ABSTRACT
In this paper, we propose a high efficiency deterministic measurement matrix for practical compressive sensing based on the combination of Logistic Chaotic system and correlation, called Chaos-Gaussian measurement matrix. Initially, deterministic Logistic system has been used to generate Chaotic sequence with good pseudo-random. Subsequently, two spread spectrum sequences have been constructed and been verified to follow Gaussian distribution. On the basis of the observation mentioned previously, Chaos-Gaussian measurement matrix is constructed. Experimental results corroborate that Chaos-Gaussian measurement matrix is superior to Gaussian and Bernoulli random measurement matrix. Furthermore, Chaos-Gaussian measurement matrix balances the randomness and the certainty.

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Compressive sensing; Logistic chaotic system; deterministic measurement matrix; Chaos-Gaussian measurement matrix

1. Introduction
In the information field, the traditional information-capturing paradigm always follows the well-known Shannon-Nyquist sampling theory (Ponuma & Amutha, 2017), which states that the signal sampling rate reaches more than twice the signal bandwidth. This paradigm increases the storage consumption and transmission bandwidth. Therefore, the compression method is used to reconstruct the signal in practice, which, in turn, results in the waste of resources for data collection. In recent years, Donoho et al. proposed a new sampling theory called compressive sensing (CS). CS (Li, Tao, & Mao, 2017) enables the synchronization of the collect and compress signals. The sampling rate is lower than the Nyquist sampling rate while preserving the information of the original signal. CS has enabled significant breakthroughs in information gathering and processing technology (Peng et al., 2017).

CS theory mainly consists of three key technologies, namely, signal sparse representation (Negri, Paterno, Muller, & Fabris, 2017), measurement matrix construction (Fan, Wu, & Li, 2017), and measurement matrix reconstruction algorithms (Shen, Zhu, Zhang, & Li, 2016). Whether the structure of the measurement matrix is good or bad is the key to the quality of signal reconstruction. The measurement matrix in CS theory is divided into two categories: one category is the random measurement matrix (Wang, Cui, Wang, Jia, & Nie, 2014), including the Bernoulli and Gaussian random measurement matrices. The reconstruction performance of these matrices is good, but the uncertainty of the random measurement matrix increases the computational complexity. The other category is the deterministic measurement matrix (Wang, Liu, & Li, 2017). When the system and the correlation coefficients are known, the matrix elements are determined, resulting in low computational complexity. Therefore, constructing the measurement matrix to balance the randomness and the certainty is significant.

Given the sensitive dependence of chaos on the initial conditions, the chaotic system can generate the pseudo-random sequence (Wang, Dong, et al., 2017). In recent years, chaos theory has been applied to the construction of the CS measurement matrix to improve this balance. For the first time, a measurement matrix that satisfies the Gaussian distribution is proposed on the basis of the chaotic sequences in (Linh-Trung et al., 2008). Yu L et al. introduced the construction of a feasible measurement matrix based on the Logistic maps in (Yu, Barbot, Zheng, & Sun, 2010). Thereafter, a measurement matrix is proposed on the basis of the deterministic random sequence obtained using the chaotic sequence in (Wang, Li, & Shen, 2013). However, the measurement matrices do not construct a high-performance deterministic measurement matrix.

In this study, we exploit the combination of the Logistic chaotic system (Gan, Song, & Zhao, 2017) and the
correlation to construct a high efficiency deterministic measurement matrix. Primarily, the chaotic system can generate the pseudo-random sequence, which is based on the deterministic equation. Thereafter, a class of spread spectrum sequences with good even cross-correlation properties is generated by symbol function mapping. The new sequences follow the Gaussian distribution. Thus, the CS measurement matrix can be constructed. The experimental results prove that the constructed Chaos-Gaussian measurement matrix is not only feasible but also better than the common random measurement matrix, such as Gaussian and Bernoulli random measurement matrices.

The rest of the paper is organized as follows. Section 2 recalls CS theory. Section 3 describes the construction of the Chaos-Gaussian measurement matrix. Section 4 tackles about the experiments that are conducted to simulate the performance of the Chaos-Gaussian measurement matrix. Finally, Section 5 concludes this paper.

