Supporting Information for

Diamagnetic response and phase stiffness for interacting isolated narrow bands

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1. Paramagnetic and diamagnetic contributions from projected Hamiltonian

In this section, we provide additional details on the computation of the electromagnetic response for the effective Hamiltonian, \( \mathcal{H}^{\text{eff}}[A] \). As noted in the main text, the effective current operator can be obtained from \( J^{\text{eff}}_x = -\delta \mathcal{H}^{\text{eff}}[A] / \delta A_x \), after expanding \( \mathcal{H}^{\text{eff}}[A] \) to first order in \( A \).

\[
J^{\text{eff}}_x(q) = P J_x(q) P + \frac{1}{2} \sum_{m, n \in \mathbb{L}, \ell' \in \mathbb{I}} (\langle m | H_o | \ell \rangle \langle \ell | J_x(q) | n \rangle + \langle m | J_x(q) | \ell \rangle \langle \ell | H_o | n \rangle) \left( \frac{1}{E_m - E_\ell} - \frac{1}{E_\ell - E_n} \right) \tag{1} \]

\[
= P J_x(q) P - \sum_{m, n \in \mathbb{L}, \ell' \in \mathbb{I}} \left( \langle m | H_o | \ell \rangle \langle \ell | J_x(q) | n \rangle E_\ell - E_n + \langle m | J_x(q) | \ell \rangle \langle \ell | H_o | n \rangle \right) + O(V^2 / \Delta). \tag{2} \]

In the \( q \to 0 \) limit, since the level mixing current can be written as \( J_x(q \to 0) = -i [\hat{X}, H_d + H_o] \) and \( |l \rangle \), \( |m \rangle \) and \( |n \rangle \) are eigenstates of \( H_d \), we have \(-i[l | [\hat{X}, H_d + H_o] | n \rangle / (E_l - E_n) = i[l | \hat{X} | n \rangle + O(V / \Delta) \). Applying this trick, Eq.2 can be simplified to yield the result in the main text.

The effective diamagnetic contribution can be obtained analogously as \( K^{\text{eff}}_{xx} = \frac{\delta^2 \mathcal{H}^{\text{eff}}[A]}{\delta A_x^2} \), after expanding \( \mathcal{H}^{\text{eff}}[A] \) to \( O(A^2) \),

\[
K^{\text{eff}}_{xx} = P K_{xx} P + \sum_{m, n \in \mathbb{L}, \ell' \in \mathbb{I}} \langle m | J_x(q \to 0) | \ell \rangle \langle \ell | J_x(-q \to 0) | n \rangle \left( \frac{1}{E_m - E_\ell} - \frac{1}{E_\ell - E_n} \right) \tag{3} \]

\[
+ \frac{1}{2} \sum_{m, n \in \mathbb{L}, \ell' \in \mathbb{I}} (\langle m | K_{xx} | \ell \rangle \langle \ell | J_x | n \rangle + \langle m | J_x | \ell \rangle \langle \ell | K_{xx} | n \rangle) \left( \frac{1}{E_m - E_\ell} - \frac{1}{E_\ell - E_n} \right). \tag{4} \]

Let us rewrite the level mixing current \( J_x(q \to 0) = -i [\hat{X}, H_d] - i [\hat{X}, H_o] \) and \( K_{xx} = -[\hat{X}, [\hat{X}, H_d] - [\hat{X}, [\hat{X}, H_o]]. \)

Since the energy denominators give \( O(1/\Delta) \) contributions Eq.3 and the level mixing current and diamagnetic term give both \( O(1) \) and \( O(\Delta) \) contributions, we have \( O(\Delta), O(1) \) and \( O(1/\Delta) \) terms in Eq.3. Keeping only the terms that are of \( O(\Delta) \) and \( O(1) \),

\[
K^{\text{eff}}_{xx} = -P [\hat{X}, [\hat{X}, H]] P \]

\[
+ P \left( -2 \hat{X} Q H_d Q \hat{X} + H_d \hat{X} Q \hat{X} + \hat{X} Q H_d \right) P \]

\[
+ 2 P \left( H_d \hat{X} Q \hat{X} Q \hat{X} H_d - \hat{X} Q H_d \right) P \]

\[
+ P \left( \hat{X} P X H_o + H_o \hat{X} P X - \hat{X} Q X H_o - H_o \hat{X} Q \hat{X} \right) P \]

\[
= -P [\hat{X} P, [P \hat{X} P, P H_d P]]. \tag{4} \]

