ON LOCAL WELL-POSEDNESS AND ILL-POSEDNESS RESULTS FOR A COUPLED SYSTEM OF MKDV TYPE EQUATIONS

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ABSTRACT. We consider the initial value problem associated to a coupled system of modified Korteweg-de Vries type equations

\begin{align*}
\partial_t v + \partial_x^3 v + \partial_x (vw^2) &= 0, \quad v(x, 0) = \phi(x), \\
\partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) &= 0, \quad w(x, 0) = \psi(x),
\end{align*}

and prove the local well-posedness results for a given data in low regularity Sobolev spaces $H^s(\mathbb{R}) \times H^k(\mathbb{R})$, $s, k > -\frac{1}{2}$ and $|s - k| \leq \frac{1}{2}$, for $\alpha \neq 0, 1$.

Also, we prove that: (I) the solution mapping that takes initial data to the solution fails to be $C^3$ at the origin, when $s < -\frac{1}{2}$ or $k < -\frac{1}{2}$ or $|s - k| > \frac{1}{2}$; (II) the trilinear estimates used in the proof of the local well-posedness theorem fail to hold when (a) $s - 2k > 1$ or $k < -1/2$ (b) $k - 2s > 1$ or $s < -1/2$; (c) $s = k = -1/2$;

1. Introduction. This paper is devoted to the initial value problem (IVP) for the system of modified Korteweg-de Vries (mKdV)-type equations

\begin{align*}
\partial_t v + \partial_x^3 v + \partial_x (vw^2) &= 0, \quad v(x, 0) = \phi(x), \\
\partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) &= 0, \quad w(x, 0) = \psi(x),
\end{align*}

where $x \in \mathbb{R}$ and $t > 0$, $v = v(x, t)$ and $w = w(x, t)$ are real-valued functions, and $\alpha \in \mathbb{R}$ is a constant.

For $\alpha = 1$, among a vast class of nonlinear evolution equations, the related system was studied by [1], in the context of inverse scattering, showing that this method provides a means of solution of the associated IVP. For existence and estability of solitary waves to the system (1.1) we refer the works [2] and [20].

The well-posedness for the IVP (1.1) with initial data in the classical Sobolev spaces $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ was studied by many authors. In 1995, following Kenig, Ponce and Vega [17], using smoothing properties of the group, Maximal functions ans Strichartz estimates, Montenegro [20] proved that the IVP (1.1) with $\alpha = 1$ is

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locally well-posed for given data \((\phi, \psi)\) in \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\), \(s \geq \frac{1}{4}\). We note that the approach in [17] implies the local well-posedness for \(s \geq 1/4\), when \(0 < \alpha < 1\).

In 1999, Alarcon, Angulo and Montenegro [2] studied some properties of the solutions for the system of nonlinear evolution equation

\[
\begin{align*}
\partial_t v + \partial_x^3 v + \partial_x (v^p u^{p+1}) &= 0, \quad u(x, 0) = \phi(x), \\
\partial_t u + \partial_x^3 u + \partial_x (v^{p+1} u^p) &= 0, \quad v(x, 0) = \varphi(x),
\end{align*}
\]

for \(p \geq 1\). In this work they proved that (1.2) is locally well-posed in \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\), for \(s \geq 1\), using the smoothing property of the linear group combined with the \(L^2 \times L^4\) Strichartz inequalities and maximal function estimates. In order to prove the global well-posedness, for a given initial data in \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\), \(s \geq 1\), they used the conservation laws

\[
I_1(v, w) := \int_{\mathbb{R}} (v^2 + w^2) \, dx \quad \text{and} \quad I_2(v, w) := \int_{\mathbb{R}} (v_x^2 + w_x^2 - v^2 w^2) \, dx,
\]

and a priori estimates; this global solution depend on the parameter \(p \geq 1\) and the size of the initial data. In fact, if \(p = 1\) they obtained global solution without any restriction. Note that the system (1.1), with \(\alpha = 1\), is a special case of (1.2) with \(p = 1\).

Alarcon, Angulo and Montenegro also proved that (1.2) has a family of solitary wave solutions, similar to those found for Korteweg de Vries(KdV)-type equations and that it can be stable or unstable depending on the range of \(p\). In fact, with respect to the existence of solitary wave solutions for system (1.2), they used the concentration and compactness method. In order to obtain the nonlinear stability they follows the abstract results in [16], as it was showed for Montenegro [20] when \(p = 1\). With relation to the instability, they followed a method established by Bona, Souganidis and Strauss in [5] to analyze the instability of solitary waves of the KdV-type equations.

In 2001, Tao [22] shows that the following trilinear estimate is valid for \(s \geq 1/4\), \(b > 1/2\) and \(b' > -1/2\)

\[
\| \partial_x (uv w) \|_{X_{s,b}} \lesssim \| u \|_{X_{s,b}} \| v \|_{X_{s,b}} \| w \|_{X_{s,b}}
\]

where \(X_{s,b}\) is the Bourgain space associated to the linear KdV equation (see [6]). This leads us to get also the local well-posedness for the system (1.1) when \(\alpha = 1\), in \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\) for \(s \geq 1/4\), in the context of Fourier restriction norm method. It is worth noting that the local well-posedness result for the system (1.1) with \(\alpha = 1\) is sharp and it can be justified in two different way; first the trilinear estimates fail if \(s < 1/4\) (see Theorem 1.7 in [18]). Second, the solution map is not uniformly continuous if \(s < 1/4\) (see Theorem 1.3 in [19]). This notion of ill-posedness is a bit strong. For further works in this direction, we refer [16]. In 2012, Corcho and Panthee in [11] improves the global result in [20], getting global well-posedness in \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\), for \(s > 1/4\), for \(\alpha = 1\), see also [8].

Recently, in 2019 Carvajal and Panthee [9] proved local well-posedness in \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\) for \(s > -1/2\), when \(\alpha \in (0, 1) \cup (1, \infty)\). They also proved that, when \(s < -1/2\), both the key trilinear estimates fails to hold and the solution map is not \(C^3\) at the origin. Observe that this result also is sharp, considering the scaling argument \(s = -1/2\) to the modified KdV equation.

Many authors have studied local well-posedness for a system with dispersive equations, when the initial data belong to diferent Sobolev spaces, i.e., in \(H^s(\mathbb{R}) \times \)
$H^k(\mathbb{R}), k \neq s$ (see, e.g., Ginibre, Tsutsumi and Velo [15]). In this context, we prove the following local well-posedness result:

**Theorem 1.1.** Let $\alpha \neq 0, 1, b = 1/2$ and $s, k$ such that $s, k > -\frac{1}{2}$ and $|s - k| \leq 1/2$. Then for any $\phi, \psi \in H^s(\mathbb{R}) \times H^k(\mathbb{R})$, there exist $\delta = \delta(\|\phi, \psi\|_{H^s \times H^k})$ (with $\delta(\rho) \to \infty$ as $\rho \to 0$) and an unique solution $(v, w) \in X_{s,b} \times X_{k,b}^{\alpha,\delta}$ to the IVP (1.1) in the time interval $[0, \delta]$. Moreover, the solution satisfies the estimate

$$
\|\langle v, w \rangle\|_{X_{s,b} \times X_{k,b}^{\alpha,\delta}} \lesssim \|\langle \phi, \psi \rangle\|_{H^s \times H^k},
$$

where the norms $\| \cdot \|_{X_{s,b}}$ and $\| \cdot \|_{X_{k,b}^{\alpha,\delta}}$ are defined in (2.3).

**Remark 1.2.** The case $\alpha \in (0, 1)$ and $k = s$, with $s > -1/2$, of this theorem, was proved in [9]. Also, they observed that the LWP in the case $0 < \alpha < 1$ is equivalent to the LWP in the case $\alpha > 1$ via the transformations $v(x, t) := \tilde{v}(\alpha^{-1/3}x, t)$ and $u(x, t) := \tilde{u}(\alpha^{-1/3}x, t)$ where

$$
\begin{cases}
\partial_t \tilde{v} + \frac{1}{\alpha} \partial_x^3 \tilde{v} + \partial_x (\tilde{v} \tilde{w}^2) = 0, \\
\partial_t \tilde{w} + \partial_x (\tilde{v}^2 \tilde{w}) = 0.
\end{cases}
$$

So we restrict ourselves to prove Theorem 1.1 in the case $\alpha \in (-\infty, 0) \cup (1, +\infty)$. We observe again that if $\alpha = 1$, the system (1.1) is locally well-posed (sharp) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1/4$. The restriction $\alpha \neq 0$ is necessary, as one can see in the proof of Proposition 3.2 (see also Remark 3.6).

The main ingredients in the proof of Theorem 1.1 are the new trilinear estimates:

**Proposition 1.3.** Let $\alpha \neq 0, 1, b = 1/2 + \epsilon$, and $b' = -1/2 + 2\epsilon$, with $0 < \epsilon < \min \{\frac{2s+1}{2s}, \frac{1}{9}\}$. Then the following trilinear estimates

$$
\|\langle v w_1 w_2 \rangle_x\|_{X_{s,b'}} \lesssim \|v\|_{X_{s,b}} \|w_1\|_{X_{s,b}} \|w_2\|_{X_{s,b}}, \quad (1.4)
$$

and

$$
\|\langle v_1 v_2 w \rangle_x\|_{X_{s,b'}} \lesssim \|v_1\|_{X_{s,b}} \|v_2\|_{X_{s,b}} \|w\|_{X_{s,b}}, \quad (1.5)
$$

holds for any $s, k$ in the following region: $s, k > -\frac{1}{2}$ and $|s - k| \leq 1/2$. Moreover (1.4) also hold if $s = -1/2$ and $-1/2 < k$ and (1.5) also hold if $k = -1/2$ and $-1/2 < s$.

Also, we establish some ill-posedness results. The first one is about the smoothness of the solution mapping associated to the system (1.1).

**Theorem 1.4.** Let $\alpha \neq 0, 1$. For any $s < -1/2$ or $k < -1/2$ or $|s - k| > 2$ and for given $(\phi, \psi) \in H^s(\mathbb{R}) \times H^k(\mathbb{R})$, there exist no time $T = T(\|\phi, \psi\|_{H^s \times H^k})$ such that the solution mapping that takes initial data $(\phi, \psi)$ to the solution $(v, w) \in C([0, T]; H^s) \times C([0, T]; H^k)$ to the IVP (1.1) is $C^2$ at the origin (see Figure 1)

**Remark 1.5.** The proof of the Theorem 1.4 follows the structure of the proofs in [12] and [13] (see also [23]). We point out that in Section 5, we prove a little bit stronger result than the Theorem 1.4, when $s < -1/2$ or $k < -1/2$: see the Proposition 5.1, (b).

The second one is about the failure of the trilinear estimates (1.4) and (1.5). We prove the following results (See Figures 2 and 3):
Proposition 1.6. Let $\alpha \neq 0, 1$.
(a) The trilinear estimate (1.4) fail to hold for any $b \in \mathbb{R}$ whenever $s - 2k > 1$ or $k < -1/2$.
(b) The trilinear estimate (1.5) fail to hold for any $b \in \mathbb{R}$ whenever $k - 2s > 1$ or $s < -1/2$.

Proposition 1.7. Let $\alpha \neq 0, 1$.
(a) The trilinear estimate (1.4) fail to hold whenever $s - k > 2$, for any $\epsilon$ such that $0 < \epsilon < \frac{2}{3}(s - k - 2)$.
(b) The trilinear estimate (1.5) fail to hold whenever $k - s > 2$, for any $\epsilon$ such that $0 < \epsilon < \frac{2}{3}(s - k - 2)$.

Also, at the endpoint we have

Proposition 1.8. Let $\alpha \neq 0, 1$, then the estimate

$$
\| (vw_1w_1)_x \|_{X_{-\frac{1}{2}, -\frac{1}{2} + \epsilon}} \lesssim \| v \|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}} \| w_1 \|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}} \| w_2 \|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}}
$$

(1.6)

and

$$
\| (v_1v_2w)_x \|_{X_{-\frac{1}{2}, -\frac{1}{2} + \epsilon}} \lesssim \| v_1 \|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}} \| v_2 \|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}} \| w \|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}}
$$

(1.7)

fails to hold whenever $\epsilon > 0$.  

