Update on 3-folds

Miles Reid

Abstract

The familiar division of compact Riemann surfaces into 3 cases

\[ g = 0, \quad g = 1 \quad \text{and} \quad g \geq 2 \]

corresponds to the well known trichotomy of spherical, Euclidean and hyperbolic non-Euclidean plane geometry. Classification aims to treat all projective algebraic varieties in terms of this trichotomy; the model is Castelnuovo and Enriques' treatment of surfaces around 1900 (reworked by Kodaira in the 1960s). The canonical class of a variety may not have a definite sign, so we usually have to beat it up before the trichotomy applies, by a minimal model program (MMP) using contractions, flips and fibre space decompositions. The classification of 3-folds was achieved by Mori and others during the 1980s.

New results over the last 5 years have added many layers of subtlety to higher dimensional classification. The study of 3-folds also yields a rich crop of applications in several different branches of algebra, geometry and theoretical physics. My lecture surveys some of these topics.

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1. Popular introduction: the great trichotomy

A trichotomy is a logical division into three cases, where we expect to win something in each case. The cases here are similar to the “much too small, just right, much too big” of Goldilocks and the Three Bears, or the geometric division of conic sections into ellipse, parabola and hyperbola due to Appollonius of Perga (200 BC), or the cosmological question of whether the universe contracts again into a big crunch, tends to an asymptotic state or continues expanding exponentially.

*Mathematics Institute, University of Warwick, Coventry CV4 7AL, England, UK. E-mail: miles@maths.warwick.ac.uk
1.1. Euclidean and non-Euclidean geometry

Euclid’s famous parallel postulate (c. 300 BC) states that

if a line falls on two lines, with interior angles on one side adding to
< 180°, the two lines, if extended indefinitely, meet on the side on
which the angles add to < 180°.

We are in plane geometry, assumed homogeneous so that any construction involving
lines, distances, angles, triangles and so on can be carried out at any point and in
any orientation with the same effect. In this context the great trichotomy is the
observation, probably due originally to Omar Khayyam (11th c.), Nasir al-Din al-
Tusi (13th c.) and Gerolamo Saccheri (1733), that two other cases besides Euclid’s
are logically coherent (see Figure 1). In spherical geometry, the two lines meet

on both sides whatever the angles, whereas in hyperbolic non-Euclidean geometry,
the two lines may diverge even though the angle sum is < 180°. Whether lines
eventually meet is a long-range question, but it reflects the local curvature of the
geometry.

1.2. Gauss and Riemann on differential geometry

A local surface $S$ in 3-space is positively curved if all its sections bend in the
same direction like the top of a sphere (see Figure 2). $S$ is flat (or developable) if

it is straight in one direction like a cylinder, and negatively curved if its sections
bend in opposite directions like a saddle or Pringle’s chip. Gauss in his Theorema
Egregium (1828) and Riemann in his Habilitationsschrift (1854) found that curva-
ture is intrinsic to the local distance geometry of $S$, independent of how $S$ sits in
3-space: living on a sphere $S$ of radius $R$, we can measure the perimeter of a disc of
radius $r$, which is $2\pi(sin^{2}r/R)R$, always less than the Euclidean value $2\pi r$. If we lived
in the hyperbolic plane, the perimeter of a disc of radius \( r \) would be \( 2\pi \sinh \frac{r}{R} R \), bigger than the Euclidean value, and growing exponentially with \( r \).

Riemann in particular generalised Gauss’ ideas on surfaces to a space given locally by an \( n \)-tuple \((x_1, \ldots, x_n)\) of real parameters (a “many-fold extended quantity” or manifold), with distance arising from a local arc length \( ds \) given by a quadratic form \( ds^2 = \sum g_{ij} dx_i dx_j \). The curvature is then a function of the second derivatives of the metric function \( g_{ij} \). Riemann’s differential geometry works with manifolds that are not homogeneous, e.g., having positive, zero, or negative curvature at different points. It was a key ingredient in Einstein’s general relativity (1915), which treats gravitation as curvature of space-time.

