Some remarks on inp-minimal and finite burden groups

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Abstract

We prove that any left-ordered inp-minimal group is abelian and we provide an example of a non-abelian left-ordered group of dp-rank 2. Furthermore, we establish a necessary condition for group to have finite burden involving normalizers of definable sets, reminiscent of other chain conditions for stable groups.

0 Introduction and preliminaries

One of the model-theoretic properties that gained a lot of interest recently is dp-minimality, which, on one hand, significantly strengthens NIP, and on the other hand, is satisfied by all strongly minimal theories, all (weakly) o-minimal theories, algebraically closed valued fields (more generally, by all C-minimal structures), and the valued field of p-adics. Several interesting results were obtained for dp-minimal structures in the algebraic contexts of groups and fields (sometimes with additional structure), see for example [12, 5, 4, 8].

Throughout this note, we work in the context of a complete first-order theory $T$, and “formula” means a first-order formula in the language of $T$.

We recall some key definitions, which are originally due to Shelah [11], though the precise form of the definitions which we give below seems to come from Usvyatsov [13].

Definition 0.1. 1. An inp-pattern of depth $\kappa$ (in the partial type $\pi(\overline{x}))$ is a sequence $\langle \varphi_i(\overline{x}; \overline{y}_i) : i < \kappa \rangle$ of formulas and an array $\{\overline{a}_{ij} : i < \kappa, j < \omega \}$ of parameters (from some model of $T$) such that:

(a) For each $i < \kappa$, there is some $k_i < \omega$ such that $\{\varphi_i,j(\overline{x}; \overline{a}_{ij}) : j < \omega \}$ is $k_i$-inconsistent; and

(b) For each $\eta : \kappa \to \omega$, the partial type

\[ \pi(\overline{x}) \cup \{\varphi_i(\overline{x}; \overline{a}_{i,\eta(i)}) : i < \kappa \} \]

is consistent.

2. The inp-rank (or burden) of a partial type $\pi(\overline{x})$ is the maximal $\kappa$ such that there is an inp-pattern of depth $\kappa$ in $\pi(\overline{x})$, if such a maximum exists. In case there are inp-patterns of depth $\lambda$ in $\pi(\overline{x})$ for every cardinal $\lambda < \kappa$ but no inp-pattern of depth $\kappa$, we say that the inp-rank of $\pi(\overline{x})$ is $\kappa_-$.

3. The inp-rank of $T$ is the inp-rank of $x = x$, and $T$ is inp-minimal if its inp-rank is 1.

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4. An ict-pattern of depth $\kappa$ (in the partial type $\pi(\overline{x})$) is a sequence $\langle \varphi_i(\overline{x}; \overline{y}_i) : i < \kappa \rangle$ of formulas and an array $\{\overline{a}_{ij} : i < \kappa, j < \omega \}$ of parameters (from some model of $T$) such that for each $\eta : \kappa \to \omega$, the partial type

$$\pi(\overline{x}) \cup \{\varphi_i(\overline{x}; \overline{a}_{i\eta(i)}) : i < \kappa \} \cup \{\neg \varphi_i(\overline{x}; \overline{a}_{ij}) : i < \kappa, j \neq \eta(i)\}$$

is consistent.

5. The dp-rank of a partial type $\pi(\overline{x})$ is the maximal $\kappa$ such that there is an ict-pattern of depth $\kappa$ in $\pi(\overline{x})$, if such a maximum exists (if not, it is “$\kappa$” exactly as in 2 above). The dp-rank of $T$ is the dp-rank of $x = x$, and $T$ is dp-minimal if its dp-rank is 1.

In spite of its name, dp-rank is really more like a cardinal-valued dimension than an ordinal-valued rank such as $SU(p)$, and in the context of stable theories, dp-minimality is equivalent to every nonalgebraic 1-type having weight 1, as observed in [10]. It turns out that a theory is dp-minimal just in case it is both inp-minimal and NIP (see [11]).

One of the context investigated in [12] is that of (bi)-ordered groups.

**Definition 0.2.** A left-ordering on a group $(G, \cdot)$ is a total ordering $\leq$ on $G$ such that for any $f, g, h \in G$, whenever $g < h$, we have that $f \cdot g < f \cdot h$. A right-ordering is defined similarly, and a bi-ordering on $G$ is an ordering which is simultaneously a left-ordering and a right-ordering.

