Simultaneous Approximation of Conformal Mappings on Smooth Domains

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Abstract. Some estimates for the simultaneous polynomial approximation of the conformal mapping of the finite simple connected domain onto the disc in the complex plane and its derivatives are obtained. The approximation rate in dependence of the differential parameters of the considered smooth domain is estimated.

1. Introduction and Background

Let G be a finite region in the complex plane, bounded by rectifiable Jordan curve \( \Gamma := \partial G \) and \( \Delta := \text{ext} \Gamma \). Let also \( T := \{ w \in \mathbb{C} : |w| = 1 \} \), \( D := \text{int} T \). By the Riemann mapping theorem, there exists a unique conformal mapping \( w = \varphi_0(z) \) of \( G \) onto the disk \( D_{r_0} := \{ w \in \mathbb{C} : |w| < r_0 \} \), normalized by \( \varphi_0(z_0) = 0 \), \( \varphi'_0(z_0) = 1 \), where \( r_0 := r_0(z_0; G) \) is called the conformal radius of \( G \) with respect to \( z_0 \) and having the inverse mapping \( \psi_0 \).

Similarly \( w = \varphi(z) \) is conformal mapping of \( \Omega \) onto \( \Delta \) with normalizations \( \varphi(\infty) = \infty \) and \( \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0 \).

We denote by \( \psi \) the inverse mapping of \( \varphi \).

For an arbitrary analytic function \( f \) given on \( G \) we set

\[
\| f \|_{L^2(G)} := \left( \int \int_G |f(z)|^2 \, d\sigma z \right)^{\frac{1}{2}}
\]

where \( d\sigma z \) stands for area measure on \( G \).

If the function \( f \) has a continuous extension to \( \overline{G} \), we use also the uniform norm

\[
\| f \|_{C(\overline{G})} := \sup \{ |f(z)| : z \in \overline{G} \}.
\]

It is well known that the function \( w = \varphi_0(z) \) minimizes the integral \( \| f' \|_{L^2(G)} \) in the class of all analytic functions in \( G \), normalized by \( f(z_0) = 0 \), \( f'(z_0) = 1 \). Definition of the Bieberbach polynomials clearly, let \( \pi_n \) be the class of polynomials \( p_n(z) \) of degree at most \( n \) and satisfying the conditions \( p_n(z_0) = 0 \), \( p'_n(z_0) = 1 \). A polynomial \( \pi_n \in \varphi_n \) is called \( n \)-th Bieberbach polynomial for pair \((G, z_0)\) if it minimizes the norm \( \| f' \|_{L^2(G)} \) in the class \( \varphi_n \). It is easy to check that \( \pi_n \) also minimizes the norm \( \| \varphi_n - p_n' \|_{L^2(G)} \) in the class \( \varphi_n \).

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As follows from the results due to Farrel [11] and Markushevich [22], if \( G \) is a Carathéodory region, then \( \| \varphi_n^\varphi - \varphi_n^\varphi \|_{L^1(G)} \) tends to zero as \( n \) approaches infinity and so of the sequence \( \{ \varphi_n \} \) converges uniformly to \( \varphi_0 \) on compact subsets of \( G \).

The first study was done by Keldych [21] on the uniform convergence \( \{ \varphi_n (z) \} \) polynomials to the function \( \varphi_0 (z) \) in the closure of \( G \). He showed that if the boundary \( L \) of \( G \) is a smooth Jordan curve with bounded curvature, then for every small \( \varepsilon > 0 \) and \( \gamma = 1 - \varepsilon \) there exists a constant \( c (\varepsilon) \) independent of \( n \) such that

\[
\| \varphi_0 - \pi_n \|_{L^1(G)} := \max_{z \in \partial G} | \varphi_0 (z) - \pi_n (z) | \leq \frac{c (\varepsilon)}{n^\gamma},
\]

where \( \gamma \) depends on the geometric properties of the boundary \( \Gamma := \partial G \), holds for every natural number \( n \).

