Inference for a mixture of symmetric distributions under log-concavity

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Abstract

In this article, we reconsider the problem of estimating the unknown symmetric density in a two-component location mixture model under the assumption that the symmetric density is log-concave. When consistent estimators for the shift locations and mixing probability are used, we show that the nonparametric log-concave Maximum Likelihood estimator (MLE) of both the mixed density and that of the unknown symmetric component are consistent in the Hellinger distance. In case the estimators for the shift locations and mixing probability are $\sqrt{n}$-consistent, we establish that these MLE's converge to the truth at the rate $n^{-2/5}$ in the $L_1$ distance. To estimate the shift locations and mixing probability, we use the estimators proposed by Hunter et al. (2007). The unknown symmetric density is efficiently computed using the R package logcondens.mode.

1 Introduction

Let us assume that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) draws from a mixture distribution, with cumulative distribution function (cdf) $G^0$ given by

$$G^0(x) = \sum_{i=1}^{k} \pi_i^0 F_i^0(x), \quad x \in \mathbb{R},$$

(1)

for some $k \in \mathbb{N}$, where $F_i^0$ are cdfs, $\pi_i^0 > 0$, and $\sum_{i=1}^{k} \pi_i^0 = 1$. Such mixture distributions are very common in statistical modeling, in part because a variety of data generating frameworks lead to mixture models; for instance, one common approach to clustering problems leads to estimation of a mixture density (Fraley and Raftery, 2002). Another reason for this popularity is that they are very flexible and many distributions can be well approximated by some mixture model (see, e.g., Everitt and Hand (1981), Titterington et al. (1985), or McLachlan and Peel (2000)). However, this flexibility comes at a price. One of the major difficulties in working with mixture models is identifiability, in that often multiple sets of parameters $\pi_1^0, \ldots, \pi_k^0, F_1^0, \ldots, F_k^0$ yield the same $G^0$ in (1). There are different approaches to dealing with the issue of identifiability. One approach, taken e.g. in Butucea and Vandekerkhove (2014), avoids the question altogether, by building estimators to be
used whenever identifiability is satisfied (thereby leaving the question of identifiability up to the user). Another approach is to add additional constraints to (1) so that identifiability holds. In particular, Bordes et al. (2006) and Hunter et al. (2007) proposed restricting the model (1) to the model where the \( F_i(x) = F^0(x - u^0_i) \) for \( u^0_i \in \mathbb{R}, \ i = 1, \ldots, k \), where \( F^0 \) is restricted to be symmetric about 0, i.e. \( F^0(-x) = 1 - F^0(x) \). This model was also studied more recently by Butucea and Vandekerkhove (2014). In fact, all of these authors focus largely on the case \( k = 2 \):

\[
G^0(x) = \pi^0 F^0(x - u^0_1) + (1 - \pi^0) F^0(x - u^0_2), \quad x \in \mathbb{R},
\]

This is still, in fact, a flexible model which is useful in many scenarios (see our three data applications in Section 6), and by making these restrictions, strong statements about identifiability can be made. Bordes et al. (2006) and Hunter et al. (2007) are both essentially able to conclude that this just-described class of symmetric two-component mixtures is identifiable (although they use different notions of identifiability; see Section 1.2 for more details on identifiability).

The main focus of Hunter et al. (2007) is then on estimation of the parametric part of (2), \( (\pi^0, u^0_1, u^0_2) \). They are able to show that their estimator of this parametric component is consistent and asymptotically normal. However, the obtained estimator of \( F^0 \) is not even guaranteed to have the properties of a genuine cdf (i.e., it is not necessarily nondecreasing).

In this paper, we build on this recent progress on symmetric two-component mixture models, and focus on estimation of \( F^0 \). We take a shape-constrained approach to modeling \( F^0 \), and in particular we assume that \( F^0 \) has a density \( f^0 \) which is log-concave, meaning \( \log f^0 \) is a closed and proper concave function (i.e., it is upper semicontinuous and always less than infinity, see Rockafellar (1970)). The assumption of log-concavity has gained much attention recently for several reasons. One of the major reasons is that the class of log-concave densities is a flexible nonparametric class, but one can still use a fully automatic (maximum likelihood) estimator that does not require tuning parameter selection. This is in contrast to alternative nonparametric approaches, generally based on making smoothness assumptions, for which tuning parameter selection is of paramount importance for successful inference. Choosing the tuning parameter can be statistically or computationally difficult to do, and furthermore, different functionals (parameters) of the underlying density require different optimal choices of tuning parameter. Bordes et al. (2006) and Butucea and Vandekerkhove (2014) use methods based on kernels for estimation of \( F^0 \). Chee and Wang (2013) also use kernel density estimators (KDEs), and they have to pick the bandwidth by using model selection procedures. The log-concave MLE automatically finds the optimal partition of the data, so we avoid choosing a tuning parameter, and we do not require any difficult-to-verify smoothness assumptions.

It is frequent in mixture modeling for practitioners to believe that the mixture components are unimodal, each mixture corresponding to a distinct subpopulation. However, the normality assumption is not always appropriate. Until now, practitioners thus had to choose between either using a poorly specified parametric approach or using an overly flexible nonparametric approach that did not take the prior
knowledge of unimodality into account. Log-concavity is often used as a surrogate for unimodality because log-concave densities are all unimodal, so our approach of using log-concave components provides a solution to this problem. Additionally the Gaussian family, as well as many other familiar parametric families are log-concave (at least for common parameter value choices), so our approach provides a unified and robust approach to handling mixtures of (symmetric versions of) all of these densities.

Another major benefit to using the log-concave maximum likelihood estimator (MLE) is that it performs well under model misspecification. Let \( \Psi(f; G) = \int_{\mathbb{R}} \log f \, dG \) for a log-concave density \( f \) and a distribution \( G \) with a finite first moment and which is not a point mass. Then Dümbgen et al. (2011) (see also Cule and Samworth (2010)) show that \( \arg\min_f \Psi(f; G) \) exists and unique, where the \( \arg\min \) is taken over all log-concave densities. It turns out that if we have \( n \) i.i.d. observations from a true distribution \( G \), then the log-concave MLE of these observations will converge (in various norms) to \( \arg\min_f \Psi(f; G) \) as \( n \to \infty \). If \( G \) has a log-concave density, then this means the log-concave MLE converges to that density, but if \( G \) does not have a log-concave density, the log-concave MLE still converges to the distribution closest to \( G \) that has a log-concave density. This is true as long as the \( G \) is not extremely heavy-tailed (i.e., as long as \( \text{E}_G(|X|) < \infty \)), even if \( G \) does not even admit a Lebesgue density. When the differentiability assumptions underlying smoothness-based techniques fail to hold, choosing a tuning parameter may be difficult or impossible and those techniques may perform very poorly. On the other hand, while one may lose efficiency by assuming log-concavity instead of a correct model when log-concavity fails to hold, procedures based on log-concavity assumptions will still often be effective without requiring additional modifications. For instance, Samworth and Yuan (2012) use the log-concave MLE to estimate marginal components in independent component analysis, and show that even when log-concavity does not hold, consistently estimating the log-concave projections allows one to perform consistent independent component analysis.

Much is already known about the large sample properties, both local and global, of the log-concave MLE. Research work on the asymptotic behavior of this estimator includes Pal et al. (2007), Dümbgen and Rufibach (2009), Balabdaoui et al. (2009), Cule and Samworth (2010), Dümbgen et al. (2010), Dümbgen and Dümbgen (2010), Chen and Samworth (2013), Doss and Wellner (2013), and Kim and Samworth (2014). More recently, In case the data are discrete, the limit distribution of the log-concave MLE of an unknown probability mass function was derived by Balabdaoui et al. (2013) in the well- and misspecified settings and was used to construct asymptotic confidence intervals of the probability mass function. Furthermore, fast active set convex programming algorithms have been developed to compute the maximum likelihood estimator and which allow the user to assign unequal weights to the observations; see for example Dümbgen et al. (2010), Rufibach (2007) and Dümbgen and Rufibach (2011).

In the present context, we need to consider the class of symmetric log-concave densities on \( \mathbb{R} \), which has not been considered before. To do so, we note that if \( f \)
is symmetric and log-concave on $\mathbb{R}$, then $f^+(t) := 2f(t)1_{t \in [0, \infty)}$ is log-concave with mode at 0. Thus, through a simple transformation of the data, it can be shown that the original estimation problem is equivalent to maximizing the log-likelihood over the class of log-concave densities on $[0, \infty)$ with mode at 0. We can then compute the maximum of the log-likelihood easily by alternating between the EM algorithm (Dempster et al., 1977) and the active set algorithm provided in the R package \texttt{logcondens.mode} which computes the log-concave MLE with a fixed mode. We use the fact that the active set algorithm allows for unequal weights to be assigned to the data points: here, the weights assigned are proportional to the posterior probabilities from the EM algorithm.

We are able to show that the symmetric log-concave MLE converges almost surely to the true symmetric log-concave component density in the Hellinger distance and in the supremum norm on sets of continuity of the true density. Furthermore, and provided that some mild assumptions hold, it can be shown that our estimator converges to the truth at the rate $n^{-2/5}$ in the $L_1$-distance. Although the risk measure we use here is different from the one considered by Butucea and Vandekerkhove (2014), it seems that the rate of convergence of our MLE, when the true mixture component is log-concave, is faster than that given in their Theorem 4 for their KDE when the smoothness parameter $\beta$ satisfies $\beta \in (1/2, 5/2)$.

We are not the first to use log-concavity in mixture modeling; Chang and Walther (2007) and Eilers and Borgdorff (2007) consider univariate mixtures of log-concave densities, and Cule et al. (2010) consider multivariate mixtures of log-concave densities. However, in none of those settings was symmetry imposed, perhaps because the authors were not worried about the (often fundamental) question of identifiability. Thus, their work does not directly apply in our setting.

We note that, as will be clear when we present our approach in detail, we do not use a pure maximum likelihood approach here. We feed in other estimators of $(\pi_0, u_{01}, u_{02})$ (for instance, we use those of Hunter et al. (2007) in our implementation) to our likelihood, which we maximize to estimate $f^0$ and thus $g^0$. An alternative approach is to estimate both the parametric and nonparametric components simultaneously by maximum likelihood, but that approach is fraught with many new difficulties, and the estimators of Hunter et al. (2007) for $(\pi^0, u_{10}, u_{20})$ are already very robust.

Finally, it is important to note that the primary goal in Bordes et al. (2006) and Hunter et al. (2007) was to study estimation in the mixture model with $k \geq 2$. Hunter et al. (2007) consider identifiability as a property of individual distributions, rather than as a property of an entire family of distributions. For $k = 3$, Hunter et al. (2007) give sufficient conditions on the mixing probabilities and mixture location parameters for a specific distribution to be identifiable. For any such case where identifiability holds, our method can be easily extended to apply. We chose not to present such extended results since it makes the presentation more onerous without adding much to the development of the theory.
1.1 Organization of the paper

The paper will be structured as follows. In Section 1.2 we describe in greater detail the existing methods for estimating $\pi^0, u^0_1, u^0_2,$ and $F^0$. In Section 2 we establish existence of the MLE and provide a sufficient condition for a candidate to be equal to the estimator. In Section 3, we establish consistency in the Hellinger distance. This implies other forms of consistency by the results of Cule and Samworth (2010). The techniques we used are adapted from Pal et al. (2007), Cule and Samworth (2010) and Schuhmacher and Dümbgen (2010). However, dealing with a mixture requires dealing with additional difficulties. Also, since consistency is mainly based on bounding from below the maximum likelihood by the likelihood obtained by replacing the symmetric density by the truth, $f^0$, special care was needed to be given to the case where $f^0$ is compactly supported. The extra difficulty in this case stems from the fact it is not excluded that the log-likelihood can be equal to $-\infty$. In Section 4, we derive the rate of convergence of our estimator and find that the MLEs of $f^0$ and $g^0$ converge to the truth at a rate of order $n^{-2/5}$. In Section 5, we give computational details related to the estimation method of Hunter et al. (2007) and active set algorithm used to compute the log-concave MLE of the symmetric density. Once the MLE is obtained, we describe how to simulate from it and also from a smoothed version. This smoothing technique was first invented by Dümbgen and Rufibach (2009) and later investigated in more theoretical depth in the multivariate context by Chen and Samworth (2013). In the same section, we consider the problem of testing the null hypothesis of absence of mixing. Our test is based on the likelihood ratio where under the null hypothesis we fit the symmetric log-concave MLE with mode at the median value of the data. The critical region is based on bootstrapping from the MLE under the null hypothesis. In our power assessment, we consider different mixing situations and the following symmetric log-concave densities: Gaussian, double exponential and uniform. The performance of our likelihood ratio test is compared to that of the trace test for log-concavity of Chen and Samworth (2013) and it is found that our test outperforms theirs, in particular when the true density is uniform. At the end of this section we include our results on clustering where the goal is to compare the performances of the Gaussian and log-concave classifiers. The algorithm we use assigns a datum to the first/second mixture component when the corresponding posterior probability to belong to the first component is greater/smaller than $1/2$. In Section 6, we present three data applications. Section 7 gathers some conclusions and remaining questions that we plan to investigate in the future. Proofs can be found in the appendix.

1.2 Overview of existing methods

The approaches of Bordes et al. (2006), Hunter et al. (2007), and Butucea and Vandekerkhove (2014) to estimation of $\pi^0$ and $u^0_1, u^0_2$ and $F^0$, are all different, but they are all based on the two following steps:

1. estimation of the mixture parameters via minimization of the empirical version
of a chosen contrast function,

2. estimation of the symmetric distribution by plugging the estimates obtained in the first step and using a nonparametric estimation method.

The contrast function of the first step is a function of the mixture parameters, and in its construction the authors use symmetry of the unknown density \( f \). The true mixture parameters are then identified using the fact that the chosen contrast function (based on the true underlying distribution) is identically equal to zero if and only if the mixture parameters coincide with the true parameters.

Bordes et al. (2006), Butucea and Vandekerkhove (2014), and Chee and Wang (2013) all use a KDE approach. Bordes et al. (2006), combine kernel density estimation of \( G_0 \) and an inversion formula giving the expression of the symmetric density \( f_0 \) in the model (2) as a function of \( g_0 \) and the mixture parameters. Under the conditions that \( \pi_0 \in (0,1/2) \) and \( u_1^0 < u_2^0 \) Bordes et al. (2006) show that

\[
f_0(x) = \frac{1}{1 - \pi_0} \sum_{k \geq 0} \left( -\frac{\pi_0}{1 - \pi_0} \right)^k g_0(x + u_1^0 + k(u_2^0 - u_1^0))
\]

for all \( x \in \mathbb{R} \). In this paper, the above formula will prove very useful in establishing both consistency and rates of convergence of our density estimator of \( f_0 \). Butucea and Vandekerkhove (2014) use a different approach based on writing the mixture model (2) in the Fourier domain. An estimator of \( f_0 \) is then obtained after estimating the mixture parameters and writing the inverse of the Fourier transform of a classical kernel estimator of the mixed density \( g_0 \) after deconvolution. Chee and Wang (2013) use kernel density estimators for the mixture component which allows them to use a maximum likelihood approach for the parameters. They write the unknown density itself as a mixture of the form

\[
f_h(x; G) = \frac{1}{2} \sum_{j=1}^{m} \lambda_j (K_h(x - \theta_j) + K_h(x + \theta_j))
\]

where \( G \) is the discrete distribution with support points \( \theta_j \) and probability masses \( \lambda_j, 1 \leq j \leq m, h > 0, K_h = h^{-1} \exp(-x^2/(2h^2)) / (\sqrt{2\pi}) \). With \( h \) fixed, the likelihood of the parameters \( (\pi^0, u_1^0, u_2^0) \) can be maximized. Then using a model selection procedure (cross-validation or Akaike or Bayesian information criterion) \( h \) and the corresponding parameters are selected.

In our implementation of the methods in this paper, we shall use the estimates of Hunter et al. (2007) and therefore we would like to give a brief overview of their method. It follows from Theorems 1 and 2 of Hunter et al. (2007) that if \( G_0 \) is the cumulative distribution function (cdf) of the observations, \( \pi^0 \notin \{0,1/2,1\} \) and \( u_1^0 < u_2^0 \) (without loss of generality) are the true mixture parameters, then for any \( p > 0 \)

\[
\int_{\mathbb{R}} \left| \pi \left( 1 - G_0(u_1 - x) - G_0(x + u_1) \right) + (1 - \pi) \left( 1 - G_0(u_2 - x) - G_0(x + u_2) \right) \right|^p dx = 0
\]

(3)
if and only if \((\pi, u_1, u_2) = (\pi^0, u_1^0, u_2^0)\). To explain better their idea, we follow here the notation of Hunter et al. (2007) and let \(\Delta(x; \pi, u_1, u_2) = \pi [x \geq u_1] + (1-\pi) [x \geq u_2] \) be the distribution of the mixture of the Dirac’s at the points \(u_1\) and \(u_2\) with respective mixing probabilities \(\pi\) and \(1-\pi\). Also, let \(\Delta^-(x; \pi, u_1, u_2) = \Delta(x; \pi, -u_1, -u_2)\). Note that the mixture model in (2) can be equivalently written under the convolution form \(G^0 = F^0 \ast \Delta(\cdot; \pi^0, u_1^0, u_2^0)\), where \(F^0(x) = \int_{-\infty}^x f^0(z) \, dz\). Now, Theorems 1 and 2 of Hunter et al. (2007) imply that if the conditions on the mixing probability and locations are satisfied, that is \(u_1 \neq u_2\) and \(\pi \notin \{0, 1/2, 1\}\), then \((\pi^0, u_1^0, u_2^0)\) is the only parameter \((\pi, u_1, u_2)\) that makes \(G^0 \ast \Delta^-(\cdot; \pi, u_1, u_2)\) symmetric. Consequently, \((\pi^0, u_1^0, u_2^0)\) is the only parameter \((\pi, u_1, u_2)\) such that

\[
\pi G^0(x + u_1) + (1-\pi) G^0(x + u_2) = 1 - \pi G^0(-x + u_1) + (1-\pi) G^0(-x + u_2)
= \pi (1 - G^0(u_1 - x)) + (1-\pi)(1 - G^0(u_2 - x))
\]

and hence the form of the contrast function given in (3). For numerical implementation, Hunter et al. (2007) use the Euclidean distance; i.e., they take \(p = 2\). To estimate the true mixture parameters, the authors minimize the same distance after replacing \(G^0\) by \(\tilde{G}_n\), the empirical distribution function based on the observations \(X_1, \ldots, X_n\). Their method will be revisited in more detail in Section 5.

In their second step, Hunter et al. (2007) estimate the distribution function, \(F^0\), by noting that for \(x \in \mathbb{R}\), \(F^0(x - u_1^0)\) and \(F^0(x - u_2^0)\) are the solution of a simple linear system; see their equation (12). Estimates of these quantities are obtained by plugging in the estimates of the mixture parameters and replacing \(G^0\) (showing up on the left side of the system) by \(\tilde{G}_n\). As mentioned earlier, the drawback of this approach is that there is no guarantee that the obtained estimate of \(F^0\) has the properties of a genuine distribution function.

