Universal relations for ultracold reactive molecules

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The realization of ultracold polar molecules in laboratories has pushed physics and chemistry to new realms. In particular, these polar molecules offer scientists unprecedented opportunities to explore chemical reactions in the ultracold regime where quantum effects become profound. However, a key question about how two-body losses depend on quantum correlations in interacting many-body systems remains open so far. Here, we present a number of universal relations that directly connect two-body losses to other physical observables, including the momentum distribution and density correlation functions. These relations, which are valid for arbitrary microscopic parameters, such as the particle number, the temperature, and the interaction strength, unfold the critical role of contacts, a fundamental quantity of dilute quantum systems, in determining the reaction rate of quantum reactive molecules in a many-body environment. Our work opens the door to an unexplored area intertwining quantum chemistry; atomic, molecular, and optical physics; and condensed matter physics.

INTRODUCTION

In a temperature regime down to a few tens of nanokelvin, highly controllable polar molecules provide scientists with a powerful apparatus to study a vast range of new quantum phenomena in condensed matter physics, quantum information processing, and quantum chemistry (1–14), such as exotic quantum phases (15–18), quantum gates with fast switching times (19, 20), and quantum chemical reactions (9–13). In all these studies, the two-body loss is an essential ingredient leading to non-Hermitian phenomena. Similar to other chemical reactions, collisions between molecules may yield certain products and release energies, which allow particles to escape the traps. For instance, a prototypical reaction, $K_Rb + K_Rb \rightarrow K_2 + Rb_2$, is the major source causing the loss of $K_Rb$ molecules. Undetectable complexes may also form, resulting in losses in the system of interest (10, 11).

Whereas chemical reactions are known for their complexities, taking into account quantum effects imposes an even bigger challenge to both physicists and chemists. The exponentially large degrees of freedom and quantum correlations built upon interactions make it difficult to quantitatively analyze the reactions. A standard approach is to consider two interacting particles, the reaction rate of which is trackable (21, 22). Although these results are applicable in many-body systems when the temperature is high enough and correlations between different pairs of particles are negligible, with decreasing the temperature, many-body correlations become profound and this approach fails. In particular, in the first realization of a degenerate Fermi gas of polar molecules, unusual behaviors of two-body losses were observed (12). In the absence of electric fields, the dipole moment vanishes and these polar molecules interact with van der Waals interactions. With decreasing the temperature, the suppression of the loss rate no longer agrees with the Bethe-Wigner threshold. Experimental results also indicated that the temperature dependence of the density fluctuation is similar to that of the loss rate (12, 14). A theory fully incorporating quantum many-body effects is, therefore, desired to understand the chemical reaction rate at low temperatures.

In this work, we show that universality exists in chemical reactions of ultracold reactive molecules. We implement contacts, the central quantity in dilute quantum systems (23–25), to establish universal relations between the two-body loss rate and other quantities including the momentum distribution and the density correlation function. Previously, two-body losses of zero-range potentials hosting inelastic $s$-wave scatterings were correlated to the $s$-wave contact (26, 27). In reality, chemical reactions happen in a finite range. Many systems are also characterized by high-$p$-partial-wave scatterings. For instance, single-component fermionic $K_Rb$ molecules interact with $p$-wave scatterings (1, 12). It is thus required to formulate a theory applicable to generic short-range reactive interactions. To concretize discussions, we focus on single-component fermionic molecules. All our results can be straightforwardly generalized to other systems with arbitrary short-range interactions.

RESULTS

Loss rate and contacts

The Hamiltonian of $N$ reactive molecules is written as

$$H = \sum_i \left[ -\frac{\hbar^2}{2M_i} \nabla_i^2 + V_{ext}(r_i) \right] + \sum_{ij} U(r_i - r_j)$$

where $M$ is the molecular mass, $V_{ext}(r)$ is the external potential, and $U(r)$ is a two-body interaction, as shown in Fig. 1. The many-body wave function, $\Psi(r_1, r_2, \ldots, r_N)$, satisfies the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(r_1, r_2, \ldots, r_N) = H\Psi(r_1, r_2, \ldots, r_N)$$

