ON PRINCIPAL COMPONENT REGRESSION IN A HIGH-DIMENSIONAL ERROR-IN-VARIABLES SETTING

BY ANISH AGARWAL 1, DEVAVRAT SHAH 1 AND DENNIS SHEN 1

1Electrical Engineering and Computer Science, Massachusetts Institute of Technology
anish90@mit.edu; devavrat@mit.edu; deshen@mit.edu

We analyze the classical method of Principal Component Regression (PCR) in the high-dimensional error-in-variables setting. Here, the observed covariates are not only noisy and contain missing data, but the number of covariates can also exceed the sample size. Under suitable conditions, we establish that PCR identifies the unique model parameter with minimum \( \ell_2 \)-norm, and derive non-asymptotic \( \ell_2 \)-rates of convergence that show its consistency. We further provide non-asymptotic out-of-sample prediction performance guarantees that again prove consistency, even in the presence of corrupted unseen data. Notably, our results do not require the out-of-samples covariates to follow the same distribution as that of the in-sample covariates, but rather that they obey a simple linear algebraic constraint. We finish by presenting simulations that illustrate our theoretical results.

1. Introduction. We consider the setup of error-in-variables regression in a high-dimensional setting. Formally, we observe a labeled dataset of size \( n \), denoted as \( \{(y_i, z_i) : i \leq n\} \). Here, \( y_i \in \mathbb{R} \) represents the response variable, also known as the label or target. For any \( i \geq 1 \), we posit that

\[
y_i = \langle x_i, \beta^* \rangle + \varepsilon_i,
\]

where \( \beta^* \in \mathbb{R}^p \) is the unknown model parameter, \( x_i \in \mathbb{R}^p \) is the associated covariate, and \( \varepsilon_i \in \mathbb{R} \) is the response noise. Unlike traditional regression settings where \( z_i = x_i \), the error-in-variables regression setting reveals a corrupted version of the covariate \( x_i \). Precisely, for any \( i \geq 1 \), let

\[
z_i = (x_i + w_i) \circ \pi_i,
\]

where \( w_i \in \mathbb{R}^p \) is the covariate measurement noise and \( \pi_i \in \{0, 1\}^p \) is a binary observation mask with \( \circ \) denoting component-wise multiplication, i.e., we observe the \( k \)-th component of \( z_i \) if \( \pi_{ik} = 1 \) and 0 otherwise. Further, we allow \( n \) to be much smaller than \( p \).

Our interest is in analyzing the performance of the classical method of Principal Component Regression (PCR) for this scenario. In a nutshell, PCR is a two-stage process: first, PCR “de-noises” the observed covariate matrix \( Z = [z_i^T] \in \mathbb{R}^{n \times p} \) via Principal Component Analysis (PCA), i.e., PCR replaces \( Z \) by its low-rank approximation. Then, PCR regresses \( y = [y_i] \in \mathbb{R}^n \) with respect to this low-rank variant to produce the model estimate \( \hat{\beta} \). We are interested in the following natural questions about the estimation quality of PCR: (1) Given that multiple models are feasible within the high-dimensional framework, what structure should be endowed on \( \beta^* \) such that \( \hat{\beta} \approx \beta^* \)? (2) Given noisy and partially observed out-of-sample covariates, can PCR accurately predict the expected response variables, i.e., under what conditions does PCR generalize?

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1.1. Contributions. As the main contribution of this work, we establish that PCR consistently learns the latent model parameter in a high-dimensional error-in-variables setting (Theorem 4.1 and Corollary 4.1). Interestingly, rather than endowing the standard sparsity structure on $\beta^*$, we establish that PCR learns the unique model parameter with minimum $\ell_2$-norm, which is of primary importance in the context of prediction. As a special case of our setting in which the spectrum of true covariates is well-balanced (Assumption 4.1), we show that the parameter estimation error decays as $O(1/n)$. This matches the best known estimation error rate in the literature, cf. [14, 10, 17].

We also establish that PCR achieves vanishing out-of-sample prediction error, even in the presence of corrupted out-of-sample covariates (Theorem 4.2 and Corollary 4.2). Notably, we do not make any distributional assumptions on the data generating process to arrive at our result, but rather introduce a natural linear algebraic condition (Assumption 2.5). In contrast, popular tools to understand the generalization behavior, such as Rademacher complexity analyses, commonly assume that both the in-sample and out-of-sample measurements are independent and identically distributed. Again, in the special case when the true covariates have a well-balanced spectra, we show that the out-of-sample prediction error rate is $O(1/n)$, which improves upon the best known error rate of $O(1/\sqrt{n})$ for PCR, established in [1, 2].

1.2. Key comparisons. We highlight a few key comparisons, both in terms of the assumptions made and algorithms furnished, between this work and prominent works in the high-dimensional error-in-variables literature, cf. [14], [10], [16], [17], [4], [5], [8], [9], [13].

Assumptions. In this work, we assume the underlying covariate matrix $X = [x_i^T] \in \mathbb{R}^{n \times p}$ is low rank, i.e., there is “sparsity” in the number of singular vectors needed to describe $X$. In comparison, prior works assume that the model parameter $\beta^*$ is sparse. These notions of sparsity are relatable. If $X$ is low-rank, then there exists a sparse $\tilde{\beta}$ that produces identical response variables, cf. [2]; meanwhile, if $\beta^*$ is sparse, then it is not hard to verify that there exists a low-rank $\tilde{X}$ that provides equivalent responses. The second key assumption of this work is that the spectra of $X$ is well-balanced. In comparison, the prior works assume that a type of restricted eigenvalue condition (see Definitions 1 and 2 in [14]) is satisfied for the empirical estimate of the covariance of $X$. We note that this estimate is typically constructed by “correcting” the covariance of $Z$ using knowledge of the latent noise covariance, e.g., $\sum_{i=1}^n (z_i z_i^T - \mathbb{E}[w_i w_i^T])$. Intuitively, both assumptions require that there is sufficient “information spread” across the rows and columns of covariates, i.e., an incoherence-like condition. See Section 3.5 in [2] for a detailed comparison of the well-balanced spectra assumption with respect to the restricted eigenvalue condition.

Algorithms. Notably, the algorithms furnished in prior works explicitly utilize knowledge of the noise covariance – or require the existence of a data-driven estimator for it, which can be too costly or simply infeasible, cf. [9] – to recover the sparse latent model parameter with respect to the $\ell_2$-error, i.e., a guarantee of the form $\|\tilde{\beta} - \beta^*\|_2$. PCR, on the other hand, is noise agnostic. More formally, the first step in PCR, which finds a low-rank approximation of $Z$, implicitly de-noises the covariates without utilizing knowledge of the noise distribution. The problem of noisy and partially observed covariates resurfaces in the context of out-of-sample predictions. More specifically, previous algorithms are not designed to de-noise out-of-sample covariates; thus, even with knowledge of the exact $\beta^*$, these works cannot provide generalization error bounds. In contrast, we provide a natural approach to handle these settings (see Section 3), which enables PCR to provably generalize.

1.3. PCR literature. PCR as a method was introduced in [12]. Despite the ubiquity of PCR in practice, the formal literature on PCR is surprisingly sparse. Notable works include [3, 6, 1, 2]. In particular, [1, 2] present finite-sample analyses for the prediction error (but
not parameter estimation error) of PCR in the high-dimensional error-in-variables setting. Specifically, in the transductive learning setting, they establish that PCR’s out-of-sample prediction error decays as $O(1/\sqrt{n})$. In such a scenario, both the in-sample (training) and out-of-sample (testing) covariates are accessible upfront. As a result, they can be simultaneously de-noised, after which only the de-noised training covariates and the associated responses are used to learn a model. In contrast, this work considers the classical supervised learning setup, where testing covariates are not revealed during training. Thus, the testing covariates must be de-noised separately, after which the linear model learnt in the training phase is applied to estimate the test responses. We further remark that [1, 2] make standard distributional assumptions on the generating process for the data, which allows them to leverage the techniques of Rademacher complexity analysis to establish their prediction error bounds.

We summarize a list of key points of comparison between this paper and notable works in both the PCR and error-in-variable literature in Table 1.

### Table 1

Comparison with a few notable works in the high-dimensional ($n \ll p$) error-in-variables regression literature under the specialized setting where the underlying covariates have well-balanced spectra (Assumption 4.1).

| Literature | Key Assumptions | Knowledge of noise distribution | Parameter estimation $1/n$ | Out-of-sample prediction error $1/\sqrt{n}$ |
|------------|-----------------|--------------------------------|---------------------------|---------------------------------------------|
| [14, 10, 17] | sparsity restricted eigenvalue cond. | Yes | $1/n$ | – |
| PCR [1, 2] | low-rank well-balanced spectra | No | – | $1/\sqrt{n}$ |
| This work | low-rank well-balanced spectra | No | $1/n$ | $1/n$ (Cor. 4.1) (Cor. 4.2) |

1.4. **Organization.** The remainder of this paper is organized as follows. We begin by formally describing our problem setup in Section 2, which includes our modeling assumptions and objectives. Next, we describe the PCR algorithm in Section 3, followed by its parameter estimation and out-of-sample prediction error bounds in Section 4. To reinforce our theoretical findings, we provide illustrative simulations in Section 5. In Sections 6 and 7, we prove Theorems 4.1 and 4.2, respectively. We conclude and discuss important future directions of research in Section 8. Lastly, we relegate standard concentration results used for our analyses to Appendix A.

1.5. **Notation.** For any matrix $A \in \mathbb{R}^{a \times b}$, we denote its operator (spectral), Frobenius, and max element-wise norms as $\|A\|_2$, $\|A\|_F$, and $\|A\|_{\max}$, respectively. For any vector $v \in \mathbb{R}^a$, let $\|v\|_p$ denote its $\ell_p$-norm. If $v$ is a random variable, we define its sub-gaussian (Orlicz) norm as $\|v\|_{\psi_2}$. Let $\circ$ denote component-wise multiplication and let $\otimes$ denote the outer product. For any two numbers $a, b \in \mathbb{R}$, we use $a \wedge b$ to denote $\min(a, b)$ and $a \vee b$ to denote $\max(a, b)$. Further, let $[a] = \{1, \ldots, a\}$ for any integer $a$.

2. **Problem Setup.** In this section, we provide a precise description of our problem, including our observations, assumptions, and objectives.

2.1. **Observation model.** As described in Section 1, we have access to $n$ labeled observations $\{(y_i, z_i) : i \leq n\}$, which we will refer to as our in-sample (training) data; recall that $x_i$ corresponds to the latent covariate with respect to $z_i$. Collectively, we assume (1) and (2) are satisfied. In addition, we observe $m \geq 1$ unlabeled out-of-sample (testing) covariates; for $i \in \{n + 1, \ldots, n + m\}$, we only observe the noisy covariates $z_i$, which again correspond to the latent covariates $x_i$, but do not have access to the associated response variables $y_i$. 
Throughout, let $X = [x_i^T : i \leq n] \in \mathbb{R}^{n \times p}$ and $X' = [x_i^T : i > n] \in \mathbb{R}^{m \times p}$ represent the underlying training and testing covariates, respectively. Similarly, let $Z = [z_i^T : i \leq n] \in \mathbb{R}^{n \times p}$ and $Z' = [z_i^T : i > n] \in \mathbb{R}^{m \times p}$ represent their observed noisy and sparse counterparts.

2.2. Modeling assumptions. We make the following assumptions.

**Assumption 2.1** (Bounded). $\|X\|_{\text{max}} \leq 1$, $\|X'\|_{\text{max}} \leq 1$.

**Assumption 2.2** (Low-rank). $\text{rank}(X) = r \ll n \wedge p$.

**Assumption 2.3** (Response noise). $\{\epsilon_i : i \leq n\}$ are a sequence of independent mean zero subgaussian random variables with $\|\epsilon_i\|_{\psi_2} \leq \sigma$.

**Assumption 2.4** (Covariate noise). $\{w_i : i \leq n + m\}$ are a sequence of independent mean zero subgaussian random vectors with $\|w_i\|_{\psi_2} \leq K$ and $\mathbb{E}[w_i \otimes w_i] \leq \gamma^2$. Further, $\pi_i$ is a vector of independent Bernoulli variables with parameter $\rho \in (0, 1]$.

**Assumption 2.5** (Subspace inclusion). The rowspace of $X'$ is contained within that of $X$, i.e., $\text{rowspan}(X') \subseteq \text{rowspan}(X)$.

2.3. Goals. There are two primary goals: (1) identify a well-defined model parameter $\beta^*$ from the labeled training data, and (2) estimate the out-of-sample responses $\{\langle x_i, \beta^* \rangle : i > n\}$ using the learned model.

3. Principal Component Regression (PCR). We describe the PCR algorithm as introduced in [12], with a variation to handle missing data. To that end, let $\hat{\rho}$ denote the fraction of observed entries in $Z$. We define $\hat{Z} = (1/\hat{\rho}) Z = \sum_{i=1}^{n+m} s_i \hat{u}_i \otimes \hat{v}_i$, where $s_i \in \mathbb{R}$ are the singular values (arranged in decreasing order) and $\hat{u}_i \in \mathbb{R}^n, \hat{v}_i \in \mathbb{R}^p$ are the left and right singular vectors, respectively.