In the three aforementioned papers, consistency results and rates of convergence of the estimators of the mixture parameters and the unknown symmetric distribution were obtained, under different assumptions. The estimators of Hunter et al. (2007) and Butucea and Vandekerkhove (2014) for the mixture parameters are shown to converge weakly to a multivariate Gaussian at the parametric rate \(n^{-1/2}\) under some regularity conditions on \(F^0\) which are related to smoothness in the case of Butucea and Vandekerkhove (2014). Bordes et al. (2006) obtain also a convergence rate under smoothness assumptions, but their rate of convergence is much slower (of order \(n^{-1/4+\alpha}\), for any \(\alpha > 0\)). Bordes et al. (2006) show that the same rates of convergence are inherited by their kernel estimator of \(F^0\) in the supremum norm, under the assumption that both locations are unknown. At the level of the density \(f^0\) (or \(g^0\) itself, Bordes et al. (2006) do not provide a rate of convergence, only almost sure consistency in the supremum norm. For their kernel estimator, Butucea and Vandekerkhove (2014) obtain, for estimating \(f^0\) at fixed point, a rate of convergence of order \(n^{-(2\beta-1)/(4\beta)}\) in the quadratic risk assuming smoothness of level \(\beta > 1/2\) and assuming that the bandwidth is chosen optimally (Butucea and Vandekerkhove (2014) suggest using cross validation).


2 Existence and form of the estimator

We begin by studying some finite sample properties of the estimator, specifically by showing its existence and demonstrating certain properties it has. First, we set up our notation.

- $G_n$ is the empirical distribution of i.i.d. observations $X_1, \ldots, X_n$, that is, $G_n(x) = n^{-1} \sum_{i=1}^{n} 1_{\{X_i \leq x\}}, x \in \mathbb{R}$.
- $P_n$ is the empirical measure based on the observations $X_1, \ldots, X_n$, that is $P_n(B) = n^{-1} \sum_{i=1}^{n} 1_{\{X_i \in B\}}$ for any measurable set $B$.
- $\mathcal{SLC}$ is the class of symmetric log-concave nonnegative functions on $\mathbb{R}$.
- $\mathcal{SLC}_1$ is the subset of $\mathcal{SLC}$ of symmetric log-concave densities on $\mathbb{R}$.
- $\mathcal{SC}$ is the class of symmetric concave functions on $\mathbb{R}$.
- If $f$ and $g$ are probability densities on $\mathbb{R}$, then $H(f, g)$ denotes the Hellinger distance between $f$ and $g$, that is $H^2(f, g) = \frac{1}{2} \int_{\mathbb{R}} \left( \sqrt{f(x)} - \sqrt{g(x)} \right)^2 dx = 1 - \int_{\mathbb{R}} \sqrt{f(x)g(x)} dx$.
- For two integrable functions $f$ and $g$ on $\mathbb{R}$, we denote the $L_1$ distance between $f$ and $g$ by $L_1(f, g) = \int_{\mathbb{R}} |f(x) - g(x)| dx$.
- $G_n$ is the empirical distribution of the independent observed data $X_1, \ldots, X_n$ from the true mixed density $g^0$.
- $|C|$ denotes the length of any convex subset $C$ of $\mathbb{R}$.
- Leb$(S)$ denotes the Lebesgue measure of any measurable subset $S$ of $\mathbb{R}$.

Now, let $X_1, \ldots, X_n$ be $n$ independent observations assumed to come from the location mixture

$$g^0(x) = \pi^0 f^0(x - u_1^0) + (1 - \pi^0) f^0(x - u_2^0),$$

with cdf $G^0$, for some $\pi^0 \in (0, 1/2)$, $u_1^0, u_2^0 \in \mathbb{R}$ such that $u_1^0 \neq u_2^0$, and $f^0 \in \mathcal{SLC}_1$.

Let us fix $\pi \in (0, 1/2)$ and $u_1, u_2 \in \mathbb{R}$ such that $u_1 < u_2$. These parameters may and will depend randomly on the data, although we will often suppress the subscript of $n$ and the commonly-used hat notation since they will be taken as fixed when we estimate the symmetric mixture component density. In the following, we consider maximizing the log-likelihood

$$\sum_{j=1}^{n} \log \left( \pi f(X_j - u_1) + (1 - \pi) f(X_j - u_2) \right)$$

(5)
over the class of symmetric log-concave densities $f$ defined on $\mathbb{R}$. This is equivalent to maximizing the criterion $\Phi_n$ defined as

$$
\Phi_n(\psi) = \frac{1}{n} \sum_{j=1}^{n} \log \left[ \pi e^{\psi(X_i - u_1)} + (1 - \pi) e^{\psi(X_i - u_2)} \right] - \int_{\mathbb{R}} e^{\psi(x)} \, dx
$$

over the class $SC$, using the Lagrange penalty term introduced by Silverman (1982). In the next proposition we establish existence of the estimator, and describe its nature. This MLE of $f$ will be denoted by $\hat{f}_n$. We remind the reader the dependence of this estimator on the choice of the parameters $\pi$ and $u_1$ and $u_2$.

**Proposition 2.1** The criterion $\Phi_n$ admits a maximizer $\hat{\psi}_n$. Furthermore, the following holds true (almost surely), letting $\hat{f}_n = e^{\hat{\psi}_n}$.

- $\hat{f}_n$ is in $SLC_1$.
- For $i = 1, \ldots, n$, let $Z_{2i-1} = |X_i - u_1|$ and $Z_{2i} = |X_i - u_2|$. Then, on $[0, \infty)$ the MLE $\hat{\psi}_n$ changes slope only at points belonging to the set

$$
\{Z_1, Z_2, \ldots, Z_{2n-1}, Z_{2n}\}.
$$

Furthermore, $\hat{\psi}'_n(0) = 0$, and $\hat{\psi}_n(x) = -\infty$ if and only if $x \notin [-Z_{(2n)}, Z_{(2n)}]$ where $Z_{(2n)}$ is the largest order statistic of $Z_1, Z_2, \ldots, Z_{2n-1}, Z_{2n}$.

In the following, we give a necessary condition for a log-concave function $f = \exp(\psi)$ to be the MLE. This condition exhibits a great similarity with the characterization of the log-concave MLE studied by Dümbgen and Rufibach (2009). However, this condition is not sufficient since it is not guaranteed that the likelihood surface is unimodal with a unique mode as is the case in estimating a log-concave density.

**Proposition 2.2** Let $\psi$ be a symmetric concave function on $\mathbb{R}$ such that $\psi(x) = -\infty$ if and only if $x \notin [-Z_{(2n)}, Z_{(2n)}]$ where $Z_{(2n)}$ is defined above, and $\psi'(0) = 0$. If $\exp(\psi) = \hat{f}_n$ is the MLE, then for any real symmetric function $\Delta$ such that $\psi + \epsilon \Delta \in SC$ for some $\epsilon > 0$ we have that

$$
\frac{1}{n} \sum_{i=1}^{n} \left\{ \hat{p}_n(X_i) \Delta(X_i - u_1) + (1 - \hat{p}_n(X_i)) \Delta(X_i - u_2) \right\} \leq \int_{\mathbb{R}} \hat{f}_n(x) \Delta(x) \, dx
$$

where

$$
\hat{p}_n(X_i) = \frac{\pi \hat{f}_n(X_i - u_1)}{\pi \hat{f}_n(X_i - u_1) + (1 - \pi) \hat{f}_n(X_i - u_2)} = \frac{\pi \hat{f}_n(X_i - u_1)}{\hat{g}_n(x)},
$$

for $i = 1, \ldots, n$. 

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Mixture problems like the one we are studying in this paper impose many challenges despite their very simple form. As noted above, the condition in (6) is only necessary, and hence cannot be viewed as something that characterizes the estimator. Also, it is not clear that the MLE is unique. These are of course unpleasant discoveries while aiming at making inference about the unknown component $f^0$. On the bright side, this seems not to matter a lot asymptotically as we will prove below.

Before studying consistency of the MLE, we would like to give another form of the condition in (6). D"umbgen and Rufibach (2009) shows that the log-concave MLE is uniquely characterized by the fact that the first integral of the cdf of the MLE stays below the first integral of the empirical distribution, while touching it exactly at the points where the logarithm of the MLE changes slope. To derive a related result, recall the definitions of the modified observations $Z_{2i-1} = |X_i - u_1|$ and $Z_{2i} = |X_i - u_2|, i = 1, \ldots, n$ and $Z_{(i)}, i = 1, \ldots, 2n$, the corresponding order statistics. Also, let $\hat{F}_n$ denote the cdf of the discrete distribution putting mass $\hat{p}_n(X_i)/n$ at $Z_{2i-1}$ and $(1 - \hat{p}_n(X_i))/n$ at $Z_{2i}$ for $i = 1, \ldots, n$, where $\hat{p}_n(X_i)$ was defined in (7). In other words,

$$\hat{F}_n = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{p}_n(X_i) \delta_{Z_{2i-1}} + (1 - \hat{p}_n(X_i)) \delta_{Z_{2i}} \right)$$

where $\delta_x(t) = 1_{\{x, \infty)\}(t)}$. Let $\hat{f}^+_n(x) = 2\hat{f}_n(x)1_{x \in [0, \infty)}$, $\hat{\psi}^+ = \log(\hat{f}^+)$ and let $\hat{F}^+_n$ be the cdf of $\hat{f}^+_n$.

Proposition 2.3 If $\hat{f}_n$ is the MLE of the true log-concave component $f^0$ then

$$\int_0^z \hat{F}^+_n(x) dx \begin{cases} \leq \int_0^z \hat{F}_n(x) dx, & \text{for } z \in [0, Z_{(2n)}] \\ = \int_0^z \hat{F}_n(x) dx, & \text{if } \hat{\psi}_n^+(z-) > \hat{\psi}_n^+(z+). \end{cases}$$

3 Consistency

The main result of this section is to establish consistency of the MLE $\hat{f}_n$ as $n \to \infty$. Our approach to the problem follows the idea of Pal et al. (2007) and D"umbgen and Rufibach (2009) but will require handling carefully the extra complexity induced by the mixture. As in Pal et al. (2007), Cule and Samworth (2010) and Schuhmacher and D"umbgen (2010), we will first need to show that the MLE of the mixed density and hence the MLE of the log-concave component are bounded. Here, the claimed boundedness will be only in probability, which is weaker than the almost sure boundedness proved in the aforementioned articles. Let $\mathcal{C}$ be the class of convex and compact subsets in $\mathbb{R}$. We start by stating, for the reader’s convenience, a Glivenko-Cantelli type of theorem.
Proposition 3.1 Fix $\alpha \in (0,1)$. Let $X_1, \ldots, X_n$ be $n$ independent observations from the density $g^0 = \pi f^0(-u_1^0) + (1 - \pi f^0(-u_2^0)$, and $u_1$ and $u_2$ be any consistent estimators of $u_1^0$ and $u_2^0$. Write $\bar{u} = (u_1 + u_2)/2$, $\bar{u}^0 = (u_1^0 + u_2^0)/2$ and let $\bar{N} = \sum_{i=1}^n 1_{X_i \leq \bar{u}}$. Then, for any $\epsilon > 0$, there exists $n_0$ depending on $\epsilon$ such that

$$P\left(\min \left\{|C| : C \in \mathcal{C} \text{ such that } \bar{N}^{-1} \sum_{i: X_i \leq \bar{u}} 1_{\{X_i \in C\}} \geq \alpha\right\} \geq \frac{\alpha G(\bar{u}^0)}{2\|g^0\|_{\infty}}\right) \geq 1 - \epsilon$$

for all $n \geq n_0$.

Lemma 3.1 Let $\psi_0 \in \mathcal{SC}$, $a < b \in \mathbb{R}$ and $\lambda \in [0,1]$. If $p_0(t) = \lambda \exp(\psi_0(t-a)) + (1 - \lambda) \exp(\psi_0(t-b))$, then

$$\int_{\mathbb{R}} |\log(p_0(t))| p_0(t) dt < \infty.$$

In the following, we introduce a condition that will prove to be useful in proving consistency of our estimator.

The condition $(\mathcal{F})$: We will say that a pair $(f, h) = (e^\psi, e^\phi)$ of log-concave densities satisfy the condition $(\mathcal{F})$ if for all $d \in \mathbb{R}$,

$$\int_{0}^{\infty} |\phi(t + d)| f(t) dt < \infty.$$
Figure 2: Left: plot of the true density of \((1/3)\mathcal{N}(0, 1) + (2/3)\mathcal{N}(4, 1)\) (solid line) and its log-concave estimator \(\hat{g}_n\) (dotted line). Right: plot of the density of \(\mathcal{N}(0, 1)\) and its symmetric log-concave MLE \(\hat{f}_n\). The MLE was based on \(n = 500\) independent data drawn from the mixture density \((1/3)\mathcal{N}(0, 1) + (2/3)\mathcal{N}(4, 1)\).

Note that \(h\) must have support \((-\infty, \infty)\).

Lemma 3.2 Let \(\psi_0\) and \(\phi\) be elements of \(SC\). Let \(a < b\) be real numbers and \(\lambda \in [0, 1]\). Let \((a_k), (b_k),\) and \((\lambda_k)\) be sequences of real numbers converging to \(a, b,\) and \(\lambda,\) respectively. Let

\[
p_k(\cdot) = \lambda_k \exp(\psi_0(\cdot - a_k)) + (1 - \lambda_k) \exp(\psi_0(\cdot - b_k)),
\]

\[
p_{k,\phi}(\cdot) = p_k(\cdot) \ast e^{\phi(\cdot)} = \lambda_k \exp(\psi_0(\cdot - a_k)) \ast e^{\phi(\cdot)} + (1 - \lambda_k) \exp(\psi_0(t - b_k)) \ast e^{\phi(\cdot)},
\]

for \(k = 0, 1, \ldots,\) where by \(a_0, b_0,\) and \(\lambda_0\) we mean \(a, b,\) and \(\lambda,\) respectively. Let \(e^{\psi_0}\) and \(e^{\phi}\) satisfy condition \((F)\). Then

\[
\int_{\mathbb{R}} \log (p_{k,\phi}(t)) p_0(t) dt \longrightarrow \int_{\mathbb{R}} \log (p_{0,\phi}(t)) p_0(t) dt \quad (9)
\]

as \(k \to \infty.\)

Note that for any log-concave density \(f = e^{\psi},\) if \(\phi\) is a polynomial function, then \(f\) and \(e^{\phi}\) satisfy condition \((F),\) since log-concave densities have finite moments of all orders. In particular, we may take \(e^{\phi}\) to be any normal density, which we will do below in the proof of consistency; see Theorem 3.2.
Proposition 3.2 Let $M > 0$ and $\mathcal{LC}_M$ be the class of log-concave functions bounded by $M$. For a fixed $b > 0$, consider the class of functions
\[
F_b = \left\{ f : f = \log(\lambda h_1 + (1-\lambda)h_2 + b), \ h_1, h_2 \in \mathcal{LC}_M, \ \lambda \in [0,1] \right\}.
\]
Then $F_b$ is a Glivenko-Cantelli class for any probability measure $P$.

Proposition 3.3 Let $\exp(\psi_0)$ be a fixed log-concave density in $\mathcal{SLLC}_1$, $\lambda \in [0,1]$, $a < b$ and let $p_0(x) = \lambda \exp(\psi_0(x-a)) + (1-\lambda) \exp(\psi_0(x-b))$, $x \in \mathbb{R}$, and $P_0$ the corresponding probability measure. Let $e^\psi \in \mathcal{SLLC}_1$ with support $(-\infty, \infty)$, and consider the class of functions
\[
H = \left\{ h : h(x) = \log(\pi \exp(\psi(x-a)) + (1-\pi) \exp(\psi(x-b))), \ \alpha \in [a-\eta/2, a+\eta/2], \beta \in [b-\eta/2, b+\eta/2] \right\},
\]
where $\eta \in (0, \min((b-a)/2$. Then $H$ is $P_0$-Glivenko-Cantelli.

Theorem 3.1 Let $\hat{g}_n$ denote again the MLE of the mixed density. Then, for a given $\epsilon > 0$ there exist an integer $n_0 > 0$ and $D > 0$ depending on $\epsilon$ such that
\[
P(\|\hat{g}_n\|_{\infty} \leq D) \geq 1 - \epsilon.
\]
for all $n \geq n_0$.

To establish the first consistency result, let $b > 0$, and $G^0$ be the cdf of the true mixed density $g^0$. Let
\[
g = \pi f^0(\cdot - u_1) + (1-\pi)f^0(\cdot - u_2).
\]
By definition of $\hat{g}_n$ we have that for any symmetric log-concave density $\tilde{g}$,
\[
0 \leq \int \log \hat{g}_n \ dG_n - \int \log \tilde{g} \ dG_n \leq \int \log(\hat{g}_n + b) \ dG_n - \int \log \tilde{g} \ dG_n.
\]
As first established by Pal et al. (2007) in their Lemma 1, the inequality above yields
\[
2H^2(\hat{g}_n, g^0) \leq \epsilon(b) + \int_{\mathbb{R}} \log(\hat{g}_n(t) + b)d(G_n(t) - G^0(t))
\]
\[
+ \int_{\mathbb{R}} \log(g^0(t) + b)dG^0(t) - \int_{\mathbb{R}} \log(\tilde{g}(t))dG^0(t)
\]
\[
- \int_{\mathbb{R}} \log(\tilde{g}(t))d(G_n(t) - G^0(t))
\]
with
\[ \epsilon(b) = 2 \int_{\mathbb{R}} \sqrt{b + g^0(t)} dG^0(t). \]

The main idea behind introducing the small positive quantity \( b \) is to avoid integration issues due to the fact that \( \log(\hat{g}_n) = -\infty \) outside a certain interval. In the problem of estimating a log-concave density, the empirical term on the right side of the previous inequality was shown to converge to zero using the fact that the maximum value of the log-concave MLE stays bounded in \( n \), see Theorem 3.2 of Pal et al. (2007) and Lemma 4 in Schuhmacher and Dümbgen (2010) for a generalization of the same result in the multivariate setting. This property of the MLE was then combined with the fact that a level set of a bounded unimodal function is convex and compact. This cannot be claimed anymore if we replace a unimodal function by a mixture of two unimodal functions, not even when those functions are log-concave. For this reason, we shall use instead the Glivenko-Cantelli results proved above.

In our proof, the natural choice for \( \tilde{g} \) would be \( g \), but this is problematic. For instance \( \int \log gdG_n \) can be \( -\infty \) for arbitrarily large \( n \) if \( g^0 \) has compact support. This might occur when the smallest/ largest order statistic is very close to the left/right end of the support if \( f^0 \) and \( u_1/u_2 \) is smaller/larger than \( u_0^1/u^0_2 \). But even if \( g^0 \) does not have compact support, if it does have too large a slope (e.g., it approximates having compact support by dropping towards 0 very quickly at some point), then even \( \int \log g dG^0 \) can be infinite. Thus, we consider the surrogate density function
\[ g_h = \pi f^0_h(\cdot - u_1) + (1 - \pi) f^0_h(\cdot - u_2) \tag{12} \]
where \( f^0_h = f^0 \ast \varphi_h \), the convolution of \( f^0 \) and the density of centered normal with standard deviation \( h > 0 \). This choice alleviates the above-mentioned problems; the risk of having a divergent log-likelihood is now excluded since \( f_h \), the component density of \( g_h \), is supported on \( \mathbb{R} \). Note that \( f_h \) is a log-concave density by stability of log-concavity under convolution, and is symmetric. More importantly, \( g^0 \) and \( g_h \) satisfy the condition \( (\mathcal{F}) \) and this will enable us to use the result of Lemma 3.2 since \( \int \log g_h dG^0 \) is indeed finite. This integral will be also shown to converge to an appropriate limit as \( h \searrow 0 \).

