Interaction Hierarchy

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Abstract

We analyse a new class of statistical systems, which simulate different systems of random surfaces on a lattice. Geometrical hierarchy of the energy functionals on which these theories are based produces corresponding hierarchy of the surface dynamics and of the phase transitions. We specially consider 3D gonihedric system and have found that it is equivalent to the propagation of almost free 2D Ising fermions. We construct dual statistical system with new matchbox spin variable $G_\xi$, high temperature expansion of which equally well describe these surfaces.
1. The correspondence between spin configurations and the surfaces of interface allows to define different theories of random surfaces on a lattice. Ising ferromagnet is equivalent to a system of random walks with the energy which is proportional to the length of the path on the two-dimensional lattice and to the area of the surface in three-dimensions [1, 2, 3, 4, 5, 6, 7, 8, 11, 12].

In the recent articles [10, 13] the authors formulated a new class of statistical systems, whose interface energy is associated with the edges of the interface. Spin systems introduced in [10, 13] represent a system of random walks, but with the energy which is proportional to a total curvature of the path on the two-dimensional lattice and to the size of the surface in three dimensions [14, 15, 16]. One motivation for the study of those statistical systems is to have well defined physical system of random surfaces which is relevant for the description of the gauge field dynamics and of the QCD [14, 15, 16].

In order to study the dynamics of the surfaces with linear-gonihedric action [14, 15, 16] we analyse statistical systems with topological and area action and compare statistical properties of these systems from the viewpoint of surface dynamics. The energy functionals on which these theories are based represent a realization of the hierarchy of geometrical functionals found by Steiner and Minkowski [14, 20, 21, 22, 23, 24, 25, 27]. The imposed structure has an important influence on the physical properties of the system and produces corresponding hierarchy of the surface dynamics and of the phase transitions. We have found that 3D gonihedric system is equivalent to the propagation of string with almost free 2D Ising fermions. We construct dual statistical system with new matchbox spin variable $G_\xi$, high temperature expansion of which equally well describe these surfaces.

2. The origin of the interaction hierarchy lies in Steiner's geometrical idea about parallel manifold $M_\rho$ [15, 14, 21, 26]. His formula represents the area of the parallel surface $M_\rho$, situated on the distance $\rho$ from $M$, in terms of the area $S$, mean integral width $A$ and the topology $T$ of the initial convex surface $M$

$$S_\rho = \int_M R_1 R_2 d\omega + \rho \int_M (R_1 + R_2) d\omega + \rho^2 \int_M d\omega,$$

where $R_{1,2}$ are the principal curvatures of the surface $M$ and $d\omega$ is the area element of the spherical map. All this functionals, $S(M)$, $A(M)$ and $T(M)$ properly defined for nonconvex case [14, 15, 16, 10] produce the systems of interacting surfaces with a very deep hierarchical structure.

At the first level one can define topological-gonimetric systems on the lattice, these are 1D Ising ferromagnet, 2D gonimetric walks and 3D gonimetric surfaces [10, 13]

$$H_{1D\text{ Ising}} = -\sum_{\text{links}} \sigma \sigma; H_{2D\text{ gonimetric}} = -\sum_{\text{plaquettes}} \sigma \sigma \sigma \sigma; H_{3D\text{ gonimetric}} = -\sum_{\text{boxes}} \sigma \sigma \sigma \sigma \sigma \sigma,$$

where the spin variables are attached to vertices of the lattice. Using high temperature expansion one can see that all topological systems (2) have the same partition function and are in a completely disordered phase [13]

$$-\beta f_{\text{gonimetric}} = \ln(\omega + \omega^{-1}),$$
Increasing the dimension of the lattices by one and leaving the Hamiltonians (2) without changes, one can see that the system (2) "moves" to next level of the hierarchy and describes now the system with gonihedric action $A(M) = \sum_{<i,j>} |X_i - X_j| \cdot |\pi - \alpha_{ij}|$ which is proportional to the linear size of the system \[14, 15, 16\].

There is mutual connection between gonimetric and gonihedric systems. Indeed in the recent article \[28\] the authors have found that the action $A(M)$ has an equivalent representation in terms of total curvature $k(E)$ of the polygons which appear in the intersection of the two-dimensional plane $E$ with the given surface $M$. This curvature $k(E)$ should then be integrated over all planes $E$ intersecting the surface $M$. This result directly connects topological-gonimetric system in two dimensions with gonihedric system in three dimensions. This connection is universal for any dimensions.

