Unbounded-Error One-Way Classical and Quantum Communication Complexity

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Abstract

This paper studies the gap between quantum one-way communication complexity \(Q(f)\) and its classical counterpart \(C(f)\), under the unbounded-error setting, i.e., it is enough that the success probability is strictly greater than \(1/2\). It is proved that for any (total or partial) Boolean function \(f\), \(Q(f) = \lceil C(f)/2 \rceil\), i.e., the former is always exactly one half as large as the latter. The result has an application to obtaining (again an exact) bound for the existence of \((m, n, p)\)-QRAC which is the \(n\)-qubit random access coding that can recover any one of \(m\) original bits with success probability \(\geq p\). We can prove that \((m, n, > 1/2)\)-QRAC exists if and only if \(m \leq 2^{2^n} - 1\). Previously, only the construction of QRAC using one qubit, the existence of \((O(n), n, > 1/2)\)-RAC, and the non-existence of \((2^{2^n}, n, > 1/2)\)-QRAC were known.

1 Introduction

Communication complexity is probably the most popular model for studying the performance gap between classical and quantum computations. Even if restricted to the one-way private-coin setting (which means no shared randomness or entanglement), several interesting developments have been reported in the last couple of years. For promise problems, i.e., if we are allowed to use the fact that inputs to Alice and Bob satisfy some special property, exponential gaps are known: Bar-Yossef, Jayram and Kerenidis \([5]\) constructed a relation to provide an exponential gap, \(\Theta(\log n)\) vs. \(\Theta(\sqrt{n})\), between one-way quantum and classical communication complexities. Recently, Gavinsky et al. \([11]\) showed that a similar exponential gap also exists for a partial Boolean function.

For total Boolean functions, i.e., if there is no available promise, there are no known exponential or even non-linear gaps: As mentioned in \([1]\), the equality function is a total Boolean function for which the one-way quantum communication complexity is approximately one half, \((1/2 + o(1)) \log n\) vs. \((1 - o(1)) \log n\), of the classical counterpart. This is the largest known gap so far. On the other hand, there are total Boolean functions for which virtually no gap exists between quantum and classical communication complexities. For example, those complexity gaps are only a smaller order additive term, \((1 - H(p))n\) vs. \((1 - H(p))n + O(\log n)\), for the index function \([4, 20]\), and \(n - 2 \log \frac{1}{2p} + 21\) vs. \(n - O(\log \frac{1}{2p})\) \([18]\) for the inner product function, where \(p\) is the success probability. Note that all the results so far mentioned are obtained under the bounded-error assumption, i.e., the success probability must be at least \(1/2 + \alpha\) for some constant \(\alpha\), being independent of the size of Boolean functions.

Thus there seem to be a lot of varieties, depending on specific Boolean functions, in the quantum/classical gap of one-way communication complexity. In this paper it is shown that such varieties completely disappear if we use the unbounded-error model where it is enough that the success probability is strictly greater than \(1/2\).
1.1 Our Contribution

We show that one-way quantum communication complexity of any (total or partial) Boolean function is always exactly (without an error of even ±1) one half of the one-way classical communication complexity in the unbounded-error setting. The study of unbounded-error (classical) communication complexity was initiated by Paturi and Simon [23]. They characterized almost tightly the unbounded-error one-way communication complexity of Boolean function $f$, denoted by $C(f)$, in terms of a geometrical measure $k_f$ which is the minimum dimension of the arrangement of points and hyperplanes. Namely, they proved that $\lceil \log k_f \rceil \leq C(f) \leq \lceil \log k_f \rceil + 1$. We show that such a characterization is also applicable to the unbounded-error one-way quantum communication complexity $Q(f)$. To this end, we need to link accurately the one-way quantum communication protocol to the arrangement of points and hyperplanes, which turns out to be possible using geometric facts on quantum states [14, 15]. As a result we show that $Q(f) = \lceil \log(k_f + 1)/2 \rceil$. Moreover, we also remove the small gap in [23], proving $C(f) = \lceil \log(k_f + 1) \rceil$. This enables us to provide the exact relation between $Q(f)$ and $C(f)$, i.e., $Q(f) = \lceil C(f)/2 \rceil$.

Our characterizations of $Q(f)$ and $C(f)$ have an application to quantum random access coding (QRAC) and classical random access coding (RAC) introduced by Ambainis et al. [4]. The $(m, n, p)$-QRAC (resp. $(m, n, p)$-RAC) is the $n$-qubit (resp. $n$-bit) coding that can recover any one of $m$ bits with success probability $\geq p$. The asymptotic relation among the three parameters $m, n, p$ was shown in [4] and [20]: If $(m, n, p)$-QRAC exists, then $n \geq (1 - H(p))m$, while there exists $(m, n, p)$-RAC if $n \leq (1 - H(p))m + O(\log m)$. This relation gives us a tight bound on $n$ when $p$ is relatively far from 1/2. Unfortunately these inequalities give us little information under the unbounded-error setting or when $p$ is very close to 1/2, because the value of $(1 - H(p))m$ become less than one. Hayashi et al. [13] showed that $(m, n, p)$-QRAC with $p > 1/2$ does not exist when $m = 2^{2n}$. Our characterization directly shows that this is tight, that is, $(m, n, > 1/2)$-QRAC exists if and only if $m \leq 2^{2n} - 1$, which solves the remained open problem in [13]. A similar tight result on the existence of $(m, n, > 1/2)$-RAC is also obtained from our characterization. Moreover, we also give concrete constructions of such QRAC and RAC with an analysis of their success probability.

