Causality, Crossing and Analyticity in Conformal Field Theories

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Abstract

Analyticity and crossing properties of four point function are investigated in conformal field theories in the frameworks of Wightman axioms. A Hermitian scalar conformal field, satisfying the Wightman axioms, is considered. The crucial role of microcausality in deriving analyticity domains is discussed and domains of analyticity are presented. A pair of permuted Wightman functions are envisaged. The crossing property is derived by appealing to the technique of analytic completion for the pair of permuted Wightman functions. The operator product expansion of a pair of scalar fields is studied and analyticity property of the matrix elements of composite fields, appearing in the operator product expansion, is investigated. An integral representation is presented for the commutator of composite fields where microcausality is a key ingredient. Three fundamental theorems of axiomatic local field theories; namely, PCT theorem, the theorem proving equivalence between PCT theorem and weak local commutativity and the edge-of-the-wedge theorem are invoked to derive a conformal bootstrap equation rigorously.

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1. Introduction

The purpose of this article is to investigate the intimate relationships between causality, analyticity as well as crossing properties of four point functions in conformal field theories. The seminal paper of Mack and Salam [1] led to vigorous research activities in conformal field theory (CFT). The rapid progress of research during that period, following work of Mack and Salam [1], has been chronicled in several books and review articles [2, 3, 4, 5, 6, 7, 8, 9].

It is well known that wide class of physical systems exhibit scale invariance at the critical points. The critical indices have universal character. Polyakov [10] had conjectured that such theories might be invariant under the conformal symmetry which encompasses the scale invariance. The conformal bootstrap program was initiated by Migdal [11], by Ferrara and his collaborators [12, 13], by Mack and Todorov [14], and by Polyakov [15]. The proposed conformal bootstrap program synthesises two important ingredients: (i) conformal invariance and (ii) operator product expansion. Moreover, crossing symmetry plays a crucial role in the derivation of the consistency conditions. We shall elaborate on some aspects of the latter and its relevance to our investigation in the sequel. The computation of the correlation functions in conformal field theories is of paramount interest. A lot of attention has been focused on the formal developments and to address important issues in conformal field theory itself. The bootstrap program is understood intuitively in the following sense [11, 12, 13, 14, 15]. The essential idea is that if we consider a 4-point correlation function with a given ordering of operators and another 4-point function where the operator orderings are permuted then the two correlation functions are related if crossing symmetry is invoked. It becomes transparent when conformal partial wave expansion is implemented in each channel. As a result, a set of consistency conditions emerge from the requirements of bootstrap hypothesis. It is accepted that the two correlation functions are analytic continuation of each other. The interests in conformal bootstrap have been revived with renewed vigor in the recent past [7, 8, 9]. Causality, crossing and analyticity are three pillars of the conformal bootstrap paradigm.

The bootstrap concept stems from the S-matrix approach [16, 17] to understand the hadronic interactions phenomenologically in an era prior to the discovery of QCD as the theory of strong interactions. It will be useful to consider an illustrative example. A vast number of meson and baryon resonances were produced in high energy accelerators. Let us consider meson-meson scatterings: (i) $a + b \rightarrow c + d$ and (ii) $a + c \rightarrow b + d$. The first process corresponds to a direct channel reaction and the latter is the crossed channel one. In the low energy region, in each channel, the two processes are dominated by exchanges of resonances. Notice, however, that crossing symmetry relates the two amplitudes. The allowed kinematical variables for each of the two reactions do not overlap. To state it more explicitly; for the direct channel process, $s \geq 4m^2$ and $t \leq 0$, whereas in case of crossed channel reaction, $s \leq 0$ and $t \geq 4m^2$. Here equal mass scattering is assumed, $m$ being the mass of the external
particle. In the first case, $s$ is c.m energy squared and $t$ is momentum transfer squared; however, for crossed channel reaction the roles are reversed. If crossing is assumed then a set of equations relating coupling constants and masses involved in two channels get related as consistent conditions. Thus the bootstrap hypothesis was tested phenomenologically. It was a challenge to construct a crossing symmetric amplitude and Veneziano [18] succeeded in presenting such an amplitude. The crossing symmetry was rigorously proved in the framework of axiomatic field theory by Bros, Epstein and Glaser[19]. Let us elaborate this aspect for future reference. The microcausality, as one of the axioms of LSZ [20] formulation, was the main ingredients to demonstrate that the absorptive parts of s-channel and u-channel amplitudes of the 4-point functions coincided in an unphysical domain of real kinematical variables. It was proved rigorously from the theory of several complex variables for analytic functions that these two absorptive amplitudes are analytic continuation of each other. The conformal bootstrap program was initiated to provide a field theoretic basis to the bootstrap paradigm of the phenomenological S-matrix philosophy. There have been stimuli for further research in conformal bootstrap proposal due to its applications to variety of physical problems. The research in conformal field theories has spread in diverse directions such as in a large class of gauge theories and supersymmetric theories. Moreover, studies of structure of two dimensional conformal field has flourished and has influenced research in several directions. It is recognized that conformal symmetry has played a key role in establishing the $\text{AdS}_5 \leftrightarrow \text{CFT}_4$ correspondence which has led to spectacular developments in our understanding of the relationships between string theory and quantum field theories [21]: a realization of the gauge-gravity duality.

A very important attribute of the conformal field theory approach is that one computes the correlations functions of the product of field operators rather than the S-matrix elements. It is worthwhile to discuss this point. In the axiomatic field theory approach [20, 22, 23, 24, 25, 26, 27, 28], it has been demonstrated that how the analyticity of the scattering amplitude is intertwined with microcausality. The analyticity properties of amplitude are rigorously proved from axioms of general field theories. Let us recall some of the important results which are derived from a set of axioms introduced by Lehmann, Symanzik and Zimmermann (LSZ)[20] who laid the foundations of general field theories. The S-matrix elements are computed from the reduction technique of LSZ.

Notice, however, that while computing the scattering amplitudes [20], the external particles are on the mass shell. The analyticity properties of the amplitude are reflected through the dispersion relations. The dispersion relations are proved from fundamental principles such as Lorentz invariance and microcausality. There are two additional important ingredients in the LSZ formulation besides the stipulated postulates: (i) The existence of (asymptotic) in and out fields so that a complete set of
operators can be constructed in terms of either in or out field. And (ii) it requires that there are massive particles in the theory. Therefore, the second assumption, intuitively implies the existence of short range forces in the theories under consideration. The analyticity and crossing symmetry of the S-matrix are proved in axiomatic frameworks for massive field theories. These results are derived in the linear program without invoking the the unitarity property of S-matrix which is a nonlinear constraint. One of the most important accomplishments of the axiomatic approach is to study analyticity property of the amplitude in order to determine the domain of holomorphy. Moreover, in order to identify the domain of analyticity of the amplitude in $t$, the momentum transferred squared, the Jost-Lehmann-Dyson [29, 30] representation played a very crucial role. This paved the way to prove the fixed-$t$ dispersion relations. However, Bogoliubov [28] has developed an alternative technique to go off mass shell for the external particles and he independently proved fixed-$t$ dispersion relations for scattering amplitudes. The analyticity properties of the S-matrix are utilized to prove rigorous theorems, usually expressed as upper and lower bounds, on experimentally measurable parameters. Therefore, if any of those bounds are violated in high energy experiments then some of the axioms will be questioned. There is no evidence for violation of any of the bounds so far.

There are formidable problems when the spectrum of a theory contains only massless particles. The pure Yang-Mills theory, Einstein’s theory of gravitation and some conformal field theories belong to this category. Our intuition guides us to the root of the problem. Notice that, in such theories, the presence of massless particles would lead to long range forces. Consequently, there are conceptual difficulties in defining the asymptotic in and out states. Therefore, the LSZ technique is not suitable, while dealing with some of the theories noted above. As we shall discuss later, conformal field theories do not admit a discrete mass parameter. Therefore, the LSZ formalism, as presented in its original formulation, is not quite suitable in the context conformal field theories. The Poincaré group is contained in the conformal group, the enlarged symmetry group. The theories are severely constrained when the conformal symmetry is enforced. One notable feature is that the states are not labelled by a $(mass)^2$, unlike the Poincare invariant conventional field theories. We recall that the generators of the spacetime translations, $P_\mu$, does not commute with the Casimir operators constructed from the generators of conformal group. Whereas, in the context of conventional QFT formulations, $P^2 = P_\mu P^\mu$ is a Casimir. The other Casimir operator is [23] $W_\mu W^\mu$ where $W_\mu = \epsilon_{\mu\nu\rho\tau} P^\nu M^{\rho\tau}$, the Pauli-Luansky vector. The eigenvalue of the second Casimir is associated with the helicity of the particle. Therefore, a state vector is represented by its mass and helicity in quantum field theories. We shall discuss in the sequel, some aspects of the unitary irreducible representations of the conformal field theories which are quite different from those of the Poincaré invariant quantum field theories.
It is evident from the preceding discussions that computation of the scattering amplitudes, as stipulated by LSZ for theories satisfying their axioms, cannot go through in a straightforward manner in case of CFT. Thus it is not possible to derive an expression for the scattering operator rigorously using the LSZ reduction technique. Recently, however, an interesting development has provided evaluation of form factor and scattering amplitude from the perspectives of LSZ technique [31] by incorporating a theorem from LSZ paper. They define form factors and scattering amplitudes in conformal field theories. These are the coefficient of singularity of the Fourier transform of time-ordered correlation functions in the limit $p_i^2 \to 0$ where $p_i$ stands for four momenta of external legs. The form factor, $F$, is extracted from the four point function. It is shown that $F$ is crossing symmetric, analytic and it admits a partial wave expansion. We note that the authors work in Lorentzian signature metric. They obtain momentum space representation for the four point function from the time ordered product of four field operators. Indeed, this investigation is an endeavor to address important issues such as analyticity and crossing symmetry from the perspective of LSZ, however, they have circumvented the notion of explicitly introducing 'in', 'out' fields as well as interacting fields in their formulation. Moreover, the idea of interpolation, how interacting field interpolates into 'in' and 'out' fields, is not utilized.

We subscribe to the philosophy that it is desirable to compute correlation functions in conformal field theories. As a consequence, we do not invoke the concept of scattering operator in our approach. We shall unravel the relationship between analyticity, causality and crossing as we proceed. In recent times, the Euclidean formulation of conformal field theory has been widely adopted. There are certain advantages in this approach while the position space description is utilized. The merits of Euclidean space formulation and its power has been discussed in [7, 8, 9]. On the other hand the Minkowski space (Lorentzian signature metric) formulation has its own merits as has been discussed in [32, 33]. We feel that the choice of the Lorentzian metric is more suitable for establishing relationship between causality and analyticity as is evident from the rigorous results derived in axiomatic field theories. As a motivation, we present an illustrative example which is a very simplified version of Toll’s work in the context of dispersion relation [34].

Let us consider a wave packet, $\psi(z, t)$, moving with velocity $c$ along the $z$-axis. The target is located at the point $z = 0$. The Fourier transform of $\psi(z, t)$ is $f(\omega)$ and we can write

$$\psi(z, t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} d\omega f(\omega) e^{i\omega(z - ct)}$$  \hspace{1cm} (1)

The spherically symmetric scattered wave has the form

$$S(r, t) = \frac{1}{r\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{S}(\omega) f(\omega) e^{i\omega(z/c - t)}$$  \hspace{1cm} (2)
Here $r$ is the radial distance from $z = 0$. We may invert eq. (1) to get

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \psi(0,t)e^{i\omega t}$$  \hspace{1cm} (3)$$

We recognize that if the wave packet does not reach the origin, $z = 0$, then there will be no observation of scattered wave; in other words if

$$\psi(0,t) = 0, \text{ for } t < 0$$  \hspace{1cm} (4)$$

then $f(\omega)$ is regular for $\text{Im} \omega > 0$; i.e. it is regular on the upper half of the complex $\omega$ plane.

For us, causality, in the present context, means no scattered wave would arrive at a distance $r$ from the origin after an interval $\frac{r}{c}$ when the incident wave has not reached $z = 0$.

$$S(r,t) = 0, \text{ for } (ct - r) < 0.$$  \hspace{1cm} (5)$$

Let us take inverse Fourier transform of (2). We conclude that $f(\omega) \tilde{S}(\omega)$ is analytic in the upper half $\omega$ plane; i.e. it is analytic when $\text{Im} \omega > 0$. Therefore, $\tilde{S}(\omega)$, the Fourier transform of the amplitude, is analytic for $\text{Im} \omega > 0$. This property is exploited to write a dispersion relation. In the context of special relativity velocity of light, $c$, is the limiting velocity and therefore, two points separated by spacelike distance are causally disconnected. In the context of quantum theory, if there are two Hermitian operators located at spacelike distance we can make observations simultaneously.

In relativistic quantum field theories the causality is stated to be $[O(x), O(x')] = 0$, if $(x - x')^2 < 0$ where $O(x)$ and $O(x')$ are two local operators. The analyticity properties of scattering amplitude crucially depend on the axiom of microcausality. The dispersion relations rest primarily on this axiom. In order to understand concepts such as causality and crossing intuitively it is desirable to work in a Minkowski spacetime manifold with a Lorentzian metric. Our adopted signature for the flat space metric is $g_{\mu\nu} = \text{diag} (+, -, -, -)$. Moreover, we shall attempt to reveal how these three fundamental properties are intertwined in the Wightman formulation of field theories in the context of conformal field theories. The necessary ingredients, to explore these aspects would be discussed at an appropriate juncture. The correlation functions are the vacuum expectation values of field or local operators i.e. the Wightman functions. However, it must be borne in mind that not all conformal fields fulfill the requirements of Wightman axioms. We shall discuss this aspect in the later part of this section. Another advantage of adopting the Wightman’s formulation is that the Wightman functions are boundary values of analytic functions. It facilitates to establish relationship between causality and analyticity and subsequently paves the way to an understanding of crossing once
we define analytic functions from the Wightman functions.

The operator product expansion (OPE) has played a very important role in the advancements of conformal field theories in recent years. Wilson [35] pioneered the technique of operator product expansion in QFT. His seminal works have profoundly influenced research in diverse branches of physics. Wilson’s theory was further advanced rigorously by Wilson and Zimmermann [36] in the frameworks of Wightman axioms [24]. It was shown that under certain conditions, the operator product expansion can be derived giving complete information on short distance behaviors. They provided procedures to construct composite field operators from the perspectives of Wightman axioms, supplemented by extra hypothesis, relevant for operator product expansions in QFT. It was argued, by Wilson and Zimmermann that, when the products of local operators are envisaged singularities invariably occur as separation between the two operators tends to vanish. These singularities appear as a consequence of relativistic invariance and positive definite metric in Hilbert space [37]. Subsequently, Otterson and Zimmermann [38], concentrated on Wilson expansion for product of two scalar field operators: \( A(x_1)A(x_2) \). They demonstrated, following the approach of [36], that this operator product could be used to define composite local operators i.e. local with respect to \((x_1 + x_2)/2\). The OPE also depends, in addition, on a vector, \( \zeta^\mu \), which is proportional to distance \((x_1 - x_2)^\mu \) between \( A(x_1) \) and \( A(x_2) \). They investigated locality and analyticity properties of appropriate matrix elements from the Wightman axiom point of view. Their conclusions are based on rigorous techniques and are quite robust.

We focus our attentions to study the consequences of locality and analyticity in conformal field theories and it will be carried out in the light of the works of [36] and [38]. Moreover, the procedures of [38] is appropriately adopted for the conformal field theories. In the context of CFT, the OPE imposes severe constraints on the structural frameworks of the theories. Furthermore, it enables us to extract important conclusions without resorting to any specific model i.e. without introducing a Lagrangian density or an action. We are aware that the issue of convergence of OPE in conformal field theory is quite pertinent. The derivation of the conformal bootstrap equation is intimately connected with the convergence of conformal partial wave expansion. It is worth while to note that, in the context of conformal field theories, supplementary postulates are needed in addition to the Wightman axioms. Microcausality is a cardinal principle of axiomatic local field theories. Therefore, for a theory with three important ingredients such as (i) an enlarged symmetry like the conformal invariance, (ii) microcausality and (iii) the power of operator product expansion; some of the important and most general attributes are extracted. For example, the two point function and the three point function of a CFT are determined up to multiplicative constant factors. Furthermore, very important characteristics and structure of four point functions are understood from these ingredients.
The crossing symmetry, in the context of conformal field theories in conjunction with the power of operator product expansion, is generally discussed in the coordinate space description. Let us consider the three point function for a scalar field where the field operators are located at spacetime points \(x_1, x_2\) and \(x_3\) and is defined as follows. \(W_3(x_1, x_2, x_3) = \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) | 0 \rangle\). If we consider a permuted field configuration then \(W_3(x_1, x_3, x_2) = \langle 0 | \phi(x_1) \phi(x_3) \phi(x_2) | 0 \rangle\); the two correlation functions are equal i.e. \(W_3(x_1, x_2, x_3) = W_3(x_1, x_3, x_2)\) when \((x_2 - x_3)^2 < 0\); that is when \(x^\mu_2\) and \(x^\mu_3\) separation is spacelike. The essential point to note is that it is necessary to prove that \(W_3(x_1, x_2, x_3)\) and \(W_3(x_1, x_3, x_2)\) are analytic continuation of each other. Similar arguments can be invoked for the four point functions as well as for \(n\)-point functions while discussing crossing.

We hold the view that the study of the analyticity properties of the correlation functions in CFT is best accomplished in the Wightman’s formulation. The Wightman functions are considered to be boundary values of analytic functions of several complex variables. Let us consider an \(n\)-point Wightman function \(W_n(x_1, x_2, ..., x_n) = \langle 0 | \phi(x_1) \phi(x_2) ... \phi(x_n) | 0 \rangle\). It is the boundary value of an analytic function. If we permute location of field operators then another corresponding Wightman function would be defined which would be boundary value of another analytic function. Our task, in the context of CFT, is to investigate analyticity properties of four point Wightman functions. Consider the four point function \(W_4 = \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle\). It is crossing symmetric if we compute \(W_4\) using standard OPE technique as long as all four points are separated by spacelike distances, modulo the issues related to the convergence of the operator product expansions. Indeed, the consistency requirements for the four point amplitude, computed by contraction of pair of field (for different configurations), leads to the so called conformal bootstrap program (see section 3 for the work in this direction). We would like to address the following question: how far we can proceed within an axiomatic framework to infer about crossing and analyticity properties of the correlation functions of conformal field theories?

