Flocking particles in a non-Newtonian shear thickening fluid

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Abstract
We prove the existence of strong solutions to the Cucker–Smale flocking model coupled with an incompressible viscous non-Newtonian fluid with the stress tensor of a power–law structure for $p \geq \frac{11}{5}$. The fluid part of the system admits strong solutions while the solutions to the CS part are weak. The coupling is performed through a drag force on a periodic spatial domain $\mathbb{T}^3$. Additionally, we construct a Lyapunov functional determining the large time behavior of solutions to the system.

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1. Introduction
Mathematical models of self-propelled agents with non-local interactions provide a way to describe a wide range of phenomena in natural sciences—physics and biology—but also in economics or even in robotics. The literature concentrates on the analysis of time asymptotics [26, 31], pattern formation [25, 39, 40] and the study of models with forces that simulate various natural factors [9, 19] (a deterministic case) or [15] (a stochastic one). The other variations of the model include forcing particles to avoid collisions [13] or to aggregate under the leadership of certain individuals [14].
We concentrate on the Cucker–Smale (CS) flocking model describing a collective self-driven motion of self-propelled particles with a tendency to flock. The system was introduced in 2007 by Cucker and Smale in [16], and it initiated intensive study of the subject from a mathematical point of view. The vast literature on the CS model refers mostly to qualitative analysis [8, 24, 40]. The simple form of the system allows us, unexpectedly, to find answers to questions concerning the structure of solutions, such as aggregation with leaders [14, 38], collision avoidance [1, 7], and cluster formation [27]. The theory also contains an examination of systems with various additional forces [19, 28] and with special cases of the communication weight: singular [1, 29, 35, 36], normalized [33] and incorporating the effect of time-delay [20]. At the same time, research on the passage from the particle CS system to the kinetic equation is performed [29, 30, 34] (see also [6, 17, 18] for general theory on the derivation of kinetic models from non-local particle systems).

The present paper considers one of the other directions of research. Our subject is the motion of agents described by the kinetic CS equation

\[
\partial_t f + v \cdot \nabla_x f + \text{div}_x F(f)f = 0, \tag{1.1}
\]

submerged in a non-Newtonian viscous incompressible fluid. Alongside the analysis of the kinetic models themselves, research on the coupling models of kinetic theory with models of hydrodynamics was conducted (see [5, 22, 23]) and they are part of the large theory called complex flows. Our motivation comes from the results for the complex flow models; here we shall mention [12, 21] concerning the Fokker–Planck equation coupled with the Navier–Stokes system. The literature on the CS model coupled with models of hydrodynamics is quite rich. It includes the coupled CS–Navier–Stokes system [2], also in the compressible case [10], and a venture towards well-posedness with small data [3]. More recent development by Choi and Sun [11] where the Navier–Stokes system is coupled with the BGK model for Boltzmann equation, shows that a similar approach can be used with a collision operator instead of the nonlocal interaction \( F \). We aim at proving global in time solvability for arbitrary large data with solutions with a regular fluid part. Note that for the classical Navier–Stokes equations, we are still not able to consider general smooth solutions, hence the application of the non-Newtonian concept of the description of flow allows us to obtain stronger results than for the Newtonian case, such as in [2], and unlike [3], it does not require the smallness of the initial data.

Our problem is motivated by the question concerning the mathematical well-posedness of the coupling of equations of fluid motion with a system describing a flocking phenomenon. Our goal is achieved for the non-Newtonian shear thickening fluid system and the classical Cucker–Smale equations. Looking at a direct interpretation of such a coupled model, one can view it as equations determining the motion of a school of fish in shallow water treated as a mixture of water and sand. Such a fluid is an example of a thickening one [4]. Of course we shall examine our problem from a broader perspective; we want to determine a class of physical models for which a coupling is mathematically well posed, at least at the level of the issue of existence for the large data case. This is the goal of the paper.

Our goal is to consider particles embedded in an incompressible viscous non-Newtonian shear thickening fluid, i.e. we aim to couple (1.1) with the system

\[
\begin{align*}
\partial_t u + (u \cdot \nabla_x)u + \nabla_x \pi - \text{div}_x(\tau) &= G_{\text{ext}}, \\
\text{div}_x u &= 0,
\end{align*}
\tag{1.2}
\]

which describes the motion of such a fluid. The function

\[ u = u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_d(t, x)) \]
represents the velocity of the fluid at position \( x \) and time \( t \). Equation (1.2)_1 expresses the conservation of mass (as well as the incompressibility constraint), while (1.2)_2 expresses the conservation of momentum. The term \( \tau \) in (1.2)_1 denotes a symmetric stress tensor that depends on \( Du \)—the symmetric part of the gradient of \( u \), i.e. \( \tau = \sigma(Du) \), where \( Du = \frac{1}{2} [\nabla u + (\nabla u)^T] \). Function \( G_{ext} \) represents an external force.

To couple (1.1) with (1.2), we introduce the following drag force

\[ F_d(t, x, v) := u(t, x) - v, \]

that influences the motion of particles and fluid. Explicitly, the coupled system reads as follows:

\[
\begin{align*}
\partial_t f + v \cdot \nabla f + \text{div}_v [(F_{CS}(f) + F_d) f] &= 0, \\
\partial_t u + (u \cdot \nabla) u + \nabla \cdot \pi - \text{div}_x (\tau) &= -\int_{\mathbb{R}^d} F_d f \, dv, \\
\text{div}_x u &= 0.
\end{align*}
\]

The system is considered over the phase-space \( \mathbb{T}^d_T \times \mathbb{R}^d \) with a set of initial data. Our main result is contained in theorem 2.1. It says that for any given initial velocity and distribution of particles assumed to be suitably regular, there exists a global in time regular solution, provided the stress tensor \( \tau(Du) \) has polynomial growth with the exponent \( p - 1 \) for \( p \geq \frac{4}{d} \)—the same as for the pure non-Newtonian fluid [32]. In addition, we construct a Lyapunov functional which shows that the energy of the system decays in time, in the special case for \( p > 3 \), in which we are able to conclude that the energy goes to zero as time goes to infinity. This type of study of large-time behavior can be found in [3], and in a particularly refined version in [10].

Let us briefly discuss the difference between the coupling of the CS model with Newtonian and non-Newtonian fluids. In [2, 5], the authors obtained the existence of weak solutions for their coupled systems and, on top of that, in [2], the authors proved asymptotic flocking (adding later, in [3], a modification of the large-time behavior part of the result in a small initial data scenario). In the case of coupling with a non-Newtonian fluid, existence, regularity and possibly uniqueness depend on the value of the exponent \( p \) and regularity of the external function \( G_{ext} \).

For the non-Newtonian system (1.2), the existence of weak solutions is known for \( p > \frac{2d}{d+2} \) and \( G_{ext} \in L^p(0, T; (W^{1,p})' (\mathbb{T}^d)) \). On the other hand, if \( p > \frac{4d}{d+2} \) and \( G_{ext} \in L^2(0, T; L^2(\mathbb{T}^d)) \), we have not only the existence of strong solutions but also their uniqueness [37]. However, for the coupled system, uniqueness is a more delicate problem since \( F_d \) in (1.3) forces the particles to move along trajectories influenced by \( u \). It results in the need to control the \( L^\infty \) norm of \( \nabla_x u \), which in the non-Newtonian case is very difficult, even with large \( p \).

The paper is organized as follows. First, we introduce the system (1.3) and formulate the main results. In section 3, the kernel of the paper, we prove theorem 2.1. In section 4, we deal with the large-time behavior of solutions. Finally, in the appendix, a number of auxiliary results is presented/proven.

## 2. Preliminaries

We introduce the notation. By \( W^{k,p}(\Omega) \) we denote the Sobolev space of functions with up to the \( k \)th weak derivative belonging to the Lebesgue space \( L^p(\Omega) \). Moreover, \( D^k(\Omega) \) denotes the space of distributions on \( \Omega \) and \( C^k(\Omega) \) —the space of the functions with up to the \( k \)th derivative belonging to the space of continuous functions, which itself is denoted as \( C(\Omega) \). The norm \( || \cdot ||_p \) denotes the \( L^p \)-norm, either over \( \mathbb{T}^3 \) or over \( \mathbb{R}^3 \times \mathbb{T}^3 \), depending on the function of the norm we have in mind, and, in case it will be necessary to distinguish between them, we will use the full notation of the norm. The same holds for the case when the time variable is considered. We also use
to emphasize that the estimate \( A \leq B \) follows by Hölder’s inequality with exponent \( q \). We use a similar notation for Young’s inequality replacing \( H \) with \( Y \). An arbitrary generic constant is denoted by \( C \); its actual value may change depending on its appearances, even in the same line.

Let us specify the structure of the main system (1.3). We start with an explanation for the equations on motion of non-Newtonian fluid. The sought elements are the velocity \( u \) and pressure \( p \) defined over the \( d \)-dimensional periodic box and time interval \([0, T]\). For the stress tensor \( \tau : \mathbb{R}^d_{\text{sym}} \rightarrow \mathbb{R}^d_{\text{sym}} \) there exist \( p \in (1, \infty) \) and positive constants \( c_1 - c_5 \), such that for all \( \xi, \eta \in \mathbb{R}^d_{\text{sym}} \)

\[
\tau_0(\xi)\xi \geq c_1(\xi^p + |\xi|^2), \quad |\tau_0(\xi)| \leq c_2(1 + |\xi|)^{p-1}, \quad (2.1)
\]

\[
(\tau_0(\xi) - \tau_0(\eta))(\xi - \eta) \geq c_3(|\xi - \eta|^2 + |\xi - \eta|^p), \quad (2.2)
\]

\[
\frac{\partial \tau_0(\eta)}{\partial \eta} \xi \xi \eta \geq c_4(1 + |\eta|)^{p-2}|\xi|^2, \quad \frac{|\partial \tau_0(\eta)|}{\partial \eta} \leq c_5(1 + |\eta|)^{p-2}. \quad (2.3)
\]

As a classical example, we point out \( \tau(\xi) = C(1 + |\xi|)^{p-2}\xi \), keeping in mind that \( \xi \) is meant as the symmetric part of the velocity gradient, i.e. \( \xi = Du = \frac{1}{2}(\nabla u + (\nabla u)^T) \).

