Boundary shape design of planar membrane structures based on an airy stress model

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Abstract
Maintaining surface accuracy is crucial to realizing the required performance of membrane structure. However, compressive stress inevitably leads to wrinkling, and the accurate analysis of the stresses becomes significant. The present study analyzed membrane structures with discrete loads. First, a simple method was proposed and the stress field was accurately described using an Airy stress model. The first and second principal stresses were obtained, and regions of wrinkling with negative stress were predicted. Second, for a square membrane structure, vertical tensions around the boundary were applied such that in-plane stresses were more evenly distributed. Finally, to reduce the wrinkling areas more effectively, an arc-edge structure was designed by removing some areas of low-stress. Simulation results showed that the in-plane stress field distribution was described accurately and the wrinkling regions were diminished effectively with the proposed method.

Keywords
Membrane, airy stress, boundary shape design, wrinkle regions, arc-edge structure

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Introduction
Membranes are used in a wide variety of engineering applications spanning from aerospace applications to civil structures owing to their tunable stiffness and light weight.1 However, a membrane is usually assumed to have zero flexural stiffness, and there is out-of-plane deformation if any of the in-plane stresses are compressive. Maintaining surface accuracy is crucial to meeting the required membrane structure performance,2,3 and accurate analysis of stresses is therefore important.

Different methods of analyzing the stress fields of membrane structures have been proposed; Wong and Pellegrino4,5 studied the in-plane stress field in a square membrane with discrete forces applied on the membrane vertices and found that the stress field comprises identical corner regions under purely radial stress and a central region under uniform biaxial stress, with wrinkles appearing within the uniaxial stress regions. Zhang and colleagues6,7 proposed a membrane model with viable Young’s modulus and Poisson’s ratio, and then adopted parametric finite element discretization and a smoothing Newton method to obtain a numerical solution. The finite element method8,9 has been widely adopted; however, its use readily leads to divergent results. Coman and Bassom10 built a numerical model to establish the nature of localized wrinkling according

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to Föppl-von Kármán nonlinear plate theory; however, the bifurcation theorem was introduced to track the bifurcation path, which is computationally expensive.

Luo et al.\textsuperscript{11} proposed a topology optimization methodology to seek the optimal wrinkle-free design of macro-scale thin membrane structures; several holes were distributed on the optimal structures, and wrinkling disappeared as the areas of the structure reduced. Edge-trimming has been confirmed as an effective method of diminishing wrinkling without destroying the central region of the membrane structure. Luo and colleagues\textsuperscript{12,13} proposed a multi-material topology optimization approach for cable-suspended membrane structures and found that topology optimization is an effective method for the layout design of products; however, this method should be applied with the finite element analysis (FEA) method, which is usually computationally expensive. Bonin and Seffen\textsuperscript{14} predicted the trimming level by converting the tensile model into a bending analogy model. This method is not intuitive and deformation gradients and any intrinsic length changes between a straight-edge model and an arc-edge model were neglected, which may lead to inaccurate results.

In our previous study,\textsuperscript{15} we analyzed regular polygonal membrane structures subjected to equal radial forces and proposed a new stress-field distribution model for an arc-edge square membrane structure to predict the optimal trimming level. However, in engineering application, the membrane structure is stressed by many forces along the boundary, as in the case of 3-meter Ka-band reflectarray antenna that was developed at the Jet Propulsion Laboratory (JPL)\textsuperscript{16} and is shown in Figure 1.

The present article proposes a simple and effective method of accurately describing the stress field of any membrane structure on the basis of Airy stress model. After eliminate the boundary effects, the first and second principal stresses are calculated using the coordinate transformation approach. Wrinkling emerges within regions of negative second principal stress. Arc-edge structures are designed to reduce the extent of wrinkling regions. Results of finite element simulation verify the effectiveness of the proposed method.

