Relativistic Bose gases at finite density

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We consider a massive relativistic Bose gas with \(N\) complex scalars at finite density. At zero temperature, we calculate the pressure, charge density and the speed of sound in the one-loop approximation. In the nonrelativistic limit, we obtain the classic results for the dilute Bose gas. We also discuss finite-temperature effects. In particular, we consider the problem of calculating the critical temperature for Bose-Einstein condensation. Dimensional reduction and effective-field-theory methods are used to perturbatively calculate the effects of the nonstatic Matsubara modes. Calculations of \(T_c\) in the effective 3d theory require nonperturbative methods. Using the Monte Carlo simulations of X. Sun [Phys. Rev. E67, 066702 (2003)] and the seven-loop variational perturbation theory (VPT) calculations of B. Kastening [Phys. Rev. A70, 043621 (2004)], we obtain \(T_c\) for \(N = 2\) to second order in the interaction.

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I. INTRODUCTION

The realization of Bose-Einstein condensation (BEC) of trapped alkali atoms almost ten years ago has created an enormous interest in the properties of the weakly interacting Bose gas \[1,2,3\]. The temperature at which these systems Bose condense is of the order \(10^{-5}\) Kelvin, which is many orders of magnitude higher than the condensation temperature and the onset of superfluidity in \(^4\)He (2.17K). Both trapped alkali gases and \(^4\)He are examples of nonrelativistic systems.

BEC in relativistic Bose systems typically takes place in matter under extreme conditions. For example, kaons may condense in the color-flavor locked phase of high-density QCD \[4,5,6\]. This phase is a superconducting phase of QCD which arises from an instability of the Fermi surface; In analogy with ordinary BCS-theory, a weak attraction among the quarks in one or more channels results in the formation of Cooper pairs and the spontaneous breakdown of the color symmetry of QCD \[7,8,9\]. Such a phase may be found in the interior of compact stars if the density is sufficiently high. The linear \(SU(2)_R \times SU(2)_L\)-symmetric sigma model at finite chemical potential \(\mu\) for the hypercharge is used as a toy model for the description of kaon condensation in the color-flavor locked phase of QCD \[10,11\].

Calculation of the critical temperature or the critical density for Bose-Einstein condensation of an ideal Bose gas has been a standard text-book calculation for a number of years \[12\]. The charge density \(n\) of excited bosons as a function of temperature is given by

\[
n = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\beta m - \mu} - 1} - \frac{1}{e^{\beta (m + \mu)} - 1},
\]

where \(\omega = \sqrt{p^2 + m^2}\), \(\beta = 1/T\) and \(\mu\) is the chemical potential. We have set \(\hbar = k_B = 1\). Bose-Einstein condensation takes place when the chemical potential is equal to the mass \(m\) of the bosons. At that temperature, all the charge can no longer be accomodated in the excited states and it condenses into the ground state. Generally, \(T_c\) can only be calculated numerically, but in various limits analytic results can be obtained. For example, in the nonrelativistic (NR) limit, the critical density \(n_c\) calculated from Eq. (1) is

\[
n_c = \frac{\zeta \left( \frac{3}{2} \right)}{(2\pi)^{3/2}} \left( \frac{mT}{2\pi} \right)^{3/2}.
\]

Inverting this equation, one obtains the well-known result \(T_c = 2\pi/m \left[ n/\zeta \left( \frac{3}{2} \right) \right]^{2/3}\). Similarly, for \(m \to 0\), one finds

\[
T_c = \left( \frac{3n}{m} \right)^{1/2}.
\]

Thus, in the ultrarelativistic limit \(m = 0\), \(T_c\) is infinite, or equivalently, the critical charge density is zero. All charge resides in the ground state irrespective of the temperature.

In the context of nonrelativistic field theory, the problem of calculating the transition temperature for Bose-Einstein condensation with a weak interaction has a very long history and conflicting results have appeared in the literature \[3\]. A first-order perturbative calculation gives no correction to the ideal-gas result, while higher-order calculations are plagued with infrared divergences. This is a typical example of infrared divergences that arise in the vicinity of a second-order phase transition. The long-distance physics in the critical region is nonperturbative and one has to sum up an infinite set of diagrams to obtain a finite result. The problem was solved only recently by Baym et al. \[13\] who realized that it can be reduced to nonperturbative calculations using a classical three-dimensional field theory. Once it was understood how to organize the problem, the calculation of \(T_c\) has later been carried out using several methods. These include \(1/N\) techniques \[14,15\], lattice simulations \[16,17\], the linear delta expansion \[18,19\], variational perturbation theory \[20,21,22\], and renormalization group methods \[23,24\]. In particular, the use of effective field theory methods to obtain an effective three-dimensional field theory combined with high precision lattice calculations has settled the issue in a very elegant way \[16,25\].

