Majority Colourings of Digraphs

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Abstract. We prove that every digraph has a vertex 4-colouring such that for each vertex \( v \), at most half the out-neighbours of \( v \) receive the same colour as \( v \). We then obtain several results related to the conjecture obtained by replacing 4 by 3.

1 Introduction

A majority colouring of a digraph is a function that assigns each vertex \( v \) a colour, such that at most half the out-neighbours of \( v \) receive the same colour as \( v \). In other words, more than half the out-neighbours of \( v \) receive a colour different from \( v \) (hence the name ‘majority’). Whether every digraph has a majority colouring with a bounded number of colours was posed as an open problem on mathoverflow [7]. In response, Ilya Bogdanov proved that a bounded number of colours suffice for tournaments. The following is our main result.

Theorem 1. Every digraph has a majority 4-colouring.

Proof. Fix a vertex ordering. First, 2-colour the vertices left-to-right so that for each vertex \( v \), at most half the out-neighbours of \( v \) to the left of \( v \) in the ordering receive the same colour as \( v \). Second, 2-colour the vertices right-to-left so that for each vertex \( v \), at most half the out-neighbours of \( v \) to the right of \( v \) in the ordering receive the same colour as \( v \). The product colouring is a majority 4-colouring.

Note that this proof implicitly uses two facts: (1) every digraph has an edge-partition into two acyclic subgraphs, and (2) every acyclic digraph has a majority 2-colouring.

The following conjecture naturally arises:

Conjecture 2. Every digraph has a majority 3-colouring.
This conjecture would be best possible. For example, a majority colouring of an odd directed cycle is proper (since each vertex has out-degree 1), and therefore three colours are necessary. There are examples with large outdegree as well. For odd $k \geq 1$ and prime $n \gg k$, let $G$ be the directed graph with $V(G) = \{ v_0, \ldots, v_{n-1} \}$ where $N^+(v_i) = \{ v_{i+1}, \ldots, v_{i+k} \}$ and vertex indices are taken modulo $n$. Suppose that $G$ has a majority 2-colouring. If some sequence $v_i, v_{i+1}, \ldots, v_{i+k}$ contains more than $\frac{k+1}{2}$ vertices of one colour, say red, and $v_i$ is the leftmost red vertex in this sequence, then more than $\frac{k-1}{2}$ out-neighbours of $v_i$ are red, which is not allowed. Thus each sequence $v_i, v_{i+1}, \ldots, v_{i+k}$ contains exactly $\frac{k+1}{2}$ vertices of each colour. This implies that $v_i$ and $v_{i+k+1}$ receive the same colour, as otherwise the sequence $v_{i+1}, \ldots, v_{i+k+1}$ would contain more than $\frac{k+1}{2}$ vertices of the colour assigned to $v_{i+k+1}$. For all vertices $v_i$ and $v_j$, if $\ell = \frac{j-i}{k+1}$ in the finite field $\mathbb{Z}_n$, then $j = i + \ell(k+1)$ and $v_i, v_{i+(k+1)}, v_{i+2(k+1)}, \ldots, v_{i+\ell(k+1)} = v_j$ all receive the same colour. Thus all the vertices receive the same colour, which is a contradiction. Hence the claimed 2-colouring does not exist.

Note that being majority $c$-colourable is not closed under taking induced subgraphs. For example, let $G$ be the digraph with $V(G) = \{ a, b, c, d \}$ and $E(G) = \{ ab, bc, ca, cd \}$. Then $G$ has a majority 2-colouring: colour $a$ and $c$ by 1 and colour $b$ and $d$ by 2. But the subdigraph induced by $\{ a, b, c \}$ is a directed 3-cycle, which has no majority 2-colouring.

The remainder of the paper takes a probabilistic approach to Conjecture 2, proving several results that provide evidence for Conjecture 2. A probabilistic approach is reasonable, since in a random 3-colouring, one would expect that a third of the out-neighbours of each vertex $v$ receive the same colour as $v$. So one might hope that there is enough slack to prove that for every vertex $v$, at most half the out-neighbours of $v$ receive the same colour as $v$. Section 2 proves Conjecture 2 for digraphs with very large minimum outdegree (at least logarithmic in the number of vertices), and then for digraphs with large minimum outdegree (at least a constant) and not extremely large maximum indegree. Section 3 shows that large minimum outdegree (at least a constant) is sufficient to prove the existence of one of the colour classes in Conjecture 2. Section 4 discusses multi-colour generalisations of Conjecture 2.

