THE INVERSE PROBLEM OF EXTERNAL BALLISTICS FOR IDENTIFICATION OF AERODYNAMIC COEFFICIENTS OF A SPIN-STABILIZED PROJECTILE WITHIN THE MODIFIED POINT-MASS TRAJECTORY MODEL

The aim of this paper is to develop a technique for the identification of the projectile aerodynamic coefficients using the free-flight-test measurements. An algebraic method for solving the inverse problem of external ballistics is proposed. As the initial mathematical model of the projectile flight, a simplified version of the modified point-mass trajectory model in explicit form is used. For all aerodynamic coefficients of the model, exact explicit analytic expressions for their dependence on the experimentally measurable trajectory parameters are derived. Importantly, within the proposed approach, the solution to the inverse problem is unique.

Keywords: identification of aerodynamic coefficients, spin stabilized projectile, free flight data, nonlinear model, inverse problem, the modified point-mass trajectory model.

INTRODUCTION

Many applied problems of external ballistics require the development of an adequate mathematical model of the projectile flight. Here, by an adequate model, it is mean a mathematical model that describes the movement of the given projectile with a predetermined accuracy (such that its nominal physical parameters remain unchanged). Currently, to describe the trajectory characteristics of the projectile, several models are used [1–4], either within simplified or quite sophisticated approaches. The most popular among them are the Modified Point-Mass Trajectory Model (MPMTM) and, the so-called Six-Degrees-of-Freedom Trajectories Model (6-DOF Model), based on the equations for motion of an absolutely rigid body. Each of these two models has several modifications. The model is represented by a system of ordinary differential equations, containing a set of parameters – coefficients of aerodynamic forces and moments, physical parameters of the projectile (mass, geometric dimensions, moments of inertia, etc.) – as well as initial conditions for the dynamic characteristics of the projectile (initial location, initial velocity, initial quadrant elevation etc.). It is understood, that to find the trajectory characteristics of a given projectile with the required accuracy one has to provide all these parameters with the corresponding accuracy. And only, in this case, one can design a mathematical model of the projectile flight.

Note that so far [3], traditionally to fit the model to the experimental data the fitting factors are used, such as the ballistic coefficient, for example. The latter is acceptable even despite the fact that the introduction of correction factors obviously deviates the mathematical model from the exact physical meaning of the studied problem, especially when it concerns the movement of a given projectile.

The paper is devoted to the identification of the coefficients of aerodynamic forces and moments of a projectile based on an analytical solution of the inverse problem within the framework of MPMTM in explicit form and the experimental free flight data measurements.

There is extensive literature on methods for the determination of the aerodynamic coefficients, with the preference currently given to the methods based on data obtained during the free flight of a projectile. The problem is formulated mainly in the form of an inverse problem, which is known to lead to essential difficulties. The earlier methods...
were based on rather rough approximate solutions of the equations of projectile motion [2, 5]. Nowadays, in a significant part of works on this topic, the problem is considered in the form of a “black box or gray box” with the use of considerable simplifications, for example, replacing the sought functions with approximating or interpolating polynomials, etc. Several studies on this topic have been conducted and most of them employ the minimization of the difference between the measured and the calculated data [6–14].

**PRELIMINARY REMARKS**

This paper develops the idea expressed in [15], where an attempt was made to identify the drag coefficient from the trajectory data of the free flight of the projectile. The mathematical basis of the proposed method is the approach described in [16].

To clarify the essence of the method and the range of issues associated with it, let us consider the simplest model

$$y'(t) = f(t, \alpha(t), y(t)),$$  \hspace{1cm} (1)

where $y(t)$ is the scalar function describing the time dependence of a certain “trajectory”, $\alpha(t)$ is the time-dependent unknown parameter of the model, the function $f(t, \alpha, y)$ is considered to be known. It is required to identify $\alpha(t)$ for $t \in [t_0, t_1]$, if some particular “solution” $\hat{y}(t)$ of equation (1) is known on this interval, at some value of the initial condition imposed on $y(t)$. If the equation

$$p = f(t, \alpha, y)$$

is an algebraically solvable with respect to $\alpha$, i.e. $\alpha = \varphi(t, y, p)$, then one obtains the analytical expression

$$\alpha(t) = \varphi(t, \hat{y}(t), \hat{y}'(t)),$$  \hspace{1cm} (2)

where $\hat{y}'(t)$ is the derivative of the identified parameter $\alpha(t)$ from the particular “solution” $\hat{y}(t)$. Concerning the problem under consideration, $\hat{y}(t)$ represents the experimentally recorded (under certain conditions) dependence of the “trajectory” $y(t)$ on time within the interval $[t_0, t_1]$.