\textbf{Stress field analysis of membrane structure}

The Airy stress function\textsuperscript{17} is applied to obtain the in-plane stress of the membrane structure. In Figure 2, any point $P$ is located at a distance $r$ and at an angle $\theta$ relative to the $Y$ axis from the loading point $O$. The radial stress is thus expressed as

$$\sigma_r = \frac{2T}{\pi rt} \cos \theta$$

where $T$ is the tension applied at point $O$, and $t$ is the thickness of the membrane structure. A circular path passes through points $O$ and $P$, with the radius of the circle being $R$. Using the relation $2R = r/\cos \theta$, equation (1) is rewritten as

$$\sigma_r = \frac{T}{\pi R t}$$

The radial stress is a constant on a circle with radius $R$; Figure 3 shows the stress contours of the radial stress $\sigma_r$. Lines of constant radial stress are circles tangent to the half-space surface at loading point $O$.

\textbf{Stress field of a circular membrane structure}

We consider a circular membrane structure with $n$ radial tensions $T_1, T_2, \ldots, T_i, \ldots, T_n$ equally spaced around the perimeter as shown in Figure 4(a). The angle between two adjacent tensions is $2\pi/n$. The radial
Equation (3) can be decomposed in $X$ and $Y$ directions as

$$\begin{align*}
\sigma_{x1} &= \sigma_r \sin^2 \beta \\
\sigma_{y1} &= \sigma_r \cos^2 \beta \\
\tau_{xy1} &= \sigma_r \sin \beta \cos \beta
\end{align*}$$

(4)

where $\sigma_{x1}$ and $\sigma_{y1}$ are, respectively, the stresses generated by $T_1$ in $X$ and $Y$ directions while $\tau_{xy1}$ is the shear stress.

Figure 4(a) shows that within the membrane structure there are the relations

$$\sin \beta = \frac{x}{r_1}, \quad \cos \beta = \frac{R - y}{r_1}$$

(5)

Substituting equations (4) and (5) into equation (3), the stresses in the $X$ and $Y$ directions at any point are written as

$$\begin{align*}
\sigma_{x1} &= \frac{2T_1}{\pi r_1^2} \cos \beta \sin^2 \beta + \frac{2T_1}{\pi r_1^2} \left(\frac{(R-y)^2}{r_1^2}\right) \\
\sigma_{y1} &= \frac{2T_1}{\pi r_1^2} \cos^2 \beta + \frac{2T_1}{\pi r_1^2} \left(\frac{(R-y)^2}{r_1^2}\right) \\
\tau_{xy1} &= \frac{2T_1}{\pi r_1^2} \cos \beta \sin \beta
\end{align*}$$

(6)

The stress generated by $T_n$ can be calculated through coordinate rotation as shown in Figure 4(b). Point $Q(x, y)$ is then converted to $Q'(x', y')$ according to

$$\begin{align*}
x' &= x \cos \left(\frac{2\pi}{n}\right) - y \sin \left(\frac{2\pi}{n}\right) \\
y' &= x \sin \left(\frac{2\pi}{n}\right) + y \cos \left(\frac{2\pi}{n}\right)
\end{align*}$$

(7)

Substituting equation (7) into equation (6), the stresses at point $Q$ generated by $T_n$ are written as

$$\begin{align*}
\sigma_{x_n} &= \frac{2T_n}{\pi r_n^2} \cos \beta \sin^2 \beta + \frac{2T_n}{\pi r_n^2} \left(\frac{(R-y')^2}{r_n^2}\right) \\
\sigma_{y_n} &= \frac{2T_n}{\pi r_n^2} \cos^2 \beta + \frac{2T_n}{\pi r_n^2} \left(\frac{(R-y')^2}{r_n^2}\right) \\
\tau_{xy_n} &= \frac{2T_n}{\pi r_n^2} \cos \beta \sin \beta
\end{align*}$$

(8)

