OPERATOR SYMBOLS

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Abstract. We consider special elliptic operators in functional spaces on manifolds with a boundary which has some singular points. Such an operator can be represented by a sum of operators, and for a Fredholm property of an initial operator one needs a Fredholm property for each operator from this sum.

1. Introduction

This paper is devoted to describing structure of a special class of linear bounded operators on a manifold with non-smooth boundary. Our description is based on Simonenko’s theory of envelopes [3] and explains why we obtain distinct theories for pseudo-differential equations and boundary value problems and distinct index theorems for such operators.

1.1. Operators of a local type. In this section we will give some preliminary ideas and definitions from [3].

Let $B_1, B_2$ be Banach spaces consisting of functions defined on compact $m$-dimensional manifold $M$, $A : B_1 \to B_2$ be a linear bounded operator, $W \subset M$, and $P_W$ be a projector on $W$ i.e.

$$(P_W u)(x) = \begin{cases} 
  u(x), & \text{if } x \in W; \\
  0, & \text{if } x \notin W.
\end{cases}$$

Definition 1. An operator $A$ is called an operator of a local type if the operator

$$P_U A P_V$$

is a compact operator for arbitrary non-intersecting compact sets $U, V \subset M$.

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1.2. Simple examples. These are two simplest examples for the illustration.

**Example 1.** If $A$ is a differential operator of the type

$$(Au)(x) = \sum_{|k|=0}^{n} a_k(x) D^k u(x), \quad D^k u = \frac{\partial^k u}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}},$$

then $A$ is an operator of a local type.

**Example 2.** If $A$ is a Calderon–Zygmund operator with variable kernel $K(x, y) \in C^1(\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ of the following type

$$(Au)(x) = \text{v.p.} \int_{\mathbb{R}^m} K(x, x - y) u(y) dy.$$
1.3.2. **Partition of unity and spaces** $H^s(M), L_p(M), C^\alpha(M)$. If $M$ is a compact manifold then there is a partition of unity \[4\]. It means the following. For every finite open covering $\{U_j\}_{j=1}^k$ of the manifold $M$ there exists a system of functions $\{\varphi_j(x)\}_{j=1}^k$, $\varphi_j(x) \in C^\infty(M)$, such that

- $0 \leq \varphi_j(x) \leq 1$,
- $\text{supp } \varphi_j \subset U_j$,
- $\sum_{j=1}^k \varphi_j(x) = 1$.

So we have

$$f(x) = \sum_{j=1}^k \varphi_j(x)f(x)$$

for arbitrary function $f$ defined on $M$.

Since an every set $U_j$ is diffeomorphic to an open set $D_j \subset \mathbb{R}^m$ we have corresponding diffeomorphisms $\omega_j : U_j \to D_j$. Further for a function $f$ defined on $M$ we compose mappings $f_j = f \cdot \varphi_j$ and as far as $\text{supp } f_j \subset U_j$ we put $\hat{f}_j = f_j \circ \omega_j^{-1}$ so that $\hat{f}_j : D_j \to \mathbb{R}$ is a function defined in a domain of $m$-dimensional space $\mathbb{R}^m$. We can consider the following spaces \[2, 1, 6\].

**Definition 5.** A function $f \in H^s(M)$ if the following norm

$$||f||_{H^s(M)} = \sum_{j=1}^k ||\hat{f}_j||_s$$

is finite.

A function $f \in L_p(M)$ if the following norm

$$||f||_{L_p(M)} = \sum_{j=1}^k ||\hat{f}_j||_p$$

is finite.

A function $f \in C^{\alpha}(M)$ if the following norm

$$||f||_{C^{\alpha}(M)} = \sum_{j=1}^k ||\hat{f}_j||_\alpha$$

is finite.

