STANDARD COMPONENTS OF A KRULL-SCHMIDT CATEGORY

SHIPING LIU AND CHARLES PAQUETTE

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Abstract. We provide criteria for an Auslander-Reiten component having sections of a Krull-Schmidt category to be standard. Specializing to the category of finitely presented representations of a strongly locally finite quiver and its bounded derived category, we obtain many new types of standard Auslander-Reiten components. An application to the module category of a finite-dimensional algebra yields some interesting new results.

Introduction

Standard Auslander-Reiten components of the module category of a finite-dimensional algebra are extremely interesting, since the maps between modules in such a component can be described in a simple combinatorial way; see [13]. This kind of component appears mainly for representation-finite algebras, hereditary algebras, tubular algebras and tilted algebras (see [13]), and each of them has at most finitely many non-periodic Auslander-Reiten orbits; see [14]. In particular, the regular ones are stable tubes or of shape $\mathbb{Z}\Delta$ with $\Delta$ a finite acyclic quiver.

On the other hand, the Auslander-Reiten theory has been extended to Krull-Schmidt categories; see [2,11]. It is natural to expect that new types of standard Auslander-Reiten components will appear in this context. Indeed, in the most general setup, we shall find various criteria for such an Auslander-Reiten component having sections to be standard. In particular, an Auslander-Reiten component which is a wing or of shape $\mathbb{N}A_+^\infty$, $\mathbb{N}^{-}\infty$ or $\mathbb{Z}A_\infty$ is standard if and only if its quasi-simple objects are pairwise orthogonal bricks. Specializing to $\text{rep}^+(Q)$, the category of finitely presented representations of a connected strongly locally finite quiver $Q$, we prove that the preprojective component and the preinjective components are standard, and every component is standard in case $Q$ is of finite or infinite Dynkin type. Applying this to the bounded derived category $D^b(\text{rep}^+(Q))$ of $\text{rep}^+(Q)$, we show that the connecting component is standard and that every component is standard in the finite or infinite Dynkin case. These results imply particularly the...
existence of standard Auslander-Reiten components which are wings or of shapes $\mathbb{N}A_\infty^+, \mathbb{N}^+A_{\infty}^-$ and $\mathbb{Z}\Delta$, where $\Delta$ is an arbitrary strongly locally finite quiver without infinite paths. Furthermore, specialized to the module category $\text{mod}\, A$ of a finite-dimensional algebra $A$, our criteria become surprisingly easy to verify; see [3.1]. As a consequence, an Auslander-Reiten component with sections of $\text{mod}\, A$ is standard if and only if it is generalized standard and if and only if it is the connecting component of a tilted factor algebra of $A$. Finally, we point out that some of our results will be applied in the future to study cluster categories of infinite Dynkin types.

1. Standard components having sections

Throughout this paper, $k$ stands for an arbitrary field. A $k$-category is a category in which the morphism sets are $k$-vector spaces and the composition of morphisms is $k$-bilinear. A $k$-category is called $\text{Hom}$-$\text{finite}$ if its morphism spaces are all finite-dimensional over $k$, and $\text{Krull-Schmidt}$ if every non-zero object is a finite direct sum of objects with a local endomorphism algebra.

For the rest of this section, let $\mathcal{C}$ stand for a $\text{Hom}$-$\text{finite}$ Krull-Schmidt additive $k$-category. The $\text{radical}$ morphisms in $\mathcal{C}$ are those in the Jacobson radical $\text{rad}(\mathcal{C})$. One calls $\text{rad}^\infty(\mathcal{C}) = \bigcap_{n \geq 1} \text{rad}^n(\mathcal{C})$ the $\text{infinite}$ radial of $\mathcal{C}$, where $\text{rad}^n(\mathcal{C})$ is the $n$-th power of $\text{rad}(\mathcal{C})$. Two objects $X, Y$ in $\mathcal{C}$ are said to be $\text{orthogonal}$ if $\text{Hom}_\mathcal{C}(X, Y) = 0$ and $\text{Hom}_\mathcal{C}(Y, X) = 0$. If $X \in \mathcal{C}$ is indecomposable, then the division algebra $k_X = \text{End}(X)/\text{rad}(X, X)$ is called the $\text{automorphism field}$ of $X$, and we shall call $X$ a $\text{brick}$ provided that $\text{End}_\mathcal{C}(X)$ is trivial, that is, $\text{End}_\mathcal{C}(X) \cong k$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. One says that $f$ is $\text{irreducible}$ if it is neither a section nor a retraction, and any factorization $f = gh$ implies that $h$ is a section or $g$ is a retraction. Moreover, $f$ is called $\text{left almost split}$ if it is not a section and every non-section morphism $g : X \to M$ in $\mathcal{C}$ factors through $f$, and $\text{left minimal}$ if every endomorphism $h$ of $Y$ such that $f = hf$ is an automorphism. In a dual manner, one defines $f$ to be $\text{right almost split}$ and $\text{right minimal}$. Further, $f$ is called a $\text{source morphism}$ for $X$ if it is left minimal and left almost split, and a $\text{sink morphism}$ for $Y$ if it is right minimal and right almost split. A sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{C}$ with $Y \neq 0$ is called $\text{almost split}$ provided that $f$ is a source morphism and a pseudo-kernel of $g$, while $g$ is a sink morphism and a pseudo-cokernel of $f$; see [11] (1.3)]. In case $\mathcal{C}$ is abelian or triangulated, the definition of an almost split sequence given here coincides somehow with the classical one; see [11] (1.5), (6.1)].

1.1. Lemma. Let $\mathcal{C}$ have an almost split sequence as follows:

$$X \xrightarrow{(f_1, f_2)} Y_1 \oplus Y_2 \xrightarrow{(g_1, g_2)} Z.$$

(1) There exists a $k$-linear isomorphism $k_X \cong k_Y$.
(2) If $u : M \to Y_1$ is a morphism in $\mathcal{C}$ such that $g_1 u = 0$, then there exists some $w : M \to X$ such that $u = f_1 w$ and $f_2 w = 0$.
(3) If $v : Y_1 \to N$ is a morphism in $\mathcal{C}$ such that $v f_1 = 0$, then there exists some $w : Z \to N$ such that $v = w g_1$ and $w g_2 = 0$. 

Proof. Statement (1) is implicitly stated and proved in the proof of [11 (2.1)]. Let \( u : M \to Y_1 \) be such that \( g_1 u = 0 \). Then \( (g_1, g_2)(u_0) = 0 \), and hence there exist some \( w : M \to X \) such that \( (u_0) = (f_1 f_2) w \). This proves Statement (2). Dually, we can show Statement (3). The proof of the lemma is completed. \( \square \)

The Auslander-Reiten quiver \( \Gamma_c \) of \( C \) is first defined to be a valued translation quiver as follows. The vertex set is a complete set of the representatives of the isomorphism classes of the indecomposable objects in \( C \). For vertices \( X \) and \( Y \), we write \( d'_{XY} \) and \( d_{XY} \) for the dimensions of \( \text{irr}(X, Y) = \text{rad}(X, Y)/\text{rad}^2(X, Y) \) over \( k_x \) and \( k_y \), respectively, and draw a unique valued arrow \( X \to Y \) with valuation \( (d_{XY}, d'_{XY}) \) if and only if \( d_{XY} > 0 \). The translation \( \tau \) is defined so that \( \tau Z = X \) if and only if \( C \) has an almost split sequence \( X \to Y \to Z \). A valuation \( (d_{XY}, d'_{XY}) \) is called symmetric if \( d_{XY} = d'_{XY} \) and trivial if \( d_{XY} = d'_{XY} = 1 \). Next, \( \Gamma_c \) is modified in such a way that each symmetrically valued arrow \( X \to Y \) is replaced by \( d_{XY} \) unvalued arrows from \( X \) to \( Y \). That is, \( \Gamma_c \) becomes a partially valued translation quiver in which all valuations are non-symmetric; see [11 (2.1)].

Let \( \Sigma \) be a convex subquiver of \( \Gamma_c \) in which every object has a trivial automorphism field. In particular, \( d_{XY} = d'_{XY} \) for all \( X, Y \in \Sigma \). By our construction, \( \Sigma \) is a non-valued translation quiver with possible multiple arrows. Thus, one can define the path category \( k\Sigma \) and the mesh category \( k(\Sigma) \) of \( \Sigma \) over \( k \); see, for example, [13 (2.1)]. In the sequel, for \( u \in k\Sigma \) we shall write \( \overline{u} \) for its image in \( k(\Sigma) \).

