ON THE RANGE OF COSINE TRANSFORM OF DISTRIBUTIONS FOR TORUS-INVARIANT COMPLEX MINKOWSKI SPACES

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Abstract. In this paper, we study the ranges of (absolute value) cosine transforms for which we give a proof for an extended surjectivity theorem by making applications of the Fredholm’s theorem in integral equations, and show a Hermitian characterization theorem for complex Minkowski metrics on $\mathbb{C}^n$. Moreover, we parametrize the Grassmannian in an elementary linear algebra approach, and give a characterization on the image of the (absolute value) cosine transform on the space of distributions on the Grassmannian $Gr_2(\mathbb{C}^2)$, by computing the coefficients in the Legendre series expansion of distributions.

1. ON THE FREDHOLM THEORY IN INTEGRAL EQUATIONS

Integral equations as different looks from differential equations appear in mathematical physics and fluid mechanics (see for instance [8]) and other fields. A groundbreaking work in the theory of integral equation was done by Fredholm, [2], in 1903. The following is one of his main theorems on the existence of solutions to Fredholm integral equations of the second kind.

Theorem 1.1. Let $K(x, y)$ and $f(x)$ be real valued functions, $\lambda \in \mathbb{R}$ and $K(x, y) \in L^2([a, b]^2)$. Then there exist solutions to the Fredholm integral equation of the second kind

$$\lambda \phi(x) - \int_a^b K(x, y) \phi(y) \, dy = f(x)$$

(1.1)

if and only if $f(x)$ satisfies

$$\int_a^b \psi(x) f(x) \, dx = 0$$

(1.2)

for any solution $\psi(x)$ to the homogeneous integral equation

$$\lambda \psi(y) - \int_a^b \psi(y) K(x, y) \, dx = 0.$$  

(1.3)

As for solving integral equations, it is not hard to solve Fredholm integral equations with separable variables, for that and some other types of integral equations one can see [1]. One can also use Fourier on convolution to express solution explicitly if the integral in (1.1) is a convolution.

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2. On $U(1) \times U(1)$-invariant Complex Finsler Metrics and $U(1) \times U(1)$-Orbits of $Gr_2(C^2)$

Given a complex Finsler space $(C^2, F)$, where $F$ is a complex Finsler metric. One of the main topics in integral geometry is to find the Crofton measures for Finsler metrics. However, there is an important class of Finsler metrics, $U(1) \times U(1)$ invariant complex ones.

If $F$ is a $U(1) \times U(1)$ invariant complex Finsler metric, then
\[
\hat{F} := F|_{(\mathbb{R} \times \{0\}) \oplus (\mathbb{R} \times \{0\})}
\]
is a Finsler metric on $\mathbb{R}^2$. Conversely, one can extend a Finsler metric on $\mathbb{R}^2$ to get a $U(1) \times U(1)$ invariant complex Finsler metric on $C^2$.

For the Crofton measure of $U(1) \times U(1)$ of invariant complex Finsler metric, we have the following

Theorem 2.1 (Invariance property of Crofton measure). The Crofton measure for $U(1) \times U(1)$ invariant complex Finsler metric $F$ on $\mathbb{C}^2$ is $U(1) \times U(1)$ invariant.

Proof. Let $\mu$ be the Crofton measure for the $U(1) \times U(1)$ invariant complex Finsler metric $F$ on $\mathbb{C}^2$ and $d\mu = f(\xi_1, \xi_2, \eta)d\xi_1d\xi_2d\eta$, then for any $(\bar{z}, \bar{w}) \in S^3$, then we have $F(z, w) = F(e^{i\xi_1} z, e^{i\xi_2} w)$ for any $(e^{i\xi_1}, e^{i\xi_2}) \in U(1) \times U(1)$.

On one hand,
\[
F(z, w) = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\cos(\xi_1 - \tilde{\xi}_1)\cos \eta + \cos(\xi_2 - \tilde{\xi}_2)\sin \eta| \cdot f(\xi_1, \xi_2, \eta)d\xi_1d\xi_2d\eta
\]
(2.2)

On the other hand, we know for any $(\bar{z}, \bar{w}) = (e^{i\xi_1} \cos \tilde{\eta}, e^{i\xi_2} \sin \tilde{\eta}) \in S^3$,
\[
F(\bar{z}, \bar{w}) = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\cos(\xi_1 - \tilde{\xi}_1 - \bar{\xi}_1)\cos \eta + \cos(\xi_2 - \tilde{\xi}_2 - \bar{\xi}_2)\sin \eta| \cdot f(\xi_1, \xi_2, \eta)d\xi_1d\xi_2d\eta
\]
(2.3)

by change of variables. Using the injectivity theorem of cosine transform, Proposition 3.4.12 in [3], from $F(z, w) = F(e^{i\xi_1} z, e^{i\xi_2} w)$ we have
\[
f(\xi_1, \xi_2, \eta) = f(\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2, \tilde{\eta})
\]
(2.4)
for any $\tilde{\xi}_1, \tilde{\xi}_2 \in [0, 2\pi]$. $\square$

Since the function $f$ is independent of $\xi_1$ and $\xi_2$ by the invariance of the complex norm under $U(1) \times U(1)$ action, so it can be denoted as $f(\eta)$.

