Abstract

We prove that the number $\gamma_N$ of the zeros of a two-parameter simple random walk in its first $N \times N$ time steps is almost surely equal to $N^{1+o(1)}$ as $N \to \infty$. This is in contrast with our earlier joint effort with Z. Shi [4]; that work shows that the number of zero crossings in the first $N \times N$ time steps is $N^{(3/2)+o(1)}$ as $N \to \infty$. We prove also that the number of zeros on the diagonal in the first $N$ time steps is $((2\pi)^{-1/2} + o(1)) \log N$ almost surely.

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1 Introduction

Let $\{X_{i,j}\}_{i,j=1}^\infty$ denote i.i.d. random variables, taking the values $\pm 1$ with respective probabilities $1/2$, and consider the two-parameter random walk $S := \{S(n, m)\}_{n, m \geq 1}$ defined by

$$S(n, m) := \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \quad \text{for } n, m \geq 1. \tag{1.1}$$

A lattice point $(i, j)$ is said to be a vertical crossing for the random walk $S$ if $S(i, j)S(i, j + 1) \leq 0$. Let $Z(N)$ denote the total number of vertical crossings in the box $[1, N]^2 \cap \mathbb{Z}^2$. A few years ago, together with Zhan Shi [4] we proved that with probability one,

$$Z(N) = N^{(3/2)+o(1)} \quad \text{as } N \to \infty. \tag{1.2}$$

We used this result to describe an efficient method for plotting the zero set of the two-parameter walk $S$; this was in turn motivated by our desire to find good simulations of the level sets of the Brownian sheet.

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The goal of the present paper is to describe the rather different asymptotic behavior of two other “contour-plotting algorithms.” Namely, we consider the total number of zeros in $[1, N]^2 \cap \mathbb{Z}^2$:

$$\gamma_N := \sum_{(i,j) \in [0,N]^2} 1_{\{ S(i,j) = 0 \}}, \quad (1.3)$$

together with the total number of on-diagonal zeros in $[1, 2N]^2 \cap \mathbb{Z}^2$:

$$\delta_N := \sum_{i=1}^N 1_{\{ S(2i,2i) = 0 \}}. \quad (1.4)$$

The main results are listed next.

**Theorem 1.1.** With probability one,

$$\gamma_N = N^{1+o(1)} \quad \text{as} \quad N \to \infty. \quad (1.5)$$

**Theorem 1.2.** With probability one,

$$\lim_{N \to \infty} \frac{\delta_N \log N}{N} = \frac{1}{(2\pi)^{1/2}}, \quad (1.6)$$

where “log” denotes the natural logarithm.

The theorems are proved in reverse order of difficulty, and in successive sections.

## 2 Proof of Theorem 1.2

Throughout, we need ordinary random-walk estimates. Therefore, we use the following notation: Let $\{\xi_i\}_{i=1}^\infty$ be i.i.d. random variables, taking the values $\pm 1$ with respective probabilities $1/2$, and consider the one-parameter random walk $W := \{W_n\}_{n=1}^\infty$ defined by

$$W_n := \xi_1 + \cdots + \xi_n. \quad (2.1)$$

We begin by proving a simpler result.

**Lemma 2.1.** As $N \to \infty$\footnote{We always write $a_N = O(1)$ to mean that $\sup_N |a_N| < \infty$. Note the absolute values.}

$$\mathbb{E} \delta_N = \frac{1}{(2\pi)^{1/2}} \log N + O(1). \quad (2.2)$$

Before we prove this, we recall some facts about simple random walks.

We are interested in the function,

$$p(n) := P\{ W_{2n} = 0 \}. \quad (2.3)$$
First of all, we have the following, which is a consequence of the inversion formula for Fourier transforms:

\[ p(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos(t)|^{2n} \, dt. \]  \hspace{1cm} (2.4)

Therefore, according to Wallis’ formula \[1\text{, eq. 6.1.49, p. 258}\], as \( n \to \infty \),

\[ p(n) = \frac{1}{(\pi n)^{1/2}} \left[ 1 - \frac{1}{8n} + \frac{1}{128n^2} - \cdots \right], \]  \hspace{1cm} (2.5)

in the sense of formal power series.\(^2\)

Next, we present a “difference estimate.”

**Lemma 2.2.** For all integers \( n \geq 1 \),

\[ 0 \leq p(n) - p(n + 1) = O(n^{-3/2}). \]  \hspace{1cm} (2.6)

**Proof.** Because \( 0 \leq \cos^2 t \leq 1 \), (2.4) implies that \( p(n) \geq p(n + 1) \). The remainder follows from (2.5) and a few lines of computations. \( \square \)

**Proof of Lemma 2.2.** Because \( S(2i, 2i) \) has the same distribution as \( W_{4i^2} \), it follows that \( E\delta_N = \sum_{1 \leq i \leq N} p(2i^2) \). The result follows readily from this and (2.5). \( \square \)

Next, we bound the variance of \( \delta_N \).

