A NEW TYPE OF DEGENERATE POLY-EULER POLYNOMIALS

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ABSTRACT. Many mathematicians have been studying various degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects. Our main focus here is a new type of degenerate poly-Euler polynomials and numbers. This focus stems from their nascent importance for applications in combinatorics, number theory and in other aspects of applied mathematics. We construct a new type of degenerate poly-Euler polynomials by using the degenerate polylogarithm functions. We also show several combinatorial identities related to this polynomials and numbers.

1. INTRODUCTION

Recently, many mathematicians have been studying various degenerate versions of special polynomials and numbers in some arithmetic and combinatorial aspects [6, 9 - 20]. These degenerate versions began when Carlitz introduced the degenerate Bernoulli polynomials and the degenerate Euler polynomials [1]. These polynomials appear in combinatorial mathematics and play a very important role in the theory and application of mathematics, thus many number theory and combinatorial experts have studied their properties, and obtained a series of interesting results. Kim et al introduced degenerate gamma random variables as well as new Jindalrae and Gaenari numbers and polynomials, and developed above mentioned polynomials and numbers related to Jindalrae and Gaenari numbers and polynomials; discrete harmonic numbers [18 - 21]. Motivated by their importance and potential for applications in number theory, combinatorics and other fields of applied mathematics, in particular, we are interested in degenerate poly-Euler polynomials and numbers. The goal of this paper is to demonstrate many explicit computational formulas and relations, involving a new type of the degenerate poly-Euler polynomials and numbers by using Kim-Kim’s the degenerate polylogarithm functions.

Now, we give some definitions and properties needed in this paper. As is known, the ordinary Euler polynomials and the ordinary Bernoulli polynomials are usually defined by the following generating functions with parallel structures [1 - 2].

\begin{equation}
\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\end{equation}

For any nonzero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

\begin{equation}
\frac{t}{e^t - 1}e^{\lambda t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2}{e^t + 1}e^{\lambda t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\end{equation}

By Taylor expansion, we get

\begin{equation}
e^{\lambda t} = (1 + \lambda t)^{\frac{1}{\lambda}}, \quad e^{\lambda t} = (1 + \lambda t)^{\frac{1}{\lambda}} = e^1_{\lambda}(t), \quad \text{(see [11 - 15, 20])}.
\end{equation}

By Taylor expansion, we get

\begin{equation}
e^{\lambda t} = \sum_{n=0}^{\infty} (\lambda t)^n \frac{n!}{n!}, \quad \text{(see [11 - 15])},
\end{equation}

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where \((x)_{0,\lambda}=1\), \((x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq1)\).

Note that
\[
\lim_{\lambda\to0} e_\lambda^k(t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^{xt}.
\]

Carlitz [1] introduced the ordinary degenerate Bernoulli polynomials and the degenerate Euler polynomials, respectively given by
\[
2 \frac{e_\lambda(t) + 1}{e_\lambda(t)} = \sum_{n=0}^{\infty} E_n(\lambda x) \frac{t^n}{n!}, \quad \frac{t}{e_\lambda(t) - 1} e_\lambda^k(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}.
\]

In 2020, Kim-Kim [9] introduced the degenerate polylogarithm function defined by
\[
l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)! n^k} x^n, \quad k \in \mathbb{Z} \quad (|x| < 1).
\]

We note that
\[
\lim_{\lambda\to0} l_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = L_k(x).
\]

From (5), we have
\[
l_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!} x^n = -\log_\lambda (1 - x).
\]

Kim-Kim also studied the new type degenerate poly Bernoulli polynomials and numbers, by using the degenerate polylogarithm function as follows:
\[
l_{k,\lambda}(1 - e_\lambda(-t)) \quad \frac{1}{1 - e_\lambda(-t)} e_\lambda^k(-t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(k) \frac{t^n}{n!}, \quad (\text{see } [9]).
\]

When \(x=0\), \(\beta_{n,\lambda}(k) = \beta_{n,\lambda}^{(k)}(0)\) are called the degenerate poly-Bernoulli numbers.