While Pierre Simon claimed that all inp-minimal “ordered groups” are abelian [12], his proof only applies to groups with a definable bi-ordering: his argument uses the fact that for any $x, y$ in a bi-orderable group and any positive $n \in \mathbb{N}$, if $x^n = y^n$ then $x = y$. But in left-orderable groups (such as in the example of the Klein bottle group below), one may have that $x^2 = y^2$ but $x \neq y$.

The main result of Section 2 of this note is that every inp-minimal left-ordered group is abelian (Theorem 2.6), which strengthens the result mentioned above from [12]. We also show that this conclusion fails already in the dp-rank 2 case by providing a suitable example (Section 1). Finally, in Section 3 we consider necessary conditions for an arbitrary (not necessarily ordered) group to have finite burden. In the stable case, this gives a simple and apparently new condition on stable groups $G$ of finite weight: such a group must contain finitely many definable abelian subgroups $A_0, \ldots, A_k$ such that $G/N[A_0] \ldots N[A_k]$ has finite exponent (Corollary 3.5).

1. A non-abelian left-ordered group of dp-rank 2

In this section, we define the “Klein bottle group” (the fundamental group of a Klein bottle) which is presented as $G = \langle x, y : x^{-1}yx = y^{-1} \rangle$. In other words, $y^{-1}x = xy$, and routine algebraic manipulation shows:

1. Every $g \in G$ can be uniquely written as $g = x^ny^m$ for some $n, m \in \mathbb{Z}$,

2. $(x^n y^m) \cdot (x^{n'} y^{m'}) = x^{n+n'} y^{m'+(-1)^n m}$, and

3. $(x^n y^m)^{-1} = x^{-n} y^{(-1)^n+1} m$.

We can define a left ordering $\leq$ on $G$ lexicographically on the exponents: $x^n y^m \leq x^{n'} y^{m'}$ iff either $n < n'$ or else $n = n'$ and $m \leq m'$. The subgroup generated by $y$ is the minimal nontrivial convex subgroup of $G$, and the order type of $G$ is $\mathbb{Z} \times \mathbb{Z}$.

We note in passing that while $G$ is non-abelian, it is abelian-by-finite: a simple calculation shows that the centralizer $C(y)$ of $y$ is $\{x^{2n}y^m : n, m \in \mathbb{Z}\}$, which is abelian, and for any $g \in G$, we have $g^2 \in C(y)$.  

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**Proposition 1.1.** The structure \((G, \cdot, \leq)\) is dp-rank 2.

**Proof.** To show that it is NIP and dp-rank at most 2, it suffices (thanks to the additivity of the dp-rank proved in [6]) to note that an isomorphic copy of \(G\) is definable in the dp-minimal structure \((\mathbb{Z}, <, +)\) with \(\mathbb{Z} \times \mathbb{Z}\) as the the universe of the group; the definition of the group operation depends on the parity of one of the coordinates, but of course \(2\mathbb{Z}\) is a definable subgroup of \(\mathbb{Z}\).

Now we compute the centralizer \(C(x)\) of the generator \(x\). For any \(a, b \in \mathbb{Z}\), we have
\[
x \cdot (x^a y^b) = x^{a+1} y^b
\]
and
\[
(x^a y^b) \cdot x = x^{a+1} y^{-b},
\]
so we conclude that \(C(x) = \langle x \rangle\).

For any \(n \in \mathbb{N}\), there are pairwise disjoint intervals \(< I_{n,k} : k < \omega \rangle\) such that for any \(k\), \(x^k \in I_{n,k}\) and \(I_{n,k}\) intersects every right coset \(C(x), C(x)y, \ldots, C(x)y^n\). So by compactness, in an \(\omega\)-saturated extension of \(G\), we can find pairwise disjoint intervals \(< I_k : k < \omega \rangle\) such that \(x^k \in I_k\) and each \(I_k\) intersects every right coset \(C(x)y^n\). Therefore the formulas expressing \(z \in I_k\) and \(z \in C(x)y^i\) (in the free variable \(z\)) give an inp-pattern of depth 2, so \(G\) is not inp-minimal, hence by NIP it is not dp-minimal.

\[\Box\]

**Remark 1.1.** In a previous version of this paper, we asked whether the group \(G\) above is dp-minimal in the pure language of groups. This was answered negatively by Halevi and Hasson [17].

## 2 Inp-minimal left-ordered groups

In this section, we prove that every inp-minimal left-ordered group is abelian. For a left-ordered group \(G\) and a subset \(A \subseteq G\), by \(h(A)\) we will denote the convex hull of \(A\) in \(G\).

