On characterization of Dirichlet-to-Neumann map of Riemannian surface with boundary

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Abstract

Let \((M, g)\) be a smooth compact orientable two-dimensional Riemannian manifold \((surface)\) with a smooth metric tensor \(g\) and smooth connected boundary \(\Gamma\). Its \(DN\)-map \(\Lambda_g : C^\infty(\Gamma) \to C^\infty(\Gamma)\) is associated with the (forward) elliptic problem \(\Delta_g u = 0\) in \(M \setminus \Gamma\), \(u = f\) on \(\Gamma\), and acts by \(\Lambda_g f := \partial_\nu u^f\) on \(\Gamma\), where \(\Delta_g\) is the Beltrami-Laplace operator, \(u = u^f(x)\) is the solution, \(\nu\) is the outward normal to \(\Gamma\). The corresponding \textit{inverse problem} is to determine the surface \((M, g)\) from its \(DN\)-map \(\Lambda_g\).

We provide the necessary and sufficient conditions on an operator acting in \(C^\infty(\Gamma)\) to be the \(DN\)-map of a surface. In contrast to the known conditions by G.Henkin and V.Michel in terms of multidimensional complex analysis, our ones are based on the connections of the inverse problem with commutative Banach algebras.

0. Introduction

- Let \((M, g)\) be a smooth compact orientable two-dimensional Riemannian manifold with a smooth metric tensor \(g\) and smooth connected boundary \(\Gamma\). In what follows, we deal with the manifolds of this class only and, for short, call them the \textit{surfaces}. The Dirichlet-to-Neumann operator \((DN\text{-map})\)
\( \Lambda : C^\infty(\Gamma) \to C^\infty(\Gamma) \) of the surface is associated with the (forward) elliptic problem

\[
\begin{align*}
\Delta_g u &= 0 \quad \text{in} \ M \setminus \Gamma, \\
u &= f \quad \text{on} \ \Gamma
\end{align*}
\]

and acts by the rule

\[
\Lambda_g f := \partial_\nu u_f \quad \text{on} \ \Gamma,
\]

where \( \Delta_g \) is the Beltrami-Laplace operator, \( u = u_f(x) \) the solution, \( \nu \) the outward normal to \( \Gamma \). The corresponding inverse problem is to recover the surface \( (M, g) \) via the operator \( \Lambda_g \). In applications it is also known as the Electric Impedance Tomography problem. More generally, one needs to answer the question: to what extent does the DN-map determine the surface?

In the paper \[8\] by M.Lassas and G.Uhlman, it is shown that the DN-map determines the surface \( M \) up to conformal equivalence. In more detail, if \( (M, g) \) and \( (M', g') \) have the common boundary \( \Gamma \), and \( \Lambda_g = \Lambda_{g'} \) holds, then there exists a diffeomorphism \( \psi : M' \to M \) provided \( \psi|_\Gamma = \text{id} \) and a smooth positive function \( \rho \) obeying \( \rho|_\Gamma \equiv 1 \), such that \( g = \rho \psi_* g' \).

In the paper \[2\] by M.I.Belishev, the same result is established by the use of the connections between the EIT problem and the holomorphic function algebra of the surface. Moreover, the formulas, which express the topological invariants of the surface (Betti numbers) in terms of the DN-map, are provided. In \[4\] these formulas are generalized on the multidimensional case. The paper \[3\] extends the algebraic approach to nonorientable surfaces.

- All the above results refer to the situation when the operator \( \Lambda_g \) is given, and the existence of the surface, for which \( \Lambda_g \) is the DN-map, is a priori assumed. Thus, for such a \( \Lambda_g \), the solvability of the EIT problem is guaranteed. However, an important question remains about the inverse data characterization, i.e., on the necessary and sufficient conditions for an operator \( \Lambda \) to be the DN-map of a surface (to satisfy \( \Lambda = \Lambda_g \)). In other words, we are talking about a criterion for solvability of the EIT problem.

Such a criterion is presented in the paper \[6\] by G.M.Henkin and V.Michel in terms of multidimensional complex analysis. In the given paper, we propose a characterization based on the connections of EIT problem with Banach algebras. So, the novelty is a new formulation of the solvability conditions. The list of our conditions is also rather long but, however, we would venture to claim that our formulation is more transparent and simpler than that proposed in \[6\].
Our approach makes the use of the classical result [1] on the existence of a complex structure on the Gelfand spectrum of a commutative Banach algebra. It is the result, which provides the sufficiency of the proposed characteristic conditions.

- In the first section, the list of the necessary and sufficient conditions for solvability of the EIT problem is specified. The second section contains the proof of necessity of these conditions, the proof being based on some general properties of the DN-map. In the third section, the sufficiency is proved. For the convenience of the reader, we present the basic definitions and minimal required information on the Banach algebras.

1. Main result

- Let $\Gamma$ be a smooth curve diffeomorphic to a circle, $d\gamma$ its length element, $\gamma$ a continuous tangent field of unit vectors on $\Gamma$, and $\Lambda : C^\infty(\Gamma; \mathbb{R}) \mapsto C^\infty(\Gamma; \mathbb{R})$ a linear map. With $\Lambda$ one associates the map $\Upsilon : C^\infty(\Gamma; \mathbb{C}) \mapsto C^\infty(\Gamma; \mathbb{C})$,

$$\Upsilon \zeta := (\Lambda \Re \zeta - \partial_\gamma \Im \zeta) + i(\Lambda \Im \zeta + \partial_\gamma \Re \zeta).$$

(3)

As is easy to verify, it is a (complex) linear operator. Also, for $\eta \in C^\infty(\Gamma; \mathbb{C})$ and $z \in \mathbb{C}\setminus\eta(\Gamma)$, introduce the map $\Upsilon_{\eta,z} : C^\infty(\Gamma; \mathbb{C}) \mapsto C^\infty(\Gamma; \mathbb{C})$ as

$$\Upsilon_{\eta,z} \zeta := \Upsilon \frac{\zeta}{\eta - ze},$$

(4)

where $e$ is the function equal to 1 on $\Gamma$.

Let $I$ be the identity operator on $C^\infty(\Gamma; \mathbb{R})$, $\partial_\gamma C^\infty(\Gamma; \mathbb{R})$ the space of smooth real-valued functions with zero mean value on $\Gamma$, $J : \partial_\gamma C^\infty(\Gamma; \mathbb{R}) \mapsto \partial_\gamma C^\infty(\Gamma; \mathbb{R})$ the integration on $\Gamma$: $J\partial_\gamma = \partial_\gamma J = I$. By $\sharp S$ we denote the cardinality of $S$.

Our main result is the following.

**Theorem 1.** The operator $\Lambda$ is the DN-map of a surface if and only if it satisfies the conditions:

i. $e \in \text{Ker}\Upsilon$ and $\zeta_1\zeta_2 \in \text{Ker}\Upsilon$ for any $\zeta_1, \zeta_2 \in \text{Ker}\Upsilon$;

ii. if $\zeta_1, \zeta_2 \in \text{Ker}\Upsilon$, $\zeta_1/\zeta_2 \in C^\infty(\Gamma; \mathbb{C})$, and there exists a polynomial $P$, deg $P \geq 1$ such that $P(\zeta_1/\zeta_2) \in \text{Ker}\Upsilon$, then $\zeta_1/\zeta_2 \in \text{Ker}\Upsilon$;

iii. $\text{Ker}\Upsilon \cap C^\infty(\Gamma; \mathbb{C}) = \text{Ker}\Upsilon$ (the closure in $C(\Gamma; \mathbb{C})$);

iv. $\dim(\partial_\gamma + \Lambda J\Lambda)C^\infty(\Gamma; \mathbb{R}) < \infty$;
v. if \( \eta \in \text{Ker}\Upsilon \) and \( z \in \mathbb{C}\setminus \eta(\Gamma) \), then
\[
\dim [\Upsilon_{\eta,z}\text{Ker}\Upsilon] = \frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_{\gamma}\eta}{\eta - ze} d\gamma;
\] (5)

vi. for any \( x \in \Gamma \), there exist a function \( \eta_x \in \text{Ker}\Upsilon \) and a neighborhood \( U_x \ni \eta_x(x) \) diffeomorphic to an open disk \( D \subset \mathbb{C} \), such that
1. \( \partial_{\gamma}\eta_x(x) \neq 0 \) is valid and there is no points on \( \Gamma \), at which all derivatives \( \partial_{\gamma}^k\eta_x, \ k \geq 1 \) vanish simultaneously, whereas \( \sharp \eta_x^{-1}(\{z\}) < \infty \) holds for all \( z \in \mathbb{C} \);
2. \( \Upsilon_{\eta_x,z,e} = 0 \) holds on one connected component of \( U_x \setminus \eta_x(\Gamma) \), whereas \( \Upsilon_{\eta_x,z,e} \neq 0 \) holds on the other connected component;
3. the equation
\[
\Upsilon\left(\frac{\zeta - ce}{\eta_x - ze}\right) = 0 \text{ on } \Gamma
\]
has a solution \( c \in \mathbb{C} \) for any \( z \in U_x \) and \( \zeta \in \text{Ker}\Upsilon \);

vii. if \( \zeta, 1/\zeta \in \text{Ker}\Upsilon \), then \( \Lambda \log |\zeta| = \partial_{\gamma}\arg \zeta \).

As a comment, note the following. Condition i means that \( \text{Ker}\Upsilon \) is an algebra, whereas vi shows that this algebra must be rich enough to contain the functions \( \eta_x \) with the required properties. By condition vii.2, the function \( \eta_x - ze \) is or is not invertible in the algebra \( \text{Ker}\Upsilon \) depending on the position of \( z \) on the complex plane. Condition vii.3, from an algebraic point of view, means that \( (\eta_x - ze) \text{Ker}\Upsilon \) is an ideal in \( \text{Ker}\Upsilon \) of codimension 1, i.e., is a maximal ideal. Also, as is easy to see, the embedding \( g \in \text{Ker}\Upsilon \) implies \( \partial_{\gamma}\Re g, \partial_{\gamma}\Im g \in \text{Ker} [I + (\Lambda J)^2] \). The operator \( I + (\Lambda J)^2 \) is a key object of the papers [2, 3, 4].

The rest of the paper is devoted to the proof of Theorem [1].

2. Necessity

Here we show that any DN-map satisfies conditions i–vii.

