The first factor of the class number of the $p$-th cyclotomic field

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Abstract. Kummer’s conjecture states that the relative class number of the $p$-th cyclotomic field follows a strict asymptotic law. Granville has shown it unlikely to be true – it cannot be true if we assume the truth of two other widely believed conjectures. We establish a new bound for the error term in Kummer’s conjecture, and more precisely we prove that

$$\log(h_p^-) = \frac{p+3}{4} \log p + \frac{p}{2} \log(2\pi) + \log(1 - \beta) + O(\log_2 p),$$

where $\beta$ is a possible Siegel zero of an $L(s, \chi)$, $\chi$ odd.

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1. Introduction

Let $h_p$ denote the class number of $\mathbb{Q}(\zeta_p)$, where $p$ is an odd prime. Let $h_p^+$ denote the class number of the totally real field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. It is well known that $h_p^+$ divides $h_p$. We denote the quotient — or so called first factor of $h_p$ — by $h_p^-$. The following formula is an application of the class number formula (see e.g. [9])

$$h_p^- = G(p) \prod_{\chi \mod p, \text{odd}} L(1, \chi),$$

(1.1)

where $G(p) = 2p \left( \frac{p}{4\pi^2} \right)^{\frac{p-1}{4}}$. Since the magnitude of the $L(1, \chi)$ is not evident, it is hoped they are insignificant. The guess that $h_p^-$ is asymptotically equivalent to $G(p)$ is known as Kummer’s Conjecture. It is opportune to study the logarithm of this equation because the orthogonality property of characters
gives us
\[ \sum_{\chi \mod p, \text{ odd}} \log(L(s, \chi)) = \frac{p-1}{2} \left( \sum_{q^m \equiv 1 (p)} \frac{1}{mq^{ms}} - \sum_{q^m \equiv -1 (p)} \frac{1}{mq^{ms}} \right). \quad (1.2) \]

One can estimate this sum to the right of \( s = 1 \), where good estimates are available, and using a zero-free region of the \( L \)-functions, one can bound the derivative in a neighbourhood of \( s = 1 \). Masley and Montgomery [5] obtained with these key ingredients that \( |\log(h_p^-/G(p))| < 7 \log p \) for \( p > 200 \), which is strong enough to solve the class number one problem for cyclotomic fields.

Puchta [8] improved this approach by using analogous bounds on higher derivatives, and using a near zero-free region, namely the open ball \( B(1, \frac{1}{c \log p}) \) with center 1 and radius \( \frac{1}{c \log p} \), where \( c \) is some big enough constant. This is a zero-free region for all but possibly one \( L \)-function mod \( p \), which is necessarily quadratic and has one zero \( \beta \) in this region, which is necessarily real and simple and goes by the name of a Siegel zero. It is worth mentioning that if \( p = 1 \mod 4 \), the odd characters are not quadratic, hence have no Siegel zero. Puchta obtained \( \log(h_p^-/G(p)) = \log(1 - \beta) + O((\log 2p)^2) \).

Our proof will follow the main ideas from [8], but our practical implementation in section 3 is of a different nature, and yields

**Theorem 1.1.** If no Siegel zero is present among the odd Dirichlet \( L \)-functions of conductor \( p \), then the relative class number of \( \mathbb{Q}(\zeta_p) \) satisfies
\[ |\log(h_p^-/G(p))| \leq 2 \log_2(p) + O(\log_3(p)) \]

If there is a Siegel zero \( \beta \) present among the odd Dirichlet \( L \)-functions of conductor \( p \), then the relative class number of \( \mathbb{Q}(\zeta_p) \) satisfies
\[ |\log(h_p^-/G(p)) - \log(1 - \beta)| \leq 4 \log_2(p) + O(\log_3(p)) \]

Since \( \log(1 - \beta) \) is negative, an upper bound without this term may be deduced. Finally, we note that this result sharpens the best known estimate, by Lepistö [3]. Indeed, he proves an upper bound for \( \log(h_p^-/G(p)) \) with main term \( 5 \log_2(p) \).

