BOHR-ROGOSINSKI PHENOMENON FOR $S^*(\psi)$ AND $C(\psi)$

KAMALJEET GANGANIA AND S. SIVAPRASAD KUMAR

Abstract. In Geometric function theory, occasionally attempts have been made to solve a particular problem for the Ma-Minda classes, $S^*(\psi)$ and $C(\psi)$ of univalent starlike and convex functions, respectively. Recently, a popular radius problem generally known as Bohr’s phenomenon has been studied in various settings, however a little is know about Rogosinski radius. In this article, for a fixed $f \in S^*(\psi)$ or $C(\psi)$, the class of analytic subordinants $S_f(\psi) := \{ g : g \prec f \}$ is studied for the Bohr-Rogosinski phenomenon in a general setting. It’s applications to the classes $S^*(\psi)$ and $C(\psi)$ are also shown.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disk $D := \{ z : |z| < 1 \}$. Using subordination [20], Ma and Minda [19] (also see [15]) introduced the unified class of univalent starlike and convex functions defined as follows:

$$S^*(\psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\}$$

and

$$C(\psi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\},$$

where $\psi$ is analytic and univalent with $\Re \psi(z) > 0$, $\psi'(0) > 0$, $\psi(0) = 1$ and $\psi(D)$ is symmetric about real axis. Note that $\psi \in \mathcal{P}$, the class of normalized Carathéodory functions. Also when $\psi(z) = (1+z)/(1-z)$, $S^*(\psi)$ and $C(\psi)$ reduces to the standard classes $S^*$ and $C$ of univalent starlike and convex functions.

In GFT, radius problems have a rich history which is being followed till today, see the recent articles [9, 10, 11, 13, 14, 15, 17, 28]. In 1914, Harald Bohr [8] proved the following remarkable radius problem related to the power series:

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Theorem 1.1 (Bohr’s Theorem, [8]). Let \( g(z) = \sum_{k=0}^{\infty} a_k z^k \) be an analytic function in \( \mathbb{D} \) and \( |g(z)| < 1 \) for all \( z \in \mathbb{D} \), then
\[
\sum_{k=0}^{\infty} |a_k||z|^k \leq 1, \quad \text{for} \quad |z| \leq \frac{1}{3}.
\]

Bohr actually proved the above result for \( r \leq 1/6 \). Further Wiener, Riesz and Shur independently sharpened the result for \( r \leq 1/3 \). Presently, the Bohr inequality for functions mapping unit disk onto different domains, other than unit disk is an active area of research. For the recent development on Bohr-phenomenon, see the articles [2, 3, 6, 7, 21, 22, 23] and references therein. The concept of Bohr phenomenon in terms of subordination can be described as:

Definition 1.2 (Muhanna, [21]). Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) are analytic in \( \mathbb{D} \) and \( f(\mathbb{D}) = \Omega \). For a fixed \( f \), consider a class of analytic functions \( S(f) := \{g : g < f\} \) or equivalently \( S(\Omega) := \{g : g(z) \in \Omega\} \). Then the class \( S(f) \) is said to satisfy Bohr-phenomenon, if there exists a constant \( r_0 \in (0, 1] \) satisfying the inequality \( \sum_{k=1}^{\infty} |b_k|^r \leq d(f(0), \partial \Omega) \) for all \( |z| = r \leq r_0 \) and \( g \in S(f) \), where \( d(f(0), \partial \Omega) \) denotes the Euclidean distance between \( f(0) \) and the boundary of \( \Omega = f(\mathbb{D}) \). The largest such \( r_0 \) is called the Bohr-radius.

In 2014, Muhanna et al. [23] proved the Bohr phenomenon for \( S(W_{\alpha}) \), where \( W_{\alpha} := \{w \in \mathbb{C} : |\arg w| < \alpha \pi/2, 1 \leq \alpha \leq 2\} \), which is a Concave-wedge domain (or exterior of a compact convex set) and the class \( R(\alpha, \beta, h) \) defined by \( R(\alpha, \beta, h) := \{f \in A : f(z) = g(z) + \alpha zg'(z) + \beta z^2 g''(z) < h(z), g \in A\} \), where \( h \) is a convex function (or starlike) and \( R(\alpha, \beta, h) \subset S(h) \). In 2018, Bhowmik and Das [6] proved the Bohr-phenomenon for the classes: \( S(f) = \{g \in A : g < f \text{ and } f \in \mu(\lambda)\} \), where \( \mu(\lambda) = \{f \in A : |z/f(z)|^2 f'(z) < \lambda, 0 < \lambda \leq 1\} \) and \( S(\alpha) = \{g \in A : g < f \text{ and } f \in S^*(\alpha), 0 \leq \alpha \leq 1/2\} \), where \( S^*(\alpha) \) is the well-known class of starlike functions of order \( \alpha \).