• Suppose that \( \Lambda = \Lambda_g \) is the DN-map of some surface \((M, g)\). Recall that \( \gamma \) and \( \nu \) are the tangent and normal unit vector fields at the boundary \( \Gamma \).
Choose a continuous family of rotations $M \ni x \mapsto \Phi_x \in \text{End} \, T_x M$, 
\[ g(\Phi_x a, \Phi_x b) = g(a, b), \quad g(\Phi_x a, a) = 0, \quad a, b \in T_x M, \quad x \in M, \]
such that $\Phi \nu = \gamma$ on $\Gamma$. A function $w \in C^\infty(M; \mathbb{C})$ is called holomorphic if the Cauchy-Riemann condition $\nabla_g \Im w = \Phi \nabla_g \Re w$ holds in $M$. Let $w$ be holomorphic and $\zeta = w|_\Gamma$ be its trace on the boundary. The real functions $\Re w$ and $\Im w$ are harmonic in $\text{int} \, M$ and provide the solutions $\Re w = u^\Re$ and $\Im w = u^\Im$ to (1), (2). Restricting the Cauchy-Riemann conditions on $\Gamma$, one obtains
\[ \Lambda \Re \zeta = \partial_\nu \Re w = \partial_\gamma \Im \zeta, \quad \Lambda \Im \zeta = \partial_\gamma \Re w = -\partial_\nu \Re \zeta \quad \text{on } \Gamma, \quad (7) \]
which implies $\Upsilon(\zeta) = 0$ according to (3).

Now, suppose that $\zeta \in C^\infty(\Gamma; \mathbb{C})$ and $\Upsilon(\zeta) = 0$. Then the function $w := u^\Re + i u^\Im$ is holomorphic in $\text{int} \, M$. Indeed, since $\Upsilon(\zeta) = 0$, one has (7), i.e., $\nabla_g \Im w = \Phi \nabla_g \Re w$ holds on $\Gamma$. Let $U$ be an arbitrary neighborhood in $M$ diffeomorphic to the disc, and $\partial U \cap \Gamma$ contains a segment $\Gamma'$ of non-zero length. Since $\partial_\nu \Re w = u^\Re$ is harmonic in $U$, there exists a function $v$ such that $\nabla_g v = \Phi \nabla_g \Re w$ in $U$. Thus, $\partial_\nu \Re w = \partial_\gamma v$ and $\partial_\nu v = -\partial_\gamma \Re w$ on $\Gamma'$. Comparing with (7), one obtains $v = \Im w + \text{const}$, $\partial_\nu v = \partial_\gamma \Re w$ on $\Gamma'$. So, $v$ and $\Im w + \text{const}$ are harmonic in $U$ and have the same Cauchy data on $\Gamma'$. Due to the uniqueness of the solution to the Cauchy problem for the second order elliptic equations, $v$ coincides with $\Im w + \text{const}$ in $U$, and $\nabla_g \Im w = \nabla_g v = \Phi \nabla_g \Re w$ in $U$. Since $U$ is arbitrary, $\nabla_g \Im w = \Phi \nabla_g \Re w$ holds in $M$, and $w$ is holomorphic. So, we have proved that $\text{Ker} \, \Upsilon$ coincides with the set of traces on $\Gamma$ of all holomorphic smooth functions on $M$. Obviously, such a set is an algebra with respect to point-wise multiplication and $e \in \text{Ker} \, \Upsilon$. So, $i$ is valid.

- Let $\zeta_1, \zeta_2 \in \text{Ker} \, \Upsilon$, $\zeta = \zeta_1 / \zeta_2 \in C^\infty(\Gamma; \mathbb{C})$, and $P(\zeta) \in \text{Ker} \, \Upsilon$, where $P$ is a polynomial of degree $p > 1$. In view of the already proven, there exist holomorphic functions $w_1, w_2, w_P$ such that $w_1|_\Gamma = \zeta_1$, $w_2|_\Gamma = \zeta_2$, and $w_P|_\Gamma = P(\zeta)$. Then the function $w := w_1 / w_2$ is meromorphic in $\text{int} \, M$ and $w|_\Gamma = \zeta \in C^\infty(\Gamma; \mathbb{C})$. The last implies that the poles of $w$ do not accumulate to $\Gamma$ and the number of them is finite. The function $P(w)$ is also meromorphic and its poles coincide with those of $w$, while their multiplicities are $p$ times greater than those of $w$. Since $P(w) = P(\zeta) = w_P$ on $\Gamma$, the function $P(w)$ coincides with $w_P$ outside the poles of $w$ due to uniqueness of analytic
continuation. Then \( P(w) = w_P \) everywhere on \( M \). Thus, \( w \) is holomorphic and its trace \( \zeta \) belongs to Ker\( \Upsilon \). This proves ii.

- Due to the maximum principle, the set Ker\( \Upsilon \) coincides with the set of traces on \( \Gamma \) of all holomorphic continuous functions. Since the smooth elements of Ker\( \Upsilon \) are the traces of holomorphic smooth functions, one obtains iii.

The property iv follows from the equality

\[
\dim(\partial_\gamma + \Lambda J \Lambda)C^\infty(\Gamma; \mathbb{R}) = 1 - \mathcal{E}(M)
\]  

(see formula (1.6), [2]), where \( \mathcal{E}(M) \) is the Euler characteristics of \( M \).

- Suppose that \( \eta \in \text{Ker} \Upsilon \) and \( z \in \mathbb{C}\setminus \eta(\Gamma) \). Then there exists the holomorphic in int\( M \) function \( w_0 \) such that \( w_0|_\Gamma = \eta \). Denote by \( x_1, \ldots, x_l \) all the zeroes of \( w_0 - z \) and by \( m_1, \ldots, m_l \) their multiplicities. We make use of the argument principle\(^2\):

\[
\frac{1}{2\pi i} \int_\Gamma \frac{\partial_\gamma \eta}{\eta - z} d\gamma = \sum_{k=1}^l m_k.
\]  

(9)

Since \( \Gamma \) is smooth, the manifold \( (M, g) \) can be embedded into a larger non-compact smooth manifold \( (M', g') \), \( g'|_M = g \). For each \( k = 1, \ldots, l \) and \( s = 0, \ldots, m_k - 1 \), choose a holomorphic into \( M' \) function \( w_{k,s} \) such that \( x_1, \ldots, x_l \) are all zeroes of \( w_{k,s} \) on \( M' \) and multiplicity of \( x_j \) is equal to \( s \) if \( j = k \) and to \( m_j \) if \( j \neq k \). The existence of such \( w_{k,s} \) follows from Proposition 26.5, [5]. The linear combination \( \sum_{k,s} c_{k,s} w_{k,s} \) has no poles in \( M \) only if all \( c_{k,s} \) equal to zero. Denote \( \eta_{k,s} := w_{k,s}|_\Gamma \); then \( \Upsilon_{\eta,z}(\sum_{k,s} c_{k,s} \eta_{k,s}) = 0 \) only if all \( c_{k,s} \) are zeros. Hence, the functions

\[
\Upsilon_{\eta,z}(\eta_{k,s}), \quad k = 1, \ldots, l, \quad s = 0, \ldots, m_k - 1
\]

are linearly independent.

Now, suppose that \( w \in C^\infty(\Gamma; \mathbb{C}) \) is holomorphic in \( M \) and \( \zeta = w|_\Gamma \). For any \( k = 1, \ldots, l \), there exist \( d_{k,s} \in \mathbb{C} \) (\( l = 1, \ldots, m_k \)) such that \( w - \sum_s d_{k,s} w_{k,s} \) has a zero of multiplicity \( \geq m_k \) at \( x = x_k \). Since \( x_k \), \( k' \neq k \) is a zero of multiplicity \( m_k \) for each each \( w_{k',s} \), the function \( w - \sum_{k,s} d_{k,s} w_{k,s} \) have at each \( x_k \) a zero of multiplicity \( \geq m_k \). Therefore, the ratio

\[
\frac{w - \sum_{k,s} d_{k,s} w_{k,s}}{w_0 - z}
\]

\(^2\)for a compact Riemann surface with boundary, the argument principle can be obtained by simple modification of the proofs of Theorem 3.17 and Corollary 3.18, [7].
is holomorphic in $M$ and, hence, $\Upsilon_{\eta,z}(\zeta - \sum_{k,s} d_{k,s} \eta_{k,s}) = 0$. This means that (10) is a basis in $\Upsilon_{\eta,z}Ker\Upsilon$. In particular, $\dim \Upsilon_{\eta,z}Ker\Upsilon = \sum_{k=1}^{l} m_k$.

Comparing with (9), one arrives at $v_0$.

• Let $x$ be an arbitrary point of $\Gamma$. According to Proposition 26.5, there exists the holomorphic in $M'$ function $w_x$ such that $x$ is unique zero of $w_x$ and its multiplicity is equal to one. For any $c \in \mathbb{C}$, the function $w_x - c$ has only finite number of zeros on $M$ (otherwise, there would be an accumulation point of such zeros due to the compactness of $M$) and each zero of $w_x - c$ is of finite multiplicity. This implies vi.1 for the function $\eta_x := w_x |_\Gamma \in Ker\Upsilon$.

Next, since $\nabla \Re w_x(x) \neq 0$, the map $w_x : M \to \mathbb{C}$ is a bijection of a neighborhood $V_0$ of $x$ and neighborhood $w_x(V_0)$ of the zero, and $|w_x(x')| > 0$ holds for any $x' \in M \setminus \{x\}$. Let $K$ be a compact in $M'$ that contains $M \cup V_0$. Then the set $K \setminus V_0$ is also compact and $|w_x(x')| > c_0 > 0$ for any $x' \in K \setminus V_0$. Choose a neighborhood $V_1 \subset V_0$ sufficiently small to obey $|w_x(x')| < c_0/2$ for any $x \in V_1$. Then the pre-image $w_x^{-1}(\{z\})$ of any $z \in w_x(V_1)$ is contained in $V_0$ and, since $w_x$ is a bijection of $V_0$ and $w_x(V_0)$, it consists of a single element. Denote $U_x := w_x(V_1)$, $U_{x,1} := U_x \setminus w_x(M)$, and $U_{x,2} := U_x \cap w_x(M)$. The function $\frac{1}{w_x - z}$ has no poles on $M$ for any $z \in U_{x,1}$ and has a simple pole on $M$ for any $z \in U_{x,1}$. Thus, $\Upsilon_{\eta_x,z} e = \Upsilon_{w_x^{-1}(z)} = 0$ for all $z \in U_{x,1}$ and $\Upsilon_{w_x^{-1}(z)} \neq 0$ for any $z \in U_{x,2}$. This yields vi.2.

