This presents an elementary discussion generalizing a Weyl type geometry to allow quaternion valued gauge transformations. Classical Yang-Mills fields are a result. Most of this material is condensed from earlier paper arXiv:1101.3606 which includes many additional topics not needed for this simple introduction to quaternionic Weyl type geometries with Yang-Mills Fields. This development will assume that the symmetric metric tensor is real in some gauge, and will develop the left and right handed approaches to quaternion valued gauge transformations. It will also utilize natural definitions of genuinely gauge invariant, dimensionless variables suitable for physics, and for use as a general formalism to describe these geometries, including General Relativity, in rather general circumstances.

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I. INTRODUCTION

This work introduces a quaternion valued generalization of Weyl-like geometries which can include General Relativity in a natural combination with quaternionic, classical Yang-Mills fields. Since quaternions have four components, and conventional SU(2) Yang-Mills fields operate with three\cite{2}, this is a more general Yang-Mills structure which actually contains the more standard SU(2) case as the subset with no real component.

Exposition will be elementary such that anyone who can understand the original works of Weyl\cite{3, 4} and Eddington\cite{5}, the basics of classical Yang-Mills Fields\cite{2}, and elementary quaternions\cite{6, 7} can follow this. Moreover, defining simple, gauge invariant, dimensionless variables will make it possible to carry through this program relatively simply in the quaternions where otherwise, the division into right and left handed expressions, and the presence of noncommuting quantities produce cumbersome expressions.

In all expressions, unless noted otherwise, the partial derivative with respect to coordinate $x^\mu$ is simply denoted by $\partial^\mu$, and $x^\mu$ itself will be considered to be dimensionless through the introduction of a universal scale factor $b_0$, which relates natural dimensionless Lorentzian coordinates $x^\mu$ to Lorentzian lab coordinates $x^\mu_{LAB}$ via $x^\mu = \sqrt{b_0} x^\mu_{LAB}$. The constant $b_0$ is not initially assigned a specific value but is assumed to be definite. Ideally, an obvious value would eventually emerge through comparison of equations to physics. Other quantities will also be assumed to be dimensionless by use of similar devices. This is illustrated later.

II. QUATERNIONIC GAUGES AND CURVATURES IN WEYL-LIKE GEOMETRIES

In order to make sense of Weyl-like geometries allowing quaternionic gauge transformations, one foundation rule is necessary. There must exist a gauge or gauges in which the metric tensor is real (with signature $(\mp\pm\mp\mp)$). Such a metric is considered to be a given. That simplifies the model enough to proceed fairly easily.

A. Quaternions

In this presentation, the basic quaternions will be taken as abstract mathematical objects similar to the number 1 and the imaginary unit $i$ in complex numbers. Specifically, they are taken to be the four quantities $Q_\mu$, where the subscript does not imply the quantities are a four vector, and where

$$Q_0 = \sigma_0$$

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and

\[ Q_k = - i \sigma_k \]  

(2)

for \( k = 1, 2, 3 \), where the \( \sigma_k \) are the standard Pauli spin matrices\(^8\), and \( \sigma_0 \) is the unit matrix. The basic properties of quaternions are reviewed in many references, such as Adler\(^6\), and Morse and Feshbach\(^7\), and there are many representations of them which may differ from those of equations \( 1 \) and \( 2 \), yet which are algebraically isomorphic to those quantities. Such possibly isomorphic representations will be denoted here by \( Q'_\mu \) if needed, and they could be given particular coordinate transformation properties for convenience, unlike the \( Q_\mu \) which are mathematical invariants.

The \( Q_k \) have an obvious vector form

\[ \vec{Q} = \sum_{k=1}^{3} \hat{e}_k Q_k \]  

(3)

where the \( \hat{e}_k \) are the Cartesian unit vectors. A completely analogous form exists for the \( Q'_k \),

\[ \vec{Q}' = \sum_{k=1}^{3} \hat{e}'_k Q'_k \]  

(4)

although now the \( \hat{e}'_k \) may be unit vectors in one of the curvilinear coordinate systems in common use\(^7\). A general quaternion \( A \) is given by \( A = Q_0 A_0 + \vec{Q} \cdot \vec{A} \) where \( A_0 \) is the real part, and \( \vec{A} \) is the imaginary part. These quaternion forms are called real quaternions because \( A_0 \) and \( \vec{A} \) are themselves real numbers.

### B. Basics of the Geometry

The basic Weyl-like geometry is understood to be based on a symmetric metric tensor and a Weyl four vector combined into a more general affine connection\(^3\)\(^5\). The reason for the term “Weyl-like” here in place of just “Weyl” will become apparent shortly, when it will be seen that the connection must include some torsion through an unbalancing of the original Weyl connection in order for the overall geometry to be nontrivial and reasonably neat in the quaternions.

In the gauge in which the metric is real, \( g_{\mu\nu} \) will be denoted by \( \tilde{g}_{\mu\nu} \). Tensor indices are lowered and raised using \( g_{\mu\nu} \) and its inverse.

Since \( g_{\mu\nu} \) is real in some gauge, it is always possible to write

\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} \gamma \]  

(5)

where \( \gamma \) is some quaternionic scalar. This allows the various operations with \( g_{\mu\nu} \) to be set up through easy correspondence with operations on real forms. For example, \( g'^{\mu\nu} \) is easily calculated through the standard requirement that

\[ g'^{\mu\nu} g_{\alpha\nu} = \delta^\mu_\nu \]  

(6)

where \( \delta^\mu_\nu \) is the Kronecker delta. In fact,

\[ g'^{\mu\nu} = \gamma^{-1} \tilde{g}^{\mu\nu} \]  

(7)

Furthermore, equations \( 5 \) and \( 7 \) give

\[ g'^{\alpha\tau} g_{\mu\nu} = \gamma^{-1} \tilde{g}^{\alpha\tau} \tilde{g}_{\mu\nu} \gamma \]

\[ = \tilde{g}^{\alpha\tau} \tilde{g}_{\mu\nu} \]

\[ = \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\tau} \]

\[ = g_{\mu\nu} \gamma \gamma^{-1} \tilde{g}^{\alpha\tau} \]

\[ = g_{\mu\nu} g^{\alpha\tau} \]  

(8)

That metric combination is always real, gauge invariant, and commutes as a unit with everything.
Since gauge transformations are to be quaternionic, they can generally be applied to \( g_{\mu \nu} \) from either the left or right, as denoted by

\[
\bar{g}_{\mu \nu} = \lambda g_{\mu \nu} \rho \tag{9}
\]

where \( \lambda \) is the left gauge transformation, and \( \rho \) is the right. However, more restricted cases where one of these two multipliers is always taken to be 1 will be most useful here. Since equation (9) can always be written as

\[
\bar{g}_{\mu \nu} = \lambda \tilde{g}_{\mu \nu} \rho \tag{10}
\]
according to equation (5), and since \( \tilde{g}_{\mu \nu} \) commutes with every scalar multiplier, restricting either \( \lambda \) or \( \rho \) to be 1 may not be a serious restriction in practice.