In the aforesaid work, the role of the sharp coefficient’s bound of \( f \) was prominent to achieve the respective Bohr radius for the class \( S(f) \), see [3, 15, 16]. But in general, the sharp coefficient’s bounds for functions in a given class are not available, for example see [3, 13, 14, 15, 28], thus certain power series inequalities are needed. In this direction, Bhowmik and Das obtained the following important inequality to achieve the Bohr radius for the class \( S(f) \), where \( f \in \mu(\lambda) \) and \( S^*(\alpha), 0 \leq \alpha \leq 1/2 \) respectively:

Lemma 1.1 ([6]). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) be analytic in \( \mathbb{D} \) and \( g < f \). Then
\[
\sum_{k=0}^{\infty} |b_k|r^k \leq \sum_{n=0}^{\infty} |a_n|r^n, \quad \text{for} \quad |z| \leq \frac{1}{3}.
\]

Motivated by the class \( S(f) \), Kumar and Gangania in [16, Sec. 5] further used the above Lemma 1.1 in the absence of the sharp coefficient’s bounds.
of $f$ to study the Bohr phenomenon for the class $S_f(\psi)$, which eventually holds for the class $S^*(\psi)$:

**Definition 1.3.** Let $f \in S^*(\psi)$ or $\mathcal{C}(\psi)$ be fixed. Then the class of subordinants functions $g$ is defined as:

$$S_f(\psi) := \left\{ g(z) = \sum_{k=1}^{\infty} b_k z^k : g \prec f \right\}.$$ 

**Theorem 1.4.** [16 Theorem 5.1] Let $r_*$ be the Koebe-radius for the class $S^*(\psi)$, $f_0(z)$ be given by the equation (2.2) and $g(z) = \sum_{k=1}^{\infty} b_k z^k \in S_f(\psi)$.

Assume $f_0(z) = z + \sum_{n=2}^{\infty} z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n|r^n$. Then $S_f(\psi)$ satisfies the Bohr-phenomenon

$$\sum_{k=1}^{\infty} |b_k|r^k \leq d(f(0), \partial \Omega), \quad \text{for } |z| = r \leq r_b,$$

where $r_b = \min\{r_0, 1/3\}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the least positive root of the equation

$$\hat{f}_0(r) = r_*.$$

The result is sharp when $r_b = r_0$ and $t_n > 0$.

Note that Muhanna et al. [24] recently discussed the Bohr type of inequalities for the k-th section for the analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ using the Bohr Operator

$$M_r(f) = \sum_{n=0}^{\infty} |a_n||z^n| = \sum_{n=0}^{\infty} |a_n|r^n.$$ 

Paulsen and Singh [25] used this operator provided an simple elementary proof of the Bohr’s Theorem [11] and extended it to the Banach algebras (for the basic important discussion, see [24, 25]). Now for the simplicity and further discussion, we define the following basic operator for $f$, where $S^N(f(z)) = \sum_{n=N}^{\infty} a_n z^n$:

$$M_r^N(f) = \sum_{n=N}^{\infty} |a_n||z^n| = \sum_{n=N}^{\infty} |a_n|r^n,$$

and thus the following observations hold for $|z| = r$ for each $z \in \mathbb{D}$

(i) $M_r^N(f) \geq 0$, and $M_r^N(f) = 0$ if and only if $f \equiv 0$

(ii) $M_r^N(f + g) \leq M_r^N(f) + M_r^N(g)$

(iii) $M_r^N(\alpha f) = |\alpha|M_r^N(f)$ for $\alpha \in \mathbb{C}$

(iv) $M_r^N(f \cdot g) \leq M_r^N(f) M_r^N(g)$

(v) $M_r^N(1) = 1$.

Using this operator, we now can get similar type of results as obtained by Muhanna et al. [24] for the interim k-th sections $S^N_k(f(z)) = \sum_{n=N}^{k} a_n z^n$ and the function $S^N(f(z))$. 

BOHR-ROGOSINSKI PHENOMENON FOR $S^*(\psi)$ AND $\mathcal{C}(\psi)$
In analogy with Bohr’s Theorem, there is also the notion of Rogosinski radius, however a little is known about Rogosinski radius as compared to Bohr radius, which is defined as follows, also see [18, 26, 27]:

**Theorem 1.5** (Rogosinski Theorem). If \( g(z) = \sum_{k=0}^{\infty} b_k \) with \( |f(z)| < 1 \), then for every \( N \geq 1 \) we have

\[
\left| \sum_{k=0}^{N-1} b_k z^k \right| \leq 1, \quad \text{for } |z| \leq \frac{1}{2}.
\]

The radius \( 1/2 \) is called the Rogosinski radius.

