle \leq e^{-C_1 n} \}) = \int_{\mathbb{S}^{d-1}} (Q_s^x)^n (\langle f, x \rangle \leq e^{-C_1 n}) \pi_s(dx) \leq e^{-C_2 n},
\]

(3.20)

where \( C_1 \) and \( C_2 \) are positive constants given in step 1. For \( n \geq 1 \), denote \( B_{f,n} := \{ x \in \mathbb{S}^{d-1} : e^{-C_1 (n+1)} \leq \langle f, x \rangle \leq e^{-C_1 n} \} \). Choosing \( \alpha \in (0, C_2/C_1) \), we deduce from (3.20) that, uniformly in \( f \in \mathbb{S}^{d-1} \),

\[
\int_{\mathbb{S}^{d-1}} \frac{1}{\langle f, x \rangle^\alpha} \pi_s(dx) = \int_{\{ x : \langle f, x \rangle > e^{-C_1 n_0} \}} \frac{1}{\langle f, x \rangle^\alpha} \pi_s(dx) + \sum_{n=n_0}^{\infty} \int_{B_{f,n}} \frac{1}{\langle f, x \rangle^\alpha} \pi_s(dx) \\
\leq e^{\alpha C_1 n_0} + \sum_{n=n_0}^{\infty} e^{\alpha C_1 e^{-(C_2-\alpha C_1)n}} < +\infty.
\]

(3.21)

This concludes the proof of (3.13). Using the Markov inequality, we can easily deduce (3.14) from (3.13).

**3.3. Proof of Theorem 2.3.** It turns out that the Hölder regularity of the stationary measure \( \nu \) established in subsection 3.2 plays a crucial role for proving Theorem 2.3.
Proof of Theorem 2.3. Without loss of generality, we assume that the target function \( \varphi \) is non-negative. We first prove the Berry-Esseen type bound (2.7) for the scalar product \( \langle f, G_n x \rangle \).

The lower bound has been shown in (3.7). The upper bound is a consequence of (3.16) together with the Berry-Esseen bound (3.1). In fact, using (3.16) with \( s = 0 \), we get that there exist constants \( C_1, C_2 > 0 \) and \( k_0 \in \mathbb{N} \) such that for all \( n \geq k \geq k_0 \),

\[
\mathbb{P} \left( \langle f, X_n^x \rangle \leq e^{-C_1 k} \right) \\
\leq \int \mathbb{P} \left( \langle f, (g_n \ldots g_{n-k+1}) \cdot X_{n-k}^x \rangle \leq e^{-C_1 k} \right) \mu(dg_1) \ldots \mu(dg_{n-k}) \\
\leq e^{-C_2 k}.
\]

(3.22)

It follows that

\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1} \left\{ \frac{\log \langle f, G_n x \rangle}{\sigma_n} - n \lambda \leq y \right\} \right] \\
\leq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1} \left\{ \frac{\log \langle f, G_n x \rangle}{\sigma_n} - n \lambda \leq y \right\} \mathbb{1} \left\{ \log \langle f, X_n^x \rangle > -C_1 k \right\} \right] + e^{-C_2 k} \| \varphi \|_{\infty} \\
\leq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1} \left\{ \frac{\log \| G_n \| - C_1 k - n \lambda}{\sigma_n} \leq y \right\} \right] + e^{-C_2 k} \| \varphi \|_{\infty}.
\]

Taking \( k = \lfloor C_3 \log n \rfloor \) with \( C_3 = \frac{1}{2C_2} \), we get that \( e^{-C_2 k} \leq \frac{C}{\sqrt{n}} \), for some constant \( C > 0 \). Using the Berry-Esseen bound (3.1) with \( y \) replaced by \( y_1 := y + \frac{C_1 k}{\sigma_n} \), we obtain the following upper bound: there exists a constant \( C > 0 \) such that for all \( x \in S_{d-1}^+ \), \( y \in \mathbb{R} \), \( \varphi \in B_\gamma \), and \( n \geq k_0 \) with \( k_0 \) large enough,

\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1} \left\{ \frac{\log \langle f, G_n x \rangle}{\sigma_n} - n \lambda \leq y \right\} \right] \\
\leq \nu(\varphi) \Phi(y_1) + \frac{C \log n}{\sqrt{n}} \| \varphi \|_{\gamma}.
\]

By calculations similar to (3.4), it can be seen that for any \( y \in \mathbb{R} \),

\[
\Phi(y_1) \leq \Phi(y) + \frac{C \log n}{\sqrt{n}}.
\]

This concludes the proof of (2.7).