2. **Operators on a compact manifold**

On the manifold $M$ we fix a finite open covering and a partitions of unity corresponding to this covering $\{U_j, f_j\}_{j=1}^n$ and choose smooth
functions \( \{g_j\}_{j=1}^n \) so that \( \text{supp } g_j \subset V_j, \ \overline{U_j} \subset V_j, \) and \( g_j(x) \equiv 1 \) for \( x \in \text{supp } f_j, \ \text{supp } f_j \cap (1 - g_j) = \emptyset. \)

**Proposition 1.** The operator \( A \) on the manifold \( M \) can be represented in the form

\[
A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,
\]

where \( T : B_1 \rightarrow B_2 \) is a compact operator.

**Proof.** The proof is a very simple. Since

\[
\sum_{j=1}^{n} f_j(x) \equiv 1, \quad \forall x \in M,
\]

then we have

\[
A = \sum_{j=1}^{n} f_j \cdot A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + \sum_{j=1}^{n} f_j \cdot A \cdot (1 - g_j),
\]

and the proof is completed. \( \square \)

**Remark 1.** It is obviously such operator is defined uniquely up to a compact operators which do not influence on an index.

By definition for an arbitrary operator \( A : B_1 \rightarrow B_2 \)

\[
||| A ||| \equiv \inf ||| A + T |||,
\]

where \( \text{infimum} \) is taken over all compact operators \( T : B_1 \rightarrow B_2. \)

Let \( B_1', B_2' \) be Banach spaces consisting of functions defined on \( \mathbb{R}^m, \)

\( \tilde{A} : B_1' \rightarrow B_2' \) be a linear bounded operator.

Since \( M \) is a compact manifold, then for every point \( x \in M \) there exists a neighborhood \( U \ni x \) and diffeomorphism \( \omega : U \rightarrow D \subset \mathbb{R}^m, \omega(x) \equiv y. \) We denote by \( S_\omega \) the following operator acting from \( B_k \) to \( B_k', k = 1, 2. \) For every function \( u \in B_k \) vanishing out of \( U \)

\[
(S_\omega u)(y) = u(\omega^{-1}(y)), \quad y \in D, \quad (S_\omega u)(y) = 0, \quad y \notin D.
\]

**Definition 6.** A local representative of the operator \( A : B_1 \rightarrow B_2 \) at the point \( x \in M \) is called the operator \( \tilde{A} : B_1' \rightarrow B_2' \) such that \( \forall \varepsilon > 0 \) there exists the neighborhood \( U_j \) of the point \( x \in U_j \subset M \) with the property

\[
||| g_j f_j - S^{-1}_{\omega_j} \tilde{g}_j \tilde{f}_j S_{\omega_j}||| < \varepsilon.
\]
3. Algebra of symbols

Definition 7. Symbol of an operator $A$ is called the family of its local representatives $\{A_x\}$ at each point $x \in \overline{M}$.

One can show like [3] this definition of an operator symbol conserves all properties of a symbolic calculus. Namely, up to compact summands

- product and sum of two operators corresponds to product and sum of their local representatives;
- adjoint operator corresponds to its adjoint local representative;
- a Fredholm property of an operator corresponds to a Fredholm property of its local representative.

4. Operators with symbols. Examples.

It seems not every operator has a symbol, and we give some examples for operators with symbols.

Example 3. Let $A$ be the differential operator from example 1, and functions $a_k(x)$ be continuous functions on $\mathbb{R}^m$. Then its symbol is an operator family consisting of multiplication operators on the function

$$\sum_{|k|=0}^{n} a_k(x)\xi^k,$$

where $\xi^k = \xi_1^{k_1} \cdots \xi_m^{k_m}$.

Example 4. Let $A$ be the Calderon–Zygmund operator from Example 2 and $\sigma(x, \xi)$ be its symbol in sense of [1], then its symbol is an operator family consisting of multiplication operators on the function $\sigma(x, \xi)$.

More important point is that symbol of an operator should be more simple to verify its Fredholm properties. For two above examples a Fredholm property of an operator symbol is equivalent to its invertibility.