Also, at the endpoint we have
2. Function spaces and preliminary estimates. In this section we fix some notations, define the function spaces and recall some preliminary results. First, we introduce the integral equations associated to the system (1.1),

\[ v(t) = U(t)\phi - \int_0^t U(t-t')\partial_x(vw^2)(t')dt', \]  
\[ w(t) = U^\alpha(t)\psi - \int_0^t U^\alpha(t-t')\partial_x(v^2w)(t')dt', \]  

where \( U^\alpha(t) := e^{-it\alpha\partial_x^2} \) is the unitary group associated to the linear problem \( \partial_t u + \alpha\partial_x^2 u = 0 \) and defined via Fourier transform by \( U^\alpha(t)\phi = \{ e^{it\alpha(\cdot)^2} \phi(\cdot) \} \). Here \( U(t) \) denotes \( U^1(t) \). In order to use the Fourier restriction norm method and prove the local result, we introduce the Bourgain space \( X^\alpha_{s,b} \), for \( s, b \in \mathbb{R} \), to be the completion of the Schwartz class \( \mathcal{S}(\mathbb{R}^2) \) under the norm

\[ \| f \|_{X^\alpha_{s,b}} := \| U^\alpha(t)f \|_{H^s_t(\mathbb{R}^2)} = \| \langle \xi \rangle^s \langle \tau - \alpha\xi^2 \rangle^b \hat{f}(\tau,\xi) \|_{L^2_{\tau,\xi}}, \]  

where \( \langle \cdot \rangle := 1 + |\cdot| \) and \( \hat{f} \) is the Fourier transform in \((t,x)\) variable

\[ \hat{f}(\tau,\xi) := c \int_{\mathbb{R}^2} e^{-ix\xi + i\tau} f(t,x) dtdx. \]

Hereafter, for \( \alpha = 1 \) we will use \( X_{s,b} \) instead of \( X^1_{s,b} \). If \( b > 1/2 \), we have that \( X^\alpha_{s,b} \to C(\mathbb{R} : H^s_t(\mathbb{R})) \) (see [14], Lemme 3.3) and thus for an interval \( I = [-\delta, \delta] \), we can define the restricted bourgain spaces \( X^\alpha_{s,b,\delta} \) endowed with the norm

\[ \| f \|_{X^\alpha_{s,b,\delta}} = \inf \{ \| g \|_{X^\alpha_{s,b}} ; \ g|_{[-\delta, \delta]} = f \}. \]

For simplicity, we write \( X_{s,b,\delta} \) instead of \( X^1_{s,b,\delta} \).

Now we recall some linear estimates, which are very useful when we apply the Fourier Restriction norm method. Let \( \eta \) be a smooth function supported on the interval \([-2, 2]\) such that \( \eta(t) = 1 \) for all \( t \in [-1, 1] \). We denote, for each \( \delta > 0 \), \( \eta_\delta(t) = \eta(t/\delta) \). The following estimates holds (see e.g. [18] or [15])

**Lemma 2.1.** Let \( \delta > 0 \), \( s \in \mathbb{R} \) and \(-1/2 < b' \leq 0 \leq b \leq b' + 1 \). Then we have

(i) \( \| \eta(t)U^\alpha(t)\phi \|_{X^\alpha_{s,b}} \lesssim \| \phi \|_{H^s_t} \),

(ii) \( \left\| \eta_\delta(t) \int_0^t U^\alpha(t-t')f(t')dt' \right\|_{X^\alpha_{s,b}} \lesssim \delta^{1-b+b'}\| f \|_{X^\alpha_{s,b,b'}} \).

The following lemma will be useful in the proof of the trilinear estimates

**Lemma 2.2.** (i) If \( a, b > 0 \) and \( a + b > 1 \), we have

\[ \int_{\mathbb{R}} \frac{dx}{\langle x - \alpha \rangle^a (x - \beta)^b} \lesssim \frac{1}{(\alpha - \beta)^c}, \quad c = \min\{a, b, a + b - 1\}. \]  

(2.4)
holds for all test functions $f$.

Let $a, \eta \in \mathbb{R}$, $a, \eta \neq 0$, $b > 1$, then
\[
\int_{\mathbb{R}} \frac{dx}{\langle x^2 - \eta^2 \rangle^b} \lesssim \frac{1}{|a\eta|},
\]  
(2.5)

(iii) Let $a, \eta \in \mathbb{R}$, $a, \eta \neq 0$, $b > 1$, then
\[
\int_{\mathbb{R}} \frac{|x \pm \eta| dx}{\langle x^2 \pm \eta^2 \rangle^k} \lesssim \frac{1}{|a|},
\]  
(2.6)

(iv) For $l > 1/3$,
\[
\int_{\mathbb{R}} \frac{dx}{(x^3 + a_2 x^2 + a_1 x + a_0)^l} \lesssim 1.
\]  
(2.7)

Proof. The proof of (2.4) can be found in [21], (2.5) and (2.6) in [7] and (2.7) in [4].

3. Trilinear estimates: Proof of Proposition 1.3. In this section, the ideas in [22] play a central role in the proof of Proposition 1.3. First, we recall some notations, results and arguments contained in this paper. Let $k \geq 2$ be an integer, a $[k; \mathbb{R}^{d+1}]$-multiplier is any function $m : \Gamma_k(\mathbb{R}^{d+1}) \rightarrow \mathbb{C}$, where $\Gamma_k(\mathbb{R}^{d+1})$ denotes the hyperplane $\Gamma_k(\mathbb{R}^{d+1}) = \{ (\xi_1, \ldots, \xi_k) \in (\mathbb{R}^{d+1})^k : \xi_1 + \cdots + \xi_k = 0 \}$ endowed with the measure
\[
\int_{\Gamma_k(\mathbb{R}^{d+1})} f := \int_{(\mathbb{R}^{d+1})^k} f(\xi_1, \ldots, \xi_k, -\xi_1 - \cdots - \xi_k) d\xi_1 d\xi_2 \cdots d\xi_k.
\]

The norm of a $[k; \mathbb{R}^{d+1}]$-multiplier $m$, denoted by $\|m\|_{[k; \mathbb{R}^{d+1}]}$, is the best constant such that
\[
\left| \int_{\Gamma_k(\mathbb{R}^{d+1})} m(\xi) \prod_{j=1}^n f_j(\xi_j) \right| \leq \|m\|_{[k; \mathbb{R}^{d+1}]} \prod_{j=1}^k \|f_j\|_{L^2(\mathbb{R}^{d+1})},
\]
holds for all test functions $f_j$ on $\mathbb{R}^{d+1}$.

Now we start to work on our trilinear estimates. By duality and Plancherel (see e.g. [22] or [9]), one can see that the estimate (1.4) is equivalent to
\[
\left| \int_{\xi_1 + \cdots + \xi_4 = 0} m(\xi_1, \tau_1, \cdots, \xi_4, \tau_4) \prod_{j=1}^4 f_j(\xi_j, \tau_j) \right| \lesssim \prod_{j=1}^4 \|f_j\|_{L^2},
\]  
(3.1)

where
\[
m(\xi_1, \tau_1, \cdots, \xi_4, \tau_4) = \frac{\xi_4(\xi_4)^s}{(\xi_1)^s(\xi_2)^s(\xi_3)^s(\xi_4)^s(\xi_1 - \xi_2^2 - \alpha \xi_2^2)\frac{1}{2} + (\tau_2 - \alpha \xi_2^2)\frac{1}{2} + (\tau_3 - \alpha \xi_3^2)\frac{1}{2} + (\tau_4 - \xi_4^2)\frac{1}{2} - 2r}.
\]  
(3.2)

In this way, recalling the definition of the norm $\|m\|_{[4, \mathbb{R}^2]}$ of the multiplier $m$, the whole matter reduces to showing that
\[
\|m\|_{[4, \mathbb{R}^2]} \lesssim 1.
\]  
(3.3)

Observe that
\[
\xi_4(\xi_4)^s \leq (\xi_4)^{s+1} \leq (\xi_4)^{1/2}(\xi_4)^{s+1/2} \leq (\xi_4)^{1/2}(\xi_4)^{s+1/2} + (\xi_4)^{s+1/2} + (\xi_4)^{s+1/2},
\]  
(3.4)

We define
\[
m_1(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{(\xi_1)^{\frac{1}{2}}}{(\tau_1 - \xi_2^2 - 2r)(\xi_2)^k(\tau_2 - \alpha \xi_2^2)^{\frac{1}{2} + r}},
\]  
(3.5)
Observe that the whole matter reduces to showing that it is enough to bound

We define

From (3.4) and (3.2), we get

Therefore, we have

Now, using comparison principle, permutations and composition properties (see respectively Lemmas 3.1, 3.3 and 3.7 in [22]), it is enough to bound $\|m_j\|_{L^2(\mathbb{R}^2)}$, $j = 1, 2$, or equivalently, to show the following bilinear estimates

This equivalence can be proved using again duality and similar calculations as the ones used to obtain (3.1).

Similarly, the estimate (1.5) is equivalent to

where

In this way, recalling the definition of the norm $\|M\|_{L^2(\mathbb{R}^2)}$ of the multiplier $M$, the whole matter reduces to showing that

Observe that

We define

\[
M_1(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{\langle \xi_1 \rangle^{\frac{1}{2}}}{\langle \tau_1 - \alpha \xi_1^3 \rangle^{\frac{1}{2} - 2\epsilon}} \langle \xi_2 \rangle^{\frac{1}{2} + \epsilon} \langle \xi_2^3 \rangle^{\frac{1}{2} - \epsilon} \langle \tau_2 - \alpha \xi_2^3 \rangle^{\frac{1}{2} + \epsilon}.
\]  

\[
M_2(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{\langle \xi_1 \rangle^{k + \frac{1}{2}}}{\langle \tau_1 - \alpha \xi_1^3 \rangle^{\frac{1}{2} + \epsilon}} \langle \xi_2 \rangle^{\frac{1}{2} + \epsilon} \langle \xi_2^3 \rangle^{\frac{1}{2} - \epsilon} \langle \tau_2 - \alpha \xi_2^3 \rangle^{\frac{1}{2} + \epsilon}.
\]
From (3.16) and (3.14), we get

\[ M \leq \frac{\langle \xi_4 \rangle^{\frac{1}{2}}}{\langle \tau_4 - \alpha \xi_4^3 \rangle^{\frac{1}{2}}} \langle \tau_2 - \xi_2^3 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2}} \]

(3.19)

\[ + \frac{\langle \xi_4 \rangle^{\frac{1}{2}}}{\langle \tau_4 - \alpha \xi_4^3 \rangle^{\frac{1}{2}}} \langle \tau_2 - \xi_2^3 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \]

\[ + \frac{\langle \xi_4 \rangle^{\frac{1}{2}}}{\langle \tau_4 - \alpha \xi_4^3 \rangle^{\frac{1}{2}}} \langle \tau_2 - \xi_2^3 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}} \]

\[ =: I_1 + I_2 + I_3. \]

Therefore, we have

\[ I_1 \leq M_1(\xi_4, \tau_4, \xi_2, \tau_2) M_2(\xi_1, \tau_1, \xi_3, \tau_3), \]

(3.20)

\[ I_2 \leq M_1(\xi_4, \tau_4, \xi_1, \tau_1) M_2(\xi_2, \tau_2, \xi_3, \tau_3), \]

(3.21)

\[ I_3 \leq M_1(\xi_4, \tau_4, \xi_1, \tau_1) M_1(\xi_3, \tau_3, \xi_2, \tau_2). \]

(3.22)

Analogously, to prove that \(|M_j|_{[2, \mathbb{R}^2]} \lesssim 1, j = 1, 2\) is respectively equivalent to show the following bilinear estimates

\[ \|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X^{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 2}} \|v\|_{X^{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 2}}, \]

(3.23)

and

\[ \|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X^{-k, -\frac{1}{2}, \frac{1}{2}, 2}} \|v\|_{X^{-k, -\frac{1}{2}, \frac{1}{2}, 2}}. \]

(3.24)

Therefore, the proof of Proposition 1.3 follows if we proof the four bilinear estimates (3.11), (3.12), (3.23) and (3.24). In fact we prove the following propositions

**Proposition 3.1.** Let \( s > -1/2, \alpha \neq 1 \) and \( 0 < \epsilon < \min \big\{ \frac{2s+1}{15}, \frac{1}{6} \big\} \). Then we have the bilinear estimates

\[ \|fg\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{X^{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 2}} \|g\|_{X^{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 2}}, \]

(3.25)

and

\[ \|fg\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{X^{-k, -\frac{1}{2}, -\frac{1}{2}, 2}} \|g\|_{X^{-k, -\frac{1}{2}, \frac{1}{2}, 2}}. \]

(3.26)

**Proposition 3.2.** Let \( \alpha \in (-\infty, 0) \cup (1, +\infty) \) and \( 0 < \epsilon < \min \big\{ \frac{2s+1}{15}, \frac{1}{6} \big\} \). Then

(a) The inequality (3.12) holds for any \((s, k)\) in the region:

\[ R_1 = \{(s, k); \ k, s > -1/2, \ s - k \leq 1/2 \} \cup \{(s, k); \ -1/2 < k, \ s = -1/2 \}. \]

(b) The inequality (3.24) holds for any \((s, k)\) in the region:

\[ R_2 = \{(s, k); \ k, s > -1/2, \ s - k \geq -1/2 \} \cup \{(s, k); \ -1/2 < s, \ k = -1/2 \}. \]

Before we prove these results, we establish some preliminary results.

