ON THE GENERALIZED BURGERS-HUXLEY EQUATION:
EXISTENCE, UNIQUENESS, REGULARITY, GLOBAL
ATTRACTORS AND NUMERICAL STUDIES

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Abstract. In this work, we consider the forced generalized Burgers-Huxley equation and establish the existence and uniqueness of a global weak solution using a Faedo-Galerkin approximation method. Under smoothness assumptions on the initial data and external forcing, we also obtain further regularity results of weak solutions. Taking external forcing to be zero, a positivity result as well as a bound on the classical solution are also established. Furthermore, we examine the long-term behavior of solutions of the generalized Burgers-Huxley equations. We first establish the existence of absorbing balls in appropriate spaces for the semigroup associated with the solutions and then show the existence of a global attractor for the system. The incompressible limits of the Burgers-Huxley equations to the Burgers as well as Huxley equations are also discussed. Next, we consider the stationary Burgers-Huxley equation and establish the existence and uniqueness of weak solution by using a Faedo-Galerkin approximation technique and compactness arguments. Then, we discuss about the exponential stability of stationary solutions. Concerning numerical studies, we first derive error estimates for the semidiscrete Galerkin approximation. Finally, we present two computational examples to show the convergence numerically.

1. Introduction. Generalized Burgers-Huxley equation characterizes a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports. The generalized Burgers-Huxley equation is given by (see [36])

\[
\frac{\partial u}{\partial t} + \alpha u^\delta \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u^\delta)(u^\delta - \gamma),
\]

(1)

where \( \alpha, \beta, \nu, \delta \) are parameters such that \( \alpha > 0 \) is the advection coefficient, \( \beta > 0 \), \( \delta \geq 1 \) and \( \gamma \in (0, 1) \) are parameters. Solitary and traveling wave solutions of the generalized Burgers-Huxley equation using a relevant nonlinear transformation is obtained in [18, 36, 46, 47], etc. In the past several years, many other mathematical methods were developed to solve the equation (1), such as Adomian decomposition.
method [21, 22, 25], spectral methods [15, 26, 27], the tanh-coth method [48], homotopy analysis method [31], Hopf-Cole transformation method [50], variational iteration method [3, 4], etc. Numerical studies of the generalized Burgers-Huxley equation have been carried out by several authors, for a sample literature, we refer the interested readers to [29, 35, 37, 44, 51], etc and references therein. A positive and bounded finite element approximation of the generalized Burgers-Huxley equation in \( \mathbb{R}^2 \) is obtained in [19].

For \( \delta = 1, \alpha \neq 0 \) and \( \beta \neq 0 \), the equation (1) becomes
\[
\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u)(u - \gamma),
\]
which is known as the Burgers-Huxley equation (see [46, 49], etc). For \( \alpha = 0 \) and \( \delta = 1 \), the equation (1) takes the form
\[
\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u)(u - \gamma),
\]
which is known as the Huxley equation and it describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals ([45]). For \( \beta = 0, \delta = 1 \) and \( \alpha = 1 \), the equation (1) can be reduced to
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0,
\]
which is the well-known viscous Burgers’ equation (in [8], Burgers studied the model from (4) for mathematical modeling of the turbulence phenomena, see [2, 9] also). Burgers’ equation describes the far field of wave propagation in nonlinear dissipative systems, shock wave theory, vorticity transportation, heat conduction, wave processes in thermoelastic medium, dispersion in porous media, hydrodynamic turbulence, elasticity, gas dynamics, etc. The equation (4) consists of a nonlinear convection term \( u \partial_x u \) and a viscosity term of higher order \( \partial_{xx} u \), which regularizes the equation and produces a dissipation effect of the solution near a shock. The viscous Burgers equation (4) can be converted to the linear heat equation and then it can be explicitly solved by the Hopf-Cole transformation (see [11, 23], etc). The existence, uniqueness and regularity of weak solutions to the viscous inhomogeneous Burgers equation on bounded domains with Dirichlet boundary conditions is obtained in [6]. The authors have also established the existence of solutions of viscous Burgers’ equations in space-time non-rectangles domains that can be transformed to rectangles. For more details on the theoretical results for viscous Burgers’ equations, the interested readers are referred to see [7, 32], etc. The systems (3) and (4) are of special significance for studying nonlinear phenomena in physics. When the viscosity coefficient \( \nu \) becomes zero, the Burgers equation is reduced to the transport equation:
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,
\]
which represents the inviscid Burgers equation.

In this work, we consider the forced generalized Burgers-Huxley equation defined on \( \Omega \times (0,T) = (0,1) \times (0,T) \):
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\frac{\partial u(x,t)}{\partial t} + \alpha u(x,t)^\delta \frac{\partial u(x,t)}{\partial x} - \nu \frac{\partial^2 u(x,t)}{\partial x^2} = \beta u(x,t)(1 - u(x,t)^\delta)(u(x,t)^\delta - \gamma) \\
+ f(x,t), \quad (x,t) \in \Omega \times (0,T), \\
u(0,t) = u(1,t) = 0, \quad t \in (0,T), \\
\end{array} \right.
\]
\]
\[
\begin{aligned}
&u(x,0) = u_0(x), \quad x \in \Omega, \\
&u(x,t) \geq 0, \quad x \in \Omega, \\
\end{aligned}
\]
(5)

where \( f(\cdot, \cdot) \) is an external forcing. Using a Faedo-Galerkin method, we show the existence and uniqueness of weak solution to the system (5). Regularity results of the weak solution are obtained under smoothness assumptions on the initial data and external forcing. A positivity result as well as a bound on the classical solution is established, whenever the external forcing is zero. Such a result for the two dimensional Burgers equation is established in [19]. By taking the external forcing to be independent of time in (5) (autonomous case), we then establish the existence of a global attractor. In order to achieve this, we first establish the existence of absorbing balls in appropriate spaces for the semigroup associated with the solutions of the generalized Burgers-Huxley equation and then use compact embedding theorems.

The inviscid limits of the Burgers-Huxley equation to the viscous Burgers equation as well as Huxley equation are also discussed in this work. To the best of authors’ knowledge, this paper appears to be the first one in which the existence, uniqueness and regularity of weak solutions as well as the asymptotic behavior (global dynamics) of solutions of the inhomogeneous generalized Burgers-Huxley equation is considered. Moreover, we establish the existence of weak solution to the stationary Burgers-Huxley equations by using Faedo-Galerkin approximation and compactness arguments. Then, we discuss about the exponential stability of stationary solutions. Finally, we carry out numerical studies for the system (5).

The rest of the paper is organized as follows. In the next section, we provide an abstract formulation of the problem and give the necessary function spaces needed to obtain the global solvability results of the system (5). Whenever the initial data \( u_0 \in L^2(\Omega) \) and external forcing \( f \in L^2(0,T;H^{-1}(\Omega)) \), the existence and uniqueness of global weak solution to the system (5) is obtained in section 3 by using a Faedo-Galerkin approximation technique (Theorem 3.1). Regularity results for the global weak solution are also established under further regularity assumptions on initial data and external forcing (Theorems 3.2 and 3.3). Whenever \( f = 0 \) and the initial data is bounded between 0 and 1, we show that the classical solution is also bounded between the same limits in Theorem 3.4. We also consider the generalized Burgers-Huxley equation in a space-time non-rectangular domain with Dirichlet boundary conditions and discuss about global solvability results. We use the change of variables to transform the space-time non-rectangular domain into a rectangle and then using the results available for space-time rectangular domain (that is, \((0,1) \times (0,T)\)), we claim the global solvability results in the space-time non-rectangular domain also (Remark 3). The existence of a global attractor for the system (5) is discussed in section 4. We show the existence of absorbing balls for the semigroup associated with the system (5) in \( L^2(\Omega) \) and \( H^1_0(\Omega) \) in Propositions 1 and 2, respectively. Then, the existence of a global attractor for the system (Theorem 4.1) follows by applying Theorem I.1.12, [42]. For \( \delta = 1 \), we discuss about the inviscid limits of the equation (5) as \( \beta \to 0 \) as well as \( \alpha \to 0 \) in section 5. We show that the unique weak solution of the Brugers-Huxley equation (see (5) with \( \delta = 1 \)) converges to the unique weak solution to the viscous Brugers equation, as \( \beta \to 0 \)
(Proposition 3). Furthermore, as $\alpha \to 0$, we show that the weak solution of the Brugers-Huxley equation tends to the weak solution of the Huxley equation (Proposition 4). In section 6, we consider the stationary Burgers-Huxley equation and establish the existence and uniqueness of weak solution by using a Faedo-Galerkin approximation technique and compactness arguments (Theorem 6.1). In the same section, we show that the stationary solution is exponentially stable (Theorem 6.3). Numerical studies have been carried out in sections 7 and 8. In section 7, we derive error estimates for the semidiscrete Galerkin approximation (Theorem 7.1). In the final section, we present two numerical examples to show the convergence.

2. Mathematical formulation. In this section, we present the necessary function spaces, and properties of linear and nonlinear operators used to obtain the global solvability results of the system (5), and provide the definition of weak and strong solution also.

2.1. Functional setting. Let $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions with compact support on $\Omega$. For $p \in [2, \infty)$, the Lebesgue spaces are denoted by $L^p(\Omega)$ and the Hilbertian Sobolev spaces are denoted by $H^k(\Omega)$. The norm in $L^p(\Omega)$ is denoted by $\| \cdot \|_{L^p}$ and for $p = 2$, the inner product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. Let $H^1_0(\Omega)$ denotes closure of $C_0^\infty(\Omega)$ in $\| \cdot \|_{L^2}$ norm.

As $\Omega$ is a bounded domain, note that $\| \partial_x \cdot \|_{L^2}$ defines an equivalent norm on $H^1_0(\Omega)$ (by using the Poincaré inequality) and we have the continuous embedding $H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$. Remember that the embedding of $H^1_0(\Omega) \subset L^2(\Omega)$ is compact. The duality paring between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$. In one dimension, we have the following continuous embedding: $H^1_0(\Omega) \subset L^\infty(\Omega) \subset L^p(\Omega)$, for all $p \in [1, \infty)$.

2.2. Linear operator. Let $A$ denotes the self-adjoint and unbounded operator on $L^2(\Omega)$ defined by

$$Au := -\frac{\partial^2 u}{\partial x^2},$$

with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. The eigenvalues and the corresponding eigenfunctions of $A$ are given by

$$\lambda_k = k^2 \pi^2 \text{ and } w_k(x) = \sqrt{\frac{2}{\pi}} \sin(k\pi x), \ k = 1, 2, \ldots$$

Since $\Omega$ is a bounded domain, $A^{-1}$ exists and is a compact operator on $L^2(\Omega)$. Moreover, one can define the fractional powers of $A$ and

$$\|A^{1/2}u\|_{L^2}^2 = \sum_{j=1}^{\infty} \lambda_j \langle u, w_j \rangle \geq \lambda_1 \sum_{j=1}^{\infty} \langle u, w_j \rangle^2 = \lambda_1 \| u \|_{L^2}^2 = \pi^2 \| u \|_{L^2}^2,$$

which is the Poincaré inequality. Note also that $\| u \|_{H^s} = \| A^{s/2} u \|_{L^2}$, for all $s \in \mathbb{R}$. An integration by parts yields

$$(Au, v) = (\partial_x u, \partial_x v) = : a(u, v), \text{ for all } v \in H^1_0(\Omega),$$

so that $A : H^1_0(\Omega) \to H^{-1}(\Omega)$.

2.3. Nonlinear operators. We define two nonlinear operators in this subsection.

\[1\] Strictly speaking one has to use $\frac{d^2}{dx^2}$ instead of $\frac{\partial^2}{\partial x^2}$.\]
2.3.1. The operator \( B(\cdot) \). Let us define the operator \( b : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{R} \) as

\[
b(u, v, w) = \int_0^1 (u(x))^\delta \frac{\partial v(x)}{\partial x} w(x) dx.
\]

For all \( u, v \in H^1_0(\Omega) \), performing an integration by parts, we have

\[
b(u, u, v) = (u^\delta \partial_x u, v) = \frac{1}{\delta + 1} \int_0^1 \frac{\partial}{\partial x} (u(x))^{\delta+1} v(x) dx
\]

\[= - \frac{1}{\delta + 1} \int_0^1 (u(x))^{\delta+1} \frac{\partial v(x)}{\partial x} dx = - \frac{1}{\delta + 1} (u^{\delta+1}, \partial_x v). \tag{6}
\]

Using an integration by parts and boundary conditions, it can be easily seen that

\[
b(u, u, u) = (u^\delta \partial_x u, u) = \int_0^1 (u(x))^\delta \frac{\partial u(x)}{\partial x} u(x) dx = \frac{\delta}{\delta + 2} \int_0^1 \frac{\partial}{\partial x} (u(x))^{\delta+2} dx = 0. \tag{7}
\]

In general, for all \( p \geq 2 \) and \( u \in H^1_0(\Omega) \), we consider

\[
b(u, u, |u|^{p-2} u) = (u^\delta \partial_x u, |u|^{p-2} u) = \frac{1}{\delta + p} \int_0^1 \frac{\partial}{\partial x} (u(x))^{\delta+p} |u(x)|^{p-2} dx
\]

\[= - \frac{1}{\delta + 2} \int_0^1 (u(x))^{\delta+2} \frac{\partial}{\partial x} |u(x)|^{p-2} dx
\]

\[= \frac{p-2}{\delta + 2} \int_0^1 (u(x))^{\delta+2} |u(x)|^{p-4} u(x) \frac{\partial u(x)}{\partial x} dx
\]

\[= - \frac{p-2}{\delta + 2} (u^\delta \partial_x u, |u|^{p-2} u). \tag{8}
\]

The above expression implies

\[
b(u, u, |u|^{p-2} u) = (u^\delta \partial_x u, |u|^{p-2} u) = 0,
\]

for all \( p \geq 2 \) and \( u \in H^1_0(\Omega) \). The equality (8) is used in the sequel, when we establish \( L^{2^*+2} \)-energy estimate for the system (5).

For \( w \in L^2(\Omega) \), we define an operator \( B(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow L^2(\Omega) \) by

\[
B(u, v, w) = b(u, v, w) \leq ||u||_L^2 ||\partial_x v||_{L^2} ||w||_{L^2} \leq ||u||_H^1 ||v||_H^1 ||w||_{L^2},
\]

so that \( ||B(u, v)||_{L^2} \leq ||u||_H^1 ||v||_H^1 \). We denote \( B(u) = B(u, u) \), and thus one can easily obtain \( ||B(u)||_{L^2} \leq ||u||_H^{\delta+1} \). Let us now show that the operator \( B(\cdot) \) is locally Lipschitz. This property is used to obtain a local solution for the finite dimensional Galerkin approximation to the system (5) (see Step 1, Theorem 3.1 below). Using Taylor’s formula, Hölder’s and Gagliardo-Nirenberg’s interpolation (see Theorem 2.1, [33]) inequalities, for \( w = u - v \in H^1_0(\Omega) \) and \( 0 < \theta_1 < 1 \), we have

\[
(B(u) - B(v), w)
\]

\[= (u^\delta \partial_x (u - v), w) + ((u^\delta - v^\delta) \partial_x v, w)
\]

\[= -\delta \frac{1}{2} (u^{\delta-1} \partial_x u, w^2) + \delta((\theta_1 u + (1 - \theta_1) v)^{\delta-1} \partial_x v, w^2)
\]

\[\leq \frac{\delta}{2} ||u||_{L^\infty}^{\delta-1} ||\partial_x u||_{L^2} ||w||_{L^2} + \delta 2^\delta (||u||_{L^\infty}^{\delta-1} + ||v||_{L^\infty}^{\delta-1}) ||\partial_x v||_{L^2} ||w||_{L^\infty} ||w||_{L^2}.
\]
so that
\[ \|B(u) - B(v)\|_{L^2} \leq \frac{C\delta}{2} \|u\|_{H^1_0}^\delta \|v\|_{H^1_0} + C\delta^2 \left( \|u\|_{H^1_0}^{\delta-1} + \|v\|_{H^1_0}^{\delta-1} \right) \|v\|_{H^1_0} \|w\|_{H^1_0} \]
\leq C\delta(1 + 2^\delta)\|w\|_{H^1_0}, \quad (9)
for \( \|u\|_{H^1_0}, \|v\|_{H^1_0} \leq r \). Hence, the operator \( B : H^1_0(\Omega) \to L^2(\Omega) \) is locally Lipschitz.

2.3.2. The operator \( c(\cdot) \). Let us define the operator \( c : H^1_0(\Omega) \to L^2(\Omega) \) as \( c(u) = u(1 - u^\delta)(u^\delta - \gamma) \). It should be noted that
\[ (c(u), u) = (u(1 - u^\delta)(u^\delta - \gamma), u) = ((1 + \gamma)u^{\delta+1} - \gamma u - u^{2\delta+1}, u) \]
\[ = (1 + \gamma)(u^{\delta+1}, u) - \gamma \|u\|_{L^2}^2 - \|u\|_{L^2(\delta+1)}^{2(\delta+1)}, \quad (10) \]
for all \( u \in L^{2(\delta+1)}(\Omega) \subset H^1_0(\Omega) \). Using Taylor’s formula and Hölder’s inequality, for \( u, v \in H^1_0(\Omega) \), \( 0 < \theta_2 < 1 \) and \( 0 < \theta_3 < 1 \), we get
\[ \|c(u) - c(v)\|_{L^2} = \|(1 + \gamma)(u^{\delta+1} - v^{\delta+1}) - \gamma(u - v) - (u^{2\delta+1} + v^{2\delta+1})\|_{L^2} \]
\[ \leq (1 + \gamma)\delta + 1 \| (\theta_2 u + (1 - \theta_2)v) \|_{L^2}^\delta + \gamma \|u\|_{L^2} \]
\[ + (2\delta + 1) \frac{1}{\|u\|_{L^2}^{\delta+1} + \|v\|_{L^2}^{\delta+1}} \|w\|_{L^2} \]
\[ \leq \frac{C}{\pi}((1 + \gamma)(\delta + 1)2^{\delta-1}(\|u\|_{L^\infty}^{\delta+1} + \|v\|_{L^\infty}^{\delta+1})\|w\|_{L^2} + \gamma \|w\|_{L^2} \]
\[ = \frac{1}{\pi}((1 + \gamma)(\delta + 1)2^{\delta-1}(\|u\|_{L^\infty}^{\delta+1} + \|v\|_{L^\infty}^{\delta+1})\|w\|_{L^2} + \gamma \|w\|_{L^2} \]
\[ \leq \frac{r}{\pi}((1 + \gamma)(\delta + 1)2^{\delta-1}(\|u\|_{L^\infty}^{\delta+1} + \|v\|_{L^\infty}^{\delta+1})\|w\|_{L^2} + \gamma \|w\|_{L^2} \]
\[ \leq C(1 + \gamma)(\delta + 1)2^{\delta-1} + \gamma + (2\delta + 1)2^{2\delta} \|w\|_{L^2}, \quad (11) \]
for \( \|u\|_{H^1_0}, \|v\|_{H^1_0} \leq r \). Thus the operator \( c : H^1_0(\Omega) \to L^2(\Omega) \) is locally Lipschitz.

2.4. Abstract formulation. With the above notations, we can rewrite the abstract formulation of the system (5) as
\[
\begin{cases}
\frac{du(t)}{dt} + \nu Au(t) = -\alpha B(u(t)) + \beta c(u(t)) + f(t), & \text{in } H^{-1}(\Omega), \\
u(0) = u_0 \in L^2(\Omega), & \text{for a.e. } t \in [0, T].
\end{cases}
\quad (12)
\]
Let us now provide the definition of weak solution to the system (5).

