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Low-rank Sachdev-Ye-Kitaev models

Jaewon Kim,1 Xiangyu Cao,1 and Ehud Altman1

1Department of Physics, University of California, Berkeley, CA 94720, USA
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Motivated by recent proposals of experimental realization of fast scramblers, we study a family of
solvable variants of the \( (q = 4) \) Sachdev-Ye-Kitaev model in which the rank and eigenvalue dis-
tribution of the coupling matrix \( J_{ij,kl} \) are tuneable. When the rank is proportional to the number
of fermions, the low temperature behavior is sensitive to the eigenvalue distribution. We obtain a
complete classification of the possible non-Fermi liquid quantum phases. These include two previ-
ously studied phases whose fermion scaling dimension depends continuously on the rank; we show
that they are maximally chaotic, but necessitate an extensively degenerate or negative semidefinite
coupling matrix. More generic distributions give rise to “almost Fermi liquids” with a scaling dimen-
sion \( \Delta = 1/2 \), but which differ from a genuine Fermi-liquid in quasi-particle decay rate, quantum
Lyapunov exponent and/or specific heat.

I. INTRODUCTION

The Sachdev-Ye-Kitaev \(^ {1,2} \) model, in its simplest
form, describes a large number of Majorana fermions
with all-to-all random interactions:

\[
H = \sum_{ijkl} J_{ij,kl} \gamma_i \gamma_j \gamma_k \gamma_l . \tag{1}
\]

At low temperatures, this exactly solvable model
describes a peculiar non Fermi liquid which has a large
symmetry, and a quantum Lyapunov exponent that
saturates the universal bound on chaos \(^ 3 \). These features
made it an attractive platform to study a wide range
of topics, e.g., strongly correlated electrons, many-body
quantum chaos, and black hole information scrambling,
each generating a flurry of recent activities \(^{1,2,4-20} \).

Historically, the SYK model originated from the
Sachdev-Ye (SY) model of quantum random spin magnet
\(^ 4 \):

\[
H = \frac{1}{\sqrt{NM}} \sum_{ab} U_{ab} S_x^a \cdot S_x^b , \tag{2}
\]

where \( S_x \) are some \( SU(M) \) spin operators. The SYK
Hamiltonian was conceived by Kitaev as a variant of the
fermionic representation of \(^ 1 \) in the double scaling limit
\( M, N \to \infty \): schematically, a spin operator is represented
by a fermion bilinear, and the coupling matrix \( U_{ab} \) by
\( J_{ij,kl} \). Although the SY model beyond the double-scaling
limit is not exactly solvable, it is more amenable to ex-
perimental realization. In particular, coupling cold atom
ensembles to optical cavity modes provides a promising
way of generating the all-to-all interaction between
atomic spins \(^ {21,31} \). In these platforms, the rank of the
matrix \( U_{ab} \) is controlled by the number of coupled cavity
modes, which is usually rather small. The effect of hav-
ing a low-rank matrix has been studied in detail in \(^ {31} \),
where it was shown that the resulting quantum dynamics
is integrable even at infinite temperature. These findings
leave one wondering how large a rank is necessary to ac-

FIG. 1. A qualitative phase diagram (main plot) and a sketch
(inset) of the low-rank SYK model. \( N \) Majorana fermions
(blue dots) are coupled by random all-to-all 4-body interac-
tions, mediated by \( R \) boson modes; \( R \) is also the rank of
the coupling matrix \( J \). The model is non-interacting when
\( R \ll N \) and equivalent to SYK when \( R \gg N \). When \( R \propto N \),
the IR fixed point, governing \( T \lesssim T_c \), depends on the eigen-
value distribution of the coupling matrix, see Table I.

by the double scaling limit: in the standard SYK model
\(^ 1 \), \( J_{ij,kl} \) has independent coefficients and is a matrix of
super-extensive rank \( \propto N^2 \), whereas in the SY model
\(^ 2 \) with a fixed \( M \), \( U_{ab} \) has an extensive rank \( \propto N \). There-
fore, a solvable variant of the SYK model where \( J_{ij,kl} \) has
tuneable rank should be beneficial to better understanding
random quantum magnets beyond large \( M \).

Such a model has recently been considered by several
authors in different contexts: for example, to showcase
the instability of the SYK fixed point towards a Fermi-
liquid phase \(^ {10} \), and to model Cooper pairing in non-
Fermi liquids \(^ {32,33} \). In the latter context, the rank
equals the number of phonon modes coupled to the elec-
trons. So far, it has been understood that the extensive
rank (\( R \sim N \)) regime is the most interesting, whereas
\( R \gg N \) leads back to the standard SYK model and
\( R \ll N \) to a non-interacting model \(^ {10,24} \), see Fig. 1.

What was overlooked, however, is the role of the eigen-
value distribution of the coupling matrix, or equivalently,
the distribution of fermion-boson/spin-boson couplings.

\[ \text{trivial} \quad \ln \frac{T_c}{T} \sim \sqrt{\frac{R}{N}} \quad \text{free-fermion} \quad \text{new QCPs} \]

\[ 1 \ll N \quad N \quad 4N \gg N \quad N^2 \]
In this paper, we fill in this gap by solving a family of “low-rank SYK models” where $J_{ij,kl}$ has a tuneable eigen-distribution. Our main contribution is an essentially complete classification table (Table II) of four universality classes of distributions, which give rise to distinct gapless quantum phases. Among them, previous works [10, 32, 33] studied two classes (III and IV in our classification), which we show are indeed SYK-like fast scramblers with extensive residual entropy. The new classes (I and II), corresponding to more generic distributions, exemplify quantum phases that are almost, but not quite, Fermi liquids.

As an application, we revisit the proposal put forward in Ref. [35] of realizing the SYK$_4$ model with electrons in the zeroth Landau level of a graphene flake with irregular boundaries. In this system the random four fermion interactions arise from the coulomb interactions, projected onto the zeroth Landau level. We argue that the model realized is actually a low-rank SYK, with extensive rank and of Class IV.

The rest of the paper is organized as follows. Sec. II defines the model and provides some preliminary discussions, including the Schwinger-Dyson equations. Sec. III discuss the low and high rank limits of the sub-extensive and super-extensive rank regimes. Sec. IV mark the start of our discussion of the extensive rank regime: the four universality classes are introduced, and the correlation function of each class is studied. Sec. V presents both the normal modes are to be integrated out: $S_f = S_f + S_b$, where

$$S_f = \frac{N}{2} \sum_{\omega_f} [\Sigma(\omega_f) - \ln(-i\omega_f - \Sigma(\omega_f))]$$

$$S_b = \frac{1}{2\beta} \sum_{n,\omega_b} \left(1 - \lambda_n|G|^2(\omega_b)\right)|\phi_n(\omega_b)|^2$$

where $G$ and $\Sigma$ are the fermion propagator and self-energy, $\omega_f/\omega_b$ are fermion/boson Matsubara frequencies, and $|G|^2(\omega_b)$ is $G(\tau)^2$ in frequency domain.

