CONTINUOUS RANDOM WALKS AND FRACTIONAL POWERS OF OPERATORS

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Abstract. We derive a probabilistic representation for the Fourier symbols of the generators of some stable processes.

1. Introduction and main results

The connection between fractional operator in space and diffusion with long jumps has been pointed out by many researchers (see for example [1; 5; 7] and the references therein). It is well known that the compound Poisson process is a continuous time stochastic process with jumps which arrive, according to a Poisson process, with specific probability law for the size. Our aim is to characterize the jumps distribution in order to obtain singular limit measure characterizing fractional powers of operators.

Let $N(t)$, $t > 0$ be a Poisson process with rate $\lambda > 0$. Let $Y_j$, $0 \leq j \leq n$ be $n+1$ i.i.d. random jumps such that $Y_j \sim Y$ for all $j$, where the symbol "$\sim$" stands for equality in law. It is well known that

$$Z_t = \sum_{j=0}^{N(t)} Y_j - \lambda t Y, \quad t > 0$$

(1.1)

is the compensated Poisson process with generator

$$\mathcal{A} f(x) = \lambda \int_{\mathbb{R}} (f(x+y) - f(x) - y f'(x)) \nu_Y(dy)$$

(1.2)

where $\nu_Y : \Omega \subseteq \mathbb{R} \mapsto [0, 1]$ is the density law of $Y \in \Omega$. The latter is quite familiar in the representation of the fractional power of the Laplacian. Indeed, the fractional Laplace operator can be defined pointwise:

$$-(-\Delta)^{\alpha} f(x) = \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - y \cdot \nabla f(x) 1_{\{|y| \leq 1\}} \right) \frac{C_d(\alpha) dy}{|y|^{2\alpha + d}}$$

(1.3)

where $C_d(\alpha)$ is a constant depending on $d$ and $\alpha \in (0, 1)$, $f$ is a suitable test function, $C^2$ function with bounded second derivative for instance.

In this short paper, we construct continuous random walks with exponential and Gaussian jumps driven by pseudo-differential operators with Fourier multiplier $\Phi_\gamma(\xi)$ which converges to $|\xi|^\beta$ with $\beta \in (0, 2)$ as $\gamma \to 0$. In particular, we first consider the random jump $Y = \gamma e^X \in [\gamma, \infty)$ where $X \sim Exp(\alpha)$ and $\alpha, \gamma > 0$.

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0. We have that \( P\{Y \in A\} = \int_A \nu_Y(dy) \) with \( \nu_Y(y) = \alpha y^{1-\alpha}1_{\{y > 1\}} \). By "symmetrizing", we get that
\[
(1.4) \quad \nu^*_Y(y) = q \nu_Y(-y)1_{\{y \leq -1\}} + p \nu_Y(y)1_{\{y > 1\}}
\]
is the density of \( Y^* = \epsilon Y \) with Rademacher law
\( P\{\epsilon = +1\} = p, \ P\{\epsilon = -1\} = q \)
for the random variable \( \epsilon \) where, we obviously assume that \( p + q = 1 \). For \( p = q \), formula (1.4) takes the form
\[
(1.5) \quad \nu^*_Y(y) = \frac{1}{2} \nu_Y(|y|) = \frac{\alpha |y|^{1-\alpha}}{2} 1_{|y| > 1}
\]
and \( Y^* \) is therefore written as
\[
(1.6) \quad Y^* = \begin{cases} \gamma e^{X}, & \text{with probability } 1/2, \\ -\gamma e^{X}, & \text{with probability } 1/2. \end{cases}
\]
We write the corresponding compound Poisson process as follows
\[
(1.7) \quad A(t) = \sum_{j=0}^{N(t)} Y_j^* = \sum_{j=0}^{N(t)} \epsilon_j \gamma_j e^{X_j}, \quad t > 0
\]
with \( X_j \sim X, \ \epsilon_j \sim \epsilon \) and \( \gamma_j = \gamma \) for all \( j = 0, 1, 2, \ldots \). We also assume that all the random variables we are dealing with are taken to be independent, that is \( \mathbb{E} \epsilon_j \epsilon_j' = 0, \ \forall j, j' \) such that \( j \neq j' \). Observe that, for all \( \epsilon > 0 \), \( P\{\gamma e^{X} < \epsilon\} \rightarrow P\{e^{X} < +\infty\} = 1 \) as \( \gamma \to 0 \).

