Rademacher Chaos, Random Eulerian Graphs and The Sparse Johnson-Lindenstrauss Transform

Vladimir Braverman∗ Rafail Ostrovsky† Yuval Rabani‡

November 12, 2010

Abstract

The celebrated dimension reduction lemma of Johnson and Lindenstrauss has numerous computational and other applications. Due to its application in practice, speeding up the computation of a Johnson-Lindenstrauss style dimension reduction is an important question. Recently, Dasgupta, Kumar, and Sarlos (STOC 2010) constructed such a transform that uses a sparse matrix. This is motivated by the desire to speed up the computation when applied to sparse input vectors, a scenario that comes up in applications. The sparsity of their construction was further improved by Kane and Nelson (ArXiv 2010).

We improve the previous bound on the number of non-zero entries per column of Kane and Nelson from $O(1/\epsilon \log(1/\delta) \log(k/\delta))$ (where the target dimension is $k$, the distortion is $1 \pm \epsilon$, and the failure probability is $\delta$) to

$$O\left(\frac{1}{\epsilon} \left(\frac{\log(1/\delta) \log \log(1/\delta)}{\log \log(1/\delta)}\right)^2\right).$$

We also improve the amount of randomness needed to generate the matrix. Our results are obtained by connecting the moments of an order 2 Rademacher chaos to the combinatorial properties of random Eulerian multigraphs. Estimating the chance that a random multigraph is composed of a given number of node-disjoint Eulerian components leads to a new tail bound on the chaos. Our estimates may be of independent interest, and as this part of the argument is decoupled from the analysis of the coefficients of the chaos, we believe that our methods can be useful in the analysis of other chaoses.

∗University of California Los Angeles, Computer Science Department. Email: vova@cs.ucla.edu.
†University of California Los Angeles, Computer Science and Mathematics Departments. Email: rafail@cs.ucla.edu.
‡The Rachel and Selim Benin School of Computer Science and Engineering, The Hebrew University of Jerusalem, Jerusalem 91904, Israel. Email: yrabani@cs.huji.ac.il. Research supported by ISF grant 1109-07 and BSF grant 2008059.
1 Introduction

The celebrated flattening lemma of Johnson and Lindenstrauss [13] has numerous applications in pure mathematics, data analysis, signal processing, computational linear algebra, and machine learning. Informally, the lemma states that a random linear transformation mapping $\mathbb{R}^d$ to $\mathbb{R}^k$, where $k = O\left(\frac{1}{\epsilon} \log(1/\delta)\right)$, preserves the $L_2$-norm of any $x \in \mathbb{R}^d$ up to a factor of $(1 \pm \epsilon)$ with probability at least $1 - \delta$. The original argument uses a projection onto a random linear subspace. However, it turns out that many simpler transformations work just as well [10] [9] [12] [1] [17]. In particular, a $k \times d$ matrix of $\{-1, 0, +1\}$ i.i.d. entries, and in fact any sub-Gaussian i.i.d. entries, works [1] [17]. What makes the lemma particularly useful are its linearity and the fact that the target dimension $k$ depends only on $\epsilon$ and $\delta$ but not on $d$. Alon [5] gave a lower bound on $k$ demonstrating that the above upper bound is nearly the best possible.

Due to its application in practice, speeding up the computation of a Johnson-Lindenstrauss style dimension reduction beyond the trivial $O(dk)$ arithmetic operations per vector is an important question. Ailon [1], then Matoušek [17], gained constant factors by using a sparse matrix. In their groundbreaking work, Ailon and Chazelle [2] designed a fast Johnson-Lindenstrauss transform (FJLT) that asymptotically beats the $O(dk)$ bound. Their approach of first applying a preconditioner that “smears” input vectors to some extent, then using a structured linear transformation that works well on smeared vectors, is prevalent in followup work. Ailon and Liberty [3] gave a better FJLT, whose running time is $O(d \log k)$ arithmetic operations per input vector. Further results in this vein were given in [16] [3].

Recently, Dasgupta, Kumar, and Sarlos [8] revisited the question of designing a sparse JL transform. This is motivated by the desire to speed up the computation when applied using a small $\epsilon$ to sparse input vectors, a scenario that comes up in applications. They construct a random $k \times d$ transformation matrix with $c = O\left(\frac{1}{\epsilon} \log(1/\delta) \log^2(k/\delta)\right)$ non-zero entries per column. They use a trivial deterministic preconditioner $P$ that duplicates each coordinate $c$ times and rescales. The choice of $c$ governs the sparsity of the matrix. The novelty of their approach lies in the construction of the projection matrix, whose entries are not independent. This allows them to overcome a lower bound of $\tilde{\Omega}(\epsilon^{-2})$ on the sparsity of a JL transform matrix with independent entries [17]. They construct the projection matrix as follows: pick $\zeta \in \{-1, 1\}^d$ and a hash function $h : [cd] \to [k]$ uniformly at random. The $k \times cd$ projection matrix $H$ has $H_{i,j} = \zeta_i 1_{h(i)=j}$. Notice that $H$ has a single non-zero entry per column, and the entire transformation $HP$ has $c$ non-zero entries per column. Kane and Nelson [14] improve the analysis of this scheme. They show that taking $c = O\left(\frac{1}{\epsilon} \log(1/\delta) \log(1/\delta)\right)$ is sufficient.

