von Neumann Lattices in Finite Dimensions Hilbert spaces

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Abstract

The prime number decomposition of a finite dimensional Hilbert space reflects itself in the representations that the space accommodates. The representations appear in conjugate pairs for factorization to two relative prime factors which can be viewed as two distinct degrees freedom. These, Schwinger’s quantum degrees of freedom, are uniquely related to a von Neumann lattices in the phase space that characterizes the Hilbert space and specifies the simultaneous definitions of both (modular) positions and (modular) momenta. The area in phase space for each quantum state in each of these quantum degrees of freedom, is shown to be exactly $\hbar$, Planck’s constant.

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I Introduction

Studies of finite dimensional Quantum Mechanics were undertaken in the early days of the development of quantum mechanics by Weyl [1] who showed thereby the connection of the commutation relation to Schrodinger’s wave equation. A systematic study of this, finite dimensional quantum mechanics problem, was initiated 30 years later by Schwinger [2]. His work introduced to the field some of the basic issues that are under intensive investigations in the most recent literature. Among these are the relevance to the physics of finite dimensional quantum mechanics of whether the dimensionality, $M$, is a prime number or is made up of product of distinct primes. In the latter case, which is our main concern in the present work, Schwinger [2] noted what he termed ”quantum degrees of freedom” (see also in [3]) which pertain to the relatively prime number factors of $M$ - these are given a concrete representation in this paper. He also considered sets of complete and orthonormal bases
that span the M- dimensional space and which are what he termed of "maximum degree of incompatibility". Such vector bases under their modern name of "mutually unbiased bases" (MUB)[4, 5] or "conjugate bases" [6, 7] have led, with the recent developments in our understanding of the foundation of quantum mechanics, to intensive research in a great variety of problems where implementations seems to involve finite dimensions. These include the pioneering studies of von Neumann lattices and magnetic orbitals on a finite phase space [8], quantum measurements [9], teleportation [10] and others. A cogent review is given, e.g., by Vourdas [11].

The kq representation was introduced by Zak [12, 13] in his study of the continuum (infinite dimensional space) to conveniently handle electrons in a periodic potential subjected to an external magnetic field. Zak’s work was based on Weyl’s [1] unitary displacement operators in both coordinates and momenta which, for some particular choice of the parameters involve commuting operators whose eigenvalues may be labelled by both the space position and the momentum of the electron. (No violation of the uncertainty principle incurs since the values of both the coordinates and momenta are modular, cf. [14].) The Zak transform formalism proves of wide use also in signal processing problems [15] where the corresponding modular variables are frequency and time. The transform is of considerable theoretical interest in conjunction with it being parametrized by both the coordinate and momentum. Only recently [16] the conjugate basis to the kq rep was explicitly given. The finite dimensional kq representation was formulated in [17]. This can be done only for cases wherein the dimensionality, M, of the Hilbert space is not prime [17, 19, 18]. For extended dimensionality (i.e. $M \to \infty$), already at the very early stages of the development of quantum mechanics von Neumann [20] suggested a physical way of accounting for states in phase space by discretizing the phase space with an area of $\hbar$ (in the units $\hbar = 1$ this equals $2\pi$) for each state. This problem was studied extensively, e.g. [21]. Recently it was shown [22] that on a finite phase plane the kq-coordinates and the sites on a von Neumann lattice are closely related.

In his studies of finite dimensional Quantum Mechanics, Schwinger [2, 3] showed that if M, the dimensionality of the Hilbert space under consideration, is not a prime or a power of a prime, then states in the space may be viewed as having "quantum degrees of freedom". Thus, if $M = M_1 M_2$ with $M_1$ and $M_2$ relative primes, the state may be considered as accounting for two distinct degrees of freedom one of dimension $M_1$ and the other of dimension $M_2$. This mapping of the the one degree of freedom, M dimensional Hilbert space on a line (i.e. spanned by $|q\rangle$, $q = 1, \ldots, M$) onto two (Schwinger’s “quantum”) degrees of freedom on a torus (spanned by $|q_1, q_2\rangle$, $q_1 = 1, \ldots, M_1$; $q_2 = 1, \ldots, M_2$) and their conjugate representations (reps- henceforth) were introduced earlier [19]. These reps are reviewed herewith and the equivalence of these partially localized state (PLS) and states over the von Neumann lattice is established. Thus we show that a “quantum degree of freedom”, much like a proper degree of freedom, occupies a phase space area equals to $\hbar$, Planck’s constant. These states

1strictly speaking factorizability of the dimensionality to relatively prime factors is required for the definitions of the mapping from the discrete line to the discrete torus.
are distinct form the finite dimension Zak’s kq rep [13] although when \( M_1, M_2 \) are co primes both bases are eigenstates of the commuting modular operators, one pertaining to the position (q) and the other to the momentum (k). The latter (i.e. Zak’s) involve Bloch like symmetry in one of the variables. The phase relation between Zak’s reps (kq) and the reps used by us when both are applicable (i.e. when the factors \( M_1, M_2 \) are co-primes) is given in the appendix. The present work utilizes Schwinger’s [2, 3] quantum degrees of freedom for finite dimensional Hilbert space to consider states with two quantum degrees of freedom as being labelled by the position of one and the momentum of the other.

