Competitive Algorithms for Online Weighted Bipartite Matching and its Variants

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Abstract

Online bipartite matching has been extensively studied. In the unweighted setting, Karp et al. [19] gave an optimal \( (1 - 1/e) \)-competitive randomized algorithm. In the weighted setting, optimal algorithms have been achieved only under assumptions on the edge weights. For the general case, little was known beyond the trivial \( 1/2 \)-competitive greedy algorithm. Recently, Fahrbach et al. [10] have presented an 0.5086-competitive algorithm (for the problem in a model, namely free-disposal), overcoming the long-standing barrier of \( 1/2 \). Besides, in designing competitive algorithms for the online matching problem and its variants, several techniques have been developed, in particular the primal-dual method. Specifically, Devanur et al. [7] gave a primal-dual framework, unifying previous approaches and Devanur and Jain [6] provided another scheme for a generalization of the online matching problem.

In this paper, we present competitive algorithms for the online weighted bipartite matching in different models; in particular we achieve the optimal \( (1 - 1/e) \) competitive ratio in the free-disposal model and in other model, namely stochastic reward. Our work also unifies the approaches by Devanur et al. [7] and Devanur and Jain [6] by the mean of the primal-dual technique with configuration linear programs.

1 Introduction

Matching is fundamental in combinatorial optimization and operations research with wide applications from students/colleges admission, kidney exchange to ad-auctions. Online bipartite matching, motivated by advertising markets, labor markets, etc., has been intensively studied. Informally, in the online bipartite matching, there are a set of agents (advertisers) given in advance and a set of items (impressions) that are released online one by one. When an item arrives, its edges to the agents are revealed and one needs to assign the item irrevocably to an agent (or assign to no one). For unweighted bipartite graphs, Karp et al. [19] gave an elegant algorithm RANKING which always outputs a matching of size at least \( (1 - 1/e) \) times that of the optimum solution [19, 3]. They also proved that was the best achievable competitive ratio. The competitive ratio of an algorithm is defined as the worst ratio between the objective of the algorithm solution and that of the optimum solution. However, for edge-weighted bipartite graphs, no algorithm is competitive with the objective of maximizing the total (edge-) weight of the output matching (see for example, [10]).

In order to circumvent the issue due to edge weights and also to abstract more appropriately the practical motivations, in particular the advertising, adwords settings, new models have been proposed. In the free-disposal model [11], multiple items can be assigned to an agent but only the maximum weight is acknowledged for that agent. The objective in this model is to maximize the sum of the heaviest edge weighted assigned to each agent. In the additive-budget model (Ad-auctions) [23], each agent has additionally an budget and the revenue received from an agent is
defined to be the minimum between the total edge weight assigned to the agent and the budget of the agent. The objective is to maximize the total revenue received from all agents. In the stochastic reward model [22], each agent has a weight and each item has additionally a successful probability to be matching to an agent. The objective is to maximize the expected number of successfully matched agents, multiplied by their weights. In the concave return model [6], a generalization of the additive-budget model, the revenue of each agent is a concave function on the total edge weight assigned to the agent. The objective is again to maximize the total revenue received from all agents.

The online matching problem has been extensively studied in those models. In the free-disposal model, Fahrbach et al. [10] have recently made a breakthrough by providing a 0.5086-competitive algorithm, breaking the long-standing competitive ratio barrier of 1/2. It revives the hope of potential improvements toward the upper bound of \((1 - 1/e)\) on this problem. In the additive-budget model, \((1 - 1/e)\)-competitive algorithms have been given with additional assumptions: either the edge weights are small compared to the agents’s budgets [23, 4] or the weight of each edge is the same for every agent (agent-independent) [1]. In this model, the existence of an \((1 - 1/e)\)-competitive algorithm has been conjectured but still remains open. In the concave-return model, Devanur and Jain [6] gave an optimal online algorithm where the competitive ratio is characterized by a system of differential equations.

The primal-dual method has been widely used to study the online bipartite matching problem and its generalizations (including the aforementioned models). Devanur et al. [7] provided an elegant online primal-dual framework that unified previous results by showing how they arose from essentially the same dual update function. That unifying technique paves a way for many developments on online matching and its variants. However, there is an exception, another primal-dual scheme by Devanur and Jain [6] in the concave-return model, which is not encompassed by the framework in [7]. It is intriguing to understand the nature of those different schemes, that potentially leads to improved results for the online matching problems.

1.1 Our Contributions

In this paper, we present an unified primal-dual approach based on configuration linear programs and give optimal algorithms with competitive ratio of \((1 - 1/e)\) for online matching problem in the free-disposal and the stochastic reward models, resolving long-standing open questions. Moreover, we provide competitive algorithms for a general model that captures the concave return model as a special case. The variable update schemes are built on previous work, in particular [7, 8]. The key element is the use of the primal-dual method with configuration LPs that allows improvements and the unification. This can be seen as the last piece to complete the picture which have been widely drawn by previous works.

In the following, we define a generalization of the online bipartite matching problem, present our approach and the results.

1.1.1 Model and Approach

General problem. We are given a bipartite graph \(G(L \cup R, E)\) where vertices \(L\) on the left-hand side are given in advance and vertices \(R\) on the right-hand side are released in an online manner. When an online vertex \(j \in R\) arrives, its incident edges are revealed and an algorithm decides to assign vertex \(j\) to an offline neighbour in \(L\) or not to assign \(j\) to any neighbor. The reward function \(c : 2^E \to \mathbb{R}_{\geq 0}\) is given so that if \(M\) is an assignment of online vertices to offline vertices, the reward received from this assignment is \(c(M)\). Note that if \(M\) is a infeasible assignment (an online vertex
In our approach, we directly study the reward function $c$ in which the reward function is “linearized” by using additional constraints (the second constraint). The changing point of view in our approach, compared to the previous ones, is to directly deal with the non-linear objective functions. Consider the additive-budget model with the weight of the heaviest edge counts in the final matching. Hence, the reward function can be expressed as $c(M) = \sum_{i \in L} \min\{W_i, \sum_{j \in M_i} w_{ij}\}$. In the previous approaches, the problem is typically formulated as

$$\max \sum_{i,j} w_{ij} x_{ij} \quad \text{s.t.} \quad \sum_i x_{ij} \leq 1 \quad \forall j, \sum_j w_{ij} x_{ij} \leq W_i \quad \forall i, x_{ij} \geq 0 \quad \forall i, j,$$

in which the reward function is “linearized” by using additional constraints (the second constraint). In our approach, we directly study the reward function $c(M) = \sum_{i \in L} \min\{W_i, \sum_{j \in M_i} w_{ij}\}$. The latter is not linear and it raises issues to current techniques. In order to circumvent this obstacle, we consider the primal-dual approach based on configuration LPs [25]. The configuration LPs have been used by Huang and Zhang [14] for the stochastic-reward matching problem but their approach is different to ours.

First, we formulate an LP for the general problem. Let $x_{ij}$ be a variable indicating whether $j \in R$ is assigned to $i \in L$. Let $z_M$ be a variable indicating whether an assignment $M \subseteq E$ is selected (output assignment). Consider the following formulation and the dual of its relaxation.

$$\max \sum_{M \subseteq E} c(M) z_M \quad \min \sum_{j \in R} \beta_j + \beta$$

$$(\alpha_j) \sum_{i \in L} x_{ij} \leq 1 \quad \forall j \in R$$

$$(\beta) \sum_{M \subseteq E} z_M = 1$$

$$(\gamma_{i,j}) \sum_{M : (i,j) \in M} z_M = x_{ij} \quad \forall i \in L, j \in R$$

$$x_{ij}, z_M \in \{0, 1\} \quad \forall i \in L, S \subseteq R$$

In the formulation, the first constraint ensures that an online vertex $j$ can be assigned to at most one online vertex $i$. The second constraint guarantees that there must be an output assignment.
The third constraint imposes that if an edge \((i, j)\) is chosen then among all assignments containing \((i, j)\), exactly one will be the output assignment.