**Lemma 3.3.** Let \( \alpha < 1, \ s > -1/2 \) and \( 0 < \epsilon < \min \big\{ \frac{2s+1}{15}, \frac{1}{6} \big\} \). Then we have

\[ \sup_{\xi, \tau} \int_{\mathbb{R}^t} (\xi_2 - \xi_1) \langle \xi_2 \rangle \langle \xi_1 \rangle^{2s}(\tau - \xi_2^3 - \alpha \xi_1^3)^{1-\epsilon} d\xi_1 \lesssim 1, \]

(3.27)

where \( \xi_2 := \xi - \xi_1 \), for a fixed \( \xi \in \mathbb{R} \).
Proof. For the case $0 < \alpha < 1$, see in [9], Lemma 3.2, the estimative of the item labeled by them as (3.12). We will prove the case $\alpha < 0$. We denote by $L_1$ the integral in (3.27). For a fixed $\xi$ and $\tau$, let

$$H(\xi_1) := \tau - \xi_2^3 - \alpha \xi_1^3 = \tau - \xi_2^3 + |\alpha|\xi_1^3.$$

(3.28)

We have

$$H'(\xi_1) = 3[\xi_2^2 + |\alpha|\xi_1^2] \geq 0.$$

Thus, the function $\xi_1 \mapsto H(\xi_1)$ is monotone on $\mathbb{R}$. We divide the proof into the following two cases:

**Case 1.** ($|\xi| < 2|\xi_1|$) In this case, we have $\langle \xi_2 \rangle \lesssim \langle \xi_1 \rangle$. Thus

$$\chi_{(|\xi_1| < 2|\xi_1|)} L_1 = \int_{|\xi_1| > |\xi|/2} \frac{\langle \xi_2 \rangle \langle \xi_1 \rangle}{\langle \xi_1 \rangle^2} d\xi_1 \lesssim \int_{\mathbb{R}} \frac{\langle \xi_1 \rangle^2}{\langle \xi_1 \rangle^2 + |\alpha|\xi_1^2} d\xi_1$$

$$= \int_{\mathbb{R}} \frac{1}{\langle \xi_1 \rangle^2 + |\alpha|\xi_1^2} d\xi_1 + \int_{\mathbb{R}} \frac{\xi_1^2}{\langle \xi_1 \rangle^2 + |\alpha|\xi_1^2} d\xi_1$$

$$= J_1 + J_2. \quad (3.29)$$

Using (2.7) we have $J_1 \lesssim 1$ provided $0 < \epsilon < \frac{1}{3}$. In what follows, we estimate $J_2$ integrating over: (a) $|\xi_1| < 1$ and (b) $|\xi_1| \geq 1$, separately. In the first situation, using that $2s + 1 \geq 0$, we have

$$\chi_{(|\xi_1| < 1)} J_2 \lesssim \int_{|\xi_1| < 1} \frac{1}{H(\xi_1)^{1 - 4\epsilon}} d\xi_1 \lesssim 1. \quad (3.30)$$

For the second case, considering the sets

$$A = \{ \xi_1 : \langle H(\xi_1) \rangle \lesssim |\xi_1|^3 \} \quad \text{and} \quad B = \{ \xi_1 : \langle H(\xi_1) \rangle \gtrsim |\xi_1|^3 \},$$

we have

$$\chi_{(|\xi_1| \geq 1)} J_2 = \int_{|\xi_1| \geq 1} \frac{\xi_1^2}{\langle \xi_1 \rangle^{2s+1} + H(\xi_1)^{1 - 4\epsilon}} \chi_a(\xi_1) d\xi_1 + \int_{|\xi_1| \geq 1} \frac{\xi_1^2}{\langle \xi_1 \rangle^{2s+1} + H(\xi_1)^{1 - 4\epsilon}} \chi_b(\xi_1) d\xi_1$$

$$\lesssim \int_{|\xi_1| \geq 1} \frac{H'(\xi_1)|\xi_1|}{\langle \xi_1 \rangle^{2s+1} + H(\xi_1)^{1 - 4\epsilon}} \chi_a(\xi_1) d\xi_1 + \int_{|\xi_1| \geq 1} \frac{|\xi_1|^2}{|\xi_1|^{2s+1} + |\xi_1|^{3 - 12\epsilon}} d\xi_1$$

$$\lesssim \int_{\mathbb{R}} \frac{H'(\xi_1)|\xi_1|}{\langle \xi_1 \rangle^{2s+1} + H(\xi_1)^{1 - 4\epsilon}} d\xi_1 + \int_{|\xi_1| \geq 1} \frac{|\xi_1|^2}{|\xi_1|^{2s+1} + |\xi_1|^{3 - 12\epsilon}} d\xi_1, \quad (3.31)$$

where in the first integral we use that $\xi_1^2 \lesssim H'(\xi_1)$. Obviously, the second integral is $\lesssim 1$, provided $0 < \epsilon < \frac{2s+1}{12}$. For the first integral, via the change of variables $x = H(\xi_1)$ on $\mathbb{R}$, if $0 < \epsilon < \frac{2s+1}{15}$, we obtain

$$\int_{\mathbb{R}} \frac{H'(\xi_1)|\xi_1|}{\langle \xi_1 \rangle^{2s+1} + H(\xi_1)^{1 - 4\epsilon}} d\xi_1 \lesssim \int_{\mathbb{R}} \frac{dx}{(x)^{1+\epsilon}} \lesssim 1. \quad (3.32)$$

**Case 2.** ($2|\xi_1| \leq |\xi|$) Since $\xi_1 + \xi_2 = \xi$, we have

$$\max \{ \langle \xi_1 \rangle, \langle \xi_2 \rangle \} \lesssim \langle \xi \rangle, \quad (3.33)$$

$$\langle \xi_2 \rangle \langle \xi \rangle \lesssim \langle \xi_2 \rangle^2 \lesssim 1 + H'(\xi_1). \quad (3.34)$$
Fix $0 < \epsilon < 1/9$, $b = (1 - 4\epsilon)^{-1}$ and $a$ such that $a + b = 3$. Considering the sets

$$A = \left\{ \xi_1 : \langle H(\xi_1) \rangle \lesssim \langle \xi_1 \rangle^a \langle \xi \rangle^b \right\} \quad \text{and} \quad B = \left\{ \xi_1 : \langle H(\xi_1) \rangle \gtrsim \langle \xi_1 \rangle^a \langle \xi \rangle^b \right\},$$

we have

$$\chi_{|\xi_1| \leq |\xi|} L_1 = \int_{|\xi_1| \leq |\xi|/2} \frac{\langle \xi_2 \rangle}{\langle \xi_2 \rangle} \frac{\langle \xi \rangle}{\langle \xi \rangle} \chi_2(\xi_1) d\xi_1 + \int_{|\xi_1| \leq |\xi|/2} \frac{\langle \xi_2 \rangle}{\langle \xi \rangle} \frac{\langle \xi \rangle}{\langle \xi \rangle} \chi_2(\xi_1) d\xi_1 \leq \int_{|\xi_1| \leq |\xi|/2} \frac{1}{\langle \xi_2 \rangle} \frac{\langle \xi \rangle}{\langle \xi \rangle} \chi_2(\xi_1) d\xi_1 + \int_{|\xi_1| \leq |\xi|/2} \frac{\langle \xi \rangle}{\langle \xi \rangle} \chi_2(\xi_1) d\xi_1 \leq J_1 + J_2 + J_3.$$

By (2.7) we have $J_1 \lesssim 1$ provided $0 < \epsilon < \frac{1}{6}$ and $2s + 1 \geq 0$. For $J_3$, recalling the definitions of $a$ and $b$, if $0 < \epsilon < (2s + 1)/12$, we have

$$J_3 \lesssim \int_{\mathbb{R}^1} \frac{d\xi_1}{\langle \xi_1 \rangle^{2s + 2s - 12\epsilon}} \lesssim 1.$$

For $J_2$, taking account (3.33), if $0 < \epsilon < 1/9$, we have $1 - 5be > 0$, and so $\langle \xi_1 \rangle^{1 - 5be} \lesssim \langle \xi \rangle^{1 - 5be}$. Thus, considering that $2s + 1 \geq 0$ and the definition of $a$ and $b$, if $0 < \epsilon < (2s + 1)/15$ we have

$$J_2 = \int_{|\xi_1| \leq |\xi|/2} \frac{\langle \xi \rangle}{\langle \xi_1 \rangle} \chi_2(\xi_1) d\xi_1 \lesssim \int_{|\xi_1| \leq |\xi|/2} \frac{\langle \xi \rangle}{\langle \xi_1 \rangle} \chi_2(\xi_1) d\xi_1 \lesssim \int_{|\xi_1| \leq |\xi|/2} \frac{\langle \xi \rangle}{\langle \xi_1 \rangle} \chi_2(\xi_1) d\xi_1 \lesssim \int_{\mathbb{R}^1} \frac{\langle \xi \rangle}{\langle \xi_1 \rangle} d\xi_1.$$

Again, making the change of variables $x = H(\xi_1)$, we get the desired bound.

Now we are in position to prove Proposition 3.1.

**Proof of the Proposition 3.1.** First of all, we recall that the case $0 < \alpha < 1$ was proved in [9]. So, let us assume that $\alpha \in (-\infty, 0) \cup (1, \infty)$. We start to prove that the inequality (3.25) hold if $\alpha > 1$. Let $u \in X^{-\frac{1}{2}, \frac{1}{2} - 2\epsilon}$ and $v \in X_{s, \frac{1}{2} + \epsilon}^\alpha$ with $\epsilon > 0$ and $s > -\frac{1}{2}$. Considering $f$ and $g$ such that

$$u(x, t) = f(\alpha^{-1/3} x, t), \quad v(x, t) = g(\alpha^{-1/3} x, t),$$

using that $\langle a \xi \rangle \sim \langle \xi \rangle$, for $a \neq 0$, and scaling properties of the Fourier transform, we have

$$\|f\|_{X^{-\frac{1}{2}, \frac{1}{2} - 2\epsilon}} \sim \|u\|_{X^{-\frac{1}{2}, \frac{1}{2} - 2\epsilon}} \quad \text{and} \quad \|g\|_{X_{s, \frac{1}{2} + \epsilon}^\alpha} \sim \|v\|_{X_{s, \frac{1}{2} + \epsilon}^\alpha},$$

thus $f \in X^{1/\alpha - \frac{1}{2}, \frac{1}{2} - 2\epsilon}$ and $g \in X_{s, \frac{1}{2} + \epsilon}^\alpha$. Because $1/\alpha \in (0, 1)$ we can apply the estimate (3.26) to obtain

$$\|fg\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{X^{1/\alpha - \frac{1}{2}, \frac{1}{2} - 2\epsilon}} \|g\|_{X_{s, \frac{1}{2} + \epsilon}^\alpha} \lesssim \|u\|_{X^{-\frac{1}{2}, \frac{1}{2} - 2\epsilon}} \|v\|_{X_{s, \frac{1}{2} + \epsilon}^\alpha}.$$
We conclude that (3.25) holds for \( \alpha > 1 \), observing that \( \|fg\|_{L^2(\mathbb{R}^2)} = \alpha^{\frac{2}{3}} \|uv\|_{L^2(\mathbb{R}^2)} \).