1.3. Riemann surfaces

The story moves on from real manifolds (e.g., surfaces depending on 2 real variables) to Riemann surfaces, parametrised instead by a single complex variable. The point here is Cauchy’s discovery (c. 1815) that differentiable functions of a complex variable are better behaved than real functions, and much more amenable to algebraic treatment. Riemann discovered that a compact Riemann surface \( C \) has an embedding \( C \cong \mathbb{P}^N_C \) into complex projective space whose image is defined by a set of homogeneous polynomial equations.

A projective algebraic curve \( C \subset \mathbb{P}^N_C \) is nonsingular if at every point \( P \in C \) we can choose \( N - 1 \) local equations \( f_1, \ldots, f_{N - 1} \) so that the Jacobian matrix \( \frac{\partial f_i}{\partial x_j} \) has maximal rank \( N - 1 \). It follows from the implicit function theorem that one of the linear coordinates \( z = z_1 \) of \( \mathbb{P}^N \) can be chosen as a local analytic coordinate on \( C \). In other words, a compact Riemann surface is analytically isomorphic to a nonsingular complex projective curve.

1.4. The genus of an algebraic curve

The canonical class \( K_C = \Omega^1_C = T_C^* \) of a curve \( C \) is the holomorphic line bundle of 1-forms on \( C \); it has transition functions on \( U \cap U' \) the Jacobian of the coordinate change \( \frac{\partial z'}{\partial z} \), where \( z, z' \) are local analytic coordinates on \( U, U' \). If \( z \) is a rational function on \( C \) that is an analytic coordinate on an open set \( U \subset C \) then a 1-form on \( U \) is \( f(z)dz \) with \( f \) a regular function on \( U \). That is, \( \Omega^1_C = \mathcal{O} \cdot dz \), or \( dz \) is a basis of \( \Omega^1_C \) on \( U \).

The genus \( g(C) \) can be defined in several ways: topologically, a compact Riemann surface is a sphere with \( g \) handles (see Figure 3). It has Euler number

\[
e(C) = 2 - 2g,\]

which equals \( \deg T_C \). The most useful formula for our purpose is
\[ \text{deg} K_C = 2g - 2. \] We see that
\[ K_C < 0 \iff g = 0, \quad K_C = 0 \iff g = 1, \quad K_C > 0 \iff g \geq 2. \]

This trichotomy is basic for the study of a curve \( C \) from every point of view, including topology, differential geometry, complex function theory, moduli, all the way through to algebraic geometry and Diophantine number theory. To relate this briefly to curvature as discussed in Section 1.2, for an arbitrary Riemannian metric, the average value of curvature over \( C \) equals \(-\text{deg} K_C\) by the Gauss–Bonnet theorem; moreover, by the Riemann mapping theorem, there exists a metric on \( C \) in the conformal class of the complex structure with constant positive, zero or negative curvature in the three cases.

### 2. Classification of 3-folds

The great trichotomy also drives classification in higher dimensions. The meaning of “higher dimensions” is time-dependent: \( \dim 2 \) was worked out around 1900 by Castelnuovo and Enriques, \( \dim 3 \) during the 1980s by Mori and others, and \( \dim 4 \) is just taking off with Shokurov’s current work. I concentrate on \( \dim 3 \), where these issues first arose systematically.

#### 2.1. Preliminaries: the canonical class \( K_X \)

An \( n \)-dimensional projective variety \( X \) can be embedded \( X \hookrightarrow \mathbb{P}^N_C \), and is given there by homogeneous polynomial equations; nonsingular means that at every point \( P \in X \), we can choose \( N - n \) of the defining equations so that the Jacobian matrix \( \frac{\partial f_i}{\partial x_j} \) has rank \( N - n \), with \( n \) linear coordinates of \( \mathbb{P}^N \) providing local analytic coordinates on \( X \).

The canonical class of \( X \) is \( K_X = \Omega^n_X = \wedge^n \Omega^1_X \). It has many interpretations: it is the line bundle obtained as the top exterior power of the holomorphic cotangent bundle; it has transition functions on \( U \cap U' \) the Jacobian determinant \( \det \left| \frac{\partial x'}{\partial x} \right| \), where \( x, x' \) are systems of local analytic coordinates on open sets \( U, U' \subset X \); its sections are holomorphic \( n \)-forms; at a nonsingular point \( P \in X \), its sections are generated by the holomorphic volume form \( dx_1 \wedge \cdots \wedge dx_n \), so that \( \Omega^n_X = \mathcal{O}_X \cdot dx_1 \wedge \cdots \wedge dx_n \).

