Spectral decimation of the magnetic Laplacian on the Sierpinski gasket: Hofstadter’s butterfly, determinants, and loop soup entropy

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The Sierpinski gasket (SG)

We denote SG on level $N$ by $G_N = (V_N, E_N)$ where $V_N$ is the vertex set, and $E_N$ is the edge set.
The Sierpinski gasket (SG)

We denote SG on level $N$ by $G_N = (V_N, E_N)$ where $V_N$ is the vertex set, and $E_N$ is the edge set.

![Graphs of level 0, 1, and 2 of the Sierpinski gasket]

**Remark**

1. Let $F_i$ be the contraction mappings for $i = 0, 1, 2$. Then the infinite SG is the unique nonempty compact set $K$ such that
   \[ K = \bigcup_{i=0}^{2} F_i(K) \]
2. $\# V_N = \frac{3^{N+1}+3}{2}$
3. SG is self-similar
The combinatorial graph Laplacian

Example: $G = (V, E)$, the level 1 SG

$$D_G(i,j) = \begin{cases} 
\text{deg}(i) & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}$$

$$A_G(i,j) = \begin{cases} 
1 & \text{if } i \sim j \\
0 & \text{otherwise} 
\end{cases}$$

$$
\begin{bmatrix}
\begin{array}{c}
a_0 \\
a_1 \\
a_2 \\
b_0 \\
b_1 \\
b_2
\end{array}
\end{bmatrix}
\begin{bmatrix}
a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
\begin{array}{c}
a_0 \\
a_1 \\
a_2 \\
b_0 \\
b_1 \\
b_2
\end{array}
\end{bmatrix}
\begin{bmatrix}
a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
$$

The combinatorial graph Laplacian of level 1 SG is $\Delta_G = D_G - A_G$. 
We can normalize $\Delta_G$ by the degree to obtain the **probabilistic graph Laplacian** $\mathcal{L}_G = D_G^{-1} \Delta_G$, or

$$(\mathcal{L}_G u)(x) = \frac{1}{\deg_G(x)} \sum_{y \sim x} (u(x) - u(y)), \quad u \in \mathbb{R}^V$$
We can normalize $\Delta_G$ by the degree to obtain the \textbf{probabilistic graph Laplacian} $L_G = D_G^{-1} \Delta_G$, or

$$(L_G u)(x) = \frac{1}{\text{deg}_G(x)} \sum_{y \sim x} (u(x) - u(y)), \quad u \in \mathbb{R}^V$$

To obtain the magnetic Laplacian, we assign a set of unit complex values \textbf{(complex line bundle)} to replace the 1's in the adjacency matrix such that $\omega_{ij} = \omega_{ji}^{-1}$ for all $i, j$ in $V_N$. The magnetic Laplacian on the level-$N$ gasket graph $G_N$ endowed with the set of weights $\omega$ is defined as

$$(L_N^\omega u)(x) = \sum_{y \sim x} \frac{1}{\text{deg}_{G_N}(x)} (u(x) - \omega_{xy} u(y)), \quad u \in \mathbb{C}^V$$
The magnetic Laplacian

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To obtain the magnetic Laplacian, We assign a set of unit complex values (**complex line bundle**) to replace the 1's in the adjacency matrix such that $\omega_{ij} = \omega_{ji}^{-1}$ for all $i, j$ in $V_N$. The magnetic Laplacian on the level-$N$ gasket graph $G_N$ endowed with the set of weights $\omega$ is defined as

$$(\mathcal{L}_N^\omega u)(x) = \sum_{y \sim x} \frac{1}{\deg_{G_N}(x)} (u(x) - \omega_{xy} u(y)), \quad u \in \mathbb{C}^V$$

**Remark**

$\mathcal{L}_N^\omega$ is self-adjoint on $L^2(V_N, \deg_{G_N})$, so it has real eigenvalues.
Magnetic fluxes

Definition

The magnetic flux through each smallest upright (resp. downright) triangle on level $N$ equals $\alpha_N$ (resp. $\beta_N$).

Suppose that the figure below is part of a level $N$ SG:

\[
\begin{align*}
e^{2\pi i \alpha_N} &= \omega_{a_1} \omega_{b_0} \omega_{b_2} \omega_{b_1} a_1 = \omega_{b_0} a_2 \omega_{a_2} b_1 \omega_{b_1} b_0 \\
e^{2\pi i \beta_N} &= \omega_{b_0} \omega_{b_1} \omega_{b_2} \omega_{b_0} \omega_{b_1} b_0 \\
e^{-2\pi i \beta_N} &= \omega_{b_0} \omega_{b_2} \omega_{b_1} \omega_{b_1} b_0
\end{align*}
\]
The magnetic flux through each smallest upright (resp. downright) triangle on level $N$ equals $\alpha_N$ (resp. $\beta_N$).

