A PROOF OF GRÜNBAUM’S LOWER BOUND CONJECTURE FOR GENERAL POLYTOPES

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ABSTRACT

In 1967, Grünbaum conjectured that any $d$-dimensional polytope with $d + s \leq 2d$ vertices has at least

$$
\phi_k(d + s, d) = \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}
$$

$k$-faces. We prove this conjecture and also characterize the cases in which equality holds.

1. Introduction

The paper is devoted to the proof of Grünbaum’s general lower bound conjecture for polytopes with few vertices.

In the last fifty years a lot of effort has gone into trying to understand face numbers of polytopes. For instance, McMullen [McM70] established the Upper Bound Theorem in 1970, which provides tight upper bounds on the number of $k$-faces a $d$-dimensional polytope with $n$ vertices can have. A couple of years later, Barnette (see [Bar71], [Bar73a], and [Bar73b]) proved the Lower Bound Theorem for simplicial polytopes; his result provides tight lower bounds on the number of $k$-faces a $d$-dimensional simplicial polytope with $n$ vertices can have. Furthermore, in 1980, Billera and Lee [BL80] and Stanley [Sta80]

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completely characterized the face numbers of all simplicial (and by duality also simple) polytopes. Their result is known as the \(g\)-theorem. Billera and Lee [BL80] established sufficiency of the conditions while Stanley [Sta80] proved their necessity.

Despite these spectacular advances, to date no Lower Bound Theorem is known for general \(d\)-dimensional polytopes with an arbitrary number of vertices; in fact, there is not even a plausible conjecture. However for general \(d\)-dimensional polytopes with \(d + s \leq 2d\) vertices, Grünbaum conjectured in [Grü03, p. 184] (see also [GS69, p. 265]) that the number of \(k\)-faces is at least

\[
\phi_k(d + s, d) = \binom{d + 1}{k + 1} + \binom{d}{k + 1} - \binom{d + 1 - s}{k + 1}.
\]

He proved this conjecture for the cases of \(s = 2, 3,\) and \(4.\) The conjecture remained completely open for \(s \geq 5\) until very recently Pineda-Villavicencio, Ugon and Yost [PVUY19] proved this conjecture for the number of edges, i.e., they verified the \(k = 1\) case.

In this paper we prove the conjecture in full generality. Our results can be summarized as follows:

**Theorem 3.2:** Let \(P\) be a \(d\)-polytope with \(d + s\) vertices where \(s \geq 2\) and \(d \geq s.\) Then \(f_k(P) \geq \phi_k(d + s, d)\) for every \(k\).

**Theorem 4.3:** Let \(P\) be a \(d\)-polytope with \(d + s\) vertices where \(s \geq 2\) and \(d \geq s.\) If \(f_k(P) = \phi_k(d + s, d)\) for some \(1 \leq k \leq d - 2\), then \(P\) is \(\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})\)—the polytope that is a \((d - s)\)-fold pyramid over a prism over an \((s - 1)\)-simplex.

The main novelty of our approach is that instead of focusing on contributions coming from facets, we look at sets of potentially unrelated vertices and bound the number of \(k\)-faces containing one or more of them.

2. Background and Preliminaries

Before starting the proof we recall some definitions and introduce some notation. We refer the reader to books by Grünbaum [Grü03] and Ziegler [Zie95] for all undefined notions. By a polytope we mean the convex hull of finitely many points in \(\mathbb{R}^d.\) A \(d\)-simplex, denoted as \(\Delta^d,\) is the convex hull of \(d + 1\) affinely independent points. A face of a polytope \(P\) is the intersection of \(P\) with a supporting hyperplane. It is known that a face of a polytope is a polytope.
The dimension of a polytope is the dimension of its affine span. For brevity, we refer to a $k$-dimensional face as a $k$-face and to a $d$-dimensional polytope as a $d$-polytope. The 0-faces are called vertices. The $(d-1)$-faces of a $d$-polytope are called facets. We denote by $f_k(P)$ the number of $k$-faces of a polytope $P$.

Let $P \subset \mathbb{R}^d$ be a $d$-polytope and $v$ a vertex of $P$. The vertex figure of $P$ at $v$, $P/v$, is obtained by intersecting $P$ with a hyperplane $H$ that separates $v$ from the rest of the vertices of $P$. One property of vertex figures that will be very useful for us is that $(k-1)$-faces of $P/v$ are in bijection with $k$-faces of $P$ that contain $v$.

