

HOMOGENEOUS RICCI SOLITONS

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Abstract. In this work, we study metrics which are both homogeneous and Ricci solitons. We prove that such metrics must be semi-algebraic Ricci solitons in the sense that they evolve under the Ricci flow by dilations and pullback by automorphisms of the isometry group.

If there exists a transitive semi-simple group of isometries on a Ricci soliton, we show that such a space is in fact Einstein. As a corollary, we obtain that all compact homogeneous Ricci solitons are necessarily Einstein.

If there exists a transitive solvable group of isometries on a Ricci soliton, we show that it is isometric to a solvsoliton. Moreover, unless the manifold is flat, it is necessarily simply-connected and diffeomorphic to $\mathbb{R}^n$.

1. Introduction

A Ricci soliton metric $g$ on a manifold $M$ is a Riemannian metric satisfying

\begin{equation}
\text{ric}_g = cg + \mathcal{L}_X g
\end{equation}

for some complete, smooth vector field $X$ on $M$ and $c \in \mathbb{R}$. This condition is equivalent to $g_t = (-2ct + 1)\varphi_t^* g$ being a solution to the Ricci flow

\begin{equation}
\frac{\partial}{\partial t} g = -2\text{ric}_g
\end{equation}

where $\varphi_t$ is the family of diffeomorphisms generated by the vector field $X$ which we reparameterize by $s(t) = \frac{1}{c} \ln(-2ct + 1)$. We are interested in homogeneous solutions to the Ricci flow.

Homogeneous solutions to the Ricci flow have long been studied for both their relative simplicity and their appearance as limits of the flow. This has been explored by many authors (e.g., [IJJ92, LLL06, GIK06, Lot07, BD07, GP09, PW09, Pay10, Lau10, Lot10, Lau11b], see also [CK04, Chapter 1]).

On non-compact homogeneous manifolds, there are many structural restrictions on the nature of soliton metrics (see Introduction of [Lau11c]) and all known examples of Ricci soliton metrics on non-compact homogeneous manifolds are isometric to left-invariant metrics $g$ on solvable Lie groups $G$ satisfying the seemingly stronger condition

\begin{equation}
\text{Ric}_g = cI + D
\end{equation}

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for some $c \in \mathbb{R}$ and $D \in \text{Der}(g)$, where $\text{Ric}_g$ is the $(1,1)$-Ricci tensor and $g = \text{Lie } G$. On a Lie group $G$, a left-invariant metric satisfying Eqn. [1.3] is called an algebraic Ricci soliton. When the group $G$ is solvable, such metrics have been called solvsolitons in the literature.

Algebraic Ricci solitons are Ricci solitons which evolve by automorphisms, instead of just diffeomorphisms. More precisely, consider the family of automorphisms $\Psi_t \in \text{Aut}(G)$ such that $d(\Psi_t)_e = \exp(tD) \in \text{Aut}(g)$. The family of metrics $g_t = (−2ct + 1)\Psi^*_s(t)g$ with $s(t) = \frac{1}{2} \ln(−2ct + 1)$ is now a solution to Eqn. [1.2] Our main result is that these are essentially the only examples that arise, in the presence of a transitive solvable group of isometries.

A Riemannian manifold $M = (M, g)$ is called a solvmanifold if there exists a transitive solvable group of isometries.

**Theorem 1.1.** Consider a solvmanifold $M$ which is a Ricci soliton. Then $M$ is isometric to a solvsoliton and the transitive solvable group may be chosen to be completely solvable. Moreover, unless $M$ is flat, it is simply-connected and diffeomorphic to $\mathbb{R}^n$.

**Remark 1.2.** All the rigid geometric results for algebraic Ricci solitons on solvable groups obtained in [Lau11c] now apply to Ricci soliton solvmanifolds. For example, Ricci solitons on solvmanifolds are now unique, up to scaling and isometry.

Special cases of this theorem have appeared in the literature. The above theorem has been obtained for nilpotent groups in [Lau01, Lau11b]. However, we warn the reader that a nilmanifold is able to admit quotients which are non-nilpotent solvmanifolds.

In the special case of solvsolitons, it was previously known that only simply-connected solvable Lie groups can appear, see [Lau11c] Remark 4.12]. However, there do exist solvmanifolds which are not solvable groups.

In the special case of Einstein solvmanifolds, the fact that these must be simply-connected seems to be new. (In the further special case of negative curvature, this result follows from [Ale75].)

In the theorem above, the statement that a Ricci soliton solvmanifold is isometric to a solvsoliton cannot be replaced with the statement “is a solvsoliton”, as the next example demonstrates. This example seems to have escaped the literature. More examples exhibiting strange structural phenomena are given in Section 6.

**Example 1.3.** Let $\mathfrak{s}_1$ be a completely solvable Lie algebra which admits a non-trivial soliton metric. Consider $\mathbb{R}^n$ and an abelian subalgebra $t$ of $\mathfrak{so}(m)$. Construct the semi-direct product $\mathfrak{s}_2 = t \ltimes \mathbb{R}^n$. The simply-connected Lie group $S$ with Lie algebra $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ admits a Ricci soliton metric, but this soliton cannot be algebraic relative to the Lie structure of $S$.

The fact that this group cannot admit an algebraic soliton follows immediately from the structural constraints given in Theorem 4.8 (iv) of [Lau11c].
To see that this space admits a Ricci soliton metric, first one recognizes that $S_2$ admits a flat metric by an old result of Milnor [Mil76, Thm. 1.5]. $S_2$ is now isometric to $\mathbb{R}^n$, for some $n$. To build the soliton metric on $S$, we take the soliton metric on $S_1$ and direct sum the flat metric on $S_2$. Now $S$ is isometric to $S_1 \times \mathbb{R}^n$. This is a solvsoliton since $c \text{Id}$ is a derivation of the abelian algebra $\text{Lie} \mathbb{R}^n$.

**Homogeneous Ricci solitons.** In addition to solvmanifolds, we investigate the general homogeneous setting. A natural weak generalization of algebraic Ricci solitons on Lie groups to homogeneous spaces is the following (for a true generalization, see Definition 2.3). Consider a homogeneous space $G/K$ with $G$-invariant metric. Let $\Phi_t \in \text{Aut}(G)$ be a family of automorphisms of $G$ such that $\Phi_t(K) = K$; each $\Phi_t$ gives rise to a well-defined diffeomorphism $\phi_t$ of $G/K$ defined by

$$\phi_t(hK) = \Phi_t(h)K \quad \text{for} \quad h \in G$$

**Definition 1.4.** Consider a Riemannian homogeneous space $(G/K, g)$. We say this space is a semi-algebraic Ricci soliton if there exists a family of automorphisms $\Phi_t \in \text{Aut}(G)$ such that $\Phi_t(K) = K$ and

$$g_t = c(t)\phi_t^*g$$

is a solution to the Ricci flow (Eqn. 1.2) on $G/K$ starting at $g$.

The notion of being a semi-algebraic Ricci soliton is relative to a fixed choice of transitive group $G$. For example, the Ricci soliton given in Example 1.3 cannot be semi-algebraic relative to the given transitive solvable group (cf. Prop. 2.3 and Lemma 4.3), however, it is semi-algebraic relative to its full isometry group, by following general result.

**Proposition.** Every Ricci soliton on a homogeneous space is a semi-algebraic Ricci soliton relative to its isometry group.

For more details, see Proposition 2.2. As the Ricci flow on a homogeneous space now evolves by automorphisms of its isometry group, its Ricci tensor has a special form similar to Eqn. 1.3 see Proposition 2.3.

As a consequence, we are able to reduce the problem of finding and classifying homogeneous Ricci solitons to a problem on generalized metric Lie algebras. We do not explore this avenue here and direct the interested reader to [Lau11a].

**Remark 1.5.** We do not have any examples of non-compact homogeneous Ricci solitons which are not isometric to solvsolitons.

In addition to manifolds admitting a transitive solvable group of isometries, we study the case that there exists a transitive semi-simple group of isometries. Our main result here is the following.

**Theorem 1.6.** Let $(M, g)$ be a homogeneous Ricci soliton and assume there exists a transitive semi-simple group of isometries. Then $(M, g)$ is Einstein.
Corollary 1.7. Every compact homogeneous Ricci soliton must be Einstein (with non-negative scalar curvature).

