Greedy solution of ill-posed problems: error bounds and exact inversion

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Abstract
The orthogonal matching pursuit (OMP) is a greedy algorithm to solve sparse approximation problems. Sufficient conditions for exact recovery are known with and without noise. In this paper we investigate the applicability of the OMP for the solution of ill-posed inverse problems in general, and in particular for two deconvolution examples from mass spectrometry and digital holography, respectively. In sparse approximation problems one often has to deal with the problem of redundancy of a dictionary, i.e. the atoms are not linearly independent. However, one expects them to be approximatively orthogonal and this is quantified by the so-called incoherence. This idea cannot be transferred to ill-posed inverse problems since here the atoms are typically far from orthogonal. The ill-posedness of the operator probably causes the correlation of two distinct atoms to become huge, i.e. that two atoms look much alike. Therefore, one needs conditions which take the structure of the problem into account and work without the concept of coherence. In this paper we develop results for the exact recovery of the support of noisy signals. In the two examples, mass spectrometry and digital holography, we show that our results lead to practically relevant estimates such that one may check a priori if the experimental setup guarantees exact deconvolution with OMP. Especially in the example from digital holography, our analysis may be regarded as a first step to calculate the resolution power of droplet holography.

1. Introduction
We consider linear inverse problems, i.e. we are given a bounded, injective, linear operator $K : B \rightarrow H$ mapping from a Banach space $B$ into a Hilbert space $H$. Moreover, we assume...
that for an unknown \( v \in \text{rg} \, K \), we are given a noisy observation \( v^e \) with \( \| v - v^e \| \leq \varepsilon \) and try to reconstruct the solution of

\[
K u = v
\]  

from the knowledge of \( v^e \). We are particularly interested in the case where the unknown solution \( u \) may be expressed sparsely in a known dictionary, i.e. we consider that there is a family \( \mathcal{E} := \{ e_i \}_{i \in \mathbb{Z}} \subset B \) of unit-normed vectors which span the space in which we expect the solution and which we call dictionary. By sparse we mean here that there exists a finite decomposition of \( u \) with \( N \) atoms \( e_i \in \mathcal{E} \).

In the following we denote with \( I \) the support of \( \alpha \), i.e. \( I = \{ i \in \mathbb{Z} | \alpha_i \neq 0 \} \). For any subset \( J \subset \mathbb{Z} \) we denote \( \mathcal{E}(J) := \{ e_i | i \in J \} \).

This setting appears in several signal processing problems, e.g. in mass spectrometry [22] where the signal is modeled as a sum of Dirac peaks (so-called impulse trains):

\[
u = \sum_{i \in \mathbb{Z}} \alpha_i \delta(\cdot - x_i)
\]

Other applications, for instance, can be found in astronomical signal processing problems or digital holography, cf [40], where images arise as superposition of characteristic functions of balls with different centers \( x_i \) and radii \( r_j \),

\[
u = \sum_{i,j \in \mathbb{Z}} \alpha_{i,j} \chi_{B_{r_j}}(\cdot - x_i)
\]

Typically \( K \) does not have a continuous inverse and hence, the solution of the operator equation (1) does not depend continuously on the data. This turns out to be a challenge for the case where only noisy data \( v^e \) with noise level \( \| v - v^e \| \leq \varepsilon \) are available—as it is always the case in praxis. First a small perturbation \( \varepsilon \) can cause an arbitrarily large error in the reconstruction \( u \) of \( Ku = v^e \) and second no solution \( u \) exists if \( v^e \) is not in the range of \( K \).

Inverse problems formulated in Banach spaces have been of recent interest and there are several results which deal with solving inverse problems formulated in Banach spaces, e.g. results concerning error estimates [1, 8, 17, 20, 27, 28, 36] or Landweber-like iterations or minimization methods for Tikhonov functionals, see e.g. [2–5, 9, 16, 37, 38].

In the following, an approximate solution of \( Ku = v^e \) shall be found by deriving iteratively the correlation between the residual and the unit-normed atoms of the dictionary:

\[
D := \{ d_i | i \in \mathbb{Z} \} := \left\{ \frac{Ke_i}{\| Ke_i \|} \right\}_{i \in \mathbb{Z}}
\]

Note that since the operator \( K \) is injective, we get that \( Ke_i \neq 0 \) for all \( i \in \mathbb{Z} \) and hence the dictionary \( D \) is well defined. In any step we select that unit-normed atom from the dictionary \( D \) which is mostly correlated with the residual, hence the name ‘greedy’ method. To stabilize the solution of \( Ku = v^e \), the iteration has to be stopped early enough.

For solving the operator equation (1) with noiseless data and the case where only noisy data \( v^e \) with noise-bound \( \| v - v^e \| \leq \varepsilon \) are available, we use the orthogonal matching pursuit, first proposed in the signal processing context by Davis et al [30] and Pati et al [35] as an
improvement upon the matching pursuit algorithm [31]:

**Algorithm 1.1 Orthogonal Matching Pursuit**

Set $k := 0$ and $I^0 := \emptyset$. Initialize $r^0 := v^\varepsilon$ (resp. $r^0 := v$ for $\varepsilon = 0$) and $\hat{u}^0 := 0$.

while $\|r^k\| > \varepsilon$ (resp. $\|r^k\| \neq 0$) do

\[ k := k + 1, \]

\[ i_k := \arg\sup \{|\langle (r^{k-1}, d_i) | d_i \in D\}, \]

\[ I^k := I^{k-1} \cup \{i_k\}. \]

Project $u$ onto span $\mathcal{E}(I^k)$

\[ \hat{u}^k := \arg\min \{|\|v^\varepsilon - K \hat{u}\|^2 | \hat{u} \in \text{span} \mathcal{E}(I^k)\}, \]

\[ r^k := v^\varepsilon - K \hat{u}^k. \]

end while

Remark that in infinite-dimensional Hilbert spaces the supremum

\[ \sup\{|\langle (r^{k-1}, d_i) | d_i \in D\} \] (2)

does not have to be realized. Because of that OMP has a variant—called weak orthogonal matching pursuit (WOMP)—which does not choose the optimal atom in the sense of (2), but only one that is nearly optimal, i.e. for some fixed $\omega \in (0, 1]$ it chooses some $i_k \in \mathbb{Z}$ with

\[ |\langle (r^{k-1}, d_{i_k}) \rangle| \geq \omega \sup\{|\langle (r^{k-1}, d_i) | d_i \in D\}. \]

In [42] a sufficient condition for exact recovery with algorithm 1.1 is derived, and in [10] it is transferred to noisy signals with the concept of coherence, which quantifies the magnitude of redundancy. This idea cannot be transferred to ill-posed inverse problems directly since the operator typically causes the correlation of two distinct atoms to become huge. Therefore, in [11, 15] the authors derive a recovery condition which works without the concept of coherence. For a comprehensive presentation of OMP, cf e.g. [29].

The paper is organized as follows. In section 2 we reflect the conditions for exact recovery for OMP derived in [11, 15, 42] and rewrite them in the context of infinite-dimensional inverse problems. Section 3 contains the main theoretical results of the paper, namely the generalization of these results to noisy signals. In section 4 we apply the deduced recovery conditions in the presence of noise to an example from mass spectrometry. Here, the data are given as sums of Dirac peaks convolved with a Gaussian kernel. To the end of this section we utilize the deduced condition for simulated data of an isotope pattern. Another example from digital holography is concerned in section 5. The data are given as sums of characteristic functions convolved with a Fresnel function. This turns out to be a challenge because the convolution kernel oscillates. Similar to section 4, we apply the theoretical condition to simulated data, namely to digital holograms of particles. The two examples from mass spectrometry and digital holography illustrate that our conditions for exact recovery lead to practically relevant estimates such that one may check \textit{a priori} if the experimental setup guarantees exact deconvolution with OMP. Especially in the example from digital holography our analysis may be regarded as a first step to calculate the resolution power of droplet holography.