**Proposition 2.3.** As \( N \to \infty \),

\[ \text{Var} \delta_N = \frac{1}{(2\pi)^{1/2}} \log N + O(1). \]  \hspace{1cm} (2.7)

**Proof.** Evidently,

\[ E[\delta_N^2] = E\delta_N + 2 \sum_{1 \leq i < j \leq N} P(i, j), \]  \hspace{1cm} (2.8)

where

\[ P(i, j) := P\{S(2i, 2i) = 0, S(2j, 2j) = 0\}, \]  \hspace{1cm} (2.9)

for \( 1 \leq i < j < \infty \). But \( S(2j, 2j) = S(2i, 2i) + W_{i,j} \), where \( W_{i,j} \) is a sum of \( 4(j^2 - i^2) \)-many i.i.d. Rademacher variables, and is independent of \( S(2i, 2i) \). Therefore,

\[ P(i, j) = p(2i^2)p(2j^2 - i^2)). \]  \hspace{1cm} (2.10)

According to Lemma 2.2, \( P(i, j) \geq p(2i^2)p(2j^2) \). Therefore, by (2.8),

\[ E[\delta_N^2] \geq E\delta_N + 2 \sum_{1 \leq i < j \leq N} p(2i^2)p(2j^2) \]

\[ = E\delta_N + (E\delta_N)^2 - \sum_{1 \leq i \leq N} p^2(2i^2). \]  \hspace{1cm} (2.11)

\(^2\)Suppose \( a_1, a_2, \ldots \) are non-negative series which \( a_1(n) \leq a_2(n) \leq \cdots \). Then please recall that “\( p(n) = a_1(n) - a_2(n) + a_3(n) - \cdots \)” is short-hand for “\( a_1(n) - a_2(n) \leq p(n) \leq a_1(n) - a_2(n) + a_3(n) \)” etc.

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Thanks to (2.5), the final sum is $O(1)$. Therefore, Lemma 2.1 implies that

$$\text{Var} \delta_N \geq \frac{1}{(2\pi)^{1/2}} \log N + O(1).$$

(2.12)

In order to bound the converse bound, we use Lemma 2.2 to find that

$$p \left(2(j^2 - i^2)\right) - p(2j^2) = \sum_{2(j^2 - i^2) \leq \ell < 2j^2} [p(\ell) - p(\ell + 1)]$$

$$\leq c \sum_{2(j^2 - i^2) \leq \ell < 2j^2} \frac{1}{\ell^{1/2}},$$

(2.13)

where $c$ is positive and finite, and does not depend on $(i,j)$. From this we can deduce that

$$p \left(2(j^2 - i^2)\right) - p(2j^2) \leq c' \frac{i^2}{(j^2 - i^2)^{3/2}}$$

$$\leq c'' \frac{i^2}{j^3(j - i)^{3/2}},$$

(2.14)

where $c'$ and $c''$ are positive and finite, and do not depend on $(i,j)$. Thus, we can find a positive and finite constant $c'''$ such that for all $n \geq 1$,

$$\sum_{1 \leq i < j \leq N} p(2i^2) [p \left(2(j^2 - i^2)\right) - p(2j^2)] \leq c''' \sum_{1 \leq i < j \leq N} \frac{i}{j^3(j - i)^{3/2}}.$$  

(2.15)

We split the sum according to whether or not $j < 2i$. First, we note that

$$\sum_{1 \leq i < j < \infty} \frac{i}{j^3(j - i)^{3/2}} \leq \sum_{1 \leq i < \infty} \frac{1}{i} \sum_{i < j < \infty} \frac{1}{(j - i)^{3/2}} < \infty.$$  

(2.16)

Next, we note that

$$\sum_{1 \leq i < 2i < j < \infty} \frac{i}{j^3(j - i)^{3/2}} \leq 2^{3/2} \sum_{1 \leq i < 2i} \sum_{j < \infty} \frac{i}{j^{9/2}} < \infty.$$  

(2.17)

This and (2.15) together prove that

$$\sum_{1 \leq i < j \leq N} P(i,j) \leq \sum_{1 \leq i < j \leq N} p(2i^2)p(2j^2) + O(1)$$

$$= (E\delta_N)^2 - \sum_{1 \leq i \leq N} p^2(2i^2) + O(1)$$

(2.18)

$$= (E\delta_N)^2 + O(1).$$

See (2.3). This and (2.8) together imply that the variance of $\delta_N$ is at most $O(1) + E\delta_N$. Apply Lemma (2.1) and (2.12), in conjunction, to finish the proof. □
Proof of Theorem 1.2. Thanks to Proposition 2.3 and the Chebyshev inequality, we can write the following: For all \( \epsilon > 0 \),

\[
P \{ |\delta_N - E\delta_N| \geq \epsilon \log N \} = O \left( \frac{1}{\log N} \right).
\]  

