Point-Like Defect on Schrödinger Particles Confined by AB-Flux Field With Harmonic Oscillator Plus Mie-type Potential : Application to Diatomic Molecules

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Abstract: In this analysis, we study a non-relativistic Schrödinger particle confined by the Aharonov-Bohm (AB) flux field with harmonic oscillator plus Mie-type potential in a conical singularity space-time background via point-like global monopole (PGM). We determine the eigenvalue solution analytically and discuss the effects of the topological defects, and the magnetic flux field with this superposed potential. This eigenvalue solution is then utilised in some diatomic molecular potential models (harmonic oscillator plus Kratzer potential, harmonic oscillator plus modified Kratzer potential, and harmonic oscillator plus attractive Coulomb potential) and analyzes the effects on the energy levels and the radial wave function. Afterwards, we consider a general potential form which is the superposition of pseudoharmonic plus Cornell-type potential (or harmonic oscillator with Cornell-type plus inverse quadratic potential) in the quantum system and analyze the effects of various factors on the eigenvalue solution. We see that the eigenvalue solutions shifted due to the topological defects and the magnetic flux field in comparison to flat space results with these potentials.

Keywords: Topological defect, Non-relativistic wave equation, solutions of wave equations: bound-state, geometric quantum phase, special function, potentials

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1 Introduction

Topology plays an important role in various physical systems in different areas of physics, such as gravitational physics, condensed matter physics and cosmology. The presence of topological defects changes the geometrical property of space-time. These defects had formed during a phase transition in the early universe through a spontaneous symmetry-breaking mechanism Refs. [1]. Topological defects are classified into Cosmic strings [2], Domain walls [3], and Global monopole [4] (see also, Ref. [5] for detailed discussions). In condensed matter physics, the
topological defect by vortices in superconductor or super-fluid and domain walls in magnetic materials [6], soliton in 1D polymers [7], and dislocations or disclinations in solids or liquid crystals [8]. Topology change of a medium introduced by a linear defect, such as disclination, dislocation or dispiration in an elastic media produces some effects on the physical properties of the medium [9].

In quantum mechanical systems, the effects of topological defects produced by cosmic string or spinning cosmic string space-time have been studied by several authors in literature in the relativistic limit (see, Refs. [10–14] and related references therein). Researchers solved the wave equations without or with external magnetic and quantum flux fields subject to interaction potentials of various kinds and obtained the eigenvalue solutions using different methods or techniques. In addition, cosmic string space-time has also been studied in the context of the Kaluza-Klein theory by many authors in the literature (see, Refs. [15, 16]) and related references therein). Some other investigations of topological defects are Dirac fermions with Kratzer-like potential under Lorentz symmetry violation in space-time with cosmic screw dislocation [17], a static composite structure under magnetic field in the spiral dislocation space-time [18], the influence of cosmic string space–time with distortion of a radial line into a spiral on the scalar field [19], free fermions in the presence of spiral dislocation of space–time with distortion of a radial line into spiral [20], the interaction of an electron with a nonuniform electric field under cut-off point induced by the spiral dislocation topology [21], spiral dislocation topology on the confinement of a point charge under rotating frame of reference [22], the interaction of an electron with magnetic field in the spiral dislocation space-time [23] etc..

Furthermore, conical singularity effects via point-like global monopole had studied in quantum mechanical systems, for example, in the relativistic limit, the quantum motion of a charged spin-0 particle in the presence of a dyon, Aharonov–Bohm magnetic field and scalar potential [10], quantum oscillators via the Klein-Gordon oscillator [24] and with rainbow gravity [25], the generalized Klein–Gordon oscillator [26], and with scalar potential [27]. In the non-relativistic limit, only a handful of works are known in the literature. These works are harmonic oscillator problem [28], and with physical potential [29], non-relativistic particle interacts with various potential of interest, such as Kratzer and Morse potential [30], generalized Morse potential [31], and diatomic molecular potential [32]. The study of topological defects in the non-relativistic quantum system has some importance and significance because the eigenvalue solutions are influenced by it and gets shifted which breaks the degeneracy and the results modified in comparison to flat space with various potential. As we know, only a few potential has been used to obtain the eigenvalue solutions of the non-relativistic wave equation in a point-like defect. So, the study of the non-relativistic Schrodinger particles or harmonic oscillator problem with other known potentials of physical interest in point-like defects has some significance that is our motivation in this article. If one introduces
the Aharonov-Bohm flux field in a quantum system, then the eigenvalue solutions get more shifted in addition to the topological defects of the geometry under consideration. It is worth mentioning that harmonic oscillator problems in space-time with distortion of a vertical line into a vertical line [33] and non-relativistic particles in space-time with distortion of a vertical line into vertical line [34] have also been studied.

The exact or approximate eigenvalue solutions of the non-relativistic Schrödinger equation (SE) using different techniques or methods with various physical potentials have been investigated in flat space background by many authors. These potentials include a general potential form [35–37], Mie-type potential [38–40], Kratzer potential (KP) [17, 30, 38–49], Morse potential [30, 50], modified Kratzer potential (MKP) [38–40, 49, 51], highly singular potentials [52], hyperbolic Potential [53], Yukawa potential [54, 55], generalized inversely quadratic Yukawa potential [56], non-central potential [57], modified Kratzer plus ring-shaped potential [58], Hulthen potential [59, 60], Manning-Rosen (MR) potential [61–63], Rosen-Morse potential [64, 65], Deng–Fan potential [66], Hyperbolic Poschl–Teller potential [67], pseudo-harmonic potential [68–71] and many more potentials known in literature which have great important in different branches of physics and chemistry. The hydrogen atom and harmonic oscillator are usually given in many textbooks as these two are several exactly solvable problems [72–74]. According to the Schrödinger formulation of quantum mechanics, the total wave function provides all relevant information about the behaviour of a physical system. Hence, if it is exactly solvable for a given potential, the wave function can describe such a system completely.

