SOME REMARKS ON THE CASSINIAN METRIC

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Abstract. Some sharp inequalities between the triangular ratio metric and the Cassinian metric are proved in the unit ball.

1. Introduction

Geometric function theory makes use of metrics in many ways. In the distortion theory, which is a significant part of function theory, one seeks to estimate the distance of \( f(z) \) and \( f(w) \) for a given analytic function \( f \) of the unit disk \( \mathbb{B}^2 \) in terms of the distance of \( z \) and \( w \) and their position in \( \mathbb{B}^2 \) [B1, KL]. Distances are often measured in terms of hyperbolic or, in the case of multidimensional theory, hyperbolic type metrics, see [GP, HIMPS, K]. Some examples of recurrent metrics are the quasihyperbolic, distance ratio, and Apollonian metrics, see [GP, B2, HIMPS].

In his paper [H], P. Hästö studied a general family of metrics. A particular case is the Cassinian metric defined as follows for a domain \( G \subset \mathbb{R}^n \) and \( x, y \in G \):

\[
c_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|x - z||z - y|}.
\]

The term "Cassinian metric" was introduced by Z. Ibragimov in [I], and the geometry of the Cassinian metric including geodesics, isometries, and completeness was first studied there. Another, similar metric is the triangular ratio metric, which we studied in [CHKV]. It is defined as follows for a domain \( G \subset \mathbb{R}^n \) and \( x, y \in G \):

\[
s_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|x - z| + |z - y|} \in [0, 1].
\]

The triangular ratio metric is also a particular case of the metrics considered in [H].

Very recently, the Cassinian metric and its relation to other metrics, in particular, to the hyperbolic metric, were discussed by Ibragimov, Mohapatra, Sahoo, and Zhang in [IMSZ]. Our goal here is to continue this study. A part of this process is to compare the Cassinian metric...
to several other widely known metrics such as the triangle ratio metric and the distance ratio metric of the unit ball $\mathbb{B}^n$.

Our main result is the following sharp theorem.

**Theorem 1.3.** Suppose that $D$ is a subdomain of $\mathbb{B}^n$. Then for $x, y \in D$ we have

$$2s_D(x, y) \leq c_D(x, y).$$

In the case $D = \mathbb{B}^n$, the constant 2 on the left-hand side is best possible.

2. Preliminary results

In this section we will prove some sharp inequalities between the Cassinian and other metrics.

2.1. **Distance ratio metric.** For a proper open subset $G \subset \mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric $j_G$ is defined as

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}\right).$$

The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [GP] and recently studied in [CHKV, HIMPS, IMSZ, K]. If confusion seems unlikely, then we also write $d(x) = d(x, \partial G)$.

For a domain $G \subset \subset \mathbb{R}^n$ we define the quantity

$$\hat{c}_G(x, y) = \frac{|x - y|}{|x - z||z - y|},$$

where $x, y \in G \subset \subset \mathbb{R}^n$ and

- $z \in \partial G \cap S^{n-1}(x, d(x))$ such that $|z - y|$ is minimal, if $d(x) \leq d(y)$,
- $z \in \partial G \cap S^{n-1}(y, d(y))$ such that $|z - x|$ is minimal, if $d(y) < d(x)$.

Clearly for all domains $G$ and for all points $x, y \in G$ there holds $\hat{c}_G(x, y) \leq c_G(x, y)$.

Our first goal is to prove lower bounds in terms of the distance ratio metric for the Cassinian metric. For this purpose we need the following technical lemma.

**Lemma 2.2.** (1) The function $f(x) = x^{-1}\log(1 + x)$ is decreasing on $(0, \infty)$.

(2) Let $a > 0$. The function

$$g(x) = \frac{\log ax}{a - \frac{1}{x}}$$

is increasing on $(0, \infty)$.