In the above derivation, the term in the first line comes from \( P K_{xx} P \) in Eq.3 and the rest terms come from the other terms in Eq.3. By expanding \( P K_{xx} P \), we also note that the \( O(\Delta) \) piece is \( 2 P \hat{X} Q H_d Q \hat{X} \), which cancels exactly the first term in the second line in Eq.4.

2. Explicit expression for diamagnetic contribution to phase stiffness

In this section, we express \( K^{\text{eff}}_{xx} \) explicitly in terms of the fields, \( c_m, c^+_m \), defined in the active bands, where \( m \) is the band index. As already emphasized in the main text, the expectation value, \( \langle K^{\text{eff}}_{xx} \rangle \), still depends on the many-body state of interest and is in general difficult to evaluate exactly for a generic model. However, the following exercise will still lead to new insights into the general structure of the theory that controls the diamagnetic contribution.

First, we consider the effect of the unitary transformation on the field \( c_m \):

\[
e^{i \alpha P X P} c_m e^{-i \alpha P X P} = \sum_{m', \ell' \in \text{act}} c_{m'} \langle k, m | e^{-i \alpha \hat{x}} | k', m' \rangle, \text{ where} \]

\[
\hat{x} \equiv \sum_{k, m, \ell' \in \text{act}} | k, m, \ell | x | k', m' \rangle \langle k', m' | \]

is the single particle “projected” position operator. Here “act” is a short form to denote the “active” bands. We define \( | k, m \rangle \equiv e^{i k x} | u_{k,m} \rangle \) as the Bloch wave function. Note that we will be interested in the terms in the above expansion up to \( O(\alpha^2) \) and the limit of \( \alpha \to 0. \)
It is readily seen that,

\[ \langle k, m | e^{i\alpha \hat{P}' \hat{X}} | k', m' \rangle = \langle k, m | e^{i\alpha \hat{P}' \hat{X}} | k', m' \rangle = \frac{1}{2} \alpha^2 \langle k, m | \hat{P}' (\sum_{m'' \not\in \text{act}} |k'', m''\rangle \langle k'', m''|) \hat{X} | k', m' \rangle + O(\alpha^3) \]  \[ \text{[6]} \]

\[ g^{mn'}_{\mu\nu}(k) = \left[ \langle \partial^\mu_{\hat{P}' \hat{X}} u_{k,m} | \partial^\nu_{\hat{P}' \hat{X}} u_{k,m'} \rangle - \sum_{n \in \text{act}} \langle \partial^\mu_{\hat{P}' \hat{X}} u_{k,m} | u_{k,n} \rangle \langle u_{k,n} | \partial^\nu_{\hat{P}' \hat{X}} u_{k,m'} \rangle \right], \]  \[ \text{[8]} \]

with \( g^{mn'}_{\mu\nu}(k) \) the quantum-metric generalized to multiple orbitals. Therefore, we have

\[ e^{i\alpha \hat{P}' \hat{X}} c_{km} c^{\dagger} e^{-i\alpha \hat{P}' \hat{X}} = \sum_{m \in \text{act}} \left\{ c_{k + \alpha e_x, m'} \langle u_{k,m} | u_{k + \alpha e_x, m'} \rangle + \frac{1}{2} \alpha^2 g^{mn'}_{xx}(k) c_{km'} \right\} + O(\alpha^3), \]  \[ \text{[9]} \]

where \( e_x \) is the unit vector along \( x \) direction.

