NEW EXACT SOLUTIONS FOR SOME FRACTIONAL ORDER DIFFERENTIAL EQUATIONS VIA IMPROVED SUB-EQUATION METHOD

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Abstract. In this paper, improved sub-equation method is proposed to obtain new exact analytical solutions for some nonlinear fractional differential equations by means of modified Riemann Liouville derivative. The method is applied to time-fractional biological population model and space-time fractional Fisher equation successfully. Finally, simulations of new exact analytical solutions are presented graphically.

1. Introduction. Fractional calculus, defined by the generalization of the order of classical (traditional) calculus to the arbitrary real or complex order, is a concept which is as old and deep-rooted as classic calculus [5]. Towards the end of the 17th century, it has arisen with some spec between L’Hospital and Leibnitz, then developed with the studies of the renowned mathematicians such as Laplace, Abel, Fourier, Liouville, Riemann, Grunwald and Letnikov [19]. In addition to these developments recently, Atangana and Baleanu suggested a new fractional order derivative, the new derivative based on the generalized Mittag-Leffler function and the fractional derivative has non-singular and nonlocal kernel [3]. With the help of the new derivative, many new papers are seen in the literature, some of these are, Alkahtani analyse chaotic behavior on the Chaos circuit model [20]. Atangana and Koca, consider the modification of Lorenz attractor model. In this study, new properties were presented of the fractional derivatives [4].

Obtaining solution of fractional differential equations has an important role because of its application area such as dynamical systems in control theory, electrical circuits with fractance, generalized voltage divider, viscoelasticity, electro chemistry and model of neurons in biology.

Thus, several methods have been developed for the purpose of obtaining numerical and exact solutions for these equations. These: Adomian Decomposition Method [10, 21], Homotopy Perturbation Method [16, 17], Variational Iteration Method [18, 24], $\left(G'/G\right)$—expansion method [7, 8], Finite Difference Method [9, 25], Finite Element Method [14, 11], fractional sub-equation method [15, 2] and Improved fractional sub-equation method [23, 22].

2010 Mathematics Subject Classification. 83C15, 34A08.
Key words and phrases. Improved sub-equation method, modified Riemann Liouville derivative, exact analytical solution, time-fractional biological population model, space-time fractional Fisher equation.
In this study, our aim is to carry out new exact analytical solutions of the time fractional biological population model and the space-time fractional Fisher equation by the aid of improved sub-equation method. The observed fractional derivative in the equations is symbolized by \( \alpha \)-order modified Riemann Liouville derivative. Here are aforesaid equations

i) Time fractional biological population model is given by [10, 16]

\[
D_\alpha^u \left( u^2 \right)_x + u^2 (u^2)_y + h (u^2 - r) \quad 0 < \alpha \leq 1
\]

(1)

where \( h \) and \( r \) are any real constants. \( u \) and \( h (u^2 - r) \) denote population density and the amount of population due to death and birth, respectively. Eq. (1.1) has an important role to understand the dynamic process of population changes and it is also an assistant to achieve precision about it.

ii) Space-time fractional Fisher equations given by the following equation

\[
D_\alpha^u \left( u^2 \right)_x = 2u \left( 1 - u^2 \right) + \epsilon (1 - u^2) \quad 0 < \alpha \leq 1
\]

(2)

The original equation (\( u_t = u_{xx} - u \left( 1 - u \right) \)) is a model for the spatial and temporal propagation of a virile gene in an infinite medium, proposed by Fisher [12].

