SEMICORINGS AND SEMICOMODULES

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In this paper, we introduce and investigate semicorings over associative semirings and their categories of semicomodules. Our results generalize old and recent ones on corings over rings and their categories of comodules. The generalization is not straightforward and even subtle at several places due to the nature of the base category of commutative monoids which is neither Abelian (not even additive) nor homological, and has no nonzero injective objects. To overcome these and other difficulties, a combination of methods and techniques from categorical, homological and universal algebra is used including a new notion of exact sequences of semimodules over semirings.

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INTRODUCTION

Coalgebraic structures in general, and categories of comodules for comonads in particular, are gaining recently increasing interest [55]. Although comonads can be defined in arbitrary categories, nice properties are usually obtained in case the comonad under consideration is isomorphic to $- \otimes C$ (or $C \otimes -$) for some comonoid $C$ [41] in a monoidal category $(\mathcal{V}, \otimes, I)$ (acting on some category $\mathcal{X}$ in a nice way [40]).

However, it can be noticed that the most extensively studied concrete examples are—so far—the categories of coacts (usually called coalgebras) of an endo-functor $F$ on the category $\textbf{Set}$ of sets (motivated by applications in theoretical computer science [31] and universal (co)algebra [11]) and categories of comodules for a coring over an associate algebra $A$ [20], [19]. i.e., a comonoid in the monoidal category $(\mathcal{M}_A, \otimes_A, A)$ of $(A, A)$-bimodules which acts on the category $\mathcal{M}_A(\mathcal{M}, A)$ of modules in the obvious way [42, p. 229].

The main goal of this paper is to investigate categories of semicomodules for a semicoring, which can be seen as comodules of comonads associated to a comonoid in the monoidal category $(\mathcal{S}_A, \otimes_A, A)$ of $(A, A)$-biseimodules over an associative semialgebra $A$ with the natural tensor product $- \otimes_A -$ [33]. This does not only
add a new concrete example where the general theory of comonads and comonoids applies, but also provides an interesting context where a combination of techniques and methods from categorical algebra, homological algebra, and universal algebra applies naturally and harmonically.

Semicorings over semirings are of particular importance for theoretical and practical reasons: On one hand, and in contrast to categories of modules over a ring, categories of semimodules over a semiring are not so nice, as the following ones (in general):

1) They are not Grothendieck: a category of semimodules over a semiring does not necessarily have enough injectives (for instance, 0 is the only injective object in the category \( \text{AbMonoid} \cong S_{\mathbb{N}_0} \), the category of semimodules over the semiring \( \mathbb{N}_0 \) of non-negative integers [28, 17.21]).

2) They are not Abelian: a semimodule does not necessarily posses a projective presentation (objects with a projective presentation are called normal [51]); moreover, one cannot make free use of some nice properties of adjoint functors between Abelian categories (e.g., [27, Proposition 6.28]); see Remark 2.34.

3) They are not additive; the hom sets are monoids which are not necessarily groups.

4) They are not Puppe-exact [44]. The notion of exact sequences of semimodules is subtle; this led to several different notions of exactness for sequences of semimodules over a semiring [2] (all of which coincide for modules over rings).

5) They are not homological since they are not protomodular [13]: several basic homological lemmas do not apply (e.g., the short five lemma). Moreover, epimorphisms are not necessarily surjective and subsemimodules are not necessarily kernels.

6) Several flatness, projectivity and injectivity properties which are equivalent for modules over rings are apparently different for semimodules over semirings [1].

7) Some notions cannot be easily checked as in the case of modules over rings; for example, to prove that a given \( S \)-semimodule is flat, one has to show that \( M \otimes_S \mathbb{N}_0 \rightarrow \text{AbMonoid} \) preserves all pullbacks [34] (or all equalizers) and not only the monomorphisms.

This has the impact that generalizing results on corings (comodules) to semicorings (semicomodules) is neither trivial nor straightforward as the first impression might be. We could overcome some of the difficulties mentioned above by introducing a new notion for exact sequence of semimodules over semirings which we used to prove restricted versions of the short five lemma [1] (the nine lemma and the snake lemma [2]). We also introduced and used suitable notions of flatness, projectivity and injectivity for semimodules. Moreover, we made use of recent developments in the theory of comonads [14] especially those associated to comonoids in monoidal categories [41, 43].

On the other hand, semirings and semimodules proved to have a wide spectrum of significant applications in several aspects of mathematics like optimization theory [22], tropical geometry [45], idempotent analysis [36], physics [32], theoretical computer science (e.g., Automata Theory [25, 26]), and many more [28]. Moreover, corings over rings showed to have important applications in areas like noncommutative ring theory, category theory, Hopf algebras, differential graded algebras, and noncommutative geometry [19]. This suggests that
investigating semicorings and semicomodules will open the door for many new applications in the future (see [56] for recent applications to Automata Theory).

Before proceeding, we mention that in many (relatively old) papers, researchers used the so called Takahashi’s tensor-like product, which we denote by $- \boxtimes_A -$ [50] (see also [28]). This product has the defect that, for a semialgebra $A$, the category $(\mathcal{A}_A, \mathcal{B}_A, A)$ of $(A, A)$-bicomodules is not necessarily monoidal. The author [5] introduced a notion of semiunital semimonoidal categories with prototype $(\mathcal{A}_A, \mathcal{B}_A, A)$ with $A$ as a semiunit; he also presented a notion of semicounital semicomonoids in such categories with semicounital $A$-semicorings as an illustrating example. The relation between the two products has been clarified in [1].

This paper is organized as follows: after this introduction, and for convenience of the readers not familiar with semirings and semimodules, we include in Section 1 some basic definitions, properties and some results (mostly without proof) related to such algebraic structures. In Section 2, we introduce the notion of a semicoring $\mathcal{C}$ over a semialgebra $A$ and study basic properties of the category $\mathcal{S}^\mathcal{C}$ of right $\mathcal{C}$-semicomodules. We present a reconstruction result in Theorem 2.21. Moreover, we apply results of Porst et al. (e.g., [11, 41–43]) to obtain a generalization of the Fundamental Theorem of Coalgebras over fields to semicoalgebras over commutative semirings in Proposition 2.11 and to semicorings over arbitrary rings in Proposition 2.12. Let $\mathcal{A}$ be a semicoring over the semialgebra $A$. We apply results of Porst et al. to $\mathcal{S}^\mathcal{C}$ (Theorem 2.22). In Section 3, we introduce and investigate the category $\text{Rat}^\mathcal{C}(\mathcal{S}_\mathcal{A})$ of $\mathcal{C}$-rational right $\mathcal{A}$-semimodules, where $\mathcal{A}$ is an $A$-semiring with a morphism of $A$-semirings $\kappa: \mathcal{A} \rightarrow^* \mathcal{C} (= \text{Hom}_A(\mathcal{C}, A)$, the left dual ring of $\mathcal{C})$ and $P = (\mathcal{A}, \mathcal{C})$ is a left $\mathcal{A}$-pairing. In this case, we prove that $\text{Rat}^\mathcal{C}(\mathcal{S}_\mathcal{A}) \simeq \mathcal{S}_{\mathcal{A}^\mathcal{C}}$ (Theorem 3.16) extending our main result in [4] on the category of right comodules for a coring over an associative algebra (see also [20]). Moreover, and assuming a uniformity condition on $\mathcal{A}$, we show in Theorem 3.23 that $\mathcal{S}^\mathcal{C} = \sigma[\mathcal{C}_{\mathcal{C}}]$ (the Wisbauer category of $\mathcal{C}$-subgenerated right $^*\mathcal{C}$-semimodules) if and only if $\mathcal{A}^\mathcal{C}$ is a mono-flat $\mathcal{A}$-semimodule and $\mathcal{S}^\mathcal{C}$ is closed under $^*\mathcal{C}$-subsemimodules. Under some suitable conditions on the semialgebra $A$ and a uniformity condition $\mathcal{A}^\mathcal{C}$, we prove in Theorem 3.26 that $\mathcal{S}^\mathcal{C} = \mathcal{S}_{\mathcal{C}_{\mathcal{C}}}$ if and only if $\mathcal{A}^\mathcal{C}$ is finitely generated projective and $\mathcal{S}^\mathcal{C}$ is closed under $^*\mathcal{C}$-subsemimodules.

1. PRELIMINARIES

1.1. Let $(G, +)$ be an Abelian semigroup. We say that $G$ is cancellative iff

$$g + g' = g + g'' \Rightarrow g' = g''$$

for all $g, g', g'' \in G$. For say that a subsemigroup $L \leq G$ is subtractive iff $L = \mathcal{T}$, where

$$\mathcal{T} = \{g \in G | g + l = l'' \text{ for some } l', l'' \in L\}.$$ 

We say that $G$ is completely subtractive iff every subsemigroup $L \leq G$ is subtractive. A morphism of semigroups $f: G \rightarrow G'$ is said to be subtractive iff $f(G) \leq G'$ is a subtractive subsemigroup.
Remark 1.2. Let \( f : G \to G' \) be a morphism of Abelian monoids. If \( L' \leq G' \) is a subtractive submonoid, then \( f^{-1}(L) \leq G \) is a subtractive submonoid.

Semirings and Semimodules

In this section, we present some basic definitions and results on semirings and semimodules. Our main reference is [28]; however, we use a different notion of exact sequences of semimodules introduced in [2]. With \( \text{AbMonoid} \), we denote the category of Abelian monoids.

1.3. A semiring is a monoid in the monoidal category (\( \text{AbMonoid} \), \( \otimes \), \( \mathbb{N}_0 \)) of Abelian monoids, or roughly speaking a ring not necessarily with subtraction, i.e., a non-empty set \( S \) with two binary operations “+” and “*” such that \((S, +, 0)\) is an Abelian monoid and \((S, *, 1)\) is a monoid such that

\[
  s \cdot (s_1 + s_2) = s \cdot s_1 + s \cdot s_2, \quad (s_1 + s_2) \cdot s = s_1 \cdot s + s_2 \cdot s \quad \text{and} \quad s \cdot 0 = 0 = 0 \cdot s \quad \text{for all} \ s, s_1, s_2 \in S.
\]

A morphism of semirings \( f : S \to T \) is a map such that \( f : (S, +_S, 0_S) \to (T, +_T, 0_T) \) and \( f : (S, \cdot_S, 1_S) \to (T, \cdot_T, 1_T) \) are morphisms of monoids. We say that the semiring \( S \) is commutative (cancellative) iff \((S, \cdot)\) is commutative \((S, +)\) is cancellative). If \( S \) is a commutative ring and \( \eta : S \to A \) is a morphism of semirings, then we call \( A \) an (associative) \( S \)-semialgebra.

1.4. Let \( S \) be a semiring. A right \( S \)-semimodule \( M \) is roughly speaking a right \( S \)-module not necessarily with subtraction (i.e., \((M, +_M, 0_M)\) is an Abelian monoid rather than a group) for which

\[
m0_s = 0_m = 0_ms \quad \text{for all} \ m \in M \text{ and } s \in S.
\]

The category of right (left) \( S \)-semimodules and \( S \)-linear maps, which respect addition and scalar multiplication, is denoted by \( \mathcal{S}_S (\mathcal{S}_S) \). A right (left) \( S \)-semimodule \( M \) is said to be cancellative (completely subtractive) iff \((M, +)\) is cancellative (every \( S \)-subsemimodule \( L \leq_S M \) is subtractive). With \( \mathcal{CS}_S \subseteq \mathcal{S}_S (\mathcal{SC}_S \subseteq \mathcal{S}_S) \), we denote the full subcategory of cancellative right (left) \( S \)-semimodules. For semirings \( S \) and \( T \), an object in the category \( \mathcal{S}_S \otimes \mathcal{S}_T \) of \((S, T)\)-bsemimodules is a left \( S \)-semimodule \( _S M \) which is also right \( T \)-semimodule \( M_T \) with \( s(mt) = (sm)t \) for all \( s \in S, m \in M \) and \( t \in T \); the arrows are the \( S \)-linear \( T \)-linear maps, and its full subcategory of cancellative \((S, T)\)-bsemimodules is denoted by \( \mathcal{SCS}_T \). We call \( _SM_T \) a completely subtractive \((S, T)\)-bsemimodule iff every \((S, T)\)-subbsemimodule \( L \leq_{(S,T)} M \) is subtractive.

Examples 1.5.

1. Every ring is a semiring.
2. The set \( \mathbb{N}_0 \) of non-negative integers is a cancellative semiring with the usual addition and multiplication. The category of (cancellative) \( \mathbb{N}_0 \)-semimodules is isomorphic to the category of (cancellative) Abelian monoids.
3. An example of a semiring due to Dedekind [24] is \((\text{Ideal}(R), \cup, \cap)\), where \(R\) is a ring and \(\text{Ideal}(R)\) is the set of ideals of \(R\). More generally, every distributive bounded (complete) lattice is a semiring.

4. \(\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)\) and \(\mathbb{R}_{\min} := (\mathbb{R} \cup \{\infty\}, \min, +)\) are **semifields** (every non-zero element has a multiplicative inverse) [22].

5. \(B = \{0, 1\}, +, \cdot\) is a *semi-field*, where \(1 + 1 = 1 \neq 0\) [28, p. 7], called the **Boolean semiring**.

### 1.6. Let \(M\) be a right \(S\)-semimodule. Each \(L \leq S\) \(M\) defines two \(S\)-congruence relations [28] on \(M\), namely \(\equiv_{L}\) and \([\equiv]_{L}\), where

\[
\begin{align*}
\text{for some } l_1, l_2 \in L; \\
\text{for some } l_1, l_2 \in L \text{ and } m' \in M.
\end{align*}
\]

We define the quotient semimodule \(M/L := M/\equiv_{L};\) notice that \(M/[\equiv]_{L}\) is cancellative for each \(L \leq S\) \(M\). In fact, we have a functor

\[
c(-) : S_{S} \rightarrow \mathcal{C}S_{S}, M \rightarrow M/[\equiv]_{0}.
\]

**Remark 1.7.** The tensor product \(- \otimes_{S} -\) of semimodules we adopt is that in the sense of [33] (see also [34, 37]) and *not* in the tensor-like product introduced by Takahashi [50], which we denote by \(- \boxtimes_{S} -\). As clarified in [1], for every right (left) \(S\)-semimodule \(M\), we have an isomorphism of Abelian monoids \(M \otimes_{S} S \cong M (S \otimes_{S} M \cong M)\), where

\[
M \boxtimes_{S} S \cong c(M \otimes_{S} S) \cong c(M)(S \boxtimes_{S} M \cong c(S \otimes_{S} M) \cong c(M)).
\]

**Definition 1.8.** Let \(\mathcal{A}\) and \(\mathcal{B}\) be categories and \(F : \mathcal{A} \rightarrow \mathcal{B}\) a covariant functor.

1. \(F\) preserves limits iff for every diagram \(D : \mathfrak{Z} \rightarrow \mathcal{A}\), we have

\[
\begin{align*}
\left( L \xrightarrow{\ell_{l}} D_{l}\right)_{l \in \text{Obj}(\mathfrak{Z})} \text{ is a limit of } D \Rightarrow (F(L) \xrightarrow{\ell_{l}} F(D_{l}))_{l \in \text{Obj}(\mathfrak{Z})} \text{ is a limit of } F \circ D;
\end{align*}
\]

2. \(F\) creates limits iff for every diagram \(D : \mathfrak{Z} \rightarrow \mathcal{A}\) and every limit \(\mathcal{L} = (L' \xrightarrow{f_{l}} D_{l})_{l \in \text{Obj}(\mathfrak{Z})}\) of \(F \circ D\) in \(\mathcal{B}\), if any, there exists a unique cone \(\mathcal{F} = (L \xrightarrow{f_{l}} D_{l})_{l \in \text{Obj}(\mathfrak{Z})}\) in \(\mathcal{A}\) with

\[
(F(L) \xrightarrow{F(f_{l})} F(D_{l}))_{l \in \text{Obj}(\mathfrak{Z})} = (L' \xrightarrow{f_{l}} D_{l})_{l \in \text{Obj}(\mathfrak{Z})}\text{ and } \mathcal{F}\text{ is a limit of } D\text{ in } \mathcal{A}.
\]

Dually, one defines functors preserving (creating) colimits.

**Lemma 1.9** (cf. [16, Proposition 3.2.2]). Let \(\mathcal{A}, \mathcal{B}\) be arbitrary categories and \(F : \mathcal{A} \rightarrow \mathcal{B} \xrightarrow{G} \mathcal{E}\) be functors such that \((F, G)\) is an adjoint pair.

1. \(F\) preserves all colimits which turn out to exist in \(\mathcal{A}\).
2. \(G\) preserves all limits which turn out to exist in \(\mathcal{B}\).
Definition 1.10. An object $G$ in a cocomplete category $\mathcal{A}$ is said to be a (regular) generator iff for every $X \in \mathcal{A}$, there exists a canonical (regular) epimorphism $f_x : \bigsqcup_{x \in (G, X)} G \to X$ [21, p. 199] (see also [35, 52]); recall that an arrow in $\mathcal{A}$ is said to be a regular epimorphism iff it is a coequalizer (of its kernel pair).

The following result collects some properties of the categories of semimodules over a semiring (cf. [1, 2, 28, 37]).

Proposition 1.11. Let $S$ and $T$ be semirings.