The paper is organized as follows. The next two sections, section II and III outline some previous results and briefly derive the formulae needed in our later discussion. Section IV contains our main result, i.e. relating partially localized states (PLS), the kq reps states and states on von Neumann’s lattices. These relations bring to the fore the role of the quantum degrees of freedom that were introduced by Schwinger [2, 3] and give a new meaning to the localization in both the coordinates and momenta in these variables (the price is complete delocalization in their conjugate variables) and relate it to states over the von Neumann lattice. The last section gives the conclusions and some remarks. The appendix gives the phase relation among different representations.

II Mapping of Discrete Line to Discrete Torus

Schwinger [2] showed that M-dimensional vector spaces allow the construction of two unitary operators, U and V (in his notation), which form a complete operator basis. This means that if an operator commutes with both U and V it is, necessarily, a multiple of the unit operator. These operators have a period M, i.e.

\[
U^M = V^M = 1,
\]

where M is the smallest integer for which this equality holds. The eigenvalues of both U and V are distinct: they are the M roots of unity, i.e. with \( |x\rangle \) the eigenfunction of U,

\[
U|q\rangle = e^{i(2\pi M/q)}|q\rangle, \quad |q + M\rangle = |q\rangle, \quad q = 1, ..., M.
\]

The operator V is defined over these eigenvectors as

\[
V|q\rangle = |q - 1\rangle.
\]

Schwinger then showed that the absolute value of the overlap between any eigenfunction of U, \( |q\rangle \) and any one of V, \( |k\rangle \), is a constant :

\[
|\langle k|q\rangle| = \frac{1}{\sqrt{M}}.
\]

Vector bases with this attribute are referred [4] to as mutually unbiased (MUB) or conjugate vector bases (we use these terms interchangeably).
A specific example of the $M$-dimensional space is the following: consider $M$ points on a line, i.e., consider discretized and truncated spatial coordinate $x$ and its conjugate momentum $p$ as our $M$-dimensional space. This may be realized by imposing boundary conditions on the spatial coordinate, $x$, of the wave functions under study, $\psi(x)$, and on their Fourier transforms, $F(p)$ (we take $\hbar = 1$) [13]:

$$\psi(x + Mc) = \psi(x), \quad F(p + \frac{2\pi c}{M}) = F(p).$$

Here $M$ is an integer - it is the dimensionality of the Hilbert space, and we term $c$ the "quantization length". As a consequence of the above boundary conditions we have that the value of the spatial coordinate, $x$, and the value of the momentum, $p$, are discrete and finite:

$$x = sc, \quad s = 1, ..., M; \quad p = \frac{2\pi}{Mc}t, \quad t = 1, ..., M.$$  

In this case we may replace the operators $x$ and $p$ by the unitary operators

$$\tau(M) = e^{i\frac{2\pi}{Mc}x}; \quad T(c) = e^{ipc}. \quad (4)$$

These operators satisfy the basic commutator relation

$$\tau(M)T(c) = T(c)\tau(M)e^{-i\frac{2\pi}{Mc}}. \quad (5)$$

They exhibit the dimensionality (i.e. periodicity) automatically

$$[\tau(M)]^M = [T(c)]^M = 1, \quad (6)$$

and we may associate Schwinger’s operator $U$ with $\tau(M)$ and his $V$ with $T(c)$ (henceforth we take $c=1$).

For our analysis where we take $M$ as as factorizable: $M = M_1 \cdot M_2$, it is convenient to represent the number $M$ in terms of prime numbers, $P_j$,

$$M = \prod_{j=1}^{N} P_j^{n_j}, \quad P_j \neq P_i, \quad j \neq i, \quad (7)$$

where the $n_j$ are integers, and more concisely we denote $P_j^{n_j}$ by $m_j$, i.e.

$$M = \prod_{j=1}^{N} m_j. \quad (8)$$

We find thus that the greatest common divisor (gcd) among the $m_j$s is 1:

$$gcd(m_j, m_i) = 1, \quad \forall j \neq i, \quad (9)$$
i.e. distinct \( m_i \)'s are relatively prime. In our study we consider bi partitioning of the product that represents \( M \) (Eq. (8)) into two factors,

\[
M = M_1 M_2. \tag{10}
\]

Here \( M_1 \) incorporates one part of the \( N \) factors of Eq. (8) and \( M_2 \) contains the other part. Our way of bi-partitioning implies that the two numbers, \( M_1 \) and \( M_2 \), are relatively prime, viz. \( \gcd(M_1, M_2) = 1 \). In our discussion of the kq representation [12, 17, 19] the above was used to show that the number of kq representations, \( \chi(M) \), each with its conjugate, which form a complete basis can be accommodated in the \( M \) dimensional space, is simply related to the number of primes, \( N \), that appear in \( M \):

\[
\chi(M) = 2^{N-1}. \tag{11}
\]

(It should be noted that the familiar finite dimensional Fourier representation is included in this counting.) The mapping of the \( M \) dimensional, one degree of freedom (a line) to (fake) two degrees of freedom (a torus) whose dimensions are \( M_1 \) and \( M_2 \) can be accomplished in \( \chi(M) - 1 \) ways (the finite dimensional Fourier transformation should not be included). We now introduce:

\[
L_1 = \frac{M}{M_1} = M_2; \quad L_2 = \frac{M}{M_2} = M_1. \tag{12}
\]

