Metrics of quantum states

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Abstract

In this work we study metrics of quantum states, which are natural generalizations of the usual trace metric and Bures metric. Some useful properties of the metrics are proved, such as the joint convexity and contractivity under quantum operations. Our result has a potential application in studying the geometry of quantum states as well as the entanglement detection.

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1. Introduction

The trace metric and the Uhlmann–Jozsa fidelity play important roles in quantum information theory [1]. Given two quantum states $\rho$ and $\sigma$, the trace metric is given by

$$d_t(\rho, \sigma) = \frac{1}{2} \text{tr}|\rho - \sigma|,$$

and the Uhlmann–Jozsa fidelity reads

$$F(\rho, \sigma) = \text{tr}\sqrt{\frac{1}{2}\rho \sigma \frac{1}{2}}.$$

Moreover, based on the Uhlmann–Jozsa fidelity the Bures metric is defined as

$$d_B(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}.$$

The trace metric and the Bures metric are two metrics frequently used in quantum information theory. Apparently, they look quite different in comparison with equations (1) and (3). However, a family of metrics is proposed in [2] such that these two important metrics can be expressed in a unified form. The definition given in [2] is as follows.

**Definition 1.** Let $\rho$ and $\sigma$ be quantum states; then, a family of metrics can be defined as

$$d_p(\rho, \sigma) = \sup \left(\sum_{k=1}^{N} \left(\frac{|\text{tr}(\rho P_k)|^2 - |\text{tr}(\sigma P_k)|^2|^p\right)^{\frac{1}{p}}\right).$$
where $p$ is a fixed positive integer, the supremum is taken over all finite families $\{P_k : k = 1, 2, \ldots, N\}$ of mutually orthogonal projectors and $\sum_{k=1}^{N} P_k = I$. We call the projectors $P_k$ attaining the supremum in equation (4) as the optimal projectors.

It has been shown in [2] that when $p = 1$, the metric $d_1(\rho, \sigma) = 2d_1(\rho, \sigma)$, i.e. $d_1(\rho, \sigma)$ equals two times the trace metric; and when $p = 2$, the metric $d_2(\rho, \sigma)$ equals exactly the Bures metric. Thus, equation (4) unifies the trace metric and the Bures metric in a common frame.

In this paper, we advance the study of the metrics $d_p(\rho, \sigma)$ in [2] and also present a family of the related metrics $D_p(\rho, \sigma)$. Some useful properties of the metrics are proved, such as the joint convexity and contractivity under quantum operations as well as the majorization relation. Our result has a potential application in studying the geometry of quantum states as well as the entanglement detection.

2. Contractivity and joint convexity of the metric $d_p(\rho, \sigma)$

Let $\rho$ be a quantum state. A positive-operator-valued measurement (POVM) is defined as a set of non-negative, Hermitian operators $E_k$ which are complete in the sense that $\sum_{k} E_k = I$, while a projective measurement (PM) requires that $E_k$ are all projectors [3]. In this section, we study whether the metric $d_p(\rho, \sigma)$ is contractive under quantum operations, which are completely positive trace preserving (CPTP) maps. We have the following theorem.

**Theorem 1** (Contractivity of the metric under CPTP map). The metric $d_p(\rho, \sigma)$ is contractive under quantum operations for all $p$. That is, suppose $T$ is a CPTP map, and $\rho, \sigma$ are density operators; then, we have the following inequality:

$$d_p(T(\rho), T(\sigma)) \leq d_p(\rho, \sigma).$$

**Proof.** Note that $M_n$ becomes a Hilbert space with inner product $\langle X, Y \rangle := \text{tr}(XY^*)$, $X, Y \in M_n$. A linear map $T$ induces its adjoint map as $\langle T(X), Y \rangle = \langle X, T^*(Y) \rangle$. If $T$ is a positive map, then its adjoint map $T^*$ is also a positive map. The trace preserving property of $T$ means that $T^*$ is unital, namely $T^*(I) = I$.