In the absence of electric fields, $U(r)$ is a short-range interaction with a characteristic length scale, $r_0$. When $|r| > r_0$, $U(r) = 0$. Chemical reactions happen in an even shorter length scale, $r^* < r_0$. We adopt the one-channel model using a complex $U(r) = U_{R}(r) + iU_{I}(r)$ to describe the chemical reaction (22), where $U_{R}(r) \leq 0$. When $|r| > r^*$, $U(r) = 0$. Using the Lindblad equation that models the losses by jump operators, the same universal relations can also be derived (Materials and Methods).
Universal relations arise from a length scale separation in dilute quantum systems, \( r^* < r_0 \ll k_F^{-1} \), where \( k_F^{-1} \) is the inverse of the Fermi momentum, captures the average interparticle separation. When the distance between two molecules is much smaller than \( k_F^{-1} \), we obtain

\[
\psi(r_1, r_2, \ldots, r_N) \approx \psi_{\text{F}}(k_F) \sum_{m, \epsilon} \psi_m(r_{ij}; \epsilon) G_m(r_{ij}; E - \epsilon)
\]

where \( r_{ij} = r_i - r_j \) denotes the relative coordinates of the \( i \)th and the \( j \)th molecules and \( R_0 = \{(r_i + r_j)/2, r_{i,j} \} \) is a short-hand notation including coordinates of their center of mass and all other particles. \( \psi_m(r_{ij}; \epsilon) \) is a p-wave function with a magnetic quantum number \( m = 0, \pm 1 \), which is determined solely by the two-body Hamiltonian, \( H_2 = -\frac{\hbar^2}{2M} \nabla^2 + U(r_{ij}) \), as all other particles are far away from the chosen pair in the regime, \( |r_{ij}| \ll k_F^{-1} \). \( G_m(r_{ij}; E - \epsilon) \) includes all many-body effects, which depend on the center of mass of the \( i \)th and the \( j \)th molecules and all other \( N - 2 \) molecules, and can be viewed as a "normalization factor" of the two-body wave function \( \psi_m(r_{ij}; \epsilon) \). \( E \) is the total energy of the many-body system. \( \epsilon \), the colliding energy, is no longer a good quantum number in a many-body system, and a sum shows up in Eq. 3. Since both a continuous spectrum and discrete bound states may exist, we use the notion of sum other than an integral.

Although \( \psi_m(r_{ij}; \epsilon) \) depends on the details of \( U(r_{ij}) \) when \( |r_{ij}| < r_0 \), it is universal when \( r_0 < |r_{ij}| \ll k_F^{-1} \), as a result of vanishing interaction in this regime. We define \( \psi_m(r_{ij}; \epsilon) = \psi_m(r_{ij}; \epsilon) Y_{lm}(\hat{r}_{ij}) \), where \( Y_{lm}(\hat{r}_{ij}) \) is the p-wave spherical harmonics. Whereas many resonances exist, the phase shift of the scattering between KRB molecules is still a smooth function of the energy, due to the large average line width of these resonances, which far exceeds the mean level spacing of the bound states (21). The phase shift, \( \eta_l \), then has a well-defined expansion, \( q_k^* \cos(\eta_l q_k^*) = -1/v_P + q_k^*/r_F \), where \( v_P \) and \( r_F \) are the p-wave scattering volume and effective range, respectively, both of which are complex for reactive interactions. \( q_k^* = (\epsilon M/\hbar^2)^{1/2} \). Consequently, \( \psi_m(r_{ij}; \epsilon) = \psi_m^{(0)}(r_{ij}) + q_k^*/q_k^* \psi_m^{(1)}(r_{ij}) + O(q_k^3) \), where

\[
\psi_m^{(1)}(r_{ij}) \approx \frac{r_0}{\nu_p} \left( \frac{1}{r_{ij}^3} + \frac{1}{\nu_p} \left( \frac{r_{ij}^3}{3} \right) \right)
\]

To simplify expressions, we have considered isotropic p-wave interactions, \( \psi(r_{ij}) = \psi_m(r_{ij}) \) and \( G(R_{ij}; E - \epsilon) = G_m(R_{ij}; E - \epsilon) \), and suppressed other partial waves in the expressions, which do not show up in universal relations relevant for single-component fermionic molecules.

