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The Segal–Bargmann transform for the heat equation associated with root systems✩

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Abstract

We study the heat equation associated to a multiplicity function on a root system, where the corresponding Laplace operator has been defined by Heckman and Opdam. In particular, we describe the image of the associated Segal–Bargmann transform as a space of holomorphic functions. In the case where the multiplicity function corresponds to a Riemannian symmetric space $G/K$ of non-compact type, we obtain a description of the image of the space of $K$-invariant $L^2$-function on $G/K$ under the Segal–Bargmann transform associated to the heat equation on $G/K$, thus generalizing (and reproving) a result of B. Hall and J. Mitchell for spaces of complex type.

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0. Introduction

Let \((M, g)\) be a Riemannian manifold. The heat equation on \(M\) is the partial differential equation \(\Delta u(x, t) = \partial_t u(x, t)\), where \(\Delta\) is the Laplace–Beltrami operator. Consider the corresponding Cauchy problem \(\Delta u(x, t) = \partial_t u(x, t), \lim_{t \searrow 0} u(x, t) = f(x)\), where \(f \in L^2(M, dv)\) and where \(dv\) is the volume form. The Cauchy problem is solved by the family of linear contraction maps

\[ H_t : L^2(M, dv) \ni f \mapsto u(\cdot, t) = e^{t\Delta} f \in L^2(M, dv). \]

Each map \(H_t\) has the form of an integral operator \(H_t f(x) = \int_M f(y)k_t(x, y)dv(y)\), where \(k_t\) is the heat kernel. The literature contains an extensive study of \(H_t\) and its kernel. We refer to [3,5] for introduction and an overview. One of the natural questions is to describe the image of \(H_t\). This can be done either using real analysis or, in several special cases, complex methods.

Let \(M = \mathbb{R}^r\) for a moment, so that we have the Fourier transform and a natural complexification at our disposal. It is easy to see that

\[ H_t(L^2(\mathbb{R}^r)) = \{ f \in L^2(\mathbb{R}^r) \mid e^{t|.|^2} \hat{f} \in L^2(\mathbb{R}^r) \}. \tag{0.1} \]

But \(H_t\) is also a smoothing operator, in fact the image consists of real analytic functions, which can be extended to entire functions on \(C^r\). Hence \(H_t\) can also be considered as a linear map \(H_t : L^2(\mathbb{R}^r) \to O(C^r)\), sometimes called the Segal–Bargmann transform (after [2,20]). It should however be noted that originally the Hilbert space \(L^2(\mathbb{R}^r)\) was replaced by \(L^2(\mathbb{R}^r, d\mu_t)\), where \(d\mu_t^r(x) = d\mu_t(x) = (4\pi t)^{-r/2}e^{-|x|^2/(4t)}\,dx\) is the heat kernel measure. The reason is, that \(d\mu_t^r\) forms a projective family of probability measures, which then defines a probability measure on the infinite dimensional space \(\mathbb{R}^\infty\). The image in \(O(C^r)\) of the Segal–Bargmann transform is the Fock space

\[ \mathcal{F}_t(C^r) := \left\{ F \in O(C^r) \mid \| F \|_t^2 := (2\pi t)^{-r/2} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} |F(x + iy)|^2 e^{-|y|^2/(2t)}\,dx\,dy < \infty \right\} \tag{0.2} \]

and \(H_t : L^2(\mathbb{R}^r) \to \mathcal{F}_t(C^r)\) is a unitary isomorphism, cf. [2,20].

The obvious problem in the general case is that there is no “natural” complexification of \(M\). An important class of spaces, where such a complexification exists, is the class of Riemannian symmetric spaces. The first work in this direction was the article by Hall [4]. Here \(\mathbb{R}^r\) is replaced by a connected compact semisimple Lie group \(U\), and \(C^r\) is replaced by its complexification \(U_C\). This was put into a general framework using polarization of a restriction map in [16]. The results of Hall were extended to compact semisimple symmetric spaces \(U/K\) by Stenzel in [21]. In this case the complexification is given by \(U_C/K_C\). It is important to note that in the compact case, every eigenfunction of the algebra of invariant differential operators as well as the heat kernel itself, extends to a holomorphic function on \(U_C/K_C\). This is related to the fact, that each irreducible representation of \(U\) extends to a holomorphic representation of \(U_C\). In the non-compact case this does not hold, which makes the situation more complicated. The natural complexification in this case is the Akhiezer–Gindikin domain \(\Xi \subset G_C/K_C\), see [1]. Using results from [14] it was shown in [15] that the image of the Segal–Bargmann transform on \(G/K\) can be identified as a Hilbert-space of holomorphic functions on \(\Xi\). It was also shown that the norm on this space was not given by a density function. Some special cases have also been considered.
in [6,7], but without using the Akhiezer–Gindikin domain explicitly. In particular, in [7] B. Hall and J. Mitchell give a description of the image of the $K$-invariant functions in $L^2(G/K)$ in case $G$ is complex. The image is in fact isomorphic to the Fock space $\mathcal{F}_t(a_{\mathbb{C}})$ described in (0.2), where $a$ is the Lie algebra of a maximal abelian vector subgroup $A$ in $G$.

Here we shall also study the case of $K$-invariant functions, but in a generalized setting. The main tool in the study of $K$-invariant functions is the reduction to analysis on $A$ and $a$ by restriction. This is made possible by the Cartan decomposition $G = KAK$. On $A$ the radial part of the Laplacian is a singular differential operator with leading part the Laplacian of $A$. The main ingredients depend on the system of restricted roots and the multiplicities $m_\alpha$, i.e., the dimension of the root spaces. The setting of analysis on Riemannian symmetric spaces has been generalized by Heckman and Opdam by allowing arbitrary values for the multiplicities $m_\alpha$, which are thus replaced by a Weyl-group invariant function $m: \Delta \to \mathbb{R}$, see [10]. The spherical functions are replaced by the Heckman–Opdam hypergeometric functions, and the spherical Fourier transform becomes the hypergeometric Fourier transform. In order to have a Plancherel decomposition without discrete parts, it is common to assume that $m_\alpha \geq 0$ for all $\alpha \in \Delta$.