Using (2.7) together with the Collatz-Wielandt formula, the proof of (2.8) can be carried out in the same way as that of (2.6). We omit the details. \( \Box \)

4. Proofs of moderate deviation expansions

The aim of this section is to establish Theorems 2.4, 2.6 and 2.7 on moderate deviation asymptotics, and Proposition 2.8 about the expressions of the asymptotic variance \( \sigma^2 \).
4.1. Proof of Theorems 2.4 and 2.6. To establish Theorems 2.4 and 2.6, we need the following Cramér type moderate deviation expansion for the norm cocycle \( \log |G_n x| \).

**Lemma 4.1.** Assume conditions \( A1, A2 \) and \( A5 \). Then, as \( n \to \infty \), we have, uniformly in \( x \in S^d_{d-1} \), \( y \in [0, o(\sqrt{n})] \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\mathbb{E} \left[ \varphi \left( X^x_n \right) 1_{\{ \log |G_n x| - n \lambda \geq \sqrt{n} \sigma y \}} \right] = e^{\frac{y^2}{2} \left( \frac{y}{\sqrt{n}} \right)} \left[ \nu(\varphi) + \| \varphi \|_\gamma O\left( \frac{y + 1}{\sqrt{n}} \right) \right].
\]  

(4.1)

\[
\mathbb{E} \left[ (-\varphi \left( X^x_n \right) 1_{\{ \log |G_n x| - n \lambda \leq -\sqrt{n} \sigma y \}} \right] = e^{-\frac{y^2}{2} \left( -\frac{y}{\sqrt{n}} \right)} \left[ \nu(\varphi) + \| \varphi \|_\gamma O\left( \frac{y + 1}{\sqrt{n}} \right) \right].
\]  

(4.2)

Lemma 4.1 has been recently established in [25] by developing a new smoothing inequality, applying a saddle point method and spectral gap properties of the transfer operator corresponding to the Markov chain \( (X^x_n)_n \geq 0 \). Note that condition \( A3 \) is not assumed in Lemma 4.1 and the expansions (4.1) and (4.2) hold uniformly with respect to the starting point \( x \) on the whole projective space \( S^d_{d-1} \).

We now prove Theorem 2.4 using Theorem 2.1, Lemmas 3.1 and 4.1.

**Proof of Theorem 2.4.** Without loss of generality, we assume that \( \varphi \) is non-negative on \( S^d_{d-1} \).

We first prove the moderate deviation expansion (2.11). In the case where \( y \in [0, 1] \), the expansion (2.11) follows from the Berry-Esseen bound (2.4) together with the fact that there exists a constant \( C > 0 \) such that for all \( n \geq 1 \), \( x \in S^d_{d-1} \), \( y \in [1, o(\sqrt{n})] \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\sup_{x \in S^d_{d-1}} \mathbb{E} \left[ \varphi \left( X^x_n \right) \right] - \nu(\varphi) \leq \frac{C}{\sqrt{n}} \| \varphi \|_\gamma.
\]  

(4.3)

It remains to establish the expansion (2.11) in the case where \( y \in (1, o(\sqrt{n})) \). The proof consists of lower and upper bounds.

The lower bound is an easy consequence of Lemma 4.1. In fact, using the expansion (4.1) together with the fact \( \log \|G_n\| \geq \log |G_n x| \), there exists a constant \( C > 0 \) such that for all \( n \geq 1 \), \( x \in S^d_{d-1} \), \( y \in (1, o(\sqrt{n})) \) and \( \varphi \in \mathcal{B}_\gamma \),

\[
\mathbb{E} \left[ \varphi \left( X^x_n \right) 1_{\{ \log \|G_n\| - n \lambda \geq \sqrt{n} \sigma y \}} \right] \geq e^{\frac{y^2}{2} \left( \frac{y}{\sqrt{n}} \right)} \left[ \nu(\varphi) - \frac{Cy + 1}{\sqrt{n}} \| \varphi \|_\gamma \right].
\]  

(4.4)