**Fact 2.1** ([12]). Let \(G\) be an inp-minimal group. Then there is a definable normal abelian subgroup \(H\) of \(G\) such that \(G/H\) has finite exponent.

Note that (in contrast to the bi-ordered groups), in a left-ordered group \(G\), the convex hull of a subgroup \(H\) need not be a subgroup of \(G\):

**Example 2.2.** Consider the left-ordered group \(G = (\text{Aut}(\mathbb{Q}, <), \prec)\), where \(\prec\) is a standard left-order on \(\text{Aut}(\mathbb{Q}, <)\) coming from a well-order on \(\mathbb{Q}\), i.e., \(f \prec g\) if \(f(x) < g(x)\) for \(x\) being the smallest (in the sense of the well-order) element on which \(f\) differs from \(g\). Choose \(a < b < c < d < e \in \mathbb{Q}\), where \(a\) is the first element of \(\mathbb{Q}\) in the fixed well-order, and \(f, g \in G\) such that \(f(a) = c, f(d) = d, g(a) = b\) and \(g(b) = e\). Then, \(g^2(a) = e\), and, for any \(k \in \mathbb{Z}\), \(f^k(a) < e\), so \(f^k \prec g^2\). So we have that \(g^2 \notin h(< f >)\), but, clearly, \(g \in h(< f >)\). So we get that \(h(< f >)\) is not a subgroup of \(G\).

Nevertheless, in left-ordered groups, \(h(H)\) is always a union of right \(H\)-cosets, and, as an analogue of Lemma 3.2 from [12], we obtain:

**Fact 2.3.** Let \(G\) be an inp-minimal left-ordered group. Let \(H\) be a definable subgroup of \(G\) and let \(C\) be the convex hull of \(H\). Then \(C\) is a union of finitely many right \(H\)-cosets.
Proof. All we need to repeat the proof from [12] is to prove that all cosets of $H$ contained in $C$ are cofinal in $C$. So take any $c \in C$ and fix $h \in H$. Choose $h_1 \in H$ such that $h_1 < c$. Then, by the left-invariance of the order, $h = (hh_1^{-1})h_1 < hh_1^{-1}c \in Hc$, so $Hc$ is cofinal in $C$. 

We will also use a group-theoretic fact about FC-groups.

**Definition 2.4.** An FC-group is a group in which the centralizer of every element has finite index.

Note that if $[G : Z(G)]$ is finite, then $G$ is an FC-group. The following is Theorem 6.24 from [7]:

**Fact 2.5.** Every torsion-free FC-group is abelian.

**Theorem 2.6.** Every left-ordered inp-minimal group is abelian.

Proof. By Fact 2.5, it is enough to show that $[G : Z(G)]$ is finite. Let $H$ be given by Fact 2.1. Notice that $G = h(H)$: if $a \in G$, say $a > e$, then for $l$ equal to the exponent of $G/H$, we have $e < a < a^2 < \cdots < a^l \in H$, so $a \in h(H)$. Hence, by Fact 2.3 we get that $n := [G : H]$ is finite.

**Claim 2.7.** For any positive $x \in G$, the interval $[e, x]$ is covered by finitely many right cosets of a central subgroup of $G$.

Proof of Claim 2.7. We can assume that $G$ is non-trivial (hence it is infinite so also $H$ is non-trivial). Notice that any coset $Hg$ in $G$ has a positive representative (if $g$ is negative then one can take $g^{-(l-1)}$ as such a representative). It follows (thanks to the normality of $H$) that for any $y, g \in G$ we can find an element $z \in G$ such that $Hg = Hz$ and $z > y$ (by choosing $z := yw$, where $w$ is a positive element of $Hy^{-1}g$).

Now, fix any positive $x = x_0 \in G$. Using the above observation, we can choose $x_0 < x_1 < \cdots < x_{n-1} \in G$ such that $G = \bigcup_{i<n} H x_i$. By Fact 2.3 each set $h(C(x_i))$ is covered by finitely many right $C(x_i)$-cosets; call them $C(x_i)k_{i,0}, \ldots, C(x_i)k_{i,m(i)}$. Thus if

$$C := \bigcap_{i<n} h(C(x_i)),$$

then any $y \in C$ belongs to some intersection

$$Hx_j \cap \bigcap_{i<n} C(x_i)k_{i,\eta(i)}$$

for some $j < n$ and some $\eta : n \to \omega$. But the above intersection of right cosets is a right coset of

$$A := H \cap \bigcap_{i<n} C(x_i),$$

hence $C$ is covered by finitely many $A$-cosets.