Finally, suppose that $w \in C^\infty(M; \mathbb{C})$ is holomorphic in $M$ and $\zeta := w |_\Gamma$. If $z \in U_{x,1}$ and $c \in \mathbb{C}$, then the function $\frac{w - c}{w_x - z}$ is holomorphic in $M$. Hence, any $c \in \mathbb{C}$ is a solution of (6). Now, suppose that $z \in U_{x,2}$. Since $\frac{1}{w_x - z}$ has a simple pole at the point $w_x^{-1}(z)$ and no other poles on $M$, the function $\frac{w - c}{w_x - z}$ is holomorphic in $M$ if and only if $c = w(w_x^{-1}(z))$. So, (6) has a unique solution $c = w(w_x^{-1}(z))$ for any $z \in U_{x,1}$. This proves vi.3.

• Suppose that $\zeta, 1/\zeta \in Ker\Upsilon$. Then $\zeta = w |_\Gamma$, where $w, 1/w$ are holomorphic functions in $M$. Let $U$ be an arbitrary simply connected neighborhood in $M$. Since $w$ have no zeroes in $U$, each branch of $\log w$ is holomorphic function in $U$. In particular, $\log |w| = \Re \log w$ is harmonic in $U$. Also, $\log |w|$ is single-valued function on the whole $M$. Then $\log |w| = w \log |\zeta|$ is a solution of (1). Hence, $\partial_\zeta \log |w| = \Lambda \log |\zeta|$ on $\Gamma$. Now choose $U$ in such a way that $\overline{U} \cap \Gamma$ is a segment $\Gamma' \subset \Gamma$ of nonzero length. Since each branch of $\log w$ is holomorphic in $U$ and smooth up to $\Gamma'$, from Cauchy-Riemann conditions it follows that

$$\partial_\zeta \log |w| = \partial_\zeta \Re \log w = \partial_\zeta |w| = \partial_\zeta \arg w = \partial_\zeta \arg \zeta$$
The necessity is proved.

3. Sufficiency

Here we assume that $\Lambda$ obeys i.-vii. and construct a Riemannian surface $(M, g)$ such that its DN-map is $\Lambda$, i.e., $\Lambda = \Lambda_g$ holds. Before that, we recall the known facts and definitions that will be used in the construction.

- A commutative Banach algebra is a (complex) Banach space $(\mathfrak{A}, \| \cdot \|)$ equipped with the multiplication operation $\mathfrak{A} \times \mathfrak{A} \ni \eta, \zeta \mapsto \eta \zeta \in \mathfrak{A}$ satisfying $\eta \zeta = \zeta \eta$, $\| \eta \zeta \| \leq \| \eta \| \| \zeta \|$ for all $\eta, \zeta \in \mathfrak{A}$. Algebra $\mathfrak{A}$ is unital if there exists $e \in \mathfrak{A}$ such that $e \eta = \eta$ holds for all $\eta \in \mathfrak{A}$. Element $\eta \in \mathfrak{A}$ is invertible if there exists $\eta^{-1} \in \mathfrak{A}$ such that $\eta^{-1} \eta = e$. The set of all $z \in \mathbb{C}$ for which $\eta - ze$ is noninvertible is called the spectrum of $\eta$ and is denoted by $\text{Sp}_\mathfrak{A} \eta$, such a set being compact.

A character of the commutative Banach algebra $\mathfrak{A}$ is a nonzero homomorphism $\chi : \mathfrak{A} \mapsto \mathbb{C}$. Each character $\chi$ is a continuous map: one has

$$|\chi(\eta)| \leq \| \eta \|, \quad \eta \in \mathfrak{A}. \quad (11)$$

The set of characters $\mathfrak{A}$ is called the spectrum of algebra $\mathfrak{A}$. For an $\eta \in \mathfrak{A}$, its Gelfand transform $\hat{\eta} : \mathfrak{A} \mapsto \mathbb{C}$ is defined as

$$\hat{\eta}(\chi) := \chi(\eta), \quad \chi \in \mathfrak{A}.$$ 

For any $\hat{\eta}$, the image $\hat{\eta}(\mathfrak{A}) \subset \mathbb{C}$ coincides with the spectrum $\text{Sp}_\mathfrak{A} \eta$.

Spectrum $\hat{\mathfrak{A}}$ is endowed with the canonical Gelfand ($\ast$-weak) topology, with respect to which it is a compact Hausdorff space. The Gelfand transforms $\{ \hat{\eta} \mid \eta \in \mathfrak{A} \}$ constitute a subalgebra in $C(\hat{\mathfrak{A}})$, which separates points of $\hat{\mathfrak{A}}$. The space $\hat{\mathfrak{A}}$ is connected if and only if there is no nontrivial idempotents $\eta = \eta^2$, $\eta \neq 0, e$ in $\mathfrak{A}$.

A closed subset $B \subset \mathfrak{A}$ is called a boundary of $\mathfrak{A}$ if $\max_B |\hat{\eta}| = \max_{\mathfrak{A}} |\hat{\eta}|$ for any $\eta \in \mathfrak{A}$. The intersection of all boundaries is called the Shilov boundary of $\mathfrak{A}$ and denoted by $b\mathfrak{A}$.

The key fact that we use in the proof of sufficiency is the fundamental Bishop-Aupetit-Wermer analytic structure theorem: see Theorem 2.2, [1] or Chapter 11, [9].
Theorem 2. Let \( \eta \in \mathfrak{A} \), the set \( \hat{\eta}(\hat{\mathfrak{A}}) \setminus \hat{\eta}(\mathfrak{bA}) \) is non-empty, and \( V \) is its connected component. Suppose that the set \( \{ z \in V \mid \| \hat{\eta}^{-1}(\{z\}) \| < \infty \} \) is of nonzero Lebesgue measure. Then \( \| \hat{\eta}^{-1}(\{z\}) \| \leq N < \infty \) for any \( z \in V \) and the subset \( \hat{\eta}^{-1}(V) \subset \hat{\mathfrak{A}} \) has the structure of 1-dim complex analytic manifold, on which all functions \( \hat{\zeta} \ (\zeta \in \mathfrak{A}) \) are holomorphic.

The rest of the proof of Theorem 1 is as follows. We construct a Riemann surface \( M \) as the spectrum \( \hat{\mathfrak{A}} \) of some Banach function algebra \( \mathfrak{A} \) provided by conditions i and iii. Then, using Theorem 2 and the condition v, we endow a part \( \Omega_\eta \subset \hat{\mathfrak{A}} \) with the structure of Riemannian surface, and this part depends on the element \( \eta \in \mathfrak{A} \). The condition vi enables one, by varying the elements \( \eta \in \mathfrak{A} \), to cover the whole spectrum \( \hat{\mathfrak{A}} \) by its analytic parts \( \Omega_\eta \) and thus endow the \( \hat{\mathfrak{A}} \) with the structure of Riemannian surface. Also, due to Theorem 2 the Gelfand transforms of elements of \( \mathfrak{A} \) form a subalgebra in the algebra of holomorphic smooth functions on \( M \). By conditions ii and iv, this subalgebra coincides with the set of all holomorphic smooth functions on \( M \). By the latter, we show that \( \Lambda \) coincides with the DN-map of the surface \( M \) on all traces on \( \Gamma \) of real parts of holomorphic smooth functions on \( M \), the set of such traces being of finite codimension. To check that \( \Lambda \) coincides with the DN-map of \( M \) on all other functions from \( C^\infty(\Gamma; \mathbb{R}) \), we use the remaining condition vii.

So, we proceed to prove the sufficiency of the conditions of Theorem 1.

• In view of i, the set \( \operatorname{Ker} \Upsilon \) is a unital (sub)algebra in \( C(\Gamma; \mathbb{C}) \). The closure

\[
\mathfrak{A} := \overline{\operatorname{Ker} \Upsilon} \subset C(\Gamma; \mathbb{C})
\]

is a unital commutative Banach algebra with the norm \( \| \zeta \| := \max_{\Gamma} |\zeta| \). We denote its spectrum \( \hat{\mathfrak{A}} \) by \( M \). Recall that \( M \) is a Hausdorff compact space. Also, \( M \) is connected: indeed, if \( \eta^2 = \eta \) on \( \Gamma \), then \( \eta(x) = 0 \) or \( 1 \) for any \( x \in \Gamma \); since \( \eta \) is continuous and \( \Gamma \) is connected, this means that \( \eta = 0 \) or \( 1 \). Note that the set of smooth elements of \( \mathfrak{A} \) coincides with \( \operatorname{Ker} \Upsilon \) due to property iii.

The Dirac measures \( \delta_x : \mathfrak{A} \ni \zeta \mapsto \zeta(x) \in \mathbb{C} \), \( x \in \Gamma \) constitute a subset \( \delta_\Gamma \subset M \). In view of (11) and the definition of the norm \( \| \cdot \| \), one has \( |\hat{\eta}(\chi)| = |\chi(\eta)| \leq \| \eta \| = \max_{x \in \Gamma} |\delta_x(\eta)| \) for any \( \chi \in M \) and \( \eta \in \mathfrak{A} \). Hence, \( \delta_\Gamma \) is a boundary of \( \mathfrak{A} \) and thus it contains the Shilov boundary \( \mathfrak{bA} \) of \( \mathfrak{A} \).

• Our first goal is to endow \( M \setminus \delta_\Gamma \) with the structure of an analytic manifold via Theorem 2. To verify the conditions of Theorem 2 we prove that
\( \hat{\eta}^{-1}(\{z\}) \) is finite for any \( \eta \in \text{Ker} \Upsilon = \mathfrak{A} \cap C^\infty(\Gamma; \mathbb{C}) \) and \( z \in \hat{\eta}(\mathfrak{A}) \setminus \eta(\Gamma) \). The proof is based on a bijection between the characters from \( \hat{\eta}^{-1}(\{z\}) \) and the characters over a certain finite-dimensional factor-algebra \( \mathfrak{A}_{\eta, z} \) which is constructed below. Let’s get down to implementing this plan.