**Theorem 3.2** Let \( g^0 \) be as in (4) and \( \hat{g}_n \) be the MLE of \( g^0 \). Then we have that
\[ H(\hat{g}_n, g^0) = o_p(1). \]

Consistency of the log-concave component, \( \hat{f}_n \), follows now from Theorem 3.2. In the proof, we will use the well-know fact that for any two densities \( p_1 \) and \( p_2 \)
\[ \frac{1}{4} \left( \int_{\mathbb{R}} |p_1(t) - p_2(t)| dt \right)^2 \leq H^2(p_1, p_2) \leq \int_{\mathbb{R}} |p_1(t) - p_2(t)| dt. \tag{13} \]
Corollary 3.1 Let $f^0$ denote again the true log-concave symmetric density. Then,

$$H(\hat{f}_n, f^0) = o_p(1),$$

and for any $a \in (0, a_0)$ such that $f^0(x) \leq \exp(-a_0x + b)$ for some $b \in \mathbb{R}$, then

$$\int_{\mathbb{R}} e^{at}|\hat{f}_n(t) - f^0(t)| = o_p(1),$$

and

$$\sup_{t \in [-A,A]} e^{at}|\hat{f}_n(t) - f^0(t)| = o_p(1).$$

on any continuity set $[-A,A]$ of $f^0$.

4 Rates of convergence

In this section, we aim at refining the convergence result obtained in the previous section. To this goal, we need first to recall some definitions from empirical processes theory. Given a class of functions $F$, the bracketing number of $F$ under some distance $\| \cdot \|$ is defined as

$$N_{[]}(|\epsilon, F, \| \cdot \|) = \min \{k : \exists f_1, \bar{f}_1, \ldots, f_k, \bar{f}_k \text{ s.t. } \| f_i - \bar{f}_j \| \leq \epsilon, F \subset \bigcup_{i=1}^{k}[f_i, \bar{f}_i]\}$$

where $[l, u] = \{f : f \in F : l \leq f \leq u\}$. In this section, we refine the consistency result above by deriving the rate of convergence of the MLE’s $\hat{g}_n$ and $\hat{f}_n$ of the mixed density and the symmetric log-concave component respectively.

For fixed $M > 0$, $a_0 < b_0$ and $\delta \in (0, (b_0 - a_0)/2)$, consider the class of functions

$$G = \left\{ \lambda f(\cdot - a) + (1 - \lambda)f(\cdot - b), f \in SLC, \ f(0) \in [1/M, M], \lambda \in [0,1], \right. \left. (a,b) \in [a_0 - \delta, a_0 + \delta] \times [b_0 - \delta, b_0 + \delta] \right\}.$$

Here, the parameters $a_0$ and $b_0$ play the role of the true location shifts $u^0_1$ and $u^0_2$. Consistency of the estimates $u_1$ and $u_2$ ensures that they are stay within distance $2\delta$ from the truth with increasing probability. Also, uniform consistency of the log-concave MLE, $\hat{f}_n$, on continuity sets of $f^0$ implies consistency at the point 0 (the common mode of $f^0$ and $\hat{f}_n$). Thus, we can find $M > 0$ such that $\hat{f}_n(0) \in [1/M, M]$ with increasing probability. The following proposition gives a bound on the bracketing entropy for the class $G$. 

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Proposition 4.1 For $\epsilon \in (0, \epsilon_0]$, we have that
\[ \log N(\epsilon, G, H) \lesssim \frac{1}{\sqrt{\epsilon}} \]
where $\epsilon_0$ and $\lesssim$ depend only on $a_0, b_0, \delta$ and $M$.

Now, we are ready to state our main theorem.

Theorem 4.1 Let $\hat{f}_n$ and $\hat{g}_n$ be again the MLE’s of the symmetric log-concave component and mixed density respectively. If $\sqrt{n}(u_1 - u_1^0) = O_p(1)$ and $\sqrt{n}(u_1 - u_2^0) = O_p(1)$, then
\[ L_1(\hat{f}_n, f^0) = O_p(n^{-2/5}), \quad \text{and} \quad L_1(\hat{g}_n, g^0) = O_p(n^{-2/5}). \]

5 Computations, testing and clustering

5.1 Consistent estimation of the mixture parameters and computation of the MLE

As mentioned in the introduction, Bordes et al. (2006), Hunter et al. (2007) and Butucea and Vandekerkhove (2014) in their more recent work propose three different ways of estimating the mixture parameters $\pi^0, u_1^0, u_2^0$. As we are interested here in $\sqrt{n}$-consistent estimators of these parameters, the work by Hunter et al. (2007) and Butucea and Vandekerkhove (2014) is of more interest to us. Note that the approach proposed by Bordes et al. (2006) yields a $\sqrt{n}$-consistent estimator for $\pi^0$ in case $u_1^0$ and $u_2^0$ are known. Also, due to some numerical instabilities encountered when computing the estimators proposed by Butucea and Vandekerkhove (2014), we adopt the approach of Hunter et al. (2007) which has been already implemented in R; one could either used the code posed at http://www.stat.psu.edu/~dhunter/code or the function in the the mixtools package. The latter option was kindly brought to the attention of the first author by David Hunter in a private communication. We first give a brief description of the estimation approach as well as the resulting estimators. Recall that the independent observations $X_1, \ldots, X_n$ are assumed to have the common density
\[ g^0(x) = \pi^0 f^0(x - u_1^0) + (1 - \pi^0) f^0(x - u_2^0). \]
Assuming that $\pi^0 \in (0, 1/2)$ and $u_1^0 < u_2^0$, Theorem 2 of Hunter et al. (2007) implies that the above mixture is identifiable. Adopting a similar notation to the one used by the authors, let us write $\Delta(\lambda, \mu_1, \mu_2)$ the discrete distribution charging $\mu_1$ and $\mu_2$ with the respective probability masses $\lambda$ and $1 - \lambda$. Then, if $F^0$ and $G^0$ are the
true cumulative distribution functions associated with \( f^0 \) and \( g^0 \), then the mixture model can also be rewritten
\[
G^0 = F^0 \star \Delta(\pi^0, u_1^0, u_2^0).
\]

Theorem 1 of Hunter et al. (2007) implies that identifiability in this case is equivalent to saying that \( \Delta^-(\pi^0, u_1^0, u_2^0) \), the reflection of \( \Delta(\pi^0, u_1^0, u_2^0) \) over the origin, is the unique distribution that yields a zero symmetric distribution when convoluted with \( \Delta(\pi^0, u_1^0, u_2^0) \). Note that \( \Delta^-(\pi^0, -u_1^0, -u_2^0) = \Delta(\pi^0, -u_1^0, -u_2^0) \). This in turn implies that \( \Delta(\pi^0, -u_1^0, u_2^0) \) is the unique distribution such that
\[
G^0 \star \Delta(\pi^0, -u_1^0, -u_2^0)
\]
is zero symmetric. The idea of Hunter et al. (2007) is to exploit this property of \( \Delta(\pi^0, -u_1^0, -u_2^0) \) and look for the parameters \( \pi, u_1, u_2 \) that minimize the distance between \( G_n \star \Delta(\pi, -u_1, -u_2) \) and \( G_n^- \star \Delta(\pi, u_1, u_2) \), with \( G_n^- \) is the reflection of \( G_n \) over the origin. When this distance is chosen to be the \( L_2 \) distance, then Hunter et al. (2007) show that the minimizer \( (\pi, u_1, u_2) \) can be viewed as a generalization of the Hodges-Lehmann estimator. Furthermore, if we define
\[
a_j(t; \mu_1, \mu_2) = \frac{1}{n} \sum_{i=1}^{n} 1\{\mu_j - X_i \leq t\} - \frac{1}{n} \sum_{i=1}^{n} 1\{X_i - \mu_j \leq t\}
\]
for \( j = 1, 2 \), then the estimator \( (\pi, u_1, u_2) \) minimizes
\[
(\lambda, \mu_1, \mu_2) \mapsto \|\lambda a_1 + (1 - \lambda) a_2\|^2.
\]
The dimension of the minimization problem can be reduced after noting that the “profile” mixing probability is given by
\[
\lambda(\mu_1, \mu_2) = -\frac{<a_1 - a_2, a_2>}{\|a_1 - a_2\|^2}.
\]
Thus, the estimators of the mixture locations \( u_1 \) and \( u_2 \) minimize the criterion
\[
(\mu_1, \mu_2) \mapsto \frac{\|a_1\|^2\|a_2\|^2 - <a_1, a_2>^2}{\|a_1 - a_2\|^2}.
\]
To find the minimizer \( (u_1, u_2) \), we use the R function “sm.optimize” of the code posted at the URL address indicated above, which is based on the function “optim”. To increase accuracy, we use 100 different starting values obtained by increasingly ordering two randomly generated numbers from \([X_1, X_n]\). We choose the minimizing solution with the smallest value of the criterion function.

Once the estimates of \( \pi^0, u_1^0 \) and \( u_2^0 \) are computed, we use the EM algorithm to find the maximum of the likelihood over the class of log-concave symmetric densities. More precisely, let \( \pi, u_1 \) and \( u_2 \) be the obtained estimates. If we denote \( f^+ = \ldots \)
$2f1_{[0,\infty)}$, $Z_{2i-1} = |X_i - u_1|$ and $Z_{2i} = |X_i - u_2|$ for $i = 1, \ldots, n$, then maximizing the log-likelihood in (5) is equivalent to maximizing

$$
\frac{1}{n} \sum_{j=1}^{n} \log \left( \pi f^+(Z_{2i-1}) + (1 - \pi) f^+(Z_{2i}) \right)
$$

over the class of log-concave and decreasing densities on $[0, \infty)$. To initialize the EM algorithm, we start with $\hat{f}^{+,(0)} = 2f_01_{[0,\infty)}$ where $f_0$ is the density of a centered Gaussian distribution with variance equal to the estimate given in formula (11) of Hunter et al. (2007) for the true variance of the symmetric component, that is $\hat{f}^+(Z_i)$.

In case this estimate is negative (this may occur for moderate sample sizes), we replace it simply by the value 1. Then, using the function ‘activeSetLogCon.mode’ available from the R package logcondens.mode, we compute $\hat{f}^{+, (1)}$, the log-concave MLE on $[0, \infty)$ constrained to have mode at 0 which maximizes the weighted log-likelihood

$$
l_n^{(1)} = \frac{1}{n} \sum_{j=1}^{n} \left( \hat{p}_{n,j}^{(1)} f^+(Z_{2i-1}) + (1 - \hat{p}_{n,j}^{(1)}) f^+(Z_{2i}) \right)
$$

where

$$
\hat{p}_{n,i}^{(1)} = \frac{\pi \hat{f}^{+,(0)}(Z_{2i-1})}{\pi \hat{f}^{+,(0)}(Z_{2i-1}) + (1 - \pi) \hat{f}^{+,(0)}(Z_{2i})}
$$

is the posterior probability for the transformed datum $Z_{2i-1}$. The posterior probabilities are updated at each step in the usual way: if $f^{+,(r)}$ is the log-concave maximizer obtained at the $r$-step of the algorithm, then the new posterior probabilities are

$$
\hat{p}_{n,i}^{(r)} = \frac{\pi \hat{f}^{+,(r)}(Z_{2i-1})}{\pi \hat{f}^{+,(r)}(Z_{2i-1}) + (1 - \pi) \hat{f}^{+,(r)}(Z_{2i})}, \quad \text{and} \quad 1 - \hat{p}_{n,i}^{(r)} = \frac{(1 - \pi) \hat{f}^{+,(r)}(Z_{2i})}{\pi \hat{f}^{+,(r)}(Z_{2i-1}) + (1 - \pi) \hat{f}^{+,(r)}(Z_{2i})}
$$

which are then used, at the $(r + 1)$-th step, to maximize

$$
l_n^{(r+1)} = \frac{1}{n} \sum_{j=1}^{n} \left( \hat{p}_{n,j}^{(r)} f^+(Z_{2i-1}) + (1 - \hat{p}_{n,j}^{(r)}) f^+(Z_{2i}) \right).
$$

The algorithm is stopped when both $|l_n^{(r+1)} - l_n^{(r)}|$ and $\max_{1 \leq j \leq 2n} |\hat{f}^{+,(r+1)}(Z_j) - \hat{f}^{+,(r)}(Z_j)|$ are smaller than some chosen thresholds. In all our simulations, these thresholds were chosen to be equal to $10^{-6}$. 

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5.2 How to simulate from the log-concave MLE

Let $X_1, \ldots, X_n$ be fixed. To generate $X^*$ with density equal to the MLE $\tilde{g}_n$, we need first to generate $Y^* \sim f_n$. We will use the same the simulation method of Dümbgen and Rufibach (2009) (implemented for log-concave density estimates in the R package logcondens (Dümbgen and Rufibach, 2010)). Let $y_0, \ldots, y_m$ be the knots of $\tilde{\varphi}_n = \log f_n$. The simulation steps are as follows:

1. generate $K \in \{1, \ldots, m\}$ with probability $P(K = k) = \tilde{F}_n(y_k) - \tilde{F}_n(y_{k-1})$, and define $\Delta := \tilde{\varphi}_n(y_K) - \tilde{\varphi}_n(y_{K-1})$.

2. generate $U \sim U[0, 1]$ independently of $K$.

3. set

$$Y^* = \begin{cases} y_{K-1} + (y_K - y_{K-1}) \frac{\log(1+U(\exp(\Delta-1)))}{\Delta}, & \text{if } \Delta \neq 0 \\ y_{K-1} + (y_K - y_{K-1})U, & \text{otherwise.} \end{cases}$$

If $Z \sim \text{Bernoulli}(\pi)$ such that $Z$ and $Y^*$ are independent, then

$$X^* := Z(Y^* + u_1) + (1 - Z)(Y^* + u_2) \sim \tilde{g}_n.$$  

Let us denote by $\tilde{\sigma}_n^2$ the variance of $\tilde{f}_n$. Let $y_0 = 0 < y_1 < \ldots < y_m = Z(2n)$ the knots of $\tilde{\psi}_n$. Recall that $\tilde{\psi}_n = \log(\tilde{f}_n)$ with $\tilde{f}_n^*(x) = 2\tilde{f}_n(x)1_{[0,\infty)}(x)$. We have that

$$\tilde{\sigma}_n^2 = \int_0^{y_m} x^2 \exp(\tilde{\psi}_n^*(x))dx$$

where

$$\int_{y_j}^{y_{j+1}} x^2 \exp(\tilde{\psi}_n^*(x))dx$$

$$= (y_{j+1} - y_j) \int_0^1 ((y_{j+1} - y_j)x + y_j)^2 \exp((1-x)\tilde{\psi}_n^*(y_j) + x\tilde{\psi}_n^*(y_{j+1}))dx$$

$$= (y_{j+1} - y_j)^3 \int_0^1 x^2 \exp((1-x)\tilde{\psi}_n^*(y_j) + x\tilde{\psi}_n^*(y_{j+1}))dx$$

$$+ 2(y_{j+1} - y_j)^2 y_j \int_0^1 x \exp((1-x)\tilde{\psi}_n^*(y_j) + x\tilde{\psi}_n^*(y_{j+1}))dx$$

$$+ (y_{j+1} - y_j)y_j^2 \int_0^1 \exp((1-x)\tilde{\psi}_n^*(y_j) + x\tilde{\psi}_n^*(y_{j+1}))dx.$$  

Put $\delta_j = y_{j+1} - y_j$. Then, the above calculations yield

$$\tilde{\sigma}_n^2 = \frac{y_1^3}{3} \tilde{f}_n^*(0) + \sum_{j=1}^{m-1} \left\{ \delta_j^3 J_{0,2}(\tilde{\psi}_n^*(y_j), \tilde{\psi}_n^*(y_{j+1})) + 2\delta_j^2 y_j J_{0,1}(\tilde{\psi}_n^*(y_j), \tilde{\psi}_n^*(y_{j+1})) \right\}$$

$$+ \delta_j y_j^2 J_{0,0}(\tilde{\psi}_n^*(y_j), \tilde{\psi}_n^*(y_{j+1})).$$
where $J_{a,b}$ is defined in the Appendix.

We generate $X^\ast_1, \ldots, X^\ast_n$ independently from the smoothed log-concave MLE $\hat{g}^s_n = \pi \hat{f}^s_n (\cdot - u_1) + (1 - \pi) \hat{f}^s_n (\cdot - u_2)$ where $\hat{f}^s_n = \hat{f}_n \ast \phi_n$, with $\phi_n$ the density of centered Gaussian density with standard deviation $\sigma$ and $\hat{n}$ is chosen so the variance of $\hat{f}^s_n$ is equal to

$$\hat{S}^2_n = \sum_{j=1}^n \left( \hat{p}_n(X_j)(X_j - u_1)^2 + (1 - \hat{p}_n(X_j))(X_j - u_2)^2 \right). \tag{14}$$

As will be shown below, the variance of the non-smooth log-concave MLE is always smaller than $\hat{\sigma}^2$. This is a consequence of the fact that $\hat{f}_n$ maximizes the likelihood among all concave symmetric functions. Note that a similar inequality occurs in the log-concave MLE and results also from the characterization of the estimator. This fact has been first by Dümbgen and Rufibach (2009) for univariate data and was later extended to arbitrary dimensions by Chen and Samworth (2013). Below, we explain how the bandwidth $\hat{n}$ can be computed.

To show that $\hat{\sigma}^2_n \leq \hat{S}^2_n$ defined above in (14), it is enough to note that the function

$$t \mapsto \hat{\psi}_n(t) - \epsilon t^2 = \hat{\psi}_n(t)$$

is an element in $\mathcal{SC}$ for any $\epsilon > 0$. Passing $\epsilon \downarrow 0$ yields

$$0 \geq \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\epsilon} \left( \Phi_n(\hat{\psi}_\epsilon) - \Phi_n(\hat{\psi}_n) \right)$$

$$= -\frac{1}{n} \sum_{i=1}^n \left( \hat{p}_n(X_i)(X_i - u_1)^2 + (1 - \hat{p}_n(X_i))(X_i - u_2)^2 \right) + \int_0^\infty t^2 \exp(\hat{\psi}_n^+(t)) dt$$

which yields the claimed inequality.

Hence,

$$\hat{n} = \sqrt{\hat{S}^2_n - \hat{\sigma}^2_n}. \tag{15}$$

Now, if $W \sim N(0,1)$ such that $W$ and $X^\ast$ are independent, then

$$X^{\ast,s} := X^\ast + \hat{n} W \sim \hat{g}^s_n.$$  

### 5.3 Testing the absence of mixing

Recall that the mixing model we consider in this paper is given by

$$g^0 = \pi^0 f^0 (\cdot - u^0_1) + (1 - \pi^0) f^0 (\cdot - u^0_2).$$

with $f^0$ a log-concave symmetric density on $\mathbb{R}$, $\pi^0 \notin \{0, 1/2, 1\}$ and $u^0_1 < u^0_2$. We now use our log-concave MLE to test for the absence of mixing, i.e. to test for the null
hypothesis that \( u_0^1 = u_0^2 \), against the alternative that \( u_0^1 \neq u_0^2 \) and \( \pi^0 \neq 1/2 \). Hunter et al. (2007) show that two-component symmetric mixture densities are identifiable, in the sense of their Definition 1, when \( \pi^0 \notin \{0, 1/2, 1\} \) and \( u_0^1 \neq u_0^2 \). In particular, this implies that in this case the mixture density \( g^0 \) cannot be itself symmetric around its mode, so that the null and alternative hypotheses are disjoint. Note that we need to exclude the case \( \pi^0 = 1/2 \) because, for example, if we take \( f^0 \) to be a uniform density on \( [-|u_2^0 - u_1^0|/2, (u_2^0 - u_1^0)/2] \), then the mixture \( g^0 \) is symmetric and log-concave, so we cannot identify it as being a mixture.

