The maximum deviation of the Sine\(_\beta\) counting process

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Abstract

In this paper, we consider the maximum of the Sine\(_\beta\) counting process from its expectation. We show the leading order behavior is consistent with the predictions of log–correlated Gaussian fields, also consistent with work on the imaginary part of the log–characteristic polynomial of random matrices. We do this by a direct analysis of the stochastic sine equation, which gives a description of the continuum limit of the Prüfer phases of a Gaussian \(\beta\)–ensemble matrix.

The Sine\(_\beta\) point process ([VV09]), which arises as the local point process limit of the eigenvalues of \(\beta\)–ensembles, can be defined in terms of the SDE

\[
d\alpha_{x,t} = \frac{\beta}{4} e^{-\frac{\alpha}{2t}} dt + \text{Re} \left[ \left( e^{-i\alpha_{x,t}} - 1 \right) dZ_t \right], \quad \alpha_{x,0} = 0.
\]

Specifically, sending \(t \to \infty\), \(\alpha_{x,t}/(2\pi)\) converges for all \(x\) to an integer valued limit, which is the counting function of the Sine\(_\beta\) point process.

This function is an example of a process that should satisfy log–correlated field predictions. Indeed, it is very strongly related to the maximum of the imaginary part of the characteristic polynomial of random matrices. In the context of random matrices, similar theorems have been proven by [ABB17, PZ17, CMN16, LP18]. Indeed recent results of [VV17a] give a coupling between the Sine\(_\beta\) process and the eigenvalues of a C\(_\beta\)E point process with sufficient precision that one could hope to transfer results between the two processes.

We consider the process \(N(x) = \lim_{t \to \infty} \frac{\alpha_{x,t} - \alpha_{-x,t}}{2\pi}\), which counts the number of points in the Sine\(_\beta\) point process between \([-x, x]\) for any \(x > 0\). This process exhibits a purer analogy with log–correlated fields (see Remark 5 for details). We show that:

**Theorem 1.**

\[
\max_{0 \leq \lambda \leq x} \left[ N(\lambda) - \frac{\lambda}{2\pi} \right] \xrightarrow{\log x} \frac{2}{\sqrt{\beta \pi}}.
\]

Moreover, we do this by a direct argument for the Sine\(_\beta\) process that avoids a Gaussian coupling.

Observe that as the process \(N(\lambda)\) is almost surely non–decreasing, we may immediately replace this maximum over all \(0 \leq \lambda \leq x\) by the maximum over any discrete net of \([0, x]\) with
maximum spacing \(o(\log x)\). Likewise, we may assume that \(x\) is an integer. Going forward, we will take \(\lambda\) and \(x\) to be integers.

It should be noted there is another SDE description due to [KS+09] (only recently proven to give rise to the same process by [Nak14], while another proof follows from [VV17b]), which can be related to (1) by a time–reversal. This arises due to an order reversal of the Prüfer phases, due to this the correlation structure is reversed from the previously studied \(C_\beta E\) model. The processes \(\alpha_{x,t}\) and \(\alpha_{y,t}\) are strongly correlated for large times and weakly correlated for small times. We elaborate upon the correlation structure in (6).

**Heuristic**

We will name the martingale part of \(\alpha_{\lambda,t} - \alpha_{-\lambda,t}\) diffusion:

\[
M_{\lambda,t} = \text{Re} \int_0^t (e^{-i\alpha_{\lambda,s}} - e^{-i\alpha_{-\lambda,s}})dZ_s.
\]

As the process \(\alpha_{x,t}\) converges for all \(x \in \mathbb{R}\) when \(t \to \infty\), so does \(M_{\lambda,t}\) converge for all \(\lambda \in \mathbb{R}\) when \(t \to \infty\). Moreover,

\[
2\pi N(\lambda) - 2\lambda = \text{Re} \int_0^\infty (e^{-i\alpha_{\lambda,s}} - e^{-i\alpha_{-\lambda,s}})dZ_s = M_{\lambda,\infty}.
\]

Therefore we can reformulate Theorem 1 as

\[
\max_{0 \leq \lambda \leq x} \frac{M_{\lambda,\infty}}{\log x} \xrightarrow{p} \frac{4}{\sqrt{\beta}}.
\]

Let \(T_\lambda = \frac{4}{\beta} \log \lambda\). This is heuristically the length of time that \(M_{\lambda,t}\) needs to evolve so that it is within bounded distance of its limit. Specifically, the variables \(M_{\lambda,\infty} - M_{\lambda,T_\lambda}\) have a uniform–in–\(\lambda\) exponential tail bound:

**Proposition 2.** There is a constant \(C = C_\beta\) so that for all \(\lambda, r \geq 0\),

\[
P\left[ M_{\lambda,\infty} - M_{\lambda,T_\lambda} \geq C + r \right] \leq e^{-r/C}.
\]

Using the monotonicity of \(N(\lambda)\), we can also show that:

**Proposition 3.**

\[
\max_{0 \leq \lambda \leq x} \left| \frac{M_{\lambda,\infty} - M_{\lambda,T_\lambda}}{\log x} \right| \xrightarrow{p} 0, \quad x \to \infty.
\]