Regarding the CS part of the system, we look for the distribution function \( f \) defined over the phase-space \( T^d 	imes \mathbb{R}^d \) for \( t \in [0, T] \). The function is required to be non-negative. The equation on \( f \) is coupled through the force term \( F(f) = F_{\text{CS}}(f) + F_d(f) \), where

\[
F_d(t, x, v) := u(t, x) - v, \quad (2.4)
\]

and

\[
F_{\text{CS}}(f)(t, x, v) = \int_{T^d \times \mathbb{R}^d} \psi(|x-y|) f(t, y, w) dy dw, \quad (2.5)
\]

where \( \psi(\cdot) \)—the communication weight is non-negative, non-increasing and smooth, with \( \|\psi\|_{C^1} \leq c_6 \). It follows \( F_{\text{CS}}(f)(t, x, v) = a(t, x) - b(t, x)v \), with

\[
a(t, x) := \int_{T^d \times \mathbb{R}^d} \psi(|x-y|) w f(t, y, w) dy dw, \quad (2.6)
\]

\[
b(t, x) := \int_{T^d \times \mathbb{R}^d} \psi(|x-y|) f(t, y, w) dy dw. \quad (2.7)
\]

System (1.3) is supplemented by the initial data \( u_0 \) and \( f_0 \) for the velocity field and distribution function, respectively.

Throughout the paper, we assume, without loss of generality, that the total mass of \( f_0 \), i.e. \( \int_{T^d \times \mathbb{R}^d} f_0 dx dv = 1 \) which, due to the conservation of mass, means that the total mass of the particles is always equal to 1 and thus may disappear in the computations.

For non-negative and integrable functions \( f \), we denote:

\[
M_\alpha f(t) := \int_{T^d \times \mathbb{R}^d} |v|^{\alpha} f(t, x, v) dx dv, \quad m_\alpha f(t, x) := \int_{\mathbb{R}^d} |v|^{\alpha} f(t, x, v) dv,
\]

with an obvious remark that \( M_0 f = \|f\|_{L^1} = 1 \) and that for \( 1 \leq q \leq \infty \),

\[
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\]
\[ m_\alpha (t, x) \leq C(R) \| f(t, x, \cdot) \|_q, \]  
provided that \( \text{supp} f(t, x, \cdot) \subset B(R) \), where \( B(R) \) is a ball centered at 0 with radius \( R \). Note that

\[ \| a \|_\infty \leq c_6 M_{\alpha f}, \quad \| b \|_\infty \leq c_6 M_{\alpha f}, \]  
and hence

\[ \| F_{\text{CS}} (f)(t, x, \nu) \| \leq \| a \|_\infty + |\nu| \| b \|_\infty \leq c_6 (M_{\alpha f} + |\nu| M_{\alpha f}), \]  
and \( \text{div}_x F_{\text{CS}} (f)(t, x, \nu) = -db(t, x) \).

### 2.1 Weak formulation

First, let us fix the physical space dimension \( d = 3 \). We introduce the basic function spaces:

\[ L^2_{\text{div}} (\mathbb{T}^3) := \{ \omega \in L^2 (\mathbb{T}^3) : \text{div} \omega = 0 \}, \]
\[ \dot{W}^1_{\text{div}} (\mathbb{T}^3) := \{ \omega \in \mathcal{D}'(\mathbb{T}^3) : \nabla \omega \in L^3 (\mathbb{T}^3), \text{div} \omega = 0 \}, \]
\[ W^1_{1,2} (\mathbb{T}^3) := \{ \omega \in W^{1,2} (\mathbb{T}^3) : \text{div} \omega = 0 \}, \]
\[ \mathcal{H} := L^\infty (0, T; \dot{W}^1_{\text{div}} (\mathbb{T}^3)) \cap C([0, T]; L^2_{\text{div}} (\mathbb{T}^3)) \cap L^2 (0, T; W^{2,2} (\mathbb{T}^3)) \cap L^\infty (0, T; W^{1,3p} (\mathbb{T}^3)), \]
\[ \mathcal{X} := L^\infty ([0, \tau) \times \mathbb{T}^3 \times \mathbb{R}^3) \cap L^\infty (0, T; L^1 (\mathbb{T}^3 \times \mathbb{R}^3)). \]

The spaces are endowed with the standard norms stemming from the definitions.

Next, we define weak solutions to (1.3). Note that in order to have a finite convective term in (iv), we need at least a certain lower bound on \( p \).

**Definition 2.1.** Let \( p \geq \frac{12}{7} \) and \( T > 0 \). The couple \((f, u)\) is a weak solution of (1.3) on the time interval \([0, T]\) if and only if the following conditions are satisfied:

(i) \( f \geq 0 \), \( f \in \mathcal{X} \) and \( M_{\alpha f} \in L^\infty ([0, T]) \); the function \( v \mapsto f(t, x, \cdot) \) is compactly supported for a.a. \( t \in [0, T] \) and \( x \in \mathbb{T}^3 \).

(ii) \( u \in \mathcal{H} \) and \( \partial_t u \in L^2 (0, T; L^2 (\mathbb{T}^3)) \).

(iii) For all \( \phi \in C^\infty_c ([0, T) \times \mathbb{T}^3 \times \mathbb{R}^3) \) such that \( \phi |_{t=T} = 0 \), the following identity holds (the lower index \( b \) means that the function is bounded on \( \mathbb{T}^3 \times \mathbb{R}^3)\)

\[ \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f[\partial_t \phi + v \cdot \nabla \phi + F(f) \cdot \nabla \phi] \, dx \, dv \, dt = -\int_{T^3} f \phi (0, \cdot, \cdot) \, dx \, dv. \]

(iv) For all \( \varphi \in W^{1,2} (\mathbb{T}^3) \cap \dot{W}^{1,p}_{\text{div}} (\mathbb{T}^3) \)

\[ \int_{\mathbb{T}^3} \left[ \frac{\partial u}{\partial t} \varphi + (u \cdot \nabla u) u \cdot \varphi + \tau (Du) : D(\varphi) \right] dx = -\int_{T^3} (u - v) \cdot \varphi f \, dx \, dv \]

a.e. is satisfied in \([0, T]\) and \( \lim_{t \to 0^+} u(t, \cdot) = u_0 \) in \( L^2 (\Omega) \).

**Remark 2.1.** In definition 2.1, the regularity of \( f \) (in particular, the boundedness of \( M_{\alpha f} \)) enables us to test (iv) with \( \varphi = |\nu|^\alpha \) for \( \alpha \in [0, 2] \). This observation will be useful in the large-time behavior part of the paper.
2.2. Main result

We present the main results of the paper.

**Theorem 2.1.** Let $p \geq \frac{11}{5}$ and $T > 0$. Suppose that the initial data $(f_0, u_0)$ satisfy

(i) $0 \leq f_0 \in (L^1 \cap L^\infty)(T^3 \times \mathbb{R}^3)$, $\text{supp} f_0(x, \cdot) \subset B(R)$ for some $R > 0$ and a.a. $x \in T^3$, where $B(R)$ is a ball centred at 0 with radius $R$,

(ii) $u_0 \in W^{1,2}_{\text{div}}(T^3)$.

Then there exists a solution of (1.3) in the sense of definition 2.1.

**Remark 2.2 (Energy inequality and conservation of momentum).** Solutions to (1.3) satisfy the following energy estimate:

$$
(M_2 f + \|u\|_2^2)(t) + c_1 \kappa \int_0^t \|\nabla_x u\|_p^p ds + \int_0^t \int_{T^3 \times \mathbb{R}^3} |u - v|^2 f dx dv ds \leq (M_2 f + \|u\|_2^2)(0).
$$

(2.11)

To see it on the formal level, one needs to add two instances of (1.3) tested with $u$ to (1.3) tested with $|v|^2$ and use (2.1). Estimate (2.11) is a crucial part of our considerations and is rigorously proved in sections 3.3 and 3.4. Moreover, (1.3) conserves the momentum:

$$
\frac{d}{dt} \left( \int_{T^3} u dx + M_1 f \right) = 0.
$$

(2.12)

Indeed, integrating (1.3) over $T^3$ by integration by parts, and thanks to (1.3)$_3$, we have

$$
\frac{d}{dt} \int_{T^3} u dx = - \int_{T^3 \times \mathbb{R}^3} (u - v) f dx dv.
$$

(2.13)

On the other hand, testing (1.3)$_1$ with $v$ reveals that

$$
\frac{d}{dt} M_1 f = \int_{T^3 \times \mathbb{R}^3} (u - v) f dx dv.
$$

Here we use the fact that

$$
\int_{T^3 \times \mathbb{R}^3} Fcs(f) f dx dv = 0,
$$

(2.14)

which is easy to see by Fubini’s theorem thanks to the anti-symmetry of $\psi(|x - y|)(w - v)$ with respect to the change of variables $(x, v)$ with $(y, w)$. Adding (2.13) to (2.14) leads to (2.12).