Indeed the intersection of the two-dimensional plane $E$ with the polyhedral surface $M$ is the union of the polygons

$$P_1(E), ..., P_k(E)$$

and as it was shown in \[28\], the absolute total curvature $k(E)$ of the polygons (4) in the intersection is

$$k(E) = \sum_{i=1}^{k} k(P_i) = \sum_{<i,j>} |\pi - \alpha_{ij}^E|,$$  \hspace{1cm} (5)

where $\alpha_{ij}^E$ are the angles of the polygons (4) and they are defined as the angles in the intersection of the two-dimensional plane $E$ with the edge $<X_i, X_j>$, that is with the dihedral angle $\alpha_{ij}$ \[28\]. Particularly for the plane $E$ which is perpendicular to the edge $<X_i, X_j>$ we will have that $\alpha_{ij}^E = \alpha_{ij}$. The meaning of (5) is that it measures the total revolution of the tangent vectors to polygons (4) $P_1(E), ..., P_k(E)$ \[28, 30, 31, 32\]. The summation in (5) can be extended to all edges of the surface $M$ if we take $\alpha_{ij}^E = \pi$ for the edges which are not intersected by the given plane $E$. Finally to get an action $A(M)$ one should integrate the total curvature $k(E)$ (5) over all intersecting planes $E$ \[28\]

$$A(M) = \frac{1}{2\pi} \int_{\{E\}} k(E)dE,$$  \hspace{1cm} (6)

where the integral is extended over all planes$\{E\}$ intersecting the surface $M$. The measure $dE$ in (6) is equal to

$$dE = dpd\Omega$$  \hspace{1cm} (7)

where we define the plane $E$ by its normal vector $\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi)$ and its distance to fixed origin. $d\Omega$ denotes the area element of the unit sphere corresponding to the end point of the normal vector $\vec{n}$. In the next section we will apply (6) to lattice surfaces.

3. Singular surfaces $\{M\}$ of interface $\[1\]$ on a cubic lattice $Z^3$ can be considered as a collection of plaquettes with the restriction that only an even number of plaquettes can intersect at a given edge ($2r = 0, 2, 4$.) and that only one plaquette is on a given place $\[8, 10, 13\]$.\]
As a first step we should find the same representation (6) for the action \(A(M)\) on a cubic lattice \(Z^3\). This can be easily done if we introduce a set of planes \(\{E_x\}, \{E_y\}, \{E_z\}\) on the dual lattice \(Z^3_d\) which are perpendicular to \(x, y\) and \(z\) axis correspondingly.

These planes intersect a given surface \(M\) and on each of these planes we will have an image of the surface \(M\). Every such image is represented as a collection of closed polygons \(\{P(E)\}\) appearing in the intersection of the plane with surface.

The energy of the surface \(M\) is equal now to a sum of the total curvatures \(k(E)\) of all these polygons on different planes

\[
A(M) = \sum_{\text{over all dual planes } \{E_x, E_y, E_z\}} k(E).
\]  

(8)

On the lattice the total curvature \(k(E)\) (5) is simply the total number of polygons right angles. In general, the surface on the lattice may have self-intersections and one should associate the energy with self-intersection edges of the surface and therefore with the self-intersections of polygons. Depending on how we will count the right angles in the self-intersection vertices we will get different theories \([16, 10, 13]\).

In the present article we will consider the case when the self-intersection coupling constant \(\kappa\) is zero \([13]\). This means that when we compute the total curvature \(k(E)\) of the polygons in (8) we should ignore the right angles at the self-intersection points.

In terms of Ising spin variables \(\sigma_r\) the Hamiltonian of this system has the form \([10, 13]\)

\[
H^{3D}_{gonihedric} = - \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r}+\vec{\alpha}} \sigma_{\vec{r}+\vec{\beta}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}},
\]  

(2a)

where \(\vec{r}\) is a three-dimensional vector whose components are integer and \(\vec{\alpha}, \vec{\beta}\) are unit vectors parallel to axis. The gonihedric system (2a) has an extra symmetry: one can independently flip spins on any combination of planes (spin layers).