This implies that the equation,

\[
q = q_1 L_1 + q_2 L_2 \quad [\text{mod } M]; \quad q = 1, \ldots, M; \quad q_1 = 1, \ldots, M_1; \quad q_2 = 1, \ldots, M_2, \tag{13}
\]

has a unique solution \( q \) for every pair \([q_1, q_2]\), with \( q \) running over its whole range of \( M \) values. We will now [2, 9, 11] modify Eq. (13) to attain this simpler relation among the solutions. This is obtained by applying the Chinese remainder theorem [9] to the solution of the two congruences

\[
\begin{align*}
q &= q_1 \quad [\text{mod } M_1] \\
q &= q_2 \quad [\text{mod } M_2].
\end{align*} \tag{14}
\]

The solution of these is

\[
q = q_1 N_1 L_1 + q_2 N_2 L_2 \quad [\text{mod } M], \tag{15}
\]

with

\[
N_2 = L_2^{-1} \quad [\text{mod } M_2], \quad \text{and} \quad N_1 = L_1^{-1} \quad [\text{mod } M_1]. \tag{16}
\]

For example, with \( M = 15 \), we have

\[
L_1 = 5; \quad L_2 = 3 \rightarrow N_1 = N_2 = 2. \tag{17}
\]

We have then,

\[
e^{i \frac{2\pi}{15}} = e^{i \frac{2\pi}{15} N_1} e^{i \frac{2\pi}{15} N_2} \rightarrow e^{i \frac{2\pi}{9}} x = e^{i \frac{2\pi}{3} N_1} x \cdot e^{i \frac{2\pi}{2} N_2} x, \tag{18}
\]
and,
\[ e^{ip} = e^{ipN_1L_1}e^{ipN_2L_2}. \]

Further, we may label
\[ |q\rangle = |q_1N_1L_1 + q_2N_2L_2\rangle; \]
\[ |k\rangle = |k_1N_1L_1 + k_2N_2L_2\rangle, \]

We have thus that the (complete) operator basis pair may be replaced by the two pairs (note the removal of \(N_i\) from the \(V\) terms) as follows:
\[
\begin{align*}
\tau(M) &= e^{i\frac{2\pi}{M}} \rightarrow \tau(M_1)\tau(M_2) = e^{i\frac{2\pi}{M_1}}e^{i\frac{2\pi}{M_2}} \equiv e^{i\frac{2\pi}{M}} \\
T(1) &= e^{ip}; \rightarrow T(L_1)T(L_2) = e^{ipL_1}e^{ipL_2}. \tag{19}
\end{align*}
\]

While the basis \(|q\rangle\) may be expressed via \(|q_1\rangle|q_2\rangle\) with
\[
\begin{align*}
e^{\frac{2\pi}{M}}|q_i\rangle &= e^{\frac{2\pi}{M}q_i}|q_i\rangle; \quad e^{ipL_i}|k_i\rangle &= e^{\frac{2\pi}{M}k_i}|k_i\rangle. \tag{20}
\end{align*}
\]

Defining
\[
\Delta^{M_i}(x) = \begin{cases} 1, & x = 0 \text{ mode}[M_i] \\ 0, & \text{otherwise} \end{cases}
\]

allows equating,
\[
\begin{align*}
<q|q_1N_1L_1 + q_2N_2L_2> &= \Delta^M(q - q_1N_1L_1 - q_2N_2L_2) = \Delta^{M_1}(q - q_1)\Delta^{M_2}(q - q_2) = <q|[q_1\rangle|q_2\rangle]. \tag{21}
\end{align*}
\]

And,
\[
\begin{align*}
<k|k_1N_1L_1 + k_2N_2L_2> &= \Delta^M(k - k_1N_1L_1 - k_2N_2L_2) = \Delta^{M_1}(k - k_1)\Delta^{M_2}(k - k_2) = <k|[k_1\rangle|k_2\rangle]. \tag{22}
\end{align*}
\]

The formulae
\[
\begin{align*}
|q\rangle &\rightarrow |q_1\rangle|q_2\rangle \\
|k\rangle &\rightarrow |k_1\rangle|k_2\rangle \tag{23}
\end{align*}
\]