In this article we study the heat equation on $A$ corresponding to a non-negative multiplicity function. In particular our results include the description of the Segal–Bargmann transform on $L^2(G/K)_K$ for a Riemannian symmetric space of non-compact type. The paper is organized as follows. In the first section we introduce the necessary facts from the Heckman–Opdam theory. We discuss two special cases, first of all the geometric case, where the set up corresponds to a Riemannian symmetric space of the non-compact type, and secondly the case where the root system is reduced and $m_\alpha \in 2\mathbb{N}$ for all $\alpha$. The second section starts off by a short discussion of the Euclidean case $\mathbb{R}^r \simeq a$, and then we discuss the heat equation corresponding to a non-negative multiplicity function. In Theorem 2.4 we give a description of the image of the Segal–Bargmann transform similar to (0.1). The main result, Theorem 2.6, describes the image in terms of holomorphic function on $a_{\mathbb{C}}$ using the standard Fock space (0.2).

1. The hypergeometric functions and hypergeometric Fourier transform

In this section we collect basic facts about the Heckman–Opdam hypergeometric functions, the corresponding hypergeometric Fourier transform, and the Plancherel formula. We then specify those results to our main example of semisimple symmetric spaces and to the case where all multiplicities are even. Our standard references are [10,18].

1.1. The Heckman–Opdam hypergeometric functions

Let $a$ be a $r$-dimensional real vector space with inner product $(\cdot, \cdot)$. Let $\Delta \subset a^*$ be a root system, not necessarily reduced. We fix a positive system $\Delta^+ \subset \Delta$ and set

$$a^+ = \left\{ H \in a \mid \forall \alpha \in \Delta: \alpha(H) > 0 \right\}. \quad (1.1)$$

For $\lambda \in a^*$ define $h_\lambda \in a$ by $\alpha(H) = (H, h_\lambda)$ for all $H \in a$. We define an inner product on $a^*$ by $(\lambda, \mu) = (h_\lambda, h_\mu)$. If $\lambda \neq 0$ let $H_\lambda = \frac{2}{(\lambda, \lambda)} h_\lambda$. Then $\alpha(H_\alpha) = 2$. Note that $H_\lambda$ is independent of the inner product $(\cdot, \cdot)$. We extend $(\cdot, \cdot)$ to a $\mathbb{C}$-bilinear form on $a_{\mathbb{C}}$ and $a_{\mathbb{C}}^*$. The associated Weyl group, which is generated by the reflections $r_\alpha(H) = H - \alpha(H)H_\alpha$, is denoted $W$. The Weyl group acts on $a^*$ (and $a_{\mathbb{C}}^*$) by duality: $w\lambda(H) = \lambda(w^{-1}H)$. Recall that
Wa^+ = \{H \in a \mid \forall \alpha \in \Delta: \alpha(H) \neq 0\} is open and dense in a. Furthermore, if wa^+ \cap a^+ \neq \emptyset then w = e, and a \setminus Wa^+ has measure zero.

A multiplicity function on \Delta is a function \alpha : \Delta \rightarrow \mathbb{C}, invariant under the Weyl group. We denote by \mathcal{M}_+ the set of non-negative, i.e., \alpha(\alpha) \geq 0, multiplicity functions. If not otherwise stated, then we will always assume that \alpha \in \mathcal{M}_+. We write \alpha_a for \alpha(\alpha). We note, that our notation differs from that of Heckman and Opdam as their root system is \{2\alpha \mid \alpha \in \Delta\}. The present notation is adapted to the special case of a Riemannian symmetric space explained in Example 1.5, and it is the same as in [17]. Let

\[
A_C = a_C/\mathbb{Z}[i\pi H_a \mid \alpha \in \Delta].
\]

Then \(A_C\) is a -dimensional complex torus and \(A_C = AT\), where \(A = a\) and \(T = i\pi a/\mathbb{Z}[i\pi H_a \mid \alpha \in \Delta]\) is compact. Denote by \(\exp : a_C \rightarrow A_C\) the canonical projection.

Define

\[
A_C^{\text{reg}} = \exp\{H \in a_C \mid \forall \alpha \in \Delta: \alpha(H) \neq i\pi \mathbb{Z}\},
\]

\[
A^{\text{reg}} = A \cap A_C^{\text{reg}} = \exp\{H \in a \mid \forall \alpha \in \Delta: \alpha(H) \neq 0\}
\]

and \(A^+ = \exp a^+ \subset A^{\text{reg}}\). Then \(WA^+ = A^{\text{reg}}\) is open and dense in \(A\), \(A \setminus A^{\text{reg}}\) has measure zero and \(wA^+ \cap A^+ = \emptyset\) if \(w \neq e\).

For \(H \in a\) denote by \(\partial(H)\) the directional derivative \(\partial(H)f(a) = \partial_i f(a \exp(tH))|_{t=0}\). Let \(H_1, \ldots, H_r\) be an orthonormal basis of \(a\). We define a Weyl group invariant differential operator \(L(m)\) on \(A^{\text{reg}}\) by

\[
L(m) = \sum_{j=1}^r \partial(H_j)^2 + \sum_{\alpha \in \Delta^+} m_{\alpha} \left(1 + e^{-2\alpha} - e^{-2\alpha}\right) \partial(h_\alpha).
\]

Note that the first part is just \(L_A\), the Laplace operator on \(A\).

Let \(\Pi = \{\alpha_1, \ldots, \alpha_r\}\) be a set of simple roots in \(\Delta^+\). Set \(\Gamma_+ := \{\sum_{j=1}^r n_j \alpha_j \mid n_j \in \mathbb{N}_0\}\). Define \(\rho(m)\) as usually by \(2\rho(m) = \sum_{\alpha \in \Delta^+} m_{\alpha} \alpha\). For \(\mu \in \Gamma_+\) such that \((\mu, \mu - 2\lambda) \neq 0\) for all \(\mu \in A \setminus \{0\}\), define \(\Gamma_\mu(m; \lambda)\) inductively by

\[
\Gamma_0(m; \lambda) = 1
\]

and

\[
(\mu, \mu - 2\lambda) \Gamma_\mu(m; \lambda) = 2 \sum_{\alpha \in \Delta^+} m_{\alpha} \sum_{k \in \mathbb{N}} \Gamma_{\mu - 2k\alpha}(m; \lambda)(\mu + \rho(m) - 2k\alpha - \lambda, \alpha).
\]

For \(B \subseteq A\) and \(\omega \subseteq a\) we set \(B(\omega) = B \exp i\omega\). Let \(\Omega := \{H \in a \mid \forall \alpha \in \Delta: |\alpha(H)| < \pi/2\}\), then \(\exp\) is bijective \(a + i2\Omega \rightarrow A(2\Omega)\). We set \(a^\lambda := e^{\lambda(H)} \in \mathbb{C}\) for \(a = \exp H \in A(2\Omega)\) and \(H \in a + i2\Omega\).