In the next, we consider the action of torus $U(1) \times U(1)$ on the space of real 2-planes in the complex plane, $Gr_2(C^2)$. The following proposition about the orbits of torus action was proposed by Joe Fu, but here we provide a proof with linear algebra flavor

Proposition 2.2 (Orbits parametrization of the Grassmannian). The orbits of $Gr_2(C^2)$ acted by torus actions can be parametrized as
\[
\{ \text{span}_{\mathbb{R}}((\cos \psi, \sin \psi), (\sqrt{-1}\cos(\theta + \psi), \sqrt{-1}\sin(\theta + \psi))) : (\theta, \psi) \in [0, \pi/2]^2 \}.
\]
(2.5)
Since a torus action preserves the argument differences of each component of any two vectors in $\mathbb{C}^2$, so to prove Proposition 2.2, it suffices to show the following

**Lemma 2.3.** For any plane $P \in Gr_2(\mathbb{C}^2)$, either there exist some $(z_0, w_0) \in P \setminus \{0\}$ and $r, s \in \mathbb{R}$ such that $(\sqrt{-1}rz_0, \sqrt{-1}sw_0) \in P$, in other words,

$$P = \text{span}_{\mathbb{R}}((z_0, w_0), (\sqrt{-1}rz_0, \sqrt{-1}sw_0)), \tag{2.6}$$

or there exists a pair of vectors $(z_1, w_1), (z_2, w_2) \in P \setminus \{0\}$ such that $z_1w_2 = z_2w_1 = 0$.

**Remark 2.4.** We call the vector $(z_0, w_0)$ a quasi-$J$-characteristic vector of the plane $P$. In particular, every non-zero vector in a complex line $L$ in $\mathbb{C}^2$ is a quasi-$J$-characteristic vector of $L$.

Let $T^2 := \{\text{span}_{\mathbb{R}}((z, 0), (0, w)) : z, w \in U(1)\} \cong U(1) \times U(1)$, then in fact the latter part of the conclusion in Lemma 2.3 is derived from the planes in $T^2$. For planes which are not in $T^2$, we need to show that they generate the former part of the conclusion in Lemma 2.3, which is geometrically equivalent to

**Lemma 2.5** (Intersection lemma). For any $P \in Gr_2(\mathbb{C}^2) \setminus T^2$, there exist $r, s \in \mathbb{R}$ such that $\dim(P_{r,s} \cap P) > 0$ where $P_{r,s} := \{(\sqrt{-1}rz, \sqrt{-1}sw) : (z, w) \in P\}$.

**Proof.** Let $P = \text{span}_{\mathbb{R}}((z_1, w_1), (z_2, w_2)) \in Gr_2(\mathbb{C}^2) \setminus T^2$, and $z_i = x_i + \sqrt{-1}y_i$ and $w_i = u_i + \sqrt{-1}v_i$ for $i = 1, 2$. Using the determinants of block matrices by partitioning a matrix, one can obtain that

$$\det\begin{pmatrix}
x_1 & y_1 & u_1 & v_1 \\
x_2 & y_2 & u_2 & v_2 \\
-ry_1 & rx_1 & -sv_1 & sv_1 \\
-ry_2 & rx_2 & -sv_2 & sv_2
\end{pmatrix} = Ar^2 + Brs + Cs^2 \tag{2.7}$$

for some $A, B, C \in \mathbb{R}$ with $A = -C = \det(M_{11})\det(M_{22})$ where $M_{11} = \begin{pmatrix} x_1 & y_1 \\
x_2 & y_2 \end{pmatrix}$ and $M_{22} = \begin{pmatrix} -v_1 & u_1 \\
-v_2 & u_2 \end{pmatrix}$. Therefore, there exist $r, s \in \mathbb{R}$ and either $r$ or $s$ is not 0, such that the determinant (2.7) is identical to 0. It follows that

$$P \oplus P_{r,s} = \text{span}_{\mathbb{R}}((z_1, w_1), (z_2, w_2), (\sqrt{-1}rz_1, \sqrt{-1}sw_1), (\sqrt{-1}rz_2, \sqrt{-1}sw_2)) \subseteq \mathbb{C}^2, \tag{2.8}$$

and then we have $\dim(P_{r,s} \cap P) > 0$ by the inclusion-exclusion principle. \quad \square

Thus we have shown Lemma 2.3. Furthermore, one can choose appropriate $(\theta, \psi) \in [0, \frac{\pi}{2}]^2$ such that $\text{span}_{\mathbb{R}}((\cos \psi, \sin \psi), (\sqrt{-1}\cos(\theta + \psi), \sqrt{-1}\sin(\theta + \psi)))$ and $P$ are on the same orbit of $Gr_2(\mathbb{C}^2)$ acted by torus actions. So we have finished the proof for Proposition 2.2.

3. **A Surjectivity Theorem**

We want to extend the surjectivity theorem on the cosine transform to functions which are not even differentiable away from zero by making applications of Fredholm’s theorem on integral equations.
Theorem 3.1 (Surjectivity theorem). For any \( U(1) \times U(1) \)-invariant function \( F : \mathbb{C}^2 \rightarrow \mathbb{R} \) with homogeneity of magnitude, there is some function \( f \) on \( S^n \), such that

\[
F(\cdot) = \int_{S^n} |\langle \xi, \cdot \rangle| f(\xi) d\xi
\]  

(3.1)

Proof. Let

\[
K(\eta, \bar{\eta}) := \int_0^{2\pi} \int_0^{2\pi} |\cos(\xi_1 - \bar{\xi}_1) \cos \eta \cos \bar{\eta} + \cos(\xi_2 - \bar{\xi}_2) \sin \eta \sin \bar{\eta}| d\xi_1 d\xi_2
\]  

(3.2)

because the double integral is independent of \( \xi_1 \) and \( \xi_2 \). Considering the integral equation

\[
\int_0^{2\pi} K(\eta, \bar{\eta}) f(\eta) \, d\eta = F(\eta),
\]  

(3.3)

and applying Theorem 1.1 to it, we know that there exists some \( f(\eta) \) satisfying integral equation (3.3).

\[\square\]

In the theory of convex bodies, [7], the support function of the unit ball in a Minkowski space is actually the metric function, and the ball is called a generalized zonoid if its support function is in the range of cosine transform on the functions on \( S^3 \). Hence we have the following

Corollary 3.2. The unit ball of any complex Minkowski plane \((\mathbb{C}^2, F)\) with \( U(1) \times U(1) \)-invariant complex Minkowski metric \( F \) is a generalized zonoid.

