Convexification method for a coefficient inverse problem and its performance for experimental backscatter data for buried targets

Michael V. Klibanov† Aleksandr E. Kolesov†‡ Dinh-Liem Nguyen§

Abstract

We present in this paper a novel numerical reconstruction method for solving a 3D coefficient inverse problem with scattering data generated by a single direction of the incident plane wave. This inverse problem is well-known to be a highly nonlinear and ill-posed problem. Therefore, optimization-based reconstruction methods for solving this problem would typically suffer from the local-minima trapping and require strong a priori information of the solution. To avoid these problems, in our numerical method, we aim to construct a cost functional with a globally strictly convex property, whose minimizer can provide a good approximation for the exact solution of the inverse problem. The key ingredients for the construction of such functional are an integro-differential formulation of the inverse problem and a Carleman weight function. Under a (partial) finite difference approximation, the global strict convexity is proven using the tool of Carleman estimates. The global convergence of the gradient projection method to the exact solution is proven as well. We demonstrate the efficiency of our reconstruction method via a numerical study of experimental backscatter data for buried objects.

Keywords. Carleman weight function, Carleman estimates, reconstruction method, convexification, global convergence, coefficient inverse problem, experimental data

AMS subject classification. 35R30, 78A46, 65C20

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erated by a single direction of the incident plane wave at multiple frequencies. More precisely, the goal of this CIP is to recover a coefficient in the Helmholtz equation from boundary measurements of its solutions for a single direction of the incident plane wave at multiple frequencies.

This CIP arises in a wide range of applications including non-destructive testing, detection of explosives, medical imaging, geophysics, etc. It is also well-known that any CIP is a highly nonlinear and ill-posed problem causing substantial challenges in the design of numerical algorithms for solving it. Optimization-based reconstruction methods can be considered as the most studied approach for solving CIPs in general. However, these methods suffer from the fact that they might converge to a local minimum, which is not the true solution of the CIP. Moreover, these methods typically require strong priori information of the solution, which is not always available in practice.

The goal of the so-called globally convergent method (GCM), recently developed by the first author and coauthors (see e.g. [6]) is to overcome the drawbacks mentioned above when solving CIPs. This method aims to provide a point in a sufficiently small neighborhood of the true solution of the CIP without any advanced knowledge of this neighborhood. The size of this neighborhood should depend only on approximation errors and the level of noise in the data.

The numerical method we develop in this paper can be considered as the second type of GCMs, which has certain advantages compared with the first type of GCMs in [6, 35, 36]. More precisely, we do not impose in the convergence analysis here the assumption on a small interval of wavenumbers. Neither we do not iterate here with respect to the so-called tail functions. The combination of the latter two features with the globally strictly convex cost functional (below) are the main improvements of the convexification over the first type of globally convergent methods.

This second type of GCMs is also called convexification methods, which was studied for the 1D case in [27]. The present work can be considered as a generalization to the 3D case of the cited 1D version. Convexification methods are based on the minimization of the weighted cost functional with a Carleman weight function (CWF) in it. The CWF is the function which is involved in the Carleman estimate for the corresponding PDE operator. The CWF can be chosen in such a way that the cost functional becomes strictly convex. Note that the majority of known numerical methods of solutions of nonlinear ill-posed problems minimize conventional least squares cost functionals (see, e.g. [11, 13, 14]), which are usually non convex and have multiple local minima and ravines, see, e.g. [39] for a good numerical example of multiple local minima.

We work in this paper with a semidiscrete version of the convexification, which is more realistic for computations than continuous versions used in previous works on the convexification of the first author with coauthors [3, 20, 24, 25, 27, 29]. “Semidiscrete” means that we develop the theory for the case when the differential operator we work with is written in finite differences with respect to two out of three spatial variables and in the continuous form with respect to the third variable. We impose a computationally reasonable assumption that the grid step size does not tend to zero (unlike the case of some forward problems). The fully discrete case, i.e. when derivatives with all three variables are written in finite differences, is not investigated yet. Indeed, it is well known that this case is quite a complicated one for ill-posed problems for PDEs, especially in nonlinear cases, such as we work with. There are known only a few results for.
the fully discrete cases of linear ill-posed problems, see, e.g. [8, 19]. We also refer to the recent publication [29] of the first two authors about a 3D version of the convexification method. In [29] convexification was numerically tested on some computationally simulated data. This is unlike the current paper in which testing is done for a significantly more challenging case of experimental data. The theory in [29] is developed for the continuous case. Although the idea of the semidiscrete version is briefly outlined in [29], corresponding theorems are neither formulated nor proved there, unlike the current paper.

We point out that the CIP considered in this paper is also called a inverse scattering problem in some contexts. There is a vast literature on both theoretical and numerical studies on this inverse problem and its variations, see, e.g. [1, 2, 10, 12, 15–18, 30–34]. These cited papers have considered the cases of multiple measurements and/or shape reconstructions. We recall that we consider in this paper the CIP with a single measurement which is both different and more challenging than the configurations considered in those cited papers.

In the next section, we provide a statement of the forward and inverse problems. In Section 3 we present an integro-differential equation formulation of the CIP. Section 4 involves the approximation of the tail function which is an important component in the integro-differential equation. We introduce in Section 5 the partial finite difference approximation and related function spaces for the integro-differential formulation. We describe in Section 6 the weighted cost functional with the Carleman weight function in it. Section 7 is dedicated to the theoretical analysis, including a Carleman estimate and proofs of global strict convexity of that functional as well as convergence results for the optimization problem. Finally, our numerical study is presented in Section 8.

2 Problem Statement

Let $x = (x, y, z) \in \mathbb{R}^3$ and consider positive numbers $b > 0$ and $d > 0$. For the convenience for our numerical study (Section 8), we define from the beginning the domain of interest $\Omega$ and the backscatter part $\Gamma$ of its boundary as

$$
\Omega = \{(x, y, z) : |x|, |y| < b, z \in (-\xi, d)\}, \quad \Gamma = \{(x, y, z) : |x|, |y| < b, z = -\xi\}.
$$

(2.1)

Let the function $c(x)$ be the spatially distributed dielectric constant and $k$ be the wavenumber. We consider the forward scattering problem for the Helmholtz equation:

$$
\Delta u + k^2 c(x) u = 0, \quad x \in \mathbb{R}^3,
$$

(2.2)

$$
u(x, k) = u_s(x, k) + u_i(x, k),
$$

(2.3)

$$
\lim_{r \to \infty} r \left( \partial u_s / \partial r - iku_s \right) = 0, \quad r = |x|,
$$

(2.4)

where $u(x, k)$ is the total wave, $u_i(x, k)$ is the incident wave and $u_s(x, k)$ is the scattered wave satisfying the Sommerfeld radiation condition. This condition means that the scattered field behaves like a outgoing spherical wave far away from the scattering medium.

Here we consider $u_i(x, k)$ as the incident plane wave propagating along the positive direction of the $z$-axis:

$$
u_i(x, k) = e^{ikz}.
$$

(2.5)
Also, the function $c(x)$ satisfies with the following conditions:

$$c(x) = 1 + \beta(x), \quad \beta(x) \geq 0, \quad x \in \mathbb{R}^3,$$

and $c(x) = 1, \quad x \notin \Omega$. \hfill (2.6)

The assumption of (2.6) $c(x) = 1$ in $\mathbb{R}^3 \setminus \Omega$ means that we have vacuum outside of the domain $\Omega$. Finally, we assume that $c(x) \in C^{15}(\mathbb{R}^3)$. This smoothness condition was imposed to derive the asymptotic behavior of the solution of the Helmholtz equation (2.2) (see [26]). We also note that extra smoothness conditions are usually not of a significant concern when a CIP is considered, see, e.g. Theorem 4.1 in [37]. Also, it follows from Lemma 3.3 of [28] that the derivative $\partial_k u(x,k)$ exists for all $x \in \mathbb{R}^3, k > 0$ and satisfies the same smoothness condition as the function $u(x,k)$.

**Coefficient Inverse Problem (CIP).** Let $\Omega$ and $\Gamma \subset \partial \Omega$ be as in (2.1). Let the wavenumber $k \in [k, \overline{k}]$, where $[k, \overline{k}] \subset (0, \infty)$ is an interval of wavenumbers. Determine the function $c(x), \quad x \in \Omega$, given the boundary data $g_0(x,k)$ as

$$u(x,k) = g_0(x,k), \quad x \in \Gamma, \quad k \in [k, \overline{k}]. \hfill (2.7)$$

In addition to the data (2.7) we can obtain the boundary conditions for the derivative of the function $u(x,k)$ in the $z$–direction using the data propagation procedure (see [35]),

$$u_z(x,k) = g_1(x,k), \quad x \in \Gamma, \quad k \in [k, \overline{k}]. \hfill (2.8)$$

Even though we use the data propagation procedure in our computations below, we do not describe it here for brevity. Instead, we refer to detailed descriptions in [35, 36]. In fact, this procedure is widely used in Optics under the name the angular spectrum representation.

In addition, we complement Dirichlet (2.7) and Neumann (2.8) boundary conditions on $\Gamma$ with the heuristic Dirichlet boundary condition at the rest of the boundary $\partial \Omega$ as:

$$u(x,k) = e^{ikz}, \quad x \in \partial \Omega \setminus \Gamma, \quad k \in [k, \overline{k}]. \hfill (2.9)$$

The boundary condition (2.9) coincides with the one for the uniform medium with $c(x) \equiv 1$. To justify (2.2), we recall that, using the tail functions method, it was demonstrated in sections 7.6 and 7.7 of [36] that (2.2) does not affect much the reconstruction accuracy as compared with the correct Dirichlet boundary condition. Besides, (2.2) has always been used in works [36] with experimental data, where accurate results were obtained by the tail functions globally convergent method.

