How to Fully Exploit the Degrees of Freedom in the Downlink of MISO Systems With Opportunistic Beamforming?

Minghua Xia, Wenkun Wen, and Soo-Chang Kim

Abstract

The opportunistic beamforming in the downlink of multiple-input single-output (MISO) systems forms $N$ transmit beams, usually, no more than the number of transmit antennas $N_t$. However, the degrees of freedom in this downlink is as large as $N_t^2$. That is, at most $N_t^2$ rather than only $N_t$ users can be simultaneously transmitted and thus the scheduling latency can be significantly reduced. In this paper, we focus on the opportunistic beamforming schemes with $N_t < N \leq N_t^2$ transmit beams in the downlink of MISO systems over Rayleigh fading channels. We first show how to design the beamforming matrices with maximum number of transmit beams as well as least correlation between any pair of them as possible, through Fourier, Grassmannian, and mutually unbiased bases (MUB) based constructions in practice. Then, we analyze their system throughput by exploiting the asymptotic theory of extreme order statistics. Finally, our simulation results show the Grassmannian-based beamforming achieves the maximum throughput in all cases with $N_t = 2, 3, 4$. However, if we want to exploit overall $N_t^2$ degrees of freedom, we shall resort to the Fourier and MUB-based constructions in the cases with $N_t = 3, 4$, respectively.

Index Terms

Degrees of freedom, downlink, multiple-input single-output (MISO), opportunistic beamforming

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Minghua Xia and Soo-Chang Kim are with the ETRI Beijing R&D Center, Beijing 100027, China (e-mail: xia_minghua@yahoo.com.cn; sckim@etri.re.kr).

Wenkun Wen is with Guangdong-Nortel R&D Center, Guangzhou 510665, China (e-mail: wenkun@gmail.com).
I. INTRODUCTION

Multiple-input multiple-output (MIMO) system holds promise for the next generation wireless communications due to its high spectral efficiency [1], [2]. In a single-user MIMO system, its capacity has been extensively investigated, assuming different channel state information (CSI) is known at the transmitter and/or receiver [3], [4]. In the multi-user scenario, the multi-user diversity was introduced as a new dimension of degrees of freedom to further increase the capacity [5]–[8]. In this paper, we focus on the downlink of multi-user MIMO systems, i.e., broadcast channels (BCs). By using dirty paper coding strategy at the transmitter [9], the optimal sum-rate capacity region of MIMO BCs was well established from the information-theoretic viewpoint, with the assumption that CSI is perfectly known at the transmitter and all the receivers [10], [11]. This region can be numerically evaluated by using the duality between BCs and multi-access channels (MACs), though it is extremely computationally intensive [12]–[14]. In practice, the optimal sum-rate capacity region of MIMO BCs can be approached using the nested lattices or trellis beamforming scheme [15], [16], which generalizes the Tomlinson-Harashima beamforming [17], [18]. Unfortunately, perfect CSI at the transmitter is almost infeasible in practical communication systems with large number of users, and also the non-linear beamforming is usually impractical for the real-time traffic. Therefore, designing linear beamforming schemes with lower feedback complexity is of great interest [19]–[21].

The opportunistic beamforming system (OBS), also known as the random beamforming system, is shown in [22] to achieve the maximum sum-rate capacity with the minimum amount of feedback, provided that the number of users is not smaller than the number of transmit antennas. This condition is surely satisfied in the practical cellular systems. The single-beam OBS is proposed in [23], in which the conceptual idea of multi-beam OBS is also presented in [23, Appendix B]. The detailed analysis on the throughput of OBS with multiple orthogonal transmit beams is performed in [24], [25]. Moreover, the opportunistic beamforming with only signal-to-interference-plus-noise ratio (SINR) feedback is generalized in [26] to the case with composite feedback consisting of quantized channel directional information and channel quality information (channel magnitude or SINR).

In the literature with respect to multi-beam OBS [24]–[29], random vector quantization (RVQ) limited feedback MIMO systems [26], [30]–[32], or the 3GPP Long Time Evolution (LTE) of 3G systems [33], it is always assumed that the number of transmit beams $N$ is identical to the number of transmit antennas $N_t$ and thus there are at most $N_t$ users can be simultaneously transmitted. In other words, the beamforming matrix is a square matrix with size $N_t \times N_t$. However, it is shown that the optimal transmission strategy regarding the sum-capacity criterion in MIMO broadcast channels with large number of users involves more than $N_t$ transmit beams at the same time but upper bounded by $N_t^2$, i.e., $N_t < N \leq N_t^2$ [34]. If each user is equipped with $N_r > 1$ receive antennas, he/she can receive up to $N_t^2$ data streams [34]. This allows the user to increase his/her own data rate but it prevents the simultaneous transmission by other users, so that the number of simultaneously transmitted users is limited to be $\lceil N_t^2/N_r^2 \rceil$, where we assume that each user receives exactly $N_r^2$ data streams and $\lceil x \rceil$ denotes the integer ceiling operator [34].

1In this paper, the term “throughput” refers to the average sum rate capacity, and the link-adaptive techniques, such as adaptive coding/modulation, finite constellation and dynamic power allocation, are not taken into account.
In other words, there are at most $N^2$ degrees of freedom in the extreme case with $N_r = 1$. Throughout this paper, we suppose there is only $N_r = 1$ receive antenna for each user and thus there are at most $N^2$ users that can be simultaneously transmitted. The beamforming matrix is now oblong with size $N_t \times N^2$. Unfortunately, [34] does not show us the implementation of beamforming schemes with $N_t < N \leq N^2$ simultaneously transmitted users. To the best of authors’ knowledge, only the case with $N = N_t + 1$ scheduled users is addressed in [35] by exploiting the tight Grassmannian frames.

In this paper, we show how to schedule $N_t < N \leq N^2$ users simultaneously. Specifically, we design the opportunistic beamforming schemes with $N_t < N \leq N^2$ transmit beams in which one user is scheduled at each beam. In particular, $N_t$ is supposed no larger than 4, just as that in 3GPP LTE [33]. Unlike the orthogonal transmitting case [24], the orthogonality between different transmit beams is not retained again if $N > N_t$. More precisely, the rank of beamforming matrix $B \in \mathbb{C}^{N_t \times N}$ is certainly no larger than $N_t$, where $\mathbb{C}$ stands for the field of complex numbers. That is, there are at least $N - N_t$ transmit beams that are no longer be orthogonal with the others. However, if the transmit beams are generated as at least correlated as possible, more transmit beams benefit to schedule more users as soon as possible and hence decrease the scheduling latency. Unfortunately, the increased multi-user interferences and the loss of orthogonality between transmit beams will inevitably deteriorate system throughput. Therefore, there must be a tradeoff between more and more transmit beams and increased multi-user interferences as well as disappearing orthogonality. In this paper, we first show how to construct the beamforming matrices and then the system throughput is rigorously investigated.