In $S_f$, the boson mode $\phi_n(\omega_b)$ is governed by a quadratic potential, which cannot be unstable:

$$1 - \lambda_n|G|^2(\omega_b) > 0.$$ (9)

The mode $\phi_n(\omega_b)$ becomes condensed when equality is attained above, in the thermodynamic limit. Note that since $0 \leq |G|^2(\omega_b) \leq |G|^2(\omega_b = 0)$ for all $\omega_b$, only the modes with $\lambda_n = \lambda_{\text{max}}$ and $\omega_b = 0$ can condense.

To proceed further, we separate the condensed and normal modes:

$$S_b = S_{b,N} + S_{b,C}.$$ (10)

The normal modes are to be integrated out:

$$S_{b,N} = \frac{N}{2\beta} \int \sum_{\omega_b} \ln(1 - \lambda_n|G|^2(\omega_b))\rho(\lambda) d\lambda$$ (11)

where the sum $\sum'$ excludes the condensed modes. The latter have macroscopic occupation and can be treated classically:

$$S_{b,C} = N \left(1 - \lambda_{\text{max}}|G|^2(\omega_b)\right) \Phi,$$ (12)

where $\Phi := \frac{1}{N\beta} \sum_{n,\lambda_n = \lambda_{\text{max}}} |\phi_n(\omega_b = 0)|^2$.
\begin{equation}
G(\omega_f) = \frac{1}{-i\omega_f - \Sigma(\omega_f)} \tag{14a}
\end{equation}
\begin{equation}
G_\lambda(\omega_b) = \frac{1}{1 - \lambda[G^2](\omega_b)} \tag{14b}
\end{equation}
\begin{equation}
\Sigma(\tau) = 2\gamma G(\tau) \int d\lambda G_\lambda(\tau)\rho(\lambda) d\lambda + 2\lambda_{\text{max}} \Phi G(\tau) \tag{14c}
\end{equation}
\begin{equation}
\Phi = 0 \text{ or } 1 - \lambda_{\text{max}} [G^2](0) = 0. \tag{14d}
\end{equation}

Above, we denote by \( G_\lambda \) the propagator of normal bosons with \( \lambda_n = \lambda \). The large-\( N \) action and the SD equations can be summarized by the Feynman diagrams in Fig. 2 and will be the starting point of all subsequent analyses.

III. NON-EXTENSIVE RANKS

In this section, we review the cases where the rank \( R \) is either much smaller or much larger than \( N \). Although their physics are known from previous works, the analysis will provide useful insights to the study of the extensive rank regime.

A. Sub-Extensive Ranks

Let us first consider the regime of sub-extensive ranks, where the rank \( R \ll N, \gamma \to 0 \). In this regime, the nontrivial behavior of the model is completely determined by boson condensation. Indeed, the fermion self energy has only a condensate contribution, as (14c) reduces to

\begin{equation}
\Sigma = 2\lambda_{\text{max}} \Phi G.
\end{equation}

Consequently, the only way to obtain a nontrivial solution is to let \( \lambda_{\text{max}} > 0 \), which we shall assume in the rest of this subsection.

Then, the trivial solution \( \Phi = 0 \), \( G(\tau) = \text{sign}(\tau)/2 \) is valid as long as \( T > T_c := \lambda_{\text{max}}/4 \). At \( T_c \), a boson condensation transition takes place. Below that, \( \Phi > 0 \) and we have

\begin{equation}
G(\omega_f) = \frac{2i}{\omega_f + \text{sign}(\omega_f) \sqrt{8\lambda_{\text{max}} \Phi + \omega_f^2}}.
\end{equation}

In turn, the value of \( \Phi \) is determined by \( \Phi > 0 \) for any \( T < T_c \). At low temperatures, \( G(\tau) \) has a power-law decay with a SYK \(_2\) (free fermion) exponent:

\begin{equation}
|G(\tau)| \sim \frac{1}{|\tau|^{2\Delta}} \quad \Delta = \frac{1}{2},
\end{equation}

B. Super-Extensive Ranks

Now let us consider the super-extensive rank regime. It is convenient to redefine how \( R \) scales with \( N \) as follows:

\begin{equation}
R = \gamma N^\alpha, \quad \alpha > 1, \quad \gamma = O(1).
\end{equation}

The random couplings \( u_{ij}^{(n)} \) should also be normalized differently:

\begin{equation}
u_{ij}^{(n)} u_{kl}^{(m)} = \frac{1}{N^\alpha} \delta_{ijkl} \delta_{nm}, \quad a = \frac{\alpha + 3}{2} > 2.
\end{equation}

The last relation will turn out necessary and sufficient to ensure an extensive free energy for \( \alpha > 1 \). Indeed, the fermionic action \( \mathcal{S}_f \) is intact, and in the bosonic one \( \mathcal{S}_b \), \( \lambda_n[G^2](\omega_b) \) is replaced by \( \lambda_n[G^2](\omega_b) N^{2-a} \ll 1 \) at large-\( N \) since \( a > 2 \). So, no condensation is possible. Moreover, we can expand the normal boson action:

\begin{equation}
\mathcal{S}_b = \mathcal{S}_{b,N} = \frac{1}{2\beta} \sum_{n,\omega_b} \ln \left( 1 - \lambda_n N^2 - [G^2](\omega_b) \right)
= -\frac{1}{2} \sum_{\ell=1}^\infty \sum_{\omega_k,n} \frac{1}{\ell} \lambda_n^\ell [G^2](\omega_b) N^{2-a} \ell
\end{equation}

and keep only the first non-trivial term. That turns out to be \( \ell = 2 \), because the \( \ell = 1 \) term is a constant \( E_0 \beta = -\beta N^{2-a} \sum_n \lambda_n/8 \). Therefore, we have

\begin{equation}
\mathcal{S}_b = -\frac{N \beta}{2} \mu_2 \int_\tau^1 \frac{1}{4} G(\tau)^4, \quad \mu_2 = 2\gamma \int \rho(\lambda) \lambda^2 d\lambda.
\end{equation}

This action is identical to that of the standard SYK\(_q=4\) model \[10\] \[34\]. Thus, at low temperatures, we have the well-known conformal solution \[4\]

\begin{equation}
|G(\tau)| \sim \frac{b}{|\tau|^{2\Delta}}, \quad b = \frac{1}{\sqrt{8\pi \mu_2}}, \quad \Delta = \frac{1}{4}.
\end{equation}

IV. EXTENSIVE RANKS

In the last section, we have shown that the low-rank SYK models reduce to SYK\(_4\) at super-extensive ranks, and SYK\(_2\) (or trivial) at sub-extensive ranks. To look for novel low-temperature behaviors, we shall focus on the regime of extensive rank, \( R = \gamma N \), and resume the normalization of Sec. III.
A. Crossover temperature

We start by determining the temperature regime where we expect new physics, as a function of rescaled rank \( \gamma = R/N \).