In this analysis, We study a non-relativistic particle confined by the Aharonov-Bohm flux field in point-like defect with potential of physical interest which is different from those potentials considered in Refs. [29, 30]. The general form of considered potential here is given by [35–37]

\[ V(r) = \beta r^2 + \left( \beta_1 \frac{r}{r^2} + \beta_2 \frac{1}{r^2} + V_0 \right), \]

(1)

where \( \beta = \frac{1}{2} M \omega^2 \) (will be discussed in the subsequent section), \( V_0 \) is a constant, and \( \beta_i, i = -2, -1, 1 \) are parameters characterise the different potential strengths. One can recover a number of well-know physical potential from this general potential form. We solve the non-relativistic wave equation analytically and determine the eigenvalue solutions and analyze the effects of various factors on the energy levels and the wave functions. Note that the first term in this general potential expression is a harmonic oscillator. Alternately, one can see a harmonic oscillator problem in point-like defect with a potential of the general form \( \left( \beta_1 \frac{r}{r^2} + \beta_2 \frac{1}{r^2} + V_0 \right) \) under the Aharonov-Bohm flux field. One can see that this general form of potential (1) is the superposition of pseudo-harmonic plus Cornell-type potential with a constant term \( V_0 \).
This paper is organised as follows: in section 2, we will discuss the Schrödinger wave equation in three dimension in the presence of the Aharonov-Bohm flux field in point-like global monopole space-time background. Then, we solve the radial wave equation with harmonic oscillator plus Mie-type potential and obtain the eigenvalue solution analytically; in section 3, we utilized this eigenvalue solution to some diatomic molecular potential models; in section 4, we solve the radial wave equation with the general potential (1) (harmonic oscillator with Cornell-type plus inverse quadratic potential) and obtain the eigenvalue solution; in section 5, we present our results. We have used the natural units $c = 1 = \hbar$.

2 Non-Relativistic Particles in Point-like Defect with Harmonic Oscillator Plus Mie-Type Potential Under AB-Flux Field

We study the quantum motions of a Schrödinger particle in point-like global monopole (PGM) defect with a physical potential in the presence of the Aharonov-Bohm flux field taking into account the effects of background curvature produced by the geometry. We solve the non-relativistic wave equation analytically and discuss the effects of various factors, such as the topological defects, the magnetic quantum flux and the potential on the eigenvalue solution.

We begin this section with a static and spherically symmetric space-time describing a point-like global monopole in the spherical coordinates $(t, r, \theta, \phi)$ in the context of Einstein’s general relativity given by [10, 27–30, 32–34]

$$ds^2 = -dt^2 + \frac{dr^2}{\alpha^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where $\alpha < 1$ represents the topological defect parameter. Here $(r, \theta, \phi)$ with $0 \leq r < \infty$, $0 \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \phi < 2\pi$ are the spatial coordinates. One of the interesting features of this point-like geometry is that it possesses a curvature singularity on the axis given by

$$R = R_\mu^\mu = \frac{2(1 - \alpha^2)}{r^2}$$

which depends on the topological defect characterised by the parameter $\alpha$. Other properties of this conical singularity space-time were discussed in detail in Refs. [10, 27, 28].

The time-dependent Schrödinger wave equation taking into account the effects of background curvature $R$ of the geometry coupled non-minimally with the field is described by the wave equation [10, 28–30, 32–34, 72–74]

$$-\frac{1}{2M} \left( \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \right) \right) + V(r) \right) \Psi = i \frac{\partial \Psi}{\partial t},$$
where $M$ is the mass of the non-relativistic particle. $\Psi = \Psi(t, r, \theta, \phi)$ is the total wave function, $D_i \equiv \left( \partial_i - ieA_i \right)$ [10, 27, 30, 32, 72, 73], $i = 1, 2, 3$ with $e$ is the electric charges, $A_i$ is the electromagnetic three-vector potential, $g = |g_{ij}|$ is the determinant of the metric tensor with $g^{ij}$ its inverse. For the considered point-like geometry (2) under consideration, its determinant is given by $g = r^2 \sin^2 \theta \frac{\partial^2}{\partial z^2}$.

By the method of separation of variables, one can express the total wave function $\Psi(t, r, \theta, \phi)$ in terms of different variables. Suppose, a possible total wave function in terms of a radial wave function the following radial, azimuthal and polar equations:

$$E \Phi(r) = \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right] \psi(r),$$

where $E$ is the energy of the particles, $Y_{l,m}(\theta, \phi) = A_{l,m}(\theta) B_m(\phi)$ is the standard spherical harmonic functions, and $l, m$ are respectively the angular momentum and magnetic moment quantum numbers.

For the present investigation, we have chosen the following electromagnetic three-vector potential $\vec{A}$ given by Refs. [10, 27, 32]

$$A_r = 0 = A_\theta, \quad A_\phi = \frac{\Phi_{AB}}{2\pi r \sin \theta}, \quad \Phi_{AB} = \Phi \Phi_0, \quad \Phi_0 = 2\pi e^{-1},$$

where $\Phi_{AB} = \text{const}$ is the Aharonov-Bohm magnetic flux, $\Phi_0$ is the quantum of magnetic flux, and $\Phi$ is the amount of magnetic flux which is a positive integer.

Thereby, using Eqs. (5)–(6) into the Eq. (4) and writing in the space-time background (1), we have obtained the following radial, azimuthal and polar equations:

$$\psi''(r) + \frac{1}{\alpha^2} \left[ 2M(E - V(r)) - \frac{(l - \Phi)(l - \Phi + 1)}{r^2} \right] \psi(r) = 0,$$

$$\left( \frac{\partial}{\partial \theta} - i\Phi \right)^2 A_{m'}(\phi) + m^2 A_m(\phi) = 0,$$

$$\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} + \lambda' \right] B_{l', m'}(\theta) = 0, \quad \lambda' = l'(l' + 1),$$

where $m \to m' = (m - \Phi)$ and $l \to l' = (l - \Phi)$ due to the presence of the magnetic flux field in the quantum system. Note that for zero magnetic flux field $\Phi_{AB} \to 0$, one will get back the standard angular equations which were given in many textbooks [72–74]. It’s known that the angular momentum quantum number $l$ is related with the magnetic quantum number $m$ by $l = \kappa + |m|$, where $\kappa = 0, 1, 2, \ldots$. Thus, the presence of the magnetic flux field in the quantum system shifts the angular momentum quantum number $l$ by the same amount $\Phi$ as that of the magnetic quantum number $m$.