In the approach, given variables \(x_{ij}\) such that \(\sum_i x_{ij} \leq 1\) for all \(j \in R\), we always maintain \(z_M = \prod_{(i,j) \in M} x_{ij} \prod_{(i,j) \notin M} (1 - x_{ij})\) for all \(M \subseteq E\). By that, the primal constraints \(\sum_M z_M = 1\) and \(\sum_i \sum_{M: (i,j) \in M} z_M \leq 1\) always hold true (see Section 2 for the detail). Let \(x\) be the vector \((x_{ij})_{(i,j) \in E}\). The primal objective can be expressed as \(C(x)\) where \(C : [0,1]^{|E|} \to \mathbb{R}_{\geq 0}\) is the multilinear extension of the reward function \(c\), defined as

\[
C(x) = \sum_{M \subseteq E} c(M) \prod_{(i,j) \in M} x_{ij} \prod_{(i,j) \notin M} (1 - x_{ij}).
\]

Note that \(C(x)\) can be seen as \(\mathbb{E}_M[c(M)]\) where an edge \((i, j)\) is randomly included in \(M\) with probability \(x_{ij}\). This view helps us to derive updating schemes for dual variables and also makes the rounding scheme obvious. Given assignment variables \(x_{ij}\)'s to \(i\), it is sufficient to independently round the variables in order to get the objective value of \(C(x)\).

In the primal-dual method, the dual variables often guide the primal assignment via complementary slackness conditions. In particular, one complementary slackness condition reads that \(x_{ij} > 0\) implies \(\alpha_j = \gamma_{i,j}\). This indicates the allocation of \(j\) to \(\arg \max_{k} \gamma_{i,k}\). That guides the strategy of allocating online arrival vertex to offline vertices which are \(\arg \max\) of some terms (corresponding to \(\gamma_{i,j}\)) in previous algorithms for online matching problems. We also adopt this strategy in our algorithms. However, for simplicity and for a better unification/comparison with previous works, we consider the following formulation, which is more compact but equivalent to the previous one (by combining the first and last primal constraints) and prove bounds using this formulation. (Even though the aforementioned strategy is less clear from the new dual.)

\[
\begin{align*}
\max & \sum_{M \subseteq E} c(M) z_M \\
(\alpha_j) & \sum_{i \in L} \sum_{M: (i,j) \in M} z_M \leq 1 \quad \forall j \in R \\
(\beta) & \sum_{M} z_M = 1 \\
& z_M \in \{0, 1\} \quad \forall M \subseteq E
\end{align*}
\]

\[
\begin{align*}
\min & \sum_{j \in R} \alpha_j + \beta \\
& \beta + \sum_{i} \sum_{j: (i,j) \in M} \alpha_j \geq c(M) \quad \forall M \subseteq E \\
& \alpha_j \geq 0 \quad \forall j \in R
\end{align*}
\]

1.1.2 Results

Building on our approach and the primal-dual schemes of previous works, we provide the following results.

- An optimal \((1 - 1/e)\)-competitive algorithm in the free-disposal model.

- We revisit the problem in the additive-budget model and give an \((e^{-R_{\text{max}}} - 1/e)\)-competitive algorithm where \(R_{\text{max}} = \max_{i,j} w_{ij}/W_i\). This slightly improves the bound of \((1 - R_{\text{max}})\left(1 - \left(1 + R_{\text{max}}\right)^{-1/R_{\text{max}}}\right)\) [4]. More importantly, the algorithm yields the optimal competitive ratio of \((1 - 1/e)\) for the online stochastic-reward matching problem with vanishing probability.
• A $(1 - \kappa)(1 - 1/e)$-competitive algorithm for the general problem where the reward functions are sub-additive\(^1\) and \(\kappa\) is the curvature of those functions (defined in Section 5).

• A competitive fractional algorithm for the general problem with a concavity assumption. We characterize the competitive ratio by system of differential equations. This result recovers the one provided by Devanur and Jain [6] in the concave-return model.

1.2 Related works

There is an extensive literature on online weighted bipartite matching problems. In terms of techniques, significant efforts have been investigated in order to unify different approaches. Specifically, Devanur et al. [7] provided an elegant online primal-dual framework showing how different approaches arise from essentially the same dual update function. That unifying technique paves a way for many current developments on online matching and its variants [8, 15, 10, 16, 14]. In the following, we summarize the most relevant works to ours and refer the readers to the survey of [21].

**Online matching with free-disposal.** Having been introduced by Feldman et al. [11] in the context of display advertising, the problem is widely applied due to its natural economic interpretation [20]. However, it had been a long-standing open question whether there exists an algorithm with competitive ratio strictly larger than the obvious bound of \(1/2\). Recently, in their breakthrough, Fahrbach et al. [10] provided a 0.5086-competitive algorithm, resolving this question. It revives the hope for improvements towards the upper bound of \(1 - 1/e\). However, as mentioned in [10], their approach would not lead to a bound better than \(5/9\) and in order to obtain a bound closer to \(1 - 1/e\), fundamentally new ideas are required.

**Online matching with additive-budget.** Mehta et al. [23] introduced the problem (Adwords problem) and gave an optimal \((1 - 1/e)\) competitive ratio when \(R_{\text{max}} = \max_{i,j} w_{ij}/W_i\) is small. Buchbinder et al. [4] simplified the analysis by a primal-dual analysis. Aggarwal et al. [1] studied another particular case in which for each \(i\), the weights \(w_{ij}\)'s are the same for every \(j\). They obtained the optimal \((1 - 1/e)\) competitive ratio with the generalization of the RANKING algorithm [19]. In this problem (without any assumption), the existence of an \((1 - 1/e)\)-competitive algorithm has been conjectured but still remains open. Huang et al. [16] recently presented a 0.5016-competitive algorithm for this problem.

**Online matching with stochastic reward.** Mehta and Panigrahi [22] initiated the study this problem and gave a 0.567-competitive algorithm for the uniform weights and identical vanishing probabilities. Moreover, they showed that no algorithm, even in the setting of identical vanishing probabilities, has a competitive ratio better than 0.621 < \((1 - 1/e)\) against a natural LP. Recently, Goyal and Udwani [13] gave an \((1 - 1/e)\)-competitive algorithm for the setting where the (vanishing) probabilities \(p_{ij}\) can be decomposed as \(p_{ij} = p_i p_j\) for all \(i, j\) (so this includes the identical vanishing probability setting as a particular case). Independently, Huang and Zhang [14] provided algorithms with competitive ratios of 0.576 and 0.572 in the settings of vanishing equal probabilities and vanishing unequal probabilities, respectively.

**Online matching with concave return.** Devanur and Jain [6] considered a generalization of the Adwords problem in which fractional allocation is allowed and the rewards are arbitrary

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\(^1\)A function \(c : 2^E \to \mathbb{R}^+\) is sub-additive if \(c(M_1 \cup M_2) \leq c(M_1) + c(M_2)\) for all \(M_1, M_2 \subseteq E\)
monotone concave function. They characterized the optimal achievable competitive ratio by system of differential equations and provided matching upper and lower bounds. The primal-dual scheme by Devanur and Jain [6] is not captured by the framework of [7]. It is intriguing to understand the nature of those algorithms and potentially unify them in a principle approach for online matching problems.

2 Preliminaries

We provide some useful facts and notations. We use bold letters, for example $\mathbf{x}, \mathbf{v}$, to denote vectors. Let $\mathcal{M}$ be a set of all feasible sub-assignments of online vertices to offline vertices. Recall that for any sub-assignment $M \in \mathcal{M}$, each online vertex is assigned to at most one offline vertex. Given $M \in \mathcal{M}$, denote $M_i := \{j \in R : (i, j) \in M\}$ (so $M_i \cap M_j = \emptyset$ for any feasible $M \in \mathcal{M}$). In the paper, we consider the following formulation and its dual.

| PRIMAL | DUAL |
|--------|------|
| \[
\begin{align*}
\max \sum_{M} c(M) z_M = \sum_{i \in L} \sum_{M: (i, j) \in M} (\alpha_j) z_M = 1 \\
\sum_{M \in \mathcal{M}} z_M & \leq 1 \quad \forall j \in R \\
\end{align*}
\] | \[
\begin{align*}
\min \sum_{j \in R} \alpha_j + \beta = \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq c(M) \\
\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j & \geq 0 \quad \forall M \in \mathcal{M} \\
\end{align*}
\] |

In order to prove the competitive ratio of an algorithm, we will bound the objective value of the primal (due to the decisions of the algorithm) and that of the dual (by a dual feasible solution).