### C. Left and Right Covariant Derivatives / Christoffel Symbols

Since \( g_{\mu \nu} \) can now have a limited quaternionic nature, the expression

\[
g_{\mu \nu, \gamma} = g_{\mu \nu, \gamma} - g_{\alpha \nu} \{ \alpha \mu \} - g_{\mu \alpha} \{ \alpha \nu \} = 0 \tag{11}
\]
is not necessarily the same as

\[
g_{\mu \nu, \gamma} = g_{\mu \nu, \gamma} - \{ \alpha \mu \} g_{\alpha \nu} - \{ \alpha \nu \} g_{\mu \alpha} = 0 \tag{12}
\]

Thus, equation (11) defines the covariant derivative of \( g_{\mu \nu} \) with respect to the “right handed Christoffel Symbol” \( \{ \alpha \mu \} \), while equation (12) defines the covariant derivative of \( g_{\mu \nu} \) with respect to the “left handed Christoffel Symbol” \( [ \alpha \mu ] \). Both equations actually define the associated Christoffel Symbols, giving

\[
\{ \alpha \mu \} = \frac{1}{2} g^{\sigma \tau}(g_{\mu \sigma \tau, \nu} + g_{\nu \sigma \tau, \mu} - g_{\mu \nu, \sigma \tau}) \tag{13}
\]
and

\[
[ \alpha \mu ] = \frac{1}{2} (g_{\mu \sigma \tau, \nu} + g_{\nu \sigma \tau, \mu} - g_{\mu \nu, \sigma \tau}) g^{\sigma \tau} \tag{14}
\]
respectively.

Furthermore, equation (9) and \( \delta_{\mu \tau} = 0 \) give

\[
g^{\beta \mu, \tau} = -g^{\beta \mu} g_{\nu \alpha, \tau} g^{\alpha \nu} \tag{15}
\]
This and equations (8), (11), and (12) then give

\[
g^{\beta \xi, \tau} = g^{\beta \xi, \tau} + \{ \beta \alpha \} g^{\alpha \xi} - \{ \xi \alpha \} g^{\beta \alpha} = 0 \tag{16}
\]
and

\[
g^{\beta \xi, \tau} = g^{\beta \xi, \tau} + g^{\alpha \xi} [ \alpha \tau ] + g^{\beta \alpha} [ \xi \tau ] = 0 \tag{17}
\]
This indicates that the contravariant metric’s indices interact with their associated Christoffel symbols on the opposite side from the covariant metric’s indices in the definition of the covariant derivative of the metric. That same convention is now adopted as well here for the form of all covariant derivatives, and also more general affine derivatives, of any tensor quantity, where the affine derivative simply uses the full affine connections which also include the Weyl four vector \( R \Gamma_{\alpha \mu \nu} \) and \( L \Gamma_{\alpha \mu \nu} \) in place of the Christoffel Symbols \( \{ \alpha \mu \} \) and \( [ \alpha \mu ] \). The more general affine connections are prefixed with the lowered “\( R \)” and “\( L \)” to maintain different right and left handed forms on the more general level, just as there are right and left handed Christoffel Symbols.

For completeness, now define the “\( \tilde{\tau} \)” derivative as the reversal of the convention just given for the “\( \tau \)” derivative. That means that all the tensor - Christoffel Symbol positions are reversed for each term for each tensor index in the covariant derivative expression. For example,

\[
g_{\mu \nu, \gamma} = g_{\mu \nu, \gamma} - \{ \alpha \mu \} g_{\alpha \nu} - \{ \alpha \nu \} g_{\mu \alpha} ( \neq 0) \tag{18}
\]
By definition, the Christoffel Symbols are always defined using a “normal” covariant derivative of the metric tensor. Clearly, tensor - Christoffel Symbol positions in these expressions are determined both by the type of Christoffel Symbol (“{}” or “[]”), and the use of “;” or “;” in the derivative.

With these facts established, the “left handed” Christoffel Symbols and more general affine connection \( L_\mu^\rho \) will now be arbitrarily dropped, and the right handed cases used in what follows. However, it should also be noted that any action principle in this extended structure will eventually involve a quaternion conjugate (“QC”) term to keep the action real. That quaternion conjugate term will tend to involve left handed forms to balance the right handed forms that are now being chosen in the first part of the action. Because of that, the basic left hand should not be suppressed any action principle in this extended structure will eventually involve a quaternion conjugate (“QC”) term.

For general quaternionic \( M_\mu \) and \( N_\nu \),

\[
(M_\mu N_\nu)_{;\tau} \neq M_{\mu;\tau} N_\nu + M_{\mu} N_{\nu;\tau} \tag{19}
\]

Additionally, contraction on tensor indices inside an already evaluated covariant derivative will not necessarily equal the covariant derivative of the contracted quantity. Examples such as these limit the usefulness of these gauge varying, generalized covariant derivatives outside of gauges in which quantities commute easily in products. However, in other cases, these covariant derivatives will still be helpful, and will be used. On the other hand, any genuine physics in this structure will require gauge invariant constructions, including gauge invariant covariant derivatives. Those will be developed later, and they will involve real Christoffel Symbols that thus avoid these limitations just noted.

### D. Weyl-Like Connections, Gauge Properties, and Curvatures

Since right handed forms are now chosen, such as equation (13), specialize equation (9) to \( \lambda = 1 \), or

\[
\tilde{g}_{\mu\nu} = g_{\mu\nu} \rho \tag{20}
\]

Corresponding to that,

\[
\tilde{g}^{\mu\nu} = \rho^{-1} g^{\mu\nu} \tag{21}
\]

These together with equation (13) then give that

\[
\{ \alpha_{\mu\nu} \} = \rho^{-1} \{ \alpha_{\mu\nu} \} \rho + \frac{1}{2} \delta_\mu^\alpha \rho^{-1} \rho_{\nu\rho} + \frac{1}{2} \delta_\nu^\alpha \rho^{-1} \rho_{\mu\rho} - \frac{1}{2} \rho^{-1} g^{\alpha\tau} g_{\mu\nu} \rho_{\tau\rho} \\
= \rho^{-1} \{ \alpha_{\mu\nu} \} \rho + \frac{1}{2} \delta_\mu^\alpha \rho^{-1} \rho_{\nu\rho} + \frac{1}{2} \delta_\nu^\alpha \rho^{-1} \rho_{\mu\rho} - \frac{1}{2} g^{\alpha\tau} g_{\mu\nu} \rho^{-1} \rho_{\tau\rho} \tag{22}
\]

where the second line follows from equation (8).