We recall that the symbol \( \overset{d}{\to} \) stands for "converges in distribution" and state the following results.

**Theorem 1.** Let \( \delta_j^\alpha(t), t > 0, j = 1, 2 \) be two independent stable subordinators. For given \( p, q \geq 0 \) such that \( p + q = 1, \ \alpha \in (0, 1) \),
\[
(1.8) \quad A(t/\gamma^\alpha) \overset{d}{\to} \delta_1^\alpha(pt^*) - \delta_2^\alpha(qt^*)
\]
with \( t^* : t \mapsto \lambda \Gamma(1 - \alpha) t \) and generator
\[
(1.9) \quad Af(x) = -\lambda \Gamma(1 - \alpha) \left( p \frac{d^\alpha}{dx^\alpha} + q \frac{d^\alpha}{d(-x)^\alpha} \right) f(x).
\]
The Weyl's fractional derivatives appearing in (1.9) are defined as follows:
\[
\frac{d^\alpha f}{dx^\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^\infty f(x-y) \frac{dy}{y^\alpha} = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (f(x) - f(x-y)) \frac{dy}{y^{\alpha+1}};
\]
\[
\frac{d^\alpha f}{d(-x)^\alpha}(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^\infty f(x+y) \frac{dy}{y^\alpha} = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (f(x) - f(x+y)) \frac{dy}{y^{\alpha+1}}.
\]
We consider "good" functions \( f : \mathbb{R} \mapsto [0, 1] \) whereas, we obtain the Riemann-Liouville derivatives (left-handed and right-handed respectively) by considering \( f : \mathbb{R}_+ \mapsto [0, 1] \) and \( f : \mathbb{R}_- \mapsto [0, 1] \) respectively. In the integrals above, for example, one can write \( f(z)1_{\{z \geq 0\}} \) and \( f(z)1_{\{z \leq 0\}} \) and obtain the operator governing the
(totally) positively and negatively skewed stable process, that is $q = 0$ and $p = 0$
respectively in (1.9).

**Remark 1.** A stable subordinator is a one-dimensional non-decreasing Lévy process
with $\mathbb{E} \exp(-\mu \tau^q) = \exp(-t\mu^q)$, $\alpha \in (0, 1)$. We observe that $S^q_j(0) = 0$ for $j = 1, 2$
and therefore the process (1.8) converges (in distribution) to a totally positively
(if $q = 0$, $P\{\epsilon_j = +1\} = 1$, $\forall j$) or negatively (if $p = 0$, $P\{\epsilon_j = -1\} = 1$, $\forall j$) skewed
stable process. Furthermore, for a given process $X_t$, $t > 0$ we notice that $X_{\theta t}$ runs
slower than $X_t$ as well as $\theta$ is less than 1.

**Theorem 2.** Let $\mathcal{S}^{\beta}(t)$, $t > 0$ be a symmetric stable process with $\beta \in (0, 2)$. Then,
for $p = q = 1/2$, $\alpha \in (0, 2)$,

\begin{equation}
A(t/\gamma^\alpha) \xrightarrow{\gamma \to 0} \mathcal{S}^\alpha(t^*)
\end{equation}

with $t^* : t \mapsto \alpha \lambda C t$ and generator

\begin{equation}
Af(x) = -\alpha \lambda C \frac{d^\alpha f}{d|x|^\alpha}(x)
\end{equation}

where

\begin{equation}
C = \frac{1}{2} \int_\mathbb{R} 1 - \cos y |y|^\alpha \frac{1}{y^\alpha+1} dy.
\end{equation}