Now, if \( \alpha < 0 \), using Plancherel’s identity, one can see that the estimate (3.25) is equivalent to

\[
\|B_s(f, g)\|_{L^2} \leq C \|f\|_{L^2} \|g\|_{L^2},
\]

where

\[
B_s(f, g) = \int_{\mathbb{R}^2} \frac{1}{\langle \xi_1 \rangle^{\frac{2}{3}} \langle \xi_2 \rangle} \langle \xi_1 \rangle^{\frac{1}{3}} \langle \xi_2 \rangle^{\frac{1}{3}} \langle \xi_1 \rangle^{\frac{1}{3}} \langle \tau_1 \rangle^{\frac{1}{3}} \langle \tau_2 \rangle^{\frac{1}{3}} \frac{d\xi_1 d\tau_1}{\langle \tau_2 - \xi_1 \rangle^{\frac{1}{3}}},
\]

with \( \xi_2 = \xi - \xi_1, \tau_2 = \tau - \tau_1 \). Using Cauchy-Schwarz inequality we note that (3.36) holds if

\[
L_1 := \sup_{\xi, \tau} \int_{\mathbb{R}^2} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2\alpha} \langle \tau - \xi_1 \rangle^{\frac{1}{3}} \langle \tau_2 - \xi_2 \rangle^{\frac{1}{3}} \langle \xi_1 \rangle^{1+\alpha} d\xi_1 \lesssim 1.
\]

In order to see that the estimate (3.38) holds, we apply the estimate (2.4) (Lemma 2.2) to the integral in \( \tau_1 \), and so we obtain

\[
L_1 \lesssim \sup_{\xi, \tau} \int_{\mathbb{R}^2} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2\alpha} \langle \tau - \xi_1 \rangle^{\frac{1}{3}} \langle \tau_2 - \xi_2 \rangle^{\frac{1}{3}} \langle \xi_1 \rangle^{1+\alpha} d\xi_1,
\]

if \( 0 < \epsilon < \frac{1}{4} \). Applying now (3.27), we get the desired bound and finish this case. In the same way we can prove the inequality (3.26) for \( \alpha < 0 \) or \( \alpha > 1 \). 

The next results are useful in the proof of the Proposition 3.2.

**Lemma 3.4.** Let \( l \geq -1/2 \) and \( b > 1/2 \). Considering \( F \) a monotone function defined on a Lebesgue-measurable set \( X \subset \mathbb{R} \) such that

\[
|F'(\xi)| \gtrsim \max\{\xi_1^2, \xi_2^2\}, \quad \forall \xi_1 \in X,
\]

where \( \xi_2 = \xi - \xi_1 \), for a fixed \( \xi \in \mathbb{R} \). We have

\[
\int_X \frac{\langle \xi_1 \rangle}{\langle \xi_2 \rangle^{2l} \langle F(\xi_1) \rangle^{2b}} d\xi_1 \lesssim \int_X \frac{1}{\langle F(\xi_1) \rangle^{2b}} d\xi_1 + 1,
\]

for all \( i, j \in \{1, 2\} \).

**Proof.** Considering \( F \) a monotone nondecreasing function, we have that \( F'(\xi_1) \gtrsim \xi_2^2 \), for \( k \in \{1, 2\} \). So

\[
\langle \xi_i \rangle \langle \xi_j \rangle \lesssim \langle \xi_i \rangle^2 + \langle \xi_j \rangle^2 \lesssim 1 + F'(\xi),
\]

for all \( i, j \in \{1, 2\} \). Hence

\[
\int_X \frac{\langle \xi_i \rangle}{\langle \xi_j \rangle^{2l} \langle F(\xi_1) \rangle^{2b}} d\xi_1 = \int_X \frac{\langle \xi_i \rangle \langle \xi_j \rangle}{\langle \xi_j \rangle^{2l+1} \langle F(\xi_1) \rangle^{2b}} d\xi_1
\]

\[
\lesssim \int_X \frac{1}{\langle \xi_1 \rangle^{2l+1} \langle F(\xi_1) \rangle^{2b}} d\xi_1 + \int_X \frac{F'(\xi)}{\langle \xi_1 \rangle^{2l+1} \langle F(\xi_1) \rangle^{2b}} d\xi_1
\]

\[
\lesssim \int_X \frac{1}{\langle F(\xi_1) \rangle^{2b}} d\xi_1 + \int_X \frac{F'(\xi)}{\langle F(\xi_1) \rangle^{2b}} d\xi_1
\]

and making the change of variable \( x = F(\xi_1) \), we finish the proof.

**Lemma 3.5.** Let \( b > 1/2 \), \( l \geq -1/2 \) and \( \alpha \in (-\infty, 0) \cup (1, +\infty) \), then

\[
J := \int_B \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2l} \langle H(\xi) \rangle^{2b}} d\xi_1 \lesssim 1,
\]

(4.41)
where \( B = \{ \xi_1; \langle \xi_2 \rangle > \frac{1+i}{\langle \xi_1 \rangle} \} \), \( \iota \) as defined in (3.46), \( \xi_2 = \xi - \xi_1 \) and
\[
H(\xi_1) := \tau - \xi_1^3 - \alpha \xi_2^3.
\] (3.42)

Proof. Of course, (3.41) holds when \( \alpha < 0 \) (see the proof of Lemma 3.4). Thus we will suppose \( \alpha > 1 \). First we will consider \( |\xi| > 1 \) and let
\[
X = \{ \xi_1; \ |\xi_1| = \iota |\xi| \},
\]
where \( \iota \in (0, 1) \) will be chosen later, we have
\[
J \leq \int_{B} \frac{\langle \xi_2 \rangle}{\langle \xi_2 \rangle^{2i}(H(\xi_1))^{2i}} \chi_X(\xi_1) d\xi_1 + \int_{B} \frac{\langle \xi_2 \rangle}{\langle \xi_2 \rangle^{2i}(H(\xi_1))^{2i}} \chi_{R\setminus X}(\xi_1) d\xi_1.
\]
Note that in \( R \setminus X = \{ \xi_1; \ |\xi_1| \geq \iota |\xi| \} \), we have \( \langle \xi_2 \rangle \leq \frac{1+i}{\iota} \langle \xi_1 \rangle \), then \( B \cap (R \setminus X) = \emptyset \) and therefore \( J_2 \equiv 0 \). Now, for \( J_1 \), we have
\[
\langle \xi_1 \rangle \lesssim \langle \xi_2 \rangle \lesssim \langle \xi \rangle.
\] (3.43)

We choose \( \lambda \in (0, 3) \) such that
\[
H'(\xi_1) = 3(\alpha - 1)\xi_1^2 - 6\alpha \xi_1 \xi + 3\alpha \xi^2 \geq \lambda \alpha \xi^2.
\] (3.44)
So, we have that \( H \) is increasing in \( X \) and \( |H'(\xi_1)| \gtrsim \xi^2 \geq \alpha \xi_1 \xi \max \{ \xi_1^2, \xi_2^2 \} \), for all \( \xi_1 \in X \). Thus, combining Lemma 3.4 and Lemma 2.2 item (iv), we get the desired estimation.

Now, we determine \( \iota \in (0, 1) \) and \( \lambda \in (0, 3) \) such that (3.44) is valid. Note, the inequality (3.44) is equivalent to
\[
\frac{3(1 - \alpha)\xi_1^2 + 6\alpha \xi_1 \xi}{\text{constant function}} \leq \frac{(3 - \lambda)\alpha \xi^2}{\text{quadratic function}}.
\] (3.45)
So we choose \( \iota \) such that the equality (3.45) is true in the interval \( |\xi_1| \leq \iota |\xi| \). Thus, \( \iota \) is a root of the quadratic equation
\[
3(1 - \alpha)\iota^2 + 6\alpha \iota + (\lambda - 3)\alpha = 0.
\]
One can see that
\[
\iota = (6\alpha - \sqrt{36\alpha^2 - 12\alpha(1 - \alpha)(\lambda - 3)})/6(\alpha - 1)
\] (3.46)
belongs to \( (0, 1) \) if \( \lambda \in (0, 3) \).

Now if \( |\xi| \leq 1 \), we have
\[
\frac{1}{2} \langle \xi_2 \rangle \leq \langle \xi_1 \rangle \leq 2 \langle \xi_2 \rangle
\]
and using Lemma 3.3 we get
\[
J \leq \int_{R} \frac{\langle \xi_2 \rangle}{\langle \xi_2 \rangle^{2i}(\tau - \xi_1^3 - \alpha \xi_2^3)^{2i}} d\xi_1 \lesssim \int_{R} \frac{\langle \xi_2 \rangle}{\langle \xi_2 \rangle^{2i}(\tau - \frac{1}{\alpha} \xi_1^3 - \frac{1}{\alpha^3} \xi_2^3)^{2i}} d\xi_1 \lesssim 1.
\] (3.47)

\[\square\]

Remark 3.6. Observe that in (3.44) we need \( \alpha \neq 0 \), because \( \alpha \) is a multiplicative constant, we have similar situations in the successive proofs. Moreover, note that if \( \alpha = 0 \), (3.50) is only true when \( s > 1/2 \), but we suppose that \( s > 1/2 \).

With these Lemmas in hands, we can prove Proposition 3.2 and therefore the trilinear estimates in Proposition 1.3.
Proof of the Proposition 3.2. We only provide a detailed proof of (a), because (b) is analogous. Suppose that $s, k > -\frac{1}{2}$ and $s - k \leq 1/2$. As before, using Plancherel identity and Cauchy-Schwarz inequality we need to estimate

$$\int_{\mathbb{R}^2} \langle \xi_2 \rangle^{2(s-k)+1} \langle \tau_2 - \alpha \xi_2^b \rangle^{2b} \langle \xi_1 \rangle^{2s} \langle \tau_1 - \xi_1^b \rangle^{2b} d\xi_1 d\tau_1,$$

where $\xi_2 = \xi - \xi_1$, $\tau_2 = \tau - \tau_1$, for $\tau$ and $\xi$ fixed and $b = 1/2 + \epsilon$. Integrating in $\tau_1$ and applying (2.4), we need to estimate the following integral

$$L_2 = \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{1+2(s-k)}}{\langle \xi_1 \rangle^{2s} \langle H(\xi_1) \rangle^{2b}} d\xi_1,$$

where $H(\xi_1)$ is given by (3.42). Our goal is to prove $L_2 \lesssim 1$. First considering the case $s - k \leq 0$, we have

$$L_2 \lesssim \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2s} \langle H(\xi_1) \rangle^{2b}} d\xi_1.$$

If $\alpha > 1$, we write $\frac{1}{\alpha} H(\xi_1) = \hat{\tau} - \xi_2^3 - \beta \xi_1^3$, where $\hat{\tau} = \tau/\alpha$ and $\beta = 1/\alpha$. Applying (3.27), with $\beta \in (0, 1)$ in place of $\alpha$, we get $L_2 \lesssim 1$, remembering that $2b > 1 - 4\epsilon$. On the other hand, if $\alpha < 0$ we can see that $H'(\xi_1) = 3\alpha \xi_2^2 - 3\xi_1^2 < 0$ and $|H'(\xi_1)| \gtrsim_{\alpha} \max(\xi_2^2, \xi_1^2)$. So, combining Lemma 3.4 and Lemma 2.2 item (iv), we get the same bound.