Weyl’s original theory[3] would now suggest an affine connection \( R{\Gamma^{\alpha}_{\mu\nu}} \) (hereafter denoted by \( \Gamma^{\alpha}_{\mu\nu} \))

\[
\Gamma^{\alpha}_{\mu\nu} = \{ \alpha_{\mu\nu} \} + \delta_\mu^\alpha v_\nu + \delta_\nu^\alpha v_\mu - g^{\alpha\tau} g_{\mu\nu} v_\tau \tag{23}
\]

where \( v_\mu \) is the Weyl four vector, and

\[
\tilde{v}_\mu = \rho^{-1} (v_\mu - \frac{1}{2} \rho_{\nu\rho} \rho^{-1}) \rho \\
= \rho^{-1} v_\mu \rho - \frac{1}{2} \rho^{-1} \rho_{\nu\rho} \rho \tag{24}
\]

These give

\[
\tilde{\Gamma}^{\alpha}_{\mu\nu} = \rho^{-1} \Gamma^{\alpha}_{\mu\nu} \rho \tag{25}
\]

as the quaternionic analog of the gauge invariance of the Weyl Connection in his original geometry.

Equation (24) will indeed be adopted, but in place of equations (25) and (23), adopt

\[
\tilde{\Gamma}^{\alpha}_{\mu\nu} = \rho^{-1} \Gamma^{\alpha}_{\mu\nu} \rho + k \delta_\mu^\alpha \rho^{-1} \rho_{\nu\rho} \tag{26}
\]

where \( k \) is a constant, and

\[
\Gamma^{\alpha}_{\mu\nu} = \{ \alpha_{\mu\nu} \} + n \delta_\mu^\alpha v_\nu + \delta_\nu^\alpha v_\mu - g^{\alpha\tau} g_{\mu\nu} v_\tau \tag{27}
\]
where \( n \) is a constant, and

\[
k = (1 - n) / 2
\]  

(28)

For \( n \neq 1 \), some amount of torsion appears, and furthermore, \( n = 0 \) gives a vanishing affine derivative for the metric, a “metric compatible” case. Note that the cases \( n \neq 1 \) produce a form of Einstein’s “lambda transformation” of the connection\[9\] in equation \( (26) \), which as he notes, leaves the curvature tensor invariant when the lambda transformation involves real (or complex) quantities.

Now both of the connections \( \Gamma^\mu_{\mu\nu} \) and \( \{ \mu \nu \} \) have associated curvature tensors, but those have both a “right handed” and a “left handed” form themselves corresponding to use of normal or reversed covariant derivative conventions such as those used in “;” or “\( \tilde{;} \)”, irrespective of the right-left nature of the underlying connection used in them. Specifically,

\[
\begin{align*}
R\gamma B^\gamma_{\mu\tau\sigma} &= \Gamma^\gamma_{\mu\sigma,\tau} - \Gamma^\gamma_{\mu\tau,\sigma} + \Gamma^\gamma_{\eta\tau} \Gamma^n_{\nu\mu\sigma} - \Gamma^\gamma_{\eta\sigma} \Gamma^n_{\nu\mu\tau} \\
L\gamma B^\gamma_{\mu\tau\sigma} &= \Gamma^\gamma_{\mu\sigma,\tau} - \Gamma^\gamma_{\mu\tau,\sigma} + \Gamma^\gamma_{\mu\sigma} \Gamma^n_{\eta\tau} - \Gamma^\gamma_{\mu\tau} \Gamma^n_{\eta\sigma}
\end{align*}
\]  

(29)

(30)

\[
\begin{align*}
R\gamma^\mu_{\mu\tau\sigma} &= \{ \mu \nu \}_\tau - \{ \mu \rho \}_\tau + \{ \gamma \}_\tau \{ \mu \nu \} - \{ \gamma \}_\tau \{ \mu \rho \} \\
L\gamma^\mu_{\mu\tau\sigma} &= \{ \mu \nu \}_\tau - \{ \mu \rho \}_\tau + \{ \gamma \}_\tau \{ \mu \nu \} - \{ \gamma \}_\tau \{ \mu \rho \}
\end{align*}
\]  

(31)

(32)

When equation \( (26) \) is substituted into equations \( (29) \) and \( (30) \) to obtain the gauge properties of those curvature tensors, generally neither curvature form transforms in a particularly neat manner by itself. Surprisingly however, the combination

\[
B^\gamma_{\mu\tau\sigma} = [(k + 1) / (2k)] R\gamma B^\gamma_{\mu\tau\sigma} + [(k - 1) / (2k)] L\gamma B^\gamma_{\mu\tau\sigma}
\]  

(33)

does have neat gauge transformation properties (reverse roles of \( \rho \) and \( \lambda \) if left handed forms with \( \rho = 1 \) and \( \lambda \neq 1 \) are adopted initially rather than right). Specifically

\[
\bar{B}^\gamma_{\mu\tau\sigma} = \rho^{-1} B^\gamma_{\mu\tau\sigma} \rho
\]  

(34)

which is basically the same form as the gauge transformation of a Yang-Mills Field in SU(2) gauge theory\[2\].

Furthermore, for real (or complex) quantities instead of quaternionic quantities, products like \( \Gamma \) and \( \gamma \) commute internally, and \( B^\gamma_{\mu\tau\sigma} \) clearly reduces to the usual curvature tensor form, and becomes gauge invariant like Weyl’s curvature tensor\[3\]. Thus, it becomes the appropriate generalization of the curvature tensor in the quaternions. To facilitate its use in what follows, the coefficients in the definition of this tensor in equation \( (33) \) are given their own symbols,

\[
k_+ = (k + 1) / (2k)
\]  

(35)

and

\[
k_- = (k - 1) / (2k)
\]  

(36)

Note that \( k_+ + k_- = 1 \), and \( k_+ - k_- = 1/k \).

Now note that if one attempts to use a full Weyl connection analog by setting \( n = 1 \) in equation \( (26) \), that causes \( k \) to become zero in equation \( (26) \), and no suitable generalized curvature \( B^\gamma_{\mu\tau\sigma} \) emerges at all. Rather, as \( k \) approaches 0 in equation \( (33) \), the coefficients of \( R\gamma B^\gamma_{\mu\tau\sigma} \) and \( L\gamma B^\gamma_{\mu\tau\sigma} \) approach equal but opposite infinite values. Essentially, equation \( (33) \) must then be replaced by

\[
B^\gamma_{\mu\tau\sigma} = R\gamma B^\gamma_{\mu\tau\sigma} - L\gamma B^\gamma_{\mu\tau\sigma}
\]  

(37)

or any simple multiple of the right side of this equation, but that will lead to the disappearance of the derivatives of \( \Gamma^\mu_{\nu \rho} \) from the result. Without the derivative terms, this quantity cannot reduce to anything at all like the curvature tensor of a Weyl Geometry when quantities commute, reducing to zero instead. In other words, this structure actually discriminates against the exact quaternionic analog of Weyl’s original theory\[3\], and favors the cases which have some
torsion. This is one primary reason the original Weyl connection must be unbalanced in order to generalize to the quaternions. Clearly \( n \neq 1 \) is necessary for a nontrivial structure.