1. $\mathcal{S}_S$ is a variety, in the sense of Universal Algebra, i.e., a class of objects which is closed under homomorphic images, direct products, and subobjects.
2. $\mathcal{S}_S$ is complete (i.e., has all small limits), equivalently $\mathcal{S}_S$ has equalizers, pullbacks, and products. The kernel of an $S$-linear map $f : M \to N$ is

$$\text{Ker}(f) := \text{Eq}(f, 0) = \{m \in M \mid f(m) = 0\}.$$ 

3. $\mathcal{S}_S$ is cocomplete (i.e., has all small colimits), equivalently $\mathcal{S}_S$ has coequalizers and products. The cokernel of an $S$-linear map $f : M \to N$ is $\text{Coeq}(f, 0) = N/\text{Ker}(f)$.

4. Every monomorphism in $\mathcal{S}_S$ is injective. A morphism in $\mathcal{S}_S$ is surjective if and only if it is a regular epimorphism.
5. $\mathcal{S}_S$ is a Barr-exact category [12] (see also [7]) with canonical factorization system given by $(\text{RegEpi, Mono}) = (\text{Surj, Inj})$.
6. $\mathcal{S}_S$ is a regular generator in $\mathcal{S}_S$.
7. For all right $S$-semimodule $M$ and $N$, we have a natural isomorphism of Abelian monoids $\text{Hom}_S(c(M), N) \simeq \text{Hom}_S(M, N)$, i.e., the embedding $\mathcal{C} \mathcal{S}_S \hookrightarrow \mathcal{S}_S$ is right adjoint to $c(-)$; so, $\mathcal{C} \mathcal{S}_S$ is a reflective subcategory of $\mathcal{S}_S$.
8. For every $(S, T)$-bimodule $\mathcal{S} M$, we have functors $M \otimes_T - : \mathcal{S} S \to \mathcal{S}$ and $- \otimes_S M : \mathcal{S} S \to \mathcal{S}$. Moreover, we have adjoint pairs of functors

$$(M \otimes_T -, \text{Hom}_{\mathcal{S}_S}(M, -)) \text{ and } (- \otimes_S M, \text{Hom}_{\mathcal{T}}(M, -)),$$

whence $M \otimes_T -$ and $- \otimes_S M$ preserve colimits.
9. $(\mathcal{S}_S, \otimes_S, S)$ is a biclosed monoidal category.
10. If $S$ is commutative, then $(\mathcal{S}_S, \otimes_S, S; \tau)$ is a symmetric braided monoidal category where $\tau$ is the flipping natural transformation

$$M \otimes_S N \cong N \otimes_S M, \quad m \otimes_S n \mapsto n \otimes_S m.$$ 

Definition 1.12. Let $M$ and $N$ be $S$-semimodules. We call an $S$-linear map $f : M \to N$:

- $i$-uniform (image-uniform) iff $f(M) = \overline{f(M)}$;
- $k$-uniform (kernel-uniform) iff for all $m, m' \in M$ we have

$$f(m) = f(m') \Rightarrow m + k = m' + k' \quad \text{for some } k, k' \in \text{Ker}(f);$$

uniform iff $f$ is $i$-uniform and $k$-uniform.
We call $L \leq_{S} M$ a uniform subsemimodule iff the embedding $L \hookrightarrow M$ is $(r)$-uniform, or equivalently iff $L \leq_{s} M$ is subtractive. If $\equiv$ is an $S$-congruence on $M$ [28], then we call $M/\equiv$ a uniform quotient iff the projection $\pi_{M}: M \rightarrow M/\equiv$ is $(k)$-uniform.

1.13 ([2]). We say that a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $S$-semimodules is exact (resp. semi-exact, proper-exact, quasi-exact) iff $f(X) = \text{Ker}(g)$ and $g$ is $k$-uniform (resp. $\overline{f(X)} = \text{Ker}(g)$). $f(X) = \text{Ker}(g)$. $\overline{f(X)} = \text{Ker}(g)$ and $g$ is $k$-uniform. A (possibly infinite) sequence of $S$-semimodules $\cdots \rightarrow X_{i-1} \xrightarrow{f_{i-1}} X_{i} \xrightarrow{f_{i}} X_{i+1} \xrightarrow{f_{i+1}} X_{i+2} \rightarrow \cdots$ is said to be exact (resp. semi-exact, proper-exact, quasi-exact) iff each three-term subsequence $X_{i-1} \xrightarrow{f_{i-1}} X_{i} \xrightarrow{f_{i}} X_{i+1}$ is exact (resp. semi-exact, proper-exact, quasi-exact).

Lemma 1.14 ([2, Lemma 2.7]). Let $X$, $Y$ and $Z$ be $S$-semimodules.

1. $0 \rightarrow X \xrightarrow{f} Y$ is exact if and only if $f$ is injective.
2. $Y \xrightarrow{g} Z \rightarrow 0$ is exact if and only if $g$ is surjective.
3. $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ is semi-exact and $f$ is uniform if and only if $f$ induces an isomorphism $X \simeq \text{Ker}(g)$.
4. $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is semi-exact and $g$ is uniform if and only if $g$ induces an isomorphism $Z \simeq \text{Coker}(f)$.
5. $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is exact if and only if $f$ induces an isomorphism $X \simeq \text{Ker}(g)$ and $g$ induces an isomorphism $Z \simeq \text{Coker}(f)$.

Lemma 1.15.

1. An $S$-linear map $f: M \rightarrow N$ induces an isomorphism $M/\text{Ker}(f) \simeq f(M)$ if and only if $f$ is $k$-uniform.
2. For every $L \leq_{S} M$, we have an exact sequence of $S$-semimodules

$$0 \rightarrow L \xrightarrow{\pi} M \xrightarrow{\pi_{M}} M/L \rightarrow 0.$$

Since $S_{x}$ is (Surj, Inj)-structured [7] (and not (Epi, Mono)-structured), the natural notions of projective objects, generators etc. in this category are defined relative to the class Surj of surjective $S$-linear maps (=regular epimorphisms) rather than the class Epi of all epimorphisms.

Definition 1.16. We say that an $S$-semimodule $X$ (uniformly) generates $M_{S}$ iff there exists an index set $\Lambda$ and a (uniform) surjective $S$-linear map $X^{(\Lambda)} \xrightarrow{\pi} M \rightarrow 0$. With Gen$(X)$ we denote the class of $S$-semimodules generated by $X_{S}$.

Definition 1.17. We say that $M_{S}$ is uniformly (finitely) generated iff there exists a (finite) index set $\Lambda$ and a uniform surjective $S$-linear map $S^{(\Lambda)} \xrightarrow{\pi} M \rightarrow 0$.

Remark 1.18. Every $S$-semimodule $M$ is generated by $S$ : there exists a surjective $S$-linear map $S^{(\cdot)} \xrightarrow{\pi} M \rightarrow 0$ [28, Proposition 17.11]. However, it is not guaranteed
that we can find $\Lambda$ for which $\pi$ is uniform. Uniformly generated semimodules were called $k$-semimodules in [9]; we prefer the terminology introduced above since it is more informative. Takahashi [51] defined an $S$-semimodule $X$ to be normal if there exists a projective $S$-semimodule $P$ and a uniform surjective $S$-linear map $P \rightarrow X \rightarrow 0$ (called a projective presentation of $X$). Indeed, every uniformly generated $S$-semimodule is normal.

1.19. Let $M$ be a right $S$-semimodule. With $\sigma[M_s]$ ($\sigma_u[M_s]$) we denote the closure of $\text{Gen}(M_s)$ under (uniform) $S$-subsemimodules, i.e., the smallest full subcategory of $S$ which contains $M_s$ and is closed under direct sums, homomorphic images, and (uniform) $S$-subsemimodules. We say that $M_s$ is a (uniformly) subgenerator for $\sigma[M_s]$ ($\sigma_u[M_s]$). Notice that $\text{Gen}(M_s) \subseteq \sigma_u[M_s] \subseteq \sigma[M_s]$.

Remark 1.20. Let $M$ be an $S$-semimodule. Notice that, by Lemma 1.15 (2), the uniform subsemimodules of a given $S$-semimodule are precisely the kernels of $S$-linear maps with domain the semimodule under consideration. It follows that $X \in \sigma_f[M_s]$ if and only if $X \cong \text{Ker}(g)$, where $g : Y \rightarrow Z$ is $S$-linear and $Y, Z \in \text{Gen}(M_s)$, or equivalently if and only if there exist exact sequences of $S$-semimodules $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in which $Y, Z \in \text{Gen}(M_s)$.

Proposition 1.21 (cf. [53, 15.4]). Let $M$ be a faithful $S$-semimodule and $T = \text{End}(M_s)$. If $TM$ is finitely generated, then $\sigma[M_s] = S$.

Definition 1.22. Let $\mathcal{A}$ be a category with finite limits and finite colimits. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be left exact (right-exact) iff $F$ preserves finite limits (finite colimits). Moreover, $F$ is called exact iff $F$ is left-exact and right-exact.

Remark 1.23. Let $M$ be a right $S$-semimodules.

1. The covariant functor $\text{Hom}_S(M, -) : S \rightarrow \text{AbMonoid}$ is left exact, since it is a right adjoint, whence it preserves all finite limits (e.g., it sends equalizers to equalizers and pullbacks to pullbacks). In particular, it preserves kernels; this explains [50, Theorem 2.6 (1)] in light of Lemma 1.14.
2. The contravariant functor $\text{Hom}_S(-, M) : S \rightarrow \text{AbMonoid}$ is left exact, whence it converts all finite colimits into finite limits, e.g., it converts coequalizers into equalizers and pushouts into pullbacks. In particular, it sends cokernels to kernels; this explains [50, Theorem 2.6 (2)] in light of Lemma 1.14.
3. The covariant functor $M \otimes_S - : S \rightarrow \text{AbMonoid}$ is right exact, since it is a left adjoint, whence it preserves all finite colimits, e.g., it sends coequalizers to coequalizers and pushouts to pushouts. In particular, it sends cokernels to kernels; this explains the analog of [50, Theorem 5.5] in light of Lemma 1.14.

Definition 1.24. We say that $M_s$ is as follows:

Injective iff for every monomorphism of $S$-semimodules (i.e., injective $S$-linear map) $X \rightarrow Y$, every $S$-linear map $h : X \rightarrow M$ can be extended to an $S$-linear map $\hat{h} : Y \rightarrow M$ (such that $\hat{h} \circ f = h$);
Uniformly injective iff \( \text{Hom}_S(-, M): S \to \text{AbMonoid} \) converts uniform monomorphisms into uniform surjective maps (equivalently, \( \text{Hom}_S(-, M) \) preserves short exact sequences);

**u-Injective** iff \( \text{Hom}_S(-, M): S \to \text{AbMonoid} \) sends (uniform) monomorphisms to (uniform) surjective maps.

**Definition 1.25.** We say that \( M_S \) is as follows:

*Projective* iff for every surjective \( S \)-linear map \( Y \to Z \) and every \( S \)-linear map \( h: M \to Z \), there exists an \( S \)-linear map \( \tilde{h}: M \to Y \) such that \( h = g \circ \tilde{h} \);

*Uniformly projective* iff \( \text{Hom}_S(M, -): S \to \text{AbMonoid} \) preserves uniform surjective morphisms, or equivalently iff it preserves short exact sequences;

**u-Projective** iff \( \text{Hom}_S(M, -): S \to \text{AbMonoid} \) sends (uniform) surjective morphisms to (uniform) surjective maps.

**Definition 1.26.** We say that \( M_S \) is as follows:

*Cogenerator* iff for every \( N_S \), there exist an index set \( \Lambda \) and an \( S \)-linear embedding \( N \to M^\Lambda \);

*Uniformly cogenerator* iff \( \text{Hom}_S(-, M): S \to \text{AbMonoid} \) reflects short exact sequences.

**Lemma 1.27.**

1. The following statements are equivalent for an \( S \)-semimodule \( P_S \):

   (a) \( P_S \) is projective;
   (b) \( P_S \) is a retract of a free \( S \)-semimodule, i.e., there exists a free \( S \)-semimodule \( F \) and \( S \)-linear maps \( F \leftrightarrows g P \) with \( g \circ f = \text{id}_P \);
   (c) \( P_S \) has a dual basis: there exists a subset \( \{(p_i, f_i)\} \subseteq P \times P^\ast \) such that: for each \( p \in P \), the set \( \Lambda(p) = \{ \lambda \mid f_i(p) \neq 0 \} \) is finite, and \( p = \sum p_i f_i(p) \).

2. If \( P_S \) is uniformly generated and uniformly projective, then \( P_S \) is projective.

**Proof.**

1. (a) \( \iff \) (b) This is [28, Proposition 17.16].
   (c) \( \iff \) (d) The proof is similar to that of [53, 18.6].

2. Let \( S^\Lambda \to P \to 0 \) be a uniform presentation of \( P_S \). Considering \( \text{id}_P : P \to P \), we find an \( S \)-linear map \( h: P \to S^\Lambda \) such that \( \pi \circ h = \text{id}_P \), i.e., \( P_S \) is a retract of a free \( S \)-semimodule and so \( P_S \) is projective by (1).

**Definition 1.28.** Let \( M_S \) be an \( S \)-semimodule, and set \( M^\ast := \text{Hom}_S(M, S) \). We say that \( M_S \) is **finitely projective** iff for every finite subset \( \{m_1, \ldots, m_l\} \subseteq M \), there exists a subset \( \{(\tilde{m}_i, f_i)\}_{i=1}^l \subseteq M \times M^\ast \) such that \( m_i = \sum_{j=1}^l \tilde{m}_j f_j(m_i) \) for each \( i = 1, \ldots, l \);

**Definition 1.29** ([1, 34]). We call a right \( S \)-semimodule \( M \):

*Flat* iff \( M \otimes_A - \) is left exact, i.e., it preserves finite limits, equivalently \( M \cong \text{lim} F_S, \) a filtered limit of finitely generated free right \( S \)-semimodules;

**Uniformly flat** iff \( M \otimes_A -: A S \to \text{AbMonoid} \) preserves uniform subobjects;

*Mono-flat* iff \( M \otimes_A -: A S \to \text{AbMonoid} \) preserves monomorphisms;
Remark 1.30. Let $M$ be a right $S$-semimodule. Since $M \otimes_S \rightarrow \text{AbMonoid}$ preserves direct sums, we conclude that $M_S$ is flat if and only if $M \otimes_S$ preserves equalizers. Moreover, $M_S$ is flat if and only if $M \otimes_S$ preserves pullbacks [34]. Flat $S$-semimodules are mono-flat and uniformly flat [1]. On the other hand, if $M_S$ is a mono-flat $S$-semimodule, then $M_S$ is uniformly flat (whence $u$-flat) if and only if $M \otimes_S \rightarrow \text{AbMonoid}$ preserves kernels. It is not known yet if there are examples of uniformly flat semimodules which are not flat.

Definition 1.31. Let $M$ be a right (left) $S$-semimodule. We say that a (uniform) $S$-subsemimodule $L \leq_S M$ is (uniformly) $W$-pure for some left (right) $S$-semimodule $W$ iff $L \otimes_S W \leq M \otimes_S W$ is a (uniform) submonoid. We call $L \leq_S M$ (uniformly) pure iff $L \leq_S M$ is (uniformly) $W$-pure for every left (right) $S$-semimodule $W$. If $M$ is an $(S, T)$-bissemimodule and $L \leq_{(S, T)} M$, then we call $L \leq_{(S, T)} M$ (uniformly) pure iff $L \hookrightarrow M$ is (uniformly) pure as a left $S$-subsemimodule as well as a right $T$-subsemimodule.

The proof of the following result is along the lines of that of [18, Proposition 3.6].

Lemma 1.32. Let $L$ be a right $S$-semimodule, $N$ a left $S$-semimodule, $K \leq_S L$, $M \leq_S N$ and consider the exact sequences of $S$-semimodules

$$0 \rightarrow K \overset{\iota}{\rightarrow} L \overset{\pi}{\rightarrow} L/K \rightarrow 0 \text{ and } 0 \rightarrow M \overset{\iota}{\rightarrow} N \overset{\pi}{\rightarrow} N/M \rightarrow 0.$$

We have an exact sequence of Abelian monoids

$$0 \rightarrow (\iota_\pi \otimes_S N)(\overline{K} \otimes_S N) + (L \otimes_S \iota_\pi)(L \otimes_S \overline{M}) \rightarrow L \otimes_S N \overset{\pi \otimes_S \overline{M}}{\rightarrow} L/K \otimes_S N/M \rightarrow 0.$$

If, moreover, $K \leq_S L$ is $N$-pure and $M \leq_S N$ is $L$-pure, then we have an exact sequence of Abelian monoids

$$0 \rightarrow \overline{K} \otimes_S N + L \otimes_S \overline{M} \rightarrow L \otimes_S N \overset{\pi \otimes_S \overline{M}}{\rightarrow} L/K \otimes_S N/M \rightarrow 0.$$

Definition 1.33. We say that an $S$-semimodule $X$ is as follows:

Finitely presented iff $\text{Hom}_S(X, -) \colon S_S \rightarrow \text{AbMonoid}$ preserves directed colimits (i.e., $X \in S_S$ is a finitely presentable object in the sense of [10]);

Uniformly finitely presented iff $X$ is uniformly finitely generated and for every exact sequence of $S$-semimodules

$$0 \rightarrow K \overset{f}{\rightarrow} \overline{S^n} \overset{g}{\rightarrow} X \rightarrow 0,$$

the $S$-semimodule $K \simeq \text{Ker}(g)$ is finitely generated.
Notation. Let $M$ be a right $S$-semimodule. For every family $F = \{X_\lambda\}_\Lambda$ of left $S$-semimodules, we have a morphism of Abelian monoids

$$\varphi_{(M,F)} : M \otimes_S \prod_{\lambda \in \Lambda} X_\lambda \longrightarrow \prod_{\lambda \in \Lambda} (M \otimes_S X_\lambda), \quad m \otimes_S \{f_\lambda\}_\Lambda \mapsto \{m \otimes_S f_\lambda\}_\Lambda.$$ 

If $X_\lambda = S$ for every $\lambda \in \Lambda$, then we set $\tilde{\varphi}_M = \varphi_{(M,S^{\Lambda})} : M \otimes_S S^{\Lambda} \longrightarrow (M \otimes_S S)^{\Lambda}$.

