TREND TO THE EQUILIBRIUM FOR THE FOKKER-PLANCK SYSTEM WITH AN EXTERNAL MAGNETIC FIELD

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Abstract. We consider the Fokker-Planck equation with an external magnetic field. Global-in-time solutions are built near the Maxwellian, the global equilibrium state for the system. Moreover, we prove the convergence to equilibrium at exponential rate. The results are first obtained on spaces with an exponential weight. Then they are extended to larger functional spaces, like certain Lebesgue spaces with polynomial weights and modified weighted Sobolev spaces, by the method of factorization and enlargement of the functional space developed in [Gualdani, Mischler, Mouhot, 2017].

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1. Introduction and main results.

1.1. Introduction. In this article, we are interested in inhomogeneous kinetic equations. These equations model the dynamics of a charged particle system described by a probability density $F(t, x, v)$ representing at time $t \geq 0$ the density of particles at position $x \in \mathbb{T}^3$ and at velocity $v \in \mathbb{R}^3$.

In the absence of force and collisions, the particles move in a straight line at constant speed according to the principle of Newton, and $F$ is the solution of the Vlasov equation
\[
\partial_t F + v \cdot \nabla_x F = 0,
\]
where $\nabla_x$ is the gradient operator with respect to the variable $x$, and the symbol “$\cdot$” represents the scalar product in the Euclidean space $\mathbb{R}^3$. When there are forces and collisions, this equation must be corrected. This leads to various kinetic equations, the most famous being those of Boltzmann, Landau and Fokker-Planck. The general model for the dynamics of charged particles, assuming that they undergo collision modulated by a collision kernel $Q$ and under the action of an external force $F \in \mathbb{R}^3$, is written by the following kinetic equation:
\[
\partial_t F + v \cdot \nabla_x F + F(t, x) \cdot \nabla_v F = Q(F),
\]
where $Q$, possibly non-linear, acts only in velocity and where $F$ can even depend on $F$ via Poisson or Maxwell equations.

According to the $H$-theorem of Boltzmann in 1872, there exists a quantity $H(t)$ called entropy which varies monotonically in time, while the gas relaxes towards the thermodynamic equilibrium characterized by the Maxwellian: it is a solution of equation (1), independent in time and having the same mass as the initial system. The effect of the collisions will bring the distribution $F(t)$ to the Maxwellian with time. A crucial question is then to know the rate of convergence and this question has been widely studied over the past 15 years, in particular with the so called hypocoercive strategy (see [28] or [14] for an introductive papers).

1.2. Fokker-Planck equation with a given external magnetic field.

1.2.1. Presentation of the equation. We are interested in the Fokker-Planck inhomogeneous linear kinetic equation with a fixed external magnetic field $x \mapsto B_0(x) \in \mathbb{R}^3$ which depends only on the spatial variables $x \in \mathbb{T}^3 \equiv [0, 2\pi]^3$. The Cauchy problem is the following:
\[
\begin{cases}
\partial_t F + v \cdot \nabla_x F - (v \wedge B_0) \cdot \nabla_v F = \nabla_v \cdot (\nabla_v + v) F \\
F(0, x, v) = F_0(x, v),
\end{cases}
\]
(2)
Here $F$ is the distribution function of the particles, and represents the probability density of finding a particle with velocity $v \in \mathbb{R}^3$ and position $x \in \mathbb{T}^3$ at time $t \geq 0$. (Where “$\wedge$” indicates the vector (cross) product.)

We define the Maxwellian
\[
\mu(v) := \frac{1}{(2\pi)^{3/2}} e^{-v^2/2}.
\]
It is a time independent solution of the system (2), since
\[
\partial_t \mu + v \cdot \nabla_x \mu = 0, \quad \nabla_v \cdot (\nabla_v + v) \mu = 0 \quad \text{and} \quad (v \wedge B_0) \cdot \nabla_v \mu = 0.
\]
For the Fokker-Planck system with external magnetic field (2), the entropy functional in relation with the so-called $H$-Theorem

$$H(h, \mu) = \iint h \ln \left( \frac{h}{\mu} \right) \, dx \, dv,$$

for a smooth sufficiently decaying (in space and velocity) positive probability density $h$. Then, we directly check that $H(\mu, \mu) = 0$. According to the non-negativity of the function $s \mapsto s \ln s - s + 1$, $H(h, \mu)$ is non-negative since

$$H(h, \mu) = \iint \left( \frac{h}{\mu} \ln \left( \frac{h}{\mu} \right) - \frac{h}{\mu} + 1 \right) \mu \, dx \, dv \geq 0.$$

When $F(t, \cdot)$ is a solution of problem (2)

$$\frac{d}{dt} H(F(t), \mu) = -\iint \frac{|(\nabla_v + v)F|^2}{F} \, dx \, dv \leq 0,$$

see [7, 14]. This is a version of the $H$-Theorem about decay of entropy. Let us now check that the equilibrium $\mu$ is in fact unique.

Let us suppose that $F$ is a positive probability density, smooth, rapidly-decaying and a time independent solution of (2), $F(t, \cdot) = F(\cdot)$. Then, we get

$$0 = \frac{d}{dt} H(F, \mu) = -\iint \frac{|(\nabla_v + v)F|^2}{F} \, dx \, dv.$$

So that

$$\nabla_v \left( \frac{F}{\mu} \right) = 0,$$

and there exist $\varphi$ such that $F = \mu(v) \varphi(x)$, and putting this into (4) yields

$$\nabla_x \varphi = 0,$$

since $(v \wedge B_e) \cdot \nabla_v \mu = 0$. (Note that this fact will be of use through this article). By positivity and the fact that $F$ is a probability density, we get $F = \mu$.

Concerning (2), we are interested in the return to the global equilibrium $\mu$, in the sense that exponential decay to equilibrium for perturbations around the Maxwellian is considered, and the convergence of $F$ to $\mu$ in norms in the largest possible spaces. Now, we fix some notation for the weighted Lebesgue spaces.

**Notation 1.1.** For some given Borel weight function $m = m(v) > 0$ on $\mathbb{R}^3$, let us define $L^p(m) \ 1 \leq p \leq 2$, as the weighted Lebesgue space associated with the norm

$$\|F\|_{L^p(m)} := \|F m\|_{L^p} = \left( \int_{\mathbb{R}^3 \times \mathbb{T}^3} F^p(x, v) m^p(v) \, dxdv \right)^{\frac{1}{p}}. \quad (3)$$

In particular, we will work in $L^2(\mu^{1/2})$ and $H^1(\mu^{1/2})$ the real Hilbert spaces defined specifically by

$$\forall h \in L^2(\mu^{1/2}), \ \|h\|^2_{L^2(\mu^{1/2})} = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |h(x, v)|^2 \mu(v) \, dxdv,$$

$$\forall h \in H^1(\mu^{1/2}), \ \|h\|^2_{H^1(\mu^{1/2})} = \|h\|^2_{L^2(\mu^{1/2})} + \|\nabla_x h\|^2_{L^2(\mu^{1/2})} + \|\nabla_v h\|^2_{L^2(\mu^{1/2})},$$

and the (real) Hilbertian scalar product $\langle \cdot, \cdot \rangle$ on the space $L^2(\mu^{1/2})$ defined by

$$\forall g, h \in L^2(\mu^{1/2}), \ \langle g, h \rangle = \iint hg \, dxdv.$$
To answer such questions, when \( F \) is close to the equilibrium \( \mu \), we define \( f \) to be the standard perturbation of \( \mu \)
\[
F = \mu + \mu f.
\]
We then rewrite equation (2) in the following form:
\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f - (v \wedge B_e) \cdot \nabla_v f &= -(-\nabla_v + v) \cdot \nabla_v f \\
f(0, x, v) &= f_0(x, v)
\end{align*}
\]
(4)

1.2.2. The main results. First we will show that the problem (4) is well-posed in the \( L^2(\mu^{1/2}) \) space, in the sense of the associated semi-group (See [27]). We associate with the problem (4) the operator \( P_1 \) defined by
\[
P_1 := X_0 + L,
\]
(5)
where \( X_0 = v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v \) and \( L = -(-\nabla_v + v) \cdot \nabla_v \) (6)
The problem (4) is then written
\[
\begin{align*}
\partial_t f + P_1 f &= 0 \\
f(0, x, v) &= f_0(x, v)
\end{align*}
\]
(7)

Theorem 1.2. If \( B_e \in C^\infty(T^3) \) and \( f_0 \in C^{\infty}_0(T^3 \times \mathbb{R}^3) \), then the problem (7) admits a unique solution \( f \in C^1([0, +\infty[; L^2(\mu^{1/2})) \).

Remark 1.3. We note that in a work in progress, we show that operator \( P_1 \) is maximal accretive with non-regular magnetic field and more precisely we suppose just that \( B_e \in L^\infty(T^3) \), which matches Hypothesis 1.8. For the time being, we rely on Theorem 1.2 which forces us to assume that \( B_e \) is smooth.

We also show the exponential convergence towards equilibrium in the norm \( L^2(\mu^{1/2}) \).

Theorem 1.4. Let \( f_0 \in L^2(\mu^{1/2}) \) such that \( \langle f_0 \rangle_\mu = \int \int f_0(x, v) \mu(v) \, dx \, dv = 0 \). If \( B_e \in C^\infty(T^3) \), then there exist \( \kappa > 0 \) and \( c > 0 \) (two explicit constants independent of \( f_0 \)) such that
\[
\forall t \geq 0, \quad \|f(t)\|_{L^2(\mu^{1/2})} \leq ce^{-\kappa t} \|f_0\|_{L^2(\mu^{1/2})},
\]

Note that in the preceding statement the mean \( \langle f(t) \rangle_\mu \) is preserved over time.

We give a result about the return to the global equilibrium \( \mu \) with an exponential decay rate in the space \( H^1(\mu^{1/2}) \).

Theorem 1.5. We suppose that \( B_e \in C^\infty(T^3) \). There exist \( c, \kappa > 0 \) such that for all \( f_0 \in H^1(\mu^{1/2}) \) with \( \langle f_0 \rangle_\mu = 0 \), the solution \( f \) of the system (4) satisfies
\[
\forall t \geq 0, \quad \|f(t)\|_{H^1(\mu^{1/2})} \leq ce^{-\kappa t} \|f_0\|_{H^1(\mu^{1/2})},
\]

Herda and Rodrigues in [19] have showed an estimate of type \( H^1 \) for the solution of the Vlasov-Fokker-Planck equation with a constant magnetic field by the hypocoercivity method. Their goal was not to obtain a large-time behaviour result of solutions but to have a uniform estimates for a parameter limit (parameter of diffusion) (see also [3, 18] for the study of the Vlasov-Poisson-Fokker-Planck with constant magnetic field). Kinetic equations with magnetic field have also been
studied by Zhang and Yin in [31] who have showed existence theorems for the initial value problem of the cometary flow equation with an external electromagnetic force (see also [6] for the study of the cometary flow equation with a self-consistent electromagnetic field).

Remark 1.6. We note that Theorems 1.4 and 1.5 give convergence to the equilibrium \( \mu \) at exponential rate for the solution \( F \) on spaces with an exponential weight \( \mu^{-1/2} \). Indeed,

\[
f = (F - \mu) \mu^{-1},
\]

with the notation given in (3), so

\[
\|f\|_{L^2(\mu^{1/2})}^2 = \|(F - \mu)\mu^{-1}\|_{L^2(\mu^{1/2})}^2
= \int |F - \mu|^2 \mu^{-2} \mu \, dx \, dv
= \int |F - \mu|^2 \mu^{-1} \, dx \, dv =: \|F - \mu\|_{L^2(\mu^{1/2})}^2.
\]

We are interested in extending the results about the exponential decay of the semi-group to much larger spaces, following the work of Gualdani-Mischler-Mouhot in [10]. The following result gives convergence in \( L^p(m) \) norms of \( F \) to \( m = \langle v \rangle^k \). So that there is no ambiguity, we denote the mean with respect to the usual \( L^1 \) norm by

\[
\langle \langle H \rangle \rangle = \iint H \, dx \, dv,
\]

The main result of this paper in this direction is the following.

Theorem 1.7. Let \( p \in [1, 2] \), let \( m = \langle v \rangle^k := (1 + |v|^2)^{k/2} \), \( k > 3(1 - \frac{1}{p}) \), and suppose \( B_\epsilon \in C^\infty(\mathbb{T}^3) \). Then for all \( 0 > a > 3(1 - \frac{1}{p}) - k \) and for all \( F_0 \in L^p(m) \), there exists \( c_{k,p} > 0 \) such that the solution \( F \) of the problem (2) satisfies the decay estimate

\[
\forall t \geq 0, \quad \|F(t) - \mu \langle \langle F_0 \rangle \rangle\|_{L^p(m)} \leq c_{k,p} e^{at} \|F_0 - \mu \langle \langle F_0 \rangle \rangle\|_{L^p(m)}.
\]

It is also possible to obtain the same type of results in the weighted Sobolev space \( \tilde{W}^{1,p}(m) \) which is defined by

\[
\tilde{W}^{1,p}(m) = \{ h \in L^p(m) \mid \langle v \rangle h, \nabla_v h \in L^p(m) \}.
\]

We equip the previous space with the following standard norm:

\[
\|h\|_{\tilde{W}^{1,p}(m)} = \left( \|h\|_{L^p(m)}^p + \|\nabla_v h\|_{L^p(m)}^p + \|\nabla_x h\|_{L^p(m)}^p \right)^{\frac{1}{p}}.
\]

Hypothesis 1.8. Let \( p \in [1, 2] \), the polynomial weight \( m(v) = \langle v \rangle^k \) is such that

\[
k > 3(1 - \frac{1}{p}) + \frac{7}{2} + \max \left( \|B_\epsilon\|_{L^\infty(\mathbb{T}^3)}, \frac{1}{2}\|\nabla_x B_\epsilon\|_{L^\infty(\mathbb{T}^3)} \right).
\]

The second main result of this paper is the following.

Theorem 1.9. Let \( m \) be a weight that satisfies Hypothesis 1.8 with \( p \in [1, 2] \) and suppose that \( B_\epsilon \in C^\infty(\mathbb{T}^3) \). If \( F_0 \in \tilde{W}^{1,p}(m) \), then there is a solution \( F \) of the
problem (2), such that $F(t) \in \tilde{W}^{1,p}(m)$ for all $t \geq 0$, and it satisfies the following decay estimate:

$$\forall t \geq 0, \quad \|F(t) - \mu \langle F_0 \rangle\|_{\tilde{W}^{1,p}(m)} \leq Ce^{at} \|F_0 - \mu \langle F_0 \rangle\|_{\tilde{W}^{1,p}(m)} \quad (11)$$

with $0 > a > \max(a_{m,1}^i, a_{m,2}^i, -\kappa), i \in \{1, 2, 3\}$, where $a_{m,1}^i$ and $a_{m,2}^i$ are constants defined afterwards in (47)-(49) and (56)-(58) and $\kappa$ is defined in Theorem 1.5.

Note that we need to consider a non-classical weighted Sobolev space (more weight is needed on the function than on its derivatives in $x$ and $v$), which is necessary to close the dissipativity estimates in $\tilde{W}^{1,p}(m)$. Note that this trick is not necessary in exponentially weighted spaces. (See also Remark 4.17.)

We will end this part by a brief review of the literature related to the analysis of kinetic PDEs using hypocoercivity methods. In some studies [15, 26, 28, 29], the treated hypocoercivity method is very close to that of hypoellipticity following the method of Kohn, which deals simultaneously with regularity properties and trend to the equilibrium.