The Harish-Chandra series \(\Phi_\lambda(m; a)\) is defined on \(A^+(2\Omega)\) by

\[
\Phi_\lambda(m; a) = a^{-\rho(m)} \sum_{\mu \in \Gamma_+} \Gamma_\mu(m; \lambda)a^{-\mu}
\]
for \( \lambda \in \mathfrak{a}_C^+ \) such that \( \lambda_\alpha := \frac{1}{2}\lambda(H_\alpha) \notin \mathbb{Z} \) for all \( \alpha \in \Delta \). It follows in the same way as Corollary 4.2.3 in [10] that the sum converges and that \( \Phi_\lambda(m; \cdot) \) is holomorphic on \( \mathfrak{a}^+(2\Omega) \). Let \( \Delta^+_i = \{ \alpha \in \Delta^+ \mid 2\alpha \notin \Delta \} \). Define the c-function by the Gindikin–Karpelevic formula

\[
c(m; \lambda) = \kappa_0 \prod_{\alpha \in \Delta^+_i} \frac{2^{-\lambda_\alpha} \Gamma(\lambda_\alpha)}{\Gamma\left(\frac{1}{2}(\lambda_\alpha + m_\alpha/2 + 1)\right) \Gamma\left(\frac{1}{2}(\lambda_\alpha + m_\alpha/2 + m_2\alpha)\right)},
\]

where the constant \( \kappa_0 \) is chosen so that \( c(m; \rho(m)) = 1 \). The hypergeometric function \( \varphi_\lambda(m; a) \), associated to the triple \((a, \Delta, m)\) is defined on \( \mathfrak{a}^+(2\Omega) \) by

\[
\varphi_\lambda(m; a) := \sum_{w \in W} c(m; w\lambda) \Phi_{w\lambda}(m; a).
\]

Although the Harish-Chandra series is well defined on \( \mathfrak{a}^+(2\Omega) \), the estimates we need can only be found in the literature for \( \mathfrak{a}(\Omega) \). We will therefore restrict our attention to this domain in the following, though it might not be maximal as some examples indicate.

**Theorem 1.1** (Heckman–Opdam). Let \( m \in \mathcal{M}_+ \). Then the following holds:

1. The function \( (\lambda, a) \mapsto \varphi_\lambda(m; a) \) extends to a holomorphic function on \( \mathfrak{a}_C^+ \times \mathfrak{a}_C^+ \).
2. We have \( L(m)\varphi_\lambda(m; \cdot) = ((\lambda, \lambda) - (\rho(m), \rho(m)))\varphi_\lambda(m; \cdot) \) for all \( \lambda \in \mathfrak{a}_C^+ \).
3. Let \( \lambda \in \mathfrak{a}_C^+ \). There exists a constant \( C > 0 \) such that for all \( a = \exp(H_R + iH_I) \in \mathfrak{a}(\Omega) \) with \( H_R \in \mathfrak{a} \) and \( H_I \in \Omega \)

\[
|\varphi_\lambda(m; a)| \leq Ce^{-\min_{w \in W} \text{Im} w\lambda(H_I) + \max_{w \in W} w\rho(m)(H_I) + \max_{w \in W} \text{Re} w\lambda(H_R)}.
\]

In particular, for all \( a = \exp(H) \in \mathfrak{a} \) with \( H \in \mathfrak{a} \)

\[
|\varphi_\lambda(m; a)| \leq Ce^{\max_{w \in W} \text{Re} w\lambda(H)}.
\]

**Proof.** See [10, Corollaries 4.2.6 and 4.3.13]. For the last part see [18, Proposition 6.1].

1.2. The hypergeometric Fourier transform

Recall that the action of \( W \) on \( \mathfrak{a}^+ \) gives rise to an action on functions on \( \mathfrak{a}^+ \) by \( w \cdot f(a) = f(w^{-1}a) \), respectively \( w \cdot F(\lambda) = F(w^{-1}\lambda) \). Define a density function \( \delta(m; \cdot) \) on \( \mathfrak{a}^+ \) by

\[
\delta(m; a) := \prod_{\alpha \in \Delta^+} |a^\alpha - a^{-\alpha}|^{m_\alpha}.
\]

Then \( \delta(m; \cdot) \) is \( W \)-invariant and \( \delta(m; a) = \prod_{\alpha \in \Delta^+} (a^\alpha - a^{-\alpha})^{m_\alpha} \) on \( \mathfrak{a}^+ \) and \( d\mu(m; a) = \delta(a) \, da \) is a positive measure on \( \mathfrak{a} \) (or \( \mathfrak{a}^+ \)). Denote the corresponding \( L^p \)-spaces of \( W \)-invariant functions

\[
L^p(\mathfrak{a}, d\mu_m)^W \simeq L^p(\mathfrak{a}^+, |W| \, d\mu(m; a)), \quad f \mapsto f|_{\mathfrak{a}^+}.
\]
Let
\[ a^*_+ = \{ \lambda \in a^* \mid \forall \alpha \in \Delta^+ : (\lambda, \alpha) > 0 \} = \{ \lambda \in a^+ \mid h_\lambda \in a^+ \}. \]

Then \( W a^*_+ \) is open and dense in \( a^* \), \( a^* \setminus W a^*_+ \) has measure zero, and \( w a^*_+ \cap a^*_+ = \emptyset \) if \( w \neq e \). On \( a^* \) we consider the measure \( d\nu(m; \lambda) = |c(m; i\lambda)|^{-2} d\lambda \) and note that
\[
L^p \left( a^*, d\nu(m; \lambda) \right)^W \simeq L^p \left( a^*_+, |W| d\nu(m; \lambda) \right), \quad f \mapsto f|_{a^*_+}.
\]

For \( f \in L^1(A, d\mu(m; a))^W \cap L^2(A, d\mu(m; a))^W \) define \( \mathcal{F}(m; f) = \hat{f} : a^* \to \mathbb{C} \) by
\[
\mathcal{F}(m; f)(\lambda) = \int_A f(a) \varphi_{-i\lambda}(a) d\mu(m; a) = |W| \int_A f(a) \varphi_{-i\lambda}(a) d\mu(m; a).
\]  
(1.9)

Equation (1.7) in Theorem 1.1 shows that the integral converges absolutely and that \( \mathcal{F}(m; f) \) is bounded by \( C \| f \|_1 \). We call \( \mathcal{F}(m; f) \) for the hypergeometric Fourier transform of \( f \) and the linear map \( \mathcal{F}(m) \) the hypergeometric Fourier transform.