3. The proof of the inequality

We start with the following formulas, most of which are straightforward consequences of Pascal’s relation: \binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}. For all integers $k$, $d$, and $a > b$,

$$
\phi_k(d + a, d) - \phi_k(d + b, d) = \binom{d+1-b}{k+1} - \binom{d+1-a}{k+1} = \sum_{i=1}^{a-b} \binom{d+1-b-i}{k};
$$

(3.1)

$$
\phi_k(d+1, d) = \binom{d+1}{k+1};
$$

(3.2)

$$
\phi_k(d+s, d-1) + \phi_{k-1}(d-1,d-2) + \phi_{k-1}(d,d-1)
$$

$$
= \phi_k(d+s, d-1) + \binom{d-1}{k} + \binom{d}{k}
$$

$$
= \phi_k(d+s+2, d).
$$

(3.3)

Let $\mathcal{D}(d+s, d)$ be the set of all $d$-polytopes with $d+s$ vertices. The main ingredient of the proof is the following.

**Proposition 3.1:** Let $P$ be a $d$-polytope and let \{\(v_1, v_2, \ldots, v_m\)\} be a subset of vertices of $P$, where $m \leq d$. Then the number of $k$-faces of $P$ that contain at least one of the $v_i$’s is bounded from below by

$$
\sum_{i=1}^{m} \binom{d-i+1}{k},
$$
Proof. We proceed by induction on \( m \) to show that there exists a sequence of faces, \( \{F_1, \ldots, F_m\} \), such that

1. each \( F_i \) has dimension \( d - i + 1 \),
2. \( F_i \) contains \( v_i \) but does not contain any \( v_j \) with \( j < i \).

The base case is \( m = 1 \), and we simply pick \( F_1 = P \). Inductively we assume that for every \( p \leq m - 1 \) and any \( p \)-set of vertices \( \{v_1, \ldots, v_p\} \), there exists a sequence \( \{F_1, \ldots, F_p\} \) such that for \( 1 \leq i \leq p \), conditions (1) and (2) are satisfied.

Let \( m > 1 \) and let \( v_1, \ldots, v_m \) be \( m \) given vertices of \( P \). By the inductive hypothesis, for \( \{v_1, \ldots, v_{m-1}\} \) there exist faces \( F_1, \ldots, F_{m-1} \) satisfying conditions (1) and (2). Similarly, by considering \( \{v_1, \ldots, v_{m-2}, v_m\} \), there also exists a \((d-m+2)\)-face \( F \) that contains \( v_m \) but not \( v_1, \ldots, v_{m-2} \). Regardless of whether \( v_{m-1} \) is in \( F \) or not, there must exist a facet of \( F \), call it \( F_m \), that contains \( v_m \) but not \( v_{m-1} \). Then \( v_i \in F_m \) if and only if \( i = m \), and \( F_1, \ldots, F_{m-1}, F_m \) is a desired sequence.

For each \( i \), the \( k \)-faces of \( F_i \) that contain \( v_i \) correspond to the \((k-1)\)-faces of the vertex figure \( F_i/v_i \). Since \( \dim(F_i/v_i) = \dim(F_i) - 1 = d - i \), we obtain that

\[
\#k\text{-faces of } P \text{ that contain some } v_i (1 \leq i \leq m) \\
\geq \# \bigcup_{i=1}^{m} \{k\text{-faces of } F_i \text{ containing } v_i \} \\
\geq \# \bigcup_{i=1}^{m} \{(k-1)\text{-faces of } F_i/v_i \} \\
\geq \sum_{i=1}^{m} \binom{d - i + 1}{k}.
\]

The result follows.

We are now ready to prove our first main result.

**Theorem 3.2:** Let \( s \geq 3 \) and \( d \geq s \). Then for all \( d \)-polytopes \( P \) with \( d + s \) vertices and for all \( 1 \leq k \leq d - 1 \),

\[
f_k(P) \geq \phi_k(d + s, d).
\]

The statement clearly holds for \( s = 1 \), and the cases of \( s = 2, 3, 4 \) were proved by Grünbaum (see [Grü03, 10.2.2]).
Proof. The proof is by induction on $s$. We fix $s \geq 2$. The following argument will show that if the statement holds for all pairs $(s', d')$ such that $s' < s$ and $d' \geq s'$, then for all $d \geq s$, it also holds for the pair $(s, d)$. Thus, consider $d \geq s$, and let $P \in \mathcal{P}(d + s, d)$.

