Implementing a Two-Photon Three-Degrees-of-Freedom Hyper-Parallel Controlled Phase Flip Gate Through Cavity-Assisted Interactions

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Hyper-parallel quantum information processing is a promising and beneficial research field. Herein, a method to implement a hyper-parallel controlled-phase-flip (hyper-CPF) gate for frequency-, spatial-, and time-bin-encoded qubits by coupling flying photons to trapped nitrogen vacancy (NV) defect centers is presented. The scheme, which differs from their conventional parallel counterparts, is specifically advantageous in decreasing against the dissipate noise, increasing the quantum channel capacity, and reducing the quantum resource overhead. The gate qubits with frequency, spatial, and time-bin degrees of freedom (DOF) are immune to quantum decoherence in optical fibers, whereas the polarization photons are easily disturbed by the ambient noise.

1. Introduction

In recent years, great advancements have been made in many areas of quantum information processing (QIP), including quantum teleportation,[1,2] quantum secret sharing,[3] quantum key distribution,[4,5] quantum secure direct communication,[6,7] quantum dense coding,[8] quantum algorithms,[9–12] and quantum gates.[13–15] Because quantum communication utilizes quantum coherent superposition and quantum entanglement effect, its propagation rate and reliability are higher than those of conventional communication methods.[16] Further, quantum computing exhibits a higher performance than its conventional counterparts to efficiently search the target items in an unsorted date base and factor large integers.[16] Recently, numerous sophisticated approaches have been proposed to improve the conventional methods by employing multiple degrees of freedom (DOFs). Multiple DOFs are beneficial for a wide range of applications, including the implementation of hyper-parallel quantum computation,[17] quantum communication,[18] simplification of quantum computation,[19] high-dimensional QIP,[20] and completion of specific deterministic tasks that cannot be solved by single DOF systems, such as deterministic linear optical quantum algorithms,[21] deterministic linear optical quantum gates,[22] linear optical teleportation,[23] and quantum key distribution without a shared reference frame.[24] Furthermore, hyper-parallel QIP has been gaining great attention because of its excellent advantages, making it an interesting and potential candidate for long-distance quantum secure communication and quantum computers.

Hyper-parallel QIP, whose operations are simultaneously executed in two or more distinct DOFs, is potentially robust against photonic dissipation noise, and enhances the quantum channel capacity, improves the security of quantum communication, reduces experimental requirements and resource overheads, augments the success rate of protocols, and improves the speed of quantum computation. Recently, various hyper-entangled states have been reported; for example, polarization-spatial-energy hyper-entangled states,[25] polarization-time-bin hyper-entangled states,[26] spin-motion hyper-entangled states,[27] polarization-momentum hyper-entangled states,[28] polarization-time-frequency hyper-entangled states,[29] and multiple-path hyper-entangled states.[30] These resources can help us implement many important quantum tasks with one DOF, such as completing entangled states analysis using linear optics,[31,32] entanglement purification and concentration,[33] one-DOF cluster state preparation and one way quantum computing,[34] quantum error-correcting,[35] teleportation,[36] linear photonic superdense coding,[37] enhanced violation of local realism,[36] and quantum algorithm.[38] Moreover, hyper-entanglement has provided other important applications in hyper-parallel photonic quantum computing,[39] one-DOF cluster state analysis, and hyper-entangled states analysis, hyper-entangled states purification,[44] and hyper-entanglement concentration.[45]