We have the following important result, cf. [18, Theorem 9.13]. The Lebesgue measures used on \( A \) and \( a^* \) are assumed to be regularly normalized, see below (1.13).

**Theorem 1.2.** Assume that \( m \in M_+ \). The operator \( \frac{1}{|W|} \mathcal{F} \) extends to a unitary isomorphism
\[
L^2 \left( A, d\mu(m; \cdot) \right)^W \simeq L^2 \left( a^*, d\nu(m; \cdot) \right)^W.
\]

Let \( f \in C^\infty_c(A)^W \), then
\[
\forall N \in \mathbb{N} \exists C > 0 \forall \lambda \in a^* : \quad |\mathcal{F}(m; f)(\lambda)| \leq C \left( 1 + |\lambda| \right)^{-N} \tag{1.10}
\]
and the following Fourier inversion formula holds
\[
f(a) = \frac{1}{|W|^2} \int_{a^*} \mathcal{F}(m; f)(\lambda) \varphi_{i\lambda}(m; a) d\nu(m; \lambda)
= \frac{1}{|W|} \int_{a^*_+} \mathcal{F}(m; f)(\lambda) \varphi_{i\lambda}(m; a) d\nu(m; \lambda). \tag{1.11}
\]

Furthermore,
\[
\mathcal{F}(m; L(m)f)(\lambda) = -(|\lambda|^2 + |\rho(m)|^2) \mathcal{F}(m; f)(\lambda), \quad \lambda \in a^*. \tag{1.12}
\]

Note however, that in general \( \mathcal{F} \) does not map the convolution product of two functions on \( A \) into the product of the Fourier transforms.
We now relate the hypergeometric Fourier transform to the usual Fourier transform $F_A$ on the abelian group $A$. Recall that $F_A$ is defined on $L^1(A, da)$ by

$$F_A(f)(\lambda) = \int_A f(a) a^{-i\lambda} da,$$

(1.13)

and that a constant multiple of $F_A$ extends to a unitary isomorphism $L^2(A, da) \simeq L^2(a^*, d\lambda)$. The measures are said to be regularly normalized when the mentioned constant is 1, as will be assumed here.

For $s, t \in W$ define $c_{s,t} : a^* \to \mathbb{C}$ by $c_{s,t}(m; \lambda) = c(m; s^{-1}i\lambda)/c(m; t^{-1}i\lambda)$. Then $|c_{s,t}(m; \lambda)| = 1$ for all $\lambda \in a^*$. It follows that we can define an isometric map $\tau_s : L^p(a^*, d\lambda) \to L^p(a^*, d\lambda)$ ($s \in W$, $1 \leq p \leq \infty$) by

$$\tau_s F(\lambda) = c_{s,e}(m; \lambda) F(s^{-1}\lambda).$$

(1.14)

Note that $\tau_s$ depends on the multiplicity function $m$ although it is not indicated in the notation. A simple calculation shows that $\tau_{st} = \tau_s \tau_t$ and hence we have an action of $W$ on $L^p(a^*, d\lambda)$. As the Fourier transform is a unitary isomorphism $L^2(A, da) \simeq L^2(a^*, d\lambda)$, we can carry this action over to $L^2(A, da)$. We denote the subspaces of $\tau(W)$-invariant functions by $L^2(a^*, d\lambda) \tau(W)$ and $L^2(A, da) \tau(W)$.

**Lemma 1.3.** The multiplication operator

$$L^2(a^*, dv(m; \lambda)) \ni F \mapsto \Psi_a(F) := c(m; -i\cdot)^{-1}F \in L^2(a^*, d\lambda)$$

is a unitary isomorphism such that $\Psi_a(t \cdot F) = \tau_s \Psi_a(F)$.

In particular $\Psi_a : L^2(a^*, dv(m; \lambda))^W \to L^2(a^*, d\lambda) \tau(W)$ is a unitary isomorphism.

**Proof.** This follows by simple computation. \(\square\)

Set

$$A = F_A^{-1} \circ F(m) : L^2(A, d\mu(m; a))^W \to L^2(A, da)^W.$$  

(1.15)

In particular, for $f \in C_c^\infty(A)^W$ it follows from (1.10) that

$$Af(a) = \int_{a^*} F(m; f)(\lambda)a^{i\lambda} d\lambda.$$  

(1.16)

Denote by $\Psi_A$ the pseudo-differential operator

$$\Psi_A := F_A^{-1} \circ \Psi_a \circ F_A := F_A^{-1} \circ \frac{1}{c(m; -i\cdot)} \circ F_A.$$  

(1.17)
Lemma 1.4. The map
\[
\Lambda := \frac{1}{|W|} \Psi_A \circ A = \frac{1}{|W|} \mathcal{F}^{-1}_A \circ \Psi_a \circ \mathcal{F}(m)
\] (1.18)
is a unitary isomorphism \(L^2(A, d\mu(m; a))^W \rightarrow L^2(A, da)^\tau(W)\). Furthermore, if \(f \in C_c^\infty(A)^W\), then
\[
\Lambda(L(m)f) = (L_A - |\rho(m)|^2)\Lambda(f).
\] (1.19)

Proof. Equation (1.18) is immediate, and the unitarity of \(\Lambda\) follows directly from Lemma 1.3. Finally, (1.19) follows from (1.12) and the corresponding Euclidean expression \(\mathcal{F}_A(LAf)(\lambda) = -|\lambda|^2 \mathcal{F}_A(f)(\lambda)\).

Example 1.5 (The geometric case). The motivating example for the previous notation and theory is the case where \((a, \Delta, m)\) corresponds to a Riemannian symmetric space of non-compact type. For that, let \(G\) be a connected semisimple Lie group with finite center and \(\theta: G \rightarrow G\) a Cartan involution. Then \(K = G^\theta = \{x \in G \mid \theta(x) = x\}\) is a maximal compact subgroup of \(G\) and \(G/K\) is a Riemannian symmetric space of the non-compact type. Denote by \(\theta\) the derived involution on \(g\), the Lie algebra of \(G\). Then \(g = \mathfrak{k} \oplus \mathfrak{s}\) where \(\mathfrak{k} = \mathfrak{g}^\theta = \{X \in \mathfrak{g} \mid \theta(X) = X\}\) and \(\mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}\). Note that \(\mathfrak{s}\) can be identified with the tangent space \(T_{x_0}(G/K)\), where \(x_0 = eK \in G/K\). Let \(a\) be a maximal abelian subspace of \(\mathfrak{s}\). For \(\alpha \in a^*\) let \(g^\alpha := \{X \in g \mid \forall H \in a: [H, X] = \alpha(H)X\}\) and
\[
\Delta := \{\alpha \in a^* \setminus \{0\} \mid g^\alpha \neq \{0\}\}.
\]
Then \(\Delta\) is a root system and the function \(m: \Delta \rightarrow \mathbb{R}^+\) defined by \(m_\alpha := \dim g^\alpha\) is a positive multiplicity function. We say that the triple \((a, \Delta, m)\) is geometric if it corresponds to a Riemannian symmetric space in this way. In this case, when \(G/K\) is fixed, the multiplicity function \(m\) is omitted from the notation.