1.2. Definition. Let \( \Sigma \) be a convex subquiver of \( \Gamma_c \), and let \( C(\Sigma) \) be the full subcategory of \( C \) generated by the objects in \( \Sigma \). We shall say that \( \Sigma \) is standard provided that every object in \( \Sigma \) has a trivial automorphism field and there exists a \( k \)-equivalence \( F : k(\Sigma) \xrightarrow{\sim} C(\Sigma) \), which acts identically on the objects.

1.3. Lemma. Let \( \Sigma \) be a convex subquiver of \( \Gamma_c \), and let \( F : k(\Sigma) \xrightarrow{\sim} C(\Sigma) \) be a \( k \)-equivalence acting identically on the objects. If \( X, Y \in \Sigma \), then the classes \( F(\alpha) + \text{rad}^2(X, Y) \) form a \( k \)-basis of \( \text{irr}(X, Y) \), where \( \alpha \) ranges over the set of arrows from \( X \) to \( Y \).

Proof. Let \( X, Y \in \Sigma \). For \( 1 \leq i \leq 2 \), consider the \( k \)-subspace \( I^{(i)}(X, Y) \) of \( k(\Sigma)(X, Y) \) generated by \( \overline{p} \), where \( p \) ranges over the set of paths of length \( \geq i \) from \( X \) to \( Y \). Write \( \Sigma_1(X, Y) \) for the set of arrows from \( X \) to \( Y \). Since the mesh relations are sums of paths of length two, the classes \( \overline{\alpha} + I^{(2)}(X, Y) \), with \( \alpha \in \Sigma_1(X, Y) \), are \( k \)-linearly independent, and hence they form a \( k \)-basis for \( I^{(1)}(X, Y)/I^{(2)}(X, Y) \). Thus, \( I^{(1)}(X, Y)/I^{(2)}(X, Y) \) and \( \text{irr}(X, Y) \) are of the same \( k \)-dimension. Since \( F \) induces a \( k \)-isomorphism \( F : k(\Sigma)(X, Y) \to \text{Hom}_{C}(X, Y) \), it is easy to see that \( F \) induces a \( k \)-epimorphism \( F : I^{(1)}(X, Y) \to \text{rad}(X, Y) \). In particular, \( F \) maps \( I^{(2)}(X, Y) \) into \( \text{rad}^2(X, Y) \). This yields a \( k \)-epimorphism \( F : I^{(1)}(X, Y)/I^{(2)}(X, Y) \to \text{irr}(X, Y) : u + I^{(2)}(X, Y) \to F(u) + \text{rad}^2(X, Y) \), which is necessarily an isomorphism. The proof of the lemma is completed. \( \square \)

Given a quiver \( \Sigma \) with no oriented cycle, one constructs a stable translation quiver \( Z\Sigma \); see, for example, [13 (2.1)]. We denote by \( N\Sigma \) the full translation subquiver of \( Z\Sigma \) generated by the vertices \( (n, x) \) with \( n \geq 0 \) and \( x \in \Sigma \), and by \( N\Sigma \) the one generated by the vertices \( (n, x) \) with \( n \leq 0 \) and \( x \in \Sigma \). Now, let \( \Gamma \) be a connected component of \( \Gamma_c \). A connected full subquiver \( \Delta \) of \( \Gamma \) is
called a section if it is convex in $\Gamma$, contains no oriented cycle, and meets every $\tau$-orbit in $\Gamma$ exactly once. In this case, every object in $\Gamma$ is uniquely written as $\tau^nX$ with $n \in \mathbb{Z}$ and $X \in \Delta$, and there exists a translation-quivver embedding $\Gamma \to \mathbb{Z}\Delta : \tau^nX \mapsto (-n,X)$; see [10] (2.3). We denote by $\Delta^+$ the full subquiver of $\Gamma$ generated by the vertices $\tau^nX$ with $n > 0$ and $X \in \Delta$, and by $\Delta^-$ the one generated by the vertices $\tau^nX$ with $n < 0$ and $X \in \Delta$. The section $\Delta$ is called right-most if $\Delta^+ = \emptyset$ and left-most if $\Delta^- = \emptyset$. Observe that $\Gamma$ has at most one right-most section and at most one left-most section.

In order to state and prove the main result of this section, we need some terminology and notation. Firstly, an infinite path in a quiver is called left infinite if it has no starting point and right infinite if it has no ending point. Secondly, given two (possibly empty) subquivers $\Sigma, \Omega$ of $\Gamma_{c}$, we shall write $\text{Hom}_{\mathcal{C}}(\Sigma, \Omega) = 0$ in case $\text{Hom}_{\mathcal{C}}(X,Y) = 0$ for all possible objects $X \in \Sigma$ and $Y \in \Omega$.

1.4. Theorem. Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt additive $k$-category, and let $\Gamma$ be a connected component of $\Gamma_{c}$ having a section $\Delta$. If $\Delta^+$ has no left infinite path and $\Delta^-$ has no right infinite path, then $\Gamma$ is standard if and only if $\Delta$ is standard such that $\text{Hom}_{\mathcal{C}}(\Delta^+, \Delta \cup \Delta^-) = 0$ and $\text{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$.

Proof. Suppose that $\Delta^+$ has no left infinite path and $\Delta^-$ has no right infinite path. Assume first that $\Gamma$ is standard. In particular, $\Delta$ is standard. Since $\Gamma$ embeds in $\mathbb{Z}\Delta$, we see that $\Gamma$ has no path from $X$ to $Y$ in case $X \in \Delta^+$ and $Y \in \Delta \cup \Delta^-$, or $X \in \Delta$ and $Y \in \Delta^-$. This shows the necessity.

Assume conversely that $\Delta$ is standard such that $\text{Hom}_{\mathcal{C}}(\Delta^+, \Delta \cup \Delta^-) = 0$ and $\text{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$. In particular, every object in $\Delta$ has a trivial endomorphism algebra. Being of the form $\tau^nX$ with $n \in \mathbb{Z}$ and $X \in \Delta$, by Lemma [11], every object in $\Gamma$ has a trivial automorphism field. By definition, every object in $\Delta^+$ is the ending term of an almost split sequence in $\mathcal{C}$. Let $X$ be an object lying in $\Delta^+$. Admitting a sink morphism, $X$ has only finitely many immediate predecessors in $\Gamma$, and in particular, it has at most finitely many immediate predecessors in $\Delta^+$. Moreover, if $Y$ is an immediate successor of $X$ in $\Delta^+$, then $\tau Y$ is an immediate predecessor of $X$ in $\Gamma$. Therefore, $X$ has at most finitely many immediate successors in $\Delta^+$. That is, $\Delta^+$ is locally finite. Further, since $\Delta^+$ has no left infinite path, it follows from Königs’s Lemma that $\Delta^+$ has only finitely many paths ending in any pre-fixed object. Thus, for each object $M \in \Delta \cup \Delta^+$, we may define an integer $n_M \geq 0$ in such a way that $n_{\lambda M} = 0$ if $M \in \Delta$; otherwise, $n_{\lambda M} - 1$ is the maximal length of the paths in $\Delta^+$ ending in $M$. The following statement is evident.

(1) Let $p : X \rightsquigarrow Y$ be a non-trivial path in $\Gamma$. If $X \in \Delta \cup \Delta^+$, then $Y \in \Delta \cup \Delta^+$ with $n_X \leq n_Y$, and the equality occurs if and only if $X, Y \in \Delta$.

For each $n \geq 0$, denote by $\Gamma^n$ the full subquiver of $\Gamma$ generated by the vertices $X \in \Delta \cup \Delta^+$ with $n_X \leq n$, which is clearly convex in $\Gamma$. Moreover, denote by $\Gamma^+$ the union of the $\Gamma^n$ with $n \geq 0$, that is, the full subquiver of $\Gamma$ generated by $\Delta^+ \cup \Delta$. The following statement is an immediate consequence of Statement (1).

(2) If $p : X \rightsquigarrow Y$ is a non-trivial path in $\Gamma^{n+1}$ with $n \geq 0$, then $X \in \Gamma^n$, and consequently, $p \notin \Gamma^n$ if and only if $Y \notin \Gamma^n$.

Now, let $F^0 : k(\Delta) \rightarrow \mathcal{C}(\Delta)$ be a $k$-linear equivalence, acting identically on the objects. Since $\Delta$ contains no mesh of $\Gamma$, we have $k(\Delta) = k\Delta$. Assume that $n \geq 0$ and $F^0$ extends to a full $k$-linear functor $F^n : k\Gamma^n \rightarrow \mathcal{C}(\Gamma^n)$, acting identically.
on the objects and having a kernel generated by the mesh relations. In order to extend $F^n$ to $k\Gamma^{n+1}$, we shall need the following statement.

