RADII OF CONVEXITY OF INTEGRAL OPERATORS

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ABSTRACT. The object of the present paper is to study of radius of convexity of two certain integral operators as follows

\[ F(z) := \int_0^z \prod_{i=1}^n (f_i'(t))^{\gamma_i} \, dt \]

and

\[ J(z) := \int_0^z \prod_{i=1}^n (f_i'(t))^{\gamma_i} \prod_{j=1}^m \left( \frac{g_j(z)}{z} \right)^{\lambda_j} \, dt, \]

where \( \gamma_i, \lambda_i \in \mathbb{C} \), \( f_i \) (\( 1 \leq i \leq n \)) and \( g_j \) (\( 1 \leq j \leq m \)) belong to the certain subclass of analytic functions.

1. Introduction

Let \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( \mathcal{H}(\Delta) \) be the set of all functions analytic in \( \Delta \) and let

\[ \mathcal{A}_p := \{ f \in \mathcal{H}(\Delta) : f(z) = z^p + a_{p+1}z^{p+1} + \cdots \}, \]

for all \( z \in \Delta \) and \( p \in \mathbb{N} := \{ 1, 2, \ldots \} \) with \( \mathcal{A}_1 \equiv \mathcal{A} \). Also let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of univalent functions.

If \( f \) and \( g \) are two of the functions in \( \mathcal{A} \), we say that \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \), if there exists a Schwarz function \( w \) such that \( f(z) = g(w(z)) \), for all \( z \in \Delta \). Furthermore, if the function \( g \) is univalent in \( \Delta \), then we have the following equivalence:

\[ f(z) \prec g(z) \iff (f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta)). \]

Let \( \mathcal{S}^*(\varphi) \) denote the class of functions \( f \) in \( \mathcal{S} \) satisfying

\[ \frac{zf'(z)}{f(z)} < \varphi(z) \quad (z \in \Delta), \]

where \( \varphi \) is an analytic function with positive real part on \( \Delta \), \( \varphi(0) = 1 \), \( \varphi'(0) > 0 \) and \( \varphi \) maps \( \Delta \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class \( \mathcal{S}^*(\varphi) \) was introduced by Ma and Minda (see [10]). We note that the class \( \mathcal{S}^*(\alpha) \) consisting of starlike functions of order \( \alpha \) and the class \( \mathcal{S}^*[A,B] \) of Janowski starlike functions are special cases of \( \mathcal{S}^*(\varphi) \) where \( \varphi(z) = (1+(1-2\xi)z)/(1-z) \) (\( 0 \leq \xi < 1 \)) and \( \varphi(z) = (1+Az)/(1+Bz) \) (\( -1 \leq B < A \leq 1 \)), respectively.

We also use the following well known notations:

\[ \mathcal{S}_p^*(\xi) := \left\{ f \in \mathcal{A}_p : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \xi, \quad 0 \leq \xi < p, |z| < 1 \right\}, \]

for the functions of \( p \)-valent starlike of order \( \xi \) and

\[ \mathcal{K}_p(\xi) := \left\{ f \in \mathcal{A}_p : \text{Re} \left( 1 + \frac{zf'(z)}{f(z)} \right) > \xi, \quad 0 \leq \xi < p, |z| < 1 \right\}, \]

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for the functions of p-valent convex of order $\xi$. We put $S^*_p(\xi) \equiv S^*(\xi)$ and $K_1(\xi) \equiv K(\xi)$, the class of starlike and convex functions of order $\xi$, respectively.

Following [3], for $\beta \in \mathbb{R}$ we consider the class $G(\beta)$ consisting of locally univalent functions $f \in A$ which satisfy the condition

\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{\beta}{2}, \quad z \in \Delta. \]

In [9], Ozaki introduced the class $G \equiv G(1)$ and proved that functions in $G$ are univalent in $\Delta$. In [13], Umezawa generalized Ozaki's result for a version of the class $G$ (convex functions in one direction). It is also known that the functions in the class $G(1)$ are starlike in $\Delta$. The function class $G(\beta)$ was studied extensively by Kargar et al. [7].