Kayumov et al. [12] considered a new quantity, called Bohr-Rogosinski sum, which is described as follows:

\[
|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k, \quad |z| = r.
\]

For the case \( N = 1 \), note that this sum is similar to the Bohr’s sum, where \( g(0) \) is replaced by \( |g(z)| \). We also refer the readers to see [1, 4]. Now we say the family \( S(f) \) has Bohr-Rogosinski phenomenon, if there exists \( r_f \in (0, 1] \) such that the inequality:

\[
|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq |f(0)| + d(f(0), \partial \Omega)
\]

holds for \( |z| = r \leq r_f \). The largest such \( r_f \) is called the Bohr-Rogosinski radius. Authors [12] also proved the following interesting results:

**Theorem 1.6.** [12, Theorem 5-6] Let \( g \in S(f) \), where \( f \) is univalent in \( \mathbb{D} \). Then for each \( m, N \in \mathbb{N} \), the inequality

\[
|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq |f(0)| + d(f(0), \partial \Omega)
\]

holds for \( |z| = r \leq r_{m,N}^f \), where \( r_{m,N}^f \) is the smallest positive root of:

\[
4r^m - (1 - r^m)^2 + 4r^N(N(1 - r) + r) \left( \frac{1 - r^m}{1 - r} \right)^2 = 0.
\]

The radius is sharp for the Koebe function \( z/(1 - z)^2 \). Moreover, if \( f \) is convex (univalent) in \( \mathbb{D} \), then \( r_{m,N}^f \) is the smallest positive root of:

\[
3r^m - 1 + 2r^N \left( \frac{1 - r^m}{1 - r} \right) = 0.
\]

The radius is sharp for the convex function \( z/(1 - z) \).

Motivated by the above work, let us now introduce the Bohr-Rogosinski phenomenon for the class of analytic subordinants \( S_f(\psi) \):
Definition 1.7. The class $S_f(\psi)$ has a Bohr-Rogosinski phenomenon, if there exists an $0 < r_0 \leq 1$ such that

$$|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(f(0), \partial \Omega)$$

for $|z| = r \leq r_0$, where $N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $d(f(0), \partial \Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega$.

Note that $S^*(\psi) \subset \bigcup_{f \in S^*(\psi)} S_f(\psi)$. Further, the connection between the Bohr-Rogosinski and Bohr phenomenon can be seen through Definition 1.7, if we replace $|g(z)|$ by $|g(z^m)|$, where $m \in \mathbb{N}$, and then consider the special case by taking $m \to \infty$ with $N = 1$. In Section 2, for a fixed $f \in S^*(\psi)$ or $C(\psi)$, the class of subordinants $S_f(\psi) := \{ g : g \prec f \}$ is studied for the Bohr-Rogosinski phenomenon in general settings along with its applications to the standard classes of univalent starlike and convex functions.

2. Bohr-Rogosinski Phenomenon

The following fundamental result is an extension of the Lemma 1.1:

Lemma 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be analytic in $\mathbb{D}$ and $g \prec f$, then

$$\sum_{k=N}^{\infty} |b_k|r^k \leq \sum_{n=N}^{\infty} |a_n|r^n$$

(2.1)

for $|z| = r \leq \frac{1}{3}$ and $N \in \mathbb{N}$.

Proof. Since $g \prec f$, we have $g(z) = f(\omega(z))$, where $\omega$ is a Schwarz function. For the case $\omega(z) = cz$, $|c| = 1$, the function $g$ is a rotation of $f$ or $g = f$, and the inequality (2.1) easily holds. So consider the case: $\omega(z) \neq cz$, $|c| = 1$. Now the coefficient $b_k$ of the function $g$ is given by: for any $k \geq N \in \mathbb{N}$

$$b_k = \sum_{n=N}^{k} a_n \beta_k^{(n)},$$

where the $t$-th power of the analytic function $\omega$ is represented as $\omega^t(z) = \sum_{l \geq t} \beta_l^{(t)} z^l$, $t \in \mathbb{N}$. Now we see that

$$\sum_{k=N}^{m} |b_k|r^k = \sum_{k=N}^{m} \left| \sum_{n=N}^{n} a_n \beta_k^{(n)} \right| r^k \leq \sum_{k=N}^{m} \sum_{n=N}^{n} |a_n||\beta_k^{(n)}|r^k = \sum_{n=N}^{m} |a_n|M_m^{(n)}(r),$$
where \( M_m(n)(r) = \sum_{k=n}^{\infty} |\beta_k(n)| r^k \) and \( m \in \mathbb{N} \). Since \(|\omega^n(z)/z^n| < 1\) for any \( n \geq 1 \), using Bohr’s Theorem \(1.1\) we have

\[
\sum_{k=n}^{m} |\beta_k(n)| r^{k-n} \leq \sum_{k=n}^{\infty} |\beta_k(n)| r^{k-n} \leq 1, \quad r \leq \frac{1}{3},
\]

that is, \( M_m(n)(r) \leq r^n \) holds for \( r \leq 1/3 \). Hence, for any \( m \geq N \geq 1 \) and \( r \leq 1/3 \)

\[
\sum_{k=N}^{m} |b_k| r^k \leq \sum_{n=N}^{m} |a_n| r^n.
\]

The result now follows by taking \( m \to \infty \). \(\square\)