3.1. Parameter estimation. For a given parameter $k \in [n \wedge p]$, PCR estimates the model parameter as

$$\hat{\beta} = \left( \sum_{i=1}^k \frac{1}{\hat{s}_i} \hat{v}_i \otimes \hat{u}_i \right) \beta.$$

3.2. Out-of-sample prediction. Let $\hat{\rho}'$ denote the proportion of observed entries in $Z'$. As before, let $\hat{Z}' = (1/\hat{\rho}') Z' = \sum_{i=1}^{m+n} s'_i \hat{u}'_i \otimes \hat{v}'_i$, where $s'_i \in \mathbb{R}$ are the singular values (arranged in decreasing order) and $\hat{u}'_i \in \mathbb{R}^m, \hat{v}'_i \in \mathbb{R}^p$ are the left and right singular vectors, respectively. Given parameter $\ell \in [m \wedge p]$, let $\hat{Z}^\ell = \sum_{i=1}^{\ell} s'_i \hat{u}'_i \otimes \hat{v}'_i$, and define the test response estimates as $\hat{y}' = \hat{Z}^\ell \hat{\beta}$.

If the responses are known to belong to a bounded interval, say $[-b, b]$ for some $b > 0$, then the entries of $\hat{y}'$ are truncated as follows: for every $i > n$,

$$\hat{y}_i = \begin{cases} -b & \text{if } \hat{y}_i \leq -b, \\ \hat{y}_i & \text{if } -b < \hat{y}_i < b, \\ b & \text{if } b \leq \hat{y}_i. \end{cases}$$
3.3. Properties of PCR. We state some useful properties of PCR, which we will use extensively throughout this work. These are well-known results, discussed at length in Chapter 17 of [15] and Chapter 6.3 of [18].

**Property 3.1.** Let \( \hat{Z}^k = \sum_{i=1}^k \hat{s}_i \hat{u}_i \otimes \hat{v}_i \). Then \( \hat{\beta} \), as given in (3), also satisfies

1. \( \hat{\beta} \) is the unique solution of the following program:
   \[
   \text{minimize} \quad \|\beta\|_2 \quad \text{over} \quad \beta \in \mathbb{R}^p \\
   \text{such that} \quad \beta \in \arg\min_{\beta' \in \mathbb{R}^p} \|y - \hat{Z}^k \beta'\|_2^2.
   \]

2. \( \hat{\beta} \in \text{rowspan}(\hat{Z}^k) \).

3.4. Choosing \( k \). In general, the correct number of principal components \( k \) to use is not known a priori. However, under reasonable signal-to-noise scenarios, Weyl’s inequality implies that a “sharp” threshold or gap should exist between the top \( r \) singular values and remaining singular values of the observed data \( \hat{Z} \). This gives rise to a natural “elbow” point and suggests choosing a threshold within this gap. Another standard approach is to use a “universal” thresholding scheme that preserves singular values above a precomputed threshold ([7] and [11]). Data-driven approaches developed around cross-validation can also be employed.

4. Main Results. We state PCR’s parameter estimation and generalization properties in this section. For the remainder of the paper, \( C(K, \gamma) > 0 \) will denote any constant that depends only on \( K \) and \( \gamma \), and \( C, c > 0 \) will denote absolute constants. The values of \( C(K, \gamma), C, \) and \( c \) may change from line to line or even within a line.

4.1. Parameter estimation. Since we work within the high-dimensional framework, our first objective in recovering the underlying parameter is ill-posed without additional structure. Consequently, among all feasible models, we consider the unique model that satisfies (1) with minimum \( \ell_2 \)-norm, i.e., \( \beta^* \in \text{rowspan}(X) \); this follows since every element in the column space of a matrix is associated with a unique element in its row space coupled with any element in its null space. Thus, for the purposes of prediction, it suffices to consider this particular \( \beta^* \) (see [15], [18] for details). Also, recall from Property 3.1 that PCR enforces \( \hat{\beta} \in \text{rowspan}(\hat{Z}^k) \). Hence, if \( k = r \) and the rowspace of \( \hat{Z}^r \) is “close” to the rowspace of \( X \), then this suggests \( \hat{\beta} \approx \beta^* \). We formalize this intuition through Theorem 4.1.

**Theorem 4.1.** Let Assumptions 2.1, 2.2, 2.3, and 2.4 hold. Consider \( \beta^* \in \text{rowspan}(X) \) and PCR with \( k = r \). Let \( \rho \geq c \log^2 np \). Then with probability at least \( 1 - O(1/(np)^{10}) \),

\[
\|\hat{\beta} - \beta^*\|_2^2 \leq C(K, \gamma) \left( \frac{n + p}{\rho^2 s_r^2} \|\beta^*\|_2^2 + \frac{(n + p)(n + \sqrt{n \log np})}{\rho^4 s_r^2 s_r^2} \|\beta^*\|_1^2 \\
+ \frac{r + \sqrt{r \log np}}{\rho^2 s_r^2} \|\beta^*\|_1^2 + \frac{\log np}{\rho ps_r^2} \|\beta^*\|_1^2 \right) \\
+ C \sigma^2 (r + \log np) + \sigma \sqrt{n \log np} \|\beta^*\|_1,
\]

(4)

where

\[
|\hat{s}_r - s_r| \leq C(K, \gamma) \frac{\sqrt{n + \sqrt{p}}}{\rho} + C \frac{\log np}{\sqrt{np}} s_r,
\]

(5)

and \( s_r, \hat{s}_r \) denote the \( r \)-th singular values of \( X \) and \( \hat{Z} \), respectively.
We specialize the above result under specific conditions on the spectral characteristics of $X$. To that end, consider the following assumption.

**ASSUMPTION 4.1 (Balanced spectra: training covariates).** The $r$ nonzero singular values $s_i$ of $X$ satisfy $s_i = \Theta(\sqrt{np/\rho})$.

**COROLLARY 4.1.** Let the setup of Theorem 4.1 and Assumption 4.1 hold, and let $\rho \geq C(K, \gamma) \sqrt{\log np}/\sqrt{np}$. Then with probability at least $1 - O(1/(np)^{10})$,

$$\|\hat{\beta} - \beta^*\|_2^2 \leq C(K, \gamma) \frac{r^2 \log np}{\rho^4 (n \wedge p)} \|\beta^*\|_2^2 + C \frac{\sigma^2 (r^2 + r \log np)}{np} + C \frac{\sigma r \sqrt{\log np}}{\sqrt{np}} \|\beta^*\|_2.$$

**PROOF.** By Assumption 4.1, we have $s_r = \Theta(\sqrt{np}/\sqrt{\rho})$. We also have $\rho \geq C(K, \gamma) \sqrt{\log np}/\sqrt{np}$. Therefore,

$$C(K, \gamma) \frac{\sqrt{n} + \sqrt{p}}{\rho} \leq \frac{\sqrt{np}}{\sqrt{r} \log np}.$$  

It then follows from (5) that $\hat{s}_r = \Theta(s_r)$. We also have $\|\beta^*\|_1^2 \leq p \|\beta^*\|_2^2$ by standard norm inequality. Using these in (4) completes the proof.

Corollary 4.1 implies that if $n \ll p$, then the parameter estimation error scales as $O(\frac{r^2 \log np}{\rho^4 (n \wedge p)} \|\beta^*\|_2^2)$. Therefore, ignoring log factors on $p$ as well as dependencies on $\rho$ and $r$, the error decays as $1/n$, which matches the best known rate (albeit, with respect to a sparse parameter).

4.2. Out-of-sample prediction error. Next, we bound PCR’s out-of-sample prediction error in the presence of corrupted unseen data, defined as

$$\text{MSE}_{\text{test}} := \frac{1}{m} \sum_{i=1}^m (\hat{y}_{n+i} - \langle x_{n+i}, \beta^* \rangle)^2.$$

We define some more useful quantities. Let $s_i, s'_i \in \mathbb{R}$ be the $\ell$-th singular values of $X$ and $X'$, respectively. Recall from Section 3 that $\hat{s}_i, \hat{s}'_i$ are the $\ell$-th singular values of $\tilde{Z}$ and $\tilde{Z}'$, respectively. Further, let

\[
\Delta_1 = \frac{C(K, \gamma)}{\rho^2} \left( 1 + \frac{p}{m} + \left( \frac{s'_1}{s_r} \right) \frac{2}{m} \frac{n \vee p}{m} \right) \|\hat{\beta} - \beta^*\|_2^2 \\
+ \frac{C(K, \gamma) \log((n \wedge m)p) \|\beta^*\|_2^2}{\rho^2} \left( \frac{s'_1}{\hat{s}_r} \right) \left( \frac{n(n \vee p)}{\rho^2 s_r^2} + r \right) + \frac{m(m \vee p)}{\rho^2 (s'_r)^2} \\
\Delta_2 = C \left( \frac{s'_1}{\hat{s}_r} \right) \left( \frac{\sigma^2 (r + \log np)}{m} + \sigma \sqrt{n \log np} \|\beta^*\|_1 \right) \\
\Delta_3 = C \left( \frac{s'_1}{\hat{s}_r} \right) \frac{2 \sigma^2 r}{m} + C b^2 \left( 1/(np)^{10} + 1/(mp)^{10} \right),
\]

where the bounds on $\|\hat{\beta} - \beta^*\|_2$ and $\hat{s}_r$ are given in (4) and (5), respectively. In Theorem 4.2, we bound $\text{MSE}_{\text{test}}$ both in probability and in expectation with respect to these quantities.
**Theorem 4.2.** Let the setup of Theorem 4.1 and Assumption 2.5 hold, and let $\rho \geq c \log^2 mp$. Then, with probability at least $1 - O(1/(np)^{10}) - O(1/(mp)^{10})$,

$$\text{MSE}_{\text{test}} \leq \Delta_1 + \Delta_2.$$  \hspace{1cm} (6)

Further, if $\{\langle x_i, \beta^* \rangle \in [-b, b] : i > n\}$ and $\hat{y}$ is appropriately truncated, then

$$\mathbb{E}[\text{MSE}_{\text{test}}] \leq \Delta_1 + \Delta_3.$$  \hspace{1cm} (7)

As before, we specialize the above result under specific conditions on the spectral characteristics of $X'$.

**Assumption 4.2 (Balanced spectra: testing covariates).** The $r'$ nonzero singular values $s'_i$ of $X'$ satisfy $s'_i = \Theta(\sqrt{mp/r'})$.

**Corollary 4.2.** Let the setup of Corollary 4.1 and Theorem 4.2 hold. Further, let Assumption 4.2 hold. Then,

$$\Delta_1 \leq C(K, \gamma) \left( \left( \frac{r}{\rho^2} + \frac{r \rho p}{\rho^2 (n \wedge m)} \right) \| \hat{\beta} - \beta^* \|_2^2 + \frac{r^2 \log((n \vee m)p)}{\rho^4 n} \| \beta^* \|_1^2 \right)$$

$$\Delta_2 \leq C \left( \frac{\sigma^2 (r^2 + r \log np)}{n} + \frac{\sigma r \sqrt{\log np}}{\sqrt{n}} \| \beta^* \|_1 \right)$$

$$\Delta_3 \leq C \left( \frac{\sigma^2 r}{n} + \frac{b^2}{((n \wedge m)p)^{10}} \right).$$

**Proof.** From Corollary 4.1, we have that $\hat{s}_r = \Theta(s_r) = \Theta(\sqrt{mp/r'})$. Also, Assumption 4.2 implies $s'_r = \Theta(s'_1) = \Theta(\sqrt{mp/r'})$. Plugging these into the definitions of $\Delta_1, \Delta_2, \Delta_3$, and simplifying completes the proof.

Notably, Theorem 4.2 and Corollary 4.2 do not require any distributional assumptions relating the in- and out-of-sample covariates, but rather rely on the linear algebraic condition given by Assumption 2.5. Because we consider $\beta^* \in \text{rowspan}(X)$, we require the row space of $X'$ to lie within that of $X$. Intuitively, this condition restricts the out-of-sample covariates to be at most as “rich” or “complex” as the in-sample covariates used for learning.

For the following discussion, we only consider the scaling with respect to $n, m, p$, but ignore log factors. Now, recall that Corollary 4.1 implies $\| \hat{\beta} - \beta^* \|_2^2 = O(1/(n \wedge p))$. Hence, Corollary 4.2 implies that if $p \ll n(n \wedge m)$, then the out-of-sample prediction error vanishes to zero both in probability and in expectation. If we make the additional assumption that $m, p = \Theta(n)$, then Corollary 4.2 implies that the error scales as $O(1/n)$ in expectation. This improves upon the best known rate of $O(1/\sqrt{n})$, established in [1, 2].

5. Simulations. In this section, we present illustrative simulations to support our theoretical results.

5.1. Parameter estimation. The purpose of this simulation is to demonstrate that PCR does indeed identify the unique linear model of minimum $\ell_2$-norm.
(a) $\ell_2$-norm error of $\hat{\beta}$ with respect to the min. $\ell_2$-norm solution of (1), i.e., $\beta^\ast$.

(b) $\ell_2$-norm error of $\hat{\beta}$ with respect to a random solution of (1).

