Inference of the Universal Constancy of Planck Constant based on First Principles

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Abstract. Since its discovery by Max Planck in 1900, the Planck constant $h$ has been demonstrated to be a universal constant for all (quasi-) stationary dynamical processes of all particles and fields, whenever an energy quantum $h \nu$ is exchanged or transmitted. Its numerical value has been accurately determined based on experiments. The physical origin of this fundamental constant $h$ has however not been well understood up to the present, and its numerical value has not been \textit{ab initio} predicted. This paper presents a theoretical inquiry into the first respect, a mathematical inference of the universal constancy of $h$ based on the second law of thermodynamics and the principle of least action combined with probability theory. We consequently conclude that the universal constancy of $h$ is the result of Maupertuis-Jacobi’s least action and entropy maximum principles governing the global system, the observable part of the cosmos at least.

1. Introduction
Max Planck hypothesised in 1900\cite{1, 2, 3} that electromagnetic radiation is ascribed to an indivisible smallest energy or quantum $h \nu$ for a given frequency $\nu$, where $h$ is the Planck constant. Based on this hypothesis Max Planck brought theoretical prediction of the black body radiation spectrum into satisfactory agreement with experiment for the high frequency end, a region where classical kinetic theory prediction presented large discrepancy, known as the ”ultraviolet catastrophe”. Planck’s work laid the foundation of quantum mechanics (QM). Through the formal QM formalism subsequently developed by E Schrödinger, W Heisenberg, P M Dirac and others in the 1920s-30s, along with a broad range of experiments performed, it has become well established that $h$ is a universal constant for all (quasi-) stationary processes of all particles and fields, whenever a quantum $h \nu$ is exchanged or transmitted.

It has been furthermore established that, at the scale $h$ for momentum times a characteristic distance and energy times the corresponding time (the Heisenberg relations), a classically point matter particle ($i$) turns to behave as a wave, of a wave function $\psi$ in general being complex in nature. And, given an applied potential $V_a$, assuming non-relativistic, $\psi$ is governed by the Schrödinger wave equation, written in a generalised form here for the particle of mass $m_i$,

$$i \hbar \partial_t \psi = H \psi, \quad H = -\frac{\hbar^2}{2m_i} \nabla^2 + V_a. \tag{1}$$

To facilitate the discussion in this paper, in place of $\hbar = \frac{h}{2\pi}$ an arbitrary real positive parameter $\pi_{n\alpha} = \frac{\hbar}{2\pi}$ has been used for a specified energy level $n$ and (position space) degree of freedom $\alpha$. (1) is a general QM equation to the extent that from it the overall other QM relations can
in principle all be derived (see Sec 4), and also that it describes electromagnetic (EM) radiation fields, including internal ones comprising a particle (see Sec. 3), indirectly through describing the motion of its generating source, an accelerating charged particle. We can thus through (1) seek out the distinct significance of \( h \), or \( a_{na} \), against two other basic aspects of QM reflected in (1): First, (1) in essence describes the \textit{stationary} state of a particle wave \( \psi(x, t) \). This is rooted in the fact that (1) is a decomposed form of the continuity equation for the probability density current \( \rho(x)v = |\psi(x, t)|^2 v \) with a velocity \( v \), \( \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \) which describes a stationary state. Second, in a confined space, the mathematical solutions to (1) in general render the dynamical variables to be quantised – with respect to a specified \( a_{na} \) in (1). But evidently in both the respects, (1) is formal and scalable; a quantised stationary-state solution to it will result irrespective of the value of \( a_{na} \). \( a_{na} \) can in fact be an arbitrary real-valued parameter for each \( n \) and \( \alpha \). More generally, \( a_{na} \) can be dependent on the property of each particle and its environment, as Planck had originally suggested\cite{2}. Indeed therefore, in virtue of its factually universal constancy as opposed to arbitrary, \( h \), or \( a_{na} \) in (1), must itself possess a fundamental significance, as opposed to being merely a parameter abiding by the Schrödinger wave equation.

Today, \( h \) is taken for granted to be a universal constant. Its numerical value has hitherto been determined by fitting such theoretical (or empirical) functions as the blackbody radiation spectrum and photo-electric work function to experimental data. These have all resulted in the value \( (a_{na} =)h = 6.6260 \times 10^{-34} \text{ Js} \). The physical origin for the universal constancy and the specific value of \( h \) has hitherto remained an open question. There is no known logical reason why all \( a_{na} \)'s are equal, and to the particular value of \( h \). The universal constancy of \( h \) is also responsible for a basic concept in statistical mechanics, namely that, in the phase space of say \( N \) particles having \( D \) position space degrees of freedom each, there exists an indivisible volume \( h^{ND} \) which can be accessed by each microscopic state of the \( N \)-particles. This concept was also originally introduced by Planck\cite{2}, based on a hypothesis of absolute entropy and postulate of equal \textit{a priori} probability (namely that all possible states of an isolated system in or approaching equilibrium are equally accessible). Similarly, equal \textit{a priori} probability is a postulate of Boltzmann and Planck for which there exists hitherto no first principles derivation, in spite of a recent attempt of proving it based on dissipation theorem\cite{4}.

An understanding of the origin of the universal constancy of \( h \) will shed light on one of the key issues regarding the origin of QM, the very presence of the universal \( h \) itself (another being what is waving with a matter wave \( \psi \)). And this is also a first step needed for an \textit{ab initio} prediction of the numerical value of \( h \), supposing that one of the dynamical processes is original and has set the \( h \) value compulsorily for the rest. In this paper, starting with a thermodynamic system composed of particles obeying (1) each, with \( a_{na} \) an arbitrary real parameter, we shall infer Boltzmann’s postulate of equal \textit{a priori} probability (Secs. 2–3) and the universal constancy of \( a_{na} \), hence \( h \) (Sec. 4), based on the second law of thermodynamics and Maupertuis-Jacobi’s principle of least action, severing as our first principles here, combined with probability theory. We summarise the conclusions in Sec 5.