**Definition 2.1** (Weak solution). A function
\[ u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \cap L^2(\delta+1)(0, T; L^{2(\delta+1)}(\Omega)) \]
with
\[ \partial_t u \in L^{1+\frac{1}{\gamma}}(0, T; H^{-1}(\Omega)) \]
is called a weak solution to the system (12), if for \( u_0 \in L^2(\Omega) \), \( f \in L^2(0, T; H^{-1}(\Omega)) \) and \( \nu(u) \) satisfies:
\[
\begin{cases}
\langle \partial_t u(t), v \rangle + \nu a(u(t), v) + \alpha b(u(t), u(t), v) = \beta(c(u(t)), v) + \langle f(t), v \rangle, \\
u(u) = (u_0, v),
\end{cases}
\quad (13)
\]
for a.e. \( t \in [0, T] \).

**Definition 2.2** (Strong solution). A function \( u \in C([0, T]; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega)) \) with \( \partial_t u \in L^2(0, T; L^2(\Omega)) \) is called a strong solution to the system (12), if for \( u_0 \in H^1_0(\Omega) \), \( f \in L^2(0, T; L^2(\Omega)) \), \( u(\cdot) \) satisfies (12) in \( L^2(\Omega) \) for a.e. \( t \in [0, T] \).
3. Global solvability. In this section, we prove the existence and uniqueness of weak solution to the system (12) by making use of a Faedo-Galerkin approximation technique. We obtain the regularity results on the weak solution by taking the initial data and forcing more regular. Whenever, the external forcing is zero, we also show that the classical solution to the system (12) is bounded and positive whenever the initial data is bounded and positive.

3.1. Existence and uniqueness of weak solution. We use a Faedo-Galerkin approximation technique to obtain the existence and uniqueness of weak solutions to the forced generalized Burgers-Huxley system (12).

Theorem 3.1. For \( u_0 \in L^2(\Omega) \) and \( f \in L^2(0,T;H^{-1}(\Omega)) \), there exists a weak solution to the system (12) in the sense of Definition 2.1. For \( \beta \nu > (2\delta \alpha)^2 \), the uniqueness of weak solution holds. Moreover, for \( 1 \leq \delta \leq 2 \), weak solution is unique for any \( \beta, \nu \) and \( \alpha \).

Proof. We establish the existence of a weak solution to the system (12) in the following steps.

Step 1: Faedo-Galerkin approximation. Let the functions \( w_k = w_k(x), k = 1, 2, \ldots \), be smooth, the set \( \{w_k(x)\}_{k=1}^{\infty} \) be an orthogonal basis of \( H_0^1(\Omega) \) and orthonormal basis of \( L^2(\Omega) \) (page 504, [16]). One can take \( \{w_k(x)\}_{k=1}^{\infty} \) as the complete set of normalized eigenfunctions of the operator \( -\partial_{xx} \) in \( H_0^1(\Omega) \). For a fixed positive integer \( m \), we look for a function \( u_m : [0,T] \to H_0^1(\Omega) \) of the form \( u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k \), where the coefficients \( d_m^k(t) \), for \( 0 \leq t \leq T \) and \( k = 1, \ldots, m \) are selected so that \( d_m^k(0) = (u_0, w_k) \), for \( k = 1, 2, \ldots, m \) and

\[
(\partial_t u_m, w_k) + \nu(\partial_x u_m, \partial_x w_k) + \alpha((u_m)\delta \partial_x u_m, w_k) = \beta(u_m(1 - (u_m)\delta) \partial_x u_m, w_k) + (f, w_k),
\]

for \( 0 \leq t \leq T \) and \( k = 1, \ldots, m \). Using the local Lipschitz properties of the nonlinear operators (see (9) and (11)) and Carathéodory’s existence theorem, one can show that the finite dimensional problem (14) has a local solution for a.e. \( t \in [0,T^*] \), for \( 0 < T^* < T \) in \( H_0^1(\Omega) \), where \( H_m = \text{span}\{w_1, \ldots, w_m\} \), that is, \( u_m \in C([0,T^*]; H_m) \). In the sequel, we show that this time \( T^* \) can be extended to \( T \) by showing that the estimates on \( u_m(t) \) does not depend on \( m \).

Step 2: \( L^2 \)-energy estimate. Multiplying (14) by \( d_m^k(t) \), and then summing over \( k = 1, \ldots, m \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{L^2}^2 + \nu \|\partial_x u_m(t)\|_{L^2}^2 + \alpha \|(u_m)\delta \partial_x u_m, u_m\) = \beta \|(u_m(1 - (u_m)\delta) \partial_x u_m, u_m\) + \beta \|f(t), u_m(t)\),
\]

for a.e. \( t \in [0,T] \). Using (7) and (10) in (15), we get

\[
\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{L^2}^2 + \nu \|\partial_x u_m(t)\|_{L^2}^2 + \beta \|u_m(t)\|_{L^2}^{2(\delta + 1)} + \frac{1}{2} \|\partial_x u_m(t)\|_{L^2}^2 \\
\leq \beta \|f(t), u_m(t)\) + \beta \|f(t)\|_{H^{-1}} \|\partial_x u_m(t)\|_{L^2} \\
\leq \frac{\beta}{2} \|u_m(t)\|_{L^2}^{2(\delta + 1)} + \frac{\beta(1 + \gamma)^2}{2} \|u_m(t)\|_{L^2}^2 + \frac{\nu}{2} \|\partial_x u_m(t)\|_{L^2}^2 + \frac{1}{2\nu} \|f(t)\|_{H^{-1}}^2.
\]
Integrating the above inequality from 0 to \( t \), we find
\[
\|u_m(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x u_m(s)\|_{L^2}^2 ds + \beta \int_0^t \|u_m(s)\|_{L^2}^{2 (\delta + 1)} ds \\
\leq \|u_0\|_{L^2}^2 + \frac{1}{\nu} \int_0^t \|f(s)\|_{H^{-1}}^2 ds + \beta (1 + \gamma^2) \int_0^t \|u_m(s)\|_{L^2}^2 ds,
\]
for all \( t \in [0, T] \), where we used the fact that \( \|u_m(0)\|_{L^2} \leq \|u_0\|_{L^2} \). An application of Gronwall’s inequality in (17) yields
\[
\|u_m(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x u_m(s)\|_{L^2}^2 ds + \beta \int_0^t \|u_m(s)\|_{L^2}^{2 (\delta + 1)} ds \\
\leq \left( \|u_0\|_{L^2}^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{H^{-1}}^2 dt \right) e^{\beta (1 + \gamma^2) T},
\]
for all \( t \in [0, T] \). Taking supremum over time \( 0 \leq t \leq T \), we get
\[
\sup_0 \leq t \leq T \|u_m(t)\|_{L^2}^2 + \nu \int_0^T \|\partial_x u_m(t)\|_{L^2}^2 dt + \beta \int_0^T \|u_m(t)\|_{L^2}^{2 (\delta + 1)} dt \\
\leq \left( \|u_0\|_{L^2}^2 + \frac{1}{\nu} \int_0^T \|f(t)\|_{H^{-1}}^2 dt \right) e^{\beta (1 + \gamma^2) T},
\]
and the right hand side of the inequality (19) is independent of \( m \).

**Step 3:** Estimate for time derivative. For any \( v \in H^1_0(\Omega) \), with \( \|v\|_{H^1_0} \leq 1 \), we write \( v = v^1 + v^2 \), where \( v^1 \in \text{span}\{w_k\}^m_{k=1} \) and \( \langle v^2, w_k \rangle = 0 \), for \( k = 1, 2, \ldots, m \). The orthogonality of the functions \( \{w_k\}^m_{k=1} \) in \( H^1_0(\Omega) \) easily gives \( \|v^1\|_{H^1_0} \leq \|v\|_{H^1_0} \leq 1 \). From (14), for a.e. \( t \in [0, T] \), we find
\[
\langle \partial_t u_m, v^1 \rangle = \langle \partial_t u_m, v^1 \rangle = -\nu \langle \partial_x u_m, \partial_x v^1 \rangle - \alpha \langle (u_m)^\delta \partial_x u_m, v^1 \rangle + \beta \langle (u_m(1 - (u_m)^\delta)) ((u_m)^\delta - \gamma), v^1 \rangle + \langle f, v^1 \rangle.
\]
An application of the Cauchy-Schwarz inequality and Hölder’s inequality yields
\[
\|\langle \partial_t u_m, v^1 \rangle \| \leq \nu \|\partial_x u_m\| \|\partial_x v^1\| + \alpha \|((u_m)^\delta \partial_x u_m, v^1)\| + \beta \|u_m(1 - (u_m)^\delta)((u_m)^\delta - \gamma), v^1\| + \|f, v^1\| \\
\leq \left( \nu \|\partial_x u_m\|_{L^2} + \frac{\alpha}{\delta + 1} \|u_m\|_{L^{2 (\delta + 1)}} + \|f\|_{H^{-1}} \right) \|\partial_x v^1\|_{L^2} \\
+ \beta \left( (1 + \gamma) \|u_m\|_{L^{2 (\delta + 1)}} + \gamma \|u_m\|_{L^2} \right) \|v^1\|_{L^2} + \beta \|u_m\|_{L^{2 (\delta + 1)}} \|v^1\|_{L^\infty},
\]
where we used (6) also. Since \( \|v^1\|_{L^2} \leq \frac{1}{\pi} \|v^1\|_{H^1_0}, \|v^1\|_{L^\infty} \leq C \|v^1\|_{H^1_0} \) and \( \|v^1\|_{H^1_0} \leq 1 \), from (21), it is immediate that
\[
\|\partial_t u_m\|_{H^{-1}} \leq \left( \nu \|\partial_x u_m\|_{L^2} + \frac{\alpha}{\delta + 1} \|u_m\|_{L^{2 (\delta + 1)}} + \|f\|_{H^{-1}} \\
+ \beta \left( (1 + \gamma) \|u_m\|_{L^{2 (\delta + 1)}} + \gamma \|u_m\|_{L^2} \right) + C \beta \|u_m\|_{L^{2 (\delta + 1)}} \right).
\]
Using interpolation inequality, we estimate the final term from the inequality (22) as
\[
\int_0^T \| u_m(t) \|_{L^{2(\delta+1)}}^{2(\delta+1)} dt \leq \sup_{t \in [0,T]} \| u_m(t) \|_{L^2}^{\frac{2\delta}{\delta+1}} \int_0^T \| u_m(t) \|_{L^{2(\delta+1)}}^{2(\delta+1)} dt
\]
\[
\leq C(\|u_0\|_{L^2}, \nu, \beta, \gamma, T, \|f\|_{L^2(0,T;H^{-1})}).
\]
Hence, by using Young's inequality, we have
\[
\int_0^T \| \partial_t u_m(t) \|_{H^{-2}}^{\frac{2\delta}{\delta+1}} dt
\]
\[
\leq C \left\{ \frac{\nu^{2\delta}}{\delta+1} \int_0^T \| \partial_x u_m(t) \|_{L^2}^{\frac{2\delta}{\delta+1}} dt + \left( \frac{\alpha}{\delta+1} \right)^{\frac{2\delta}{\delta+1}} \int_0^T \| u_m(t) \|_{L^{2(\delta+1)}}^{\frac{2(\delta+1)}{2}} dt 
+ \int_0^T \| f(t) \|_{H^{-1}}^{\frac{2\delta}{\delta+1}} dt + C \beta \frac{2\delta}{\delta+1} \int_0^T \| u_m(t) \|_{L^{2(\delta+1)}}^{\frac{2(\delta+1)}{2}} dt 
+ \left( \frac{\beta}{\pi} \right)^{\frac{2\delta}{\delta+1}} (1 + \gamma) \frac{2\delta}{\delta+1} \int_0^T \| u_m(t) \|_{L^{2(\delta+1)}}^{\frac{2(\delta+1)}{2}} dt + \gamma \frac{2\delta}{\delta+1} \int_0^T \| u_m(t) \|_{L^2}^{\frac{2\delta}{\delta+1}} dt \right\}
\]
\[
\leq C(\|u_0\|_{L^2}, \nu, \beta, \gamma, T, \|f\|_{L^2(0,T;H^{-1})}),
\]
so that \( \partial_t u_m \in L^{1+\frac{1}{\delta+1}}(0,T;H^{-2}(\Omega)) \), since
\[
\int_0^T \| f(t) \|_{H^{-2}}^{\frac{2\delta}{\delta+1}} dt \leq T^{\frac{2\delta}{\delta+1}} \left( \int_0^T \| f(t) \|_{H^{-2}}^{2} dt \right)^{\frac{\delta-1}{\delta+1}},
\]
for \( \delta > 1 \).

**Step 4: Passing to limit.** From the energy estimates (19) and (24), we know that the sequence \( \{u_m\}_{m=1}^{\infty} \) is bounded uniformly and is independent of \( m \). Using the Banach-Alaoglu theorem, we can extract a subsequence \( \{u_{m_j}\}_{j=1}^{\infty} \) of \( \{u_m\}_{m=1}^{\infty} \) such that
\[
\begin{align*}
&\left\{ u_{m_j} \right\}_{j=1}^{\infty} \overset{w^*}{\rightarrow} u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \\
&\left\{ u_{m_j} \right\}_{j=1}^{\infty} \overset{w}{\rightarrow} u \quad \text{in} \quad L^2(0,T;H^1_0(\Omega)), \\
&\left\{ u_{m_j} \right\}_{j=1}^{\infty} \overset{w}{\rightarrow} u \quad \text{in} \quad L^{2\delta+1}(0,T;L^{2\delta+1}(\Omega)), \\
&\partial_t u_{m_j} \overset{w}{\rightarrow} \partial_t u \quad \text{in} \quad L^{1+\frac{1}{\delta+1}}(0,T;H^{-2}(\Omega)),
\end{align*}
\]
as \( j \to \infty \), for all \( \delta \geq 1 \). Since the embedding of \( H^1_0(\Omega) \subset L^2(\Omega) \) is compact (page 274, [20]), an application of the Aubin-Lions compactness lemma (Theorem 1, [38]) yields the following strong convergence:
\[
u m_j \to u \quad \text{in} \quad L^2(0,T;L^2(\Omega)),
\]
as \( j \to \infty \). The above strong convergence also implies \( u_{m_k} \to u \), for a.e. \( (x,t) \in \Omega \times [0,T] \) as \( j \to \infty \) (that is, along a subsequence). Let us now take limit in (14) along the subsequence \( \{u_{m_j}\}_{j=1}^{\infty} \). Let us first fix and integer \( N \) and choose a function
\( v \in C^1([0,T];H^1_0(\Omega)) \) having the form
\[
v(t) = \sum_{k=1}^{N} a_m^k(t)w_k,
\]
where \( \{a_m^k(t)\}_{k=1}^{N} \) are given smooth functions. Let us choose \( m \geq N \), multiply (14) by \( d^k(t) \), sum from
Let us show that $u(0) = u_0$. From (28), we note that
\[
\int_0^T \left[ -\langle u(t), \partial_t v(t) \rangle + \nu(\partial_x u(t), v(t)) + \alpha((u(t))^\delta \partial_x u(t), v(t)) \right] dt
\]
\[
= \int_0^T \left[ \beta(u(t)(1 - (u(t))^\delta))((u(t))^\delta - \gamma), v(t) \right] + \langle f(t), v(t) \rangle] dt + (u(0), v(0)), \quad (31)
\]
for each $v \in C^1([0, T]; H^1_0(\Omega))$ with $v(T) = 0$. In a similar fashion, we deduce from (27) that
\[
\int_0^T \left[ -\langle u_m(t), \partial_t v(t) \rangle + \nu(\partial_x u_m(t), v(t)) + \alpha((u_m(t))^\delta \partial_x u_m(t), v(t)) \right] dt
\]
\[
= \int_0^T \left[ \beta(u_m(t)(1 - (u_m(t))^\delta))((u_m(t))^\delta - \gamma), v(t) \right] + \langle f(t), v(t) \rangle] dt + (u_m(0), v(0)). \quad (32)
\]

**Step 5: Initial data.** Let us now show that $u(0) = u_0$. From (28), we note that
\[
\int_0^T \left[ -\langle u(t), \partial_t v(t) \rangle + \nu(\partial_x u(t), v(t)) + \alpha((u(t))^\delta \partial_x u(t), v(t)) \right] dt
\]
\[
= \int_0^T \left[ \beta(u(t)(1 - (u(t))^\delta))((u(t))^\delta - \gamma), v(t) \right] + \langle f(t), v(t) \rangle] dt + (u(0), v(0)), \quad (31)
\]
for each $v \in C^1([0, T]; H^1_0(\Omega))$ with $v(T) = 0$. In a similar fashion, we deduce from (27) that
\[
\int_0^T \left[ -\langle u_m(t), \partial_t v(t) \rangle + \nu(\partial_x u_m(t), v(t)) + \alpha((u_m(t))^\delta \partial_x u_m(t), v(t)) \right] dt
\]
\[
= \int_0^T \left[ \beta(u_m(t)(1 - (u_m(t))^\delta))((u_m(t))^\delta - \gamma), v(t) \right] + \langle f(t), v(t) \rangle] dt + (u_m(0), v(0)). \quad (32)
\]
Taking $m = m_j$ in the above expression, using the convergences (25), (26) and (29), and then passing to limit, we find
\[
\int_0^T \left[ -\langle u(t), \partial_x v(t) \rangle + \nu (\partial_x u(t), v(t)) + \alpha ((u(t))^\delta \partial_x u(t), v(t)) \right] dt
= \int_0^T \left[ \beta (u(t)(1 - (u(t))^\delta)((u(t))^\delta - \gamma), v(t)) + (f(t), v(t)) \right] dt + (u_0, v(0)),
\]
(33)
since $u_m(0) \to u_0$ as $m \to \infty$ in $L^2(\Omega)$. As $v(0)$ is arbitrary, comparing (31) and (33), one can easily conclude that $u(0) = u_0 \in L^2(\Omega)$.