If there exists a facet $Q$ of $P$ with

$$d \leq f_0(Q) \leq d + s - 2,$$

or equivalently if $Q \in \mathcal{P}(d + s - m, d - 1)$ where $2 \leq m \leq s$, then there are $m$ vertices of $P$ outside of $Q$. We denote them by $\{v_1, v_2, \ldots, v_m\}$. The $k$-faces of $P$ fall into two disjoint categories: the $k$-faces of $Q$ and the $k$-faces of $P$ that contain some $v_i$. By the inductive hypothesis,

$$f_k(Q) \geq \phi_k(d + s - m, d - 1).$$

Therefore by Proposition 3.1,

$$f_k(P) \overset{(\odot)}{=} \phi_k(d + s - m, d - 1) + \sum_{i=1}^{m} \binom{d - i + 1}{k}$$

(by (3.3))

$$\overset{(\odot \odot)}{=} \phi_k(d + s - m + 2, d) + \sum_{i=3}^{m} \binom{d - i + 1}{k}$$

(by (3.1))

$$= \phi_k(d + s, d) + \sum_{j=1}^{m-2} \left[ - \binom{d - j - 1 - (s - m)}{k} + \binom{d - j - 1}{k} \right] \geq 0 \text{ (since } s \geq m)$$

(by (3.1))

$$\geq \phi_k(d + s, d).$$

This completes the proof of this case. The inequalities $(\odot)$ and $(\odot \odot)$ will be discussed later in the proof of Theorem 4.3.

Otherwise, all facets in $P$ have $d + s - 1$ vertices. But this implies that each vertex of $P$ is not in the affine span of the rest, hence the vertex set of $P$ is affinely independent, and so $P$ can only be a simplex, contradicting our assumption that $P$ has $d + s$ vertices and $s \geq 2$. \hfill \blacksquare
4. Treatment of equality

In this section we discuss the cases of equality in the Lower Bound Theorem. We first review some definitions relevant to the proof below. For more details, see for example [Zie95, Chapter 1]. Let $P \subset \mathbb{R}^{d+1}$ be a $d$-polytope, and let $x \in \mathbb{R}^{d+1}$ be a point that does not lie in the affine hull of $P$. The pyramid over $P$ with apex $x$ is the convex hull of $P \cup \{x\}$. A pyramid over a $d$-polytope $P$ is a $(d + 1)$-polytope, denoted as $\text{Pyr}(P)$. An $s$-fold pyramid over $P$ is a pyramid over an $(s - 1)$-fold pyramid over $P$, denoted as $\text{Pyr}^s(P)$.

For every $d$-polytope $P$, there exists a polytope of the same dimension, denoted by $P^*$, whose face lattice is the opposite of the face lattice of $P$. In particular, vertices of $P^*$ correspond to facets of $P$. The polytope $P^*$ (or more precisely, its combinatorial type) is called the dual polytope of $P$.

Let $P \subseteq \mathbb{R}^d$ be a $d$-polytope with $n$ vertices and $P' \subseteq \mathbb{R}^{d'}$ be a $d'$-polytope with $n'$ vertices. Then the product $P \times P'$ is a $(d + d')$-polytope with $n \cdot n'$ vertices, defined as

$$P \times P' = \{(v, u) \in \mathbb{R}^{d+d'} | v \in P, u \in P'\}.$$  

Assume that both $P$ and $P'$ have the origin in their interiors, then the direct sum $P \oplus P'$ is the following $(d + d')$-polytope with $n + n'$ vertices:

$$P \oplus P' = \text{conv}(\{(v, 0) \in \mathbb{R}^{d+d'} | v \in P\} \cup \{(0, u) \in \mathbb{R}^{d+d'} | u \in P'\}).$$

When both $P$ and $P'$ have the origin in their interiors, the product and the direct sum are “dual constructions”. In particular,

$$(P \oplus P'^*)^* = P^* \times P'^*.$$  

A vertex of a $d$-polytope is simple if it is contained in exactly $d$ facets (equivalently, if it is adjacent to exactly $d$ vertices). A polytope $P$ is simple if all vertices of $P$ are simple. The dual polytope of a simple polytope is a simplicial polytope, and vice versa.

Grünbaum [Grü03, Section 6.1] proved the following results, which will be used in the proof of the main result of this section. Recall that $\Delta^d$ is a $d$-simplex.

