Long Paths and Cycles Passing Through Specified Vertices Under the Average Degree Condition

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Abstract Let G be a k-connected graph with k ≥ 2. In this paper we first prove that: For two distinct vertices x and z in G, it contains a path connecting x and z which passes through its any k − 2 specified vertices with length at least the average degree of the vertices other than x and z. Further, with this result, we prove that: If G has n vertices and m edges, then it contains a cycle of length at least 2m/(n − 1) passing through its any k − 1 specified vertices. Our results generalize a theorem of Fan on the existence of long paths and a classical theorem of Erdős and Gallai on the existence of long cycles under the average degree condition.

Keywords Long path · Long cycle · Average degree

1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider finite simple graphs only.
Let $G$ be a graph and $H$ a subgraph of $G$. We use $V(H)$ and $E(H)$ to denote the set of vertices and edges of $H$, respectively, and use $e(H)$ for the number of the edges of $H$. For a vertex $v \in V(G)$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of neighbors of $v$ in $H$. We call $d_H(v)$ the degree of $v$ in $H$. Let $x$ and $z$ be two distinct vertices of $G$. A path connecting $x$ and $z$ is called an $(x, z)$-path. For a subset $Y$ of $V(G)$, an $(x, z)$-path passing through all the vertices in $Y$ is called an $(x, Y, z)$-path, and a cycle passing through all the vertices in $Y$ is called a $Y$-cycle. If $Y$ contains only one vertex $y$, an $(x, \{y\}, z)$-path and a $\{y\}$-cycle are simply denoted by an $(x, y, z)$-path and a $y$-cycle, respectively. The distance between $x$ and $z$ in $H$, denoted by $d_H(x, z)$, is the length of a shortest $(x, z)$-path with all its internal vertices in $H$. If no such a path exists, we define $d_H(x, z) = \infty$. The codistance between $x$ and $z$ in $H$, denoted by $d_H^*(x, z)$, is the length of a longest $(x, z)$-path with all its internal vertices in $H$. If no such a path exists, we define $d_H^*(x, z) = 0$. We remark that in the definitions of $d_H(x, z)$ and $d_H^*(x, z)$, the vertices $x$ and $z$ is not necessarily in $H$. When no confusion occurs, we use $N(v)$, $d(v)$, $d(x, z)$ and $d^*(x, z)$ instead of $N_G(v)$, $d_G(v)$, $d_G(x, z)$ and $d_G^*(x, z)$, respectively.

Long path and cycle problems are interesting and important in graph theory and have been deeply studied, see [1, 7]. The following theorem by Erdős and Gallai [5] opened the study on long paths with specified end vertices.

**Theorem 1** (Erdős and Gallai [5]) Let $G$ be a 2-connected graph and $x$ and $z$ be two distinct vertices of $G$. If $d(v) \geq d$ for every vertex $v \in V(G) \setminus \{x, z\}$, then $G$ contains an $(x, z)$-path of length at least $d$.

Theorem 1 has a stronger extension due to Enomoto [4].

**Theorem 2** (Enomoto [4]) Let $G$ be a 2-connected graph and $x$ and $z$ be two distinct vertices of $G$. If $d(v) \geq d$ for every vertex $v \in V(G) \setminus \{x, z\}$, then for every given vertex $y \in V(G) \setminus \{x, z\}$, $G$ contains an $(x, y, z)$-path of length at least $d$.

Another direction of extending Theorem 1 is to weaken the minimum degree condition to the average degree condition. Fan [6] finished this work as follows.

**Theorem 3** (Fan [6]) Let $G$ be a 2-connected graph and $x$ and $z$ be two distinct vertices of $G$. If the average degree of the vertices other than $x$ and $z$ is at least $r$, then $G$ contains an $(x, z)$-path of length at least $r$.

The following graph shows that one cannot replace the minimum degree condition in Theorem 2 by the average degree condition. Let $H$ be the complete graph on $n - 1$ vertices and $x, z \in V(H)$, and $G$ be the graph obtained from $H$ by adding a new vertex $y$ and two edges $xy, yz$. Then the length of the longest $(x, y, z)$-path in $G$ is 2, less than the average degree of the vertices other than $x$ and $z$ when $n \geq 5$.

Our first result in this paper is a generalization of Theorem 3.

**Theorem 4** Let $G$ be a $k$-connected graph with $k \geq 2$, and $x$ and $z$ be two distinct vertices of $G$. If the average degree of the vertices other than $x$ and $z$ is at least $r$, then for any subset $Y$ of $V(G)$ with $|Y| = k - 2$, $G$ contains an $(x, Y, z)$-path of length at least $r$. 

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We remark here that the size of $Y$ cannot be replaced by $k - 1$. Let $H$ be a complete graph on $n - k + 1$ vertices with $n > 3k$ and $u_1 = x, u_2, \ldots, u_k = z$ be $k$ vertices of $H$, and $Y = \{y_1, y_2, \ldots, y_{k-1}\}$ be a set of vertices not in $V(H)$. We construct a graph $G$ with $V(G) = V(H) \cup Y$ and $E(G) = E(H) \cup \{u_i y_j : 1 \leq i \leq k, 1 \leq j \leq k - 1\}$. Then $G$ is a $k$-connected graph and the longest $(x, Y, z)$-path has length $2k - 1$, which is less than

$$\frac{\sum_{v \in V(G) \setminus \{x, z\}} d(v)}{n - 2} = \frac{(k - 1)k + (k - 2)(n - 1) + (n - 2k + 1)(n - k)}{n - 2} = \frac{n^2 - 2kn + n + 3k^2 - 3k}{n - 2}.$$ 

Besides, the complete graph $K_n$ with $n \geq k + 1$ shows that the bound $r$ on the length of the $(x, Y, z)$-path is sharp.