Remark 1.8. Our proof of the corollary does not require the vector field $X$ satisfying Eqn. (1.1) to be complete and so the above result holds more generally for any compact homogeneous space satisfying Eqn. (1.1).

We remind the reader that the content of this corollary is in the case that the Ricci soliton is a so-called shrinker (i.e. $c > 0$ in Eqn. (1.1)). In general, any steady or expanding Ricci soliton (i.e. $c \leq 0$ in Eqn. (1.1)) on a compact manifold is necessarily Einstein, see [Ive93]. The fact that the scalar curvature of a compact homogeneous Einstein space must be non-negative is due to Bochner [Boc46].

The above corollary is well-known to the experts and can be deduced from the work of Petersen-Wylie [PW09] where homogeneous gradient Ricci solitons are studied. To use that work, one must apply a result of Perelman [Per02, Remark 3.2] which states that compact Ricci solitons are gradient solitons.

Our proof of the corollary is algebraic in nature and does not depend on [Per02].

To put Theorem 1.6 in context, we remind the reader that the only known examples of homogeneous spaces with Einstein metrics are when either 1) $G/K$, and hence $G$, is compact or 2) when $G$ is non-compact and $K$ a maximal compact subgroup. In the first case, Einstein metrics are not unique and it is still an open question to determine if there are a finite number of these metrics (up to isometry and rescaling), see [BWZ04]. In the second case, if Alekseevsky’s conjecture is true, then these are the only possible examples, see also [Lau11a].

Remark 1.9. For the most part, the techniques used to prove our main results are not special to the Ricci flow and could be used to study solitons (i.e. self-similar solutions) of any reasonably well-behaved geometric evolution on homogeneous manifolds.

Our work is organized as follows. In the following section, we analyze the general homogeneous setting and show that homogeneous Ricci solitons are all semi-algebraic (relative to their isometry group). In Section 3 we study the case that there exists a transitive semi-simple group of isometries and prove Theorem 1.6 and Corollary 1.7. In Section 4 we prove Theorem 1.1 in the case that our manifold is simply-connected; the reduction to the simply-connected case is given in Section 5. In the last section we consider structural and existence questions.

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2. General Homogeneous Setting

**Preservation of the isometry group.** Although uniqueness of solutions to the Ricci flow is not known in general (on non-compact manifolds), much more can be said for homogeneous metrics.

**Lemma 2.1.** Consider a homogeneous Riemannian manifold \((M,g)\).

(i) There exists a homogeneous solution to the Ricci flow starting at the given homogeneous metric \(g\).

(ii) Solutions to the Ricci flow, among homogeneous metrics, are unique and, denoting the isometry group of the solution \(g_t\) by \(\text{Isom}(g_t)\), we have

\[
\text{Isom}(g_t) = \text{Isom}(g)
\]

for all \(t\) such that the solution \(g_t\) exists.

**Proof.** The first statement is well-known and we do not give a proof. Proving (i) amounts to analyzing an ODE on the space of inner products on a single tangent space of \(M\).

The second statement is a special case of [CZ06] and [Kot10]. In these works, the metrics of interest are those which are complete and have bounded curvature. Clearly homogeneous metrics satisfy these conditions. For uniqueness of solutions see [CZ06] Thm. 1.1] and [Kot10] Thm. 1.1], for preservation of the isometry group see [CZ06] Cor. 1.2.] and [Kot10] Thm. 1.2].

We observe that if one were just considering \(G\)-invariant metrics, uniqueness of the flow could be determined by analyzing the associated ODE on the space of inner products on a single tangent space. However, to get uniqueness among all homogeneous metrics, that approach is not sufficient.

**Remark.** In the special case of Ricci solitons, one can obtain the lemma above using only [CZ06].

To use only that work, one simply recognizes that the size of the isometry group does not change along a soliton solution to Eqn. 1.2 (which is true since these isometry groups are all conjugate).

**G-invariant metrics.** Let \((M,g)\) be a homogeneous Riemannian manifold with a transitive group of isometries \(G\). Fixing a point \(p \in M\), and denoting the isotropy at \(p\) by \(K = G_p\), we may naturally identify \(M\) with \(G/K\). Every \(G\)-invariant metric on \(M\) arises in the following way.

Let \(g\) denote the Lie algebra of \(G\) and fix an \(\text{Ad}(K)\)-invariant decomposition \(\mathfrak{p} \oplus \mathfrak{k}\), where \(\mathfrak{k}\) is the Lie algebra of \(K\). The subspace \(\mathfrak{p}\) is naturally identified with \(T_p M\).

The restriction of an \(\text{Ad}(K)\)-invariant inner product \(\langle \cdot, \cdot \rangle\) on \(g\) to \(\mathfrak{p}\) gives rise to a \(G\)-invariant metric \(g\) on \(M \simeq G/K\) defined by

\[
g(v,v)_p = \langle X_v, X_v \rangle_e \quad \text{for} \quad v \in T_p M
\]
where $X_v \in \mathfrak{p}$ is the unique vector in $\mathfrak{p}$ such that $v = \frac{d}{dt} \big|_{t=0} \exp(tX_v) \cdot p$.

Each automorphism $\Phi \in \text{Aut}(G)$ which fixes $K$ gives rise to a well-defined diffeomorphism $\phi$ of $M$. More precisely, $\phi$ is defined by

$$\phi(h \cdot p) = \Phi(h) \cdot p$$

for $h \in G$.

Denote by $\text{Aut}(G)^K$ the subgroup of $\text{Aut}(G)$ which stabilizes $K$. As described above, this is naturally a subgroup of $\text{Diff}(M)$.

Let $g$ be a $G$-invariant metric on $M$ corresponding to $\langle \cdot , \cdot \rangle$ on $G$ (as above). Given $\Phi \in \text{Aut}(G)^K$ with corresponding $\phi \in \text{Diff}(M)$, we have

$$\phi^*g$$

on $M$ corresponds to $\Phi^*\langle \cdot , \cdot \rangle$ on $G$

where we are restricting $\Phi^*\langle \cdot , \cdot \rangle$ to the $\text{Ad}(K)$-invariant subspace $\Phi(p)$.

**Proposition 2.2.** Let $(M, g)$ be a homogeneous Ricci soliton. The Ricci flow starting at $(M, g)$ evolves by automorphisms of the isometry group $G$.

More precisely, fix a point $p \in M$ and let $K = G_p$ denote the isotropy subgroup of $G$. Let $\langle \cdot , \cdot \rangle$ be a $G$-invariant metric on $G$, which is also right $K$-invariant, and which submerses to the $G$-invariant metric $g$ on $M \simeq G/K$.

There exists a 1-parameter family of automorphisms $\Phi_t \in \text{Aut}(G)$ such that

(i) $\Phi_t(K) = K$ and the associated family of diffeomorphisms $\phi_t \subset \text{Diff}(M)$ yields a solution to the Ricci flow given by

$$g_t = c(t)\phi_t^*g$$

(ii) One can choose $\Phi_t \in \text{Aut}(G)^K$ so that $\Phi_t|_{K_0} = \text{id}$, where $K_0$ denotes the identity component of $K$,

(iii) Additionally, $\Phi_t$ can be chosen to be a 1-parameter subgroup of $\text{Aut}(G)$ (after reparameterizing $t$), hence there exists $D \in \text{Der}(g)$ such that $d\Phi_t = \exp(tD) \in \text{Aut}(g)$.

An immediate consequence of the proof of this proposition is the following. Let $pr : g \to \mathfrak{p}$ denote the orthogonal projection onto $\mathfrak{p}$.

**Proposition 2.3.** Let $(M, g)$ be a homogeneous Ricci soliton. Let $G$ be a transitive group of isometries such that the solution to the Ricci flow on $(M, g)$ evolves by automorphisms of $G$ as in Definition 1.4 (e.g. when $G$ is the full isometry group). Then there exists a derivation $D \in \text{Der}(g)$ such that

$$\text{Ric} = c\text{Id} + \frac{1}{2}[\text{pr} \circ D + (\text{pr} \circ D)^t]$$

In this presentation, we have naturally identified $\mathfrak{p} \subset g$ with $T_pM$ and $\text{Ric}$ is the $(1,1)$-Ricci tensor evaluated at $T_pM$.

Moreover, we may assume $D|_{\mathfrak{k}} = 0$, where $\mathfrak{k} = \text{Lie } K$.