The \(O(2)\)-symmetric relativistic Bose gas at finite temperature and chemical potential has been studied in detail by Benson, Bernstein, and Dodelson \[26,27\], while the problem of calculating \(T_c\) was addressed in a paper by Bedingham and Evans employing the linear delta expansion \[28\]. In the
In this section, we briefly discuss the perturbative framework for a massive Bose gas with \( N \) charged scalars at finite chemical potential \( \mu \). The action is

\[
S = \int dt \int d^3x \left[ (\partial_0 + i\mu)\Phi^\dagger (\partial_0 - i\mu)\Phi - \left( (\partial_\mu)^2 \right) (\Phi^\dagger \Phi) - \lambda (\Phi^\dagger \Phi)^2 \right],
\]

where \( \Phi = (\Phi_1, \Phi_2, ..., \Phi_N) \) and \( \Phi_i \) is a complex scalar field. We first parametrize the quantum field \( \Phi_1 \) in terms of a time-independent vacuum expectation value \( \phi_0 \) and two real quantum fluctuating fields:

\[
\Phi_1 = \phi_0 + \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2).
\]

Similarly, the remaining complex fields \( \Phi_2, ..., \Phi_N \) are parametrized in terms of \( 2N - 2 \) real fields \( \phi_1, ..., \phi_{2N} \). Substituting Eq. (5) into Eq. (1), the action can be written as

\[
S = S_0 + S_{\text{free}} + S_{\text{int}},
\]

where

\[
S_0 = \int dt \int d^3x \left[ (\mu^2 - m^2) \phi_0^2 - \lambda \phi_0^4 \right],
\]

\[
S_{\text{free}} = \int dt \int d^3x \left\{ \frac{1}{2} \phi_0^2 - \frac{\partial^2}{\partial t^2} + \nabla^2 + \mu^2 - m^2 - 2\lambda \phi_0^2 - 4\delta_3 \delta_1 \lambda \phi_0^2 \phi_1 + i\mu \left[ \frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t} \right] + ... + \phi_{2N} \frac{\partial \phi_{2N-1}}{\partial t} - \phi_{2N-1} \frac{\partial \phi_{2N}}{\partial t} \right\},
\]

\[
S_{\text{int}} = -\int dt \int d^3x \left[ \sqrt{2} (m^2 - \mu^2 + 2\lambda \phi_0^2) \phi_1 \phi_0 + \sqrt{2} \lambda \left( \phi_1^2 + \phi_2^2 + ... + \phi_{2N}^2 \right) \phi_0 + \frac{1}{4} \lambda (\phi_0 \phi_1)^2 \right].
\]

The propagators that correspond to the free part \( S_{\text{free}} \) of the action are given by

\[
D_1(\omega, p) = \frac{i}{(\omega^2 - \omega_{1+}^2)(\omega^2 - \omega_{1-}^2)} \begin{pmatrix} \omega^2 - p^2 - m_1^2 & 2i\mu \omega \\ -2i\mu \omega & \omega^2 - p^2 - m_2^2 \end{pmatrix},
\]

\[
D_2(\omega, p) = \frac{i}{(\omega^2 - \omega_{2+}^2)(\omega^2 - \omega_{2-}^2)} \begin{pmatrix} \omega^2 - p^2 - m_2^2 & 2i\mu \omega \\ -2i\mu \omega & \omega^2 - p^2 - m_2^2 \end{pmatrix},
\]

where the dispersion relations are

\[
\omega_{1\pm}(p) = \sqrt{p^2 + 2\mu^2 + \frac{1}{2} (m_1^2 + m_2^2)} \pm \frac{1}{2} \sqrt{(m_1^2 + m_2^2)^2 + 2\mu (2m_1^2 + m_1^2 + m_2^2) + 4\mu^2p^2},
\]

\[
\omega_{2\pm}(p) = \sqrt{p^2 + p^2 + m_2^2 \pm \mu}.
\]

Here the tree-level masses \( m_1^2 \) and \( m_2^2 \) are

\[
m_1^2 = -\mu^2 + m_2^2 + 6\lambda \phi_0^2,
\]

\[
m_2^2 = -\mu^2 + m_2^2 + 2\lambda \phi_0^2.
\]

In the minimum of the classical action, \( m_1^2 = 2(\mu^2 - m^2) \) and \( m_2^2 = 0 \), and so the dispersion relations reduce to

\[
\omega_{1\pm}(p) = \sqrt{p^2 + 3\mu^2 - m^2 \pm \sqrt{3\mu^2 - m^2)^2 + 4\mu^2p^2}},
\]

\[
\omega_{2\pm}(p) = \sqrt{p^2 + \mu^2 \pm \mu}.
\]

From these equations, we see that there are two massless modes that in the long-wavelength behave as

\[
\omega_{1-}(p) = \sqrt{\frac{\mu^2 - m_1^2}{3\mu^2 - m_1^2}} p,
\]

\[
\omega_{2-}(p) = \frac{p}{2\mu}.
\]

The other excitations \( \omega_1^+(p) \) and \( \omega_2^+(p) \) are gapped with gaps \( \Delta_1 = \sqrt{2(3\mu^2 - m^2)} \) and \( \Delta_2 = 2\mu \). In the case \( N = 2 \), the gapless particles \( \omega_1^+ \) and \( \omega_2^+ \) carry the quantum numbers of \( K^0 \) and \( K^\ast \), while the massive modes \( \omega_1^- \) and \( \omega_2^- \) carry those
of \( K^- \) and \( \bar{K}^0 \). Note that there are only \( N \) massless modes despite the fact that the potential has \( 2N - 1 \) flat directions which also is the number of broken generators. This is in agreement with the counting rule derived by Nielsen and Chadba \[29\], which states that the modes with a quadratic dispersion relation must be counted twice. Secondly, due to the quadratic dispersion relation for small \( p \), the Landau criterion \[30\] for superfluidity can never be satisfied, except for \( N = 1 \). Thus despite the presence of a Bose condensate, the system is not a superfluid.