The uniqueness of the solution of this CIP is an open and long standing problem. In fact, uniqueness of a similar coefficient inverse problem can be currently proven only in the case if the right hand side of equation (2.2) is a function which is not vanishing in $\Omega$. This can be done by the Bukhgeim-Klibanov method [9], also see, e.g. [7,21,22] and references cited therein for this method. Hence, for the computational purpose, we assume below the uniqueness of our CISP.

In this last part of this section we want to briefly describe the travel time $\tau(x)$ which is important in our analysis. The Riemannian metric generated by the function $c(x)$ is:

$$d\tau(x) = \sqrt{c(x)}|dx|, \quad |dx| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}.$$
For a fixed number $a > 0$, consider the plane $P_a = \{(x, y, -a) : x, y \in \mathbb{R}\}$. We assume that $\Omega \subset \{z > -a\}$ and impose everywhere below the following condition on the function $c(x)$:

**Regularity Assumption.** For any point $x \in \mathbb{R}^3$ there exists a unique geodesic line $\Gamma(x, a)$, with respect to the metric $d\tau$, connecting $x$ with the plane $P_a$ and perpendicular to $P_a$ near the intersection point.

A sufficient condition of the regularity of geodesic lines is [38]:

$$\sum_{i,j=1}^{3} \frac{\partial^2 \ln(c(x))}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \text{ for all } x \in \overline{\Omega}, \xi \in \mathbb{R}^3.$$ 

We introduce the travel time $\tau(x)$ from the plane $P_a$ to the point $x$ as [26]

$$\tau(x) = \int_{\Gamma(x, a)} \sqrt{c(\xi)} d\sigma.$$ 

# 3 The Integro-Differential Equation

In this section we reformulate our coefficient inverse problems as an integro-differential equation, which is one of the main ingredients in our reconstruction method. To this end, we first need a result on (high frequency) asymptotic behavior of the total field $u(x, k)$ in [26]. It was shown in this cited paper that

$$u(x, k) = A(x) e^{ik(\tau(x) - a)} [1 + s(x, k)], \quad x \in \overline{\Omega}, k \to \infty,$$ (3.1)

where $A(x) > 0$ and $s(x, k)$ satisfies

$$s(x, k) = O \left(\frac{1}{k}\right), \quad \partial_k s(x, k) = O \left(\frac{1}{k}\right), \quad x \in \overline{\Omega}, k \to \infty.$$ (3.2)

Here $\tau(x)$ is the length of the geodesic line generated by the function $c(x)$ in the Riemannian metric. Define

$$w(x, k) = \frac{u(x, k)}{u_i(x, k)}.$$ (3.3)

From (2.5), (3.1) and (3.3), we have

$$w(x, k) = A(x) e^{ik(\tau(x) - z - a)} [1 + s(x, k)], \quad x \in \overline{\Omega}, k \to \infty.$$ (3.4)

From (3.1) and (3.4), and for $x \in \Omega, k \in [k, \overline{k}]$, we can uniquely define the function $\log w(x, k)$ for sufficiently large values of $k$ as

$$\log w(x, k) = \ln A(x) + ik(\tau(x) - z - a) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (s(x, k))^n.$$ (3.5)

It is clear that, with $\log w(x, k)$ defined as above, $\exp[\log w(x, k)]$ equals to the right hand side of (3.4). Thus, we assume below that the number $k$ is sufficiently large.
Now we are ready to derive the integro-differential equation. For \( x \in \Omega, k \in [k, k] \) we define the function \( v(x, k) \),

\[
v(x, k) = \frac{\log w(x, k)}{k^2}.
\]  

(3.6)

Then

\[
\Delta v + k^2 \nabla v \cdot \nabla v = -c(x).
\]  

(3.7)

Setting \( q(x, k) \) as

\[
q(x, k) = \partial_k v(x, k),
\]  

(3.8)

we obtain

\[
v(x, k) = -\int_k^\pi q(x, \kappa) d\kappa + V(x).
\]  

(3.9)

Here we call \( V(x) \) the tail function,

\[
V(x) = v(x, \vec{k}).
\]  

(3.10)

Combining (2.2), (2.5), (2.6) and (3.3), we obtain

\[
\Delta w + k^2 \beta w + 2ik \frac{\partial w}{\partial z} = 0.
\]  

(3.11)

Taking into account (3.6), equation (3.11) becomes

\[
\Delta v + k^2 \nabla v \cdot \nabla v + 2ik \frac{\partial v}{\partial z} + \beta(x) = 0.
\]  

(3.12)

To eliminate the function \( \beta(x) \) we differentiate (3.12) with respect to \( k \),

\[
\Delta q + 2k \nabla v \cdot (k \nabla q + \nabla v) + 2i \left( k \frac{\partial q}{\partial z} + \frac{\partial v}{\partial z} \right) = 0.
\]  

(3.13)

Substituting (3.9) into (3.13) leads to the following integro-differential equation

\[
L(q) = \Delta q + 2k \left( \nabla V - \int_k^\pi \nabla q(\kappa) d\kappa \right) \cdot \left( k \nabla(q + V) - \int_k^\pi \nabla q(\kappa) d\kappa \right)
+ 2i \left( kq_z + V_z - \int_k^\pi q_\kappa(\kappa) d\kappa \right) = 0.
\]  

(3.14)

This equation is complemented with the overdetermined boundary conditions:

\[
q(x, k) = \phi_0(x, k), \quad q_\kappa(x, k) = \phi_1(x, k), \quad x \in \Gamma, k \in [k, k],
q(x, k) = 0, \quad x \in \partial \Omega \setminus \Gamma, k \in [k, k],
\]  

(3.15)

where the functions \( \phi_0 \) and \( \phi_1 \) are computed from the functions \( g_0 \) and \( g_1 \) in (2.7), (2.8). The third boundary condition (3.15) follows from (2.5), (3.3), (3.6) and (3.8).
Note that in (3.14) we have two unknowns \( q(x,k) \) and \( V(x) \). Hence, we will solve the problem (3.14), (3.15) using a predictor-corrector method. Here we find some approximation of \( V(x) \) first and use it as a predictor, and then solve for \( q(x,k) \). One can see that if certain approximations of \( q(x,k) \) and \( V(x) \) are found, then an approximation for the unknown coefficient \( c(x) \) can be found via (3.9) and (3.7) for a certain value of \( k \in [k_0,K] \). In our computations we use \( k = k_0 \) for that value. Therefore, we focus below on approximating functions \( q(x,k), V(x) \).

4 Approximation of the tail function

In this section we present a method for finding an approximation of the tail function \( V(x) \). We note that this method is different the one studied in [27].

It follows from (3.5) and (3.10) that there exists a function \( p(x) \) such that

\[
v(x,k) = \frac{p(x)}{k} + O\left(\frac{1}{k^2}\right), \quad q(x,k) = -\frac{p(x)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty, \quad x \in \Omega.
\]

(4.1)

For sufficiently large \( K \), we drop \( O(1/k^2) \) and \( O(1/k^3) \) in (4.1) and set

\[
v(x,k) = \frac{p(x)}{k}, \quad q(x,k) = -\frac{p(x)}{k^2}, \quad k \geq K, \quad x \in \Omega.
\]

(4.2)

Next, substituting (4.2) in (3.14) and setting \( k = K \), we obtain

\[
\Delta V(x) = 0, \quad x \in \Omega.
\]

(4.3)

This equation is supplemented by the following boundary conditions:

\[
V(x) = \psi_0(x), \quad V_z(x) = \psi_1(x), \quad x \in \Gamma, \quad V(x) = 0, \quad x \in \partial \Omega \setminus \Gamma,
\]

(4.4)

where functions \( \psi_0 \) and \( \psi_1 \) are computed using (2.7) and (2.8). Boundary conditions (4.4) are over-determined ones. Due to the approximate nature of (4.2), we have observed that the obvious approach of finding the function \( V(x) \) by dropping the second boundary condition (4.4) and solving the resulting Dirichlet boundary value problem for Laplace equation (4.3) with the boundary data (4.4) does not provide satisfactory results. The same observation was made in [27] for the 1D case.

We show in section 5 how do we approximately solve the problem (4.3)–(4.4).