Remark 3.3. To apply the integral equation theory, one does not need any smoothness condition on the metric. However, the approach of integral equation theory cannot be generalized to Minkowski metric on \( \mathbb{R}^n \) for any \( n \), in which the unit ball could be not a generalized zonoid, for example, the octahedron, as pointed out by Joe Fu, in \( \mathbb{R}^3 \) with \( l^1 \) metric.

4. On a Complex Minkowski Metric To Be Hermitian on \( \mathbb{C}^n \)

From the perspective of complex integral geometry, the following theorem on a characterization of complex Minkowski metric \( \mathbb{C}^n \) to be Hermitian is established

Theorem 4.1 (Characterization of Hermitian metric). Suppose that \((\mathbb{C}^n, F)\) is a complex Minkowski space. Then the Holmes-Thompson valuation, that is extended from the Holmes-Thompson area on \((\mathbb{C}^n, F)\), restricted on \( \mathbb{C}P^{n-1} \) is in the range of the cosine transform on \( C(\mathbb{C}P^{n-1}) \) if and only if the complex Minkowski metric \( F \) is Hermitian.

Proof. For any fixed complex line \( L \in \mathbb{C}P^{n-1} \), let \( U \) be the rectangle spanned by \( v := (z_1, \cdots, z_n) \in L \) and \( \sqrt{-1}v \in L \). Since \( F \) is \( U(1) \)-invariant on \( L \), then the Holmes-Thompson area of \( U \) is

\[
HT^2(U) = F^2(v).
\]  

(4.1)

On the other hand, for any complex line \( \tilde{L} := \text{span}_\mathbb{C}(\tilde{e}) \in \mathbb{C}P^{n-1} \) where \( \tilde{e} := (\tilde{z}_1, \cdots, \tilde{z}_n) \in \mathbb{C}^n \) with \( |\tilde{e}| = (\sum_{i=1}^n |\tilde{z}_i|^2)^{1/2} = 1 \), we know that

\[
\text{area}(\pi_{\tilde{L}}(U)) = |\det\begin{pmatrix}
\text{Re}(\langle v, \tilde{e} \rangle_\mathbb{C}) & \text{Im}(\langle v, \tilde{e} \rangle_\mathbb{C}) \\
\text{Im}(\langle v, \tilde{e} \rangle_\mathbb{C}) & -\text{Re}(\langle v, \tilde{e} \rangle_\mathbb{C})
\end{pmatrix}| = |\langle v, \tilde{e} \rangle_\mathbb{C}|^2,
\]  

(4.2)
in which $\langle v, \bar{e} \rangle_C$ is the complex inner product, and $\text{area}(\pi_L(U))$ is independent of the choice of unit vector $\bar{e}$ in $\mathcal{L}$.

If $HT^2$ is in the range of cosine transform on $C(\mathbb{CP}^{n-1})$, then there exists some function $f : \mathbb{CP}^{n-1} \to \mathbb{R}$, such that

$$\int_{\mathbb{CP}^{n-1}} \text{area}(\pi_L(U)) f(\bar{L}) d\bar{L} = F^2(v). \tag{4.3}$$

Since $\mathbb{CP}^{n-1} = S^{2n-1}/U(1)$, then by (4.2) we have

$$\int_{\mathbb{CP}^{n-1}} \text{area}(\pi_L(U)) f(\bar{L}) d\bar{L} = \int_{S^{2n-1}/U(1)} |\langle v, \bar{e} \rangle_C|^2 f(\bar{e}) d\bar{e}. \tag{4.4}$$

Written in terms of components of $\bar{e}$ and $v$,

$$\int_{\mathbb{CP}^{n-1}} \text{area}(\pi_L(U)) f(\bar{L}) d\bar{L} = \int_{S^{2n-1}/U(1)} \sum_{i=1}^n |\bar{z}_i\bar{z}_i|^2 f(\bar{e}) d\bar{e} = \int_{S^{2n-1}/U(1)} \sum_{i,j=1}^n \bar{z}_i\bar{z}_j f(\bar{e}) d\bar{e} \tag{4.5}$$

Let $h_{ij} := \int_{S^{2n-1}/U(1)} \bar{z}_i\bar{z}_j f(\bar{e}) d\bar{e}$, then $h_{ij} = h_{ji}$ since $f(\bar{e}) \in \mathbb{R}$. Thus it follows from (4.3) that $F^2(v) = \sum_{i,j=1}^n h_{ij} \bar{z}_i\bar{z}_j$ is Hermitian.

Conversely, one can see, by the fact that $\{ \bar{z}_i\bar{z}_j : i, j = 1, \cdots, n \}$ are linearly independent in the Hilbert space $L^2(S^{2n-1}/U(1))$ and

$$F|_{\mathbb{CP}^{n-1}} \in C(\mathbb{CP}^{n-1}) \subset L^2(S^{2n-1}/U(1)), \tag{4.6}$$

that if $F$ is Hermitian then the Holmes-Thompson valuation restricted on $\mathbb{CP}^{n-1}$ is in the range of the cosine transform on $C(\mathbb{CP}^{n-1})$. □

**Remark 4.2.** Form the proof of Theorem 4.1, we know that the range of the cosine transform on $C(\mathbb{CP}^{n-1})$ is finite dimensional.