2. Maximum entropy

We consider until Sec. 3 a volume \( V \) of \( N \) (identical) particles that are (a) isolated, (b) weakly interacting only, and (c) distinguishable. Here, (b) implies that the particles may interact so as to exchange energies but introduce no correlations in occupancy of states; (c) can be subject to correction for indistinguishability. The \( N \)-particle system is in or approaching thermodynamic equilibrium, spontaneously owing to cond. (a), and is thus of or approaching a constant internal energy \( U \). Each particle is assumed (intrinsically) a wave obeying (1), with \( a_{na} \) arbitrary real and positive.

Assume that the \( N \) particles have each \( D \) (position space) degrees of freedom, or dimensions. The quantum state of a particle \( i \) \( (= 1, 2, \ldots, N) \) at time \( t \) is thus completely specified by \( D \)
eigen functions \( \psi_{n_{\alpha}}(x_{\alpha}, t) \)'s, with \( \alpha = iD - (D - 1), iD - (D - 2), \ldots, iD = (1, 2, \ldots, ND) \). In a confined space \([0, R_{\alpha}]\) the eigen energy \( E_{n_{\alpha}} = \sum_{\alpha=1}^{iD} E_{n_{\alpha}} \) from solving (1) for it is in general space quantised, with \( n_{\alpha} \) a discrete number and having a total \( \mathfrak{n}_{\alpha 0} \) possible values; \( n_{i} = n_{i}(n_{iD} - (D - 1), \ldots, n_{iD}) \). For example, for a spin zero free particle \( i \) of mass \( m_{i} \) in a cube \((D = 3, R_{\alpha} = V^{1/3})\), \( \alpha = 1, \ldots, 3N \) moving at velocity \( v_{n_{\alpha}} \) along \( \alpha \), then

\[
\psi_{n_{\alpha}}(x_{\alpha}, t) = \frac{1}{\sqrt{\pi k_{dn_{\alpha}}}} e^{-\frac{1}{2k_{dn_{\alpha}}^2} \sum \alpha = 1} \psi_{n_{\alpha}}(x_{\alpha})e^{-i\left(E_{n_{\alpha}}(\pi \alpha a_{n_{\alpha}})^2\right)},
\]

\[
\psi_{n_{\alpha}}(x_{\alpha}) = C \sin(k_{dn_{\alpha}}x_{\alpha}), \text{ where } k_{dn_{\alpha}}R_{\alpha} = n_{\alpha},
\]

\( n_{\alpha} = 0, 1, \ldots, \mathfrak{n}_{\alpha 0} \), given by boundary condition \( \psi_{n_{\alpha}}(0) = \psi_{n_{\alpha}}(R_{\alpha}); \mathfrak{n}_{\alpha 0} = V^{1/3}/(\lambda_{dn_{\alpha}}/2) \) for a cut-off de Broglie wavelength \( \lambda_{dn_{\alpha}} = 2\pi/k_{dn_{\alpha}} \) and \( E_{n_{\alpha}} = p_{n_{\alpha}}^2/(2m_i) \), \( p_{n_{\alpha}} = m_i \psi_{n_{\alpha}} = \pi_{n_{\alpha}a_{n_{\alpha}}} \). The microscopic state, or simply state, of the \( ND \)-particles is thus at time \( t \) completely specified by \( ND \) eigen functions, with a distinct set of \( n_{1}, \ldots, n_{\alpha}, \ldots, n_{ND} \) values. And each state has a definite total internal energy \( U = \sum_{\alpha=1}^{ND} E_{n_{\alpha}} = \sum_{i=1}^{N} E_{n_{i}} \). There can be \( \mathfrak{n}_{\alpha} \) \((\leq \mathfrak{n}_{\alpha 0}) \) states given by \( n_{\alpha} \)'s lying in a narrow range \( [n_{\alpha a}, n_{\alpha b}] \) for particle \( i \) in \( \alpha \) dimension which contribute to a fixed \( U \). There are then a total \( \Omega = \prod_{\alpha=1}^{N} \mathfrak{n}_{\alpha} \) number of possible states of the \( N \)-particles of a constant \( N, V, U \), which make up a microcanonical ensemble, expressed by \( \Omega \) sets of eigen functions as

\[
\mathfrak{E} = \left\{ \psi_{n_{1}}, \psi_{n_{2}}, \ldots, \psi_{n_{ND}} | n_{\alpha a} \leq n_{\alpha} \leq n_{\alpha b} \right\} = \left\{ \psi_{\nu 1}, \psi_{\nu 2}, \ldots, \psi_{\nu \alpha a}, \ldots, \psi_{\nu \alpha b, \nu(ND)} | \nu = 1, 2, \ldots, \Omega \right\} \quad (2)
\]

Each state labelled by \( n_{\alpha} \)'s in the first of Eqs. (2) is in the second re-labelled by a running index \( \nu \) which is a unique value for each distinct state and arranged in an arbitrary order to \( n_{\alpha} \)'s. We assume that, as cond. (d), \( \Omega \gg N \) so that no more than one particle can occupy the same state. The \( \Omega \) sets of wave functions (2) span a Hilbert space \( \mathcal{H} \) (or Fock space) of the \( N \) particles in momentum states representation, in which the (total or ensemble) state vector is \( \psi_{enS} = \sum_{\nu=1}^{\Omega} c_{\nu} \psi_{\nu} \), where \( c_{\nu} \) is the amplitude of \( \nu \)th state and \( \psi_{\nu} = (\text{sgn}) \sqrt{\mathfrak{E}} \psi_{\nu 1}(x_{1}) \ldots \psi_{\nu(ND)}(x_{ND}) \) is the state vector of the \( N \) independent distinguishable \((r = 1)\) or indistinguishable \((r = 1/N!)\) particles in \( \nu \)th state.