1.2 Related Work

We mainly focus on the gap between classical and quantum communication complexities.

**Partial/Total Boolean Functions.** For total functions, the one-way quantum communication complexity is nicely characterized or bounded below in several ways. Klauck [16] characterized the one-way communication complexity of total Boolean functions by the number of different rows of the communication matrix in the exact setting, i.e., the success probability is one, and showed that it equals to the one-way deterministic communication complexity. Also, he gave a lower bound of bounded-error one-way quantum communication complexity of total Boolean functions by the VC dimension. Aaronson [1, 2] presented lower bounds of the one-way quantum communication complexity that are also applicable for partial Boolean functions. His lower bounds are given in terms of the deterministic or bounded-error classical communication complexity and the length of Bob’s input, which are shown to be tight by using the partial Boolean function of Gavinsky et al. [11].

**One-way/Two-way/SMP Models.** Two-way communication model is also popular. It is known that the two-way communication complexity has a non-linear quantum/classical gap for total functions in the bounded-error model. The current largest gap is quadratic. Buhrman, Cleve and Wigderson [6] showed that the almost quadratic gap, $O(\sqrt{n} \log n)$ vs. $\Omega(n)$, exists for the disjointness function. This gap was improved to $O(\sqrt{n})$ vs. $\Omega(n)$ in [3], which turned out to be optimal within a constant factor for the disjointness function [24]. On the contrary, in the unbounded-error setting, two-way communication model can be simulated by one-way model with only one bit additional communication [23]. In the simultaneous message passing (SMP) model where we have a referee other than Alice and Bob, an exponential quantum/classical gap for total functions was shown by Buhrman et al. [7].

**Private-coin/Public-coin Models.** The exponential quantum/classical separations in [5] and [11] still hold under the public-coin model where Alice and Bob share random coins, since the one-way classical public-coin model can be simulated by the one-way classical private-coin model with additional $O(\log n)$-bit communication [19]. However, exponential quantum/classical separation for total functions remains open for all of the bounded-error two-way, one-way and SMP models. Note that the public-coin model is too powerful in the unbounded-error model: we can easily see that the unbounded-error one-way (classical or quantum)
communication complexity of any function (or relation) is 1 with prior shared randomness.

Unbounded-error Models. Since the seminal paper \cite{23}, the unbounded-error (classical) one-way communication complexity has been developed in the literature \cite{8, 9, 10}. (Note that in the classical setting, the difference of communication cost between one-way and two-way models is at most 1 bit.) Klauck \cite{17} also studied a variant of the unbounded-error quantum and classical communication complexity, called the weakly unbounded-error communication complexity: the cost is communication (qu)bits plus \log \frac{1}{\epsilon} where \frac{1}{2} + \epsilon is the success probability. He characterized the discrepancy, a useful measure for bounded-error communication complexity \cite{18}, in terms of the weakly unbounded-error communication complexity.

2 Preliminaries

For basic notations of quantum computing, see \cite{22}. In this paper, a “function” represents both total and partial Boolean functions.

Communication Complexity. The two-party communication complexity model is defined as follows. One party, say Alice, has input \(x\) from a finite set \(X\) and another party, say Bob, input \(y\) from a finite set \(Y\). One of them, say Bob, wants to compute the value \(f(x, y)\) for a function \(f\). (In some cases, relations are considered instead of functions.) Their communication process is called a quantum (resp. classical) protocol if the communication is done by using quantum bits (resp. classical bits). In particular, the protocol is called one-way if the communication is only from Alice to Bob. The communication cost of the protocol is the maximum number of (qu)bits needed over all \((x, y)\in X \times Y\) by the protocol. The unbounded-error one-way quantum (resp. classical) communication complexity of \(f\), denoted by \(Q(f)\) (resp. \(C(f)\)), is the communication cost of the best one-way quantum (resp. classical) protocol with success probability strictly larger than \(1/2\). In what follows, the term “classical” is often omitted when it is clear from the context. We denote the communication matrix of \(f\) by \(M_f = ((-1)^{f(x,y)})\). (We use the bold font letters for denoting vectors and matrices.)