With (8) the partition function of the system can be written in the form

\[
Z(\beta) = \sum_{\text{over all } \{M\}} \exp\{-\beta \sum_{\{E\}} k(E)\}.
\]  

(9)

Let us represent the sum in the exponent as a product

\[
\exp\{-\beta \sum_{\{E\}} k(E)\} = \prod_{\{E\}} e^{-\beta k(E)} = \prod_{\{E_x\}} e^{-\beta k(E_x)} \prod_{\{E_y\}} e^{-\beta k(E_y)} \prod_{\{E_z\}} e^{-\beta k(E_z)}
\]  

(10)

It is true that these products are not independent for a given surface \(M\) and our goal is to express the initial energy (8) and the product (10) in terms of independent quantities, that is only through the product over all two-dimensional planes in one fixed direction, let’s say through \(\{E_z\}\).

The question is: what kind of information do we need to know on these planes \(\{E_z\}\) to recover the values of the total curvature \(k(E_x)\) and \(k(E_y)\) on the planes \(\{E_y\}\) and \(\{E_x\}\) for the given surface \(M\)?
With this aim let us consider the sequence of two planes \( E^i_z \) and \( E^{i+1}_z \) and denote by \( P_i \) the polygon-image of the surface \( M \) on the plane \( E^i_z \) and by \( P_{i+1} \) the polygon-image of \( M \) on the plane \( E^{i+1}_z \).

With this one can compute the contribution to the total curvature \( k(E_x) \) which comes from the pieces of the polygon-images on the planes \( \{ E_x \} \) which lie between two planes \( E^i_z \) and \( E^{i+1}_z \). This contribution is equal to the number of bonds of the polygons \( P_i \) and \( P_{i+1} \) which are parallel to the \( x \) axis without common bonds.

In the same way the contribution to the \( k(E_y) \) from the polygons on the planes \( \{ E_y \} \) which lie between \( E^i_z \) and \( E^{i+1}_z \) is equal to the length of the non-common bonds of the \( P_i \) and \( P_{i+1} \) which are parallel the \( y \) axis.

Therefore the total contribution to the curvature \( k(E_x) + k(E_y) \) of the polygons which are on the perpendicular planes between \( E^i_z \) and \( E^{i+1}_z \) is equal to the length of the polygons \( P_i \) and \( P_{i+1} \) without length of the common bonds

\[
l(P_i) + l(P_{i+1}) - 2 \cdot l(P_i \cap P_{i+1}). \tag{11}\]

This formula represents an important fact that the curvature of the polygons which lie on the perpendicular planes \( \{ E_y \} \) and \( \{ E_x \} \) is equal to the length of the polygons on the \( \{ E_z \} \) planes

\[
\left( \begin{array}{c} \text{Contribution to } k(E_x) + k(E_y) \\
\text{from planes } \{ E_x \}, \{ E_y \} \\
\text{between } E^i_z \text{ and } E^{i+1}_z \end{array} \right) = l(P_i) + l(P_{i+1}) - 2 \cdot l(P_i \cap P_{i+1}). \tag{12}\]

Where \( l(1) + l(2) - 2 \cdot l(1 \cap 2) = l(1 \cup 2 \setminus 1 \cap 2) \). Now the energy functional (8) is reduced to an independent sum over the polygon loops only on the \( E_z \) planes

\[
A(M) = \sum_{\{ E \}} k(E) = \sum_{\{ E_i \}} k(P_i) + l(P_i) + l(P_{i+1}) - 2 \cdot l(P_i \cap P_{i+1}) \tag{8a}\]

which we will consider as a sum of the free action proportional to the total curvature plus the length

\[
A_0(P) = k(P)/2 + l(P) \tag{8b}\]

and of the interaction term which is proportional to the length of overlapping of the polygon loops \( P_i \) and \( P_{i+1} \)

\[
A_{int}(P_i, P_{i+1}) = -2 \cdot l(P_i \cap P_{i+1}). \tag{8c}\]

With this formula one can represent the product (10) in the form

\[
\prod_{\{ E \}} e^{-\beta k(E)} = \prod_{\{ E_i \}} \exp\{-\beta (A_0(P_i) + A_0(P_{i+1}) + A_{int}(P_i, P_{i+1}))\} \tag{13}\]

and the partition function as

\[
Z(\beta) = \sum_{\{ P_i \} \text{ on } \{ E_i \}} \prod_i \exp\{-\beta (A_0(P_i) + A_0(P_{i+1}) + A_{int}(P_i, P_{i+1}))\}, \tag{14}\]
where an independent summation is extended over all closed polygons on the different planes $E_z$.