For \(n \geq 0\), the Stirling numbers of the first kind are defined by
\[
(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (\text{see } [8 - 22]).
\]

where \((x)_0 = 1\), \((x)_n = x(x-1)\cdots(x-n+1), \ (n \geq 1)\).

From (8), it is easy to see that
\[
\frac{1}{k!} (\log(1 + t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad (\text{see } [3, 22]).
\]

In the inverse expression to (8), for \(n \geq 0\), the Stirling numbers of the second kind are defined by
\[
x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (\text{see } [3, 22]).
\]

From (10), it is easy to see that
\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}, \quad (\text{see } [2]).
\]
Kim et al. [13] introduced the degenerate Stirling numbers of the second kind as follows:

\[(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n, l)(x)^l \quad (n \geq 0).\]  

(12)

As an inversion formula of (12), the degenerate Stirling numbers of the first kind are defined by

\[(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n, l)(x)^l \quad (n \geq 0), \quad \text{(see [9]).} \]  

(13)

From (12) and (13), it is well known that

\[\frac{1}{k!}(e^{\lambda(t)} - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k)\frac{t^n}{n!} \quad (k \geq 0), \quad \text{(see [9, 13]),} \]  

(14)

and

\[\frac{1}{k!}(\log \lambda(1 + t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k)\frac{t^n}{n!} \quad (k \geq 0), \quad \text{(see [9]).} \]  

(15)

In 1997, Kaneko [7] introduced the poly-Bernoulli numbers \(B_n^{(k)}\) represented by the following generating function

\[Li_k(1 - e^{-t})\]  

\[\frac{e^t - 1}{e^t - 1} = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!}, \]  

where

\[Li_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k} \quad \text{(see [24]).} \]  

(16)

If \(k = 1\), we get \(B_n^{(1)} = (-1)^nB_n\) for \(n \geq 0\) where \(B_n\) are the Bernoulli numbers.

The poly-Bernoulli polynomials of index \(k\) were defined by the generating function

\[Li_k(1 - e^{-t})e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} \quad \text{(see [25]).} \]  

(17)

Ohno and Sasaki [26] defined poly-Euler numbers as

\[Li_k(1 - e^{-t})\]  

\[\frac{4t\cosh(t)}{e^{4t}\cosh(t)} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}, \]  

where

\[E_n^{(k)}(x) = \sum_{n=0}^{\infty} \frac{z^n}{n^k} \quad \text{(see [24]).} \]  

(16)

It was recently extended to

\[\frac{2Li_k(1 - e^{-t})}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!} \]  

in polynomial form by Yoshinori [27].

Lee et al. [23] introduced the type 2 degenerate poly-Euler polynomials constructed from the modified polyexponential function

\[\frac{Ei_k(\log(1 + 2t))}{t(e^\lambda(t) + 1)} e^\lambda(t) = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}. \]  

(18)

An outline of this paper is as follows. In section 2, we construct a new type of poly-Euler polynomials and numbers, by using the polylogarithm function. We also show several combinatorial identities related to the poly-Euler polynomials and numbers. Some of them include some special polynomials and numbers such as the Stirling numbers of the second kind, ordinary Euler numbers, ordinary Bernoulli polynomials and numbers, poly Bernoulli polynomials, etc. In section 3, we also
consider the degenerate poly-Euler polynomials, by using the degenerate polylogarithm function and investigate some identities for those polynomials.

2. A NEW TYPE OF POLY-EULER NUMBERS AND POLYNOMIALS

In this section, we define a new type of poly-Euler polynomials and numbers, by using the polylogarithm functions. We also show several combinatorial identities related to the poly-Euler polynomials and numbers.