We provide alternative, tighter, analysis of this scheme and show that it is sufficient to set

$$c = O\left(\frac{1}{\epsilon} \left(\frac{\log(1/\delta) \log \log(1/\delta)}{\log \log(1/\delta)}\right)^2\right).$$

In both previous works, as well as this work, the starting point is the same: the argument boils down to analyzing the distribution of an order 2 Rademacher chaos $Z = \sum_{1 \leq i < j \leq d} a_{ij} \zeta_i \zeta_j$, where the coefficients $a_{ij}$ are derived from the hash function $h$ and the projected vector $x$. In particular, showing that the transform works for a particular choice of $c$ boils down to proving a tail inequality bounding the probability that $Z$ deviates from 0. We prove such a tail inequality by bounding a judiciously chosen large even moment of $Z$.

Notice that the monomials in the expansion of $E[Z^{2m}]$ are (sums of) products of terms in the sum defining $Z$. As each term involves two indices $i, j$, there is a correspondence between monomials and graphs on $\{1, 2, \ldots, d\}$. The non-zero monomials correspond to graphs where all nodes have even degree, in other words: unions of node-disjoint Eulerian graphs. The previous papers resorted to existing measure concentration inequalities. They implicitly related the moments to the weight of a subset of the monomials.
where the graphs are composed of pairs of parallel edges, and thus used the combinatorial structure only partially. This approach seems to hit a barrier when \( c = o \left( \frac{1}{\epsilon} \log^2 (1/\delta) \right) \).

In order to overcome this barrier, we fully exploit the combinatorial structure of the monomial terms in the expansion of \( E[Z^{2m}] \). In particular, we prove non-trivial bounds on the probability that a random multigraph is the union of a given number of disjoint Eulerian components (the difficulty stems from the fact that this is not a monotone property). These bounds may be of independent interest. Moreover, our analysis of the combinatorial structure of the monomials is decoupled from the use of the specific properties of the coefficients of the chaos that lead to the specific tail inequality that we get. Therefore, our methods are likely to be useful in the analysis of other order 2 Rademacher chaoses.

Kane and Nelson \[14\] also reduce the required amount of randomness, as compared to the original construction of \[3\]. Our analysis also further improves slightly the bound on the randomness needed. We need \( O(\log(1/\delta)) \)-wise independent vectors, whereas Kane and Nelson use \( O(\log(k/\delta)) \)-wise independent vectors.

**Definitions, Assumptions and Main Results**

Let \( 0 < \delta, \epsilon < 1 \) be two parameters. We assume that \( \epsilon \leq \log^{-2} (\delta^{-1}) \). Define \( m = O(\log \delta^{-1}) \) and \( k = O(\epsilon^{-2} m) \). Define \( C = O(\epsilon^{-1} \left( \frac{m}{F(m)} \right)^2 ) \) for some function \( F \) such that \( F(m) = O(\frac{\log m}{\log \log (m)}) \). Let \( H : [d] \to [k] \) be a random function and let \( \zeta \in \{-1,1\}^d \) be a random vector. Both vectors have \( O(m) \)-wise independent entries. Let \( x \in \mathbb{R}^d \) be a fixed vector such that \( ||x||_2 = 1 \) and \( ||x||_\infty \leq C^{-0.5} \). Define

\[
Z_t = \sum_{i \neq j \in [d]} x_i x_j \zeta_i \zeta_j 1_{\{H(i)=H(j)=t\}}, \quad Z = \sum_{t=1}^k Z_t.
\]

We note that for fixed \( H \) each variable \( Z_t \) can be seen as a particular case of Rademacher chaos. Rademacher chaos of order 2 is defined as a random variable of the form \( \sum_{i \neq j \in [d]} a_{i,j} \zeta_i \zeta_j \). Thus, we consider a special case when \( a_{i,j} = x_i x_j \). There are many bounds for Rademacher chaos, such as Bonami inequality \[7\] and others, see e.g., Blei and Janson \[6\], Hanson and Wright \[11\], Latala \[15\]. In particular, they can be applied for each \( Z_t \) for fixed \( H \). However, there are two issues with applying general inequality in our setting. First, we might lose precision, when applied directly to a random sum of (defined by \( H \)) of Rademacher chaoses \( Z \). Second, we can employ the structure of \( a_{i,j} \) to achieve better bounds. Our main technical result is a new tail probability inequality for a random sum of Rademacher chaoses of the special form as above. In particular we prove:

**Theorem 1.1.** There exists an absolute constant \( \alpha \) such that if \( C > \alpha \epsilon^{-1} \left( \frac{m}{F(m)} \right)^2 \) and \( k > \alpha \epsilon^{-2} m \) then:

\[
E[Z] \leq (0.1\epsilon)^{2m}. \quad \text{Further, there exists an absolute constant } \gamma \text{ such that}
\]

\[
P(|Z| \geq \epsilon) \leq \gamma \delta.
\]

Thus, we give an improvement to Theorem 2 from \[8\] and Theorem 10 from \[14\]. It is important to emphasize the difference between our approach and that of \[8, 14\]. Both previous works first bound \( Z_t \) using known tail bounds for a Rademacher chaos and then take a union bound for summing the error of all \( Z_{1 \leq i \leq t} \) in order to upper bound \( Z \). We, in contrast provide a new tail inequality.

Next, we note that theorem \[1\] immediately implies the following, by repeating the arguments from \[8\]:

\[2\]
Theorem 1.2. There exists a universal constant $\gamma$ and a distribution $\mathcal{D}$ over $k \times d$ matrices with real-valued elements such that if $M \sim \mathcal{D}$ then for any fixed $x \in \mathbb{R}^d$ the following is true. First,

$$P((1 - \epsilon)||x||_2 \leq ||Mx||_2 \leq (1 + \epsilon)||x||_2) \geq 1 - \gamma \delta.$$  

Second, $Mx$ can be computed in time

$$O\left(\frac{1}{\epsilon} \left( \frac{\log(1/\delta) \log \log(1/\delta)}{\log \log(1/\delta)} \right)^2 \|x\|_0 \right).$$

Third, $M$ can be constructed using vectors with $O(\log(1/\delta))$-independent entries.

1.1 An Informal Explanation

We take a direct approach to the above problem and try to estimate the moments of $Z$ directly. That is, we write

$$Z = \sum_{1 \leq i < j \leq d} x_i x_j \zeta_i \zeta_j \left( \sum_{t=1}^k \mathbf{1}_{\{H(i)=H(j)=t\}} \right).$$

Further, $Z^{2m}$ can be seen as a sum of all possible monomials which can be constructed from $2m$ elements of the form $x_i x_j \zeta_i \zeta_j \left( \sum_{t=1}^k \mathbf{1}_{\{H(i)=H(j)=t\}} \right)$. Thus, we group the terms according to certain criteria and estimate the expectation of term inside each group differently. It turns out that each monomial with positive expectation corresponds to a multigraph with positive and even degrees. The expectation depends on the number of connected components of such graphs. That is, we reduce the problem of estimating moments of $Z$ to the question of how many multigraphs can be constructed for a given subset of vertices $\{1, 2, \ldots, i\}$ and a given number of connected components $t$. It is not hard to see that $t \leq i/2$ for graphs with even degrees. Also, note that there is a direct upper bound on the number of such sequences that is $i^{2m}$.

Informally, we employ the following intuitive fact. If the multigraph has a small number of connected components, then the total probability of such a graph is very small. On the other hand, if there are many connected components, then the graph should be sparse with $o(i^2)$ edges and thus better bounds are possible. The main technical work is to prove that for any number of components, the combined influence of probability and sparsity in fact gives the required bound.

2 Reduction to Graphs

Let $S$ be a sequence of pairs $S = \{S_1, \ldots, S_{2m}\}$ where $S_i = \{S_{i,1}, S_{i,2}\}$ such that $1 \leq S_{i,1} < S_{i,2} \leq d$. Define $A$ to be a set of all such sequences. Define a random variable

$$R_S = \prod_{i=1}^{2m} \left( x_{S_{i,1}} x_{S_{i,2}} \zeta_{S_{i,1}} \zeta_{S_{i,2}} \left( \sum_{t=1}^k \mathbf{1}_{\{H(S_{i,1})=H(S_{i,2})=t\}} \right) \right).$$

Fact 2.1. $E(Z^{2m}) = 2^{2m} \sum_{S \in A} E(R_S)$.

Proof. We can rewrite:

$$Z = 2 \sum_{1 \leq i < j \leq d} x_i x_j \zeta_i \zeta_j \left( \sum_{t=1}^k \mathbf{1}_{\{H(i)=H(j)=t\}} \right).$$

The fact follows. \qed
Definition 2.2. Let $G$ be an undirected connected multigraph with $G = (V, E)$ and $V \subseteq [d]$. Define $\text{WEIGHT}(G) = 0$ if $G$ has at least one vertex with an odd degree and otherwise define

$$\text{WEIGHT}(G) = \frac{1}{k|V| - 1} \prod_{v \in V} x_v^{\deg(v)}.$$ 

Let $G$ be an undirected multigraph and let $G_1, \ldots, G_t$ be the connected components of $G$. Define $\text{WEIGHT}(G) = \prod_{i=1}^t \text{WEIGHT}(G_i)$.