Hence we need only consider the process \(M_{\lambda,t}\) up to time \(t = T_\lambda\). We delay the proofs of these propositions to Section 1.
Another representation for $M_{\lambda,t}$ is given by, for all $t \geq 0$

$$M_{\lambda,t} = \text{Re} \int_0^t (e^{-\frac{i}{2}(\lambda_s - \alpha_{-\lambda_s})} - e^{-\frac{i}{2}(\alpha_{-\lambda_s} - \alpha_{\lambda_s})}) e^{-\frac{i}{2}(\alpha_{\lambda_s} + \alpha_{-\lambda_s})} dZ_s$$

$$= \text{Re} \int_0^t (e^{-\frac{i}{2}(\lambda_s - \alpha_{-\lambda_s})} - e^{-\frac{i}{2}(\alpha_{-\lambda_s} - \alpha_{\lambda_s})}) (dV^{(\lambda)}_s + idW^{(\lambda)}_s)$$

$$= \int_0^t 2 \sin \left( \frac{\alpha_{\lambda_s} - \alpha_{-\lambda_s}}{2} \right) dW^{(\lambda)}_s. \quad (4)$$

where $dV^{(\lambda)}_s + idW^{(\lambda)}_s = e^{-\frac{i}{2}(\alpha_{\lambda_s} + \alpha_{-\lambda_s})} dZ_s$ is a standard complex Brownian motion.

Hence, the bracket process is given by

$$[M_\lambda]_t = \int_0^t 4 \sin \left( \frac{\alpha_{\lambda_s} - \alpha_{-\lambda_s}}{2} \right)^2 ds. \quad (5)$$

Applying the trig identity $2 \sin(x)^2 = 1 - \cos(2x)$, and treating the oscillating the term as negligible, we can consider $[M_\lambda]_t \approx 2t$, for $t \leq T_\lambda$. This allows us to roughly consider $M_{\lambda,T_\lambda}$, for the purpose of moderate deviations, as a centered Gaussian of variance $2T_\lambda$.

As for the correlation structure,

$$[M_\lambda, M_\mu]_t = \text{Re} \int_0^t (e^{-i\alpha_{\lambda,s}} - e^{-i\alpha_{-\lambda,s}})(e^{i\alpha_{\mu,s}} - e^{i\alpha_{-\mu,s}}) ds \quad (5)$$

Approximating $\alpha_{\lambda,t}$ by its drift, we are led to the heuristic that $M_\lambda$ and $M_\mu$ behave approximately independently for $t \leq \frac{4}{\beta} \log_+ |\lambda - \mu|$ and are maximally correlated for larger $t$. This leads to the cross variation heuristic:

$$[M_\lambda, M_\mu]_{T_\lambda \wedge T_\mu} \approx 2(T_{\lambda} \wedge T_{\mu} - \frac{4}{\beta} \log_+ |\lambda - \mu|). \quad (6)$$

We can define a Gaussian process that has the exact correlation structure suggested by the heuristics in (6):

$$G_{\lambda,t} = \text{Re} \int_0^t (e^{-i\alpha_{\lambda,s}} - e^{-i\alpha_{-\lambda,s}})dZ_s. \quad (7)$$

For this process, we have correlation given by

$$[G_\lambda, G_\mu]_t = 4 \int_0^t \sin \left( \lambda(1 - e^{-\frac{\beta}{4}s}) \right) \sin \left( \mu(1 - e^{-\frac{\beta}{4}s}) \right) ds. \quad (7)$$

On the supposition that the maximum of the field $(G_{\lambda,T_\lambda}, 0 \leq \lambda \leq x)$ and the maximum of $\lambda \mapsto M_{\lambda,\infty}$ agree, we are led to the following conjecture.

**Conjecture 4.** There is a random variable $\xi$ so that

$$\max_{0 \leq \lambda \leq x} (M_{\lambda,\infty}) - \frac{1}{\sqrt{\beta}} \left( \log x - \frac{3}{4} \log \log x \right) \xrightarrow{d} \xi \text{ as } x \to \infty.$$ 

Indeed by a theorem of [DRZ+17], this full convergence could be proven for the $G_{\lambda,T_\lambda}$ field.
Remark 5. If we instead considered the one–sided problem, we would instead see
\[
\max_{0 \leq \lambda \leq x} \frac{[\alpha_{\lambda,\infty} - \lambda]}{\log x} \xrightarrow{p} \frac{4}{\sqrt{2\beta}} \quad \text{as} \quad x \to \infty.
\]
We would be led to considering the martingale
\[
V_{\lambda,t} = \text{Re} \int_0^t (e^{-i\lambda t} - 1) dZ_s.
\]
which has quadratic variation \([V_{\lambda}]_t \approx 2t \) for \( t < T_\lambda \) and cross variation:
\[
[V_{\lambda}, V_\mu]_{T_\lambda \wedge T_\mu} = \text{Re} \int_0^t (e^{-i\lambda s} - 1)(e^{i\mu s} - 1) ds \approx T_\lambda \wedge T_\mu + \frac{1}{2}[M_\lambda, M_\mu]_{T_\lambda \wedge T_\mu}.
\] (8)
Thus, the process has an additional positive correlation, which is heuristically equivalent to adding a common standard normal of variance \( \frac{4}{\sqrt{2\beta}} \log x \) to every \( V_{\lambda,\infty} \) for \( \delta x \leq \lambda \leq x \). In particular this is too small to change the behavior of the maximum. As working with \( V_{\lambda,t} \) does not materially change the argument, we have not pursued it here.

1 Background tools

We begin with the proofs of Propositions 2 and 3. These rely heavily on basic properties of the diffusion established in [VV09, Proposition 9].