We are now in a position to directly generalize the notion of algebraic Ricci soliton on Lie groups to homogeneous spaces, cf. Eqn. 1.3. The notion of algebraic Ricci soliton given here is consistent with that used by Lauret, cf. [Lau11a, Lau11b].
**Definition 2.4.** A homogeneous Ricci soliton \((G/K, g)\) is called an algebraic Ricci soliton if

\[
\text{Ric} = c\text{Id} + pr \circ D
\]

for some \(c \in \mathbb{R}\) and some \(D \in \text{Der}(g)\).

**Remark.** Although every homogeneous Ricci soliton is semi-algebraic relative to its isometry group (Proposition 2.2), it is not clear if every Ricci soliton is actually algebraic relative to its isometry group.

**Proof of Prop. 2.2.** We prove (i) first. Let \(g\) be a homogeneous Ricci soliton satisfying \(\text{ric} = cg + \mathcal{L}_X g\) with corresponding solution of the Ricci flow (Eqn. \(1.2\)) given by \(g_t = (-2ct + 1)\varphi_s(t)g\), where \(s(t) = \frac{1}{c} \ln(-2ct + 1)\) and \(\varphi_s\) is the 1-parameter group of diffeomorphisms generated by \(X\).

Fix a point \(p \in M\). We may assume \(\varphi_t\) fixes \(p\). To see this, consider the smooth curve \(\varphi_t(p) \subset M\). By taking a section of the right \(K\)-action on \(G\), there exists a smooth curve \(h(t) \in G\) such that \(L_{h(t)} \cdot p = \varphi_t(p) \subset M \simeq G/K\).

As left-translation by elements of \(G\) is an isometry of \((M, g_t)\) for all \(t\) (see Lemma 2.1), we have a solution to the Ricci flow given by

\[
g_t = (-2ct + 1)\psi_{s(t)}^* g
\]

where \(\psi_s = L_{h(s)}^{-1} \circ \varphi_s\). (It is not clear that this new family is a 1-parameter group of diffeomorphisms of \(M\), even after reparameterizing time, as \(L_{h(s)}^{-1}\) and \(\varphi_s\) may not commute.)

Given \(\sigma \in \text{Isom}(g) = G\), we have \(\psi_{s(t)} \circ \sigma \circ \psi_{s(t)}^{-1} \in \text{Isom}(g_t) = G\) due to Lemma 2.1. Thus, we have a smooth family of automorphisms \(\Phi_s \in \text{Aut}(G)\) defined by

\[
\Phi_s(\sigma) = \psi_s \circ \sigma \circ \psi_s^{-1} \quad \text{for} \quad \sigma \in G
\]

Let \(K = G_p\). Since \(\psi_s\) fixes \(p\), we see that \(\Phi_s(K) = K\), i.e. \(\Phi_s \in \text{Aut}(G)^K\).

Let \(\phi_s \in \text{Diff}(M)\) be the diffeomorphism of \(M\) associated to \(\Phi_s\). Observe that \(\phi_s = \psi_s\). This proves (i), we prove (iii) next.

Thus far, we have shown our solution to the Ricci flow satisfies

\[
g_t \subset \mathbb{R}_{>0} \times \text{Aut}(G)^K \cdot g
\]

The set of metrics \(\mathbb{R}_{>0} \times \text{Aut}(G)^K \cdot g\) is a homogeneous manifold and so its tangent space is generated by 1-parameter subgroups of \(\mathbb{R}_{>0} \times \text{Aut}(G)^K\).

More precisely, there exists a derivation \(D \in \text{Der}(g)\) such that \(D(\mathfrak{t}) \subset \mathfrak{t}\) (where \(\mathfrak{t} = \text{Lie } K\)) so that

\[
\text{ric} = cg + \mathcal{L}_Y g
\]

where \(Y = \frac{d}{dt} \big|_{t=0} \phi_t\) and \(\phi_t \in \text{Diff}(M)\) is defined by \(\exp(tD) \in \text{Aut}(G)^K\).

It is clear that \(Y\) is a complete vector field as 1-parameter subgroups of a Lie group are defined for all time and we obtain the desired solution to the Ricci flow by reparameterizing by \(s(t) = \frac{1}{c} \ln(-2ct + 1)\) in the family \(\Phi_s = \exp(sD) \in \text{Aut}(G)^K\). This proves (iii).
We finish the proof of Prop. 2.2 by showing \( \Phi_s \) may be chosen so that \( \Phi_s|_{K_0} = id \). First observe that \( \Phi_s|_K \in Aut(K) \), since \( \Phi_s \in Aut(G)^K \). And this subgroup \( K = G_p \) is compact since \( G \) is the isometry group of \((M,g)\).

Since \( K_0 \) is compact and connected, it may be written as a product
\[
K_0 = K_{ss}Z(K_0)
\]
of its maximal semi-simple subgroup \( K_{ss} = [K_0, K_0] \) and its center \( Z(K_0) \).
(The intersection \( K_{ss} \cap Z(K_0) \) is finite.)

The center of a group is always preserved by automorphisms and so \( \Phi_s|_{Z(K_0)} \in Aut(Z(K_0)) \). Being the automorphism group of a compact torus, \( Aut(Z(k_0)) \) is finite and since \( \Phi_0 = id \), we see that \( \Phi_s|_{Z(K_0)} = id \).

Since \( \Phi_s(K_{ss}) \) is semi-simple and \( K_{ss} \) is a maximal semi-simple subgroup of \( K_0 \), we see that \( \Phi_s|_{K_{ss}} \in Aut(K_{ss}) \). The group \( K_{ss} \) being semi-simple implies the connected component of the identity of \( Aut(K_{ss}) \) consists of inner automorphisms and so \( \Phi_s|_{K_{ss}} = C_{exp(sX)} \) for some \( X \in Lie K_{ss} \), where \( C_h \) denotes conjugation by \( h \in K_{ss} \).

As conjugation is the identity on the center, we see that
\[
\Phi_s|_{K_0} = C_{exp(sX)}
\]
for some \( X \in Lie K_{ss} \).

Consider the composition \( \Phi_s \circ C_{exp(sX)}^{-1} \). By construction, this is the identity on \( K_0 \). Moreover, since \( \Phi_s \) and \( C_{exp(sX)} \) commute, this composition is a 1-parameter subgroup of \( Aut(G)^K \). Lastly, \( C_{exp(sX)} \in Aut(G)^K \) corresponds to the diffeomorphism of \( M \) which is left-translation by \( exp(sX) \in K \). As this is an isometry, the diffeomorphisms of \( M \) corresponding to the family \( \Phi_s(t) \circ C_{exp(s(t)X)}^{-1} \) give rise to a solution to the Ricci flow satisfying (ii). This completes the proof of the proposition.

\[\square\]

**Proof of Prop. 2.3.** Consider a homogeneous Ricci soliton \((M,g)\) with transitive group of isometries \( G \) such that the Ricci flow (Eqn. 1.2) on \( M \) evolves by automorphisms of \( G \) as in Definition 1.4.

The proof of Proposition 2.2 shows that we have a solution to the Ricci flow given by \( g_t = (-2ct + 1)\phi_s(t)g \) where \( s(t) = \frac{t}{c} \ln(-2ct + 1) \) and \( \phi_s \in \text{Diff}(M) \) comes from \( \Phi_s = exp(sD) \in Aut(G)^K \), for some \( D \in Der(g) \). Differentiating at \( t = 0 \) gives the desired result.

Choosing \( \Phi_s \) to be the identity on \( K_0 \) yields \( D|_{\mathfrak{k}} = 0 \). \( \square \)

**Transitive unimodular group of isometries.** Let \((M,g)\) be a homogeneous Riemannian manifold with transitive group of isometries \( G \). Fix a point \( p \in M \) and let \( K = G_p \) denote the isotropy subgroup at \( p \). Fix an \( Ad(K) \)-invariant compliment \( \mathfrak{p} \) of \( \mathfrak{k} \) in \( \mathfrak{g} \).

As discussed earlier, the set of \( G \)-invariant metrics on \( M \) is naturally identified with the set of \( Ad(K) \)-invariant inner products on \( \mathfrak{p} \). Thus, the
set of $G$-invariant metrics on $M$ is identified with

$$GL(p)^K / O(p)^K$$

where $GL(p)^K$ denotes the matrices of $GL(p)$ which commute with $Ad(k)$ for all $k \in K$, and $O(p)^K = O(p) \cap GL(p)^K$.