Arrangements. The notion of arrangement has often been used as one of the basic concepts in computer science such as computational geometry and learning theory. The arrangement of points and hyperplanes has two well-studied measures: the minimum dimension and margin complexity. We use the former, as in \cite{23}, to characterize the unbounded-error one-way communication complexity (while the latter was used in \cite{12} to give a lower bound of bounded-error one-way quantum communication complexity under prior shared entanglement). A point in \(\mathbb{R}^n\) is denoted by the corresponding \(n\)-dimensional real vector. Also, a hyperplane \(\{(a_i) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i h_i = h_{n+1}\}\) on \(\mathbb{R}^n\) is denoted by the \((n+1)\)-dimensional real vector \(h = (h_1, \ldots, h_n, h_{n+1})\), meaning that any \((a_i)\) on the plane satisfies the equation \(\sum_{i=1}^{n} a_i h_i = h_{n+1}\). A \{\(-1\)\}-valued matrix \(M\) on \(X \times Y\) is realizable by an arrangement of a set of \(|X|\) points \(p_x = (p_x^1, \ldots, p_x^k)\) and a set of \(|Y|\) hyperplanes \(h_y = (h^1_y, \ldots, h^k_y, h_{k+1})\) in \(\mathbb{R}^k\) if for any \(x \in X\) and \(y \in Y\), \(\delta(p_x, h_y) := \text{sign}(\sum_{i=1}^{k} p_x^i h^i_y - h^i_{k+1})\) is equal to \(M(x, y)\). Here, \(\text{sign}(a) = 1\) if \(a > 0\), \(-1\) if \(a < 0\), and \(0\) otherwise. Intuitively, the point lies above, below, or on the plane if \(\delta(p_x, h_y) = 1, -1, \text{ and } 0\) respectively. The value \(k\) is called the dimension of the arrangement. Let \(k_M\) denote the smallest dimension of all arrangements that realize \(M\). In particular, if \(M = M_f\) then we denote \(k_M\) by \(k_f\), and say that \(f\) is realized by the arrangement.

Bloch Vector Representations of Quantum States. Mathematically, the \(N\)-level quantum state is represented by an \(N \times N\) positive matrix \(\rho\) satisfying \(\text{Tr}(\rho) = 1\). (Note that if \(N = 2^n\) then \(\rho\) is considered as a quantum state that consists of \(n\) qubits.) In this paper we use \(N \times N\) matrices \(I_N, \lambda_1, \ldots, \lambda_{N^2-1}\), called generator matrices, as a basis to represent \(N\)-level quantum states. Here, \(I_N\) is the identity matrix (the subscript \(N\) is often omitted), and \(\lambda_i\)'s are the generators of \(SU(N)\) satisfying (i) \(\lambda_i = \lambda_i^\dagger\), (ii) \(\text{Tr}(\lambda_i) = 0\) and (iii) \(\text{Tr}(\lambda_i \lambda_j) = 2 \delta_{ij}\). Then, the following lemma is known (see, e.g., \cite{15}).

**Lemma 2.1** For any \(N\)-level quantum state \(\rho\) and any \(N \times N\) generator matrices \(\lambda_i\)'s, there exists an \((N^2 - 1)\)-dimensional vector \(r = (r_i)\) such that \(\rho\) can be written as

\[
\rho = \frac{1}{N} \left( I + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{N^2-1} r_i \lambda_i \right).
\]

The vector \(r\) in this lemma is often called the Bloch vector of \(\rho\). Note that \(\lambda_i\) can be any generator matrices satisfying the above conditions. In particular, it is well-known \cite{22} that for \(N = 2\) one can choose
\[ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \] of Pauli matrices as \( \lambda_1, \lambda_2, \text{ and } \lambda_3 \), respectively. Generally for \( N = 2^n \), one can choose the tensor products of Pauli matrices, including \( I \), for \( \lambda_1, \ldots, \lambda_{N^2-1} \).

Note that Lemma 2.1 is a necessary condition for \( \rho \) to be a quantum state. Although our knowledge of the sufficient condition is relatively weak (say, see \[13\] \[15\]), the following lemma, shown similarly as Lemma 2.1, is a necessary condition for and the corresponding POVM over \( \text{Tr}(N) \) there exists an \( r \) constant factor to obtain \( r \) by Lemma 3.4 we can simply multiply \( r \) to obtain \( \rho \) is a quantum state if and only if \( r \leq \sqrt{\frac{2}{N(N-1)}} \left( \frac{1}{m(A)} \right) \), where \( m(A) \) denotes the minimum of eigenvalues of a matrix \( A \), and \( \lambda \)'s are any generator matrices.

**Lemma 2.2** \([15]\) Let \( r = \sqrt{\sum_{i=1}^{N^2-1} r_i^2} \). Then, \( \rho = \frac{1}{N} \left( I + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{N^2-1} r_i \lambda_i \right) \) is a quantum state for \( \rho \leq \frac{1}{N} \left( I + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{N^2-1} r_i \lambda_i \right) \), where \( m(A) \) denotes the minimum of eigenvalues of a matrix \( A \), and \( \lambda \)'s are any generator matrices.