2. Bounds around \( s = 1 \)

In this section we exploit formula (1.2), which gives a representation in terms of splitting behaviour in \( \mathbb{Q}(\zeta_p)/\mathbb{Q} \). We define
\[ \Pi(x, p, a) = \sum_{q^m \leq x, q^m \equiv a(p)} \frac{1}{mq^{ms}}, \]

where \( q^m \) ranges over the primepowers. A Brun-Titchmarsh style bound is given by the following lemma.

**Lemma 2.1.** For \( x > p \), and \( p > 500 \) we have that
\[ \Pi(x, p, \pm 1) \leq \frac{2x}{(p - 1)\log(x/p)} \]
Proof. When $x \geq p^2$, we start from the following inequality (see [5], Lemma 1)
\[
\Pi(x, p, \pm 1) \leq \pi(x, p, \pm 1) + \frac{4\sqrt{x}}{p} + \log x.
\]
In [7], the following strong version of the Brun-Titchmarsh inequality is proven.
\[
\pi(x, p, \pm 1) \leq \frac{2x}{(p-1)(\log(x/p) + 5/6)}
\]
Thus we only need to prove that
\[
\frac{4\sqrt{x}}{p} + \log x < \frac{2x}{(p-1)} \left( \frac{1}{\log(x/p)} - \frac{1}{\log(x/p) + 5/6} \right).
\]
By setting $x = pX$, $X \geq p$, it suffices to prove that
\[
g(X) := \frac{4}{\sqrt{p}} + \frac{\log(pX)}{\sqrt{X}} < h(X) := \frac{5\sqrt{X}}{3(\log X + 5/6)^2}.
\]
Now, $g(X)$ decreases for $X - e^2$ and $h(X)$ increases for $X \geq e^{19/6}$, hence it suffices to check that
\[
g(p) = \frac{4}{\sqrt{p}} + \frac{2\log(p)}{\sqrt{p}} < h(p) = \frac{5\sqrt{p}}{3(\log p + 5/6)^2}
\]
for $p \geq 500$. Now, $g(p)$ decreases for $p \geq 2$ and $h(p)$ increases for $p \geq e^{19/6}$, hence it suffices to check that $g(500) < h(500)$, which is clear.

When $p < x < p^2$, any two primepowers in the sum $\Pi(x, p, \pm 1)$ are necessarily coprime. Indeed, their quotient would be $1 \mod p$, so at least $p + 1$, implying that the smallest one should be less than $p^2/p + 1$. The only option then is that $p - 1 = 2^m$ and $p^2 - 1 = 2^k$, but except for $p = 3$ this is impossible. Thus, $\Pi(x, p, \pm 1) \leq N(x, Q, p, \pm 1) + \pi(Q)$, where $N(x, Q, p, a)$ is the number of integers $n \equiv a \mod p$, $n \leq x$ such that $n$ is not divisible by any prime number less than $Q$. We may bound $\pi(Q)$ trivially by $Q$, so that the quantity to be bounded is $N(x, Q, p, \pm 1) + Q$.

In the proof of the Brun-Titchmarsh inequality
\[
\pi(x, q, \pm 1) \leq \frac{2x}{(p-1)(\log(x/p))}
\]
using the large sieve, as in [6, p.42-44], the first step is to bound $\pi(x, q, \pm 1)$ by exactly the quantity $N(x, Q, p, \pm 1) + Q$. This shows that in this range of $x$, the large sieve method for the Brun-Titchmarsh inequality can be applied with the same success for primepowers as for primes.

Let us define $f(s)$ by
\[
f(s) = \left( \sum_{\chi(-1)=-1} \log L(s, \chi) \right) - \log(s - \beta),
\]
in case that any of the $L$-functions with $\chi$ odd has a so-called Siegel zero $\beta$ in $\left[ 1 - \frac{1}{c \log p}, 1 \right]$, where $c$ is some big enough constant. Otherwise, we leave out the term with the Siegel zero. In any case $f$ is holomorphic in $B(1, \frac{1}{c \log p})$. 