Before proceeding, we mention some related topics in the literature:

- For undirected graphs, the situation is much simpler. Lovász [4] proved that for every undirected graph $G$ and integer $k \geq 1$, there is a $k$-colouring of $G$ such that every vertex $v$ has at most $\frac{1}{k} \deg(v)$ neighbours receiving the same colour as $v$. The proof is simple. Consider a $k$-colouring of $G$ that minimises the number of monochromatic edges. Suppose that some vertex $v$ coloured $i$ has greater than $\frac{1}{k} \deg(v)$ neighbours coloured $i$. Thus less than $\frac{k-1}{k} \deg(v)$ neighbours of $v$ are not coloured $i$, and less than $\frac{1}{k} \deg(v)$ neighbours of $v$ receive some colour $j \neq i$. Thus, if $v$ is recoloured $j$, then the number of monochromatic edges decreases. Hence no vertex $v$ has greater than $\frac{1}{k} \deg(v)$ neighbours with the same
colour as $v$.

- Seymour [6] considered digraph colourings such that every non-sink vertex receives a colour different from some outneighbour, and proved that a strongly-connected digraph $G$ admits a 2-colouring with this property if and only if $G$ has an even directed cycle. The proof shows that every digraph has such a 3-colouring, which we repeat here: We may assume that $G$ is strongly connected. In particular, there are no sink vertices. Choose a maximal set $X$ of vertices such that $G[X]$ admits a 3-colouring where every vertex has a colour different from some outneighbour. Since any directed cycle admits such a colouring, $X \neq \emptyset$. If $X \neq V(G)$, then choose an edge $uv$ entering $X$ and colour $u$ different from the colour of $v$, contradicting the maximality of $X$. So $X = V(G)$. (The same proof shows two colours suffice if you start with an even cycle.)

- Alon [1, 2] posed the following problem: Is there a constant $c$ such that every digraph with minimum outdegree at least $c$ can be vertex-partitioned into two induced digraphs, one with minimum outdegree at least 2, and the other with minimum outdegree at least 1?

- Wood [8] proved the following edge-colouring variant of majority colourings: For every digraph $G$ and integer $k \geq 2$, there is a partition of $E(G)$ into $k$ acyclic subgraphs such that each vertex $v$ of $G$ has outdegree at most $\left\lceil \frac{\deg^+(v)}{k-1} \right\rceil$ in each subgraph. The bound $\left\lceil \frac{\deg^+(v)}{k-1} \right\rceil$ is best possible, since in each acyclic subgraph at least one vertex has outdegree 0.

2 Large Outdegree

We now show that minimum outdegree at least logarithmic in the number of vertices is sufficient to guarantee a majority 3-colouring. All logarithms are natural.

**Theorem 3.** Every graph $G$ with $n$ vertices and minimum outdegree $\delta > 72 \log(3n)$ has a majority 3-colouring. Moreover, at most half the out-neighbours of each vertex receive the same colour.

**Proof.** Randomly and independently colour each vertex of $G$ with one of three colours $\{1, 2, 3\}$. Consider a vertex $v$ with out-degree $d_v$. Let $X(v, c)$ be the random variable that counts the number of out-neighbours of $v$ coloured $c$. Of course, $E(X(v, c)) = d_v/3$. Let $A(v, c)$ be the event that $X(v, c) > d_v/2$. Note that $X(v, c)$ is determined by $d_v$ independent trials and changing the outcome of any one trial changes $X(v, c)$ by at most 1. By the simple concentration bound\(^1\),

$$
P(A(v, c)) \leq \exp\left(-\frac{(d_v/6)^2}{2d_v}\right) = \exp\left(-\frac{d_v}{72}\right) \leq \exp\left(-\frac{\delta}{72}\right).$$

\(^1\) The simple concentration bound says that if $X$ is a random variable determined by $d$ independent trials, such that changing the outcome of any one trial affects $X$ by at most $c$, then $P(X > E(X) + t) \leq \exp\left(-t^2/2c^2d\right)$; see [5, Chapter 10]. With $E(X_v) = d_v/3$ and $t = d_v/6$ and $c = 1$ we obtain the desired upper bound on $P(X_v > d_v/2)$. 

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The expected number of events $A(v, c)$ that hold is
\[
\sum_{v \in V(G)} \sum_{c \in \{1, 2, 3\}} P(A(v, c)) \leq 3n \exp(-\delta/72) < 1,
\]
where the last inequality holds since $\delta > 72 \log(3n)$. Thus there exists colour choices such that no event $A(v, c)$ holds. That is, a majority 3-colouring exists.