For MMP to work in \( \dim \geq 3 \), we are eventually forced to allow certain mild singularities. The theory in \( \dim 3 \) is now standard and not very hard (see [YPG] and compare the foreword to [CR]). We always insist that the first Chern class of \( K_X \) restricted to the nonsingular locus \( X^0 \subset X \) comes from an element of \( H^2(X, \mathbb{Q}) \), that I continue to denote by \( K_X \). This ensures that the pullback \( f^* K_X \) by a morphism \( f : Y \rightarrow X \) is defined, together with the intersection number \( K_X C \) with every curve \( C \subset X \) (obtained by evaluating \( K_X \in H^2 \) against the class \([C]\) \( \in H_2(X, \mathbb{Q}) \)). Note that \(-K_X C\) is an integral or average value of Ricci curvature (a 2-form) calculated over a 2-cycle \([C]\) corresponding to a holomorphic curve; we have taken several steps back from varieties of constant curvature suggested by the colloquial pictures in Section 1.
In many contexts, the canonical class of a variety is closely related to the discrepancy divisor. If $f: Y \to X$ is a birational morphism, its discrepancy $\Delta_f$ is defined by $K_Y = f^* K_X + \Delta_f$; if $X$ and $Y$ are nonsingular this is the divisor of zeros $\Delta_f = \text{div} \left( \det \left| \begin{array}{cc} \frac{\partial x_i}{\partial y_j} \\ \frac{\partial y_j}{\partial y_j} \end{array} \right| \right)$ of the Jacobian determinant of $f$, or its appropriate generalisation if $X$ and $Y$ are singular. Since the components of $\Delta_f$ are exceptional, it follows that if $\Delta_f > 0$, then there exists a component $E$ of $\Delta_f$ such that $K_Y C < 0$ for almost every curve $C \subset E$. It is known that every section $s \in H^0(Y, nK_Y)$ vanishes along $\Delta_f$ for every $n \geq 0$. A morphism $f$ is crepant if $\Delta_f = 0$; then $K_Y = f^* K_X$, so that $K_Y$ is numerically zero relative to $f$.

2.2. The trichotomy: $K_X < 0$, $K_X = 0$ or $K_X > 0$?

The naive section heading is misleading: $K_X$ may have “different sign” at different points of $X$ and in different directions. The aim is not to apply the trichotomy to $X$ itself, but to modify it first to a variety $X'$ by a MMP. We need to be more precise; we say that $K_X$ is nef or numerically nonnegative if $K_X C \geq 0$ for every $C \subset X$ (nef is an acronym for numerically eventually free – we hope that $|nK_X|$ is a free linear system for some $n > 0$). As we saw at the end of Section 2.1, a discrepancy divisor $\Delta_f > 0$ for a birational morphism $f: Y \to X$ is a local obstruction to the nefdom of $K_Y$. Mori theory (or the MMP) is concerned with the case that $K_X$ is not nef.

2.3. Results of MMP for 3-folds

The Mori category consists of (quasi-)projective $n$-folds $X$ with $\mathbb{Q}$-factorial terminal singularities; see [YPG] for details. For $X$ in the Mori category, an elementary contraction is a morphism $\varphi: X \to X_1$ such that

(i) $X_1$ is a normal variety and $\varphi$ has connected fibres.
(ii) All curves $C \subset X$ contracted by $\varphi$ have classes in a single ray in $H_2(X, \mathbb{R})$, and $K_X C < 0$. This implies that $-K_X$ is relatively ample.

An elementary contraction $X \to S$ with $\dim S < \dim X$ is a Mori fibre space (Mfs). The case to bear in mind is when $S = \text{pt}$.; then (ii) implies that $-K_X$ is ample and $\rho(X) = \text{rank Pic} X = 1$, that is, $X$ is a Fano 3-fold with $\rho = 1$. If $\dim S = \dim X - 1$ then $X \to S$ is a conic bundle.

**Theorem 1** (see for example [KM]) An elementary contraction exists if and only if $K_X$ is not nef. For any 3-fold $X$ in the Mori category there is a chain of birational transformations

$$X \dasharrow X_1 \dasharrow \cdots \dasharrow X_n = X'$$

where (1) each step $X_i \dasharrow X_{i+1}$ is an elementary divisorial contraction or flip of the Mori category, and (2) the final object $X'$ either has $K_X$, nef, or has a Mfs structure $X' \to S$.