Our goal is modest. We intend to study only a class of four dimensional conformal field theories which respect Wightman axioms. The generators of the conformal group are: (i) the ten generators of the Poincaré group i.e. spacetime translations and Lorentz transformations respectively denoted by \(P_\mu\) and \(M_{\mu\nu}\). (ii) The five additional generators are: the dilatation operator, \(D\), (associated with the scale transformation) and the four generators corresponding to special conformal transformations are denoted by \(K_\mu\). Thus there are altogether fifteen generators in \(D = 4\). The scalar field, \(\phi(x)\), transforms as

\[
[P_\mu, \phi(x)] = i\partial_\mu \phi(x), \quad [M_{\mu\nu}, \phi(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \phi(x) \quad (6)
\]

\[
[D, \phi(x)] = (d + x^\nu \partial_\nu) \phi(x), \quad [K_\mu, \phi(x)] = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - x_\mu d) \phi(x) \quad (7)
\]
where $d$ is the scale dimension of the field $\phi(x)$. The above transformation rules are modified appropriately when we consider tensor fields. The fifteen generators of the conformal group satisfy the algebra of $SO(4,2)$. There are three Casimir operators which are constructed from these generators. We give below the expressions for the Casimir operators for the sake of completeness.

$$C_2 = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} - K_\mu P^\mu - 4iD - D^2$$

$$C_3 = -\frac{1}{4} \left( W_\mu K^\mu + K_\mu W^\mu \right) - \frac{1}{8} \epsilon_{\mu\nu\rho\tau} M^{\mu\nu} M^{\rho\tau}$$

$$C_4 = \frac{1}{4} \left[ K_\mu K^\mu P_\nu P^\nu - 4K_\mu M^{\mu\nu} M_{\nu\rho} P^\rho - 4K_\mu M^{\mu\nu} P_\nu (D + 6i) 
+ \frac{3}{4} (M_{\mu\nu} M^{\mu\nu})^2 + \frac{1}{16} (\epsilon_{\mu\nu\rho\tau} M^{\mu\nu} M^{\rho\tau})^2 
+ M_{\mu\nu} M^{\mu\nu} (D^2 + 8iD - C_2 - 22) - D^4 - 16iD^3 
+ 80D^2 + 128iD + 36C_2 - 16iC_2 D - 2C_2 D^2 \right]$$

where $W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\tau} P^\nu M^{\rho\tau}$. The expressions for the Casimir operators for four dimensional Minkowski space, as given above, are according to the notations and conventions of Fradkin and Plachek [5, 39].

Let us discuss Wightman approach in the context of conformal field theories. The analyticity properties structure of conformal field theories was systematically investigated by Lüscher and Mack [40] extensively. We shall discuss some of the salient aspects in the next section. Let $\hat{\phi}(p)$ denote the Fourier transform of $\phi(x)$. Notice that the Fourier transform, $\hat{\phi}(p)$, of every conformal field does not satisfy spectrality condition of Wightman axioms [22]; i.e. $p^2 \geq 0$ and $p^0 \geq 0$. A special type of conformal fields fulfill the spectrality condition. These are designated as nonderivative fields [41]. They satisfy the constraint $[\phi(0), K_\mu] = 0$. Note that $x$-dependence of the field can be accomplished through the transformation: $\phi(x) = e^{-iP\cdot x} \phi(0) e^{iP\cdot x}$. We shall consider the nonderivative conformal fields, denoted by $\phi(x)$, throughout this investigation. A state belonging to a unitary irreducible representation is specified by appropriate quantum numbers of the covering group of $SO(2,4)$. The classification of the representations have been studied several decades ago [42, 43, 44, 45]. A desirable and useful line to pursue is to envisage the universal covering group of $SO(4,2)$ to be $SU(2,2)$. We mention below the following postulates of the Wightman formulation which are quite important in the context of conformal field theories. A summary of Wightman axioms and its implications will be presented in the next section. The two important ingredients are:

(i) The vacuum: The vacuum is denoted as $|0\rangle$. The vacuum is conformally invariant
and therefore, the generators of the conformal group annihilate the vacuum.

(ii) The Hilbert Space: A radically new approach is invoked in the study of conformal field theories (CFT). The structure of the Hilbert space, in the case of conformal field theories, is different which would be alluded to later [5].

We discuss briefly the correspondence between local operators and states. This concept plays a crucial role in understanding of the structure of the Hilbert space in conformal field theories. The state may be constructed either in the momentum space representation or in the coordinate representation as well. Consider a local operator, \( \mathcal{O}(x) \), acting on the vacuum. The operator ↔ state correspondence is

\[
\mathcal{O}(x)|0> = |\mathcal{O}(x)>
\]

(11)

In fact we could consider the operator at \( x = 0 \), i.e. \( \mathcal{O}(0) \) and generate \( x \)-dependence by a translation operation. The momentum space state vector is

\[
|\mathcal{O}(p)> = \int d^4x e^{ip.x} |\mathcal{O}(x)>
\]

(12)

Therefore, the vacuum expectation values of product of operators can be evaluated either for those in \( p \)-space representations or those in \( x \)-space representations. In the framework of conformal field theories, we have a set of fields. One does not envisage a model or a Lagrangian density with a single field or a set of fields while investigating general structure of CFT. Thus the Hilbert space is constructed from such a set of fields:

\[
\{\Phi_m\} : \Phi_0(x), \Phi_1(x), .......
\]

(13)

Each field, \( \Phi_m(x) \), carries a scale dimension, \( d_m \), and might be endowed with its own tensor structure; moreover, it might be characterized with internal quantum numbers. The product of two conformal fields is expanded in terms of complete set of local fields with C-number coefficients. Therefore, we need an infinite set of local fields which belong to unitary irreducible representation of the conformal group. Generically we express the product of a pair as

\[
\Phi(x_1)\Phi(x_2) = \sum_{m=0}^{\infty} A_m \int C_m(x : x_1, x_2) \Phi_m(x)d^4x
\]

(14)

where \( \{\Phi_m(x)\} \) are the set of fields belonging to irreducible representation of the conformal group and \( C_m(x; x_1, x_2) \), the C-number functions, which have singular behavior in the short distance limit. \( A_m \) are a set of constants, may be interpreted as coupling constants and these are determined from the dynamical inputs of the specific theory under considerations. We may associate a state vector with each of the fields in \( \{\Phi_m(x)\} \) from the hypothesis of state ↔ operator correspondence alluded to
already. A state vector is created when a field operator acts on the conformal vacuum as noted earlier. Thus a state in the Hilbert space is defined
\[ |\Phi_m > = \Phi_m |0> \] (15)
The full Hilbert space is decomposed into direct sum of mutually orthogonal spaces since two normalized states \(|\Phi_m >\) and \(|\Phi_n >\) with respective scale dimensions \(d_m\) and \(d_n\) are orthogonal i.e. \(<\Phi_m |\Phi_n > = \delta_{m,n}\). In fact each of the states belonging to unitary irreducible representations of the conformal group constitute subvector spaces. Therefore, the full Hilbert space, \(\mathcal{H}\), decomposes into direct sums as
\[ \mathcal{H} = \oplus \mathcal{H}^\chi \] (16)
\(\mathcal{H}^\chi\) is the subspace where the complete set of state vectors are created by complete set of fields \(\Phi^\chi\) which belong to an irreducible representation of the conformal group. Here \(\chi\) stands, collectively, for all the quantum numbers that characterize the irreducible representation. A noteworthy feature is the closure of the algebra in the sense that the product of two fields can expanded in terms fields of conformal field theory: symbolically; \(\Phi_m \Phi_n \sim \sum_l A^l_{mn} \Phi_l\); where \(A^l_{mn}\) are set of C-number coefficients.

Let us consider the three point Wightman function for scalar fields with scale dimension \(d\). Our interest lies in the study of analyticity properties. Therefore, we adopt the \(i\epsilon\) prescription for all the Wightman functions in what follows: (see more details in subsequent next section when we complexify the coordinates). After suitably implementing the conformal transformations on each of the scalar fields, the three point Wightman function assumes the following form.
\[ W_{123}(x_1, x_2, x_3) = <0|\phi(x_1)\phi(x_2)\phi(x_3)|0> \]
\[ = g_3(x_{12}^2 - i\epsilon x_{12}^0)^{-d/2}(x_{23}^2 - i\epsilon x_{23}^0)^{-d/2}(x_{13}^2 - i\epsilon x_{13}^0)^{-d/2} \] (17)
The above expression is presented for the scalar field, \(\phi(x)\); however, the expression for three point function of three arbitrary field is already known in the literature. Here \(g_3\) is interpreted a the coupling constant.
When fields are separated by spacelike distances i.e. \((x_1 - x_2)^2 < 0\), \((x_2 - x_3)^2 < 0\) and \((x_1 - x_3)^2 < 0\) then the Wightman functions are related to each other: 
\[
W_{123}(x_1, x_2, x_3) = W_{132}(x_1, x_3, x_2) = W_{312}(x_3, x_1, x_2) \text{ as noted earlier; expressed explicitly}
\]
\[
< 0|\phi(x_1)\phi(x_2)\phi(x_3)|0 > = < 0|\phi(x_1)\phi(x_3)\phi(x_2)|0 > = < 0|\phi(x_3)\phi(x_1)\phi(x_2)|0 > \tag{18}
\]
This is a consequence of the axiom of microcausality. Indeed, the equation relating different 3-point function (18) for the permutations of the field operators is the crossing relation for vertex functions. In order to recognize the importance of the study of analyticity, let us examine the expressions for a pair of three point functions. In the case of the Wightman functions these are boundary values of analytic functions (see later). Consider the expressions for \(W_{123}\) and \(W_{132}\)
\[
W_{123} = g(x_{12}^2 - i\epsilon x_{12}^0)^{-d/2}(x_{23}^2 - i\epsilon x_{23}^0)^{-d/2}(x_{13}^2 - i\epsilon x_{13}^0)^{-d/2} \tag{19}
\]
and
\[
W_{132} = g(x_{13}^2 - i\epsilon x_{13}^0)^{-d/2}(x_{12}^2 - i\epsilon x_{12}^0)^{-d/2}(x_{32}^2 - i\epsilon x_{32}^0)^{-d/2} \tag{20}
\]
We note that \(W_{123}\) and \(W_{132}\) do not necessarily coincide when \(x_{12}\), \(x_{23}\) and \(x_{13}\) are timelike. Let us compare each term of (19) and (20) on the r.h.s and especially focus attention on the second term of (19) and the third term of (20). The former is \((x_{23}^2 - i\epsilon(x_2 - x_3)^0)^{-d/2}\) whereas the latter is \((x_{32}^2 + i\epsilon(x_{23})^0)^{-d/2}\). Therefore, when we approach the real axis in the complex plane we must approach it from opposite directions. Thus in order to provide a proof of crossing we have to appeal to the the method of analytic completion. In recent years, the study of the analyticity properties in CFT have attracted attentions [46, 47, 48, 49, 50, 32, 33].

It is more interesting to consider the a pair of 4-point Wightman function, with the \(i\epsilon\) prescription, which have the following form in the convention of [9]
\[
W_4(x_1, x_2, x_3, x_4) = < 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 > \nonumber
\]
\[
= \frac{1}{(z_{12}^2 - i\epsilon z_{12}^0)(z_{34}^2 - i\epsilon z_{34}^0)}[d\mathcal{F}(Z_1, Z_2)] \tag{21}
\]
for the case when we have only one scalar field, \(\phi(x)\), with scale dimension \(d\); and \(\tilde{z}_{12} = x_1 - x_2, \tilde{z}_{34} = x_3 - x_4, \tilde{z}_{13} = x_1 - x_3, \tilde{z}_{24} = x_2 - x_4, \tilde{z}_{14} = x_1 - x_4, \tilde{z}_{23} = x_2 - x_3\). Here \(Z_1\) and \(Z_2\) are the cross ratios
\[
Z_1 = \frac{(z_{12}^2 - i\epsilon z_{12}^0)(z_{34}^2 - i\epsilon z_{34}^0)}{(z_{13}^2 - i\epsilon z_{13}^0)(z_{24}^2 - i\epsilon z_{24}^0)} \tag{22}
\]
and
\[
Z_2 = \frac{(z_{14}^2 - i\epsilon z_{14}^0)(z_{23}^2 - i\epsilon z_{23}^0)}{(z_{13}^2 - i\epsilon z_{13}^0)(z_{24}^2 - i\epsilon z_{24}^0)} \tag{23}
\]
F is a function which depends on cross ratios and its form is determined by the
model under considerations. The discussions of crossing operation is to be treated
with care. For example, in order to establish crossing, it is desirable to show how
\( W_4(x_1, x_2, x_3, x_4) \) is analytically continued to \( W_4(x_1, x_2, x_4, x_3) \). They coincide for
real \( x_2 \) and \( x_3 \) when \( (x_3 - x_4)^2 < 0 \) i.e. when the two spacetime points are separated
by spacelike distance. The holomorphic properties of scattering amplitudes, in the
momentum space representation, has been thoroughly studied in the past. We shall
focus primarily on the momentum space representation of three point functions in
Section 2 and study their analyticity properties.

The article is organized as follows. In the next section, Section 2, we first present
a summary of the important results in the Wightman formulation of general field the-
ory. Next, in this section, we study the domain of holomorphy of four point functions
and a simple method is adopted to accomplish analytic completion. The domain of
analyticity of three point Wightman function was investigated in detail by Kallén and
Wightman [51]. The problem of analytic completion for four point function deserves
some discussions. There are several attempts to accomplish analytic completion for
four point function [52, 53, 54] and there has been some progress. We mention, in
passing, that these endeavors have not been able to accomplish as much as what have
been achieved exhaustively for the three point function by Kallén and Wightman
[51]. Our goal, in this article, is to address issues associated with crossing. Therefore,
following the line of arguments and technique of our previous work [55], in the con-
text of three point function, we investigate crossing for the four point function. We
employed the analytic completion technique for a pair of permuted Wightman four
point functions and showed that they are analytic continuation of each other. Thus
all Wightman functions are shown to be analytic continuation of each other pairwise.
However, the holomorphic envelope might be much larger. We shall next consider the
domain function for a nonderivative conformal field theory. It will be continuation
of our previous effort to study crossing symmetry. We investigated the analyticity
property and crossing for three point Wightman function in conformal field theories
[55]. We applied the technique to a pair of Wightman functions taken at a time. In
other words if we permute a pair of field to obtain a new three point function from a
given configuration it possible to implement analytic continuation in a simple manner.
Our work synthesized work of Dyson [56] and its modified version adopted by Streater
[57]. Dyson [56] had obtained a representation for double commutator of three scalar
fields: \( A(x), B(x) \) and \( C(x) \). This proposal was suitably modified by Streater [57] to
implement the technique of analytic completion for a pair of Wightman functions. We
adopted Streater’s prescription to prove crossing for a pair of three point functions
by in CFT applying method of analytic completion. We recall, if we permute a pair
of field to obtain a new three point function from a given configuration then, it was
shown that, the two vertex functions are analytic continuation of each other. One
remark is quite pertinent at this stage. The Wightman functions are boundary values
of analytic functions of several complex variables of complexified coordinates as will be defined in the next section. The important point is these analytic functions are defined over a domain known as extended tubes (see Section 2 for details). There is a Wightman function defined for a given ordering of fields. There a Wightman function for each permutation of the field ordering. Moreover, corresponding to each Wightman function there is an analytic function, defined over an extended tube and the Wightman function is its boundary value. The domain of holomorphy is the union of the extended tubes on which the permuted Wightman functions are defined. It is argued that the set of analytic functions so obtained are analytic continuation of each other. Notice that Wightman function, defined as vacuum expectation value of product of field operators is not related to another Wightman function obtained from permutation of the fields. Thus it might not be possible, in our approach, to identify the envelope of holomorphy when the union of domain of analyticities of all permuted Wightman functions taken. The reason is that the envelope of holomorphy might be larger than the union of the domains of analyticities. We shall elaborate on this aspect in Section 2. We carry out a detailed study, adopting an argument of Jost which provides a relationship between the retarded three point function and the three point Wightman function. This result is appropriately tailored for conformal field theory.

Section 3 deals with operator product expansion in conformal field theories adopting the formalism of Wilson and Zimmermann. The prescription laid down by Ottersen and Zimmerman [38] for operator product expansion of two field operator in the frameworks of Wilson is quite rigorous. We employ the procedure for operator product expansion of nonderivative conformal fields. The matrix elements of the composite field are investigated to examine consequences of microcausality. It is shown how a theorem, analogous to the Jost-Lehmann-Dyson theorem can be proved. Furthermore, the analyticity property of the matrix elements are derived to prove crossing for the four point function. In the later part of third section (in the subsection), we appeal to PCT theorem to derive conformal bootstrap equation in a novel way. The PCT operation, applied to a four point Wightman function, transforms into another Wightman function. These two four point functions are equal if PCT is a symmetry. The equivalence between PCT theorem and weak local commutativity (WLC) plays a very important role in the derivation of the equation. First, the conformal partial wave (CPW) expansion technique is employed to the each of the four point equations which are related by WLC. The two functions coincide at the Jost point for real values of spacetime coordinates when their separation is spacelike. As we shall explain in Section 3, the two Wightman functions are boundary values of analytic functions and they are holomorphic as complex valued functions. We argue that these two four point functions are analytic continuation of each other. Thus, we present a rigorous derivation of the conformal bootstrap equation. In a short communication [58], we have reported part of this result. The summary of our work and conclusions are
incorporated in Section 4.

2. The Wightman Functions in Conformal Field Theory.

We study the crossing and analyticity properties of the four point function of a conformal field, $\phi(x)$, in this section. The crossing symmetry relates two permuted Wightman functions. We have noted, in Section 1, the motivations for the study of crossing in CFT. It was pointed out that two Wightman functions, where a pair of fields get interchanged, coincide when the corresponding pair of spacetime coordinates are separated by spacelike distance. The conformal bootstrap equation is derived under such a condition. This is not the full story. The proof of crossing in QFT, from the axiomatic standpoint, demands more. It is necessary to prove that the pair of functions have analytic continuations to their domain of holomorphy although the pair coincide at certain spacetime point. Therefore, the domain of holomorphy of each of the functions have to be identified. This is the first task. The role of OPE in CFT has been emphasized already. In turn, as was noted in the previous section, the Hilbert space structure of CFT is different from that of the conventional (axiomatic) fields theories. We discuss this aspect very briefly in the sequel. We have emphasized the relationship between microcausality and analyticity. A simple way to bring out their intimate relationship is to consider crossing property of the three point Wightman function. This problem has been studied recently [55]. We briefly recall essence of this work, later in this section and incorporate our new results. A momentum space formulation is presented and analyticity properties are studied by appealing to works of Jost and Ruelle (see subsection 2.2 for details). It will set a background to study relation between microcausality and analyticity properties of the four point function. Moreover, we invoke the arguments of [55] to prove crossing for four point function.