The microscopic states of the \( N \)-particles can be alternatively described in a usual \( 2ND \)-dimensional phase space \( \mathcal{P} \) spanned by \( ND \) space- and \( ND \) momentum- coordinates \( x_{1}, \ldots, x_{ND} \) and \( p_{1}, \ldots, p_{ND} \). A \( \nu \)th distinct state is in \( \mathcal{P} \) completely specified by a volume element \((\Delta x_{\nu 1}, \ldots, \Delta x_{\nu(ND)}; \Delta p_{\nu 1}, \ldots, \Delta p_{\nu(ND)}) \) located about a fixed point \((x_{\nu 1}, \ldots, x_{\nu(ND)}; p_{\nu 1}, \ldots, p_{\nu(ND)}) \) which are bound between adjacent states each \([\text{Eqs (20)} \rightarrow \text{(21)}]\); and this has a one to one correspondence with a state vector \( \psi_{\nu} \) in \( \mathcal{H} \). The volume occupied by \( \nu \)th state \( (b_{\nu}) \), and the total volume by all \( \Omega \) states \( (B) \), in \( \mathcal{P} \) are

\[
b_{\nu} = \prod_{\alpha=1}^{ND} \Delta x_{\nu \alpha} \Delta p_{\nu \alpha} = \prod_{\alpha=1}^{ND} a_{\nu \alpha}, \quad a_{\nu \alpha} = \Delta x_{\nu \alpha} \Delta p_{\nu \alpha}, \quad \nu = 1, 2, \ldots, \Omega;
\]

\[
B = \sum_{\nu=1}^{\Omega} b_{\nu} \quad (3.1)
\]

At any instant of time \( t \) the \( N \)-particles statistically lies in a definite state \( \psi_{\nu} \) in \( \mathcal{H} \), or volume \( b_{\nu} \) in \( \mathcal{P} \), with the probability \( |c_{\nu}|^2 \). Over long time, as the result of particle–particle interactions, the \( N \)-particles will statistically explore all the \((\leq \Omega)\) accessible possible states. To observe all of the \( \Omega \) states at least once requires long time. To facilitate a one time only measurement, we can instead employ a Gibbs ensemble consisting of \( \mathfrak{G}(\leq \Omega) \) replicas of the original \( N \)-particles,

\[
\mathfrak{G} = \left\{ \psi_{\mu 1}, \psi_{\mu 2}, \ldots, \psi_{\mu \alpha}, \ldots, \psi_{\mu \nu(ND)} | \mu = 1, 2, \ldots, \Xi \right\} \quad (4)
\]
which are in different (microscopic) states and yet of identical macroscopic properties, \( N, V, U \) here. Assuming ergodicity, to attempt to access all \( \Omega \) states over time thus amounts to try to accommodate the \( \Xi \) replicas of the \( N \)-particles in the phase space \( \mathcal{P} \). We proceed by imagining the \( \Xi \) replicas as \( \Xi \) objects "thrown" into \( \mathcal{P} \), which will spontaneously attain their equilibrium positions after a relaxation time. We do not \textit{a priori} assume here, as the postulate of equal \textit{a priori} probability instead does, that all the \( \Omega \) possible states in \( \mathcal{H} \) are equally accessible to the \( N \)-particles, nor (accordingly) that all \( b_\nu \)'s in \( \mathcal{P} \) are equal and can be readily uniformly occupied by the \( \Xi \) replicas; this in the first place leads to \( \Xi \neq \Omega \) (or \( \Xi < \Omega \)). The actual accessibility or occupancy of \( \mathbb{B} \) by \( \mathbb{G} \), hence the actual correspondence between \( \mathbb{B} \) and \( \mathbb{G} \), is to be determined.

The degree of accessibility of a \( \nu \)th state can be described by the probability \( P_\nu \) for the \( N \)-particles to be found in the \( \nu \)th state upon a measurement at time \( t \), or equivalently, for a replica \( \mu \) to be found in the \( \nu \)th volume element in \( \mathcal{P} \). We assume that the 2ND-dimensional phase space \( \mathcal{P} \) for accommodating a replica pertains to a geometric nature in the same sense as the position space for accommodating say, a dart in the dart game; this is formally supported by the analogous mathematical operations (3) for the volume(s) in \( \mathcal{P} \) as in position space. Then to attempt to accommodate or "throw" a replica \( \mu \) in a volume element \( b_\nu \) out of \( B \) in \( \mathcal{P} \) is like to attempt to throw a dart into a small area element \( \sigma(r, dr, d\theta) \) about a position \( r, dr, d\theta \) on a dartboard \( \mathcal{D} \). Based on dart game experiment (see e.g. [5]) performed in an effectively homogeneous space (assuming gravity field etc. is negligible), the larger the area \( \sigma \) is, the more likely the dart will hit \( \sigma \) on \( \mathcal{D} \). The probability \( P_\sigma \) for a successful hit at the target area \( \sigma \) is thus \( P_\sigma \propto \sigma \). Making direct analogy, granted with a homogeneous phase space in volume \( B \) where \( U \) is everywhere the same, then the larger the volume \( b_\nu \) of the \( \nu \)th volume element is, the more likely a replica \( \mu \) will "hit", or be accommodated in \( b_\nu \). That is, \( P_\nu \propto b_\nu \); and the normalised probability is

\[
P_\nu = \frac{b_\nu}{B}, \quad \nu = 1, 2, \ldots, \Omega \quad \sum_{\nu=1}^{\Omega} P_\nu = \frac{1}{B} \sum_{\nu=1}^{\Omega} b_\nu = 1
\]  

Cond. (d) restrains two or more replicas \( \mu, \mu' \) from occupying the same state \( \nu \) at the same time; so \( \mu = \nu = 1, 2, \ldots, \Xi \).

Now according to the second law of thermodynamics, the disorderliness, hence the entropy \( S \), of the isolated system here in (or approaching) thermodynamic equilibrium is (or approaches) maximum. To achieve a maximum disorderliness, hence \( S \), we obviously must first require that all the \( \Omega \) possible states are at any time \( t \) simultaneously accessible to the \( N \)-particle system. Assuming ergodicity, this can be restated as

The \( \Omega \) volume elements in \( \mathcal{P} \) are simultaneously occupied by \( \Xi = \Omega \) replicas, with one and only one (ensured by cond. (d)) replica occupying one volume element at a time.