2. Description modified Riemann Liouville derivative and improved sub-equation method. Recently, in many studies, fractional Riemann Liouville derivative has been used as the modified Riemann Liouville derivative. According to this derivative, derivative of a constant is equal to zero and it is applicable for any continuous (nondifferentiable) functions. So by the help of this new concept, it has overcome the shortcomings of previous ones. Modified Riemann Liouville derivative is expressed as

\[
D_\alpha^u f(x) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{\alpha-1} [f(\xi) - f(0)] d\xi & \alpha < 0 \\
\frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi & 0 < \alpha < 1 \\
\left[ f(x) \right]^{(n)} & n \leq \alpha < n + 1, n \geq 1
\end{array} \right.
\]

(3)

and some useful properties of the derivative are given below;

\[
\begin{align*}
D_\alpha^u x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha} & \gamma > 0, \\
D_\alpha^u \left[ f(x) g(x) \right] &= g(x) D_\alpha^u f(x) + f(x) D_\alpha^u g(x), \\
D_\alpha^u f \left[ g(x) \right] &= f' \left[ g(x) \right] D_\alpha^u g(x) &= D_\alpha^u f \left[ g(x) \right] \left( g'(x) \right)^\alpha,
\end{align*}
\]

(4)

Eqs. (3) and (4) which are going to be used in the application of the improved sub-equation method, which are important tools for fractional calculus. In the rest of this section, basic steps of the method are going to be present. At the third section, application of the method and obtained results is going to take place.

Let us assume a fractional order partial differential equation with is presented polynomial \( P \) including various order derivatives as

\[
P \left( u, u_t, u_x, u_y, D_\alpha^u u, D_\beta^u u, D_\gamma^u u, ... \right) = 0
\]

(5)

where \( x, y, t \) are independent variables and \( u(x, y, t) \) is an unknown function and polynomial \( P \) includes the highest order derivative and nonlinear term of \( u(x, y, t) \). Also, \( D_\alpha^u (\cdot) \), symbolizes modified Riemann Liouville fractional derivation.

**Step 1.** First of all, using a suitable fractional complex transform

\[
u(x, y, t) = u(\xi) \quad \xi = \xi(x, y, t)
\]

(6)
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$P$ polynomial given in Eq. (5) converts into nonlinear ordinary differential equation given below

$$P(u, u', u'', u''' ... ) = 0.$$  \hspace{1cm} (7)

**Step 2.** Assuming, the solution form of Eq. (7) is

$$u = \sum_{i=-n}^{-1} a_i \varphi^i + a_0 + \sum_{i=1}^{n} a_i \varphi^i.$$  \hspace{1cm} (8)

During the process, constants $a_i$ ($i = -n, ..., n$) are going to be determined. Here, $n$ is a positive integer and it is obtained using the homogeneous balance of the highest order derivative and the nonlinear term seen in Eq. (7). Function $\varphi = \varphi(\xi)$ is the solution of Riccati equation

$$\varphi' = \sigma + \varphi^2$$  \hspace{1cm} (9)

where $\sigma$ is a constant and the solutions of the equation (9) are obtained by Zhang [26] using the generalized Exp-function method as follow

$$\varphi(\xi) = \begin{cases} 
-\sqrt{-\sigma} \tanh(\sqrt{-\sigma} \xi) & \sigma < 0 \\
-\sqrt{-\sigma} \coth(\sqrt{-\sigma} \xi) & \sigma < 0 \\
\sqrt{\sigma} \tan(\sqrt{\sigma} \xi) & \sigma > 0 \\
-\sqrt{\sigma} \cot(\sqrt{\sigma} \xi) & \sigma > 0 \\
-\frac{\Gamma(1+\alpha)}{\xi^{\alpha+w}} & \text{w is a const., } \sigma = 0
\end{cases}$$  \hspace{1cm} (10)

**Step 3.** In this step of the method, using necessary derivatives of Eq. (8) with Eq. (9) in Eq. (7), we obtain a new polynomial in terms of $\varphi$. This new polynomial is arranged according to the powers of $\varphi^k$ ($k = 0, 1, 2, ..., -1, -2, ...$), then all coefficients of $\varphi^k$ are set equal to zero.

**Step 4.** Finally, algebraic equations are obtained in the previous step for $a_i$ ($-n \leq i \leq n$), and $\sigma$ are solved with the help of symbolic programming MAPLE. Then the solutions of fractional differential equation given in (5) are obtained using the newly obtained values, Riccati equation given in Eq. (9) and (10).