To test for mixing, we use a likelihood ratio as our test statistic. Under the null hypothesis, we take the estimator of the true density to be equal to the log-concave MLE which is symmetric around the median of the data. If \( \tilde{g}_n^0 \) denotes this estimator, then our test statistic is given by

\[
\Lambda_n = \frac{\prod_{i=1}^{n} \tilde{g}_n(X_i)}{\prod_{i=1}^{n} g_n(X_i)}. \tag{16}
\]

The null hypothesis is then rejected when \( \Lambda_n \) is too large. We use the null hypothesis estimator to find critical values; that is, we bootstrap from the symmetric log-concave estimator \( \tilde{g}_n^0 \). We refer to Section 5.2 for the details on how to sample from \( \tilde{g}_n^0 \). The critical values of \( \Lambda_n \) are then computed in the usual way: based on the bootstrapped samples from \( \tilde{g}_n^0 \), we compute the estimators of the mixing probability and mixture locations and the corresponding estimator \( \tilde{g}_n \). The order statistics of the bootstrapped values of the likelihood ratio are then obtained to compute upper empirical quantiles of a given order.

We compare our test for mixing against a procedure using the trace test of Chen and Samworth (2013). Note that the trace test aims to test the null hypothesis that the underlying density is log-concave, rather than the null hypothesis that it is symmetrically log-concave. The results of Hunter et al. (2007) show that symmetric log-concave densities are identifiable i.e., the mixture density \( g^0 \) in (2) cannot be symmetric around its mode. However, it is possible that it is log-concave. For instance, it is known that for any \( \pi \in [0, 1] \), the corresponding mixture of normal densities with variances 1 is log-concave if \( |u_2 - u_1| \leq 2 \). Thus, we expect the likelihood ratio test and the trace test to perform differently, as they indeed do.

In assessing the power, we take the true symmetric component \( f^0 \) to be one of the following distributions:

1. a standard Gaussian
2. a double exponential
3. a uniform on \([-1, 1]\]
4. a zero-symmetric Beta on \([-1, 1]\) with parameters \( \alpha = \beta = 2 \), that is

\[
f^0(t) = \frac{\Gamma(4)}{2\Gamma(2)^2} |t|(1 - |t|) 1_{[-1,1]}.
\]
Also, we take the true parameters to be

\[ \pi_0 \in \{0.20, 0.40\}, \]

\[ (u_0^1, u_0^2) \in \{(0, 0), (0, 1), (0, 3)\}. \]

We give the estimated probability of rejecting the null hypothesis based on \( R = 100 \) replications with \( B = 50 \) bootstrap samples in Table 1, for \( n = 100 \), and in Table 2, for \( n = 500 \). The simulation results show that, ignoring for the moment the Laplace cases, the LR test is more powerful across the board for detecting mixtures of symmetric densities than the TR test is, which is as expected. In some settings neither test has much power, whereas in others the LR test has some power and the TR test has very little. In the easier-to-distinguish cases in which \( u_0^2 - u_0^1 = 3 \), the LR test generally has power close to 1, especially with \( n = 500 \). The LR test is noticeably less powerful in these scenarios. The LR test is slightly anti-conservative when \( n = 100 \), but its level is close to the nominal .05 level in many scenarios. The trace test appears to be much more anti-conservative, and, in all the Laplace cases, behaves very erratically. The actual level appears to differ from the specified level of 0.05 quite noticeably. This effect appears to diminish as sample size increases, but it is unclear what causes this strange behavior.

In the cases with a normal density, notice that with \( u_0^2 - u_0^1 = 1 \), the mixture distribution has a log-concave density. Thus we do not expect the TR test to have power different than the type I error rate, and it does not. The LR test also has low power in that regime, since the mixture density is nearly symmetric. Similarly, when \( u_0^2 - u_0^1 = 1 \), the mixtures of Laplace densities (with \( \pi_0 = .2 \) or \( \pi_0 = .4 \)) are very close to being symmetrically log-concave, and so are difficult for the LR test to detect, regardless of the chosen sample size.

Perhaps counter-intuitively, the mixtures with mixture probability \( \pi_0 = .4 \) are, in many cases, more difficult to distinguish than those with \( \pi_0 = .2 \). However, this is explained by the following. Let \( d_{KL}(g_1, g_2) = \int \log(g_1(x)/g_2(x))g_1(x) \, dx \) be the Kullback-Leibler divergence between two densities \( g_1 \) and \( g_2 \), and let \( d_{KL}(g_1, \mathcal{G}) = \inf_{g_2 \in \mathcal{G}} d_{KL}(g_1, g_2) \) be the Kullback-Leibler divergence from \( g_1 \) to a class of densities \( \mathcal{G} \). Then if \( \pi_0 = .2 \), \( d_{KL}(g_0, \mathcal{SLC}) \) is generally larger than when \( \pi_0 = .4 \). A similar statement holds replacing \( \mathcal{SLC} \) by the class of all log-concave densities. So, that is, when \( \pi_0 = .2 \), the corresponding \( g_0 \) is farther from the null hypotheses.

The intuitive reason for this larger divergence is as follows. Let \( g^* \) be the log-concave density achieving the minimum Kullback-Leibler divergence from \( g_0 \) to the class of log-concave densities, which exists by the results of Dümbgen et al. (2011). Then \( g^*(x) \) tends to be small for \( x \) values near \( u_0^1 \) when \( \pi_0 = .2 \), whereas \( g^*(x) \) is not as small for \( x \) values near \( u_0^1 \) when \( \pi_0 = .4 \). The former case tends to lead to larger values of \( d_{KL}(g_0, g^*) \); for instance, the Kullback-Leibler divergence \( d_{KL}(g_1, g_2) \) is infinite when the support of \( g_2 \) does not contain the support of \( g_1 \). Thus, distinguishing many of the cases with \( \pi_0 = .2 \) should, in fact, be less difficult than those with \( \pi_0 = .4 \).
$$\pi_0 \in \{0.2, 0.4\}, \quad f_0 \text{ is the symmetric log-concave density shown in the first column.}$$

The values of the power are based on a sample drawn from the true mixture of size $n = 100$. The number of bootstraps and the number of replications were taken to be $B = 50$ and $R = 100$, respectively.

| Distribution | $\pi_0$ | $u_2^0 - u_1^0 =$ | Test | 0  | 1  | 3  |
|--------------|---------|-------------------|------|----|----|----|
| $\mathcal{N}(0, 1)$ | .2      | LR                | 0.04 | 0.04 | 0.94 |
|               |         | TR                | 0.07 | 0.07 | 0.26 |
|               | .4      | LR                | 0.04 | 0.14 | 0.53 |
|               |         | TR                | 0.07 | 0.08 | 0.26 |
| $\mathcal{L}(1)$ | .2      | LR                | 0.06 | 0.09 | 0.78 |
|               |         | TR                | 0.56 | 0.38 | 0.51 |
|               | .4      | LR                | 0.06 | 0.07 | 0.51 |
|               |         | TR                | 0.56 | 0.40 | 0.72 |
| $\mathcal{U}[-1, 1]$ | .2      | LR                | 0.08 | 0.50 | 0.99 |
|               |         | TR                | 0.12 | 0.38 | 0.91 |
|               | .4      | LR                | 0.08 | 0.33 | 0.95 |
|               |         | TR                | 0.12 | 0.29 | 0.95 |

Table 1: Values of the bootstrapped power for the Likelihood Ratio (LR) and Trace (TR) tests for the alternatives $\pi_0 f_0(\cdot - u_1^0) + (1 - \pi_0) f_0(\cdot - u_2^0)$, where $u_1^0 = 0$, $u_2^0 - u_1^0 \in \{0, 1, 3\}$, $\pi^0 \in \{0.2, 0.4\}$, and $f_0$ is the symmetric log-concave density shown in the first column.
\[ \pi^0 \left( u_2^0 - u_1^0 = \right) 0 \quad 1 \quad 3 \]

| Distribution | \(\pi^0\) | Test | 0 | 1 | 3 |
|--------------|-----------|------|---|---|---|
| \(N(0,1)\)  | .2        | LR   | 0.07 | 0.05 | 1.00 |
|              |           | TR   | 0.11 | 0.05 | 0.58 |
|              | .4        | LR   | 0.07 | 0.12 | 0.99 |
|              |           | TR   | 0.11 | 0.07 | 0.75 |
| \(L(1)\)    | .2        | LR   | 0.04 | 0.14 | 1.00 |
|              |           | TR   | 0.37 | 0.15 | 0.95 |
|              | .4        | LR   | 0.04 | 0.05 | 0.97 |
|              |           | TR   | 0.37 | 0.40 | 1.00 |
| \(U[-1,1]\) | .2        | LR   | 0.06 | 0.99 | 1.00 |
|              |           | TR   | 0.14 | 0.74 | 1.00 |
|              | .4        | LR   | 0.06 | 0.85 | 1.00 |
|              |           | TR   | 0.14 | 0.86 | 1.00 |

Table 2: Values of the bootstrapped power for the Likelihood Ratio (LR) and Trace (TR) tests for the alternatives \(\pi^0 f^0(\cdot - u_1^0) + (1 - \pi^0) f^0(\cdot - u_2^0)\), where \(u_1^0 = 0\), \(u_2^0 - u_1^0 \in \{0, 1, 3\}\), \(\pi^0 \in \{0.2, 0.4\}\), and \(f^0\) is the symmetric log-concave density shown in the first column. The values of the power are based on a sample drawn from the true mixture of size \(n = 500\). The number of bootstraps and the number of replications were taken to be \(B = 50\) and \(R = 100\), respectively.

| Method | G | HG | SLC |
|--------|---|----|-----|
| \(N(0,1)\) | 160 | 167 | 168 |
| \(L(1)\)  | 232 | 172 | 171 |
| \(U[-1,1]\) | 145 | 66  | 55  |
| 2· Beta(2, 2) − 1 | 174 | 56  | 54  |

Table 3: Comparison of three different clustering methods. The first method uses pure Gaussian likelihood (G). The next two methods estimate \(\pi^0\) and \(u_i^0\), \(i = 1, 2\) via the tools of Hunter et al. (2007), and then estimate the symmetric component with either a Gaussian density (HG), or with a symmetric log-concave density (SLC). In all cases, we took \(n = 500\), \(u_2^0 - u_1^0 = 1\), and \(\pi^0 = .2\), with \(R = 1000\) replications.
5.4 Gaussian versus symmetric log-concave clustering

We now consider the problem of clustering, i.e., of assigning to each observation in a dataset a label without being given any “training” labels. We will assume that the data can be clustered into two groups, which we will do by by fitting the two-component mixture (2) and assigning a label to an observation \( X \) based on whether our estimate of the posterior probability

\[
\pi^0 f^0(X - u_0^1) \\
\pi^0 f^0(X - u_1) + (1 - \pi^0) f(X - u_2^0)
\]

is larger than our estimate of

\[
(1 - \pi^0) f(X - u_2^0) \\
\pi f^0(X - u_1) + (1 - \pi^0) f(X - u_2^0)
\]

or not, that is, whether the estimate of the posterior probability (17) is greater than \( 1/2 \) or not.

We fit the mixture three different ways. As a baseline, our first approach is to use a Gaussian approach (denoted “G”) wherein we maximize the likelihood (5) under the assumption that the component \( f \) is a normal density. We use the EM algorithm (Dempster et al., 1977) to maximize the likelihood. Our next two approaches both use the method of Hunter et al. (2007) to estimate the mixture components \( u_0^1, u_0^2, \) and \( \pi^0 \). Then we either fit the components using a normal density (denoted “HG”), with variance estimate also given by Hunter et al. (2007), or we use the symmetric log-concave density estimator (denoted “SLC”) for the components.

We record the average number of missclassifications when the true density is one of the densities listed in Section 5.3. In all cases we took the sample size to be \( n = 500, \quad u_2^0 - u_1^1 = 1, \quad \pi^0 = .2, \) and ran \( R = 1000 \) simulations. Since the EM algorithm is only guaranteed to converge to a local maximum, in each simulation we ran the EM 20 different times with different starting values. The EM algorithm was not allowed to take more than 8000 iterations. These settings seemed to find the maximum in most cases, but in some situations wherein the log-likelihood was very flat, it would not necessarily find the true maximum. Thus, the results we report are best described, strictly speaking, as results from running the EM algorithm rather than as maximum likelihood estimates. That said, in the cases where the EM algorithm with these settings had difficulty the log-likelihood was very flat, so we think most numeric approaches would have difficulty and that the reported results are indicative of what one might expect to experience.

The performances of the three approaches were then compared, and the results are reported in Table 3. One can see that SLC outperforms HG by 18% when the true density is the discontinuous uniform. In the other cases, HG and SLC perform similarly. All three methods define the two cluster regions by dividing the real line into two half-lines. HG and SLC have the same mixture components so the shape of the component density estimates have to be dramatically different (e.g., uniform instead of normal) in order to noticeably change the results; otherwise,
even if the cutoff point defining the two clusters moves somewhat, it is usually in
a low-density region for the true distribution, and so the difference does not affect
the labeling of very many points. Note this somewhat deceiving outcome is not
totally in contradiction with the finding of Cule et al. (2010) about the performance
of their two-dimensional log-concave classifier applied to the Breast cancer data
of Wisconsin; see Cule et al. (2010) for details. The authors found that the log-
concave MLE reduces the percentage of missclassification from 10.36% obtained for
the Gaussian estimate to only 8.43% for that particular data set. However, the
posterior probabilities of cluster membership, which can be used as a measurement
of uncertainty, can differ more noticeably between the two methods. This might
be particularly useful if instead of clustering by assigning labels when the posterior
probability (17) is greater or less than 1/2, respectively, one used a cutoff value
that is much closer to 1 (or 0) than 1/2 is. This setup can arise when there is
an asymmetric loss, e.g., in screening patients for cancer risk one wants to err on
the side of safety rather than just minimizing misclassification probability. In these
cases, the shape of the density component estimator near its center is of greater
import (because the point dividing the two clusters will be nearer to the center of
one of the mixture components). Thus, we see that when normality is inappropriate,
log-concavity can indeed be a useful surrogate.

6 Data application

In this section, we apply our new estimation approach to three different datasets.

6.1 Old Faithful data

The data to which we first apply our estimation procedure are the times, in minutes,
between eruptions of the Old Faithful geyser in Yellowstone National park. There
are many forms of the Old Faithful data. As far as we know, the oldest version of
the data was collected by S. Weisberg from R. Hutchinson in August 1978. The data
we analyze were collected between August 1 and August 15, 1985 continuously, and
are from Azzalini and Bowman (1990). The following explanation from Weisberg
(2005) motivates interest in the data:

Old Faithful Geyser is an important tourist attraction, with up to several
thousand people watching it erupt on pleasant summer days. The park
service uses data like these to obtain a prediction equation for the time
to the next eruption.

In Figure 3, we have two plots related to the Old Faithful data. First, we discuss
the plot on the left. It has a histogram of the data, with around 30 bins. We
then have three mixture density estimates plotted on top of the histogram. The
“SLC” (symmetric log-concave) estimate is the mixture model where \( u_0^0, u_1^0 \) and \( \pi^0 \)
are estimated using the method of Hunter et al. (2007), and then the components
are estimated using our symmetric log-concave estimator. The “IHG” (Hunter et al.
Figure 3: Time between eruptions of Old Faithful Geyser (min). The “SLC” and “HG” estimates both use the method of Hunter et al. (2007) to estimate the mixture parameters $u_1^0, u_2^0$, and $\pi^0$. “SLC” then fits with symmetric log-concave components and “HG” fits with Gaussian components. The “G” estimate is the maximum likelihood estimate of a mixture model with two Gaussian components with equal variances.

and Gaussian components) estimate is given by again using the method of Hunter et al. (2007) to find $u_1^0, u_2^0$, and $\pi^0$, but then fitting with Gaussian components (the same Gaussian density for both mixture components). The estimates for $u_1^0, u_2^0$, and $\pi^0$ given by Hunter et al. (2007) are 55.5, 80.5, and .33, respectively. The “G” (Gaussian) estimate in the plot is based on simply using a Gaussian mixture model with two components with equal variances. Assuming equal variances forces the two components to be identical, which makes the model analogous to the others. In this case, we estimated $u_1^0, u_2^0$, and $\pi^0$ by the EM algorithm (Dempster et al., 1977), with estimated values of 55.3, 81.0, and .339. The normal components are slightly more peaked than the log-concave ones, but the overall fit is fairly similar; in large part this is because the locations and weights are very similar.

The plot on the right is that of the symmetric log-concave component, centered at 0, used in the mixture density. As expected from the known theoretical properties of this estimator, it has a flat interval about 0, and is the exponential of a concave piecewise linear and symmetric function.
6.2 Acidity data

The data we consider in this section are acidity measurements on the log scale for 155 lakes in north-central Wisconsin from the Eastern Lake Survey, a part of the National Surface Water Survey; the measurements are acid neutralizing capacity (ANC) on the log scale; specifically, \( \log(\text{ANC} + 50) \). From Crawford (1994), “ANC describes the capability of a lake to absorb acid; low ANC values can lead to a loss of biological resources.” These data have been previously studied in Crawford et al. (1992), Crawford (1994), Richardson and Green (1997), and Yao and Lindsay (2009), amongst other places. The sequential bootstrap likelihood ratio statistic procedure of McLachlan and Peel (2000) at the 5% level finds evidence for 2 normal components. The Bayesian approach of Richardson and Green (1997) gives equal posterior weight to 3-5 normal components.

We present plots related to the acidity data in Figure 4. The plots are analogous to those we used for the Old Faithful data. The plot on the left has a histogram with around 40 bins, and three mixture density estimates, each found using the same three methods we describe for the Old Faithful data. The estimates from Hunter et al. (2007) for \( u_1^0, u_2^0, \) and \( \pi^0 \) are 4.30, 6.45, and .622. Those from the Gaussian method are 4.37, 6.32, and .623.

The symmetric log-concave component is again plotted to the right. Here, the component has a somewhat more pronounced flat modal region. Thus, while the overall shape of the density estimate appears similar in the three methods, the symmetric log-concave component estimate does not resemble a normal component in its center region. We think it worth noting that the posterior density estimate (averaging over all numbers of components) of Richardson and Green (1997) is somewhat flat on the region between the acidity measurements of 6 and 7; our method with symmetric log-concave components naturally reproduces this effect with its elongated flat modal region with just two components.

6.3 Height data

We next examine 1766 human height observations. We look at the heights of the population of Campora, a village in the south of Italy. This population is studied by the “Genetic Park of Cilento and Vallo di Dano Project” (Ciullo, 2009), which is interested in identifying geographically and genetically isolated populations. Such populations are of particular interest because in addition to “genetic homogeneity,” they have a “uniformity of diet, life style and environment.” These homogeneities are valuable in the study of genetic risk factors for complex pathologies such as “hypertension, diabetes, obesity, cancer, and neurodegenerative diseases,” by allowing for a “simplification of the complexity of genetic models” involved, because

the features of isolated populations [...] may reduce the number of susceptibility genes and increase genetic homogeneity. In addition, as individuals share a similar life-style and environment, the relative weight of environmental variation can be controlled (Ciullo, 2009).
Figure 4: Acidity measurements on the log scale for 155 lakes in north-central Wisconsin. The “SLC” and “HG” estimates both use the method of Hunter et al. (2007) to estimate the mixture parameters $u_1^0, u_2^0,$ and $\pi^0$. “SLC” then fits with symmetric log-concave components and “HG” fits with Gaussian components. The “G” estimate is the maximum likelihood estimate of a mixture model with two Gaussian components with equal variances.