Given a reward function on the assignments $c : \mathcal{M} \to \mathbb{R}_{\geq 0}$, let $C : [0,1]^{E} \to \mathbb{R}_{\geq 0}$ be the multilinear extension of $c$, defined as

\[
C(\mathbf{x}) = \sum_{M \in \mathcal{M}} c(M) \prod_{(i,j) \in M} x_{ij} \prod_{(i,j) \notin M} (1 - x_{ij}),
\]

where $\mathbf{x} = (x_{ij})_{(i,j) \in E}$. $C(\mathbf{x})$ can be seen as $\mathbb{E}_M[c(M)]$ where an edge $(i,j)$ is randomly included in $M$ with probability $x_{ij}$. The increasing rate of $C(\mathbf{x})$ while varying $x_{ij}$ is

\[
\frac{\partial C(\mathbf{x})}{\partial x_{ij}} = C(\mathbf{x}_{-(i,j)}) \cdot 1 - C(\mathbf{x}_{-(i,j)}) \cdot 0 \\
= \mathbb{E}_{M' \sim \mathbf{x}_{-(i,j)}}[c(M' \cup (i,j)) - c(M')] \\
\]

(1)

where $\mathbf{x}_{-(i,j)}$ denote the vector $\mathbf{x}$ without coordinate $(i,j)$. Here, $M' \sim \mathbf{x}_{-(i,j)}$ is a random assignment in $\mathcal{M}$ in which an edge $(i',j') \neq (i,j)$ is included with probability $x_{i'j'}$.

Given variables $x_{ij}$ such that $\sum_i x_{ij} \leq 1$ for all $j \in R$, in our algorithms we always maintain $z_M = \prod_{(i,j) \in M} x_{ij} \prod_{(i,j) \notin M} (1 - x_{ij})$. The variable $z_M$ can be interpreted as the probability that the assignment $M$ is selected. By that, $\sum_{M \in \mathcal{M}} z_M = 1$. Moreover, for every $j \in R,$

\[
\sum_{i \in L} \sum_{M: (i, j) \in M} z_M = \sum_i \left( x_{ij} \sum_{M': (i, j) \notin M'} z_{M'} \right) \leq \sum_i x_{ij} \leq 1.
\]
Hence, given $x_{ij}$’s satisfying $\sum_i x_{ij} \leq 1$ for all $j \in R$, the variables $z_M$ are feasible.

In the paper, we use functions $g(y) = \frac{e^{y-1}}{1-1/e}$ and $G(y) = \frac{e^{y-1}-1}{1-1/e}$, the primitive integral of $g(y)$. Note that both $g, G$ are increasing, $G(0) = 0$, $G(1) = 1$, and $1 - G(y) + g(y) = \frac{1}{1-1/e}$. Moreover, in the paper, we denote random variables by capital letters (e.g., $A$) and realizations of random assignments by calligraphic letters (e.g., $\mathcal{E}$). Given a realization $\mathcal{E}$, we denote $A | \mathcal{E}$ as the variable $A$ given $\mathcal{E}$.

3 Online Weighted Matching with Free-Disposal

In this section, we consider the online weighted matching problem in the free-disposal model in which the reward function $c(M) = \sum_i \max_{(i,j) \in M} b_{ij}$ for $M \in \mathcal{M}$. For each $i \in L$, define a function $c_i : R \rightarrow \mathbb{R}^+$ such that $c_i(S) = \max_{j \in S} b_{ij}$ for $S \subseteq R$. By this notation, $c(M) = \sum_{i \in L} c_i(M_i)$.

**Algorithm.** Let $X_{ij}$ be a $0-1$ random variable indicating whether $j \in R$ is assigned to vertex $i \in L$, and $x_{ij} = P[X_{ij} = 1]$. Let $Y_i(w)$ be a $0-1$ random variable indicating whether at least one online vertex $j$ such that $w_{ij} \geq w$ is assigned to $i \in L$. In other words, $P[Y_i(w)]$ is the probability that $i$ receives a reward larger than $w$.

Informally, when an online vertex arrives, given the realization $\mathcal{E}$ of the random assignment, we select vertices $i \in L$ that maximize $w_{ij} - \{B_j | \mathcal{E}\}$ among all $i$ such that $w_{ij}$ is larger than the heaviest weight $w_{i',j'}$ currently assigned to $i'$ in $\mathcal{E}$. (Note that the selection of $i$ depends on the realization $\mathcal{E}$ of the rounding.) We continuously increase $x_{ij}$ by $dx$. By doing that, $P[Y_i(w) = 1]$ is increased by $dx$ for $w_{i,\sigma(i)} < w \leq w_{ij}$ (and remain unchanged for other $w$). In the algorithm, we also maintain two random variables $A_j$ and $B_i$ in the algorithm. We increase variables $A_j$ and $B_i$ by rules in Step 9. Intuitively, $A_j, B_i$ represent the dual variables to be defined later. In the end of the while loop when the online vertex $j$ is completely considered, $j$ is assigned to $i$ independently with probability $x_{ij}$.

**Algorithm 1** Algorithm for Edge-Weighted Matching with Free Disposal.

1: All primal and dual variables are initially set to 0.
2: for each arrival of a new vertex $j$ do
3: Let $\mathcal{E}$ be the realization of the random assignment before arrival of $j$.
4: For each $i \in L$, let $\sigma(i)$ be the online vertex with heaviest weight currently assigned to $i$ in $\mathcal{E}$.
5: while $\sum_i x_{ij} < 1$ and $w_{i,\sigma(i)} < w_{ij}$ for some $i \in L$ do
6: for every $i \in L$ in $\arg \max_{i'} \{w_{i',j} - \{B_j | \mathcal{E}\} : w_{i',j} > w_{i',\sigma(i')}\}$ do
7: Increase the probability of assigning $j$ to $i$ by $dx$, i.e., $x_{ij} \leftarrow x_{ij} + dx$.
8: Maintain $Y_i(w) = X_{ij}$ for $w_{i,\sigma(i)} < w \leq w_{ij}$.
9: Increase $B_i$ and $A_j$ by the following rules respectively,

$$d B_i = \left( \int_{w_{i,\sigma(i)}}^{w_{ij}} g(Y_i(w)) dw | \mathcal{E} \right) dx$$

and

$$d A_j = (wi_{ij} - B_i) dx.$$ 

10: end for
11: end while
12: For every $i$, assign $j$ to $i$ with probability $x_{ij}$.
13: end for
Primal/dual variables. Given $x_{ij}$’s, we always maintain primal variables $z_M = \prod_{(i,j) \in M} x_{ij} \prod_{(i,j) \notin M} (1-x_{ij})$ for every $M \subseteq \mathcal{M}$. As argued in Section 2, these primal variables are feasible. Moreover, we define dual variables as $\alpha_j := \mathbb{E}[A_j]$ and $\beta := \sum_{i \in L} \beta_i$ where $\beta_i := \mathbb{E}[B_i]$.