Fig 1: Plots of $\ell_2$-norm errors, i.e., $\|\hat{\beta} - \beta^\ast\|_2$ in (1a) and $\|\hat{\beta} - \beta\|_2$ in (1b), versus the rescaled sample size $n/(r^2 \log p)$ after running PCR with rank $r = p^{1/3}$. As predicted by Theorem 4.1, the curves for different values of $p$ under (1a) roughly align and decay to zero as $n$ increases.

5.1.1. Generative model. We construct covariates $X \in \mathbb{R}^{n \times p}$ via the classical probabilistic PCA model, cf. [19]. That is, we first generate $X_r \in \mathbb{R}^{n \times r}$ by independently sampling each entry from a standard normal distribution. Then, we sample a transformation matrix $Q \in \mathbb{R}^{r \times p}$, where each entry is uniformly and independently sampled from $\{-1/\sqrt{r}, 1/\sqrt{r}\}$. The final matrix then takes the form $X = X_r Q$. We choose $\text{rank}(X) = r = p^{1/3}$, where $p \in \{128, 256, 512\}$.

Next, we generate $\beta \in \mathbb{R}^p$ by first sampling from a multivariate standard normal vector with independent entries and then scale each coordinate by $5$. The noiseless response vector $a \in \mathbb{R}^n$ is defined to be $a = X \beta$. Finally, as motivated by Property 3.1, the minimum $\ell_2$-norm model of interest, $\beta^\ast$, is computed as $\beta^\ast = X^{\dagger} a$, where $X^{\dagger}$ denotes the pseudo-inverse of $X$.

We consider an additive noise model. Specifically, the entries of $\epsilon \in \mathbb{R}^n$ are sampled i.i.d. from a normal distribution with mean $0$ and variance $\sigma^2 = 0.2$. The entries of $W = [w_i^T] \in \mathbb{R}^{n \times p}$ are sampled in an identical fashion. We then define our observed response vector as $y = a + \epsilon$ and observed covariate matrix as $Z = X + W$. For simplicity, we do not mask any of the entries.

5.1.2. Results. Using the observations $(y, Z)$, we perform PCR as in Section 3.1 to yield $\hat{\beta}$. To show that PCR can accurately recover $\beta^\ast$, we compute the $\ell_2$-norm parameter estimation error, or root-mean-squared-error (RMSE), with respect to $\beta^\ast$ and $\hat{\beta}$ in Figures 1a and 1b, respectively. As suggested by Figure 1a, the RMSE with respect to $\beta^\ast$ roughly aligns for different values of $p$, after rescaling the sample size as $n/(r^2 \log p)$, and decays to zero as the sample size increases; this is predicted by Theorem 4.1. On the other hand, Figure 1b shows that the RMSE with respect to $\hat{\beta}$ stays roughly constant across different values of $p$. Therefore, as established in [1], PCR performs implicit regularization by not only de-noising the observed covariates, but also finding the minimum-norm solution.

5.2. Out-of-sample prediction: PCR vs. Ordinary Least Squares. The purpose of this simulation is to demonstrate the benefit of the implicit de-noising effect of PCR vs. ordinary least squares (OLS).
Fig 2: MSE plot of $\hat{y}_{\text{pcr}}$ (blue) versus $\hat{y}_{\text{ols}}$ (orange) as we increase the level of covariate and response noises. While PCR’s error scales gracefully with the level of noise, OLS suffers large amounts of bias, even in the presence of small amounts of measurement error.

5.2.1. Generative model. For each experiment, we let $n = m = p = 1000$. We generate training and testing covariates $X, X' \in \mathbb{R}^{1000 \times 1000}$, respectively, with $\text{rank}(X) = \text{rank}(X') = 10$ and $\text{rowspan}(X') \subseteq \text{rowspan}(X)$, i.e., Assumption 2.5 holds. To do so, we sample $U, U', V \in \mathbb{R}^{1000 \times 10}$ by independently sampling each entry from a standard normal distribution. Then, we define $X = UV^T$ and $X' = U'V^T$.

We then generate $\beta \in \mathbb{R}^{1000}$ as in Section 5.1, and use it to produce $a = X\beta$ and $a' = X'\beta$. Similarly, we generate the response noise $\varepsilon \in \mathbb{R}^{1000}$ and covariate noises $W, W' \in \mathbb{R}^{1000 \times 10}$ by independently sampling each entry from a normal distribution with mean zero and variance $\sigma^2$, where $\sigma^2 \in \{0.1, 0.2, \ldots, 1.0\}$. Again, for simplicity, we do not mask any of the entries. We then define our observed response as $y = a + \varepsilon$, and the observed training and testing covariates as $Z = X + W$ and $Z' = X' + W'$, respectively.

5.2.2. Results. Using the observations $(y, Z, Z')$, we perform PCR as in Section 3.2 to produce $\hat{y}_{\text{pcr}} \in \mathbb{R}^{1000}$. The OLS out-of-sample estimates are produced using the same algorithm as in Section 3.2 without the singular value thresholding step on either $Z$ or $Z'$, i.e., we do not de-noise the training nor testing covariates. The estimates produced from OLS are defined as $\hat{y}_{\text{ols}} \in \mathbb{R}^{1000}$. In both PCR and OLS, we do not truncate the estimated entries. For any estimate $\hat{y} \in \mathbb{R}^{1000}$, we define the out-of-sample mean squared error (MSE) as $(1/1000)\|\hat{y} - a'\|_2^2$. In Figure 2, as we vary the level of response and covariate noise $\sigma^2$, we plot the MSE of $\hat{y}_{\text{pcr}}$ versus that of $\hat{y}_{\text{ols}}$. The MSE of OLS is between three to four orders of magnitude larger than that of PCR across all noise levels. We remark that even when $\sigma^2 = 0.1$, the error of OLS is almost three orders of magnitude larger than PCR – this indicates the significant level of bias that is introduced even with minimal measurement error. In essence, this stresses the importance of de-noising the training and testing covariates via singular value thresholding.

5.3. Out-of-sample prediction: robustness of PCR to distribution shifts. The purpose of this simulation is to demonstrate that PCR can generalize even when the testing covariates are not only corrupted, but also sampled from a different distribution than the training covariates.

5.3.1. Generative model. Throughout, we let $n = m = p = 1000$. We generate the training covariates as in Section 5.2, i.e., $X = UV^T$, where the entries of $U, V$ are sampled independently from a standard normal distribution. Next, we generate four different out-of-sample covariates, defined as $X'_{N_1}, X'_{N_2}, X'_{U_1}, X'_{U_2}$ via the following procedure: We independently sample the entries of $U'_{N_1}$ from a standard normal distribution, and define
$X'_{N_1} = U'_{N_1} V^T$. We define $X'_{N_2} = U'_{N_2} V^T$ similarly with the entries of $U'_{N_2}$ sampled from a normal distribution with mean zero and variance 5. Next, we independently sample the entries of $U'_{U_1}$ from a uniform distribution with support $[-\sqrt{3}, \sqrt{3}]$, and define $X'_{U_1} = U'_{U_1} V^T$. We define $X'_{U_2} = U'_{U_2} V^T$ similarly with the entries of $U'_{U_2}$ sampled from a uniform distribution with support $[-\sqrt{15}, \sqrt{15}]$.

By construction, we note that the mean and variance of the entries in $X'_{U_1}$ match that of $X'_{N_1}$; an analogous relationship holds between $X'_{U_2}$ and $X'_{N_2}$. Further, while $X'_{N_1}$ follows the same distribution as that of $X$, we note that there is a clear distribution shift from $X$ to $X'_{U_1}, X'_{N_2}, X'_{U_2}$.

We proceed to generate $\beta$ as in Section 5.2. We then define $a'_{N_1} = \beta X'_{N_1}$, and define $a'_{N_2}, a'_{U_1}, a'_{U_2}$ analogously. Further, the response noise $\varepsilon$ and covariate noises $W, W'$ are constructed in the same fashion as described in Section 5.2, where the variance again follows $\sigma^2 \in \{0.1, 0.2, \ldots, 1.0\}$. We define the training responses as $y = X \beta + \varepsilon$ and observed training covariates as $Z = X + W$. The first set of observed testing covariates is defined as $Z'_{N_1} = X'_{N_1} + W'$, with analogous definitions for $Z'_{N_2}, Z'_{U_1}, Z'_{U_2}$.

5.3.2. Results. Using the observations $(y, Z, Z'_{N_1})$, we perform PCR to produce $\hat{y}'_{N_1}$. We produce $\hat{y}'_{N_2}, \hat{y}'_{U_1}, \hat{y}'_{U_2}$ analogously. We define MSE as in Section 5.2 with each estimate compared against its corresponding latent response, e.g., $\hat{y}'_{N_1}$ against $a'_{N_1}$. Figure 3 shows the MSE of $\hat{y}'_{N_1}, \hat{y}'_{U_1}, \hat{y}'_{U_2}$, and $\hat{y}'_{U_2}$ as we vary $\sigma^2$. Pleasingly, despite the changes in the data generating process of the out-of-sample responses we evaluate on, the MSE for all four experiments closely matches across all noise levels. This motivates Assumption 2.5 as the key requirement for generalization, at least for PCR, rather than distributional invariance between the training and testing covariates.

5.4. Out-of-sample prediction: subspace inclusion vs. distributional invariance. The purpose of this simulation is to further illustrate that subspace inclusion (Assumption 2.5) is the key structure that enables PCR to successfully generalize, and not necessarily distributional invariance between the training and testing covariates.

5.4.1. Generative model. As before, we let $n = m = p = 1000$. We continue to generate the training covariates as $X = U V^T$ following the procedure in Section 5.2. We now generate two different testing covariates. First, we generate $X'_{1} = U' V^T$, where the entries of $U'$ are independently sampled from a normal distribution with mean zero and variance 5. As such, it follows that Assumption 2.5 immediately holds between $X'_{1}$ and $X$, though they do...
Fig 4: Plots of PCR’s MSE under two situations: when Assumption 2.5 holds but distributional invariance is violated (blue), and when Assumption 2.5 is violated but distributional invariance holds (orange). Across varying levels of noise, the former condition achieves a much lower MSE.

not obey the same distribution. Next, we generate \( X'_2 = UV'^T \), where the entries of \( V' \) are independently sampled from a standard normal (just as in \( V \)). In doing so, we ensure that \( X'_2 \) and \( X \) follow the same distribution, though Assumption 2.5 no longer holds.

We generate \( \beta \) as in Section 5.2, and define \( a'_1 = X'_1 \beta \) and \( a'_2 = X'_2 \beta \). We also generate \( \varepsilon, W, \) and \( W' \) as in Section 5.2. In turn, we define the training data as \( y = X \beta + \varepsilon \) and \( Z = X + W \), and testing data as \( Z'_1 = X'_1 + W' \) and \( Z'_2 = X'_2 + W' \).

5.4.2. Results. We apply PCR under two scenarios. First, we apply PCR using \( (y, Z, Z'_1) \) to yield \( \hat{y}'_1 \), and once again using \( (y, Z, Z'_2) \) to yield \( \hat{y}'_2 \). We define MSE as in Section 5.2 with each estimate compared against its corresponding latent response, e.g., \( \hat{y}'_1 \) against \( a'_1 \).

Figure 4 shows the MSE of \( \hat{y}'_1 \) and \( \hat{y}'_2 \) across varying levels of noise. As we can see, when Assumption 2.5 holds yet distributional invariance is violated, the corresponding MSE of \( \hat{y}'_1 \) is almost three orders of magnitude smaller than that of \( \hat{y}'_2 \), where Assumption 2.5 is violated but distributional invariance holds. This reinforces that the key structure required for PCR (and possibly other linear estimators) to generalize is Assumption 2.5, and not necessarily distributional invariance, as is typically assumed in the statistical learning literature.

6. Proof of Theorem 4.1. We start with some useful notations. Let \( y = X \beta^* + \varepsilon \) be the vector notation of (1) with \( y = [y_i : i \leq n] \in \mathbb{R}^n, \varepsilon = [\varepsilon_i : i \leq n] \in \mathbb{R}^n \). Throughout, let \( X = USV^T \). Recall that the SVD of \( \hat{Z} = 1/\hat{\rho}Z = \hat{U}\hat{S}\hat{V}^T \). Its truncation using the top \( k \) singular components is denoted as \( \hat{Z}_k = \hat{U}_k\hat{S}_k\hat{V}_k^T \).

Further, we will often use the following bound: for any \( A \in \mathbb{R}^{a \times b}, v \in \mathbb{R}^b \),

\[
\|Av\|_2 = \left\| \sum_{j=1}^b A_{:j}v_j \right\|_2 \leq \left( \max_{j \leq b} \|A_{:j}\|_2 \right) \left( \sum_{j=1}^b |v_j| \right) = \|A\|_{2,\infty} \|v\|_1 ,
\]

where \( \|A\|_{2,\infty} = \max_j \|A_{:j}\|_2 \) with \( A_{:j} \) representing the \( j \)-th column of \( A \).