Each birational step $X_i \dasharrow X_{i+1}$ removes a subvariety of $X$ on which $K_X$ is negative. A divisorial contraction contracts an irreducible surface in $X$ to a curve.
or a point. A flip is a surgery operation that cuts out a finite number of curves in $X_i$ on which $K$ is negative, replacing them with curves on which $K$ is positive. At the end of the MMP comes the dichotomy: either $K_{X'}$ is nef, or $-K_{X'}$ is ample on a global structure of $X'$.

The main theorem on varieties with $K_X$ nef is the existence of an Iitaka–Kodaira fibration $X \to Y$, with fibres the curves $C \subset X$ with $K_X C = 0$. This gives a natural case division according to $\dim Y$. The extreme cases are Calabi-Yau varieties (CY), where $K_X = 0$, and varieties of general type, where $X \to Y$ is birational to a canonical model $Y$ having canonical singularities and ample $K_Y$.

This takes my story up to around 1990; for more details, see Kollár and Mori [KM] or Matsuki [M].

3. Lots of recent progress

3.1. Extension of MMP to dimension 4

Already from the mid 1980s, it was understood that the MMP could in large parts be stated in all dimensions as a string of conjectures (or the log MMP, where we proceed in like manner, but directed by a log canonical class $K_X + D$). The difficult parts in $\dim \geq 3$ are the existence of flips (or log flips), and the termination of a chain of flips. Recent work of Shokurov [Sh] has established the existence of log flips in $\dim 4$; the key idea is the reduction to prelimiting flips, already prominent in Shokurov’s earlier work (see [FA], Chapter 18).

3.2. Rationally connected varieties

A variety $X$ is rational if it is birationally equivalent to $\mathbb{P}^n$. That is, there are dense Zariski open sets $X_0 \subset X$ and $U \subset \mathbb{P}^n$, and an isomorphism $X_0 \cong U$ such that both $\varphi$ and $\varphi^{-1}$ are given by rational maps. In other words, $X$ has a one-to-one parametrisation by rational functions. By analogy with curves and surfaces, one might hope that rational varieties have nice characterisations, and that rationality behaves well under passing to images or under deformation. Unfortunately, in $\dim \geq 3$, our experience is that this is not the case, and we are obliged to give up on the question of rationality.\footnote{This is of course exaggerated. Rationality itself remains the major issue in many contexts, in particular the rationality of GIT quotients. Iskovskikh’s conjectured rationality criterion for conic bundles remains one of the driving forces of 3-fold birational geometry. Thanks to Slava Shokurov for reminding me of this important point.}

However, it turns out that the notion of rationally connected variety developed independently by Campana and by Kollár, Miyaoka and Mori is a good substitute. $X$ is rationally connected if there is a rational curve through any two points $P, Q \in X$. See [Ca], [KMM], [Ko] and [GHS] for developments of this notion.
3.3. Explicit classification results for 3-folds

Section 2.3 discussed the Mori category and its elementary contractions. The explicit classification manifesto of the foreword of [CPR] calls for the abstract definitions and existence results to be translated into practical lists of normal forms. The ideal result here is Mori’s theorem [YPG], Theorem 6.1, that classifies 3-fold terminal singularities into a number of families; these relate closely to cyclic covers between Du Val singularities, and deform to varieties having only the terminal cyclic quotient singularities $\frac{1}{r}(1, a, -a)$.

To complete our grasp of Mori theory, we hope for explicit classification results in this style for divisorial contractions, flips and Mfs. The last few years have seen remarkable progress by Kawakita [Ka1], [Ka2] on divisorial contractions to points. A guiding problem in this area was Corti’s 1994 conjecture ([Co2], p. 283) that every Mori divisorial contraction $\varphi: X \to Y$ to a nonsingular point $P \in Y$ is a $(1, a, b)$ weighted blowup. Kawakita proved this, and went on to classify explicitly the divisorial contractions to compound Du Val singularities of type A. There are also results of Tziolas on contractions of surfaces to curves. For progress on Mfs see Section 4.3.