When \( \gamma \) is large, we expect the model to reduce to SYK\(_q\) in some temperature regime, by consistency with Sec. III B above. To find the crossover temperature, we apply the results there, extrapolated to the extensive regime \( \alpha = 1 \) and \( \alpha = 2 \). The truncation of the Taylor series in (20) is valid if and only if \( |\lambda_n| [G^2(\omega_b) | \ll 1 \) for any \( n \) and \( \omega_b \). This is equivalent, assuming (22), to

\[
T \gg T_\ast = \sqrt{\mu_2} \exp \left[ -\frac{2\pi \mu_2}{\max_n |\lambda_n|} \right]
\]

where \( \mu_2 \) is defined in (21). For a fixed distribution \( \rho(\lambda) \), \( T_\ast \) depends on the rank in a stretched exponential fashion:

\[
T_\ast = \sqrt{c_1 \gamma e^{-\sqrt{c_2 \gamma}}}
\]

where \( c_1 \) and \( c_2 \) depend on \( \rho(\lambda) \).

In summary, the model is governed by an unstable SYK\(_q\) fixed point at intermediate temperatures \( \sqrt{\mu_2} \gg T \gg T_\ast \). This transient regime exists only for large \( \gamma \). For \( \gamma \leq 1 \), there is only one crossover at \( T_\ast \sim \max_n |\lambda_n| \), from the trivial UV fixed point directly to a novel IR fixed point. The rest of the section will be devoted to characterizing the latter.

B. Four-fold way

The low-temperature behavior in the extensive-rank regime depends strongly on the shape of the distribution \( \rho(\lambda) \). To prepare for a systematic study, we shall describe and motivate the classification table I.

For that, let us recall the SD equation (14c). The integral over boson modes on the RHS can be split into a condensate part (\( \lambda = \lambda_{\max}, \omega_b = 0, G_\lambda = \infty \)) and a normal part (\( G_\lambda \ll \infty \)), as follows:

\[
\Sigma(\tau) = 2\pi G(\tau) F(\tau) + 2G(\tau) \lambda_{\max} \Phi
\]

\[
F(\omega_b) = f([G^2(\omega_b)], f(y) := \int \frac{\lambda \rho(\lambda)}{1 - \lambda y} d\lambda.
\]

Above, \( F \) is a weighted sum of the propagator of non-condensed bosons. It depends on \( \rho(\lambda) \) via \( f(y) \). The classification of \( \rho(\lambda) \) will be based on the analytical properties of \( f(y) \).

First, Class IV is defined by \( \lambda_{\max} \leq 0 \). Such distributions are clearly distinct from the rest in that \( f(y) \) is analytical on the positive real axis \([0, +\infty)\). On the other hand, when \( \lambda_{\max} > 0 \), \( f(y) \) increases with \( y \) and becomes maximal at the singularity at

\[
y_* := 1/\lambda_{\max}.
\]

The nature of the singularity is completely determined by the right edge of \( \rho(\lambda) \) near \( \lambda_{\max} \). There are three possibilities/classes:

I. \( \lim_{y \to y_*} f(y) < +\infty \),

II. \( \lim_{y \to y_*} f(y) = +\infty \) but \( f(y) \ll 1/(y_* - y) \),

III. \( f(y) \sim c_0/(y_* - y), y \to y_* \), \( c_0 \in (0, 1) \).

In terms of the right edge of \( \rho(\lambda) \), these classes are exemplified by the following (see Table I for a cartoon):

I. \( \rho \sim (\lambda_{\max} - \lambda)^\eta, \eta > 0 \) (vanishing edge)

II. \( \rho \sim (\lambda_{\max} - \lambda)^\eta, -1 < \eta \leq 0 \), (non-vanishing edge)

III. \( \rho = c_0 \delta(\lambda - \lambda_{\max}) + \ldots \) (delta peak)

Note that, although the above example distributions do not exhaust all the possibilities (there can be log corrections to power laws), the classification in terms of \( f \) is exhaustive.

Let us provide some further rationale for Class I-III. Class I is distinguished by \( f(y \to y_*) < +\infty \), which is a necessary condition for condensation at finite \( T \). Indeed, recall from (14c) that condensation requires \( [G^2(\omega_b) = 0 = 1/\lambda_{\max} = y_* \), and thus \( F(\omega_b) = f(y_*) \) must be finite. Thus, finite-\( T \) condensation only happens in Class I, and not in Classes II and III. Amongst the latter two, Class III is distinguished by a macroscopic degeneracy of the softest boson modes. This prevents condensation at even zero-\( T \), making Class III rather resemble Class IV. In contrast, Class II is closer to Class I; as we shall see later, the softest boson modes do condense at zero-\( T \).

C. Class III & IV: SYK\(_q\)-like

The low-temperature behavior of these classes have been partially studied in Refs. [11] and [32, 33], respectively. It was shown that the fermion Green’s function \( G(\tau) \) becomes conformal invariant at low temperatures:

\[
G(\tau) = A \mathrm{sign}(\tau)|\tau|^{-\Delta}, 1 \ll |\tau| \ll \beta,
\]

where \( \Delta \) depends continuously on the rescaled rank \( \gamma = R/N \).

Let us review how to find such a solution in Class IV. We claim that the SD equations have the following conformal approximations:

\[
-G(\omega_f) \Sigma(\omega_f) = 1 \quad (30a)
\]

\[
[G^2(\omega_b)] F(\omega_b) = 1 \quad (30b)
\]

\[
\Sigma(\tau) = 2\pi F(\tau) G(\tau) \quad (30c)
\]

Above, \( 30a \) comes from \( G(\omega_f)^{-1} = -i\omega_f - \Sigma(\omega_f) \) by dropping \( i\omega_f \) at low frequencies, while \( 30b \) assumes \( [G^2(\omega_b) \to \infty \) as \( |\omega_b| \to 0 \), which will be verified below, and which implies

\[
F(\omega_b) = f([G^2(\omega_b)] \sim \frac{1}{[G^2(\omega_b)]} \int \rho(\lambda) d\lambda = \frac{1}{[G^2(\omega_b)]}
\]
by (30c) is implied by (29) and the absence of condensation. Using standard Fourier transform formulae, one finds that eqs. (30) are satisfied by (29) if the scaling dimension \( \Delta \) satisfies
\[
\gamma = \frac{(2\Delta - 1)(\sec(2\pi \Delta) - 1)}{8\Delta - 2}, \quad \Delta \in (0,1/4),
\]
see Appendix 2 for details, and Fig. 3 for a numerical check. The fact that \( \Delta < 1/4 \) ensures the assumption behind (30b) above. In the limits \( \gamma \to 0 \) and \( \gamma \to +\infty \), \( \Delta \) tends to 0 (the SYK \( \gamma \to \infty \) value) and 1/4 (the SYK4 value), respectively.

A nice byproduct of the above analysis is that all boson propagators are equal in the scaling regime:
\[
\lambda G_\lambda(\tau) = F(\tau) \sim |\tau|^{−2\Delta_b}
\]
where
\[
\Delta_b = 1 - 2\Delta
\]
is the bosonic scaling dimension. Furthermore, the approximate SD equations (30) enjoy a reparametrization symmetry, just as those of SYK4. Upon rewriting (30a) and (30b) in time domain, it is readily checked that any reparametrization \( \tau \to f(\tau) \), transforms a solution (30) to another one in the following way:
\[
\begin{align*}
G(\tau, \tau') &\to [f'(\tau)f'(\tau')]^{\Delta_b}G(f(\tau), f(\tau')) \\
F(\tau, \tau') &\to [f'(\tau)f'(\tau')]^{\Delta_b}F(f(\tau), f(\tau')) \\
\Sigma(\tau, \tau') &\to [f'(\tau)f'(\tau')]^{\Delta_b}\Sigma(\tau, \tau').
\end{align*}
\]
Adapting the argument of Ref. [7], one may show that the reparametrization symmetry is broken explicitly and spontaneously, giving rise to a Schwartzian action of soft modes. A consequence of this broken symmetry [4, 5, 7] is the maximal out-of-time order correlator growth (Lyapunov exponent), which we will show in Sec. VII below.