(3) The function

$$h(x) = \frac{\log \frac{1+x}{1-x}}{\frac{1-x}{1+x}}$$

is decreasing on $(0, 1)$. 
(4) Let $x \in (0, 1)$. The function

$$f(b) = \frac{\log \left( 1 + \frac{b}{1-x} \right)}{\log \left( 1 + \frac{b}{(1-x)(b+1-x)} \right)},$$

is increasing on $(0, 2)$.

**Proof.** (1) By [AS 4.1.33], we easily obtain that

$$f'(x) = \frac{x}{1+x} - \frac{\log(1+x)}{x^2} = \frac{f_1(x)}{x^2} < 0.$$  

(2) Now

$$g'(x) = \frac{ax - (1 + \log(ax))}{(1-ax)^2} = \frac{g_1(x)}{(1-ax)^2}$$

and it is easily seen by [AS 4.1.33] that $g_1(x) > 0$ and hence $g'(x) > 0$ for all $x > 0$.

(3) Recall first that $\log(1+x) > \frac{2x}{2+x}$, for $x > 0$. Using this inequality we see that

$$h'(x) = \frac{2x - (1 + x^2) \log \frac{1+x}{1+x}}{2x^2} < 0.$$  

(4) We have

$$f'(b) = \frac{(1-x) \log \left( 1 + \frac{b}{1-x} \right) + (b(x-2) - (x-1)^2)C}{(b(x-2) - (x-1)^2)(1+b-x)C^2} = \frac{A}{B},$$

where

$$C = \log \left( 1 + \frac{b}{b+(x-1)^2-bx} \right).$$

Clearly $B < 0$, therefore it is enough to show that $A < 0$.

$$A'(b) = (x-2) \log \left( 1 + \frac{b}{b+(x-1)^2-bx} \right),$$

and

$$A''(b) = -\frac{(x-2)(x-1)}{(b(x-2) - (x-1)^2)(1+b-x)}$$

that is negative, therefore $A'(b)$ is decreasing and $A'(b) < A'(0) = 0$.

So $A(b)$ also is decreasing and $A(b) < A(0) = 0$. □

**Theorem 2.3.** For all $x, y \in B^n$ we have

$$j_{B^n}(x, y) \leq a \log(1 + c_{B^n}(x, y)),$$

where

$$a = \frac{\log \left( \frac{1+\alpha}{1-\alpha} \right)}{\log \left( \frac{1+2\alpha-\alpha^2}{1-\alpha^2} \right)} \approx 1.3152$$

and $\alpha \in (0, 1)$ is the solution of the equation

$$(1 + t^2) \log \frac{1 + t}{1 - t} + (t^2 - 2t - 1) \log \frac{1 + 2t - t^2}{1 - t^2} = 0.$$
Proof. By the definition of $\hat{c}_{B^n}(x, y)$, it is enough to show that
\[ j_{B^n}(x, y) \leq a \log(1 + \hat{c}_{B^n}(x, y)). \]
Assume $|y| \leq |x|$, and denote $y' = |x| - |x - y|$ then by geometry
\[
\frac{\log \left( 1 + \frac{|x-y|}{1-|x|} \right)}{\log \left( 1 + \frac{|x-y|}{(1-|x|)|y-e_1|} \right)} \leq \frac{\log \left( 1 + \frac{|x-y|}{1-|x|} \right)}{\log \left( 1 + \frac{|x-y|}{1-|x|} \right)} = \frac{\log \left( 1 + \frac{|x-y|}{1-|x|} \right)}{\log \left( 1 + \frac{|x-y|}{(1-|x|)(|x-y|+1-|x|)} \right)}
\]
If we denote $b = |x - y|$, then by 2.2 (4),
\[
f(b) = \frac{\log \left( 1 + \frac{b}{1-|x|} \right)}{\log \left( 1 + \frac{b}{(1-|x|)(b+1-|x|)} \right)}
\]
is increasing. Thus
\[
f(b) \leq f(2|x|) = \frac{\log \left( \frac{1+|x|}{1-|x|} \right)}{\log \left( \frac{1+2|x|-|x|^2}{1-|x|^3} \right)} := m(|x|).
\]
The function $m(t)$ attains its maximum when
\[
(1 + t^2) \log \frac{1+t}{1-t} + (t^2 - 2t - 1) \log \frac{1 + 2t - t^2}{1 - t^2} = 0.
\]
and by numerical computation we see that $m(|x|)$ has its maximal value $m(\alpha) \approx 1.3152 = a$ when $|x| = \alpha \approx 0.6564$.