In the following lemma, we collect some properties of finitely presented semimodules over semirings.

Lemma 1.34. Let $M$ be a right $S$-semimodule.

1. If $\tilde{\varphi}_M : M \otimes_S S^{\Lambda} \longrightarrow M^{\Lambda}$ is surjective for every index set $\Lambda$, then $M_S$ is finitely generated.
2. If $M \otimes_S S^{\Lambda} \tilde{\cong} M^{\Lambda}$ is bijective and $- \otimes_S S^{\Lambda} : S_S \longrightarrow \text{AbMonoid}$ preserves $i$-uniform morphisms for every index set $\Lambda$, then $M_S$ is uniformly finitely presented.
3. If $M_S$ is uniformly finitely presented, then $M_S$ has a finite presentation through an exact sequence of $S$-semimodules

$$S^n \overset{f}{\longrightarrow} S^m \overset{g}{\longrightarrow} M \longrightarrow 0. \quad (2)$$

4. If $M_S$ is uniformly finitely presented, then $M_S$ is finitely presentable.
5. If $M_S$ is finitely presented and flat, then $M_S$ is projective.

Proof. The proofs are similar to those for modules over rings.

1. The proof is similar to that of [53, 12.9 (1)].
2. The proof is similar to that of [53, 12.9 (2)] using a restricted version of the Short Five Lemma [1, Lemma 1.22] for semicomodules over semirings.
3. This is [1, Proposition 2.25].
4. Given an arbitrary directed system $\{X_i, \{f_i\}\}$ of $S$-semimodules, we apply the contravariant functor $\text{Hom}_S(-, \lim X_i)$ to a finite presentation (2) of $M_S$, and then use a restricted version of the Short Five Lemma [1, Lemma 1.22] for semimodules over semirings to prove that $\text{Hom}_S(M, \lim X_i) \tilde{\cong} \lim \text{Hom}_S(M, X_i)$.
5. Assume that $M_S$ is a finitely presented flat $S$-semimodule. By definition of flat semimodules, $M = \lim M_i$, where $\{M_i\}_I$ is a directed system of free (projective) $S$-semimodules. Since $M_S$ is finitely presented, we have $\text{End}_S(M) = \text{Hom}_S(M, \lim M_i) = \lim \text{Hom}_S(M, M_i)$, and so $\text{id}_M$ factorizes through some $M_i$, whence a retract of the projective $S$-semimodule $M_i$ (cf. [17, Proof of Theorem 2.6]). Since a retract of a projective $S$-semimodule is projective [28], we conclude that $M_S$ is projective. \qed

2. SEMICORINGS

In this section, we introduce and investigate semicorings over semirings and their categories of semicomodules.
Throughout, $S$ is a commutative semiring with $1_s \neq 0_s$ and $(S_S, \otimes_S, S)$ is the symmetric monoidal category of $S$-semimodules [1]. Moreover, $A$ is an $S$-semialgebra, i.e., a monoid in $(S_S, \otimes_S, S)$, or equivalently a semiring $A$ along with a morphism of semirings $\eta_A : S \rightarrow A$. With $(S_S, S)$, we denote the category of left (right) $A$-semimodules and with $(A_S, \otimes_A, A)$ the monoidal category of $(A, A)$-bisemimodules.

2.1. By an $A$-semiring we mean a monoid in $(A_S, \otimes_A, A)$, i.e., triple $(\mathcal{A}, \mu_A, \eta_A)$ in which $\mathcal{A}$ is an $(A, A)$-bisemimodule and $\mu_A : \mathcal{A} \otimes_A \mathcal{A} \rightarrow \mathcal{A}$, $\eta_A : A \rightarrow \mathcal{A}$ are $(A, A)$-bilinear maps such that the following diagrams are commutative:

We call $\mu_A$ the multiplication and $\eta_A$ the unity of $\mathcal{A}$. If $A$ is commutative and $\mathcal{A}$ is an $A$-semiring with $xa = ax$ for all $x \in \mathcal{A}$ and $a \in A$, then $\mathcal{A}$ is an $A$-semialgebra. For $A$-semirings $\mathcal{A}$ and $\mathcal{B}$, we call an $(A, A)$-bilinear map $f : \mathcal{A} \rightarrow \mathcal{B}$ a morphism of $A$-semirings iff $f \circ \mu_A = \mu_B \circ (f \otimes_A f)$ and $f \circ \eta_A = \eta_B$; the set of morphisms of $A$-semirings form $\mathcal{A}$ to $\mathcal{B}$ is denoted by $\text{SRng}_A(\mathcal{A}, \mathcal{B})$. The category of $A$-semirings will be denoted by $\text{SRng}_A$.

Corings over (associative) algebras were introduced by Sweedler [48] as algebraic structures that are dual to rings. This suggests defining semicorings over semialgebras as algebraic structures dual to semirings.

2.2. An $A$-semicoring is a comonoid in $(A_S, \otimes_A, A)$, equivalently a triple $(\mathcal{C}, \Delta_C, \epsilon_C)$ in which $\mathcal{C}$ is an $(A, A)$-bisemimodule and $\Delta_C : \mathcal{C} \otimes_A \mathcal{C} \rightarrow \mathcal{C}$, $\epsilon_C : \mathcal{C} \rightarrow A$ are $(A, A)$-bilinear maps such that the following diagrams are commutative:

We call $\Delta_C$ the comultiplication and $\epsilon_C$ the counity of $\mathcal{C}$.

2.3. For $A$-semicorings $(\mathcal{C}, \Delta_C, \epsilon_C)$, $(\mathcal{D}, \Delta_D, \epsilon_D)$, we call an $(A, A)$-bilinear map $f : \mathcal{D} \rightarrow \mathcal{C}$ an $A$-semicoring morphism iff the following diagrams are commutative:

The set of $A$-semicoring morphisms from $\mathcal{D}$ to $\mathcal{C}$ is denoted by $\text{SCog}_A(\mathcal{D}, \mathcal{C})$. The category of $A$-semicorings is denoted by $\text{SCorng}_A$. 
Notation. Let \((\mathcal{C}, \Delta, \varepsilon)\) be an \(A\)-semicoring. We use Sweedler–Heyneman’s \(\Sigma\)-notation, and write for \(c \in \mathcal{C}\):

\[
\Delta(c) = \sum c_1 \otimes_A c_2 \in \mathcal{C} \otimes_A \mathcal{C}.
\]

Example 2.4. Let \(\varphi : B \rightarrow A\) be a morphism of \(S\)-semialgebras and consider \(A\) as a \((B, B)\)-bisemimodule with actions given by \(b \rightarrow a = \varphi(b)a\) and \(a \leftarrow b = a\varphi(b)\) for all \(b \in B\) and \(a \in A\). It follows that \((\mathcal{C} := A \otimes_B A, \Delta, \varepsilon)\) is an \(A\)-semicoring where

\[
\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}, \quad a \otimes_B a' \mapsto (a \otimes_B 1_A) \otimes_A (1_A \otimes_B a') = a \otimes_B 1_A \otimes_B a';
\]

\[
\varepsilon : \mathcal{C} \rightarrow A, \quad a \otimes_B a' \mapsto aa'.
\]

We call \((A \otimes_B A, \Delta, \varepsilon)\) Sweedler’s semicoring.

Example 2.5. Let \(M\) be an \((A, A)\)-bisemimodule. We have an \(A\)-semicoring structure \(\mathcal{C} = (A \oplus M, \Delta, \varepsilon)\), where

\[
\Delta : (a, m) \mapsto (a, 0) \otimes_A (1, 0) + (1, 0) \otimes_A (0, m) + (0, m) \otimes_A (1, 0);
\]

\[
\varepsilon : (a, m) \mapsto a.
\]

Notice that there are many properties such that \(\mathcal{C}\) has Property \(\mathbb{P}\) if (and only if) \(M\) has Property \(\mathbb{P}\), e.g., being flat, (finitely) projective, finitely generated [57, Example 10 (1)].

2.6. Let \((C, \Delta, \varepsilon)\) be a \(S\)-semiring with \(cs = sc\) for all \(s \in S\) and \(c \in C\). We call \(C\) an \(S\)-semicoalgebra. An \(S\)-semicoalgebra is a comonoid in the symmetric monoidal category \((\mathbb{S}, \otimes_S, S)\) of \(S\)-semimodules. If, moreover, \(\sum c_1 \otimes_S c_2 = \sum c_2 \otimes_S c_1\) for all \(c \in C\), then we say that \(C\) is a cocommutative \(S\)-semicoalgebra. We denote the category of \(S\)-semicoalgebras by \(\mathbf{SCoalg}_S\) and its full subcategory of cocommutative \(S\)-semicoalgebras by \(\mathbf{cscSCoalg}_S\).

Example 2.7. Let \(X\) be any set, and consider the free \(S\)-semimodule with basis \(X\). One can easily see that \((S(X), \Delta, \varepsilon)\) is an \(S\)-semicoalgebra where \(\Delta\) and \(\varepsilon\) are defined by extending linearly the following assignments:

\[
\Delta : S(X) \mapsto S(X) \otimes_S S(X), \quad a \mapsto a \otimes_S a;
\]

\[
\varepsilon : S(X) \mapsto S, \quad x \mapsto 1_S.
\]

Notice that \(S[X] \simeq S^{(\infty)} \simeq S[x]\), the polynomial semiring in one indeterminate. So, \((S[x], \Delta_1, \varepsilon_1)\) is an \(S\)-semicoalgebra with

\[
\Delta_1 : S[x] \mapsto S[x] \otimes_S S[x], \quad \sum_{i=0}^n s_i x^i \mapsto \sum_{i=0}^n s_i x^i \otimes_S x^i;
\]

\[
\varepsilon_1 : S[x] \mapsto S, \quad \sum_{i=0}^n s_i x^i \mapsto \sum_{i=0}^n s_i.
\]
Moreover, \((S[x], \Delta_2, e_2)\) is an \(S\)-semialgebra where
\[
\Delta_2 : S[x] \to S[x] \otimes S[x], \quad \sum_{i=0}^{n} s_i x^i \mapsto \sum_{i=0}^{n} s_{i} \left(\sum_{j=0}^{i} \binom{i}{j} x^j \otimes x^{i-j}\right);
\]
\[
e_2 : S[x] \to S, \quad \sum_{i=0}^{n} s_i x^i \mapsto s_0.
\]

**Example 2.8** ([56]). Consider the idempotent Boolean semiring \(B = \{0, 1\}\) (with \(1 + 1 = 1 \neq 0\)). Let \(P = B(x, y)\), the \(B\)-semimodule of formal sums of words formed from the noncommuting letters \(x\) and \(y\). In fact, \(P\) is a \(B\)-semialgebra with multiplication given by the *concatenation* of words. It can be seen that \(P\) has three structures of a \(B\)-semialgebra given as follows:

1. \((P, \Delta_1, e_1)\) is a \(B\)-semialgebra with \(\Delta_1\) and \(e_1\) defined on monomials and extended linearly
\[
\Delta_1 : P \to P \otimes_B P, \quad w \mapsto w \otimes_B w;
\]
\[
e_1 : P \to B, \quad w \mapsto w(1, 1).
\]

2. \((P, \Delta_2, e_2)\) is a \(B\)-semialgebra with \(\Delta_2\) and \(e_2\) defined on monomials and extended linearly
\[
\Delta_2 : P \to P \otimes_B P, \quad w \mapsto \sum_{w_1w_2=w} w_1 \otimes_B w_2;
\]
\[
e_2 : P \to B, \quad w \mapsto w(0, 0).
\]

3. \((P, \Delta_3, e_3)\) is a \(B\)-semialgebra with \(\Delta_3\) and \(e_3\) defined on monomials and extended as semialgebra morphisms
\[
\Delta_3 : P \to P \otimes_B P, \quad \Delta(x) = 1 \otimes_B x + x \otimes_B 1, \quad \Delta(y) = 1 \otimes_B y + y \otimes_B 1;
\]
\[
e_3 : P \to B, \quad w \mapsto w(0, 0).
\]

In what follows, we mean by *locally presentable* categories those in the sense of [10].

**Definition 2.9** ([10, Definition 1.17]). Let \(\mathcal{A}\) be a category and \(\lambda\) a regular cardinal. We say that an object \(X \in \mathcal{A}\) is *locally \(\lambda\)-presentable* iff \(\mathcal{A}(X, -)\) preserves \(\lambda\)-directed colimits. A category \(\mathcal{A}\) is said to be *locally presentable* iff \(\mathcal{A}\) is cocomplete and has a set \(P\) of \(\lambda\)-presentable objects, for some regular cardinal \(\lambda\), such that every object in \(\mathcal{A}\) is a \(\lambda\)-directed colimit of objects from \(P\).

**2.10.** Applying results of Porst [43] iteratively, we have the following: The category \((\text{Set}, \times, [\ast])\) of sets is an admissible symmetric monoidal category which is locally presentable. It follows that the category \(\text{Monoid} = \text{Mon(Set)}\) of monoids is finitary monadic over \(\text{Set}\) and is locally presentable; the full subcategory \(\text{AbMonoid} = \text{Mon(\text{Set})}\) of Abelian (commutative) monoids is finitary monadic over
Set and locally presentable. Notice that \((\text{AbMonoid}, \otimes, \mathbb{N}_0) \simeq (\mathbb{S}_{\mathbb{N}_0}, \otimes_{\mathbb{N}_0}, \mathbb{N}_0)\), the category of semimodules over the semiring \(\mathbb{N}_0\) of non-negative integers, is a biclosed symmetric monoidal category and it follows that the category of semirings \(\text{SRng} = \text{Mon}(\text{AbMonoid})\) is finitary monadic over \(\text{AbMonoid}\) and locally presentable; the full subcategory of commutative semirings \(\text{SRng} \simeq \text{Mon}(\text{AbMonoid})\) is finitary monadic over \(\text{AbMonoid}\) and locally presentable. Moreover, for every (commutative) semiring \(A\) (\(S\)), the category \(\mathcal{A}_A\) (\(\mathcal{S}_A\)) of \((A, A)\)-bismimodules (\(S\)-semimodules) is locally presentable since it is a variety [10, 1.10 (2)].

The Fundamental Theorem of Coalgebras [47] states that every coalgebra over a field is a directed limit of finite dimensional (equivalently locally presentable [41, Proposition 1]) subcoalgebras. This result was generalized to comonoids in a locally presentable symmetric monoidal category by Porst [43] (see also [41, 42]). The results of Porst apply in particular to semicoalgebras over commutative semirings.

**Proposition 2.11.** Consider the categories \(\mathcal{S}_{\text{Alg}}\) of \(S\)-semialgebras and \(\mathcal{S}_{\text{Coalg}}\) of \(S\)-semicoalgebras.

1. \(\mathcal{S}_{\text{Alg}}\) is finitary monadic over \(\mathcal{S}_{\mathbb{S}}\) and locally presentable.
2. \(\mathcal{S}_{\text{Alg}}\) is reflective in \(\mathcal{S}_{\mathbb{S}}\), finitary monadic over \(\mathcal{S}_{\mathbb{S}}\), and locally presentable.
3. \(\mathcal{S}_{\text{Alg}}\) is closed in \(\mathcal{S}_{\mathbb{S}}\) under limits, directed colimits, and absolute colimits.\(^1\)
4. \(\mathcal{S}_{\text{Coalg}}\) is comonadic over \(\mathcal{S}_{\mathbb{S}}\) and locally presentable.
5. \(\mathcal{S}_{\text{Coalg}}\) is coreflective in \(\mathcal{S}_{\mathbb{S}}\), comonadic over \(\mathcal{S}_{\mathbb{S}}\), and locally presentable.
6. \(\mathcal{S}_{\text{Coalg}}\) is closed in \(\mathcal{S}_{\mathbb{S}}\) under colimits and absolute limits.

**Proof.** The result is an immediate application of the main results of [43] taking into consideration that \((\mathcal{S}_{\mathbb{S}}, \otimes_{\mathbb{S}}, \mathcal{S})\) is a biclosed (whence admissible) symmetric monoidal category and that we have isomorphisms of categories \(\mathcal{S}_{\text{Alg}} \simeq \text{Mon}(\mathcal{S}_{\mathbb{S}})\) and \(\mathcal{S}_{\text{Coalg}} \simeq \text{Comonoid}(\mathcal{S}_{\mathbb{S}})\). \(\square\)

The proof of the following result is essentially the same as that for corings over an algebra [41].