The upper bound can be deduced from Lemmas 3.1 and 4.1. From Lemma 3.1, we have seen that the inequality (3.2) holds for some constant \( C_1 > 0 \).
For any \( y \in (1, o(\sqrt{n})) \), we denote
\[
y_1 = y - \frac{C_1}{\sigma \sqrt{n}}.
\]
Since \( y_1 \in [0, o(\sqrt{n})] \) for sufficiently large \( n \), we are allowed to apply the moderate deviation expansion (4.1) with \( y \) replaced by \( y_1 \). Specifically, using (4.1) and (3.2), we obtain that for any compact set \( K \subset (S_{d-1}^d)^o \), there exists a constant \( C > 0 \) such that, as \( n \to \infty \), uniformly in \( x \in K \), \( y \in (1, o(\sqrt{n})) \) and \( \varphi \in \mathcal{B}_\gamma \),
\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n \lambda \geq \sqrt{n} \sigma y_1\}} \right] \leq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n \lambda \geq \sqrt{n} \sigma y\}} \right] \leq \frac{e^{y_3 \sqrt{n} \zeta(y_1)} - e^{y_3 \sqrt{n} \zeta(y)}}{1 - \Phi(y_1)} \leq e^y \zeta(\frac{y}{\sqrt{n}}) \left[ \nu(\varphi) + C \frac{y_1 + 1}{\sqrt{n}} \|\varphi\|_\gamma \right]. \tag{4.5}
\]
Since the Cramér series \( \zeta \) is convergent and analytic in a small neighborhood of 0, there exist constants \( c, C > 0 \) such that for all \( y \in (1, o(\sqrt{n})) \),
\[
\left| \zeta\left(\frac{y_1}{\sqrt{n}}\right) - \zeta\left(\frac{y}{\sqrt{n}}\right) \right| \leq c \frac{|y_1 - y|}{\sqrt{n}} \leq \frac{C}{n}. \tag{4.6}
\]
By simple calculations, it follows that uniformly in \( y \in (1, o(\sqrt{n})) \),
\[
\exp \left\{ \frac{y_3^3}{\sqrt{n}} \zeta\left(\frac{y_1}{\sqrt{n}}\right) - \frac{y_3^3}{\sqrt{n}} \zeta\left(\frac{y}{\sqrt{n}}\right) \right\} = \exp \left\{ \left[ - \frac{3C_1 y_1^2}{\sigma n} + \frac{3C_1^2 y^2}{\sigma^2 n^{3/2}} - \frac{C_3^2}{\sigma^3 n^2} \right] \zeta\left(\frac{y_1}{\sqrt{n}}\right) \right\} \times \exp \left\{ \left[ - \frac{3C_1 y_1^2}{\sigma n} + \frac{3C_1^2 y^2}{\sigma^2 n^{3/2}} - \frac{C_3^2}{\sigma^3 n^2} \right] \zeta\left(\frac{y}{\sqrt{n}}\right) \right\} \leq \exp \left\{ C_2 \left(\frac{y_1^2}{n} + \frac{1}{n^2}\right) \right\} \exp \left\{ C_3 \frac{y_3^3}{n^{3/2}} \right\} \leq 1 + C_4 y_2 + 1 \frac{n^2}{n}. \tag{4.7}
\]
Note that
\[
\frac{1 - \Phi(y_1)}{1 - \Phi(y)} = 1 + \left( \int_{y - \frac{C_1}{\sigma \sqrt{n}}}^{y} e^{-\frac{t^2}{2}} dt \right) \left( \int_{y}^{\infty} e^{-\frac{t^2}{2}} dt \right)^{-1}.
\]
Using the basic inequality 
\[
\frac{y}{y^2 + 1}e^{-\frac{y^2}{2}} < \int_y^\infty e^{-\frac{t^2}{2}}dt, \quad y > 1,
\]
we obtain that uniformly in \(y \in (1, o(\sqrt{n})]\),
\[
1 < \frac{1 - \Phi(y_1)}{1 - \Phi(y)} < 1 + \frac{y^2 + 1}{y} e^{\frac{2}{\sigma \sqrt{n}}} e^{-\frac{1}{\sqrt{2\sigma}}(y - \frac{C_1}{\sigma \sqrt{n}})^2}
= 1 + \left(y + \frac{1}{\sqrt{n}}\right) \frac{C_1 \sqrt{y}}{\sigma \sqrt{n}} \cdot \frac{C_1^2 y}{2 \sigma^2 n} = 1 + O\left(\frac{y}{\sqrt{n}}\right).
\]
This implies that uniformly in \(y \in (1, o(\sqrt{n})]\),
\[
\frac{1 - \Phi(y_1)}{1 - \Phi(y)} = 1 + O\left(\frac{y + 1}{\sqrt{n}}\right). \tag{4.8}
\]
Note that \(\frac{y + 1}{\sqrt{n}} = O\left(\frac{y + 1}{\sqrt{n}}\right)\). Combining this with (4.5), (4.7) and (4.8), we obtain that there exists a constant \(C > 0\) such that, as \(n \to \infty\), uniformly in \(x \in K, y \in (1, o(\sqrt{n})\) and \(\varphi \in B_\gamma\),
\[
\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n \lambda \geq \sqrt{n} \sigma y\}}\right] \leq e^{-\frac{y^3}{\sqrt{n}} \zeta\left(\frac{y}{\sqrt{n}}\right)} \left[\nu(\varphi) + C \frac{y + 1}{\sqrt{n}} \|\varphi\|_\gamma\right].
\]
Together with (4.4), this concludes the proof of the expansion (2.11).

We next prove the moderate deviation expansion (2.12). The proof consists of upper and lower bounds.

