Mutation classes of $\tilde{A}_n$—quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$

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Abstract

We give an explicit description of the mutation classes of quivers of type $\tilde{A}_n$. Furthermore, we provide a complete classification of cluster tilted algebras of type $\tilde{A}_n$ up to derived equivalence. We show that the bounded derived category of such an algebra depends on four combinatorial parameters of the corresponding quiver.

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1 Introduction

A few years ago, Fomin and Zelevinsky introduced the concept of cluster algebras [14] which rapidly became a successful research area. Cluster algebras nowadays link various areas of mathematics, like combinatorics, Lie theory, algebraic geometry, representation theory, integrable systems, Teichmüller theory, Poisson geometry and also string theory in physics (via recent work on quivers with superpotentials [12], [18]).

In an attempt to 'categorify' cluster algebras, which a priori are combinatorially defined, cluster categories have been introduced by Buan, Marsh, Reineke, Reiten and Todorov [6]. For a quiver $Q$ without loops and oriented 2-cycles and the corresponding path algebra $KQ$ (over an algebraically closed field $K$), the cluster category $C_Q$ is the orbit category of the bounded derived category $D^b(KQ)$ by the functor $\tau^{-1}[1]$, where $\tau$ denotes the Auslander-Reiten translation and $[1]$ is the shift functor on the triangulated category $D^b(KQ)$.

Important objects in cluster categories are the cluster-tilting objects. The endomorphism algebras of such objects in the cluster category $C_Q$ are called cluster tilted algebras of type $Q$ [8]. Cluster tilted algebras have several interesting properties, e.g. their representation theory can be completely understood in terms of the representation theory of the corresponding path algebra of a quiver (see [8]). These algebras have been studied by various authors, see for instance [2], [3], [7] or [10].

In recent years, a focal point in the representation theory of algebras has been the investigation of derived equivalences of algebras. Since a lot of properties and invariants of rings and algebras are preserved by derived equivalences, it is important for many purposes to classify classes of algebras up to derived equivalence, instead of Morita equivalence. For selfinjective algebras the representation type is preserved under derived equivalences (see [17] and [20]). It has been also proved in [21] that the class of symmetric algebras is closed under derived equivalences. Additionally, we note that derived equivalent algebras have the same number of pairwise nonisomorphic simple modules and isomorphic centers.

In this work, we are concerned with the problem of derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$. Such a classification was done for cluster tilted algebras of type $A_n$ by...
Buan and Vatne in 2007 [9]; see also the work of Murphy on the more general case of $m$-cluster tilted algebras of type $A_n$ [19]. Since the quivers of cluster tilted algebras of type $\tilde{A}_n$ are exactly the quivers in the mutation classes of $\tilde{A}_n$–quivers; these mutation classes are known to be finite (for example see [13]). The second purpose of this note is to describe, when two cluster tilted algebras of type $Q$ have equivalent derived categories, where $Q$ is a quiver whose underlying graph is $\tilde{A}_n$.

In Definition 3.4 we present a class $Q_n$ of quivers with $n+1$ vertices which includes all non-oriented cycles of length $n+1$. To show that this class contains all quivers mutation equivalent to some quiver of type $\tilde{A}_n$ we first prove that this class is closed under quiver mutation. Furthermore, we define parameters $r_1$, $r_2$, $s_1$ and $s_2$ for any quiver $Q \in Q_n$ in Definition 3.8 and prove that every quiver in $Q_n$ with parameters $r_1$, $r_2$, $s_1$ and $s_2$ can be mutated to a normal form, without changing the parameters.

With the help of the above result we can show that every quiver $Q \in Q_n$ with parameters $r_1$, $r_2$, $s_1$ and $s_2$ is mutation equivalent to some non-oriented cycle with $r := r_1 + 2r_2$ arrows in one direction and $s := s_1 + 2s_2$ arrows in the other direction. Hence, if two quivers $Q_1$ and $Q_2$ of $Q_n$ have the parameters $r_1$, $r_2$, $s_1$, $s_2$, respectively $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{s}_1$, $\tilde{s}_2$ and $r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2$, $s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$ (or vice versa), then $Q_1$ is mutation equivalent to $Q_2$.

The converse of this result, i.e., an explicit description of the mutation classes of quivers of type $\tilde{A}_n$, can be shown with the help of Lemma 6.8 in [13].

The main result of the derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$ is the following theorem:

**Theorem 5.5.**

Two cluster tilted algebras of type $\tilde{A}_n$ are derived equivalent if and only if their quivers have the same parameters $r_1$, $r_2$, $s_1$ and $s_2$ (up to changing the roles of $r_i$ and $s_i$, $i \in \{1, 2\}$).

We prove that every cluster tilted algebra of type $\tilde{A}_n$ with parameters $r_1$, $r_2$, $s_1$ and $s_2$ is derived equivalent to a cluster tilted algebra corresponding to a quiver in normal form. Furthermore, we compute the parameters $r_1$, $r_2$, $s_1$ and $s_2$ as combinatorial derived invariants for a quiver $Q \in Q_n$ with the help of an algorithm defined by Avella-Alaminos and Geiß in [5].

The paper is organized as follows. In Section 2 we collect some basic notions about quiver mutations. In Section 3 we present the set $Q_n$ of quivers which can be obtained by iterated mutation from quivers whose underlying graph is of type $\tilde{A}_n$. Moreover, we describe, when two
quivers of $Q_n$ are in the same mutation class. In the fourth section we describe the cluster tilted algebras of type $\tilde{A}_n$ and their relations (as shown in [1]). In Section 5 we first briefly review the fundamental results on derived equivalences. Afterwards, we prove our main result, i.e., we show, when two cluster tilted algebras of type $\tilde{A}_n$ are derived equivalent.

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2 Quiver mutations

First, we present the basic notions for quiver mutations. A quiver is a finite directed graph $Q$, consisting of a finite set of vertices $Q_0$ and a finite set of arrows $Q_1$ between them.

Let $Q$ be a quiver and $K$ be an algebraically closed field. We can form the path algebra $KQ$, where the basis of $KQ$ is given by all paths in $Q$, including trivial paths $e_i$ of length zero at each vertex $i$ of $Q$. Multiplication in $KQ$ is defined by concatenation of paths. Our convention is to read paths from right to left. For any path $\alpha$ in $Q$ let $s(\alpha)$ denote its start vertex and $t(\alpha)$ its end vertex. Then the product of two paths $\alpha$ and $\beta$ is defined to be the concatenated path $\alpha \beta$ if $s(\alpha) = t(\beta)$. The unit element of $KQ$ is the sum of all trivial paths, i.e., $1_{KQ} = \sum_{i \in Q_0} e_i$.

We now recall the definition of quiver mutation which was introduced by Fomin and Zelevinsky in [14].

**Definition 2.1.** Let $Q$ be a quiver without loops and oriented 2-cycles. The mutation of $Q$ at a vertex $k$ to a new quiver $Q^*$ can be described as follows:

1. Add a new vertex $k^*$.
2. If there are $r > 0$ arrows $i \to k$, $s > 0$ arrows $k \to j$ and $t \in \mathbb{Z}$ arrows $j \to i$ in $Q$, there are $t - rs$ arrows $j \to i$ in $Q^*$. (Here, a negative number of arrows means arrows in the opposite direction.)
3. For any vertex $i$ replace all arrows from $i$ to $k$ with arrows from $k^*$ to $i$, and replace all arrows from $k$ to $i$ with arrows from $i$ to $k^*$.
4. Remove the vertex $k$.

Note that mutation at sinks or sources only means changing the direction of all incoming or outgoing arrows. Two quivers are called mutation equivalent (sink/source equivalent) if one can be obtained from the other by a finite sequence of mutations (at sinks and/or sources). The mutation class of a quiver $Q$ is the class of all quivers mutation equivalent to $Q$.

**Example 2.2.**

i) First, we consider the following quiver $Q$

\[
\begin{array}{c}
\text{i} \\
& \text{j} \\
\text{k} \\
\end{array}
\]
Here, we have $r = 1$ arrow from $i$ to $k$, $s = 2$ arrows from $k$ to $j$ and $t = 1$ arrow from $j$ to $i$. Thus, we have $t - rs = -1$ arrow from $j$ to $i$ in $Q^*$, i.e., there is one arrow from $i$ to $j$:

![Diagram](https://example.com/diagram1.png)

ii) Now consider the quiver $Q$ below

![Diagram](https://example.com/diagram2.png)

There is $r = 1$ arrow $i \to k$, $s = 1$ arrow $k \to i$ and one arrow $i \to j$, i.e., $t = -1$. Thus, we have $t - rs = -2$ arrows $j \to i$ in $Q^*$, i.e., there are two arrows from $i$ to $j$:

![Diagram](https://example.com/diagram3.png)

3 Mutation classes of $\tilde{A}_n$–quivers

**Remark 3.1.** Quivers of type $\tilde{A}_n$ are just cycles with $n + 1$ vertices. If all arrows go clockwise or all arrows go anti-clockwise, then we get the mutation class of $D_{n+1}$ (see [11], [13] and Type IV in type $D$ in [23]). If the cycle is non-oriented, we get what we call the mutation classes of $\tilde{A}_n$.