### III. ZERO TEMPERATURE

In this section, we apply perturbation theory at zero temperature to calculate the pressure, charge density, and the speed of sound in the one-loop approximation.

The partition function \( Z \) is given by the path integral

\[
Z = \int D\Phi^\dagger D\Phi e^{iS}
\]

where the action \( S \) is given by \[41\]. The pressure \( P \) is

\[
P(\mu) = -i \frac{\partial}{\partial \mu} \ln \frac{Z}{\sqrt{VT}}.
\]

where \( VT \) is the space-time volume of the system. The charge density can be found by differentiating the pressure with respect to \( \mu \):

\[
n(\mu) = \frac{\partial P(\mu)}{\partial \mu}.
\]

The speed of sound \( c \) is given by the coefficient of \( \omega_\mu(p) \) as \( p \to 0 \). Corrections to the tree-level result can be found by calculating the dispersion relation in the long-wavelength limit including the self-energy function \( \Pi_\mu(\omega, p) \). It can also be derived once we know the charge density and is given by

\[
c^2 = \frac{n(\mu)}{\mu} \frac{\partial \mu}{\partial n}.
\]

The chemical potential measures the amount of energy needed to add a particle to the system, and in the nonrelativistic limit, we introduce the nonrelativistic chemical potential \( \mu_{NR} \) by \( \mu = \mu + \mu_{NR} \). In the NR limit, Eq. \[23\] is therefore replaced by

\[
c^2 = \frac{n(\mu_{NR})}{m} \frac{\partial \mu}{\partial n}.
\]

#### A. Pressure

The mean-field pressure \( P_0 \) is found by evaluating minus the classical thermodynamic potential \( \Omega_0(\mu, \phi_0) \) at the minimum of the classical action \( S_0 \):

\[
P_0(\mu) = \frac{1}{4\pi} \left( \mu^2 - m^2 \right)^2.
\]

The one-loop contribution to the effective potential is

\[
\Omega_1(\mu, \phi_0) = \frac{1}{2} \int dp \int d\omega_\mu \ln \det D_1(\omega, p) + \Delta_1 m^2 \phi_0^2 + \Delta_1 \lambda \phi_0^4 + \Delta_1 \lambda \phi_0^4 + \Delta_1 \epsilon.
\]

where \( \Delta_1 m^2, \Delta_1 \lambda, \) and \( \Delta_1 \epsilon \) are the one-loop mass counterterm, coupling constant counterterm, and vacuum counterterm, respectively. After integrating over the energy \( \omega \), we obtain

\[
\Omega_1(\mu, \phi_0) = \frac{1}{2} \int dp \left[ \omega_\mu(p) + (N - 1)\omega_\mu(p) \right] + \Delta_1 m^2 \phi_0^2 + \Delta_1 \lambda \phi_0^4 + \Delta_1 \epsilon.
\]

The integral involving \( \omega_\mu(p) \) can be calculated analytically in dimensional regularization, but the integral of \( \omega_\mu(p) \) cannot. In order to extract the divergences analytically, we make subtractions in the integrand that render the integral finite in \( d = 3 \) dimensions and then extract the poles in \( d - 3 \) from the subtracted integrals. The subtraction term \( \Omega_{sub} \) should not introduce any infrared divergences. Our choice for the subtracted integral is

\[
\Omega_{sub} = \int dp \left[ \frac{m^2 + 4\lambda \phi_0^2}{2p} - \frac{m^2 + 8\lambda \phi_0^2 + 20\lambda \phi_0^2}{8(p^2 + \mu^2)^{1/2}} \right].
\]

The first two terms in \( \Omega_{sub} \) vanish identically in dimensional regularization since there is no mass scale in the integrand. The last term is given in Eq. \[A.3\]. The one-loop thermodynamics potential can then be written as

\[
\Omega_1 = -\frac{1}{2(4\pi)^2} \left\{ m^4 \left[ \frac{N}{\epsilon} + 2L + \frac{3}{2}(N - 1) \right] + 4m^2 \lambda \phi_0^2 \left[ (N + 1) \left( \frac{1}{\epsilon} + 2L \right) + \frac{3}{2}(N - 1) \right] 
+ 4\lambda \phi_0^4 \left[ (N + 4) \left( \frac{1}{\epsilon} + 2L \right) + \frac{3}{2}(N - 1) \right] \right\} + \Delta_1 m^2 \phi_0^2 + \Delta_1 \lambda \phi_0^4 + \Delta_1 \epsilon,
\]

where \( L = \ln \frac{\Lambda^2}{\mu} \) and \( g \) is a function of the ratio \( m/\mu \) that must be evaluated numerically:

\[
g(m/\mu) = \frac{1}{2} \int dp \omega_\mu - \Omega_{sub}.
\]