For $a \in \Delta$, let $\text{Aut}(\Delta)$ be the set of all automorphisms $\phi(z) = e^{i\theta}z + a_1 + z\overline{a}$, where $\theta \in \mathbb{R}$. Following [11], we recall a definition:

**Definition 1.1.** A subclass $M \subset A$ is said to be a linear invariant family if:

(i) every $f \in M$ is locally univalent in $\Delta$ and

(ii) for any $f \in M$ and $\phi \in \text{Aut}(\Delta)$ we have

\[ F_{\phi}(f)(z) = f(\phi(0)) - f(\phi(z)) - f'(\phi(0)) \phi'(0) \in M. \]

Define also the order of $f \in A$ as $\text{ord } f = \sup_{\phi \in \text{Aut}(\Delta)} |a_2(F_{\phi})|$ and the universal linear invariant family of order $\alpha \geq 1$ as

\[ U_{\alpha} = \{ f \in A : \text{ord } f \leq \alpha \}. \]

It is well known that $U_1 = K$, whereas $S \subset U_2$. The following lemma will be required.

**Lemma 1.1.** (see [5]) If $f \in U_{\alpha}$ and $\alpha \geq 1$, then

\[ \left| z f''(z) \frac{2|z|^2}{1 - |z|^2} \right| \leq 2\alpha |z| \frac{1 - |z|^2}{1 - |z|^2}, \quad z \in \Delta. \]

Over the years, study on the integral operators have been investigated by many authors, including (for example) the following cases.

In [12], Silverman obtained the order of starlikeness of functions given by

\[ z \prod_{i=1}^m \left( \frac{f_i(z)}{z} \right)^{a_i} \prod_{i=1}^n \left( g_i(z) \right)^{b_i}, \]

where $f \in S^*(\alpha)$, $g \in K(\alpha)$ and $a_i, b_i \geq 0$. Also, Dimkow (see [3]) studied the operator

\[ z \prod_{i=1}^m \left( \frac{f_i(z)}{z} \right)^{a_i} \]

and found the radii of starlikeness and convexity as well as orders of starlikeness and convexity. Again, Dimkov and Dziołk [4] considered the functions of the type

\[ z^p \prod_{i=1}^m \left( \frac{f_i(z)}{z^p} \right)^{a_i}, \]

where $f_i \in S^*_p(\alpha_i)$ and $a_i$ are the complex numbers. They found that the conditions for the centre and the radius of the disc $\{ z \in \mathbb{C} : |z - w| < r \}$, contained in the unit disc $\Delta$ and containing the origin, so that its transformation by the function (1.3)
be a domain starlike with respect to the origin. In 2008, also Breaz et al. (see [2]) introduced a new integral operator as follows:

\[
F(z) := \int_0^z \prod_{i=1}^n (f'_i(t))^\gamma_i \, dt
\]

and studied some properties of it where \(\gamma_i\) are the complex numbers. For example, they showed that under certain conditions, the integral operator \(F(z)\) is univalent, starlike and convex function.

In this paper we shall consider the following integral operator:

\[
J(z) := \int_0^z \prod_{i=1}^n (f'_i(t))^\gamma_i \prod_{j=1}^m \left( \frac{g_j(z)}{z} \right)^{\lambda_j} \, dt,
\]

where \(\gamma_i, \lambda_j \in \mathbb{C}\) and \(f_i, g_j \in \mathcal{A}\). Note that by taking \(g_j = f_j\) and \(m = n\) in (1.5), we have the integral operator \(F(z)\) that studied by Frasin (see [6]).

In the next Section 2, we obtain the radius of convexity of the integral operator (1.4) where \(f_i\) belonging to the classes \(\mathcal{U}_\alpha, \mathcal{K}, \mathcal{S}\) and \(\mathcal{G}(\beta)\). Moreover, the radii of convexity of the integral operator (1.5) is obtained where \(f_i\) are the universal linear invariant, convex and locally convex functions and \(g_j\) belong to the class of starlike functions of order \(\alpha_j\).