The hypocoercive results in $L^2$ specifically were developed in [8, 13] for linear collisional kinetic models. The hypocoercivity $L^2$ and $H^1$ methods were developed in [14, 28] for linear collisional kinetic operators with one-dimensional kernels. See also [24] for an introductory paper for the hypocoercivity methods for the general collisional kinetic models. The methods used were close in spirit to the ones developed by Guo in [11, 12] in functional spaces with exponential weights.

In recent years, the theory of factorization and enlargement of Banach spaces was introduced in [10] and [22]. We note that Mouhot in [23] initiated this type of strategy which has then been developed in an abstract setting by Gualdani, Mischler and Mouhot in [10]. This theory allows us to extend hypocoercivity results into much larger spaces with polynomial weights. We refer for example to [4] and [22], where the authors show, using a factorization argument, the return to equilibrium with an exponential decay rate for the Fokker-Planck equation with an external electrical potential, or [17] for the inhomogeneous Boltzmann equation without angular cutoff case.

The main objective of this article is to obtain exponential decay results for solutions of the Fokker-Planck equation with magnetic field in the largest possible space. To achieve this goal, we first develop both $L^2$ and $H^1$ hypocoercivity methods. Then by applying the enlargement method we extend the results to much larger spaces like $L^1$ with polynomial weights. We emphasize that, on the way to proving the previously mentioned results, we also prove quantitative regularity estimates.

We conclude this section with some comments on our result. For the proof of Theorem 1.4, we follow the micro-macro method proposed in [14]. Note that for the proof of Theorem 1.4, the black box method proposed in [8] (see also [4]) could perhaps be employed, anyway the presence of the Magnetic field induces some difficulties. To prove Theorem 1.7 and 1.9, we apply the abstract theorem of enlargement from [10, 22] to our Fokker-Planck-Magnetic linear operator. We deduce the semi-group estimates of Theorem 1.4 on large spaces like $L^p(\langle v \rangle^k)$ and $\tilde{W}^{1,p}(\langle v \rangle^k)$ with $p \in [1, 2]$.

We hope that this first work will help in future investigations of non-linear problems like the Vlasov-Poisson-Fokker-Planck or Vlasov-Maxwell-Fokker-Planck equations (see [11, 16] and [12, 30]).
Plan of the paper: This article is organized as follows. In Section 2, we prove that the Fokker-Planck-magnetic operator $P_1$ is a generator of a strongly continuous semi-group over the space $L^2(\mu^{1/2})$. In section 3, we show hypocoercivity in the weighted spaces $L^2$ and $H^1$ with an exponential weight. Finally, section 4 is devoted to the proofs of Theorems 1.7 and 1.9 with factorization and enlargement of the functional space arguments.

2. Study of the operator $P_1$. In this part, we show that the problem (7) is well-posed in the space $L^2(\mu^{1/2})$ in the sense of semi-groups. By the Hille-Yosida Theorem, it is sufficient to show that $P_1$ is maximal accretive in the space $L^2(\mu^{1/2})$.

Notation 2.1. We define $P_0$ by

$$P_0 = v \cdot \nabla_x - (v \wedge B_c) \cdot \nabla_v - \nabla_v \cdot (\nabla_v + v).$$

The linearization around the equilibrium $\mu$ of the Cauchy problem (2) reduces the study of the operator $P_1$ defined in (5) to the study of $P_0$, since $P_1$ is obtained via a conjugation of the operator $P_0$ by the function $\mu$, that is to say

$$P_1 u = (\mu^{-1} P_0 \mu) u \quad \forall u \in D(P_1).$$

Similarly, we can define the operator $P_\theta$ as the conjugation of the operator $P_1$ by the function $\mu^\theta$ with $\theta \in [0,1]$. Note that any result on the operator $P_0$ is also true on the operator $P_1$ in the corresponding conjugated space.

We will work in this section on operator $P_{1/2}$ which is defined by

$$P_{1/2} := v \cdot \nabla_x - (v \wedge B_c) \cdot \nabla_v - \nabla_v \cdot (\nabla_v + v) = X_0 + (-\nabla_v + \frac{v}{2})(\nabla_v + \frac{v}{2}),$$

we note that $(-\nabla_v + \frac{v}{2})$ is the adjoint of $(\nabla_v + \frac{v}{2})$ in $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ and $X_0$ is defined in (6). We now show that operator $P_{1/2}$ is maximal accretive in the space $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ and note that this gives the same result for $P_1$ in the space $L^2(\mu^{1/2})$. We study the following problem:

$$\begin{cases} 
\partial_t u + P_{1/2} u = 0 \\
u(0, x, v) = u_0(x, v).
\end{cases}$$

Proposition 2.2. Suppose that $B_c \in L^\infty(\mathbb{T}^3)$. Then the closure with respect to the norm $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ of the magnetic-Fokker-Planck operator $P_{1/2}$ on the space $C_0^\infty(\mathbb{R}^3 \times \mathbb{T}^3)$ is maximally accretive.

Proof. We adapt here the proof given in [26, page 44]. We apply the abstract criterion by taking $H = L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ and the domain of $P_{1/2}$ defined by $D(P_{1/2}) = C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$. First, we show the accretivity of the operator $P_{1/2}$. When $u \in D(P_{1/2})$, we have to show that $\langle P_{1/2} u, u \rangle \geq 0$ where we note by $\langle \cdot, \cdot \rangle$ the scalar product in the space $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ and $\| \cdot \|$ the associated norm. Indeed,

$$\langle P_{1/2} u, u \rangle = \langle v \cdot \nabla_x u - (v \wedge B_c) \cdot \nabla_v u + (-\nabla_v + \frac{v}{2})(\nabla_v + \frac{v}{2}) u, u \rangle$$

$$= \iint v \cdot \nabla_x u \times u \, dx \, dv - \iint (v \wedge B_c) \cdot \nabla_v u \times u \, dx \, dv + \| (\nabla_v + \frac{v}{2}) u \|^2$$

$$= \| (\nabla_v + \frac{v}{2}) u \|^2 \geq 0,$$
since operators $(v \wedge B_\epsilon) \cdot \nabla v$ and $v \cdot \nabla x$ are skew-adjoint see Remark A.2.

Let us now show that there exists $\lambda_0 > 0$ such that the operator

$$P_{1/2} + \lambda_0 \text{Id}$$

has dense image in $H$. We take $\lambda_0 = \frac{3}{2} + 1$ (following [26]). Let $u \in H$ satisfy

$$\langle u, (P_{1/2} + \lambda_0 \text{Id}) h \rangle = 0, \quad \forall h \in D(P_{1/2}).$$  

(13)

We have to show that $u = 0$.

First, we observe that equality (13) implies that

$$(-\Delta + \frac{v^2}{4} + 1 - X_0)u = 0, \quad \text{in } D'(\mathbb{R}^3 \times \mathbb{T}^3).$$

Under Hypothesis that $B_\epsilon \in L^\infty(\mathbb{T}^3)$, and following Hormander [20, 21] or Helffer-Nier [26, Chapter 8], operator $-\Delta + \frac{v^2}{4} + 1 - X_0$ is hypoelliptic, so $u \in C^\infty(\mathbb{R}^3 \times \mathbb{T}^3)$.

Now we introduce the family of truncation functions $\xi_k$ indexed by $k \in \mathbb{N}^*$ and defined by

$$\xi_k(v) := \xi(\frac{v}{k}) \quad \forall k \in \mathbb{N}^*,$$

where $\xi$ is a $C^\infty$ function satisfying $0 \leq \xi \leq 1$, $\xi = 1$ on $B(0,1)$, and $\text{Supp} \xi \subset B(0,2)$. We note that in [26], a cut-off in $x$ and $v$ was necessary to develop the argument whereas here it suffices to perform a cut-off in $v$ because $x \in \mathbb{T}^3$. For all $u, w \in C^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, we have

$$\int \int \nabla_v (\xi_k u) \cdot \nabla_v (\xi_k w) \, dx dv + \int \int \xi_k^2 (\frac{v^2}{4} + 1) w u \, dx dv + \int \int u X_0 (\xi_k^2 w) \, dx dv$$

$$= \int \int |\nabla_v \xi_k|^2 w u \, dx dv + \int \int (u \nabla_v w - w \nabla_v u) \cdot \xi_k \nabla_v \xi_k \, dx dv + \langle u, T(\xi_k^2 w) \rangle.$$

When $u$ satisfies (13) in particular, when $h = \xi_k^2 w$, we get for all $w \in C^\infty$

$$\int \int \nabla_v (\xi_k u) \cdot \nabla_v (\xi_k w) \, dx dv + \int \int \xi_k^2 (\frac{v^2}{4} + 1) w u \, dx dv + \int \int u X_0 (\xi_k^2 w) \, dx dv$$

$$= \int \int |\nabla_v \xi_k|^2 w u \, dx dv + \int \int (u \nabla_v w - w \nabla_v u) \cdot \xi_k \nabla_v \xi_k \, dx dv.$$

In particular, we take the test function $w = u$, so

$$\langle \nabla_v (\xi_k u), \nabla_v (\xi_k u) \rangle + \int \int \xi_k^2 (\frac{v^2}{4} + 1) u^2 \, dx dv + \int \int u X_0 (\xi_k^2 u) \, dx dv$$

$$= \int \int |\nabla_v \xi_k|^2 u^2 \, dx dv.$$

By an integration by parts, we obtain

$$\langle \nabla_v (\xi_k u), \nabla_v (\xi_k u) \rangle + \int \int \xi_k^2 (\frac{v^2}{4} + 1) u^2 \, dx dv + \int \int \xi_k u^2 X_0 (\xi_k) \, dx dv$$

$$= \int \int |\nabla_v \xi_k|^2 u^2 \, dx dv.$$

Which gives the existence of a constant $c > 0$ such that, for all $k \in \mathbb{N}^*$,

$$\|\xi_k u\|^2 + \frac{1}{4} \|\xi_k vu\|^2$$

$$\leq \frac{c}{k^2} \|u\|^2 + \frac{c}{k} \|(v \wedge B_\epsilon) \xi_k u\| \|u\|. $$
This leads to, for $\eta > 0$ to be chosen later,
\[ \|\xi_k u\|^2 + \frac{1}{4}\|\xi_k vu\|^2 \leq c\left(\frac{1}{k^2} + \frac{c_\eta}{k^2}\|Be\|^2_\infty\right)\|u\|^2 + \eta\|\xi_k vu\|^2. \]
Choosing $\eta \leq \frac{1}{4}$, we get
\[ \|\xi_k u\|^2 \leq c\left(\frac{1}{k^2} + \frac{c_\eta}{k^2}\|Be\|^2_\infty\right)\|u\|^2. \] (14)
Taking $k \to +\infty$ in (14), leads to $u = 0$. \(\square\)

**Proof of Theorem 1.2.** According to Remark 2.1, the operator $P_1$ has a closure $\overline{P_1}$ from $C^\infty_0(\mathbb{T}^3 \times \mathbb{R}^3)$. This gives Theorem 1.2, by a direct application of Hille-Yosida’s theorem (cf. [27] for more details for the semi-group theory) to the problem (4), with $D(P_1) = C^\infty_0(\mathbb{T}^3 \times \mathbb{R}^3)$ and $H = L^2(\mu^{1/2})$. \(\square\)

From now on, we write $P_\theta$ for the closure of the operator $P_\theta$ from the space $C^\infty_0(\mathbb{T}^3 \times \mathbb{R}^3)$ with respect the norm $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$.

3. **Trend to the equilibrium.**

3.1. **Hypocoercivity in the space $L^2(\mu^{1/2})$**. The purpose of this subsection is to show the exponential time decay of the $L^2(\mu^{1/2})$ entropy for $P_1$, based on macroscopic equations. First, we try to find the macroscopic equations associated with system (4). We write $f$ in the following form:
\[ f(x, v) = r(x) + h(x, v), \] (15)
where $r(f)(x) = \int f(x, v) \mu(v) dv$ and $m(f)(x) = \int v f(x, v) \mu(v) dv$ will be used later.

**Definition 3.1.** In the following, we define
\[ \Lambda_x = (1 - \Delta_x)^{1/2}, \]
and introduce a class of Hilbert spaces
\[ \mathbb{H}^\alpha := \{u \in \mathcal{S}', \Lambda_x^\alpha u \in L^2(dx)\} \quad \text{with} \quad \alpha \in \mathbb{R}, \]
where $\mathcal{S}'$ is the space of temperate distributions.

We recall that the operator $\Lambda_x^\alpha$ is an elliptic, self-adjoint, invertible operator from $\mathbb{H}^2(dx)$ to $L^2(dx)$ and $\Lambda_x \geq \text{Id}$. (cf [15, section 6] for a proof of these properties).

**Lemma 3.2.** Let $f$ be the solution of the system (4), with the decomposition given in (15). Then we have
\[ \partial_t r = Op_1(h), \]
\[ \partial_t m = -\nabla_x r - m \wedge B_e + Op_1(h). \] (16) (17)
Where $Op_1$ denotes a bounded generic operator from $L^2(\mu^{1/2})$ to $\mathbb{H}^{-1}(dx)$.

**Proof.** We suppose is $f$ is a Schwarz function. In order to show equation (16), we integrate equation (4) with respect to the measure $d\mu := \mu(v) dv$. We get
\[ \partial_t \int f d\mu + \int v \cdot \nabla_x f d\mu - \int (v \wedge B_e) \cdot \nabla_v f d\mu = -\int (-\nabla_v + v) \cdot \nabla_v f d\mu \]
\[ = (Lf, 1) = (f, L1) = 0, \]
since, $L_1 = 0$, $L$ is a self-adjoint operator and

$$(v \land B_c) \cdot \nabla_v f = \nabla_v \cdot (v \land B_c)f,$$

where we noted by $(\cdot, \cdot)$ the scalar product in the space $L^2(\mu^{1/2})$ and $\|\cdot\|$ the associated norm. Then, by using the equality $\int v f d\mu = \int v h d\mu$ with $d\mu = \mu dv$ we obtain

$$\partial_t r = \nabla_x \cdot \int vh d\mu = Op_1(h),$$

hence equality (16).

To show (17), we multiply equation (4) by $v$ before performing an integration with respect to the measure $d\mu$, we obtain

$$\partial_t \int v f d\mu + \nabla_x \cdot \int (v \otimes v) f d\mu - \int v((v \land B_c) \cdot \nabla_v f) d\mu = \int v L f d\mu,$$

where $(\nabla_x \cdot \int (v \otimes v) f d\mu)_{1 \leq j \leq 3} = \left( \sum_{i=1}^{3} \int v_j v_i \partial_{x_i} f d\mu \right)_{1 \leq j \leq 3}$. Now, we will calculate term by term the left-hand side of the equality (18). We use that

$$\sum_{i=1}^{3} \int v_j v_i \partial_{x_i} f d\mu = \int v_j^2 \partial_{x_j} f d\mu = \int (v_j^2 - 1) \partial_{x_j} f d\mu + \partial_{x_j} r$$

$$= \int (v_j^2 - 1) \partial_{x_j} h d\mu + \partial_{x_j} r,$$

because $\forall j \neq i$, performing an integration by parts we get

$$\int v_j v_i \partial_{x_i} f d\mu = \int (v_j^2 - 1) \partial_{x_j} r d\mu = 0.$$

By using the equality (19), then we have

$$\nabla_x \cdot \int (v \otimes v) f d\mu = Op_1(h) + \nabla_x r.$$

Furthermore,

$$\int v L f d\mu = \int L v f d\mu = \int f v d\mu = \int h v d\mu.$$

It remains to compute component by component $\int v ((v \land B_c \cdot \nabla_v f)) d\mu$. We have for all $1 \leq j \leq 3$,

$$\int v_j((v \land B_c) \cdot \nabla_v f) d\mu = \int v_j \nabla_v \cdot ((v \land B_c) f) d\mu$$

$$= \int (-\nabla_v + v)(v_j) \cdot (v \land B_c) f d\mu$$

$$= -\delta_j \cdot \int (v \land B_c) f d\mu$$

$$= -\delta_j \cdot (\int v f d\mu) \land B_c$$

$$= -\delta_j \cdot (m \land B_c)$$

$$= -(m \land B_c)_j.$$
Therefore \( \int v ((v \cdot B_e \cdot \nabla_x f)) \, d\mu = -m \wedge B_e \), where \( m \) is defined in (15).