Delayed proofs from introduction

Proof of Proposition 2. Observe first by integrating the drift
\[
M_{\lambda,\infty} - M_{\lambda,T_\lambda} = \alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda} - 1.
\] (9)
Consider the process \( v \) that satisfies
\[
dv_t = \frac{\beta}{4} e^{-\frac{\beta}{2} t} \mathbf{1} \{ t \leq T_\lambda \} dt + \text{Re} \left[ (e^{-ivy} - 1) dZ_t \right], \quad v_0 = 0.
\]
Then \( \alpha_{\lambda,t} \) and \( u_t \) are equal until \( T_\lambda \). After this time, \( v \) never crosses another multiple of 2\( \pi \). Moreover, it eventually converges to a multiple of 2\( \pi \) ([VV09, Proposition 9(iv)]). Hence we have
\[
|v_\infty - \alpha_{\lambda,T_\lambda}| \leq 2\pi.
\] (10)
On the other hand \( \alpha_{\lambda,\infty} - v_\infty \) has the same law at time \( T_\lambda \) as \( \alpha_{1,\infty} \) started at time 0. By [VV09, Proposition 9(viii)], this has an exponential tail bound. \( \square \)

Proof of Proposition 3. By (9), it suffices to show the same for \( \alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda} \). The diffusion \( \alpha_{\lambda,t} \) can not cross below an integer multiple of 2\( \pi \). Hence if \( s \leq t \), for all \( \lambda \geq 0 \) \( \alpha_{\lambda,s} \leq \alpha_{\lambda,t} + 2\pi \).
This implies
\[
\min_{0 < \lambda \leq x} (\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}) \geq -2\pi,
\]
and it suffices to consider an upper bound. For $x/2 \leq \lambda \leq x$, we can estimate

$$\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda} \leq \alpha_{\lambda,\infty} - \alpha_{\lambda,T_{x/2}} + 2\pi$$

Let $v_\lambda$ satisfy

$$dv_{\lambda,t} = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4} t} 1 \{t \leq T_{x/2}\} \, dt + \text{Re} \left[ (e^{-iv_{\lambda,t}} - 1) \, dZ_t \right], \quad v_{\lambda,0} = 0.$$

As $v_\lambda$ can not cross multiples of $2\pi$, for any $\lambda \in \mathbb{R}$, after $T_{x/2}$, we have

$$\alpha_{\lambda,\infty} - \alpha_{\lambda,T_{x/2}} + 2\pi \leq \alpha_{\lambda,\infty} - v_{\lambda,\infty} + 4\pi.$$  

On the other hand $\alpha_{\lambda,t} - v_{\lambda,t}$ is monotone increasing in $\lambda$ almost surely (as the difference for parameters $\lambda_1 > \lambda_2$ satisfies an SDE that can not cross below 0, c.f. [VV09, Proposition 9(ii)].) Combining the work so far, we have the bound

$$\max_{x/2 \leq \lambda \leq x} (\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}) \leq \alpha_{x,\infty} - v_{x,\infty} + 4\pi.$$  

Using the equality in law given by

$$\left( \alpha_{x,t+T_{x/2}} - v_{x,t+T_{x/2}}, t \geq 0 \right) \overset{\mathcal{L}}{=} (\alpha_{2,t}, t \geq 0),$$

and by [VV09, Proposition 9(viii)], $\alpha_{2,\infty}$ has an exponential tail bound depending only on $\beta$. Applying the same argument for $j \in \mathbb{N}$ and $x 2^{-j-1} \leq \lambda \leq x 2^{-j}$, we may use a union bound up to $j$ on the order of $\log x$ to conclude that there is a constant $C_\beta$ so that

$$\max_{0 < \lambda \leq x} (\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}) \leq C_\beta \log \log x$$  

with probability going to 1 as $x \to \infty$. \hfill \square

**Oscillatory integrals**

For each $\lambda \in \mathbb{R}$, suppose that $A_{\lambda,t}$ is an adapted finite variation process so that $|A_{\lambda,t}| \leq \xi \in (0, \infty)$ for all time almost surely and suppose that $X_{\lambda,t}$ is a martingale satisfying $d[X_{\lambda,t}] \leq 2$. Suppose that

$$du_{\lambda,t} = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4} t} dt + A_{\lambda,t} dt + dX_{\lambda,t}, \quad u_{\lambda,0} = 0.$$  

**Proposition 6.** Let $u_{\lambda,t}$ satisfy (12) and let $f(t) = \frac{\beta}{4} e^{-\frac{\beta}{4} t}$, then for each fixed $\beta > 0$ there exist constants $R$ and $\gamma$ uniform in $T$ and $\lambda, a \in \mathbb{R}$ such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t e^{iau_{\lambda,s}} ds \right| \right] \leq \frac{R(1 + |\xi|)}{|a\lambda| f(T)},$$  

and for all $C > 0$

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \int_0^t e^{iau_{\lambda,s}} ds \right| - \frac{R(1 + |\xi|)}{|a\lambda| f(T)} \geq C \right) \leq \exp \left[ -\gamma C^2 a^2 \lambda^2 f(T)^2 \right].$$  