There also exist results on long cycles passing through specified vertices in graphs. Theorem 5 shows the existence of long cycles in $2$-connected graph under the minimum degree condition, and Theorem 6 extends Theorem 5 to graphs with higher connectivity.

Theorem 5 (Locke [8]) Let $G$ be a $2$-connected graph. If the minimum degree of $G$ is at least $d$, then for any two vertices $y_1$ and $y_2$ of $G$, $G$ contains either a $(y_1, y_2)$-cycle of length at least $2d$ or a Hamilton cycle.

Theorem 6 (Egawa et al. [3]) Let $G$ be a $k$-connected graph with $k \geq 2$. If the minimum degree of $G$ is at least $d$, then for any subset $Y$ of $V(G)$ with $|Y| = k$, $G$ contains either a $Y$-cycle of length at least $2d$ or a Hamilton cycle.

On the existence of long cycles in graphs with a given number of edges, Erdős and Gallai [5] gave the following result.

Theorem 7 (Erdős and Gallai [5]) Let $G$ be a $2$-edge-connected graph on $n$ vertices. Then $G$ contains a cycle of length at least $2e(G)/(n - 1)$.

In this paper, as an application of Theorem 4, we give the following theorem on long cycles passing through specified vertices of graphs with a given number of vertices and edges.

Theorem 8 Let $G$ be a $k$-connected graph on $n$ vertices with $k \geq 2$. Then for any subset $Y$ of $V(G)$ with $|Y| = k - 1$, $G$ contains a $Y$-cycle of length at least $2e(G)/(n - 1)$.

In Theorem 8, one cannot expect a cycle passing through $k$ specified vertices of length at least $2e(G)/(n - 1)$. Let $H$ be a complete graph on $n - k$ vertices with $n > 3k$ and $u_1, u_2, \ldots, u_k$ be $k$ vertices of $H$, and $Y = \{v_1, v_2, \ldots, v_k\}$ be a set of vertices not in $V(H)$. We construct a graph $G$ with $V(G) = V(H) \cup Y$ and $E(G) = E(H) \cup \{u_i v_j : 1 \leq i, j \leq k\}$. Then $G$ is a $k$-connected graph and the longest $Y$-cycle has length $2k$, which is less than

$$\frac{2e(G)}{n - 1} = \frac{(n - k)(n - k - 1) + 2k^2}{n - 1}.$$
In the following section we will give some further notations and preliminary results that will be used later. The proofs of Theorems 4 and 8 are given in Sects. 3 and 4, respectively.

2 Preliminaries

Let $G$ be a graph and $P$, $H$ two disjoint subgraphs of $G$. We use $E(P, H)$ to denote the set, and $e(P, H)$ the number, of edges with one vertex in $P$ and the other in $H$. If $E(P, H) \neq \emptyset$, then we call $P$ and $H$ are joined. We use $N_P(H)$ to denote the set of vertices in $P$ which are joined to $H$. If $x$ is a vertex in $G - P$, we say that $x$ is locally $k$-connected to $P$ (in $G$) if there are $k$ paths connecting $x$ and vertices in $P$ such that any two of them have only the vertex $x$ in common. We say that $H$ is locally $k$-connected to $P$ (in $G$) if for every vertex $x \in V(H)$, $x$ is locally $k$-connected to $P$. Note that if $H$ is locally $k$-connected to $P$, then $H$ is locally $l$-connected to $P$ for all $l, 0 \leq l \leq k$; and, if $G$ is $k$-connected and $|V(P)| \geq k$, then $H$ is locally $k$-connected to $P$ in $G$.

The following propositions on local $k$-connectedness are proved in [6].

**Proposition 1** (Fan [6]) Let $H$ and $P$ be two disjoint subgraphs of a graph $G$. If $H$ is locally $k$-connected to $P$ in the subgraph induced by $V(H) \cup V(P)$, then $E(P, H)$ contains an independent set of $t$ edges, where $t \geq \min\{k, |V(H)|\}$.

**Proposition 2** (Fan [6]) Let $H$ and $P$ be two disjoint subgraphs of a graph $G$. Let $u \in N_P(H)$ and $G'$ be the graph obtained from $G$ by deleting all edges from $u$ to $H$. If $H$ is locally $k$-connected to $P$ in $G$, then $H$ is locally $(k - 1)$-connected to $P$ in $G'$.

**Proposition 3** (Fan [6]) Let $H$ and $P$ be two disjoint subgraphs of a graph $G$, and $B$ a block of $H$. Let $H'$ be the subgraph obtained from $H$ by contracting $B$. If $H$ is locally $k$-connected to $P$ in $G$, then $H'$ is also locally $k$-connected to $P$ in the resulting graph.

Next we introduce the concept of local maximality for paths.

Let $P$ be a path of a graph $G$, and $u, v \in V(P)$. We use $P[u, v]$ to denote the segment of $P$ from $u$ to $v$, and $P(u, v)$ the segment obtained from $P[u, v]$ by deleting the two end vertices $u$ and $v$. Let $H$ be a component of $G - P$. We say that $P$ is a locally longest path with respect to $H$ if we cannot obtain a longer path than $P$ by replacing the segment $P[u, v]$ by a $(u, v)$-path with all its internal vertices in $H$ for any $u, v \in V(G)$. In other words, $P$ is locally longest with respect to $H$ if, for any $u, v \in V(P)$,

$$e(P[u, v]) \geq d_H^*(u, v).$$

If $P$ is an $(x, Y, z)$-path of $G$, where $x, z \in V(G)$ and $Y \subset V(G)$, then we say that $P$ is a locally longest $(x, Y, z)$-path with respect to $H$ if we cannot obtain an $(x, Y, z)$-path longer than $P$ by replacing the segment $P[u, v]$ with $Y \cap V(P(u, v)) = \emptyset$ by a $(u, v)$-path with all its internal vertices in $H$. Note that if $P$ is a longest path [longest $(x, Y, z)$-path] in a graph $G$, then, of course, $P$ is a locally longest path [locally
longest \((x, Y, z)\)-path] with respect to any component of \(G - P\). If two vertices \(u\) and \(u'\) in \(V(P)\) are joined to \(H\) by two independent edges, then we call \(\{u, u'\}\) a strong attached pair of \(H\) to \(P\). A strong attachment of \(H\) to \(P\) (in \(G\)) is a subset 
\[T = \{u_1, u_2, \ldots, u_t\} \subset NP(H),\]
where \(u_i, 1 \leq i \leq t\), are in order along \(P\), such that each ordered pair \(\{u_i, u_{i+1}\}\), \(1 \leq i \leq t - 1\), is a strong attached pair of \(H\) to \(P\). A strong attachment \(T\) of \(H\) to \(P\) is maximum if \(T\) has maximum cardinality over all strong attachments of \(H\) to \(P\).