2. Exact recovery conditions

In [42], Tropp gives a sufficient and necessary condition for exact recovery with OMP. Next, we list this result in the language of infinite-dimensional inverse problems.
Define the linear continuous synthesis operator for the dictionary \( \mathcal{D} = \{ d_i \} = \{ Ke_i / \| Ke_i \| \} \) via

\[
D : \ell^1 \to H, \\
(\beta_i)_{i \in \mathbb{Z}} \mapsto \sum_{i \in \mathbb{Z}} \beta_i d_i = \sum_{i \in \mathbb{Z}} \beta_i \frac{Ke_i}{\| Ke_i \|}.
\]

Since \( D \) is linear and bounded, the Banach space adjoint operator

\[
D^* : H \to (\ell^1)^* = \ell^\infty
\]

exists and arises as

\[
D^* v = ((v, d_i))_{i \in \mathbb{Z}} = \left( \left( v, \frac{Ke_i}{\| Ke_i \|} \right) \right)_{i \in \mathbb{Z}}.
\]

Note that the use of \( \ell^1 \) and its dual \( \ell^\infty \) arises naturally in this context. Furthermore, for \( J \subseteq \mathbb{Z} \) we denote with \( P_J : \ell^1 \to \ell^1 \) the projection onto \( J \) and with \( A^\dagger \) the pseudoinverse operator of \( A \). With this notation we state the following theorem.

**Theorem 2.1** (Tropp [42]). Let \( \alpha \in \ell^0 \) with \( \text{supp} \alpha = I \), \( u = \sum_{i \in \mathbb{Z}} \alpha_i e_i \) be the source and \( v = Ku \) the measured signal. If the operator \( K : B \to H \) and the dictionary \( \mathcal{E} = \{ e_i \}_{i \in \mathbb{Z}} \) fulfill the exact recovery condition (ERC)

\[
\sup_{d \in \ell^1} \|(DP_I) d\|_\ell < 1,
\]

then OMP with its parameter \( \ell \) set to 0 recovers \( \alpha \) exactly.

Theorem 2.1 gives a sufficient condition for exact recovery with OMP. In [42], Tropp shows that condition (3) is even necessary in the sense that if

\[
\sup_{d \in \ell^1} \|(DP_I) d\|_\ell \geq 1,
\]

then there exists a signal with support \( I \) for which OMP does not recover \( \alpha \) with \( v = Ku = K \sum \alpha_i e_i \).

The ERC (3) is hard to evaluate. Therefore, Dossal and Mallat [11] and Gribonval and Nielsen [15] derive a weaker sufficient but not necessary recovery condition that depends on inner products of the dictionary atoms of \( \mathcal{D}(I) \) and \( \mathcal{D}(I^c) \) only.

**Proposition 2.2** (Dossal and Mallat [11], Gribonval and Nielsen [15]). Let \( \alpha \in \ell^0 \) with \( \text{supp} \alpha = I \), \( u = \sum_{i \in \mathbb{Z}} \alpha_i e_i \) be the source and \( v = Ku \) the measured signal. If the operator \( K : B \to H \) and the dictionary \( \mathcal{E} = \{ e_i \}_{i \in \mathbb{Z}} \) fulfill the Neumann ERC

\[
\sup_{i \in I} \sum_{j \in I} |\langle d_i, d_j \rangle| + \sup_{i \in I^c} \sum_{j \in I} |\langle d_i, d_j \rangle| < 1,
\]

then OMP with its parameter \( \ell \) set to 0 recovers \( \alpha \).

The proof uses a Neumann series estimate for \( P_I D^* D P_I \)—this clarifies the term ‘Neumann’ ERC. The proof is contained in [15].

**Remark 2.3.** Obviously the Neumann ERC (4) is not necessary for exact recovery. A demonstrative example can be found in \( \mathbb{R}^4 \) with the signal \( v = (1, 1, 1, 0)^T \) and the unit-normed dictionary \( \mathcal{D} = \{ d_1 := (1, 0, 0, 0)^T, d_2 := 2^{-1/2}(1, 1, 0, 0)^T, d_3 := 2^{-1/2}(0, 0, 1, 0)^T, d_4 := (0, 0, 0, 1)^T \} \). Here with \( I = \{ 1, 2, 3 \} \) and \( I^c = \{ 4 \} \) we get

\[
|\langle d_1, d_4 \rangle| + |\langle d_2, d_4 \rangle| + |\langle d_3, d_4 \rangle| = 0,
\]
but
\[ |\langle d_1, d_2 \rangle| + |\langle d_1, d_3 \rangle| = \sqrt{2} > 1, \]
hence the Neumann ERC is not fulfilled. The ERC (3) is nevertheless fulfilled since in that case \( \| (DP_1) d_4 \|_\ell^1 = 0 \). OMP will then recover exactly, as one could expect by considering that just \( \{d_1, d_2, d_3\} \) span the \( \mathbb{R}^3 \).

This counter example may be generalized by considering \( I \subset \mathbb{Z} \) such that
\[ \sup_{i \in I} \sum_{j \in I, j \neq i} |\langle d_i, d_j \rangle| = 0 \quad \text{and} \quad \sup_{i \in I} \sum_{j \in I, j \neq i} |\langle d_i, d_j \rangle| \geq 1. \]
Here the Neumann ERC fails, but for any signal with support \( I \), OMP will recover exactly since the atoms \( d_i, i \in I \), and \( d_j, j \in I^c \), are uncorrelated and OMP never chooses an atom twice.

**Remark 2.4.** The sufficient conditions for WOMP with weakness parameter \( \omega \in (0, 1] \) are
\[ \sup_{d \in D(I^c)} \| (DP_I) d \|_\ell^1 < \omega \]
and
\[ \sup_{i \in I} \sum_{j \in I, j \neq i} |\langle d_i, d_j \rangle| + \frac{1}{\omega} \sup_{i \in I} \sum_{j \in I} |\langle d_i, d_j \rangle| < 1, \]
according to theorem 2.1 and proposition 2.2, respectively. They are proved analogously to the OMP case—same as all other following WOMP results.

Usually for sparse approximation problems, the behavior of the dictionary is characterized as follows.

**Definition 2.5.** Let \( \mathcal{F} := \{f_i\}_{i \in \mathbb{Z}} \) be a dictionary. Then the corresponding coherence parameter \( \mu \) and cumulative coherence \( \mu_1(m) \) for a positive integer \( m \) are defined as
\[ \mu := \sup_{i \neq j} |\langle f_i, f_j \rangle| \]
and
\[ \mu_1(m) := \sup_{\Lambda \subset \mathbb{Z} \atop |\Lambda| = m} \sum_{j \in \Lambda} |\langle f_i, f_j \rangle|, \]
respectively. Note that \( \mu_1(1) = \mu \) and \( \mu_1(m) \leq m \mu \) for all \( m \in \mathbb{N} \).

Since
\[ \sum_{i \in I} \sum_{j \in I, j \neq i} |\langle d_i, d_j \rangle| \leq \mu_1(N - 1) \]
and
\[ \sum_{i \in I} \sum_{j \in I} |\langle d_i, d_j \rangle| \leq \mu_1(N), \]
we get another condition in terms of the cumulative coherence, which is even weaker than the Neumann ERC.

**Proposition 2.6** (Tropp [42]). Let \( \alpha \in \ell^0 \) with \( \text{supp}\ \alpha = I \), \( u = \sum_{i \in \mathbb{Z}} \alpha_i e_i \) be the source and \( v = Ku \) the measured signal. If the operator \( K : B \to H \) and the dictionary \( \mathcal{E} = \{e_i\}_{i \in \mathbb{Z}} \) lead to a dictionary \( D \) which fulfills the condition
\[ \mu_1(N - 1) + \mu_1(N) < 1, \]
then OMP with its parameter \( \varepsilon \) set to 0 recovers \( \alpha \).

Remark that the condition in proposition 2.6 for ill-posed inverse problems might be unsuitable, since the typically compact operator causes that the coherence parameter \( \mu \) is
probably close to 1. Therefore, the cumulative coherence can grow large with increasing support.

Remark 2.7. Another major approach for solving sparse approximation problems is the basis pursuit (BP). Here one solves the convex optimization problem

$$\min_{\alpha \in \ell^2} \|\alpha\|_{\ell^1} \quad \text{subject to} \quad K \sum \alpha_i e_i = v.$$  

This idea is closely related to Tikhonov regularization with sparsity constraints. Here, the basic idea is to minimize least squares with $\ell^1$-penalty,

$$\min_{\alpha \in \ell^2} \|K \sum \alpha_i e_i - v\|_2^2 + \gamma \|\alpha\|_{\ell^1}.$$  

In [42] it is shown that the ERC (3) also ensures the exact recovery by means of BP. Since the propositions 2.2 and 2.6 are estimates for the ERC (3) and the proofs do not take into account any properties of the OMP algorithm these results hold here, too.