**Lemma 4.1:** If $P$ is a simplicial $d$-polytope with $d + 2$ vertices, then$^1$

$$P \simeq \Delta^m \oplus \Delta^{d-m} \quad \text{for some } 1 \leq m \leq d - 1.$$  

$^1$ The polytope $\Delta^m \oplus \Delta^{d-m}$ is denoted by $T^d_m$ in [Grü03, Section 6.1].
Lemma 4.2: For all $0 \leq k \leq d - 1$,

$$f_k(\text{Pyr}^{d-a}(\Delta^m \oplus \Delta^{a-m})) = \binom{d + 2}{d - k + 1} - \binom{d - a + m + 1}{d - k + 1} - \binom{d - m + 1}{d - k + 1} + \binom{d - a + 1}{d - k + 1}.$$  

In particular, $f_{d-1}(\Delta^m \oplus \Delta^{d-m}) = d + 1 + m(d - m)$.

Now we are ready to state the main result of this section.

Theorem 4.3: Let $P \in \mathcal{P}(d + s, d)$ where $s \geq 2$ and $d \geq s$. If

$$f_k(P) = \phi_k(d + s, d) \quad \text{for some } k \text{ with } 1 \leq k \leq d - 2,$$

then

$$P \simeq \text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1}).$$

First it is easy to verify that

$$f_k(\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})) = \phi_k(d + s, d) \quad \text{for all } d \geq s, 1 \leq k \leq d - 1.$$

Assuming $f_k(P) = \phi_k(d + s, d)$ for some $k$ with $1 \leq k \leq d - 2$, we will prove this theorem using the following corollary of Theorem 3.2.

Corollary 4.4 (Corollary of Theorem 3.2): If $f_k(P) = \phi_k(d + s, d)$ for some $k$ with $1 \leq k \leq d - 2$, then each facet of $P$ has $d$, $d + s - 2$, or $d + s - 1$ vertices, and $P$ has $d + 2$ facets.

Proof. Notice that (3.4) holds independently of the choice of a facet (with at most $d + s - 2$ vertices) in $P$ or the ordering of the vertices that lie outside of this facet. Thus for $f_k(P) = \phi_k(d + s, d)$ to hold, both inequalities in (3.4) must be satisfied as equalities for any chosen facet with at most $d + s - 2$ vertices. The inequality $(\diamond)$ of (3.4) holds as equality for some $k < d - 1$ if and only if $m = 2$ or $s$. This implies that for the equality to hold, each facet of $P$ can only have $d$, $d + s - 2$, or $d + s - 1$ vertices. The first inequality $(\diamond)$ in (3.4) holds as equality only if, for every facet in $P$ that has at most $d + s - 2$ vertices and for each of the remaining vertices $v_1, v_2, \ldots$,

$$(4.1) \quad \#\{k\text{-faces containing } v_i \text{ but not any } v_j \text{ for } j < i \in P\} = \binom{d - i + 1}{k}.$$ 

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2 The polytope $\text{Pyr}^{d-a}(\Delta^m \oplus \Delta^{a-m})$ is denoted by $T^{d-a}_m$ in [Grü03, Section 6.1].

3 Notice that $\text{Pyr}^{d-a}(\Delta^m \oplus \Delta^{a-m}) \simeq (\text{Pyr}^{d-a}(\Delta^m \times \Delta^{a-m}))^s$. 
Particularly, the number of $k$-faces containing $v_1$ is $\binom{d}{k}$, hence the number of edges containing $v_1$ is $\binom{d}{1} = d$, so $v_1$ is simple. Since $v_1$ is arbitrary, this means that all of the vertices that are not in the chosen facet are simple. For each vertex $v$ of $P$ that is not an apex, there is a facet (of size $\leq d + s - 2$) that does not contain $v$, so we conclude that every non-apex vertex of $P$ is simple.

We saw in the proof of Theorem 3.2 that it is impossible for all facets of $P$ to contain $d + s - 1$ vertices. This means that there must exist some facet with $d + s - p$ vertices where $2 \leq p \leq s$. Pick such a facet $F$, and label the vertices outside of $F$ as $v_1, v_2, \ldots, v_p$. We will show that $f_{d-1}(P) = d + 2$. The facets of $P$ fall into the following disjoint categories:

(0) $F$;
(1) facets containing $v_1$;
(2) facets containing $v_2$, but not $v_1$;
(3) facets containing $v_3$, but not $v_1, v_2$;
\vdots
(p) facets containing $v_p$, but not $v_1, \ldots, v_{p-1}$.