The following result due to Fan is useful in our proofs.

**Lemma 1** (Fan [6]) Let \(G\) be a graph and \(P\) an \((x, z)\)-path of \(G\). Suppose that \(H\) is a component of \(G - P\) and \(T = \{u_1, u_2, \ldots, u_t\}\) is a maximum strong attachment of \(H\) to \(P\). Set \(S = NP(H) \setminus T\). Then the following statements are true:

1. Every vertex in \(S\) is adjacent to exactly one vertex in \(H\).
2. For each segment \(P[u_i, u_{i+1}]\), \(1 \leq i \leq t - 1\), suppose that

\[NP(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \ldots, a_q, a_{q+1}\},\]

where \(a_0 = u_i\), \(a_{q+1} = u_{i+1}\) and \(a_j, 0 \leq j \leq q + 1\), are in order along \(P\). Then there is a subscript \(m\), \(0 \leq m \leq q\), such that

\[NH(a_j) = NH(a_0), \text{ for } 0 \leq j \leq m,\]

and

\[NH(a_j) = NH(a_{q+1}), \text{ for } m + 1 \leq j \leq q + 1.\]

Besides, if

\[NP(H) \cap V(P[x, u_1]) = \{a_1, \ldots, a_q, a_{q+1}\},\]

where, \(a_{q+1} = u_1\), then

\[NH(a_j) = NH(a_{q+1}), \text{ for } 1 \leq j \leq q + 1;\]

and if

\[NP(H) \cap V(P[u_t, z]) = \{a_0, a_1, \ldots, a_q\},\]

where, \(a_0 = u_t\), then

\[NH(a_j) = NH(a_0), \text{ for } 0 \leq j \leq q.\]

3. If \(H\) is locally \(k\)-connected to \(P\) in \(G\), then

\[t \geq \min\{k, h + d_2\},\]
where $h = |V(H)|$ and $d_2$ is the number of vertices in $N_P(H)$ each of which has at least two neighbors in $H$.

Lemma 1 (2) is somewhat different from that in [6], but the proofs of them are similar.

For a path $P$, we use $l(P)$ to denote the length of $P$.

**Lemma 2** Let $G$ be a graph, $P$ an $(x, y, z)$-path of $G$, where $x, z \in V(G)$ and $Y \subseteq V(G)$, $H$ a component of $G - P$ and $T = \{u_1, u_2, \ldots, u_t\}$ a maximum strong attachment of $H$ to $P$. Set $S = N_P(H) \setminus T$ and $s = |S|$. Suppose that $P$ is a locally longest $(x, y, z)$-path with respect to $H$, and $\theta = |\{x, z\} \cap N_P(H)|$. Set

$$T_r = \{u_i \in T \setminus \{u_t\} : Y \cap V(P(u_i, u_{i+1})) = \emptyset\} \text{ and } t_r = |T_r|.$$

Then

$$l(P) \geq \sum_{u_i \in T_r} d_H^*(u_i, u_{i+1}) + 2(s + t - t_r) - \theta.$$

**Proof** If $t = 0$, then $s = 0$ and the statement is trivially true. Suppose now that $t \geq 1$. Consider a segment $P[u_i, u_{i+1}]$, $1 \leq i \leq t - 1$. Suppose that

$$N_P(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \ldots, a_q, a_{q+1}\},$$

where $q = |S \cap V(P[u_i, u_{i+1}])|$, $a_0 = u_i$, $a_{q+1} = u_{i+1}$, and $a_j, 0 \leq j \leq q + 1$, are in order along $P$.

If $Y \cap V(P[u_i, u_{i+1}]) = \emptyset$, then by Lemma 1 (2), there is a subscript $m$, $0 \leq m \leq q$, such that

$$N_H(a_0) = N_H(a_m) \quad \text{and} \quad N_H(a_{q+1}) = N_H(a_{m+1}).$$

Therefore

$$d_H^*(a_m, a_{m+1}) = d_H^*(a_0, a_{q+1}) = d_H^*(u_i, u_{i+1}).$$

Since $P$ is a locally longest $(x, y, z)$-path with respect to $H$, we have

$$l(P[u_i, u_{i+1}]) \geq \sum_{j=0}^{q} d_H^*(a_j, a_{j+1}) = d_H^*(a_m, a_{m+1}) + \sum_{j=0 \atop j \neq m}^{q} d_H^*(a_j, a_{j+1})$$

$$= d_H^*(u_i, u_{i+1}) + \sum_{j=0 \atop j \neq m}^{q} d_H^*(a_j, a_{j+1}).$$

Note that $d_H^*(a_j, a_{j+1}) \geq 2$, for every $j$, $0 \leq j \leq q$, we have

$$l(P[u_i, u_{i+1}]) \geq d_H^*(u_i, u_{i+1}) + 2q.$$
If $Y \cap V(P(u_i, u_{i+1})) \neq \emptyset$, then noting that $l(P[a_j, a_{j+1}]) \geq 2$, we have