3. Exact recovery in the presence of noise

In [10], Donoho, Elad and Temlyakov transfer Tropp’s result [42] to noisy signals. They derive a condition for exact recovery in terms of the coherence parameter $\mu$ of a dictionary. This condition is—just as remarked in [10]—an obvious weaker condition. As already mentioned, in particular for ill-posed problems, this condition is too restrictive. In the following we will give exact recovery conditions in the presence of noise which are closer to the results of theorem 2.1 and proposition 2.2.

Assume that instead of exact data $v = Ku \in H$ only a noisy version

$$v^\varepsilon = v + \eta = Ku + \eta$$  

with noise level $\|v - v^\varepsilon\| = \|\eta\| \leq \varepsilon$ can be observed. Now, the OMP has to stop as soon as the representation error $r_k$ is smaller or equal to the noise level $\varepsilon$, i.e. if $\varepsilon \geq \|r_k\|$.

**Theorem 3.1** (ERC in the presence of noise). Let $\alpha \in \ell^0$ with $\text{supp} \alpha = I$. Let $u = \sum_{i \in \mathbb{Z}} \alpha_i e_i$ be the source and $v^\varepsilon = Ku + \eta$ the noisy data with noise level $\|\eta\| \leq \varepsilon$ and noise-to-signal ratio

$$r_{\varepsilon/\alpha} := \sup_{i \in \mathbb{Z}} \frac{|\eta_i, d_i|}{\min_{i \in I} |\alpha_i| \|K e_i\|}.$$  

If the operator $K : B \rightarrow H$ and the dictionary $E = \{e_i\}_{i \in \mathbb{Z}}$ fulfill the Exact Recovery Condition in the Presence of Noise ($\varepsilon$ERC)

$$\sup_{d \in D} \| (D P_I)^d \|_{\ell^1} < 1 - 2 r_{\varepsilon/\alpha} \frac{1}{1 - \sup_{i \in I} \sum_{j \neq i} |\langle d_i, d_j \rangle|},$$  

and $\sup_{i \in I} \sum_{j \neq i} |\langle d_i, d_j \rangle| < 1$, then OMP recovers the support $I$ of $\alpha$ exactly.

**Proof.** We prove the $\varepsilon$ERC analogously to [42, theorem 3.1] by induction. Assume that OMP recovered the correct patterns in the first $k$ steps, i.e.

$$\hat{\alpha}^k = \sum_{i \in I^k} \hat{\alpha}_i e_i.$$
with $I^k \subset I$. Then we get for the residual

$$r^k := v^k - K\hat{u}^k = v + \eta - K\hat{u}^k = K \left( \sum_{i \in I} \left( \alpha_i - \alpha_i^k \right) e_i \right) + \eta$$

$$\quad = \sum_{i \in I} \|Ke_i\| \left( \alpha_i - \alpha_i^k \right) d_i + \eta,$$

hence the noiseless residual $s^k := \sum \|Ke_i\| \left( \hat{\alpha}_i^k - \alpha_i \right) d_i$ has support $I$. The correlation $\langle r^k, d_i \rangle$, $i \in \mathbb{Z}$, can be estimated from below and above, respectively, via

$$|\langle r^k, d_i \rangle| = |\langle s^k + \eta, d_i \rangle| \geq |\langle s^k, d_i \rangle| \pm |\langle \eta, d_i \rangle|.$$

Hence with

$$\sup_{i \in I} |\langle r^k, d_i \rangle| \leq \sup_{i \in \mathbb{Z}} |\langle s^k, d_i \rangle| + \sup_{i \in I} |\langle \eta, d_i \rangle|$$

and

$$\sup_{i \in I} |\langle s^k, d_i \rangle| - \sup_{i \in I} |\langle \eta, d_i \rangle| \leq \sup_{i \in I} |\langle r^k, d_i \rangle|,$$

we get the condition

$$\|P_{I^k} D^* s^k\|_{\ell^\infty} + \sup_{i \in \mathbb{Z}} |\langle \eta, d_i \rangle| \leq \|P_{I^k} D^* s^k\|_{\ell^\infty} - \sup_{i \in I} |\langle \eta, d_i \rangle|,$$

which ensures a right choice in the $(k+1)$th step. Since $(P_{I^k} D^*)^\dagger P_{I^k} D^*$ is the orthogonal projection onto $D(I)$ and $\sup s^k = I$, we can write

$$s^k = (P_{I^k} D^*)^\dagger P_{I^k} D^* s^k.$$

With this identity we get the sufficient condition for OMP in the presence of noise

$$\frac{\|P_{I^k} D^* (P_{I^k} D^*)^\dagger P_{I^k} D^* s^k\|_{\ell^\infty}}{\|P_{I^k} D^* s^k\|_{\ell^\infty}} < 1 \cdot \frac{2}{\|P_{I^k} D^* s^k\|_{\ell^\infty}}.$$

Consequently, since $\|P_{I^k} D^* (P_{I^k} D^*)^\dagger P_{I^k} D^* s^k\|_{\ell^\infty} \leq \|P_{I^k} D^* (P_{I^k} D^*)^\dagger\|_{\ell^\infty, \ell^\infty} \|P_{I^k} D^* s^k\|_{\ell^\infty}$ and with the definition of the adjoint operator $D^*$, we get the equivalent sufficient conditions for a correct choice in the $(k+1)$th step:

$$\|P_{I^k} D^* (P_{I^k} D^*)^\dagger\|_{\ell^\infty, \ell^\infty} = \|(D P_{I^k})^\dagger D P_{I^k}\|_{\ell^\infty, \ell^\infty} < 1 \cdot \frac{2}{\|P_{I^k} D^* s^k\|_{\ell^\infty}}.$$

Obviously, on the one hand we get

$$\sup_{i \in I^k} \|\langle (D P_{I^k})^\dagger d_i \rangle\|_{\ell^\infty} \leq \sup_{\|\beta_i\|_{\ell^\infty} = 1} \sum_{i \in \mathbb{Z}} |\beta_i| d_i \|_{\ell^\infty} = \|(D P_{I^k})^\dagger D P_{I^k}\|_{\ell^\infty, \ell^\infty}$$

and on the other hand, since $(D P_{I^k})^\dagger$ is linear, we get

$$\sup_{\|\beta_i\|_{\ell^\infty} = 1} \|\langle (D P_{I^k})^\dagger \beta_i d_i \rangle\|_{\ell^\infty} \leq \sup_{\|\beta_i\|_{\ell^\infty} = 1} \sum_{i \in \mathbb{Z}} |\beta_i| \sup_{d \in D(I^k)} \|\langle (D P_{I^k})^\dagger d \rangle\|_{\ell^\infty}$$

$$\quad = \sup_{d \in D(I^k)} \|\langle (D P_{I^k})^\dagger d \rangle\|_{\ell^\infty}.$$

This shows that

$$\|\langle (D P_{I^k})^\dagger D P_{I^k}\|_{\ell^\infty, \ell^\infty} = \sup_{d \in D(I^k)} \|\langle (D P_{I^k})^\dagger d \rangle\|_{\ell^\infty} < 1 \cdot \frac{2}{\|P_{I^k} D^* s^k\|_{\ell^\infty}}.$$
is another equivalent condition for exact recovery. The last thing we have to afford to finish the proof is an estimation for the term $\|P_D^*s_k\|_\infty$ from below.