(3) If $f : X \to Y$ is a non-zero radical morphism in $C(\Gamma^{n+1})$, then $\Gamma^{n+1}$ has a non-trivial path from $X$ to $Y$. Let $f : X \to Y$ be a non-zero radical morphism in $C(\Gamma^{n+1})$. Assume on the contrary that $\Gamma^{n+1}$ has no non-trivial path from $X$ to $Y$. We then claim that $\Delta$ has an object $M$, which is a predecessor of $Y$ in $\Gamma^{n+1}$, such that $\text{Hom}_C(X,M) \neq 0$. Indeed, suppose that this claim was false. In particular, $Y \notin \Delta$, and hence $Y \in \Delta^+$. Since $\Delta$ is a section of $\Gamma$, every immediate predecessor of an object in $\Delta^+$ lies in $\Delta^+ \cup \Delta$. Since every object in $\Delta^+$ admits a sink morphism in $C$, by factorizing the radical morphism $f$ and using our assumption and the non-validity of our claim, we obtain a left infinite path

$$\cdots \longrightarrow Y_i \longrightarrow Y_{i-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y$$

in $\Delta^+$ such that $\text{Hom}_C(X,Y_i) \neq 0$ for all $i > 0$, contrary to the hypothesis on $\Delta^+$. Thus, $\Delta$ has the claimed object $M$. Since $\text{Hom}_C(\Delta^+,\Delta) = 0$, we have $X \in \Delta$. Since $k\Delta \cong C(\Delta)$, there exists a path in $\Delta$ from $X$ to $M$. This yields a non-trivial path in $\Gamma^{n+1}$ from $X$ to $Y$, contrary to our assumption. Statement (3) is established.

Fix $Z \in \Gamma^{n+1}\backslash \Gamma^n$. Observe that $Z \in \Delta^+$ and $\tau Z \in \Delta^+ \cup \Delta$. Thus, $k\Gamma^{n+1}$ has a mesh relation $\delta_Z = \sum_{i=1}^r \beta_i \alpha_i$, where $\alpha_i : \tau Z \to Y_i$, $i = 1, \ldots, r$, are the arrows starting in $\tau Z$ and $\beta_i : Y_i \to Z$, $i = 1, \ldots, r$, are the arrows ending in $Z$. By Statement (2), $\tau Z, Y_1, \ldots, Y_r \in \Gamma^n$. Since $\tau Z$ admits a source morphism in $C$, it follows from Lemma 1.3 that $f = (F^n(\alpha_1), \ldots, F^n(\alpha_r))^T : \tau Z \to Y_1 \oplus \cdots \oplus Y_r$ is a source morphism, which embeds in an almost split sequence

$$\tau Z \xrightarrow{f} Y_1 \oplus \cdots \oplus Y_r \xrightarrow{(g_1, \ldots, g_r)} Z$$

in $C$; see [11 (1.4)]. Set $F^{n+1}(Z) = Z$, $F^{n+1}(\varepsilon_Z) = 1_Z$, where $\varepsilon_Z$ is the trivial path at $Z$, and $F^{n+1}(\beta_i) = g_i$, for $i = 1, \ldots, s$. In view of Statement (2), we have defined $F^{n+1}$ on the vertices, the trivial paths, and the arrows in $\Gamma^{n+1}$. In an evident manner, we may extend $F^n$ to a $k$-functor $F^{n+1} : k\Gamma^{n+1} \to C(\Gamma^{n+1})$, acting identically on the objects.

Let $u : Y \to Z$ be a non-zero radical morphism in $C(\Gamma^{n+1})$. By Statement (3), $\Gamma^{n+1}$ has a non-trivial path from $Y$ to $Z$, and hence $Y \in \Gamma^n$ by Statement (2). If $Z \in \Gamma^n$ then, by the induction hypothesis, $u = F^n(\rho)$ for some morphism $\rho : Y \to Z$ in $k\Gamma^n$. Otherwise, $Z$ is the ending term of an almost split sequence $(\ast)$ as stated above. Then $u = \sum_{i=1}^r g_i u_i$, with morphisms $u_i : Y \to Y_i$ in $C$. Since $Y_i \in \Gamma^n$, there exists $\rho_i : Y \to Y_i$ in $k\Gamma^n$ such that $u_i = F^n(\rho_i)$, for $i = 1, \ldots, r$. This yields $u = F^{n+1}(\sum_{i=1}^r \beta_i \rho_i)$; that is, $F^{n+1}$ is full.

Next we shall show, for $\theta \in k\Gamma^{n+1}$, that $F^{n+1}(\theta) = 0$ if and only if $\theta$ lies in the mesh ideal of $k\Gamma^{n+1}$. In view of the induction hypothesis, we may assume that $\theta$ is non-zero of the form $\theta : Y \to Z$ with $Z \in \Gamma^{n+1}\backslash \Gamma^n$. In particular, $\Gamma^{n+1}$ has a non-trivial path from $Y$ to $Z$. By Statement (2), $Y \in \Gamma^n$. Suppose first that $\theta$ lies in the mesh ideal of $k\Gamma^{n+1}$. For simplicity, we may assume that $\theta = \zeta \delta \sigma$, where $\sigma, \delta, \zeta \in k\Gamma^{n+1}$ with $\delta$ a mesh relation. If $\zeta$ has as a non-zero summand a multiple of a non-trivial path, then $\delta \in k\Gamma^n$ by Statement (2). Hence, $F^{n+1}(\theta) = 0$ by the induction hypothesis. Otherwise, $\delta$ is the mesh relation $\delta_Z$ as stated above, and $\theta = (\sum_{i=1}^r \beta_i \alpha_i) \eta$, where $\eta : Y \to \tau Z$ is a morphism in $k\Gamma^n$. Since $(\ast)$ is an almost split sequence, we obtain $F^{n+1}(\theta) = 0$. 

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Suppose conversely that $F^{n+1} (\theta) = 0$. Consider the mesh relation $\delta_Z$ and the almost split sequence $(\ast)$ as stated above. Then $\theta = \sum_{i=1}^r \beta_i \theta_i$, where $\theta_i : Y \to Y_i$ is in $k\Gamma^n$. Since $\sum_{i=1}^r F^{n+1} (\beta_i) F^n (\theta_i) = F^{n+1} (\theta) = 0$, there exists $v : Y \to \tau Z$ in $\mathcal{C}$ such that $F^n (\theta_i) = F^n (\alpha_i) v$, for $i = 1, \ldots, r$. Since $F^n$ is full, $v = F^n (\eta)$ with $\eta : Y \to \tau Z$ in $k\Gamma^n$. Hence $F^n (\theta_i) = F^n (\alpha_i \eta)$, and by the induction hypothesis, $\theta_i - \alpha_i \eta$ lies in the mesh ideal of $k\Gamma^n$, $i = 1, \ldots, r$. As a consequence, 
\[
\theta = \sum_{i=1}^r \beta_i (\theta_i - \alpha_i \eta) + (\sum_{i=1}^r \beta_i \alpha_i) \eta
\]
lies in the mesh ideal of $k\Gamma^{n+1}$. This shows that $F^{n+1}$ is full and its kernel is generated by the mesh relations. By induction, $F^0$ extends to a full $k$-functor $F^+ : k\Gamma^+ \to \mathcal{C}(\Gamma^+)$, acting identically on the objects and having a kernel generated by the mesh relations.

Finally, for each object $N \in \Gamma$, we may define $m_N \geq 0$ so that $m_N = 0$ if $N \in \Delta^-$; otherwise, $m_N - 1$ is the maximal length of the paths in $\Delta^-$ which start in $N$. For $m \geq 0$, denote by $\Gamma^{(m)}$ the full subquiver of $\Gamma$ generated by the objects $Y$ with $m_Y \leq m$. Then $\Gamma$ is the union of $\Gamma^{(m)}$ with $m \geq 0$. In a dual manner, we may apply the induction on $m$ to show that $F^+$ extends to a full $k$-functor $F : k\Gamma \to \mathcal{C}(\Gamma)$, which acts identically on the objects and has a kernel generated by the mesh relations. The proof of the theorem is completed. $\square$

The following result is useful for verifying the conditions stated in Theorem 1.3.

1.5. **Lemma.** Let $\Gamma$ be a connected component of $\Gamma_c$, containing a section $\Delta$.

(1) If $\Delta$ has no left infinite path, then $\Delta^+$ has no left infinite path.

(2) If $\Delta$ has no right infinite path, then $\Delta^-$ has no right infinite path.