Similarly, the total stress at any point $Q$ inside the membrane structure generated by all tensions is obtained as

$$\begin{align*}
\sigma_x &= \sum_{i=1}^{n} \sigma_{x_i} = \sum_{i=1}^{n} \frac{2T_i}{\pi r_i^2} \left(\frac{R-x \sin \left(\frac{2\pi i}{n}\right) + y \cos \left(\frac{2\pi i}{n}\right)}{r_i^2}\right)^2 \\
\sigma_y &= \sum_{i=1}^{n} \sigma_{y_i} = \sum_{i=1}^{n} \frac{2T_i}{\pi r_i^2} \left(\frac{R-x \sin \left(\frac{2\pi i}{n}\right) + y \cos \left(\frac{2\pi i}{n}\right)}{r_i^2}\right)^2 \\
\tau_{xy} &= \sum_{i=1}^{n} \tau_{xy_i} = \sum_{i=1}^{n} \frac{2T_i}{\pi r_i^2} \left(\frac{R-x \sin \left(\frac{2\pi i}{n}\right) + y \cos \left(\frac{2\pi i}{n}\right)}{r_i^2}\right)^2
\end{align*}$$

(9)
Equation (9) is derived from equation (2) and is applicable to an infinite half space. However, a circular membrane structure is a free body and the stress outside the membrane structure should thus be zero. The stresses around the perimeter of the circle need to be calculated to counteract the edge effect.

\[
\begin{align*}
\sigma_x &= \sum_{i=1}^{n} \frac{T_i}{\pi R^2} \\
\sigma_y &= \sum_{i=1}^{n} \frac{T_i}{\pi R^2} \\
\tau_{xy} &= \sum_{i=1}^{n} \frac{T_i}{\pi R^2}
\end{align*}
\]

On the basis of equation (2), the radial stress around the perimeter of the circle membrane is

\[\sigma = \sum_{i=1}^{n} \frac{T_i}{\pi R^2} \]  

(10)

Considering the edge effect, a compressive stress \(-\sigma\) needs to be added to equation (9), and the final results are summarized as

\[
\begin{align*}
\sigma_x &= -\sigma + \sum_{i=1}^{n} \sigma_{xi} \\
\sigma_y &= -\sigma + \sum_{i=1}^{n} \sigma_{yi} \\
\tau_{xy} &= -\sigma + \sum_{i=1}^{n} \tau_{xyi}
\end{align*}
\]

(11)

For the vertical tensions, the radial stress is a constant on the inscribed circle having radius \(R_m\). The stress generated by the vertical tensions can be calculated through coordinate translation as shown in Figure 6(b), where \(\delta\) is the distance of translation of axes. The stresses generated by \(F_i\) are then written as

\[
\begin{align*}
\sigma_{xF_i} &= \frac{2F_i}{\pi d} \left( \frac{(R_m-x)(y-\delta)}{r_i^2} \right) \\
\sigma_{yF_i} &= \frac{2F_i}{\pi d} \left( \frac{(R_m-y)(x-\delta)}{r_i^2} \right) \\
\tau_{xyF_i} &= \frac{2F_i}{\pi d} \left( \frac{(R_m-y)(x-\delta)}{r_i^2} \right)
\end{align*}
\]

(13)

Considering the edge effect, the total stresses generated by all tensions are
Simulation results

For a circular membrane structure with radius $R = 300$ mm, which is stressed by six equal radial tensions, $T = 3$ N. The analytical stress result is shown in Figure 7(a). For the same loading, finite element software ANSYS is used in linear stress analysis; the first principal stress field is shown in Figure 7(b). The analytical result compares well with the FEA result in terms of shape and magnitude. The local enlargement in Figure 7(a) shows that the areas between two loading portions and near the boundary are weakly tensioned, whereas areas closer to the loading points are highly tensioned.