Denote by $\mathcal{M}_G$ the space of $G$-invariant metrics on $M$. As we will demonstrate, this space of $G$-invariant metrics on $M$ is endowed with a natural Riemannian metric so that $GL(p)^K$ acts isometrically. Let $Q$ be a $G$-invariant metric on $M$. The tangent space at $Q$ is

$$T_Q \mathcal{M}_G = \{ Ad(K) - \text{invariant symmetric, bilinear forms on } p \}$$

and the natural Riemannian metric on $\mathcal{M}_G$ is defined by

$$\langle v, w \rangle_Q = tr \; vw = \sum_i v(e_i, e_i)w(e_i, e_i)$$

where $v, w \in T_Q \mathcal{M}_G$ and $\{ e_i \}$ is a $Q$-orthonormal basis of $T_pM$. This inner product on $T_Q \mathcal{M}_G$ can also be realized as follows. Given $v \in T_Q \mathcal{M}_G$, we may consider the associated symmetric endomorphism of $T_pM$ defined by $Q(V(x), y) = v(x, y)$. Then $\langle v, w \rangle_Q = tr \; VW$, where $tr$ denotes the trace.

**Lemma 2.5.** Let $(M, g)$ be a homogeneous Riemannian manifold with transitive unimodular group of isometries $G$. Given $Q \in \mathcal{M}_G$, denote by $ric_Q$ and $sc(Q)$ the Ricci and scalar curvatures of $(G, Q)$, respectively. The gradient of the function $sc : \mathcal{M}_G \to \mathbb{R}$ is

$$(\text{grad } sc)_Q = -ric_Q$$

relative to the above Riemannian metric on $\mathcal{M}_G$.

For a proof of this result, see [Heb98, Section 3] or [Nik98].

3. **Semi-simple homogeneous spaces**

In this section, we consider the case that the isometry group $G$ contains a transitive semi-simple group $H$. Such spaces include all compact homogeneous spaces with finite fundamental group and familiar non-compact spaces such as symmetric spaces. This section is devoted to proving Theorem 1.6 and Corollary 1.7.

As in the case of solvmanifolds, the proof of Theorem 1.6 uses the fact that there is a very special transitive group contained in the isometry group $G$. We may assume our homogeneous space $M$ is connected. The following is Theorem 4.1 of [Gor80].

**Theorem 3.1** (Gordon). Suppose a connected Lie group $H$ with compact radical acts transitively and effectively by isometries on a Riemannian manifold $M$. Then the connected component of the isometry group is reductive.

This theorem clearly applies when the transitive group $H$ is semi-simple as the radical of a semi-simple group is trivial.
Remark. The following proof of Thm. 1.6 is inspired by the case of a left-invariant metric on a semi-simple group satisfying the stronger condition $Ric = cId + D$ which was worked out by J. Lauret, see Thm. 5.1 of [Lau01].

Proof of Thm. 1.6. Let $G$ denote the isometry group of $(M,g)$ and $H$ a transitive semi-simple subgroup of $G$. Let $G_0$ denote the connected component of $G$.

Recall, the maximal semi-simple subgroups of a connected Lie group are all conjugate by elements of the nilradical. Since the nilradical of $G_0$ is contained in the center (cf. Thm. 3.1), we see that $G_0$ contains a unique maximal semi-simple subgroup which is normal in $G_0$. As $H$ is contained in this maximal, semi-simple subgroup, this maximal subgroup acts transitively on $M$ and we may replace $H$ with said maximal, semi-simple subgroup of $G_0$.

Let $\Phi_t$ be the automorphisms of $G$ by which the Ricci flow on $M$ is evolving (cf. Prop. 2.2). Since $\Phi_t(H)$ is a semi-simple subgroup of $G_0$, we have $\Phi_t(H) = H$, by maximality. Thus, $\Phi_t|_H \in Aut(H)$ and by Prop. 2.3, there exists $X \in h$ such that

(3.1) $Ric = cId + \frac{1}{2}(pr \circ ad X + (pr \circ ad X)^t)$

Here we have used the fact that the derivations of a semi-simple Lie algebra are all inner. To finish the proof of the theorem, we present a lemma whose proof we postpone until the end of the current proof.

Lemma 3.2. Let $X \in h$ be as above. Then $tr(pr \circ ad X) = 0$.

Denote by $\phi_t$ the diffeomorphism of $M$ induced by the automorphism $\Phi_t$ of $H$, as in Section 2. As scalar curvature is an invariant, $sc(g) = sc(\phi_t^* g)$. Using Lemma 2.5 together with Eqn. 3.1 and Lemma 3.2 we have

$$0 = \frac{d}{dt} \bigg|_{t=0} sc(\phi_t^* g) = \langle \text{grad sc, pr } \circ \text{ ad } X \rangle_g$$

$$= c tr(pr \circ ad X) + tr \frac{1}{2}(pr \circ ad X + (pr \circ ad X)^t)(pr \circ ad X)$$

$$= tr \frac{1}{4}(pr \circ ad X + (pr \circ ad X)^t)^2$$

This implies $\frac{1}{2}(pr \circ ad X + (pr \circ ad X)^t) = 0$ and hence the metric is Einstein.

We finish by proving the last lemma.

Proof of Lemma 3.2. Let $K$ denote the isotropy of the $H$-action on $M$ at $p$. Let $\mathfrak{p}$ denote an $Ad(K)$-invariant compliment of $\mathfrak{k}$ in $\mathfrak{h}$, as above. Recall, $K$ is invariant under the 1-parameter family of automorphisms $\Phi_t = \exp(t \text{ ad } X)$ of $H$ and we have

$$tr \text{ ad } X = tr(pr \circ ad X) + tr \text{ ad } X|_\mathfrak{k}$$

where $pr : g \rightarrow \mathfrak{p}$ is the projection onto $\mathfrak{p}$, as before.
However, using Prop. 2.2 (ii), we may assume that $\text{ad} X |_k = 0$. Since $\mathfrak{h}$ is semi-simple, we have $0 = \text{tr} \text{ad} X = \text{tr}(pr \circ \text{ad} X)$, which proves the lemma.

Next we prove Corollary 1.7. While this can be deduced from Theorem 1.6 using topological properties of compact Ricci solitons, there is a direct algebraic proof which is shorter. This is the proof we give.

**Proof of Corollary 1.7.** Consider a compact homogeneous Ricci soliton $(M, g)$. As $M$ is compact, the isometry group $G$ of $(M, g)$ is also compact.

Let $\Phi_t \in Aut(G)$ be the family of automorphisms of $G$ which induces the soliton metric (see Prop. 2.2). Since $G$ is compact, we see that there exists $X \in [\mathfrak{g}, \mathfrak{g}]$ such that $\Phi_t|_{G_0} = \exp(t \text{ad} X)$ (see the proof of Prop. 2.2 (ii)), where $G_0$ is the identity component of $G$.

As compact groups are unimodular, we may use Lemma 2.5 and finish the proof of the corollary as in the proof of Theorem 1.6.

**4. Solvmanifolds**

We prove Theorem 1.1 first assuming $M = \tilde{M}$ is simply-connected.

**A canonical presentation - standard groups.** To study solvmanifolds, we first produce a preferred transitive solvable group of isometries which is in so-called standard position.

**Definition 4.1.** Let $G$ be a Lie group and $S$ a subgroup of $G$. We say that $S$ is in standard position in $G$ if, among Lie subgroups of $G$ containing $S$, the normalizer $N_G(S)$ is maximal with respect to the property that it contains no non-trivial noncompact semi-simple subgroups. When $G$ is understood, we will just say $S$ is in standard position.

Let $\mathcal{M} = (M, g)$ be a solvmanifold. There is a unique (up to conjugacy in the isometry group) solvable Lie group $S < \text{Isom}(\mathcal{M})_0$ which acts almost simply transitively and is in standard position in $\text{Isom}(\mathcal{M})_0$, see Section 1 and Definition 1.10 of [GW88].

The definition of $S$ being in standard position in $G$ is purely Lie theoretic, depending only on the embedding of $S$ in $G$. Consequently, by Lemma 2.1, we have the following.

**Lemma 4.2.** Let $\mathcal{M}$ be a solvmanifold and $\mathcal{M}_t$ a homogeneous solution to the Ricci flow. If $S$ is in standard position for $\mathcal{M}$, then it is in standard position for $\mathcal{M}_t$.