Lemma 1 For any realization $\mathcal{E}$ of the assignment during the execution of the algorithm, the following inequality holds.

$$\mathbb{E}[B_i | \mathcal{E}] \geq \mathbb{E} \left[ \int_0^\infty G(Y_i(w)) dw | \mathcal{E} \right]$$

**Proof** We prove by induction on $R$. Initially, when $R = \emptyset$, both sides are 0 so the identity hold trivially. Assume that the inequality holds for any realization of the random assignment over online vertices released before $j$. Let $\mathcal{E}$ be an arbitrary realization of the random assignment before $j$ arrives. Conditional on $\mathcal{E}$, for every $i \in L$, we have

$$\mathbb{E}[dB_i | \mathcal{E}] = \mathbb{E} \left[ \int_{w_{i,\sigma(i)}}^{w_{ij}} g(Y_i(w)) dw | \mathcal{E} \right] dx = \mathbb{E} \left[ \int_0^\infty g(Y_i(w)) dw | \mathcal{E} \right] = \mathbb{E} \left[ \int_0^\infty dG(Y_i(w)) dw | \mathcal{E} \right].$$

The second equality holds since, conditional on $\mathcal{E}$, during the increase of $x_{ij}$ $Y_i(w) = 1$ for $w \leq w_{i,\sigma(i)}$; so $dY_i(w) = 0$ for $w \leq w_{i,\sigma(i)}$ during this period. (Recall that if $i$ is not selected in Step 6 then the increase of $x_{ij}$ is 0.)

Integrating both sides of the above equality and note that $G(0) = 0$ and $B_i \geq 0$ at any time, we get

$$\mathbb{E}[B_i | \mathcal{E}] \geq \mathbb{E} \left[ \int_0^\infty G(Y_i(w)) dw | \mathcal{E} \right]$$

\qed

Lemma 2 The dual solution is feasible, i.e., $\beta + \sum_i \sum_{j \in M_i} \alpha_j \geq c(M) \forall M \subseteq \mathcal{M}$.

**Proof** Recall that for $M \subseteq \mathcal{M}$, $M_i = \{ j \in R : (i,j) \in M \}$ and $c_i(S) = \max_{j \in S} b_{ij}$ for $S \subseteq R$. By definition of the dual variables and the reward function, we need to prove that for every $M \subseteq \mathcal{M}$

$$\sum_{i \in L} \beta_i + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq \sum_{i \in L} \max_{(i,j) \in M} b_{ij} = \sum_{i \in L} c_i(M_i)$$

Indeed, we will prove a stronger inequality. That is, for any vertex $i \in L$ and subset $S \subseteq R$, it always holds that

$$\beta_i + \sum_{k \in S} \alpha_k \geq c_i(S)$$

This inequality subsequently implies the feasibility of the dual variables.

Fix a vertex $i \in L$ and a subset $S \subseteq R$. We prove the above inequality by induction on $R$. Initially, when $R = \emptyset$, the inequality holds since both sides are 0. Assume that the inequality holds before the arrival of vertex $j$.

**Case 1:** $j \notin S$ or $j \notin \text{arg max}_{k \in S} w_{ik}$. Then $c_i(S) = c_i(S \setminus \{j\})$. As the inequality holds before the arrival of $j$, i.e., $\beta_i + \sum_{k \in S \setminus \{j\}} \alpha_k \geq c_i(S \setminus \{j\})$, and $\beta_i$ is non-decreasing and $\alpha_j \geq 0$, the inequality after the arrival of $j$ also holds, $\beta_i + \sum_{k \in S} \alpha_k \geq c_i(S) = c_i(S \setminus \{j\})$.

**Case 2:** $j \in S$ and $w_{ij} \geq w_{ik}$ $\forall j \neq k \in S$ and $\beta_i \geq w_{ij}$. The constraint immediately follows again by the non-negativity of $\alpha_k$ and $c_i(S) = w_{ij}$.
Case 3: $j \in S$ and $w_{ij} \geq w_{ik}$ $\forall k \neq i \in S$ and $w_{ij} > \beta_i$. We will prove a stronger statement: $\beta_i + \alpha_j \geq w_{ij}$. This will imply $\beta_i + \sum_{k \in S} \alpha_k \geq \beta_i + \alpha_j \geq w_{ij} = c_i(S)$.

Let $\mathcal{F}$ be the event that $w_{i,\sigma(i)} < w_{ij}$. In this event, the while loop must have ended with $\sum_i x_{ij} = 1$. Throughout the loop (by the condition of the for loop), $dA_j/dx$ is always at least $(w_{ij} - B_i)$. Therefore,

$$\left(A_j \mid \mathcal{F}\right) = \int_0^1 \frac{d(A_j \mid \mathcal{F})}{dx} dt \geq \int_0^1 (w_{ij} - (B_i \mid \mathcal{F})) dt = w_{ij} - (B_i \mid \mathcal{F}).$$

Hence, $(A_j + B_i \mid \mathcal{F}) \geq w_{ij}$.

In case of negated event to $\mathcal{F}$, by Lemma 1, we have

$$\mathbb{E}\left[B_i \mid \neg \mathcal{F}\right] \geq \mathbb{E}\left[\int_0^\infty G(Y_i(w))dw \mid \neg \mathcal{F}\right] \geq \mathbb{E}\left[\int_0^{w_{i,\sigma(i)}} G(Y_i(w))dw\right] = \int_0^{w_{i,\sigma(i)}} G(1)dw = w_{i,\sigma(i)} \geq w_{ij}$$

Combining both cases, we deduce $\beta_i + \alpha_j = \mathbb{E}[A_j + B_i] \geq w_{ij}$. \hfill \Box

**Theorem 1** The randomized Algorithm 1 is $(1 - 1/e)$-competitive.

**Proof** We prove that for every online vertex $j$, the increase of the primal is at least $(1 - 1/e)$ that of the dual. Fix an arbitrary realization $\mathcal{E}$ before the arrival of vertex $j$. When $x_{ij}$ increases, by Equation (1), the increasing rate of the primal is

$$\mathbb{E}_{M' \sim x_{-i,j}} \left[c(M' \cup (i, j)) - c(M') \mid \mathcal{E}\right] = w_{ij} - w_{i,\sigma(i)}$$

where $w_{i,\sigma(i)}$ is the heaviest weight assigned to $i$ in $\mathcal{E}$.

Besides, given a realization $\mathcal{E}$, $\int_0^{w_{i,\sigma(i)}} G(Y_i(w))dw = \int_0^{w_{i,\sigma(i)}} G(1)dw = \int_0^{w_{i,\sigma(i)}} 1dw$. Therefore, conditional on the realization $\mathcal{E}$,

$$\mathbb{E}\left[\frac{dA_j}{dx} \mid \mathcal{E}\right] = w_{ij} - \mathbb{E}\left[B_i \mid \mathcal{E}\right] \leq w_{ij} - \mathbb{E}\left[\int_0^\infty G(Y_i(w))dw \mid \mathcal{E}\right] = \int_0^{w_{ij}} 1dw - \mathbb{E}\left[\int_0^{w_{i,\sigma(i)}} G(1)dw\right] - \mathbb{E}\left[\int_{w_{i,\sigma(i)}}^{\infty} G(Y_i(w))dw \mid \mathcal{E}\right]$$

$$\leq \mathbb{E}\left[\int_{w_{i,\sigma(i)}}^{w_{ij}} (1 - G(Y_i(w)))dw - \int_{w_{ij}}^{\infty} G(Y_i(w))dw \mid \mathcal{E}\right]$$

where the first inequality is due to Lemma 1.

We deduce that, conditional on the realization $\mathcal{E}$,

$$\mathbb{E}\left[\frac{dA_j}{dx} + \frac{dB_i}{dx} \mid \mathcal{E}\right] \leq \mathbb{E}\left[\int_{w_{i,\sigma(i)}}^{w_{ij}} (1 - G(Y_i(w)) + g(Y_i(w)))dw \mid \mathcal{E}\right] = \int_{w_{i,\sigma(i)}}^{w_{ij}} \frac{dw}{1 - 1/e} = \frac{w_{ij} - w_{i,\sigma(i)}}{1 - 1/e}$$

Note that whenever $x_{ij}$ increases, only $\beta_i$ increases whereas $\beta_{i'}$‘s for $i' \neq i$ remain unchanged. Hence, the increase of the primal is at least $(1 - 1/e)$ that of the dual for any realization $\mathcal{E}$. As it holds for any realization, it holds in expectation and the theorem follows. \hfill \Box
Remark 1 Algorithm 1 has the structure similar to the algorithm for the fractional version of the online matching with free-disposal [8, Algorithm 2]. However, there is a small but crucial difference. In [8, Algorithm 2], the update rate of $\beta$-variable is $\int_{w_{i,j'}} g(y_i(w))dw$ where $w_{i,j'}$ is the smallest edge-weight among the weights already assigned to $i$ with positive probability. This update is based on the complementary cumulative distribution function viewpoint. Our update in Step 9 (on the random variable counterparts) is guided by the configuration LP approach. Specifically, we update $B_i$ at the rate of $\int_{w_{i,\sigma(i)}} g(Y_i(w))dw$ where $w_{i,\sigma(i)}$ is the heaviest edge-weight among the weights already assigned to $i$ (in the current realization).