5 Partial Finite Differences

5.1 Grid points

We now write differential operators in (3.14) and (4.3) in finite differences with respect to \( x,y \). Let the domain \( \Omega_1 \subset \mathbb{R}^2 \) be the orthogonal projection of the domain \( \Omega \subset \mathbb{R}^3 \) in (2.1) on the plane \( \{z = 0\} \),

\[
\Omega_1 = \{(x,y,z) : |x| < b, |y| < b, z = 0\}.
\]
Consider a finite difference grid in $\Omega_1$ with the uniform grid step size $h$. This grid consists of points \{$(x_j, y_s)_{j,s=1}^{N_h} \subset \overline{\Omega}_1$, $z \in (-\xi, d)$\}. \hspace{1cm} (5.1)

For every interior point $(x_j, y_s, z) \in \overline{\Omega} \setminus \partial \Omega$ four neighboring points are:

$$
(x_{j+1}, y_s, z) = (x_j + h, y_s, z), \quad (x_{j-1}, y_s, z) = (x_j - h, y_s, z),
$$

$$
(x_j, y_{s+1}, z) = (x_j, y_s + h, z), \quad (x_j, y_{s-1}, z) = (x_j, y_s - h, z).
$$

The corresponding Laplace operator written in partial finite differences is

$$
\Delta^h u = u_{zz} + u_{xx}^h + u_{yy}^h,
$$

where $u_{xx}^h$ and $u_{yy}^h$ are finite difference analogs of continuous derivatives $u_{xx}$ and $u_{yy}$,

$$
u_{xx}^h(x_j, y_s, z) = \frac{u(x_j - h, y_s, z) - 2u(x_j, y_s, z) + u(x_j + h, y_s, z)}{h^2}
$$

and similarly for $u_{yy}^h$. Next,

$$
\nabla^h u = (\partial^h u, \partial^h u, \partial_z u),
$$

where

$$
\partial^h_x u(x_j, y_s, z) = \frac{u(x_j + h, y_s, z) - u(x_j - h, y_s, z)}{2h}
$$

and similarly for $\partial^h_y u(x_j, y_s, z)$.

### 5.2 Problems (3.14)–(3.15) and (4.3)–(4.4) in partial finite differences

We now rewrite problem (3.14)–(3.15) in partial finite differences. To this end, we keep in mind that only interior grid points are involved in differential operators below. Using (5.1)–(5.4), we obtain for $x \in \Omega_h$

$$L^h(q) = \Delta^h q + 2k \left( \nabla^h V - \int_k^\overline{k} \nabla^h q(\kappa) d\kappa \right) \cdot \left( k \nabla^h (q + V) - \int_k^\overline{k} \nabla^h q(\kappa) d\kappa \right) + 2i \left( k q_z + V_z - \int_k^\overline{k} q_z(\kappa) d\kappa \right) = 0,$$

$$q(x, k) = \phi_0(x, k), \quad q_z(x, k) = \phi_1(x, k), \quad x \in \Gamma, \ k \in [k, \overline{k}],
$$

$$q(x, k) = 0, \quad x \in \partial \Omega \setminus \Gamma, \ k \in [k, \overline{k}].$$

Similarly, problem (4.3)–(4.4) becomes

$$\Delta^h V(x) = 0, \quad x \in \Omega.$$

$$V(x) = \psi_0(x), \quad V_z(x) = \psi_1(x), \quad x \in \Gamma, \quad V(x) = 0, \quad x \in \partial \Omega \setminus \Gamma.$$
Remark 5.1.

1. From now on functions \( q(x, k) \) and \( V(x) \), and other functions we consider are semidiscrete, i.e. they are defined on \( \Omega_h \). This means that, e.g. \( q(x, k) = \{q(x_j, y_s, z)\}_{j,s=1}^{N_h} \), \( V(x) = \{V(x_j, y_s, z)\}_{j,s=1}^{N_h} \), etc. Boundary conditions at \( \partial \Omega \) for the functions \( q \) and \( V \) are also defined only on grid points which belong to the boundary \( \partial \Omega \).

2. Since the grid step size \( h \) is not changing in our arrangement, we will not indicate below for brevity the dependence of some parameters on \( h \), although they do depend on \( h \). Thus, for example below \( C = C(\xi, d) > 0 \) denotes different positive constants depending only on numbers \( \xi, d \) and \( h \).

5.3 Some functional spaces

Denote by \( \overline{z} \) the complex conjugate of \( z \in \mathbb{C} \). It is convenient for us to consider any complex valued function \( U = \text{Re}U + i\text{Im}U = U_1 + iU_2 \) as the 2D vector function \( U = (U_1, U_2) \). Furthermore, each component \( U_j \) of this vector function is, in turn, another vector function defined on the above grid, \( U_j = U_j(x_j, y_s, z, k) \).

Hence, below any Banach space of complex valued functions is actually the space of these real valued vector functions with the well known definitions of norms and scalar products (if in Hilbert spaces). For brevity we do not differentiate below between complex valued functions and corresponding vector functions. These things are always clear from the context.

We introduce the Hilbert spaces \( H^{2,h}(\Omega_h) \), \( L^2_2(\Omega_h) \) and \( H^n_h \) of semidiscrete complex valued functions as

\[
H^{2,h}(\Omega_h) = \{ f(x_j, y_s, z) : \| f \|^2_{H^{2,h}(\Omega_h)} = \sum_{j,s=1}^{N_h} \sum_{r=0}^{2} h^2 \int_{-\xi}^{\xi} |\partial^r_{x} f(x_j, y_s, z)|^2 \, dz < \infty \},
\]

\[
L^2_2(\Omega_h) = \{ f(x_j, y_s, z) : \| f \|^2_{L^2_2(\Omega_h)} = \sum_{j,s=1}^{N_h} h^2 \int_{-\xi}^{\xi} |f(x_j, y_s, z)|^2 \, dz < \infty \},
\]

\[
H^n_h = \{ f(x_j, y_s, z, k) : \| f \|^2_{H^n_h} = \int \| f(x, k) \|^2_{H^{n,h}(\Omega_h)} \, dk < \infty \}, \quad n = 2, 3.
\]

Denote \( [\cdot, \cdot] \) the scalar product in the space \( H^{2,h}(\Omega_h) \). We also define subspaces \( H^{2,h}_0(\Omega_h) \) and \( H^{2,h}_{0,2} \) as

\[
H^{2,h}_0(\Omega_h) = \{ f(x_j, y_s, z) \in H^{2,h}(\Omega_h) : f(x) |_{\partial \Omega} = 0, f_x(x) |_{\Gamma} = 0 \},
\]

\[
H^{2,h}_{0,2} = \{ f(x_j, y_s, z, k) \in H^2_2 : f(x, k) |_{\partial \Omega} = 0, f_z(x, k) |_{\Gamma} = 0, \forall k \in [k, k_0] \}.
\]

Note that since, for all \( f \in H^{2,h}_{0,2} \),

\[
f(x_j, y_s, z, k) = \int_{-\xi}^{\xi} f_z(x_j, y_s, \rho, k) \, d\rho, \quad f_z(x_j, y_s, z, k) = \int_{-\xi}^{\xi} f_{zz}(x_j, y_s, \rho, k) \, d\rho,
\]

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then the norm in $H_{0,2}^h$ is equivalent with
\[
\|f(x)\|^2_{H_{0,2}^h(\Omega_h)} = \sum_{j,s=1}^{N_h} h^2 \int_{-\xi}^d \left| \Delta^h f(x_j, y_s, z) \right|^2 dz.
\] (5.9)

In addition, for $l = 0, 1$
\[
C_l^t(\Omega_h) = \{ f(x, y, z) : \|f\|_{C_l^t(\Omega_h)} = \max_{j,s} \|f(x_j, y_s, z)\|_{C_l^t(\Omega_h)} < \infty \},
\]
\[
C_l^h = \{ f(x, y, z, k) : \|f\|_{C_l^h} = \max_{k \in [l, h]} \|f(x, y, z, k)\|_{C_l^t(\Omega_h)} < \infty \}.
\]

By embedding theorem $H_{2,0}^h(\Omega_h) \subset C^1(\Omega_h), H_{h}^n \subset C_{h-1}^n$ and
\[
\|f\|_{C^1(\Omega_h)} \leq C \|f\|_{H_{2,0}^h(\Omega_h)}, \text{ for all } f \in H_{2,0}^h(\Omega_h), \tag{5.10}
\]
\[
\|f\|_{C_{h-1}^n} \leq C \|f\|_{H_{h}^n}, \text{ for all } f \in H_{h}^n. \tag{5.11}
\]

6 Two Cost Functionals with CWFs

It is our computational experience for the 1D case [3, 25, 27] that one should use for computations such a CWF which would be a simple one. A similar conclusion can be found on page 1581 of [5]. Thus, the CWF we use in this paper is:
\[
\varphi_\lambda(z) = e^{-2\lambda z}. \tag{6.1}
\]

6.1 Problem (5.7)–(5.8)

First, we present the cost functional for the solution of problem (5.7)–(5.8) which is about the tail function. Non-zero boundary conditions in (5.8) are inconvenient for us. Hence, we assume that there exists a function $Q(x) \in H_{2,0}^h(\Omega_h)$ such that
\[
Q(x) = \psi_0(x), \quad \partial_z Q(x) = \psi_1(x), \quad x \in \Gamma; \quad Q(x) = 0, \quad x \in \partial \Omega \setminus \Gamma. \tag{6.2}
\]

Define
\[
W(x) = V(x) - Q(x) \in H_{0,2}^h(\Omega_h). \tag{6.3}
\]

Hence, we consider the following minimization problem:

**Minimization Problem 1.** For $W \in H_{0,2}^h(\Omega_h)$, minimize the functional $I_\mu(W)$,
\[
I_\mu(W) = e^{2\mu d} \sum_{j,s=1}^{N_h} h^2 \int_{-\xi}^d \left| (\Delta^h W + \Delta^h Q)(x_j, y_s, z) \right|^2 \varphi_\mu(z)dz. \tag{6.4}
\]

The multiplier $e^{2\mu d}$ is introduced here to ensure that $e^{2\mu d} \min_{[-\xi, \xi]} \varphi_\mu(z) = 1$.