$$l(P[u_i, u_{i+1}]) = \sum_{j=0}^{q} l(P[a_j, a_{j+1}]) \geq 2q + 2.$$  

Besides, consider the two segments $P[x, u_1]$ and $P[u_t, z]$. Suppose that

$$N_P(H) \cap V(P[x, u_1]) = \{a_0, a_1, \ldots, a_m\}$$

and

$$N_P(H) \cap V(P[u_t, z]) = \{a_{m+1}, a_{m+2}, \ldots, a_{q+1}\},$$

where $m = |S \cap V(P[x, u_1])|$, $q - m = |S \cap V(P[u_t, z])|$, $a_m = u_1, a_{m+1} = u_t$, and $a_j, 0 \leq j \leq q + 1$ are in order along $P$. Note that $l(P[x, a_0]) + l(P[a_{q+1}, z]) \geq 2 - \theta$ and $l(P[a_j, a_{j+1}]) \geq 2$, for every $0 \leq j \leq q$, and $j \neq m$, we have

$$l(P[x, u_1]) + l(P[u_t, z]) \geq 2q + 2 - \theta.$$  

Thus summing over the lengths of all the segments, yields

$$l(P) = l(P[x, u_1]) + \sum_{i=1}^{t-1} l(P[u_i, u_{i+1}]) + l(P[u_t, z])$$

$$\geq 2(|S \cap V(P[x, u_1])| + |S \cap V(P[u_t, z])|) + 2 - \theta$$

$$+ \sum_{u_i \in T_r} (d^n_H(u_i, u_{i+1}) + 2|S \cap V(P[u_i, u_{i+1}])|)$$

$$+ \sum_{u_i \in T_r} (2|S \cap V(P[u_i, u_{i+1}])| + 2)$$

$$= \sum_{u_i \in T_r} d^n_H(u_i, u_{i+1}) + 2(s + t - t_r) - \theta.$$  

This ends the proof.

For a strong attachment $T = \{u_1, u_2, \ldots, u_t\}$, the pairs $\{u_j, u_{j+1}\}, 1 \leq j \leq t - 1$, are called strong attached pairs supported by $T$, and we call a strong attached pair $\{u_j, u_{j+1}\}$ of $H$ to $P$ transitive if $Y \cap V(P(u_j, u_{j+1})) = \emptyset$.

A connected graph is separable if it has at least one cut-vertex.

**Lemma 3** Let $G$ be a graph and $P$ an $(x, z)$-path of $G$. Suppose that $H$ is a separable component of $G - P$, $B$ is an endblock of $H$, $b$ is the cut vertex of $H$ contained in $B$, and $M = B - b$. Let $T = \{u_1, u_2, \ldots, u_t\}$ be a maximum strong attachment of $H$ to $P$. If $H$ is locally $k$-connected to $P$, then
(1) \[ |N_P(M) \cap T| \geq \min\{k - 1, m + d'_2\}; \text{ and} \]
(2) \text{there exist at least } \min\{k - 1, m + d'_2\} \text{ strong attached pairs supported by } T \text{ which are joined to } M,

where \( m = |V(M)| \) and \( d'_2 \) is the number of vertices in \( N_P(M) \) each of which has at least two neighbors in \( H \).

\textbf{Proof} \text{ Since } H \text{ is locally } k\text{-connected to } P, \text{ } |V(P)| \geq k. \text{ It is easy to know that } M \text{ is locally } (k - 1)\text{-connected to } P \text{ in the subgraph induced by } V(P) \cup V(M). \text{ By Proposition 1, there are } \min\{k - 1, m\} \text{ independent edges in } E(P, M). \text{ Let } u_iw_i, 1 \leq i \leq \min\{k - 1, m\} \text{ be such edges, where } u_i \in V(P) \text{ and } w_i \in V(M).

If \( v_i \) has at least two neighbors in \( H \), then by Lemma 1 (1), \( v_i \in T \). If \( v_i \) has only one neighbor \( w_i \) in \( H \), then by Lemma 1 (2), there exists a vertex \( v'_i \) (maybe equal to \( u_i \)) in \( T \) which also has only one neighbor \( w_i \) in \( H \). This implies that \( |N_P(M) \cap T| \geq \min\{k - 1, m\} \).

Now, we prove (1) by induction on \( d'_2 \). If \( d'_2 = 0 \), then by the analysis above, the assertion is true. Thus we assume that \( d'_2 \geq 1 \).

Let \( u_j \) be a vertex in \( N_P(M) \) which has at least two neighbors in \( H \) \([u_j \text{ is of course in } T \text{ by Lemma 1 (1)}]\). Let \( G' \) be the graph obtained from \( G \) by deleting all edges from \( u_j \) to \( H \). By Proposition 2, \( H \) is locally \((k - 1)\)-connected to \( P \) in \( G' \). If \( u_j = u_1 \) or \( u_t \), or \( \{u_{j-1}, u_{j+1}\} \) is joined to \( H \) by two independent edges, then \( T' = T\setminus\{u_j\} \) is a strong attachment of \( H \) to \( P \) in \( G' \). Since \( u_j \) is joined to at least two vertices of \( H \) in \( G \), any strong attachment of \( H \) to \( P \) in \( G' \) together with \( u_j \) is a strong attachment of \( H \) to \( P \) in \( G \). Since \( |T'| = t - 1 \), we see that \( T' \) is a maximum strong attachment of \( H \) to \( P \) in \( G' \). By the induction hypothesis,

\[ |N_P(M) \cap T'| \geq \min\{k - 2, m + d'_2 - 1\}. \]

Therefore

\[ |N_P(M) \cap T| \geq \min\{k - 1, m + d'_2\}, \]

as required.