\]
Lemma 2.2. For any $c, p \geq 500$, and $\sigma \in [1, 1 + \frac{1}{c \log p}]$, we have the following estimates.

$$|f(\sigma)| \leq (1 + 1_\beta) \log \left( \frac{1}{\sigma - 1} \right) + \frac{3}{2}$$ (2.1)

$$|f^{(\nu)}(\sigma)| \leq (1 + 1_\beta + c_{p,\nu}) \frac{(\nu - 1)!}{(\sigma - 1)^\nu}$$ (2.2)

Where the notation $1_\beta$ stands for 1 if a Siegel zero is present and 0 otherwise, and we may choose the $c_{p,\nu}$ to be equal to $\log(2) \frac{c}{2} \nu \left( \frac{\nu}{\nu - 1} \right)! \log p + \log(2) + e^{-1} + \frac{c_{p,\nu}}{c^{\log \nu} \log(\nu)!}$.

Proof. The case $\nu = 0$ can be proven as in [5]. The estimates for the derivatives are stated in [8], but the statement is slightly incorrect and the proof omitted, so we will prove them here in full. We bound the sums occurring in the $\nu$-th derivative of (1.2) using Lemma 2.1 and partial summation.

$$\frac{p - 1}{2} \sum_{q = 1 \ (p)}^{2^m} m \frac{\log q\nu}{mq^{m\sigma}} = \frac{p - 1}{2} \int_{2p}^\infty \frac{(\log x)\nu d(\Pi(x, p, 1))}{x\sigma}$$

$$= \frac{p - 1}{2} \int_{2p}^\infty \frac{\sigma x^{\sigma - 1}(\log x)^\nu - \nu x^{\sigma - 1}(\log x)^{\nu - 1}}{x^{2\sigma}} \frac{x^\sigma}{\Pi(x, p, 1)dx}$$

$$\leq \int_{2p}^\infty \frac{\sigma (\log x)^\nu}{x^\sigma \log(x/p)} dx$$

$$= \frac{p\sigma}{p^\sigma} \int_{2}^\infty \frac{(\log x + \log p)^\nu}{x^\sigma \log x} dx =: I,$$

where we possibly omitted the first term $\frac{(p - 1)^{\log(p + 1)^\nu}}{2m(p + 1)^\nu}$ if $p + 1$ is a primepower $q^m$. If this is the case, then $q = 2$ and $m = \log(p + 1)/\log(2)$. This term is smaller than $\epsilon_1 \frac{(\nu - 1)!}{(\sigma - 1)!}$ for all $\sigma$ in the desired range for $\epsilon_1 = \frac{\log(2)}{2c^{\nu} (\nu - 1)! \log p}$.

We expand the integrand with the binomial theorem, and get

$$I = \frac{p\sigma}{p^\sigma} (\log p)^\nu \int_{2}^\infty \frac{1}{x^\sigma \log x} dx + \frac{p\sigma}{p^\sigma} \sum_{i = 0}^{\nu - 1} \frac{\nu! (\log p)^i}{(\nu - i)! i!} \int_{1}^\infty \frac{(\log x)^{\nu - i - 1}}{x^\sigma} dx$$

$$= \frac{p\sigma}{p^\sigma} (\log p)^\nu \int_{2}^\infty \frac{1}{x^\sigma \log x} dx + \frac{p\sigma}{p^\sigma} \sum_{i = 0}^{\nu - 1} \frac{\nu! (\log p)^i}{(\nu - i)! i!} \frac{((\sigma - 1) \log p)^i}{i!},$$

where we have used the identity

$$\int_{1}^\infty \frac{(\log x)^a}{x^\sigma} dx = \int_{0}^\infty \frac{t^a}{e^{(\sigma - 1)t}} dt = \frac{a!}{(\sigma - 1)^{a + 1}}.$$
We consider first the term
\[
\frac{p\sigma}{p^\sigma} (\log p)^\nu \int_2^\infty \frac{1}{x^\sigma \log x} \, dx = \frac{p\sigma}{p^\sigma} (\log p)^\nu \int_{\log 2}^\infty e^{-(\sigma-1)t} \frac{dt}{t} \\
\leq \frac{p\sigma}{p^\sigma} (\log p)^\nu \left( \int_1^1 \frac{1}{t} \, dt + \int_1^\infty e^{-t} \, dt \right) \\
\leq (\log p)^\nu \left( \log\left(\frac{1}{\sigma-1}\right) - \log_2(2) + e^{-1} \right).
\]