By combining all the previous equalities in (18), we obtain

\[
\partial_t m = -\nabla_x r - m \wedge B_e + Op_1(h).
\]

\( \Box \)

**Remark 3.3.** If \( B_e \in L^\infty(\mathbb{T}^3) \), then

\[
m \wedge B_e = Op_1(h),
\]

so the macroscopic equation (15) takes the following form

\[
\partial_t m = -\nabla_x r + Op_1(h).
\]

Now we are ready to build a new entropy, defined for any \( u \in L^2(\mu^{1/2}) \) by

\[
F_\varepsilon(u) = \| u \|^2 + \varepsilon \langle \Lambda_x^{-2} \nabla_x r(u), m(u) \rangle, \quad r(u) := \int u \, d\mu \text{ and } m(u) := \int u \, d\mu,
\]

where \( d\mu = \mu(v) \, dv \). Using the Cauchy-Schwarz inequality gives us directly that

**Lemma 3.4.** If \( \varepsilon \leq \frac{1}{2} \), then

\[
\frac{1}{2} \| u \|^2 \leq F_\varepsilon(u) \leq 2\| u \|^2
\]

Now, we can prove the main result of hypocoercivity leading to the proof of Theorem 1.4.

**Proposition 3.5.** There exists \( \kappa > 0 \) such that, if \( f_0 \in L^2(\mu^{1/2}) \) and \( \langle f_0 \rangle = 0 \), then the solution of system (4) satisfies

\[
\forall t \geq 0, \quad F_\varepsilon(f(t)) \leq e^{-\kappa t} F_\varepsilon(f_0).
\]

**Proof.** We write

\[
\frac{d}{dt} F_\varepsilon(f(t)) = \frac{d}{dt} \| f \|^2 + \varepsilon \frac{d}{dt} \langle \Lambda_x^{-2} r, m \rangle.
\]

We will omit the dependence of \( f \) with respect to \( t \). For the first term, we notice that

\[
\frac{d}{dt} \| f \|^2 = 2 \langle Lf, f \rangle = -2 \| \nabla_x f \|^2 \leq -2 \| h \|^2,
\]

by Poincaré’s inequality and the spectral gap property of the operator \( L \) (see [9, Lemma 2.1] for more details on this subject). For the second term, using the macroscopic equations, we get

\[
\frac{d}{dt} \langle \Lambda_x^{-2} \nabla_x r, m \rangle = \langle \Lambda_x^{-2} \nabla_x \partial_t r, m \rangle + \langle \Lambda_x^{-2} \nabla_x r, \partial_t m \rangle
\]

\[
= -\langle \Lambda_x^{-2} \nabla_x r, \nabla_x r \rangle + \langle \Lambda_x^{-2} \nabla_x Op_1(h), m \rangle + \langle \Lambda_x^{-2} \nabla_x r, Op_1(h) \rangle
\]

\[
\leq \| \Lambda_x^{-1} \nabla_x r \|^2 + C \| \Lambda_x^{-1} Op_1(h) \| (\| \Lambda_x^{-1} \nabla_x r \| + \| \Lambda_x^{-1} \nabla_x m \|).
\]

Now, using \( \| m \| \leq \| h \| \), the Cauchy-Schwarz inequality and the following estimate:

\[
\| \Lambda_x^{-1} \nabla_x \phi \| \leq \| \phi \|, \quad \forall \phi \in L^2(\mu^{1/2}),
\]

we obtain

\[
\frac{d}{dt} \langle \Lambda_x^{-2} r, m \rangle \leq -\frac{1}{2} \| \Lambda_x^{-1} \nabla_x r \|^2 + C \| h \|^2.
\]
Poincaré’s inequality on $L^2(dx)$ takes the form
\[ \forall \phi \in L^2(dx), \text{ such that } \langle \phi \rangle = 0, \quad \| \Lambda_x^{-1} \nabla_x \phi \|^2 \geq \frac{c_P}{c_P + 1} \| \phi \|^2, \]
where $\langle \phi \rangle = \int \phi(x) \, dx$ and $c_P > 0$ is the spectral gap of $-\Delta_x$ on the torus (see [14, Lemma 2.6] for the proof of the previous inequality). Using this, we obtain, by applying the previous estimate to $r$ (since $(r) = (f) = (f_0) = 0$),
\[ \frac{d}{dt} \langle \Lambda_x^{-2} \nabla_x r, m \rangle \leq -\frac{1}{2} \frac{c_P}{c_P + 1} \| r \|^2 + C\| h \|^2. \tag{22} \]

This completes the proof of Theorem 1.4.

We can deduce the proof of Theorem 1.4.

Proof of Theorem 1.4. Starting from Lemma 3.4 and Proposition 3.5, we have, for $f$ the solution of the system (4),
\[ \| f \|^2 \leq 2 \mathcal{F}_e(f) \leq 2e^{-\alpha t} \mathcal{F}_e(f_0) \leq 4e^{-\alpha t} \| f_0 \|^2. \]
This completes the proof of Theorem 1.4. \hfill \square

3.2. Hypocoercivity in the space $H^1(\mu^{1/2})$. We will establish some technical lemmas, which will help us to deduce the exponential time decay of the norm $H^1(\mu^{1/2})$, noting that we work in three dimensions.

The following lemma gives the exact values of some commutators will be used later.

Lemma 3.6. The following equalities hold:
1. $[\partial_{v_i}, v \cdot \nabla_x] = \partial_{v_i}, \quad \forall i \in \{1, 2, 3\}$.
2. $[\partial_{v_i}, (-\partial_{v_j} + v_j)] = \delta_{ij}, \quad \forall i, j \in \{1, 2, 3\}$.
3. $[\nabla_x, (v \wedge B_e) \cdot \nabla_v] = B_e \wedge \nabla_v$.
4. $[\nabla_x, (v \wedge B_e) \cdot \nabla_v] = (v \wedge \nabla_x B_e) \cdot \nabla_v$.

Proof. Let $f \in C_0^\infty(T^3 \times \mathbb{R}^3)$. The first two equalities are obvious. We go directly to the proof of 3. Writing $B_e = (B_1, B_2, B_3)$,
\[ [\partial_{v_1}, (v \wedge B_e) \cdot \nabla_v] f = \partial_{v_1} ((v \wedge B_e) \cdot \nabla_v) f - ((v \wedge B_e) \cdot \nabla_v) \partial_{v_1} f = \partial_{v_1} (v_2 B_3 - v_3 B_2) \partial_{v_1} f + (v_3 B_1 - v_1 B_3) \partial_{v_2} f + (v_1 B_2 - v_2 B_1) \partial_{v_3} f - ((v \wedge B_e) \cdot \nabla_v) \partial_{v_1} f = (B_2 \partial_{v_2} f - B_3 \partial_{v_3} f) = (B_e \wedge \nabla_v)_1 f. \]

Similarly we can show that, for all $1 \leq i \leq 3$,
\[ [\partial_{v_i}, (v \wedge B_e) \cdot \nabla_v] = (B_e \wedge \nabla_v)_i. \]
This proves the equality. Now, we will show (4),

\[
\nabla_x ((v \wedge B_e) \cdot \nabla_v) f = \nabla_x ((v \wedge B_e) \cdot \nabla_v) f - ((v \wedge B_e) \cdot \nabla_v) \nabla_x f
\]
\[
= (v \wedge \nabla_x B_e) \cdot \nabla_v f.
\]

Now, we are ready to build a new entropy that will allow us to show the exponential decay of the norm \(H^1(\mu^{1/2})\). We define this modified entropy by

\[
E(u) = C \|u\|^2 + D \|\nabla_v u\|^2 + E \langle \nabla_x u, \nabla_v u \rangle + \|\nabla_x u\|^2, \quad \forall u \in H^1(\mu^{1/2}),
\]

where \(C > D > E > 1\) are constants fixed below. We first show that \(E(u)\) is equivalent to the norm \(H^1(\mu^{1/2})\) of \(u\).

**Lemma 3.7.** If \(E^2 < D\), then \(\forall u \in H^1(\mu^{1/2})\)

\[
\frac{1}{2} \|u\|^2_{H^1(\mu^{1/2})} \leq E(u) \leq 2C \|u\|^2_{H^1(\mu^{1/2})}.
\]

**Proof.** Let \(u \in H^1(\mu^{1/2})\). Using the Cauchy-Schwarz inequality, we get

\[
|E(\nabla_x u, \nabla_v u)| \leq \frac{E^2}{2} \|\nabla_v u\|^2 + \frac{1}{2} \|\nabla_x u\|^2,
\]

which implies

\[
C \|u\|^2 + (D - \frac{E^2}{2}) \|\nabla_v u\|^2 + (1 - \frac{1}{2}) \|\nabla_x u\|^2 \leq E(u)
\]
\[
\leq C \|u\|^2 + (D + \frac{E^2}{2}) \|\nabla_v u\|^2 + (1 + \frac{1}{2}) \|\nabla_x u\|^2.
\]

This implies (23) if \(E^2 < D\).

Note that using the same approach as in Section 3, we can show the existence of a solution of the problem (4), which will be denoted as \(f\), in the space \(H^1(\mu^{1/2})\) in the sense of an associated semi-group. Using the preceding results, we are able to study the decrease of the modified entropy \(E(f(t))\).

**Proposition 3.8.** Suppose that \(B_e \in L^\infty(\mathbb{T}^3)\), then there exist \(C, D, E\) and \(\kappa > 0\), such that for all \(f_0 \in H^1(\mu^{1/2})\) with \(\langle f_0 \rangle = 0\), the solution \(f\) of the system (4) satisfies

\[
\forall t > 0, \, \, E(f(t)) \leq E(f_0) e^{-\kappa t}.
\]

**Proof.** The time derivatives of the four terms defining \(E(f(t))\) will be calculated separately. For the first term we have

\[
\frac{d}{dt} \|f\|^2 = -2 \langle \partial_t f, f \rangle
\]
\[
= -2 \langle v \cdot \nabla_x f, f \rangle + 2 \langle (v \wedge B_e) \cdot \nabla_v f, f \rangle - 2 \langle -\nabla_v + v \cdot \nabla_v f, f \rangle
\]
\[
= -2 \|\nabla_v f\|^2.
\]
The time derivative of the third term can be calculated as follows:

\[
\frac{d}{dt} ||\nabla_v f||^2 = 2(\nabla_v \partial_t f, \nabla_v f) \\
= -2(\nabla_u (v \cdot \nabla_x f), \nabla_v f) + 2(\nabla_v ((v \wedge B_v) \cdot \nabla_v f), \nabla_v f) \\
- 2(\nabla_u (-\nabla_u + v) \cdot \nabla_v f, \nabla_v f) \\
= -2(\nabla_u (v \cdot \nabla_x f), \nabla_v f) - 2(\nabla_u (v \wedge B_v) \cdot \nabla_v f, \nabla_v f) \\
+ 2(\nabla_u (-\nabla_u + v) \cdot \nabla_v f, \nabla_v f) - 2(-\nabla_v + v) \cdot \nabla_v f. \\
\]

We used the fact that the operators \( v \cdot \nabla_x \) and \( (v \wedge B_v) \cdot \nabla_v \) are skew-adjoint in \( L^2(\mu^{1/2}) \) by Lemma A.1. According to equalities (1) and (3) of Lemma 3.6, we then obtain

\[
\frac{d}{dt} ||\nabla_v f||^2 = -2(\nabla_x f, \nabla_v f) + 2((B_e \wedge \nabla_v) f, \nabla_v f) - 2(-\nabla_v + v) \cdot \nabla_v f. \\
\]

The time derivative of the third term can be calculated as follows:

\[
\frac{d}{dt} (\nabla_v f, \nabla_v f) = (\nabla_v \partial_t f, \nabla_v f) + (\nabla_v f, \nabla_v \partial_t f). \\
\]

We calculate each term of equality (24). For the first term, using equalities (1), (2) and (3) of Lemma 3.6, we obtain

\[
(\nabla_v \partial_t f, \nabla_v f) = -(\nabla_v (v \cdot \nabla_x f - (v \wedge B_v) \cdot \nabla_v f + (-\nabla_v + v) \cdot \nabla_v f), \nabla_v f) \\
= -||\nabla_x f||^2 - (v \cdot \nabla_x (\nabla_v f), \nabla_v f) - (\nabla_v f, \nabla_v f) - (\Delta_v f, v \cdot \nabla_v f) \\
+ ((B_e \wedge \nabla_v) f, \nabla_v f) + (v \wedge B_v) \cdot \nabla_v f, \nabla_v f). \\
\]

For the second term of equality (24), using equality (4) of Lemma 3.6, we have

\[
(\nabla_v f, \nabla_v \partial_t f) = -((v \wedge B_v) \cdot \nabla_v (v \cdot \nabla_x f) - (v \wedge B_v) \cdot \nabla_v f + (-\nabla_v + v) \cdot \nabla_v f)) \\
= -((v \wedge B_v) \cdot \nabla_x (\nabla_v f)) + (v \wedge B_v) \cdot \nabla_v f, \nabla_v f \\
+ (v \wedge B_v) \cdot \nabla_v f, \nabla_v f - (\nabla_v f, \nabla_v f) - (\Delta_v f, v \cdot \nabla_v f). \\
\]

Combining the preceding equalities of the two terms in (24), we get

\[
\frac{d}{dt} (\nabla_v f, \nabla_v f) = -||\nabla_x f||^2 - (v \cdot \nabla_x (\nabla_v f), \nabla_v f) + (v \cdot \nabla_x (\nabla_v f), \nabla_v f) \\
- (\nabla_v f, v \cdot \nabla_x (\nabla_v f)) + (v \cdot \nabla_x (\nabla_v f), \nabla_v f) \\
+ ((B_e \wedge \nabla_v) f, \nabla_v f) + (v \wedge B_v) \cdot \nabla_v f, \nabla_v f \\
+ (v \wedge B_v) \cdot \nabla_v f, \nabla_v f + (v \wedge B_v) \cdot \nabla_v f, \nabla_v f). \\
\]

According to Lemma A.1, the operators \( v \cdot \nabla_x \) and \( (v \wedge B_v) \cdot \nabla_v \) are skew-adjoint in \( L^2(\mu^{1/2}) \), so we have

\[
(\nabla_v f, v \cdot \nabla_x (\nabla_v f)) = 0 \quad (25) \\
(\nabla_v f, (v \wedge B_v) \cdot \nabla_v f) = 0. \quad (26) \\
\]