Suppose that the figure below is part of a level $N$ SG:

![Diagram](image)

$$e^{2\pi i \alpha_N} = \omega_{a_1 b_0} \omega_{b_0 b_2} \omega_{b_2 a_1} = \omega_{b_0 a_2} \omega_{a_2 b_1} \omega_{b_1 b_0}$$

$$e^{2\pi i \beta_N} = \omega_{b_0 b_1} \omega_{b_1 b_2} \omega_{b_2 b_0}$$

$$e^{-2\pi i \beta_N} = \omega_{b_0 b_2} \omega_{b_2 b_1} \omega_{b_1 b_0}$$

**Remark**

Having uniform magnetic field over SG implies $\alpha_N = \beta_N$. 
The magnetic spectrum

Question

What is the magnetic spectrum when SG is subject to uniform magnetic field?

1 Bellissard, 1990; Ghez et. al, 1987
The magnetic spectrum

Question

What is the magnetic spectrum when SG is subject to uniform magnetic field?

Answer:¹

¹ Bellissard, 1990; Ghez et. al, 1987
Case I: magnetic spectrum under (half-) integer flux, $\alpha, \beta \in \{0, \frac{1}{2}\}$ (Chen–G. ’19)

$$\mathcal{L}^{(0,0)}_N \xrightarrow{R(0,0,\cdot)} \mathcal{L}^{(0,0)}_{N-1} \xrightarrow{R(0,0,\cdot)} \mathcal{L}^{(0,0)}_{N-2} \xrightarrow{\cdots} \mathcal{L}^{(0,0)}_0$$

| $\alpha, \beta$ | $\sigma(\mathcal{L}^{(\alpha,\beta)}_N)$ | Respective multiplicity |
|----------------|---------------------------------|------------------------|
| $(0, 0)^2$     | $0, \frac{3}{2}, R(0,0,\cdot), \frac{3}{4}, R(0,0,\cdot), \frac{5}{4}$ | $1, \frac{3^{N+3}}{2}, \frac{3^{N-k-1}+3}{2}, \frac{3^{N-k-1}}{2}$ |
| $(\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{3}{4}, \frac{5}{4}, 2, \{\frac{3}{4}, \frac{5}{4}\})^{-1} (R_1 \cup R_2)$ | $\frac{3^{N+3}}{2}, \frac{3^{N-1}-1}{2}, \frac{3^{N-1}+3}{2}, \frac{3^{N-2}-1}{2}, \frac{3^{N-2}+3}{2}, \frac{3^{N-k-3}}{2}$ |
| $(\frac{1}{2}, 0)$ | $(\frac{1}{2}, 1, \frac{5}{4}, \frac{7}{4}, \{\frac{3}{4}, \frac{5}{4}\})^{-1} (\{\frac{3}{4}, \frac{5}{4}\}), (R_3 \cup R_4)$ | $\frac{3^{N+3}}{2}, \frac{3^{N-1}-1}{2}, \frac{3^{N-1}+3}{2}, \frac{3^{N-2}-1}{2}, \frac{3^{N-2}+3}{2}, \frac{3^{N-k-3}}{2}$ |
| $(0, \frac{1}{2})$ | $(\frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}, \{\frac{3}{4}, \frac{5}{4}\})^{-1} (\{\frac{3}{4}, \frac{5}{4}\}), (R_3 \cup R_4)$ | $\frac{3^{N-1}+3}{2}, \frac{3^{N-1}-1}{2}, \frac{3^{N+3}}{2}, \frac{3^{N-2}-1}{2}, \frac{3^{N-2}+3}{2}, \frac{3^{N-k-3}}{2}$ |

where $R_1 = \bigcup_{k=0}^{N-2} (R(0,0,\cdot))^{-k} \left(\frac{3}{4}\right)$, $R_2 = \bigcup_{k=0}^{N-3} (R(0,0,\cdot))^{-k} \left(\frac{5}{4}\right)$, $R_3 = \bigcup_{k=0}^{N-3} (R(0,0,\cdot))^{-k} \left(\frac{3}{4}\right)$, $R_4 = \bigcup_{k=0}^{N-4} (R(0,0,\cdot))^{-k} \left(\frac{5}{4}\right)$

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$R(\alpha, \beta, \lambda)$ is the decimation function, $k = \{0, 1, \cdots N-1\}$, Fukushima & Shima, 1992