**Remark 2.3.** Assumption (i) in the above theorem immediately implies that $M_2 f_0 \leq C$ for some positive constant $C$ and all $\alpha \geq 0$ (from the point of view of definition 2.1 we need at least $M_2 f_0 \leq C$). In fact, we could replace the boundedness of the support of $f_0$ by the assumption that $M_2 f_0 \leq C$. Then, instead of working with the local second a priori estimate in section 3.3, it is possible to put the weight of the proof onto estimating $\frac{d}{dt} M_1 f$ for $\alpha \in [2, 5]$ to obtain local existence. Then, global existence follows from the first a priori estimate in a similar way to that in section 3.4. This approach is viable but seems more involved. Note also that the assumption on the support of $f_0$ is not an assumption on the smallness, since a function
with finite $M_0 f_0$ can be approximated by a sequence of functions with bounded support.

The second result concerns the time-asymptotic behavior of solutions to (1.3). We express the asymptotics in the language introduced in [3], where the authors introduced the functional $E$ that measures the deviation of the velocity of the fluid and the velocity of the particles from their average velocities. The functional is defined as follows

$$
E(t) = 2E_p(t) + 2E_f(t) + E_d(t),
$$

$$
E_p(t) = \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v - v_c(t)|^2 f dv, \quad E_f(t) = \int_{\mathbb{T}^3} |u - u_c(t)|^2 dx, \quad E_d(t) = |u_c(t) - v_c(t)|^2,
$$

where

$$
\begin{align*}
u_c(t) &= \int_{\mathbb{T}^3} u dx, \quad u_c(t) = \int_{\mathbb{T}^3 \times \mathbb{R}^3} v f dx dv.
\end{align*}
$$

**Theorem 2.2.** Suppose that $T, f_0, and u_0$ satisfy

$$
T \in (0, \infty), \quad E(0) < \infty.
$$

Then the solution to (1.3) in the sense of definition 2.1 satisfies the following exponential estimate:

$$
E(t) \leq E(0) e^{-\gamma t}, \quad t \in [0, T),
$$

where $\gamma := \min\{2 \psi(\sqrt{2}) + \frac{2\eta}{1 + \eta}, c_1 \kappa \varpi, \frac{4\eta}{1 + \eta}\}$ and $\eta$ (a positive constant depending on $T$) is equal to $\eta := \frac{\kappa_0}{\sup_{v \in \mathbb{R}^3} |m v| f}$. Here $\kappa$ is the constant from Korn’s inequality and $\varpi$ is the constant from Poincare’s inequality for the torus $\mathbb{T}^3$.

Moreover, if $p > 3$, then (2.16) holds for $E(t) \to 0$ as $t \to \infty$.

To better understand the meaning of theorem 2.2, we shall look at the $E(t)$ as a Lyapunov functional. Decay (2.16) shows that $E(t)$ decreases in time, although it seems that the result is local since the supremum norm of $m f$ may increase in time (but for all $T$ it is well defined). As $p$ is greater than the dimension thanks to Sobolev imbeddings, we are able to show that, indeed, the energy described by $E(t)$ vanishes to zero as time goes to infinity.

### 3. Existence of solutions

Our first goal is to prove theorem 2.1. The idea of proving existence is based on the analysis of an approximative system and the suitable application of the Schauder fixed-point theorem.

#### 3.1. Regularized system

Note first there is a need to control the support of $f$ in $\nu$ in the external force $G_{\text{ext}}(t, x) = \int_{\mathbb{R}^3} (u(t, x) - v) f(t, x, v) dv$. We introduce a cut-off function $\gamma_0 : \mathbb{R}^3 \to \mathbb{R}$, such that the support in $v$ of $f$ is contained in a ball of the radius $\frac{1}{2}$. Then we define

$$
G_\epsilon(t, x) = \int_{\mathbb{R}^3} (\theta_\epsilon * u(t, x) - v) \gamma_0(v) f(t, x, v) dv,
$$

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where $\gamma_\epsilon \in C^\infty(\mathbb{R}^3)$, supp $\gamma_\epsilon \subset B(1/\epsilon)$, $0 \leq \gamma_\epsilon \leq 1$, $\gamma_\epsilon = 1$ on $B(1/2\epsilon)$, $\gamma_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0^+$, and $\theta_\epsilon$ is the standard mollifier, i.e. $\theta_\epsilon(x) := \frac{1}{\epsilon^3} \theta \left( \frac{x}{\epsilon} \right)$, for some $0 \leq \theta \in C^\infty_c(\mathbb{R}^3)$ with $\int_\mathbb{R} \theta dx = 1$. We also further regularize the drag force in the CS equation. For $\epsilon > 0$, we denote the regularized force $F_\epsilon(f_\epsilon)$, where

$$F_\epsilon(f_\epsilon; u) := F_{\text{CS}}(f) + (\theta_\epsilon * u - v)\gamma_\epsilon.$$  

We now write down the regularized system. For $\epsilon > 0$, we consider

$$\begin{align*}
\partial_t f_\epsilon + v \cdot \nabla f_\epsilon + \text{div}_x[F_\epsilon(f_\epsilon; u_\epsilon)f_\epsilon] &= 0, \\
\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)x u_\epsilon + \nabla_\epsilon \pi_\epsilon - \text{div}_x(\tau(Du_\epsilon)) &= - \int_{\mathbb{R}^3} f_\epsilon(\theta_\epsilon \ast (u_\epsilon - v))\gamma_\epsilon dv, \\
\text{div}_x u_\epsilon &= 0,
\end{align*}$$  

(3.1)

with a smooth, compactly supported (in the variable $\nu$) initial data $f_0(x)$, where $0 \leq f_0 \rightarrow f_0$ strongly in $L^p(\mathbb{T}^3 \times \mathbb{R}^3)$ for all $p \geq 1$ and weakly * in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, $u_{0\epsilon} = u_0$.

To solve the regularized problem (3.1), we apply the following Schauder-type scheme. Given $\epsilon > 0$ and for suitably chosen $T$ (defined in proposition 3.3), we define a set

$$V = \{u \in L^\infty(0, T; L^3(\mathbb{T}^3)) : \text{div}_x u = 0 \text{ and } \|u\|_{L\infty([0, T]; L^3)} \leq C\},$$  

(3.2)

where $C$ is a chosen constant greater than $\|u_0\|_2$.

We take a function $\bar{u} \in V$. We define

$$F_d = (\theta_\epsilon \ast \bar{u} - v)\gamma_\epsilon,$$

which is at this point a given function. Next we solve the Vlasov-type equation:

$$\begin{align*}
\partial_t f + v \cdot \nabla f + \text{div}_x[(F_{\text{CS}}(f) + F_d)f] &= 0, \\
\text{with initial datum } f(0, x, v) &= f_0(x, v).
\end{align*}$$  

(3.3)

Next, we define $u$ as the solution of the system

$$\begin{align*}
\partial_t u + (u \cdot \nabla)x u + \nabla_\epsilon \pi - \text{div}_x(\tau(Du)) &= - \int_{\mathbb{R}^3} f_\epsilon(\theta_\epsilon \ast \bar{u} - v)\gamma_\epsilon dv, \\
\text{div}_x u &= 0,
\end{align*}$$  

(3.4)

with the initial datum $u(0, x) = u_0(x)$, noting that in this system, the right-hand side depends on $f$ and $\bar{u}$, which are at this point given functions. Thus, in fact, we solve (1.2) with a given external force.

The existence of $f$ and $u$ is guaranteed by the following propositions belonging to classical theory.

**Proposition 3.1.** Let $T > 0$. There exists a solution in the sense of definition 2.1 to the problem

$$\begin{align*}
\partial_t f + v \cdot \nabla f + \text{div}_x[(F_{\text{CS}}(f) + (\theta_\epsilon \ast u - v))\gamma_\epsilon(v)f] &= 0, \\
\text{as long as } 0 \leq f_0 \in C^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \text{ is compactly supported in } v \text{ and } u \in L^\infty(0, T; L^2_{\text{div}}(\mathbb{T}^3)).
\end{align*}$$  

This solution $f$ belongs to the space $C^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. Moreover,

$$\|f\|_{L^\infty([0, T]; L^3(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C, \quad \|f\|_{C^1} \leq C(\epsilon).$$  

(3.6)

where $C$ is a positive constant depending on $\|f_0\|_{L^3(\mathbb{T}^3 \times \mathbb{R}^3)}$ and $\|f_0\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$, while $C(\epsilon)$ also depends on $\epsilon$ and $\|u\|_{L^\infty([0, T]; L^3)}$ (both constants also depend on $T$). Furthermore, $f \geq 0$ in $[0, T] \times \mathbb{T}^3 \times \mathbb{R}^3$, provided that $f_0 \geq 0$ in $\mathbb{T}^3 \times \mathbb{R}^3$.
Proof. This proposition, along with its proof, can be found in [2, appendix]. It is based on the fact that both \( F_{CS}(f) \) and \( F_{d} \) in (3.5) are smooth. Local existence in proposition 3.1 is shown by a standard method of characteristics combined with a fixed point argument. Then, to conclude the global existence, the \( a \ priori \) \( C^1 \) estimate for \( f \) is derived. It can be done because the nonlinearity in (3.5) that comes from the multiplication by \( F_{CS}(f) + F_{d} \) is smooth (here, the regularity of the communication weight \( \psi \) and the mollifier \( \theta_{\epsilon} \) plays a crucial role).