Let \(\Delta^+\) be a fixed positive system and put \(n := \bigoplus_{\alpha \in \Delta^+} g^\alpha\). Then \(n\) is a nilpotent Lie algebra. Let \(N = \exp n\) and \(A = \exp a\), then the multiplication map
\[
N \times A \times K \ni (n, a, k) \mapsto nak \in G
\]
is an analytic diffeomorphism the inverse of which is the Iwasawa decomposition
\[
G \ni x \mapsto (n(x), a(x), k(x)) \in N \times A \times K.
\]
As before, we set \(a^\lambda = e^{\lambda}(a) = e^{\lambda(H)}\) if \(a = \exp H \in A\). Denote by \(dk\) the normalized Haar measure on \(K\). The spherical functions on \(G/K\) are given by
\[
\varphi_\lambda(x) = \int_K a(kx)^{\lambda + \rho} dk,
\]
where $\lambda \in \mathfrak{a}_C^*$, with $\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' \in W\lambda$, see [8,9] and [13]. The hypergeometric function $\varphi_\lambda(m; \cdot)$ associated to $(\mathfrak{a}, \Delta, m)$ is exactly the function $a \mapsto \varphi_\lambda(a)$.

Recall that $G = KAK$, that $KA^+K$ is open, dense and with complement of measure zero in $G$, and that $K/M \times A^+ \ni (km, a) \mapsto kax_0$ is a diffeomorphism onto its image, see [12, Chapter IX, Theorem 1.1 and Corollary 1.2]. We identify $A$ with the subset $A \cdot x_0 \subset G/K$. It follows that $K$-bi-invariant measurable functions are determined by their restriction to $A^+$. We obtain from [13, Chapter I, Theorem 5.8]:

**Lemma 1.6.** The invariant measure on $G/K$ can be normalized so that

$$
\int \frac{f(x)}{G} \, dx = \int \frac{f(a)\delta(a)}{A} \, da = |W| \int \frac{f(a)\delta(m; a)}{A^+} \, da
$$

for all $f \in L^1(G/K)^K$. In particular, the restriction map $f \mapsto f|_A$ defines a unitary isomorphism $L^2(G/K)^K \simeq L^2(A, d\mu(m; a))^W \simeq L^2(A^+, |W| d\mu(m; a))$.

The spherical Fourier transform $\mathcal{F} : L^2(G/K)^K \to L^2(\mathfrak{a}_c^*, d\nu(m; \lambda))^W$ is defined by

$$
\mathcal{F}(f)(\lambda) := \int_{G/K} f(x)\varphi_{-i\lambda}(x) \, dx, \quad f \in L^2(G/K)^K \cap L^1(G/K)^K.
$$

By Lemma 1.6 it follows that the spherical Fourier transform is a special case of the hypergeometric Fourier transform.

For $f \in C_c^\infty(G/K)^K$ the Abel transform of $f$ is defined by

$$
\mathcal{A}f(a) = a^\rho \int_N f(an) \, dn.
$$

Then $\mathcal{A}f$ is $W$-invariant, cf. [8, Lemma 17] or [13, Chapter I, Theorem 5.7]. Here we normalize the Haar measure $dn$ on $N$ such that the measure in Lemma 1.6 is given by $dx = a^{-2\rho}dn\,da\,dk = da\,dn\,dk$. Then

$$
\mathcal{F} = \mathcal{F}_A \circ \mathcal{A},
$$

see [13, p. 450, Eq. (7)]. This explains our definition of $\mathcal{A}$ in (1.15). An integral formula as (1.20) seems not available in general.

If $L \subset G$ and $f$ is a function defined on $G$, then we set $\text{Res}_L(f) = f|_L$. By [13, Chapter II, Corollary 5.11], we have that $\text{Res}_A : C^\infty(G/K)^K \to C^\infty(A)^W$ is an isomorphism. Denote by $\mathcal{D}(G/K)$ the commutative algebra of $G$-invariant differential operators on $G/K$. If $D \in \mathcal{D}(G/K)$ then there exists a unique $W$-invariant differential operator $\text{Rad}(D)$ on $A$ such that

$$
\text{Res}_A \circ D = \text{Rad}(D) \circ \text{Res}_A.
$$

The differential operator $\text{Rad}(D)$ is called the radial part of $D$. Denote by $L_{G/K}$ the Laplace–Beltrami operator on $G/K$. Then, according to [13, Chapter II, Proposition 3.9], we have

$$
\text{Rad}_A(L_{G/K}) = L(m).
$$
Example 1.7 (The even multiplicity case). We now consider the (not necessarily geometric) special case where \( m_\alpha \in 2\mathbb{Z} \) and \( 2\alpha \notin \Delta \) for all \( \alpha \in \Delta \). We refer to [17] for details. It follows from the definition in (1.4) that \( 1/c(m; \lambda) \) is a polynomial. In fact we have:

\[
\frac{1}{c(m; \lambda)} = \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_\alpha/2-1} \frac{\lambda_\alpha + k}{\rho(m)_\alpha + k}.
\]

(1.24)

We therefore get the following lemma in this case:

Lemma 1.8. The operator \( \Psi_A \) is a constant coefficient differential operator on \( A \) given by

\[
\Psi_A = \kappa(m) \prod_{\alpha \in \Delta^+} m_\alpha/2 - 1 \prod_{k=0}^{\lambda_\alpha} \left( -\frac{1}{2} \partial(H_\alpha) + k \right),
\]

where the constant \( \kappa(m) \) is given by

\[
\kappa(m) = \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_\alpha/2-1} \frac{1}{\rho(m)_\alpha + k}.
\]

The following theorem is obtained in [17, Theorem 5.1]:

Theorem 1.9. Assume \( m_\alpha \in 2\mathbb{Z} \) and \( 2\alpha \notin \Delta \) for all \( \alpha \in \Delta \). Let \( \psi_\lambda := \sum_{w \in W} e^{w\lambda} \). There exists a \( W \)-invariant differential operator \( D \) on \( A \) with analytic coefficients such that

\[
\delta(m; a) \varphi_\lambda(m; a) = c(m; \lambda) c(m; -\lambda) D\psi_\lambda(a) = \prod_{\alpha \in \Delta^+} \prod_{k=0}^{m_\alpha/2-1} \frac{k^2 - \rho(m)_\alpha^2}{k^2 - \lambda_\alpha^2} D\psi_\lambda(a)
\]

for all \( \lambda \in a^*_C \). Moreover, these expressions are holomorphic in \( \lambda \).