**Remark 6.1.** Since the operator $\Delta^h$ is linear, then, in principle at least, one can apply straightforwardly the quasi-reversibility method to find an approximate solution of the problem $\Delta W + \Delta Q = 0$ for $W \in H_{0,2}^h(\Omega_h)$ [23]. This means that one can use $\lambda = 0$ in (6.4). However, it was observed in [3] that the involvement of the CWF like in (6.4) leads to a better solution accuracy.
We now follow the classical Tikhonov regularization concept [4,40]. By this concept, we should assume that there exists an exact solution $V_*(x)$ of the problem (5.7)–(5.8) with the noiseless data $\psi_0(x), \psi_1(x)$. Below the subscript "*" is related only to the exact solution. In fact, however, the data $\psi_0(x)$ and $\psi_1(x)$ contain noise. Let $\delta \in (0,1)$ be the level of noise in the data $\psi_0(x)$ and $\psi_1(x)$. Again, following the same concept, we should assume that the number $\delta \in (0,1)$ is sufficiently small. Assume that there exists the function $Q_*(x) \in H^{2,5}_b(\Omega_h)$ such that

\begin{align*}
Q_*(x) &= \psi_0(x), \quad \partial_2 Q_*(x) = \psi_1(x), \quad x \in \Gamma; \quad Q_*(x) = 0, \quad x \in \partial \Omega \setminus \Gamma, \tag{6.5}
\end{align*}

\begin{align*}
\|Q - Q_*\|_{H^{2,5}_b(\Omega_h)} < \delta, \tag{6.6}
\end{align*}

where $Q$ is defined in (6.2). We will choose in Theorem 7.2 of section 6 a certain dependence $\mu = \mu(\delta)$ of the parameters $\mu$ on the noise level $\delta$. Denote $W_{\mu(\delta)}(x) = W_{\text{min}}(x)$ the unique minimizer of the functional $I_{\mu(\delta)}(W)$ (Theorem 7.2) and by (6.3) let

\begin{align*}
V_{\mu(\delta)}(x) = W_{\mu(\delta)}(x) + Q(x) = W_{\text{min}}(x) + Q(x). \tag{6.7}
\end{align*}

### 6.2 Problem (3.14)–(3.15)

Suppose that there exists a function $F(x,k) \in H^5_0$ such that (see (3.15)):

\begin{align*}
F(x,k) = \phi_0(x,k), \quad F_2(x,k) = \phi_1(x,k), \quad x \in \Gamma, \quad F(x,k) = 0, \quad x \in \partial \Omega \setminus \Gamma. \tag{6.8}
\end{align*}

Also, assume that there exists an exact solution $c_*(x)$ of our CIP satisfying the above conditions imposed on the coefficient $c(x)$ and generating the noiseless boundary data $\phi_0$, $\phi_1$ in (3.15). Also, assume that there exists the function $F_*(x,k) \in H^5_0$ satisfying the following analog of boundary conditions (6.8):

\begin{align*}
F_*(x,k) = \phi_0(x,k), \quad \partial_2 F_*(x,k) = \phi_1(x,k), \quad x \in \Gamma, \quad F_*(x,k) = 0, \quad x \in \partial \Omega \setminus \Gamma. \tag{6.9}
\end{align*}

We assume that

\begin{align*}
\|F - F_*\|_{H^5_0} < \delta. \tag{6.10}
\end{align*}

Let $q_* \in H^5_0$ be the function $q$ generated by the exact coefficient $c_*(x)$. We define functions $p$ and $p_*$ as

\begin{align*}
p(x,k) = q(x,k) - F(x,k), \quad p_*(x,k) = q_*(x,k) - F_*(x,k). \tag{6.11}
\end{align*}

Hence, the functions $p,p_*$ $H^5_0$. Let $R > 0$ be an arbitrary number. Consider the ball $B(R) \subset H^5_0$ of the radius $R$,

\begin{align*}
B(R) = \{ r \in H^5_0 : \|r\|_{H^5_0} < R \}. \tag{6.12}
\end{align*}

Using the integro-differential equation (3.14), boundary conditions (3.15) for it, (6.8), (6.9) and (6.11), we construct our cost functional $J_\lambda(p)$ with the CWF
\( J_{\lambda}(p) = e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_{-\xi}^{d} |L^h(p + F)(x_j, y_s, z, \kappa)|^2 \varphi_{\lambda}(z) dz dk, \quad p \in B(R), \) 

(6.13)

where the tail function in \( L^h \) is defined in (6.7). Similarly with (6.4), the multiplier \( e^{2\lambda d} \) is introduced to balance two terms in the right hand side of (6.13). We consider the following minimization problem:

**Minimization Problem 2.** Minimize the functional \( J_{\lambda}(p) \) on the set \( p \in B(R) \).

### 7 Carleman Estimate and Global Strict Convexity

In this section we formulate theorems about the minimization problems 1 and 2 of section 6. First, we are concerned with the Carleman estimate with the CWF (6.1).

**Theorem 7.1** (Carleman estimate). For \( \lambda > 0 \) let

\[
\begin{align*}
B_h(u, \lambda) &= \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} |\Delta^h u(x_j, y_s, z)|^2 \varphi_{\lambda}(z) dz. \\
\end{align*}
\]

Then there exists a sufficiently large number \( \lambda_0 = \lambda_0(\xi, d) > 1 \) such that for all \( \lambda \geq \lambda_0 \) the following estimate is valid for all functions \( u \in H^2_{0,h}(\Omega_h) \)

\[
B_h(u, \lambda) \geq C \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} |u_{zz}(x_j, y_s, z)|^2 \varphi_{\lambda}(z) dz + C\lambda \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} |u_z(x_j, y_s, z)|^2 \varphi_{\lambda}(z) dz \\
+ C\lambda^3 \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} |u(x_j, y_s, z)|^2 \varphi_{\lambda}(z) dz.
\]

(7.1)

**Proof.** Recall that we do not indicate the dependence of neither constants \( C \) nor other constants on \( h \) (second item in Remarks 5.1). Since \( |v|^2 = (\text{Re} v)^2 + (\text{Im} v)^2 \) for \( v \in \mathbb{C} \), then it is sufficient to prove estimate (7.1) for real valued functions \( u \in H^2_{0,h}(\Omega_h) \). The following Carleman estimate was proven in lemma 3.1 of [27] for all real valued functions \( w(\xi, d) \) such that \( w(-\xi) = w'(\xi) = 0 \):

\[
\int_{-\xi}^{d} (w^\prime)^2 \varphi_{\lambda}(z) dz \geq \tilde{C} \int_{-\xi}^{d} (w^\prime)^2 \varphi_{\lambda}(z)dz + \tilde{C}\lambda \int_{-\xi}^{d} (w)^2 \varphi_{\lambda}(z)dz + \tilde{C}\lambda^3 \int_{-\xi}^{d} w^2 \varphi_{\lambda}(z)dz,
\]

(7.2)

for all \( \lambda \geq \lambda_0(\xi, d) \), where the number \( \tilde{C} = \tilde{C}(\xi, d) > 0 \) depends only on \( \xi \) and
Next, it follows from (5.2) and (5.3) that
\[ B_h(u, \lambda) = \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} [(u_{zz} + u_{xz} + u_{yz})(x_j, y_s, z)]^2 \varphi_{\lambda}(z) dz \]
\[ \geq \frac{1}{2} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} [u_{zz}(x_j, y_s, z)]^2 \varphi_{\lambda}(z) dz - C \sum_{j,s=1}^{M_h} \int_{-\xi}^{d} [u(x_j, y_s, z)]^2 \varphi_{\lambda}(z) dz. \]

Hence, using (7.2), we obtain
\[ B_h(u, \lambda) \geq C \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} [u_{zz}(x_j, y_s, z)]^2 \varphi_{\lambda}(z) dz + C \lambda \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} [u_{zz}(x_j, y_s, z)]^2 \varphi_{\lambda}(z) dz - C \sum_{j,s=1}^{M_h} \int_{-\xi}^{d} [u(x_j, y_s, z)]^2 \varphi_{\lambda}(z) dz. \]

(7.3)

Now choosing \( \lambda_0 \) so large that \( \lambda_0 h^2 > C/2 \), we obtain from (7.3) the target estimate (7.1) for all \( \lambda > \lambda_0 \). \( \square \)

The next theorem is about the functional \( I_\mu(W) \) in (6.4).