with 19 constitutes the transcription of the \(M\) dimensional, one degree of freedom system to the (Schwinger's quantum) two degrees of freedom one of \(M_1\) and the other of \(M_2\) dimensions \((M = M_1M_2)\) which is our mapping of the \(M\) dimensional line to a torus. Our numerical subscripts designates the dimensionality; thus, e.g., \(|q_1\rangle\) refers to the spatial coordinate with \(q_1 = 1,2,...M_1\).
We shall now discuss briefly some attributes of what is termed the representation (rep henceforth). The conjugate basis viz \(|k_1||k_2\rangle\) may be evaluated directly via the result
\[ \langle q|k \rangle = \frac{e^{iqk}}{\sqrt{M}} \]
to give
\[ \langle q_1|\langle q_2|\langle k_1||k_2\rangle = \frac{e^{i(q_1k_1L_1+q_2k_2L_2)N^2}}{\sqrt{M}}, \tag{24} \]
and, by direct evaluation
\[ \langle q_i|k_i \rangle = \frac{e^{i\frac{2\pi}{M}N_iq_ik_i}}{\sqrt{M^i}}, \tag{25} \]
Within the description in terms of two (quantum) degrees of freedom (QDF). E.g. when accounting for \(|q\rangle\) via \(|q_1\rangle|q_2\rangle\) (cf. Eq.(23)) we may construct eigen functions the above operators by Fourier transformation in either one of the variables (we show in the appendix that we need not differentiate between the two forms)
\[ |q_1, k_2;\rangle = \langle q_1|k_2\rangle = \frac{1}{\sqrt{M_2}} \sum_{q_2} e^{i\frac{2\pi}{M_2}k_2q_2N_2} |q_1\rangle|q_2\rangle, \]
\[ = \frac{1}{\sqrt{M_1}} \sum_{k_1} e^{-i\frac{2\pi}{M_1}k_1q_1N_1} |k_1\rangle|k_2\rangle. \tag{26} \]
One can readily check that these are indeed the eigenfunctions of the operators Eq.(42) with the same eigenvalues. In this form the state \(|q_1, k_2;\rangle\) can be described as partially localized (PLS) as we have
\[ \langle q_1|\langle q_2|q_{01}, k_{02};\rangle = \frac{\Delta^{M_1}(q_1 - q_{01})e^{i\frac{2\pi}{M_1}k_2q_2N_2}}{\sqrt{M_2}}, \tag{27} \]
thus it is localized in the \(q_1\) variable while completely delocalized in the \(q_2\) variable. (In [17] we used \(e^{ipM_2}\) instead of the present \(T(N_1L_1)\); these two operators have the same eigenvalues and eigenstates, but enumerated differently.)
In the next section we expand our presentation, [22], to show that these states are states over the von Neumann lattice.

### III von Neumann lattices

We now consider the von Neumann lattice for the M dimensional Hilbert space. To this end we first describe the phase space of our system: The spatial like coordinates, \(q\), are now discrete, \(q = 0, 1, ..., M - 1\) (this in dimensionless units: \(q = cM\) is the "size" of our Hilbert space) and label the eigenfunction \(|q\rangle\) of the operator \(exp[i\frac{2\pi}{M}x]\). These label the abscissa of our phase space. The momentum like coordinates are \(p = \frac{2\pi}{M}k\); \(k = 0, 1, ..., M - 1\), the label \(k\) of \(|k\rangle\), the eigenfunctions of \(exp[i\frac{2\pi}{M}p]\) are used for the ordinate of our phase space.
We now consider the case wherein $M = M_1 \cdot M_2$, $gcd[M_1, M_2] = 1$. Since as $k$ runs from 1 to $M$ the momenta runs from $p = \frac{2\pi}{M} c \bar{h}$ to $\frac{2\pi}{c} \bar{h}$, the "distance" $\delta k = 1$ represents $\delta p = \frac{2\pi}{M} \bar{h}$. (In dimensionless units - i.e. $c = \bar{h} = 1$ - $p$ runs from $\frac{2\pi}{M}$ to $2\pi$ and $\delta p = \frac{2\pi}{M}$.)

The "distance" $\delta q = 1$ along the abscissa is, obviously, $c(1$ in dimensionless units). In this way we see that each "point" marked on our phase space by $(q,k)$ may be viewed as representing an "area", $\delta q \delta k = \frac{2\pi}{M}$.

Our aim in this section is to show that $|q_1, k_2\rangle$ states, i.e. the eigen functions of $e^{i\frac{2\pi}{M}x}e^{i[pM_1]}$ as given above, Eq. (42) have the attractive physical properties of being states over the von Neumann lattice, a term that will also defined herewith.

A von Neumann lattice in this phase space are the $M$ points whose coordinates are

$$ q = nM_1, k = mM_2, n = 0, 1, 2..., M_2 - 1, m = 0, 1, 2...M_1 - 1. \tag{28} $$

A displaced von Neumann lattice are the $M$ points - here the points are defined mode[$M$].

$$ q = q_{01} + nM_1; \quad n, k_2 = 0, 1, 2..., M_2 - 1, $$
$$ k = k_{02} + mM_2; \quad m, q_{01} = 0, 1, 2...M_1 - 1. \tag{29} $$

We now define a "state over a von Neumann lattice" to be the state whose density matrix representative, $\rho$ is given by

$$ |\langle q | \rho | k \rangle| = \begin{cases} 
\frac{1}{\sqrt{M}} & \text{on a lattice point} \\
0 & \text{otherwise}. \end{cases} \tag{30} $$

(This requirement implies that only $M$ out of the $M^2$ matrix elements are non vanishing.) It is easily verified that the partially localized state (PLS) Eq. (27) with localization coordinates at the origin ($q_{01} = k_{02} = 0$) is such a state (summation over repeated indices is implied):

$$ \langle q | \rho | k \rangle = \langle q | q_1 q_2 | \psi \rangle \langle \psi | k_1 k_2 \rangle \langle k_1 k_2 | p \rangle $$
$$ = \Delta(x - q_1 N_1 L_1 - q_2 N_2 L_2) \frac{\Delta M_1(q_1)}{\sqrt{M_2}} \frac{\Delta M_2(k_2)}{\sqrt{M_1}} \Delta(p - k_1 L_1 - k_2 L_2). \tag{31} $$

Thus the PLS is a state over a von Neumann lattice. For example, $M = 15, M_1 = 3, M_2 = 5; L_2 = 3, N_2 = 2$ leads to non vanishing matrix elements for

$$ q = q_2 N_2 L_2 \rightarrow q = 0, 6, 12, 3, 9; \quad k = k_1 L_1 \rightarrow k = 0, 5, 10. $$

We take it as obvious that with each von Neumann lattice we have its conjugate obtained by interchanging $q$ with $k$.