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implement probabilistic photonic QIP. It has been demonstrated that the deterministic QIP can be achieved by encoding computing qubits in different DOFs in a photonic architecture. However, scalability is a major challenge in this approach because it only allows for one photon system. Currently, cross-Kerr medium, atoms, atom assemble, and artificial atoms (such as quantum dot, superconductor, and diamond nitrogen vacancy defect center) are often employed as a prominent platform to address the scalability problem that overcomes the intrinsic weak interactions between individual photons. In recent years, diamond nitrogen vacancy (NV) in defect centers has gained great attention because of its exceptional features, such as excellent scalability, optical property, and ultra-long coherence time even at room temperature. \[56,57\] Electronic spin confined in NV center enables generation of spin–photon entanglement; in addition, assistance using NV center, polarization-spatial hyper-parallel quantum computing, hyperentanglement purification, hyperentanglement concentration, and hyperentangled state analysis have been proposed. Cavity quantum electrodynamics significantly enhances photon–matter interactions, and it is essential for quantum networks and distributed quantum computation. \[56,60\] Conditional transmitters/reflectors of the NV-cavity platform have a wide range of applications both in the strong-coupling regime of the high-Q cavity and the weak-coupling region of the low-Q cavity. Nowadays, conditional transmission and reflection techniques based on photon–NV interaction have been extensively employed for QIP ranging from implementations of quantum gates to measurement-device-independent quantum key distribution. \[54,65\] Quantum networks, quantum networks, quantum networks, \[66,67\] hyper-parallel quantum gates, \[68,69\] and multiple DOFs entanglement distribution. \[70\]

In this paper, we design a scheme to efficiently implement optical hyper-parallel controlled phase flip (hyper-CPF) gate; a hyper-parity gate simultaneously acts on frequency, spatial, and time-bin DOFs through cavity-assisted interactions. The CPF gate or similar logic operations are essential for quantum communication and quantum computation, and our gate mechanism is deterministic in principle. It is of note that polarization photons are highly susceptible to decoherence in optical fibers because of polarization mode dispersion caused by thermal fluctuation, vibration, and imperfects of the fiber. \[71,72\] On the contrary, frequency, spatial, and time-bin photons are immune to these decoherence effects. Furthermore, frequency photons can efficiently transfer quantum information at telecommunications wavelengths, \[73,74\] and time-bin photons can minimize the effects of detector dead-time and have the advantage of relative insensitivity to inhomogeneities in transmission media. \[76,78\] Moreover, in contrast to conventional quantum computation with one DOF, our constructions are not only more robust against the photonic dissipation noise, but can also increase the capacity of the quantum channel and reduce the demand for quantum resources.

2. Hyper-CPF Gate and Hyper-Parity-Check

2.1. The Optical Property of a Double-Sided NV-Cavity Platform

In recent years, constant efforts have been made toward the diamond NV center. As shown in Figure 1, the electron-spin triple ground states of a negatively charged NV\(^{-}\) center are split into \(|m_z = 0\rangle\) (marked as \(|0\rangle\)), and \(|m_z = \pm 1\rangle\) (marked as \(|\pm \rangle\)) by 2.88 GHz with zero magnetic field due to the crystal field. \[77,78\] The degeneracy levels \(|\pm\rangle\) can be further split according to Zeeman effect by applying a magnetic field (\(2\pi \times 200 \text{ MHz}\)) to the sample along one of the NV axes. \[80\] The six excited states \(|A_\pm\rangle\) of polarized photons \(|A\rangle\) of polarized photons are highly susceptible to decoherence in optical fibers because of polarization mode dispersion caused by thermal fluctuation, vibration, and imperfections of the fiber. \[71,72\] On the contrary, frequency, spatial, and time-bin photons are immune to these decoherence effects. Furthermore, frequency photons can efficiently transfer quantum information at telecommunications wavelengths, \[73,74\] and time-bin photons can minimize the effects of detector dead-time and have the advantage of relative insensitivity to inhomogeneities in transmission media. \[76,78\] Moreover, in contrast to conventional quantum computation with one DOF, our constructions are not only more robust against the photonic dissipation noise, but can also increase the capacity of the quantum channel and reduce the demand for quantum resources.
Here, \(\omega, \omega_c, \text{and } \omega_{\text{C}}\) are the frequencies of the incident single photon, cavity mode, and \(X^+\)-dipole transition, respectively; \(g\) denotes the coupling constant of the \(X^+\)-cavity combination; \(\kappa/2\) and \(\gamma/2\) are the decay rates of the \(X^+\)-dipole, cavity field, and side leakage, respectively; \(\hat{H}\) and \(\hat{C}\) are the noise operators; \(\alpha_2\) represents the Pauli operator; \(\hat{a}_0\) (\(\hat{a}_1^+\)) and \(\hat{a}_1\) (\(\hat{a}_0^+\)) denote the input and output field operators, respectively. When \(X^+\) stays predominantly in the ground state; that is, taking \(\langle \sigma_x \rangle \approx -1\), the reflection/transmission coefficients \(r(\omega)/t(\omega)\) of the diamond NV center can be calculated as \(^{(87)}\)