For the upper bound, in a similar way as in the proof of (4.4), using the expansion (4.2) together with the fact \(\log \|G_n\| \geq \log |G_n x|\), we immediately get that there exists a constant \(C > 0\) such that, as \(n \to \infty\), uniformly in \(x \in S_{d-1}^+, y \in [0, o(\sqrt{n})]\) and \(\varphi \in B_\gamma\),
\[
\mathbb{E} \left[\varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n \lambda \leq -\sqrt{n} \sigma y\}}\right] \leq e^{-\frac{y^3}{\sqrt{n}} \zeta\left(-\frac{y}{\sqrt{n}}\right)} \left[\nu(\varphi) + \|\varphi\|_\gamma O\left(\frac{y + 1}{\sqrt{n}}\right)\right]. \tag{4.9}
\]

For the lower bound, recall that by Lemma 3.1, the inequality (3.2) holds for some constant \(C_1 > 0\). For any \(y \in [0, o(\sqrt{n})]\), we denote
\[
y_2 = y + \frac{C_1}{\sigma \sqrt{n}},
\]
and it holds that \(y_2 \in [0, o(\sqrt{n})]\). Applying the inequality (3.2) and the moderate deviation expansion (4.1) with \(y\) replaced by \(y_2\), we obtain that for any compact set \(K \subset (S_{d-1}^+)^\circ\), there exists a constant \(C > 0\) such that,
as $n \to \infty$, uniformly in $x \in K$, $y \in [0, o(\sqrt{n})]$ and $\varphi \in B_k$,

\[
\begin{align*}
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log \|G_n\| - n\lambda \leq -\sqrt{n\sigma y}\}} \right] & \geq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\{\log |G_n, x| - n\lambda \leq -\sqrt{n\sigma y_2}\}} \right] \\
& \geq e^{-\frac{y^3}{\sqrt{n}}(\frac{-y}{\sqrt{n}})} \left[ \varphi(\varphi) + \|\varphi\|_1 O\left(\frac{y_2^2 + 1}{\sqrt{n}}\right) \right].
\end{align*}
\]

(4.10)

Similarly to (4.6) and (4.7), by simple calculations, we get that uniformly in $y \in [0, o(\sqrt{n})]$,

\[
\left| \zeta\left(-\frac{y}{\sqrt{n}}\right) - \zeta\left(-\frac{y_2}{\sqrt{n}}\right) \right| \leq C \frac{|y_2 - y|}{\sqrt{n}} \leq C_n,
\]

(4.11)

and

\[
\begin{align*}
\exp & \left\{ \frac{y^3}{\sqrt{n}} \zeta\left(-\frac{y}{\sqrt{n}}\right) - \frac{y^3}{\sqrt{n}} \zeta\left(-\frac{y_2}{\sqrt{n}}\right) \right\} \\
& = \exp \left\{ \frac{y^3}{\sqrt{n}} - \frac{y^3}{\sqrt{n}} \zeta\left(-\frac{y}{\sqrt{n}}\right) \right\} \exp \left\{ \frac{y^3}{\sqrt{n}} \left[ \zeta\left(-\frac{y}{\sqrt{n}}\right) - \zeta\left(-\frac{y_2}{\sqrt{n}}\right) \right] \right\} \\
& = \exp \left\{ \left[ - \frac{3C_1 y^2}{\sigma} n - \frac{3C_1^2 y}{\sigma^2 n^{3/2}} - \frac{C_3}{\sigma^3 n^2} \right] \zeta\left(-\frac{y}{\sqrt{n}}\right) \right\} \\
& \times \exp \left\{ \frac{y^3}{\sqrt{n}} \left[ \zeta\left(-\frac{y}{\sqrt{n}}\right) - \zeta\left(-\frac{y_2}{\sqrt{n}}\right) \right] \right\} \\
& \geq \exp \left\{ -C_2 \frac{y^2 + 1}{n} \right\} \exp \left\{ -C_3 \frac{y^3 + 1}{n^{3/2}} \right\} \\
& \geq 1 - C_4 \frac{y^2 + 1}{n}.
\end{align*}
\]

(4.12)

Notice that

\[
\frac{\Phi(-y_2)}{\Phi(-y)} = 1 - \frac{\Phi(y_2)}{\Phi(y)} = 1 - \left( \int_y^{\infty} \frac{C_1}{\sigma \sqrt{n}} e^{-\frac{t^2}{2}} dt \right) \left( \int_y^{\infty} e^{-\frac{t^2}{2}} dt \right)^{-1}.
\]

