A New Type of String Field Theory*

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Abstract

We propose a new way of second quantizing string theory. The method is based on considering the Fock space of strings described by constituents which make up the $X_R^\mu$ and the $X_L^\mu$ i.e. the right and left mover modes separately. A state with any number of strings get represented by the Cartesian product of two free particle Fock spaces, one for right mover degrees of freedom, and one for left. The resulting string field theory is a free theory.

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1. Introduction

There exist already several variants of string field theories along the line of the Kaku-Kikkawa’s one\(^1\), which has for any state single-string creation and annihilation operators so that various numbers of strings can be present in the different single-string states. In models of this kind of second quantized string theories one can distinguish two two-string states which are denoted as \(|1\rangle\) and \(|2\rangle\) (see Figure 1) although they look somewhat similar in the following way:

1) The state \(|1\rangle\) is a two-particle state in which two open strings are present in such a configuration that the two strings lie just along the same curve for a piece somewhere in the middle of the strings.

2) The state \(|2\rangle\) is a corresponding two-string state to the one under 1), but the two strings follow each other somewhere in the middle by permuting so to speak the “tails” of the two strings in the Fock-space state \(|1\rangle\). That is to say that the two-particle state \(|2\rangle\) in the Fock space describes two strings one of which “half” coincides with a piece of string number one in Fock state \(|1\rangle\) while the other “half” instead coincides with the “tail” part of the second string in Fock state \(|1\rangle\).

The two Fock states \(|1\rangle\) and \(|2\rangle\) have some string material present – in single or double amounts – in just the same curve pieces in space, so that they can only be distinguished if one can find out how the string pieces hang together. Nevertheless string field theories such as Kaku-Kikkawa’s one\(^1\), Kyoto group’s (HIKKO’s) one\(^2\), Witten’s cubic one\(^3\), and Zwiebach’s one\(^4\) have two-particle states \(|1\rangle\) and \(|2\rangle\) as mentioned that are counted as quite different, distinguishable Fock space states.

It is the purpose of the present work to present ideas to make a string field theory model representing the class of theories in which the Fock states \(|1\rangle\) and \(|2\rangle\) are not distinguishable but rather represent the same physical state. This class of models has not been much studied unless one counts that the strings of QCD as well as the strings of matrix models\(^5, 6\) are really of this type.

In QCD you have only local fields to describe where a string is present and it would seem very hard to see how two QCD-strings lying on top of each other along a piece of curve could get their “heads” and “tails” associated with each
other by QCD degrees of freedom information. So it seems much more likely that
QCD develops the type of strings where the states \(|1\rangle\) and \(|2\rangle\) just described
above must be identified. This conclusion becomes even more obvious if we use
a strong coupling approximation as the method for implementing the strings into
the lattice QCD or just Yang Mills theories. Then, the strings become flux quanta
of color electric flux and there is no way to keep track of or identify parts of the
same string.

QCD or Yang Mills theories as well as also matrix models provides models of
string field theories of the same type as we are going to propose in the present
article.

It is, however, our goal to make string field theory model that does not need
a very hard and non-linear calculation to connect to the string picture as QCD
needs.

We shall indeed see that our model is inspired by an infinite set of seemingly
conserved quantities noticeable in classical (i.e. non quantum mechanical) string
theory, as we shall explain in the following section II. Then we shall start the de-
scription of our string field theory in section III. A crucial complication of our model
is that it needs a constraint ensuring that each “constituent” in \(X_R\) - or \(X_L\)-space
has a successor constituent as shall be described in section IV. Since our model has
at first some bad features because too many states have been made identical it is
far from obvious that our model is indeed an acceptable string field theory. It is
therefore absolutely crucial that it could be used to deduce the Veneziano-model
scattering amplitude. That we shall briefly sketch in section V. Finally in section
VI we shall resume and conclude among other things that our model is a free the-
ory and that it thus becomes important for judging the validity of string theory as
a model for nature, if really a free theory could be the model for nature.

2. Inspiration by the conservation of right and left
moving patterns

The crucial observation that has inspired our proposal for string field theory
originates from considering classical string “scattering” which takes place by a
couple of permuting their “tails” when strings touch in one point. Hereby we
understand that, say two strings come along in such a way that in a moment of
time they have one point in common, but that then after this moment the strings
develop as if they were a different pair of strings. Namely the one obtained by
combining the first part of the string number 1 with the second part of string
number 2, and vice versa. We call that the strings get their tails permuted when
the beginnings and ends of the original strings are combined with the ends in an
different way.
Consider – in classical approximation – two strings described before the collision by:

The first string:

\[ X^\mu_I(\sigma, \tau) = X^\mu_{RI}(\sigma, \tau) + X^\mu_{LI}(\sigma, \tau) \]
\[ = X^\mu_{RI}(\tau - \sigma) + X^\mu_{LI}(\tau + \sigma) \]  

(1)

The second string:

\[ X^\mu_{II}(\hat{\sigma}, \hat{\tau}) = X^\mu_{RII}(\hat{\tau} - \hat{\sigma}) - X^\mu_{LII}(\hat{\tau} + \hat{\sigma}) \]  

(2)

It is important that the two strings are described with conformal gauge choice and with Minkowskian metric. This gauge choice does not fix the gauge (of reparametrization) freedom completely, but only poses the restrictions.

\[ \frac{\partial X^\mu_I}{\partial \tau} \frac{\partial X^\mu_I}{\partial \tau} - \frac{\partial X^\mu_I}{\partial \sigma} \frac{\partial X^\mu_I}{\partial \sigma} = 0 \]  

(3)

and

\[ \frac{\partial X^\mu_I}{\partial \sigma} \frac{\partial X^\mu_I}{\partial \sigma} = 0 \]  

(4)

and analogous ones for string II, i.e. for \( X^\mu_{II} \) instead of \( X^\mu_I \). With this reparametrization gauge choice the equations of motions become

\[ \left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^\mu_I = 0 \]  

(5)

\[ \left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^\mu_{II} = 0 \]  