First, we have to fix one drawing of this non-oriented cycle, i.e., one embedding into the plane. Thus, we can speak of clockwise and anti-clockwise oriented arrows. But we have to consider that this notation is only unique up to reflection of the cycle, i.e., up to changing the roles of clockwise and anti-clockwise oriented arrows.

The following proposition is well-known. Since we could not find a suitable reference, we provide a proof for the convenience of the reader.

**Proposition 3.2.** Let $Q$ be a non-oriented cycle of length $n + 1$. Let $s$ be the number of arrows in $Q$ which are oriented in the clockwise direction, and let $r$ be the number of arrows in $Q$ which are oriented in the anti-clockwise direction. Then $Q$ is sink/source equivalent to a quiver of the following form.
Proof. First, we note that mutating at sinks or sources does not change the numbers of clockwise and anti-clockwise oriented arrows.
Since the cycle is not oriented there exist at least one sink and one source. Furthermore, the number of sinks equals the number of sources in $Q$ and they appear alternately on the cycle.
We begin with an arbitrary source $S_1$ and move along the cycle in the clockwise direction until we find the next source $S_2$.
If $S_2 = S_1$, then $Q$ is of the required form.
Thus, let $S_2 \neq S_1$. Then there is one sink $x_1$ between $S_1$ and $S_2$. If we mutate at $S_2$, the vertex $x_2$ right of $S_2$, i.e., the vertex which follows $S_2$ in the anti-clockwise direction, is a new source or it is already $x_1$. If it is $x_1$, this first step is finished. If it is a new source, we mutate at this source and continue this procedure until the right vertex of the mutated source is not a source, i.e., the right vertex is the sink $x_1$: 

\[ \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array} \]
Hence, the sink $x_1$ goes one arrow forward in the clockwise direction, i.e., there is one more arrow in the (oriented) path between $S_1$ and the new sink $x_1$. Afterwards, we move again from $S_1$ along the cycle in the clockwise direction and search for the next source. Doing this iteratively, the sink $x_1$ goes one arrow forward in the clockwise direction after each step, i.e., this procedure ends with the required form.

Thus, if two non-oriented cycles of length $n+1$ have the same parameters $r$ and $s$ (up to changing the roles of $r$ and $s$), then they are mutation equivalent. Lemma 6.8 in [13] proves that the converse also holds:

**Lemma 3.3** (Fomin, Shapiro and Thurston [13]). Let $C_1$ and $C_2$ be two $(n+1)$-cycles with parameters $r, s$, respectively $\tilde{r}, \tilde{s}$. Then $C_1$ and $C_2$ are mutation equivalent if and only if the unordered pairs $\{r, s\}$ and $\{\tilde{r}, \tilde{s}\}$ coincide.

### 3.1 Description of the mutation classes of $\tilde{A}_n$–quivers

In this section we will provide an explicit description of the mutation classes of $\tilde{A}_n$–quivers. For this we need the description of the mutation class of quivers of type $A_k$ which is given in [9] as follows:

- each quiver has $k$ vertices,
- all non-trivial cycles are oriented and of length 3,
- a vertex has at most four incident arrows,
- if a vertex has four incident arrows, then two of them belong to one oriented 3-cycle, and the other two belong to another oriented 3-cycle,
- if a vertex has three incident arrows, then two of them belong to an oriented 3-cycle, and the third arrow does not belong to any oriented 3-cycle.

By a *cycle* in the first condition we mean a cycle in the underlying graph, not passing through the same edge twice. In particular, this condition excludes multiple arrows.

Now we can formulate the description of the mutation classes of $\tilde{A}_n$–quivers which is a similar description as for Type IV in type $D$ in [23].

**Definition 3.4.** Let $Q_n$ be the class of quivers with $n + 1$ vertices which satisfy the following conditions:

i) There exists precisely one full subquiver which is a non-oriented cycle of length $\geq 2$. Thus, if the length is two, it is a double arrow.

ii) For each arrow $x \xrightarrow{\alpha} y$ in this non-oriented cycle, there may (or may not) be a vertex $z_\alpha$ which is not on the non-oriented cycle, such that there is an oriented 3-cycle of the form

```
\begin{tikzpicture}
  \node (x) at (0,0) {$x$};
  \node (y) at (1,0) {$y$};
  \node (z) at (0.5,0.866) {$z_\alpha$};
  \draw (x) -- (y);
  \draw (z) -- (x);
  \draw (z) -- (y);
\end{tikzpicture}
```

Apart from the arrows of these oriented 3-cycles there are no other arrows incident to vertices on the non-oriented cycle.
iii) If we remove all vertices in the non-oriented cycle and their incident arrows, the result is a disconnected union of quivers $Q_1, Q_2, \ldots$, one for each $z_\alpha$ (which we call $Q_\alpha$ in the following). These are quivers of type $A_{k_\alpha}$ for $k_\alpha \geq 1$, and the vertices $z_\alpha$ have at most two incident arrows in these quivers. Furthermore, if a vertex $z_\alpha$ has two incident arrows in such a quiver, then $z_\alpha$ is a vertex in an oriented 3-cycle in $Q_\alpha$.

![Diagram of quivers]

Our convention is to choose only one of the double arrows to be part of the oriented 3-cycle in the following case:

![Diagram of oriented 3-cycle]

**Notation 3.5.** Note that whenever we draw an edge

\[
\begin{array}{c}
 j \\
 \rightarrow \\
 k
\end{array}
\]

the direction of the arrow between $j$ and $k$ is not important for this situation; and whenever we draw a cycle

![Diagram of oriented 3-cycle]

it is an oriented 3-cycle.

**Lemma 3.6.** $Q_n$ is closed under quiver mutation.
Proof. Let \( Q \) be a quiver in \( \mathcal{Q}_n \) and let \( i \) be some vertex.

If \( i \) is a vertex in one of the quivers \( Q_1, Q_2, \ldots \) of type \( A \), but not one of the vertices \( z_\alpha \) connecting this quiver of type \( A \) to the rest of the quiver \( Q \), then the mutation at \( i \) leads to a quiver \( Q^* \in \mathcal{Q}_n \) since type \( A \) is closed under quiver mutation.

It therefore suffices to check what happens when we mutate at the other vertices, and we will consider the following four cases.

1) Let \( i \) be one of the vertices \( z_\alpha \), i.e., not on the non-oriented cycle. For the situation where \( z_\alpha \) has only two incident arrows, i.e., the quiver of type \( A \) consists only of one vertex, we can look at the first mutated quiver in case 2) below since quiver mutation is an involution.

Thus, we have the following three cases,

\[
\begin{align*}
Q_1 & \ \\
Q_2 & \ \\
\text{or} & \\
\text{or} & \ \\
\text{or}
\end{align*}
\]

where \( Q_1 \) and \( Q_2 \) are quivers of type \( A \).

Then the mutation at \( i \) leads to the following three quivers which have a non-oriented cycle one arrow longer than for \( Q \), and this is indeed a non-oriented cycle since the arrows \( j \rightarrow i \rightarrow k \) have the same orientation as \( \alpha \) before.

\[
\begin{align*}
Q_1 & \ \\
Q_2 & \ \\
\text{or} & \\
\text{or} & \ \\
\text{or}
\end{align*}
\]

The vertices \( l \) and \( m \) have at most two incident arrows in the quivers \( Q_1 \) and \( Q_2 \) since they had at most four resp. three incident arrows in \( Q \) (see the description of quivers of type \( A \)). Furthermore, if \( l \) or \( m \) has two incident arrows in the quiver \( Q_1 \) or \( Q_2 \), then these two arrows form an oriented 3-cycle as in \( Q \).

Thus, the mutated quiver \( Q^* \) is also in \( \mathcal{Q}_n \).

2) Let \( i \) be a vertex on the non-oriented cycle, and not part of any oriented 3-cycle. Then the following three cases can occur:
Then the mutation at $i$ leads to the following three quivers

If $i$ is a sink or a source in $Q$, the non-oriented cycle in $Q^*$ is of the same length as before and $Q^*$ is in $Q_n$.

If there is a path $j \rightarrow i \rightarrow k$ in $Q$, then the mutation at $i$ leads to a quiver $Q^*$ which has a non-oriented cycle one arrow shorter than in $Q$, and this is indeed a non-oriented cycle since $\alpha$ has the same orientation as the arrows $j \rightarrow i \rightarrow k$ before.

Note that in this case the non-oriented cycle in $Q$ consists of at least three arrows and thus, the non-oriented cycle in $Q^*$ has at least two arrows. Moreover, $i$ is a new vertex $z_\alpha$ in $Q^*$ with two incident arrows. Thus, the mutated quiver $Q^*$ is also in $Q_n$.