The following result shows that large outdegree (at least a constant) and not extremely large indegree is sufficient to guarantee a majority 3-colouring.

**Theorem 4.** Every digraph with minimum out-degree $\delta \geq 1200$ and maximum in-degree at most $\exp(\delta/72)/12\delta$ has a majority 3-colouring. Moreover, at most half the out-neighbours of each vertex receive the same colour.

**Proof.** We assume $\delta \geq 1200$, as otherwise the minimum out-degree $\delta$ is greater than the maximum in-degree $\exp(\delta/72)/12\delta$, which does not make sense.

We use the following weighted version of the Local Lemma [3, 5]: Let $A := \{A_1, \ldots, A_n\}$ be a set of ‘bad’ events, such that each $A_i$ is mutually independent of $A \setminus (D_i \cup \{A_i\})$, for some subset $D_i \subseteq A$. Assume there are numbers $t_1, \ldots, t_n \geq 1$ and a real number $p \in [0, \frac{1}{4}]$ such that for $1 \leq i \leq n$,
\[
(a) \quad \mathbb{P}(A_i) \leq p^{t_i} \quad \text{and} \quad (b) \quad \sum_{A_j \in D_i} (2p)^{t_j} \leq t_i/2.
\]

Then with positive probability no event $A_i$ occurs.

Define $p := \exp(-\delta/72)$. Since $\delta \geq 1200$ we have $p \in [0, \frac{1}{4}]$. Randomly and independently colour each vertex of $G$ with one of three colours $\{1, 2, 3\}$. Consider a vertex $v$ with out-degree $d_v$. Let $X(v, c)$ be the random variable that counts the number of out-neighbours of $v$ coloured $c$. Of course, $\mathbb{E}(X(v, c)) = d_v/3$. Let $A(v, c)$ be the event that $X(v, c) > d_v/2$. Let $A := \{A(v, c) : v \in V(G), c \in \{1, 2, 3\}\}$ be our set of events. Let $t(v, c) := t_v := d_v/\delta$ be the associated weight. Then $t_v \geq 1$. It suffices to prove that conditions (a) and (b) hold.

Note that $X(v, c)$ is determined by $d_v$ independent trials and changing the outcome of any one trial changes $X(v, c)$ by at most 1. By the simple concentration bound,
\[
\mathbb{P}(A(v, c)) \leq \exp(-d_v/6)^2/2d_v) = \exp(-d_v/12) = \exp(-\delta t_v/72) = p^{t_v}.
\]

Thus condition (a) is satisfied. For each event $A(v, c)$ let $D(v, c)$ be the set of all events $A(w, c') \in A$ such that $v$ and $w$ have a common out-neighbour. Then $A(v, c)$ is mutually
independent of \( A \setminus (D(v, c) \cup \{A(v, c)\}) \). Since \( t_w \geq 1 \),

\[
\sum_{A(w, c') \in D(v, c)} (2p)^{t_w} \leq \sum_{A(w, c') \in D(v, c)} (2p)^1 = 2p|D(v, c)|. 
\]

Since each out-neighbour of \( v \) has in-degree at most \( \exp(\delta/72)/12\delta \), we have \( |D(v, c)| \leq d_v \exp(\delta/72)/4\delta \) and

\[
\sum_{A(w, c') \in D(v, c)} (2p)^{t_w} \leq pd_v \exp(\delta/72)/2\delta = \exp(-\delta/72)t_v \exp(\delta/72)/2 = t_v/2. 
\]

Thus condition (b) is satisfied. By the local lemma, with positive probability, no event \( A(v, c) \) occurs. That is, a majority 3-colouring exists. \( \square \)

Note that the conclusion in Theorem 3 and Theorem 4 is stronger than in Conjecture 2. We now show that such a conclusion is impossible (without some extra degree assumption).

**Lemma 5.** For all integers \( k \) and \( \delta \), there are infinitely many digraphs \( G \) with minimum outdegree \( \delta \), such that for every vertex \( k \)-colouring of \( G \), there is a vertex \( v \) such that all the out-neighbours of \( v \) receive the same colour.