(6)

and it is these equations that are solved by writing

\[ X^\mu_I(\sigma, \tau) = X^\mu_{RI}(\tau - \sigma) + X^\mu_{LI}(\tau + \sigma) \]  

(7)

\[ X^\mu_{II}(\hat{\sigma}, \hat{\tau}) = X^\mu_{RII}(\hat{\tau} - \hat{\sigma}) + X^\mu_{LII}(\hat{\tau} + \hat{\sigma}) \]  

(8)

In fact will any 26 pairs of functions \( X^\mu_{RI} \) and \( X^\mu_{LI} \) only depending on \( \tau_R = \tau - \sigma \) and \( \tau_L = \tau + \sigma \) respectively lead to solution of the equation of motion.
When at a moment string I and string II have a common point it means that there exist two sets of timetrack surface coordinates \((\sigma_o, \tau_o)\) and \((\tilde{\sigma}_o, \tilde{\tau}_o)\) such that

\[
X_I^\mu(\sigma_o, \tau_o) = X_{II}^\mu(\tilde{\sigma}_o, \tilde{\tau}_o) \tag{9}
\]

After the collision we imagine that there are in fact two new strings which we may call III and IV composed from pieces of the original strings I and II and that the development goes on as if III and IV are the strings then. Formally the relation – and we here think locally at first ignore for simplicity the problems of boundary conditions – among the strings III and IV to I and II are simply at \(\tau = \tau_o\) and \(\tilde{\tau} = \tilde{\tau}_o\) described as \(\tau_{III} = \tau_{IIIo} = \tau_o\) moment for string III and \(\tau_{IV} = \tau_{IVo} = \tau_o\) too say.

\[
X_{III}^\mu(\sigma_{III}, \tau_{IIIo}) = \begin{cases} 
X_I^\mu(\sigma_{III}, \tau_o) & \text{for } \sigma_{III} > \sigma_o \\
X_{II}^\mu(\sigma_{III} - \sigma_o + \tilde{\sigma}_o, \tilde{\tau}_o) & \text{for } \sigma_{III} < \sigma_o
\end{cases} \tag{10}
\]

\[
X_{IV}^\mu(\sigma_{IV}, \tau_{IVo}) = \begin{cases} 
X_{II}^\mu(\sigma_{IV} - \sigma_o + \tilde{\sigma}_o, \tilde{\tau}_o) & \text{for } \sigma_{IV} > \sigma_o \\
X_I^\mu(\sigma_{IV}, \tau_o) & \text{for } \sigma_{IV} < \sigma_o
\end{cases} \tag{11}
\]

Also the \(\tau_{III}\) and \(\tau_{IV}\) derivatives obey the analogous relations; you just take the \(\tau\)-derivatives. Using both equations (11) and the corresponding \(\frac{\partial}{\partial \tau}\) relations we have information enough to put the solutions for the development of the strings III and IV in terms of the left and right mover functions from the strings I and II. Indeed we may even simply argue that from causality at finite distance from the point of collision \(X_I^\mu(\sigma_o, \tau_o) = X_{II}^\mu(\sigma_o, \tilde{\tau}_o) = X_{III}^\mu(\sigma_o, \tau_o) = X_{IV}^\mu(\sigma_o, \tau_o)\) tight in \(\tau\) to the “moment of collision” in \(\tau_{III}, \tau_{IV}\) etc. compared to the \(\sigma\)-distance to the collision point solutions in string III and string IV must be identical to the corresponding ones in I and II locally. What goes on has simply no knowledge of whether the collision took place. Hence the solutions for all later “time” or better “\(\tau\)” are:

For string III with \(\tau_{III} \geq \tau_{IIIo} = \tau_o\)

\[
X_{III}^\mu(\sigma_{III}, \tau_{III}) = X_{RIII}^\mu(\tau_{III} - \sigma_{III}) + X_{LIII}^\mu(\tau_{III} + \sigma_{III}) \tag{12}
\]

where

\[
X_{RIII}^\mu(\tau_{III} - \sigma_{III}) = \begin{cases} 
X_{RI}^\mu(\tau_{III} - \sigma_{III}) & \text{for } \tau_{III} - \sigma_{III} > \tau_o - \sigma_o \\
X_{RII}^\mu(\tau_{III} - \sigma_{III} - \sigma_o + \tilde{\sigma}_o - \tilde{\tau}_o) & \text{for } \tau_{III} - \sigma_{III} < \tau_o - \sigma_o
\end{cases} \tag{13}
\]
and

\[ X_{\mu}^{\nu}(\tau_{III} - \sigma_{III}) = \begin{cases} X_{LIII}^{\mu}(\tau_{III} + \sigma_{III}) & \text{for } \tau_{III} + \sigma_{III} < \tau_{o} + \sigma_{o} \\ X_{LIII}^{\mu}(\tau_{III} + \sigma_{III} - \tau_{o} - \hat{\tau}_{o} - \sigma_{o} - \hat{\sigma}_{o}) & \text{for } \tau_{III} + \sigma_{III} > \tau_{o} + \sigma_{o} \end{cases} \tag{14} \]

For string IV:

\[ X_{IV}^{\mu}(\sigma_{IV}, \tau_{IV}) = X_{RIV}^{\mu}(\tau_{IV} - \sigma_{IV}) + X_{LIV}^{\mu}(\tau_{IV} + \sigma_{IV}) \tag{15} \]

where

\[ X_{RIV}^{\mu}(\tau_{IV} - \sigma_{IV}) = \begin{cases} X_{RIII}^{\mu}(\tau_{IV} - \sigma_{IV} - \tau_{o} + \sigma_{o} + \hat{\tau}_{o} - \hat{\sigma}_{o}) & \text{for } \tau_{IV} - \sigma_{IV} > \tau_{o} - \sigma_{o} \\ X_{RI}^{\mu}(\tau_{IV} - \sigma_{IV}) & \text{for } \tau_{IV} - \sigma_{IV} < \tau_{o} - \sigma_{o} \end{cases} \tag{16} \]

and

\[ X_{LIV}^{\mu}(\tau_{IV} + \sigma_{IV}) = \begin{cases} X_{LIII}^{\mu}(\tau_{IV} + \sigma_{IV} - \tau_{o} - \sigma_{o} + \hat{\tau}_{o} + \hat{\sigma}_{o}) & \text{for } \tau_{IV} + \sigma_{IV} < \tau_{o} + \sigma_{o} \\ X_{LI}^{\mu}(\tau_{IV} + \sigma_{IV}) & \text{for } \tau_{IV} + \sigma_{IV} > \tau_{o} + \sigma_{o} \end{cases} \tag{17} \]