Colonna et al. (2007) provide evidence that this population is indeed genetically isolated. Because of this feature, the distribution of heights of this population is not necessarily the same as that of the global population at large, so estimating its distribution is of interest. Height data are often modeled as mixtures of two components, corresponding to the two sexes, so the symmetric log-concave density estimate is of interest in this context, since it gives the estimate of the distribution for either sex considered alone.

We present plots related to the heights data in Figure 5. The plot on the left is analagous to the plots we used for the other two datasets. It has a histogram with around 80 bins, and three mixture density estimates, each found using the same three methods we describe for the other datasets. The height data do not exhibit multi-modality, but two-component mixtures still fit the data well. The three approaches fit similarly, but the log-concave components are able to capture a bit more asymmetry near to the mode.

The plot on the right includes the mixture component density (labeled “All”), in black. The data include the sex of each individual, so, using this extra information we can also estimate the true component densities separately. We did that, using the symmetric log-concave MLE, and added the resulting density estimates to the
Figure 5: Height data of the population of Campora, in the south of Italy. The “SLC” and “HG” estimates both use the method of Hunter et al. (2007) to estimate the mixture parameters $u_0^1$, $u_0^2$, and $\pi^0$. “SLC” then fits with symmetric log-concave components and “HG” fits with Gaussian components. The “G” estimate is the maximum likelihood estimate of a mixture model with two Gaussian components with equal variances.

The plot on the right. To center the data, we used the medians. The medians were not quite equal to the estimates given by the method of Hunter et al. (2007); the medians were 168.7 and 156, for the men and women, respectively. The estimates given by Hunter et al. (2007) for $\pi^0$, $\mu_1^0$, and $\mu_2^0$ were 0.719, 157.5, and 170.5. The true proportion of women was 0.569.

The components estimated by using the labels for men and women differ from that using the mixture model without the labels especially towards the center. This is in large part because the values of $\pi^0$, $\mu_1^0$, and $\mu_2^0$ are estimated in the mixture model, whereas when we have the labels there is no need to estimate $\pi^0$ and taking the medians to be the center points gives a slightly different fit to the data, which especially effects the estimates near those center points. It does appear that the distributions of heights of men and women are slightly different, especially near the center of those distributions, with men having a more peaked density and women having a flatter one. Thus, in the mixture model, without using the labels, the component density estimate is somewhere in between the two shapes.

7 Conclusions

The goal in this paper is to make use of the log-concavity constraint to estimate the unknown density component in a semi-parametric location mixture model assuming
that this unknown density is symmetric around the origin. The main motivation for choosing this approach is to build an estimation procedure that is totally free of any bandwidth selection, in contrast with other nonparametric methods based on kernel estimators. From a computational point of view, selection of the kernel bandwidth that achieves the desired optimality is usually a time consuming step. Given the fact that many known models are log-concave, our log-concave maximum likelihood estimator of the unknown component density offers a real flexibility, as can already be seen from our results in the data application section. Our estimator enjoys also the merit of being easily computable using the EM algorithm in combination with an active set algorithm already implemented in the R package logcondens.mode. The latter enables the user to compute the log-concave maximum likelihood estimator of a density constrained to have a known mode. A simple transformation of the data enables us to put the original problem in the framework of computing the log-concave maximum likelihood estimator of a decreasing log-concave density on the positive half-line, hence with mode constrained to be at the origin.

In our estimation procedure, the parameters of the location mixture, i.e., the mixture probability and locations, are first estimated using the method of Hunter et al. (2007). These estimates are then plugged-in in the expression of the log-likelihood as if they were the true parameters. Hence, the estimation procedure first estimates the parametric part of the model, and then finds the estimate of the nonparametric part under the assumption of log-concavity. An alternative method would have been to find the maximum likelihood estimator of both the parametric and nonparametric components of the model. However, such an approach imposes more theoretical challenges and deriving upper bounds on the rates of convergence of the estimator seems to be much more involved. Indeed, the estimates are described by a set a (Fenchel) conditions whose expressions involve unfortunately the unknown maximum likelihood estimates. Sophisticated empirical process arguments would need to be developed to overcome the difficulty induced by the implicit form of the estimates. We conjecture that bound on the rates of convergence, $n^{-2/5}$, should remain the same with this alternative estimation procedure and it would be worthwhile to investigate the validity in a future work.

We mentioned in the Introduction that our method is not to be applied when the component mixture distributions may have very heavy tails. In such cases, there are other shape constraints that may be appropriate, specifically, $s$-concavity, as studied in Koenker and Mizera (2010) and Doss and Wellner (2013). However, the theoretical development of estimators of $s$-concave densities is much less well developed than that of log-concave MLEs, which provides a barrier to using $s$-concavity in our current context. In other (non heavy-tailed) cases where the true distribution deviates from log-concavity, our estimator still appears to give sensible results. (We do not present the results of all of our numerical investigations, to keep the manuscript at a reasonable length).

Finally, as mentioned in the Introduction, Hunter et al. (2007) give sufficient conditions on the mixing probabilities and mixture locations for the model to be 3-identifiable (recall that this notion is property of individual distribution and not
of the model as a whole), and in such cases our method can be easily extended. Recently, Balabdaoui and Butucea (2014) proved that the number of components \( k \), the mixture parameters and the unknown density are identifiable provided that the density is Pólya frequency (of infinite order) such that its expectation is equal to 0. For a precise definition of Pólya frequency functions, we refer to Schoenberg (1951). The obtained identifiability result can be used of course in the case of symmetry. The obtained identifiability result can be used of course in the case of symmetry.

**Proof of Proposition 2.1.** To show existence, we first start by proving that \( \hat{\psi}_n \) is necessarily a piecewise linear function that is flat in the neighborhood of zero. Let \( \psi \in \mathcal{SC} \). Also, let \( Z(1) \) denote the first order statistic of the transformed data \( Z_i \)’s defined above. Consider \( \tilde{\psi} \) the unique concave function such that \( \psi(Z_i) = \psi(Z_i), i = 1, \ldots, 2n, \tilde{\psi}(0) = \tilde{\psi}(Z(1)) \), \( \tilde{\psi} \) is piecewise linear between the points \( Z_i, i = 1, \ldots, 2n, \tilde{\psi}(t) = -\infty \) for \( t > Z(2n) \) and \( \tilde{\psi}(t) = \psi(-t) \) for \( t \in \mathbb{R} \). Clearly, \( \tilde{\psi} \in \mathcal{SC} \) and admits the properties \( \pi e^{\tilde{\psi}(X_i - u_1)} + (1 - \pi) e^{\tilde{\psi}(X_i - u_2)} = \pi e^{\psi(X_i - u_1)} + (1 - \pi) e^{\psi(X_i - u_2)} \) for \( i = 1, \ldots, n \) and \( \tilde{\psi} \leq \psi \). This implies that \( \Phi_n(\tilde{\psi}) \geq \Phi_n(\psi) \) and the logarithm of the MLE has to be necessarily piecewise linear between the transformed data points \( Z_i \) and \( -Z_i, i = 1, \ldots, 2n \) with a flat piece in the neighborhood of zero and support \([-Z(2n), Z(2n)]\). Further, if a maximizer \( \hat{f}_n = \exp(\hat{\psi}_n) \) exists then it has to be a density. This follows from the fact that \( \hat{\psi}_n + \epsilon \in \mathcal{SC} \) for all \( \epsilon \in \mathbb{R} \), and

\[
0 = \lim_{\epsilon \to 0} \frac{\Phi_n(\hat{\psi}_n + \epsilon) - \Phi_n(\hat{\psi}_n)}{\epsilon} = 1 - \int_{\mathbb{R}} \exp(\hat{\psi}_n).
\]

Given the results obtained above, a function \( \psi \in \mathcal{SC} \) can be identified with the vector

\[
\bar{\psi} \equiv \left( \psi(X_i - u_1), \psi(X_i - u_2) \right)_{i=1}^n = \left( \psi(Z_{2i-1}), \psi(Z_{2i}) \right)_{i=1}^n.
\]

Let \( \mathcal{SC}_n \) denote the set of such vectors. It is clear that \( \Phi_n \) is continuous on \( \mathcal{SC}_n \). Let \( \tilde{\psi}^{(p)} \) be a maximizing sequence, such that \( \int \exp(\tilde{\psi}^{(p)}(t)) dt = 1 \), and as \( p \to \infty \)

\[
\max_{1 \leq i \leq n} |\psi^{(p)}(Z(i))| \to \infty, \text{ and } |\psi^{(p)}(Z(i))| \to l_i \in [-\infty, \infty], \text{ for } 1 \leq i \leq n
\]

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where $Z_{(i)}$ denotes the $i$-th order statistic of $Z_1, \ldots, Z_{2n}$.

For $1 \leq i \leq n$ write $\psi_{i}^{(p)} := \psi^{(p)}(Z_{(i)})$. Suppose that for all $i \in \{1, \ldots, n\}$

$$
\lim_{p \to \infty} \psi_{i}^{(p)} < \infty.
$$

This implies that there exists $j \in \{1, \ldots, n\}$ such that $\lim_{p \to \infty} \psi_{j}^{(p)} = -\infty$. If there exists $i \in \{1, \ldots, n\}$ such that $Z_{(j)} = \max(|X_i - u_1|, |X_i - u_2|)$ then

$$
\lim_{p \to \infty} \log \left( \pi \exp(\psi^{(p)}(|X_i - u_1|)) + (1 - \pi) \exp(\psi^{(p)}(|X_i - u_2|)) \right) \leq \lim_{p \to \infty} \psi_{j}^{(p)} = -\infty,
$$

implying that $\lim_{p \to \infty} \Phi_n(\psi^{(p)}) = -\infty$ which is in contradiction with the definition of $\psi^{(p)}$.

Let us assume now that for any $j$ such that $\lim_{p \to \infty} \psi_{j}^{(p)} = -\infty$, there exists $i \in \{1, \ldots, n\}$ such that

$$
Z_{(j)} = \min(|X_i - u_1|, |X_i - u_2|) \text{ and } \lim_{n \to \infty} \psi^{(p)}(\max(|X_i - u_1|, |X_i - u_2|)) > -\infty. \quad (18)
$$

Suppose that $j = 2n$ is the only integer for which this divergence is occurring. Let $i \in \{1, \ldots, n\}$ such that we have $Z_{(2n)} = \min(|X_i - u_1|, |X_i - u_2|)$. Without loss of generality, assume that $|X_i - u_1| < |X_i - u_2|$. Put

$$
a^{(p)} = (1 - \pi) \exp\left(\psi^{(p)}(|X_i - u_2|)\right), \text{ and } b^{(p)} = \psi_{2n-1}^{(p)}
$$

and call $a$ and $b$ their respective limits as $p \to \infty$. Note that both $a$ and $b$ are in $\mathbb{R}$ by assumption. Consider now the function defined by

$$
h(x) = \log (\pi e^x + a) - 2(Z_{(2n)} - Z_{(2n-1)}) \frac{\exp(b) - \exp(x)}{b - x}
$$
on $(-\infty, 0]$. Put $\Delta_{2n-1} = Z_{(2n)} - Z_{(2n-1)}$. We have that

$$
h'(x) = \frac{\pi \exp(x)}{\pi \exp(x) + a} - \frac{2\Delta_{2n-1}}{b - x} \left( \frac{\exp(b) - \exp(x)}{b - x} - \exp(x) \right)
$$

which has the same sign as

$$
\pi - 2\Delta_{2n-1} \exp(b) \left( \pi \exp(x) + a \right) \frac{1}{b - x} \left( \frac{\exp(-x) - \exp(-b)}{b - x} - \exp(-b) \right) = \pi - 2\Delta_{2n-1} \exp(b) \left( \pi \exp(x) + a \right) \frac{\exp(-x) - \exp(-b)}{b - x} \left( 1 - \frac{(b - x) \exp(-b)}{\exp(-x) - \exp(-b)} \right)
$$

$\to -\infty$, as $x \to -\infty$. 

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This implies that as \( p \to \infty \)

\[
\log \left( \pi \exp(\psi(p)(|X_i - u_1|) + (1 - \pi) \exp(\psi(p)(|X_i - u_2|)) \right) - \int_{Z_{(2n-1)}}^{Z_{(2n)}} \exp(\psi(p))(t) dt
\]

must be decreasing, and hence bigger values of the log-likelihood are to be obtained if \( \psi(p)(Z_{(2n)}) \) were not divergent. This implies that \( \psi_{2n}(p) \) cannot diverge to \(-\infty\). The same reasoning can be applied if (18) is satisfied by other integers \( k \in \{1, \ldots, 2n-1\} \), and let \( j \) be the smallest one. Let \( i \) be such that \( Z_{(j)} = \min(|X_i - u_1|, |X_i - u_2|) \).

Note that \( j > 1 \). Also, note that by definition of \( j \) \( \lim_{p \to \infty} \psi(p)(Z_{(j-1)}) > -\infty \). Thus, we obtain the same conclusion as before, that is, the term

\[
\log \left( \pi \exp(\psi(p)(|X_i - u_1|) + (1 - \pi) \exp(\psi(p)(|X_i - u_2|)) \right) - \int_{Z_{(j-1)}}^{Z_{(j)}} \exp(\psi(p))(t) dt
\]

can be increased (and hence the log-likelihood) if \( \psi_j(p) \) were not diverging to \(-\infty\). By a recursive reasoning, we obtain the same conclusion about the remaining integers.

Now suppose that \( \lim_{p \to \infty} \max_{1 \leq i \leq n} \psi(p)(Z_{(i)}) = \infty \). Since \( \psi(p) \) is decreasing on \([0, \infty)\), \( \max_{1 \leq i \leq n} \psi(p)(Z_{(i)}) = \psi(p)(Z_{(1)}) \to \infty \) as \( p \to \infty \). For \( x \in [Z_{(1)}, Z_{(2)}] \), we have

\[
1 \geq \int_{Z_{(1)}}^{Z_{(2)}} \exp(\psi_1(p)) \exp\left( \frac{\psi_2(p) - \psi_1(p)}{Z_{(2)} - Z_{(1)}} (t - Z_{(1)}) \right) dt
\]

\[
= \exp(\psi_1(p))(Z_{(2)} - Z_{(1)}) \frac{1 - \exp(\psi_1(p) - \psi_2(p))}{\psi_1(p) - \psi_2(p)}
\]

\[
\geq \exp(\psi_1(p))(Z_{(2)} - Z_{(1)}) \frac{1}{\psi_1(p) - \psi_2(p) + 1},
\]

where the last inequality follows from the fact that \( \frac{1 - \exp(-t)}{t} \geq \frac{1}{t+1} \) on \([0, \infty)\). This implies that

\[
\psi_1(p) - \psi_2(p) \geq \exp(\psi_1(p))(Z_{(2)} - Z_{(1)}) - 1.
\]

Thus

\[
\psi_1(p) + \psi_2(p) = 2\psi_1(p) + \psi_2(p) - \psi_1(p) \leq 2\psi_1(p) - \exp(\psi_1(p))(Z_{(2)} - Z_{(1)}) + 1 \to -\infty
\]

as \( p \to \infty \), and hence \( \psi_2(p) \leq (\psi_1(p) + \psi_2(p))/2 \to -\infty \). Since a maximizer of \( \Phi_n \) is necessarily constant on \([-\tau_1, \tau_1] \) with \( \tau_1 \) the first kink point, here equal to \( Z_{(1)} \), we can write

\[
1 = 2\exp(\psi_1(p)Z_{(1)}) + \sum_{j=2}^{n} \int_{Z_{(j)}}^{Z_{(j+1)}} \exp(\psi(p))(t) dt \geq 2\exp(\psi_1(p)Z_{(1)})
\]

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But this implies using the results obtained above that $\exp(\psi(p)) \leq 1/(2Z(1))$, which contradicts the assumption that $\psi(p) \to \infty$. \hfill \Box

**Proof of Proposition 2.2.** This follows from arguments similar to those used in the proof of Theorem 2.2 of Dümbgen and Rufibach (2009). \hfill \Box

**Proof of Proposition 2.3:** Using symmetry of $\hat{f}_n$ and $\Delta$, and the definition of $\hat{F}_n$, the inequality in (6) can be re-written as

$$\int_0^\infty \Delta(x)d\hat{F}_n(x) \leq \int_0^\infty \Delta(x)\hat{f}_n^+(x)dx,$$

with equality for $\Delta$ satisfying $\hat{\psi}_n^+ + \epsilon \Delta \in SC$ for $\epsilon > 0$ small enough. For $z \in [0, \infty)$, consider the concave perturbation function defined on $[0, \infty)$ as

$$\Delta_z(x) = -(x-z)_+$$

$$= \begin{cases} z - x & \text{if } x \geq z \\ 0 & \text{otherwise.} \end{cases}$$

Using Fubini’s theorem, the inequality in (19) yields

$$Z(2n) - z - \int_z^{Z(2n)} \hat{F}_n(x)dx \geq Z(2n) - z - \int_z^{Z(2n)} \hat{F}_n^+(x)dx.$$ 

This yields

$$\int_0^z \hat{F}_n^+(x)dx \leq \int_0^z \hat{F}_n(x)dx + \int_0^{Z(2n)} \hat{F}_n^+(x)dx - \int_0^{Z(2n)} \hat{F}_n(x)dx.$$ 

Since for $z = 0$, the inequality in (19) becomes an equality ($\Delta_0$ is a straight line, and hence satisfies $\hat{\psi}_n^+ + \epsilon \Delta_0 \in SC$ for $\epsilon > 0$ small enough), we have that

$$\int_0^{Z(2n)} \hat{F}_n(x)dx = \int_0^{Z(2n)} \hat{F}_n^+(x)dx$$

from which follows the inequality part in (8). The equality part follows from noting that when $z$ is a knot of $\hat{\psi}_n^+ = \log(\hat{f}_n^+)$ we have $\hat{\psi}_n^+ \pm \epsilon \Delta_z \in SC$ for $\epsilon > 0$ small enough. \hfill \Box

The following theorem of Bhattacharya and Ranga Rao (1976) will be an important component of the proof of Proposition 3.1.