The new type of poly-Euler polynomials is defined by

\[
\frac{\text{Li}_k(1 - e^{-2t})}{t(e^t + 1)} e^t = \sum_{n=0}^{\infty} E_n(k)(x) \frac{t^n}{n!}.
\] (19)

When \( x = 0 \), \( E_n(k)(x) = E_n(k)(0) \) are called the poly-Euler numbers.

When \( k = 1 \), as \( \text{Li}_1(1 - e^{-2t}) = -\log(1 - 1 + e^{-2t}) = 2t \), we see that \( E_n(1)(x) = E_n(x) \ (n \geq 0) \) are the Euler polynomials.

Lemma 1. For \( n \geq 0 \), \( k \in \mathbb{Z} \), we have

\[
\frac{\text{Li}_k(1 - e^{-2t})}{t(e^t + 1)} e^t = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \frac{2^n(-1)^n m!}{m^k} S_2(n,m) \right) \frac{t^n}{n!}.
\] (20)

Proof.

\[
\text{Li}_k(1 - e^{-2t}) = \sum_{m=1}^{\infty} \frac{1}{m^k} (1 - e^{-2t})^m = \sum_{m=1}^{\infty} \frac{(-1)^m m!}{m^k} \frac{1}{m!} (e^{-2t} - 1)^m
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^m m!}{m^k} \sum_{n=m}^{\infty} S_2(n,m) (2t)^n \frac{n!}{n!}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \frac{2^n(-1)^n m!}{m^k} S_2(n,m) \right) \frac{t^n}{n!}.
\]

\[\square\]

Theorem 2. For \( n \geq 0 \), \( k \in \mathbb{Z} \), we have

\[
E_n^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \sum_{j=1}^{m+1} \binom{n}{l} \binom{l}{m} \frac{2^{m+n-l}(-1)^{m+1+j}}{(l-m+1) j^k (m+1)} S_2(m+1,j) B_{n-l} \left( \frac{x}{2} \right).
\] (21)
Proof. From (11) and (20), we have

\[ \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{L_i k(1-e^{-2t})}{t(e^t+1)} e^{ux} \]

\[ = \frac{e^{ut}}{t(e^t+1)} \left( \prod_{m=1}^{\infty} \left( \sum_{j=1}^{m} \frac{2^m(-1)^{m+j} t^j}{j^k} S_2(m,j) \right) \right) \frac{t^n}{n!} \]

\[ = \frac{e^{ut}}{e^{2t}-1} \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} \right) \left( \sum_{j=1}^{m+1} \frac{2^m(-1)^{m+1+j} t^j}{j^k} S_2(m+1,j) \right) \frac{t^n}{(m+1)!} \]

\[ = \frac{2te^{ut}}{e^{2t}-1} \left( \sum_{m=0}^{\infty} \frac{1}{m+1} \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} \left( \sum_{j=1}^{m+1} \frac{2^m(-1)^{m+1+j} t^j}{j^k} S_2(m+1,j) \right) \frac{t^n}{(m+1)!} \right) \]

\[ = \left( \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \right) \left( \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} \left( \sum_{j=1}^{m+1} \frac{2^m(-1)^{m+1+j} t^j}{j^k} S_2(m+1,j) \right) \frac{t^n}{(m+1)!} \right) \right) \frac{t^n}{n!} . \]

Therefore, by comparing the coefficients on both sides of (22), we get the desired result. \[\square\]

Corollary 3. For \( n \geq 0, \ k \in \mathbb{Z}, \) and \( x = 0, \) we have

\[ E_n^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{m} \frac{2^m(-1)^{m+1+l} t^m}{l^k (m+1)} S_2(m+1,l) B_{n-l}. \]

Theorem 4. For \( n \geq 0, \ k \in \mathbb{Z}, \) we have

\[ E_n^{(k)}(x) = \sum_{m=0}^{\infty} \sum_{l=1}^{m+1} \binom{n}{m} \frac{2^m(-1)^{m+1+l} t^m}{l^k (m+1)} S_2(m+1,l) E_{n-m}(x). \]