From the above radial equation (7), one can easily find the effective potential of the quantum system given by

$$V_{eff} = \left[ \frac{(l - \Phi)(l - \Phi + 1)}{2M \alpha^2 r^2} + \frac{V(r)}{\alpha^2} \right].$$
One can see that for a given potential \( V(r) \), the effective potential of the quantum system depends on the topological defects of the geometry characterised by the parameter \( \alpha \), and the magnetic flux field \( \Phi_{AB} \).

In this analysis, we consider a potential superposition of harmonic oscillator potential \( (\beta r^2) \) [28, 29] and Mie-type potential \( \left( \frac{\beta_{-1}}{r} + \frac{\beta_{-2}}{r^2} + V_0 \right) \) [38–40]. This superposed potential is of the following form

\[
V(r) = \beta r^2 + \left( \frac{\beta_{-1}}{r} + \frac{\beta_{-2}}{r^2} + V_0 \right). \tag{9}
\]

One can see that for \( \beta_{-1} \to 0 \), one will have pseudoharmonic potential [68–71]. For \( \beta_{-2} \to 0 \), the potential reduces to harmonic oscillator plus Coulomb-type potential that has been studied in Ref. [29] in the same spacetime background. Furthermore, for \( \beta \to 0 \), we have Mie-type potential [38–40] from which one can recover some known molecular potential. We have plotted few graphs (fig. 1) of the effective potential of the quantum system.
with different values of the topological defect parameter \( \alpha \), the magnetic flux \( \Phi \), the angular momentum quantum number \( l \), and the non-minimal coupling parameter with this potential (9).

Thereby, substituting the potential (9) in Eq. (7), we have obtained the following radial wave equation:

\[
\psi''(r) + \left( \Lambda - \frac{\gamma_2}{r^2} - \frac{\gamma_1}{r} - \frac{\gamma_2}{r^2} \right) \psi(r) = 0, \tag{10}
\]

where we set the parameters

\[
\Lambda = \frac{2M(E - V_0)}{\alpha^2}, \quad \gamma_2 = \frac{2M\beta}{\alpha^2}, \quad \gamma_1 = \frac{2M\beta + 1}{\alpha^2}, \quad \gamma_2 = \frac{(l - \Phi)(l - \Phi + 1) + 2M\beta - 2}{\alpha^2}. \tag{11}
\]

Let us perform a change of variable via \( x = \left( \gamma_2 \right)^{1/4} r \) into the Eq. (10), we have obtained the following second order differential equation:

\[
\psi''(x) + \left[ \Delta - x^2 - \frac{\gamma_2}{x^2} - \frac{\kappa}{x} \right] \psi(x) = 0, \tag{12}
\]

where

\[
\Delta = \frac{\Lambda}{\sqrt{\gamma_2}}, \quad \kappa = \frac{\gamma_1}{\gamma_2^{1/4}}. \tag{13}
\]

Equation (12) is the one-dimensional Schrödinger wave equation which can be solved using different methods or techniques. Several researchers have employed or applied various methods or techniques, such as the asymptotic iteration method (AIM) \([52, 53]\), the Nikiforov-Uvarov (NU) method \([75]\), supersymmetric quantum mechanics (SUSYQM) \([55–57]\), path integral method (PIM) \([76]\), factorization method \([58]\), exact quantization rule \([41]\) and many more in order to find the exact and approximate solutions of the Schrödinger equation. In this analysis, we approach another method where the eigenvalue solution of Eq. (12) can express as the biconfluent Heun (BCH) functions. The biconfluent Heun polynomial \( H(\rho, \sigma, \lambda, \mu; x) \) can obtain as a Frobenius solution to the BCH equation \([77, 78]\) computed as a power series expansion around the origin \([15, 16, 29, 79]\).

Let us choose a possible solution to the Eq. (12) as follows:

\[
\psi(x) = x^A e^{-Bx^2} H(x), \tag{14}
\]

where \( H(x) \) is an unknown function.

Thereby, substituting Eq. (14) into the Eq. (12), we have arrived the following differential equation

\[
H''(x) + \left[ \frac{2A}{x} - 4Bx \right] H'(x) + \left[ \frac{A^2 - A - \gamma_2}{x^2} - \frac{\kappa}{x} + (4B^2 - 1)x^2 + \left( \Delta - 2B - 4AB \right) \right] H(x) = 0. \tag{15}
\]
Equating the coefficients of $x^{-2}, x^2$ from the third term equals to zero, we have (taking positive values)

$$A^2 - A - \gamma - 2 = 0 \Rightarrow A = \frac{1}{2} (1 + \sqrt{1 + 4 \gamma - 2}) = \frac{1}{2} (1 + 2 j) \quad , \quad j = \sqrt{\gamma - 2 + \frac{1}{4}},$$

$$4B^2 = 1 \Rightarrow B = 1/2. \quad (16)$$

Thereby, using Eq. (16) into the Eqs. (14), we have obtained the following radial wave function

$$\psi(x) = x^{\frac{1}{2}} x^l e^{-\frac{x^2}{2}} H(x) \quad , \quad j = \sqrt{(l - \Phi) (l - \Phi + 1) + 2\gamma\beta - \frac{1}{4}}. \quad (17)$$

And the differential equation from Eq. (15) becomes

$$H''(x) + \left[ \frac{1 + 2 j}{x} - 2x \right] H'(x) + \left[ -\frac{\kappa}{x} + \Pi \right] H(x) = 0. \quad (18)$$

where

$$\Pi = \Delta - 2 (1 + j). \quad (19)$$

Equation (18) is the biconfluent Heun differential equation form [15, 16, 29, 77, 78] and $H(x)$ is the Heun polynomial function.