**Step 6: Uniqueness.** Let $u_1(\cdot)$ and $u_2(\cdot)$ be two weak solutions of the system (12) with the same external forcing $f(\cdot, \cdot)$ and initial data $u_0(\cdot)$. Then, $w = u_1 - u_2$ satisfies:

\[
\begin{cases}
\frac{\partial w(t)}{\partial t} + \alpha \left[ u_1(t)^\delta \partial_x u_1(t) - u_2(t)^\delta \partial_x u_2(t) \right] - \nu \frac{\partial^2 w(t)}{\partial x^2} \\
= \beta \left[ (u_1(t)(1 - u_1(t)^\delta)(u_1(t)^\delta - \gamma) - u_2(t)(1 - u_2(t)^\delta)(u_2(t)^\delta - \gamma) \right],
\end{cases}
\]

in $H^{-1}(\Omega)$, for a.e. $t \in [0, T]$. In the weak formulation, one can write the above system as

\[
\langle \partial_t w(t), v \rangle + \nu (\partial_x w(t), \partial_x v) - \frac{\alpha}{\delta + 1} (u_1(t)^\delta + 1 - u_2(t)^\delta + 1, \partial_x v)
= \beta \left[ (u_1(t)(1 - u_1(t)^\delta)(u_1(t)^\delta - \gamma) - u_2(t)(1 - u_2(t)^\delta)(u_2(t)^\delta - \gamma), v \right],
\]

for all $v \in H^1_0(\Omega)$ and a.e. $t \in [0, T]$. Taking $v = w$ in the above expression, we find

\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_{H^2}^2 + \nu \|\partial_x w(t)\|_{L^2}^2
= \frac{\alpha}{\delta + 1} (u_1(t)^\delta + 1 - u_2(t)^\delta + 1, \partial_x w(t))
+ \beta \left[ (u_1(t)(1 - u_1(t)^\delta)(u_1(t)^\delta - \gamma) - u_2(t)(1 - u_2(t)^\delta)(u_2(t)^\delta - \gamma), w(t) \right].
\]

(36)

It can be easily seen that

\[
\beta \left[ (u_1(t)(1 - u_1(t)^\delta)(u_1(t)^\delta - \gamma) - u_2(t)(1 - u_2(t)^\delta)(u_2(t)^\delta - \gamma), w \right]
= -\beta \|w\|^2_{H^2} - \beta (u_1^{2\delta+1} - u_2^{2\delta+1}, w) + \beta (1 + \gamma) (u_1^{1+\delta} - u_2^{1+\delta}, w).
\]

(37)

Let us take the term $-\beta (u_1^{2\delta+1} - u_2^{2\delta+1}, w)$ from (37) and estimate it as

\[
-\beta (u_1^{2\delta+1} - u_2^{2\delta+1}, w)
= -\beta(u_1^{2\delta}, w^2) - \beta(u_2^{2\delta}, w^2) - \beta(u_1^{2\delta} u_2 - u_2^{2\delta} u_1, w)
= -\beta\|u_1^{2\delta} w\|^2_{L^2} - \beta\|u_2^{2\delta} w\|^2_{L^2} - \beta(u_1 u_2, u_1^{2\delta} + u_2^{2\delta}) + \beta(u_1^{2\delta}, u_2^{2\delta}) + \beta(u_2^{2\delta}, u_1^{2\delta})
= -\frac{\beta}{2} \|u_1^{2\delta} w\|^2_{L^2} - \frac{\beta}{2} \|u_2^{2\delta} w\|^2_{L^2} - \beta\|u_1^{2\delta} u_2 - u_2^{2\delta} u_1, w\|^2_{L^2}
\leq -\frac{\beta}{2} \|u_1^{2\delta} w\|^2_{L^2} - \frac{\beta}{2} \|u_2^{2\delta} w\|^2_{L^2}.
\]

(38)
Next, we take the term $\beta(1 + \gamma)(u^{1+\delta} - u^{\delta+1}, w)$ from (37) and estimate it using Taylor’s formula, Hölder’s and Young’s inequalities as

\[
\beta(1 + \gamma)(u^{1+\delta} - u^{\delta+1}, w) = \beta(1 + \gamma)(\delta + 1)((\theta u_1 + (1 - \theta)u_2)^\delta w, w) \\
\leq \beta(1 + \gamma)(\delta + 1)2^{\delta-1}(||u_1^\delta w||_{L^2} + ||u_2^\delta w||_{L^2})||w||_{L^2} \\
\leq \frac{\beta}{4}||u_1^\delta w||_{L^2}^2 + \frac{\beta}{4}||u_2^\delta w||_{L^2}^2 + \frac{\beta}{2}2^{2\delta}(1 + \gamma)^2(\delta + 1)^2||w||_{L^2}^2.
\]

Combining (38)-(39) and substituting it in (37), we obtain

\[
\beta[(u_1(1 - u_1^\delta)(u_1^\delta - \gamma) - u_2(1 - u_2^\delta)(u_2^\delta - \gamma), w)] \\
\leq -\beta\gamma||w||_{L^2}^2 - \frac{\beta}{4}||u_1^\delta w||_{L^2}^2 - \frac{\beta}{4}||u_2^\delta w||_{L^2}^2 + \frac{\beta}{2}2^{2\delta}(1 + \gamma)^2(\delta + 1)^2||w||_{L^2}^2.
\]

Once again using Taylor’s formula, Hölder’s and Young’s inequalities, we estimate $\frac{\alpha}{\delta+1}(u_1^{\delta+1} - u_2^{\delta+1}, \partial_x w)\partial_x w$ as

\[
\frac{\alpha}{\delta+1}(u_1^{\delta+1} - u_2^{\delta+1}, \partial_x w)\partial_x w = \alpha((\theta u_1 + (1 - \theta)u_2)^\delta w, \partial_x w) \\
\leq \alpha 2^{\delta-1}(||u_1^\delta w||_{L^2} + ||u_2^\delta w||_{L^2})||\partial_x w||_{L^2} \\
\leq \frac{\nu}{2}||\partial_x w||_{L^2}^2 + \frac{\alpha^2}{4\nu^2}2^{2\delta}||u_1^\delta w||_{L^2}^2 + \frac{\alpha^2}{4\nu^2}2^{2\delta}||u_2^\delta w||_{L^2}^2.
\]

Applying (40) and (41) in (36), we find

\[
\frac{1}{2} \frac{d}{dt} ||w(t)||_{L^2}^2 + \frac{\nu}{2}||\partial_x w(t)||_{L^2}^2 + \frac{\beta}{4}||u_1^\delta(t)w(t)||_{L^2}^2 + \frac{\beta}{4}||u_2^\delta(t)w(t)||_{L^2}^2 + \beta\gamma||w(t)||_{L^2}^2 \\
\leq \frac{\beta}{2}2^{2\delta}(1 + \gamma)^2(\delta + 1)^2||w||_{L^2}^2 + \frac{\alpha^2}{4\nu^2}2^{2\delta}||u_1^\delta w||_{L^2}^2 + \frac{\alpha^2}{4\nu^2}2^{2\delta}||u_2^\delta w||_{L^2}^2.
\]

From the above expression, it is immediate that

\[
\frac{d}{dt} ||w(t)||_{L^2}^2 + \nu||\partial_x w(t)||_{L^2}^2 \\
+ \left(\frac{\beta}{2} - \frac{\alpha^2}{2\nu^2}2^{2\delta}\right)||u_1^\delta(t)w(t)||_{L^2}^2 + \left(\frac{\beta}{2} - \frac{\alpha^2}{2\nu^2}2^{2\delta}\right)||u_2^\delta(t)w(t)||_{L^2}^2 \\
\leq \beta 2^{2\delta}(1 + \gamma)^2(\delta + 1)^2||w(t)||_{L^2}^2.  
\]

For $\beta\nu > (2^\delta\alpha)^2$, an application of Gronwall’s inequality in (42) yields

\[
||w(t)||_{L^2}^2 \leq ||w(0)||_{L^2}^2 e^{\beta 2^{2\delta}(1 + \gamma)^2(\delta + 1)^2t},
\]

for all $t \in [0, T]$. Since $w(0) = 0$, and $u_1(\cdot)$ and $u_2(\cdot)$ are weak solutions of the system (13), uniqueness follows from (43) easily.

For the case $1 \leq \delta \leq 2$, we can get the uniqueness without any restriction on $\beta, \nu$ and $\alpha$. Using Gagliardo-Nirenberg’s, Hölder’s and Young’s inequalities, one
can estimate the term $\frac{\alpha}{\delta+1}(u_1^{\delta+1} - u_2^{\delta+1}, \partial_x w)$ as

$$\frac{\alpha}{\delta+1}(u_1^{\delta+1} - u_2^{\delta+1}, \partial_x w) = \alpha((\theta_2 u_1 + (1-\theta_2)u_2)^\delta, \partial_x w) \leq \alpha 2^{\delta-1}(\|u_1\|_{L^{(2+)\delta}}^\delta + \|u_2\|_{L^{(2+)\delta}}^\delta)\|w\|_{L^{(2+)\delta}} \leq C\alpha 2^{\delta-1}(\|u_1\|_{L^{(2+)\delta}}^\delta + \|u_2\|_{L^{(2+)\delta}}^\delta)\|w\|_{L^{(2+)\delta}}^{\frac{3\delta+2}{\delta}} \|\partial_x w\|_{L^{(2+)\delta}}^{\frac{3\delta+2}{\delta}}.$$

Combining (40) and (44), and then substituting it in (36), we obtain

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 + \nu \|\partial_x w(t)\|_{L^2}^2 + \beta \|u_1(t)w(t)\|_{L^2}^2 + \frac{\beta}{2} \|u_2(t)w(t)\|_{L^2}^2 + 2\beta \|u_1(t)w(t)\|_{L^2}^2 \leq 2^{\delta} \beta (1+\gamma)^2(\delta+1)^2 \|w(t)\|_{L^2}^2 + C 2^{\frac{4\delta+4}{\delta+2}} \alpha \|u_2(t)\|_{L^{(2+)\delta}}^{\frac{4\delta+4}{\delta+2}} \left(3\beta + \frac{2\beta}{2\nu(\delta+1)}\right)^{\frac{3\delta+2}{\delta+2}}.$$

An application of Gronwall’s inequality in (45) gives

$$\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 e^{2^{\delta}(1+\gamma)^2(\delta+1)^2 T} \times \exp \left\{ C(\alpha, \delta, \nu) \int_0^T \left(\|u_1(t)\|_{L^{(2+)\delta}}^{\frac{4\delta+4}{\delta+2}} + \|u_2(t)\|_{L^{(2+)\delta}}^{\frac{4\delta+4}{\delta+2}}\right) dt \right\},$$

(46)

and the integral appearing in the exponential is finite, provided $1 \leq \delta \leq 2$. Thus, the uniqueness follows from (46), for any $\nu, \beta, \alpha > 0$ and $1 \leq \delta \leq 2$.

Let us now establish the regularity of the weak solution obtained in Theorem 3.1 under the smoothness assumptions on initial data and external forcing. Note that if there exists a strong solution, then any weak solution with the same initial data coincides with it. That is, strong solutions are unique in a much larger class of weak solutions (see Step 6, proof of Theorem 3.1 for uniqueness of strong solutions).

**Theorem 3.2.** Let $u_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$ be given. Then, we have

$$u \in C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$$

(47)

with $\partial_t u \in L^2(0,T;L^2(\Omega))$ and the following system is satisfied:

$$\begin{cases}
\frac{du(t)}{dt} + \nu Au(t) = -\alpha B(u(t)) + \beta c(u(t)) + f(t), & \text{in } L^2(\Omega), \\
u(0) = u_0 \in L^2(\Omega),
\end{cases}$$

(48)

for a.e. $t \in [0,T]$. Furthermore, the strong solution is unique for any $\nu, \beta, \alpha, \delta > 0$.

**Proof.** Note that $H_0^1(\Omega) \subset L^\infty(\Omega) \subset L^{2\delta+2}(\Omega)$, and hence $u_0 \in H_0^1(\Omega)$ implies $u_0 \in L^{2\delta+2}(\Omega)$. Thus, our first aim is obtain an $L^{2\delta+2}$-energy estimate.
Step 1: $L^{2\delta+2}$-energy estimate. Multiplying (14) by $d^k_m(t)|u_m(t)|^{2\delta}$, and then summing over $k = 1, \ldots, m$, we obtain
\[
\frac{1}{2\delta + 2} \frac{d}{dt} \|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} + \nu(2\delta + 1) \|u_m(t)\|_2^2 \|\partial_x u_m(t)\|_{L^2}^2 \\
+ \beta \|u_m(t)\|_{L^{4\delta+2}}^{4\delta+2} + \beta \gamma \|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} \\
= \beta(1 + \gamma)(\|(u_m(t))^{\delta+1}, u_m(t)\|_{L^{2\delta+2}}^{2\delta+2}) + (f(t), u_m(t)) \|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} \\
\leq \beta(1 + \gamma)(\|u_m(t)\|_{L^{4\delta+2}}^{4\delta+1} + \|f(t)\|_{L^2} + \|u_m(t)\|_{L^{4\delta+2}}^{4\delta+1} + \frac{1}{\beta} \|f(t)\|_{L^2}^2) \\
\leq \beta \frac{2}{2} \|u_m(t)\|_{L^{4\delta+2}}^{4\delta+2} + \beta(1 + \gamma)^2 \|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} + \frac{1}{\beta} \|f(t)\|_{L^2}^2, \tag{49}
\]
where we used (8), the Cauchy-Schwarz inequality, Holder’s as well as Young’s inequalities. Integrating the inequality (49) from 0 to $t$, we find
\[
\frac{1}{2\delta + 2} \|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} + \nu(2\delta + 1) \int_0^t \|u_m(s)\|_2^2 \|\partial_x u_m(s)\|_{L^2}^2 ds \\
+ \beta \int_0^t \|u_m(s)\|_{L^{4\delta+2}}^{4\delta+2} ds \\
\leq \frac{1}{2\delta + 2} \|u_0\|_{L^{2\delta+2}}^{2\delta+2} + \frac{\nu}{2\beta} \int_0^T \|f(t)\|_{L^2}^2 dt + \beta(1 + \gamma + \gamma^2) \int_0^t \|u_m(s)\|_{L^{2\delta+2}}^{2\delta+2} ds. \tag{50}
\]
Applying Gronwall’s inequality in (50), we get
\[
\|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} \leq \left(\|u_0\|_{L^{2\delta+2}}^{2\delta+2} + \frac{\nu}{2\beta} \int_0^T \|f(t)\|_{L^2}^2 dt\right) e^{\beta(1+\gamma+\gamma^2)T}, \tag{51}
\]
for all $t \in [0, T]$. Thus, it is immediate that
\[
\sup_{0 \leq t \leq T} \|u_m(t)\|_{L^{2\delta+2}}^{2\delta+2} + \nu(2\delta + 1)(2\delta + 2) \int_0^T \|u_m(t)\|_2^2 \|\partial_x u_m(t)\|_{L^2}^2 dt \\
+ \beta(2\delta + 2) \int_0^T \|u_m(t)\|_{L^{4\delta+2}}^{4\delta+2} dt \\
\leq \left(\|u_0\|_{L^{2\delta+2}}^{2\delta+2} + \frac{\nu}{2\beta} \int_0^T \|f(t)\|_{L^2}^2 dt\right) e^{2\beta(1+\gamma+\gamma^2)T}, \tag{52}
\]
and the right hand side of the above inequality is independent of $m$.

Step 2: $H^1_0$-energy estimate. Let $\lambda_k$ denotes the $k^{th}$ eigenvalue of $-\partial_{xx}$ in $H^1_0(\Omega)$. Multiplying the identity (14) by $\lambda_k d^k_m(t)$ and summing it from $k = 1, 2, \ldots, m$, we find
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u_m(t)\|_{L^2}^2 + \nu\|\partial_x u_m(t)\|_{L^2}^2 \\
= \alpha(u_m(t)^\delta \partial_x u_m(t), \partial_x u_m(t)) - \beta(1 - u_m(t)^\delta)(u_m(t)^\delta - \gamma), \partial_x u_m(t)) \tag{53}
\]
Using an integration by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u_m(t)\|_{L^2}^2 + \nu\|\partial_x u_m(t)\|_{L^2}^2 + \beta \gamma \|\partial_x u_m(t)\|_{L^2}^2 \\
+ \beta(2\delta + 1) \|u_m(t)^\delta \partial_x u_m(t)\|_{L^2}^2 
\]
\[
= \alpha(u_m(t)^\delta \partial_x u_m(t), \partial_x u_m(t)) - \beta(1 + \gamma)(u_m(t)^\delta, \partial_x u_m(t)) + (f(t), \partial_x u_m(t)). \tag{54}
\]
Using Hölder’s, Gagliardo-Nirenberg’s and Young’s inequalities, we estimate the first term from the right hand side of the inequality (54) as

\[
\alpha\|u_m^\delta \partial_x u_m, \partial_{xx} u_m\| \leq \alpha\|u_m^\delta \partial_x u_m\|_{L^2} \|\partial_{xx} u_m\|_{L^2} \\
\leq \alpha\|u_m^\delta\|_{L^{4 \delta + 2}} \|\partial_x u_m\|_{L^{4 \delta + 2}} \|\partial_{xx} u_m\|_{L^2} \\
\leq C\alpha\|u_m\|_{L^{4 \delta + 2}} \|\partial_{xx} u_m\|_{L^2}^{\delta/2} \\
\leq \frac{\nu}{4} \|\partial_{xx} u_m\|_{L^2}^2 + \frac{27C\alpha^2}{4\nu^3} \|u_m\|_{L^{4 \delta + 2}}^{4 \delta + 2}. \tag{55}
\]

Similarly, we estimate the second term from the right hand side of the inequality (54) as

\[
\beta(1 + \gamma)\|u_m^{\delta + 1}, \partial_{xx} u_m\| \leq \beta(1 + \gamma)\|u_m^{\delta + 1}\|_{L^2} \|\partial_{xx} u_m\|_{L^2} \\
\leq \frac{\nu}{4} \|\partial_{xx} u_m\|_{L^2}^2 + \frac{\beta^2(1 + \gamma)^2}{\nu} \|u_m\|_{L^{4 \delta + 2}}^{4 \delta + 2}. \tag{56}
\]

We estimate the final term from the inequality (54) as

\[
\|(f, \partial_{xx} u_m)\| \leq \|f\|_{L^2} \|\partial_{xx} u_m\|_{L^2} \leq \frac{\nu}{4} \|\partial_{xx} u_m\|_{L^2}^2 + \frac{1}{\nu} \|f\|_{L^2}^2. \tag{57}
\]

Combining (55)-(57), substituting it in (54), and the integrating from 0 to t, we find

\[
\|\partial_x u_m(t)\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\partial_{xx} u_m(s)\|_{L^2}^2 ds + 2\beta\gamma \int_0^t \|\partial_x u_m(s)\|_{L^2}^2 ds \\
+ 2\beta(2\delta + 1) \int_0^t \|u_m(s)^\delta \partial_x u_m(s)\|_{L^2}^2 ds \\
\leq \|u_0\|_{H^\delta_0}^2 + \frac{27C\alpha^2}{2\nu^3} \int_0^t \|u_m(s)\|_{L^{4 \delta + 2}}^{4 \delta + 2} ds + \frac{2\beta^2(1 + \gamma)^2}{\nu} \int_0^t \|u_m(s)\|_{L^{4 \delta + 2}}^{4 \delta + 2} ds \\
+ \frac{2}{\nu} \int_0^t \|f(s)\|_{L^2}^2 ds \\
\leq C\left(\|u_0\|_{H^\delta_0}, \|f\|_{L^2(0,T;L^2(\Omega))}, \alpha, \beta, \gamma, \delta, \nu, T\right), \tag{58}
\]

since \(u_m(\cdot)\) satisfies the energy estimates (19) and (52). Whenever \(u_0 \in H^\delta_0(\Omega)\) and \(f \in L^2(0,T;L^2(\Omega))\), from (58), we have

\[
u \in L^\infty(0,T;H^\delta_0(\Omega)) \cap L^2(0,T;H^2(\Omega)).
\]

Step 3: Estimate for time derivative. For \(m \geq 1\), multiplying (14) by \(d_{m}'(t)\) and summing it over \(k = 1, 2, \ldots, m\), we discover

\[
\|\partial_t u_m(t)\|_{L^2}^2 + \frac{\nu}{2} \|\partial_x u_m(t)\|_{L^2}^2 \\
= -\alpha((u_m(t)^\delta \partial_x u_m(t), \partial_t u_m(t)) + \beta(u_m(t)(1 - (u_m(t))^\delta)((u_m(t))^{\delta} - \gamma), \partial_t u_m(t)) \\
+ (f(t), \partial_t u_m(t)) \\
= -\alpha((u_m(t)^\delta \partial_x u_m(t), \partial_t u_m(t)) + \beta(1 + \gamma)(u_m(t)^{\delta + 1}, \partial_t u_m(t)) - \beta \gamma \frac{d}{dt} \|u_m(t)\|_{L^2}^2 \\
- \frac{\beta}{2(\delta + 1)} \frac{d}{dt} \|u_m(t)\|_{L^2(\delta + 1)}^2 + (f(t), \partial_t u_m(t)). \tag{59}
\]
Using Hölder’s and Young’s inequalities, we estimate the first term from the right hand side of the equality (59) as
\[
\alpha \| (u_m)^\delta \partial_x u_m, \partial_t u_m \| \leq \alpha \| u_m \|_{L^\infty} \| \partial_x u_m \| L^2 \| \partial_t u_m \| L^2 \\
\leq \frac{1}{4} \| \partial_t u_m \| L^2 + \alpha^2 \| u_m \|_{L^\infty}^\delta \| \partial_x u_m \| L^2.
\] (60)

Similarly, we estimate the term \( \beta (1 + \gamma) (u_m^{\delta+1}, \partial_t u_m) \) as
\[
\beta (1 + \gamma) (u_m^{\delta+1}, \partial_t u_m) \leq \beta (1 + \gamma) \| u_m \|_{L^2(\delta+1)} \| \partial_t u_m \| L^2 \\
\leq \frac{1}{4} \| \partial_t u_m \| L^2 + \beta^2 (1 + \gamma)^2 \| u_m \|_{L^2(\delta+1)}^2.
\] (61)

The final term from the right hand side of the equality (59) can be estimated as
\[
| (f, \partial_t u_m) | \leq \| f \| L^2 \| \partial_t u_m \| L^2 \leq \frac{1}{4} \| \partial_t u_m \| L^2 + \| f \| L^2.
\] (62)

Combining (60)-(62), substituting it in (59) and then integrating from 0 to \( t \), we obtain
\[
2\nu \| \partial_x u_m(t) \| L^2 + \frac{2\beta}{\delta + 1} \| u_m(t) \|_{L^2(\delta+1)}^\delta + 4\beta \gamma \| u_m(t) \| L^2 + \int_0^t \| \partial_t u_m(s) \| L^2 ds \\
\leq 2\nu \| u_0 \|_{H^\delta} + \frac{2\beta}{\delta + 1} \| u_0 \|_{L^2(\delta+1)}^\delta + 4\beta \gamma \| u_0 \| L^2 + 4 \int_0^t \| f(s) \| L^2 ds \\
+ 4\alpha^2 \int_0^t \| u_m(s) \|_{L^2}^\delta \| \partial_x u_m(s) \| L^2 ds + 4\beta^2 (1 + \gamma)^2 \int_0^t \| u_m(s) \|_{L^2(\delta+1)}^2 ds.
\] (63)

Applying Gronwall’s inequality in (63), we arrive at
\[
2\nu \| \partial_x u_m(t) \| L^2 + \frac{2\beta}{\delta + 1} \| u_m(t) \|_{L^2(\delta+1)}^\delta + 4\beta \gamma \| u_m(t) \| L^2 + \int_0^t \| \partial_t u_m(s) \| L^2 ds \\
\leq \left( 2\nu \| u_0 \|_{H^\delta} + \frac{2\beta}{\delta + 1} \| u_0 \|_{L^2(\delta+1)}^\delta + 4\beta \gamma \| u_0 \| L^2 + 4 \int_0^t \| f(t) \| L^2 dt \right) \\
\times \exp \left( 4\alpha^2 \int_0^T \| u_m(t) \|_{L^2}^\delta dt \right) e^{4\beta (1 + \gamma)^2 (1 + \delta) T},
\] (64)

for all \( t \in [0, T] \). But since \( H^\delta(\Omega) \subset L^\infty(\Omega) \subset L^p(\Omega) \), for \( 1 \leq p < \infty \), and using the estimate given in (58), we obtain that the right hand side of (64) is independent of \( m \). Thus, it is immediate that \( \partial_t u_m \in L^2(0, T; L^2(\Omega)) \). Using arguments similar to Steps 4 and 5 in Theorem 3.1, we finally obtain the existence of a \( u(\cdot) \) such that
\[
u \in L^\infty(0, T; H^\delta(\Omega)) \cap L^2(0, T; \mathbb{H}^2(\Omega)) \quad \text{and} \quad \partial_t u \in L^2(0, T; L^2(\Omega)),
\]
so that we get \( u \in C([0, T]; H^\delta(\Omega)) \) (see Theorem 3, section 5.9.2, [20]) and is a strong solution to the system (12).