Now the observation which is so important for inspiring the proposed attempt in this article to second quantize string theory is the following:

On string III and IV together you will find realization of any value of \( X_{RIII}^{\mu} \) and \( X_{RIV}^{\mu} \) and any pattern just once for each time; you find such value or pattern for the right mover position on I and II together i.e. on \( X_{IR}^{\mu} \) and \( X_{IIR}^{\mu} \). The same result follows quite analogously for the \( X_{L}^{\mu} \)’s.

It should be noted that since only \( X_{R}^{\mu} + X_{L}^{\mu} \) or \( \dot{X}_{R}, \dot{X}_{L} \) have physical meaning you always make a kind of gauge transformation by making the transformation

\[ X_{R}^{\mu} \rightarrow X_{R}^{\mu} + k \]
\[ X_{L}^{\mu} \rightarrow X_{L}^{\mu} - k \tag{18} \]

As we formulated our observation just now it is only true by an appropriate adjustment of this freedom in values. The solutions which we just proposed were, however, put with the choice that made our observation work as stated. If we prefer to state our observation in a way not suffering from this need for adjustment of notation of \( X_{R}^{\mu} \) and \( X_{L}^{\mu} \) we may state it for the derivatives with respect to \( \tau_{R} = \tau - \sigma \) and \( \tau_{L} = \tau + \sigma \) respectively instead:
The strings III and IV contain together a value-spectrum for their \( \frac{dX^\mu_{RIII}}{d\tau_{RIII}} \) and \( \frac{dX^\mu_{RIV}}{d\tau_{RIV}} \) which is just the same as that for the incoming strings I and II, i.e. for their \( \frac{dX^\mu_{RI}}{d\tau_{RI}} \) and \( \frac{dX^\mu_{RII}}{d\tau_{RII}} \).

More formally stated we may formulate this observation in the following:

\[
\{ \dot{X}^\mu_{RIII}(\tau_{RIII}) | \tau_{RIII} = \tau_{III} - \sigma_{III} \} \text{ value realized at “some later moment” on string III} \]
\[
U \{ \dot{X}^\mu_{RIV}(\tau_{RIV}) | \tau_{RIV} = \tau_{IV} - \sigma_{IV} \} \text{ value realized at “some later moment” on string IV} \]
\[
= \{ \dot{X}^\mu_{RI}(\tau_{R}) | \tau_{R} = \tau - \sigma \} \text{ a value realized at “some earlier moment” on string I} \]
\[
U \{ \dot{X}^\mu_{RIII}(\hat{\tau}_{R}) | \hat{\tau}_{R} = \hat{\tau} - \hat{\sigma} \} \text{ a value realized at “some earlier moment” on string II} \]

Of course we have the analogous result for the \( X^\mu_{L} \)'s.

Also this result generalizes to the case of successive scatterings of the type described – i.e. a common point at one moment and a tail exchange – and we may loosely state the general result:

Any piece of pattern of the \( \dot{X}^\mu_{R} \)-values (or with appropriate adjustments of \( X^\mu_{R} \) itself) in the “incoming” set of strings will reappear just once on the \( X^\mu_{R} \)'s of the outgoing strings. In other words such \( \dot{X}^\mu_{R} \) (or \( X^\mu_{R} \)) patterns are conserved objects.

In this formulation we had in mind an S-matrix-like situation of classical strings, i.e. a set of classically treated strings come in from far way and scatter by what locally looks like hit in one point with tail-exchange. Also the strings are imagined to separate infinitely at the end so that we are allowed to use the concepts of “incoming” and “outgoing” strings.

The formal formulation of this general version of the observation is

\[
U \{ \text{“incoming” strings} \} \{ \dot{X}^\mu_{RI}(\tau_{Ri}) | \tau_{Ri} = \tau_{i} - \sigma_{i} \} \text{ an early realized } \tau_{Ri} \text{-value on string i} \}
\[
= U \{ \text{“outgoing” strings} \} \{ \dot{X}^\mu_{Rj}(\tau_{Rj}) | \tau_{Rj} = \tau_{j} - \sigma_{j} \} \text{ a late realized } \tau_{Rj} \text{-value on string j} \}
\]

It should be admitted that strictly speaking there is a lack of proving the observation for these very values in \( \tau_{Ri} \) and \( \tau_{Rj} \) which correspond to the hit-points – the common points for crossing strings. So strictly speaking the statement is only valid modulo this supposedly measure null set of hit-points. By continuity of the functions it should not matter so much though.

Really the theorem as stated is only true for a theory with only closed strings in which case we have also the analogous one for left mover i.e. \( X^\mu_{L} \)'s. But \( X^\mu_{R} \)-waves can run to the end of the string and we now want to ensure and remind the reader
that the pattern in $X_R^\mu$ is at the end of the string reflected as an $X_L^\mu$-pattern of same sort.