**Lemma 2.3** \([14]\) Let \( B(\mathbb{R}^{N^2-1}) \) be the set of Bloch vectors of all \( N \)-level quantum states. Let \( D_{r_s}(\mathbb{R}^{N^2-1}) = \{ r \in \mathbb{R}^{N^2-1} \mid r \leq \frac{1}{N-1} \} \) (called the small ball), and \( D_{r_s}(\mathbb{R}^{N^2-1}) = \{ r \in \mathbb{R}^{N^2-1} \mid r \leq 1 \} \) (called the large ball). Then, \( D_{r_s}(\mathbb{R}^{N^2-1}) \subseteq B(\mathbb{R}^{N^2-1}) \subseteq D_{r_s}(\mathbb{R}^{N^2-1}) \).

### 3 Quantum Tight Bound

In \([13]\), we gave a geometric view of the quantum protocol on random access coding. It turns out that this view together with the notion of arrangements is a powerful tool for characterizing the unbounded-error one-way quantum communication complexity.

**Theorem 3.1** \( Q(f) = \lceil \log(k_f + 1)/2 \rceil \) for every function \( f : X \times Y \rightarrow \{0, 1\} \).

The outline of the proof is as follows: In Lemma 3.2 we first establish a relation similar to Lemma 2.1 between a POVM (Positive Operator-Valued Measure) \( \{E, I - E\} \) over \( n \) qubits and a \((2^{2n} - 1)\)-dimensional (Bloch) vector \( h(E) \). Then, we prepare Lemma 3.3 to show that the arrangement results of POVM \( \{E, I - E\} \) on a state \( \rho \) correspond to the arrangement operation \( \delta(r(\rho), h(E)) \), where \( r(\rho) \) is the Bloch vector for \( \rho \).

Now in order to prove \( Q(f) \geq \lceil \log(k_f + 1)/2 \rceil \), suppose that there is a protocol whose communication complexity is \( n \). This means for any \( x \in X \) and \( y \in Y \), we have \( n \)-qubit states \( \rho_x \) and POVMs \( \{E_y, I - E_y\} \) such that: (i) the dimensions of \( r(\rho_x) \) and \( h(E_y) \) are \( 2^{2n} - 1 \) and \( 2^{2n} \) (by Lemmas 2.1 and 3.2 and note that \( N = 2^n \)), and (ii) \( M_f(x, y) = \text{sign}(\text{Tr}(E_y \rho_x)) \) (the first equality by the assumption and the second one by Lemma 3.3. By (ii) we can conclude that the arrangement of points \( r(\rho_x) \) and hyperplanes \( h(E_y) \) realize \( f \), and by (i) its dimension is \( 2^{2n} - 1 \). Thus, \( k_f \) is at most \( 2^{2n} - 1 \), implying that \( n \) (or \( Q(f) \)) \( \geq \lceil \log(k_f + 1)/2 \rceil \).

To prove the converse, suppose that there exists an \((N^2 - 1)\)-dimensional arrangement of points \( r_x \) and hyperplanes \( h_y \) realizing \( f \). For simplicity, suppose that \( N^2 - 1 = k_f \) (see the proof of Theorem 3.1 for the details). Let us fix some generator matrices \( \lambda \)'s. However, \( \rho_x \) obtained directly from \( \lambda \)'s and \( r_x \) by Eq. 1 may not be a valid quantum state. Fortunately, by Lemma 3.2 we can simply multiply \( r_x \) by a fixed constant factor to obtain \( r_x' \) such that \( r_x' \) lies in the small ball in Lemma 2.3 and therefore corresponds to an \( n \)-qubit state \( \rho(r_x') \). Similarly, by Lemma 3.3 we can get \( h_y' \) corresponding to POVM \( \{E(h_y'), I - E(h_y')\} \) (called the small ball), and \( D_{r_s}(\mathbb{R}^{N^2-1}) \) are an \( N \)-level (or \( \lceil \log N \rceil \)-qubit) quantum state and a POVM over \( N \)-level quantum states, respectively. Now, by Lemma 3.3 we can compute \( f(x, y) \) by \( \text{sign}(\text{Tr}(E(h_y') \rho(r_x'))) \) (the first equality by the assumption and the second one by Lemma 3.3). By (ii) we can conclude that the arrangement of points \( r(\rho_x) \) and hyperplanes \( h(E_y) \) realizes \( f \), its dimension is the same \( N^2 - 1 \) and the corresponding \( \rho(r_x') \) and \( \{E(h_y'), I - E(h_y')\} \) are an \( N \)-level (or \( \lceil \log N \rceil \)-qubit) quantum state and a POVM over \( N \)-level quantum states, respectively. Now, by Lemma 3.3 we can compute \( f(x, y) \) by \( \text{sign}(\text{Tr}(E(h_y') \rho(r_x'))) \) (the first equality by the assumption and the second one by Lemma 3.3).
\{E, I - E\} with their Bloch vectors.