**Theorem A.1 (Theorem 1.11, page 22, Bhattacharya and Ranga Rao (1976))**

Let $P$ be a probability measure on $\mathbb{R}$. Then, as $n \to \infty$

$$\sup_{C \in \mathcal{C}} |P_n(C) - P(C)| \to 0$$

almost surely.
Proof of Proposition 3.1: Let \( C \) denote the class of all convex and compact subsets in \( \mathbb{R} \). Fix \( \epsilon > 0 \). For \( C \in C \) we have that

\[
\frac{1}{N} \sum_{i: X_i \leq \bar{u}} 1_{\{X_i \in C\}} = \frac{\mathbb{P}_n(C \cap [-\infty, \bar{u}])}{G_n(\bar{u})} = \frac{\mathbb{P}_n(C_{\bar{u}})}{G_n(\bar{u})}.
\]

where \( C_{\bar{u}} = C \cap [-\infty, \bar{u}] \) is also compact and convex. Let \( \bar{u}^0 = (u_1^0 + u_2^0)/2 \). To simplify notation, we denote \( C \) for \( C_{\bar{u}} \), and \( g \), \( G \) and \( P \) the true density, distribution function and associated measure respectively.

\[
\frac{\mathbb{P}_n(C) - P(C)}{G(\bar{u}^0)} = \frac{\mathbb{P}_n(C) - P(C)[G(\bar{u}) - G(\bar{u}^0)]}{G_n(\bar{u})G(\bar{u}^0)}.
\]

Consistency of \( u_1 \) and \( u_2 \), the fact that the class \( \{ -\infty, x \} \) is Glivenko-Cantelli (see Example 2.5.4 of van der Vaart and Wellner (1996)) and continuity of \( g \) imply that the event

\[
\left\{ G(\bar{u}) \geq \frac{G(\bar{u}^0)}{2}, G_n(\bar{u}) \geq \frac{G(\bar{u}^0)}{2}, \max_{t \in [\bar{u}^0, \bar{u}^0 \vee \bar{u}]} g(t) \leq 2g(\bar{u}^0) \right\}
\]

occurs with probability converging to one. On the other hand, we have that

\[
\left| \frac{\mathbb{P}_n(C) - P(C)}{G(\bar{u}^0)} \right| \leq \sup_{C \in C} \left| \frac{\mathbb{P}_n(C) - P(C)}{G_n(\bar{u})} \right| + \sup_{x \in \mathbb{R}} \left| \frac{G_n(x) - G(x)}{G_n(\bar{u})G(\bar{u}^0)} \right|
\]

\[
\leq \sup_{C \in C} \left| \frac{\mathbb{P}_n(C) - P(C)}{G_n(\bar{u})} \right| + \sup_{x \in \mathbb{R}} \left| \frac{G_n(x) - G(x)}{G_n(\bar{u})G(\bar{u}^0)} \right|
\]

\[
+ \frac{g(\xi)|\bar{u} - \bar{u}^0|}{G(\bar{u})G(\bar{u}^0)}, \text{ for some } \xi \in [\bar{u}^0, \bar{u}^0 \vee \bar{u}]
\]

\[
\leq D \left( \sup_{C \in C} \left| \frac{\mathbb{P}_n(C) - P(C)}{G_n(\bar{u})} \right| + \sup_{x \in \mathbb{R}} \left| \frac{G_n(x) - G(x)}{G_n(\bar{u})G(\bar{u}^0)} \right| + |\bar{u} - \bar{u}^0| \right)
\]

probability increasing to one, where \( D = 4G(\bar{u}^0)^{-2} \max(2g(\bar{u}^0), 1) \). Using Theorem A.1, and again consistency of \( \bar{u} \), we conclude that

\[
\sup_{C \in C} \left| \frac{\mathbb{P}_n(C) - P(C)}{G_n(\bar{u})} \right| \rightarrow 0 \text{ in probability.}
\]

In particular, this implies that for any fixed \( \epsilon > 0 \) there exists \( n_0 \) depending on \( \epsilon \) such that

\[
P \left( \frac{\mathbb{P}_n(C)}{G_n(\bar{u})} \leq \frac{P(C)}{G(\bar{u}^0)} + \alpha/2 \text{ for all } C \in C \right) \geq 1 - \epsilon.
\]
Let us denote by $E_{n,\alpha/2}$ the event $\\{P_n(C) \leq \frac{P(C)}{G(\bar{u})} + \frac{\alpha}{2} \text{ for all } C \in \mathcal{C}\}$. Also, consider the subclass

$$\mathcal{C}_{\alpha/2} = \left\{ C \in \mathcal{C} : \frac{P(C)}{G(\bar{u})} \geq \frac{\alpha}{2} \right\}.$$

Clearly, for any $C \in \mathcal{C}_{\alpha/2}$ we have that

$$|C| \geq \frac{\alpha G(\bar{u})}{2\|g\|_{\infty}}.$$

On the other hand,

$$\left\{ C \in \mathcal{C} : \bar{N}^{-1} \sum_{i: X_i \in C} 1_{\{X_i \in C\}} \geq \alpha \right\} = \left\{ C \in \mathcal{C} : \frac{P_n(C)}{G_n(\bar{u})} \geq \frac{\alpha}{2} \text{ for } C \in \mathcal{C}\right\}$$

$$= \left\{ C \in \mathcal{C} : \frac{P_n(C)}{G_n(\bar{u})} \geq \alpha \right\} \cap E_{n,\alpha/2} \cup \left\{ C \in \mathcal{C} : \frac{P_n(C)}{G_n(\bar{u})} \geq \alpha \right\} \cap \mathcal{E}^c_{n,\alpha/2}$$

$$\subseteq \mathcal{C}_{\alpha/2} \cup \left\{ C \in \mathcal{C} : \frac{P_n(C)}{G_n(\bar{u})} \geq \alpha \right\} \cap \mathcal{E}^c_{n,\alpha/2}$$

implying that for $n \geq n_0$

$$P \left( \min \left\{ |C| : C \in \mathcal{C} \text{ such that } \bar{N}^{-1} \sum_{i: X_i \leq \bar{u}} 1_{\{X_i \in C\}} \geq \alpha \right\} < \frac{\alpha G(\bar{u})}{2\|g\|_{\infty}} \right) \leq \epsilon$$

completing the proof of the lemma.

**Proof of Lemma 3.1.** Note that for $t \in \mathbb{R}$ we have that

$$|t - a| \leq |t - b| \iff t \leq (a + b)/2.$$

Using the fact that the function $t \mapsto |\log(t)|t$ is increasing on $(0, e^{-1}]$ and that $\psi_0$ is decreasing on $[0, \infty)$, we have for $A$ taken to be larger than $(a + b)/2$,

$$\int_A^{\infty} |\log(p_0(t))|p_0(t)dt \leq \int_A^{\infty} |\psi_0(t - a)| \exp(\psi_0(t - a))dt < \infty$$

where the second inequality follows e.g. from Lemma 2 in Schuhmacher and Dümbgen (2010). By symmetry, the integral on $(-\infty, -A]$ is also finite.

**Proof of Lemma 3.2.** It is enough to show that for $k$ large, $|\log p_{k,\phi}|$ is bounded by a $p_0$-integrable function to apply the Lebesgue Dominated Convergence Theorem.
Since \( p_{k,\phi} \) is uniformly bounded above, to upper bound \(|\log p_{k,\phi}|\) we need to lower bound \( p_{k,\phi} \). Now,

\[
p_{k,\phi}(x) = \int_{\mathbb{R}} p_k(y)e^{\phi(x-y)} \, dy \geq \int_{\alpha}^{\beta} \left( \min_{z \in [\alpha, \beta]} p_k(z) \right) e^{\phi(x-y)} \, dy
\]

for any \( \alpha \) and \( \beta \). Taking \( \alpha \) and \( \beta \) to be in the interior of the support of \( p_0 \), we have that, for all \( k \) large enough, \( \min_{x \in [\alpha, \beta]} p_k(z) \geq c \) for some \( c > 0 \). Now because \( \phi \) is concave, if \( x \geq A \) where \( A \) is large enough, \( \phi \) is decreasing on \([x - \beta, x - \alpha]\), i.e. \( \phi(x - y) \) is increasing in \( y \) for \( y \in [\alpha, \beta] \), so that then \( \phi(x - y) \geq \phi(x - \alpha) \). Taking \( A \) large enough that \( \phi(x - \alpha) < 0 \), so \( |\phi(x - y)| \leq |\phi(x - \alpha)| \), we have that, for \( k \) large, and \( x > A \), (20) is bounded below by

\[
c \int_{\alpha}^{\beta} e^{\phi(x-\alpha)} \, dy = (\beta - \alpha)c e^{\phi(x-\alpha)}.
\]

Recall that if \( d_1 < d_2 \), then

\[
|t - d_1| \leq t - d_2 |\iff t \leq (d_1 + d_2)/2.
\]

Thus, taking \( A > (a + b)/2 \),

\[
\int_{A}^{\infty} |\phi(x - \alpha)| p_0(x) \, dx \leq \int_{A}^{\infty} |\phi(x - \alpha)| e^{\psi_0(x-b)} \, dx = \int_{A+b}^{\infty} |\phi(x - \alpha + b)| e^{\psi_0(x)} \, dx,
\]

and this is less than infinity, by the condition \((F)\). Thus, for \( x \geq A \) and \( k \) large,

\[
|\log p_{k,\phi}(x)| \leq |\log ((\beta - \alpha)c) + |\phi(x - \alpha)|,
\]

so we have shown that \( |\log p_{k,\phi}| \) is bounded by a function \( p_0 \)-integrable on \([A, \infty)\). A similar argument shows it is bounded by a function integrable on \((-\infty, -A)\). Finally, since \( p_0,\phi \) has support \((-\infty, \infty)\) (since \( e^{\phi} \) has support \((-\infty, \infty)\) by the condition \((F)\)),

\[
\sup_{x \in [-A, A]} |\log p_{k,\phi}(x)| \leq \sup_{x \in [-A - \eta, A + \eta]} |\log p_0(x)| < \infty,
\]

for all \( k \) large enough that \( \max(|a - a_k|, |b - b_k|) < \eta \) for some \( \eta > 0 \). Thus for \( k \) large we have shown that \( |\log p_{k,\phi}| \) is bounded by a \( p_0 \)-integrable function on \( \mathbb{R} \), and \( p_{k,\phi}(t) \to p_{0,\phi}(t) \) for all \( t \in \mathbb{R} \), so (9) holds.

\[\square\]

**Proof of Proposition 3.2:** Let \( f \in \mathcal{F}_b \) and \( m_1 \) and \( m_2 \) be the respective modes of the component functions \( h_1 \) and \( h_2 \) assumed to be supported on \((L_1, U_1)\) and \((L_2, U_2)\) respectively. Also, denote by \( c_i \) and \( d_i \) be the restriction of \( h_i \) on \((L_i, m_i)\) and \((m_i, U_i)\) respectively for \( i = 1, 2 \). Note that \( c_i \) and \( d_i \) are increasing and decreasing functions on their domain of definition. Let us assume that \( m_1 \leq m_2 \), and \( L_1 \leq L_2 \leq U_1 \leq U_2 \).
Although the arguments below are developed only for this case, they can easily be modified in the other possible configurations. We have that

\[
\lambda h_1(x) + (1 - \lambda)h_2(x) = \begin{cases} 
0, & \text{if } x \leq L_1 \text{ or } x \geq U_2 \\
\lambda c_1(x)1_{x \leq m_1} + \lambda d_1(x)1_{x > m_1}, & \text{if } L_1 < x \leq L_2 \\
\left(\lambda c_1(x) + (1 - \lambda)c_2(x)\right)1_{x \leq m_1} \\
+ \left(\lambda d_1(x) + (1 - \lambda)c_2(x)\right)1_{m_1 < x \leq m_2} \\
+ \left(\lambda d_1(x) + (1 - \lambda)d_2(x)\right)1_{x > m_2}, & \text{if } L_2 < x < U_1 \\
(1 - \lambda)c_2(x)1_{x \leq m_2} + (1 - \lambda)d_2(x)1_{x > m_2}, & \text{if } U_1 \leq x < U_2.
\end{cases}
\]

Note that the function \( f = \log(b) \) on \([-\infty, L_1] \cup [U_2, \infty) \). We will show now that on \((L_1, U_2)\) it can be decomposed into a finite sum of functions \( k_j \) where \( k_j \) is defined on some convex set and is either monotone (increasing/decreasing) on this set or can be written the difference of two monotone increasing functions. This is clear on all intervals except for \( I \equiv (m_1, m_2) \cap (L_2, U_1) \) on which we have

\[
f(x) = \log\left(\lambda d_1(x) + (1 - \lambda)c_2(x) + b\right).
\]

Hence,

\[
\int_I \left| f'(x) \right| dx \leq \int_I \frac{\lambda |d'_1(x)| + (1 - \lambda)|c'_2(x)|}{\lambda d_1(x) + (1 - \lambda)c_2(x) + b} dx \\
\leq \frac{1}{b} \left(\lambda \int_I -d'_1(x) dx + (1 - \lambda) \int_I c'_2(x) dx\right) \leq \frac{2M}{b}.
\]

Above, \( f', d'_1 \) and \( c'_2 \) denoted either the left or right derivative. Thus, \( f \) admits a bounded variation on \( I \) and by the Jordan decomposition there exist two increasing functions \( f_1 \) and \( f_2 \) such that \( |f_1| \leq (2M/b + M)/2 \), \( |f_2| \leq (2M/b + M)/2 \) and \( f = f_1 - f_2 \). Now, the class of functions that are monotone (increasing or decreasing) on some convex set and bounded by some fixed constant \( D > 0 \) is Glivenko-Cantelli for any probability measure \( P \). Indeed, let \( g \) be a monotone function on some convex set such that \( |g| \leq D \). We can assume without loss of generality that \( g \) is increasing and that the convex set is bounded of the form \([a, b]\) (the other cases can be handled similarly). Then we can consider \( h \) the monotonic extension of \( g \):

\[
h(x) = g(a)1_{x < a} + g(x)1_{a \leq x < b} + g(b)1_{x \geq b}.
\]

Note that \( |h| \leq D \). Let \( \mathcal{M}_D \) denote the class of increasing functions on \( \mathbb{R} \) that are bounded by \( D \). We have that

\[
\left| \int f(x) d(P_n(x) - P(x)) \right| \leq \left| \int h(x) d(P_n(x) - P(x)) \right| + 3D\|P_n - P\|_\infty \\
= \sup_{h \in \mathcal{M}_D} \left| \int h(x) d(P_n(x) - P(x)) \right| + o_p(1) \\
= o_p(1)
\]
using Theorem 2.7.5 and Theorem 2.4.1 in van der Vaart and Wellner (1996). Since there are finitely many components, in the decomposition of \( f \), we have
\[
\sup_{f \in \mathcal{F}_b} \left| \int_{\mathbb{R}} f(x) d(\mathbb{P}_n(x) - P(x)) \right| \leq 2 \log b \| \mathbb{P}_n - P \|_\infty + o_P(1) = o_P(1)
\]
which completes the proof. \( \square \)

**Proof of Proposition 3.3:** The arguments go along the same line as those in the proof of Proposition 3.2. Suppose we are in the first case. Let
\[
\text{Fix } a \in \{a - \eta/2, a + \eta/2\} \text{ and } \beta \in \{b - \eta/2, b + \eta/2\}, \text{ we have that}
\]
\[
h(x) = \log \left( \pi \exp(\psi(x - \alpha)) + (1 - \pi)\exp(\psi(x - \beta)) \right)
\]
\[
= \log \left( \pi c_0(x - \alpha) + (1 - \pi)c_0(x - \beta) \right) \mathbb{1}_{\{x \leq \alpha\}}
\]
\[
+ \log \left( \pi d_0(x - \alpha) + (1 - \pi)c_0(x - \beta) \right) \mathbb{1}_{\{\alpha < x \leq \beta\}}
\]
\[
+ \log \left( \pi d_0(x - \alpha) + (1 - \pi)d_0(x - \beta) \right) \mathbb{1}_{\{x > \beta\}}
\]
\[
= h_1(x) \mathbb{1}_{\{x \leq \alpha\}} + h_2(x) \mathbb{1}_{\{\alpha < x \leq \beta\}} + h_3(x) \mathbb{1}_{\{x > \beta\}}.
\]
The functions \( h_1 \) and \( h_3 \) are increasing and decreasing on \((-\infty, \alpha]\) and \((\beta, \infty)\) respectively. Now,
\[
\int_\alpha^\beta |h'_3(x)| dx = \int_\alpha^\beta \frac{\pi |d'_0(x - \alpha) + (1 - \pi)c'_0(x - \beta)|}{d_0(x - \alpha) + (1 - \pi)c_0(x - \beta)} dx
\]
\[
\leq \int_\alpha^\beta \frac{\pi c'_0(\alpha - x) + (1 - \pi)c'_0(x - \beta)}{c_0(x - \alpha) + (1 - \pi)c_0(\beta - x)} dx
\]
\[
\leq \frac{c_0(0) - c_0(\beta - \alpha)}{c_0(\beta - \alpha)} \leq \frac{c_0(0)}{c_0(b - a + 2\eta)} - 1.
\]
Hence, \( h_3 \) admits a bounded total variation on the interval \([\alpha, \beta]\) with a bound depending only on \( \psi, a, b \) and \( \eta \). Thus, we can argue as in the proof of Proposition 3.2 that \( \mathcal{H} \) is \( P \)-Glivenko-Cantelli class for any probability measure \( P \), in particular for \( P_0 \). \( \square \)

**Proof of Theorem 3.1:** Fix \( \epsilon > 0 \). Recall that if \( a < b \), then for \( t \in \mathbb{R} \)
\[
|t - a| \leq |t - b| \iff t \leq (a + b)/2.
\]
Write \( \bar{u} = (u_1 + u_2)/2 \). Using the symmetry of \( \hat{\psi}_n \) and the fact that it is decreasing on \([0, \infty)\), it follows that
\[
\Phi_n(\log(\hat{g}_n)) \leq \frac{1}{n} \sum_{i: X_i \leq \bar{u}} \hat{\psi}_n(X_i - u_1) + \frac{1}{n} \sum_{i: X_i > \bar{u}} \hat{\psi}_n(X_i - u_2) - 1. \tag{21}
\]
Without loss of generality, we can assume that \( \{X_1, X_2, \ldots, X_N\} = \{X_i : X_i \leq \bar{u}\} \) and that \( \psi_n(X_1 - u_1) < \ldots < \hat{\psi}_n(X_N - u_1) \), and that
\[
\max_{i : X_i > \bar{u}} \hat{\psi}_n(X_i - u_2) \leq \max_{i : X_i \leq \bar{u}} \psi_n(X_i - u_1)
\]
otherwise the same reasoning above has to be applied to the second term of the right side of the inequality in (21). In the following, let
\[
M_n = \hat{\psi}_n(X_n - u_1)
\]
the largest value taken by \( \hat{\psi}_n(X_i - u_1) \) and \( \hat{\psi}_n(X_i - u_2) \) for \( i = 1, \ldots, n \).