2. Main Results

Our first result is contained in the following:

**Theorem 2.1.** Let \(f_i\) belong to the class \(\mathcal{U}_{\alpha_i}\) where \(\alpha_i \geq 1\) \((1 \leq i \leq n)\) and \(\alpha := \max\{\alpha_1, \ldots, \alpha_n\}\). Also, let \(M > 0\) and \(\sum_{i=1}^n |\gamma_i| \leq M\) \((\gamma_i \in \mathbb{C})\). Then the radius of convexity of the integral operator \(F(z)\) defined by (1.4) is

\[
r_F(F) = \frac{\sqrt{\alpha^2 M^2 + 2M + 1} - \alpha M}{2M + 1}.
\]

**Proof.** At first for a fixed \(M > 0\) we denote by \(\mathfrak{R}(M)\), the class of all integral operators of the form (1.4) such that

\[
\sum_{i=1}^n |\gamma_i| \leq M
\]

and we define the subclass \(\mathfrak{R}(m, \gamma)\) as follows

\[
\mathfrak{R}(m, \gamma) := \left\{ F \in \mathfrak{R}(M) : \sum_{i=1}^n \mathfrak{Re}\{\gamma_i\} = \gamma, \ -m \leq \gamma \leq m, 0 < m < M \right\}.
\]

So \(\bigcup_{m \in (0, M]} \mathfrak{R}(m, \gamma) \subseteq \mathfrak{R}(M)\). Using the analytic definition of convexity, the radius \(r\) of convexity of the class \(\mathfrak{R}(M)\) is the largest number \(0 < r < 1\) for which

\[
\min_{|z|=r} \mathfrak{Re}\left\{ 1 + \frac{zF''(z)}{F(z)} \right\} \geq 0 \quad (F \in \mathfrak{R}(m, \gamma)).
\]

By (1.4) for every \(r \in (0, 1)\) we have

\[
\min_{|z|=r} \mathfrak{Re}\left\{ 1 + \frac{zF''(z)}{F(z)} \right\} = 1 + \min_{|z|=r} \mathfrak{Re}\left\{ \sum_{i=1}^n \gamma_i \left( \frac{zf_i''(z)}{f_i'(z)} \right) \right\}
\]

\[
= 1 + \min_{|z|=r} \sum_{i=1}^n \mathfrak{Re}\left\{ \gamma_i \left( \frac{zf_i''(z)}{f_i'(z)} \right) \right\}
\]

\[
\geq 1 + \sum_{i=1}^n \min_{|z|=r} \mathfrak{Re}\left\{ \gamma_i \left( \frac{zf_i''(z)}{f_i'(z)} \right) \right\}.
\]
Therefore, \( \psi (2.4) \) 

\[ r \alpha \] 

the proof. \( \square \)

Let \( \text{Theorem 2.3.} \)

omit the details. \( \square \)

The remain of the proof is similar to the proof of the \( \text{Theorem 2.1} \) and we thus

Let \( \text{Corollary 2.1.} \)

\( r (2.6) \) is \( (1.4) \) 

\( \epsilon \) \( (2.3) \) \( r \) \( (1.4) \) 

Since an inequality \( |a| \leq c \) implies \( -c \leq \Re \{a\} \leq c \), it follows from the last inequality that

\[ \Re \left\{ \gamma_i \left( \frac{zf''(z)}{f'(z)} \right) \right\} > \frac{2\alpha z|z|}{1-|z|^2} \leq \frac{2\alpha \gamma_i |z|}{1-|z|^2} \] 

\( (z \in \Delta) \).