2.1 Properties of Wightman Functions.

The Wightman functions [22] are vacuum expectation values of product of field operators. They are envisaged as boundary values of analytic function of several complex variables. Wightman [59] has argued that if all the vacuum expectation values $<0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0>$ for all the permutations of $\phi(x_1)\ldots\phi(x_n)$ are given then the operator $\phi(x)$ is determined. Our intent is to consider the Wightman functions for the scalar conformal field $\phi(x)$ and study their analyticity properties and crossing symmetry. The axioms are:

A1. Invariance of the theory under proper inhomogeneous Lorentz group.

A2. The existence of vacuum. There exists a Hilbert space, $\mathcal{H}$, spanned by the physical state vectors. The states have nonnegative energy spectrum i.e. $p_0 \geq 0$ and $p^2 \geq 0$. There exists a vacuum, $|0>$, the lowest energy state such that if $P_\mu$ is the energy momentum operator, $P_\mu|0>=0$. The vacuum is unique and stable. The vacuum is annihilated by all the generators of the conformal group.
There exist field operators which are tempered. In other words, vacuum expectation values of operators are tempered distributions in the Schwartzian sense.

Local commutativity. Expressed in another way; local operators commute (for bosons) or anticommute (for fermions) when they are separated by spacelike distance. For bosonic local operators,

\[ [\mathcal{O}(x), \mathcal{O}(x')] = 0, \text{ if } (x - x')^2 < 0 \]  

In the context of conformal field theories, these axioms are to be appropriately interpreted and additional hypothesis might be added. The Wightman function \( W_n(x_1, x_2, ... x_n) \) is

\[ W_n(x_1, x_2, ... x_n) = \langle 0 | \phi(x_1) \phi(x_2) ... \phi(x_n) | 0 \rangle \]  

We know that these vacuum expectation values are not ordinary functions but are distributions. They are to be interpreted as linear functionals as defined below

\[ W[f] = \int d^4 x_1 ... d^4 x_n W_n(x_1, x_2, ... x_n) f(x_1, x_2, ... x_n) \]  

Thus we assign a complex number with the introduction of \( f(x_1, x_2, ... x_n) \) in the above the functional. Moreover, \( f(x_1, x_2, ... x_n) \) are infinitely differentiable functions. They they vanish outside a bounded region of \( 4n \) dimensional space. It is worth while to note that \( W_n(f_1, f_2, ... f_n) \) (indeed they are generally denoted as functional \( W[f] \)) are well defined. The limit and continuity are defined for such objects. We work with Wightman functions, \( W_n(x_1, x_2, ... x_n) \), in this article. We emphasize the fact that these are distributions. Thus when we discuss convergence and boundedness etc. of Wightman functions, they are to be understood in the sense that these are distributions, as defined above, with well behaved weight functions. Note that fields are operator valued distributions. Consequently, the spacetime average of operators are interpreted as observables. Thus operators of the form

\[ \phi[f] = \int d^4 x \phi(x) f(x) \]  

are meaningful. It is assumed that \( \phi[f] \) is defined as the class of all infinitely differentiable functions of compact support in spacetime. In this light, note that \( W_{\phi}[f] = \langle 0 | \phi[f] | 0 \rangle \) is a linear functional with respect to \( f \). More refined statements can be made by introducing a sequence of test functions \( \{ f_n(x) \} \) and by specifying convergence properties [25]. In the optics of the preceding discussions the n-point function \( W_n(x_1, x_2, ... x_n) = \langle 0 | \phi(x_1) \phi(x_2) ... \phi(x_n) | 0 \rangle \) is a distribution in each of the variables \( x_i \)

\[ W_n[f_1, f_2, ..., f_n] = \langle 0 | \phi(f_1) \phi(f_2) .... \phi(f_n) | 0 \rangle \]  

We can give a precise interpretation to \( W_n(x_1, x_2, ... x_n) \) in terms of the infinitely differentiable test functions \( \{ f_n \} \). Now on, when we deal with Wightman functions
$W_n(x_1, x_2, \ldots x_n)$, it is understood that they have interpretations as alluded to above.

As a consequence of translational invariance of the theory; we conclude that $W_n(x_1, x_2, \ldots x_n)$ depends on the difference of coordinates

$$W_n(x_1, x_2, \ldots x_n) = W_n(y_1, y_2 \ldots y_{n-1})$$

where $y_i = x_i - x_{i+1}$. Moreover, $W_n(y_1, y_2 \ldots y_{n-1})$ are invariant under inhomogeneous Lorentz transformations; for a real orthochronous Lorentz transformation

$$W_n(y_1, y_2 \ldots y_{n-1}) = W_n(\Lambda_r y_1, \Lambda_r y_2 \ldots \Lambda_r y_{n-1})$$

where $\Lambda_r$ is a real Lorentz transformations, det $\Lambda_r = 1$.

**Local commutativity:** It follows from axiom (A4) that i.e. $[\phi(x), \phi(x')] = 0$ if $(x-x')^2 < 0$. Consequently,

$$W_n(x_1, x_2, \ldots x_j, x_{j+1}, \ldots x_n) = W_n(x_1, x_2, \ldots x_{j+1}, x_j, \ldots x_n), \text{ if } (x_j - x_{j+1})^2 < 0$$

Let $\tilde{W}(p_1, p_2, \ldots p_{n-1})$ denote the Fourier transform of $W_n(y_1, y_2 \ldots y_{n-1})$ then

$$W_n(y_1, y_2 \ldots y_{n-1}) = \int d^4p_1 d^4p_2 \ldots d^4p_{n-1} e^{-i\sum_{j=1}^{n-1} p_j y_j} \tilde{W}_n(p_1, p_2 \ldots p_{n-1})$$

From the temperedness property we know that $W_n$-functions have at most polynomial growth at infinity \(^2\). It is required that spectrum of the physical states must have timelike four momenta and positive energy. Therefore, from the stability of vacuum and support condition

$$\tilde{W}_n(p_1, p_2 \ldots p_{n-1}) = 0, \text{ unless } p_i^2 \geq 0, p_i^0 \geq 0, i = 1, 2 \ldots n - 1.$$  \hspace{1cm} (33)

The analyticity structure of $W_n(x_1, x_2, \ldots x_n)$, in the coordinate space is related to the support properties of the Fourier transformed Wightman functions, $\tilde{W}_n$, in the momentum space.

The function $\mathcal{W}_n(\xi_1, \xi_2 \ldots \xi_{n-1})$ of complex variables $\xi_j = y_j^\mu - i\eta_j^\mu, j = 1, 2, \ldots n-1$ is defined as analytic continuation of the vacuum expectation values $W_n(y_1, y_2 \ldots y_{n-1})$. The set of complex variables $\{\xi_j^\mu\}$ are defined as follows: the real pair $\{y_j^\mu, \eta_j^\mu\}$ are such that $\eta_j^\mu \in V^+$, i.e. $\eta_j^2 \geq 0$, $\eta_j^0 \geq 0$. Thus $\{\eta_j\}$ is in the forward lightcone; moreover, $-\infty < y_{ij}^\mu < +\infty$. This is the definition of forward tube $T_{n-1}$. The primitive domain is now identified. Wightman functions, the distributions, are boundary values of analytic functions i.e.

$$W_n(y_1, y_2 \ldots y_{n-1}) = \lim_{\{\eta_j \to 0\}} \mathcal{W}_n(\xi_1, \xi_2 \ldots \xi_{n-1})$$

\(^2\)see Froissart [87] for detailed arguments.
Note that \( W_n(\xi_1, \xi_2...\xi_{n-1}) \) are also invariant under real orthochronous Lorentz transformations; \( \det \Lambda_r = 1. \)

\[
W_n(\xi_1, \xi_2...\xi_{n-1}) = W_n(\Lambda_r \xi_1, \Lambda_r \xi_2...\Lambda_r \xi_{n-1})
\]  

(35)

\( \Lambda_r \) is real proper Lorentz transformation. Moreover, according to Hall and Wightman \([60]\), if \( W_n(\xi_1, \xi_2...\xi_{n-1}) \) is analytic in the tube, \( T_{n-1} \), and is invariant under real orthochronous Lorentz transformations then \( W_n(\xi_1, \xi_2...\xi_{n-1}) \) is invariant under complex Lorentz transformations where \( (\xi_1, \xi_2...\xi_{n-1}) \rightarrow (\Lambda \xi_1, \Lambda \xi_2...\Lambda \xi_{n-1}) \); note that \( (\xi_1, \xi_2...\xi_{n-1}) \in T_{n-1} \). Here \( \Lambda \in SL_+(2\mathbb{C}) \), \( \det \Lambda = 1 \). The set of points \( (\Lambda \xi_1, \Lambda \xi_2...\Lambda \xi_{n-1}) \), for arbitrary \( \Lambda \in SL_+(2\mathbb{C}) \), define the extended tube \( T'_{n-1} \). The enlargement of domain of holomorphy is achieved through this procedure. Important point to note is that \( T_{n-1} \) does not contain the real points of \( \{\xi_j\} \). The extended tube contains real points \( \{y_i\} \). The axioms: Lorentz invariance, uniqueness of vacuum, stability of vacuum and local commutativity lead to the following assertion. The function \( W_n(\xi_1, \xi_2...\xi_{n-1}) \) exists and is analytic in the domain specified above. It is also single valued. It is analytic in the domain which is union of the permuted extended tubes.

The Wightman functions, \( W_n(\{y_i\}) \) are analytic functions of real variables \( \{y_i\} \) when all of them are spacelike \([60, 61]\). The result of Hall and Wightman are very important for our purpose. We have noted earlier that when we define \( W_n(\{\xi_i\}) \) in the extended tube they are analytic functions of complex four vectors \( \{\xi_i^\mu\} \). Hall and Wightman proved that if \( W_n(\xi_1, \xi_2...\xi_{n-1}) \) is analytic in the four vector variables \( \{\xi_i^\mu\} \) which is invariant under the Lorentz transformations then it is an analytic function of scalar products of those complex four vectors. The statement, intuitively, seems to be reasonable; however, the proof is quite formidable when mathematical rigor is enforced. There are two significant implications of this theorem in the context of our work. First, the number of variables that the analytic function depends on is reduced considerably. For example, the three point function, \( W_3(x_1, x_2, x_3) \) depends on two four vectors, \( y_1^\mu = (x_1 - x_2)^\mu \) and \( y_2^\mu = (x_2 - x_3)^\mu \), from translation invariance of the theory. Thus it is a function of eight complex four vectors. In the light of Hall-Wightman theorem, now we know that \( W_3(\xi_1, \xi_2) \) depends on the invariants \( z_{ij} = \xi_i . \xi_j, i, j = 1, 2 \). In order to bring out the essence of the second implication let us recall the following facts. Notice that \( z_{ij} \) is a complex symmetric matrix \( (\xi_k = y_k - i \eta_k, k = 1, 2) \) in the forward light cone and \( \eta_j \) is in the interior of the lightcone. This set \( \{z_{ij}\} \) is a domain of analyticity of an invariant function. Moreover, the Wightman function is the boundary value when \( \eta_j \rightarrow 0 \). It has been proved by Hall and Wightman \([60]\) that there are set of points \( \{\xi_i\} \) on the boundary of the tube with following properties. These vectors can be used to construct matrices \( \xi_i . \xi_j \) which lie in the interior. Moreover, they have argued that an invariant analytic function in the tube will not admit an arbitrary invariant distribution as boundary value. It has also been proved that the boundary value is an analytic function of real
variables $\xi_i, \xi_j, i, j = 1, 2, ... n - 1$, where $\eta_j = 0$, in a certain domain. The analytic function is uniquely determined once its values are known in some subdomain of the boundary of the tube. These remarks might look as if they are out of context to be interjected at this juncture. The importance is intimately related to the Wightman theorem which we shall encounter in the discussion of crossing.

The next question to ask is where do the real points of $T_{n-1}'$ reside and what are their attributes? Jost proved the following: [62] The real points $\{y_1^\mu, y_2^\mu, ... y_{n-1}^\mu\}$ lie in the extended tube, if and only if, the convex hull of $\{y_1^\mu, y_2^\mu, ... y_{n-1}^\mu\}$ only contains spacelike points. To elaborate a little bit, the convex hull of points $\{y_1^\mu, y_2^\mu, ... y_{n-1}^\mu\}$ is the set of all four vectors of the form $\lambda_1 y_1^\mu + \lambda_2 y_2^\mu + ... \lambda_{n-1} y_{n-1}^\mu$ where the set $\{\lambda_i\}$ take positive real values, $\lambda_i \geq 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. Therefore, the real points of the extended tube are the ones for which if we take an arbitrary convex linear combination $\left(\sum_{i=1}^{n-1} \lambda_i y_i^\mu, \lambda_i \geq 0, \sum_{i=1}^{n-1} \lambda_i = 1\right)$, are always spacelike i.e. $(\sum_{i=1}^{n-1} \lambda_i y_i^\mu)^2 < 0, \lambda_i \geq 0, \sum_{i=1}^{n-1} \lambda_i = 1)$. This is the Jost point. The following remarks are pertinent to appreciate the importance of the theorem of Jost. We recognize that the determination of Jost point implies the existence of a domain consisting of points $y_1^\mu, y_2^\mu, ... y_{n-1}^\mu$ where $\{y_{\mu i}\}$ are real and spacelike. These points lie in the interior of the extended tube $T_{n-1}'$. However, they reside on the boundary of the forward tube, $T_{n-1}$. The Wightman function $W_n(y_1, y_2, ... y_{n-1})$ is an analytic function of the set of variables $\{y_{\mu i}\}$ at the Jost points. We can expand $W_n$ in a convergent power series in these variables. We argue that if we know the Wightman function in the neighborhood of $y_1^\mu, y_2^\mu, ... y_{n-1}^\mu$, the Jost points, then the Wightman function is uniquely determined for the set of variables $y_1^\mu, y_2^\mu, ... y_{n-1}^\mu$ of its argument. We shall utilize this fact in the sequel. Moreover, their importance is realized in the study of crossing and in the context PCT theorem. This is a very powerful result. We have been discussing crossing symmetry. We have alluded to the fact that the permuted Wightman functions coincide at spacelike point. This theorem has far reaching consequences as would be evident in the later part of this section as well as in the next section.

The importance of Wightman formulation in conformal field theories has been recognized long ago cite [2, 15, 40, 41, 45, 5]. Fradkin and Palchik [5] have presented a very comprehensive exposition in their book. Subsequently, Mack [41] has rigorously investigated the convergence of operator product expansion in conformal field theories utilizing the Wightman formulation. We have noted, in the last section, that not all conformal field theories satisfy the spectrality condition of Wightman axioms. All the representations of $SU(2, 2)$, the covering group of the conformal group, have been classified quite sometime ago [42, 43, 45, 5]. According to the classifications, a conformal field belongs to unitary irreducible representation of the covering group. The spectrality properties have been also investigated. Of importance, from our perspective, is a class of conformal field theories known as nonderivative conformal field theories which satisfy the Wightman axioms. They belong to the discrete representation of
the covering group. Let us consider the following situation: \( \{ \phi^i \} \) be a set of conformal fields of dimensions \( d_i \) and transform as finite dimensional representations of the Lorentz group, \( \mathcal{L} = SL(2\mathbb{C}) \). Let \( U \approx SU(2) \) the rotation subgroup of \( \mathcal{L} \); \( \bar{U} \) stands for set all finite dimensional irreducible representations, denoted by \( l \). Therefore, a representation, characterizing the field, would be collectively denoted by \( \chi = (l, d) \). As an illustration, consider the unitary irreducible representations of \( SU(2, 2) \), labeled by \( l \), which is a finite dimensional representation of the Lorentz group \( SL(2\mathbb{C}) \). Suppose the highest weight representation of \( l \) is \( (2j_1, 2j_2) \) (note that \( 2j_1 \) and \( 2j_2 \) are nonnegative integers). Now, \( \chi = (l, d) \) and \( d_{\min} = 2j_1 + 2j_2 + 2 \) for nonzero \( j_1 \) and \( j_2 \). In addition, the field might be endowed with internal quantum numbers. However, all along, we consider only a single Hermitian nonderivative conformal scalar field, \( \phi \). The Wightman axioms are respected by \( \phi(x) \). We had discussed the structure of the Hilbert space and we recall that it decomposes into several disjoint subspaces.

\[ \mathcal{H} = \sum_{\chi} \bigoplus \mathcal{H}^\chi \]  

(36)

Here \( \mathcal{H}^\chi \) stands for subspaces where each vector is characterized by \( \chi \) which are the quantum number belonging to the irreducible representations of the covering group alluded to earlier. It follows from the algebra in OPE that even if we start with a single field, \( \phi(x) \), the operator product expansion of a pair of this field needs infinite set of composite operator of nonderivative type[41]. Therefore, we need the states associated with them when we construct the full Hilbert space, \( \mathcal{H} \), which has subspaces \( \mathcal{H}^\chi \).