It is easy to see that $1 > \frac{\Phi(-y_2)}{\Phi(-y)} > 1 - \frac{C}{\sqrt{n}}$, uniformly in $y \in [0, 1]$. From the basic inequality $\frac{y}{y^2+1} e^{-\frac{y^2}{2}} < \int_y^{\infty} e^{-\frac{t^2}{2}} dt, y > 1$, we deduce that uniformly in $y \in (1, o(\sqrt{n}))$,

\[
1 > \frac{\Phi(-y_2)}{\Phi(-y)} > 1 - \frac{y^2 + 1}{y} e^{-\frac{C_1}{\sigma \sqrt{n}} y} \frac{C_1}{\sigma \sqrt{n}} e^{-\frac{C_1}{2 \sigma \sqrt{n}} y^2} \\
= 1 - \left( y + \frac{1}{y} \right) \frac{C_1}{\sigma \sqrt{n}} e^{-\frac{C_1}{2 \sigma \sqrt{n}} y} \frac{C_1}{2 \sigma \sqrt{n}} = 1 + O\left( \frac{y}{\sqrt{n}} \right).
\]
Hence we get that uniformly in \( y \in [0, o(\sqrt{n})] \),
\[
\Phi(-y_2) = 1 + O\left(\frac{y + 1}{\sqrt{n}}\right).
\]  

(4.13)

Note that \( \frac{y_2 + 1}{\sqrt{n}} = O\left(\frac{y + 1}{\sqrt{n}}\right) \). Combining this with (4.10), (4.12) and (4.13), we obtain that there exists a constant \( C > 0 \) such that, as \( n \to \infty \), uniformly in \( x \in K \), \( y \in [0, o(\sqrt{n})] \) and \( \varphi \in \mathcal{B}_\gamma \),
\[
\mathbb{E} \left[ \phi(X_n^x) \mathbb{1}_{\{\log|G_n| - n\lambda > -\sqrt{n}\sigma y\}} \right] \geq e^{-\frac{3}{\sqrt{n}}\zeta\left(\frac{y}{\sqrt{n}}\right)} \left[ \nu(\varphi) + C \frac{y + 1}{\sqrt{n}} \|\varphi\|_{\gamma} \right].
\]

This, together with the upper bound (4.9), concludes the proof of the moderate deviation expansion (2.12). \( \square \)

We next prove Theorem 2.6 based on Lemmas 3.2 and 4.1.

**Proof of Theorem 2.6.** Without loss of generality, we assume that \( \varphi \) is non-negative on \( S^d_+ \). We first prove (2.14). The proof consists of upper and lower bounds.

**Upper bound.** Since \( \log \langle f, G_n x \rangle \leq \log |G_n x| \), applying Lemma 4.1, this implies that there exists a constant \( C > 0 \) such that, as \( n \to \infty \), uniformly in \( f, x \in S^d_+ \), \( y \in [0, o(\sqrt{n})] \) and \( \varphi \in \mathcal{B}_\gamma \),
\[
\mathbb{E} \left[ \phi(X_n^x) \mathbb{1}_{\{\log|f, G_n x| - n\lambda > -\sqrt{n}\sigma y\}} \right] \leq e^{\frac{3}{\sqrt{n}}\zeta\left(\frac{y}{\sqrt{n}}\right)} \left[ \nu(\varphi) - c \frac{y_1 + 1}{\sqrt{n}} \|\varphi\|_{\gamma} \right].
\]  

(4.14)

**Lower bound.** Using (3.8) and applying (4.1) in Lemma 4.1, with \( y_1 = y + \frac{C_1}{\sqrt{n}} \), we obtain that there exists a constant \( c > 0 \) such that
\[
\mathbb{E} \left[ \phi(X_n^x) \mathbb{1}_{\{\log|\langle f, G_n x \rangle| - n\lambda > -\sqrt{n}\sigma y\}} \right] \geq e^{\frac{3}{\sqrt{n}}\zeta\left(\frac{y_1}{\sqrt{n}}\right)} \left[ \nu(\varphi) - c \frac{y_1 + 1}{\sqrt{n}} \|\varphi\|_{\gamma} \right].
\]

(4.15)

In an analogous way as in the proof of the upper bound in Theorem 2.4, one can verify that \( |\zeta\left(\frac{y}{\sqrt{n}}\right) - \zeta\left(\frac{y_1}{\sqrt{n}}\right)| \leq \frac{C_2}{n} \), uniformly in \( y \in [0, o(\sqrt{n})] \). Moreover, elementary calculations yield that uniformly in \( y \in [0, o(\sqrt{n})] \), it holds that
\[
e^{\frac{3}{\sqrt{n}}\zeta\left(\frac{y}{\sqrt{n}}\right)} e^{-\frac{3}{\sqrt{n}}\zeta\left(\frac{y_1}{\sqrt{n}}\right)} = 1 + O\left(\frac{y^2 + 1}{n}\right), \quad e^{\frac{1}{\Phi(y)} - \frac{1}{\Phi(y_1)}} = 1 + O\left(\frac{y + 1}{\sqrt{n}}\right) \quad \text{and} \quad \frac{y_1 + 1}{\sqrt{n}} = O\left(\frac{y + 1}{\sqrt{n}}\right).
\]