If we define transition amplitude from configuration $P_i$ on the plane $E_z^i$ to the configuration $P_{i+1}$ on the plane $E_z^{i+1}$ as

$$K(P_i, P_{i+1}) = \exp\{-\beta(A_0(P_i) + A_0(P_{i+1}) + A_{int}(P_i, P_{i+1})\},$$

then

$$Z(\beta) = \sum_{\{P_i\}} \prod_i K(P_i, P_{i+1})$$

which we can interpret as the propagation of the polygon-loop in the z-direction.

To compare with 3D Ising system \[33, 34, 35, 36, 37, 41\] we will represent the energy of the interface in the form

$$S(M) = \frac{1}{2} \sum_{\{E\}} l(E) = \sum_{\{E_z\}} l(P_i) + s(P_i) + s(P_{i+1}) - 2 \cdot s(P_i \cap P_{i+1})$$

with the free action $A_0$ which is proportional to the total length plus oriented area

$$A_0 = l(P)/2 + s(P)$$

and interaction which is now proportional to overlapping area of the polygon loops

$$A_{int} = -2 \cdot s(P_i \cap P_{i+1})$$

where $s$ denotes the area. We see, that compared with (8), the interaction is much more stronger.

Description of the spin systems in terms of interface factors part of the symmetry: two spin configurations connected by the global $Z^2$ transformation correspond to the same surface. Description in terms of polygon loops (8a,b,c) factors the number of surface configurations by the factor $2^N$, when $\kappa = 0$. This fact is connected with the layer symmetry of the system when $\kappa = 0$ (see section 3).

The 3D Ising ferromagnet does not have this symmetry, therefore in (17),(18) and (19) we have to ascribe consistent orientation to every polygon loop on the $E_z^i$ planes and then to sum over $2^N$ different orientations.

Using (8a,b,c) let us define

$$A(1, 2) = A_0(1) + A_0(2) + A_{int}(1, 2)$$

where $P_i \equiv i$ so that (15) becomes equal to

$$K(1, 2) = e^{-\beta A(1, 2)}.$$ (21)

Let us consider intermediate summation over all polygon configurations between $E_z^1$ and $E_z^3$

$$\sum_{\{2\}} K(1, 2)K(2, 3) = \sum_{\{2\}} e^{-\beta A(1, 2) - \beta A(2, 3)}$$ (22)

and represent the result in the form
where
\[ F(1, 3) = \sum_{\{2\}} e^{-2\beta A_0(2)} V(1, 2, 3), \]
and
\[ V(1, 2, 3) = e^{-\beta (-A_{int}(1,3)+A_{int}(1,2)+A_{int}(2,3))}. \]
The last term in (24) represents the interaction which is proportional to the overlapping length.

6. To simplify the system and to have crude approximation we will ignore the interaction term (24). In that case \( F(1, 3) \) does not depend on the loops 1 and 3. Therefore denoting this amplitude simply by \( F \) we will have
\[ F = \sum_{\{P\}} e^{-2\beta A_0(P)} = \sum_{\{P\}} e^{-\beta(k(P) + 2l(P))} \]
and that the partition function reduces to two-dimensional partition function
\[ Z_0(\beta) = F^N. \]
So in this approximation the partition function coincides with the two-dimensional model (25) with the energy which is proportional to the sum of the lengths of the loops plus the total curvature. This system coincides with the sum of the 2D Ising model \( Z(\beta) = \sum_{\{P\}} e^{-\beta l(P)} \) and of the 2D gonimetric walks. In short one can say that in this approximation the system is equivalent to 2D Ising model in which the paths are weighted by the total curvature. Our aim is to evaluate (14) in this approximation.

7. The two-dimensional topological model (2), 2D gonimetric walk, has the partition function (3) and is in the disordered regime \([13]\).