Set \( n_k := \exp(q^k) \) for an arbitrary but fixed \( q > 1 \), and apply the Borel–Cantelli lemma to deduce that

\[
\lim_{k \to \infty} \frac{\delta_{nk}}{\log n_k} = \frac{1}{(2\pi)^{1/2}}.
\]  

Let \( m \to \infty \) and find \( k = k(m) \) such that \( n_k \leq m < n_{k+1} \). Evidently, \( \delta_{nk} \leq \delta_m \leq \delta_{nk+1} \). Also, \( \log n_k \leq \log n_{k+1} = (q + o(1)) \log n_k \). Therefore, a.s.,

\[
\lim sup_{m \to \infty} \frac{\delta_m}{\log m} \leq \lim sup_{k \to \infty} \frac{\delta_{nk+1}}{\log n_k} = \frac{q}{(2\pi)^{1/2}}.
\]  

Similarly, a.s.,

\[
\lim inf_{m \to \infty} \frac{\delta_m}{\log m} \geq \lim inf_{k \to \infty} \frac{\delta_{nk}}{\log n_k+1} \geq \frac{1}{q(2\pi)^{1/2}}.
\]  

Let \( q \downarrow 1 \) to finish.

3 Proof of Theorem 1.1

We begin by proving the easier half of Theorem 1.1; namely, we first prove that with probability one, \( \gamma_N \leq N^{1+o(1)} \).

Proof of Theorem 1.1: First Half. We apply (2.5) to deduce that as \( N \to \infty \),

\[
E\gamma_N = \sum_{i=1}^{N} \sum_{j=1}^{N} P\{ S(i,j) = 0 \} = \sum_{i=1}^{N} \sum_{j=1}^{N} p(ij/2) \leq \text{const} \cdot \left( \sum_{i=1}^{N} i^{-1/2} \right)^2,
\]  

and this is \( \leq \text{const} \cdot N \). By Markov’s inequality,

\[
P\{ \gamma_N \geq N^{1+\epsilon} \} \leq \text{const} \cdot N^{-\epsilon},
\]  

where the implied constant is independent of \( \epsilon > 0 \) and \( N \geq 1 \). Replace \( N \) by \( 2^k \) and apply the Borel–Cantelli lemma to deduce that with probability one, \( \gamma_{2^k} < 2^{(1+\epsilon)} \) for all \( k \) sufficiently large. If \( 2^k \leq N \leq 2^{k+1} \) is sufficiently large [how large might be random], then a.s.,

\[
\gamma_N \leq \gamma_{2^k+1} < 2^{(k+1)(1+\epsilon)} \leq 2^{k(1+2\epsilon)} \leq N^{1+2\epsilon}.
\]  

Since \( \epsilon > 0 \) is arbitrary, this proves half of the theorem.
The proof of the converse half is more delicate, and requires some preliminary estimates.

For all $i \geq 1$ define

$$\rho_1(i) := \min \{ j \geq 1 : S(i, j)S(i, j + 1) \leq 0 \},$$
$$\rho_2(i) := \min \{ j \geq \rho_1(i) : S(i, j)S(i, j + 1) \leq 0 \},$$
$$\vdots$$
$$\rho_\ell(i) := \min \{ j \geq \rho_{\ell-1}(i) : S(i, j)S(i, j + 1) \leq 0 \}, \ldots .$$

These are the successive times of “vertical upcrossings over time-level $i$.” For all integers $i \geq 1$ and all real numbers $t \geq 1$, let us consider

$$f(i; t) := \max \{ k \geq 1 : \rho_k(i) \leq t \} .$$

Then, it should be clear that

$$\sum_{i=1}^{N} f(i; N) = Z(N) .$$

where $Z(N)$ denotes the total number of vertical upcrossings in $[1, N]^2$; see the introduction.

**Lemma 3.1.** With probability one, if $N$ is large enough, then

$$\max_{1 \leq i \leq N} f(i; N) \leq N^{1/2+o(1)} .$$

**Remark 3.2.** It is possible to improve the “$\leq$” to an equality. In fact, one can prove that $f(1; N) = N^{1/2+o(1)}$ a.s., using the results of Borodin [2]; for further related results see [3]. We will not prove this more general assertion, as we shall not need it in the sequel.

**Proof.** Choose and fix two integers $N \geq 1$ and $i \in \{1, \ldots, N\}$.

We plan to apply estimates from the proof of Proposition 4.2 of [4], whose $\zeta_i(0, N)$ is the present $f(i; N)$.