The rest of this paper is organized as follows. We present the system model and scheduling strategy in Section II. In Section III, we show how to design the beamforming matrices with constrained correlation property. Then, the system throughput is analyzed in Section IV. Simulation results and discussion are presented in Section V, and finally, Section VI concludes the paper.

II. SYSTEM MODEL AND SCHEDULING STRATEGY

A. System Model

In this paper, we consider the downlink of a homogeneous single-cell cellular system where the base station with $N_t$ antennas transmits packets to $K$ single-antenna users, that is, the number of receive antennas $N_r = 1$ for each user. The number of users $K$ is assumed no less than $N^2$ and all users are scattered geographically and do not cooperate; moreover, their average SNRs are identical. Block flat Rayleigh fading channels are supposed and all the time indices are omitted for the sake of notation brevity if no other specific statement; furthermore, different channels among users are mutually independent. In addition, we suppose that the transmission time is divided into consecutive and equal time slots, and each time slot is less than the possible time delay but long enough so that there is a coding strategy available that operates closely to Shannon channel capacity. Moreover, each time slot is

2The case in which the number of users and the number of transmit antennas are of the same order is addressed in [30], [36] and the references therein.
Fig. 1. The block diagram of OBS with multiple transmit beams

divided into a number of equal sized mini-slots and several initial ones are used to transmit common pilot symbols, so that the base station can determine which users shall be chosen for data transmission in the rest mini-slots according to the feedback of each user.

The OBS with $N$ transmit beams is illustrated in Fig. 1. At the base station, $N$ different beams are simultaneously transmitted during one time slot, where $N \in [1, N_t^2]$. When $N = 1$, it denotes the single-beam transmission [23]. When $N = N_t$, it refers to the conventional multi-beam orthogonal transmission [24]–[26]. In this paper, we concentrate on the cases with $N_t < N \leq N_t^2$.

When pilot symbols are transmitted during the first several mini-slots, $x = [x_1, \cdots, x_N]^T \in \mathbb{C}^{N \times 1}$ comprises $N$ different elements simultaneously transmitted at $N$ different beams. Furthermore, $x_n$, $n = 1, \cdots, N$ is simultaneously sent out from $N_t$ transmit antennas with each being multiplied by a beamforming coefficient $\sqrt{\alpha_{i,n}} e^{j\theta_{i,n}}$ at Antenna $i$, $1 \leq i \leq N_t$ and Beam $n$, $1 \leq n \leq N$. That is, $N$ different pilot symbols are needed to distinguish $N$ transmit beams. On the other hand, when data is transmitted during the later mini-slots, $x_n$ refers to user data transmitted at Beam $n$. Then, the received symbol of User $k$, $y_k \in \mathbb{C}^{1 \times 1}$, is given by

$$y_k = \sqrt{\frac{\rho}{N}} \sum_{n=1}^{N} h_{k,n} b_n x_n + z_k$$

(1)

where $\rho$ is the average received SNR for each user, $z_k \in \mathbb{C}^{1 \times 1}$ stands for additive white Gaussian noise with zero mean and unit variance; $h_{k,n} \in \mathbb{C}^{1 \times N_t}$ denotes the complex channel vector between User $k$ and the base station.

\footnote{It is shown that on an average 2.5 mini-slots is required to find the scheduled users [37].}

\footnote{In order to make a fair comparison between different transmit schemes, the transmit power in our proposal is normalized such that it is independent of the number of transmit beams $N$ as shown in [4], whereas in the conventional orthogonal opportunistic beamforming systems the transmit power is assumed to be identical with $N$ [24, Footnote 3].}
and it agrees with Rayleigh fading with zero mean and variance $1/m$. Moreover, the instantaneous beamforming matrix $\mathbf{B}(t) \in \mathbb{C}^{N_t \times N}$ at time slot $t$ can be written as

$$\mathbf{B}(t) = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \end{bmatrix}$$

(2)

where

$$\mathbf{b}_n = \left[ \sqrt{\alpha_{1,n}} e^{j\theta_{1,n}} \quad \sqrt{\alpha_{2,n}} e^{j\theta_{2,n}} \quad \cdots \quad \sqrt{\alpha_{N_t,n}} e^{j\theta_{N_t,n}} \right]^T, \quad n = 1, \ldots, N$$

(3)

is the beamforming vector at Beam $n$, and $(\cdot)^T$ denotes the transpose operator.

In the case with $N = N_t$, the amplitudes $\alpha_{i,n}, i = 1, \cdots, N_t$ in (3) are uniformly distributed over $[0, 1)$ such that $\sum_{i=1}^{N_t} \alpha_{i,n} = 1$, and the phases $\theta_{i,n}, i = 1, \cdots, N_t$ are independent and uniformly distributed over $[0, 2\pi)$. Moreover, different beamforming vectors are orthogonal with each other, that is,

$$\mathbf{b}_l^H \mathbf{b}_n = \begin{cases} 1, & l = n \\ 0, & l \neq n \end{cases}, \quad l, n = 1, \cdots, N_t$$

(4)

where $(\cdot)^H$ denotes the Hermitian transpose operator. Now, $\mathbf{B}(t)$ is a unitary matrix and it can be generated according to an isotropic distribution. However, in the cases with $N_t < N \leq N_{t}^2$, the beamforming vectors $\mathbf{b}_n, n = 1, \cdots, N$ are no longer orthogonal with each other.

**B. Scheduling Strategy**

As far as the scheduling strategy is concerned, we assume that each user sends them back to the base station, his/her maximum received SINR and its corresponding beam index among $N$ different beams. The feedback beam-index $\hat{n}_k \in [1, N]$ of User $k$ is determined by

$$\hat{n}_k = \arg \max_{n=1, \cdots, N} |\mathbf{h}_k \mathbf{b}_n|$$

(5)

where $|x|$ denotes the amplitude of $x$.