Hence we have

\[ (2.3) \quad \min_{|z|=r} \Re \left\{ 1 + \frac{zF''(z)}{f'(z)} \right\} > \frac{-2(M+1)r^2 - 2\alpha M r + 1}{1-r^2} =: \psi (r). \]

Therefore, \( \psi (r) > 0 \) if and only if \( \omega (r) := -(2M+1)r^2 - 2\alpha M r + 1 > 0 \). Since 

\( \alpha^2 M^2 + 2M + 1 > 0 \), the zeros of \( \omega (r) \) are real. It is easy to see that \( \omega (r) > 0 \) when 

\[ 0 < |z| < r_c, \] 

where

\[ r_c = \frac{\sqrt{\alpha^2 M^2 + 2M + 1} - \alpha M}{2M+1} \]

and \( r_c \in (0,1) \). Also, \( \omega (r) \) has a unique root in the interval \( (0,1) \). This completes the proof. \( \square \)

If we put \( \alpha_i = 1 \) \((i = 1,2,\ldots,n)\) in the \( \text{Theorem 2.3.} \) then we have.

**Corollary 2.1.** Let \( f_i (1 \leq i \leq n) \) belong to the class \( \mathcal{K} \). Also let \( \sum_{i=1}^{n} |\gamma_i| \leq M \). Then the radius of convexity of the integral operator \( (1.4) \) is

\[ (2.4) \quad r_c' := \frac{1}{2M+1} \quad (M > 0). \]

**Theorem 2.2.** Let \( f_i (1 \leq i \leq n) \) be univalent functions. Then the radius of convexity of the integral operator \( (1.4) \) is

\[ (2.5) \quad r := \frac{1}{\sqrt{4M^2 + 2M + 1} + 2M} \quad (M > 0). \]

**Proof.** From \[ 1 \], if \( f \in \mathcal{S} \), then

\[ \left| \frac{zf''(z)}{f'(z)} \right| - \frac{2r^2}{1-r^2} \leq \frac{4r}{1-r^2} \quad (|z| = r < 1). \]

The remain of the proof is similar to the proof of the \( \text{Theorem 2.1} \) and we thus omit the details. \( \square \)

**Theorem 2.3.** Let \( f_i \) belong to the class \( \mathcal{G}(b_i) \), where \( 0 < b_i \leq 1 \) and \( 1 \leq i \leq n \). Also let \( \beta := \max\{b_1, \ldots, b_n\} \). Then the radius of convexity of the integral operator \( (1.4) \) is

\[ (2.6) \quad r := \frac{1}{\beta M + 1} \quad (M > 0). \]

**Proof.** From \[ 3 \] proof of Theorem 1, if \( f_i \) belong to the class \( \mathcal{G}(b_i) \), then we have

\[ \left| \gamma_i \left( \frac{zf''(z)}{f'(z)} \right) \right| - \frac{\beta_i |z|}{1-|z|} \leq \frac{\beta |z|}{1-|z|} \]

\[ (2.7) \]
From now (2.9), we obtain
\[
1 + \frac{zJ''(z)}{J'(z)} = 1 + \sum_{i=1}^{n} \min_{|z|=r} \Re \left\{ \frac{zf_i''(z)}{f_i'(z)} \right\} + \sum_{j=1}^{m} \lambda_j \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right).
\]
We shall to show that
\[
\min_{|z|=r} \Re \left\{ 1 + \frac{zJ''(z)}{J'(z)} \right\} \geq 0.
\]
From now (2.9), we obtain
\[
\min_{|z|=r} \Re \left\{ 1 + \frac{zJ''(z)}{J'(z)} \right\} \geq 1 + \sum_{i=1}^{n} \min_{|z|=r} \Re \left\{ \frac{zf_i''(z)}{f_i'(z)} \right\} + \sum_{j=1}^{m} \min_{|z|=r} \Re \left\{ \frac{zg_j'(z)}{g_j(z)} - 1 \right\}.
\]
Since \( f_i \in U_{\alpha_i} \), using the proof of Theorem 2.1 we get
\[
1 + \sum_{i=1}^{n} \min_{|z|=r} \Re \left\{ \frac{zf_i''(z)}{f_i'(z)} \right\} \geq \frac{-(2M+1)r^2 - 2\alpha Mr + 1}{1-r^2}.
\]
Also, because \( g_j \in S^*(\xi_j) \) we have
\[
\frac{zg_j'(z)}{g_j(z)} \leq \frac{1 + (1 - 2\xi_j)z}{1 - z} \quad (z \in \Delta).
\]
The subordination principle it follows that
\[
\lambda_j \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right) - \frac{2(1 - \xi_j)\lambda_j r^2}{1 - r^2} \leq \frac{2(1 - \xi_j)|\lambda_j| r}{1 - r^2} \leq \frac{2(1 - \xi)\lambda_j |r|}{1 - r^2} \quad (|z| = r < 1).
\]
In the next theorem, we obtain the radii of convexity of the integral operator (1.5) in special cases and at first we assume that \( f_i \in U_{\alpha_i} \) and \( g_j \in S^*(\xi_j) \).