After Komlós, Major, and Tusnády [5], we can—after a possible enlargement of the underlying probability space—find three finite and positive constants $c_1, c_2, c_3$ and construct a standard Brownian motion $w := \{w(t)\}_{t \geq 0}$ such that for all $z > 0$,

$$\max_{1 \leq i \leq N} \mathbb{P} \{|S(i, j) - w(ij)| > c_1 \log(ij) + z\} \leq c_2 e^{-c_3 z} .$$

The Brownian motion $w$ depends on the fixed constant $i$, but we are interested only in its law, which is of course independent of $i$. In addition, the constants $c_1, c_2, c_3$ are universal.

Fix $\epsilon \in (0, 1/2)$ and $\delta \in (0, \epsilon/2)$, and consider the event

$$\mathcal{E}_N := \left\{ \max_{1 \leq i \leq N} |S(i, j) - w(ij)| \leq N^{\delta} \right\} .$$

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[We are suppressing the dependence of $\mathcal{E}_N$ on $i$, as $i$ is fixed.] By (3.8), we can find a constant $c_4$—independent of $N$ and $i$—such that

$$P(\mathcal{E}_N) \geq 1 - c_4 N^{-4}. \tag{3.10}$$

Let $S(i, 0) := 0$ for all $i$. Then, almost surely on $\mathcal{E}_N$, we have

$$\sum_{j=0}^{N-1} \mathbf{1}_{\{S(i,j) \geq 0, S(i,j+1) \leq 0\}} \leq \sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij) \geq -N^\delta, w(i(j+1)) \leq N^\delta\}} \tag{3.11}$$

$$\leq \sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij) \geq 0, w(i(j+1)) \leq 0\}} + 2 \sup_{a \in \mathbb{R}} \sum_{j=0}^{N} \mathbf{1}_{\{a \leq w(ij) \leq a + N^\delta\}}.$$

This is equation (6.6) of [4]. Now we use eq. (1.13) of Borodin [2] to couple $w$ with another Brownian motion $B := \{B(t)\}_{t \geq 0}$ such that

$$P \left\{ \left| \sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij) \geq 0, w(i(j+1)) \leq 0\}} - \mu(N/i)^{1/2} L_1^0(B) \right| \geq c_5 N^{1/4} \log N \right\} \leq (c_5 N)^{-4}, \tag{3.12}$$

where $\mu := \mathbb{E}(\{B^+(1)\})$, $c_5 \in (0, 1)$ does not depend on $(i, N)$, and $L_1^0(B) := \lim_{\eta \to 0} (2\eta)^{-1} \int_0^1 \mathbf{1}_{\{|B(s)| \leq \eta\}} ds$ denotes the local time of $B$ at time 1 at space value 0. See also the derivation of [4] eq. (6.10)] for some detailed technical comments.

It is well known that $P\{L_1^0(B) \geq \lambda\} \leq 2e^{-\lambda^2/2}$ for all $\lambda > 0$ [7]. In particular, $P\{L_1^0(B) \geq N^\delta\} \leq 2 \exp(-N^\delta/2)$. Since $\delta < 1/4$, this, (3.10), and (3.12) together imply that

$$P \left\{ \sum_{j=0}^{N-1} \mathbf{1}_{\{w(ij) \geq 0, w(i(j+1)) \leq 0\}} \geq \frac{N^{(1/2)+\delta}}{\sqrt{i^{1/2}}} \right\} \leq c_6 N^{-4}, \tag{3.13}$$

where $c_6 \in (1, \infty)$ is independent of $N$ and $i$. On the other hand, eq. (6.20) of [4] tells us that we can find a constant $c_7 \in (1, \infty)$—independent of $N$ and $i$—such that

$$P \left\{ \sum_{a \in \mathbb{R}} \sum_{j=0}^{N} \mathbf{1}_{\{a \leq w(ij) \leq a + N^\delta\}} \geq \frac{N^{(1/2)+\delta}}{\sqrt{i^{1/2}}} \right\} \leq c_7 N^{-4} + 2 \exp\left(-N^{2\delta}\right). \tag{3.14}$$

Since $i \geq 1$ and $\delta < 1/4 < 1/2$, this implies that

$$P \left\{ \sum_{a \in \mathbb{R}} \sum_{j=0}^{N} \mathbf{1}_{\{a \leq w(ij) \leq a + N^\delta\}} \geq N^{(1/2)+\delta} \right\} \leq c_7 N^{-4} + 2 \exp\left(-N^{2\delta}\right). \tag{3.15}$$