Now let us consider two four point Wightman functions for the conformal field theory: \( W_4(x_1, x_2, x_3, x_4) = \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle \) and \( W_4(x_1, x_2, x_3, x_4) = \langle 0 | \phi(x_1) \phi(x_2) \phi(x_4) \phi(x_3) | 0 \rangle \). Where would they coincide? We associate a tube (see the precise definition given earlier in this section) with respect to the first \( W_4(x_1, x_2, x_3, x_4) \) and another tube with the second \( W_4(x_1, x_2, x_3, x_4) \). They are continued to one another as regular functions in the union of two extended tubes associated with each of the functions. Note that the Jost point has a real neighborhood in the extended tube. Consider, for the case at hand, \( f_1(\xi, \xi_2, \xi_3) \) and \( f_2(\xi, \xi_2', \xi_3') \) which are analytic in their corresponding extended tubes, \( T^\prime \). To remind the reader, from translational invariance argument each 4-point Wightman function depends only on three coordinates, say \( y_1, y_2, y_3 \) and \( y_1', y_2', y_3' \) and we can define corresponding extended tubes. Let these two functions coincide for a real neighborhood in the extended tube

\[ f_1(y_1'', y_2'', y_3'') = f_2(y_1'', y_2'', y_3'') \]  

(37)

for \( y_1'', y_2'', y_3'' \) in a real neighborhood of a Jost point. The essential conclusion of the Jost theorem is

\[ f_1(\xi, \xi_2, \xi_3) = f_2(\xi, \xi_2, \xi_3) \]  

(38)

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This is the edge-of-the-wedge theorem [63, 64, 65] and the implications of the theorem in the context of conformal bootstrap equation will be discussed in Section 3. This theorem was first proved, in the context of dispersion relations from LSZ axiomatic field theory [62]. We recapitulate a few points for the sake of motivations and historical considerations. If we consider a four point scattering amplitude, it has a right hand cut and a left hand cut. The right hand cut originates from the direct, $s$-channel, reaction; whereas, the left hand cut arises from the crossed channel, the $u$-channel, process. The discontinuities across the cuts are related to the absorptive parts of the respective amplitudes. The edge-of-the-wedge theorem proves that the two absorptive amplitudes are analytic continuation of each other [63, 64]. Thus crossing was proved from analyticity property of the amplitude and a dispersion relation could be written down. We recall that the theorem was proved for S-matrix elements where external particles are on the mass shell and furthermore, equations of motion were utilized in obtaining matrix elements of source current commutators. The same arguments may be extended to the cases of retarded and advanced commutators of currents. In the context of conformal field theory, we deal with Wightman functions. Therefore, a different route has to be chosen when we intend to prove analyticity and crossing in the present context. The edge-of-the-wedge theorem has been proved for Wightman functions by Epstein [65]. Subsequently, the holomorphicity and envelop of holomorphy were studied by Streater [66, 68] and Tomozawa [67]. The preceding statements in the context of (38) is a qualitative and intuitive argument about the edge-of-the-wedge theorem. We shall discuss this aspect in more details in the next section.

2.2 Analyticity and Crossing Properties of Three Point Function.

Our objective is to discuss analyticity and crossing properties of the four point Wightman function for Hermitian scalar nonderivative conformal field. The three point Wightman function is

$$W_3(x_1, x_2, x_3) = \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) | 0 \rangle$$

A crossed three point function, for example, is $W_3(x_1, x_3, x_2)$ and the two coincide when $(x_2 - x_3)^2 < 0$.

We shall adopt the method employed in [55] to investigate analyticity and crossing properties of the three point function and report our further progress. There are six permuted three point functions. Our goal is to derive crossing relation for a pair of permuted Wightman functions and we address a simplified problem. We employ the analytic completion technique for the problem at hand. We utilize the same prescription to study analyticity and crossing for the four point function later in.
this section. In what follows, we summarize the method employed by us to study
the three point function. \( W_3(x_1, x_2, x_3) \) depends on two variables, \( y_1 = x_1 - x_2 \)
and \( y_2 = x_2 - x_3 \), as a consequence of translational invariance. The function of
interest is \( W_3(x_1, x_2, x_3) - W_3(x_1, x_3, x_2) \). The recent progress made in this direction
by us is presented in the sequel. The above difference, \( W_3(x_1, x_2, x_3) - W_3(x_1, x_3, x_2) \),
may be expressed as commutator \( < 0 | \phi(x_1) \phi(x_2) \phi(x_3) | 0 > \) and it vanishes when
\( (x_2 - x_3)^2 < 0 \). The goal is to study its analyticity property which is intimately
connected with crossing. We adopted a variance of the representation, due to Dyson
[56], of the double commutator of three scalar fields: \( < 0 | [C(x_3), [B(x_2), A(x_1)]] | 0 > \)
where \( A, B, C \) are the three scalar fields. In fact Dyson’s technique together with the
arguments of Streater [57] were suitably adopted by us to obtain a representation for
the VEV of our interest. Define
\[
F(y_1, y_2) = \left( W_3(x_1, x_2, x_3) - W_3(x_1, x_3, x_2) \right)
\]
(40)
Note that \( F(y_1, y_2) = 0 \) for \( y_2^2 < 0 \) from microcausality. The Fourier transform,
\( \tilde{F}(p, q) \), of \( F(y_1, y_2) \) has a representation [57, 55]
\[
\tilde{F}(p, q) = \int \Psi(p, u, s) \delta((u - q)^2 - s^2) \epsilon((u - q)_0) d^4uds^2
\]
(41)
where \( \epsilon((u - q)_0) \) is the Heaviside step function. This is a generalized version of the
Jost-Lehmann-Dyson representation [29, 30]. The function \( \Psi(p, u, s) \) has following
properties: It vanishes unless the hyperbola in the \( q \)-space i.e. \( (u - q)^2 = s^2 \) lies in
the union of two domains characterized as \( q \in V^+ \cup (p - q) \in V^+ \). It was concluded
that \( \Psi(p, u, s) = 0 \) except when the conditions \( p, u \in (u \in V^+ \cap (p - u) \in V^+) \) are
fulfilled.
It is necessary to identify the extended tubes for the two Wightman functions: (i)
\( W_3(\xi_1, \xi_2) \) is regular in the extended tube \( T'_2(\xi_1, \xi_2) \). (ii) Similarly, \( W_3(\xi_1 + \xi_2, -\xi_2) \)
is regular in the extended tube \( T'_2(\xi_1 + \xi_2, -\xi_2) \). If \( \tilde{W}_3(p, q) \) and \( \tilde{W}'_3(p, q) \) denote the
Fourier transforms of the Wightman functions \( W_3(\xi_1, \xi_2) \) and \( W_3(\xi_1 + \xi_2, -\xi_2) \) respectively; note that the latter corresponds to the crossed 3-point Wightman function.
The point to note is that \( W_3(\xi_1, \xi_2) = W_3(\xi_1 + \xi_2, -\xi_2) \) in a domain where they are
regular since these are Jost points i.e. they corresponds to real points separated by
spacelike distance. We conclude that they analytically continue to one another in the
domain
\[
T' = T'_2(y_1, y_2) \cup T'_2(y_1 + y_2, -y_2)
\]
(42)
Let us denote the Fourier transforms of \( W_3(y_1, y_2) \) and \( W_3(y_1 + y_2, -y_2) \) respectively
by \( \tilde{W}_3(p, q) \) and \( \tilde{W}'_3(p, q) \). We can read off the support properties to be
\[
\begin{align*}
\tilde{W}_3(p, q) &= 0, \text{ unless } p^2 > 0, p_0 > 0, \text{ and } q^2 > 0, q_0 > 0 \\
\tilde{W}'_3(p, q) &= 0, \text{ unless } p^2 > 0, p_0 > 0 \text{ and } (p - q)^2 > 0, (p - q)_0 > 0
\end{align*}
\]
(43)
Now recall the expression for the three point function (19) of the conformal field, $\phi(x)$. Note that $y_1 = (x_1 - x_2)$, $y_2 = (x_2 - x_3)$ and $x_1 - x_3 = y_1 + y_2$. In terms of $\xi_1$ and $\xi_2$, it assumes the form,

$$W_3(\xi_1, \xi_2) = \text{const.} \left[ \frac{1}{\xi_1^2 \xi_2^2 (\xi_1 + \xi_2)^2} \right]^{d/2} \quad (44)$$

Let us now consider the scale transformation: $\xi_i \rightarrow \lambda \xi_i$. Then

$$W_3(\xi_1, \xi_2) \rightarrow W_3(\lambda \xi_1, \lambda \xi_2) = \left[ \left( \frac{1}{\lambda} \right)^3 \left[ \frac{1}{\xi_1^2 \xi_2^2 (\xi_1 + \xi_2)^2} \right]^{d/2} \right]^{\lambda^d} = \left[ \left( \frac{1}{\lambda} \right)^3 \right]^{d/2} W_3(\xi_1, \xi_2) \quad (45)$$

We recall equations (19) and (20) for real spacelike coordinate differences i.e. $(x_i - x_j)^2 < 0$, for $i, j = 1, 2, 3$. It is quite transparent from the expression that the three point functions are analytic at the Jost point. Moreover, the denominator of (44) is analytic whenever, for real $\xi_i, \xi_1^2,$ and $\xi_2^2$ are negative and it is more transparent when we consider $x_i$-variables. Thus, expressed in terms of $x_i$’s the two three point functions $W_3(x_1, x_2, x_3)$ and $W_3(x_1, x_3, x_2)$ coincide since both can be considered as boundary values of corresponding analytic functions at the Jost point. These two functions are analytic functions in the real environment for spacelike separated points. It follows from the Jost theorem and the edge-of-the-wedge theorem that they are analytic continuation of each other. Thus the crossing is established.

The preceding discussions lead to the conclusion that the two Wightman functions are boundary values of analytic functions with known support properties. Thus crossing is proved for a pair of Wightman functions such that one is obtained from the other from interchange of a pair of fields. Let us invoke the Hall-Wightman theorem [60] to discuss the analytic continuation. The three point function $W_3(x_1, x_2, x_3)$ depends on two variables $y_i, i = 1, 2$. It is boundary value of an analytic function of two complex variables, $\xi_i, i = 1, 2$. Thus it depends on eight four-vectors. On the other hand, it follows from the Hall-Wightman theorem that $W_3$ is a function of Lorentz invariant variables constructed from $\xi_1$ and $\xi_2$: it depends on three complex variables: $z_{jk} = \xi_j^\mu \xi_{\mu k}$, $j, k = 1, 2$; expressed explicitly: $z_{11} = \xi_1^2$, $z_2 = \xi_2^2$ and $z_3 = \xi_1 \xi_2$. The Jost points are real and spacelike; moreover, they belong to $T_2'$. If the Jost points are denoted by $v_i, i = 1, 2$ that is $v_i^2 < 0$, and $v_i + \lambda v_2$ is also a Jost point with $0 < \lambda < 1$. Thus for real $\xi_i, i = 1, 2$

$$(\xi_1 + \lambda \xi_2)^2 < 0, \quad \xi_1, \xi_2 \text{ real} \quad (46)$$

We derive a relationship among the variables $z_{jk}$. Reality of $\lambda$ and condition (46) imply

$$z_{12} > \sqrt{z_{11} z_{22}} \quad (47)$$
Obviously the two Wightman functions $W_3(x_1, x_2, x_3)$ and $W_3(x_1, x_3, x_2)$ are equal at
the Jost point, The corresponding extended tube for $W_3(x_1, x_3, x_2)$ is $T'_2(\xi_1 + \xi_2, -\xi_2)$. Invoking Jost’s theorem, we conclude that the two Wightman functions are analytic
continuation of each other. We have shown that the crossing holds for Wightman
functions while considered pairwise. Therefore, all the six permuted three point
Wightman functions are analytic continuations of each other when one pair is con-
sidered at a time. It is important to note, however, that the envelope of holomorphy
could be a much larger domain. We were contented to prove crossing in this simple
approach.

It is worth while to dwell upon the analyticity properties of the three point function
in the momentum space representation. The analyticity properties in momentum
space representation of n-point functions have been investigated in the past in the
frameworks of axiomatic field theories [69, 70, 71, 72]. The retar
ded functions, defined in the coordinate space are endowed with retard
edness and causal
properties, are defined to be

\[ R \phi(x)\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n) = (-1)^n \sum_{\theta} \theta(x_0 - x_{10})\theta(x_{10} - x_{20})...\theta(x_{n-10} - x_{n0}) \]

\[ \left[[[\phi(x), \phi_1(x_{i1}), \phi_2(x_{i2})], ..., \phi_{in}(x_{in})]\right] \quad (48) \]

with $R\phi(x) = \phi(x)$. Here P stands for all permutations $(i_1, i_2, ..., i_n)$ of 1, 2, ..., $n$. The
R-product is hermitial for hermitial fields $\phi_i(x_i)$ and the product is symmetric under
exchange of any fields $\phi_1(x_1), ..., \phi_n(x_n)$. Notice that the field $\phi(x)$ is kept where it is
located in its position. We mention some of the important attributes of R-products.
(i) $R \phi(x)\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n) \neq 0$ only if $x_0 > \max \{x_{10}, ..., x_{n0}\}$.
(ii) Another property of the R-product is that

\[ R \phi(x)\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n) = 0 \quad (49) \]

whenever the time component $x_0$, appearing in the argument of $\phi(x)$ whose position
is held fix, is less than time component of any of the four vectors $(x_1, ..., x_n)$ appearing
in the arguments of $\phi(x_1)\phi_2(x_2)...\phi_n(x_n)$.

(iii) Under Lorentz transformation

\[ \phi(x_i) \rightarrow \phi(\Lambda x_i) = U(\Lambda, 0)\phi(x_i)U(\Lambda, 0)^{-1} \quad (50) \]

Therefore, under Lorentz transformation $U(\Lambda, 0)$

\[ R \phi(\Lambda x)\phi(\Lambda x_1)\phi(\Lambda x_2)...\phi_n(\Lambda x_n) = U(\Lambda, 0)R \phi(x)\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)U(\Lambda, 0)^{-1} \quad (51) \]

And

\[ \phi_i(x_i) \rightarrow \phi_i(x_i + a) = e^{ia.P} \phi_i(x_i)e^{-ia.P} \quad (52) \]
under spacetime translations. Consequently,

\[ R \phi(x + a)\phi(x_1 + a)\ldots\phi_n(x_n + a) = e^{iaP}R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n)e^{-iaP} \]  

(53)

We conclude, therefore, that the vacuum expectation value of the R-product depends only on difference between pair of coordinates: in other words it depends on the following set of coordinate differences: \( \xi_1 = x_1 - x, \xi_2 = x_2 - x_1 \ldots \xi_n = x_{n-1} - x_n \) as a consequence of translational invariance. It has been demonstrated that the Fourier transform of the VEV of R-Products are boundary values of analytic functions of complexified momenta. There is a close analogy between the three point Wightman function and the retarded three point function from the analysis of Jost [73] which has been further studied by Brown [74].

We study the analyticity properties of three point function of conformal field theory in the momentum space. Recently, there has been quite a bit of interest in understanding momentum space descriptions of correlation functions in conformal field theories [75, 76, 77, 78, 79, 80, 32, 33, 81, 82, 83, 84, 85, 86, 55]. In particular, one of our interests is the study of the analyticity properties of three point function in the momentum space description [32, 33, 55]. We were motivated to undertake this study by the recent two papers [32, 33]. We know, from Jost theorem, that Wightman functions are analytic in the spacelike regions i.e. when the coordinate separations are spacelike. Moreover, the Fourier transform of a Wightman function for spacelike coordinate differences would have conjugate momenta lying in the spacelike region.

Let us define the vacuum expectation value of \( R \)-product of \( n \) scalar field to be \( r_n(y_1, y_2, \ldots y_{n-1}) \) where \( y_i = x_i - x_{i+1} \). Therefore, for a single type of real scalar field, \( \phi(x) \)

\[ r_n(y_1, y_2, \ldots y_{n-1}) = <0| R \phi(x)\phi(x_1)\ldots\phi(x_n)|0> \]  

(54)

The Green function is

\[ G_n(p_1, p_2, \ldots p_{n-1}) = \int d^4y_1d^4y_2\ldots d^4y_{n-1}e^{i\sum_{j=1}^{n-1} p_j \cdot y_j} r_n(y_1, y_2, \ldots y_{n-1}) \]  

(55)

The next step is to define an analytic function of complexified momentum variables: \( p_j^\mu \rightarrow k_j^\mu = (p_j^\mu + iq_j^\mu) \), \( j = 1, 2, \ldots n - 1 \). The \( (n - 1) \) real four vectors \( (p_j^\mu, q_j^\mu) \) are such that \( q_j \in V^+ \) and \( p_j^\mu \) is unrestricted. The Fourier transform

\[ G_n(k_1, k_2, \ldots k_{n-1}) = \int d^4y_1d^4y_2\ldots d^4y_{n-1}e^{i\sum_{j=1}^{n-1} k_j \cdot y_j} r_n(y_1, y_2, \ldots y_{n-1}) \]  

(56)

has good convergence property. Now the Green function \( G_n(p_1, p_2, \ldots p_{n-1}) \) is boundary value of an analytic function

\[ G_n(p_1, p_2, \ldots p_{n-1}) = \lim_{q_n \rightarrow 0} G(k_1, k_2, \ldots k_{n-1}) \]  

(57)
Ruelle [70] has proved analog of Jost theorem in the momentum space. Since \( \{q^\mu_i\} \in V^+ \), we may choose a coordinate frame such that \( q^\mu_i = (q^0_i, \mathbf{0}) \) and there are no restrictions on \( p^\mu_i \). A simplified version of Ruelle’s theorem can be expressed as follows: The function \( G(k^\mu_i, \mathbf{p}_i) \) can have singularities if \( q^\mu_i = 0 \) and \( p^\mu_i \) is not spacelike. This theorem is valuable for us in what follows.