For the three-dimensional case, we obtain an additional perimeter term to gonimetric walks (8b), (25) and this results to the change of the phase structure of the system in three dimensions. We can expect that because of the perimeter term, the linear system (6), (8) will show the phase transition in 3D which should be of the same nature as it is in the 2D Ising ferromagnet.

To find partition function one has to represent the corresponding weights in terms of eight-vertex model \([9, 38, 39, 11]\). The 2D Ising system has the weights
\[ \omega_1 = 1, \quad \omega_2 = w^4, \quad \omega_3 = \omega_4 = \omega_5 = \omega_6 = \omega_7 = \omega_8 = w^2, \]
where \( w = \exp(-\beta J) \) and \( \omega_\xi = \exp(-\beta \epsilon_\xi) \) and \( \epsilon_\xi \) is the energy assigned to the \( \xi \)th type of vertex configuration (\( \xi = 1, \ldots, 8 \)) \( \epsilon_1 = 0, \quad \epsilon_2 = 4J, \quad \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \epsilon_8 = 2J \). The weights of the gonimetric walks with \( \kappa = 0 \) are \([10, 13]\)
\[ \omega_1 = \omega_2 = \omega_3 = \omega_4 = 1, \quad \omega_5 = \omega_6 = \omega_7 = \omega_8 = \omega, \]
where \( \omega = \exp(-\beta \Theta(\pi/2)) \) and \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0, \quad \epsilon_5 = \epsilon_6 = \epsilon_7 = \epsilon_8 = \Theta(\pi/2) \).

With this notations the system (25), (26) has the weights

\[
\Omega_1 = 1, \quad \Omega_2 = w^4, \quad \Omega_3 = \Omega_4 = w^2, \quad \Omega_5 = \Omega_6 = \Omega_7 = \Omega_8 = \omega \cdot w^2. \quad (29)
\]

Following [13, 6, 40, 38, 11] one can rewrite the partition function (25) as a product of the fermion operators and obtain

\[
-\beta f_{\text{gonihedric}}^0 = \frac{1}{8\pi^2} \int_0^{2\pi} \ln[(1 + w^4)^2 - 4w^8\omega^2(1 - \omega^2) + 4w^4(1 - \omega^2) \cos \theta \cos \phi + 2(w^6 + w^2 - 2w^6\omega^2)(\cos \theta + \cos \phi)]d\theta d\phi,
\]

which confirms our expectation. One can also compute the partition function with the Kac and Ward [5] combinatorial approach. To each oriented closed polygon one can correspond a term of the matrix determinant and vice versa. For that we should define the matrix \( A((i, j)(X, Y)) \), where \( (i, j) \) - indicates which points of the lattice are connected by the polygon loop and \( X = (\text{Right, Left, Down, Up}) \equiv (R, L, D, U) \) - indicate the direction in which the bond joining the vertices of the loop is traversed and \( Y = (R, L, D, U) \) - subsequent direction of "motion". In our case \( RR = LL = UU = DD = w, \ RU = LD = UL = DR = w\omega \alpha, \ RD = LU = DL = UR = w\omega \alpha^- \), where \( \alpha = \exp(-i\pi/4) \) [5]. The computation of the determinant can be easily done and coincide with the expression (30). We conclude from this that only boundary terms can change this result and that the system describes the propagation of almost free fermionic string.

8. We have found also the dual system to (2a), high temperature expansion of which equally well describe those surfaces. The high temperature expansion of (2a) is

\[
Z(\beta) = \sum_{\{\sigma\}} \prod_{\text{plaquettes}} \text{ch} \beta \cdot \{1 + \text{th} \beta \cdot (\sigma\sigma\sigma\sigma)\}. \quad (31)
\]

Opening the brackets and summing over \( \sigma \) one can see that only such terms produce nonzero contribution which contain an even number of plaquettes on every given vertex, therefore

\[
Z(\beta) = (2\text{ch} \beta)^{3N^3} \sum_{\{\Sigma\}} (\text{th} \beta)^{s(\Sigma)}, \quad (32)
\]

where the summation is extended over all surfaces \( \{\Sigma\} \) with an even number of plaquettes at any given vertex. The \( s(\Sigma) \) is the number of plaquettes of \( \Sigma \), e.g. the area of the surface.