Indeed let us remember the usual boundary condition at the end of the string – let us say that $\sigma$ there is 0,

$$\frac{\partial}{\partial \sigma} X^\mu(0, \tau) = X'^\mu(\sigma = 0, \tau) = 0 \quad (21)$$

which implies

$$\dot{X}_R^\mu(\tau_R = \tau) - \dot{X}_L^\mu(\tau_L = \tau) = 0 \quad (22)$$

This equation must hold for all $\tau$ and thus $X_R^\mu(\tau)$ can only deviate by an additive constant

$$X_R^\mu(\tau) = X_L^\mu(\tau) + \text{const} \quad (23)$$

from $X_L^\mu(\tau)$. So really since this constant could be shuffled away by a transformation of the type (18) we can simply say that:

For open string models we can take

$$X_L^\mu = X_R^\mu \quad (24)$$

The boundary constraint at the other end, where according to the usual convention $\sigma|_{\text{second boundary}} = \pi$ we get rather

$$X_R^\mu(\tau - \pi) = X_L^\mu(\tau + \pi) \quad (\text{modulo constant}) \quad (25)$$

which together with (24) leads to the requirement of periodicity of $X_R^\mu = X_L^\mu$ as a function of the argument, $\tau$ (up to an additive constant).

Actually even in closed string case where $X_R^\mu$ and $X_L^\mu$ are not connected they have to be periodic (up to an additive constant), for the string to close as a circle. This is because there shall be periodicity with respect to $\sigma$ (for fixed $\tau$) and say $\tau_R = \tau - \sigma$ so that periodicity with respect to $\tau_R$ is also needed.

Our formulated observations for closed string above will for theories involving also open strings instead be:

Patters – or say simply $\dot{X}_R^\mu$ and $\dot{X}_L^\mu$ values – found on the “incoming” strings $X_L^\mu$ or $X_R^\mu$ (all counted together) will reappear just once each on the combined set of “outgoing” strings counting for them both $X_R^\mu$ and $X_L^\mu$. 

\[ -8 - \]
The inspiration to make our string field theory from this conservation of patterns on $X^\mu_R$ and $X^\mu_L$ is the following: If we represent – as we do in our model – the $X^\mu_R$ or $X^\mu_L$ values taken on by what we call “constituents” placed at those points in the $X^\mu_L$- or $X^\mu_R$-spaces, then these “constituents” sit at quite the same places in correspondence to the incoming set of strings as corresponding to the outgoing set. In other words, although scatterings as described goes on constituents representing $X^\mu_R$ or $X^\mu_L$ do not change their “position” say. In the case of open string theories the $X^\mu_R$- and $X^\mu_L$-spaces are combined to one common space. These constituents does not do anything. They are just sitting undisturbed and changing neither position nor momentum.

So at least we have a timeless description telling a lot of information about the states and developments in a string theory with classically treated strings if we know the “constituents” in $X^\mu_R$- and $X^\mu_L$-spaces in the closed case, or the combined $X^\mu_L$- and $X^\mu_R$-space in the also open string case.

Although it is actually the point of view of our string field theory to throw away as only imagination all other information than that of the just introduced constituents, it must be admitted that there is at least some information about the strings which is not described even if one get to know the positions (and momenta) of all the constituents in the $X^\mu_R$ etc. This lacking information includes at least the information about how the different pieces of strings hang together.

For example we stressed above that the $X^\mu_R$ and $X^\mu_L$ patterns found, and thus the constituents representing them would be quite the same if we just had string I and string II as if we had instead string III and string IV say moving undisturbed at all times. These two thinkable string developments a)I+II happening not to interact and b)III+IV also happening not to interact, would have quite the same combined patterns or constituents. They would therefore be quite undistinguishable if one has no further markings to distinguish these two situations, as shall in fact be seen to be the case in our string field theory. The reader may have good reason to worry if we are throwing away too much information, since after all one would expect that it ought to make sense to distinguish I+II from III+IV existing without scattering. Actually it is even worse, since only knowing the constituents would also not distinguish the two just mentioned developments, I+II and III+IV from the scattering development I+II→III+IV nor from III+IV→I+II.

3. Setting up our string field theory

One of the basic ideas of our string field theory is to construct an actually timeless (i.e. Heisenberg picture) Fock space or Fock space analogue geared to describe – only! – the conserved patterns or image curves in the $X^\mu_R$- and $X^\mu_L$-spaces for classically treated strings. Obviously the sets of values taken on in these
$X^\mu_R$- and $X^\mu_L$-spaces are (continuous) curves, rather than discrete points, since the $X^\mu_R(\tau_R)$ and $X^\mu_L(\tau_L)$ are functions of one variable $\tau_R = \tau - \sigma$ or $\tau_L = \tau + \sigma$ respectively. Nevertheless we are allowed as our special formulation or model to let them be represented by a very dense chain of point positioned constituents. In the next section we shall go a bit more into a rather detailed constraint which we shall propose that these constituents must indeed lie in long chains, thereby to some extent enforcing the one dimensional curve nature of the constituent chains. We shall use the same constraint to impose that crudely speaking $X^\mu_R$ only carry half a degree of freedom in as far as $X^\mu_L$ carry the other half of the original $X^\mu = X^\mu_L + X^\mu_R$. A priori we shall just make creation and annihilation operators for a Fock space filling in or removing “constituents” in the say $X^\mu_R$-space. Concerning the details of the construction of this Fock space it will turn out to be a more familiar task if we first notice that the constraints on the $X^\mu(\sigma, \tau)$ in the single string formulation $\dot{X}^\mu \dot{X}_\mu - X^\mu X'_\mu \approx 0$ and $\dot{X}^\mu X'_\mu \approx 0$ to be “weakly implemented” in terms of the $X^\mu_R$ and $X^\mu_L$ become $\dot{X}^\mu_R \dot{X}_{R\mu} \approx 0$ and $\dot{X}^\mu_L \dot{X}_{L\mu} \approx 0$. Or if we think of $\dot{X}^\mu$ as a 26-momentum density then the 26-momentum densities function as the 26-momentum for massless onshell particles. Part of the momentum density of the single string will be in $\dot{X}^\mu_R$ part in $\dot{X}^\mu_L$. When we go to the constituents it would be the natural suggestion that integrating say $\dot{X}^\mu_R$ over the bit of $\tau_R$ corresponding to that “constituent” should give the 26-momentum of this constituent. With such an interpretation into the “constituent” language the condition of constraint $(\dot{X}^\mu_R)^2 \approx 0$ (as “weak constraint”) becomes the onshell condition (i.e. equation of motion) for the constituent. So if we make the Fock space for the constituents in $X^\mu_R$-space a completely usual one with only onshell particles possible to create or to annihilate should be applicable. The restriction to onshell should only imply that the constraint $(\dot{X}^\mu_R)^2 \approx 0$ gets ensured at the end.