The case of three point function will be taken up now in the light of the preceding observations of Jost [73]. We deal with the \( R \)-product where the product is of a single conformal scalar field, \( \phi(x) \) defined to be

\[
R \phi(x)\phi(x_1)\phi(x_2) = \theta(x_0 - x_1)\theta(x_1 - x_2)[[\phi(x), \phi(x_1)], \phi(x_2)]
+ \theta(x_0 - x_2)\theta(x_2 - x_1)[[\phi(x), \phi(x_2)], \phi(x_1)]
\]

(58)

Let us consider the implications of causality on the first double commutator on the \( r.h.s \) of (58), suppressing the presence of the two \( \theta \)-functions,

\[
[[\phi(x), \phi(x_1)], \phi(x_2)] = 0 \quad (x - x_1)^2 < 0, \quad \text{or} \quad ((x - x_2)^2 < 0 \text{ and } (x_1 - x_2)^2 < 0)(59)
\]

Noting that \( y_1 = x - x_1 \) and \( y_2 = x_1 - x_2 \), we may express these constraints in terms of \( y_i \)-variables. Similar constraints follow for the second terms of (58). We concentrate on the three point function \( <0|\phi(x_1)\phi(x_2)\phi(x_3)|> \) to study momentum space analyticity properties. It depends on two variables \( y_1 \) and \( y_2 \) and we introduce complexified coordinates \( \xi_1, \xi_2 \). Moreover, \( W_3(\xi_1, \xi_2) = \int dp^4_1 dp^4_2 e^{-i(\xi_1 q_1 + \xi_2 p_2)}W(p_1, p_2) \) and \( p_1, p_2 \in V^+ \). We also know from works of Ruelle [70] that, in the coordinate space description, we may go to a frame where \( \eta_i = (\eta^0_i, 0, 0, 0) \), \( i = 1, 2 \); this is permitted since \( \eta_i \in V^+ \). Note that \( -\infty < y_i < +\infty, i = 1, 2 \), the real part of \( \xi_i \) (we remind the reader that \( \xi_j = y_j - i\eta_j, j = 1, 2 \)). We also know that, for real \( \xi_i \), each of the three point functions \( W_3(\{\xi\}) \) (appearing in the \( R \)-product) are analytic whenever \( \{y_i\} \in T^d_2 \) and thus are spacelike, from the Jost theorem. We may adopt another theorem of Ruelle [70] for this case; \( W_3(\eta^0, \eta^0_2, y_1, y_2) \) can only have singularities if two of its arguments \( (\eta^0_1, y_1) \) and \( (\eta^0_2, y_2) \) are such that \( \eta^0_1 = \eta^0_2 \) and \( y^\alpha_1 - y^\alpha_2 \) is not spacelike. In what follows, we shall consider a three point Wightman function \( W_3(\xi_1, \xi_2) \) taking the clue from analysis of Jost [73]. Let us consider the momentum space representation of three point function

\[
\tilde{W}_3(p_1, p_2) = \int d^4y_1 d^4y_2 e^{i(p_1 \cdot y_1 + p_2 \cdot y_2)}W(y_1, y_2)
\]

(60)

Now if we want this Fourier transform to be convergent for real \( y_1 \) and \( y_2 \) then we complexify\(^3\) the momentum variables \( (p_1, p_2) \). As before, \( (p_1, p_2) \rightarrow (k_1 = p_1 + iq_1, k_2 = p_2 + iq_2) \). Thus the tube, \( \tilde{T}_2 \) is defined in the momentum space and it is the primitive domain; moreover, the integral (60) is convergent since \( q_1 \) and \( q_2 \in V^+ \).

The momentum-space 3-point function is the boundary value of an analytic function:

\[
\tilde{W}_3(k_1, k_2)
\]

\[
\tilde{W}_3(p_1, p_2) = \lim_{\{q_i\} \rightarrow 0} \tilde{W}_3(k_1, k_2)
\]

(61)

\(^3\)see the discussions of Froissart [87].
Assuming that Ruelle’s theorem is valid for $\tilde{W}_3(k_1, k_2)$ we may invoke the momentum space theorem [70]. Thus \( \lim_{\mu \to 0} \tilde{W}_3(k_1, k_2) \) is analytic when real momenta \( p_1^\mu \) and \( p_2^\mu \) lie in the spacelike region. We draw attention to a few points. Let us consider the difference \( R \phi(x)\phi(x_1)\phi(x_2) - A \phi(x)\phi(x_1)\phi(x_2) \) where the \( A \)-product is defines such that \( \theta((x_i - x_j)\mu) \) is replaced by \( \theta((x_j - x_i)\mu) \). Now consider the VEV of this difference and denote the VEV of \( R \)-product as \( r(x, x_1, x_2) \) and correspondingly VEV of \( A \)-product as \( a(x, x_1, x_2) \)

\[
r(x, x_1, x_2) - a(x, x_1, x_2) = <0[[\phi(x), \phi(x_1)], \phi(x_2)]0 \>
\]

We arrive at above expression using the properties of \( \theta \)-functions and also the fact that \( \theta(x_0) + \theta(-x_0) = 1 \). This is the double commutator we have encountered before. If we open up the double commutator it will be sum of four three point functions. We also know the constraints due to microcausality. Note that there are altogether six permuted Wightman functions and they are different. Let us consider the momentum space functions. Then we complexify the momenta and define the analytic functions in these complex variables. Now we appeal to Hall-Wightman theorem [60] for these functions. Obviously, we have function which depends on Lorentz invariants \( z_{ij} = k_i k_j, i, j = 1, 2, 3 \). We also know \( k_1^\mu + k_2^\mu + k_3^\mu = 0 \) due to translational invariance and the complexified momenta are conjugate to \( x, x_1, x_2 \). The point to note is that now we can define a function which is analytic in the union of the extended tubes associated with the three point analytic functions. If we go to boundary values in each of the real valued momenta the corresponding Fourier transformed Wightman functions are different. We see the power of analytic completion in this simple example.

It is intuitively quite appealing. If a Wightman function is defined for spacelike coordinate, in the coordinate space representation, the conjugate momenta are also spacelike when we take the Fourier transform and study analyticity. Our intuition guides that if those coordinate points correspond to Jost point, we expect that the conjugate momenta would be spacelike. Moreover, the momentum space Green function will be analytic in momentum variables in the real environment of these Jost points (momentum space Jost points). It is quite safe to conjecture that our aforementioned arguments regarding analyticity of \( \tilde{W}_3(k_1, k_2) \) are valid. Let us start from the momentum space Jost point where \( \tilde{W}_3(k_1, k_2) \) is analytic for \( \text{Re} (k_1, k_2) \). Moreover, \( \tilde{W}_3(k_1, k_2) \) is analytic in the momentum space extended tube, \( T'_2 \). If we extend the argument of Jost, for momentum space three point function, \( \tilde{W}_3(k_1, k_2) \), it is analytic for real spacelike \( p_1, p_2 \) in their neighborhood. Let us implement an infinitesimal complex Lorentz transformation on these real momenta. The momenta \( p_1, p_2 \) will assume complex values and arguments of \( \tilde{W}_3(p_1, p_2) \) will be complex (denote them as \( k_1 \) and \( k_2 \). The resulting \( \tilde{W}_3(k_1, k_2) \) would be analytically continued in \( T'_2 \). Therefore, the function is analytic in the domain belonging to \( T'_2 \). We can say more. Note that \( W_3(\xi_1, \xi_2) \) is a tempered distribution. The Fourier transform of a tempered
distribution is also a tempered distribution as proved by Froissart [87]. Consequently, it is bounded as is defined for a distribution i.e. the boundedness is to be understood in the sense that $\tilde{W}_3(k_1, k_2)$ is a distribution (see Froissart [87] for detail discussion).

We examine the implications of these general arguments in what follows. Recently, the properties of three point function in the Lorentzian metric description has drawn attentions [32, 33]. Baurista and Godazgar [32] have systematically investigated the three point function in momentum space for the Euclidean metric as well as for the Lorentzian signature metrics. They derive an expression for the three point function as an integral which includes product of Bessel function as well as modified Bessel functions. Their work revealed several interesting analyticity properties of the Wightman function. Gillioz [33] studied the properties of three point function in the momentum space with Lorentzian signature metric. He employed operator product expansion in momentum space representation and derived Ward identities. One advantage of this approach is that he constructed states, utilizing state ↔ correspondence, in the momentum space in order to compute three point function. He has demonstrated that the three point function for scalars is expanded in a double hypergeometric series. Moreover, he explored its behavior in various kinematical limits. Our objective is to study the analyticity properties in the momentum space in the light of the preceding discussions.

Let us consider the three point function given by (44). It is bounded and analytic for spacelike $\{\xi_i\}, i = 1, 2, 4$. The Fourier transform of the coordinate space three point function has been investigated in detail in [32, 33]. Recall that the Fourier transform of $W_3(x_1, x_2, x_3)$ would depend on three momentum variable, however, the total momentum conservation implies that only two of the three momentum variables are independent. It follows from Lorentz invariance that the three point function $\tilde{W}_3(p_1, p_2, p_3)$ depends on the Lorentz invariants constructed from $p_i^\mu, i = 1, 2, 3$. Furthermore, conformal invariance imposes restrictions on its structure. We recall the results of Gillioz [33] for our purpose. The expression for the three point function assumes the following generic form (for $D = 4$ in our metric convention)

$$<0|\phi(p_1)\phi(p_2)\phi(p_3)|0> = \theta(-p_1^0)\theta(p_3^0)\delta^{(4)}(p_1 + p_2 + p_3)(p_1^2 + p_2^2 + p_3^2)^{(3d-8)/2}F(\frac{\mu^2}{p_1^2}, \frac{\mu^2}{p_2^2})$$

(63)

where the energy momentum conserving $\delta$-function implies $p_1 + p_2 + p_3 = 0$. Moreover, $F$ is a function of ratios of squares of momenta. Eventually, $F$ gets related to Appell’s

This boundedness property is to be understood as pertinent to distributions. Such a concept of boundedness is quite common in axiomatic S-matrix theory. For example, when the notion of polynomial boundedness is mentioned for matrix element of commutator of currents in the proof of Jost-Lehmann-Dyson theorem theorem [29, 30] it is interpreted that the matrix element is a distribution. Moreover, it fulfils polynomial boundedness so that the integral requires only finite number of subtractions. Similar arguments are advanced in proving the dispersion relations for 4-point amplitude where absorptive parts are argued to be polynomially bounded.
A few comments are in order at this stage: (i) Following Ruelle, we argue that the momenta can be complexified as \( p_1^\mu \rightarrow k_1^\mu = (p_1^\mu + iq_1^\mu) \) to define an analytically continued three point function in the complex momentum variables. We identify the primitive domain to be \( \mathcal{T}_2 \) and the corresponding extended tube as \( \mathcal{T}_2' \). (ii) It follows from momentum conservation that \( p_2^2 = (p_1 + p_3)^2 \) and the dimensionless ratio of the momenta, appearing as arguments of the function \( F \) go over to \( \frac{p_1^2}{p_2^2} \rightarrow \frac{p_1^2}{(p_1 + p_3)^2} \) and \( \frac{p_2^2}{p_1^2} \rightarrow \frac{p_2^2}{(p_1 + p_3)^2} \). The function, \( \mathcal{F}\left(\frac{p_1^2}{(p_1 + p_3)^2}, \frac{p_2^2}{(p_1 + p_3)^2}\right) \) is conformally invariant as is evident from the structure. (iii) Now consider \( p_1 \) and \( p_3 \) to be spacelike. Furthermore, as \( p_1^2 \) and \( p_3^2 \) tend to asymptotic values, with \( \frac{p_1^2}{p_3^2} \rightarrow \text{constant} \), in the spacelike region, then ratios \( \frac{p_2^2}{(p_1 + p_3)^2} \) and \( \frac{p_1^2}{(p_1 + p_3)^2} \) tend to constants. Therefore, in this region, the function, \( \mathcal{F}\left(\frac{p_1^2}{(p_1 + p_3)^2}, \frac{p_2^2}{(p_1 + p_3)^2}\right) \), will tend to constant. Moreover, we note that the three point function is analytic for spacelike momenta and in its a real neighborhood belonging to \( \mathcal{T}_2' \). Furthermore, this function admits analytic continuation to the extended tube. (iv) It is evident that the three point function depends on Lorentz invariant variables and their ratios. This is analog of the Hall-Wightman theorem [60] for three point function of our conformal field theory. The function of the complex variables \( k_i \) is a tempered distribution as the coordinate space three point function is a tempered distribution.

### 2.3 Analyticity and Crossing for Four point Wightman Function.

We proceed to study analyticity of four point Wightman function following the prescription discussed above. The four point function

\[
W_4(x_1, x_2, x_3, x_4) = \langle 0| \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 \rangle
\]  
(64)

will be considered along with another permuted Wightman function. We shall proceed to investigate analyticity and crossing adopting the spirit adopted for three point functions. The translational invariance implies that \( W_4(x_1, x_2, x_3, x_4) = W_4(y_1, y_2, y_3) \) where \( y_i = x_i - x_{i+1} \). Let us recall the expression for the four point function (21), (22) and (23). In the light of the preceding discussion for the three point function, we examine the analyticity of \( W_4(x_1, x_2, x_3, x_4) \) for real values of the coordinates \( \xi_1, \xi_2, \xi_3 \) and when the separations are spacelike. Recall that four vectors, \( y_i, i = 1, 2, 3, \) are all spacelike and their linear combinations with positive coefficients are also spacelike; a more precise statement is that for real \( \xi_i, \xi_i^2 < 0 \) and \( \sum_{i=1}^{3}(\lambda_i, \xi_i^2)^2 < 0, \quad 0 < \lambda \sum_{i} \lambda_i = 1. \) It follows from the definition of \( Z_1 \) and \( Z_2 \) that these are ratio of \( y_i^4 \)'s and the ratio is positive as long as each vector is spacelike. Moreover, \( Z_1 \) and \( Z_2 \) are scale invariant. The denominator appearing as prefactor of
\( \mathcal{F}(Z_1, Z_2) \) is also positive. We invoke the Jost theorem and once again, the edge-of-the-wedge theorem when we consider the two four point functions \( W_4(x_1, x_2, x_3, x_4) \) and \( \tilde{W}_4(x_1, x_2, x_3, x_4) \) and argue that they are analytic continuation of each other. Thus crossing is demonstrated from this perspective.

We intend to study the support properties from the Fourier transform of the four point function. We aim at deriving an integral representation for the function, analogous to the Jost-Lehmann-Dyson representation as our next step. Thus

\[
\tilde{W}_4(p_1, p_2, p_3) = \int d^4y_1d^4y_2d^4y_3e^{i\sum_{j=1}^{3}p_j\cdot y_j}W_4(y_1, y_2, y_3)
\]  

(65)

Note, from the spectral condition that

\[
\tilde{W}_4(p_1, p_2, p_3) = 0, \text{ unless } p_j \in V^+, \ j = 1, 2, 3
\]  

(66)

It is worth while to mention that the Fourier transform, \( \tilde{W}_4(p_1, p_2, p_3) \), is also a distribution in the momentum variables. Note that \( \tilde{W}_4(p_1, p_2, p_3) \) is a boundary value of a holomorphic function as \( W_4(y_1, y_2, y_3) \) is. We argue that \( W_4(\xi_1, \xi_2, \xi_3) \), recall(\( \xi_i = y_i - i\eta_i \)) has a Laplace transform

\[
W_4(\xi_1, \xi_2, \xi_3) = \int d^4p_1d^4p_2d^4p_3e^{-i\sum_{j=1}^{3}p_j\cdot (y_j - i\eta_j)}\tilde{W}_4(p_1, p_2, p_3)
\]  

(67)

It is holomorphic in the upper half-plane (note that \( \eta_i \in V^+ \) ensures convergence). Moreover, the derivatives \( \frac{dW_4(\xi_i)}{d\xi_j} \) exists and does not depend on direction. We remind that the function is holomorphic in the forward tube and it is Laplace transform of a distribution. The distributions \( \tilde{W}_4(p_1, p_2, p_3) \) vanish for \( p_i^\mu \notin V^+ \). The purpose of interjecting these remarks is to convey that our next steps are based on the known analyticity properties of the Fourier transformed four point function.

We have introduced three complex variables and we have defined the forward tube. Thus \( W_4(y_1, y_2, y_3) \) is boundary value of analytic function where the complex coordinates are in \( T_3(\xi_1, \xi_2, \xi_3) \). Next, the extended tube \( T'_3 \) is obtained by implementing orthochronous complex Lorentz transformations \( SL_+(2\mathbb{C}) \) on \( \{\xi_j\} \). In order to proceed further, we define the permuted Wightman function where the locations of the two fields \( \phi(x_3) \) and \( \phi(x_4) \) are interchanged (recall the case of three point function).

\[
W'_4(x_1, x_2, x_4, x_3) = <0|\phi(x_1)\phi(x_2)\phi(x_4)\phi(x_3)|0>
\]  

(68)

The new complexified coordinates are \( \xi'_1 = \xi_1, \xi'_2 = \xi_2 + \xi_3, \xi'_3 = -\xi_3, \) expressed in terms of \( \{\xi_j\} \). The associated forward tube is \( T_3(\xi_1, \xi_2 + \xi_3, -\xi_3) \). We can obtain the extended tube \( T'_3 \) once we have constructed \( T'_3(\xi_1, \xi_2 + \xi_3, -\xi_3) \). The support properties of the Fourier transform \( \tilde{W}'_4(p_1, p_2, p_3) \) of \( W'_4(x_1, x_3, x_2, x_4) \) are

\[
\tilde{W}'_4(p_1, p_2, p_3) \neq 0, \text{ for } p_1 \in V^+, \ p_2 \in V^+, \ p_2 - p_3 \in V^+, \text{ and } p_3 \notin V^+
\]  

(69)
which can be inferred from the expression for the Fourier transform. We now define a function

\[ \tilde{F}(p_1, p_2, p_3) = \tilde{W}_4(p_1, p_2, p_3) - \tilde{W}_4'(p_1, p_2, p_3) \]

\[ = \tilde{W}_4 \neq 0, \text{ if } p_1 \in V^+, \ p_2 \in V^+, \ p_3 \in V^+, \text{ and} \]

\[ -\tilde{W}_4'(p_1, p_2, p_3) \neq 0, \text{ if } p_1 \in V^+, \ p_2 \in V^+, \ p_2 - p_3 \in V^+, \]

and \( p_3 \notin V^+ \) \hspace{1cm} (70)

We consider the two functions \( W_4(\xi_1, \xi_2, \xi_3) \) and \( W_4'(\xi_1, \xi_2 + \xi_3, -\xi_3) \) to be analytic continuation of each other. The corresponding domain is the union of two extended tubes

\[ T' = T_3'(\xi_1, \xi_2, \xi_3) \cup \bar{T}_3'(\xi_1, \xi_2 + \xi_3, -\xi_3) \] \hspace{1cm} (71)

We define the momentum space function

\[ \tilde{F}(p_1, p_2, p_3) = \int <0|\phi(x_1)\phi(x_2)[\phi(x_3), \phi(x_4)]|0> e^{i(p_1.y_1+p_2.y_2)}e^{+ip_3.y_3}d^4y_1d^4y_2d^4y_3 \] \hspace{1cm} (72)

Notice that, from microcausality,

\[ \int d^4p_3 e^{-ip_3.y_3} \tilde{F}(p_1, p_2, p_3) = 0 \text{ if } y_3^2 < 0 \] \hspace{1cm} (73)

We may use similar arguments as we used for the 3-point function to introduce an integral representation for \( \tilde{F}(p_1, p_2, p_3) \), following the arguments of [68]

\[ \tilde{F}(p_1, p_2, p_3) = \int \Phi(u, p_2, v, s, \kappa)\delta[(p_1 - u)^2 - s]\epsilon((p_1 - u)_0) \]

\[ \delta[(p_3 - v)^2 - \kappa]\epsilon((p_3 - v)_0)d^4u d^4v ds dk \] \hspace{1cm} (74)

The function \( \Phi(u, p_2, v, s, \kappa) \) satisfies following properties: (i) Note that

\[ \Phi(u, p_2, v, s, \kappa) = 0, \text{ unless } (p_1 - u)^2 > 0, p_2^2 > 0, (p_3 - v)^2 > 0 \] \hspace{1cm} (75)

(ii) The condition \((p_1 - u)^2 = s\) and \((p_3 - v)^2 = k\) defines a pair of hyperboloids with four momenta satisfying above constraints. The function \( \Phi(u, p_2, v, s, \kappa) \) vanishes unless these conditions are fulfilled. This is analog of the Jost-Lehmann-Dyson representation and was considered by Streater [68, 66].