**Proof.** It suffices to prove Statement (1). Suppose that $\Delta^+$ has a left infinite path
\[
\cdots \to \tau^{-n_1} X_i \to \cdots \to \tau^{-n_1} X_1 \to \tau^{-n_0} X_0,
\]
where $X_i \in \Delta$ and $n_i > 0$. Since $\Gamma$ embeds in $\mathbb{Z}\Delta$ (see [10, (2.3)]), we see that $n_i \leq n_{i-1}$ for all $i > 0$. As a consequence, there exists $r \geq 0$ such that $n_i = n_r$ for $i \geq r$. This yields a left infinite path
\[
\cdots \to X_i \to \cdots \to X_r
\]
in $\Delta$. The proof of the lemma is completed. $\square$

We shall say that a sink morphism in $\mathcal{C}$ is **proper** if it either is a monomorphism or fits in an almost slit sequence; dually, a source morphism is **proper** if it either is an epimorphism or fits in an almost slit sequence. Observe that sink or source morphisms in an abelian category are all proper. The following result is a generalization of Lemma 3 stated in [13] (2.3).

1.6. **Theorem.** Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt additive $k$-category. Let $\Gamma$ be a connected component of $\Gamma_c$, and let $\Delta$ be a section of $\Gamma$ in which every object has a trivial automorphism field and admits a proper sink morphism as well as a proper source morphism. If $\Delta$ has no infinite path, then $\Gamma$ is standard if and only if $\operatorname{Hom}_C(\Delta^+, \Delta^-) = 0$.

**Proof.** Suppose that $\Delta$ has no infinite path. By Lemma 1.5 $\Delta^+$ has no left infinite path and $\Delta^-$ has no right infinite path. We shall need the following statement.
Sub-lemma. Let $M \in H$ with $\operatorname{Hom}_C(M, \Delta^-) = 0$, and let $N \in \Delta$. If $C$ has a non-zero radical morphism $f : M \to N$, then $\Delta$ has a non-trivial path $M \rightsquigarrow N$.

Indeed, suppose that $\Delta$ has no non-trivial path from $M$ to $N$. By assumption, $N$ admits a sink morphism $g = (g_1, \ldots, g_r) : N_1 \oplus \cdots \oplus N_r \to N$, where $N_i \in \Delta$. If $f : M \to N$ is non-zero and radical, then $f = \sum_{i=1}^r g_if_i$, with $f_i : M \to N_i$ in $C$. We may assume that $f_1$ is non-zero. Since $\Delta$ is a section, $N_1 \in \Delta \cup \tau \Delta$; see \cite[(2.2)]{10}. Since $\operatorname{Hom}_C(M, \Delta^-) = 0$, we have $N_1 \in \Delta$. Since $\Delta$ has no path from $M$ to $N_1$, we see that $f_1$ is radical. Repeating this process, we see that $\Delta$ contains an infinite path ending in $N$, a contradiction. This proves the sub-lemma.

Now, assume that $\operatorname{Hom}_C(\Delta^+, \Delta^-) = 0$. We deduce from the above sub-lemma that $\operatorname{Hom}_C(\Delta^+, \Delta^-) = 0$. Using the dual statement, we obtain $\operatorname{Hom}_C(\Delta, \Delta^-) = 0$. It remains to construct a $k$-linear equivalence $F : k\Delta \to C(\Delta)$. Since every object in $\Delta$ has a trivial automorphism field, so do the objects in $\Gamma$. Set $F(X) = X$ and $F(\xi_X) = 1_X$ for $X \in \Delta$. Let $X, Y \in \Delta$ with $d = d_{XY} > 0$. If $\alpha_i : X \to Y$, $i = 1, \ldots, d$, are the arrows from $X$ to $Y$, then we choose irreducible morphisms $f\alpha_i : X \to Y$ such that $f\alpha_i + \operatorname{rad}^2(X,Y), \ldots, f\alpha_r + \operatorname{rad}^2(X,Y)$ form a $k$-basis of $\operatorname{irr}(X,Y)$, and set $F(\alpha_i) = f\alpha_i$, $i = 1, \ldots, d$. In an evident manner, we obtain a $k$-linear functor $F : k\Delta \to C(\Delta)$.

We claim that $F$ induces a $k$-isomorphism $F_{XY} : \operatorname{Hom}_{k\Delta}(X,Y) \to \operatorname{Hom}_C(X,Y)$, for any $X, Y \in \Delta$. Since every object in $\Delta$ admits a sink morphism and a source morphism, $\Delta$ is locally finite. Having no infinite path, by König’s Lemma, $\Delta$ has at most finitely many paths from $X$ to $Y$. Define an integer $n_{XY}$ in such a way that $n_{XY} = -1$ if $\Delta$ has no path from $X$ to $Y$; otherwise, $n_{XY}$ is the maximal length of the paths from $X$ to $Y$. If $n_{XY} = -1$, then the claim follows easily from the above statement. If $n_{XY} = 0$, then $\operatorname{Hom}_{k\Delta}(X,Y) = k\xi_Y$. On the other hand, $\operatorname{Hom}_C(X,Y) = k1_X$ by the above sub-lemma, and the claim follows.

Suppose that $n_{XY} > 0$. Let $\beta_i : Z_i \to Y$, $i = 1, \ldots, s$, be the arrows in $\Delta$ ending in $Y$. Then $n_{XZ_i} < n_{XY}$, and $(f_{\beta_1}, \ldots, f_{\beta_s}) : Z_1 \oplus \cdots \oplus Z_s \to Y$ is irreducible; see \cite[(3.4)]{2}. Since $Y$ admits a proper sink morphism, there exists a morphism $u : U \to Y$ such that $v = (f_{\beta_1}, \ldots, f_{\beta_s}, u) : Z_1 \oplus \cdots \oplus Z_s \oplus U \to Y$ is a proper sink morphism. Let $h : X \to Y$ be a morphism in $C$. Being radical, $h$ factors through $v$. Since $\Delta$ is a section, every indecomposable summand of $U$ lies in $\tau \Delta$, and since $\operatorname{Hom}_C(X, \Delta^-) = 0$, we have $h = f_{\beta_1}h_1 + \cdots + f_{\beta_s}h_s$, with morphisms $h_i : X \to Z_i$ in $C$. For each $1 \leq i \leq s$, by the induction hypothesis, $h_i$ is a sum of composites of the chosen irreducible morphisms. Therefore, $h$ is a sum of composites of the chosen irreducible morphisms. Hence, $F_{XY}$ is surjective. Next, let $\rho : X \to Y$ be in $k\Delta$ such that $F(\rho) = 0$. Then $\rho = \beta_1\rho_1 + \cdots + \beta_s\rho_s$, where the $\rho_i : X \to Z_i$ are in $k\Delta$. Set $w = (F(\rho_1), \ldots, F(\rho_s))^T : X \to Z_1 \oplus \cdots \oplus Z_s$. Then $(f_{\beta_1}, \ldots, f_{\beta_s})w = F(\rho) = 0$. If $C$ has an almost split sequence ending in $Y$, then, by Lemma \cite[(11)]{11}, $w$ factors through $\tau Y$, and since $\operatorname{Hom}_C(X, \Delta^-) = 0$, we have $w = 0$. Otherwise, $v$ is a monomorphism, and hence, $w = 0$. That is, in any case, $F(\rho_i) = 0$, and by the inductive hypothesis, $\rho_i = 0$, $i = 1, \ldots, s$. As a consequence, $\rho = 0$. Thus, $F_{XY}$ is injective. This implies that $F$ is an equivalence. By Theorem \cite[4]{1} $\Gamma$ is standard. This establishes the sufficiency, and the necessity is evident. The proof of the theorem is completed.

Let $\Sigma$ be a convex subquiver of $\Gamma$. We shall say that $\Sigma$ is schurian if, for any objects $X, Y$ in $\Sigma$, the $k$-space $\operatorname{Hom}_C(X,Y)$ is of dimension at most one, and it vanishes whenever $Y$ is a not successor of $X$ in $\Sigma$. Moreover, we call $\Sigma$ a wing of
rank \( n \) if it is trivially valued of the following shape:

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{shape.png}}
\end{array}
\]

where the dotted arrows indicate the action of \( \tau \), the objects are pairwise distinct and the number of \( \tau \)-orbits is \( n \); see \cite[(3.3)]{13}. In this case, the object on the top is called the wing vertex, and the objects at the bottom are said to be quasi-simple.

1.7. Lemma. Let \( \mathcal{W} \) be a wing of \( \Gamma_c \). If the quasi-simple objects in \( \mathcal{W} \) are pairwise orthogonal bricks, then \( \mathcal{W} \) is schurian.