The counterterms necessary to cancel the poles in \( \epsilon \) are \[20\]:

\[
\Delta_1 \epsilon = \frac{N m^4}{2(4\pi)^2 \epsilon},
\]

\[
\Delta_1 m^2 = \frac{2(N + 1)m^2 \lambda}{(4\pi)^2 \epsilon},
\]

\[
\Delta_1 \lambda = \frac{2(N + 4)\lambda^2}{(4\pi)^2 \epsilon}.
\]

The one-loop contribution to the pressure \( P_1 \) is given by \( -\Omega_1 \) evaluated at the classical minimum. After renormalization, the pressure through one loop reduces to

\[
P_{0+1}(\mu) = \frac{1}{4\pi} \left( \mu^2 - m^2 \right)^2 + \frac{1}{(4\pi)^2} \left\{ 2m^4 L - 6m^2 \mu^2 L 
+ \mu^4 \left[ (N + 4)L + \frac{3}{4}(N - 1) \right] \right\} + \mu^4 g(m/\mu).
\]

We next consider the NR limit of the pressure. In this limit, \( \mu_{NR} \ll m \). Moreover, the kinetic energy is much smaller than
\[ \omega_1^+(p) = \frac{p^2}{4m^2} \left( p^2 + 4m \mu_{NR} \right) , \]
\[ \omega_2^-(p) = \frac{p^4}{4m^2} . \]

The other quasiparticle excitations have \( \omega_{1+} = \omega_{2+} = 2m \) and so their contribution can be neglected. In NR field theory, it is customary to set \( 2m = 1 \) and \( \omega_{1-} = 0 + 1 \) by a vacuum counterterm. The pressure becomes
\[ P_{0+1} = \frac{\mu_{NR}^2}{16\pi a} \left[ 1 - 32\sqrt{2\mu_{NR} a^2} \right] . \]

Note that \( \omega_{2-} \) does not contribute to the pressure since there is no scale in the integral and so it set to zero in dimensional regularization. Using a simple ultraviolet cutoff \( \Lambda \) to regulate the integral, the divergence would be cancelled by a vacuum counterterm \(^1\). Using Eq. (A.4), we obtain
\[ P_{0+1} = \frac{\mu_{NR}^2}{16\pi a} \left[ 1 - 32\sqrt{2\mu_{NR} a^2} \right] . \]

\[ \text{B. Charge density and the speed of sound} \]

We next consider the speed of sound, which is given by Eq. (42). For simplicity we consider only the ultrarelativistic and nonrelativistic limits. In these limits, the tree-level results are \( c = 1/\sqrt{3} \) and \( c = \sqrt{2\mu} \), respectively.

The charge density can be calculated using Eqs. (22) and (39):
\[ n = \frac{\mu^3}{\lambda} \left\{ 1 + \left( \frac{\lambda}{(4\pi^2)^2} \right) \left[ 4(N + 4)L + 2N - 7 + 64\pi^2 g(0) \right] \right\} . \]

Inverting (39) and using (28), we obtain the speed of sound due to interactions in the medium:
\[ c = \frac{1}{\sqrt{3}} \left[ 1 + \frac{(N + 4)\lambda}{24\pi^2} \right] . \]

The sign of the correction is determined by the beta-function. In the NR limit, the charge density follows from Eqs. (22) and (38):
\[ n = \frac{\mu_{NR}^2}{8\pi a} \left[ 1 - 8\sqrt{2\mu_{NR} a^2} \right] . \]

The speed of sound then becomes
\[ c = 4\sqrt{\pi a n} \left[ 1 + 8\sqrt{n a^3} \right] , \]
where we have eliminated \( \mu_{tr, NR} \) in favor of the density \( n \). Note that the expansion parameter in the NR limit is the dimensionless quantity \( \sqrt{na^2} \) which is referred to as the the gas parameter. This result was first derived by Beliaev \(^2\), who calculated the leading corrections to the dispersion relation \(^3\) in the low-momentum limit \(^2\).

\[ \text{IV. FINITE TEMPERATURE} \]

We now discuss the behavior of the system defined by Eq. (4) at finite temperature.

\[ \text{A. Low-temperature effects} \]

We first consider the thermal corrections to the pressure at temperatures \( T \) much lower than the chemical potential \( \mu \). In this regime, the thermodynamics is dominated by the massless modes. We can then approximate the dispersion relations by \( \omega_1(p) \) and \( \omega_2(p) \) by their low-momentum limits \(^1\) and \(^2\). The pressure in the one-loop approximation is
\[ P_{0+1} = -\frac{1}{2} \sum_{p} \ln \left[ P_0^2 + \omega_1^2(p) \right] \]
\[ -\frac{1}{2} \left( N - 1 \right) \sum_{p} \ln \left[ P_0^2 + \omega_2^2(p) \right] . \]