Let \( \eta \in \text{Ker} \Upsilon \) and \( z \in \mathbb{C} \setminus \eta(\Gamma) \); then the function \( \eta - ze \) is invertible in \( C^\infty(\Gamma; \mathbb{C}) \) (but not necessarily in \( \mathfrak{A} \)). Consider the main ideal \( I_{\eta, z} := (\eta - ze) \mathfrak{A} \) in \( \mathfrak{A} \). It is closed in \( \mathfrak{A} \); indeed, if \( I_{\eta, z} \ni \zeta_k \to \zeta \) in \( \mathbb{C} \setminus \eta(\Gamma) \), then the convergence \( \mathfrak{A} \ni \frac{\zeta_k}{\eta - ze} \to \frac{\zeta}{\eta - ze} \) holds by \( \frac{1}{\eta - ze} \in C^\infty(\Gamma; \mathbb{C}) \). Since \( \mathfrak{A} \) is Banach, \( \frac{\zeta}{\eta - ze} \in \mathfrak{A} \) and \( \zeta \in I_{\eta, z} \). Since \( \text{Ker} \Upsilon \) is dense in \( \mathfrak{A} \), the set \( \mathfrak{A}_\infty := \{ \zeta + I_{\eta, z} \mid \zeta \in \text{Ker} \Upsilon \} \) is dense in \( \mathfrak{A}_{\eta, z} \). The function \( \zeta \in \text{Ker} \Upsilon \) belongs to \( \mathfrak{A}_\infty \) if and only if \( 0 = \Upsilon(\frac{\zeta}{\eta - ze}) = \Upsilon_{\eta, z}(\zeta) \).

Introduce the factor-algebra

\[
\mathfrak{A}_{\eta, z} := \mathfrak{A} / I_{\eta, z}
\]

with the factor-norm \( \| \zeta + I_{\eta, z} \|_{\eta, z} := \inf_{\zeta \in I_{\eta, z}} \| \zeta + \tilde{\zeta} \| \); here and in what follows we denote by \( \zeta + I_{\eta, z} \) the equivalence class in \( \mathfrak{A}_{\eta, z} \) of element \( \zeta \in \mathfrak{A} \). Due to definition of the factor-norm and the equality \( \text{Ker} \Upsilon = \mathfrak{A} \), the set \( \mathfrak{A}_\infty := \{ \zeta + I_{\eta, z} \mid \zeta \in \text{Ker} \Upsilon \} \) is dense in \( \mathfrak{A}_{\eta, z} \). Let us prove that the algebra \( \mathfrak{A}_{\eta, z} \) is finite-dimensional. To this end, consider a linear map \( \mathcal{G}_{\eta, z} : \mathfrak{A}_\infty \to C^\infty(\Gamma; \mathbb{C}) \) defined by the rule

\[
\mathcal{G}_{\eta, z}(\zeta + I_{\eta, z}) = \Upsilon_{\eta, z}(\zeta).
\]

The map \( \mathcal{G}_{\eta, z} \) is well-defined and its kernel is trivial. Indeed, if \( \zeta_1 + I_{\eta, z} = \zeta_2 + I_{\eta, z} \in \mathfrak{A}_\infty \), then \( \zeta_1 - \zeta_2 \in I_{\eta, z} \cap C^\infty(\Gamma; \mathbb{C}) = I_{\eta, z} \) and \( \Upsilon_{\eta, z}(\zeta_1) - \Upsilon_{\eta, z}(\zeta_2) = 0 \). Similarly, if \( \mathcal{G}_{\eta, z}(\zeta + I_{\eta, z}) = 0 \), then \( \Upsilon_{\eta, z}(\zeta) = 0 \) and thus \( \zeta \in I_{\eta, z} \) i.e. \( \zeta + I_{\eta, z} \) is the zero element in \( \mathfrak{A}_{\eta, z} \). Note that \( \mathcal{G}_{\eta, z} \mathfrak{A}_\infty = \Upsilon_{\eta, z} \text{Ker} \Upsilon \). Since the map \( \mathcal{G}_{\eta, z} \) is a bijection of \( \mathfrak{A}_\infty \) and \( \mathfrak{A}_{\eta, z} \mathfrak{A}_\infty \), one has

\[
\dim \mathfrak{A}_\infty = \dim \left[ \Upsilon_{\eta, z} \text{Ker} \Upsilon \right].
\]

In view of condition \( v \), the right-hand side is equal to the integral

\[
\frac{1}{2\pi i} \int_\Gamma \frac{\partial_\gamma \eta}{\eta - ze} \, d\gamma.
\]

Since the functions \( \eta \) and \( \frac{1}{\eta - ze} \) are smooth, this integral is finite. So, \( \dim \mathfrak{A}_\infty \) is finite and, since \( \mathfrak{A}_\infty \) is dense in \( \mathfrak{A}_{\eta, z} \), one has

\[
\dim \mathfrak{A}_{\eta, z} = \frac{1}{2\pi i} \int_\Gamma \frac{\partial_\gamma \eta}{\eta - ze} \, d\gamma.
\]
Note that the right-hand side is the number $d(z)$ of revolutions of the image $\eta(\Gamma) \subset \mathbb{C}$ around the point $z$; this number depends only on the connected component $V$ of $\mathbb{C}\setminus\eta(\Gamma)$ that contains $z$. If $d(z) = 0$, then $\dim \mathfrak{A}_{\eta,z} = 0$ and $e \in \mathcal{I}_{\eta,z} = \mathfrak{A}$. This means that $\eta - ze$ is invertible in $\mathfrak{A}$ and $z \not\in \text{Sp}_{\mathfrak{A}}\eta = \hat{\eta}(M)$. Thus,

$$
\hat{\eta}(M)\setminus\eta(\Gamma) = \{z \in \mathbb{C}\setminus\eta(\Gamma) \mid d(z) > 0\}.
$$

- Now, we show that the set $\hat{\eta}^{-1}\{z\}$ is finite, $z \in V$ being the same as before. Let $\bar{\chi}$ be a character on the algebra $\mathfrak{A}_{\eta,z}$; then the rule

$$
\chi(\zeta) := \bar{\chi}(\zeta + \mathcal{I}_{\eta,z}) \tag{12}
$$

defines a character $\chi \in M$ that vanishes on $\mathcal{I}_{\eta,z}$. Hence, we have $\chi(\eta - ze) = 0$ and $\chi(\eta) = z$. Conversely, suppose that $\chi \in M$ and $\chi(\eta) = z$ (then, obviously, $\chi(\mathcal{I}_{\eta,z}) = \{0\}$). Then the same rule (12) defines the character $\bar{\chi}$ on $\mathfrak{A}_{\eta,z}$. Thus, we have

$$
\#\hat{\mathfrak{A}}_{\eta,z} = \#\hat{\eta}^{-1}\{z\}. \tag{13}
$$

Suppose that $\bar{\chi}_1, \ldots, \bar{\chi}_N$ are the distinct characters in $\hat{\mathfrak{A}}_{\eta,z}$. Since the Gelfand transforms of the elements $\zeta + \mathcal{I}_{\eta,z} \in \mathfrak{A}_{\eta,z}$ separate the points of $\hat{\mathfrak{A}}_{\eta,z}$, there exists $\zeta_{ij} \in \mathfrak{A}$ such that $\bar{\chi}_i(\zeta_{ij} + \mathcal{I}_{\eta,z}) = 1$ and $\bar{\chi}_j(\zeta_{ij} + \mathcal{I}_{\eta,z}) = 0$. Denote $q_i := \prod_{j \neq i}(\zeta_{ij} + \mathcal{I}_{\eta,z})$, then $\bar{\chi}_j(q_i) = \delta_{ij}$. In particular, $q_i$, $i = 1, \ldots, N$ are linearly independent in $\mathfrak{A}_{\eta,z}$. Therefore, $N \leq \dim \mathfrak{A}_{\eta,z} = d(z)$. So, we see that $\#\hat{\mathfrak{A}}_{\eta,z} \leq d(z)$ and, by (13), arrive at $\#\hat{\eta}^{-1}\{z\} \leq d(z) < \infty$.

- Suppose that $z \in \hat{\eta}(M)\setminus\eta(\Gamma)$ (to provide $\hat{\eta}(M)\setminus\eta(\Gamma) \neq \emptyset$, one can take $\eta = \eta_x$ for any function $\eta_x$ obeying vi.1). Denote by $V$ the connected component of $\mathbb{C}\setminus\eta(\Gamma)$ that contains $z$. Obviously, $V$ is open and, hence, it has a nonzero Lebesgue measure. Moreover, $1 \leq d(z) < \infty$ and $d(z') = d(z)$ is valid for any $z' \in V$. For such a $z'$, the dimension $d(z')$ of algebra $\mathfrak{A}_{\eta,z'}$ is finite and nonzero. Hence, $\mathcal{I}_{\eta,z'} \neq \mathfrak{A}$, i.e., $\eta - z'e$ is noninvertible. Therefore $z' \in \hat{\eta}(M)$ and, thus, for the whole $V$ the embedding $V \subset \hat{\eta}(M)\setminus\eta(\Gamma)$ holds.

Since $\eta(\Gamma) = \hat{\eta}(\delta_T)$ and $\partial\mathfrak{A} \subset \delta_T$, the set $V$ does not intersect with $\hat{\eta}(\partial\mathfrak{A})$. In view of Theorem 2, the set $\hat{\eta}^{-1}(V) \subset \hat{\mathfrak{A}}$ has the structure of 1-dim complex analytic manifold on which all functions $\hat{\zeta}$ ($\zeta \in \mathfrak{A}$) are analytic. Thus, for any character $\chi \in \hat{\eta}^{-1}(V)$ there exist an open (in the Gelfand topology) neighborhood $U \ni \chi$ and a homeomorphism $\kappa : U \to D$ onto an open disk $D \subset \mathbb{C}$ such that any function $\hat{\zeta} \circ \kappa^{-1}$ ($\zeta \in \mathfrak{A}$) is holomorphic on $D$. In other words, every $\chi \in \hat{\eta}^{-1}(V)$ does possess a local analytic coordinate $\hat{\eta}$. 
As a result, we can represent the spectrum $M$ as the disjoint union:

$$M = M' \cup \delta \cup \tilde{M},$$

where

$$M' := \bigcup_{\eta \in \text{Ker} \Upsilon} \hat{\eta}^{-1}(C\setminus \eta(\Gamma))$$

is the set of characters that can be provided with the local coordinate by the choice of a suitable $\eta \in \text{Ker} \Upsilon$, and $\tilde{M} := M \setminus (M' \cup \delta)$.