But, since $H$ is abelian, and $G$ is generated by $H, x_0, \ldots, x_{n-1}$, we have that $A \subseteq Z(G)$. Also, since $x_0 < x_1 < \cdots < x_{n-1}$ and $\forall_{i<n} x_i \in C(x_i)$, we get that $x = x_0 \in C$, so, by convexity of $C$, $[e, x_0] \subseteq C$, which proves the claim. □

Now, suppose for a contradiction that $[G : Z(G)]$ is infinite. Note that if some coset $Z(G)g$ contains only negative elements, then the coset $Z(G)g^{-1}$ contains only positive elements, so in any case we may choose infinitely many positive representatives $y_0, y_1, y_2, \ldots$ of pairwise distinct right cosets of $Z(G)$ in $G$. Without loss of generality, $G$ is $\omega$-saturated, and there is an element $x \in G$ greater than all the $y_i$’s. Then $[e, x]$ cannot be covered by finitely many right cosets of $Z(G)$, so it cannot be covered by finitely many right cosets of any central subgroup of $G$, contradicting the Claim. □
Corollary 2.8. If \((G, \cdot, <)\) is a left-ordered group which is inp-minimal (in the pure language of ordered groups), then it is dp-minimal.

Proof. By Theorem 2.6 \(G\) is abelian, and any ordered abelian group is NIP, as shown in [3]; since NIP and inp-minimality imply dp-minimality, we are done.

3 Some observations on groups of finite inp-rank

The example from Section 1, as every group definable in the Presburger arithmetic, is abelian-by-finite (see [9]). It seems natural to ask the following general question:

Problem 3.1. What can be said about ordered groups of finite inp-rank (possibly under some additional model-theoretic assumptions)?

To apply some ideas from the proof of Theorem 2.6 it seems necessary to prove some variant of the following property, which was essentially observed in the proof of Proposition 3.1 from [12]:

Fact 3.2. If \(G\) is an inp-minimal group and \(H, K < G\) are definable, then either \([H : H \cap K]\) or \([K : H \cap K]\) is finite.

Below, we make an observation of this kind in the context of finite inp-rank, but we need to work with normal subgroups.

If \(G\) is a group and \(A \subseteq G\), then by \(N[A]\) we shall denote the normal subgroup of \(G\) generated by \(A\). If \(H\) is a subgroup of \(G\), then we put \(A/H := \{aH : a \in A\}\). For any elements \(g, h \in G\), by \(g^h\) we mean the conjugate \(h^{-1}gh\) of \(g\) by \(h\).

Proposition 3.3. If \(G\) is a group of burden \(n \in \omega\), then there do not exist definable sets \(D_0, D_1, \ldots, D_n\) such that, if we put \(N_i = N[D_i]\), then

\[(\forall i \leq n)((D_i/N_0N_1 \ldots N_{i-1}N_{i+1}N_{i+2} \ldots N_n) \geq \omega).\]

Moreover, we can replace the above condition by: for each \(i \leq n\), there is an infinite subset \(E_i\) of \(D_i\) such that

\[(\forall e_0, e_1 \in E_i)(e_0e_1^{-1} \in ((D_0D_1 \ldots D_{i-1}D_{i+1}D_{i+2} \ldots D_n)^G)^{2n} \Rightarrow e_0 = e_1).\]

Proof. Clearly, it is enough to prove the “moreover” part. Suppose for a contradiction that there exist sets \((D_i)_{i \leq n}\) and \((E_i)_{i \leq n}\) as above. For each \(i \leq n\), let \((e_{i,j})_{j<\omega}\) be a sequence of pairwise distinct elements of \(E_i\). We claim that the formulas

\[(\phi_i(x, e_{i,j}) := x \in D_0D_1 \ldots D_{i-1}e_{i,j}D_{i+1}D_{i+2} \ldots D_n)_{i \leq n, j<\omega}\]