Omitting the contribution from the massive modes, neglecting the zero-temperature pieces, and using (A.14)–(A.16), we obtain
\[ P_{0+1}^T = \frac{\sqrt{3}\pi^2 T^4}{30} + (N - 1)T \left( \frac{\pi T}{\mu} \right)^{3/2} \zeta \left( \frac{3}{2} \right) . \]

Similarly, in the nonrelativistic limit, one finds
\[ P_{0+1}^T = \frac{\pi^2 T^4}{90(2\mu_{NR})^{3/2}} + (N - 1)T \left( \frac{\pi T}{\mu} \right)^{3/2} \zeta \left( \frac{3}{2} \right) , \]
where we again have set \( 2m = 1 \). For \( N = 1 \), this reduces to the old result of Lee and Yang \(^3\).

\[ \text{B. Dimensional reduction} \]

Effective field theory methods can conveniently be used to organize the calculation of physical quantities whenever two or more momentum scales are well separated. The condensation temperature for BEC is an ideal problem for applying effective field theory. At finite temperature there are two characteristic scales in the system. The first is the correlation length which is associated with the effective chemical potential. Since the phase transition is second order, the correlation length becomes infinite at \( T_c \). The second scale is
\[ \text{1 In the NR limit, all the divergences at the one-loop level are power divergences and hence the parameters require no renormalization if one uses dimensional regularization. See Ref. \( \Delta \) for a thorough discussion.} \]

\[ \text{2 In the original derivation, Beliaev expressed his result in terms of the condensate density \( n_0 \), which is different from the total density due to the depletion of the condensate caused by quantum fluctuations. We have } n = n_0 \left[ 1 + \frac{3}{2} \sqrt{\frac{a^3}{\mu}} \right].} \]
associated with the nonzero Matsubara modes and is of order $T$. For distances much larger than the inverse temperature and for temperatures sufficiently close to the critical temperature, so that the effective chemical potential is much smaller than the temperature, the nonstatic Matsubara modes decouple and the long-distance physics can be described in terms of an effective three-dimensional field theory for the zeroth Matsubara mode. The action for this effective theory is

$$S_{3d} = - \int d^3x \left[ (\partial_i \Phi^\dagger)(\partial_i \Phi) + m_3^2 \Phi^\dagger \Phi + \lambda_3 (\Phi^\dagger \Phi)^2 + f + ... \right], \quad (46)$$

where the dots indicate higher-order operators. $f$ is a referred to as the coefficient of the unit operator and represents the contribution to the free energy density from the nonzero Matsubara modes. The parameters in Eq. (46) are functions of $T$ and the coefficients of the underlying theory (4), and are renormalized due to their coupling to the nonstatic Matsubara frequencies. These parameters can be determined by integrating out nonstatic modes explicitly as done by Bedingham and Evans in Ref. 22, but perhaps a more streamlined way of calculating them is by matching static Green’s functions 31.

For the purpose of matching, the chemical potential can formally be treated as a perturbation on the same footing as the quartic coupling. In the effective theory $m_3^2$ is also treated as a perturbation. The matching is carried out by calculating Green’s functions perturbatively in the two theories and demand they be the same for distances $R$ much larger than $1/T$. This way of calculating static correlators introduces infrared divergences at an intermediate stage and must be regularized. Note that these infrared divergences that appear are the same in the two theories and hence they cancel in the matching procedure. We use dimensional regularization as discussed in the appendix.

![FIG. 1: One -and two-loop vacuum diagrams for $\Phi$. A dot indicates an insertion of the operator $-(\mu^2 - m^2) \Phi^\dagger \Phi + \mu(\Phi^\dagger \partial_0 \Phi - \partial_0 \Phi^\dagger \Phi)$.](image)

In Fig. 1 we show the vacuum diagrams through two loops in the full theory and the expression is

$$\mathcal{F} \approx N \sum_p \ln P^2 - N (\mu^2 - m^2) \sum_p \left[ \frac{1}{P^2} - 2 \frac{P_0^2}{P^4} \right] - \frac{1}{2} N \mu^2 \sum_p \left[ \frac{1}{P^4} - \frac{P_0^2}{P^6} + \frac{8 P_0^4}{P^8} \right] + N \mu^2 m^2 \sum_p \left[ \frac{1}{P^4} - 4 \frac{P_0^2}{P^6} \right] + N (N + 1) \lambda_\Phi \sum_{pQ} \frac{1}{P^2Q^2}$$

$$+ 2 N (N + 1) \lambda_\Phi m^2 \sum_{pQ} \left[ \frac{1}{P^4Q^2} - 4 \frac{P_0^2}{P^6Q^2} \right], \quad (47)$$

where we have omitted terms of order $m^4$, $\mu^6$ etc. The sign $\approx$ is reminder that we are neglecting infrared physics which will be taken care of by the effective theory. Loop correction to the free energy in the 3d theory vanish since there is no momentum scale in the loop integrals 3. Thus $f$ is directly given by (47).