3.4. Calabi-Yau 3-folds and mirror symmetry

A CY manifold $X$ is a Kähler manifold with $K_X = 0$, usually assumed simply connected, or at least having $H^1(\mathcal{O}_X) = 0$. A popular recipe for constructing CY 3-folds is due to Batyrev, based on resolving the singularities of toric complete intersections. This gives some 500,000,000 families of CY 3-folds, so much more impressive than a mere infinity (see the website [KS]). There are certainly many more; I believe there are infinitely many families, but the contrary opinion is widespread, particularly among those with little experience of constructing surfaces of general type.

Calabi-Yau 3-folds are the scene of exciting developments related to the Strominger-Yau-Zaslow special Lagrangian approach to mirror symmetry. For lack of space, I refer to Gross [Gr] for a recent discussion.

3.5. Resolution of orbifolds and McKay correspondence

Klein around 1870 and Du Val in the 1930s studied quotient singularities $\mathbb{C}^2/G$ for finite groups $G \subset \text{SL}(2, \mathbb{C})$. Du Val characterised them as singularities that “do not affect the condition of adjunction”, that is, as surface canonical singularities. Quotient singularities $\mathbb{C}^3/G$ by finite subgroups $G \subset \text{SL}(3, \mathbb{C})$ were studied by many authors around 1990; they proved case-by-case that a crepant resolution exists, and that its Euler number is equal to the number of conjugacy classes of $G$, as predicted by string theorists. The McKay correspondence says that the geometry of the crepant resolution of $\mathbb{C}^3/G$ can be described in terms of the representation theory of $G$. This has now been worked out in a number of contexts; see my Bourbaki talk [Bou].
3.6. The derived category as an invariant of varieties

The derived category $D(A)$ of an Abelian category $A$ was introduced by Grothendieck and Verdier in the 1960s as a technical tool for homological algebra. A new point of view emerged around 1990 inspired by results of Beilinson and Mukai: for a projective nonsingular variety $X$ over $\mathbb{C}$, write $D(X)$ for the bounded derived category of coherent sheaves on $X$; following Bondal and Orlov, we consider $D(X)$ up to equivalence of $\mathbb{C}$-linear triangulated category as an invariant of $X$, somewhat like a homology theory; the Grothendieck group $K_0(X)$ is a natural quotient of $D(X)$.

The derived category $D(X)$ is an enormously complicated and subtle object (already for $\mathbb{P}^2$); in this respect it is like the Chow groups, that are usually infinite dimensional, and contain much more information than anyone could ever use. Despite this, there are contexts, usually involving moduli constructions, in which “tautological” methods give equivalences of derived categories between $D(X)$ and $D(Y)$. An example is the method of [BKR] that establishes the McKay correspondence on the level of derived categories by Fourier–Mukai transform. There is no such natural treatment for the McKay correspondence in ordinary (co-)homology (see [Cr]).

The following conjectural discussion is based on ideas of Bondal, Orlov and others, as explained by Bridgeland (and possibly only half-understood by me). As I said, classification divides up all varieties into $K > 0$, $K = 0$, $K < 0$ and constructions made from them. Current work with $D(X)$ assumes that $X$ is nonsingular, but I ignore this technical point. There must be some sense in which the derived category of a variety with $K < 0$ is “small” or “discrete”; for example, a semi-orthogonal sum of discrete pieces arising from smaller dimension. A contraction of the MMP should break off a little $K < 0$ semi-orthogonal summand; for nonsingular blowups, this is known [O], and also for certain flips [K]. For a variety $X$ with $K = 0$, we expect $D(X)$ to have enormous symmetry, like a K3 or CY 3-fold; and for a variety with $K > 0$, $D(X)$ should be very infinite but rigid and indecomposable. Bondal and Orlov [BO] have proved that $D(X)$ determines $X$ uniquely if $\pm K_X$ is ample, but as far as I know, they have not established a qualitative difference between the two cases.

Right up to Kodaira’s work on surfaces in the 1960s, minimal models were seen in terms of tidying away $-1$-curves to make a really neat choice of model in a birational class, that eventually turns out to be unique. In contrast, starting from around 1980, the MMP in Mori theory sets itself the direct aim of making $K$ nef if possible. Derived categories give us a revolutionary new aim: each step of the MMP chops off a little semi-orthogonal summand of $D(X)$.