**Proposition 2.12.** The category \(\mathcal{S}_{\text{Coring}}\) of \(A\)-semicorings is comonadic over \(\mathcal{A}_{\mathcal{S}_A}\), locally presentable and a covariety (in the sense of [11]).

**2.13.** Let \((\mathcal{C}, \Delta_c, \varepsilon_c)\) be an \(A\)-semiring and \(\mathcal{D} \subseteq_{(A,A)} \mathcal{C}\). We say that \(\mathcal{D}\) is an \(A\)-subsemiring of \(\mathcal{C}\) iff \(\mathcal{D}\) is an \(A\)-semiring and the embedding \(i : \mathcal{D} \hookrightarrow \mathcal{C}\) is a morphism of \(A\)-semirings. If \(\mathcal{D} \subseteq_{(A,A)} \mathcal{C}\) is pure, then \(\mathcal{D}\) is an \(A\)-subsemiring of \(\mathcal{C}\) if and only if \(\Delta_c(\mathcal{D}) \subseteq \mathcal{D} \otimes_A \mathcal{D} \subseteq \mathcal{C} \otimes_A \mathcal{C}\) and \(\varepsilon_c\) is the restriction of \(\varepsilon\) to \(\mathcal{D}\).

**2.14.** Let \(\mathcal{C}\) be a coassociative \(A\)-semiring. Associated to \(\mathcal{C}\) are the following three dual \(A\)-semirings:

\[\mathcal{C}^* := (\text{Hom}_{A-}(\mathcal{C}, A), \star)\] is an \(A\)-semiring, where

\[f \star g)(c) = \sum g(c_i f(c_j)) \quad \text{for all } f, g \in \mathcal{C}^* \text{ and } c \in \mathcal{C};\]

\(^1\)A limit (colimit) \(\mathcal{X}\) is said to be absolute iff \(\mathcal{X}\) is preserved by every functor \(G : \mathcal{A} \to \mathcal{C}\), where \(\mathcal{C}\) is an arbitrary category [7, 20.14 (3)].
\[ \mathcal{C}^* := (\text{Hom}_{\mathcal{A}}(\mathcal{C}, A), \ast) \] is an \( A \)-semiring, where
\[ (f \ast g)(c) = \sum f(g(c_1)c_2) \quad \text{for all } f, g \in \mathcal{C}^* \text{ and } c \in \mathcal{C}; \]
\[ \ast \mathcal{C}^* := (\text{Hom}_{\mathcal{A}}(\mathcal{C}, A), \ast) \] is an \( A \)-semiring, where
\[ (f \ast g)(c) = \sum g(c_1)f(c_2) \quad \text{for all } f, g \in \ast \mathcal{C}^* \text{ and } c \in \mathcal{C}. \]
The counity \( \epsilon_\mathcal{C} \) is a unity for \( \ast \mathcal{C}, \mathcal{C}^* \) and \( \ast \mathcal{C}^* \).

**Definition 2.15.** Let \( \mathcal{C} \) be an \( A \)-semiring. We call
\[ K \leq \mathcal{C}_A \text{ a right } \mathcal{C}\text{-coideal iff } \Delta_\mathcal{C}(K) \subseteq \text{Im}(i_K \otimes_A \mathcal{C}), \]
\[ K \leq_A \mathcal{C} \text{ a left } \mathcal{C}\text{-coideal iff } \Delta_\mathcal{C}(K) \subseteq \text{Im}(\mathcal{C} \otimes_A i_K), \]
\[ K \leq_{(A,A)} \mathcal{C} \text{ a } \mathcal{C}\text{-bicoideal iff } \Delta_\mathcal{C}(K) \subseteq \text{Im}(i_K \otimes_A \mathcal{C}) \cap \text{Im}(\mathcal{C} \otimes_A i_K), \]
\[ K \leq_{(A,A)} \mathcal{C} \text{ a } \mathcal{C}\text{-coideal iff } K = \text{Ker}(f) \text{ for some uniform surjective morphism of } \mathcal{C}\text{-semimodules } f: \mathcal{C} \twoheadrightarrow \mathcal{C}. \]

**Proposition 2.16.** Let \((\mathcal{C}, \Delta_\mathcal{C}, \epsilon_\mathcal{C})\) be an \( A \)-semiring, \( K \leq_{(A,A)} \mathcal{C} \) be uniform and consider the canonical (uniform) surjection \( \pi_K: \mathcal{C} \twoheadrightarrow \mathcal{C}/K \). The following are equivalent:

1. \( K \) is a \( \mathcal{C} \)-coideal;
2. There exists a morphism of \( \mathcal{C} \)-semimodules \( f: \mathcal{C} \rightarrow \mathcal{C}' \) and an exact sequence of \( (A,A) \)-bimodules
\[ 0 \rightarrow K \rightarrow \mathcal{C} \xrightarrow{f} \mathcal{C}' \rightarrow 0; \]
3. \( \mathcal{C}/K \) is an \( A \)-semiring and the \( \pi_K: \mathcal{C} \rightarrow \mathcal{C}/K \) is a morphism of \( A \)-semimodules;
4. \( \Delta_\mathcal{C}(K) \subseteq (i_K \otimes_A \mathcal{C})(K \otimes_A \mathcal{C}) + (\mathcal{C} \otimes_A i_K)(\mathcal{C} \otimes_A K) \) and \( \epsilon_\mathcal{C}(K) = 0. \)
5. \( \Delta_\mathcal{C}(K) \subseteq K \otimes_A \mathcal{C} + \mathcal{C} \otimes_A K \) and \( \epsilon_\mathcal{C}(K) = 0. \)

**Proof.** First of all, notice that the uniform subsemimodules are precisely the subtractive ones, whence \( K = K \).

(1) \iff (2) Follows directly from the definition and Lemma 1.14.
(2) \implies (3) By Lemma 1.15 (1), \( f \) induces an isomorphism of \( (A,A) \)-bimodules \( \mathcal{C}/K \cong \mathcal{C} \). One can easily check that this isomorphism provides \( \mathcal{C}/K \) with a structure of an \( A \)-semiring and that \( \pi_K = T^{-1} \circ f : \mathcal{C} \rightarrow \mathcal{C}/K \) is a morphism of \( A \)-semimodules.

(3) \implies (2) Since \( K \leq_{(A,A)} \mathcal{C} \) is uniform, it follows by Lemma 1.14 that \( K \cong \text{Ker}(\pi_K) \), whence \( K \) is a coideal (notice that \( \pi_K \) is uniform).

(3) \implies (4) Consider the following diagram of \( (A,A) \)-bimodules:
\[ \begin{array}{ccc}
0 & \rightarrow & K & \xrightarrow{i_K} & \mathcal{C} & \xrightarrow{\pi_K} & \mathcal{C}/K & \rightarrow & 0 \\
& | & \downarrow{\epsilon} & | & \downarrow{\Delta} & | & \downarrow{\Delta} & | & \\
0 & \rightarrow & \text{Ker}(\pi_K \otimes_A \pi_K) & \xrightarrow{\epsilon} & \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\pi_K \otimes_A \pi_K} & \mathcal{C}/K \otimes_S \mathcal{C}/K & \rightarrow & 0
\end{array} \]
By assumption, the second square is commutative and so \((\pi_K \otimes_A \pi_K) \circ \Delta_e \circ i_K = \Delta \circ \pi_K \circ i_K = 0\). By the universal property of kernels, there exists an \((A, A)\)-bilinear map \(\kappa : K \to \text{Ker}(\pi_K \otimes_A \pi_K)\) such that the first square is commutative, equivalently \(\Delta_e(K) \subseteq \text{Ker}(\pi_K \otimes_A \pi_K) = (i_K \otimes_A \epsilon) (K \otimes_A \epsilon) + (\epsilon \otimes_A i_K)(\epsilon \otimes_A \kappa)\) by Lemma 1.32. Moreover, we have \(\bar{e} \circ \pi_K = e_e\), whence \(e_e(K) = (\bar{e} \circ \pi_K)(K) = 0\).

(4) \(\Rightarrow\) (3) Consider Diagram (3). By assumption, the first square is commutative and so \((\pi_K \otimes_A \pi_K) \circ \Delta_e \circ i_K = (\pi_K \otimes_A \pi_K) \circ i \circ \kappa = 0\). By the universal property of cokernels, there exists a unique \((A, A)\)-bilinear map \(\bar{\Delta} : \mathcal{C}/K \to \mathcal{C}/K \otimes_A \mathcal{C}/K\) such that the second square is commutative. Moreover, since \(e_e(K) = 0\), the assignments

\[\bar{e} : \mathcal{C}/K \to A, \bar{c} \mapsto e_e(c)\]

is a well-defined \((A, A)\)-bilinear map. One can easily check that \((\mathcal{C}/K, \bar{\Delta}, \pi)\) is an \(A\)-semicoring and that \(\pi_K : \mathcal{C} \to \mathcal{C}/K\) is a morphism of \(A\)-semicorings.

If \(K \leq (A, A)\) \(\mathcal{C}\) is uniformly \(\mathcal{C}\)-pure, then \(\text{Ker}(\pi_K \otimes_A \pi_K) = K \otimes_A \epsilon + \epsilon \otimes_A K\) by Lemma 1.32 and so the last assertion follows.

2.17. Let \((\mathcal{C}, \Delta, \epsilon)\) be an \(A\)-semicoring. A right \(\mathcal{C}\)-semicomodule is a right \(A\)-semimodule \(M\) associated with an \(A\)-linear map (called \(\mathcal{C}\)-coaction)

\[\rho^M : M \to M \otimes_A \mathcal{C}, \quad m \mapsto \sum m_{(0)} \otimes_A m_{(1)},\]

such that the following diagrams are commutative:

\[M \xrightarrow{\rho^M} M \otimes_A \mathcal{C} \xrightarrow{M \otimes_A \Delta} M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \]

Let \(M\) and \(N\) be right \(\mathcal{C}\)-semicomodules. We call an \(A\)-linear map \(f : M \to N\) a \(\mathcal{C}\)-semicomodule morphism (or \(\mathcal{C}\)-colinear) iff the following diagram is commutative:

\[M \xrightarrow{\rho^M} M \otimes_A \mathcal{C} \xrightarrow{f \otimes_A \mathcal{C}} N \otimes_A \mathcal{C}\]

The set of \(\mathcal{C}\)-colinear maps from \(M\) to \(N\) is denoted by \(\text{Hom}^\mathcal{C}(M, N)\). The category of right \(\mathcal{C}\)-semicomodules and \(\mathcal{C}\)-colinear maps is denoted by \(\mathcal{S}^\mathcal{C}\). For a right \(\mathcal{C}\)-semicomodule \(M\), we call \(L \leq_A M\) a \(\mathcal{C}\)-subsemicomodule iff \((L, \rho^L) \in \mathcal{S}^\mathcal{C}\) and the embedding \(L \hookrightarrow M\) is \(\mathcal{C}\)-colinear. Symmetrically, we define the category \(\mathcal{S}\) of
left \mathcal{C}\text{-semicomodules}. For two left \mathcal{C}\text{-semicomodules }M \text{ and } N, \text{ we denote by } \text{^tHom}(M, N) \text{ the set of } \mathcal{C}\text{-colinear maps from } M \text{ to } N.

**Remark 2.19.** Let \((\mathcal{C}, \Delta, \varepsilon)\) be an \(A\)-semiringing. If \((M, \rho^M)\) is a right \(\mathcal{C}\)-semicomodule, then \(\rho^M\) is a splitting monomorphism in \(\mathcal{S}_A\) (but \(M\) is not necessarily a direct summand of \(M \otimes_A \mathcal{C}\); see [28, 16.6]).

Although every \(S\)-semialgebra \(A\) is a regular generator in \(\mathcal{S}_A\) and in \(\mathcal{A} \mathcal{S}\), it is not evident that \(A\) is a generator in \(\mathcal{A} \mathcal{S}_A\) (even if \(S\) is a commutative ring and \(A\) is an \(A\)-algebra [54, 28.1]).

**Definition 2.20.** We define the centroid of the \(S\)-semialgebra \(A\) as

\[ C(A) := \{ f \in \text{End}_A(A) \mid af(b) = f(ab) = f(a)b \text{ for all } a, b \in A \}. \]

We say that \(A\) is a central \(S\)-semialgebra iff \(S \cong C(A)\), where

\[ \varphi : S \to C(A), s \mapsto [a \mapsto sa]. \]

We say that \(A\) is an Azumaya \(S\)-semialgebra iff \(A\) is a central \(S\)-semialgebra such that \(A\) is a regular generator in \(\mathcal{A} \mathcal{S}_A\).

We present now the main reconstruction result.

**Theorem 2.21.**

1. **Let \(\mathcal{C}\) be an \((A, A)\)-bisemimodule. The following statements are equivalent:**

   (a) \(\mathcal{C}\) is an \(A\)-semiringing;
   (b) \(\mathcal{C} \otimes_A - : \mathcal{A} \mathcal{S} \to \mathcal{A} \mathcal{S}\) is a comonad;
   (c) \(- \otimes_A \mathcal{C} : \mathcal{S}_A \to \mathcal{S}_A\) is a comonad.

2. **If \(A\) is an Azumaya \(S\)-semialgebra, then there is a bijective correspondence between the structures of \(A\)-semirings on \(\mathcal{C}\), the comonad structures on \(\mathcal{C} \otimes_A - : \mathcal{A} \mathcal{S} \to \mathcal{A} \mathcal{S}\) and the comonad structures on \(- \otimes_A \mathcal{C} : \mathcal{S}_A \to \mathcal{S}_A\).**

3. **Let \(\mathcal{C}\) be an \(A\)-semiringing and \(\mathcal{D}\) a \(B\)-semiringing (for some \(S\)-semialgebra \(B\)). We have isomorphisms of categories**

\[ \mathcal{D} \mathcal{S} \cong (B \mathcal{S})^\mathcal{D}@\mathcal{B}, \quad \mathcal{C} \mathcal{S} \cong (S_A)^{-\mathcal{D}@A}; \]

\[ ((B\mathcal{S}_A)^{-\mathcal{D}@A})^\mathcal{D}@\mathcal{B} \cong \mathcal{D} \mathcal{S} \cong ((B\mathcal{S}_A)^{\mathcal{D}@B})^{-\mathcal{D}@A}. \]
Proof. (1) and (3) follow directly from the definitions [20, 18.28]. The proof of the bijective correspondence in (2) is similar to that of [52, Theorem 3.9] taking into consideration that $A$ is regular generator in $A$ (by our assumption that $A$ is an Azumaya $S$-semialgebra which yields also that $A$ is braided [8, Theorem 2.6]) and the fact that $\otimes_A X$ and $X \otimes_A -$ preserve colimits in $S^A$ for every $(A, A)$-bisemimodule $X$. □

The main setting in [42, p. 228] applies perfectly to our context. In particular, the category $(\mathcal{C}, \otimes_A, A)$ is a pointed monoidal category, $A$ is a variety whence a locally presentable category [10]. Moreover, for each $X \in \mathcal{C}$, the functor $X \otimes_A -$ has a right adjoint given by $\text{Hom}_A(X, -)$ and the functor $- \otimes_A X: \mathcal{C} \to \mathcal{C}$ has a right adjoint given by $\text{Hom}_A(-, X)$. So, the following result are essentially the same as in the proof of the corresponding ones in [41].

**Proposition 2.22.** Let $\mathcal{C}$ be an $A$-semicoring and $\mathcal{C}: S^\mathcal{C} \to S_A$ the forgetful functor:

1. $S^\mathcal{C}$ is comonadic, locally presentable and a covariety.
2. $\mathcal{C}$ creates all colimits and isomorphisms.
3. $S^\mathcal{C}$ is cocomplete, i.e., $S^\mathcal{C}$ has all (small) colimits, e.g., coequalizers, cokernels, pushouts, directed colimits and direct sums. Moreover, the colimits are formed in $S_A$.
4. $S^\mathcal{C}$ is complete, i.e., $S^\mathcal{C}$ has all (small) limits, e.g., equalizers, kernels, pullbacks, inverse limits and direct products. Moreover, $\mathcal{C}$ creates all limits preserved by $- \otimes_A \mathcal{C}: S_A \to S_A$.

**Remark 2.23.** Although the existence of equalizers (kernels) in $S^\mathcal{C}$ is guaranteed for every $A$-semicorings $\mathcal{C}$, they are not necessarily formed in $S_A$ (compare with [41, Problem 16]). For sufficient conditions for forming equalizers (kernels) of $\mathcal{C}$-colinear maps in $S_A$, see Proposition 2.26 below.

The proof of the following result is essentially the same as that for comodules of corings (e.g., [20, 23]).

**Proposition 2.24.** Let $(\mathcal{C}, \Delta_\mathcal{C}, \epsilon_\mathcal{C})$ be an $A$-semicoring, and consider the forgetful functor $\mathcal{C}: S^\mathcal{C} \to S_A$.

1. For every $M \in S^\mathcal{C}$, we have a functor

   
   $\otimes_A M: S_A \to S^\mathcal{C}, \quad X \mapsto (X \otimes_A M, X \otimes_A \rho^M)$.