As discussed in Section 4.1, we shall denote \( \beta^* \) as the unique minimum \( \ell_2 \)-norm model parameter satisfying (1); equivalently, this can be formulated as \( \beta^* \in \text{rowspan}(X) \). As a result, it follows that

\[
V_\perp^T \beta^* = 0 ,
\]

where \( V_\perp \) represents a matrix of orthonormal basis vectors that span the nullspace of \( X \).
Similarly, let $\hat{V}_{k,\perp} \in \mathbb{R}^{p \times (p-k)}$ be a matrix of orthonormal basis vectors that span the nullspace of $\tilde{Z}^k$; thus, $\hat{V}_{k,\perp}$ is orthogonal to $\hat{V}_k$. Then,

\[
\|\hat{\beta} - \beta^*\|_2^2 = \|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*) + \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T (\hat{\beta} - \beta^*)\|_2^2 \\
= \|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2 + \|\hat{V}_{k,\perp} \hat{V}_{k,\perp}^T (\hat{\beta} - \beta^*)\|_2^2 \\
= \|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2 + \|\hat{V}_{k,\perp} \hat{V}_{k,\perp}^T \beta^*\|_2^2.
\]

(10)

Note that in the last equality we have used Property 3.1, which states that $\hat{V}_{k,\perp}^T \beta = 0$. Next, we bound the two terms in (10).

**Bounding** $\|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2$. To begin, note that

\[
\|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2 = \|\hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2,
\]

since $\hat{V}_k$ is an isometry. Next, consider

\[
\|\tilde{Z}^k (\hat{\beta} - \beta^*)\|_2^2 \leq 2\|\tilde{Z}^k \hat{\beta} - X \beta^*\|_2^2 + 2\|X \beta^* - \tilde{Z}^k \beta^*\|_2^2 \\
\leq 2\|\tilde{Z}^k \hat{\beta} - X \beta^*\|_2^2 + 2\|X - \tilde{Z}^k\|_{2,\infty}\|\beta^*\|_1^2,
\]

where we used (8). Recall that $\tilde{Z}^k = \tilde{U}_k \tilde{S}_k \tilde{V}_k^T$. Therefore,

\[
\|\tilde{Z}^k (\hat{\beta} - \beta^*)\|_2^2 = (\hat{\beta} - \beta^*)^T \tilde{V}_k \tilde{S}_k^2 \tilde{V}_k^T (\hat{\beta} - \beta^*) \\
= (\hat{V}_k^T (\hat{\beta} - \beta^*))^T \tilde{S}_k^2 (\hat{V}_k^T (\hat{\beta} - \beta^*)) \\
\geq s_k^2 \|\hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2.
\]

Therefore using (11), we conclude that

\[
\|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2 \leq \frac{2}{s_k^2} \left( \|\tilde{Z}^k \hat{\beta} - X \beta^*\|_2^2 + \|X - \tilde{Z}^k\|_{2,\infty}\|\beta^*\|_1^2 \right).
\]

Next, we bound $\|\tilde{Z}^k \hat{\beta} - X \beta^*\|_2$.

\[
\|\tilde{Z}^k \hat{\beta} - y\|_2^2 = \|\tilde{Z}^k \hat{\beta} - X \beta^* - \varepsilon\|_2^2 \\
= \|\tilde{Z}^k \hat{\beta} - X \beta^*\|_2^2 + \|\varepsilon\|_2^2 - 2\langle \tilde{Z}^k \hat{\beta} - X \beta^*, \varepsilon \rangle.
\]

(13)

By Property 3.1 we have,

\[
\|\tilde{Z}^k \hat{\beta} - y\|_2^2 \leq \|\tilde{Z}^k \beta^* - y\|_2^2 = \|(\tilde{Z}^k - X) \beta^* - \varepsilon\|_2^2 \\
= \|(\tilde{Z}^k - X) \beta^*\|_2^2 + \|\varepsilon\|_2^2 - 2\langle (\tilde{Z}^k - X) \beta^*, \varepsilon \rangle.
\]

(14)

From (13) and (14), we have

\[
\|\tilde{Z}^k \hat{\beta} - X \beta^*\|_2^2 \leq \|(\tilde{Z}^k - X) \beta^*\|_2^2 + 2\langle (\tilde{Z}^k - X) \hat{\beta} - \beta^*, \varepsilon \rangle \\
\leq \|X - \tilde{Z}^k\|_{2,\infty}\|\beta^*\|_1^2 + 2\langle (\tilde{Z}^k - X) \beta^* - \varepsilon \rangle,
\]

where we used (8). From (12) and (15), we conclude that

\[
\|\hat{V}_k \hat{V}_k^T (\hat{\beta} - \beta^*)\|_2^2 \leq \frac{4}{s_k^2} \left( \|X - \tilde{Z}^k\|_{2,\infty}\|\beta^*\|_1^2 + \langle (\tilde{Z}^k - X) \beta^* - \varepsilon \rangle \right).
\]

(16)
Bounding $\| \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T \beta^* \|_2^2$. Consider

$$
\| \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T \beta^* \|_2 = \| (\hat{V}_{k,\perp} \hat{V}_{k,\perp} - V_{\perp} V_{\perp}^T) \beta^* + V_{\perp} V_{\perp}^T \beta^* \|_2 \\
= (a) \| (\hat{V}_{k,\perp} \hat{V}_{k,\perp} - V_{\perp} V_{\perp}^T) \beta^* \|_2 \\
\leq \| \hat{V}_{k,\perp} \hat{V}_{k,\perp} - V_{\perp} V_{\perp}^T \|_2 \| \beta^* \|_2,
$$

where (a) follows from $V_{\perp}^T \beta^* = 0$ due to (9). Then,

$$
\hat{V}_{k,\perp} \hat{V}_{k,\perp}^T - V_{\perp} V_{\perp}^T = (I - V_{\perp} V_{\perp}^T) - (I - \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T)
$$

(18)

From (17) and (18), it follows that

$$
\| \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T \|_2 \leq \| \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T \|_2 \| \beta^* \|_2^2.
$$

Bringing together (10), (16), and (19). Collectively, we obtain

$$
\| \beta - \beta^* \|_2^2 \leq \| \hat{V} \hat{V}^T - \hat{V}_{k,\perp} \hat{V}_{k,\perp}^T \|_2 \| \beta^* \|_2^2 + \frac{4}{s_k^2} \left( \| X - \hat{Z}^k \|_{2,\infty}^2 \| \beta^* \|_1^2 + \langle \hat{Z}^k (\hat{\beta} - \beta^*), \varepsilon \rangle \right).
$$

Key lemmas. We state the key Lemmas bounding each of the terms on the right hand side of (20). This will help us conclude the proof of Theorem 4.1. The proofs of these Lemmas are presented in Sections 6.1, 6.2, 6.3, 6.4.

**Lemma 6.1.** Consider the setup of Theorem 4.1, and PCR with parameter $k = r = \text{rank}(X)$. Then, for any $t > 0$, the following holds with probability at least $1 - \exp(-t^2)$:

$$
\| UU^T - \hat{U}_r \hat{U}_r^T \|_2 \leq C(K, \gamma) \frac{\sqrt{n} + \sqrt{p} + t}{\rho s_r},
$$

$$
\| \hat{V} \hat{V}^T - \hat{V}_r \hat{V}_r^T \|_2 \leq C(K, \gamma) \frac{\sqrt{n} + \sqrt{p} + t}{\rho s_r}.
$$

Here, $s_r > 0$ represents the $r$-th singular value of $X$.

**Lemma 6.2.** Consider PCR with parameter $k = r$ and $\rho \geq c \frac{\log^2 np}{np}$. Then with probability at least $1 - O(1/(np)^{10})$,

$$
\| X - \hat{Z}^r \|_{2,\infty}^2 \leq C(K, \gamma) \left( \frac{(n + p)(n + \sqrt{n} \log np)}{\rho^2 s_r^2} + r + \sqrt{r \log np} \right) + C \frac{\log np}{\rho p}.
$$

**Lemma 6.3.** If $\rho \geq c \frac{\log^2 np}{np}$, then for any $k \in [n \wedge p]$, we have with probability at least $1 - O(1/(np)^{10})$,

$$
| \hat{s}_k - s_k | \leq C(K, \gamma) \frac{\sqrt{n} + \sqrt{p}}{\rho} + C \frac{\sqrt{\log np}}{\sqrt{\rho np}} s_k.
$$

**Lemma 6.4.** Given $\hat{Z}^r$, the following holds with probability at least $1 - O(1/(np)^{10})$ with respect to randomness in $\varepsilon$:

$$
\langle \hat{Z}^r (\hat{\beta} - \beta^*), \varepsilon \rangle \leq \sigma^2 r + C \sigma \sqrt{\log np} (\sigma \sqrt{r} + \sigma \sqrt{\log np} + \| \beta^* \|_1 (\sqrt{n} + \| \hat{Z}^r - X \|_{2,\infty})) .
$$
Completing the proof of Theorem 4.1. Using Lemma 6.4, the following holds with probability at least $1 - O(1/(np)^{10})$:

$$\|X - \tilde{Z}^r\|_{2,\infty}^2 \|\beta^*\|_1^2 + \langle \tilde{Z}^r(\hat{\beta} - \beta^*), \varepsilon \rangle$$

$$\leq \|X - \tilde{Z}^r\|_{2,\infty}^2 \|\beta^*\|_1^2 + C\sigma \sqrt{\log np}\|X - \tilde{Z}^r\|_{2,\infty} \|\beta^*\|_1 + C\sigma^2 \log np$$

$$+ C\sigma \sqrt{\log np}(\sqrt{n} \|\beta^*\|_1 + s\sigma \sqrt{r}) + \sigma^2 r$$

$$\leq C(\|X - \tilde{Z}^r\|_{2,\infty} \|\beta^*\|_1 + \sigma \sqrt{\log np})^2 + C\sigma \sqrt{\log np}(\sqrt{n} \|\beta^*\|_1 + \sigma \sqrt{r}) + \sigma^2 r$$

$$\leq C\|X - \tilde{Z}^r\|_{2,\infty} \|\beta^*\|_1^2 + C\sigma^2 (\log np + r) + C\sigma \sqrt{n \log np} \|\beta^*\|_1.$$ (21)

Using (20) and (21), we have with probability at least $1 - O(1/(np)^{10})$,

$$\|\hat{\beta} - \beta^*\|_2^2 \leq \|VV^T - \hat{V}_k \hat{V}_k^T\|_2^2 \|\beta^*\|_2^2$$

$$+ C\|X - \tilde{Z}^r\|_{2,\infty} \|\beta^*\|_1^2 + \sigma^2 (\log np + r) + \sigma \sqrt{n \log np} \|\beta^*\|_1.$$ (22)

Using Lemma 6.1 in (22), we have with probability at least $1 - O(1/(np)^{10})$,

$$\|\hat{\beta} - \beta^*\|_2^2 \leq C(K, \gamma)^n + \frac{p}{\beta^2 s_r^2} \|\beta^*\|_2^2 + C\|X - \tilde{Z}^r\|_{2,\infty} \|\beta^*\|_1^2$$

$$+ C\frac{\sigma^2 (\log np + r) + \sigma \sqrt{n \log np} \|\beta^*\|_1}{s^2}.$$ (23)

Incorporating Lemmas 6.2 and 6.3 into (23) concludes the proof.

6.1. Proof of Lemma 6.1. Recall that $U, V$ denote the left and right singular vectors of $X$ (equivalently, $\rho X$), respectively; meanwhile, $\hat{U}_k, \hat{V}_k$ denote the top $k$ left and right singular vectors of $\hat{Z}$ (equivalently, $Z$), respectively. Further, observe that $E[Z] = \rho X$ and let $\tilde{W} = Z - \rho X$. To arrive at our result, we recall Wedin’s Theorem [21].

**Theorem 6.1 (Welin’s Theorem).** Given $A, B \in \mathbb{R}^{n \times p}$, let $A = USV^T$ and $B = \hat{U} \hat{S} \hat{V}^T$ be their respective SVDs. Let $U_k, V_k$ (respectively, $\hat{U}_k, \hat{V}_k$) correspond to the truncation of $U, V$ (respectively, $\hat{U}, \hat{V}$) that retains the columns corresponding to the top $k$ singular values of $A$ (respectively, $B$). Let $s_k$ denote the $k$-th singular value of $A$. Then,

$$\max \left( \|UU_k^T - \hat{U}_k \hat{U}_k^T\|_2, \|VV_k^T - \hat{V}_k \hat{V}_k^T\|_2 \right) \leq \frac{2\|A - B\|_2}{s_k - s_{k+1}}.$$ (24)

Using Theorem 6.1 for $k = r$, it follows that

$$\max \left( \|UU^T - \hat{U}_r \hat{U}_r^T\|_2, \|VV^T - \hat{V}_r \hat{V}_r^T\|_2 \right) \leq \frac{2\|\tilde{W}\|_2}{\rho s_r},$$

where $s_r$ is the smallest nonzero singular value of $X$. Next, we obtain a high probability bound on $\|\tilde{W}\|_2$. To that end,

$$\frac{1}{n} \|\tilde{W}\|_2^2 = \frac{1}{n} \|\tilde{W}^T\tilde{W}\|_2 \leq \frac{1}{n} \|W^T \tilde{W} - E[\tilde{W}^T\tilde{W}]\|_2 + \frac{1}{n} \|E[\tilde{W}^T\tilde{W}]\|_2.$$ (25)

We bound the two terms in (25) separately. We recall the following lemma, which is a direct extension of Theorem 4.6.1 of [20] for the non-isotropic setting, and we present its proof for completeness in Section 6.5.
Lemma 6.5 (Independent sub-gaussian rows). Let $A$ be an $n \times p$ matrix whose rows $A_i$ are independent, mean zero, sub-gaussian random vectors in $\mathbb{R}^p$ with second moment matrix $\Sigma = (1/n)\mathbb{E}[A_i^T A_i]$. Then for any $t \geq 0$, the following inequality holds with probability at least $1 - \exp(-t^2)$:

$$\frac{1}{n} A^T A - \Sigma_2 \leq K^2 \max(\delta, \delta^2), \quad \text{where} \quad \delta = C \sqrt{\frac{p}{n}} \frac{t}{\sqrt{n}};$$

here, $K = \max_i \|A_i\|_{\psi_2}$.