form an inp-pattern of depth \(n+1\), which will contradict the assumption. Obviously, for any \(\eta \in \omega^{n+1}\), the element \(\prod_{i \leq n} e_{i,\eta(i)}\) satisfies \(\bigwedge_{i \leq n} \phi_i(x, e_{i,\eta(i)})\). On the other hand, if there is some \(g\) satisfying both \(\phi_i(x, e_{i,j})\) and \(\phi_i(x, e_{i,j'}):=\) then for some \((d_k, d'_k)_{k \in \{0, 1, \ldots, n\} \setminus \{i\}}\) with \(d_k, d'_k \in D_k\), we have

\[d_0d_1 \ldots d_{i-1}e_{i,j}d_{i+1} \ldots d_n = g = d'_0d'_1 \ldots d'_{i-1}e_{i,j'}d'_{i+1} \ldots d'_n,\]

so

\[e_{i,j}e_{i,j}^{-1} = d_{i-1}^{-1}d_{i-2}^{-1} \ldots d_0^{-1}d'_0 \ldots d'_{i-1}(d'_{i+1}d_{i+2} \ldots d_n^{-1}d_{n-1} \ldots d_{i+1}^{-1})e_{i,j}^{-1}\]

is an element of \(((D_0D_1 \ldots D_{i-1}D_{i+1}D_{i+2} \ldots D_n)^G)^{2n}\), hence, by the assumption on \(E_i\), we get that \(e_{i,j} = e_{i,j'}\). This completes the proof.
Example 3.4. Note that the above proposition does not follow from the (somewhat similarly looking) chain condition \([2, \text{Proposition } 4.5\ (2)]\), as the latter is satisfied in a non-abelian free group \(F\) (since the only non-trivial definable proper subgroups are the cyclic groups which are not normal), and the conclusion of Proposition \(3.3\) is not satisfied in \(F\).

To see this, we may assume (as the failure of the conclusion of Proposition is \(\bigwedge\)-expressible) that \(F\) is the free group on generators \(x_0, x_1, x_2, \ldots\). Put \(D_i = \{x_i^m : m < \omega\}\). Then, clearly,

\[(\forall i \leq n)(|D_i/N[D_0]N[D_1] \ldots N[D_{i-1}]N[D_{i+1}]N[D_{i+2}] \ldots N[D_n]| \geq \omega).\]

Using the above chain condition we get in the stable context:

**Corollary 3.5.** If \(G\) is a stable group of finite weight, then there are finitely many definable abelian subgroups \(A_0, \ldots, A_k\) of \(G\) such that the quotient \(G/N[A_0]N[A_1] \ldots N[A_k]\) has finite exponent.

**Proof.** It follows from the assumptions that \(G\) has a finite burden, say \(n\). For any \(g \in G\) put \(A_g = C(C(g))\). Note that \(A_g\) is a definable abelian group, containing the group generated by \(g\). Suppose for a contradiction that the conclusion does not hold. Then, using compactness, we can find inductively a sequence \((g_i)_{i<\omega}\) of elements of \(G\) such that for each \(i, m < \omega\), \(g_i^m \notin N[A_{g_0}]N[A_{g_1}] \ldots N[A_{g_{i-1}}]\). Since the latter is a type-definable condition on the sequence \((g_i)_{i<\omega}\) (as \(N[A_{g_0}]N[A_{g_1}] \ldots N[A_{g_{i-1}}]\) is \(\forall\)-definable over \(g_0, g_1, \ldots, g_{i-1}\)), we can additionally assume that \((g_i)_{i<\omega}\) is an indiscernible sequence. Now, by Proposition \(3.3\) there is some \(i \leq n\) such that for some \(m < \omega\), \(g_i^m \in ((A_0A_1 \ldots A_{i-1}A_{i+1}A_{i+2} \ldots A_n)^G)^{2n}\) (otherwise, putting \(D_i = A_i\) and \(E_i = \{g_i^m : m < \omega\}\), we contradict the conclusion of the Proposition). But this is expressible by a sentence \(\phi(g_i; g_0, g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+2}, \ldots, g_n)\), and by the choice of the \(g_i\)’s, the sentence \(\phi(g_n; g_0, g_1, \ldots, g_{n-1})\) is not true in \(G\), so the sequence \((g_i)_{i<\omega}\) is not totally indiscernible. This contradicts stability. \(\square\)

We end by stating a question about relaxing the assumption of stability in the last theorem to some settings which allow the existence of a definable order:

**Question 3.6.** Is the conclusion of Theorem \(3.3\) true for:

1) rosy groups of finite burden? (in particular, for simple groups of finite weight and groups definable in o-minimal structures?)
2) distal groups of finite burden?

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