In the first step this is easy, since $s^0 = v^\epsilon$ resp. $s^0 = v$ with $v = K\sum_{i\in I} \alpha_i e_i$. With that we get

$$\|P_D^*s_k\|_\infty = \|P_D^*v\|_\infty = \sup_{j \in I} |\langle v, d_j \rangle| = \sup_{j \in I} \left| \sum_{i \in I} \alpha_i \|Ke_i\| \langle d_i, d_j \rangle \right| \geq \left| \alpha_l \|Ke_l\| \left( 1 - \sup_{i \in I \setminus j \neq i} |\langle d_i, d_j \rangle| \right) \right|$$

for all $l \in I$, hence in particular

$$\|P_D^*v\|_\infty \geq \min_{i \in I} |\alpha_i| \|Ke_i\| \left( 1 - \sup_{i \in I \setminus j \neq i} |\langle d_i, d_j \rangle| \right).$$

To prove this for general $k$ we successively apply this estimation. Here, again, we get

$$\|P_D^*s_k\|_\infty = \sup_{j \in I} |\langle s^k, d_j \rangle| = \sup_{j \in I} \left| \sum_{i \in I} (\hat{\alpha}_i^k - \alpha_i) \|Ke_i\| \langle d_i, d_j \rangle \right| \geq \left| \alpha_l \|Ke_l\| \left( 1 - \sup_{i \in I \setminus j \neq i} |\langle d_i, d_j \rangle| \right) \right|$$

for all $l \in I, l \neq I^k$, since OMP never chooses an atom twice, in particular

$$\|P_D^*s_k\|_\infty \geq \min_{i \in I} |\alpha_i| \|Ke_i\| \left( 1 - \sup_{i \in I \setminus j \neq i} |\langle d_i, d_j \rangle| \right).$$

In particular, to ensure the $\varepsilon$ERC (6) one has necessarily for the noise-to-signal ratio $r_{\varepsilon/\alpha} < 1/2$. For a small noise-to-signal ratio the $\varepsilon$ERC (6) approximates the ERC (3). A rough upper bound for $\sup_{i \in \mathbb{Z}} |\langle \eta, d_i \rangle|$ is $\varepsilon$ and hence, one may use

$$r_{\varepsilon/\alpha} \leq \min_{i \in I} \frac{\varepsilon}{\|Ke_i\|}.$$

Similar to the noiseless case, the $\varepsilon$ERC (6) is hard to evaluate. Analogously to section 2 we now give a weaker sufficient recovery condition that depends on inner products of the dictionary atoms. It is proved analogously to proposition 2.2.

**Proposition 3.2 (Neumann ERC in the presence of noise).** Let $\alpha \in \ell^0$ with $\text{supp} \, \alpha = I$. Let $u = \sum_{i \in \mathbb{Z}} \alpha_i e_i$ be the source and $v^\epsilon = Ku + \eta$ the noisy data with noise level $\|\eta\| \leq \varepsilon$ and noise-to-signal ratio $r_{\varepsilon/\alpha} < 1/2$. If the operator $K : B + H$ and the dictionary $E = \{e_i\}_{i \in \mathbb{Z}}$ fulfill the Neumann $\varepsilon$ERC

$$\sup_{i \in I} \sum_{j \in I \setminus j \neq i} |\langle d_i, d_j \rangle| + \sup_{i \in I^c} \sum_{j \in I} |\langle d_i, d_j \rangle| < 1 - 2r_{\varepsilon/\alpha},$$

then OMP recovers the support $I$ of $\alpha$ exactly.

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Remark 3.3. The according sufficient conditions for exact recovery with WOMP with weakness parameter \( \omega \in (0, 1] \) for the case of noisy data with noise-to-signal ratio \( r_{\varepsilon/\alpha} < \omega / 2 \) are

\[
\sup_{d \in \mathcal{D}(I^c)} \| (DP_j) d \|_{\ell^1} < \omega - 2r_{\varepsilon/\alpha} \frac{1}{1 - \sup_{i \in I} \sum_{j \neq i} |\langle d_i, d_j \rangle|},
\]

and

\[
\sup_{i \in I} \sum_{j \neq i} |\langle d_i, d_j \rangle| + \frac{1}{\omega} \sup_{i \in I} \sum_{j \neq i} |\langle d_i, d_j \rangle| < 1 - \frac{2r_{\varepsilon/\alpha}}{\omega},
\]

analog to theorem 3.1 and proposition 3.2, respectively.

Proposition 3.4. Let \( \alpha \in \ell^0 \) with \( \text{supp} \alpha = I \). Let \( u = \sum_{i \in \mathbb{Z}} \alpha_i e_i \) be the source and \( v^\varepsilon = Ku + \eta \) the noisy data with noise level \( \| \eta \| \leq \varepsilon \) and noise-to-signal ratio \( r_{\varepsilon/\alpha} < 1/2 \). If the operator \( K : B \rightarrow H \) and the dictionary \( \mathcal{E} = \{ e_i \}_{i \in \mathbb{Z}} \) lead to dictionary \( \mathcal{D} \) which fulfills the condition

\[
\mu_1(N - 1) + \mu_1(N) < 1 - 2r_{\varepsilon/\alpha},
\]

then OMP recovers the support \( I \) of \( \alpha \) exactly.

Remark 3.5. We remark again on an exact recovery condition for BP. Unlike section 2 where the results can be transferred to BP, see remark 2.7, this is not possible for the presence of
noise. To prove theorem 3.1, we used properties of the OMP algorithm which are not valid for BP.

For the case of noisy data \( v^\varepsilon \) in [10], an exact recovery condition for BP is derived. This condition depends on the coherence parameter \( \mu \). Since in this paper the focus is on the greedy solution of inverse problems, we give up on deriving stronger results for BP which are closer to the results of this section.

4. Resolution bounds for mass spectrometry

Granted to apply the Neumann conditions of propositions 2.2 and 3.2, respectively, one has to know the support \( I \). In this case there would be no need to apply OMP—one may just solve the restricted least squares problem, i.e. project onto \( \mathcal{E}(I) \). For deconvolution problems, however, with certain prior knowledge the Neumann ERC (4) and Neumann \( \varepsilon \)ERC (7) are easier to evaluate than the ERC (3) and \( \varepsilon \)ERC (6), respectively, especially when the support \( I \) is not known exactly. In the following we will use the weaker conditions exemplarily with impulse trains convolved with Gaussian kernel as e.g. occurs in mass spectrometry, cf [22].

Analysis. In mass spectrometry the source \( u \) is given—after simplification—as sum of Dirac peaks at integer positions \( i \in \mathbb{Z} \):

\[
u = \sum_{i \in \mathbb{Z}} \alpha_i \delta(\cdot - i),
\]

with \( \text{supp} \alpha = |I| = N \). Since the measuring procedure is influenced by Gaussian noise, the measured data can be modeled by a convolution operator \( K \) with Gaussian kernel

\[
\kappa(x) = \frac{1}{\pi \frac{1}{4\sigma^2}} \exp \left( -\frac{x^2}{2\sigma^2} \right),
\]

i.e. the operator under consideration is \( Ku = \kappa \ast u \). As Banach space \( B \) we may use the space \( \mathcal{M} \) of regular Borel measures on \( \mathbb{R} \) (which contains impulse trains if the coefficients \( \alpha_i \) are summable), and as Hilbert space \( H \) the space \( L^2(\mathbb{R}) \). We form the dictionary \( \mathcal{E} \) of Dirac peaks at integer positions and hence, we have \( \mathcal{D} = \{\delta(\cdot - i) \ast \kappa = \{\kappa(\cdot - i)\} \) since \( \|\kappa(\cdot - i)\|_{L^2} = 1 \).

To verify the Neumann ERC (4) and Neumann \( \varepsilon \)ERC (7), we need the autocorrelation of two atoms \( \kappa(\cdot - i) \) and \( \kappa(\cdot - j) \). In \( L^2(\mathbb{R}, \mathbb{R}) \) it arises as

\[
\langle \kappa(\cdot - i), \kappa(\cdot - j) \rangle_{L^2} = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi \sigma}} \exp \left( -\frac{(x - i)^2}{2\sigma^2} \right) \times \exp \left( -\frac{(x - j)^2}{2\sigma^2} \right) \, dx
= \exp \left( -\frac{(j - i)^2}{4\sigma^2} \right),
\]

which is positive and monotonically decreasing in the distance \( |i - j| \). If we additionally assume that the peaks of any source \( u \) have the minimal distance

\[
\rho := \min_{i, j \in \text{supp} \alpha} |i - j|,
\]

then w.l.o.g. we can estimate the sums of correlations from above as follows. For \( \rho \in \mathbb{N} \), we get for the correlations of support atoms

\[
\sup_{i \in I} \sum_{j \neq i} |\langle d_i, d_j \rangle| \leq 2 \sum_{j=1}^{\lfloor N/2 \rfloor} \langle \kappa, \kappa(\cdot - j \rho) \rangle = 2 \sum_{j=1}^{\lfloor N/2 \rfloor} \exp \left( -\frac{(j \rho)^2}{4\sigma^2} \right).
\]
For the correlations of support atoms and non-support atoms we have to distinguish between two cases for \(\rho\). For \(\rho \geq 2\) we get

\[
\sup_{i \in \mathbb{Z}} \sum_{j \in I} |(d_i, d_j)| \leq \sup_{1 \leq i < \rho} \sum_{j=-[N/2]}^{[N/2]} (\kappa(-i), \kappa(-j\rho)) = \sup_{1 \leq i < \rho} \sum_{j=-[N/2]}^{[N/2]} \exp\left(-\frac{(i - j\rho)^2}{4\sigma^2}\right),
\]

and for \(\rho = 1\)

\[
\sup_{i \in \mathbb{Z}} \sum_{j \in I} |(d_i, d_j)| \leq 2 \sum_{j=1}^{[N/2]} \langle \kappa, \kappa(-j)\rangle = 2 \sum_{j=1}^{[N/2]} \exp\left(-\frac{j^2}{4\sigma^2}\right).
\]

With that we can formulate the Neumann ERC and the Neumann \(\varepsilon\)ERC for Dirac peaks convolved with Gaussian kernel.