Following the notation of Schuhmacher and Dümbgen (2010) in the proof of their Lemma 4, let \( k = \lceil \bar{N}\alpha \rceil \) for some fixed \( \alpha \in (0, 1] \), \( C_k \) the convex hull of the set \( \{X_i - u_1, i \geq k\} \) and \( m = \hat{\psi}_n(X_k - u_1) \). Also, let
\[
\eta = \frac{\alpha G(\bar{u}^0)}{2\|g^0\|_\infty}.
\]
In the sequel, \( C_k + u_1 = \{t + u_1 : t \in C_k\} \). Note that \( C_k + u_1 \) is convex and compact such that
\[
\sum_{i : X_i \leq \bar{u}} 1_{\{X_i \in C_k + u_1\}} = \sum_{i = 1}^N 1_{\{X_i \in C_k + u_1\}} = \bar{N} - k + 1 \geq \bar{N}(1 - \alpha).
\]
By taking \( 1 - \alpha \) in place of \( \alpha \) in Proposition 3.1, we can find an integer \( n_0 > 0 \) such that for \( n \geq n_0 \).
\[
P(\{|C_k| \geq \eta\}) \geq 1 - \epsilon/5. \quad (22)
\]
Consider the event
\[
\left\{ M_n > B_0 := \max(\log(1/\eta) + 1, 0) \right\}. \quad (23)
\]
Using the same argument as in Schuhmacher and Dümbgen (2010), we can write
\[
1 = \int_{\mathbb{R}} e^{\hat{\psi}_n(t)} dt \geq \int_{C_k} e^{\hat{\psi}_n(t)} dt \geq e^m |C_k| \quad (24)
\]
using concavity of \( \hat{\psi}_n \) on the convex set \( C_k \). By (23). Hence,
\[
\left\{ M_n > B_0, |C_n| > \eta \right\} \subset \{M_n > m + 1\}. \quad (25)
\]
Indeed, this follows from the fact that
\[
e^{M_n - 1} > 1/\eta, \quad \text{by (23)}
\]
\[
> 1/|C_k| \quad \text{by (24)}
\]
\[
\geq e^m \quad \text{by (24)}.
\]
Also, on the same event (25), we have for $t$ such that $t \in C_k + u_1$

\[
\hat{\psi}_n\left(X_n - u_1 + \frac{t - X_n}{M_n - m}\right) \geq \frac{1}{M_n - m} \hat{\psi}_n(t - u_1) + \frac{M_n - m - 1}{M_n - m} \hat{\psi}_n(X_n - u_1)
\]

\[
\geq \frac{m}{M_n - m} + \frac{M_n(M_n - m - 1)}{M_n - m} = M_n - 1.
\]

Thus,

\[
\text{Leb}(x \in \mathbb{R} : \hat{\psi}_n(x) \geq M_n - 1) \geq \text{Leb}\left(\frac{M_n - m - 1}{M_n - m}(X_n - u_1) + \frac{1}{M_n - m} C_k\right) = \frac{|C_k|}{M_n - m},
\]

and hence

\[
1 = \int_{\mathbb{R}} e^{\hat{\psi}_n(t)} dt \geq \frac{e^{M_n - 1}|C_k|}{M_n - m}.
\]

Hence, the event in (25) is included in

\[
\left\{ m \leq M_n - e^{M_n - 1}|C_k| < M_n - e^{M_n - 1}\eta \right\}.
\]

Let $B_1 > 0$ be large enough so that $B_1 > B_0$, $B_1 - e^{B_1 - 1}\eta < -e^{B_1 - 1}\eta/2$ and $x \mapsto x - e^{x - 1}\eta/2$ is decreasing on $[B_1, \infty)$, and consider the event

\[
\left\{ M_n > B_1, |C_k| > \eta \right\}.
\]

Then, this event is included in

\[
\left\{ m \leq -e^{M_n - 1}\eta/2 \right\}
\]

and occurrence of the latter implies in turn that

\[
\Phi_n(\log(\hat{g}_n)) \leq \frac{1}{n} \left[ (\lfloor \alpha \bar{N} \rfloor - 1) M_n - (\bar{N} - \lfloor \alpha \bar{N} \rfloor + 1) \frac{1}{2} e^{M_n - 1}\eta + \frac{n - \bar{N}}{n} n M_n \right],
\]

and consequently

\[
\Phi_n(\log(\hat{g}_n)) \leq \left( 1 - (1 - \alpha) \frac{\bar{N}}{n} \right) M_n - \frac{(1 - \alpha) \bar{N}}{2n} e^{M_n - 1}\eta.
\]

Using similar arguments as in the proof of our Proposition 3.1, we can increase $n_0$ to ensure that for $n \geq n_0$

\[
P\left( \frac{\bar{N}}{n} \geq \frac{G^0(\bar{u}^0)}{2} \right) \geq 1 - \epsilon/5.
\]

Hence, occurrence of the event

\[
\left\{ M_n > B_1, |C_k| > \eta, \frac{\bar{N}}{n} \geq \frac{G^0(\bar{u}^0)}{2} \right\}.
\]
implies that
\[ \Phi_n(\log(\hat{g}_n)) \leq \left( 1 - (1 - \alpha) \frac{G^0(\hat{u})}{2} \right) M_n - \frac{(1 - \alpha)G^0(\hat{u})^2}{4} \eta e^{M_n - 1} \]  
(26)

Next, if we take \( e^u = \varphi_\sigma \) the density of standard Normal random variable. Then,
\[ \Phi_n(\log(\hat{g}_n)) \geq \Phi_n \left( \pi e^{u(\cdot - u_1)} + (1 - \pi)e^{u(\cdot - u_2)} \right) \]
\[ = \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) d(G_n(x) - G^0(x)) \]
\[ + \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) g^0(x)dx - 1. \]

For \( \eta > 0 \) small enough, it follows from consistency of \( u_1, u_2 \) and \( \pi \) that the event
\[ \mathcal{E}^0 = \left\{ \pi - \pi^0 \leq \eta/2, |u_1 - u_1^0| \leq \eta/2, |u_2 - u_2^0| \leq \eta/2 \right\} \]
occurs with probability greater than \( 1 - \epsilon/5 \) for \( n \geq n_0 \) (at the cost of increasing the previous \( n_0 \)). By Proposition 3.3, this implies that the event
\[ \mathcal{E}_{\text{emp}} = \left\{ \left| \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) d(G_n(x) - G^0(x)) \right| \leq \frac{1}{2} \right\} \]
occurs with probability greater than \( 1 - \epsilon/5 \) for \( n \geq n_0 \). Also, there exists a constant \( D > 0 \) (depending on \( \epsilon \)) such that the event
\[ \mathcal{E}_{KL} = \left\{ \left| \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) g^0(x)dx \right| \leq D \right\} \]
with probability greater than \( 1 - \epsilon/5 \). Indeed, note that \( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \leq 1 \) for all \( x \in \mathbb{R} \) so that
\[ \left| \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) g^0(x)dx \right| = \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) g^0(x)dx. \]

Using this fact and concavity of the logarithm, it follows that
\[ 0 \leq - \int_{\mathbb{R}} \log \left( \pi e^{u(x-u_1)} + (1 - \pi)e^{u(x-u_2)} \right) g^0(x)dx \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}} \left( \pi(x-u_1)^2 + (1 - \pi)(x-u_2)^2 \right) g^0(x)dx \]
\[ < \infty \]
since log-concave densities have finite moments of any order. Put \( L = -3/2 - D \), and let \( B_2 > B_1 \) such that
\[ \left( 1 - (1 - \alpha) \frac{G(\hat{u})}{2} \right) B_2 - \frac{(1 - \alpha)G(\hat{u})^2}{4} \eta e^{B_2 - 1} < L \]

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and the function \( x \mapsto \left(1 - (1 - \alpha)\frac{G(\hat{u}^0)}{2}\right)x - \frac{(1-\alpha)G(\hat{u}^0)}{4}\eta e^{x-1} \) is decreasing on \([B_2, \infty)\).

By the choice of \( B_2 \) above, we see that

\[
\mathcal{E}_\cap = \left\{ M_n > B_2, |C_k| > \eta, \frac{\bar{N}}{n} \geq \frac{G^0(\bar{u}^0)}{2} \right\} \cap \mathcal{E}_0^c \cap \mathcal{E}_{\text{emp}} \cap \mathcal{E}_{KL} = \emptyset
\]

Hence,

\[
P\left(\|\hat{g}_n\|_\infty > B_2\right) = P(M_n > B_2) = P(M_n > B_2, \mathcal{E}_\cap) \leq P(\mathcal{E}_0^c) \leq \epsilon.
\]

Note that \( B_2 \) depends on \( \epsilon \) through \( L \).

\textbf{Proof of Theorem 3.2:} Using the inequality in (11), we will first show that the empirical process

\[
\int_{\mathbb{R}} \log(\hat{g}_n(t) + b)d(G_n(t) - G^0(t))
\]

converges to 0 in probability. By Theorem 3.1, we have

\[
A_n = \sup_{t \in \mathbb{R}} |\log(\hat{g}_n(t) + b)| = O_p(1).
\]

This implies by Proposition 3.2 that

\[
\left| \int_{\mathbb{R}} \log(\hat{g}_n(t) + b)d(G_n(t) - G^0(t)) \right| = o_p(1).
\]

(27)

Consider again the density \( g_h \) defined in (12), where \( h \) will eventually be made arbitrarily small. It follows from Lemma 3.2 that

\[
\int_{\mathbb{R}} \log(g_h(t))dG^0(t) \to n \to \infty \int_{\mathbb{R}} \log(g_h(t))dG^0(t)
\]

in probability. The last term in (11) is \( o_p(1) \) by Proposition 3.3. Thus, for an arbitrary subsequence \( (n') \) we can extract a further subsequence \( (n'') \) such that

\[
\limsup_{n'' \to \infty} 2H^2(\hat{g}_{n''}, g^0) \leq \epsilon(b) + \int_{\mathbb{R}} \left(\log(g^0(t) + b) - \log(g_h(t))\right)dG^0(t)
\]

\[
\to \int_{\mathbb{R}} \left(\log(g^0(t)) - \log(g_h(t))\right)dG^0(t)
\]

(28)

as \( b \searrow 0 \) by the dominated convergence theorem. Now, for \(|t|\) large enough, \( g_h(t) \geq g^0(t) \), so \( |\log g_h(t)| \leq |\log g^0(t)|\), and the latter is \( g^0 \)-integrable (Lemma 3.4, Seregin and Wellner (2010)). Thus, as \( h \searrow 0 \), (28) converges to 0. As \( (n') \) was arbitrary, we conclude again that \( H(\hat{g}_n, g^0) = o_p(1) \).
Proof of Corollary 3.1. Recall the inequalities in (13), and that $g = \pi f^0(\cdot - u_1) + (1 - \pi)f^0(\cdot - u_2)$. By consistency of $\pi$ we have $\pi < 1/2$ with increasing probability. Hence, the inversion formula (9) in Bordes et al. (2006) yields

$$\int_{\mathbb{R}} |\hat{f}_n(t) - f^0(t)| dt \leq \frac{1}{1 - \pi} \sum_{k=0}^{\infty} \left( \frac{\pi}{1 - \pi} \right)^k \int_{\mathbb{R}} |\hat{g}_n(t) - g(t)| dt$$

$$= \frac{1}{1 - 2\pi} \int_{\mathbb{R}} |\hat{g}_n(t) - g(t)| dt.$$  (29)

From the first inequality in (13) and Theorem 3.2 above, it follows that

$$\int_{\mathbb{R}} |\hat{g}_n(t) - g^0(t)| dt = o_p(1)$$

which in turn implies that

$$\int_{\mathbb{R}} |\hat{g}_n(t) - g(t)| dt = o_p(1).$$

Indeed,

$$\int_{\mathbb{R}} |\hat{g}_n(t) - g(t)| dt \leq \int_{\mathbb{R}} |\hat{g}_n(t) - g^0(t)| dt + \int_{\mathbb{R}} |g^0(t) - g(t)| dt$$

with

$$\int_{\mathbb{R}} |g^0(t) - g(t)| dt \leq 2|\pi - \pi^0| + \int_{\mathbb{R}} |f^0(t - u_1) - f^0(t - u_1^0)| dt$$

$$+ \int_{\mathbb{R}} |f^0(t - u_2) - f^0(t - u_2^0)| dt$$

$$= o_p(1)$$

by consistency of $\pi$, $u_1$ and $u_2$ and Lemma A.1. It follows now from (29) and the inequalities in (13) that

$$H(\hat{f}_n, f^0) = o_p(1), \text{ and } \int_{\mathbb{R}} |\hat{f}_n(t) - f^0(t)| dt = o_p(1).$$

Note that convergence of the MLE $\hat{f}_n$ in probability to 0 in the $L_1$ distance implies its weak convergence to $f^0$ with increasing probability. Hence, for any arbitrary sequence $(n')$ we can extract a further subsequence $(n'')$ such that $\hat{f}_n''$ converges weakly to $f^0$ almost surely. Hence, the assertion (c) in Proposition 2 of Cule and Samworth (2010) holds almost surely for $f_n = \hat{f}_n''$ and $f = f^0$, and the remaining claims of our lemma now follow since $(n')$ was chosen arbitrarily. □

Proof of Proposition 4.1: It follows from the recent result of Doss and Wellner (2013); see their Theorem 4.1 for $s = 0$, that the classes

$$\left\{ f(\cdot - a), \ f \in SLC, \ f(0) \in [1/M, M], \ a \in [a_0 - \delta, a_0 + \delta] \right\}$$

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and

\[
\left\{ f(\cdot - b), \ f \in \mathcal{SLLC}, \ f(0) \in [1/M, M], \ b \in [b_0 - \delta, b_0 + \delta] \right\}
\]

have both the same bracketing entropy \(\log(N) \lesssim 1/\sqrt{\epsilon}\). Let \(\{\epsilon, \ldots, K\epsilon\}\), a \(\epsilon\)-net for \([0,1]\), where \(K = [1/\epsilon] + 1\). For \(i \in \{1, \ldots, N\}\), let \([l_i, u_i]\) and \([l'_i, u'_i]\) an \(\epsilon\)-bracket for the first and second class respectively. Note that since the brackets are in the Hellinger sense, we have that \(u_i \geq l_i \geq 0\) and \(u'_i \geq l'_i \geq 0\). Now, there exist \(i \in \{1, \ldots, N\}\) and \(j \in \{1, \ldots, K\}\) such that

\[
g \equiv (j - 1)\epsilon l_i + (1 - j\epsilon)l'_i \leq \lambda f(x - a) + (1 - \lambda)f(x - b) \leq \bar{g} \equiv j\epsilon u_i + (1 - (j - 1)\epsilon)u'_i
\]

with

\[
H^2(g, \bar{g}) = \frac{1}{2} \int_{\mathbb{R}} \left\{ \left( \frac{j\epsilon u_i(t) + (1 - (j - 1)\epsilon)u'_i(t)}{\sqrt{2}} \right)^{1/2} - \left( \frac{(j - 1)\epsilon l_i(t) + (1 - j\epsilon)l'_i(t)}{\sqrt{2}} \right)^{1/2} \right\}^2 dt
\]

\[
\leq 2H^2((j - 1)\epsilon l_i + (1 - (j - 1)\epsilon)u'_i, j\epsilon u_i + (1 - (j - 1)\epsilon)u'_i)
\]

\[
+ 2H^2((1 - j\epsilon)l'_i + (j - 1)\epsilon l_i, (1 - (j - 1)\epsilon)u'_i + (j - 1)\epsilon l_i)
\]

\[
\leq 2H^2((j - 1)\epsilon l_i, j\epsilon u_i) + 2H^2((1 - j\epsilon)l'_i, (1 - (j - 1)\epsilon)u'_i), \text{ by Lemma A.2}
\]

\[
\leq 4(j - 1)\epsilon \ H^2(l_i, u_i) + 4(\sqrt{j} - \sqrt{j - 1})^2 \epsilon
\]

\[
+ 4(1 - j\epsilon) \ H^2(l'_i, u'_i) + 4(\sqrt{1 - (j - 1)\epsilon} - \sqrt{1 - j\epsilon})^2 \epsilon, \text{ applying again Lemma A.2.}
\]

Using the fact that \(0 \leq j\epsilon \leq 1\) and \(1 - (j - 1)\epsilon \geq \epsilon\) for all \(j \in \{1, \ldots, K\}\), we conclude from the preceding calculations that

\[
H^2(g, \bar{g}) \leq 4H^2(l_i, u_i) + 4H^2(l'_i, u'_i) + 8\epsilon \lesssim \epsilon.
\]

The proof is complete by noting that

\[
\log N[1](\epsilon, \mathcal{G}, H) \leq \log K + \log N \leq \log \left( \frac{1}{\epsilon} + 1 \right) + \log N
\]

\[
\leq \frac{1}{\sqrt{\epsilon}} + \log N \lesssim \frac{1}{\sqrt{\epsilon}}
\]

using the fact that \(\log(x + 1) \leq \sqrt{x}\) for all \(x \in [0, \infty)\). \(\square\)

To prepare for the proof of Theorem 4.1, we recall that \(\pi, u_1\) and \(u_2\) are estimates of \(\pi^0, u^0_1\) and \(u^0_2\) respectively, that are converging at the rate \(1/\sqrt{n}\). Recall also that

\[
g_n = \pi f^0(\cdot - u_1) + (1 - \pi)f^0(\cdot - u_2), \text{ and that } \bar{g}_n = \pi \hat{f}_n(\cdot - u_1) + (1 - \pi)\hat{f}_n(\cdot - u_2),
\]
where $\hat{f}_n$ is the log-concave MLE. As done in van der Vaart and Wellner (1996) (page 326, Section 3.4), we consider the criterion function

$$r_{n,g} = \log \frac{g + \tilde{g}_n}{2\tilde{g}_n}$$

for $g$ and

$$\tilde{g}_n = \pi \hat{f}_n (\cdot - u_1) + (1 - \pi) \hat{f}_n (\cdot - u_2)$$

with $\hat{f}_n \in \mathcal{SG}_1$ to be constructed. Note that

$$r_{n,\tilde{g}_n} = 0,$$

and

$$P_n r_{n,\hat{g}_n} \geq \frac{1}{2} P_n \log \frac{\tilde{g}_n}{\hat{g}_n} \geq 0 \quad (30)$$

where the second claim follows from the definition of the MLE $\hat{g}_n$ and concavity of the logarithm.

Consider now the class of functions

$$\mathcal{R}_{n,\eta} = \left\{ r_{n,g} - r_{n,\tilde{g}_n} : g \in \mathcal{G}, H(g, \tilde{g}_n) < \eta \right\} = \left\{ r_{n,g} : g \in \mathcal{G}, H(g, \tilde{g}_n) < \eta \right\}.$$ 

If $P^0$ denotes again the true probability measure associated with $g^0$, let

$$\mathbb{G}_n(r_{n,g}) = \sqrt{n} \left( P_n - P^0 \right) r_{n,g} = \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} r_{n,g}(X_i) - \int r_{n,g}(x) dP^0(x) \right)$$

and denote

$$\| \mathbb{G}_n \|_{\mathcal{R}_{n,\eta}} = \sup_{g \in \mathcal{R}_{n,\eta}} |\mathbb{G}_n(r_{n,g})|.$$ 

Finally, define

$$\mathcal{J}_{([\delta], \mathcal{G}, H)} = \int_{0}^{\delta} \sqrt{1 + \log N_1(\epsilon, \mathcal{G}, H)} d\epsilon.$$ 

Theorem 3.4.4 of van der Vaart and Wellner (1996) gives sufficient conditions to obtain control on $\| \mathbb{G}_n \|_{\mathcal{R}_{n,\eta}}$ in the mean. This control will involve the bracketing entropy bound obtained for the class $\mathcal{G}$. One of the crucial conditions to be fulfilled is that the sequence of densities $\tilde{g}_n$ need to be chosen such that $\tilde{g}_n$ approximates the truth $g^0$ and

$$\frac{g^0}{\tilde{g}_n} \leq M \quad (31)$$

for some $M > 0$. Note that the reason we cannot choose $\tilde{g}_n = g^0$ is that in this problem maximization of the log-likelihood involve the random variables $\pi, u_1$ and
u_2$, hence it is not at all straightforward to compare the values taken by the criterion at \( \tilde{g}_n \) and \( g^0 \). As will be shown in Proposition A.1, we will exhibit an approximating sequence \( \tilde{g}_n \) that will satisfy the condition in (31) with increasing probability. Based on this proposition, we give now the proof of Theorem 4.1 along the same lines of the proof of Theorem 3.25 in van der Vaart and Wellner (1996); see page 290.