Proof. Assume that the quasi-simple objects in \( \mathcal{W} \) are pairwise orthogonal bricks. Let \( n \) be the rank of \( \mathcal{W} \). If \( n = 1 \), then the lemma holds trivially. Suppose that \( n > 1 \) and the lemma holds for wings of rank \( n - 1 \). Write the objects in \( \mathcal{W} \) as \( X_{ij} \) with \( 1 \leq i \leq n \) and \( 1 \leq j \leq i \) so that \( X_{11} \) is the wing vertex, the \( X_{nj} \) with \( 1 \leq j \leq n \) are the quasi-simple objects, and \( \tau X_{ij} = X_{i,j+1} \) for \( 1 < i \leq n \) and \( 1 \leq j < i \). Observe that \( X_{21} \) is the wing vertex of a schurian wing \( \mathcal{W}_1 \), while \( X_{22} \) is the wing vertex of a schurian wing \( \mathcal{W}_2 \). It is evident that we may choose irreducible morphisms \( f_{ij} : X_{ij} \to X_{i+1,j} \) for \( j \leq i < n \) and \( 1 \leq j < n \) and irreducible morphisms \( g_{pq} : X_{pq} \to X_{p-1,q-1} \) for \( q \leq p \leq n \) and \( 2 \leq q \leq n \) such that

\[
\mathcal{E}(X_{nj}) : X_{n,j+1} \xrightarrow{g_{n,j+1}} X_{n-1,j} \xrightarrow{f_{n-1,j}} X_{nj}
\]
is an almost split sequence for \( j = 1, \ldots, n - 1 \), and

\[
\mathcal{E}(X_{ij}) : X_{i,j+1} \xrightarrow{(g_{i,j+1}, f_{i,j+1})} X_{i-1,j} \oplus X_{i+1,j+1} \xrightarrow{(f_{i-1,j}, g_{i+1,j+1})} X_{ij}
\]
is an almost split sequence for \( 1 \leq j < n \) and \( j < i < n \). Next, we shall divide the proof into several sub-lemmas.

(1) \( \text{Hom}_C(X_{n1}, X_{ii}) = 0 \) and \( \text{Hom}_C(X_{i1}, X_{nn}) = 0 \), for \( 1 \leq i \leq n \). Suppose that \( C \) has a non-zero morphism \( f : X_{n1} \to X_{rr} \) for some \( 1 \leq r \leq n \). Assume that \( r \) is maximal. Since \( X_{n1}, X_{nn} \) are orthogonal, we have \( r < n \). Since \( \mathcal{W}_1 \) is schurian, \( f_{rr}, f = 0 \). Applying Lemma\((\ref{14.2})\) to the almost split sequence \( \mathcal{E}(X_{r+1,r}) \), we see that \( f \) factors through \( g_{r+1,r+1} : X_{r+1,r+1} \to X_{rr} \), which contradicts the maximality of \( r \). The first part of the statement is established. In a dual manner, we may prove the second part.

(2) \( \text{Hom}_C(X_{i1}, \mathcal{W}_2) = 0 \) and \( \text{Hom}_C(\mathcal{W}_1, X_{ii}) = 0 \), for \( 1 \leq i \leq n \). Suppose that \( f : X_{s1} \to X \) is a non-zero morphism with \( 1 \leq s \leq n \) and \( X \in \mathcal{W}_2 \), which is necessarily radical. If \( X \neq X_{jj} \) for any \( 2 \leq j \leq n \), then \( X \) admits a sink morphism whose domain is a direct sum of one or two objects in \( \mathcal{W}_2 \). Factorizing \( f \) through this sink morphism, we obtain a non-zero morphism \( g : X_{s1} \to X_{ii} \) with \( 2 \leq i \leq n \). Assume that \( s \) is maximal for this property. By Statement (1), \( s < n \). Since \( \mathcal{W}_2 \) is schurian, \( gg_{s+1,2} = 0 \). Applying Lemma\((\ref{14.3})\) to \( \mathcal{E}(X_{s+1,1}) \), we see that \( g \) factors through \( f_{s1} : X_{s1} \to X_{s+1,1} \), which contradicts the maximality of \( s \). The first part of the statement is established. In a dual fashion, we may establish the second part.
The first part of the statement is established, and the second part follows dually.

(4) If \( f : X_i \to X_{i+1} \) with \( 1 \leq i < n \) is such that \( f_{i+1,i+1} \cdots g_{nn} = 0 \), then \( f = 0 \).
Dually, if \( g : X_{i+1} \to X_i \) is a morphism with \( 1 \leq i < n \) such that \( f_{n-1,i+1} \cdots f_{i1} g = 0 \), then \( g = 0 \).
Suppose that \( f_{i+1,i+1} \cdots g_{nn} = 0 \), but \( f \neq 0 \). Let \( r \) with \( i+1 \leq r \leq n \) be minimal such that \( f_{g_{i+1,i+1} \cdots g_{rr}} = 0 \). Write \( f_{g_{i+1,i+1} \cdots g_{rr}} = g_{rr}g_{r1} \), where \( g : X_{r-1,r-1} \to X_{i+1} \) is a non-zero morphism. Applying Lemma \( \text{(3)} \) to \( \mathcal{E}(X_{r-1,r-1}) \), we see that \( g \) factors through \( f_{r-1,r-1} \), which contradicts Statement (2).
This establishes the first part of the statement.

(5) \( \text{Hom}_C(X_{ii}, X_{ii}) \) and \( \text{Hom}_C(X_{ii}, X_{ii}) \) are one-dimensional, for \( 1 \leq i \leq n \).
It suffices to prove the first part of the statement, since the second part follows dually.
Let \( f : X_{nn} \to X_{11} \) be a morphism. By Statement (3), \( f_{11} f = 0 \).
Applying Lemma \( \text{(2)} \) to \( \mathcal{E}(X_{21}) \), we obtain some \( f_1 : X_{nn} \to X_{22} \) such that \( f = g_{22} f_1 \).
Since \( g_{22} f_1 = 0 \) by Statement (3), we may repeat this process to obtain a morphism \( f_{n-1} : X_{nn} \to X_{nn} \) such that \( f = g_{22} \cdots g_{nn} f_{n-1} \).
Since \( W_2 \) is schurian, \( f_{n-1} = I_{X_{nn}} \) for some \( k \), and hence, \( f = \lambda g_{22} \cdots g_{nn} \).
Since \( g_{22} \cdots g_{nn} \neq 0 \), we see that \( \{ g_{22} \cdots g_{nn} \} \) is a \( k \)-basis for \( \text{Hom}_C(X_{nn}, X_{ii}) \).
Write \( g_{ii} = I_{X_{ii}} \).
If \( g : X_{ii} \to X_{ii} \) is a morphism with \( 1 \leq i < n \), then \( g_{ii} g_{ii} \cdots g_{nn} = \mu g_{22} \cdots g_{nn} = \mu g_{ii} \cdots g_{nn} \), for some \( \mu \).
This yields that \( \{ g_{ii} \cdots g_{ii} \} \) is a \( k \)-basis for \( \text{Hom}_C(X_{ii}, X_{ii}) \).
Being non-zero, \( g_{ii} \cdots g_{ii} \) forms a \( k \)-basis for \( \text{Hom}_C(X_{ii}, X_{ii}) \).

Now, suppose that \( \text{Hom}_C(X,Y) \neq 0 \) for some \( X,Y \in W \).
We claim that \( Y \) is a successor of \( X \) and \( \text{Hom}_C(X,Y) \) is one-dimensional.
If \( X \in W_1 \), then \( Y \in W_1 \) by Statement (2).
Since \( W_1 \) is schurian, our claim follows.
Otherwise, \( X = X_{ss} \) for some \( 1 \leq s < n \).
If \( s = n \), then, by Statement (3), \( Y = X_{ii} \) for some \( 1 \leq i < n \).
Combining Statement (5) and the fact that \( W_2 \) is schurian, we see that \( \text{Hom}_C(X,Y) \) is one-dimensional.
If \( s = 1 \) then, by Statement (2), \( Y = X_{j1} \) for some \( 1 \leq j < n \), and hence, \( \text{Hom}_C(X,Y) \) is one-dimensional by Statement (5).
Finally, suppose that \( 1 < s < n \).
If \( Y \in W_2 \), then our claim follows, since \( W_2 \) is schurian.
Otherwise, by Statement (3), \( Y = X_{t1} \) for some \( 1 \leq t < n \).
If \( t = 1 \), then \( \text{Hom}_C(X,Y) \) is one-dimensional by Statement (5).