Using Eqs. 1 to 3, we find that the decay of the total particle number is captured by

\[
\partial_t N = -\frac{\hbar^2}{8\pi^2 M} \sum_{\kappa, \nu} \kappa \nu C_{\nu}
\]

where the three contacts are written as

\[
C_1 = 3(4\pi)^2 N(N - 1) \int dR_{ij} |g(r)^{00}|^2
\]

\[
C_2 = 6(4\pi)^2 N(N - 1) \int dR_{ij} Re(g(r)^{01}) g(r)^{10}
\]

\[
C_3 = 6(4\pi)^2 N(N - 1) \int dR_{ij} Im(g(r)^{01}) g(r)^{10}
\]

where \( |r| = \sum_{i,j} r_{ij} \), and \( g(r) = \sum_k \hat{q}_k^2 G(R_{ij}; E - \epsilon) \). As shown later, \( C_1 \) determines the leading term in the large momentum tail, similar to systems without losses (28–31). In contrast, \( C_{2,3} \) are new quantities in systems with two-body losses.

\( \kappa_\nu \), in Eq. 6 are microscopic parameters determined purely by the two-body physics. In our one-channel model, their explicit expressions are given by

\[
k_1 = -\frac{M}{\hbar^2} \int_0^\infty U(r) (\psi^{(0)}(r))^2 r^2 dr
\]

\[
k_2 = -\frac{M}{\hbar^2} \int_0^\infty U(r) (\psi^{(0)}(r) \psi^{(1)}(r))^2 r^2 dr
\]

\[
k_3 = \frac{M}{\hbar^2} \int_0^\infty U(r) (\psi^{(0)}(r) \psi^{(1)}(r))^2 r^2 dr
\]

where \( |r| = \sum_{i,j} r_{ij} \). If \( U(r) \) is modeled by two square well potentials, one for its real part and the other for its imaginary part, then \( k_{1,2,3} \) can be evaluated explicitly. For simplicity, we set \( U(r) = -U_R - iU_I \) when \( |r| < r_0 \) and \( 0 \) elsewhere. Changing the ratio \( r_0/r^* \) does not change any results qualitatively. Figure 2 shows how \( k_{1,2,3} \) depend on \( U_I \) when \( U_R \) is fixed at various values including those corresponding to small and divergent \( \nu_P \) in the absence of \( U_I \). When \( U_I = 0 \), \( k_{1,2,3} = 0 \). With increasing \( U_I \), \( k_{1,2,3} \) change non-monotonically and all approach zero when \( U_I \) is large, indicating a vanishing reaction rate in the extremely large \( U_I \) limit.

Equation 6 is universal for any particle number and any short-range interactions with arbitrary interaction strengths, as well as any real external potential. It separates \( C_\nu \), which fully capture the many-body physics, from two-body parameters, \( k_\nu \), which are independent on the particle number and the temperature. Therefore, even when microscopic details of the reactive interaction, for instance, the exact expression of \( U(r) \), are unknown, \( k_\nu \) can still be accessed in systems whose \( C_\nu \) are easily measurable (Supplementary Materials). Equation 6 also holds for any many-body eigenstates, and a thermal average does not change its form. Therefore, Eq. 6 is
and the thermal wavelength. We define the total angular averaged $\kappa_1$. In particular, when $\lambda_1$ crosses zero, the location of the maximum of $\kappa_1$ first approaches and then leaves the origin, and $\kappa_1$ remains positive (negative). In contrast, $\kappa_2$ quickly changes from positive to negative at small values of $U_1$ when $\lambda_1$ crosses zero. In the large $U_1$ limit, all three parameters vanish, as shown by the insets.

does apply for any finite temperatures, provided that the reaction rate is slow compared to the time scale of establishing quasi-equilibrium in the many-body system, i.e., the many-body system has a well-defined temperature at any time. Under this situation, $C_v$ should be understood as their thermal averages.