Moreover, $\otimes_A M$ is left adjoint to $\text{Hom}^\mathcal{C}(M, -): S^\mathcal{C} \to S_A$; we have natural isomorphisms for all $X \in S_A$ and $Y \in S^\mathcal{C}$:

   
   $\text{Hom}^\mathcal{C}(X \otimes_A M, Y) \cong \text{Hom}_A(X, \text{Hom}^\mathcal{C}(M, Y)), \quad f \mapsto [x \mapsto [m \mapsto f(x \otimes_A m)]]$

with inverse $g \mapsto [x \otimes_A m \mapsto g(x)(m)]$. 

2. \(- \otimes_A \mathcal{C} : \mathcal{S}_A \rightarrow \mathcal{S}^e\) is right adjoint to \(\mathcal{F}\). We have a natural isomorphism for all \(X_A\) and \(Y \in \mathcal{S}^e\):

\[
\text{Hom}^e(Y, X \otimes_A \mathcal{C}) \simeq \text{Hom}_A(\mathcal{F}(Y), X), \quad f \mapsto [y \mapsto \phi_Y \circ (X \otimes_A e)(f(y))]
\]

with inverse \(g \mapsto [y \mapsto \sum g(y(0)) \otimes_A y(1)]\).

**Corollary 2.25.** Let \((\mathcal{C}, \Delta, e, e)\) be an \(A\)-semicoring.

1. Let \(M \in \mathcal{S}^e\). The functor \(- \otimes_A M : \mathcal{S}_A \rightarrow \mathcal{S}^e\) preserves all colimits, whence right exact. In particular, it preserves coequalizers (cokernels), pushouts, (regular) epimorphisms, direct sums and directed colimits. On the other hand, the functor \(\text{Hom}^e(M, -) : \mathcal{S}^e \rightarrow \mathcal{S}_A\) preserves all limits, whence left exact. In particular, it preserves equalizers (kernels), pullbacks, monomorphisms, direct products, and inverse limits.

2. \(- \otimes_A \mathcal{C} : \mathcal{S}_A \rightarrow \mathcal{S}^e\) preserves all colimits and all limits, whence exact. In particular, it preserves coequalizers (cokernels), equalizers (kernels), pushouts, pullbacks, (regular) epimorphisms, monomorphisms, direct sums, direct products, directed colimits, and inverse limits.

3. The forgetful functor \(\mathcal{F} : \mathcal{S}^e \rightarrow \mathcal{S}_A\) does as follows:

   \begin{enumerate}
   \item Creates and preserves all colimits, whence right exact. In particular, it creates and preserves coequalizers (cokernels), pushouts, (regular) epimorphisms, direct sums, and directed colimits.
   \item Creates all limits which are preserved by \(- \otimes_A \mathcal{C} : \mathcal{S}_A \rightarrow \mathcal{S}_A\).
   \end{enumerate}

**Proposition 2.26.** Let \((\mathcal{C}, \Delta, e)\) be \(A\)-semicorings.

1. Coequalizers of \(\mathcal{S}^e\) are formed in \(\mathcal{S}_A\). In particular, for every morphism \(f : M \rightarrow N\) in \(\mathcal{S}^e\), we have \(\text{Coker}(f) = N/f(M)\).

2. If \(\mathcal{A}\) is flat, then equalizers of \(\mathcal{S}^e\) are formed in \(\mathcal{S}_A\).

3. If \(\mathcal{A}\) is \(u\)-flat, then kernels of \(\mathcal{S}^e\) are formed in \(\mathcal{S}_A\).

**Proof.** 1. The forgetful functor \(\mathcal{F}\) creates and preserves all colimits, and in particular coequalizers, by Proposition 2.22. It follows that coequalizers (cokernels) are formed in \(\mathcal{S}_A\). In what follows, we provide an elementary direct proof. Let \(f, g : M \rightarrow N\) be morphisms in \(\mathcal{S}^e\) and let \(\text{Coeq}(f, g)\) be the coequalizer of \(f, g\) in \(\mathcal{S}_A\). Since \(- \otimes_A \mathcal{C} : \mathcal{S}_A \rightarrow \mathcal{S}_A\) preserves coequalizers, we have the following commutative diagram of right \(A\)-semimodules:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\rho^M \downarrow & & \downarrow \rho^N \\
M \otimes_A \mathcal{C} & \xrightarrow{f \otimes_A \mathcal{C}} & N \otimes_A \mathcal{C} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Coeq}(f, g) \quad \downarrow \rho^{\text{Coeq}(f, g)} \\
\end{array}
\]
The left square is commutative since $f$ is a morphism of right $\mathcal{C}$-semicomodules. It follows that
\[
(\pi \otimes_{A} \mathcal{C}) \circ \rho^{N} \circ f = (\pi \otimes_{A} \mathcal{C}) \circ (f \otimes_{A} \mathcal{C}) \circ \rho^{M}
\]
\[
= ((\pi \circ f) \otimes_{A} \mathcal{C}) \circ \rho^{M}
\]
\[
= ((\pi \circ g) \otimes_{A} \mathcal{C}) \circ \rho^{M}
\]
\[
= (\pi \otimes_{A} \mathcal{C}) \circ (g \otimes_{A} \mathcal{C}) \circ \rho^{M}
\]
\[
= (\pi \otimes_{A} \mathcal{C}) \circ \rho^{N} \circ g.
\]

By the universal property of coequalizers, there exists a unique $A$-linear map $\rho^{\text{Coeq}(f,g)} : \text{Coeq}(f,g) \rightarrow \text{Coeq}(f,g) \otimes_{A} \mathcal{C}$ such that the right square is commutative. Consider the following diagram with commutative trapezoids and inner rectangle:

\[
\begin{array}{cccc}
\text{Coeq}(f,g) & \xrightarrow{\rho^{\text{Coeq}(f,g)}} & \text{Coeq}(f,g) \otimes_{A} \mathcal{C} \\
N & \xrightarrow{\rho^{N}} & N \otimes_{A} \mathcal{C} & \xrightarrow{\pi \otimes \mathcal{C}} \\
N \otimes_{A} \mathcal{C} & \xrightarrow{\rho^{N} \otimes \mathcal{C}} & N \otimes_{A} \mathcal{C} \otimes_{A} \mathcal{C} & \xrightarrow{\pi \otimes \mathcal{C} \otimes \mathcal{C}} \\
\text{Coeq}(f,g) \otimes_{A} \mathcal{C} & \xrightarrow{\rho^{\text{Coeq}(f,g) \otimes_{A} \mathcal{C}}} & \text{Coeq}(f,g) \otimes_{A} \mathcal{C} \otimes_{A} \mathcal{C}.
\end{array}
\]

Notice that
\[
(\text{Coeq}(f, g) \otimes_{A} \Delta) \circ \rho^{\text{Coeq}(f,g)} \circ \pi = (\text{Coeq}(f, g) \otimes_{A} \Delta) \circ (\pi \otimes_{A} \mathcal{C}) \circ \rho^{N}
\]
\[
= (\pi \otimes_{A} \mathcal{C} \otimes_{A} \mathcal{C}) \circ (N \otimes_{A} \Delta) \circ \rho^{N}
\]
\[
= (\pi \otimes_{A} \mathcal{C} \otimes_{A} \mathcal{C}) \circ (\rho^{N} \otimes_{A} \mathcal{C}) \circ \rho^{N}
\]
\[
= (\rho^{\text{Coeq}(f,g)} \otimes_{A} \mathcal{C}) \circ (\pi \otimes_{A} \mathcal{C}) \circ \rho^{N}
\]
\[
= (\rho^{\text{Coeq}(f,g)} \otimes_{A} \mathcal{C}) \circ \rho^{\text{Coeq}(f,g)} \circ \pi.
\]

Since $\pi$ is an epimorphism, we conclude that
\[
(\text{Coeq}(f, g) \otimes_{A} \Delta) \circ \rho^{\text{Coeq}(f,g)} = (\rho^{\text{Coeq}(f,g)} \otimes_{A} \mathcal{C}) \circ \rho^{\text{Coeq}(f,g)}.
\]

Moreover, we have
\[
\theta^{\text{Coeq}(f,g)}_{\text{Coeq}(f,g)} \circ (\text{Coeq}(f, g) \otimes_{A} e) \circ \rho^{\text{Coeq}(f,g)} \circ \pi
\]
\[
= \theta^{\text{Coeq}(f,g)}_{\text{Coeq}(f,g)} \circ (\text{Coeq}(f, g) \otimes_{A} e) \circ (\pi \otimes_{A} \mathcal{C}) \circ \rho^{N}
\]
\[
= \pi \circ \theta^{\text{Coeq}(f,g)}_{\text{Coeq}(f,g)} \circ (N \otimes_{A} e) \circ \rho^{N}
\]
\[
= \pi.
\]

Since $\pi$ is an epimorphism, we conclude that $\theta^{\text{Coeq}(f,g)}_{\text{Coeq}(f,g)} \circ (\text{Coeq}(f, g) \otimes_{A} e) \circ \rho^{\text{Coeq}(f,g)} = \text{id}_{\text{Coeq}(f,g)}$. Consequently, $(\text{Coeq}(f, g), \rho^{\text{Coeq}(f,g)})$ is a right $\mathcal{C}$-semicomodule.
2. Notice that $\mathbb{S}_A$ has equalizers and that $\text{Eq}(f, g) = \{m \in M \mid f(m) = g(m)\}$ for all $A$-linear maps $f, g$. Since $A\mathcal{C}$ is flat, we have $\text{Eq}(f \otimes_A \mathcal{C}, g \otimes_A \mathcal{C}) = (f, g) \otimes_A \mathcal{C}$ in $\mathbb{S}_A$. Consider the following diagram of right $A$-semimodules:

$$
\begin{array}{c}
\text{Eq}(f, g) \xrightarrow{\epsilon} M \xrightarrow{f} N \\
\rho^{\text{Eq}(f, g)} \\
\downarrow \\
\text{Eq}(f, g) \otimes_A \mathcal{C} \xrightarrow{\iota \otimes_A \mathcal{C}} M \otimes_A \mathcal{C} \xrightarrow{g \otimes_A \mathcal{C}} N \otimes_A \mathcal{C}.
\end{array}
$$

(6)

Since $f$ is a morphism of right $\mathcal{C}$-semicomodules, the right square is commutative. It follows that

$$
(f \otimes_A \mathcal{C}) \circ \rho^M \circ \iota = \rho^N \circ (f \circ \iota) = \rho^N \circ (g \circ \iota) = (g \otimes_A \mathcal{C}) \circ \rho^M \circ \iota,
$$

and so there exists, by the universal property of equalizers, a unique $A$-linear map $\rho^{\text{Eq}(f, g)} : \text{Eq}(f, g) \rightarrow \text{Eq}(f, g) \otimes_A \mathcal{C}$ such that the left square is commutative. Consider the following diagram with commutative trapezoids and outer rectangle:

$$
\begin{array}{c}
M \xrightarrow{\iota} \text{Eq}(f, g) \xrightarrow{\iota \otimes_A \mathcal{C}} M \otimes_A \mathcal{C} \xrightarrow{\iota \otimes_A \mathcal{C}} \text{Eq}(f, g) \otimes_A \mathcal{C} \xrightarrow{\iota \otimes_A \mathcal{C}} M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \\
\rho^M \xrightarrow{\rho^{\text{Eq}(f, g)} \circ \rho^M} \rho^N \xrightarrow{\rho^{\text{Eq}(f, g)} \circ \rho^M} \rho^N \xrightarrow{\rho^{\text{Eq}(f, g)} \circ \rho^M} \rho^N
\end{array}
$$

(7)

Notice that

$$
(\iota \otimes_A \mathcal{C} \otimes A \mathcal{C}) \circ (\text{Eq}(f, g) \otimes_A \Delta) \circ \rho^{\text{Eq}(f, g)} = (M \otimes_A \Delta) \circ (\iota \otimes_A \mathcal{C}) \circ \rho^{\text{Eq}(f, g)}
$$

$$
= (M \otimes_A \Delta) \circ \rho^M \circ \iota
$$

$$
= (\rho^M \otimes_A \mathcal{C}) \circ \rho^M \circ \iota
$$

$$
= (\rho^M \otimes_A \mathcal{C}) \circ (\iota \otimes_A \mathcal{C}) \circ \rho^{\text{Eq}(f, g)}
$$

$$
= (\iota \otimes_A \mathcal{C} \otimes A \mathcal{C}) \circ (\rho^{\text{Eq}(f, g)} \otimes_A \mathcal{C}) \circ \rho^{\text{Eq}(f, g)}.
$$

Since $A\mathcal{C}$ is flat, it follows that $A\mathcal{C}$ is mono-flat and so $\iota \otimes_A \mathcal{C} \otimes A \mathcal{C}$ is injective, whence

$$
(\text{Eq}(f, g) \otimes_A \Delta) \circ \rho^{\text{Eq}(f, g)} = (\text{Eq}(f, g) \otimes_A \mathcal{C}) \circ \rho^{\text{Eq}(f, g)}.
$$
Moreover, we have
\[
\begin{align*}
1. \quad & t \circ \delta_{Eq(f, g)}^t \circ (Eq(f, g) \otimes_A \varepsilon) \circ \rho_{Eq(f, g)}^t = \delta_M^t \circ (M \otimes_A \varepsilon) \circ (t \otimes_A \varepsilon) \circ \rho_{Eq(f, g)}^t \\
& = \delta_M^t \circ (M \otimes_A \varepsilon) \circ \rho^M \circ t \\
& = t \circ Eq(f, g).
\end{align*}
\]

Since \( Eq(f, g) \hookrightarrow M \) is a monomorphism, we conclude that \( \delta_{Eq(f, g)}^t \circ (Eq(f, g) \otimes_A \varepsilon) \circ \rho_{Eq(f, g)}^t = id_{Eq(f, g)} \). It follows that \( (Eq(f, g), \rho_{Eq(f, g)}) \) is a right \( \mathcal{C} \)-semicomodule.

3. Since \( A \mathcal{C} \) is \( u \)-flat, \( - \otimes_A \mathcal{C} : \mathcal{S}_A \to \mathcal{S}_A \) preserves kernels (see Remark 1.30). The proof is along the lines of that of (2).

**Notation.** Let \( \mathcal{C} \) be an \( A \)-semicoring. In addition to the forgetful functor \( \mathcal{F} : \mathcal{S}^\mathcal{C} \to \mathcal{S}_A \), we consider the following functors:

\[ \mathcal{G} := - \otimes_A \mathcal{C} : \mathcal{S}_A \to \mathcal{S}^\mathcal{C} \quad \text{and} \quad \mathcal{M} := - \otimes_A \mathcal{C} : \mathcal{S}_A \to \mathcal{S}_A. \]

**Remark 2.27.** Let \( \mathcal{C} \) be an \( A \)-semicoring. Recall that the functor \( \mathcal{G} \) is exact, whence it preserves monomorphisms and kernels. If \( \mathcal{F} \) preserves monomorphisms (kernels), then \( \mathcal{M} = \mathcal{F} \circ \mathcal{G} \) preserves monomorphisms (kernels) as well.

**Proposition 2.28.** Let \( \mathcal{C} \) be an \( A \)-semicoring, and consider the forgetful functor \( \mathcal{F} : \mathcal{S}^\mathcal{C} \to \mathcal{S}_A \).

1. \( A \mathcal{C} \) is flat if and only if \( \mathcal{F} \) is (left) exact.

2. Assume that \( A \mathcal{C} \) is mono-flat. The following statements are equivalent:

   (a) \( A \mathcal{C} \) is uniformly flat;

   (b) \( A \mathcal{C} \) is \( u \)-flat;

   (c) \( \mathcal{F} \) creates and preserves kernels;

   (d) \( \mathcal{F} \) preserves uniform monomorphisms (i.e., every uniform monomorphism in \( \mathcal{S}^\mathcal{C} \) is injective).

**Proof.** Recall first that \( (\mathcal{F}, \mathcal{G}) \) and \( (\mathcal{G}, \text{Hom}^\mathcal{G}(\mathcal{C}, -)) \) are adjoint pairs, whence \( \mathcal{F} \) is right exact and \( \mathcal{G} \) is exact. Moreover, notice that \( \mathcal{M} = \mathcal{F} \circ \mathcal{G} : \mathcal{S}_A \to \mathcal{S}_A \).

1. \((\Rightarrow)\) Notice that we have two (left) exact functors \( \mathcal{S}_A \to \mathcal{S}^\mathcal{C} \to \mathcal{S}_A \), whence \( \mathcal{M} = \mathcal{F} \circ \mathcal{G} \) is (left) exact, i.e., \( A \mathcal{C} \) is flat.

\((\Leftarrow)\) Since \( A \mathcal{C} \) is flat, all finite limits are preserved by the (left) exact functor \( \mathcal{G} \), and these are consequently created and preserved by \( \mathcal{F} \), i.e., \( \mathcal{F} \) is (left) exact.

2. Assume that \( A \mathcal{C} \) is mono-flat.

   (a \iff b) This follows by Remark 1.30.

   (b \Rightarrow c) This follows by Proposition 2.26.