It remains to consider the case where \( 1 < t < n \).
Let \( f : X_{ss} \to X_{t1} \) be a non-zero morphism with \( 1 < s,t < n \).
Factorizing \( f \) along the \( \mathcal{E}(X_{j1}) \) with \( 2 \leq j \leq t \), we get \( g : X_{ss} \to X_{t+1,2} \) and \( h : X_{ss} \to X_{ii} \) such that \( f = g_{t+1,2} + f_{t-1,1} \cdots f_{i1} h \).
By Statement (5), \( h = \lambda g_{22} \cdots g_{ss} \) with \( \lambda \in k \).
This yields \( f = g_{t+1,2} u \), where \( u : X_{ss} \to X_{t+1,2} \) is a non-zero morphism.
Since \( W_2 \) is schurian, \( X_{t+1,2} \) is a successor of \( X_{ss} \) and \( \text{Hom}_C(X_{ss}, X_{t+1,2}) \) has a \( k \)-basis \( \{ v \} \).
Therefore, \( f = \mu g_{t+1,2} v \) with \( \mu \in k \).
This shows that \( \{ g_{t+1,2} v \} \) is a \( k \)-basis for \( \text{Hom}_C(X_{ss}, X_{t1}) \).
This establishes our claim.
The proof of the lemma is completed.

Let \( \mathbb{A}^+_{\infty} \) and \( \mathbb{A}^-_{\infty} \) denote the linearly oriented quivers of type \( \mathbb{A}_\infty \) having a unique source and having a unique sink, respectively.
If \( \Gamma \) is a connected component of \( \mathbb{A}^+_\infty \) of shape \( \mathbb{Z} \mathbb{A}_\infty \), \( \mathbb{N} \mathbb{A}^+_{\infty} \) or \( \mathbb{N}^- \mathbb{A}^-_{\infty} \), then the objects in \( \Gamma \) having at most one immediate predecessor and at most one immediate successor are called quasi-simple.
1.8. Theorem. Let $\mathcal{C}$ be a Hom-finite Krull-Schmidt additive $k$-category, and let $\Gamma$ be a connected component of $\Gamma_\mathcal{C}$. If $\Gamma$ is a wing or of shape $ZA_{\infty}$, $NA_\infty^<$ or $N^-A^-_{\infty}$, then it is standard if and only if its quasi-simple objects are pairwise orthogonal bricks.

Proof. We shall need only to prove the sufficiency, since the necessity is trivial. Let $\Gamma$ be a wing or of shape $ZA_{\infty}$, $NA_\infty^<$ or $N^-A^-_{\infty}$ with the quasi-simple objects being pairwise orthogonal bricks. Then any two objects in $\Gamma$ lie in a wing whose quasi-simples are pairwise orthogonal bricks. By Lemma 1.7, $\Gamma$ is schurian. Choose a section $\Delta$ of $\Gamma$ so that $\Delta$ is the right-most section if $\Gamma$ is a wing or of shape $N^-A^-_{\infty}$, $\Delta$ is the left-most section if $\Gamma$ is of shape $NA_\infty^<$, and $\Delta$ is any section with an alternating orientation if $\Gamma$ is of shape $ZA_{\infty}$. Then $\Delta^-$ has no right infinite path and $\Delta^+$ has no left infinite path such that $\text{Hom}_{\mathcal{C}}(\Delta^+, \Delta \cup \Delta^-) = 0$ and $\text{Hom}_{\mathcal{C}}(\Delta, \Delta^-) = 0$.

For each arrow $\alpha : X \to Y$ in $\Delta$, choose an irreducible morphism $f_\alpha : X \to Y$ in $\mathcal{C}$. Since every path in $\Delta$ is sectional, the composite of any chain of the chosen irreducible morphisms is non-zero; see [11, (2.7)]. Therefore, for any $M, N \in \Delta$, $\text{Hom}_{\mathcal{C}}(M, N)$ is one-dimensional if and only if $N$ is a successor of $M$ in $\Delta$, and in this case, the composite of the chain of the chosen irreducible morphisms corresponding to the path from $M$ to $N$ forms a $k$-basis for $\text{Hom}_{\mathcal{C}}(X, Y)$. It is easy to see that $k\Delta \cong \mathcal{C}(\Delta)$. By Theorem 1.4, $\Gamma$ is standard. The proof of the theorem is completed. □

2. Specialization to representation categories of quivers

Throughout this section, we fix a connected quiver $Q = (Q_0, Q_1)$, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows, which is assumed to be strongly locally finite; that is, $Q$ is locally finite such that the number of paths between any two given vertices is finite. A $k$-representation $M$ of $Q$ consists of a family of $k$-spaces $M(x)$ with $x \in Q_0$, and a family of $k$-maps $M(\alpha) : M(x) \to M(y)$ with $\alpha : x \to y \in Q_1$. For such a representation $M$, one defines its support $\text{supp} M$ to be the full subquiver of $Q$ generated by the vertices $x$ for which $M(x) \neq 0$, and one calls $M$ locally finite-dimensional if $\dim_k M(x)$ is finite for all $x \in Q_0$, and finite-dimensional if $\sum_{x \in Q_0} \dim_k M(x)$ is finite. The locally finite-dimensional $k$-representations of $Q$ form a hereditary abelian $k$-category $\text{rep}(Q)$. The subcategory of $\text{rep}(Q)$ of finite-dimensional representations is written as $\text{rep}^b(Q)$.

For each $x \in Q_0$, one constructs an indecomposable projective representation $P_x$ and an indecomposable injective representation $I_x$; see [8] Section 1]. Since $Q$ is strongly locally finite, $P_x$ and $I_x$ lie in $\text{rep}(Q)$. One says that $M \in \text{rep}(Q)$ is finitely presented if $M$ has a minimal projective presentation $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, where $P_1, P_0$ are finite direct sums of some $P_x$ with $x \in Q_0$, and finitely co-presented if $M$ has a minimal injective co-presentation $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$, where $I_0, I_1$ are finite direct sums of some $I_x$ with $x \in Q_0$. Let $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ be the full subcategories of $\text{rep}(Q)$ generated by the finitely presented representations and by the finitely co-presented representations, respectively. Then $\text{rep}^b(Q)$ is the intersection of $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$; see [8] (1.15)]. In particular, $I_x \in \text{rep}^+(Q)$ if and only if $I_x \in \text{rep}^b(Q)$. We shall denote by $Q^+$ the full subquiver of $Q$ generated by the vertices $x$ for which $I_x \in \text{rep}^b(Q)$. 

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It is known that $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ are hereditary, abelian and Hom-finite; see [3, (1.15)]. In particular, they are Krull-Schmidt. The shapes of their Auslander-Reiten components have been well described. Indeed, the Auslander-Reiten quiver $\Gamma_{\text{rep}^+(Q)}$ of $\text{rep}^+(Q)$ has a unique preprojective component, which has a left-most section generated by the $P_x$ with $x \in Q_0$; see [3] (4.6) and [13] (2.4)]. The connected components of $\Gamma_{\text{rep}^+(Q)}$ containing some of the $I_x$ with $x \in Q^+$ are called preinjective and correspond bijectively to the connected components of the quiver $Q^+$. Note that every preinjective component has a unique right-most section generated by its injective representations $I_x$; see [13] (2.4]) and [3] (4.7)]. The other connected components of $\Gamma_{\text{rep}^+(Q)}$ are called regular, and they are wings, stable tubes or of shapes $\mathbb{Z}A_\infty, \mathbb{N}A_\infty^+$ and $\mathbb{N}^-A_\infty^-$; see [3] (4.14], [12] and [13].

The following easy fact is well known in the finite case.

2.1. Lemma. Let $X$ and $Y$ be representations lying in $\Gamma_{\text{rep}^+(Q)}$. If $\tau X$ and $\tau Y$ are defined in $\Gamma_{\text{rep}^+(Q)}$, then $\text{Hom}_{\text{rep}^+(Q)}(X, Y) \cong \text{Hom}_{\text{rep}^+(Q)}(\tau X, \tau Y)$.

Proof. Assume that $\tau X$ and $\tau Y$ are defined in $\Gamma_{\text{rep}^+(Q)}$. In view of the proof stated in [3] (2.8)], we have $\text{Hom}(\tau X, \tau Y) \cong D\text{Ext}^1(Y, \tau X)$. Dually, since $\tau X$ is not injective and finite-dimensional (see [3] (3.6]), $\text{Hom}(X, Y) \cong D\text{Ext}^1(Y, \tau X)$. The proof of the lemma is completed. $\square$

Recall that $Q$ is of infinite Dynkin type if its underlying graph is $A_\infty$, $A_\infty^+$ or $D_\infty$. In this case, a reduced walk is called a string if it contains at most finitely many, but at least one, sinks or sources. To each string $w$, one associates a string representation $M_w$ defined as follows: for $x \in Q_0$, one sets $M_w(x) = k$ if $x$ appears in $w$, and otherwise, $M_w(x) = 0$; and for $\alpha \in Q_1$, one sets $M_w(\alpha) = 1$ if $\alpha$ appears in $w$, and otherwise, $M_w(\alpha) = 0$; see [3] Section 5]. It is easy to see that every string representation has a trivial endomorphism algebra.