The simultaneous occupancies of the \( \Omega \) possible states (6) represent \( \Omega \) simultaneous events that are, due to cond. (b), independent with one another (hence no correlated occupancies in \( \mathcal{P} \)). Thus the probability for the occurrence of the \( \Omega \) simultaneous independent events (6) is according to probability theory

\[
P_{\text{ens}} = \prod_{\nu=1}^{\Omega} P_\nu = \prod_{\nu=1}^{\Omega} b_\nu = \frac{b_\nu}{B^{\Omega}}, \quad b_\nu = \prod_{\nu=1}^{\Omega} b_\nu.
\]  

The \( \Omega \) simultaneous events can be maximally achieved if \( P_{\text{ens}} \) is maximum (\( P_{\text{ens,max}} \)),

\[
\delta P_{\text{ens}} = 0, \quad \delta^2 P_{\text{ens}} < 0.
\]  

4
To (universally) determine $\mathcal{P}_{\text{ens},\text{max}}$ using calculus requires a (universal) analytical function $\mathcal{P}_{\text{ens}}$ and is inevitably unfeasible since the geometries and dynamics of the physical systems, and accordingly the geometries of the volume elements in $\mathcal{P}$, can be arbitrary. A maximum $\mathcal{P}_{\text{ens}}$ solution however can be readily obtained by means of algebraic methods (see e.g. [11]) as follows. Observe that $\mathcal{P}_{\text{ens}}$ given by (7) is equivalent to the geometric mean of $\mathcal{P}_\nu$’s

$$
\mathcal{P}_\nu = \left( \prod_{\nu=1}^{\Omega} \mathcal{P}_\nu \right)^{1/\Omega} = \frac{\left( \prod_{\nu=1}^{\Omega} b_\nu \right)^{1/\Omega}}{B} \tag{9}
$$

raised to the power $\Omega$, i.e. $\mathcal{P}_{\text{ens}} = \left( \mathcal{P}_\nu \right)^{\Omega}$. And the arithmetic mean of the $\mathcal{P}_\nu$’s is

$$
\langle \mathcal{P}_\nu \rangle = \frac{1}{\Omega} \sum_{\nu=1}^{\Omega} \mathcal{P}_\nu = \frac{1}{\Omega} \sum_{\nu=1}^{\Omega} b_\nu = \frac{1}{\Omega} \tag{10}
$$

It follows from the theorem of inequality of the arithmetic-geometric means that $\mathcal{P}_\nu < \langle \mathcal{P}_\nu \rangle = \left( \frac{1}{\Omega} \right)$, and hence $\mathcal{P}_{\text{ens}} = \mathcal{P}_\nu^{\Omega} < \langle \mathcal{P}_\nu \rangle^{\Omega} = \left( \frac{1}{\Omega} \right)^{\Omega}$, if the $\mathcal{P}_\nu$’s, being nonnegative and real, are not all equal. And $\mathcal{P}_\nu$, and hence $\mathcal{P}_{\text{ens}}$, are maxima each,

$$
\mathcal{P}_{\nu,\text{max}} = \langle \mathcal{P}_\nu \rangle = \frac{1}{\Omega}, \quad \mathcal{P}_{\text{ens},\text{max}}(= \mathcal{P}_{\nu,\text{max}}^{\Omega}) = \langle \mathcal{P}_\nu \rangle^{\Omega} = \left( \frac{1}{\Omega} \right)^{\Omega}, \tag{11}
$$

if and only if the nonnegative real $\mathcal{P}_\nu$’s are all equal, $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_\Omega$. Or equivalently, (11) holds if and only if, on substituting (5) for $\mathcal{P}_\nu$, all $b_\nu$’s are equal to one another and hence to a common value denoted by $b_0$ for the specified $N,D$,

$$
b_1 = b_2 = \ldots = b_\Omega = b_0. \tag{12}
$$

With (12), (3.2) becomes $B = \sum_{\nu=1}^{\Omega} b_\nu = \Omega b_0$ and (5a) becomes

$$
\mathcal{P}_\nu = \frac{b_0}{\Omega b_0} = \frac{1}{\Omega}, \quad \nu = 1, 2, \ldots, \Omega. \tag{13}
$$

(13) states that, for the fixed $N,V,U$ system under conds. (a)–(d), all the $\Omega$ possible states are equally accessible, which is just the statement of the fundamental postulate of equal $a \text{ priori}$ probability. Using (13) for $\mathcal{P}_\nu$ in the Gibbs entropy formula $S = -k_B \sum_{\nu=1}^{\Omega} \mathcal{P}_\nu \ln \mathcal{P}_\nu$ gives the entropy of the $N,V,U$ system

$$
S = -k_B \sum_{\nu=1}^{\Omega} \frac{1}{\Omega} (-\ln \Omega) = k_B \ln \Omega, \tag{14}
$$

where $k_B$ is Boltzmann’s constant. (11) essentially has maximised all the $\Omega$ states, in contrast to some lesser $\Omega' < \Omega$ states, to be equally accessible at a time. $S$ given in (14) is thus in contrast to some lesser quantity $S' = k_B \ln \Omega'$, and hence is maximum.

3. Global system

The basic conclusions of Sec. 2, Eq. (12) in particular, have been arrived at upon no restrictions on the kinds or species of the $N$ particles. $N$ can thus include say $N_1$ atoms, $N_2$ electrons (free or bound), $N_3$ photons (EM radiation energy quanta), $N_4$ phonons (sound energy quantum), and so forth; suppose these have $D_s$ degrees of freedom and $\Omega_s$ states, $s = 1, 2, \ldots$. Then,
\[ N = \sum_s N_s, \quad ND = \sum_s N_s D_s, \quad \Omega = \prod_s \Omega_s, \text{ etc.} \]

Of the \( D_s \) dimensions, \( \alpha \) can refer to position co-ordinates but also spatial orientations of (internal) orbital or spin angular momentum. If two different species' \( D_{s'} \)'s represent the same energy, such as the energy levels of a charged oscillator and its emitted radiation quanta, evidently only one should be considered for entropy. If some \( \alpha \) indexes internal motion, this motion would not (directly) contribute to entropy. In any such cases, the properties of \( b_{\alpha} \)'s being our central concern, we can consider one of the motions at a time that does contribute to entropy, to eventually have considered all possible motions.