5. Stratification of manifolds and operators

5.1. Sub-manifolds. The above definition of an operator on a manifold supposes that all neighborhoods $\{U_j\}$ have the same type. But if a manifold has a smooth boundary even then there are two types of neighborhoods related to a place of neighborhood, namely inner neighborhoods and boundary ones. For inner neighborhood $U$ such that $\overline{U} \subset \overline{M}$ we have the diffeomorphism $\omega : U \to D$, where $D \in \mathbb{R}^m$ is an open set. For a boundary neighborhood such that $U \cap \partial M \neq \emptyset$ we have another diffeomorphism $\omega_1 : U \to D \cap \mathbb{R}_+^m$, where $\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > 0\}$. May be this boundary $\partial M$ has some
singularities like conical points and wedges. The conical point at the boundary is a such point, for which its neighborhood is diffeomorphic to the cone

$$C^a_+ = \{ x \in \mathbb{R}^m : x_m > a|x'|, x' = (x_1, \ldots, x_{m-1}), a > 0 \},$$

the wedge point of codimension $k, 1 \leq k \leq m - 1$, is a such point for which its neighborhood is diffeomorphic to the set

$$\{ x \in \mathbb{R}^m : x = (x', x''), x'' \in \mathbb{R}^{m-k}, x' = (x_1, \ldots, x_{m-k-1}), x_{m-k-1} > a|x''|, x'' = (x_1, \ldots, x_{m-k-2}), a > 0 \}. $$

So if the manifold $M$ has such singularities we suppose that we can extract certain $k$-dimensional sub-manifolds, namely $(m - 1)$-dimensional boundary $\partial M$, and $k$-dimensional wedges $M_k, k = 0, \ldots, m - 2$; $M_0$ is a collection of conical points.

5.2. Enveloping operators. If the family $\{ A_x \}_{x \in M}$ is continuous in operator topology, then according to Simonenko’s theory there is an enveloping operator, i.e. such an operator $A$ for which every operator $A_x$ is the local representative for the operator $A$ in the point $x \in M$.

**Example 5.** If $\{ A_x \}_{x \in M}$ consists of Calderon–Zygmund operators in $\mathbb{R}^m \mathbb{I}$ with symbols $\sigma_x(\xi)$ parametrized by points $x \in M$ and this family smoothly depends on $x \in M$ then Calderon–Zygmund operator with variable kernel and symbol $\sigma(x, \xi)$ will be an enveloping operator for this family.

**Example 6.** If $\{ A_x \}_{x \in M}$ consists of null operators then an enveloping operator is a compact operator $\mathbb{E}$.

**Theorem 1.** Operator $A$ has a Fredholm property iff its all local representatives $\{ A_x \}_{x \in M}$ have the same property.

This property was proved in $\mathbb{F}$. But we will give the proof (see Lemma 2) including some new constructions because it will be used below for a decomposition of the operator.

5.3. Hierarchy of operators. We will remind here the following definition and Fredholm criteria for operators $\mathbb{G}$.

**Definition 8.** Let $B_1, B_2$ be Banach spaces, and $A : B_1 \to B_2$ be a linear bounded operator. The operator $R : B_2 \to B_1$ is called a regularizer for the operator $A$ if the following properties

$$RA = I_1 + T_1, \quad AR = I_2 + T_2$$

hold, where $I_k : B_k \to B_k$ is identity operator, $T_k : B_k \to B_k$ is a compact operator, $k = 1, 2$.

**Proposition 2.** The operator $A : B_1 \to B_2$ has a Fredholm property iff there exists a linear bounded regularizer $R : B_2 \to B_1$.

**Lemma 1.** Let $f$ be a smooth function on the manifold $M, U \subset M$ be an open set, and $\text{supp} \ f \subset U$. Then the operator $f \cdot A - A \cdot f$ is a compact operator.
Proof. Let \( g \) be a smooth function on \( M, \text{supp} \, g \subset V \subset M \), moreover \( \overline{U} \subset V, g(x) \equiv 1 \) for \( x \in \text{supp} \, f \). Then we have
\[
f \cdot A = f \cdot A \cdot g + f \cdot A \cdot (1 - g) = f \cdot A \cdot g + T_1,
\]
\[
A \cdot f = g \cdot A \cdot f + (1 - g) \cdot A \cdot f = g \cdot A \cdot f + T_2,
\]
where \( T_1, T_2 \) are compact operators. Let us denote \( g \cdot A \cdot g \equiv h \) and write
\[
f \cdot A \cdot g = f \cdot g \cdot A \cdot g = f \cdot h, \quad g \cdot A \cdot f = g \cdot A \cdot g \cdot f = h \cdot f,
\]
and we obtain the required property. \( \square \)

Definition 9. The operator \( A \) is called an elliptic operator if its operator symbol \( \{ A_x \}_{x \in M} \) consists of Fredholm operators.

