GENERALIZED RABINOWITZ FLOER HOMOLOGY AND COISOTROPIC INTERSECTIONS

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Abstract. In this paper we generalize the Rabinowitz Floer theory which has been established in the hypersurfaces case. We apply it to the coisotropic intersection problem which interpolates between the Lagrangian intersection problem and the closed orbit problem. More specifically, we study leafwise intersections on a contact submanifold and the displacement energy of a stable submanifold. Moreover we prove that the Rabinowitz action functional is generically Morse, so that Rabinowitz Floer homology is well-defined. The chain complex in Rabinowitz Floer homology is generated by leafwise coisotropic intersection points and the boundary map is defined by counting solutions of a nonlinear elliptic PDE. In the extremal case that is, when the coisotropic submanifold is Lagrangian, it is foliated by only one leaf. Therefore Rabinowitz Floer homology is also relevant to the Lagrangian intersection problem.

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1. Introduction and main results

In this paper, we generalize the Rabinowitz Floer theory which was developed by Cieliebak-Frauenfelder. The ordinary Rabinowitz Floer theory has worked on a hypersurface in a symplectic manifold. We extend this theory to an arbitrary codimensional coisotropic submanifold. Coisotropic intersection was first studied in depth by Ginzburg [Gi1], and have been explored by many experts. (See subsection 1.6.) Generalized Rabinowitz Floer theory helps us study about the coisotropic intersection problem. To be more specific, we shall investigate leafwise intersections on a contact submanifold and the displacement energy of a stable submanifold. Furthermore we define a generalized Rabinowitz Floer homology and compute it in the easiest case.

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We consider a symplectically aspherical symplectic 2n dimensional manifold \((M, \omega)\) which is convex at infinity with a closed and coisotropic codimension \(k\) submanifold \(\Sigma\) in \(M\). For given submanifold \(\Sigma\) of \((M, \omega)\), we define a symplectic orthogonal space of \(\Sigma\) as \(T\Sigma_\omega = \ker \omega|_\Sigma\). Then \(\Sigma\) is said to be coisotropic if \(T_x\Sigma_\omega\) is a \(k\) dimensional subspace of \(T_x\Sigma\) for every \(x \in \Sigma\). We note that \(0 \leq k \leq n\) by definition of coisotropic. We shall recall the definition of stable, contact and restricted contact type for higher codimension cases in Section 2 which were introduced by Bolle. Throughout this paper, we treat coisotropic submanifolds which have one of those types.

**Definition 1.1.** We call a symplectic manifold \((M, \omega)\) convex at infinity if \((M, \omega)\) is symplectomorphic to the symplectization of a compact contact manifold at infinity. Furthermore \((M, \omega)\) is called symplectically aspherical if one has the equality \(\omega|_{\pi^2(M)} = 0 = c_1|_{\pi^2(M)}\).

We will see in Lemma 2.10 that every contact manifold \(\Sigma\) of codimension \(k\) is an intersection of \(k\) contact hypersurfaces, namely \(\Sigma_i\) for \(1 \leq i \leq k\). We denote by \(R_i\) the Reeb vector field on \(\Sigma_i\) for \(1 \leq i \leq k\). The following two formulas (Bo) and (Eq) play crucial roles in the generalized Rabinowitz Floer theory.

\[
\partial_t v(t) = \sum_{i=1}^{k} \eta_i R_i(v(t)) \quad t \in S^1 = \mathbb{R}/\mathbb{Z}, \quad \eta_i \in \mathbb{R}. \quad \text{(Bo)}
\]

It can be viewed as a generalized version of the equation of the Reeb orbit, in other words when \(k = 1\), this equation is exactly coincide with the Reeb vector field equation. In this paper, we are interested in the solutions of (Eq) and (Bo) lying on a leaf (i.e. tangent to the characteristic foliation \(F\) of \(\Sigma\)). Equation (Bo) was considered by Bolle [Bo2]. He investigated non-contractible solutions of (Bo). For contact type submanifolds and contractible solutions, equation (Bo) can be reduced to the following equation.

\[
\partial_t v(t) = \eta \sum_{i=1}^{k} R_i(v(t)) \quad t \in S^1 = \mathbb{R}/\mathbb{Z}, \quad \eta \in \mathbb{R}. \quad \text{(Eq)}
\]

It is noteworthy that (Eq) is rather meaningful than (Bo) since \(\eta\) becomes to a period of \(v_\eta(t) := v(t/\eta)\) \(t \in \mathbb{R}\), a solution for \(\partial_t v(t) = \sum R_i(v(t))\) if \((v(t), \eta)\) solves equation (Eq). (See Lemma 3.2 and equation (3.9).)

1.1. **Leafwise coisotropic intersections.** Let \((M, \omega)\) be a symplectic manifold and \(\Sigma\) be a coisotropic submanifold of codimension \(k\). Then the symplectic structure \(\omega\) determines a symplectic orthogonal bundle \(T\Sigma_\omega \subset T\Sigma\) as follows:

\[
T\Sigma_\omega := \{(x, \xi) \in T\Sigma \mid \omega_\Sigma(\xi, \zeta) = 0 \text{ for all } \zeta \in T_x\Sigma\} \quad (1.1)
\]

Since \(\omega\) is closed, \(T\Sigma_\omega\) is integrable, thus \(\Sigma\) is foliated by the leaves of the characteristic foliation and we denote by \(L_x\) the leaf through \(x\). In the case \(\Sigma = \bigcap_{i=1}^{k} H_i^{-1}(0)\) where \(H_i\)'s are Poisson-commuting Hamiltonian functions and have 0 as a regular value, \(L_x\) is written by

\[
L_x = \{\phi_{H_1}^{t_1} \circ \phi_{H_2}^{t_2} \circ \cdots \circ \phi_{H_k}^{t_k}(x) \mid t_1, \ldots, t_k \in \mathbb{R}\}. \quad (1.2)
\]
We call \( x \in \Sigma \) a leafwise intersection point of \( \phi \in \text{Ham}(M,\omega) \) if \( x \in L_x \cap \phi(L_x) \). In the extremal case \( k = n \), Lagrangian submanifold consists of only one leaf. Thus a Lagrangian intersection point coincides with a leafwise intersection point in the Lagrangian case.

One of the fundamental questions in the intersection theory of coisotropic submanifolds is about the leafwise intersection property.

**Question 1.2.** Given a Hofer-small Hamiltonian diffeomorphism, does it carry a leafwise coisotropic intersection point?

In general, there is a counterexample when a coisotropic submanifold does not admit contact structure. (See [Gi1] and [Gü].) Nevertheless, if a coisotropic submanifold does have contact type and a Hamiltonian diffeomorphism is small in the sense of Hofer-norm, the answer to the question is affirmative.

Alber-Frauenfelder [AF1] realized that a perturbed Rabinowitz action functional enable us to approach variationally to the study of leafwise intersections. When we show an existence of a leafwise intersection point for codimension one case by means of the Rabinowitz action functional [AF1], one of the tough assumptions needed is that a contact hypersurface \( \Sigma \) bounds a compact region in a symplectic manifold \( M \). In the case that \( \Sigma \) does not split \( M \), we are not able to find a defining Hamiltonian function on \( M \) so that its zero locus is exactly equal to \( \Sigma \). For such a reason, a natural model on which the generalized Rabinowitz action functional works is a contact type (stable) submanifold with global Bolle’s coordinates. (see Definition 2.8). Thereby we encounter a problem about finding an appropriate concept replacing the splitting assumption as a codimension of \( \Sigma \) becomes bigger. But it is difficult to find such a concept. Therefore we ought to have overcome such a splitting assumption and eventually was able to remove it in many results in this paper.

**Definition 1.3.** We denote by \( \phi(\Sigma) > 0 \) the minimal period of a orbit lying on \( \Sigma \) which solves equation (Bo) and contractible in \( M \). If there is no such a orbit we set \( \phi(\Sigma) = \infty \).

**Theorem A.** Let \( \Sigma \) be of restricted contact type in \( M \). If the Hofer norm of \( \phi \in \text{Ham}_c(M,\omega) \) is less than \( \phi(\Sigma) \), then there exists a leafwise coisotropic intersection point for \( \phi \).

**Remark 1.4.** Gürel [Gü] also settled this kind of theorem in quite different methods.

We are able to give an affirmative answer due to [Ka] even if the coisotropic submanifold \( \Sigma \) is of unrestricted contact type. But the proof of Theorem B is considerably modified from the proof of Theorem A. Before we state Theorem B, we introduce some notations. We will meet these again in the Sections 2 and 5. Set \( \delta_0 := \sup\{r \in \mathbb{R} \mid \psi : \Sigma_r \hookrightarrow M \} \) and \( \Sigma_r := \{(q,p) = (q,p_1, \ldots, p_k) \in \Sigma \times \mathbb{R}^k \mid |p_i| < r, \ 1 \leq i \leq k \} \).

**Definition 1.5.** For a time dependent Hamiltonian function \( F \in C^\infty(M \times S^1) \) we define a support of the Hamiltonian vector field \( X_F \) defined by \( dF = i_{X_F} \omega \) as

\[
\text{Supp}X_F := \{x \in M \mid X_F(x,t) \neq 0 \text{ for some } t \in S^1 \}. \tag{1.3}
\]

We call a Hamiltonian function \( F \in C^\infty(M \times S^1) \) is admissible if \( \text{Supp}X_F \subset \Sigma_{\delta_0} \). In other words \( F \) is constant outside of \( \Sigma_{\delta_0} \). Furthermore we denote by \( \mathfrak{F} \) the set of all admissible Hamiltonian functions.

**Theorem B.** Let \( \Sigma \) has a contact type submanifold of codimension \( k \) in \( M \). Then for a time dependent Hamiltonian function \( F \in \mathfrak{F} \) such that \( F \) is constant outside of \( \Sigma_{\delta_2} \) for some \( \delta_2 < \delta_0 \), \( \phi_F \in \text{Ham}_c(M,\omega) \) has a leafwise intersection point provided \( ||F|| < \frac{1}{\delta_0 - \delta_2} \phi(\Sigma) \).
Remark 1.6. If a contact type submanifold does not have restricted contact type, the core of the difference is that our ambient symplectic manifold can be closed.

When we slightly modify the proof of Theorem B, we obtain the following corollary.

Corollary B. A Lagrangian torus $T^n$ embedded in $(\mathbb{C}^*)^n$ has a self intersection point for every compactly supported Hamiltonian diffeomorphisms $\phi$.

Remark 1.7. Even though the symplectic manifold $(\mathbb{C}^*)^n$ does not have a nice infinity, we can show that the gradient flow lines of Rabinowitz action functional do not go to the infinity. Then Theorem A helps us. Furthermore all orbits solving the main equation (Eq) does not contractible in $(\mathbb{C}^*)^n$ and thus $\psi(\Sigma) = \infty$.

Next, we are also interested in the number of leafwise intersection points. In [AF1], they showed that the following theorem for the hypersurface case. Using their proof, we obtain a lower bound of the number of leafwise coisotropic intersection points in general case. (Albers-Momin [AM] recently obtained an another lower bound.)

Theorem C. The number of leafwise coisotropic intersection points is bounded below the sum of $\mathbb{Z}$/2-betti numbers of $\Sigma$ of restricted contact type with global Bolle’s coordinates for a Hamiltonian diffeomorphism $\phi$ satisfying $||\phi|| < \psi(\Sigma)$.

Proving Theorem C, we use the Rabinowitz Floer homology. In order to define the Rabinowitz Floer homology, the condition that $\Sigma$ admits global Bolle’s coordinates is needed and thus so does Theorem C.

Albers-Frauenfelder [AF2, AF3], Abbondandolo-Schwarz [AS] showed that there are infinitely many leafwise intersections in the (unit) cotangent bundle. (Merry [Me] proved it for the twisted cotangent bundle.) In Theorem D, we show that there are infinitely many leafwise coisotropic intersection points of Hofer-small Hamiltonian diffeomorphisms under the additional condition on $\Sigma$.

Definition 1.8. We call $\Sigma$ is non-degenerate if the set $R$ of solutions of equation (Bo) form a discrete set.

Definition 1.9. The nonconstant solution $(v, \eta)$ where $\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{R}^k$ of (Bo) is called a multiple cover, if there exist $(v_0, \eta_0)$ solving (Bo) such that $(v(t), \eta) = n_*(v_0(t), \eta_0) := (v_0(nt), n\eta_0)$ for some $2 \leq n \in \mathbb{N}$. Otherwise we call $(v, \eta)$ is primitive.

Theorem D. Assume that there are only finitely many primitive solutions of (Bo) on non-degenerate restricted contact submanifold $\Sigma$, namely $(v^1, \eta^1), \ldots, (v^m, \eta^m)$ and all $\eta^1, \ldots, \eta^m$ are rational numbers. Then there exists $\gamma > 0$ such that if $||F|| < \gamma$ (for $F \in \mathfrak{F}$ in the unrestricted case) then a generic $\phi_F$ have infinitely many leafwise coisotropic intersection points.

Remark 1.10. The inspiration for Theorem D came from Usher’s paper [Us]. For the positivity of $\gamma$ in Theorem E, the assumptions are needed. If $\Sigma$ is a hypersurface, then we know that generic $\Sigma$ is non-degeneracy [CF]. Moreover we expect that the genericity assumption can be removed by using spectral invariants in the Rabinowitz Floer homology [AF3].
1.2. **Displacement energy.** Another significant problem in coisotropic intersections is the positivity of the displacement energy of a displaceable coisotropic submanifold. In other words,

**Question 1.11.** Can we displace a displaceable coisotropic submanifold by a Hamiltonian diffeomorphism arbitrarily Hofer-small?

**Definition 1.12.** The displacement energy of $\Sigma$ is given by

$$e(\Sigma) := \inf_{F \in C_\infty^0(M \times S^1)} \{|F| | \phi^1_F(\Sigma) \cap \Sigma = \emptyset\}.$$  \hspace{1cm} (1.4)

We set $e(\Sigma) = \infty$ for the infimum of the empty set. That is, the displacement energy of a non-displaceable submanifold is infinity.

**Theorem E.** Let $\Sigma$ be a displaceable stable submanifold in $M$. Then there exists a loop $v$ lying on a leaf, contractible in $M$, such that

$$0 < |\Omega(v)| < e(\Sigma).$$  \hspace{1cm} (1.5)

Here $\Omega$ stands for the area functional that is, $\Omega(v) = \int_{D^2} \bar{v}^* \omega$.

**Remark 1.13.** This is a generalization of Schlenk’s theorem [Schl] to higher codimension. Ginzburg [Gi1] proved Theorem E for the restricted contact type case. (See also [Gi2].)

1.3. **Coisotropic Liouville class.** Let an ambient symplectic manifold $M$ be exact with a symplectic structure $\omega = d\lambda$. Then $\lambda|_F = \lambda_F$, the restriction of $\lambda$ to the characteristic foliation $F$ of $\Sigma$ is closed. The cohomology class $\lambda_F \in H^1_{dR}(F)$ in the foliated de Rham cohomology is called the **coisotropic Liouville class**. In the case that the dimension of $\Sigma$ is a half of the dimension of $M$, it coincides with the ordinary Liouville class defined on Lagrangian submanifolds. This generalized version of the Liouville class was considered by Ginzburg and he also deduced Corollary E [Gi1].

For given contractible loop $v : S^1 \rightarrow F$, choose any map $\bar{v} : D^2 \rightarrow M$ such that $\bar{v}|_{S^1} = v$ so called a filling disk of $v$. In the sense of the following formula, we refer to $(\lambda, v)$ as the **symplectic area of** $v$.

$$(\lambda, v) = \int_{S^1} v^* \lambda_F = \int_{D^2} \bar{v}^* \omega.$$  \hspace{1cm} (1.6)

Accordingly, to show $\lambda_F \neq 0$ is equivalent to find an loop tangent to the foliation which has nonzero symplectic area. Therefore Corollary E below is an immediate consequence of Theorem E.

**Corollary E.** A stable submanifold $\Sigma$ is nondisplaceable provided that $[\lambda_F] = 0$.

1.4. **Rabinowitz Floer homology.** In order to define a Rabinowitz Floer homology, we need two ingredients. One is the Morse property of the Rabinowitz action functional and the other is a compactness of moduli space of gradient flow lines. The last one has already established in the proof of Theorem A. Thus the following theorem proved in section 7 enable us to define the Rabinowitz Floer homology.

**Theorem 1.14.** For a generic Hamiltonian function $F \in C^\infty(M \times S^1)$ the perturbed Rabinowitz action functional $A^H_F$ is Morse.
Due to Theorem 1.14 and Theorem 4.9 the perturbed Rabinowitz Floer homology can be defined. One important thing to remark is that the Rabinowitz Floer homology is defined for a contact submanifold with global Bolle’s coordinates. In such a reason we tacitly assume that our contact submanifold admits global Bolle’s coordinates when we deal with the Rabinowitz Floer homology. In the unperturbed case, generally Rabinowitz action functional is Morse-Bott, we also obtain the Rabinowitz Floer homology by Frauenfelder’s Morse-Bott homology \( \text{Fr} \). We denote by
\[
RFH(\Sigma, M) = HF(A^H), \quad RFH(\Sigma, M, F) = HF(A^H_F).
\]

Moreover many properties in the Floer homology theory follow immediately. (See Section 8.) For example, above two Rabinowitz Floer homologies are isomorphic by the standard argument of continuation homomorphism in the Floer theory. Even for the case that \( \Sigma \subset M \) is an unrestricted contact type coisotropic submanifold, we are still able to define \( RFH(\Sigma, M) \) and \( RFH(\Sigma, M, F) \) for \( F \in \mathfrak{F} \).

**Remark 1.15.** In the extremal case, where \( i : \Sigma = T^n \hookrightarrow M \) is an embedded Lagrangian torus of contact type and the induced homomorphism \( i_* : \pi_1(T^n) \rightarrow \pi_1(M) \) is injective, then there is no contractible orbit solving the equation \( \text{Bo} \). Thus \( \text{Crit}A^H = \Sigma = T^n \), moreover \( RFH(T^n, M) = H(T^n) \neq 0 \). Therefore Theorem F below guarantees the existence of self intersection point of a Lagrangian torus \( T^n \).

We can deduce the same consequence using the Lagrangian Floer theory. Consider any map \( u \in C^\infty(D^2, M) \) satisfying the boundary condition, \( u(\partial D^2) \subset T^n \). Then \( u|_{\partial D^2} \) is contractible in \( M \), indeed in \( T^n \) since \( i_* : \pi_1(T^n) \rightarrow \pi_1(M) \) is an injection. Thus we can regard \( u \) as an element of \( C^\infty(S^2, M) \) up to homotopy, that is \( [u] \in \pi_2(M) \). But we assumed that \( (M, \omega) \) is symplectically aspherical so that \( \int_{S^2} u^*\omega = 0 \). Hence \( \omega|_{\pi_2(M,T^n)} = 0 \) and the Lagrangian Floer homology is well-defined. Then we use the well-known property in the Lagrangian Floer theory that \( HF(T^n, T^n) = H(T^n) \neq 0 \), and eventually obtain the same result that \( T^n \cap \phi(T^n) \neq \emptyset \) for any \( \phi \in \text{Ham}_c(M, \omega) \).

**Proposition 1.16.** For a Lagrangian torus \( T^n \) of contact type in \( M \), if an embedding \( i : \Sigma = T^n \hookrightarrow M \) induces an injective homomorphism on \( \pi_1 \)-level, then the Lagrangian Floer homology is well-defined, moreover
\[
RFH(T^n, M) \cong H(T^n) \cong HF(T^n, T^n).
\]

Therefore there always exists a self intersection point of \( T^n \).

**Proof.** See above Remark 1.15. \( \square \)

**Proposition 1.17.** More generally if a contact submanifold \( \Sigma \) carries no solutions of equation \( \text{Bo} \) which is noncontractible in \( M \), then
\[
RFH(\Sigma, M) \cong H(\Sigma).
\]

**Proof.** It easily follows from the same argument as in Remark 1.15. \( \square \)

**Theorem F.** If the Rabinowitz Floer homology does not vanish, then there exists a leafwise coisotropic intersection point. In particular if the coisotropic submanifold \( \Sigma \) is displaceable in \( M \), then the Rabinowitz Floer homology vanishes.
Corollary F. There always exists a nonconstant contractible orbit solving (Bo), provided that \( \Sigma \) is displaceable in \( M \). In the case that \( \Sigma \) is a unrestricted contact type we have the same theorem if \( \Sigma \) is displaced by an element of \( \mathfrak{F} \).

1.5. Organization of the article. At first, we recall the notions of stable, contact and restricted contact type for a coisotropic submanifold and introduce its properties in section 2. In section 3, we introduce the generalized Rabinowitz action functional defined with several Lagrange multipliers and investigate its critical points. We treat the perturbed Rabinowitz action functional in section 4 and show that its critical points give rise to leafwise coisotropic intersections. Additionally we prove that the moduli space which is composed of gradient flow lines of Rabinowitz action functional is compact modulo breaking. Furthermore we prove Theorem A in section 4. In section 5, we work on a unrestricted contact submanifold. Nevertheless, we deduce exactly the same results for admissible perturbations and prove Theorem B. Section 6 comprises the work on a stable submanifold and includes the proof of Theorem E. In section 7, we show that the Rabinowitz action functional is generically Morse. Therefore we are able to define the Rabinowitz Floer homology in section 8. Moreover we study the Filtered Rabinowitz Floer homology and the Local Rabinowitz Floer homology. These give the proof of Theorem C and F. Finally we guarantee infinitely many leafwise coisotropic intersections under the special assumption in section 9 (Theorem D).

1.6. History and related results. The aim of this paper is to generalize the Rabinowitz Floer theory to higher codimension contact type submanifolds. This Rabinowitz Floer theory was introduced by Cieliebak and Frauenfelder in [CF] and has been well studied for the hypersurface cases by Albers, Cieliebak, Frauenfelder, Kang, Momin, Oancea, Paternain [AF1, AF2, AF3, AF4, AF5, AM, CF, CFO, CFP, Ka], Abbondandolo, Schwarz [AS] and Merry [Me]. The framework of this paper and many results were inspired by their remarkable achievements. (including Ginzburg’s work [Gi1]..) The generalized Rabinowitz Floer theory is a useful approach to the coisotropic intersection problem similar to the Lagrangian Floer theory do in the Lagrangian case. The problem of the existence of a leafwise intersection point was addressed by Moser [Mo]. Moser obtained the result for simply connected \( M \) and \( C^1 \)-small \( \phi \). Banyaga [Ba] removed the assumption of simply connectedness. But these days in many papers the assumption of \( C^1 \)-smallness can be reduced by constraints on the Hofer norm. Hofer [Ho] and Ekeland-Hofer [EH] replaced the assumption of \( C^1 \)-smallness by boundedness of the Hofer norm below a certain symplectic capacity for restricted contact type in \( \mathbb{R}^{2n} \). Ginzburg [Gi1] generalized Ekeland-Hofer’s results for a restricted contact type in subcritical Stein manifolds with homological capacity. Dragnev [Dr] obtained this result to closed contact type submanifold in \( \mathbb{R}^{2n} \). Albers-Frauenfelder [AF1] proved the existence of leafwise intersection points for restricted contact hypersurfaces whenever the Hamiltonian diffeomorphisms are close to the identity in the Hofer norm. By a different approach Gürel [Gü] also proved existence on a restricted contact submanifold (not necessarily of codimension one) under the Hofer norm smallness. Ziltener [Zi1, Zi2] also studied the question in a different way and obtained a lower bound of the number of leafwise intersection points under the assumption that the characteristic foliation is a fibration. In a different aspect, the displacement energy of coisotropic submanifolds is also an integral part of the coisotropic intersection theory. Bolle [Bo1, Bo2] proved that the displacement energy for stable coisotropic submanifolds of \( \mathbb{R}^{2n} \) is positive. Ginzburg also extend Bolle’s result to wide (or closed) and geometrically bounded manifold [Gi1, Gi2]. Recently Kerman [Ke] generalize this to closed,
rational symplectic manifold, Usher [Us] prove it more generally that the displacement energy of a stable coisotropic submanifold of closed or convex symplectic manifold is positive.