$$f = - \frac{N \pi^2 T^4}{45} \left( 1 - \frac{5(N + 1)\lambda}{16} \right) - \frac{1}{6} \frac{N \mu^2 T^2}{\pi^2} \left( 1 - \frac{\mu^2}{4\pi^2 T^2} + \frac{(N + 1)\lambda}{8\pi^2} \right) + \frac{1}{6} \frac{N \mu^2 T^2}{\pi^2} \left( 1 - \frac{3\mu^2}{4\pi^2 T^2} \right) \quad (48)$$

![FIG. 2: One -and two-loop Feynman diagrams for the self-energy $\Pi(p, p)$. A dot indicates an insertion of the operator $-(\mu^2 - m^2) \Phi^\dagger \Phi + \mu(\Phi^\dagger \partial_0 \Phi - \partial_0 \Phi^\dagger \Phi)$.](image)

The mass parameter is found by matching the two-point functions in the two theories at zero external momenta. The diagrams that contribute to the self-energy function in the full theory through two loops are shown in Fig. 2. The self-energy in the effective theory vanishes for the same reason as did the loop corrections to the free energy. This implies that the mass parameter $m_3^2$ is given directly by evaluating the Feynman diagrams in Fig. 2. We obtain

$$m_3^2 \approx -\mu^2 + m^2 + 2(N + 1)Z_\lambda \lambda \sum_\mu \frac{1}{P^2}$$

$$+ 2(N + 1) \left( \mu^2 - m^2 \right) \lambda \sum_\mu \left[ \frac{1}{P^4} - 4 \frac{P_0^2}{P^6} \right]$$

$$- 4(N + 1) \lambda_\Phi \sum_\mu \frac{1}{P_0 Q^4}$$

$$- 4(N + 1) \lambda_\Phi \sum_{\mu \nu} \frac{1}{P_0 Q^4 (P + Q)^2}, \quad (49)$$

where we again have neglected terms of higher-order terms. $Z_\lambda$ is the renormalization constant for the coupling $\lambda$:

$$Z_\lambda = 1 + \frac{(N + 4)\lambda}{8\pi^2 \epsilon}. \quad (50)$$

After renormalization, the mass term reduces to

$$m_3^2 = -\mu^2 + m^2 + \frac{(N + 1)\lambda}{6} T^2 \left[ 1 - \frac{3(\mu^2 - m^2)}{2\pi^2 T^2} \right.$$

$$+ \frac{3\lambda}{8\pi^2} \left( \frac{1}{\epsilon} + 2 \frac{2(N + 1)}{3} \gamma + 2 \lambda \left( -1 \right) \right)$$

$$\left. + \frac{(4 - 2N)}{3} \ln \frac{\Lambda}{4\pi T} \right]. \quad (51)$$

3 Recall that $m_3^2$ for the purpose of matching is treated as a perturbation and thus the propagators are massless.
Note that the mass parameter has a UV divergence after renormalization. The remaining divergence is exactly the one arising at the two-loop level in the 3d effective theory (see also Sec. IV C).

Finally, we need \( \lambda_3 \) at the tree level. By comparing the coefficients of the operator \( (\Phi^4)^2 \) in the two theories and taking the different normalization of the fields into account, one finds

\[
\lambda_3 = \lambda T .
\]  

(52)

C. Critical temperature

After having determined the parameters in the effective theory, the strategy for calculating \( T_c \) is as follows. First we determine the critical chemical potential \( \mu_c \) as a function of temperature and the critical value of the \( m_3^2 \). Then we calculate the density as a function of \( T \) and \( \mu \) and obtain the critical density as a function of \( T \) by substituting the expression for \( \mu_c \). Finally, the critical temperature is determined by inverting the critical density as a function of \( T \).

The relation between the bare mass and the renormalized mass is

\[
m_3^2 = m_{3,\text{ren}}^2 + \frac{(N+1)\lambda_3^2}{(4\pi)^2} \ln \Lambda .
\]  

(53)

This relation is exact due to the fact that the effective 3d theory is superrenormalizable. Since the bare mass is independent of the renormalization scale, the renormalized mass satisfies an evolution equation. This equation relates the value of \( m_{3,\text{ren}}^2 \) evaluated at two different normalization points \( \Lambda \) and \( \Lambda_0 \):

\[
m_{3,\text{ren}}^2(\Lambda_0) = m_{3,\text{ren}}^2(\Lambda) + \frac{(N+1)\lambda_3^2}{4\pi^2} \ln \frac{\Lambda_0}{\Lambda} .
\]  

(54)

Using Eq. (53), Eq. (51) for the chemical potential becomes

\[
\left( \mu^2 - m^2 \right) \left( 1 + \frac{(N+1)\lambda}{4\pi^2} \right) = \frac{(N+1)\lambda}{6} \left[ 1 - \frac{3\lambda}{8\pi^2} \left( 2 - \frac{2(N+1)}{3} \gamma + \frac{\zeta(1')}{\zeta(1)} \right) + \left( \frac{4-2N}{3} \ln \frac{\Lambda}{4\pi T} - \frac{16\pi^2}{N+1} m_{3,\text{ren}}^2(\Lambda) \right) \right].
\]  

(55)