The above equation (18) can be solved by using the Frobenius series solution method. Writing the solution as a power series expansion around the origin [79]:

$$H(x) = \sum_{i=0}^{\infty} d_i x^i. \quad (20)$$

Substituting the power series solution (20) into the Eq. (18), one will find the following recurrence relation:

$$d_{n+2} = \frac{1}{(n + 2)(n + 2 + 2 j)} \left[ \kappa d_{n+1} - (\Pi - 2n) d_n \right]. \quad (21)$$

With the few coefficients

$$d_1 = (\frac{\kappa}{1 + 2 j}) d_0 \quad , \quad d_2 = \frac{1}{4(1 + j)} \left( \kappa d_1 - \Pi d_0 \right) \quad , \quad d_3 = \frac{1}{6(j + \frac{1}{2})} \left( \kappa d_2 - 2d_1 \right). \quad (22)$$

In quantum theory, it is required that the wave function $\psi(x)$ must be well-behaved and regular everywhere for $x \to 0$ and $x \to \infty$. One can find the eigenvalue solution by imposing a condition on the Heun function $H(x)$ that it must be a finite degree polynomial of $x$ with degree $n$. Through the recurrence expression (16), we can see that this power series expansion $H(x)$ becomes a polynomial of degree $n$ provided the following conditions fulfilled [15, 16, 29]

$$\Pi = 2n(n = 1, 2, 3, \ldots) \quad , \quad d_{n+1} = 0 \quad (23)$$
such that from Eq. (21), we have \( d_{n+2} = 0 \) and now the Heun function is a polynomial solution of finite degree, 
\[
H(x) = d_0 + d_1 x + d_2 x^2 + \ldots + d_n x^n.
\]

After simplification of the condition \( \Pi = 2n \), we have obtained the following expression of the energy eigenvalues \( E_{n,l} \) given by

\[
E_{n,l} = V_0 + \alpha \sqrt{\frac{2\beta}{M}} \left( n + 1 + \sqrt{\frac{(l - \Phi)(l - \Phi + 1) + 2M\beta_{-2}}{\alpha^2} + \frac{1}{4}} \right). \tag{24}
\]

Equation (24) is the energy eigenvalue expression (non-compact) of a non-relativistic particle confined by the AB-flux field with harmonic oscillator plus Mie-type potential in point-like defect taking into account the background curvature associated with the topological defects. To have the complete information about a quantum system, one must analyze the second condition \( d_{n+1} = 0 \) for each radial mode as done in Refs. [15, 16, 29]. As for example, for the radial mode \( n = 1 \), we have \( \Pi = 2 \) and \( d_2 = 0 \) which implies from Eq. (22) that

\[
\frac{2}{\kappa} d_0 = \left( \frac{\kappa}{1 + 2j} \right) d_0 \Rightarrow \beta^{1,j} = \frac{M^3}{2\alpha^6} \left( \frac{\beta_{-1}^2}{j + \frac{1}{2}} \right)^2 \tag{25}
\]

a constraint on the parameter \( \beta \to \beta^{1,j} \) that depends on the topological defects characterise by the parameter \( \alpha \), the magnetic flux \( \Phi \), and the potential parameter \( \beta_{-1} \).

Therefore, the ground state energy level defined by the radial mode \( n = 1 \) is given by

\[
E_{1,l} = V_0 + \frac{M}{\alpha^2} \frac{\beta_{-1}^2}{\sqrt{\left(\frac{(l - \Phi)(l - \Phi + 1) + 2M\beta_{-2}}{\alpha^2} + \frac{1}{4}\right)}} \left( 2 + \sqrt{\frac{(l - \Phi)(l - \Phi + 1) + 2M\beta_{-2}}{\alpha^2} + \frac{1}{4}} \right). \tag{26}
\]

And the corresponding radial wave function is

\[
\psi_{1,l}(x) = x^{\frac{1}{2}(1+2j)} e^{-\frac{x^2}{2}} (d_0 + d_1 x), \quad d_1 = \frac{1}{\sqrt{j + \frac{1}{2}}} d_0, \tag{27}
\]

where \( j \) is defined in (17).

Similarly, for the radial mode \( n = 2 \), we have \( \Pi = 4 \) and \( d_3 = 0 \) which implies from Eq. (22) that

\[
d_2 = \frac{2}{\kappa} d_0 \Rightarrow \kappa = 4 \sqrt{j + \frac{3}{4}} \Rightarrow \beta^{1,j} = \frac{M^3}{32\alpha^6} \left( \frac{\beta_{-1}^2}{j + \frac{3}{4}} \right)^2 \tag{28}
\]

another constraint on the parameter \( \beta \to \beta^{2,j} \). Thus, we can see that for each radial mode, we have a different relation of the potential parameter \( \beta \to \beta^{n,j} \) that depends on the topological defects, the magnetic flux field, and the potential parameter \( \beta_{-1} \).
Therefore, the energy level for the radial quantum number \( n = 2 \) is given by

\[
E_{2,l} = V_0 + \frac{M}{4 \alpha^2} \left( \frac{\beta_{-1}^2}{\sqrt{(l-\Phi)(l-\Phi+1)+2M\beta_{-2}}} + \frac{1}{4} + \frac{3}{2} \right) \left( 3 + \sqrt{\frac{(l-\Phi)(l-\Phi+1)+2M\beta_{-2}}{\alpha^2}} + \frac{1}{4} \right). \tag{29}
\]

And the corresponding bound-state radial wave function is

\[
\psi_{2,l}(x) = x^{\frac{j+3}{4}} e^{-\frac{x^2}{2}} (d_0 + d_1 x + d_2 x^2), \tag{30}
\]

where the coefficients are

\[
d_1 = 2 \sqrt{\frac{j+3}{4}} d_0, \quad d_2 = \frac{1}{\left( j + \frac{3}{2} \right)} d_0. \tag{31}
\]

We can see that the energy levels \( E_{1,l}, E_{2,l}, E_{3,l}, \ldots \) and the radial wave functions \( \psi_{1,l}, \psi_{2,l}, \psi_{3,l}, \ldots \) are influenced by the topological defects characterised by the parameter \( \alpha \) and the magnetic flux field. The presence of the magnetic flux field \( \Phi_{AB} \) shifts the energy levels and the wave function more in addition to the topological defects which shows an analogue of the Aharonov-Bohm effect \([80, 81]\).