In [21] the author also constructed an example of a starlike region, bounded by piecewise analytic curve with one singular point, where Bieberbach polynomials diverge. Therefore, the uniform convergence of the sequence \( \{ \varphi_n (z) \} \) in \( G \) depends on the properties of \( \Gamma := \partial G \). Later, Keldych’s counterexample was given by Andrievskii and Pritsker [9], for more generalized region. Bisedes, its geometry is made more clearly than Keldych’s counterexample.

Furthermore, Mergelyan [23] showed that \( \gamma = \frac{1}{2} - \varepsilon \) for arbitrary small \( \varepsilon > 0 \), whenever \( \Gamma := \partial G \) is a smooth Jordan curve. Additionally, Mergelyan stated it as a conjecture that the exponent \( \gamma = \frac{1}{2} - \varepsilon \) in (1) could be replaced by \( \gamma = 1 - \varepsilon \). Mergelyan’s conjecture was proved for a smooth domain of bounded boundary rotation by Israfilov in [17].

A considerable progress in this area has been achieved by Mergelyan [23], Suetin [26], Simonenko [25], Wu [29], Andrievskii [7, 8], Gaier [12, 13], Abdullayev [1, 2, 4, 5], Israfilov [16, 17] and the others.

In the paper [23] S.N. Mergelyan also noted without an estimate the convergence of any derivatives of the Bieberbach polynomials \( \pi_n \) to the phenoma is also true for derivatives of conformal mapping function \( \varphi_0 \), so called the simultaneous approximation.

In our opinion, first results related on the simultaneous approximation of Bieberbach polynomials were obtained by Suetin [26] and Israfilov [20].

A smooth Jordan curve \( \Gamma \) is called Dini-smooth if \( \delta (s) \) be its tangent direction angle expressed via arclenght of \( \Gamma \), satisfying the condition

\[
\int_0^c \frac{\omega (\delta, u)}{u} du < \infty
\]

where \( \omega (\delta, u) \) is the modulus of continuity of \( \delta (s) \), for some \( c > 0 \).

**Definition 1.1.** [18] Let \( r = 0, 1, 2, ... , \alpha \in (0, 1] \) and \( \beta \in [0, \infty) \). If the tangent direction angle \( \delta \) of \( \Gamma \) fulfills

\[
\omega (\delta^r, \delta) \leq c \delta^\alpha \ln^\beta \left( \frac{4}{\delta} \right), \quad \delta \in (0, \pi]
\]

with a positive constant \( c \) independent of \( \delta \), then we say that the Jordan curve \( \Gamma \) belongs to the class \( C^{r, \alpha, \beta} \).

We also say that \( f \in C^{r, \alpha, \beta} \) if \( \omega (f, \delta) \leq c \delta^\alpha \ln^\beta \left( \frac{4}{\delta} \right), \quad \alpha \in (0, 1] , \beta \in [0, \infty) \) for some constant \( c \).

The class \( C^{r, \alpha, \beta} \) is generalization of the class \( B (\alpha, \beta) \), defined in [19]. In particular, the class \( C^{0, \alpha, \beta} \) coincides with \( B (\alpha, \beta) \) and the class \( C^{0, 0, \alpha} \), \( \alpha \in (0, 1) \), coincides with the class of Lyapunov curves.

The aim of this article, we study the estimation (1) for simultaneous approximation in domains with a subclass of smooth Jordan curves.