To be explicit, let us list several features in the physical setting of the considered problem. First, as a rule, the dependence $\hat{y}(t)$ is represented in tabular form by its values at discrete time moments. Second, the registration process is always accompanied by experimental (hardware) errors, which, in turn, will be transferred to the identified function $\alpha(t)$. Since expression (2) contains the derivative $\hat{y}'(t)$ of the experimental discrete function, the use of numerical differentiation can lead to destructive consequences for the method. In such a case, the numerical differentiation operation is an ill-conditioned problem. Our experience reveals that for such problems, the preliminary approximation of the initial discrete experimental data by one expression of certain smooth functions over the entire interval $[t_0, t_1]$ essentially solves the problem of differentiation since in this case continuous functions that are already smoothed during approximation are analytically differentiated.

Often solution (2) is not unique. To choose the correct branch of solutions, one has to use a priori conditions or refer to the physical meaning of the identified parameter.

Now let’s complicate the formulation of the problem. Consider a more realistic model

$$y'(t) = f(t, \alpha_1(t), \alpha_2(t), y(t)),$$  \hspace{1cm} (3)

which differs from (1) only by the number of identified parameters: now there are two of them $\alpha_1(t), \alpha_2(t)$ per one equation (3). A slight modification is in order here, which, strictly speaking, can lead to noticeable complications, first of all, concerning the uniqueness and accuracy of reconstruction. Namely, two particular solutions, which were determined experimentally (with different initial conditions) $\hat{y}_1(t)$ and $\hat{y}_2(t)$, are in demand here. Then we arrive with a system of two equations

$$y'_1(t) = f(t, \alpha_1(t), \alpha_2(t), \hat{y}_1(t)),$$

$$y'_2(t) = f(t, \alpha_1(t), \alpha_2(t), \hat{y}_2(t))$$

for two variables and the required parameters will be the solution of this algebraic system with respect to $\alpha_1(t), \alpha_2(t)$. By default, the arguments of the experimental functions $\hat{y}_1(t)$ and $\hat{y}_2(t)$ must be synchronized with each other. The rest of the recovery procedure remains unchanged.

Finally, let us consider an additional version, in which the derivative of the identified parameter stands within the mathematical model

$$y'(t) = f(t, \alpha(t), y(t)).$$  \hspace{1cm} (4)

At first glance, the problem of recovering the parameter $\alpha(t)$ from a known particular solution becomes much more complicated, since (4) is a differential equation for $\alpha(t)$ and, strictly speaking, it is not always clear how to get the initial condition for $\alpha(t)$. It is true, that the complications are indeed present, but in some cases they can be simply overcome. Note that the substitution $\alpha'(t) = \beta(t)$ sometimes transforms problem (4) into the problem (3) and it suffices to use two experimental functions $\hat{y}_1(t)$ and $\hat{y}_2(t)$, to reduce the problem to solving a system of two algebraic equations.
We emphasize that in all the cases considered above, it is assumed that the model is adequate in relation to the corresponding physical phenomenon, and it is this phenomenon that is investigated experimentally. For this reason, in particular, the replacement of \( \alpha'(t) \) by \( \beta(t) \) during the identification of \( \alpha(t) \) does not affect the physical phenomenon, since it still occurs in accordance with (4). Then, after reconstruction, it should automatically turn out that \( \beta(t) = \alpha'(t) \) with a certain degree of accuracy. The simplest example where this trick does not work is a model of the form

\[
y'(t) = f(t, \alpha'(t)/\alpha(t), y(t)),
\]

where in fact \( \alpha'(t)/\alpha(t) \) is a single parameter. Consequently, the differential equation cannot be reduced to an algebraic problem. We will meet such a situation further below.