2. Compressive sensing

The conventional CS algorithm can be described simply as follows. We let $x$ be the original signal with length $N$. The original signal $x$ can be sparse in a known transform domain. The sparsifying transform can be denoted as $\Psi \in \mathbb{R}^{N \times N}$. Thus, the signal can be expressed as follows:

$$X = \sum_{i=1}^{N} \theta_i \psi_i,$$

where $\Theta$ is a column vector of sparse coefficients, having merely $k < N$ non-zero elements. $x$ and $\Theta$ are the same signal representations in the time/space domain and $\phi$ domain, respectively. The signal $x$ is said to be $K$-sparse if $\Theta$ consists of only $K$ non-zero $\Theta_j$ values. Thus, $x$ can be represented as a linear combination of only $K$ basis vectors. In this case, $x$ is called a compressible signal, and sparse approximation methods can be applied.

Subsequently, the data acquisition process of conventional CS theory is defined as follows:

$$Y = \Phi X = \Phi \Psi \Theta,$$

where $\Phi_{M \times N}(M < N)$ is defined as the measurement matrix and $Y \in \mathbb{R}^{M \times 1}$ is the measurement vector.

To reconstruct original signal $x$, E. Candes and T. Tao corroborated that the measurement matrix must satisfy the restricted isometry property (RIP) (Zhou, Jing, Zhang, Huang, & Li, 2017): For matrix $\Psi \in \mathbb{R}^{N \times N}$, the smallest constant satisfying is defined, such that, for the any $K$-sparse vector $v$, the following inequality can be derived:

$$1 - \varepsilon \leq \frac{||\Phi v||_2}{||v||_2} \leq 1 + \varepsilon \quad \varepsilon > 0$$

The related condition for RIP is incoherence, which requires that the rows of $\Phi$ cannot sparsely represent the columns of $\Psi$. The properties RIP and incoherence can be achieved by selecting $\Phi$ as a random matrix. The measurement matrix that satisfies certain RIP conditions mainly includes the Bernoulli, Gaussian, sparse, partial Fourier random matrices.

Intuitively, Equation (2) can be solved by searching the sparsest vector on the null surface of $\Phi$, i.e.

$$\min ||\theta||_0, s.t. Y = \Phi \Theta$$

However, the problem was shown to be NP–Hard. Thus, the problem was practically relaxed to solve the following equation:

$$\min ||\theta||_1, s.t. Y = \Phi \Theta,$$

where $||\theta||_1$ is the $l_1$ norm and denotes the absolute sum of all values of $\Theta$.

For the reconstruction algorithm, the commonly used algorithms include Base Pursuit algorithm, orthogonal matching pursuit (OMP) algorithm (Zeng, Zhang, Chen, Cao, & Yang, 2015), gradient pursuit algorithm, and iterative threshold method.

3. Construction of the Chaos-Gaussian measurement matrix

3.1. One-dimensional logistic system

The chaotic system has excellent pseudo-random properties and can produce uncertain trajectories based on deterministic rules. The chaotic system has a high sensitivity to the initial condition. Therefore, the chaotic system is the unity of certainty and randomness. In recent years, the chaotic system has been exploited by a wide range of applications in the fields of nonlinear control, signal processing, and secure communication. The chaotic system is a deterministic system, but it presents certain random characteristics. The one-dimensional Logistic system is defined as follows:

$$x_{n+1} = 1 - ux_n^2 = f(x_n) \quad x \in [-1, 1]$$

The Logistic system (Ling & Sun, 1999) is a chaotic system with the parameter $u \in [1.872, 2.0]$.

3.2. Even cross-correlation function

We assume that $\{a_i\}_{i=1, 2, \ldots N}$ and $\{b_i\}_{i=1, 2, \ldots N}$ are two binary sequences with period $N$ and the corresponding sequences of elements of $\{+1, -1\}$. 
The even cross-correlation function $R_{ab}$ is defined as follows:

$$R_{ab}(\tau) = C_{ab}(\tau) + C_{ab}(\tau - N),$$

where the discrete aperiodic cross-correlation function $C_{ab}$ for the sequences $\{a_i\}$ and $\{b_i\}$ is defined as follows:

$$C_{ab}(\tau) = \begin{cases} \sum_{n=0}^{N-1} a_n b_{n+\tau} & 0 \leq \tau \leq N - 1 \\ \sum_{n=0}^{N-1} a_n - \tau b_n & 1 - N \leq \tau < 0 \\ 0 & |\tau| \geq N \end{cases}$$

Thereafter, the autocorrelation function $R_{aa}$ can be considered the cross-correlation of two identical binary sequences with period $N$ as follows:

$$R_{aa}(\tau) = C_{aa}(\tau) + C_{aa}(\tau - N)$$

The autocorrelation peak is obtained with parameter $\tau = 0$.