**Theorem 7.2.** Assume that there exists a function \( Q \in H^{2,h}(\Omega_h) \) satisfying conditions (6.5). Introduce the function \( W \in H^{2,h}(\Omega_h) \) via (6.3). Then for each \( \mu > 0 \) there exists unique minimizer \( W_\mu \in H^{2,h}(\Omega_h) \) of the functional (6.4). Suppose now that there exists an exact solution \( V_* \in H^{2,h}(\Omega_h) \) of equation (4.3) with the boundary data \( \psi_{0,\mu}(x) \) and \( \psi_{1,\mu}(x) \) in (4.4). Also, assume that there exists a function \( Q_* \in H^{2,h}(\Omega_h) \) satisfying conditions (6.5) and such that inequality (6.6) holds, where \( \delta \in (0,1) \) is the noise level in the data. Let \( \lambda_0 > 0 \) be the number of Theorem 6.1. Choose a number \( \delta_0 \in (0, e^{-2(d+\varepsilon)\lambda_0}) \). For any \( \delta \in (0, \delta_0) \) let
\[ \mu = \mu(\delta) = \ln(\delta^{-1/(2(d+\varepsilon))}). \]

Let the function \( V_{\mu(\delta)}(x) \) be defined via (6.7). Then the following convergence estimate of \( V_{\mu(\delta)}(x) \) to the exact solution \( V_*(x) \) holds as \( \delta \to 0 \)
\[ \| V_{\mu(\delta)} - V_* \|_{H^{2,h}(\Omega_h)} \leq C \sqrt{\delta}. \]

(7.5)

In addition, \( V_{\mu(\delta)} \in C^1(\Omega_h) \) and
\[ C \| \nabla V_{\mu(\delta)} \|_{C^1(\Omega_h)} \leq \| V_{\mu(\delta)} \|_{H^{2,h}(\Omega_h)} \leq C \left[ 1 + \| V_* \|_{H^{2,h}(\Omega_h)} \right]. \]

(7.6)

**Proof.** It follows from (6.4) and the variational principle that the vector function \( W_{\text{min}} = (W_{1,\text{min}}, W_{2,\text{min}}) \in H^{2,h}_0(\Omega_h) \) is a minimizer of the functional
where the constants 

\[ I_{\mu, \alpha}(W) \]

if and only if

\[
e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left( \Delta_{h} W_{1,\min} \Delta_{h} r_1 + \Delta_{h} W_{2,\min} \Delta_{h} r_2 \right)(x_j, y_s, z) \varphi_{\mu}(z) dz
\]

\[
= -e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left( \Delta_{h} Q_1 \Delta_{h} r_1 + \Delta_{h} Q_2 \Delta_{h} r_2 \right)(x_j, y_s, z) \varphi_{\mu}(z) dz,
\]

(7.7)

for all \( r = (r_1, r_2) \in H^2_{0,h}(\Omega_h) \). For any vector function \( P = (P_1, P_2) \in H^2_{0,h}(\Omega_h) \) consider the expression in the left hand side of (7.7) in which the vector function \( (W_{1,\min}, W_{2,\min}) \) is replaced with \( (P_1, P_2) \). Then (5.9) implies that this expression defines a new scalar product \( \{P, r\} \) in the space \( H^2_{0,h}(\Omega_h) \), and the corresponding norm \( \{P, P\}^{1/2} \) is equivalent to the norm in the space \( H^2_{0,h}(\Omega_h) \). Next, for all \( r = (r_1, r_2) \in H^2_{0,h}(\Omega_h) \), we have

\[
- e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left( \Delta_{h} Q_1 \Delta_{h} r_1 + \Delta_{h} Q_2 \Delta_{h} r_2 \right)(x_j, y_s, z) \varphi_{\mu}(z) dz \leq D \|Q\|_{H^2_{0,h}(\Omega_h)} \|r\|_{H^2_{0,h}(\Omega_h)}
\]

\[
\leq D \sqrt{\{Q, Q\} \sqrt{\{r, r\}}},
\]

where the constants \( D, D_1 \) do not depend on \( Q \) and \( r \). Hence, by Riesz theorem there exists unique vector function \( \hat{Q} = (\hat{Q}_1, \hat{Q}_2) = \hat{Q}(Q) \in H^2_{0,h}(\Omega_h) \) such that

\[
\{\hat{Q}, r\} = -e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left( \Delta_{h} Q_1 \Delta_{h} r_1 + \Delta_{h} Q_2 \Delta_{h} r_2 \right)(x_j, y_s, z) \varphi_{\mu}(z) dz,
\]

for all \( r = (r_1, r_2) \in H^2_{0,h}(\Omega_h) \). Hence, by (7.7) \( \{W_{\min}, r\} = \{\hat{Q}, r\}, \forall r \in H^2_{0,h}(\Omega_h) \). This implies that \( W_{\min} = \hat{Q} \). Thus, both existence and uniqueness of the minimizer of the functional \( I_{\mu}(W) \) are established.

We now prove convergence estimate (7.5). Let \( W_* = (W_{*,1}, W_{*,2}) = V_0 - Q_* \). Then \( W_* \in H^2_{0,h}(\Omega_h) \). Denote \( \hat{W} = W_{\min} - W_* \) and \( \hat{Q} = Q - Q_* \). Since

\[
e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left( \Delta_{h} W_{*,1} \Delta_{h} r_1 + \Delta_{h} W_{*,2} \Delta_{h} r_2 \right)(x_j, y_s, z) \varphi_{\mu}(z) dz
\]

\[
= -e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} \left( \Delta_{h} Q_* \Delta_{h} r_1 + \Delta_{h} Q_* \Delta_{h} r_2 \right)(x_j, y_s, z) \varphi_{\mu}(z) dz,
\]

(7.8)
then subtracting (7.8) from (7.7) and setting \( r = \tilde{W} \), we obtain

\[
e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\Delta^h \tilde{W})^2 \varphi_\mu(z) dz
\]

\[
= -e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\Delta^h \tilde{Q}_1 \Delta^h \tilde{W}_1 + \Delta^h \tilde{Q}_2 \Delta^h \tilde{W}_2) \varphi_\mu(z) dz.
\]

Using the Cauchy-Schwarz inequality, taking into account (6.6) and (7.4), we obtain

\[
e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\Delta^h \tilde{W}(x_j, y_s, z))^2 \varphi_\mu(z) dz \leq C e^{2\mu(d+\xi)} \delta^2.
\]  

(7.9)

By (7.4) \( \mu = \ln(\delta^{-1/(2(d+\xi))}) \), which implies \( e^{2\mu(d+\xi)} \delta^2 = \delta \). Hence, (7.9) implies that

\[
e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\Delta^h \tilde{W}(x_j, y_s, z))^2 \varphi_\mu(z) dz \leq C \delta.
\]  

(7.10)

We now apply Theorem 7.1 to the left hand side of (7.10). We obtain for all \( \mu \geq \mu_0 \)

\[
\delta \geq C e^{2\mu d} \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\tilde{W}_{zz}(x_j, y_s, z))^2 \varphi_\mu(z) dz
\]

\[
+ C e^{2\mu d} \left[ \mu \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\tilde{W}_z(x_j, y_s, z))^2 \varphi_\mu(z) dz + \mu^3 \sum_{j,s=1}^{M_h} h^2 \int_{-\xi}^{d} (\tilde{W}(x_j, y_s, z))^2 \varphi_\mu(z) dz \right].
\]  

(7.11)

Since \( e^{2\mu d} \varphi_\mu(z) \geq e^{2\mu d} \varphi_\mu(d) = 1 \), then (7.11) implies that

\[
\|\tilde{W}\|_{H^{2,s}(\Omega_h)} \leq C \sqrt{\delta}.
\]

Hence, (6.6), (6.7) and triangle inequality imply that

\[
\|V_\mu(\delta) - V_s\|_{H^{2,s}(\Omega_h)} = \|\tilde{W} + (Q - Q_*)\|_{H^{2,s}(\Omega_h)} \leq C \sqrt{\delta},
\]  

(7.12)

which proves (7.5). Next, by (7.5) and triangle inequality imply the right estimate (7.6). The left estimate (7.6) follows from (5.10). \( \Box \)

The main analytical result of this paper is Theorem 7.3.

**Theorem 7.3 (globally strict convexity).** Assume that conditions of Theorem 6.2 hold. Let \( \lambda_1 \geq \lambda_0 \) be the number defined below in the formulation of this theorem. Assume that there exist functions \( F(x,k), F_*(x,k) \in H^1_+ \) satisfying conditions (6.8)–(6.10), where \( \delta \in (0, \delta_1) \) and \( \delta_1 \in (0, e^{-2(d+\xi)\lambda_1}) \). Set in (6.13) \( V = V_\mu(\delta) \), where the function \( V_\mu(\delta) \) is defined in Theorem 7.2. First, the
Second, there exist numbers

\[ \lambda_1 = \lambda_1(\Omega_h, R, \|F_*\|_{H^1_0}, \|V_\lambda\|_{H^2,\alpha(\Omega_h)}, \kappa, R, F) \] \geq \lambda_0, \\
\[ C_1 = C_1(\Omega_h, R, \|F_*\|_{H^1_0}, \|V_\lambda\|_{H^2,\alpha(\Omega_h)}, \kappa, R, F) > 0, \]

depending only on listed parameters, such that for any \( \lambda \geq \lambda_1 \) the functional \( J_\lambda(p) \) is strictly convex on \( \overline{B(R)} \). In other words, the following estimate holds:

\[ J_\lambda(p + r) - J_\lambda(p) - J_\lambda'(p)(r) \geq C_1 \|r\|_{H^1_0}, \text{ for all } p, p + r \in \overline{B(R)}. \] (7.13)

Proof. In this proof \( C_1 = C_1(\Omega_h, R, \|F_*\|_{H^1_0}, \|V_\lambda\|_{H^2,\alpha(\Omega_h)}, \kappa, R, F) > 0 \) denotes different positive constants. In addition, in this proof we denote for brevity \( V(\mathbf{x}) = V_{\mu(\delta)}(\mathbf{x}) \) and also sometimes we do not indicate the dependencies on \( (x_j, y_k, z, r) \). Note that (5.10), (5.11), (6.10)-(6.12) and (7.6) imply that

\[ \|\nabla V\|_{C^1(\Omega_h)} \leq C_1, \quad \|F\|_{C^1_h} \leq C_1, \] (7.14)
\[ \|\nabla^h r\|_{C^1_h} \leq C_1. \] (7.15)