3) Let $i$ be a vertex on the non-oriented cycle which is part of exactly one oriented 3-cycle. Then four cases can occur, but two of them have been dealt with by the second and third mutated quiver in case 1) since quiver mutation is an involution. Thus, we only have to consider the following two situations and their special cases where the non-oriented cycle is a double arrow.
Here, $Q_1$ is a quiver of type $A$.
The non-oriented cycle has the same length as before. Moreover, $l$ has the same number of incident arrows as before. Thus, $Q^*$ is in $Q_n$.

4) Let $i$ be a vertex on the non-oriented cycle which is part of two oriented 3-cycles. Then three cases can occur, but one of them has been dealt with by the first mutated quiver in case 1). Thus, we only have to consider the following two situations and their special cases where the non-oriented cycle is a double arrow.

Here, $Q_1$ and $Q_2$ are quivers of type $A$.
The non-oriented cycle has the same length as before. Moreover, $l$ and $m$ have the same number of incident arrows as before. Thus, the mutated quiver $Q^*$ is in $Q_n$.

Remark 3.7. It is easy to see that all orientations of a circular quiver of type $\tilde{A}_n$ are in $Q_n$ (except the oriented case; but this leads to the mutation class of $D_{n+1}$). Since $Q_n$ is closed under quiver mutation every quiver mutation equivalent to some quiver of type $\tilde{A}_n$ is in $Q_n$, too.

Now we fix one drawing of a quiver $Q \in Q_n$, i.e., one embedding into the plane, without arrow-crossing. Thus, we can again speak of clockwise and anti-clockwise oriented arrows of the non-oriented cycle. But we have to consider that this notation is only unique up to reflection of the non-oriented cycle, i.e., up to changing the roles of clockwise and anti-clockwise oriented arrows. We define four parameters $r_1$, $r_2$, $s_1$ and $s_2$ for a quiver $Q \in Q_n$ as follows:

Definition 3.8. Let $r_1$ be the number of arrows which are not part of any oriented 3-cycle and which fulfill one of the following two conditions:
1) These arrows are part of the non-oriented cycle and they are oriented in the anti-clockwise direction.
2) These arrows are not part of the non-oriented cycle, but they are attached to an oriented 3-cycle $C$ which shares one arrow $\alpha$ with the non-oriented cycle and $\alpha$ is oriented in the anti-clockwise direction. In this sense, 'attached' means that these arrows are part of the quiver $Q_\alpha$ of type $A$ which shares the vertex $z_\alpha$ with the cycle $C$ (see Definition 3.4).

Let $r_2$ be the number of oriented 3-cycles which fulfill one of the following two conditions:

1) These cycles share one arrow $\alpha$ with the non-oriented cycle and $\alpha$ is oriented in the anti-clockwise direction.

2) These cycles are attached to an oriented 3-cycle $C$ sharing one arrow $\alpha$ with the non-oriented cycle and $\alpha$ is oriented in the anti-clockwise direction. Here, 'attached' is in the same sense as above.

Similarly we define the parameters $s_1$ and $s_2$ with 'clockwise' instead of 'anti-clockwise'.

**Example 3.9.** We denote the arrows which count for the parameter $r_1$ by $\vDash$ and the arrows which count for $s_1$ by $\circlearrowleft$. Furthermore, the oriented 3-cycles of $r_2$ are denoted by $\circlearrowright$ and the oriented 3-cycles of $s_2$ are denoted by $\circlearrowright$.

i) Consider the following quiver $Q_1 \in \mathcal{Q}_6$

![Diagram](image1)

Here, we have $r_1 = 1$, $r_2 = 0$, $s_1 = 2$ and $s_2 = 2$.

ii) Consider the quiver $Q_2 \in \mathcal{Q}_8$ of the following form

![Diagram](image2)
Now, we have \( r_1 = 1, \) \( r_2 = 2, \) \( s_1 = 0 \) and \( s_2 = 2. \)

iii) The last quiver \( Q_3 \in Q_{16} \) is of the following form

\[
\begin{array}{c}
\text{Diagram of } Q_3 \in Q_{16} \text{ is shown here}.
\end{array}
\]

and we have \( r_1 = 3, \) \( r_2 = 3, \) \( s_1 = 4 \) and \( s_2 = 2. \)

Thus, we can formulate the following Lemma:

**Lemma 3.10.** If \( Q_1 \) and \( Q_2 \) are quivers in \( Q_n \), and \( Q_1 \) and \( Q_2 \) have the same parameters \( r_1, r_2, s_1 \) and \( s_2 \) (up to changing the roles of \( r_i \) and \( s_i, i \in \{1, 2\} \)), then \( Q_2 \) can be obtained from \( Q_1 \) by iterated mutation, where all the intermediate quivers have the same parameters as well.

**Proof.** It is enough to show that all quivers in \( Q_n \) with parameters \( r_1, r_2, s_1 \) and \( s_2 \) can be mutated to a quiver of the following form, without changing the parameters \( r_1, r_2, s_1 \) and \( s_2. \)

\[
\begin{array}{c}
\text{Diagram of normal form quiver is shown here.}
\end{array}
\]

Here, \( r_1 \) is the number of arrows in the anti-clockwise direction which do not share any arrow with an oriented 3-cycle and \( s_1 \) is the number of arrows in the clockwise direction which do not share any arrow with an oriented 3-cycle. Furthermore, \( r_2 \) is the number of oriented 3-cycles sharing one arrow \( \alpha \) with the non-oriented cycle and \( \alpha \) is oriented in the anti-clockwise direction and \( s_2 \) is the number of oriented 3-cycles sharing one arrow \( \beta \) with the non-oriented cycle and \( \beta \) is oriented in the clockwise direction (see Definition 3.8).

We call such a quiver a **normal form**.

We divide this process into five steps.
Step 1: Let $Q$ be a quiver in $Q_n$. We move all oriented 3-cycles of $Q$ sharing no arrow with the non-oriented cycle towards the oriented 3-cycle which is attached to them and which shares one arrow with the non-oriented cycle.

Method: Let $C$ and $C'$ be a pair of neighbouring oriented 3-cycles in $Q$ (i.e., no arrow in the path between them is part of an oriented 3-cycle) such that the length of the path between them is at least one. We want to move $C$ and $C'$ closer together by mutation.

In the picture the $Q_i$ are subquivers of $Q$. Mutating at $d$ will produce a quiver $Q^*$ which looks like this:

Thus, the length of the path between $C^*$ and $C'$ decreases by 1 and there is a path of length one between $C^*$ and $Q_c$.

(The arguments for the quiver

are analogous.)

These are the mutations we use for moving oriented 3-cycles closer together. Note that these mutations can also be used if the arrows between $d$ and $f$ are part of the non-oriented cycle (see step 4).

In this procedure, the parameters $r_1$, $r_2$, $s_1$ and $s_2$ are left unchanged since we are not changing the number of arrows and the number of oriented 3-cycles which are attached to an oriented 3-cycle sharing one arrow with the non-oriented cycle.

Step 2: We move all oriented 3-cycles onto the non-oriented cycle.

Method: Let $C$ be an oriented 3-cycle which shares one vertex $z_\alpha$ with an oriented 3-cycle $C_\alpha$ sharing an arrow $\alpha$ with the non-oriented cycle. Then we mutate at the vertex $z_\alpha$: 13
Hence, both of the oriented 3-cycles share one arrow with the non-oriented cycle and these arrows are oriented as α before. Thus, the parameters $r_1$, $r_2$, $s_1$ and $s_2$ are left unchanged. Furthermore, the length of the non-oriented cycle increases by 1. By iterated mutation of that kind, we produce a quiver $Q^*$, where all the oriented 3-cycles share an arrow with the non-oriented cycle.

**Step 3:** We move all arrows onto the non-oriented cycle.

**Method:** Let $C_i$ be an oriented 3-cycle (which can be assumed to share one arrow $β$ with the non-oriented cycle after step 2) and let $α$ be an arrow which is not part of the non-oriented cycle and which shares one vertex with $C_i$. Thus, $α$ can be assumed to be no part of an oriented 3-cycle by step 2. Then there are two different situations:

**a)** $C_i \xrightarrow{\text{mutation}} C_i' \xrightarrow{\text{at } z_α}$

Hence, $β'$ has the same orientation as $β$ and the new arrow $α'$ is part of the non-oriented cycle. Furthermore, it is also oriented as $β$, respectively $β'$. Thus, the parameters $r_1$, $r_2$, $s_1$ and $s_2$ are left unchanged. By iterated mutation of that kind, we produce a quiver $Q^*$, where all the oriented 3-cycles share one arrow with the non-oriented cycle and all arrows which are not part of such an oriented 3-cycle are part of the non-oriented cycle.
Step 4: Move oriented 3-cycles along the non-oriented cycle.

Method: As in step 1, we can move one oriented 3-cycle towards another one, without changing the orientation of the arrows, i.e., without changing the parameters $r_1$, $r_2$, $s_1$ and $s_2$.