We have found that $\kappa_1$ and $\kappa_2$ can be rewritten as familiar parameters. $\kappa_1 = \text{Im} \left( \frac{v_p}{p} \right)$ and $\kappa_2 = \text{Im} \left[ -1/(2\lambda_2) \right]$ (Materials and Methods). In contrast, to our best knowledge, $\kappa_3$ is a new parameter that has not been addressed in previous works. Similar to $\kappa_2$, $\kappa_3$ can be expressed as the difference between the extrapolation of the two-body wave function in the regime $|r| > r_0$ toward the origin and the realistic wave function at short distance, $|r| < r_0$ (Materials and Methods). Equation 6 can be rewritten as

$$\partial_t N = -\frac{\hbar}{8\pi^2M} \left[ \text{Im} \left( \frac{v_p}{p} \right) C_1 - \frac{1}{2} \text{Im} \left( \frac{r}{r} \right) C_2 + \kappa_3 C_3 \right].$$

For s-wave inelastic scatterings due to complex zero-range interactions, the first term on the right-hand side of Eq. 13 was previously derived, with $v_p$ replaced by the complex s-wave scattering length ($26$). For a generic short-range interaction, all three contacts and all three microscopic parameters are required, as shown in Eqs. 6 and 13. In particular, when $\kappa_{2,3}$ are comparable to or even larger than $\kappa_1$, the other two terms cannot be ignored. This expression allows us to directly connect the two-body loss rate to a wide range of physical quantities.

**Universal relations with other physical quantities**

We first consider the momentum distribution, which has a universal behavior when $|k| \ll 1/r_0$ but is much larger than all other momentum scales, including $k_P$, the inverses of the scattering length and the thermal wavelength. We define the total angular averaged momentum, $n(|k|) = \sum_{m=0, \pm 1} \int d\Omega |m(k)|$, where $\Omega$ is the solid angle

$$n(|k|) \sim \frac{C_1}{|k|^2}$$

Once $n(|k|)$ is measured, the first term in Eqs. 6 and 13 is known. For radio frequency (RF) spectroscopy in molecules, similar to that for atoms, Eq. 14 also indicates that such spectroscopy has a universal tail, $\Gamma(\omega) \sim [(\Omega_{RF} V)/(8\pi^2)] C_1 (\hbar/M)^{-1/2}$, where $\omega$ is the RF frequency, $\Omega_{RF}$ is the RF Rabi frequency, and $V$ is the volume of the system. It is worth mentioning that, for atoms with elastic p-wave interactions, Eq. 14 describes the leading term of the large momentum tail. We have not found that the subleading term $\sim |k|^{-4}$ has connections to two-body losses.

Another fundamentally important quantity in condensed matter physics is the density correlation function, $S(r) = \int dR n(R + r/2)(n(R - r/2))$, which measures the probability of having two particles separated by a distance $r$. Using Eqs. 3 to 5, $S(r)$ can be evaluated explicitly in the regime, $r_0 < |r| < k_F$. To enhance the signal-to-noise ratio, $S(r)$ can be integrated over a shell with inner and outer radii, $x$ and $x + D$, respectively. Such an integrated density correlation is given by $P(x,D) = \int_{x}^{x+D} dr S(r)$, and

$$\frac{\partial P(x,D)}{\partial D} \bigg|_{D=0} = \frac{1}{16\pi^2} \left\{ C_1 \frac{1}{x^2} + \frac{1}{2} C_2 \right\}$$

$$- \left[ 2 \text{Re} \left( \frac{1}{v_p} \right) C_1 - \text{Re} \left( \frac{1}{\lambda_2} \right) C_2 + \text{Im} \left( \frac{1}{\lambda_2} \right) C_3 \right] \frac{x}{3}. \right.$$  

Again, other partial waves have been suppressed in the expression, as their contributions are given by different spherical harmonics. Fitting $\partial P(x,D)/\partial D |_{D=0}$ measured in experiments using the power series in Eq. 15 allows one to obtain all three contacts, $C_{1,2,3}$, provided that $v_p$ and $\lambda_2$ are known. If these two parameters are unknown, then it is necessary to include higher-order terms in the expansion (Materials and Methods).