Finally, the four vector \( v_\mu \) of equation (24) has its own directly associated Yang-Mills field tensor. The gauge properties give that

\[
y_{\mu \nu} = v_{\nu, \mu} - v_{\mu, \nu} + 2(v_\nu v_\mu - v_\mu v_\nu)
\]

(38)
gauge transforms just as \( B_{\mu \tau \sigma}^\gamma \) does, or

\[
y_{\mu \nu} = \rho^{-1} y_{\mu \nu} \rho
\]

(39)

Comparison of the transformation rules of equation (24) and the corresponding equation in Guidry then relates his \( A_\mu \) to \( v_\mu \) via

\[
v_\mu = -ig^2 A_\mu
\]

(40)

where \( g \) is the coupling constant, and the \(-i\) is absorbed into \( \vec{\sigma} \) to produce the quaternion \( \vec{Q} \). Additionally, his \( U = \rho^{-1} \) to complete the matchup with his \( SU(2) \) Yang-Mills theory. As a bonus, one sees that the absorption of \((g/2)\) into \( A_\mu \) is what renders it dimensionless, and suitable for this structure.

E. The Makeup of the Curvature Tensor

Equation (27) can be written

\[
\Gamma^\alpha_{\mu \nu} = \{\alpha_{\mu \nu}\} + U^\alpha_{\mu \nu}
\]

(41)

where

\[
U^\alpha_{\mu \nu} = n \delta^\alpha_\mu v_\nu + \delta^\alpha_\nu v_\mu - g^{\alpha \tau} g_{\mu \nu} v_\tau
\]

(42)

Substituting these into equations (29) and (30) then gives the surprisingly neat results

\[
R^\gamma_{\mu \tau \sigma} = R^\gamma_{\mu \sigma \tau} - U^\gamma_{\mu \tau \sigma} + U^\gamma_{\mu \tau} U^\sigma_{\mu \sigma} - U^\gamma_{\nu \tau} U^\sigma_{\nu \sigma}
\]

(43)

and

\[
L^\gamma_{\mu \tau \sigma} = L^\gamma_{\mu \sigma \tau} - U^\gamma_{\mu \tau \sigma} + U^\gamma_{\mu \tau} U^\sigma_{\mu \sigma} - U^\gamma_{\nu \tau} U^\sigma_{\nu \sigma}
\]

(44)

These equations are one example (perhaps the best) in which the “ ; ” and “ ˜ ; ” covariant derivatives give results that are both compact, and express useful information.

However, in order to proceed further with the evaluation of \( B_{\mu \tau \sigma}^\gamma \) via equations (33), (43), and (44), the covariant derivatives should be written out as partial derivatives and Christoffel Symbol terms, and equation (42) should be substituted into the result. Furthermore, any resulting partial derivatives of \( g_{\mu \nu} \) or \( g^{\mu \nu} \) should be evaluated using equations (11) and (16) to substitute terms with Christoffel Symbols in place of the partial derivative terms. In practice, the combination \((g^{\gamma \eta} g_{\mu \sigma})_\tau \) always appears as a unit, and can be eliminated using

\[
(g^{\gamma \eta} g_{\mu \sigma})_\tau = g^{\gamma \eta} g_{\alpha \sigma}\{\alpha_\tau\} + g^{\gamma \eta} g_{\mu \alpha}\{\alpha_\tau\} - \{\gamma_\tau\} g^{\gamma \eta} g_{\mu \sigma} - \{\eta_\tau\} g^{\gamma \eta} g_{\mu \sigma}
\]

(45)

keeping equation (8) in mind for the result. The fact that the \( g^{\gamma \eta} g_{\mu \sigma} \) terms are real then allows equation (45) to have more than one valid form simply by varying the position of such terms in its products. However, the same form should consistently be chosen internally throughout evaluation of either one of the separate tensors in the pair \( R^\gamma_{\mu \tau \sigma} \) or \( L^\gamma_{\mu \tau \sigma} \) to avoid possibly encountering extraneous terms that should evaluate to zero with some effort, but are more easily avoided from the outset. Additionally, the full expression of equation (45) itself should be real, and could be moved around as a unit in products in its containing equation if necessary. However, all this flexibility leads to more than one expansion of \( B_{\mu \tau \sigma}^\gamma \) in gauge varying quantities like \( v_\mu \), although all the expansions are equivalent, and all will lead to the same, unique, gauge invariant result in what follows. Since the gauge invariant result contains any real physics, its uniqueness is what is important.
Thus equations (48) and (49) give

$$B_{\mu\tau\sigma} = k_+ R_{\mu\tau\sigma} + k_- L R_{\mu\tau\sigma}$$

$$+ (k_+ v_{\mu\tau} + k_- v_{\mu\tau}) \delta^\gamma_\sigma - (k_+ v_{\mu\sigma} + k_- v_{\mu\sigma}) \delta^\gamma_\tau$$

$$- (k_+ v_{\eta\tau} + k_- v_{\eta\tau}) \eta^\gamma g_{\mu\sigma} + (k_+ v_{\eta\sigma} + k_- v_{\eta\sigma}) \eta^\gamma g_{\mu\tau}$$

$$+ (1/k) [v_{\eta\gamma}, \{\gamma_{\alpha\beta}\}] \eta^{\alpha\beta} g_{\mu\sigma} - (1/k) [v_{\eta\gamma}, \{\gamma_{\alpha\beta}\}] \eta^{\alpha\beta} g_{\mu\tau}$$

$$+ (k_+ v_{\sigma\mu} + k_- v_{\sigma\mu}) \delta^\gamma_\tau - (k_+ v_{\tau\nu} + k_- v_{\tau\nu}) \delta^\gamma_\sigma$$

$$- \nu v_{\rho\beta} g^{\eta\beta} g_{\mu\sigma} \delta^\gamma_\tau + \nu v^{\rho\gamma} g^{\eta\beta} g_{\mu\tau} \delta^\gamma_\sigma$$

$$+ (k_+ v_{\nu\tau} + k_- v_{\nu\tau}) \eta^{\gamma\nu} g_{\mu\sigma} - (k_+ v_{\nu\sigma} + k_- v_{\nu\sigma}) \eta^{\gamma\nu} g_{\mu\tau}$$

$$+ n [k_+ (\nu_{\sigma\tau} - \nu_{\tau\sigma}) + k_- (\nu_{\tau\nu} - \nu_{\nu\tau})] \delta^\gamma_\nu$$

$$- (n/k) [v_{\nu\sigma}, \{\gamma_{\alpha\beta}\}] + (n/k) [v_{\nu\tau}, \{\gamma_{\alpha\beta}\}$$

$$- (n^2/k) (\nu_{\sigma\nu} - \nu_{\nu\sigma}) \delta^\gamma_\nu$$

$$- (n/k) (\nu_{\tau\nu} - \nu_{\nu\tau}) \delta^\gamma_\nu + (n/k) (\nu_{\mu\sigma} - \nu_{\sigma\mu}) \delta^\gamma_\tau$$

$$+ (n/k) (\nu_{\tau\nu} - \nu_{\nu\tau}) \eta^{\gamma\nu} g_{\mu\sigma} - (n/k) (\nu_{\sigma\nu} - \nu_{\nu\sigma}) \eta^{\gamma\nu} g_{\mu\tau}$$

(46)

where the “[,]” terms are conventional commutators. Those commutators will clearly vanish in gauges in which \{\gamma_{\alpha\beta}\} is real.