We now introduce the reciprocal gamma random variable $E_\alpha$, $\alpha > 0$ with
$P\{E_\alpha \in dx\} = \frac{e^{-\alpha x}}{\gamma(x)} \alpha^{-1/\gamma}(\alpha) dx$ (the reciprocal gamma process has interesting
connections with stable subordinators and Bessel processes, see for example [3]).
We also consider the (normal) random vector $Y \sim N(0, \sigma^2_\alpha)$, $Y \in \mathbb{R}^d$, with random variance $\sigma^2_\alpha = 2E_\alpha$ for some $\gamma > 0$ and define the process

\begin{equation}
A(t) = \sum_{j=0}^{N(t)} \epsilon_j Y_j
\end{equation}

where $\epsilon_j \sim \epsilon \forall j$, $\epsilon$ has Rademacher law as above, $Y_j \sim Y \forall j$ and $\mathbb{E}\epsilon_j \epsilon_{j'} = 0$ for
all $j, j'$ such that $j \neq j'$. We notice that

\begin{align*}
P\{N(0, \sigma^2_\alpha) > x\} & \leq \frac{1}{x} \int_x^\infty y \left(\int_0^\infty e^{-\frac{y^2}{2s}} P\{\sigma^2_\alpha \in ds\}\right) dy \approx \gamma^\alpha x^{-2\alpha}
\end{align*}

and therefore, for large $x$,

\begin{align*}
P\{N(0, \sigma^2_\alpha) \in dx\}/dx & \approx 2\alpha \gamma^\alpha x^{-2\alpha-1}.
\end{align*}

After some calculations we explicitly write the law of $Y \in \mathbb{R}^d$ as follows

\begin{equation}
\nu_Y(y) = \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^{d/2} \Gamma(\alpha)} \frac{\gamma^\alpha}{(|y|^2 + \gamma)^{\alpha + d/2}}, \quad y \in \mathbb{R}^d.
\end{equation}

We are now ready to present the next result.

**Theorem 3.** Let $\mathcal{S}^{\beta}(t) \in \mathbb{R}^d$, $t > 0$ be an isotropic stable process with $\beta \in (0, 2)$.
Then, for $\alpha \in (0, 1)$,

\begin{equation}
A(t/\gamma^\alpha) \xrightarrow{\gamma \to 0} \mathcal{S}^{2\alpha}(t^*)
\end{equation}
with \( t^* : t \mapsto \lambda \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^\frac{d}{2} \Gamma(\alpha)} C t \) and infinitesimal generator

\[
Af(x) = -\lambda \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^\frac{d}{2} \Gamma(\alpha)} C(-\Delta)\alpha f(x)
\]

where

\[
C = \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{2\alpha + d}} dy.
\]

2. COMPENSATED POISSON AND FRACTIONAL LAPLACE OPERATOR

Let us consider the Lévy process \( F_t, t > 0, \) with associated Feller semigroup \( T_t f(x) = \mathbb{E} f(F_t - x) \) solving \( \partial_t u = Au, u_0 = f. \) The operator \( A \) is the infinitesimal generator of \( F_t, t > 0 \) and the following representation holds

\[
(Af)(x) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i \xi \cdot x} \Phi(\xi) \hat{f}(\xi) d\xi
\]

for all functions in the domain

\[
D(A) = \left\{ f \in L^2(\mathbb{R}^d, dx) : \int_{\mathbb{R}^d} \Phi(\xi)|\hat{f}(\xi)|^2 d\xi < \infty \right\}
\]

where \( \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{i \xi \cdot x} f(x) dx \) is the Fourier transform of \( f, \Phi(\cdot) \) is continuous and negative definite. We say that \( T_t \) is a pseudo-differential operator with symbol \( \exp(-t\Phi) \) and, \( \Phi \) is the Fourier multiplier (or Fourier symbol) of \( A, (A\hat{f})(\xi) = -\hat{\Phi}(\xi)\hat{f}(\xi). \) Furthermore (as in [4]), we write

\[
-\partial_t \mathbb{E} e^{i \xi \cdot F_t} \bigg|_{t=0} = \Phi(\xi).
\]