We emphasize that, no matter whether thermodynamic quantities and correlation functions can be computed accurately in theories, Eqs. 6 and 13 to 15 allow experimentalists to explore how contacts determine chemical reactions in interacting few-body and many-body systems. In the strongly interacting regime where exact theoretical results are not available, these universal relations become most powerful.

**Temperature dependence of the loss rate**

It is useful to illuminate our results using some examples. For a two-body system in free space, the center of mass and the relative motion are decoupled. $\delta$ in Eqs. 7 to 9 becomes a good quantum number, i.e., $G(R; \delta = \epsilon)$ becomes a delta function in the energy space. For scattering states with $\epsilon > 0$, we consider the wave function,
\[ \Psi^{[2]}(r_1, r_2) = \phi_r(R_{12}) \psi(r_1) \psi(r_2), \text{where } \phi_r(R_{12}) \text{ is a normalized wavefunction of the center of mass and } \psi(r_1) = \sqrt{8\pi} / \sqrt{i((\cot \eta - i) \cot \eta_1 (q_1 | R_{12}) - m | q_{12}) \Sigma_m y_{1m}(\mathbf{r}_1)} \] 

Figure 3 shows the dependence of \( C_1 \) on \( U_I \) when \( V_p \) is fixed at various values. Results for a bound state are also shown. With increasing \( U_I \), \( C_1 \) approaches a nonzero constant in both cases. Here, \( C_2 = 2\ c_1 \text{ Re}(q_{12}^*). \ C_3 = 0 \) if we consider a scattering state. In contrast, \( C_1 = 2\ c_1 \text{ Im}(q_{12}^* \) for a bound state. Analytical results in the limits, \( V_p = 0^+ \), are shown in Table 1.

We use the second-order virial expansion to study a thermal gas at high temperatures. The partition function is written as \( Z = Z_0 + e^{2u/(k_BT)} \sum e^{(iE + \epsilon_i)/(k_BT)} - e^{-(E + \epsilon_i)/(k_BT)} \), where \( Z_0 \) is the partition function of noninteracting fermions, \( \mu \) is the chemical potential, and \( E_i = h^2 K_i^2 / (4M) \) is the energy of the center of mass motion carrying a momentum \( K_i \). \( \epsilon_i \) and \( \epsilon_{12} \) are the eigenvalues of the relative motion with and without interactions, respectively. On the basis of the results of the two-body problem, thermal averaged contacts are derived using \( \langle C_m \rangle_T = Z_0^{-1}(e^{2u/(k_BT)} \sum e^{(iE + \epsilon_i)/(k_BT)} (\sum C_m(c_1) e^{-\epsilon_i/(k_BT)}) \). Using \( N = k_BT_c \), in \( Z \), we eliminate \( \mu \) and \( \langle C_m \rangle_T \) as a function of \( N \) and \( T \). Analytical expressions in the limits, \( V_p = 0^+ \), are shown in Table 2.

Table 2 may shed light on some recent experiments conducted in the weakly interacting regime (9, 12). Although \( V_p > 0 \), it is likely that bound states are not occupied, i.e., the system is prepared at the upper branch. Therefore, \( C_3 = 0 \). In a homogenous system, we obtain

\[ \partial_t N = \frac{144\pi^2}{h} \text{Im}(V_p) N n_c \frac{k_B T}{M} \]

A previous work derived the first term in Eq. 16 using a different approach (22). However, a complete expression needs to include the contribution from \( r_{cm} \), which leads to a different power of the dependence on \( T \). A recent experiment has shown the deviation from the linear dependence on \( T \) (12). However, it is worth investigating whether such deviation comes from the second term in Eq. 16 or some other effects, particularly correlations beyond the description of the second-order virial expansion.

As a harmonic trap exists, the dependence on \( T \) could be completely different. We use the local density approximation to obtain the total contacts by integrating local contacts. As a result, \( C_{\text{trap}} = [(\pi k_BT)/(M\omega^2)]^{1/2} C_1(0) \), where \( \omega \) is trapping frequency and \( C_1(0) \) are the contact densities at the center of the trap (Supplementary Materials).