**Proof.** [Alternate proof of the Lemma 2.1] Since \( g(z) = f(\omega(z)) \), where \( \omega \) is the Schaurz function, we have

\[
M_r^n(g) = M_r^n \left( \sum_{k=N}^{\infty} a_k(\omega(z))^k \right) \\
\leq \sum_{k=N}^{\infty} |a_k| (M_r(\omega(z)))^k \\
\leq \sum_{k=N}^{\infty} |a_k||z|^k
\]

for \(|z| = r \leq 1/3\). \(\square\)

**Remark 2.1.** In Lemma \( 2.1 \) taking \( N \to 1 \) and the fact the \( g(0) = f(0) \) we obtain Lemma \( 1.1 \)

Moreover, the following results is obtained using the properties of the operator \( M_r^N(f) \) and Lemma \( 2.1 \)

**Corollary 2.1.** Let the analytic functions \( f, g \) and \( h \) satisfies \( g(z) = h(z)f(\omega(z)) \) in \( \mathbb{D} \), where \( \omega \) is the Schaurz function. Assume \(|h(z)| \leq \tau\) for \(|z| < \tau \leq 1\). Then

\[
M_r^N(g) \leq \tau M_r^N(f), \quad 0 \leq |z| = r \leq \frac{\tau}{3}.
\]

**Corollary 2.2.** Let \( \tau = 1 \) in Theorem \( 2.1 \). Then

\[
M_r^N(g) \leq M_r^N(f), \quad 0 \leq |z| = r \leq \frac{1}{3}.
\]

**Lemma 2.2.** (\cite{19}) Let \( f \in S^*(\psi) \) and \(|z_0| = r < 1\). Then \( f(z)/z \prec f_0(z)/z \) and

\[
-f_0(-r) \leq |f(z_0)| \leq f_0(r).
\]

Equality holds for some \( z_0 \neq 0 \) if and only if \( f \) is a rotation of \( f_0 \), where \( zf_0(z)/f_0(z) = \psi(z) \) such that

\[
f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} \, dt. \quad (2.2)
\]
Our next results discuss Bohr-Rogosinski phenomenon for the classes $S_f(\psi)$ and $S^*(\psi)$, respectively.

**Theorem 2.3.** Let $r_*$ be the Koebe-radius for the class $S^*(\psi)$, $f_0(z)$ be given by the equation (2.2) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\psi)$. Assume $f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n|r^n$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial \Omega)$$

(2.3)

holds for $|z| = r_0 \leq \min \{ \frac{1}{3}, r_0 \}$, where $m, N \in \mathbb{N}, \Omega = f(\mathbb{D})$ and $r_0$ is the unique positive root of the equation:

$$\hat{f}_0(r^m) + \hat{f}_0(r) - p_{f_0}(r) = r_*,$$

(2.4)

where

$$p_{f_0}(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2 \\ r + \sum_{n=2}^{N-1} |t_n|r^n, & N \geq 3 \end{cases}$$

The result is sharp when $r_0 = r_0$ and $t_n > 0$.

**Proof.** Let $g(z) = \sum_{k=1}^{\infty} b_k z^k \prec f(z)$, where $f \in S^*(\psi)$. Now by Lemma 2.1, for $r \leq 1/3$, we have

$$\sum_{k=N}^{\infty} |b_k|r^k \leq \sum_{n=N}^{\infty} |a_n|r^n.$$

Again applying Lemma 2.1 on $f(z)/z \prec f_0(z)/z$ (Lemma 2.2), we get that

$$\sum_{k=N}^{\infty} |b_k|r^k \leq \sum_{n=N}^{\infty} |a_n|r^n \leq \sum_{n=N}^{\infty} |t_n|r^n, \quad r \leq \frac{1}{3}.$$ (2.5)

Now $g \prec f$ implies that $g(z) = f(\omega(z))$, which using the Lemma 2.2 yields

$$|g(z)| = |f(\omega(z))| \leq f_0(r)$$

for $|z| = r$, where $\omega$ is a Schwarz function. Moreover,

$$|g(z^m)| \leq \hat{f}_0(r^m).$$ (2.6)

Also, by letting $r$ tends to 1 in Lemma 2.2, we obtain the Koebe-radius $r_* = -f_0(1)$. Therefore, the open ball $B(0, r_*) \subset f(\mathbb{D})$, which implies that for $|z| = 1$

$$r_* \leq d(0, \partial \Omega).$$ (2.7)

Now using the equations (2.5), (2.6) and (2.7), we have

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq \hat{f}_0(r^m) + \sum_{n=N}^{\infty} |t_n|r^n$$

$$= \hat{f}_0(r^m) + \hat{f}_0(r) - p_{f_0}(r)$$

$$\leq r_* \leq d(0, \partial \Omega).$$
holds whenever \(|z| = r \leq \min\{\frac{1}{3}, r_0\}\), where \(r_0\) is the smallest positive root of the equation:

\[
G(r) := \hat{f}_0(r^m) + \hat{f}_0(r) - p_{f_0}(r) - r_* = 0.
\]

