Spherical Collapse Model

Ding-fang Zeng and Yi-hong Gao

1Institute of Theoretical Physics, Chinese Academy of Science.

We studied the spherical collapse model in the flat ΛCDM cosmology and provided exact and analytical formulae for the calculation of the two important parameters in the application of Press-Schechter theory, the critical density contrast and the ratio of cluster/background densities at the virialization point in terms of a media variable which can be solved precisely by numerical methods.

INTRODUCTION

Press-Schechter theory has been written into textbooks, but two important parameters δc and ∆c of it extracted from the studying of spherical collapse models are not determined well enough. [3, 4] are two early works studying this model in the flat ΛCDM or open cosmologies. Our purpose here is to provide a simple and direct method for the calculation of δc(Ωm0, a0) and ∆c(Ωm0, a0). Our background is flat ΛCDM cosmology but the method can be easily generalized to the QCDM [5] as well as non-flat cosmologies. Comparing with the existed results on this model, our results are analytical and exact.

PRESS-SCHECHTER THEORY

Press-Schechter Theory [4] predicts that the fraction of volume which has collapsed at a certain red-shift z is

\[ f_{coll}(M(R), z) = \frac{2}{\sqrt{2\pi} \sigma(R, z)} \int_{\delta_c}^{\infty} d\delta e^{-\delta^2/2\sigma^2(R, z)}. \] (1)

Here, R is the radius over which the density field has been smoothed, σ(R, z) is the rms of the smoothed density field [1]. δc is the threshold of density contrast at time ta (count stopping time) beyond which objects collapse.

In existing literatures, there are two conventions for the definition of σ(R, z) and δc. In the first convention [1, 3, 10], σ(R, z) contains growth factor, while δc only depends on the partition of the cosmological component today and the count stopping epoch ac. In the second convention [3, 4], σ(R) does not contain the growth factor, but δc contains both the cosmological component partition effects and the growth factor effects. We will take the first convention in this paper. Our δc here will be corresponded to the δc(0) of [3], which was plotted in the upper panel of figure.1 of it.

Operationally, δc is obtained by extrapolating the primordial perturbation to the collapse epoch using the growth law of linear perturbation theory, i.e.,

\[ \delta_c = \left[ \frac{\rho_{mc}(a, a_c)}{\rho_{mb}(a)} - 1 \right] \frac{1}{D_1(a)} \int_0^{a_c} da' H(a') \] (2)

Where \( \rho_{mc} \) and \( \rho_{mb} \) are the mass density of the clusters and the background respectively; while \( D_1(a) \) is the growth function of linear perturbation theory [1],

\[ D_1(a) = \frac{5\Omega_{m0} H_0^2}{2} H(a) \int_0^{a} da' [a' H(a')]^{-3}. \] (3)

It can be shown that \( D_1(a)_{a \to 0} \to a \). Using the method of [6] we can show that

\[ \left[ \frac{r}{a} \right]_{a \to 0} \propto (1 - \alpha \cdot a) \] (4)

So,

\[ \delta_c = 3\alpha \cdot D_1(a_c). \] (5)

In a given cosmological model, \( D_1(a) \) are known [1], the purpose of studying spherical collapse model is to determine \( \alpha \).

It should be noted that besides the partition of the cosmological components and the observational epoch, the value of \( \delta_c \) is also quite dependent on the choice of smoothing window used to obtain the dispersion \( \sigma(R, z) \) [6]. We will not consider this effect in this paper.

SPHERICAL COLLAPSE MODEL

To calculate parameter \( \alpha \) of eq[5], let us start with the Friedmann equations for both the over-dense region and background cosmology:

\[ \left( \frac{\dot{r}}{r} \right)^2 = \frac{8\pi G}{3} \left[ \frac{\rho_{mc, \tau a} r_{ta}^3}{r^3} + \rho_{\Lambda 0} \right] - \frac{\kappa}{r^2} \] (6)

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left[ \frac{\rho_{mb, \tau a} a_{ta}^3}{a^3} + \rho_{\Lambda 0} \right] \] (7)

where \( r \) and \( a \) denote the radius of the over-dense region and scale factor of background universe respectively; \( \kappa \) is a constant; while subscript \( \tau a \) and \( c \) denote the turn around and collapse point time. Because at the turn around time \( \dot{r} = 0 \),

\[ \frac{\kappa}{r_{\tau a}^2} = \frac{8\pi G}{3} \left[ \rho_{mc, \tau a} + \rho_{\Lambda 0} \right]. \] (8)
Dividing eq(6) by (7) and using (8) we get
\[ \frac{r^2 r'_t}{a^2 a'_t} = \frac{\rho_{mc,ta} r^{-1} r'_t + \rho_{\Lambda 0} r^2 r'_t - (\rho_{mc,ta} + \rho_{\Lambda 0})}{\rho_{mb,ta} a^{-1} a'_t + \rho_{\Lambda 0} a^2 a'_t} \] (9)