If \( u_j \in \{u_2, \ldots, u_{t-1}\} \), and \( \{u_{j-1}, u_{j+1}\} \) are not joined to \( H \) by two independent edges, i.e.,

\[ N_H(u_{j-1}) = N_H(u_{j+1}) = \{w\}, \]

for some \( w \in V(H) \), then

\[ T' = T\setminus\{u_j, u_{j+1}\} = \{u_1, \ldots, u_{j-1}, u_{j+2}, \ldots, u_t\} \]

is a strong attachment of \( H \) to \( P \) in \( G' \). We prove now that \( T' \) is maximum by showing that any strong attachment of \( H \) to \( P \) in \( G' \) has cardinality at most \( t - 2 = |T'| \).

Let \( v_1, v_2 \) (\( \neq u_j \)) be the two vertices in \( N_P(H) \) which are closest to \( u_j \) on \( P \), say \( v_1 \) preceding, and \( v_2 \) following, \( u_j \) on \( P \) (but not necessarily adjacent to \( u_j \) on \( P \)). Since \( |N_H(u_j)| \geq 2 \), it follows from Lemma 1 (2) that

\[ N_H(v_1) = N_H(u_{j-1}) = \{w\} = N_H(u_{j+1}) = N_H(v_2). \]
By the choices of $v_1$ and $v_2$, for any maximum strong attachment $\{a_1, a_2, \ldots, a_p\}$ of $H$ to $P$ in $G'$, there is an integer $l$, $0 \leq l \leq p$, such that $v_1, v_2 \in V(P[a_1, a_{l+1}])$, where $a_0 = x$ and $a_{p+1} = z$. Since $N_H(v_1) = \{w\} = N_H(v_2)$, it follows from Lemma 1 (2) that either $N_H(a_i)$ or $N_H(a_{l+1}) = \{w\}$. The former implies a strong attachment $\{a_1, \ldots, a_l, v_2, a_{l+1}, \ldots, a_p\}$, the latter a strong attachment $\{a_1, \ldots, a_l, v_1, u_j, a_{l+1}, \ldots, a_p\}$, of $H$ to $P$ in $G$; in either case we have that $p+2 \leq t$, that is, $p \leq t - 2 = |T'|$. This shows that $T'$ is a maximum strong attachment of $H$ to $P$ in $G'$, as claimed. As before, by the induction hypothesis,

$$|N_P(M) \cap T'| \geq \min\{k - 2, m + d'_2 - 1\}.$$ 

Consequently

$$|N_P(M) \cap T| \geq \min\{k - 1, m + d'_2\},$$

which completes the proof of (1).

Now we prove (2). Clearly for every vertex $u_j \in N_P(M) \cap T\{u_i\}$, the strong attached pair $\{u_j, u_{j+1}\}$ supported by $T$ is joined to $M$. If $|N_P(M) \cap T\{u_i\}| \geq \min\{k - 1, m + d'_2\}$, then the assertion is true. By (1), we assume that $|N_P(M) \cap T| = \min\{k - 1, m + d'_2\}$ and $u_i \in N_P(M) \cap T$.

By Lemma 1 (3), $t \geq \min\{k, h + d_2\} \geq \min\{k - 1, m + d'_2\} + 1$. This implies that there exists at least one vertex in $T \setminus N_P(M)$. We choose a vertex $u_i \in T \setminus N_P(M)$ such that $u_{i+1} \in N_P(M) \cap T$. Then $\{u_i, u_{i+1}\}$ together with $\{u_j, u_{j+1}\}$ for $u_j \in N_P(M) \cup T\{u_i\}$ are $\min\{k - 1, m + d'_2\}$ strong attached pairs supported by $T$ joined to $M$.

Let $P, H, B, M$ be defined as in Lemma 3. In the following, we call a strong attached pair which is joined to $M$ a good pair (with respect to $M$). Let $\{u_j, u_{j+1}\}$ be a good attached pair. If one of the vertices in $\{u_j, u_{j+1}\}$ is joined to $M$, and the other to $H - M$, then we call it a better pair (with respect to $M$); and if one of the vertices in $\{u_j, u_{j+1}\}$ is joined to $M$, and the other to $H - B$, then we call it a best pair (with respect to $M$).

3 Proof of Theorem 4

If $k = 2$, then the assertion is Theorem 3. So we assume that $k \geq 3$. Since $G$ is $k$-connected and $|Y| = k - 2$, $G$ contains an $(x, Y, z)$-path. In order to prove the theorem, we choose a longest $(x, Y, z)$-path $P$ in $G$. Clearly $|V(P)| \geq |Y| + 2 = k$. Moreover, by the $k$-connectedness of $G$, for each component $H$ of $G - P$, $H$ is locally $k$-connected to $P$, and $P$ is a locally longest $(x, Y, z)$-path with respect to $H$. So it is sufficient to prove that:

**Proposition 4** Let $G$ be a graph, $P$ an $(x, Y, z)$-path of $G$, where $x, z \in V(G), Y \subseteq V(G)$, and $|Y| = k - 2$. Suppose that the average degree of vertices in $V(G) \setminus \{x, z\}$ is $r$. If for each component $H$ of $G - P$, $H$ is locally $k$-connected to $P$, and $P$ is a locally longest $(x, Y, z)$-path with respect to $H$, then $l(P) \geq r$.  

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Proof We prove this proposition by induction on \(|V(G - P)|\). If \(V(G - P) = \emptyset\), noting that \(r \leq |V(G)| - 1\), the result is trivially true. So we assume that \(V(G - P) \neq \emptyset\).