**Proposition 4.1.** An estimation from above for the ERC (i.e. \(r_{\varepsilon/\alpha} = 0\)) and \(\varepsilon\)ERC (i.e. \(0 < r_{\varepsilon/\alpha} < \frac{1}{2}\)) for Dirac peaks convolved with Gaussian kernel is for \(\rho \geq 2\)

\[
2 \sum_{j=1}^{[N/2]} \exp\left(-\frac{(j\rho)^2}{4\sigma^2}\right) + \sup_{1 \leq i < \rho} \sum_{j=-[N/2]}^{[N/2]} \exp\left(-\frac{(i - j\rho)^2}{4\sigma^2}\right) < 1 - 2r_{\varepsilon/\alpha},
\]

and for \(\rho = 1\)

\[
2 \sum_{j=1}^{[N/2]} \exp\left(-\frac{j^2}{4\sigma^2}\right) + 2 \sum_{j=1}^{[N/2]} \exp\left(-\frac{j^2}{4\sigma^2}\right) < 1 - 2r_{\varepsilon/\alpha}.
\]

This means that we are able to recover the support of the impulse train with OMP exactly from the convolved data if the above conditions are fulfilled.

**Remark 4.2.** Remark that the case \(\rho = 1\) of proposition 4.1 coincides more or less with the recovery condition in terms of the cumulative coherence since for odd \(N\) we get \(\mu_1(N) = 2 \sum_{j=1}^{[N/2]} \exp(-j^2/(4\sigma^2))\). Summing up just over a subset of \(\rho \mathbb{Z} := \{ j \in \mathbb{Z} | j/\rho \in \mathbb{Z}\}\) is not a feasible estimation since for the support \(I\) we allow any point \(i \in \mathbb{Z}\) and not only atoms of the sub-dictionary \(D(\rho \mathbb{Z})\). This turns out to be the main disadvantage of the coherence condition: it does not distinguish between support and non-support atoms, and hence in some applications it is a clearly weaker estimation.

**Remark 4.3.** If the cardinality of the support \(N\) is unknown, one could replace the finite sums by infinite sums. Obviously these sums exist since the geometric series is a majorizing series. With \(i\) representing the imaginary unit they can be expressed in terms of the Jacobi theta function of the third kind, \(\vartheta_3(z, q) := \sum_{j=\infty}^{\infty} q^j \exp(2jiz)\).

The condition of proposition 4.1 is plotted for some combinations of \(\sigma\), \(\rho\) and \(r_{\varepsilon/\alpha}\) with unknown \(N\) in figure 1. The colored areas describe the combinations where the Neumann ERC is fulfilled.

Often for deconvolution problems the autocorrelation of two atoms \(|(d(-i), d(-j))|\) is not monotonically decreasing in the distance \(|i - j|\), and it obviously depends on the kernel \(\kappa\). However, if the correlation of two atoms can be estimated from above via a monotonically decreasing function w.r.t. an appropriate distance, then we can use a similar estimate. We do this exemplarily for an oscillating kernel in section 5, namely for Fresnel-convolved characteristic functions as appear in digital holography.
Remark 4.4. We remark on a possible fully continuous formulation of OMP. We assume that we are given some data

\[ v = \kappa \ast u = \sum_{i \in \mathbb{Z}} \alpha_i \kappa(\cdot - x_i), \]

and that we do not know the positions \( x_i \). We allow our dictionary to be uncountable, i.e. we search for peaks at every real number. Note that here \( i \in \mathbb{Z} \) does not represent the set of possible positions for peaks, but it is an index set for continuous positions \( x_i \in \mathbb{R} \).

In the first step of the matching pursuit we correlate \( v \) with \( \kappa(\cdot - x) \) and take that \( x \) which gives maximal correlation. In the special case of the Gaussian blurring kernel (9), this amounts in finding the maximum of the function

\[ f(x) = |\langle v, \kappa(\cdot - x) \rangle| = \sum_{i \in \mathbb{Z}} \alpha_i |\langle \kappa(\cdot - x_i), \kappa(\cdot - x) \rangle|. \]

From (10) we see that this is

\[ f(x) = \sum_{i \in \mathbb{Z}} \alpha_i \exp \left( -\frac{(x - x_i)^2}{4\sigma^2} \right). \]

It is clear that any maximum of \( f \) is unlikely to be precisely at some of the \( x_i \)'s, albeit very close. A detailed study of this effect goes beyond the scope of this paper and we present a simple example.

Let us assume that we have two peaks, one at 0 and one at \( x_1 \):

\[ u = \alpha_0 \delta(\cdot) + \alpha_1 \delta(\cdot - x_1). \quad (11) \]

Moreover, we assume that \( \alpha_0 > \alpha_1 \), i.e. the peak in zero is higher. The matching pursuit will find the first peak at the maximum of the function

\[ f(x) = \alpha_0 \exp \left( -\frac{x^2}{4\sigma^2} \right) + \alpha_1 \exp \left( -\frac{(x - x_1)^2}{4\sigma^2} \right). \]
and hence at some root of
\[ f'(x) = -\frac{1}{2\sigma^2} \left( \alpha_0 x \exp \left( -\frac{x^2}{4\sigma^2} \right) + \alpha_1 (x - x_1) \exp \left( -\frac{(x - x_1)^2}{4\sigma^2} \right) \right). \]

The error that the matching pursuit makes is hence the error \( \varrho \) in the root of \( f' \) near zero. In figure 2 it is shown how the root of \( f' \) close to zero depends on the variance \( \sigma \). One observes that the error \( \varrho \) is smaller than the variance \( \sigma \) by some orders of magnitude.

As a final remark we mention that we measured the error not in some norm but only the distance of the \( \delta \)-peaks. This corresponds to the so-called Prokhorov-metric which is a metrization for the weak-* convergence in measure space.

**Numerical examples.** We apply the Neumann \( \varepsilon \)ERC of proposition 4.1 to simulated data of an isotope pattern. Here the data consist of equidistant peaks with different heights. In our example we use four peaks with a distance of \( \rho = 5 \) and heights of 130, 220, 180 and 90, cf the balls at the top of figure 3. After convolving with the Gaussian kernel with \( \sigma = 1.125 \), we apply a Poisson noise model. This is realistic because in mass spectrometry a finite number of particles are counted.

In the first example with low noise (mean and variance of 1.5 for regions without peaks) the Neumann \( \varepsilon \)ERC is fulfilled and hence OMP recovered the support exactly, see the middle of figure 3. However, the condition is restrictive: for the second example the signal is disturbed with huge noise (mean and variance of 30 for regions without peaks) and the Neumann \( \varepsilon \)ERC is not fulfilled. Certainly, OMP recovered the support exactly, see the bottom of figure 3.

5. Resolution bounds for digital holography

In digital holography, the data correspond to the diffraction patterns of the objects [12, 24]. Under Fresnel’s approximation, diffraction can be modeled by a convolution with a ‘chirp’ kernel. In the context of holograms of particles [18, 19, 44], the objects can be considered opaque (i.e. binary) and the hologram recorded on the camera corresponds to the convolution...
of disks with Fresnel’s chirp kernels. The measurement of particle size and location therefore amounts to an inverse problem \[39, 40\].