One can notice that above we chose the number of necessary particular (independent) solutions (experimental functions) according to the number of parameters to be identified. Then, a question arises, what if we use more experimental functions? The answer is simple: if and only if the mathematical model is adequate to the physical process, then using different experimental functions we will get the same results, of course, taking into account the natural errors of both the model and the experiment. This almost obvious result opens up possibilities, simultaneously with the reconstruction of the model, to establish its adequacy due to the excess of the experimental data array.

An increase in the number of equations in the mathematical model, the number of parameters to be identified, and the differential order of equations hardly affects the described approach. Only the cumbersomeness of the calculations increases. Fortunately, such a formulation of external ballistics problems, as a rule, does not require real-time data processing.

**SYSTEM OF EQUATIONS FOR MPMTM IN EXPLICIT FORM**

In this section we present the description of the identification process of the aerodynamic coefficients of a spin stabilized projectile. The simplest currently accepted model is the Modified Point-Mass Trajectory Model (MPMTM) [3] recommended by the STANAG 4355 NATO standard. A significant drawback of this model, from the point of view of the approach we propose, is that this model is implicit. Recently, in [4], an equivalent MPMTM modification of the system of equations was proposed in an explicit form, which is most suitable to our purposes. The main difference of this latter form of the model in comparison with the MPMTM is that functions for the transverse angles of the projectile location are accurately deduced from the system of equations.

For the sake of clarity of the presentation, we will use a slightly simplified version of this system, which does not take into account the presence of wind, i.e. \( \mathbf{u} = \mathbf{v} \), Coriolis acceleration, etc. We follow the system of notation adopted in [3] and partially in [4], though the index \( \alpha \) was excluded from the notation in the places where it does not lead to misunderstandings.

![Fig. 1. Used coordinate system](image)

\[
\begin{align*}
\dot{x} &= \nu \cos \gamma \cos \chi; & y &= \nu \sin \gamma; & \dot{z} &= \nu \cos \gamma \sin \chi; \\
\dot{p} &= \frac{\rho \nu^2}{2I_x} \dot{p} \alpha \cos \gamma; \\
\dot{v} &= -\frac{\rho \nu^2 S}{2m} C_{D_0} - \frac{2mg^2}{\rho \nu^2 S} \times \\
& \quad \left( \frac{\hat{I}_x \dot{p}^2 \cos \gamma}{\left[ 1 - \hat{I}_x \dot{p}^2 C_{mag-f} \right]^2} + \left( \hat{I}_x \dot{p} \hat{C}_L \right)^2 \right) \frac{C_D}{\alpha^2}; \\
\dot{\theta} &= \frac{g \nu \cos \gamma}{v^2} \left( \frac{1 - \hat{I}_x \dot{p}^2 C_{mag-f}}{\left[ 1 - \hat{I}_x \dot{p}^2 C_{mag-f} \right]^2} + \left( \hat{I}_x \dot{p} \hat{C}_L \right)^2 \right)^2; \\
\dot{\chi} &= \frac{g \nu \cos \gamma}{v^2} \left( \frac{1 - \hat{I}_x \dot{p}^2 C_{mag-f}}{\left[ 1 - \hat{I}_x \dot{p}^2 C_{mag-f} \right]^2} + \left( \hat{I}_x \dot{p} \hat{C}_L \right)^2 \right)^2 \hat{I}_x \dot{p} \hat{C}_L,
\end{align*}
\]

where \( r = (x, y, z) \) is the three-dimensional position vector, \( \nu \) is the velocity of the projectile with respect to the ground-fixed axes, \( v \) is the speed of the projectile or rocket with respect to air, \( p \) is the axial spin rate of projectile, \( m \) is the mass density of air, \( m \) is the fuzed projectile mass, \( d \) is the reference diameter of projectile, \( s = \pi d^2/4 \), \( g \) is gravitational acceleration, \( \chi \) is the difference between the weapon azimuth (bearing) and the 1 axis measured clockwise, \( \gamma \) is the quadrant elevation, \( \hat{I}_x \) is axial moment of inertia.