Now \(B_{\mu\tau\sigma}\) can be contracted to give

$$B_{\mu\tau} = B^{\omega}_{\mu\tau\omega}$$

(47)

and clearly equation (34) gives that

$$\bar{B}_{\mu\tau} = \rho^{-1} B_{\mu\tau} \rho$$

(48)

The similarity between this equation and equation (39) might then raise expectations that the antisymmetric part of \(B_{\mu\tau}\) will be proportional to \(y_{\mu\tau}\). However, this is generally not quite the case. A check reveals that the antisymmetric part of \(k_+ R^{\omega}_{\mu\tau\omega} + k_- L R^{\omega}_{\mu\tau\omega}\) equals \(\{1 - [1/(4k)]\} [\gamma^{-1}_{\mu\tau}, \gamma^{-1}_{\tau\sigma}]\) where \(\gamma\) is the gauge function in \(g_{\mu\nu} = \tilde{g}_{\mu\nu} \gamma\). Since this commutator does not generally vanish, then the antisymmetric part of \(k_+ R^{\omega}_{\mu\tau\omega} + k_- L R^{\omega}_{\mu\tau\omega}\) is generally not zero, and that antisymmetric component must be gauge balanced elsewhere by antisymmetric terms, even though it vanishes in gauges in which \(g_{\mu\nu}\) is real, and also when \(\gamma\) remains in the complex plane. However, the obvious exception is the case \(k = 1/4\), which causes the term to vanish even when the commutator is nonzero. That special value of \(k\) corresponds to \(n = 1/2\). The consequences of all this will become clearer in the next section where a simple method is developed to express contractions of the expansion of \(B^\gamma_{\mu\tau\sigma}\) given by equation (46).

Finally define the scalar curvature

$$B = B_{\mu\tau} g^{\mu\tau}$$

(49)

Since

$$\tilde{g}^{\mu\tau} = \rho^{-1} g^{\mu\tau}$$

(50)

then equations (48) and (49) give

$$\bar{B} = \rho^{-1} B$$

(51)

Thus \(B\) is the key quantity needed to define gauge invariant variables. As conceived by Weyl and Eddington, it is basically an intrinsic yardstick provided by the spacetime structure itself to reduce equations to dimensionless, gauge invariant quantities that can correspond to actual physics. For this purpose, it is assumed to be nonzero.

F. Gauge Invariant Variables and Their Fundamental Identity

Define

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} (B/C)$$

(52)
and its inverse
\[ \hat{g}^{\mu\nu} = CB^{-1}g^{\mu\nu} \]  

(53)

where \( C \) is a constant to be explained shortly. Using equations (20) and (51), these are seen to have the same value in all gauges. They are gauge invariant forms of the metric and its inverse. As such, they are immediately assumed to be real, giving a real metric tensor that can be used in physics. The constant \( C \) is necessary if all quantities are real, and the scalar curvature \( B < 0 \) [10][12]. Then in order to keep the signatures of \( g_{\mu\nu} \) and \( \hat{g}_{\mu\nu} \) from being opposite, clearly \( C = -1 \) should be used. Thus having established its right to exist, \( C \) is retained as a (dimensionless) constant in more general cases.

There is a gauge invariant \( \hat{\{\hat{\alpha}_{\mu\nu}\}} \) based on the real \( \hat{g}_{\mu\nu} \), and it is real,

\[
\{\hat{\alpha}_{\mu\nu}\} = \frac{1}{2} \hat{\gamma}^{\sigma\tau}(\hat{g}_{\mu\tau,\nu} + \hat{g}_{\nu\tau,\mu} - \hat{g}_{\mu\nu,\tau})
\]

(54)

without a right-left nature any longer. The covariant derivative with respect to it is indicated by \( \| \)”, and it is now quite well behaved, including obeying the product rule since \( \{\hat{\alpha}_{\mu}\} \) commutes with everything. Both the “;" and the “:” covariant derivative conventions will reduce to it. If equations (52) and (53) are substituted into equation (54), the result gives

\[
\{\hat{\alpha}_{\mu\nu}\} = B^{-1}\{\hat{\alpha}_{\mu}\}B + \frac{1}{2} \delta^\sigma_\mu B^{-1}B_{\nu,\sigma} + \frac{1}{2} \delta^\sigma_\nu B^{-1}B_{\mu,\sigma} - \frac{1}{2} B^{-1}\hat{\gamma}^{\sigma\tau}g_{\mu\nu,\tau}B_{\sigma
\]

(55)

These real, commuting Christoffel Symbols now give us a normal, real, gauge invariant Riemannian geometry on which we can impose a form of General Relativity. They define \( \hat{R}_{\mu\nu\tau} \) as the Riemann Curvature Tensor, and the conventions used to define \( R_{\mu\nu\tau\sigma} \) and \( L\hat{R}_{\mu\nu\tau\sigma} \) now both reduce to this same tensor. There is a (now symmetric) \( \hat{R}_{\mu\nu} = \hat{R}_{\nu\mu} \), and a scalar \( \hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} \).

The gauge invariant Weyl vector is

\[ \hat{\psi}_\mu = B^{-1}(v_\mu - \frac{1}{2} B_{\mu,\nu}B^{-1})B \]

(56)

which is fully quaternionic generally. Then in analogy to equation (27), define the gauge invariant

\[
\hat{\Gamma}^\alpha_{\mu\nu} = \{\hat{\alpha}_{\mu\nu}\} + n\delta^\alpha_\mu \hat{v}_\nu + \delta^\alpha_\nu \hat{v}_\mu - \hat{\gamma}^{\alpha\tau}g_{\mu\nu,\tau} \hat{v}_\tau
\]

(57)

Note that since \( \hat{\Gamma}^\alpha_{\mu\nu} \) is fully quaternionic, the full affine derivative of a quantity using \( \hat{\Gamma}^\alpha_{\mu\nu} \) is not as well behaved as the covariant derivative using only the real \( \{\hat{\alpha}_{\mu\nu}\} \).