As we are assuming $\mathcal{M}$ is simply connected, $S$ acts simply transitively on $\mathcal{M}$ and so $\mathcal{M} \simeq \{S, g\}$ where $g$ is a left-invariant metric on $S$. The solvmanifold $\mathcal{M}_t$ can be written as $\{S, g_t\}$, where $g_t$ is a left-invariant metric on $S$. Now let $\mathcal{M}$ be a Ricci soliton with metric $g_0$ and consider the homogeneous solution $g_t$ to the Ricci flow given above: $g_t = (-2ct + 1)\varphi^*_s(t)g_0$ where
\[ s(t) = \frac{1}{c} \ln(-2ct + 1) \] and \( \varphi_s \in \text{Diff}(M) \) is generated by \( X \). We consider the dilution \( h_t = \frac{1}{-2ct + 1} g_t = \varphi_t^* g_0 \).

As dilations do not change the isometry group, \( S \) is in standard position for both \( g_0 \) and \( h_t \), and we may apply Theorem 5.2 of [GW88] to see that there exists a 1-parameter family \( \Psi_t \in \text{Aut}(S) \), each of which is an isometry \( \Psi_t : \{S, g_0\} \to \{S, h_t\} \); that is, the Ricci flow is evolving by automorphisms relative to \( S \).

**Remark.** A priori, one might be concerned about the smoothness of the family \( \Psi_t \). However, one can see that \( h_t \) is tangent (first at \( t = 0 \) and, hence, for all \( t \)) to \( \text{Aut}(S)^* g_0 \) and so \( \Psi_t \) may be chosen to be smooth.

Moreover, as in the general homogeneous setting (Proposition 2.2), after reparameterizing time we have \( d(\Psi_t)_e = \exp(tD) \in \text{Aut}(\mathfrak{s}) \) for some \( D \in \text{Der}(\mathfrak{s}) \).

Thus far, we have shown that a simply-connected Ricci soliton solvmanifold \( \mathcal{M} \) is isometric to \( \{S, g\} \) (where \( S \) is in standard position) and the Ricci flow evolves by automorphisms, i.e.

\[
(4.1) \quad \text{Ric}_g = c \text{Id} + \frac{1}{2}(D + D^t)
\]

where \( \text{Ric}_g \) is the (1,1)-Ricci tensor of the left-invariant metric \( g \) on \( S \). Here we have evaluated the tensor on the tangent space \( T_e S \simeq \mathfrak{s} \).

**Evolving by automorphisms implies algebraic.** The second part of the proof of Theorem 1.1 is to show that if the Ricci flow evolves by automorphisms, then the soliton is actually algebraic.

**Lemma 4.3.** Let \( \{S, g\} \) be a solvable Lie group with left-invariant metric. If \( \text{Ric}_g \) satisfies Eqn. 4.1 then

\[
\frac{1}{2}(D + D^t) \in \text{Der}(\mathfrak{s})
\]

and hence \( \{S, g\} \) is an algebraic Ricci soliton.

The proof of this lemma amounts to a careful reading of [Lau11c]; while we do not provide full details, we make note of what modifications need to be done to that work.

We divide the proof into two cases. Given \( D \in \text{Der}(\mathfrak{s}) \), denote by \( S(D) = \frac{1}{2}(D + D^t) \) the symmetric part of \( D \), relative to \( g \).

**Case \( c \geq 0 \).** The proof of this case follows the proof of [Lau11a, Prop. 4.6]. The analysis there with \( \text{Ric} = c \text{Id} + D \) works with \( \text{Ric} = c \text{Id} + S(D) \) upon noting that \( D(\mathfrak{s}) \subset \mathfrak{n} \), the nilradical of \( \mathfrak{s} \), for any derivation \( D \in \text{Der}(\mathfrak{s}) \).
The conclusion of [Lau11c, Prop. 4.6] is that the manifold is Ricci flat, hence one may take $c = 0$ and $D = 0$.

**Remark.** By a result of Alekseevsky-Kimel’fel’d [AK75], the solvmanifold $(S, g)$ must be flat and hence $S$ is abelian (being in standard position).

**Case $c < 0$.** The proof of this case follows the proof of [Lau11c, Thm. 4.8]. The work there remains valid with the hypothesis $Ric = cId + D$ replaced with $Ric = cId + S(D)$. The only modification that the proof needs is the term $F$ should now be $S(ad H + D)$ instead of $S(ad H) + D$. Consequently, [Lau11c, Thm 4.8] now shows that $S(D) \in Der(s)$.

So far we have shown that any simply-connected Ricci soliton solvmanifold is isometric to an algebraic Ricci soliton relative to some, possibly different, solvable Lie structure. The fact that $S$ may be chosen to be completely solvable is the content of [Lau11c, Cor. 4.10]. We note that completely solvable groups are in standard position, see [GW88, Thm. 4.3]. This proves Theorem 1.1 in the case that $\mathcal{M}$ is simply-connected.

### 5. Simple Connectivity

Next we show that any Ricci soliton solvmanifold is simply-connected, unless it is flat. Let $\mathcal{M}$ be a non-flat solvmanifold with simply-connected cover $\tilde{\mathcal{M}}$.

Assume that $\mathcal{M}$ is a Ricci soliton. As the projection $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ is a local isometry, $\tilde{\mathcal{M}}$ is also a Ricci soliton. The Ricci soliton on $\mathcal{M}$ corresponds to the family of diffeomorphisms $\tilde{\varphi}_t$ which satisfy the following commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{M}} & \xrightarrow{\tilde{\varphi}_t} & \tilde{\mathcal{M}} \\
\downarrow\pi & & \downarrow\pi \\
\mathcal{M} & \xrightarrow{\varphi_t} & \mathcal{M}
\end{array}
\]

where $\tilde{\varphi}_t$ is the lift of $\varphi_t$. Moreover, $\mathcal{M}$ is homogeneous.

**Proposition 5.1.** The degree of the cover is 1, hence $\mathcal{M}$ is simply-connected.

As the proposition suggests, we will show that $\mathcal{M}$ is simply-connected by showing that the $\pi^{-1}(p)$ consists of a single point for one, hence any, point $p \in \mathcal{M}$. A more technical version of this proposition is Prop. 5.4.

**The soliton diffeomorphisms.** By definition, $\mathcal{M}$ admits a transitive solvable group of isometries and, in fact, it can be realized as $\mathcal{M} \simeq R/C$ where $R$ is a simply-connected solvable group acting almost simply transitively and $C$ is a discrete subgroup of $\{x \in R \mid Ad(x) \in O(\tau, (\cdot, \cdot)')\}$, where $(\cdot, \cdot)'$ is the induced inner product on $\tau \simeq T_eR$ (see [GW88, Lemma 1.2]). The group $R$ acts simply transitively on $\mathcal{M} \simeq R$. 
Lemma 5.2. Fix any point \( p \in \mathcal{M} \). Let \( \varphi_t \) be the 1-parameter group generated by the vector field \( X \) appearing in Eqn. (1.1). We may replace \( \varphi_t \) with a family which fixes \( p \). Moreover, this new family of diffeomorphisms exists for all \( t \).

Proof. The first claim follows from the transitivity of \( R \) on \( \mathcal{M} \). Take \( r(\varphi_t(p)) \in R \) which maps to \( \varphi_t(p) \in R/C \) under the usual quotient \( R \rightarrow R/C \). As \( C \) is discrete, \( R/C \) is locally diffeomorphic to \( R \) and this choice can be made smoothly in \( t \). The translation \( L_{r(\varphi_t(p))}^{-1} \) is an isometry of \( R/C \) by Lemma 2.1. Upon composing \( \varphi_t \) with the isometry \( L_{r(\varphi_t(p))}^{-1} \) we have that \( p \) is fixed. (Note, this new family yields a solution of the Ricci flow (Eqn. (1.2)) as Ricci flow is invariant under isometries.)

The second claim follows immediately from the definition of the vector field \( X \) being complete, i.e., the family of diffeomorphisms \( \tilde{\varphi}_t \) exists for all time. Composing \( \varphi_t \) with the time dependent translations \( L_{r(\varphi_t(p))}^{-1} \) does not change the time interval on which the family exists.