Theorem 1.9 implies the following inversion formula for the Abel transform in terms of \( D \) and \( \delta(a) \):

Corollary 1.10. Assume that \( m_\alpha \in 2\mathbb{N} \) and that \( f \in C^\infty_c(A)^W \). Then \( Af \in C^\infty(A) \) and

\[
DAf(a) = |W| \delta(m; a) f(a), \quad a \in A.
\]

Proof. The smoothness of \( Af \) follows from (1.10) and (1.16). Moreover, we can carry \( D \) under the integral sign in the latter equation. From \( W \)-invariance of \( \mathcal{F}(m; f) \) and \( D \) we therefore obtain

\[
DAf(a) = \int_{a^*} \mathcal{F}(m; f)(\lambda) Da^i_\lambda d\lambda
\]

\[
= \frac{1}{|W|} \int_{a^*} \mathcal{F}(m; f)(\lambda) D\psi_\lambda(a) d\lambda
\]
\[
\frac{1}{|W|} \delta(m; a) \int_{a^*} \mathcal{F}(m; f)(\lambda) \varphi_{i\lambda}(m; a) \frac{d\lambda}{|c(m; i\lambda)|^2}
\]

\[
= |W| \delta(m; a) f(a),
\]

where (1.11) was used in the last step. \[ \square \]

**Remark 1.11.** The differential operator \( D \) is closely related to the Heckman–Opdam shift operator \( D_+(m) \). We refer to [19] for discussion on inversion of the Abel transform using shift operators.

**Example 1.12** *(The geometric case with \( G \) complex).* Let us now consider the simple case where \( m_\alpha = 2 \) for all \( \alpha \). This corresponds to the geometric case where \( G \) has a complex structure. Let \( \pi(\lambda) = \prod_{\alpha \in \Delta^+} (\alpha, \lambda) \), then \( c(\lambda) = \pi(\rho)/\pi(\lambda) \) by (1.24). It is well known that

\[
\varphi_\lambda(a) = c(\lambda) \delta(a)^{-1/2} \sum_{w \in W} \text{sign}(w)a^{w\lambda},
\]

where \( \delta(a)^{1/2} : = \prod_{\alpha \in \Delta^+} (a^\alpha - a^{-\alpha}) \). Let \( \pi(\partial_\alpha) = \prod_{\alpha \in \Delta^+} \partial(h_\alpha) \). Then

\[
\Psi_A = (-1)^{|\Delta^+|} \pi(\rho)^{-1} \pi(\partial_\alpha)
\]

by Lemma 1.8.

**Lemma 1.13.** Assume that \( m_\alpha = 2 \) for all \( \alpha \in \Delta^+ \). Then the differential operator \( D \) can be taken as

\[
D = (-1)^{|\Delta^+|} \pi(\rho)^{1/2} \pi(\partial_\alpha).
\]

Furthermore, with the notation of (1.3),

\[
D = \delta^{1/2} \Psi_A.
\]

**Proof.** We have \( \pi(\partial_\alpha) a^{w\lambda} = \text{sign}(w) \pi(\lambda) a^{w\lambda} \) and \( c(-\lambda) = (-1)^{|\Delta^+|} c(\lambda) \). Hence

\[
(-1)^{|\Delta^+|} \pi(\rho)^{-1} \delta(a)^{1/2} \pi(\partial_\alpha) \psi_\lambda(a) = (-1)^{|\Delta^+|} \pi(\rho)^{-1} \delta(a)^{1/2} \pi(\lambda) \sum_{w \in W} \text{sign}(w)a^{w\lambda}
\]

\[
= c(-\lambda)^{-1} \delta(a)^{1/2} \sum_{w \in W} \text{sign}(w)a^{w\lambda}
\]

\[
= [c(\lambda) c(-\lambda)]^{-1} \delta(a) \varphi_\lambda(a).
\]

The second part follows from (1.26). \[ \square \]

**Lemma 1.14.** Assume that \( m_\alpha = 2 \) for all \( \alpha \). The representation \( \tau \) is given on functions on \( \mathfrak{a}^* \) and \( A \) by \( \tau(w) F(\lambda) = \text{sign}(w) F(w^{-1}\lambda) \) and \( \tau(w) f(a) = \text{sign}(w) f(w^{-1}a) \), respectively.
Proof. This follows from the fact, that $c(w\lambda) = \text{sign}(w)c(\lambda)$. □

Lemma 1.15. Assume that $m_\alpha = 2$ for all $\alpha$. The isometry $\Lambda: L^2(A, d\mu)^W \to L^2(A, d\mu)^{\tau(W)}$ of Lemma 1.4 is given by multiplication with $\delta^{1/2} = \prod_{\alpha \in \Delta^+} (a^\alpha - a^{-\alpha})$.

Proof. By the definition of $\Lambda$ in Lemma 1.4, by Lemma 1.13 and by Corollary 1.10,

$$\Lambda = \frac{1}{|W|} \Psi_A \circ A = \frac{1}{|W|} \delta^{-1/2} D \circ A = \delta^{1/2}. \quad \square$$

2. The heat equation

In this section we study the heat equation on $A^\text{reg}$ associated with the operator $L(m)$ of (1.3). We start with the classical case which is well known. We will identify $A$ and $a$ without further comments.