According to (1), the received SINR of User $k$ at Beam $n$ is

$$\gamma_{n,k} = \frac{\mathbf{h}_k^H \mathbf{b}_n^2}{1 + \frac{\rho}{N} \sum_{l=1, l \neq n}^{N} |\mathbf{h}_k \mathbf{b}_l|^2}$$

(6)

Hence, his/her maximum SINR among $N$ beams is

$$\hat{\gamma}_k = \max_{n=1, \cdots, N} \gamma_{n,k}$$

(7)

Combining (5) and (7), the feedback information of User $k$ can be shown as $(\hat{n}_k, \hat{\gamma}_k)$. At the base station, there are $N$ different data sets $\mathcal{S}_n$, where $n \in [1, N]$, corresponding to $N$ different beams to store the feedback information of all users. That is, for any feedback $(\hat{n}_k, \hat{\gamma}_k)$, if $\hat{n}_k = n$, then $\hat{\gamma}_k \in \mathcal{S}_n$. Moreover, the maximum

Actually, it is not necessary for each user to offer his/her feedback information. Instead, the same system throughput can be nearly achieved by only allowing the strongest 10% users to provide feedback, whose received SINRs are above a predefined threshold level. This is the so-called selective multi-user diversity beneficial to greatly decrease the feedback complexity [19].
SINR scheduling strategy is adopted at the base station to choose a user for transmission at each beam. Therefore, the index of scheduled user at Beam \( n \) is

\[
\hat{k}_n = \arg \max_{\hat{\gamma}_k \in \mathcal{S}_n} \hat{\gamma}_k, \quad n = 1, \cdots, N
\]  

(8)

Finally, the maximum SINR directing the transmission at Beam \( n \) is \( \hat{\gamma}_{\hat{k}_n} \).

III. THE IMPLEMENTATION OF OBS WITH \( N_t < N \leq N_t^2 \)

In this section, we show three practical beamforming schemes with \( N_t < N \leq N_t^2 \). In general, the instantaneous beamforming matrix \( B(t) \) shown in (2) is constructed with a fixed initial matrix \( B \in \mathbb{C}^{N_t \times N} \) and a time-variable vector

\[
c(t) = \begin{bmatrix} e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_N} \end{bmatrix}
\]

(9)
in which \( \theta_n, n = 1, \cdots, N \) are fixed in time slot \( t \) but varied from time slot \( t \) to \( t+1 \), furthermore, they are independent and uniformly distributed over \([0, 2\pi)\). More accurately, \( B(t) \) is generated as

\[
B(t) = \begin{bmatrix} b_1 & b_2 & \cdots & b_N \\ e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_N} \end{bmatrix} B(:, \cdot)
\]

(10)

where \( B(:, n) \) refers to the \( n^{th} \) column of \( B \). Equation (11) implies only a phase rotation is performed on each column of \( B \) to get \( B(t) \). Therefore, the correlation property of \( B \) is remained.

From a purely information-theoretic point of view, using the deterministic initial beamforming matrix \( B \) yields the same system throughput as that if the time-variable \( B(t) \) shown in (11) is applied in fast fading environment. The artificial randomness introduced by \( c(t) \) shown in (9) is to ensure fairness between in fast and slow fading environments [23]. The introduction of \( c(t) \) changes neither the correlation property between any pair of columns of \( B \) nor the distribution function of the received SINR.

In what follows, the key point is how to design \( B \in \mathbb{C}^{N_t \times N} \) to accommodate more users (larger \( N \)) and maximize system throughput. A straightforward idea is first to generate a unitary matrix with size \( N \times N \), and then choose its first \( N_t \) rows. Despite its simplicity, the main drawback of this construction is the correlation between different beamforming vectors \( b_n, n = 1, \cdots, N \) is not guaranteed at all. Moreover, the transmit power \( \|b_n\|^2, n = 1, \cdots, N \) at different beams is randomized.

Now, we present three different methods to construct \( B \) with constrained correlation property.

A. Fourier-Based Construction

It is well known that a Fourier matrix \( F \in \mathbb{C}^{N_t^2 \times N_t^2} \) is an orthogonal basis in a \( N_t^2 \)-dimensional complex space. Its projection into a \( N_t \)-dimensional complex space forms a tight frame whose elements have the broadest scattering, and this projection simply retains the first \( N_t \) rows of \( F \) [38]. Inspired by this observation, we propose to set the
TABLE 1

THE COMPARISON OF MINIMUM MAXIMUM CROSS-CORRELATION OF THE FOURIER, GRASSMANNIAN, AND MUB-BASED CONSTRUCTIONS. THE GRASSMANNIAN-BASED ONE HAS THE BEST PERFORMANCE IF \( N = 4, 7, 13 \). ALTHOUGH ONLY FOURIER-BASED ONE FUNCTIONS IF \( N = 9 \), THE MUB-BASED ONE OUTPERFORMS IT IF \( N = 16 \).

| \( N_t \) | \( N \) | \( \delta_0 \) | # selected rows | \( \delta(\mathbf{B}_F) \) | \( \delta(\mathbf{B}_{G}) \) | \( \delta(\mathbf{B}_{M}) \) | Lower bound | \( \delta^2 \) |
|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 0.7071 | \{2, 3\} | 0.7071 | 0.5774 | \( \) | \( \) | 1 |
| 3 | 7 | 0.7490 | \{1, 2, 4\} | 0.4714 | 0.4714 | \( \) | \( \) | 1.3333 |
| 3 | 9 | 0.8440 | \{3, 7, 9\} | 0.6565 | \( \) | \( \) | 0.5 | 2 |
| 4 | 13 | 0.8597 | \{1, 3, 4, 8\} | 0.4330 | \( \) | \( \) | \( \) | 2.2499 |
| 4 | 16 | 0.9061 | \{1, 10, 12, 13\} | 0.5817 | \( \) | \( \) | 0.4472 | 3 |

initial beamforming matrix \( \mathbf{B}_F \) as, where the subscript \( F \) refers to the Fourier-based beamforming,

\[
\mathbf{B}_F = \frac{1}{\sqrt{N_t}} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & w & \cdots & w^{N_t^2-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & w^{N_t-1} & \cdots & w^{(N_t-1)(N_t^2-1)}
\end{bmatrix}
\]

(12)

in which \( w = e^{-j2\pi/N_t^2} \).

For this choice, the correlation between transmit Beams \( l \) and \( n \) is

\[
c_{l,n} = \left| \mathbf{B}_F(:, l)^H \mathbf{B}_F(:, n) \right| = \begin{cases}
1, & l = n \\
\frac{\sin \left( \frac{\pi(l-n)}{N_t} \right)}{N_t}, & l \neq n
\end{cases}
\]

(14)

Roughly speaking, (14) suggests the correlation of \( \mathbf{B}_F \) behaves like a sinc function and hence all the cross-correlations between a specific beam and the others are smaller than its auto-correlation. Therefore, different users’ channels can be well matched by different beamforming vectors.