The matrix $\tilde{W} = Z - \rho X$ has independent rows by Assumption 2.4. We state the following Lemma about the distribution property of the rows of $\tilde{W}$, the proof of which can be found in Section 6.6.

Lemma 6.6. Let Assumption 2.4 hold. Then, $z_i - \rho x_i$ is a sequence of independent, mean zero, sub-gaussian random vectors satisfying $\|z_i - \rho x_i\|_{\psi_2} \leq C(K + 1)$.

From Lemmas 6.5 and 6.6, with probability at least $1 - \exp(-t^2)$,

$$\frac{1}{n} \|\tilde{W}^T \tilde{W} - \mathbb{E}[\tilde{W}^T \tilde{W}]\|_2 \leq C(K + 1)^2 (1 + \frac{p}{n} + \frac{t^2}{n}).$$

Finally, we claim the following bound on $\|\mathbb{E}[\tilde{W}^T \tilde{W}]\|_2$, the proof of which is in Section 6.7.

Lemma 6.7. Let Assumption 2.4 hold. Then, we have

$$\|\mathbb{E}[\tilde{W}^T \tilde{W}]\|_2 \leq C(K + 1)^2 n(\rho - \rho^2) + n\rho^2 \gamma^2.$$

From (25), (27) and Lemma 6.7, it follows that with probability at least $1 - \exp(-t^2)$ for any $t > 0$, we have

$$\|\tilde{W}\|_2^2 \leq C(K + 1)^2 (n + p + t^2) + n(\rho(1 - \rho)(K + 1)^2 + \rho^2 \gamma^2).$$

For this, we conclude the following lemma.

Lemma 6.8. For any $t > 0$, the following holds with probability at least $1 - \exp(-t^2)$:

$$\|Z - \rho X\|_2 \leq C(K, \gamma)(\sqrt{n} + \sqrt{p} + t).$$

Using the above and (24), we conclude the proof of Lemma 6.1.

6.2. Proof of Lemma 6.2. We want to bound $\|X - \tilde{Z}^k\|_{2, \infty}$. To that end, let $\Delta_j = X_j - \tilde{Z}^k_j$ for any $j \in [p]$. Our interest is in bounding $\|\Delta_j\|_2^2$ for all $j \in [p]$. Consider,

$$\tilde{Z}^k_j - X_j = (\tilde{Z}^k_j - \hat{U}_k \hat{U}_k^T X_j) + (\hat{U}_k \hat{U}_k^T X_j - X_j).$$

Now, note that $\tilde{Z}^k_j - \hat{U}_k \hat{U}_k^T X_j$ belongs to the subspace spanned by column vectors of $\hat{U}_k$, while $\hat{U}_k \hat{U}_k^T X_j - X_j$ belongs to its orthogonal complement with respect to $\mathbb{R}^n$. As a result,

$$\|\tilde{Z}^k_j - X_j\|_2^2 = \|\tilde{Z}^k_j - \hat{U}_k \hat{U}_k^T X_j\|_2^2 + \|\hat{U}_k \hat{U}_k^T X_j - X_j\|_2^2.$$
By Assumption 2.1, we have that $\rho$ Bounding $16$. Therefore, for $X$ we now state Lemmas 6.9 and 6.10. Their proofs are in Sections 6.8 and 6.9, respectively.

Therefore, we have

$$\tilde{Z}_j^k - \tilde{U}_k \tilde{U}_k^T X_j = \frac{1}{\rho} \tilde{U}_k \tilde{U}_k^T Z_j - \tilde{U}_k \tilde{U}_k^T X_j$$

$$= \frac{1}{\rho} \tilde{U}_k \tilde{U}_k^T (Z_j - \rho X_j) + \left( \frac{\rho - \bar{\rho}}{\rho} \right) \tilde{U}_k \tilde{U}_k^T X_j.$$

Thus, we have

$$\| \tilde{Z}_j^k - \tilde{U}_k \tilde{U}_k^T X_j \|_2^2 \leq \frac{4}{\rho^2} \| \tilde{U}_k \tilde{U}_k^T (Z_j - \rho X_j) \|_2^2 + 2 \left( \frac{\rho - \bar{\rho}}{\rho} \right)^2 \| \tilde{U}_k \tilde{U}_k^T X_j \|_2^2$$

$$\leq 2 \| \tilde{U}_k \tilde{U}_k^T (Z_j - \rho X_j) \|_2^2 + 2 \left( \frac{\rho - \bar{\rho}}{\rho} \right)^2 \| X_j \|_2^2,$$

where we have used the fact that $\| \tilde{U}_k \tilde{U}_k^T \|_2 = 1$. Recall that $U \in \mathbb{R}^{n \times r}$ represents the left singular vectors of $X$. Thus,

$$\| \tilde{U}_k \tilde{U}_k^T (Z_j - \rho X_j) \|_2^2 \leq 2 \| \tilde{U}_k \tilde{U}_k^T - UU^T \|_2^2 \| Z_j - \rho X_j \|_2^2 + 2 \| UU^T (Z_j - \rho X_j) \|_2^2$$

$$\leq 2 \| \tilde{U}_k \tilde{U}_k^T - UU^T \|_2^2 \| Z_j - \rho X_j \|_2^2 + 2 \| UU^T (Z_j - \rho X_j) \|_2^2.$$

By Assumption 2.1, we have that $\| X_j \|_2^2 \leq n$. This yields

$$\| \tilde{Z}_j^k - \tilde{U}_k \tilde{U}_k^T X_j \|_2^2 \leq \frac{4}{\rho^2} \| \tilde{U}_k \tilde{U}_k^T - UU^T \|_2^2 \| Z_j - \rho X_j \|_2^2$$

$$+ 2 \left( \frac{\rho - \bar{\rho}}{\rho} \right)^2 \| Z_j - \rho X_j \|_2^2 + 2 \rho \| \tilde{U}_k \tilde{U}_k^T X_j \|_2^2.$$

We now state Lemmas 6.9 and 6.10. Their proofs are in Sections 6.8 and 6.9, respectively.

**Lemma 6.9.** For any $\alpha > 1$,

$$\mathbb{P} \left( \frac{\rho}{\alpha} \leq \bar{\rho} \leq \alpha \rho \right) \geq 1 - 2 \exp \left( - \frac{(\alpha - 1)^2 n p \rho}{2 \alpha^2} \right).$$

Therefore, for $\rho \geq C \log^2 np$, we have with probability $1 - O(1/(np)^{10})$

$$\frac{\rho}{2} \leq \bar{\rho} \leq 2 \rho \quad \text{and} \quad \left( \frac{\rho - \bar{\rho}}{\rho} \right)^2 \leq C \log np \rho.$$

**Lemma 6.10.** Consider any matrix $Q \in \mathbb{R}^{n \times \ell}$ with $1 \leq \ell \leq n$ such that its columns $Q_{j}$ for $j \in [\ell]$ are orthonormal vectors. Then for any $t > 0$,

$$\mathbb{P} \left( \max_{j \in [p]} \| QQ^T (Z_j - \rho X_j) \|_2^2 \geq \ell C(K + 1)^2 + t \right)$$

$$\leq p \cdot \exp \left( - c \min \left( \frac{t^2}{C(K + 1)^2 \ell}, \frac{t}{C(K + 1)^2} \right) \right).$$
\[
\max_{j \in [p]} \left\| \sum_{i=1}^p \sigma_i \hat{U}_k^T \hat{S}_k \right\|_2 \leq C(K, \gamma) \left( \frac{n + p(n + \sqrt{n} \log np)}{\rho^4 s^2} + \frac{r + \sqrt{r} \log np}{\rho^2} \right) + C \frac{\log np}{\rho p}.
\]

Bounding \( \left\| \hat{U}_k^T \hat{S}_k - I \right\|_2 \). Recalling \( \hat{X} = USV^T \), we claim with probability at least \( 1 - O(\sqrt{np}) \) that
\[
\left\| \hat{U}_k^T \hat{S}_k - I \right\|_2 \leq C(K, \gamma) \left( \frac{n + p(n + \sqrt{n} \log np)}{\rho^4 s^2} + \frac{r + \sqrt{r} \log np}{\rho^2} \right) + C \frac{\log np}{\rho p}.
\]

Concluding. From (28), (32), and (33), we claim with probability at least \( 1 - O(1/(np)^{10}) \)
\[
\left\| \hat{X} - \hat{S}_k \right\|_{2, \infty} \leq C(K, \gamma) \left( \frac{n + p(n + \sqrt{n} \log np)}{\rho^4 s^2} + \frac{r + \sqrt{r} \log np}{\rho^2} \right) + C \frac{\log np}{\rho p}.
\]

This completes the proof of Lemma 6.2.

6.3. Proof of Lemma 6.3. To bound \( \hat{s}_k \), we recall Weyl's inequality.

**Lemma 6.11 (Weyl's inequality).** Given \( A, B \in \mathbb{R}^{m \times n} \), let \( \sigma_i \) and \( \hat{\sigma}_i \) be the \( i \)-th singular values of \( A \) and \( B \), respectively, in decreasing order and repeated by multiplicities. Then for all \( i \in [m \wedge n] \),
\[
|\sigma_i - \hat{\sigma}_i| \leq \left\| A - B \right\|_2.
\]

Let \( \hat{s}_k \) be the \( k \)-th singular value of \( \hat{X} \). Then, \( \hat{s}_k = (1/\rho)\hat{\sigma}_k \) since it is the \( k \)-th singular value of \( \hat{X} = (1/\rho)\hat{X} \). By Lemma 6.11, we have
\[
|\hat{s}_k - \rho \hat{s}_k| \leq \left\| \hat{X} - \rho \hat{X} \right\|_2;
\]
recall that \( s_k \) is the \( k \)-th singular value of \( X \). As a result,
\[
|\tilde{s}_k - s_k| = \frac{1}{\rho} |\tilde{s}_k - \hat{\rho}s_k|
\leq \frac{1}{\rho} |\tilde{s}_k - \hat{\rho}s_k| + \frac{|\rho - \hat{\rho}|}{\hat{\rho}} s_k
\leq \frac{\|Z - \rho X\|_2}{\hat{\rho}} + \frac{|\rho - \hat{\rho}|}{\hat{\rho}} s_k.
\]
From Lemma 6.8 and Lemma 6.9, it follows that with probability at least \( 1 - O(1/(np)^{10}) \),
\[
|\tilde{s}_k - s_k| \leq \frac{C(K, \gamma)(\sqrt{n} + \sqrt{p})}{\rho} + C\frac{\sqrt{\log np}}{\sqrt{\rho np}} s_k.
\]
This completes the proof of Lemma 6.3.

6.4. Proof of Lemma 6.4. We need to bound \( \langle \tilde{Z}^k(\tilde{\beta} - \beta^*), \varepsilon \rangle \). To that end, we recall that \( \tilde{\beta} = \tilde{V}_k \tilde{S}_k^{-1} \tilde{U}_k^T y \), \( \tilde{Z}^k = \tilde{U}_k \tilde{S}_k \tilde{V}_k^T \), and \( y = X \beta^* + \varepsilon \). Thus,
\[
\tilde{Z}^k \tilde{\beta} = \tilde{U}_k \tilde{S}_k \tilde{V}_k^T \tilde{V}_k \tilde{S}_k^{-1} \tilde{U}_k^T y = \tilde{U}_k \tilde{U}_k^T X \beta^* + \tilde{U}_k \tilde{U}_k^T \varepsilon.
\]
Therefore,
\[
\langle \tilde{Z}^k(\tilde{\beta} - \beta^*), \varepsilon \rangle = \langle \tilde{U}_k \tilde{U}_k^T X \beta^*, \varepsilon \rangle + \langle \tilde{U}_k \tilde{U}_k^T, \varepsilon \rangle - \langle \tilde{U}_k \tilde{S}_k \tilde{V}_k^T \beta^*, \varepsilon \rangle.
\]
Now, \( \varepsilon \) is independent of \( \tilde{U}_k, \tilde{S}_k, \tilde{V}_k \) since \( \tilde{Z}^k \) is determined by \( Z \), which is independent of \( \varepsilon \). As a result,
\[
E[\langle \tilde{U}_k \tilde{U}_k^T, \varepsilon \rangle] = E[\varepsilon^T \tilde{U}_k \tilde{U}_k^T \varepsilon]
= E[\text{tr}(\varepsilon^T \tilde{U}_k \tilde{U}_k^T \varepsilon)] = E[\text{tr}(\varepsilon \varepsilon^T \tilde{U}_k \tilde{U}_k^T)]
= \text{tr}[E[\varepsilon \varepsilon^T] \tilde{U}_k \tilde{U}_k^T] \leq C \text{tr}(\sigma^2 \tilde{U}_k \tilde{U}_k^T)
= C\sigma^2 \|\tilde{U}_k\|^2_F = C\sigma^2 k.
\]
Therefore, it follows that
\[
E[\langle \tilde{Z}^k(\tilde{\beta} - \beta^*), \varepsilon \rangle] \leq C\sigma^2 k,
\]
where we used the fact \( E[\varepsilon] = 0 \). To obtain a high probability bound, using Lemma (A.3) it follows that for any \( t > 0 \)
\[
P(\langle \tilde{U}_k \tilde{U}_k^T X \beta^*, \varepsilon \rangle \geq t) \leq \exp\left( -\frac{ct^2}{n\|\beta^*\|_1^2 \sigma^2} \right)
\]
due to Assumption 2.3, and
\[
\|\tilde{U}_k \tilde{U}_k^T X \beta^*\|_2 \leq \|X \beta^*\|_2 \leq \|X\|_{2,\infty} \|\beta^*\|_1 \leq \sqrt{n} \|\beta^*\|_1;
\]
note that we have used the fact that \( \tilde{U}_k \tilde{U}_k^T \) is a projection matrix and \( \|X\|_{2,\infty} \leq \sqrt{n} \) due to Assumption 2.1. Similarly, for any \( t > 0 \)
\[
P(\langle \tilde{U}_k \tilde{S}_k \tilde{V}_k^T \beta^*, \varepsilon \rangle \geq t) \leq \exp\left( -\frac{ct^2}{\sigma^2 (n + \|Z^k - X\|_{2,\infty}^2 \|\beta^*\|_1^2)} \right),
\]
As a result, we can apply Bernstein’s inequality (see Theorem A.1) to obtain
\[ \| \tilde{U}_k \tilde{S} \tilde{V}_k^T \beta^* \|_2 = \| (\tilde{Z}^k - X) \beta^* + X \beta^* \|_2 \leq \| (\tilde{Z}^k - X) \beta^* \|_2 + \| X \beta^* \|_2 \]
\[ \leq (\| \tilde{Z}^k - X \|_{2,\infty} + \| X \|_{2,\infty}) \| \beta^* \|_1. \]