\[
\begin{align*}
\frac{r(\omega)}{t(\omega)} &= \frac{\left[ i(\omega_X - \omega) + \frac{\gamma}{2} \right]}{\left[ i(\omega_X - \omega) + \frac{\gamma}{2} \right]} \frac{\left[ i(\omega_X - \omega) + \frac{\gamma}{2} \right]}{\left[ i(\omega_X - \omega) + \frac{\gamma}{2} \right]} + g^2 \\
\end{align*}
\]

By adjusting the single-photon pulse to resonate with the cavity mode and the \(X^+\)-dipole transition (\(\omega = \omega_c = \omega_{\text{C}}\)), and considering the ideal conditions that the Purcell factor \(g^2/(\kappa \gamma) \gg 1\) with \(\kappa /\omega \approx 0\) in the cold cavity mode (\(g = 0\)), one can obtain \(r(\omega) \rightarrow 1\) and \(t(\omega) \rightarrow 0\); \(\kappa /\omega \ll 1\) in the cold cavity mode (\(g = 0\)), one can obtain \(r_0(\omega) \rightarrow 0\) and \(t_0(\omega) \rightarrow -1\). That implicit mechanism is that when the high cavity modes, it is reflected by the cavity mode without any phase shift; when the coupling strength \(g\) and \(\kappa\) are negligible, the transmitted photon experiences a \(\pi\)-phase shift. Thus, the spin-dependence optical transition rules can be summarized as

\[
\begin{align*}
| R^1, \omega_1, + \rangle &\rightarrow -| R^1, \omega_1, + \rangle, \quad | R^1, \omega_1, + \rangle &\rightarrow | L^1, \omega_1, + \rangle, \\
| R^1, \omega_2, + \rangle &\rightarrow -| R^1, \omega_2, + \rangle, \quad | R^1, \omega_2, + \rangle &\rightarrow -| R^1, \omega_2, + \rangle, \\
| R^1, \omega_2, - \rangle &\rightarrow -| R^1, \omega_2, - \rangle, \quad | R^1, \omega_2, - \rangle &\rightarrow -| R^1, \omega_2, - \rangle, \\
| L^1, \omega_1, + \rangle &\rightarrow | R^1, \omega_1, + \rangle, \quad | L^1, \omega_1, + \rangle &\rightarrow | L^1, \omega_1, + \rangle, \\
| L^1, \omega_2, + \rangle &\rightarrow -| L^1, \omega_2, + \rangle, \quad | L^1, \omega_2, + \rangle &\rightarrow -| L^1, \omega_2, + \rangle, \\
| L^1, \omega_2, - \rangle &\rightarrow -| L^1, \omega_2, - \rangle, \quad | L^1, \omega_2, - \rangle &\rightarrow -| L^1, \omega_2, - \rangle, \\
| L^1, \omega_2, - \rangle &\rightarrow -| L^1, \omega_2, - \rangle, \quad | L^1, \omega_2, - \rangle &\rightarrow -| R^1, \omega_2, - \rangle
\end{align*}
\]

