Power function inflation potential analysis for cosmological model with Gauss-Bonnet term

Getbogi Hikmawan\textsuperscript{1}, Agus Suroso\textsuperscript{1,2} and Freddy P Zen\textsuperscript{1,2}

\textsuperscript{1} Theoretical Physics Laboratory, THEPI Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung 40132, Indonesia
\textsuperscript{2} Indonesia Center of Theoretical and Mathematical Physics (ICTMP), Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung 40132, Indonesia
E-mail: getbogihikmawan@itb.ac.id

Abstract. Inflation is still an interesting topic in the study of our universe. An interesting inflation scenario named Gauss-Bonnet inflation proposed without inflation potential has been shown unstable. In this work, we consider the general power function inflation potential, $V(\phi) = m\phi^n$ in the model, then the solution is analyzed according to the inflation scenario. Using the stability condition, inflation potential with $m$ positive and $0 \leq n < 5$ give proper solution for the inflation scenario.

1. Introduction
Inflationary universe scenario at early time by Guth \cite{Guth} in 1981 has become a possible solution to the horizon and flatness problems in cosmology. There are a lot of ways developed to explain these phenomena, such as slow-roll inflation model \cite{Guth}, Dirac-Born-Infeld (DBI) model \cite{DBI}, ghost inflation model \cite{Ghost}, effective field theory \cite{Effective} and many more. However, those models are still speculative because only few observations show that inflation happens.

Until recently, cosmological models with scalar field have given good solutions for some problems in cosmology, such as for late-time acceleration \cite{Late, Late2}, dark matter and dark energy \cite{Dark, Dark2}, and inflation \cite{Inflation, Inflation2}. Kanti \textit{et al} \cite{Kanti1, Kanti2} analyzed the cosmological model with nonminimal coupling between scalar field $\phi$ and Gauss-Bonnet term (GB term), or Einstein-scalar-Gauss-Bonnet theory, without inflation potential and found that the model gives inflationary solution. However, Hikmawan \textit{et al} \cite{Hikmawan} show that this model is unstable for tensor perturbation modes. In four-dimensional cosmology, GB term arises as a one-loop string correction \cite{String1, String2}. Based on this fact, Einstein-scalar-Gauss-Bonnet theory is studied, and it leads to the nonsingular cosmological solutions \cite{Nonsingular1, Nonsingular2}. The solutions have superinflation phase but generally unstable because of the violation of the weak energy condition \cite{Weak1, Weak2, Weak3, Weak4}.

The paper is organized as follows. In Section 2, we review the result of Kanti \textit{et al} \cite{Kanti1, Kanti2} and Hikmawan \textit{et al} \cite{Hikmawan} of the Einstein-scalar-Gauss-Bonnet theory and the stability condition required. In Section 3, we include the power function inflation potential, $V(\phi) = m\phi^n$, in the model and numerically solve the field equations for some cases based on the choice of $m$ and $n$ variables. The final section is for conclusion and discussion.
2. Stability condition of cosmological model with Gauss-Bonnet term

In this section, we review the main result of Kanti et al [11, 12] and the instability analysis for this model [13]. In particular, we point out the stability condition required for this model to give stable inflation solution.

The action considered in this model is a non-minimally coupling between Gauss-Bonnet term (GB term), $R^2_{\text{GB}}$ and a scalar field $\phi$ via function coupling $f(\phi)$,

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} (\nabla \phi)^2 + \frac{1}{8} f(\phi) R^2_{\text{GB}} \right],$$  

where $\kappa^2 \equiv 8\pi G$ is the gravitational coupling constant, $g$ is determinant of the metric $g_{\mu\nu}$ and $R$ is the Ricci scalar. GB term is given by

$$R^2_{\text{GB}} = R^\mu_\nu^\rho_\lambda R^\nu_\mu_\rho_\lambda - 4 R^\mu_\nu R^\nu_\mu + R^2.$$  

The variation of this action with respect to the scalar field and the homogeneous and isotropic flat spacetime metric,