However, it is not necessarily constrained to choose the first \( N_t \) rows, but instead the maximum cross-correlation between different beams can be further lowered by appropriately choosing another set of \( N_t \) components. Unfortunately, the optimal choice with the lowest maximum cross-correlation

\[
\delta = \min \max_{l \neq n} c_{l,n}
\]

(15)

requires exhaustive searching [38]. In Table I we list the number of selected rows with the minimum \( \delta(\mathbf{B}_F) \) as shown in (15), where \( \delta_0 \) stands for the maximum cross-correlation with the fist \( N_t \) rows. For example, when \( N_t = 3 \) and \( N = 9 \), \( c_{i,n} \) shown in (14) is plotted in Fig. 2. We observe that, with the best choice of \( \{3, 7, 9\} \) rows, the maximum cross-correlation is decreased from 0.8440 to 0.6565.
Fig. 2. The correlation coefficient $c_{l,n}$ shown in (14) of Fourier-based constructions with $N_t = 3$ and $N = 9$, as a function of $|l - n|$. The left-hand panel corresponds to the beamforming matrix composed of the first three rows of Fourier matrix with size $9 \times 9$. The right-hand panel refers to our optimal beamforming matrix with selected $\{3, 7, 9\}$ rows. Obviously, the maximum cross-correlation is decreased from 0.8440 to 0.6565.

B. Grassmannian-Based Construction

Our intention to find $B \in \mathbb{C}^{N_t \times N}$ that has the minimum maximum cross-correlation between any pair of $N$ beamforming vectors, is equivalent to the Grassmannian line packing problem in the space $\mathbb{C}^{N_t}$, which is to find a set of $N$ lines that the minimum distance between any pair of lines is as large as possible [39]. Although the Grassmannian packing methodology has already been widely applied in the codebook design [40]–[43], it has seldom been employed in the design of opportunistic beamforming. To the best of authors’ knowledge, only the tight Grassmannian frames are exploited to construct the beamforming matrix in the case with $N = N_t + 1$ [35]. In this subsection, however, we focus on the generalized cases with $N_t < N \leq N_t^2$, making the connections between Grassmannian frames and opportunistic beamforming design more transparent.

The Grassmannian frame $\{b_n\}$, $n = 1, \cdots, N$ minimizes the maximum correlation between frame elements among all unit norm frames which have the same redundancy defined by

$$\eta = \frac{N}{N_t}$$

(16)

Furthermore, if $b_n \in \mathbb{C}^{N_t \times 1}$, $n = 1, \cdots, N$, then the maximum frame correlation is lower bounded by [39, Theorem 6]

Note please that the number of lines $N$ is of no any constraint in the separable infinite-dimensional Hilbert space for the Grassmannian line packing problem [39]. But in our opportunistic beamforming design, $N \leq N_t^2$ is imposed because of the limited degrees of freedom in the downlink of MISO systems [34].
Moreover, the equality in (17) can only hold if \( N \leq N_t^2 \), and also now \( \{b_n\}, n = 1, \ldots, N \) is an equiangular tight frame. One achieving the equality in (17) is called optimal Grassmannian frame. Unfortunately, although there are at most \( N_t^2 \) degrees of freedom in the downlink of MISO systems, the optimal Grassmannian frame does not always exist for any choices of \( N_t \) and \( N \). For example, if \( N_t = 3 \), there are at most \( N = 7 \) frame elements for the optimal Grassmannian frame. In what follows, we give the initial beamforming matrix with maximum redundancy \( \eta \), that is, the number of transmit beams \( N \) is maximized while achieving the equality in (17), according to the following Lemma 1.

**Lemma 1: (König [44])** Let \( p \) be a prime number and \( l \) over the field \( \mathbb{N} \) of positive numbers, we set \( N_t = p^l + 1 \) and \( N = N_t^2 - N_t + 1 \). Then there exist integers \( 0 \leq d_1 < \cdots < d_N, i < q \) such that all numbers \( 1, \ldots, N - 1 \) occur as residues mod \( N \) of the \( N_t(N_t-1) \) differences \( d_i - d_q, 1 \leq i, q \leq N_t \). For \( n = 1, \ldots, N \), we define

\[
    b_n = \frac{1}{\sqrt{N_t}} \left[ e^{i2\pi nd_1/N} \quad e^{i2\pi nd_2/N} \quad \cdots \quad e^{i2\pi nd_{N_t}/N} \right]^T
\]

and then the vectors \( b_n, n = 1, \ldots, N \) form a harmonic optimal Grassmannian frame with maximum frame correlation \( \sqrt{N_t-1}/N_t \).

1) \( N_t = 2, N = 4 \): In this case, according to [40, Table II], the initial beamforming matrix \( B_G \) where the subscript \( G \) denotes the Grassmannian-based beamforming, can be given by

\[
    B_G = \begin{bmatrix}
        -0.1612 - 0.7348j & -0.0787 - 0.3192j & -0.2399 + 0.5985j & -0.9541 \\
        -0.5135 - 0.4128j & -0.2506 + 0.9106j & -0.7641 - 0.0212j & 0.2996
    \end{bmatrix}
\]

We can easily verify that the columns of \( B_G \) form an equiangular unit norm frame. Furthermore, the equality of lower bound in (17) is attained with \( \delta(B_G) = 0.5774 \). Therefore, the columns of \( B_G \) in (19) make an optimal Grassmannian frame.

2) \( N_t = 3, N = 7 \): We get \( d_1 = 0, d_2 = 1 \), and \( d_3 = 5 \) through exhaustive searching, and then substituting them into (18), we have

\[
    B_G = \begin{bmatrix}
        0.5774 & 0.3600 + 0.4514j & -0.1285 - 0.5629j \\
        0.5774 & -0.1285 + 0.5629j & -0.5202 + 0.2505j \\
        0.5774 & -0.5202 + 0.2505j & 0.3600 + 0.4514j \\
        0.5774 & -0.5202 - 0.2505j & 0.3600 - 0.4514j \\
        0.5774 & -0.1285 - 0.5629j & -0.5202 - 0.2505j \\
        0.5774 & 0.3600 - 0.4514j & -0.1285 + 0.5629j \\
        0.5774 & 0.5774 & 0.5774
    \end{bmatrix}
\]

which achieves the equality in (17) with \( \delta(B_G) = 0.4714 \).
3) \( N_t = 4, N = 13 \): We show that \( d_1 = 0, d_2 = 1, d_3 = 3, \) and \( d_4 = 9 \) through exhaustive searching, and then substituting them into (18), we get

\[
B_G = \begin{bmatrix}
0.5 & 0.4427 + 0.2324j & 0.0603 + 0.4964j & -0.1773 - 0.4675j \\
0.5 & 0.2840 + 0.4115j & -0.4855 + 0.1197j & -0.3743 + 0.3316j \\
0.5 & 0.0603 + 0.4964j & -0.1773 - 0.4675j & 0.4427 + 0.2324j \\
0.5 & -0.1773 + 0.4675j & 0.4427 - 0.2324j & 0.0603 - 0.4964j \\
0.5 & -0.3743 + 0.3316j & 0.2840 + 0.4115j & -0.4855 + 0.1197j \\
0.5 & -0.4855 + 0.1197j & -0.3743 + 0.3316j & 0.2840 + 0.4115j \\
0.5 & -0.3743 - 0.3316j & -0.2840 - 0.4115j & -0.4855 - 0.1197j \\
0.5 & -0.1773 - 0.4675j & 0.4427 + 0.2324j & 0.0603 + 0.4964j \\
0.5 & 0.0603 - 0.4964j & -0.1773 + 0.4675j & 0.4427 - 0.2324j \\
0.5 & 0.2840 - 0.4115j & -0.4855 - 0.1197j & -0.3743 - 0.3316j \\
0.5 & 0.4427 - 0.2324j & 0.0603 - 0.4964j & -0.1773 + 0.4675j \\
0.5 & 0.5 & 0.5 & 0.5 \\
\end{bmatrix}^T
\]  

which achieves the equality in (17) with \( \delta(B_G) = 0.4330 \).