- \( C_{D_0} \) is the zero-yaw drag force, \( C_D \) is the quadratic drag force coefficient, \( C_{\text{spin}} \) is the spin damping moment, \( C_L \) is the lift force coefficient, \( C_{\text{mag-f}} \) is the Magnus force coefficient, \( C_M \) is the overturning moment coefficient.
The dimensionless, i.e. “hatted”, coefficients were given as in [4]:

\[ i_x = \frac{I_x}{m}, \quad \hat{p} = \frac{pd}{v}, \quad \hat{C}_D = \frac{C_D}{(C_M)^2}, \quad \hat{C}_L = \frac{C_L}{C_M}; \]

\[ \hat{C}_{mag-f} = \frac{C_{mag-f}}{C_M}. \]

** SOLUTION OF THE SIMPLIFIED INVERSE PROBLEM OF IDENTIFICATION THE AERODYNAMIC COEFFICIENTS OF A PROJECTILE BASED ON MPMTM IN AN EXPLICIT FORM **

Again, it is worth to start with a simplified problem. Sufficiently accurate experimental free flight data of the trajectory parameters are most often obtained only at the initial part of the trajectory, where the change in the projectile rotation speed is small and can be neglected. For this reason, we set \( \hat{p}(t) = \hat{p}_0 = 2\pi/t_0 \), where \( t_0 \) is the twist of rifling at muzzle. To proceed we have to transform the system of equations (7)-(9) into a form suitable for identification of the aerodynamic coefficients according to the scheme described above in Section 2.

Solving equations (5d) and (5e) with respect to \( \hat{C}_L(v(t)) = \hat{C}_L, \) and \( \hat{C}_{mag-f}(v(t)) = \hat{C}_{mag-f} \), we obtain

\[ \hat{C}_L = \frac{g\gamma \cos^2 \gamma}{v \sqrt{x \hat{p}_0}} \left( \frac{2}{\gamma^2 \cos^2 \gamma + \gamma^2} \right), \]

\[ \hat{C}_{mag-f} = \frac{1}{I_x \hat{p}_0} \left[ \frac{1}{1 + \frac{g\gamma \cos \gamma}{v \sqrt{\gamma^2 \cos^2 \gamma + \gamma^2}}} \right]. \]

Further, substituting (6) and (7) into equation (5c), we obtain the following equation for the identification of \( C_D(v(t)) = C_D_0 \) and \( \hat{C}_D = \hat{C}_D_0 \):

\[ \rho^2 v^2 \sin^2 \gamma C_D_0 + 4I_x^2 m^2 \hat{p}_0^2 \left[ \frac{\gamma^2 \cos^2 \gamma + \gamma^2}{\gamma^2 \cos^2 \gamma + \gamma^2} \right] \]

\[ + 2m \rho \varphi (g \sin \gamma + \dot{v}) = 0. \]

Hereinafter we denote \( \rho = \rho(y) \).

System of equations (6)-(8) is equivalent to system (5c)-(5e) if one replaces \( \hat{p} \) in (5c)-(5e) by \( \hat{p}_0 \). Now let us turn directly to the problem of reconstructing the aerodynamic coefficients from the results of experimental data on the trajectory parameters of the projectile at the free flight. As the registering parameters one can chose the time readings of the Cartesian coordinates of the center of mass of the projectile \( [x(t), y(t), z(t)] \) or spherical coordinates \( [v(t), \gamma(t), \chi(t)] \), obtained for certain (initial) firing data. Importantly, we do not introduce special notations for the experimental parameters, since this difference follows from the context. The Cartesian coordinates of the center of mass of the projectile are converted into spherical coordinates by the conventional transformations:

\[ v = \sqrt{x^2 + y^2 + z^2}; \quad \gamma = \arcsin \left( \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right); \]

\[ \chi = \arctan \left( \frac{z}{x} \right), \]

and we will assume that such a transformation has already been done. It should be noted, that there is no principal problem to perform the reconstruction procedure directly from Cartesian coordinates, but this does not bring anything new, while results in more cumbersome expressions.

Note that equation (8) contains the function \( \rho = \rho(y(t)) \) as a parameter. With available meteorological and experimental data \( [x(t), y(t), z(t)] \) its determination does not imply any principal difficulties. If the experimental data are available in the form of \( [v(t), \gamma(t), \chi(t)] \), after their approximation, one can integrate the second of equations (5a) and obtain \( y(t) \) and, then, to restore \( \rho \).