Now substituting into equation (57) from equations (55) and (56), and using equation (27), one sees

\[ \hat{\Gamma}^\alpha_{\mu\nu} = B^{-1}\hat{\Gamma}^\alpha_{\mu\nu} B + k\delta^\alpha_\mu B^{-1}B_{\nu} \]

(58)

But this is exactly the same form as a gauge transformation on \( \hat{\Gamma}^\alpha_{\mu\nu} \) as defined in equation (26). Thus, if one defines \( R\hat{\tilde{B}}^\gamma_{\mu\tau\sigma} \) and \( L\hat{B}^\gamma_{\mu\tau\sigma} \) using \( \hat{\Gamma}^\alpha_{\mu\nu} \) in full analogy to the use of \( \Gamma^\alpha_{\mu\nu} \) in \( R\tilde{B}^\gamma_{\mu\tau\sigma} \) and \( L\tilde{B}^\gamma_{\mu\tau\sigma} \), the result gives finally that

\[
\hat{B}^\gamma_{\mu\tau\sigma} = k_+ R\hat{\tilde{B}}^\gamma_{\mu\tau\sigma} + k_- L\hat{B}^\gamma_{\mu\tau\sigma} - B^{-1}\hat{B}^\gamma_{\mu\tau\sigma}B
\]

(59)

This can be expanded just like equation (46), but now with so many quantities real, the much simpler result is

\[
\hat{B}^\gamma_{\mu\tau\sigma} = \hat{R}^\gamma_{\mu\tau\sigma} + \hat{\psi}_{\mu||\tau}\delta^\gamma_\sigma - \hat{\psi}_{\mu||\sigma}\delta^\gamma_\tau - \hat{\psi}_{\sigma||\tau}\delta^\gamma_\mu + \hat{\psi}_{\sigma||\mu}\delta^\gamma_\tau
\]

\[-\hat{\psi}_{\mu||\tau}\hat{g}^{\nu\sigma}\hat{g}_{\mu\sigma} + \hat{\psi}_{\sigma||\tau}\hat{g}^{\nu\sigma}\hat{g}_{\mu\tau}
\]

\[ + (k_+ \hat{\psi}_\sigma \hat{v}_\mu + k_- \hat{\psi}_\mu \hat{v}_\sigma) \delta^\gamma_\tau - (k_+ \hat{\psi}_\tau \hat{v}_\mu + k_- \hat{\psi}_\mu \hat{v}_\tau) \delta^\gamma_\sigma
\]

\[-\hat{\psi}_\mu \hat{\psi}_\nu \hat{g}^{\nu\sigma}\hat{g}_{\mu\sigma} \delta^\gamma_\tau + \hat{\psi}_\tau \hat{\psi}_\mu \hat{g}^{\nu\sigma}\hat{g}_{\mu\tau} \delta^\gamma_\sigma
\]

\[ + (k_+ \hat{\psi}_\mu \hat{v}_\sigma + k_- \hat{\psi}_\sigma \hat{v}_\mu) \hat{g}^{\alpha\gamma}\hat{g}_{\mu\sigma} - (k_+ \hat{\psi}_\sigma \hat{v}_\mu + k_- \hat{\psi}_\mu \hat{v}_\sigma) \hat{g}^{\alpha\gamma}\hat{g}_{\mu\tau}
\]

\[ + n (\hat{\psi}_{\sigma||\tau} - \hat{\psi}_{\tau||\sigma}) \delta^\gamma_\mu
\]

\[ - (n^2/k) (\hat{\psi}_\tau \hat{v}_\sigma - \hat{\psi}_\sigma \hat{v}_\tau) \delta^\gamma_\mu
\]

\[ - (n/k) (\hat{\psi}_\mu \hat{v}_\nu - \hat{\psi}_\nu \hat{v}_\mu) \delta^\gamma_\sigma + (n/k) (\hat{\psi}_\mu \hat{v}_\tau - \hat{\psi}_\tau \hat{v}_\mu) \delta^\gamma_\tau
\]

\[ + (n/k) (\hat{\psi}_\sigma \hat{v}_\nu - \hat{\psi}_\nu \hat{v}_\sigma) \hat{g}^{\alpha\gamma}\hat{g}_{\mu\sigma} - (n/k) (\hat{\psi}_\sigma \hat{v}_\tau - \hat{\psi}_\tau \hat{v}_\sigma) \hat{g}^{\alpha\gamma}\hat{g}_{\mu\tau}
\]

(60)

Much of the right-left distinction of equation (46), along with the commutators, is now gone. The main left-right
distinction remaining is in the terms involving products of \( \hat{v}_\mu \), because that quantity is fully quaternionic still. 

Now using equations (28), (35), and (36), equation (60) and equation (59) then contract to give

\[
\hat{B}_{\mu \tau} = B_{\mu \sigma}^{\alpha} \rho^\sigma_{\tau \omega} \\
= B^{-1} B_{\mu \tau} B \\
= \hat{R}_{\mu \tau} + (\hat{v}_{\mu || \tau} + \hat{v}_{\tau || \mu}) + \hat{v}_\alpha || \alpha \hat{g}_{\mu \tau} \\
- (\hat{v}_\mu \hat{v}_\tau + \hat{v}_\tau \hat{v}_\mu) + 2 \hat{v}_\alpha \hat{v}_\alpha \hat{g}_{\mu \tau} \\
+ (1 + n) \left( \hat{v}_{\mu || \tau} - \hat{v}_{\tau || \mu} \right) + \left[ (4 - 4n - 2n^2)/(1 - n) \right] \left( \hat{v}_\mu \hat{v}_\tau - \hat{v}_\tau \hat{v}_\mu \right) \\
\]

(61)

where the symmetric and antisymmetric parts have been clearly separated with the antisymmetric part all on the last line. Because \( \hat{v}_{\mu || \tau} - \hat{v}_{\tau || \mu} = \hat{v}_{\mu \tau} - \hat{v}_{\tau \mu} \), that antisymmetric part is

\[
- \hat{w}_{\mu \tau} = (1 + n) \left( \hat{v}_{\mu || \tau} - \hat{v}_{\tau || \mu} \right) + \left[(4 - 4n - 2n^2)/(1 - n)\right] \left( \hat{v}_\mu \hat{v}_\tau - \hat{v}_\tau \hat{v}_\mu \right) \\
= (1 + n) \left( \hat{v}_{\mu \tau} - \hat{v}_{\tau \mu} \right) + \left[(4 - 4n - 2n^2)/(1 - n)\right] \left( \hat{v}_\mu \hat{v}_\tau - \hat{v}_\tau \hat{v}_\mu \right) \\
= -(1 + n) \left\{ \hat{v}_{\mu \tau} - \hat{v}_{\tau \mu} + 2 \left( \hat{v}_\tau \hat{v}_\mu - \hat{v}_\mu \hat{v}_\tau \right) + \left[(2 - 4n)/(1 - n^2)\right] \left( \hat{v}_\tau \hat{v}_\mu - \hat{v}_\mu \hat{v}_\tau \right) \right\} \\
= -(1 + n) \left\{ \hat{g}_{\mu \tau} + \left[(2 - 4n)/(1 - n^2)\right] \left( \hat{v}_\tau \hat{v}_\mu - \hat{v}_\mu \hat{v}_\tau \right) \right\} \\
\]