Remark 1: With the Fourier-based construction, we minimize the maximum cross-correlation between different transmit beams. On the other hand, different transmit beams are forced to be equiangular with the Grassmannian-based construction, and also they have the maximum distance between any pair of beams. However, we claimed in Section I that there are at least \( N - N_t \) transmit beams that are no longer orthogonal with the others. Therefore, a natural question to ask is: Can we design a beamforming matrix \( B \in C^{N_t \times N} \) with \( N_t \) orthogonal vectors while simultaneously they have the same cross-correlation with the rest \( N - N_t \) vectors? The answer is yes, but we have to rely on the concept of mutually unbiased bases (MUB) elaborated in the next subsection.

C. MUB-Based Construction

Let \( U = \{ u_1, \cdots, u_{N_t} \} \) and \( V = \{ v_1, \cdots, v_{N_t} \} \) be orthonormal bases of \( \mathbb{C}^{N_t} \). \( U \) and \( V \) are mutually unbiased if the cross-correlation of vectors satisfies

\[
|u_i^H v_n| = \frac{1}{\sqrt{N_t}}, \quad 1 \leq i, n \leq N_t
\]

Furthermore, the set \( B = \{ U_1, \cdots, U_s \} \) is known as an MUB. It is reported in [45] that \( B \) can be constructed according to the following Lemma 2.

Lemma 2: (Gow [45]) Let \( N_t \) be a power of 2 and let \( X \) consisting of unitary matrices be an irreducible complex representation of \( G_{N_t} \) of degree \( N_t \), where \( G_{N_t} \) denotes a finite group of order \( N_t^2 \). Let \( D \) be a \( N_t \times N_t \) matrix that satisfies \( D^{N_t+1} = I \) and \( D^{-1} X(x) D = X(\delta(x)) \) for all \( x \) in \( G_{N_t} \). Then the powers \( D, D^2, \cdots, D^{N_t+1} = I \) define \( N_t + 1 \) pairwise mutually unbiased bases. Furthermore, all entries of \( D \) are in the field \( \mathbb{Q} (\sqrt{-1}) \).
Based on Lemma 2, our initial beamforming matrix $B_M$ in which the subscript $M$ denotes the MUB-based beamforming, can be given by

$$B_M = \begin{bmatrix} D & D^2 & \cdots & D^{N_t} \end{bmatrix}$$  \hspace{1cm} (23)$$

Note that $D^{N_t+1} = I$ corresponding to the case of transmit antenna selection is abandoned, due to the limitation of $N_t^2$ degrees of freedom in the downlink of MISO systems. Unfortunately, for the cases under consideration with $N_t \leq 4$, the powerful Lemma 2 can only be exploited in the cases with $N_t = 2, 4$, rather than the case with $N_t = 3$.

1) $N_t = 2, N = 4$: In this case, $D$ is given by

$$D = \frac{1+j}{2} \begin{bmatrix} -1 & j \\ 1 & j \end{bmatrix}$$  \hspace{1cm} (24)$$

Substituting it into (23), we have

$$B_M = \frac{1+j}{2} \begin{bmatrix} -1 & j & j & -j \\ 1 & j & -1 & -1 \end{bmatrix}$$  \hspace{1cm} (25)$$

2) $N_t = 4, N = 16$: Based on [46], we can easily show that $D$ can be given by

$$D = \frac{1}{2} \begin{bmatrix} -j & -j & -j & -j \\ 1 & -1 & 1 & -1 \\ -j & -j & j & j \\ -1 & 1 & 1 & -1 \end{bmatrix}$$  \hspace{1cm} (26)$$

Substituting it into (23) yields

$$B_M = \frac{1}{2} \begin{bmatrix} -j & -j & -j & -j & -1 & -1 & j & -1 & j & 1 & j & 1 & j & -1 \\ 1 & -1 & 1 & 1 & -j & -1 & -1 & j & -j & -1 & j & 1 & -1 & j \\ -j & -j & j & j & -j & -1 & -1 & j & -1 & -1 & -1 & j & 1 & -j \\ 1 & 1 & 1 & 1 & -1 & j & -j & 1 & -1 & -1 & j & -1 & -j & -1 \end{bmatrix}$$  \hspace{1cm} (27)$$

In Table I, the minimum maximum cross-correlations $\delta(B_G)$ and $\delta(B_M)$ of Grassmannian and MUB-based constructions, respectively, as well as the lower bound shown in (17) are also listed, with respect to different number of transmit antennas $N_t$ and number of transmit beams $N$.

IV. ASYMPTOTIC THROUGHPUT ANALYSIS

In this section, we investigate the system throughput and thus give definite answer to which kind of beamforming scheme is most preferable for a specific $(N_t, N)$ configuration, among Fourier, Grassmannian and MUB-based constructions.

A. Received SINR of User $k$

For any two non-orthogonal beamforming vectors $b_l$ and $b_n$ where $l \neq n$, $b_l$ can be expressed in reference to $b_n$ through their cross-correlation coefficient $\delta_l$, that is,

$$b_l = \delta_l b_n + \sqrt{1 - \delta_l^2} b_n^\perp, \quad 1 \leq l, n \leq N$$  \hspace{1cm} (28)$$
where $b_n^\perp$ stands for the orthonormal vector to $b_n$. Substituting (28) into (6), the received SINR of User $k$ at Beam $n$ can be rewritten as

$$\gamma_{n,k} = \frac{\frac{\rho}{N}|h_k b_n|^2}{1 + \frac{\rho}{N} \sum_{l=1, l \neq n}^N |\delta_l h_k b_n + \sqrt{1 - \delta^2_l} h_k b_n^\perp|^2}$$

$$\approx \frac{\frac{\rho}{N}|h_k b_n|^2}{1 + \frac{\rho}{N} \delta^2 |h_k b_n|^2}$$

where we explored the approximation $h_k b_n^\perp \approx 0$ in (30), with the assumption that the beamforming vector $b_n$ matches perfectly with the channel $h_k$ when the number of active users is large enough. The parameter $\hat{\delta}^2$ is a constant determined by the correlation structure of the beamforming matrix, which can be calculated respectively as follows.