The uniqueness of strong solution follows easily from the estimate (46).

**Remark 1.** The existence of a strong solution ensures that the weak solution satisfies the following energy equality:
\[
\| u(t) \| L^2 + 2\nu \int_0^t \| \partial_x u(s) \| L^2 ds + 2\beta \gamma \int_0^t \| u(s) \| L^2 ds + 2\beta \int_0^t \| u(s) \|_{L^2(\delta+1)}^2 ds \\
= \| u_0 \| L^2 + \beta (1 + \gamma) \int_0^t \langle (u(s))^{\delta+1}, u(s) \rangle ds + \int_0^t \langle f(s), u(s) \rangle ds,
\] (65)
for all \( t \in [0, T] \), and hence \( u \in C([0, T]; L^2(\Omega)) \).

**Theorem 3.3.** For \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( f \in H^1(0, T; L^2(\Omega)) \), we have

\[
\begin{align*}
(67) \\
\frac{1}{2} \frac{d}{dt} \| \partial_t u_m \|^2_{L^2} + \nu \| \partial_x u_m \|^2_{L^2} + \alpha \| \partial_t u_m \|^2_{L^2} + \beta(\gamma + 1)(\delta + 1) \| u_m \|^2_{L^2} + \beta(\gamma + 1)(\delta + 1) \| u_m \|^2_{L^2} \leq \| \partial_t f \|^2_{L^2} + \frac{1}{2} \| \partial_t u_m \|^2_{L^2}.
\end{align*}
\]

Differentiating (14) with respect to \( t \), we find

\[
\begin{align*}
(68) \\
(\partial_{tt} u_m, w_k) + \nu(\partial_x u_m, (w_k)_x) + \alpha(\partial_t u_m, \partial_x u_m) + \beta u_m^{\delta-1} \partial_t u_m \partial_x u_m, w_k
\end{align*}
\]

Multiplying (67) with \( \frac{d}{dt} \| \partial_t u_m \|^2_{L^2} + \nu \| \partial_x u_m \|^2_{L^2} + \beta(\gamma + 1)(\delta + 1) \| u_m \|^2_{L^2} \) and summing it over \( k = 1, 2, \ldots, m \), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \partial_t u_m \|^2_{L^2} + \nu \| \partial_x u_m \|^2_{L^2} + \beta \| \partial_t u_m \|^2_{L^2} + \beta(\gamma + 1)(\delta + 1) \| u_m \|^2_{L^2} = -\alpha(\partial_t u_m, \partial_t u_m) + \beta(\gamma + 1)(\delta + 1) \| u_m \|^2_{L^2} + \| \partial_t f \|^2_{L^2} + \frac{1}{2} \| \partial_t u_m \|^2_{L^2},
\end{align*}
\]

where we used an integration by parts also. We estimate the first term from the right hand side of the equality (68) using Hölder’s and Young’s inequalities as

\[
\alpha \| (u_m)^\delta \partial_t u_m, \partial_t u_m \| \leq \alpha \| u_m \|^\gamma_{L^\infty} \| \partial_x u_m \|_{L^2} \| \partial_t u_m \|_{L^2}
\]

Similarly, we estimate the terms \( \beta(\gamma + 1)(\delta + 1)(u_m^\delta \partial_t u_m, \partial_t u_m) \) and \( (\partial_t f, \partial_t u_m) \) as

\[
\begin{align*}
\beta(\gamma + 1)(\delta + 1)(u_m^\delta \partial_t u_m, \partial_t u_m) &\leq \beta(\gamma + 1)(\delta + 1) \| u_m \|^\gamma_{L^\infty} \| \partial_t u_m \|_{L^2}^2, \\
|\langle \partial_t f, \partial_t u_m \rangle | &\leq \| \partial_t f \|_{L^2} \| \partial_t u_m \|_{L^2} \leq \frac{1}{2} \| \partial_t f \|^2_{L^2} + \frac{1}{2} \| \partial_t u_m \|^2_{L^2}.
\end{align*}
\]

Combining (69)-(71), applying it in (68) and then integrating from 0 to \( t \), we get

\[
\begin{align*}
\| \partial_t u_m(t) \|^2_{L^2} + \nu \int_0^t \| \partial_x u_m(s) \|^2_{L^2} ds + \beta \gamma \int_0^t \| \partial_t u_m(s) \|^2_{L^2} ds
\end{align*}
\]

An application of Gronwall’s inequality in (72) yields

\[
\begin{align*}
\| \partial_t u_m(t) \|^2_{L^2} + \nu \int_0^t \| \partial_x u_m(s) \|^2_{L^2} ds + \beta \gamma \int_0^t \| \partial_t u_m(s) \|^2_{L^2} ds
\end{align*}
\]

\[
\leq \left( \| \partial_t u_m(0) \|^2_{L^2} + \int_0^T \| \partial_t f(t) \|^2_{L^2} dt \right) \times \exp \left( 2 \int_0^T \left[ \frac{\alpha^2}{\nu} \| u_m(t) \|^2_{L^\infty} + 2\beta(\gamma + 1)(\delta + 1) \| u_m(t) \|^\gamma_{L^\infty} + 1 \right] dt \right)
\]

\[
\leq C(\| u_0 \|_{H^2}, \| f \|_{H^1(0,T;L^2(\Omega))}, \alpha, \nu, \beta, \gamma, \delta, T),
\]

\[
\begin{align*}
(72)
\end{align*}
\]
for all $t \in [0, T]$, where we employed (48) and (64) in the final inequality. Note that $\|u_m(0)\|_{H^3} \leq C\|u_0\|_{H^3}$ (see page 363, [20]). Therefore, from (73), we infer that

$$\sup_{0 \leq t \leq T} \|\partial_t u_m(t)\|_{L^2}^2 + \nu \int_0^T \|\partial_{xx} u_m(s)\|_{L^2}^2 \, ds \leq C\left(\|u_0\|_{H^3}, \|f\|_{H^{j}(0,T;L^2(\Omega))}, \alpha, \beta, \gamma, \delta, \nu, T\right).$$

It should be noted that $\frac{1}{2} \frac{d}{dt} \|\partial_x u_m(t)\|_{L^2}^2 = -\langle \partial_t u_m(t), \partial_{xx} u_m(t) \rangle$. From (54), we have

$$\nu\|\partial_{xx} u_m(t)\|_{L^2}^2 + \beta\gamma\|\partial_x u_m(t)\|_{L^2}^2 + \beta(2\delta + 1)\|u_m(t)\|_{L^2}^2 \delta\|\partial_x u_m(t)\|_{L^2}^2 = \alpha\|u_m(t)\|_{L^2}^2 + \beta(1 + \gamma)(u_m(t)\|_{L^2}^2)\|\partial_x u_m(t)\|_{L^2}^2 + (f(t), \partial_{xx} u_m(t)) + (\partial_t u_m(t), \partial_{xx} u_m(t)).$$

(75)

We estimate the terms on the right hand side of the equality (75) using Hölder’s and Young’s inequalities as

$$|\alpha\|u_m\|_{L^\infty}\|\partial_x u_m\|_{L^2}\|\partial_{xx} u_m\|_{L^2}| \leq \alpha\|u_m\|_{L^\infty}\|\partial_x u_m\|_{L^2}\|\partial_{xx} u_m\|_{L^2},$$

$$\leq \frac{\nu}{8}\|\partial_{xx} u_m\|_{L^2}^2 + \frac{2\alpha^2}{\nu}\|u_m\|_{L^\infty}\|\partial_x u_m\|_{L^2}^2,$$

$$|\beta(1 + \gamma)(u_m(t)\|_{L^2}^2)\|\partial_x u_m(t)\|_{L^2}^2| \leq \beta(1 + \gamma)\|u_m\|_{L^\infty}\|u_m\|_{L^2}\|\partial_x u_m\|_{L^2}^2,$$

$$\leq \frac{\nu}{8}\|\partial_{xx} u_m\|_{L^2}^2 + \frac{2\beta^2(1 + \gamma)^2}{\nu}\|u_m\|_{L^\infty}\|\partial_x u_m\|_{L^2}^2,$$

$$|(f, \partial_{xx} u_m)| \leq \|f\|_{L^2}\|\partial_{xx} u_m\|_{L^2},$$

$$\leq \frac{\nu}{8}\|\partial_{xx} u_m\|_{L^2}^2 + \frac{2}{\nu}\|f\|_{L^2}^2,$$

Thus, from (75), it can be easily seen that

$$\nu\|\partial_{xx} u_m(t)\|_{L^2}^2 + 2\beta\gamma\|\partial_x u_m(t)\|_{L^2}^2 + 2\beta(2\delta + 1)\|u_m(t)\|_{L^2}^2 \delta\|\partial_x u_m(t)\|_{L^2}^2 \leq \frac{4\alpha^2}{\nu}\|u_m(t)\|_{L^\infty}\|\partial_x u_m(t)\|_{L^2}^2 + \frac{4\beta^2(1 + \gamma)^2}{\nu}\|u_m(t)\|_{L^\infty}\|\partial_x u_m(t)\|_{L^2}^2,$$

$$+ \frac{4}{\nu}\|f(t)\|_{L^2}^2 + \frac{4}{\nu}\|\partial_t u_m(t)\|_{L^2}^2,$$

(76)

for all $t \in [0, T]$. Using the energy estimates (64) and (74), we have

$$\sup_{0 \leq t \leq T} \left\|\partial_t u_m(t)\right\|_{L^2}^2 + \nu \int_0^T \|\partial_{xx} u_m(s)\|_{L^2}^2 \, ds \leq C\left(\|u_0\|_{H^3}, \|f\|_{H^{j}(0,T;L^2(\Omega))}, \alpha, \beta, \gamma, \delta, \nu, T\right).$$

(77)

Passing limit along a subsequence $m_j \rightarrow \infty$, we obtain the required bound for $u$. It is only left to show that $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$. Once again, we take $v \in H^1_0(\Omega)$, with $\|v\|_{H^1} \leq 1$, then $v = v^1 + v^2$, as in the previous case. Then, for a.e. $t \in [0, T]$, form (67), we have

$$\langle \partial_t u_m, v \rangle = \langle \partial_t u_m, v \rangle = -\nu\|\partial_{xx} u_m\|_{L^2}^2 - \alpha\|u_m\|_{L^2}^2 \delta\|\partial_x u_m\|_{L^2}^2 - \beta\|u_m\|_{L^2}^2 \delta\|\partial_x u_m\|_{L^2}^2 + \gamma\|\partial_t u_m\|_{L^2}^2 - (2\delta + 1)\|u_m\|_{L^2}^2 \delta\|\partial_x u_m\|_{L^2}^2,$$

(78)
Thus, for all $v \in H^1_0(\Omega)$, we find
\[
|\langle \partial_t u_m, v \rangle| \leq \nu \|\partial_x u_m\|_{L^2} \|v\|_{L^1} + \alpha \|u_m\|_{L^\infty} \|\partial_x u_m\|_{L^2} \|v\|_{L^1} \\
+ \beta \|u_m\|_{L^\infty} \|\partial_x u_m\|_{L^2} \|v\|_{L^1} \\
+ \beta \|u_m\|_{L^\infty} \|\partial_t u_m\|_{L^2} \|v\|_{L^1}.
\]
(79)
so that we get
\[
\|\partial_t u_m\|_{H^{-1}} \leq \left( \nu + \frac{\alpha}{\pi} \|u_m\|_{L^1} \right) \|\partial_x u_m\|_{L^2} \\
+ \frac{\alpha}{\pi} \|u_m\|_{L^2} \|\partial_x u_m\|_{L^2} \|\partial_t u_m\|_{L^2},
\]
(80)
Using the bounds given in (74) and (77), we obtain that $\partial_t u_m$ is bounded uniformly in $L^2(0,T;H^{-1}(\Omega))$ and hence we get
\[
\partial_t u_m \in L^2(0,T;H^{-1}(\Omega)).
\]
\[\square\]

**Remark 2.** Note that $C^k(\Omega) \subset H^s(\Omega)$, for $s > k + 1/2$. Thus, one can obtain a classical solution to the system (12), if we take $u_0 \in H^3(\Omega) \cap H^1_0(\Omega)$ and $f \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ (Theorem 6, page 386, [20]). Then, we have the following regularity:
\[
u \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^4(\Omega)), \quad \partial_t u \in L^2(0,T;H^2(\Omega)), \quad \partial_{tt} u \in L^2(0,T;L^2(\Omega)),
\]
(81)
with $\partial_t u \in C((0,T);H^1(\Omega))$ and $u \in C([0,T];H^3(\Omega))$ (Theorem 4, page 288, [20]), and hence we get
\[
u \in C^1(0,T;C(\Omega)) \cap C([0,T];C^2(\Omega)),
\]
for $u_0 \in C^2(\Omega)$.

If we take $f = 0$ and $u_0 \in H^3(\Omega) \cap H^1_0(\Omega)$, then one can obtain a classical solution to (12) and we have the following result on positivity of classical solutions. We follow the work [19] to get the next result (see Theorem 2 and Corollary 2, [19]).

**Theorem 3.4.** If $u(x,t)$ is a classical solution of (5) satisfying $0 \leq u(x,0) \leq 1$, for $x \in \overline{\Omega}$, then $0 \leq u(x,t) \leq 1$, for $x \in \overline{\Omega}$ and $t \geq 0$.

**Proof.** Let us first show that $u(x,t) \geq 0$, for $x \in \overline{\Omega}$ and $t \geq 0$. On the contrary, we assume that there exists $t_0$ and $x_0$ such that $u(t_0,x_0) < 0$. From this, and the boundary conditions, there exists $\eta, t_0$ and $x_0$ such that $0 < \eta < \gamma$ and $u(x_0,t_0) = -\eta$ with the property that $u(x_0,t_0)$ is a local minimum for $u(x,t_0)$. Note that
\[
I = u(x_0,t_0) = 0 - \eta(1 - (\eta)^{\delta} - \gamma - 1 - (\eta)^{\delta}),
\]
(82)
Also note that \( u(x, 0) \geq 0 \), for \( x \in \overline{\Omega} \), one can assume that \( t_\delta \) is the first time at which such a local minimum occurs. That is, for some \( \varepsilon > 0 \), we must have that \( u(x_n, t) \) is strictly decreasing for \( t \in (t_n - \varepsilon, t_n) \), which is a contraction to (82). Thus, we have \( u(x, t) \geq 0 \), for \( x \in \Omega \) and \( t > 0 \).

Let us now prove that \( u(x, t) \leq 1 \). Let \( 0 \leq u(x, 0) < \gamma^{1/\delta} \), for \( x \in \Omega \). Note that if \( u(x, t) \) satisfying \( 0 < u(x, t) < \gamma^{1/p} \) and (5), has an interior local maximum at \( x_\xi \in \Omega, t = t_\xi \), then

\[
\frac{\partial u(x_\xi, t_\xi)}{\partial t} = -\alpha u(x_\xi, t_\xi)^\delta \partial_x u(x_\xi, t_\xi) + \nu \partial_x u(x_\xi, t_\xi) \\
\leq -\alpha u(x_\xi, t_\xi)^\delta \times 0 + \nu \partial_x u(x_\xi, t_\xi)(1 - u(x_\xi, t_\xi)) (u(x_\xi, t_\xi)^\delta - \gamma) \leq 0.
\]

Hence, \( u(x, t) \) needs to be strictly decreasing at any such interior maximum point. Since \( 0 \leq u(x, 0) < \gamma^{1/\delta} \), for \( x \in \Omega \), and \( 0 = u(x, t) \), for \( x \in \partial\Omega \) and \( t > 0 \), then is follows that \( 0 \leq u(x, t) < \gamma^{1/\delta} \), for \( x \in \Omega \) and \( t > 0 \). Let us now take \( \gamma^{1/\delta} < u(x, 0) \leq 1 \), for \( x \in \Omega \). It should be noted that if \( u(x, t) \) satisfying \( \gamma^{1/p} < u(x, t) \leq 1 \) and (5), has an interior local maximum at \( x_\xi \in \Omega, t = t_\xi \), then

\[
\frac{\partial u(x_\xi, t_\xi)}{\partial t} = -\alpha u(x_\xi, t_\xi)^\delta \partial_x u(x_\xi, t_\xi) + \nu \partial_x u(x_\xi, t_\xi) \\
\leq -\alpha u(x_\xi, t_\xi)^\delta \times 0 + \nu \partial_x u(x_\xi, t_\xi)(1 - u(x_\xi, t_\xi)) (u(x_\xi, t_\xi)^\delta - \gamma) \leq 0.
\]

Once again we see that, \( u(x, t) \) needs to be strictly decreasing at any such interior maximum point. Since \( \gamma^{1/\delta} < u(x, 0) \leq 1 \), for \( x \in \Omega \), then it is immediate that \( \gamma^{1/\delta} < u(x, t) \leq 1 \), for \( x \in \Omega \) and \( t > 0 \). Combining the above two cases, we obtain the required result \( 0 \leq u(x, t) \leq 1 \), for \( x \in \Omega \) and \( t \geq 0 \).