Combining this with (4.15), we obtain
\[
\mathbb{E} \left[ \phi(X_n^x) \mathbb{1}_{\{\log|\langle f, G_n x \rangle| - n\lambda > -\sqrt{n}\sigma y\}} \right] \geq e^{\frac{3}{\sqrt{n}}\zeta\left(\frac{y}{\sqrt{n}}\right)} \left[ \nu(\varphi) - c \frac{y + 1}{\sqrt{n}} \|\varphi\|_{\gamma} \right].
\]

Together with the upper bound (4.14), this concludes the proof of (2.14). The proof of (2.15) is similar to that of (2.14) by using (4.2) and Lemma 3.2.
The proof of the expansions (2.16) and (2.17) for the spectral radius $\rho(G_n)$ can be carried out in an analogous way using Theorem 2.4, Lemma 4.1 and inequality (3.10). We omit the details. \hfill \Box

4.2. **Proof of Theorem 2.7.** We establish Theorem 2.7 on the moderate deviation principles for the entry $G_{i,j}^n$ and the spectral radius $\rho(G_n)$. Under conditions \textbf{A1, A3 and A5}, the results are direct consequences of Theorem 2.6. Under conditions \textbf{A1, A4 and A6}, the proof relies on the Hölder regularity of the stationary measure $\nu$ shown in Proposition 3.3.

**Proof of Theorem 2.7.** As mentioned above, it remains to establish Theorem 2.7 under conditions \textbf{A1, A4 and A6}. We first prove the assertion (1) on the moderate deviation principle for the scalar product $\langle f, G_n x \rangle$.

Let $\varphi \in \mathcal{B}_\gamma$ be any real-valued function satisfying $\nu(\varphi) > 0$. By Lemma 4.4 of [19], it suffices to prove the following moderate deviation asymptotics: for any $y > 0$, uniformly in $f, x \in \mathbb{S}^{d-1}$,

$$
\lim_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log(f,G_n x)}{b_n} - n\lambda \geq y \right\}} \right] = -\frac{y^2}{2\sigma^2}, \quad (4.16)
$$

$$
\lim_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log(f,G_n x)}{b_n} - n\lambda \leq -y \right\}} \right] = -\frac{y^2}{2\sigma^2}. \quad (4.17)
$$

We first prove (4.16). The upper bound follows immediately from Lemma 4.1 and the fact that $\langle f, G_n x \rangle \leq |G_n x|$: for any $y > 0$, uniformly in $f, x \in \mathbb{S}^{d-1}$,

$$
\limsup_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log(f,G_n x)}{b_n} - n\lambda \geq y \right\}} \right] \leq -\frac{y^2}{2\sigma^2}. \quad (4.18)
$$

The lower bound can be deduced from Lemma 4.1 together with Proposition 3.3. Specifically, using (3.22), we obtain that there exist constants $C_1, C_2 > 0$ and $k_0 \in \mathbb{N}$ such that for all $n \geq k \geq k_0$,

$$
I_n := \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log(f,G_n x)}{b_n} - n\lambda \geq y \right\}} \right] \\
\geq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log(f,G_n x)}{b_n} - n\lambda \geq y \right\}} \mathbb{1}_{\left\{ \log(f,G_n x) - \log |G_n x| \geq -C_1 k \right\}} \right] \\
\geq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \log |G_n x| - n\lambda \geq y b_n + C_1 k \right\}} \right] - e^{-C_2 k} \|\varphi\|_\infty. \quad (4.19)
$$

In the sequel, we take

$$
k = \left\lfloor C_3 \frac{b_n^2}{n} \right\rfloor, \quad (4.20)$$
where $C_3 > 0$ is a constant whose value will be chosen large enough. From the moderate deviation expansion (4.1), it follows that for any $y > 0$ and $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$E\left[\varphi\left(X_n^x\right)\mathbb{1}_{\left\{ \frac{\log|G_n x| - n\lambda}{b_n} \geq y \right\}}\right] \geq e^{-\frac{b_n^2}{2\sigma^2}(\frac{y^2}{2\sigma^2} + \eta)}. \tag{4.21}$$

Set

$$b'_n = b_n + \frac{C_1 k}{y}.$$

We easily see that the sequence $(b'_n)_{n \geq 1}$ satisfies $\frac{b'_n}{\sqrt{n}} \to \infty$ and $\frac{b'_n}{n} \to 0$, as $n \to \infty$. Using (4.21), we get that uniformly in $f, x \in S^{d-1}$,

$$E\left[\varphi\left(X_n^x\right)\mathbb{1}_{\left\{ \log|G_n x| - n\lambda \geq yb_n + \varepsilon k \right\}}\right] \geq e^{-\frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2} + \eta\right)}.$$