**Theorem 2.4.** Let \( f_i \) belong to the class \( U_{\alpha_i} \) (1 ≤ i ≤ n), where \( \alpha_i \geq 1 \), \( \alpha := \max\{\alpha_1, \ldots, \alpha_n\} \) and \( g_j \in S^*(\xi_j) \) (1 ≤ j ≤ m) where 0 ≤ \( \xi_j < 1 \) and \( \xi = \max\{\xi_1, \ldots, \xi_m\} \). Also, let \( \sum_{i=1}^{n} |\gamma_i| \leq M \, (M > 0, \gamma_i \in \mathbb{C}) \) and \( \sum_{j=1}^{m} |\lambda_j| \leq N \, (N > 0, \lambda_j \in \mathbb{C}) \). Then the radius of convexity of the integral operator \( J(z) \) defined by (1.5) is
\[
r_c(M, N) := \sqrt{\left[ (\xi - 1)N - \alpha M \right]^2 - 2[(\xi - 1)N - M - 1] - (\xi - 1)N + \alpha M}.
\]

**Proof.** Let \( J(z) \) be defined by (1.5). It is easy to see that \( J(z) \in A \) and
\[
1 + \frac{zJ''(z)}{J'(z)} = 1 + \sum_{i=1}^{n} \gamma_i \frac{zf_i''(z)}{f_i'(z)} + \sum_{j=1}^{m} \lambda_j \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right).
\]
We shall to show that
\[
\min_{|z|=r} \Re \left\{ 1 + \frac{zJ''(z)}{J'(z)} \right\} \geq 0.
\]
From now (2.9), we obtain
\[
\min_{|z|=r} \Re \left\{ 1 + \frac{zJ''(z)}{J'(z)} \right\} \geq 1 + \sum_{i=1}^{n} \min_{|z|=r} \Re \left\{ \frac{zf_i''(z)}{f_i'(z)} \right\} + \sum_{j=1}^{m} \min_{|z|=r} \Re \left\{ \frac{zg_j'(z)}{g_j(z)} - 1 \right\}.
\]
Since \( f_i \in U_{\alpha_i} \), using the proof of Theorem 2.1 we get
\[
1 + \sum_{i=1}^{n} \min_{|z|=r} \Re \left\{ \frac{zf_i''(z)}{f_i'(z)} \right\} \geq \frac{-(2M+1)r^2 - 2\alpha Mr + 1}{1-r^2}.
\]
Also, because \( g_j \in S^*(\xi_j) \) we have
\[
\frac{zg_j'(z)}{g_j(z)} \leq \frac{1 + (1 - 2\xi_j)z}{1 - z} \quad (z \in \Delta).
\]
The subordination principle it follows that
\[
\lambda_j \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right) - \frac{2(1 - \xi_j)\lambda_j r^2}{1 - r^2} \leq \frac{2(1 - \xi_j)|\lambda_j| r}{1 - r^2} \leq \frac{2(1 - \xi)\lambda_j |r|}{1 - r^2} \quad (|z| = r < 1).
\]
Thus
\[
\min_{|z|=r} \Re \left\{ \lambda_j \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right) \right\} \geq \frac{2(1 - \xi) \Re \{ \lambda_j \} r^2 - 2(1 - \xi) |\lambda_j| r}{1 - r^2},
\]
with the equality for the function \( \frac{zg_j'(z)}{g_j(z)} \). We denote by \( \mathcal{G}(N) \), the class of all integral operators of the form (1.5) such that
\[