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Proof. The theorem is vacuous if $a\lambda = 0$, so we may assume this is not the case. Writing $u_t$ in its integrated form, we have

$$u_t = \lambda \left(1 - \frac{4}{\beta} t(t)\right) + \mathcal{R}_t$$

$$\mathcal{R}_t = \int_0^t \{A_{\lambda,s} ds + dX_{\lambda,s}\}.$$

Let $H(t) = 1 - \frac{4}{\beta} t(t)$ and $\Lambda(t) = \int_0^t e^{ia\lambda H(s)} ds$, then we may use Itô integration by parts to get

$$\int_0^t e^{ia\lambda H(s)} ds = \int_0^t e^{ia\lambda H(s)} e^{ia\mathcal{R}_s} ds = e^{ia\mathcal{R}_t} \Lambda(t) - \int_0^t \Lambda(s) e^{ia\mathcal{R}_s} d\mathcal{R}_s + \frac{a^2}{2} \int_0^t \Lambda(s) e^{ia\mathcal{R}_s} d[\mathcal{R}]_s. \tag{15}$$

Now observe that $\Lambda(t)$ may be bounded in the following way:

$$
\int_0^t e^{ia\lambda H(s)} ds = \int_0^t \frac{1}{i a \lambda f(s)} \frac{d}{ds} e^{ia\lambda H(s)} ds = \frac{4 e^{\frac{2}{\beta} t}}{\beta i a \lambda} \{e^{ia\lambda H(t)} - 1\} - \frac{1}{i a \lambda} \int_0^t e^{\frac{2}{\beta} s} \{e^{ia\lambda H(s)} - 1\} ds.
$$

This gives us $|\Lambda(s)| \leq \frac{16}{\beta |a\lambda|} e^{\frac{2}{\beta} t}$. Applying this to our integrated equation we get for the finite variation terms

$$\left| \int_0^t \Lambda(s) e^{ia\mathcal{R}_s} a A_{\lambda,s} ds + \frac{a^2}{2} \int_0^t \Lambda(s) e^{ia\mathcal{R}_s} d[\mathcal{R}]_s \right| \leq \frac{16}{\beta a \lambda} e^{\frac{2}{\beta} t} (|a| + a^2).$$

By (15) and the triangle inequality, it remains to show the desired tail bound and supremum bound for the martingale $V_t$ given by

$$V_t = \int_0^t \Lambda(s) i a e^{ia\mathcal{R}_s} \cdot dX_{\lambda,s}$$

Note we have an easy bracket bound, for $\sigma \in \{1, i\}$ given by

$$[\mathbb{R}(\sigma V)]_t \leq \int_0^t 2 \Lambda(s) a^2 ds \leq \frac{C_\beta}{\lambda^2} |a| e^{\frac{2}{\beta} t}$$

for some constant $C_\beta$. Hence the desired bounds follow immediately from the Dambis–Dubins–Schwarz theorem ([RY99, Theorem V.1.6] or [Pro05, Theorem II.42]) and Doob’s inequality.

\[\square\]

**Tilting**

We now want to look at the measure tilted so that $W^{(\lambda)}$ (see (4)) has a drift. In particular for deterministic $\xi \in \mathbb{R}$, we consider the measure $Q_{\xi,\lambda}$ so that

$$dX_s = dW^{(\lambda)}_s - \xi \sin \left(\frac{\alpha_{\lambda,s} - \alpha_{\lambda,s}}{2}\right) ds$$
is a standard Brownian motion up to time $T$ under $Q_{\xi,\lambda}$. By Girsanov (see e.g. [Pro05, Theorem III.8.46]) we get that

$$\frac{dQ_{\xi,\lambda}}{dP} = \mathcal{E}(\xi M) = \exp(\xi M_{\lambda,T} - \frac{\xi^2}{2}[M]_T)$$

(16)

Since $\sin^2(x) \leq 1$ we have that the bracket process of $[M]_t \leq T$ almost surely for all $t \geq 0$. In particular, the exponential martingale is uniformly integrable by Novikov’s condition for all $\xi \in \mathbb{R}$.

Under $Q_{\xi,\lambda}$ the law of $\alpha_{\lambda,t} - \alpha_{-\lambda,t}$ changes; it can be succinctly described as the solution to

$$du_{\lambda,\xi,t} = 2\lambda^2 e^{-\frac{\beta}{4}} dt + 2\xi \sin \left( \frac{\mu_{\lambda,t}}{2} \right) dt + 2 \sin \left( \frac{\mu_{\lambda,t}}{2} \right) dX_t, \quad u_0 = 0$$

(17)

for a Brownian motion $dX$, which we call the accelerated stochastic sine equation with acceleration $\xi$. Let $M_{\lambda,\xi,t}$ be the martingale part of $u_{\lambda,\xi,t}$.

**Martingale bounds**

Using the Girsanov transformation, we now give a nearly sharp tail bound for $M_{\lambda}$.

**Proposition 7.** For any $\eta \in \mathbb{R}$, there is an $R > 0$ so that for all $\lambda > 0$, all $T \leq T_{\lambda}$

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} M_{\lambda,\eta,t} \geq C \right) \leq \exp \left[ \frac{-C^2}{4(T + R)} \left( 1 - \frac{C^2 R}{2(T + R)^3} \right) \wedge \frac{-C^4}{4T^{1/3}} \right].$$

and

$$\mathbb{P} \left( \inf_{0 \leq t \leq T} M_{\lambda,\eta,t} \leq -C \right) \leq \exp \left[ \frac{-C^2}{4(T + R)} \left( 1 - \frac{C^2 R}{2(T + R)^3} \right) \wedge \frac{-C^4}{4T^{1/3}} \right].$$

**Remark 8.** For $C$ up to the order of magnitude of $T^{3/2}$ the Gaussian tail majorizes the martingale tail. For larger $C$, the second term majorizes the martingale tail. For much much larger $C$ (on the order $T^2$) a small change in the proof gives decay of order $e^{-cC^{4/3}}$. A large deviations principle for $N_{\lambda}$ is proven in [HV15] which suggests a stronger tail bound ought to be true.