Equations (6) and (7) allow one to identify \( \hat{C}_L \) and \( \hat{C}_{mag-f} \) by an algebraic method from one set of experimental data \( [v(t), \gamma(t), \chi(t)] \) on the interval of variation of the velocity \( v \in \left[v(t_0), v(t_1)\right] \), where \( t_0, t_1 \) is the data time interval.

Since expressions (10) and (11) contain the derivatives of the measured parameters, then, as we mentioned above, for such problems, preliminary approximation of the initial discrete experimental data by certain smooth functions is required over the entire interval \( [t_0, t_1] \). By the substitution of the above mentioned smooth functions into expressions (6) and (7), one obtains the dependence of the corresponding aerodynamic coefficient on time. However, also one needs their dependence on the speed \( v \). The easiest way to do this for the discrete time is to relate \( t_i \) in the tabular form with the corresponding value of the speed \( v(t_i) \). In addition to this, it is convenient to use the approximation of the obtained aerodynamic coefficient by a single smooth function over the entire range of \( v \) values, as we proposed in [17]. The remarks given here are valid for all cases considered below in this paper.

Equation (8), an analogue of equation (3), contains two parameters to be recovered. Note that all aerodynamic coefficients depend only on the velocity modulus \( v(t) = v \) (or, more precisely, on the Mach number, but we will not complicate the expressions, assuming that such a transformation has been already done) while being independent of the quadrant elevation, etc. In other words, for any initial conditions of projectile, aerodynamic coefficients depend on the speed (Mach number) in the same way. Similarly to case (3), two sets (vectors) of particular solutions with different initial conditions determined experimentally \( [v_1(t_1), \gamma_1(t_1), \chi_1(t_1)] \) are needed. After the following notation: \( v_1(t_1) = v_1 \), and for derivatives \( \frac{dv_1(t_1)}{dt} = \dot{v}_1 \) and similarly for all used functions, we come to a system of two equations...
\[
\rho_1^2 \nu_1^2 S^2 \mathcal{C}_{D_0} (v_1) + 4 \rho_2^2 m^2 \nu_1^2 \left( x_2^2 \cos^2 \gamma_1 + \gamma_1^2 \right) x \times \mathcal{C}_{D_{\alpha_2}} \left( v_2 \right) = 2 m S \nu_1 (g \sin \gamma_1 + v_1) = 0, \ i = 1, 2.
\]

It is now important to synchronize the variables in the equations. If one determines the moments of time \( t_j \) from the conditions \( v_i (t_j) = \hat{v} \), where \( \hat{v} \) is a variable parameter (velocity) and substitutes these moments of time in the experimental data in an approximated form, then this system reduces to an algebraic system of two equations for two variables \( C_{D_0} (\hat{v}) \) and \( \mathcal{C}_{D_{\alpha_2}} (\hat{v}) \), such that \( C_{D_0} (\hat{v}) \) and \( \mathcal{C}_{D_{\alpha_2}} (\hat{v}) \) are independent of \( i \), which in turn means, that they are the same in all equations. In the latter expressions, we use the “hat” symbol over the speed variable only to emphasize that this parameter is related to the synchronization of the \( t_j \) variables when using several particular solutions.

It should be emphasized that the derivatives of the velocities \( \dot{v}_i \) depend on \( i \).

The solution for system (9) is

\[
C_{D_0} (\hat{v}) = -\frac{2 m}{\hat{v}^2 S} \rho_1^2 p_0 \left( \rho_2^2 p_0 \left( x_2^2 \cos^2 \gamma_2 + \gamma_2^2 \right) g \sin \gamma_1 + v_1 \right) - \rho_2^2 p_0 \left( x_2^2 \cos^2 \gamma_2 + \gamma_2^2 \right) \rho_2^2 p_0 \left( x_2^2 \cos^2 \gamma_1 + \gamma_1^2 \right)
\]

\[
\mathcal{C}_{D_{\alpha_2}} (\hat{v}) = -\frac{\rho_1 (g \sin \gamma_2 + v_2) - \rho_2 (g \sin \gamma_1 + v_1)}{2 \rho_1^2 \rho_2^2 (x_2^2 \cos^2 \gamma_2 + \gamma_2^2) - \rho_2^2 p_0 \left( x_2^2 \cos^2 \gamma_1 + \gamma_1^2 \right)}.
\]

Therefore, we have demonstrated in a simplified manner how, based on the experimental trajectory data of the projectile flight and the system of equations (6), (7), (10), (11), one can identify the aerodynamic coefficients by algebraic methods. Importantly, the solutions are unique.