After letting
\[ x = \frac{a}{a_t}, y = \frac{r}{r_t}, \zeta = \frac{\rho_{mc,ta}}{\rho_{mb,ta}}, \nu = \frac{\rho_{\Lambda 0}}{\rho_{mb,ta}} \] (10)

eq(9) becomes
\[ (\frac{dy}{dx})^2 = \frac{\zeta y^{-1} + \nu y^2 - (\zeta + \nu)}{x^{-1} + \nu x^2}. \] (11)

We will not try to solve this equation, we derive it here because we need to use its asymptotic behavior to calculate \( \delta_c \).

In the \( \Lambda \)CDM cosmologies, the quantity \( \zeta \) and \( \nu \) in eq(11) are only functions of \( \Omega_{m0} \) and \( a_c \). They do not depend on \( x \) or \( y \). Using eq(11), we can write
\[ \left[ \frac{y}{x} \right]_{x \to 0} = \left[ \frac{rr'_t}{aa'_t} \right]_{a \to 0} = \zeta^\frac{1}{2}(1 - \alpha \cdot a_t x). \] (12)

So,
\[ \frac{d}{dx} [y_{x \to 0}] = \zeta^\frac{1}{2}(1 - 2\alpha \cdot a_t x). \] (13)

Substitute eqs(12) and (13) into eq(11), expand it and keep only the linear term, we get
\[ \alpha = \frac{1}{5} a_t^{-1} [\zeta^\frac{1}{2} + \nu \cdot a_t \zeta^{-\frac{1}{2}}]. \] (14)

Substitute this result into eq(11), we get:
\[ \delta_c(\Omega_{m0}, a_c) = \frac{3}{5} a_t^{-1} [(\zeta^\frac{1}{2} + \nu \cdot a_t \zeta^{-\frac{1}{2}})] D_1(a_c) \] (15)

This is the first important result of this paper. We will derive the differential equation for determining \( \zeta(\Omega_{m0}, a_c) \) and the fitting formula eq(29) for it as well as the analytical solution for \( \nu(\Omega_{m0}, a_c) \) eq(32) (and the notes there) in the technique details section. Comparing with the method of \( \delta_c \), our method here is more easier to operate and the result is more simply-looking.

In FIG.1 we plotted \( \delta_c \)'s dependence on \( \Omega_{m0} \) and \( a_c \). From the figure we see that if \( \Omega_{m0} \) is fixed, then \( \delta_c \) is a decreasing function of \( a_c \). Physically this is because, in the given cosmological model, \( \Omega_{m0} \) fixed, the earlier an objects collapse, the more denser it is required to assure collapse occur successfully. On the other hand, if \( a_c \) is fixed, then \( \delta_c \) increases as \( \Omega_{m0} \) increases. Superficially this means that in those lower matter (thus high dark energy) percentage universes, collapse occurs more easily. This is not the fact. Because in the lower matter percentage universes, density perturbation grows more slowly. As the result of linear extrapolation of the primary perturbations eq(2), the more smaller growth factor comes from the lower matter percentage suppresses the primary perturbations more strongly so gives more smaller \( \delta_c \).

To compare our results with that of \( \delta_c \), we plotted \( \delta_c \)'s dependence on \( \Omega_{m0} \) and \( \Omega_{mb,c} \) in FIG.2. From this figure we see that what was shown in the upper panel of figure 1 of \( \delta_c \) is the projection of our results in the \( \delta_c \)-\( \Omega_{mb,c} \) plane.