**Proof.** Let $X_t$ be a standard Brownian motion, and let $w$ solve (17) the accelerated stochastic sine equation with acceleration $\eta$. Let $M$ be the martingale part of $w$. Let $\xi \in \mathbb{R}$, and apply Doob’s inequality to the submartingale $e^{\xi M_t}$ to get

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} M_t \geq C \right) \leq e^{-\xi C} \mathbb{E}(e^{\xi M_T}).$$

Applying (16), we have that

$$\mathbb{E}(e^{\xi M_T}) = \mathbb{E} \left( \mathcal{E}(\xi M_T) e^{\frac{\xi^2}{2}[M]_T} \right) = \hat{Q} \left( e^{\frac{\xi^2}{2}[M]_T} \right),$$

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with \( \hat{Q}_E(\cdot) \) the expectation under the probability measure \( \hat{Q} \) defined by
\[
\frac{d\hat{Q}}{dP} = E(\xi M_T).
\]
By the Girsanov theorem,
\[
dY_s = dX_s - \xi \sin \left(\frac{w_s}{2}\right) \, ds
\]
is a \( \hat{Q} \)-Brownian motion. Hence,
\[
M_t = \int_0^t 2 \sin \left(\frac{w_s}{2}\right) dY_s + \int_0^t 2\xi \sin \left(\frac{w_s}{2}\right)^2 \, ds.
\]
Further, the law of \( w_s \) changes under \( \hat{Q} \), as we have that
\[
dw_t = 2\lambda \beta e^{-\beta t} dt + 2(\xi + \eta) \sin \left(\frac{w_t}{2}\right) dt + 2 \sin \left(\frac{w_t}{2}\right) dY_t, \quad w_0 = 0.
\]
Hence, under \( \hat{Q} \), \( w \) is a solution of the accelerated stochastic sine equation with acceleration \( \xi + \eta \).

As for the bracket, we have that for \( t \leq T \)
\[
[M_\lambda]_t = \int_0^t 4 \sin \left(\frac{w_s}{2}\right)^2 \, ds = 2t - \int_0^t 2 \cos (w_s) \, ds.
\]
Using Proposition 6, we have that for \( T \leq T_\lambda \), there is an \( R \) independent of \( \xi \) and \( \eta \) so that for all \( C > 0 \)
\[
\hat{Q} \left( \int_0^T -2 \cos (w_s) \, ds \geq R(1 + |\xi + \eta|) + C \right) \leq e^{-C^2/R}.
\]
Therefore, we have that for \( T \leq T_\lambda \)
\[
\hat{Q}_E \left( e^{\frac{\xi^2}{2} [M_\lambda]_T} \right) = e^{\xi^2 T} \hat{Q}_E \left( \exp \left( \int_0^T -\xi^2 \cos (w_s) \, ds \right) \right) \leq e^{\xi^2 (T+S) + S\xi^4}
\]
for some constant \( S > 0 \) independent of \( \xi, \lambda \) or \( T \) but depending on \( \eta \).

There remains to optimize in \( \xi \). From the work so far, we have
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} M_t \geq C \right) \leq e^{-\xi C} \mathbb{E}(e^{\xi M_T}) \leq e^{-\xi C + \xi^2 (T+S) + S|\xi|^3}.
\]
Taking \( \xi = \frac{C}{2(T+S)} \),
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} M_t \geq C \right) \leq \exp \left[- \frac{C^2}{4(T+S)} + \frac{SC^4}{8(T+S)^3} \right],
\]
and taking \( \xi = \left( C/(4T + 4S) \right)^{1/3} \) gives
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} M_t \geq C \right) \leq \exp \left[- \frac{3C^{4/3}}{4(4(T+S))^{1/3}} + \frac{C^{2/3}(T+S)^{1/3}}{4^{2/3}} \right].
\]
Hence the desired bound holds by taking the second bound for \( C > P(T+S) \) and \( P \) sufficiently large, and the first bound for \( C \leq P(T+S) \).

The statement about the infimum may be proved in an identical fashion by recognizing that the statement is equivalent to at statement on the supremum of \( -M_\lambda \). We then use the submartingale \( e^{-\xi M_s} \) and use \( [M_\lambda]_t = [-M_\lambda]_t \).
2 Main theorem

The one-point upper bound

Using Proposition 7, we can give the upper bound in (3).