Ruoyu Guo  | Sierpinski-Hofstadter problem
Case I: magnetic spectrum under (half-) integer flux, $\alpha, \beta \in \{0, \frac{1}{2}\}$ (Chen–G. '19)

\[
\begin{align*}
\mathcal{L}_N^{(0, 0)} & \xrightarrow{R(0, 0, \cdot)} \mathcal{L}_N^{(0, 0)} & \mathcal{L}_N^{(0, 0)} & \xrightarrow{R(0, 0, \cdot)} \mathcal{L}_{N-1}^{(0, 0)} & \mathcal{L}_{N-2}^{(0, 0)} & \xrightarrow{\ldots} \mathcal{L}_0^{(0, 0)}
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}_N^{(\frac{1}{2}, \frac{1}{2})} & \xrightarrow{R(\frac{1}{2}, \frac{1}{2}, \cdot)} \mathcal{L}_{N-1}^{(\frac{1}{2}, \frac{1}{2})}
\end{align*}
\]

| $(\alpha, \beta)$ | $\sigma(\mathcal{L}_N^{(\alpha, \beta)})$ | Respective multiplicity |
|-------------------|---------------------------------|------------------------|
| $(0, 0)^2$       | $0, \frac{3}{2}, R(0, 0, \cdot)^{-k} \left(\frac{3}{4}\right), R(0, 0, \cdot)^{-k} \left(\frac{5}{4}\right)$ | $1, \frac{3N+3}{2}, \frac{3N-k-1+3}{2}, \frac{3N-k-1-1}{2}$ |
| $(\frac{1}{2}, \frac{1}{2})$ | $(R \left(\frac{1}{2}, \frac{1}{2}, \cdot\right))^{-1} (R_1 \cup R_2)$ | $\frac{3N+3}{2}, \frac{3N-1+1}{2}, \frac{3N-1+3}{2}, \frac{3N-k-2+1}{2}, \frac{3N-k-2-1}{2}$ |
| $(\frac{1}{2}, 0)$ | $(\frac{1}{2}, 1, \frac{5}{4}, \frac{7}{4}, (R \left(\frac{1}{2}, 0, \cdot\right))^{-1} \left(\left\{\frac{3}{4}, \frac{5}{4}\right\}\right), (R \left(\frac{1}{2}, 0, \cdot\right))^{-1} \circ (R \left(\frac{1}{2}, \frac{1}{2}, \cdot\right))^{-1} (R_3 \cup R_4)$ | $\frac{3N+3}{2}, \frac{3N-1+1}{2}, \frac{3N-1+3}{2}, \frac{3N-k-2+1}{2}, \frac{3N-k-2-1}{2}$ |
| $(0, \frac{1}{2})$ | $\left\{\frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}\right\}, (R \left(0, 1, \cdot\right))^{-1} \left(\left\{\frac{3}{4}, \frac{5}{4}\right\}\right)$, $(R \left(0, 1, \cdot\right))^{-1} \circ (R \left(1, \frac{1}{2}, \cdot\right))^{-1} (R_3 \cup R_4)$ | $\frac{3N+3}{2}, \frac{3N-1+1}{2}, \frac{3N-1+3}{2}, \frac{3N-k-2+1}{2}, \frac{3N-k-2-1}{2}$ |

where

- $R_1 = \bigcup_{k=0}^{N-2} (R(0, 0, \cdot))^{-k} \left(\frac{3}{4}\right)$
- $R_2 = \bigcup_{k=0}^{N-3} (R(0, 0, \cdot))^{-k} \left(\frac{5}{4}\right)$
- $R_3 = \bigcup_{k=0}^{N-3} (R(0, 0, \cdot))^{-k} \left(\frac{3}{4}\right)$
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\[ R(\alpha, \beta, \lambda) \text{ is the decimation function, } k = \{0, 1, \ldots, N-1\}, \text{ Fukushima & Shima, 1992} \]
Case I: magnetic spectrum under (half-) integer flux, $\alpha, \beta \in \{0, \frac{1}{2}\}$ (Chen–G. ’19)

\[
\begin{align*}
\mathcal{L}_N^{(0,0)} & \xrightarrow{R(0,0,\cdot)} \mathcal{L}_N^{(0,0)} \xrightarrow{R(0,0,\cdot)} \mathcal{L}_N^{(0,0)} \xrightarrow{\cdots} \mathcal{L}_0^{(0,0)} \\
\mathcal{L}_N^{(\frac{1}{2},\frac{1}{2})} & \xrightarrow{R(0,0,\cdot)} \mathcal{L}_N^{(\frac{1}{2},\frac{1}{2})} \xrightarrow{R(0,0,\cdot)} \mathcal{L}_N^{(0,\frac{1}{2})}
\end{align*}
\]