Finally, using Lemma A.4 and (36), it follows that for any \( t > 0 \)
\[ \mathbb{P} \left( \langle \tilde{U}_k \tilde{V}_k^T \varepsilon, \varepsilon \rangle \geq \sigma^2 k + t \right) \leq \exp \left( -c \min \left( \frac{t^2}{2k\sigma^2}, \frac{t}{\sigma^2} \right) \right), \]
since \( \tilde{U}_k \tilde{V}_k^T \) is a projection matrix and by Assumption 2.3.

From (34), (37), (38), and (39), we conclude that with probability at least \( 1 - O(1/(np)^{10}) \),
\[ \langle \tilde{Z}^k (\hat{\beta} - \beta^*), \varepsilon \rangle \leq \sigma^2 k + C \sigma \sqrt{\log np} \left( \sigma \sqrt{k} + \sigma \sqrt{\log np} + \| \beta^* \|_1 (\sqrt{n} + \| \tilde{Z}^k - X \|_{2,\infty}) \right). \]
This completes the proof of Lemma 6.4.

6.5. Proof of Lemma 6.5. As mentioned earlier, the proof presented here is a natural extension of that for Theorem 4.6.1 in [20] for the non-isotropic setting. Recall that
\[ \| A \| = \max_{x \in S^{n-1}, y \in S^{n-1}} \langle Ax, y \rangle, \]
where \( S^{n-1} \), \( S^{n-1} \) denote the unit spheres in \( \mathbb{R}^p \) and \( \mathbb{R}^n \), respectively. We start by bounding the quadratic term \( \langle Ax, y \rangle \) for a finite set \( x, y \) obtained by placing \( 1/4 \)-net on the unit spheres, and then use the bound on them to bound \( \langle Ax, y \rangle \) for all \( x, y \) over the spheres.

**Step 1: Approximation.** We will use Corollary 4.2.13 of [20] to establish a \( 1/4 \)-net of \( \mathcal{N} \) of the unit sphere \( S^{n-1} \) with cardinality \( |\mathcal{N}| \leq 9^p \). Applying Lemma 4.4.1 of [20], we obtain
\[ \| \frac{1}{n} A^T A - \Sigma \|_2 \leq 2 \max_{x \in \mathcal{N}} \left| \langle \frac{1}{n} A^T A - \Sigma \rangle x, x \rangle \right| = 2 \max_{x \in \mathcal{N}} \left| \frac{1}{n} \| Ax \|_2^2 - x^T \Sigma x \right|. \]
To achieve our desired result, it remains to show that
\[ \max_{x \in \mathcal{N}} \left| \frac{1}{n} \| Ax \|_2^2 - x^T \Sigma x \right| \leq \frac{\epsilon}{2}, \]
where \( \epsilon = K^2 \max(\delta, \delta^2) \).

**Step 2: Concentration.** Let us fix a unit vector \( x \in S^{n-1} \) and write
\[ \| Ax \|_2^2 - x^T \Sigma x = \sum_{i=1}^n \left( \langle A_i, x \rangle^2 - \mathbb{E}[(A_i, x)^2] \right) = \sum_{i=1}^n (Y_i^2 - \mathbb{E}[Y_i^2]), \]
Since the rows of \( A \) are assumed to be independent sub-gaussian random vectors with \( \| A_i \|_{\psi_2} \leq K \), it follows that \( Y_i = \langle A_i, x \rangle \) are independent sub-gaussian random variables with \( \| Y_i \|_{\psi_2} \leq K \). Therefore, \( Y_i^2 - \mathbb{E}[Y_i^2] \) are independent, mean zero, sub-exponential random variables with
\[ \| Y_i^2 - \mathbb{E}[Y_i^2] \|_{\psi_1} \leq C \| Y_i^2 \|_{\psi_1} \leq C \| Y_i \|_{\psi_2}^2 \leq C K^2. \]

As a result, we can apply Bernstein’s inequality (see Theorem A.1) to obtain
\[ \mathbb{P} \left( \left| \frac{1}{n} \| Ax \|_2^2 - x^T \Sigma x \right| \geq \frac{\epsilon}{2} \right) = \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^n (Y_i^2 - \mathbb{E}[Y_i^2]) \right| \geq \frac{\epsilon}{2} \right) \]
Here, \( \psi \) is continuous with respect to \( u \) and \( \| \psi \|_2 \leq K \) is independent and bounded in absolute value by 1. Assumption 2.4. The proof of Lemma 6.6 is complete by choosing a large enough \( C \).

### Step 3: Union bound.

We now apply a union bound over all elements in the net. Specifically,

\[
\Pr \left( \max_{x \in \mathcal{N}} \left| \frac{1}{n} \| A x \|_2^2 - x^T \Sigma x \right| \geq \frac{\epsilon}{2} \right) \leq 2 \exp\left(-cC^2(p + t^2)\right),
\]

for large enough \( C \). This concludes the proof.

#### 6.6. Proof of Lemma 6.6.

Recall that \( z_i = (x_i + w_i) \circ \pi_i \), where \( w_i \) is an independent mean zero subgaussian vector with \( \| w_i \|_{\psi_2} \leq K \) and \( \pi_i \) is a vector of independent Bernoulli variables with parameter \( \rho \). Hence, \( E[z_i - \rho x_i] = 0 \) and is independent across \( i \in [n] \). The only remaining item is a bound on \( \| z_i - \rho x_i \|_{\psi_2} \). To that end, note that

\[
\| z_i - \rho x_i \|_{\psi_2} = \| x_i \circ \pi_i + w_i \circ \pi_i - \rho x_i \|_{\psi_2} \leq \| x_i \circ (\rho - 1) \|_{\psi_2} + \| w_i \circ \pi_i \|_{\psi_2}. \]

Now, \((\rho - 1) \circ \pi_i \) is independent, zero mean random vector whose absolute value is bounded by 1, and is component-wise multiplied by \( x_i \) which are bounded in absolute value by 1 as per Assumption 2.1. That is, \( x_i \circ (\rho - 1) \circ \pi_i \) is a zero mean random vector where each component is independent and bounded in absolute value by 1. That is, \( \| \cdot \|_{\psi_2} \leq C \).

For \( w_i \circ \pi_i \), note that \( w_i \) and \( \pi_i \) are independent random vectors and the coordinates of \( \pi_i \) have support \{0, 1\}. Therefore, from Lemma 6.12, it follows that \( \| w_i \circ \pi_i \|_{\psi_2} \leq \| w_i \|_{\psi_2} \leq K \) by Assumption 2.4. The proof of Lemma 6.6 is complete by choosing a large enough \( C \).

#### Lemma 6.12.

Suppose that \( Y \in \mathbb{R}^n \) and \( P \in \{0, 1\}^n \) are independent random vectors. Then,

\[
\| Y \circ P \|_{\psi_2} \leq \| Y \|_{\psi_2}.
\]

**Proof.** Given a binary vector \( P \in \{0, 1\}^n \), let \( I_P = \{ i \in [n] : P_i = 1 \} \). Observe that

\[
Y \circ P = \sum_{i \in I_P} e_i \otimes e_i Y.
\]

Here, \( \circ \) denotes the Hadamard product (entry-wise product) of two matrices. By definition of the \( \psi_2 \)-norm,

\[
\| Y \|_{\psi_2} = \sup_{u \in S^{n-1}} \| u^T Y \|_{\psi_2} = \sup_{u \in S^{n-1}} \inf \{ t > 0 : \mathbb{E}_Y[\exp(|u^T Y|^2/t^2)] \leq 2 \}.
\]

Let \( u_0 \in S^{n-1} \) denote the maximum-achieving unit vector (such a \( u_0 \) exists because \( \inf \{ \cdots \} \) is continuous with respect to \( u \) and \( S^{n-1} \) is compact). Now,

\[
\| Y \circ P \|_{\psi_2} = \sup_{u \in S^{n-1}} \| u^T Y \circ P \|_{\psi_2} = \sup_{u \in S^{n-1}} \inf \{ t > 0 : \mathbb{E}_{Y,P}[\exp(|u^T Y \circ P|^2/t^2)] \leq 2 \}.
\]
Therefore, taking supremum over \( u \in \mathbb{S}^{n-1} \), observe that
\[
\mathbb{E}_Y[\exp(\| (\sum_{i \in I_p} e_i \otimes e_i u)^T Y \|_2^2 / t^2) | P] \leq \mathbb{E}_Y[\exp(\| u_0^T Y \|_2^2 / t^2)].
\]

Therefore, taking supremum over \( u \in \mathbb{S}^{n-1} \), we obtain
\[
\| Y \circ P \|_{\psi_2} \leq \| Y \|_{\psi_2}.
\]

\[ \square \]

6.7. Proof of Lemma 6.7. Consider
\[
\mathbb{E}[\tilde{W}^T \tilde{W}] = \sum_{i=1}^{n} \mathbb{E}[(z_i - \rho x_i) \otimes (z_i - \rho x_i)]
\]
\[
= \sum_{i=1}^{n} \mathbb{E}[z_i \otimes z_i] - \rho^2(x_i \otimes x_i)
\]
\[
= \sum_{i=1}^{n} (\rho - \rho^2) \text{diag}(x_i \otimes x_i) + (\rho - \rho^2) \text{diag}(\mathbb{E}[w_i \otimes w_i]) + \rho^2 \mathbb{E}[w_i \otimes w_i].
\]

Note that \( \| \text{diag}(X^T X) \|_2 \leq n \) due to Assumption 2.1. Using Assumption 2.4, it follows that \( \| \text{diag}(\mathbb{E}[w_i \otimes w_i]) \|_2 \leq C K^2 \). By Assumption 2.4, we have \( \| \mathbb{E}[w_i \otimes w_i] \|_2 \leq \gamma^2 \). Therefore,
\[
\| \mathbb{E}[\tilde{W}^T \tilde{W}] \|_2 \leq C n (\rho - \rho^2)(K + 1)^2 + n \rho^2 \gamma^2.
\]

This completes the proof of Lemma 6.7.

6.8. Proof of Lemma 6.9. By the Binomial Chernoff bound, for \( \alpha > 1 \),
\[
\mathbb{P}(\hat{\rho} > \alpha \rho) \leq \exp \left( -\frac{(\alpha - 1)^2}{\alpha + 1} npp \right) \quad \text{and} \quad \mathbb{P}(\hat{\rho} < \rho / \alpha) \leq \exp \left( -\frac{(\alpha - 1)^2}{2 \alpha^2} npp \right).
\]

By the union bound,
\[
\mathbb{P}(\rho / \alpha \leq \hat{\rho} \leq \alpha \rho) \geq 1 - \mathbb{P}(\hat{\rho} > \alpha \rho) - \mathbb{P}(\hat{\rho} < \rho / \alpha).
\]

Noticing \( \alpha + 1 < 2 \alpha < 2 \alpha^2 \) for all \( \alpha > 1 \), we obtain the desired bound claimed in Lemma 6.9. To complete the remaining claim of Lemma 6.9, we consider an \( \alpha \) that satisfies
\[
(\alpha - 1)^2 \leq C \frac{\log n p}{p n p},
\]
for a constant \( C > 0 \). Thus,
\[
1 - C \frac{\sqrt{\log n p}}{\sqrt{p n p}} \leq \alpha \leq 1 + C \frac{\sqrt{\log n p}}{\sqrt{p n p}}.
\]
Then, with $\rho \geq c\frac{\log^2 np}{np}$, we have that $\alpha \leq 2$. Further by choosing $C > 0$ large enough, we have

$$\frac{(\rho - \hat{\rho})^2}{\rho^2} \leq C \frac{\log np}{np}.$$

holds with probability at least $1 - O(1/(np)^{10})$. This completes the proof of Lemma 6.6.