Substituting this into (4.19), we obtain

$$I_n \geq e^{-\frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2} + \eta\right)} \left[1 - e^{-C_3 k + \frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2} + \eta\right)}\|\varphi\|_{\infty}\right].$$

In view of (4.20), choosing $C_3 > \frac{1}{C_3}(\frac{y^2}{2\sigma^2} + \eta)$, by elementary calculations, we get

$$\lim_{n \to \infty} \frac{(b'_n)^2}{kn} \left(\frac{y^2}{2\sigma^2} + \eta\right) = \frac{1}{C_3} \left(\frac{y^2}{2\sigma^2} + \eta\right) < C_2.$$

Thus, for some constant $C_4 > 0$,

$$I_n \geq e^{-\frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2} + \eta\right)} \left[1 - e^{-C_4 k}\|\varphi\|_{\infty}\right].$$

Hence, recalling that $k = \left\lfloor \frac{C_3 b_n^2}{n} \right\rfloor \to \infty$ as $n \to \infty$, we obtain

$$\liminf_{n \to \infty} \frac{n}{b_n^2} \log I_n \geq \lim_{n \to \infty} \frac{n}{b_n^2} \left[ -\frac{(b'_n)^2}{n}\left(\frac{y^2}{2\sigma^2} + \eta\right) \right] + \lim_{n \to \infty} \frac{n}{b_n^2} \log(1 - e^{-C_4 k}\|\varphi\|_{\infty})$$

$$= \lim_{n \to \infty} \left[ -\left(1 + \frac{C_1 k}{yb_n}\right)^2 \left(\frac{y^2}{2\sigma^2} + \eta\right) \right] + 0$$

$$= -\left(\frac{y^2}{2\sigma^2} + \eta\right).$$

Letting $\eta \to 0$, the desired lower bound follows: for any $y > 0$, uniformly in $f, x \in S^{d-1}$,

$$\liminf_{n \to \infty} \frac{n}{b_n^2} \log E\left[\varphi\left(X_n^x\right)\mathbb{1}_{\left\{ \frac{\log(f, G_n x) - n\lambda}{b_n} \geq y \right\}}\right] \geq -\frac{y^2}{2\sigma^2}.$$
We next prove (4.17). By (4.2) and the fact that \( \langle f, G_n x \rangle \leq |G_n x| \), the lower bound easily follows: for any \( y > 0 \), uniformly in \( f, x \in S_{d-1}^d \),

\[
\liminf_{n \to \infty} \frac{n}{b_n^2} \log \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \langle f, G_n x \rangle - n \lambda}{b_n} \leq -y \right\}} \right] \geq -\frac{y^2}{2\sigma^2}. \tag{4.22}
\]

For the upper bound, by (3.22), there exist constants \( C_5, C_6 > 0 \) and \( k_0 \in \mathbb{N} \) such that for all \( n \geq k \geq k_0 \),

\[
J_n := \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \langle f, G_n x \rangle - n \lambda}{b_n} \leq -y \right\}} \right] = \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \langle f, G_n x \rangle - n \lambda}{b_n} \leq -y \right\}} \mathbb{1}_{\left\{ \log \langle f, G_n x \rangle - \log |G_n x| \geq -C_5 k \right\}} \right]
\]

\[
+ \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \langle f, G_n x \rangle - n \lambda}{b_n} \leq -y \right\}} \mathbb{1}_{\left\{ \log \langle f, G_n x \rangle - \log |G_n x| \leq -C_5 k \right\}} \right]
\]

\[
\leq \mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \log |G_n x| = \log |G_n x| - n \lambda \leq -y b_n + C_5 k \right\}} \right] + e^{-C_6 k \|\varphi\|_\infty}.
\]

As in the proof of (4.16), we choose

\[
k = \left\lfloor C_7 \frac{b_n^2}{n} \right\rfloor, \tag{4.23}
\]

where \( C_7 > 0 \) is a constant whose value will be chosen large enough. From (4.2), it follows that for any \( \eta > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for any \( n \geq n_0 \),

\[
\mathbb{E} \left[ \varphi(X_n^x) \mathbb{1}_{\left\{ \frac{\log \langle f, G_n x \rangle - n \lambda}{b_n} \leq -y \right\}} \right] \leq e^{-\frac{y^2}{2\sigma^2} \left( \frac{\|\varphi\|_\infty}{2\sigma^2} - \eta \right)}. \tag{4.24}
\]

Denote \( b'_n = b_n - \frac{C_5 k}{y} \). Then, by (4.23), it holds that \( \frac{b'_n}{\sqrt{n}} \to \infty \) and \( \frac{b'_n}{n} \to 0 \), as \( n \to \infty \). From (4.24), it follows that uniformly in \( f, x \in S_{d-1}^d \),

\[
J_n \leq e^{-\frac{(b'_n)^2}{b_n^2} \left( \frac{y^2}{2\sigma^2} - \eta \right)} + e^{-C_6 k \|\varphi\|_\infty}. \tag{4.25}
\]