The situation in Class III is formally similar, although physically different. Indeed, the equations (30a) and (30c) still hold (the latter does so because of no condensation), while (30b) becomes
\[
(1 - \lambda_{\max}G^2(\omega_b))F(\omega_b) = c_0\lambda_{\max},
\]
according to (28c). By a similar analysis, we find a conformal solution such that \( \lambda_{\max}(G^2(\omega_b)) = 1 - C|\omega_b|^{4\Delta-1} \), where \( \Delta \) is determined by \( c_0 \) as:
\[
\gamma c_0 = \frac{(2\Delta - 1)(\sec(2\pi \Delta) - 1)}{8\Delta - 2}, \quad \Delta \in (1/4, 1/2),
\]
which we plot and test in Fig. 3. Unlike Class IV, as \( \gamma \) decreases to 0, \( \Delta \to 1/2 \) increases to the SYK2 value. Concerning the bosons, only the soft modes, with \( \lambda_n = \lambda_{\max} \), have a power-law propagator satisfying (32) and (33).

D. Class I and II: almost Fermi liquids

The analysis of Class I and II involves the shape of the distribution \( \rho(\lambda) \) to a greater extent. For simplicity, we shall focus on the following family of power-law edge singularities: \( \rho(\lambda) \sim (\lambda_{\max} - \lambda)^\eta \), so that
\[
f(y) \sim \begin{cases} (y_* - y)^\eta & \eta < 0 \\
y_* - C(y_* - y)^\eta & \eta > 0 \end{cases},
\]
where \( y_* \) is the quantum Lyapunov exponent.
We shall assume $0 < \eta < 1$ in Class I and $-1 < \eta < 0$ in Class II, although our analysis can be easily applied to other situations, e.g., a uniform distribution $\rho(\lambda) = \text{const}$, which is a marginal case of Class II with $f(y) \sim -\ln(y_s - y)$.

In contrast to Class III and IV, the fermion scaling dimension is always

$$\Delta = 1/2.$$  \hspace{1cm} (38)

This can be seen by a simple argument: any Class I/II distribution can be approached from Class III, by taking a $c_0 \to 0$ limit. Then (36) implies $\Delta \to 1/2$. Physically, however, Class I and II are far from being the limit cases of Class III. Let’s discuss them in turn.

For Class I, a boson condensation must form at low temperature. To see why it must be so, recall that the singularity of $\Sigma$ at low frequencies. The situation of Class II is similar, except for the following subtlety: condensation is impossible at finite temperature. To see why it must be so, recall that the singularity of $\Sigma$ at low temperatures, and $\beta \Delta \to 0$ implies that $\Sigma \to |\lambda|^{-1/2}$ at low frequencies.

Now, a consequence of the condensation is that $\Sigma(\omega) \sim |\omega|^{1/2}$, which must come from the condensate contribution $\Sigma_c(\omega) \sim |\omega|^{1-2}$, which is dominant.

Moreover, $\Sigma(\omega) \sim |\omega|^{1/2}$, which must come from the condensate contribution $\Sigma_c(\omega) \sim |\omega|^{1-2}$, which is dominant.

Above, we redefined $\Sigma_c$ and $\Sigma_N$ in terms of the effective condensate $\hat{\Sigma}$. A similar argument as above shows that, at low temperature, $\hat{\Sigma}$ remains positive. This means that

$$y_s - [G^2](\omega_b = 0) \sim T^{-1/\eta} \ll T, T \to 0$$  \hspace{1cm} (42)

(since $0 > \eta > -1$) is negligibly small, so that (39) and (40) still hold (of course, the meaning of $\Sigma_N$ and the value of $\eta$ are different). So, like Class I, Class II realizes an almost Fermi liquid, with a higher quasi-particle decay rate at low frequency. The physical consequences of this will be studied below.

To close, let us discuss the spontaneous breaking of time reversal symmetry $\mathcal{T}$, which is closely related to soft boson modes. Indeed, while the Hamiltonian is even under $\mathcal{T}$, the fermion bilinears $Q_n$ are odd. Therefore, a condensed boson mode generates a term $\phi_nQ_n$, which breaks $\mathcal{T}$. It follows immediately that $\mathcal{T}$ is broken at low temperatures in Class I (and also in the sub-extensive regime). In Class II, although no condensation takes place at finite temperature, $\mathcal{T}$ is broken at zero temperature. To see this, note that the softest boson mode with $\lambda_n = \lambda_{\max}$ has the following propagator:

$$G_{\lambda_{\max}}(\omega_b = 0) = T^{1/\eta} \gg 1/T,$$

according to (12). This means that the following order parameter diverges

$$\frac{1}{\beta} \int_0^{\beta} \langle \phi_n(\tau)\phi_n(0) \rangle \, d\tau \to \infty$$

as $T \to 0$, which implies the breaking of $\mathcal{T}$ symmetry at zero temperature [10]. Repeating the analysis for Class III and IV, using the results in Sec. [IVC], it is not hard to show that $\mathcal{T}$ is unbroken even at zero temperature in both SYK$_q$ classes.

V. THERMODYNAMICS

In this section, we study the low temperature thermodynamics of the four classes of the extensive-rank regime both analytically and numerically. The free energy of the model is given by the saddle-point action $F = S_{\text{saddle}}/\beta$, where $S$ is as defined in (8a). From that, it is not hard to obtain the energy density:

$$-\frac{\beta E}{N} = \frac{1}{\beta} \Phi + \frac{1}{2} \sum_{\omega_b} [G^2](\omega_b)F(\omega_b)$$  \hspace{1cm} (43)

We shall study the low-temperature thermodynamics in both sub-extensive and extensive regimes, by a combination of analytical and numerical methods.