It is well known that, for \( \Phi(\xi) = |\xi|^\alpha, \) formula (2.1) gives us the fractional power of the Laplace operator which can be also expressed as

\[
-(-\Delta)^\alpha f(x) = C_d(\alpha) \text{ p.v.} \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|x - y|^{2\alpha + d}} dy
\]

\[
= C_d(\alpha) \text{ p.v.} \int_{\mathbb{R}} \frac{f(x + y) - f(x)}{|y|^{2\alpha + d}} dy
\]

where \( "\text{p.v.}" \) stands for the "principal value" of the singular integrals above near the origin. For \( \alpha \in (0, 1), \) the fractional Laplace operator can be defined, for \( f \in \mathcal{S} \) (the space of rapidly decaying \( C^\infty \) functions), as follows

\[
-(-\Delta)^\alpha f(x) = \frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d} \frac{f(x + y) + f(x - y) - 2f(x)}{|x - y|^{2\alpha + d}} dy
\]

\[
\text{where } \forall x \in \mathbb{R}^d.
\]

This representation comes out by considering straightforward calculations and removes the singularity at the origin ([2]). Indeed, from the second order Taylor expansion of the smooth function \( f (f \in \mathcal{S}) \) we obtain

\[
\frac{f(x + y) + f(x - y) - 2f(x)}{|y|^{2\alpha + d}} \leq \frac{\|D^2 f\|_{L^\infty}}{|y|^{2\alpha + d - 2}}
\]

which is integrable near the origin provided that \( \alpha \in (0, 1). \) The constant \( C_d(\alpha) \) must be considered in order to obtain \( -\Delta^\alpha f(\xi) = |\xi|^\alpha \hat{f}(\xi). \)
Remark 2. Let $\mathcal{S}^\alpha$, $t > 0$ be a stable subordinator. The generator of $\mathbf{F}_t$, $t > 0$ is given by the beautiful formula

\begin{equation}
-(-A)^\alpha f(x) = \alpha \int_0^\infty \left( T_s f(x) - f(x) \right) \frac{ds}{s^{\alpha+1}}
\end{equation}

for all $f \in \mathcal{S}$ ($T_s = e^{sA}$ is the Feller semigroup of $\mathbf{F}_t$, $t > 0$).

Formula (1.2) can be obtained by considering the following characteristic function

\begin{equation}
\mathbb{E} e^{i\xi Z_t} = \mathbb{E} \prod_{j=0}^{N(t)} e^{i\xi Y_j} e^{-i\xi \alpha t Y}
= \mathbb{E} \left( \mathbb{E} e^{i\xi Y} \right)^{N(t)} e^{-i\xi \alpha t Y}
= \exp \left( \lambda t \left( e^{i\xi Y} - 1 - i\xi Y \right) \right).
\end{equation}

Therefore, we get that

\[ \partial_t \mathbb{E} e^{i\xi Z_t} \bigg|_{t=0} = \lambda \mathbb{E} (e^{i\xi Y} - 1 - i\xi Y) = \lambda \int_\mathbb{R} (e^{i\xi y} - 1 - i\xi y) \nu_Y(dy) = -\Phi(\xi). \]

If $Y_j \sim Y$ are symmetric random variables such that $EY_j = EY = 0$ for all $j = 1, 2, \ldots$, then $\nu_Y(y) = \nu_Y(-y)$ and

\begin{equation}
\int_\mathbb{R} y f'(x) \nu_Y(dy) = f'(x) \int_\mathbb{R}^B y \nu_Y(dy) + f'(x) \int_{\mathbb{R}\setminus B} y \nu_Y(dy) = 0
\end{equation}

where we also include those density law $\nu_Y(\cdot)$ for which (2.8) holds as principal value. If (2.8) holds true, then formula (1.2) takes the form

\[ (Af)(x) = \lambda \int_\mathbb{R} (f(x+y) - f(x)) \nu_Y(dy) \]

and the integral converges depending on $\nu_Y(\cdot)$. If we choose $\nu_Y(dy) = 2\alpha |y|^{-2\alpha-1}dy$ for instance, then the integral must be understood in the principal value sense and we get the fractional Laplace operator as formula (2.4) entails.