**Proof.** Start with a digraph \( G_0 \) with at least \( k\delta \) vertices and minimum outdegree \( \delta \). For each set \( S \) of \( \delta \) vertices in \( G_0 \), add a new vertex with out-neighbourhood \( S \). Let \( G \) be the digraph obtained. In every \( k \)-colouring of \( G \), at least \( \delta \) vertices in \( G_0 \) receive the same colour, which implies that for some vertex \( v \in V(G) \setminus V(G_0) \), all the out-neighbours of \( v \) receive the same colour. \( \square \)

3 Stable Sets

A set \( T \) of vertices in a digraph \( G \) is a stable set if for each vertex \( v \in T \), at most half the out-neighbours of \( v \) are also in \( T \). A majority colouring is a partition into stable sets. Of course, if a digraph has a majority 3-colouring, then it contains a stable set with at least one third of the vertices. The next lemma provides a sufficient condition for the existence of such a set.

**Theorem 6.** Every digraph \( G \) with \( n \) vertices and minimum outdegree at least 22 has a stable set with at least \( \frac{n}{3} \) vertices.

Theorem 6 is proved via the following more general lemma.
Lemma 7. For $0 < \alpha < p < \beta < 1$, every digraph $G$ with minimum outdegree at least

$$\delta := \left\lfloor \frac{(\beta + p) \log \left( \frac{p - \alpha}{p - \beta} \right)}{(\beta - p)^2} \right\rfloor$$

contains a set $T$ of at least $\alpha n$ vertices, such that $|N^+_G(v) \cap T| \leq \beta |N^+_G(v)|$ for every vertex $v \in T$.

Proof. Let $d_v := |N^+_G(v)|$ be the outdegree of each vertex $v$ of $G$. Initialise $S := \emptyset$. For each vertex $v$ of $G$, add $v$ to $S$ independently and randomly with probability $p$. Let $X_v := |N^+_G(v) \cap S|$. Note that $X_v \sim \text{Bin}(d_v, p)$ and

$$P(X_v > \beta d_v) = \sum_{k \geq \lceil \beta d_v \rceil + 1} \binom{d_v}{k} p^k (1 - p)^{d_v - k}. \quad (1)$$

By the Chernoff bound$^2$,

$$P(X_v > \beta d_v) \leq \exp \left( -\frac{(\beta - p)^2}{\beta + p} d_v \right) \leq \exp \left( -\frac{(\beta - p)^2}{\beta + p} \delta \right) \leq \frac{p - \alpha}{p}. \quad (2)$$

where the last inequality follows from the definition of $\delta$. Let $B := \{v \in S : X_v > \beta d_v\}$. Then

$$E(|B|) = \sum_{v \in V(G)} P(v \in S \text{ and } X_v > \beta d_v).$$

Since the events $v \in S$ and $X_v > \beta d_v$ are independent,

$$E(|B|) = \sum_{v \in V(G)} P(v \in S) P(X_v > \beta d_v) = p \sum_{v \in V(G)} P(X_v > \beta d_v) \leq (p - \alpha)n.$$

Let $T := S \setminus B$. Thus $|N^+_G(v) \cap T| \leq \beta d_v$ for each vertex $v \in T$, as desired. By the linearity of expectation,

$$E(|T|) = E(|S|) - E(|B|) = pn - E(|B|) \geq \alpha n.$$

Thus there exists the desired set $T$. \qed

Proof of Theorem 6. The proof follows that of Lemma 7 with one change. Let $\alpha := \frac{1}{3}$ and $\beta := \frac{1}{2}$ and $p := 0.38$. Then $\delta = 129$. If $22 \leq d_v \leq 128$ then direct calculation of the formula in (1) verifies that $P(X_v > \beta d_v) \leq \frac{p - \alpha}{p}$, as in (2). For $d_v \geq 129$ the Chernoff bound proves (2). The rest of the proof is the same as in Lemma 7. \qed

$^2$ The Chernoff bound implies that if $X \sim \text{Bin}(d, p)$ then $P(X \geq (1 + \epsilon)pd) \leq \exp(-\frac{\epsilon^2}{3} pd)$ for $\epsilon \geq 0$. With $\epsilon = \frac{p - 1}{p} \leq 1$ we have $P(X > \beta d) \leq \exp(-\frac{(\beta - p)^2}{p + \beta} d)$. 

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Note the following corollary of Lemma 7 obtained with $\alpha = \frac{1}{2} - \epsilon$ and $p = \frac{1}{2} - \frac{\epsilon}{2}$. This says that graphs with large minimum outdegree have a stable set with close to half the vertices.

Proposition 8. For $0 < \epsilon < \frac{1}{2}$, every $n$-vertex digraph $G$ with minimum outdegree at least $2\epsilon^{-2}(2 - \epsilon) \log(\frac{1}{1 - \epsilon})$ contains a stable set of at least $(\frac{1}{2} - \epsilon)n$ vertices.