**Lemma 3.3** Let \( r = (r_i) \in \mathbb{R}^{N^2-1} \) and \( e = (e_i) \in \mathbb{R}^{N^2} \) be the Bloch vectors of an \( N\)-level quantum state \( \rho \) and a POVM \( \{E, I - E\} \). Then, the probability that the measurement value 0 is obtained is

\[
\text{Tr}(E\rho) = e_{N^2} + \sqrt{\frac{2(N-1)}{N}} \sum_{i=1}^{N^2-1} r_i e_i.
\]

The last two lemmas provide a shrink-and-shift mapping from any real vectors and hyperplanes to, respectively, Bloch vectors of quantum states lying in the small ball of Lemma 2.3 and POVMs.

**Lemma 3.4** (1) For any \( r = (r_1, r_2, \ldots, r_k) \in \mathbb{R}^k \) and \( N \) satisfying \( N^2 \geq k + 1 \),

\[
\rho(r) = \frac{1}{N} \left(I + \sqrt{\frac{N(N-1)}{2}} \sum_{i=1}^{k} \left(\frac{r_i}{|r|(N-1)}\right) h_i\right)
\]

is an \( N\)-level quantum state.

(2) If \( \rho(r) \) is a quantum state, then \( \rho(\gamma r) \) is also a quantum state for any \( \gamma \leq 1 \).

**Lemma 3.5** For any hyperplane \( h = (h_1, \ldots, h_k, h_{k+1}) \in \mathbb{R}^{k+1} \), let \( N \) be any number such that \( N^2 \geq k + 1 \), and let \( \alpha, \beta \) be two positive numbers that are at most \( 2^\frac{1}{N^2-k} \), then, the N\(^2\)-dimensional vector defined by \( h(\alpha, \beta) = (\beta h_1, \ldots, \beta h_k, 0, \ldots, 0, 1/2 - \alpha h_{k+1}) \) is the Bloch vector of a POVM \( \{E_0, E_1\} \) over \( N\)-level quantum states, where \( E_0 \) and \( E_1 \) are given as

\[
E_0 = \left(\frac{1}{2} - \alpha h_{k+1}\right) I + \beta \sum_{i=1}^{k} h_i \lambda_i \quad \text{and} \quad E_1 = \left(\frac{1}{2} + \alpha h_{k+1}\right) I - \beta \sum_{i=1}^{k} h_i \lambda_i,
\]

(2)

Now we prove our main theorem in this section.

**Proof of Theorem 3.1** \( k_f \) is simply written as \( k \) in this proof.

\( (Q(f) \geq \lfloor \log(k + 1/2) \rfloor \). Let \( n = Q(f) \) and \( N = 2^n \). Assume that there is an \( n\)-qubit protocol for \( f \). That is, Alice on input \( x \) sends an \( n\)-qubit state \( \rho_x \) to Bob with input \( y \). He then measures \( \rho_x \) with a POVM \( \{E_y, I - E_y\} \) so that \( \text{sign}(\text{Tr}(E_y\rho_x) - 1/2) = M_f(x, y) \). From Lemmas 2.4 and 3.2 we can define the points \( p_x = (p_x^y) \in \mathbb{R}^{N^2-1} \) and hyperplanes \( h_y = (h_y^x) \in \mathbb{R}^N \) so that \( p_x \) is the Bloch vector of \( \rho_x \), and \( h_y = \left(\sqrt{\frac{2(N-1)}{N}}, \ldots, \sqrt{\frac{2(N-1)}{N}} \right) e_y^{x_1}, 1/2 - e_y^{x_N} \) where \( e_y = (e_y^x) \) is the Bloch vector of the POVM \( \{E_y, I - E_y\} \). Notice that by Lemma 3.3, \( \text{Tr}(E_y\rho_x) = e_{N^2} + \sqrt{\frac{2(N-1)}{N}} \sum_{i=1}^{N^2-1} p_i^x e_i^y \), which is \( > 1/2 \) if \( M_f(x, y) = 1 \) and \( < 1/2 \) if \( M_f(x, y) = -1 \) by assumption. Thus, we can see that

\[
\delta(p_x, h_y) = \text{sign} \left( e_{N^2} + \sqrt{\frac{2(N-1)}{N}} \sum_{i=1}^{N^2-1} p_i^x e_i^y - 1/2 \right) = M_f(x, y),
\]

meaning that there exists an arrangement of points and hyperplanes in \( \mathbb{R}^{N^2-1} \) which realizes \( f \). Thus, by definition, \( k \) is at most \( N^2 - 1 = 2^{2n} - 1 \) which implies \( Q(f) = n \geq \lfloor \log(k + 1/2) \rfloor \).