### 2. Main Results

We consider domains \( C^{r, \alpha, \beta} \) with \( r = 0, 1, 2, ... , 0 < \alpha \leq 1 \) and \( \beta \geq 0 \) in this section. The following main results contain estimates for the rates of uniform convergence of the derivatives of Bieberbach polynomials.
Theorem 2.1. Let $\Gamma \in C^{\alpha, \beta}$ with $r = 2, 3, \ldots$, $0 < \alpha < 1$ and $0 \leq \beta < r + \alpha - \frac{3}{2}$. If $1 \leq k \leq r - 1$, then

$$\|\phi_0^{(k)} - \pi_n^{(k)}\|_{C(\Gamma)} \leq \frac{c \ln(n)}{n^{r-a-k+\frac{1}{2}}}, \quad n \geq k$$

with some positive constant $c = c(r)$.

Corollary 2.2. Let $\Gamma \in C^{\alpha, 0}$ with $r = 2, 3, \ldots$ and $0 < \alpha < 1$. If $1 \leq k \leq r - 1$ then

$$\|\phi_0^{(k)} - \pi_n^{(k)}\|_{C(\Gamma)} \leq \frac{c}{n^{r-a-k+\frac{1}{2}}}, \quad n \geq k$$

with some positive constant $c = c(r)$.

Theorem 2.3. Let $\Gamma \in C^{\beta, \alpha}$ with $r = 3, 4, \ldots$, and $0 \leq \beta < r - \frac{5}{2}$. If $1 \leq k \leq r - 1$ then

$$\|\phi_0^{(k)} - \pi_n^{(k)}\|_{C(\Gamma)} \leq \frac{c \ln^{\beta+1}(n)}{n^{r-k+\frac{1}{2}}}, \quad n \geq k$$

with some positive constant $c = c(r)$.

Theorem 2.4. Let $\Gamma \in C^{\alpha, \beta}$ with $r = 1, 2, 3, \ldots$, $\frac{1}{2} < \alpha < 1$ and $0 \leq \beta < \alpha - \frac{1}{2}$. Then

$$\|\phi_0^{(r)} - \pi_n^{(r)}\|_{C(\Gamma)} \leq \frac{c \ln^\beta(n)}{n^{a+\frac{1}{2}}}, \quad n \geq r$$

with some positive constant $c = c(r)$.

3. Some Auxiliary Facts

The following some auxiliary results are given spaces $L^p(G)$ and $E^p(G)$ with $p > 1$, but in this work, our interest is focused on the Hilbert spaces $L^2(G)$ and $E^2(G)$. Throughout this paper $c, c_1, c_2, \ldots$ are positive constants which in general depend on $G$. By $L^p(G)$ and $E^p(G)$ we denote the set of all measurable complex valued functions such that $|f|^p$ is lebesgue integrable with respect to arclengt, and Simirnov class of analytic functions in $G$, respectively.

Each function $f \in E^p(G)$ has a non-tangential limit almost everywhere on $\Gamma$ and if we use the same notation for the non-tangential limit of $f$, then $f \in L^p(\Gamma)$.

For $p \geq 1$, $L^p(G)$ and $E^p(G)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(G)} = \|f\|_{L^p(G)} := \left( \int_{\Gamma} |f(z)|^p |dz| \right)^{\frac{1}{p}}.$$

For the further fundamental properties see [15, p.438-453]. For the mapping $\phi_0$ and a weight function $\omega$ defined on $\Gamma$ we set

$$\varepsilon_n(\phi_0) := \inf_{p_n} \|\phi_0 - p_n\|_{L^p(G)} \quad \text{and} \quad E_0^{\omega}(\phi_0, \omega) := \inf_{p_n} \|\phi_0 - p_n\|_{L^p_{\omega}(\Gamma)}.$$

where infimum is taken over all algebraic polynomials $p_n$ of degree at most $n$ and

$$L^p(\Gamma, \omega) := \left\{ f \in L^1(\Gamma) : \|f\|^p \omega \in L^1(\Gamma) \right\}, \quad E^p(G, \omega) := \left\{ f \in E^1(G) : f \in L^p(\Gamma, \omega) \right\}.$$
According to Dynkin’s result [10], in case of \( \omega := \|q^r\|^{-1} \) between the best approximation numbers \( \varepsilon_n(q^r) \) and \( E_n(q^r, \|q^r\|^{-1}) \), the following relation holds

\[
\varepsilon_n(q^r) \leq c n^{-\frac{1}{2}} E_n(q^r, \|q^r\|^{-1}).
\]