Proof. From (11) and (20), we have

\[ \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{L_i k(1-e^{-2t})}{t(e^t+1)} e^{ux} \]

\[ = \frac{e^{ut}}{t(e^t+1)} \left( \prod_{m=1}^{\infty} \left( \sum_{j=1}^{m+1} \frac{2^m(-1)^{m+1+l} t^j}{j^k} S_2(m,j) \right) \frac{t^n}{m!} \right) \]

\[ = \frac{2e^{ut}}{e^{2t}-1} \left( \sum_{m=0}^{\infty} \frac{1}{m+1} \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} \left( \sum_{j=1}^{m+1} \frac{2^m(-1)^{m+1+l} t^j}{j^k} S_2(m+1,l) \right) \frac{t^n}{m!} \right) \]

\[ = \left( \sum_{j=0}^{\infty} \frac{t^j}{j!} \right) \left( \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} \left( \sum_{j=1}^{m+1} \frac{2^m(-1)^{m+1+l} t^j}{j^k (m+1)} S_2(m+1,l) E_{n-m}(x) \right) \right) \frac{t^n}{n!} \right) . \]

Therefore, by comparing the coefficients on both sides of (24), we get the desired result. \[\square\]

Corollary 5. For \( n \geq 0, \ k \in \mathbb{Z}, \) and \( x = 0, \) we have

\[ E_n^{(k)}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{2^m(-1)^{m+1+l} t^m}{l^k (m+1)} S_2(m+1,l) E_{n-m}. \]
Theorem 6. For \( n \geq 0, k \in \mathbb{Z} \), we have

\[
E_n^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(k)} x^l.
\]

Proof. From (11) and (19), we get

\[
\sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{Li_k(1 - e^{-2t})}{t(e^t + 1)} e^{xt}
\]

Therefore, by comparing the coefficients on both sides of (26), we get the desired result. \( \square \)

Theorem 7. For \( n \geq 0, k \in \mathbb{Z} \), we have

\[
\frac{d}{dx} E_n^{(k)}(x) = nE_{n-1}^{(k)}(x).
\]

Proof. From (25), we have

\[
\frac{d}{dx} E_n^{(k)}(x) = \frac{d}{dx} \left( \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(k)} x^l \right)
\]

\[
= \sum_{l=1}^{n} \binom{n}{l} E_{n-l}^{(k)} l x^{l-1}
\]

Thus, we get the desired result. \( \square \)

Theorem 8. For \( n \geq 0, k \in \mathbb{Z} \), we have

\[
E_n^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_m S_2(m, l) E_{n-l}^{(k)}.
\]

Proof. From (11) and (19), we get

\[
\sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{Li_k(1 - e^{-2t})}{t(e^t + 1)} (e^t - 1 + 1)^x
\]

\[
= \sum_{l=0}^{\infty} E_l^{(k)} \frac{t^l}{l!} \sum_{m=0}^{\infty} (x)_m \frac{(e^t - 1)^m}{m!}
\]

Thus, we get the desired result.
Therefore, by comparing the coefficients on both sides of (29), we have the desired result. □

**Theorem 9.** For \( n \geq 0, \ k \in \mathbb{Z} \), we have

\[
 nE_{n-1}^{(k)}(x+1) + nE_{n-1}^{(k)}(x) = 2^n \left( \beta_n^{(k)}(x+2) - \beta_n^{(k)}(x) \right).
\]