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2. Coisotropic submanifolds

We denote by the Hamiltonian tuple $\mathcal{H} := (H_1, \ldots, H_k)$ for Hamiltonian functions $H_i \in C^\infty(M, \mathbb{R})$. We sometimes interpret $\mathcal{H}$ as an element of $C^\infty(M, \mathbb{R}^k)$.

Definition 2.1. Given Hamiltonian functions $F$ and $G$, the Poisson bracket $\{\cdot, \cdot\}$ is defined by the adjoint action as follows:

$$\{F, G\} := \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\phi^t_F} G.$$  (2.1)

This expression is equivalent to $\{F, G\} = \omega(X_F, X_G)$.

Remark 2.2. $[X_F, X_G] = -X_{\{F, G\}}$ where $[\cdot, \cdot]$ is the Lie bracket and the flows $\phi^t_F$ and $\phi^t_G$ are commute if and only if $\{F, G\} = 0$.

Definition 2.3. The Hamiltonian tuple $\mathcal{H}$ is said to be Poisson-commute (or in involution) if $\{H_i, H_j\} = 0$ for all $1 < i, j < k$.

If Hamiltonian tuple $\mathcal{H}$ Poisson-commutes and $0$ is a regular value of $\mathcal{H}$, then $\nabla H_1, \ldots, \nabla H_k$ are linearly independent on the tangent space of a smooth manifold $\bigcap_{i=1}^k H_i^{-1}(0) = \mathcal{H}^{-1}(0)$. Therefore $T\mathcal{H}^{-1}(0)$ is spanned by their Hamiltonian vector fields, namely $X_{H_1}, \ldots, X_{H_k}$ and $\mathcal{H}^{-1}(0)$ becomes a smooth coisotropic submanifold.

We recall the notion of contact (stable) condition on coisotropic submanifolds as it was introduced by Bolle in [Bo1, Bo2].

Definition 2.4. The coisotropic submanifold $\Sigma$ of codimension $k$ in $M$ is called stable if there exist one forms $\alpha_1, \ldots, \alpha_k$ on $\Sigma$ which satisfy

(i) $\ker d\alpha_i \supset \ker \omega_{\Sigma}$ for $i = 1, \ldots, k$;

(ii) $\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \omega^{n-k}_{\Sigma} \neq 0$.

We call $\Sigma$ is of contact type if $\alpha_i$ is primitives of $\omega$ for $1 \leq i \leq k$. Furthermore we say $\Sigma$ has restricted contact type if all $\alpha_i$ is defined globally on $M$. We occasionally say $\Sigma$ is of unrestricted contact type to emphasize $\Sigma$ needs not be of restricted contact type.

Example 2.5. Although the contact condition is restrictive, we have numerous examples as follows [Bo2, Gi1].

(i) A coisotropic submanifold which is $C^1$-close to a coisotropic submanifold of contact type also has contact type.

(ii) A hypersurface has a contact type if and only if it is of contact type in the standard sense.

(iii) A Lagrangian torus is of contact type with contact one forms $d\theta_1, \ldots, d\theta_n$ where $\theta_1, \ldots, \theta_n$ are angular coordinates on the $n$ dimensional torus. Indeed it turns out that a Lagrangian torus of contact type is necessarily a torus.

(iv) Let $\Sigma \subset M_1^{2n_1}$ be a contact type submanifold and $T^{n_2} \subset M_2^{2n_2}$ be a Lagrangian torus. Then $\Sigma \times T^{n_2}$ is a submanifold in $M_1 \times M_2$ of contact type.
(v) Consider the Hopf fibration \( \pi : S^{2n-1} \rightarrow \mathbb{C}P^{n-1} \). According to Marsden-Weinstein-Meyer reduction, we know that there is a canonical symplectic form \( \omega_{\mathbb{C}P^{n-1}} \) on \( \mathbb{C}P^{n-1} \)

satisfying \( \pi^{*} \omega_{\mathbb{C}P^{n-1}} = \omega_{\mathbb{R}^{2n}} \) where \( \omega_{\mathbb{R}^{2n}} \) is the standard symplectic form on \( \mathbb{R}^{2n} \).

For a contact hypersurface \((\Delta, \alpha) \subset \mathbb{C}P^{n-1}, \pi^{-1}(\Delta) \) is a contact submanifold in \( \mathbb{R}^{2n} \) of codimension 2.

### Remark 2.6.

A product of contact type coisotropic manifold need not be of contact type. See [Bo2] or [Gi1]. However a product of stable manifolds is stable.

In the case that \( \Sigma \) is of contact (stable) type, the normal space of \( \Sigma \subset M \) is isomorphic to \( N_{x} \cong TL_{x} \cong \Sigma \times \mathbb{R}^{k} \) for every \( x \in \Sigma \). Thus we use the same symbols \( \omega_{\Sigma} \) and \( \alpha_{i} \) for differential forms in \( \Sigma \) and for their pullback to \( \Sigma \times \mathbb{R}^{k} \). From the Weinstein neighborhood theorem, we have

### Proposition 2.7. [Bo1, Bo2]

Let \( \Sigma \) be a closed, contact (stable) codimension \( k \) submanifold of \( M \). Then there exist \( \epsilon > 0 \), a neighborhood \( V \) of \( \Sigma \) which is symplectomorphic by \( \psi : U_{\epsilon} \rightarrow V \) to \( U_{\epsilon} := \{(q, p) = (q, p_{1}, \ldots, p_{k}) \in \Sigma \times \mathbb{R}^{k} \mid |p| < \epsilon \} \) with \( \psi^{*} \omega = \omega_{\Sigma} + \sum_{i=1}^{k} d(p_{i} \alpha_{i}) \).

Then we have \( X_{p_{i}} \in \ker \omega_{\Sigma} \), \( dp_{j}(X_{p_{i}}) = 0 \) and \( \alpha_{j}(X_{p_{i}}) = \delta_{ij} \) on \( \Sigma \) for \( 1 \leq i, j \leq k \) since \( iX_{p_{i}} \omega = dp_{i} \) by definition. Moreover the Hamiltonian tuple \( p = (p_{1}, \ldots, p_{k}) \) Poisson-commutes since \( \{X_{p_{i}}, \ldots, X_{p_{k}}\} \) forms a basis for \( \ker \omega_{\Sigma} \).

### Definition 2.8.

We call a coisotropic submanifold \( \Sigma \) is of contact (stable) type with global Bolle’s coordinates if we are able to extend its coordinate functions \( p_{1}, \ldots, p_{k} \) globally on \( M \) so that Poisson-commute each other.

In order to define the Rabinowitz action functional, we need global Bolle’s coordinate functions defined on \( M \). Therefore we encounter the problem that when we extend Bolle’s coordinates globally. This problem of extending is not easy, thus we came up with many of the results in this paper even though our contact (stable) manifold does not admit global Bolle’s coordinates. However a contact manifold with global Bolle’s coordinates is a natural model on which the Rabinowitz action functional works nevertheless. Accordingly the Rabinowitz Floer homology is only defined on such a natural model.

In general, an intersection of contact hypersurface is not of contact type. The following proposition yields the additional condition which makes such an assertion hold.

### Proposition 2.9.

Let \( \Sigma_{1}, \ldots, \Sigma_{k} \) be hypersurfaces which have contact structures near \( \Sigma := \bigcap_{i=1}^{k} \Sigma_{i} \) with linearly independent Liouville vector fields \( Y_{1}, \ldots, Y_{k} \) such that \( \omega(Y_{i}, Y_{j}) = 0 \) for all \( 1 \leq i, j \leq k \). Then \( \Sigma \) is of contact type.

**Proof.** We are able to choose \( k \) Hamiltonian functions \( H_{1}, \ldots, H_{k} : U_{\epsilon} \rightarrow \mathbb{R} \) defined locally near \( \Sigma \) so that \( H_{i}^{-1}(0) = \Sigma_{i}|_{U_{\epsilon}} \) and \( \nabla_{q} H_{i} = Y_{i} \) for each \( i \). Here we choose an \( \omega \)-compatible almost complex structure \( J \) which interchanges the Reeb vector field of \( \Sigma_{i} \) with \( Y_{i} \) on \( \Sigma \) for all \( i \in \{1, \ldots, k\} \). Then we have a metric \( g = \omega(\cdot, J \cdot) \) on \( M \). Since \( \omega(X_{H_{i}}, X_{H_{j}}) = \omega(\nabla H_{i}, \nabla H_{j}) = 0 \) for all \( i, j \in \{1, \ldots, k\} \), we can decompose the tangent space at \( x \in \Sigma \subset U_{\epsilon} \).
as follows:

\[ T_x U_\epsilon = T_x \Sigma \bigoplus \langle Y_1, \ldots, Y_k \rangle \]

\[ = T_x \Sigma^\omega \bigoplus \left( \bigcap_{i=1}^k \ker \alpha_i \right) \bigoplus \langle Y_1, \ldots, Y_k \rangle \]  

\[ = \langle X_{H_1}, \ldots, X_{H_k} \rangle \bigoplus \left( \bigcap_{i=1}^k \ker \alpha_i \right) \bigoplus \langle Y_1, \ldots, Y_k \rangle. \]

On the neighborhood \( U_\epsilon \), \( k \) contact one forms \( \alpha_1, \ldots, \alpha_k \) defined as \( \alpha_i := i_{Y_i^*} \omega \), are linearly independent since \( Y_1, \ldots, Y_k \) are so. Since \( \alpha_i(Y_j) = \omega(Y_i, Y_j) = 0 \) by assumption, we know that all \( \alpha_i \) vanishes on \( \left( \bigcap_{i=1}^k \ker \alpha_i \right) \bigoplus \langle Y_1, \ldots, Y_k \rangle \). Therefore the restrictions to \( \ker \omega \Sigma \) of \( \alpha_1, \ldots, \alpha_k \) are linearly independent and \( \Sigma \) is of contact type. \( \square \)

The converse of the above assertion also holds.

**Lemma 2.10.** If \( \Sigma \) is a contact type submanifold of codimension \( k \), then there exist \( k \) contact hypersurfaces which have defining Poisson-commuting Hamiltonian functions defined locally.

**Proof.** Hypersurfaces \( p^{-1}_i(0), \ldots, p^{-1}_k(0) \) in \( U_\epsilon \subset M \) satisfies the assertion of the lemma. \( \square \)

 Accordingly, \( X_{p_i} \) is a Reeb vector field on \( p^{-1}_i(0) \). We sometimes use the notations \( X_{p_i} = R_i \) with \( p_i^{-1}(0) = \Sigma_i \) appeared in equations (Bo) and (Eq) in the introduction.

**Remark 2.11.** In fact, a closed contact type codimension \( k \) submanifold \( \Sigma \) in a symplectic manifold \( M \) with an infinity part can be considered as a zero level set of \( k \) Hamiltonian functions defined globally and not compactly supported. In order to unfolding our story, however, we use Hamiltonian functions defining \( \Sigma \) which are constant outside of a compact set. Otherwise the gradient flow lines of the Rabinowitz action functional may escape to infinity. However, though Hamiltonian vector fields of defining Hamiltonian functions are not compactly supported, there is still a possibility that the gradient flow lines are bounded in \( L^\infty \). For example, [CFO] used a Hamiltonian which is linear with respect to the coordinate of \( \mathbb{R} \) in infinity part. Moreover [AS] showed an \( L^\infty \)-bound for a Hamiltonian function growing at most polynomially outside of a compact set. However, these arguments are needed hard analysis work, thus we postpone this enlarging problem for the future.

### 3. Rabinowitz action functional with several Lagrange multipliers

**3.1. Rabinowitz action functional.** We denote by the tuple \( \eta := (\eta_1, \ldots, \eta_k) \) for the several Lagrange multipliers \( \eta_i \in \mathbb{R} \) for \( i = 1, \ldots, k \) and \( \mathcal{L} \in C^\infty(S^1, M) \) the component of contractible loops in \( M \). For an arbitrary Poisson-commute Hamiltonian tuple \( \mathcal{H} \) which has \( 0 \in \mathbb{R}^k \) as a regular value, the generalized Rabinowitz action functional \( \mathcal{A}^\mathcal{H}(v, \eta) : \mathcal{L} \times \mathbb{R}^k \to \mathbb{R} \) is defined as follows:

\[
\mathcal{A}^\mathcal{H}(v, \eta) = -\int_{D^2} \bar{v}^* \omega - \sum_{i=1}^k \eta_i \int_0^1 H_i(v(t)) dt
\]

(3.1)

where \( \bar{v} |_{\partial D^2} = v(t) \) for \( t \in S^1 \).

Using the standard scalar product \( \langle \cdot, \cdot \rangle \) in \( \mathbb{R}^k \), we occasionally interpret the last term in
equation (3.1) as
\[ \sum_{i=1}^{k} \eta_i \int_0^1 H_i(v(t))dt = \int_0^1 \langle \eta, \mathcal{H} \rangle(v(t))dt. \]  

(3.2)

3.2. Critical points. Critical points of Rabinowitz action functional, \((v, \eta) \in \text{Crit}_\mathcal{A} \mathcal{H}\) satisfy following equations.

\[ \begin{aligned}
\partial_t v &= \sum_{i=1}^{k} \eta_i X_{H_i}(v(t)) \\
\int_0^1 H_i(v(t))dt &= 0 \quad 1 \leq \forall i \leq k
\end{aligned} \]  

(3.3)

Lemma 3.1. If \((v, \eta) \in \text{Crit}_\mathcal{A} \mathcal{H}\), then \(v(t)\) lies in the codimension \(k\) contact submanifold \(\mathcal{H}^{-1}(0)\).

Proof. Using the first equation in (3.3),

\[ \frac{d}{dt} H_i(v(t)) = dH_i(v(t))[\partial_t v] \]

\[ = dH_i \left( \sum_{j=1}^{k} \eta_j X_{H_j}(v(t)) \right) \]

\[ = \sum_{j=1}^{k} \eta_j \{ H_i, H_j \}(v(t)) \]

\[ = 0 \]  

(3.4)
since the Hamiltonian tuple \(\mathcal{H}\) is Poisson-commuting.

What we know from the above computation is that all Hamiltonian functions \(H_i\) are stationary along \(v(t)\). The second equation in (3.3) implies \(H_i(v(t)) = 0\) for all \(1 \leq i \leq k\). This completes the proof of lemma. \(\square\)

From now on, we tacitly assume that \(\Sigma\) is a restricted contact submanifold with global Bolle’s coordinates. We define \(\delta > 0\) as follows:

\[ \delta_0 := \sup \left\{ r \in \mathbb{R} \mid \psi : (\Sigma_r, \omega_\Sigma + \sum_{i=1}^{k} d(e^{p_i} \alpha_i)) \xrightarrow{\cong} (M, \omega) \right\} \]  

(3.5)

where

\[ \Sigma_r := \{(q, p) \in \Sigma \times \mathbb{R}^k \mid |p_i| < r, \quad 1 \leq \forall i \leq k \} \cong \Sigma \times (-r, r)^k. \]  

(3.6)

Then we modify global Bolle’s coordinate functions \(p_1, \ldots, p_k \in C^\infty(M, \mathbb{R})\) to

\[ \tilde{p}_i : M \to \mathbb{R} \text{ by } \tilde{p}_i = \begin{cases} p_i & \text{on } \Sigma_{\delta_0/2} \\ \text{locally constant} & \text{outside } \Sigma_{\delta_0}. \end{cases} \]  

(3.7)

\(\tilde{p}_i\) is the new global Bolle’s coordinates and we relabel \(p_i\) for notational convenience. Let \(\chi \in C^\infty(S^1, \mathbb{R})\) be a smooth function such that \(\int_0^1 \chi(t)dt = 1\) and \(\text{supp}(\chi) \subset (\frac{1}{2}, 1)\). Using \(\chi\) and Bolle’s coordinate functions \(p = (p_1, \ldots, p_k) \in C^\infty(M, \mathbb{R}^k)\) defining \(\Sigma\) which has restricted contact type of codimension \(k\) with global Bolle’s coordinates, we can define a weakly time dependent Hamiltonian \(H_i : M \times S^1 \to \mathbb{R}\) by \(H_i(t, x) = \chi(t)p_i(x)\) for all \(1 \leq i \leq k\).
i.e $\mathcal{H}(t, x) = \chi(t)p(x)$. We note that the Rabinowitz action functional defined by arbitrary Poisson-commuting Hamiltonian tuple and Lemma 3.1 holds for arbitrary Hamiltonian functions, yet from now on, $H_i$'s are the Hamiltonian functions defined above. In other words, henceforth, $\mathcal{H} = \{H_1, \ldots, H_k\}$ is the extension of the Bolle’s coordinates of given $\Sigma$.

In the case of hypersurfaces, we know that if $(v, \eta) \in \mathcal{L} \times \mathbb{R}$ is a critical point of the Rabinowitz action functional then $v_\eta(t) := v(t/\eta)$ is the Hamiltonian orbit lying in the hypersurface with period $\eta$. For the general codimension $k$ contact case, we have an extended result.

**Lemma 3.2.** Let $(v, \eta) = (v, \eta_1, \ldots, \eta_k) \in \text{Crit}\mathcal{A}^\mathcal{H}$ and $\mathcal{H}(t, x) = \chi(t)(p_1(x), \ldots, p_k(x)) \in C^\infty(S^1 \times M, \mathbb{R}^k)$ be the time dependent Hamiltonian tuple, then $\eta_1 = \cdots = \eta_k$.

**Proof.** For each $i = 1, \ldots, k$, we calculate

$$\mathcal{A}^\mathcal{H}(v, \eta) = -\int_{D^2} v^* \omega - \int_0^1 \langle \eta, \mathcal{H}(t, v(t)) \rangle dt$$

$$= -\int_0^1 v^* \alpha_i - \sum_{j=1}^k \int_0^1 H_i(t, v(t)) dt$$

$$= -\int_0^1 \alpha_i(\partial_t v) dt$$

$$= -\int_0^1 \alpha_i \left( \sum_{j=1}^k \eta_j X_{H_j}(v) \right) dt$$

$$= -\eta_i \int_0^1 \alpha_i(X_{H_i}(v)) dt$$

$$= -\eta_i$$

We note that the second equality follows with any $\alpha_i$ since all $\alpha_i$ is the primitive of $\omega$. Thus we know that all $\eta_i$, $i = 1, \ldots, k$ are same. \(\square\)

The upshot is that if $(v, \eta) = (v, \eta_1, \ldots, \eta_k) \in \text{Crit}\mathcal{A}^\mathcal{H}$, then $v_\eta(t) := v(t/\eta)$ is a solution of

$$\partial_t v_\eta(t) = \sum_{i=1}^k R_i(v_\eta(t)) \quad t \in \mathbb{R}/\eta \mathbb{Z}.$$  

(3.9)

with period $\eta$.

4. A perturbation of the generalized Rabinowitz action functional

4.1. **Hofer norm.** We briefly recall the definition of Hofer norm. Let $\text{Ham}_c(M, \omega)$ be the group of compactly supported Hamiltonian diffeomorphisms.

**Definition 4.1.** Let $F \in C^\infty(M \times S^1, \mathbb{R})$ be a compactly supported Hamiltonian function. We set

$$|F|_+ := \int_0^1 \max_{x \in M} F(t, x) dt \quad |F|_- := -\int_0^1 \min_{x \in M} F(t, x) dt = | - F|_+$$  

(4.1)
and
\[ |F| = |F|_+ + |F|_- \tag{4.2} \]
For \( \phi \in \text{Ham}_c(M, \omega) \) the Hofer norm is
\[ ||\phi|| = \inf\{|F| \mid \phi = \phi_F \}. \tag{4.3} \]

**Lemma 4.2.** For all \( \phi \in \text{Ham}_c(M, \omega) \)
\[ ||\phi|| = \inf\{|F| \mid \phi = \phi_F, F(t, \cdot) = 0 \ \forall t \in [\frac{1}{2}, 1] \}. \tag{4.4} \]

**Proof.** To prove \( ||\phi|| \geq ||\phi|| \), pick a smooth monotone increasing map \( r : [0, 1] \to [0, 1] \) with \( r(0) = 0 \) and \( r(\frac{1}{2}) = 1 \). For \( F \) with \( \phi_F = \phi \) we set \( F_r(t, x) := r'(t) F(r(t), x) \). Then a direct computation shows \( \phi_{F_r} = \phi_F, ||F_r|| = ||F||, \) and \( F_r(t, x) = 0 \) for all \( t \in [\frac{1}{2}, 1] \). The reverse inequality is obvious. \( \square \)

Thanks to the previous lemma, henceforth, we assume that a time dependent Hamiltonian function \( F \) has a time support \((0, \frac{1}{2})\).

**Note.** The time support of \( \mathcal{H} = (H_1, \ldots, H_k) \) and the time support of \( F \) are disjoint.

### 4.2. Perturbed Rabinowitz action functional

Given Hamiltonian tuple \( \mathcal{H} \) and time dependent Hamiltonian function \( F \) defined so far, we consider the perturbed Rabinowitz action functional.

\[ \mathcal{A}_F^\mathcal{H}(v, \eta) = -\int_{D^2} \bar{v}^*\omega - \int_0^1 F(t, v(t))dt - \int_0^1 \langle \eta, \mathcal{H}(t, v(t)) \rangle dt. \tag{4.5} \]

where \( \bar{v} : D^2 \to M \) is a filling disk of \( v \).

A critical points of the perturbed Rabinowitz action functional \((v, \eta) \in \text{Crit}\mathcal{A}_F^\mathcal{H}\) satisfies the following equations.

\[ \begin{align*}
\partial_t v &= X_F(t, v) - \sum_{i=1}^k \eta_i X_{H_i}(t, v(t)) \\
\int_0^1 H_i(t, v(t))dt &= 0 \quad 1 \leq \forall i \leq k
\end{align*} \tag{4.6} \]

In the next proposition, we observe that a critical point of \( \mathcal{A}_F^\mathcal{H} \) gives rise to a leafwise intersection point. [AF1] proved that the following proposition for the case that \( \Sigma \) is a contact hypersurface when the Rabinowitz action functional is defined with one Lagrange multiplier. We generalize their remarkable proposition using the same proof as before.