First, we number all oriented 3-cycles by $C_1, \ldots, C_{r_2+s_2}$ in such a way that $C_{i+1}$ follows $C_i$ in the anti-clockwise direction. We begin with the oriented 3-cycle $C_1$ and move the next oriented 3-cycle $C_2$ towards this cycle. If the non-oriented cycle includes the vertex $c$ in the pictures of step 1, then the arrows between the two cycles are inserted between $C_2$ and $C_3$ in the non-oriented cycle. If the non-oriented cycle includes the vertex $a$ in the pictures of step 1, the arrows between the two cycles move to the top of $C_2$, i.e., they are no longer part of the non-oriented cycle. But then, we can reverse their directions by mutating at sinks, respectively sources, and insert these arrows into the non-oriented cycle between $C_2$ and $C_3$ by mutations like in step 3. After $C_1$ and $C_2$ share one vertex, we move the next oriented 3-cycle $C_3$ towards $C_2$. Doing this iteratively, we produce a quiver $Q^*$ as follows,

![Diagram](attachment:quiver.png)

where $r_1$ of the arrows which are not part of any oriented 3-cycle are oriented in the anti-clockwise direction and $s_1$ of the arrows not being part of any oriented 3-cycle are oriented in the clockwise direction. Moreover, $r_2$ of the oriented 3-cycles share one arrow with the non-oriented cycle which is oriented in the anti-clockwise direction and $s_2$ of the oriented 3-cycles share one arrow with the non-oriented cycle which is oriented in the clockwise direction.

Step 5: Changing orientation on the non-oriented cycle to the orientation of figure (1).

Method: The part of the non-oriented cycle without oriented 3-cycles can be mutated as follows to reach the desired orientation:

We only mutate at sinks. Thus, the parameters $r_1$ and $s_1$ are left unchanged. If there is no sink, this part is oriented in the required form. Thus, we can assume that there is at least one sink. If we have more than one sink, we begin with the sink which is closest to $C_1$. We mutate at this sink $S_1$ and thus, the vertex which follows $S_1$ in the clockwise direction is a new sink or the next source $x_1$: 
Note that the part of the non-oriented cycle without oriented 3-cycles can be without sources, if there is only one sink $S_1$.

If it is $x_1$, this first step is finished. If it is a new sink, we mutate at it and continue this procedure until the vertex which follows in the clockwise direction is not a sink, i.e., if it is either the next source $x_1$ or it is the connecting vertex of the part with oriented 3-cycles and the part without these cycles:

If there is no sink after this first step, we have finished with the desired orientation. If there are other sinks, we search the one which is closest to $C_1$, and do the same procedure as above.

Thus, there is one more arrow $\alpha_1$ which follows the vertex $S_1$ in the clockwise direction and which is oriented in the clockwise direction after each step. Furthermore, there is one more arrow $\alpha_2$ which follows $S_1$ in the anticlockwise direction and which is oriented in the anti-clockwise direction:

If no new sink arises after one step which follows $S_1$ in the anti-clockwise direction, then there is one sink less than before.

If a new sink arises after one step which follows $S_1$ in the anti-clockwise direction we have the same number of sinks as before.

Thus, we have $\#\{\text{sinks in } Q^*\} \leq \#\{\text{sinks in } Q\}$, where $Q^*$ is the quiver which we achieve after one step.

But it can not happen that $\#\{\text{sinks in } Q^*\} = \#\{\text{sinks in } Q\}$ after each step since the part of the non-oriented cycle without oriented 3-cycles is bounded in the anti-clockwise direction (by $C_1$). Thus, we either arise the following situation after a finite number of mutations:
or we arise the situation (*). Hence, the number of sinks decreases by 1 after mutating at $S_1$.

Doing this iteratively, the number of sinks decreases to zero and thus, we reach the desired orientation.

Note that this is a similar process as in the proof of Proposition 3.2.

Every oriented 3-cycle shares one arrow with the non-oriented cycle. If all of these arrows are oriented in the same direction, the quiver is in the required form. Thus, we can assume that there are at least two arrows of two oriented 3-cycles $C_i$ and $C_{i+1}$ which are oriented in converse directions. If we mutate at the connecting vertex of $C_i$ and $C_{i+1}$, the directions of these arrows are changed:

Thus, these mutations act like sink/source mutations at the non-oriented cycle and the parameters $r_2$ and $s_2$ are left unchanged. Since we want the converse orientation as in the part without oriented 3-cycles, we mutate at sources instead of sinks. But then, we can mutate at such connecting vertices as in the part without oriented 3-cycles to reach the desired orientation of figure (1).

**Theorem 3.11.** Let $Q \in \mathcal{Q}_n$ with parameters $r_1$, $r_2$, $s_1$ and $s_2$. Then $Q$ is mutation equivalent to a non-oriented cycle of length $n + 1$ with parameters $r = r_1 + 2r_2$ and $s = s_1 + 2s_2$.

**Proof.** We can assume that $Q$ is a normal form (see Lemma 3.10):
Mutation at the vertex $x_i$ of an oriented 3-cycle

leads to two arrows of the following form

Thus, after mutating in all the $x_i$, the parameter $r_2$ is zero and we have a new parameter $r = r_1 + 2r_2$.

Similarly, we get $s = s_1 + 2s_2$.

Hence, mutating in all the $x_i$ and $y_i$ leads to a quiver with underlying graph $\tilde{A}_n$ as follows:

Since there is a non-oriented cycle in every $Q \in Q_n$, these parameters $r$ and $s$ are non-zero. Thus, the graph $\tilde{A}_n$ above is also non-oriented. Hence, $Q$ is mutation equivalent to some quiver of type $\tilde{A}_n$ with parameters $r = r_1 + 2r_2$ and $s = s_1 + 2s_2$. 

\[ \square \]
Corollary 3.12. Let \( Q_1, Q_2 \in \mathcal{Q}_n \) with parameters \( r_1, r_2, s_1 \) and \( s_2 \), respectively. If \( r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2 \) and \( s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2 \) (or vice versa), then \( Q_1 \) is mutation equivalent to \( Q_2 \).

Theorem 3.13. Let \( Q_1, Q_2 \in \mathcal{Q}_n \) with parameters \( r_1, r_2, s_1 \) and \( s_2 \), respectively. Then \( Q_1 \) is mutation equivalent to \( Q_2 \) if and only if \( r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2 \) and \( s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2 \) (or \( r_1 + 2r_2 = \tilde{s}_1 + 2\tilde{s}_2 \) and \( s_1 + 2s_2 = \tilde{r}_1 + 2\tilde{r}_2 \)).

Note that the only-if-part follows from Theorem 3.11 and Lemma 3.3.

4 Cluster tilted algebras of type \( \tilde{A}_n \)

In general, cluster tilted algebras arise as endomorphism algebras of cluster-tilting objects in a cluster category. Since a cluster tilted algebra \( A \) is finite dimensional over an algebraically closed field \( K \), there exists a quiver \( Q \) which is in the mutation classes of \( \tilde{A}_n \) and an admissible ideal \( I \) of the path algebra \( KQ \) of \( Q \) such that \( A \cong KQ/I \). A non-zero linear combination \( k_1\alpha_1 + \cdots + k_m\alpha_m, k_i \in K \setminus \{0\} \), of paths \( \alpha_i \) of length at least two, with the same starting point and the same end point, is called a relation in \( Q \). If \( m = 1 \), we call such a relation a zero-relation. Any admissible ideal of \( KQ \) is generated by a finite set of relations in \( Q \).

From [1] and [4] we know that a cluster tilted algebra \( A \) is gentle. Thus, recall the definition of gentle algebras:

Definition 4.1. We call \( A = KQ/I \) a special biserial algebra if the following properties hold:

1) Each vertex of \( Q \) is starting point of at most two arrows and end point of at most two arrows.

2) For each arrow \( \alpha \) in \( Q \) there is at most one arrow \( \beta \) such that \( \alpha\beta \notin I \), and at most one arrow \( \gamma \) such that \( \gamma\alpha \notin I \).

A is gentle if moreover:

3) The ideal \( I \) is generated by paths of length 2.

4) For each arrow \( \alpha \) in \( Q \) there is at most one arrow \( \beta' \) with \( t(\alpha) = s(\beta') \) such that \( \beta'\alpha \in I \), and there is at most one arrow \( \gamma' \) with \( t(\gamma') = s(\alpha) \) such that \( \alpha\gamma' \in I \).

Furthermore, all relations in a cluster tilted algebra \( A \) of type \( \tilde{A}_n \) occur in the oriented 3-cycles, i.e., in cycles of the form

\[
\begin{array}{c}
\alpha \\
\gamma \\
\beta
\end{array}
\]

with (zero-)relations \( \alpha\gamma, \beta\alpha \) and \( \gamma\beta \) (see [1] and [4]).