**Proof.** From (17) and (19), we get

\[
 \frac{L_i(k)(1-e^{-2t})}{t(e^t+1)}t(e^t+1)e^\mu
\]

\[
 = \frac{L_i(k)(1-e^{-2t})}{t(e^t+1)}e^{(x+1)\mu} + \frac{L_i(k)(1-e^{-2t})}{t(e^t+1)}te^\mu
\]

\[
 = \sum_{n=0}^{\infty} E_n^{(k)}(x+1) \frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^{n+1}}{n!}
\]

\[
 = \sum_{n=1}^{\infty} \left( nE_{n-1}^{(k)}(x+1) + nE_{n-1}^{(k)}(x) \right) \frac{t^n}{n!}.
\]

On the other hand,

\[
 \frac{L_i(k)(1-e^{-2t})}{e^{2\mu}-1}(e^{2\mu}-1)e^\mu
\]

\[
 = \frac{L_i(k)(1-e^{-2t})}{e^{2\mu}-1}e^{(x+2)\mu} - \frac{L_i(k)(1-e^{-2t})}{e^{2\mu}-1}e^\mu
\]

\[
 = \sum_{n=0}^{\infty} \left( \beta_n^{(k)}(x+2) - \beta_n^{(k)}(x) \right) \frac{(2\mu)^n}{n!}
\]

\[
 = \sum_{n=0}^{\infty} \beta_n^{(k)}(x+2) \frac{(2\mu)^n}{n!}.
\]

Therefore, by comparing the coefficients of (30) and (31), we get the desired result. □

### 3. A NEW TYPE OF DEGENERATE POLY-EULER NUMBERS AND POLYNOMIALS

In this section, we construct a new type of degenerate poly-Euler polynomials, by using the degenerate polylogarithm function. We also show several combinatorial identities related to the degenerate poly-Euler polynomials and numbers.

We define the degenerate poly-Euler polynomials, by using the degenerate polylogarithm function as follows:

\[
l_{k,\lambda}(1-e_{\lambda}(-2t)) \frac{1}{t(e_{\lambda}(t) + 1)}e_{\lambda}(t) = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}.
\]

When \( x = 0 \), \( E_n^{(k)}(0) \) are called the degenerate poly-Euler numbers.

When \( k = 1 \), from (12), we see that \( E_n^{(1)}(x) = E_{n,\lambda}(x) \) \( (n \geq 0) \) are the degenerate Euler polynomials because of

\[
l_{1,\lambda}(1-e_{\lambda}(-2t)) = -\log_{\lambda}(1 - 1 + e_{\lambda}(-2t)) = 2t.
\]
Theorem 10. For $n \geq 0$, $k \in \mathbb{Z}$, we have

$$E_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} (x)_{l,\lambda} E_{n-l,\lambda}^{(k)}.$$ 

Proof. From (3) and (32), we get

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{\lambda_{k,\lambda} \left( 1 - e_{\lambda}^{(-2t)} \right)}{t(e_{\lambda}(t) + 1)} e_{\lambda}(t)$$

$$= \sum_{m=0}^{\infty} \frac{E_{m,\lambda}^{(k)} t^m}{m!} \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} (x)_{l,\lambda} E_{n-l,\lambda}^{(k)} \right) \frac{t^n}{n!}.$$ 

(34)

Therefore, by comparing the coefficients on both sides of (34), we have the desired result.

\[\square\]

Theorem 11. For $n \geq 0$, $k \in \mathbb{Z}$, we have

$$E_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} (x)_{m} S_{2,\lambda}(m,l) E_{n-l,\lambda}^{(k)}.$$ 

Proof. From (14) and (32), we get

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{\lambda_{k,\lambda} \left( 1 - e_{\lambda}^{(-2t)} \right)}{t(e_{\lambda}(t) + 1)} e_{\lambda}(t)$$

$$= \sum_{m=0}^{\infty} \frac{E_{i,\lambda}^{(k)} t^i}{i!} \sum_{m=0}^{\infty} (x)_{m} \frac{(e_{\lambda}(t) - 1)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} (x)_{m} S_{2,\lambda}(m,l) E_{n-l,\lambda}^{(k)} \right) \frac{t^n}{n!}.$$ 

(35)