4. Fano 3-folds: biregular and birational geometry

4.1. The Sarkisov program
The modern view of MMP and classification of varieties is as a \textit{biregular} theory: although we classify varieties up to birational equivalence, the aims and the methods are stated in biregular terms. Beyond the MMP, the main birational problems are the following:

1. If $X$ is birational to a Mfs as in Theorem 1, then \textit{in how many different ways} is it birational to a Mfs?
2. Can we decide when two Mfs are birationally equivalent?
3. Can we determine the group of birational selfmaps of a Mfs?

The Sarkisov program gives general answers to these questions, at least in principle. It untwists any birational map $\varphi: X \to Y$ between the total spaces of two Mfs $X/S$ and $Y/T$ as a chain of links, generalising Castelnuovo's famous treatment of birational maps of $\mathbb{P}^2$. A Sarkisov link of Type II consists of a Mori divisorial extraction, followed by a number of antiflips, flops and flips (in that order), then a Mori divisorial contraction.

More generally, the key idea is always to reduce to a \textit{2-ray game} in the Mori category (see [Co2], 269–272). By definition of Mfs, we have $\rho(X/S) = 1$, but for a 2-ray game we need a contraction $X' \to S'$ with $\rho(X'/S') = 2$. A Sarkisov link starts in one of two ways (depending on the nature of the map $\varphi$ we are trying to untwist): either blow $X$ up by a Mori extremal extraction $X' \to X$ and leave $S' = S$; or find a contraction $S \to S'$ of $S$ so that $\rho(X/S) = 2$ and leave $X = X'$. In either case, the Mori cone of the new $X'/S'$ is a wedge in $\mathbb{R}^2$ with a marked Mori extremal ray, and we can play a 2-ray game that contracts the other ray, flipping it whenever it defines a small contraction. It is proved that, given $\varphi: X \to Y$, one or other of these games can be played, and the link ends as it began in a Mori divisorial contraction or a change of Mfs structure, making four types of links. Each link decreases a (rather complicated) invariant of $\varphi$, and it is proved that a chain of links terminates. See [Co] and Matsuki [M] for details.

\section*{4.2. Birational rigidity}

While the Sarkisov program factors birational maps as a chain of links that are elementary in some categorical sense, an explicit description of general links is still a long way off. To obtain generators of the Cremona group of $\mathbb{P}^3$ would involve classifying every Mfs $X/S$ that is rational, and every Sarkisov link between these; for the time being, this is an impossibly large problem. There is, however, a large and interesting class of Mfs for which there are rather few Sarkisov links.

A Mori fibre space $X \to S$ is \textit{birationally rigid} if for any other Mfs $Y \to T$, a birational map $\varphi: X \dasharrow Y$ can only exist if it lies over a birational map $S \dasharrow T$ such that $X/S$ and $Y/T$ have isomorphic general fibres (but $\varphi$ need not induce an isomorphism of the general fibres – this is a tricky definition). If $S = \text{pt}$, so that $X$ is a Fano variety with $\rho(X) = 1$, the condition means that the only Mfs $Y/T$ birational to $X$ is $Y \cong X$ itself. For example, $\mathbb{P}^2$ is not rigid, since it is birational to all the scrolls $\mathbb{F}_n$. Following imaginative but largely non-rigorous work of Fano in the 1930s, Iskovskikh and Manin proved in 1971 that a nonsingular quartic 3-fold $X_4 \subset \mathbb{P}^4$ is birationally rigid. This proof has since been simplified and reworked.
by many authors. The main result of [CPR] is that a general element $X$ of any of the famous 95 families of Fano hypersurface $X_d \subset \mathbb{P}(1,a_1,\ldots,a_4)$ is likewise birationally rigid.