2.2. Theorem. Let $Q$ be a connected quiver which is strongly locally finite.

(1) The preprojective component and the preinjective components of $\Gamma_{\text{rep}^+(Q)}$ are standard.

(2) If $Q$ is of finite or infinite Dynkin type, then every connected component of $\Gamma_{\text{rep}^+(Q)}$ is standard.

(3) If $Q$ is infinite but not of infinite Dynkin type, then $\Gamma_{\text{rep}^+(Q)}$ has infinitely many non-standard regular components.

Proof. (1) The preprojective component $\mathcal{P}_Q$ of $\Gamma_{\text{rep}^+(Q)}$ has a unique left-most section $\Delta$ which is generated by $P_x$ with $x \in Q_0$ and isomorphic to $Q^\text{op}$; see [3] (4.6]) and [13] (2.4]). Hence, $\Delta^- = \emptyset$. Moreover, $\Delta^+$ has no left infinite path; see [3] (4.8)]. If $f : X \rightarrow Y$ is a non-zero morphism with $X \in \mathcal{P}_Q$ and $Y \in \Delta$, then $X$ is a predecessor of $Y$ in $\mathcal{P}_Q$ (see [3] (4.9]), and hence, $X \in \Delta$. Therefore, $\text{Hom}_{\text{rep}^+(Q)}(\Delta^+, \Delta) = 0$. Let $\mathcal{P}(Q)$ be the full subcategory of $\text{rep}^+(Q)$ generated by $P_x$ with $x \in Q_0$. For each arrow $\alpha : y \rightarrow x$ in $Q$, denote by $P_\alpha : P_x \rightarrow P_y$ the morphism given by the right multiplication by $\alpha$. It is easy to see that

$$F : kQ^\text{op} \rightarrow \mathcal{P}(Q) : x \mapsto P_x; \alpha^\circ \mapsto P_\alpha$$

is a faithful $k$-functor, which is also full by Proposition 1.3 stated in [3]. Thus, $\Delta$ is standard. By Theorem 1.4, $\mathcal{P}_Q$ is standard. Dually, the preinjective component $\mathcal{I}_Q$ of $\Gamma_{\text{rep}^-(Q)}$ is standard. Now, the preinjective components of $\Gamma_{\text{rep}^+(Q)}$ are
precisely the connected components of the full subquiver of $\mathcal{I}_Q$ generated by the
finite-dimensional representations; see the remark following Theorem 4.7 in [3]. On
the other hand, by the dual of Lemma 4.5(1) stated in [3], the possible infinite-dimensional
representations lying in $\mathcal{I}_Q$ form a left-most section. Therefore, the
possible preinjective components of $\Gamma_{\text{rep}^+ (Q)}$ are convex translation subquivers of
$\mathcal{I}_Q$, and in particular, they are standard.

(2) Suppose that $Q$ is of infinite Dynkin type. Let $\Gamma$ be a regular component of
$\Gamma_{\text{rep}^+ (Q)}$. Then $\Gamma$ is a wing or of shape $\mathbb{Z}A_{\infty}$, $\mathbb{N}A_{\infty}$ or $\mathbb{N}A_{\infty}^+$; see [3] (4.14)].
Moreover, $Q$ is of type $\mathbb{A}_{\infty}^+$ or $D_{\infty}$; see [3] (5.16)]. Assume first that $Q$ is of
type $\mathbb{A}_{\infty}^+$. By Proposition 5.9 stated in [3], the representations in $\Gamma_{\text{rep}^+ (Q)}$ are all
string representations, and hence, they are all bricks. Moreover, the quasi-simple
representations in $\Gamma$ have pairwise disjoint supports; see [3] (5.15)]. In particular,
they are pairwise orthogonal. By Theorem 1.5, $\Gamma$ is standard.

Assume next that $Q$ is of type $D_{\infty}$. Then $\Gamma$ is of shape $\mathbb{Z}A_{\infty}$, $\mathbb{N}A_{\infty}^+$ or $\mathbb{N}A_{\infty}^-$; see [3] (5.22)]. In particular, $\tau$ or $\tau^-$ is defined everywhere in $\Gamma$. We shall consider
only the first case, since the second case can be treated in a dual manner. Let
$a \in Q_0$ be one of the two vertices of degree one, which lies in the support of at most
two quasi-simple representations; see [3] (5.20)]. Thus, there exists a quasi-simple
representation $S \in \Gamma$ such that $(\tau^S)(a) = 0$, for all $n \geq 0$.

Let $M, N$ be quasi-simple representations in $\Gamma$. There exists $m \geq 0$ such that
$\tau^mM = \tau^r S$ and $\tau^mN = \tau^sS$ with $r, s \geq 0$. We may assume that $r \geq s$. By
Lemma 2.1, Hom$(M, N) \cong$ Hom$(\tau^mM, \tau^mN) =$ Hom$(\tau^r S, \tau^sS)$. Since $(\tau^r S)(a) = 0$, we see that $\tau^r S$ is a string representation; see [3] (5.19)]. Thus, $\tau^r S$ is a brick.
Taking $N = M$, we see that $M$ is a brick. Suppose that $M \neq N$. Then $r > s$. Set
t = r − s. Then $\Gamma$ has a sectional path $S_t \rightarrow S_{t−1} \rightarrow \cdots \rightarrow S_1 \rightarrow \tau^s S$. For
$x \in Q_0$, we have dim$S_t(x) = \sum_{i=0}^{t} \dim \tau^i S(x)$. Since $(\tau^i S)(a) = 0$ for $i \geq 0$, we
obtain dim$S_t(a) = 0$. Hence, $S_t$ is a string representation; see [3] (5.19)]. If the
supports of $\tau^r S$ and $\tau^s S$ have a common vertex $b$, then
dim$S_t(b) \geq \dim \tau^r S(b) + \dim \tau^s S(b) \geq 2$,
contrary to $S_t$ being a string representation. Thus, $\tau^r S$ and $\tau^s S$ have disjoint
supports. In particular, they are orthogonal, and so are $M$ and $N$. By Theorem 1.3,
$\Gamma$ is standard. In view of Statement (1), we have established Statement (2).

(3) Suppose that $Q$ is infinite but not of infinite Dynkin type. Then $Q$ has a finite
subquiver $\Sigma$ of Euclidean type. Then we can find a homogeneous tube $T$ in $\Gamma_{\text{rep}^+ (\Sigma)}$
see, for example, [3] (6.3)]. Let $M_i$ with $i \geq 1$ be the representations in $T$ which
are not quasi-simple. Regarded as representations of $Q$, the $M_i$ are distributed in
infinitely many regular components of $\Gamma_{\text{rep}^+ (Q)}$; see [3] (6.1), (6.2)]. These regular
components are not standard, since the $M_i$ have non-trivial endomorphism algebras.
The proof of the theorem is completed.

Remark. (1) In view of Theorem 5.17 stated in [3], we see that wings and the
translation quivers $\mathbb{Z}A_{\infty}$, $\mathbb{N}A_{\infty}^+$ and $\mathbb{N}A_{\infty}^-$ all occur as standard Auslander-Reiten
components of Krull-Schmidt categories.

(2) Let $Q$ be finite of Euclidean type. If $k$ is not algebraically closed, then some
indecomposable $k$-representations of $Q$ have a non-trivial automorphism field; see
the proof in [3] (6.3)]. As a consequence, every connected component of $\Gamma_{\text{rep}^b (Q)}$ is
standard if and only if $k$ is algebraically closed.
We conclude this section with an application to the bounded derived category $D^b(\text{rep}^+(Q))$ of $\text{rep}^+(Q)$. Since $\text{rep}^+(Q)$ is hereditary, the vertices of $\Gamma_{D^b(\text{rep}^+(Q))}$ can be chosen to be the shifts of those in $\Gamma_{\text{rep}^+(Q)}$. If $Q$ is not of finite Dynkin type, then the connected components of $\Gamma_{D^b(\text{rep}^+(Q))}$ are the shifts of the regular components of $\Gamma_{\text{rep}^+(Q)}$ and the shifts of the connecting component, which is obtained by gluing the preprojective component together with the shift by $-1$ of the preinjective components of $\Gamma_{\text{rep}^+(Q)}$; see [3, (4.4)] and [3, (7.10)]. In case $Q$ is of finite Dynkin type, $\Gamma_{D^b(\text{rep}^+(Q))}$ is connected of shape $\mathbb{Z}Q^{\text{op}}$, which is obtained by gluing, for each integer $i$, the shift by $i$ of $\Gamma_{\text{rep}^+(Q)}$ together with its shift by $i+1$; see [3, (4.5)]. In this case, we also call $\Gamma_{D^b(\text{rep}^+(Q))}$ the connecting component.