Consider an arbitrary vector function \( p = (p_1, p_2) \in \overline{B(R)} \) and an arbitrary function \( r = (r_1, r_2) \in H^1_0, \) such that \( p + r \in \overline{B(R)} \). By (6.13) we need to consider \( A \), where

\[ A = |L^h(p + r + F)|^2 - |L^h(p + F)|^2. \] (7.16)

First, we will single out such a part of \( A \), which is linear with respect to \( r \). This will lead us to the Frechét derivative \( J'_\lambda \). Next, we will single out \( |\Delta^h r|^2 \). Based on this, we will apply the Carleman estimate of Theorem ???. For all \( z_1, z_2 \in \mathbb{C} \), we have

\[ |z_1|^2 - |z_2|^2 = (z_1 - z_2)\overline{z_1} + (\overline{z_1} - \overline{z_2})z_2. \] (7.17)

Denote

\[ z_1 = L^h(p + r + F), \quad z_2 = L^h(p + F). \] (7.18)

Then

\[ A_1 = (z_1 - z_2)\overline{z_1}, \quad A_2 = (\overline{z_1} - \overline{z_2})z_2, \quad A = A_1 + A_2. \] (7.19)

Using (5.5), (6.11) and (7.18), we obtain

\[ z_1 - z_2 = \Delta^h r - 2k^2 \nabla^h r \cdot \left[ \nabla^h V - \int_k^{\kappa} \nabla^h (p + F) dk \right] + 2i \left[ r_2 - \int_k^{\kappa} r_2 dk \right]. \]
\[ + 2k \int_k^{\kappa} \nabla^h r \cdot \left[ 2\nabla^h V - \int_k^{\kappa} \nabla^h (p + F) dk + k \nabla^h (p + F) \right]. \] (7.20)
Next,

\[ z_1 = \Delta^h(r + p + F) - 2k \left[ \nabla^h \nabla - \int_k \nabla^h (r + pr + F) \, dk \right] \cdot \left[ k \nabla^h (r + pr + F) \right] + \nabla^h \nabla - \int_k \nabla^h (r + pr + F) \, dk \]  

Hence,
\[ A_1 = (z_1 - z_2)Z_1 = |\Delta^h r|^2 + Z_{l,1}(r, k) + Z_1(r, k), \]  

where \( Z_{l,1}(h, k) \) is linear with respect to the vector function \( r = (r_1, r_2) \),

\[ Z_{l,1}(r, k) = \Delta^h r \cdot Y_1 + \nabla^h r \nabla^h Y_2 \cdot Y_3 + \nabla^h r Y_4 \cdot Y_5 + \left( \int_k \nabla^h r \, dk \right) \nabla^h Y_6 \cdot Y_7 + \left( \int_k \nabla^h r \, dk \right) \nabla^h Y_8 \cdot Y_9 + \left( \int_k r_2 \, dk \right) Y_{10} + \left( \int_k r_2 \, dk \right) Y_{11}, \]  

where explicit expressions for functions \( Y_j(x, k), j = 1, ..., 11 \) can be written in an obvious way. Also, it follows from those formulae as well as from (7.14) that \( Y_1, Y_2, Y_4, Y_6 \in C^1_0 \) and \( Y_3, Y_5, Y_7, Y_9, Y_{10}, Y_{11} \in C^0_0 \). In addition,

\[ \begin{cases} 
\|Y_1\|_{C^1_0}, \|Y_2\|_{C^1_0}, \|Y_4\|_{C^1_0}, \|Y_6\|_{C^1_0} \leq C_1, \\
\|Y_3\|_{C^0_0}, \|Y_5\|_{C^0_0}, \|Y_7\|_{C^0_0}, \|Y_9\|_{C^0_0}, \|Y_{10}\|_{C^0_0}, \|Y_{11}\|_{C^0_0} \leq C_1. 
\end{cases} \]  

The term \( Z_1(r, k) \) in (7.21) is nonlinear with respect to \( r \). Applying the Cauchy-Schwarz inequality and (7.15), we obtain

\[ Z_1(r, k) \geq \frac{1}{2} |\Delta^h r|^2 - C_1 \int_k |\nabla^h r|^2 \, dk. \]  

Similarly with (7.21)–(7.24) we obtain

\[ A_2 = (z_1 - z_2)Z_2 = Z_{l,2}(r, k) + Z_2(r, k), \]  

where the term \( Z_{l,2}(r, k) \) is linear with respect to \( r \) and has the form similar with the one in (7.22), although with different functions \( Y_j \), which still satisfy direct analogs of estimates (7.23). As to the term \( Z_2(r, k) \), it is nonlinear with respect to \( r \) and, as in (7.24),

\[ Z_2(r, k) \geq -C_1 |\nabla^h r|^2 - C_1 \int_k |\nabla^h r|^2 \, dk. \]
In addition, the following upper estimate is valid

\[ |Z_1(r,k)| + |Z_2(r,k)| \leq C_1 \left( |\Delta_h^r|^2 + |\nabla_h r|^2 + \int_k^\kappa |\nabla_h r|^2 d\kappa \right). \]  

(7.27)

Thus, it follows from (6.13) and (7.18)-(7.26) that

\[ J_\lambda(p + r) - J_\lambda(p) = \]

\[ e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_k^\kappa d [ (S_1 \Delta_h r + S_2 \cdot \nabla_h r)(x_j, y_s, z) ] \varphi_\lambda(z) d\kappa \]

\[ + e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_k^\kappa d Z(r,k)(x_j, y_s, z) \varphi_\lambda(z) d\kappa, \]  

where

\[ Z(r,k)(x_j, y_s, z) = Z_1(r,k)(x_j, y_s, z) + Z_2(r,k)(x_j, y_s, z). \]  

(7.29)

The second line of (7.28),

\[ \text{Lin}(r) = e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_k^\kappa d [ (S_1 \Delta_h r + S_2 \cdot \nabla_h r)(x_j, y_s, z) ] \varphi_\lambda(z) d\kappa \]  

(7.30)

is linear with respect to \( r \). Also, the vector functions \( S_1(x, k) \) and \( S_2(x, k) \) are such that

\[ |S_1(x, k)|, |S_2(x, k)| \leq C_1 \quad \text{in} \quad \Omega_h \times [k, \kappa]. \]  

(7.31)

As to the third line of (7.28), it can be estimated from the below as

\[ e^{2\lambda d} \sum_{j,s=1}^{M_h} h^2 \int_k^\kappa d Z(r,k)(x_j, y_s, z) \varphi_\lambda(z) d\kappa \]

\[ \geq e^{2\lambda d} \sum_{j,s=1}^{M_h} h^2 \left[ \frac{1}{2} \int_k^\kappa d |\Delta_h^r(x_j, y_s, z)|^2 \varphi_\lambda(z) d\kappa \right] \]

\[ - C_1 e^{2\lambda d} \int_k^\kappa d |\nabla_h^r(x_j, y_s, z)|^2 \varphi_\lambda(z) d\kappa. \]  

(7.32)
In addition, using (7.27) and (7.29), we obtain

$$e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_{\Omega}^{d} \int_{-\xi}^{\xi} |Z(r,k)(x_j,y_s,z)|\varphi_\lambda(z)dzdk$$

$$\leq C_1 e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_{\Omega}^{d} \int_{-\xi}^{\xi} (|\Delta h r|^2 + |\nabla h r|^2) + \int_{\Omega}^{d} (x_j,y_s,z)\varphi_\lambda(z)dzdk.$$  \hspace{1cm} (7.33)

The functional $\text{Lin}(h)$ in (7.30) is linear with respect to $r$. Also, by (7.30) and (7.31)

$$|\text{Lin}(r)| \leq C_1 e^{2\lambda(d+\xi) ||r||_{H_0^h}} , \text{ for all } r \in H_{0,2}^h.$$ 

Hence, $\text{Lin}(r) : H_{0,2}^h \to \mathbb{R}$ is a bounded linear functional. Hence, by Riesz theorem for each pair $\lambda > 0$ there exists a vector function $X_\lambda \in H_{0,2}^h$ independent on $r$ such that

$$\text{Lin}(r) = [X_\lambda, r], \text{ for all } r \in H_{0,2}^h.$$ \hspace{1cm} (7.34)

In addition, (7.28), (7.33) and (7.34) imply that

$$|J_\lambda(p_1 + r) - J_\lambda(p_1) - [X_\lambda, r]| \leq C_1 e^{2\lambda(d+\xi)} ||r||_{H_0^h}^2, \text{ for all } r \in H_{0,2}^h.$$ \hspace{1cm} (7.35)

Thus, (7.28)–(7.35) imply that $X_\lambda \in H_{0,2}^h$ is the Frechét derivative of the functional $J_\lambda(p)$ at the point $p_1$, i.e. $X_\lambda = J'_\lambda(p_1)$.