We are now prepared for the set up of our string field theory model in a couple of steps:

1) In the first step we set up the simple and usual Fock space with constituents which are able at first also not to sit in chains (essentially curves with the only half a degree of freedom).

2) In the next step, we impose the constraints so that only “chains” of constituents can actually be created at a time.

At this step ends in principle the set up of the Fock space, but it must be admitted that although the translation of the one or two Fock spaces into strings is relatively simple, it is not at all obvious that one would get the idea of interpreting the model that way if we really happened to live in such a world. So we rather strongly need a third step in explaining our model as a string theory:

3) Interpretation of the one or two Fock space model as a string (field) theory, by interpreting sums of $X^\mu_R$’s (and $X^\mu_L$’s) for two constituents as meaning that a string passes the space time at the event with this sum as coordinates.
3.1 Steps of setting up the model

3.1.1 Particle Fock space

1) The basis for our string field theory is in the case of the only closed string model the Cartesian product of two Fock spaces, \( H_R \) and \( H_L \) each of which is simply the ordinary particle Fock space for massless free – scalar in the case of the bosonic 26-dimensional string theory – particles that can be created and annihilated into all the onshell states of a single massless scalar.

One point to be thought about is that this scalar particle shall in fact like a \( \pi^0 \) meson be its own antiparticle (i.e. the analogue of a majorana particle for fermion case). One indication for this is that the strings have no charge proportional to the length.

It means that the creation and annihilation operators are defined only over positive \( \dot{X}_R^\mu \) states while the ones with \( \dot{X}_R^\mu \) negative are related to the ones with positive – by hermitian conjugation –.

2) Second step is that we impose a constraint telling that if we have one constituent we have also a “successor” in a single particle state that is obtained from the first one by action with a certain operator

\[
\exp \left( i 2 \pi \alpha' p^2 + i f(X) \right)
\]  

(26)

This is to be understood that if there is a particle \( \Psi_A \) in the single particle state \( \Psi_A \) there must be its successor \( \Psi_B \) in the state \( \Psi_B \) obtained from \( \Psi_A \) by

\[
\Psi_B = \exp \left( i 2 \pi \alpha' p^2 + i f(X) \right) \Psi_A
\]

(27)

In this operator \( X^\mu \) and \( p^\mu \) are the position and momentum operators for the \( \mu \)-th coordinate and \( p^2 = p^\mu p_\mu \).

The function \( f \) can be chosen. Therefore there is a freedom to choose the operator by choosing \( f \) differently, all the time obtaining a satisfactory successor \( B \). If there were no such freedom in the successor producing operator \( (26) \) the successor \( B \) would be totally determined by \( A \) and there would be no way of having different chains with the same starting constituent. It should, however, be noted that \( f \) is just one real function and that there is no corresponding function to vary depending on \( p^\mu \), while a constituent \( B \) in a wave packet would in analogy to classical physics expectedly be possible to change a bit compared to the foregoing constituent \( A \) by two free parameters per dimension. We imagine to take \( f(X) \) linear \( f(X) = f_\mu X^\mu \) so that it is only one parameter per dimension. It is this limited amount of freedom in setting up the successors that we refer to by saying that the chain of constituents only corresponds “half a degree of freedom” – You
can adjust the momentum of the successor $B$ by adjusting $f$ or $f^\mu$, but you cannot adjust its position (directly).

The reason for the specific form, and especially the funny appearance of the parameter in string theory $\alpha'$ will be postponed to next section.

To make more precise the meaning of the requirement of having successors to all constituents it may be best to describe the allowed subspace $H_R$ allowed. It consists of those Fock states, i.e. states from $H_R$ for which all the particles have their appropriate successor that is produced from a vacuum state by acting with products of creation operators, creating whole chains of constituents.

We may in fact think of a product of creation operators in which each factor is some creation operator creating a particle/constituent in some wave packet like state $\psi_i$. Then the product

$$\Pi_\it\ "chain" a^+(\psi_i)$$

is defined to be an allowed chain of creating operators, provided that the series of single particle states $\psi_0, \psi_1, \psi_2, \cdots, \psi_i, \cdots$ (presumably a closed chain, or a to both sides infinite one) obey

$$\psi_{i+1} = \exp \left( i 2 \pi \alpha' p^2 + i f^{(i)}_\mu X^\mu \right) \psi_i$$

for all $i$ counted cyclically for the case of a closed chain.

3) Third step is the interpretation of the model into a string language.

As the model in step 1) and 2) is set up we have constructed constituent chains in the $X^\mu_R$- and $X^\mu_L$-space (or in the combined one in open string model case) which are of dimension 1 in spaces with same dimension as space time, in the well working bosonic string case $d = 26$. Thus the chains defined under 2) are in the high density limit of constituents similar in dimension as time tracks of particles a priori. (it must though be contemplated that the onshell condition from 1) actually enforces them to be in a superposition extended infinitely in some direction.)

The strings, i.e. the string time tracks which are 2-dimensional embeddings into 26-dimensions, come out by asking for the set

$$\{ X^\mu_R(A) + X^\mu_L(B) | A, B \text{ constituents} \}$$

which become two-dimensional, once the chains in $X^\mu_R$- and $X^\mu_L$-spaces are one dimensional tracks.

This interpretation actually represents a worrisome point for the model of ours because we have no information in our formalism telling which chains to combine
with which. So a priori we obtain string-time-tracks from all possible combinations of one curve in $X_R^\mu$-space with one in $X_L^\mu$-space. This is what we call the “cross combination problem” of our model and it is clearly not a property of a good physical model and it is not contained in conventional string theory either.

The hope that this “problem” is not really a problem may run like this:

Physicists living in a world of strings would typically make (thought) experiments of the sort that they arrange or find out the state of some strings and then some time may pass and they look for another set of strings and ask the theory for the probability (density) for such happening. This type of experiment is really the $S$-matrix or an approximate $S$-matrix type experiment.