We pick \( p = [C] \in \mathcal{M} \). As \( \varphi_t \) fixes \( p \), \( \tilde{\varphi}_t \) stabilizes \( \pi^{-1}(p) \). This set is discrete and, since \( \tilde{\varphi}_0 \) is the identity map, we see that \( \tilde{\varphi}_t(q) = q \) for all \( q \in \pi^{-1}(p) \). The manifold \( \tilde{\mathcal{M}} \) is isometric to \( R \), with some left-invariant metric, and we fix \( \tilde{p} \in \pi^{-1}(p) \) which corresponds to the identity element \( e_R \in R \).

The manifold \( \tilde{\mathcal{M}} \) is also acted upon by a group \( S \) in so-called standard position, as in Section 4 and is isometric to \( S \) endowed with a left-invariant metric. We will identify \( \tilde{p} \) with the identity \( e_S \in S \) and we denote the induced inner product on \( \mathfrak{s} \) by \( \langle \cdot , \cdot \rangle \).

As shown in Section 4 there exists a 1-parameter group of automorphisms \( \Psi_s \subset \text{Aut}(S) \) such that \( \tilde{\varphi}_{s(t)}^*g = \Psi_s^*g \) where \( s(t) = \frac{1}{c} \ln(-2ct + 1) \). This equality holds true as long as our solutions to the Ricci flow exist, which is on the interval \( (\frac{1}{2c}, \infty) \).

Lemma 5.3. Given \( \tilde{\varphi}_s \) and \( \Psi_s \) as above, \( \tilde{\varphi}_s^*g = \Psi_s^*g \) for all \( s \in \mathbb{R} \). Consequently, there exists a family of isometries \( I_s \) defined for all \( s \) such that \( \tilde{\varphi}_s = \Psi_s \circ I_s \). Moreover, \( I_s \) is contained in the isotropy at \( \tilde{p} \), which is compact.

Proof. As \( t \) ranges over \( (\frac{1}{2c}, \infty) \), \( s(t) = \frac{1}{c} \ln(-2ct + 1) \) ranges over all real numbers. This proves the first claim.

The equality \( \tilde{\varphi}_s^*g = \Psi_s^*g \) implies that \( I_s = (\Psi_s)^{-1}\tilde{\varphi}_s \) is an isometry. Moreover, since \( \tilde{\varphi}_s \) and \( \Psi_s \) both fix \( \tilde{p} = e_S \), we see that \( I_s \) is contained in the isotropy at \( \tilde{p} \), which is compact.

The preimage \( \pi^{-1}(p) \subset \tilde{\mathcal{M}} \) as an orbit. As \( R \) acts transitively on \( \tilde{\mathcal{M}} \), it is a ‘modification’ of a group in standard position, which we may choose to be \( S \) (see GW88 Thm. 3.1). In the notation of GW88, the Lie algebra \( \mathfrak{r} \subset \mathfrak{k} + \mathfrak{s} \), where \( \mathfrak{k} = N_l(\mathfrak{s}) \) is the set of skew-symmetric derivations of \( \mathfrak{s} \), relative to \( \langle \cdot , \cdot \rangle \). As \( R \) is connected, it is a subgroup of \( K \ltimes S \), where
$K = N_L(S)$ is the set of orthogonal automorphisms of $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$. (Here we are making the usual identification between $\text{Aut}(S)$ and $\text{Aut}(\mathfrak{s})$: $\Phi \in \text{Aut}(S)$ corresponds to $d\Phi_e \in \text{Aut}(\mathfrak{s})$.)

The group $R$ acts on $S$ by isometries via the usual action of $K \ltimes S$ on $S$, and we can think of the manifold $\tilde{M}$ as either $R$ or $S$ with the appropriate left-invariant metric. Recall, $e_R = \tilde{p} = e_S$.

To show that $\mathcal{M}$ is simply-connected, we will show that the number of elements in $\pi^{-1}(p) \subset \tilde{M}$ is one. As we have chosen $p = [C] \in R/C$, $\pi^{-1}(p) = C \subset R$. The set $C$ in $R$ is the orbit $C \cdot e_R = C \cdot \tilde{p}$ which on $S$ is the set $C \cdot \tilde{p} = C \cdot e_S$, here $C$ is acting on $S$ as a subgroup of $K \ltimes S$.

**Proposition 5.4.** $C \cdot e_S = e_S$. Consequently, the number of elements in $\pi^{-1}(p)$ is one and $\mathcal{M}$ is simply-connected.

To prove the proposition, we must carefully analyze the relationship between the two different homogeneous structures on $\tilde{M}$ coming from $R$ and $S$.

Recall that the Lie group exponential $\exp : \mathfrak{s} \to S$ is a diffeomorphism since $\mathfrak{s}$ is completely solvable. As such, it has an inverse, $\log$, which is also a diffeomorphism. (Unless otherwise stated, $\exp$ and $\log$ refer only to those for $S$.) We study the condition that $\tilde{\varphi}_t = \Psi_t \circ I_t$ fixes every element $c \cdot \tilde{p} \in \pi^{-1}(p)$, where $c \in C < R$.

**Lemma 5.5.** For $c \in C$, $\log(c \cdot \tilde{p}) \in \ker D$, where $D$ is the derivation of $\mathfrak{s}$ which generates the family of automorphisms $d(\Psi_t)_e = \exp(tD)$ of $\mathfrak{s}$.

**Proof.** By taking the log, the statement that $\tilde{\varphi}_t = \Psi_t \circ I_t$ fixes $c \cdot \tilde{p} \in \pi^{-1}(p) \subset S$ is equivalent to $\exp(-tD)(\log(c \cdot \tilde{p})) = \log(I_t(c \cdot \tilde{p})) \in \mathfrak{s}$.

As $I_t$ is contained in the compact isotropy group at $\tilde{p}$, $\log(I_t(c \cdot \tilde{p}))$ is contained in a compact set of $\mathfrak{s}$. Using Lemma 5.3, we may let $t \to \pm \infty$. As $D$ is symmetric, $\exp(-tD)(\log(c \cdot \tilde{p}))$ is contained in a compact set as $t \to \pm \infty$ if and only if $\log(c \cdot \tilde{p}) \in \ker D$. \qed

**Remark.** From this point forward, $D$ will only refer to the derivation above.

**Decomposing $\mathfrak{s}$ and the kernel of $D$.** Decompose the completely solvable algebra $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$ and $\mathfrak{a} = \mathfrak{n}^\perp$, relative to the solvsoliton metric $\langle \cdot, \cdot \rangle$. The set $\mathfrak{a}$ is abelian and $\text{ad} A : \mathfrak{n} \to \mathfrak{n}$ is symmetric for $A \in \mathfrak{a}$, see [Lau11c, Thm. 4.8].

Consider the symmetric derivation $D$ which generates the family $\psi_t = \exp(tD) \subset \text{Aut}(\mathfrak{s})$. Using the above decomposition, we have $D|_{\mathfrak{a}} = 0$ and $D|_{\mathfrak{n}} = D_1 - \text{ad} H$ where $H$ is the ‘mean curvature vector’ satisfying $\langle H, X \rangle = \text{tr} \text{ad} X$, for all $X \in \mathfrak{s}$, and $D_1$ is the pre-Einstein derivation of $\mathfrak{n}$ (see [Lau11c, Prop. 4.3]). As the kernel of this derivation is important, we study it more closely.

**Lemma 5.6.** $\ker D \subset \mathfrak{a} + \text{Im}(\text{ad} H)$, where $\text{Im}(\text{ad} H)$ denotes the image of $\text{ad} H : \mathfrak{s} \to \mathfrak{n}$. 


Proof. As $a \subset \text{Ker } D$, it suffices to understand $\text{Ker } D \cap n$. For $X \in \text{Ker } D \cap n$, we have

$$D_1X = \text{ad } H(X)$$

The derivation $D_1$ is positive definite on $n$ ([Nik11, Thm. 1]) and all symmetric and skew-symmetric derivations of $n$ commute with $D_1$ ([Jab11, Lemma 5.2]). The derivation $\text{ad } H \in \text{Der}(n)$ is a symmetric derivation and hence commutes with $D_1$. Decomposing $X = \sum X_\lambda$ into a sum of eigenvectors of $D_1$, we see that $\text{ad } H(X_\lambda) = \lambda X_\lambda$, as well.

Letting $Y = \sum \frac{X}{\lambda}X_\lambda$, we see that $X = \text{ad } H(Y)$. \hfill \Box

We now write $s = a \oplus n = a \oplus n_1 \oplus n_0$, where $n_1 = \text{Im}(\text{ad } H)$ and $n_0 = n \cap \text{Ker } \text{ad } H$. Define $s_1 = a \oplus n_1 \subset s$.