The next two results refine [IMSZ, Corollary 3.5] and give the sharp constant.

**Theorem 2.4.** For all $x, y \in \mathbb{B}^n$ we have
\[ j_{B^n}(x, y) \leq \hat{c}_{B^n}(x, y). \]
Moreover, the right hand side cannot be replaced with $\lambda \hat{c}_{B^n}(x, y)$ for any $\lambda \in (0, 1)$.

Proof. We denote $G = \mathbb{B}^n$ and may assume $d(x) \leq d(y)$. We first fix $|x|$. Now by writing $t = |x - y|/(1 - |x|) > 0$ we obtain
\[
j_G(x, y) \hat{c}_G(x, y) = \log \left( 1 + \frac{|x-y|}{1-|x|} \right) = \log(1 + t) \frac{|y - x|}{t}.
\]
Next we fix $|y - x|/|x|$ and by Lemma 2.2 (1) and the triangle inequality it is clear that $|x - y| \geq |y - x|/|x| - (1 - |x|)$. We denote $s = |y - x|/|x| \in \mathbb{R}$. 

□
\[
(1 - |x|, 1 + |x|) \text{ and obtain }
\]
\[
\frac{j_G(x,y)}{\hat{c}_G(x,y)} = \frac{\log(1 + t)}{t} s \leq \frac{\log \left(1 + \frac{s-1}{1-|x|}\right)}{\frac{s-1}{1-|x|}} s = \frac{1}{1-|x|} - \frac{1}{s}.
\]

Next we find an upper bound for this expression in terms of \(s\). Since \(s \leq 1 + |x|\) we have by Lemma 2.2 (2)
\[
\frac{j_G(x,y)}{\hat{c}_G(x,y)} \leq \frac{\log s}{1-|x|} - \frac{1}{s} \leq \frac{\log \left(1 + \frac{1+|x|}{1-|x|}\right)}{1-|x|} = 1,
\]
and the assertion follows.

Finally, suppose that \(\lambda \in (0,1)\) and \(j_{\mathbb{B}^n}(x,y) \leq \lambda \hat{c}_{\mathbb{B}^n}(x,y)\) for all \(x,y \in \mathbb{B}^n\). This yields
\[
j_{\mathbb{B}^n}(x,0) = \log \left(1 + \frac{|x|}{1-|x|}\right) \leq \lambda \hat{c}_{\mathbb{B}^n}(x,0) = \lambda \frac{|x|}{1-|x|}.
\]
Letting \(|x| \to 0\) yields a contradiction. \(\square\)

**Corollary 2.5.** For all \(x,y \in \mathbb{B}^n\) we have
\[
j_{\mathbb{B}^n}(x,y) \leq c_{\mathbb{B}^n}(x,y).
\]
Moreover, the right hand side cannot be replaced with \(\lambda c_{\mathbb{B}^n}(x,y)\) for any \(\lambda \in (0,1)\).

### 3. A Formula for Triangular Ratio Metric

It seems to be a challenging problem to give an explicit formula for \(s_{\mathbb{B}^n}(x,y)\) for given \(x,y \in \mathbb{B}^n\). We shall give in this section a simple formula for \(s_{\mathbb{B}^2}(a,b)\) in the case when \(|a| = |b| < 1\).

