A Recursive Equations Based Representation for the $G/G/m$ Queue

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Abstract

New recursive equations designed for the $G/G/m$ queue are presented. These equations describe the queue in terms of recursions for the arrival and departure times of customers, and involve only the operations of maximum, minimum and addition.

Key-Words: multi-server queues, recursive equations, ordering operator.

1 Introduction

As a representation of dynamics of queueing systems, recursive equations have been introduced by Lindley in his classical investigation of the $G/G/1$ queue [1]. In the last few years the representations based on recursive equations have been extended to a variety of systems which consist of single-server queues and operate according the first-come, first-served (FCFS) discipline. Specifically, there are the equations designed for open and close tandem queues with both infinite and finite buffers [2, 3, 4]. Recursive equations have been also derived to represent more complicated systems of $G/G/1$ queues, including acyclic fork-join networks [5, 6] and closed networks with a general deterministic routing mechanism [7, 8, 9].

Recursive equations find a wide application in recent works on both analytical study and simulation of queueing systems. As an analytical tool, they were exploited to investigate system performance measures [2, 7, 10] and estimates of their gradients [7, 8, 9, 11]. In [5, 6] recursive equations based representations have provided the means for establishing the stability conditions and deriving bounds on system performance in a class of queueing systems. Finally, these representations made it possible to develop efficient algorithms of queueing systems simulation [8, 11] as well as powerful methods of estimating gradients of system performance measures [7, 9, 11].

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In the queueing theory, the $G/G/m$ queue is normally represented by using the recursions introduced by Kiefer and Wolfowitz in [12]. Expressed in rather general terms, these recursions may be inconvenient to use if an explicit form of representation is required. The purpose of this paper is to present new recursive equations designed for the $G/G/m$ queue. These equations describe the queue in terms of the arrival and departure times of customers, and involve only the operations of maximum, minimum and addition.

The rest of the paper is organized as follows. In Section 2 we briefly describe both the $G/G/1$ and $G/G/m$ queues, and give preliminary analysis of their representations. Section 3 includes two technical lemmas which offer useful representations for the ordering operator. Finally, in Section 4 we present our main result providing an explicit recursive representation for the $G/G/m$ queue.

2 Definitions and Preliminary Analysis

Recursive equations as a representation of queueing systems were originally expressed in terms of waiting times [1, 12]. Equations of this classical type continue in use (see, e.g., [10]); however in many recent works [2, 3, 4, 5, 6, 7, 8, 9, 11] one can find the equations describing system dynamics through recursions for arrival and departure times. The last approach to the representation of queues may be considered as a useful generalization of the classical one, and it will be applied below to derive recursive equations for the $G/G/m$ queue.

We start with the equations for the $G/G/1$ queue, which provide the basis for representing more complicated systems including the $G/G/m$ queue. To set up these equations, consider a single server queue with infinite buffer capacity and the FCFS queue discipline. Denote the interarrival time between the $k$th customer and his predecessor by $\alpha_k$, and the service time of the $k$th customer by $\tau_k$. We assume $\alpha_k \geq 0$ and $\tau_k > 0$, for any $k = 1, 2, \ldots$. Furthermore, let $A_k$ be the $k$th arrival epoch to the queue, and $D_k$ be the $k$th departure epoch from the queue. As is customary, the sequences $\{\alpha_k\}_{k \geq 1}$ and $\{\tau_k\}_{k \geq 1}$ are assumed to be given, whereas $\{A_k\}_{k \geq 1}$ and $\{D_k\}_{k \geq 1}$ are considered unknown. Finally, provided that the queue starts operating at time zero, it is convenient to set $A_k \equiv 0$ and $D_k \equiv 0$ for all $k \leq 0$.

Using these notations, the recursive equations for the $G/G/1$ queue may be written as [2, 3, 4]

$$A_k = A_{k-1} + \alpha_k$$
$$D_k = (A_k \vee D_{k-1}) + \tau_k,$$

where $\vee$ denotes the maximum operator, $k = 1, 2, \ldots$
The first equation in (1) is trivial. To understand the second recursion it is sufficient to see that the term \( A_k \lor D_{k-1} \) determines the service initiation time for the \( k \)th customer. Clearly, the service of this customer may be initiated not earlier than he arrives at the server. If the \( k \)th customer finds the server busy, he has to wait until service of the \((k - 1)\)st customer completes. In other words, the time of the \( k \)th initiation of service coincides with the maximum out of \( A_k \) and \( D_{k-1} \).