In usual applications, (1), hence \( E_{n_i}, p_{n_i}, \text{ etc.} \), describe the kinetic motion of a matter particle in an applied \( V_a \) field. Alternatively, (1) can also describe the oscillatory motion \( u_{n_{i\alpha}} \) of a charge \( q_i \), of a dynamical mass \( M_{q_i} \), in a (quadratic) vacuum potential field \( V_q (= \frac{1}{2} \beta_q u_{i\alpha}^2) \) (of extension \( \sim 10^{-15} \text{ m} \)); \( u_i \), together with the resultant EM radiation field, represents the total, internal motion of a simple matter particle \( i \) such as an electron, or proton (the IED model[6, 7]). At the scale \( \sim 10^{-15} \text{ m} \), \( q_i \) is extensive and has a distribution function \( \psi_{n_{i\alpha}}(x_i, t) \), or displacement \( u_{n_{i\alpha}}(x_i, t) = \psi_{n_{i\alpha}}(x_i, t) \), for a mass element of \( q_i \) about \( x_i \), with \( \psi_{n_{i\alpha}} \) the amplitude. So, using \( H = -(\nabla u_{n_{i\alpha}}/2M_{q_i})V^2 + V_q \) in (1), the eigen energy and momentum (in places of \( E_{n_i}, p_{n_i} \) for \( n_{i\alpha} = 1 \)th level, \( \varepsilon_{q_1} = \frac{1}{2} \beta_q \alpha, \quad \varepsilon_{q_2} = 1/a_{1\alpha}, \quad p_{q_1} = \varepsilon_{q_1}/c \) (in zero EM radiation limit), give the (total) internal energy and linear momentum of the resultant IED matter particle of mass \( m_i \), where \( \nu_i = (1/2\pi)\sqrt{\beta_{q_i}/2M_{q_i}} = m_i c^2/a_{1\alpha} \). The properties of \( a_{1\alpha} \)'s and \( b_{\nu} \)'s of the so created protons and electrons can be e.g. studied by considering a dilute plasma gas of single protons and electrons of fixed \( N, V, U, \) to which the basic conclusions of Sec. 2 apply.

(12) has been otherwise arrived at for a fixed \( N, V, U \) system under conds. (a)–(d) that are also the conditions for the direct application of the postulate of equal a priori probability in statistical mechanics. So (12) may be extended to other (important) variant systems following the same well established ”correction” procedures (see e.g. [8, 9, 10]) in statistical mechanics, as are briefly outlined in (i)–(iv) below. (i) The \( N \) particles may be correlated with one another, hence a deviation from cond. (b). The usual method to extend beyond (b) is virial expansion of the equation of state. (ii) The \( N \) particles may be indistinguishable with one another, hence a deviation from cond. (c), when confined in a small geometry where their \( \psi_{n_{i\alpha}} \)'s mutually overlap to a high degree. The standard correction to (c) is, as done in (15), to divide out \( N! \) number of indistinguishable states from \( \Omega \), giving \( \Omega_{1,D} = \Omega/N! \). (iii) \( T \) may be very low such that \( \Omega \sim N \) and the situation of more than one particle occupying the same states becomes frequent, hence a deviation from cond. (d). The standard approach to such system is the replacement of the Boltzmann statistics by quantum statistics (Fermi-Dirac or Bose-Einstein statistics) where restrictions are made to the number of accessible states by applying symmetry properties of the wave functions of particles. Of the systems of (i)–(iii), the extents or forms of the contributions of their \( \Omega \) states to the \( \mathcal{P}_\nu \)'s are modified in certain fashion and thus corrected for. The corrections may be appreciated as being to scale the systems back to obeying conditions (a)–(d).

And finally (iv), two important variations from the ”isolated system” of cond. (a) are the ”closed isothermal system”, whose \( U \) may fluctuate and \( N, V, T \) are fixed, and ”open isothermal system”, whose \( U, N \) may fluctuate and \( \mu, V, T \) (\( \mu \) denotes the chemical potential) are fixed. Their \( \mathcal{P}_\nu \)'s for states \( \nu \)'s are now given by the canonical (or Boltzmann) and grand canonical distributions

\[
\mathcal{P}_\nu(U_{\nu})|_{U_{\nu}=U} = \mathcal{P}(U) \frac{e^{-U_{\nu}/k_BT}}{Q(N, V, T)}, \quad \mathcal{P}_\nu(U_{\nu}, N_{\nu})|_{U_{\nu}=U, N_{\nu}=N} = \mathcal{P}(U, N) \frac{e^{-(U_{\nu}-\mu N_{\nu})/k_BT}}{Z(\mu, V, T)},
\]