| $(\alpha, \beta)$ | $\sigma(\mathcal{L}_N^{(\alpha,\beta)})$ | Respective multiplicity |
|-------------------|---------------------------------|------------------------|
| $(0, 0)$          | $0, \frac{3}{2}, R(0, 0, \cdot)^{-k} \left(\frac{3}{4}\right), R(0, 0, \cdot)^{-k} \left(\frac{5}{4}\right)$ | $1, \frac{3^N+3}{2}, \frac{3^N-k-1+3}{2}, \frac{3^N-k-1-1}{2}$ |
| $(\frac{1}{2}, \frac{1}{2})$ | $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}, 2,$ | $\frac{3^N+3}{2}, \frac{3^N-k-2-3}{2}, \frac{3^N-k-2-1}{2}$ |
| $(\frac{1}{2}, 0)$ | $\left(R \left(\frac{1}{2}, 0, \cdot\right)\right)^{-1} \left(\frac{3}{4}\right), \left(R \left(\frac{1}{2}, 0, \cdot\right)\right)^{-1} \left(\frac{5}{4}\right)$ | $\frac{3^N+3}{2}, \frac{3^N-k-3-3}{2}, \frac{3^N-k-3-1}{2}$ |
| $(0, \frac{1}{2})$ | $\left(R \left(0, \frac{1}{2}, \cdot\right)\right)^{-1} \left(\frac{3}{4}\right), \left(R \left(0, \frac{1}{2}, \cdot\right)\right)^{-1} \left(\frac{5}{4}\right)$ | $\frac{3^N+3}{2}, \frac{3^N-k-3-3}{2}, \frac{3^N-k-3-1}{2}$ |

where $R_1 = \bigcup_{k=0}^{N-2} (R(0, 0, \cdot))^{-k} \left(\frac{3}{4}\right)$, $R_2 = \bigcup_{k=0}^{N-3} (R(0, 0, \cdot))^{-k} \left(\frac{5}{4}\right)$, $R_3 = \bigcup_{k=0}^{N-3} (R(0, 0, \cdot))^{-k} \left(\frac{3}{4}\right)$, $R_4 = \bigcup_{k=0}^{N-4} (R(0, 0, \cdot))^{-k} \left(\frac{5}{4}\right)$

$R(\alpha, \beta, \lambda)$ is the decimation function, $k = \{0, 1, \cdots N-1\}$, Fukushima & Shima, 1992.
Other cases

**Theorem:** Magnetic spectra under non-(half-)integer fluxes (Chen–G. ’19)

Let $\mathcal{E}(\alpha_N, \beta_N)$ be the **exceptional set for spectral decimation**. Suppose not both of $\alpha_N$ and $\beta_N$ are in $\{0, \frac{1}{2}\}$. Then

$$\sigma \left( \mathcal{L}_N^{(\alpha_N, \beta_N)} \right) = \left\{ \lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma \left( \mathcal{L}_{N-1}^{(\alpha_{N-1}, \beta_{N-1})} \right) \right\}$$

$$\quad \sqcup \left\{ \lambda : \mathcal{D}(\beta_N, \lambda) = 0, \ \text{mult} \left( \mathcal{L}_N^{(\alpha_N, \beta_N)}, \lambda \right) > 0 \right\} \sqcup \left\{ \begin{array}{ll}
3/2, & \text{if } \alpha_N = 0 \\
1/2, & \text{if } \alpha_N = 1/2
\end{array} \right\},$$
**Spectral decimation** is a process in which we project the eigenspace of $L^\Omega_N$ to that of $L^\Omega_{N-1}$. We do so by computing the Schur complement.

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**Schur complement**

Define the **Schur complement** of $L^\omega_N - \lambda I$ with respect to the minor $D - \lambda I$ as

$$S^\omega_N(\lambda) := (A - \lambda I) - B(D - \lambda I)^{-1}C,$$

where

- $A : \ell(V_{N-1}) \to \ell(V_{N-1})$,
- $B : \ell(V_N \setminus V_{N-1}) \to \ell(V_{N-1})$,
- $C : \ell(V_{N-1}) \to \ell(V_N \setminus V_{N-1})$,
- $D : \ell(V_N \setminus V_{N-1}) \to \ell(V_N \setminus V_{N-1})$,

and make the connection by writing

$$S^\omega_N(\alpha, \beta, \lambda) := \phi(\alpha, \beta, \lambda)(L^\Omega_{N-1} - R(\alpha, \beta, \lambda)). \quad \lambda \in \mathbb{C},$$