Let \( f \in E^p(G) \) and let

\[
\omega_p(f, \delta) = \sup_{\|\psi\| \leq \delta} \left\| (f \circ \psi)\left(e^{(i\theta + i\delta)}\right) - (f \circ \psi)\left(e^{i\theta}\right) \right\|_{L^p[0, 2\pi]} = \sup_{\|\psi\| \leq \delta} \left\{ \int_0^{2\pi} \left| (f \circ \psi)\left(e^{(i\theta + i\delta)}\right) - (f \circ \psi)\left(e^{i\theta}\right) \right|^p d\theta \right\}^{\frac{1}{p}}
\]

be the generalized modulus of continuity of \( f \). We use the following approximation theorem by polynomials in the Simirnov class \( E^p(G) \), \( 1 < p < \infty \).

**Theorem 3.1.** [6] Let \( k \in \mathbb{N} \) and \( f^{(k)} \in E^p(G) \) with \( 1 < p < \infty \). If \( \Gamma \) satisfies the condition (2), then for an arbitrary algebraic polynomial \( p_n(z, f) \) we have

\[
\|f - p_n(z, f)\|_{L^p(\Gamma)} \leq \frac{c}{n^k} \omega_p(f^{(k)}, \delta),
\]

where \( \omega_p(f^{(k)}, \delta) := \sup_{\|\psi\| \leq \delta} \left\| (f^{(k)} \circ \psi)\left(e^{(i\theta + i\delta)}\right) - (f^{(k)} \circ \psi)\left(e^{i\theta}\right) \right\|_{L^p[0, 2\pi]} \).

**Definition 3.2.** [24] A bounded Jordan region \( G \) is called a \( k-\)quasidisk, \( 0 \leq k < 1 \), if any conformal mapping \( \psi_0 \) can be extended to a \( K-\)quasiconformal, \( K = \frac{1+k}{1-k} \), homeomorphism of the plane \( \overline{\mathbb{C}} \) on \( \overline{\mathbb{C}} \). In that case the curve \( \Gamma := \partial G \) is called a \( K-\)quasicircle. The region \( G \) (curve \( \Gamma \)) is called a quasidisk (quasicircle), if it is \( k-\)quasidisk (\( k-\)quasicircle) with some \( 0 \leq k < 1 \).

**Theorem 3.3.** [3] Let \( G \) be a \( k-\)quasidisk, \( 0 \leq k < 1 \). Then for arbitrary \( q_n \in \varphi_n \) and any \( m = 0, 1, 2, \ldots \) we have

\[
\|q_n\|_{L^p(\Gamma)} \leq c n^{(m+\frac{1}{2})k^{(1+k)}} \|q_n\|_{L^p(\Gamma)},
\]

\( p > 1 \).

**Corollary 3.4.** Let \( \Gamma \in C^{\alpha, \beta} \), \( 0 < \alpha < 1, \beta \geq 0 \) and \( r = 1, 2, \ldots \). Then for arbitrary \( p \in \varphi_n \) we have

\[
\|q_n\|_{L^p(\Gamma)} \leq c n^r p_{\varphi_n}\|\psi|_{L^p(\Gamma)}.
\]

**Proof.** Since \( q_0 \) is a conformal mapping we can get \( k = 0 \) by taking \( K = 1 \) into account in Definition 3.2. Moreover substituting \( q_n = p_n, p = 2 \) and \( m = r - 1 \) into the Theorem 3.3, we easily obtain Corollary 3.4. \( \square \)