Therefore, by comparing the coefficients on both sides of (35), we have the desired result.

\[\square\]

Theorem 12. For $n \geq 0$, $k \in \mathbb{Z}$, we have

$$E_{n-1,\lambda}^{(k)}(1) + E_{n-1,\lambda}^{(k)} = \frac{2^n}{n} \sum_{l=1}^{n} \binom{n}{1} \left( -1 \right)^{n-1} \lambda^{l-1} \frac{S_{2,\lambda}(n,l)}{l}. $$
Proof. From (5), (14) and (32), we have

\[ I_{k, \lambda} \left( 1 - e_{\lambda} (-2t) \right) = t(e_{\lambda}(t) + 1) \sum_{l=0}^{\infty} \frac{E_{l, 1}^{(k)} t^l}{l!} \]

\[ = t \left( \sum_{m=0}^{\infty} \frac{m!}{m!} + 1 \right) \sum_{l=0}^{\infty} \frac{E_{l, 1}^{(k)} t^l}{l!} \]

\[ = t \left( \sum_{n=0}^{\infty} \frac{n!}{n!} \right) \left( 1 \right) E_{n, 1, \lambda}^{(k)} + E_{n, 1, \lambda}^{(k)} \frac{t^n}{n!} \]

\[ = \sum_{n=1}^{\infty} r \left( \sum_{l=0}^{n} \frac{(n-1)}{l!} \left( 1 \right) E_{n-1, 1, \lambda}^{(k)} + E_{n-1, 1, \lambda}^{(k)} \frac{t^n}{n!} \right) \]

On the other hand,

\[ I_{k, \lambda} \left( 1 - e_{\lambda} (-2t) \right) = \sum_{l=1}^{\infty} \frac{(1)_{l, 1, \lambda} (-\lambda)^{l-1}}{(l-1)!} \left( 1 - e_{\lambda} (-2t) \right)^l \]

\[ = \sum_{l=1}^{\infty} \frac{(1)_{l, 1, \lambda} (-\lambda)^{l-1}}{l!} \left( 1 - e_{\lambda} (-2t) \right)^l \]

\[ = \sum_{l=1}^{\infty} \frac{(1)_{l, 1, \lambda} (-\lambda)^{l-1} (1 - e_{\lambda} (-2t) - 1)^l}{l!} \]

\[ = \sum_{n=1}^{\infty} r \left( \sum_{l=1}^{n} \frac{(1)_{l, 1, \lambda} (-\lambda)^{l-1} (1 - e_{\lambda} (-2t) - 1)^l}{l!} S_{2, \lambda} (n, l) \frac{(-2t)^n}{n!} \right) \]

Therefore, by comparing the coefficients of (36) and (37), we have the desired result. \( \square \)

Theorem 13. For \( n \geq 0, k \in \mathbb{Z} \), we have

\[ \sum_{i=1}^{n} \sum_{m=0}^{n-i} \left( \frac{n!}{l!} \right) (1)_{i, \lambda} 2^n (-1)^{m+n+1} S_{2, \lambda} (n - i, m) E_{m, 1, \lambda}^{(k)} \]

\[ = \sum_{m=1}^{n} \sum_{l=0}^{n-m} \left( \frac{n!}{m!} \right) \lambda^m (-1)^{m-1} \frac{2^n (-1)^{m-1}}{m^k-1} S_{2, \lambda} (n - m, l) E_{l, 1, \lambda}. \]
Proof. Replace $t$ by $1 - e_x(-2t)$, we observe

\[
(1 - e_x(-2t)) \sum_{m=0}^{\infty} E_{m,\lambda}^{(k)} \frac{(1 - e_x(-2t))^m}{m!} = \frac{l_{k,\lambda}(t)}{e_x(1 - e_x(-2t)) + 1}
\]

\[
= \frac{1}{2} \frac{2}{e_x(1 - e_x(-2t)) + 1} \sum_{m=1}^{\infty} \frac{(1)_{m,1/\lambda}(-1)^{m-1}\lambda^{m-1}}{m^2(m-1)!} t^m
\]