\( (Q(f) \leq \lfloor \log(k + 1/2) \rfloor \). Suppose that there is a \( k\)-dimensional arrangement of points \( p_x = (p_x^y) \in \mathbb{R}^k \) and hyperplanes \( h_y = (h_y^x) \in \mathbb{R}^{k+1} \) that realizes \( M_f \). That is, \( \delta(p_x, h_y) = M_f(x, y) \) for every \( (x, y) \in X \times Y \). By carefully shrinking-and-shifting this arrangement into Bloch vectors in the small ball, we will show the construction an \( n\)-qubit protocol for \( f \), that is, \( n\)-qubit states \( \rho_x \) for Alice and POVMs \( \{E_y, I - E_y\} \) for Bob with the smallest \( n \) satisfying \( k \leq 2^{2n} - 1 \), and hence obtain \( Q(f) \leq n = \lfloor \log(k + 1/2) \rfloor \).

Let \( \gamma_x = \min \left\{ \frac{1}{|p_x^y|\sqrt{2^{2n-1} - 1}}, \frac{1}{2^{2n-1}} \right\} \) for each \( x \in X \). Then, since \( (2^n)^2 \geq k + 1 \), Lemma 3.4 implies that \( \frac{1}{2} \left(I + \sqrt{\frac{2^{2n-2} - 1}{2}} \sum_{i=1}^{k} \gamma_x p_i^x \lambda_i \right) \) is an \( n\)-qubit state, and hence \( \gamma_x p_x \) is the Bloch vector of its qubit.
state. Moreover, Lemma 3.5 implies that by taking \( \beta_y = \frac{1}{2} \left( h_{k+1}^y + \sqrt{\sum_{i=1}^k (h_i^y)^2 + \sqrt{2(n-1)}} \right) \), \( h_y(\beta_y, \beta_y) = (\beta_y h_1^y, \ldots, \beta_y h_k^y, 0, \ldots, 0, 1/2 - \beta_y h_{k+1}^y) \) is the Bloch vector of a POVM over \( n \)-qubit states.

Now let \( \gamma = \frac{1}{\sqrt{\min_{x \in X} \gamma_x}} \), \( \beta = \min_y \gamma_y \), and \( \alpha = \sqrt{\frac{2(n-1)}{2^n}} \). Since \( \gamma \leq \gamma_x \) for any \( x \in X \) and \( 0 < \alpha < \beta \leq \beta_y \) for any \( y \in Y \), Lemmas 3.2 and 3.3 show that \( \gamma p_x \) and \( h_y(\beta, \alpha) \) are also the Bloch vectors of an \( n \)-qubit state \( \rho_x \) and a POVM \( \{ E_y, I - E_y \} \) over \( n \)-qubit states, respectively. By Lemma 3.3 the probability that the measurement value 0 is obtained is

\[
\begin{align*}
\text{Tr}(E_y \rho_x) &= \frac{1}{2} - \alpha h_{k+1}^y + \sqrt{\frac{2(n-1)}{2^n}} \gamma \sum_{i=1}^k p_i^y h_i^y = \frac{1}{2} + \alpha \left( \sum_{i=1}^k p_i^y h_i^y - h_{k+1}^y \right) \\
&= \begin{cases} 
> \frac{1}{2} \text{ if } M_f(x, y) = 1 \\
< \frac{1}{2} \text{ if } M_f(x, y) = -1,
\end{cases}
\end{align*}
\]

where the last inequality comes from the assumption. Therefore, the states \( \rho_x \) and POVMs \( \{ E_y, I - E_y \} \) can be used to obtain an \( n \)-qubit protocol for \( f \).

Combined with the results in [8, 10], Theorem 3.1 gives us a nontrivial bound for the inner product function \( IP_n \) (i.e., \( IP_n(x, y) = \sum_{i=1}^n x_i y_i \mod 2 \) for any \( x = x_1 \cdots x_n \in \{0, 1\}^n \) and \( y = y_1 \cdots y_n \in \{0, 1\}^n \)). Note that the bounded-error quantum communication complexity is at least \( n - O(1) \), and \( n/2 - O(1) \) even if we allow two-way protocol and prior entanglement [21].

**Corollary 3.6** \( [n/4] \leq Q(IP_n) \leq [(\log 3)n + 2)/4] \).

### 4 Classical Tight Bound

Paturi and Simon [23] shows that for every function \( f : X \times Y \rightarrow \{0, 1\} \), \( [\log k_f] \leq C(f) \leq [\log k_f] + 1 \). We remove this small gap as follows.