We can now obtain the corresponding transformation of the Hamiltonian and the associated diamagnetic response in the main text. For the kinetic energy, \( H^{\text{kin}}_{\text{eff}} = \sum_{k,m \in \text{act}} \epsilon_k c_{km} c^{\dagger}_{km} \), where \( \epsilon_k \) includes both the bare dispersion and the interaction induced renormalization,

\[ e^{i\alpha \hat{P}' \hat{X}} H^{\text{kin}}_{\text{eff}} e^{-i\alpha \hat{P}' \hat{X}} = \sum_{n_1, n_2 \in \text{act}} c_{n_1, k}^\dagger c_{n_2, k} \left[ \sum_{m \in \text{act}} \langle u_{k,n_1} | u_{k-\alpha e_x, m} \rangle c_{k-\alpha e_x, m} \langle u_{k-\alpha e_x, m} | u_{k,n_2} \rangle \right. \]

\[ + \frac{1}{2} \alpha^2 g^{n_1 n_2}_{xx}(k) (\epsilon_{k,n_1} + \epsilon_{k,n_2}) \right] + O(\alpha^3). \]  \[ \text{[10]} \]

Therefore,

\[ \langle K^{\text{kin}}_{\text{eff}} \rangle_{\text{kinetic}} = \partial_n^2 \left( e^{i\alpha \hat{P}' \hat{X}} H^{\text{eff}}_{\text{kin}} e^{-i\alpha \hat{P}' \hat{X}} \right) = \sum_{n_1, n_2 \in \text{act}} \langle \epsilon_{n_1, k}^\dagger c_{n_2, k} \rangle \left[ \langle u_{k,n_1} | \partial_{e_x}^2 \epsilon \hat{X} | u_{k,n_2} \rangle + g^{n_1 n_2}_{xx}(k) (\epsilon_{k,n_1} + \epsilon_{k,n_2}) \right], \]  \[ \text{[11]} \]

where the operator \( \epsilon_k \equiv \sum_{m \in \text{act}} \mathbb{P}_{km} \epsilon_{k,m} \) and \( \mathbb{P}_{km} \equiv |u_{k,m}\rangle \langle u_{k,m}| \), is a single body projector defined in terms of the Bloch functions. Note that the above quantity does not depend on the bare energy levels \( \epsilon_{k,n} \) of the active bands; if we shift \( \epsilon_{k,n} \) by a constant, the two terms in Eq.11 give opposite contributions and cancel each other. When there is only one active band, there is a significant simplification, leading to the familiar results.

Next, we consider the interaction term \( H^{\text{int}}_{\text{eff}} \). To simplify the notation, the repeated indices are summed over and the summation over the band indices only includes the active bands if not specified.

\[ e^{i\alpha \hat{P}' \hat{X}} H^{\text{int}}_{\text{eff}} e^{-i\alpha \hat{P}' \hat{X}} = \sum_{q,k_1,k_2} V(q) c_{k_1,\alpha_1,n_1}^\dagger c_{k_2,\beta_1,m_1} c_{k_2+q,\beta_2,m_2} c_{k_1-q,\alpha_2,n_2} \]

\[ \times \left[ \langle u_{k_1,n_1} | u_{k_1-\alpha_1,n} \rangle \langle u_{k_1-\alpha_1,n} | u_{k_1-\alpha_2,n} \rangle \langle u_{k_1-\alpha_2,n} | u_{k_1-q,n_2} \rangle \right. \]

\[ \times \left[ \langle u_{k_2,m_1} | u_{k_2-\alpha_2,m} \rangle \langle u_{k_2-\alpha_2,m} | u_{k_2-\alpha_1,m} \rangle \langle u_{k_2-\alpha_1,m} | u_{k_2+q,m_2} \rangle \right. \]

\[ + \frac{1}{2} \alpha^2 \left( g^{n_1,n_2}_{xx}(k_1) \langle u_{k_1,n_1} | u_{k_1-q,n_2} \rangle + g^{n_1,n_2}_{xx}(k_1-q) \langle u_{k_1,n_1} | u_{k_1-q,n_1} \rangle \right) \langle u_{k_2,m_1} | u_{k_2+q,m_1} \rangle \delta_{m_1,m_2} \]