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Now we combine (3.11), (3.13), and (3.15) to deduce the following:

\[
\sum_{N=1}^{\infty} P \left( \max_{1 \leq i \leq N} \sum_{j=0}^{N-1} 1 \{ S(i,j) \geq 0, S(i,j+1) \leq 0 \} \geq 2N^{(1/2)+\delta} ; \mathcal{E}_N \right) 
\leq \sum_{N=1}^{\infty} \sum_{i=1}^{N} P \left( \sum_{j=0}^{N-1} 1 \{ S(i,j) \geq 0, S(i,j+1) \leq 0 \} \geq 2N^{(1/2)+\delta} ; \mathcal{E}_N \right) 
\leq \sum_{N=1}^{\infty} \left( c_6 N^{-3} + c_7 N^{-3} + 2N \exp(-N^{2\delta}) \right) 
< \infty.
\]

(3.16)

This and (3.10), in turn, together imply that

\[
\sum_{N=1}^{\infty} P \left\{ \max_{1 \leq i \leq N} \sum_{j=0}^{N-1} 1 \{ S(i,j) \geq 0, S(i,j+1) \leq 0 \} \geq 2N^{(1/2)+\delta} \right\} < \infty.
\]

(3.17)

Since \(-S\) is another simple walk on \(\mathbb{Z}\), it follows that

\[
\sum_{N=1}^{\infty} P \left\{ \max_{1 \leq i \leq N} f(i;N) \geq 2N^{(1/2)+\delta} \right\} < \infty.
\]

(3.18)

The lemma follows the Borel–Cantelli lemma, because \(\epsilon\), and hence \(\delta\), can be made arbitrarily small. \(\blacksquare\)
Consider the following random set of times:
\[
H_N(\alpha, \beta) := \{ 1 \leq i \leq N^{1-\alpha} : f(i; N) > N^{(1/2)-\beta} \}.
\] (3.19)

**Lemma 3.3.** Choose and fix three positive constants \(\alpha, \beta, \epsilon\) such that \(\beta > (\alpha/2) + \epsilon\). Then, the following happens a.s.: For all but a finite number of values of \(N\),
\[
|H_N(\alpha, \beta)| \geq N^{1-(3\alpha/2)-2\epsilon},
\] (3.20)
where \(|\cdots|\) denotes cardinality.

**Proof.** We apply (1.2), via (3.6) and Lemma 3.1, to see that with probability one, the following holds for all but a finite number of values of \(N\):
\[
N^{3(1-\alpha)/2-\epsilon} = \sum_{1 \leq i \leq N^{1-\alpha}} f(i; N^{1-\alpha}) \\
\leq \sum_{1 \leq i \leq N^{1-\alpha}} f(i; N) \\
= \sum_{i \in H_N(\alpha, \beta)} f(i; N) + \sum_{1 \leq i \leq N^{1-\alpha} : f(i, N) \leq N^{(1/2)-\beta}} f(i; N) \\
\leq |H_N(\alpha, \beta)| \cdot N^{(1/2)+\epsilon} + N^{1-\alpha+(1/2)-\beta}.
\] (3.21)

The lemma follows because \(\beta > (\alpha/2) + \epsilon\). \(\square\)

Define
\[
U(i; \ell) := 1_{\{S(i, \rho(\ell))S(i, 1+\rho(\ell)) = 0\}}.
\] (3.22)

The following is a key estimate in our proof of Theorem 1.1.

**Proposition 3.4.** There exists a finite constant \(c > 0\) such that for all integers \(i, M \geq 1\),
\[
P \left\{ \sum_{\ell=1}^{M} U(i; \ell) \leq \frac{cM}{i^{1/2}} \right\} \leq \exp \left( -\frac{cM}{4i^{1/2}} \right).
\] (3.23)

Our proof of Proposition 3.4 begins with an estimate for the simple walk.

**Lemma 3.5.** There exists a constant \(K\) such that for all \(n \geq 1\) and positive even integers \(x \leq 2n\),
\[
P \left( W_{2n} = x \mid W_{2n} \geq x \right) \geq \frac{K}{n^{1/2}}.
\] (3.24)

**Proof.** Let \(P_n(x)\) denote the conditional probability in the statement of the lemma. Define the stopping times \(\nu(x) := \min\{j \geq 1 : W_{2j} = x\}\), and write
\[
P_n(x) = \sum_{j=x/2}^{n} \frac{P \left( W_{2n} = x \mid \nu(x) = 2j \right) \cdot P \{ \nu(x) = 2j \}}{P \{ W_{2n} \geq x \}}.
\] (3.25)
We first recall (2.3), and then apply the strong markov property to obtain
\[ P(W_{2n} = x \mid \nu(x) = 2j) = p(n - j) \]. Thanks to (2.5), we can find two constants
\( K_1 \) and \( K_2 \) such that
\[ p(n - j) \geq K_1 (n - j)^{-1/2} \geq K_1 n^{-1/2} \] if \( n - j \geq K_2 \). On
the other hand, if \( n - j < K_2 \), then
\[ p(n - j) \geq K_3 \geq K_3 n^{-1/2} \]. Consequently,
\[ P_n(x) \geq \frac{K_4}{n^{1/2} \cdot \sum_{j=x/2}^{n} P\{\nu(x) = 2j\}} \cdot \sum_{j=x/2}^{n} P\{\nu(x) = 2j\} \] (3.26)
and this last quantity is at least \( K_4 n^{-1/2} \) since
\( \{\nu(x) \leq 2n\} \supseteq \{W_{2n} \geq x\} \).