- Let us show that $\tilde{M} = \emptyset$. Suppose, on the contrary, that $\chi \in \tilde{M}$ and $\eta$ satisfies condition vi.1 (as such $\eta$, one can choose any $\eta_x$ from vi). Then, $z := \hat{\eta}(\chi) \in \eta(\Gamma)$ and the set $\delta \cap \hat{\eta}^{-1}(\{z\}) = \{\delta_x \mid \eta(x) = z\}$ is finite. Denote by $\delta_{x_1}, \ldots, \delta_{x_l}$ all characters from $\delta \cap \hat{\eta}^{-1}(\{z\})$. Choose $\eta_{x_k}$ from condition vi in such a way that $\eta_{x_k}(x_k) = z$ (this condition can always be satisfied since $\eta_x$ in vi are determined up to a constant). In view of vi.3, any $\zeta \in \text{Ker} \Upsilon$ can be represented as $\zeta = c_{\zeta} e + \zeta'_k$, where $\Upsilon_{\eta_{x_k} \zeta}(\zeta'_k) = 0$. Then $\tilde{\zeta}'_k := \frac{\zeta'_k}{\eta_{x_k} - z e}$ belongs to Ker $\Upsilon = \mathfrak{A} \cap C^\infty(\Gamma; \mathbb{C})$, whence $\zeta'_k(x_k) = 0$ and $c_{\zeta,k} = \zeta(x_k)$. Thus, $\zeta = \zeta(x_k) e + (\eta_{x_k} - z e) \tilde{\zeta}'_k$ with $\tilde{\zeta}'_k \in \text{Ker} \Upsilon$ and one has

$$\chi(\zeta) - \zeta(x_k) = [\chi(\eta_{x_k}) - z] \chi(\tilde{\zeta}'_k).$$

In particular, if $\chi(\eta_{x_k}) = z$, then $\chi$ coincides with $\delta_{x_k}$ on the dense set Ker $\Upsilon$ in $\mathfrak{A}$ and, hence, on the whole $\mathfrak{A}$. Thus, the embedding $\chi \in \tilde{M}$ shows that $\chi(\eta_{x_k}) \neq z$ for any $k = 1, \ldots, l$. Then the element

$$\theta = \prod_{k=1}^l [\eta_{x_k} - \eta_{x_k}(x_k) e]$$

satisfies $\chi(\theta) \neq 0$ and $\theta(x_k) = 0$ for any $k = 1, \ldots, l$.

Denote

$$\eta_{\varepsilon, \varphi} := \eta + \varepsilon^2 e^{i \varphi} \chi(\theta)^{-1} \theta,$$

where $\varepsilon > 0$ and $\varphi \in [0, 2\pi)$. Due to condition vi.1, the function $\eta - ze$ has a zero of multiplicity $\leq m < \infty$ at each $x_k$ and there are no other zeros of $\eta - ze$ on $\Gamma$. Thus, the pre-image $\eta^{-1}(B_{\varepsilon})$ of the $\varepsilon$-neighborhood $U_{\varepsilon}$ of $z$ is contained in $O(\varepsilon^{1/m})$-neighborhood of the set $\{x_1, \ldots, x_M\}$ in $\Gamma$. Therefore,

$$|\eta_{\varepsilon, \varphi}(x) - \eta(x)| = \varepsilon^2 |\chi(\theta)^{-1} \theta(x)| = \varepsilon^2 |\theta(x_k) + O(\text{dist}(x, x_k))| =
$$

$$= \varepsilon^2 |0 + O(\varepsilon^{1/m})| = O(\varepsilon^{2+1/m}),$$

(14)
where \( x_k \) is chosen to be the nearest point to \( x \). For \( x \in \Gamma \backslash \eta^{-1}(B_\varepsilon) \), one has \( |\eta(x)| \geq \varepsilon \), whence

\[
|\eta_{\varepsilon, \varphi}(x)| = |\eta(x) + \varepsilon^2 O(1)| \geq \varepsilon - O(\varepsilon^2).
\]  

Estimates (14), (15) show that, for sufficiently small \( \varepsilon \), the set \( B_{\varepsilon/2} \) does not intersect with the fragment \( \eta_{\varepsilon, \varphi}(\Gamma \backslash \eta^{-1}(B_\varepsilon)) \) of \( \eta_{\varepsilon, \varphi}(\Gamma) \) while the fragment \( \eta_{\varepsilon, \varphi}(\eta^{-1}(B_\varepsilon)) \) is contained in \( O(\varepsilon^{2+1/m}) \)-neighbourhood of \( \eta(\eta^{-1}(B_\varepsilon)) \). Thus, it is possible to choose \( \varepsilon \) and \( \varphi \) such that \( \hat{\eta}_{\varepsilon, \varphi}(\chi) = \chi(\eta_{\varepsilon, \varphi}) = z + \varepsilon^2 e^{i\varphi} \notin \eta_{\varepsilon, \varphi}(\Gamma) \). This means that \( \chi \in \tilde{\eta}_{\varepsilon, \varphi}^{-1}(\mathbb{C} \backslash \eta(\Gamma)) \subset M' \), so that we arrive at the contradiction and prove that \( \tilde{M} = \emptyset \).

- Now, we coordinatize \( \delta_\Gamma \). Let \( \delta_x (x \in \Gamma) \) be an arbitrary character from \( \delta_\Gamma \). Consider the map

\[
\hat{\eta}_x : \hat{\eta}_x^{-1}(U_x) \mapsto \mathbb{C},
\]

where \( \eta_x, U_x \) are the same as in condition vi. Due to condition vi.2, for any \( z \) from one connected component \( U_{x,1} \) of \( U_x \eta_x(\Gamma) \) one has \( \frac{1}{\eta_x - ze} \in \text{Ker} \Upsilon \subset \mathfrak{A} \).

Hence \( z \notin \text{Sp}_\mathfrak{A} \eta_x = \hat{\eta}_x(M) \). Now, suppose that \( z \in U_x \backslash U_{x,1} \) and \( \zeta \in \text{Ker} \Upsilon \). By vi.2, one has \( \frac{1}{\eta_x - ze} \notin \text{Ker} \Upsilon \), i.e., either \( z \in \eta_x(\Gamma) \) or \( \frac{1}{\eta_x - ze} \) is smooth on \( \Gamma \) and does not belong to \( \mathfrak{A} \). Thus, \( z \in \text{Sp}_\mathfrak{A} \eta_x = \hat{\eta}_x(M) \). Also, due to vi.3, there exists \( c_{\zeta, z} \in \mathbb{C} \) such that \( \frac{c_{\zeta, z} x e}{\eta_x - ze} =: \tilde{\zeta}_{\zeta, z} \in \text{Ker} \Upsilon \subset \mathfrak{A} \). If \( \chi \in \hat{\eta}_x^{-1}(\{z\}) \), then one has

\[
\chi(\zeta) = \chi(c_{\zeta, z} e + (\eta_x - ze)\tilde{\zeta}_{\zeta, z}) = c_{\zeta, z} + (\chi(\eta_x) - z)\chi(\tilde{\zeta}_{\zeta, z}) =
\]

\[
= c_{\zeta, z} + (\hat{\eta}_x(\chi) - z)\chi(\tilde{\zeta}_{\zeta, z}) = c_{\zeta, z}.
\]

Since \( \zeta \in \text{Ker} \Upsilon \) is arbitrary and \( \text{Ker} \Upsilon \) is dense in \( \mathfrak{A} \), this means that \( \chi(\zeta) = \chi'(\zeta) \) for any \( \chi, \chi' \in \hat{\eta}_x^{-1}(z) \) and \( \zeta \in \mathfrak{A} \). Thus, \( \# \hat{\eta}_x^{-1}(\{z\}) = 1 \) and the map \( \hat{\eta}_x : \hat{\eta}_x^{-1}(U_x) \mapsto U_x \backslash U_{x,1} \) is a bijection.

Since the expansion

\[
\zeta = c_{\zeta, z} e + (\eta_x - ze) \tilde{\zeta}_{\zeta, z} \quad \text{with} \quad c_{\zeta, z} = \tilde{\zeta} \circ \hat{\eta}_x^{-1}(z) = \tilde{\zeta} \left( \hat{\eta}_x^{-1}(z) \right) \quad \text{and} \quad \tilde{\zeta}_{\zeta, z} \in \text{Ker} \Upsilon
\]

holds for any \( z \in U_x \backslash U_{x,1} \) and \( \zeta \in \text{Ker} \Upsilon \), one can iterate it to obtain

\[
\zeta = c_{\zeta, z} e + (\eta_x - ze) \left[ c'_{\zeta, z} e + (\eta_x - ze) \tilde{\zeta}'_{\zeta, z} \right] \quad \text{with} \quad c_{\zeta, z} = \tilde{\zeta} \circ \hat{\eta}_x^{-1}(z),
\]

\[
c'_{\zeta, z} = \tilde{\zeta}_{\zeta, z} \circ \hat{\eta}_x^{-1}(z) \quad \text{and} \quad \tilde{\zeta}'_{\zeta, z} \in \text{Ker} \Upsilon
\]

13
for any \( z \in U_x \setminus U_{x,1} \). Then one has
\[
\hat{\zeta} \circ \hat{n}_x^{-1}(z') - \hat{\zeta} \circ \hat{n}_x^{-1}(z) = \\
\begin{aligned}
&\left[\hat{n}_x^{-1}(z')\right] \left[\hat{\zeta} \circ \hat{n}_x^{-1}(z)e + (\eta_x - ze)(\hat{\zeta}_{\zeta,z} \circ \hat{n}_x^{-1}(z)e + (\eta_x - ze)\hat{\zeta}_{\zeta,z}'\right] - \\
&\hat{\zeta} \circ \hat{n}_x^{-1}(z) = (z' - z) \left[\hat{\zeta}_{\zeta,z} \circ \hat{n}_x^{-1}(z) + (z' - z) : \hat{\zeta}_{\zeta,z}' \circ \hat{n}_x^{-1}(z')\right]
\end{aligned}
\]  

(16)

for any \( z, z' \in U_x \setminus U_{x,1} \). In view of (11),
\[
|\hat{\zeta}' \circ \hat{n}_x^{-1}(z)| = ||\hat{n}_x^{-1}(z)(\zeta')|| \leq \|\zeta'\|, \quad \zeta' \in \mathbb{A}.
\]

Then the function \( z' \mapsto \hat{\zeta}_{\zeta,z}' \circ \hat{n}_x^{-1}(z') \) is bounded on \( U_x \setminus U_{x,1} \) and (16) implies that the function \( z \mapsto \hat{\zeta} \circ \hat{n}_x^{-1}(z) \) is continuous on \( U_x \setminus U_{x,1} \) for any \( \zeta \in \text{Ker} \Upsilon \).

