The square root of the Dirac operator on the superspace
and the Maxwell equations

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Abstract

We re-consider the procedure of “taking a square root of the Dirac equation” on the
superspace and show that it leads to the well known superfield $W_\alpha$ and to the proper
equations of motion for the components, i.e. the Maxwell equations and the massless
Dirac equation.

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1 Introduction

The Dirac equation is often “derived” by requiring that it squares to the Klein-Gordon
equation [1]. In this sense one can view the Dirac operator as a square root of the Klein-
Gordon operator. It is then natural to ask if this procedure can be taken one step further,
i.e. if one can give a sensible and useful definition of the square root of the Dirac operator.

Such a construction within a $\mathcal{N} = 1$ superspace was given in [2] and extended to $\mathcal{N} = 2$
case in [3]. The main conclusion that followed (expressed implicitly in [2] and quite explicitly
in [3]) was a bit disappointing: the equations that result from taking a square root of the
Dirac operator are just constraints imposed on the superfield which eliminates some of the
components, but do not yield the dynamics to the remaining ones.

The main goal of this letter is to re-analyze the situation and to show that (providing
one requires that the obtained equations square to the Dirac equation and not to its “gen-
eralization”) such a conclusion is not correct: properly understood Szwed equations lead to
dynamical equations for the component fields. We discuss in details the simplest case, with
the solution to the Szwed equations being the Maxwell superfield with the vector compo-


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2 The square root of the Dirac operator

Using the two-component notation and the chiral representation for the Dirac matrices (for
the conventions see [4]) one can write the Dirac equation as

$$- \left( \begin{array}{cc} i\bar{\sigma}^{\mu\dot{\alpha}}\partial_{\mu} & m \\ m & i\sigma^{\mu\alpha\dot{\alpha}}\partial_{\mu} \end{array} \right) \left( \begin{array}{c} \varphi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right) \equiv \mathcal{D} \left( \begin{array}{c} \varphi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{array} \right) = 0. \quad (1)$$

We are looking for an operator $A$ which satisfies

$$A^\dagger A = \mathcal{D}. \quad (2)$$

If one requires $A$ to be local and to contain space-time derivatives, then the only possibility
of satisfying (2) is to allow $A$ to depend on the anticommuting variables. Indeed there are
no second order space-time derivatives in the Dirac operator, and one can check that the
coefficient at $\partial_{\mu}$ in $A$ has to square to zero. We are therefore lead to consider the operator $A$
as acting on the superspace with the coordinates $(x^{\mu}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$.

A solution to (2) is given by

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} D^{\alpha} & -\bar{D}_{\dot{\alpha}} \\ \bar{D}^{\dot{\alpha}} & D_{\alpha} \end{pmatrix} \quad (3)$$

where the differential operators

$$D_{\alpha} = \partial/\partial\theta^{\alpha} + i\sigma^{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu},$$
$$\bar{D}_{\dot{\alpha}} = -\partial/\partial\bar{\theta}^{\dot{\alpha}} - i\theta^{\alpha}\sigma^{\mu\alpha\dot{\alpha}}\partial_{\mu}, \quad (4)$$

satisfy the algebra

$$\{D_{\alpha}, D_{\beta}\} = \{\bar{D}_{\alpha}, \bar{D}_{\beta}\} = 0,$$
$$\{D_{\alpha}, \bar{D}_{\dot{\beta}}\} = -2i\sigma^{\mu\alpha\dot{\beta}}\partial_{\mu}. \quad (5)$$

From (3) we get

$$A^\dagger A = - \left( \begin{array}{cc} i\bar{\sigma}^{\mu\dot{\alpha}}\partial_{\mu} & M \\ M & i\sigma^{\mu\alpha\dot{\alpha}}\partial_{\mu} \end{array} \right) \quad (6)$$

with

$$M = -\frac{1}{4} (DD + \bar{D}\bar{D}). \quad (7)$$

We conclude that the operators $D$ and $A^\dagger A$ are equal on the space of fields $F$ which
satisfy

$$(DD + \bar{D}\bar{D}) F + 4m F = 0. \quad (8)$$

On this space we can therefore consider the equation

$$A F = 0 \quad (9)$$

as a square root of the Dirac equation.
The spinorial derivatives $D_\alpha, \bar{D}_{\dot{\alpha}}$ anticommute with the supersymmetry generators

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \quad (10)$$

Consequently, the SUSY transformations generated by $Q_\alpha, \bar{Q}_{\dot{\alpha}}$,

$$\delta F = (\epsilon Q + \bar{\epsilon} \bar{Q}) F,$$

where $\epsilon$ is an infinitesimal, constant, anticommuting spinor, transform one solution of the system of equations (8), (9) into another.