Here and throughout, let \( F(i; \ell) \) denote the \( \sigma \)-algebra generated by the ran-
memonic variables \( \{\rho_i(j)\}_{j=1}^{\ell} \) and \( \{S(i, m)\}_{m=1}^{\rho_i(\ell)} \) [interpreted in the usual way, since
\( \rho_i(\ell) \) is a stopping time for the infinite-dimensional walk \( i \mapsto S(i, \bullet) \)]. Then we
have the following.

Lemma 3.6. For all \( i, \ell \geq 1, \)
\[ P(S(i, 1 + \rho_i(i)) = 0 \mid F(i; \ell)) \geq \frac{K}{i^{1/2}}, \] (3.27)
where \( K \) was defined in Lemma 3.5.

Proof. Let \( \xi := -S(i, \rho_i(i)) \), for simplicity. According to the definition of the
\( \rho_i(i) \)'s,
\[ S(i, 1 + \rho_i(i)) \geq 0 \] almost surely on \( \{\xi > 0\} \). (3.28)
Consequently,
\[ \Delta_{i,\ell} := S(i, 1 + \rho_i(i)) - S(i, \rho_i(i)) \geq \xi \] almost surely on \( \{\xi > 0\} \). (3.29)
Clearly, the strong markov property of the infinite dimensional random walk
\( i \mapsto S(i, \bullet) \) implies that with probability one,
\[ P(S(i, 1 + \rho_i(i)) = 0 \mid F(i; \ell)) = P(\Delta_{i,\ell} = \xi \mid F(i; \ell)) \]
\[ \geq P(\Delta_{i,\ell} = \xi \mid \Delta_{i,\ell} \geq \xi) \mathbf{1}_{\{\xi > 0\}}. \] (3.30)
Therefore, we can apply Lemma 3.5 together with to deduce that (3.27) holds
a.s. on \( \{\xi > 0\} \). Similar reasoning shows that the very same bound holds also
a.s. on \( \{\xi < 0\} \). \( \square \)

We are ready to derive Proposition 3.4.

Proof of Proposition 3.4. We recall the following form of Bernstein’s inequal-
ity, as found, for example, in [5, Lemma 3.9]: Suppose \( J_1, \ldots, J_n \) are random
variables, on a common probability space, that take values zero and one only.
If there exists a nonrandom \( \eta > 0 \) such that \( E(J_{k+1} | J_1, \ldots, J_k) \geq \eta \) for all \( k = 1, \ldots, n-1 \). Then, that for all \( \lambda \in (0, \eta) \),
\[
P \left\{ \sum_{i=1}^{n} J_i \leq \lambda n \right\} \leq \exp \left( -\frac{n(\eta - \lambda)^2}{2\eta} \right). \tag{3.31}
\]

We apply the preceding with \( J_\ell := U(i; \ell) \); Lemma 3.6 tells us that we can use (3.31) with \( \eta := K i^{-1/2} \) and \( \lambda := \eta/2 \) to deduce the Proposition with \( c := K/2 \).

**Lemma 3.7.** Choose and fix two constants \( a, b > 0 \) such that \( 1 > a > 2b \). Then with probability one,
\[
\min_{1 \leq i \leq N^{1-a}} \sum_{1 \leq \ell \leq N^{1/a}} U(i; \ell) \geq c N^{(a/2) - b}, \tag{3.32}
\]
for all \( N \) sufficiently large, where \( c \) is the constant in Proposition 3.4.

**Proof.** Proposition 3.4 tells us that
\[
P \left\{ \min_{1 \leq i \leq N^{1-a}} \sum_{1 \leq \ell \leq N^{1/a}} U(i; \ell) \leq c N^{(a/2) - b} \right\}
\leq P \left\{ \sum_{1 \leq \ell \leq N^{(1/2) - b}} U(i; \ell) \leq \frac{c N^{(1/2) - b}}{2^{1/2}} \right\} \quad \text{for some } 1 \leq i \leq N^{1-a}
\leq \sum_{1 \leq i \leq N^{1-a}} \exp \left( -\frac{c N^{(1/2) - b}}{4^{1/2}} \right)
\leq N^{1-a} \exp \left( -\frac{c N^{(a/2) - b}}{4} \right). \tag{3.33}
\]
An application of the Borel–Cantelli lemma finishes the proof.