Now suppose $X_k$'s are optimal projectors for quantum states $T(\rho)$ and $T(\sigma)$, so we obtain $d_p(T(\rho), T(\sigma)) = \left(\sum_{k} |\text{tr}(T(\rho)X_k)|^p - |\text{tr}(T(\sigma)X_k)|^p\right)^{\frac{1}{p}}$. Let $Y_k = T^*(X_k)$; then, $Y_k \geq 0$ and $\sum_{k} Y_k = \sum_{k} T^*(X_k) = T^*(\sum_{k} X_k) = T^*(I) = I$. So we have

$$\sum_{k} |\text{tr}(T(\rho)X_k)|^\frac{1}{p} - |\text{tr}(T(\sigma)X_k)|^\frac{1}{p} = \sum_{k} |\text{tr}(\rho Y_k)|^\frac{1}{p} - |\text{tr}(\sigma Y_k)|^\frac{1}{p} \leq d_p(\rho, \sigma)^p.$$

Thus, we have $d_p(T(\rho), T(\sigma)) \leq d_p(\rho, \sigma)$; this ends the proof.

As for the joint convexity of the metric $d_p(\rho, \sigma)$, we have the following theorem and leave its proof in the appendix.

**Theorem 2.** $[d_p(\rho, \sigma)]^p$ is joint convex if and only if $p = 1, 2$; in other words, for $p \neq 1$ and 2, $[d_p(\rho, \sigma)]^p$ is not joint convex.
3. A related metrics $D_p(\rho, \sigma)$

We know that for $p = 1, 2$, there are operational forms for $d_p(\rho, \sigma)$, i.e. $d_1(\rho, \sigma) = \text{tr}|\rho - \sigma|$, and $d_2(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}$; then, we can obtain the value of $d_1(\rho, \sigma)$ and $d_2(\rho, \sigma)$ directly from the matrix entries of $\rho$ and $\sigma$. However, for other $p$, we cannot enjoy this advantage, since $d_p(\rho, \sigma)$ is defined via taking the supremum of projectors, so a natural question arises: just like $p = 1$, $2$, can we obtain the operational form for all $d_p(\rho, \sigma)$?

This problem is difficult and we would like to leave it as a future topic. What we want to do in this paper is to introduce a new family of related metrics $D_p(\rho, \sigma)$, which have the advantage of being easy to calculate.

**Definition 2.** Similar to definition 1, we define $D_p(\rho, \sigma) := [\text{tr}(|\rho^\frac{p}{2} - \sigma^\frac{p}{2}|^p)]^{\frac{1}{p}}$.

Based on the definition, we have the following theorem.

**Theorem 3.** Let $\rho$ and $\sigma$ be quantum states; then, $D_p(\rho, \sigma)$ is a metric on $S(H)$, where $S(H)$ denote the set of all quantum states on the Hilbert space $H$.

**Proof.** It is easy to show that $D_p(\rho, \sigma) = D_p(\sigma, \rho)$, $D_p(\rho, \sigma) \geq 0$ and $D_p(\rho, \rho) = 0$. If $D_p(\rho, \sigma) = 0$, then $|\rho^\frac{p}{2} - \sigma^\frac{p}{2}| = 0$, so we obtain $\rho = \sigma$. Recall that the Schatten $p$-norm $\|\cdot\|_p$ for an operator $y$ is defined as [4]: $\|y\|_p = [\text{tr}(|y^\frac{p}{2}|^p)]^{\frac{1}{p}}$. Now we define two matrices $y_1$ and $y_2$, $y_1 := \rho^\frac{1}{p}$, $y_2 := \sigma^\frac{1}{p}$. After using the triangle inequality for the Schatten $p$-norm $\|\cdot\|_p$:

$$\|y_1 - y_2\|_p \leq \|y_1\|_p + \|y_2\|_p = 2,$$

we obtain the triangle inequality for $D_p$. Therefore, $D_p(\rho, \sigma)$ is a metric on $S(H)$; theorem 3 is proved. In addition, from $(UDU^*)_p = UD^\frac{p}{2}U^*$, one easily knows that the metric $D_p(\rho, \sigma)$ is unitary invariant.