1) $N_t = 2, N = 4$: In this case, we observe from Table I that the Grassmannian-based beamforming is better than Fourier or MUB-based construction, since its minimum maximum cross-correlation $\delta(B_G)$ achieves the lower bound 0.5774. Furthermore, the Grassmannian-based beamforming matrix $B_G$ in (19) is equiangular, so that

$$\hat{\delta}^2 = 3 \times 0.5774^2 = 1$$

2) $N_t = 3, N = 7$: In this case, it is observed from Table I that the beamforming matrix with Grassmannian-based construction has the same performance as that of the Fourier-based one with selected $\{1, 2, 4\}$ rows. They both achieve the low bound 0.4714 and thus

$$\hat{\delta}^2 = 6 \times 0.4714^2 = 1.3333$$

3) $N_t = 3, N = 9$: We find from Table I that only the Fourier-based construction functions in this case, though $\delta(B_F) = 0.6565$ is larger than the lower bound 0.5. From the right-hand panel of Fig. 2, we have

$$\hat{\delta}^2 = 0.2280^2 + 0.4285^2 + 0.5774^2 + 0.6565^2 + 0.6565^2 + 0.5774^2 + 0.4285^2 + 0.2280^2 = 2$$

4) $N_t = 4, N = 13$: In this case, it is shown in Table I that the beamforming matrix with Grassmannian-based construction has the same performance as that of the Fourier-based one with selected $\{1, 3, 4, 8\}$ rows and the lower bound 0.4330 is achieved, hence

$$\hat{\delta}^2 = 12 \times 0.4330^2 = 2.2499$$

5) $N_t = 4, N = 16$: In this case, we observe from Table I that the MUB-based beamforming matrix outperforms the Fourier-based one, though they both don’t arrive at the lower bound 0.4472 but $\delta(B_{su}) = 0.5$ of the former is much closer to it than $\delta(B_F) = 0.5817$ of the latter. Therefore,

$$\hat{\delta}^2 = 12 \times 0.5^2 = 3$$

All the above values of $\hat{\delta}^2$ are also listed in the last column of Table I.
B. Asymptotic Distribution of N Maximum Received SINRs

At the base station, we arrange the $K$ received SINRs in Set $S_n$ corresponding to Beam $n$ as $\gamma_1, \cdot \cdot \cdot, \gamma_K$ in an ascending order\footnote{There are at most $K$ SINR values in Set $S_n$ and meanwhile the other $N-1$ sets are all empty, which means all users simultaneously have their maximum received SINRs at Beam $n$.} where $\gamma_k$, $k = 1, \cdot \cdot \cdot, K$ has the same meaning as $\gamma_{n,k}$ in (6) but the beam index $n$ is ignored here for the sake of notation brevity. Then, we turn to find the limiting distribution of the $N$ upper extremes of order statistics $\gamma_1, \cdot \cdot \cdot, \gamma_K$, by applying the asymptotic theory of extreme order statistics.

It is straightforward to show that $z = |h_kb_n|^2$ in (30) is of the chi-square distribution with two degrees of freedom, that is, its PDF can be given by

$$f_Z(z) = \frac{m}{2} e^{-\frac{m}{2}z}, \quad z \geq 0$$

Thus, after some manipulations, the PDF and CDF of $\gamma_k$ in (30) can be shown respectively as,

$$f_{\Gamma_k}(\gamma) = \frac{mN}{\rho(1-\delta^2\gamma)} \left[ 1 - \right]^{1/\delta^2} \gamma < 1$$

and

$$F_{\Gamma_k}(\gamma) = 1 - \exp \left[ -\frac{mN\gamma}{\rho(1-\delta^2\gamma)} \right], \quad \gamma < 1$$

Resorting to the well-known von Mises’s sufficient conditions in the asymptotic theory of extreme order statistics [47], [48], we substitute (37) and (38) into the growth function defined by

$$g(\gamma) = \frac{1 - F_{\Gamma_k}(\gamma)}{f_{\Gamma_k}(\gamma)}$$

and then it is straightforward to show the limit of the derivative of $g(\gamma)$ is, as $\gamma \rightarrow 1/\delta^2$,

$$\lim_{\gamma \rightarrow 1/\delta^2} \frac{dg(\gamma)}{d\gamma} = 0$$

Therefore, $F_{\Gamma_k}(\gamma)$ is in the domain of attraction of Gumbel-type limiting distribution $H_{3,0}(\gamma)$, where [47, p. 296]

$$H_{3,0}(\gamma) = \exp \left( -e^{-\gamma} \right)$$

That is, the limiting CDF of the maximum received SINR $\gamma_K$ over $\gamma_1, \cdot \cdot \cdot, \gamma_K$ is

$$\lim_{K \rightarrow +\infty} F_{\Gamma_K}(\gamma) = \lim_{K \rightarrow +\infty} \left[ F_{\Gamma_k}(\gamma) \right]^K$$

$$= \lim_{K \rightarrow +\infty} \left[ 1 - \exp \left( -\frac{mN\gamma}{\rho(1-\delta^2\gamma)} \right) \right]^K$$

$$= H_{3,0} \left( \gamma - \frac{a}{b} \right)$$

in which we used (38) in (43). Moreover, the position parameter $a$ is the solution to [48, Theorem 2.1.3]

$$1 - F_{\Gamma_k}(a) = \frac{1}{K}$$

$$\quad (45)$$
Substituting (38) into (45) yields

$$a = \frac{\rho \ln K}{mN + \rho \delta^2 \ln K}$$

(46)

On the other hand, the scale factor $b$ can be obtained as [48, Remark 2.7.1]

$$b = g(a) = \frac{1 - F_{\Gamma_k}(a)}{f_{\Gamma_k}(a)}$$

(47)

$$= \frac{\rho m N}{(mN + \rho \delta^2 \ln K)^2}$$

(49)

Moreover, based on [48, Theorem 2.8.1], their respective limiting CDFs of $N$ upper extremes of $\gamma_1, \cdots, \gamma_K$, that is, $\gamma_K, \gamma_{K-1}, \cdots, \gamma_{K-N+1}$, can be given by, as $K \to +\infty$,

$$F_{\Gamma_{K-n+1}}(a + b\gamma) = \exp \left( -e^{-\gamma} \sum_{l=0}^{n-1} \frac{e^{-l\gamma}}{l!} \right), \quad n = 1, \cdots, N$$

(50)

Finally, it is straightforward to show that their limiting PDFs are, respectively, as $K \to +\infty$,

$$f_{\Gamma_{K-n+1}}(a + b\gamma) = \frac{e^{-n\gamma}}{\Gamma(n)} \exp \left( -e^{-\gamma} \right), \quad n = 1, \cdots, N$$

(51)

where $\Gamma(.)$ refers to the Gamma function.