We are ready to complete the proof of our first theorem.

**Proof of Theorem 1.1, Second Half.** Let us begin by choosing and fixing a small constant \( \epsilon \in (0, 1/2) \). Next, we choose and fix two more constants \( a \) and \( b \) such that
\[
b \in (0, 1/2) \quad \text{and} \quad a \in (2b, 1). \tag{3.34}
\]
Finally, we choose and fix yet two more constants \( \epsilon \) and \( \alpha \) such that
\[
\alpha \in (a, 1), \quad \beta \in \left( \frac{\alpha}{2} + \epsilon, b \right), \quad \text{and} \quad \frac{3\alpha}{2} - \frac{\alpha}{2} + b \leq \epsilon. \tag{3.35}
\]
It is possible to verify that we can pick such \( a, b, \alpha, \text{and} \beta \), regardless of how small \( \epsilon \) is.
Because $\alpha \in (a, 1)$,

$$
\bigcap_{1 \leq i \leq N^{1-a}} \left\{ \sum_{1 \leq \ell \leq N^{(1/2)-b}} U(i; \ell) > cN^{(a/2)-b} \right\} \subseteq \left\{ \sum_{i \in \mathcal{H}_N(\alpha, \beta)} \sum_{1 \leq \ell \leq N^{(1/2)-b}} U(i; \ell) \geq cN^{(a/2)-b} |\mathcal{H}_N(\alpha, \beta)| \right\}. \tag{3.36}
$$

According to Lemma 3.3 and since $\beta > (\alpha/2) + \epsilon$, $|\mathcal{H}_N(\alpha, \beta)|$ is at least $N^{1-(3\alpha/2)-2\epsilon}$, for all $N$ large. The preceding and Lemma 3.7 together imply that with probability one, for all $N$ sufficiently large.

$$
\sum_{i \in \mathcal{H}_N(\alpha, \beta)} \sum_{1 \leq \ell \leq N^{(1/2)-b}} U(i; \ell) \geq cN^{1-(3\alpha/2)+(a/2)-b-2\epsilon}, \tag{3.37}
$$

for all $N$ sufficiently large. Consequently, the following holds almost surely: For all but a finite number of values of $N$,

$$
\gamma_N = \sum_{i=1}^{N} \sum_{\ell=1}^{f(i,N)} U(i; \ell) \geq \sum_{i \in \mathcal{H}_N(\alpha, \beta)} \sum_{1 \leq \ell \leq N^{(1/2)-b}} U(i; \ell). \tag{3.38}
$$

Since $\beta < b$ (3.37) implies that with probability one, the following holds for all but finitely-many values of $N$:

$$
\gamma_N \geq cN^{1-(3\alpha/2)+(a/2)-b-2\epsilon}, \tag{3.39}
$$

which is $\geq cN^{1-2\epsilon}$, thanks to the last condition of (3.35). Since $\epsilon$ is arbitrary, this completes our proof. \hfill \Box

### 4 Questions on the distribution of zeros

We conclude this paper by asking a few open questions:

1. Let us call a point $(i, j) \in \mathbb{Z}_2^2$ even if $ij$ is even. Define $Q_N$ to be the largest square in $[0, N]^2$ such that $S(i,j) = 0$ for every even point $(i, j)$ in $Q_N$. What is the asymptotic size of the cardinality of $Q_N \cap \mathbb{Z}^2$, as $N \to \infty$ along even integers? The following shows that this is a subtle question: One can similarly define $\tilde{Q}_N$ to be the largest square in $[0, N]^2$—with one vertex equal to $(N, N)$—such that $S(i,j) = 0$ for all even $(i, j) \in \tilde{Q}_N$. [Of course, $N$ has to be even in this case.] In the present case, we estimate the size of $\tilde{Q}_N$ by first observing that if $N$ is even, then

$$
P \{S(N,N) = S(N+2, N+2) = 0\} = P \{S(N,N) = 0\} \cdot P \{S(N+2, N+2) - S(N,N) = 0\} \tag{4.1}
$$

$$
= (\text{const} + o(1))N^{-3/2} \quad \text{as } N \to \infty \text{ along evens.}
$$
Since the preceding defines a summable sequence, the Borel–Cantelli lemma tells us that \( \# \tilde{Q}_N \leq 1 \) for all sufficiently-large even integers \( N \).

2. Consider the number \( D_N := \sum_{i=1}^{N} 1_{\{S(i,N-\bar{)}=0\}} \) of “anti-diagonal” zeros. It it the case that with probability one,

\[
0 < \limsup_{N \to \infty} \frac{\log D_N}{\log \log N} < \infty \tag{4.2}
\]

At present, we can prove that \( D_N \leq (\log N)^{1+o(1)} \).