Remark 4.2. According to our convention in Definition 3.4 there are only three (zero-)relations in the following quiver

\[
\begin{array}{c}
\beta \\
\gamma \\
\delta
\end{array}
\]

and here, these are \( \alpha\delta, \beta\alpha \) and \( \delta\beta \).
4.1 Cartan matrices

For the next section, we need the notion of Cartan matrices of an algebra \( A \) (for example see \[16\]). Let \( K \) be a field and \( A = KQ/I \). Since \( \sum_{i \in Q^0} e_i \) is the unit element in \( A \) we get \( A = A \cdot 1 = \bigoplus_{i \in Q^0} Ae_i \), hence the (left) \( A \)-modules \( P_i := Ae_i \) are the indecomposable projective \( A \)-modules. The Cartan matrix \( C = (c_{ij}) \) of \( A \) is a \(|Q^0| \times |Q^0|\)-matrix defined by setting \( c_{ij} = \text{dim}_K \text{Hom}_A(P_j, P_i) \). Any homomorphism \( \varphi : Ae_j \rightarrow Ae_i \) of left \( A \)-modules is uniquely determined by \( \varphi(e_j) \in e_j Ae_i \), the \( K \)-vector space generated by all paths in \( Q \) from vertex \( i \) to vertex \( j \) which are non-zero in \( A \). In particular, we have \( c_{ij} = \text{dim}_K e_j Ae_i \). That means, computing entries of the Cartan matrix for \( A \) reduces to counting paths in \( Q \) which are non-zero in \( A \).

Example 4.3. We consider two quivers which are in the mutation class of \( \tilde{A}_2 \), i.e., the corresponding algebras are cluster tilted algebras of type \( \tilde{A}_2 \).

i) First, we have a look at an algebra \( A \) which corresponds to the following quiver \( Q \)

```
1 —-> 3 —-> 2
 \ ^ \ ^ \ ^
  \ |   \ |   \ |  \alpha_3 \alpha_2 \alpha_1
```

Here, we have no relations since there is no oriented 3-cycle in \( Q \). Hence, we can compute the Cartan matrix of \( A \) to be
\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

ii) Now consider the algebra \( B \) which corresponds to the quiver below

```
1 —-> 3 —-> 2
 \ ^ \ ^ \ ^
  \ |   \ |   \ |  \alpha_3 \alpha_2 \alpha_1
```

Here, we have three (zero-)relations \( \alpha_1 \alpha_3, \alpha_3 \alpha_1 \) and \( \alpha_3 \alpha_2 \). Note that the paths \( \alpha_3 \alpha_4 \) and \( \alpha_4 \alpha_1 \) are not zero. Thus, we can compute the Cartan matrix of \( B \) to be
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 0 & 1
\end{pmatrix}
\]

5 Derived equivalence classification of cluster tilted algebras of type \( \tilde{A}_n \)

First, we briefly review the fundamental results on derived equivalences.

For a \( K \)-algebra \( A \) the bounded derived category of \( A \)-modules is denoted by \( D^b(A) \). Recall that two algebras \( A, B \) are called derived equivalent if \( D^b(A) \) and \( D^b(B) \) are equivalent as triangulated categories. By a theorem of Rickard \[22\] derived equivalences can be found using the concept of tilting complexes.

Definition 5.1. A tilting complex \( T \) over \( A \) is a bounded complex of finitely generated projective \( A \)-modules satisfying the following conditions:
i) $\text{Hom}_{D^b(A)}(T, T[i]) = 0$ for all $i \neq 0$, where $[.]$ denotes the shift functor in $D^b(A)$;

ii) the category $\text{add}(T)$ (i.e. the full subcategory consisting of direct summands of direct sums of $T$) generates the homotopy category $K^b(P_A)$ of projective $A$-modules as a triangulated category.

We can now formulate Rickard’s celebrated result.

**Theorem 5.2** (Rickard [22]). Two algebras $A$ and $B$ are derived equivalent if and only if there exists a tilting complex $T$ for $A$ such that the endomorphism algebra $\text{End}_{D^b(A)}(T) \cong B$.

For calculating the endomorphism ring $\text{End}_{D^b(A)}(T)$ we can use the following alternating sum formula which gives a general method for computing the Cartan matrix of an endomorphism ring of a tilting complex from the Cartan matrix of the algebra $A$.

**Proposition 5.3** (Happel [15]). For an algebra $A$ let $Q = (Q^r)_{r \in \mathbb{Z}}$ and $R = (R^s)_{s \in \mathbb{Z}}$ be bounded complexes of projective $A$-modules. Then

$$\sum_i (-1)^i \dim \text{Hom}_{D^b(A)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim \text{Hom}_A(Q^r, R^s).$$

In particular, if $Q$ and $R$ are direct summands of the same tilting complex then

$$\dim \text{Hom}_{D^b(A)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim \text{Hom}_A(Q^r, R^s).$$

**Lemma 5.4.** Let $A = KQ/I$ be a cluster tilted algebra of type $\tilde{A}_n$. Let $r_1$, $r_2$, $s_1$ and $s_2$ be the parameters of $Q$ which are defined in 3.8. Then $A$ is derived equivalent to a cluster tilted algebra corresponding to the following quiver in normal form:

![Quiver Diagram]

**Proof.** First we note that the number of oriented 3-cycles with full relations is invariant under derived equivalence for gentle algebras (see [10]), i.e., the number $r_2 + s_2$ is an invariant. Furthermore, from Proposition B in [5] we know that the number of arrows is also invariant under derived equivalence, i.e., the number $r_1 + s_1$ is an invariant, too. Later, we show in the proof of Theorem 5.5 that the single parameters $r_1$, $r_2$, $s_1$ and $s_2$ are invariants of derived equivalence.
Our strategy in this proof is to go through the proof of Lemma 3.10 and define a tilting complex for each mutation in the steps 1 and 2. We can omit the three other steps since these are just the same situations as in the first two steps. We show that if we mutate at some vertex of the quiver $Q$ and obtain a quiver $Q'$, then the two corresponding cluster tilted algebras are derived equivalent.

Step 1

Let $A$ be a cluster tilted algebra with corresponding quiver

We can compute the Cartan matrix to be

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & \ddots \\
0 & 1 & 1 & 0 & \ddots \\
1 & 0 & 1 & 0 & \ddots \\
1 & 0 & 1 & 1 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Since we deal with left modules and read paths from right to left, a non-zero path from vertex $i$ to $j$ gives a homomorphism $P_j \to P_i$ by right multiplication. Thus, two arrows $\alpha : i \to j$ and $\beta : j \to k$ give a path $\beta \alpha$ from $i$ to $k$ and a homomorphism $\alpha \beta : P_k \to P_i$.

Let $T = \bigoplus_{i=1}^{n} T_i$ be the following bounded complex of projective $A$-modules, where $T_i : 0 \to P_i \to 0$, $i \in \{1, 2, 4, \ldots, n\}$, are complexes concentrated in degree zero and $T_3 : 0 \to P_3 \xrightarrow{(\alpha_3, \alpha_4)} P_2 \oplus P_4 \to 0$ is a complex concentrated in degrees $-1$ and $0$.

Now we want to show that $T$ is a tilting complex. We only show that condition $i)$ of Definition 5.1 is fulfilled since condition $ii)$ holds for all such two-term-complexes of indecomposable projective modules we need.

Since condition $i)$ is obvious for all $|i| \geq 2$ we begin with possible maps $T_3 \to T_3[1]$ and $T_3 \to T_3[-1]$.

$$
\begin{align*}
0 & \to P_3 \xrightarrow{(\alpha_3, \alpha_4)} P_2 \oplus P_4 \to 0 \\
& \downarrow \psi \\
0 & \to P_3 \xrightarrow{(\alpha_3, \alpha_4)} P_2 \oplus P_4 \to 0 \\
& \downarrow 0 \\
0 & \to P_3 \xrightarrow{(\alpha_3, \alpha_4)} P_2 \oplus P_4 \to 0
\end{align*}
$$

where $\psi \in \text{Hom}(P_3, P_2 \oplus P_4)$ and $(\alpha_3, 0), (0, \alpha_4)$ is a basis of this two-dimensional space.

The first homomorphism is homotopic to zero (as we can easily see). In the second case there is no non-zero homomorphism $P_2 \oplus P_1 \to P_3$ (as we can see in the Cartan matrix of $A$).
Now consider possible maps $T_3 \to T_j[-1]$, $j \neq 3$. These maps are given by a map of complexes as follows

$$
0 \to P_3 \xrightarrow{(\alpha_3, \alpha_4)} P_2 \oplus P_4 \to 0
$$

where $Q$ can be any $P_i$ such that there is a path from vertex $i$ to vertex 3 not containing a zero-relation, or direct sums of these. There exist non-zero homomorphisms of complexes, but they are all homotopic to zero since every path from vertex $i$ to vertex 3 ends with $\alpha_3$ or $\alpha_4$. Thus, every homomorphism from $P_3$ to $Q$ starts with a scalar multiple of $\alpha_3$ or $\alpha_4$ and hence, every homomorphism $P_3 \to Q$ can be factored through the map $(\alpha_3, \alpha_4) : P_3 \to P_2 \oplus P_4$.