In quantum information theory, majorization has become a powerful tool to detect entanglement. It was proved in [5] that any separable state $\rho$ acting on $\mathbb{C}^d \otimes \mathbb{C}^d$ is majorized by its reduced state $\rho_A$:

$$\rho_A \succ \rho,$$

i.e. $\forall k \leq d : \sum_{i=1}^{k} \lambda_i^{(A)} \geq \sum_{i=1}^{k} \lambda_i$,

where $\{\lambda_i\}$ and $\{\lambda_i^{(A)}\}$ are the decreasingly ordered eigenvalues of $\rho$ and $\rho_A$, respectively. □

Now we give the following majorization relation for the metric $D_p(\rho, \sigma)$:

**Theorem 4.** If $1 \leq p \leq q$, and $\rho, \sigma$ are density operators, define the vectors $\lambda(|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^p), \lambda(|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^q)$ as the vectors of eigenvalues for $|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^p$, $|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^q$, then the following majorization relation holds:

$$\lambda(|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^q) \prec \lambda(|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^p).$$

(7)

That is, $\lambda(|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^q)$ is majorized by $\lambda(|\sqrt[2]{\rho} - \sqrt[2]{\sigma}|^p)$. In particular, the following inequality holds:

$$D_q(\rho, \sigma)^q \leq (D_p(\rho, \sigma))^p.$$  

(8)

**Proof.** In [6], Ando proved that if $f(t)$ is a non-negative operator-monotone function on $[0, \infty)$ and $\|\cdot\|$ is a unitary invariant norm, then

$$\|f(A) - f(B)\| \leq \|f(A - B)\|,$$

$A, B \geq 0,$

$$D_p(\rho, \sigma)^q \leq (D_p(\rho, \sigma))^p.$$  

(8)
or in the majorization language as
\[ \lambda (| f(A) - f(B) |) \prec_w \lambda (| A - B |). \]

Since the function \( f(t) = t^\frac{p}{q} \) is operator-monotone for \( 1 \leq p \leq q \), one has
\[ \lambda (| \rho^\frac{p}{q} - \sigma^\frac{p}{q} |) \prec_w \lambda (| \rho - \sigma |^{\frac{p}{q}}). \] (9)

Consider the Schatten-\( q \)-norm from equation (9) to obtain
\[ \text{tr}(| \rho^\frac{p}{q} - \sigma^\frac{p}{q} |^q) \leq \text{tr}(| \rho - \sigma |^p). \] (10)

Replace \( \rho, \sigma \) in equation (10) by \( \rho^\frac{1}{p}, \sigma^\frac{1}{p} \) respectively; then we complete the proof. \( \square \)

As for the convex property of \( D_p(\rho, \sigma) \), similar to the proof of theorem 2, we have that \( (D_p(\rho, \sigma))^p \) is joint convex if and only \( p = 1, 2 \). In the following, we would like to discuss whether the metric \( D_p(\rho, \sigma) \) is contractive under quantum operation.

Especially, for \( p = 1 \), we know that \( D_1(\rho, \sigma) \) equal \( d_1(\rho, \sigma) \), so it is contractive under quantum operation. And for \( p = 2 \), we know that \( (D_2(\rho, \sigma))^2 = [\text{tr}(| \rho^\frac{1}{2} - \sigma^\frac{1}{2} |^2)] = 2 - 2\text{tr}(\rho^\frac{1}{2} \sigma^\frac{1}{2}). \) The Uhlmann–Jozsa fidelity was widely studied and plays a key role in quantum information theory, but it is not easy to calculate, so some alternative fidelity measures were introduced, see [8–10]. The new fidelity introduced in [8–10] is all proved to be a good fidelity measure. On the other hand, we know that Uhlmann–Jozsa fidelity can be rewritten as \( F(\rho, \sigma) = \text{tr}(| \rho^\frac{1}{2} \sigma^\frac{1}{2} |) \), that means, Uhlmann–Jozsa fidelity is the trace of the modulus of the operator \( \rho^\frac{1}{2} \sigma^\frac{1}{2} \). However, \( \text{tr}(\rho^\frac{1}{2} \sigma^\frac{1}{2}) \) is exactly the trace of the operator \( \rho^\frac{1}{2} \sigma^\frac{1}{2} \). They only differ from a phase factor!