Remark: We have considered two permuted Wightman functions and it is argued that their domain of holomorphy is the union two extended tube \( T_3'(\xi_1, \xi_2, \xi_3) \) and \( \bar{T}_3'(\xi_1, \xi_2 + \xi_3, -\xi_3)' \). Let us construct the Lorentz invariant variables from the set \( \{\xi_i\} \) for the application of Hall-Wightman theorem. \( z_{ij} = \xi_i, \xi_j, i, j = 1, 2, 3 \). We remind the reader that the three point Wightman function has six permutations of the ordering of the field. Similarly, the four point function has twelve permutations. The determination of the envelope of holomorphy for this case is a formidable task.
The problem for this case has not been analyzed in so much of details as has been done for three point functions by Kalleén and Wightman [51]. We recall that Lorentz invariance and local commutativity are the two principles which are instrumental to study the general structures. In the case of three point function (the VEV) can be continued to a function regular in a certain domain \( \mathcal{M}_3 \) in the space of scalar products of \( \{\xi_i\} \). We have seen that local commutativity plays an important role to identify the union of domains which are formed when we consider permuted field configurations. There have been several attempts [52, 53, 23] to implement similar prescription for the four point function and identify the domain \( \mathcal{M}_4 \) and they have achieved some success; however, the detail analysis carried out for the three point function [51] has not been accomplished for the four point function. We shall be content with the analysis for a pair of functions as envisaged above. We consider a simplified situation guided by our previous attempt [55]. We consider two spacelike vectors on \( \mathcal{M}_3 \) which is a subspace of \( \mathcal{M}_4 \). A third spacelike vector is considered which lies in \( \mathcal{M}_4 \) but not in \( \mathcal{M}_3 \). Then we determine the Jost point. This procedure enables us to determine points of \( \mathcal{M}_4 \) similar to the condition derived for three point functions. Now there will be more inequalities (see 47) compared to the case of \( W_3(\xi_1, \xi_2) \).

We begin with the set of complexified four vectors \( \{\xi_i^\mu\}, i = 1, 2, 3 \) which are defined in manifold \( \mathcal{M}_4 \). Let us consider a submanifold \( \mathcal{M}_3 \) in \( \mathcal{M}_4 \). Consider, two spacelike real vectors which lie in \( \mathcal{M}_3 \). Their linear combination, with a positive real coefficient, bounded by unity, is also a spacelike vector. Now choose a spacelike vector in the compliment of \( \mathcal{M}_3 \). We should be in a position to obtain constraints on these three real spacelike vectors which would be similar to (47). For the case at hand, we have to consider the decomposition of \( \mathcal{M}_4 \) into \( \mathcal{M}_3 \oplus \mathcal{M} \); \( \mathcal{M} \) lies in the space compliment to \( \mathcal{M}_3 \), for three different configurations as discussed below.

We go through the following steps:

(i) Let \( \xi_1 \) be a real spacelike vector in \( \mathcal{M} \) which is compliment of corresponding \( \mathcal{M}_3 \) i.e. \( \mathcal{M}_4 = \mathcal{M}_3 \oplus \mathcal{M} \).

(ii) \( \xi_2 \) and \( \xi_3 \) are two spacelike vectors which are lying in \( \mathcal{M}_3 \). Thus \( \xi_2 + \lambda \xi_3, \ 0 < \lambda < 1 \) is also spacelike where \( \lambda \) is positive and real. Let us consider the properties of the three vectors in three different configurations.

Case (a): Recall that \( \xi_1^2 < 0 \) and \( \xi_1^2 (\xi_2 + \lambda \xi_3)^2 > 0 \) and therefore, \( \sqrt{[\xi_1^2 (\xi_2 + \lambda \xi_3)^2]} > 0 \) with \( \lambda > 0 \). We conclude that that

\[
[\xi_1^2 (\xi_2 + \lambda \xi_3)^2] < \xi_1^2 (\xi_2 + \lambda \xi_3)^2
\]

(76)

Note that the r.h.s of the above equation is positive. As before, define the Lorentz invariant Hall-Wightman variables \( z_{ij} = \xi_i \cdot \xi_j, \ i, j = 1, 2, 3 \). We choose \( \{\xi_i\} \) to be real in the light of the preceding discussion to derive relations among \( z_{ij} \) from the constraint that \( \lambda > 0 \). When expressed in terms of \( z_{ij} \), The inequality (76) translate to

\[
z_{11}z_{22} - z_{12}^2 + 2\lambda (z_{11}z_{23} - z_{12}z_{13}) + \lambda^2 (z_{11}z_{33} - z_{13}^2) > 0
\]

(77)
We can derive an inequality, similar to (47) if we demand that $\lambda$ be positive and real (recall $0 < \lambda < 1$ here). Instead of obtaining such constraints in a case by case basis let us consider the other two cases. We choose two spacelike vectors in a subspace $\mathcal{M}_3$ and another one which lies in its compliment. The other two case are

(b) $\xi_1$ and $\xi_3$ lie in an $\mathcal{M}_3$ and $\xi_2$ in its compliment, $\mathcal{M}$.

(c) Here we have another permuted scenario for $\{\xi_1, \xi_2, \xi_3\}$.

We shall obtain two more equations from the case (b) and (c) which will be analogous to (77). We can derived the set of constraints in a more efficient and elegant manner. Define the matrix

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \quad (78)$$

Note that $Z$-matrix is symmetric, $Z^T = Z$ as was introduced by Hall and Wightman [60]. Let us define a matrix

$$\tilde{M}_{ij} = (\det Z)(Z^{-1})_{ij} \quad (79)$$

The constraint equations arising from the requirement that $\lambda > 0$ is expressed in terms of the elements of the $\tilde{M}$. Notice that we have to solve for the analog of equations like (77) to derive the requisite constraints in terms of the matrix elements of $\tilde{M}$. Moreover, there will be altogether three equations. The conditions are given below:

$(a')$ We start with the configurations $\xi_1^2 < 0$ and $(\xi_2 + \lambda \xi_3)^2 < 0, 0 < \lambda < 1$ where $\xi_2$ and $\xi_3$ are spacelike. Positivity of $\lambda$ leads to following inequality: $\tilde{M}_{23}^2 > \tilde{M}_{33}\tilde{M}_{22}$.

$(b')$ For the case $\xi_2$, $\xi_3$ and $\xi_1$ spacelike with $(\xi_1 + \lambda \xi_3)^2 < 0, 0 < \lambda < 1$, the corresponding the condition is: $\tilde{M}_{13}^2 > \tilde{M}_{33}\tilde{M}_{11}$.

$(c')$ For the third case i.e. $\xi_3$ real and spacelike and two real spacelike vectors $\xi_1$ and $\xi_2$ lying in a a complement. Thus $(\xi_1 + \lambda \xi_2)^2 < 0$ for $0 < \lambda < 1$. The condition becomes: $\tilde{M}_{21}^2 > \tilde{M}_{11}\tilde{M}_{22}$.

Notice that same procedure can be adopted to derive constraints for the real values of the the complexified variables, $\xi_1', \xi_2', \xi_3'$: $(\xi_1' = \xi_1 + \xi_2, \xi_2' = -\xi_2, \xi_3' = \xi_2 + \xi_3)$. Moreover, $W_4(x_1, x_2, x_3, x_4)$ and $W'_4(x_1, x_3, x_2, x_4)$ are equal for $(x_2 - x_3)^2 < 0$ which is a Jost point and corresponds to $(\text{Re } \xi_2)^2 < 0$. We have proved crossing for the pair of Wightman functions and identified the analyticity regions.

Let us recall the structure of the four point function for the conformal field theory, expressed in terms of the real $x_i$-variables

$$W_4(\xi_1, \xi_2, \xi_3) = \left[ \frac{1}{\xi_1^2 \xi_3^2} \right] d \tilde{F}(\tilde{Z}_1, \tilde{Z}_2) \quad (80)$$

32
Now $\tilde{Z}_1$ and $\tilde{Z}_2$ are cross ratios of variables $\{\xi_i\}$.

$$\tilde{Z}_1 = \frac{\xi_1^2 \xi_2^2}{(\xi_1 + \xi_2)^2(\xi_2 + \xi_3)^2}$$  \hspace{1cm} (81)$$

and

$$\tilde{Z}_2 = \frac{(\xi_1 + \xi_2 + \xi_3)^2 \xi_2^2}{(\xi_1 + \xi_3)^2(\xi_2 + \xi_3)^2}$$  \hspace{1cm} (82)$$

Notice also that the cross ratios are conformally invariant. The prefactor of the function of cross ratio transforms according to the known rule. Thus

$$W_4(\lambda \xi_1, \lambda \xi_2, \lambda \xi_3) = (\lambda)^{-4d} W_3(\xi_1, \xi_2, \xi_3)$$  \hspace{1cm} (83)$$

If we now recall equations (21), (22) and (23)

$$W_4(x_1, x_2, x_3, x_4) = \langle 0 \mid \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \mid 0 \rangle \hspace{1cm}$$

$$= \frac{1}{(\tilde{z}_{12}^2 - i\epsilon \tilde{z}_{12}^0)(\tilde{z}_{34}^2 - i\epsilon \tilde{z}_{34}^0)} \mathcal{F}(Z_1, Z_2)$$  \hspace{1cm} (84)$$

with $\tilde{z}_{12} = x_1 - x_2$, $\tilde{z}_{34} = x_3 - x_4$, $\tilde{z}_{13} = x_1 - x_3$, $\tilde{z}_{24} = x_2 - x_4$, $\tilde{z}_{14} = x_1 - x_4$, $\tilde{z}_{23} = x_2 - x_3$.

$$Z_1 = \frac{(\tilde{z}_{12}^2 - i\epsilon \tilde{z}_{12}^0)(\tilde{z}_{34}^2 - i\epsilon \tilde{z}_{34}^0)}{(\tilde{z}_{13}^2 - i\epsilon \tilde{z}_{13}^0)(\tilde{z}_{24}^2 - i\epsilon \tilde{z}_{24}^0)}$$  \hspace{1cm} (85)$$

and

$$Z_2 = \frac{(\tilde{z}_{14}^2 - i\epsilon \tilde{z}_{14}^0)(\tilde{z}_{23}^2 - i\epsilon \tilde{z}_{23}^0)}{(\tilde{z}_{13}^2 - i\epsilon \tilde{z}_{13}^0)(\tilde{z}_{24}^2 - i\epsilon \tilde{z}_{24}^0)}$$  \hspace{1cm} (86)$$

$\mathcal{F}$ is a function which depends on cross ratios and its form is determined by the model under considerations. Now we consider real $x_i$ such that $(x_i - x_j)^2 < 0$ $i, j = 1, 2, 3, 4$ (without $i\epsilon$ factors) then this is a Jost point. The four point function is analytic at the Jost point. Moreover, the permuted four point function will coincide with (84) at the Jost point it is clear that the four point function Jost point. Therefore, the crossing will be valid in this region. We are not in a position to make more accurate statements since the functional dependence of cross ratios are not known unless we appeal a specific model.

3. Analyticity and Causality in Operator Product Expansions and Conformal Bootstrap Equation.

In this section we discuss analyticity and causality properties of the four point Wightman functions in the context of conformal bootstrap equation. First, we focus attention on operator product expansion of a pair of conformal fields. The proposal of
Otterson and Zimmermann [38] is appropriately modified in the context of CFT. Their investigation is based on the work of Wilson and Zimmermann [36] who rigorously studied operator expansion in QFT from the Wightman axioms. The analyticity property of the matrix elements of composite operators appearing in OPE are investigated. Mack [41] had initiated the study of the properties OPE in CFT from Wightman axiom view point. When we invoke microcausality for the commutator of two operator product expansions, the matrix element of the (difference) of two composite operators are constrained. Therefore, the analog of the Jost-Lehmann-Dyson representation can be derived.

The second part of this section is devoted to derive bootstrap equation in a novel way through the PCT theorem. The equivalence between PCT theorem and weak local commutativity (WLC) is invoked to related two four point functions. The conformal partial wave expansion is implemented on each the four point Wightman functions to derive the bootstrap equation. We draw attention to following points to highlight our approach. The first point to note is that the two four point functions coincide at the Jost point. It should be noted that the corresponding Fourier transformed Wightman functions depend on conjugate momenta which are spacelike and we term them as 'unphysical'. Moreover, each four point function is boundary value of an analytic function. These Wightman functions are analytic in real neighborhood of Jost point. They are analytic in corresponding extended tubes. Thus it will be demonstrated, with chain of arguments, that the two four point functions are analytic continuation of each other. The second point is the following. The conformal bootstrap equation hold at real points where coordinates have spacelike separations. As will be shown later, the bootstrap equations can be interpreted as boundary values of analytic functions defined over extended tubes. We feel that this proof is quite novel in the sense that the power of PCT theorem and its equivalence with WLC enables us to derive the bootstrap equation in CFT rigorously.

3.1. Operator Product Expansion, Causality and Analyticity.

Let us recapitulate the Wilson’s operator product expansion proposal for the product of two scalar fields $A_1(x_1)A_2(x_2)$ as envisaged in [38], based on the works of Wilson and Zimmermann [36]. In the case of scalar fields

$$A_1(x_1)A_2(x_2) = \sum_{j=1}^{k} f_j(\rho)C_j(x, \zeta, \rho) + R(x, \zeta, \rho)$$

(87)

where $x_1^\mu = x^\mu + \rho \zeta^\mu$, $x_2^\mu = x^\mu - \rho \zeta^\mu$, $\rho > 0$ and $R(x, \zeta, \rho)$ stands for the remainder of the series collectively. The series can be so organized that coefficients would satisfy

$$\lim_{\rho \to 0} \frac{f_{j+1}(\rho)}{f_j(\rho)} = 0, \quad \lim_{\rho \to 0} \frac{R}{f_j(\rho)} = 0$$

(88)
Note that \( \{ f_j(\rho) \} \) are C-numbers and they become singular as \( \rho \to 0 \). The operators, \( C_j \), depend on the vector \( x^\mu \) which is identified as the center of mass point. It also depends on another vector \( \zeta^\mu \) which is proportional to the distance between the two operators \( A(x_1) \) and \( A_2(x_2) \). We note that the \( \zeta \)-dependence is connected with the directional dependence of the set of operators \( \{ C_j \} \). It becomes obvious from the relations
\[
\zeta^\mu = \frac{\kappa^\mu}{\sqrt{\kappa^2}}, \quad \rho = \sqrt{\kappa^2}
\] (89)

We can reexpress (87) as
\[
A_1(x + \kappa)A_2(x - \kappa) = \sum_{j=1}^{k} f_j(\sqrt{\kappa^2})C_j(x, \frac{\kappa}{\sqrt{\kappa^2}}) + R
\] (90)
The scalar fields, \( A(x_1) \) and \( A_2(x_2) \) respect the Wightman axioms. The operators \( C_j(x, \zeta) \) are local in \( x \) for a given \( \zeta \). Otterson and Zimmermann [38] have investigated the relationship between causality and analyticity rigorously for OPE of two scalar fields.