Proposition 3.2. Let \( p \geq \frac{11}{5} \) and \( T > 0 \). There exists a unique solution in the sense of definition 2.1 to the problem

\[
\partial_t u + (u \cdot \nabla_t) u + \nabla_t \pi - \text{div}_x(\tau(Du)) = G,
\]

provided \( u_0 \in W_{0, \text{div}}^{1,2}(\mathbb{T}^3) \) and \( G \in L^2(0, T; L^2(\mathbb{T}^3)) \). Moreover,

\[
\|u\|_{H^t} \leq C, \quad \|\partial_t u\|_{L^1(0,T;L^2(\mathbb{T}^3))} \leq C,
\]

where \( C \) is a positive constant depending on \( \|u_0\|_{W^{1,2}(\mathbb{T}^3)}, \|G\|_{L^2(0,T;L^2(\mathbb{T}^3))} \), \( p \), and \( T \).

Proof. The proof can be found in [32, theorem 4.5]. The proof is based on the structure of \( \tau(\cdot) \). We may consider different types of approximations for which it is not difficult to construct solutions. To obtain \( a \ priori \) estimates allowing us to move from the approximate problem to the original one, we first test the approximate problem by the velocity. The pressure and the stress tensor (property (2.1)) yield the estimates of the velocity in \( L^\infty(0,T;L^2(\mathbb{T}^3)) \) and in \( L^p(0,T;W^{1,p}(\mathbb{T}^3)) \). The next step consists of testing by \((1 + \|\nabla u(t)\|_{L^3(\mathbb{T}^3)})\Delta u \) for suitable \( \lambda \in [0,1] \). The time derivative and the structure of the stress tensor (more precisely, property (2.3)) provide now estimates in \( L^\infty(0,T;W^{1,2}(\mathbb{T}^3)) \), \( L^p(0,T;W^{1,3p}(\mathbb{T}^3)) \) and \( L^2(0,T;W^{2,2}(\mathbb{T}^3)) \). However, the convective term does not disappear now, and we need to control a term of the form \( \sim |\nabla u|^3 \) on the right-hand side using estimates from the first step together with the form of the left-hand side. It is possible to estimate the cubic term for \( p \geq \frac{11}{5} \) and obtain the following bound

\[
\sup_{t \in [0,T]} \|u\|_{W^{1,2}(\mathbb{T}^3)} + \|\nabla u\|_{L^2(0,T;L^6(\mathbb{T}^3))} + \|\nabla u\|_{L^p(0,T;L^{3p}(\mathbb{T}^3))} \leq C(\|G\|_{L_{t}^{2}(\mathbb{T}^3)} + \|u_0\|_{W^{1,2}(\mathbb{T}^3)}), \tag{3.7}
\]

where the constant \( C \) also depends on the estimates from the first step, i.e. on the norms of the velocity in \( L^\infty(0,T;L^2(\mathbb{T}^3)) \) and in \( L^p(0,T;W^{1,p}(\mathbb{T}^3)) \). Moreover, the velocity \( u \) can be used as a test function in the weak formulation which allows us to prove the uniqueness of the solution. Finally, using \( \partial_t u \) as a test function, we deduce the estimates of \( u \) in \( L^\infty(0,T;\dot{W}^{1,p}(\mathbb{T}^3)) \) and \( \partial_t u \) in \( L^2((0,T) \times \mathbb{T}^3) \).

In a sense, we repeat the idea of the proof for the two dimensional Navier–Stokes system, but with better integrability given by the features of \( \tau(\cdot) \) for \( p \geq 11/5 \).

The coupling is realized by the force \( G \). Let \( u \in L^\infty(0,T;L^2_{\text{div}}(\mathbb{T}^3)) \) and \( f \in C^1([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3) \), then
\[
\int_0^T \|G\|^2_{L^2(T)} = \int_0^T \int_{\mathbb{R}^3} \left( \theta_\epsilon \ast \bar{u} - v \right) \gamma_j f \, dv \, dx \, dt \\
= \int_0^T \int_{\mathbb{R}^3} \left( \theta_\epsilon \ast \bar{u} - v \right) \gamma_j f \, dv \, dx \, dt \\
\leq C(T, \epsilon) \int_0^T \int_{\mathbb{R}^3} \left( \|\theta_\epsilon \ast u - v\|^2 f \right) \, dv \, dx \, dt \\
\leq C(T, \epsilon) \|f\|_{L^2(0,T;L^2(\mathbb{T}))}^2 + 1. \tag{3.8}
\]

Therefore \( G \) belongs to \( L^2(0, T; L^2(\mathbb{T})) \) with its norm depending on \( T, \epsilon, \|f\|_{L^\infty(0,T;L^2(\mathbb{T}))} \) and \( u \|_{L^\infty(0,T;L^2(\mathbb{T}))} \). Thus, by proposition 3.2, there exists a unique \( u \)—a solution to (3.4) in the sense of definition 2.1. The existence of a unique \( f \)—a solution to (3.3) belonging additionally to the space \( C^1([0,T] \times \mathbb{T} \times \mathbb{R}^3) \)—follows then by proposition 3.1. Note that \( C(T, \epsilon) \) decreases to zero as \( T \to 0 \).

The definition of \( f \) and \( u \) constructed by \( \bar{u} \) determines a map \( T : V \to L^\infty(0,T;L^2(\mathbb{T})) \) such that \( T(u) = u \), where \( u \) is the solution to (3.4) and \( f \) to (3.3). We need to show that \( T \) maps \( V \) into itself, and it is continuous and compact. There is no need to explain that the set \( V \) is convex.

### 3.2. Compactness

Our next step is to prove that map \( T \) has a fixed point in \( V \) and it defines a solution of (3.1).

We begin with estimates for \( u \) and \( f \) in \( H \) and \( X \), respectively.

**Proposition 3.3.** Given \( \epsilon > 0 \), let \( C > \|u_0\|_2 \) and \( \bar{u} \in V \). Then there exists \( T > 0 \) such that
\[
\|u\|_{L^\infty(0,T;L^2(\mathbb{T}))} \leq C. \tag{3.9}
\]

Moreover, there exist positive constants \( C(\epsilon) \) and \( C \) such that \( \{u, f\} \) satisfy the following bounds:

\begin{enumerate}
  \item \( ||u||_H \leq C(\epsilon) \),
  \item \( ||\partial_t u||_{L^2(0,T;L^2(\mathbb{T}))} \leq C(\epsilon) \),
  \item \( ||f||_X \leq C \),
  \item \( ||f||_{C^1} \leq C(\epsilon) \),
\end{enumerate}

where \( C \) is independent of \( \epsilon \). Moreover, there exists a non-decreasing function \( \mathcal{R}_\epsilon : [0,T] \to [0,\infty) \) such that
\[
\text{supp}(t, x, \cdot) \subset B(\mathcal{R}_\epsilon(t)), \quad \text{for all } t \text{ and a.a. } x. \tag{3.10}
\]

**Proof.** By proposition 3.2 and the definition of \( u \), it is clear that to obtain an estimate of \( u \) in \( H \), it suffices to estimate \( \|G\|_{L^2(0,T;L^2(\mathbb{T}))} \). By testing the weak formulation by \( u \) (which by proposition 3.2 is a suitable test function), applying Korn’s inequality and (2.1), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_2 + c_1 \kappa \|\nabla u\|_p^p \leq \|u\|^2_2 + \|G\|^2_2,
\]
which, by inequality (3.8) and (3.6), implies that
\[ \frac{1}{2} \frac{d}{dt} \|u\|_2^2 + c_1 \kappa \|\nabla_x u\|_p^p \leq \|u\|_2^2 + C(T, \epsilon) \|u\|_2^2 + C(T, \epsilon). \]

Therefore, by Gronwall’s lemma, there exists \( T \) depending on \( \epsilon \), such that
\[ \|u\|_{L^\infty(0,T, L^2(\mathbb{R}))} \leq C. \]

Using proposition 3.2, we finish the proof of (i).

The proof of (ii) follows similarly to the proof of (i) by testing the weak formulation for \( u \) with \( \partial_t u \) and using the previously proved estimates.

We continue with estimates of \( f \). The key point concerns the propagation of the support. The estimate of the support of \( f \) is proved in lemmas 3.1 and 3.2 below. Lemma 3.1 shows that \( R_\epsilon(t) \) depends on \( \|u\|_{L^2(0,T, W^{2,2}(\mathbb{T}))} \) and \( \|Mf\|_{L^\infty(0,T)} \). On the other hand, in lemma 3.2, we prove that \( \|Mf\|_{L^\infty(0,T)} \) is uniformly bounded in terms of \( \|u\|_{L^2(0,T, W^{2,2}(\mathbb{T}))} \). Therefore, by (i) from proposition 3.3, the function \( R_\epsilon \) is independent of \( n \) but depends on \( \epsilon \). This observation concludes the proof of (3.10).

**Lemma 3.1 (Propagation of the support of velocity).** Let \( f \) be a solution to (3.5) subject to the initial data with the support in \( v \) contained in the ball \( B(R) \). Then there exists a non-decreasing function \( R : [0, T] \to [0, \infty) \) such that for all \( t \in [0, T] \) and almost all \( x \in \mathbb{T}^d \), the support of \( f(t, x, \cdot) : \mathbb{R} \to \mathbb{R} \) is contained in a ball of radius \( R(t) \). Moreover, for each \( t \in [0, T] \) the value \( R(t) \) depends only on \( t, \|u\|_{L^2(0,T, W^{2,2}(\mathbb{T}))} \), \( \|Mf\|_{L^\infty(0,T)} \) and \( R \).