**Remark 3.** We use the notation \( R = (0, 1) \times (0, T) \) for space-time rectangular domain. As in [6], one can consider the following generalized Burgers-Huxley equation:

\[
\begin{aligned}
\frac{\partial u(x, t)}{\partial t} + \alpha u(x, t)^\delta \frac{\partial u(x, t)}{\partial x} + \frac{\nu}{\partial x^2} &\quad = \beta u(x, t)(1 - u(x, t)^\delta)(u(x, t)^\delta - \gamma) \\
&\quad + f(x, t), \quad (x, t) \in Q,
\end{aligned}
\]

\[
\frac{\partial u(x, t)}{\partial t} \Big|_{x = \varphi_i(t)} = 0, \quad \text{for} \quad i = 1, 2, \quad t \in (0, T),
\]

\[
u u(x, 0) = u_0(x), \quad x \in [\varphi_1(0), \varphi_2(0)],
\]

in a space-time non-rectangular domain

\[
Q = \{(x, t) \in \mathbb{R}^2 : \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T\},
\]

where \( \varphi_1, \varphi_2 \in C^1(0, T) \) (see [10] also). We assume that \( \varphi_1(t) < \varphi_2(t) \), for all \( t \in (0, T) \). In order to establish the existence and uniqueness of the weak solution to the system (85), one has to impose the assumption

\[
|\varphi'(t)| \leq C, \quad \text{for all} \quad t \in [0, T],
\]
where $C$ is a positive constant and $\varphi(t) = \varphi_1(t) - \varphi_2(t)$, for all $t \in [0, T]$. Using the change of variables

$$(x, t) \mapsto (y, t) = \left( \frac{x - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}, t \right),$$

the domain $Q$ can be transformed into the rectangle $R = (0, 1) \times (0, T)$. As in Theorems 3.1, 3.2 and 3.3, one can obtain the existence, uniqueness and regularity of the following quasilinear parabolic problem:

$$
\begin{align*}
\frac{\partial u(x, t)}{\partial t} + \alpha(t)u(x, t)\delta \frac{\partial u(x, t)}{\partial x} - \nu(t)\frac{\partial^2 u(x, t)}{\partial x^2} + \eta(x, t)\frac{\partial u(x, t)}{\partial x} &= \beta u(x, t)(1 - u(x, t)^\delta)(u(x, t)^\delta - \gamma) + f(x, t), \ (x, t) \in \Omega \times (0, T), \\
u_0(t) &= u(1, t) = 0, \ t \in (0, T), \\
u(x, 0) &= u_0(x), \ x \in \Omega,
\end{align*}
$$

(88)

where the functions $\nu, \alpha$ depend only on $t$ and the function $\eta$ depends on $x$ and $t$. We assume that there exists positive constants $\{\nu_i\}_{i=1}^2$, $\{\alpha_i\}_{i=1}^2$ and $\eta_1$ such that

$$
\begin{align*}
\nu_1 \leq \nu(t) \leq \nu_2, \text{ for all } t \in (0, T), \\
\alpha_1 \leq \alpha(t) \leq \alpha_2, \text{ for all } t \in (0, T), \\
|\partial_x \eta(x, t)| \leq \eta_1 \text{ or } |\eta(x, t)| \leq \eta_1, \text{ for all } (x, t) \in \Omega \times (0, T).
\end{align*}
$$

(89)

Substituting $u(x, t) = v(y, t)$ and $f(x, t) = g(y, t)$ in (85), we find

$$
\begin{align*}
\frac{\partial v(y, t)}{\partial t} + \frac{\alpha}{\varphi(t)} v(y, t)\delta \frac{\partial v(y, t)}{\partial y} - \frac{\nu}{\varphi^2(t)} \frac{\partial^2 v(y, t)}{\partial y^2} + \eta(t, y)\frac{\partial v(t, y)}{\partial y} &= \beta v(y, t)(1 - v(y, t)^\delta)(v(y, t)^\delta - \gamma) + g(y, t), \ (y, t) \in \Omega \times (0, T), \\
v(0, t) &= v(1, t), \ t \in (0, T), \\
v(y, 0) &= v_0(y) = u_0(\varphi_1(0) + \varphi(0)y), \ y \in [0, 1],
\end{align*}
$$

(90)

where

$$
\eta(t, y) = -\frac{y \varphi'(t) + \varphi'(t)}{\varphi(t)}.
$$

Taking $\alpha(t) = \frac{\alpha}{\varphi(t)}$ and $\nu(t) = \frac{\nu}{\varphi(t)}$ in (90), the problem (90) can be reduced to (88) (where $u(x, t)$ needs to be replaced by $v(y, t)$). Note that the change of variables preserves the spaces and the hypotheses (89) are satisfied. Thus, the global solvability results of the system (88) easily implies the global solvability of the system (90).

4. Global attractors. In this section, we show the existence of a global attractor for the semigroup associated with the system (12) with $f$ independent of $t$ (autonomous case) and for large $\nu$. For $f(\cdot)$ independent of $t$ in $L^2(\Omega)$, using Theorem 3.1, we know that there exists a unique weak solution for the system (12) (for any $\beta, \nu, \alpha$, for $1 \leq \delta \leq 2$, and $\nu > \frac{(\alpha^2\nu)^2}{\beta}$, for $\delta > 2$) and the solution can be represented through a one parameter family of continuous semigroup. Thanks to Theorems 3.1 and 3.2 (see Remark 1 also), we can define a continuous semigroup $\{S(t)\}_{t \geq 0}$ in $L^2(\Omega)$ by

$$S(t)u_0 = u(t), \ t \geq 0,$$

(91)

where $u(\cdot)$ is the unique weak solution of the system (12) with $f(t) = f \in L^2(\Omega)$ and $u(0) = u_0 \in L^2(\Omega)$. 

4.1. A bounded absorbing set in $L^2(\Omega)$. Let us first show the existence of an absorbing ball in $L^2(\Omega)$.

**Proposition 1.** Let us assume that $\nu > \frac{\beta(1+\gamma^2)}{\pi^2}$. For any $f \in L^2(\Omega)$, let

$$g_0^2 := \frac{1}{\kappa \pi^2} \|f\|_{L^2}^2,$$

where $\kappa = \nu \pi^2 - \beta(1 + \gamma^2) > 0$. Then, for any $\rho_0 > g_0$ and $u_0 \in L^2(\Omega)$, there exists a time $t_{\rho_0}(\|u_0\|)$ such that

$$\|S(t)u_0\|_{L^2} \leq \rho_0, \text{ for all } t \geq t_{\rho_0}(\|u_0\|),$$

that is, the ball in $L^2(\Omega)$ of radius $\rho_0$ is absorbing. Furthermore, we have

$$\frac{\kappa}{\pi^2} \int_t^{t+r} \|\partial_x u(s)\|_{L^2}^2 ds + \beta \int_t^{t+r} \|u(s)\|_{L^2(\Omega)}^{2(\delta+1)} ds \leq \rho_0^2 + \frac{r}{\nu \pi^2} \|f\|_{L^2}^2,$$  \hspace{1cm} (93)

for any $r > 0$ and for all $t \geq t_{\rho_0}(\|u_0\|)$, and

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \|\partial_x u(t)\|_{L^2}^2 dt \leq \frac{1}{\nu \kappa} \|f\|_{L^2}^2,$$

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \|u(t)\|_{L^2(\Omega)}^{2(\delta+1)} dt \leq \frac{1}{\nu \beta \pi^2} \|f\|_{L^2}^2.$$  \hspace{1cm} (94)

**Proof.** If $f$ is independent of $t$, then from (17), we know that the weak solution $u(\cdot)$ to the system (12) satisfies:

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + \nu \pi^2 \|u(t)\|_{L^2}^2 + \beta \|u(s)\|_{L^2(\Omega)}^{2(\delta+1)} \leq \frac{1}{\nu} \|f\|_{L^2}^2 + \beta(1 + \gamma^2) \|u(t)\|_{L^2}^2,$$  \hspace{1cm} (95)

for a.e. $t \in [0, T]$. For $\nu > \frac{\beta(1+\gamma^2)}{\pi^2}$, using the variation of constants formula, we further have

$$\|u(t)\|_{L^2}^2 \leq e^{-\kappa t} \|u_0\|_{L^2}^2 + \frac{1}{\kappa \nu} \|f\|_{H^{-1}}^2.$$  \hspace{1cm} (96)

for all $t \geq 0$, where $\kappa = \nu \pi^2 - \beta(1 + \gamma^2) > 0$. Furthermore, we find

$$\limsup_{t \to \infty} \|u(t)\|_{L^2}^2 \leq \frac{1}{\kappa \nu} \|f\|_{H^{-1}}^2 \leq \frac{1}{\kappa \nu \pi^2} \|f\|_{L^2}^2.$$  \hspace{1cm} (97)

From the above relation it is clear that for any $\rho_0 > g_0$, there exists a time $t_0$, which depends on $\rho_0$ and $\|u_0\|_{L^2}$, such that

$$\|u(t)\|_{L^2}^2 \leq \rho_0^2, \text{ for all } t \geq t_{\rho_0}(\|u_0\|_{L^2}).$$  \hspace{1cm} (98)

From (95), we also have

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + \left(\nu - \frac{\beta(1+\gamma^2)}{\pi^2}\right) \|\partial_x u(t)\|_{L^2}^2 + \beta \|u(s)\|_{L^2(\Omega)}^{2(\delta+1)} \leq \frac{1}{\nu} \|f\|_{H^{-1}}^2,$$  \hspace{1cm} (99)

for a.e. $t \in [0, T]$. Integrating the above inequality from $t$ to $t + r$, we obtain

$$\frac{\kappa}{\pi^2} \int_t^{t+r} \|\partial_x u(s)\|_{L^2}^2 ds + \beta \int_t^{t+r} \|u(s)\|_{L^2(\Omega)}^{2(\delta+1)} ds \leq \|u(t)\|_{L^2}^2 + \frac{r}{\nu} \|f\|_{H^{-1}}^2,$$  \hspace{1cm} (100)

for all $r \geq 0$. The inequality (93) follows by taking $t \geq t_{\rho_0}(\|u_0\|_{L^2})$. The inequality (94) follows by performing a similar integration in (99).

The inequality (96) implies that the semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$, $t \geq 0$ associated with the weak solution to the problem (12) has an absorbing ball. \hfill \Box
Remark 4. For $1 \leq \delta \leq 2$, the condition $\nu > \frac{\beta(1 + \gamma^2)}{\pi \rho^2}$ is enough in Proposition 1, as the existence of unique weak solution to the system (12) is known for any $\beta, \nu$ and $\alpha$. But for $\delta > 2$, the uniqueness of weak solution is known only for $\nu > \frac{(2\delta^2 \alpha^2)}{\beta}$.

Thus, one has to take $\nu > \max \left\{ \frac{\beta(1 + \gamma^2)}{\pi \rho^2}, \frac{(2\delta^2 \alpha^2)}{\beta} \right\}$ in Proposition 1, for getting an absorbing ball in $L^2(\Omega)$.

4.2. A bounded absorbing set in $H^1_0(\Omega)$. Next, we show the existence of an absorbing ball in $H^1_0(\Omega)$.

Proposition 2. Given any $f \in L^2(\Omega)$ and $\rho_0 > \varrho_0$, let us define

$$\rho_1^2 := \left( \frac{2}{\nu} + \frac{(2\delta + 2)\beta}{\nu} \right) \|f\|_{L^2}^2$$

$$+ \left[ \frac{\pi^2}{\kappa} + \frac{2\beta(1 + \gamma)^2}{\nu} + \frac{1}{\beta} (1 + \beta(2\delta + 2)(1 + \gamma + \gamma^2)) \right] \left( \rho_0^2 + \frac{1}{\nu \pi^2} \|f\|_{L^2}^2 \right).$$

Then, for any $u_0 \in L^2(\Omega)$, we have

$$\|S(t)u_0\|_{H^1} \leq \rho_1, \quad \text{for all} \quad t \geq t_{\rho_0} \left( \|u_0\|_{L^2} \right) + 2,$$

where $t_{\rho_0}(\|u_0\|_{L^2})$ same as in Proposition 1, that is, the ball in $H^1_0(\Omega)$ of radius $\rho_1$ is absorbing. Moreover, we have

$$\frac{\nu}{2} \int_t^{t + r} \|\partial_{xx} u(s)\|_{L^2}^2 ds$$

$$\leq \rho_1^2 + \frac{2\nu}{\nu} \|f\|_{L^2}^2 + \frac{27C\alpha^2 r}{4\nu^2} \rho_0 \delta + \frac{2\beta(1 + \gamma)^2}{\nu} \left( \rho_0^2 + \frac{r}{\nu \pi^2} \|f\|_{L^2}^2 \right),$$

for all $t \geq t_{\rho_0} \left( \|u_0\|_{L^2} \right) + 2$ and

$$\limsup_{r \to \infty} \frac{1}{r} \int_t^{t + r} \|\partial_{xx} u(s)\|_{L^2}^2 ds \leq \frac{4}{\nu^2} \left( \|f\|_{L^2}^2 + \frac{27C\alpha^2 \rho_1 \delta + \frac{2\beta(1 + \gamma)^2}{\nu \pi^2} \|f\|_{L^2}^2 \right).$$

Proof. Let us first show the existence of an absorbing ball in $L^{2\delta+2}(\Omega)$. In order to establish this, we use the double integration trick (see [34]), which can be formalized as the “uniform Gronwall lemma” given in Lemma 1.1, Chapter III, [42]. Taking inner product with $|u(\cdot)|^{2\delta}u(\cdot)$ to the first equation in (12) and then calculating similarly as in (58), we find

$$\frac{1}{2\delta + 2} \frac{d}{dt} \|u(t)\|_{L^{2\delta+2}}^2 + \nu (2\delta + 1) \|u(t)\|^{2\delta} \partial_x u_m(t) \|_{L^2}^2 + \frac{\beta}{2} \|u(t)\|_{L^{4\delta+2}}^{4\delta+2}$$

$$\leq \beta (1 + \gamma + \gamma^2) \|u(t)\|_{L^{2\delta+2}}^{2\delta+2} + \frac{1}{\beta} \|f\|_{L^2}^2,$$

for $t \geq t_{\rho_0} \left( \|u_0\|_{L^2} \right) + 2$ and

$$\limsup_{r \to \infty} \frac{1}{r} \int_t^{t + r} \|\partial_{xx} u(s)\|_{L^2}^2 ds \leq \frac{4}{\nu^2} \left( \|f\|_{L^2}^2 + \frac{27C\alpha^2 \rho_1 \delta + \frac{2\beta(1 + \gamma)^2}{\nu \pi^2} \|f\|_{L^2}^2 \right).$$

Dropping the term $\nu (2\delta + 1) \|u(t)\|^{2\delta} \partial_x u_m(t) \|_{L^2}^2$ from (104), we get

$$\frac{d}{dt} \|u(t)\|_{L^{2\delta+2}}^2 + \frac{\beta}{2} \|u(t)\|_{L^{4\delta+2}}^{4\delta+2}$$

$$\leq \beta (2\delta + 2)(1 + \gamma + \gamma^2) \|u(t)\|_{L^{2\delta+2}}^{2\delta+2} + \frac{(2\delta + 2)}{\beta} \|f\|_{L^2}^2.$$
Integrating the above inequality from \( s \) to \( t + 1 \), with \( t \leq s < t + 1 \), we obtain
\[
\|u(t + 1)\|_{L_{2q+2}}^{2q+2} + \frac{\beta}{2} \int_{s}^{t+1} \|u(\tau)\|_{L_{4q+2}}^{4q+2}d\tau \\
\leq \|u(s)\|_{L_{2q+2}}^{2q+2} + \beta(2\delta + 2)(1 + \gamma + \gamma^2) \int_{s}^{t+1} \|u(\tau)\|_{L_{4q+2}}^{4q+2}d\tau + \frac{(2\delta + 2)}{\beta} \|f\|_{L_2}^2
\]
(105)
since \( 0 < t + 1 - s \leq 1 \). Integrating both sides of (105) with respect to \( s \) between \( t \) and \( t + 1 \), we find
\[
\|u(t + 1)\|_{L_{2q+2}}^{2q+2} + \frac{\beta}{2} \int_{t}^{t+1} (\tau - t)\|u(\tau)\|_{L_{4q+2}}^{4q+2}d\tau \\
\leq \|u(t)\|_{L_{2q+2}}^{2q+2} + \frac{\beta(2\delta + 2)(1 + \gamma + \gamma^2)}{\beta} \int_{t}^{t+1} (\tau - t)\|u(\tau)\|_{L_{4q+2}}^{4q+2}d\tau \\
\leq \frac{(2\delta + 2)}{\beta} \|f\|_{L_2}^2 + \frac{1}{\beta} (1 + \beta(2\delta + 2)(1 + \gamma + \gamma^2)) \int_{t}^{t+1} \|u(s)\|_{L_{4q+2}}^{4q+2}ds \\
\leq \frac{(2\delta + 2)}{\beta} \|f\|_{L_2}^2 + \frac{1}{\beta} \left( 1 + \beta(2\delta + 2)(1 + \gamma + \gamma^2) \right) \left( \rho_0^2 + \frac{1}{\nu \pi^2} \|f\|_{L_2}^2 \right),
\]
(106)
for all \( t \geq t_0 \) (\( \|u_0\|_{L_2} \)), where we used (93).