The same is true for the function \( z' \mapsto \hat{\zeta}_{\zeta,z}' \circ \hat{n}_x^{-1}(z') \). Then (16) implies also that, for any \( \zeta \in \text{Ker} \Upsilon \), the function \( z \mapsto \hat{\zeta} \circ \hat{n}_x^{-1}(z) \) is holomorphic on \( U_x \setminus \overline{U_{x,1}} \) and its partial derivatives are continuous on \( U_x \setminus U_{x,1} \). In view of definition of the Gelfand topology, the map \( \eta_x : \hat{n}_x^{-1}(U_x) \mapsto U_x \setminus U_{x,1} \) is a homeomorphism. So, any character \( \delta_x \in \mathfrak{t} \) is coordinatizable in the following sense: there exists a neighbourhood \( \Gamma := \hat{n}_x^{-1}(U_x) \) (in the Gelfand topology) of \( \delta_x \) and the local coordinate \( \hat{\eta}_x : V \mapsto U_x \setminus U_{x,1} \) in which all functions \( \hat{\zeta} \) (\( \zeta \in \text{Ker} \Upsilon \)) are holomorphic on \( U_x \setminus \overline{U_{x,1}} \) and continuous differentiable up to (smooth) curve \( U_x \cap \eta_x(\Gamma) \). Note that, in view of vi, a., the map \( \eta_x^{-1}(U_x) \ni x' \mapsto (\Re \eta_x(\delta_{x'}), \Im \eta_x(\delta_{x'})) \) is a diffeomorphism.

- We have proved above that the whole \( M \) is coordinatizable i.e. each character \( \chi \in M \) has the open (in the Gelfand topology) neighbourhood \( V_\chi \) and the homeomorphism \( \kappa_\chi : V_\chi \mapsto U_\chi \subset \mathbb{C} \) such that 1) the set \( U'_\chi := \kappa_\chi(V_\chi \setminus \chi(\mathfrak{t})) \) is open, 2) \( U'_\chi \setminus \overline{U'_\chi} \subset \partial U'_\chi \) is empty or it is the fragment of smooth curve, and 3) each function \( \hat{\zeta} \circ \kappa_\chi^{-1} (\zeta \in \text{Ker} \Upsilon) \) is holomorphic on \( U'_\chi \) and continuous differentiable up to \( U'_\chi \subset \partial U'_\chi \).

Now, we construct a biholomorphic atlas on \( M \) using \( \{V_\chi, \kappa_\chi\}_{\chi \in M} \). The collection \( \{V_\chi\}_{\chi \in M} \) is an open cover of \( M \) and, since \( M \) is compact, there exists a finite subcover \( \{V_{\chi_k}\}_{k=1}^L \). Denote \( V_k := V_{\chi_k} \) and \( \kappa_k := \kappa_{\chi_k} \). Suppose that \( V_k \cap V_l \neq \emptyset \) and denote \( W_k := \kappa_k((V_k \cap V_l) \setminus \mathfrak{t}) \), \( W_l := \kappa_l((V_k \cap V_l) \setminus \mathfrak{t}) \).

Choose an arbitrary nonconstant \( \zeta \in \text{Ker} \Upsilon \) (for example, one of \( \eta_x \) from condition iii.), then \( \hat{\zeta} \circ \kappa_k^{-1}, \hat{\zeta} \circ \kappa_l^{-1} \) are holomorphic on \( W_k \) and \( W_l \), respectively. In particular, any zero of \( \nabla \Re(\hat{\zeta} \circ \kappa_k^{-1}) \) on \( W_k \) is isolated. If \( \kappa_k(\chi) \in W_k \) does not coincide with zero of \( \nabla \Re(\hat{\zeta} \circ \kappa_k^{-1}) \), then there exists the neighbourhood
\(V\) of \(\chi\) such that \(\zeta \circ \chi_k^{-1} : \kappa_k(V') \mapsto \zeta \circ \chi_k^{-1}(W')\) is biholomorphic map. So, the function

\[\kappa_k \circ \kappa_l^{-1} = \kappa_k \circ \zeta^{-1} \circ \zeta \circ \kappa_l = (\zeta \circ \kappa_l^{-1})^{-1} \circ (\zeta \circ \kappa_l^{-1})\]

is holomorphic on \(\kappa_l(V')\). So, \(\kappa_k \circ \kappa_l^{-1}\) is holomorphic on \(W_l\) except for only some isolated points. Since \(\kappa_k \circ \kappa_l^{-1}\) is continuous on \(W_l\), we find that \(\kappa_k \circ \kappa_l^{-1}\) is holomorphic on the whole \(W_l\). The same reason shows that \(\kappa_l \circ \kappa_k^{-1}\) is holomorphic on \(W_k\) and, thus, the transition function \(\kappa_k \circ \kappa_l^{-1}\) is biholomorphic. So, we have proved that \(\{V_k := V_{\chi_k}, \kappa_k := \kappa_k\}_{k=1}^L\) is a biholomorphic atlas on \(M\). Endowed with this atlas, \(M\) is a Riemann surface with boundary \(\delta\). Moreover, the map \(\delta : \Gamma \ni x \mapsto \delta_x \in \delta\Gamma\) is diffeomorphism. In what follows, we identify \(\Gamma\) and \(\delta\Gamma\) via the map \(\delta\).

- Now, we introduce the metric \(g\) and the rotation \(\Phi\) on \(M\) which are consistent with the metrics and the tangent field \(\gamma\) on \(\Gamma\). Endow \(M\) with the metric tensor \(g' = \sum_{k=1}^L \psi_k g_k\), where \(g_k^{ij} = \delta^{ij}\) in local coordinates \(\kappa_k\), and \(\{\psi_k\}_{k=1}^L\) is a partition of unity on \(M\): \(\psi_k \circ \kappa_l^{-1}\) is smooth for any \(l, \psi_k \geq 0\), \(\text{supp} \psi_k \subset V_k\), and \(\sum_{k=1}^L \psi_k = 1\). Since the transition functions are biholomorphic, the tensor \(g'\) is of the form \(\sum_{k=1}^L \psi_k \nabla \Re(\kappa_k \circ \kappa_l^{-1})^2 \delta^{ij}\) in any local coordinates \(\kappa_k\). Tensor \(g'\) induces the new metrics \(d\gamma' = q(x)d\gamma\) on \(\Gamma \equiv \delta\Gamma\), where \(q > 0\) is smooth on \(\Gamma\) due to condition vi.1. Introducing a smooth conformal multiplier \(\rho\), such that \(\rho = q^{-1}\) on \(\Gamma\), we obtain the new metric tensor \(g = \rho g'\) which is consistent with the original metric on \(\Gamma\).

Choose a continuous family of rotations \(M \ni x \mapsto \Phi_x \in \text{End} T_x M\),

\[g(\Phi_x a, \Phi_x b) = g(a, b), \quad g(\Phi_x a, a) = 0, \quad \forall a, b \in T_x M, \quad x \in M\]

such that \(\Phi_1^1 = \Phi_2^1 = 0, \Phi_1^2 = -\Phi_2^2 = 1\) in local coordinates \(x_1 = \Re \kappa_k, x_2 = \Im \kappa_k\). For any \(k\) and \(\zeta \in \text{Ker} \hat{\Upsilon}\), the function \(\hat{\zeta} \circ \kappa_k\) is holomorphic, whence

\[\nabla \Re \hat{\zeta} = \hat{\Phi} \nabla \Re \hat{\zeta} \in M \setminus \Gamma. \quad (17)\]

So, any function \(\hat{\zeta}\) (\(\zeta \in \text{Ker} \hat{\Upsilon}\)) is holomorphic in \(M \setminus \Gamma\) (in the sense of Cauchy-Riemann conditions [17]) and continuously differentiable up to \(\Gamma\). In particular, any functions \(\Re \hat{\zeta}\) and \(\Im \hat{\zeta}\) are harmonic in \(M \setminus \Gamma\) and continuously differentiable up to \(\Gamma\). Let \(\nu\) be the outward normal on \(\Gamma\). Then \(\gamma' = \Phi \nu\) is the tangent field on \(\Gamma\) and, hence, it coincides with \(s\gamma\), where \(s = 1\) or \(-1\). Choose some \(\eta \in \text{Ker} \hat{\Upsilon}\) and \(z \in \text{Sp}_\Re \eta \setminus \eta(\Gamma)\), then \(\hat{\eta} = ze\) have at least one
zero on \( M \). Note that \( \hat{\eta} = \eta \) on \( \Gamma \). Consider the integral
\[
\int_{\Gamma} \frac{1}{2\pi i} \frac{\partial_{\gamma} \hat{\eta}}{\hat{\eta} - z} d\gamma = s \int_{\Gamma} \frac{1}{2\pi i} \frac{\partial_{\gamma} \eta}{\eta - ze} d\gamma.
\]
In view of the argument principle, the integral in the left-hand side coincides with the number of zeroes of \( \hat{\eta} \) counted with their multiplicities, and, thus, it is positive. The integral in the right-hand side is positive in view of (5). Therefore, \( s = 1 \) and \( \gamma = \gamma' \).

• Suppose that \( f \in \text{Ker}(\partial_{\gamma} + \Lambda J\Lambda) \). Denote \( h := J\Lambda f (= J\Lambda J\partial_{\gamma} f) \) and \( \zeta = f + ih \). Then \( \partial_{\gamma} h = \Lambda f, \partial_{\gamma} f = -\Lambda h \) and, hence, \( \zeta \in \text{Ker}\Upsilon \). In view of what already proven, the functions \( u := \Re \zeta, v := \Im \zeta \) are harmonic in \( M \setminus \Gamma \) and continuously differentiable up to \( \Gamma \). In particular \( u = u^f \) and \( \partial_{\nu} u = \Lambda_{g} f \), where \( u^f \) is the solution of (1), (2) and \( \Lambda_{g} \) is the DN-map of the above constructed \( (M, g) \). Moreover, the Cauchy-Riemann condition (17) holds. Passing in (17) to the trace on \( \Gamma \), one obtains
\[
\Lambda_{g} f = \partial_{\nu} u = \partial_{\gamma} h = \Lambda f, \quad \partial_{\nu} v = -\partial_{\gamma} f.
\]
Since \( f \) is arbitrary, we have proved that \( \text{Ker}(\partial_{\gamma} + \Lambda J\Lambda) \subset \text{Ker}(\partial_{\gamma} + \Lambda_{g} J\Lambda_{g}) \) and \( \Lambda f = \Lambda_{g} f \) for any \( f \in \text{Ker}(\partial_{\gamma} + \Lambda J\Lambda) \).