3 APPLICATIONS TO DIATOMIC MOLECULAR POTENTIALS

The above result of the quantum system with harmonic oscillator plus Mie-type potential is now being utilized to develop solutions to some specific types of interacting molecular potentials which have wide application in practical problems.

3.1 HARMONIC OSCILLATOR PLUS KRATZER POTENTIAL

The harmonic oscillator plus Kratzer potential can be recovered by setting the parameters \( \beta = \frac{1}{2} M \omega^2, \beta_{-1} = -2 D_e r_0, \beta_{-2} = D_e r_0^2, \) and \( V_0 = 0 \) in the potential expression (9), we have the following potential form

\[
V(r) = \frac{1}{2} M \omega^2 r^2 - \frac{2 D_e r_0}{r} + \frac{D_e r_0^2}{r^2}. \tag{32}
\]

That may be written as

\[
V(r) = \frac{1}{2} M \omega^2 r^2 + 2 D_e \left[ \frac{1}{2} \left( \frac{r_0}{r} \right)^2 - \frac{r_0}{r} \right]. \tag{33}
\]

Here \( \omega \) is the oscillator frequency, \( D_e \) is the dissociation energy between two atoms in a solid, \( r_0 \) is the equilibrium inter-nuclear separation. This Kratzer potential has of great importance in molecular physics and quantum
chemistry [42, 43], in inter-nuclear vibration of diatomic molecules [44, 45] as well as other branches of physics
and chemistry [38–40, 46–49].

Thus, using the above potential Eq. (33) in the radial Eq. (7) and following the previous procedure, one will
have the following energy eigenvalue expression

$$E_{n,l} = \alpha \omega_{n,l} \left( n + 1 + \sqrt{\frac{(l - \Phi)(l - \Phi + 1) + 2MD_e r_0^2}{\alpha^2} + \frac{1}{4}} \right), \quad (34)$$

Equation (34) is the energy eigenvalue expression of a non-relativistic particle confined by the Aharonov-
Bohm flux field with harmonic oscillator plus Kratzer potential in point-like global monopole geometry.

As done earlier, one can evaluate the individual energy levels and the radial wave functions one by one. For
example, for the radial mode $n = 1$, we have a constraint on the oscillator frequency given by

$$\omega_{1,l} = \left( \frac{4MD_e^2 r_0^2}{\alpha^2} \right) \frac{1}{(\zeta + \frac{1}{2})}. \quad (35)$$

Its value changes with change in the quantum numbers \{\(n, l\}\} and depends on the topological defects characterise
by the parameter $\alpha$, and the magnetic flux $\Phi$.

Therefore, the ground state energy level and the radial wave function for the radial mode $n = 1$ are given by

$$E_{1,l} = \left( \frac{4MD_e^2 r_0^2}{\alpha^2} \right) \frac{(\zeta + 2)}{(\zeta + \frac{1}{2})}, \quad \psi_{1,l}(x) = x^{\frac{1}{2} + \zeta} e^{-\frac{x^2}{2}} (d_0 + d_1 x), \quad d_1 = \frac{1}{\sqrt{\zeta + \frac{1}{2}}} d_0, \quad (36)$$

where

$$\zeta = \sqrt{\frac{(l - \Phi)(l - \Phi + 1) + 2MD_e r_0^2}{\alpha^2} + \frac{1}{4}}. \quad (37)$$

Similarly, for the radial quantum number $n = 2$, we have another constraint on the oscillator frequency $\omega \rightarrow \omega_{2,l}$ given by

$$\omega_{2,l} = \frac{MD_e^2 r_0^2}{\alpha^2 \left( \zeta + \frac{3}{4} \right)}. \quad (38)$$

Therefore, the energy level and the corresponding radial wave function for the radial mode $n = 2$ are given by

$$E_{2,l} = \left( \frac{MD_e^2 r_0^2}{\alpha^2} \right) \frac{(\zeta + 3)}{(\zeta + \frac{3}{2})}, \quad \psi_{2,l}(x) = x^{\frac{1}{2} + (1 + 2\zeta)} e^{-\frac{x^2}{2}} (d_0 + d_1 x + d_2 x^2),$$

$$d_1 = \frac{2}{\left( \frac{1}{\sqrt{\zeta + \frac{3}{4}}} \right) d_0}, \quad d_2 = \frac{1}{\left( \frac{1}{\sqrt{\zeta + \frac{1}{2}}} \right) d_0}, \quad (39)$$

where $\zeta$ is given in (37).
Thus, we can see that the energy levels $E_{1,l}, E_{2,l}, E_{2,l}, \ldots$ and the radial wave function $\psi_{1,l}, \psi_{2,l}, \psi_{2,l}, \ldots$ of harmonic oscillator are influenced by the topological defects characterised by the parameter $\alpha$, and the magnetic flux field $\Phi_{AB}$ in point-like global monopole and gets modified in comparison to flat space result with this harmonic oscillator plus Kratzer potential.

### 3.2 Harmonic Oscillator plus Modified Kratzer or Kratzer-Fues Potential

The harmonic oscillator plus modified Kratzer or Kratzer-Fues potential can be recovered by setting the parameters $eta_2 = \frac{1}{4} M \omega^2$, $\beta_{-1} = -2 D_e r_0$, $\beta_{-2} = D_e r_0^2$, and $V_0 = D_e$ in potential (9). Thus, we have the following potential form

$$V(r) = \frac{1}{2} M \omega^2 r^2 - \frac{2 D_e r_0}{r} + \frac{D_e r_0^2}{r^2} + D_e.$$  

That may be written as

$$V(r) = \frac{1}{2} M \omega^2 r^2 + D_e \left( \frac{r}{2} \right).$$  

The second term in the above potential is called the modified Kratzer or Kratzer-Fues potential and has been used by several authors in literature [38, 40, 49, 51].