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,$$

and assuming further that the scalar field depends solely on the time coordinate, $\phi = \phi(t)$, give explicit field equations for this model,

$$\ddot{\phi} + 3H \dot{\phi} - 3 \frac{\partial f}{\partial \phi} H^2 (H^2 + \dot{H}) = 0,$$

$$6H^2 (1 + H \dot{f}) = \dot{\phi}^2,$$

$$2(1 + H \dot{f})(H^2 + \dot{H}) + H^2 (1 + \dot{f}) = -\frac{1}{2} \dot{\phi}^2,$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and the dot denotes a derivative with respect to time. Kanti et al [11, 12] found that the quadratic function $f = \lambda \phi^2$, with $\lambda$ being a constant, gives rise to the inflationary solutions if $\lambda < 0$ and takes very large value. As the result, the GB term will dominate the Ricci term so that the latter can be neglected.

Because of the existence of ghost, this model is unstable under the tensor perturbation [13]. However, there is a possible way to find a stable cosmological model with the GB term, that is, when the inflation potential $V(\phi)$ is included in the action function:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} (\nabla \phi)^2 - V(\phi) + \frac{1}{8} f(\phi) R^2_{\text{GB}} \right].$$

Therefore, the field equations can be obtained as

$$\ddot{\phi} + 3H \dot{\phi} - 3 f' H^2 (H^2 + \dot{H}) + \frac{dV(\phi)}{d\phi} = 0,$$

$$6H^2 (1 + H \dot{f}) = \dot{\phi}^2 + 2V(\phi),$$

$$2(1 + H \dot{f})(H^2 + \dot{H}) + H^2 (1 + \dot{f}) = -\frac{1}{2} \dot{\phi}^2 + V(\phi).$$

As indicated in [13], the stability condition for cosmological model with GB term is given by

$$Q \equiv \frac{V(\phi)}{4H^2} - \frac{1}{8} \left( \frac{\dot{H}}{H^2} + 5 \right)(1 + H \dot{f}) > 0.$$  

Therefore, one can find a viable model for cosmological model with GB term by choosing the inflation potential $V(\phi)$. 


3. Power function inflation potential analysis

Now, let us consider $f(\phi) = \lambda \phi^2$, as in [11, 12], and power function potential

$$V(\phi) = m\phi^n,$$

with $n$ constant integer and $m$ arbitrary constant; thus (8)-(10) can be written as

$$\ddot{\phi} + 3H\dot{\phi} - 6\lambda \phi H^2 (H^2 + \dot{H}) + mn\phi^{n-1} = 0$$

with

$$6H^2 (1 + 2\lambda \phi \dot{\phi}) = \dot{\phi}^2 + 2m\phi^n,$$  \hspace{2cm} (14)

and

$$2(1 + 2\lambda H \phi \dot{\phi})(H^2 + \dot{H}) + H^2 (1 + 2\lambda \phi^2 + 2\lambda \ddot{\phi}) = -\frac{1}{2} \dot{\phi}^2 + m\phi^n.$$  \hspace{2cm} (15)

If we combine (14) and (15), we get

$$(5H^2 + 2\dot{H})(1 + 2\lambda H \phi \dot{\phi}) + H^2 (1 + 2\lambda \phi^2 + 2\lambda \ddot{\phi}) = 2m\phi^n.$$  \hspace{2cm} (16)

The stability condition (11) can be written as

$$Q_m \equiv m\phi^n [12 - 2\left(\frac{2\ddot{H}}{H^2} + 5\right)] - \left(\frac{2\ddot{H}}{H^2} + 5\right)\dot{\phi}^2 > 0.$$  \hspace{2cm} (17)

We consider two cases from this problem, i.e., $m = -1$ and $m = 1$. After that we do a numerical calculation for each $n$ value until we find the value that makes the model stable. In this numerical calculation, we consider $\lambda = -10^{15}$ to make sure that the GB term is dominant enough.