Proposition 9. For any $\delta > 0$

$$\lim_{x \to \infty} \mathbb{P} \left( \max_{0 \leq \lambda \leq x} M_{\lambda,T} > \left( \frac{4}{\sqrt{\beta}} + \delta \right) \log x \right) = 0$$

Proof. As commented, it suffices to bound the probability for natural numbers $\lambda$ and $x$. For any $\delta > 0$ sufficiently small there is an $\epsilon > 0$ and an $x_0$ sufficiently large so that for all $x > x_0$ and all $x > \lambda > \exp((\log x)^{3/4})$

$$\mathbb{P} \left( M_{\lambda,T} > \left( \frac{4}{\sqrt{\beta}} + \delta \right) \log x \right) \leq \exp \left( - (\log x)^2 \frac{4}{\log x} \right) \leq \exp \left( - (\log x) (1 + \epsilon) \right).$$

For smaller $\lambda$, we have, taking the $4/3$–power bound in Proposition 7, that for some $C_{\beta,\delta}$

$$\mathbb{P} \left( M_{\lambda,T} > \left( \frac{4}{\sqrt{\beta}} + \delta \right) \log x \right) \leq \exp \left( - (\log x)^{13/12} C_{\beta,\delta} \right)$$

Hence, taking a union bound over all natural numbers $\lambda$ less than $x$ gives the desired bound.

Remark 10. In fact, the proof is easily modified to give

$$\limsup_{\lambda \to \infty} \left( \frac{M_{\lambda,T}}{\log \lambda} \right) \leq \frac{4}{\sqrt{\beta}}, \text{ a.s.}$$

The tube event and the lower bound

Let $x$ be a natural number, and let $R$ be a large parameter to be chosen later. Let $T'_x = T_x - R^2 \sqrt{\log \lambda}$. Define an event $\mathcal{A}_\lambda$ given by

$$\mathcal{A}_\lambda = \left\{ |M_{\lambda,t} - \sqrt{\beta} t| \leq R \sqrt{\log x}, \forall 0 \leq t \leq T'_x; \right\}.$$

Let $x$ be a natural number, and define

$$S_x = \sum_{\lambda=x}^{2x} \mathcal{E}(\sqrt{\beta} M_{\lambda,T'_x}) \mathbf{1} \{ \mathcal{A}_\lambda \}$$

Notice that with this definition of $S_x$ we will have that $S_x > 0$ if and only if the event $\mathcal{A}_\lambda$ occurs for some integer $\lambda \leq x$. Using the Payley–Zygmund inequality,

$$\mathbb{P}(S_x > 0) \geq \frac{(\mathbb{E}S_x)^2}{\mathbb{E}S_x^2}.$$
We wish to show that this has probability going to 1 as \( \lambda \to \infty \) for any \( \delta > 0 \). Hence, we need to produce a lower bound of the form
\[
\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda,T'_\lambda}) \mathbf{1}\{A_\lambda\}] = Q_{\sqrt{\beta} \lambda}(A_\lambda) \geq 1 - C_\beta e^{-R^{4/3}/C_\beta},
\]
and we need to produce a similar upper bound on
\[
\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda_1,T'_\lambda}) \mathbf{1}\{A_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2,T'_\lambda}) \mathbf{1}\{A_{\lambda_2}\}].
\]
From these bounds we will be able to show that as \( x \to \infty \)
\[
(\mathbb{E} S^2_x)/x^2 \to 0 \quad \text{and} \quad \mathbb{E} S_x \geq x(1 - C_\beta e^{-R^{4/3}/C_\beta}). \quad (19)
\]
Hence, we conclude using Payley-Zygmund that for any \( \epsilon > 0 \) there is an \( R \) sufficiently large and an \( x_0 \) sufficiently large so that for all \( x > x_0 \)
\[
\mathbb{P}(S_x > 0) \geq \frac{\mathbb{E} S^2_x}{\mathbb{E} S^2_x} \geq 1 - \epsilon.
\]
We have therefore shown that by letting \( R_x \) tend arbitrarily slowly to infinity
\[
\max_{x \leq \lambda \leq 2x} \{M_{\lambda,T'_\lambda}\} \geq \sqrt{\beta} T'_x - R_x \sqrt{\log x}, \quad (20)
\]
with probability going to 1 as \( x \to \infty \).

**One point lower bound**

We need to find a lower bound on
\[
\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda,T'_\lambda}) \mathbf{1}\{A_\lambda\}] = Q_{\sqrt{\beta} \lambda}(A_\lambda),
\]
which is on the order of unity. Recall that under \( Q_{\sqrt{\beta} \lambda} \) the process \( \alpha_{\lambda,-} - \alpha_{-\lambda,+} \) follows the accelerated stochastic sine equation (17) with \( \xi = \sqrt{\beta} \). The process \( M_{\lambda,t} \) referenced in the event \( A_\lambda \) can be expressed as
\[
M_{\lambda,t} = u_{\lambda,\xi,t} - 2\lambda(1 - \frac{4}{\beta} f(t)).
\]
Meanwhile, the performing the Doob decomposition on \( u_{\lambda,\xi,t} \), we have
\[
M_{\lambda,\xi,t} = u_{\lambda,\xi,t} - 2\lambda(1 - \frac{4}{\beta} f(t)) - \int_0^t 2\xi \sin \left( \frac{u_{\lambda,\xi,s}}{2} \right)^2 ds
\]
The bracket process \( [M_{\lambda,\xi}]_t \) is given as before by
\[
[M_{\lambda,\xi}]_t = \int_0^t 4 \sin \left( \frac{u_{\lambda,\xi,s}}{2} \right)^2 ds = 2t - \int_0^t 2 \cos (u_{\lambda,\xi,s}) ds.
\]
Hence we can write
\[
Q_{\xi,\lambda}(A_{\lambda}) \geq 1 - Q_{\xi,\lambda} \left( \sup_{0 \leq t \leq T_\lambda^*} \left| M_{\lambda,\xi,t} + \int_0^t \xi \cos(u_{\lambda,\xi,s}) \, ds \right| > R \sqrt{\log x} \right)
\]
\[
- Q_{\xi,\lambda} \left( \sup_{0 \leq t \leq T_\lambda^*} \left| \int_0^t 2 \cos(u_{\lambda,\xi,s}) \, ds \right| > R \right).
\]

By Propositions 6 and 7, we conclude that
\[
Q_{\xi,\lambda}(A_{\lambda}) \geq 1 - C_\beta e^{-R^{4/3}/C_\beta}
\]
for some $C_\beta$ sufficiently large and all $\lambda$ sufficiently large.