Using equalities (25)-(26), we obtain

\[
\frac{d}{dt} (\nabla_v f, \nabla_v f) = -||\nabla_x f||^2 - (\nabla_v f, \nabla_v f) + 2((-\nabla_v + v) \nabla_v f, \nabla_v \cdot (\nabla_v f)) \\
+ (B_e \wedge \nabla_v) f, \nabla_v f + (v \wedge B_v) \cdot \nabla_v f). \\
\]
Finally, the time derivative of the last term takes the following form
\[
\frac{d}{dt} \| \nabla_x f \|^2 = 2 \langle \nabla_x \partial_t f, \nabla_x f \rangle \\
= -2 \langle \nabla_x (v \cdot \nabla_x f), \nabla_x f \rangle + 2 \langle \nabla_x ((v \wedge B_e) \cdot \nabla_v f), \nabla_x f \rangle \\
- 2 \langle \nabla_x (-\nabla_v + v) \cdot \nabla_v f, \nabla_x f \rangle \\
= -2 \| \nabla_x \nabla_v f \|^2 + 2 \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle + 2 \langle (v \wedge B_e) \cdot \nabla_v f, \nabla_x f \rangle \\
= -2 \| \nabla_x \nabla_v f \|^2 + 2 \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle,
\]
Tying together all these computations, we get
\[
\frac{d}{dt} \mathcal{E}(f) = -2 C \| \nabla_v f \|^2 - 2 D \| (-\nabla_v + v) \cdot \nabla_v f \|^2 - E \| \nabla_x f \|^2 - 2 \| \nabla_x \nabla_v f \|^2 \\
- (2 D + E) \langle \nabla_x f, \nabla_v f \rangle + 2 E \langle (\nabla_v - v) \cdot \nabla_v f, \nabla_v f \cdot (\nabla_x f) \rangle \\
+ 2 D \langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle + E \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle \\
+ E \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle + 2 \langle (v \wedge \nabla_x B_e \cdot \nabla_v f, \nabla_x f \rangle.
\]
Now, using Lemma A.3, the time derivative of \( \mathcal{E}(f(t)) \) takes the following form:
\[
\frac{d}{dt} \mathcal{E}(f(t)) = -2 C \| \nabla_v f \|^2 - 2 D \| (-\nabla_v + v) \cdot \nabla_v f \|^2 - E \| \nabla_x f \|^2 - 2 \| \nabla_x \nabla_v f \|^2 \\
- (2 D + E) \langle \nabla_x f, \nabla_v f \rangle + 2 E \langle (\nabla_v - v) \cdot \nabla_v f, \nabla_v f \cdot (\nabla_x f) \rangle \\
+ 2 D \langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle + E \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle \\
- E \langle (\nabla_v \wedge (\nabla_x B_e \cdot \nabla_v), f \rangle \nabla_v f \rangle - 2 \langle (v \wedge \nabla_x B_e \cdot \nabla_v), f \rangle \nabla_v f \rangle.
\]
Now we estimate the scalar products in the previous inequality in \( L^2(\mu^{1/2}) \). For all \( \eta, \eta' \) and \( \eta'' > 0 \), we have
\[
|\langle 2 D + E \rangle \langle \nabla_x f, \nabla_v f \rangle| \leq \frac{1}{2} \| \nabla_x f \|^2 + \frac{1}{2} (2 D + E)^2 \| \nabla_v f \|^2,
\]
\[
|2 E \langle (\nabla_v - v) \cdot \nabla_v f, \nabla_v f \cdot (\nabla_x f) \rangle| \leq \| \nabla_v \cdot (\nabla_x f) \|^2 + E^2 \| (\nabla_v - v) \cdot \nabla_v f \|^2,
\]
\[
|2 D \langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle| \leq 2 D \| B_e \|_{\infty} \| \nabla_v f \|^2,
\]
\[
|E \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle| \leq \frac{E \eta}{2} \| \nabla_x f \|^2 + \frac{E}{2 \eta} \| B_e \|^2 \| \nabla_v f \|^2,
\]
and using then \( \| \nabla_v f \|^2 \leq \| (-\nabla_v + v) \cdot \nabla_v f \|^2 + \| \nabla_v f \|^2 \), we obtain
\[
|E \langle (\nabla_v \wedge (\nabla_x B_e \cdot \nabla_v)) f, \nabla_v f \rangle| \leq \frac{\eta'}{2} \| \nabla_v f \|^2 + \left( \frac{E^2}{2 \eta} \| \nabla_x B_e \|^2 \| \nabla_v f \|^2 \right)
\leq \frac{\eta'}{2} \| (-\nabla_v + v) \cdot \nabla_v f \|^2 + \left( \frac{E^2}{2 \eta} \| \nabla_x B_e \|^2 \| \nabla_v f \|^2 + \frac{\eta'}{2} \| \nabla_v f \|^2 \right).
\]
The last scalar product is bounded by
\[
|2 \langle (\nabla_v \wedge (\nabla_x B_e \cdot \nabla_v)) f, \nabla_v f \rangle| \leq \eta'' \| \nabla_x \nabla_v f \|^2 + \left( \frac{1}{\eta} \| \nabla_x B_e \|^2 \| \nabla_v f \|^2 \right).
\]
Combining all the previous estimates, we have
\[
\frac{d}{dt} \mathcal{E}(f) \leq -2C + \frac{1}{2}(2D + E)^2 + 2D\|B_e\|_\infty + \frac{E}{2\eta}\|B_e\|_\infty^2 + \frac{E^2}{2\eta}\|\nabla_x B_e\|_\infty^2 + \frac{1}{\eta}\|\nabla_x B_e\|_\infty^2 \|\nabla_v f\|^2 + (-2D + \frac{E^2}{2})\|(-\nabla_v + v) \cdot \nabla_v f\|^2 + (-E + \frac{1}{2} + \frac{E}{2})\|\nabla_x f\|^2 + (-2 + 1 + \eta)\|\nabla_x \nabla_v f\|^2.
\]

We notice that
\[
A = \frac{1}{2}(2D + E)^2 + 2D\|B_e\|_\infty + \frac{E}{2\eta}\|B_e\|_\infty^2 + \frac{E^2}{2\eta}\|\nabla_x B_e\|_\infty^2 + \frac{1}{\eta}\|\nabla_x B_e\|_\infty^2.
\]

We choose \(\eta, \eta'', E, D\) and \(C\) such that
1. \(\eta \leq 1\) and \(\eta'' \leq 1\).
2. \(E \geq 2\).
3. \(D \geq \frac{1}{2}(E^2 + \frac{\eta''}{\eta})\).
4. \(C \geq A\).

Under the previous conditions, we get
\[
\frac{d}{dt} \mathcal{E}(f) \leq -C\|\nabla_v f\|^2 - \frac{E}{4}\|\nabla_x f\|^2 \leq -\frac{E}{4}(\|\nabla_v f\|^2 + \|\nabla_x f\|^2).
\]

Using the Poincaré inequality in space and velocity variables, we then obtain
\[
\frac{d}{dt} \mathcal{E}(f) \leq -\frac{E}{8}(\|\nabla_v f\|^2 + \|\nabla_x f\|^2) = -\frac{E}{8}c_p\|f\|^2 \leq -\frac{E}{8} \frac{c_p}{2C}\mathcal{E}(f).
\]

Which completes Proposition 3.5 with \(\kappa = \frac{E}{8} \frac{c_p}{2C} > 0\).

**Proof of Theorem 1.5.** Using Lemma 3.7 and Proposition 3.8, we get \(\kappa > 0\) and \(1 < E < D < C\) such that
\[
\|f\|^2_{\mathcal{H}^1(\mu^{1/2})} \leq 2\mathcal{E}(f) \leq 2Ce^{-\kappa t}\mathcal{E}(f_0) \leq 4Ce^{-\kappa t}\|f_0\|^2_{\mathcal{H}^1(\mu^{1/2})}.
\]

This completes the proof of Theorem 1.5.

4. **Enlargement of the functional space.**

4.1. **Intermediate results.** In this section, we extend the results of exponential time decay of the semi-group to enlarged spaces (which we will define later), following the recent work of Gualdani, Mischler, Mouhot in [10].

**Notation:** Let \(E\) be a Banach space.
- We denote by \(\mathcal{C}(E)\) the space of unbounded, closed operators with dense domains in \(E\).
- We denote by \(B(E)\) the space of bounded operators in \(E\).
- Let $a \in \mathbb{R}$. We define the complex half-plane
  \[ \Delta_a = \{ z \in \mathbb{C}, \text{Re } z > a \}. \]
- Let $L \in \mathcal{C}(E)$. $\Sigma(L)$ denote the spectrum of the operator $L$ and $\Sigma_d(L)$ its
discrete spectrum.
- Let $\xi \in \Sigma_d(L)$, for $r$ sufficiently small we define the spectral projection asso-
ciated with $\xi$ by
  \[ \Pi_{L,\xi} := \frac{1}{2\pi i} \int_{|z-\xi|=r} (L-z)^{-1} \, dz. \]
- Let $a \in \mathbb{R}$ be such that $\Delta_a \cap \Sigma(L) = \{ \xi_1, \xi_2, \ldots, \xi_k \} \subset \Sigma_d(L)$. We define $\Pi_{L,a}$
as the operator
  \[ \Pi_{L,a} = \sum_{j=1}^{k} \Pi_{L,\xi_j}. \]

We need the following definition on the convolution of semigroups (corresponding
to composition at the level of the resolvent operators).

**Definition 4.1 (Convolution of time dependent operators).** Let $X_1$, $X_2$ and $X_3$ be
Banach spaces. For two given functions
\[ S_1 \in L^1(\mathbb{R}^+; B(X_1, X_2)) \text{ and } S_2 \in L^1(\mathbb{R}^+; B(X_2, X_3)), \]
we define the convolution $S_2 * S_1 \in L^1(\mathbb{R}^+; B(X_1, X_3))$ by
\[ \forall t \geq 0, (S_2 * S_1)(t) := \int_0^t S_2(s)S_1(t-s) \, ds. \]

When $S = S_1 = S_2$ and $X_1 = X_2 = X_3$, we define inductively $S^{(s+1)} = S$ and $S^{(s+\ell)} = S * S^{(s+\ell-1)}$ for any $\ell \geq 2$.

**Definition 4.2.** Consider a Banach space $(E, \| \cdot \|_E)$ and some operator $L \in \mathcal{C}(E)$.
We say that $L$ is hypodissipative if it is dissipative for some norm equivalent to the
canonical norm of $E$ and we say that $L$ is dissipative for the norm $\| \cdot \|_E$ on $E$ if
for all $f \in D(L)$ and $f^* \in E^*$ such that
\[ (f, f^*) = \|f\|^2_E = \|f^*\|^2_{E^*}, \]
we have $\Re(Lf, f^*) \leq 0$.

The concept of hypodissipativity will be needed later. We give a practical crite-
ion to prove that an operator $L$ is hypodissipative in $E$.

**Corollary 4.3.** [27, Chapter 1] Consider $E$ a Banach space, $L$ the generator of a
$C^0$-semi-group $(S_L(t))_{t \geq 0}$, and $a \in \mathbb{R}$, then the operator $L - a$ is hypodissipative on
$E$ if and only if the semi-group $S_L(t)$ satisfies the growth estimate
\[ \forall t \geq 0, \| S_L(t) \|_{B(E)} \leq M e^{\alpha t}, \quad \text{where } M > 0. \]

We refer to the paper [10, Section 2.3] for an introduction to this subject. Now, we
recall the crucial Theorem of enlargement of the functional space.

**Theorem 4.4 (Theorem 2.13 in [10]).** Let $E$ and $\mathcal{E}$ be two Banach spaces such that
$E \subset \mathcal{E}$, $L \in \mathcal{C}(E)$ and $\mathcal{L} \in \mathcal{C}(\mathcal{E})$ such that $\mathcal{L}|_E = L$. We suppose that there exist
$\mathcal{A}$ and $\mathcal{B} \in \mathcal{C}(\mathcal{E})$ such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ (with corresponding restrictions $\mathcal{A}, \mathcal{B}$ on $E$).
Suppose there exists $a \in \mathbb{R}$ and $n \in \mathbb{N}$ such that
(H₁) Locating the spectrum of \( L \):

\[ \Sigma(L) \cap \Delta_a = \{0\} \subset \Sigma_{\rho}(L), \]

and \( L - a \) is dissipative on \( \text{Im}(\text{Id}_E - \Pi_{L,0}) \)

(H₂) Dissipativity of \( \mathcal{B} \) and bounded character of \( \mathcal{A} \): \( \mathcal{B} - a \) is hypodissipative on \( \mathcal{E} \) and \( \mathcal{A} \in \mathcal{B}(\mathcal{E}) \) and \( \mathcal{A} \in \mathcal{B}(E) \).

(H₃) Regularization properties of \( T_n(t) = (\mathcal{A}\mathcal{S}_n(t))^{(*)n} \):

\[ \|T_n(t)\|_{B(\mathcal{E},E)} \leq C_{a,n} e^{at}, \]

where \( S_B \) is the semi-group associated to the operator \( \mathcal{B} \) acting on \( \mathcal{E} \). Then for all \( 0 > a' > a \), we have the following estimate:

\[ \forall t \geq 0, \quad \|S_{\mathcal{L}}(t) - S_{\mathcal{L}}(t)\Pi\|_{B(\mathcal{E},E)} \leq C_{a',E} e^{a't}, \]

where \( S_{\mathcal{L}} \) is the semi-group associated to the operator \( \mathcal{L} \) acting on \( \mathcal{E} \).

In the application of the previous theorem in the case of the Fokker-Planck operator with magnetic field, the key step of the proof is to obtain decay estimates verifying dissipation and regularization properties. In order to prove the dissipativity of the operator we apply Corollary 4.3. We give a lemma providing a practical criterion to prove hypothesis (H₃) of the previous theorem.

**Lemma 4.5** (Lemma 2.4 in [22]). Let \( E \) and \( \mathcal{E} \) be two Banach spaces with \( E \subset \mathcal{E} \) dense with continuous embedding, and consider \( L \in \mathcal{C}(E) \) and \( \mathcal{L} \in \mathcal{C}(\mathcal{E}) \) with \( \mathcal{L}|_{E} = L \) and \( a \in \mathbb{R} \). Let us assume that:

a) \( \mathcal{B} - a \) is hypdissipative on \( \mathcal{E} \) and \( B - a \) on \( E \).

b) \( \mathcal{A} \in \mathcal{B}(\mathcal{E}) \) and \( A \in \mathcal{B}(E) \).

c) There are constants \( b \in \mathbb{R} \) and \( \Theta \geq 0 \) such that

\[ \|\mathcal{A}\mathcal{S}_n(t)\|_{B(\mathcal{E},E)} \leq C_{\epsilon b t} t^{-\Theta} \quad \text{and} \quad \|\mathcal{S}_B(t)\mathcal{A}\|_{B(\mathcal{E},E)} \leq C_{\epsilon b t} t^{-\Theta}. \]

Then for all \( a' > a \), there exist some explicit constants \( n \in \mathbb{N} \) and \( C_{a'} \geq 1 \), such that

\[ \forall t \geq 0, \quad \|T_n\|_{B(\mathcal{E},E)} \leq C_{a'} e^{a't}. \]

Note that, in the application of the previous lemma in the case of the Fokker-Planck operator with magnetic field, only regularization properties of the operator \( \mathcal{A}\mathcal{S}_n(t) \) are needed to deduce the hypothesis (H₃) of Theorem 4.4. (See the proof of Lemma 2.4 in [22] and Lemma 2.17 in [10]).

### 4.2. Study of the magnetic-Fokker-Planck operator on the spaces \( L^p(m) \) and \( W^{1,p}(m) \): This part consists in building the general framework of the problem.