Considering now $0 < s - k \leq 1/2$, let

$$A = \{\xi_1; \langle \xi_2 \rangle \lesssim \langle \xi_1 \rangle\} \quad \text{and} \quad B = \{\xi_1; \langle \xi_1 \rangle \lesssim \langle \xi_2 \rangle\}.$$

Thus

$$L_2 \leq \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{1+2(s-k)}}{\langle \xi_1 \rangle^{2s} \langle H(\xi_1) \rangle^{2b}} \chi_A(\xi_1) d\xi_1 + \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{1+2(s-k)}}{\langle \xi_1 \rangle^{2s} \langle H(\xi_1) \rangle^{2b}} \chi_B(\xi_1) d\xi_1.$$

For the first integral, we have

$$L_2^A \lesssim \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2k} \langle H(\xi_1) \rangle^{2b}} d\xi_1,$$

and this integral has already been estimated in (3.50). For the second integral $L_2^B$, similarly as above one can see that if $s \leq 0$, using Lemma 3.5 we have

$$L_2^B \lesssim \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2(k-s)} \langle H(\xi_1) \rangle^{2b}} d\xi_1 \lesssim 1.$$

On the other hand, if $s > 0$ again using Lemma 3.5 we have

$$L_2^B \lesssim \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2(k-s)} \langle H(\xi_1) \rangle^{2b}} d\xi_1 \lesssim 1,$$

since $k - s \geq -1/2$. Now, we prove the same estimate in the range: $s = -1/2$ and $-1/2 < k$. Indeed, we need to prove that

$$\|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X_{k, \frac{1}{2} + \epsilon}} \|v\|_{X_{-\frac{1}{2}, \frac{1}{2} + \epsilon}}.$$

This estimate follows from the estimate (3.25), noting that $1/2 - 2\epsilon < 1/2 + \epsilon$. □
4. Local well-posedness result: Proof of Theorem 1.1. In view of the previous sections, we are in position to prove the local well-posedness result given in Theorem 1.1. Here we understand that a solution of the system (1.1) is in fact a solution of the associated integral equations (2.1) and (2.2). We use standard arguments, so we give the proof for the sake of completeness.

Proof of Theorem 1.1. Let $s, k \in \mathbb{R}$ such that $s, k > -1/2$ and $|s - k| < 1/2$. Let $b = 1/2 + \epsilon$ and $b' = -1/2 + 2\epsilon$, with $\epsilon > 0$. Let fix $\alpha \in (-\infty, 0) \cup (1, \infty)$ (see Remark 1.2). For $a > 0$, to be chosen later, let

$$B_a = \{(v, w) \in X_{s,b} \times X_{k,b}^\alpha; \|(v, w)\|_{X_{s,b} \times X_{k,b}^\alpha} = \|v\|_{X_{s,b}} + \|w\|_{X_{k,b}^\alpha} < a\},$$

a complete metric spaces. For $\delta > 0$ (chosen later), we define the map $\mathfrak{G} = \mathfrak{G}_\delta : B_a \to X_{s,b} \times X_{k,b}^\alpha$ such that the solution mapping

$$\mathfrak{G}(v, w) = \mathfrak{G}_1(v, w),$$

where

$$\mathfrak{G}_1(v, w) = \eta(t)U(t)\phi - \eta_0(t) \int_0^t U(t - t')\partial_x(vw^2)(t') dt'. $$

From the linear estimates in Lemma 2.1 and the trilinear estimates in Proposition 1.3, for all $(v, w) \in B_a$ we get

$$\|\mathfrak{G}(v, w)\|_{X_{s,b} \times X_{k,b}^\alpha} \leq C\|\phi, \psi\|_{H^s \times H^k} + C\delta^{b + b'} \left(\|v\|_{X_{s,b}} + \|w\|_{X_{k,b}^\alpha}\right) \leq 2C\|\phi, \psi\|_{H^s \times H^k} + 2C\delta^\alpha a^3. $$

Now, taking $a = 2C\|\phi, \psi\|_{H^s \times H^k}$ and $\delta > 0$ such that $2C\delta^\alpha < 1/2$ we conclude that

$$\|\mathfrak{G}(v, w)\|_{X_{s,b} \times X_{k,b}^\alpha} \leq a, \quad \text{for all} \quad (v, w) \in B_a,$$

i.e., for $a > 0$ and $\delta > 0$ as before, $\mathfrak{G}(B_a) \subset B_a$. Also, with a similar arguments and taking $\delta > 0$ smaller, if necessary, we can conclude that

$$\|\mathfrak{G}(v, w) - \mathfrak{G}(\tilde{v}, \tilde{w})\|_{X_{s,b} \times X_{k,b}^\alpha} \leq \|(v, w) - (\tilde{v}, \tilde{w})\|_{X_{s,b} \times X_{k,b}^\alpha},$$

i.e., $\mathfrak{G} : B_a \to B_a$ is a contraction, therefore it has an unique fixed point, establishing an unique solution $(v, w)$ satisfying (2.1) and (2.2) for every $t \in [-\delta, \delta]$. Since $b > 1/2$ we have the persistence property $(v, w) \in C([-\delta, \delta]; H^s) \times C([-\delta, \delta]; H^k)$ and also, following a similar arguments, we can conclude that the solution mapping is locally Lipchitz from $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ into $C([-\delta, \delta]; H^s) \times C([-\delta, \delta]; H^k)$. The uniqueness in the class $X_{s,b}^\delta \times X_{k,b}^\alpha$ can be proved with standard arguments (see e.g. [4] or [12]).

5. Ill-posedness results. In this section we prove some ill-posedness results related to the system (1.1).

5.1. The solution mapping is not $C^3$. Here we will prove the Theorem 1.4. It is well known that if the LWP results in $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ for (1.1) is obtained by means of contraction method, then for a fixed $r > 0$ there is a $T = T(r, s, k) > 0$ such that the solution mapping

$$S : \quad B_r \quad \rightarrow \quad C\left([0, T]; H^s\right) \times C\left([0, T]; H^k\right) \quad (5.1)$$

$$(\phi, \psi) \quad \mapsto \quad S(\phi, \psi) = (v_{(\phi, \psi)}, w_{(\phi, \psi)}),$$

where $\mathfrak{G}(v, w) = (\phi, \psi)$.
is analytic (see Theorem 3 in [3]), where $B_r$ is the $r$-ball centered at the origin of $H^r(\mathbb{R}) \times H^k(\mathbb{R})$ and $v = v_{(\phi, \psi)}$ and $w = w_{(\phi, \psi)}$ satisfies, respectively, the integral equations (2.1) and (2.2) for initial data $\phi$ and $\psi$, in the time interval $[0, T]$. In this way, if we show that for a certain indices $(s, k)$ the solution mapping is not three times differentiable at the origin $(0, 0)$ for all $T > 0$ fixed, the contraction method can not be applied to get LWP for these indices $(s, k)$.

Fixing $t \in [0, T]$, we define the flow mapping associated to the system (1.1) the map

$$S^t : B_r \rightarrow H^s(\mathbb{R}) \times H^k(\mathbb{R})$$

$$(\phi, \psi) \mapsto S^t_{(\phi, \psi)} = S_{(\phi, \psi)}(t) = (v_{(\phi, \psi)}(t), w_{(\phi, \psi)}(t)).$$

The Theorem 1.4 follows from the next proposition:

**Proposition 5.1.** Assume that the system (1.1) is locally well-posed in the time interval $[0, T]$. Then we have

(a) The solution mapping (5.1) is not 3-times Fréchet differentiable at the origin in $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ if $|s - k| > 2$

(b) The flow mapping (5.2) is not 3-times Fréchet differentiable at the origin in $H^s(\mathbb{R}) \times H^k(\mathbb{R})$ if $s < -1/2$ or $k < -1/2$.

**Remark 5.2.** We point out that the result in (b) implies that the solution mapping is not 3-times differentiable at $(0, 0)$ for the same indices $s$ and $k$. For a more detailed discussion we refer Remarks 1.4 and 1.5 in [13].

Before we start to prove our results, we need to do some calculations. If $S^t$ is 3-times Fréchet differentiable at the origin in $H^s(\mathbb{R}) \times H^k(\mathbb{R})$, then its third derivative $D^3 S^t_{(0,0)}$ belongs to $B_1$, the normed space of bounded trilinear applications from $(H^s \times H^k) \times (H^s \times H^k)$ to $H^s \times H^k$ and we have the following estimate for the third Gâteaux derivative of $S^t$

$$\left\| \frac{\partial^3 S^t_{(0,0)}}{\partial \Phi_0 \partial \Phi_1 \partial \Phi_2} \right\|_{H^s \times H^k} = \left\| D^3 S^t_{(0,0)}(\Phi_0, \Phi_1, \Phi_2) \right\|_{H^s \times H^k} \leq \left\| D^3 S^t_{(0,0)} \right\|_{B_1} \left\| \Phi_0 \right\|_{H^s \times H^k} \left\| \Phi_1 \right\|_{H^s \times H^k} \left\| \Phi_2 \right\|_{H^s \times H^k},$$

for all $\Phi_0, \Phi_1, \Phi_2 \in H^s \times H^k$. Also, if $S$ is 3-times Fréchet differentiable at the origin, we have a similar estimate:

$$\sup_{t \in [0, T]} \left\| \frac{\partial^3 S^t_{(0,0)}}{\partial \Phi_0 \partial \Phi_1 \partial \Phi_2} \right\|_{H^s \times H^k} = \left\| D^3 S_{(0,0)}(\Phi_0, \Phi_1, \Phi_2) \right\|_{H^s \times H^k} \leq \left\| D^3 S_{(0,0)} \right\|_{B_2} \left\| \Phi_0 \right\|_{H^s \times H^k} \left\| \Phi_1 \right\|_{H^s \times H^k} \left\| \Phi_2 \right\|_{H^s \times H^k},$$

for all $\Phi_0, \Phi_1, \Phi_2 \in H^s \times H^k$, where $B_2$ is the normed space of bounded trilinear applications from $(H^s \times H^k) \times (H^s \times H^k)$ to $C([0, T] ; H^s) \times C([0, T] ; H^k)$.

For $\Phi_m = (\phi_m, \psi_m) \in S(\mathbb{R}) \times S(\mathbb{R})$, $m = 0, 1, 2$, we can calculate the third Gâteaux derivative of each component of $S^t$:

$$\frac{\partial^3 \psi_{(0,0)}}{\partial \Phi_0 \partial \Phi_1 \partial \Phi_2} = -2 \int_0^t U(t-t') \partial_x(U(t') \phi_0 U^{\alpha}(t') \psi_1 U^{\alpha}(t') \psi_2 + U(t') \phi_1 U^{\alpha}(t') \psi_2 U^{\alpha}(t') \psi_0) dt'$$
and
\[
\frac{\partial^3 w_{(0,0)}}{\partial \Phi_0 \partial \Phi_1 \partial \Phi_2} = -2 \int_0^t U^\alpha(t-t') \partial_x \left\{ U(t') \phi_0 U^\alpha(t') \psi_1 U^\alpha(t') \psi_2 + U(t') \phi_0 U(t') \phi_2 U^\alpha(t') \psi_1 + U(t') \phi_1 U(t') \phi_2 U^\alpha(t') \psi_0 \right\} dt.
\]

So, for directions \( \Phi_0 = (\phi_0, 0) \), \( \Phi_1 = (0, \psi_1) \) and \( \Phi_2 = (0, \psi_2) \) in \( \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \) we get
\[
\frac{\partial^3 v_{(0,0)}}{\partial \Phi_0 \partial \Phi_1 \partial \Phi_2} = -2 \int_0^t U(t-t') \partial_x \left\{ U(t') \phi_0 U^\alpha(t') \psi_1 U^\alpha(t') \psi_2 \right\} dt' \tag{5.5}
\]
and for directions \( \Phi_0 = (\phi_0, 0) \), \( \Phi_1 = (\phi_1, 0) \) and \( \Phi_2 = (0, \psi_2) \)
\[
\frac{\partial^3 w_{(0,0)}}{\partial \Phi_0 \partial \Phi_1 \partial \Phi_2} = -2 \int_0^t U^\alpha(t-t') \partial_x \left\{ U(t') \phi_0 U(t') \phi_1 U^\alpha(t') \psi_2 \right\} dt'. \tag{5.6}
\]

With these in hand we also need a elementary result, proved in [12]:

**Lemma 5.3.** Let \( A, B, R \) Lebesgue-measurable subsets of \( \mathbb{R}^n \) such that\(^1 \) \( R - B \subset A \). Then\(^2 \)
\[
\| \chi_A * \chi_B \|_{L^2(\mathbb{R})} \gtrsim |B||R|^{1/2}.
\]

The following lemma, which is a version of the elementary lemma above, plays a central role in the proof of the Proposition 5.1.