Directly from the definition we see that $\text{Hom}(T, T_j[-1]) = 0$ for $j \neq 3$ and thus we have shown that $\text{Hom}(T, T[-1]) = 0$.

Finally we have to consider maps $T_j \to T_3[1]$ for $j \neq 3$. These are given as follows

$$
0 \to Q \to 0
\downarrow
0 \to P_3 \xrightarrow{(\alpha_3, \alpha_4)} P_2 \oplus P_4 \to 0
$$

where $Q$ can be any $P_i$ such that there is a path from vertex 3 to vertex $i$ not containing a zero-relation, or direct sums of these. But no non-zero map can be zero when composed with both $\alpha_3$ and $\alpha_4$ since the path $\alpha_1\alpha_4$ is not a zero-relation. So the only homomorphism of complexes $T_j \to T_3[1]$, $j \neq 3$, is the zero map.

It follows that $\text{Hom}_{\mathcal{D}^b(A)}(T, T[i]) = 0$ in the derived category. Hence, $T$ is indeed a tilting complex for $A$.

By Rickard’s Theorem 5.2, $E := \text{End}_{\mathcal{D}^b(A)}(T)$ is derived equivalent to $A$. Using the alternating sum formula of the Proposition 5.3 of Happel we can compute the Cartan matrix of $E$ to be

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
& 0 & 1 & 1 & \cdots \\
& & 1 & 0 & 0 & \cdots \\
& & & & & \cdots
\end{pmatrix}
$$

We define homomorphisms in $E$ as follows

$$
\begin{array}{c}
Q_1 \\
α_3α_4 \\
\alpha_2, 0 \\
2(id, 0)
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
Q_z \quad x \\
\alpha_4α_1 \\
(0, id) \\
4
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
Q_2 \\
\cdots
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
y \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
z \\
\cdots \\
\cdots \\
\cdots
\end{array}
\begin{array}{c}
Q_z
\end{array}
$$

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Now we have to check the relations, up to homotopy. Clearly, the homomorphism \((\alpha_4\alpha_1\alpha_2, 0)\) in the oriented 3-cycle containing the vertices 1, 3 and 4 is zero since \(\alpha_1\alpha_2\) was zero in \(A\). Furthermore, the composition of \((\alpha_2, 0)\) and \((0, \text{id})\) yields to a zero-relation. The last zero-relation in this oriented 3-cycle is the concatenation of \((0, \text{id})\) and \(\alpha_4\alpha_1\) since this homomorphism is homotopic to zero:

\[
\begin{array}{ccc}
0 & \xrightarrow{\alpha_1} & P_1 \\
\downarrow & & \downarrow \\
(0, \alpha_4\alpha_1) & \xrightarrow{} & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{} & P_1 \\
\downarrow & & \downarrow \\
(\alpha_3, \alpha_4) & \xrightarrow{} & P_2 \oplus P_4 \\
\end{array}
\]

The relations in all the other oriented 3-cycles of this quiver are the same as in the quiver of \(A\).

Thus, we defined homomorphisms between the summands of \(T\) corresponding to the arrows of the quiver which we obtain after mutating at vertex 3 in the quiver of \(A\). We have shown that they satisfy the defining relations of this algebra and the Cartan matrices agree. Thus, \(A\) is derived equivalent to \(E\) and \(A^{op}\) is derived equivalent to \(E^{op}\), where the quiver of \(E\) is the same as the quiver we obtain after mutating at vertex 3 in the quiver of \(A\). Furthermore, the quivers of \(A^{op}\) and \(E^{op}\) are the quivers in the other case in step 1.

**Step 2** Let \(A\) be a cluster tilted algebra with corresponding quiver

We define a tilting complex \(T\) as follows: Let \(T = \bigoplus_{i=1}^{n} T_i\) be the following bounded complex of projective \(A\)-modules, where \(T_i : 0 \rightarrow P_i \rightarrow 0\), \(i \in \{1, 2, 4, \ldots, n\}\), are complexes concentrated in degree zero and \(T_3 : 0 \rightarrow P_1 \rightarrow P_2 \oplus P_4 \rightarrow 0\) is a complex concentrated in degrees \(-1\) and \(0\).

By Rickard’s theorem, \(E := \text{End}_{D^b(A)}(T)\) is derived equivalent to \(A\). Using the alternating sum formula of the Proposition of Happel we can compute the Cartan matrix of \(E\) to be

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & \ldots \\
1 & 1 & 1 & 0 & 0 & \ldots \\
1 & 0 & 1 & 0 & 1 & \ldots \\
0 & 1 & 0 & 1 & 0 & \ldots \\
0 & 0 & 1 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

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We define homomorphisms in $E$ as follows

$$
\begin{align*}
\alpha_1 & \rightarrow 2 \\
\alpha_2 & \rightarrow 3 \\
\alpha_3 & \rightarrow 4 \\
\alpha_4 & \rightarrow 5 \\
(\alpha_1,0) & \rightarrow (0,\alpha_2) \\
(0,\alpha_3) & \rightarrow (\alpha_4,0)
\end{align*}$$

Thus, $A$ is derived equivalent to $E$ and $A^{op}$ is derived equivalent to $E^{op}$, where the quiver of $E$ is the same as the quiver we obtain after mutating at 3.

In the steps 3, 4 and 5 of the proof of Lemma 3.10 we are just in the same situations as in the first two steps. In the steps 3 and 4 we mutate at a vertex with three incident arrows as in step 1. In step 5 we mutate at sinks and at vertices with four incident arrows as in step 2. Thus, we obtain a quiver of a derived equivalent cluster tilted algebra by all mutations in the proof of Lemma 3.10. Hence, every cluster tilted algebra $A = KQ/I$ of type $A_n$ is derived equivalent to a cluster tilted algebra with a quiver in normal form which has the same parameters as $Q$. □

Our next aim is to prove the main result:

**Theorem 5.5.** Two cluster tilted algebras of type $A_n$ are derived equivalent if and only if their quivers have the same parameters $r_1$, $r_2$, $s_1$ and $s_2$ (up to changing the roles of $r_1$ and $s_i$, $i \in \{1, 2\}$).

But first, we recall some background from [3]. Let $A = KQ/I$ be a gentle algebra, where $Q = (Q_0, Q_1)$ is a connected quiver. A **permitted path** of $A$ is a path $C = \alpha_1 \ldots \alpha_2 \alpha_1$ which contains no zero-relations. A permitted path $C$ is called a non-trivial permitted thread if for all $\beta \in Q_1$ neither $\beta C$ nor $C \beta$ is a permitted path. Similarly a **forbidden path** of $A$ is a sequence $\Pi = \alpha_2 \ldots \alpha_2 \alpha_1$ formed by pairwise different arrows in $Q$ with $\alpha_{i+1} \alpha_i \in I$ for all $i \in \{1, 2, \ldots, n-1\}$. A forbidden path $\Pi$ is called a non-trivial forbidden thread if for all $\beta \in Q_1$ neither $\beta \Pi$ nor $\Pi \beta$ is a forbidden path. Let $v \in Q_0$ such that $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = t(\beta)$ then $\beta \gamma \notin I$. Then we consider $e_v$, a trivial permitted thread in $v$ and denote it by $h_v$. Let $H_A$ be the set of all permitted threads of $A$, trivial and non-trivial. Similarly, for $v \in Q_0$ such that $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = t(\beta)$ then $\gamma \beta \notin I$, we consider $e_v$ a trivial forbidden thread in $v$ and denote it by $p_v$. Note that certain paths can be permitted and forbidden threads simultaneously.

Now, define two functions $\sigma, \varepsilon : Q_1 \rightarrow \{1, -1\}$ by:

1) If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.

2) If $\gamma_1 \neq \gamma_2$ are arrows with $t(\gamma_1) = t(\gamma_2)$, then $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$.

3) If $\beta$ and $\gamma$ are arrows with $s(\gamma) = t(\beta)$ and $\gamma \beta \notin I$, then $\sigma(\gamma) = -\varepsilon(\beta)$.

We can extend these functions to threads of $A$ as follows: For a non-trivial thread $H = \alpha_n \ldots \alpha_2 \alpha_1$ of $A$ define $\sigma(H) := \sigma(\alpha_1)$ and $\varepsilon(H) := \varepsilon(\alpha_n)$. If there is a trivial permitted thread $h_v$ for some $v \in Q_0$, the connectivity of $Q$ assures the existence of some $\gamma \in Q_1$ with $s(\gamma) = v$ or some $\beta \in Q_1$ with $t(\beta) = v$. In the first case, we define $\sigma(h_v) = -\varepsilon(h_v) := -\sigma(\gamma)$, for the second case $\sigma(h_v) = -\varepsilon(h_v) := \varepsilon(\beta)$. If there is a trivial forbidden thread $p_v$ for some $v \in Q_0$, we know
that there exists $\gamma \in Q_1$ with $s(\gamma) = v$ or $\beta \in Q_1$ with $t(\beta) = v$. In the first case, we define $\sigma(p_v) = \varepsilon(h_v) := -\varepsilon(\gamma)$, for the second case $\sigma(p_v) = \varepsilon(h_v) := -\varepsilon(\beta)$.