5. **Revolutions of Spheres and Torus Actions**

We have shown the following

**Proposition 5.1.** The orbits of $Gr_2(\mathbb{C}^2)$ acted by torus actions can be parametrized as

$$\left\{ \text{span}_\mathbb{R}((\cos \theta, \sin \theta), (\sqrt{-1}\cos(\theta + \psi), \sqrt{-1}\sin(\theta + \psi))) : (\theta, \psi) \in [0, \frac{\pi}{2}] \right\}. \tag{5.1}$$

Gluck and Warner in [5] gave an isomorphism from $Gr_2^+(\mathbb{R}^4)$ to $S^2 \times S^2$ in (5.1), that is expressed explicitly by

$$\iota(v_1 \wedge v_2) := (\sqrt{2} (v_1 \wedge v_2 + (v_1 \wedge v_2)^\perp), \sqrt{2} (v_1 \wedge v_2 - (v_1 \wedge v_2)^\perp)), \tag{5.2}$$

in which $\{(v_1, v_2)\}$ is an orthonormal basis of the plane spanned by them in $Gr_2^+(\mathbb{R}^4)$ and $(v_1 \wedge v_2)^\perp$ here denotes the wedge of the orthonormal basis of the complement of $v_1 \wedge v_2$, and so we have the Cartesian product decomposition

$$Gr_2^+(\mathbb{R}^4) \cong S^2 \times S^2. \tag{5.3}$$

In [4], Goodey and Howard described the bases

$$b_1^+ = \frac{\sqrt{2}}{2} (e_1 \wedge e_2 + e_3 \wedge e_4), \quad b_2^+ = \frac{\sqrt{2}}{2} (e_1 \wedge e_2 - e_3 \wedge e_4), \quad b_3^+ = \frac{\sqrt{2}}{2} (e_1 \wedge e_4 + e_2 \wedge e_3), \tag{5.4}$$
and
\[ b_1^- = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 - e_3 \wedge e_4), \quad b_2^- = \frac{\sqrt{2}}{2}(e_1 \wedge e_3 + e_2 \wedge e_4), \quad b_3^- = \frac{\sqrt{2}}{2}(e_1 \wedge e_4 - e_2 \wedge e_3), \quad (5.5) \]

where \( \{e_1, e_2, e_3, e_4\} \) is the basis for \( \mathbb{R}^4 \), for the components \( \wedge^2_1(\mathbb{R}^4) \) and \( \wedge^2(\mathbb{R}^4) \) respectively in the vector space decomposition
\[ \wedge^2(\mathbb{R}^4) = \wedge^2_1(\mathbb{R}^4) \oplus \wedge^2(\mathbb{R}^4). \quad (5.6) \]

In \( \mathbb{C}^2 \), we set \( \{e_1, e_2, e_3, e_4\} \) to be a basis of \( \mathbb{C}^2 \) such that \( (z, w) = Re(z) e_1 + Im(z) e_2 + Re(w) e_3 + Im(w) e_4 \) for any \( (z, w) \in \mathbb{C}^2 \). Thus, we can identify the orbit space \( (5.1) \) with a subspace of \( S^2 \times S^2 \), and it turns out that

**Lemma 5.2.** The orbit space \( (5.1) \) is in the quotient space of the Cartesian product \( S^1 \times S^1 \) by identifying antipodal points, where the circles \( S^1 \) are the equators of the spheres \( S^2 \) with \( b_2^+ \) and \( b_2^- \) as north poles in \( (5.3) \).

**Proof.** Let
\[ P_{\theta, \psi} := \text{span}_\mathbb{R}((\cos \psi, \sin \psi), (\sqrt{1 - \cos(\theta + \psi)}, \sqrt{1 - \sin(\theta + \psi)})) \quad (5.7) \]
in the orbit space \( (5.1) \). Then from \( (5.2) \) the first component of \( \iota(P_{\theta, \psi}) \), denoted by \( \iota_1(P_{\theta, \psi}) \), is
\[ \iota_1(P_{\theta, \psi}) = \frac{\sqrt{2}}{2}((\cos \psi e_1 + \sin \psi e_3) \wedge (\cos \theta + \psi e_2 + \sin \theta + \psi e_4)
+ \frac{\sqrt{2}}{2}((\sin \psi e_1 - \cos \psi e_3) \wedge (\sin \theta + \psi e_2 - \cos \theta + \psi e_4)
= \frac{\sqrt{2}}{2}\cos \theta(e_1 \wedge e_2 + e_3 \wedge e_4) + \frac{\sqrt{2}}{2}\sin \theta(e_1 \wedge e_4 - e_2 \wedge e_3)
= \cos \theta b_1^+ + \sin \theta b_3^+. \quad (5.8)\]
and analogously the second component of \( \iota(P_{\theta, \psi}) \), denoted by \( \iota_2(P_{\theta, \psi}) \), is
\[ \iota_2(P_{\theta, \psi}) = \cos(2\psi + \theta) b_1^- + \sin(2\psi + \theta) b_3^-. \quad (5.9)\]
Hence \( \iota_1(P_{\theta, \psi}) \) and \( \iota_2(P_{\theta, \psi}) \) don’t have \( b_2^+ \) or \( b_2^- \) component, then the claim follows. \[ \square \]

When acting 2-planes in \( \text{Gr}_2(\mathbb{C}^2) \) by torus actions, we have

**Lemma 5.3.** The torus action of a 2-plane in \( \text{Gr}_2(\mathbb{C}^2) \) is a revolution along the axes \( b_1^+ \) and \( b_1^- \) on the component spheres \( S^2 \) in \( (5.3) \).

**Proof.** Let \( P_{\theta, \psi}^{\alpha, \beta} \) be the 2-plane obtained from a torus action \( (e^{\sqrt{1-\alpha}}, e^{\sqrt{1-\beta}}) \) on the 2-plane \( P_{\theta, \psi}, (5.7) \), then
\[ P_{\theta, \psi}^{\alpha, \beta} = \text{span}_\mathbb{R}((\cos \psi e^{\sqrt{1-\alpha}}, \sin \psi e^{\sqrt{1-\beta}}), \sqrt{1}(e^{\sqrt{1-\alpha}} \cos(\theta + \psi), e^{\sqrt{1-\beta}} \sin(\theta + \psi))). \quad (5.10)\]
and it orthogonal complement plane \( (P_{\theta, \psi}^{\alpha, \beta})^\perp \) can be expressed as
\[ \text{span}_\mathbb{R}((\sin \psi e^{\sqrt{1-\alpha}}, -\cos \psi e^{\sqrt{1-\beta}}), (e^{\sqrt{1-\alpha}} e^{i\theta}), e^{\sqrt{1-\beta}} e^{i\theta} \sin(\theta + \psi))). \quad (5.11)\]