**Theorem 3.1.** Let \(a = \alpha + i\beta\), \(\alpha, \beta > 0\), be a point in the unit disk. If \(|a - 1/2| > 1/2\), then \(s_{\mathbb{B}^2}(a,\bar{a}) = |a|\) and otherwise
\[
(3.2) \quad s_{\mathbb{B}^2}(a,\bar{a}) = \frac{\beta}{\sqrt{(1-\alpha)^2 + \beta^2}}.
\]

**Proof.** From the definition of the triangular ratio metric it follows that
\[
s_{\mathbb{B}^2}(a,\bar{a}) = \frac{|a - \bar{a}|}{|a - z| + |\bar{a} - z|} = \frac{2\text{Im}(a)}{|a - z| + |\bar{a} - z|}
\]
for some point \(z = u + iv\). In order to find \(z\) we consider the ellipse
\[
E(c) = \{w : |a - w| + |\bar{a} - w| = c\}
\]
and require that 1) $E(c) \subset \overline{B}^2$, 2) $E(c) \cap \partial B^2 \neq \emptyset$ and the $x$-coordinate of the point of contact of $E(c)$ and the unit circle is unique. Both requirements (1) and (2) can be met for a suitable choice of $c$. The major and minor semiaxes of the ellipse are $c/2$ and $\sqrt{(c/2)^2 - \beta^2}$, respectively. The point of contact can be obtained by solving the system

\[
\begin{align*}
x^2 + y^2 &= 1 \\
\frac{(x-\alpha)^2}{(c/2)^2 - \beta^2} + \frac{y^2}{(c/2)^2} &= 1.
\end{align*}
\]

Solving this system yields a quadratic equation for $x$ with the discriminant

\[D = 64(c^2 - 4\beta^2)(\alpha^2c^2 + \beta^2(c^2 - 4)).\]

The uniqueness requirement for $x$ requires that $D = 0$ and hence

\[c = \frac{2\beta}{\sqrt{\alpha^2 + \beta^2}}.
\]

In this case

\[x = \frac{1}{32\beta^2}8\alpha c^2 = \frac{\alpha}{\alpha^2 + \beta^2}.
\]

Consider first the case when $\frac{\alpha}{\alpha^2 + \beta^2} = 1$. These points define the circle $|w - 1/2| = 1/2$ and we have $\frac{\alpha}{\alpha^2 + \beta^2} > 1$ if and only if $|w - 1/2| < 1/2$.

In the case $\frac{\alpha}{\alpha^2 + \beta^2} > 1$ the contact point is $z = (1, 0)$, by symmetry, whereas in the case $\frac{\alpha}{\alpha^2 + \beta^2} < 1$ the point is

\[z = (x, \sqrt{1 - x^2}) = \left(\frac{\alpha}{\alpha^2 + \beta^2}, \frac{\sqrt{(\alpha^2 + \beta^2) - \alpha^2}}{\alpha^2 + \beta^2}\right).
\]

We now compute the focal sum $c$ in both cases

\[
\begin{cases}
    c = \frac{2\beta}{\sqrt{\alpha^2 + \beta^2}} = \frac{2\mathrm{Im}a}{|a|}, & \text{if } |a - 1/2| \geq 1/2, \\
    c = 2|a - (1, 0)| = 2\sqrt{\beta^2 + (1 - \alpha)^2}, & \text{if } |a - 1/2| \leq 1/2.
\end{cases}
\]

Finally we see that

\[s_{\mathbb{B}^2}(a, \bar{a}) = \frac{|a - \bar{a}|}{c} = |a|, \quad \text{if } |a - 1/2| \geq 1/2,
\]

otherwise

\[s_{\mathbb{B}^2}(a, \bar{a}) = \frac{|a - \bar{a}|}{c} = \frac{\beta}{\sqrt{\beta^2 + (1 - \alpha)^2}} = \frac{\mathrm{Im}a}{\sqrt{(1 - \mathrm{Re}a)^2 + (\mathrm{Im}a)^2}}.
\]