Now there is a combinatorial algorithm (stated in [5]) to produce certain pairs of natural numbers, by using only the quiver with relations which defines a gentle algebra. In the algorithm we are going forward through permitted threads and backwards through forbidden threads in such a way that each arrow and its inverse is used exactly once.

**Definition 5.6.** The algorithm is as follows:

1) a) Begin with a permitted thread $H_0$ of $A$.

   b) If $H_i$ is defined, consider $\Pi_i$ the forbidden thread which ends in $t(H_i)$ and such that $\varepsilon(H_i) = -\varepsilon(\Pi_i)$.

   c) Let $H_{i+1}$ be the permitted thread which starts in $s(\Pi_i)$ and such that $\sigma(H_{i+1}) = -\sigma(\Pi_i)$.

   The process stops when $H_n = H_0$ for some natural number $n$. Let $m = \sum_{1 \leq i \leq n} l(\Pi_{i-1})$, where $l()$ is the length of a path, i.e., the number of arrows of the path. We obtain the pair $(n, m)$.

2) Repeat the first step of the algorithm until all permitted threads of $A$ have been considered.

3) If there are directed cycles in which each pair of consecutive arrows form a relation, we add a pair $(0, m)$ for each of those cycles, where $m$ is the length of the cycle.

4) Define $\phi_A : \mathbb{N}^2 \rightarrow \mathbb{N}$, where $\phi_A(n, m)$ is the number of times the pair $(n, m)$ arises in the algorithm.

This function $\phi$ is invariant under derived equivalence, i.e.,

**Lemma 5.7** (Avella-Alaminos and Geiß [5]). *Let $A$ and $B$ be gentle algebras. If $A$ and $B$ are derived equivalent, then $\phi_A = \phi_B$.*

**Example 5.8.** Consider the following quiver of a cluster tilted algebra $A$ of type $\tilde{A}_{18}$,
where \( r_1 = 2 \), \( r_2 = 3 \), \( s_1 = 3 \) and \( s_2 = 4 \) and thus, \( r := r_1 + r_2 = 5 \), \( s := s_1 + s_2 = 7 \).

Now, we define the functions \( \sigma \) and \( \varepsilon \) for all arrows in \( Q \):

\[
\begin{align*}
\sigma(\alpha_1) &= 1, \quad \varepsilon(\alpha_1) = -1 \quad \text{for all } i = 1, \ldots, 5 \\
\sigma(\alpha_1) &= -1, \quad \varepsilon(\alpha_1) = 1 \quad \text{for all } i = 6, \ldots, 12 \\
\sigma(\beta_{j,1}) &= 1, \quad \varepsilon(\beta_{j,1}) = 1 \quad \text{for all } j = 1, \ldots, 3 \\
\sigma(\beta_{j,2}) &= -1, \quad \varepsilon(\beta_{j,2}) = 1 \quad \text{for all } j = 1, \ldots, 3 \\
\sigma(\gamma_{l,1}) &= -1, \quad \varepsilon(\gamma_{l,1}) = -1 \quad \text{for all } l = 1, \ldots, 4 \\
\sigma(\gamma_{l,2}) &= 1, \quad \varepsilon(\gamma_{l,2}) = -1 \quad \text{for all } l = 1, \ldots, 4
\end{align*}
\]

In this case \( \mathcal{H}_A \) is formed by \( h_{v_1}, h_{v_2}, h_{v_3}, \gamma_{4,2} \alpha_5 \alpha_3 \alpha_2 \alpha_1, \beta_{3,2} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6, \beta_{1,1}, \beta_{1,2} \beta_{2,1}, \beta_{2,2} \beta_{3,1}, \gamma_{1,1}, \gamma_{1,2} \gamma_{2,1}, \gamma_{2,2} \gamma_{3,1} \) and \( \gamma_{3,2} \gamma_{4,1} \).

The forbidden threads of \( A \) are \( p_{x_1}, p_{x_2}, p_{x_3}, h_{y_1}, h_{y_2}, p_{y_3}, p_{y_4}, \alpha_1, \alpha_2, \alpha_6, \alpha_7, \alpha_8 \) and all the oriented 3-cycles.

Moreover, we can write

\[
\begin{align*}
\sigma(h_{v_1}) &= -\varepsilon(h_{v_1}) = -\sigma(\alpha_2) = \varepsilon(\alpha_1) = -1 \\
\sigma(h_{v_2}) &= -\varepsilon(h_{v_2}) = -\sigma(\alpha_7) = \varepsilon(\alpha_6) = 1 \\
\sigma(h_{v_3}) &= -\varepsilon(h_{v_3}) = -\sigma(\alpha_8) = \varepsilon(\alpha_7) = 1 \\
\sigma(p_{x_1}) &= \varepsilon(p_{x_1}) = -\sigma(\beta_{1,1}) = -\varepsilon(\beta_{1,2}) = -1 \quad \text{for all } i = 1, 2, 3 \\
\sigma(p_{y_1}) &= \varepsilon(p_{y_1}) = -\sigma(\gamma_{l,1}) = -\varepsilon(\gamma_{l,2}) = 1 \quad \text{for all } i = 1, 2, 3, 4
\end{align*}
\]

for the trivial permitted threads and

\[
\begin{align*}
\sigma(\varepsilon) &= \varepsilon(\sigma) = -\varepsilon(\varepsilon) = 1
\end{align*}
\]

for the trivial forbidden threads.

Let \( H_0 = h_{v_1} \) and \( \Pi_0 = \alpha_1 \) with \( \varepsilon(h_{v_1}) = -\varepsilon(\alpha_1) = 1 \). Then \( H_1 \) is the permitted thread which starts in \( s(\Pi_0) = v_0 \) and \( \sigma(H_1) = \sigma(\alpha_6) = -\sigma(\Pi_0) = -1 \), that is \( \beta_{3,2} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \).

Now \( \Pi_1 = p_{x_2} \) since it is the forbidden thread which ends in \( \beta_{3,2} \) and \( \varepsilon(\Pi_1) = -\varepsilon(H_1) = -\varepsilon(\beta_{3,2}) = -1 \). Then \( H_2 = \beta_{3,2} \beta_{3,1} \) is the permitted thread starting in \( x_3 \) and \( \sigma(\Pi_1) = -\sigma(H_2) = -\sigma(\beta_{3,1}) = -1 \). Thus, \( \Pi_2 = p_{x_2} \) with \( \sigma(\Pi_2) = \sigma(H_2) = \sigma(\beta_{3,2}) = -1 \). Now \( H_3 = \beta_{3,2} \beta_{2,1} \) is the permitted thread which starts in \( x_2 \) and \( \sigma(\Pi_2) = -\sigma(H_3) = -\sigma(\beta_{2,1}) = -1 \). Then \( \Pi_3 = p_{x_1} \) is the trivial forbidden thread in \( x_1 \) with \( \varepsilon(H_3) = \varepsilon(\beta_{2,1}) = -\varepsilon(\Pi_3) = 1 \). Then \( H_4 \), the permitted thread starting in \( s(\Pi_3) = x_1 \) and such that \( \sigma(\Pi_4) = -\sigma(\Pi_3) = 1 \) is the path \( \beta_{1,1} \). Thus, \( \Pi_4 = p_{x_1} \) for the forbidden thread which ends in \( v_2 \) with \( \varepsilon(\Pi_4) = -\varepsilon(H_4) = -\varepsilon(\alpha_2) = 1 \). Then \( H_5 = h_{v_1} = H_0 \), \( n = 5 \) and \( m = l(\Pi_0) + l(\Pi_1) + l(\Pi_2) + l(\Pi_3) + l(\Pi_4) = 1 + 0 + 0 + 0 + 1 = 2 \).

The corresponding pair is \((5,2) = (r,r_1)\). We can write this as follows:

\[
\begin{align*}
H_0 &= h_{v_1} & \Pi_0^{-1} &= \alpha_1^{-1} \\
H_1 &= \beta_{3,2} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 & \Pi_1^{-1} &= p_{x_3} \\
H_2 &= \beta_{3,2} \beta_{3,1} & \Pi_2^{-1} &= p_{x_2} \\
H_3 &= \beta_{2,1} \beta_{2,1} & \Pi_3^{-1} &= p_{x_1} \\
H_4 &= \beta_{1,1} & \Pi_4^{-1} &= \alpha_2^{-1} \\
H_5 &= H_0 \\
\rightarrow (5,2)
\end{align*}
\]

where \( \alpha_1^{-1} \) is defined by \( s(\alpha_1^{-1}) := e(\alpha_1), e(\alpha_1^{-1}) := s(\alpha_1) \) and \( (\alpha_1^{-1})^{-1} = \alpha_1 \).

If we continue with the algorithm we obtain the second pair \((7,3) = (s,s_1)\) in the following way:
Finally, we have to add seven pairs \((0, 3)\) for the seven oriented 3-cycles. Thus, we have \(\phi_A(5, 2) = 1\), \(\phi_A(7, 3) = 1\) and \(\phi_A(0, 3) = 7\).