Recall first the equation of Fokker-Planck (2) written in original variable:

\[ \partial_t F = -P_0 F, \quad F(0,x,v) = F_0(x,v), \]  

where \(-P_0 F = \nabla_v \cdot (\nabla_x F + K F) - v \cdot \nabla_x F,\)

and where we recall that \( P_0 \) was introduced in Section 2 and with

\[ K(x,v) = v + v \wedge B_e(x) = \nabla_v \Phi + v \wedge B_e(x), \]

where \( \Phi(v) = \frac{|v|^2}{2} \)

and \( B_e \) is the external magnetic field satisfying \( B_e \in W^{1,\infty}(\mathbb{T}^3) \). As mentioned in Section 2, the Maxwellian \( \mu \) is a solution of the system (2). We will need the
following modified Poincaré inequality:
\[
\int_{\mathbb{T}^3 \times \mathbb{R}^3} \left| \nabla v \left( \frac{F}{\mu} \right) \right|^2 \mu(v) dx dv \\
\geq 2\lambda_p \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left( F - \int_{\mathbb{R}^3} F(v') dv' \right)^2 (1 + |\nabla_v \Phi|^2) \mu^{-1}(v) dx dv,
\]
where \( \lambda_p > 0 \) which depends on the dimension (see [22, Lemma 3.6]). See also [24], [2] and [1].

Now we will define the expanded functional space.

**Definition 4.6.** Let \( m = m(v) > 0 \) on \( \mathbb{R}^3 \) be a weight of class \( C^\infty \) and recall that

- The space \( L^p(m) \) for \( p \in [1, 2] \), is the Lebesgue space with weight associated with the norm
  \[
  \|F\|_{L^p(m)} := \|Fm\|_{L^p} = \left( \int_{\mathbb{T}^3 \times \mathbb{R}^3} F^p(x,v) m^p(v) dv dx \right)^{\frac{1}{p}}.
  \]
- We define the technical function \( \Psi_{m,p} \) by
  \[
  \Psi_{m,p} := (p - 1) \frac{|\nabla_v m|^2}{m^2} + \frac{\Delta_v m}{m} + (1 - \frac{1}{p}) \nabla_v \cdot K - K \cdot \nabla_v m,
  \]
  where \( K(x,v) = v \wedge B_e(x) + v \).

**Remark 4.7.** We note that \( \nabla_v \cdot (v \wedge B_e) = 0 \) (because \( B_e \) is independent of \( v \)) and for all \( m \) which are radial in \( v \) and independent of \( x \), we obtain
\[
(v \wedge B_e) \cdot \nabla_v m = 0, \text{ since } (v \wedge B_e) \cdot v = 0.
\]
Therefore, we obtain that \( \Psi_{m,p} \) is independent of the magnetic field when \( m \) is radial in \( v \).

We will show the decay of the semi-group associated with the problem (2) in the spaces \( L^p(m) \) where \( p \in [1, 2] \), when \( m \) verifies the following hypothesis:

(W\( _p \)) The weight \( m \) is defined such that \( L^2(\mu^{-1/2}) \subset L^p(m) \) with continuous injection and
\[
\limsup_{|v| \to +\infty} \Psi_{m,p} =: a_{m,p} < 0.
\]

**Remark 4.8.** In the following, we note \( m_0 = \mu^{-1/2} \) the exponential weight. By direct computation, \( L^2(\mu^{-1/2}) \subset L^q(m_0) \) for any \( q \in [1, 2] \) with continuous injection and there exists \( b \in \mathbb{R} \) such that
\[
\left\{ \begin{array}{l}
\sup_{q \in [1,2], v \in \mathbb{R}^3} \Psi_{m_0,q} \leq b \\
\sup_{v \in \mathbb{R}^3} \left( \frac{\Delta_v m_0}{m_0} - \frac{|\nabla_v m_0|^2}{m_{0^2}} \right) \leq b.
\end{array} \right.
\]
(See Lemma 3.7 in [10] for a proof of the previous property). Under the previous hypothesis, by direct computation we obtain that the semi-group \( S_{L_0} \) associated to the operator \( L_0 := -P_0 \) defined in (27) is bounded from \( L^p(m_0) \) to \( L^p(m_0) \).

We work now in \( L^p(m) \) with a polynomial weight \( m \) satisfying Hypothesis (W\( _p \)).
Lemma 4.9. Let $m = \langle v \rangle^k := (1 + |v|^2)^{k/2}$ and $p \in [1, 2]$. Then hypothesis $(W_p)$ is true when $k$ satisfies the following estimate:

$$k > 3(1 - \frac{1}{p}).$$

Proof. We consider $m = \langle v \rangle^k$, $k > 0$, and we compute

$$\nabla_v m = k v (v)^{k-2} \text{ and } \Delta_v m = k(k+1)(v)^{k-2}.$$ 

We recall that

$$\Psi_{m,p} = (p-1) \frac{|\nabla_v m|^2}{m^2} + \frac{\Delta_v m}{m} + (1 - \frac{1}{p}) \nabla_v \cdot K - K \cdot \nabla_v m.$$ 

Using Remark 4.7

$$\nabla_v \cdot (v \wedge B) = 0 \text{ and } (v \wedge B) \cdot \nabla_v m = 0,$$

As $|v| \to +\infty$,

$$\Psi_{m,p} = 3(1 - \frac{1}{p}) - v \cdot \frac{\nabla_v m}{m} + \mathcal{O}(|v|^{-1})$$

$$= 3(1 - \frac{1}{p}) - k \frac{|v|^2 (v)^{k-2}}{(v)^k} + \mathcal{O}(|v|^{-1}).$$

We deduce that

$$a_{m,p} = \limsup_{|v| \to \infty} \Psi_{m,p} = 3(1 - \frac{1}{p}) - k,$$

and $a_{m,p}$ is negative if $k > 3(1 - \frac{1}{p})$.

By applying Holder’s inequality, we can easily show the continuous injection $L^2(\mu^{-1/2}) \subset L^p(\langle v \rangle^k)$ for any $p \in [1, 2]$. (Cf. [10, Lemma 3.7] for more details on this subject). 

4.2.1. Proof of Theorem 1.7. From now on, we write $L_0$ for the operator $-P_0$, the Fokker-Planck operator defined in (27) considered on the space $L^2(\mu^{-1/2})$ defined in (27), respectively $L_0$ for $-P_0$ the Fokker-Planck operator considered on the space $E = L^p(m)$, with $m = \langle v \rangle^k$, where $k > 3(1 - \frac{1}{p})$ and $p \in [1, 2]$. We will prove Theorem 1.7 by applying Theorem 4.4 to $L_0$. To verify Hypotheses $(H_2)$ and $(H_3)$ of Theorem 4.4, we need two lemmas about the dissipativity and regularization properties of $L_0$ following [10].

Definition 4.10. We split the operator $L_0$ into two pieces: for $M, R > 1$, we define the operator $\mathcal{B}$ by

$$\mathcal{B} = L_0 - A \text{ with } Af = M \chi_R f, \quad (30)$$

where $\chi_R(v) = \chi(v/R)$, and $0 \leq \chi \in C_0^\infty (T^3 \times \mathbb{R}^3)$ is such that $\chi(v) = 1$ when $|v| \leq 1$. We also denote by $A$ and $B$ the restriction of the operators $A$ and $B$ to the space $E$. 

Lemma 4.11 (Dissipativity of $\mathcal{B}$). Under Assumption $(W_p)$, for all $0 > a > a_{m,p}$, we can choose $R, M > 1$ such that the operator $\mathcal{B} - a$ satisfies the dissipativity estimate for some $C > 0$

$$\forall t \geq 0, \quad \|S_\mathcal{B}(t)F\|_{L^p(m)} \leq C e^{at} \|F\|_{L^p(m)}.$$
We introduce the following entropy defined for all $t$

**Proof.** The proof follows the one given in Lemma 3.8 in [10]. Let $F$ be smooth, rapidly decaying and positive function $F$. Since of $\Psi_{m,p}$ is independent of the magnetic field (see Remark 4.7), by performing an integration by parts with respect to $x$ and $v$ and using Remark A.2 (specifically the fact that the operator $v \cdot \nabla x$ is skew-adjoint in $L^2(m)$ with $m = (v)^k$ and $m$ is independent of $x$), we have

$$
\frac{1}{p} \frac{d}{dt} \|F\|_{L^p(m)}^p = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\mathcal{L}_0 F - M \chi_R(v) F) |F|^{p-2} F m^p(v) \, dx \, dv
\]

$$
= -(p-1) \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\nabla_v F|^2 |F|^{p-2} m^p(v) \, dx \, dv
\]

$$
+ \int_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p \Psi_{m,p} m^p(v) \, dx \, dv - \int_{\mathbb{T}^3 \times \mathbb{R}^3} M \chi_R(v) |F|^p m^p(v) \, dx \, dv
\]

$$
\leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p (\Psi_{m,p} - M \chi_R) m^p(v) \, dx \, dv.
\]

Let now take $a > a_{m,p}$. As $m$ satisfies the hypothesis ($W_p$), there exist $M$ and $R$ two large constants such that

$$
\forall v \in \mathbb{R}^3, \quad \Psi_{m,p} - M \chi_R \leq a,
\]

and we obtain

$$
\frac{1}{p} \frac{d}{dt} \|F\|_{L^p(m)}^p \leq a \int_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p m^p(v) \, dx \, dv.
\]

This completes the proof of Lemma 4.11. \qed

From now on, $a$, $M$ and $R$ are fixed. We note that $B^*$ is the dual operator of $B$ relative to the pivot space $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$, which is defined as follows:

$$
B^* F := \nabla_v \cdot (\nabla_v F - K F) + v \cdot \nabla x F - M \chi_R F.
\]

**Lemma 4.12** (Regularization properties). There exists $b \in \mathbb{R}$ and $C > 0$ such that, for all $t \geq 0$,

$$
\forall 1 \leq p \leq q \leq 2, \quad \|S_B(t) F_0\|_{L^q(m_0)} \leq C e^{bt} (3d+1)(\frac{1}{2} - \frac{1}{4}) \|F_0\|_{L^p(m_0)},
\]

$$
\forall 2 \leq q' \leq p' \leq +\infty, \quad \|S_B^\ast(t) F_0\|_{L^{q'}(m_0)} \leq C e^{bt} (3d+1)(\frac{1}{2} - \frac{1}{4}) \|F_0\|_{L^{p'}(m_0)},
\]

where $p'$ and $q'$ are the conjugates of $p$ and $q$ respectively and $m_0 = \mu^{-1/2}$.

**Proof.** We consider $F(t)$ the solution of the evolution equation

$$
\partial_t F(t) = BF(t), \quad F|_{t=0} = F_0.
\]

We introduce the following entropy defined for all $t \in [0, T]$, with $T \ll 1$ and $r > 1$ to be fixed later:

$$
\mathcal{H}(t, h) = B \|h\|_{L^1(m_0)}^2 + t^r \mathcal{G}(t, h),
\]

with

$$
\mathcal{G}(t, h) = \alpha \|h\|_{L^2(m_0)}^2 + \delta t \|\nabla_v h\|_{L^2(m_0)}^2 + E t^2 \langle \nabla x h, \nabla v h \rangle_{L^2(m_0)} + \beta t^3 \|\nabla x h\|_{L^2(m_0)}^2,
\]

where $B > \alpha > D, \beta, E < \sqrt{3D}$ and $r$ is an integer that will be determined later.

We will omit the dependence of $F$ on $t$. Using the methods and computations of the proof of Proposition 3.8 and adapting the techniques used in [14], we choose
the constants $\alpha, D$ and $E > 0$ large enough such that there exist a constant $C_G > 0$ (depending on $\|B_c\|_{L^\infty(T^3)}$ and $\|\nabla_x B_c\|_{L^\infty(T^3)}$) such that

$$
\frac{d}{dt} G(t, F) \leq -C_G (\|\nabla_v F\|_{L^2(m_0)}^2 + t^2 \|\nabla_x F\|_{L^2(m_0)}^2)
+ \left( \frac{M}{2} \|\Delta_v \chi_n\|_{L^\infty(T^3)}^2 + \frac{M}{2} \|\chi n\|_{L^\infty(T^3)}^2 + M \|\nabla \chi n\|_{L^2(T^3)}^2 \right) \|F\|_{L^2(m_0)}^2
\leq -C_G (\|\nabla_v F\|_{L^2(m_0)}^2 + t^2 \|\nabla_x F\|_{L^2(m_0)}^2) + C_\chi \|F\|_{L^2(m_0)}^2.
$$

Here, $C_\chi > 0$ is a uniform constant in $R > 1$ but depends on $M$.

$$
\frac{d}{dt} H(t, F) = B \frac{d}{dt} (\|F\|_{L^2(m_0)}^2)
+ r t^{-1} G(t, F) + t^r \frac{d}{dt} G(t, F)
\leq B \frac{d}{dt} (\|F\|_{L^2(m_0)}^2)
+ r t^{-1} G(t, F)
- C_G t^r (\|\nabla_v F\|_{L^2(m_0)}^2 + t^2 \|\nabla_x F\|_{L^2(m_0)}^2)
+ C_\chi t^r \|F\|_{L^2(m_0)}^2.
$$

We choose the constants $\beta$ and $T > 0$ such that

$$
\beta < \frac{C_G}{2r} \quad \text{and} \quad T \leq \frac{C_G}{2r} \left( \frac{1}{D} + \frac{1}{\beta} \right).
$$

We deduce that

$$
\frac{d}{dt} H(t, F) \leq B \frac{d}{dt} (\|F\|_{L^2(m_0)}^2)
- C_G t^r \left( \|\nabla_v F\|_{L^2(m_0)}^2 + t^2 \|\nabla_x F\|_{L^2(m_0)}^2 \right)
+ \frac{C_\chi}{2} t^r \|F\|_{L^2(m_0)}^2.
$$

Now, the Nash inequality [25] implies that there exists $C_d > 0$ such that

$$
\iint_{T^d \times \mathbb{R}^d} |F(x, v)|^2 m_0^2 \, dx \, dv \leq C_d \left( \iint_{T^d \times \mathbb{R}^d} |\nabla_{x,v} F_m_0|^2 \, dx \, dv \right)^{\frac{1}{2}}
\times \left( \iint_{T^d \times \mathbb{R}^d} |F| m_0 \, dx \, dv \right)^{\frac{1}{2}}.
$$

We need to have an estimate based on $\|\nabla_{x,v} F\|_{L^2(m_0)}$. Firstly,

$$
\iint |\nabla_v (F m_0)|^2 \, dx \, dv = \iint |\nabla_v F + \frac{v}{2} F|^2 m_0^2 \, dx \, dv
\leq 2 \left( \iint |\nabla_v F|^2 m_0^2 \, dx \, dv + \iint |F|^2 |v|^2 m_0^2 \, dx \, dv \right)
\leq 2 \left( \|\nabla_v F\|_{L^2(m_0)}^2 + \|F\|_{L^2(m_0)}^2 \right).
$$