3. Proof of Theorem 1

The characteristic function of (1.7) is written as follows

\[ \mathbb{E} \exp \left( i\xi \sum_{j=0}^{N(t)} \epsilon_j Y_j \right) = \mathbb{E} \left( \mathbb{E} e^{i\xi Y} \right)^{N(t)}
= \exp \left( \lambda t (\mathbb{E} e^{i\xi Y} - 1) \right)
= \exp \left( \lambda t \left( p \mathbb{E} e^{i\xi Y} + q \mathbb{E} e^{-i\xi Y} - (p + q) \right) \right)
= \exp \left( \lambda t \left( p(\mathbb{E} e^{i\xi Y} - 1) + q(\mathbb{E} e^{-i\xi Y} - 1) \right) \right). \]

From this, we immediately get

\[ \mathbb{E} \exp \left( i\xi A(t/\gamma^\alpha) \right) = \exp \left( \frac{\lambda t}{\gamma^\alpha} \left( p(\mathbb{E} e^{i\xi Y} - 1) + q(\mathbb{E} e^{-i\xi Y} - 1) \right) \right). \]
We recall that the Lévy symbol of a stable subordinator is a mapping from $\mathbb{R} \mapsto \mathbb{C}$ which takes the form
\begin{equation}
-(-i\xi)^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (e^{i\xi y} - 1) \frac{dy}{y^{\alpha+1}}
\end{equation}
for $\alpha \in (0,1)$. The Fourier symbol (depending on $\gamma$) of the characteristic function of $A(t/\gamma)$ is therefore given by
\[
\Phi_{\gamma}(\xi) = -\partial_t \mathbb{E} \exp (i\xi A(t/\gamma)) \bigg|_{t=0} = -\frac{\lambda}{\gamma^\alpha} \left( p \int_0^\infty (e^{i\xi y} - 1) \nu_Y(dy) + q \int_0^\infty (e^{-i\xi y} - 1) \nu_Y(dy) \right)
\]

\[
= -\lambda p \int_\gamma^\infty (e^{i\xi y} - 1) \frac{\alpha dy}{y^{\alpha+1}} - \lambda q \int_\gamma^\infty (e^{-i\xi y} - 1) \frac{\alpha dy}{y^{\alpha+1}}.
\]

For $\gamma \to 0$ we obtain
\begin{equation}
\Phi_{\gamma}(\xi) \to \Phi(\xi) = \lambda \Gamma(1-\alpha) \left( p(-i\xi)^\alpha + q(i\xi)^\alpha \right), \quad \alpha \in (0,1)
\end{equation}
and, from (2.1) we arrive at
\begin{equation}
A f(x) = -\lambda \Gamma(1-\alpha) \left( p \frac{d^\alpha}{dx^\alpha} + q \frac{d^\alpha}{d(-x)^\alpha} \right) f(x).
\end{equation}

The fact that the Lévy process (1.8) has infinitesimal generator (1.9) comes directly from the characteristic function
\[
\mathbb{E} \exp \left( i\xi \mathcal{D}_t^\alpha (pt^*) - i\xi \mathcal{D}_t^\alpha (qt^*) \right) = \exp \left( -t^* p(-i\xi)^\alpha - t^* q(i\xi)^\alpha \right)
\]
where $t^* = \lambda \Gamma(1-\alpha)t > 0$. Thus, we get that
\[
-\partial_t \mathbb{E} \exp \left( i\xi \mathcal{D}_t^\alpha (pt^*) - i\xi \mathcal{D}_t^\alpha (qt^*) \right) \bigg|_{t=0} = \lambda \Gamma(1-\alpha) \left( p(-i\xi)^\alpha + q(i\xi)^\alpha \right)
\]
which coincides with $\Phi(\xi)$ in (3.2).