2.3. Theorem. Let $Q$ be a connected quiver which is strongly locally finite.

(1) The connecting component of $\Gamma_{D^b(\text{rep}^+(Q))}$ is standard.

(2) If $Q$ is of finite or infinite Dynkin type, then every connected component of $\Gamma_{D^b(\text{rep}^+(Q))}$ is standard.

Proof. We denote by $C_Q$ the connecting component of $\Gamma_{D^b(\text{rep}^+(Q))}$. Let $\Delta$ be the full subquiver of $C_Q$ generated by the representations $P_x \in \Gamma_{\text{rep}^+(Q)}$ with $x \in Q_0$, which is isomorphic to $Q^{\text{op}}$. It follows from Lemma 7.8 stated in [3] that $\Delta$ is a section of $C_Q$. Since $\text{rep}^+(Q)$ fully embeds in $D^b(\text{rep}^+(Q))$, by Theorem 2.2 $\Delta$ is standard. Let $M, N \in \text{rep}^+(Q)$. Since $\text{rep}^+(Q)$ is hereditary, $\text{Hom}_{D^b(\text{rep}^+(Q))}(M[m], N[n]) = 0$ for $m > n$; see [7, (3.1)]. Combining this fact with the standardness of the preprojective component of $\Gamma_{\text{rep}^+(Q)}$, we deduce that $\text{Hom}_{D^b(\text{rep}^+(Q))}(\Delta^+, \Delta) = 0$ and $\text{Hom}_{D^b(\text{rep}^+(Q))}(\Delta \cup \Delta^+, \Delta^-) = 0$.

If $Q$ is not of finite Dynkin type, then $\Delta^+$ coincides with the full subquiver of the preprojective component of $\Gamma_{\text{rep}^+(Q)}$ generated by the non-projective representations, while $\Delta^-$ coincides with the shift by $-1$ of the preinjective components of $\Gamma_{\text{rep}^+(Q)}$. Thus, $\Delta^+$ contains no left infinite path and $\Delta^-$ contains no right infinite path; see [3, (4.8)]. This is also the case if $Q$ is of finite Dynkin type; see [15]. Thus $C_Q$ is standard by Theorem 2.2. This establishes Statement (1). Combining this with Theorem 2.2, we obtain Statement (2). The proof of the theorem is completed.

Remark. Let $Q$ have no infinite path. If $Q$ is not of finite Dynkin type, then $\Gamma_{\text{rep}^+(Q)}$ has a unique preinjective component of shape $\mathbb{N}Q^{\text{op}}$ and its proprojective component is of shape $\mathbb{N}^{-1}Q^{\text{op}}$; see [3, (4.7)]. Thus the connecting component of $\Gamma_{D^b(\text{rep}^+(Q))}$ is always of shape $\mathbb{Z}Q^{\text{op}}$.

3. Specialization to module categories of algebras

Throughout this section, assume that $k$ is algebraically closed. Let $A$ stand for a finite-dimensional $k$-algebra and mod$A$ for the category of finite-dimensional left $A$-modules. In this classical situation, we have the following easy criteria for an Auslander-Reiten component with sections to be standard.

3.1. Theorem. Let $A$ be a finite-dimensional algebra over an algebraically closed field, and let $\Gamma$ be a connected component of $\Gamma_{\text{mod}A}$. If $\Delta$ is a section of $\Gamma$, then $\Gamma$ is standard if and only if $\text{Hom}_A(\Delta, \tau \Delta) = 0$, and if and only if $\text{Hom}_A(\tau^{-1} \Delta, \Delta) = 0$. 
Proof. Let $\Delta$ be a section of $\Gamma$. Note that every module in $\Delta$ admits a proper sink map and a proper source map. Moreover, since the base field is algebraically closed, every module in $\Delta$ has a trivial automorphism field.

Suppose that $\text{Hom}_A(\Delta, \tau \Delta) = 0$. Then $\Delta$ is finite; see [12] (2.1). By Lemma 1.4, $\Delta^+\setminus \Delta$ has no left infinite path and $\Delta^-\setminus \Delta$ has no right infinite path. Assume that $\text{Hom}_A(X,Y) \neq 0$ for some $X \in \Delta^+$ and $Y \in \Delta^-$. Since every module in $\Delta^+$ admits a sink epimorphism, we obtain an arrow $X_1 \to X$ in $\Gamma$ such that $\text{Hom}_A(X_1, Y) \neq 0$. Observe that $X_1 \in \Delta \cup \Delta^+$. If $X_1 \in \Delta^+$, then $\Gamma$ has an arrow $X_2 \to X_1$ such that $\text{Hom}_A(X_2, Y) \neq 0$. Since $\Delta^+$ has no left infinite path, there exists a module $M$ in $\Delta$ such that $\text{Hom}_A(M, Y) \neq 0$. Similarly, since $\Delta^-$ has no right infinite path and every module in $\Delta^-$ has a source monomorphism, there exists a module $N$ in $\tau \Delta$ such that $\text{Hom}_A(M, N) \neq 0$, a contradiction. This shows that $\text{Hom}_A(\Delta^+, \Delta^-) = 0$. By Theorem 1.6, $\Gamma$ is standard. If $\text{Hom}_A(\tau^-, \Delta) = 0$, one shows in a dual manner that $\Gamma$ is standard. Conversely, it is evident that $\text{Hom}_A(\Delta, \tau \Delta) = 0$ and $\text{Hom}_A(\tau^-, \Delta) = 0$ if $\Gamma$ is standard. The proof of the theorem is completed. \hfill \Box

Let $\Gamma$ be a connected component of $\Gamma_{\mathfrak{mod}A}$. Recall that $\Gamma$ is generalized standard if $\text{rad}^\infty(\mathfrak{mod}A)$ vanishes in $\Gamma$; see [14]. It is known that $\Gamma$ is generalized standard if it is standard (see [9]), and the converse holds true in case $\Gamma$ has no projective module or no injective module; see [16]. Observing that the conditions on $\Delta$ stated in Theorem 3.1 are trivially verified in case $\Gamma$ is generalized standard, we obtain the following consequence.

3.2. Corollary. Let $\Gamma$ be a connected component of $\Gamma_{\mathfrak{mod}A}$. If $\Gamma$ has a section, then it is standard if and only if it is generalized standard.

The algebra $A$ is called tilted if $A = \text{End}_H(T)$, where $H$ is a finite-dimensional hereditary algebra and $T$ is a tilting $H$-module. In this case, $\mathfrak{mod}A$ contains slices (see [6]), and a connected component of $\Gamma_{\mathfrak{mod}A}$ containing the indecomposable modules of a slice is called a connecting component. It is shown that a connecting component of a tilted algebra is standard; see [1] (5.7).

3.3. Corollary. If $\Gamma$ is a connected component of $\Gamma_{\mathfrak{mod}A}$, then $\Gamma$ is standard with sections if and only if $\Gamma$ is a connecting component of a tilted factor algebra of $A$.

Proof. Let $\Gamma$ be a connected component of $\Gamma_{\mathfrak{mod}A}$. Suppose first that $\Gamma$ is standard with a section $\Delta$. In particular, we have $\text{Hom}_A(\Delta, \tau \Delta) = 0$. If $I$ is the intersection of the annihilators of the modules in $\Gamma$, then $B = A/I$ is a tilted algebra with $\Gamma$ a connecting component of $\Gamma_{\mathfrak{mod}B}$; see [8] (2.2) and also [15].

Suppose next that there exists a tilted algebra $B = A/I$ with $\Gamma$ being a connecting component of $\Gamma_{\mathfrak{mod}B}$. Then $\Gamma$ has a section $\Delta$ generated by the non-isomorphic indecomposable modules of a slice of $\mathfrak{mod}B$. By the defining property of a slice, $\text{Hom}_B(\Delta, \tau \Delta) = 0$. Thus, by Theorem 3.3, $\Gamma$ is a standard component of $\Gamma_{\mathfrak{mod}B}$. Since $\mathfrak{mod}B$ fully embeds in $\mathfrak{mod}A$, we see that $\Gamma$ is a standard component of $\Gamma_{\mathfrak{mod}A}$. The proof of the corollary is completed. \hfill \Box

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