On the other hand, we use the fact that $v m_0^2 = \nabla_v (m_0^2)$ to estimate $\|v F\|_{L^2(m_0)}$. We get

$$
\iint |F|^2 |v|^2 m_0^2 \, dx \, dv = \iint v |F|^2 \cdot v m_0^2 \, dx \, dv
= \iint v |F|^2 \cdot \nabla_v (m_0^2) \, dx \, dv,
$$
and integrating by parts in \( v \) in the previous estimate, we obtain
\[
\iint |F|^2 |v|^2 m_0^2 \, dx \, dv \leq - \iint \nabla_v \cdot (v |F|^2) \, m_0^2 \, dx \, dv
\]
\[
= -3 \iint |F|^2 m_0^2 \, dx \, dv - 2 \iint v \cdot F \nabla_v F \, m_0^2 \, dx \, dv
\]
\[
\leq -2 \iint v \cdot F \nabla_v F \, m_0^2 \, dx \, dv.
\]
Applying the Cauchy-Schwarz inequality, we get
\[
\iint |F|^2 |v|^2 m_0^2 \, dx \, dv \leq 2 \left( \iint |v|^2 |F|^2 m_0^2 \, dx \, dv \right)^{1/2} \times \left( \iint |\nabla_v F|^2 m_0^2 \, dx \, dv \right)^{1/2}
\]
\[
\leq 8 \iint |\nabla_v F|^2 m_0^2 \, dx \, dv + \frac{1}{2} \iint |v|^2 |F|^2 m_0^2 \, dx \, dv.
\]
Therefore
\[
\iint |F|^2 |v|^2 m_0^2 \, dx \, dv \leq 16 \iint |\nabla_v F|^2 m_0^2 \, dx \, dv.
\]
Using the previous estimate and inequality (33), we have
\[
\iint |\nabla_v (F_{m_0})|^2 \, dx \, dv \leq 34 \iint |\nabla_v F|^2 m_0^2 \, dx \, dv.
\]
Using the previous inequality and the fact that \( \nabla_x (F_{m_0}) = m_0 \nabla_x F \) (since \( m_0 \) does not depend on \( x \)), there exists \( C_d' > 0 \) such that the estimate (32) becomes
\[
\iint_{T^d \times \mathbb{R}^d} |F(x, v)|^2 m_0^2 \, dx \, dv \leq C_d' \left( \iint_{T^d \times \mathbb{R}^d} |\nabla_{x,v} (F)|^2 m_0^2 \, dx \, dv \right)^{2/3} \times \left( \iint_{T^d \times \mathbb{R}^d} |F| m_0 \, dx \, dv \right)^{2/3}.
\]
Using Young’s inequality with \( p = (d + 1) \) and \( q = (d + 1)/d \), we get, for all \( \varepsilon > 0 \),
\[
\iint_{T^d \times \mathbb{R}^d} |F(x, v)|^2 m_0^2 \, dx \, dv \leq C_d' t^{-d/3 + 1} \left( \iint_{T^d \times \mathbb{R}^d} |F| m_0 \, dx \, dv \right)^{2/3} \times \left( \iint_{T^d \times \mathbb{R}^d} |\nabla_{x,v} F|^2 m_0^2 \, dx \, dv \right)^{2/3} \leq C_{\varepsilon,d} t^{-3d} \|F\|^2_{L^1(m_0)} + \varepsilon t^3 \|\nabla_{x,v} F\|^2_{L^2(m_0)}.
\]
Using the previous estimate, we choose \( \varepsilon > 0 \) small enough that there is a \( C'' > 0 \)
\[
\frac{d}{dt} \mathcal{H}(t, F) \leq B \frac{d}{dt} \|F\|^2_{L^1(m_0)} + C'' t^{r-1-3d} \|F\|^2_{L^1(m_0)}.
\]
According to Remark 4.8 there exists \( b \in \mathbb{R} \) such that \( \forall p \in [1, 2] \)
\[
\frac{d}{dt} \|F\|_{L^p(m_0)} \leq b \|F\|_{L^p(m_0)}, \quad \forall t \geq 0,
\]
Finally, using the previous estimate when \( p = 1 \) and choosing \( r = 3d + 1 \), we deduce that there exists \( B'' > 0 \) such
\[
\frac{d}{dt} \mathcal{H}(t, F) \leq B'' \|F\|^2_{L^1(m_0)} \leq \frac{B''}{B} \mathcal{H}(t, F).
\]
Thanks to Gronwall’s Lemma, there exists $B'' > 0$ such that
\[
\forall t \in [0, T], \quad \mathcal{H}(t, F) \leq B'' \mathcal{H}(0, F_0) \leq C\|F_0\|_{L^2(m_0)}^2.
\]
Then,
\[
\forall t \in (0, T], \quad \|F\|_{L^2(m_0)}^2 \leq \frac{\alpha}{t^r} \mathcal{H}(t, F) \leq \frac{C}{t^{3d+1}} \|F_0\|_{L^2(m_0)}^2.
\]
As a consequence, using the continuity of $S_\mathcal{B}(t)$ on $L^p(m_0)$ with $p = 2$,
\[
\forall t \in (T, +\infty), \quad \|F\|_{L^2(m_0)}^2 = \|S_\mathcal{B}(t - T)F_0\|_{L^2(m_0)}^2 \leq C e^{(t-T)h} \|S_\mathcal{B}(T)F_0\|_{L^2(m_0)}^2,
\]
and eventually for all $t \in (0, +\infty)$
\[
\|F\|_{L^2(m_0)}^2 \leq \frac{C}{t^{3d+1}} \|F_0\|_{L^2(m_0)}^2.
\]

Let us now consider $p$ and $q$ satisfying $1 \leq p \leq q \leq 2$. $S_\mathcal{B}(t)$ is continuous from $L^p(m_0)$ into $L^q(m_0)$ using the Riesz-Thorin Interpolation Theorem. Moreover, if we denote by $C_{p,q}(t)$ the norm of $S_\mathcal{B}(t) : L^p(m_0) \to L^q(m_0)$, we get the following estimate:
\[
C_{p,q}(t) \leq C_{2,2}^{2-p} C_{1,1}^{1-p} C_{1,2}^{p-1} (t) \leq C \frac{e^{bt}}{t^{(3d+1)(1/p-1/q)}}.
\]
This shows the first estimate.

Now we will show the second estimate. According to the first estimate, we have
\[
\forall 1 \leq p \leq q \leq 2, \quad \|S_\mathcal{B}(t)F_0\|_{L^q(m_0)} \leq C e^{bt} t^{-(3d+1)(1/p-1/q)} \|F_0\|_{L^p(m_0)},
\]
which means
\[
\|S_{m_0} B_{m_0}^{-1}(t)h\|_{L^q} \leq C e^{bt} t^{-(3d+1)(1/p-1/q)} \|h\|_{L^p},
\]
where $h = m_0 F_0$. Then by duality, we get
\[
\|S_{m_0} B_{m_0}^{-1}(t)h\|_{L^{q'}} \leq C e^{bt} t^{-(3d+1)(1/p-1/q)} \|h\|_{L^{p'}},
\]
where $p'$ and $q'$ are the conjugates of $p$ and $q$ respectively. Which gives the result by reusing the definition of weighted dual spaces
\[
\|S_{B_0} (t)F_0\|_{L^{q'}(m_0)} \leq C e^{bt} t^{-(3d+1)(1/p-1/q)} \|F_0\|_{L^{q'}(m_0)}.
\]
This completes the proof.

**Corollary 4.13.** Let $m$ be a weight that satisfies Hypothesis 1.8, then there exists $\Theta \geq 0$ such that for all $F_0 \in L^p(m)$ with $p \in [1, 2]$, we have the following estimate
\[
\forall t \geq 0, \quad \|A S_\mathcal{B}(t)F_0\|_{L^2(m_0)} \leq C e^{bt} t^{-\Theta} \|F_0\|_{L^p(m)},
\]
\[
\forall t \geq 0, \quad \|S_\mathcal{B}(t)AF_0\|_{L^2(m_0)} \leq C e^{bt} t^{-\Theta} \|F_0\|_{L^p(m)},
\]
where $m_0 = \mu^{-1/2}$.

**Proof.** We first prove the second inequality. Let $F_0 \in L^p(m)$ with $m$ a polynomial weight satisfying Hypothesis 1.8. For all $1 \leq p \leq 2$ and for all $t \in [0, 1]$ and $v \in \mathbb{R}^d$,
using Lemma 4.12 with \( q = 2 \), we get
\[
\| S_B(t)A F_0 \|_{L^2(m_0)} \leq C e^{b t} t^{-(3d+1)(\frac{1}{p} - \frac{1}{2})} \| A F_0 \|_{L^p(m_0)}
\]
\[
\leq C e^{b t} t^{-(3d+1)(\frac{1}{p} - \frac{1}{2})} \left( \| A F_0 \|_{L^p(m)} \right)
\]
\[
\leq C M e^{b t} t^{-(3d+1)(\frac{1}{p} - \frac{1}{2})} \left( \sup_{v \in B(0,R)} \frac{m_0(v)}{m(v)} \right) \| F_0 \|_{L^p(m)}
\]
\[
\leq C' e^{b t} t^{-(3d+1)(\frac{1}{p} - \frac{1}{2})} \| F_0 \|_{L^p(m)} \leq C' e^{b t} t^{-\Theta} \| F_0 \|_{L^p(m)},
\]
where \( \Theta = (3d + 1)(1/p - 1/2) > 0 \).

To show the first estimate, we proceed step by step.

**Step 1:** First, we will show the following estimate:
\[
\| S_B(t)g \|_{L^{q'}}(m) \leq C e^{b t} t^{-\Theta} \| g \|_{L^2(m)}, \quad \forall t \geq 0.
\]  
(35)

Indeed, using the continuous and dense injection \( L^{q'}(m_0) \subset L^{q'}(m) \), we obtain
\[
\| S_B(t)g \|_{L^{q'}(m)} \leq \| S_B(t)g \|_{L^{q'}(m_0)},
\]
then using Lemma 4.12 with \( q' = 2 \), we obtain
\[
\| S_B(t)g \|_{L^{q'}(m_0)} \leq C e^{b t} t^{-\Theta} \| g \|_{L^2(m_0)}, \quad \forall t \geq 0,
\]  
(36)

where \( \Theta = (3d + 1)(1/p - 1/2) \).

**Step 2:** Of the inequality (36), it follows that for \( g = A F_0 \), we get
\[
\| S_B(t)AF_0 \|_{L^{q'}(m)} \leq C e^{b t} t^{-\Theta} \| AF_0 \|_{L^2(m)},
\]
which means, denoting \( h = m F_0 \)
\[
\| S_{mBm^{-1}}A h \|_{L^{q'}} \leq C e^{b t} t^{-\Theta} \| A h \|_{L^2} \leq C' e^{b t} t^{-\Theta} \| h \|_{L^2},
\]
by a duality argument and noting that \( A^* = A \), we get
\[
\| A S_{mBm^{-1}}h \|_{L^2} \leq C e^{b t} t^{-\Theta} \| h \|_{L^p}.
\]

Finally, according to our definition of weighted dual spaces and replacing \( h \) by \( m F_0 \), we obtain
\[
\| A S_B(t) F_0 \|_{L^2(m)} \leq C e^{b t} t^{-\Theta} \| F_0 \|_{L^p(m)},
\]  
(37)

To obtain the result, we notice that
\[
\| A S_B(t) F_0 \|_{L^2(m)} \leq \| A S_B(t) F_0 \|_{L^2(m)},
\]
and we combine the previous estimate with the estimate (37), which completes the proof of the first estimate.

Now we prove Theorem 1.7.

**Proof of Theorem 1.7.** For \( p \in [1, 2] \). We consider \( \mathcal{E} = L^p(m) \), \( E = L^2(m_0) \), and denote \( L_0 \) and \( L_0 \) the Fokker-planck operator considered respectively on \( \mathcal{E} \) and \( E \) (defined in (27)). We split the operator as \( L_0 = A + B \) as in (30). Let us proceed step by step:

- **Step 1:** Verification of condition \((H_1)\) of Theorem 4.4

Theorem 1.4 shows us the existence of the semi-group \( S_{L_0}(t) \), associated with the
By interpolation, we get
\[ \forall t \geq 0, \quad \| F(t) \|_{L^2(m_0)} \leq c e^{-\kappa t} \| F_0 \|_{L^2(m_0)}. \]  
(38)
Which implies the dissipativity of the operator \( L_0 - a \) on \( E \), for all \( 0 > a > -\kappa \).

- **Step 2: Verification of condition \((H_2)\) of Theorem 4.4.**
  According to Lemma 4.11 and Corollary 4.3, the operator \( B - a \) is dissipative on \( E \), for all \( 0 > a > a_{m,p} \), and by definition of the operator \( A \) and \( A \), we have \( A \in B(E) \) and \( A = B(E) \).

- **Step 3: Verification of condition \((H_3)\) of Theorem 4.4.** According to Corollary 4.13, the operators \( A S_B \) and \( S_B A \) satisfy the property \( c \) of Lemma 4.5. By applying Lemma 4.5,
\[ \| S_B(t) A \|_{B(E)} \leq C e^{bt} t^{-\Theta} \quad \text{and} \quad \| A S_B(t) \|_{B(E)} \leq C e^{bt} t^{-\Theta}. \]
Then for all \( a' > a \), there exist constructible constants \( n \in \mathbb{N} \) and \( C_{a'} \geq 1 \), such that
\[ \forall t \geq 0, \quad \| T_n(t) \|_{B(E)} \leq C_{a'} e^{a't}. \]

- **Step 4: End of the Proof**
  All the hypotheses of Theorem 4.4 are satisfied. We deduce that \( L_0 - a \) is a dissipative operator on \( E \) for all \( a > \max(a_{m,p}, -\kappa) \), with the semi-group \( S_{L_0}(t) \) satisfying estimate \( (8) \).

**Remark 4.14.** It is possible to give an alternative proof of results of decay in polynomially weighted \( L^p \) spaces (Theorem 1.7) using an enlargement argument, whose starting point is the result of decay in \( H^1(m_0) \) where \( m_0 = \mu^{-1/2} \), since \( H^1(m_0) \subset L^p(m) \) with \( m = (v)^k \) and \( k > 3(1-\frac{1}{p}) \). Indeed, the conditions \((H_1)\) and \((H_2)\) of Theorem 4.4 are satisfied by Theorem 1.5 and Lemma 4.11 respectively. It remains to verify the condition \((H_3)\). In the proof of Lemma 4.12, it is shown that
\[ \mathcal{H}(t, F) \leq C \| F_0 \|_{L^1(m_0)}^2, \quad \forall t \in (0, T]. \]
Allong with the definition of \( \mathcal{H} \), we obtain for all \( t \in (0, T] \)
\[ \| S_B(t) F_0 \|_{L^2(m_0)}^2 \leq \frac{C'}{t^d+1} e^{bt} \| F_0 \|_{L^1(m_0)}^2, \]
\[ \| \nabla_x S_B(t) F_0 \|_{L^2(m_0)}^2 \leq \frac{C''}{t^{d+2}} e^{bt} \| F_0 \|_{L^1(m_0)}^2, \]
\[ \| \nabla_x S_B(t) F_0 \|_{L^2(m_0)}^2 \leq \frac{C'''}{t^{d+4}} e^{bt} \| F_0 \|_{L^1(m_0)}^2. \]
Following in the proof of Lemma 4.12, we obtain
\[ \forall t \in (0, +\infty), \quad \| S_B(t) F_0 \|_{L^2(m_0)}^2 \leq \frac{\tilde{C}}{t^d+4} e^{bt} \| F_0 \|_{L^1(m_0)}^2. \]
By interpolation, we get
\[ \forall t \in (0, +\infty), \forall 1 \leq p \leq 2, \quad \| S_B(t) F_0 \|_{H^1(m_0)} \leq C e^{bt} t^{-(d+4)(\frac{1}{p}-\frac{1}{2})} \| F_0 \|_{L^p(m_0)}. \]
Therefore, using the same techniques as in the proof of Corollary 4.13, we prove that the property \( c \) of Lemma 4.5 is satisfied. Then by applying Lemma 4.5 we obtain that the condition \((H_3)\) is satisfied. Finally, by applying Theorem 4.4 we deduce the proof of Theorem 1.7.
One could try to use Lemma 2.17 in [10] to prove the hypothesis \((H_3)\) in Theorem 4.4. But the presence of the magnetic field creates a lot of difficulties in adapting the proof of this lemma. We note also that in the hypothesis of Lemma 2.17 include an estimate of type
\[ \|T_n\|_{B(L^p(m); L^2(m_0))} \leq C e^{bt / t^\alpha}, \]
where \(T_n(t) = (A S_B(t))^{(n)}\) for some \(n\) with \(\alpha \in (0, 1]\) whereas our estimates are of the form \(\alpha = 3d + 1 > 1\) and \(n = 1\).