It is well known that the Kullback-Leibler distance $D(f\|g)$ which behaves like the square of the Euclidean distance [49, p. 299] is a measure of the inefficiency of an approximate distribution $g$ to its true distribution $f$. Thus, we may exploit it to check the inefficiency of the Gumbel-type limiting distribution (44). Specifically, in our numerical evaluation, we compare the true PDF

$$f(\gamma) = K f_{\Gamma_k}(\gamma) \left[ F_{\Gamma_k}(\gamma) \right]^{K-1}$$

(52)
with its limiting PDF

\[ g(\gamma) = \frac{1}{b} \exp \left( -\frac{\gamma - a}{b} \right) H_{3,0} \left( \frac{\gamma - a}{b} \right) \]

(53)
of the maximum received SINR \( \gamma_K \), and the Kullback-Leibler distance is defined as [49, p. 231]

\[ D(f||g) = \int_0^{+\infty} f(\gamma) \log_2 \frac{f(\gamma)}{g(\gamma)} \]

(54)

In Fig. 3, we show the numerical results of (54) with \( \rho = 0 \) dB, \( m = 0.5 \) and 3. We observe that the Kullback-Leibler distance is only 0.14 bits if \( K = 8 \) and \( m = 0.5 \), and it decreases as increasing \( m \). For example, it is about 0.025 bits if \( K = 8 \) and \( m = 3 \). Furthermore, it further decreases as increasing \( K \) and finally it approaches zero as \( K > 23 \). Therefore, the Gumbel-type limiting distribution (44) is a good approximation to its true distribution of the maximum received SINR.

C. Throughput Analysis

We suppose that all \( N \) scheduled users have simultaneously the maximum SINR at \( N \) different transmit beams, then the system throughput is upper bounded by

\[ R_u \leq N \mathbb{E} \left\{ \log_2 (1 + \gamma_K) \right\} \]

(55)

\[ = \frac{N}{b} \int_0^{+\infty} \log_2 (1 + \gamma) e^{\bar{\gamma}} \exp \left( -e^{\bar{\gamma}} \right) d\gamma \]

(56)

where \( \bar{\gamma} = -(\gamma - a)/b \).

On the other hand, if \( N \) scheduled users always have different SINR at \( N \) different transmit beams, that is, if we ignore the small probability that at least two scheduled users obtain the same SINR, then the system throughput is lower bounded by

\[ R_l \geq \mathbb{E} \left\{ \sum_{k=K-N+1}^{K} \log_2 (1 + \gamma_k) \right\} \]

(57)

\[ = \frac{1}{b} \int_0^{+\infty} \log_2 (1 + \gamma) \left( \sum_{n=1}^{N} e^{\bar{\gamma}} \right) \exp \left( -e^{\bar{\gamma}} \right) d\gamma \]

(58)

\[ = \frac{1}{b} \sum_{n=1}^{N} \frac{1}{\Gamma(n)} \int_0^{+\infty} \log_2 (1 + \gamma) e^{\bar{\gamma}} \exp \left( -e^{\bar{\gamma}} \right) d\gamma \]

(59)
in which we used (51) in (58). Unfortunately, \( R_u \) and \( R_l \) above can only be calculated by numerical integration.

Actually, when the number of users is large enough, all \( N \) scheduled users have almost the same SINR at \( N \) different transmit beams and hence the system throughput approaches the upper bound. Therefore, the upper bound
$R_u$ shown in (55) can be analytically reformulated as

$$R_u \leq NE \{\log_2 (1 + \gamma_K)\}$$

(60)

$$\leq N \log_2 (1 + E\{\gamma_K\})$$

(61)

$$= N \log_2 (1 + a + b\Upsilon)$$

(62)

$$= N \log_2 \left[ 1 + \frac{pmN(\Upsilon + \ln K) + \rho^2 \hat{\delta}^2 (\ln K)^2}{mN + \rho \hat{\delta}^2 \ln K} \right]$$

(63)

where we used the Jensen’s inequality in (61), and also in (62) we explored the Gumbel-type limiting distribution function as shown in (44), which has a mean $\Upsilon = 0.5772 \cdots$, corresponding to the Euler-Mascheroni constant. Moreover, we exploited (46) and (49) in (63).

**Remark 2:** In comparison with the orthogonal counterpart with $N = N_t$, we find from (29) that the received SINR in our proposed scheme with $N_t < N \leq N_t^2$ is greatly decreased due to increased multi-user interferences as well as their mutual non-orthogonality, which will dramatically deteriorate the system throughput. However, this deterioration will be compensated by the increased spatial multiplexing gain $N$, as shown in (63). Anyway, the most important characteristic of our proposal is able to serve as large as $N_t^2$ users simultaneously, which benefits to significantly decrease the scheduling latency. Moreover, we ignored the minimum data-rate requirement of each user in this paper. If we take it into account, the number of simultaneously transmitted users will possibly be decreased.

**V. SIMULATION RESULTS AND DISCUSSION**

**A. The Effectiveness of Closed-form Upper Bound**

In this section, we first show the accuracy of our closed-form upper bound (63) in comparison with its asymptotic counterpart (56) as well as the Monte-Carlo simulation results. In our simulations, the minimum number of users is 16, since there are at most 16 transmit beams if $N_t = 4$. On the other hand, the maximum number of users is set to be 2048. Although there will not be so many users in practical cellular communication systems, we are able to confirm the validity of our throughput analysis by comparing the numerical results with the simulation ones for such a large number of users.

In Fig. 4 we show the system throughput of OBS with Grassmannian-based beamforming matrix as shown in (20), where $N_t = 3$, $N = 7$, and $m = 0.5$. We observe that the closed-form upper bound (63) almost always overlaps with the asymptotic (56), no matter how many users there are or whatever SNR is 0 or 5 dB. However, although we claimed in Section IV-C that all the scheduled users have almost the same maximum SINR as the number of users approaches infinity, there is always a very small gap between the simulation results and the upper bound as shown in Figs. 4 and 5. For example, when $m = 0.5$, $\rho = 0$ dB and the number of users $K = 64$, it is observed from the left-hand panel of Fig. 4 that the difference between (63) and the simulations results is about 0.06 bit/s/Hz, or 1.6% in relative to the simulation result 3.93 bit/s/Hz. Moreover, this gap becomes smaller and smaller as the number of users or the average SNR increases. The same observation can be attained from Fig. 5 where the beamforming matrix is based on MUB construction as shown in (27), $N_t = 4$, $N = 16$, and $m = 3$. 
Fig. 4. The system throughput of OBS with $N_t = 3, N = 7$, and Grassmannian-based beamforming; $m = 0.5$.