where \( Q(N, V, T) = r \sum_{U'} \Omega(U') e^{-U'/k_BT} = r \sum_{U'} e^{-U'/k_BT} \), and \( Z(\mu, V, T) = r \sum_{\nu, N_{\nu}} e^{-(U_{\nu}-\mu N_{\nu})/k_BT} \), with \( r = 1 \) and \( 1/N! \) for \( N \) distinguishable and indistinguishable particles. For the two systems above or the like, one can continue to write down (5) for each given \( U' \) or \( U', N', \) \( \mathcal{P}_\nu' = b_{\nu}'/B \); one gets a range of \( \mathcal{P}_\nu' \), \( b_{\nu}' \), values for \( U' \) or \( U', N' \) fluctuate in a range. (For each \( \mathcal{P}_\nu \), the entropy continues to be given by \( S = -k_B \sum_\nu \mathcal{P}_\nu \ln \mathcal{P}_\nu \).) But \( \mathcal{P} \) is no
longer homogenous, and the $b'_s$ are now scaled by the Boltzmann factors $e^{-U'/k_BT}$, etc. in (15). Nevertheless, each of the systems (iv) and its immense reservoir together are isolated, rendering again a fixed $U, V, N$ system. The immense number of states of the reservoir permits one to ignore the local fluctuation in $U'$ or $U', N'$, and perceive a homogeneous phase space instead. For the properties of $b'_s$ of a greatest possible globe system being our eventual interest, we can most generally choose the entire cosmos as our thermodynamic system, or at least the part of it we can be certain, which is isolated, and is at or approaching thermodynamic equilibrium, hence of fixed $N, V, U$; its total entropy thus is or approaches maximum. For this part of the cosmos, provided that the corrections (i)-(iii) or the like are made, the basic conclusions of Sec 2, $b'_\nu \cosm = v'_\cosm = \prod_{\alpha=1}^{(ND)\cosm} a_{\nu\alpha} \text{ (assuming } (ND)\cosm \text{ dimensions) of Eq. (12) in particular, are valid.}$

4. Least action

Insofar as all $b'_s$ are equal, to $b_0 = \prod_{\alpha=1}^{ND} a_{\nu\alpha} \text{ [Eq (3)]}$ for a specified $ND$, it is irrelevant that the $a_{\nu\alpha}$ values are unless otherwise constrained. In the following we shall not assume any pre-established knowledge of the $a_{\nu\alpha}$'s (esp. their being all equal and to $h$), but shall investigate the characteristics of the $a_{\nu\alpha}$'s, or the $a_{\nu\alpha}$'s below, regarding their constancy based on Maupertuis-Jacobi's least action principle.

Of the $N$ particles of Sec. 2, we first focus on an individual particle $(i)$ in $n$th level in motion at a component velocity $\nu_{na}$ in $\alpha$th dimension confined in a space interval $[0, R_{\alpha}]$. The particle is assumed (quasi-) stationary and described by an eigen wave function $\psi_{na}(x_{\alpha}, t)$ governed by Eq. (1); by writing the suffices $na$ here we stress the state $n, \alpha$ rather than the value $n_{\alpha}$. As the mathematical function $\psi_{na}(x_{\alpha}, t)$ is, the particle is generally extensive across $[0, R_{\alpha}]$ at any time $t$. Its dynamical functions, such as linear momentum $p'_{na}(x_{\alpha}, t) = m_{\nu} \nu'_{na}$, kinetic energy $T'_{na}(x_{\alpha}, t) = \frac{1}{2} p'_{na}(x_{\alpha}, t) \nu'_{na}(x_{\alpha}, t)$, potential energy $V'_{\nu_{na}}(x_{\alpha}(t))$, and Hamiltonian $E'_{\nu_{na}}(x_{\alpha}, t) = T'_{na}(x_{\alpha}, t) + V'_{\nu_{na}}(x_{\alpha}(t))$, are accordingly distributed functions in $[0, R_{\alpha}]$.

For this indulatory stationary particle extensive in $[0, R_{\alpha}]$, the one-loop distance $2R_{\alpha}$ thus in general specifies a characteristic space interval $\Delta x_{na}$, such that only over $\Delta x_{na}$ (the dynamical variables of) the particle can be wholly and meaningfully defined. The latter respect is in general described by one or multiple integral periodical distance. The space interval with the two features combined in principle is the same as given by the usual boundary conditions. For examples, for the free particles confined in a cube of sides $R$ in Sec 2, $\Delta x_{na} = n_{\alpha} \lambda_{d_{\alpha}} = 2R$; for an electron of wavelength $\lambda_{d_{\alpha}}$ orbiting along a circle of radius $r_{\alpha}$ in a hydrogen-like atom, the one-loop, characteristic space interval is the circumference of the circle (in place of $\Delta x_{na}$) $\Delta \delta_{n} = n_{\alpha} \lambda_{d_{\alpha}} = 2\pi r_{\alpha}$; and for a harmonic oscillator, the one-loop distance is $4 \times$ the oscillation amplitude. And, for each $\Delta x_{na}$, there exists a characteristic time interval for (a tagged wave front of) the particle to traverse $\Delta x_{na}$, $\Delta t_{na} = \Delta x_{na}/(\nu'_{na})$.

We define for the particle the Maupertuis-Jacobi's action integral$^1$ across $\Delta x_{na}$ and $\Delta t_{na} = \Delta x_{na}/(\nu'_{na})$ as

$$A_{na} = \int_{0}^{\Delta x_{na}} p'_{na}(x_{\alpha}, t)dx_{\alpha} = \int_{0}^{\Delta x_{na}} \frac{2T'_{na}(x_{\alpha}, t)}{\nu'_{na}(x_{\alpha}, t)}dx_{\alpha} = \int_{0}^{\Delta t_{na}} 2T'_{na}(x_{\alpha}, t)dt,$$

(16) where $dx_{\alpha}/\nu'_{na} = dt$. (16) can be re-written as

$$A_{na} = p_{na} \Delta x_{na}, \quad A_{na} = (2T_{na}) \Delta t_{na},$$

(17)$^1 \ A_{na}$ may be re-written as $A_{na} = \int [T'_{na} - V'_{\nu_{na}} + \nu'_{\nu_{na}}] \nu'_{na} \nu_{\nu_{na}} dt = \int [L'_{na} + E'_{\nu_{na}}] dt$, where $L'_{na} = T'_{na} - V'_{\nu_{na}}$ is the Lagrangian function. $\int L'_{na} dt = S_{na}$ defines a distinct action integral which is widely in use today and which, as may be easily seen, is yet unsuited for the present problem.
where $p_{na}$, $T_{na}$ are the expectation values of $p'_{na}$, $T'_{na}$ or of the corresponding operators $\hat{p}_{na}$, $\hat{T}_{na}$ acting on $\psi_{na}$ (assuming $\psi_{na}$ is normalised across $\Delta x_{na}$):