Our intent is to investigate relationship between causality and analyticity in conformal field theory in the frameworks of Wightman axioms and adopt the formalism introduced by [38]. We remind that not all conformal field theories fulfill the requirements of Wightman axioms. The Hermitian scalar nonderivative field, \( \phi(x) \), respects Wightman axioms. Its Fourier transform, \( \tilde{\phi}(p) \), satisfies the spectrality condition i.e. \( p \in V^+ \). We consider a single conformal scalar field, \( \phi(x) \). The operator product expansion is
\[
\phi(x_1)\phi(x_2) = \sum_\chi \sum_j f_j^\chi(\rho)C_j^\chi(x, \zeta)
\] (91)
Mack [41] has investigated the convergence of the operator product expansion rigorously for nonderivative scalar conformal field theory. The coefficients \( \{ f_j^\chi(\rho) \} \) are C-number functions which become singular as \( \rho \to 0 \). For the nonderivative field, \( \phi(x) \), in the OPE, the complete set of local operators, \( \{ C_j^\chi \} \), are also nonderivative operators [41]. It is to be understood that in the operator product expansion there might be derivative of field satisfying the desired properties. We mention that Mack [41] considered operator product expansion for nonderivative field and convergence property of a matrix element of the type \( \langle \psi(\phi(x/2))\phi(-x/2)|0 \rangle \). Here \( |\psi(\phi)\rangle \) is a vector in the Hilbert space, \( \mathcal{H} \), defined to be
\[
|\psi(\phi)\rangle = \sum_{l} \int dx_1 dx_2, ... dx_l f_l(x_1, x_2, ... x_l)\phi(x_1)\phi(x_2)...\phi(x_l)|0 \rangle
\] (92)
Note that here one considers a single nonderivative field \( \phi(x) \) and \( f(x_1, x_2, ... x_l) \) is the usual weight function. He presented the matrix element of the operator product
of two fields. In eq. (91), the sum over $\chi$ goes over all unitary irreducible finite dimensional representations of the covering group of the conformal group, $SO(4,2)$, usually identified as $SU(2,2)$. We adopt a notation $\chi = [l, \delta]$; $l$ corresponds to finite dimensional irreducible representation of $SL(2,\mathbb{C})$; it is the 'Lorentz spin' and $\delta \geq \delta_{min}$, real dimension. OPE, as noted in section 1, requires infinite set of operators belonging to irreducible representations of the covering group. Therefore, a Hilbert space is associated with each of these local fields characterized by $\chi$. Thus the full Hilbert space, $\mathcal{H}$, is decomposed as $\mathcal{H} = \bigoplus \chi \mathcal{H}^\chi$ as discussed already. It will suffice to consider a series expansion for a given $\chi$ (i.e. fixed $\chi$) in order to investigate causality and analyticity properties. We pick up a generic term in the double series expansion (91) and define

$$\sum_j f_j^{\chi}(\rho) C_j^{\chi}(x, \zeta) = \sum_j \tilde{f}_j \tilde{C}_j(x, \zeta), \text{ for a given } \chi$$  

(93)

Note that the r.h.s of the OPE, (91), is a double sum. It suffices for us to analyze properties of (91) in the present context. Therefore, we investigate the causality and analyticity properties of a single sum over $j$ in (93) i.e. look at the summed over $j$ term for a given $\chi$. The conclusion drawn from here will hold for each of term of (91) in the sum over $\chi$ of the double sum. Here $\tilde{f}_j(\rho)$ and $\tilde{C}_j(x, \zeta)$ stand for $f_j^{\chi}(\rho)$ and $C_j^{\chi}(x, \zeta)$ respectively so that we do not carry the index $\chi$ everywhere. The series is so organized, as was adopted in [38], for a given sector of $\chi$, that coefficients of the operator $\tilde{C}_j(x, \zeta)$ satisfy a condition analogous to (88); i.e.

$$\lim_{\rho \to 0} \frac{\tilde{f}_{j+1}(\rho)}{\tilde{f}_j(\rho)} = 0$$  

(94)

We define

$$P_j(x, \zeta, \rho) = \tilde{f}_j(\rho) \tilde{C}_j(x, \zeta)$$  

(95)

following the clue from Otterson and Zimmermann [38]. Consequently,

$$\tilde{C}_j(x, \zeta) = \lim_{\rho \to 0} \frac{P_j(x, \zeta, \rho)}{\tilde{f}_j(\rho)}$$  

(96)

Define Fourier transform of $\tilde{C}_j(x, \zeta)$ as

$$\tilde{\tilde{C}}_j(x, u) = \frac{1}{2\pi^2} \int d^4 \zeta e^{i\zeta \cdot u} \tilde{C}_j(x, \zeta)$$  

(97)

Let us envisage two state vectors $|p>$ and $|q>$ in the Hilbert space $\mathcal{H}^\chi$ for a given $\chi$ in the momentum space representation. Thus $P_\mu |p> = p_\mu |p>$ and $P_\mu |q> = q_\mu |q>$. Now consider the matrix element which satisfies

$$<p|\tilde{C}_k(x, u)|q> = 0, \text{ unless } u \in V^+$$  

(98)
In fact if $< \phi(p)|$ and $|\psi(q)>$ are arbitrary states which are superposition of momentum states the matrix element $< \phi(p)|\bar{C}_k(x, u)|\psi(q)> = 0$, unless $u \in V^+$. We recall that Wilson and Zimmermann [36] introduced additional hypothesis when they envisaged operator product expansion in the Wightman formulation for QFT. The same hypothesis was invoked in [38]. In the present case, we also assumes that similar properties are satisfied by the matrix elements of operator product expansion of two conformal fields taken between two momentum space state $|p>$ and $|q>$ obtained from state operator correspondence with $\tilde{\phi}(p)$ and $\tilde{\phi}(q)$. Moreover, $\tilde{\phi}(p)$ and $\tilde{\phi}(q)$ satisfy the spectrality condition i.e. $p \in V^+$ and $q \in V^+$. The microcausality constraint is

$$[\phi(x_1), \phi(x_2)] = 0, \text{ if } (x_1 - x_2)^2 < 0 \quad (99)$$

This condition, for the local composite operators in the OPE, translates to

$$\tilde{C}_j(x, \zeta) = \tilde{C}_j(x, -\zeta), \text{ for } \zeta^2 < 0 \quad (100)$$

Thus the function

$$F_j(x, \zeta) = < \phi(p)|(\tilde{C}_j(x, \zeta) - \tilde{C}_j(x, -\zeta))|\psi(q)> = 0, \text{ for } \zeta^2 < 0 \quad (101)$$

In the Fourier space,

$$\tilde{F}_j(x, u) = < \phi(p)|(\tilde{C}_j(x, u) - \tilde{C}_j(x, -u))|\psi(q)> = 0, \text{ if } u^2 < 0 \quad (102)$$

Notice that $\tilde{F}_j(u)$ satisfies all the conditions of Jost-Lehmann-Dyson theorem [29, 30]. It admits the representation

$$\tilde{F}_j(u) = \int ds \int d^4 u' \Sigma(x, u - u', s) \quad (103)$$

The function $\Sigma(x, u - u', s)$ vanishes unless the hyperboloid $(u - u')^2 = s$ lies in the region $u^2 \geq 0$; otherwise it is arbitrary. We recall that $\tilde{C}_j(x, \zeta)$ is local in $x$ for every fixed $\zeta$. Moreover, from the Fourier transform of $\tilde{C}_j(x, \zeta)$, we know that $u \in V^+$ which is conjugate to $\zeta$.

It is necessary to discuss further the analyticity properties in the present context. Let us complexify $\zeta$, i.e. define $\tilde{\zeta} = \zeta - i\alpha, \alpha \in V^+$. We employ the arguments developed in the previous section to consider the complex $\zeta$-plane. This defines the forward tube, $T(\tilde{\zeta})$. Thus, $\tilde{C}_j(x, \zeta)$ is boundary value of an analytic function $G_j(x, \tilde{\zeta})$ such that

$$\tilde{C}_j(x, \zeta) = \lim_{\alpha \to 0} G_j(x, \zeta - i\alpha) \quad (104)$$

We obtain the corresponding extended tube $T'(x, \tilde{\zeta})$ by implementing orthochronous complex Lorentz transformations, $SL_+(2\mathbb{C})$, on the points of the forward tube $T(\tilde{\zeta})$. Moreover,

$$G_j(x, \tilde{\zeta}) = 0 \text{ for } \tilde{\zeta}^2 < 0, \tilde{\zeta} \text{ real} \quad (105)$$

37
Note that the real points are separated by spacelike distance. Therefore, $\tilde{C}_j(x, \zeta)$ and $\tilde{C}_j(x, -\zeta)$ are analytic continuation of each other.

We may interpret this conclusion as a proof of crossing in the following sense. (i) The two local operators $\tilde{C}_j(x, \zeta)$ and $\tilde{C}_j(x, -\zeta)$ coincide at the Jost point. (ii) Now we consider the difference of two Wightman functions $W_4(x_1, x_2, x_3, x_4) - W'_4(x_1, x_3, x_2, x_4)$ which is

$$< 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0> - < 0|\phi(x_1)\phi(x_3)\phi(x_2)\phi(x_4)|0>$$

Once we implement OPE for the fields $\phi(x_2)\phi(x_3)$ and $\phi(x_3)\phi(x_2)$ we arrive at the JLD representation and finally see that two functions coincide for spacelike separated points.

3.2. PCT Theorem and Conformal Bootstrap Equation.

We proceed to derive bootstrap equation by invoking PCT theorem and its equivalence with weak local commutativity as alluded to in the beginning of this section. Let us briefly recapitulate important aspects of PCT theorem for motivations. The PCT theorem is very profound. The theorem, known as Pauli-Lüder theorem, was proved by Pauli [88] and by Lüder [89, 90]. Furthermore, when the theorem was proved by Pauli and Lüder, they had not considered the possibilities of the violation of discrete symmetries: P, C and T. Moreover, in their proof, they considered Lagrangian field theories. The violation of parity in weak interactions was proposed by Yang and Lee [91] later. The parity violation was experimentally observed very soon by Wu and collaborators [92]. Therefore, with hindsight, we may say that the proof of Pauli-Lüder PCT theorem did not consider the most general case at that juncture. Jost [62] proved PCT theorem rigorously for axiomatic local field theories and established the equivalence of the theorem with weak local commutativity of Wightman functions. Dyson [93] as well as Ruelle [94] have further investigated consequences of WLC and analyticity properties of Wightman functions. Greenberg [95] has proved that violation of PCT theorem by Wightman functions implies violation of Lorentz invariance of the local field theory. One of the consequences of the PCT theorem is that the masses of particle and antiparticle be equal. The best experimental test is from the $K^0$ and $\bar{K}^0$ mass difference [96]: $-4 \times 10^{-19} \text{GeV} < m_{K^0} - m_{\bar{K}^0} < +4 \times 10^{-19} \text{GeV}$. It is obvious why so much of premium is placed on CPT theorem. Let us consider the four point Wightman function to derive the conformal bootstrap equation from the PCT theorem.

The discrete spacetime transformations: parity, P, and time reversal, T, have following properties. Under P, $(t, \mathbf{x}) \rightarrow (t, -\mathbf{x})$, whereas under time reversal, T, $(t, \mathbf{x}) \rightarrow (-t, \mathbf{x})$. All the additive quantum numbers characterizing a field reverse
their signs under $C$. Moreover, $T$ is an antilinear operator. The product of the three
discrete operators is denoted as $\Theta = PCT$. For local field theories, if it is invariant
under proper Lorentz transformation, the existence of $\Theta$ can be proved. Consider a
complex scalar field, $\Phi(x)$. The action of the operator is

$$\Theta \Phi(x) \Theta^{-1} = \eta \Phi(-t, -x)^\dagger, \quad |\eta|^2 = 1$$

(107)

$\eta$, satisfying the constraint, is a phase and it is so chosen that the theory is PCT
invariant. A Wightman function of complex scalar field $\Phi(x)$ transforms as follows under $\Theta$

$$< 0|\Phi(x_1)\Phi(x_2)...\Phi(x_n)|0 > \rightarrow \quad < 0|\Phi(-x_1)\Phi(-x_2)...\Phi(-x_n)|0 >^*$$

$$= < 0|\Phi(-x_n)\Phi(-x_{n-1})...\Phi(-x_1)|0 >$$

(108)

We deal with a real scalar conformal field, $\phi(x)$. We recall that the four point function,
$W_4(x_1, x_2, x_3, x_4) = < 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 >$ which transforms as follows under the
PCT operation

$$< 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 > \rightarrow < 0|\phi(-x_4)\phi(-x_3)\phi(-x_2)\phi(-x_1)|0 >$$

(109)

If the theory is invariant under PCT symmetry then

$$< 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 > = < 0|\phi(-x_4)\phi(-x_3)\phi(-x_2)\phi(-x_1)|0 >$$

(110)

We have noted, in the previous section, that $W_4(x_1, x_2, x_3, x_4)$ depends on difference
of the coordinates, $y_j = x_j - x_{j+1}$ due to translational invariance. Recall our defini-
tion of complexified coordinates, $\xi_j = y_j - i\eta_j, \eta_j \in V^+$. We argue, as before, that
$W_4(\{y_j\}), j = 1, 2, 3$ is the boundary value of the analytic function $W(\{\xi_j\}), j = 1, 2, 3$
of complex variables $\{\xi_j\}$ and $\xi_j \in T_3$, the forward tube. We remind that $W_4(\{\xi_j\})$ is
also invariant under orthochronous complex Lorentz transformation $SL_+(2C)$. The set of points of $\xi_j \in T_3$ which are generated under arbitrary complex Lorentz trans-
formations, $\Lambda \in SL_+(2C)$ define an extended tube $T'_3$. In other words, the points
$\{\Lambda \xi_j\}$ obtained from $\{\xi_j\}$ belong to the extended tube. Moreover, there is a single
valued continuation of $W_4(\{\xi_j\})$ to the extended tube $[60]$. We have emphasized before
that $T_3$ contains only the complex points of the forward tube. On the other hand $T'_3$
contains the real points, $\{y_j\}$, as well. These points are spacelike; the Jost points.
The Jost points are spacetime points in which all convex combinations of successive

differences are spacelike. A Jost point is an ordered set $(x_1, x_2, x_3, x_4)$.

The Jost theorem $[62]$ : A real point of $\xi_1, \xi_2, \xi_3$ lies in the extended tube, $T'_3$, if and
only if all real four vectors of the form $\sum_3^3 \lambda_j \xi_j^\mu, \lambda_j \geq 0, \sum_1^3 \lambda_j = 1$ are spacelike i.e.
$(\sum_3^3 \lambda_j \xi_j^\mu)^2 < 0, \lambda_j > 0, \sum_1^3 \lambda_j = 1$.

The equivalence between PCT theorem and WLC:
If the PCT theorem holds for all $x_1, x_2, x_3, x_4$ then for every $x_1, x_2, x_3, x_4$ such that
each of the $y_j = x_j - x_{j+1}$ is a Jost point. The WLC condition leading to

$$< 0|\phi(x_1)\phi(x_2), \phi(x_3)\phi(x_4)|0 > = < 0|\phi(x_4)\phi(x_3), \phi(x_2)\phi(x_1)|0 >$$

(111)
is satisfied. The converse statement of the Jost theorem is paraphrased as if WLC holds in a real neighborhood of (111), a Jost point, then the PCT condition (110) is valid everywhere. Moreover, WLC implies validity of PCT symmetry for the conformal scalar. We are in a position to address the conformal bootstrap proposal in the present perspective and go through the following steps.

I. The validity of the PCT theorem is assumed for conformal field theory. Moreover, we know that \( \mathcal{W}_4(\xi_1, \xi_2, \xi_3) \) is a holomorphic function and (111) holds in the extended tube \( T_3' \). Furthermore, the four point function is boundary value of an analytic function

\[
\lim_{\eta_i \to 0} \mathcal{W}_4(\xi_1, \xi_2, \xi_3) = W_4(y_1, y_2, y_3) \tag{112}
\]

We also know from [60] that \( \mathcal{W}_4(\xi_1, \xi_2, \xi_3) \) is invariant under proper complex Lorentz transformations, \( SL_+(2\mathbb{C}) : \{\xi_i\} \to \Lambda\{\xi_i\}, \xi_i \in T_3' \). Let us choose a \( \Lambda \) such that the set of complex four vectors \( \xi_i^\mu \to -\xi_i^\mu \). Consequently,

\[
\mathcal{W}_4(\xi_1, \xi_2, \xi_3) = \mathcal{W}_4(-\xi_1, -\xi_2, -\xi_3) \tag{113}
\]

II. We recall that the r.h.s. of the equation (110) is a statement of PCT invariance of the four point function. It is also boundary value of an analytic function.

\[
\lim_{\eta_i \to 0} \mathcal{W}_4(\xi_3, \xi_2, \xi_1) = W_4(y_3, y_2, y_1) = <0|\phi(-x_4)\phi(-x_3)\phi(-x_2)\phi(-x_1)|0> \tag{114}
\]

III. Now consider the difference of two four point functions: \( \mathcal{W}_4(\xi_1, \xi_2, \xi_3) - \mathcal{W}_4(\xi_3, \xi_2, \xi_1) \). This is holomorphic in the domain \( T_3' \). We know that this difference vanishes for Re \( \xi_i, i = 1, 2, 3 \) by CPT theorem (110). Let us invoke the edge-of-the-wedge theorem [65, 67]. Consider two real open cones, \( C_1 \) and \( C_2 \), in \( \mathbb{R}^n \) for generality. Let the functions \( f_1(z) \) and \( f_2(z) \), with \( z = x + iy \) (note \( x, y \) carry no indices and they have no relationship with \( \{y_i^\mu\} \) defined above) satisfy following properties: (i) The two functions, \( f_1(z) \) and \( f_2(z) \), are defined and analytic in the intersection of the tube over \( C_\alpha, \alpha = 1, 2 \); that is \( z : Im z \in C_\alpha \) and also analytic in a certain neighborhood of \( z = 0 \). (ii) When \( y \) tends to 0 from inside \( C_\alpha \) the two functions \( f_1(x + iy) \) and \( f_2(x + iy) \) tend to distributions \( T_1(x) \) and \( T_2(x) \) respectively; in \( D_N \) where \( N \) is certain real neighborhood of the point \( z = 0 \). Recall that the Wightman function, a distribution, is boundary value of the analytic function. And (iii) \( T_1 = T_2 \). Then \( f_1(z) \) and \( f_2(z) \) have a common analytic extension \( f(z) \) on the intersection of the neighborhoods of \( z = 0 \) and the convex closure of \( C_1 \cup C_2 \). Consider a situation where \( C_1 \) and \( C_2 \) are completely opposite i.e. \( C_1 \cap (-C_2) \) contains an open cone, then \( f(z) \) is analytic in a neighborhood of \( z = 0 \). It follows, in our context, that if the coordinate differences, \( x_i - x_{i+1}, i = 1, 2, 3, 4 \) are all spacelike, the two Wightman functions are analytic at the real points \( \{y_i\}, i = 1, 2, 3 \) and they are equal. Consequently, they are analytic continuations of each other.
Therefore, PCT theorem and WLC together with the edge-of-the-wedge theorem will be crucial to what follows. The immediate conclusion is

\[ \mathcal{W}_4(\xi_1, \xi_2, \xi_3) = \mathcal{W}_4(\xi_3, \xi_2, \xi_1) \]  

(115)

The converse of the above statement is the following. It is a consequence of Hall and Wightman theorem [60] that if (115) holds good in an arbitrary neighborhood of \( T_3' \) it also holds good in the extended tube. Moreover, if it is also valid for passing into the boundary in the tube \( T_3 \) then we recover the condition of PCT invariance (110).

In the historical context, note that the first proof of the edge-of-the-wedge theorem [63] was presented to prove dispersion relations for pion-nucleon scattering. The amplitude was obtained by adopting the LSZ [20] reduction technique. Subsequently, the matrix element of causal commutator of the source currents was envisaged. Thus the equations of motion were implicitly used. For the present case we are dealing with Wightman functions and the edge-of-the-wedge theorem is invoked to prove (115). Thus the conclusion is that PCT invariance is equivalent to WLC in CFT. It follows from equations (113) and (115) that

\[ \mathcal{W}_4(\xi_1, \xi_2, \xi_3) = \mathcal{W}_4(-\xi_1, -\xi_2, -\xi_3) \]  

(116)

**Remark:** Let us try to pass to the boundary in the above equation for any set of \( y_1, y_2, y_3 \) in (116). A problem arises. We shall not be able to obtain a relation between the two functions (in the above equation) for any set of \( \{ y_i \} \) for the following reason. On the l.h.s. \( \xi_1, \xi_2, \xi_3 \) approach the real vectors which are in \( V^+ \). Note that the real vectors of \(-\xi_1, -\xi_2, -\xi_3 \) would be in \( V^- \). The equality holds for for \( \text{Re} \ (\xi_1, \xi_2, \xi_3) \) and \( \text{Re} \ (-\xi_1, -\xi_2, -\xi_3) \) when we are at the Jost point.