Therefore, applying the map \( (5.2) \) to \( P_{\theta, \psi}^{\alpha, \beta} \), we get the first component of \( \iota(P_{\theta, \psi}^{\alpha, \beta}) \),
\[ \iota_1(P_{\theta, \psi}^{\alpha, \beta}) = \frac{\sqrt{2}}{2}\cos \theta (e_1 \wedge e_2 - \sin \theta \sin(\alpha + \beta) e_1 \wedge e_3 + \sin \theta \cos(\alpha + \beta) e_1 \wedge e_4
+ \sin \theta \cos(\alpha + \beta) e_2 \wedge e_3 + \sin \theta \sin(\alpha + \beta) e_2 \wedge e_4 + \cos \theta e_3 \wedge e_4)
= \cos \theta b_1^+ - \sin \theta \sin(\alpha + \beta) b_2^- + \sin \theta \cos(\alpha + \beta) b_3^+, \quad (5.12)\]
and the second component of $\iota(P^{\alpha,\beta}_{\theta,\psi})$,

$$
\iota_2(P^{\alpha,\beta}_{\theta,\psi}) = \cos(2\psi + \theta) b_1 - \sin(2\psi + \theta) \sin(\alpha - \beta) b_2 + \sin(2\psi + \theta) \cos(\alpha - \beta) b_3.
$$

(5.13)

Thus we can see that the claim follows from the expressions of $\iota_1(P^{\alpha,\beta}_{\theta,\psi})$ and $\iota_2(P^{\alpha,\beta}_{\theta,\psi})$ in (5.12) and (5.13).

One can parametrize the two spheres by

$$
\xi(x, \phi_1) := (\sqrt{1 - x^2} \cos \phi_1, \sqrt{1 - x^2} \sin \phi_1, x) \quad (5.14)
$$

and

$$
\eta(y, \phi_2) := (\sqrt{1 - y^2} \cos \phi_2, \sqrt{1 - y^2} \sin \phi_2, y). \quad (5.15)
$$

In [4], Goodey and Howard characterized the kernel of general cosine transform

$$
C(f)(P) := \int_{Q \in Gr_2(\mathbb{R}^4)} |\langle P, Q \rangle| f(Q) dQ \quad (5.16)
$$

by Legendre polynomials

$$
p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (5.17)
$$

However, if $f$ is a $\mathcal{U}(1) \times \mathcal{U}(1)$-invariant function on $Gr_2(\mathbb{C}^2)$, then it is independent of $\phi_1$ and $\phi_2$. We have

**Theorem 5.4.** The kernel of cosine transform

$$
C_T(f)(P) := \int_{Q \in Gr_2(\mathbb{C}^2)} |\langle P, Q \rangle| f(Q) dQ, \quad (5.18)
$$

in which $f \in L^2(Gr_2(\mathbb{C}^2))$ is torus invariant, is

$$
\left\{ \sum_{m - n \text{ even}, |m - n| \geq 3} c_{m,n} p_m(\cos \theta) p_n(\cos(2\psi + \theta)) \right\}, \quad (5.19)
$$

in which $(\theta, \psi)$ are the parameters of the orbit in (5.1) that $Q$ belongs to.

**Proof.** From the expressions (5.12) and (5.13), we know that $x = \cos \theta$ and $y = \cos(2\psi + \theta)$ in (5.14) and (5.15). On the other hand, we also know that the kernel of cosine transform (5.4) is exactly the sum of functions $p_m(x)p_n(y)$, the product of Legendre polynomials, for $m - n$ even and $|m - n| \geq 3$. So the claim follows. \( \square \)

From Theorem 5.4, we know that the cosine transform annihilates the pieces of $p_m(\sin \theta)p_n(\sin(2\psi + \theta))$, $|m - n| \geq 3$, in the Legendre series expansions of functions in $L^2(Gr_2(\mathbb{C}^2))$, and it follows from from the self-adjoint property of the cosine transform operator that

**Corollary 5.5.** The image of cosine transform

$$
C_T(f)(P) := \int_{Q \in Gr_2(\mathbb{C}^2)} |\langle P, Q \rangle| f(Q) dQ, \quad (5.20)
$$
in which \( f \in L^2(Gr_2(C^2)) \) is torus invariant, is the space of torus invariant continuous functions on \( Gr_2(C^2) \) which are not in \( \mathcal{M} \), in other words,

\[
\text{im}(C_T) = \left\{ \sum_{|m-n| = 0 \text{ or } 2} c_{m,n} p_m(\cos \theta) p_n(\cos(2\psi + \theta)) \right\}.
\] (5.21)

6. ON THE VOLUME OF A CONVEX BODY OF MOKOWSKI SUM

Now we consider \( HT_4(K + \sqrt{-1}K) \) for a convex body \( K \) in a plane \( P \) in \( Gr_2(C^2) \). First, we know that there is a function \( f \) on \( Gr_2(C^2) \) such that

\[
HT_4(K + \sqrt{-1}K) = f(P)HT_2(K)^2,
\] (6.1)

for any convex body \( K \subset P \in Gr_2(C^2) \). Furthermore, \( f(P) = 0 \) for all \( P \in \mathbb{CP}^1 \). In general, by the property of Euclidean volumes, we know that

\[
f(P) = \frac{c \sin^2 \theta}{(Kl_{HT^2}(P))^2}
\] (6.2)

where \( c = \frac{HT^4(B_+^2)}{\text{vol}_4(B_+^2)} \) for the unit ball \( B_+^2 \) of \( (C^2, F) \) is a constant with respect to \( P \) and \( Kl_{HT^2} \) is the Klain function of the Holmes-Thompson area, \( HT^2 \).