To this point in this section, we imagined the evolution of an over-dense region as: as long as it is over-dense enough, it will grow from an \( r = 0 \) point, to the maximum radius, then collapse to \( r = 0 \) point. Factually, before the region collapse to the \( r = 0 \) point, pressures from the random moving of the materials inside the region will balance their self-gravity and the system will enter virialization status. At this point, the second im-
important parameter $\Delta_c := \frac{\rho_{mc,c}}{\rho_{mb,c}}$ in the application of Press-Schechter theory [8, 9, 10] appears. Assuming that at the collapse point, the system has virialized fully, we can write:

$$\Delta_c = \frac{\rho_{mc}(a_0)^3 r_{ta}^3 r_{c}^{-3}}{\rho_{mb}(a_0)^3 r_{ta}^3 a_0^3} = \frac{r_{ta}^2 a_0^3}{r_{c}^2 a_0^3}. \quad (16)$$

According to virial theorem and energy conservation law [9],

$$E_{\text{kinetic}} = -\frac{1}{2} U_G + U_\Lambda$$

$$\frac{1}{2} U_{G,c} + 2 U_{\Lambda,c} = U_{G,ta} + U_{\Lambda,ta} \quad (17)$$

we have

$$-\frac{1}{2} \frac{3GM^2}{5r_c} - \frac{4\pi GM \rho_{\Lambda,cr^3}}{5} = -\frac{3GM^2}{5r_{ta}} - \frac{4\pi GM \rho_{\Lambda,ta} r_{ta}^2}{5}$$

$$\Rightarrow \frac{r_{ta}}{r_c} = \frac{2(\rho_{mc,ta} + \rho_{\Lambda,ta})}{(\rho_{mc,c} + 4\rho_{\Lambda,c})\rho_{mc,c}}$$

$$\frac{2(\rho_{mc,ta} + \rho_{\Lambda,ta})}{(\rho_{mc,c} + 4\rho_{\Lambda,c})\rho_{mc,c}} = \frac{1}{1 + 4\frac{\Omega_{\Lambda,c}}{\Omega_{mb,c}}\left[\frac{r_{ta}^2}{r_c^2}\right]^{-1}.} \quad (18)$$

Looking as an equation for $\frac{r_{ta}}{r_c}$, eq(18) can be solved analytically. Substituting the solution into eq(16), we get

$$\Delta_c(\Omega_{m0}, a_c) = \zeta \left[ \frac{a_c}{a_{ta}} \right]^3 \times \left( \frac{2}{3} \frac{(\mu_t + 1)^2}{f(\mu_t, \mu_c)} + (\mu_t + 1) + f(\mu_t, \mu_c) \right)^3 \quad (19)$$

$$f(\mu_t, \mu_c) = [(\mu_t + 1)^3 - \frac{27}{4} \mu_c + \sqrt{\frac{3}{4} \mu_c(27\mu_c - 8(\mu_t + 1)^3)}]^{1/3} \quad (20)$$

$$\mu_t = \frac{\Omega_{\Lambda,c} a_{ta}^4}{\Omega_{mb,c}}, \quad \mu_c = \frac{\Omega_{\Lambda,c} a_{ta}^3}{\Omega_{mb,c} a_c^3} \quad (21)$$

Eq(16) is the second important result of this paper. In $\Lambda$CDM cosmology, all the quantities on the right hand side of eq(16) are known functions of $\Omega_{m0}$ and $a_c$. We will give the relevant formulae eq(28), eq(29) and eq(32) in the technique section. Comparing with [8], our results here is exact instead of approximated.

To see the physical meaning of eq(16) clearly, we plotted $\Delta_c$'s dependence on $\Omega_{m0}$ and $a_c$ in FIG. 4. From the figure, we see that $\Delta_c$ is an increasing function of $a_c$, i.e., the latter a region collapses, the more denser should it be. This can be understood physically. Because at more later times, dark energy's percentage will be more larger. To cancel its counter-collapse effects, more denser matter is required to assure collapse. It is worth noting that

![FIG. 3: Left panel: $\Delta_c$'s dependence on the collapse epoch $a_c$, different curves denote different cosmological models indexed by $\Omega_{m0}$. Right panel: $\Delta_c$'s dependence on the cosmological models, different curves denote different collapse epoch indexed by $a_c$.](image)

![FIG. 4: Right panel: $\Delta_c$'s dependence on $\Omega_{m0}$ and the collapse point $r_{mb,c}$, the same curve has the same collapse epoch. Right panel: projection of three space curves from the right panel.](image)

used a different definition of $\Delta_c(\text{ECF}) := \frac{\rho_{mc,c}}{\rho_{mb,c}}$ (so different mass-temperature relations, please comparing eq(2.2) of [8] and eq(4) of [10]), we reproduce the results there in FIG. 4. From the right panel of this figure we see that, what was plotted in the lower panel of figure 1 of [8] is the projection of our results on the $\Delta_c$-$\Omega_{mb,c}$ plane.