The total spontaneous emission rate \(\gamma_{\text{total}} = 2\pi \times 15\) MHz of the NV center was demonstrated in 2010.\(^{(88)}\) In 2009, Barclay et al.\(^{(86)}\) experimentally demonstrated the NV center relevant parameters \(g_{\text{ZPL}}, \kappa, \gamma_{\text{ZPL}} / 2\pi \approx 0.03, 0.013, 0.0004\) GHz. Here, \(g_{\text{ZPL}}\) is the coupling strength of the single microdisk photon and the NV zero phonon line (ZPL), and \(\gamma_{\text{ZPL}}\) is the total ZPL spontaneous optical decay rates of the NV center, respectively. Fortunately, \(\gamma_{\text{ZPL}} / \gamma_{\text{total}}\) has been enhanced from 3–4% to 70%\(^{(89)}\). In the following, we will employ the coherence emission within the narrow-band ZPL described by Equation (4) to construct hyper-CPF gate.

### 2.2. Implementation of the Hyper-CPF Gate

Based on the above photon-NV platform, we design a quantum circuit to a two-photon three-DOF CPF gate in which the gate quibits are independently encoded in the frequency, spatial, and time-bin DOFs of the single photon system (see Figure 2). The electron spins confined in the NV centers act as ancilla qubits.

Suppose that the initial states of the photons \(a\) and \(b\), and the three electronic spins \(e_1, e_2,\) and \(e_3\) in diamond NV centers are

\[
\begin{align*}
|\varphi\rangle_a &= |\varphi\rangle_a \otimes |\varphi\rangle_b \otimes |\varphi\rangle_{e_1} \otimes |\varphi\rangle_{e_2} \otimes |\varphi\rangle_{e_3}, \\
|\psi\rangle_{e_1} &= \frac{1}{\sqrt{2}}(|\downarrow\rangle_{e_1} + |\downarrow\rangle_{e_1}), \quad |\psi\rangle_{e_2} = \frac{1}{\sqrt{2}}(|\downarrow\rangle_{e_2} + |\downarrow\rangle_{e_2}), \\
|\varphi\rangle_b &= |\varphi\rangle_b \otimes |\varphi\rangle_b \otimes |\varphi\rangle_b \otimes |\varphi\rangle_b \otimes |\varphi\rangle_b \otimes |\varphi\rangle_b, \\
|\varphi\rangle_a &= |\varphi\rangle_a \otimes |\varphi\rangle_a \otimes |\varphi\rangle_a \otimes |\varphi\rangle_a \otimes |\varphi\rangle_a \otimes |\varphi\rangle_a
\end{align*}
\]

Here, \(|\varphi\rangle_a\) and \(|\varphi\rangle_b\) are the noise operators; \(|\varphi\rangle_{e_1}\) and \(|\varphi\rangle_{e_2}\) are the two frequency-qubit states, spatial-qubit states, and time-bin-qubit states of photon \(a\) and \(b\), respectively; \(|\varphi\rangle_{e_1}\) (\(|\varphi\rangle_{e_2}\)) denotes the \(R\)-polarized state of the photon \(a\) (\(b\)). The complex coefficients \(a_1, a_2, \text{and } a_3\) satisfy the normalization condition \(|a_1|^2 + |a_2|^2 = 1, |\gamma|^2 + |\gamma|^2 = 1, |\gamma|^2 + |\gamma|^2 = 1, |\gamma|^2 + |\gamma|^2 = 1, |\gamma|^2 + |\gamma|^2 = 1, |\gamma|^2 + |\gamma|^2 = 1\), respectively.