**Lemma 5.7.** Using the notation above, we have

(i) For $E \in k$ (a skew-symmetric derivation of $s$) and $A \in a$,

$$E \circ \text{ad } A = \text{ad } A \circ E$$

(ii) $n_1$ is an ideal, $s_1$ is a subalgebra, and both are stable under $k$, the set of skew-symmetric derivations.

Proof. We prove (i). This is a standard result and follows from the fact that any derivation $E$ sends $s$ to $n$; consequently, derivations preserve $n$. If $E$ is skew-symmetric, it must also preserve $a = n^\perp$ and hence vanish on $a$. The above equation is now just the Jacobi identity.

We prove (ii). Let $V_\lambda$ denote the $\lambda$-eigenspace of $\text{ad } H$ and observe $\text{Ker } \text{ad } H = a \oplus n_0$. Observe that $[V_\lambda, V_\mu] \subset V_{\lambda+\mu}$. As $n_1 = \sum_{\lambda \neq 0} V_\lambda$, it is clearly an ideal. Clearly $s_1$ is now a subalgebra.

To see that $n_1$ is stable under $k$, we use the fact that a skew-symmetric derivation of $s$ commutes with $\text{ad } H$ since $H \in a$ (by part (i)) and hence preserves the eigenspaces of $\text{ad } H$. Since skew-symmetric derivations vanish on $a$, $s_1$ is also stable under $k$. \hfill \Box

**Analysis on the modification $R$ of $S$.** As stated before, the group $R$ is a so-called modification of $S$, by a main result of Gordon-Wilson [GW88, Thm. 3.1]. This means that the Lie algebra $\tau$ of $R$ is obtained from the Lie algebra $s$ of $S$ from a map $\phi : s \to k = sl(n, k) \cap \text{Der}(s)$ (called a modification map) and $\tau = (\text{Id} + \phi)s$, see Section 2 of [GW88]. There are minimal conditions on $\phi$, only enough to insure that $R$ will be solvable and act almost simply-transitively.

One condition on $\phi(s)$ worth noting is that this set of skew-symmetric derivations is abelian, see [GW88, Prop. 2.4]. Consequently, $\tau < s \rtimes t$, where $t < k$ is a set of commuting skew-symmetric derivations of $s$. This implies $R < S \rtimes T$ where $T$ is an abelian group of orthogonal automorphisms of $S$.

**Lemma 5.8.** Let $\phi$ be a modification map of $s$ with $\tau = (\text{Id} + \phi)s$.

(i) For $X \in s_1$, $\phi(X)|_{s_1}$ is a derivation of $s_1$.

(ii) $n_1 = [s_1, s_1] \subset \text{Ker } \phi$
iii) Let $H$ be the mean curvature vector of $\mathfrak{s}$ with image $\tilde{H} \in \mathfrak{r}$, then $\text{ad} \tilde{H} = \text{ad} H + \phi(H)$ is non-singular on $\mathfrak{n}_1 \subset \mathfrak{r}$.

Proof. (i) is true by the second claim of the lemma above.

To prove (ii), consider Theorem 2.7 of [GW88] which states that any modification can be realized as two successive normal modifications performed one after the other. (A normal modification of $\mathfrak{s}$ is one such that $[\mathfrak{s}, \mathfrak{s}] \subset \text{Ker} \text{ad} \phi$.)

Our first step is to perform a normal modification by $\mathfrak{a}$. Observe that $\mathfrak{n}_1 = [\mathfrak{s}_1, \mathfrak{s}_1]$ since $\text{ad} H$ is non-singular on $\mathfrak{n}_1$ by construction. Thus $\phi(X) = 0$ for $X \in \mathfrak{n}_1$. The image of $\mathfrak{s}_1$ under the modification will be denoted $\mathfrak{s}_1 = \mathfrak{n}_1 + \mathfrak{a}$.

Observe that $\mathfrak{n}_1 = [\mathfrak{s}_1, \mathfrak{s}_1]$. To see this, consider $H + \phi(H) \in \mathfrak{a}$. The derivations $\text{ad} \ H$ and $\phi(H)$ are symmetric and skew-symmetric, respectively, and they commute (Lemma 5.7). As such, the kernel of $\text{ad} H + \phi(H)$ is contained in $\text{Ker} \text{ad} H \cap \text{Ker} \phi(H)$. But $\text{ad} H$ is non-singular on $\mathfrak{n}_1$ and, hence, $\text{ad} \tilde{H}$ is non-singular on $\mathfrak{n}_1$ which implies $[\tilde{H}, \mathfrak{n}_1] = \mathfrak{n}_1$.

Applying the second normal modification to $\mathfrak{a}$ still leaves $\mathfrak{n}_1 = [\mathfrak{s}_1, \mathfrak{s}_1]$ fixed, which proves (ii).

The proof of (iii) appears in the proof of (ii) above. \qed

In the following subsection, we study $C$ more closely. Thus far, we have shown the following. Write $c \in C$ as $c = s \alpha$ where $\alpha \in K$, an orthogonal automorphism, and $s \in S$. Then $c \cdot e_S = s$, as $\alpha$ fixes $e_S$. However, $\log(c \cdot e_S) = \log(s) \in \text{Ker} \text{D} \subset \mathfrak{s}_1$ (Lemmas 5.5 & 5.6). Hence $s \in S_1$, the subgroup of $S$ with Lie algebra $\mathfrak{s}_1$. As the algebra $\mathfrak{s}_1$ is preserved by $K$ (Lemma 5.7), we see that $C$ is a subgroup of $T \times S_1$. Recall, $T$ is an abelian subgroup of $K$, the orthogonal automorphisms of $\mathfrak{s}$.

The two homogeneous structures on $\tilde{M}$. Recall, the solvmanifold $\tilde{M}$ is isometric to $S$ endowed with a left-invariant metric. This left-invariant metric is completely determined by the induced inner product $\langle \ , \ \rangle$ on $\mathfrak{s} = T_{e_S} S$. To study the geometry of the quotient $M$, we are required to understand $\tilde{M}$ with a different homogeneous structure $R$. This group $R$ is a subgroup of $K \times S$ and acts isometrically on $S$. This endows $R$ with a left-invariant metric which is denoted on $\mathfrak{r} \simeq T_{e_R} R$ by $\langle \ , \ \rangle'$. The inner products on $\mathfrak{r} \simeq T_{e_R} \tilde{M} \simeq \mathfrak{s}$ are related as follows

$$\langle X + \phi(X), X + \phi(X) \rangle' = \langle X, X \rangle$$

for $X \in \mathfrak{s}$

where $\phi : \mathfrak{s} \rightarrow \mathfrak{r} = \text{Der}(\mathfrak{s}) \cap \text{so}(\langle \ , \ \rangle)$ is the modification map with $\mathfrak{r} = \langle Id + \phi \rangle \mathfrak{s}$. This can also be stated as $\langle Y, Y \rangle' = \langle Y_s, Y_s \rangle$ for $Y \in \mathfrak{r}$, where $Y_s$ is the $\mathfrak{s}$-component of $Y \in \mathfrak{r} + \mathfrak{s}$. For more details, see [GW88] Def. 3.4.

We are now in a position to investigate the condition $\text{Ad}(c) \in O(\mathfrak{r}, \langle \ , \ \rangle')$ for $c \in C$. Given $c \in C$, write $c = nak$ where $n \in N_1 = \text{exp}(\mathfrak{n}_1)$, $a \in \text{exp}(\mathfrak{a})$, and $k \in T < K$. (This decomposition is possible, since $S_1 < S$ is completely solvable and $\mathfrak{s}_1 = \mathfrak{n}_1 \oplus \mathfrak{a}$.)
Lemma 5.9. Given the presentation $c = n a k$, as above, we have $n = e$.

To prove the lemma, we need the following technical result.

Sublemma 5.10. Let $X \in a$ and $c \in C$. Writing $c = n a k$, as above, we have

$$Ad(n)(X + \phi(X)) = X + \phi(X)$$

Proof. By Lemma 5.7 and the fact that $a$ is abelian, we see that $Ad(c)X = Ad(n)Ad(a)Ad(k)X = Ad(n)X$, for $X \in a$. Again, using that $a$ and $\phi(s)$ commute, together with $\phi(s)$ being abelian, we see that $Ad(c)\phi(X) = Ad(n)Ad(a)Ad(k)\phi(X) = Ad(n)\phi(X)$.