$$p_{na} = \frac{1}{\Delta x_{na}} \int_{0}^{\Delta x_{na}} p'_{na}(x_{a}, t) dx_{a} = \int_{0}^{\Delta x_{na}} \psi_{na}(x_{a}, t) \hat{p}_{na} \psi_{na}(x_{a}, t) dx_{a},$$

$$T_{na} = \frac{1}{\Delta t_{na}} \int_{0}^{\Delta t_{na}} T'_{na}(x_{a}, t) dt = \int_{0}^{\Delta t_{na}} \psi_{na}(x_{a}, t) \hat{T}_{na} \psi_{na}(x_{a}, t) dt;$$

and $E_{na} = T_{na} + V_{na}$. $(p_{na}$ and $T_{na}$ of the typical stationary particle systems in applications, and $E_{na}$ of any such systems, are in fact independent of $x_{a}$, $t$.) The absolute difference Maupertuis-Jacobi’s action integral between $A_{na}$’s of two adjacent levels $n + 1$ and $n$ is then given as

$$a_{na} = |A_{(n+1)\alpha} - A_{n\alpha}|; \quad \text{or} \quad a_{na} = \Delta p_{na} \Delta x_{na} \equiv \int_{0}^{\Delta x_{na}} \Delta p'_{na}(x_{a}, t) dx_{a},$$

$$a_{na} = (2\Delta T_{na}) \Delta t_{na} \equiv \int_{0}^{\Delta t_{na}} 2\Delta T'_{na}(x_{a}, t) dt \equiv \Delta E_{na} \Delta t_{na},$$

where $\Delta x_{na}$ and $\Delta t_{na}$ are the mean characteristic space and time intervals given by

$$\Delta x_{na} = \frac{|p_{(n+1)\alpha} \Delta x_{(n+1)\alpha} - p_{na} \Delta x_{na}|}{\Delta p_{na}}, \quad \Delta t_{na} = \frac{|T_{(n+1)\alpha} \Delta t_{(n+1)\alpha} - T_{na} \Delta t_{na}|}{\Delta T_{na}};$$

$$\Delta p_{na} = |p_{(n+1)\alpha} - p_{na}|, \quad \Delta T_{na} = |T_{(n+1)\alpha} - T_{na}|, \quad \Delta E_{na} = |E_{(n+1)\alpha} - E_{na}|.$$}

$\Delta p'_{na}$ and $\Delta T'_{na}$ are similarly defined. The $a_{na}$, $\Delta p_{na}$, $\Delta x_{na}$, etc are in the above each defined positively since their magnitudes only, not their signs, will be relevant. $A_{na}$ as defined above pertains evidently to a characteristic dynamical variable of the state $na$, and $a_{na}$ to that of the difference of two adjacent states $na$, $n'\alpha$ of the particle. It can be checked against the typical quantum systems in applications that, using $\Delta x_{na}, \Delta t_{na}$ defined in (20) as combined with the eigen solutions for $p_{na}, T_{na}$ will indeed yield the usual Heisenberg relations. For a spatially confined particle, the $\Delta p_{na}, \Delta x_{na},$ etc, are quantised as are the $p_{na}, T_{na}$ and $E_{na}$ (Sec. 2) and $A_{na}$. The mechanical energy difference $\Delta E_{na}$ corresponds to an energy quantum of harmonic radiation field, or a photon, emitted upon transition from $n + 1$ to $n$, for which there always holds $\Delta T_{na} = D_{V_{na}}$, so $\Delta E_{na} = \Delta T_{na} + \Delta V_{na} = 2\Delta T_{na}.$

The least action principle requires $A_{na}$, accordingly $a_{na}$, to be minimum each, hence

$$(a): \quad \delta A_{na} = 0, \quad \delta^{2}A_{na} > 0; \quad (b): \quad \delta a_{na} = \delta A_{(n+1)\alpha} - \delta A_{na} = 0, \quad \delta^{2}a_{na} > 0. \quad (22)$$

Since $\Delta x_{na}, \Delta t_{na}, \Delta p_{na}$ and $\Delta E_{na}$ are finite for the $\Delta x_{na}$ and $\Delta t_{na}$ being finite – intrinsically for an intrinsically extensive IED particle[6], (22) must have nontrivial, finite valued solutions. Since $\sum_{\alpha} a_{na} = A_{na}$, to achieve a minimum $A_{na}$ solution for (22a) it suffices, ultimately for the final solution form, that each $a_{na}$ is minimised according to (22b). For $N$ particles in (quasi) stationary states, least action should ultimately be satisfied by all of the $a_{na}$’s of all energy levels of all $N$ particles, and hence by the sum of $a_{na}$’s first only over all $\alpha$ values:

$$I_{\nu} = \sum_{\alpha=1}^{ND} a_{\nu\alpha} = \sum_{\alpha=1}^{ND} \Delta p_{\nu\alpha} \Delta x_{\nu\alpha} = \sum_{\alpha=1}^{ND} 2\Delta T_{\nu\alpha} \Delta t_{\nu\alpha}; \quad \delta I_{\nu} = \delta \sum_{\alpha=1}^{ND} a_{\nu\alpha} = 0, \quad \delta^{2}I_{\nu} > 0. \quad (23)$$

Similarly as discussed for $\Psi_{\nu}$ earlier, a general differentiable function for $I_{\nu}$ for all particles in a global system, and thus a general solution for (23) using calculus, will be absent. However, a minimum $I_{\nu}$ can be readily obtained using algebraic method as follows. For the $N$ particles
in $ND$ position space dimensions the product of all $a_{\nu\alpha}$ over all $\alpha$’s was given by (3) and, with (12), is rewritten as

$$b_{\nu} = \prod_{\alpha=1}^{ND} a_{\nu\alpha} = b_0, \quad \nu = 1, \ldots, \Omega.$$  \hfill (24)