First, the photon \(a\) is injected; then, it interacts with “Block,” composed of HWP1, PBS1, NV, PBS2, and HWP2. The polarizing beam splitters PBS1 and PBS2 transmit the R-polarized photon and reflect the L-polarized photon, respectively. The half-wave plates HWP1 and HWP2 are rotated to 22.5° to complete the Hadamard transformations as follows:

\[
\begin{align*}
|R\rangle &\rightarrow \frac{1}{\sqrt{2}} (|R\rangle + |L\rangle), \quad |L\rangle &\rightarrow \frac{1}{\sqrt{2}} (|R\rangle - |L\rangle)
\end{align*}
\]

Combining the above-mentioned facts and Equation (4), it can be seen that the “Block” completes the following transformations:

\[
\begin{align*}
| R^1, \omega_1, + \rangle &\rightarrow | R^1, \omega_1, + \rangle, \quad | R^1, \omega_1, - \rangle &\rightarrow -| R^1, \omega_1, - \rangle, \\
| R^1, \omega_2, + \rangle &\rightarrow -| R^1, \omega_2, + \rangle, \quad | R^1, \omega_2, - \rangle &\rightarrow -| R^1, \omega_2, - \rangle, \\
| L^1, \omega_1, + \rangle &\rightarrow | L^1, \omega_1, + \rangle, \quad | L^1, \omega_1, - \rangle &\rightarrow | L^1, \omega_1, - \rangle, \\
| L^1, \omega_2, + \rangle &\rightarrow -| L^1, \omega_2, + \rangle, \quad | L^1, \omega_2, - \rangle &\rightarrow -| L^1, \omega_2, - \rangle
\end{align*}
\]

Therefore, the “Block,” (HWP1 → PBS1 → NV → PBS2 → HWP2) evolves the state of the entire system \(|\varphi\rangle_a\) into \(|\varphi\rangle_1\).

Here,

\[
|\varphi\rangle_1 = |\varphi\rangle_a \otimes |\varphi\rangle_b \otimes |\varphi\rangle_{e_1} \otimes |\varphi\rangle_{e_2} \otimes |\varphi\rangle_{e_3}
\]
Figure 2. A schematic for implementing the two-photon three-DOF hyper-CPF gate. PBS, \((i = 1, 2, \ldots, 6)\) are circularly polarizing beam splitters that transmit the R-polarized photon and reflect the L-polarized photon. WDM, \((i = 1, 2, \ldots, 6)\) represent the polarization independent wavelength division multiplexers, which lead the photons to different spatial modes according to their frequencies. FS, \((i = 1, 2, \ldots, 6)\) are frequency shifters, which are used to complete the qubit flip operation on the frequency of a single photon, that is, \(\omega_{1} \leftrightarrow \omega_{2}\). The Pockels cells \(PC_{i} (i = 1, 2, 3, 4)\) perform bit-flip operations on the polarization DOF of the photons when the l-time-bin component appears. The half-wave plates \(HPW_{1}\) and \(HPW_{2}\) are oriented at 22.5° to complete the Hadamard transformations.

\[
|\varphi_{1}\rangle = \frac{1}{\sqrt{2}} \left[ |\varphi_{e}^{+}\rangle + |\varphi_{e}^{-}\rangle \right] = \frac{1}{\sqrt{2}} \left[ |\varphi_{e}^{+}\rangle + |\varphi_{e}^{-}\rangle \right]
\]

\[
|\varphi_{2}\rangle = \frac{1}{\sqrt{2}} \left[ |\varphi_{e}^{+}\rangle + |\varphi_{e}^{-}\rangle \right] \otimes \left( |\varphi_{s}^{+}\rangle + |\varphi_{s}^{-}\rangle \right)
\]

After Hadamard operations \(H_{S}\) are performed on the NV center \(e_{1}\), the state \(|\varphi_{1}\rangle\) becomes

\[
|\varphi_{3}\rangle = |R\rangle_{e} |R\rangle_{s} \left[ -|\varphi_{e}^{+}\rangle |\varphi_{s}^{+}\rangle + |\varphi_{e}^{-}\rangle |\varphi_{s}^{-}\rangle \right]
\]

Here, \(H_{S}\) executes the following operations:

\[
|+\rangle \rightarrow \frac{1}{\sqrt{2}} \left( |+\rangle + |-\rangle \right), \quad |\rangle \rightarrow \frac{1}{\sqrt{2}} \left( |+\rangle - |\rangle \right)
\]