**Proof.** Let \( f \) be a solution to (3.5). Consider the solution of the system of ODE’s:
\[
\begin{align*}
\frac{dx}{dt}(t) &= v(t), \\
\frac{dv}{dt}(t) &= F_{CS}(f)(t, x(t), v(t)) + [(\theta_\epsilon * u)(t, x(t)) - v(t)] \gamma_\epsilon(v(t)), \\
x(0) &= x_0, \\
v(0) &= v_0.
\end{align*}
\]

Then the function \( \tilde{f}(t, x_0, v_0) := f(t, x(t), v(t)) \) satisfies the equation
\[ \partial_t \tilde{f} = - (\text{div}_x F_{CS}(f) + \text{div}_v [(\theta_\epsilon * u)(t, x(t)) - v] \gamma_\epsilon(v)) \tilde{f}. \]

Note that we are required to look at the three terms stemming from the divergence: \( \gamma_\epsilon(v) \), \( \theta_\epsilon * u \nabla_x \gamma_\epsilon(v) \) and \( v \cdot \nabla_v \gamma_\epsilon(v) \). By the definition of \( \gamma_\epsilon(\cdot) \) from the beginning of section 3.1, we see that
\[ |\gamma_\epsilon(v)| + |v \cdot \nabla_v \gamma_\epsilon(v)| \leq C \]
with \( C \) independent of \( \epsilon \). On the other hand, using the explicit form of \( \theta_\epsilon \), it is possible to compute that
\[ |(\theta_\epsilon * u)(t, x)| \leq \|(\theta_\epsilon * u)(t, \cdot)\|_{L^\infty} \leq C \epsilon^{-\frac{1}{2}} \|u(t, \cdot)\|_{L^2(\mathbb{T})}, \]
so we conclude
\[ |(\theta_\epsilon * u)(t, x) \nabla_x \gamma_\epsilon(v)| \leq C \epsilon^{\frac{1}{2}}. \]

Hence, recalling the fact that \( b \) is defined in (2.7), we obtain
\[ \tilde{f}(t, x(t), v(t)) = e^{\epsilon^2 \int_0^t (b + b)dt}_0 \tilde{f}_0(x(t), v(t)), \text{ where } \|B(t)\|_{L^\infty} \leq C(\|u(t, \cdot)\|_{L^2(\mathbb{T})} + C) \]
for sufficiently small \( \epsilon \). Let us note at this point that the above estimate provides a proof of \( L^\infty \) bound from (3.6).
Therefore, \( \overline{f}(t,x_0,v_0) = 0 \) whenever \( f_0(x_0,v_0) = 0 \), which implies that \( \overline{f}(t,x,v) = 0 \) whenever the characteristic that contains point \((x,v)\) starts at \((x_0,v_0)\) such that \( f_0(x_0,v_0) = 0 \). We solve (3.11) to obtain
\[
v(t) = e^{- \int_0^t [b(x,v(t)) + B(x,v(t),v(t)))]ds} \times \left( v_0 + \int_0^t e^{\int_0^s [b(x,v(r)) + B(x,v(r),v(r)))]dr} [a(s,x(x(s)) + (\theta_{x} * u)(s,x(x(s)))]\gamma_{x}(v(s))ds \right),\]
which, since by (2.9) \( 1 \leq b + 1 \leq c_M f + 1 \), by (3.6) \( M_0 f = 1 \), and \( B \) is bounded in terms of the norm \( L^2(0,T;W^{2,2}(T^3)) \), implies that
\[
|v(t)| \leq C e^{Ct} \left( |v_0| + \int_0^t |a(s,x(s))|ds + \int_0^t \|u(s)\|_{\infty}ds \right)
\leq C e^{Ct} \left( |v_0| + t\|M_0 f\|_{L^\infty(0,T)} + \|u\|_{L^2(0,T;W^{2,2}(T^3))} \right)
\leq C e^{Ct} \left( R + t\|M_0 f\|_{L^\infty(0,T)} + \|u\|_{L^2(0,T;W^{2,2}(T^3))} \right) =: \mathcal{R}(t),
\]
where we also used the embedding \( L^2(0,T;W^{2,2}(T^3)) \hookrightarrow L^1(0,T;L^\infty(T^3)) \). We will underline the fact that, at this stage, the estimate depends on \( \epsilon \), but at the end of the proof of existence, this estimate will imply that the support of \( f \) is bounded independently of \( \epsilon \).

**Lemma 3.2.** Let \( f \) be a solution to (3.5) subjected to the initial data with the support in \( v \) contained in the ball \( B(R) \). Then
\[
M_1 f \leq C(\epsilon),
\]
for some positive \( \epsilon \)-dependent constant \( C(\epsilon) \).

**Proof.** First, we integrate (3.5) to see that \( M_0 f = 1 \). Next we multiply (3.5) by \( |v| \) and integrate to obtain
\[
0 = \frac{d}{dt} M_1 f + \int_{T^3 \times \mathbb{R}} |v| \cdot \nabla f dv + \int_{T^3 \times \mathbb{R}} |v| dv [ (F_{\text{CS}}(f) + (\theta_{x} * u - v) \gamma_{x}) f ] dv
= 0
= \frac{d}{dt} M_1 f - \int_{T^3 \times \mathbb{R}} \frac{v}{|v|} \cdot (F_{\text{CS}}(f) + (\theta_{x} * u - v) \gamma_{x}) f dv.
\]
Thus
\[
\frac{d}{dt} M_1 f = \int_{T^3 \times \mathbb{R}} \frac{v}{|v|} \cdot F_{\text{CS}}(f) f dv + \int_{T^3 \times \mathbb{R}} \frac{v}{|v|} \cdot \theta_{x} * u f dv - \int_{T^3 \times \mathbb{R}} |v| f \gamma_{x} dv
\leq \int_{T^3 \times \mathbb{R}} |F_{\text{CS}}(f)| f dv + \int_{T^3 \times \mathbb{R}} |\theta_{x} * u| f dv
\leq C M_1 f M_0 f + C \|u\|_{W^{2,2}(T^3)} M_0 f.
\]
Since \( M_0 f = 1 \), it implies that \( M_1 f \) is bounded on \([0,T]\) if \( M_0 f_0 \) is finite.

**Proof of (iii)** and (iv). Even though it follows directly from proposition 3.1, it is worthwhile noting that the conservation of mass is trivial, while the \( L^\infty \) bound was shown by (3.12).
With the estimates provided by proposition 3.3, we are ready to prove the following proposition that states the existence of solutions to the regularized system (3.1).

**Proposition 3.4.** Given $C$ defined as in (3.2) and $T$ given by proposition 3.3, then there exists a fixed point of $T$ belonging to $V$. In addition, $u$ fulfills all regularity described by proposition 3.3.

**Proof.** The continuity of map $T$ follows from the stability of solvability given by propositions (3.3) and (3.4). As $u \in V$ then $G \in L^2(0, T; L^2(\mathbb{T}^3))$, hence $u \in H$ and $u_t \in L^2(0, T; L^2(\mathbb{T}^3))$. These facts imply compactness. By Schauder’s theorem, we found a fixed point of map $T$ belonging to $V$. The extra regularity comes from proposition 3.3. Properties of $f$ are concluded from proposition 3.3. We are done.

3.3. Local convergence with the regularized solutions

Until now we have proved the existence of solutions to the regularized system (3.1). The next goal is to converge with $\varepsilon$ to 0 and to obtain local-in-time existence for (1.3).

**Proposition 3.5.** Let $p \geq \frac{11}{5}$ and $(f_\varepsilon, u_\varepsilon)$ be a solution to system (3.1) constructed as a limit of the approximate solutions as proved in proposition 3.4. Then there exists $T^* \in (0, T]$ such that $(f_\varepsilon, u_\varepsilon)$ satisfies the following estimates
\[
\|M_\alpha f_\varepsilon\|_{L^\infty(0, T^*]} \leq C(T), \quad 0 \leq \alpha \leq 2,
\]
\[
\|u_\varepsilon\|_{L^\infty(0, T^*; L^2(\mathbb{T}^3))} \leq C(T),
\]
\[
\left\|\int_{\mathbb{T}^3} (\theta_\varepsilon u_\varepsilon - v) \gamma_\varepsilon f_\varepsilon \, dv\right\|_{L^2(0, T^*; L^2(\mathbb{T}^3))} \leq C(T),
\]
where $C(T)$ is a positive constant depending on the initial data and $T$.