**Two point bound**

Following the heuristic (6), we treat $M_{\lambda_1,t}$ and $M_{\lambda_2,t}$ as uncorrelated until $T_\ast = \frac{1}{\beta} \log \lambda - \lambda_2$. Without loss of generality, suppose that $\lambda_2 \geq \lambda_1$. On the event $A_{\lambda_2}$, we can estimate
\[
E(\sqrt{\beta} M_{\lambda_2,t^*}) = E(\sqrt{\beta} M_{\lambda_2, T_\ast}) \exp \left( \sqrt{\beta} (M_{\lambda_2, t^*} - M_{\lambda_2, T_\ast}) - \frac{\beta}{2} (\sup_{0 \leq t \leq T_\lambda^*} |M_{\lambda_2, t^*} - [M_{\lambda_2}]_{T_\ast}|) \right)
\]
\[
\leq E(\sqrt{\beta} M_{\lambda_2, T_\ast}) \exp \left( 2 \sqrt{\beta} R \sqrt{\log x} + \beta R \right).
\]

Hence, we have the estimate
\[
\mathbb{E} \left[ E(\sqrt{\beta} M_{\lambda_1,T_\ast}^*) 1 \{ A_{\lambda_1} \} E(\sqrt{\beta} M_{\lambda_2,T_\ast}^*) 1 \{ A_{\lambda_2} \} \right]
\]
\[
\leq \mathbb{E} \left[ E(\sqrt{\beta} M_{\lambda_1,T_\ast}^*) E(\sqrt{\beta} M_{\lambda_2,T_\ast}^*) \exp \left( 2 \sqrt{\beta} R \sqrt{\log x} + \beta R \right) \right]. \tag{22}
\]

We now observe that
\[
E(\sqrt{\beta} M_{\lambda_1,T_\ast}^*) E(\sqrt{\beta} M_{\lambda_2,T_\ast}^*) = E(\sqrt{\beta} (M_{\lambda_1,T_\ast}^* + M_{\lambda_2,T_\ast}^*)) \exp (\beta [M_{\lambda_1, T_\ast}^* - [M_{\lambda_2}]_{T_\ast}^*]). \tag{23}
\]

By the Girsanov theorem, under the measure $\mathbb{S}$ with Radon–Nikodym derivative
\[
\frac{d\mathbb{S}}{d\mathbb{P}} = \mathbb{E}(\sqrt{\beta} (M_{\lambda_1,T_\ast}^* + M_{\lambda_2,T_\ast}^*))
\]
we have that there is a finite variation process $A_t$ bounded almost surely by an absolute constant so that
\[
dU_t = dZ_t - \sqrt{\beta} A_t \, dt
\]
is a standard complex $\mathbb{S}$–Brownian motion. Here $Z_t$ is the standard complex Brownian motion used in equation (1) under the measure $\mathbb{P}$. Meanwhile (1) (also c.f. (5)) shows that $[M_{\lambda_1}, M_{\lambda_2}]_t$ is a sum of integrals of $e^{i \sigma_j \alpha_{2,1} \alpha_{2,1} + \sigma_j \alpha_{2,2} \alpha_{2,2}}$ with $\sigma_j \in \{1, -1\}$. Applying Proposition 6 to each of these integrals, we can conclude
\[
\mathbb{P} \left( [M_{\lambda_1}, M_{\lambda_2}]_{T_\ast}^* > t + C \right) \leq e^{-t^2/C}
\]
for sufficiently large $C$. Hence we conclude using (23) and (22) that there is some constant $C_\beta$ so that for any $R > 0$
\[
\mathbb{E} \left[ E(\sqrt{\beta} M_{\lambda_1,T_\ast}^*) 1 \{ A_{\lambda_1} \} E(\sqrt{\beta} M_{\lambda_2,T_\ast}^*) 1 \{ A_{\lambda_2} \} \right] \leq e^{C_\beta + 2 R \sqrt{\beta} \log x + \beta R}. \tag{24}
\]
Fine estimate

We also need an estimate that improves when \( \lambda_1 \) and \( \lambda_2 \) are well separated. Once more, we estimate by dropping the indicators and writing

\[
\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda_1, T_x}) \mathbf{1} \{A_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2, T_x}) \mathbf{1} \{A_{\lambda_2}\}] \leq S (\exp(\beta [M_{\lambda_1}, M_{\lambda_2}] T_x)), \tag{25}
\]

where

\[
\frac{dS}{dP} = \mathcal{E}(\sqrt{\beta} (M_{\lambda_1, T_x} + M_{\lambda_2, T_x})).
\]

Now, on applying Proposition 6, we have a tail bound of the form

\[
\mathbb{P}(\{ [M_{\lambda_1}, M_{\lambda_2}] T_x > t + C_\beta / \Delta \}) \leq e^{-t^2 \Delta^2 / C_\beta}
\]

where \( \Delta = |\lambda_1 - \lambda_2| f(T_x) \) and \( C_\beta > 0 \) is a constant. This leads to an estimate of the form

\[
\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda_1, T_x}) \mathbf{1} \{A_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2, T_x}) \mathbf{1} \{A_{\lambda_2}\}] \leq \exp(C_\beta / \Delta). \tag{26}
\]

for some other \( C_\beta \) and all \( \Delta \geq 1 \).