In the next, we are going to compute the cosine transform to get the Klain function of the Holmes-Thompson area, \( HT^2 \). First, the projection area of a square \( E \) spanned by \( (\cos \psi_0, \sin \psi_0) \) and \( (\sqrt{-1}\cos(\theta_0 + \psi_0), \sqrt{-1}\sin(\theta_0 + \psi_0)) \) from one plane,

\[
P_{\theta_0, \psi_0} := \text{span}_\mathbb{R}(\cos \psi_0, \sin \psi_0, \sqrt{-1}\cos(\theta_0 + \psi_0), \sqrt{-1}\sin(\theta_0 + \psi_0))
\] (6.3)

in \( \mathbb{RP}^1 \) to another arbitrary plane

\[
P^*_{\theta, \psi} := \text{span}_\mathbb{R}(e^{i\alpha} \cos \psi, e^{i\beta} \sin \psi, \sqrt{-1}e^{i\alpha} \cos(\theta + \psi), \sqrt{-1}e^{i\beta} \sin(\theta + \psi))
\] (6.4)

is

\[
\pi_{P_{\theta, \psi}}(E) := \left| \langle (\cos \psi_0, \sin \psi_0, e^{i\alpha} \cos \psi, e^{i\beta} \sin \psi) \rangle \mathbb{R} \cdot (\sqrt{-1}(\cos \phi_0, \sin \phi_0), \sqrt{-1}(e^{i\alpha} \cos \phi, e^{i\beta} \sin \phi)) \rangle \mathbb{R} \cdot (\sqrt{-1}(\cos \phi_0, \sin \phi_0), (e^{i\alpha} \cos \psi, e^{i\beta} \sin \psi)) \rangle \mathbb{R} \cdot (\cos \psi_0, \sin \psi_0, \sqrt{-1}(e^{i\alpha} \cos \phi, e^{i\beta} \sin \phi)) \rangle \mathbb{R} \right|
\] (6.5)

for \( \phi_0 := \theta_0 + \psi_0 \) and \( \phi := \theta + \psi \). By spherical harmonics, there exists a Crofton measure for any complex Finsler metric under some smoothness condition, and we have shown the following

**Proposition 6.1.** The Crofton measure for \( U(1) \times U(1) \) invariant complex Finsler metric \( F \) on \( C^2 \) is \( U(1) \times U(1) \) invariant.

Furthermore, one can obtain \( U(1) \times U(1) \) invariant Crofton measure for \( HT^2 \) from the one for the metric.
7. THE COSINE TRANSFORM OF THE COMPLEX $l^1$ SPACE’S SINGULAR MEASURE

Let us consider the case of complex $l^1$ for the Kain function of $HT^2$. As we know, the Crofton measure $\mu$ for $HT^2$ is induced from the intersection map

$$\pi : \text{Gr}_3(\mathbb{R}^4) \times \text{Gr}_3(\mathbb{R}^4) \setminus \triangle \rightarrow \text{Gr}_2(\mathbb{R}^4),$$

(7.1)

and indeed

$$\mu = \delta_{\{C \times \{0\}\}} + \delta_{\{\{0\} \times C\}} + \frac{1}{4\pi} \lambda_T$$

(7.2)

where $\lambda_T$ is the uniform measure on

$$T := \{\text{span}(v, w) : v \in C \times \{0\}, w \in \{0\} \times C\}.$$ (7.3)

Applying the cosine transform,

$$C_T(f)(P) = \int_{Q \in \text{Gr}_2(C^2)} |\langle P, Q \rangle| d\mu(Q)$$

$$= |\langle P, C \times \{0\} \rangle| + |\langle P, \{0\} \times C \rangle| + \frac{1}{4\pi} \int_{Q \in T} |\langle P, Q \rangle| d\lambda_T(Q).$$ (7.4)

Let

$$P = \text{span}_\mathbb{R}((\cos \psi, \sin \psi), (\sqrt{-1}\cos(\theta + \psi), \sqrt{-1}\sin(\theta + \psi)))$$

(7.5)

and

$$Q = \text{span}_\mathbb{R}(e^{\sqrt{-1}x}, 0), (0, e^{\sqrt{-1}y}),$$

(7.6)

then

$$|\langle P, C \times \{0\} \rangle| = |\cos(\theta + \psi)\cos \psi|,$$

(7.7)

$$|\langle P, \{0\} \times C \rangle| = |\sin(\theta + \psi)\sin \psi|$$

(7.8)

and

$$I := \int_{Q \in T} |\langle P, Q \rangle| d\lambda_T(Q) = \int_0^{2\pi} \int_0^{2\pi} |\cos \psi \sin(\theta + \psi) \cos x \sin y + \sin \psi \cos(\theta + \psi) \sin x \cos y| dx dy.$$ (7.9)

Thus, we have

$$C_T(f)(P) = |\cos(\theta + \psi)\cos \psi| + |\sin(\theta + \psi)\sin \psi|$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} |\cos \psi \sin(\theta + \psi) \cos x \sin y + \sin \psi \cos(\theta + \psi) \sin x \cos y| dx dy. (7.10)$$

Example 7.1. If $P = C \times \{0\}$ or $\{0\} \times C$, then we know that $P$ is a Euclidean 2-plane, so the Kain function of $HT^2$ at P is 1. On the other hand, we also get $C_T(f)(P) = 1$ from the right hand side of (7.4): If $P$ is a complex line, $\text{span}_\mathbb{R}((\cos \psi, \sin \psi), \sqrt{-1}(\cos \psi, \sin \psi))$, then $P$ is still a Euclidean 2-plane. By (7.10) we have

$$C_T(f)(P) = \cos^2 \psi + \sin^2 \psi + 2|\cos \psi \sin \psi| = (|\cos \psi| + |\sin \psi|)^2.$$ (7.11)

On the other hand, the rectangle spanned by $(\cos \psi, \sin \psi)$ and $\sqrt{-1}(\cos \psi, \sin \psi)$ has Holmes Thompson area $(|\cos \psi| + |\sin \psi|)^2$ and so is its Kain function at $P$.