   (c \Rightarrow d) Let \( f : X \to Y \) be a uniform monomorphism in \( \mathcal{S}^\mathcal{C} \), and consider the \( \ker(f) : \text{Ker}(f) \to X \) in \( \mathcal{S}^\mathcal{C} \). Since \( f \circ \ker(f) = 0 = f \circ 0 \), we conclude that \( \text{Ker}(f) = 0 \) in \( \mathcal{S}^\mathcal{C} \). Since \( \mathcal{F} \) preserves kernels, \( \text{Ker}(f) = \{0\} \) in \( \mathcal{S}_A \), whence \( f \) is injective being a \( k \)-uniform \( A \)-linear map with zero kernel.
(d → a) Let \( X \to Y \) be a uniform \( A \)-subsemimodule, whence \( X = \overline{X} = \text{Ker}(Y \to Y/X) \). Since \(- \otimes_A C : \mathcal{S}_A \to \mathcal{S}^e\) is left exact, it preserves kernels and so \( X \otimes_A C = \text{Ker}(Y \otimes_A C \to Y/X \otimes_A C) \) in \( \mathcal{S}^e \); in particular, \( i \otimes_A C : X \otimes_A C \to Y \otimes_A C \) is a uniform monomorphism in \( \mathcal{S}^e \). By our assumption on \( \mathcal{F} \), the map \( i \otimes_A C : X \otimes_A C \to Y \otimes_A C \) is a uniform monomorphism in \( \mathcal{S}_A \), i.e., \( X \otimes_A C \leq Y \otimes_A C \). We conclude that \( A \otimes_A C \) is uniformly flat. \( \square \)

**Remark 2.29.** Let \( (M, \rho^M) \) be a right \( C \)-semicomodule and \( N \leq_A M \). If \( A \otimes_A C \) is mono-flat and \( \rho^N_1, \rho^N_2 : N \to N \otimes_A C \) make \( N \) a \( C \)-subsemicomodule of \( M \), then one can easily see that \( \rho^N_1 = \rho^N_2 \). However, if \( A \otimes_A C \) is not mono-flat, then it might happen that \( N \) has two different structures as a \( C \)-subsemicomodule of \( M \).

The following example appeared originally in [46] (and cited in [57]) with \( \mathbb{Z} \) at the place of \( \mathbb{N}_0 \):

**Example 2.30.** Let \( C = (\mathbb{N}_0 \oplus \mathbb{Z}/n\mathbb{Z}, \Delta, \varepsilon) \) be the \( \mathbb{N}_0 \)-semicoalgebra whose comultiplication and counity are given by

\[
\Delta((l, m)) := (l, 0) \otimes_{\mathbb{N}_0} (1, 0) + (1, 0) \otimes_{\mathbb{N}_0} (0, m) + (0, m) \otimes_{\mathbb{N}_0} (1, 0) + (0, m) \otimes_{\mathbb{N}_0} (0, \overline{1}),
\]

\[
\varepsilon((l, m)) := l.
\]

Consider the Abelian monoid \( M = \mathbb{Q}/\mathbb{Z} \) and the embedding of monoids

\[
i : \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}, \quad \overline{z} \mapsto \left[ \frac{z}{n} \right] \quad \text{(where } z \equiv r \text{ mod } n \text{ and } r \in \{0, 1, \ldots, n-1\}).
\]

We have a structure of a right \( C \)-semicomodule \((M, \rho^M)\) and two different \( C \)-subsemicomodule structures \((N, \rho^N_1), (N, \rho^N_2)\), where

\[
\rho^M : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{N}_0} C, \quad \overline{q} \mapsto \overline{q} \otimes_{\mathbb{N}_0} (1, 0),
\]

\[
\rho^N_1 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{N}_0} C, \quad \overline{z} \mapsto \overline{z} \otimes_{\mathbb{N}_0} (1, 0),
\]

\[
\rho^N_2 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{N}_0} C, \quad \overline{z} \mapsto \overline{z} \otimes_{\mathbb{N}_0} (1, 0) + (0, \overline{1}) \otimes_{\mathbb{N}_0} \overline{z}.
\]

Notice that \( \mathbb{N}_0 \oplus \mathbb{Z}/n\mathbb{Z} \) is not mono-flat in \textbf{AbMonoid}.

**Proposition 2.31.** Let \( C \) be an \( A \)-semicoring.

1. If \( Q \) is a cogenerator in \( \mathcal{S}_A \), then \( Q \otimes_A C \) is a cogenerator in \( \mathcal{S}^e \).
2. If \( A_A \) is a cogenerator in \( \mathcal{S}_A \), then \( C \) is a cogenerator in \( \mathcal{S}^e \).

**Proof.** 1. It is well known that right adjoint functors preserve cogenerators. However, we provide a direct proof: let \( f, g : M \to N \) be morphisms in \( \mathcal{S}^e \) with \( f \neq g \). Since \( \mathcal{S}_A \) has products and \( Q_A \) is a cogenerator, there exists an index set \( \Lambda \) such that \( N \to Q^\Lambda \). By Corollary 2.25, \(- \otimes_A C\) preserves monomorphisms and direct products. It follows that we have a monomorphism \( \gamma : N \otimes_A C \to \)
1. If every (uniform) monomorphism in \( Q \) is uniformly injective, then \( \pi : (Q \otimes_A \mathcal{C})^A \longrightarrow Q \otimes_A \mathcal{C} \). If \( \pi \circ f = \pi \circ g \), then we have \( (\gamma \circ f)(\lambda) = (\gamma \circ g)(\lambda) \) for every \( \lambda \in \Lambda \), whence \( \gamma \circ f = \gamma \circ g \) and this yields \( f = g \) (a contradiction). Setting \( h := \pi \circ \gamma : N \longrightarrow Q \otimes_A \mathcal{C} \), we have \( h \circ f \neq h \circ g \), and we conclude that \( Q \otimes_A \mathcal{C} \) is a cogenerator in \( S^\mathcal{C} \).

2. This follows directly from (1) and the canonical isomorphism \( A \otimes_A \mathcal{C} \simeq \mathcal{C} \).

2.32. Let \( \mathcal{C} \) be an \( A \)-semiring with \( \lambda \mathcal{C} \) be flat, so that the forgetful functor \( \mathcal{F} : S^\mathcal{C} \longrightarrow S_A \) is exact (by Proposition 2.28 (1)). It follows that \( S^\mathcal{C} \) has kernels (as well as cokernels) formed in \( S_A \) and that monomorphisms are injective while regular epimorphisms are surjective. One can prove that in this case the category \( S^\mathcal{C} \) has a (Surj, Inj)-factorization system [7]. The arguments in [2] about the natural definition of exact sequences of semimodules apply to the category \( S^\mathcal{C} \) as well. So, we call a sequence \( X \longrightarrow Y \longrightarrow Z \) of right \( \mathcal{C} \)-semicomodules exact if \( f(X) = \text{Ker}(g) \) and \( g \) is \( k \)-uniform. A sequence \( 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \) will be called a short exact sequence if \( f \) induces an isomorphism \( X \simeq \text{Ker}(g) \) and \( g \) induces an isomorphism \( Z \simeq \text{Coker}(f) \).

**Definition 2.33.** Let \( A \mathcal{C} \) be a semiring. We say that a right \( \mathcal{C} \)-semicomodule \( E \) is uniformly injective if for every uniform monomorphism \( f : M \longrightarrow N \) in \( S^\mathcal{C} \), the induced map of Abelian monoids

\[
(f, E) : \text{Hom}^\mathcal{C}(N, E) \longrightarrow \text{Hom}^\mathcal{C}(M, E), \quad h \mapsto h \circ f
\]

is surjective and uniform.

**Remark 2.34.** It is well-known that functors between Abelian categories with an exact left adjoint preserve injective objects [27, 6.28]. We extend this result to the functor \( \mathcal{G} := - \otimes_A \mathcal{C} : S_A \longrightarrow S^\mathcal{C} \), with \( \lambda \mathcal{C} \) flat, which is right adjoint to the exact forgetful functor. Please notice that the categories under consideration are, in general, far away from being Abelian (not even additive).

**Definition 2.35.** We say that \( S_A \) has enough (uniformly) injective objects if every \( A \)-semimodule is an \( A \)-subsemimodule of a (uniformly) injective \( A \)-semimodule.

**Proposition 2.36.** Let \( \mathcal{C} \) be an \( A \)-semiring and consider the functor \( \mathcal{G} := - \otimes_A \mathcal{C} : S_A \longrightarrow S^\mathcal{C} \).

1. If every (uniform) monomorphism in \( S^\mathcal{C} \) is injective, then \( \mathcal{G} \) preserves (uniformly) injective objects.
2. Assume that \( S_A \) has enough (uniformly) injective objects. Every uniform monomorphism in \( S^\mathcal{C} \) is injective if and only if \( \mathcal{G} \) preserves (uniformly) injective objects.

**Proof.** 1. Let \( E \) be a (uniformly) injective \( A \)-semimodule. Let \( i : L \longrightarrow M \) be a (uniform) monomorphism in \( S^\mathcal{C} \). By our assumptions \( \mathcal{F} \) preserves (uniform)
monomorphisms, whence $L \leq_A M$ ($L \leq^u_A M$). By Proposition 2.24, we have natural isomorphisms $\text{Hom}^\Phi(L, E \otimes_A \mathcal{C}) \cong \text{Hom}_A(\mathcal{F}(L), E)$, and $\text{Hom}^\Phi(L, E \otimes_A \mathcal{C}) \cong \text{Hom}_A(\mathcal{F}(L), E)$ where $\mathcal{F} : \mathcal{S}^\mathcal{C} \to \mathcal{S}_A^\mathcal{C}$ is the forgetful functor. Consider the following commutative diagram of Abelian monoids:

$$
\begin{array}{c}
\text{Hom}^\mathcal{C}(M, E \otimes_A \mathcal{C}) \\
\downarrow
\end{array}
\begin{array}{c}
\text{Hom}^\Phi(L, E \otimes_A \mathcal{C}) \\
\downarrow
\end{array}
\begin{array}{c}
\text{Hom}^\mathcal{C}(L, E \otimes_A \mathcal{C})
\end{array}
\begin{array}{c}
\to 0
\end{array}
$$

By assumption, $(i, E)$ is a (uniform) surjective map, whence $(i, E \otimes_A \mathcal{C})$ is a (uniform) surjective map. Consequently, $E \otimes_A \mathcal{C}$ is (uniformly) injective in $\mathcal{S}_A^\mathcal{C}$.

2. The proof is along the lines of that of the corresponding result for comodules of coalgebras over a commutative ring [57, Proposition 8]. Assume that $\mathcal{C}$ preserves (uniformly) injective objects. Let $h : L \to M$ be a (uniform) monomorphism of right $\mathcal{C}$-comodules. We claim that $h$ is injective. By assumption, there exists a (uniformly) injective right $\mathcal{C}$-semicomodule $E$ such that $L \hookrightarrow E$. By assumption, $\mathcal{C}$ preserves (uniformly) injective objects, whence $E \otimes_A \mathcal{C}$ is (uniformly) injective in $\mathcal{S}^\mathcal{C}$. Notice that we have a morphism of $\mathcal{C}$-colinear maps

$$(i_L \otimes \mathcal{C}) \circ \rho^L : L \to L \otimes_A \mathcal{C} \to E \otimes_A \mathcal{C}.$$

Since $E \otimes_A \mathcal{C}$ is (uniformly) injective, there exists a unique $\mathcal{C}$-colinear map $g : M \to E \otimes_A \mathcal{C}$ such that $g \circ h = (i_L \otimes \mathcal{C}) \circ \rho^L$. So, we have

$$
i_L = i_L \circ \text{id}_L
= i_L \circ \partial'_L \circ (L \otimes_A \mathcal{C}) \circ \rho^L
= \partial'_E \circ (E \otimes_A \mathcal{C}) \circ (i_L \otimes \mathcal{C}) \circ \rho^L
= \partial'_E \circ (E \otimes_A \mathcal{C}) \circ g \circ h.
$$

It follows that $h$ is injective, and we are done. □

Combining Proposition 2.28 (2) and Proposition 2.36, we get the following corollary.

**Corollary 2.37.** Let $\mathcal{C}$ be an $A$-semicoring, and consider the functor $\mathcal{C} := - \otimes_A \mathcal{C} : \mathcal{S}_A \to \mathcal{S}^\mathcal{C}$.

1. If $\mathcal{S}_A^\mathcal{C}$ is $u$-flat, then $\mathcal{S}$ preserves uniformly injective objects.
2. Assume that $\mathcal{S}_A^\mathcal{C}$ has enough uniformly injective objects and that $\mathcal{S}_A^\mathcal{C}$ is mono-flat. The following statement are equivalent:
   
   (a) $\mathcal{S}_A^\mathcal{C}$ is $u$-flat;
   
   (b) Every uniform monomorphism in $\mathcal{S}^\mathcal{C}$ is injective;
   
   (c) $\mathcal{S}$ preserves uniformly injective objects.
Proposition 2.38. Let \( \mathcal{C} \) be an \( A \)-semicoring and assume that \( A \mathcal{C} \) is flat.

1. If \( E_A \) is a (uniformly) injective cogenerator, then \( E \otimes A \mathcal{C} \) is a (uniformly) injective cogenerator in \( \mathcal{S}^e \).
2. If \( A_A \) is a (uniformly) injective cogenerator, then \( \mathcal{C} \) is a (uniformly) injective cogenerator in \( \mathcal{S}^e \).

3. MEASURING \( \alpha \)-PAIRINGS

In this section, we introduce and investigate the \( \mathcal{C} \)-rational \( A \)-semimodules associated with a measuring left (right) \( \alpha \)-pairing \( (\mathcal{A}, \mathcal{C}) \).

The \( \alpha \)-Condition

3.1. We say that \( P = (V, W) \) is a left \( \alpha \)-pairing iff \( V \) is a right \( A \)-semimodule, \( W \) is a left \( A \)-semimodule and there exists an \( A \)-linear map \( \kappa_P : V \rightarrow W \) is \( A \)-linear. A left \( A \)-pairing \( P = (V, W) \); satisfies the \( \alpha \)-condition, or is a left \( \alpha \)-pairing iff the following map is injective and subtractive (whence uniform):

\[
\sigma_P^\alpha : M \otimes_A W \rightarrow \text{Hom}_A(V, M), \quad \sum m_i \otimes_A w_i \mapsto \sum [v \mapsto \sum m_i(v, w_i)].
\]

Right \( A \)-pairings are defined symmetrically. A right \( A \)-pairing \( P = (V, W) \) is said to satisfy the (right) \( \alpha \)-condition, or to be a right \( \alpha \)-pairing, iff for every left \( A \)-semimodule \( M \), the canonical map \( \sigma_P^\alpha : W \otimes_A M \rightarrow \text{Hom}_A(V, M) \) is injective and subtractive.

Definition 3.2. We say that \( AW \) is a left \( \alpha \)-semimodule iff the left \( A \)-pairing \( (\star W, W) \) satisfies the \( \alpha \)-condition, equivalently iff the canonical map

\[
\sigma_M^\alpha : M \otimes_A W \rightarrow \text{Hom}_A(\star W, M), \quad m \otimes_A w \mapsto [f \mapsto mf(w)]
\]

is injective and subtractive (uniform). Symmetrically, one defines right \( \alpha \)-semimodules. Moreover, we say that \( AWB \) is an \( \alpha \)-bisemimodule iff \( AW \) and \( WB \) are \( \alpha \)-semimodules.

Remarks 3.3.

1. If \( P = (V, W) \) is a left \( \alpha \)-pairing, then \( W \leq^u V^* \) (take \( M = A \)).
2. If \( A \) is finitely projective, then \( \text{Ker}(\sigma_M^\alpha) = 0 \) for every \( M_A \).

Examples 3.4. A left \( A \)-semimodule \( W \) is an \( \alpha \)-semimodule if, for example, \( W \) satisfies any of the following conditions:

1. \( AW \) is a free \( A \)-semimodule;
2. \( AW \) is a direct summand of a free \( A \)-semimodule;
3. \( AW \) is finitely projective and \( \sigma_M^\alpha \) is subtractive for every \( M_A \).

Lemma 3.5. If \( P = (V, W) \) is a measuring left \( \alpha \)-pairing, then \( AW \) is uniformly flat.
Proof. Let $M$ be any right $A$-semimodule, $L \leq_A M$, and consider the commutative diagram of Abelian monoids:

\[
\begin{array}{ccc}
L \otimes_A W & \xrightarrow{\alpha_L} & \text{Hom}_A(V, L) \\
\downarrow{\iota_L \otimes_A W} & & \downarrow{(V, \iota)} \\
M \otimes_A W & \xrightarrow{\alpha_M} & \text{Hom}_A(V, M).
\end{array}
\]

It is easy to see that $\text{Hom}_A(V, -)$ preserves uniform morphisms. By assumption, $\alpha_L^p$ is injective and uniform, whence $(V, i) \circ \alpha_L^p$ is injective and, moreover, uniform by Lemma [2, Lemma 1.15]. It follows that $\alpha_M \circ (L \otimes_A W) = (V, i) \circ \alpha_L^p$ is injective and uniform, whence $L \otimes_A W \leq_A M \otimes_A W$. \hfill \square

The following technical lemma plays an important role in the investigations of rational semimodules.