4.2.2. Proof of Theorem 1.9. This part is dedicated to the proof of the exponential time decay estimates of the semi-group associated with the Cauchy problem (2) with an external magnetic field \(B_e\), with an initial datum in \(W^{1,p}(m)\) defined in (9).

For the proof of Theorem 1.9, we consider the space \(E = W^{1,p}(m)\) and \(E = H^1(m_0)\).

**Definition 4.15.** We split operator \(\mathcal{L}_0\) into two pieces and define for all \(R, M > 0\)
\[ \mathcal{B}u = \mathcal{L}_0u - Au \quad \text{with} \quad Au = M \chi_R u, \quad (39) \]
where \(M > 0\), \(\chi_R(v) = \chi(v/R) \quad R > 1\), and \(\chi \in C_0^\infty(\mathbb{R}^3)\) such that \(\chi(v) = 1 \quad |v| \leq 1\). We also denote \(A\) and \(B\) the restriction of operators \(A\) and \(B\) on the space \(E\) respectively.

**Lemma 4.16** (Dissipativity of \(B\)). Under Assumption 1.8 and suppose that \(B_e \in W^{1,\infty}(T^3)\), there exists \(M \quad R > 0\) such that for all \(0 > a > \max \{a_{1,1}, a_{1,2}, a_{1,3}\} \) (defined in (47)-(49) and (56)-(58)) such that operator \(B - a\) is dissipative in \(W^{1,p}(m)\) where \(p \in [1, 2]\). In other words, the semi-group \(S_B\) satisfies the following estimate:
\[ \forall t \geq 0, \quad \|S_B(t)F_0\|_{W^{1,p}(m)} \leq e^{at} \|F_0\|_{W^{1,p}(m)}, \quad \forall F_0 \in W^{1,p}(m). \]

**Proof.** Let \(F_0 \in W^{1,p}(m)\). We consider \(F\) the solution of the evolution equation
\[ \partial_t F = BF, \quad F_{t=0} = F_0. \quad (40) \]
Recall that the norm on the space \(W^{1,p}(m)\) is given by
\[ \|F\|_{W^{1,p}(m)}^p = \|F\|^p_{L^p(m)} + \|\nabla_v F\|^p_{L^p(m)} + \|\nabla_x F\|^p_{L^p(m)}, \]
where \(\bar{m} = m(v)\). Differentiating the previous equality with respect to \(t\), we get
\[ \frac{d}{dt} \frac{1}{p} \|F\|^p_{W^{1,p}(m)} = \frac{d}{dt} \frac{1}{p} \|F\|^p_{L^p(m)} + \frac{d}{dt} \frac{1}{p} \|\nabla_v F\|^p_{L^p(m)} + \frac{d}{dt} \frac{1}{p} \|\nabla_x F\|^p_{L^p(m)}. \quad (41) \]

We now estimate each term of the equality (41).
For the first term in (41), we apply Lemma 4.11 and get
\[ \frac{1}{p} \frac{d}{dt} \|F\|^p_{L^p(m)} \leq \iint_{T^3} |F|^p (\Psi_{\bar{m},p} - M \chi_R) \bar{m}^p(v) dxdv, \]
Secondly, we differentiate the equation (40) with respect to \(v\), and then we use the equalities of Lemma 3.6. We get the following equation (recall \(d = 3\)):
\[ \partial_t \nabla_v F = B(\nabla_v F) + 3\nabla_v F + (B_e \land \nabla_v)F - \nabla_x F - M (\nabla_v \cdot \chi_R) F_t. \quad (42) \]
This gives

\[
\frac{d}{dt} \left\| \nabla_v F \right\|_{L_p(m)}^p = \int \int \partial_t \nabla_v F \left| \nabla_v F \right|^{p-2} \cdot \nabla_v F m^p \, dx \, dv
\]

\[
= \int \int B(\nabla_v F) \left| \nabla_v F \right|^{p-2} \cdot \nabla_v F m^p \, dx \, dv + 3 \left\| \nabla_v F \right\|_{L_p(m)}^p
\]

\[
- \int \int \nabla_x F \left| \nabla_v F \right|^{p-2} \cdot \nabla_v F m^p \, dx \, dv
\]

\[
+ \int \int (B_e \wedge \nabla_x) F \left| \nabla_v F \right|^{p-2} \cdot \nabla_v F m^p \, dx \, dv
\]

\[
- M \int \int (\nabla_v \chi_R) F \left| \nabla_v F \right|^{p-2} \cdot \nabla_v F m^p \, dx \, dv.
\]

Then, proceeding exactly as in the proof of Lemma 4.11 and applying Young’s inequality, we obtain for all \( \eta_1 > 0 \)

\[
\frac{d}{dt} \left\| \nabla_v F \right\|_{L_p(m)}^p
\]

\[
\leq \int \int \left| \nabla_v F \right|^p (\Psi_{m,p} - M \chi_R) m^p \, dx \, dv + 3 \left\| \nabla_v F \right\|_{L_p(m)}^p
\]

\[
+ \frac{1}{2} \left\| \nabla_v F \right\|_{L_p(m)}^p + \frac{1}{2} \left\| \nabla_v F \right\|_{L_p(m)}^p + \frac{M}{R} C \eta_1 \left\| \nabla_v \chi \right\|_{L_\infty(\mathbb{R}^3)} \left\| F \right\|_{L_p(m)}^p
\]

\[
+ \frac{M}{R} \eta_1 \left\| \nabla_v \chi \right\|_{L_\infty(\mathbb{R}^3)} \left\| \nabla_v F \right\|_{L_p(m)} + \left\| B_e \right\|_{L_\infty(\mathbb{R}^3)} \left\| \nabla_v F \right\|_{L_p(m)}^p
\]

\[
\leq \int \int \left( \left| \nabla_v F \right|^p (\Psi_{m,p} - M \chi_R + 3 + \frac{1}{2} \frac{M}{R} \left\| \nabla_v \chi \right\|_{L_\infty(\mathbb{R}^3)} \eta_1 + \left\| B_e \right\|_{L_\infty(\mathbb{R}^3)} \right) m^p \, dx \, dv
\]

\[
+ \frac{1}{2} \left\| \nabla_v F \right\|_{L_p(m)}^p + \frac{M}{R} C \eta_1 \left\| \nabla_v \chi \right\|_{L_\infty(\mathbb{R}^3)} \left\| F \right\|_{L_p(m)}^p.
\]

Finally, we estimate the last term of the equality \((41)\). We treat two cases, and then we use an interpolation argument to complete the proof.

**Case 1:** \( p = 1 \).

We differentiate the equation \((40)\) with respect to \( x_i \) for all \( i = 1, 2, 3 \), then we use the equalities of Lemma 3.6. We will have the following equation:

\[
\partial_t \partial_{x_i} F = B(\partial_{x_i} F) + (v \wedge \partial_{x_i} B_e) \cdot \nabla_v F.
\]  

(43)

Using the previous equation, we obtain

\[
\frac{d}{dt} \left\| \partial_{x_i} F \right\|_{L^1(m)} = \int \int \partial_t \left| \partial_{x_i} F \right| m \, dx \, dv
\]

\[
= \int \int (\partial_{x_i} \partial_t F) \partial_{x_i} F \left| \partial_{x_i} F \right|^{-1} m \, dx \, dv
\]

\[
= \int \int B(\partial_{x_i} F) \partial_{x_i} F \left| \partial_{x_i} F \right| m \, dx \, dv
\]

\[
+ \int \int (v \wedge \partial_{x_i} B_e) \cdot \nabla_v F \partial_{x_i} F \left| \partial_{x_i} F \right|^{-1} m \, dx \, dv.
\]
Using the computations made in Lemma 4.11 for $p = 1$, using Lemma B.1 in the appendix B, and performing an integration by parts with respect to $v$, we get

$$
\frac{d}{dt} \| \partial_x F \|_{L^1(m)} \leq \iint (\Psi_{m,1} - M\chi_R) |\partial_x F| m \, dx \, dv - \iint (v \wedge \partial_x B_e) F \partial_x F |\partial_x F|^{-1} \nabla_v m \, dx \, dv,
$$

where, we used the fact that $(v \wedge \partial_x B_e) \cdot \nabla_v m = 0$. Then, defining the norm

$$
\| \nabla_x F \|_{L^p(m)} := \sum_{i=1}^{3} \| \partial_x i F \|_{L^p(m)},
$$

and using the previous definition, we have

$$
\frac{d}{dt} \| \nabla_x F \|_{L^1(m)} \leq \iint (\Psi_{m,1} - M\chi_R) |\nabla_x F| m \, dx \, dv.
$$

Collecting all the estimates, we obtain

$$
\frac{d}{dt} ||F||_{\tilde{W}^{1,1}(m)} \leq \iint (\Psi_{m,1} - M\chi_R) |\nabla_x F| m \, dx \, dv.
$$

We define then (for $M$ and $R$ to be fixed below).

$$
\Psi_{m,1} := \Psi_{\tilde{m},1} - M\chi_R + \frac{M}{R} C_{\eta_1} \| \nabla_v \chi \|_{L^\infty(\mathbb{R}^3)}, \quad (44)
$$

$$
\Psi_{m,2} := \Psi_{m,1} - M\chi_R + 3 + \frac{1}{2} + \frac{M}{R} \eta_1 \| \nabla_v \chi \|_{L^\infty(\mathbb{R}^3)} + \| B_e \|_{L^\infty(\mathbb{T}^3)}, \quad (45)
$$

$$
\Psi_{m,3} := \Psi_{m,1} - M\chi_R + \frac{1}{2}. \quad (46)
$$

(Recall that $\limsup_{|v| \to +\infty} \Psi_{m,1} = -k$). We denote then

$$
a_{m,1}^1 = -k - 1, \quad (47)
$$

$$
a_{m,1}^2 = -k + \frac{7}{2} + \| B_e \|_{L^\infty(\mathbb{T}^3)}, \quad (48)
$$

$$
a_{m,1}^3 = -k + \frac{1}{2}. \quad (49)
$$

We now assume that $k$ satisfies

$$
k > \frac{7}{2} + \| B_e \|_{L^\infty(\mathbb{T}^3)}. \quad (50)
$$

Hypothesis (50) implies that $a_{m,1}^i < 0$, for all $i = 1, 2, 3$. Consequently, for $\eta_1$ sufficiently small, we may then find $M$ and $R > 0$ large enough so that, for all $0 > a > \max(a_{m,1}^1, a_{m,1}^2, a_{m,1}^3)$, we have

$$
\frac{d}{dt} ||F(t)||_{\tilde{W}^{1,1}(m)} \leq a ||F(t)||_{\tilde{W}^{1,1}(m)}. \quad (51)
$$
Hence the operator $B - a$ is dissipative on $\tilde{W}^{1,1}(m)$.

**Case 2:** $p = 2$.

Again, we differentiate the equation (40) with respect to $x$, and we use the equalities of Lemma 3.6 to obtain the following equation:

$$\partial_t \nabla_x F = B(\nabla_x F) + (v \wedge \nabla_x B_e) \cdot \nabla_v F.$$  \hfill (52)

Using the calculations made in Lemma 4.11 and the previous equation, we obtain

$$\frac{d}{dt} \frac{1}{2} \| \nabla_x F \|^2_{L^2(m)} = - \iint | \nabla_v \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \iint (\Psi_{m,2} - M \chi_R) | \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \iint (v \wedge \nabla_x B_e) \cdot \nabla_v F \nabla_x F m^2 \, dx \, dv.$$

Then, by integration by parts with respect to $v$, we get

$$\frac{d}{dt} \frac{1}{2} \| \nabla_x F \|^2_{L^2(m)} \leq - \iint | \nabla_v \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \iint (\Psi_{m,2} - M \chi_R) | \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \iint |v \wedge \nabla_x B_e|^2 |F| |\nabla_v F| m^2 \, dx \, dv.$$

According to the Cauchy-Schwarz inequality, for every $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that

$$\frac{d}{dt} \frac{1}{2} \| \nabla_x F \|^2_{L^2(m)} \leq - \iint | \nabla_v \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \iint (\Psi_{m,2} - M \chi_R) | \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \varepsilon \iint | \nabla_v \nabla_x F |^2 m^2 \, dx \, dv + C_\varepsilon \iint |v \wedge B_e|^2 |F|^2 m^2 \, dx \, dv.$$

We choose $\varepsilon = \frac{1}{4}$, and we finally get

$$\frac{d}{dt} \frac{1}{2} \| \nabla_x F \|^2_{L^2(m)} \leq \iint (\Psi_{m,2} - M \chi_R) | \nabla_x F |^2 m^2 \, dx \, dv$$
$$+ \frac{1}{2} \| \nabla_x B_e \|_{L^\infty(T^3)} \| F \|_{L^2(m)}^2 + \frac{1}{2} \| \nabla_x F \|^2_{L^2(m)}.$$

Collecting all the estimates, we thus obtain

$$\frac{d}{dt} \frac{1}{2} \| F \|^2_{\tilde{W}^{1,2}(m)}$$
$$\leq \iint (\Psi_{m,2} - M \chi_R + \frac{M}{R} C_{\xi_1} \| \nabla_v \chi \|_{L^\infty(\mathbb{R}^3)} + \frac{1}{2} \| \nabla_x B_e \|_{L^\infty(T^3)}) |F|^2 m^2 \, dx \, dv$$
$$+ \iint (\Psi_{m,2} - M \chi_R + 3 + \frac{1}{2} + M \eta \| \nabla_v \chi \|_{L^\infty(\mathbb{R}^3)} + \| B_e \|_{L^\infty(T^3)}) |\nabla_v F|^2 m^2 \, dx \, dv$$
$$+ \iint (\Psi_{m,2} - M \chi_R + \frac{1}{2} + \frac{1}{2}) |\nabla_x F|^2 m^2 \, dx \, dv.$$
Again, we define then, for $M$ and $R$ to be fixed in the next paragraph
\[\Psi_{m,2} = \Psi_{m,2} - M \chi_R + \frac{M}{R} C_{\eta_1} \| \nabla_v \chi \|_{L^\infty(\mathbb{R}^3)} + \frac{1}{2} \| \nabla_x B_e \|_{L^\infty(T^3)},\]  
(53)
\[\Psi_{m,2} = \Psi_{m,2} - M \chi_R + 3 + \frac{M}{R} \eta_1 \| \nabla_v \chi \|_{L^\infty(\mathbb{R}^3)} + \| B_e \|_{L^\infty(T^3)},\]  
(54)
\[\Psi_{m,2} = \Psi_{m,2} - M \chi_R + 1.\]  
(55)
Again, we denote
\[a_{m,2}^1 = \frac{3}{2} - k - 1 + \frac{1}{2} \| \nabla_x B_e \|_{L^\infty(T^3)}\]  
(56)
\[a_{m,2}^2 = \frac{3}{2} - k + \frac{7}{2} + \| B_e \|_{L^\infty(T^3)}\]  
(57)
\[a_{m,2}^3 = \frac{3}{2} - k + 1,\]  
(58)
Assuming $k$ satisfies
\[k > 5 + \max \left( \| B_e \|_{L^\infty(T^3)}, \frac{1}{2} \| \nabla_x B_e \|_{L^\infty(T^3)} \right),\]  
(59)
we obtain that $a_{m,2}^i < 0$ for all $i = 1, 2, 3$. Consequently, we may find $M, R > 0$ large enough so that for all $0 > a > \max(a_{m,2}^1, a_{m,2}^2, a_{m,2}^3)$
\[\frac{d}{dt} \frac{1}{2} \| F(t) \|_{L^p_{1,2}(m)}^p \leq a \| F(t) \|_{L^p_{1,2}(m)}^2.\]
Hence the operator $\mathcal{B} - a$ is dissipative on $\tilde{W}^{1,2}(m)$ for such $M$ and $R$.