The charge density is given by

\[
n = -\left\langle \frac{\partial S_{3d}}{\partial \mu} \right\rangle = -\frac{\partial f}{\partial \mu} + \frac{\partial \left( \Phi^4 \right)}{\partial \mu} \frac{\partial m^2}{\partial \mu} + \lambda_3 \left\langle (\Phi^4)^2 \right\rangle .
\]  

(56)

The quantity \( \left\langle (\Phi^4)^2 \right\rangle \) is by dimensional analysis proportional to \( \lambda_3^2 \). Its contribution to the density is therefore third order in the interaction and can be omitted in a second-order calculation. Using Eqs. (18), (31), and (36), the charge density becomes

\[
n = \frac{1}{3} N \mu T^2 \left[ 1 + \frac{3m^2}{4\pi^2 T^2} - \frac{\mu^2}{2\pi^2 T^2} + \frac{(N+1)\lambda}{8\pi^2} \right] - 2\mu \left\langle \Phi^4 \right\rangle .
\]  

(57)

At the critical point, the renormalized mass is by dimensional analysis proportional to \( \lambda_3^2 \). The case \( N = 2 \) is relevant to kaon condensates in stars and we therefore consider this case in the following. In the remainder of this section, we also restrict ourselves to the ultrarelativistic limit. Its value was determined by Sun [34] using lattice simulations:

\[
m_{3,\text{ren}}^2(\Lambda = \lambda_3/3) = \frac{0.002558(16)}{\lambda_3^2},
\]  

(58)

where the renormalization scale was chosen to be \( \Lambda = \lambda_3/3 \). The critical chemical potential then reduces to

\[
\mu_c = \sqrt{\frac{3}{2} \lambda T} \left[ 1 - \frac{3\lambda}{4\pi^2} \left( 2 - 2\gamma + \frac{\zeta(1')}{\zeta(1)} - 0.1346 \right) \right].
\]  

(59)

The expectation value \( \left\langle \Phi^4 \right\rangle \) cannot be calculated in perturbation theory due to infrared divergences. They depend on nonperturbative physics and can e.g. be determined using lattice simulations or the 1/N-expansion. At the critical point and for \( N = 2 \), it was computed by Sun [34] using Monte Carlo calculations:

\[
\left\langle \frac{\Phi^4}{\lambda_3} \right\rangle = -0.00289(18).
\]  

(60)

Inserting Eqs. (59) and (60) into Eq. (57), the critical density becomes

\[
n_c = \sqrt{\frac{2}{9} \lambda T^3} \left[ 1 - \frac{3\lambda}{4\pi^2} \left( 2 - 2\gamma + \frac{\zeta(1')}{\zeta(1)} + 0.3217 \right) \right].
\]  

(61)

Inverting this equation, we obtain the critical temperature as a function of the density

\[
T_c = \frac{9}{2\lambda} n^{1/3} \left[ 1 - \frac{\lambda}{(4\pi)^2} \left( 2 - 2\gamma + \frac{\zeta(1')}{\zeta(1)} + 0.3217 \right) \right].
\]  

(62)

The leading-order result is the usual perturbatively calculable high-temperature result, while the second-order term involves nonperturbative physics. In contrast, the first order correction to \( T_c \) in the nonrelativistic Bose gas cannot be determined in perturbation theory [12]. Note also that in accordance with Eq. (6), \( T_c \) becomes infinite in the absence of interactions. Finally, the impressive seven-loop VPT calculations of Kastening [21, 22] give \( 0.002586(17) \) and \( -0.002796(19) \) for the quantities in Eqs. (18) and (36). Thus the critical temperature is within errors in complete agreement with the lattice prediction.

V. SUMMARY

In this paper, we have discussed the thermodynamics of relativistic Bose gases at zero and finite temperature. At zero temperature, thermodynamic quantities can be expanded in a loop expansion, and we calculated the pressure, charge density and speed of sound in the one-loop approximation. In the nonrelativistic limit, one easily obtains the standard results...
for the dilute Bose gas. In the critical region, perturbation theory breaks down due to infrared divergences. However, one can take advantage of the fact that there is a separation of scales to simplify the problem of calculating static quantities such as $T_c$. The effects of the nonstatic modes can be calculated perturbatively employing dimensional reduction, while the effective three-dimensional theory must be treated nonperturbatively. It is interesting to note that the leading correction to $T_c$ in the ultrarelativistic limit, is calculable in perturbation theory while in the NR limit it is not. The reason is simply that the chemical potential couples quadratically to $\Phi\Phi$ in the first case and linearly in the latter.

By gauging the linear $SU(2)_L \times SU(2)_R$-symmetric sigma model in various ways, one obtains a number of interesting gauge theories [10, 11]. The Higgs mechanism and the Goldstone mechanism are both realized in the conventional manner. The interest in these models is partly due to the fact that the rotational symmetry is broken as well and leads to a directional dependence of the dispersion relation which is linear for small wave vectors and roton-like for larger wave vectors. These models may therefore be of interest for condensed matter systems such as superfluid helium.