In our string field theory the knowledge about the “incoming” strings – the ones in the initial state – will mean that we have to have those chains or curves of constituents in the $X_R^\mu$ and $X_L^\mu$-spaces that can give these strings.

As we argued in the classical approximation in the second section the $X_R^\mu$ and $X_L^\mu$ patterns are conserved so that the outgoing and the ingoing string systems are indeed sets of strings not distinguishable if one only keeps the information of the constituent chains as in our string field theory. So typically the states with the constituents and their chains which split up and combine in a new way corresponds to what could be scattering results of the strings. So one might hope that finding the cross combined strings could be interpreted as seeing the scattering of strings.

It must be admitted though that this hope may not quite work in the case of two closed strings which have simply their $X_L^\mu$-mover degrees of freedom permuted. In fact but there is a somewhat lucky occurrence of the common points of the strings: It could be that the strings in the final state classically just could turn out to be the same as the one obtained by such permutation of the $X_L^\mu$-degrees of freedom. So should one really observe such a pair of strings that have resulted from the permutation of the $X_L^\mu$-mover degrees of freedom it could (in some cases, with sufficient delay) be interpreted as the scattering of the first pair of strings. But if it occurred in the same moment of time in some frame it does not seem easy to interpret them this way.

Another idea that could help on the problem is to make use of the gauge-like transformation (18) \footnote{\textsuperscript{13}}

\begin{align}
X_R^\mu &\to X_R^\mu + k^\mu \\
X_L^\mu &\to X_L^\mu - k^\mu
\end{align}

which can be used on the right and left constituents contributing to a single string. For another string you may choose another 26-vector for $k^\mu$. Considering two sets of this type of transformation for two (say incoming) strings, the cross
combined strings will not have their positions invariant but will rather be displaced
by \( \pm (k^\mu_1 - k^\mu_2) \) where \( k^\mu_1 \) and \( k^\mu_2 \) are the shiftings for the first and the second
string respectively. If one somehow thinks of these \( k^\mu_1 \) and \( k^\mu_2 \) as random and
spreading over all 26-space we would almost certainly get the cross combined strings
out of sight in practice.

In this philosophy one expects scattering amplitudes (=\( S \)-matrix elements) to
be computed basically by putting up the state in our Fock-space(s) [both \( H_R \) and
\( H_L \) in the only closed string case and the identified space \( H_R = H_L \) in the open
string case] corresponding to the incoming strings and then simply take the Hilbert
inner product overlap with the corresponding outgoing system

\[
\langle \text{outgoing} | S | \text{ingoing} \rangle = \left\langle \begin{array}{c}
\text{outgoing} \\
\text{in our string field theory} \\
\end{array} \left| \begin{array}{c}
\text{ingoing} \\
\text{in our string field theory} \\
\end{array} \right. \right\rangle \tag{32}
\]

The scattering so to speak is immaterial and nothing really happens in our
string field theory. It is totally “free”, scattering is all phantasy!

4. The successor operator and the commutation rules of \( X^\mu_R \) with itself

There can be considered to be two motivations for imposing the condition
mentioned under step 2) in the foregoing section:

a) we like the constituents to form chains/curves in \( X^\mu_R \)-space (or \( X^\mu_L \)-space).

b) we must implement a restriction corresponding to the feature

\[
\Pi^\mu_R = \frac{d}{d\tau_R} X^\mu_R \cdot \frac{1}{\pi\alpha'} \tag{33}
\]

which occurs for a physically reasonable assignment of the momentum density
\( \Pi^\mu \) to be a sum \( \Pi^\mu_L + \Pi^\mu_R \) of terms associated with the left- and right-mover degrees
of freedom. This relation \( (33) \) really tells us that the \( X^\mu_R \)-degrees of freedom are
only one half degrees of freedom for each dimension, as can also be seen from the
commutation relation

\[
[X^\mu_R(\tau_R), X^\nu_R(\tau'_R)] = -ig^{\mu\nu} \theta(\tau_R - \tau'_R) \tag{34}
\]

which shows that the \( X^\mu_R \) for different \( \tau_R \)'s do not commute.

Translated into the language of constituents we must think of a discrete but very
dense chain of them, each constituent covering (so to speak) a very small interval
in the $\tau_R$-variable say, of length $\Delta \tau_R$. Then we should identify the momentum $p^\mu$ of the constituent at $\tau_R$ with

$$p^\mu = \Delta \tau_R \Pi^\mu_R(\tau_R) \cdot \frac{1}{2} \quad (35)$$

(the $\frac{1}{2}$ comes because of our normalization of $\Pi^\mu_R$ so that $[\Pi^\mu_R, X^\nu] = -i\frac{1}{2} g^{\mu\nu} \delta(\sigma' - \sigma)$)

On the other hand

$$- X^\mu_R(\tau - \sigma) = \dot{X}^\mu_R(\tau - \sigma) = \frac{d}{d\tau_R} X^\mu_R(\tau_R) \quad (36)$$

also means the differentiation as you go along in $\tau_R$ and therefore $\frac{1}{\Delta \tau_R}$ (the step between constituents) $\approx \frac{X^\mu_R(i+1) - X^\mu_R(i)}{\Delta \tau_R}$ where $i$ is the number of a constituent along the chain. We must thus have

$$\frac{X^\mu_R(\tau_R + \Delta \tau_R)^{i+1} - X^\mu_R(i)}{\Delta \tau_R} = \Pi^\mu_R \cdot 2 \pi \alpha' = \frac{2 \rho^\mu 2 \pi \alpha'}{\Delta \tau_R} \quad (37)$$

and thus we have

$$X^\mu_R(i+1) - X^\mu_R(i) = 4 \rho^\mu \pi \alpha' \quad (38)$$

Here the “successor” of the $i$th constituent is the $(i + 1)$th and we may seek to construct an operator $O$ that can bring the state $\psi_i$ of the $i$th constituent into that for the successor, the $i + 1$th,

$$\exp \left(i \frac{\pi \alpha'}{2} \cdot 4 \rho^2 \right) = \exp \left(i 2 \pi \alpha' \rho^2 \right) \quad (39)$$