Then, $L^\omega_N$ and $L^\Omega_{N-1}$ are said to be spectrally similar, and if $\lambda \notin E(\alpha_N, \beta_N)$, then

$$\lambda \in \sigma(L^\omega_N) \iff R(\alpha_N, \beta_N, \lambda) \in \sigma(L^\Omega_{N-1})$$
Recall that we write
\[ S_N^\omega(\alpha, \beta, \lambda) = \phi(\alpha, \beta, \lambda)(L_N^\Omega - R(\alpha, \beta, \lambda)) \]
and if \( \lambda \notin \mathcal{E}(\alpha_N, \beta_N) \), then
\[ \lambda \in \sigma(L_N^\omega) \iff R(\lambda) \in \sigma(L_N^\Omega) \]

**Computations**

\[ R(\alpha, \beta, \lambda) = 1 + \frac{A(\alpha, \beta, \lambda) - 64D(\beta, \lambda)(1 - \lambda)}{16|\Psi(\alpha, \beta, \lambda)|}, \]
\[ \phi(\alpha, \beta, \lambda) = \frac{|\Psi(\alpha, \beta, \lambda)|}{4D(\beta, \lambda)}, \]
\[ A(\alpha, \beta, \lambda) = 16\lambda^2 - (32 + 4\cos(2\pi\alpha))\lambda + 15 + 4\cos(2\pi\alpha) + \cos(2\pi(\alpha + \beta)), \]
\[ D(\beta, \lambda) = -\lambda^3 + 3\lambda^2 - \frac{45}{16}\lambda + \frac{13}{16} - \frac{1}{32}\cos(2\pi\beta), \]
\[ \Psi(\alpha, \beta, \lambda) = (1 - \lambda)^2 - \frac{1}{16} + \frac{1 - \lambda}{4}(2e^{-2\pi i\alpha} + e^{-2\pi i(2\alpha + \beta)}) \]
\[ + \frac{1}{16}(e^{-4\pi i\alpha} + 2e^{-2\pi i(\alpha + \beta)}), \]
\[ \mathcal{E}(\alpha, \beta) = \{ \lambda \in \mathbb{R} : \Psi(\alpha, \beta, \lambda) = 0 \text{ or } D(\beta, \lambda) = 0 \} \]
Flux changes in spectral decimation

\[ \Omega_{a_1a_2}(\alpha, \beta, \lambda) = \omega_{a_1b_0} \omega_{b_0a_2} e^{2\pi i \theta(\alpha, \beta, \lambda)} \]

Therefore,

\[ \theta(\alpha, \beta, \lambda) = \frac{\arg \Psi(\alpha, \beta, \lambda)}{2\pi} \quad \text{(arg : } \mathbb{C} \rightarrow [0, 2\pi)) \}, \]

\[ \alpha_{N-1} = \alpha_\downarrow(\alpha_N, \beta_N, \lambda) \quad \text{and} \quad \beta_{N-1} = \beta_\downarrow(\alpha_N, \beta_N, \lambda), \]

\[ \alpha_\downarrow(\alpha, \beta, \lambda) = 3\alpha + \beta - 3\theta(\alpha, \beta, \lambda) \quad \text{(mod 1)}, \]

\[ \beta_\downarrow(\alpha, \beta, \lambda) = 3\beta + \alpha + 3\theta(\alpha, \beta, \lambda) \quad \text{(mod 1)} \]

3-parameter non-rational function

\[ U(\alpha, \beta, \lambda) = (3\alpha + \beta - 3\theta, 3\beta + \alpha + 3\theta, R(\alpha, \beta, \lambda)) \]
Other cases

Theorem: Magnetic spectra under non-(half-)integer fluxes (Chen–G. ’19)

Let $\mathcal{E}(\alpha_N, \beta_N)$ be the **exceptional set for spectral decimation**. Suppose not both of $\alpha_N$ and $\beta_N$ are in $\{0, \frac{1}{2}\}$. Then

$$
\sigma \left( \mathcal{L}_N^{(\alpha_N, \beta_N)} \right) = \left\{ \lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma \left( \mathcal{L}_{N-1}^{(\alpha_{N-1}, \beta_{N-1})} \right) \right\}
$$

$$
\sqcup \left\{ \lambda : D(\beta_N, \lambda) = 0, \ \text{mult} \left( \mathcal{L}_N^{(\alpha_N, \beta_N)}, \lambda \right) > 0 \right\} \sqcup \left\{ \begin{array}{ll}
\frac{3}{2}, & \text{if } \alpha_N = 0 \\
\frac{1}{2}, & \text{if } \alpha_N = \frac{1}{2}
\end{array} \right\},
$$
The exceptional set for spectral decimation

Question (Bellissard, 1990)

Is the dynamical spectrum equal to the actual spectrum of the original operator? This is a question with no answer yet.

\[ S_{\omega N}(\alpha, \beta, \lambda) = \phi(\alpha, \beta, \lambda)(L_{\Omega N} - 1 - R(\alpha, \beta, \lambda)) \]

so naturally,

\[ E(\alpha, \beta) = \{ \lambda \in \mathbb{R} : \Psi(\alpha, \beta, \lambda) = 0 \text{ or } D(\beta, \lambda) = 0 \} \]