The first term decreases with increasing \( T \), in sharp contrast to the homogeneous case. In a trap, the molecular cloud expands when the temperature increases such that densities and the total contacts decrease for a fixed \( N \). Similarly, the second term increases slower than the result in homogenous systems with increasing \( T \). Alternatively, we could consider the density at the center of the trap, the decay rate of which linearly depends on \( T \) again (Supplementary Materials).

The temperature dependence of the loss rate in the trap can be qualitatively obtained from the result in a homogenous system by considering a temperature-dependent volume \( \sim T^{3/2} \) (12), a rigorous calculation as aforementioned is required to obtain the exact numerical factors in Eq. 17.

**DISCUSSION**

Although we have used the high-temperature regime as an example to explain Eqs. 6 and 13 to 15, we need to emphasize that these universal relations are powerful tools at any temperatures. In particular, at lower temperatures, contacts are no longer proportional to \( N^2 \), directly reflecting the critical roles of many-body correlations in determining the reaction rate. For instance, below the superfluid transition temperature, contacts may be directly related to superfluid-order parameters (32, 33). Universal relations constructed here thus offer us a unique means to explore the interplay between the chemical reaction and symmetry breaking in quantum many-body systems.

We also would like to point out that the density fluctuation measured in experiments (12, 14), \( f(r) = \langle n^2(r) \rangle - \langle n(r) \rangle^2 \), is different from the density-density correlation, \( S(r) \), studied in our work. Whereas \( S(r) \) directly tells us all contacts by capturing the probability of having two particles as a function of \( r \), their relative coordinate, \( f(r) \) traces the compressibility \( \partial n / \partial m \) as a function of \( r \), the single-particle coordinate. Since the pressure, \( P \), is controlled by contacts and other thermodynamical quantities can be derived from \( P \) (34), how the compressibility and \( f(r) \) are related to contacts.

**Table 1. Analytical expressions for contacts \( C_2 \) of two particles in different limits.** Lines 1 and 2 show the results in the weakly interacting regime and those at resonance, respectively. \( V_p \rightarrow 0 \) means \( V_p \rightarrow 0 + \infty \) on the complex plane. Line 3 includes the results for bound states, in which a single angular momentum \( m \) is considered.

\[
\begin{align*}
\text{Line 1} & : & V_p & \rightarrow 0 & & 2C_1 |q_{12}^*|^2 & & 2C_1 \text{ Re}(q_{12}^*^2) & & 0 \\
\text{Line 2} & : & V_p & \rightarrow \infty & & 2C_1 |q_{12}^*|^2 & & 2C_1 \text{ Re}(q_{12}^*^2) & & 0 \\
\text{Bound} & : & & & & 2C_1 \text{ Re}(q_{12}^*^2) & & 2C_1 \text{ Im}(q_{12}^*^2) & &
\end{align*}
\]
and the loss rate remains an interesting open question worthy of exploration.

In current experiments (12, 14), the electric field is absent and the unpolarized molecules interact with each other with the van der Waals potential. Once an electric field is turned on, the dipole-dipole interaction between polarized molecules, \( r \approx 1/|r|^3 \), decays slower and has a longer range. For any power-law potentials, \( A/|r|^n \), if \( n > 2 \), then the scattering theory applies and a characteristic length of the range of the interaction can be defined as \( r = (M|A|/\hbar^2)^{1/(n-2)} \) (35). In dilute systems, once \( k_F r \ll 1 \) is satisfied, universal relations rise from such a length scale separation. The details of the interaction, such as \( n \) of the power-law potential, determine the low energy expansion of the phase shift and microscopic parameters in the universal relations. It will be interesting to study how the electric field influences contacts, universal relations, and the decay rate of molecules. It is also worth mentioning a subtlety in the p-wave scattering when \( n = 6 \). It has been theoretically predicted that an extra term linearly dependent on the momentum should exist in the phase shift (36, 37). Whereas current experiments have not found evidence for extra contacts associated with this term yet (31), it will be useful to explore whether such a term may affect contacts and universal relations to bridge quantum chemistry; atomic, molecular, and optical physics; and condensed matter physics.