Note that \(G(0) < 0\), and since \(\hat{f}_0(1) \geq |f_0(1)| \geq r_*\), we see that

\[
2\hat{f}_0(1) - \sum_{n=1}^{N-1} |t_n| - r_* = (\hat{f}_0(1) - \sum_{n=1}^{N-1} |t_n|) + (\hat{f}_0(1) - r_*) > 0
\]

where \(t_1 = 1\), which implies \(G(1) > 0\). Clearly, for \(0 \leq r \leq 1\)

\[
G'(r) = \hat{f}'_0(r^m) + (\hat{f}'_0(r) - p'_{f_0}(r)) > 0,
\]

which implies \(G\) is a continuous increasing function in \([0, 1]\). Thus \(G(r) = 0\) has a root in the interval \((0, 1)\). The sharpness follows for the function \(f_0\) as

\[
f_0(r^m_0) + \sum_{n=N}^{\infty} t_n r_b^n = r^* = d(0, \partial\Omega)
\]

when \(r_b = r_0\) and \(t_n > 0\).

\(\square\)

**Remark 2.2.** Let \(\psi(z) = (1 + z)/(1 - z)\), then Theorem 2.3 reduces to [12, Theorem 5].

**Remark 2.3.** Observe that if we take \(m \to \infty\) and \(N = 1\), then Theorem 2.3 reduces to [16, Theorem 5.1].

**Corollary 2.4.** Let \(r_*\) be the Koebe-radius for the class \(S^s(\psi)\), \(f_0(z)\) be given by the equation (2.2). Assume \(f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n\) and \(\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n\). If \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^s(\psi)\). Then

\[
|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(0, \partial\Omega)
\]

(2.8)

holds for \(|z| = r_b \leq \min\{\frac{1}{3}, r_0\}\), where \(m, N \in \mathbb{N}\), \(\Omega = f(\mathbb{D})\) and \(r_0\) is the unique positive root of the equation:

\[
\hat{f}_0(r^m) + \hat{f}_0(r) - p_{f_0}(r) = r_*,
\]

where \(p_{f_0}\) is as defined in Theorem 2.3. The radius is sharp for the function \(f_0\) when \(r_b = r_0\) and \(t_n > 0\).

**Corollary 2.5.** Let \(\psi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2\), \(f_0(r) = r \exp\left(\frac{4}{3}r + \frac{r^2}{3}\right)\) and \(m = 1\). If \(g \in S_f(\psi)\). Then the inequality (2.3) holds for \(|z| = r \leq r_N\), where \(N \in \mathbb{N}\) and \(r_N < 1/3\) is the unique positive root of the equation:

\[
2r \exp\left(\frac{4}{3}r + \frac{r^2}{3}\right) - p_{f_0}(r) - \exp(-1) = 0,
\]

where \(p_{f_0} = p_{\hat{f}_0}\) is as defined in Theorem 2.3 with \(|t_n| = t_n = f_0^n(0)/n!\). Moreover, if \(f \in S^s(\psi)\). Then the inequality (2.8) also holds for \(r \leq r_N\). The radius \(r_N\) is sharp.
Remark 2.4. In Corollary 2.5 we observe that the radius $r_N$ approaches $r_0 = 0.25588 \cdots$ for large value of $N$, where $r_0$ is the unique positive root of
\[
r \exp \left( \frac{4}{3}r + \frac{r^2}{3} \right) - \exp(-1) = 0.
\]
Moreover, if $m \geq 2$ then the inequalities (2.3) and (2.8) hold for $r \leq 1/3$.

Corollary 2.6. Let $\psi(z) = 1 + z e^z$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (2.3) holds for $|z| = r \leq r_N = \{r_0, 1/3\}$, where $N \in \mathbb{N}$ and $r_0$ is the unique positive root of the equation:
\[
2r \exp(e^r - 1) - T(r) - \exp(e^{-1} - 1) = 0,
\]
where
\[
T(r) = \begin{cases} 
0, & N = 1; \\
r, & N = 2; \\
\sum_{n=1}^{N-1} \frac{B_{n+1}}{(n-1)!} r^n, & N \geq 3
\end{cases}
\]
and $B_n$ are the bell numbers such that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Moreover, if $f \in S^*(\psi)$. Then the inequality (2.8) also holds for $r \leq r_N$. The radius $r_N < 1/3$ is sharp for $N \leq 3$.