Our next aim is to show the existence of an absorbing ball in \( H_0^1(\Omega) \). Taking inner product with \( -\partial_{xx}u(\cdot) \) to the first equation in (12) and then calculating similarly as in (58), we obtain
\[
\frac{d}{dt} \|\partial_x u(t)\|_{L_2}^2 + \frac{\nu}{2} \|\partial_{xx} u(t)\|_{L_2}^2 + 2\beta \gamma \|\partial_x u(t)\|_{L_2}^2 + 2\beta(2\delta + 1) \|u(t)\|_{L_{4q+2}} \|\partial_x u(t)\|_{L_2}^2 \\
\leq \frac{2}{\nu} \|f\|_{L_2}^2 + \frac{27C\alpha^2}{2\nu^3} \|u(t)\|_{L_{4q+2}}^{4q+2} + \frac{2\beta^2(1 + \gamma)^2}{\nu} \|u(t)\|_{L_{4q+2}}^{2q+2}.
\]
(107)
Dropping the term \( \frac{\nu}{2} \|\partial_{xx} u(t)\|_{L_2}^2 + 2\beta \gamma \|\partial_x u(t)\|_{L_2}^2 + 2\beta(2\delta + 1) \|u(t)\|_{L_{4q+2}} \|\partial_x u(t)\|_{L_2}^2 \) from (107), we find
\[
\frac{d}{dt} \|\partial_x u(t)\|_{L_2}^2 \leq \frac{2}{\nu} \|f\|_{L_2}^2 + \frac{27C\alpha^2}{2\nu^3} \|u(t)\|_{L_{4q+2}}^{4q+2} + \frac{2\beta^2(1 + \gamma)^2}{\nu} \|u(t)\|_{L_{4q+2}}^{2q+2}.
\]
(108)
Integrating the inequality (108) from \( s \) and \( t + 1 \), with \( t \leq s < t + 1 \), we obtain
\[
\|\partial_x u(t + 1)\|_{L_2}^2 \leq \|\partial_x u(s)\|_{L_2}^2 + \frac{2}{\nu} \|f\|_{L_2}^2 + \frac{27C\alpha^2}{2\nu^3} \int_{s}^{t+1} \|u(\tau)\|_{L_{4q+2}}^{4q+2}d\tau \\
+ \frac{2\beta^2(1 + \gamma)^2}{\nu} \int_{s}^{t+1} \|u(\tau)\|_{L_{4q+2}}^{2q+2}d\tau,
\]
(109)
since \( 0 < t + 1 - s \leq 1 \). Let us now integrate both sides of (109) with respect to \( s \) between \( t \) and \( t + 1 \) to get
\[
\|\partial_x u(t + 1)\|_{L_2}^2 \\
\leq \frac{2}{\nu} \|f\|_{L_2}^2 + \int_{t}^{t+1} \|\partial_x u(s)\|_{L_2}^2 ds + \frac{27C\alpha^2}{2\nu^3} \int_{t}^{t+1} (\tau - t)\|u(\tau)\|_{L_{4q+2}}^{4q+2}d\tau \\
+ \frac{2\beta^2(1 + \gamma)^2}{\nu} \int_{t}^{t+1} (\tau - t)\|u(\tau)\|_{L_{4q+2}}^{2q+2}d\tau
\]
\[
\leq \frac{2}{\nu} \|f\|^2_{L^2} + \int_t^{t+1} \|\partial_x u(s)\|^2_{L^2} ds + \frac{27 C\alpha^2}{2\nu^3} \int_t^{t+1} (\tau-t) \|u(\tau)\|_{L^{2\delta+2}}^2 d\tau \\
+ \frac{2\beta^2(1+\gamma)^2}{\nu} \int_t^{t+1} \|u(\tau)\|_{L^{2\delta+2}}^2 d\tau \\
\leq \frac{2}{\nu} \|f\|^2_{L^2} + \left(\frac{\pi^2}{\kappa} + \frac{2\beta(1+\gamma)^2}{\nu}\right) \left(\rho_0^2 + \frac{1}{\nu \pi^2} \|f\|^2_{L^2}\right) \\
+ \frac{(2\delta + 2)}{\beta} \|f\|^2_{L^2} + \frac{1}{\nu} \left(1 + \beta(2\delta + 2)(1+\gamma + \gamma^2)\right) \left(\rho_0^2 + \frac{1}{\nu \pi^2} \|f\|^2_{L^2}\right),
\]
(110)

for all \( t \geq t_{\rho_0}(\|u_0\|_{L^2}) + 1 \), where we used (93) and (106). Thus (101) follows easily. Integrating (107) from \( t \) to \( t + r \), we further have

\[
\|\partial_x u(t + r)\|^2_{L^2} + \frac{\nu}{2} \int_t^{t+r} \|\partial_x u(s)\|^2_{L^2} ds + 2\beta \gamma \int_t^{t+r} \|\partial_x u(s)\|^2_{L^2} ds \\
+ 2\beta(2\delta + 1) \int_t^{t+r} \|u(s)\|^2_{L^2} ds \\
\leq \|\partial_x u(t)\|^2_{L^2} + \frac{2r}{\nu} \|f\|^2_{L^2} + \frac{27 C\alpha^2}{2\nu^3} \int_t^{t+r} \|u(s)\|_{L^{2\delta+2}}^2 ds \\
+ \frac{2\beta^2(1+\gamma)^2}{\nu} \int_t^{t+r} \|u(s)\|_{L^{2\delta+2}}^2 ds \\
\leq \rho_1^2 + \frac{2r}{\nu} \|f\|^2_{L^2} + \frac{27 C\alpha^2}{2\nu^3} \int_t^{t+r} \|u(s)\|_{L^{2\delta+2}}^2 ds \\
+ \frac{2\beta(1+\gamma)^2}{\nu} \left(\rho_0^2 + \frac{r}{\nu \pi^2} \|f\|^2_{L^2}\right),
\]
(111)

for all \( t \geq t_{\rho_0}(\|u_0\|_{L^2}) + 2 \). Using the Gagliardo-Nirenberg interpolation inequality, Hölder’s inequality, (92) and (101), we obtain

\[
\int_t^{t+r} \|u(s)\|_{L^{2\delta+2}}^2 ds \leq C \int_t^{t+r} \|\partial_x u(s)\|_{L^2}^{2\delta} \|u(s)\|_{L^2}^{2\delta+2} ds \\
\leq C \sup_{s \in [t, t+r]} \|\partial_x u(s)\|_{L^2}^{2\delta} \sup_{s \in [t, t+r]} \|u(s)\|_{L^2}^{2\delta+2} \leq C r \rho_1^{2\delta} \rho_0^{2\delta+2}.
\]
(112)

Thus, from (111), it is immediate that

\[
\|\partial_x u(t + r)\|^2_{L^2} + \frac{\nu}{2} \int_t^{t+r} \|\partial_x u(s)\|^2_{L^2} ds + 2\beta \gamma \int_t^{t+r} \|\partial_x u(s)\|^2_{L^2} ds \\
\leq \rho_1^2 + \frac{2r}{\nu} \|f\|^2_{L^2} + \frac{27 C\alpha^2}{2\nu^3} \rho_1^{2\delta} \rho_0^{2\delta+2} + \frac{2\beta(1+\gamma)^2}{\nu} \left(\rho_0^2 + \frac{r}{\nu \pi^2} \|f\|^2_{L^2}\right),
\]
(113)

for all \( t \geq t_{\rho_0}(\|u_0\|_{L^2}) + 1 \). Moreover, we find

\[
\limsup_{r \to \infty} \frac{1}{r} \int_t^{t+r} \|\partial_x u(s)\|^2_{L^2} ds \\
\leq \frac{2}{\nu} \left(\frac{2}{\nu} \|f\|^2_{L^2} + \frac{27 C\alpha^2}{2\nu^3} \rho_1^{2\delta} \rho_0^{2\delta+2} + \frac{2\beta(1+\gamma)^2}{\nu \pi^2} \|f\|^2_{L^2}\right),
\]
(114)

and the estimate (103) follows. \( \square \)
4.3. Global attractors. Let us now show that the semigroup associated with the system (12) possesses a global attractor.

**Theorem 4.1.** Let the assumptions of Proposition 1 and Remark 4 be satisfied. Let $f \in L^2(\Omega)$ be given. Then the semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ has an absorbing ball $B_1 = \{ v \in L^2(\Omega) : ||v||_{L^2} \leq \rho_0 \}$ and a global attractor $A \in L^2(\Omega)$. Moreover, the attractor $A$ is compact, connected and invariant. That is, $A$ is compact in $L^2(\Omega)$, strictly invariant with respect to $L^2(\Omega)$, and

$$\text{dist}_{L^2(\Omega)}(S(t)B, A) \to 0 \quad \text{as} \quad t \to +\infty,$$

for any bounded subset $B \subset L^2(\Omega)$ of initial data, where $\text{dist}_{L^2(\Omega)}(X, Y)$ is the Hausdorff semidistance between two sets $X$ and $Y$, that is,

$$\text{dist}_{L^2(\Omega)}(X, Y) = \sup_{x \in X} \inf_{y \in Y} ||x - y||_{L^2}.$$

Proof. From Proposition 2, it is clear that the ball $B_2 = \{ v \in H^1_0(\Omega) : ||v||_{H^1_0} \leq \rho_1 \}$ is an absorbing set in $H^1_0(\Omega)$ for the semigroup $S(t)$. Moreover, if $B$ is any bounded set of $L^2(\Omega)$, then $S(t)B \subset B_2$, for $t \geq t_{\rho_1} ||u_0||_{L^2} + 2$. This shows the existence of an absorbing set in $H^1_0(\Omega)$ and also that the operators $S(t)$ are uniformly compact (see Chapter I, [42]). Thus, all the assumptions of Theorem I.1.1, [42] are satisfied and we deduce from Theorem I.1.1, [42] that there exists a global attractor. \hfill \Box

5. The inviscid limit. In this section, we take $\delta = 1$ and discuss the inviscid limit of the equation (12) as $\beta \to 0$. Let $u(\cdot)$ be the unique weak solution of the system (12). We consider the following Brugers equation:

$$\begin{cases}
\frac{\partial v(x, t)}{\partial t} + \alpha v(x, t) \frac{\partial v(x, t)}{\partial x} - \nu \frac{\partial^2 v(x, t)}{\partial x^2} = f(x, t), & (x, t) \in \Omega \times (0, T), \\
v(0, t) = v(1, t) = 0, & t \in (0, T), \\
v(x, 0) = u_0(x), & x \in \Omega.
\end{cases}$$  

(115)

In the abstract form, the above system can be written as

$$\begin{cases}
\frac{\partial v(t)}{\partial t} + \alpha v(t) \frac{\partial v(t)}{\partial x} - \nu \frac{\partial^2 v(t)}{\partial x^2} = f(t), & \text{in } H^{-1}(\Omega), \\
v(0) = u_0 \in L^2(\Omega).
\end{cases}$$  

(116)

The existence and uniqueness of weak solution of the above system can be established by using a Faedo-Galerkin technique as in section 3 (see [6] also). For $u_0 \in L^2(\Omega)$ and $f \in L^2(0, T; H^{-1}(\Omega))$, the unique weak solution of the system (116) satisfies the energy inequality:

$$\sup_{0 \leq t \leq T} ||v(t)||_{L^2}^2 + \frac{\nu}{2} \int_0^T ||\partial_x v(t)||_{L^2}^2 dt \leq ||u_0||_{L^2}^2 + \frac{1}{\nu} \int_0^T ||f(t)||_{H^{-1}}^2 dt =: K_T. \quad (117)$$

Also $u(\cdot)$ has the regularity

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)),$$

with $u_t \in L^2(0, T; H^{-1}(\Omega))$ and $u \in C([0, T], L^2(\Omega))$.

**Proposition 3.** Let $u(\cdot)$ be the unique weak solution of the Brugers-Huxley equation (see (12) with $\delta = 1$). As $\beta \to 0$, the weak solution of the system (12) tends to the weak solution of the Brugers equation (116).
Proof. Let us define \( w = u - v \). Then \( w(\cdot) \) satisfies:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial w(t)}{\partial t} + \alpha \left( u(t) \frac{\partial w(t)}{\partial x} + w(t) \frac{\partial v(t)}{\partial x} \right) - \nu \frac{\partial^2 w(t)}{\partial x^2} \\
\quad = \beta u(t)(1 - u(t))(u(t) - \gamma), \quad \text{in} \ H^{-1}(\Omega), \\
\quad w(0) = 0,
\end{array} \right.
\end{aligned}
\]  

(118)

for a.e. \( t \in [0, T] \). Taking inner product with \( w(\cdot) \) to the first equation in (118) and then applying integration by parts, we find

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2_{L^2} + \nu \|\partial_x w(t)\|^2_{L^2} \\
= -\alpha(u(t)\partial_x w(t), w(t)) - \alpha(w(t)\partial_x v(t), w(t)) + \beta(u(t)(1 - u(t))(u(t) - \gamma), w(t)) \\
= \frac{\alpha}{2}(w(t)^2, \partial_x u(t)) - \alpha(w(t)\partial_x v(t), w(t)) + \beta(u(t)^3, w(t)) - \beta(\gamma + 1)(u(t)^2, w(t)) \\
+ \beta\gamma(u(t), w(t)) =: \sum_{j=1}^{5} J_j,
\end{aligned}
\]  

(119)

where \( J_j, j = 1, \ldots, 5 \) are the terms appearing in the right hand side of the equality (119). We estimate \( J_1 \) using Hölder’s and Young’s inequalities as

\[
\begin{aligned}
J_1 &\leq \frac{\alpha}{2} \|w\|_{L^\infty} \|w\|_{L^2} \|\partial_x u\|_{L^2} \leq \frac{\alpha C}{2} \|\partial_x w\|_{L^2} \|w\|_{L^2} \|\partial_x u\|_{L^2} \\
&\leq \frac{\nu}{8} \|\partial_x w\|^2_{L^2} + \frac{C_1}{\nu} \|\partial_x u\|^2_{L^2} \|w\|^2_{L^2}.
\end{aligned}
\]  

(120)

Similarly, we estimate \( J_j, j = 2, \ldots, 5 \) as

\[
\begin{aligned}
J_2 &\leq \frac{\nu}{8} \|\partial_x w\|^2_{L^2} + \frac{C_2}{\nu} \|\partial_x v\|^2_{L^2} \|w\|^2_{L^2},
\end{aligned}
\]  

(121)

\[
\begin{aligned}
J_3 &\leq \beta \|w\|_{L^\infty} \|u\|^3_{L^2} \leq C \beta \|\partial_x w\|_{L^2} \|\partial_x u\|^1_{L^2} \|u\|^3_{L^2} \\
&\leq \frac{\nu}{8} \|\partial_x w\|^2_{L^2} + \frac{C_3}{\nu} \|\partial_x u\|^2_{L^2} \|u\|^4_{L^2},
\end{aligned}
\]  

(122)

\[
\begin{aligned}
J_4 &\leq \beta(\gamma + 1) \|w\|_{L^\infty} \|u\|^2_{L^2} \leq C \beta(\gamma + 1) \|\partial_x w\|_{L^2} \|u\|^2_{L^2} \\
&\leq \frac{\nu}{8} \|\partial_x w\|^2_{L^2} + \frac{C_4}{\nu} \|\partial_x u\|^2_{L^2} \|u\|^2_{L^2},
\end{aligned}
\]  

(123)

\[
\begin{aligned}
J_5 &\leq \beta\gamma \|w\|_{L^2} \|u\|_{L^2} \|w\|_{L^2} \leq \frac{1}{2} \|w\|^2_{L^2} + \frac{\beta^2\gamma^2}{2} \|u\|^2_{L^2},
\end{aligned}
\]  

(124)

where in second step we employed Gagliardo-Nirenberg’s and Poincaré’s inequalities also. Combining (120)-(124), substituting it in (119) and then integrating it from 0 to \( t \), we obtain

\[
\begin{aligned}
\|w(t)\|^2_{L^2} + \nu \int_0^t \|\partial_x w(s)\|^2_{L^2} ds \\
\leq \|w(0)\|^2_{L^2} + C \beta^2 \int_0^t \left[ \left( \frac{\gamma^2}{2} + \frac{(\gamma + 1)^2}{\nu} \right) \|u(s)\|^2_{L^2} + \frac{1}{\nu \pi^2} \|\partial_x u(s)\|^2_{L^2} \|u(s)\|_{L^2} \right] ds \\
+ \frac{C_5}{\nu} \int_0^t (\|\partial_x u(s)\|^2_{L^2} + \|\partial_x v(s)\|^2_{L^2}) \|w(s)\|^2_{L^2} ds.
\end{aligned}
\]  

(125)
An application of Gronwall’s inequality in (125) gives
\[
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x w(s)\|_{L^2}^2 ds \\
\leq C \beta^2 \left( \frac{\gamma^2}{2} + \frac{\nu}{\nu \pi^2} \right) \int_0^t \|u(s)\|_{L^2}^2 ds + \frac{1}{\nu \pi^2} \sup_{s \in [0, t]} \|u(s)\|_{L^2}^2 \int_0^t \|\partial_x u(s)\|_{L^2}^2 ds \\
\times \exp \left( C \alpha^2 \nu \int_0^t (\|\partial_x u(s)\|_{L^2}^2 + \|\partial_x v(s)\|_{L^2}^2) ds \right),
\]  
(126)
since \(w(0) = 0\). A calculation similar to (19) provides
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 + \nu \int_0^T \|\partial_x u(s)\|_{L^2}^2 ds + \beta \int_0^T \|u(s)\|_{L^4}^4 ds \\
\leq \left( \|u_0\|_{L^2}^2 + \frac{1}{T} \int_0^T \|f(s)\|_{H^1}^2 ds \right) e^{\beta(1+\gamma)^2 T} = K_T e^{\beta(1+\gamma)^2 T}.
\]  
(127)
Applying (117) and (127) in (126), we obtain
\[
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x w(s)\|_{L^2}^2 ds \\
\leq C \beta^2 K_T e^{\beta(1+\gamma)^2 T} \left[ \left( \frac{\gamma^2}{2} + \frac{(\gamma + 1)^2}{\nu} \right) + \frac{1}{\nu \pi^2} K_T^2 e^{2\beta(1+\gamma)^2 T} \right] \\
\times \exp \left( C \alpha^2 K_T (1+e^{\beta(1+\gamma)^2 T}) \right),
\]  
(128)
for all \(t \in [0, T]\). Passing \(\beta \to 0\) in (128), one can easily obtain the required result.

**Remark 5.** For \(\delta = 1\) and \(\beta = 0\), one can obtain the global attractor for the autonomous viscous Burgers equation (116) in \(L^2(\Omega)\) (that is, \(f\) is independent of \(t\)). Firstly, one has to establish a bounded absorbing ball in \(L^2(\Omega)\) for the Burgers equation (116) by taking inner product with \(v(\cdot)\) in (116). It is easy to show the existence of bounded absorbing ball in \(H^1(\Omega)\) by taking inner product with \(\partial_x v(\cdot)\) in (116) and then applying the classical Sobolev and interpolation inequalities, and uniform Gronwall’s lemma (see [42]). It leads to the existence of a global attractor, which is a compact invariant set in \(L^2(\Omega)\) that attracts all bounded sets in \(L^2(\Omega)\) ([5, 24, 42], etc). A computer assisted proof of the existence of globally attracting fixed points of viscous Burgers equation with constant forcing is obtained in [12] (see [13] for non-autonomous forcing). The authors in [39] studied numerically the long-time dynamics the viscous forced Burgers equation. For \(\delta = 1\), if we denote the global attractors for the Burgers-Huxley equation as \(A_\beta\) and if \(A\) denote the global attarctor for Burgers’ equation, then one can show that
\[
\lim_{\beta \to 0} \text{dist}_{L^2(\Omega)} (A_\beta, A) = 0.
\]
For \(\delta = 1\), let us now discuss the inviscid limit of the equation (12) as \(\alpha \to 0\). We consider the following Huxley equation for \((x, t) \in \Omega \times (0, T)\):
\[
\begin{cases}
\frac{\partial z(x, t)}{\partial t} - \nu \frac{\partial^2 z(x, t)}{\partial x^2} = \beta z(x, t)(1 - z(x, t))(z(x, t) - \gamma) + f(x, t), \\
z(0, t) = z(1, t) = 0, \ t \in (0, T), \\
z(x, 0) = u_0(x), \ x \in \Omega.
\end{cases}
\]  
(129)
In the abstract form, the above system can be written as

\[
\begin{aligned}
\frac{\partial z(t)}{\partial t} - \nu \frac{\partial^2 z(t)}{\partial x^2} &= \beta z(t)(1 - z(t))(z(t) - \gamma) + f(t), \quad \text{in } H^{-1}(\Omega), \\
z(0) &= u_0, \quad \text{in } L^2(\Omega).
\end{aligned}
\] (130)

The existence and uniqueness of weak solutions \( z \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \) to the system (130) can be proved in a similar way as in Theorem 3.1. Moreover, \( z(\cdot) \) satisfies:

\[
\sup_{0 \leq t \leq T} \|z(t)\|^2_{L^2} + \nu \int_0^T \|\partial_x z(s)\|^2_{L^2} ds + \beta \int_0^T \|z(s)\|^2_{L^2} ds \\
\leq \left(\|u_0\|^2_{L^2} + \frac{1}{\nu} \int_0^T \|f(s)\|^2_{H^{-1}} ds\right) e^{\beta(1+\gamma)^2 T} = K T e^{\beta(1+\gamma)^2 T}.
\] (131)

Then, we have the following result:

**Proposition 4.** Let \( u(\cdot) \) be the unique weak solution of the Burgers-Huxley equation (see (12) with \( \delta = 1 \)). As \( \alpha \to 0 \), the weak solution of the Burgers-Huxley equation (12) tends to the weak solution of the Huxley equation (130).