Because \( p\sigma \leq p^\sigma \). We now seek the \( \varepsilon_2 \) such that
\[
(\log p)^\nu \left( \log\left(\frac{1}{\sigma-1}\right) - \log_2(2) + e^{-1} \right) \leq \varepsilon_2 (\nu - 1)!.
\]

If we put \( \varepsilon_2 = \frac{\log_2(p) + \log(c) - \log_2(2) + e^{-1}}{e^{\nu(\sigma-1)!}} \), the inequality holds for \( \sigma \to 1 \) and for \( \sigma = 1 + \frac{1}{c \log p} \). One may check that the derivative of the difference does not have a zero in the interval under consideration if \( p > e^e \). Thus the difference is monotone, and the inequality holds throughout.

To deal with the rest of the terms efficiently, write \( X = (\sigma-1) \log p \leq 1/c \). Then we have for any integer \( B \geq 1 \)
\[
\frac{p\sigma}{p^\sigma} \sum_{i=0}^{\nu-1} \frac{X^i}{\nu - i}! \leq \frac{p\sigma}{p^\sigma} \sum_{i=0}^{B-1} \frac{X^i}{\nu - B}! + \frac{p\sigma}{p^\sigma} X^B \sum_{i=0}^{\nu-1} \frac{X^{i-B}}{(i-B)!} \\
\leq \frac{p\sigma}{p^\sigma} \frac{\nu}{\nu - B} e^X + \frac{p\sigma}{p^\sigma} \frac{\nu}{c^\nu B!} e^X = \frac{\nu \sigma}{\nu - B} + \frac{\nu \sigma}{c B!}
\]

We now put \( B = \lfloor \log \nu \rfloor \), and see that the to be bounded sum is bounded by
\[
(1 + \varepsilon_3) \frac{(\nu - 1)!}{(\sigma-1)!}, \text{ where } \varepsilon_3 = \frac{1}{c \log p} + \frac{\sigma}{\nu \log \nu} + \frac{\sigma \nu}{c \log \nu \log \nu}!
\]

One may now bound the \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \) by the coefficient of \( \frac{(\nu - 1)!}{(\sigma-1)!} \) except the \( 1/\beta \) in the statement of the lemma. We note that the sum over the primepowers congruent to \(-1 \mod p\) obeys the same bound with the same proof as above. One of the sums is strictly positive and the other is strictly negative, thus we have proven that
\[
|f^\nu(s) + (\log(\sigma - \beta))^{(\nu)}| \leq (1 + c_{p,\nu}) \frac{(\nu - 1)!}{(\sigma-1)!},
\]
or since \( \frac{(\nu - 1)!}{(\sigma-1)!} \leq \frac{(\nu - 1)!}{(\sigma-1)!} \),
\[
|f^\nu(s)| \leq (1 + 1/\beta + c_{p,\nu}) \frac{(\nu - 1)!}{(\sigma-1)!}.
\]

\[\Box\]

On the other hand we can prove the following bound on the derivatives of \( f \) to the right of \( s = 1 \), using the holomorphic property of \( f \) on \( B(1, \frac{1}{c \log p}) \), when \( c \) is big enough. We note that due to Kadiri ([2], Theorem 12.1) the value \( c = 6.4355 \) is big enough.
Lemma 2.3. For \( c > 6.4355, \frac{p-1}{\log p} > c, \) and \( \sigma \in [1, 1 + \frac{2}{c \log p}] \), we have that
\[
|f^{(\nu)}(\sigma)| \leq 2c^{\nu} \nu! \log^{\nu+1} p
\] (2.3)

Proof. Recall the lemma of Borel-Caratheodory (see [1], p. 12) which states that if \( g \) is holomorphic and \( \Re(g(s)) \leq M \) in \( B(\sigma_0, R) \) and \( g(\sigma_0) = 0 \), then
\[
|g^{\nu}(s)| \leq \frac{2M \nu!}{(R-r)^{\nu}}, \quad s \in B(\sigma_0, r).
\]
We wish to apply this to \( f(s) - f(\sigma_0) \). This function vanishes at \( \sigma_0 \), and is holomorphic as long as \( R \leq \sigma_0 - \frac{1}{c \log p} \).