Notice the important fact: at the real point of holomorphy, at the Jost point, we have the following relation

\[ \mathcal{W}_4(\xi_1, \xi_2, \xi_3) = <0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0> = W(-\xi_3, -\xi_2, -\xi_1) = <0|\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0> \]  

(117)

This equation has important implications for the conformal bootstrap proposal.

Let us proceed to envisage the four point Wightman function \(<0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0>\). We employ the conformal partial wave expansion. Introduce a complete set of states, \( \{|\Psi>\} \), between the product of the two pair of operators: \( \phi(x_1)\phi(x_2) \) and \( \phi(x_3)\phi(x_4) \). The states \( \{|\Psi>\} \) belong to the full Hilbert space, \( \mathcal{H} \). Therefore, all irreducible representations of the conformal group are included as 'intermediate' states. Now

\[ \mathcal{W}_4 = <0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0> = \sum_{|\Psi>} <0|\phi(x_1)\phi(x_2)|\Psi><\Psi|\phi(x_3)\phi(x_4)|0> \]  

(118)

The sum \( \sum |\Psi> \) also includes integration over coordinates as is the custom when we insert complete set of intermediate states. We resort to the state ↔ operator
correspondence and then interpret $<0|\phi(x_1)\phi(x_2)|\Psi>$ as a three point function $<0|\phi(x_1)\phi(x_2)\Psi|0>$. Let us identify $|\Psi|:=\hat{\Psi}|0>$; $\hat{\Psi}$ represents the complete set of operator belonging to irreducible representations of the conformal group. Notice that the second matrix element of the r.h.s. of the above equation is another three point function and $<\Psi|=<0|\hat{\Psi}$; $\hat{\Psi}$ is the adjoint of $\hat{\Psi}$. We may reexpress (118) as

$$<0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0>=\sum_{\hat{\Psi},\hat{\Psi}}\sum_{\alpha,\beta,\lambda,\phi_1,\phi_2,\phi_3,\phi_4}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta}\tag{119}$$

where $\lambda_\phi^{\alpha\beta}$ and $\lambda_\phi^{\beta}$ can be read off from the above equation. We have adopted the notation that $\phi_i$ stands for $\phi(x_i)$ for brevity. Moreover, $\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}$ is the conformal partial waves (CPW) [13, 7, 8, 9]. The sum over complete set of operators include all the allowed irreducible representations of the conformal group such as Lorentz spin, scale dimensions etc. We have discussed the analyticity and its close relationship with causality in the previous section in the Wightman’s formulation for CFT. Let us briefly recall the analyticity properties of the commutator of a pair of composite operators that appear in OPE. We consider the CPW expansion of another familiar four point function

$$<0|\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0>=\sum_{\hat{\Psi},\hat{\Psi}}\sum_{\alpha,\beta,\lambda,\phi_1,\phi_2,\phi_3,\phi_4}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta}\tag{120}$$

The two expressions for Wightman functions (119) and (120) are equal at those Jost points and these are conformal bootstrap conditions [13, 15, 7, 8, 9].

We remind that the Wightman functions, satisfying (119) and (120) are analytic functions in the extended tubes. Let us invoke Jost’s theorem [62] and Dyson’s arguments [93] for the proof of analyticity of the Wightman functions for real $\{\xi\}$ and when the point corresponds to Jost point. It follows from WLC condition that

$$\sum_{\hat{\Psi},\hat{\Psi}}\sum_{\alpha,\beta,\lambda,\phi_1,\phi_2,\phi_3,\phi_4}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta} = \sum_{\hat{\Psi},\hat{\Psi}}\sum_{\alpha,\beta,\lambda,\phi_1,\phi_2,\phi_3,\phi_4}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta}\tag{121}$$

This is a conformal bootstrap equation. We may discuss an $s-u$ crossing relation and obtain a bootstrap equation. The point is to note that two corresponding Wightman functions are permutation of field configuration of one another as alluded to in Section 2.3. The four point function of interest is $W_4(x_1,x_2,x_3,x_4) = <0|\phi(x_1)\phi(x_2)\phi(x_3)|0>$. If we follow the known prescription we obtain the following equation.

$$<0|\phi(x_1)\phi(x_2)\phi(x_3)|0> = \sum_{\zeta} <0|\phi(x_1)\phi(x_2)|\hat{\zeta}> <\hat{\zeta}|\phi(x_3)|0>\tag{122}$$

As before, $\hat{\zeta}$ is a complete set of operators created from a complete set of intermediate set of states. Notice that at the Jost point, $(x_3-x_4)^2 < 0$ the two equations (118) and (122) are equal. Now the bootstrap equation, at the Jost point, is

$$\sum_{\hat{\Psi},\hat{\Psi}}\sum_{\alpha,\beta,\lambda,\phi_1,\phi_2,\phi_3,\phi_4}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta}\mathcal{W}^{\alpha\beta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\alpha\beta} = \sum_{\zeta,\gamma,\delta}\sum_{\phi_1,\phi_2,\phi_3,\phi_4}\mathcal{W}^{\gamma\delta}_{\phi_1,\phi_2,\phi_3,\phi_4}\lambda_\phi^{\gamma\delta}\lambda_\phi^{\delta}\tag{123}$$
Each expression on \textit{l.h.s} and \textit{r.h.s} of the above equation corresponds to a boundary values of an analytic function. Moreover, each analytic function is defined in a domain corresponding to its extended tube. The two analytic functions coincide at the Jost point as expressed in (123). Thus by invoking the edge-of-the-wedge theorem we conclude that the two (Wightman) functions are analytic continuation of each other.

We have alluded to the importance and significance of this equation. Let us deliberate on a few points. Notice that the equation holds when points are separated by spacelike distance. If we considered Fourier transform of the two Wightman functions, \( W_4(\xi_1, \xi_2, \xi_3) \) and \( W_4(\xi_3, \xi_2, \xi_1) \) the conjugate momenta are spacelike i.e. for \( \xi_3^2 < 0 \). Therefore, the two functions coincide in the region where momenta are spacelike. This is the situation in case of the absorptive amplitudes of \( s \) and \( u \) channels of a scattering amplitude derived from the LSZ reductions. The absorptive parts of the two amplitude are equal when momenta are real and lie in the unphysical region. It was necessary to prove that the two absorptive parts are analytic continuation of each other. The situation is the same here. It is necessary to identify the domain of analyticity of each of the Wightman functions and argue that they are analytic continuation of each other. We have proved this through the chain of arguments presented earlier. We have persuasively argued in the previous section that a pair of Wightman functions are holomorphic in the union of their extended tubes. We think that the PCT theorem together with its equivalence of WLC provide a very strong basis to derive the conformal bootstrap equation. It is needless to emphasize the crucial role played by the edge-of-the-wedge theorem.

\textit{Remark:} The above bootstrap condition is not specifically valid for a nonderivative scalar conformal field theory. If we consider four point Wightman function, for nonderivative conformal fields, which belong to finite dimensional irreducible representation of conformal group then the above proof will go through with appropriate modifications. Now the corresponding Wightman function will carry tensor indices as the fields would transform according to the representations of \( SL(2\mathbb{C}) \otimes SL(2\mathbb{C}) \) and they will carry their conformal dimensions \cite{5}. Thus the \( W_4 \) will transform covariantly under \( SL(2\mathbb{C}) \otimes SL(2\mathbb{C}) \). The preceding arguments will essentially go through with adequate technical modifications only. Consequently, the analyticity properties and bootstrap equations will continue to hold for the general four point functions as long as the fields are of nonderivative types. Therefore, we conclude that the two resulting four point functions, in general, will be analytic continuation of each other.

To briefly summarize the contents of this section: (i) We considered OPE of a pair of nonderivative scalar conformal fields and expanded them in a set of composite (also nonderivative) conformal fields. Next, we considered OPE for the commutator of the pair of fields and noted that the commutator vanishes when they are separated by spacelike distance. Then we argue that the matrix element of the Fourier transforms
of the difference of the two composite field operators enjoy certain support properties. Consequently, a representation, for the matrix elements could derived which is analogous to Jost-Lehmann-Dyson representation. The analyticity properties of the matrix elements are discussed. The connections with bootstrap equation are presented.

In the second part, we invoked PCT theorem to aim at derivation of the bootstrap equation. The equivalence of weak local commutativity (WLC) with the PCT theorem plays a crucial role in arriving at the bootstrap equation. We went through a number of steps. It is important, in our view, to note that although the two four point Wightman functions coincide at the Jost point, one has to demonstrate that the two four point functions are analytic functions. It is necessary to identify their domain of holomorphy and then invoke the edge-of-the-wedge-theorem to show that these two functions are analytic continuation of each other.

4. Summary and Conclusions.

The objective of this investigation was to study analyticity and crossing properties of four point correlation functions of conformal field theory. We adopted Wightman’s formulation. It is well known that Wightman axioms are not respected by all conformal field theories. Therefore, we considered a nonderivative real scalar field, \( \phi(x) \), which has the desired properties [41]. We invoked the arguments that Wightman functions are boundary values of analytic functions of several complex variables and focused on the four point function. The primitive domain of analyticity of the corresponding analytic function was identified to be the forward tube, \( T_3 \) since the four point function depends on three independent coordinates due to translational invariance. Next, the extended tube, \( T_3' \) was defined following the standard procedure where the function is analytic and is single valued. A permuted four point Wightman function was considered. The two four point functions are related by crossing. As a prelude, we presented analyticity and crossing properties of the three point function in some detail incorporating essential results of [55].

Moreover, we have studied the analyticity properties of three point function in the momentum space representation. The work of Jost [73] was crucial. The R-product of three point function was shown to be related to Wightman functions. We utilized this fact to investigate analyticity of momentum space three point function.

Next, we analyzed the crossing properties of four point function. In general, each of the permuted Wightman function is boundary value of the corresponding analytic function. The domain of analyticity is union of the domain of holomorphy of four point functions. However, the entire domain of holomorphy of the collection all the Wightman function might be much larger. We have considered a pair of Wightman
functions at a time and then found the domain of holomorphicity. The Fourier transforms of the two four point functions were considered and we read off their support properties in the momentum space. We adopted the prescriptions of [55] to derive representation of the Fourier transform of the four point Wightman function. This representation is similar to the Jost-Lehmann-Dyson representation derived for the matrix element of causal current commutators in field theory. It might be useful to recall a few important results on analyticity of four point scattering amplitude from axiomatic field theory viewpoint. One of the most important result is the existence of the Jost-Lehmann-Dyson representation [29, 30]. This is derived from microcausality and temperedness of the amplitude. The next important result for the proof of the existence of Lehmann ellipses which crucially depends on Jost-Lehmann-Dyson result. Moreover, the polynomial boundedness of the nonforward scattering amplitude follows from the linear program without invoking unitarity of S-matrix. It is, sometimes, stated that polynomial boundedness and existence of Lehmann ellipses are underlying assumptions in the proof of dispersion relations. However, it is not the case - the two properties are proved from general axioms. Martin invoked the nonlinear constraint (unitarity of S-matrix) to demonstrate the existence of enlarged domain of analyticity, i.e. existence of an ellipse known as Lehmann-Martin ellipse. The celebrated Froissart bound is derived rigorously from axioms of general field theory without any extra assumptions. In other words, polynomial boundedness of the amplitude, existence of Lehmann ellipses and enlarged domain of analyticity (the existence of Martin ellipse) have been proved. Consequently, the fixed-t dispersion relations are proved in the domain mentioned above. The purpose of this digression is to discuss analyticity of the Wightman function (say four point function) in the light of preceding remarks. First of all, in conformal field theories, we deal with Green functions and not scattering amplitudes. The 'external' legs are off the mass shell. Polykov [15] has discussed this aspect very succinctly while deriving his bootstrap equation. All the external legs have large spacelike momenta. The momentum transfers are spacelike and he assigns large spacelike values to some of them while computing discontinuity to obtain the desired equation. In order to derive a dispersion relation, we propose the following strategy. Bogoliubov [28] had proved fixed-t dispersion relation by taking external legs to large spacelike region and then analytically continued the function to onshell values. It might be worth while to adopt his prescription. However, we are aware that we do not deal with a scattering amplitude, there is no passage to the mass shell in a rigorous manner. We may conjecture, optimistically, that a dispersion relation might be proved for the conformal field theory in future. We turn to crossing symmetry. Indeed, starting from this point, we get insights into crossing symmetry for a pair of permuted Wightman function at a point where the two functions coincide. Next step was to derive the domain of analyticity in the coordinate space. We depended on two important results. First, the Hall-Wightman theorem was invoked to argue that the analytic function depend on Lorentz invariants constructed from the complexified four vectors. The second ingredient was to appeal to the Jost theorem. As far as
we are aware, the envelope of holomorphy have not been constructed for four point Wightman functions completely in QFT [52, 53, 54] i.e. as exhaustively as for the three point function [51]. Our principal goal was to establish crossing for four point functions. Therefore, it suffices, for our purpose, to consider a pair of Wightman functions and identify the domain of holomorphy. Consequently, we can proceed to prove crossing for Wightman functions taken pairwise. There was one more technical obstacle. In case of the three point functions there were only two Hall-Wightman Lorentz invariant complex variables for all practical purposes (although there are three of them $z_{11}, z_{22}, z_{12}$; and crossing was proved [55]). Note that for the four point functions, the number of Hall-Wightman variables increase and the prescription of [55] does not go through in a straightforward manner. Therefore, we simplified the task a little bit and derive constraints on (real) Hall-Wightman invariants since we go to a domain where Jost theorem is applicable. Thus in Section 2, we established crossing for a pair of permuted Wightman functions.

The third section was devoted to study analyticity properties of the matrix elements of composite operators which arise in the OPE. In this section the microcausality plays a crucial role in establishing the analyticity properties. The OPE of a pair of nonderivative conformal fields has been investigated by Mack [41]. In the OPE, the composite fields are also of nonderivative type and they belong to the irreducible representations of the conformal group. Consequently, they respect Wightman axioms. We considered, OPE of the commutator of the two scalar fields. Thus we obtain difference of two composite fields as has been demonstrated in section 3. We considered the Fourier transforms and then took matrix elements of the composite operators. This matrix element is constrained from the microcausality arguments. The constraints are the same as those utilized to derive the Jost-Lehmann-Dyson representation (JLD) [29, 30]. We recall that, JLD representation was crucial to derive crossing in QFT. Moreover, we use the techniques developed in section 2 to study the analyticity properties of these matrix elements. We presented a derivation of the conformal bootstrap equation from a novel perspective. We invoked two very powerful theorems of axiomatic field theory to accomplish our goal. We first appeal to the PCT theorem. PCT theorem is profound and is respected by all local field theories which obey Wightman axioms. We noted that two Wightman functions are equal if they are PCT transform of each other. We utilized the equivalence of PCT theorem and weak local commutativity which was rigorously proved by Jost. Therefore, we were able to relate two Wightman functions at Jost point. Next, we presented a series of steps to relate the two Wightman functions. The conformal partial wave expansion technique was applied to arrive at the conformal bootstrap equation. It is not adequately emphasized that the equality between two four point functions holds when a pair of coordinates, for their real values, are separated by spacelike distance. It is essential to prove that the four point functions are analytic at that point. Moreover, from the perspectives of Wightman axioms, it is required that we identify their
respective extended domain of holomorphicity of the two functions in the complex domain. Dyson [93] and Ruelle [94] have proved the analyticity of the Wightman functions at Jost point in the context of WLC [62]. Furthermore, as has been argued in section 3, the bootstrap equation is not special to the case of Wightman function of four scalar field. Indeed, a four point functions can be defined as product of four conformal fields belonging to irreducible representations of the conformal group, in a general scenario, as long as they satisfy Wightman axioms. Then the corresponding bootstrap equation holds. It is important to note that three fundamental theorems of axiomatic local field theories such as PCT theorem, the theorem stating equivalence between PCT theorem and weak local commutativity and the edge-of-the-wedge theorem, are invoked to derive the conformal bootstrap equation rigorously. We presented arguments for the $s - u$ bootstrap equation.

We conclude the article with following remarks. It is tempting to suggest that crossing and analyticity of n-point function can be derived following the techniques introduced here. It must be noted that proof of crossing and analyticity for n-point functions in axiomatic QFT is a formidable task. Bros [97] has comprehensively reviewed the progress on crossing and analyticity properties of n-point amplitude in axiomatic QFT at that juncture. Our understanding is that several issues have remained unresolved in this topic. Another avenue to explore, in order to derive crossing relations for n-point functions in the context of CFT, is to follow a clue from QFT. We should look for a generalization of Jost-Lehmann-Dyson representation for the n-point function in CFT. The JLD representation was the principal ingredient since the coincident region was identified through this technique. Thus the singularity free region was identified. There have been attempts, in the past, to obtain integral representations for the VEV of product of field operators and VEV of the commutators of string of field operators in the axiomatic QFT [66, 68]. Therefore, those results might be utilized to prove crossing for, at least, a pair of permuted Wightman functions. However, we have not established existence of JLD representation to n-point functions in CFT. Therefore, it might be premature to speculate that problem of crossing and analyticity could be solved in a straight forward manner in CFT for n-point functions. However, conformal symmetry is very powerful. It is well known that two point and three point functions get fixed in CFT up to constant coefficients. Moreover, conformal symmetry imposes constraints on the structure of the n-point functions ($n > 3$). In principle, given the two point and three point functions of a theory, the structure of n-point functions could be inferred. There are reasons to be optimistic that CFT might provide more insights into the analyticity and crossing properties of n-point functions.\(^5\).

\(^5\)Witten has suggested that it might be worth while to investigate questions about analytic continuations and crossing properties of n-point correlation functions of conformal field theories. [98]
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