Corollary 2.7. Let $\psi(z) = 1 + \frac{x}{k} \left( \frac{k+x}{k-x} \right)$ with $k = \sqrt{2} + 1$. If $g \in S_f(\psi)$. Then the inequality (2.3) holds for $|z| = r \leq r_b = \min\{1/3, r_0\}$, where $N \in \mathbb{N}$ and $r_0$ is the unique positive root of the equation:
\[
r^{m} \exp \left( \frac{k}{k-r} \right) - \frac{r}{e^r} \left( \frac{k}{k-r} \right)^{2k} - p_{f_0}(r) - e \left( \frac{k}{k+1} \right)^{2k} = 0,
\]
where $p_{f_0} = p_{f_0}$ is as defined in Theorem 2.3 and $t_n = |t_n|$ are the Taylor coefficients of the function $f_0(r) = r \exp \left( \frac{k}{k-r} \right)$. Moreover, if $f \in S^*(\psi)$. Then the inequality (2.8) also holds for $r \leq r_b$. The radius $r_b$ is sharp when $m = 1$ and $N \leq 4$.

Since all the Taylor coefficients of the function $1 + \sin z$ are not positive, $f_0 \neq f_0$. So we consider the radius $r_N$ upto three decimal places only, which also reveals the connection of positive coefficients of $\psi$ to the sharp Bohr-Rogosinski radius.

Corollary 2.8. Let $\psi(z) = 1 + \sin z$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (2.3) holds for $|z| = r \leq r_N$, where $N \in \mathbb{N}$ and $r_N(<1/3)$ is the unique positive root of the equation:
\[
2r \exp(Si(r)) - \exp(Si(-1)) - p_{f_0}(r) = 0,
\]
where $f_0(r) = r \exp(Si(r))$, where $Si(x)$ is the Sin Integral defined as:
\[
Si(x) := \int_0^x \frac{\sin(x)}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}
\]
Moreover, if $f \in S^*(\psi)$. Then the inequality (2.8) also holds for $r \leq r_N$. 

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Remark 2.5. In Corollary 2.8, the numerical computations reveal that the Bohr-Rogosinski radius $r_N \approx 0.290 \cdots < 1/3$ for any $N > 4$, where $* = 6$ or 7. Also $r_N < 1/3$ for $N \leq 4$. Moreover, as $N \to \infty$, the required radius $r_0 \approx 0.290 \cdots$ is the unique positive root of

$$r \exp(Si(r)) - \exp(Si(-1)) = 0.$$ 

Next we discuss the Bohr-Rogosinski phenomenon for the celebrated Janowski class of univalent starlike functions. For this, we first need the following: for simplicity write $S_*((1+Dz)/(1+Ez)) \equiv S[D,E]$, where $-1 \leq E < D \leq 1$.

Lemma 2.3. [5, Theorem 3] If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S[D,E].$$

Then for $n \geq 2$, the following sharp bounds occur:

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|E-D+Ek|}{k+1}.$$

Corollary 2.9. Let

$$\psi(z) = (1+Dz)/(1+Ez), \quad -1 \leq E < D \leq 1.$$ 

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\psi)$. Then

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(0,\partial \Omega)$$

holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the unique positive root of the equations:

$$r^m(1+Er^m)\frac{D-E}{E-D} + A(r) + \sum_{n=N}^{\infty} \prod_{k=0}^{n-2} \frac{|E-D+Ek|}{k+1} r^n - (1-E) \frac{D-E}{E-D} = 0, \quad \text{if } E \neq 0,$$

where $A(r) = r$ for $N = 1$ and 0 otherwise, and

$$r^m e^{Dr^m} + (1-E) D - J(r) - e^{-D} = 0, \quad \text{if } E = 0,$$

where

$$J(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2; \\ \sum_{n=2}^{N-1} \prod_{k=0}^{n-2} \frac{D}{k+1} r^n, & N \geq 3. \end{cases}$$

The radius $r_0$ is sharp.

Proof. Let us consider the function $f_0$ such that $zf_0'(z)/f_0(z) = (1+Dz)/(1+Ez)$, which is given by

$$f_0(z) = \begin{cases} z(1+Ez) \frac{D-E}{E-D}, & E \neq 0; \\ ze^{Dz}, & E = 0. \end{cases}$$

(2.11)

Now using the Lemma 2.2 and Lemma 2.3 we have

$$|f(z^m)| \leq f_0(r^m), \quad r_* = -f_0(-1)$$

and

$$\sum_{n=N}^{\infty} |a_n||z|^n \leq \sum_{n=N}^{\infty} \prod_{k=0}^{n-2} \frac{|E-D+Ek|}{k+1} r^n, \quad N \geq 2.$$
Now proceeding as in Theorem 2.3 for $r_0$ as defined in the statement, the result follows. To prove the sharpness of the radius $r_0$, we see that at $|z| = r = r_0$ and $f = f_0$ given in (2.11):

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n$$

$$= \begin{cases} (r_0)^m(1 + E(r_0)^m)\frac{D-E}{E} + A(r_0) + \sum_{n=N}^{\infty} n^{-2} \prod_{k=0}^{n-2} \frac{|E-D+E_k|}{k+1}(r_0)^n, & E \neq 0; \\ (r_0)^m e^{D(r_0)^m} + (r_0) e^{Dr_0} - J(r_0), & E = 0. \end{cases}$$

$$= \begin{cases} (1 - E)\frac{D-E}{E}, & E \neq 0; \\ e^{-D}, & E = 0. \end{cases}$$

$$= -f_0(-1) = d(0, \partial \Omega),$$

where $J(r)$ is as defined in (2.10), and $A(r) = r$ for $N = 1$ and 0 otherwise for the case $E \neq 0$. \hfill \Box$

**Remark 2.6.** Taking $m \to \infty$ and $N = 1$ in Corollary 2.9, we obtain the Bohr radius for the class $S[D, E]$, which covers many classical cases.