**Lemma 5.4.** Let \( A, B, C, R \) Lebesgue-measurable subsets of \( \mathbb{R}^n \) such that \( R - B - C \subset A \). Then
\[
\| \chi_A * \chi_B * \chi_C \|_{L^2(\mathbb{R})} \gtrsim |B||C||R|^{1/2}.
\]

Now we move to the

**Proof of Proposition 5.1.** (a) Suppose that \( S' \) is 3-times differentiable at the origin in \( H^s \times H^k \). Because (5.5) and (5.6) are the components of the third Gâteaux derivative at the origin, we have the same estimate with the \( H^s \times H^k \) norm of (5.5) or (5.6) in the right-hand side of (5.4). We start with the first component. Considering \( A, B, C \subset \mathbb{R} \) bounded subsets and choosing \( \phi_0, \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}) \) such that\(^3 \) \( \langle \cdot \rangle^s \phi_0 \sim \chi_A, \langle \cdot \rangle^k \psi_1 \sim \chi_B \) and \( \langle \cdot \rangle^k \psi_2 \sim \chi_C \), we have that\(^4 \)
\[
(5.4)_L \gtrsim \left\| \int_0^t \frac{\langle \xi \rangle^s \langle \xi \rangle^k \langle \xi_3 \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k \langle \xi_3 \rangle^s} \cos(tQ) \chi_A(\xi_1) \chi_B(\xi_2) \chi_C(\xi_3) d\xi_1 d\xi_2 dt \right\|_{L^2_x} \tag{5.7}
\]
where \( \xi_3 := \xi - \xi_1 - \xi_2 \) and
\[
Q_\alpha = Q_\alpha(\xi, \xi_1, \xi_2) := \xi^3 - \xi_1^3 - \alpha \xi_2^3 - \alpha \xi_3^3. \tag{5.8}
\]
So combining (5.7) with (5.4) we get
\[
\sup_{t \in [0,T]} \left\| \int_0^t \frac{\langle \xi \rangle^s \langle \xi \rangle^k}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k \langle \xi_3 \rangle^s} \cos(tQ) \chi_A(\xi_1) \chi_B(\xi_2) \chi_C(\xi_3) d\xi_1 d\xi_2 dt \right\|_{L^2_x} \lesssim |A|^{1/2} |B|^{1/2} |C|^{1/2}. \tag{5.9}
\]

In view of Lemma 5.4, now we must choose the sets \( A, B, C, R \) and a sequence of times \( t_N \in [0,T] \) in this way: for \( N \in \mathbb{N} \)
\[
A_N = \{ \xi_1 \in \mathbb{R} : |\xi_1| < 1/2 \}, \quad B_N = \{ \xi_2 \in \mathbb{R} : |\xi_2| < 1/4 \},
\]
\(^1\)Here \( X - Y = \{ x - y ; x \in X \text{ and } y \in Y \} \).
\(^2\)|\(X|\) denotes Lebesgue measure of the set \( X \).
\(^3\)For \( B, \) bounded subset of \( \mathbb{R}, \langle \cdot \rangle^k \hat{\varphi} \sim \chi_B \) means \( \chi_B \leq \langle \cdot \rangle^k \hat{\varphi} \) with \( \| \varphi \|_{H^l} \leq 2 \| \chi_B \|_{L^2} \).
\(^4\)\( (\ast, \ast)_R \) (or \( (\ast, \ast)_L \)) denotes the right(or left)-hand side of an inequality numbered by \( (\ast, \ast) \).
In this case we are not able to take a sequence \( \xi \). Also, for \( s \) and \( c \), similarly what we did to get (5.9), we have that

\[
N^{s-k+1} \quad \text{and} \quad \cos(t'Q_\alpha) > 1/2, \ \forall t' \in [0, t_N].
\]

From Lemma 5.4 and (5.9) yields

\[
t_N |R_N|^2 |B_N||C_N| N^{s-k+1} \lesssim |A_N|^{1/2} |B_N|^{1/2} |C_N|^{1/2}, \ \forall N \in \mathbb{N}. \quad \text{(5.10)}
\]

Taking account that \( |A_N| \sim |B_N| \sim |C_N| \sim 1 \) and \( t_N \sim N^{-3} \), then we have \( s - k < 2 \).

Now, dealing with the second component (5.6), analogously as we did before, we have

\[
\sup_{t \in [0, T]} \left\| \int_0^t \frac{\langle \xi \rangle^s \langle \xi \rangle}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k \langle \xi_3 \rangle^k} \cos(t'P_\alpha) \chi_\alpha(\xi_1) \chi_b(\xi_2) \chi_c(\xi_3) d\xi_1 d\xi_2 dt' \right\| \lesssim |A|^{1/2} |B|^{1/2} |C|^{1/2},
\]

where

\[
P_\alpha = P_\alpha(\xi, \xi_1, \xi_2) = \xi^3 - \xi_1^3 - \xi_2^3 - \alpha \xi_3^3. \quad \text{(5.12)}
\]

Interchanging the rules of the sets \( B_N \) and \( C_N \) given before we can conclude that \( k - s < 2 \). We finishes the proof of item (a).

(b) In this case we are not able to take a sequence \( t_N \to 0 \), because \( t \) is fixed number. We need to control \( \text{de} \) argument in the terms \( \cos(t'Q_\alpha) \) and \( \cos(t'P_\alpha) \), and we get this by making both \( Q_\alpha \) and \( P_\alpha \) sufficiently small. Fixing \( t \in [0, T] \), by (5.3), similarly what we did to get (5.9), we have that

\[
\left\| \int_0^t \frac{\langle \xi \rangle^s \langle \xi \rangle}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k \langle \xi_3 \rangle^k} \cos(t'Q_\alpha) \chi_\alpha(\xi_1) \chi_b(\xi_2) \chi_c(\xi_3) d\xi_1 d\xi_2 dt' \right\| \lesssim |A|^{1/2} |B|^{1/2} |C|^{1/2}.
\]

(5.13)

For \( N \in \mathbb{N} \), considering the sets

\[
A_N = \{ \xi_1 \in \mathbb{R} : |\xi_1 - aN| < \varepsilon(t)^{-1}N^{-2} \}, \quad B_N = \{ \xi_2 \in \mathbb{R} : |\xi_2 - bN| < \varepsilon(t)^{-1}N^{-2} \},
\]

\[
C_N = \{ \xi_3 \in \mathbb{R} : |\xi_3 - cN| < \varepsilon(t)^{-1}N^{-2} \} \quad \text{and} \quad R_N = \{ \xi \in \mathbb{R} : |\xi - N| < \varepsilon(t)^{-1}N^{-2} \}
\]

where the constant \( \varepsilon > 0 \) will be small, but fixed, and the positive constants \( a, b \) and \( c \) satisfies the conditions

\[
\begin{cases}
    a + b + c = 1, \\
    a^3 + ab^3 + ac^3 = 1.
\end{cases}
\]

(5.14)

First we note that \( R_N - B_N - C_N \subset A_N \) and \( |A_N| \sim |B_N| \sim |C_N| \sim |R_N| \sim N^2 \).

Also, for \( \xi_1 \in A_N, \xi_2 \in B_N, \xi_3 \in C_N \) yields

\[
\frac{\langle \xi \rangle^s \langle \xi \rangle}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^k \langle \xi_3 \rangle^k} \sim N^{1-2k}
\]

and

\[
|Q_\alpha| = |\xi^3 - N^3 + \xi_1^3 - \xi_2^3 - \xi_3^3 - 0| \lesssim |\xi^3 - N^3| + |(a^3 + ab^3 + ac^3)N^3 - \xi_1^3 - \xi_2^3 - \xi_3^3| \\
\lesssim |\xi^3 - N^3| + |(aN)^3 - \xi_1^3| + |bN^3 - \xi_2^3| + |cN^3 - \xi_3^3|
\]

\[
\lesssim_{a,b,c} (t')^{-1} \varepsilon_{a,b,c},
\]

where

\[
T = 2N^3(1 + T).
\]
where we use the elementary identity $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$. This implies the desired estimate

$$\cos(t'tQ_a) > 1/2, \text{ for all } t' \in [0, t].$$

From the Lemma 5.4 and (5.13) we get

$$t(N^{-2})^{1/2}N^{-1}N^{1/2}N^{-2k} \lesssim (N^{-2})^{1/2}(N^{-2})^{1/2}(N^{-2})^{1/2},$$

and so $k \geq -1/2$. For the second component (5.6), in the same way as before, we have for a fixed $t$ in $[0, T]$ the following inequality

$$\left\| \int_0^t \frac{(\xi^3)(\xi^3)(\xi^3)}{\xi^3} \cos(t'P_{a})f(t', \xi)g(t', \xi)h(t', \xi) \right\|_{L^2_t} \lesssim |A|^{1/2}|B|^{1/2}|C|^{1/2}.$$  \hspace{1cm} (5.15)

Of course, with the same choice of subsets $A_N$, $B_N$, $C_N$, etc. but with

$$\left\{ \begin{array}{l}
a + b + c = 1, \\
a^3 + b^3 + c^3 = 1,
\end{array} \right.$$ \hspace{1cm} (5.16)

we can conclude that $s \geq -1/2$ and we finish the proof of the item (b) and also the proof of the Theorem.

\section*{5.2. Failure of trilinear estimates.}

In this subsection we will prove the failure of the trilinear estimates (1.4) and (1.5).

\textbf{Proof of Proposition 1.6.} We prove only item (a), because the item (b) follows analogously. Using definition of the $X_{s,a}$-norm and Plancherel’s identity, the estimate (1.4) is equivalent to

$$\| \mathcal{T}_s(f, g, h) \|_{L^2_t L^2_x} \lesssim \|f\|_{L^2_x} \|g\|_{L^2_x} \|h\|_{L^2_x},$$ \hspace{1cm} (5.17)

where $\mathcal{T}_s(f, g, h)$ is defined by

$$\left\| \langle \xi \rangle^s (\tau - \xi^3) \hat{f} \hat{g} \hat{h} \right\|_{L^2_t} \lesssim \|f\|_{L^2_x} \|g\|_{L^2_x} \|h\|_{L^2_x}.$$ \hspace{1cm} (5.18)

with $\xi_3 = \xi - \xi_1 - \xi_2$, $\tau_3 = \tau - \tau_1 - \tau_2$.