It is interesting to take a result of Corti and Mella [CM] as an example going beyond the framework of [CPR]. The codim 2 complete intersection $X_{3,4} \subset \mathbb{P}^5(1,1,1,1,2,2)$ is a Fano 3-fold; write $x_1, \ldots, x_4, y_1, y_2$ for homogeneous coordinates and $f_3 = g_4 = 0$ for the equations of $X_{3,4}$. By a minor change of coordinates, I can assume that $g_4 = y_1 y_2 + y'(x_1, \ldots, x_4)$. Then $X_{3,4}$ has $2 \times \frac{1}{2}(1,1,1)$ quotient singularities at the $y_1, y_2$ coordinate points. [CM] shows that blowing up either of these point leads to a Sarkisov link

$$X_{3,4} \dashrightarrow Y_5 \dashleftarrow Z_4$$

$$\bigcap \mathbb{P}^5(1^4,2^2) \bigcap \mathbb{P}^4(1^4,2) \bigcap \mathbb{P}^4$$

(4.1)

Here the midpoint $Y_5$ of the link is a general quintic containing the plane $\Pi = \mathbb{P}^2$, say given by $\Pi : (x_4 = y_1 = 0)$. Thus $Y_5 : (A_4 x_4 - B_3 y_1 = 0)$, where $A_4, B_3$ are quartic and cubic; note that $Y_5$ itself is not in the Mori category, because it is not factorial. We obtain $X_{3,4}$ by adding $y_2 = \frac{A_4}{B_3}$ to its homogeneous coordinate ring, and $Z_4$ by adding $x_0 = \frac{A_4}{x_4} = \frac{B_3}{x_4}$.

This example makes several points: $X_{3,4}$ and $Z_4$ are both Mori Fano 3-folds with $\rho = 1$. They are not birationally rigid, since they are birational to one another. [CM] proves that they are not birational to any MFs other than $X_{3,4}$ and $Z_4$, so they form a bi-rigid pair. $X_{3,4}$ is general in its family, whereas $Z_4$ has in general a double point locally isomorphic to $x^2 + y^2 + z^3 + t^3$. This is a new kind of phenomenon that arises many times as soon as we go beyond the Fano hypersurfaces.

4.3. Explicit classification of Fano 3-folds

The anticanonical ring $R(X, -K_X) = \bigoplus H^0(-nK_X)$ of a Fano 3-fold $X$ is a Gorenstein ring. Choosing a minimal set of homogeneous generators $x_0, \ldots, x_N$ of $R$ with $\text{wt} x_i = a_i$ defines an embedding $X \hookrightarrow \mathbb{P}(a_0, \ldots, a_N)$ as a projectively normal variety. The codimension of $X$ is its codimension $N - 3$ in this embedding. If $N \leq 3$ the equations defining $X$ are well understood, and we can describe $X$ explicitly. For example, Altınok [Al] gives 69 families of Fano 3-folds whose general element has anticanonical ring of codim 3, given by the $4 \times 4$ Pfaffians of a $5 \times 5$ matrix, that is, a section of a weighted Grassmannian $\text{wGr}(2,5)$ in the sense of [CR2].

The paper [ABR] explains how to use the formulas of [YPG] and the ideas of [Al] to make a computer database that includes all possible Hilbert series for $R(X, -K_X)$. In most cases the rings themselves can be studied by projection methods, as described in [Ki], in fact usually by projections of the simplest type. In other words, as in (4.1), we can make a weighted blowup $Y \to X$ of a terminal quotient singularity of $X$ of type $\frac{1}{2}(1,a,-a)$. If we know $R(Y, -K_Y)$ and the ideal of the blown up $\mathbb{P}(1,a,-a)$ in it, we can reconstruct $X$ by Kustin–Miller unprojection [PR]. Takagi’s examples in [Ki], 6.4 and 6.8 is a warning that this process
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is entertaining and nontrivial: there are two different families of Fano 3-folds in codim 4 with the same Hilbert series, obtained by unprojections that are numerically identical, and that differ only in the way that their unprojection planes embed \( \Pi = \mathbb{P}^2 \hookrightarrow \text{wGr}(2,5) \) in the weighted Grassmannian. These are the Tom and Jerry unprojections of [Ki], Section 8. The K3 surface sections of the two families form a single unobstructed family, but their extension to Fano 3-folds break up into two families; this is reminiscent of the extension-deformation theory of the del Pezzo surface of degree \( S_6 \), which has both \( \mathbb{P}^2 \times \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) as extensions.

References

[Al] S. Altınok, Graded rings corresponding to polarised K3 surfaces and Q-Fano 3-folds, Univ. of Warwick Ph.D. thesis, Sep. 1998, vii+93, get from www.maths.warwick.ac.uk/~miles/doctors/Selma.