**Lemma 3.5.** [18] If \( \Gamma \in C^{\alpha, \beta} \), with \( r = 0, 1, 2, \ldots, \alpha \in (0, 1], \beta \in [0, \infty), \) then for \( \Phi^{(\alpha)}(\psi) := \psi^{(\alpha)}(\psi(\psi)) \), we have

\[
\omega_1(\Phi^{(\alpha)}(\psi), \delta) \leq \begin{cases} \cd \ln^\alpha \left( \frac{\delta}{\alpha} \right) & ; \text{if } 0 < \alpha < 1 \\ \cd \ln^{\beta+1} \left( \frac{\delta}{\beta} \right) & ; \text{if } \alpha = 1. \end{cases}
\]

**Lemma 3.6.** [27] Suppose that \( \sum_{k=1}^n a_k \) converges and \( s \) is the value of the series. If \( r_n := \frac{a_n}{a_{n+1}} \) is a decreasing sequence and \( r_{n+1} < 1 \), then

\[
0 \leq R_n = s - \sum_{k=1}^n a_k \leq \frac{a_{n+1}}{1 - r_{n+1}}.
\]
4. Proof of the Main Results

For the proofs of the main results we use a traditional method based on the extremal property of Bieberbach polynomials and also the inequality (7)

**Proof of Theorem 2.1.** Let $1 \leq k \leq r - 1$. Since $\pi_n \to q_0$, as $n \to \infty$, uniformly in $G$, for any $z \in G$, $n \in \mathbb{N}$ with $n \geq k$ and $2^l \leq n \leq 2^{l+1}$ we have

$$q_0(z) - \pi_n(z) = [\pi_{2^{l+1}}(z) - \pi_n(z)] + \sum_{m=j+1}^{\infty} [\pi_{2^{m+1}}(z) - \pi_{2^{m}}(z)]$$

and

$$q_0^{(k)}(z) - \pi_n^{(k)}(z) = [\pi_{2^{l+1}}^{(k)}(z) - \pi_n^{(k)}(z)] + \sum_{m=j+1}^{\infty} [\pi_{2^{m+1}}^{(k)}(z) - \pi_{2^{m}}^{(k)}(z)].$$

Therefore, the inequality

$$\|q_0^{(k)} - \pi_n^{(k)}\|_{C(\overline{G})} \leq \|q_0 - \pi_n\|_{L^2(G)} + \sum_{m=j+1}^{\infty} 2^{m+1} \|\pi_m^{(k)} - \pi_{m-1}^{(k)}\|_{L^2(G)}$$

holds. If we use Corollary 3.4, also we have

$$\|q_0^{(k)} - \pi_n^{(k)}\|_{C(\overline{G})} \leq c_2 2^{(j+1)} \|\pi_{2^{l+1}}^{(k)} - \pi_n^{(k)}\|_{L^2(G)} + c_2 \sum_{m=j+1}^{\infty} 2^{m+1} \|\pi_m^{(k)} - \pi_{m-1}^{(k)}\|_{L^2(G)}.$$  (8)

Setting

$$Q_n(z) := \int_{z_0}^{z} q_n(t) \, dt \quad \text{and} \quad t_n(z) := Q_n(z) + [1 - q_n(z_0)] (z - z_0)$$

for the polynomial $q_n$, best approximating $q_0$ in the norm $\|\cdot\|_{L^2(G)}$. We have $t_n(z_0) = 0$ and $t_n'(z_0) = 1$. Then,

$$\|q_0 - t_n^{(k)}\|_{L^2(G)} = \|q_0 - q_n - 1 + q_n(z_0)\|_{L^2(G)} \leq \|q_0 - q_n\|_{L^2(G)} + \|1 - q_n(z_0)\|_{L^2(G)}$$

$$= \epsilon_n(q_0) + \|1 - q_n(z_0)\|_{L^2(G)}.$$  (9)