Therefore, using the above potential (41) in the radial Eq. (7) and following the previous procedure, one will have the following energy eigenvalue expression

$$E_{n,l} = D_e + \alpha \omega_{n,l} \left( n + 1 + \sqrt{\frac{(l - \Phi)(l - \Phi + 1) + 2 M D_e r_0^2}{\alpha^2} + \frac{1}{4}} \right).$$  

Equation (42) is the energy spectra of a harmonic oscillator confined by the Aharonov-Bohm flux field with this modified Kratzer potential in point-like global monopole.

Here also, one can evaluate the individual energy level and the corresponding radial wave function one by one as done earlier. The ground state energy level $E_{1,l}$ and the radial wave function $\psi_{1,l}$ for the radial mode $n = 1$ are given by

$$E_{1,l} = D_e + \left( \frac{4 M D_e r_0^2}{\alpha^2} \right) \left( \frac{2 + \zeta}{\zeta + \frac{1}{2}} \right), \quad \psi_{1,l}(x) = x^{\frac{1}{2}(1+2\zeta)} e^{-\frac{1}{2} x} (d_0 + d_1 x), \quad d_1 = \frac{1}{\sqrt{\zeta + \frac{1}{2}}} d_0,$$

where $\zeta$ is given by (37).
Similarly, for the radial mode \( n = 2 \) the energy level \( E_{2,l} \) and the radial wave function \( \psi_{2,l} \) are given by

\[
E_{2,l} = D_n + \left( \frac{MD_0^2r_0^2}{\alpha^2} \right) \left( \frac{\xi + 3}{\xi + \frac{3}{2}} \right), \quad \psi_{2,l}(x) = x^\frac{1}{2}(1+2\xi) e^{-\frac{x^2}{2\tau}} (d_0 + d_1 x + d_2 x^2),
\]

\[
d_1 = 2 \frac{\sqrt{\xi + \frac{3}{2}}}{(\xi + \frac{1}{2})^2} d_0, \quad d_2 = \frac{1}{(\xi + \frac{1}{2})^2} d_0.
\]

(44)

Thus, we can see that the energy levels \( E_{1,l}, E_{2,l}, E_{3,l}, \ldots \) and the radial wave function \( \psi_{1,l}, \psi_{2,l}, \psi_{3,l}, \ldots \) of harmonic oscillator are influenced by the topological defects characterise by the parameter \( \alpha \), the background curvature associated with topological defects, and the magnetic flux field \( \Phi_{AB} \) in point-like global monopole and gets modified in comparison to flat space results with this harmonic oscillator plus modified Kratzer potential.

3.3 HARMONIC OSCILLATOR PLUS COULOMB POTENTIAL

The harmonic oscillator with attractive Coulomb potential can be recovered from the potential (9) by setting the parameters \( \beta_2 = \frac{1}{2} M \omega^2, \beta_{-1} = -\eta_c, \beta_{-2} = 0, \) and \( V_0 = 0 \). Thus, we have the following potential form

\[
V(r) = \frac{1}{2} M \omega^2 r^2 + \left( -\frac{\eta_c}{r} \right).
\]

(45)

Thereby, substituting this potential (45) in the radial Eq. (7) and following the same procedure done earlier, one will have the following energy eigenvalue expression

\[
E_{n,l} = \omega_{n,l} \left[ (n + 1) \alpha + \sqrt{(l - \Phi)(l - \Phi + 1) + \frac{\alpha^2}{4}} \right].
\]

(46)

The corresponding radial wave function

\[
\psi_{n,l}(x) = x^\frac{1}{2} \tau^x e^{-\frac{x^2}{2\tau}} H(x), \quad \tau = \sqrt{(l - \Phi)(l - \Phi + 1) + \frac{\alpha^2}{4}}.
\]

(47)

Now, we evaluate the individual energy levels and radial wave function one by one. For example, for the radial mode \( n = 1 \), one will find a constraint on the oscillator frequency given by

\[
\omega_{1,l} = \frac{M \eta_c^2}{\alpha^3 \left( \tau + \frac{1}{2} \right)}.
\]

(48)

The ground state eigenvalue solution is given by

\[
E_{1,l} = \omega_{1,l} \left[ 2 \alpha + \sqrt{(l - \Phi)(l - \Phi + 1) + \frac{\alpha^2}{4}} \right], \quad \psi_{1,l}(x) = x^\frac{1}{2}(1+2\tau) e^{-\frac{x^2}{2\tau}} (d_0 + d_1 x), \quad d_1 = \frac{1}{\sqrt{\tau + \frac{1}{2}}} d_0.
\]

(49)
where $\tau$ is given in Eq. (47) and $\omega_{1,l}$ in (48).

Similarly, for the radial mode $n = 2$, we have another constraint on the oscillator frequency given by

$$\omega_{2,l} = \frac{M n^2}{4 \alpha^3 \left( \frac{\tau}{2} + \frac{3}{4} \right)}.$$  \hspace{1cm} (50)

The energy level $E_{2,l}$ and the radial wave function $\psi_{2,l}$ for the radial mode $n = 2$ are given by

$$E_{2,l} = \omega_{2,l} \left[ \frac{3}{2} \frac{\alpha + \sqrt{(l - \Phi)(l - \Phi + 1) + \frac{\alpha^2}{4}}}{(l + \frac{1}{2})} \right], \quad \psi_{2,l}(x) = x^{l + \frac{1}{2}} e^{-\frac{x^2}{4}} (d_0 + d_1 x + d_2 x^2),$$

$$d_1 = 2 \sqrt{\frac{\tau + \frac{3}{4}}{(\tau + \frac{1}{2})}} d_0, \quad d_2 = \frac{1}{(\tau + \frac{1}{2})} d_0,$$  \hspace{1cm} (51)

where $\tau$ is given in Eq. (47) and $\omega_{2,l}$ in (50).