Let us attach plaquette variables \( U_P \) to each plaquette \( P \) of \( Z^3 \)

\[
U_P = -1 \quad \text{if} \quad P \in M \quad \text{and} \quad U_P = 1 \quad \text{if} \quad \notin M \quad (33)
\]

The constraint on the plaquette variables \( U_P \) in every vertex

\[
\prod_{\text{12 plaquettes incident to vertex}} U_P = 1, \quad (34)
\]
uniquely characterizes our set of surfaces \( \{ \Sigma \} \). Now one can introduce the group structure on this set of surfaces \( \{ \Sigma \} \). Let us consider two surfaces \( \Sigma^1 \) and \( \Sigma^2 \) and denote their plaquette variables as \( U^1_P \) and \( U^2_P \) respectively. Let us define group product of these two surfaces as

\[
U_P = U^1_P \cdot U^2_P. \tag{35}
\]

According to this definition the set of surfaces \( \{ \Sigma \} \) (34) forms an Abelian group \( G \). The whole group \( G \) is a direct product of the local groups \( G_\xi \). This group \( G_\xi \) has four elements-elementary surfaces, \( G_\xi = \{ e(\xi), g_\chi(\xi), g_\eta(\xi), g_\varsigma(\xi) \} \) with the multiplication table \( e \cdot g_{\chi,\eta,\varsigma} = g_{\chi,\eta,\varsigma} \); \( g_\chi \cdot g_\chi = g_\eta \cdot g_\eta = g_\varsigma \cdot g_\varsigma = 1; \) \( g_\chi \cdot g_\eta = g_\varsigma \), which follows from the fact that elementary surfaces are match box surfaces with different orientations and from the multiplication law (35).

Any set of elementary surfaces \( e, g_\chi, g_\eta, g_\varsigma \) distributed independently over the lattice \( Z^3 \) describes some allowed surface \( \Sigma \) and any given surface from \( \{ \Sigma \} \) (34) can be uniquely decomposed into the product of \( G_\xi \)

\[
\Sigma = \prod_\xi G_\xi. \tag{36}
\]

This approach allows to describe the original surface \( \Sigma \) in terms of a new independent variable \( G_\xi = \{ e(\xi), g_\chi(\xi), g_\eta(\xi), g_\varsigma(\xi) \} \) which should be attached to center of the cube \( \xi \) of the original lattice \( Z^3 \) or which is the same to the vertices \( \xi \) of the dual lattice \( Z^* 3 \).

The group \( G_\xi \) is an Abelian group of the fourth order and therefore has four one-dimensional irreducible representations \( E = \{ 1, 1, 1, 1 \}, \) \( R^\chi = \{ 1, 1, -1, -1 \}, \) \( R^\eta = \{ 1, -1, 1, -1 \}, \) \( R^\varsigma = \{ 1, -1, -1, 1 \} \) which express algebraically the "matchbox spin" variable \( G_\xi \).

The dual Hamiltonian is nonhomogeneous in the directions \( \chi, \eta \) and \( \varsigma \)

\[
H_{\text{dual}} = \sum_\xi H_{\xi,\chi+\xi} + H_{\xi,\eta+\xi} + H_{\xi,\varsigma+\xi}, \tag{37}
\]

where \( \chi, \eta \) and \( \varsigma \) are unit vectors in the corresponding directions of the dual lattice and

\[
H_{\xi,\chi+\xi} \equiv H(G_\xi, G_{\chi+\xi}) = -R^\chi(\xi) \cdot R^\chi(\xi + \chi),
\]

\[
H_{\xi,\eta+\xi} \equiv H(G_\xi, G_{\eta+\xi}) = -R^\eta(\xi) \cdot R^\eta(\xi + \eta),
\]

\[
H_{\xi,\varsigma+\xi} \equiv H(G_\xi, G_{\varsigma+\xi}) = -R^\varsigma(\xi) \cdot R^\varsigma(\xi + \varsigma). \tag{38}
\]

The partition function of the dual system (37),(38) can be written in the form

\[
Z(\beta^*) = \sum_{\{ G_\xi \}} \exp\{-\beta^* H_{\text{dual}} \}. \]

One can check now, that the high temperature expansion of the dual system (37), (38) indeed coincides with the low temperature expansion of the original system (2a) and provides a new realization of gonihedric system on the lattice. This dual representation can be compared with the dual representation of the 3D Ising ferromagnet in terms of 3D Gauge Wegner system.