Second, a polarization-independent wavelength division multiplexer WDM, separates the wavepackets emitted from spatial model \(a_{1}\) into two arms \(a_{11}\) and \(a_{12}\) according to their frequencies, and a frequency shifter FS flips the frequency of the photon emitted from the spatial \(a_{12}\), that is, \(|\omega_{1}\rangle|a_{12}\rangle \rightarrow |\omega_{2}\rangle|a_{12}\rangle\). Subsequently, the wavepackets converge at WDM. The Pockels cells \(PC_{1}\) and \(PC_{2}\) perform bit-flip operations on the polarization DOF of the passing photon when the l-time-bin component occurs. The PBS, \(PBS_{6}\) directly reflects the polarization components to PBS, \(PBS_{6}\) and transmits the R-polarization components to the operations consisting of WDM, FS, “Block”, FS, WDM, and PBS. Thus, these operations WDM, FS, “Block”, FS, WDM, PC, PBS, PC, PBS, WDM, FS, “Block”, FS, WDM, PBS, \(H_{o}\) and \(PC_{6}\). The PBS, \(PBS_{6}\), WDM, FS, “Block”, FS, WDM, PBS, \(H_{o}\) and \(PC_{6}\) change the state \(|\varphi_{3}\rangle\) into \(|\varphi_{4}\rangle\).

\[
|\varphi_{4}\rangle = |R\rangle_{e} |R\rangle_{s} \left[ -|\varphi_{e}^{+}\rangle |\varphi_{s}^{+}\rangle + |\varphi_{e}^{-}\rangle |\varphi_{s}^{-}\rangle \right]
\]

Finally, we measure the outcomes of the three NV spins based on \(|\pm\rangle\) basis, and perform some feed-forward operations on the exiting photons according to Table 1. A deterministic two-photon...
2.3. Deterministic Hyper-Parity Gate

The parity gate is the key to implementing entanglement swapping and quantum repeater, measuring the inequality of the Bell, designing quantum algorithms, constructing quantum gates, and entanglement purification and concentration.\[90–94\]

Figure 2 can additionally perform hyper-parity gate in the frequency, spatial, and time-bin DOFs simultaneously. Further, as described in Figure 2, after the photon a is injected, the initial state $\vert \varphi \rangle_4$ of the entire system is transformed into the state $\vert \varphi \rangle_1$. Before and after the photon b passes through the “Block1,” “Block2,” and “Block3,” the $H_2$ are executed on three electron-spins successively, which transforms the state $\vert \varphi \rangle_1$ into

$$
\vert \varphi \rangle_4 = (R_a R_b) \vert \varphi \rangle_4 (\alpha_1 \beta_1 a_1 \omega_1 b_1 + \alpha_1 \beta_1 a_2 \omega_2 b_2) + (\alpha_1 \beta_1 a_1 \omega_1 b_1 + \alpha_1 \beta_1 a_2 \omega_2 b_2)$$

$$
= [\alpha_2 a_2 b_2 + \alpha_2 a_2 b_2] + (\alpha_2 a_2 b_2) + (\alpha_2 a_2 b_2) + (\alpha_2 a_2 b_2)
$$

On detecting the electronic spin states $\vert 1,1 \rangle_2$, the joint state $\vert \varphi \rangle_4$ collapses into the even hyper-parity-state

$$
(\alpha_1 \beta_1 a_1 \omega_1 b_1 + \alpha_1 \beta_1 a_2 \omega_2 b_2) \otimes (\alpha_2 a_2 b_2 + \alpha_2 a_2 b_2)
$$

$$
= \alpha_2 a_2 b_2 b_2 + \alpha_2 a_2 b_2 b_2 + \alpha_2 a_2 b_2 b_2 + \alpha_2 a_2 b_2 b_2
$$

It is noted that Equations (17)–(22) can be transformed into Equations (15) and (16) by performing single-qubit feed-forward operations.