\[ + \frac{1}{2} \alpha^2 \langle u_{k_1,n_1} | u_{k_1-q,n_1} \rangle \delta_{n_1,n_2} \left( g^{m_1,m_2}_{xx}(k_2) \langle u_{k_2,m_2} | u_{k_2+q,m_2} \rangle + g^{m_1,m_2}_{xx}(k_2+q) \langle u_{k_2,m_2} | u_{k_2+q,m_1} \rangle \right) \]  \[ \text{[12]} \]
where $\alpha, \beta$ label the spin indices and repeated band indices are summed over active bands. The contribution to $(K^\text{eff})_\text{int}$ is,

$$
(K^\text{eff})_\text{int} \equiv \partial^2 \left( e^{i\alpha \mathbf{X} \cdot \mathbf{P}} H^\text{int} e^{-i\alpha \mathbf{X} \cdot \mathbf{P}} \right)
$$

$$
= \sum_{q,k_1,k_2} \mathcal{V}(q) \langle \hat{c}^+_k \alpha_n |\hat{c}^+_k \beta_m \hat{c} \beta_{m+1} \hat{c} \alpha_{n+1} \rangle
$$

$$
\times 2 \langle u_{k_1,n_1} | \partial_x (\mathbb{P}_{k_1} \mathbb{P}_{k_1} - q) | u_{k_1,n_2} \rangle \langle u_{k_2,m_1} | \partial_x (\mathbb{P}_{k_2} \mathbb{P}_{k_2} + q) | u_{k_2,m_2} \rangle
$$

$$
+ \langle \langle u_{k_1,n_1} | \partial_x^2 (\mathbb{P}_{k_1} \mathbb{P}_{k_1} - q) | u_{k_1,n_2} \rangle \langle u_{k_2,m_1} | u_{k_2,m_2} \rangle \delta_{m_1,m_2} + \langle \langle u_{k_1,n_1} | u_{k_1,n_2} \rangle \delta_{n_1,n_2} \langle u_{k_2,m_1} | \partial_x^2 (\mathbb{P}_{k_2} \mathbb{P}_{k_2} + q) | u_{k_2,m_2} \rangle
$$

$$
+ g_{xx}^{m_1,m_2} (k_1) \langle u_{k_1,n_1} | u_{k_1,n_2} \rangle + g_{xx}^{m_1,m_2} (k_1 - q) \langle u_{k_1,n_1} | u_{k_1,n_1} \rangle \langle u_{k_2,m_1} | u_{k_2,m_1} \rangle \delta_{m_1,m_2}
$$

$$
+ \langle \langle u_{k_1,n_1} | u_{k_1,n_2} \rangle \delta_{n_1,n_2} \langle u_{k_2,m_1} | u_{k_2,m_2} | u_{k_2,m_2} \rangle + g_{xx}^{m_1,m_2} (k_1 - q) \langle u_{k_2,m_1} | u_{k_2,m_1} \rangle \rangle
$$

$$
\equiv (K^\text{eff})_\text{int,1} + (K^\text{eff})_\text{int,2}
$$

[13]

where the terms in the first line in the square bracket is denoted as $(K^\text{eff})_\text{int,1}$ and the rest are denoted as $(K^\text{eff})_\text{int,2}$.

If there is only one active band (possibly degenerate), we have,

$$
(K^\text{eff})_\text{int,1} = \sum_{q,k_1,k_2} \mathcal{V}(q) \langle \hat{c}^+_k \alpha_n |\hat{c}^+_k \beta_m \hat{c} \beta_{m+1} \hat{c} \alpha_{n+1} \rangle
$$

$$
\times 2 \langle \partial_{k_1,x} (u_{k_1,n_1}) | u_{k_1,n_2} \rangle + i \langle u_{k_1,n_1} | (A_{k_1,x} - A_{k_1,-x}) \rangle \langle \partial_{k_2,x} (u_{k_2,m_1}) | u_{k_2,m_2} + q \rangle
$$

$$
+ \langle \langle u_{k_1,n_1} | u_{k_1,n_2} \rangle \delta_{n_1,n_2} \langle u_{k_2,m_1} | u_{k_2,m_2} | u_{k_2,m_2} \rangle + g_{xx}^{m_1,m_2} (k_1 - q) \langle u_{k_2,m_1} | u_{k_2,m_1} \rangle \rangle,
$$