Definition 2.3. Let $V \subseteq [d]$. Define $\text{SQUARES}(V) = \prod_{v \in V} x_v^2$.

Definition 2.4. Let $S \in A$. Define $G(S)$ to be the following undirected multigraph. Vertices of the graph are the numbers that appear in the sequence $S$. That is, the set of vertices of $G(S)$ is $\{v \in [d] : \exists i \in [2m], j \in \{1, 2\} S_{i,j} = v\}$. The multiset of edges of $G(S)$ consists of all edges of the form $(S_{i,1}, S_{i,2})$.

Definition 2.5. Let $G$ be a multigraph with vertices in $[d]$. Define $\text{Ver}(G)$ to be the set of all vertices of $G$ with positive degree. Define $\text{Edg}(G)$ to be a multiset of all edges of $G$.

Lemma 2.6. $E(R_S) = \text{WEIGHT}(G(S))$.

Proof. Definition 2.4 implies that all vertices of $G(S)$ have positive degree. It follows that $G(S)$ has a vertex $v$ with an odd degree if and only if $x_v$ has an odd degree in $R_S$. In this case we can write $R_S$ as $\zeta L$ where $L$ is independent of $\zeta$ and thus $E(R_S) = 0 = \text{WEIGHT}(G(S))$.

Consider the case when $G(S)$ has only vertices with positive and even degree. First, let us assume that $G(S)$ is connected.

$$E(R_S) = \prod_{v \in V} x_v^{\deg(v)} E\left(\prod_{i=1}^{2m} \left(\sum_{t=1}^k 1_{\{H(S_{i,1})=H(S_{i,2})=t\}}\right)\right).$$

Since $G(S)$ is connected we have

$$E\left(\prod_{i=1}^{2m} \left(\sum_{t=1}^k 1_{\{H(S_{i,1})=H(S_{i,2})=t\}}\right)\right) = E\left(\sum_{t=1}^k \prod_{v \in V} 1_{H(v)=t}\right) = \frac{1}{k|V| - 1}.$$

The case when $G$ has more than one connected component is proven by repeating the above arguments for each connected component and by noting that the random variables that correspond to components are independent.

Definition 2.7. For $Q \subseteq [d]$ and define $W_{Q,t}$ to be set of all sequences $S$ such that $\text{Ver}(G(S)) = Q$, such that $G(S)$ has $t$ connected components and such that all degrees in $G(S)$ are positive and even. By symmetry, for any $Q \neq Q'$ such that $|Q| = |Q'|$ we have $|W_{Q,t}| = |W_{Q',t}|$. 


Lemma 2.8. Let \( S \in W_{[i],t} \). Then

\[
WEIGHT(G(S)) \leq \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}} SQUARVES(Ver(G(S))).
\]

Proof. By Definition

\[
WEIGHT(G(S)) = \frac{1}{k^{i-t}} \prod_{v \in Ver(G(S))} x_v^{\deg(v)}.
\]

Next, note the following. For every \( v \) it is true that: \( \deg(v) \geq 2 \) and \( x_v^2 \leq C \). Also, \( \sum_{v \in V} \deg(v) = 4m \).
Thus, we conclude:

\[
WEIGHT(G(S)) \leq \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}} \prod_{v \in Ver(G(S))} x_v^2 = \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}} SQUARVES(Ver(G(S))).
\]

Fact 2.9. Let \( S \not\in \bigcup_{t=1}^{i/2} \bigcup_{i=1}^{2m} W_{[i],t} \). Then \( E(R_S) = 0 \).

Proof. Consider \( S \not\in \bigcup_{t=1}^{i/2} \bigcup_{i=1}^{2m} W_{[i],t} \). Then \( G(S) \) has at least one node of odd degree. It follows that \( E(R_S) = 0 \).

Further, we show that \( W_{[i],t} = \emptyset \) for \( t > i/2 \). Indeed, consider \( S \in W_{[i],t} \). It follows that at least one of the connected components of \( G(S) \) has exactly one node. This contradicts the definition of sequences \( S \).
Thus, \( W_{[i],t} = \emptyset \) and the fact follows.