For the bound on the real part, consider
\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = s \int_{1}^{\infty} \frac{\sum_{n \leq x} \chi(n)}{x^{s+1}} dx.
\]
Since \( |\sum_{n=1}^{x} \chi(n)| \leq \frac{p-1}{2} \), we have that
\[
|L(s, \chi)| \leq |s| \int_{1}^{\infty} \frac{|\sum_{n \leq x} \chi(n)|}{x^{s+1}} dx \leq \frac{|s| p}{2 \sigma}.
\]
This means that
\[
\Re(f(s)) \leq \frac{p-1}{2} (\log p + \log(|s|/2\sigma)) - \log(|s - \beta|).
\]
For \( s \) on the border of the domain determined by \( 3/4 < \Re(s) < 2, |\Im(s)| \leq \frac{1}{4}, |s|/2\sigma \leq \sqrt{10}/6 \) and say \( |s - \beta| > 1/8 \), thus this bound is smaller than \( \frac{p-1}{2} \log p \). Since \( f(s) \) is harmonic with at most logarithmic singularities in which \( \Re(f) \to -\infty \), the same bound holds also inside the domain. In the region \( \sigma > 1 \), consider the following estimation.
\[
|\Re(\log L(s, \chi))| = |\Re\left( \sum_{q^m} \frac{\chi(q^m)}{mq^m s} \right)| \leq \sum_{q^m} \frac{1}{mq^m s} = \log \zeta(\sigma) \leq \log\left(\frac{\sigma}{\sigma - 1}\right),
\]
thus if \( \sigma_0 > p/(p-1) \), then \( |\Re(f(\sigma_0))| \leq \frac{p-1}{2} \log(p) + \log(p-1) \). In conclusion, as long as \( \sigma_0 > p/(p-1) \),
\[
\Re(f(\sigma) - f(\sigma_0)) \leq p \log p.
\]
One retrieves the statement of the theorem by putting \( \sigma_0 = 1 + \frac{1}{c \log p}, R = \frac{2}{c \log p}, r = \frac{1}{c \log p} \). \( \square \)

3. Worst case scenario

Among all functions that satisfy the bounds from the preceding section, what is the largest value \( f(1) \) can attain? We define \( \sigma_\nu \) to be the point where the bound (2.2) and the absolute bound (2.3) coincide. We note that
\[
\sigma_\nu - 1 = \frac{1}{c \log p} \sqrt{\frac{1 + 1/\beta + c_p \nu}{2\nu p \log p}} \geq \frac{1}{c \log p} \sqrt{2\nu p \log p}.
\] (3.1)
The first factor of the class number of the \( p \)-th cyclotomic field

**Theorem 3.1.** For all \( p > 500 \), and \( c > 6.4355 \),

\[
|f(1)| \leq (1 + 1/2 + e^{1/c}) \log_2(p) + O(1),
\]

where the \( O(1) \)-term is bounded by \( (3 + e^{1/c}) \log(c) + 0.791e^{1/c} + 10.720 + \frac{0.943}{c} \).