3. Discussion and Summary

Optical QIP has been extensively studied in recent years, and previous studies are mainly focused on single DOF of the photon. The KLM scheme\[13\] was a cornerstone in linear optical quantum computing with significant success probability. Matter qubits, ranging from cross-Kerr to natural and artificial atoms (quantum dot, superconducting, NV center in diamond), are often employed to ensure deterministic gate reciprocity between isolated individual photons. Great progress has been made in such entangled photon–matter platform.

However, the giant Kerr nonlinear remains a challenge in experiments. Further, millions neutral atom can trapped in microscopic arrays, long coherence time of neutral atoms can be
achieved at very low temperatures (nK–μK), while individual manipulation and readout of neutral atoms in optical lattices are not possible. Superconducting qubits have μs-scale coherence time and operate at mK temperature. A semiconductor quantum dot circuit operates at a few K and supports μs-scale coherence time. The NV center in diamond has an ultra-long coherence time (ns), moreover, it could operate even at room temperature. The initialization and readout (ps) and the side leakage of the cavity reduces the performance of the cavity. Mismatch, and the finite coupling rate between the photon and the cavity reduces the performance of the cavity. However, the inevitable nonzero photon bandwidth, mismatch, and the finite coupling rate between the photon and the cavity mode induce imperfect birefringence of the cavity, and the side leakage of the cavity reduces the performance of our scheme. Thus, the interactions between the incident photon pulse and the NV center in Equation (4) should be rewritten as

$$\begin{align*}
[R^2, \alpha_1, +] & \rightarrow t_j |R^2, \alpha_2, +\rangle + r_j |L^1, \alpha_2, +\rangle \\
[R^2, \alpha_1, +] & \rightarrow t_j |R^2, \alpha_2, +\rangle + r_j |L^1, \alpha_2, +\rangle \\
[R^2, \alpha_1, -] & \rightarrow t_j |R^2, \alpha_2, -\rangle + r_j |L^1, \alpha_2, -\rangle \\
[R^2, \alpha_1, -] & \rightarrow t_j |R^2, \alpha_2, -\rangle + r_j |L^1, \alpha_2, -\rangle \\
[R^1, \alpha_1, -] & \rightarrow t_j |R^1, \alpha_2, -\rangle + r_j |L^1, \alpha_2, -\rangle \\
[R^1, \alpha_1, +] & \rightarrow t_j |R^1, \alpha_2, +\rangle + r_j |L^1, \alpha_2, +\rangle \\
[R^1, \alpha_1, +] & \rightarrow t_j |R^1, \alpha_2, +\rangle + r_j |L^1, \alpha_2, +\rangle \\
[R^1, \alpha_1, -] & \rightarrow t_j |R^1, \alpha_2, -\rangle + r_j |L^1, \alpha_2, -\rangle \\
[R^1, \alpha_1, -] & \rightarrow t_j |R^1, \alpha_2, -\rangle + r_j |L^1, \alpha_2, -\rangle \\
\end{align*}$$

To evaluate the performance of the hyper-CPF gate and the hyper-parity gate, we simulated the results of the average fidelity and efficiency of the “Block”. It is known that the fidelity of arbitrary two quantum states ρ and σ is defined as(F(ρ, σ) ≡ tr√ρ(σρ)1/2 √ρ(σρ)1/2) (23)