Lemma 2.10.

\[
E(Z^{2m}) \leq 2^{2m} \sum_{i=1}^{2m} \frac{1}{i!} \sum_{t=1}^{i/2} |W_{[i],t}| \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}}.
\]

Proof.

\[
E(Z^{2m}) = 2^{2m} \sum_{S \in A} E(R_S) = (\text{By Fact 2.9})
\]

\[
2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} \sum_{Q \in d_{[i],|Q|=i}} \sum_{S \in W_{Q,t}} E(R_S) =
\]

\[
2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} \sum_{Q \in d_{[i],|Q|=i}} \sum_{S \in W_{Q,t}} \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}} SQUARVES(G(S)) \leq
\]

\[
2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} \sum_{Q \in d_{[i],|Q|=i}} \sum_{S \in W_{Q,t}} \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}} SQUARVES(G(S)) \leq
\]

\[
2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} |W_{[i],t}| \frac{1}{k^{i-t}} \frac{1}{C^{2m-i}} \sum_{Q \in d_{[i],|Q|=i}} SQUARVES(Q) \leq
\]

5
\[2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} |W_{[i,t]}| \frac{1}{k^{i-t}} C^{2m-i} \frac{1}{i!} \left( \sum_{j \in [d]} x_j^2 \right)^i =
\]

\[2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} |W_{[i,t]}| \frac{1}{k^{i-t}} C^{2m-i} \frac{1}{i!} \]

\[\Box\]

### 2.1 Proof of Theorem 1.1

**Proof.** Let \( \epsilon < m^{-2} \). Then by Lemma 2.12 and by Fact 2.11 there exists an absolute constant \( \alpha \) such that if \( C > \alpha \epsilon^{-1} \left( \frac{m}{F(m)} \right)^2 \) and \( k > \alpha \epsilon^{-2} m \) then the following is true. For any \( 1 \leq i \leq 2m \) and for any \( 1 \leq t \leq i/2 \):

\[|W_{[i,t]}| \leq (0.01)^{2m} m^{2m} k^{i-t} C^{2m-i}.\]

Thus, by Lemma 2.10 for sufficiently large \( m \):

\[E(Z^{2m}) \leq 2^{2m} \sum_{i=1}^{2m} \sum_{t=1}^{i/2} |W_{[i,t]}| \frac{1}{k^{i-t}} C^{2m-i} \leq \epsilon^{2m} m^2 (0.02)^{2m} \leq (0.1\epsilon)^{2m}.\]

To show the second claim, note that \( P(|Z| \geq \epsilon) \leq P(Z^{2m} \geq \epsilon^{2m}) \). Also, recall that \( m = O(\log(1/\delta)) \).

Since \( Z^{2m} \) is a non-negative random variable, the second claim follows from Markov inequality and the first claim of the theorem. \( \Box \)

### Fact 2.11.

\[k^{i-t} C^{2m-i} \geq \alpha^m \frac{1}{\epsilon^{2m}} m^{4m+i-5t} \left( \frac{1}{F(m)} \right)^{4m-2i}.\]

**Proof.** Recall that \( \epsilon \leq m^{-2} \) and that \( t \leq i/2 \leq m \).

\[k^{i-t} C^{2m-i} \geq \frac{1}{\epsilon^{2i-2t}} (\alpha m)^{i-t} \frac{1}{\epsilon^{2m-i}} \left( \frac{m}{F(m)} \right)^{4m-2i} \]

\[= \alpha^m \frac{1}{\epsilon^{2m-i}} m^{4m+i-5t} \left( \frac{1}{F(m)} \right)^{4m-2i}.\]

\( \Box \)

In the remainder of our paper we prove the following main technical lemma.

### Lemma 2.12.

Let \( \epsilon < m^{-2} \). There exists an absolute constant \( CONST = O(1) \) such that for any \( 1 \leq i \leq 2m \) and for any \( 1 \leq t \leq i/2 \):

\[|W_{[i,t]}| \leq (CONST)^{2m} m^{4m+i-5t} \frac{1}{F(m)}^{4m-2i}.\]

**Proof.** The lemma follows directly from Lemma 3.2, Lemma 3.10 and Lemma 3.12. \( \Box \)
3 Bounding \(W[i,t].\)

**Fact 3.1.** There exists a constant \(v\) such that for \(F(m) \leq \frac{v \log(m)}{\log(\log(m))}\) and for any \(x > 0:\)

\[ F(m) \leq m^{0.01}. \]

**Proof.** Follows from the fact that for small constant \(v:\)

\[ F(m) \log(F(m)) \leq 0.01 \log(m). \]

\[ \square \]

3.1 Small \(t\)

**Lemma 3.2.** Let \(t < 0.39i\) and \(\epsilon < m^{-2}.\) There exists an absolute constant \(\text{CONST} = O(1)\) such that for any \(1 \leq i \leq 2m\) and for any \(1 \leq t \leq i/2:\)

\[ |W[i,t]| \leq \left(\text{CONST}\right)2m \frac{i^{4m+i-5t}}{m^{4m+i-5t}} \left(\frac{1}{F(m)}\right)^{4m-2i}. \]