**Proof.** Let us define \( w = u - z \). Then \( w(\cdot) \) satisfies:

\[
\begin{aligned}
\frac{\partial w(t)}{\partial t} - \nu \frac{\partial^2 w(t)}{\partial x^2} &= \beta u(t)(1 - u(t))(u(t) - \gamma) \\
&\quad + \beta z(t)(1 - z(t))(z(t) - \gamma) = \alpha u(t) \frac{\partial u(t)}{\partial x}, \quad \text{in } H^{-1}(\Omega), \\
w(0) &= 0,
\end{aligned}
\] (132)

for a.e. \( t \in [0,T] \). Taking inner product with \( w(\cdot) \) to the first equation in (118) and then applying integration by parts, we find

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2_{L^2} + \nu \|\partial_x w(t)\|^2_{L^2} \\
&= -\alpha(u(t)\partial_x u(t), w(t)) \\
&\quad + \beta(u(t)[(u(t) + z(t))(1 + \gamma) - (\gamma + u(t)^2 + u(t)z(t) + z(t)^2)], w(t)) \\
&= -\alpha(u(t)\partial_x u(t), w(t)) - \beta \|w(t)\|^2_{L^2} - \beta \|u(t)w(t)\|^2_{L^2} - \beta \|z(t)w(t)\|^2_{L^2} \\
&\quad + \beta(1 + \gamma)(w(t)(u(t) + z(t)), w(t)) - \beta(u(t)z(t)w(t), w(t)).
\end{aligned}
\] (133)

The above equality implies

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2_{L^2} + \nu \|\partial_x w(t)\|^2_{L^2} + \beta \|w(t)\|^2_{L^2} + \beta \|u(t)w(t)\|^2_{L^2} + \beta \|z(t)w(t)\|^2_{L^2} \\
&= -\alpha(u(t)\partial_x u(t), w(t)) + \beta(1 + \gamma)(w(t)(u(t) + z(t)), w(t)) - \beta(u(t)z(t)w(t), w(t)) \\
&=: \sum_{j=1}^4 K_j,
\end{aligned}
\] (134)
where \( K_j, j = 1, \ldots, 4 \) are the terms appearing in the right hand side of the equality (134). We estimate \( K_1, \ldots, K_4 \) using Hölder’s and Young’s inequalities as

\[
K_1 \leq \alpha \|u\|_{L^\infty} \|\partial_x u\|_{L^2} \|w\|_{L^2} \leq \frac{C \alpha}{2} \|\partial_x u\|_{L^2}^2 (1 + \|w\|_{L^2}^2),
\]

(135)

\[
K_2 \leq \beta (1 + \gamma) \|u\|_{L^\infty} \|w\|_{L^2}^2 \leq C \beta (1 + \gamma) \|u\|_{H^1} \|w\|_{L^2}^2,
\]

(136)

\[
K_3 \leq \beta (1 + \gamma) \|z\|_{L^\infty} \|w\|_{L^2}^2 \leq C \beta (1 + \gamma) \|z\|_{H^1} \|w\|_{L^2}^2,
\]

(137)

\[
K_4 \leq \beta \|u\|_{L^\infty} \|z\|_{L^\infty} \|w\|_{L^2}^2 \leq C \beta \|u\|_{H^1} \|z\|_{H^1} \|w\|_{L^2}^2.
\]

(138)

Combining (135)-(138), substituting it in (134) and then integrating it from 0 to \( t \), we obtain

\[
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x w(s)\|_{L^2}^2 ds + \beta \gamma \int_0^t \|w(s)\|_{L^2}^2 ds + \beta \int_0^t \|u(s)w(s)\|_{L^2}^2 ds
\]

\[
+ \beta \int_0^t \|z(s)w(s)\|_{L^2}^2 ds
\]

\[
\leq \|w(0)\|_{L^2}^2 + C \alpha \int_0^t \|u(s)\|_{H^1}^2 ds
\]

\[
+ \int_0^t \left( C \alpha + C \beta (1 + \gamma) (\|u(s)\|_{H^1}^2 + \|z(s)\|_{H^1}^2) \right) \|w(s)\|_{L^2}^2 ds.
\]

(139)

An application of Gronwall’s inequality in (125) gives

\[
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x w(s)\|_{L^2}^2 ds + \beta \gamma \int_0^t \|w(s)\|_{L^2}^2 ds + \beta \int_0^t \|u(s)w(s)\|_{L^2}^2 ds
\]

\[
+ \beta \int_0^t \|z(s)w(s)\|_{L^2}^2 ds
\]

\[
\leq C \alpha \int_0^t \|u(s)\|_{H^1}^2 ds \exp \left\{ \int_0^t \left( C \alpha + C \beta (1 + \gamma) (\|u(s)\|_{H^1}^2 + \|z(s)\|_{H^1}^2) \right) ds \right\},
\]

(140)

since \( w(0) = 0 \). Applying (131) and (127) in (126), we obtain

\[
\|w(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_x w(s)\|_{L^2}^2 ds + \beta \gamma \int_0^t \|w(s)\|_{L^2}^2 ds + \beta \int_0^t \|u(s)w(s)\|_{L^2}^2 ds
\]

\[
+ \beta \int_0^t \|z(s)w(s)\|_{L^2}^2 ds
\]

\[
\leq C \alpha K T e^{\beta (1 + \gamma)^2 T} \exp \left\{ C \alpha T + C \beta (1 + \gamma) K_T e^{\beta (1 + \gamma)^2 T} \right\},
\]

(141)

for all \( t \in [0, T] \). Passing \( \alpha \to 0 \) in (141), one can easily obtain the required result.

6. Exponential stability. In this section, we consider the Burgers-Huxley equation and discuss about its stationary solution as well as exponential stability results.

Let us consider the following stationary Burgers-Huxley equation:

\[
\begin{align*}
- \nu \frac{\partial^2 u_\infty}{\partial x^2} + \alpha u_\infty \frac{\partial u_\infty}{\partial x} - \beta u_\infty (1 - u_\infty) (u_\infty - \gamma) & = f, \\
 u_\infty (0) & = u_\infty (1) = 0.
\end{align*}
\]

(142)
Let equation (142).

Let us first discuss about the existence and uniqueness of weak solutions of the problem.

Our problem is to find a $u$ satisfying:

$$
u \partial_t u\big|_{\Omega} + \alpha B(u) - \beta c(u) = f,$$

where $B(u) = u \partial_x u$ and $c(u) = u(1-u)(u-\gamma)$. Given any $f \in H^{-1}(\Omega)$, our problem is to find a $u$ satisfying:

$$
u(\partial_t u_{\infty}, \partial_x v) + \alpha (B(u_{\infty}), v) - \beta (c(u_{\infty}), v) = \langle f, v \rangle, \quad \text{for all } v \in H^1_0(\Omega). \quad (144)$$

Let us first discuss about the existence and uniqueness of weak solutions of the equation (142).

**Theorem 6.1.** Let $f$ be given and

$$\nu > \frac{\beta}{\pi^2}(1 + \gamma^2). \quad (145)$$

Then the following holds true:

(i) For every $f \in H^{-1}(\Omega)$, there exists at least one solution of the system (142).

(ii) If $f \in L^2(\Omega)$, then all solutions belong to $D(A)$.

(iii) If

$$\nu > \frac{\beta(1 + \gamma + \gamma^2)}{\pi^2} + \frac{\alpha}{\sqrt{2\pi\nu\kappa}} \| f \|_{H^{-1}}, \quad (146)$$

where $\kappa = \left(\frac{1}{\frac{\gamma^2}{\pi^2} + 1}\right)^2$, then the solution of (144) is unique. Furthermore, if

$$\nu > \max\left\{\frac{2\beta(1 + \gamma + \gamma^2)}{\pi^2}, \frac{\alpha^2}{2\beta\nu}\right\}, \quad (147)$$

then also the uniqueness of weak solution holds.

**Proof.** (i) We show the existence of weak solution to the equation (142) by implementing a Faedo-Galerkin approximation method. Let the functions $w_k = w_k(x)$, $k = 1, 2, \ldots$, be smooth, the set $\{w_k(x)\}_{k=1}^{\infty}$ be an orthogonal basis of $H^1_0(\Omega)$ and orthonormal basis of $L^2(\Omega)$ (page 504, [16]). One can take $\{w_k(x)\}_{k=1}^{\infty}$ as the complete set of normalized eigenfunctions of the operator $-\partial_x^2$ in $H^1_0(\Omega)$. For a fixed positive integer $m$, we look for a function $u_m \in H^1_0(\Omega)$ of the form

$$u_m = \sum_{k=1}^{m} \xi_m^k w_k, \quad \xi_m^k \in \mathbb{R}, \quad (148)$$

and

$$\nu(\partial_x u_m, \partial_x w_k) + \alpha ((u_m)\partial_x u_m, w_k) - \beta (u_m(1 - (u_m))((u_m) - \gamma), w_k) = \langle f, w_k \rangle, \quad (149)$$

for $k = 1, \ldots, m$. The equation (149) is also equivalent to

$$\nu A u_m + \alpha P_m B(u_m) - \beta P_m c(u_m) = P_m f. \quad (150)$$

The equations (148)-(149) are system of nonlinear equations for $\xi_1^m, \ldots, \xi_m^m$ and the existence of solutions is proved in the following way. We use Lemma 1.4, Chapter 2, [40] to get the existence of solution to the system of equations (148)-(149). Let $W = \text{Span}\{w_1, \ldots, w_m\}$ and the scalar product on $W$ is the scalar product $\langle \cdot, \cdot \rangle = (\partial_x (\cdot), \partial_x (\cdot))$ induced by $H^1_0(\Omega)$ and $P = P_m$ is defined by

$$[P_m(u), v] = (\partial_x P_m(u), \partial_x v) = \nu(\partial_x u, \partial_x v) + \alpha b(u, u, v) - \beta (c(u), v) - \langle f, v \rangle, \quad (151)$$

One can write down the abstract formulation of the equation (142) as

$$\nu A u_{\infty} + \alpha B(u_{\infty}) - \beta c(u_{\infty}) = f, \quad (143)$$

where $B(u) = u \partial_x u$ and $c(u) = u(1-u)(u-\gamma)$. Given any $f \in H^{-1}(\Omega)$, our problem is to find a $u_{\infty}$ satisfying:

$$\nu(\partial_t u_{\infty}, \partial_x v) + \alpha (B(u_{\infty}), v) - \beta (c(u_{\infty}), v) = \langle f, v \rangle, \quad \text{for all } v \in H^1_0(\Omega). \quad (144)$$
for all \( u, v \in W \). The continuity of the mapping \( P_m : H^1_0(\Omega) \to H^1_0(\Omega) \) is easy to verify. In order to apply Lemma 1.4, Chapter 2,\(^\text{[40]}\), we need to show that
\[
[P_m(u), u] > 0, \quad \text{for } [u] = k > 0,
\]
where \([ \cdot ]\) denotes the norm on \( W \). In fact, it is the norm induced by \( H^1_0(\Omega) \). Let us now consider
\[
[P_m(u), u] = \nu \|\partial_x u\|^2_{L^2} + \nu \|u\|^4_{L^4} + \beta \|\partial_x u\|^2_{L^2} + \beta \|u\|^4_{L^4} - \beta (1 + \gamma) (u^2, u) - (f, u)
\]
\[
\geq \nu \|\partial_x u\|^2_{L^2} + \nu \|u\|^4_{L^4} + \beta \|\partial_x u\|^2_{L^2} + \beta \|u\|^4_{L^4} - \beta (1 + \gamma) \|u\|^3_{L^3} - \|f\|_{H^{-1}} \|\partial_x u\|_{L^2}
\]
\[
\geq \frac{\nu}{2} \|\partial_x u\|^2_{L^2} + \beta \|u\|^2_{L^2} + \frac{\beta}{2} \|u\|^4_{L^4} - \frac{\beta}{2} (1 + \gamma)^2 \|u\|^2_{L^2} - \frac{1}{2\nu} \|f\|^2_{H^{-1}}
\]
\[
\geq \left( \frac{\nu}{2} - \frac{\beta}{2\pi^2} (1 + \gamma^2) \right) \|u\|^2_{H^1} - \frac{1}{2\nu} \|f\|^2_{H^{-1}}.
\]
It follows that \([P_m(u), u] > 0\), for \( \|u\|_{H^1_0} = k \) and \( k \) is sufficiently large, more precisely \( k > \nu \sqrt{1 - \frac{1}{2\pi^2} (1 + \gamma^2)} \|f\|_{H^{-1}} \). Thus the hypotheses of Lemma 1.4, Chapter 2,\(^{[40]}\) are satisfied and a solution \( u_m \) of (149) exists.

Now, we show the uniform bounds for the solution \( u_m \) of (149). Multiplying (149) by \( \xi_m^k \) and then adding from \( k = 1, \ldots, m \), we find
\[
\nu \|\partial_x u_m\|^2_{L^2} + \beta \|u_m\|^4_{L^4} + \beta \|u_m\|^2_{L^2}
\]
\[
= \beta (1 + \gamma) (u_m^2, u_m) + (f, u_m)
\]
\[
\leq \beta (1 + \gamma) \|u_m\|^2_{L^4} \|u_m\|_{L^2} + \|f\|_{H^{-1}} \|u_m\|_{H^1_0}
\]
\[
\leq \frac{\beta}{2} \|u_m\|^2_{L^4} + \frac{\beta}{2} (1 + \gamma)^2 \|u_m\|^2_{L^2} + \frac{\nu}{2} \|u_m\|^2_{H^1_0} + \frac{1}{2\nu} \|f\|^2_{H^{-1}},
\]
where we used Hölder’s and Young’s inequalities. From (153), we deduce that
\[
\left( \frac{\nu}{2} - \frac{\beta}{2\pi^2} (1 + \gamma^2) \right) \|u_m\|^2_{H^1_0} + \frac{\nu}{2} \|u_m\|^4_{L^4} \leq \frac{1}{2\nu} \|f\|^2_{H^{-1}}.
\]
Using the condition given in (145), we have \( \|u_m\|_{H^1_0} \) is bounded uniformly and independent of \( m \). Since \( H^1_0(\Omega) \) is reflexive, using the Banach-Alaoglu theorem, we can extract a subsequence \( \{u_{m_k}\} \) of \( \{u_m\} \) such that
\[
u \begin{array}{c}
\text{w-}\to u_\infty, \quad \text{in} \ H^1_0(\Omega), \quad \text{as} \ k \to \infty.
\end{array}
\]
Since the embedding of \( H^1_0(\Omega) \subset L^2(\Omega) \) is compact, one can extract a subsequence \( \{u_{m_{k_j}}\} \) of \( \{u_{m_k}\} \) such that
\[
|u_{m_{k_j}}| \to u_\infty, \quad \text{in} \ L^2(\Omega), \quad \text{as} \ j \to \infty.
\]
Passing to limit in (149) along the subsequence \( \{m_{k_j}\} \), we find that \( u_\infty \) is a solution to (144) and \( u_\infty \) satisfies
\[
\|u_\infty\|_{H^1_0} \leq \frac{1}{\sqrt{2\nu\kappa}} \|f\|_{H^{-1}},
\]
where \( \kappa = \left( \frac{\nu}{2} - \frac{\beta}{2\pi^2} (1 + \gamma^2) \right) \).
(ii) For $f \in L^2(\Omega)$, clearly $Au_\infty, B(u_\infty), c(u_\infty) \in L^2(\Omega)$ and (143) is satisfied as an equality in $L^2(\Omega)$. Taking inner product with $Au_\infty$ in (143), we find

$$
\nu \|Au_\infty\|_{L^2}^2 = -\alpha (B(u_\infty), Au_\infty) + \beta (c(u_\infty), Au_\infty) + (f, Au_\infty)
$$

$$
\leq \alpha \|\partial_x u_\infty\|_{L^2}^2 + \beta \|u_\infty\|_{L^2} \|Au_\infty\|_{L^2} + \|f\|_{L^2} \|Au_\infty\|_{L^2}
$$

$$
\leq \frac{3\nu}{4} \|Au_\infty\|_{L^2}^2 + \frac{\alpha^2}{2\nu} \|u_\infty\|^2_{H^1} + \frac{\beta^2}{2\nu} \|f\|_{L^2}^2
$$

$$
+ \beta \gamma \|u_\infty\|_{L^\infty}^2 + \beta \|u_\infty\|_{L^\infty} + \|u_\infty\|_{H^\infty}^3.
$$

(158)

Thus, from (158), we deduce that

$$
\frac{\nu}{4} \|Au_\infty\|_{L^2}^2
$$

$$
\leq \frac{C\alpha^2}{2\nu} \|u_\infty\|_{H^1}^3 + \frac{1}{2\nu} \|f\|_{L^2}^2 + \frac{C\beta^2}{2\nu} \|u_\infty\|_{H^2}^2 + \gamma \|u_\infty\|_{H^2} + \|u_\infty\|_{H^3}^3 < \infty,
$$

since $u_\infty \in H^2(\Omega)$ satisfies (157) and hence $u_\infty \in D(A)$.

(iii) For uniqueness, we take $u_\infty$ and $v_\infty$ as two solutions of (144). Let us define $w_\infty := u_\infty - v_\infty$. Then $w_\infty$ satisfies:

$$
\nu (\partial_x u_\infty, \partial_x v) + \alpha (B(u_\infty) - B(v_\infty), v) - \beta (c(u_\infty) - c(v_\infty), v) = 0,
$$

(159)

for all $v \in H^1(\Omega)$. Taking $v = w_\infty$ in (159), we have

$$
\nu \|\partial_x w_\infty\|_{L^2}^2 = -\alpha (w_\infty \partial_x u_\infty, w_\infty) - \alpha (v_\infty \partial_x w_\infty, w_\infty) - \alpha (v_\infty \partial_x w_\infty, w_\infty) + \beta (c(u_\infty) - c(v_\infty), w_\infty).
$$

(160)

We estimate

$$
\alpha \|w_\infty \partial_x u_\infty, w_\infty\|_{L^2} - \alpha (v_\infty \partial_x w_\infty, w_\infty)
$$

$$
= -\alpha (w_\infty \partial_x u_\infty, w_\infty) - \alpha (w_\infty \partial_x v_\infty, w_\infty) - \alpha (v_\infty \partial_x w_\infty, w_\infty)
$$

$$
= \frac{\alpha}{2} (v_\infty \partial_x u_\infty, w_\infty^2) = -\frac{\alpha}{2} \|\partial_x v_\infty\|_{L^2} \|w_\infty\|_{L^4}^2
$$

$$
\leq \frac{\alpha}{\pi} \|v_\infty\|_{H^1} \|w_\infty\|_{L^2} \|w_\infty\|_{H^1} \leq \frac{\alpha}{\pi} \|v_\infty\|_{H^1} \|w_\infty\|_{H^1}^2,
$$

(161)

where we used Hölder’s inequality and the fact that $\|w_\infty\|_{L^4}^2 \leq 2 \|w_\infty\|_{L^2} \|w_\infty\|_{H^1}$. Similarly, we estimate $\beta (c(u_\infty) - c(v_\infty), u_\infty - v_\infty)$ as

$$
\beta (c(u_\infty) - c(v_\infty), u_\infty - v_\infty)
$$

$$
= \beta ((1 + \gamma) (u_\infty^2 - v_\infty^2)) - (u_\infty^3 - v_\infty^3), u_\infty - v_\infty)
$$

$$
= \beta (1 + \gamma) (u_\infty v_\infty - v_\infty u_\infty - v_\infty^2 - \beta \gamma \|u_\infty - v_\infty\|_{L^2}^2
$$

$$
- \beta (u_\infty^2 - u_\infty v_\infty + v_\infty^2 (u_\infty - v_\infty), u_\infty - v_\infty)
$$

$$
= \beta (1 + \gamma) ((u_\infty + v_\infty) u_\infty - v_\infty, u_\infty - v_\infty - \beta \gamma \|u_\infty - v_\infty\|_{L^2}^2
$$

$$
- \beta \|u_\infty (u_\infty - v_\infty)\|_{L^2}^2 - \beta \|v_\infty (u_\infty - v_\infty)\|_{L^2}^2 - \beta (u_\infty v_\infty (u_\infty - v_\infty), u_\infty - v_\infty)
$$

$$
\leq \beta (1 + \gamma) ((u_\infty u_\infty - v_\infty) \|u_\infty - v_\infty\|_{L^2} + \|v_\infty (u_\infty - v_\infty)\|_{L^2}) \|u_\infty - v_\infty\|_{L^2}
$$

$$
- \beta \gamma \|u_\infty - v_\infty\|_{L^2}^2 - \beta \|u_\infty (u_\infty - v_\infty)\|_{L^2}^2 - \beta \|v_\infty (u_\infty - v_\infty)\|_{L^2}^2
$$

$$
+ \frac{\beta}{2} (u_\infty v_\infty + v_\infty^2 (u_\infty - v_\infty), u_\infty - v_\infty)
$$

$$
\leq \beta (1 + \gamma) \|u_\infty - v_\infty\|_{L^2}^2,
$$

(162)
Combining (161) and (162), and substituting it in (160), we further have
\[
\nu \| \partial_x w_\infty \|^2_{L^2} \leq \frac{\alpha}{\pi} \| v_\infty \|_{H^1_0} \| w_\infty \|^2_{H^1_0} + \beta (1 + \gamma + \gamma^2) \| w_\infty \|^2_{L^2}. \tag{163}
\]
From (163), we get
\[
\left\{ \left( \nu - \frac{\beta (1 + \gamma + \gamma^2)}{\pi^2} \right) - \frac{\alpha}{\pi} \| v_\infty \|_{H^1_0} \right\} \| w_\infty \|^2_{H^1_0} \leq 0. \tag{164}
\]
Since \( v_\infty \) satisfies (157), from (164), we obtain
\[
\left\{ \left( \nu - \frac{\beta (1 + \gamma + \gamma^2)}{\pi^2} \right) - \frac{\alpha}{\sqrt{2\pi i\nu}} \| f \|_{H^{-1}} \right\} \| w_\infty \|^2_{H^1_0} \leq 0. \tag{165}
\]
If the condition (146) is satisfied, then we have \( u_\infty = v_\infty \).