**Proposition 4.3.** Let \((v, \eta) \in \text{Crit}\mathcal{A}_F^\mathcal{H}\). Then \( x = v(0) \) satisfies \( \phi_F(x) \in \mathcal{L}_x \). Thus, \( x \) is a leafwise coisotropic intersection point.

**Proof.** Since the time support of \( F \) is \((0, \frac{1}{2})\), for \( t \geq \frac{1}{2} \) and all \( i = 1, \ldots, k \),

\[ \frac{d}{dt} H_i(t, v(t)) = dH_i(t, v(t))[\partial_t v] \]
\[ = dH_i(t, v(t))[X_F(t, v) + \sum_{j=1}^k \eta_j X_{H_j}(t, v)] \]
\[ = 0 \tag{4.7} \]
The last equality follows from the Poisson-commutativity of $H$. As in the proof of Proposition 3.3, $\int_0^1 H_i(t, v(t)) dt = 0$ implies $v(t) \in H^{-1}(0) = \Sigma$ for $t \in [\frac{1}{2}, 1]$. On the other hand $H$ has the time support on $(\frac{1}{2}, 1)$, $v$ solves the equation $\partial_t v = X_H(t, v)$ on $[0, \frac{1}{2}]$. Therefore $v(\frac{1}{2}) = \frac{1}{\sqrt{2}} v(0) = \frac{1}{\sqrt{2}} \phi_F(v(0))$ since $F = 0$ for $t \geq \frac{1}{2}$. For $t \in (\frac{1}{2}, 1)$, $\partial_t v = \sum_{i=1}^k \eta_i X_{\xi_i}(t, v)$ implies $x = v(0) = v(1) \in L_v(\frac{1}{2})$. Thus we conclude that $x \in L_{\phi_F(x)}$, this is equivalent to $\phi_F(x) \in L_x$.

From now on, we impose a $s$-dependence on $F$ as follows. Let $\{F_s\}_{s \in \mathbb{R}}$ be a family of Hamiltonian functions varies only for a finite interval in $\mathbb{R}$. More specifically, we assume $F_s(t, x) = F_{-s}(t, x)$ for $s \leq 1$ and $F_s(t, x) = F_+(t, x)$ for $s \geq 1$. We also choose a family of compatible almost complex structure $\{J(s, t)\}_{s \in \mathbb{R}}$ on $M$ such that $J(s, t)$ is invariant outside of the interval $[-1, 1]$.

**Definition 4.4.** A map $w = (v, \eta) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ which solves

$$\partial_s w(s) + \nabla_s A^H_{F_s}(w(s)) = 0. \quad (4.8)$$

is called a gradient flow line of $A^H_{F_s}$.

In the above ODE (4.8), the gradient is taken with respect to metric $m$ defined by the $L^2$-metric on the loop space with the standard metric on $\mathbb{R}^k$. To be more specific, on the tangent space $T_{(v, \eta)}(\mathcal{L} \times \mathbb{R}^k) = T_v \mathcal{L} \times T_\eta \mathbb{R}^k$ for $(v, \eta) \in \mathcal{L} \times \mathbb{R}^k$,

$$m_{(v, \eta)}((\vec{v}^1, \vec{\eta}^1), (\vec{v}^2, \vec{\eta}^2)) := \int_0^1 g_v(\vec{v}^1, \vec{v}^2) dt + \langle \vec{\eta}^1, \vec{\eta}^2 \rangle_\eta. \quad (4.9)$$

where $g(\cdot, \cdot) := \omega(\cdot, J \cdot)$ is a metric on $M$. Here $\vec{\eta}^1$ and $\vec{\eta}^2$ are elements in $T_\eta \mathbb{R}^k \cong \mathbb{R}^k$, write $\vec{\eta}^1 = (\vec{\eta}^1_1, \ldots, \vec{\eta}^1_k)$ and $\vec{\eta}^2 = (\vec{\eta}^2_1, \ldots, \vec{\eta}^2_k)$.

According to Floer’s interpretation, gradient flow equation 4.8 can be interpreted as $v(s, t) : \mathbb{R} \times S^1 \to M$ and $\eta_i(s) : \mathbb{R} \to \mathbb{R}$ solve

$$\begin{align*}
\partial_s v + J_s(t, v)(\partial_t v - \sum_{i=1}^k \eta_i X_{\xi_i}(t, v) - X_{F_s}(t, v)) &= 0 \\
\partial_s \eta_i - \int_0^1 H_i(s, t) dt &= 0 \quad 1 \leq i \leq k.
\end{align*} \quad (4.10)$$

**Definition 4.5.** The energy of a map $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ is defined as

$$E(w) := \int_{-\infty}^\infty \|\partial_s w\|^2 ds. \quad (4.11)$$

**Lemma 4.6.** Let $w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k)$ be a gradient flow line of $\nabla_s A^H_{F_s}$ with $w_\pm = w(\pm \infty)$. Then we have the following estimation.

$$E(w) \leq A^H_{F_s}(w_-) - A^H_{F_s}(w_+) + \int_{-\infty}^\infty \|\partial_s F_s\| ds. \quad (4.12)$$

Moreover, equality hold if $\partial_s F_s = 0$. 

Proof. It follows from the gradient flow equation (4.8)

\[ E(w) = -\int_{-\infty}^{\infty} dA_F^H(w(s)) (\partial_s w) ds \]

\[ = -\int_{-\infty}^{\infty} \frac{d}{ds} (A_F^H(w(s))) ds + \int_{-\infty}^{\infty} (\partial_s A_F^H)(w) ds \]

\[ = A_F^H(w_-) - A_F^H(w_+) - \int_{-\infty}^{\infty} \int_{0}^{1} \partial_s F_s(t,v) dt ds \]

\[ \leq A_F^H(w_-) - A_F^H(w_+) + \int_{-\infty}^{\infty} \|\partial_s F_s\|_- ds . \] (4.13)

□

Remark 4.7. We note that \( \int_{-\infty}^{\infty} \|\partial_s F_s\|_- ds \) has a finite value since \( \partial_s F_s \) has a compact support by construction.

Proposition 4.8. \( A_F^H \) has a uniform bound along the gradient flow lines.

Proof. For any gradient flow line \( w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \) of \( A_F^H \), we calculate

\[ 0 \leq \int_{s_1}^{s_2} \|\partial_s w\|_g^2 ds \]

\[ = -\int_{s_1}^{s_2} dA_F^H(w(s)) (\partial_s w) ds \]

\[ = A_{F,s_1}^H(w(s_1)) - A_{F,s_2}^H(w(s_2)) - \int_{s_1}^{s_2} \int_{0}^{1} \partial_s F_s(t,v) dt ds \]

\[ \leq A_{F,s_1}^H(w(s_1)) - A_{F,s_2}^H(w(s_2)) + \int_{s_1}^{s_2} \|\partial_s F_s\|_- ds . \] (4.14)

From above inequality we get

\[ A_{F,s_2}^H(w(s_2)) \leq A_{F,s_1}^H(w_-) + \int_{-\infty}^{\infty} \|\partial_s F_s\|_- ds, \]

\[ A_{F,s_1}^H(w(s_1)) \geq A_{F,s_2}^H(w_+) - \int_{-\infty}^{\infty} \|\partial_s F_s\|_- ds . \] (4.15)

Therefore for any \( s_0 \in \mathbb{R} \),

\[ |A_{F,s}^H(w(s_0))| \leq \max \{ A_{F,s}^H(w_-), -A_{F,s}^H(w_+) \} + \int_{-\infty}^{\infty} \|\partial_s F_s\|_- ds . \] (4.16)

□

4.3. Compactness of the moduli space. In this section we prove Theorem 4.9 which is one of the vital ingredients for our main results.

Theorem 4.9. Let \( \mathcal{M} \) be a moduli space of gradient flow lines of \( A_F^H \). Then this moduli space is compact modulo breaking.

Proof. Once we establish the following facts, the proof of the theorem follows from standard arguments in Floer theory. For a sequence of elements \( \{(v_n, \eta_{1,n}, \ldots, \eta_{1,n})\}_{n \in \mathbb{N}} \) in \( \mathcal{M} \), we have
(i) a uniform \( L^\infty \) bound on \( v_n \),
(ii) a uniform \( L^\infty \) bound on \( \eta_{i,n} \),
(iii) a uniform \( L^\infty \) bound on the derivatives of \( v_n \).

(i) follows from the assumption ‘convex at infinity’. Once the \( L^\infty \) bound on \( \eta_{i,n} \) is established, the \( L^\infty \) bound on the derivatives of \( v_n \) follows from the bubbling argument and the symplectic asphericity of \((M,\omega)\). Hence Theorem 4.12 finishes the proof. \( \square \)

Recall \( \alpha_i(X_{p_j}) = \delta_{ij} \), see Proposition 2.7 with next paragraph. For the following fundamental lemma we choose \( \delta_1 > 0 \), so that on \( \Sigma_{\delta_1} = p^{-1}(-\delta_1,\delta_1) = \mathcal{H}^{-1}(-\delta_1,\delta_1) \),
\[
\frac{3}{4} < \alpha_i(X_{p_i}) < \frac{5}{4} \quad \text{and} \quad \frac{1}{4(k-1)} < \alpha_i(X_{p_i}) < \frac{1}{4(k-1)}.
\] (4.17)

**Lemma 4.10.** There exist \( \epsilon > 0 \) and \( C > 0 \) such that for \( w = (v,\eta) \in \mathcal{L} \times \mathbb{R}^k \),
\[
||\nabla A^H_{F_k}(v,\eta)|| < \epsilon \quad \text{implies} \quad |\eta_i| \leq C(|\mathcal{A}^H_{F_k}(v,\eta)| + 1).
\] (4.18)

**Proof.** The proof proceeds in three steps.

**Step 1:** Assume \( v(t) \in \Sigma_\delta \) for \( t \in (0,\frac{1}{2}) \), where \( \delta = \min\{1,\delta_0/2,\delta_1\} \). Then there exists \( C_0 > 0 \) satisfying the following inequality for all \( i = 1,\ldots,k \).
\[
|\eta_i| \leq C_0(|\mathcal{A}^H_{F_k}(v,\eta)| + ||\nabla A^H_{F_k}(v,\eta)|| + 1).
\] (4.19)

Proof of Step 1. For each \( i = 1,\ldots,k \) we estimate,
\[
|\mathcal{A}^H_{F_k}(v,\eta)| = \left| \int_0^1 v^* \alpha_i + \sum_{j=1}^k \eta_j \int_0^1 H_j(v) dt + \int_0^1 F_s(v) dt \right| \\
\geq \left| \sum_{j=1}^k \eta_j \int_0^1 \alpha_i(v)(X_{H_j}(v)) dt \right| - \left| \int_0^1 \alpha_i(v)(X_{F_s}(v)) dt \right| - \left| \int_0^1 F_s(v) dt \right| \\
- \int_0^1 \alpha_i(v)(\partial_t v - \sum_{j=1}^k \eta_j X_{H_j}(v) - X_{F_s}(v)) dt \right| - \left| \sum_{j=1}^k \eta_j \int_0^1 H_j(v) dt \right| \\
\geq \eta_i \int_0^1 \alpha_i(v)(X_{H_i}(v)) dt - \left| \sum_{j \neq i} \eta_j \int_0^1 \alpha_i(v)(X_{H_j}(v)) dt \right| - \left| \int_0^1 \alpha_i(v)(X_{F_s}(v)) dt \right| \\
- C_i ||\nabla A^H_{F_k}(v,\eta)|| - \left| \sum_{j=1}^k \eta_j \int_0^1 H_j(v) dt \right| - \left| \int_0^1 F_s(v) dt \right| \\
\geq \frac{3}{4} |\eta_i| - \frac{1}{4(k-1)} \sum_{j \neq i} |\eta_j| - C_i ||\nabla A^H_{F_k}(v,\eta)|| - \frac{1}{4k} \sum_{j=1}^k |\eta_j| - C_i,F
\] (4.20)

where \( C_i := ||\alpha_i|\Sigma_\delta||_{L^\infty} < \infty \) and \( C_i,F := ||F_s||_{L^\infty} + C_i ||X_{F_s}||_{L^\infty} < \infty \).

Combine above inequality for all \( i = 1,\ldots,k \), then eventually we obtain
\[
\frac{1}{4} \sum_{i=1}^k |\eta_i| \leq k|\mathcal{A}^H_{F_k}(v,\eta)| + \sum_{i=1}^k C_i ||\nabla A^H_{F_k}(v,\eta)|| + C_i,F.
\] (4.21)
Therefore step 1 follows with $C_0 := \max\{4k, 4kC_1, \ldots, 4kC_k, 4C_i, \epsilon\}$.

**Step 2:** If there exists $t \in (\frac{1}{2}, 1)$ such that $v(t) \not\in \Sigma_\delta$ then $||\nabla_s A^H_{\Sigma_\delta}(v, \eta)|| \geq \epsilon$.

Proof of Step 2. The assumption $v(t) \not\in \Sigma_\delta$ means that there exists $i \in \{1, \ldots, k\}$ such that $v(t) \not\in \Sigma_\delta^i = H_i^{-1}(-\delta, \delta)$. If $v(t) \in M - \Sigma_\delta^i$ for all $t \in (\frac{1}{2}, 1)$ then easily we have

$$||\nabla_s A^H_{\Sigma_\delta}(v, \eta)|| \geq \left| \int_0^1 H_i(t, v(t))dt \right| = \left| \int_0^{\frac{1}{2}} H_i(t, v(t))dt \right| \geq \delta \quad (4.22)$$

Otherwise there exists $t' \in (\frac{1}{2}, 1)$ such that $v(t') \in \Sigma_\delta^i$. Thus we can find $t_0, t_1 \in (\frac{1}{2}, 1)$ such that

$\begin{align*}
v(t_0) &\in \partial \Sigma_\delta^i, v(t_1) &\in \partial \Sigma_\delta^i \\
&\ &\ &\text{&} \quad v(s) \in \Sigma_\delta^i - \Sigma_\delta^i &\text{for } \forall s \in [t_0, t_1],
\end{align*}$

(4.23)

or

$\begin{align*}
v(t_1) &\in \partial \Sigma_\delta^i, v(t_0) &\in \partial \Sigma_\delta^i \\
&\ &\ &\text{&} \quad v(s) \in \Sigma_\delta^i - \Sigma_\delta^i &\text{for } \forall s \in [t_1, t_0].
\end{align*}$

(4.24)

We treat only the first case. The later case is analogous.

With $\kappa := \max_{x \in \Sigma_\delta} ||\nabla H_i(x)||_g < \infty$ we estimate,

$$\kappa ||\nabla_s A^H_{\Sigma_\delta}(v, \eta_1, \eta_2)|| \geq \kappa ||\partial_t v - \sum_{j=1}^k \eta_j X_{H_j}(v) - X_{F_i}(v)||_{L^2}$$

$$\geq \kappa ||\partial_t v - \sum_{j=1}^k \eta_j X_{H_j}(v) - X_{F_i}(v)||_{L^1}$$

$$\geq \int_{t_0}^{t_1} ||\partial_t v - \sum_{j=1}^k \eta_j X_{H_j}(v) - X_{F_i}(v)||_g ||\nabla H_i(v(t))||_g dt$$

$$\geq \left| \int_{t_0}^{t_1} \langle \nabla H_i(v(t)), \partial_t v(t) - \sum_{j=1}^k \eta_j X_{H_j}(v) - X_{F_i}(v) \rangle dt \right|$$

(4.25)

$$= \left| \int_{t_0}^{t_1} dH_i(v(t)) (\partial_t v(t) - \sum_{j=1}^k \eta_j X_{H_j}(v) - X_{F_i}(v)) dt \right|$$

$$\geq \left| \int_{t_0}^{t_1} \frac{d}{dt} H_i(v(t)) dt - dH_i(\sum_{j=1}^k \eta_j X_{H_j}) v(t) \right|$$

$$\geq \left| H_i(v(t_1)) - H_i(v(t_0)) \right|$$

$$= \frac{\delta}{2}.$$ 

Thus Step 2 follows with $\epsilon = \min\{\frac{\delta}{2}, \frac{\delta}{2\kappa}\}$.

**Step 3:** Proof of the lemma.

Proof of Step 3. According to Step 2, $v(t) \in \Sigma_\delta$ for all $t \in (0, \frac{1}{2})$. Then Step 1 completes the proof of lemma with $C = C_0 + \epsilon + 1$. \qed
Lemma 4.11. For given a gradient flow line $w$ of $A^H_F$ and $\sigma \in \mathbb{R}$ we define
\[
\tau(\sigma) := \inf\{\tau \geq 0 \mid ||\nabla A^H_F(w(\sigma + \tau))|| \leq \epsilon\}
\] (4.26)
and
\[
C_F := \int_{-\infty}^{\infty} \int_{0}^{1} \max_{x \in M} ||\partial_s F_s(t, x)||_g dt ds < \infty.
\] (4.27)
Then we obtain a bound on $\tau(\sigma)$
\[
\tau(\sigma) \leq \frac{A^H_F(w(-\infty)) - A^H_F(w(\infty)) + C_F}{\epsilon^2}.
\] (4.28)

Proof. Compute
\[
\epsilon^2 \tau(\sigma) \leq \int_{\sigma}^{\sigma + \tau(\sigma)} ||\nabla A^H_F(w)||^2 ds
\]
\[
\leq \int_{-\infty}^{\infty} -dA^H_F(w)(\partial_s w) ds - C_F + C_F
\] (4.29)
\[
\leq \int_{-\infty}^{\infty} -\frac{d}{ds}(A^H_F(w(s))) ds + C_F
\]
\[
= A^H_F(w(-\infty)) - A^H_F(w(\infty)) + C_F
\]
We get a bound on $\tau(\sigma)$ by dividing $\epsilon^2$ in the above formula. \(\Box\)

Theorem 4.12. Given two critical points $w_-$ and $w_+$, there exists constant $\Lambda > 0$ depending only on $w_-$ and $w_+$ such that every gradient flow line $w = (v, \eta) = (v, \eta_1, \ldots, \eta_k)$ of $A^H_F$ with $w(\pm \infty) = w_{\pm}$ satisfies
\[
||\eta_i||_{L^\infty} \leq \Lambda \quad \text{for all} \ i = 1, \ldots, k.
\] (4.30)

Proof. Using Lemma 4.8 and Lemma 4.11,
\[
|\eta_i(\sigma)| \leq |\eta_i(\sigma + \tau(\sigma))| + \int_{\sigma}^{\sigma + \tau(\sigma)} ||\partial_s \eta_i(s)|| ds
\]
\[
\leq C\left(||A^H_F(w(\sigma + \tau(\sigma)))|| + 1 + \tau(\sigma)||H_i||_{L^\infty}\right)
\] (4.31)
\[
\leq C(\Upsilon(w_-, w_+) + 1) + \left(\frac{A^H_F(w_-) - A^H_F(w_+) + C_F}{\epsilon^2}\right)||H_i||_{L^\infty}.
\]
where $0 < \Upsilon(w_-, w_+) < \infty$ is a uniform bound depending only on the asymptotic ends which comes from Proposition 4.8. \(\Box\)

4.4. Proof of Theorem A. The proof proceeds in two steps. In Step 1, we prove Theorem A under the assumption that $\Sigma$ admits global Bolle’s coordinate. Then we remove this additional assumption in Step 2.

Step 1: There always exists a critical point $(v, \eta)$ of $A^H_F$ provided $||F|| < \varphi(\Sigma)$ with additional condition that $\Sigma$ is of restricted contact type with global Bolle’s coordinates. Moreover an action value of that critical point is uniformly bounded as
\[
-||F|| < A^H_F(v, \eta) < ||F||.
\] (4.32)
Proof of Step 1. For $0 \leq r$, we choose a smooth family of functions $\beta_r \in C^\infty(\mathbb{R}, [0, 1])$ satisfying
Then set $K_r(s,t,x) := \beta_r(s)F(t,x)$ where $F$ such that $\phi_F = \phi$ and $F(t,x) = 0$ for $t \in [\frac{1}{2},1]$ as before, and $||F|| \leq \varphi(\Sigma)$. We fix a point $p \in \Sigma$ and consider the moduli space

$$
\mathcal{M} := \left\{ (r, w) \in [0, \infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \mid w \text{ solves (4.10) for } K_r \right. \\
\left. \lim_{s \to -\infty} w(s) = (p,0), \lim_{s \to \infty} w(s) \in \Sigma \times \{0\} \right\}.
$$

(4.33)

We can consider the moduli space $\mathcal{M}$ as the zero set of a Fredholm section of a Banach bundle over a Banach manifold. (See Section 7.) Moreover, Fredholm section is regular and already transversal at the boundary point since the boundary is a constant curve. Therefore we can perturb a Fredholm section away from the boundary point to get a transverse Fredholm section whose zero set is an one dimensional manifold with a single boundary point $(0,p,0)$. We maintain the notation $\mathcal{M}$ for the zero locus of perturbed Fredholm section. Thus $\mathcal{M}$ itself is never compact, so there has to be an another limit point of $\mathcal{M}$ under the breaking process. To check breaking along a sequence of $\mathcal{M}$ occurs, we need an energy bound. For $(r, w) \in \mathcal{M}$, we estimate

$$
E(w) = -\int_{-\infty}^{\infty} dA^H_{\beta_r(s)F}(w(s))(\partial_s w) ds \\
\leq A^H_0(p,0) - A^H_0(p,0) + \int_{-\infty}^{\infty} ||\partial_s K_r||_s ds \\
= \int_{-\infty}^{\infty} ||\beta_r'(s)F||_s ds \\
= \int_{-\infty}^{0} \beta_r'(s)||F||_s ds - \int_{0}^{\infty} \beta_r'(s)||F||_s ds \\
= \beta_r(0)(||F||_s + ||F||_s) \\
\leq ||F||.
$$

Accordingly we can also estimate,

$$
-||F|| < A^H_{\beta_{r_n}F}(w_n(s)) < ||F|| \quad (r_n, w_n) \in \mathcal{M}.
$$

(4.35)

Due to the energy bound, a sequence $\{(r_n, w_n)\}_{n \in \mathbb{N}}$ converges (after choosing a subsequence) up to breaking, see Theorem 4.9. As $r_n$ converges, there might be

(i) a non constant gradient flow line $v$ of $A^f_0$ with one asymptotic end being $(p,0)$.

But this case is ruled out by the assumption that $||F|| < \varphi(\Sigma)$. To be precise, another asymptotic end has to be of the form $(\gamma, \eta) = (\gamma, \eta_1, \ldots, \eta_k)$ with nonzero $\eta_1 = \cdots = \eta_k$ since otherwise $E(v) = 0$. Thus we can derive the following estimate which contradicts to the definition of $\varphi(\Sigma)$.

$$
|\eta| = |A^H_0(\gamma, \eta)| = E(v) \leq \limsup E(w_n) \leq ||F|| < \varphi(\Sigma, \alpha).
$$

(4.36)
On the other hand if \( r_n \rightarrow \infty \), we have

(ii) a gradient flow line \( v \) of \( \mathcal{A}_{\beta_{\infty}^F} \).