6.9. Proof of Lemma 6.10. By definition $QQ^T \in \mathbb{R}^{n \times n}$ is a rank $\ell$ matrix. Since $Q$ has orthonormal column vectors, the projection operator has $\|QQ^T\|_2 = 1$ and $\|QQ^T\|_F^2 = \ell$. For a given $j \in [p]$, the random vector $Z_j - \rho X_j$ is such that it has zero mean, independent components that are sub-gaussian by Assumption 2.4. For any $i \in [n], j \in [p]$, we have by property of $\psi_2$ norm, $\|z_{ij} - \rho x_{ij}\|_{\psi_2} \leq \|z_i - \rho x_i\|_{\psi_2}$ which is bounded by $C(K+1)$ using Lemma 6.6. Recall the Hanson-Wright inequality ([20]):

**Theorem 6.2** (Hanson-Wright inequality). Let $\zeta \in \mathbb{R}^n$ be a random vector with independent, mean zero, sub-gaussian coordinates. Let $A$ be an $n \times n$ matrix. Then for any $t > 0$,

$$P(\|\zeta^T A \zeta - \mathbb{E}[\zeta^T A \zeta]\| \geq t) \leq 2 \exp \left(-c \min \left(\frac{t^2}{L^1\|A\|_F^2}, \frac{t}{L^2\|A\|_2^2}\right)\right),$$

where $L = \max_{i \in [n]} \|\zeta_i\|_{\psi_2}$.

Now with $\zeta = Z_j - \rho X_j$ and the fact that $QQ^T = I \in \mathbb{R}^{\ell \times \ell}$, $\|QQ^T \zeta\|_2 = \zeta^T QQ^T \zeta$. Therefore, by Theorem 6.2, for any $t > 0$,

$$\|QQ^T \zeta\|_2^2 \leq \mathbb{E}[(\zeta^T QQ^T \zeta) + t],$$

with probability at least $1 - \exp \left(-c \min \left(\frac{t}{C(K+1)^2}, \frac{t^2}{C(K+1)^4}\right)\right)$. Now,

$$\mathbb{E}[\zeta^T QQ^T \zeta] = \sum_{m=1}^{\ell} \mathbb{E}[\|Q_m \zeta\|^2]$$

$$= \sum_{m=1}^{\ell} \text{Var}(Q_m \zeta)$$

$$= \sum_{m=1}^{\ell} \sum_{i=1}^{n} Q_{im}^2 \text{Var}(\zeta_i)$$

$$\leq C(K+1)^2 \ell,$$

where $\zeta = Z_j - \rho X_j$, and hence (a) follows from $\mathbb{E}[\zeta] = \mathbb{E}[Z_j - \rho X_j] = 0$, (b) follows from $\zeta$ having independent components and (c) follows from each component of $\zeta$ having $\psi_2$-norm bounded by $C(K+1)$. Therefore, it follows by union bound that for any $t > 0$,

$$P\left(\max_{j \in [p]} \|QQ^T (Z_j - \rho X_j)\|_2^2 \geq \ell C(K+1)^2 + t\right)$$

$$\leq p \cdot \exp \left(-c \min \left(\frac{t^2}{C(K+1)^4 \ell}, \frac{t}{C(K+1)^2}\right)\right).$$

This completes the proof of Lemma 6.10.
7. Proof of Theorem 4.2. Recall that $X'$ and $Z'$ denote the latent and observed testing covariates, respectively. We denote the SVD of the former as $X' = U'S'V'T$. Let $s'_{\ell}$ be the $\ell$-th singular value of $X'$. Further, recall that $\tilde{Z}' = (1/\rho')Z'$, and its rank $\ell$ truncation is denoted as $\tilde{Z}'_{\ell}$. Our interest is in bounding $\|\tilde{Z}'_{\ell}\bar{\beta} - X'\beta^*\|_2$. Towards this, consider

$$
\|\tilde{Z}'_{\ell}\bar{\beta} - X'\beta^*\|_2^2
\leq 2\|\tilde{Z}'_{\ell}(\tilde{\beta} - \beta^*)\|_2^2 + 2\|\tilde{Z}'_{\ell} - X'\|_2^2.
$$

We shall bound the two terms on the right hand side of (40) next.

Bounding $\|\tilde{Z}'(\tilde{\beta} - \beta^*)\|_2^2$. Since $\tilde{Z}' = (1/\rho')Z'$, we have

$$
\|\tilde{Z}'(\tilde{\beta} - \beta^*)\|_2^2 = \frac{1}{\rho'^2}\|Z'(\tilde{\beta} - \beta^*)\|_2^2
\leq \frac{1}{\rho'^2}\|Z' - \rho X' + \rho X'(\tilde{\beta} - \beta^*)\|_2^2
\leq \frac{2}{\rho'^2}\|Z' - \rho X'(\tilde{\beta} - \beta^*)\|_2^2 + 2\left(\frac{\rho}{\rho'}\right)^2\|X'(\tilde{\beta} - \beta^*)\|_2^2.
$$

Now, note that $\|Z' - Z'|_2$ is the $(\ell + 1)$-st largest singular value of $Z'$. Therefore, by Weyl’s inequality (Lemma 6.11), we have for any $\ell \geq r'$,

$$
\|Z' - Z'|_2 \leq \|Z' - \rho X'\|_2.
$$

In turn, this gives

$$
\|Z' - \rho X'\|_2 \leq \|Z' - Z'|_2 + \|Z' - \rho X'\|_2 \leq 2\|Z' - \rho X'\|_2.
$$

Thus, we have

$$
\|(Z' - \rho X')(\tilde{\beta} - \beta^*)\|_2^2 \leq 4\|Z' - \rho X'\|_2^2 \|\tilde{\beta} - \beta^*\|_2^2.
$$

Recall that $V$ and $V_{\perp}$ span the rowspace and nullspace of $X$, respectively. By Assumption 2.5, it follows that $V'V_{\perp} = 0$ and hence $X'V_{\perp}V_{\perp}' = 0$. As a result,

$$
\|X'(\tilde{\beta} - \beta^*)\|_2^2 = \|X'(VV'T + V_{\perp}V_{\perp}'\tilde{\beta} - \beta^*)\|_2^2
\leq \|X'\|_2^2 \|VV'T(\tilde{\beta} - \beta^*)\|_2^2.
$$

Recalling that $\hat{V}_{\beta}$ denotes the top $r$ right singular vectors of $\hat{Z}^r$, consider

$$
\|VV'T(\tilde{\beta} - \beta^*)\|_2^2 = \|(VV'T - \hat{V}_{\beta}\tilde{\beta} + \hat{V}_{\beta}\beta^*)\|_2^2
\leq 2\|VV'T - \hat{V}_{\beta}\|_2^2 \|\beta^*\|_2^2 + 2\|\hat{V}_{\beta}\|_2^2 \|\tilde{\beta} - \beta^*\|_2^2.
$$

From (16) and above, we obtain

$$
\|VV'T(\tilde{\beta} - \beta^*)\|_2^2 \leq C\|VV'T - \hat{V}_{\beta}\|_2^2 \|\beta^*\|_2^2 + \frac{C}{s_{r'}}(\|X - \hat{Z}^r\|_{2,\infty}\|\beta^*\|_2^2 + \langle \hat{Z}^r(\tilde{\beta} - \beta^*), \varepsilon \rangle).
$$

Thus,

$$
\|X'(\tilde{\beta} - \beta^*)\|_2^2 \leq C\|X'\|_2^2 \|VV'T - \hat{V}_{\beta}\|_2^2 \|\beta^*\|_2^2 + \frac{C}{s_{r'}}(\|X - \hat{Z}^r\|_{2,\infty}\|\beta^*\|_2^2 + \langle \hat{Z}^r(\tilde{\beta} - \beta^*), \varepsilon \rangle).
$$
In summary, plugging (42) and (43) into (41), we have
\[
\| \tilde{Z}^\ell (\tilde{\beta} - \beta^*) \|_2^2 \leq \frac{C}{(\bar{\rho})^2} \| Z' - \rho X' \|_2^2 \| \tilde{\beta} - \beta^* \|_2^2 \\
+ C \left( \frac{\rho}{(\bar{\rho})^2} \right)^2 \| X' \|_2^2 \| V V^T - \tilde{V} \tilde{V}^T \|_2^2 \| \tilde{\beta} - \beta^* \|_2^2 \\
+ \frac{C \rho^2 \| X' \|_2^2}{(\bar{\rho})^2 s_r^2} \left( \| X - \tilde{Z}^r \|_{2,\infty} \| \beta^* \|_1^2 + \langle \tilde{Z}^r (\tilde{\beta} - \beta^*), \varepsilon \rangle \right).
\] (44)

Bounding \( \| (\tilde{Z}^\ell - X') \beta^* \|_2^2 \). Using inequality (8),
\[
\| (\tilde{Z}^\ell - X') \beta^* \|_2 \leq \| \tilde{Z}^\ell - X' \|_{2,\infty} \| \beta^* \|_1^2.
\] (45)

Combining. Incorporating (44) and (45) into (40) with \( \ell = r' \) yields
\[
\| \tilde{Z}^{r'} - X' \beta^* \|_2^2 \leq \Delta_1 + \Delta_2,
\] (46)

where
\[
\Delta_1 = \frac{C}{(\bar{\rho})^2} \| Z' - \rho X' \|_2^2 \| \tilde{\beta} - \beta^* \|_2^2 \\
+ C \left( \frac{\rho s_r}{(\bar{\rho})^2} \right)^2 \| V V^T - \tilde{V} \tilde{V}^T \|_2^2 \| \tilde{\beta} - \beta^* \|_2^2 \\
+ 2 \| X' - \tilde{Z}^{r'} \|_{2,\infty} \| \beta^* \|_1^2;
\]
\[
\Delta_2 = \frac{C (\rho s_r)}{(\bar{\rho})^2 s_r^2} \left( \| X - \tilde{Z}^r \|_{2,\infty} \| \beta^* \|_1^2 + \langle \tilde{Z}^r (\tilde{\beta} - \beta^*), \varepsilon \rangle \right).
\]

Note that (46) is a deterministic bound. We will now proceed to bound \( \Delta_1 \) and \( \Delta_2 \), first in high probability then in expectation.

Bound in high-probability. We first bound \( \Delta_1 \). By adapting Lemma 6.8 for \( Z', X' \) in place of \( Z, X \), we have with probability at least \( 1 - O(1/(mp)^{10}) \),
\[
\| Z' - \rho X' \|_2 \leq C(K, \gamma) (\sqrt{m} + \sqrt{p}).
\] (47)

Recall Lemma 6.1, which states with probability at least \( 1 - O(1/(np)^{10}) \)
\[
\| V V^T - \tilde{V} \tilde{V}^T \|_2^2 \leq C(K, \gamma) \frac{n + p}{\rho^2}.
\] (48)

Adapting Lemma 6.2 for \( \tilde{Z}', X' \) in place of \( \tilde{Z}, X \) with \( \ell = r' \), we obtain that if \( \rho \geq c \frac{\log^2 mp}{mp} \), then with probability at least \( 1 - O(1/(mp)^{10}) \)
\[
\| X' - \tilde{Z}^{r'} \|_{2,\infty} \leq C(K, \gamma) \frac{\log mp}{\rho^2} \left( \frac{m(m \lor p)}{\rho^2 (s_r')^2} + r' + \frac{\rho}{p} \right).
\] (49)

Finally, adapting Lemma 6.9 with \( \bar{\rho}' \) in place of \( \bar{\rho} \), we obtain with probability at least \( 1 - O(1/(mp)^{10}) \),
\[
\rho/2 \leq \bar{\rho}' \leq \rho.
\] (50)

Using (47), (48), (49), and (50), we conclude that with probability at least \( 1 - O(1/(np)^{10}) - O(1/(mp)^{10}) \)
\[
\Delta_1 \leq C(K, \gamma) \left( \frac{m + p}{\rho^2} + \frac{(p + n)(s_r')^2}{\rho^2 s_r^2} \right) \| \tilde{\beta} - \beta^* \|_2^2 \\
+ C(K, \gamma) \frac{\log mp \| \beta^* \|_1^2}{\rho^2} \left( \frac{m(m \lor p)}{\rho^2 (s_r')^2} + r' \right).
\] (51)
We will now bound $\Delta_2$. As per (21), with probability at least $1 - O(1/(np)^{10})$
\[
\|X - \mathbf{Z}^r\|_2 \leq C(K)\log np \left(\frac{n(n \lor p)}{p^2s_r^2} + r + \frac{\rho}{p}\right).
\]

Therefore, (50) and (52) imply that with probability at least $1 - O(1/(np)^{10}) - O(1/(mp)^{10})$
\[
\Delta_2 \leq \left(\frac{s'_1}{s_r}\right)^2 \left(C(K, \gamma)\log np \frac{\|\beta^*\|_2^2}{\rho^2} \left(\frac{n(n \lor p)}{p^2s_r^2} + r\right) + C\sigma^2(r + \log np) + C\sigma\sqrt{n \log np}\|\beta^*\|_1\right).
\]

Incorporating (51) and (53) into (46), normalizing by $1/m$, and appropriately redefining
$\Delta_1, \Delta_2$ as in the statement of Theorem 4.2 concludes the high-probability bound.