To get the factors 2 right here we should be careful with the commutation rules in our notation

$$\Pi^\mu = \Pi^\mu_R + \Pi^\mu_L \quad (40)$$

$$X^\mu = X^\mu_R + X^\mu_L$$

and

$$[\Pi^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = -i g^{\mu\nu} \delta(\sigma - \sigma') \quad (41)$$
and thus using also $[X^\mu_R, X^\nu_L] = 0$, $[\Pi^\mu_R, \Pi^\nu_L] = 0$ having

$$[\Pi^\mu_R, X^\nu_R] = -\frac{i}{2} g^{\mu\nu} \delta(\sigma - \sigma') \quad (42)$$

The operator (39) will do the job of creating a state displaced by $4\pi \alpha' p^\mu$ in $X^\mu_R$-space, but any operator of exponential form with a function of the $X^\mu_R$-operator, say $f(X^\mu_R)$ added will do this job too. The appearance of such a freedom in the choice of the operator, which could thus be

$$\exp \left( i 2\pi \alpha' p^2 + f(X) \right) \quad (43)$$

for each coordinate in $X^\mu_R$-space or generally formulated

$$\exp \left( i 2\pi \alpha' p^\mu p^\mu + f(X^1_R, X^2_R, \cdots X^{25}_R, X^O_R) \right) \quad (44)$$

is welcome and not unexpected. Indeed there should be what we called “half a degree of freedom” per point in choosing how the chain of constituents should be embedded into $X^\mu_R$-space. In the classical approximation there should be one real parameter to determine the state of one constituent, once the state of the foregoing is known. We thus expect the operator $O = \exp \left( i 2\pi \alpha' p^2 + f(X_R) \right)$ to have one real free parameter with which implement this “half” a degree of freedom in the classical approximation. In classical approximation we consider the constituents in wave packet states with so small extension in both $X^\mu_R$-space and its conjugate that we can consider interesting functions as slowly varying over such small distances. So in this classical approximation we would like to Taylor expand the function $f(X^\mu_R)$ and approximate it by its term linear in $X^\mu_R$ and that would just allow one real parameter to use to parameterize the state of the chain of constituents. It could be tempting and fun – but it ought not to be of much importance in classical limit – to play with the second order term in the Taylor expansion of $f(X^\mu_R)$ and put it in “for beauty” with a small once for all settled coefficient. Then we can combine it with the $2\pi \alpha' p^2$ term and make out of the whole exponent a harmonic oscillator Hamiltonian multiplied by the imaginary unit $i$. We then propose for elegance – hoping that the details do not matter much – the operator $O$ of the form

$$O = \exp(i H) \quad (45)$$

where $H$ is the harmonic oscillator Hamiltonian with the kinetic term enforced to be of the form $2\pi \alpha' p^2$. The mass $M_{osc}$ of the oscillating particle is given by
\[
\frac{1}{2M_{\text{osc}}} = 2\pi\alpha' \quad (46)
\]
i.e. \(M_{\text{osc}} = 1/(4\pi\alpha')\)

Therefore the Hamiltonian is given by

\[
H = 2\pi\alpha' p^2 + \frac{1}{2}K(X^\mu_R - X^\mu_{Ro})^2 \quad (47)
\]

The frequency of this formal oscillator is

\[
\omega = \sqrt{\frac{K}{M_{\text{osc}}}} = \sqrt{\frac{4\pi\alpha'K}{4}} \quad (48)
\]

and for the use of the “Hamiltonian” \(H\) in the exponent \(\exp(iH)\) the dimensions shall be so that \(\omega \hbar\) is dimensionless, or with \(\hbar = 1\) \(\omega\) should be dimensionless. That is to say the dimensions should be

\[
[M_{\text{osc}}] = \left[\frac{1}{4\pi\alpha'}\right] = [\text{GeV}^2]
\]
\[
[K] = \left[\frac{1}{\hbar^2 \cdot \frac{1}{4\pi\alpha'}}\right] = [\text{GeV}^2]
\]

We could for instance choose \(K\) so that \(\omega \hbar\) the spacing between levels would be \(2\pi\) divided by some large natural number, \(q\) say, so that the \(\hat{O} = e^{-i\frac{1}{2}\omega \hbar} O\) deviating only from \(O\) by a constant phase factor would have the property

\[
\hat{O}^q = 1 \quad (50)
\]

Only in the limit \(q \to \infty\) we would really get the string theory, but it would be interesting to see if this special idea could be relevant to connect to the string theories with \(q\)-adic numbers if \(q\) were a prime. We would obtain \(\omega \hbar = \hbar\sqrt{K/4\pi\alpha'} = \frac{1}{q}\) for \(K = \frac{1}{q^2 4\pi\alpha'}\). The parameter \(X^\mu_{Ro}\) that denotes the bottom of the potential for the (analogue) oscillator is the one that gives the “half” degree of freedom.

Now the application of the operator \(O\) or \(\hat{O}\) should be that the allowed subspace of the Fock-Hilbert space for the \(X^\mu_R\)-space constituents \(H_{R \text{ allowed}} \subseteq H_R\) is built up by inserting products of a large (in principle in the infinitely many limit) number of creation operators, each creating the particles in a state connected by \(O\) to the foregoing state. The states in \(H_{R \text{ allowed}}\) are constructed from a “vacuum” state in \(H_R\) by action with products of creation operators of the form
\[ \Pi_k a^+ \left( \mathcal{H}_{l=0}^k \hat{O}_l(X^\mu_{Rol})\Psi \right) \] (51)

Here the state of the \( k \)th creation operator \( a^+ \left( \mathcal{H}_{l=0}^k \hat{O}_l(X^\mu_{Rol})\Psi \right) \), which we could call

\[ \Psi_R = \mathcal{H}_{l=0}^k \hat{O}_l(X^\mu_{Rol})\Psi \] (52)

is obtained from a starting state \( \Psi \) by a series of successive applications of the “going to the successor” operator \( \hat{O}_l(X^\mu_{Rol}) \). Here a bottom of the oscillator point \( X^\mu_{Rol} \) varies – in a smooth way – as one goes along with the chain-link enumerating integer \( l \). Also \( k \) is used to enumerate constituents along the chain. The product sign \( \mathcal{H} \) means product with respect to the function composition \( \circ \). But the functions \( \hat{O}_l(X^\mu_{Rol}) \) composed are really just linear operators acting on the single particle Hilbert space so that we would really denote usually the product without using the function composition sign \( \circ \). To start the chain at some point is not allowed. It is meant that either \( k \) must run infinitely in both positive and negative directions – so that it should be made sense also of \( k \) being negative – or it should make up a closed loop chain.