A. Sub-extensive ranks

The low-temperature thermodynamics can be calculated exactly in the sub-extensive regime. The only contribution to the energy is the condensate:

$$\varepsilon := E/N = -\frac{1}{2} \Phi,$$  \hspace{1cm} (44)
which is determined by (14d) and (16), rewritten as:
\[
T \sum_{k=0}^{\infty} g_\Phi(\pi T + 2\pi k T) = 1, \tag{45}
\]
where \( g_\Phi(\omega_f) := \frac{8\lambda_{\text{max}}}{(\omega_f + \sqrt{8\lambda_{\text{max}} \Phi + \omega_f^2})^2} \). \tag{46}

For small \( T \), the sum can be estimated with the Euler-McLaurin formula,
\[
1 = \int_{\pi T}^\infty \frac{d\omega_f}{2\pi} g_\Phi(\omega_f) + \frac{T}{2} g_\Phi(\pi T) - \frac{T^2}{6} g_\Phi'(\pi T) + \cdots \tag{47}
\]
\[
= \int_0^\infty \frac{d\omega_f}{2\pi} g_\Phi(\omega_f) + \frac{T^2}{12} g_\Phi'(\pi T) + \cdots \tag{48}
\]
Above, we denoted \( g' := \partial_{\omega_f} g \); in the second line, we approximated the integral \( \int_{\pi T}^\infty g \) by expanding \( g \) at \( \omega_f = \pi T \); throughout, the omitted terms \( \in O(T^3) \). Equating the first term in (48) to 1, and evaluating some integrals, we obtain
\[
\Phi = \Phi_0 + c_V T^2 + O(T^3) \text{ where} \tag{49}
\]
\[
\Phi_0 := \frac{8}{9\pi^2} \lambda_{\text{max}}, \quad c_V := \frac{\pi^2}{8\lambda_{\text{max}}}. \tag{50}
\]
Consequently, by (44), the specific heat
\[
C_V = c_V T + O(T^2). \tag{51}
\]
is linear in \( T \). Note that, only the numerical value of \( \Phi_0 \) and \( c_V \) depend on the exact form of \( g \), while \( C_V \propto T \) only depends on the fact that \( \partial_{\omega_f} g \) and \( \partial_{\Phi} g \) both exist, are continuous and nonzero whenever \( \Phi > 0 \).

B. Class I and II

We now extend the above exact analysis to Classes I and II, by making some approximations. As the method is similar for both Classes, let us explain it just for Class I in some detail.

To start, we observe that the SD equations (14a) and (25) imply that
\[
G(\omega_f) = \frac{2i}{J + \text{sign}(\omega_f) \sqrt{8\lambda_{\text{max}} \Phi + J^2}} \tag{52}
\]
where \( J := \omega_f - i\Sigma_N(\omega_f) \). \tag{53}

To make progress, we make two approximations. First, by (40), \( J \approx \omega_f \) is independent of \( \Phi \) at low frequencies.

1. We ignore the \( \Phi \) dependence of \( J \), and approximate it by its leading small \( \omega_f \) behavior. In Class I, we have \( J \sim \omega_f \) by (40).

2. We approximate the energy by \( \varepsilon \approx -\frac{1}{2} \Phi \), ignoring the \( \propto \gamma \) terms in (43).

These approximations renders the problem nearly identical to the sub-extensive case. Indeed, \( \varepsilon \approx -\Phi/2 \) is determined by the same equation (45) with the same \( g_\Phi \). Therefore, we predict that
\[
C_V \propto T \text{ (Class I)}, \tag{54}
\]
at least for small \( \gamma \).

We now apply the same approximations to Class II, while switching \( \Phi \) for \( \hat{\Phi} \) everywhere. Now, notice that since \( \eta < 0 \) in Class II, \( J \sim \vert \omega_f \vert^{1+\eta} \) by (40). Thus, \( g_{\Phi} \) is non-analytical in \( \omega_f \) at \( \omega_f = 0 \); the derivative \( g_{\Phi}'(\omega_f) \sim \vert \omega_f \vert^\nu \) is divergent as \( \omega_f \to 0 \), so that (47) now implies
\[
1 - \int_0^\infty \frac{d\omega_f}{2\pi} g_{\Phi}(\omega_f) \sim T^2 g_{\Phi}'(\pi T) \sim T^{2+\eta}. \tag{55}
\]
Consequently, we predict that the specific heat is anomalously large at low-\( T \):
\[
C_V \sim T^{1+\eta} \text{ (Class II)}. \tag{56}
\]

C. Numerical Results

We now compute the temperature dependence of entropy in all four classes in the extensive rank regime, by solving the large-\( N \) SD equations numerically. Representative results are given in Fig. 4.

In Classes III and IV, the data is well described by
\[
S/N = S_0 + c_V T + \cdots, \tag{57}
\]
where the zero-temperature entropy is positive \( S_0 > 0 \). This nonvanishing zero-temperature entropy imply that
the Class III and IV models are reminiscent of the SYK$_q$ model.

In stark contrast, we find that neither Class I nor II has an extensive residual entropy, and the entropy obeys a power law

$$S/N \sim T^{-n}$$

which are consistent with the predictions (54) and (56) above, since $C_V = T\partial S/\partial T$. We computed the exponent more thoroughly, albeit for relatively small ranks, and found a good quantitative agreement with (56), see Fig. 5.

VI. OUT-OF-TIME ORDER CORRELATOR

We now study the growth of the out-of-time order correlator (OTOC):

$$\text{Tr} \left[ e^{-\beta H/4} y^{(t_1) y^{(t_2)} y^{(t_3)} y^{(t_4)}} \right] = \frac{e^{-\beta H/4}}{\text{Tr} (e^{-\beta H})}.$$  \hfill (59)

Following closely the approach of Refs [2, 3, 8], we focus on the $O(1/N)$ and exponentially growing part of the OTOC, given by the sum of a series of ladder diagrams generated by two types of ladder rungs. The ladder kernel is $K = K_b + K_f$, where (see Fig. 6):

$$K_b(t_1,\ldots,t_4) = \frac{2}{N} \sum_n G_R(t_1)G_R(t_2)G_{\lambda_R}(t_34)$$

$$K_f(t_1,\ldots,t_4) = \frac{4}{N} \sum_n \int dt_5 dt_6 G_R(t_1)G_R(t_2) \times$$

$$\lambda^2 G_{\lambda_R}(t_{35})G_{\lambda_R}(t_{46})G_{\eta_R}(t_{34})G_{\eta_R}(t_{56}).$$

$$\int dt_1 dt_2 K(t_1,\ldots,t_4)F(t_1, t_2) = kF(t_3, t_4)$$  \hfill (61)

Above, $t_{ij} := t_i - t_j$, the subscript “$R$” indicates a retarded propagator, and “$b$” a Wightman correlator [3]; both can be obtained from the Euclidean-time correlator.

We then compute the quantum Lyapunov exponent $\lambda_L$ by finding an eigenfunction

$$G_R(t) = A \frac{\sin \beta |\tau|}{\sinh \beta \sin \frac{\pi |\tau|}{\beta}}$$  \hfill (62)

The constant $A$ will drop out in the final results. The retarded and wightman correlators are obtained by analytical continuations to real time [3];

$$G_{\eta_R}(t) = A \frac{\sin \beta |\tau|}{\sinh \beta \sin \frac{\pi |\tau|}{\beta}}$$  \hfill (63)

and similarly for the bosons. Therefore, the summed terms in (60) are independent of $n$, so the sum $\sum_{n=1}^N \sum_{n=1}^N$ can be simply replaced with $R/N = \gamma$:

$$K_b(t_1,\ldots,t_4) = 2\gamma G_R(t_1)G_R(t_2)F_{\eta_R}(t_{34})$$

$$K_f(t_1,\ldots,t_4) = 4\gamma \int dt_5 dt_6 G_R(t_1)G_R(t_2)$$

$$F_{\eta_R}(t_{35})F_{\eta_R}(t_{46})F_{\eta_R}(t_{34})G_{\eta_R}(t_{56}).$$

(a) 1 2 2 2 2 2 2 . . .
(b) 1 2 2 2 2 2 2 2 2 . . .
(c) $K_b = \begin{array}{c|c}
\end{array}$ $K_f = \begin{array}{c|c}
\end{array}$

FIG. 5. Numerical test of the prediction [56] for Class II. Main plot: Entropy density $S/N$ as function of temperature $T$, with $f(y) = y(1-y)^\gamma$ for $\gamma = -0.8, -0.7, \ldots, -0.2$ (top to bottom), and $\gamma = 0.2$ (except that $\gamma = 0.1$ for $\gamma = -0.8$). The dots are from numerical solution of the SD equation. The dashed lines are best fits to a power law $S/N = cT^\nu$. Inset: the fit exponent $\nu$ (dots, same color code as main plot), compared to the prediction (56) (solid line).