In the last calculations we have used the fact that
\[
\mathbb{E} e^{i\xi \mathcal{D}_t^\alpha} = \exp \left( -t(-i\xi)^\alpha \right) = \exp \left( -t|\xi|^\alpha e^{-i\frac{\pi\alpha}{4}} \right), \quad \xi \in \mathbb{R}, \ t \geq 0
\]
and thus,
\[
\mathbb{E} e^{-i\xi \mathcal{D}_t^\alpha} = \exp \left( -t|\xi|^\alpha e^{i\frac{\pi\alpha}{4}} \right) = \exp \left( -t(i\xi)^\alpha \right).
\]
The Fourier transforms of the Weyl’s fractional derivatives, for $\alpha \in (0,1)$, are given by (6)
\begin{equation}
\int_{\mathbb{R}} e^{i\xi x} \frac{d^n}{d(\pm x)^\alpha} f(x) dx = (\mp i\xi)^\alpha \hat{f}(\xi), \quad f \in L^1(\mathbb{R}).
\end{equation}

4. Proof of Theorem 2

For $p = q = 1/2$ and $\alpha \in (0,2)$ we obtain that
\[
\Phi_{\gamma}(\xi) = -\partial_t \mathbb{E} \exp (i\xi A(t/\gamma)) \bigg|_{t=0} = -\frac{\lambda}{2\gamma^\alpha} \left( \mathbb{E} e^{i\xi Y} + \mathbb{E} e^{-i\xi Y} - 2 \right)
\]
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\[- \frac{\lambda}{2^\alpha} \int_0^\infty (e^{i\xi y} + e^{-i\xi y} - 2) \nu_\gamma(dy) \]

\[- \frac{\lambda}{\gamma^\alpha} \int_\mathbb{R} \nu_\gamma^\ast(dy) \]

where, we recall that

\[ \nu_\gamma^\ast(y) = \frac{\alpha \gamma^\alpha}{2} |y|^{-\alpha - 1} 1_{|y| \geq \gamma}. \]

We explicitly have that

\[ \Phi_\gamma(\xi) = - \frac{\lambda}{\gamma^\alpha} \int_\mathbb{R} \nu_\gamma^\ast(dy) \]

\[ = - \frac{\alpha \lambda}{2} \int_{\mathbb{R}\setminus B_\gamma} (\cos(\xi y) - 1) |y|^{-\alpha - 1} dy. \]

By taking the limit for \( \gamma \to 0 \), we obtain

\[ (4.1) \quad \Phi_\gamma(\xi) \to \Phi(\xi) = - \frac{\alpha \lambda}{2} \int_{\mathbb{R}\setminus B_\gamma} \cos(\xi y) - 1 |y|^{-\alpha - 1} dy. \]

where, due to the fact that \( (\cos(y) - 1) |y|^{-\alpha - 1} \leq y^2 |y|^{-\alpha - 1} \) by Taylor expansion near the origin, we obtain that

\[ 0 < C = \frac{1}{2} \int_\mathbb{R} (1 - \cos(y)) |y|^{-\alpha - 1} dy < \infty \]

and \( |\xi|^\alpha \) is the Fourier multiplier of the infinitesimal generator of a stable symmetric process. Indeed, for the symmetric stable process \( \mathcal{G}^\beta(t), t > 0, \beta \in (0, 2) \), we have that

\[ -\partial_t \mathbb{E} e^{i\xi \mathcal{G}^\beta(t)} \bigg|_{t=0} = |\xi|^\beta \]

and (2.1) holds with

\[ (4.2) \quad \mathcal{A} f(x) = - \frac{\partial^\beta f}{d|x|^\beta}(x) = - \frac{\sigma}{2} \left( \frac{\partial^\beta f}{d(x)} + \frac{\partial^\beta f}{d(-x)}(x) \right) \]

where \( \sigma = (\cos \pi \beta/2)^{-1} \). The Fourier symbol of the Riesz operator (4.2) is written as (see formula (3.4))

\[ \int_\mathbb{R} e^{i\xi x} \frac{\partial^\beta f}{d|x|^\beta}(x) dx = \frac{\sigma}{2} \left( (-i\xi)^\beta + (i\xi)^\beta \right) \hat{f}(\xi) = |\xi|^\beta \hat{f}(\xi). \]