Note that \( \frac{b'_n}{b_n} \to 1 \) as \( n \to \infty \). Choosing \( C_7 > \frac{1}{C_6} \left( \frac{y^2}{2\sigma^2} - \eta \right) \), we get

\[
\lim_{n \to \infty} \frac{(b'_n)^2}{kn} \frac{y^2}{2\sigma^2} \leq \frac{1}{C_7} \left( \frac{y^2}{2\sigma^2} - \eta \right) < C_6.
\]

Hence,

\[
\limsup_{n \to \infty} \frac{n}{b_n^2} \log J_n \leq \limsup_{n \to \infty} \frac{n}{b_n^2} \log e^{-\frac{(b'_n)^2}{b_n^2} \left( \frac{y^2}{2\sigma^2} - \eta \right)}
\]

\[
= - \lim_{n \to \infty} \left( \frac{b'_n}{b_n} \right)^2 \frac{y^2}{2\sigma^2} \left( \frac{y^2}{2\sigma^2} - \eta \right) = - \left( \frac{y^2}{2\sigma^2} - \eta \right).
\]
Since $\eta > 0$ can be arbitrary small, we obtain the desired upper bound:

$$
\limsup_{n \to \infty} \frac{n}{b_n^2} \log J_n \leq -\frac{y^2}{2\sigma^2}.
$$

Combining this with the lower bound (4.22), we finish the proof of (4.17).

Combining (4.16) and (4.17), we get the assertion (1). Using the assertion (1) and the Collatz-Wielandt formula, one can obtain the assertion (2). \hfill $\Box$

### 4.3. Proof of Proposition 2.8

We prove Proposition 2.8 based on Lemmas 3.1, 3.2 and the Collatz-Wielandt formula (3.9).

**Proof of Proposition 2.8.** We first prove part (1). For fixed $x \in K \subset (S^d_+)^c$, we denote

$$
A_n = E\left[ \left( \log |G_n x| - n\lambda \right)^2 \right], \quad B_n = E\left[ \left( \log \|G_n\| - n\lambda \right)^2 \right].
$$

Since $\frac{1}{n}A_n \to \sigma^2$ as $n \to \infty$ (see (2.3)), it suffices to show that $\frac{1}{n}(B_n - A_n) \to 0$ as $n \to \infty$. Using Minkowski’s inequality, we see that there exists a constant $C > 0$ independent of $x \in K$ such that

$$
|\sqrt{B_n} - \sqrt{A_n}| \leq \sqrt{E\left[ \left( \log \|G_n\|/\|G_n x\| \right)^2 \right]} \leq C,
$$

where the last inequality holds by Lemma 3.1. Consequently, it follows that

$$
|B_n - A_n| \leq |\sqrt{B_n} - \sqrt{A_n}| \left( |\sqrt{B_n} - \sqrt{A_n}| + 2\sqrt{A_n} \right) \leq C(C + O(\sqrt{n})), \hfill (4.26)
$$

which leads to the desired assertion in part (1).

Now we proceed to prove part (2). Denote

$$
D_n = E\left[ \left( \log \langle f, G_n x \rangle - n\lambda \right)^2 \right], \quad E_n = E\left[ \left( \log \rho(G_n) - n\lambda \right)^2 \right].
$$

As in the proof of part (1), by Minkowski’s inequality, we have, uniformly in $f, x \in S^d_+$,

$$
|\sqrt{D_n} - \sqrt{A_n}| \leq \sqrt{E\left[ (\log \langle f, X_n \rangle)^2 \right]} \leq C,
$$

where the last inequality holds by Lemma 3.2. In the same way as in the proof of (4.26), one can verify that $\frac{1}{n}(D_n - A_n) \to 0$, as $n \to \infty$, uniformly in $f, x \in S^d_+$. This ends the proof of the first equality in part (2). To prove the second one in part (2), using again the Minkowski inequality, we have

$$
|\sqrt{E_n} - \sqrt{B_n}| \leq \sqrt{E\left[ (\log \|G_n\|/\rho(G_n))^2 \right]}.
$$
Taking into account the Collatz-Wielandt formula (3.9) with $i = 1$ and $x_0 = (1, 1, \ldots, 1)^T$, we get that $\rho(G_n) \geq \langle e_1, G_n x_0 \rangle$. Since $\rho(G_n) \leq \|G_n\|$ and $\|G_n\| \leq C|G_n x_0|$ (see Lemma 3.1), it follows from Lemma 3.2 that

$$|\sqrt{E_n} - \sqrt{B_n}| \leq C + \sqrt{E \left[ (\log \langle e_1, X_n^2 \rangle)^2 \right]} \leq C.$$

Together with part (1), this proves the second equality in part (2).

□

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