Next, using (7.28) and (7.32), we obtain

$$J_\lambda(p + r) - J_\lambda(p) - J'_\lambda(p)(r) \geq e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \left[ \frac{1}{2} \int_{\Omega}^{d} \int_{-\xi}^{\xi} |\Delta h r(x_j,y_s,z)|^2 \varphi_\lambda(z)dzdk - C_1 \int_{\Omega}^{d} \int_{-\xi}^{\xi} |\nabla h r(x_j,y_s,z)|^2 \varphi_\lambda(z)dzdk \right].$$

We now apply Carleman estimate of Theorem 7.1 for $\lambda \geq \lambda_0$,

$$e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \left[ \frac{1}{2} \int_{\Omega}^{d} \int_{-\xi}^{\xi} |\Delta h r(x_j,y_s,z)|^2 \varphi_\lambda(z)dzdk - C_1 \int_{\Omega}^{d} \int_{-\xi}^{\xi} |\nabla h r(x_j,y_s,z)|^2 \varphi_\lambda(z)dzdk \right]$$

$$\geq e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \left[ \int_{\Omega}^{d} \int_{-\xi}^{\xi} |r_{zz}(x_j,y_s,z)|^2 \varphi_\lambda(z)dzdk + C\lambda \int_{\Omega}^{d} \int_{-\xi}^{\xi} |r_{zz}(x_j,y_s,z)|^2 \varphi_\lambda(z)dz \right.$$  

$$+ \lambda^3 \int_{-\xi}^{\xi} |r(x_j,y_s,z)|^2 \varphi_\lambda(z)dz \bigg] - C_1 e^{2\lambda d} \int_{\Omega}^{d} \int_{-\xi}^{\xi} |\nabla h r(x_j,y_s,z)|^2 \varphi_\lambda(z)dzdk.$$ 

Hence, from these two equations it follows that for sufficiently large $\lambda_1$

$$\lambda_1 = \lambda_1(\Omega_h, R, \|F_\star\|_{H_0^h}, \|V_\star\|_{H^2(\Omega_h), E, k}) \geq \lambda_0$$
and for all \( \lambda \geq \lambda_1 \)

\[
J_{\lambda}(p_n + r) - J_{\lambda}(p_n) - J_{\lambda}'(p_n)(r) \geq c^{2d} \sum_{j=0}^{N_0} h^2 \int_{-\xi}^{\xi} \int_{\mathbb{R}^d} |r_{zz}(x_j, y_n, z)|^2 \varphi_{\lambda}(z) \, dz \, d\kappa
\]

\[
+ C_1 \lambda \int_{-\xi}^{\xi} |r(x_j, y_n, z)|^2 \varphi_{\lambda}(z) \, dz + C_1 \lambda^3 \int_{-\xi}^{\xi} |r(x_j, y_n, z)|^2 \varphi_{\lambda}(z) \, dz \geq C_1 \| r \|_{H^2_0}^2,
\]

which establishes (7.13).

**Theorem 7.4.** Assume that the conditions of Theorems 7.2 and 7.3 regarding the tail function \( V = V_{\mu(\delta)} \) and the functions \( F \) and \( F_n \) are satisfied. Then the Fréchet derivative \( J_{\alpha}' \) of the functional \( J_{\alpha} \) satisfies the Lipschitz continuity condition in any ball \( B(R') \) as in (6.12) with an arbitrary \( R' > 0 \). More precisely, the following inequality holds with the constant \( M = M(\Omega_\h, R', \| F_s \|_{H^1_{g}(\Omega_\h)}, \lambda, \xi, \bar{\kappa}, \bar{\lambda}) > 0 \) depending only on listed parameters:

\[
\| J_{\alpha}'(p_1) - J_{\alpha}'(p_2) \|_{H^2_0} \leq M \| p_1 - p_2 \|_{H^2_0}, \quad \text{for all } p_1, p_2 \in B(R').
\]

The proof of this theorem is completely similar with that of Theorem 3.1 of [3] and is, therefore, omitted.

Denote \( P_{\mathbb{R}} : H^h_{0,2} \to B(\mathbb{R}) \) the projection operator of the Hilbert space \( H^h_{0,2} \) on \( B(\mathbb{R}) \subset H^h_{0,2} \). Let \( p_0 \in B(\mathbb{R}) \) be an arbitrary point of the ball \( B(\mathbb{R}) \). Let the number \( \gamma \in (0, 1) \) be the number of Theorem 7.3, \( \delta \in (0, \delta_1) \).

The following theorem follows immediately from the combination of Theorems 7.3 and 7.4 with lemma 2.1 and Theorem 2.1 of [3].

**Theorem 7.5.** Assume that conditions of Theorems 7.2 and 7.3 are satisfied. Let \( \lambda \geq \lambda_1 \), where \( \lambda_1 \) is defined in Theorem 7.3. Then there exists unique minimizer \( p_{\min, \lambda} \in B(\mathbb{R}) \) of the functional \( J_{\lambda}(p) \) on the set \( B(\mathbb{R}) \) and

\[
J_{\lambda}'(p_{\min, \lambda})(y - p_{\min, \lambda}) \geq 0, \quad \text{for all } y \in H^h_{0,2}.
\]

Also, there exists a sufficiently small number \( \gamma_0 = \gamma_0(\Omega_\h, R, \| F_s \|_{H^2_0}, \| V_s \|_{H^2_{\text{g,}\lambda}(\Omega_\h)}, \xi, \bar{\kappa}, \bar{\lambda}) \in (0, 1) \) depending only on listed parameters such that for any \( \gamma \in (0, \gamma_0) \) the sequence (7.36) converges \( p_{\min, \lambda} \):

\[
\| p_{\min, \lambda} - p_n \|_{H^2_0} \leq \theta^n \| p_{\min, \lambda} - p_0 \|_{H^2_0}, \quad n = 1, 2, \ldots
\]

where the number \( \theta = \theta(\Omega_\h, R, \| F_s \|_{H^2_0}, \| V_s \|_{H^2_{\text{g,}\lambda}(\Omega_\h)}, \xi, \bar{\kappa}, \bar{\lambda}, \gamma) \in (0, 1) \) depends only on listed parameters.

Thus, (7.38) estimates the convergence rate of the sequence (7.36) to the minimizer \( p_{\min, \lambda} \). We now need to estimate the convergence rate of this sequence to the exact solution. To do this, we follow the Tikhonov regularization concept [4, 48] in Theorem 7.6 via assuming that the exact solution \( p_* \in B(\mathbb{R}) \).

**Theorem 7.6.** Assume that conditions of Theorems 7.2 and 7.3 are satisfied. Let \( \lambda_1 \) be the number of Theorem 7.3, \( \delta_1 \in (0, e^{-4(d+\xi)\lambda_1}) \) and \( \delta \in (0, \delta_1) \).
Set $\lambda = \lambda(\delta) = \ln(\delta^{-1/(4(d+\xi))}) > \lambda_1$. Furthermore, assume that the function $p_* \in B(R)$. Then there exists a number

$$
C_2 = C_2(\Omega_h, R, \|d_*\|_{H^2}, \|V_*\|_{H^2}^2, x_k) > 0
$$

depending only on listed parameters such that

$$
\|p_* - p_{\min, \lambda(\delta)}\|_{H^2} \leq C_2 \delta^{1/4},
$$

(7.39)

$$
\|c_* - c_{\min, \lambda(\delta)}\|_{L^2(\Omega_h)} \leq C_2 \delta^{1/4},
$$

(7.40)

In addition, the following convergence estimates hold

$$
\|p_* - p_n\|_{H^2} \leq C_2 \delta^{1/4} + \theta^n \|p_{\min, \lambda(\delta)} - p_0\|_{H^2}, \ n = 1, 2, \ldots
$$

(7.41)

$$
\|c_* - c_n\|_{L^2(\Omega_h)} \leq C_2 \delta^{1/4} + \theta^n \|p_{\min, \lambda(\delta)} - p_0\|_{H^2}, \ n = 1, 2, \ldots
$$

(7.42)

where $\theta \in (0, 1)$ is the number of Theorem 7.5 and functions $c_{\min, \lambda(\delta)}(x)$ and $c_n(x)$ is reconstructed from functions $p_{\min, \lambda(\delta)}$ and $p_n(x, k)$ respectively using (3.7)-(3.10) and (6.11).

**Remark 7.1.** Since $R > 0$ is an arbitrary number and $p_0$ is an arbitrary point of the ball $B(R)$, then Theorems 7.5 and 7.6 ensure the global convergence of the gradient projection method for our case, see section 1. We note that if a functional is non-convex, then the convergence of a gradient-like method of its minimization can be guaranteed only if the starting point of iterations is located in a sufficiently small neighborhood of its minimizer.

**Proof.** We temporarily denote $J_\lambda(p) := J_\lambda(p + F)$, see (6.13). We have

$$
J_\lambda(p_* + F) = e^{2\lambda d} \sum_{j,s=1}^{N_h} h^2 \int_\Omega \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \varphi_j^s(z) dz dt.
$$

(7.43)

It follows from (5.5), (5.6), (6.8)-(6.13), (7.5) (7.43) that

$$
J_\lambda(p_* + F) \leq C_2 \delta e^{2\lambda(d+\xi)}.
$$

(7.44)

Next, using (7.13) and , we obtain

$$
J_\lambda(p_* + F) - J_\lambda(p_{\min, \lambda(\delta)} + F) - J_\lambda(p_{\min, \lambda(\delta)} + F)(p_* - p_{\min, \lambda}) \geq C_2 \|p_* - p_{\min, \lambda}\|_{H^2}^2.
$$

Hence, since $-J_\lambda(p_{\min, \lambda(\delta)} + F) \leq 0$ and by (7.37) $-J_\lambda(p_{\min, \lambda(\delta)} + F)(p_* - p_{\min, \lambda(\delta)}) \leq 0$, we obtain, using (7.44) and recalling that $\lambda = \ln(\delta^{1/(4(d+\xi))})$:

$$
\|p_* - p_{\min, \lambda(\delta)}\|_{H^2}^2 \leq C_2 \delta^3,
$$

which implies (7.39). Estimate (7.40) follows immediately from (3.7)-(3.10), (6.11), (7.5) and (7.39).