It is of importance for the appearance of scattering at all, even if it gets somehow phantasy only in our model, that the same state in the allowed Fock space \( H_R \) allowed may be created by several different combinations of chain operators (51). At least it is of importance that states made by different chain combinations may have a nonzero overlap. Otherwise there could not even in some point of view – of “phantasy” – be any scattering. But that is also possible since two different chains could even have a constituent each so that these two constituents would be in exactly the same single particle state. Then one could make an “overlapping” pair of chains to a given two-chain state where the two chains have such a common constituent state in this way: take a new pair obtained by tail exchange from the first pair. Thus we can construct another two-string state in which compared to the first pair the strings are constructed from one “half” from each string in the first pair. The switch over between the two interpretations could be done at the common state constituents. Note how the switching of tails lacks of significance which was announced to characterize our type of string field theory. It is in the just given example used to argue that we could make examples of the same Fock state of e.g. two chains created on it, being constructable in more than one way.

You may also bear in mind that this property of our allowed Fock space, which has same state constructable in several ways, i.e. from several chain (= “half strings”) combinations, is the one that allows scattering to seemingly take place without anything having to happen in the Fock space language in our model. It is the property that scattering is allowed to take place as pure phantasy.
5. Idea of Deriving the Veneziano Model

Since it has now been suggested that fake scattering should be possible, it is very important to see if such “fake” scattering indeed will go according to the Veneziano model as its scattering amplitude. Otherwise our string field theory would not correspond to the string theory.

We shall limit ourselves to a sketch of an argument only by suggesting that at least it can look likely that the Veneziano model will result.

What we have to do in order to look for a Veneziano model scattering amplitude is to think of a number of incoming strings – in some mass eigenstates say $|1, p_1; 2, p_2; \cdots ; q, p_q >$ and then look for overlap of this state with the outgoing state $|q + 1, -p_{q+1}; \cdots ; N, -p_N >$. This overlap should then hopefully turn out to be the Veneziano model scattering amplitude.

The main idea in the derivation of the Veneziano model as we hope to perform it from our model is that the “summation” over the different ways of combining ingoing and outgoing particle constituents becomes the integration in the Veneziano model expressions. We can say that this integration/summation over various assignments of ingoing to outgoing constituents roughly speaking should become the Koba-Nielsen variables.

Although we have not yet completed the calculation\textsuperscript{[7]}, it would be strange if we did not get Veneziano model when calculating the overlaps proposed.

6. Conclusion and resume

We have put forward ideas for second quantizing string theory into a string field theory in a way that is suggested to be probably different from string field theory schemes in the cases of Kaku-Kikkawa, Kyoto group, the cubic theory of Witten and Zwiebach. Rather our formulation or model is in class of strings in QCD or matrix model strings. Indeed our model is in a class we could call “constituent string field theory.” One of the consequences in this class is that two strings following each other for a piece in the middle (or just meeting somewhere) cannot be distinguished from the string pair obtained by tail-exchange. Furthermore our string field theory is characterized by the fact that it has right mover and left mover degrees of freedom expressed by constituents separately. In the case of theories with open strings the Hilbert space of all the possible numbers of strings is suggested to be described by a certain subspace – the space of allowed states $H_{\text{allowed}}$ – of the Fock space for massless (scalar in bosonic string case) particles. This “allowed” subspace consists of state-vectors that can be constructed by operators creating chains of constituents. One constituent has its state obtained from the foregoing one (along the chain) by action of an operator with a parameter. Thus the freedom
of inserting such chains corresponds so to speak to “half a degree of freedom per constituent”. In this way the model gets the right number of degrees of freedom since there is locally both $X^\mu_R$ and $X^\mu_L$ which makes up $X^\mu = X^\mu_L + X^\mu_R$ although in the open string globally $X^\mu_R$ and $X^\mu_L$ are constructed from the same constituents.

It is to be stressed that in our string field theory the scattering, meaning that strings exchange parts with each other as time passes is not represented as anything happening in the fundamental language of our model! That is to say that in our model – fundamentally speaking – no scattering happens; it is rather just a fake. The point is that we have declared the features of the multistring states which change under a scattering process for non-existing as fundamental degrees of freedom. We do not distinguish the scattered and yet unscattered systems of strings. This may seem at first to look like throwing too much information out of our scheme but we suggest that it should nevertheless be possible to obtain the Veneziano model as the transition amplitude between the incoming and outgoing multistring states in the Fock space in our model. If this is confirmed it will support the thesis that our model is indeed a description of string theory. Remarkably enough our model is really a free particle Fock space theory.

So from our model point of view the hypothesis is that Nature should be described by string theory would raise the question: Could Nature indeed be successfully described by a free theory? Presumably it can, but one might be worried how a Thuring machine could be embedded in a totally free theory. How could all the complicated computations be done by a machine finally described by a “free” model?

What would be helpful in making the working of our scheme more trustworthy would be if we find how some of the features of string theory like branes, favoured dimensions and gauge groups, are to be seen in our scheme. Since our model is in many ways much simpler – really a free massless particle theory, with some constraints on the allowed states though – one could hope that such feature, D-brane, dimension, etc., might be seen rather differently and perhaps more easily.

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