So this leads to the following idea: if we define another fidelity, called A-fidelity in this paper, as
\[ F_A(\rho, \sigma) = [\text{tr}(\sqrt{\rho} \sqrt{\sigma})]^2, \] (11)
then we ask: Can \( F_A(\rho, \sigma) \) be a good fidelity measure? The answer is yes. In fact, in [7], the author has shown that \( F_A(\rho, \sigma) \) has the following appealing properties.

**Property 1:** CPTP expansive property. If \( \rho \) and \( \sigma \) are density matrices, \( \Phi \) is a CPTP map, then \( F_A(\Phi(\rho), \Phi(\sigma)) \geq F_A(\rho, \sigma) \).

**Property 2.** When \( \rho = |\phi\rangle \langle \phi| \) and \( \sigma = |\psi\rangle \langle \psi| \) are two pure states, Uhlmann–Jozsa fidelity and A-fidelity both reduce to the inner product, that is, \( F(\rho, \sigma) = F_A(\rho, \sigma) = |\langle \phi | \psi \rangle|^2 \).

Now we know that if \( p = 1 \) or \( p = 2 \), \( D_p(\rho, \sigma) \) is joint convex and also is contractive under quantum operation. We will show that they are the only two cases satisfying the CPTP contractive property.

For others \( p \neq 1, 2 \), using the numerical method [11], we can get that \( D_p(\rho, \sigma) \) is neither decreasing nor increasing under quantum operation. Therefore, we conclude as following: \( D_p(\rho, \sigma) \) is contractive under quantum operation if and only \( p = 1, 2 \).

To prove that for others \( p \), \( [d_p(\rho, \sigma)]^p \) is not joint convex, we need the following simple example.

**Example 1.** When \( p \neq 1, 2 \), and let
\[
\rho = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.6 \end{pmatrix},
\]
we can prove that \( [d_p(\rho, \sigma)]^p \) is not joint convex. This example also shows that \( [D_p(\rho, \sigma)]^p \) is not joint convex for \( p \neq 1, 2 \).
4. Conclusions and applications

Our conclusion is as follows.

(1) The metric $d_p(\rho, \sigma)$ is contractive under quantum operation for all $p$, while the metric $D_p(\rho, \sigma)$ is merely contractive under quantum operation if and only if $p = 1, 2$.

(2) Both $[d_p(\rho, \sigma)]^p$ and $[D_p(\rho, \sigma)]^p$ are joint convex if and only if $p = 1, 2$.

Since the metrics $d_p(\rho, \sigma)$ and $D_p(\rho, \sigma)$ are natural generalizations of the trace metric and the Bures metric, we wish that they can be also used to study the geometrical structure of quantum states [13, 14], such as the volume of quantum states [15–19], which was traditionally studied by using the trace metric and the Bures metric as tools.

Moreover, our result may be used to study quantum entanglement. We know that the geometrical entanglement measure is a famous entanglement measure; it shares some appealing properties [12, 20–23]. The idea of geometrical entanglement measure is based on the following: the set of all separable states is a convex set denoted by $S$; if we have a state $\rho$, then the closer the state $\rho$ to the set $S$, the less entangled it is. So the entanglement measure is defined as the minimal distance of the state $\rho$ to any state of $S$:

$$E(\rho) = \min_{\sigma \in S} D(\rho, \sigma).$$

Usually, we use the Bures metric $d_2(\rho, \sigma)$ to obtain the geometrical entanglement measure; we wish that the metric $D_2(\rho, \sigma)$ is also a good candidate for the geometrical entanglement measure. This interesting work will be done subsequently.

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Appendix. proof of theorem 2

Now we give the proof of theorem 2.

Proof. For simplicity, we only discuss $2 \times 2$ density matrices $\rho$ and $\sigma$; there is no difficulty to prove the $N \times N$ case by applying the same method.