**Proof.** We use the Taylor expansion of \( f \) with error term in integral form,

\[
f(1) = f(\sigma_\nu) + (1 - \sigma_\nu)f'(\sigma_\nu) + \frac{(1 - \sigma_\nu)^2}{2} f^{(2)}(\sigma_\nu) + \ldots
\]

\[+ \int_{\sigma_\nu}^{1} \frac{f^{(\nu)}(x)}{(\nu - 1)!} (1 - x)^{\nu - 1} dx.
\]

Now note that \( |f^{(\nu)}(x)| \) is bounded above by the bound (2.3) for all \( x \) between 1 and \( \sigma_\nu \), which is equal to \( |f^{(\nu)}(\sigma_\nu)| \). Using (2.1), (2.2) and (3.1), we get

\[
|f(1)| \leq |f(\sigma_\nu)| + \sum_{i=1}^{\nu} \frac{(\sigma_\nu - 1)^i}{i!} |f^{(i)}(\sigma_\nu)|.
\]

\[
\leq (1 + 1/2) \log\left(\frac{1}{\sigma_\nu - 1}\right) + 3/2 + \sum_{i=1}^{\nu} \frac{1 + 1/2 + c_{p,i}}{i}
\]

\[
\leq (1 + 1/2) \left( \log_2(p) + \log(c) + \frac{\log(2\nu p \log p)}{\nu} \right) + 3/2 + \sum_{i=1}^{\nu} \frac{1 + 1/2 + c_{p,i}}{i}.
\]

Upon taking \( \nu = \log p \), this first contribution is bounded by

\[
(1 + 1/2) \left( \log_2(p) + \log(c) + 1 + \frac{\log(2(\log p)^2)}{\log p} \right) + 3/2.
\]

In the rest of the terms, we find the first \( \nu \) terms of some converging series;

\[
\sum_{i=1}^{\nu} \frac{1}{c^i i!} \leq e^{1/c} - 1, \quad \sum_{i=1}^{\nu} \frac{\log(\nu)}{\nu(\nu - \log \nu)} \leq 1.90, \quad \sum_{i=1}^{\nu} \frac{1}{c^{\log \nu} [\log \nu]!} \leq 1.13.
\]

Using this and the well-known estimate \( \sum_{i=1}^{\nu} \frac{1}{i} \leq \log(\nu) + 1 \) we bound the last contribution as follows

\[
\sum_{i=1}^{\nu} \frac{1 + 1/2 + c_{p,i}}{i} \leq (1 + 1/2 + 1/c \log p)(\log(\nu) + 1) + (1 + 1/c \log p)3.03
\]

\[+ \left( \frac{\log(2)}{2 \log p} + \log_2(p) + \log(c) - \log_2(2) + e^{-1} \right)(c^{1/c} - 1).
\]

Gathering everything and filling in \( p = 500 \) for the terms converging to zero, we recover the statement of the theorem. \( \square \)

To finish the proof of Theorem 1.1, one now needs to plug the above estimate of \( f(1) \) into the logarithm of the formula (1.1), and check that the choice of \( c = \log_2(p)^{6.4355 / \log_2(500)} \) is permitted.

**Remark 3.2.** It is quite counterintuitive that a bigger value of \( c \) gives a better estimate in Theorem 3.1 while a smaller value of \( c \) means a bigger zero-free region, thus means a stronger input. In truth there is a tradeoff between
having \( \sigma_\nu \) big to control the main term coming from Lemma 2.2 and at the same time not too big to bound the term coming from \( \varepsilon_1 \) in the proof of Lemma 2.2. This \( \varepsilon_1 \) cannot be efficiently bounded by a lack of good bounds on the number of primes of the form \( ap + 1 \), where \( a \) is a small integer.

Remark 3.3. From (1.1) it is now clear that the general behaviour of \( h_p^- \) is dominated by \( G(p) \) and that the \( L \)-values can perturb this term only slightly. It is somewhat common (see [4]) to state upper bounds for \( h_p^- \) in terms of \( G(p) \), where \( 4\pi^2 \) is replaced by a smaller constant.

**Corollary 3.4.** We have that \( h_p^- \leq 2p\left( \frac{p}{39} \right)^{\frac{p+1}{4}} \), for \( p > 9649 \).

*Proof.* This follows from plugging in \( c = \frac{6.4355}{\log_2(500)} \) = 3.523 \( \log_2(p) \) in Theorem 3.1 and checking that

\[
|f(1)| \leq e^{\frac{p-1}{4} \log\left(\frac{4\pi^2}{39}\right)},
\]

whenever \( p > 9649 \). \( \square \)

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