To study the case of displaced von Neumann lattices we consider briefly the PLS with (at least) one coordinate not at the origin. For example

$$ \langle q_1 q_2 | \psi \rangle = \frac{\Delta M_1(q_1 - q_{10})e^{i[q_2 k_{20} \bar{h}^2]}}{\sqrt{M_2}}; \quad q_{01}, k_{02} \neq 0. \tag{32} $$
The non vanishing elements of the density matrix are now the M points

\[
q = q_{01}N_1L_1 + q_{02}N_2L_2; \quad q_2 = 0, \ldots, M_2 - 1; \\
k = k_1L_1 + k_02L_2, \quad k_1; = 0, \ldots, M_1 - 1.
\]  

(33)

This is a von Neumann lattice with its origin shifted to \(q_{01}N_1L_1, k_{02}L_2\). Note that corresponding to PLS \(|q_{01}, k_{02}\rangle\) the non vanishing terms involve the coordinates of the conjugate state, \(|k_1, q_2\rangle\). This corresponds to a localized state at x say implies all momentum states equally probable. This state is orthogonal to the von Neumann state considered above. Indeed each of the M pairs of coordinates \(q_{01}, k_{02}\) represents a state over the von Neumann lattice with its origin shifted to the designated phase space point. These states are easily shown to be orthogonal. Thus, considering the overlap between arbitrary such states (summation convention implied)

\[
\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | q_1 q_2 \rangle \langle q_1 q_2 | \psi_2 \rangle \\
= \frac{\Delta^{M_1}(q_1 - q_{01}) e^{i(q_2 k_{02} \frac{\pi}{2})} \Delta^{M_2}(q_1 - q_{01}) e^{-i(q_2 k_{02} \frac{\pi}{2})}}{\sqrt{M_1} \sqrt{M_2}} \\
= \Delta^{M_1}(q_0 - q_{01}') \Delta^{M_2}(k_{02} - k_{02}').
\]  

(34)

Thus these M states span the space and form a complete orthonormal basis. We shall show below that these are the PLS considered above. In closing this section we wish to emphasize that by construction each von Neumann lattice point represents an area of \(2\pi\) in our phase space diagram. The total number of such points within a rectangle labelled by \((q_{01}, k_{02})\) is M. Thus, e.g. for the rectangle labeled by \((q_{01} = 0, k_{02} = 0)\) includes the phase space coordinates within the area contained in the rectangle \((0, 0); (M_1, 0); (0, M_2); (M_1, M_2)\). There are M possible values for \((x | \rho | p)\) within this rectangle. Thus the \(2\pi\) area that contain M points - each labels a state over the von Neumann lattice (there are M such states) each state includes M ”points”: one in each rectangle - this point is discussed further below.

We now show that PLS are \(kq\) states, i.e. are the eigen functions of

\[e^{i \frac{2\pi}{M_2} k_q} e^{i p M_1} \]

(repeated indices are summed over):

\[
\langle q_1 q_2 | e^{i \frac{2\pi}{M_1} q_1} e^{i p M_1} | \psi \rangle = e^{i \frac{2\pi}{M_1} q_1} \langle q_1 q_2 | k_1 k_2 \rangle e^{i \frac{2\pi}{M_2} k_2} \langle k_1 k_2 | \psi \rangle \\
= e^{i \frac{2\pi}{M_1} q_1} e^{i \frac{2\pi}{M_2} k_2} \langle q_1 q_2 | \psi \rangle.
\]  

(35)

We have then that the \(|k_1 q_2\rangle\) rep (as well as their concomitant conjugates, \(|k_2 q_1\rangle\)), PLS (at the corresponding points) and the states over the von Neumann lattice (shifted to the corresponding spots), are one and the same states. (We note that these states differ from Zak’s [12] states. The latter are defined regardless whether \(M_1, M_2\) are co primes or not.) The completeness and orthogonality of the \(|k_1 q_2\rangle\) states assures their validity for the other
two. The states over the von Neumann lattice were shown to occupy precisely an area of $h$, Planck’s constant in phase space for the finite dimensional case. While the coordinates of the non vanishing density matrix for the states (partially) localized at $k_{02} = q_{01} = 0$ were at the appropriate von Neumann lattice (as well as their conjugates $k_{01} = q_{02} = 0$). We thereby accounted for the connection between the $q_1, q_2$ reps and von Neumann lattices that were obtained recently [22].

We now return to the more general case: we consider an arbitrary pair with partially localized phase points: $q_{01}, k_{02}$. These, in turn, lead to different marked points in our phase space - it specifies a lattice point within the rectangles considered above and may serve as a label for a partially localized state localized at $q_{01}$ and having quasi momenta $k_{02}$. One can readily verify that this state may be viewed as a state over the von Neumann lattice where the M points for which the the matrix elements 

$$\langle k | q_{01}, k_{02} \rangle \langle q_{01}, k_{02} | q \rangle$$

do not vanish are given by,

$$q = q_{01} + nM_1, \ k = k_{02} + mM_2, \ n = 0, 1, 2..., M_2 - 1, \ m = 0, 1, 2...M_1 - 1. \ (36)$$

Thus these states are states over the von Neumann lattice with the whole lattice origin being shifted to the point $q_{01}, k_{02}$.