**Bound in expectation.** Here, we assume that $\{\langle x_i, \beta^* \rangle \in [-b, b] : i > n\}$. As such, we enforce
$\{\hat{y}_i \in [-b, b] : i > n\}$. With (46), this yields
\[
\text{MSE}_{\text{test}} \leq \frac{1}{m} \|\mathbf{Z}^r\hat{\beta} - X^T\beta^*\|_2^2 \leq \frac{1}{m}(\Delta_1 + \Delta_2).
\]

We define $\mathcal{E}$ as the event such that (47), (48), (49), (50), (52), and Lemma 6.3 hold. Thus, if $\mathcal{E}$ occurs, then (51) implies that
\[
\mathbb{E}[\Delta_1|\mathcal{E}] \leq C(K, \gamma)\left(\frac{m+p}{\rho^2} + \frac{(p+n)(s'_1)^2}{\rho^2s_r^2}\right)\|\hat{\beta} - \beta^*\|_2^2
\]
\[
+ C(K, \gamma)\log mp \frac{\|\beta^*\|_2^2}{\rho^2} \left(\frac{m(m \lor p)}{p^2(s'_r)^2} + r'\right).
\]

Next, we bound $\mathbb{E}[\Delta_2|\mathcal{E}]$. To do so, observe that $\epsilon$ is independent of the event $\mathcal{E}$. Thus, by
(35), we have
\[
\mathbb{E}[\langle \mathbf{Z}^r(\hat{\beta} - \beta^*), \epsilon \rangle|\mathcal{E}] = \mathbb{E}[\langle \hat{U}_r\hat{U}_r^T X \beta^* , \epsilon \rangle + \langle \hat{U}_r\hat{U}_r^T, \epsilon \rangle - \langle \hat{U}_r\hat{S}_r\hat{V}_r^T \beta^* , \epsilon \rangle|\mathcal{E}]
\]
\[
= \mathbb{E}[\langle \hat{U}_r\hat{U}_r^T, \epsilon \rangle|\mathcal{E}] \leq C\sigma r.
\]

Combining the above inequality with (52),
\[
\mathbb{E}[\Delta_2|\mathcal{E}] \leq \left(\frac{s'_1}{s_r}\right)^2 \left(C(K, \gamma)\log np \frac{\|\beta^*\|_2^2}{\rho^2} \left(\frac{n(n \lor p)}{p^2s_r^2} + r\right) + C\sigma^2 r\right).
\]

Due to truncation, observe that $\text{MSE}_{\text{test}}$ is trivially bounded by $4b^2$. Thus,
\[
\mathbb{E}[\text{MSE}_{\text{test}}] \leq \mathbb{E}[\text{MSE}_{\text{test}}|\mathcal{E}] + \mathbb{E}[\text{MSE}_{\text{test}}|\mathcal{E}^c] \cdot P(\mathcal{E}^c)
\]
\[
\leq \frac{1}{m}\mathbb{E}[\Delta_1 + \Delta_2|\mathcal{E}] + Cb^2 \left(1/(np)^{10} + 1/(mp)^{10}\right).
\]

Plugging (54) and (55) into the inequality above completes the proof.
8. Conclusion. In the high-dimensional error-in-variables setting, we establish that PCR identifies the unique model parameter with minimum $\ell_2$-norm and achieves desirable out-of-sample prediction performance. More formally, when the singular values of the covariates are well-balanced and $n \ll p$, we prove that both the parameter estimation and out-of-sample prediction errors vanish at a non-asymptotic rate of $O(1/n)$; for prediction, we require the additional assumption that $p \ll n(n \wedge m)$. To the best of our knowledge, out-of-sample prediction guarantees have been elusive for the other estimators proposed in the literature.

As an important future direction of research, it remains to establish bounds when the covariates are only approximately low-rank, i.e., there exists a matrix $A$ such that $\text{rank}(A) = r$ and $X \approx A$ in some norm. Using PCR in such a setting will induce an additional error term of the form $\|V_{r,\perp} V^T_{r,\perp} \beta^*\|_2$; here, the columns of $V_{r,\perp} \in \mathbb{R}^{p \times (p-r)}$ form an orthonormal basis that is orthogonal to the top $r$ right singular vectors of $X$, and $\beta^*$ is again the minimum norm model. To justify our postulation, recall that $\beta^* \in \text{rowspan}(X)$. Thus, it follows that $\|V_{r,\perp} V^T_{r,\perp} \beta^*\|_2$ is precisely the unavoidable parameter estimator error by taking a rank $r$ approximation of $X$. Hence, it stands to reason that soft singular value thresholding (SVT) may be the more appropriate algorithmic approach as opposed to the hard SVT approach in PCR.

Lastly, we believe another important future line of research is to bridge our out-of-sample prediction error analysis with recent exciting work on analyzing the generalization of over-parameterized estimators. Our key enabling assumption is that the rowspace of the test covariates lies within that of the training covariates, i.e., the test covariates are no more “complex” than the training covariates in a linear algebraic sense. In comparison, recent techniques to bound the generalization error of modern statistical estimators focus on the complexity of the learning algorithm itself, and assume the data generating process produces i.i.d. samples. Hence, a likely fruitful approach to produce tighter generalization error bounds for more complex non-linear settings would be to exploit both the complexity of the learning algorithm and the relative complexity of the test covariates compared to the training covariates – possibly by adapting our subspace inclusion condition with an appropriate non-linear notion.

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APPENDIX A: HELPFUL CONCENTRATION INEQUALITIES.

In this section, we state and prove a number of helpful concentration inequalities used to establish our primary results.

**Lemma A.1.** Let $X$ be a mean zero, sub-gaussian random variable. Then for any $\lambda \in \mathbb{R}$,

$$
\mathbb{E} \exp(\lambda X) \leq \exp\left( C \lambda^2 \|X\|_\psi^2 \right).
$$

**Lemma A.2.** Let $X_1, \ldots, X_n$ be independent, mean zero, sub-gaussian random variables. Then,

$$
\left\| \sum_{i=1}^n X_i \right\|_\psi^2 \leq C \sum_{i=1}^n \|X_i\|_\psi^2.
$$

**Theorem A.1 (Bernstein’s inequality).** Let $X_1, \ldots, X_n$ be independent, mean zero, sub-exponential random variables. Then, for every $t \geq 0$, we have

$$
\mathbb{P}\left( \left| \sum_{i=1}^n X_i \right| \geq t \right) \leq 2 \exp\left( -c \min\left( \frac{t^2}{\sum_{i=1}^n \|X_i\|_\psi^1}, \frac{t}{\max_i \|X_i\|_\psi^1} \right) \right),
$$

where $c > 0$ is an absolute constant.

**Lemma A.3 (Modified Hoeffding Inequality).** Let $X \in \mathbb{R}^n$ be random vector with independent mean-zero sub-Gaussian random coordinates with $\|X_i\|_\psi^2 \leq K$. Let $a \in \mathbb{R}^n$ be another random vector that satisfies $\|a\|_2 \leq b$ almost surely for some constant $b \geq 0$. Then for all $t \geq 0$,

$$
\mathbb{P}\left( \left| \sum_{i=1}^n a_i X_i \right| \geq t \right) \leq 2 \exp\left( - \frac{ct^2}{K^2b^2} \right),
$$

where $c > 0$ is a universal constant.
PROOF. Let $S_n = \sum_{i=1}^{n} a_i X_i$. Then applying Markov’s inequality for any $\lambda > 0$, we obtain
\[
\mathbb{P}(S_n \geq t) = \mathbb{P}(\exp(\lambda S_n) \geq \exp(\lambda t)) \\
\leq \mathbb{E}[\exp(\lambda S_n)] \cdot \exp(-\lambda t) \\
= \mathbb{E}_a[\mathbb{E}[\exp(\lambda S_n) | a]] \cdot \exp(-\lambda t).
\]

Now, conditioned on the random vector $a$, observe that
\[
\mathbb{E}[\exp(\lambda S_n)] = \prod_{i=1}^{n} \mathbb{E}[\exp(\lambda a_i X_i)] \leq \exp(C K^2 \lambda^2 \|a\|^2_2) \leq \exp(C K^2 \lambda^2 b^2),
\]
where the equality follows from conditional independence, the first inequality by Lemma A.1, and the final inequality by assumption. Therefore,
\[
\mathbb{P}(S_n \geq t) \leq \exp(C K^2 \lambda^2 b^2 - \lambda t).
\]

Optimizing over $\lambda$ yields the desired result:
\[
\mathbb{P}(S_n \geq t) \leq \exp\left(-\frac{ct^2}{K^2 b^2}\right).
\]

Applying the same arguments for $-\langle X, a \rangle$ gives a tail bound in the other direction. \hfill \Box

**Lemma A.4 (Modified Hanson-Wright Inequality).** Let $X \in \mathbb{R}^n$ be a random vector with independent mean-zero sub-Gaussian coordinates with $\|X_i\|_{\psi_2} \leq K$. Let $A \in \mathbb{R}^{n \times n}$ be a random matrix satisfying $\|A\| \leq a$ and $\|A\|^2_F \leq b$ almost surely for some $a, b \geq 0$. Then for any $t \geq 0$,
\[
\mathbb{P}\left(\|X^T A X - \mathbb{E}[X^T A X]\| \geq t\right) \leq 2 \cdot \exp\left(-c \min\left(\frac{t^2}{K^4 b}, \frac{t}{K^2 a}\right)\right).
\]

**Proof.** The proof follows similarly to that of Theorem 6.2.1 of [20]. Using the independence of the coordinates of $X$, we have the following useful diagonal and off-diagonal decomposition:
\[
X^T A X - \mathbb{E}[X^T A X] = \sum_{i=1}^{n} (A_{ii} X_i^2 - \mathbb{E}[A_{ii} X_i^2]) + \sum_{i \neq j} A_{ij} X_i X_j.
\]

Therefore, letting
\[
p = \mathbb{P}\left(X^T A X - \mathbb{E}[X^T A X] \geq t\right),
\]
we can express
\[
p \leq \mathbb{P}\left(\sum_{i=1}^{n} (A_{ii} X_i^2 - \mathbb{E}[A_{ii} X_i^2]) \geq t/2\right) + \mathbb{P}\left(\sum_{i \neq j} A_{ij} X_i X_j \geq t/2\right) =: p_1 + p_2.
\]

We will now proceed to bound each term independently.

**Step 1: diagonal sum.** Let $S_n = \sum_{i=1}^{n} (A_{ii} X_i^2 - \mathbb{E}[A_{ii} X_i^2])$. Applying Markov’s inequality for any $\lambda > 0$, we have
\[
p_1 = \mathbb{P}(\exp(\lambda S_n) \geq \exp(\lambda t/2)) \\
\leq \mathbb{E}_A[\mathbb{E}[\exp(\lambda S_n) | A]] \cdot \exp(-\lambda t/2).
\]
Since the $X_i$ are independent, sub-Gaussian random variables, $X_i^2 - \mathbb{E}[X_i^2]$ are independent mean-zero sub-exponential random variables, satisfying

$$\|X_i^2 - \mathbb{E}[X_i^2]\|_{\psi_1} \leq C\|X_i^2\|_{\psi_1} \leq C\|X_i\|^2_{\psi_2} \leq CK^2.$$ 

Conditioned on $A$ and optimizing over $\lambda$ using standard arguments, yields

$$p_1 \leq \exp\left(-c \min\left(\frac{t^2}{K^4b}, \frac{t}{K^2a}\right)\right).$$

**Step 2: off-diagonals.** Let $S = \sum_{i \neq j} A_{ij} X_i X_j$. Again, applying Markov’s inequality for any $\lambda > 0$, we have

$$p_2 = \mathbb{P}(\exp(\lambda S) \geq \exp(\lambda t/2)) \leq \mathbb{E}_A [\mathbb{E} [\exp(\lambda S) | A]] \cdot \exp(-\lambda t/2).$$

Let $g$ be a standard multivariate gaussian random vector. Further, let $X'$ and $g'$ be independent copies of $X$ and $g$, respectively. Conditioning on $A$ yields

$$\mathbb{E}[\exp(\lambda S)] \leq \mathbb{E} [\exp(4\lambda X^T A X')]$$

(by Decoupling Remark 6.1.3 of [20])

$$\leq \mathbb{E} [\exp(C_1 \lambda g^T A g')]$$

(by Lemma 6.2.3 of [20])

$$\leq \exp\left(C_2 \lambda^2 \|A\|^2_F\right)$$

(by Lemma 6.2.2 of [20])

$$\leq \exp(C_2 \lambda^2 b),$$

where $|\lambda| \leq c/a$. Optimizing over $\lambda$ then gives

$$p_2 \leq \exp\left(-c \min\left(\frac{t^2}{K^4b}, \frac{t}{K^2a}\right)\right).$$

**Step 3: combining.** Putting everything together completes the proof. 

\[\square\]