**For the general case $1 \leq p \leq 2$:** The cases 1 and 2 show us that the operator $S_{\mathcal{B}}(t)$ is continuous on $\tilde{W}^{1,1}(m)$ (on $\tilde{W}^{1,2}(m)$) with the operator $\mathcal{B}$ is given by
\[\mathcal{B} = L_0 - M \chi_R,\]
where $M$ and $R > 0$ agree with the conditions given in case 1 and case 2. Applying the Riesz-Thorin interpolation Theorem and using Hypothesis 1.8, we obtain that the operator $S_{\mathcal{B}}(t)$ is continuous on $\tilde{W}^{1,p}(m)$ for all $1 \leq p \leq 2$, with the following dissipative estimate:
\[\forall 0 > a > \max(a_{m,1}^i, a_{m,2}^i, i = 1, 2, 3), \quad \| S_{\mathcal{B}}(t) F_0 \|_{\tilde{W}^{1,p}(m)} \leq C e^{at} \| F_0 \|_{L^p_{1,2}(m)}.\]
This completes the proof. \(\square\)

**Remark 4.17.** We note that the only step where we needed to suppose $F \in L^2((v) m)$ and $\nabla_v F, \nabla_x F \in L^2(m)$, was in the estimate of
\[\iint (v \wedge \nabla_x B_e) \cdot \nabla_v F \nabla_x F m^2 dv.\]  
(60)
But this problem is not encountered in Sobolev space with exponential weight $\mu^{-1/2}$, because we have the equality
\[\nabla_v \mu^{-1/2} = \frac{v}{2} \mu^{-1/2},\]
that allowed us to estimate the term (60) with $m = m_0 = \mu^{-1/2}$ without the need to add a weight. (See Lemma A.3). This is no longer true with a polynomial weight because
\[\nabla_v (v)^k = k v (v)^{k-2}.\]
From now on, $M$ and $R$ are fixed as in Lemma 4.16.
Lemma 4.18 (Property of regularization). There exist $b$ and $C > 0$ such that, for all $p, q$ with $1 \leq p \leq q \leq 2$, we have

$$\forall t \geq 0, \quad \|S_B(t)F_0\|_{\dot{W}^{1,q}(m_0)} \leq Ce^{bt}t^{-(3d+4)(\frac{1}{p} - \frac{1}{q})} \|F_0\|_{\dot{W}^{1,p}(m_0)},$$

(61)

$$\forall t \geq 0, \quad \|S_B^*(t)F_0\|_{\dot{W}^{-1,p'}(m_0)} \leq C'e^{bt}t^{-(3d+4)(\frac{1}{p'} - \frac{1}{q})} \|F_0\|_{\dot{W}^{-1,q'}(m_0)},$$

(62)

Here $2 \leq q' \leq p' \leq +\infty$ are the conjugates of $p$ and $q$ respectively.

Proof. Let $F$ be the solution of the evolution equation

$$\partial_t F = BF, \quad F_{|t=0} = F_0.$$

In to the proof of Lemma 4.12, the following relative entropy has been introduced

$$\mathcal{H}(t, h) = B\|h\|_{L^1(m_0)}^2 + t^2G(t, h),$$

with $G$ defined by

$$\mathcal{G}(t, h) = C\|h\|_{L^2(m_0)}^2 + D t\|\nabla_v h\|_{L^2(m_0)}^2 + E t^2\|\nabla_x h, \nabla_v h\| + t^3\|\nabla_x h\|_{L^2(m_0)}^2.$$

We have shown, for constants $\alpha, D, E$ and $\beta > 0$ well chosen, that there exist $C > 0$ and $r = 3d + 1$ such that

$$\forall t \geq 0, \quad \mathcal{H}(t, F) \leq B''' \mathcal{H}(0, F_0) \leq C\|F_0\|_{L^1(m_0)}^2.$$

Using the previous estimate and the definition of $\mathcal{H}$ and $\mathcal{G}$, we get

$$\|S_B(t)F_0\|_{L^2(m_0)}^2 \leq \frac{\alpha}{t^{3d+1}} \mathcal{H}(t, F) \leq \frac{C'}{t^{3d+1}} e^{bt} \|F_0\|_{L^2(m_0)}^2,$$

$$\|\nabla v S_B(t)F_0\|_{L^2(m_0)}^2 \leq \frac{D}{t^{3d+2}} \mathcal{H}(t, F) \leq \frac{C''}{t^{3d+2}} e^{bt} \|F_0\|_{L^2(m_0)}^2,$$

$$\|\nabla_x S_B(t)F_0\|_{L^2(m_0)}^2 \leq \frac{\beta}{t^{3d+4}} \mathcal{H}(t, F) \leq \frac{C'''}{t^{3d+4}} e^{bt} \|F_0\|_{L^2(m_0)}^2.$$

Therefore,

$$\forall t \in [0, +\infty), \quad \|S_B(t)F_0\|_{H^{1}(m_0)}^2 \leq \frac{C}{t^{3d+4}} e^{bt} \|F_0\|_{\dot{W}^{1,1}(m_0)}^2.$$

Finally, to complete the proof, we use the Riesz-Thorin Interpolation Theorem in the real case on the operator $S_B(t)$. We obtain the continuity of $S_B(t)$ from $\dot{W}^{1,p}(m_0)$ to $\dot{W}^{1,q}(m_0)$, with $S_B$ satisfying the estimate (61).

The estimate (62) follows from (61) by duality.

Corollary 4.19. Let $m$ be a weight that satisfies Hypothesis 1.8. Then there exists $\Theta > 0$ such that for all $F_0 \in \dot{W}^{1,p}(m)$ with $p \in [1,2]$

$$\forall t \geq 0, \quad \|A S_B(t)F_0\|_{H^{1}(m_0)} \leq Ce^{bt}t^{-\Theta} \|F_0\|_{\dot{W}^{1,p}(m)},$$

$$\forall t \geq 0, \quad \|S_B(A)F_0\|_{H^{1}(m_0)} \leq Ce^{bt}t^{-\Theta} \|F_0\|_{\dot{W}^{1,p}(m)}.$$

Proof. The proof is similar to that of Corollary 4.13.

Proof of Theorem 1.9. The estimate (11) is an immediate consequence of Theorem 4.4 together with Theorem 1.5, Lemma 4.16, Lemma 4.18 and Corollary 4.19.
Remark A.2. We can generalize the results of the preceding Lemma for all function $m$. In the following Lemma we show that operator $(v \wedge B_e) \cdot \nabla_v$ and $v \cdot \nabla_x$ are formally skew-adjoint operators in the space $L^2(\mu^{1/2})$.

Lemma A.1. Let $B_e$ be the external magnetic field, then, with adjoints in the space $L^2(\mu^{1/2})$,
$$((v \wedge B_e) \cdot \nabla_v)^* = -(v \wedge B_e) \cdot \nabla_v,$$
and
$$(v \cdot \nabla_x)^* = -v \cdot \nabla_x.$$  

Proof. Let $f$ and $g \in C_0^\infty (\mathbb{R}^3 \times \mathbb{T}^3)$. We have
$$\langle((v \wedge B_e) \cdot \nabla_v)^* f, g\rangle = \langle f, ((v \wedge B_e) \cdot \nabla_v)g \rangle = \iint f((v \wedge B_e) \cdot \nabla_v)g \, dx \, d\mu,$$
where $d\mu = \mu(v) \, dv$. Using the fact
$$\langle(v \wedge B_e) \cdot \nabla_v f, g \rangle = \nabla_v \cdot (v \wedge B_e)f,$$
we obtain
$$\langle((v \wedge B_e) \cdot \nabla_v)^* f, g\rangle = \iint f \nabla_v \cdot (v \wedge B_e)g \, dx \, d\mu$$
$$= -\iint f(-\nabla_v + v) \cdot (v \wedge B_e)g \, dx \, d\mu,$$
since $(v \wedge B_e) \cdot v = 0$. By integration by parts, we have then
$$\langle((v \wedge B_e) \cdot \nabla_v)^* f, g\rangle = -\iint \nabla_v f \cdot (v \wedge B_e)g \, dx \, d\mu$$
$$= -\iint g(v \wedge B_e) \cdot \nabla_v f \, dx \, d\mu$$
$$= -\langle((v \wedge B_e) \cdot \nabla_v)f, g\rangle.$$  

For the second equality, we obtain
$$\langle(v \cdot \nabla_x)^* f, g\rangle = \langle f, v \cdot \nabla_x g \rangle = \iint f(v \cdot \nabla_x g) \, dx \, d\mu,$$
by integration by parts with respect to $x$. Since $\mu$ is independent of $x$, we have then
$$\langle(v \cdot \nabla_x)^* f, g\rangle = -\iint (v \cdot \nabla_x f) g \, dx \, d\mu = -\langle v \cdot \nabla_x f, g \rangle.$$

This completes the proof. \hfill $\square$

Remark A.2. We can generalize the results of the preceding Lemma for all function $m$ which are radial in $v$ and independent of $x$. We obtain that $v \cdot \nabla_x$ and $(v \wedge B_e) \cdot \nabla_v$ are formally skew-adjoint operators in the space $L^2(m)$.

Now, we give the following technical lemma.

Lemma A.3. We have the following equalities in $L^2(\mu^{1/2})$:

i. $\langle(v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_v f\rangle = -\langle\nabla_v \cdot (\nabla_x B_e \cdot \nabla_x f), \nabla_v f\rangle$.

ii. $\langle(v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f\rangle = -\langle\nabla_v \cdot (\nabla_x B_e \cdot \nabla_x f), \nabla_v f\rangle$.

where $\nabla_x B_e$ is the Jacobian matrix of the function $x \mapsto B_e(x) = (B_1(x), B_2(x), B_3(x))$. 

Proof. Let \( f \in C_0^\infty(T^3 \times \mathbb{R}^3) \). We will show (i), using the fact that

\[-v \mu(v) = \nabla_v(\mu(v)), \forall v \in \mathbb{R}^3,\]

we obtain

\[
\langle (v \wedge \nabla_x B_c) \cdot \nabla_v f, \nabla_v f \rangle = \\
\int ((v \wedge \nabla_x B_c) \cdot \nabla_v f) \cdot \nabla_v f \, dx \, d\mu \\
= -\sum_{i=1}^3 \int ((v \mu(v) \wedge \nabla_x B_c) \cdot \nabla_v f)_i \, \partial_{v_i} f \, dx \, d\mu \\
= -\sum_{i=1}^3 \left[ \int \partial_{v_i} \mu(v) \left( (\nabla_x B_c)_{12} \partial_{v_3} f - (\nabla_x B_c)_{13} \partial_{v_2} f \right) \partial_{v_i} f \, dx \, d\mu \right] \\
- \sum_{i=1}^3 \left[ \int \partial_{v_i} \mu(v) \left( (\nabla_x B_c)_{13} \partial_{v_1} f - (\nabla_x B_c)_{11} \partial_{v_3} f \right) \partial_{v_i} f \, dx \, d\mu \right] \\
- \sum_{i=1}^3 \left[ \int \partial_{v_i} \mu(v) \left( (\nabla_x B_c)_{11} \partial_{v_2} f - (\nabla_x B_c)_{12} \partial_{v_1} f \right) \partial_{v_i} f \, dx \, d\mu \right],
\]

where \( d\mu = \mu(v) \, dv \). Then, by an integration by parts with respect to \( v \), we get

\[
\langle (v \wedge \nabla_x B_c) \cdot \nabla_v f, \nabla_v f \rangle = \\
\sum_{i=1}^3 \left[ \int (\nabla_x B_c)_{12} \partial_{v_3} f - (\nabla_x B_c)_{13} \partial_{v_2} f \right] \partial_{v_i} \partial_{v_i} f \, dx \, d\mu \\
+ \sum_{i=1}^3 \left[ \int (\nabla_x B_c)_{13} \partial_{v_1} f - (\nabla_x B_c)_{11} \partial_{v_3} f \right] \partial_{v_i} \partial_{v_i} f \, dx \, d\mu \\
+ \sum_{i=1}^3 \left[ \int (\nabla_x B_c)_{11} \partial_{v_2} f - (\nabla_x B_c)_{12} \partial_{v_1} f \right] \partial_{v_i} \partial_{v_i} f \, dx \, d\mu.
\]

Then we rewrite the integrals as mixed products between vectors, so we get

\[
\langle (v \wedge \nabla_x B_c) \cdot \nabla_v f, \nabla_v f \rangle = \\
\sum_{i=1}^3 \left[ \int \partial_{v_3} \partial_{v_i} f \, (\nabla_x B_c)_{12} f - \partial_{v_2} \partial_{v_i} f \, (\nabla_x B_c)_{13} f \right] \partial_{v_i} \partial_{v_i} f \, dx \, d\mu \\
+ \sum_{i=1}^3 \left[ \int \partial_{v_1} \partial_{v_i} f \, (\nabla_x B_c)_{13} f - \partial_{v_3} \partial_{v_i} f \, (\nabla_x B_c)_{11} f \right] \partial_{v_i} \partial_{v_i} f \, dx \, d\mu \\
+ \sum_{i=1}^3 \left[ \int \partial_{v_2} \partial_{v_i} f \, (\nabla_x B_c)_{11} f - \partial_{v_1} \partial_{v_i} f \, (\nabla_x B_c)_{12} f \right] \partial_{v_i} \partial_{v_i} f \, dx \, d\mu.
\]
Finally, we deduce the equality (i) by the following equalities

\[(v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_v f \rangle = -\int (\nabla_v \wedge (\nabla_x B_e \cdot \nabla_v f)) \cdot \nabla_v f \, dx \, d\mu \]

Similarly, we can show (ii). This completes the proof.

Appendix B. Non positivity of a certain integral. The following well-known lemma is used in the proof of the dissipativity of the operator $B - a$ in the spaces $L^p(m)$ and $\tilde W^{1,p}(m)$ in Section 4. This lemma is a special case of the general study done in the article [5].

Lemma B.1. Let $g$ be a smooth function and let $p \geq 1$. Then the following integral is well-posed and satisfy the following estimate

\[\iint_{T^3 \times \mathbb{R}^3} (\Delta_v g) |g|^{p-2} g \, dx \, dv \leq 0.\]

Proof. Formal integration by parts with respect to $v$ justifies the property for all $p > 1$. For $p = 1$, we regularize and use convexity of the function $\Psi : s \to |s|$. □

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