**Theorem 3.3.** Let $x, y \in \mathbb{B}^2$ with $|x| = |y|$ and $z \in \partial \mathbb{B}^2$ such that $|y - z| < |x - z|$ and

\[\angle(y, z, 0) = \angle(0, z, x) = \gamma.
\]

Then $\cos \gamma = (|x - z| + |y - z|)/2$ and hence $|y - z| < 1$. Moreover, $0, x, y, z$ are concyclic.
Figure 1. The ellipse with foci $x$ and $y$ internally tangent to the unit circle at $z$. The points $0, x, z, y$ are concyclic by Theorem 3.3.

Proof. By the Law of Cosines

$$|x|^2 = |x - z|^2 + 1 - 2|z| \cos \gamma$$

and

$$|y|^2 = |y - z|^2 + 1 - 2|z| \cos \gamma.$$

Because $|x| = |y|$ these equalities yield

$$\cos \gamma = \frac{|x - z| + |y - z|}{2}.$$  \hfill (3.4)

By Ptolemy’s Theorem $0, x, z, y$ are concyclic if and only if

$$|y - z||x| + |y||x - z| = 1 \cdot |x - y|,$$

which is equivalent to

$$|y - z| + |x - z| = \frac{|x - y|}{|x|}. $$  \hfill (3.5)

By Theorem 3.1 we see that

$$s_{B^2}(x, y) = \frac{|x - y|}{|x - z| + |z - y|} = |x|,$$

which proves (3.5). \hfill \square

Corollary 3.6. Let $D \subset B^2$ be a domain and let $x, -x \in D$. Then

$$s_D(x, -x) \geq |x|.$$  

Proof. It follows from Theorem 3.1 that

$$s_D(x, -x) \geq s_{B^2}(x, -x) = |x|.$$  \hfill \square
Theorem 3.7. Let \( x \in (0, 1), y \in \mathbb{B}^2 \setminus \{0\}, \) \( \text{Im} y \geq 0, \) with \( |y| = |x| \) and denote \( \omega = \angle(x, 0, y). \) Then the supremum in (1.2) is attained at \( z = e^{i\theta} \) for
\[
\theta = \begin{cases} 
\frac{\omega}{2}, & \text{if } \sin \frac{\pi - \omega}{2} \geq |x|, \\
\frac{\omega - \pi}{2} + \arcsin \sin \frac{\pi - \omega}{|x|}, & \text{if } \sin \frac{\pi - \omega}{2} < |x|.
\end{cases}
\]

Proof. By (1.2) and geometry it is clear that the supremum is attained at a point \( z = e^{i\theta} \) with \( \angle(x, 0, z) = \angle(y, 0, z). \) We denote this angle by \( \gamma. \) Since \( \gamma = \angle(x, 0, z) = \angle(y, 0, z) \) and \( |x| = |y| \) we obtain by the Law of Sines
\[
\sin \left( \pi - \theta - \gamma \right) = \frac{|x|}{\sin \gamma} = \frac{1}{\sin \left( \pi - \omega + \theta - \gamma \right)},
\]
which is equivalent to
\[
\sin \left( \pi - \theta - \gamma \right) = \sin \left( \pi - \omega + \theta - \gamma \right).
\]
This has two solutions: \( a + b = \pi, \) where \( a = \pi - \theta - \gamma \) and \( b = \pi - \omega + \theta - \gamma. \) The solution \( a = b \) gives
\[
\theta = \frac{\omega}{2}.
\]
The solution \( a + b = \pi \) gives \( \omega = \pi - 2\gamma. \) In this case by (3.8) we obtain
\[
\frac{1}{\sin \left( \frac{\pi + \omega}{2} - \theta \right)} = \frac{|x|}{\sin \left( \frac{\pi}{2} \right)},
\]
which gives
\[
\theta = \frac{\pi + \omega}{2} - \arcsin \frac{\sin \frac{\pi - \omega}{2}}{|x|}.
\]