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References

[1] B.L.van der Waerden. Z.Physik 118 (1941) 473
[2] H.A.Kramers and G.H.Wannier. Phys.Rev. 60 (1941) 252
[3] E.Ising. Z.Physik 31 (1925) 253
[4] L.Onsager. Phys.Rev. 65 (1944) 117
[5] M.Kac and J.C.Ward. Phys.Rev. 88 (1952) 1332
[6] C.A.Hurst and H.S.Green. J.Chem.Phys. 33 (1960) 1059
[7] T.D.Schultz, D.C.Mattis and E.H.Lieb. Rev. Mod. Phys. 36 (1964) 856
[8] F.J. Wegner, J. Math. Phys.12 (1971) 2259
[9] R.J.Baxter, Exactly solved models in statistical mechanics. Academic Press, London 1982
[10] G.K.Savvidy and F.J.Wegner. Nucl.Phys.B413(1994)605.
[11] C.Fan and F.Y.Wu. Phys.Rev.B2 (1970) 723
[12] L.P.Kadanoff and F.J.Wegner. Phys.Rev. B4 (1971) 3989
[13] G.K.Savvidy and K.G.Savvidy. Phys.Lett. B324 (1994) 72
[14] R.V. Ambartzumian, G.K. Savvidy, K.G. Savvidy and G.S. Sukiasian. Phys. Lett. B275 (1992) 99
[15] G.K. Savvidy and K.G. Savvidy. Int. J. Mod. Phys. A8 (1993) 3993
[16] G.K. Savvidy, K.G. Savvidy. Mod.Phys.Lett. A8 (1993) 2963
[17] B.Durhuus and T.Jonsson. Phys.Lett. B297 (1992) 271
[18] C.F.Baillie and D.A.Johnston. Phys.Rev D 45 (1992) 3326
[19] J.Steiner. Über parallele Flächen. Gesammelte Werke Band 2. (Berlin, 1882) S. 171-176
[20] H.Minkowski. Volumen und Oberfläche. Math. Ann. B57 (1903) 447
[21] W.Blaschke. Kreis und Kugel. Berlin, 1956
[22] S.S.Chern. Differential geometry and integral geometry. In: Proc. Inter. Congress of Mathematicians (Edinburgh,1958) (Cambridge U.P., Cambridge, 1960) pp.441-449.
[23] L.A.Santalo. Integral geometry and geometric probability (Addison-Wesley. Reading, MA, 1976)
[24] H. Hadwiger. Vorlesungen über inhalt, oberflache und isoperimetric (Springer-Verlag, Berlin 1957)

[25] R. V. Ambarzumian. Combinatorial integral geometry (Wiley, New York, 1982)

[26] W. Blaschke. Griechische und Anschauliche Geometrie. München, 1953; Peri isoperimetron schematwn. 150 B.C.

[27] R. Schneider and W. Weil. Integralgeometrie. (Stuttgart: Teubner, 1992)

[28] G. K. Savvidy and R. Schneider. Comm. Math. Phys. 161 (1994) 283

[29] W. Fenchel. Math. Ann. 101 (1929) 238

[30] K. Borsuk. Ann. de la Soc. Polonaise 20 (1947) 251

[31] J. W. Milnor. Ann. Math. 52 (1950) 247

[32] S. S. Chern and R. K. Lashof. Amer. J. Math. 79 (1957) 306

[33] E. Fradkin, M. Srednicki and L. Susskind. Phys. Rev. D21 (1980) 2885

[34] C. Itzykson. Nucl. Phys. B210 (1982) 477

[35] A. Casher, D. Foerster and P. Windey. Nucl. Phys. B251 (1985) 29

[36] A. Polyakov. Gauge fields and String. (Harwood Academic Publishers, 1987)

[37] V. S. Dotsenko and et. al. Phys. Rev. Lett. 71 (1993) 811

[38] C. Fan and F. Y. Wu. Phys. Rev. 179 (1969) 560

[39] F. Y. Wu. Phys. Rev. 183 (1969) 604

[40] C. A. Hurst. J. Math. Phys. 7 (1966) 305

[41] P. Orland, Int. J. Mod. Phys. 5 (1991) 2401