**MATERIALS AND METHODS**

The **Lindblad equation**

We consider a Lindblad master equation

\[
\frac{d\rho}{dt} = -i[H,\rho]+D[\rho]
\]

where \( H \) is the Hamiltonian that describes the unitary part of the time evolution and the dissipator \( D \) describes the loss due to inelastic collisions

\[
D[\rho] = -\frac{1}{\hbar}\int d\mathbf{x}_1 d\mathbf{x}_2 \Gamma((\mathbf{x}_1 - \mathbf{x}_2)) \left\{ 2\Psi^\dagger(\mathbf{x}_2)\Psi(\mathbf{x}_1)\rho\Psi^\dagger(\mathbf{x}_1)\Psi(\mathbf{x}_2), \Psi^\dagger(\mathbf{x}_1)\Psi(\mathbf{x}_2), \rho \right\}
\]

(19)

\( \Psi(\mathbf{x}) \) is the fermionic field operator satisfying \( \{\Psi(\mathbf{x}), \Psi^\dagger(\mathbf{x}')\} = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \). \((1/2)\Gamma((\mathbf{x}_1 - \mathbf{x}_2))\) describes a finite range dissipation. The loss rate of the total particle number, \( dN/dt = \int d\mathbf{x}(dN/dt)\Gamma(n(x)\rho) \), \( n(x) = \Psi^\dagger(x)\Psi(x) \), is written as

\[
\frac{dN}{dt} = -\frac{1}{\hbar}\int d\mathbf{x}\Psi^\dagger(\mathbf{x})\Psi(\mathbf{x})(\int d\mathbf{x}_1 d\mathbf{x}_2 \Gamma((\mathbf{x}_1 - \mathbf{x}_2)) \left\{ 2\Psi^\dagger(\mathbf{x}_2)\Psi(\mathbf{x}_1)\rho\Psi^\dagger(\mathbf{x}_1)\Psi(\mathbf{x}_2), \Psi^\dagger(\mathbf{x}_1)\Psi(\mathbf{x}_2), \rho \right\})
\]

(20)

This equation is valid for any finite range dissipator. In the approximation of zero-range dissipators, \( \Gamma = g\delta^{(3)}(\mathbf{x} - \mathbf{x}') \), it reduces to (43–45)

\[
\frac{dN}{dt} = \frac{2}{\hbar}g\int d\mathbf{x}(\Psi^\dagger(\mathbf{x})\Psi^\dagger(\mathbf{x})\Psi(\mathbf{x})\Psi(\mathbf{x}))
\]

(21)

The work in (43) considered two-component fermions and obtained \( d(N_1)/dt = d(N_2)/dt = -[\hbar/(2\pi m)] Im(1/aC) \), where \( N_1 \) (\( N_2 \)) is the number of spin-up (spin-down) fermions, \( a \) is the s-wave scattering length, and \( C \) is the s-wave contact.
Decay rate
In the presence of complex short-range interactions, the many-body wave function satisfies
\[ i\hbar \partial_t \Psi(r_1, r_2, \ldots, r_N) = \left[ \sum_{m=1}^{N} \left( -\frac{\hbar^2}{2M} \nabla^2 + V_{ext}(r_i) + \sum_{ij} U(r_{ij}) \right) \right] \Psi(r_1, r_2, \ldots, r_N) \]

(22)

For any finite-size system, net current vanishes at the boundary. We obtain
\[ \partial_t N = \frac{\hbar}{8\pi^2 M} \left[ \text{Im} \left( \sum_{i<j} d(r_{ij}) \left| \Psi(r_{ij}, r_{ij}) \right| \right) \right] \]

which leads to Eqs. 10 to 12,
\[ \kappa_1 = \text{Im} \left( v_p^2 \right) = -M \int_0^\infty dr r^2 \left| \phi^{(0)}(r) \right|^2 U_f(r) \]

(29)
\[ \kappa_2 = \text{Im}(-r_e^{-1}/2) = -M \frac{\hbar}{2} \text{Re} \left( \int_0^\infty dr d^2 r \phi^{(0)*}(r) \phi^{(1)}(r) U_f(r) \right) \]