Now we can extend this example to general quivers of cluster tilted algebras of type \(\tilde{A}_n\) in normal form.

Proof of Theorem 5.5

We know from Lemma 5.4 that every cluster tilted algebra \(A = KQ/I\) of type \(\tilde{A}_n\) with parameters \(r_1, r_2, s_1\) and \(s_2\) is derived equivalent to a cluster tilted algebra with a quiver in normal form, where \(r_1\) is the number of arrows in the anti-clockwise direction which do not share any arrow with an oriented 3-cycle and \(s_1\) is the number of arrows in the clockwise direction which do not share any arrow with an oriented 3-cycle. Moreover, \(r_2\) is the number of oriented 3-cycles which share one arrow \(\alpha\) with the non-oriented cycle and \(\alpha\) is oriented in the anti-clockwise direction and \(s_2\) is the number of oriented 3-cycles which share one arrow \(\beta\) with the non-oriented cycle and \(\beta\) is oriented in the clockwise direction (see Definition 3.8). Thus, \(r := r_1 + r_2\) is the number of arrows of the non-oriented cycle in the anti-clockwise direction and \(s := s_1 + s_2\) is the number of arrows of the non-oriented cycle in the clockwise direction.

Consider the quiver \(Q\) in normal form with the following notations:

\[
\begin{align*}
H_0 &= h_{v_6} & \Pi_0^{-1} &= \alpha_6^{-1} \\
H_1 &= \gamma_{4,2}0504030201 & \Pi_1^{-1} &= p_{y_4} \\
H_2 &= \gamma_{3,2}74,1 & \Pi_2^{-1} &= p_{y_3} \\
H_3 &= \gamma_{2,2}73,1 & \Pi_3^{-1} &= p_{y_2} \\
H_4 &= \gamma_{1,2}72,1 & \Pi_4^{-1} &= p_{y_1} \\
H_5 &= \gamma_{1,1} & \Pi_5^{-1} &= \alpha_8^{-1} \\
H_6 &= h_{v_2} & \Pi_6^{-1} &= \alpha_7^{-1} \\
H_7 &= H_0 \\
\rightarrow (7, 3)
\end{align*}
\]
Thus, we can apply the algorithm 5.6 as follows:

Now, we define the functions \( \sigma \) and \( \varepsilon \) for all arrows in \( Q \):

\[
\begin{align*}
\sigma(\alpha_i) &= 1, & \varepsilon(\alpha_i) &= -1 \quad \text{for all} \quad i = 1, \ldots, r \\
\sigma(\alpha_i) &= -1, & \varepsilon(\alpha_i) &= 1 \quad \text{for all} \quad i = r + 1, \ldots, r + s \\
\sigma(\beta_{j,1}) &= 1, & \varepsilon(\beta_{j,1}) &= 1 \quad \text{for all} \quad j = 1, \ldots, r_2 \\
\sigma(\beta_{j,2}) &= -1, & \varepsilon(\beta_{j,2}) &= 1 \quad \text{for all} \quad j = 1, \ldots, r_2 \\
\sigma(\gamma_{l,1}) &= -1, & \varepsilon(\gamma_{l,1}) &= -1 \quad \text{for all} \quad l = 1, \ldots, s_2 \\
\sigma(\gamma_{l,2}) &= 1, & \varepsilon(\gamma_{l,2}) &= -1 \quad \text{for all} \quad l = 1, \ldots, s_2
\end{align*}
\]

In this case \( \mathcal{H}_A \) is formed by

\[
\begin{align*}
h_{v_1}, \ldots, h_{v_{r-1}}, h_{v_r+1}, \ldots, h_{v_{r+s-1}}, \gamma_{s_2,2} \alpha_r \alpha_{r-1} \ldots \alpha_2 \alpha_1, \\
\beta_{r_2,2} \alpha_{r+s} \alpha_{r+s-1} \ldots \alpha_{r+2} \alpha_{r+1}, \beta_{1,1}, \beta_{1,2} \beta_{2,1}, \ldots, \beta_{r_2-1.2} \beta_{r_2-1.1}, \gamma_{1,1}, \gamma_{1,2} \gamma_{2,1}, \ldots, \gamma_{s_2-1.2} \gamma_{s_2-1.1}.
\end{align*}
\]

The forbidden threads of \( A \) are \( p_{x_1}, \ldots, p_{x_{r_2}}, p_{y_1}, \ldots, p_{y_{s_2}}, \alpha_1, \ldots, \alpha_{r_1}, \alpha_{r+1}, \ldots, \alpha_{r+s} \) and all the oriented 3-cycles.

Moreover, we can write

\[
\begin{align*}
\sigma(h_{v_1}) &= -\varepsilon(h_{v_1}) = -\sigma(\alpha_2) = \varepsilon(\alpha_1) = -1 \\
\cdots \\
\sigma(h_{v_{r-1}}) &= -\varepsilon(h_{v_{r-1}}) = -\sigma(\alpha_{r_1}) = \varepsilon(\alpha_{r_1-1}) = -1 \\
\sigma(h_{v_r+1}) &= -\varepsilon(h_{v_r+1}) = -\sigma(\alpha_{r+2}) = \varepsilon(\alpha_{r+1}) = 1 \\
\cdots \\
\sigma(h_{v_{r+s-1}}) &= -\varepsilon(h_{v_{r+s-1}}) = -\sigma(\alpha_{r+s_1}) = \varepsilon(\alpha_{r+s_1-1}) = 1
\end{align*}
\]

for the trivial permitted threads and

\[
\begin{align*}
\sigma(p_{x_i}) &= \varepsilon(p_{x_i}) = -\sigma(\beta_{i,1}) = -\varepsilon(\beta_{i,2}) = -1 \quad \text{for all} \quad i = 1, \ldots, r_2 \\
\sigma(p_{y_i}) &= \varepsilon(p_{y_i}) = -\sigma(\gamma_{i,1}) = -\varepsilon(\gamma_{i,2}) = 1 \quad \text{for all} \quad i = 1, \ldots, s_2
\end{align*}
\]

for the trivial forbidden threads.

Thus, we can apply the algorithm 5.6 as follows:
If we continue with the algorithm we obtain the second pair \((s, s_1)\) in the following way:

\[
\begin{align*}
H_0 &= h_{r+1} \\
H_1 &= \gamma_{s_2,2}\alpha_r\alpha_{r-1}\ldots\alpha_2\alpha_1 \\
H_2 &= \gamma_{s_2-1,2}\gamma_{s_2,1} \\
& \vdots \\
H_{s_2} &= \gamma_{1,2}\gamma_{2,1} \\
H_{s_2+1} &= \gamma_{1,1} \\
H_{s_2+2} &= h_{v_{r+1}} \\
& \vdots \\
H_{s-1} &= h_{v_s} \\
H_s &= H_0
\end{align*}
\]

\[
\begin{align*}
\Pi_0^{-1} &= \alpha_r^{-1} \\
\Pi_1^{-1} &= p_{s_2} \\
\Pi_2^{-1} &= p_{v_{s_2}} \\
& \vdots \\
\Pi_{s_2}^{-1} &= p_{v_1} \\
\Pi_{s_2+1}^{-1} &= \alpha_{r_1-1} \\
\Pi_{s_2+2}^{-1} &= \alpha_{r_1-1} \\
& \vdots \\
\Pi_{s-1}^{-1} &= \alpha_{r_2-1} \\
\Pi_s^{-1} &= \alpha_{r_2-1}
\end{align*}
\]

\[
\rightarrow (s, s_1)
\]

Finally, we have to add \(r_2 + s_2\) pairs \((0, 3)\) for the oriented 3-cycles. Thus, we have \(\phi_A(r, r_1) = 1\), \(\phi_A(s, s_1) = 1\) and \(\phi_A(0, 3) = r_2 + s_2\), where \(r = r_1 + r_2\) and \(s = s_1 + s_2\).

Now, let \(A\) and \(B\) be two cluster tilted algebras of type \(\tilde{A}_n\) with parameters \(r_1, r_2, s_1, s_2\), respectively \(\tilde{r}_1, \tilde{r}_2, \tilde{s}_1, \tilde{s}_2\). From above we can conclude that \(\phi_A = \phi_B\) if and only if \(r_1 = \tilde{r}_1, r_2 = \tilde{r}_2, s_1 = \tilde{s}_1\) and \(s_2 = \tilde{s}_2\) or \(r_1 = \tilde{s}_1, r_2 = \tilde{s}_2, s_1 = \tilde{r}_1\) and \(s_2 = \tilde{r}_2\) (which ends up in the same quiver).

Hence, if \(A\) is derived equivalent to \(B\), we know from Theorem 5.7 that \(\phi_A = \phi_B\) and thus, that the parameters are the same.

Otherwise, if \(A\) and \(B\) have the same parameters, they are both derived equivalent to the same cluster tilted algebra with a quiver in normal form. \(\square\)
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