In closing this section we would like to point out that the results above are based on the particular choice of labelling, viz. on having achieved a mapping of the M dimensional line geometry ($q = 1, 2...., M$) to a torus like geometry with one torus radius $M_1$ dimensional ($q_1 = 1, 2,...M_1$) and the other $M_2$ dimensional ($q_2 = 1, 2,...M_2$). This was shown to be possible for $M = M_1M_2$ with $\gcd[M_1, M_2] = 1$.

**IV Concluding Remarks**

In a finite, M dimensional Hilbert space with $M = M_1M_2$ and $M_1, M_2$ are co primes i.e. $g.c.d \ [M_1, M_2] = 1$, one may, ala Schwinger, view the states as made of two quantum degrees of freedom (QDF): one of dimensionality $M_1$ and the other $M_2$. This was utilized to consider the $|q_1, q_2\rangle$ with $q_1$ being $M_1$, while $q_2$ the $M_2$ dimensional coordinates. We analyzed the two QDF states as eigenfunctions of the two commuting operators $e^{ip_{M_2}}, e^{i\frac{\pi}{M_2}z}$, which, in the present context may be viewed as referring to the two QDF. We showed that the eigenstates, of these operators may be obtained via Fourier transformation in one of the QDF. e.g. $|k_1, q_2; C\rangle$ is the Fourier like transform in one of these QDF variables ($q_1$ in this example). (Either of these sets spanning the M dimensional space under study.)

Thus the basis $|k_1, q_2; C\rangle$ specify $k_1$ as the (modular) momenta of the $M_1$ dimensional QDF and $q_2$ the (modular) position of the second, $[mode \ M_2]$, variable. (Hence the conjugate basis is $|k_2, q_1\rangle$ with analogous meaning for the subscripted labels.) The M orthogonal basis
vectors span the space - as does the conjugate basis. These states are termed partially localized states as therein (in the above example) the second QDF is localized while the first has its momenta defined. We termed these states partially localized states (PLS).

We discuss briefly the distinction of these from Zak’s kq states in the appendix. In this, finite dimensional quantum mechanics, one may draw the phase space of the system as a whole (i.e. M dimensional) by marking the abscissa by the eigenvalues of \( \exp[i\frac{2\pi}{M}x] \) by the discrete, position like, eigenvalue of the state \(|q\rangle\), \( (q = 0, 1, 2, ..., M - 1) \), and the ordinate by eigenvalues of \( \exp[ip] \) by the momentum like eigenvalue of the state, \(|p\rangle\), \( (p = \frac{2\pi}{M}k; k = 0, 1, 2, ..., M - 1) \). (Giving thereby \( M^2 \) points forming a square, with each point designating an area of \( \frac{2\pi}{M} \)) The von Neumann lattice in this phase space and pertaining to the factorization \( M = M_1M_2; \ \text{g.c.d} \ [M_1, M_2] = 1 \), is given by the points

\[
q = nM_2, \ k = mM_1; \ n = 0, 1, ..., M_1 - 1, \ m = 0, 1, ..., M_2 - 1.
\]

We refer to these points as vn points. Clearly each vn point designates an area containing M phase space points i.e. it designates an area of \( 2\pi \). (Recall that we work in units of \( \hbar = 1 \), i.e. the area in more physical units is \( \hbar \), Planck’s constant.) We then defined ”states over the von Neumann lattice” as those density matrices whose only non vanishing matrix elements are

\[
\langle x|\rho|k\rangle = \frac{1}{\sqrt{M}}; \ x, p \text{ on von Neumann lattice}.
\]

Since the total number of von Neumann lattice points is M and each point occupies and area of \( \frac{2\pi}{M} \) each von Neumann state occupies exactly an area of \( 2\pi \) (i.e. an area of \( \hbar \), Planck’s constant). We then showed that the state \(|k_1 = 0, q_2 = 0\rangle\) (as well as its conjugate state) are states over von Neumann lattice and thus occupy an area of \( \hbar \), Planck’s constant. This led to the demonstration that concomitant to this state over the von Neumann lattice each of the other vectors in this basis designates a shifted von Neumann lattice - the shift being by the coordinates of the designated vector. Thus the vector \( k_1 = k_01 \neq 0, q_2 = q_02 \neq 0 \) is a state over the von Neumann lattice shifted to \((k_01 \text{ [mode } M_1], q_02 \text{ [mode } M_2])\). Here too the area occupied by each of the M points is \( \frac{2\pi}{M} \).

Thus the states over von Neumann lattice are labelled by the the coordinates of one of the QDF and by the momentum of the other.