Given any fluxes \( \alpha \) and \( \beta \), the exceptional set (for spectral decimation of \( L_{\omega N} \)) \( E(\alpha, \beta) \) consists of:

- The three zeros of \( D(\beta, \cdot) \);
- The corresponding values \( x \) in the table below if any of the conditions in the first column is met.

| Condition | Value |
|-----------|-------|
| \( \alpha = 0 \) | 3/2 |
| \( \alpha = \frac{1}{2} \) | 1/2 |
| \( \alpha + \beta = \frac{1}{2} \) (mod 1) | \( 1 + \frac{1}{2} \cos(2\pi\alpha) \) |

where \( D(\beta, \lambda) = -\lambda^3 + 3\lambda^2 - \frac{45}{16}\lambda + \frac{13}{16} - \frac{1}{32}\cos(2\pi\beta) \).
Question (Bellissard, 1990)

Is the dynamical spectrum equal to the actual spectrum of the original operator? This is a question with no answer yet.

Recall that we write

\[ S^\omega_N(\alpha, \beta, \lambda) = \phi(\alpha, \beta, \lambda)(L^\Omega_{N-1} - R(\alpha, \beta, \lambda)), \quad \phi(\alpha, \beta, \lambda) = \frac{|\Psi(\alpha, \beta, \lambda)|}{4D(\beta, \lambda)}, \]

so naturally,

\[ E(\alpha, \beta) = \{ \lambda \in \mathbb{R} : \Psi(\alpha, \beta, \lambda) = 0 \text{ or } D(\beta, \lambda) = 0 \} \]

Given any fluxes \( \alpha \) and \( \beta \), the exceptional set (for spectral decimation of \( L^\omega_N \)) \( E(\alpha, \beta) \) consists of:

- The three zeros of \( D(\beta, \cdot) \); and
- The corresponding values \( x \) in the table below if any of the conditions in the first column is met.

| Condition                  | Value \( x \) to be added to \( E(\alpha, \beta) \) |
|---------------------------|-----------------------------------------------|
| \( \alpha = 0 \)          | \( \frac{3}{2} \)                                      |
| \( \alpha = \frac{1}{2} \) | \( \frac{1}{2} \)                                      |
| \( 3\alpha + \beta = \frac{1}{2} \) (mod 1) | \( 1 + \frac{1}{2} \cos(2\pi\alpha) \) |

where \( D(\beta, \lambda) = -\lambda^3 + 3\lambda^2 - \frac{45}{16} \lambda + \frac{13}{16} - \frac{1}{32} \cos(2\pi\beta) \).
Additional analysis on the exceptional set

\[ \mathcal{E}(\alpha, \beta) = \{ \lambda \in \mathbb{R} : \Psi(\alpha, \beta, \lambda) = 0 \text{ or } D(\beta, \lambda) = 0 \} \]

**Case I**: \( \alpha, \beta \in \{0, \frac{1}{2}\} \). Spectral decimation can be carried out explicitly.

**Case II**: Only one of \( \alpha \) and \( \beta \) is in \( \{0, \frac{1}{2}\} \). There is only one \( \mathbb{R} \)-valued zero of \( \Psi(\alpha, \beta, \cdot) \).

**Case III**: \( 3\alpha + \beta = \frac{1}{2} \pmod{1} \), excluding flux values already discussed in Cases I & II. There is only one \( \mathbb{R} \)-valued zero of \( \Psi(\alpha, \beta, \cdot) \).

**Case IV**: The remaining case. There are no \( \mathbb{R} \)-valued zeros of \( \Psi(\alpha, \beta, \cdot) \).

There is a standard way to analyze the exceptional set using complex analysis. However, it is necessary to use real analysis in our case.

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3 Bajorin et. al, 2008 - Ruoyu Guo

Sierpinski-Hofstadter problem
Other cases

\[ \mathcal{E}(\alpha, \beta) = \{ \lambda \in \mathbb{R} : \Psi(\alpha, \beta, \lambda) = 0 \text{ or } D(\beta, \lambda) = 0 \} \]

**Case I**: \( \alpha, \beta \in \{0, \frac{1}{2}\} \). Spectral decimation can be carried out explicitly.
**Case II**: Only one of \( \alpha \) and \( \beta \) is in \( \{0, \frac{1}{2}\} \). There is only one \( \mathbb{R} \)-valued zero of \( \Psi(\alpha, \beta, \cdot) \).
**Case III**: \( 3\alpha + \beta = \frac{1}{2} \pmod{1} \), excluding flux values already discussed in Cases I & II. There is only one \( \mathbb{R} \)-valued zero of \( \Psi(\alpha, \beta, \cdot) \).
**Case IV**: The remaining case. There are no \( \mathbb{R} \)-valued zeros of \( \Psi(\alpha, \beta, \cdot) \).