(30)
\[ \kappa_3 = -\int_0^\infty \left[ |\text{Im} \phi^{(0)}(r)|^2 - |\text{Im} \phi^{(0)}(r)|^2 \right] r^2 dr \]

\[ = \frac{M}{\hbar^2} \text{Im} \left( \int_0^\infty dr d^2 r \phi^{(0)*}(r) \phi^{(1)}(r) U_f(r) \right) \]

(31)

where $\phi^{(0)}(r)$ is a wave function obtained from the actual wave function $\psi^{(0)}(r)$ outside the potential ($r > r_0$) into the regime $r < r_0$. We obtain Eq. 13
\[ \partial_t N = -\frac{\hbar}{8\pi^2 M} \text{Im} \left( v_p^2 \right) C_1 - \frac{1}{2} \text{Im} \left( r_e^{-1} \right) C_2 + \kappa_3 C_3 \]

(32)

Momentum distribution
Similar to systems with real interactions (30), using $n(k) = \sum_{m=1}^{N} \int d^3 r \langle \Psi(r_1, r_2, \ldots, r_N) \rangle e^{-ik \cdot r} \right|^2$, we obtain
\[ n(k) \propto |k| < k_r^{-1} \]

\[ \frac{C_1}{3 |k|^2} \sum_m Y_{1m}(k) \]

(33)

which leads to
\[ n(|k|) = \int d\Omega n(k) = \frac{C_1}{|k|^2} \]

(34)

Density correlations
The density correlation function $S(r) = \int dR n(R + r/2)n(R - r/2)$ can be rewritten as
\[ S(r) = N(N-1) \left( \prod_{k=i,j} dR_{k} \right) \left( R_{1i} \ldots R_{1j} = R - \frac{r}{2} \right) \left( R_{2i} \ldots R_{2j} = R + \frac{r}{2} \right) \]

(35)
\[ \times \left| \psi \left( R_{1i} \ldots R_{1j} = R + \frac{r}{2} , \ldots , R_{N} = R - \frac{r}{2} \right) \right|^2 \]

In the regime, $r \ll k_r^{-1}$, $S(r)$ is written as
\[ S(r) = N(N-1) \int dR_0 \sum_m Y_{1m}(R) \left[ \left| \phi^{(0)}(r) \right|^2 \right] \left[ \phi^{(0)}(r) \right] \]

(36)
\[ + \phi^{(0)}(r) \phi^{(0)}(r) \phi^{(1)}(r) \phi^{(1)}(r) \]

where $r = |r|$. The integral over a shell allows us to obtain
\[ P(x, D) = \int_x^{x+D} dS(r) = \int_x^{x+D} d^2 r \left| \phi^{(0)}(r) \right|^2 \left| \phi^{(0)}(r) \right|^2 C_1 \]

(37)
Using Eqs. 4 and 5, we obtain
\[
\frac{\partial P(x,D)}{\partial D} \bigg|_{D=0} = -\frac{1}{16\pi} \left( C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} \right) + \left[ -2\text{Re} \left( \frac{1}{|v|} \right) C_1 + \text{Re} \left( \frac{1}{|R|} \right) C_2 - 2\text{Im} \left( \frac{1}{|v|} \right) C_3 \right] \frac{x^2}{2} + \left[ -\frac{2}{3} \text{Re} \left( \frac{1}{|v|} \right) C_2 - \text{Im} \left( \frac{1}{|v|} \right) C_3 \right] \frac{x^3}{5} + \left[ \frac{1}{|v|} \right] C_1 - \left( \frac{1}{|v|} \right) C_2 + \text{Im} \left( \frac{1}{|v|} \right) C_3 \right] \frac{x^4}{9} - \frac{C_2}{|v|^2} x^6 \right)
\]

(38)

where the first line recovers Eq. 15 and the second line includes higher-order terms. When $v_p$ and $r_e$ are known, the first line readily allows experimentalists to obtain $C_{1,2,3}$ by fitting the experimental data. When $v_p$ and $r_e$ are unknown, the second line is required to obtain $C_{1v}$, $v_p$, and $r_e$.

SUPPLEMENTARY MATERIALS
Supplemental material for this article is available at http://advances.sciencemag.org/cgi/content/full/6/5/eabd4699/DC1

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