Taking (1) as the starting point, we now turn to the discussion of multi-server queues. Consider a queueing system with an infinite buffer and \( m \), \( 1 \leq m < \infty \), servers operating in parallel. When a customer arrives, he occupies any one of those servers which are not busy. If there are no free servers at his arrival, the customer takes his place in the buffer and starts waiting to be served. His service begins as soon as any one of the servers is unoccupied, provided that all his predecessors have left the buffer.

We may extend all the definitions introduced above to this queueing system. Note, however, that in the multi-server queue the \( k \)th departure time may not coincide with the completion time of the \( k \)th customer. One can consider the first customer as an example. Being the first to initiate service, he may prove to be the \( k \)th to depart from the system, for any \( k \geq 1 \). To take account of the distinction between these times, we further introduce the notation \( C_k \) for the completion time of the \( k \)th customer, \( k = 1, 2, \ldots \)

Since upon their service completions the customers immediately leave the system, it is easy to see that both the sequences of completion times and departure times are constituted by the same elements. In fact, the sequence \( \{D_k\}_{k \geq 1} \) is simply the sequence \( \{C_k\}_{k \geq 1} \) arranged in ascending order. Therefore, an operator which produces ordered values is required to express the relation between these sequences.

The idea to apply some ordering operator in recursive equations based representations of the \( G/G/m \) queue had its origin in [12]. However, this operator, as it has been introduced in [12] and occurs in recent works (see, e.g., [6]), is expressed in general terms rather than in an explicit form. It will be shown in the next section how the ordering operator may be represented in closed form as a function of values being ordered.

### 3 Representations of the Ordering Operator

Let \( \{r_j\}_{j=1}^n = \{r_1, \ldots, r_n\} \) be a finite set (sequence) of real numbers. Suppose that we arrange its elements in order of increase, and denote the \( k \)th smallest element by \( r^n_{(k)} \). If there are elements of an equal value, we count them repeatedly in an arbitrary order. Finally, we introduce the notation \( \land \) for the minimum operator, and set \( r^n_{(k)} = -\infty \) for all \( k \leq 0 \).
Lemma 1. For each \( k = 1, \ldots, n \), the value of \( r_{(k)}^n \) is given by

\[
r_{(k)}^n = \bigwedge_{1 \leq j_1 < \cdots < j_k \leq n} (r_{j_1} \lor \cdots \lor r_{j_k}).
\] (2)

The proof of this statement can be found in [9]. Now suppose that a new element \( r_{n+1} \) is added to \( \{r_j\}_{j=1}^n \). For the expanded set \( \{r_j\}_{j=1}^{n+1} \), we denote the \( k \)th smallest element by \( r_{(k)}^n \). The next lemma is intended to relate the ordered values of the set \( \{r_j\}_{j=1}^n \) to those of \( \{r_j\}_{j=1}^{n+1} \).

Lemma 2. For each \( k = 1, \ldots, n \), it holds

\[
r_{(k)}^{n+1} = r_{(k)}^n \land (r_{(k-1)}^n \lor r_{n+1}).
\] (3)

Proof. To derive (3), we first apply Lemma 1 to represent \( r_{(k)}^{n+1} \) in the form

\[
r_{(k)}^{n+1} = \bigwedge_{1 \leq j_1 < \cdots < j_k \leq n+1} (r_{j_1} \lor \cdots \lor r_{j_k}).
\]

The terms being minimized may be rearranged to write this relation as

\[
r_{(k)}^{n+1} = \left( \bigwedge_{1 \leq j_1 < \cdots < j_k \leq n} (r_{j_1} \lor \cdots \lor r_{j_k}) \right) \land \left( \bigwedge_{1 \leq j_1 < \cdots < j_{k-1} \leq n+1} (r_{j_1} \lor \cdots \lor r_{j_{k-1}}) \lor r_{n+1} \right).
\]

Clearly, the first term on the right-hand side of the above relation just represents \( r_{(k)}^n \). It remains to factor out the common member \( r_{j_k} = r_{n+1} \) from the second term, and to apply Lemma 1 once again so as to get (3):

\[
r_{(k)}^{n+1} = r_{(k)}^n \land \left( \bigwedge_{1 \leq j_1 < \cdots < j_{k-1} \leq n} (r_{j_1} \lor \cdots \lor r_{j_{k-1}}) \lor r_{n+1} \right) = r_{(k)}^n \land (r_{(k-1)}^n \lor r_{n+1}).
\]

To conclude this section, note that the above representations of the ordering operator involve only the operations of maximum and minimum, and appear to be particularly suited for use in the recursive equations under discussion, which are actually expressed in terms of similar operations.