• Let us show that \( \text{Ker}(\partial_{\gamma} + \Lambda J\Lambda) = \text{Ker}(\partial_{\gamma} + \Lambda_{g} J\Lambda_{g}) \). By iv, the dimension \( q \) of the factor-space \( \text{Ker}(\partial_{\gamma} + \Lambda_{g} J\Lambda_{g})/\text{Ker}(\partial_{\gamma} + \Lambda J\Lambda) \) is finite. In view of (3), one has
\[
\text{Ker}\Upsilon = \{ f + iJ\Lambda f + ic \mid f \in \text{Ker}(\partial_{\gamma} + \Lambda J\Lambda), \ c \in \mathbb{R} \}. \tag{18}
\]
Denote by \( \mathfrak{A}^\infty \) the algebra of traces of all holomorphic smooth functions on \( M \); obviously, \( \text{Ker}\Upsilon \) is a subalgebra of \( \mathfrak{A}^\infty \). From Cauchy-Riemann conditions on \( \Gamma \), the representation
\[
\mathfrak{A}^\infty := \{ f + iJ\Lambda f + ic \mid f \in \text{Ker}(\partial_{\gamma} + \Lambda_{g} J\Lambda_{g}), \ c \in \mathbb{R} \}.
\]
is valid. Comparison of the last two formulas shows that the algebra \( \mathfrak{A}^\infty \) is a finite-dimensional extension of the algebra \( \text{Ker}\Upsilon \), and dimension of the factor-space \( \mathfrak{A}^\infty/\text{Ker}\Upsilon \) is equal to \( q \).

Suppose that \( q > 0 \) and choose the elements \( \theta_1, \ldots, \theta_q \in \mathfrak{A}^\infty \) linearly independent modulo \( \text{Ker}\Upsilon \). Then any \( \theta \in \mathfrak{A}^\infty \) can be represented as
\[
\theta = \sum_{k=1}^{q} c_k(\theta) \theta_k + \tilde{\theta}, \tag{19}
\]
where \( c_q(\theta) \in \mathbb{C} \) and \( \tilde{\theta} \in \ker \Upsilon \). Take any nonconstant \( \eta \in \ker \Upsilon \). Representation (19) implies
\[
\eta \theta_l = \sum_{k=1}^{q} T_{kl} \theta_k + \tilde{\theta}_l, \tag{20}
\]
where \( T \) is a complex \( q \times q \)-matrix and \( \tilde{\theta}_l \in \ker \Upsilon \). Choose an arbitrary eigenpair \( \lambda, X = (X_1, \ldots, X_q)^{tr} \) of \( T \) and denote
\[
\Theta := \sum_{k} X_k \theta_l, \quad \tilde{\Theta} := \sum_{l=1}^{q} X_l \tilde{\theta}_l.
\]
From (20) it follows that
\[
\eta \Theta = \sum_{l=1}^{q} X_l \eta \theta_l = \sum_{k=1}^{q} \left( \sum_{l=1}^{q} T_{kl} X_k \right) \theta_k = \lambda \sum_{k=1}^{q} X_k \theta_k + \sum_{l=1}^{q} X_l \tilde{\theta}_l = \lambda \Theta + \tilde{\Theta}.
\]
Note that \( \eta - \lambda e \) does not vanish identically on any segment \( \Gamma' \) of \( \Gamma \) of non-zero length (indeed, since \( \hat{\eta} \) is holomorphic and smooth on \( M \), the equality \( \eta = \lambda e \) on \( \Gamma' \) implies \( \eta = \lambda e \) on the whole \( \Gamma \)). So,
\[
\Theta := \frac{\tilde{\Theta}}{\eta - \lambda e}, \tag{21}
\]
holds on \( \Gamma \), where both numerator and denominator are elements of \( \ker \Upsilon \). Note that \( X \neq 0 \) and \( \Theta \notin \ker \Upsilon \). Similarly, representation (19) yields
\[
\Theta' := \sum_{k=1}^{q} N_{kl} \theta_k + \tilde{\Theta}_l, \quad l = 1, \ldots, q, \tag{22}
\]
where \( N \) is a complex \( q \times q \)-matrix and \( \tilde{\Theta}_l \in \ker \Upsilon \).

If \( \det N = 0 \), then there exists a non-zero \( Y = (Y_1, \ldots, Y_q)^{tr} \in \ker N \) and the polynomial \( P(\Theta) = \sum_{l=1}^{q} Y_l \Theta' \) admits the representation
\[
P(\Theta) = \sum_{k=1}^{q} \left( \sum_{l=1}^{q} N_{kl} Y_l \right) \theta_k + \sum_{l=1}^{q} Y_l \tilde{\Theta}_l = 0 + \sum_{l=1}^{q} Y_l \tilde{\Theta}_l.
\]
Therefore \( P(\Theta) \in \ker \Upsilon \) and, due to (21) and condition ii, one has \( \Theta \in \ker \Upsilon \), which leads to a contradiction.
If \( \det N \neq 0 \) and \( N' \) is the inverse matrix to \( N \), then (22) implies

\[
\theta_s - \sum_{l=1}^{q} N'_l \Theta^l = \sum_{l=1}^{q} N'_l \tilde{\Theta}_l \in \text{Ker } \Upsilon.
\]

This means that \( \Theta, \Theta^2, \ldots, \Theta^q \) are linearly independent modulo \( \text{Ker } \Upsilon \). So, we can assume that \( \theta_k = \Theta_k \). Now, formula (19) renders

\[
\mathcal{R}(\Theta) := \Theta^{q+1} - \sum_{k=1}^{q} c_k \theta_k \in \text{Ker } \Upsilon,
\]

where \( c_k \in \mathbb{C} \). Since the polynomial \( \mathcal{R} \) is of degree \( q + 1 > 0 \), the inclusion \( \mathcal{R}(\Theta) \in \text{Ker } \Upsilon \), formula (21) and condition ii yield \( \Theta \in \text{Ker } \Upsilon \). This contradiction means that \( A_\infty = \text{Ker } \Upsilon \) and \( q = 0 \). Thus, it is proved that \( \text{Ker } \Upsilon \) is algebra of traces of all holomorphic smooth functions on \( M \) and \( \text{Ker } (\partial + \Lambda J \Lambda) = \text{Ker } (\partial + \Lambda g J \Lambda g) \). In particular, from (8) it follows that \( \dim (\partial + \Lambda J \Lambda) C^\infty(\Gamma; \mathbb{R}) = 1 - \mathcal{X}(M) \), where \( \mathcal{X}(M) \) is the Euler characteristic of \( M \).

Thus, we have proved that \( \Lambda \) coincides with DN-map \( \Lambda_g \) of the surface \((M, g)\) on the subspace

\[
\mathfrak{R} := \text{Ker } (\partial + \Lambda J \Lambda) = \text{Ker } (\partial + \Lambda_g J \Lambda_g)
\]

(23) of codimension \( r := 1 - \mathcal{X}(M) \) in \( C^\infty(\Gamma; \mathbb{R}) \). To complete the proof of sufficiency, it remains to show that \( \Lambda f_1 = \Lambda_g f_1, \ldots, \Lambda f_r = \Lambda_g f_r \), where \( f_1, \ldots, f_r \) are some functions from \( C^\infty(\Gamma; \mathbb{R}) \) linearly independent modulo \( \mathfrak{R} \). Before that, recall the terminology associated with vector fields on the Riemannian manifolds and some well-known facts.

The vector fields are the \( TM_x \)-valued functions on \( M \) (the cross-sections of \( TM \)). A field of the form \( b = \nabla_g \varphi \) is called potential, \( \varphi \) being a potential. A field \( a \) is harmonic if \( \text{div}_g a = \text{div}_g (\Phi a) = 0 \) holds. The rotation \( \Phi \) preserves the harmonicity. Each harmonic field is locally potential. If \( b = \nabla_g \varphi \) is harmonic then the potential \( \varphi \) is a harmonic function: \( \Delta_g \varphi = 0 \), the opposite being also true.

So, let \( f_1, \ldots, f_r \) be linearly independent modulo \( \mathfrak{R} \), \( u_j \) the solution of problem (1), (2) with \( f = f_j \). The vector-fields \( a_j := \Phi \nabla_g u_j \) are harmonic in \( M \). Note that any non-zero linear combination of \( a_j \) is not a potential field in \( M \). Indeed, if \( \sum_{j=1}^{r} c_j a_j = \nabla_g v \), then the function \( w := u + iv \), where
$u := \sum_{j=1}^{r} c_j u_j$, is holomorphic in $M$. Then $w|_\Gamma \in \mathfrak{A}^\infty(M) = \text{Ker} \Upsilon$ and $\Re w|_\Gamma = \sum_{j=1}^{r} c_j f_j \in \mathfrak{K}$ in view of (18). Since $f_k$ are linearly independent modulo $\mathfrak{K}$, all $c_j$ equal zero.

Although $a_j$ are not potential on $M$, they can be represented as gradients of some multi-valued functions $V_j$ which are defined on an appropriate cover $\hat{M}$ of the surface $M$. The cover $\hat{M}$ is constructed in the following way. Let $D$ be a surface diffeomorphic to an open disk in $\mathbb{R}^2$ and $\partial D = \Gamma$. Gluing up the boundaries of $M$ and $D$, we obtain the closed compact surface $M' = M \cup D$ of genus $\text{gen} M' = 1 - \frac{2 \chi(M')}{2} = 1 - \frac{2 \chi(M) + 1}{2} = \frac{r}{2}$.

As is well known, the metric tensor $g$ and rotation $\Phi$ on $M$ can be extended to the (smooth) metric tensor $g'$ and rotation $\Phi'$ on the whole $M'$.

Let $\hat{M}'$ be the universal covering of $M'$ (see, for definition, §5, [5]), which is a simply connected Riemann surface, and $\pi' : \hat{M}' \to M'$ the projection, which is a local homeomorphism. Tensor $g'$ and rotation $\Phi'$ on $M'$ induce the tensor $g := g'|_M$ and rotation $\Phi := \Phi'|_M$ on $M$. As a result, $\pi' : (\hat{M}', g') \to (M', g')$ turns out to be a local isometry. At last, we get the required cover for $(M, g, \Phi)$ as the collection $(\hat{M}, \pi, g, \Phi)$, where $\hat{M} := \hat{M}' \setminus \pi'^{-1}(\partial D)$ is the surface with the boundary $\partial \hat{M} = \pi'^{-1}(\Gamma)$, $g := g'|_{\hat{M}}$, $\Phi := \Phi'|_{\hat{M}}$, and $\pi := \pi'|_{\hat{M}}$.