[ABR] S. Altınok, G. Brown and M. Reid, Fano 3-folds, K3 surfaces and graded rings, in Singapore Internat. Symp. in Topology and Geometry (NUS, 2001), Ed. A.J. Berrick and others, Contemp. Math. AMS, 2002; preprint math.AG/0202092, 29.

[BO] Alexei Bondal and Dmitri Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, Comp. Math. 125 (2001) 327–344.

[BKR] Tom Bridgeland, Alastair King and Miles Reid, Mukai implies McKay, J. Amer. Math. Soc. 14 (2001) 535–554.

[Ca] F. Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. École Norm. Sup. 25 (1992) 539–545.

[Co] Alessio Corti, Factoring birational maps of threefolds after Sarkisov, J. Alg. Geom. 4 (1995) 223–254.

[Co2] Alessio Corti, Singularities of linear systems and 3-fold birational geometry, in [CR], 259–312.

[CM] Alessio Corti and Massimiliano Mella, Birational geometry of terminal quartic threefolds. I, can.dpmms.cam.ac.uk/~corti/cm.pdf, 41.

[CPR] Alessio Corti, Sasha Pukhlikov and Miles Reid, Fano 3-fold hypersurfaces, in [CR], 175–258.

[CR] Alessio Corti and Miles Reid, Explicit birational geometry of 3-folds, CUP, 2000.

[CR2] Alessio Corti and Miles Reid, Weighted Grassmannians, in memorial volume for Paolo Francia (Genova, Sep 2001), M. Beltrametti Ed., de Gruyter 2002, 22, preprint math.AG/0206011, 27.

[Cr] Alastair Craw, An explicit construction of the McKay correspondence for A-Hilb \( \mathbb{C}^3 \), math.AG/0010053, 30.

[GHS] Tom Graber, Joe Harris and Jason Starr, Families of rationally connected varieties, math.AG/0109220, 21.

[Gr] Mark Gross, Examples of special Lagrangian fibrations, math.AG/0012002, 29.

[K] Kawamata Yujiro, Francia’s flip and derived categories math.AG/0111041,
23.

[Ka1] Kawakita Masayuki, Divisorial contractions in dimension three which contract divisors to smooth points, Invent. Math. 145 (2001) 105–119.

[Ka2] Kawakita Masayuki, Divisorial contractions in dimension three which contract divisors to compound $A_1$ points, Comp. Math. to appear, math.AG/0010207, 23.

[Ko] Kollár János, Rational curves on algebraic varieties, Springer 1996.

[FA] Kollár János and others, Flips and abundance for algebraic threefolds, Astérisque 211 (1992), SMF 1992.

[KM] Kollár János and Mori Shigefumi, Birational geometry of algebraic varieties, CUP 1998.

[KMM] Kollár János, Miyaoka Yoichi and Mori Shigefumi, Rational connectedness and boundedness of Fano manifolds, J. Diff. Geom. 36 (1992) 765–779.

[KS] Maximilian Kreuzer and Harald Skarke, Calabi-Yau data, website tph16.tuwien.ac.at/~kreuzer/CY.

[M] Matsuki Kenji, Introduction to the Mori program, Springer, 2002.

[O] Dmitri Orlov, Projective bundles, monoidal transformations, and derived categories of coherent sheaves, Izv. 56 (1992) 852–862 = Russian Acad. Sci. Izv. Math. 41 (1993) 133–141.

[PR] S. Papadakis and M. Reid, Kustin–Miller unprojection without complexes, to appear in J. Algebraic Geometry, math.AG/0011094, 18.

[YPG] Miles Reid, Young person’s guide to canonical singularities, in Algebraic geometry (Bowdoin, 1985), Proc. Sympos. Pure Math. 46, Part 1, AMS 1987, 345–414.

[Ki] Miles Reid, Graded rings and birational geometry, in Proc. of algebraic geometry symposium (Kinosaki, 2000), K. Ohno Ed., 1–72, get from www.maths.warwick.ac.uk/~miles/3folds.

[Bou] Miles Reid, La correspondance de McKay, Séminaire Bourbaki, Astérisque 276, SMF 2002, 53–72.

[Sh] V.V. Shokurov, Prelimiting flips, to appear in Proc. Steklov Inst., first draft Aug 1999, Mar 2002 draft 247, available from www.maths.warwick.ac.uk/~miles/3folds.