3.1. $m = -1$ case

If $m = -1$, (13) and (16) can be written as

$$\ddot{\phi} + 3H\dot{\phi} - 6\lambda \phi H^2 (H^2 + \dot{H}) - n\phi^{n-1} = 0,$$

$$\left(5H^2 + 2\dot{H}\right)(1 + 2\lambda H \phi \dot{\phi}) + H^2 (1 + 2\lambda \phi^2 + 2\lambda \ddot{\phi}) = -2\phi^n,$$  \hspace{2cm} (18)

and stability condition function becomes

$$Q_{m-} = \phi^n \left[2\left(\frac{2\ddot{H}}{H^2} + 5\right) - 12\right] - \left(\frac{2\ddot{H}}{H^2} + 5\right)\dot{\phi}^2 > 0.$$  \hspace{2cm} (20)

We do numerical calculation for some value of $n$ such as positive, zero and negative, and then plot the solution in e-fold number, $N(t) = \log a(t)$. In doing so, we take $a(0) = 1$, $H(0) = 1$, $\phi(0) = 1$ and $\dot{\phi}(0) = 1$ as boundary conditions because we want to get zero initial e-fold number and the decreasing scalar field after inflation generated. Natural units ($\hbar = c = 1$) are used, so each scale in time coordinate corresponds the time scale in natural units.

We can see in figures 1 and 2 that, for every $n$ value, e-fold number naturally expands in the early times (small $t$), indicating that inflation happens for this model. For positive $n$ value, the plots of e-fold number for each $n$ value coincide in small $t$, but they differ in large $t$ where we define large $t$ as $t > 10^3 \text{ (GeV)}^{-1}$. However, we can see in figures 3 and 4 that for all $n$ values, the model is unstable because $Q_{m-}$ is always negative, meaning that the stability condition is not satisfied.
3.2. $m = 1$ case

If $m = 1$, (13) and (16) can be written as

\[ \ddot{\phi} + 3H\dot{\phi} - 6\lambda\phi H^2 (H^2 + \dot{H}) + n\phi^{n-1} = 0 \tag{21} \]

\[ (5H^2 + 2\dot{H})(1 + 2\lambda H\dot{\phi}) + H^2 (1 + 2\dot{\phi}^2 + 2\lambda\phi\ddot{\phi}) = 2\phi^n, \tag{22} \]

and stability condition function becomes

\[ Q_{m+} \equiv \phi^n [12 - 2\left(2\frac{\dot{H}}{H^2} + 5\right)] - \left(2\frac{\dot{H}}{H^2} + 5\right)\phi^2 > 0. \tag{23} \]

We do same analysis as in negative $m$ case.

As shown in figures 5 and 6, the plot of e-fold number as a function of time gives similar result as the negative $m$ case. However, the difference with the negative $m$ case comes from the stability condition. We can see from figure 7 that, for negative $n$, the model is unstable, but for zero and positive $n$ (figures 8 and 9) there exist some stable cases, that is, for $0 \leq n < 5$. For $n \geq 5$ case, we can see that initially $Q_{m+}$ is positive, but as $t$ increases, the value becomes negative.

**Figure 1.** Plot of e-fold number, $N(t) = \log a(t)$ as a function of time for various positive values of $n$ for $m = -1$ case.
Figure 2. Plot of e-fold number, $N(t) = \log a(t)$ as a function of time for negative values of $n$ for $m = -1$ case.

Figure 3. Plot of $Q_{m^\sim}$ of the model as a function of time for various positive values of $n$ for $m = -1$ case.
Figure 4. Plot of $Q_m$ of the model as a function of time for various negative values of $n$ for $m = -1$ case.

Figure 5. Plot of e-fold number, $N(t) = \log a(t)$ as a function of time for various positive values of $n$ for $m = 1$ case.
Figure 6. Plot of e-fold number, $N(t) = \log a(t)$ as a function of time for various negative values of $n$ for $m = 1$ case.