The double integral (7.9) can be transformed into an elliptic integral,

$$I = 4 \int_0^{2\pi} \sqrt{\cos^2 \psi \sin^2(\theta + \psi) - \sin \theta \sin(\theta + 2\psi) \sin^2 \psi} d\psi$$

$$= 4 \cos \psi \sin(\theta + \psi) \int_0^{2\pi} \sqrt{1 - \frac{\sin \theta \sin(\theta + 2\psi)}{\cos^2 \psi \sin^2(\theta + \psi)}} \sin^2 \psi d\psi.$$ (7.12)
Assume \( \theta + 2\psi \leq \pi \) and let 
\[
k^2 = \frac{\sin \theta \sin(\theta+2\psi)}{\cos \psi \sin(\theta+\psi)},
\]
then by the series expansion of the incomplete elliptic integral of the second kind,
\[
I = 4 \cos \psi \sin(\theta + \psi) \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{1}{1-2m} \left( -\frac{1}{2} \right)^2 k^{2m} 
= 4 \cos \psi \sin(\theta + \psi) \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{1}{1-2m} \left( -\frac{1}{2} \right)^2 \left( \frac{\sin \theta \sin(\theta+2\psi)}{\cos \psi \sin(\theta+\psi)} \right)^m 
= \pi (\sin(\theta + 2\psi) + \sin \theta) \sum_{m=0}^{\infty} \frac{1}{1-2m} \left( -\frac{1}{2} \right)^2 \frac{2^{2m} \sin^m \theta \sin^m(\theta+2\psi)}{(\sin(\theta+2\psi) + \sin \theta)^{2m}}.
\]
(7.13)

So we have
\[
K_{\mathcal{HT}}(P) = \frac{1}{4\pi} I + \left| \cos \psi \cos(\theta + \psi) \right| + \left| \sin \psi \sin(\theta + \psi) \right| 
= \frac{1}{4} (\sin(\theta + 2\psi) + \sin \theta) \sum_{m=0}^{\infty} \frac{1}{1-2m} \left( -\frac{1}{2} \right)^2 \frac{2^{2m} \sin^m \theta \sin^m(\theta+2\psi)}{(\sin(\theta+2\psi) + \sin \theta)^{2m}} 
+ \max(\left| \cos \theta \right|, \left| \cos(2\psi + \theta) \right|).
\]
(7.14)

8. The Picture on the Two Spheres

In this section, we are going to describe the cosine transform of the torus invariant \( l^2 \) space’s singular measure in terms of the picture of Gluck and Warner’s decomposition of the Grassmannian \( \text{Gr}_2(\mathbb{R}^4) \), \([5,2]\), and Goody and Howard’s kernel characterization in \([4]\).

First, we can find the following correspondences under the isomorphism \([5.2]\) in terms of the spherical coordinates \([5.14] \) and \([5.15]\): \( \mathbb{C} \times \{0\} \) corresponds to \((0, 0, 1)\) on the first sphere \( S^2_+ \) and \((0, 0, 1)\) on the second sphere \( S^2_2 \), \( \{0\} \times \mathbb{C} \) to \((0, 0, -1)\) on the first sphere \( S^2_+ \) and \((0, 0, -1)\) on the second sphere \( S^2_2 \), and the torus \( T \) in \([7.3]\) to the torus \( S^1_+ \times S^1_+ \), where \( S^1_+ \) and \( S^1_- \) are equators of \( S^2_+ \) and \( S^2_2 \) whose \( b^+ \) and \( b^- \) components in \([5.3]\) and \([5.5]\) are zeros.

For any fixed point \((\xi(x, 0), \eta(y, 0)) \in S^2_+ \times S^2_2 \) representing a plane \( P \) in the orbit of \( \text{Gr}_2(\mathbb{R}^4) \) acted by torus actions, then we have
\[
\langle P, Q \rangle = \langle (\xi(x, 0), \eta(y, 0)), (\xi(0, \phi_1), \eta(0, \phi_2)) \rangle 
= |\cos \phi_1 \sqrt{1-x^2} + \cos \phi_2 \sqrt{1-y^2}|
\]
for any \((\xi(0, \phi_1), \eta(0, \phi_2)) \in S^1_+ \times S^1_+ \) representing a plane \( Q \) in torus \( T \) in \([7.3] \). So the integral in \([7.9]\) appears as
\[
I' := \int_0^{2\pi} \int_0^{2\pi} |\cos \phi_1 \sqrt{1-x^2} + \cos \phi_2 \sqrt{1-y^2}| d\phi_1 d\phi_2
\]
(8.2)
in this picture.

Moreover, by \([5.13]\) and \([5.15]\),
\[
\langle P, \mathbb{C} \times \{0\} \rangle = |x + y|
\]
(8.3)
and
\[
\langle P, \{0\} \times \mathbb{C} \rangle = |x - y|,
\]
(8.4)
and then
\[
\langle P, \mathbb{C} \times \{0\} \rangle + |P, \{0\} \times \mathbb{C} \rangle = 2\max(|x|, |y|).
\]
(8.5)
So the Klain function of $HT^2$ can be also expressed as

$$Kl_{HT^2}(P) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} |\cos \phi_1 \sqrt{1-x^2} + \cos \phi_2 \sqrt{1-y^2}| d\phi_1 d\phi_2 + 2max(|x|,|y|)$$

(8.6)

for any $(\xi(x,0), \eta(y,0)) \in S^2_+ \times S^2_-$.

The term $max(|x|,|y|)$ can be expanded as a Legendre series in ascending total degree of the Legendre polynomial of $x$ and $y$,

$$max(|x|,|y|) = \sum_{n=0}^{\infty} \frac{4^n}{(2n)!} \left( p_n(x) + p_n(y) \right)$$

(8.7)

Furthermore, it is not hard to rigorously show that $max(|x|,|y|)$ has only the components of $p_m(x)p_n(y)$, in which $m$ and $n$ are even and $|m-n| = 0$ or $2$, in its Legendre series expansion. By Lemma 3.4 in [4],

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{x+y} p_m(x)p_n(y) dxdy = 0$$

(8.8)

for $m$ and $n$ even and $|m-n| = 0$ or $2$, and by substitution

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{x+y} p_m(x)p_n(y) dxdy = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{x-y} p_m(x)p_n(y) dxdy$$

(8.9)

for $m$ and $n$ even. Therefore,

$$\int_{-1}^{1} \int_{-1}^{1} max(|x|,|y|) p_m(x)p_n(y) dxdy = 0$$

(8.10)

for $m$ and $n$ even and $|m-n| = 0$ or $2$.