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**Theorem**: Magnetic spectra under non-(half-)integer fluxes (Chen–G. ’19)

Let \( \mathcal{E}(\alpha_N, \beta_N) \) be the **exceptional set for spectral decimation**. Suppose not both of \( \alpha_N \) and \( \beta_N \) are in \( \{0, \frac{1}{2}\} \). Then

\[
\sigma \left( \mathcal{L}_N^{(\alpha_N, \beta_N)} \right) = \left\{ \lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma \left( \mathcal{L}_{N-1}^{(\alpha_{N-1}, \beta_{N-1})} \right) \right\}
\]

\[ \cup \left\{ \lambda : D(\beta_N, \lambda) = 0, \ \text{mult} \left( \mathcal{L}_N^{(\alpha_N, \beta_N)}, \lambda \right) > 0 \right\} \cup \begin{cases} 
\frac{3}{2}, & \text{if } \alpha_N = 0 \\
\frac{1}{2}, & \text{if } \alpha_N = \frac{1}{2} 
\end{cases} \]
Theorem: Magnetic spectra under non-(half-)integer fluxes (Chen–G. '19)

Let $\mathcal{E}(\alpha_N, \beta_N)$ be the exceptional set for spectral decimation. Suppose not both of $\alpha_N$ and $\beta_N$ are in $\{0, \frac{1}{2}\}$. Then

$$\sigma \left( \mathcal{L}_N^{(\alpha_N, \beta_N)} \right) = \left\{ \lambda \in \mathbb{R} \setminus \mathcal{E}(\alpha_N, \beta_N) : R(\alpha_N, \beta_N, \lambda) \in \sigma \left( \mathcal{L}_{N-1}^{(\alpha_N-1, \beta_N-1)} \right) \right\}$$

$$\sqcup \left\{ \lambda : D(\beta_N, \lambda) = 0, \mult \left( \mathcal{L}_N^{(\alpha_N, \beta_N)}, \lambda \right) > 0 \right\} \sqcup \left\{ \begin{array}{ll}
\frac{3}{2}, & \text{if } \alpha_N = 0 \\
\frac{1}{2}, & \text{if } \alpha_N = \frac{1}{2}
\end{array} \right\},$$

Theorem: Magnetic spectra under (half-)integer fluxes (Chen–G. '19)

| $(\alpha, \beta)$ | $\sigma(\mathcal{L}_N^{(\alpha, \beta)})$ | Respective multiplicity |
|-------------------|-----------------------------------|------------------------|
| $(0, 0)^*$        | $0, \frac{3}{2}, R(0, 0, \cdot)^{-1} \left( \frac{3}{4} \right), R(0, 0, \cdot)^{-1} \left( \frac{5}{4} \right)$ | $1, \frac{3N+3}{2}, \frac{3N-k-1+3}{2}, \frac{3N-k-1-1}{2}$ |
| $(\frac{1}{2}, \frac{1}{2})$ | $\left( R \left( \frac{1}{2}, \frac{1}{2}, \cdot \right) \right)^{-1} \left( R_1 \cup R_2 \right)$ | $\frac{3N+3}{2}, \frac{3N-1-1}{2}, \frac{3N-1+3}{2}, 1, \frac{3N-k-2^2+3}{2}, \frac{3N-k-2^2-1}{2}$ |
| $(\frac{1}{2}, 0)$ | $\left( R \left( \frac{1}{2}, 0, \cdot \right) \right)^{-1} \left( \frac{3}{4}, \frac{5}{4} \right), \left( R \left( \frac{1}{2}, 0, \cdot \right) \right)^{-1} \left( \frac{3}{4}, \frac{5}{4} \right), \left( R \left( \frac{1}{2}, \frac{1}{2}, \cdot \right) \right)^{-1} \left( R_3 \cup R_4 \right)$ | $\frac{3N+3}{2}, \frac{3N-1-1}{2}, \frac{3N-1+3}{2}, \frac{3N-k-2^2-1}{2}, \frac{3N-k-3^2+3}{2}, \frac{3N-k-3^2-1}{2}$ |
| $(0, \frac{1}{2})$ | $\left( R \left( 0, \frac{1}{2}, \cdot \right) \right)^{-1} \left( \frac{3}{4}, \frac{5}{4} \right), \left( R \left( 0, \frac{1}{2}, \cdot \right) \right)^{-1} \left( \frac{3}{4}, \frac{5}{4} \right)$ | $\frac{3N+3}{2}, \frac{3N-1-1}{2}, \frac{3N-1+3}{2}, 1, \frac{3N-k-2^2+3}{2}, \frac{3N-k-2^2-1}{2}, \frac{3N-k-3^2+3}{2}, \frac{3N-k-3^2-1}{2}$ |
Determinants of the magnetic Laplacian under (half-) integer fluxes (Chen–G. ’19)