Lemma 3.6. Let $P = (V, W)$ be a left $\alpha$-pairing. If $L$ is a right $A$-semimodule and $K \leq_A L$ is an $A$-subsemimodule, then we have for every $\sum l_i \otimes_A w_i \in L \otimes_A W$

\[\sum l_i \otimes_A w_i \in K \otimes_A W \iff \sum l_i \langle v, w_i \rangle \in K \text{ for all } v \in V.\]

Proof. Notice that we have an exact sequence of right $A$-semimodules

\[0 \rightarrow K \xrightarrow{\iota_K} L \xrightarrow{\pi_K} L/K \rightarrow 0.\]

Consider the commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & K \otimes_A W \\
\downarrow{\alpha_K} & & \downarrow{\alpha_K} \\
L \otimes_A W & \xrightarrow{\pi_K \otimes_A W} & L/K \otimes_A W \\
\downarrow{\alpha_L} & & \downarrow{\alpha_L} \\
\text{Hom}_A(V, K) & \xrightarrow{(V, \pi_K)} & \text{Hom}_A(V, L) \\
\downarrow{\alpha_L} & & \downarrow{\alpha_L} \\
\text{Hom}_A(V, L/K) & & \text{Hom}_A(V, L/K)
\end{array}
\]

By Lemma 3.5, $A W$ is uniformly flat and so the first row is exact. Clearly, $\sum l_i \langle v, w_i \rangle \in K$ for every $v \in V$ if and only if $\sum l_i \otimes_A w_i \in \text{Ker}((V, \pi_K) \circ \alpha_L^p) = \text{Ker}(\alpha_L^p \circ (\pi_K \otimes_A W)) = \text{Ker}(\pi_K \otimes_A W) = K \otimes_A W$. \hfill \square

The proof of the following result is similar to that of [6, Proposition 2.5].

Lemma 3.7. Let $V, W$ be $(A, A)$-bisemimodules.

1. If $P = (V, W)$, $P' = (V', W')$ are left $\alpha$-pairings, then $P \otimes_A P' := (V' \otimes_A V, W \otimes_A W')$ is a left $\alpha$-pairing, where

\[\kappa_{P \otimes_A P'}(v' \otimes_A v)(w' \otimes_A w') = \langle v, w \langle v', w' \rangle \rangle = \langle \langle v', w' \rangle v, w \rangle.\]

2. If $P = (V, W)$, $P' = (V', W')$ are right $\alpha$-pairings, then $P \otimes'_A P' := (V \otimes_A V', W' \otimes_A W)$ is a right $\alpha$-pairing, where

\[\kappa_{P \otimes'_A P'}(v \otimes_A v')(w' \otimes_A w) = \langle v, \langle v', w' \rangle w \rangle = \langle v(v', w'), w \rangle.\]
Measuring Pairings

3.8. Let \( \mathcal{C} \) be an \( A \)-semicoring and consider the left dual \( A \)-semiring \( ^*\mathcal{C} := \text{Hom}_A(\mathcal{C}, A) \). If \( \mathcal{A} \) is an \( A \)-semiring with a morphism of \( A \)-semirings \( \kappa : \mathcal{A} \rightarrow ^*\mathcal{C} \), \( a \mapsto [c \mapsto \langle a, c \rangle] \), then we call \( P := (\mathcal{A}, \mathcal{C}) \) a measuring left \( A \)-pairing. A measuring right \( A \)-pairing \( P = (\mathcal{A}, \mathcal{C}) \) consists of an \( A \)-semiring \( \mathcal{A} \) and an \( A \)-semicoring \( \mathcal{C} \) with a morphism of \( A \)-semirings \( \kappa_P : \mathcal{A} \rightarrow ^*\mathcal{C} \). If \( \mathcal{A} \) is an \( A \)-semiring with a morphism of \( A \)-semirings \( \kappa_P : \mathcal{A} \rightarrow ^*\mathcal{C} \), then we call \( (\mathcal{A}, \mathcal{C}) \) a measuring \( A \)-pairing.

3.9. If \( P = (\mathcal{A}, \mathcal{C}) \) is a measuring left (right) \( A \)-pairing, then \( \mathcal{C} \) is a right (left) \( A \)-semimodule with \( \mathcal{A} \)-action given by

\[
c \leftarrow a := \sum c_1 \langle a, c_2 \rangle \quad (a \rightarrow c := \sum \langle a, c_1 \rangle c_2) \tag{8}
\]

for all \( a \in \mathcal{A} \) and \( c \in \mathcal{C} \). If \( P = (\mathcal{A}, \mathcal{C}) \) is a measuring \( A \)-pairing, then \( \mathcal{C} \) is an \( (\mathcal{A}, \mathcal{A}) \)-bisemimodule with the right and the left \( \mathcal{A} \)-actions in (8).

Rational Semimodules

In what follows, we introduce and investigate the category \( \text{Rat}^{\mathcal{C}}(\mathcal{S}_A) \) of \( \mathcal{C} \)-rational right \( \mathcal{A} \)-semimodules associated with a measuring left \( \mathcal{C} \)-pairing \( (\mathcal{A}, \mathcal{C}) \).

3.10. Let \( P = (\mathcal{A}, \mathcal{C}) \) a measuring left \( \mathcal{C} \)-pairing and \( M \) a right \( \mathcal{A} \)-semimodule. Since \( \mathcal{S}_A \) is complete, it has pullbacks. We define \( \text{Rat}^{\mathcal{C}}(M_A) \) as the pullback of the following diagram of (canonical) right \( \mathcal{A} \)-semimodules and \( \mathcal{A} \)-linear maps:

\[
\begin{array}{ccc}
\text{Rat}^{\mathcal{C}}(M_A) & \xrightarrow{\rho_M} & M \otimes_A \mathcal{C} \\
\downarrow & & \downarrow \alpha_M^C \\
M^\mathcal{C} & \xrightarrow{\rho_M} & \text{Hom}_A(\mathcal{A}, M)
\end{array}
\]

Clearly, \( \text{Rat}^{\mathcal{C}}(M_A) := (\rho_M)^{-1}(\mathcal{A}^\mathcal{C}(M \otimes_A \mathcal{C})) \), i.e., \( m \in \text{Rat}^{\mathcal{C}}(M_A) \) iff there exists a uniquely determined element \( \sum m_i \otimes_A c_i \in M \otimes_A \mathcal{C} \) such that \( ma = \sum m_i \langle a, c_i \rangle \) for every \( a \in \mathcal{A} \). We say that \( M_A \) is \( \mathcal{C} \)-rational iff \( \text{Rat}^{\mathcal{C}}(M_A) = M \) and set

\[
\text{Rat}^{\mathcal{C}}(\mathcal{S}_A) := \{ M_A \mid \text{Rat}^{\mathcal{C}}(M_A) = M \}.
\]

Symmetrically, if \( Q = (\mathcal{B}, \mathcal{C}) \) is a measuring right \( \mathcal{C} \)-pairing and \( M \) is a left \( \mathcal{A} \)-semimodule, then we set \( \text{Rat}^{\mathcal{C}}(M) := (\rho_M)^{-1}(\mathcal{B}^\mathcal{C}(\mathcal{C} \otimes_A M)) \); similarly, we say that \( \mathcal{A}M \) is \( \mathcal{C} \)-rational iff \( \text{Rat}^{\mathcal{C}}(\mathcal{A}M) = M \) and set

\[
\text{Rat}^{\mathcal{C}}(\mathcal{M}) := \{ \mathcal{A}M \mid \text{Rat}^{\mathcal{C}}(\mathcal{A}M) = M \}.
\]

3.11. Let \( P = (\mathcal{A}, \mathcal{C}) \) be a measuring left \( \mathcal{C} \)-pairing and \( Q = (\mathcal{B}, \mathcal{D}) \) a measuring right \( \mathcal{C} \)-pairing. For each \( (\mathcal{B}, \mathcal{A}) \)-bisemimodule \( (M, \rho_M^B, \rho_M^A) \), we have

\[
\text{Rat}^{\mathcal{C}}((\text{Rat}^{\mathcal{D}}(\mathcal{A}M))_\mathcal{B}) = \mathcal{B} \text{Rat}^{\mathcal{C}}(\mathcal{B}M) \cap \text{Rat}^{\mathcal{C}}(M) = \mathcal{D} \text{Rat}^{\mathcal{C}}(\text{Rat}^{\mathcal{C}}(M_A))). \tag{9}
\]
Moreover, we set

\[ \mathcal{R} \text{at}^\mathcal{E}(\mathcal{S}_\mathcal{A}) := \{ \mathcal{A}M_{\mathcal{A}} | \mathcal{R} \text{at}^\mathcal{E}(\mathcal{R} \text{at}(\mathcal{A}M)) = M \}. \]

The following technical lemma plays an important role in our investigations.

**Lemma 3.12.** Let \( P = (\mathcal{A}, \mathcal{C}) \) be a measuring left \( \mathcal{A} \)-pairing. For every \( (M, \rho_M) \in \mathcal{S}_{\mathcal{A}}, \) we have the following statements:

1. \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}) \subseteq M \) is an \( \mathcal{A} \)-subsemimodule;
2. \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}) = \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}); \)
3. For every \( L \subseteq \mathcal{A} M \), we have \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{L}_\mathcal{A}) = \mathcal{L} \cap \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}); \)
4. \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A})) = \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}); \)
5. For every \( N \in \mathcal{S}_{\mathcal{A}} \) and \( f \in \text{Hom}_{\mathcal{A}}(M, N), \) we have \( f(\mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A})) \subseteq \mathcal{R} \text{at}^\mathcal{E}(\mathcal{N}_\mathcal{A}). \)

**Proof.**

1. This follows directly from the definition since \( \mathcal{S}_{\mathcal{A}} \) has pullbacks.
2. This follows from Remark 1.2 (note that \( M \otimes_\mathcal{A} \mathcal{C} \hookrightarrow \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \) is subtractive by our definition of left \( \mathcal{A} \)-pairings).
3. Let \( m \in \mathcal{L} \cap \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}) \) with \( \rho_M(m) = \sum m_i \otimes_\mathcal{A} c_i \). For every \( a \in \mathcal{A} \) we have \( \sum m_i \langle a, c_i \rangle = ma \in \mathcal{L}, \) whence \( \sum m_i \otimes_\mathcal{A} c_i \in \mathcal{L} \otimes_\mathcal{A} \mathcal{C} \) by Lemma 3.6, i.e., \( m \in \mathcal{R} \text{at}^\mathcal{E}(\mathcal{L}). \) The reverse inclusion is obvious.
4. This follows directly from (2) and (3).
5. Consider the following commutative diagram:

\[ \begin{array}{ccc}
\mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}_\mathcal{A}) & \mathcal{R} \text{at}^\mathcal{E}(\mathcal{N}_\mathcal{A}) & \mathcal{R} \text{at}^\mathcal{E}(\mathcal{N}_\mathcal{A}) \\
\downarrow f & \downarrow f & \downarrow f \\
M & N & \text{Hom}_{\mathcal{A}}(\mathcal{A}, N).
\end{array} \]

The equality \( \rho_N \circ f \circ i = x_N^\mathcal{E} \circ (f \otimes_\mathcal{A} \mathcal{C}) \circ \rho_M^\mathcal{E} \) and the fact that the inner rectangle is a pullback, by our definition of \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{N}_\mathcal{A}), \) imply the existence of a unique \( \mathcal{A} \)-linear map \( \tilde{f} : \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}) \rightarrow \mathcal{R} \text{at}^\mathcal{E}(\mathcal{N}) \) which completes the diagram commutatively. Indeed, \( \tilde{f} = f_{\mathcal{R} \text{at}^\mathcal{E}(\mathcal{M})} \) and we are done. \( \square \)

**Remarks 3.13.** Let \( P = (\mathcal{A}, \mathcal{C}) \) be a measuring left \( \mathcal{A} \)-pairing.

1. If \( M \) is a \( \mathcal{C} \)-rational right \( \mathcal{A} \)-semimodule and \( L \subseteq \mathcal{M} \), then it follows from Lemma 3.12 (3) that \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{L}) = L \cap \mathcal{R} \text{at}^\mathcal{E}(\mathcal{M}) = L \cap M = L \), i.e., \( \mathcal{L} \) is \( \mathcal{C} \)-rational. So, \( \mathcal{R} \text{at}^\mathcal{E}(\mathcal{S}_{\mathcal{A}}) \) is closed under uniform subobjects.
Proposition 3.14. Let \( P = (\mathcal{A}, \mathcal{C}) \) be a left measuring \( \alpha \)-pairing.

1. We have an embedding

\[
i : \mathcal{S}^\mathcal{C} \longrightarrow \mathcal{S}_\mathcal{A}, \quad (M, \rho^M) \mapsto (M, \mathcal{A}^{\rho} \circ \rho^M).
\]

In particular, \( \text{Hom}^\mathcal{C}(M, N) \subseteq \text{Hom}_\mathcal{A}(M, N) \) for all \( M, N \in \mathcal{S}^\mathcal{C} \).

2. If \( P \) satisfies the \( \alpha \)-condition, then we have a functor

\[
\text{Rat}^\mathcal{C}(-) : \mathcal{S}_\mathcal{A} \longrightarrow \mathcal{S}^\mathcal{C}, \quad (M, \rho^M) \mapsto (M, (\mathcal{A}^{\rho^M})^{-1} \circ \rho^M).
\]

In particular, \( \text{Hom}^\mathcal{C}(M, N) = \text{Hom}_\mathcal{A}(M, N) \) for all \( M, N \in \mathcal{S}^\mathcal{C} \).

Proposition 3.15. If \( (\mathcal{A}, \mathcal{C}) \) is a left measuring \( \alpha \)-pairing. The full subcategory \( \text{Rat}^\mathcal{C}(-) \hookrightarrow \mathcal{S}_\mathcal{A} \) is reflective (i.e., \( (i, \text{Rat}^\mathcal{C}(-)) \) is an adjoint pair of functors).

We are ready now to present our first main result in this section.

Theorem 3.16. Let \( \mathcal{A} \) be an \( \mathcal{A} \)-semiring and \( \mathcal{C} \) an \( \mathcal{A} \)-semicoring.

1. If \( P = (\mathcal{A}, \mathcal{C}) \) is a measuring left \( \alpha \)-pairing, then \( \mathcal{S}^\mathcal{C} \cong \text{Rat}^\mathcal{C}(\mathcal{S}_\mathcal{A}) \).

2. If \( P = (\mathcal{A}, \mathcal{C}) \) is a measuring right \( \alpha \)-pairing, then \( \mathcal{A} \mathcal{S} \cong \text{Rat}^\mathcal{C}(\mathcal{A} \mathcal{S}_\mathcal{A}) \).

3. If \( P = (\mathcal{A}, \mathcal{C}) \) is a measuring left \( \alpha \)-pairing and \( Q = (\mathcal{B}, \mathcal{D}) \) is a measuring right \( \alpha \)-pairing, then \( \mathcal{A} \mathcal{S} \cong \text{Rat}^\mathcal{C}(\mathcal{A} \mathcal{S}_\mathcal{A}) \).

3.17. For every \( \mathcal{A} \)-semiring \( \mathcal{C} \), we have an isomorphism of \( \mathcal{A} \)-semirings \( (\mathcal{C}^*, \cdot, \cdot) \cong \text{End}^\mathcal{C}(\mathcal{C}) \) via \( f \mapsto [c \mapsto \sum f(c) \cdot c_2] \) with inverse \( g \mapsto \mathcal{C} \circ g \) (compare with Proposition 2.24 (1)). Symmetrically, \( (\mathcal{C}^*, \cdot, \cdot) \cong (\mathcal{C})^{\text{op}} \) and \( (\mathcal{C}^*, \cdot, \cdot) \) as \( \mathcal{A} \)-semirings. If \( P = (\mathcal{A}, \mathcal{C}) \) is a measuring left \( \alpha \)-pairing, then we have by Proposition 3.14 (2) \( \mathcal{C}^* \cong \text{End}^\mathcal{C}(\mathcal{C}) = \text{End}(\mathcal{C}_\mathcal{A}) \). On the other hand, if \( P \) is a measuring right \( \alpha \)-pairing, then \( \mathcal{C}^* \cong (\mathcal{C})^{\text{op}} = \text{End}(\mathcal{C})^{\text{op}} \). In particular, if \( \mathcal{C} \) satisfies the left and the right \( \alpha \)-conditions, then we have

\[
\text{End}(\mathcal{C}, \mathcal{C}_{\mathcal{A}^*}) = \mathcal{C} \text{End}(\mathcal{C}) \cong Z(\mathcal{C}^*) = Z(\mathcal{C}^{\mathcal{A}^*}),
\]

where \( Z(\mathcal{C}^*) \) and \( Z(\mathcal{C}^{\mathcal{A}^*}) \) are the centers of \( \mathcal{C}^* \) and \( \mathcal{C}^{\mathcal{A}^*} \), symmetrically.

An important role by studying the category of rational representations related to a left measuring \( \alpha \)-pairing is played by the following finiteness result which holds for the restricted class of completely subtractive semicomodules.
**Lemma 3.18.** Let \( P = (\mathcal{A}, \mathcal{C}) \) be a measuring left \( \mathcal{C} \)-pairing. If \( M \in \text{Rat}^\mathcal{C}(\mathcal{S}_\mathcal{A}) \) is completely subtractive, then there exists for every finite subset \( \{m_1, \ldots, m_k\} \subset M \) some \( N \in \text{Rat}^\mathcal{C}(\mathcal{S}_\mathcal{A}) \), such that \( N \subset M \) and \( N \) is finitely generated.