3 A solution

We shall be interested in the solutions of (9) having the form

$$F = \begin{pmatrix} W_\alpha \\ \bar{\mathcal{H}}^{\dot{\alpha}} \end{pmatrix}. \quad (11)$$

Using (3) we get

$$D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}} \bar{\mathcal{H}}^{\dot{\alpha}} = 0,$$

$$\bar{D}^{\dot{\alpha}} W_\alpha + D_\alpha \bar{\mathcal{H}}^{\dot{\alpha}} = 0. \quad (12)$$

Let us now work out some of the consequences of (12) and (8).

It is not difficult to show that $W_\alpha$ and $\bar{\mathcal{H}}^{\dot{\alpha}}$ are a fully fledged superfields, i.e. their components at the non-zero powers of $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$ are non-vanishing. Indeed if we assume that

$$W_\alpha = \lambda_\alpha(x), \quad \bar{\mathcal{H}}^{\dot{\alpha}} = \bar{h}^{\dot{\alpha}}(x),$$

then it follows from (12) that

$$\partial_\mu \lambda_\alpha = \partial_\mu \bar{h}^{\dot{\alpha}} = 0,$$

so the only allowed solutions are constants.

Acting with the $D^\alpha$ derivative on the equations (12) and using the equality

$$i\sigma'^{\mu\dot{\alpha}} \partial_\mu W_\alpha + M \bar{\mathcal{H}}^{\dot{\alpha}} = 0, \quad (13)$$

(which follows from (6) and the fact that $AF = 0 \Rightarrow A^\dagger AF = 0$), we get

$$D^2 \bar{\mathcal{H}}^{\dot{\alpha}} = 0. \quad (14)$$

Similarly one shows that

$$\bar{D}^2 W_\alpha = 0. \quad (15)$$

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3We shall exclude another form of solution discussed in [2] and [3]

$$B = \begin{pmatrix} \Phi \\ V^{\alpha\dot{\alpha}} \end{pmatrix}$$

from our considerations, as the equation $A^\dagger A B = 0$ is not equivalent to the Dirac equation.
After some more algebra one checks that the relations
\[ \partial_\mu (\bar{D}^\alpha W_\alpha) = D^\beta (\bar{D}^\beta W_\alpha) = \bar{D}^\beta (\bar{D}^\alpha W_\alpha) = 0 \]
and
\[ \partial_\mu (D_\alpha \bar{H}^\dot{\alpha}) = D^\beta (D_\alpha \bar{H}^\dot{\alpha}) = \bar{D}^\beta (D_\alpha \bar{H}^\dot{\alpha}) = 0 \]
hold, what proves the equality
\[ \bar{D}^\alpha W_\alpha = -D_\alpha \bar{H}^\dot{\alpha} = \text{const.} \] (16)

Let us denote this constant by \( \eta^\dot{\alpha}_\alpha \) and notice that (8) and (12) are invariant under the shift
\[ W_\alpha \rightarrow W_\alpha + \bar{\theta}_\alpha \eta^\dot{\alpha}_\alpha, \]
\[ \bar{H}^\dot{\alpha} \rightarrow \bar{H}^\dot{\alpha} - \theta^\alpha \eta^\dot{\alpha}_\alpha. \] (17)
The shifted fields thus satisfy
\[ \bar{D}^\alpha W_\alpha = 0, \quad D_\alpha \bar{H}^\dot{\alpha} = 0. \] (18)

It is worth to emphasize that the result above does not depend on the equation (8).