Since $v_1$ is simple, it is contained in $d$ facets. These facets together with $F$ account for $d + 1$ facets of $P$. Next we show that there is a unique facet in category (2), i.e., a unique facet that contains $v_2$, but not $v_1$. Suppose not, and let $F_2$ and $F_2'$ be two distinct facets that contain $v_2$ but not $v_1$. Then there must be a $k$-face of $F_2'$ that contains $v_2$ and is not a face of $F_2$. Therefore

$$\# \{k\text{-faces containing } v_2 \text{ but not } v_1 \text{ in } P\}$$

$$> \# \{k\text{-faces containing } v_2 \text{ but not } v_1 \text{ in } F_2\}$$

$$\geq \binom{d-1}{k},$$

which contradicts our assumption in (4.1).

We have shown that the number of facets of $P$ in categories (0), (1), and (2) is $1 + d + 1 = d + 2$. If $F$ has $d + s - 2$ vertices (and so $p = 2$), we are done. In the case that $p > 2$, it suffices to show that for all $v_i$ with $i \geq 3$, there exist no facets that contain $v_i$ but not $v_1$ and $v_2$. Moreover, by reordering the vertices, it suffices to prove this statement for $v_3$. 

Suppose there exists a facet $G$ that contains $v_3$, but not $v_1, v_2$. Then
\[
\#\{\text{$k$-faces containing $v_3$ but not $v_1, v_2$ in $P$}\} \geq \#\{\text{$k$-faces containing $v_3$ in $G$}\} \\
= \#\{(k-1)\text{-faces of $G/v_3$}\} \\
\geq \binom{d-1}{k}. \\
> \binom{d-2}{k} \quad \text{(since $1 \leq k \leq d-2$)}.
\]
This again contradicts our assumption in (4.1).

Now we are ready to prove Theorem 4.3.

**Proof of Theorem 4.3.** Suppose that $P$ has $d-a$ facets that have $d+s-1$ vertices. By Corollary 4.4, $P$ has $d+2$ facets. Then $P$ is a $(d-a)$-fold pyramid over an $a$-polytope $Q$ with $a+s$ vertices and $a+2$ facets. Since the vertex set of $Q$ consists of non-apex vertices of $P$, by the argument above, $Q$ is a simple polytope that is not a simplex. According to Lemma 4.1, $Q \simeq \Delta^m \times \Delta^{a-m}$ for some $1 \leq m \leq a-1$. We will show that $Q \simeq \Delta^1 \times \Delta^{s-1}$.

Since each facet of $P$ has $d+s-1, d+s-2$, or $d$ vertices, and since $Q$ is not a pyramid, each facet of $Q$ has either $a+s-2$ or $a$ vertices. Since $Q \simeq \Delta^m \times \Delta^{a-m}$, there are only two possible types of facets of $Q$: $\Delta^{m-1} \times \Delta^{a-m}$ and $\Delta^m \times \Delta^{a-m-1}$. Let $F$ be a facet of $Q$ with $a+s-2$ vertices (such a facet must exist as $Q$ is not simplicial). As for any two polytopes $P_1, P_2$,
\[
f_0(P_1 \times P_2) = f_0(P_1) \cdot f_0(P_2),
\]
it follows that $f_0(Q) = (m+1)(a-m+1)$ and $f_0(F) = (m+1)(a-m)$ or $m(a-m+1)$. But we also know that $f_0(Q) = a+s$ and $f_0(F) = a+s-2$. Comparing the equalities, we conclude that $m = 1$ or $a-1$ and $a = s$, so $Q \simeq \Delta^1 \times \Delta^{s-1}$. Therefore $P \simeq \text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$ as desired.

**Remark 4.5:** Our proof shows that for $d + 2 \leq s \leq d$ and $1 \leq k \leq d-2$, $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$ is the unique polytope in $\mathcal{P}(d+s,d)$ that has $\phi_k(d+s,d)$ many $k$-faces. This is in general not true in the case of $k = d-1$ (where $\phi_{d-1}(d+s,d) = d+2$), i.e., there might exist more than one polytope $\text{Pyr}^{d-a}(\Delta^m \times \Delta^{a-m}) \in \mathcal{P}(d+s,d)$ with $d+2$ facets. Among those polytopes, by the theorem above, $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$ has the componentwise minimal $f$-vector. This result was also proved in [PVUY19, Theorem 24].
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