**Analysis.** We consider the case of spherical particles which is of significant interest in applications such as fluid mechanics \[32, 45\]. We model the particles \(j \in \{1, \ldots, N\}\) as opaque disks \(B_r(x_j, y_j, z_j)\) with center \((x_j, y_j, z_j) \in \mathbb{R}^3\), radius \(r\) and disk

---

**Figure 3.** Simulated isotope pattern. Top: support and Gaussian-convolved data without noise. Middle: low noise, Neumann εERC satisfied. Bottom: high noise, Neumann εERC not satisfied but still exact recovery possible.
orientation orthogonal to the optical axis \((Oz)\). Hence the source \(u\) is given as a sum of characteristic functions:

\[
u = \sum_{j=1}^{N} \alpha_j \chi_{B_j} (\cdot - x_j, \cdot - y_j, \cdot - z_j) =: \sum_{j=1}^{N} \alpha_j \chi_{j}.
\]

The real values \(\alpha_j\) are amplitude factors of the diffraction pattern that in praxis depend on experimental parameters, cf \([40, 43]\).

To an incident laser beam of (complex) amplitude \(A_0\) and wavelength \(\lambda\), the amplitude \(A\) in the observation plane, i.e. at a depth \(z = 0\), is well modeled by a bidimensional convolution \(\ast\) w.r.t. \((x, y)\). In the following \(\iota\) represents the imaginary unit. Then, with \(\delta_{x_j, y_j}\) denoting Dirac’s peak located at \((x_j, y_j)\) and \(h_{z_j}(x, y) = \frac{1}{\iota \lambda z_j} \exp(i \frac{\pi}{\lambda z_j} R^2)\), with \(R^2 := x^2 + y^2\), the amplitude \(A\) arises as

\[
A = A_0 \left[ 1 - \sum_{j=1}^{N} \alpha_j \left( \chi_{j} \ast \Re(h_{z_j}) \ast \delta_{x_j, y_j} \right) \right].
\]

Remark that \(\Re(h_{z_j}) \ast \delta_{x_j, y_j}\) denotes the shifted Fresnel function.

One difficulty occurring at digital holography inverse problems is that in praxis only the absolute value of \(A\) can be measured by the detector and the phase gets lost. The measured intensity consequently arises as

\[
G = |A|^2 = |A_0|^2 \left[ 1 - 2 \sum_{j=1}^{N} \alpha_j \left( \Re(h_{z_j}) \ast \delta_{x_j, y_j} \right) \right]
\]

\[
+ \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \left( \chi_{i} \ast h_{z_i} \ast \delta_{x_i, y_i} \right) \alpha_j \left( \chi_{j} \ast h_{z_j} \ast \delta_{x_j, y_j} \right) \right].
\]

Since the second term is dominant over the third one for \(\chi\) small, the intensity is classically linearized \([43, 40]\):

\[
G \approx |A_0|^2 \left[ 1 - 2 \sum_{j=1}^{N} \alpha_j \left( \Re(h_{z_j}) \ast \delta_{x_j, y_j} \right) \right]. \tag{12}
\]

Analogously to section 4 we will next derive the Neumann ERC and the Neumann ERC in the presence of noise for the operator equation (12), \(\sum \alpha_j \chi_{j} \mapsto G\). Here for fixed \((x_j, y_j, z_j)\) the associated (not necessarily unit-normed) atoms \(\widetilde{d}_{z_j}(\cdot - x_j, \cdot - y_j) := \chi_{B_j} (\cdot - x_j, \cdot - y_j) \ast \Re(h_{z_j}) \ast \delta_{x_j, y_j}\). (13)

As before the first step is to calculate the norm of an atom and the correlation of two distinct ones. Therefore, we need some properties of the Fresnel function.

**Proposition 5.1.** For the convolution of the Fresnel function we have the properties \([26]\)

\[
h_{z_1} \ast h_{z_2} = h_{z_1 + z_2}, \quad \text{for all} \quad z_1, z_2 \in \mathbb{R},
\]

\[
h_z \ast h_{-z} = \delta, \quad \text{for all} \quad z \in \mathbb{R}.
\]

With that and \(h_{z_1} \ast \delta_{x_j, y_j} = \delta_{x_j, y_j}\) we get for the real part of the Fresnel function

\[
\Re(h_{z_1}) \ast \Re(h_{z_2}) = \frac{1}{2} (\Re(h_{z_1 + z_2}) + \Re(h_{z_1 - z_2})), \quad \text{for all} \quad z_1, z_2 \in \mathbb{R},
\]

\[
\Re(h_z) \ast \Re(h_z) = \frac{1}{2} (\delta + \Re(h_{2z})), \quad \text{for all} \quad z \in \mathbb{R}.
\]
Another important property is that the convolution of a function with the Fresnel function—the so-called Fresnel transform—can be related to a direct multiplication with its Fourier transform which is defined by

\[ \mathcal{F} f(\xi, \eta) = \int_{\mathbb{R}^2} f(x, y) \exp(-2\pi i (x\xi + y\eta)) \, dx \, dy. \]

**Proposition 5.2.** Let \( f \in L^2(\mathbb{R}^2) \) and \( h_z \) be a Fresnel function. Then

\[ (f \circledast h_z)(\xi, \eta) = \mathcal{F}\{i\lambda zh_z f\} \left( \frac{\xi}{\lambda z}, \frac{\eta}{\lambda z} \right) h_z(\xi, \eta). \]

**Proof.** Let \( f \in L^2(\mathbb{R}^2) \), \( z \in \mathbb{R} \) and \( h_z \) the corresponding Fresnel function. Then rearranging yields the statement

\[
(f \circledast h_z)(\xi, \eta) = \int_{\mathbb{R}^2} f(x, y) \frac{1}{i\lambda z} \exp \left( \frac{i\pi}{\lambda z} (x^2 + y^2) \right) \, dx \, dy
\]

\[
= \frac{1}{i\lambda z} \exp \left( \frac{i\pi}{\lambda z} (\xi^2 + \eta^2) \right) \int_{\mathbb{R}^2} f(x, y) \exp \left( \frac{i\pi}{\lambda z} (x^2 + y^2) \right) \, dx \, dy
\]

\[
\times \exp \left( -2\pi i \left( \frac{x\xi}{\lambda z} + \frac{y\eta}{\lambda z} \right) \right)
\]

\[
= \mathcal{F}\{i\lambda zh_z f\} \left( \frac{\xi}{\lambda z}, \frac{\eta}{\lambda z} \right) h_z(\xi, \eta).
\]

\[ \square \]

**Remark 5.3.** In praxis \( f \) has a bounded and small support w.r.t. \( \sqrt{\lambda z} \). With \((x^2 + y^2)_{\text{max}} \) denoting the maximal spatial dimension of \( f \) resp. the maximal spatial extend of the corresponding particle the so-called far-field condition \((x^2 + y^2)_{\text{max}} / (\lambda z) \ll 1 \) holds in the proof of proposition 5.2, cf [43]. In [40], e.g. particles of radius at about 50 \( \mu \)m are illuminated with a red laser beam (wavelength 630 nm) and distance to camera of about 250 mm. Thus the term \((x^2 + y^2)_{\text{max}} / (\lambda z) \approx 3 \times 10^{-4} \) and hence \( \exp \left( \frac{i\pi (x^2 + y^2)}{\lambda z} \right) \) is approximately 1. Under the far-field condition we can estimate

\[
(f \circledast h_z)(\xi, \eta) \approx \mathcal{F} f \left( \frac{\xi}{\lambda z}, \frac{\eta}{\lambda z} \right) h_z(\xi, \eta).
\]

With that for the complex valued diffraction, with \( \rho^2 := \xi^2 + \eta^2 \) and \( J_\nu \) denoting the first kind Bessel function of order \( \nu \) we get

\[
(\chi_B \circledast h_z)(\rho) \approx \frac{r}{i\rho} J_1 \left( \frac{2\pi r}{\lambda z} \rho \right) \exp \left( \frac{i\pi}{\lambda z} \rho^2 \right),
\]

since \( \mathcal{F} x_B(\rho) = 2\pi r \left[ \frac{\lambda (2\pi \rho)}{2\pi r} \right] \) holds (Airy’s pattern, vide infra). With that for a real valued intensity atom we get

\[
\chi_B \circledast \text{Re}(h_z) = \text{Re}(\chi_B \circledast h_z) \approx \frac{r}{\rho} J_1 \left( \frac{2\pi r}{\lambda z} \rho \right) \sin \left( \frac{\pi}{\lambda z} \rho^2 \right),
\]

which corresponds to the model given by Tyler and Thompson in [43].