To compactify moduli space \( \mathcal{M} \), this case has to occur, then we know that one asymptotic end of \( v \), let \( v_{\infty} \), is a critical point of \( \mathcal{A}_{\beta_{\infty}^F} \). It gives rise to a leafwise coisotropic intersection point. Furthermore we have an uniform action bound on \( v_{\infty} \) by taking limits on \( s \) and \( n \) in an inequality (4.35). It finishes the proof of Step 1.

\[ \square \]

**Step 2:** Proof of Theorem A.

Proof of Step 2. In Step 2, our restricted contact submanifold \( \Sigma \) needs not admit global Bolle’s coordinates. We consider a family of Hamiltonian functions \( H_{i,\nu} \in C^\infty(M, \mathbb{R}) \) for \( \nu \in \mathbb{N} \) such that

(i) \( \epsilon_\nu > 0 \) for all \( \nu \in \mathbb{N} \) and \( \epsilon_\nu \rightarrow 0 \) as \( \nu \) goes to infinity,

(ii) \( H_{i,\nu}|_{\Sigma_{\delta_0/2}} = r^2 - \epsilon_\nu \),

(iii) \( H_{i,\nu}|_{M-\Sigma_{\delta_0}} \) = constant,

(iv) \( H_{\nu}^{-1}(0) = \bigcup_{k} \Sigma \times (\pm \epsilon_\nu, \ldots, \pm \epsilon_\nu) = \mathbb{S}^k \bigcup \ldots \bigcup \mathbb{S}^2 \).

\( X_{H_{i,\nu}|_{\Sigma \times (\pm \epsilon_\nu, \ldots, \pm \epsilon_\nu)}} = \pm 2\epsilon_\nu R_{i,\nu} \). And we denote by \( \phi^{\nu}_{R_{i,\nu}} \), the flow of the Reeb vector field \( R_{i,\nu} \).

Then according to Proposition 4.3, we have

\[ \phi^{\nu}_{F}((v_{\nu}(\frac{1}{2})) = v_{\nu}(0) = \phi^{2\nu \eta_{i,\nu}}_{R_{1,\nu}} \circ \cdots \circ \phi_{R_{k,\nu}}^{-2\nu \eta_{k,\nu}}(v_{\nu}(\frac{1}{2})) \] (4.37)

Moreover Step 1 guarantees the existence of critical points \( (v_{\nu}, \eta_{\nu}) \). Then formula (4.32) in Step 1 implies the following lemma.

**Lemma 4.13.** \( \{2\epsilon_\nu \eta_{i,\nu}\}_{\nu \in \mathbb{N}} \) for each \( i \in \{1, \ldots, k\} \) are uniformly bounded in terms of \( \alpha_1, \ldots, \alpha_k \) and \( F \).

**Proof.** We estimate

\[ \|F\| \geq \|A_{H_{\nu}}(v_{\nu}, \eta_{\nu})\| \]

\[ = \left| \int_0^1 \nu^* \alpha_i + \int_0^1 \langle \eta, H_{\nu}\rangle(t, v_{\nu}(t))dt + \int_0^1 F(t, v_{\nu}(t))dt \right| \]

\[ = \left| \int_0^1 \alpha_i \left( \sum_{i=1}^k 2\epsilon_\nu \eta_{i,\nu} R_{i,\nu} + X_F \right) + \int_0^1 F(t, v_{\nu}(t))dt \right| \] (4.38)

\[ = |2\epsilon_\nu \eta_{i,\nu} + \int_0^1 \alpha_i(X_F) + \int_0^1 F(t, v_{\nu}(t))dt| \]

Therefore we conclude

\[ |2\epsilon_\nu \eta_{i,\nu}| \leq \|F\| + \|\alpha_i\|_{L^\infty} \cdot \|X_F\|_{L^\infty} + \|F\|_{L^\infty} \] (4.39)

\[ \square \]

Two sequences of points \( \{v_{\nu}(0)\}_{\nu \in \mathbb{N}} \) and \( \{v_{\nu}(1/2)\}_{\nu \in \mathbb{N}} \) converge and we denote by

\[ x_0 := \lim_{\nu \rightarrow \infty} v_{\nu}(0), \]

\[ x_{1/2} := \lim_{\nu \rightarrow \infty} v_{\nu}(\frac{1}{2}). \] (4.40)
Obviously $x_0$ and $x_{1/2}$ are points in $\Sigma$. Moreover we know that
\[
    x_0 = \lim_{\nu \to \infty} v_\nu(0) = \lim_{\nu \to \infty} \phi^1_F (v_\nu(\frac{1}{2})) = \phi^1_F (\lim_{\nu \to \infty} v_\nu(\frac{1}{2})) = \phi^1_F (x_{1/2}). \tag{4.41}
\]
On the other hand, due to Lemma 4.13, we have limits
\[
    \lim_{\nu \to \infty} 2\epsilon_\nu \eta_i,\nu =: n_i \quad \forall i \in \{1, \ldots, k\}. \tag{4.42}
\]
Thus we conclude that $x_0$ and $x_{1/2}$ lie on a same leaf:
\[
    x_0 = \lim_{\nu \to \infty} v_\nu(0) = \lim_{\nu \to \infty} \phi^2_{R_1,\nu} \circ \cdots \circ \phi^k_{R_k,\nu} (v_\nu(\frac{1}{2})) = \phi^{n_1}_{R_1} \circ \cdots \circ \phi^{n_k}_{R_k} (x_{1/2}). \tag{4.43}
\]
It immediately follows
\[
    \phi^{n_1}_{R_1} \circ \cdots \circ \phi^{n_k}_{R_k} (x_{1/2}) = \phi^1_F (x_{1/2}) \tag{4.44}
\]
from equation (4.41) together (4.43). It completes the proof of Theorem A. \qed

5. On unrestricted contact submanifolds

In this section we develop Rabinowitz Floer homology theory in a unrestricted contact manifold. As we already mentioned in the introduction, a unrestricted contact submanifold is notably differ from a restricted contact submanifold because an ambient symplectic manifold need not be exact, so that can be closed. We can achieve the same results for the unperturbed Rabinowitz action functional $A^H$. More precisely, although $\alpha_i$ are defined locally around $\Sigma$, we can bound the Lagrange multipliers $\eta_i$ with an aid of the auxiliary Rabinowitz action functional. However in the perturbed case $A^H_F$, we insist constraints not only on the Hofer norm of $F$ but also on the support of $X_F$. In other word, we adapt perturbations only in $\mathcal{F}$ defined in Definition 1.5. To prove Theorem B, we firstly show that the Lagrange multipliers are uniformly bounded and then we apply the proof of Theorem A with minor modification. The strategy is similar to [CFP] and [Ka].

Let us recall $\delta_0 \in \mathbb{R}$ and $\mathfrak{F}$:
\[
    \delta_0 := \sup \left\{ r \in \mathbb{R} \bigg| \psi : \Sigma_r = \{(q, p) \in \Sigma \times \mathbb{R}^k \mid |p_i| < r, \quad 1 \leq \forall i \leq k \} \cong \mathbb{R} \right\} \tag{5.1}
\]

**Definition 5.1.** We call a Hamiltonian function $F \in C^\infty (M \times S^1)$ is admissible if $\text{Supp} X_F \subset \Sigma_{\delta_2}$. Furthermore we abbreviate $\mathfrak{F}$ the set of all admissible Hamiltonian functions.

For given perturbation $F \in \mathfrak{F}$, choose $\delta_2 < \delta_0$ such that $F$ is constant outside the region $\text{cl}(\Sigma_{\delta_2})$. Then we modify Hamiltonian functions $H_i(t, x) = \chi(t) p_i(x)$ to
\[
    \tilde{H}_i : M \to \mathbb{R} \quad \text{by} \quad \tilde{H}_i = \left\{ \begin{array}{ll}
    H_i & \text{locally constant on } \Sigma_{\delta_2/2} \\
    \text{outside } & \Sigma_{\delta_2}.
    \end{array} \right. \tag{5.2}
\]

We denote $\tilde{H}_i$ by $H_i$ again for convenience. Indeed the behavior of $H_i$ outside of small neighborhood of $\Sigma$ is not important. Now we introduce a cut-off function to extend stable one forms globally. For $\delta_3 < \delta_2$
\[
    (i) \quad \rho(r) = \frac{1}{r} \quad r \in [-\delta_2, \delta_2],
\]
\[
    (ii) \quad \text{Supp}(\rho) \subset (-\delta_0, \delta_0),
\]
\[
    (iii) \quad \rho'(r) \leq 1 + \epsilon \quad r \in \mathbb{R} \quad \text{for some } \epsilon > 0 \text{ satisfying}
\]
\[
    \frac{1 - \delta_2}{\delta_0 - \delta_2} < 1 + \epsilon. \tag{5.3}
\]
Furthermore we will use the difference of two action functionals: $J$ as interval to cutoff with any slope. Then we obtain a global one form

$$
β(y) := \begin{cases} 
\sum_{i=1}^k \rho(r_i)\alpha_i(x) & y = ψ(x, r_1, \ldots, r_k) \in \Sigma \times (-δ_0, δ_0)^k \\
0 & y \in M - (\Sigma \times (-δ_0, δ_0)^k)
\end{cases}
$$

(5.4)

Furthermore we adapt an almost complex structure $J$ on $M$ which splits on $Σ_{δ_0}$ with respect to

$$
TΣ_{δ_0} = \left( \bigcap_{i=1}^k \ker α_i \right) \bigoplus \left( \ker ω \oplus \frac{∂}{∂p_1} \oplus \cdots \oplus \frac{∂}{∂p_k} \right)
$$

as $J|_Σ$ is an almost complex structure which interchanges the Reeb vector fields $R_i$ with $\frac{∂}{∂p_i}$ for $1 \leq i \leq k$.

**Proposition 5.2.** For every $v \in TM$, the following inequality holds

$$
dβ(v, Jv) \leq (1 + ε)ω(v, Jv).
$$

(5.6)

**Proof.** Outside of $Σ_{δ_0}$, the inequality is obvious since $dβ$ vanishes by construction but $ω(\cdot, J\cdot) = g(\cdot, \cdot)$ is positive definite. For $v \in TΣ_{δ_0}$, we can write $v = v_1 + v_2$ with respect to the decomposition $TΣ_{δ_0} = (\bigcap_{i=1}^k \ker α_i) \bigoplus Ξ$ as above. Recall that we have chosen $φ_i$ as $φ_i(p_i) ≤ p_i + \frac{1}{k}$ and $φ'_i(p_i) ≤ 1 + ε$ so that

$$
dβ(v, Jv) \leq \sum_{i=1}^k φ_i(p_i) ω_Σ(v_1, Jv_1) + \sum_{i=1}^k φ'_i dp_i \wedge α_i(v_2, Jv_2)
$$

$$
\leq \sum_{i=1}^k \left( p_i + \frac{1}{k} \right) ω_Σ(v_1, Jv_1) + (1 + ε) \sum_{i=1}^k dp_i \wedge α_i(v_2, Jv_2)
$$

$$
\leq (1 + ε)\left( \sum_{i=1}^k \left( p_i + \frac{1}{k} \right) ω_Σ(v_1, Jv_1) + \sum_{i=1}^k dp_i \wedge α_i(v_2, Jv_2) \right)
$$

$$
= (1 + ε)(ω_Σ + \sum_{i=1}^k d(p_i α_i))(v, Jv)
$$

$$
= (1 + ε)ω(v, Jv).
$$

(5.7)

This inequality finishes the proof. □

We defined a bilinear form $\hat{m}$ on $T(\mathcal{L} \times \mathbb{R}^k)$ which is not necessarily positive definite.

$$
\hat{m}( (\hat{v}^1, \hat{η}^1), (\hat{v}^2, \hat{η}^2) ) := \int_0^1 dβ(\hat{v}^1, J\hat{v}^2)dt + \hat{η}^1\hat{η}^2.
$$

(5.8)

Using $β$, we also define an auxiliary Rabinowitz action functional:

$$
\hat{A}^H_{F_⁺}(v, η) = -\int_{D^2} \hat{v}^*dβ - \int_0^1 F_⁺(t, v(t))dt - \int_0^1 \langle η, \mathcal{H} \rangle(t, v(t))dt.
$$

(5.9)

Furthermore we will use the difference of two action functionals:

$$
\mathcal{A} := \hat{A}^H_{F_⁺} - A^H_{F_⁺} = \int_{D^2} \hat{v}^*(ω - dβ).
$$

(5.10)
Proposition 5.3. If \((v, \eta) \in \mathcal{L} \times \mathbb{R}^k\) and \((\dot{v}, \dot{\eta}) \in T_{(v, \eta)}(\mathcal{L} \times \mathbb{R}^k) = \Gamma(S^1, v^* TM) \times \mathbb{R}^k\) then the following assertion holds
\[
d\widehat{A}_F^H(v, \eta)(\dot{v}, \dot{\eta}) = \widehat{m}\left(\nabla_m A_F^H(v, \eta), (\dot{v}, \dot{\eta})\right).
\] (5.11)

Proof. On the region \(\text{cl}(\Sigma_{\delta_2})\), \(\varphi_1(p_t)\) is equal to \(p_t + \frac{1}{\delta_2}\), thus \(\omega = d\beta\) follows. On the other hand we constructed \(X_{H_f}\) and \(X_{F}\) to vanish outside of \(\text{cl}(\Sigma_{\delta_2})\). Therefore we have \(i_{X_H}\omega = i_{X_H}d\beta\) and \(i_{X_F}\omega = i_{X_F}d\beta\) on \(T_{(v, \eta)}(\mathcal{L} \times \mathbb{R}^k)\). Now the following simple calculation finishes the proof of the assertion.

\[
d\widehat{A}_F^H(v, \eta)(\dot{v}, \dot{\eta}) = \int_0^1 d\beta(\dot{v}, J\dot{v}) - \omega(\sum_{i=1}^k \eta_i X_{H_i}(v) + X_{F}(v), J\dot{v})dt + \int_0^1 \langle \dot{\eta}, \mathcal{H}(v)\rangle dt
\]
\[
= \int_0^1 d\beta(\dot{v} - \sum_{i=1}^k \eta_i X_{H_i}(v) - X_{F}(v), J\dot{v})dt + \int_0^1 \langle \dot{\eta}, \mathcal{H}(v)\rangle dt
\]
\[
= \widehat{m}\left(\nabla_m A_F^H(v, \eta), (\dot{v}, \dot{\eta})\right).
\] (5.12)

\[\square\]

Proposition 5.4. Along a gradient flow line \(w = (v, \eta)\) of \(A_F^H\), the following inequality holds
\[
|A(w(s))| \leq \max\{|A(w_+)|, |A(w_-)|\} + \varepsilon E(w).
\] (5.13)

where \(w_\pm = w(\pm \infty)\).

Proof. Using Proposition 5.2 and Proposition 5.3,
\[
\frac{d}{ds} A(w) = d\widehat{A}_F^H(w)(\partial_s w) - d\widehat{A}_F^H(w)(\partial_s w) + \partial_s \widehat{A}_F^H(w) - \partial_s A_F^H(w)
\]
\[
= \int_0^1 (\omega - d\beta)(\partial_s v, J\partial_s v)dt + \int_0^1 \partial_s F_s dt - \int_0^1 \partial_s F_s dt
\]
\[
\geq - \int_0^1 \varepsilon \omega(\partial_s v, J\partial_s v)dt.
\] (5.14)

Integrate both side of (5.14) with respect to \(s\) from \(-\infty\) to \(s_0\)
\[
A(w(s_0)) - A(w_-) = \int_{-\infty}^{s_0} \frac{d}{ds} A(w(s)) ds
\]
\[
\geq -\varepsilon \int_{-\infty}^{s_0} \int_0^1 \omega(\partial_s v, J\partial_s v)dt ds
\]
\[
\geq -\varepsilon E(w).
\] (5.15)

Similar way, we have
\[
A(w(s_0)) - A(w_+) \leq \varepsilon E(w).
\] (5.16)

The Proposition follows by combining above two inequalities. \(\square\)

Proposition 5.5. \(\widehat{A}_F^H\) has a uniform bound along gradient flow lines of \(A_F^H\).

Proof. By the definition of \(A\), we know
\[
|\widehat{A}_F^H(w(s))| \leq |A_F^H(w(s))| + |A(w(s))|.
\] (5.17)
But the uniform bound of the righthand side of the above equation follows from the Proposition 5.4 and Proposition 5.4.

The next task is to find a uniform bound on \( \eta_i \) for \( i = 1, \ldots, k \). Since we already know that \( \mathcal{A}_F^H \) is uniformly bounded, the argument about obtaining a uniform bound on Lagrange multipliers is analogous to the case of restricted contact type.

**Lemma 5.6.** There exist \( \epsilon > 0 \) and \( C > 0 \) such that for \( w = (v, \eta) \in \mathcal{L} \times \mathbb{R}^k \) a gradient flow line of \( \mathcal{A}_F^H \),
\[
||\nabla \mathcal{A}_F^H(v, \eta)|| < \epsilon \quad \text{implies} \quad |\eta_i| \leq C(|\hat{\mathcal{A}}_F^H(v, \eta)| + 1).
\]

**Proof.** We reformulate Step 1 in Lemma 4.10.

**Step 1:** Assume \( v(t) \in \Sigma_{\hat{\delta}} \) for \( t \in (0, \frac{1}{2}) \), where \( \hat{\delta} = \min\{1, \delta_1, \frac{24}{7} \} \). Then there exists \( C_0 > 0 \) satisfying the following inequality for all \( i = 1, \ldots, k \).
\[
|\eta_i| \leq C_0(|\hat{\mathcal{A}}_F^H(v, \eta)| + ||\nabla \mathcal{A}_F^H(v, \eta)|| + 1).
\]

Start Step 1 with the \( \hat{\mathcal{A}}_F^H(v, \eta) \) but this is equal to \( \mathcal{A}_F^H(v, \eta) \) since we assumed \( v(t) \) remains \( \Sigma_{\hat{\delta}} \) where \( \omega \) equals to \( d\beta \).

\[
|\hat{\mathcal{A}}_F^H(v, \eta)| = \left| \int_0^1 v^* \beta - \sum_{j=1}^k \eta_j \int_0^1 H_j(v) dt - \int_0^1 F_s(v) dt \right|
= \left| \int_0^1 v^* \omega - \sum_{j=1}^k \eta_j \int_0^1 H_j(v) dt - \int_0^1 F_s(v) dt \right|
\geq \frac{3}{4} |\eta_i| - \frac{1}{4k} \sum_{j \neq i} |\eta_j| - C_1 ||\nabla \mathcal{A}_F^H(v, \eta)|| - \frac{1}{4(k - 1)} \sum_{j=1}^k |\eta_j| - C,F
\]

with \( C_i := ||\alpha_i||_{\Sigma_{\hat{\delta}}} < \infty \) and \( C,F := ||F_s||_{L^\infty} + C||X_{F_s}||_{L^\infty} < \infty \).

Combine above inequality for all \( i = 1, \ldots, k \), then eventually we get the inequality 5.19 in the same way as we proved in the restricted case Lemma 4.10. The same arguments Step 2 and Step 3 in Lemma 4.10 completes the proof of lemma.

**Remark 5.7.** The reason why we introduce the auxiliary Rabinowitz action functional is that we cannot achieve Step 1 in Lemma 5.6 with the ordinary Rabinowitz action functional. More precisely, since the one form \( \alpha_i \) is not globally defined, thus we do not have the equality \( \int_{\mathcal{D}^2} v^* \omega = \int_0^1 v^* \alpha_i \).

**Lemma 5.8.** We have a bound on \( \tau(\sigma) \) for \( \sigma \in \mathbb{R} \) as follows:
\[
\tau(\sigma) \leq \frac{\mathcal{A}^H_F(w(-\infty)) - \mathcal{A}^H_F(w(\infty))}{\epsilon^2} + C_F.
\]

**Proof.** Exactly same as before. (See Proposition 4.11.)

**Theorem 5.9.** Given two critical points \( w_- \) and \( w_+ \), there exists constant \( \Lambda > 0 \) depending only on \( w_- \), \( w_+ \) such that every gradient flow line \( w = (v, \eta) = (v, \eta_1, \ldots, \eta_k) \) of \( \mathcal{A}_F^H \) satisfies
\[
||\eta_i||_{L^\infty} \leq \Lambda \quad \text{for all} \quad i = 1, \ldots, k.
\]
Proof. Using Proposition 5.5, Lemma 5.6 and Lemma 5.8,
\[ |\eta_2(\sigma)| \leq |\eta_1(\sigma + \tau(\sigma)) + \int_{\sigma}^{\sigma + \tau(\sigma)} |\partial_s \eta_1(s)|ds \]
\[ \leq C(|\hat{A}^H_{F_1}(w)| + 1) + \tau(\sigma)||H_i||_{L^\infty} \]
\[ \leq C(\hat{\Phi}(w_-, w_+) + 1) + \left( \frac{A^H_{F_1}(w_-) - A^H_{F_1}(w_+) + C_F}{\epsilon^2} \right)||H_i||_{L^\infty}. \]
(5.23)
where \(0 < \hat{\Phi}(w_-, w_+) < \infty\) is a uniform bound depending only on the asymptotic ends which comes from Proposition 5.5.

5.1. Proof of Theorem B. The strategy for Theorem B is to modify the proof of Theorem A to make it suitable for the unrestricted case. Once we prove Theorem B for unrestricted contact submanifold with global Bolle’s coordinates, the argument in Step 2 in Theorem A completes the proof of Theorem B.

If we are able to rule out the case (recall proof of Theorem A) that

(i) a non constant gradient flow line \(v\) of \(A^H_0\) with one asymptotic end being \((p, 0)\),

then the argument in the proof of Theorem A finishes the proof. The only difference comes from inequality (4.36) since we do not have inequality (4.36) in unrestricted case because of the reason that we mentioned in Remark 5.7. But we overcome this obstacle with the auxiliary Rabinowitz action functional.

\[ |\eta| = |\hat{A}^H_0(\gamma, \eta)| \]
\[ \leq |A^H_0(\gamma, \eta)| + |A(\gamma, \eta)| \]
\[ = E(v) + \left| \int_{-\infty}^{\infty} \frac{d}{ds}A(v(s))ds \right| \]
\[ \leq \limsup_{n \in \mathbb{N}} E(w_n) + \limsup_{n \in \mathbb{N}} \left| \int_{-\infty}^{s_n} \frac{d}{ds}A(w_n(s))ds \right| \]
\[ \leq ||F|| + \limsup_{n \in \mathbb{N}} \left| \max_{s \in \mathbb{R}} A(w_n(s)) - A(p, 0) \right| \]
\[ \leq ||F|| + \limsup_{n \in \mathbb{N}} \left( \max_{s \in \mathbb{R}} \{ |A(w_{+\infty}), |A(w_{-\infty})| \} + \epsilon E(w) \right) \]
\[ \leq ||F|| + \epsilon ||F||. \]
(5.24)
where \(\{s_n\}_{n \in \mathbb{N}}\) is a sequence such that \(w_n(s_n)\) converge to \(v(\infty)\).