Remark 2 As a sanity check, we consider Algorithm 1 in the online unweighted matching problem, especially applying to the graph corresponding to the upper triangular matrix — it is the worst example for the KVV algorithm of Karp et al. [19]. In this setting, the graph has vertex sets $(L = R = \{1, \ldots, n\})$. One samples a uniformly random permutation $\pi$ of the set $[n]$ and define the edge set of the graph to be $E = \{(\pi(i), j) : i \geq j\}$. Note that $\pi$ is unknown to the algorithm. The online vertices in $R$ arrive in the order $j = 1, 2, \ldots, n$. The optimal solution is the perfect matching consisting of the edges $(\pi(j), j)$ for $1 \leq j \leq n$. On this input, at the arrival of an online vertex $j$, Algorithm 1 increases $x_{ij}$ uniformly for all unmatched offline neighbors of $j$. This results in these offline nodes’ dual variables increasing uniformly, resulting in the next allocation of the next iteration again being uniform among the unmatched neighbors, and so on. The integral (random) allocation then matches each online node $j$ to each of its free neighbors uniformly random. Implementing Algorithm 1 and comparing to KVV algorithm on this example, we observe in [9] that they both have the theoretically predicted competitive ratio $(1 - 1/e \approx 0.632)$.

4 Online Weighted Matching with Additive Budgets

In this section, we consider first the online weighted matching problem in the additive-budget model in which for $M \in \mathcal{M}$, the reward function $c(M) = \sum_{i \in L} c_i(M_i)$ where $c_i(S) = \min\{\sum_{j \in S} w_{ij}, W_i\}$ for $S \subseteq R$, for all $i$. Subsequently, we deduce the performance guarantee for the problem in the stochastic-reward model as a corollary.

Algorithm. Recall $X_{ij}$ is the 0-1 random variable indicating whether $j$ is assigned to $i$. In the algorithm, we maintain a random variable $Y_i$ intuitively (but not exactly) representing the fraction of the consumed budget of $i$. Moreover, as the previous algorithm, we also maintain additional random variables $A_j$ and $B_i$ that help the algorithm’s decisions and the definitions of dual variables. Informally, when an online vertex $j$ arrives, given the current assignment $E$, we continuously increase $x_{ij}$ by $dx$ if $i$ maximizes the term $\min\{w_{i',j}, \max\{0, W_i - \sum_{j' \in \E} w_{i',j'}\}\} \cdot \frac{(B_j|\E)}{w_{i',j'}}$ among all $i'$. Here, the coefficient of $\min\{w_{i,j}, \max\{0, W_i - \sum_{j' \in \E} w_{i,j'}\}\}$, as we will argue later, is the rate of the primal when $x_{ij}$ varies. (The choice of this coefficient shows the usefulness of the configuration LP approach.) Intuitively, given an assignment $E$ before $j$ is released, $\min\{w_{i,j}, \max\{0, W_i - \sum_{j' \in \E} w_{i,j'}\}\}$ is the increase of the total reward if $j$ is (integrally) assigned to $i$. Subsequently, we increase variables $A_j$ and $B_i$ by rules in Step 7.

Primal/dual variables. As in the previous section, given $x_{ij}$’s, we define primal variables $z_{i,S} = \prod_{j \in S} x_{ij} \prod_{j \notin S} (1 - x_{ij})$ for every $i \in L$, $S \subseteq R$ and dual variables $\alpha_j = \mathbb{E}[A_j]$, $\beta_i = \mathbb{E}[B_i]$ and $\delta = \sum_{i \in L} \beta_i$. 


Consequently, it implies that for any feasible up to a factor of $F$ or every vertex $i$, $\beta_i + \sum_{j \in S} \alpha_j \geq G(1 - R_{\text{max}}) \cdot c_i(S)$. Consequently, it implies that for any $M \in \mathcal{M}$, $\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq G(1 - R_{\text{max}}) \cdot c(M)$. 

**Algorithm 2** Algorithm for Edge-Weighted Matching with Additive Budgets.

1: Initially, all primal/dual variables and the corresponding probabilities of random variables $X_{ij}, Y_i, A_j, B_i$ are set to 0.
2: for each arrival of a new vertex $j$ do
3: Let $\mathcal{E}$ be the realization of the random assignment before arrival of $j$.
4: while $\sum_i x_{ij} < 1$ and $(Y_i \mid \mathcal{E}) < 1$ for some $i \in L$ do
5: for every $i \in L$ in $\arg \max \{\min \{w_{ij}, \max \{0, W_i - \sum_{j' : (i,j') \in \mathcal{E}} w_{ij'}\}\} \cdot (1 - (B_i \mid \mathcal{E})^{-1})\}$
6: Increase the probability of assigning $j$ to $i$ by $dx$, i.e., $x_{ij} \leftarrow x_{ij} + dx$.
7: Update $Y_i$, $B_i$ and $A_j$ by the following rules respectively,

$$Y_i = (Y_i \mid \mathcal{E}) + \min \{w_{ij}, W_i - \sum_{j' : (i,j') \in \mathcal{E}} w_{ij'}\} \cdot \frac{1}{W_i} \cdot X_{ij},$$
$$dB_i = \min \{w_{ij}, \max \{0, W_i - \sum_{j' : (i,j') \in \mathcal{E}} w_{ij'}\}\} \cdot g(Y_i)dx,$$
$$dA_j = \min \{w_{ij}, \max \{0, W_i - \sum_{j' : (i,j') \in \mathcal{E}} w_{ij'}\}\} \cdot \left(1 - \frac{B_i}{W_i}\right)dx.$$
8: end for
9: end while
10: For every $i$, assign $j$ to $i$ with probability $x_{ij}$.
11: end for

**Lemma 3** For any realization $\mathcal{E}$ of the random assignment and for every $i \in L$, the following invariant always holds

$$\mathbb{E}[B_i \mid \mathcal{E}] = \mathbb{E}[W_i \cdot G(Y_i) \mid \mathcal{E}].$$

**Proof** Fix an arbitrary vertex $i \in L$. Again, we prove by induction on $R$. For the base case where $R = 0$, both side are 0 so the invariant holds trivially. Assume that the invariant holds for any realization of the assignment before the arrival of vertex $j$. Let $\mathcal{E}$ be a realization of the random assignment before the arrival of $j$. During the consideration of $j$, it holds that

$$\mathbb{E}[dB_i \mid \mathcal{E}] = \mathbb{E}[\min \{w_{ij}, \max \{0, W_i - \sum_{j' : (i,j') \in \mathcal{E}} w_{ij'}\}\} \cdot g(Y_i)dx \mid \mathcal{E}] = \mathbb{E}[W_i g(Y_i) dY_i \mid \mathcal{E}]$$

where the last equality is due to the definition of $Y_i$. Integrating both sides, the lemma follows. □

Recall that $R_{\text{max}} = \max_{i,j} w_{i,j}/W_i$. The following lemma shows that the dual constraints are feasible up to a factor of $G(1 - R_{\text{max}})$.

**Lemma 4** For every vertex $i \in L$ and subset $S \subseteq R$, it holds that

$$\beta_i + \sum_{j \in S} \alpha_j \geq G(1 - R_{\text{max}}) \cdot c_i(S).$$

Consequently, it implies that for any $M \in \mathcal{M}$, $\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq G(1 - R_{\text{max}}) \cdot c(M)$.