We now prove (7.41) and (7.42). Using (7.38), (7.39) and the triangle inequality, we obtain for $n = 1, 2, \ldots$

$$
\|p_* - p_n\|_{H^2} \leq \|p_* - p_{\min, \lambda(\delta)}\|_{H^2} + \|p_{\min, \lambda(\delta)} - p_n\|_{H^2} \leq C_2 \delta^{1/4} + \|p_{\min, \lambda(\delta)} - p_0\|_{H^2},
$$

$$
\|c_* - c_n\|_{L^2(\Omega_h)} \leq C_2 \delta^{1/4} + \theta^n \|p_{\min, \lambda} - p_0\|_{H^2},
$$

(7.45)

(7.46)
which proves (7.41). Next, using (3.7)-(3.10), (6.11), (7.5), (7.38) and (7.40), we obtain
\[
\|c^* - c_n\|_{L^2(\Omega_h)} \leq \|c^* - c_{\min, \lambda(\delta)}\|_{L^2(\Omega_h)} + \|c_{\min, \lambda(\delta)} - c_n\|_{L^2(\Omega_h)} \\
\leq C_2\delta^{1/4} + C_2 \|p_{\min, \lambda(\delta)} - p_n\|_{H^2} \leq C_2\delta^{1/4} + C_2 \|p_{\min, \lambda(\delta)} - p_n\|_{H^2} \\
\leq C_2\delta^{1/4} + C_2\theta^\alpha \|p_{\min, \lambda(\delta)} - p_0\|_{H^2}.
\]
The latter establishes (7.42).

8 Numerical Study

We present in this section a numerical study of the application of our convexification method to microwave experimental backscatter data for buried objects. One of possible applications is in the standoff detection of explosives. We note that these data were treated in [35] by a different globally convergent method. We first describe very briefly the measured data and its preprocessing which is important for the application of our convexification method. We refer to [35] for all the details of data collection and preprocessing.

8.1 Measured data and its processing

The experimental data were measured by a scattering facility at the University of North Carolina at Charlotte. We have measured the backscatter data for objects buried in a sandbox. This sandbox was filled with dry sand and contains no moisture, see Figure 1. The data were measured on a rectangular surface of dimensions 1 m × 1 m. The distance between this surface and the sandbox was about 75 centimeters (cm). The coordinate system is chosen in such a way that the x−axis and the y−axis are respectively the horizontal and the vertical axis, while the z−axis is orthogonal to the measurement surface. The direction from the measurement surface to the target is the positive direction of the z−axis.

![Figure 1: A schematic diagram of the collection of our experimental data.](image)

The measurements consist of multi-frequency backscatter data associated with 300 frequency points uniformly distributed over the range from 1 GHz to 10 GHz. However, we work with the preprocessed data which are stable on narrow intervals of frequencies centered at 2.6 GHz, 3.01 GHz or 3.1 GHz. Since the corresponding wavelength for 2.6 GHz is 11.5 cm, the distance between the
source and the buried targets was about at least 6.17 wavelengths. This distance is sufficiently large in terms of wavelengths, and therefore justifies our modeling of the source as a plane wave. The backscatter data were generated by a single direction of the incident plane wave.

Recall that these experimental data were preprocessed in [35] and we will study the performance of our inversion method on that preprocessed data instead of the raw ones. The preprocessing developed in the cited paper comprises two main goals: distill the signals reflected by our buried targets from signals reflected by the sandbox and other unwanted objects, and reduce the noise in the data as well as the computational domain.

For the convenience of the readers we briefly summarize the main steps of the data preprocessing developed in [35].

Step 1. Subtract the reference data from the measured data for buried objects. The reference data are the ones measured in the case when the sandbox contains no buried objects. This subtraction helps us to sort of extract the signals of the buried targets from the total signal and also to reduce the noise.

Step 2. The data obtained after Step 1 were back propagated to the sandbox using the data propagation process. This process aims to “move” the data closer to the target. As a result, we obtain reasonable estimates for the location of the buried targets, particularly in the \((x, y)\)-plane, see Figure 1. In addition, this step helps us reduce the computational domain.

Step 3. Determine an interval of frequencies on which the data obtained after Step 2 are stable.

Figure 2: a) Absolute value of measured experimental data for the buried target 4 (sycamore, see table 1). b) Absolute value of the propagated data of a).

8.2 Reconstruction results

In this section we present the results of reconstructions from experimental data for the objects buried in a sandbox in Table 1 using our convexification method.
Table 1: Buried objects

| Number | Description     | Size in $x \times y \times z$ directions (in cm) |
|--------|----------------|-------------------------------------------------|
| 1      | Bamboo          | $3.8 \times 11.6 \times 3.8$                    |
| 2      | Geode           | $8.8 \times 8.8 \times 8.8$                     |
| 3      | Rock            | $10.5 \times 7.5 \times 4.0$                    |
| 4      | Sycamore        | $3.8 \times 9.9 \times 3.8$                     |
| 5      | Wet wood        | $9.1 \times 5.7 \times 5.8$                     |
| 6      | Yellow pine     | $9.0 \times 8.3 \times 5.8$                     |

The experimental setup for the case of objects buried in a sandbox is shown in Figure 1.

In Table 2 we present the optimal frequencies and corresponding intervals of wavenumbers $[\frac{k}{\lambda}, \frac{K}{\lambda}]$ for our objects. We refer to [35] for the details of the determination of these intervals.

The objects with their directly measured dielectric constant $c_{\text{meas}}$ and computed coefficient $c_{\text{comp}}$ along with corresponding measurement $\varepsilon_{\text{meas}}$ and computational errors $\varepsilon_{\text{comp}} = \frac{|c_{\text{comp}} - c_{\text{meas}}|}{c_{\text{meas}}} \ast 100\%$ are listed in Table 3. Note that the coefficients $c_{\text{comp}}$ in Table 3 are the maximal values of the reconstructed functions $c(x)$. In all our numerical tests we have used reasonable values of parameters $\mu = \lambda = 3.0$.

Considering the significant amount of noise in the measured data, the computational errors $\varepsilon_{\text{comp}}$ of reconstructed coefficients are sufficiently small. The computed dielectric constant of object 3 (a piece of rock) has the biggest error $\varepsilon_{\text{comp}} = 9.63\%$, but it is lower than its measurement error 21.3%.

In Table 4 we present the propagation distance $d$ [35], estimated location of objects and location of the reconstructed objects, i.e. the location of the maximum value of computed coefficient $\text{max}(c_{\text{comp}}(x))$. Errors of locations are small comparable with the size of the computational domain where we solve our inverse problem.

Fig. 3 and 4 illustrate the exact and computed images for the objects 2 and 4, respectively. Images are obtained using the contour filter in Paraview.

8.3 Conclusion

Table 3 and Figures 3, 4 demonstrate that our numerical method accurately reconstructs both dielectric constants and locations of targets in a quite challenging case of backscatter experimental data collected for buried targets.

References

[1] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Solna, and H. Wang, Mathematical and Statistical Methods for Multistatic Imaging, vol. 2098 of Lecture Notes in Mathematics, Springer, Cham, 2013.
Figure 3: Reconstruction result for target 2: (a) exact image, (b) computed image

Figure 4: Reconstruction result for the target 4: (a) exact image, (b) computed image
Table 2: Optimal frequencies and interval of wavenumbers

| Number | Optimal frequency, GHz | Interval of wavenumbers [k, k] |
|--------|------------------------|---------------------------------|
| 1      | 3.10                   | [6.322, 6.638]                  |
| 2      | 3.01                   | [6.133, 6.448]                  |
| 3      | 3.01                   | [6.070, 6.385]                  |
| 4      | 3.10                   | [6.322, 6.638]                  |
| 5      | 2.62                   | [5.313, 5.691]                  |
| 6      | 2.62                   | [5.313, 5.691]                  |

Table 3: Measured and reconstructed coefficients of objects

| Number | $c_{meas}$ | $\varepsilon_{meas}$ | $c_{comp}$ | $\varepsilon_{comp}$ |
|--------|------------|-----------------------|------------|-----------------------|
| 1      | 4.50       | 5.99%                 | 4.69       | 4.22%                 |
| 2      | 5.45       | 1.13%                 | 5.28       | 3.12%                 |
| 3      | 5.61       | 21.3%                 | 5.07       | 9.63%                 |
| 4      | 4.89       | 2.89%                 | 4.95       | 1.23%                 |
| 5      | 7.58       | 46.9%                 | 8.06       | 6.33%                 |
| 6      | 4.89       | 15.4%                 | 5.22       | 8.75%                 |

Table 4: Estimated and reconstructed locations of objects

| Number | Estimated location in $(x, y, z)$ | Computed location in $(x, y, z)$ |
|--------|-----------------------------------|---------------------------------|
| 1      | (0.80, -0.11, 0.19)               | (0.83, 0.03, -0.05)             |
| 2      | (0.58, -0.14, 0.44)               | (0.63, 0.03, 0.16)              |
| 3      | (0.62, -0.14, 0.20)               | (0.63, 0.08, -0.20)             |
| 4      | (0.80, -0.04, 0.19)               | (1.04, 0.08, -0.30)             |
| 5      | (0.57, -0.42, 0.29)               | (0.53, -0.08, 0.16)             |
| 6      | (0.54, -0.33, 0.29)               | (0.53, -0.03, 0.21)             |

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