Note that if the function \( \tilde{p}(t) \) can be recorded experimentally, then the above expressions and (5b) are suitable for identification all aerodynamic coefficients without the assumption that \( C_{\text{spin}} \) is small.

**SOLUTION OF THE INVERSE PROBLEM OF PROJECTILE AERODYNAMIC COEFFICIENTS IDENTIFICATION BASED ON MPMTM**

Now let the speed of rotation of the projectile be variable in time, implying \( \tilde{p} = \tilde{p}(t) \) in equations (5c-e). Experimental measurement of the speed of rotation of the projectile with a required accuracy is of considerable difficulty and, thus, to avoid them, we use the method of excluding “unmeasurable” parameters.

To proceed we solve algebraically the system of two equations (5d) and (5e) for \( \tilde{p} \) and \( C_{\text{mag-f}} \). The solution is unique and is of the form

\[
\tilde{p} = \frac{g \hat{v} \cos^2 \gamma}{v \hat{v} C_L \left( \frac{x^2 \cos^2 \gamma + \gamma^2}{x^2 \cos^2 \gamma + \gamma^2} \right)},
\]

\[
C_{\text{mag-f}} = \hat{C}_L \times \frac{\frac{\hat{v}}{v} \left( \frac{x^2 \cos^2 \gamma + \gamma^2}{x^2 \cos^2 \gamma + \gamma^2} + g \hat{v} \cos \gamma \right)}{g \hat{v} \cos^2 \gamma}.
\]

By substitution of (12)-(13) into (5c), we obtain the equation

\[
C_{D_0} + \frac{2 m}{\rho \nu^2 S} (g \sin \gamma + v) + \frac{\hat{C}_{D_{\alpha_2}}}{\hat{C}_L} \times \frac{4 g \hat{v} \nu^2 \nu \cos \gamma}{\rho^2 \nu^2 S} = 0.
\]

By substitution \( \tilde{p} = p d / v \), we reduce equations (5b) and (12) to the same variables \( C_{D_0} (\hat{v}) \) and \( \hat{C}_{D_{\alpha_2}} (\hat{v}) \) are independent of \( i \), which in turn means, that they are the same in all equations. In the latter expressions, we use the “hat” symbol over the speed variable only to emphasize that this parameter is related to the synchronization of the \( t_j \) variables when using several particular solutions.
\[ C_{D_0}(v) = -\frac{2m}{\gamma S} \frac{\rho_1 x_2 \cos^4 \gamma_2 \left( x_1^2 \cos^2 \gamma_1 + \gamma_1^2 \right) (\gamma_2 \sin \gamma_1 + v_1)}{\rho_2 x_2 \cos^4 \gamma_1 \left( x_1^2 \cos^2 \gamma_1 + \gamma_1^2 \right)} - \rho_2 x_1 \cos^4 \gamma_1 \left( x_1^2 \cos^2 \gamma_1 + \gamma_1^2 \right) \; ; \]  

\[ \dot{C}_{D_2}(v) = -C_{D_2} \frac{\rho_2 v^2 S}{2mg} \left( x_1^2 \cos^2 \gamma_1 + \gamma_1^2 \right) + \frac{\rho_1 (\gamma_2 \sin \gamma_1 + v_1)}{\rho_2 (\gamma_2 \sin \gamma_1 + v_1)} \right) ; \]  