Each $b_{\nu}$ is a fixed value for the $N$-particles in or approaching equilibrium when in a given state $\nu$. All $b_{\nu}$'s are equal to a common $b_0$ as is directly derived for the fixed $N, V, U$ system in Sec 2, and as holds also for a generalised global $N$-particle system in the sense discussed in Sec 3. So $b_0$ is fixed value for a given $N \times D$ value. Then, according to the theorem (a corollary of the theorem of inequality of the arithmetic-geometric means) for finding extrema, the sum $I_{\nu}$ given in (23a) is a minimum if and only if all the positive $a_{\nu\alpha}$’s are equal to one another, and hence to a common value $a_{\nu0}$ for a given $\nu$, and furthermore to a common value $a_0$ for all $\nu$’s since $b_0$ is the same and $ND$ is fixed,

$$a_{\nu1} = a_{\nu2} = \ldots = a_{\nu(ND)} = a_{\nu0} = a_0, \quad \nu = 1, 2, \ldots, \Omega; \quad I_{\nu, \min} = \sum_{\alpha=1}^{ND} a_0 = NDa_0.$$  \hfill (25)

Substituting (19b)–(c) in (25a)–(b) gives

$$\Delta p_{\nu\alpha} \Delta x_{1\alpha} = \ldots = \Delta p_{\nu\alpha} \Delta x_{\nu\alpha} = a_0, \quad \Delta E_{\nu1} \Delta t_{1\alpha} = \ldots = \Delta E_{\nu\alpha} \Delta t_{\nu\alpha} = a_0, \quad \nu = 1, 2, \ldots, \Omega;$$

$$\sum_{\alpha=1}^{ND} \Delta p_{\nu\alpha} \Delta x_{\nu\alpha} = \sum_{\alpha=1}^{ND} 2\Delta T_{\nu\alpha} \Delta t_{\nu\alpha} = NDa_0.$$  \hfill (26)

Substituting (25) in (19a) gives (for $A_{\nu+1} > A_{\alpha}$) $A_{\nu+1} = A_{\nu0} = a_0$. Calculating $A_{\nu0}$ successively, we obtain $A_{\nu1} = A_{\nu0} + a_0, A_{\nu2} = A_{\nu1} + a_0 = A_{\nu0} + 2a_0$, and so on; and

$$A_{\nu0} (= p_{\nu0} \Delta x_{\nu0} = 2\Delta T_{\nu0} \Delta t_{\nu0}) = A_{\nu0} + na_0.$$  \hfill (27)

Conversely, if the sum $I_{\nu}$ of the positive $a_{\nu\alpha}$’s is set to a fixed value, concretely the minimum $I_{\nu, \min}$, then according again to the theorem for finding extrema stated alternatively, in reverse order to the foregoing, the product of the positive $a_{\nu\alpha}$’s, $b_0(= \prod_{\alpha=1}^{ND} a_{\nu\alpha})$, is a maximum if and only if all the positive $a_{\nu\alpha}$’s are equal, as given in (25). With (25) and (26), (24) is written as

$$b_0 = \prod_{\alpha=1}^{ND} a_{\nu\alpha} = \prod_{\alpha=1}^{ND} \Delta p_{\nu\alpha} \Delta x_{\nu\alpha} = \prod_{\alpha=1}^{ND} 2\Delta T_{\nu\alpha} \Delta t_{\nu\alpha} = a_0^{ND} = b_{0, \max}.$$  \hfill (28)

The foregoing inference of the universal constancy of $a_0$ is valid to the extent that the second law of thermodynamics, to which no violation has hitherto been observed, and the least action principle are valid. The latter, the least action principle, too is a general dynamical law as much as the former, in the sense that from it such basic mechanical laws as the Euler-Lagrange equations, Newton’s second law, the Schrödinger equation, and the de Broglie relations (see e.g. [12]), can each be derived.

By its having the dimensions “joule × second”, its universal constancy as mathematically inferred in the foregoing, and its basic relationships with particle dynamical variables as given e.g. by (1), and (26)–(28), $a_0$ is therefore identifiable with the Planck constant $h$,

$$a_0 = h.$$  \hfill (29)
Accordingly (1) identifies with the usual Schrödinger equation, (26a-b) with the Heisenberg uncertainty relations

\[ \Delta p_\alpha \Delta x_\alpha = \ldots = \Delta p_\nu \Delta x_\nu = h, \quad \Delta E_\alpha \Delta t_\alpha = \ldots = \Delta E_\nu \Delta t_\nu = h, \quad (30) \]

(27) with e.g. the de Broglie relations \( p_\alpha \lambda_{dn_\alpha} = n_\alpha h, \quad E_\alpha \tau_{dn_\alpha} = n_\alpha h \) (with \( A_{\alpha \beta} = 0 \), \( \Delta x_{dn_\alpha} = \lambda_{dn_\alpha}, \quad \Delta t_{dn_\alpha} = \tau_{dn_\alpha} \), and \( n_\alpha \) effectively continuous) of a free particle, and (28) with the usual equation for the volume element \( h^{DN} \) accessible to a state in phase space.

5. Conclusion

Based on application of probability theory to the phase space and the one-to-one correspondent Hilbert space of a (globalised) microcanonical ensemble, and by using the algebraic arithmetic-geometric mean inequality for finding extrema, we have induced the universal constancy of \( h \) to be the result of Maupertuis-Jacobi’s least action principle and entropy maximum principle governing the global system. It is by the requirement of the two referred-to first principles that, as an application to particle-wave duality, energy is exchanged between particles or transmitted between locations by one energy quantum \( h\nu \) (or its integral multiples) at a time, often through EM radiation or photon(s). \( h\nu \) has no specific connection in any sense to a geometric point entity in position space.

Acknowledgments

The author expresses thanks to Chairmen Professors C Burdik and M A del Olmo for providing the opportunity to present the author’s research at the 32nd Int Colloq Group Theo Meth Phys, Prague, July, 2018, where the author has enjoyed communications with Drs/Professors Rolf Dahm, Niels Gresnigt, H-D Doebner and other participants, to emeritus scientist P-I Johansson for private financial support of the author’s research, and to Professors B Johansson, I Lindgren, and others for giving moral support to the author’s research.

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