**Proof.** It follows from the definition of \(W[i,t]\) that \(|W[i,t]| \leq i^{4m}.\) Also note that \(i \leq 2m.\) Also, \(2i - 5t > 0.05i.\) Thus, there exists a constant \(\phi\) such that,

\[ \frac{i^{4m}}{m^{4m+i-5t}} \left(\frac{1}{F(m)}\right)^{4m-2i} \leq \phi \frac{i^{4m-i}}{m^{4m-i}} \frac{F(m)^{4m-2i}}{m^{0.05i}}. \]

First, consider the case when \(i \leq \frac{m}{F(m)}.\) Then the lemma follows immediately. Otherwise, for sufficiently large \(m\) and for some constant \(\psi:\)

\[ \frac{i^{4m}}{m^{4m+i-5t}} \left(\frac{1}{F(m)}\right)^{4m-2i} \leq \psi \frac{F(m)^{4m}}{m^{0.05i}}. \]

The lemma follows from Fact 3.1.

\[ \square \]

3.2 Some Facts

**Fact 3.3.** Let \(t\) be such that \(3t > i\) and let \(S \in W[i,t].\) Then \(G(S)\) has at least \((3t - i) > 0\) components with size exactly 2.

**Proof.** Each component must have at least 2 nodes. Thus, there are at most \((i - 2t)\) components with more than 2 nodes. Thus, there are at least \((3t - i)\) components of size exactly 2.

\[ \square \]

**Definition 3.4.** Define \(\text{SPARSE}_u\) as a set of all sequences \(S\) such that \(\text{Ver}(G(S)) = [i],\) \(G(S)\) has at least \(u\) components of size two and such that all vertices of \(G(S)\) are of even and positive degree.

**Fact 3.5.** Let \(t\) be such that \(3t > i.\) Then \(W[i,t] \subseteq \text{SPARSE}_{3t-i}.

**Proof.** Follows directly from Fact 3.3 and the definitions.

\[ \square \]
**Definition 3.6.** Let $Q$ be a set of size $(3t-i)$ of pairs of distinct numbers from $[i]$. That is

$$Q = \bigcup_{j=1}^{3t-i} \{(q_{2j-1}, q_{2j})\}$$

such that $q_j \in [i]$ and $q_j \neq q_j'$ for any $j \neq j'$. Let $Q$ be a set of all such possible $Q$. For $Q \in Q$, define $\text{CONCRETE}(Q)$ to be a set of all sequences $S$ such that $G(S)$ has connected components with the following sets of vertices: $\{q_1, q_2\}, \ldots, \{q_{2(3t-i)-1}, q_{2(3t-i)}\}$.

**Fact 3.7.**

$$|Q| \leq \binom{i}{2(3t-i)} \frac{(2(3t-i))!}{(3t-i)!}.$$

**Fact 3.8.**

$$\text{SPARSE}_{(3t-i)} \subseteq \left( \bigcup_{Q \in Q} \text{CONCRETE}(Q) \right).$$

### 3.3 Medium $t$

In the remainder of the paper we consider the case when $t > 0.39i$. Denote $z = i - 2t$. In this section we consider the case when $t$ is not very large such that $z^2 > 2(3t-i)$.

**Fact 3.9.** Let $Q$ be an ordered set of size $(3t-i)$ from Definition 3.6. Then there exists an absolute constant $\gamma$ such that

$$|\text{CONCRETE}(Q)| \leq (2m)^2(3t-i)(\gamma z)^{4(m-3t+i)}.$$

**Proof.** Let $A' = \bigcup_{j=1}^{3t-i} \{(q_{2j-1}, q_{2j})\}$. Let $B = [i] \setminus \{q_1, \ldots, q_{2(3t-i)}\}$ and $B' = A' \cup \left( \bigcup_{j \neq j' \in B} (j, j') \right)$. If $S \in \text{CONCRETE}(Q)$ then $S \in B'^{2m}$. Also,

$$|B'| \leq 2(3t-i) + (i - 2(3t-i))^2 = 2(3t-i) + (3z)^2 \leq 10z^2.$$

Also, each pair $(q_{2j-1}, q_{2j})$ must appear at least twice in the sequence $S$. We count the number of such sequences as follows. First, we choose the $2(3t-i)$ locations of the appearances for the pairs $(q_{2j-1}, q_{2j})$. For a fixed set of locations, the number of sequence $S$ that agree on these locations is bounded by $|B'|^{2m-2(3t-i)}$. The total number of different sets of locations is bounded by $(2m)^2(3t-i)$. This in an over-counting, yet it is sufficient for our goals. Thus, we conclude that there exists an absolute constant $\beta$ such that

$$|\text{CONCRETE}(Q)| \leq (2m)^2(3t-i)(\beta z)^{4(m-3t+i)}.$$

\[\Box\]