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**Proof** Fix a vertex \( i \in L \) and a subset \( S \subseteq R \). We prove that, given an arbitrary realization \( \mathcal{E} \) of the random assignment,

\[
\mathbb{E}[B_i + \sum_{j \in S} A_j | \mathcal{E}] \geq G(1 - R_{\text{max}}) \cdot c_i(S) \tag{2}
\]

If \((Y_i | \mathcal{E}) = 1\) then \((B_i | \mathcal{E}) = W_i\), so the lemma inequality holds. In the following, assume that \((Y_i | \mathcal{E}) < 1\) after the arrival of the last vertex in \( S \). By the while loop condition, \( \sum_j x_{ij} = 1 \) for every \( j \in S \). For every vertex \( j \in S \), let \( E_j \) be the assignment before the arrival of \( j \). (Note that \( E_j \subseteq E \).) By the condition of the for loop, \( dA_j/dx \) must be at least \( \min \{ w_{ij}, \max \{0, W_i - \sum_{j':(i,j') \in E_j} w_{ij'}\} \} \cdot (1 - \frac{(B_i | E_j)}{W_i}) \). Therefore, given \( \mathcal{E} \) (in particular \( E_j \)),

\[
A_j = \int_0^1 \frac{dA_j}{dx} dt \geq \int_0^1 \min \{ w_{ij}, \max \{0, W_i - \sum_{j':(i,j') \in E_j} w_{ij'}\} \} \cdot (1 - \frac{(B_i | E_j)}{W_i}) dt
\]

\[
= \min \{ w_{ij}, \max \{0, W_i - \sum_{j':(i,j') \in E_j} w_{ij'}\} \} \cdot (1 - \frac{(B_i | E_j)}{W_i})
\]

\[
\geq \min \{ w_{ij}, \max \{0, W_i - \sum_{j':(i,j') \in E_j} w_{ij'}\} \} \cdot (1 - \frac{(B_i | \mathcal{E})}{W_i})
\]

where the last inequality holds since the value of \( B_i \) is non-decreasing as long as online vertices arrive.

Assume that there exists \( k \in S \) such that \( \min \{ w_{ik}, \max \{0, W_i - \sum_{j':(i,j') \in E_k} w_{ij'}\} \} = W_i - \sum_{j':(i,j') \in E_k} w_{ij'} \), meaning that \( (Y_i | E_k) \geq 1 - \frac{w_{ik}}{W_i} \). In this case,

\[
\mathbb{E}[B_i + \sum_{j \in S} A_j | \mathcal{E}] \geq \mathbb{E}[B_i | E_k] \geq \mathbb{E}[W_i G(Y_i) | E_k]
\]

\[
\geq W_i G \left( 1 - \frac{w_{ik}}{W_i} \right) \geq W_i G(1 - R_{\text{max}}).
\]

In the remaining, assume that for every \( j \in S \), \( \min \{ w_{ij}, \max \{0, W_i - \sum_{j':(i,j') \in E_j} w_{ij'}\} \} = w_{ij} \).

Given \( \mathcal{E} \), we have

\[
\mathbb{E}\left[ B_i + \sum_{j \in S} A_j | \mathcal{E} \right] \geq \mathbb{E}\left[ B_i + \sum_{j \in S} w_{ij} \left( 1 - \frac{B_i}{W_i} \right) | \mathcal{E} \right]
\]

\[
= \mathbb{E}\left[ B_i + \left( 1 - \frac{B_i}{W_i} \right) \sum_{j \in S} w_{ij} | \mathcal{E} \right]
\]

\[
\geq \mathbb{E}\left[ B_i + \left( 1 - \frac{B_i}{W_i} \right) \cdot \min \{ W_i, \sum_{j \in S} w_{ij} \} | \mathcal{E} \right]
\]

\[
= \min \{ W_i, \sum_{j \in S} w_{ij} \} + \left( 1 - \frac{\min \{ W_i, \sum_{j \in S} w_{ij} \}}{W_i} \right) \mathbb{E}\left[ B_i | \mathcal{E} \right]
\]

\[
\geq \min \{ W_i, \sum_{j \in S} w_{ij} \} = c_i(S).
\]
where the second inequality is due to \(1 - \frac{W_i}{d_i} \geq 0\). As Inequality (2) holds for any realization \(\mathcal{E}\), we deduce that

\[
\beta_i + \sum_{j \in S} \alpha_j \geq G(1 - R_{\max}) \cdot c_i(S).
\]

Consequently, for any \(M \in \mathcal{M}\), applying the above inequality for \(S = M_i\) and summing over all \(i \in L\), we have \(\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq G(1 - R_{\max}) \cdot c(M)\).

\[\square\]

**Theorem 2** The randomized Algorithm 2 has the competitive ratio of \(G(1 - R_{\max}) \cdot (1 - 1/e) = e^{-R_{\max}} - e^{-1}\).

**Proof** We prove that for every online vertex, the increase of the primal is at least \((1 - 1/e)\) that of the dual. Fix an arbitrary realization \(\mathcal{E}\) before the arrival of vertex \(j\). When \(dx\) amount of \(j\) is allocated to \(i\), by Equation (1), the increasing rate of the primal is

\[
\mathbb{E}_{M' \sim x_{-(i,j)}} [c(M' \cup (i,j)) - c(M')] | \mathcal{E} = \min \{w_{ij}, \max\{0, W_i - \sum_{j' : (i,j') \in E} w_{ij'}\}\}.
\]

Conditional on the realization \(\mathcal{E}\), the expected increasing rate of the dual is

\[
\mathbb{E}\left[\frac{dA_i}{dx} + \frac{dB_i}{dx} | \mathcal{E}\right] = \min \{w_{ij}, \max\{0, W_i - \sum_{j' : (i,j') \in E} w_{ij'}\}\} \cdot \mathbb{E}\left[1 - \frac{B_i}{W_i} + g(Y_i) | \mathcal{E}\right]
\]

\[
= \min \{w_{ij}, \max\{0, W_i - \sum_{j' : (i,j') \in E} w_{ij'}\}\} \cdot \mathbb{E}\left[1 - G(Y_i) + g(Y_i) | \mathcal{E}\right]
\]

\[
= \min \{w_{ij}, \max\{0, W_i - \sum_{j' : (i,j') \in E} w_{ij'}\}\} \cdot \frac{1}{1 - 1/e}.
\]

So, given a realization \(\mathcal{E}\), the ratio between the increasing rates of the primal and the dual is \((1 - 1/e)\). This holds for any realization. Therefore, by Lemma 6 and the weak duality, the competitive ratio of Algorithm 2 is \(G(1 - R_{\max}) \cdot (1 - 1/e) = e^{-R_{\max}} - e^{-1}\).

\[\square\]

**Corollary 1** Algorithm 2 is \((1 - 1/e)\)-competitive for the online stochastic-reward matching problem with vanishing probability.

**Proof** In the online matching problem with stochastic rewards, the reward function \(c(M)\) can be expressed as \(\sum_{i \in L} c_i(M_i)\) where \(c_i(S) = w_i \cdot \min\{1, \sum_{j \in S} p_{ij}\} = \min\{w_i, \sum_{j \in S} p_{ij} w_i\}\) for \(S \subseteq R\). Applying Theorem 2 and using the vanishing probability property \((R_{\max} = \max p_{ij} \rightarrow 0)\), we deduce the competitive ratio of \((1 - 1/e)\).

\[\square\]

5 Online Weighted Matching with Sub-Additive Rewards

In this section, we consider the online weighted matching problem in a general model in which the reward function is sub-additive, i.e., \(c(M_1 \cup M_2) \leq c(M_1) + c(M_2)\). Without loss of generality, assume that \(c(M) \leq 1\) for all \(M\) (this can be done by scaling).