In Corollary 2.9, putting $D = 1 - 2\alpha$ and $E = -1$, where $0 \leq \alpha < 1$, we get the result for the class of univalent starlike functions of order $\alpha$, that is, $S^*(\alpha)$:

**Corollary 2.10.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha)$. Then the inequality (2.9) holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the smallest positive root of the equations:

$$\frac{r^m}{(1 - r^m)^{2(1-\alpha)}} + A(r) + \sum_{n=N}^{\infty} n^{-2} \prod_{k=0}^{n-2} k + 2(1-\alpha) \frac{r^n}{k+1} - \frac{1}{4^{1-\alpha}} = 0,$$

where $A(r) = r$ for $N = 1$ and 0 otherwise. The radius $r_0$ is sharp.

Putting $\alpha = 0$ in Corollary 2.10, we get the following:

**Corollary 2.11.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$. Then the inequality (2.9) holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the smallest positive root of the equations:

$$4r^m - (1 - r^m)^2 + 4r^N(N(1-r)+r) \left(\frac{1 - r^m}{1-r} \right)^2 = 0.$$

The radius $r_0$ is sharp.

To proceed further, we need to recall the following fundamental result:

**Lemma 4.** \cite{19} Let $f \in C(\psi)$. Then $zf''(z)/f'(z) \prec zl''_0(z)/l'_0(z)$ and $f'(z) \prec l'_0(z)$. Also, for $|z| = r$ we have

$$-l_0(-r) \leq |f(z)| \leq l_0(r),$$

where

$$zl''_0(z)/l'_0(z) = \psi(z).$$

(2.12)
Now we discuss the results for the convex analogue $C(\psi)$ of $S^*(\psi)$.

**Theorem 2.12.** Let $r_*$ be the Koebe-radius for the class $C(\psi)$, $l_0(z)$ be given by the equation (2.12) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\psi)$. Assume $l_0(z) = z + \sum_{n=2}^{\infty} l_n z^n$ and $\hat{l}_0(r) = r + \sum_{n=2}^{\infty} |l_n| r^n$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial \Omega)$$

(2.13)

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the unique positive root of the equation:

$$\hat{l}_0(r^m) + \hat{l}_0(r) - p_{l_0}(r) = r_*,$$

where

$$p_{l_0}(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2; \\ r + \sum_{n=2}^{N-1} |l_n| r^n, & N \geq 3. \end{cases}$$

The result is sharp when $r_b = r_0$ and $l_n > 0$.

**Proof.** Let $g(z) = \sum_{k=1}^{\infty} b_k z^k < f(z)$, where $f \in C(\psi)$. From the Alexander relation, it is known that $f \in C(\psi)$ if and only if

$$zf'(z) = \tilde{g}(z),$$

or equivalently $f(z) = \int_0^z \frac{\tilde{g}(t)}{t} dt$

for some $\tilde{g} \in S^*(\psi)$. Now by Lemma 2.1 for $r \leq 1/3$, we have

$$\sum_{k=N}^{\infty} |b_k|r^k \leq \sum_{n=N}^{\infty} |a_n|r^n = \sum_{n=N}^{\infty} \frac{\tilde{b}_n}{n} r^n,$$

(2.14)

where $\tilde{b}_n$ are the Taylor coefficients of $\tilde{g}$. Again applying Lemma 2.1 on $f'(z) < l'_0(z)$ (Lemma 2.3), we get that

$$M_\tilde{g}(r) - p_{\tilde{g}}(r) \leq M_h(r) - p_h(r), \quad r \leq \frac{1}{3},$$

(2.15)

where $Mg(x) := \sum_{k=1}^{\infty} |b_k|x^k$, and $h$ is given by the relation $zl'_0(z) = h(z)$. Now using the equations (2.14) and (2.15), we have for $r \leq 1/3$

$$\sum_{k=N}^{\infty} |b_k||z|^k \leq \sum_{n=N}^{\infty} \frac{\tilde{b}_n}{n} r^n$$

$$= \int_0^r \frac{M_\tilde{g}(t) - p_{\tilde{g}}(t)}{t} dt$$

$$\leq \int_0^r \frac{M_h(t) - p_h(t)}{t} dt = \sum_{n=N}^{\infty} |l_0| r^n$$

(2.16)

$$= \hat{l}_0(r) - p_{l_0}(r).$$

Now $g \preceq f$ implies that $g(z) = f(\omega(z))$, which using the Lemma 2.4 yields

$$|g(z)| = |f(\omega(z))| \leq l_0(r)$$

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for $|z| = r$, where $\omega$ is a Schwarz function. Moreover,

$$|g(z^m)| \leq \hat{l}_0(r^m).$$  \hfill (2.17)