**TECHNIQUE DETAILS**

To calculate parameter $\zeta$ appearing in eqs(14) and (10), we write down Friedman equations for the background cosmology and the space-space component of Einstein equation for the radius of the over-dense region,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_{mb} + \rho_\Lambda). \quad (22)$$

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3}(3\rho_\Lambda + \rho_{mc}). \quad (23)$$

Using the notations introduced in eq(13), eqs(22) and (23) can be rewritten as

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G \rho_{mb} a_{ta}}{3 x^3 \Omega_{mb}(a)} \quad (24)$$

$$\frac{\ddot{y}}{y} = -\frac{4\pi G \rho_{mb} a_{ta}}{3}(1 + 3w) \frac{1 - \Omega_{mb}(a)}{x^3 \Omega_{mb}(a)} + \left(\frac{c}{y}\right)^2 \quad (25)$$
, where we preserved the equation of state coefficient \( w \) as a general parameter. It can even be a function of \( a \) or \( x \). The above two equations can be translated into:

\[
\frac{d^2y}{dx^2} = \frac{dy}{dx} \frac{1}{x} + \left( \frac{d\Omega_{mb}}{dx} \frac{1}{\Omega_{mb}} \right) \frac{1}{x^2} + \frac{\zeta x \Omega_{mb}}{y^2} = 0 \quad (26)
\]

Eq 26 is a second ordinary differential equation, it contains a free parameter \( \zeta \), but satisfies three boundary condition:

\[
y \bigg|_{x=0} = \left[ \frac{d}{dx}y \right] \bigg|_{x=0} = 1 - \cdot\zeta^2, \\
y \bigg|_{x=1} = 1, \quad y \bigg|_{x=1} = 0. \quad (27)
\]

This is a two point boundary condition problem. It can also be looked as an eigen-value problem and solved numerically by the method described in §17.4 of [3]. Noting

\[
\Omega_{mb}(a) = \frac{\Omega_{m0}a^{-3}}{\Omega_{m0}a^{-3} + (1 - \Omega_{m0})} = \frac{1}{1 + \frac{1 - \Omega_{mb,ta}}{\Omega_{mb,ta}}x^3} \quad (28)
\]

we see that \( \zeta \) solved from eq 26 only depends on \( \Omega_{mb,ta} \). So we can choose \( a_{ta} = 1 \) and solve eq 26 for different values of \( \Omega_{m0} \) to get the appropriate \( \zeta \) and fit the result as \( \zeta(\Omega_{m0}) \). Then change the function into \( \zeta(\Omega_{mb,ta}) \) to get \( \zeta \)'s dependence on \( \Omega_{m0} \) and \( a_c \). We do so by software Mathematica and confirmed the results by the method of [2] and get

\[
\zeta(\Omega_{mb,ta}) = \left( \frac{3\pi}{4} \right)^2 \Omega_{mb}^{-0.7384+0.2451\Omega_{m0}} \bigg|_{ta}. \quad (29)
\]

Now the final question we need to answer is to express the scale factor of turn around point with that of the collapse point. Note that, in the unperturbed spherical cases, clusters’ formation process is symmetrical about the turn around time \( t_{ta} \), so

\[
t_c = 2t_{ta}. \quad (30)
\]

According to Friedmann equation \( H^2 = H_0^2\Omega_{m0}/a^3\Omega_m \), the time-scale-factor relation is

\[
t \propto \int_0^a da' \sqrt{\Omega_m(a')a'} \\
= \int_0^a da' \sqrt{1 + \nu_0a'^3} \\
\propto \ln \left[ \sqrt{\nu_0a^3} + \sqrt{\nu_0a^3 + 1} \right]. \quad (31)
\]

Substitute eq 31 into 30 and solve it analytically, we get

\[
a_{ta} = \left[ \frac{\sqrt{1 + \nu_0a^3} - 1}{2\nu_0} \right]^{1/3}, \quad (32)
\]

where \( \nu_0 \) is today’s ratio of dark-energy/mass density and \( \nu = \nu_0a_{ta}^3 \). [1] proposes that when eqs 24 and 25 solved, using eq(A12) and (A13) of it to calculate \( \delta_c \) directly. This method has difficulty to get high precision.

**CONCLUSIONS**

We studied the spherical collapse model in the flat ΛCDM cosmology and provided exact and analytical formulas eqs 18 and 19 for the calculation of the two important parameters \( \delta_c(\Omega_{m0}, a_c) \) and \( \Delta_c(\Omega_{m0}, a_c) \) in terms of the media variable \( \zeta \) eq 29. We reproduced the existing results in the literature, but our method is more easier to operate and the result is more simply-looking. Our result will be useful in the studying of galaxy clusters evolution and the application of Press-Schechter theory.

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