(38)

\[
= \frac{1}{2} \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} E_{i,\lambda}(-1)^i \sum_{j=1}^{\infty} S_{2,\lambda}(j,l)(-2)^{j/l} \right) \sum_{m=1}^{\infty} \frac{(1)_{m,1/\lambda}(-1)^{m-1}\lambda^{m-1}}{m^k m!} t^m
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \sum_{l=0}^{n-m} \binom{n}{m} \frac{\lambda^{m-1}(1)_{m,1/\lambda} 2^{n-m-1}(-1)^{l+n-1} S_{2,\lambda}(n-m,l)E_{l,\lambda}}{m^k m!} \right) t^n.
\]

On the other hand, we have

\[
(1 - e_x(-2t)) \sum_{m=0}^{\infty} E_{m,\lambda}^{(k)} \frac{(1 - e_x(-2t))^m}{m!}
\]

\[
= - \sum_{i=1}^{\infty} \frac{(1)_{i,\lambda}(-2t)^i}{i!} \left( \sum_{m=0}^{\infty} E_{m,\lambda}^{(k)}(-1)^m \sum_{j=m}^{\infty} S_{2,\lambda}(j,m) \frac{(-2)^j t^j}{j!} \right)
\]

(39)

\[
= - \sum_{i=1}^{\infty} \frac{(1)_{i,\lambda}(-2t)^i}{i!} \left( \sum_{j=0}^{i} \sum_{m=0}^{i} \binom{i}{j} E_{m,\lambda}^{(k)}(-1)^m S_{2,\lambda}(j,m) \frac{(-2)^j t^j}{j!} \right)
\]

\[
= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{m=0}^{n-i} \binom{n}{i} \frac{\lambda^{n-i}(1)_{n-i,1/\lambda} 2^{n-i+1} E_{i,\lambda}^{(k)} S_{2,\lambda}(n-i,m)}{n!} t^n.
\]

Therefore, by comparing the coefficients of (38) and (39), we have the desired result. \qed

Theorem 14. For $n \geq 1$, $k \in \mathbb{Z}$, we get

\[
\sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \lambda^{n-m-1}(1)_{n-m,1/\lambda}(-1)^l 2^{-l-1} S_{1,\lambda}(m,l) E_{l,\lambda}^{(k)}
\]

\[
= \sum_{m=1}^{n} \sum_{l=0}^{n-m} \binom{n}{m} (-1)^{l-1} 2^{-l-1} \frac{(1)_{m,1/\lambda} \lambda^{m-1}}{m^k m!} S_{1,\lambda}(n-m,l) E_{l,\lambda}.
\]
Proof. Replace \( t \) by \(-\frac{1}{2}\log_\lambda (1 + t)\), from (5), (15), we observe

\[
\begin{align*}
&\frac{\lambda^{k,\lambda}(-t)}{e^\lambda \left(-\frac{1}{2}\log_\lambda (1 + t)\right) + 1} \\
&= \frac{1}{2} \sum_{l=0}^\infty E_{l,\lambda} \left(-\frac{1}{2}\log_\lambda (1 + t)\right)^l \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda} (-t)^m}{(m-1)! m^\lambda} \\
&= \frac{1}{2} \sum_{l=0}^\infty E_{l,\lambda} \lambda^{l-1} (1)_{1,1/\lambda} \frac{t^l}{l!} \sum_{m=1}^\infty \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda} (-t)^m}{m^{k-1}} \\
&= \frac{1}{2} \sum_{n=1}^\infty \left( \sum_{m=1}^n \sum_{l=0}^{n-m} \binom{n}{m} (-1)^{l-1} 2^{-l-1} S_{1,\lambda} (m, l) E_{l,\lambda} (1)_{m,1/\lambda} \right) \frac{t^n}{n!}.
\end{align*}
\]