\[ C_{\text{spin}}(v) = \frac{2I_x}{v} \left( \cos \gamma_1 \cos \gamma_2 \left( x_1^2 \cos^2 \gamma_1 + \gamma_1^2 \right) x_2 \cos^2 \gamma_2 + \gamma_2^2 \right) v_1 x_2 \gamma_2 - 2 \gamma_1 x_2 \gamma_2^2 \gamma_1 \cos^2 \gamma_1 + \gamma_1^2 \right) + 2 v_1 x_2 \gamma_2 \gamma_1 \cos^2 \gamma_2 + \gamma_2^2 \right) - 2 \gamma_1 x_2 \gamma_2 v_1 \gamma_3 \cos \gamma_1 \sin \gamma_2 - v_2 \gamma_1 \gamma_2 \gamma_3 \cos \gamma_1 \sin \gamma_2 - v_2 \gamma_1 \gamma_2 \gamma_3 \cos \gamma_1 \sin \gamma_2 - v_2 \gamma_1 \gamma_2 \gamma_3 \cos \gamma_1 \sin \gamma_2 ; \]  

\[ \frac{d\dot{C}_L}{dv} = \rho_1 \left( x_2 \gamma_2 \cos^2 \gamma_1 + x_1^2 \gamma_1 \cos^2 \gamma_1 + \gamma_1^2 \right) (\gamma_2 \sin \gamma_1 + v_1) - \rho_2 (\gamma_2 \sin \gamma_1 + v_1) ; \]  

Equation (25) is the differential equation of the form  

\[ \frac{d\dot{C}_L}{dv} = F(\dot{v}) \]  

(22)  

Since the right-hand side of expression (21) contains known (experimentally measured) functions, in the discrete-time approach, by replacing \( t_i \) with the corresponding value of the velocity \( v_i(t_i) \), we obtain the function \( F(\dot{v}) \) for (26) in a tabular form. The solution to equation (22) is of the form  

\[ \dot{C}_L(\dot{v}) = \dot{C}_L(\dot{v}_0) k^\dot{v}_0 \]  

(23)  

where \( \dot{C}_L(\dot{v}_0) \) is the initial condition to be imposed on the sought function \( \dot{C}_L(\dot{v}) \).  

As we have already mentioned, the most accurate experimental data are obtained at the initial part of the trajectory, i.e. for the time interval \([t_0, t_1] \), with \( t_0 = 0 \) being the time of projectile departure from the muzzle of the gun. In this case, the initial value \( p(0) = 2\pi/t_c \), determined from the barrel rifling period, is known. Then from equation (16) using experimental data and, substituting \( t = 0 \), we obtain  

\[ p = \frac{1}{\rho(0)} \frac{g \gamma \cos \gamma}{v_1 \left( \gamma^2 \cos^2 \gamma + \gamma^2 \right)} \]  

(24)  

It remains to substitute (24) into (23), perform the integration and obtain \( \dot{C}_L(\dot{v}) \).  

Finally, we substitute the values of \( \dot{C}_L(\dot{v}) \) into expressions (13), (19), thereby obtaining all the aerodynamic coefficients considered here. Note that solutions to the inverse problem based on experimental trajectory data of the projectile flight are unique in this section as well.  

**CONCLUDING REMARKS**  
In this work, exact explicit algebraic expressions for the dependences of all aerodynamic coefficients MPMTM on the experimentally measured parameters are obtained.  
Despite the fact that the equations obtained in this paper for the identification of the aerodynamic coefficients are equivalent forms of the original equations, the results of the identification from the experimental data of the projectile flight, even in an ideal case, cannot be exact. First, any mathematical model is an approximation to the real process. Second, the MPMTM used in this paper is already simplified. It is known, for example, that the projectile flight range depends on the initial projectile nutation angles, but this condition is completely ignored in MPMTM. In the course of a real flight and, accordingly, in the experimental data, this dependence is retained, which will affect the accuracy of the identification aerodynamic coefficients.  
Strictly speaking, the errors arising in the identification process depend on the accuracy of the mathematical model, the stability of the recovery procedure, the accuracy of recording experimental data, etc., and thus, their analysis is rather complicated. Extensive studies based on numerical experiments still are needed to assess the basic requirements for the main sources of errors. Using the above mentioned possibility due to the excess of realizations of experimental data, the adequacy of the used models and the method as such should be examined. Our preliminary numerical experiments using the 6-DOF model or its simplified versions [18] show encouraging results.  

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**Ключові слова:** відновлення аеродинамічних коефіцієнтів, снаряд, що стабілізується обертанням, експериментальні дані вільного руху снаряда, обернена задача, модифікована модель точкової маси.

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The article was received by the editorial board on 27.01.2021.