Let \(d = |N_P(H)|, \theta = |\{x, z\} \cap N_P(H)|\) and \(N_P(H) = \{v_1, v_2, \ldots, v_d\}\), where \(v_i, 1 \leq i \leq d\), are in order along \(P\). Then, we have

\[
l(P) = l(P[x, v_1]) + \sum_{i=1}^{d-1} l(P[v_i, v_{i+1}]) + l(P[v_d, z]).
\]

It is easy to know that \(l(P[x, v_1]) + l(P[v_d, z]) \geq 2 - \theta\) and \(l(P[v_i, v_{i+1}]) \geq 2\) for \(1 \leq i \leq d - 1\). Thus, we have

\[
l(P) \geq 2d - \theta.
\]

Note that \(d \geq k\) by the local \(k\)-connectedness of \(H\) to \(P\) and clearly \(\theta \leq 2\). If \(r \leq 2k - 2\), then we have \(l(P) \geq 2k - 2 \geq r\), and the proof is complete. Thus we assume that

\[
r > 2k - 2.
\]

Besides, if \(d \geq (r + \theta)/2\), then \(l(P) \geq r\), and we complete the proof. Thus, we assume that

\[
d < (r + \theta)/2.
\]

Let \(T = \{u_1, u_2, \ldots, u_t\}\) be a maximum strong attachment of \(H\) to \(P\). Set \(S = N_P(H) \setminus T\) and \(s = |S|\) (note that \(s + t = d\)). Let \(T_r = \{u_i \in T \setminus \{u_t\} : \exists y \in V(P[u_i, u_{i+1}]) = \emptyset\}\) and \(t_r = |T_r|\).

Clearly, for every transitive strong attached pair \([u_j, u_{j+1}]\), where \(u_j \in T_r\), we have

\[
d_H^*(u_j, u_{j+1}) \geq 2.
\]

We distinguish two cases:

Case 1 \(H\) is nonseparable.

Let \(h = |V(H)|\) and \(r'\) the average degree of vertices in \(V(H)\). If \(r'h + e(P - \{x, z\}, H) \leq rh\), then we consider the graph \(G'\) obtained from \(G\) by deleting the component \(H\). Note that

\[
\sum_{v \in V(G') \setminus \{x, z\}} d_{G'}(v) = r(|V(G)| - 2) - r'h - e(P - \{x, z\}, H) \\
\geq r(|V(G)| - 2) - rh \\
= r(|V(G')| - 2).
\]
By the induction hypothesis, we have \( l(P) \geq r \), and the proof is complete. Thus we assume that

\[
  r'h + e(P - \{x, z\}, H) > rh. \tag{4}
\]

We use \( d_1 \) to denote the number of vertices in \( N_P(H) \) which have only one neighbor in \( V(H) \), \( d_2 = d - d_1 \), \( \theta_1 \) to denote the number of vertices in \( \{x, z\} \) which have only one neighbor in \( V(H) \) and \( \theta_2 = \theta - \theta_1 \).

Clearly,

\[
r'h \leq h(h - 1 + d_2) + d_1 \quad \text{and} \quad e(P - \{x, z\}, H) \leq h(d_2 - \theta_2) + d_1 - \theta_1.
\]

Thus, by (4), we have

\[
  h(h - 1 + 2d_2 - \theta_2) + 2d_1 - \theta_1 \geq r'h + e(P - \{x, z\}, H) > rh.
\]

Note that \( d_1 = d - d_2 \) and \( \theta_1 = \theta - \theta_2 \), we have

\[
  h(h - 1 + 2d_2 - \theta_2) + 2d - 2d_2 - \theta + \theta_2 \geq rh.
\]

By (2), we have

\[
  h(h - 1 + 2d_2 - \theta_2) + (r + \theta) - 2d_2 - \theta + \theta_2 > rh.
\]

Thus,

\[
  (h - 1)(h + 2d_2 - r - \theta_2) > 0.
\]

This implies that \( h \geq 2 \) and \( h + 2d_2 > r + \theta_2 \geq r \), and then \( 2h + 2d_2 > r + 2 \). By (1), we have \( 2h + 2d_2 > 2k \), that is

\[
  h + d_2 > k. \tag{5}
\]

By (5) and Lemma 1 (3), \( t \geq k \). Since \( |Y| \leq k - 2 \), there exists at least one transitive strong attached pair \( (u_p, u_{p+1}) \) supported by \( T \), where \( u_p \in T_r \).

Let \( G' \) be the subgraph induced by \( V(H) \cup \{u_p, u_{p+1}\} \). If \( u_pu_{p+1} \notin E(G) \), we add the edge \( u_pu_{p+1} \) in \( G' \). Thus \( G' \) is 2-connected, and by (4),

\[
  \sum_{v \in V(G') \backslash \{u_p, u_{p+1}\}} d_{G'}(v) = \sum_{v \in V(H)} d(v) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, H)
\]

\[
  = r'h - e(N_P(H) \backslash \{u_p, u_{p+1}\}, H)
\]

\[
  \geq rh - e(P - \{x, z\}, H) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, H).
\]
Note that

\[ e(P - \{x, z\}, H) \leq (s + t - \theta)h, \quad \text{and} \quad e(N_P(H)\backslash\{u_p, u_{p+1}\}, H) \leq (s + t - 2)h, \]

we have

\[
\sum_{v \in V(G')\backslash\{u_p, u_{p+1}\}} d_{G'}(v) \geq rh - (s + t - \theta)h \geq (r - 2s - 2t + \theta + 2)h.
\]

By Theorem 3, \( G' \) contains a \((u_p, u_{p+1})\)-path of length at least \( r - 2s - 2t + \theta + 2 \), which implies that

\[
d^*_H(u_p, u_{p+1}) \geq r - 2s - 2t + \theta + 2. \tag{6}
\]

Substituting (6) for \( d^*_H(u_p, u_{p+1}) \) in Lemma 2 and (3) for the other terms, we have

\[
l(P) \geq (r - 2s - 2t + \theta + 2) + 2(t_r - 1) + 2(s + t - t_r) - \theta \geq r.
\]

**Case 2** \( H \) is separable.

Let \( B \) be an endblock of \( H \), \( b \) the cut vertex of \( H \) contained in \( B \), \( M = B - b \), \( m = |V(M)| \), and \( r'' \) the average degree of the vertices in \( V(M) \).