Therefore, from (4.1), we conclude that

\[ \sum_{j=0}^{N(t/\gamma^\alpha)} Y_j \gamma \to^0 \mathcal{G}^\alpha(t^\ast) \]

in distribution, where

\[ \mathbb{E} e^{i\xi \mathcal{G}^\alpha(t^\ast)} = \exp(-t^\ast |\xi|^\alpha) \]

and \( t^\ast = \alpha \lambda C t, t > 0 \). Furthermore, the generator of \( \mathcal{G}^\alpha(t^\ast) \) with \( \alpha \in (0, 2) \) is

\[ -\alpha \lambda C \frac{d^\alpha f}{d|x|^\alpha}(x) = - \frac{1}{2\pi} \int_\mathbb{R} e^{-i\xi x} \Phi(\xi) \hat{f}(\xi) d\xi. \]

We notice that, for \( \alpha \in (0, 1) \), the constant \( \alpha \lambda C \) equals

\[ \frac{\alpha \lambda}{2} \int_\mathbb{R} (1 - \cos(y)) |y|^{-\alpha - 1} dy = 2 \int_0^\infty (1 - \cos(y)) \frac{dy}{y^{\alpha + 1}} \]
and therefore, for the process (5.2). Finally, we observe that

\[
-\partial_t \mathbb{E} e^{i\xi \cdot \mathbf{A}(t)} = -\lambda \int_{\mathbb{R}^d} (e^{i\xi \cdot \mathbf{y}} + e^{-i\xi \cdot \mathbf{y}} - 2) \nu_{\mathbf{y}}(d\mathbf{y})
\]

and therefore, for the process (1.15), we get that

\[
-\partial_t \mathbb{E} e^{i\xi \cdot \mathbf{A}(t)} = -\lambda \int_{\mathbb{R}^d} (\cos \xi \cdot \mathbf{y} - 1) \frac{\gamma^\alpha \Gamma(\alpha + \frac{d}{2})}{\pi^\frac{d}{2} \Gamma(\alpha) (|\mathbf{y}|^2 + \gamma)^{\alpha + \frac{d}{2}}} d\mathbf{y} = \Phi_\gamma(\xi).
\]

The limit for \( \gamma \to 0 \) leads to the Fourier symbol

\[
\lim_{\gamma \to 0} \Phi_\gamma(\xi) = -\lambda \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^\frac{d}{2} \Gamma(\alpha)} \int_{\mathbb{R}^d} (\cos \xi \cdot \mathbf{y} - 1) \frac{d\mathbf{y}}{|\mathbf{y}|^{2\alpha + d}} = \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^\frac{d}{2} \Gamma(\alpha)} C |\xi|^{2\alpha}
\]

where

\[
C = \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{2\alpha + d}} d\mathbf{y}.
\]

The interested reader can find in [2] a detailed computation of the integrals in (5.2) and (5.3). Finally, we observe that

\[
-\partial_t \mathbb{E} e^{i\xi \cdot \mathbf{A}(t)} = -\frac{\lambda}{2 \gamma^\alpha} \int_{\mathbb{R}^d} (e^{i\xi \cdot \mathbf{y}} + e^{-i\xi \cdot \mathbf{y}} - 2) \nu_{\mathbf{y}}(d\mathbf{y}) = \Phi_\gamma(\xi)
\]
converges, for \( \gamma \to 0 \), to the Fourier symbol

\[
\Phi(\xi) = \frac{-\lambda \Gamma(\alpha + \frac{d}{2})}{2 \pi^{\frac{d}{2}} \Gamma(\alpha)} \int_{\mathbb{R}^d} \left( e^{i\xi \cdot y} + e^{-i\xi \cdot y} - 2 \right) \frac{dy}{|y|^{2\alpha + d}}.
\]

By applying formula (2.1), we get

\[
\mathcal{A}f(x) = \frac{\lambda \Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha)} C_{d}(\alpha) \int_{\mathbb{R}^d} \left( f(x + y) + f(x - y) - 2f(x) \right) \frac{dy}{|y|^{2\alpha + d}}
\]

\[
= -\frac{\lambda \Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha)} C(-\Delta)^{\alpha} f(x)
\]

where

\[
C_{d}(\alpha) = \left( \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|y|^{2\alpha + d}} \, dy \right)^{-1}
\]

which is the generator of the isotropic stable process \( \mathcal{S}_{2\alpha}(t^*) \) with

\[
t^* = \frac{\lambda \Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha)} Ct, \quad t > 0.
\]

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