3. **Applications.** In this section of the paper, we are going to present the application of improved sub-equation method to time-fractional biological model and space-time fractional Fisher equation and obtain results for these fractional differential equations. The obtained results are compared with those available in the literature and also new results are presented graphically.

3.1. **Time-fractional biological population model.** As the first example, we consider time-fractional biological model given in (1), after applying travelling wave transformation to the Eq. (1.1)

$$u(x, y, t) = u(\xi) \quad \xi = kx + iky + \frac{ct^\alpha}{\Gamma(1+\alpha)}, \quad i^2 = -1$$  \hspace{1cm} (11)

it is reduced to ordinary differential equation as

$$cu' - h (u^2 - r) = 0.$$  \hspace{1cm} (12)

The solution of Eq. (12) is the form (8) and here $n$ is get from the homogeneous balance between the highest order derivative $u'$ and the nonlinear term $u^2$ follow;

$$n + 1 = 2n \Rightarrow n = 1$$  \hspace{1cm} (13)
we obtain the solution of Eq. (12) as

\[ u(\xi) = a_{-1}\varphi^{-1} + a_0 + a_1\varphi \]  

(14)

Substituting Eq. (14) together with its necessary derivatives into Eq. (12), then the algebraic equation is arranged according to the powers of the function \(\varphi^k(\xi)\). Thus the following coefficients are obtained

\[
\begin{align*}
\varphi^0 : & \; c\sigma a_1 - ha_1^2 + hr - ca_{-1} - 2ha_1a_{-1} \\
\varphi^1 : & \; -2ha_0a_1 \\
\varphi^2 : & \; ca_1 - ha_1^2 \\
\varphi^{-1} : & \; -2ha_0a_{-1} \\
\varphi^{-2} : & \; -ca_{-1}\sigma - ha_1^2
\end{align*}
\]  

(15)

letting the coefficients to be zero in Eq. (15), then solving the set of equations given above for \(a_{-1}, a_0, a_1\) and \(\sigma\) we obtain solution sets following as

\[
\begin{align*}
\text{set 1 : } & \{ a_{-1} = 0, \; a_0 = 0, \; a_1 = \frac{c}{h}, \; \sigma = -\frac{h^2r}{c^2}\} \\
\text{set 2 : } & \{ a_{-1} = \frac{hr}{c^2}, \; a_0 = 0, \; a_1 = 0, \; \sigma = -\frac{h^2r}{c^2}\} \\
\text{set 3 : } & \{ a_{-1} = \frac{hr}{4c}, \; a_0 = 0, \; a_1 = \frac{c}{h}, \; \sigma = -\frac{h^2r}{4c^2}\}
\end{align*}
\]  

(16)

In this study, the solutions of differential equations are symbolized as \(u_i^j(i, j \in Z^+)\). Here, \(i\) denotes obtained set number and \(j\) is the solution number of Riccati equation, respectively. Thus, using set 1 and set 2 we obtain the following solutions;

\[
\begin{align*}
u_1^1 &= u_2^2 = \mp \sqrt{r}\tanh\left[\frac{\pm h}{c}\sqrt{r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma < 0 \\
u_1^2 &= u_2^1 = \mp \sqrt{r}\coth\left[\frac{\pm h}{c}\sqrt{r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma < 0 \\
u_3^1 &= u_2^4 = \pm \sqrt{-r}\tan\left[\frac{\pm h}{c}\sqrt{-r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma > 0 \\
u_4^1 &= u_3^4 = \mp \sqrt{-r}\cot\left[\frac{\pm h}{c}\sqrt{-r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma > 0 \\
u_5^1 &= -\frac{\Gamma(1+\alpha)}{h((kx + iky + \frac{ct^n}{\Gamma(1+\alpha)})\varphi + w)}, \quad \text{w is cons.} \; \sigma = 0
\end{align*}
\]  