**Proof of Theorem 4.1.** Fix \( \epsilon > 0 \). Let \( \tilde{g}_n \) be the approximating sequence of Proposition A.1, and consider the shells

\[
S_{j,n} = \left\{ g : 2^{j-1} < n^{2/5} H(g, \tilde{g}_n) \leq 2^j \right\}
\]

for integers \( j \geq 1 \). Fix an integer \( J \geq 1 \), and consider the event

\[
\left\{ n^{2/5} H(\tilde{g}_n, \tilde{g}_n) > 2^J \right\}.
\]

Occurrence of this event implies that \( \tilde{g}_n \) belongs to some \( S_{j,n} \) with \( j \geq J \). But our remark in (30) implies that

\[
\sup_{g \in S_{j,n}} (\mathbb{P}_n r_{n,g} - \mathbb{P}_n r_{n,\tilde{g}_n}) = \sup_{g \in S_{j,n}} \mathbb{P}_n r_{n,g} \geq 0.
\]

Thus, for any \( \delta > 0 \) and \( M > 0 \) we can write

\[
P \left( n^{2/5} H(\tilde{g}_n, \tilde{g}_n) > 2^J \right) = P \left( n^{2/5} H(\tilde{g}_n, \tilde{g}_n) > 2^J, g^0 \leq M \tilde{g}_n \right) + P \left( g^0 > M \tilde{g}_n \right)
\]

\[
= \sum_{j \geq J, 2^j \leq n^{2/5} \eta} P \left( \sup_{g \in S_{j,n}} (\mathbb{P}_n r_{n,g} - \mathbb{P}_n r_{n,\tilde{g}_n}) \geq 0, g^0 \leq M \tilde{g}_n \right) + P \left( H(\tilde{g}_n, \tilde{g}_n) \geq \delta/2 \right) + P \left( g^0 > M \tilde{g}_n \right).
\]

By Proposition A.1, for any \( \eta > 0 \) and \( M \) large enough such that the second and third terms are bounded by \( \epsilon/2 \). Now, using Theorem 3.4.4 of van der Vaart and Wellner (1996), we have

\[
P^0 (r_{n,g} - r_{n,\tilde{g}_n}) \lesssim -H^2(g, \tilde{g}_n)
\]

for all densities \( g \) such that \( H(g, \tilde{g}_n) \geq 32 MH(\tilde{g}_n, g^0) \), and

\[
E_{P^0} \|G_n\|_{\mathcal{R}_{n,\eta}} \lesssim \tilde{J}_\eta(\eta, G, H) \left( 1 + \frac{\tilde{J}_\eta(\eta, G, H)}{\eta^2 \sqrt{n}} \right).
\]

From Proposition 4.1, we know that there exists a constant \( K > 0 \) (not depending on \( n \)) such that

\[
\log N_{\|\|}(\epsilon, G, H) \leq \frac{K}{\sqrt{\epsilon}}.
\]
For $\eta > 0$ small enough, $1 < K/\sqrt{\epsilon}$ and hence

$$J_{\eta}(\eta, G, H) \leq \int_0^{\eta^2} \sqrt{2K} \epsilon^{1/4} e^{\epsilon} = \frac{4\sqrt{2K}}{3} \eta^{3/4}.$$ 

Now, define

$$\phi_n(\eta) := \frac{4\sqrt{2K}}{3} \eta^{3/4} \left(1 + \frac{4\sqrt{2K}}{3\eta^{5/4}/\sqrt{n}}\right).$$

We have that

$$\frac{\phi_n(\eta)}{\eta} \propto \frac{1}{\eta^{1/4}} \left(1 + \frac{4\sqrt{2K}}{3\eta^{5/4}/\sqrt{n}}\right)$$

and hence the function $\eta \mapsto \phi_n(\eta)/\eta$ is decreasing on $(0, \infty)$. Also, if we put $r_n = n^{2/5}$ then,

$$r_n^2 \phi_n \left(\frac{1}{r_n}\right) = n^{4/5} \frac{4\sqrt{2K}}{3} n^{-3/10} \left(1 + \frac{4\sqrt{2K}}{3}\right)$$

$$= \frac{4\sqrt{2K}}{3} \left(1 + \frac{4\sqrt{2K}}{3}\right) \sqrt{n}.$$ 

Now note that on the event $\{g^0 \leq M\tilde{g}_n\}$ it follows from (32) that for all $g \in S_{n,j}$

$$P_0(r_{n,g} - r_{n,\tilde{g}_n}) \leq \frac{-2^{2j-2}}{r_n^2}$$

and hence

$$\sum_{j \geq J, 2^j \leq n^{2/5}} \left(\sup_{g \in S_{n,j}} \frac{P_0(\mathbb{P}_{g_{n}} - \mathbb{P}_{n,\tilde{g}_n})}{\|G_n\|_{S_{n,j}}} \geq \frac{-2^{2j-2}}{r_n^2}\right)$$

$$\leq \sum_{j \geq J, 2^j \leq n^{2/5}} P\left(\left\|G_n\right\|_{S_{n,j}} \geq \frac{-2^{2j-2}}{r_n^2}\right)$$

$$\leq \sum_{j \geq J} \frac{\phi_n(2^j/r_n)}{2^{2j-2} \sqrt{n}}, \text{ by Markov’s inequality}$$

$$\leq \frac{1}{4} \sum_{j \geq J} \frac{\phi_n(1/r_n)}{2^{2j} \sqrt{n}}, \text{ using the fact that } \frac{\phi_n(ct)}{ct} \leq \frac{\phi_n(t)}{t} \text{ for } c > 1$$

$$\leq \sum_{j \geq J} 2^{-j} \to 0 \text{ as } J \to \infty \quad (34)$$

which implies that $H(\tilde{g}_n, \tilde{g}_n) = O_p(n^{-2/5})$. Hence, $L_1(\tilde{g}_n, \tilde{g}_n) = O_p(n^{-2/5})$ which in turn implies that $L_1(\tilde{g}_n, g^0) = O_p(n^{-2/5})$ using the result of Proposition A.1 and the triangular inequality.
To conclude a similar rate result for \( \hat{f}_n \), we show now that \( L_1(\hat{f}_n, c_n \tilde{f}_n) = O_p(n^{-2/5}) \) where \( \hat{f}_n \) and the normalizing constant \( c_n \) are defined in Proposition A.1. Using again the inversion formula (9) in Bordes et al. (2006), we can write

\[
\int_{\mathbb{R}} |\hat{f}_n(t) - c_n \tilde{f}_n(t)| dt \leq \frac{1}{1 - 2\pi} \int_{\mathbb{R}} |\tilde{g}_n(t) - \tilde{g}_n(t)| dt = O_p(n^{-2/5}).
\]

The proof is complete using the result of Proposition A.1 and the triangle inequality.

\( \square \)

**Proposition A.1** There exist \( \tilde{f}_n \in SCG \) and \( c_n > 0 \) such that \( c_n \tilde{f}_n \in SCG_1 \) and

\[
\sup_{t \in \mathbb{R}} \frac{g^0(t)}{\tilde{g}_n(t)} = O_p(1)
\]

where

\[
\tilde{g}_n = \pi c_n \tilde{f}_n(\cdot - u_1) + (1 - \pi)c_n \tilde{f}_n(\cdot - u_2).
\]

Furthermore, if \( \sqrt{n}(u_1 - u_1^0) = O_p(1), \sqrt{n}(u_2 - u_2^0) = O_p(1) \) and \( \sqrt{n}(\pi - \pi^0) = O_p(1/\sqrt{n}) \), then

\[
L_1(\hat{f}_n, f^0) = O_p(n^{-1/2}), \text{ and } L_1(\tilde{g}_n, g^0) = O_p(n^{-1/2}).
\]

**Proof of Proposition A.1.** In the following we denote

\[
\delta_1 = u_1 - u_1^0 \text{ and } \delta_2 = u_2 - u_2^0.
\]

Suppose that \( 0 \leq \delta_2 \leq \delta_1 \). Define

\[
\hat{f}_n(t) = \begin{cases} f^0(|t| - \delta_1), & \text{if } t \leq -\delta_1 \\ f^0(0), & \text{if } -\delta_1 \leq t \leq \delta_1 \\ f^0(t - \delta_1), & \text{if } t \geq \delta_1. \end{cases}
\]

It is not difficult to see that \( \hat{f}_n \in SLC \). Now, define the ratios

\[
\tilde{R}_{n,1}(t) := \frac{f^0(t)}{f_n(t - \delta_1)}, \text{ and } \tilde{R}_{n,2}(t) := \frac{f^0(t)}{f_n(t - \delta_2)}.
\]

We have that

\[
\tilde{R}_{n,1}(t) = \begin{cases} \frac{f^0(t)}{f_n(\delta_1 - t - \delta_1)} = 1, & \text{if } t \leq 0 \\ \frac{f^0(t)}{f_n(\delta_1)} \leq 1, & \text{if } 0 \leq t \leq 2\delta_1 \\ \frac{f^0(t)}{f_n(t - 2\delta_1)} \leq 1, & \text{if } t \geq 2\delta_1. \end{cases}
\]
The third inequality follows from the fact that $f^0$ is decreasing on $[0, \infty)$. Also,

$$\tilde{R}_{n,2}(t) = \begin{cases} 
\frac{f^0(t)}{f^0(\bar{\delta}_2 - \delta_1 - t)} \leq 1, & \text{if } t \leq \delta_2 - \delta_1 \\
\frac{f^0(t)}{f^0(0)} \leq 1, & \text{if } \delta_2 - \delta_1 \leq t \leq \delta_1 + \delta_2 \\
\frac{f^0(t)}{f^0(t - \delta_1 - \delta_2)} \leq 1, & \text{if } t \geq \delta_1 + \delta_2.
\end{cases}$$

The first inequality follows from symmetry of $f^0$ which allows to write that $f^0(t)/f^0(\delta_2 - \delta_1 - t) = f^0(x)/f^0(x - (\delta_1 - \delta_2))$ with $x = -t \geq \delta_1 - \delta_2 \geq 0$. Since $f^0$ is decreasing on $[0, \infty)$, it follows that $f^0(x) \leq f^0(x - (\delta_1 - \delta_2))$. The third inequality is again a consequence of the latter property of $f^0$.

Now let $c_n = (\int_{\mathbb{R}} \tilde{f}_n(t)dt)^{-1}$. We have that

$$c_n^{-1} - 1 = \int_{-\infty}^{-\delta_1} f^0(|t| - \delta_1)dt + \int_{\delta_1}^{\infty} f^0(t - \delta_1)dt + 2\delta_1 f^0(0) - 1$$

This in turn implies that

$$0 \leq 1 - c_n = \frac{2\delta_1 f^0(0)}{1 + 2\delta_1 f^0(0)}.$$

Now, define $\tilde{g}_n := c_n \left( \pi \tilde{f}_n(\cdot - u_1) + (1 - \pi) \tilde{f}_n(\cdot - u_2) \right)$. Then,

$$\tilde{g}_n(t) = c_n^{-1} \pi f^0(t - u_1) + (1 - \pi) f^0(t - u_2)$$

$$\leq c_n^{-1} \pi f^0(t - u_1) + c_n^{-1} \frac{1 - \pi}{1 - \frac{1}{2}} f^0(t - u_2)$$

$$= c_n^{-1} \pi \tilde{R}_{n,1}(t - u_1) + c_n^{-1} \frac{1 - \pi}{1 - \frac{1}{2}} \tilde{R}_{n,2}(t - u_2)$$

By consistency of $\pi$ and $u_1$, we have $\pi^0 < \pi \leq (3/2)\pi^0$ and $c_n^{-1} \leq 3/2$ with increasing probability. Hence, we can bound the right hand side of the preceding display by $9/2$ with increasing probability. As the other cases can be handled similarly, the details are skipped but we give below the corresponding expression of $\tilde{f}_n$:

- If $0 \leq \delta_1 < \delta_2$, then we only need to switch the roles of $\delta_1$ and $\delta_2$ and hence take

$$\tilde{f}_n(t) = \begin{cases} 
f^0(|t| - \delta_2), & \text{if } t \leq -\delta_2 \\
f^0(0), & \text{if } -\delta_2 \leq t \leq \delta_2 \\
f^0(t - \delta_2), & \text{if } t \geq \delta_2.
\end{cases}$$
• If \( \delta_2 \leq 0 \leq \delta_1 \), then we can take
\[
\hat{f}_n(t) = \begin{cases}
  f^0(|t| - (\delta_1 - \delta_2)), & \text{if } t \leq -(\delta_1 - \delta_2) \\
f^0(0), & \text{if } -(\delta_1 - \delta_2) \leq t \leq \delta_1 - \delta_2 \\
f^0(t - (\delta_1 - \delta_2)), & \text{if } t \geq \delta_1 - \delta_2.
\end{cases}
\]

• If \( \delta_1 \leq 0 \leq \delta_2 \), then we only need to switch \( \delta_1 \) and \( \delta_2 \).

• If \( \delta_1 \leq \delta_2 \leq 0 \), then we can take
\[
\tilde{f}_n(t) = \begin{cases}
  f^0(t + \delta_1), & \text{if } t \geq -\delta_1 \\
f^0(0), & \text{if } \delta_1 \leq t \leq -\delta_1 \\
f^0(0), & \text{if } t \leq \delta_1.
\end{cases}
\]

• If \( \delta_2 \leq \delta_1 \leq 0 \), we again switch the roles of \( \delta_1 \) and \( \delta_2 \).

In all the cases above, one can verify that the ratios \( \tilde{R}_{n,1} \) and \( \tilde{R}_{n,2} \) as defined above stay below 1. We would like to stress the fact that the way \( \tilde{f}_n \) is constructed is not unique: one only need to exhibit examples which would give control of the ratio \( g^0/\tilde{g}_n \). To show now the second assertion, we will again consider only the first case where \( 0 \leq \delta_2 \leq \delta_1 \) since the remaining configurations can be handled similarly. We have that
\[
L_1(\tilde{f}_n, f^0) = \int_\mathbb{R} |\tilde{f}_n(t) - f^0(t)|dt
\]
\[
= \int_{-\delta_2}^{-\delta_2} |f^0(|t| - \delta_2) - f^0(t)|dt + \int_{-\delta_2}^{\delta_2} |f^0(0) - f(t)|dt
\]
\[
+ \int_{\delta_2}^{\infty} |f^0(t - \delta_2) - f^0(t)|dt
\]
\[
= 2 \int_{\delta_2}^{\infty} |f^0(t - \delta_2) - f^0(t)|dt + 2\delta_2 f^0(0)
\]
\[
\leq 2(C + 1)\delta_2
\]
where \( C \) is the constant given in Lemma A.1. By the assumption on the rate of convergence of \( \delta_2 \), this implies that \( L_1(\tilde{f}_n, f^0) = O_p(n^{-1/2}) \). Also, we have
\[
L_1(\tilde{g}_n, g^0)
\]
\[
= \int_\mathbb{R} |\tilde{g}_n(t) - g^0(t)|dt
\]
\[
= \int_\mathbb{R} \left| c_n (\pi \tilde{f}_n(t - u_1) + (1 - \pi)\tilde{f}_n(t - u_2)) - (\pi^0 f^0(t - u_1^0) + (1 - \pi^0) f^0(t - u_2^0)) \right|dt
\]
\[
\leq 1 - c_n + 4|\pi - \pi^0| + \int_\mathbb{R} |\tilde{f}_n(t - u_1) - f^0(t - u_1^0)|dt + \int_\mathbb{R} |\tilde{f}_n(t - u_2) - f^0(t - u_2^0)|dt
\]
\[
\leq 1 - c_n + 4|\pi - \pi^0| + \int_\mathbb{R} |f^0(t - \delta_1) - f^0(t)|dt + \int_\mathbb{R} |f^0(t - \delta_2) - f^0(t)|dt
\]
\[
= O_p(n^{-1/2})
\]
using again Lemma A.1, and the assumption on the rate of convergence of $\pi$ and $u_1$ and $u_2$. \hfill \square

### A.2 Auxiliary Results

**Lemma A.1** Let $f \in SLC_1$. Then, there exists a constant $C > 0$ depending only on $f$ such that
\[
\int_{\mathbb{R}} \left| f(t + \delta) - f(t) \right| dt \leq C\delta
\]
for all $|\delta| \leq 1$.

**Proof.** For $\delta \geq 0$, we have $f(t + \delta) \leq f(t)$ on $t \in [0, \infty)$, $f(t - \delta) \geq f(t)$ on $t \in [\delta/2, \infty)$ and $f(t - \delta) \leq f(t)$ on $[0, \delta/2]$. Using the symmetry of $f$, we can write
\[
\int_{\mathbb{R}} \left| f(t + \delta) - f(t) \right| dt = \int_{0}^{\infty} \left| f(t + \delta) - f(t) \right| dt + \int_{0}^{\delta/2} \left| f(t + \delta) - f(t - \delta) \right| dt + \int_{\delta/2}^{\infty} \left| f(t - \delta) - f(t) \right| dt 
\]
\[
\leq \frac{1}{2} \int_{\delta}^{\infty} f(t) dt + \int_{0}^{\delta/2} f(t) dt + \int_{-\delta/2}^{0} f(t) dt - \int_{\delta/2}^{\infty} f(t) dt 
\]
\[
= \int_{0}^{\delta} f(t) dt + \int_{0}^{\delta/2} f(t) dt + 2 \int_{0}^{\delta/2} f(t) dt 
\]
\[
\leq 3\delta \sup_{t \in [0,1]} f(t) = 3\delta f(0)
\]
provided that $\delta \leq 1$. The same reasoning can be applied for negative values of $\delta$. \hfill \square

**Lemma A.2** For any positive functions $p, q, h$ we have that
\[
H(p + h, q + h) \leq H(p, q)
\]
Proof. By definition, we have that
\[
2H^2(p + h, q + h) = \int_\mathbb{R} \left( \sqrt{p(t) + h(t)} - \sqrt{q(t) + h(t)} \right)^2 dt
\]
\[
\leq \int_\mathbb{R} \left( \frac{p(t) - q(t)}{\sqrt{p(t) + q(t) + \sqrt{q(t) + h(t)}}} \right)^2 dt
\]
\[
= \int_\mathbb{R} \left( \sqrt{p(t)} - \sqrt{q(t)} \right)^2 \left( \frac{\sqrt{p(t)} + \sqrt{q(t)}}{\sqrt{p(t) + q(t) + h(t)}} \right)^2 dt
\]
\[
\leq \int_\mathbb{R} \left( \sqrt{p(t)} - \sqrt{q(t)} \right)^2 dt = 2H^2(p, q)
\]
where the last inequality follows since \( h \geq 0 \), hence the result. \qed

The function \( J \) and its partial derivatives: As in Dümbgen and Rufibach (2009), we consider the two-dimensional function \( J \) defined by
\[
J(r, s) = \int_0^1 \exp((1 - t)r + ts)dt
\]
for \( r, s \in \mathbb{R} \). Using the same notation of these authors, define
\[
J_{a,b}(r, s) = \frac{\partial^{a+b}J(r, s)}{\partial^a s \partial^b s}.
\]
Direct calculations yield
\[
J_{a,b}(r, s) = \exp(r)J_{a,b}(0, s - r)
\]
with
\[
J_{0,0}(0, y) = J(0, y) = \begin{cases} 1, & \text{if } y = 0 \\ \frac{\exp(y) - 1}{y}, & \text{otherwise}, \end{cases}
\]
\[
J_{0,1}(0, y) = \begin{cases} \frac{1}{y}, & \text{if } y = 0 \\ \frac{\exp(y) - (\exp(y) - 1)}{y^2}, & \text{otherwise}, \end{cases}
\]
and
\[
J_{0,2}(0, y) = \begin{cases} \frac{1}{3}, & \text{if } y = 0 \\ \frac{y^2\exp(y) - 2y\exp(y) + 2(\exp(y) - 1)}{y^4}, & \text{otherwise}. \end{cases}
\]
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