**Theorem 4.1** \( C(f) = [\log(k_f + 1)] \) for every function \( f : X \times Y \rightarrow \{0, 1\} \).

**Proof.** Let \( k = k_f \) in this proof.

\( (C(f) \geq [\log(k + 1)]) \). Let \( N = 2^{C(f)} \). Suppose that there is a \( C(f) \)-bit protocol for \( f \). Paturi and Simon (in Theorem 2 in [23]) gave an \( N \)-dimensional arrangement of points \( p_x = (p_i^x) \in \mathbb{R}^N \) and hyperplanes \( h_y = (h_1^y, \ldots, h_N^y, 1/2) \in \mathbb{R}^{N+1} \), that is, \( \delta(p_x, h_y) = M_f(x, y) \) for every \( (x, y) \in X \times Y \). Noting that the points \( p_x \) are probabilistic vectors satisfying \( \sum_{i=1}^N p_i = 1 \), we can reduce the dimension of the arrangement to \( N - 1 \). We define \( q_x = (q_i^x) \in \mathbb{R}^{N-1} \) and \( h_y = (h_1^y, \ldots, h_N^y, 1/2 - h_{N+1}^y) \). From the assumption and \( p_i^x = 1 - \sum_{i=1}^{N-1} p_i^x \),

\[
\begin{align*}
\sum_{i=1}^{N-1} q_i^x l_i^y - l_N^y &= \sum_{i=1}^{N-1} p_i^x (h_i^y - h_N^y) - \frac{1}{2} + h_N^y = \sum_{i=1}^{N-1} p_i^x h_i^y - \frac{1}{2} + h_N^y - \sum_{i=1}^{N-1} p_i^x h_N^y \\
&= \sum_{i=1}^N p_i^x h_i^y - \frac{1}{2} \begin{cases} 
> 0 \text{ if } M_f(x, y) = 1 \\
< 0 \text{ if } M_f(x, y) = -1.
\end{cases}
\end{align*}
\]

Thus, \( \delta(q_x, h_y) = M_f(x, y) \) for every \( (x, y) \in X \times Y \). That is, \( M_f \) is realizable by the \( (N - 1) \)-dimensional arrangement of points \( q_x \) and hyperplanes \( h_y \). By definition, \( k \leq N - 1 = 2^{C(f)} - 1 \), which means that \( C(f) \geq [\log(k + 1)] \).

\( (C(f) \leq [\log(k + 1)]) \). The proof is also based on that of Theorem 2 of Paturi and Simon [23]. They showed the existence of a protocol where Alice (with input \( x \)) sends a probabilistic mixture of (at most) \( k + 2 \) different messages to Bob (with input \( y \)). In this proof we reduce the number of messages to \( k + 1 \). That is, we construct the following protocol using \( k + 1 \) different messages: Alice sends a message \( S_j \) with probability \( q_j^x \) where \( j \in [k + 1] \), and Bob outputs 0 with probability \( l_j^y \) upon receiving \( S_j \). Here, \( [n] := \{1, 2, \ldots, n\} \)
for any \( n \in \mathbb{N} \). We will show that the probability of Bob outputs 0, represented as \( \sum_{j=1}^{k+1} q_j^y p_j^x \), is > 1/2 if \( M_f(x,y) = 1 \) and < 1/2 if \( M_f(x,y) = -1 \).

Assume that there exists a \( k \)-dimensional arrangement of points \( p_x = (p_x^x) \in \mathbb{R}^k \) and hyperplanes \( h_y = (h_y^x) \in \mathbb{R}^{k+1} \) that realizes \( M_f \), that is, the function \( f(x,y) = M_f(x,y) \) for every \( (x,y) \in X \times Y \). Let \( s = \max_{x \in X} \max_{i \in [k]} |p_i^x|, \alpha_x = 1 + \sum_{i=1}^{k} (s + p_i^x) \) for each \( x \in X \), and \( \beta_y = \max(|h_1^y|, \ldots, |h_{k+1}^y|, |h_{k+1}^y + \sum_{i=1}^{k} h_i^y|) \) for each \( y \in Y \). Then, we define \( h_x = (h_x^i) \in \mathbb{R}^{k+1} \) and \( l_y = (l_y^i) \in \mathbb{R}^{k+1} \) by \( h_x = \left( s + p_1^x, \ldots, s + p_k^x, \frac{1}{\alpha_x}, \ldots, \frac{1}{\alpha_x} \right) \) and \( l_y = \left( \frac{1}{2} + \frac{h_1^y}{2\beta_y}, \ldots, \frac{1}{2} + \frac{h_{k+1}^y}{2\beta_y} - \frac{h_{k+1}^y + \sum_{i=1}^{k} h_i^y}{2\beta_y} \right) \). It can be easily checked that \( 0 \leq q_i^x \leq 1 \) for all \((x,i) \in X \times [k], \sum_{i=1}^{k+1} q_i^x = 1 \), and \( 0 \leq l_i^y \leq 1 \) for all \((y,i) \in Y \times [k+1] \). Moreover,

\[
\sum_{i=1}^{k+1} q_i^x l_i^y = \sum_{i=1}^{k} \left( \frac{s + p_i^x}{\alpha_x} \right) \left( \frac{1}{2} + \frac{h_i^y}{2\beta_y} \right) + \frac{1}{\alpha_x} \left( \frac{1}{2} - \frac{h_{k+1}^y + \sum_{i=1}^{k} h_i^y}{2\beta_y} \right)
\]

\[
= \frac{1}{2} + \frac{1}{2\alpha_x\beta_y} \left( \sum_{i=1}^{k} h_i^y p_i^x - h_{k+1}^y \right) = \left\{ \begin{array}{ll}
> 1/2 & \text{if } M_f(x,y) = 1 \\
< 1/2 & \text{if } M_f(x,y) = -1.
\end{array} \right.
\]