Now we will show that each elliptic operator really has a Fredholm property. Our proof in general follows the book [3], but our constructions are more stratified and we need such constructions below.

Lemma 2. Let \( A \) be an elliptic operator. Then the operator \( A \) has a Fredholm property.

Proof. To obtain the proof we will construct the regularizer for the operator \( A \). For this purpose we choose two coverings like proposition 1 and write the operator \( A \) in the form
\[
A = \sum_{j=1}^{n} f_j \cdot A \cdot g_j + T,
\]
where \( T \) is a compact operator. Without loss of generality we can assume that there are \( n \) points \( x_k \in U_k \subset V_k, k = 1, 2, ..., n \). Moreover, we can construct such coverings by balls in the following way. Let \( \varepsilon > 0 \) be enough small number. First, for every point \( x \in M_0 \) we take two balls \( U_x, V_x \) with the center at \( x \) of radius \( \varepsilon \) and \( 2\varepsilon \) and construct two open coverings for \( M_0 \) namely \( \mathcal{U}_0 = \bigcup_{x \in M_0} U_x \) and \( \mathcal{V}_0 = \bigcup_{x \in M_0} V_x \).

Second, we consider the set \( L_1 = \overline{M} \setminus \mathcal{V}_0 \) and construct two coverings \( \mathcal{U}_1 = \bigcup_{x \in L_1 \cap M_1} U_x \) and \( \mathcal{V}_1 = \bigcup_{x \in L_1 \cap M_1} V_x \). Further, we introduce the set \( L_2 = \overline{M} \setminus (\mathcal{V}_0 \cup \mathcal{V}_1) \) and two coverings \( \mathcal{U}_2 = \bigcup_{x \in L_2 \cap M_2} U_x \) and \( \mathcal{V}_2 = \bigcup_{x \in L_2 \cap M_2} V_x \). Continuing these actions we will come to the set \( L_{m-1} = \overline{M} \setminus (\bigcup_{k=0}^{m-2} \mathcal{U}_k) \) which consists of smoothness points of \( \partial M \) and inner points of \( M \), construct two covering \( \mathcal{U}_{m-1} = \bigcup_{x \in L_{m-1} \cap \partial M} U_x \) and \( \mathcal{V}_{m-1} = \bigcup_{x \in L_{m-1} \cap \partial M} V_x \). Finally, the set \( L_m = \overline{M} \setminus (\bigcup_{k=0}^{m-1} \mathcal{U}_k) \) consists of inner points of the manifold \( M \) only. We finish this process by choosing the covering \( \mathcal{U}_m \) for the latter set \( L_m \). So, the covering \( \bigcup_{k=0}^{m} \mathcal{U}_k \) will be a covering for the whole manifold \( M \). According to compactness property we can take into account that this covering is finite, and centers of balls which cover \( M_k \) are placed at \( M_k \).
Now we will rewrite the formula (1) in the following way

\[ A = \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} f_{jk} \cdot A \cdot g_{jk} \right) + T, \]

where coverings and partitions of unity \( \{ f_{jk} \} \) and \( \{ g_{jk} \} \) are chosen as mentioned above. In other words the operator

\[ \sum_{j=1}^{n_k} f_{jk} \cdot A \cdot g_{jk} \]

is related to some neighborhood of the sub-manifold \( M_k \); this neighborhood is generated by covering of the sub-manifold \( M_k \) by balls with centers at points \( x_{jk} \in M_k \). Since \( A_{x_{jk}} \) is a local representative for the operator \( A \) at point \( x_{jk} \) we can rewrite the formula (2) as follows

\[ A = \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \right) + T. \]

Let us denote \( S_{\omega_j^{-1}} \hat{g}_j \equiv \tilde{g}_j, \hat{f}_j S_{\omega_j} \equiv \tilde{f}_j \). Further, we can assert that the operator

\[ R = \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} f_{jk} \right), \]

will be the regularizer for the operator \( A' \); here \( A_{x_{jk}}^{-1} \) is a regularizer for the operator \( A_{x_{jk}} \).