The heat equation on $a$ is $L_a u(X, t) = \partial_t u(X, t)$. We consider the corresponding Cauchy problem

$$L_a u(X, t) = \partial_t u(X, t), \quad \lim_{t \searrow 0} u(X, t) = f(X). \quad (2.1)$$

For $f \in L^2(a, dX)$ (and with the above limit in $L^2$-sense) the solution is given by applying the contraction semigroup $e^{t\Delta}$ to $f$, i.e., $u(X, t) = e^{t\Delta} f(X) = h_t \ast f(X)$ where $h_t(X) = (4\pi t)^{-r/2} e^{-|X|^2/(4t)}$ is the heat kernel. With the present normalization of measures this can be written as follows

$$u(X, t) = \int_{\mathbb{C}} e^{-t|\lambda|^2} \hat{f}(\lambda) e^{i\lambda(X)} d\lambda. \quad (2.2)$$

Because of the factor $e^{-t|\lambda|^2}$ we can replace $X$ in (2.2) by $X + iY$ and the result is a holomorphic function on $a_\mathbb{C}$. Define a density function

$$\omega_t^a(X + iY) = \omega_t(X + iY) := (2\pi t)^{-r/2} e^{-|Y|^2/2t} \quad (2.3)$$

and set

$$\mathcal{H}_t(a) = \left\{ F \in \mathcal{O}(a_\mathbb{C}) \mid \int_{a_\mathbb{C}} |F(Z)|^2 \omega_t^a(Z) dX dY < \infty \right\}. \quad (2.4)$$

Here $\mathcal{O}(a_\mathbb{C})$ stands for the space of holomorphic functions on $a_\mathbb{C}$. The following is well known, see [2,20]:

Theorem 2.1 (Bargmann, Segal). Let $t > 0$. The space $\mathcal{H}_t$ is a Hilbert space with a reproducing kernel. The solution $u(\cdot, t)$ belongs to $\mathcal{H}_t(a)$ for all $f \in L^2(a)$, and the map

$$H^a_t : L^2(a) \ni f \to u(\cdot, t) \in \mathcal{H}_t(a)$$

is a unitary isomorphism.
The transform $H_t^m : L^2(A) \to \mathcal{H}_t(A)$ is called the Segal–Bargmann transform.

Consider now a non-negative multiplicity function $m$ on a root system $\Delta$. In analogy we take
$L(m)u(a, t) = \partial_t u(a, t)$ where $a \in A_{\text{reg}}$, as a definition for the hypergeometric heat equation associated to $m$, and we consider the corresponding Cauchy problem

$$L(m)u(a, t) = \partial_t u(a, t), \quad \lim_{t \to 0^+} u(a, t) = f(a). \quad (2.5)$$

More precisely, we shall study this problem with the condition that $f \in L^2(A, d\mu(m; a))W$ where $d\mu(m; a) = \delta(a) \, da$, and with the above limit with respect to this $L^2$-space.

**Example 2.2.** The heat equation on a Riemannian manifold is given by (2.1), with $L_a$ replaced by the Laplace–Beltrami operator. In particular, if the Riemannian manifold is a Riemannian symmetric space $G/K$ of the non-compact type we obtain

$$L_{G/K} u(x, t) = \partial_t u(x, t), \quad \lim_{t \to 0^+} u(x, t) = f(x). \quad (2.6)$$

If we assume that $f$ is $K$-invariant, then obviously the solution $u(\cdot, t)$ is also $K$-invariant. Taking the radial part as in Example 1.5 (see (1.23)) we get exactly the problem (2.5) for the associated multiplicity function $m$, with $f \in L^2(A_{\text{reg}}, d\mu(m; a))W$.

The problem (2.5) is easily solved by means of the hypergeometric Fourier transform:

**Lemma 2.3.** Let $f \in L^2(A, d\mu(m; a))W$. The solution to the problem (2.5) is given by

$$u(a, t) = \frac{1}{|W|^2} \int_{a^*} e^{-t(|\lambda|^2 + |\rho(m)|^2)} \mathcal{F}(m; f)(\lambda) \phi_{i\lambda}(m; a) \, d\nu(m; \lambda). \quad (2.7)$$

In particular, the solution $a \mapsto u(a, t)$ is well defined and real analytic on $A$ and extends to a $W$-invariant holomorphic function on $A(\Omega)$.

**Proof.** Define $u(a, t)$ for $t > 0$ by (2.7). Then $u(\cdot, t)$ is well defined and real analytic on $A$ and extends to a $W$-invariant holomorphic function on $A(\Omega)$. This follows from (1.6), from which a uniform estimate $|\phi_{i\lambda}(m; a)| \leq Ce^{|\lambda|}$, $\lambda \in a^*$, $a \in A(\Omega)$, is easily derived.

It is now easily seen that $u(\cdot, t)$ solves the hypergeometric heat equation. Moreover, applying dominated convergence as $t \to 0$, we see that the functions $\lambda \mapsto e^{-t(|\lambda|^2 + |\rho(m)|^2)} \mathcal{F}(m; f)(\lambda) \times \phi_{i\lambda}(m; a)$ converge in $L^2(a^*, d\nu(m; \cdot))W$ to $\lambda \mapsto \mathcal{F}(m; f)(\lambda) \phi_{i\lambda}(m; a)$. Hence the limit relation in (2.5) follows by continuity of the inverse Fourier transform. \qed

The extension to $A(\Omega)$ of the solution $u(a, t)$ in (2.7) is denoted $H_t(m; f)$, and the map

$$L^2(A, d\mu(m; a))W \ni f \mapsto H_t(m; f) \in \mathcal{O}(A(\Omega))$$

is called the Segal–Bargmann transform associated with the hypergeometric heat equation. The following lemma characterizes of the image of the transform.
Theorem 2.4 (The image, first version). The image

\[ \mathcal{H}_t(m) := H_t\left(L^2\left(A, d\mu(m; a)\right)^W\right) \subset \mathcal{O}(A(\Omega)) \]

is the space of \( W \)-invariant holomorphic function \( F \) on \( A(\Omega) \) satisfying the following conditions:

1. The function \( F|_A \) is in \( L^2(A, d\mu(m; a))^W \);
2. The function \( \lambda \mapsto e^{(\|\lambda\|^2 + \|\rho(m)\|^2)t} F(m; F|_A)(\lambda) \) is in \( L^2\left(a^*, dv(m; \lambda)\right)^W \).

Proof. Let \( f \in L^2(A, da)^W \) and \( F = H_t(m; f) \). Then \( F \) is a \( W \)-invariant holomorphic function on \( A(\Omega) \), and by (2.7) we have

\[ F|_A = F(m) - \frac{1}{e^{-t(\|\lambda\|^2 + \|\rho(m)\|^2)t}} F(m; F|_A). \quad (2.8) \]

Since \( |e^{-(\|\lambda\|^2 + \|\rho(m)\|^2)t}| \leq 1 \) we have \( e^{-(\|\lambda\|^2 + \|\rho(m)\|^2)t} F(m; f) \in L^2\left(a^*, dv(m; \lambda)\right)^W \). Thus \( F|_A \in L^2(A, d\mu)^W \) and \( e^{(\|\lambda\|^2 + \|\rho(m)\|^2)t} F(m; F|_A) = F(m; f) \in L^2\left(a^*, dv(m; \lambda)\right)^W \), so \( F \) satisfies (1) and (2).