The second moment

Here we estimate \( \mathbb{E} S_x^2 \). Recalling (18), we can write

\[
\mathbb{E} S_x^2 = \sum_{\lambda_1=x}^{2x} \sum_{\lambda_2=x}^{2x} \mathbb{E} \left[ \mathcal{E}(\sqrt{\beta} M_{\lambda_1, T_x}) \mathbf{1} \{A_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2, T_x}) \mathbf{1} \{A_{\lambda_2}\} \right]. \tag{27}
\]

We partition this sum according to the magnitude of \( |\lambda_1 - \lambda_2| \). Let \( S_0 \) be all those pairs \((\lambda_1, \lambda_2)\) so that \( |\lambda_1 - \lambda_2| \geq x e^{-\frac{1}{2} R^2 \sqrt{\log x}} \). Let \( S_1 \) be the remaining pairs. Observe that the cardinality of \( S_1 \) is at most \( 2x^2 e^{-\frac{1}{2} R^2 \sqrt{\log x}} \).

For terms in \( S_0 \), we apply the fine bound (26). The term \( \Delta \) that appears for such terms can be estimated uniformly by

\[
\Delta \geq x e^{-\frac{1}{2} R^2 \sqrt{\log x}} \cdot \frac{\beta}{4} e^{-\log x + R^2 \sqrt{\log x}},
\]

which tends to \( \infty \) with \( x \). In particular, we can estimate

\[
\sum_{S_0} \mathbb{E} \left[ \mathcal{E}(\sqrt{\beta} M_{\lambda_1, T_x}) \mathbf{1} \{A_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2, T_x}) \mathbf{1} \{A_{\lambda_2}\} \right] \leq x^2 \cdot (1 + O(e^{-\frac{1}{2} R^2 \sqrt{\log x}})). \tag{28}
\]

For the remaining terms, we apply the coarse bound (24), using which we conclude that

\[
\sum_{S_1} \mathbb{E} \left[ \mathcal{E}(\sqrt{\beta} M_{\lambda_1, T_x}) \mathbf{1} \{A_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2, T_x}) \mathbf{1} \{A_{\lambda_2}\} \right] \leq x^2 e^{\beta C_\beta - \frac{1}{2} R^2 \sqrt{\log x} + 2 R \sqrt{\beta \log x} + \beta R}. \tag{29}
\]

Hence picking \( R \) sufficiently large (anything larger than \( 4 \sqrt{\beta} \) will do), we have combining (27), (28) and (29) that

\[
(\mathbb{E} S_x^2) / x^2 \to 0 \tag{30}
\]
as \( x \to \infty \).
Proof of main theorem

As in the proofs of Propositions 2 and 3, we have that \( \left( \alpha_{\lambda,t} - \alpha_{\lambda,T_x'} - 4\pi : t \geq T_x', \lambda > 0 \right) \) is stochastically dominated by \( \left( \alpha_{\lambda,4\pi(T_x'),t} : t \geq 0, \lambda > 0 \right) \). Therefore we have by Proposition 7 that there is a \( \gamma > 0 \) so that for all \( C > 0 \),

\[
\max_{x \leq \lambda \leq 2x} \mathbb{P} \left( \alpha_{\lambda,T_x} - \alpha_{\lambda,T_x'} - 2\lambda \left( \frac{4}{\lambda} \right) (f(T_x) - f(T_x')) \leq -C + 4\pi \right) \leq e^{-\gamma C^2/(T_x - T_x')}. \]

In particular we conclude that

\[
\max_{x \leq \lambda \leq 2x} \left\{ -M_{\lambda,T_x} + M_{\lambda,T_x'} \right\} \leq C_\beta R_x (\log x)^{3/4} \quad (31)
\]

with probability going to 1.

Finally, we observe that for \( 0 \leq \lambda \leq 2x \),

\[
0 \leq \alpha_{\lambda,\infty} - \alpha_{\lambda,T_x} = M_{\lambda,\infty} - M_{\lambda,T_x} + 2\lambda \left( \frac{4}{\lambda} f(T_x) \right) \leq M_{\lambda,\infty} - M_{\lambda,T_x} + \frac{16}{\beta}. \]

Therefore, we conclude that

\[
\max_{x \leq \lambda \leq 2x} \left\{ M_{\lambda,\infty} \right\} \geq \max_{x \leq \lambda \leq 2x} \left\{ M_{\lambda,T_x} \right\} - \frac{16}{\beta} \quad (32)
\]

Combining (20), (31) and (32), we conclude that

\[
\max_{x \leq \lambda \leq 2x} \left\{ M_{\lambda,\infty} \right\} \geq \frac{4}{\sqrt{\beta}} \log(x) - C_\beta R_x (\log x)^{3/4} - (R_x^2 + R_x) \sqrt{\log x} - \frac{16}{\beta}
\]

with probability going to 1 as \( x \to \infty \).

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