FIG. 6. (a,b) Examples of ladder diagrams contributing to the out-of-time order correlator [59]. Disorder lines are omitted for display. (c) The kernels generating the ladders, with disorder lines. All propagators are dressed.
The RHS of the above equations involve only known conformal propagators, and will be analyzed exactly.

Before doing so, we argue that \( 66 \) holds for Class III as well, provided we replace \( \gamma \to \gamma_0 \gamma \) [note that \( \Delta \) is also a function of \( \gamma_0 \gamma \) instead of \( \gamma \), see \( 30 \)]. This is because the sum over bosons in \( 60 \) are dominated by the softest ones, with \( \gamma_0 = \gamma \). There are \( \gamma_0 \gamma N \) of those, and their propagator still satisfies \( 63 \).

We now look for eigenfunctions of \( K = K_b + K_f \) with the following Ansatz \( 24 \) [4]:

\[ F(t_1, t_2) = e^{-\frac{\pi}{\beta} t_1} \left[ 1 - \frac{2\Delta - h}{\cosh \frac{\pi}{\beta} t_1} \right]^{2\Delta - h} , \]

where the Lyapunov exponent is related to \( \gamma \) by \( \lambda_L = -2h\pi T \). By a straightforward but tedious calculation (going back and forth between the time and frequency domains), we can show that \( F \) is indeed an eigenfunction of both \( K_b \) and \( K_f \), with the following eigenvalues:

\[ k_b(h) = \frac{(1 - 2\Delta) \sin(2\pi\Delta) \Gamma(1 - 2\Delta) \Gamma(2\Delta - h)}{\pi \Gamma(-h - 2\Delta + 2)} , \]

\[ k_f(h) = \frac{2(8\Delta^2 - 6\Delta + 1) \sin(2\pi\Delta) \sin(4\pi\Delta)}{\pi^2 \Gamma(-h - 2\Delta + 2) \Gamma(4\Delta - h)} \times (1 - 2\Delta)^2 \Gamma(4\Delta - 1)^2 \Gamma(-h - 4\Delta + 2) \Gamma(2\Delta - h) . \]

The eigenvalue of the total kernel \( k(h) := k_f(h) + k_b(h) \) has a remarkable property: for any \( \Delta \in (0, 1/2) \), \( k(h) = 1 \) if and only if \( h = 1 \). As a consequence, the low rank SYK model in the extensive regime with Class III or IV distributions is maximally chaotic:

\[ \lambda_L = 2\pi T \quad \text{Class III, IV, } T \ll \max_n |\lambda_n| . \]

**B. Class I & II: non-maximal chaos**

We now briefly discuss the cases of Class I and II. The key difference of these classes is that, determining \( \lambda_L \) requires going beyond the fermion scaling dimension \( \Delta \). Indeed, the kernel eigenvalues \( 68 \) satisfy

\[ \lim_{\Delta \to 1/2} k_b(h) = 1 , \quad \lim_{\Delta \to 1/2} k_f(h) = 0 , \]

for any \( h \). So \( k(h) = k_b(h) + k_f(h) \) also tends to 1 in that limit (this can be seen in Fig. 11), and \( \lambda_L \) cannot be determined by the above method. This situation also occurs in the Fermi-liquid phase of Ref. [8], and with the SYK_4 model in the \( q \to 2 \) limit. In all these cases, \( \lambda_L \) depends on the sub-leading terms in the propagators.

A detailed analysis along this line, which we will present in an upcoming work, leads to the following results. In Class I, The \( T \)-dependence of \( \lambda_L \) is reminiscent of the \( \omega \) dependence of the quasi-particle decay rate \( 10 \):

\[ \lambda_L \sim T^{1+\eta} , \quad 0 < \eta < 1 . \]

Therefore, Class I is more chaotic than a Fermi liquid where \( \lambda_L \propto T^2 \) [8, 35]. Naively extrapolating \( 71 \) to Class II, we would have a violation of the bound on chaos \( \lambda_L \leq 2\pi T \). Yet, a more careful analysis indicates that \( \lambda_L \propto T \), but the bound is not always saturated by the pre-factor.

**VII. AN APPLICATION**

Recently, Ref. [35] proposed an interesting realization of the SYK model in a graphene flake with irregular boundaries, using quantum Hall ferromagnetism. A strong magnetic flux \( \Phi \) is induced onto the graphene flake, creating \( \Phi/\Phi_0 \) degenerate lowest Landau levels (LL_0) in the presence of chiral symmetry. Here, \( \Phi_0 \) is a flux quanta \( hc/e \). Since the graphene boundary is irregular, the LL_0 wave-functions are pseudo-random. Hence, Projecting the Coulomb interaction onto the LL_0 then produces a disordered four-fermion interaction, which the authors of Ref. [35] claimed to be of SYK_4 nature.

Now, we argue that the realized Hamiltonian more likely a low rank SYK, in the extensive-rank regime and of Class IV. For this, let \( |\varphi_j⟩ \) be the LL_0 wave-functions, and \( c_j \) be the associated fermionic annihilation operator. Then the projected Coulomb interaction is

\[ H = \sum_{ijkl,r,r'} V(r-r') \langle r|\varphi_i⟩ \langle \varphi_j|r⟩ c_i^† c_j (r|\varphi_k⟩ \langle \varphi_l|r⟩ c_l^† c_i^† . \]

Above, \( r \) and \( r' \) runs over all the lattice sites, and \( V(r-r') \) is the Coulomb potential, which we can diagonalize as

\[ V(r-r') = -\sum_n \lambda_n U_{rn} U_{r'n} , \]

where \( U_{rn} = (n|r⟩ \in \mathbb{R} \) forms a real orthogonal matrix, and \( \lambda_n \) are the eigenvalues. Therefore,

\[ H = -\sum_n \lambda_n Q_n^2 , \]

where \( Q_n = \sum_{ij} u_{ij} c_i^† c_j , \quad u_{ij} = \sum_r \langle n|r⟩ \langle r|\varphi_i⟩ \langle \varphi_j|r⟩ \]
Note that $Q_n$ is a Hermitian fermion bilinear. At this point, if we approximate $Q_n$ by a set of independent random fermion bilinears, we will have the complex version of the low rank SYK model, with an extensive rank. Finally, the repulsive nature of Coulomb interactions implies that $\lambda_n \geq 0$ for all $n$, resulting in a Class IV distribution.