**Lemma 3.10.** Let $t > 0.39i$ such that $z^2 > 2(3t-i)$. There exists an absolute constant $\text{CONST} = O(1)$ such that for any $1 \leq i \leq 2m$ and for any $1 \leq t \leq i/2$:

$$|W_{i,t}| \leq (\text{CONST})^{2m+i-5t} \left( \frac{1}{F(m)'} \right)^{4m-2i}.$$

\footnote{in fact we only need $3t > (1 + \gamma)i$ for some constant $\gamma$}
Proof. By Fact 3.9, Fact 3.7 and Fact 3.8 there exists an absolute constant \( \beta \) such that

\[
|W_{[i], t}| \leq |\text{PARSE}(3t-i)| \leq \left( \frac{i}{2(3t-i)} \right) \left( \frac{2(3t-i)!}{(3t-i)!} \right) (2m)^{2(3t-i)} (\beta z)^{4(m-3t+i)}.
\]

Further,

\[
\left( \frac{i}{2(3t-i)} \right) \left( \frac{2(3t-i)!}{(3t-i)!} \right) (2m)^{2(3t-i)} z^{4(m-3t+i)} = \frac{i!}{(3t-i)! (3z)!} (2m)^{2(3t-i)} z^{4m-12t+4i} \leq
\]

Note that \( 3t-i > 0.1i \). Thus,

\[
(3t-i)! > \left( \frac{(3t-i)}{e} \right)^{3t-i} \geq \left( \frac{i}{10e} \right)^{3t-i}.
\]

Thus, there exists an absolute constant \( \gamma \) such that

\[
|W_{[i], t}| \leq \gamma m^i i^{3t-i} (2m)^{2(3t-i)} z^{4m-12t+4i-3z} = \gamma m^i i^{3t-i} m^6 t^i m^6 t^i z^{4m-6t+i}.
\]

To prove the lemma, we need to estimate the following quantity:

\[
m^6 t^i z^{4m-6t+i} F(m)^{4m-2i}
\]

We show that there exists a constant \( \phi \) such that:

\[
m^6 t^i z^{4m-6t+i} F(m)^{4m-2i}
\]

Rewrite:

\[
m^6 t^i z^{4m-6t+i} F(m)^{4m-2i}
\]

We consider the following three cases. If \( z \leq \frac{m}{F(m)} \) then\(^2\) there exists a constant \( \psi \):

\[
\frac{z^{4m-6t+i} F(m)^{4m-2i}}{i^{3t-i} m^{4m+2i-9t m^z}} \leq \psi m \left( \frac{F(m) z}{i^{4m+i-6t m^z}} \right)^{4m-6t+i} \leq \psi m.
\]

If \( i \leq m \) and \( \frac{m}{F(m)} < z \leq \frac{m}{F(m)} \) then there exists a constant \( \gamma \):

\[
\frac{z^{4m-6t+i} F(m)^{4m-2i}}{i^{3t-i} m^{4m+2i-9t m^z}} \leq \gamma m \left( \frac{F(m) z}{m} \right)^{4m-9t+i} \leq \gamma m \left( \frac{F(m) z}{m} \right)^{4m-9t+i} \leq \gamma m \left( \frac{F(m)}{m} \right)^{4m-9t+i}
\]

Finally, if \( \max \left( \frac{m}{F(m)}, \frac{i}{F(m)} \right) < z \) then there exists a constant \( \beta \):

\[
\frac{z^{4m-6t+i} F(m)^{4m-2i}}{i^{3t-i} m^{4m+2i-9t m^z}} \leq \beta m \left( \frac{F(m)}{m} \right)^{4m-9t+i}
\]

The lemma follows from Fact 3.1.

\(^2\)We stress that this claim is correct for any \( 1 \leq i \leq 2m \).
3.4 Large $t$

In this section we consider $t$ such that $z^2 < 2(3t - i)$. The proof of the following fact is identical to Fact 3.9 if we note that $z^2 < 2(3t - i) ≤ i$.

**Fact 3.11.** Let $Q$ be an ordered set of size $2(3t - i)$ from Definition 3.6. Then there exists an absolute constant $β$ such that:

$$|\text{CONCRETE}(Q)| ≤ (2m)^{2(3t-i)}(βi)^{2(m-3t+i)}.$$  

**Lemma 3.12.** Let $t$ be such that $z^2 < 2(3t - i)$. There exists an absolute constant $\text{CONST} = O(1)$ such that for any $1 ≤ i ≤ 2m$ and for any $1 ≤ t ≤ i/2$:

$$|W_{[i],t}| ≤ (\text{CONST}2m)!m^{4m+i-5t}\left(\frac{1}{F(m)}\right)^{4m-2i}.$$  