But \(|\eta| \leq (1 + \epsilon)||F||\), thus we can prove the claim by contradiction with a hypothesis \(||F|| \leq \frac{\delta_0 - \delta_2}{1 + \epsilon} \varphi(\Sigma, \lambda)\) for all \(\varphi\) satisfying \(\frac{1 - \delta_0}{\delta_0 - \delta_2} < 1 + \epsilon\). Taking the limit \(\epsilon \rightarrow \frac{1 - \delta_0}{\delta_0 - \delta_2} - 1\), we can deduce a contradiction with the original assumption \(||F|| < \frac{\delta_0 - \delta_2}{1 - \delta_2} \varphi(\Sigma, \lambda)\). Therefore such a case does not occur, the proof is completed.

5.2. Proof of Corollary B. Let cl(Supp\(X_F\)) be a compact subset of the \((\mathbb{C}^*)^n \cong T^n \times (-1, \infty)^n\). Recall the Bolle’s coordinate function \(p_i : \text{cl}(\text{Supp}X_F) \rightarrow (-1, \infty),\) then we
denote by
\[
\varrho^- := \min_{i,x} p_i(x) \quad \text{and} \quad \varrho^+ := \max_{i,x} p_i(x).
\]

We change the cut off functions as \( \varphi_i : \mathbb{R} \to \mathbb{R} \) and \( \text{supp} \varphi \subset [\varrho^- - \epsilon, \varrho^+ + \epsilon] \) for any small \( \epsilon > 0 \) such that \( \varphi(r) = r - |\varrho^-| \) on \([\varrho^-, \varrho^+]\) and \( \varphi'(r) \leq 1 \) for all \( r \in \mathbb{R} \). Then we have a global one form \( \beta = \sum_{j=1}^n \varphi_j(p_j) \alpha_j \). And we also any \( \delta_3 > 0 \) satisfying \( T_{\delta_3}^n \subset T^n \times [\varrho^-, \varrho^+]^n \).

Furthermore define action functionals \( A_{F_s}^H, \hat{A}_{F_s}^H \) and \( A \) with the new \( \beta \) as before.

**Proposition 5.10.** If \((v, \eta) \in \mathcal{L} \times \mathbb{R} \) and \((\hat{v}, \hat{\eta}) \in T_{(v,\eta)}(\mathcal{L} \times \mathbb{R})\) then the following two assertions hold

\begin{itemize}
  \item[(i)] \( d\hat{A}_{F_s}^H(v, \eta)(\hat{v}, \hat{\eta}) = \hat{m}(\nabla_m A_{F_s}^H(v, \eta), (\hat{v}, \hat{\eta})) \),
  \item[(ii)] \( (m - \hat{m})((\hat{v}, \hat{\eta}), (\hat{v}, \hat{\eta})) \geq 0 \).
\end{itemize}

**Proof.** Proof of assertions (i) and (ii) are almost same as Proposition 5.3 and Proposition 5.2 respectively. We have chosen \( \varphi_i \) so that \( \varphi_i'(p_i) = 1 \) on \([\varrho^-, \varrho^+]\) and we have \( \varphi_i(r) < r + 1, \varphi_i'(r) \leq 1 \) for all \( r \in \mathbb{R} \). Therefore one can easily check (i) and (ii).

**Corollary 5.11.** The functional \( A := \hat{A}_{F_s}^H - A_{F_s}^H \) is nondecreasing along gradient flow lines of \( A_{F_s}^H \).

**Proof.** Using Proposition 5.10, we estimate the following inequality.

\[
\frac{d}{ds} A(w(s)) = \frac{d}{ds} \left( \hat{A}_{F_s}^H(w(s)) \right) - \frac{d}{ds} \left( A_{F_s}^H(w(s)) \right)
= d\hat{A}_{F_s}^H(w) (\partial_s w) + (\partial_s \hat{A}_{F_s}^H)(w) - dA_{F_s}^H(w) (\partial_s w) - (\partial_s A_{F_s}^H)(w)
= m(\nabla_m A_{F_s}^H(w), \nabla_m A_{F_s}^H(w)) - \hat{m}(\nabla_m A_{F_s}^H(w), \nabla_m A_{F_s}^H(w))
+ \int_0^1 \partial_s F_s(t, v)dt - \int_0^1 \partial_s F_s(t, v)dt
\geq 0
\]

**Corollary 5.12.** Under the set-up of the proof of Theorem A, \( A(w(s)) \equiv 0 \) for all \( (r, w) \in \hat{\mathcal{M}} \).

**Proof.** We note that \( A(w(+\infty)) = A(w(-\infty)) = 0 \) since \( w(\pm \infty) \) is constant. Therefore the proof immediately follows from the previous corollary.

In the case that \( A_{F_s}^H = A_{\beta_r(s)F}^H \) as in the proof of Theorem A, we can also deduce Theorem 4.9 and apply Theorem A.

**Proposition 5.13.** Assume \((r, w) = (r, v, \eta) \in \hat{\mathcal{M}} \). Then \( v \in C^\infty(\mathbb{R}, \mathcal{L}) \) remains in \( T^n \times [\varrho^-, \varrho^+]^n \).

**Proof.** We are going to investigate the case that \( w(s, t) = (v, \eta) \) goes out of the region \( T^n \times [\varrho^-, \varrho^+]^n \). Assume that \( v(s, t) \) does not lie in \( T^n \times [\varrho^-, \varrho^+ + \epsilon]^n \) for \( s_- < s < s_+ \). It means that there exists a nonempty open subset \( U \subset \mathcal{Z} := (s_-, s_+) \times S^1 \) such that \( v(s, t) \in T^n \times ((-1, \varrho^- - \epsilon) \cup (\varrho^+ + \epsilon, \infty))^n \) for \( (s, t) \in U \).
Using the previous corollary, we calculate

\[
0 = \int_{s_-}^{s_+} \frac{d}{ds} A(w(s)) = \int_{s_-}^{s_+} \int_0^1 (\omega - d\beta)(\partial_s v, J(v)\partial_s v) dt ds
\]

\[
= \int_{Z-U} (\omega - d\beta)(\partial_s v, J(v)\partial_s v) dt ds + \int_U \omega(\partial_s v, J(v)\partial_s v) dt ds.
\]

But \((\omega - d\beta)(\partial_s v, J(v)\partial_s v)\) is bigger or equal to zero and \(\int_U \omega(\partial_s v, J(v)\partial_s v) dt ds > 0\). Thus this case can not occur and every gradient flow line of \(A_{HS}\), satisfying \(w(\pm \infty) = (p,0)\) lies in \(T^n \times [\varrho^- - \epsilon,\varrho^+ + \epsilon]^n\). Taking the limit \(\epsilon \to 0\), this finishes the proof of proposition.

\[\square\]

**End of the proof of Theorem C.** The previous proposition helps us to overcome the problem about \(L^\infty\)-bound on \(v\) and \(L^\infty\)-bound on the derivatives of \(v\), although the symplectization of \(\Sigma\) is not convex at infinity. The \(L^\infty\) bound on \(\eta\) is almost the same as what we showed (e.g Proposition 5.5, Lemma 5.6 and Lemma 5.8.) therefore, Theorem 4.9 follows.

Hence Theorem A guarantees the existence of a leafwise intersection point.

\[\square\]

For later purpose we state the following corollary. Due to this corollary we can define the Rabinowitz Floer homology even on the unrestricted contact type manifold.

**Corollary 5.14.** Let \(A^H\) be a Rabinowitz action functional defined on the symplectically aspherical and convex at infinity symplectic manifold \(M\) with an unrestricted contact type manifold \(\Sigma\). Then the moduli space of gradient flow lines of \(A^H\) is compact modulo breaking.

**Proof.** Choose any \(\delta_3 < \delta_2\) and modify \(k\) cut-off functions, namely \(\varphi_1'(p_1),\ldots,\varphi_k'(p_k)\) such that \(\varphi_i'(p_i) = p_i + \frac{1}{2k}\) on \(\Sigma_{\delta_3}\) with \(\varphi_i'(p_i) \leq 1\). With these constructions, Proposition 5.10 and Corollary 5.11 still hold. Therefore we can find an uniform bound of the Lagrangian multipliers by the same procedure as before. (e.g Proposition 5.5, Lemma 5.6 and Lemma 5.8.)

\[\square\]

### 6. On stable submanifolds

In this section we consider a stable submanifold with global Bolle’s coordinates \(\Sigma\) with its stable one forms \(\alpha_1,\ldots,\alpha_k\). In the last of the proof of Theorem E, the additional assumption about global Bolle’s coordinates will be removed. The inequality in Theorem E was first shown by Schlenk [Schl] for stable hypersurfaces. His result was reproved by Cieliebak-Frauenfelder-Paternain [CFP] by using the Rabinowitz action functional. Their proof enable us to obtain Theorem E for higher codimension.

At first, we consider the following quantity.

\[
\kappa(\alpha_i, J) := \sup_{\theta \neq v \in \xi} \frac{|d\alpha_i(v, Jv)|}{\omega(v, Jv)} \geq 0.
\]

where \(\xi = \bigcap \ker \alpha_i\). We easily notice the property of \(\kappa\) that

\[
\kappa(c\alpha_i, J) = c \kappa(\alpha_i, J) \quad c \in \mathbb{R}.
\]

(6.1)
We also consider another quantity that is
\[ \vartheta(\alpha_1, \ldots, \alpha_k) := \sup \{ r \in \mathbb{R} | \psi : \Sigma \times (-r, r)^k \to M \}. \]  
(6.3)

It also has a similar behavior about scaling
\[ \vartheta(c\alpha_1, \ldots, c\alpha_k) = \frac{1}{c} \vartheta(\alpha_1, \ldots, \alpha_k) \quad c \in \mathbb{R}. \]  
(6.4)

Next, we scale our stable one forms so that
\[ \kappa < 1 \quad \& \quad \vartheta > 1. \]  
(6.5)

For the sake of convenience, we maintain notations \( \alpha_1, \ldots, \alpha_k \) after scaled.

Now we introduce a cut-off function to extend stable one forms globally. For \( \theta \in (1, \vartheta) \)
\[
\begin{align*}
(i) \quad \rho(r) &= r + \frac{1}{\vartheta}, \quad r \in [-\theta, \theta], \\
(ii) \quad \rho'(r) &\leq 1 \quad r \in \mathbb{R}, \\
(iii) \quad \text{Supp}(\rho) &\subset (-\theta, \theta).
\end{align*}
\]

Then we obtain a global one form \( \beta \). (different from \( \beta \) in section 5.)
\[
\beta(y) := \begin{cases} 
\sum_{i=1}^{k} \rho(r_i) \alpha_i(x) & y = \psi(x, r_1, \ldots, r_k) \in \Sigma \times [-\theta, \theta]^k \\
0 & y \not\in M - (\Sigma \times [-\theta, \theta]^k)
\end{cases}
\]  
(6.6)

Using this one form \( \beta \), we define an auxiliary action functionals and bilinear form like before.
\[
\hat{A}^H(v, \eta) := -\int_{D^2} \tilde{v}^* d\beta - \int_0^1 \langle \eta, \mathcal{H}(t, v) \rangle dt
\]  
(6.7)
\[
\mathcal{A} := \hat{A}^H - A^H
\]  
(6.8)
\[
\hat{m}(\hat{v}_1, \hat{\eta}_1, \hat{v}_2, \hat{\eta}_2) := \int_0^1 d\beta(\hat{v}_1, J\hat{v}_2) dt + \langle \hat{\eta}_1, \hat{\eta}_2 \rangle
\]  
(6.9)

**Proposition 6.1.** The following two assertions hold for \( (\hat{v}, \hat{\eta}) \in T_{(v, \eta)}(\mathcal{L} \times \mathbb{R}^k) \).
\[
(i) \quad dA^H_{(v, \eta)}(\hat{v}, \hat{\eta}) = \hat{m} \left( \nabla_m A^H_{(v, \eta)}, (\hat{v}, \hat{\eta}) \right), \\
(ii) \quad (m - \hat{m})(\hat{v}, \hat{\eta}) \geq 0.
\]

**Proof.** For \( v \in T_y(\Sigma \times [-\theta, \theta]^k) \), we write \( v = v_1 + v_2 \) with respect to the decomposition
\[ T_y(\Sigma \times [-\theta, \theta]^k) = \xi_y \oplus \xi^\circ_y. \]  
(6.10)
According to (6) the construction of \( \rho \) with (6.5), we estimate

\[
d\beta(v, Jv) = \sum_{i=1}^{k} \rho(r_i) d\alpha_i + \sum_{i=1}^{k} \rho'(r_i) dr_i \wedge \alpha_i
\]

\[
\leq \max \left\{ \sum_{i=1}^{k} (r_i + \frac{1}{k}) d\alpha_i(v_1, J_\xi v_1), 0 \right\} + \omega(v_2, J_\xi v_2)
\]

\[
\leq \max \left\{ (\omega_\Sigma + \sum_{i=1}^{k} r_i d\alpha_i)(v_1, J_\xi v_1), 0 \right\} + \omega(v_2, J_\xi v_2)
\]

\[
\leq \omega(v_1, J_\xi v_1) + \omega(v_2, J_\xi v_2)
\]

\[
= \omega(v, Jv).
\]

Outside of \( \Sigma \times [-\theta, \theta]^k \), \( d\beta \) vanishes but \( \omega \) is positive definite. Therefore we conclude the lemma. \( \square \)

**Corollary 6.2.** The action functional \( A \) is non-increasing along gradient flow lines of \( A^H \).

**Proof.** Using the previous proposition, for \( w \) a gradient flow line of \( A^H \) we estimate

\[
\frac{d}{ds} A(w(s)) = dA^H(w)[\partial_s w] - dA^H(w)[\partial_s w]
\]

\[
= m(\nabla_m A^H(w), \nabla_m A^H(w)) - \tilde{m}(\nabla_m A^H(w), \nabla_m A^H(w))
\]

\[
\geq 0
\]

The corollary immediately follows from the above calculation. \( \square \)

6.1. **Proof of Theorem E.** We denote by

\[
\mathcal{G}(H) := \text{cl}\{ x \in M \mid X_{H_i}(x) \neq 0, \ldots, X_{H_k}(x) \neq 0 \}.
\]

**Lemma 6.3.** If \( \Sigma \) is displaced by \( \phi_F \), then there exist a constant \( \mu > 0 \) such that for any \( (v, \eta) \in \mathcal{L} \times \mathbb{R}^k \),

\[
\| \nabla A^H_F(v, \eta) \|_J \geq \mu.
\]

**Proof.** Define a norm on \( T_x M \) by for \( \hat{x} \in T_x M \)

\[
\| \hat{x} \|_{J,F} := \min_{t \in [0,1]} \| d\phi_F(x)[\hat{x}] \|_J
\]

and denote by associate metric \( d_{1,F} : M \times M \rightarrow \mathbb{R}_{\geq 0} \). We push forward this metric by \( \phi_F \) and obtain another one \( d_{2,F} = (\phi_F)_* d_{1,F} \). Since \( \mathcal{G} \) is compact, we know

\[
d_{2,F}^2(\mathcal{G}(H), \phi_F(\mathcal{G}(H))) =: \varepsilon_0 > 0.
\]

The following three steps complete the proof.

**Step 1.** If \( (v, \eta) \in \mathcal{L} \times \mathbb{R}^k \) satisfies

\[
\| \partial_t v - \sum \eta_i X_{H_i}(v) - X_F \| < \varepsilon_0
\]

then \( (v(0), v\left(\frac{1}{2}\right)) \notin \mathcal{G}(H) \times \mathcal{G}(H) \).

**Proof of Step 1.** We argue by contradiction with assumption \( (v(0), v\left(\frac{1}{2}\right)) \in \mathcal{G}(H) \times \mathcal{G}(H) \). Define a path

\[
v_F(t) := \phi_F^{-\frac{t+1}{2}}(v(t + \frac{1}{2})).
\]

(6.17)
Differentiate this equation and we get
\[ \partial_t v_F = \frac{1}{2} d\phi_F^{-1}(v)(\partial_t v - X_F(v)). \] (6.18)

We easily notice that
\[ (v_F(0), v_F(1)) \in \mathcal{G}(\mathcal{H}) \times \phi_F^{-1}(\mathcal{G}(\mathcal{H})). \] (6.19)

Now we deduce a contradictive inequality and it finishes the proof of Step 1.
\[
\varepsilon_0 \leq d^{2}_{J,F}(\phi_F(v_F(0)), \phi_F(v_F(1))) \\
= d^{1}_{J,F}(v_F(0), v_F(1)) \\
\leq \int_{0}^{1} ||\partial_t v_F||_{J,F} dt \\
= \int_{0}^{1} ||d\phi_F^{-1}(v)(\partial_t v - X_F(v))||_{J,F} dt \\
= \int_{0}^{1} ||\partial_t v - X_F(v)||_{J,F} dt \\
\leq ||\partial_t v - \sum_{i} \eta_i X_{H_i}(v) - X_F(v)||_{2} \\
< \varepsilon_0.
\] (6.20)

**Step 2.** The following assertion holds with some \( \varepsilon, \delta > 0 \): For all \( t \in [0, \frac{1}{2}] \),
\[
|H_i(v(t))| \geq \frac{\delta}{2} \quad i \in \{1, \ldots, k\}
\] (6.21)
provided that
\[
||\partial_t v - \sum_{i=1}^{k} \eta_i X_{H_i}(v) - X_F(v)||_{2} < \varepsilon.
\] (6.22)

Proof of Step 2. By construction of \( H_1, \ldots, H_k \) we can define
\[
\inf_{M \in \mathcal{G}(\mathcal{H})} \{H_1, \ldots, H_k\} =: \delta > 0.
\] (6.23)

On the other hand, for \( t_0, t_1 \in [0, \frac{1}{2}] \) and \( \kappa = ||\nabla H_i||_{g} \),
\[
||\partial_t v - \sum_{i=1}^{k} \eta_i X_{H_i}(v) - X_F(v)||_{2} \geq \frac{1}{\kappa} \int_{t_0}^{t_{1}} ||\partial_t v - \sum_{i=1}^{k} \eta_i X_{H_i}(v) - X_F(v)||_{g} ||\nabla H_i||_{g} dt \\
\geq \frac{1}{\kappa} \int_{t_0}^{t_{1}} \left( \nabla H_i(v(t)), \partial_t v - \sum_{i=1}^{k} \eta_i X_{H_i}(v) - X_F(v) \right) dt \] (6.24)
\[
= \frac{1}{\kappa} \int_{t_0}^{t_{1}} \frac{d}{dt} H_i(v(t)) dt \\
= \frac{1}{\kappa} |H_i(v(t_1)) - H_i(v(t_0))|.
\]

Now, we set
\[
\varepsilon := \min\{\varepsilon_0, \frac{\delta}{2\kappa}\}.
\] (6.25)
Thus we know that from Step 1 and the definition of $\epsilon$,

$$\max_{t \in [0, \frac{1}{2}]} |H_i(v(t))| \geq \delta$$  \hspace{1cm} (6.26)$$

and

$$\max_{t_1, t_1 \in [0, \frac{1}{2}]} |H_i(v(t_0)) - H_i(v(t_1))| \leq \frac{\delta}{2}.$$  \hspace{1cm} (6.27)$$

Above two inequalities reveal that

$$|H_i(v(t))| \geq \frac{\delta}{2} \quad t \in [0, \frac{1}{2}], \quad i \in \{1, \ldots, k\}$$  \hspace{1cm} (6.28)$$

and it finishes the proof of Step 2.

**Step 3**: Proof of the lemma.

Proof of Step 3. Recall the formula of $\nabla A_{HF}$.

$$\nabla A_{HF}^H = \begin{pmatrix}
\partial_t v - \sum \eta_i X_{H_i}(v) - X_F(v) \\
- \int_0^1 H_1(t, v)dt \\
\vdots \\
- \int_0^1 H_k(t, v)dt
\end{pmatrix}$$

With $\epsilon, \delta$ what we defined in Step 2, we set

$$\mu := \min\{\epsilon, \frac{\delta}{2}\}.$$  \hspace{1cm} (6.29)$$

If $||\partial_t v - \sum \eta_i X_{H_i}(v) - X_F(v)|| \geq \mu$, then lemma follows immediately from the first component of $\nabla A_{HF}^H$. Conversely, in the case that $||\partial_t v - \sum \eta_i X_{H_i}(v) - X_F(v)|| < \mu$, Step 2 completes the proof with other components of $\nabla A_{HF}^H$.  \hfill \square

We choose a smooth family of functions $\beta_r \in C^\infty(\mathbb{R}, [0, 1])$ for $r \geq 0$ satisfying

(i) $\beta_r(s) = 0 \quad ||s|| \geq r$,
(ii) $\beta_r(s) = 1 \quad ||s|| \leq r - 1$,
(iii) $s^2 \beta'_r(s) \leq 0 \quad s \in \mathbb{R}, \quad r \in [0, \infty)$.

We fix a point $p \in \Sigma$ and consider the moduli space

$$\mathcal{M} := \left\{ (r, w) \in [0, \infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \mid w \text{ solves (4.10) for } \beta_r(s) F(t, x) \right\}.$$

**Lemma 6.4.** For $(r, w) \in \mathcal{M}$, $E(w) \leq ||F||$.

**Proof.** We estimate

$$E(w) = \int_{-\infty}^{\infty} ||\partial_s w||^2 ds$$

$$= \int_{-\infty}^{\infty} \nabla A_{HF}^H(w)[\partial_s w] ds$$

$$\leq ||F||.$$  \hspace{1cm} (6.31)$$

**Lemma 6.5.** There exists $r_0 > 0$ such that for every $(w, r) \in \mathcal{M}$, $r \leq r_0$.  \hfill \square
Proof. Since $\beta_r F = F$ on $[-r + 1, r - 1]$, we have

$$||F|| \geq \int_{-r+1}^{r-1} ||\nabla A^H(w)||^2 ds \geq \mu^2(2r-2)$$

(6.32)

The proof follows with

$$r_0 := \frac{||F||}{2\mu^2} + 1.$$  

(6.33)

□

Lemma 6.6. For all $(r, w) \in \mathcal{M}$ where $w = (v, \eta_1, \ldots, \eta_k)$, the Lagrange multipliers $\eta_i$ are uniformly bounded.