(62)

where

\[
\hat{g}_{\mu \tau} = \hat{v}_{\tau \mu} - \hat{v}_{\mu \tau} + 2 \left( \hat{v}_\tau \hat{v}_\mu - \hat{v}_\mu \hat{v}_\tau \right) \\
= B^{-1} \hat{g}_{\mu \tau} B \\
\]

(63)

by equations (38) and (39), since equation (56) has the same form as the gauge transformation of \( \hat{v}_\mu \) in equation (24). Additionally, the distinct alternate contraction of \( \hat{B}^\gamma_{\mu \sigma \tau} \) gives

\[
\hat{B}^\gamma_{\mu \sigma \tau} = 4 \hat{g}_{\tau \sigma} + 4 \left[(1 - 2n)/(1 - n)\right] \left( \hat{v}_\sigma \hat{v}_\tau - \hat{v}_\tau \hat{v}_\sigma \right) \\
\]

(64)

This contraction of the curvature tensor is also important in Weyl’s original theory[3,5].

We have

\[
\hat{w}_{\mu \nu} = (1 + n) \left\{ \hat{g}_{\mu \nu} + \left[(2 - 4n)/(1 - n^2)\right] \left( \hat{v}_\nu \hat{v}_\mu - \hat{v}_\mu \hat{v}_\nu \right) \right\} \\
\]

(65)

For \( n \neq 1/2 \), the quantity \( \hat{w}_{\mu \nu} \) appears to have a tail on it in addition to \( \hat{g}_{\mu \nu} \). This was anticipated above when the antisymmetric part of \( k_+ R^\omega_{\mu \tau \sigma} + k_- l R^\omega_{\mu \tau \sigma} \) was noted to require additional antisymmetric terms to gauge balance it unless \( n = 1/2 \). This is the form those extra terms take in \( \hat{w}_{\mu \nu} \).

Given that \( \hat{B} = \hat{B}^\gamma_{\mu \sigma \tau} \hat{g}^{\sigma \tau} \), \( \hat{g}^{\mu \tau} = C B^{-1} g^{\mu \tau} \), and \( \hat{B}^\gamma_{\mu \sigma \tau} = B^{-1} \hat{B}_{\mu \tau} B \), then

\[
\hat{B} = \hat{B}^\mu_{\mu \tau} \hat{g}^{\mu \tau} \\
= C B^{-1} B_{\mu \tau} g^{\mu \tau} \\
= C \\
\]

(66)

This is a fundamental, kinematic identity the gauge invariant variables must satisfy by virtue of their definitions, and the geometry’s kinematics. Substituting from equations (61) and (62) for the expansion of \( \hat{B}_{\mu \tau} \), equation (66) becomes

\[
\hat{R} + 6 \hat{w}_{\mu || \mu} + 6 \hat{w}_{\mu} = C \\
\]

(67)

Obviously the case \( n = 1/2 \) gives the simplest expression for \( \hat{w}_{\mu \nu} \) in equation (65). Furthermore, for \( n = 1/2 \), and only for this value of \( n \), \( \hat{B}^\gamma_{\mu \sigma \tau} \) given by equation (64) is proportional to \( \hat{w}_{\mu \sigma} \) given by equation (62). That proportionality is a property that is true in Weyl’s original theory[3], so the case \( n = 1/2 \) is the only case that matches that property of Weyl’s original theory (which had \( n = 1 \), a value not allowed in this model). One can fairly say that the \( n = 1/2 \) case is the closest one can get to Weyl’s original model when generalizing to the quaternions, and it emphasizes the standard SU(2) Yang-Mills field (with optional additional real part) as a unique antisymmetric tensor in the structure, thereby unifying it with the framework of Riemannian General Relativity naturally.

The cost of all these simplifications introduced by choosing \( n = 1/2 \), is that the structure now has an equal mix of Weyl’s nonmetricity[3,5] with torsion, rather than insisting on just nonmetricity or torsion alone. That may seem
unusual for a model in which the non-Riemannian behavior is primarily based on a Weyl-like vector. Nevertheless, it does achieve a notable reduction in the complexity of the results. It’s interesting to see that quaternionic curvatures in this model seem not only to reject the quaternionic generalization of the pure Weyl model, as noted after equation [37], but that they also preferentially select this case with an equal balance of torsion and nonmetricity. That preference is expressed by the overall simplicity of this case. No such preferential selection between torsion and nonmetricity would appear with purely real or complex gauges and curvatures.

However, note that it is also true that effective nonmetricity may not vanish even when \( n = 0 \), even though the full affine derivative of \( g_{\mu\nu} \) using connection \( \hat{\Gamma}_{\mu\nu}^{\alpha} \) vanishes then, implying metric compatibility. This effective nonmetricity can be seen by looking at equation \( \hat{\Gamma}_{\mu\nu}^{\alpha} \) in the \( n = 0 \) case, and noting that the change in length of a vector transported using the full affine connection around a closed loop involves this quantity[5]. This may still be nonzero even in the \( n = 0 \) case here, because \( \hat{v}_{\mu} \) is fully quaternionic. Thus, it appears that there may be no quaternionic models in this family which are completely devoid of all aspects of nonmetricity. This result appears to follow from the fact that covariant and contravariant vectors interact with the affine connection on opposite sides of the (quaternionic) connection, and the length of a vector is a contraction of a covariant and a contravariant vector. To put this another way, the affine derivative using the full \( \hat{\Gamma}_{\mu\nu}^{\alpha} \), no longer obeys the product rule of differentiation because \( \hat{\Gamma}_{\mu\nu}^{\alpha} \) is quaternionic, not real. Thus, the calculations of Weyl and Eddington[3, 5], which would give the change in a parallel transported vector’s length around a closed loop in terms of the affine derivative of the metric (which vanishes given metric compatibility), would no longer be completely valid.

III. DISCUSSION

At this point, the kinematical framework of a quaternionic Weyl-like geometry is in place. The use of gauge invariant variables allows definition of a real metric tensor suitable for General Relativity, and also produces an \( SU(2) \) Yang-Mills Field, with an added possible real component as well, since the quaternions also have a real component. In that sense, since it is fully quaternionic, it’s really in a four dimensional Euclidian space which is probably better visualized as \( SU(2) \times SU(2) \), or \( SO(4) \).