[14]

$$
(K^\text{eff})_\text{int,2} = \sum_{q,k_1,k_2} \mathcal{V}(q) \langle \hat{c}^+_k \alpha_n |\hat{c}^+_k \beta_m \hat{c} \beta_{m+1} \hat{c} \alpha_{n+1} \rangle
$$

$$
\times \left\{ \langle \partial_{k_1,x} (i A_{k_1,x} - A_{k_1,-x}) \rangle \right\}^2 \langle u_{k_1,n_1} | u_{k_1,n_2} \rangle + \langle \langle u_{k_1,n_1} | u_{k_1,n_2} \rangle \langle \partial_{k_2,x} | i A_{k_2,x} - A_{k_2,-x} \rangle \rangle^2 \langle u_{k_2,m_1} | u_{k_2,m_2} \rangle	ext{,}
$$

[15]

where $A_k = -i \langle u_k | \partial_k u_k \rangle$ is the Berry connection. From here one can read off the expression for $F(k_1, k_2, q)$.

### 3. Partial f-sum rule

In this section, we demonstrate that the following partial f-sum rule holds for the longitudinal conductivity at temperature $T \ll \Delta$,

$$
\int_0^\Lambda d\omega \text{Re}[\sigma_{xx}(q_x \to 0, \omega)] = \frac{\pi e^2}{2} \langle K^\text{eff} \rangle,
$$

[16]

where $\Lambda$ is a cut-off frequency that lies in the gap between the active bands and the upper bands.

In the spectral representation, the real part of the longitudinal conductivity can be written as,

$$
\text{Re}[\sigma_{xx}(q_x \to 0, \omega)] = \frac{\pi e^2}{\omega} \sum_{m,n} \frac{e^{-\beta E_m} - e^{-\beta E_n}}{Z} \langle n | j_x (q_x \to 0) | m \rangle \langle m | j_x (-q_x \to 0) | n \rangle \delta(\omega - E_m + E_n),
$$

[17]

where $Z$ is the partition function. If we perform an integral over $\omega$, at temperature $T \ll \Delta$, for $|\omega| \lesssim \Delta$, we only need to consider the states $|n\rangle$ and $|m\rangle$ that belong to the low energy Hilbert space $\mathbb{H}$ and therefore,

$$
\int_{-\Lambda}^\Lambda d\omega \text{Re}[\sigma_{xx}(q_x \to 0, \omega)] \approx \pi e^2 \sum_{m,n} \frac{\langle n | j_x (q_x \to 0) | m \rangle \langle m | j_x (-q_x \to 0) | n \rangle e^{-\beta E_m} - e^{-\beta E_n}}{Z (E_m - E_n)}
$$

$$
= -\pi e^2 \sum_{m,n} \langle n | [\hat{X}, \hat{H}] | m \rangle \langle m | [\hat{X}, \hat{H}] | n \rangle \frac{e^{-\beta E_m} - e^{-\beta E_n}}{Z (E_m - E_n)}
$$

$$
= \pi e^2 \sum_{m,n} \langle n | \hat{X} | m \rangle \langle m | E_m - E_n \rangle \langle m | \hat{X} | n \rangle \frac{e^{-\beta E_m} - e^{-\beta E_n}}{Z}
$$

[18]

$$
= \pi e^2 (K^\text{eff})_\text{xx}.
$$

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Note that $\text{Re}[\sigma_{xx}(q_x \rightarrow 0, \omega)]$ is even in $\omega$ and $\langle K_{xx}^{\text{eff}} \rangle$ is the thermal expectation value of $K_{xx}^{\text{eff}}$. 