### 3.3. Even cross-correlation function probability distribution

We set $\{x_n\}_{0}^{N-1}$ as the output chaotic sequence generated by Logistic mapping (Ling & Sun, 1999) with the initial condition $x_0$ and parameter $u = 2.0$, i.e.

$$x_{n+1} = 1 - 2x_n^2 = f(x_n) \quad x \in [-1, 1]$$

We let the sequence $\{a_n\}_{0}^{N-1}$ denoting the map of the spread spectrum sequence $\{x_n\}_{0}^{N-1}$ by the symbol function $\text{sgn}(\cdot)$ as follows:

$$a_n = \text{sgn}(x_n) \quad n = 0, 1, 2 \ldots N - 1$$

In (Lin & Peng, 2013), the spread spectrum sequence is a Bernoulli sequence that satisfies the RIP. Assuming that the sequence length $N$ is long, the sequences $\{a_n\}_{0}^{N-1}$ and $\{b_n\}_{0}^{N-1}$ are generated by the previously presented formulas. The two sequences are two binary sequences with length $N$. The even cross-correlation distribution of the sequences $\{a_n\}_{0}^{N-1}$ and $\{b_n\}_{0}^{N-1}$ follows the Gaussian distribution with a mean of 0 and a variance of $N$.

**Proof:** We let $P(k)$ denote the probability that the correlation function is equal to $k$. Given the sensitive dependence of chaos on the initial conditions, $\{a_n\}$ and $\{b_n\}$ are independent Bernoulli sequences. Therefore, the even cross-correlation distribution of $\{a_n\}$ and $\{b_n\}$ follows the binomial distribution. Particularly, when $N$ is large, the even cross-correlation distribution follows the Gaussian distribution. Given the perspective of set averaging, $P(k)$ is defined as follows:

$$P(k) = \left( \frac{N}{N+k} \right) \left( \frac{1}{2} \right)^{N+k} \left( \frac{1}{2} \right)^{N-k}$$

$$\approx \frac{2}{\sqrt{2\pi N}} e^{-\frac{k^2}{2N}} k = -N, -N + 2, \ldots, N$$

According to the ergodicity of chaos, the conclusion also holds true for the time average, and the meaning of $P(k)$ becomes the frequency of occurrence of the correlation value $k$.

Given that the amount of even cross-correlation changes can only be a multiple of 4, $P(k)$ must be corrected as follows:

$$P(k) \approx \frac{4}{\sqrt{2\pi N}} e^{-\frac{k^2}{2N}} |k| \leq N, k \equiv N + e \mod 4$$

For even cross-correlation, parameter $e = 0$ or 2, depending on the sequences $\{a_n\}$ and $\{b_n\}$. Then, according to the relationship between $P(k)$ and $k$, the theoretical result is shown in Figure 1.

For the analysis of the simulation results, the even cross-correlation distribution of spread spectrum sequences with $N = 4096$ is given, and the relationship between the number of occurrences and the correlation value is denoted in Figure 1. The comparison of the theoretical results of the Gaussian distribution proves that the even cross-correlation distribution follows the Gaussian distribution with a mean of 0 and a variance of $N$.

![Figure 1. Even cross-correlation distribution of two spread spectrum sequences with $N = 4096$ by the Logistic map.](image-url)
3.4. Gaussian distribution structure measurement matrix

The Chaos-Gaussian measurement matrix is constructed as follows:

Step 1: The constructed measurement matrix is determined to be feasible and effective with parameter $u = 2.0$ and the initial values $x_0 = 0.23$ and $y_0 = 0.37$ after conducting several experimental comparisons (Lin & Peng, 2013). Therefore, the chaotic sequences $\{x_n\}_{0}^{N-1}$ and $\{y_n\}_{0}^{N-1}$ are generated by the Logistic chaotic system with the previously presented conditions, and the sequence length is $n = M \times N - 1$.