Also, by letting $r$ tends to 1 in Lemma 2.12, we obtain the Koebe-radius $r_* = -l_0(-1)$. Therefore, the open ball $B(0, r_*) \subset f(D)$, which implies that for $|z| = 1$

$$r_* \leq d(0, \partial \Omega).$$  \hfill (2.18)

Hence, using the inequalities (2.16), (2.17) and (2.18), we have

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq \hat{l}_0(r^m) + \hat{l}_0(r) - p_{l_0}(r) \leq r_* \leq d(0, \partial \Omega)$$

holds whenever $|z| = r \leq \min\{\frac{1}{2}, r_0\}$, where $r_0$ is the smallest positive root of the equation:

$$H(r) := \hat{l}_0(r^m) + \hat{l}_0(r) - p_{l_0}(r) - r_* = 0.$$  

Clearly, $H$ is continuous and $H'(r) > 0$ for $0 \leq r \leq 1$. Note that $H(0) < 0$, and since $\hat{l}_0(1) \geq |l_0(1)| \geq r_*$, we see that

$$2\hat{l}_0(1) - \sum_{n=1}^{N-1} |l_n| - r_* = (\hat{l}_0(1) - \sum_{n=1}^{N-1} |l_n|) + (\hat{l}_0(1) - r_*) > 0,$$

which implies $H(1) > 0$. Thus $H(r) = 0$ has a root in the interval $(0, 1)$. The sharpness follows for the function $l_0$ as

$$l_0(r^m_b) + \sum_{n=N}^{\infty} l_n r_n^m = r_* = d(0, \partial \Omega)$$

when $r_b = r_0$ and $l_n > 0$. \hfill $\square$

**Remark 2.7.** Let $\psi(z) = (1 + z)/(1 - z)$, then Theorem 2.12 reduces to [12, Theorem 6].

The following result is explicitly for the class $C(\psi)$.

**Corollary 2.13.** Let $r_*$ be the Koebe-radius for the class $C(\psi)$, $l_0(z)$ be given by the equation (2.12). Assume $l_0(z) = z + \sum_{n=2}^{\infty} c_n z^n$ and $\hat{l}_0(r) = r + \sum_{n=2}^{\infty} |l_n| r^n$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C(\psi)$. Then

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(0, \partial \Omega)$$  \hfill (2.19)

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(D)$ and $r_0$ is the unique positive root of the equation:

$$\hat{l}_0(r^m) + \hat{l}_0(r) - p_{l_0}(r) = r_*,$$

where $p_{l_0}$ is as defined in Theorem 2.12. The radius is sharp for the function $l_0$ when $r_b = r_0$ and $l_n > 0$.  

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Remark 2.8. The special case of taking $m \to \infty$ and $N = 1$ in Theorem 2.13 and Corollary 2.13 establish the Bohr phenomenon for the classes $S_f(\psi)$ and $C(\psi)$, respectively.

After some little computations when $\psi(z) = (1 + z)/(1 - z)$, the Corollary 2.13 yields:

**Corollary 2.14.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C$. Then the inequality (2.19) holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $r_0$ is the unique positive root of the equations:

$$3r^m - 1 + 2r^N \left( \frac{1 - r^m}{1 - r} \right) = 0.$$  

The radius $r_0$ is sharp.

**Corollary 2.15.** Let $\psi(z) = 1 + ze^z$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (2.13) holds for $|z| = r \leq r_N$, where $N \in \mathbb{N}$ and $r_N(< 1/3)$ is the unique positive root of the equation:

$$2r(1 + re^r) \exp(e^r - 1) - (1 - e^{-1})e^{e^{-1}-1} = 0,$$

where

$$H(r) = \begin{cases} 
0, & N = 1; \\
N = 2; \\
\sum_{n=0}^{N-1} \frac{(n+1)B_n}{n!} r^{n+1}, & N \geq 3.
\end{cases}$$

and $B_n$ are the bell numbers such that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Moreover, if $f \in C(\psi)$. Then the inequality (2.19) also holds for $r \leq r_N$. The radius $r_N$ is sharp.

**Conflict of interest**

The authors declare that they have no conflict of interest.

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DEPARTMENT OF APPLIED MATHEMATICS, DELHI TECHNOLOGICAL UNIVERSITY, DELHI–110042, INDIA

Email address: gangania.m1991@gmail.com

DEPARTMENT OF APPLIED MATHEMATICS, DELHI TECHNOLOGICAL UNIVERSITY, DELHI–110042, INDIA

Email address: spkumar@dce.ac.in