We have two solutions (3.9) and (3.11). Next we find out which solution gives the supremum in (1.2). First we note that (3.11) is valid only for \( \sin \frac{\pi - \omega}{2} \leq |x|. \) Thus for \( \sin \frac{\pi - \omega}{2} > |x| \) we choose (3.9). In the case \( \sin \frac{\pi - \omega}{2} = |x| \) both solutions give \( \theta = \frac{\pi}{2}. \) Thus, in the case \( \sin \frac{\pi - \omega}{2} \geq |x|, \) the supremum in (1.2) is attained at \( z = e^{i\omega/2}. \)

Finally, we consider the case \( \sin \frac{\pi - \omega}{2} < |x|. \) Let us denote \( \theta_1 = \frac{\omega}{2}, \)
\[
z_1 = e^{i\theta_1}, \quad \theta_2 = \frac{\pi + \omega}{2} - \arcsin \frac{\sin \frac{\pi - \omega}{2}}{|x|} \quad \text{and} \quad z_2 = e^{i\theta_2}. \]
Moreover let \( \omega_0 = \angle(0, x, z_1), \) \( \omega_1 = \angle(0, x, z_2), \) \( \omega_2 = \angle(0, y, z_2). \) Again by the Law of Sines, we obtain
\[
\frac{|x - z_2|}{\sin \theta_2} = \frac{1}{\sin \omega_1} = \frac{1}{\sin \omega_2} = \frac{|z_2 - y|}{\sin(\omega - \theta_2)} := k_1,
\]
and
\[
\frac{|x - z_1|}{\sin \omega_0} = \frac{1}{\sin \omega_0} := k_2.
\]
By \((3.10)\), we see that
\[ k_1 = \frac{|x|}{\sin \left( \frac{\pi - \omega}{2} \right)}. \]

By \((3.12)\) and \((3.13)\), the inequality \(|x - z_2| + |z_2 - y| < |x - z_1| + |z_1 - y|\) is equivalent to
\[ k_1 (\sin \theta_2 + \sin (\omega - \theta_2)) < 2k_2 \sin \frac{\omega}{2}. \]

By substituting \(k_1\) and \(k_2\), it is enough to show that
\[ \frac{|x|}{\sin \left( \frac{\pi - \omega}{2} \right)} \cos \left( \theta_2 - \frac{\omega}{2} \right) < \frac{1}{\sin \omega_0}, \]
which is, by substitution of \(\theta_2\), equivalent to the inequality
\[ 1 < \frac{1}{\sin \omega_0}. \]

Thus, in the case \(\sin \frac{\pi - \omega}{2} < |x|\), the supremum in \((1.2)\) is attained at \(z_2\). \(\square\)

**Remark 3.14.** By the assumptions of Lemma 3.7, if \(\sin \frac{\pi - \omega}{2} \geq |x|\) we attain
\[ s(\mathbb{B}^2, x, y) = \frac{|x - y|}{|x - e^{i\omega/2}| + |y - e^{i\omega/2}|} = \frac{|x| \sin \frac{\omega}{2}}{\sqrt{1 + |x|^2 - 2 |x| \cos \omega/2}}. \]

This formula is equivalent to \((3.2)\), if \(y = \bar{x}\), and thus by Theorem 3.1 we collect
\[ s(\mathbb{B}^2, x, y) = \begin{cases} |x|, & \cos (\omega/2) < |x|, \\ |x| \sin (\omega/2) & \cos (\omega/2) \geq |x|, \\ \frac{1}{\sqrt{1 + |x|^2 - 2 |x| \cos (\omega/2)}}, & \end{cases} \]
where \(x, y \in \mathbb{B}^2, |y| = |x|\) and \(\omega = \angle(x, 0, y)\).