Considering the inequality (see [14, p.4])

$$|f(z_0)| \leq \frac{\|f\|_{L^2(G)}}{\sqrt{\text{dist}(z_0, \Gamma)}}$$

which holds for every analytic function $f$ with $\|f\|_{L^2(G)} < \infty$, we can write $q_n - q_n$ instead of $f$, and we get

$$\|1 - q_n(z_0)\|_{L^2(G)} \leq \frac{\|q_n^{(k)} - q_n\|_{L^2(G)}}{\sqrt{\text{dist}(z_0, \Gamma)}} = c_3 \epsilon_n(q_n)_{L^2(G)}.$$  (9)

Using last inequality, the minimization property of the Bieberbach polynomials and substituting (7) into (9), we have

$$\|q_0^{(k)} - \pi_n^{(k)}\|_{L^2(G)} \leq \|q_0^{(k)} - t_n^{(k)}\|_{L^2(G)} \leq c_4 \epsilon_n(q_0)_{L^2(G)} \leq c_5 n^{-\frac{1}{2}} E_n^0 \left(\phi_{0, \phi_0^{-1}}\right)_{L^2(G)}.$$
Then for a natural number \( n \in \mathbb{N} \) with \( n \geq k \) and \( 2^j \leq n \leq 2^{j+1} \), by applying Theorem 3.1 for \( \varphi_0' \) we have
\[
\|\pi_{2^{j+1}} - \pi_n'\|_{L^2(\Gamma)} \leq \|\pi_{2^{j+1}} - \varphi_0'\|_{L^2(\Gamma)} + \|\varphi_0' - \pi_n'\|_{L^2(\Gamma)} \leq c_6 \varepsilon_n(\varphi_0') + c_5 \varepsilon_n(\varphi_0') \\leq c_7 \varepsilon_n(\varphi_0') \leq c_8 n^{-\frac{1}{2}} \mathcal{E}_n(\varphi_0') (\varphi' |\varphi'|^{-1}) \|_{2} \leq c_9 n^{-\frac{1}{2}} \|p_n(z, \varphi_0') - \varphi_0'\|_{L^2(\Gamma, |\varphi'|^{-1})}.
\]

Using these estimations in (8) and Lemma 3.6 we obtain the required estimation
\[
\|\varphi_0^{(k)} - \pi_n^{(k)}\|_{C(\overline{\Gamma})} \leq c_{16} 2^{(j+1)k} \ln^k(n) + c_{17} \sum_{m=j+1}^{\infty} \frac{2^{(m+1)k} \ln^k(2m)}{2^{m(\tau + \frac{a}{2})}} + c_{18} \frac{\ln^k(n)}{n^{\tau + a - \frac{1}{2}}} + c_{19} \sum_{m=j+1}^{\infty} \frac{\ln^k(2m)}{2^{m(\tau + a - \frac{1}{2})}} \leq c_{20} \frac{\ln^k(n)}{n^{\tau + a - \frac{1}{2}}}.
\]

Thus the proof of Theorem 2.1 is completed.

**Proof of Theorem 2.3.** As in the case of of Theorem 2.1, we obtain the following estimations
\[
\|\pi_{2^{j+1}} - \pi_n'\|_{L^2(\Gamma)} \leq \frac{c_{21} \ln^{\beta+1}(n)}{n^{\tau - \frac{1}{2}}} , \quad \|\pi_{2^{j+1}} - \pi_n'\|_{L^2(\Gamma)} \leq \frac{c_{22} \ln^{\beta+1}(2^j)}{2^{(\tau - \frac{1}{2})}}. \quad (10)
\]

Combining (8), (10) and lemma 3.6 we have
\[
\|\varphi_0^{(k)} - \pi_n^{(k)}\|_{C(\overline{\Gamma})} \leq \frac{c_{23} \ln^{\beta+1}(n)}{n^{\tau - \frac{1}{2}}},
\]
This gives the desired inequality.

**Proof of Theorem 2.4.** The proof of Theorem 2.4 is similar to that of Theorem 2.1.
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