**Proof.** Let \( \{m_1, \ldots, m_k\} \subset M \). For each \( i = 1, \ldots, n \), we have \( m_i \mathcal{A}_i \leq \mathcal{A} \), whence a \( \mathcal{C} \)-subsemicomodule by Remark 3.13 and Proposition 3.14. Moreover, \( m_i \mathcal{A}_i = m_i \mathcal{A}_i \), and consequently, there exists a subset \( \{(m_{ij}, c_{ij})\}_{j=1}^n \subset m_i \mathcal{A}_i \times \mathcal{C} \) such that \( \rho_{\mathcal{A}_i}(m_i) = \sum_{j=1}^n m_{ij} \otimes_\mathcal{A}_i \mathcal{C} \) for \( i = 1, \ldots, k \). Obviously, \( N := \sum_{i=1}^k m_i \mathcal{A}_i = \sum_{i=1}^k \sum_{j=1}^n m_{ij} \mathcal{A}_i \subset M \) is a \( \mathcal{C} \)-subsemicomodule and contains \( \{m_1, \ldots, m_k\} \). □

An application of Lemma 3.18 and its dual yields the following finiteness result.

**Proposition 3.19.** Let \( \mathcal{C} \) be an \( \mathcal{A} \)-semicoring. If \( \mathcal{C}_\mathcal{A} \) (resp. \( \mathcal{C}_\mathcal{A}, \mathcal{A} \mathcal{C}_\mathcal{A} \)) is completely subtractive, then every finite subset of \( \mathcal{C} \) is contained in a right \( \mathcal{C} \)-coideal (resp. left \( \mathcal{C} \)-coideal), which is finitely generated in \( \mathcal{S}_\mathcal{A} \).

**Lemma 3.20.** Let \( \mathcal{C} \) be an \( \mathcal{A} \)-semicoring.

1. Every right \( \mathcal{C} \)-semicomodule is a subsemicomodule of a \( \mathcal{C} \)-generated right \( \mathcal{C} \)-semicomodule.
2. \( \mathcal{S}^\mathcal{C} \subseteq \sigma[\mathcal{C}, \mathcal{C}] \).

**Proof.** 1. Let \( (M, \rho^M) \) be an arbitrary right \( \mathcal{C} \)-semicomodule. There exists a set \( \Lambda \) and a surjective morphism of right \( \mathcal{A} \)-semimodules \( A^{(\Lambda)} \rightarrow M \rightarrow 0 \) [28, Proposition 17.11]. It follows that we have a surjective morphism of right \( \mathcal{C} \)-semicomodules \( C^{(\Lambda)} \simeq A^{(\Lambda)} \otimes_\mathcal{A} \mathcal{C} \rightarrow M \otimes_\mathcal{A} \mathcal{C} \rightarrow 0 \). So, \( M \otimes_\mathcal{A} \mathcal{C} \) is generated by \( \mathcal{C} \) as an object of \( \mathcal{S}^\mathcal{C} \). Since \( \rho^M : M \rightarrow M \otimes_\mathcal{A} \mathcal{C} \) is a retraction and \( \mathcal{C} \)-colinear, we conclude that \( M \) is a subobject of \( M \otimes_\mathcal{A} \mathcal{C} \) in \( \mathcal{S}^\mathcal{C} \).

2. Since morphisms of right \( \mathcal{C} \)-semicomodules are \( \mathcal{C} \)-linear by Proposition 3.14, the result follows by (1). □

**Lemma 3.21.** Let \( (\mathcal{C}, \Delta, \varepsilon) \) be an \( \mathcal{A} \)-semiringing such that \( \varepsilon_A^\mathcal{C} : M \otimes_\mathcal{A} \mathcal{C} \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{C}, M) \) is subtractive for every \( M \). If \( \mathcal{S}^\mathcal{C} = \sigma[\mathcal{C}, \mathcal{C}] \), then \( \mathcal{C} \mathcal{A} \mathcal{C} \) is a mono-flat \( \varepsilon \)-semimodule.

**Proof.** Assume that \( \mathcal{S}^\mathcal{C} = \sigma[\mathcal{C}, \mathcal{C}] \). Notice that in this case every monomorphism in \( \mathcal{S}^\mathcal{C} \) is injective, whence \( \mathcal{C} \mathcal{A} \mathcal{C} \) is mono-flat by Remark 2.27. Let \( M \) be an arbitrary right \( \mathcal{A} \)-semimodule and consider \( (M \otimes_\mathcal{A} \mathcal{C}, M \otimes_\mathcal{A} \Delta) \in \mathcal{S}^\mathcal{C} = \sigma[\mathcal{C}, \mathcal{C}] \). For every \( L \in \sigma[\mathcal{C}, \mathcal{C}] \), we have a commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}^\mathcal{C}(L, M \otimes_\mathcal{A} \mathcal{C}) & \cong & \text{Hom}_{\mathcal{A}}(L, M \otimes_\mathcal{A} \mathcal{C}) \\
\uparrow & \cong & \downarrow \text{Hom}_{\mathcal{A}}(L, \mathcal{C} \Delta) \\
\text{Hom}_{\mathcal{A}}(L, M) & \quad & \text{Hom}_{\mathcal{C}}(L, \text{Im}(\alpha_M^\mathcal{C})) \\
\uparrow & \cong & \downarrow \\
\text{Hom}_{\mathcal{A}}(L \otimes_\mathcal{C} \mathcal{C}, M) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{C}}(L, \text{Hom}_{\mathcal{A}}(\mathcal{C}, M))
\end{array}
\]
and so \((L, \alpha_M^C)\) is injective. It follows that 
\[ \alpha_M^C : M \otimes_A C \longrightarrow \text{Im}(\alpha_M^C) \]

is a monomorphism in \(\sigma[[\mathcal{C}_v]]\), whence injective.

**Definition 3.22.** We say that an \(\mathcal{A}\)-semicoring \(\mathcal{C}\) is **left uniform** if \(\mathcal{C}\) is uniformly generated, \(\alpha_M^C : M \otimes_A C \longrightarrow \text{Hom}_\mathcal{A}(\mathcal{C}, M)\) is subtractive for every \(M_A\) and every \(\mathcal{C}\)-generated right \(\mathcal{C}\)-semimodules is completely subtractive. Symmetrically, we define **right uniform** \(\mathcal{A}\)-semicorings. A left and right subtractive \(\mathcal{A}\)-semicoring is said to be **uniform**.

We are now ready to present the second main result in this section.

**Theorem 3.23.** Let \((\mathcal{C}, \Delta, \varepsilon)\) be an \(\mathcal{A}\)-semicoring such that \(\alpha_M^C\) is subtractive for every \(M_A\).

1. The following statements are equivalent:
   
   (a) \(\mathcal{S}_\mathcal{C} = \sigma[[\mathcal{C}_v]]\);
   (b) \(\mathcal{A}_\mathcal{C}\) is a mono-flat \(\mathcal{A}\)-semimodule, and \(\mathcal{S}_\mathcal{C}\) is closed under \(\mathcal{C}\)-subsemimodules;
   (c) \(\mathcal{S}_\mathcal{C}\) is a full subcategory of \(\mathcal{S}_{\mathcal{C}_v}\) and is closed under \(\mathcal{C}\)-subsemimodules.

   In this case, we have

   \[ \mathcal{S}_\mathcal{C} \simeq \text{Rat}^\mathcal{C}(\mathcal{S}_{\mathcal{C}_v}) = \sigma [[\mathcal{C}_v]] = \sigma[[\mathcal{C}_v]]. \]  

2. If \(\mathcal{C}\) is a left uniform \(\mathcal{A}\)-semicoring, then the following statements are equivalent:

   (a) \(\mathcal{S}_\mathcal{C} = \sigma[[\mathcal{C}_v]]\);
   (b) \(\mathcal{A}_\mathcal{C}\) is a mono-flat \(\mathcal{A}\)-semimodule;
   (c) \(\mathcal{S}_\mathcal{C}\) is a full subcategory of \(\mathcal{S}_{\mathcal{C}_v}\).

**Proof.**

1. \((1) \Rightarrow (2)\) By Lemma 3.21, \(\mathcal{A}_\mathcal{C}\) is a mono-flat \(\mathcal{A}\)-semimodule. Moreover, \(\sigma[[\mathcal{C}_v]]\) is—by definition—closed under \(\mathcal{C}\)-subsemimodules.

   \((2) \Rightarrow (3)\) By Theorem 3.16 (1), \(\mathcal{S}_\mathcal{C} \simeq \text{Rat}^\mathcal{C}(\mathcal{S}_{\mathcal{C}_v}) \hookrightarrow \mathcal{S}_{\mathcal{C}_v}\) is a full subcategory.

   \((3) \Rightarrow (1)\) Since \(\mathcal{S}_\mathcal{C}\) is cocomplete and closed under homomorphic images, it follows that \(\text{Gen}(\mathcal{C}_v) \subseteq \mathcal{S}_\mathcal{C} \subseteq \sigma[[\mathcal{C}_v]]\) (the last inclusion follows by Lemma 3.20). However, \(\sigma[[\mathcal{C}_v]]\) is—by definition—the smallest subcategory of \(\mathcal{S}_{\mathcal{C}_v}\) which contains \(\text{Gen}(\mathcal{C}_v)\) and is closed under \(\mathcal{C}\)-subsemimodules, whence \(\mathcal{S}_\mathcal{C} = \sigma[[\mathcal{C}_v]]\).

2. We need only to prove \((3) \Rightarrow (1):\) As in (1), we have \(\text{Gen}(\mathcal{C}_v) \subseteq \mathcal{S}_\mathcal{C} \subseteq \sigma[[\mathcal{C}_v]]\), where the last inclusions follows by Lemma 3.20 and our assumptions on the \(\mathcal{A}\)-semicoring which imply that \(M \leq \alpha^C M \otimes_A \mathcal{C}\) for each \(M \in \mathcal{S}_\mathcal{C}\). Notice also that \(\mathcal{S}_\mathcal{C} \simeq \text{Rat}^\mathcal{C}(\mathcal{S}_{\mathcal{C}_v})\) is closed under uniform \(\mathcal{C}\)-subsemimodules by Remark 3.13 (1), whence \(\mathcal{S}_\mathcal{C} = \sigma[[\mathcal{C}_v]]\) since \(\sigma[[\mathcal{C}_v]]\) is the smallest subcategory of \(\mathcal{S}_{\mathcal{C}_v}\) which contains \(\text{Gen}(\mathcal{C}_v)\) and is closed under uniform \(\mathcal{C}\)-subsemimodules. \(\square\)

**Proposition 3.24.** Let \(\mathcal{C}\) be an \(\mathcal{A}\)-semicoring, and consider the functors \(\mathbf{R} := - \otimes_A \mathcal{C} : \mathcal{S}_\mathcal{A} \longrightarrow \mathcal{S}_{\mathcal{C}_v}\) and \(\mathbf{M} := - \otimes_A (\mathcal{C} : \mathcal{S}_\mathcal{A} \longrightarrow \mathcal{S}_A)\). If \(\mathcal{S}_\mathcal{C} = \mathcal{S}_{\mathcal{C}_v}\), then the following statements hold:

1. \(\mathbf{\#}_{\mathcal{C}_v} \simeq \mathbf{F}\) and \(\mathbf{R} \simeq - \otimes_A \mathcal{C}\) ;
2. $R$ has a (left) exact adjoint $L$ such that $LM = L \circ R$;
3. $\mathcal{A}$ is flat;
4. The forgetful functor $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}_A$ is (left) exact;
5. $\mathcal{M} := - \otimes_A \mathcal{C} : \mathcal{S}_A \longrightarrow \mathcal{S}_A$ is (left) exact;
6. If $\mathcal{A} \mathcal{C}$ is uniformly generated and $\mathcal{A}^\Lambda \otimes_A - : \mathcal{A}\mathcal{S} \longrightarrow \text{AbMonoid}$ preserves $i$-uniform morphisms, then the following statements hold:

(a) $\mathcal{A} \mathcal{C}$ is uniformly finitely presented;
(b) $\mathcal{A} \mathcal{C}$ is finitely presented;
(c) $\mathcal{A} \mathcal{C}$ is finitely generated and projective.

Proof. 1. The morphism of $A$-semirings $\eta_{\mathcal{C}} : A \longrightarrow \mathcal{C}$ induces an adjoint pair of functors $(- \otimes_A \mathcal{C}, \#_{\eta_{\mathcal{C}}})$, where $\#_{\eta_{\mathcal{C}}} : \mathcal{S}_C \longrightarrow \mathcal{S}_A$ is the so-called restriction of scalars functor. Indeed, $\#_{\eta_{\mathcal{C}}} \simeq \mathcal{F}$ in our case, and so we have $R \simeq - \otimes_A \mathcal{C}$ by the uniqueness of the left adjoint functor.

2. Notice that $\mathcal{F}$ is (left) exact and is left adjoint to $R \simeq - \otimes_A \mathcal{C} : \mathcal{S}_A \longrightarrow \mathcal{S}_C$.

3. Since $R$ has a left adjoint, it preserves equalizers, whence $LM = L \circ R$ preserves equalizers, and it follows that $\mathcal{A} \mathcal{C}$ is flat.

4. This follows by Proposition 2.28 (1).

5. Notice that both $G := - \otimes_A \mathcal{C} : \mathcal{S}_A \longrightarrow \mathcal{S}_C$ and the forgetful functor $\mathcal{F} : S_C \longrightarrow S_A$ are (left) exact, whence $LM = L \circ G : \mathcal{S}_A \longrightarrow \mathcal{S}_A$ is (left) exact.

6. Since $LM$ is left exact, it preserves products. In particular, $\mathcal{A}^\Lambda \otimes_A \mathcal{C} \simeq \mathcal{C}^\Lambda$ for every index set $\Lambda$, and it follows that $\mathcal{A} \mathcal{C}$ is uniformly finitely presented by Lemma 1.34 (2) (notice that we assumed that $\mathcal{A}^\Lambda \otimes_A -$ preserves $i$-uniform morphisms). Since $LM$ is left exact, $\mathcal{A} \mathcal{C}$ is flat (by definition). Finally, as indicated in Lemma 1.34 (5), finitely presented flat semimodules are projective. \[\] Theorem 3.25. Let $\mathcal{C}$ be a left uniform $A$-semicoring and assume that $\mathcal{A}^\Lambda \otimes_A -$ preserves $i$-uniform morphisms. The following are equivalent:

1. $S_C = S_{\eta_{\mathcal{C}}}$;
2. $\#_{\eta_{\mathcal{C}}} \simeq \mathcal{F}$ and $R \simeq - \otimes_A \mathcal{C}$;
3. $R$ has a (left) exact adjoint $L$ such that $LM = L \circ R : \mathcal{S}_A \longrightarrow \mathcal{S}_A$;
4. $\mathcal{A} \mathcal{C}$ is flat and (uniformly) finitely presented;
5. $\mathcal{A} \mathcal{C}$ is finitely generated and (finitely) projective;
6. $\mathcal{A} \mathcal{C}$ is finitely generated and $\mathcal{A} \mathcal{C}$ is an $\alpha$-semimodule.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are clear by following the proof of Proposition 3.24, which was given in this order.

$(5) \Rightarrow (6)$ Since $\mathcal{A} \mathcal{C}$ is uniformly finitely generated and $\mathcal{C}^\alpha \otimes_A -$ preserves cokernels, whence normal quotients, it follows that $\mathcal{C}^\alpha \mathcal{C}$ is uniformly finitely
generated. Moreover, since $A \mathcal{C}$ is finitely projective, we have $\ker(z_M^\alpha) = 0$ for each $M_A$, whence $z_M^\alpha$ is injective (notice that $z_M^\alpha$ is assumed to be subtractive).

(6) $\Rightarrow$ (1) Notice that $\mathcal{C}$ is a faithful and finitely generated as a left $\mathcal{C}^*$-semimodule. Since $\mathcal{C}^* \simeq \text{End}(\mathcal{C}) = \text{End}(\mathcal{C}_{\alpha\mathcal{C}})$, it follows that

$$S^\mathcal{C} = \sigma[\mathcal{C}_{\alpha\mathcal{C}}] \text{ is left uniform} \Rightarrow \sigma[\mathcal{C}_{\alpha\mathcal{C}}] = S_{\alpha\mathcal{C}}.$$  \hfill □

**Theorem 3.26.** Let $\mathcal{C}$ be an $A$-semicoring such that $A \mathcal{C}$ is uniformly generated, $z_M^\alpha$ is subtractive for every $M_A$, and assume that $\alpha A \otimes_A -$ preserves $i$-uniform morphisms. We have $S^\mathcal{C} = S_{\alpha\mathcal{C}}$ if and only if $A \mathcal{C}$ is finitely generated and projective and $S^\mathcal{C}$ is closed under $^*\mathcal{C}$-subsemimodules.

**Proof.** $(\Rightarrow)$ Follows by Proposition 3.24 and Lemma 1.34.

$(\Leftarrow)$ As in the proof of Theorem 3.25, we have $\mathcal{C}_{\alpha\mathcal{C}}$ is finitely generated and $A \mathcal{C}$ is an $\alpha$-semimodule, whence

$$S^\mathcal{C} = \sigma[\mathcal{C}_{\alpha\mathcal{C}}] \text{ is left uniform} \Rightarrow \sigma[\mathcal{C}_{\alpha\mathcal{C}}] = S_{\alpha\mathcal{C}}. \hfill □$$

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