Step 2: We let the sequences $\{a_n\}_{0}^{N-1}$ and $\{b_n\}_{0}^{N-1}$ denote the symbol map of the chaotic sequences $\{x_n\}_{0}^{N-1}$ and $\{y_n\}_{0}^{N-1}$ using Formula (11).

Step 3: The even cross-correlation distribution of the sequences $\{a_n\}_{0}^{N-1}$ and $\{b_n\}_{0}^{N-1}$ from step 2 is obtained using Formula (7). Subsequently, the even cross-correlation sequence is cut off to form the $M \times N$-dimensional measurement matrix $\Phi$.

In the proposed scheme, according to the chaotic pseudo-random characteristics of nonlinear dynamic systems, we use the deterministic logistic chaotic system to generate a chaotic sequence; then symbolic function mapping is performed; and the even cross-correlation distribution of the mapped sequence follows the Gaussian distribution, and finally, the even cross-correlation sequence is cut off to form the measurement matrix with excellent pseudo-randomness.

4. Experimental results

In this study, two-dimensional images are simulated to verify the feasibility and validity of the Chaos-Gaussian measurement matrix and compared with the Gaussian and Bernoulli random matrices. In this study, the lena,

![Figure 2](image1.png)

**Figure 2.** Comparison of image reconstruction results of each measurement matrix ($M/N = 0.5$). (a) The original image (b) Chaos-Gaussian measurement matrix (c) Bernoulli measurement matrix (d) Gaussian measurement matrix.

![Figure 3](image2.png)

**Figure 3.** Comparison of the PSNR of each image for different measurement matrices with compression ratio (a) Lena image (b) Camera image (c) Peppers image.
cameraman and peppers images with the dimension 256 × 256 are simulated under different compression ratios (M/N) (Chen, Chen, Long, & Zhu, 2017). We use wavelet transform to sparsely represent the image, construct the Chaos-Gaussian measurement matrix, and utilize the OMP algorithm to reconstruct the image.

Initially, for the sake of visual observation, we take the peppers image as an example, and the reconstruction performance of different measurement matrices is discussed under the compression ratio M/N = 0.5. Figure 2 exhibits the experimental results.

Figure 2 depicts that the Chaos-Gaussian measurement matrix reconstruction effect is better than other measurement matrices. For the further explanation of the experimental results shown in Figure 2, the peak signal-to-noise ratio (PSNR) with the compression ratio for different images under different measurement matrices is shown in Figure 3. Given that the other measurement matrices used for comparison are random matrices, the average of 20 experiments is selected as the experimental data. The PSNR of the two-dimensional image is defined as follows:

\[ \text{PSNR} = 10 \times \log \left( \frac{255 \times 255 \times W \times H}{\sum \sum (I - \hat{I})} \right) \]  

(14)

where \( I \) is the original image, \( \hat{I} \) is the reconstructed image, and \( W \) and \( H \) are the width and height of the image, respectively.

Figure 3 illustrates that the proposed Chaos-Gaussian measurement matrix has approximately 0.5 dB to 2.0 dB average improvement of the PSNR compared with the Gaussian and Bernoulli random measurement matrices. Particularly, in the compression ratio range of 0.3 to 0.4, this performance improvement of Chaos-Gaussian measurement matrix is evident.

5. Conclusion

In this study, according to the chaotic pseudo-random characteristics of nonlinear dynamics systems, the Logistic chaotic system is primarily fully mapped with a good pseudo-random chaotic sequence (Zhongeng & Chen, 2017); then, symbolic function mapping is performed; and finally, the mapped sequence achieves a good even cross-correlation.

The even cross-correlation distribution follows the Gaussian distribution of with the mean value of 0 and the variance of \( N \). Therefore, on the basis of this theory, a Chaos-Gaussian measurement matrix construction algorithm is proposed. The simulation results confirm that the Chaos-Gaussian measurement matrix is a deterministic measurement matrix, and that the reconstruction effect is better than that of the Bernoulli (Huang, Fan, & Zhu, 2017) and Gaussian random measurement matrices. The PSNR of the reconstructed signal exhibits evident improvement in the case of a large compression ratio. This construction algorithm can also be similar to the odd cross-correlation distribution, which is worth further studies and has a certain practical value.

Disclosure statement

No potential conflict of interest was reported by the authors.

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