To evaluate (8.2), we let $\sin \theta := \sqrt{1-x^2}$, $\sin \phi := \sqrt{1-y^2}$, $\phi_1 := \alpha - \beta$ and $\phi_2 := \alpha + \beta$, then

$$I' = 2\pi \int_0^{2\pi} \int_0^{2\pi} \sin \theta \cos \phi_1 + \sin \phi \cos \phi_2 |d\phi_1 d\phi_2$$

$$= \int_0^{2\pi} \int_0^{2\pi} \sin \theta \sin \phi_1 + \sin \phi \sin \phi_2 |d\phi_1 d\phi_2$$

$$= 2 \int_0^{2\pi} \int_0^{2\pi} \frac{\sin \phi_1 + \sin \phi_2}{2} + \sin \phi_1 + \sin \phi_2 |d\phi_1 d\phi_2$$

which becomes the double integral (7.9) if we let $\phi := 2\phi + \theta$, which can be transformed into the elliptic integral (7.12) and furthermore the series (7.13), which by back-substitution turns out to be

$$I' = 2\pi \left( 1 - x^2 + \sqrt{1-y^2} \right) \sum_{m=0}^{\infty} \frac{1}{1-2m} \left( -\frac{1}{m} \right)^2 \frac{2^{2m}}{(1-x^2)^m(1-y^2)^m}$$

(8.11)

9. A characterization of the Klain Function

However, we can give a characterization of the Klain function as the following

**Theorem 9.1.** The Klain function of the second Holmes-Thompson valuation in complex $t^1$ space is

$$\sum_{|k-l| = 0 \text{ or } 1} c_{k,l} p_{2k}(x)p_{2l}(y)$$

(9.1)
for some $c_{k,l} \in \mathbb{R}$.

To prove the above theorem, let’s show the following lemma first.

**Lemma 9.2.** The cosine transform is a self-adjoint operator on the space of torus invariant functions

$$\mathcal{F} := \{ f : Gr_2(\mathbb{C}^2) \to \mathbb{R} : f \text{ is } U(1) \times U(1) \text{-invariant} \} \quad (9.2)$$

**Proof.** By Fubini’s theorem,

$$\langle C_T(f), g \rangle = \int_{P \in Gr_2(\mathbb{C}^2)} g(P) \langle \int_{Q \in Gr_2(\mathbb{C}^2)} |\langle P, Q \rangle| f(Q) dQ \rangle dP$$

$$= \int_{Q \in Gr_2(\mathbb{C}^2)} f(Q) \langle \int_{P \in Gr_2(\mathbb{C}^2)} |\langle P, Q \rangle| g(P) dP \rangle dQ$$

$$= \langle f, C_T(g) \rangle$$

for any $f, g \in \mathcal{F}$. \( \square \)

**Remark 9.3.** The claim holds more generally for the space of all functions on $Gr_2(\mathbb{C}^2)$ and furthermore distributions.

Now let’s prove Theorem 9.1.

**Proof.** First we know that $ker(C_T)$ is orthogonal to $Im(C_T)$, because, by (9.3),

$$\langle f, C_T(g) \rangle = 0 \text{ for any } f \in ker(C_T) \text{ and any } g \in \mathcal{F}.$$

On the other hand, by Lemma 3.3 in [4], we know that $p_m(x)p_n(y) \in ker(C_T)$ for $m - n$ even and $|m - n| > 2$, and since $Kl_{HT2} \in Im(C_T)$, hence

$$\langle Kl_{HT2}, p_m(x)p_n(y) \rangle = 0$$

for $m - n$ even and $|m - n| > 2$, moreover, $Kl_{HT2}$ is an even function, thus it follows that $Kl_{HT2}$ has only the components of $p_m(x)p_n(y)$ for $m$ and $n$ even and $|m - n| = 0$ or $2$ in its Legendre polynomial expansion. \( \square \)

**Remark 9.4.** The double integral \( \delta_{S^1_+ \times S^1_+}(x, y) \) is the cosine transform of the delta function of $S^1_+ \times S^1_+$ on $S^1_+ \times S^2_+$, which can be expanded as a series in terms of Legendre polynomials

$$\delta_{S^1_+ \times S^1_+}(x, y) = \delta(x)\delta(y)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{p_{2k}(0)p_{2l}(0)}{p_{2k}(x)p_{2l}(x)} \right) \frac{1}{k!} \frac{1}{l!} \left( \frac{-2}{k} \right) \left( \frac{-2}{l} \right) p_{2k}(x)p_{2l}(y).$$

(9.5)

Basically, the cosine transform annihilates the components of $p_{2k}(x)p_{2l}(y)$, $|k - l| > 1$ in (9.5), and maps the components of $p_{2k}(x)p_{2l}(y)$, $|k - l| = 0$ or $1$, to those components, in other words, the cosine transform has an invariant subspace

$$\mathcal{G} := \left\{ \sum_{|k - l| = 0 \text{ or } 1} c_{k,l} p_{2k}(x)p_{2l}(y) : c_{k,l} \in \mathbb{R} \right\}.$$

(9.6)

More generally, the proof for Theorem 9.1 works for any distribution on $Gr_2(\mathbb{C}^2)$, so we have the following

**Theorem 9.5** (Image of cosine transform on distributions). The image of the cosine transform on the space $D'(Gr_2(\mathbb{C}^2))$ that consists of torus invariant distributions on $Gr_2(\mathbb{C}^2)$ is the space $\mathcal{G}$ in (9.6).

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