Indeed,

\[ RA = \left( \sum_{k=0}^{m} \left( \sum_{j=1}^{n_k} g_{jk} A_{x_{jk}}^{-1} f_{jk} \right) \right) \cdot A = \]

\[ \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} \cdot (A - A_{x_{jk}} + A_{x_{jk}}) \cdot f_{jk} + T_1 = \]

\[ \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} \cdot (A - A_{x_{jk}}) \cdot f_{jk} + \sum_{k=0}^{m} \sum_{j=1}^{n_k} f_{jk} + T_1 = I_1 + T_1 + \Theta_1, \]

\[ \Theta_1 = \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot A_{x_{jk}}^{-1} \cdot (A - A_{x_{jk}}) \cdot f_{jk}, \]

because \( f_{jk} \cdot A_{x_{jk}} = A_{x_{jk}} \cdot f_{jk} + \text{compact summand} \), and \( f_{jk} \cdot g_{jk} = f_{jk} \), and

\[ \sum_{k=0}^{m} \sum_{j=1}^{n_k} f_{jk} \equiv 1 \]
as the partition of unity. The same property

\[ AR = I_2 + T_2 + \Theta_2, \]
\[ \Theta_2 = \sum_{k=0}^{m} \sum_{j=1}^{n_k} g_{jk} \cdot (A - A_{x_{jk}}) \cdot A_{x_{jk}}^{-1} \cdot f_{jk}, \]

is verified analogously. \( \square \)

6. PIECE-WISE CONTINUOUS OPERATOR FAMILIES

Given operator \( A \) with the symbol \( \{A_x\}_{x \in \overline{M}} \) generates a few operators in dependence on a quantity of singular manifolds. We consider this situation in the following way. We will assume additionally some smoothness properties for the symbol \( \{A_x\}_{x \in \overline{M}} \).

**Theorem 2.** If the symbol \( \{A_x\}_{x \in \overline{M}} \) is a piece-wise continuous operator function then there are \( m + 1 \) operators \( A^{(k)}, k = 0, 1, \ldots, m \) such that the operator \( A \) and the operator

\[ A' = \sum_{k=0}^{m} A^{(k)} + T \]

have the same symbols, where the operator \( A^{(k)} \) is an enveloping operator for the family \( \{A_x\}_{x \in \overline{M}} \), \( T \) is a compact operator.

**Proof.** We will use the constructions from proof of Lemma 2, namely the formula (3). We will extract the operator

\[ \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} \]

which “serves” the sub-manifold \( M_k \) and consider it in details. This operator is related to neighborhoods \( \{U_{jk}\} \) and the partition of unity \( \{f_{jk}\} \). Really \( U_{jk} \) is the ball with the center at \( x_{jk} \in M_k \) of radius \( \varepsilon > 0 \), therefore \( f_{jk}, g_{jk}, n_k \) depend on \( \varepsilon \).

According to Simonenko’s ideas [3] we will construct the component \( A^{(k)} \) in the following way. Let \( \{\varepsilon_n\}_{n=1}^{\infty} \) be a sequence such that \( \varepsilon_n > 0, \forall n \in \mathbb{N}, \lim_{n \to \infty} \varepsilon_n = 0 \). Given \( \varepsilon_n \) we choose coverings \( \{U_{jk}\}_{j=1}^{n_k} \) and \( \{V_{jk}\}_{j=1}^{n_k} \) as above with partition of unity \( \{f_{jk}\} \) and corresponding functions \( \{g_{jk}\} \) such that

\[ ||| f_{jk} \cdot (A_x - A_{x_{jk}}) \cdot g_{jk} ||| < \varepsilon_n, \quad \forall x \in V_{jk}; \]