Hence, given a \( k \)-dimensional arrangement of points and hyperplanes realizing \( M_f \), we can construct a protocol using at most \( k + 1 \) different messages for \( f \). This means that \( C(f) \leq \lceil \log (k + 1) \rceil \). This completes the proof.

Now we obtain our main result in this paper.

**Theorem 4.2** For every function \( f : X \times Y \to \{0,1\} \), \( Q(f) = \lceil C(f)/2 \rceil \).

5 Applications to Random Access Coding

In this section we discuss the random access coding as an application of our characterizations of \( Q(f) \) and \( C(f) \). The concept of quantum random access coding (QRAC) and the classical random access coding (RAC) were introduced by Ambainis et al. [4]. The \((m,n,p)\)-QRAC (resp. \((m,n,p)\)-RAC) is an encoding of \( m \) bits using \( n \) qubits (resp. \( n \) bits) so that *any* one of the \( m \) bits can be obtained with probability at least \( p \). In fact, the function computed by the RAC (or QRAC) is known before as the index function in the context of communication complexity. It is denoted as \( INDEX_n(x,i) = x_i \) for any \( x \in \{0,1\}^n \) and \( i \in [n] \) (see [18]).

5.1 Existence of QRAC and RAC

First we use Theorems 3.1 and 4.1 to show the existence of RAC and QRAC. As seen in [23], the smallest dimension of arrangements realizing \( INDEX_n \) is \( n \). Thus, Theorem 4.1 gives us the following corollary for its unbounded-error one-way quantum communication complexity.

**Corollary 5.1** \( Q(INDEX_n) = \lceil \log (n + 1) \rceil / 2 \).

Similarly, Theorem 4.1 gives its classical counterpart, which is tighter than [23].

**Corollary 5.2** \( C(INDEX_n) = \lceil \log (n + 1) \rceil \).

Since random access coding is the same as \( INDEX_n \) as Boolean functions, the following tight results are obtained for the existence of random access coding.

**Corollary 5.3** \((2^{2n} - 1, n, > 1/2)\)-QRAC exists, but \((2^{2n}, n, > 1/2)\)-QRAC does not exist. Moreover, \((2^n - 1, n, > 1/2)\)-RAC exists, but \((2^n, n, > 1/2)\)-RAC does not exist.

Corollary 5.3 solves the open problem in [13] in its best possible form. It also implies the non-existence of \((2, 1, > 1/2)\)-RAC shown in [4]. Note that this fact does not come directly from the characterization of \( C(f) \) in [23].
5.2 Explicit Constructions of QRAC and RAC

In this subsection, we give an explicit construction of \((2^{2n} - 1, n, > 1/2)\)-QRAC and \((2^{n} - 1, n, > 1/2)\)-RAC that leads to a better success probability than what obtained from direct applications of Theorems 3.1 and 4.1. For the case of QRAC, the construction is based on the proof idea of Theorem 3.1 combined with the property of the index function. Their proofs are omitted due to space constraint.

**Theorem 5.4** For any \(n \geq 1\), there exists a \((2^{2n} - 1, n, p)\)-QRAC such that

\[
p \geq \frac{1}{2} + \frac{1}{2\sqrt{(2^n-1)(2^{2n}-1)}}.
\]

We can also obtain the upper bound of the success probability of \((2^{2n} - 1, n, p)\)-QRAC from the asymptotic bound by Ambainis et al. [4]: For any \((2^{2n} - 1, n, p > 1/2)\)-QRAC, \(p \leq \frac{1}{2} + \frac{\sqrt{(2n)^2}}{2^{2n-1}}\).

It remains open to close the gap between the lower bound, \(\approx 1/2 + \Omega(1/2^{1.5n})\), and the upper bound, \(\approx 1/2 + O(\sqrt{n}/2^n)\), of the success probability.

Similarly, for the case of RAC we have the following theorem.

**Theorem 5.5** There exists a \((2^{n} - 1, n, p)\)-RAC such that

\[
p \geq \frac{1}{2} + \frac{1}{2(2^n + 1 - 5)}.
\]

The success probability of \((2^{n} - 1, n, p)\)-RAC can also be bounded by the asymptotic bound in [4]: For any \((2^{n} - 1, n, p > 1/2)\)-QRAC, \(p \leq \frac{1}{2} + \sqrt{\frac{(2n)^2}{2^{2n-1}}}\).

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