Figure 8 shows the second principal stress field. The local enlargement shows the second principal stress $\sigma_2 < 0$ around the boundary, and wrinkling usually appears near the boundary owing to uneven tensioning. However, in the FEA result, the minimum stress value is zero and there are no compressive stresses. We use SHELL181 elements to model the membrane, and thus cannot obtain negative values under tension.

We consider the case of a square membrane structure that has a side length of 500 mm and is stressed by four radial tensions, $T = 3$ N. Figure 9(a) shows the first principal stress field, Figure 9(b) shows the second principal stress field, which is positive in a central circular region, and negative around the boundary, shown in Figure 9(c); wrinkling forms in the compression region, occupying as much as 33.5% of the total area.

Considering the effect of wrinkling, the initial wrinkling shown in Figure 10 is introduced into the finite element model, and the stress fields of the wrinkled membrane structure shown in Figure 11 are obtained employing the nonlinear buckling finite element method. The analytical results are closer to the FEA results, which further validates the analytical model.

In order to have in-plane stresses more evenly distributed, vertical tensions around the boundary are applied to the membrane structure, where $F_1 = F_2 = \sqrt{2}T$. On the basis of equation (14), the first principal stress field is shown in Figure 12(a), while the second principal stress field is shown in Figure 12(b). Figure 12(c) shows that the negative second principal stress regions are diminished compared with those in Figure 10, with regions of wrinkling occupying about 20.3% of the total area; however, areas between tensioning forces are very weakly tensioned and roughly circular.
Membrane shape design for the reduction of wrinkling

Design of the boundary shape

The removal of some areas of low-stress has been confirmed as an effective method of diminishing wrinkling regions, and a boundary cable is used to ensure that the membrane tension is uniform. Figure 13 shows the fundamental circular boundary shape with tensions, $T_r$, applied in the tangent direction of the cable. It is assumed that no shear is transferred between the cable and membrane. For the circular edge, the loads are equal and locally perpendicular to the edge, $q$ represents the magnitude of the uniform loads, the boundary stress of membrane is expressed as $\sigma = q/t$, and the shear stress is zero. A circular arc cable has the
potential to ensure uniform and biaxial membrane stressing.

For the square membrane structure with a side length of 500 mm as shown in Figure 11, the boundary is trimmed into circular arcs stretched with the cable shown in Figure 14. The diagonals are tangent to the circles near the vertices, and the radius of circle arcs is \( R_c = 500\sqrt{2}/8 \) mm. Tensions \( T_r \) are applied in the tangent direction of each circular arc, and the vertical tensions are then obtained as \( F = \sqrt{2}T_r \).

**Simulation results**

We use finite element software ANSYS for linear stress analysis, SHELL181 elements are used to model the membrane. Tensions \( T_r = 3 \) N, and \( F = \sqrt{2}T_r = 3\sqrt{2} \) N. Figure 15(a) shows the first principal stress

![Figure 10. Initial wrinkling.](image)

![Figure 11. FEA results: (a) the first principal stress field; (b) the second principal stress field.](image)

![Figure 12. Stress field: (a) the first principal stress field \((\sigma_1)\); (b) the second principal stress field \((\sigma_2)\); (c) The negative second principal stress.](image)
Conclusion

The Airy stress model was introduced to accurately estimate the stress field of a membrane structure. The first and second principal stresses were obtained through coordinate transformation. It was obvious that the areas between two loading portions and near the boundary were weakly tensioned, and regions of wrinkling with negative principal stress were estimated. To have stresses more evenly distributed, vertical tensions along sides were applied along the sides of a square membrane structure, and the ratio of the wrinkling area to the total area reduced from 33.5% to 20.3%. On this
foundation, the arc-edge membrane structure was designed. Simulation results showed that the ratio of the wrinkling area to the total area dropped to approximately 4.5%. However, the optimal boundary shape was not analyzed in this article, and we will study a simpler and more effective optimization method based on an analytical stress model in future work.

Declaration of conflicting interests

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