One can see that the energy levels $E_{1,l}, E_{2,l}, ...$ and the radial wave function $\psi_{1,l}, \psi_{2,l}, ...$ of a harmonic oscillator with an attractive Coulomb potential under the AB-flux field in point-like monopole.

It is worth mentioning that for zero magnetic flux field $\Phi_{AB} \to 0$, the energy levels $E_{1,l}, E_{2,l}, ...$ and the radial wave function $\psi_{1,l}, \psi_{2,l}, ...$ with the constraint on the oscillator frequency $\omega_{1,l}, \omega_{2,l}, ...$ reduces to the result obtained in Ref. [29]. Thus, the presence of the magnetic flux $\Phi$ in the quantum system modified the eigenvalue solution and shows an analogue of the Aharonov-Bohm effect [80, 81].

4 Non-Relativistic Particles Confined by AB-flux field in Point-like Defect with Pseudoharmonic plus Cornell-type Potential

In this section, we will study the non-relativistic particle in the presence of the AB-flux field in point-like defect with potential (1) given by

$$V(r) = \beta_1 r^2 + \beta_1 r + \frac{\beta_{-1}}{r} + \frac{\beta_2}{r^2} + V_0,$$  \hspace{1cm} (52)

where $\beta_1$ is a parameter characterise the linear confining potential and others are mentioned earlier. We can see that this general form of potential is the superposition of pseudo-harmonic plus Cornell-type potential or harmonic oscillator potential and inverse quadratic plus Cornell-type potential. For $\beta_{-1} \to 0$ and $\beta_1 \to 0$, the above potential reduces to a pseudo-harmonic potential [68–71]. Furthermore, for $\beta_{-2} \to 0$, the potential reduces to a harmonic oscillator plus Cornell-type potential that has been studied in Ref. [29]. Using the above potential (52) in (8), we have plotted few graphs of the effective potential of the quantum system showing the influences of various factors, such as topological defects characterise by the parameter $\alpha$, non-minimal coupling parameter $\xi$, and the magnetic flux $\Phi$ (fig2. 2).
Thereby, substituting the above potential into the Eq. (7), we have obtained the following radial wave equation:

$$\psi''(r) + \left( \Lambda - \gamma_1 r - \gamma_2 r^2 - \gamma_{-1} - \gamma_{-2} r^2 \right) \psi(r) = 0,$$

where we have defined

$$\Lambda = \frac{2 M (E - V_0)}{\alpha^2}, \quad \gamma_1 = \frac{2 M \beta_1}{\alpha^2}, \quad \gamma_{-1} = \frac{2 M \beta_{-1}}{\alpha^2}, \quad \gamma_2 = \frac{2 M \beta}{\alpha^2},$$

$$\gamma_{-2} = \frac{(l - \Phi) (l - \Phi + 1) + 2 M \beta_{-2}}{\alpha^2}. \quad (54)$$

Let us perform a change of variable via $x = (\gamma_2)^{1/4} r$ into the above Eq. (53), we have obtained the following second order differential equation:

$$\psi''(x) + \left[ \Delta - \chi x - x^2 - \frac{\gamma_{-2}}{x^2} - \frac{\kappa}{x} \right] \psi(x) = 0,$$
where

\[ \Delta = \frac{\Lambda}{\sqrt{\gamma^2}}, \quad \chi = \frac{\gamma_1}{\gamma_2^{1/4}}, \quad \kappa = \frac{\gamma_1 - 1}{\gamma_2^{1/4}}. \]  

Equation (63) is the one-dimensional Schrödinger radial wave equation. As stated earlier, we can solve this equation using the BCH procedure. Let us choose a possible solution to the Eq. (55) as follows:

\[ \psi(x) = x^A e^{-\left(Bx^2 + Cx\right)} H(x), \]  

(57)

where \( H(x) \) is an unknown function.

Substituting Eq. (57) into the Eq. (55), we have arrived the following equation

\[
H''(x) + \left[ \frac{2A}{x} - 2C - 4Bx \right] H'(x) \\
+ \left[ \frac{A^2 - A - \gamma-2}{x^2} - \frac{2AC + \kappa}{x} + (4BC - \chi) x + (4B^2 - 1)x^2 + \left( \Delta - 2B - 4AB + C^2 \right) \right] H(x) = 0.
\]  

(58)

Equating the coefficients of \( x^{-2}, x^1, x^2 \) from the third term equals to zero, we have

\[ A^2 - A - \gamma-2 = 0 \Rightarrow A = \frac{1}{2} \left( 1 + \sqrt{1 + 4 \gamma - 2} \right) = \frac{1}{2} \left( 1 + 2j \right), \quad j = \sqrt{\gamma - 2 + \frac{1}{4}}, \]

\[ 4BC = \chi \Rightarrow C = \frac{\chi}{2}, \]

\[ 4B^2 = 1 \Rightarrow B = 1/2. \]  

(59)

From Eq. (58), one will arrive at the following differential equation

\[
H''(x) + \left[ \frac{1 + 2j}{x} - 2x - \chi \right] H'(x) + \left[ - \frac{\zeta}{x} + \sum \right] H(x) = 0.
\]  

(60)

where

\[ \zeta = \kappa + \frac{\chi}{2} \left( 1 + 2j \right), \quad \sum = \Delta + \frac{\chi^2}{4} - 2(1 + j). \]  

(61)

Equation (60) is the biconfluent Heun’s differential equation form [15, 16, 29, 77, 78] and \( H(x) \) is the Heun polynomial function.