Proof. Recall definition of $A$ and compute

$${\frac{d}{ds} A(w(s))} = d\tilde{A}^H(w)[\partial_s w] - dA^H(w)[\partial_s w]$$

$$= (m - \tilde{m})(\nabla_m A^H, \partial_s w)$$

$$= (m - \tilde{m})(\partial_s w, \partial_s w) + \beta_r (m - \tilde{m})(\nabla_m F(w), \partial_s w)$$

$$\geq \beta_r (m - \tilde{m})(\nabla_m F(w), \partial_s w),$$

where $F : \mathcal{L} \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a functional defined by

$$F(v, \eta) = \int_0^1 F(t, v) dt.$$  

(6.34)

Since $F$ is compactly supported, we can find a $c > 0$ such that

$$||\beta_r (m - \tilde{m})(\nabla_m F(t, v), \partial_s w)||_{m} \leq c.$$  

(6.36)

Combine previous lemmas and estimation, we compute

$$A(w(s_0)) = \int_{-\infty}^{s_0} {\frac{d}{ds} A(w(s))} ds$$

$$= \int_{-\infty}^{s_0} \beta_r (m - \tilde{m})(\nabla_m F(w), \partial_s w) ds$$

$$\geq -c \int_{-r}^{r} ||\partial_s w||_m ds$$

$$\geq -c \int_{-r}^{r} ||\partial_s w||^2_m + 1 ds$$

$$\geq -c(2r + E(w))$$

$$\geq -c(2r_0 + ||F||).$$

(6.37)

and analogously get

$$A(w(s_0)) = \int_{s_0}^{\infty} {\frac{d}{ds} A(w(s))} ds \leq c_0(2r_0 + ||F||).$$

(6.38)

Thus we have action bounds as follows:

$$|A(w(s))| \leq c(2r_0 + ||F||),$$

$$|A_{\beta_r H}^H(w(s))| \leq ||F||.$$  

(6.39)
The second bound comes from equation (6.31). Since $\beta = 0$ outside of $(-r, r)$, above action bounds yields

$$|\mathcal{A}^H(w(s))| \leq |\mathcal{A}^H(w(s))| + |A(w(s))| \leq c(2r_0 + ||F||) + ||F||, \quad s \in (-\infty, r] \cup [r, \infty).$$

(6.40)

Now, we set for $\sigma \in \mathbb{R}$

$$\tau(\sigma) = \inf \{ \tau \in \mathbb{R} \left| ||\nabla \mathcal{A}^H(w(\sigma + \tau))|| \leq \epsilon \text{ and } \sigma + \tau \notin (-r, r) \}.$$  

(6.41)

$\tau(\sigma)$ is bounded again by

$$\tau(\sigma) \leq \frac{1}{\epsilon^2} \left( \int_\sigma^{\sigma + \tau(\sigma)} ||\nabla \mathcal{A}^H(w)||^2 ds + \int_{-r}^{r} \epsilon^2 ds \right)$$

$$\leq \frac{E(w)}{\epsilon^2} + 2r$$

$$\leq \frac{||F||}{\epsilon^2} + 2r_0.$$  

(6.42)

We are finally able to find uniform bound for each $\eta_i$’s by using the exactly same argument in section 4.

$$|\eta_i(\sigma)| \leq |\eta_i(\sigma + \tau(\sigma))| + \int_\sigma^{\sigma + \tau(\sigma)} |\partial_s \eta_i(s)| ds$$

$$\leq C\left(c(2r_0 + ||F||) + ||F|| + 1 \right) + \tau(\sigma)||H_i||_{L^\infty}$$

$$\leq C\left(c(2r_0 + ||F||) + ||F|| + 1 \right) + \frac{||F||^2}{\epsilon^2} + 2r_0||H_i||_{L^\infty}.$$  

(6.43)

\[ \square \]

**End of the proof of Theorem E.** We know that for a sequence $\{(r_\nu, v_\nu)\}_{\nu \in \mathbb{R}}$ in $\mathcal{M}$, it $C^\infty$-locally converges (after choosing a subsequence) up to breaking due to previous lemmas. Therefore the following case occurs.

(i) There is a non-constant gradient flow line $w = (0, v, \eta)$ of $\mathcal{A}^H$ with one asymptotic end being $(p, 0)$. For convenience let $v_\pm = (p, 0)$.

Again by compactness its asymptotic ends $w_\pm = (v_\pm, \eta_\pm)$ are critical points of $\mathcal{A}^H$. From equation (4.35), moreover we have

$$-||F|| \leq \mathcal{A}^H_{\beta, \nu, F}(w_\nu(s)) < ||F|| \quad \forall s \in \mathbb{R}.$$  

(6.44)

As $\nu$ goes to infinity, we are able to deduce

$$-||F|| < \mathcal{A}^H_{\nu}(v_+) < ||F||$$

(6.45)

where $\Omega$ is the area functional. Next, we claim that $v_+$ is a non-constant periodic orbit. We argue by contradiction and assume that both $v_\pm$ is constant. Then $w$ is a element of $\mathcal{M}$ and we conclude that the moduli space $\mathcal{M}$ is compact 1 dimensional manifold with boundary $(0, p, 0)$. (after perturbing a Fredholm section as before.) But such a manifold does not exist and therefore $(v_\pm)$ is nontrivial loop. Furthermore since $w$ is a gradient flow line of $\mathcal{A}^H$, action values decrease along $w$ and $\mathcal{A}^H(v_-) = 0$, thus we derive that $v_+$ bounds a positive symplectic area, i.e.

$$|\mathcal{A}^H(v_+)| > 0.$$  

(6.46)

Equations (6.45) and (6.46) prove Theorem E under the additional assumption that $\Sigma$ has a global Bolle’s coordinates.
Now we treat the case that $\Sigma$ does not have a global Bolle’s coordinate. In such a case, we overcome the problem by an approximation of copy of $\Sigma$. We choose defining Hamiltonian functions by

$$H = (H_1, \ldots, H_k) : M \rightarrow \mathbb{R}^k = \begin{cases} H_1|_{\Sigma_{\epsilon}} = p_{i}^2 - \epsilon^2, \\ H_1|_{M - \Sigma_{\epsilon/2}} = \text{constant}, \\ H^{-1}(0) = \bigcup_{2k} \Sigma \times (\pm \epsilon, \ldots, \pm \epsilon) \end{cases}$$

With this defining Hamiltonian tuple, above argument works and thus we know that there is a loop $v_\epsilon \in H^{-1}(0)$. Since $H^{-1}(0)$ is disconnected, $v_\epsilon$ lies on one of them and denote it by $\Sigma_\epsilon$. Therefore we have $0 < \Omega(v_\epsilon) = A^H(v_\epsilon) < e(\Sigma_\epsilon)$. Then a diffeomorphism between $\Sigma$ and $\Sigma_\epsilon$ helps us. We denote by

$$\psi_\epsilon : M \xrightarrow{\cong} M$$

$$\Sigma_\epsilon \xrightarrow{} \Sigma$$

Then $\psi_\epsilon(v_\epsilon)$ a loop tangent to $\mathcal{F}$, contractible in $M$ with $A(\psi_\epsilon) = A(v_\epsilon) > 0$. Moreover if we have chosen sufficiently small $\epsilon > 0$ so that

$$\{\text{displacing Hamiltonian functions for } \Sigma_\epsilon\} = \{\text{displacing Hamiltonian functions for } \Sigma\},$$

then we have by definition

$$e(\Sigma) = e(\Sigma_\epsilon).$$

Hence finally we deduce the following conclusion and it completes the proof of Theorem E.

$$0 < \Omega(\psi(v_\epsilon)) = \Omega(v_\epsilon) < e(\Sigma) = e(\Sigma_\epsilon)$$ (6.51)

7. Genericity of Morse property

In the case of hypersurfaces, [CFP, AF2] proved that $A^H_F$ is generically Morse. [CFP] showed that the Rabinowitz action functional is Morse when there is no restriction on a time support of a perturbation. In this case, we apply the standard argument that a certain linear operator is surjective. However if one insists that a perturbation has a time support in $[0, 1/2]$, we examine the linear operator more carefully. (See [AF2].) In this section, we generalize their grateful results for higher codimension cases, yet the argument is almost same as [CFP, AF2]. In this chapter, $\Sigma$ is not necessarily of contact type.

Remark 7.1. The unperturbed Rabinowitz action functional $A^H_F$ is unfortunately never Morse since there is a $S^1$-symmetry comming from time-shift on the critical points set. However $A^H_F$ is generically Morse-Bott.

Definition 7.2. Let $f$ be a $C^\infty$-function defined on a smooth manifold $N$ and $H_f$ be the Hessian of $f$. Then $f$ is called Morse-Bott if the critical point set of $f$, $\text{Crit}f$ is a submanifold of $N$ and $T_x\text{Crit}f = \ker H_f(x)$ for any $x \in \text{Crit}f$.

Definition 7.3. For given time Hamiltonian tuple $H$ and a Hamiltonian function $F$, the pair $(H, F)$ is called a Moser pair if

(i) their time supports are disjoint, i.e.

$$H(t, \cdot) = 0 \quad \text{for } \forall t \in [0, 1/2] \quad \text{and} \quad F(t, \cdot) = 0 \quad \text{for } \forall t \in [1/2, 1]$$ (7.1)
(ii) $\mathcal{H}$ is weakly autonomous Hamiltonian tuple. That is, $\mathcal{H}$ is of the form $\mathcal{H}(t,x) = \chi(t)p(x) = \chi(t)(p_1(x), \ldots, p_k(x))$ for $\chi : S^1 \to S^1$ with $\int_0^1 \chi = 1$ and $\text{Supp} \chi \subset (\frac{1}{2}, 1)$.

(iii) Moreover $\{X_{p_1}(x), \ldots, X_{p_k}(x)\}$ is linearly independent for any $x \in p^{-1}(0)$.

**Remark 7.4.** In our situation, $(\mathcal{H}, F)$ considered so far, is a Moser pair.

**Definition 7.5.** We denote by

$$\mathcal{F}^j = \left\{ F \in C^j(M \times S^1) \mid F(t, \cdot) = 0 \quad \forall t \in \left[ \frac{1}{2}, 1 \right] \right\}. \tag{7.2}$$

7.1. **Adjoint Operator.** For later purpose, we briefly review the notion of the adjoint operator. Consider two separable real Hilbert spaces $W$ and $V$ such that $W \subset V$ is dense and the inclusion is compact. Let $L : W \to V$ be a bounded linear operator. We view $L$ as an operator defined on $V$ with domain $\text{dom}(L) = W$. We set

$$\text{dom}(L^*) := \{ v \in V \mid \exists h \in V \text{ such that } \langle h, w \rangle = \langle v, Lw \rangle \text{ for any } w \in \text{dom}(L) \} \tag{7.3}$$

Then we can define the adjoint operator $L^* : \text{dom}(L^*) \to V$ by $L^*v = h$. The well-definedness of $L^*$ follows from the density of $\text{dom}(L) = V$ in $V$. By construction we have the equation

$$\langle L^*v, w \rangle_V = \langle v, Lw \rangle_V. \tag{7.4}$$

**Definition 7.6.** With the set-up as above, $L$ is called *symmetric* if $W \subset \text{dom}(L^*)$ and $\langle L^*v, w \rangle_V = \langle v, Lw \rangle_V$ for all $v, w \in W$. If $L$ is symmetric and $\text{dom}(L^*) = W$, we say $L$ is *self-adjoint*.

7.2. **Proof of Theorem 1.14.** In this section we denote by $\mathcal{L} = W^{1,2}(S^1, M)$ the loop space which is indeed a Hilbert manifold. Let $\mathcal{E}$ be a $L^2$-bundle over $\mathcal{L}$ with $\mathcal{E}_v = L^2(S^1, v^*TM)$. We consider the sections

$$S : \mathcal{L} \times \mathbb{R} \times \mathcal{F}^j \to \mathcal{E}^\vee \times \mathbb{R} \quad \text{defined by } S(v, \eta, F) := d\mathcal{A}^H_F(v, \eta). \tag{7.5}$$

and for $F \in \mathcal{F}^j$, $s_F : \mathcal{L} \times \mathbb{R} \to \mathcal{E}^\vee \times \mathbb{R}$ defined by $s_F(v, \eta) = S(v, \eta, F) \tag{7.6}$

Here and so on, the symbol $\vee$ represents the dual bundle or dual space. Then the set $s_{F^{-1}(0)}^{-1}(0)$ consists of critical points of $\mathcal{A}^H_F$ by definition of $s_F$. At $(v_0, \eta_0, F) \in S^{-1}(0)$, the vertical differential

$$DS : T_{(v_0, \eta_0, F)} \mathcal{L} \times \mathbb{R}^k \times \mathcal{F}^j \to \mathcal{E}^\vee_{v_0} \times T_{\eta_0} \mathbb{R}^k \tag{7.7}$$

via the pairing with for $(\dot{v}^1, \dot{\eta}^1, \hat{F}, [\dot{v}^2, \dot{\eta}^2]) \in \mathcal{E} \times \mathbb{R}^k$

$$\langle DS_{(v_0, \eta_0, F)}[\dot{v}^1, \dot{\eta}^1, \hat{F}], [\dot{v}^2, \dot{\eta}^2] \rangle = \mathcal{H}^H_{\hat{F}}(v_0, \eta_0)[(\dot{v}^1, \dot{\eta}^1), (\dot{v}^2, \dot{\eta}^2)] + \int_0^1 \hat{F}(t, v_0) dt. \tag{7.8}$$

Moreover at $(v_0, \eta_0) \in s_F^{-1}(0)$,

$$DS_F : T_{(v_0, \eta_0)} \mathcal{L} \times \mathbb{R}^k \to \mathcal{E}^\vee_{v_0} \times T_{\eta_0} \mathbb{R}^k \tag{7.9}$$

via the pairing with for $(\dot{v}^2, \dot{\eta}^2) \in \mathcal{E} \times \mathbb{R}^k$

$$\langle DS_F(v_0, \eta_0)[\dot{v}^1, \dot{\eta}^1, \hat{F}], [\dot{v}^2, \dot{\eta}^2] \rangle = \mathcal{H}^H_F(v_0, \eta_0)[(\dot{v}^1, \dot{\eta}^1), (\dot{v}^2, \dot{\eta}^2)]. \tag{7.10}$$

If we show that the linearized operator $DS$ is surjective, then the proof of Theorem 1.14 finishes with the following standard argument. We shall give the proof of the surjectivity of $DS$ in forthcoming Proposition 7.9. Since $DS$ is surjective, $\mathcal{M}_{\mathcal{F}^j} := S^{-1}(0)$ is a Banach
manifold by the implicit function theorem.

**Claim:** Let \( \Pi : \mathcal{M}_{\mathcal{F}} \rightarrow \mathcal{F}^j \) be a natural projection. If \( F \) is a regular value of \( \Pi \), then \( DS_{F,(v_0,\eta_0)} \) is surjective for any \( (v_0,\eta_0) \in s_F^{-1}(0) \).

**Proof of Claim.** At first, we arrange properties which follow immediately from the definitions. For \( (v_0,\eta_0), (v_0,\eta_0), F \) \( \in S^{-1}(0) \), choose any element \( \tilde{v}, \tilde{\eta} \in E_{v_0} \times T_{\eta_0} \mathbb{R} \).

(i) Then Proposition 7.9 says that there exists \( (\tilde{v}^1, \tilde{\eta}^1, \tilde{F}^1) \in T_{(v_0,\eta_0, F)} \mathcal{L} \times \mathbb{R} \times \mathcal{F}^k \) such that \( DS_{F,(v_0,\eta_0)}(\tilde{v}^1, \tilde{\eta}^1, \tilde{F}^1) = (\tilde{v}, \tilde{\eta}) \).

(ii) Since \( F \) is a regular value of \( \Pi \), there exists \( (\tilde{v}^2, \tilde{\eta}^2, \tilde{F}^2) \in T_{(v_0,\eta_0, F)} S^{-1}(0) \) satisfying \( d\Pi_{(v_0,\eta_0, F)}(\tilde{v}^2, \tilde{\eta}^2, \tilde{F}^2) = \tilde{F}^1 \). But \( d\Pi_{(v_0,\eta_0, F)}(\tilde{v}^2, \tilde{\eta}^2, \tilde{F}^2) = \tilde{F}^2 \) by definition of \( \Pi \), thus \( \tilde{F}^1 = \tilde{F}^2 \).

(iii) Moreover \( DS_{F,(v_0,\eta_0, F)}(\tilde{v}^2, \tilde{\eta}^2, \tilde{F}^2) = 0 \), since \( (\tilde{v}^2, \tilde{\eta}^2, \tilde{F}^2) \) is an element of \( T_{(v_0,\eta_0, F)} S^{-1}(0) \).

(iv) We can write the property (iii) as \( DS_{F,(v_0,\eta_0)}(\tilde{v}^2, \tilde{\eta}^2) + \int_0^1 \tilde{F}^2(t, v_0)dt = 0 \).

This preparation enable us to prove the Claim, we calculate

\[
(\tilde{v}, \tilde{\eta}) = DS_{F,(v_0,\eta_0)}(\tilde{v}^1, \tilde{\eta}^1, \tilde{F}^1)
= DS_{F,(v_0,\eta_0)}(\tilde{v}^1, \tilde{\eta}^1) + \int_0^1 \tilde{F}^1(t, v_0)dt
= DS_{F,(v_0,\eta_0)}(\tilde{v}^1, \tilde{\eta}^1) + \int_0^1 \tilde{F}^2(t, v_0)dt
= DS_{F,(v_0,\eta_0)}(\tilde{v}^1, \tilde{\eta}^1) - DS_{F,(v_0,\eta_0)}(\tilde{v}^2, \tilde{\eta}^2)
= DS_{F,(v_0,\eta_0)}(\tilde{v}^1 - \tilde{v}^2, \tilde{\eta}^1 - \tilde{\eta}^2).
\]

Therefore the element \( \tilde{v}, \tilde{\eta} \) is contained in the image of \( DS_{F,(v_0,\eta_0)} \), we conclude that the operator \( DS_{F,(v_0,\eta_0)} \) is surjective. This finishes the proof of Claim. \( \square \)

\( DS_{F,(v_0,\eta_0)} \) is equal to \( \mathcal{H}_{A_F^H}((v_0,\eta_0), v_0,\eta_0) \), the Hessian of \( A_F^H \) at \( (v_0,\eta_0) \) via the natural pairing and therefore \( A_F^H \) is Morse for the Moser pair \( (\mathcal{H}, F) \). Moreover the set of regular value of \( \Pi \), \( \mathcal{F}_\text{reg}^j \) is of second category by Sard-Smale theorem. Then an intersection of \( \mathcal{F}_\text{reg}^j \) for all \( j \in \mathbb{N} \) is also dense set. Therefore \( A_F^H \) is Morse for a generic perturbation \( F \in C^\infty(M \times S^1) \), the proof of theorem is completed. \( \square \)

**Theorem 7.7. (Sard-Smale)** Let \( X \) and \( Y \) be separable Banach manifolds and \( f : X \rightarrow Y \) be a \( C^1 \)-map. If \( df_x \) is a Fredholm map of index \( k \) and \( l \geq k + 1 \), then a generic \( y \in Y \) is a regular value of \( f \). More precisely, \( Y_{\text{reg}} \) the set of regular value of \( f \) is of second category.

**Remark 7.8.** When Sard-Smale theorem works, the Implicit function theorem implies that \( f^{-1}(y) \) is a smooth manifold of dimension \( k \).

We compute the Hessian \( \mathcal{H}_{A_F^H} \) to get the surjectivity of \( DS \) in Proposition 7.9. To compute the Hessian of \( A_F^H \) at a critical point \((v_0,\eta_0)\), we pullback the functional \( A_F^H \) to the space

\[
\tilde{\mathcal{L}} := \{ \tilde{v} \in W^{1,2}([0, 1], M) \mid \tilde{v}(0) = \phi_{\eta_0, \mathcal{H}, F}(\tilde{v}(1)) \}
\]

(7.12)
Then its differential is

\[ \text{d} \Phi_{\eta_0, H, F} : \tilde{L} \rightarrow L \] given by \( \Phi_{\eta_0, H, F}(\tilde{v})(t) = \phi^t_{\eta_0, H, F}(\tilde{v}(t)). \) (7.13)

We set

\[ \tilde{\mathcal{A}}_{(\eta_0, H)} + F = (\Phi_{\eta_0, H, F})^* \mathcal{A}_{(\eta_0, H)} + F. \] (7.14)

Here and so on, we abuse in which \( \eta_0 \) lives. We denote \( \eta_0 \) for \( \eta_0 = (\eta_0, \ldots, \eta_0) \in \mathbb{R}^k, \) then

\( \langle \eta_0, H \rangle + F = \eta_0 \sum_{i=1}^k H_i + F. \)

We suppress the dependencies on \( \Phi_{\eta_0, H, F}, \phi^t_{\eta_0, H, F} \) and \( \tilde{\mathcal{A}}_{(\eta_0, H)} + F \) if there is no confusion. We write \( \Phi_H = \Phi_{\eta_0, H, F}, \phi_H = \phi_{\eta_0, H, F} \) and \( \tilde{\mathcal{A}} = \tilde{\mathcal{A}}_{(\eta_0, H)} + F. \)

Using the formula

\[
\partial_t \{ \phi^t_H(v(t)) \} = d\phi^t_H[\partial_t v] + \frac{\partial}{\partial t} \phi^t_H(v(t)) \\
= d\phi^t_H[\partial_t v] + \eta_0 \sum_{i=1}^k X_{H_i}(t, \phi^t_H(v(t))) + X_F(t, \phi^t_H(v(t))),
\] (7.15)

we compute

\[
d\tilde{\mathcal{A}}(\tilde{v})[\tilde{\dot{v}}] = (\Phi_H)^* d\mathcal{A}_{(\eta_0, H), + F}(\tilde{v})[\tilde{\dot{v}}] \\
= d\mathcal{A}_{(\eta_0, H), + F}(\Phi_H(\tilde{v}))[d\Phi_H(\tilde{v})] \\
= \int_0^1 \omega \left( \partial_t \{ \phi^t_H(\tilde{v}(t)) \} - \eta_0 \sum_{i=1}^k (X_{H_i} - X_F)(t, \phi^t_H(\tilde{v}(t))), d\phi^t_H(\tilde{v})[\tilde{\dot{v}}] \right) dt \\
= \int_0^1 \omega \left( d\phi^t_H(\tilde{v}(t))[\partial_t \tilde{v}], d\phi^t_H(\tilde{v})(\tilde{\dot{v}}) \right) dt \\
= \int_0^1 \omega(\partial_t \tilde{v}, \tilde{v}) dt.
\] (7.16)

Therefore, the Hessian is

\[ \mathcal{K}_{\tilde{\mathcal{A}}}(\tilde{v})[\tilde{v}^1, \tilde{v}^2] = \int_0^1 \omega(\partial_t \tilde{v}^1, \tilde{v}^2) dt. \] (7.17)

Next, we split \( \mathcal{A}^H_F(\eta, \eta) \) into

\[ \mathcal{A}^H_F(\eta, \eta) = \mathcal{A}_{(\eta_0, H), + F}(t, \eta) + \int_0^1 \langle \eta_0 - \eta, H \rangle(t, \eta(t)) dt. \] (7.18)

Then its differential is

\[ d\mathcal{A}^H_F(\eta, \eta)[\tilde{v}, \hat{\eta}] = d\mathcal{A}_{(\eta_0, H), + F}(t, \eta)[\tilde{v}] - \int_0^1 \langle \eta, H \rangle(t, \eta(t)) dt + (\eta_0 - \eta) \sum_{i=1}^k \int_0^1 dH_i(t, \eta)[\tilde{v}] dt. \] (7.19)
thus the Hessian of $A^H_F$ at $(v_0, \eta_0) \in \text{Crit} A^H_F$ equals

$$
\mathcal{H}_{A^H_F}(v_0, \eta_0)[(\dot{v}^1, \dot{\eta}^1), (\dot{v}^2, \dot{\eta}^2)] = \mathcal{H}_{A^{(0, \eta_0)+F}}(v_0)[\dot{v}^1, \dot{\eta}^2] - \sum_{i=1}^k \dot{\eta}^i \int_0^1 dH_i(t, v)[\dot{v}^2] dt \\
- \sum_{i=1}^k \dot{\eta}^i \int_0^1 dH_i(t, v)[\dot{v}^1] dt
$$

(7.20)

where $\dot{\eta}^1 = (\dot{\eta}^1_1, \ldots, \dot{\eta}^1_k)$, $\dot{\eta}^2 = (\dot{\eta}^2_1, \ldots, \dot{\eta}^2_k) \in \mathbb{R}^k$.