Now, suppose that $s - 2k > 1$. Let $c_2$ and $c_3$ be two constants satisfying

$$\begin{array}{l}
c_2 + c_3 = 1, \\
c_2^3 + c_3^3 = 1,
\end{array}$$ \hspace{1cm} (5.19)

and let $\sigma_1 = \tau_1 - \alpha \xi_3^2$, $\sigma_2 = \tau_2 - \alpha \xi_3^2$, $\sigma_3 = \tau_3 - \alpha \xi_3^2$ in order to apply the Lemma 5.4, we define the sets

$$A_N = \{ (\xi_1, \tau_1) : |\xi_1| < N^{-2}, |\sigma_1| < C_0 \}, \quad B_N = \{ (\xi_2, \tau_2) : |\xi_2 - c_2 N| < \frac{N^2}{3}, |\sigma_2| < 1 \},$$

$$C_N = \{ (\xi_3, \tau_3) : |\xi_3 - c_3 N| < \frac{N^2}{3}, |\sigma_3| < 1 \}, \quad R_N = \{ (\xi, \tau) : |\xi - N| < \frac{N^2}{3}, |\tau - \xi_3| < 1 \}.$$

Then $R_N \subset B_N \subset C_N$. In fact, if $(\xi_1, \tau_1) = (\xi, \tau) - (\xi_2, \tau_2) - (\xi_3, \tau_3)$ with $(\xi, \tau) \in R_N$, $(\xi_2, \tau_2) \in B_N$ and $(\xi_3, \tau_3) \in C_N$, then using (5.19), we have

$$|\xi_1| \leq |\xi - \xi_2 - \xi_3| \leq |\xi - N| + |N - \xi_2 - \xi_3| \leq \frac{N^2}{3} + |c_2 N - \xi_2| + |c_3 N - \xi_3| < N^{-2}.$$
On the other hand using again (5.19)
\[|\sigma_1| = |\tau_1 - \xi_1^3| = |\tau - \tau_2 - \tau_3 - \xi_1^3| = |\tau - \xi^3 + \sigma_2 - \alpha_2 \xi_2 - \sigma_3 - \alpha_3 \xi_3 - \xi_3^3| \leq |\tau - \xi^3| + |\sigma_2| + |\sigma_3| + |\xi_1^3| + |\xi_2 - \alpha_2 \xi_2 - \alpha_3 \xi_3 - \xi_3^3| \leq 3 + N^{-6} + |\xi_3^3 - N^3 + \alpha_3 \xi_3 - \alpha_3 \xi_3^3 - \alpha_3 \xi_3^3 + \alpha N^3 c_3^2 - \alpha_3 \xi_3^3 - \alpha_3 \xi_3^3| \lesssim (1 + |\alpha|).

Consider \( \tilde{f} = \chi_{A_N}, \tilde{g} = \chi_{B_N}, \tilde{h} = \chi_{C_N} \). From the definition of \( A_N, B_N \) and \( C_N \) holds that \( \langle \xi_1 \rangle \sim 1, \langle \xi_2 \rangle \sim N, \langle \xi_3 \rangle \sim N, \langle \xi \rangle \sim N \) and \( \langle \sigma_j \rangle \sim 1, j = 1, 2, 3 \). By (5.17) and Lemma 5.4 we obtain
\[N^{s-2k+1} \| R_N \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \leq N^{s-2k+1} \| \chi_{A_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \leq \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \]
which implies
\[NN^{s-2k} N^{-1} N^{-4} \lesssim N^{-3}\]
and the last inequality is false if \( s - 2k > 1 \) for \( N \) sufficiently large.

Finally, if \( k < -\frac{1}{2} \), let \( c_1, c_2 \) and \( c_3 \) be three constants satisfying
\[c_1 + c_2 + c_3 = 1, \quad c_1^2 + \alpha c_2^3 + \alpha c_3^3 = 1. \tag{5.20}\]

Analogously as above, if we consider the sets
\[A_N = \{ \langle \xi_1, \tau_1 \rangle : |\xi_1 - c_1 N| < N^{-2}, |\sigma_1| < C_0 \}, \quad B_N = \{ \langle \xi_2, \tau_2 \rangle : |\xi_2 - c_2 N| < \frac{N^{-2}}{3}, |\sigma_2| < 1 \}, \quad C_N = \{ \langle \xi_3, \tau_3 \rangle : |\xi_3 - c_3 N| < \frac{N^{-2}}{2}, |\sigma_3| < 1 \}, \quad R_N = \{ \langle \xi, \tau \rangle : |\xi - N| < \frac{N^{-2}}{3}, |\tau - \xi^3| < 1 \}.

Then using the condition (5.20) we obtain
\[|\xi_1 - c_1 N| = |\xi - \xi_2 - \xi_3 - c_1 N| \leq |\xi - N| + |c_2 N - \xi_2| + |c_3 N - \xi_3| < N^{-2}, \]
as above and using the condition (5.20) again
\[|\sigma_1| \leq |\tau - \xi^3| + |\sigma_2| + |\sigma_3| + |\xi_1^3 - \alpha_2 \xi_2 - \alpha_3 \xi_3 - \alpha_3 \xi_3^3| \leq 3 + |\xi^3 - N^3| + |c_1^2 N^3 - \xi_1^3| + |\alpha| |c_2^3 N^3 - \xi_2^3| + |\alpha| |c_3^3 N^3 - \xi_3^3| \lesssim (1 + |\alpha|).

Thus \( R_N - B_N - C_N \subseteq A_N \) and \( \langle \xi_j \rangle \sim N, j = 1, 2, 3, \langle \xi \rangle \sim N \) and \( \langle \sigma_j \rangle \sim 1, j = 1, 2, 3 \). Again, by (5.17) and Lemma 5.4 we get
\[\frac{N^{1+s}}{N^{s-2k}} \| R_N \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \leq \frac{N^{1+s}}{N^{s-2k}} \| \chi_{A_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \leq \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)} \]
therefore
\[NN^{-2k} N^{-1} N^{-4} \lesssim N^{-3}\]
and the last inequality is false if \( -2k > 1 \) for \( N \) sufficiently large. \( \square \)

**Proof of Proposition 1.7.** We only give the proof of (a), because (b) can be proved in the same way. We will consider the following sets
\[A_N = \{ \langle \xi_1, \tau_1 \rangle : |\xi_1| < 1, |\sigma_1| \sim N^3 \}, \quad B_N = \{ \langle \xi_2, \tau_2 \rangle : |\xi_2| < 1/2, |\sigma_2| < 1 \}, \quad C_N = \{ \langle \xi_3, \tau_3 \rangle : |\xi_3 - N| < 1/4, |\sigma_3| < 1 \}, \quad R_N = \{ \langle \xi, \tau \rangle : |\xi - N| < 1/4, |\tau - \xi^3| < 1 \}. \]
Then $R_N - B_N - C_N \subset A_N$ and $\langle \xi_3 \rangle \sim N$, $\langle \xi_j \rangle \sim 1$, $j = 1, 2$, $\langle \xi \rangle \sim N$ and $\langle \sigma_j \rangle \sim 1$, $j = 2, 3$. Using (5.18) we obtain
\[
\frac{N^{1+s}}{N^{3b}N^k} \| R_N \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^1(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^1(\mathbb{R}^2)} \leq \frac{N^{1+s}}{N^{3b}N^k} \| \chi_{A_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{C_N} \|_{L^2(\mathbb{R}^2)}
\]
therefore
\[
\frac{N^{1+s}}{N^{3b}N^k} \lesssim N^{3/2}
\]
and the last inequality is false if $s - k > 1/2 + 3b$, $b = 1/2 + \epsilon$ for $N$ sufficiently large. \hfill \Box

For the next proof we will use another elementary result, proved in [9].

**Lemma 5.5.** Let $R_j := [a_j, b_j] \times [c_j, d_j] \subset \mathbb{R}^2$, $j = 1, \ldots, n$ be rectangles such that $b_j - a_j = N$ and $d_j - c_j = M$. Then
\[
\| \chi_{R_1} \ast \chi_{R_2} \ast \cdots \ast \chi_{R_n} \|_{L^2(\mathbb{R}^2)} \sim (NM)^{n-\frac{1}{2}} = |R_j|^{n-\frac{1}{2}}.
\]

**Proof of Proposition 1.8.** We only prove that (1.6) fails to hold. Using Plancherel’s identity, the estimate (1.6) is equivalent to showing that
\[
\| (\xi - \frac{1}{2} (\tau - \xi^3))^{-\frac{1}{2} + \epsilon} \langle \xi \rangle \langle \xi_1 \rangle \langle \xi_2 \rangle \langle \xi_3 \rangle \|_{L^2(\mathbb{R}^2)} \leq \| f \|_{L^2(\mathbb{R}^2)} \| g \|_{L^2(\mathbb{R}^2)} \| h \|_{L^2(\mathbb{R}^2)},
\]
where $f, g, h$ are such that
\[
\bar{f}(\xi, \tau) = (\xi - \frac{1}{2} (\tau - \langle \xi \rangle \langle \xi_1 \rangle \langle \xi_2 \rangle \langle \xi_3 \rangle \|_{L^2(\mathbb{R}^2)} \)
\]
and consider two rectangles $R_1$, $R_2$ and $R_3$ with centers respectively at $(c_1N, \alpha(c_1N)^3)$, $(c_2N, \alpha(c_2N)^3)$ and $(c_3N, (c_3N)^3)$, and each with dimension $N^{-(2+r)} \times N^{-r}$, where $-2 < r < 0$. Now, consider $f$ and $g$ defined, via their Fourier transform, defined by $\tilde{f} = \chi_{R_1}$, $\tilde{g} = \chi_{R_2}$, $\tilde{h} = \chi_{R_3}$. It is easy to see that
\[
\| f \|_{L^2(\mathbb{R}^2)} = \| g \|_{L^2(\mathbb{R}^2)} = \| h \|_{L^2(\mathbb{R}^2)} = N^{-(1+r)}.
\]

Also,
\[
|\xi_1 - c_1N| \leq \frac{N^{-(2+r)}}{2}, \quad |\tau_1 - \alpha(c_1N)^3| \leq \frac{N^{-r}}{2},
\]
and
\[
|\xi_2 - c_2N| \leq \frac{N^{-(2+r)}}{2}, \quad |\tau_2 - \alpha(c_2N)^3| \leq \frac{N^{-r}}{2},
\]
and
\[
|\xi_3 - c_3N| \leq \frac{N^{-(2+r)}}{2}, \quad |\tau_3 - (c_3N)^3| \leq \frac{N^{-r}}{2},
\]
We have $|\xi_j| \sim N$ and
\[
|\xi - N| = |\xi_1 + \xi_2 + \xi_3 - c_1N - c_2N - c_3N| \\
\leq |\xi_1 - c_1N| + |\xi_2 - c_2N| + |\xi_3 - c_3N| \\
\leq \frac{3N^{-(2+r)}}{2}.
\]
Also, one can prove that
\[
|\tau - \xi^3| = |\tau_1 + \tau_2 + \tau_3 - (\alpha c_1^3 + \alpha c_2^3 + c_3^3)N^3 + N^3 - \xi^3|
\leq |\tau_1 - \alpha(c_1N)^3| + |\tau_2 - \alpha(c_2N)^3| + |\tau_3 - (c_3N)^3| + |N^3 - \xi^3|
\leq \frac{3N^{-r}}{2} + \frac{N^{-(2+r)}}{2}|N^2 + N\xi + \xi^2|
\lesssim N^{-r},
\]
for \( j = 1, 2 \):
\[
|\tau_j - \alpha c_j^3| = |\tau_j - \alpha(c_jN)^3 + \alpha(c_jN)^3 - \alpha c_j^3|
\leq |\tau_j - \alpha(c_jN)^3| + |\alpha||c_j N)^3 - c_j^3|
\leq \frac{N^{-r}}{2} + \frac{N^{-(2+r)}}{2}|c_j^2 N^2 + c_j N\xi_j + \xi_j^2|
\lesssim N^{-r}.
\]
Similarly
\[
|\tau_3 - \xi_3^3| = |\tau_3 - (c_3N)^3 + (c_3N)^3 - \xi_3^3|
\leq |\tau_2 - \alpha N^3| + |c_3 N - \xi_3||(|c_3N)^3 + (c_3N)\xi_3 + \xi_3^2|
\lesssim \frac{N^{-r}}{2} + \frac{N^{-(2+r)}}{2}N^2
\lesssim N^{-r}.
\]
Thus
\[
\langle \tau_j - \alpha c_j^3 \rangle \gtrsim N^{-r}, \; j = 1, 2, \quad \langle \tau_3 - \xi_3^3 \rangle \gtrsim N^{-r}, \quad \langle \tau - \xi^3 \rangle \gtrsim N^{-r}
\]
With these considerations, we get from (5.22)
\[
(5.22)_L \sim \left\| N^{\frac{1}{2}} (\tau_1 + \tau_2 + \tau_3 - (\alpha c_1^3 + \alpha c_2^3 + c_3^3)N^3 + N^3 - \xi^3) \int_{\mathbb{R}^4} N^{\frac{3}{2}} \chi_{\mathcal{R}_1}(\xi_1, \tau_1) \chi_{\mathcal{R}_2}(\xi_2, \tau_2) \chi_{\mathcal{R}_3}(\xi_3, \tau_3) \frac{d\xi_1 d\tau_1 d\xi_2 d\tau_2}{N^{-3(\frac{3}{2} + r)}} \right\|_{L^2_\tau(L^2_\xi)}
\sim N^{2+2r-\epsilon r} \| \chi_{\mathcal{R}_1} * \chi_{\mathcal{R}_2} * \chi_{\mathcal{R}_3} \|_{L^2_\tau(L^2_\xi)}
= N^{2+2r-\epsilon r} |\mathcal{R}_j|^{-\frac{3}{2}} = N^{3+2r-\epsilon r} N^{-5-5r} = N^{-3-r(3+\epsilon)}.
\]
Now, using (5.23) and (5.27) in (5.22),
\[
N^{-3-r(3+\epsilon)} \lesssim N^{-3-3r} \iff N^{-\epsilon r} \lesssim 1.
\]
Since \( r < 0 \), if we choose \( N \) large, the estimate (5.28) fails to hold whenever \( \epsilon > 0 \) and this completes the proof of the proposition.