Figure 7. Plot of $Q_{m+}$ of the model as a function of time for various negative values of $n$ for $m = 1$ case.
Figure 8. Plot of $Q_{m-}$ of the model as a function of time for $n = 0$ for $m = 1$ case

Figure 9. Plot of $Q_{m+}$ of the model as a function of time for various positive values of $n$ for $m = 1$ case
4. Discussion and conclusion
We have analyzed the influence of power function inflation potential in cosmological model with Gauss-Bonnet term (GB term). In this model, in the case of quadratic coupling between GB term and scalar field, inflation occurs naturally for each case considered. Using the stability condition [13], it can be seen that there exist some stable cases, that is, for $0 \leq n < 5$ with $m = 1$. Although, for both cases, the e-fold number for each $n$ value coincides in small $t$, but differs in large $t$. This can happen perhaps because the difference between $n$ values in the inflation potential affects the e-fold number later in large $t$. However, it does not change the result that the addition of power function inflation potential can be a way to find the stable cosmological model with the GB term. There are other works too that can show there are stable inflationary solutions [23–27], and the result of this works can make intriguing motivation to study Einstein-scalar-Gauss-Bonnet theory in the cosmological context.

Acknowledgments
This work was supported by “Riset Inovasi KK ITB 2016”, “Riset Desentralisasi ITB 2016” and “Riset PMDSU 2016” from Ministry of Research, Technology and Higher Education of the Republic of Indonesia.

References
[1] Guth A H 1981 Phys. Rev. D 23 347
[2] Alishahiha M, Silverstein E and Tong D 2004 Phys. Rev. D 70 123505
[3] Arkani-Hamed N, Creminelli P, Mukohyama S and Zaldarriaga M 2004 J. Cosmol. Astropart. Phys. JCAP04(2004)001
[4] Cheung C, Creminelli P, Fitzpatrick A L, Kaplan J and Senatore L 2008 J. High Energy Phys. JHEP03(2008)014
[5] Caldwell R R, Dave R and Steinhardt P J 1998 Phys. Rev. D 80 103505
[6] Zen F P, Arianto, Gunara B E, Triyanta and Purwanto A 2009 Eur. Phys. J. C 63 477
[7] Amendola L, 1993 Phys. Lett. B 301 175
[8] Capozziello S, Lambiase G and Schmidt H J 2000 Annalen. Phys. 9 39
[9] Tsujikawa S 2012 Phys. Rev. D 85 083518
[10] Kanti P, Gannouji R and Dadhich N 2015 Phys. Rev. D 92 083524
[11] Antoniadis I, Gava E and Narain K S 2009 Phys. Rev. D 80 063518
[12] Antoniadis I, Gava E and Narain K S 2009 Eur. Phys. J. C 63 477
[13] Kawai S and Soda J 1999 Phys. Rev. D 59 063506
[14] Kawai S, Sakagami M and Soda J 1998 Phys. Rev. D 59 063506
[15] Kawai S, Sakagami M and Soda J 1997 Perturbative analysis of nonsingular cosmological model (Preprint gr–qc/9901065)
[16] Kawai S, Sagami M and Soda J 1998 Novel instability in superstring cosmology (Preprint gr–qc/9807056)
[17] Kawai S and Soda J 1999 Phys. Lett. B 460 41
[18] Kawai S and Soda J 1999 Structure formation from nonsingular kinetic inflation (Preprint gr–qc/9906046)
[19] Neupane I P 2007 Constraints on Gauss-Bonnet cosmologies (Preprint arXiv:0711.3234 [hep-th])
[20] Satoh M, Kanno S and Soda J 2008 Phys. Rev. D 77 023526
[21] Satoh M and Soda J 2008 J. Cosmol. Astropart. Phys. JCAP09(2008)019
[22] Guo Z K and Schwarz D 2009 Phys. Rev. D 80 063523
[23] Satoh M 2010 J. Cosmol. Astropart. Phys. JCAP11(2010)024