In general concurrence means the combined action of agents or causes, often in the form of simultaneous occurrence. It is also a form of correlation. Examples are cofiring of neurons, coexpression of genes, events happening at the same time, etc. In this paper we want to extend the concept of concurrency to a higher version fitting into the general scheme of higher structures (hyperstructures) as described and studied in [1–14]. For example neurons may fire simultaneously in groups, even groups of groups, or it could be brain domains active in various tasks being performed and measured by fMRI techniques.

We are inspired by the work of Ellis and Klein [17], and the framework for neural codes by Curto and Itskov [16].

The idea of introducing higher concurrences in general applies to all kinds of events. We will here illustrate the idea in the framework of neural codes as in [16] and introduce higher neural codes. We will show how the use of topology extends to this case as well.

By a neural code word of length $n$ we mean a sequence

$$C = (\hat{c}_1, \ldots, \hat{c}_n), \quad \hat{c}_i \in \{0, 1\}.$$ 

A code is a finite collection of such words

$$\mathcal{C} = \{c_i\}.$$ 

We follow the notation of [16].

Hence $\mathcal{C} \subseteq 2^n = \mathcal{P}[n]$ or $\mathcal{C} \in \mathcal{P}^2[n]$.

We define the support as follows:

$$\text{supp } C = \{i \in [n] \mid \hat{c}_i = 1\} \subseteq [n]$$

$$\text{supp } \mathcal{C} = \{\text{supp}(C) \mid C \in \mathcal{C}\} \subseteq 2^n = \mathcal{P}[n]$$

$$\text{supp } \mathcal{C} \in \mathcal{P}^2[n].$$

Such words come from neural data (spike trains) as follows:

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We are naturally interested in cofiring patterns, hence we pay attention to $\text{supp}\ C$.

We are looking for (topological) structure in the coding data. Let us form:

$$\Delta(C) = \{ \sigma \subseteq [n] | \sigma \subseteq \text{supp}(C) \text{ for some } C \}$$

— this is the smallest simplicial complex that contains $\text{supp}\ C$.

The question we could like to address is: *Is there some higher order (topological) information in the code data, how to extract it?*

All thinking about higher order structures in set theory, type theory, logic, higher categories, etc. is in some way often a sophisticated reflection of the basis hierarchy:

$$X, \mathcal{P}(X), \mathcal{P}^2(X), \ldots, \mathcal{P}^n(X), \ldots$$

Hyperstructures, see [1–14] for references, represent a highly refined version of this. Let us pursue this in our coding context combining with the ideas from hyperstructures, hence constructing an interesting example.

Put $C_1 = \text{supp}\ C \in \mathcal{P}^2[n]$, an element represents a cofiring or a concurency

$$(101101110\ldots0) \mapsto (111111)$$

pattern. Then we may look for patterns of patterns.
To be more precise a pattern exists when $\text{supp} C \in \mathcal{C}_1$ is realized at a certain time.

If several patterns are realized or occur at another time ($t_3$) we say that they form a second order pattern. In the language of hyperstructures

*Observer 1:* individual neurons firing (or other activity)

*Bond 1:* cofiring of neurons (coactivity)

*Observer 2:* firing of a pattern (activity of patterns)

*Bond 2:* cofiring of patterns (coactivity of patterns).

This clearly continues, and should be manageable combinatorially due to a reasonably limited number of cofiring patterns.

Hence we put

$$\mathcal{C}_2 = \text{supp} \mathcal{C}_1 \in \mathcal{P}^3[n]$$

representing cofiring patterns

$$\cdot$$

$$\mathcal{C}_3 = \text{supp} \mathcal{C}_2 \in \mathcal{P}^4[n]$$

representing
Iteratively:
\[ C_k = \text{supp } C_{k-1} \in \mathcal{P}^{k+1}[u]. \]

If we include \( k \) levels of structure then
\[ C = \{ C_1, C_2, \ldots, C_k \} \]
form a hyperstructure with “boundary” maps
\[ \partial_i : C_{i+1} \to \mathcal{P}(C_i), \quad \partial_i(c_{i+1}) = \{ c_i^j \} \quad [c_{i+1} = \{ c_i^1, \ldots, c_i^{k_i} \}] \]
dissolving the bond (as for geometric cobordisms!). Let us call the hyperstructure \( \mathcal{H}(C) \).

The hyperstructure is a useful way to organize the code data. How to extract topological information out of this?

We may associate the clique topology complex to each level
\[ \Delta(C) : \Delta(C_k) \xleftarrow{\delta_{k-1}} \Delta(C_{k-1}) \xleftarrow{\delta_{k-2}} \cdots \xleftarrow{\delta_1} \Delta(C_1) \]
where the \( \delta_i \)'s are induced from the \( \partial_i \)'s. We assume these are well-defined.

Clearly this sequence gives more information than just \( \Delta(C_1) \). For various filtrations (frequency, ...) we may apply persistent homology — \( PH \) — to the sequence and get:
\[ PH(\Delta(C_k)) \xleftarrow{\delta_{k-1}} PH(\Delta(C_{k-1})) \xleftarrow{\delta_{k-2}} \cdots \xleftarrow{\delta_1} PH(\Delta(C_1)) \]
giving a sequence of barcodes (persistent diagrams) and maps as an invariant.