$$\begin{align*}
|\psi_{\text{init}}\rangle_{\text{block}} & = \frac{1}{\sqrt{2}} |R\rangle_a |R\rangle_b (\cos \alpha |\alpha_1\rangle_a + \sin \alpha |\alpha_2\rangle_a) \\
& \text{(cos \beta |\alpha_1\rangle_b + \sin \beta |\alpha_2\rangle_b)(+ [+])} \\
|\psi_{\text{ideal}}\rangle_{\text{block}} & = \frac{1}{\sqrt{2}} |R\rangle_a |R\rangle_b (\cos \alpha \cos \beta |\alpha_1\rangle_a |\alpha_1\rangle_b) \\
& \text{− \cos \alpha \sin \beta |\alpha_1\rangle_a |\alpha_2\rangle_b [+]) + \cos \alpha \cos \beta |\alpha_1\rangle_a \\
& \text{− \sin \alpha \cos \beta |\alpha_2\rangle_a |\alpha_1\rangle_b [+] + \sin \alpha \sin \beta |\alpha_2\rangle_a} \\
& \text{− \sin \alpha \sin \beta |\alpha_2\rangle_b [+] + \sin \alpha \cos \beta |\alpha_1\rangle_a |\alpha_1\rangle_b [+] + \sin \alpha \sin \beta |\alpha_2\rangle_a} \\
& \text{− \sin \alpha \sin \beta |\alpha_2\rangle_b [+] + \sin \alpha \cos \beta |\alpha_1\rangle_a |\alpha_1\rangle_b [+] + \sin \alpha \sin \beta |\alpha_2\rangle_a} \\
& \text{(24)}
\end{align*}$$
Fidelity and efficiency can be achieved by increasing strength, cavity side leakage, and imperfect birefringence. \( g^2/\kappa \approx 8.654 \) in which \( g = g_0 \), for the larger parameter \( g^2/\kappa \geq 25 \) with \( Q \approx 10^3 \) (corresponding to \( \kappa \approx 1 \text{ GHz} \)) or \( Q \approx 10^4 \) (corresponding to \( \kappa \approx 10 \text{ GHz} \)), \( r(\omega) \) can reach nearly unity. The cavity side leakage rate \( 10\kappa \approx \kappa \) can be achieved with the current state of the art NV-cavity fabrication techniques. The average fidelity and efficiency of the “Block” operation as functions of \( \kappa/\kappa \) and \( g^2/\kappa \) are depicted in Figure 3. One can find that the NV-cavity coupling strength, cavity side leakage, and imperfect birefringence have a great impact on average fidelity and efficiency. High average fidelity and efficiency can be achieved by increasing \( g^2/\kappa \) and decreasing \( \kappa/\kappa \). When \( \kappa/\kappa = 0.1 \) and \( g^2/\kappa \approx 8.654 \), the average fidelity is \( F_{\text{Block}} = 99.999 \% \) and the efficiency is \( \eta_{\text{Block}} = 66.01 \% \).

In actuality, the major experimental imperfections that reduce the fidelity of the photon–NV block are: a) imperfections in electronic spin manipulations and readout,\(^{124,125}\) such as the electronic spin preparation (reduction < 1%), spin decoherence (reduction < 1%), off-resonant excitation errors (reduction ≈ 1%), spin-flip errors in the excited states (reduction ≈ 1%), microwave pulse errors (reduction ≈ 3.5%),\(^{102,109,126}\) and detector dark counts (reduction 3%); b) spatial mode mismatch between cavity and incident photon (reduction 3%); c) stability of the differences between the cavity resonance and the frequency of the incident photon; d) small probability of two photons in one qubit mode pulses (reduction 2%); e) errors induced by optical elements, such as PBS, HWP, and optical fiber; f) linear optical elements and fiber absorption losses. However, these limitations are not fundamental.

To summarize, we designed a compact quantum circuit to implement an two-photon three-DOF hyper-CPF gate by utilizing the significant advantages, such as frequency, spatial, and time-bin DOFs of photons. The hyper-CPF gate mechanism based on the photon–NV entangled building block is deterministic and robust against photonic dissipation noise because the gate is hyperparallel and the computing qubits are encoded in multi-DOFs. Moreover, our hyper-CPF gate not only increased the quantum channel capacity but also reduced the quantum resources overhead. Moreover, as an interesting application, we proposed the hyper-parity gate for hyper-parallel quantum computing and hyper-parallel quantum communication. Finally, the evaluations indicated that our schemes are feasible with the current experimental technology.

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