3. The preceding complements the following, which is not very hard to prove:

\[
\liminf_{N \to \infty} D_N = 0 \quad \text{almost surely} \tag{4.3}
\]

Here is the proof: According to the local central limit theorem, and after a line or two of computation, \( \lim_{N \to \infty} \mathbb{E}(D_{2N}) = (\pi/8)^{1/2} \). Therefore, by Fatou’s lemma, \( \liminf_{N \to \infty} D_{2N} \leq (\pi/8)^{1/2} < 1 \) with positive probability, whence almost surely by the Kolmogorov zero-one law [applied to the sequence-valued random walk \( \{S(i,\bullet)\}_{i \geq 1} \): (4.3) follows because \( D_N \) is integer valued. We end by proposing a final question related to (4.3): Let \( \{S(s,t)\}_{s,t \geq 0} \) denote two-parameter Brownian sheet; that is, \( S \) is a centered gaussian process with continuous sample functions, and \( \mathbb{E}[S(s,t)S(u,v)] = \min(s,u) \min(t,v) \) for all \( s,t,u,v \geq 0 \).

Define “anti-diagonal local times,”

\[
D_t := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|S(s,t-s)| \leq \epsilon\}} \, ds \quad \text{for } t > 0. \tag{4.4}
\]

(a) Does \( \{D_t\}_{t > 0} \) exist? Is it continuous?

(b) Is it true that \( Z := \{ t > 0 : D_t = 0 \} \) is almost surely nonempty? That is, does the continuum-limit analogue of (4.3) hold? If \( Z \) is nonempty, then what is its Hausdorff dimension?

4. For all \( \epsilon \in (0,1) \) and integers \( N \geq 1 \) define

\[
E(\epsilon, N) := \{(i,j) \in [\epsilon N,N]^2 : S(i,j) = 0\}. \tag{4.5}
\]

It is not hard to verify that if \( \epsilon \in (0,1) \) is fixed, then \( E(\epsilon, N) = \emptyset \) for infinitely-many \( N \geq 1 \). This is because there exists \( p \in (0,1) \) independent of \( N \)—such that for all \( N \) sufficiently large,

\[
P \left\{ \max_{\epsilon N \leq i,j \leq N} |S(i,j) - S(\epsilon N, N)| \leq N \right\} > p. \tag{4.6}
\]

Is there a good way to characterize which positive sequences \( \{\epsilon_k\}_{k=1}^\infty \), with \( \lim_{k \to \infty} \epsilon_k = 0 \), have the property that \( E(\epsilon_N, N) \neq \emptyset \) eventually?
5. Let $\gamma'_N$ denote the number of points $(i, j) \in [0, N]^2$ such that $S(i, j) = 1$. What can be said about $\gamma_N - \gamma'_N$?

6. A point $(i, j)$ is a twin zero if it is even and there exists $(a, b) \in \mathbb{Z}_+^2$ such that: (i) $0 < |i - a| + |j - b| \leq 100$ [say]; and (ii) $S(a, b) = 0$. Let $d(\epsilon, N)$ denote the number of twin zeros that lie in the following domain:

$$D(\epsilon, N) := \{(i, j) \in \mathbb{Z}_+^2 : \epsilon i < j < i/\epsilon, 1 < i < N\}.$$  
(4.7)

Is it true that $\lim_{N \to \infty} d(\epsilon, N) = \infty$ a.s. for all $\epsilon \in (0, 1)$?

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References

[1] Abramowitz, Milton and Irene A. Stegun (1972). *Handbook of Mathematical Functions*, ninth edition, Dover Publishing, New York.

[2] Borodin, A. N. (1986). On the character of convergence to Brownian local time, II, *Probab. Th. Rel. Fields* **72**, 251–277.

[3] Csörgő, M. and P. Révész, (1985). On strong invariance for local time of partial sums, *Stoch. Proc. Appl.* **20**(1), 59–84.

[4] Khoshnevisan, Davar, Pál Révész, and Zhan Shi (2005). Level crossings of a two-parameter random walk, *Stoch. Proc. Appl.* **115**, 359–380.

[5] Khoshnevisan, Davar, Pál Révész, and Zhan Shi (2004). On the explosion of the local times along lines of Brownian sheet, *Ann. de l’Instit. Henri Poincaré: Probab. et Statist.* **40**, 1–24.

[6] Komlós, J., P. Major, and G. Tusnády (1975). An approximation of partial sums of independent RV’s and the sample DF. I, *Z. Wahrsch. verw. Geb.* **32**, 111–131.

[7] Lacey, Michael T. (1990). Limit laws for local times of the Brownian sheet, *Probab. Th. Rel. Fields* **86**(1), 63–85.

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