Recall that the solutions $u_j$ and fields $a_j$ correspond to the functions $f_1, \ldots, f_r$ which are linearly independent modulo $\mathfrak{K}$ (see (23)). Introduce the vector fields $A_j := \pi_* a_j = \Phi \nabla g (u_j \circ \pi)$ on $\hat{M}$ and the functions $M \ni x \mapsto V_j(x) = \int_{\mathcal{L}} g(A_j, l) \, dl \in \mathbb{R}$, where $\mathcal{L}$ is an arbitrary curve in $\hat{M}$ that connects a fixed point $x_0 \in M$ to a point $x$. In what follows, we denote by $l$ and $dl$ the unit tangent vector and the length element on the curve, respectively. Since $\hat{M} = \hat{M}' \setminus \pi'^{-1}(\partial D)$ is no longer simply connected, one needs to check that $V_j$ are single-valued on $\hat{M}$. To this end, it suffices to show that $\int_{\Gamma} g(A_j, l) \, dl = 0$ for any connected component $\hat{\Gamma}$ of $\pi^{-1}(\Gamma)$. Since $\hat{\Gamma}$ is isometric to $\Gamma$, one needs to check only that $\int_{\Gamma} g(a_j, \gamma) \, d\gamma = 0$. By the Green formula, one has

$$\int_{\Gamma} g(a_j, \gamma) \, d\gamma = \int_{\Gamma} g(\Phi \nabla g u_j, \gamma) \, d\gamma = \int_{\Gamma} \partial_n u_j \, d\gamma = \int_{M} \Delta_g u_j \, dx = 0$$

in view of harmonicity of $u_j$. So, we have provided the functions $V_j$ such that
\[ \nabla_V g_j = A_j = \dot{\Phi} \nabla_g (u_j \circ \pi) \text{ holds on } \mathbb{M}. \] 
This means that the functions
\[ W_j := u_j \circ \pi + i V_j, \quad j = 1, \ldots, r \] 
are holomorphic on \((\mathbb{M}, g)\), whereas the Cauchy-Riemann conditions \( \nabla_g \Im W_j = \dot{\Phi} \nabla_g \Re W_j \) hold.

- We are going to show that the functions \( f_1, \ldots, f_r \) can be chosen in such a way that \( e^{W_j} = w_j \circ \pi \), where \( w_j \) are holomorphic functions in \( M \).

Introduce the groups
\[ \text{Deck}(\mathbb{M}/M) := \{ \phi \mid \phi \text{ is automorphism of } \mathbb{M}, \pi \circ \phi = \pi \}, \]
\[ \text{Deck}(\mathbb{M}'/M') := \{ \phi' \mid \phi \text{ is automorphism of } \mathbb{M}', \pi' \circ \phi' = \pi' \} \]
of fiber-wise automorphisms of \( \mathbb{M} \) and \( \mathbb{M}' \), respectively (see, e.g., [5], 5.4).

Obviously, if \( \phi' \in \text{Deck}(\mathbb{M}'/M') \), then \( \phi'|_M \in \text{Deck}(\mathbb{M}/M) \). Conversely, if \( \phi \in \text{Deck}(\mathbb{M}/M) \), then it can be lifted to \( \phi' \in \text{Deck}(\mathbb{M}'/M') \) such that \( \phi'|_M = \phi \).

Indeed, if \( x \) belongs to a connected component \( \tilde{D} \) of \( \pi'^{-1}(D) \), then \( \phi'(x) \) is uniquely determined by its projection \( \pi'(\phi'(x)) = \pi'(x) \) and by the fact that the boundary of the connected component of \( \pi'^{-1}(D) \) containing \( \phi'(x) \) must coincide with \( \phi(\partial D) \). So, the map \( \phi' \mapsto \beta \phi' = \phi'|_M \) is an isomorphism of the groups \( \text{Deck}(\mathbb{M}'/M') \) and \( \text{Deck}(\mathbb{M}/M) \).

Denote by \( \pi_1(M') \) the fundamental group of \( M' \) and by \([L]\) the homotopy class of a closed curve \( L \) in \( M' \). In view of Proposition 5.6, [5], the groups \( \pi_1(M') \) and \( \text{Deck}(\mathbb{M}'/M') \) are isomorphic. The isomorphism
\[ \alpha : \text{Deck}(\mathbb{M}'/M') \leftrightarrow \pi_1(M') \]
is constructed as follows. Let \( \phi' \in \text{Deck}(\mathbb{M}'/M') \). Choose an arbitrary point \( x \in M' \) and a curve \( \mathcal{L}_{\phi'} \) which connects \( x \) to \( \phi'(x) \). Then \( \pi'(\mathcal{L}_{\phi'}) \) is a closed curve in \( M' \) due to the equality \( \pi'(\phi'(x)) = \pi'(x) \). It turns out that the homotopy class \([\pi'(\mathcal{L}_{\phi'})]\) of the curve \( \pi'(\mathcal{L}_{\phi'}) \) does not depend on the choice of \( \mathcal{L}_{\phi'} \) and \( x \). The required isomorphism \( \alpha \) is defined by the rule
\[ \alpha(\phi') := [\pi'(\mathcal{L}_{\phi'})]. \]

Also, \( \alpha \circ \beta^{-1} \) is an isomorphism of groups \( \text{Deck}(\mathbb{M}/M) \) and \( \pi_1(M') \).

Since \( M' \) is a surface of the genus \( \text{gen}M' = r/2 \), there are \( 2 \text{gen}M' = r \) generators \([L_1], \ldots, [L_r]\) of the fundamental group \( \pi_1(M') \). Note that, since \( D \) is simply connected, one can deform the curves \( L_j \), preserving their homotopy
class, in such a way that any $L_j$ does not intersect $D$. Thus, we assume that $L_1, \ldots, L_r \subset M$. Since the groups $\text{Deck}(M/M)$ and $\pi_1(M')$ are isomorphic, the automorphisms $\phi_j := \beta \circ \alpha^{-1}([L_j]), \ j = 1, \ldots, r$ generate the group $\text{Deck}(M/M)$. Therefore, a function $V$ on $M$ can be represented as $V = v \circ \pi$ if and only if $V \circ \phi_j = V, \ j = 1, \ldots, r$.

Suppose that $V$ is a function on $M$ such that $\nabla_g V = A := \pi^* a$, where $a$ is a vector field on $M$. Then

$$V(\phi_j(x)) - V(x) = \int_{L_j} g(A, l) \, dl,$$

where $L_j$ connects $x$ to $\phi_j(x)$. Since the field $A = \pi^* a$ is invariant under action of the group $\text{Deck}(M/M)$, the right-hand side does not depend on $x$ and one can choose $L_j$ to provide $\pi(L_j) = L_j$. Then the difference $V(\phi_j(x)) - V(x)$ is equal to

$$T_j(a) := \int_{L_j} g(a, l) \, dl.$$

Thus, $V = v \circ \pi$ and $a = \nabla_g v$ if and only if $T_1(a) = \cdots = T_r(a) = 0$.

Introduce the $r \times r$-matrix $T$ with entries $T_{ij} = T_i(a_j)$. Recall that any non-zero linear combination $\sum_{j=1}^r c_j a_j$ is not potential field in $M$. This means that all $T_i(\sum_{j=1}^r c_j a_j) = \sum_j T_{ij} c_j$ are zero if and only if $c_1 = \cdots = c_r = 0$. Thus, $T$ is invertible. Denote $f'_s := 2\pi \sum_{l=1}^r R_{ls} f_l$, where $R = T^{-1}$. Then $f'_1, \ldots, f'_r$ are linear independent modulo $\mathfrak{K}$. Introduce the new functions

$$V'_s = 2\pi \sum_{l=1}^r R_{ls} V_l, \quad W'_s = 2\pi \sum_{l=1}^r R_{ls} W_l,$$

that are determined by $f'_s$ in the same way as $V_s$ and $W_s$ are determined by $f_s$ (see (24)). Then $\nabla_g V'_s = 2\pi \sum_{l=1}^r R_{ls} A_l = \pi^* a'_s$, where $a'_s = 2\pi \sum_{l=1}^r R_{ls} a_l$ and

$$T_j(a'_s) = 2\pi \sum_{l=1}^r T_{jl} R_{ls} = 2\pi \delta_{js}$$

holds. By the latter, one has

$$V'_s \circ \phi_j - V'_s = W'_s \circ \phi_j - W'_s = 2\pi \delta_{js}, \quad j = 1, \ldots, r.$$

Hence,

$$e^{W'_s \circ \phi_j} = e^{W'_s}$$

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for any $j, s = 1, \ldots, r$. This means that $e^{W_s'}$ can be represented as $e^{W_s'} = w_s \circ \pi$, where $w_s$ is a function on $M$. Since $W_s'$ is holomorphic in $\mathbb{M}$, the function $w_s$ is holomorphic in $M$. Replacing $f_s$ by $f_s'$ (what is the same, omitting ‘prime’ everywhere in the notation), one obtains $e^{W_j} = w_j \circ \pi$.

So, we have provided the functions $f_1, \ldots, f_r$ with the properties claimed at the beginning of the paragraph.

- Since $w_j$ is holomorphic on $M$, one has $\zeta_j := w_j|_{\Gamma} \in \text{Ker } \Upsilon$. Also, one has

$$\log |w_j(\pi(x))| = \Re W_j(x) = u_j(\pi(x)), \quad x \in \mathbb{M}.$$ 

In particular,

$$\log |w_j| = u_j \quad \text{and} \quad \log |\zeta_j| = f_j$$

holds on $M$ and $\Gamma$ respectively.

Since $W_j$ is holomorphic on $\mathbb{M}$, the Cauchy-Riemann conditions yield

$$(\partial_{\nu} u_j) \circ \pi = \partial_{\nu} (u_j \circ \pi) = \partial_{\nu} \Re W_j = \partial_{\gamma} \Im W_j = \partial_{\gamma} \Im \log (w_j \circ \pi) =$$

$$= \partial_{\gamma} \text{arg} (w_j \circ \pi) = (\partial_{\gamma} \text{arg} w_j) \circ \pi$$

on $\pi^{-1}(\Gamma)$. This means that

$$\Lambda g f_j = \partial_{\nu} u_j = \partial_{\gamma} \text{arg} \zeta_j \quad (25)$$

holds on $\Gamma$. In the mean time, by virtue of vii we have

$$\Lambda f_j = \Lambda \log |\zeta_j| = \partial_{\gamma} \text{arg} \zeta_j. \quad (26)$$

Comparing (25) and (26), one obtains $\Lambda f_j = \Lambda g f_j$ for any $j = 1, \ldots, r$. Together with what was proved above, this means that $\Lambda = \Lambda g$ and, hence, $\Lambda$ is the DN-map of the surface $(M, g)$.

The sufficiency of the conditions i-vii is established.

Theorem 1 is proved.

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