Back to the correlation and—as a special case—the norm of an atom, the correlation appears as the autoconvolution, namely

\[
[\vec{d}_{ij}(\cdot - x_i, \cdot - y_i)] \ast [\vec{d}_{ij}(\cdot - x_j, \cdot - y_j)]
\]

\[
= \int_{\mathbb{R}^2} \vec{d}_{ij}(x, y) \, dx \, dy
\]

\[
= (\vec{d}_{ij} \ast \vec{d}_{ij})(x_j - x_i, y_j - y_i).
\]
In the following we assume that all particles are located in a plane parallel to the detector, i.e. $z := z_i$ is constant for all $i$. Then the autoconvolution of an atom appears as

$$\vec{d}_z \ast \vec{d}_z = \chi_{B_i} \ast \chi_{B_i} \ast \text{Re}(h_z) \ast \text{Re}(h_z).$$

With proposition 5.1 and the formula

$$C(\rho) = (\chi_{B_i} \ast \chi_{B_i})(\rho) = \begin{cases} 2\rho^2 \cos^{-1} \left( \frac{\rho}{\rho_2} \right) - \frac{\rho}{2} \sqrt{4\rho^2 - \rho_2^2}, & \text{for } 4\rho^2 > \rho_2^2, \\ 0, & \text{else}, \end{cases}$$

we get

$$\vec{d}_z \ast \vec{d}_z = C \ast \frac{1}{2} [\delta + \text{Re}(h_{2z})] = \frac{1}{2} [C \ast \text{Re}(h_{2z})]. \quad (15)$$

With remark 5.3 and since $\mathcal{F}C$ is real valued we get

$$\text{C} \ast \text{Re}(h_{2z}) = \text{Re}(\mathcal{F}C (\cdot / \lambda z)) \approx \text{Re}(\mathcal{F}C (\cdot / \lambda z)) = \mathcal{F} \chi_{B_i} (\cdot / \lambda z) \mathcal{F} \chi_{B_i} (\cdot / \lambda z) \text{Re}(h_{2z}).$$

In physics it is well known that the Fourier transform of a disk is the Bessel cardinal function $\text{Jinc}(x) := J_1(x)/x$, since it is the diffraction of a circular aperture at infinite distance. Nevertheless, for the sake of completeness and mathematical beauty we will illustrate this computation: since the Fourier transform of a radial function is the Hankel transform of order zero (also known as Bessel transform of order zero), cf [41, Theorem IV.3.3, page 155], the Fourier transform of $\chi_{B_i}$ appears, for $\rho^2 = \xi^2 + \eta^2$, as

$$\mathcal{F} \chi_{B_i} (\rho) = 2\pi \int_0^\rho S J_0(2\pi \rho S) \, dS = \frac{1}{2\pi \rho^2} \int_0^{2\pi \rho} S J_0(S) \, dS.$$ 

In order to solve this definite integral we use $\int S J_0(S) \, dS = S J_1(S)$, cf [21, equation 5.52 1.], and get

$$\mathcal{F} \chi_{B_i} (\rho) = 2\pi \rho \left[ J_1(2\pi \rho) \right] / (2\pi \rho).$$

hence the Fourier transform of the circle–circle intersection $C$ appears as

$$\mathcal{F} C(\rho) = \mathcal{F} \chi_{B_i} (\rho) \mathcal{F} \chi_{B_i} (\rho) = \frac{\rho^2}{\rho^2} J^2_1(2\pi \rho).$$

With that result we can easily calculate the norm of an atom $\vec{d}_z$. Since $C(0) = \pi r^2$, $\mathcal{F} C(0) = (\mathcal{F} \chi_{B_i}(0))^2 = (\int \chi_{B_i} \, dx)^2 = \pi^2 r^4$ and $h_{2z}(0) = 0$ we obtain

$$||\vec{d}_z||^2 = |\vec{d}_z \ast \vec{d}_z|(0) \approx \frac{1}{2} \pi r^2.$$

Hence for fixed $z$ we can represent the associated unit-normed atoms $d_z \in \mathcal{D}$, with $R^2 := x^2 + y^2$, via

$$d_z := \vec{d}_z / ||\vec{d}_z|| \approx \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{R} J_1 \left( \frac{2\pi r}{\lambda z} R \right) \sin \left( \frac{\pi}{\lambda z} R^2 \right). \quad (16)$$

In figure 4 the centered atom for a particle of 50 $\mu$m radius is displayed which is illuminated with a red laser beam (wavelength 630 nm) in a distance of 250 mm to the camera. The autoconvolution for general $\rho$ and hence the correlation of two atoms $d_z(\cdot - x_i, \cdot - y_i)$ and $d_z(\cdot - x_j, \cdot - y_j)$ with distance $\rho = ((x_j - x_i)^2 + (y_j - y_i)^2)^{\frac{1}{2}}$ in digital holography emerges.
as

$$|\langle d_z(\cdot - x_i, \cdot - y_i), d_z(\cdot - x_j, \cdot - y_j) \rangle| = |d_z \ast d_z|(\rho) = \frac{1}{\|d_z\|^2} |\tilde{d}_z \ast \tilde{d}_z|(\rho)$$

$$\approx \frac{C(\rho)}{\pi r^2} + \frac{1}{4} J_1^2 \left( \frac{2\pi r}{\lambda z} \rho \right) |\text{sinc} \left( \frac{1}{2\lambda z} \rho^2 \right)|.$$  \hspace{1cm} (17)

where sinc denotes the normalized sine cardinal and is defined via sinc \((x) := \sin(\pi x)/\pi x\).

The correlation in digital holography (17) is not as easily valuable as in mass spectrometry because it is not monotonically decreasing in the distance \(\rho\) due to the oscillating Bessel and sine functions. To come to an estimate from above which is monotonically decreasing we use bounds for the absolute value of the Bessel functions \(J_1^2\). In [25] Landau gives estimates for \(|J_1^2(x)|\) for \(x > 0\) and \(\nu > 0\), namely

$$|J_1^2(x)| \leq \min \left\{ b_L \nu^{-1/3}, c_L x^{-1/3} \right\},$$  \hspace{1cm} (18)

with constants

$$b_L := \sqrt{2} \sup_{x > 0} \sqrt{x} \left( \frac{\frac{2}{3} x^2}{x^2} + \frac{\frac{2}{3} x^2}{x^2} \right) \approx 0.6748,$$

$$c_L := \sup_{x > 0} x^2 J_0(x) \approx 0.7857.$$  

In addition sine cardinal obviously is bounded from above via 1 and 1/\(x\) and hence we have

$$|d_z \ast d_z|(\rho) \leq \frac{C(\rho)}{\pi r^2} + \frac{1}{4} \min \left\{ b_L^2, c_L^2 \left( \frac{\lambda z}{2\pi r} \right) \frac{\rho^2}{\rho^2} \right\} \min \left\{ 1, \frac{2\lambda z}{\pi} \rho^{-2} \right\}.$$  \hspace{1cm} (19)

which now is monotonically decreasing in \(\rho\). Figure 5 illustrates the oscillating part of the correlation (17) and its corresponding upper bound for two particles of 50 \(\mu\)m radius which are illuminated with a red laser beam (wavelength 630 nm) in a distance of 250 mm to the camera.

Remark 5.4. In [23] Krasikov gives more precise estimation for \(J_1^2(x)\), namely for \(\nu > -1/2, \varsigma := (2\nu + 1)(2\nu + 3)\) and \(x > \sqrt{\varsigma + \varsigma^{2/3}}/2,\)

$$J_1^2(x) \leq \frac{4(4x^2 - (2\nu + 1)(2\nu + 5))}{\pi((4x^2 - \varsigma)^{3/2} - \varsigma)}.$$  

**Figure 4.** Unit-normed, centered atom of particles of radius 50 \(\mu\)m, illuminated with a red laser beam (wavelength 630 nm) and distance to camera of 250 mm.
With that (asymptotically $|d_i \oplus d_j|/(\rho) \sim \rho^{-3}$) instead of Landau’s rough bound (18) (asymptotically $|d_i \oplus d_j|/(\rho) \sim \rho^{-\frac{8}{3}}$) one can get a more precise recovery condition for digital holography. Since this technical computation is beyond the scope of this theoretical paper we postpone it here.