But it does not capture the fact that many cofiring patterns are not simplicial, in the sense that if a collection cofires, not necessarily all subcollections will cofire (at some time). This is organizationally captured in the hyperstructure.

How can this get reflected topologically?

We will associate a space — a simplicial complex, the nerve — to the hyperstructure and then study its shape via (persistent) homology. Let us return to the hyperstructure of coding (firing) patterns:
\[ \mathcal{H}(C) = \{ C_1, C_2, \ldots, C_k \}. \]
We will construct “the nerve of \( H(C) \)”. What should the simplexes be?

Let us use the construction of the classifying space of a category as a model, where the \( p \)-simplexes are compositions of morphisms of length \( p \). We have to define composition of bonds at any level. We define compositions of bonds by “gluing” similar to morphisms in categories and higher categories.

\[
\begin{array}{c}
\bullet \\
\quad f \\
\quad \rightarrow \\
\bullet \\
\downarrow \\
\bullet \\
\quad g \\
\quad \rightarrow \\
\bullet
\end{array}
\]

\( g \circ f \) is defined if target \( f = \text{source} \, g \). For \( n \)-morphisms we may have that at level \( p \):

\[
\text{target}^n_p(f) = \text{source}^n_p(g)
\]

“gluing” at level \( p \), giving a composition

\[
f_n \prod^p g
\]

For a hyperstructure we search for compatibility conditions in order to glue. If we have two \( n \)-bonds:

\[
a_n \quad \text{and} \quad b_n
\]

and

\[
\partial_p \circ \cdots \circ \partial_{n-1}(a_n) \cap \partial_p \circ \cdots \partial_{n-1}(b_n) \neq \emptyset
\]

(not necessarily equal) then we glue along this part and defines a composition

\[
a_n \prod^p b_n.
\]

See [8].

Furthermore:
Fix level $i$ in the hyperstructure $\mathcal{H}(\mathcal{C})$ and let $b_1, \ldots, b_k$ be bonds at level $i$, but gluable at level $j$, $j \leq i$. Then we form compositions

$$b_1 \square^i_j b_2 \square^i_j \cdots \square^i_j b_k$$

which will form the $k$-simplexes at level $(i, j)$, let us call this set $\Delta^{i,j}_k$.

We do this for all $i, j, k$ and form the smallest simplicial complex having $\{\Delta^{i,j}_k\}$ as simplexes and we call this the nerve of $\mathcal{H}(\mathcal{C})$:

$$N\mathcal{H}(\mathcal{C}).$$

This construction holds for any hyperstructure $\mathcal{H}$, not just $\mathcal{H}(\mathcal{C})$ and extends the notion of classifying space or geometric realization of $n$-categories.

In connection with the neural data it should be possible to compute their homology and for suitable filtrations, their persistent homology. Our suggestion is that when we have data giving rise to a code, we should look at the associated higher code or hyperstructure $\mathcal{H}(\mathcal{C})$. For topological properties one should then study the nerve $N\mathcal{H}(\mathcal{C})$, not just $\Delta(\mathcal{C})$. This ought to capture the structure of the data in a better way.

In a way forming the simplexes

$$\{b_1 \square^i_j b_2 \square^i_j \cdots \square^i_j b_k\}$$

may amount to the same (or similar) process of “geometric realization” of the sequence

$$\Delta(\mathcal{C}_k) \xleftarrow{\delta^i_{k-1}} \Delta(\mathcal{C}_{k-1}) \xleftarrow{\delta^i_{k-2}} \cdots \xleftarrow{\delta^i_1} \Delta(\mathcal{C}_1)$$

making levelwise realization compatible with the $\delta^*_i$ maps.

Also in comparing two data sets this may be a useful approach. Suppose that we have two data sets or codes, then we apply the described procedure and produce two hyperstructures:

$$\mathcal{H} = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$$

$$\mathcal{H}' = \{\mathcal{C}'_1, \ldots, \mathcal{C}'_k\}$$
with inbuilt boundary maps. Then we may just compare bonds level-wise by maps, are they for example one to one or what?

In general one may ask what kind of equivalence between $\mathcal{H}$ and $\mathcal{H}'$ is needed for $N(\mathcal{H})$ and $N(\mathcal{H}')$ to have the same shape (or homotopy type)?

It is tempting to think that neurons organized in a hierarchy of Hebb assemblies may give rise to neural coding hyperstructures. Such assemblies may possibly be detected by the inference model for neural data introduced in [21], actually detecting tiers.

Hierarchies of Hebb assemblies, see [18, 20] may metaphorically compare to gene assemblies as in regulatory models of the genome by Jacob and Monod [19] and Britten and Davidson [15].

Hyperstructures should play a role both in the analysis of neural data and genomic data. Let us give an illustration of how $N(\mathcal{H}(t))$ may contain more information than $\Delta(t)$.

| time | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ |
|------|------|------|------|------|------|------|
| neurons | 1 | $\Delta_1(\mathcal{C})$ | 2 | $\subseteq \Delta_2(\mathcal{C}) (N_{\mathcal{H}}(t))$ |

Here we get at level 2 (not existing at level 1):

In addition come lower level gluing simplexes. For one level, see [17].

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