Recall that $n_1$ is stable under $K$ (Lemma 5.7) and so $t \times n_1$ is a subalgebra of $t \times s$. As such, we see that $Ad(n)\phi(X) \in t \times n_1$. Write this element as a sum of its components $Ad(n)\phi(X) = (Ad(n)\phi(X))_t + (Ad(n)\phi(X))_{n_1}$.

Next we write $Ad(n)X = X + (Ad(n)X)_{n_1}$, where the second term is in $n_1$. To see that this decomposition is accurate, write $Ad(n) = exp(ad \log n)$ and recall that $n_1 = [s_1, s_1]$.

As $r$ is a subalgebra of $t \times s$, we have $Ad(c)(X + \phi(X)) \in r$. This element can be written as $Y + \phi(Y)$ for some $Y \in s$. The above work shows that $Y = X + (Ad(n)X)_{n_1} + (Ad(n)\phi(X))_{n_1}$ and so $\phi(Y) = \phi(X)$ by Lemma 5.8 i.e.,

$$Ad(n)\phi(X)_t = \phi(X)$$

We now exploit the condition $Ad(c) \in O(\tau, \langle, \rangle')$. Using the definition of $\langle, \rangle'$ preceding Lemma 5.9 we have

$$|X + (Ad(n)X)_{n_1} + (Ad(n)\phi(X))_{n_1}| = |Ad(c)(X + \phi(X))'| = |X + \phi(X)|'$$

As $n_1$ and $a$ are orthogonal relative to $\langle, \rangle$ on $s$, we see that $(Ad(n)X)_{n_1} + (Ad(n)\phi(X))_{n_1} = 0$. This implies

$$X + (Ad(n)X)_{n_1} + (Ad(n)\phi(X))_{n_1} + \phi(X) = X + \phi(X)$$

But $(Ad(n)\phi(X))_t = \phi(X)$, see Eqn. 5.1 which implies the left-hand side of Eqn. 5.2 is precisely $Ad(n)(X + \phi(X))$.

\[\square\]

Proof of Lemma 5.7. We apply the sublemma to $X = H$ and use the fact that $ad \tilde{H}$ is non-singular, where $\tilde{H} = H + \phi(H)$ (see Lemma 5.8).

The previous sublemma says that $Ad(n)\tilde{H} = \tilde{H}$. Upon exponentiating, we have $n \exp(t\tilde{H}) n^{-1} = \exp(t\tilde{H})$, where $exp$ denotes the exponential map of $r$. This equality holds for small $t$ and implies

$$n = \exp(t\tilde{H}) n \exp(t\tilde{H})^{-1}$$
The algebra $n_1$ is also contained in $s$, and so the exponential map of $N_1$, $exp_{N_1} : n_1 \rightarrow N_1$, is a diffeomorphism. Applying $\log_{N_1}$ to the previous equality we obtain

$$\log n = Ad(exp_t\tilde{H})\log n$$

Differentiating at $t = 0$ yields $0 = ad\tilde{H}(\log n)$. The map $ad\tilde{H} : n_1 \rightarrow n_1$ being non-singular (Lemma 5.8) now implies $\log n = 0$ and hence $n = e$. $\Box$

Thus far, we have shown $C < K exp(a)$. The proof of Proposition 5.4 will be complete upon proving one more technical lemma.

Lemma 5.11. Given $c \in C$, write $c = ak = ka$ as above; i.e., $k \in T < K$ and $a \in exp(a)$. Then $Ad(c) \in O(s, ( , ))$ and hence $Ad(a) \in O(s, ( , ))$.

Proof. The second claim follows immediately from the first since $k \in K$ is an orthogonal automorphism.

To prove the first claim, take $X \in s$ corresponding to $X + \phi(X) \in \tau$. Observe that $Ad(c)X \in s$ and $Ad(c)\phi(X) = \phi(X)$, as in the proof of the sublemma above. The condition $Ad(c) \in O(\tau, ( , )')$ gives

$$|X| = |X + \phi(X)|' = |Ad(c)(X + \phi(X))|' = |Ad(c)X|$$

As $X$ can be any element of $s$, the first claim is proven. $\square$

We now complete the proof of Proposition 5.4. Since $s$ is completely solvable, the map $Ad(a) : s \rightarrow s$ has only positive eigenvalues. However, for $a$ appearing in $c = ka$, $Ad(a)$ is orthogonal and hence must be the identity map. This says precisely that $a \in Z(S)$, the center of $S$. However, $Z(S) < N$, the nilradical of $S$, and hence $a \in N \cap exp(a) = \{e\}$. This shows that $C$ is a subgroup of $K$ and hence fixes $e_S \in S$.

6. Existence Questions

The main result of this work shows that every Ricci soliton solvmanifold is isometric to a completely solvable Lie group with solvsoliton metric.

Question 6.1. Which solvable Lie groups admit left-invariant Ricci soliton metrics? Does there exist an algebro-geometric criterion which tests for the existence of such a metric?

The main result of [Jab11] shows that the existence of a solvsoliton metric on a solvable Lie group may be determined by measuring algebraic invariants of the underlying Lie algebra and infinitesimal deformations of any initial left-invariant metric. In this sense, the existence of a solvsoliton metric becomes a local question. It is not clear that this is the case for general setting of Ricci soliton metrics on solvable groups. An analogous result in the general solvable Ricci soliton setting would be very interesting.
Structural results. In [Lau11c], it is shown that there are many algebraic restrictions on the Lie algebra $\mathfrak{s}$ of a Lie group $S$ admitting a solvsoliton metric. For example, decompose $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$ and $\mathfrak{a} = \mathfrak{n}^\perp$ (relative to the soliton metric). Then $\mathfrak{a}$ is abelian and $ad : \mathfrak{a} \to \mathfrak{n}$, for $A \in \mathfrak{a}$, consists of reductive endomorphisms. It is not clear what kind of constraints must hold for a solvable Lie group with Ricci soliton metric.

Example 6.2. There exists a solvable group with Ricci soliton metric with the property that $ad_{\mathfrak{a}}$ does not consist of reductive endomorphisms.

Remark 6.3. One can also build examples where the orthogonal complement of the nilradical is no longer abelian.

Proof of 6.2. Consider the 5-dimensional Heisenberg Lie algebra $\mathfrak{n}$ with orthonormal basis $\{X_1, Y_1, X_2, Y_2, Z\}$. The only non-trivial bracket relations are $[X_1, Y_1] = [X_2, Y_2] = Z$, and the obligatory relations from anti-symmetry of the bracket. Let $N$ denote the simply-connected nilpotent group with said Lie algebra.

There exists a modification $\mathfrak{r}$ of $\mathfrak{n}$ which is solvable, has 4-dimensional nilradical $\mathfrak{n}(\mathfrak{r})$, but $\mathfrak{n}(\mathfrak{r})^\perp$ is not $ad$-reductive. The orthonormal basis of $\mathfrak{r}$ is $\{\tilde{X}_1, Y_1, X_2, Y_2, Z\}$ where $\tilde{X}_1 = X_1 + \phi(X_1)$ and $\phi(X_1)$ acts on $\mathfrak{s}$ via $\phi(X_1) : \{X_1, Y_1, Z\} \to 0$, $\phi(X_1)X_2 = Y_2$, and $\phi(X_1)Y_2 = -X_2$. Clearly, $ad_{\tilde{X}_1}$ is not diagonalizable over $\mathbb{C}$.

Let $R$ denote the simply-connected Lie group with Lie algebra $\mathfrak{r}$. As $\mathfrak{r}$ is a modification of $\mathfrak{n}$, $R$ acts isometrically on $N$. Now $R$ has a Ricci soliton metric and its Lie algebra has the desired properties. \[\square\]

Remark. The above phenomena are special to Ricci soliton solvmanifolds and cannot happen for Einstein solvmanifolds by the structural results of Lauret, see [Lau10a].

Question 6.4. What is the relationship between solvable Ricci solitons and Geometric Invariant Theory?

The strong structural results of Lauret for solvsoliton metrics are largely due to a close relationship with Geometric Invariant Theory (GIT), i.e., the existence of such metrics is intimately related to the existence of so-called distinguished orbits in the space of Lie brackets, see [Lau11c] and [Jab11].

It is not clear how Ricci soliton metrics on solvable Lie groups are related to GIT. If there were a similar relationship, it would be very interesting and useful.

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