With this estimation we will come to a resolution bound for droplets jet reconstruction, as e.g. used in [40]. Here monodisperse droplets, i.e. they have the same size, shape and mass, were generated and emitted on a straight line parallel to the detector plane. This configuration eases the computation of the Neumann ERC and the Neumann ERC in the presence of noise. Analogously to mass spectrometry we define that the particles appear at some selected points $i \in \Delta Z := \{i \in \mathbb{Z} | \frac{i}{\Delta} \in \mathbb{Z}\}$, where the parameter $\Delta$ describe the dictionary refinement. If we additionally assume that the particles have the minimal distance $\rho \in \mathbb{N}$, then the sum of inner products of support atoms $D(I)$ and non-support atoms $D(I^\perp)$ can be estimated from above. For $\rho > \Delta$ we get

$$\sup_{i \in I} \sum_{j \in I, j \neq i} |\langle d_i, d_j \rangle| \leq \sum_{j=1}^{\lfloor N/2 \rfloor} \frac{2}{\pi r^2} C(j\rho) + \frac{1}{2} \min \left\{ b_L^2, c_L^2 \left( \frac{\lambda z}{2\pi r} \right)^{-\frac{3}{2}} (j\rho)^{-\frac{1}{2}} \right\} \min \left\{ 1, \frac{2\lambda z}{\pi} (j\rho)^{-2} \right\}$$

and

$$\sup_{i \in I^\perp} \sum_{j \in I} |\langle d_i, d_j \rangle| \leq \sup_{\Delta \leq i \leq \rho - \Delta} \sum_{l=-(\lfloor N/2 \rfloor)}^{\lfloor N/2 \rfloor} \frac{1}{\pi r^2} C(|j\rho - i|)$$

$$+ \frac{1}{4} \min \left\{ b_L^2, c_L^2 \left( \frac{\lambda z}{2\pi r} \right)^{-\frac{3}{2}} |j\rho - i|^{-\frac{1}{2}} \right\} \min \left\{ 1, \frac{2\lambda z}{\pi} |j\rho - i|^{-2} \right\}.$$ 

**Proposition 5.5.** An estimation from above for the ERC (i.e. $r_{\varepsilon/\alpha} = 0$) and $\varepsilon$ERC (i.e. $0 < r_{\varepsilon/\alpha} < \frac{1}{2}$) for characteristic functions convolved with the real part of the Fresnel kernel
is, for $\rho > \Delta$,

$$
\sum_{j=1}^{[N/2]} \frac{1}{\pi r^2} C(j \rho) + \frac{1}{2} \min \left\{ b^2_L, c^2_L \left( \frac{\lambda z}{2 \pi r} \right)^{\frac{3}{2}} (j \rho)^{-\frac{3}{2}} \right\} \min \left\{ 1, \frac{2 \lambda z}{\pi} (j \rho)^{\frac{3}{2}} \right\} \\
+ \sup_{1 \leq i < \frac{N}{2}} \left\{ \sum_{j=-[N/2]}^{[N/2]} \frac{1}{\pi r^2} C(|j \rho - i \Delta|) \right\} \\
+ \frac{1}{4} \min \left\{ b^2_L, c^2_L \left( \frac{\lambda z}{2 \pi r} \right)^{\frac{3}{2}} |j \rho - i \Delta|^{-\frac{3}{2}} \right\} \min \left\{ 1, \frac{2 \lambda z}{\pi} |j \rho - i \Delta|^{-\frac{3}{2}} \right\} \\
< 1 - 2 \frac{r \varepsilon}{\alpha}.
$$

**Remark 5.6.** Same as before for mass spectrometry. If the cardinality of the support $N$ is unknown, one could replace the finite sums by infinite sums. These sums exist and can be expressed in terms of the Hurwitz zeta function $\zeta(v, q) := \sum_{j=0}^{\infty} (q + j)^{-v}$, for $v > 1, q > 0$, and the Riemann zeta function $\zeta(v) := \zeta(v, 1) = \sum_{j=1}^{\infty} j^{-v}$, respectively.

The condition in proposition 5.5 seems not to be easy to handle. However, in praxis all parameters are known and one can compute a bound via approaching from large $\rho$. As soon as the sum is smaller than $1 - 2 \frac{r \varepsilon}{\alpha}$, it is guaranteed that OMP can recover exactly. A typical setting for digital holography of particles is the usage of a red laser of wavelength $\lambda = 0.6328 \ \mu m$ and a distance of $z = 200 \ mm$ from the camera, cf [40]. In figure 6 the condition of proposition 5.5 is plotted for particles with typical radii $r \in \{5, 15, 25, 35, 50, 75\} \ \mu m$. In the computation the asymptotic formula is used, i.e. for an unknown support cardinality $N$. For the dictionaries a corresponding refinement of $\Delta = r/2$ was chosen. The colored areas describe the combinations where the Neumann ERC is fulfilled and hence OMP recovers exactly.
Figure 7. Simulated holograms of spherical particles. In the left column the noiseless signals $v$ are displayed. For reconstruction the noisy signals $v'$ of the right column are used. The dots correspond to the location of detected particles with OMP. The algorithm recovered all particles exactly; however, the condition of proposition 5.5 was just fulfilled for the image on top right. In the image in the middle the particles have a too small distance to each other and at the bottom the image was manipulated with unrealistically high noise.
Numerical examples. We apply the Neumann $\varepsilon$ERC (7) to simulated data of droplet jets. For the simulation we use the same setting as above, i.e. a red laser of wavelength $\lambda = 0.6328 \mu m$ and a distance of $z = 200 \text{ mm}$ from the camera. The particles have a diameter of 100 $\mu m$ and for the corresponding dictionary we choose a refinement of 25 $\mu m$. Those parameters correspond to that of the experimental setup used in [39, 40].

After applying the digital holography model (12) we add Gaussian noise of different noise levels and in each case of zero mean. For the coefficients we choose $2\alpha_i = 10$ for all $i \in \text{supp } \alpha$. The figure 7 shows three simulated holograms with different distances $\rho$ and noise-to-signal levels $r_{\varepsilon/\alpha}$. For all three noisy examples in the right column all the particles were recovered exactly. However, only for the image on top ($\rho \approx 721 \mu m$) the condition of proposition 5.5 holds. In the second image in the middle of the figure the particles have a too small distance to each other ($\rho \approx 360 \mu m$) and even for the noiseless case the condition is not fulfilled. The last image ($\rho \approx 721 \mu m$) was manipulated with unrealistically huge noise so that here the condition of proposition 5.5 is violated, too, cf figure 6.

6. Conclusion and future prospects

In this paper we gave exact recovery conditions for the orthogonal matching pursuit for noisy signals that work without the concept of coherence. Our motivation was to treat ill-posed problems, and in particular, two problems of convolution type. We obtained results on exact recovery of the support for noiseless and even noisy data. Moreover, for noisy data there is a simple error bound in proposition 3.5 which shows a convergence rate of $O(\varepsilon)$. The rate of convergence resembles what is known for sparsity-enforcing regularization with $\ell^p$ penalty term for $0 < p \leq 1$ [6, 13, 14]; moreover, our results also guarantee the exact recovery of the support.

In two real-world applications we showed that these conditions lead to computable conditions and hence, are practically relevant. A main tool here was that the atoms in the dictionary are shifted copies of the same shape and that the correlation of the atom depends on the distance of the atoms only. Once there is a sufficiently decaying upper bound for the correlation, we are able to apply the Neumann ERC (4) and the Neumann $\varepsilon$ERC (7), and obtain computable conditions for exact recovery as illustrated in the examples in sections 4 and 5. However, experiments indicated that the conditions for exact recovery from theorems 2.2 and 3.2 are too restrictive. An idea to come to a tighter exact recovery condition is to bring in more prior knowledge, as e.g. a non-negativity constraint, cf [7]. We postpone this idea for future work. For the particle digital holography application even more prior knowledge may be taken into account, since the objects are not only non-negative but also all apertures have the same denseness, i.e. $\alpha_i$ is constant for all $i \in I$.

As discussed in remark 4.4, a straightforward generalization of our approach to fully continuous dictionaries runs into problems. Especially it seems that there is little hope to obtain exact recovery of the support, but one may obtain bounds on how accurate the support is localized. This is strongly related to the structure of the dictionary (e.g. that consists of shifts of the same object) and of course related to the correlations.

Finally, a further direction of research may be to investigate other types of pursuit algorithms like regularized orthogonal matching pursuit [34] or CoSAMP [33].

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