The radial wave function is given by

\[ \psi(x) = x^{\frac{1}{2}(1+2j)} e^{-\frac{1}{2}(x + \chi)x} H(x), \]  

(62)

where \( j \) is defined in Eq. (17).
Substituting the power series solution Eq. (20) into the Eq. (60), one will find a recurrence relation of the following form
\[
d_{n+2} = \frac{1}{(n+2)(n+2+2j)} \left\{ \kappa + \chi \left( n + j + \frac{3}{2} \right) \right\} d_{n+1} - \left( \sum -2n \right) d_n. \tag{63}
\]

As stated earlier, the wave-function \( \psi(x) \) must be well-behaved for \( x \to 0 \) and \( x \to \infty \). One can obtain the bound-state solutions by imposing a condition on the Heun function \( H(x) \) that it must be a finite degree polynomial of degree \( n \). Through the expression (63), one can see that the power series expansion \( H(x) \) becomes a polynomial of degree \( n \) provided \([15, 16, 29]\) we have
\[
\sum = 2n (n = 1, 2, 3, \ldots) , \quad d_{n+1} = 0. \tag{64}
\]

After simplification of the first condition \( \sum = 2n \), we have obtained the following expression of the energy \( E_{n,l} \) given by
\[
E_{n,l} = V_0 + \alpha \sqrt{\frac{2\beta}{M}} \left( n + 1 + \sqrt{\left( \frac{l - \Phi}{l - \Phi + 1} \right) \frac{2M\beta - 2}{\alpha^2} + \frac{1}{4}} \right) - \frac{\beta^2}{4\beta}. \tag{65}
\]

And the radial wave function is given by
\[
\psi_{n,l}(x) = x^{\frac{1}{2}} (1+2j)^{-\frac{1}{4}} \left[ e^{-\frac{1}{2} \left( \frac{2M\beta}{\alpha^2 \beta} \right)^{1/4}} \right] x^{j} H(x), \tag{66}
\]

where \( j = \sqrt{\left( \frac{l - \Phi}{l - \Phi + 1} \right) \frac{2M\beta - 2}{\alpha^2} + \frac{1}{4}} \).

Equation (65) is the non-compact expression of the energy profile of a non-relativistic particle in the presence of the Aharonov-Bohm flux field with a physical potential of the general form (52) in point-like defect. We can see that the energy eigenvalue Eq. (65) gets modified in comparison to the result obtained in Ref. [29] due to the presence of an additional inverse quadratic potential term \( \sim \frac{1}{r^2} \) in (52), and the Aharonov-Bohm flux field \( \Phi_{AB} \) which shows an analogue of the Aharonov-Bohm effect \([80, 81]\).

5 Conclusions

To sum up, in this paper, we have studied the non-relativistic Schrodinger equation in three dimensions in the presence of the Aharonov-Bohm flux field with potential in point-like global monopole defect. We have shown that the eigenvalue solutions of the non-relativistic particle depend on the non-trivial topological features of point-like global monopole space-time and the magnetic flux field. We have applied the wave function ansatz method to obtain analytical the eigenvalue solutions of the radial wave equation. We have verified that the global feature
of the geometry under consideration is present explicitly in the structure of states and energy levels and shifts the eigenvalue solutions in comparison to flat space results. Also, we have seen that the background curvature of the geometry coupled non-minimally with the field in Eq. (4) influences the eigenvalue solutions that were not considered in the previous analysis Refs. [28–31]. Furthermore, the presence of a magnetic flux field shifts the energy levels and the radial wave function more in addition to the topological defects. This dependence of the eigenvalue solutions on the geometric quantum phase gives us an analogue of the Aharonov-Bohm effect [80, 81].

In section 2, we have considered a harmonic oscillator plus Mie-type potential and derived the radial wave equation which becomes a biconfluent Heun (BCH) differential equation form after a few mathematical steps. We then solved this equation using a power series solution method and obtained the non-compact energy eigenvalue expression Eq. (24) and the radial wave function Eq. (17). In Minkowski flat space background, that is, $\alpha \to 1$, the energy eigenvalue of the non-relativistic particles with this harmonic oscillator plus Mie-type potential becomes

$$E_{n,l} = V_0 + \sqrt{\frac{2\beta}{M}} \left( n + 1 + \sqrt{l(l+1) + 2M\beta_2 + 1} \right).$$  \hspace{1cm} (67)

Thus, we have seen that the presence of the topological defect of point-like global monopole space-time characterises by the parameter $\alpha$ modifies the energy eigenvalue (24) as well as the wave function in comparison to flat space and breaks the degeneracy of the energy levels. In addition, the presence of the AB-flux field in the quantum system shifts the energy eigenvalue and the wave function in comparison to flat space result. As the energy eigenvalue expression (24) in non-compact, we have employed the condition $d_{n+1} = 0$ for each radial mode defined by $n = 1, 2, 3, \ldots$ as done in Refs. [15, 16, 29]. In this analysis, we have evaluated two such individual energy levels $E_{1,l}, E_{2,l}$ and the corresponding radial wave function $\psi_{1,l}, \psi_{2,l}$ for the radial mode $n = 1, 2$ and others are in the same way.

In section 3, we have utilized the above results of the quantum system under investigation to some known diatomic molecular potential models. The first one is a harmonic oscillator with Kratzer potential (sub-section: 3(a)) and using this potential, we have obtained the energy eigenvalue expression (34). The second one is a harmonic oscillator with modified Kratzer or Kratzer-Fues potential (sub-section: 3(a)) and using this potential, we have obtained the energy eigenvalue expression (42). Finally, we have considered a harmonic plus attractive Coulomb potential (sub-section: 3(a)) and obtained the energy eigenvalue expression (46). In all cases, we have evaluated the individual energy levels $E_{1,l}, E_{2,l}$ and the radial wave functions $\psi_{1,l}, \psi_{2,l}$ as done earlier. One can see that the eigenvalue solutions of non-relativistic particles with various combined potential gets modified by the various factors, such as the topological defects of the point-like global monopole, and the magnetic flux field in comparison to those results obtained in flat space.
In section 4, we have considered potential of the general form given by $V(r) = \left(\beta r^2 + \beta_1 r + \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + V_0\right)$ and solved the non-relativistic wave equation under the considered space-time background. Following the previous procedure, we obtained the energy eigenvalue expression (65) and the radial wave function Eq. (66). Here also, one can observe that the topological defects of the geometry characterises by the parameter $\alpha$, and the magnetic flux field $\Phi_{AB}$ shifts the energy levels and the radial wave functions and get modified in comparison to flat space result with this general form of potential.

Conflict of Interest

There is no conflict of interests in this paper.

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