we remained that \( U_{jk}, V_{jk} \) are balls with centers at \( x_{jk} \in \overline{M_k} \) of radius \( \varepsilon \) and \( 2\varepsilon \). This requirement is possible according to continuity of family
\{A_x\} on the sub-manifold $M_k$. Now we will introduce such constructed operator

$$A_n = \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk}$$

and will show that the sequence \{\(A_n\)\} is Cauchy sequence with respect to norm \(||| \cdot |||\). We have

$$A_l = \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik},$$

where the operator \(A_l\) is constructed for given \(\varepsilon_l\) with corresponding coverings \(\{u_{ik}\}_{i=1}^{l_k}\) and \(\{v_{ik}\}_{j=1}^{l_k}\) with partition of unity \(\{F_{ik}\}\) and corresponding functions \(\{G_{ik}\}\) so that

$$|||F_{ik} \cdot (A_x - A_{y_{ik}}) \cdot G_{ik}||| < \varepsilon_l, \quad \forall x \in v_{ik};$$

here \(u_{ik}, v_{ik}\) are balls with centers at \(y_{ik} \in M_k\) of radius \(\tau\) and \(2\tau\).

We can write

$$A_n = \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} = \sum_{i=1}^{l_k} F_{ik} \cdot \sum_{j=1}^{n_k} f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} =$$

$$\sum_{i=1}^{l_k} \sum_{j=1}^{n_k} F_{ik} \cdot f_{jk} \cdot A_{x_{jk}} \cdot g_{jk} = \sum_{i=1}^{l_k} \sum_{j=1}^{n_k} F_{ik} \cdot f_{jk} \cdot A_{y_{ik}} \cdot g_{jk} \cdot G_k + T_1,$$

and the same we can write for \(A_l\)

$$A_l = \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} = \sum_{j=1}^{n_k} f_{jk} \cdot \sum_{i=1}^{l_k} F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} =$$

$$\sum_{j=1}^{n_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} = \sum_{j=1}^{n_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot A_{y_{ik}} \cdot G_{ik} \cdot g_{jk} + T_2.$$

Let us consider the difference

$$|||A_n - A_l||| = ||| \sum_{j=1}^{n_k} \sum_{i=1}^{l_k} f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk}|||.$$  \(5\)

Obviously, summands with non-vanishing supplements to the formula \(4\) are those for which \(U_{jk} \cap u_{ik} \neq \emptyset\). A number of such neighborhoods are finite always for arbitrary finite coverings, hence we obtain

$$|||A_n - A_l||| \leq \sum_{j=1}^{n_k} \sum_{i=1}^{l_k} |||f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk}||| \leq$$
\[
\sum_{x \in U_{jk} \cap u_{ik} \neq \emptyset} ||f_{jk} \cdot F_{ik} \cdot (A_{x_{jk}} - A_x) \cdot G_{ik} \cdot g_{jk}|| + \\
\sum_{x \in U_{jk} \cap u_{ik} \neq \emptyset} ||f_{jk} \cdot F_{ik} \cdot (A_{x} - A_{y_{ik}}) \cdot G_{ik} \cdot g_{jk}|| \leq 2K \max[\varepsilon_n, \varepsilon_l],
\]
where \(K\) is a universal constant.

Thus, we have proved that the sequence \(\{A_n\}\) is a Cauchy sequence hence there exists \(\lim_{n \to \infty} A_n = A^{(k)}\). \(\square\)

**Corollary 1.** The operator \(A\) has a Fredholm property iff all operators \(A^{(k)}, k = 0, 1, \cdots, m\) have the same property.

**Remark 2.** The constructed operator \(A'\) generally speaking does not coincide with the initial operator \(A\) because they act in different spaces. But for some cases they may be the same.

7. **Conclusion**

This paper is a general concept of my vision to the theory of pseudo-differential equations and boundary value problems on manifolds with a non-smooth boundary. The second part will be devoted to applying these abstract results to index theory for such operator families and then to concrete classes of pseudo-differential equations.

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