In summary, our argument reveals an additional structure in the seemingly random four-body interaction. It will be interesting to study whether there exists further relevant structures. If there are none and the realized model is indeed a Class IV, extensive-rank SYK, the goal of Ref. [35] will be still fulfilled. Indeed, as we showed above, Class IV is a maximal chaotic scrambler almost indistinguishable from SYK_q for some $q > 4$.

VIII. DISCUSSION

We have introduced and solved the low-rank SYK models, unifying and completing previous results [10, 32]. The four classes of quantum phases that the model possesses, summarized in Table I, fall into two categories. The fast scramblers of Class III and IV are equivalent to SYK_q in all aspects we have studied, although the reparametrization symmetry in Class III is worth further elaborating. On the other hand, the almost Fermi liquids of Class I and II may not have reparametrization symmetry. However, they are stable under weak quadratic perturbations (since such a term is already generated dynamically).

The fermion-boson coupling form [7] of our model generalizes the electron-phonon coupling model of Refs [32, 33] in the normal state [34, Sec. II]. These authors considered a Class III distribution of couplings $\rho(\lambda) = \delta(\lambda - \lambda_{\text{max}})$. We showed that a non-degenerate distribution will belong to Class I or II (Class IV is impossible in this setting since $\lambda_n$ is always positive), which is almost a Fermi liquid. It will be interesting to understand the instability of such a phase into the superconducting state.

Finally, our model in the extensive regime restores the physical rank of the coupling matrix in $SU(M)$ random quantum magnets away from the large $M$ limit. Our results thus suggest that the critical low-energy state of the magnet at finite $M$ is almost a Fermi liquid, probably of Class I, which contains the semi-circle law. Yet, by engineering a coupling matrix with a Class II-IV spectrum, one can still realize faster scramblers in atom-cavity settings.

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1. Large $N$ Action and Schwinger-Dyson Equations

In this section we derive the action of the low rank SYK model. We first focus on the “replica diagonal ensemble” given by the disorder averaged partition function $Z_t$ at inverse temperature $\beta$. Before we start, however, we will relax \cite{3} and \cite{4} in order to also discuss the sub-extensive and super-extensive rank regimes. We modify \cite{3} and \cite{4} to

$$R = \gamma N^\alpha + \text{sub-leading corrections},$$

and fast scrambling. [Phys. Rev. A 99, 051803 (2019)]

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$[35]$ Anh Anh Chen, R. Ian, F. de Juan, D. I. Pikulin, and M. Franz, “Quantum Holography in a Graphene Flake with an Irregular Boundary,” Phys. Rev. Lett. 121, 036403 (2018) arXiv:1802.00802 [cond-mat.str-el]

$[36]$ It is justified to call $\lambda_i$ eigenvalues because in the large-$N$ limit, $\{u_{ij}^{(n)}\}$ has the same law as $R$ orthogonal random vectors in $\mathbb{R}^{N^2}$, so $\mathbb{E}$ is effectively an eigen-decomposition.

$[37]$ The Supplemental Material, which contains Refs. [9][10], derives the large-$N$ solution to the low-rank SYK model, discusses the super- and sub-extensive regimes, provides technical details on the conformal solutions of Class III and IV, the low-rank perturbation theory for Class I and II, thermodynamics, quantum chaos, and relation to the electron-phonon model in Ref $[33]$. $[38]$ Haoyu Guo, Yingfei Gu, and Subir Sachdev, “Transport and chaos in lattice sachdev-ye-kitaev models,” Phys. Rev. B 100, 045140 (2019)

$[39]$ Andrew Lucas, “Quantum many-body dynamics on the star graph,” arXiv:1903.01468 (2019).

$[40]$ Jian-Min Liu and Gerhard Müller, “Finite-temperature dynamics of the equivalent-neighbor xyz model,” Phys. Rev. A 42, 5854–5864 (1990)
\( \alpha \in [0, 2] \), and \( \gamma \) is an order unity constant. Note that the parameter \( a \) controls the normalization of the Hamiltonian. Requiring extensive energy fluctuation at infinite \( T \), we can find a relation between \( a \) and \( \alpha \):

\[
\text{Tr}[H^2] - \text{Tr}[H]^2 \sim N^{4-2\alpha + \alpha}.
\]

(75)

The fluctuation scales extensively with \( N \) provided

\[
a = (\alpha + 3)/2.
\]

(76)

In particular, we have \( a = 3/2 \) for a finite rank interaction \( \alpha = 0 \); For a near full rank interaction \( \alpha = 2 \), \( a = 5/2 \). For the extensive scaling in the main text, we have \( \alpha = 1 \), \( a = 2 \). In general, however, normalization of the Hamiltonian at infinite \( T \) may be different from that at finite \( T \). As we will come to later, for sub-extensive ranks \( a = 2 \) in order to have an extensive free energy at finite temperatures.

Having a rough idea of the normalization, let us get back to the large \( N \) action.

\[
\mathcal{Z} = \int [D\gamma] e^{-\int d\tau \mathcal{L}}, \quad \mathcal{L} = \sum_j \gamma_j \dot{\gamma}_j + H.
\]

(77)

As mentioned in the main text, after a Hubbard-Stratonovich (HS) decoupling, the Lagrangian is given as the following:

\[
\mathcal{L} = \sum_j \gamma_j \dot{\gamma}_j + \sum_n \left( \lambda_n^j \phi_n \dot{Q}_n + \frac{\phi_n^2}{2} \right).
\]

(78)

Then, averaging over disorder results in the bi-local effective action

\[
S = \int_\tau \left( \sum_j \gamma_j \dot{\gamma}_j + \sum_n \frac{1}{2} \phi_n^2 \right) - \frac{1}{2} \int_{\tau,\tau'} \sum_{nij} N^{-a} \lambda_n (\phi_n \dot{r}_i \gamma_j)(\tau)(\phi_n \dot{r}_i \gamma_j)(\tau').
\]

(79)

We now introduce as usual the Green function \( G(\tau, \tau') = \frac{1}{N} \sum_j \gamma_j(\tau) \gamma_j(\tau') \) and impose the relation by adding the lagrange multiplier

\[
N \Sigma(\tau, \tau') \left( G(\tau, \tau') - \sum_j \gamma_j(\tau) \gamma_j(\tau') \right)
\]
to the action, where \( \Sigma \) is the self-energy. Integrating out the fermions results in large-\( N \) actions in the main text.

### 2. Details on the Scaling Analysis of Class III & IV

In this section we derive (31) and (36). The main tool is the following Fourier transform formulae:

\[
\int e^{i\tau \omega |\tau|^{-a} \text{sign}(\tau) d\tau = -2i \text{cos} \left( \frac{\pi a}{2} \right) \Gamma(1-a) |\omega|^{a-1}, \quad \int e^{i\tau \omega |\tau|^{-a} d\tau = 2 \text{sin} \left( \frac{\pi a}{2} \right) \Gamma(1-a) |\omega|^{a-1}}
\]

It is important to notice that, when we apply the above formulas to some \( g(\tau) \) that is described by a power law only for large \( \tau \), \( g(\omega) \) will be given by the RHS plus a constant that depends on the UV details.