First, we give an inequality. Let $P_1$, $P_2$, $P_3$ and $P_4$ be two-dimensional discrete probability distributions with $P_i = \{P_{i1}, P_{i2}\}$, where $P_{i1}, P_{i2} \geq 0$, $P_{i1} + P_{i2} = 1$, $i = 1, 2, 3, 4$. Then the following holds: for all $0 \leq \lambda \leq 1$ and $p = 1, 2$,

$$\sum_{j=1}^{2} \left| \lambda P_{1j} + (1 - \lambda) P_{2j} \right|^p - \left| \lambda P_{3j} + (1 - \lambda) P_{4j} \right|^p \leq \lambda \sum_{j=1}^{2} \left| (P_{1j})^\frac{1}{p} - (P_{3j})^\frac{1}{p} \right|^p + (1 - \lambda) \sum_{j=1}^{2} \left| (P_{2j})^\frac{1}{p} - (P_{4j})^\frac{1}{p} \right|^p. \quad (A.1)$$
For $p = 1$, the above inequality always holds. In fact, suppose $P_1 = \{a_1, 1 - a_1\}$, $P_2 = \{a_2, 1 - a_2\}$, $P_3 = \{b_1, 1 - b_1\}$, $P_4 = \{b_2, 1 - b_2\}$; then, it easily follows from the absolute values inequality $|\lambda a_1 - b_1| + (1 - \lambda)(a_2 - b_2)| \leq \lambda|a_1 - b_1| + (1 - \lambda)|a_2 - b_2|$. 

For $p = 2$, the problem reduces to prove that the function $f(a, b) := |a^2 - b^2|^2 + |(1 - a)^2 - (1 - b)^2|^2$ is joint convex; here $0 \leq a, b \leq 1$. We can obtain the Hessian matrix of $f(a, b)$ as

$$H(f) = \begin{pmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{pmatrix},$$

where $f_{aa} = \frac{1}{2}a^{-2}b + \frac{1}{2}(1 - a)^{-2}(1 - b)^2$, $f_{ab} = f_{ba} = -\frac{1}{2}a^{-2}b^{-2} - \frac{1}{2}(1 - a)^{-2}(1 - b)^{-2}$, $f_{bb} = \frac{1}{2}b^{-2}a^2 + \frac{1}{2}(1 - b)^{-2}(1 - a)^2$.

The function $f(a, b)$ is joint convex if and only if the Hessian matrix $H(f)$ is non-negative definite. However, $H(f)$ is non-negative definite if and only if $f_{aa} \geq 0$, $f_{bb} \geq 0$, $f_{ab}f_{bb} - f_{aa}f_{ba} \geq 0$, $f_{aa} \geq 0$, $f_{ab} \geq 0$ always hold, so we only need to prove $f_{aa}f_{bb} - f_{ab}f_{ba} \geq 0$.

This is equivalent to the following: $a^{-2}b + (1 - a)^{-2}(1 - b)^2 + a^{-2}b^{-2} + (1 - a)^{-2}(1 - b)^{-2} \geq 2a^{-2}b^{-2} + (1 - a)^{-2}(1 - b)^{-2}$, and we know this holds from the Cauchy–Schwarz inequality.

Now we have proved the inequality (A.1) for $p = 1, 2$. We will use inequality (A.1) to prove that $(d_p(\rho, \sigma))^p$ is joint convex for $p = 1, 2$.

Suppose $X_1$ are projectors and $\sum_{k} X_k = I$; then $\text{tr}(\lambda \rho_1 + (1 - \lambda) \rho_2) X_k$, $\text{tr}(\lambda \sigma_1 + (1 - \lambda) \sigma_2) X_k$ and $\text{tr}(\rho_1 X_k)$, $\text{tr}(\rho_2 X_k)$, $\text{tr}(\sigma_1 X_k)$, $\text{tr}(\sigma_2 X_k)$ are all discrete probability distributions. Putting the four probability distributions $\text{tr}(\rho_1 X_k)$, $\text{tr}(\rho_2 X_k)$, $\text{tr}(\sigma_1 X_k)$, $\text{tr}(\sigma_2 X_k)$ in inequality (A.1), we obtain

$$\sum_{k} |\text{tr}(\lambda X_k) + (1 - \lambda) \text{tr}(\rho_2 X_k)|^2 - |\text{tr}(\lambda X_k) + (1 - \lambda) \text{tr}(\sigma_2 X_k)|^2 |^p \leq \lambda \sum_{k} |(\text{tr}(\rho_1 X_k))^{\frac{1}{p}} - (\text{tr}(\sigma_1 X_k))^{\frac{1}{p}}|^p + (1 - \lambda) \sum_{k} |(\text{tr}(\rho_2 X_k))^{\frac{1}{p}} - (\text{tr}(\sigma_2 X_k))^{\frac{1}{p}}|^p. $$