(17)

and with set 3

\[
\begin{align*}
u_1^3 &= u_2^2 = \mp \frac{c}{2}\sqrt{r}\tanh\left[\frac{\pm h}{2c}\sqrt{r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma < 0 \\
u_1^3 &= u_2^1 = \mp \frac{c}{2}\sqrt{r}\coth\left[\frac{\pm h}{2c}\sqrt{r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma < 0 \\
u_3^1 &= u_2^2 = \mp \frac{c}{2}\sqrt{r}\tanh\left[\frac{\pm h}{2c}\sqrt{r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma > 0 \\
u_4^1 &= u_3^4 = \mp \frac{c}{2}\sqrt{r}\coth\left[\frac{\pm h}{2c}\sqrt{r}(kx + iky + \frac{ct^n}{\Gamma(1+\alpha)}\right], \quad \sigma > 0 \\
u_5^1 &= \frac{\Gamma(1+\alpha)}{h((kx + iky + \frac{ct^n}{\Gamma(1+\alpha)})\varphi + w)}, \quad \text{w is cons.} \; \sigma = 0.
\end{align*}
\]  

(18)

Finding exact solutions of fractional biological population model has attracted many researchers. One can see Refs. [1, 27, 6]. However, in the present study, we have obtained different exact analytical solution according to studies mentioned. Also, some of these solutions are similar with study of Zhang and Zhang [27]. the advantage of improved sub equation method than sub-equation method given is to obtain new solutions with the help of coefficient \(a_{-1}\) that encountered in (14). At the end of this example, graphical representation of exact analytical solution \(u_1^3\) are presented for values \(k = c = 1, r = -1.4, h = 0.1, \alpha = 1, t = 5\) and interval \((x, y) \in [-15, 15]\) at Fig. 1.
3.2. **Space-time fractional Fisher equation.** In the second example, we are going to consider space-time fractional Fisher equation given in (2). Applying the following wave transformation to the equation [13],

\[
u(x, t) = \nu(\xi) \quad \xi = \frac{kx^\alpha}{\Gamma(1 + \alpha)} - \frac{ct^\alpha}{\Gamma(1 + \alpha)}
\]  

we obtain following ordinary differential equation follow

\[
cu' + k^2 u'' - 2u^3 - \varepsilon u^2 + 2u + \varepsilon = 0
\]  

with the help of homogeneous balance, we calculate \(n = 1\). Thus, we can rewrite the solution of Eq. (20) as

\[
u(\xi) = a_{-1} \varphi^{-1} + a_0 + a_1 \varphi
\]  

Similar to the previous example, if we substitute the solution and its necessary derivatives into Eq. (20) and arrange, we acquire coefficients of algebraic equation according to power of \(\varphi\) as

\[
\begin{align*}
\varphi^0 : & ( -12a_0 - 2 \varepsilon a_{-1} + ca \sigma) a_1 - 2a_0^3 - \varepsilon a_0 - ca_{-1} + 2a_0 + \varepsilon \\
\varphi^1 : & 2a_1 (k^2 \sigma - \varepsilon a_0 - 3a_0^2 - 3a_1 a_{-1} + 1) \\
\varphi^2 : & a_1 ( -6a_0 - \varepsilon a_1 + c) \\
\varphi^3 : & 2k^2 a_1 - 2a_1^3 \\
\varphi^{-1} : & 2a_{-1} (k^2 \sigma - \varepsilon a_0 - 3a_0^2 - 3a_1 a_{-1} + 1) \\
\varphi^{-2} : & - a_{-1} ((6a_0 + \varepsilon) a_{-1} + c \sigma)
\end{align*}
\]  

By solving the coefficients given above with the aid of symbolic programming, we get the solution sets as given below