6. Failure of Bilinear estimates in Section 3. In this section we will conclude that the Theorem 1.1 is sharp in the sense that we cannot use the approach developed in Section 3, to improves the Sobolev indices in Proposition 1.3.

**Proposition 6.1.** Let \( \alpha \neq 0, 1 \).

(a) If \( s - k > -\frac{1}{2} \) or \( k < -1/2 \) then the following bilinear estimate
\[
\|uv\|_{L^2_\tau(L^2_\xi)} \lesssim \|u\|_{X_{s-k, -\frac{1}{2} + \epsilon}} \|v\|_{X_{s, \frac{1}{2} + \epsilon}},
\]
\[
\|uv\|_{L^2_\tau(L^2_\xi)} \lesssim \|u\|_{X_{s-k, -\frac{1}{2} + \epsilon}} \|v\|_{X_{s, \frac{1}{2} + \epsilon}}.
\]


fails to hold.
(b) If $k - s > -\frac{1}{2}$ or $s < -1/2$ then the following bilinear estimate
\[
\|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X_{s-k}^{-\frac{1}{2}+\varepsilon}} \|v\|_{X_{s}^{-\frac{1}{2}+\varepsilon}},
\]
fails to hold.

Proof. As before, we only prove item (a). Using Plancherel’s identity, the estimate (3.12) is equivalent to showing that
\[
B_s(f,g) := \left\| \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{s-k+1/2}}{\langle \tau \rangle^{1/2}} \tilde{f}(\xi,\tau) \tilde{g}(\xi,\tau_1) \frac{d\xi d\tau_1}{\langle \tau_1 - \alpha \xi \rangle^\varepsilon \langle \tau_2 - \xi \rangle^\varepsilon} \right\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)},
\]
where $b = \frac{1}{2} + \varepsilon$.

Let $\sigma_1 = \tau_1 - \alpha \xi_1^3$, $\sigma_2 = \tau_2 - \xi_2^3$, we define the sets
\[A_N = \{ (\xi_1, \tau_1) : |\xi_1 - N| < N^{-2}, |\tau_1| < C_\alpha \}, \quad B_N = \{ (\xi_2, \tau_2) : |\xi_2| < (2N)^{-2}, |\sigma_2| < 1 \}, \]
and
\[R_N = \{ (\xi, \tau) : |\xi - N| < (2N)^{-2}, |\tau - \alpha \xi + (1 - \alpha)\xi^2 N| < 1 \}.
\]
Then $R_N - B_N \subset A_N$. In fact, if $(\xi_1, \tau_1) = (\xi, \tau) - (\xi_2, \tau_2)$ with $(\xi, \tau) \in R_N$ and $(\xi_2, \tau_2) \in B_N$, then
\[|\xi_1 - N| = |\xi - \xi_2 - N| \leq |\xi - N| + |\xi_2| < N^{-2},
\]
also observe that $\sigma_1 + \sigma_2 = \tau - \alpha \xi^3 - (1 - \alpha)\xi_1^3 + 3\alpha \xi_1 \xi_2$, thus
\[
|\sigma_1| \leq |\sigma_2| + |\tau - \alpha \xi^3 - (1 - \alpha)\xi_1^3 + 3\alpha \xi_1 \xi_2|
\leq 1 + |\tau - \alpha \xi^3 - (1 - \alpha)\xi_1^3| + |1 - \alpha| |\xi_1||\xi_2| + 3|\alpha \xi_1 \xi_2|
\leq 2 + |1 - \alpha| [N]|\xi - \xi_1||\xi + \xi_1| + \xi_2^2|N - \xi_1| + 3|\alpha \xi_1 \xi_2|
\leq C_\alpha.
\]

Consider $\tilde{f} = \chi_{A_N}$, $\tilde{g} = \chi_{B_N}$. We have $\langle \xi_1 \rangle \sim N$, $\langle \xi_2 \rangle \sim 1$, $\langle \sigma_j \rangle \sim 1$, $j = 1, 2$. From (6.3) and Lemma 5.3, we obtain
\[
N^{s-k+1/2} \|R_N\|_{L^2(\mathbb{R}^2)} \|\chi_{B_N}\|_{L^1(\mathbb{R}^2)} \leq N^{s-k+1/2} \|\chi_{A_N} \ast \chi_{B_N}\|_{L^2(\mathbb{R}^2)}
\leq C_\alpha \maple{\|\chi_{A_N}\|_{L^2(\mathbb{R}^2)} \|\chi_{B_N}\|_{L^2(\mathbb{R}^2)}}
\]
which implies
\[
N^{s-k+1/2} N^{-1} N^{-2} \lesssim N^{-2}
\]
and the last inequality is false if $s - k > 1/2$ for $N$ sufficiently large.

Analogously, if we consider the sets
\[A_N = \{ (\xi_1, \tau_1) : |\xi_1 - N| < N^{-2}, |\tau_1 - \alpha N^3| < 2 \}, \quad B_N = \{ (\xi_2, \tau_2) : |\xi_2 + N| < (2N)^{-2}, |\tau_2 + N^3| < 1 \}
\]
and
\[R_N = \{ (\xi, \tau) : |\xi| < (2N)^{-2}, |\tau - (\alpha - 1)N^3| < 1 \}
\]
then $R_N - B_N \subset A_N$ and
\[
|\tau_1 - \alpha \xi_1^3| \leq |\tau_1 - \alpha N^3 + \alpha(N^3 - \xi_1^3)| \lesssim 2 + |\alpha|, \quad |\tau_2 - \xi_2^3| \leq |\tau_2 + N^3 - (N^3 + \xi_2^3)| \lesssim 1.
\]
Noting that $\langle \xi_1 \rangle \sim N$, $\langle \xi_2 \rangle \sim N$, $\langle \sigma_j \rangle \sim 1$, $j = 1, 2$,
from (6.3) we obtain
\[ N^{-k+1/2} \| R_N \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^1(\mathbb{R}^2)} \leq N^{-k+1/2} \| \chi_{A_N} \ast \chi_{B_N} \|_{L^2(\mathbb{R}^2)} \]
\[ \leq \| \chi_{A_N} \|_{L^2(\mathbb{R}^2)} \| \chi_{B_N} \|_{L^2(\mathbb{R}^2)}, \]
which implies
\[ N^{-k+1/2} N^{-1} N^{-2} \lesssim N^{-2} \]
and the last inequality is false if \( k < -1/2 \), for \( N \) sufficiently large.

At the endpoint we have the following result.

**Proposition 6.2.** Let \( \alpha \neq 0,1 \), then the following bilinear estimate
\[ \| uv \|_{L^2(\mathbb{R}^2)} \lesssim \| u \|_{X_{\alpha, -2/3}} \| v \|_{X_{\alpha, -\frac{2}{3}+}}, \]  
(6.5)
fails to hold for all \( \epsilon > 0 \).

**Proof.** Using Plancherel’s identity, the estimate (6.1) is equivalent to showing that
\[ \left\| \int_{\mathbb{R}^2} \frac{\xi_1}{\tau_1} \frac{\xi_2}{\tau_2} \frac{\xi}{\tau} \frac{\xi_3}{\tau_3} \right\|_{L^2(\mathbb{R}^2)} \lesssim \| f \|_{L^2(\mathbb{R}^2)} \| g \|_{L^2(\mathbb{R}^2)}, \]  
(6.6)
where
\[ f(\xi, \tau) = \frac{\xi}{\tau} \frac{\xi_2}{\tau_2} \frac{\xi_3}{\tau_3}, \quad g(\xi, \tau) = \frac{\xi}{\tau} \frac{\xi_2}{\tau_2} \frac{\xi_3}{\tau_3}, \]
and \( \xi_2 = \xi - \xi_1 \) and \( \tau_2 = \tau - \tau_1 \).

We will choose functions \( f \) and \( g \) for which the estimate (6.6) fails to hold for all \( \epsilon > 0 \).

Consider two rectangles \( R_1 \) and \( R_2 \) centered respectively at \((N, \alpha N^3)\), and \((N, N^3)\), and each with dimension \( N^{-(2+r)} \times N^{-r} \), where \(-2 < r < 0\). Now, let \( f \) and \( g \) defined, via their Fourier transform, by \( f = \chi_{R_1} \) and \( g = \chi_{R_2} \). It is easy to see that
\[ \| f \|_{L^2(\mathbb{R}^2)} = \| g \|_{L^2(\mathbb{R}^2)} = N^{-(1+r)}. \]  
(6.7)
Also,
\[ |\xi - N| \leq N^{-(2+r)} \frac{2}{2}, \quad |\tau_1 - N^3| \leq N^{-r} \frac{2}{2} \]
(6.8)
and
\[ |\xi_2 - N| \leq N^{-(2+r)} \frac{2}{2}, \quad |\tau_2 - \alpha N^3| \leq N^{-r} \frac{2}{2}. \]  
(6.9)

We have \( |\xi| \sim N \). Also, we have that
\[ |\tau_1 - \xi_1^3| = |\tau_1 - N^3 + N^3 - \xi_1^3| \leq |\tau_1 - N^3| + |N^3 - \xi_1^3| \leq N^{-r} + N^3 + \xi_1^3 \]
\[ \leq N^{-r} + N^3 + N \xi_1 + \xi_1 \]
\[ \lesssim N^{-r}. \]

Similarly,
\[ |\tau_2 - \alpha N^3| = |\tau_2 - \alpha N^3 + \alpha N^3 - \alpha N^3| \leq |\tau_2 - \alpha N^3| + |\alpha| \| N^3 - \xi_2^3 \| \]
\[ \leq N^{-r} + |\alpha| \| N^{-2} + N \xi_2 + \xi_2^2 \| \]
\[ \lesssim N^{-r}. \]
With these considerations, we get from (6.6) and Lemma 5.5
\[
(6.6)_L \gtrsim \left\| N^{1+r-r\epsilon} \int_{\mathbb{R}^2} \chi R_1(\xi_1, \tau_1) \chi R_2(\xi_2, \tau_2) d\xi_1 d\tau_1 \right\|_{L^2_t(L^2)} \\
\sim N^{1+r-r\epsilon} \| \chi R_1 \ast \chi R_2 \|_{L^2(\mathbb{R}^2)} \\
= N^{1+r-r\epsilon} |R_j|^{2\epsilon - 1} = N^{1+r-r\epsilon} N^{-3-3r} = N^{-2-r(2+\epsilon)}.
\]
(6.10)

Now, using (6.7) and (6.10) in (6.3),
\[
N^{-2-r(2+\epsilon)} \lesssim N^{-2-2r} \iff N^{-r\epsilon} \lesssim 1.
\]
(6.11)

Since \( r < 0 \), if we choose \( N \) large, the estimate (6.11) fails to hold whenever \( \epsilon > 0 \) and this completes the proof of the proposition. \( \square \)

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