4 Recursive Equations for the \( G/G/m \) Queue

We are now in a position to present the main result of the paper. We start the derivation of the equations representing the \( G/G/m \) queue with the observation that the arrival process in this queue system is no different from that in the \( G/G/1 \). Therefore, the first equation at (1)

\[
A_k = A_{k-1} + \alpha_k
\]
remains unchanged.

Obviously, to calculate the completion time of the customer which is the \( k \)th to arrive to the system, one has to add his service time \( \tau_k \) to the time of his service initiation. Similarly as in the above analysis of the \( G/G/1 \) queue, let us examine the possibilities for this customer to initiate his service. Firstly, the customer may find one or more servers free at his arrival. In this case he receives service immediately at the time \( A_k \).

Suppose now that at the arrival of the \( k \)th customer all the servers are found to be busy. If there are no other customers waiting for service, he occupies that server which becomes free earlier. Clearly, the customers being served unoccupy servers according to the sequence of the departure times \( D_k, \ldots, D_{k-m} \). We may therefore conclude that, as the earliest time in this sequence, \( D_{k-m} \) just represents the service initiation time of the \( k \)th customer. It is easy to see that this conclusion remains the same in case there are previous customers still waiting for service, when the \( k \)th customer arrives. Finally, for the completion time of this customer, we have

\[
C_k = (A_k \lor D_{k-m}) + \tau_k.
\]

As an important consequence of the above equation, one can state that \( C_k > D_{k-m} \). By renaming the indices, this relation may be rewritten as

\[
D_k < C_{k+m}.
\]

To represent the \( G/G/m \) queue completely, we have to define the departure time \( D_k \) by a suitable equation. One can conclude from the discussion in Section 2 that \( D_k \) coincides with the \( k \)th smallest element of the set \( \{C_j\}_{j \geq 1} \). Moreover, as it follows from (4), it is sufficient to examine only the finite subset \( \{C_1, \ldots, C_{k+m-1}\} \) so as to determine \( D_k \). Let \( C^{k+m-1}_{(k)} \) be the \( k \)th smallest element of this set. We may now define \( D_k = C^{k+m-1}_{(k)} \).

By applying (3) and (4), and taking into account that \( D_{k-1} = C^{k+m-2}_{(k-1)} \), we successively get

\[
C^{k+m-1}_{(k)} = C^{k+m-2}_{(k)} \land (C^{k+m-2}_{(k-1)} \lor C_{k+m-1}) = C^{k+m-2}_{(k)} \land (D_{k-1} \lor C_{k+m-1}) = C^{k+m-2}_{(k)} \land C_{k+m-1}.
\]

Finally, using (2), we arrive at

\[
D_k = C^{k+m-1}_{(k)} = \bigwedge_{1 \leq j_1 < \cdots < j_k \leq k+m-2} (C_{j_1} \lor \cdots \lor C_{j_k}) \land C_{k+m-1}.
\]

We summarize the above results as follows.
Theorem 3. With the notations introduced in Section 2, the dynamics of the $G/G/m$ queue is defined by the recursive equations

\begin{align}
A_k &= A_{k-1} + \alpha_k \\
C_k &= (A_k \vee D_{k-m}) + \tau_k \\
D_k &= \bigwedge_{1 \leq j_1 < \cdots < j_k \leq k+m-2} (C_{j_1} \vee \cdots \vee C_{j_k}) \wedge C_{k+m-1},
\end{align}

for each $k = 1, 2, \ldots$

In conclusion, let us consider two consequences which follow immediately from the theorem. Clearly, if $m = 1$, it results from the last equation at (5) that $C_k \equiv D_k$, $k = 1, 2, \ldots$ In this case, the above equations are reduced to the recursions (1) representing the $G/G/1$ queue.

As another consequence of the obtained representation, one may produce the equations describing the $G/G/2$ queue. Putting $m = 2$ in (5), it is not difficult to arrive at

\begin{align}
A_k &= A_{k-1} + \alpha_k \\
C_k &= (A_k \vee D_{k-2}) + \tau_k \\
D_k &= (C_1 \vee \cdots \vee C_k) \wedge C_{k+1}.
\end{align}

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