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Appendix: Phase relation among different reps

Utilizing the general Definitions, Eqs.(10) to Eq.24) one obtains directly

\[ <q|q_1N_1L_1+q_2N_2L_2> = \Delta^M (q - q_1N_1L_1 - q_2N_2L_2) = \Delta^{M_1} (q - q_1) \Delta^{M_2} (q - q_2) = <q|[q_1 > |q_2 >], \]

\[ <k|k_1N_1L_1+k_2N_2L_2> = \Delta^M (k - k_1N_1L_1 - k_2N_2L_2) = \Delta^{M_1} (k - k_1) \Delta^{M_2} (k - k_2) = <k|[k_1 > |k_2 >]. \]

We also have

\[ <q' + q_2'N_2L_2|q_1N_1L_1 + q_2N_2L_2> = \Delta(q' - q_1N_1L_1 - (q_2 - q_2'N_2^{-1})N_2L_2) = \Delta^{M_1} (q'_1 - q_1) \Delta^{M_2} (q'_2 - q_2 + q_2'N_2^{-1}). \]

The above implies also shift operators as follows:

\[ e^{i\frac{2\pi}{M_1}L_1} |k > = |k + 1 >; \quad e^{i\nu} |q > = |q - 1 >. \]

\[ e^{i\frac{2\pi}{M_2}N_2} |k_1 > = |k_1 + M_2 >; \quad e^{ipL_1} |q_1 > = |q_1 - M_2 >. \]

With similar equations for \(|k_2 >, |q_2 >. \)

We now consider the wave functions explicitly

Define

\[ |q_1k_2; C_1 > \equiv \frac{1}{\sqrt{M_1}} \sum_{k_1} e^{-i\frac{2\pi}{M_1}k_1N_1} |k_1N_1L_1 + k_2N_2L_2 > = \frac{1}{\sqrt{M_1}} \sum_{k_1} e^{-i\frac{2\pi}{M_1}k_1N_1} |k_1 > |k_2 > = |q_1 > |k_2 >. \] (38)

We have by direct calculations that (the eigenvalue indices are deleted for brevity),

\[ e^{i\frac{2\pi}{M_1}L_1} |C_1 > = e^{i\frac{2\pi}{M_1}q_1} |C_1 >; \quad e^{ipL_2} |C_1 > = e^{i\frac{2\pi}{M_2}k_2} |C_1 >. \]

And, in similar fashion, define

\[ |q_1k_2; C_2 > \equiv \frac{1}{\sqrt{M_2}} \sum_{q_2} e^{i\frac{2\pi}{M_2}k_2N_2} |q_1N_1L_1 + q_2N_2L_2 >. \] (39)

With,

\[ e^{i\frac{2\pi}{M_1}L_1} |q_1k_2; C_2 > = e^{i\frac{2\pi}{M_1}q_1} |q_1k_2; C_2 >; \quad e^{ipL_2} |C_2 > = e^{i\frac{2\pi}{M_2}k_2} |q_1k_2; C_2 >. \]
Thus \(|q_1k_2;C_1\rangle > 0\) and \(|q_1k_2;C_2\rangle > 0\) are eigenfunctions of the complete set of commuting operators for the M dimensional space under study. To calculate the (possible) phase difference between them we evaluate the overlap,

\[
< q_1k_2; C'_1 | q_1k_2; C_2 > = \Delta^{M_1} (q'_1 - q_1) \Delta^{M_2} (k'_2 - k_2),
\]

This evaluation requires the evaluation of the Fourier transform, \(\langle k_1 | q \rangle\):

\[
\begin{align*}
< k_1 | e^{i \frac{2\pi}{M_1} x} | q_1 > &= e^{i \frac{2\pi}{M_1} q_1} < k_1 | q_1 > = < k_1 - M_2 q_1 >, \\
< k_1 | e^{i p L_2} | q_1 > &= e^{i \frac{2\pi}{M_1} k_1} < k_1 | q_1 > = < k_1 | q_1 - M_2 >.
\end{align*}
\]

This leads to (with the proper normalization, see Eq.(38)).

\[
\frac{1}{\sqrt{M_1}} e^{-i \frac{2\pi}{M_1} q_1 k_1 N_1} = < k_1 | q_1 >.
\]

We check this by evaluating (we do not include the normalization for simplicity),

\[
\begin{align*}
< k | q > &= e^{-i \frac{2\pi}{M_1} q k} \\
&= e^{-i \frac{2\pi}{M_1} (q_1 N_1 L_1 + q_2 N_2 L_2) (k_1 N_1 L_1 + k_2 N_2 L_2)} \\
&= e^{-i \frac{2\pi}{M_1} q_1 k_1 N_1} e^{-i \frac{2\pi}{M_2} q_2 k_2 N_2} \\
&= < k_1 | q_1 > < k_2 | q_2 >.
\end{align*}
\]

The \(kq\) rep for finite dimensional were considered by Zak [12, 17] and are defined as the eigenfunctions of the (commuting) operators

\[
\tau(M_1) = e^{i \left( \frac{2\pi}{M_1} x \right)}; \quad T(L_2) = e^{i p L_2},
\]

which are periodic in one variables and quasi periodic in the other. These can be written in terms of \(|q\rangle\), the eigen functions of \(\tau(M) = e^{i \left( \frac{2\pi}{M} x \right)}\) (we label these by \(E_1\)) as

\[
|q_1, k_2; E_1\rangle \equiv \frac{1}{\sqrt{M_2}} \sum_{q_2} e^{i \frac{2\pi}{M_2} k_2 q_2} |q_1 + q_2 L_2\rangle.
\]