**Proof.** We multiply equation (3.1) by $|v|^2$ and integrate to obtain
\[
0 = \frac{d}{dt}M_2 f_\varepsilon + \int_{\mathbb{T}^3} |v|^2 v \cdot \nabla f_\varepsilon \, dx \, dv + \int_{\mathbb{T}^3} |v|^2 dv [F_{CS}(f_\varepsilon), f_\varepsilon] + \gamma_\varepsilon (\theta_\varepsilon u_\varepsilon - v) f_\varepsilon \, dv
\]
\[
= \frac{d}{dt}M_2 f_\varepsilon - 2 \int_{\mathbb{T}^3} \gamma_\varepsilon v \cdot F_{CS}(f_\varepsilon) f_\varepsilon \, dx \, dv - 2 \int_{\mathbb{T}^3} \gamma_\varepsilon v \cdot (\theta_\varepsilon u_\varepsilon - v) f_\varepsilon \, dx \, dv,
\]
and since, by substituting $x$ with $y$ and $v$ with $w$ (as in the estimate of $S_1$ in the proof of proposition 3.5), we have
\[
\int_{\mathbb{T}^3} v \cdot F_{CS}(f_\varepsilon) f_\varepsilon \, dx \, dv \leq -C \int_{\mathbb{T}^3} |w|^2 \psi(|x - y|) f_\varepsilon(t, w, y) f_\varepsilon(t, u, x) \, dy \, dx \, dv \leq 0,
\]
and we deduce the inequality
\[
\frac{d}{dt}M_2 f_\varepsilon \leq 2 \int_{\mathbb{T}^3} \gamma_\varepsilon v \cdot (\theta_\varepsilon u_\varepsilon - v) f_\varepsilon \, dx \, dv. \tag{3.16}
\]
Next, we test the weak formulation for $u_\varepsilon$ by $u_\varepsilon$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_t\|^2_2 + c_1 \| \nabla_x u_t \|^2_p \leq - \int_{T^1 \times \mathbb{R}^3} f_\epsilon(\theta_x u_t - v) \cdot u_t \gamma_\epsilon \, dx \, dv \\
= \int_{T^1 \times \mathbb{R}^3} f_\epsilon(\theta_x u_t - v) \cdot (\theta_x u_t - u_t) \gamma_\epsilon \, dx \, dv - \int_{T^1 \times \mathbb{R}^3} f_\epsilon(\theta_x u_t - v) \cdot \theta_x u_t \gamma_\epsilon \, dx \, dv.
\]

We add (3.16) and two instances of (3.17), obtaining
\[
\frac{d}{dt} (M_2 f_\epsilon + \|u_t\|^2_2) + 2c_1 \| \nabla_x u_t \|^2_p + 2 \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - v|^2 f_\epsilon \, dx \, dv \\
\leq 2 \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - \theta_x u_t - u_t| f_\epsilon \, dx \, dv.
\]

Hölder’s and Young’s inequalities yield the following estimate of the right-hand side
\[
2 \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - v|^2 f_\epsilon \, dx \, dv \\
\leq \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - v|^2 f_\epsilon \, dx \, dv + C \| \theta_x u_t - u_t \|^2_\infty \| m_0 \sqrt{f_\epsilon} \|^2_\infty \\
\leq \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - v|^2 f_\epsilon \, dx \, dv + \eta \| \theta_x u_t - u_t \|^2_p + C(\eta) \| m_0 f_\epsilon \|^2_\infty.
\]

By Young’s inequality for convolutions, we have \( \| \theta_x u_t - u_t \|^2_\infty \leq 2^p \| \nabla_x u_t \|^p_2 \); thus choosing a suitable \( \eta \) we obtain
\[
\frac{d}{dt} (M_2 f_\epsilon + \|u_t\|^2_2) + c_1 \| \nabla_x u_t \|^2_p + \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - v|^2 f_\epsilon \, dx \, dv \\
\leq C(M_2 f_\epsilon)^{\frac{1}{2}} \leq C(M_2 f_\epsilon)^{\frac{1}{2}}.
\]

We apply lemma A.1 and uniform \( L^1 \) and \( L^\infty \) bounds on \( f_\epsilon \) given by (3.6) to obtain
\[
\| m_0 f_\epsilon \|^\frac{1}{2}_2 \leq C(M_2 f_\epsilon)^{\frac{1}{2}} \leq C(M_2 f_\epsilon)^{\frac{1}{2}},
\]
which yields
\[
\frac{d}{dt} (M_2 f_\epsilon + \|u_t\|^2_2) + c_1 \| \nabla_x u_t \|^2_p + \int_{T^1 \times \mathbb{R}^3} \gamma_\epsilon |\theta_x u_t - v|^2 f_\epsilon \, dx \, dv \\
\leq C(M_2 f_\epsilon)^{\frac{p}{p-1}}.
\]

By the nonlinear version of Gronwall’s lemma A.2, there exists \( T^* \in (0, T) \) (with \( T^* = T \) for \( p \geq 4 \)) such that for \( t \in [0, T^*] \), we have
\[
(M_2 f_\epsilon + \|u_t\|^2_2)(t) + c_1 \int_0^t \| \nabla_x u_t \|^2_p \, ds + \int_0^t \int_{T^1 \times \mathbb{R}^3} |\theta_x u_t - v|^2 \gamma_\epsilon f_\epsilon \, dx \, dv \, ds \leq C(T).
\]

This proves (3.13) and (3.14). It remains to prove (3.15). We do this by using (3.7): we estimate
\[
G_\epsilon = \int_{\mathbb{R}^3} (\theta_x u_t - v) \gamma_\epsilon f_\epsilon \, dv
\]
in $L^2(0, T^*; L^2(\mathbb{T}^3))$ by a combination of terms that are bounded thanks to the energy estimate (3.20) and terms that appear on the left-hand side of (3.7); then we move the bad terms to the left hand side of (3.20) and finish the estimation. We perform the computations only for $p < 3$. For $p \geq 3$, a slightly different argument is needed, but due to the higher regularity of solutions, the proof is in fact easier. We have

$$
\int_0^T ||G_i||^2 dt \leq C \left( \int_0^T \int_0^{T^*} \left( \int_0^{\theta_i} u_\epsilon \gamma f_i dv \right)^2 dx dt + \int_0^T \int_0^{T^*} \left( \int_0^{\theta_i} v \gamma f_i dv \right)^2 dx dt \right) =: A + B.
$$

First we estimate $A$:

$$
A \leq \int_0^T \int_0^{T^*} \left| \theta_i \ast u_\epsilon \right|^2 (m_0 f_i)^2 dx dt \leq \int_0^T ||u_i||^2 \cdot \sup_{t \in T^*} \int_{\mathbb{T}^3} (m_0 f_i)^2 dx. \tag{3.21}
$$

Lemma A.1 implies that

$$
\int_{\mathbb{T}^3} (m_0 f_i)^2 dx \leq \int_{\mathbb{T}^3} (m_0 f_i)^2 dx \cdot ||m_0 f_i||_\infty^\frac{4}{3} \leq CM_2 \epsilon ||f_i||_\infty^\frac{3}{2} \mathcal{R} \leq C(T^*) \mathcal{R}, \tag{3.22}
$$

where $\mathcal{R}$ denotes the radius of the support of $f_i$ in $\epsilon$. Moreover, the proof of lemma 3.1 implies that

$$
\mathcal{R} \leq Ce^{C T^*} \left( R + T^* \sqrt{M_2} + \int_0^{T^*} ||u_i||_\infty dt \right); \tag{3.23}
$$

thus, combining (3.21) and (3.22) with the energy estimate (3.20), we obtain

$$
A \leq C(T) \int_0^T ||u_i||^2 \cdot \left( \sqrt{M_2} + \int_0^{T^*} ||u_i||_\infty dt + 1 \right) \leq C(T) \left[ \left( \int_0^{T^*} ||u_i||^2 dt \right)^\frac{3}{2} + 1 \right]. \tag{3.24}
$$

To estimate $||u_i||_\infty$, we use Gagliardo–Nirenberg inequality obtaining

$$
||u_i||_\infty \leq C \left( \int_0^{T^*} \left( ||\nabla u_i||_{3p}^{\frac{2p}{3}} + ||u_i||_\infty^{\frac{2p}{3}} \right) ||u_i||_\infty^{\frac{2p}{3}} \right),
$$

which implies that (note that $\frac{p^2}{2} > 1$ for $p > \frac{3}{2}$)

$$
\int_0^{T^*} \left( \int_0^{T^*} \left( ||\nabla u_i||_{3p}^{\frac{2p}{3}} + ||u_i||_\infty^{\frac{2p}{3}} \right) \left( \int_0^{T^*} \left( ||\nabla u_i||_{3p}^{\frac{2p}{3}} + ||u_i||_\infty^{\frac{2p}{3}} \right) dt \right) dt \leq C \left( \int_0^{T^*} \left( ||\nabla u_i||_{3p}^{\frac{2p}{3}} + ||u_i||_\infty^{\frac{2p}{3}} \right) \left( \int_0^{T^*} \left( ||\nabla u_i||_{3p}^{\frac{2p}{3}} + ||u_i||_\infty^{\frac{2p}{3}} \right) dt \right) + \text{l.o.t.},
$$

where l.o.t. denotes lower order terms connected with the presence of $||u_i||_2$ which is bounded on $(0, T^*)$. Therefore
We use the fact that \( \frac{p-1}{2p-3} \leq 1 \) for \( p \geq 2 \) and Young’s inequality with exponent \( \frac{2p}{2p-3} \) (which is greater than 1 for \( p > \frac{3}{2} \)) to obtain for arbitrary \( \eta > 0 \)

\[
A \leq \eta \int_0^T \| \nabla_x u_e \|_{L_\infty}^p \, dt + C(\eta) \left( \int_0^T \| \nabla_x u_e \|_p^p \, dt \right)^{\frac{p-1}{p}} + C. \tag{3.25}
\]

On the other hand, for \( B \), we have by lemma A.1

\[
B \leq \int_0^T \int_{T^*} (m_{f_e})^\frac{5}{2} \cdot (m_{f_e})^\frac{3}{2} \leq TM_{f_e} \| m_{f_e} \|_{L_\infty} \leq C(T) R^3.
\]

We estimate \( R \) again using (3.23) obtaining

\[
B \leq C(T) \left[ \left( \int_0^T \| u_e \|_{L_\infty} \, dt \right)^3 + 1 \right] \leq C(T) \left[ \left( \int_0^T \| u_e \|_{L_\infty}^3 \, dt \right)^{\frac{1}{2}} + 1 \right]
\]

and this is the estimate with exactly the same right-hand side as (3.24). Thus, from this point, we proceed as in the estimation of \( A \) altogether obtaining

\[
\int_0^T \| G_e \|_{L_2}^2 \, dt \leq A + B \leq \eta \int_0^T \| \nabla_x u_e \|_{L_\infty}^p \, dt + C(\eta) \left[ \left( \int_0^T \| \nabla_x u_e \|_p^p \, dt \right)^{\frac{p-1}{p}} + 1 \right]. \tag{3.26}
\]

for all \( \eta > 0 \), and note that, due to the energy estimate (3.20), the second summand on the right-hand side is bounded on \([0,T^*] \). We apply this estimate to (3.7), which, after taking a sufficiently small \( \eta \), enables us to move the term with \( \| \nabla_x u_e \|_{L_\infty}^p \) to the left-hand side, which results in

\[
\sup_{t \in [0,T^*]} \| u_e \|_{H^2(T^*)}^2 + \| \nabla_x u_e \|_{L^2(0,T^*;L^2(T^*)^2)}^2 + \frac{1}{2} \| \nabla_x u_e \|_{L^p(0,T^*;L^p(T^*))}^p \leq C,
\]

where \( C \) depends on \( \| u_0 \|_{L_2}^2, M_2, f_{0,\epsilon}, R, \| f_e \|_{L_\infty}, T^*, T, p \), all of which are either fixed or bounded independently of \( \epsilon \). Finally, applying (3.27) to (3.26) finishes the proof of (3.15).