Note that the following inequalities are equivalent:
\[ |a - \frac{1}{2}| \leq \frac{1}{2} \text{ where } a \text{ is as in Theorem 3.1} \]
\[ \cos \frac{\omega}{2} \geq |x|, \quad \frac{|x-y|}{2} \leq |x|\sqrt{1-|x|^2}. \]

4. **The proof of the main result**

**Proof of Theorem 1.3.** By a simple geometric observation we see that
\[ (4.1) \quad \inf_{w \in \partial \mathbb{B}^n} |x - w| |w - y| \leq 1. \]

In fact, for given \(x, y \in \mathbb{B}^n\), let \(x', y' \in \mathbb{B}^n\) be the points such that \(y' - x' = y - x\) and \(y' = -x'\). Then the size of the maximal Cassinian oval \(C(x, y)\) with foci \(x, y\) which is contained in the closed unit ball is not greater than that of the maximal Cassinian oval \(C(x', y')\) with foci \(x', y'\), see the Figure 2. This implies that
Figure 2. The maximal Cassinian oval $C(x,y)$ is not larger than the maximal Cassinian oval $C(x',y')$.

$$\inf_{w \in \partial B^n} |x - w||w - y| \leq \inf_{w \in \partial B^n} |x' - w||w - y'|$$

$$= 1 - \left( \frac{|x - y|}{2} \right)^2 \leq 1.$$  

Therefore, for $x, y \in D \subset \mathbb{B}^n$, we have that

$$(4.2) \quad \inf_{w \in \partial D} |x - w||w - y| \leq \inf_{w \in \partial B^n} |x - w||w - y| \leq 1.$$  

For $x = y \in D$, the desired inequality is trivial. For $x, y \in D$ with $x \neq y$, it follows from the inequality of arithmetic and geometric means and the inequality (4.2) that

$$\frac{c_D(x,y)}{2s_D(x,y)} = \frac{\inf_{w \in \partial D} (|x - w| + |w - y|)}{2 \inf_{w \in \partial D} (|x - w||w - y|)}$$

$$\geq \frac{\inf_{w \in \partial D} \sqrt{|x - w||w - y|}}{\inf_{w \in \partial D} (|x - w||w - y|)}$$

$$= \sqrt{\frac{\inf_{w \in \partial D} (|x - w||w - y|)}{\inf_{w \in \partial D} (|x - w||w - y|)}}$$

$$\geq 1.$$  

For the sharpness of the constant in the case of the unit ball, let $y = -x \to 0$. It is easy to see that both the inequality of arithmetic and geometric means and the inequality (4.1) will asymptotically become equalities. This completes the proof. □
Corollary 4.3. Let $D \subset \mathbb{R}^n$ be a bounded domain. Then, for $x, y \in D$,

$$c_D(x, y) \geq \frac{2}{\sqrt{n/(2n+2)} \text{diam}(D)} s_D(x, y).$$

Proof. By the well-known Jung’s theorem [Be, Theorem 11.5.8], there exists a ball $B$ with radius $\sqrt{n/(2n+2)} \text{diam}(D)$ which contains the bounded domain $D$. Let $f$ be a similarity which maps the ball $B$ onto the unit ball $\mathbb{B}^n$. Then it is easy to see that for all $x, y \in B$,

$$|f(x) - f(y)| = \frac{|x - y|}{\sqrt{n/(2n+2)} \text{diam}(D)}.$$

By the definitions of the Cassinian metric and the triangle ratio metric, we have that for $x, y \in D$,

(4.4) $c_{fD}(f(x), f(y)) = \sqrt{n/(2n+2)} \text{diam}(D) c_D(x, y)$

and

(4.5) $s_{fD}(f(x), f(y)) = s_D(x, y)$.

Since $fD \subset \mathbb{B}^n$, by Theorem 1.3 we have

(4.6) $c_{fD}(f(x), f(y)) \geq 2s_{fD}(f(x), f(y)).$

Combining (4.4), (4.5), and (4.6), we get the desired inequality. □

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