It is intriguing to notice that even if we started from the massive Dirac equation, consistency requires \( m \) to be equal to zero. Indeed it follows from (8) that \( W_\alpha \) satisfies
\[ (DD + \bar{D} \bar{D}) W_\alpha = -4m W_\alpha \] (19)
Acting on this equation with \( D^\alpha \) we get (using (15))
\[ -4m D^\alpha W_\alpha = DDD^\alpha W_\alpha = 0. \]
If we assume that \( m \neq 0 \) then acting with \( D^\alpha \) on the equation (16), we get
\[ \bar{\sigma}^\mu \dot{\alpha} \partial_\mu W_\alpha = 0 \]
so that from (13):
\[ m \bar{H}^\dot{\alpha} = 0 \quad \Rightarrow \quad \bar{H}^\dot{\alpha} = 0. \]
Similarly one can show that the assumption \( m \neq 0 \) leads to \( W_\alpha = 0 \). Therefore, in order to obtain non-trivial solutions to the Szwed equation (9) one has to assume that
\[ m = 0. \] (20)

Let us now consider the simplest case \( W_\alpha = \mathcal{H}_\alpha \). Using (18) one can see that equations (12) now read
\[ \bar{D}^\alpha W_\alpha = 0, \quad D_\alpha \bar{W}^\dot{\alpha} = 0, \quad D^\alpha W_\alpha = \bar{D}_\alpha \bar{W}^\dot{\alpha}. \] (21)
It follows [4, 5] that \( W_\alpha \) has the form of the Maxwell superfield,
\[ W_\alpha = -i \lambda_\alpha(y) + \left[ \delta_\alpha^\beta d(y) - \frac{i}{2} (\sigma^\mu \sigma^\nu)_{\alpha}^\beta F_{\mu \nu}(y) \right] \theta_\beta + \theta \sigma^\nu \dot{\alpha} \partial_\mu \lambda^\dot{\alpha}(y). \] (22)
Here $F^{\mu\nu}$ is an antisymmetric tensor, $y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\bar\alpha} \bar{\theta}^{\bar\alpha}$, 

$$d^*(x) = d(x), \quad F^{*\mu\nu}(x) = F^{\mu\nu}(x),$$

(23)

(star denotes complex conjugation) and

$$\varepsilon^{\mu\nu\rho\lambda} \partial_\nu F^\rho_\lambda = 0.$$  

(24)

Now we may employ equation (19) which, in view of (15) and (20) takes the form

$$DD W_\alpha = 0.$$  

(25)

From (25) and its complex conjugation one can check (taking steps similar to those that lead to the equality (16)) that the relations

$$\partial_\mu (D^\alpha W_\alpha) = D^\beta (D^\alpha W_\alpha) = \tilde{D}^{\beta\tilde{\alpha}} (D^\alpha W_\alpha) = 0$$

hold, what proves the equality

$$D^\alpha W_\alpha = \tilde{D}_\tilde{\alpha} \tilde{W}^{\tilde{\alpha}} = \text{const.}$$  

(26)

For $W_\alpha$ given in (22) this boils down to the equations for the component fields of the form

$$d = \text{const.}, \quad \sigma^\mu_{\alpha\bar\alpha} \partial_\mu \bar{\lambda}^{\bar\alpha} = 0,$$

(27)

and

$$\partial_\nu F^{\nu\mu} = 0.$$  

(28)

The equations (28) and (24) form a full set of Maxwell equations for the antisymmetric, real tensor $F^{\mu\nu}$.

It may be interesting to extend the results from this letter in several directions.

Starting from the Dirac equation on the curved space–time one may hope to end up with the equations covariant with respect to the local supersymmetry transformations, “re-deriving” in this way some simple supergravity model.

It is known that by “taking a square root” of the Klein–Gordon operator one can obtain not only the Dirac equation, but also the first order equations describing the massive scalar or the massive vector fields (the Dirac matrices are in this case replaced by the matrices satisfying the Duffin–Kemmer algebra). It is tempting to try to repeat this procedure here and to check if one arrives at the equations describing other supermultiplets.

Last but not least, one may try to “take a square root” of the coupled Maxwell–Dirac equations. With a bit of luck this may lead to an interacting supersymmetric model.

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