(2.11) \quad \sum_{j=1}^{m} |\lambda_j| \leq N \quad (N > 0)
\]
and we denote the subclass \( \mathcal{G}(s, \lambda) \) as follows
\[
\mathcal{G}(s, \lambda) := \left\{ J \in \mathcal{G}(N) : \sum_{j=1}^{m} \Re \{ \lambda_j \} = \lambda, \ -s \leq \lambda \leq s, 0 < s < N \right\}.
\]
Therefore \( \mathcal{G}(N) = \bigcup_{s \in (0, N]} \mathcal{G}(s, \lambda) \). Hence we get
\[
(2.12) \quad \min_{|z|=r} \Re \left\{ \lambda_j \left( \frac{zg_j'(z)}{g_j(z)} - 1 \right) \right\} \geq \frac{-2(1 - \xi) N (r + 1) r}{1 - r^2} \quad (|z| = r < 1).
\]
Now from (2.10) and (2.12), we obtain
\[
\min_{|z|=r} \Re \left\{ 1 + \frac{zJ''(z)}{J'(z)} \right\} \geq \frac{-2[(1 - \xi)N + (M + 1)]r^2 - 2(\alpha M + (1 - \xi)N)r + 1}{1 - r^2} =: \varphi(r).
\]
Easily seen that \( \varphi(r) > 0 \) if \( 0 < |z| = r_c(M, N) < 1 \), where
\[
r_c(M, N) = \sqrt{[(\xi - 1)N - \alpha M]^2 - 2[(\xi - 1)N - M - 1] - (\xi - 1)N + \alpha M}
\]
and concluding the proof. \( \square \)

Taking \( \alpha = 1 \) in the Theorem 2.4 we get.

**Corollary 2.2.** Let \( f_i \) be convex functions for \( 1 \leq i \leq n \) and \( g_j \in S^*(\xi_j) \) \( (j = 1, 2, \ldots, m) \), \( \xi = \max\{\xi_1, \ldots, \xi_m\} \). Then the radius of convexity of the integral operator \( J(z) \) is
\[
r_c(M, N) = \sqrt{[(\xi - 1)N - M - 1]^2 - 2[(\xi - 1)N - M - 1] - (\xi - 1)N + M} \quad (M, N > 0).
\]

**Theorem 2.5.** Let \( f_i \) be locally convex univalent functions of order \( \beta_i \), where \( 0 < \beta \leq 1 \) and \( 1 \leq i \leq n \) and \( g_j \in S^*(\xi_j) \) \( (j = 1, 2, \ldots, m) \), \( \xi = \max\{\xi_1, \ldots, \xi_m\} \). Then the integral operators \( J(z) \) is convex in \( |z| < r \), where \( r \) is the positive root
\[
(2.13) \quad -[2(1 - \xi)N + \beta M + 1]r^2 - [2(1 - \xi)N + \beta M]r + 1 = 0.
\]

**Proof.** From (2.7) and (2.12), we get
\[
\min_{|z|=r} \Re \left\{ 1 + \frac{zJ''(z)}{J'(z)} \right\} \geq \frac{-2[(1 - \xi)N + \beta M + 1]r^2 - 2(1 - \xi)N + \beta M) r + 1}{1 - r^2} =: \phi(r).
\]
It is easy too see that \( \phi(r) > 0 \) if the denominator of \( \phi(r) > 0 \). The remain of proof is obvious and we omit the details. Thus the proof is completed. \( \square \)
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