On the other hand, from (15), we have

\[
\begin{align*}
-\frac{1}{2}\log_\lambda (1 + t) &= \sum_{l=0}^\infty E_{l,\lambda} \left(-\frac{1}{2}\log_\lambda (1 + t)\right)^l \\
&= -\frac{1}{2} \sum_{l=0}^\infty \lambda^{l-1} (1)_{1,1/\lambda} \frac{t^l}{l!} \sum_{m=1}^\infty \frac{(-\lambda)^{m-1}(1)_{m,1/\lambda} (-t)^m}{m^{k-1}} \\
&= -\frac{1}{2} \sum_{l=0}^\infty \lambda^{l-1} (1)_{1,1/\lambda} \frac{t^l}{l!} \sum_{m=0}^l \sum_{j=0}^m \frac{(-1)^j 2^{-l-1} S_{1,\lambda} (m, l) E_{l,\lambda} (1)_{m,1/\lambda} (-1)^{l+j+2-l-1}}{m!} \\
&= \sum_{n=1}^\infty \left( \sum_{m=0}^n \sum_{l=0}^{n-m} \binom{n}{m} \lambda^{n-m-1} (1)_{n-m,1/\lambda} (-1)^{l+j+2-l-1} S_{1,\lambda} (m, l) E_{l,\lambda} (1)_{m,1/\lambda} \right) \frac{t^n}{n!}.
\end{align*}
\]

Therefore, by comparing the coefficients of (40) and (41), we have the desired result.

\[\square\]

**Theorem 15.** For \( n \geq 0, \ k \in \mathbb{Z} \), we have

\[
E_{n,\lambda}^{(k)} = \sum_{l=0}^n \sum_{m=0}^{l} \binom{n}{l} \binom{l}{m} (-1)^j 2^{l+1} (1)_{m+1,1/\lambda} \frac{E_{l-m,\lambda} E_{n-l,\lambda}}{m+1}. \]
Proof. From (4), (7) and (10), we observe

\[
\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!} = \frac{I_{k,\lambda}(1-e^{\lambda}(-2t))}{t(e^{\lambda}(t)+1)} \frac{1-e^{\lambda}(-2t)}{1-e^{\lambda}(-2t)}
\]

\[
= \frac{1}{2t} \frac{I_{k,\lambda}(1-e^{\lambda}(-2t))}{e^{\lambda}(t)+1} \frac{1-e^{\lambda}(-2t)}{1-e^{\lambda}(-2t)} \left(1-e^{\lambda}(-2t)\right)
\]

\[
= \frac{1}{2t} \sum_{i=0}^{\infty} E_{i,\lambda} \frac{t^i}{i!} \sum_{j=0}^{\infty} \beta_{j,i,\lambda} \frac{(-2)^j}{j!} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} 2^m (1)_{m,\lambda} t^m}{m!}
\]

\[
= \sum_{i=0}^{\infty} E_{i,\lambda} \frac{t^i}{i!} \sum_{j=0}^{\infty} \beta_{j,i,\lambda} \frac{(-2)^j}{j!} \sum_{m=0}^{\infty} \frac{(-1)^{m+2} 2^m (1)_{m+1,\lambda} t^m}{m+1}
\]

\[
= \sum_{i=0}^{\infty} E_{i,\lambda} \frac{t^i}{i!} \sum_{j=0}^{\infty} \beta_{j,i,\lambda} \frac{(-2)^j}{j!} \sum_{m=0}^{\infty} \frac{(-1)^{m+2} 2^m (1)_{m+1,\lambda} E_{n-1,\lambda}}{m+1} \frac{t^m}{m!}
\]

(42)

Therefore, by comparing the coefficients on both sides of (42), we have the desired result.

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Competing interests
The authors declare no conflict of interest.

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