The following corollary combines all the necessary local in time uniform estimates of \( u_e \) and \( f_e \) proved throughout this section.

**Corollary 3.1.** The solutions \((f_e, u_e)\) satisfy estimates (i)–(iii) from proposition 3.3 and all estimates from proposition 3.5 uniformly with respect to \( \epsilon > 0 \) on the time interval \([0,T^*] \).

**Proof.** To prove estimate (i), we notice that by proposition (3.4), the \( \| \cdot \|_{H^2} \) norm of \( u_0^\epsilon \) depends only on \( \| u_0 \|_{W^{2,1}(T^*)} \), which is fixed, and on \( \| G \|_{L_2(0,T^*;L_2(T^*))} \), which, by proposition 3.5, is uniformly bounded with respect to \( \epsilon \) on the time interval \([0,T^*] \). Therefore, \( \| u_0^\epsilon \|_{H^2} \) is also uniformly bounded on \([0,T^*] \). Exactly the same argument is valid for the \( \epsilon \)-independent
estimate (ii). Estimate (iii) was already proved to be \( \epsilon \)-independent. It remains to show that \( f_\epsilon \) satisfies (3.10) with \( R \) independent of \( \epsilon \). By lemmas 3.1 and 3.2, each iterative solution \( f_\epsilon \) has a support contained in a ball of radius \( R_\epsilon \) with \( R_\epsilon(t) \) depending on \( \| u_\epsilon \|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))} \) and \( \| M f_\epsilon \|_\infty \) (and \( R \) which depends only on the initial data). However, by proposition 3.5, these quantities are uniformly bounded on \([0,T^*]\), and thus so is \( R_\epsilon \).

**Proof of theorem 2.1—local existence.** With the uniform bounds from corollary 3.1, it remains to let \( \epsilon \) to 0 and to show the compactness of the set \( \{(f_\epsilon, u_\epsilon)\}_{\epsilon>0} \) in appropriate spaces, and that the limits of \((f_\epsilon, u_\epsilon)\) solve (1.3) in the sense of definition 2.1.

By virtue of the previously proved uniform bounds, it follows that \( u_\epsilon \) is uniformly bounded in \( H \hookrightarrow L^2(0,T;W^{2,2}(\mathbb{T}^3)) \) and \( \partial_\tau u_\epsilon \) is uniformly bounded in \( L^2(0,T;L^2(\mathbb{T}^3)) \). Since it holds \( W^{2,2}(\mathbb{T}^3) \hookrightarrow W^{1,2}(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3) \), by Aubin–Lions lemma, we may extract from \( u_\epsilon \) a strongly convergent subsequence in \( L^2(0,T;W^{1,2}(\mathbb{T}^3)) \). Thus, up to a subsequence

\[
\text{for some } u \in L^2(0,T;L^2(\mathbb{T}^3)) \quad \text{(3.28)}
\]

\[
\nabla_t u_\epsilon \to \nabla_t u \text{ in } L^2(0,T;L^2(\mathbb{T}^3)). \quad \text{(3.29)}
\]

On the other hand, the compactness of \( f_\epsilon \) follows from the \( L^\infty \) bound (iii) from proposition 3.3 and Banach–Alaoglu theorem. Then, up to a subsequence, \( f_\epsilon \to f \) weakly * in \( L^\infty([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3) \).

To finish the proof, we need to show that \((f,u)\) satisfies (3.1) in the sense of definition 2.1. By (3.28)

\[
\int_0^T \int_{\mathbb{T}^3} -u_\epsilon \cdot \partial_\tau \phi dxdvdt \to \int_0^T \int_{\mathbb{T}^3} -u \cdot \partial_\tau \phi dxdvdt, \quad \text{(3.30)}
\]

for all divergence free smooth \( \phi \) with compact support in \( t \). Thus, \( \partial_\tau u_\epsilon \to \partial_\tau u \) in the distributional sense, where \( \partial_\tau u \) is the distributional derivative of \( u \). However, since \( \partial_\tau u_\epsilon \) is bounded in \( L^2(0,T;L^2(\mathbb{T}^3)) \), it actually implies that \( \partial_\tau u_\epsilon \to \partial_\tau u \) weakly in \( L^2(0,T;L^2(\mathbb{T}^3)) \).

By (3.28) and (3.29) \( u_\epsilon \to u \) and \( \nabla_t u_\epsilon \to \nabla_t u \) a.e. (up to a subsequence), which also implies that the convective term \((u_\epsilon \cdot \nabla_x)u_\epsilon \to (u \cdot \nabla_x)u \) a.e. Moreover, for a sufficiently small \( \eta > 0 \), we have

\[
\| (u_\epsilon \cdot \nabla_x)u_\epsilon \|_{L^{1+\eta}(0,T;L^{1+\eta}(\mathbb{T}^3))} \leq \| u \|_{L^\infty(0,T;L^2(\mathbb{T}^3))} \| \nabla_x u \|_{L^\infty([0,T] \times \Omega)} \leq \| u \|_{H}^2,
\]

which means that \((u_\epsilon \cdot \nabla_x)u_\epsilon \) is uniformly bounded in \( L^{1+\eta} \), and thus it is uniformly integrable. By Vitali’s convergence theorem

\[
\int_0^T \int_{\mathbb{T}^3} (u_\epsilon \cdot \nabla_x)u_\epsilon \cdot \phi dxdvdt \to \int_0^T \int_{\mathbb{T}^3} (u \cdot \nabla_x)u \cdot \phi dxdvdt, \quad \text{(3.31)}
\]

for all divergence free smooth \( \phi \) with compact support in \((0,T)\).

Similarly, up to a subsequence \( \tau(Du_\epsilon) \to \tau(Du) \) a.e. and by (2.1) it is bounded in \( L^p(0,T;L^p(\mathbb{T}^3)) \). Vitali’s convergence theorem implies that \( \tau(Du_\epsilon) \to \tau(Du) \) strongly in \( L^{p-\eta}(0,T;L^{p-\eta}(\mathbb{T}^3)) \) for some \( \eta > 0 \). On the other hand, by Banach–Alaoglu theorem, the se-
sequence \( \{\tau(Du_t)\}_{t>0} \) converges weakly in \( L^p(0;L^p(T^3)) \) to \( \tau(Du) \) and by weak sequential lower semicontinuity of the norm \( \tau(Du) \in L^p(0;L^p(T^3)) \), Whence

\[
\int_0^T \int_{T^3} \tau(Du_t) : D\phi dtdx + \int_0^T \int_{T^3} \tau(Du) : D\phi dtdx dt
\]

for all divergence free smooth \( \phi \in C^\infty \) with compact support in \( t \). The convergence and boundedness of the external force follows by similar arguments. Altogether, (3.30)–(3.32) imply that for all divergence free smooth \( \phi \) with compact support in the time variable, we have

\[
\int_0^T \int_{T^3} (u \cdot \partial_t \phi + (u \cdot \nabla \phi) u \cdot \phi + \tau(Du) : D\phi) dtdxdt = \int_0^T \int_{\mathbb{R}^3} (u - v) \cdot \phi f dtdxdt.
\]

As \( u \) is a limit of \( u_\epsilon \), we also have

\[
u \in L^\infty(0, T; \dot{W}^{1,p}(T^3)) \cap L^\infty(0, T; W^{1,2}(T^3)) \cap L^2(0, T; W^{2,2}(T^3)),
\]

and since \( \partial_t u \in L^2(0, T; L^2(T^3)) \), the well-known result on the Gelfand triplet allows us to conclude that \( u \in C([0,T]; L^2(T^3)) \), and thus \( u \in \mathcal{H} \).

Finally, due to the sufficient regularity of \( u \), we may replace (3.33) by the equation from point 4 of definition 2.1 and, by a density argument, extend the class of admissible test functions to \( W^{1,2}(T^3) \cap \dot{W}^{1,p}(T^3) \).