Let us look for the conformal solution

\[
G(\tau) \sim A \text{sign}(\tau) \tau^{-2\Delta}
\]

that is compatible with the SD equations with appropriate approximations that make everything a power law. In all cases, we make the standard approximation \( G(\omega) = -1/\Sigma(\omega) \). Note that it is crucial to keep the pre-factors (the power-laws alone do not constrain \( \Delta \)). For (31), we also approximate \( f(y) \) to be \( y^{-1} \). Straightforward computations yield

\[
G(\omega) \sim 2iA \Gamma(1-2\Delta) \cos(\pi \Delta)\omega^{2\Delta-1}, \quad [G^2](\omega) \sim 2iA^2 \Gamma(1-4\Delta) \sin(2\pi \Delta)\omega^{4\Delta-1}, \quad \Sigma(\omega) \sim \frac{-2i\gamma \cot(2\pi \Delta) \cos(\pi \Delta) \Gamma(2-4\Delta) \Gamma(2\Delta-1)}{A \pi} \frac{\Gamma(1-4\Delta)}{\Gamma(1-\Delta)} \omega^{1-2\Delta}
\]
at low frequency or long time. Imposing \( G(\omega)\Sigma(\omega) = -1 \) gives (31); the condition \( \Delta < 1/4 \) ensures \( |G^2|/\omega) \rightarrow 0 \) as \( \omega \rightarrow 0 \), justifying the approximation of \( f(y) \) by \( y^{-1} \).

The case of (30) is similar. \( f(y) \) is approximated by \( c_0(y_\ast - y)^{-1} \) where \( y_\ast = 1/\lambda_{\max} \) is the nearest positive singularity of \( f \). To approximate this, we look for solutions such that \( \Phi = 0 \) (no condensate) and that \( |G^2|/\omega) \rightarrow y_\ast \) as \( \omega \rightarrow 0 \) (this constant value depends on the UV details of \( G \)); then \( y_\ast - |G^2|/\omega) \) is a power-law that only depends on the IR limit of \( G \). With this in mind, the actual computation is almost the same as for (31) above.

The condition \( \Delta > 1/4 \) ensures that \( |G^2|/\omega) - |G^2|(0) \sim |\omega|^{1/4} \) is vanishing.

We provide some details on Fig. in the main text. For each data point, we numerically solve the SD equations for \( \beta \in [10^2,10^3] \) and extract \( \Delta \) as follows: for each \( \beta \), we compute the minimum of the log derivative \( \Delta_\beta = \Delta + a/\beta + b/\beta^2 \). The errors are comparable to the marker size.

3. A Related Boson-Fermion Model

In this section, we consider a variant of the low-rank SYK model, which allows us to make connection with Ref. [32, 33]. As aforementioned, the four-fermion interactions of low-rank SYK model can be equivalently mediated by interactions with “boson modes” that do not have a kinetic term see (1). We now consider the effect of modifying the action by making the free boson action more “realistic”:

\[
\frac{1}{2} \int d\tau \phi_n(\tau)^2 \rightarrow \frac{1}{2} \int d\tau \phi_n(\tau) \left[ m^2 - \partial^2_\tau \right] \phi_n(\tau),
\]

(80)

where \( m > 0 \). We shall focus on the extensive rank regime.

Following Appendix 1, one can show that only the bosonic action (8c) is altered:

\[
S_b = \frac{1}{2} \sum_{n,\omega_b} \left( \omega_b^2 + m^2 - \lambda_n(G^2)(\omega_b) \right) |\phi_n(\omega_b)|^2,
\]

(81a)

Integrating out the non-condensed bosons and adding the condensate contribution leads to

\[
S_b = N \frac{\gamma}{2} \int \rho(\lambda) \omega_b^2 \ln(m^2 + \omega_b^2 - \lambda(G^2)(\omega_b)) d\lambda + \frac{N\beta}{2} \Phi(m^2 - |G^2|(0)\lambda_{\max}),
\]

(82)

where the condensate fraction \( \Phi \) is still defined by (13) as only the zero-frequency modes can condense; \( \Phi > 0 \) if \( \lambda_{\max}|G^2|(0) = m^2 \). Among the Schwinger-Dyson equations, only the one involving the summed boson propagator \( F \) is changed:

\[
F(\omega_b) = \int \frac{\rho(\lambda)\lambda}{m^2 + \omega_b^2 - \lambda(G^2)(\omega_b)} d\lambda
\]

(83)

Although the relation between \( F(\omega_b) \) and \( |G^2|/\omega_b \) can no longer be encoded in a function \( f(y) \), the quantum critical behavior found in the main text, summarized in Table 1, will remain essentially intact. This is because in any case, the low-frequency singularity of \( |G^2|/\omega_b \) has a power law \( |\omega_b|^{1-D-1} \gg \omega_b^2 \) (as \( D < 1/2 \)), so we can drop the term \( \omega_b^2 \) in (83) without affecting the low-frequency behavior. Then it is not hard to check that in Classes IV and III, the critical exponent \( \Delta \) is still governed by (31) and (36), respectively, whereas \( \Delta = 1/2 \) in Classes I and II: the whole low-rank perturbative theory carries through.

On the other hand, the super-extensive rank case needs more care. Restoring the \( N^{2-a} \) factors in (83) and expanding around \( \lambda = 0 \) gives (Although \( N \) is originally the system size, it is more appropriate here to view it as a finite large parameter with which we take the high-rank limit from the extensive rank regime.)

\[
F(\omega_b) = F_1(\omega_b) + F_2(\omega_b) + \ldots
\]

\[
= \frac{\mu_1 N^{2-a}}{m^2 + \omega_b^2} + \frac{\mu_2 N^{2(2-a)}}{(m^2 + \omega_b^2)^2} |G^2|/\omega_b + \ldots
\]

(84)

where \( a > 2 \) and \( \mu_\ell = \int \rho(\lambda)\lambda^\ell d\lambda \). The self energy has a similar expansion:

\[
\Sigma(\tau) = \sum_{\ell=1}^\infty \Sigma_\ell(\tau) = \sum_{\ell=1}^\infty 2\gamma N^{\alpha-1} F_\ell(\tau) G(\tau)
\]

(85)
where $\alpha > 1$. Again, we want to determine the relation between $\alpha$ and $a$ to ensure the correct thermodynamics when $N \to \infty$.

Unlike in Section III B, the term $\ell = 1$ can no longer be ignored, and the $\ell = 2$ term is not exactly $q = 4$ SYK anymore. However, those do not affect the low temperature limit [32, 33]. Indeed, the extra factor $1/(m^2 + \omega^2_b)$ in $F_2$ does not change the low-frequency behavior of $[G^2](\omega_b)$. For the $\ell = 1$ term, (85) and (84) implies

$$
\Sigma_1(\tau) \leq \gamma F_1(\tau) = N^{1-a+\alpha} \gamma \mu_1 \frac{e^{-|\tau|m}}{2m}
$$

decays exponentially. Therefore, if we adopt the scaling $\alpha = 2a - 3$, then the $\ell = 1$ term will become subdominant when $\tau m \gg \frac{(\alpha-1)}{2} \ln N$. Meanwhile, at intermediate temperature, the model is dominated by the $\ell = 1$ term; this is reminiscent of the (un-stable) “impurity” fixed point in Ref. [33].