If \( r''m + e(P - \{x, z\}, M) + d_M(b) \leq rm \), then we consider the graph \( G' \) obtained from \( G \) by contracting \( B \). Let \( H' \) be the component of \( G' - P \) obtained from \( H \) by contracting \( B \). By Proposition 3, \( H' \) is locally \( k \)-connected to \( P \). Clearly \( P \) is a locally longest \((x, Y, z)\)-path with respect to \( H' \), and

\[
\sum_{v \in V(G')\backslash\{x, z\}} d_{G'}(v) \geq \sum_{v \in V(G)\backslash\{x, z\}} (d(v) - r''m - e(P - \{x, z\}, M) - d_M(b)) \\
\geq r(|V(G)| - 2) - rm \\
= r(|V(G')| - 2).
\]

By the induction hypothesis, \( l(P) \geq r \), and the proof is complete. Thus we assume that

\[
r''m + e(P - \{x, z\}, M) + d_M(b) > rm. \tag{7}
\]

Let \( d'_0 = |N_P(H)\backslash N_P(M)| \), \( d'_1 \) be the number of vertices in \( N_P(M) \) which have only one neighbor in \( V(H) \), \( d''_0 = d - d'_0 - d'_1; \theta'_0 = |\{x, z\} \cap N_P(H)\backslash N_P(M)|, \theta'_1 \) be the number of vertices in \( \{x, z\} \cap N_P(M) \) which have only one neighbor in \( V(H) \) and \( \theta''_1 = \theta - \theta'_0 - \theta'_1 \).
Now we prove that

\[ m + d_2' \geq k - 1. \] (8)

Let \( B' \) be an endblock of \( H \) other than \( B \), \( b' \) the cut vertex of \( H \) contained in \( B' \), \( M' = B' - b' \) and \( m' = |V(M')| \).

By the local \( k \)-connectedness of \( H \) to \( P \), \( |N_P(M')| \geq k - 1 \). If \( |N_P(M') \cap N_P(M)| \leq m \), then \( d_2' \geq |N_P(M) \cap N_P(M')| \geq k - 1 - m \), and \( m + d_2' \geq k - 1 \), and (8) holds. Thus we assume that \( |N_P(M') \cap N_P(M)| \geq m + 1 \). So we have

\[ d_0' \geq m + 1. \] (9)

Clearly,

\[ r'' m \leq m(m + d_2') + d_1', \]
\[ e(P - \{x, z\}, M) \leq m(d_2' - \theta_2') + d_1' - \theta_1', \quad \text{and} \]
\[ d_M(b) \leq m. \]

Thus, by (7),

\[ m(m + 2d_2' + 1 - \theta_2') + 2d_1' - \theta_1' \geq r'' m + e(P - \{x, z\}, M) + d_M(b) > rm. \]

Note that \( d_1' = d - d_0' - d_2' \) and \( \theta_1' = \theta - \theta_0' - \theta_2' \), we have

\[ m(m + 2d_2' + 1 - \theta_2') + 2d - 2d_0' - 2d_2' - \theta + \theta_0' + \theta_2' > rm. \]

By (2) and (9), we have

\[ m(m + 2d_2' + 1 - \theta_2') + (r + \theta) - 2(m + 1) - 2d_2' - \theta + \theta_0' + \theta_2' > rm. \]

Thus,

\[ (m - 1)(m + 2d_2' - r - \theta_2') > 2 - \theta_0' \geq 0. \]

This implies that \( m \geq 2 \) and \( m + 2d_2' > r + \theta_2' \geq r \), and then \( 2m + 2d_2' > r + 2 \). By (1), \( 2m + 2d_2' > 2k \), that is \( m + d_2' > k \), and (8) holds.

By Lemma 3 (2), there exist at least \( k - 1 \) good pairs supported by \( T \) with respect to \( M \). Since \( |Y| = k - 2 \), there exists at least one transitive good pair \( \{u_p, u_{p+1}\} \) with respect to \( M \). Similarly there exists at least one transitive good pair \( \{u_q, u_{q+1}\} \) with respect to \( M' \).

First we assume that there is a transitive best pair supported by \( T \) with respect to \( M \) or \( M' \). Without loss of generality, we assume that \( \{u_p, u_{p+1}\} \) is a best pair, where \( u_p \in N_P(M) \) and \( u_{p+1} \in N_P(H - B) \). Consider the subgraph \( G' \) induced by
If $u_p b \notin E(G)$, we add the edge $u_p b$ in $G'$. Thus $G'$ is 2-connected, and by (7),

$$\sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) = \sum_{v \in V(M)} d(v) - e(N_P(H) \setminus \{u_p\}, M)$$

$$= r'm - e(N_P(H) \setminus \{u_p\}, M)$$

$$\geq rm - e(P - \{x, z\}, M) - d_M(b) - e(N_P(H) \setminus \{u_p\}, M).$$

Note that

$$e(P - \{x, z\}, M) \leq (s + t - \theta)m,$$

$$d_M(b) \leq m, \text{ and}$$

$$e(N_P(H) \setminus \{u_p\}, M) \leq (s + t - 1)m,$$

we have

$$\sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) \geq rm - (s + t - \theta)m - m - (s + t - 1)m$$

$$= (r - 2s - 2t + \theta)m.$$

By Theorem 3, $G'$ contains a $(u_p, b)$-path of length at least $r - 2s - 2t + \theta$. It is clear that there is a $(b, u_p+1)$-path in $H - M$ of length at least 2, which implies that

$$d^*_H(u_p, u_p+1) \geq r - 2s - 2t + \theta + 2. \quad (10)$$

Substituting (10) for $d^*_H(u_p, u_p+1)$ in Lemma 2 and (3) for the other terms, we have

$$l(P) \geq (s + t - \theta)m - (s + t - \theta)m$$

as required.