As for the particle part of the solution \( f \), thanks to the regularising effect of \( \psi \) in \( F_{CS}(f) \) and of the sufficient regularity of \( u \), converging with each term of the weak formulation for \( f_\epsilon \) is straightforward.

\[\square\]

### 3.4. Global existence

This part is dedicated to showing that \( T^* = T \). The previous subsection gave the local existence for the original problem. To show the existence on the whole time interval \([0, T]\), it is sufficient to construct an \emph{a priori} estimate controlling traces in time which allow us to prolong the solution till time \( T \). As \( \theta, u_\epsilon \to u \), in the regularity class, where \( u_\epsilon \) is bounded, it is not difficult to show that estimate (3.18) takes the form

\[
(Msf + ||u||^2_2)(t) + c_1 \kappa \int_0^t \|\nabla_s u\|^p_2 ds + \int_0^t \int_{\mathbb{R}^3} |u - v|^2 f dtdxds \leq (Msf + ||u||^2_2)(0).
\]

Note that the right-hand side depends only on the initial data of our problem. We were not able to use this argument previously and this led to it being necessary to prove the local existence first.

Then, in order to apply proposition 3.2, we estimate the drag force

\[
G = \int_{\mathbb{R}^3} (u - v) f dv
\]

in the same way as we estimated \( G_\epsilon \) in the proof of proposition 3.5. So we find better integrability of \( u \). Hence, for all \( t \in [0, T^*] \), we are able to construct the \emph{a priori} estimate without dependence from \( T^* \), guaranteeing that, by continuity \( u(T^*) \in W^{2,2}(T^3) \),
4. Large-time behavior of the solutions

To prove theorem 2.2, we apply the strategy from [3], where the proof follows directly from the following lemma.

**Lemma 4.1.** Recall $\mathcal{E}$ defined in (2.15). Let $(f, u)$ be any solution of system (1.3) in the sense of definition 2.1. Then we have

(i) $\frac{d\mathcal{E}_p}{dt} \leq -2\psi(\sqrt{2})\mathcal{E}_p + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (v - v_c) \cdot (u - v) f dx dv$.

(ii) $\frac{d\mathcal{E}_f}{dt} \leq -2c_1\kappa\varpi\mathcal{E}_f + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u_c - u) \cdot (u - v) f dx dv$.

(iii) $\frac{d\mathcal{E}_d}{dt} \leq -4 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u_c - v_c) \cdot (u - v) f dx dv$.

**Proof.** The proof can be found in [3] lemma 4.1. The only slightly different part is the proof of (ii), which we present below. We test (1.3) with $u - u_c$ (which is an admissible test function) obtaining

$$\int_{\mathbb{T}^3} \partial_t u \cdot (u - u_c) dx + \int_{\mathbb{T}^3} (u \cdot \nabla_x) u \cdot (u - u_c) dx + \int_{\mathbb{T}^3} \tau(Du) : Du dx = -\int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - v) \cdot (u - u_c) f dx dv.$$

Since $\text{div}_x u = 0$, the convective term disappears, i.e.

$$\int_{\mathbb{T}^3} (u \cdot \nabla_x) u \cdot (u - u_c) dx = 0.$$

Moreover, by (2.1) and Korn’s and Poincaré’s inequalities, we have

$$\int_{\mathbb{T}^3} \tau(Du) : Du dx \geq c_1\kappa\|\nabla_x u\|^2 \geq c_1\kappa\varpi \int_{\mathbb{T}^3} |u - u_c|^2 dx = c_1\kappa\varpi\mathcal{E}_f.$$

Finally, we note that

$$\int_{\mathbb{T}^3} \partial_t u \cdot (u - u_c) dx = \int_{\mathbb{T}^3} \partial_t u \cdot (u - u_c) dx - \frac{d}{dt} u_c \cdot \int_{\mathbb{T}^3} (u - u_c) dx = \frac{1}{2} \frac{d\mathcal{E}_d}{dt},$$

and combine the above estimates to obtain

$$\frac{d\mathcal{E}}{dt} \leq -2c_1\kappa\varpi\mathcal{E}_f + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - v) \cdot (u_c - u) f dx dv. \quad \square$$

**Proof of theorem 2.2.** Fix $T > 0$. Since

$$\mathcal{E} = 2\mathcal{E}_p + 2\mathcal{E}_f + \mathcal{E}_d,$$

it follows from lemma 4.1 that

$$\frac{d\mathcal{E}}{dt} \leq -4\psi(\sqrt{2})\mathcal{E}_p - 4c_1\kappa\varpi\mathcal{E}_f - 4 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - v|^2 f dx dv \quad (4.1)$$

and it proves that $\mathcal{E}$ is nonincreasing. In order to show (2.16), we aim to apply Gronwall’s inequality to (4.1), and hence we need a term containing $\mathcal{E}_d$ on the right-hand side, which we extract from $-4 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - v|^2 f dx dv$. We may follow the approach from [3] to obtain for $\eta > 0$
\[
\frac{d\mathcal{E}}{dt} \leq -2 \left( \psi(\sqrt{2}) + 1 - \frac{1}{1 + \eta} \right) (2\mathcal{E}_p) - 2 \left( c_1 \kappa \nu - \eta \|m_0f\|_\infty \right) 2\mathcal{E}_f - 4 \left( 1 - \frac{1}{1 + \eta} \right) \mathcal{E}_d.
\]

(4.2)

Note that from the boundedness of the support in \(v\) of \(f\), for any \(T > 0\) we have \(\sup_{t \leq T} \|m_0f\|_\infty < \infty\), thus after fixing
\[
\eta := \frac{c_1 \kappa \nu}{2\sup_{t \leq T} \|m_0f\|_\infty} > 0
\]
from (4.2) and Gronwall’s inequality, we deduce that
\[
\mathcal{E}(t) \leq \mathcal{E}(0) e^{-\gamma t}, \quad t \in [0, T),
\]
where \(\gamma := \min\{2\psi(\sqrt{2}) + \frac{2\eta}{1 + \eta}, c_1 \kappa \nu, \frac{4\eta}{1 + \eta} \}\). Finally, with an additional assumption that \(\sup_{t \leq \infty} \|m_0f\|_\infty < \infty\), we can take \(T = \infty\) in (4.3) and in the definition of \(\eta\).

In the case of \(p > 3\), one can use the advantage of the imbedding theorem. Estimate (4.3) shows that \(\mathcal{E}(t)\) is indeed a Lyapunov functional, and it must decrease for all time. Now, taking the energy estimate (2.11), we know that
\[
\int_0^T \left( \int_{\mathbb{R}^3} |\nabla_x u|^p \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u - v|^2 f \, dx \, dv \right) \, dt \leq C
\]
with the right-hand side independent of \(T\). It means that there exists a sequence \(t_n \to \infty\) (increasing) such that
\[
\int_{\mathbb{R}^3} |\nabla_x u(t_n)|^p \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u(t_n) - v|^2 f(t_n) \, dx \, dv \to 0 \text{ as } n \to \infty.
\]

(4.5)

The Sobolev inequality yields
\[
\|u(t_n) - u_c(t_n)\|_{L^\infty} \to 0 \text{ and as a consequence } \mathcal{E}_f(t_n) \to 0.
\]

(4.6)

Next, we note that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - u_c(t_n)|^2 f(t_n) \, dx \, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( (u_c^2 + v^2) f - 2u_c \cdot vf \right)(t_n) \, dx \, dv.
\]

(4.7)

But the conservation of momentum (2.12) states that
\[
\int_{\mathbb{R}^3} (u + \int_{\mathbb{R}^3} v f \, dv) \, dx = 0
\]
for a.a. \(t\). So
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - u_c(t_n)|^2 f(t_n) \, dx \, dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( (u_c^2 + v^2) f \right)(t_n) \, dx \, dv + 2u_c^2(t_n).
\]

(4.9)

It means that
\[
u_c(t_n) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v^2 f(t_n) \to 0.
\]

(4.10)
This implies that \( v_r(t_n) \to 0 \). Hence, we proved that \( \mathcal{E}(t_n) \to 0 \), so by the monotonicity of \( \mathcal{E} \), we obtain
\[
\mathcal{E}(t) \to 0
\]
for a.a. \( t \to \infty \).

\[\square\]

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Appendix

We present the basic tools used throughout the paper, i.e. two crucial lemmas from [30].

**Lemma A.1.** Let \( \beta > 0 \) and \( g \) be a nonnegative function in \( L^\infty([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3) \). The following estimate holds for any \( \alpha < \beta \):
\[
 m_\alpha g(t,x) \leq \left( \frac{4}{3} \pi \|g(t,x,\cdot)\|_\infty + 1 \right) m_\beta g(t,x) \frac{3^{-\beta/3}}{3^{-\alpha/3}},
\]
for a.a. \((t,x)\).

**Proof.** The proof can be found in [30] (p 9 (lemma 1)).

We include the formulation of the classical Gronwall’s lemma with its less popular nonlinear varieties.

**Lemma A.2 (Gronwall’s lemma).** Let \( f \) be a nonnegative function satisfying inequality
\[
f(t) \leq c + \int_{t_0}^t (a(s)f(s) + b(s)f^q(s))ds, \ c \geq 0, \ q > 1,
\]
where \( a \) and \( b \) are nonnegative, integrable functions for \( t \geq t_0 \). Then we have
\[
f(t) \leq c \left[ e^{(1-q) \int_{t_0}^t a(s)ds} - e^{-1} (q-1) \int_{t_0}^t b(s)e^{(1-q) \int_{t_0}^r a(r)dr}dr \right]^{\frac{1}{q-1}},
\]
for \( t \in [t_0, h] \) for \( h > 0 \) provided that
\[
c < \left[ e^{(1-q) \int_{t_0}^{t_0+h} a(s)ds} \right]^{\frac{1}{q-1}} \left[ (q-1) \int_{t_0}^{t_0+h} b(s)ds \right]^{\frac{1}{q-1}}.
\]
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