Now, let \( F \in \mathcal{O}(A(\Omega)) \) satisfy (1), (2), and let

\[ g = F(m) - \frac{1}{e^{-t(\|\lambda\|^2 + \|\rho(m)\|^2)t}} F(m; F|_A) \in L^2(A, da)^W. \]

Let \( G = H_t(m; g) \). By definition,

\[ F(m; G|_A) = e^{-t(\|\lambda\|^2 + \|\rho(m)\|^2)} F(m; g) = F(m; F|_A). \]

Hence \( G = F \) almost everywhere on \( A \). Since \( A \) is a totally real submanifold of \( A(\Omega) \) it follows that \( G = F \). \( \square \)

We extend the \( \tau \)-action of the Weyl group on \( L^2(a, d\lambda) \) to \( \mathcal{H}_t(a) \) by defining it to be \( \tau_w(F) := H_t^a \tau_w(H_t^a)^{-1} F \). If \( F = H_t^a f \), then the action is given by

\[
\tau_s F(X + iY) = \int_a e^{-|\lambda|^2t} c_{s,e}(m; \lambda) \hat{f}(s^{-1}\lambda) e^{i\lambda(X+iY)} d\lambda \\
= \int_a e^{-|\lambda|^2t} c_{e,s^{-1}}(m; \lambda) \hat{f}(\lambda) e^{i\lambda(s^{-1}(X+iY))} d\lambda.
\]

In particular, the subspace \( \mathcal{H}_t(a)^\tau(W) \) is defined by means of this action.

Recall, that we identify \( A \) and its Lie algebra \( a \). Define a density function on \( a \subset \mathbb{C} \) by

\[ \omega_t(m; X + iY) = (2\pi t)^{-r/2} e^{2t|\rho(m)|^2 - |Y|^2/2t}. \quad (2.9) \]

Lemma 2.5. Let \( F = H_t(m; f) \in \mathcal{H}_t(m) \) with \( f \in L^2(A, d\mu(m; a))^W \). Then \( e^{t|\rho(m)|^2} \Lambda(F|_A) \) solves (2.1) on \( a \) with initial value \( A f \in L^2(a, dX)^\tau(W) \).
In particular \( \Lambda(F|_A) \) extends to a \( \tau(W) \)-invariant holomorphic function on \( \mathfrak{a}_\mathbb{C} \), denoted by \( \Lambda F \), such that

\[
\|f\|^2 = \int_{\mathfrak{a}_\mathbb{C}} |\Lambda F(X + iY)|^2 \omega_t(m; X + iY) \, dX \, dY.
\]  

(2.10)

**Proof.** The following chain of equalities is easily obtained from the definitions of \( \Lambda \), \( H_t \) and \( H^a_t \):

\[
\mathcal{F}_A\left(e^{t|\rho(m)|^2} \Lambda(F|_A)\right) = e^{t|\rho(m)|^2} c(m; -i\cdot)^{-1} \mathcal{F}(m; F|_A)
\]

\[
= e^{-t|\cdot|^2} c(m; -i\cdot)^{-1} \mathcal{F}(m; f)
\]

\[
= e^{-t|\cdot|^2} \mathcal{F}_A(Af)
\]

\[
= \mathcal{F}_A\left(H^a_tAf\right).
\]

Hence \( e^{t|\rho(m)|^2} \Lambda(F|_A) = H^a_tAf \), which is exactly the first statement of the lemma.

The \( \tau(W) \)-invariance of \( \Lambda F \) is immediate from the above definition, since \( \Lambda f \) is \( \tau(W) \)-invariant by Lemma 1.4. Finally, (2.10) follows from the unitarity in Theorem 2.1. \( \square \)

We can now define a sesquilinear product on \( \mathcal{H}_t(m) \) by

\[
(F, G) := \int_{\mathfrak{a}_\mathbb{C}} \Lambda F(X + iY) \overline{G(X + iY)} \omega_t(m; X + iY) \, dX \, dY.
\]  

(2.11)

We collect the main results in the following theorem.

**Theorem 2.6** *(The image, second version).* The space \( \mathcal{H}_t(m) \) is a Hilbert space and \( \mathcal{H}_t(m) : L^2(A, da) \to \mathcal{H}_t(m) \) is a unitary isomorphism.

**Proof.** Follows immediately from the preceding lemmas. \( \square \)

**Example 2.7** *(The geometric case with \( G \) complex).* Assume \( m_\alpha = 2 \) for all \( \alpha \), i.e., \((a, \Delta, m)\) corresponds to a Riemannian symmetric space \( G/K \) with \( G \) complex. Note that the function \( \delta^{1/2} \) has a holomorphic extension to \( A \). We then obtain the following result of B. Hall and J. Mitchell [7, Theorem 3].

**Theorem 2.8.** Let \( f \in L^2(G/K)^K \), and let \( u(x, t) = H_t f(x) \) be the solution to (2.6). The map \( X \mapsto u(\exp X, t) \), \( X \in \mathfrak{a} \) has a meromorphic extension to \( \mathfrak{a}_\mathbb{C} \), denoted \( z \mapsto U(Z) \), which is \( W \)-invariant and satisfies the following: The map \( \delta^{1/2}U \) is holomorphic and

\[
\|f\|^2 = (2\pi t)^{-r/2} \int_{\mathfrak{a}_\mathbb{C}} |(\delta^{1/2}U)(X + iY)|^2 e^{2t|\rho|^2 - |Y|^2/2t} \, dX \, dY.
\]
Conversely, any meromorphic function $U(Z)$ which is invariant under $W$ and which satisfies
\[
\int_{\mathbb{C}} \left| \left( \frac{\delta^{1/2} U}{2} (X + iY) \right)^2 e^{2t|\rho|^2 - |Y|^2/2t} \right| dX dY < \infty
\]
is the Segal–Bargmann transform $H_t f$ for some $f \in L^2(G/K)^K$.

**Proof.** Immediate from Theorem 2.6 by Corollary 1.10. The statements about $W$-invariance follows from Lemma 1.14, since $\delta^{1/2} U$ is invariant under the action mentioned there if and only if $U$ is invariant for the ordinary action. □

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