Given a sub-additive function \(f : 2^E \rightarrow [0, 1]\), define the total curvature \(\kappa_f\) of \(f\) as

\[
\kappa_f = 1 - \min_{\emptyset \neq M \subseteq E} \min_{e \in M} \frac{f(M) - f(M \setminus \{e\})}{f(\{e\})}.
\]
This definition generalizes the notion of curvature for submodular functions introduced by Conforti and Cornuéjols [5]. A function \( f : 2^E \to \mathbb{R}_{\geq 0} \) is submodular if \( f(M_1 \cup \{e\}) - f(M_1) \geq f(M_2 \cup \{e\}) - f(M_2) \) for all \( M_1, M_2 \subseteq E \). In the context of submodular functions, the curvature [5] is defined as \( 1 - \min_{e \in E} \frac{f(E) - f(E \setminus \{e\})}{f(e)} \). The latter is exactly the same as (3) since for submodular functions, \( \min_{M \subseteq E} \{ f(M) - f(M \setminus \{e\}) \} = f(E) - f(E \setminus \{e\}) \). Intuitively, the total curvature measures how far away \( f \) is from being modular. The concept of curvature is widely used in the context of submodular optimization; for example in determining both upper and lower bounds on the approximation ratios for many submodular and learning problems [5, 12, 2, 26, 17, 24].

Denote \( \kappa = \kappa_c \). We will bound the competitive ratio in this section as a function of \( \kappa \). Note that if the reward function can be decomposed as \( c(M) = \sum_{i \in L} c_i(M_i) \) for every \( M \in \mathcal{M} \) where \( c_i : 2^R \to \mathbb{R}^+ \) then \( \kappa = \kappa_c = \max_i \kappa_{c_i} \).

**Algorithm.** Recall that \( C : [0,1]^{|E|} \to [0,1] \) be the multilinear extension of \( c : 2^E \to [0,1] \). The algorithm for the online matching with sub-additive reward is a generalization of the previous algorithms. The main difference is that in this algorithm, we deal directly with fractional variables instead of random variables and realizations of random assignments. The reason is that in the previous sections, our goal is to achieve the tight competitive ratio of \((1 - 1/e)\) whereas in this general setting, we aim for a weaker guarantee, given the hardness result\(^2\) of [18].

**Algorithm 3** Algorithm for Edge-Weighted Matching with Sub-Additive Rewards.

1. All primal and dual variables are initially set to 0.
2. for each arrival of a new vertex \( j \) do
3. \hspace{1em} while \( \sum_{i \in L} x_{ij} < 1 \) and \( \beta < 1 \) do
4. \hspace{2em} for every \( i \in L \) in \( \arg \max_{j'} \{ \frac{\partial C(x)}{\partial x_{i,j'}} \} \) do
5. \hspace{3em} Increase \( x_{ij} \) by \( dx \).
6. \hspace{2em} Increase \( \beta \) and \( \alpha_j \) by the following rules respectively,
7. \hspace{2em} \hspace{1em} \( d\beta = \frac{\partial C(x)}{\partial x_{ij}} g(C(x)) dx \) and \( d\alpha_j = \frac{\partial C(x)}{\partial x_{ij}} (1 - \beta) dx \).
8. \hspace{1em} end for
9. \hspace{1em} end while
10. end for

Note that in the algorithm, \( \frac{\partial C(x)}{\partial x_{ij}} = \mathbb{E}_{M' \sim x_{-\{i,j\}}} \left[ c(M' \cup \{i,j\}) - c(M') \right] \) can be computed (up to any precision) based on already arrival vertices (since \( x_{i'j'} = 0 \) for every \( j' \) unreleased so far).

**Primal/dual variables.** Again, we define primal variables \( z_M = \prod_{(i,j) \in M} x_{ij} \prod_{(i,j) \notin M} (1 - x_{ij}) \) and dual variables \( \alpha_j, \beta \) are constructed in the algorithm.

**Lemma 5** During the execution of the algorithm, the following invariant holds

\[
\beta = G(C(x))
\]

\(^2\)No randomized algorithm is 1/2-competitive unless NP= RP, even for submodular rewards (a sub-class of sub-additive rewards).
Proof Fix a vertex \( i \in L \). By the algorithm, whenever a variable \( x_{ij} \) for some \( j \in R \) is increased by an amount \( dx \), \( \frac{d\beta}{dx} = \frac{\partial C(x)}{\partial x_{ij}} g(C(x)) = \frac{\partial G(C(x))}{\partial x_{ij}} \). Together with the property \( G(0) = G(C(0)) = 0 \), the invariant follows. \( \square \)

**Lemma 6** For every \( M \in \mathcal{M} \), it holds that

\[
\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq (1 - \kappa) c(M)
\]

In other words, the dual variables are feasible up to a factor \( (1 - \kappa) \).

Proof Fix \( M \in \mathcal{M} \). If \( \beta \geq 1 \) then the lemma inequality holds (recall that \( c(M) \leq 1 \)). In the following, assume that \( \beta < 1 \). By the conditions of the loops, for every vertex \( j \in M_i \), \( \frac{d\alpha_j}{dx} \) must be at least \( \frac{\partial C(x)}{\partial x_{ij}} (1 - \beta) \) since \( \beta \) is non-decreasing. Besides, by the definition of multilinear extension, \( C \) is linear function w.r.t \( x_{ij} \). In other words, \( \frac{\partial C(x)}{\partial x_{ij}} \) is constant when \( x_{ij} \) varies. Therefore,

\[
\alpha_j = \int_0^1 \frac{d\alpha_j}{dx} dt \geq \int_0^1 \frac{\partial C(x)}{\partial x_{ij}} (1 - \beta) dt = \frac{\partial C(x)}{\partial x_{ij}} (1 - \beta).
\]

We have

\[
\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq \beta + \sum_{i \in L} \sum_{j \in M_i} \frac{\partial C(x)}{\partial x_{ij}} (1 - \beta)
\]

\[
\geq \beta + (1 - \beta) \cdot \min \left\{ 1, \sum_{(i,j) \in M} \frac{\partial C(x)}{\partial x_{ij}} \right\}
\]

\[
= \min \left\{ 1, \sum_{(i,j) \in M} \frac{\partial C(x)}{\partial x_{ij}} \right\} + \left( 1 - \min \left\{ 1, \sum_{(i,j) \in M} \frac{\partial C(x)}{\partial x_{ij}} \right\} \right) \beta
\]

\[
\geq \min \left\{ 1, \sum_{(i,j) \in M} \frac{\partial C(x)}{\partial x_{ij}} \right\}
\]

\[
= \min \left\{ 1, \sum_{(i,j) \in M} E_{M' \sim x_{-i,j}} [c(M' \cup (i,j)) - c(M')] \right\}
\]

\[
\geq \min \left\{ 1, \sum_{(i,j) \in M} E_{M' \sim x_{-i,j}} [(1 - \kappa) c(i,j)] \right\}
\]

\[
= \min \left\{ 1, (1 - \kappa) \sum_{(i,j) \in M} c(i,j) \right\}
\]

\[
\geq (1 - \kappa) c(M).
\]

The second inequality holds since \( 1 - \beta \geq 0 \). The second equality is due to property (1) of multilinear extensions. The third inequality follows the definition of curvature. The last inequality is due to the sub-additivity of \( c \) and \( c(M) \leq 1 \). The lemma inequality follows. \( \square \)

**Theorem 3** Algorithm 3 is \((1 - \kappa)(1 - 1/e)\)-competitive.
Proof We bound the increasing rates of the primal and the dual. Assume that $dx$ amount of $j$ is allocated to $i$. The increase in the primal is \( \frac{\partial C(x)}{\partial x_{ij}} dx \). The increase of the dual is
\[
d\alpha_j + d\beta_i = \frac{\partial C(x)}{\partial x_{ij}} (1 - \beta) dx + \frac{\partial C(x)}{\partial x_{ij}} g(C(x)) dx
\]
\[
= \left(1 - G(C(x)) + g(C(x))\right) \frac{\partial C(x)}{\partial x_{ij}} dx = \frac{1}{1 - 1/e} \cdot \frac{\partial C(x)}{\partial x_{ij}} dx.
\]
By Lemma 6 and the weak duality, the competitive ratio follows. \( \square \)

6 Online Fractional Weighted Matching with Concave Rewards

The main goal of this section is to build the connection with the primal-dual scheme in [6] by showing that the latter can be described and analyzed similarly as algorithms in previous sections. Consider the online weighted matching problem in the general model in which the multilinear extension $C : [0, 1]^{|E|} \rightarrow [0, 1]$ (of the reward function $c$) is concave.