Fig. 5. The system throughput of OBS with $N_t = 4, N = 16$, and MUB-based beamforming; $m = 3$. 
Therefore, we conclude that our upper bound (63) is very tight with simulation results, and thus it can be exploited below to evaluate the system throughput effectively.

B. System Throughput of Proposed Schemes

According to (63), we compare the system throughput of proposed schemes in Figs. 6 and 7, where the beamforming construction of $N = 4, 7, 13$ are Grassmannian based, $N = 9$ is Fourier based, and $N = 16$ is MUB based, respectively. We find that the OBS with Grassmannian-based beamforming achieves the maximum system throughput whenever $N_t = 3$ or 4, corresponding to $N = 7$ or 13, respectively. But if we want to fully exploit the degrees of freedom when $N_t = 3, 4$, then we have to rely on the Fourier and MUB-based construction, that is, $N = 9, 16$ users can be simultaneously transmitted, respectively. Unfortunately, the increase of the number of simultaneously scheduled users is at the penalty of system throughput. For example, when $K = 64$ and $\rho = 0$ dB, we observe from the left-hand panel of Fig. 6 that the throughput difference between the cases with $N = 7$ and $N = 9$ is about $0.19 \text{bit/s/Hz}$, or $4.8\%$ throughput loss of the case with $N = 9$ in relative to the throughput $3.99 \text{bit/s/Hz}$ of the case with $N = 7$. Moreover, this throughput loss will slightly increase as the number of users or the average SNR increases, but it decreases fast as the the variance $1/m$ of Rayleigh fading decreases by comparing Fig. 6 with Fig. 7. Furthermore, we observe that the system throughput degrades with decreasing variance $1/m$ by comparing Fig. 6 with Fig. 7. This degradation should not come as a surprise since the multi-user


Fig. 7. The system throughput of OBS with $N = 4, 7, 9, 13, 16$, and $m = 3$.

diversity gain will be smaller and smaller as the channel fading becomes more and more stable [29].

C. The Preferred Low SNR Case

We point out that the proposed schemes with $N_t < N \leq N_t^2$ transmit beams is much beneficial to the low SNR case over the high SNR scenario. In Fig. 8 we see that the system throughput increases very slowly as the number of users increases sharply from 16 to 2048, where $m = 1$ and $\rho = 10$ dB. That is, when the SNR is high enough, the multi-user diversity gain becomes saturated soon. This phenomena can be understood as follows: We see from (30) that the received SINR $\gamma_{n,k}$ can be approximated to $1/\hat{\delta}^2$ if $\rho$ is large enough, that is, the value of $\gamma_{n,k}$ is independent of the user index $k$ and therefore the multi-user diversity gain vanishes.

D. Throughput Comparison With Orthogonal Counterpart

Although our proposed schemes can schedule much more users than the number of transmit antennas $N_t$, how about the system throughput in comparison with their conventional orthogonal counterparts in which the number of transmit beams $N$ equals $N_t$? In Figs. 9 and 10 we show their system throughput comparison. Usually, in each cell of a practical cellular communication system, there are only tens of simultaneously active users. In this regard, we can see from the right-hand panel of Fig. 9 that the system throughput of our proposed scheme with $N_t = 4, N = 13$ and its orthogonal counterpart with $N_t = 4, N = 4$ are 6.06 and 7.31bit/s/Hz, respectively, if
Fig. 8. The system throughput of OBS with $N = 4, 7, 9, 13, 16$, $m = 1$, and SNR $= 10$ dB.

$m = 0.5$ and $K = 128$. In other words, there is 18% throughput loss but the scheduled users is of 225% increase! Furthermore, when the channel becomes more and more flat (as $m$ increases), the throughput loss turns to be smaller and smaller, and even if $m = 3$ and $K \leq 128$ as shown in Fig. 10, the system throughput of our proposed scheme with $N_t = 4$, $N = 13$ outperforms that of its orthogonal counterpart as well as any other cases with $N_t < 4$. The underlying reason is that, as the number of active users is small, for example, in any practical cellular system, the multi-user diversity gain is strictly limited and thus larger spatial multiplexing gain of our scheme leads to larger system throughput. Therefore, the proposed schemes, especially, the case with $N_t = 4$, $N = 13$, is of great interest in practical employment.

**Remark 3:** Actually, the proposed schemes can be generalized to the cases with the number of receive antennas $N_r > 1$. Although he/she has the potential to use up to $N_r^2$ degrees of freedom, we can employ a combining strategy to reduce effectively each user with $N_r > 1$ to a single-dimensional receive terminal [34], [50]. That is, the rank of received signal of each user is forced to be 1, and thus the number of simultaneously transmitted users remains $N_t^2$ all the same.

**VI. Conclusion**

Inspired by the $N_t^2$ degrees of the freedom in the downlink of MISO systems, we demonstrated how to transmit to more than $N_t$ users simultaneously, whereas at most $N_t$ users can be simultaneously scheduled in the conventional MISO beamforming systems. We proposed three different opportunistic beamforming schemes:
Fig. 9. Throughput comparison between proposed scheme and their orthogonal counterpart, $m = 0.5$.

Fig. 10. Throughput comparison between proposed scheme and their orthogonal counterpart, $m = 3$. 

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Throughput, bit/s/Hz

# users (K)

$N_t=2$, $N=4$

$N_t=3$, $N=7$

$N_t=4$, $N=13$

$N_t=2$, $N=2$

$N_t=3$, $N=3$

$N_t=4$, $N=4$
Fourier, Grassmannian, and MUB-based constructions. The Grassmannian-based scheme achieves the maximum system throughput with the number of transmit beams $N = 4, 7, 13$ in the cases with $N_t = 2, 3, 4$, respectively, by taking the optimal Grassmannian frames as the beamforming matrices. However, it can not exploit all $N^2$ degrees of freedom when $N_t > 2$. On the other hand, if we want to fully exploit 9 and 16 degrees of freedom in the cases with $N_t = 3$ and 4, we may resort to the Fourier and MUB-based schemes, respectively, despite a little penalty on system throughput. Finally, the special Grassmannian-based case with $N_t = 4$ and $N = 13$ was shown to be promising for practical employment in cellular systems, since it outperforms its orthogonal counterpart in terms of the number of simultaneously scheduled users but without any throughput loss.

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