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APPENDIX A: FORMULAS

Dimensional regularization can be used to regularize both the ultraviolet divergences and infrared divergences in three-dimensional integrals over momenta. The spatial dimension is generalized to $d = 3 - 2\epsilon$ dimensions. Integrals are evaluated at a value of $d$ for which they converge and then analytically continued to $d = 3$. We use the integration measure

$$\int_p \equiv \left(\frac{e^\gamma \Lambda^2}{4\pi}\right)^\epsilon \int \frac{d^3 p}{(2\pi)^{3-2\epsilon}}.$$  \hspace{1cm} (A.1)

where $\Lambda$ is an arbitrary momentum scale. The factor $(e^\gamma/4\pi)^\epsilon$ is introduced so that, after minimal subtraction of the poles in $\epsilon$ due to ultraviolet divergences, $\Lambda$ coincides with the renormalization scale of the MS renormalization scheme.

We need the regularized integrals

$$\int_p \sqrt{p^2 + \mu^2} = -\mu^4 \left(\frac{\Lambda}{\mu}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + \frac{3}{2}\right],$$ \hspace{1cm} (A.2)

$$\int_p \left(p^2 + \mu^2\right)^{3/2} = \frac{4}{(4\pi)^2} \left(\frac{\Lambda}{\mu}\right)^{2\epsilon} \left[\frac{1}{\epsilon}\right],$$ \hspace{1cm} (A.3)

where $\Lambda$ is an arbitrary momentum scale.

In the imaginary-time formalism for thermal field theory, the 4-momentum $P = (\omega_n, \mathbf{p})$ is Euclidean with $P^2 = \omega_n^2 + \mathbf{p}^2$. The Euclidean energy $\omega_n$ has discrete values: $\omega_n = 2n\pi T$ for bosons, where $n$ is an integer. Loop diagrams involve sums over $\omega_n$ and integrals over $\mathbf{p}$. We define the dimensionally regularized sum-integral by

$$\sum\int_P = \left(\frac{e^\gamma \Lambda^2}{4\pi}\right)^\epsilon T \sum_{\omega_n = 2n\pi T} \int \frac{d^3 p}{(2\pi)^3}.$$  \hspace{1cm} (A.5)

The specific sum-integrals needed are

$$\sum\int_P \ln P^2 = -\frac{\pi^2 T^2}{45},$$ \hspace{1cm} (A.6)

$$\sum\int_P \frac{P^2}{P^4} = \frac{T^2}{12} \left[1 + \left(2 + 2\frac{\zeta^\prime(-1)}{\zeta(-1)}\right)\epsilon\right],$$ \hspace{1cm} (A.7)

$$\sum\int_P \frac{P^2_0}{P^4_0} = \frac{T^2}{24} \left[1 + 2\frac{\zeta^\prime(-1)}{\zeta(-1)}\epsilon\right],$$ \hspace{1cm} (A.8)

$$\sum\int_P \frac{1}{P^4} = \frac{1}{(4\pi T)} \left(\frac{\mu}{4\pi T}\right)^{2\epsilon} \frac{1}{\epsilon} + 2\gamma,$$ \hspace{1cm} (A.9)

$$\sum\int_P \frac{P^2_0}{P^4_0} = \frac{1}{(4\pi T)} \left(\frac{\mu}{4\pi T}\right)^{2\epsilon} \frac{1}{8} \left[\frac{1}{\epsilon} + 8 + 3 + 2\gamma\right],$$ \hspace{1cm} (A.10)

$$\sum\int_P \frac{P^4_1}{P^4_0} = \frac{1}{(4\pi T)} \left(\frac{\mu}{4\pi T}\right)^{2\epsilon} \frac{1}{8} \left[\frac{1}{\epsilon} + 8 + 3 + 2\gamma\right],$$ \hspace{1cm} (A.11)

$$\sum\int_{PQ} \frac{1}{P^2Q^2(P + Q)^2} = 0.$$ \hspace{1cm} (A.12)

We also need to expand some sum-integrals about zero temperature. The phonon part of the spectrum then dominates the temperature-dependent part of the sum-integral. We can therefore approximate the dispersion relations $\omega_1(p)$ and $\omega_2(p)$ by their low-momentum limits (13-19), and this gives the leading temperature correction. These are

$$\sum\int_P \ln [P_0^2 + \omega_1^2(p)] = \int_P \omega_1(p) + \frac{T}{\pi^2} \int_0^\infty dp p^2 \ln [1 - e^{-\beta \omega_1(p)}]$$ \hspace{1cm} (A.13)

$$= \int_P \omega_1(p) - \frac{\pi^2 T^4}{45} \left(3\mu^2 - m^2\right)^{3/2} + \ldots,$$ \hspace{1cm} (A.14)

$$\sum\int_P \ln [P_0^2 + \omega_2^2(p)] = \int_P \omega_2(p) + \frac{T}{\pi^2} \int_0^\infty dp p^2 \ln [1 - e^{-\beta \omega_2(p)}]$$ \hspace{1cm} (A.15)

$$= \int_P \omega_2(p) - 2T \left(\frac{\mu}{T}\right)^{3/2} \zeta\left(\frac{3}{2}\right) + \ldots.$$ \hspace{1cm} (A.16)

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