We take the supremum in both sides of the above inequality, but the optimal projector for $[d_p(\lambda \rho_1 + (1 - \lambda) \rho_2, \lambda \sigma_1 + (1 - \lambda) \sigma_2)]^p$ may not be the optimal projector for $[d_p(\rho_1, \sigma_1)]^p$ and $[d_p(\rho_2, \sigma_2)]^p$; then we can obtain the needed result. Theorem 1 is proved. \hfill $\square$

References

[1] Uhlenbeck A 1976 Rep. Math. Phys. 9 273

[2] Jozsa R 1994 J. Mod. Opt. 41 2315

[3] Schumacher B 1995 Phys. Rev. A 51 2738

[4] Wang X B, Oh C H and Kwek L C 1998 Phys. Rev. A 58 4186

[5] Yang S, Shi-Jian G, Sun C-P and Lin H-Q 2008 Phys. Rev. A 78 012304

[6] Zanardi P, Quan H T, Wang Xiaoguang and Sun C P 2007 Phys. Rev. A 75 032109

[7] Xiao-Ming L, Sun Z, Wang X and Zanardi P 2008 Phys. Rev. A 78 032309

[8] Wang X, Sun Z and Wang Z D 2009 Phys. Rev. A 79 012105

[9] Nicolas H 1986 Linear Algebra Appl. 84 281

[10] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)

[11] Pisier G and Xu Q 2003 Non-Commutative $L_p$-Spaces (Amsterdam: North-Holland)

[12] Nielsen M A and Kempe J 2001 Phys. Rev. Lett. 86 5184

[13] Ando T 1988 Math. Z. 197 403

[14] Raggio G A 1984 Generalized Transition Probabilities and Applications Quantum Probability and Applications to the Quantum Theory of Irreversible Processes (Lecture Notes in Mathematics vol 1055) (Berlin: Springer) p 327

[15] Chen J L, Fu L, Ungar A A and Zhao X G 2002 Phys. Rev. A 65 054304

[16] Chen J L, Fu L, Ungar A A and Zhao X G 2002 Phys. Rev. A 65 024303
[9] Miszczak J A, Puchala Z, Horodecki P, Uhlmann A and Życzkowski K 2009 *Quantum Inf. Comput.* 9 0103
[10] Mendonça Paulo E M F, Napolitano R d J, Marchiolli M A, Foster C J and Liang Y C 2008 *Phys. Rev. A* **78** 052330
[11] Bruzda W, Cappellini V, Sommers H and Życzkowski K 2009 *Phys. Lett. A* **373** 320
[12] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
[13] Fuchs C A and van de Graaf J 1999 *IEEE Trans. Inf. Theory* **45** 1216
[14] Garca D, Wolf M, Petz D and Ruskai M 2006 *J. Math. Phys.* **47** 083506
[15] Ye D 2010 *J. Phys. A: Math. Theor.* **43** 315301
[16] Ye D 2009 *J. Math. Phys.* **50** 083502
[17] Sommers H J and Życzkowski K 2003 *J. Phys. A: Math. Gen.* **36** 10083
[18] Życzkowski K and Sommers H J 2003 *J. Phys. A: Math. Gen.* **36** 10115
[19] Andai A 2006 *J. Phys. A: Math. Gen.* **39** 13641
[20] Wei T-C and Goldbart P M 2003 *Phys. Rev. A* **68** 042307
[21] Wei T-C, Ericsson M, Goldbart P M and Munro W J 2004 *Quantum Inf. Comput.* **4** 252
[22] Hubener R, Kleinmann M, Wei T-C, González-Guillén C and Gühne O 2009 *Phys. Rev. A* **80** 032324
[23] Cao Y and Wang A 2007 *J. Phys. A: Math. Theor.* **40** 3507