- **set1**: \(a_{-1} = 0 \quad a_0 = 0 \quad a_1 = k \quad c = \varepsilon k \quad \sigma = -\frac{1}{k^2}\)
- **set2**: \(a_{-1} = \frac{1}{k} \quad a_0 = 0 \quad a_1 = 0 \quad c = \varepsilon k \quad \sigma = -\frac{1}{k^2}\)
- **set3**: \(a_{-1} = 0 \quad a_0 = -\frac{\varepsilon}{4} - \frac{1}{2} \quad a_1 = -k \quad c = \frac{k^2}{2} + 3k \quad \sigma = -\frac{(\varepsilon - 2)^2}{64k^2}\)
- **set4**: \(a_{-1} = -\frac{(\varepsilon - 2)^2}{64k^2} \quad a_0 = -\frac{\varepsilon}{4} - \frac{1}{2} \quad a_1 = -k \quad c = \frac{k^2}{2} + 3k \quad \sigma = -\frac{(\varepsilon - 2)^2}{64k^2}\)

using solution sets and Eqs.(10) into (21), we get Using set 1 and 2 for \(c = \varepsilon k\)

\[
\begin{align*}
u_1^1 = u_1^1 = u_2^2 = u_3^2 = -\tanh \left[ \frac{kx^\alpha - ct^\alpha}{\Gamma(1 + \alpha)} \right] \\
u_2^1 = u_1^2 = u_3^2 = -\coth \left[ \frac{kx^\alpha - ct^\alpha}{\Gamma(1 + \alpha)} \right]
\end{align*}
\]  

**Figure 1.** The numerical simulations of real and imaginal part for fractional Biological Population Model for \(u_1^3\), respectively.
set 3 for $c = \frac{\epsilon k}{2} + 3k, \sigma = -\frac{(\epsilon - 2)^2}{64k^2}$

$$u_3^3 = u_3^2 = \frac{1}{4} \left(-2 - \epsilon \pm (\epsilon - 2) \tanh \left[ \pm \frac{(-2)(\frac{k\alpha^\alpha - \epsilon}{1+\alpha})}{4k} \right] \right)$$

$$u_2^3 = u_4^3 = \frac{1}{4} \left(-2 - \epsilon \pm (\epsilon - 2) \coth \left[ \pm \frac{(-2)(\frac{k\alpha^\alpha - \epsilon}{1+\alpha})}{8k} \right] \right)$$

(25)

set 4 for $c = (\frac{\epsilon k}{2} + 3k), \sigma = -\frac{(\epsilon - 2)^2}{64k^2}$, respectively

$$u_1^4 = u_2^4 = u_3^4 = u_4^4 = \frac{1}{8} \left(-2(2 + \epsilon) \pm (\epsilon - 2) \coth \left[ \pm \frac{(-2)(\frac{k\alpha^\alpha - \epsilon}{1+\alpha})}{8k} \right] \right)$$

$$- (\epsilon - 2) \tanh \left[ \pm \frac{(-2)(\frac{k\alpha^\alpha - \epsilon}{1+\alpha})}{8k} \right] \right)$$

(26)

In the end, we obtained the results given below for $\sigma = 0$ and $w$ is a constant

$$u_3^5 = u_5^4 = -1 + \frac{k\Gamma(1+\alpha)}{(\alpha + w)^{1+\alpha}}, \quad \text{for} \quad \epsilon = 2 \quad (27)$$

At the last, the simulations of some new exact analytical solutions of fractional Fisher equation are presented for $k = 0.5, \alpha = 1, \epsilon = 0.1$ and $x \in [-15, 15], t \in [0, 5]$ in Fig. 2.

![Figure 2](image-url)  

**Figure 2.** The numerical simulations of fractional Fisher equation for $u_1^1, u_2^1, u_3^1$ and $u_4^1$, respectively.

As a result of this example, to the best of our knowledge, the solutions which carry out in this paper, have not been seen in the literature.

4. **Conclusions.** In summary, Improved sub-equation method has been employed for finding new exact analytical solutions of Time-fractional biological population model and Space-time fractional fisher equation. Applying the method, fractional order differential equations are converted into ordinary differential equations. Then,
with the help of Riccati equation, generalized hyperbolic function and rational function solutions are obtained. Some of these experience solutions shown in figures are new in the literature. Consequently, the obtained results illustrate that the method is reliable and efficient method for solving a wide range of fractional order differential equations.

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Received for publication June 2017.

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