Given a function $v : [0, 1]^{|E|} \rightarrow [0, 1]^{|E|}$, for $x \in [0, 1]^{|E|}$, we can write $v(x) = (v_{i,j}(x))_{(i,j) \in E}$. Define $r > 0$ as the largest constant such that the following system of differential equations has a solution $v : [0, 1]^{|E|} \rightarrow [0, 1]^{|E|}$
\[
\frac{1}{r} \cdot \frac{\partial C(x)}{\partial x_{ij}} = \frac{\partial C(v(x))}{\partial v_{ij}(x)} - \left\langle v(x), \frac{\partial}{\partial x_{ij}} \nabla_v C(v(x)) \right\rangle \quad \forall i \in L, j \in R, \tag{4}
\]
\[
\frac{\partial v(x)}{\partial x_{ij}} \geq 0 \quad \forall i \in L, j \in R, \tag{5}
\]
\[
\frac{\partial^2 C(v(x))}{\partial x_{ij} \partial v_{ij}(x)} \leq 0 \quad \forall i \in L, j \in R, \tag{6}
\]
with boundary conditions $v(0) = 0$. Note that the last inequality guarantees that $\frac{\partial C(v(x))}{\partial v_{ij}(x)}$ is non-increasing when $x_{ij}$ increases.

Let $v$ be a solution of the system with $r$. Denote $\nabla_v C(v(x)) = (\frac{\partial C(v(x))}{\partial v_{ij}(x)})_{(i,j) \in E}$. Consider the following algorithm, building from the salient ideas of the previous ones.

Lemma 7 For every $M \in \mathcal{M}$, it holds that
\[
\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq c(M)
\]

Proof Fix $M \in \mathcal{M}$. For any vertex $i \in L$ and $j \in M_i$, at every time during the execution of the algorithm, the increasing rate of $\alpha_j$ is at least the current value of $\frac{\partial C(v(x))}{\partial v_{ij}(x)}$. Moreover, the latter is non-increasing (by (6)). Therefore,
\[
\alpha_j = \int_0^1 \frac{\partial C(v(x))}{\partial v_{ij}(x)} dt \geq \frac{\partial C(v(x))}{\partial v_{ij}(x)}.
\]
We have
\[
\beta + \sum_{i \in L} \sum_{j \in M_i} \alpha_j \geq C(v(x)) - \langle v(x), \nabla_v C(v(x)) \rangle + \sum_{(i,j) \in M} \frac{\partial C(v(x))}{\partial v_{ij}(x)}
\]
\[
= C(v(x)) + \langle \nabla_v C(v(x)), 1_M - v(x) \rangle
\]
\[
\geq C(v(x)) + C(1_M) - C(v(x))
\]
\[
= C(1_M) = c(M)
\]
Algorithm 4 Algorithm for Edge-Weighted Matching with Concave Rewards.

1: All primal and dual variables are initially set to 0.
2: \textbf{for} each arrival of a new vertex \( j \) \textbf{do}
3: \hspace{1em} \textbf{while} \( \sum_{i \in L} x_{ij} < 1 \) \textbf{do}
4: \hspace{2em} \textbf{for} every \( i \in L \) in \arg \max_i \{ \frac{\partial C(v(x))}{\partial v_{ij}(x)} \} \textbf{do}
5: \hspace{3em} Allocate a \( dx = dx_{ij} \) amount of \( j \) to \( i \).
6: \hspace{2em} Increase \( \alpha_j \) by the following rule:
7: \hspace{3em} \( \frac{d\alpha_j}{dx_{ij}} = \frac{\partial C(v(x))}{\partial v_{ij}(x)} \).
8: \hspace{2em} \textbf{end for}
9: \hspace{1em} Always maintain \( \beta = C(v(x)) - \langle v(x), \nabla v C(v(x)) \rangle \).
10: \textbf{end while}

where \( 1_M \) is the indicator vector of \( M \), i.e., \((1_M)_e = 1 \) if \( e \in M \) and \((1_M)_e = 0 \) otherwise. The last inequality is due to the concavity of \( C \).

\[ \text{Theorem 4} \quad \text{Algorithm 4 is } r\text{-competitive for the fractional online matching problem under the assumption that } C \text{ is concave.} \]

\textbf{Proof} Initially, when no online vertex is released, the objective values of the primal and the dual are 0. At any time in the execution of the algorithm when an online vertex \( j \) is released, the increasing rate of the dual is

\[ \frac{d\alpha_j}{dx_{ij}} + \frac{d\beta}{dx_{ij}} = \frac{\partial C(v(x))}{\partial v_{ij}(x)} - \frac{\partial}{\partial x_{ij}} \left( C(v(x)) - \langle v(x), \nabla v C(v(x)) \rangle \right) \]

\[ = \frac{\partial C(v(x))}{\partial v_{ij}(x)} - \langle v(x), \frac{\partial}{\partial x_{ij}} \nabla v C(v(x)) \rangle. \]

Besides, the increasing rate of the primal is \( \frac{\partial C(v)}{dx_{ij}} \). The competitive ratio follows the definition of \( r \). \[ \square \]

\textbf{Connection to the scheme of Devanur and Jain [6].} Consider the setting in [6] in which each edge between \( i \in L \) and \( j \in R \) has weight \( w_{ij} \geq 0 \) and the reward function of each offline vertex \( i \in L \) can be expressed as \( P(\sum_j w_{ij} x_{ij}) \) where \( P : \mathbb{R}^+ \to \mathbb{R}^+ \) is a 1-dim concave function and \( x_{ij} \) is the fraction of online vertex \( j \) assigned to \( i \). The total reward is \( \sum_{i \in L} P(\sum_j w_{ij} x_{ij}) \). In this setting, using \( \sum_{i \in L} P \) instead of \( C \) in (4, 5, 6) and Algorithm 4, after a simplification (due to the separability of the reward on each offline vertex in the total reward), the system of differential equations characterizing the competitive ratio reads

\[ \frac{1}{r} P'(u) = P'(v(u)) - [v(u)P''(v(u))] \frac{dv(u)}{du} \geq 0 \]

\[ \frac{dv(u)}{du} \geq 0 \]
with boundary condition \( v(0) = 0 \). The system becomes simpler as \( P \) is one-variable function. Note that in this system, we name (1-dim) variable \( u \) (instead of \( x \)) and the condition (6), written as

\[
\frac{d}{du} \left( \frac{\partial P(v(u))}{\partial v(u)} \right) = \frac{P''(v(u))}{u} \leq 0,
\]

always holds (by the concavity of \( P \) and the condition \( \frac{dv(u)}{du} \geq 0 \)). This system is exactly the one given in [6]. For completeness, we give it below.

\[
\frac{1}{r} \cdot P'(u) = P'(v(u)) + Y'(v(u)) \frac{dv(u)}{du}
\]

\[
\frac{dv(u)}{du} \geq 0
\]

with boundary condition \( Y(0) = 0 \) where the function \( Y(v) = P(v) - vP'(v) \).

7 Conclusion

In this paper, we have presented a primal-dual framework with configuration LPs for the online bipartite matching problem that unifies previous approaches and provides optimal \((1 - 1/e)\) competitive ratio in some models, resolving long-standing open questions. We believe that the use of primal-dual with configuration LPs (for example, to circumvent hard constraints, etc) will find other applications and lead to improvements in different problems. Related to our paper, an open question is to study whether the bounds given in Sections 5 and 6 are tight.

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