In terms of the eigen functions of \(T(L_2)\) these can be expressed via (the extra label here is \(E_2\)),

\[
|q_1, k_2; E_2\rangle \equiv \frac{1}{\sqrt{M_1}} \sum_{k_1} e^{-i \frac{2\pi}{M_1} k_1 q_1} |k_2 + k_1 M_2\rangle.
\]

The variables in these states can not be considered as referring to two degrees of freedom. Indeed they are well defined whether or not \(M_1\) and \(M_2\) are relative primes (in the latter case, of course, \(N_1, N_2\) do not exist).
We now consider

$$|q_1k_2; E_1 > \equiv \frac{1}{\sqrt{M_1}} \sum_{k_1} e^{-i\frac{2\pi}{M_1}k_1q_1}|k_2 + k_1M_2 > .$$

(45)

It is easily shown that

$$e^{i\frac{2\pi}{M_1}q_1}|q_1k_2; E_1 > = e^{i\frac{2\pi}{M_1}q_1}|q_1k_2; E_1 > ; \quad e^{i\rho L_2}|q_1k_2; E_1 > = e^{i\frac{2\pi}{M_2}k_2}|q_1k_2; E_1 > .$$

Thus this wave function can differ at most by phase from $|q_1k_2; C_1 >$. This phase difference is gotten by evaluation of the overlap. In the following we suppress the (common) eigenvalues indices e.g. $|q_1k_2; C_1 > \rightarrow C_1 >$:

$$< C'_1|E_1 > = \frac{1}{M_1} \sum_{k_1'q_1} k'_1N_1L_1 + k'_2N_2L_2|k_2 + k_1L_1 > \exp [\frac{2\pi}{M_1}(k'_1q_1N_1 - k_1q_1)],$$

$$= \Delta^{M_2}(k_2 - k_2')\Delta^{M_1}(q_1 - q_1')e^{i\frac{2\pi}{M_1}k_2q_1N_1} ,$$

The phase difference between $|C_1 >$ and $|C_2 >$ may be obtained by evaluating the overlap between these two wave functions,

$$< C'_1|C_2 > = \frac{1}{\sqrt{M}} \sum_{k_1'q_2} e^{i\frac{2\pi}{M_1}k_1'q_1N_1} k'_1N_1L_1...|q_1N_1L_1... > e^{i\frac{2\pi}{M_2}k_2q_2N_2} .$$

Inserting

$$< k'_1N_1L_1...|q_1N_1L_1... > = \frac{1}{\sqrt{M}} \exp -i[\frac{2\pi}{M_1}(k'_1q_1N_1) + \frac{2\pi}{M_2}(k'_2q_2N_2)]$$

We get

$$< C'_1|C_2 > = \Delta^{M_1}(q_1 - q_1')\Delta^{M_2}(k_2 - k_2') .$$

Thus confirming Eq. (40). This can be checked by evaluating the overlap $< C'_2|E_1 >$: it gives the same result that we got above for $< C'_1|E_1 >$.

We now consider

$$|E_2 > \equiv \frac{1}{\sqrt{M_2}} \sum_{q_2} e^{i\frac{2\pi}{M_2}k_2q_2}|q_1 + q_2L_2 > .$$

One can check directly that

$$e^{i\frac{2\pi}{M_2}q_2}|E_2 > = e^{i\frac{2\pi}{M_2}q_1}|E_2 > ; \quad e^{i\rho L_2}|E_2 > = e^{i\frac{2\pi}{M_2}k_2}|E_2 > .$$

Thus this function may differ from those above by phase only. We calculate the overlap to get,

$$< C'_2|E_2 > = \Delta^{M_2}(k_2 - k_2')\Delta^{M_1}(q_1 - q_1')e^{-i\frac{2\pi}{M_2}k_2q_1N_2} .$$

This implies that the phase difference between $|E_1 >$ and $|E_2 >$ is,

$$e^{-i\frac{2\pi}{M_2}k_2q_1N_2} e^{-i\frac{2\pi}{M_1}k_2q_1N_1} .$$
We now evaluate this overlap directly to confirm this result,

\[
\langle E_1' | E_2 \rangle = \frac{1}{\sqrt{M}} \sum_{k_1', q_2} e^{\frac{2\pi}{M} k_1' q_1'} \langle k_2' + k_1 L_1 | q_1 + q_2 L_2 \rangle e^{\frac{2\pi}{M} k_2 q_2}. \quad (47)
\]

The overlap,

\[
\langle k_2' + k_1' L_1 | q_1 + q_2 L_2 \rangle = \frac{1}{\sqrt{M}} e^{-\frac{i \pi}{M} (k_2 q_1 + k_1 q_2 L_2 + k_1' q_1 L_1)}.
\]

Substituting this into the Eq.(47) we get

\[
\langle E_1' | E_2 \rangle = \frac{1}{M} e^{\frac{2\pi}{M} k_2 q_1} \sum_{k_1'} e^{\frac{2\pi}{M} k_1' (q_1 - q_1')} \sum_{q_2} e^{\frac{2\pi}{M} k_1' (q_1 - q_1')} = \Delta M_1 (q_1 - q_1') \Delta M_2 (k_2 - k_2') e^{\frac{2\pi}{M} k_2 q_1}.
\]

Thereby confirming the previous anticipation.