\[
\det(L_N^{(1/2, 1/2)}) = \frac{1}{\kappa(G_N)} \cdot 2 \cdot 3^{N-2} \cdot \frac{3}{2} \cdot 3 \cdot \frac{3}{2} \cdot N - \frac{3}{2} \cdot 5 \cdot \frac{3}{2} + \frac{3}{2} \\
\times \left[ \prod_{k=0}^{N-2} \left( H(k) + \frac{1}{2} \right) \right]^{3N-k-2+3} \left[ \prod_{k=0}^{N-3} \left( H(k) + \frac{5}{2} \right) \right]^{3N-k-2-1},
\]

where \( H(0) = 26.5 \), and for \( k \geq 1 \), \( H(k) = [H(k - 1)]^2 - \frac{15}{4} \).

\[
\det(L_N^{(1/2, 0)}) = \frac{1}{\kappa(G_N)} \cdot 2 \cdot 13 \cdot 3^{N-1} \cdot \frac{5}{2} \cdot 3 \cdot \frac{3}{2} \cdot N - \frac{3}{2} \cdot 5 \cdot \frac{3}{2} \cdot 3^{N-2} - 1 \cdot 7 \cdot \frac{3}{2} + \frac{3}{2} \cdot 17 \cdot \frac{3}{2} + \frac{3}{2} \\
\times \left[ \prod_{k=0}^{N-3} \left( \tilde{H}(k) + \frac{1}{2} \right) \right]^{3N-k-3+3} \left[ \prod_{k=0}^{N-4} \left( \tilde{H}(k) + \frac{5}{2} \right) \right]^{3N-k-3-1},
\]

where \( \tilde{H}(0) = 302.5 \), and for \( k \geq 1 \), \( \tilde{H}(k) = [\tilde{H}(k - 1)]^2 - \frac{15}{4} \).

\[
\det(L_N^{(0, 1/2)}) = \frac{1}{\kappa(G_N)} \cdot 2 \cdot 13 \cdot 3^{N-1} \cdot \frac{5}{2} \cdot 3 \cdot \frac{3}{2} \cdot 3^{N-1} - N + 3 \cdot 7 \cdot \frac{3}{2} - 1 \frac{1}{2} \\
\times \left[ \prod_{k=0}^{N-3} \left( \hat{H}(k) + \frac{1}{2} \right) \right]^{3N-k-3+3} \left[ \prod_{k=0}^{N-4} \left( \hat{H}(k) + \frac{5}{2} \right) \right]^{3N-k-3-1},
\]

where \( \hat{H}(0) = 86.5 \), and for \( k \geq 1 \), \( \hat{H}(k) = [\hat{H}(k - 1)]^2 - \frac{15}{4} \).
Loop soup entropy

A cycle-rooted spanning forest (CRSF) is a spanning forest whose connected components are unicycles (a tree plus an edge to form a single cycle).

Matrix-CRSF Theorem\(^4\): Let \( \mathcal{L}_{(G,c)}^{\omega} \) be the line bundle Laplacian, then

\[
\det (\mathcal{L}_{(G,c)}^{\omega}) = \sum_{\text{OCRFSs}} \prod_{e \in \text{bushes}} c(e) \prod_{\gamma \in \text{cycles}} C(\gamma) (1 - \omega(\gamma)).
\]

Asymptotic complexity (tree entropy\(^5\)):

\[
h(G_{\infty}, \mathcal{L}_{\infty}^{\omega}) := \lim_{N \to \infty} \frac{\log (\kappa(G_N) \det(\mathcal{L}^\omega_N))}{|V_N|}
\]

Loop soup entropy:

\[
h_{\text{loop}}(G_{\infty}, \mathcal{L}_{\infty}^{\omega}) := h(G_{\infty}, \mathcal{L}_{\infty}^{\omega}) - h(G_{\infty}, \mathcal{L}_{\infty}^{\text{Id}}).
\]

Probabilistic interpretation:

\[
\lim_{N \to \infty} \lim_{c \downarrow 0} \frac{1}{|V_N|} \log P_{N,c}^{(\alpha,\beta)} [\text{no loops}] = -h_{\text{loop}}(SG, \mathcal{L}_{\infty}^{(\alpha,\beta)})
\]

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\(^4\) Kenyon, 2011

\(^5\) Lyons, 2005
Thank you for your attention!