**Proof.** By Fact 3.11 Fact 3.7 and Fact 3.8 there exists an absolute constant $β$ such that:

$$|W_{[i],t}| ≤ |\text{SPARSE}_{(3t-i)}| ≤ \left(\frac{i}{2(3t-i)}\right)\left(\frac{2(3t-i)}{(3t-i)!}\right)(2m)^{2(3t-i)}(βi)^{2(m-3t+i)}.$$  

Further, there exists a constant $γ$:

$$\left(\frac{i}{2(3t-i)}\right)\left(\frac{2(3t-i)}{(3t-i)!}\right)(2m)^{2(3t-i)}i^{2(m-3t+i)} ≤ γ^m(2m)^{2(3t-i)}i^{2m-9t+3i}.$$  

Thus,

$$\frac{|W_{[i],t}|}{i!m^{4m+i-5t}\left(\frac{1}{F(m)}\right)^{4m-2i}} ≤ γ^m\frac{(2m)^{2(3t-i)}i^{2m-9t+3i}F(m)^{4m-2i}}{m^{4m+i-5t}} ≤ γ^m\frac{(i/m)^{2m-9t+3i}F(m)^{4m-2i}}{m^{2m-i}} ≤ ψ^m$$

for a constant $ψ$.  

References

[1] Dimitris Achlioptas. Database-friendly random projections: Johnson-lindenstrauss with binary coins. *J. Comput. Syst. Sci.*, 66(4):671–687, 2003.

[2] Nir Ailon and Bernard Chazelle. Approximate nearest neighbors and the fast johnson-lindenstrauss transform. In *STOC ’06: Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pages 557–563, New York, NY, USA, 2006. ACM.

[3] Nir Ailon and Edo Liberty. Fast dimension reduction using rademacher series on dual bch codes. In *SODA ’08: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 1–9, Philadelphia, PA, USA, 2008. Society for Industrial and Applied Mathematics.

[4] Nir Ailon and Edo Liberty. An almost optimal unrestricted fast johnson-lindenstrauss transform. In *SODA ’11: To appear in proceedings of the twenty second annual ACM-SIAM symposium on Discrete algorithms*, 2011.
[5] Noga Alon. Problems and results in extremal combinatorics–i. *Discrete Mathematics*, 273(1-3):31 – 53, 2003. EuroComb’01.

[6] Ron Blei and Svante Janson. Rademacher chaos: tail estimates versus limit theorems. *Arkiv fr Matematik*, 42:13–29, 2004. 10.1007/BF02385577.

[7] A Bonami. Etude des coefficients de fourier des fonctions de lp(g.). 1970.

[8] Anirban Dasgupta, Ravi Kumar, and Tamás Sarlos. A sparse johnson lindenstrauss transform. In *STOC ’10: Proceedings of the 42nd ACM symposium on Theory of computing*, pages 341–350, New York, NY, USA, 2010. ACM.

[9] Sanjoy Dasgupta and Anupam Gupta. An elementary proof of a theorem of johnson and lindenstrauss. *Random Struct. Algorithms*, 22(1):60–65, 2003.

[10] P. Frankl and H. Maehara. The johnson-lindenstrauss lemma and the sphericity of some graphs. *J. Comb. Theory Ser. A*, 44(3):355–362, 1987.

[11] D. L. Hanson and F. T. Wright. A bound on tail probabilities for quadratic forms in independent random variables. *The Annals of Mathematical Statistics*, 42(3):pp. 1079–1083, 1971.

[12] Piotr Indyk and Rajeev Motwani. Approximate nearest neighbors: towards removing the curse of dimensionality. In *STOC ’98: Proceedings of the thirtieth annual ACM symposium on Theory of computing*, pages 604–613, New York, NY, USA, 1998. ACM.

[13] William B. Johnson and Joram Lindenstrauss. Extensions of Lipschitz mappings into a Hilbert space. *Contemp. Math.*, 26:189–206, 1984.

[14] Daniel M. Kane and Jelani Nelson. A derandomized sparse johnson-lindenstrauss transform. *CoRR*, abs/1006.3585, 2010.

[15] Rafał Latała. Tail and moment estimates for some types of chaos. *Studia Math.*, 135(1):39–53, 1999.

[16] Edo Liberty, Nir Ailon, and Amit Singer. Dense fast random projections and lean walsh transforms. In *APPROX ’08 / RANDOM ’08: Proceedings of the 11th international workshop, APPROX 2008, and 12th international workshop, RANDOM 2008 on Approximation, Randomization and Combinatorial Optimization*, pages 512–522, Berlin, Heidelberg, 2008. Springer-Verlag.

[17] Jiří Matoušek. On variants of the johnson–lindenstrauss lemma. *Random Struct. Algorithms*, 33(2):142–156, 2008.