Using the map $\Phi_{\eta_0, H, F}$ in (7.13), we pullback $A^H_F$ to the space $\tilde{L} \times \mathbb{R}^k$ and call

$$
\tilde{A}^H_F = (\Phi_{\eta_0, H, F} \times \text{id}_{\mathbb{R}^k})^* A^H_F.
$$

(7.21)

Combine two equations (7.17) and (7.20), we get

$$
\mathcal{H}_{\tilde{A}^H_F}(\tilde{v}_0, \eta_0)[(\tilde{\dot{v}}^1, \tilde{\dot{\eta}}^1), (\tilde{\dot{v}}^2, \tilde{\dot{\eta}}^2)] = \int_0^1 \omega(\partial_t \tilde{\dot{v}}^1, \tilde{\dot{v}}^2) dt - \sum_{i=1}^k \tilde{\dot{\eta}}^i \int_0^1 d\tilde{H}_i(t, v)[\tilde{v}^2] dt \\
- \sum_{i=1}^k \tilde{\dot{\eta}}^i \int_0^1 d\tilde{H}_i(t, v)[\tilde{v}^1] dt
$$

(7.22)

where

$$
\int_0^1 \tilde{H}_i(t, v(t)) dt = (\Phi_{\eta_0, H, F})^* \int_0^1 H_i(t, v(t)) dt = \int_0^1 H_i \circ \Phi_{\eta_0, H, F}(t, v(t)) dt.
$$

(7.23)

But we have chosen a Moser pair $(H, F)$ so that

$$
\int_0^1 \tilde{H}_i(t, v(t)) dt = \int_0^1 H_i(t, \phi^t_{\eta_0, H, F}(v(t))) dt \\
= \int_0^{\frac{1}{2}} H_i(t, \phi^t_{\eta_0, H, F}(v(t))) dt \\
= \int_0^{\frac{1}{2}} H_i(t, \phi^t_{\eta_0, H}(v(t))) dt \\
= \int_0^{\frac{1}{2}} H_i(t, v(t)) dt \\
= \int_0^1 H_i(t, v(t)) dt
$$

(7.24)

where $\phi^t_{\eta_0, H}$ is the flow of the Hamiltonian vector field $\eta_0 \sum_{i=1}^k X_{H_i}$. The third equality in (7.24) follows since we assumed that the Hamiltonian tuple $\mathcal{H}$ is Poisson-commute so that $H_i$ is constant along the flow of $X_{H_j}$ for any $1 \leq i, j \leq k$. 
We change the Hessian of \( \widetilde{A}_F^H \) to another formula using the integration by parts, then we easily derive that the Hessian is symmetric.

\[
\mathcal{H}_{\widetilde{A}_F^H}(\tilde{v}_0, \eta_0)[(\tilde{v}^1, \hat{\eta}^1), (\tilde{v}^2, \hat{\eta}^2)] = \int_0^1 \omega(\partial_t \tilde{v}^1, \hat{\eta}^2)dt - \sum_{i=1}^k \eta_i^1 \int_0^1 dH_i(t, \tilde{v}_0)[\hat{\eta}^2]dt - \sum_{i=1}^k \eta_i^2 \int_0^1 dH_i(t, \tilde{v}_0)[\hat{\eta}^1]dt
\]
\[
= \frac{1}{2} \int_0^1 \omega(\partial_t \tilde{v}^1, \hat{\eta}^2)dt - \sum_{i=1}^k \eta_i^1 \int_0^1 dH_i(t, \tilde{v}_0)[\hat{\eta}^2]dt + \frac{1}{2} \int_0^1 \omega(\partial_t \tilde{v}^2, \hat{\eta}^1)dt
\]
\[
= \frac{1}{2} \int_0^1 \left\{ \omega(\partial_t \tilde{v}^1 - 2 \sum_{i=1}^k \eta_i^1 X_{H_i}(t, \tilde{v}_0), \hat{\eta}^2) + \omega(\partial_t \tilde{v}^2 - 2 \sum_{i=1}^k \eta_i^2 X_{H_i}(t, \tilde{v}_0), \hat{\eta}^1) \right\} dt
\]
\[
= \frac{1}{2} \omega(\tilde{v}^1(1), \hat{\eta}^2(1)) + \frac{1}{2} \omega(\tilde{v}^1(0), \hat{\eta}^2(0)).
\]

We define the \( L^2 \)-bundle \( \widetilde{\mathcal{E}} \) over \( \widetilde{\mathcal{L}} \) by \( \widetilde{\mathcal{E}} : = L^2([0, 1], \tilde{v}^*TM) \). Moreover we define the linear operator

\[
\widetilde{DS} : T_{(\tilde{v}_0, \eta_0, F)}\widetilde{\mathcal{L}} \times \mathbb{R}^k \times \mathcal{F} \longrightarrow (\widetilde{\mathcal{E}}_{\tilde{v}_0})^\vee \times T_{\tilde{v}_0} \mathbb{R}^k
\]

via the pairing with for \((\tilde{v}^2, \hat{\eta}^2) \in \widetilde{\mathcal{E}}_{\tilde{v}_0} \times \mathbb{R}^k \)

\[
\langle \widetilde{DS}_{(\tilde{v}_0, \eta_0, F)}[\tilde{v}^1, \hat{\eta}^1, \tilde{F}], [\tilde{v}^2, \hat{\eta}^2] \rangle = \mathcal{H}_{\widetilde{A}_F^H}(\tilde{v}_0, \eta_0)[(\tilde{v}^1, \hat{\eta}^1), (\tilde{v}^2, \hat{\eta}^2)] + \int_0^1 d((\Phi_H)^*\tilde{F}(t, \tilde{v}_0))[\hat{\eta}^2(t)]dt.
\]

where \( \tilde{v}_0 = \Phi_{\eta_0, H, F}(v_0) \).

Then \( \widetilde{DS} \) is the pullback of the linearized operator \( DS \) under the diffeomorphism \( \Phi_{\eta_0, H, F} \times \text{id}_{\mathbb{R}^k} \times \text{id}_{\mathcal{F}} \). Therefore our strategy is to show that \( \widetilde{DS} \) is surjective.

**Proposition 7.9.** The linearized operator \( DS \) is surjective. In fact, \( DS \) is surjective when restricted to the space

\[
\mathcal{V} : = \{(\hat{v}, \hat{\eta}, \tilde{H}) \in T_{(v_0, \eta_0, H)}L \times \mathbb{R} \times \mathcal{H} | \hat{v}(\frac{1}{2}) = 0 \}.
\]

**Proof.** It is enough to show that \( \widetilde{DS} \) is surjective. The operator \( \widetilde{DS} \) is Fredholm since \( L^2 \)-Hessian is a self-adjoint Fredholm operator. Thus the image of \( DS \) is closed, moreover \((\mathcal{E}_{\tilde{v}_0})^\vee \times T_{\tilde{v}_0} \mathbb{R}^k \) splits into \( \text{im}(\widetilde{DS}) \) and its orthogonal complement. Therefore it suffices to show that the annihilator of the image of \( DS \) vanishes. Pick an annihilator \((\hat{v}^2, \hat{\eta}^2) \in \text{Ann}(\text{im}(\widetilde{DS}))\), then for all \((\tilde{v}^1, \hat{\eta}^1, \tilde{F}) \in T_{(v_0, \eta_0, F)}\widetilde{\mathcal{L}} \times \mathbb{R}^k \times \mathcal{F} , \)

\[
\langle \widetilde{DS}_{(\tilde{v}_0, \eta_0, F)}[\tilde{v}^1, \hat{\eta}^1, \tilde{F}], [\hat{v}^2, \hat{\eta}^2] \rangle = 0
\]
Using the formula (7.27), this is equivalent to
\begin{equation}
\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0)[(\hat{v}^1, \hat{v}^1), (\hat{v}^2, \hat{v}^2)] = 0
\end{equation}
(7.30)
and
\begin{equation}
\int_0^1 d((\Phi_H)^* \hat{F}(t, \tilde{v}_0))[\hat{v}^2(t)]dt = 0.
\end{equation}
(7.31)
From the equation (7.31), we know that
\begin{equation}
\hat{v}^2(t) = 0 \quad \text{for all } t \in [0, \frac{1}{2}].
\end{equation}
(7.32)
From equation (7.30), \((\hat{v}^2, \hat{v}^2)\) can be also viewed as an annihilator of the image of the Hessian \(\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0)\), we easily notice that \(\text{Ann}(\text{im}(\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0))) \subset \text{dom}(\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0)*)\). Since the Hessian \(\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0)\) is a self-adjoint operator, \(\text{dom}(\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0)*) = \text{dom}(\mathcal{H}_{\mathcal{A}^H}(v_0, \eta_0)) = W^{1,2}([0,1], \tilde{v}_0^*TM)\) so that the expression \(\partial \hat{v}^2\) is well-defined. In particular, when \(\hat{v}^1 = 0\), equation (7.30) equals to
\begin{equation}
\int_0^1 \omega(\partial_t \hat{v}^2 - \sum_{i=1}^k \hat{\eta}^2_i X_{H_i}(t, \tilde{v}_0), \hat{v}^1) = 0 \quad \forall \hat{v}^1 \in T_{v_0}S.
\end{equation}
(7.33)
Thus \(\hat{v}^2 \in W^{1,2}([0,1], \tilde{v}_0^*TM)\) is a weak solution of the equation
\begin{equation}
\partial_t \hat{v}^2 - \sum_{i=1}^k \hat{\eta}^2_i X_{H_i}(t, \tilde{v}_0) = 0
\end{equation}
(7.34)
Since \(X_{H_i}(t, \tilde{v}_0)\) is an element of \(C^j([0,1], \tilde{v}_0^*TM)\), the bootstrapping argument shows that \(\hat{v}^2 \in C^{j+1}([0,1], \tilde{v}_0^*TM)\) is a strong solution of equation (7.34). In fact, when the Hessian is restricted to \(\mathcal{V}\) then equation (7.34) holds for all \(t \neq \frac{1}{2}\), since the Hessian is a local operator. Thus, by continuity, it holds for all \(t\) in any case.
Equation (7.34) equals to
\begin{equation}
\partial_t \hat{v}^2 - \sum_{i=1}^k \hat{\eta}^2_i \chi(t)X_{p_i}(\tilde{v}_0) = 0.
\end{equation}
(7.35)
Solving this ODE, we get
\begin{equation}
\hat{v}^2(t) = \hat{v}^2(q) - \left( \int_t^1 \chi(x)dx \right) \sum_{i=1}^k \hat{\eta}^2_i X_{p_i}(\tilde{v}_0).
\end{equation}
(7.36)
Since \(\int_t^1 \chi(x)dx = 1\) for all \(0 \leq t \leq \frac{1}{2}\), we deduce the following equation with (7.32)
\begin{equation}
0 = \hat{v}^2(t) = \hat{v}^2(q) - \sum_{i=1}^k \hat{\eta}^2_i X_{p_i}(\tilde{v}_0) \quad \forall t \leq \frac{1}{2}.
\end{equation}
(7.37)
In particular, \(\hat{v}^2(0) = 0\), moreover we know \(\hat{v}^2(t)\) satisfies the linearized boundary condition
\begin{equation}
\hat{v}^2(0) = d\phi_H(\tilde{v}_0)[\hat{v}^2(1)],
\end{equation}
(7.38)
eventually we derive that \( \hat{v}^2(1) = 0 \). Therefore we can rewrite equation (7.37)

\[
\sum_{i=1}^{k} \hat{\eta}_i^2 X_{p_i}(\tilde{v}_0) = 0. \tag{7.39}
\]

But \((H, F)\) is a Moser pair so that \( \{X_{p_1}(\tilde{v}_0), \ldots, X_{p_k}(\tilde{v}_0)\} \) is linearly independent, thus \( \hat{\eta}^2 = (\hat{\eta}_1^2, \ldots, \hat{\eta}_k^2) = 0 \). Furthermore, \( \hat{v}^2(t) = \hat{v}^2(0) = 0 \) from formula (7.36). Hence \( (\hat{v}^2, \hat{\eta}^2) = 0 \), \( DS \) is surjective. \( \Box \)

**Remark 7.10.** The additional assertion of the surjectivity of \( DS|_{\mathcal{V}} \) is used in Proposition 9.6.

**Remark 7.11.** If we works with \( W^{1,2} \)-Hessian, instead of \( L^2 \)-Hessian, Proposition 7.9 would be failed. Because \( W^{1,2} \)-Hessian does not have closed image and not Fredholm in general. For example, the image of \( W^{1,2} \)-Hessian of the area functional is not closed.

### 8. Rabinowitz Floer Homology

In the previous section, we observed that the Rabinowitz Floer action functional \( A_F^H \) is Morse for a generic perturbation \( F \in C^\infty(M \times S^1) \). Furthermore we know that the moduli space of the gradient flow lines is compact modulo breaking from Theorem 4.9. (In the case that the \( \Sigma \) is of unrestricted contact type, Theorem 5.9 and Corollary 5.14 help us for admissible perturbations). Thus we can define a Floer homology of \( A_F^H \) as usual. We denote this homology \( RFH(\Sigma, M, F) \). If there is no perturbation i.e. \( F = 0 \), \( A_F^H \) is unfortunately never Morse as we have already mentioned. Although \( A_F^H \) is generically Morse-Bott, we can compute the Morse-Bott homology of \( A_F^H \) by counting gradient flow lines with cascades. (See [Fr].) In this case the Rabinowitz Floer homology is denoted by \( RFH(\Sigma, M) \). As one expects, these two Rabinowitz Floer homologies are isomorphic.

#### 8.1. Boundary Operator

We define a \( \mathbb{Z}/2 \)-vector space

\[
\text{CF}(A_F^H) := \left\{ \xi = \sum_{(v, \eta) \in \text{Crit}A_F^H} \xi_{(v, \eta)}(v, \eta) \left| \xi_{(v, \eta)} \in \mathbb{Z}/2 \right\} \tag{8.1}
\]

where \( \xi_{(v, \eta)} \) satisfy the finite condition

\[
\#\{(v, \eta) \in \text{Crit}A_F^H \mid \xi_{(v, \eta)} \neq 0, A_F^H((v, \eta)) \geq \kappa\} < \infty \quad \forall \kappa \in \mathbb{R}. \tag{8.2}
\]

To define the boundary operator, we consider the moduli space

\[
\mathcal{M}(w_-, w_+) := \left\{ w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \left| w \text{ solves } (4.10) \text{ with } \lim_{s \to \pm \infty} w(s) = w_{\pm} \right. \right\} \tag{8.3}
\]

and divide out the \( \mathbb{R} \)-action from shifting the gradient flow lines in the \( s \)-variable. Then we obtain the moduli space of unparametrized gradient flow lines, denote by

\[
\widehat{\mathcal{M}} := \mathcal{M}/\mathbb{R}. \tag{8.4}
\]

From the abstract perturbation theory, we know that this moduli space is a smooth manifold for generic choice of the almost complex structure and the metric. Next, we recall our starting point. Theorem 4.9 tells us that \( \widehat{\mathcal{M}}(w_-, w_+) \) is a zero dimensional compact manifold when the index difference of \( w_- \) and \( w_+ \) is equal to 1. Critical points of \( A_F^H \) are endowed with some indices namely, Conley-Zehnder indices \( \mu_{CZ} \), yet we omit the definition of Conley-Zehnder index and refer to the reader any other papers. (e.g. [RS, SZ].) Therefore \( \widehat{\mathcal{M}}(w_-, w_+) \) is a
finite set and we let $n(w_-, w_+)$ be the number of element in this moduli space. We define the boundary map $\partial$ as follows:

$$
\partial : \text{CF}(A^H_F) \rightarrow \text{CF}(A^H_F)
$$

$$
\begin{align*}
\partial & : \text{CF}(A^H_F) \rightarrow \text{CF}(A^H_F) \\
&w_- \mapsto \sum_{w_+} n(w_-, w_+)d
\end{align*}
$$

(8.5)

for $d \in \text{Crit}_F A^H_F$ satisfying $\mu_{CZ}(w_-) - \mu_{CZ}(w_+) = 1$. Then due to the Floer’s central theorem, we know $\partial \circ \partial = 0$ so that $(\text{CF}(A^H_F), \partial)$ is a complex. We define a Rabinowitz Floer homology by

$$
\text{RFH}(\Sigma, M, F) := \text{HF}(A^H_F) = \text{H}(\text{CF}(A^H_F), \partial).
$$

(8.6)

8.2. Continuation Homomorphism. Given any two non-autonomous Hamiltonian functions $F$ and $K$, we define the homotopies

$$
G^+_s(t, x) := K(t, x) + \beta^+_\infty(s)(F(t, x) - K(t, x))
$$

(8.7)

and

$$
G^-_s(t, x) := K(t, x) + \beta^-\infty(s)(F(t, x) - K(t, x))
$$

(8.8)

where $\beta^\pm_\infty(s)$ are cut-off functions defined in the proof of Theorem A. For convenience, we recall the construction of a smooth family of cut-off functions $\beta_r \in C^\infty(\mathbb{R}, [0, 1])$ with $0 \leq r$

(i) for $r \geq 1$: $\beta'_r(s) \cdot s \leq 0$ for all $s \in \mathbb{R}$, $\beta_r(s) = 1$ for $|s| \leq r - 1$, and $\beta_r(s) = 0$ for $|s| \geq r$,

(ii) for $r \leq 1$: $\beta_r(s) \leq r$ for all $s \in \mathbb{R}$ and supp$\beta_r \subset [-1, 1]$,

(iii) $\lim_{r \rightarrow \infty} \beta_r(s \mp r) =: \beta^\pm_\infty(s)$ exists, where the limit is taken with respect to the $C^\infty$ topology.

We consider the equations

$$
\begin{align*}
\partial_s v + J_s(t, v)(\partial_t v - \sum_{i=1}^k \eta_i X_{H_i}(t, v) - X_{G^+_s}(t, v)) &= 0 \\
\partial_s \eta_i - \int_0^1 H_i(t, v)dt &= 0 \quad 1 \leq i \leq k.
\end{align*}
$$

(8.9)

The solution of (8.9) with the limit conditions form a moduli space

$$
\mathcal{M}(w_K, w_F) := \left\{ w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}^k) \mid w \text{ solves (8.9) with } \lim_{w \rightarrow \pm \infty} w_{F/K} = w_{F/K} \right\}
$$

(8.10)

for $w_K \in \text{Crit}_F A^H_K$ and $w_F \in \text{Crit}_F A^H_F$.

This moduli space is a smooth manifold of dimension dim $\mathcal{M}(w_K, w_F) = \mu_{CZ}(w_K) - \mu_{CZ}(w_F)$ for a generic homotopy.

We denote the number of moduli space $\mathcal{M}(w_K, w_F)$ by $n(w_K, w_F)$ when the dimension of $\mathcal{M}(w_K, w_F)$ equal to zero that is, $w_K$ and $w_F$ have same index. Theorem 4.9 guarantees the finiteness of this number. Thus the following continuation homomorphism is well-defined.

$$
\Phi^F_K : \text{CF}(A^H_K) \rightarrow \text{CF}(A^H_F)
$$

$$
\begin{align*}
\Phi^F_K & : \text{CF}(A^H_K) \rightarrow \text{CF}(A^H_F) \\
w_K & \mapsto \sum_{w_F} n(w_K, w_F)w_F
\end{align*}
$$

(8.11)

for $w_F \in \text{Crit}_F A^H_F$ satisfying $\mu_{CZ}(w_K) - \mu_{CZ}(w_F) = 0$. In a similar way we also define $\Phi^K_F : \text{CF}(A^H_F) \rightarrow \text{CF}(A^H_K)$ using the homotopy $G^-_s$. 


Then we get an invariance of Rabinowitz Floer homology via the continuation homomorphism using the standard argument in Floer theory.

**Theorem 8.1.** The Rabinowitz Floer homology is independent of the perturbation. In particular perturbed Rabinowitz Floer homology is isomorphic to unperturbed Rabinowitz Floer homology. That is,

\[
\text{RFH}(\Sigma, M, K) \cong \text{RFH}(\Sigma, M, F) \cong \text{RFH}(\Sigma, M).
\]

for any Hamiltonian functions \( K \) and \( F \).

**Proof.** We have chosen homotopies \( G_s^+ \) from \( K \) to \( F \) and \( G_s^- \) from \( F \) to \( K \). Then we know that \( \text{id}_K : \text{CF}(A^H_K) \rightarrow \text{CF}(A^H_F) \) and \( \Phi_K^F \circ \Phi_F^K : \text{CF}(A^H_K) \rightarrow \text{CF}(A^H_K) \) are chain homotopic by applying a homotopy of homotopies \( G_s^+(t, x) := K(t, x) + \beta_r(s)(F(t, x) - K(t, x)) \). Conversely, there is also a chain homotopy between \( \Phi_F^K \circ \Phi_K^F \) and \( \text{id}_F \). Therefore the continuation homomorphism \( \Phi_K^F \) induces an isomorphism on the homology level. \( \square \)

### 8.3. Proof of Theorem F

Suppose that there is no leafwise coisotropic intersection points. Then the set \( \text{Crit} A^H_K \) is empty for any Hamiltonian function \( F \) since otherwise a critical point of \( A^H_M \) give rise to a coisotropic leafwise intersection point. Therefore \( \text{RFH}(\Sigma, M) \cong 0 \). It completes the proof of Theorem F by contradiction. \( \square \)

### 8.4. Proof of Corollary F

Suppose there are only constant solutions for the equation (Bo). Due to Proposition 1.17, we know that the Rabinowitz Floer homology is isomorphic to the singular homology. The singular homology of \( \Sigma \) never vanishes, however the Rabinowitz Floer homology vanishes by Theorem F because of the assumption that \( \Sigma \) is displaceable. It contradicts, thus the proof of Corollary F is finished. \( \square \)

For later purpose we compare the action values of \( A^H_K \) and \( A^H_F \) via the energy estimate.