4 Multi-Colour Generalisation

The following natural generalisation of Conjecture 2 arises.

Conjecture 9. For $k \geq 2$, every digraph has a vertex $(k + 1)$-colouring such that for each vertex $v$, at most $\frac{1}{k}\deg^+(v)$ out-neighbours of $v$ receive the same colour as $v$.

The proof of Theorem 1 generalises to give an upper bound of $k^2$ on the number of colours in Conjecture 9. It is open whether the number of colours is $O(k)$. This conjecture would be best possible, as shown by the following example. Let $G$ be the $k$-th power of an $n$-cycle, with arcs oriented clockwise, where $n \geq 2k + 3$ and $n \not\equiv 0 \pmod{k + 1}$. Each vertex has outdegree $k$. Say $G$ has a vertex $(k + 1)$-colouring such that for each vertex $v$, at most $\epsilon k$ out-neighbours of $v$ receive the same colour as $v$. If $\epsilon k < 1$ then the underlying undirected graph of $G$ is properly coloured, which is only possible if $n \equiv 0 \pmod{k + 1}$. Hence $\epsilon \geq \frac{1}{k}$.

Lemma 7 with $\alpha = \frac{1}{k} - \epsilon$ and $\beta = \frac{1}{k}$ and $p = \frac{1}{k} - \frac{\epsilon}{2}$ implies the following ‘stable set’ version of Conjecture 9 for digraphs with large minimum outdegree.

Proposition 10. For $k \geq 2$ and $\epsilon \in (0, \frac{1}{k})$, every $n$-vertex digraph $G$ with minimum outdegree at least $2\epsilon^{-2}(\frac{k}{2} - \epsilon) \log(\frac{2k}{2k - 1})$ contains a set $T$ of at least $(\frac{1}{k} - \epsilon)n$ vertices, such that for every vertex $v \in T$, at most $\frac{1}{k}\deg^+(v)$ out-neighbours of $v$ are also in $T$.

5 Open Problems

In addition to resolving Conjecture 2, the following open problems arise from this paper:

1. Is there a constant $\beta < 1$ for which every digraph has a 3-colouring, such that for every vertex $v$, at most $\beta \deg^+(v)$ out-neighbours receive the same colour as $v$?

2. Does every tournament have a majority 3-colouring?

3. Does every Eulerian digraph have a majority 3-colouring? Note that for an Eulerian digraph $G$, if each vertex $v$ has in-degree and out-degree $\deg(v)$, then by the result for undirected graphs mentioned in Section 1, the underlying undirected graph of $G$ has a
4-colouring such that each vertex $v$ has at most $\frac{1}{2} \deg(v)$ in- or out-neighbours with the same colour as $v$. In particular, $G$ has a majority 4-colouring. By an analogous argument every Eulerian digraph has a 3-colouring such that each vertex $v$ has at most $\frac{2}{3} \deg(v)$ in- or out-neighbours with the same colour as $v$, thus proving a special case of the first question above.

4. Does every digraph in which every vertex has in-degree and out-degree $k$ have a majority $3$-colouring? A variant of Theorem 4 proves this result for $k \geq 144$.

5. Is there a characterisation of digraphs that have a majority $2$-colouring (or a polynomial time algorithm to recognise such digraphs)?

6. Does every digraph have a $O(k)$-colouring such that for each vertex $v$, at most $\frac{1}{k} \deg^+(v)$ out-neighbours receive the same colour as $v$ (for all $k \geq 2$)?

7. A digraph $G$ is majority $c$-choosable if for every function $L : V(G) \to \mathbb{Z}$ with $|L(v)| \geq c$ for each vertex $v \in V(G)$, there is a majority colouring of $G$ with each vertex $v$ coloured from $L(v)$. Is every digraph majority $c$-choosable for some constant $c$? The proof of Theorem 1 shows that acyclic digraphs are majority 2-choosable, and obviously Theorem 3 and Theorem 4 extend to the setting of choosability.

8. Consider the following fractional setting. Let $S(G)$ be the set of all stable sets of a digraph $G$. Let $S(G, v)$ be the set of all stable sets containing $v$. A fractional majority colouring is a function that assigns each stable set $T \in S(G)$ a weight $x_T \geq 0$ such that $\sum_{T \in S(G, v)} x_T \geq 1$ for each vertex $v$ of $G$. What is the minimum number $k$ such that every digraph $G$ has a fractional majority colouring with total weight $\sum_{T \in S(G)} x_T \leq k$? Perhaps it is less than 3.

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