So, we assume that there are no transitive best pairs supported by $T$ with respect to $M$ or $M'$. Now we assume that there is a transitive better pair (but not best pair) supported by $T$ with respect to $M$ or $M'$. Without loss of generality, we assume that $\{u_p, u_p+1\}$ is a better pair, where $u_p \in N_P(M)$ and $u_p+1 \in N_P(b)$. Consider the subgraph $G'$ induced by $V(B) \cup \{u_p\}$. If $u_p b \notin E(G)$, we add the edge $u_p b$ in $G'$. Thus $G'$ is 2-connected and

$$\sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) \geq rm - e(P - \{x, z\}, M) - d_M(b) - e(N_P(H) \setminus \{u_p\}, M).$$
Note that

\[ e(P - \{x, z\}, M) \leq (s + t - \theta)m, \quad \text{and} \]

\[ d_M(b) \leq m, \]

and since at least one vertex of \(u_q\) and \(u_{q+1}\) is not joined to \(M\) [otherwise, \(\{u_q, u_{q+1}\}\) will be a best pair], we have

\[ e(N_P(H) \backslash \{u_p\}, M) \leq (s + t - 2)m. \]

Thus we have

\[
\sum_{v \in V(G') \backslash \{u_p, b\}} d_{G'}(v) \geq rm - (s + t - \theta)m - m - (s + t - 2)m
\]

\[
= (r - 2s - 2t + \theta + 1)m.
\]

By Theorem 3, \(G'\) contains a \((u_p, b)\)-path of length at least \(r - 2s - 2t + \theta + 1\), and then, by \(bu_{p+1} \in E(G)\),

\[ d_H^*(u_p, u_{p+1}) \geq r - 2s - 2t + \theta + 2. \]

Thus we also have \(l(P) \geq r\).

So, we assume that there are no transitive better pairs supported by \(T\) with respect to \(M\) or \(M'\). Thus \(\{u_p, u_{p+1}\} \cap \{u_q, u_{q+1}\} = \emptyset\), and \(\{u_p, u_{p+1}\}\) and \(\{u_q, u_{q+1}\}\) are two distinct strong attached pairs.

Note that \(u_p\) and \(u_{p+1}\) are joined to \(M\) by two independent edges. Consider the subgraph \(G'\) induced by \(V(B) \cup \{u_p, u_{p+1}\}\). If \(u_pu_{p+1} \notin E(G)\), we add the edge \(u_pu_{p+1}\) in \(G'\). Thus \(G'\) is 2-connected, and by (7),

\[
\sum_{v \in V(G') \backslash \{u_p, u_{p+1}\}} d_{G'}(v)
\]

\[
= \sum_{v \in V(M)} d(v) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, M) + d_M(b) + |\{u_p, u_{p+1}\} \cap N_P(b)|
\]

\[
= r''m + d_M(b) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, M) + |\{u_p, u_{p+1}\} \cap N_P(b)|
\]

\[
\geq rm - e(P - \{x, z\}, M) - e(N_P(H) \backslash \{u_p, u_{p+1}\}, M).
\]

Note that

\[ e(P - \{x, z\}, M) \leq (s + t - \theta)m, \]

and since neither \(u_q\) and \(u_{q+1}\) has neighbors in \(M\) [otherwise \(\{u_q, u_{q+1}\}\) will be a better pair], we have

\[ e(N_P(H) \backslash \{u_p, u_{p+1}\}, M) \leq (s + t - 4)m. \]
Thus, we have,
\[
\sum_{v \in V(G') \setminus \{u_p, u_{p+1}\}} d_{G'}(v) \geq rm - (s + t - \theta)m - (s + t - 4)m
= (r - 2s - 2t + \theta + 4)m.
\]

By Theorem 3, \(G'\) contains a \((u_p, u_{p+1})\)-path of length at least \((r - 2s - 2t + \theta + 4)m/(1 + m) \geq (r - 2s - 2t + \theta + 4)/2\) (note that \(m \geq 1\)), which implies that
\[
d_H^{\ast}(u_p, u_{p+1}) \geq (r - 2s - 2t + \theta + 4)/2.
\]

and similarly,
\[
d_H^{\ast}(u_q, u_{q+1}) \geq (r - 2s - 2t + \theta + 4)/2.
\]

Then,
\[
d_H^{\ast}(u_p, u_{p+1}) + d_H^{\ast}(u_q, u_{q+1}) \geq r - 2s - 2t + \theta + 4.
\]

Thus, by Lemma 2, we have
\[
l(P) \geq (r - 2s - 2t + \theta + 4) + 2(t_r - 2) + 2(s + t - t_r) - \theta \geq r.
\]

The proof is complete.

4 Proof of Theorem 8

By the \(k\)-connectedness of \(G\), it contains a \(Y\)-cycle. If \(2e(G)/(n - 1) \leq 3\), then the result is trivially true. Thus we assume that \(2e(G)/(n - 1) > 3\).

We choose a vertex \(y \in Y\), and construct a graph \(G'\) such that \(V(G') = V(G) \cup \{y'\}\), where \(y' \notin V(G)\) and \(E(G') = E(G) \cup \{vy' : v \in N_G(y)\}\). Clearly, \(G'\) is \(k\)-connected. Besides, we have that
\[
e(G') = e(G) + d_G(y) \quad \text{and} \quad d_{G'}(y) = d_{G'}(y') = d_G(y),
\]
and the order of \(G'\) is \(n + 1\). Now, by Theorem 4, there exists a \((y, Y \setminus \{y\}, y')\)-path \(P\) of length at least
\[
\frac{2e(G') - d_{G'}(y) - d_{G'}(y')}{(n + 1) - 2} = \frac{2(e(G) + d_G(y)) - 2d_G(y)}{n - 1} = \frac{2e(G)}{n - 1}.
\]

Let \(uy'\) be the last edge of \(P\). Then \(uy \in E(G)\) and \(C = P[y, u]uy\) is a cycle of \(G\) passing through all the vertices in \(Y\) of length at least \(2e(G)/(n - 1)\), which completes the proof.
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