**Proposition 8.2.** If the moduli space \( \mathcal{M}(w_K, w_F) \) is not empty, then we have the following estimates.

\[
A^H_F(w_F) \leq A^H_K(w_K) + ||F - K||_-.
\]

and

\[
A^H_K(w_K) \leq A^H_F(w_F) + ||F - K||_+.
\]

**Proof.** Choose \( w \in \mathcal{M}(w_K, w_F) \) and estimate its energy

\[
0 \leq E(w) = -\int_{-\infty}^{\infty} dA^H_{G^+} (w(s)) [\partial_su] ds
\]

\[
= -\int_{-\infty}^{\infty} \frac{d}{ds} (A^H_{G^+} (w(s))) ds - \int_{-\infty}^{\infty} \int_0^1 \beta^+_r(s)(F(t, w(s)) - K(t, w(s))) dt ds.
\]

\[
\leq A^H_{G^+} (w_{-\infty}) - A^H_{G^+} (w_{\infty}) - \int_{-\infty}^{\infty} \int_0^1 \beta^+_r(s)(F(t, w(s)) - K(t, w(s))) dt ds.
\]

\[
\leq A^H_K(w_K) - A^H_F(w_F) + ||F - K||_-.
\]

Conversely, switch the roles of \( K \) and \( F \). Then we obtain the inequality

\[
0 \leq A^H_F(w_F) - A^H_K(w_K) + ||K - F||_-.
\]
But we recall the Definition 4.1, it follows that \( ||K - F||_+ = ||F - K||_+ \). This finishes the proof. \( \square \)

8.5. **Filtered Rabinowitz Floer Homology.** For \( a < b \in \mathbb{R} \), we define the \( \mathbb{Z}/2 \)-module

\[
\text{CF}^{(a,b)}(\mathcal{A}_F^H) := \text{Crit}^{(a,b)}(\mathcal{A}_F^H) \otimes \mathbb{Z}/2. \tag{8.17}
\]

where

\[
\text{Crit}^{(a,b)}(\mathcal{A}_F^H) := \left\{ (v, \eta) \in \text{Crit} \mathcal{A}_F^H | \mathcal{A}_F^H(v, \eta) \in (a, b) \right\}. \tag{8.18}
\]

We also define \( \text{CF}^{[a,b]} \), \( \text{CF}^{(a,b]} \) and \( \text{CF}^{[a,b)} \) in a similar way.

Then \( (\text{CF}^{(-\infty,b]}(\mathcal{A}_F^H), \partial^b) \) is a sub-complex of \( (\text{CF}(\mathcal{A}_F^H), \partial) \) since the gradient flow lines of \( \mathcal{A}_F^H \) flow downhill. Here \( \partial^b := \partial_{|\text{CF}^{(-\infty,b]}}. \)

There are canonical homomorphisms

\[
i^{b,c}_{a,b} : \text{CF}^{(a,b)}(\mathcal{A}_F^H) \rightarrow \text{CF}^{(a,c)}(\mathcal{A}_F^H) \quad a \leq b \leq c \tag{8.19}
\]

and

\[
\pi^{c}_{a,b} : \text{CF}^{(a,c)}(\mathcal{A}_F^H) \rightarrow \text{CF}^{(b,c)}(\mathcal{A}_F^H) \quad a \leq b \leq c. \tag{8.20}
\]

We note that

\[
\text{CF}^{(a,c)}(\mathcal{A}_F^H) = \text{CF}^{(a,b)}(\mathcal{A}_F^H) \oplus \text{CF}^{(b,c)}(\mathcal{A}_F^H), \tag{8.21}
\]

\( \pi^{c}_{a,b} \) is a projection along \( \text{CF}^{(a,b)}(\mathcal{A}_F^H) \). We suppress the indices \( a, b \) and \( c \) if there is no confusion. Moreover we have a short exact sequence

\[
0 \rightarrow \text{CF}^{(-\infty,a]}(\mathcal{A}_F^H) \xrightarrow{i} \text{CF}^{(-\infty,b]}(\mathcal{A}_F^H) \xrightarrow{\pi} \text{CF}^{(a,b)}(\mathcal{A}_F^H) \rightarrow 0. \tag{8.22}
\]

Thus the complex \( (\text{CF}^{(a,b)}(\mathcal{A}_F^H), \partial^b) \) naturally induces a homology group, namely filtered Rabinowitz Floer homology

\[
\text{RFH}^{(a,b)}(\Sigma, M, F) = H(\text{CF}^{(a,b)}(\mathcal{A}_F^H), \partial^b). \tag{8.23}
\]

More generally for \( a \leq b \leq c \), we have

\[
0 \rightarrow \text{CF}^{(a,b]}(\mathcal{A}_F^H) \xrightarrow{i} \text{CF}^{(a,c)}(\mathcal{A}_F^H) \xrightarrow{\pi} \text{CF}^{(b,c)}(\mathcal{A}_F^H) \rightarrow 0. \tag{8.24}
\]

The canonical homomorphism \( i, \pi \) and the boundary map \( \partial \) are compatible each other so that they induce canonical homomorphisms on the homology level, denote \( i_* \) and \( \pi_* \). Therefore we obtain the long exact sequence

\[
\cdots \rightarrow \text{RFH}^{(b,c)}_{j+1} \xrightarrow{\partial^b} \text{RFH}^{(a,b]}_{j} \xrightarrow{i_*} \text{RFH}^{(a,c)}_{j} \xrightarrow{\pi_*} \text{RFH}^{(b,c)}_{j} \xrightarrow{\partial^b} \text{RFH}^{(a,b]}_{j-1} \rightarrow \cdots. \tag{8.25}
\]

**Corollary 8.3.** The canonical homomorphism for the filtered case is defined as

\[
(\Phi^F_K)_*: \text{RFH}^{(a,b]}(\Sigma, M, K) \rightarrow \text{RFH}^{(a-||F-K||_+, b+||F-K||_-)}(\Sigma, M, F) \tag{8.26}
\]

**Proof.** It follows from the comparison of the action values of \( \mathcal{A}_K^H \) and \( \mathcal{A}_F^H \). (See Proposition 8.2.) \( \square \)
8.6. Local Rabinowitz Floer Homology. \(^1\) For \(||F|| < \varphi(\Sigma)\) we define

\[
\text{Crit}_{\text{loc}}(\mathcal{A}_F^H) := \left\{ (v, \eta) \in \text{Crit}_{\text{loc}}(\mathcal{A}_F^H) \mid -||F||_+ \leq \mathcal{A}_F^H(u, \eta) \leq ||F||_+ \right\}.
\] (8.27)

We note that the set \(\text{Crit}_{\text{loc}}(\mathcal{A}_F^H)\) is finite. This follows from the Arzela-Ascoli theorem since the Lagrange multipliers \(\eta_i\) is uniformly bounded according to Theorem 4.12. We define the finite dimensional \(\mathbb{Z}/2\) vector space

\[
\text{CF}_{\text{loc}}(\mathcal{A}_F^H) := \text{Crit}_{\text{loc}}(\mathcal{A}_F^H) \otimes \mathbb{Z}/2.
\] (8.28)

\((\text{CF}_{\text{loc}}(\mathcal{A}_F^H), \partial_{\text{loc}})\) is a differential complex since the action along gradient flow lines is decreasing. Define local Rabinowitz Floer homology by

\[
\text{RFH}_{\text{loc}}(\Sigma, M, F) := H(\text{CF}_{\text{loc}}(\mathcal{A}_F^H), \partial_{\text{loc}}).
\] (8.29)

**Definition 8.4.** We denote the number of leafwise intersection points of \(\phi_H \in \text{Ham}(M, \omega)\) by \(\nu_{\text{leaf}}(\phi_H)\).

**Proposition 8.5.** If \(||F|| < \varphi(\Sigma)\), the following inequalities hold.

\[
\nu_{\text{leaf}}(\phi_H) \geq \dim \text{CF}_{\text{loc}}(\mathcal{A}_F^H) \geq \dim \text{RFH}_{\text{loc}}(\Sigma, M, F).
\] (8.30)

**Proof.** The last inequality is obvious. In order to show the first inequality, it suffices to show that different critical points give rise the distinct leafwise intersection points. Suppose that two critical points \((v, \eta) \neq (v', \eta') \in \text{Crit}_{\text{loc}}(\mathcal{A}_F^H)\) give rise the same leafwise intersection point. That is, \(v(1) = v(0) = v'(0) = v'(1)\) and according to Proposition 4.3 \(v(\frac{1}{2}) = \phi_F(v(0)) = \phi_F(v'(0)) = v'(\frac{1}{2})\). Recall that \(F(t, \cdot) = 0\) for \(t \geq \frac{1}{2}\), \(v\) and \(v'\) satisfy \(\partial_t v = \sum_{i=1}^{k} \eta_i X_{H_i}(v(t))\) and \(\partial_t v' = \sum_{i=1}^{k} \eta_i X_{H_i}(v'(t))\) for \(t \in [\frac{1}{2}, 1]\). Thus the map

\[
\gamma := v' \big|_{[\frac{1}{2}, 1]} \# v \big|_{[\frac{1}{2}, 1]}
\] (8.31)

is a nontrivial closed orbit solving the equation (Bo) where \(\bar{v}\) is the path \(v\) traversed in the opposite direction that is, \(\bar{v}(t) = v(1-t)\) for \(0 \leq t \leq 1\). The closed orbit \(\gamma\) is homotopic to \(v' \# v\), therefore \(\gamma\) is contractible since \(v' \# v\) is the concatenation of two contractible loops. With the examined so far, we compute the difference of action values of \((v, \eta)\) and \((v', \eta)\),

\[
\left| \mathcal{A}_F^H(v, \eta) - \mathcal{A}_F^H(v', \eta') \right| = \left| \int_{\frac{1}{2}}^{1} v^* \alpha_i + \int_{0}^{\frac{1}{2}} v^* \alpha_i - \int_{0}^{\frac{1}{2}} F(t, v) dt - \int_{\frac{1}{2}}^{1} \langle \eta, \mathcal{H}(t, v) \rangle dt \right|
\]

\[
- \left( \int_{0}^{\frac{1}{2}} v^* \alpha_i - \int_{\frac{1}{2}}^{1} (v')^* \alpha_i + \int_{\frac{1}{2}}^{1} F(t, v) dt - \int_{\frac{1}{2}}^{1} \langle \eta', \mathcal{H}(t, v) \rangle dt \right)\]

\[
= \left| \int_{\frac{1}{2}}^{1} v^* \alpha_i - \int_{\frac{1}{2}}^{\frac{1}{2}} (v')^* \alpha_i \right|
\]

\[
= \left| \int_{S^1} \gamma^* \alpha_i \right| \geq \varphi(\Sigma) > ||F||
\]

\(^1\)All of lemmas and propositions in this subsection were established on hypersurfaces in [AF1].
But by definition of $\text{Crit}_{\text{loc}} \mathcal{A}^H_F$, we have
\[ ||F|| = ||F||_+ + ||F||_- \geq |\mathcal{A}^H_F(v, \eta) - \mathcal{A}^H_F(v', \eta')| > ||F||. \] (8.33)
It contradicts the assumption that $(v, \eta) \neq (v', \eta')$ give rise the same leafwise intersection point, therefore the proof follows. □

**Proposition 8.6.** The local Rabinowitz Floer homology of $\mathcal{A}^H$ is isomorphic to the singular homology of $\Sigma$. In other word, we have an isomorphism $\Theta$ as follows:
\[ H(\Sigma; \mathbb{Z}/2) \cong \text{RFH}_{\text{loc}}(\Sigma, M). \] (8.34)

**Proof.** By definition, $\text{Crit}_{\text{loc}} \mathcal{A}^H$ consists of critical points of $\mathcal{A}^H$ whose action values are zero. According to Proposition 3.2, (see (3.9) together) action value of the critical point is equal to the period of the solution of $(\text{Eq})$. Therefore $\text{Crit}_{\text{loc}} \mathcal{A}^H = \Sigma$, $\mathcal{A}^H$ is Morse-Bott locally around the action value zero. Since there is just one critical manifold $\Sigma$, there is no cascades and finally $HF_{\text{loc}}(\mathcal{A}^H)$ equals to the Morse homology on the critical manifold $\Sigma$. □

**Lemma 8.7.** For any $(a, b) \subset (-\wp(\Sigma), \wp(\Sigma))$, we have an isomorphism
\[ \text{RFH}^{(a,b)}(\Sigma, M) \cong \text{RFH}_{\text{loc}}(\Sigma, M). \] (8.35)

**Proof.** By the property of critical points of $\mathcal{A}^H$ we notice that $\text{Crit}_{\text{loc}} \mathcal{A}^H = \Sigma = \text{Crit}^{(a,b)}(\mathcal{A}^H)$. (See the proof of Proposition 8.6.) □

**Proposition 8.8.** For the Moser pair $(H, F)$ satisfying $||F|| < \wp(\Sigma)$, there exists an injective homomorphism
\[ \iota : H(\Sigma; \mathbb{Z}/2) \rightarrow \text{RFH}_{\text{loc}}(\Sigma, M, F). \] (8.36)
In particular, $\dim \text{RFH}_{\text{loc}}(\Sigma, M) \geq \dim H(\Sigma; \mathbb{Z}/2)$.

**Proof.** We have the continuation homomorphism by Corollary 8.3.
\[ (\Phi^F_0)_* : \text{RFH}_{\text{loc}}(\Sigma, M) \rightarrow \text{RFH}^{[-||F||_+, +||F||_-]}(\Sigma, M, F) = \text{RFH}_{\text{loc}}(\Sigma, M, F). \] (8.37)
Conversely, we have
\[ (\Phi^F_0)_* : \text{RFH}_{\text{loc}}(\Sigma, M, F) \rightarrow \text{RFH}^{[-||F||, ||F||]}(\Sigma, M). \] (8.38)

By Lemma 8.7 we have equalities
\[ \text{RFH}^{[-||F||, ||F||]}(M, \Sigma) = \text{RFH}_{\text{loc}}(\Sigma, M). \] (8.39)
Using the homotopy of homotopies $G^r_\epsilon(t, x) = \beta_r(s)(F(t, x)$, we know
\[ (\Phi^F_0)_* \circ (\Phi^F_0)_* = r_* \circ i_* = \text{id}_{\text{RFH}_{\text{loc}}(\Sigma, M)}. \] (8.40)
Therefore $(\Phi^F_0)_*$ is an injection, the Proposition follows with $\iota = (\Phi^F_0)_* \circ \Theta$. □

8.7. **Proof of Theorem C.** Theorem C immediately follows from Proposition 8.5 and Proposition 8.8. □
9. Infinitely many leafwise coisotropic intersections

In [AF2], Albers-Frauenfelder showed the interesting result that there are infinitely many leafwise intersection points on \((T^*B, S^*B)\) the (unit) cotangent bundle of closed manifold \(B\). (See also [AF3, AS, Me].) In this subsection, we see that if there is a nontrivial co-tractible solution for \((B_0)\) there are infinitely many leafwise coisotropic intersection points provided the Hofer norm of Hamiltonian diffeomorphism is less than certain value. This value unfortunately might equal to zero, in that case we cannot tell anything. However this value can be positive under the additional assumption. The main idea came from Usher’s paper [Us].

In order to describe our result, we need to remind the following well-known property in the standard Floer theory.

**Proposition 9.1.** For given two perturbation \(F\) and \(K\), consider the composition of the canonical homomorphisms \(\Phi^F_K \circ \Phi^K_F : CF(A^K_H) \longrightarrow CF(A^F_H)\). Then there exists a chain homotopy

\[
\mathfrak{h} : CF_*(A^K_H) \longrightarrow CF_{*-1}(A^F_H)
\]

between \(\Phi^K_F \circ \Phi^K_F\) and \(id_{CF(A^K_H)}\). Furthermore we have the following estimate.

\[
L_K(\mathfrak{h}\xi) \leq L_K(\xi) + ||F - K|| \quad \text{for} \quad \xi = \sum_{(v,\eta)\in \text{Crit}A^F_H} \xi_{(v,\eta)}(v,\eta) \in CF(A^F_H)
\]

where \(L_K\) is defined as

\[
L_K(\xi) := \max\{A^H_K(v,\eta) \mid \xi_{(v,\eta)} \neq 0\}.
\]

**Definition 9.2.** We denote by

\[
\nu := \inf \left\{ E(w) \mid w \text{ is a gradient flow line interchanging two critical points of index difference one.} \right\}.
\]

We are now in a position to introduce a proposition in [Us]. For the sake of completeness, we restate his proposition to fit in our situation.

**Proposition 9.3.** [Us] For given Hamiltonian function \(F\), if \(||F|| < \nu\) then the continuation homomorphism \(\Phi^0_F : CF(A^H) \longrightarrow CF(A^H)\) is injective on the chain level.

**Proof.** As we have mentioned in the previous Proposition, \(\mathfrak{h}\) is a chain homotopy between \(\Phi^0_F \circ \Phi^F_0\) and \(id_{CF(A^H_H)}\). In other word, we have

\[
\Phi^K_F \circ \Phi^F_0 - \text{id}_{CF(A^H_H)} = \partial \circ \mathfrak{h} - \mathfrak{h} \circ \partial.
\]

Let abbreviate \(A = \partial \circ \mathfrak{h} - \mathfrak{h} \circ \partial\), then we know that \(L_0(A^j\xi) \leq L_K(\xi) - j(\nu - ||F||)\). Next, we define

\[
B := \sum_{j=0}^{\infty} (-A)^j : CF(A^H) \longrightarrow CF(A^H).
\]

This map is well defined since \(L_K(A^j\xi)\) goes to \(-\infty\) negative infinity as \(j \to \infty\). We operate \(B\) on the both side of equation (9.5), then we have

\[
B \circ \Phi^0_F \circ \Phi^F_0 = B \circ (id_{CF(A^H_H)} + A) = id_{CF(A^H_H)}.
\]

Therefore \(\Phi^F_0\) is injective, the proof is completed. □
9.1. Proof of Theorem D. Since there is at least one nonconstant solution \((v, \eta)\) of equation (Bo), we have infinitely many nontrivial solutions, namely \(n_* (v, \eta)\) for all \(n \in \mathbb{N}\). It means that the cardinality of \(\mathrm{CF}(\mathcal{A}_F^\mathcal{H})\) is infinite. By setting \(\gamma = \nu\), Proposition 9.3 implies that \(\mathrm{CF}(\mathcal{A}_F^\mathcal{H})\) also have infinitely many elements. Thus there are infinitely many critical points of \(\mathcal{A}_F^\mathcal{H}\). To complete the proof it suffices to show that \(\nu\) is strictly bigger than zero and critical points give rise distinct leafwise coisotropic intersection points for a generic Hamiltonian diffeomorphism. In order to show the inequality \(\nu > 0\) let \(\eta^l = \frac{\eta}{a^l}\) with \(\gcd(a^l, b^l) = 1\) for all \(a_i, b_i \in \mathbb{N}\). Suppose there is a gradient flow line \(w\) interchanging between \(n_*(v^j, \eta^j)\) and \(m_*(v^m, \eta^m)\) for \(1 \leq l, j \leq m\) and \(n, m \in \mathbb{N}\), then the energy of \(w\) always positive and equals to the difference of Lagrange multipliers. Thus we estimate

\[
E(w) = |n l^l - m m^j| = \left| n \frac{b^l}{a^l} - m \frac{b^j}{a^j} \right| = \frac{|\gcd(b^l, b^j)|}{|\text{lcm}(a^l, a^j)|} |n a^l b^j - m a^j b^j| \geq \frac{|\gcd(b^l, b^j)|}{|\text{lcm}(a^l, a^j)|}.
\]

Here \(a^d := \frac{\text{lcm}(a^l, a^j)}{a^l}, a^l := \frac{\text{lcm}(a^l, a^j)}{a^l}, b^l := \frac{b^l}{\gcd(a^l, b^l)}\) and \(b^j := \frac{b^j}{\gcd(a^j, b^j)}\). Therefore we have

\[
\nu \geq \max \left\{ \frac{|\gcd(b^l, b^j)|}{|\text{lcm}(a^l, a^j)|} \right\} \quad \text{for all } 1 \neq j \leq k
\]

Therefore Lemma 9.5 with the non-degeneracy of \(\Sigma\) and Proposition 9.6 concludes the proof of Theorem D. □

Lemma 9.4. (Salamon) Let \(\mathcal{E} \rightarrow \mathcal{B}\) be a Banach bundle and \(s : \mathcal{B} \rightarrow \mathcal{E}\) a smooth section. Moreover, let \(\phi : \mathcal{B} \rightarrow N\) be a smooth map into the Banach manifold \(N\). We fix a point \(x \in s^{-1}(0) \subset \mathcal{B}\) and set \(K := \ker d\phi(x) \subset T_x\mathcal{B}\) and assume the following two conditions.

(i) The vertical differential \(d\phi(x) : T_x\mathcal{B} \rightarrow T_{\phi(x)}N\) is surjective.

(ii) \(d\phi(x) : T_x\mathcal{B} \rightarrow T_{\phi(x)}N\) is surjective.

Then \(d\phi(x)|_{\ker d\phi(x)} : \ker d\phi(x) \rightarrow T_{\phi(x)}N\) is surjective.

Proof. We fix \(\xi \in T_{\phi(x)}N\). Condition (ii) implies that there exists \(\eta \in T_x\mathcal{B}\) satisfying \(d\phi(x) \eta = \xi\). Condition (i) implies that there exists \(\zeta \in K \subset T_x\mathcal{B}\) satisfying \(D\phi(x) \zeta = D\phi(x) \eta\). We set \(\tau := \eta - \zeta\) and compute

\[
D\phi(x) \tau = D\phi(x) \eta - D\phi(x) \zeta = 0
\]

thus, \(\tau \in \ker D\phi(x)\). Moreover,

\[
d\phi(x) \tau = d\phi(x) \eta - d\phi(x) \zeta = d\phi(x) \eta - \eta = \xi
\]

proving the Lemma. □

Lemma 9.5. Any two critical points of \(\mathcal{A}_F^\mathcal{H}\) give rise to distinct leafwise coisotropic intersection points unless the leaf a close orbit solving equation pass through (Eq) all leafwise coisotropic intersection point \(x \in \Sigma\).

Proof. It easily follows from the proof of Proposition 8.5. □

Proposition 9.6. [AF2] Let \(\Sigma \subset M\) be a non-degenerate of contact type submanifold. If \(\dim M \geq 4\) then the set

\[
\mathcal{F}_\Sigma := \{ F \in \mathcal{F}^\infty \mid \text{im}(v) \cap \text{im}(\gamma) = \emptyset \quad \forall (v, \eta) \in \text{Crit}_F^\mathcal{H}, \gamma \in \mathcal{R} \}
\]

is dense in \(\mathcal{F}^\infty = \bigcap_{i=1}^k \mathcal{F}^j\).
Proof. Consider the evaluation map
\[
ev : S^{-1}(0) \to \Sigma
\]
\[
(v, \eta, F) \mapsto v(\frac{1}{2}).
\]
(9.13)

From Proposition 7.9 and Lemma 9.4, the evaluation map
\[
ev_F := \ev(\cdot, \cdot, F) : s^{-1}_F(0) = \text{Crit}^H_F \to \Sigma
\]
(9.14)
is a submersion for generic \(F \in \mathcal{F}_j\).

Thus the preimage of the one dimensional set
\[
R^\tau := \{ \gamma \text{ solves (Eq) with period } \leq \tau \}
\]
under \(\ev_F\) does not intersect \(\text{Crit}^H_F\) using that \(\dim M \geq 4\). Therefore, the set
\[
\mathcal{F}_\Sigma := \{ F \in \mathcal{F}_j \mid \text{im}(v) \cap \text{im}(\gamma) = \emptyset \quad \forall (v, \eta) \in \text{Crit}^H_F, \quad \gamma \in R^j \}
\]
(9.16)
is generic in \(\mathcal{F}\) for all \(j \in \mathbb{N}\). We note that \(\mathcal{F}_\Sigma = \bigcap_{j=1}^{\infty} \mathcal{F}_j^j\), it finishes the proof of Proposition. \(\square\)

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