This framework is so far general, blank slate beyond the above, since an action principle must define a particular dynamics to proceed further. In that regard, it is perfectly legitimate to phrase the action in terms of the gauge variables, and indeed, it may be the only easy way to formulate an action without having to keep track of the dynamics to proceed further. In that regard, it is perfectly legitimate to phrase the action in terms of the gauge variables, and indeed, it may be the only easy way to formulate an action without having to keep track of the dynamics to proceed further. In that regard, it is perfectly legitimate to phrase the action in terms of the gauge variables, and indeed, it may be the only easy way to formulate an action without having to keep track of the dynamics to proceed further.

To best illustrate this, if everything is restricted to the real numbers, and the original Weyl action is translated into gauge invariant variables, then generalized slightly, it becomes for this structure

\[
I = \int \sqrt{-\hat{g}} \left[ (\hat{R} - 2\sigma) - \frac{1}{2} j^2 (\hat{y}_{\mu\nu} \hat{y}^{\mu\nu}) + 6\hat{v}_\mu + (\hat{R} + 6\hat{v}_\mu + 6\hat{v}_\mu - C)\hat{J} \right] d^4 x \tag{68}
\]

where \( j^2 \) is a dimensionless gravitational constant. The fact that this is a modified, unbalanced Weyl structure does not affect this form. Indeed, the contraction of the torsion with itself has the value \( (3/8)\hat{v}_\mu \hat{v}_\mu \), so its form is already contributed to the action through other terms. For \( \sigma = 1/4 \) and \( C = 1 \), this is the original Weyl action[13, 14], a fact seen by substituting for \( \hat{R} \) in it everywhere by using the constraint, discarding any total divergence terms, and expanding surviving terms into the unhatted Weyl variables. The Yang-Mills tensor reduces to the curl of the Weyl vector here, or essentially what Weyl called the electromagnetic field. In practice, the constraint of equation [67] is used to determine \( \hat{B} \), given \( \hat{g}_{\mu\nu} \) and a value of \( v_\mu \) in a particular gauge[13, 15].

Among other results, this action will give that \( \hat{J} = 1 - [(4\sigma)/C] \), and that the electromagnetic four current is proportional to \( \hat{v}_\mu \), thus giving \( \hat{v}_\mu \parallel_{\mu} = 0 \) from conservation of charge. In the original Weyl case, \( \hat{J} = \hat{B} \), and there is no contribution to the stress tensor from the constraint terms. Because \( \hat{v}_\mu = v_\mu - [(1/2)\ln B]_\mu \), now, if \( C - \hat{R} \approx C \), the constraint itself clearly takes the form of the Hamilton-Jacobi equation for a nonzero rest mass charged particle in the combined gravitational and electromagnetic field[15]. In order to make sense of the large dimensionless cosmological constant of \( 1/4 \), the scale factor \( b_0 = \Lambda/\sigma \) where \( \Lambda \) is the usual cosmological constant in laboratory units. Thus, \( b_0 \) must be tiny to make sense of the original Weyl action, although a much smaller value for \( \sigma \) relieves that difficulty, while only deviating slightly from Weyl’s original action[14]. However, if the charge in the coupling constant is electronic, the \( 6\hat{v}_\mu \hat{v}_\mu \) in the action with no gravitational constant in front of it will then restrict the value of \( b_0 \) to near Planck scale values if that term is to be of the same magnitude as the \( j^2 (\hat{y}_{\mu\nu} \hat{y}^{\mu\nu}) \) term (using \( j^2 = (8/g^2)G(b_0/c^4) \) with \( G \) being Newton’s Constant). Still, one interesting consequence of this action is that the continuous electromagnetic
sources, which are proportional to \( \hat{\nu}_\mu \), everywhere move like a particle whose charge is fixed by the coupling constant absorbed into \( \hat{\nu}_\mu \) to make it dimensionless, and whose rest mass is that of the rest mass term in the Hamilton-Jacobi Equation in the constraint (near Planck scale rest mass for near Planck scale \( b_0 \), unless the \( \hat{R} \) term is large enough to change that). Evidently, the sign of that charge is effectively reversed by reversing the sign of the energy in the Hamilton-Jacobi Equation.

However, this action is dependent upon the term \( 6\hat{\nu}^\mu \hat{\nu}_\mu \) outside the constraint to achieve those neat results (\( \hat{R} + 6\hat{\nu}^\mu \hat{\nu}_\mu \), ignoring a noncontributing, divergence term), and that term is unusual. If it is omitted, that is equivalent to injecting the negative of this term times \( B^2 \) into the original Weyl action, and expanding that into the gauge dependent Weyl variables. That’s then a higher order action, and it has a tendency to produce (via the constraint term contributions from the above viewpoint) both positive energy density fields, plus additional, ghost, negative energy density fields rather generally\[14, 16\]. On the one hand, the singularity theorems that afflict classical General Relativity\[17\] then no longer need apply, but on the other hand, it is still unusual, even if some papers are exploring such fields\[18\]. Even more unusual, the constraint now takes on the form of the Klein-Gordon equation from wave mechanics with the standard added conformal term\[17\] and a nonzero rest mass, rather than a Hamilton-Jacobi type form\[11–15\]. Correctly formed wave function stress tensor and four current terms also appear, and appear consistent with atomic physics if \( b_0 \) is set using atomic instead of Planck level scales. Furthermore with an added self dual, antisymmetric part to the metric, the second order Dirac Equation form can be seen, and in quaternions, there are enough correct degrees of freedom to correspond to a two component spinor\[1, 13, 14\]. Those issues will not be explored further here, but the interested reader can check the references for more. This general framework remains regardless.

IV. COMMENT ON TORSION AND TRANSLATIONS

The presence of gauge potentials in torsion has been criticized on general grounds as being contrary to the geometric relation between torsion and translations, and thus to energy-momentum\[19\]. The model above achieves its generalization of Weyl’s geometry into the quaternions in a neat manner by necessarily shifting some of Weyl’s nonmetricity into torsion, and so appears to conflict with this general criticism. This requires an additional comment.

As Einstein noted, torsion is not invariant under his “lambda transformation”\[9\], which is essentially a treatment of the affine connection as a gauge potential rather than as the gauge field that would be the case in equation (25). This difference is crucial to generalizing Weyl’s structure into quaternions. Thus it seems odd that a gauge potential could not be part of the torsion to correspond to this fact.

Fortunately, a straightforward resolution of this conflict is already suggested in section II using the simple real variable action given in equation (68), and its resulting dynamics. As noted there, the constraint then becomes the Hamilton-Jacobi equation for a nonzero rest mass charged particle in the combined electromagnetic and gravitational fields. A moment’s reflection then reveals that the gauge invariant potential \( \hat{\nu}_\mu \) is essentially the mechanical portion of the four momentum of the “particle”\[20\]. Thus the gauge invariant torsion will be an antisymmetric tensor directly constructed from that four momentum, and the Kronecker Delta. Then the torsion is indeed related to energy-momentum, and this would seem to resolve the conflict in a novel manner.

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