For \( \nu > \max \left\{ \frac{2\beta (1 + \gamma + \gamma^2)}{\pi^2}, \frac{\alpha^2}{2\beta} \right\} \), the uniqueness of weak solution can be obtained in the following way also. From (161), we obtain
\[
-\alpha (w_\infty \partial_x u_\infty, w_\infty) - \alpha (v_\infty \partial_x w_\infty, w_\infty) = \frac{\alpha}{2} (v_\infty, \partial_x w_\infty^2) = \alpha (v_\infty, w_\infty \partial_x w_\infty) \\
\leq \alpha \| v_\infty w_\infty \|_{L^2} \| \partial_x w_\infty \|_{L^2} \\
\leq \nu \| \partial_x w_\infty \|^2_{L^2} + \frac{\alpha^2}{2\nu} \| v_\infty w_\infty \|^2_{L^2}. \tag{166}
\]
From (162), we further have
\[
\beta (c(u_\infty) - c(v_\infty), u_\infty - v_\infty) \leq -\beta \| v_\infty w_\infty \|^2_{L^2} + \beta (1 + \gamma + \gamma^2) \| w_\infty \|^2_{L^2}. \tag{167}
\]
Combining (166) and (167), substituting it in (160), we get
\[
\left( \nu - \frac{\beta (1 + \gamma + \gamma^2)}{\pi^2} \right) \| \partial_x w_\infty \|^2_{L^2} + \left( \beta - \frac{\alpha^2}{2\nu} \right) \| v_\infty w_\infty \|^2_{L^2} \leq 0, \tag{168}
\]
and hence the uniqueness follows.

6.1. Exponential stability. Let us now assume that \( f \) is independent of \( t \) in the following equation:
\[
\begin{cases}
\frac{du(t)}{dt} + \nu Au(t) = -\alpha B(u(t)) + \beta c(u(t)) + f, & \text{in } H^{-1}(\Omega), \\
uu(t) = u_0 \in L^2(\Omega).
\end{cases} \tag{169}
\]
for a.e. \( t \in [0, T] \) and discuss about the exponential stability results for the stationary solution \( u_\infty \). In (169), the operators \( B(\cdot) \) and \( c(\cdot) \) are defined as in section 2 with \( \delta = 1 \). Exponential stability results for the 2D Navier-Stokes equations have been established in [41] and for the 2D viscoelastic fluid flow equations, arising from the Oldroyd model of order one has been obtained in the work [30].

Definition 6.2. A weak solution \( u(t) \) of the system (169) converges to \( u_\infty \) is exponentially stable in \( L^2(\Omega) \), if there exist a positive number \( a > 0 \), such that
\[
\| u(t) - u_\infty \|_{L^2} \leq \| u_0 - u_\infty \|_{L^2} \exp^{-at}, \quad t \geq 0.
\]
In particular, if \( u_\infty \) is a stationary solution of system (169), then \( u_\infty \) is called exponentially stable in \( L^2(\Omega) \) provided that any weak solution to (169) converges to \( u_\infty \) at the same exponential rate \( a > 0 \).
Theorem 6.3. Let $u_\infty$ be the unique weak solution of the system (143). If $u(\cdot)$ is any weak solution of the system (169) with $u_0 \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$ arbitrary, then we have $u_\infty$ is exponentially stable in $L^2(\Omega)$ and

$$u(t) \to u_\infty \text{ in } L^2(\Omega) \text{ as } t \to \infty,$$

for either

$$\nu > \frac{2\beta(1 + \gamma + \gamma^2)}{\pi^2} + \frac{\alpha^2}{\nu^2 \pi^2} \|u_\infty\|_{H^1_0}^2,$$  \hspace{1cm} (171)

or

$$\nu > \max \left\{ \frac{2\beta(1 + \gamma + \gamma^2)}{\pi^2}, \frac{\alpha^2}{2\beta \nu} \right\}$$  \hspace{1cm} (172)

where $u_\infty$ satisfies (157).

Proof. Let us define $w = u - u_\infty$, so that $w$ satisfies the following:

$$\left\{ \begin{array}{ll}
\frac{dw(t)}{dt} + \nu Aw(t) &= -\alpha(B(u(t)) - B(u_\infty)) + \beta(c(u(t)) - c(u_\infty)), & \text{in } H^{-1}(\Omega),
\end{array} \right.$$  \hspace{1cm} (173)

for a.e. $t \in [0, T]$. Taking inner product with $w(\cdot)$ to the first equation in (173), we find

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \nu \|\partial_z w(t)\|_{L^2}^2$$

$$= -\alpha((B(u(t)) - B(u_\infty)), w(t)) + \beta((c(u(t)) - c(u_\infty)), w(t)).$$  \hspace{1cm} (174)

Calculations similar to (161) and (162) yield

$$-\alpha(B(u) - B(u_\infty)), w) \leq \frac{\alpha}{2} \|u_\infty\|_{H^1_0} \|w\|_{L^2}^2 \leq \frac{\nu}{2} \|w\|_{H_0^1}^2 + \frac{\alpha^2}{2\nu} \|u_\infty\|_{H^1_0}^2 \|w\|_{L^2}^2,$$  \hspace{1cm} (175)

$$\beta((c(u) - c(u_\infty)), w) \leq \beta(1 + \gamma + \gamma^2) \|w\|_{L^2}^2.$$  \hspace{1cm} (176)

Combining (175)-(176) and substituting it in (174), we obtain

$$\frac{d}{dt} \|w(t)\|_{L^2}^2 + 2 \left[ \frac{\nu \pi^2}{2} - \beta(1 + \gamma + \gamma^2) \right] - \frac{\alpha^2}{2\nu} \|u_\infty\|_{H^1_0}^2 \|w\|_{L^2}^2 \leq 0.$$  \hspace{1cm} (177)

For the condition given in (171), an application of the variation of constants formula in (177) yields

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 e^{-\tilde{\kappa} t},$$  \hspace{1cm} (178)

for all $t \in [0, T]$, where

$$\tilde{\kappa} = \left[ \frac{\nu \pi^2}{2} - \beta(1 + \gamma + \gamma^2) \right] - \frac{\alpha^2}{\nu} \|u_\infty\|_{H^1_0}^2,$$

and the exponential stability follows.

For the condition provided in (172), we derive the exponential stability in the following way. Calculations similar to (166) and (167) gives

$$-\alpha(B(u) - B(u_\infty)), w) \leq \frac{\nu}{2} \|\partial_z w\|_{L^2}^2 + \frac{\alpha^2}{2\nu} \|u_\infty w\|_{L^2}^2,$$  \hspace{1cm} (179)

$$\beta((c(u) - c(u_\infty)), w) \leq -\beta \|u_\infty w\|_{L^2}^2 + \beta(1 + \gamma + \gamma^2) \|w\|_{L^2}^2.$$  \hspace{1cm} (180)
Thus, we easily have
\[
\frac{d}{dt} \| w(t) \|_{L^2}^2 + 2 \left\{ \frac{\nu r^2}{2} - \beta(1 + \gamma + \gamma^2) \right\} \| w \|_{L^2}^2 + \left( \beta - \frac{\alpha^2}{2\nu} \right) \| u_\infty w \|_{L^2}^2 \leq 0.
\] (181)

Dropping the term \( \left( \beta - \frac{\alpha^2}{2\nu} \right) \| u_\infty w \|_{L^2}^2 \) and using the variation of constants formula, we finally obtain
\[
\| w(t) \|_{L^2}^2 \leq \| w_0 \|_{L^2}^2 e^{-\hat{\kappa} t},
\] (182)
where \( \hat{\kappa} = 2 \left\{ \frac{\nu r^2}{2} - \beta(1 + \gamma + \gamma^2) \right\} \), provided (172) is satisfied.

7. Semidiscrete Galerkin approximation and error estimates. In this section, we derive error estimates in the semidiscrete Galerkin approximation. Let \( S^h \) be a finite dimensional subspace of \( H^1_0(\Omega) \) associated with a small mesh parameter \( h \). We are seeking the numerical solution in the family \( \{ S^h \} \subset H^1_0(\Omega) \), \( 0 < h < 1 \) having the property (see [43])
\[
\inf_{\chi \in S^h} \left\{ \| u - \chi \|_{L^2}^2 + h \| \partial_x (u - \chi) \|_{L^2}^2 \right\} \leq C h^k \| u \|_{H^k},
\]
for all \( u \in H^r(\Omega) \cap H^1_0(\Omega), \) \( 1 \leq k \leq r \), where \( r \) is the order of accuracy of the family \( \{ S^h \} \). Note that for linear elements \( r = 2 \). Let us take \( S^h = \text{span} \{ \{ \varphi_j \} \}_{j=0}^n \subset H^1_0(\Omega) \), where \( \{ \varphi_j \} \) \( {N_0} \) is a standard basis for \( S^h \). One can consider a space of piecewise continuous Lagrange elements. The continuous time Galerkin approximation of the problem (12) is defined in the following way. Find \( u^h(\cdot,t) \in S^h \), such that for \( t \in (0,T) \):
\[
\left\{ \begin{array}{l}
\langle \partial_t u^h(t), \chi \rangle + \nu a(u^h(t), \chi) + ab(u^h(t), u^h(t), \chi) = \beta(c(u^h(t)), \chi) + \langle f(t), \chi \rangle, \\
\langle u^h(0), \chi \rangle = \langle u_0^h, \chi \rangle, \quad \text{for} \quad \chi \in S^h,
\end{array} \right.
\] (183)
where \( u_0^h \) approximates \( u_0 \) in \( S^h \). Let us define \( R^h \) as the elliptic or Ritz projection onto \( S^h \) (see [43]) such that it satisfies
\[
\langle \partial_x R^h v, \partial_x \chi \rangle = \langle \partial_x v, \partial_x \chi \rangle, \quad \text{for all} \quad \chi \in S^h \quad \text{for} \quad v \in H^1_0(\Omega).
\]
By setting \( \chi = R^h v \) in the above equality we obtain that the Ritz projection is stable, that is, \( \| \partial_x R^h v \|_{L_2} \leq \| \partial_x v \|_{L_2} \), for all \( v \in H^1_0(\Omega) \). Moreover, using Lemma 1.1, Chapter 1, [43], we have
\[
\| R^h v - v \|_{L^2}^2 + h \| \partial_x (R^h v - v) \|_{L^2} \leq C h^s \| v \|_{H^s},
\] (184)
for all \( v \in H^s(\Omega) \cap H^1_0(\Omega), 1 \leq s \leq r \). For all \( t \in [0,T] \), let us set \( W(t) = R^h u(t) \). Now, we shall derive some error estimates in this semidiscretization.

**Theorem 7.1.** Let \( S^h \) be a finite dimensional subspace of \( H^1_0(\Omega) \) with parameter \( h \). Assume that \( u(\cdot) \) satisfies (13) and has the following regularity:
\[
u \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega)), \quad \partial_t \nu \in L^2(0,T; H^2(\Omega)).
\]

Then the error in semidiscretization satisfies the following estimate:
\[
\| u^h - u \|_{L^\infty(0,T; L^2(\Omega))}^2 + \| u^h - u \|_{L^2(0,T; H^1_0(\Omega))}^2 + \| u^h - u \|_{L^{2s+2}(0,T; L^{2s+2}(\Omega))}^2 \leq C \{ \| u_0^h - u_0 \|_{L^2}^2 + h^2 \},
\]
(185)
where the constant $C$ is independent of $h$ and $C$ depends on $\|u_0\|_{H^2}, \nu, \alpha, \beta, \gamma, \delta, T$, $\|f\|_{L^2(0,T;H^1(\Omega))}$, $\|\partial_t f\|_{L^2(0,T;L^2(\Omega))}$, etc.

**Proof.** Using (13) and (183), we know that $(u^h - u)(\cdot)$ satisfies
\[
\langle \partial_t (u^h(t) - u(t)), \chi \rangle + \nu a(u^h(t) - u(t), \chi) \\
= -\alpha [b(u^h(t), u^h(t), \chi) - b(u(t), u(t), \chi)] + \beta [c(u^h(t), \chi) - c(u(t), \chi)],
\]
for all $\chi \in S^h$. Let us choose $\chi = u^h - W \in S^h$ in (186) to obtain
\[
\langle \partial_t (u^h(t) - u(t)), u^h(t) - W(t) \rangle + \nu a(u^h(t) - u(t), u^h(t) - W(t)) \\
= -\alpha [b(u^h(t), u^h(t), W(t)) - b(u(t), u(t), W(t))] \\
+ \beta [c(u^h(t), u^h(t) - W(t)) - c(u(t), u^h(t) - W(t))].
\]

Next, we write $u^h - W$ as $u^h - u + u - W$ in (187) to find
\[
\frac{1}{2} \frac{d}{dt} \|u^h(t) - u(t)\|_{L^2}^2 + \nu \|\partial_x (u^h(t) - u(t))\|_{L^2}^2 \\
= -\langle \partial_t (u^h(t) - u(t)), u(t) - W(t) \rangle - \langle \partial_x (u^h(t) - u(t)), \partial_x (u(t) - W(t)) \rangle \\
- \alpha [u^h(t) - u(t)] \partial_x u^h(t) - u(t)] \partial_x u(t) \\
- \alpha [u^h(t) - u(t)] \partial_x u^h(t) - u(t)] \partial_x u(t) \\
+ \beta [u^h(t)(1 - u^h(t)) - u(t)(1 - u(t))] \partial_x u^h(t) - u(t) \\
+ \beta [u^h(t)(1 - u^h(t)) - u(t)(1 - u(t))] \partial_x u^h(t) - u(t).
\]

Using calculations similar to (45) yields
\[
\frac{d}{dt} \|u^h(t) - u(t)\|_{L^2}^2 + \nu \|\partial_x (u^h(t) - u(t))\|_{L^2}^2 + \frac{\beta}{2} \|u^h(t)(u^h(t) - u(t))\|_{L^2}^2 \\
+ \frac{\beta}{2} \|u(t)(u^h(t) - u(t))\|_{L^2}^2 + 2\beta \gamma \|u^h(t) - u(t)\|_{L^2}^2 \\
\leq C(\beta, \alpha, \delta) \|u^h(t) - u(t)\|_{L^2}^2 \\
+ C(\nu, \alpha, \delta) \left( \|u^h(t)\|_{L_{x}^{2\delta+2}}^{2\delta+1} + \|u(t)\|_{L_{x}^{2\delta+2}}^{2\delta+1} \right) \|u^h(t) - u(t)\|_{L^2}^2 \\
- 2\partial_t (u^h(t) - u(t), u(t) - W(t)) + 2(u^h(t) - u(t), \partial_t u(t) - W(t)) \\
- 2(\partial_x (u^h(t) - u(t)), \partial_x (u(t) - W(t))) \\
- 2\alpha (u^h(t) - u(t)) \partial_x u^h(t) - u(t)] \partial_x u(t) \\
+ 2\beta (u^h(t)(1 - u^h(t)) - u(t)(1 - u(t))] \partial_x u^h(t) - u(t) \\
= C(\beta, \alpha, \delta) \|u^h(t) - u(t)\|_{L^2}^2 \\
+ C(\nu, \alpha, \delta) \left( \|u^h(t)\|_{L_{x}^{2\delta+2}}^{2\delta+1} + \|u(t)\|_{L_{x}^{2\delta+2}}^{2\delta+1} \right) \|u^h(t) - u(t)\|_{L^2}^2 \\
- 2\partial_t (u^h(t) - u(t), u(t) - W(t)) + \sum_{i=1}^{4} J_i,
\]

(189)
where the positive constants \( C(\beta, \alpha, \delta) = \beta 2^{2\delta}(1 + \gamma)^2(\delta + 1)^2 \) and \( C(\nu, \alpha, \delta) = C2^{\frac{16(\delta + 1)}{\delta + 2}} \alpha^{\frac{4\delta + 2}{4\delta + 1}} \left( \frac{3\delta + 2}{2(\delta + 1)} \right)^{\frac{3\delta + 2}{3\delta + 1}} \left( \frac{\delta + 2}{4(\delta + 1)} \right) \). We estimate \( J_1 \) and \( J_2 \) using the Cauchy-Schwarz inequality and Young’s inequality as

\[
|J_1| \leq 2\|u^h - u\|^2_{L^2} \|\partial_t (u-W)\|_{L^2} \leq \|u^h - u\|^2_{L^2} + \|\partial_t (u-W)\|^2_{L^2},
\]

\[
|J_2| \leq 2\|\partial_x (u^h - u)\|_{L^2} \|\partial_x (u-W)\|_{L^2} \leq \frac{\nu}{2} \|\partial_x (u^h - u)\|^2_{L^2} + \frac{2}{\nu} \|\partial_x (u-W)\|^2_{L^2}.
\]

Using an integration by parts, Taylor’s formula, Hölder’s and Young’s inequalities, we rewrite \( J_3 \) as

\[
J_3 = -\frac{2\alpha}{\delta + 1} (\partial_x u^h \delta^1 - \delta^{1+1}, u - W) = \frac{2\alpha}{\delta + 1} \left( u^{h\delta^1} - \delta^{1+1}, \partial_x (u-W) \right)
\]

\[
\leq 2\alpha (\|\theta u^h + (1 - \theta)u\|^{\delta^2}(u^h - u, \partial_x (u-W))
\]

\[
\leq 2\delta^2 \alpha \left( \|u^h\|_{L^2}^2 + \|u^{\delta^2} - u\|_{L^2}^2 \right) \|\partial_x (u-W)\|_{L^2}
\]

\[
\leq \frac{\beta}{8} \|u^h - u\|^2_{L^2} + \frac{\beta}{8} \|u^{\delta^2} - u\|^2_{L^2} + \frac{2\delta(1 + \gamma)^2}{\beta} \|\partial_x (u-W)\|^2_{L^2}.
\]

Let us rewrite \( J_4 \) as

\[
J_4 = 2\beta(1 + \gamma)(u^{h\delta^1} - \delta^{1+1}, u - W) - 2\beta\gamma(u^h - u, u - W)
\]

\[
- 2\beta(u^{h2\delta^1} - u^{2\delta^1}, u - W) := \sum_{i=5}^{7} J_i.
\]

We estimate \( J_5 \) using Taylor’s formula, Hölder’s and Young’s inequalities as

\[
J_5 = 2\beta(1 + \gamma)(\delta + 1)((\theta u^h + (1 - \theta)u)^\delta (u^h - u), u - W)
\]

\[
\leq 2\beta(1 + \gamma)(\delta + 1) \left( \|u^h\|_{L^2}^2 + \|u^{\delta^2} - u\|_{L^2}^2 \right) \|u^h - u\|_{L^2}
\]

\[
\leq \frac{\beta}{8} \|u^h - u\|^2_{L^2} + \frac{\beta}{8} \|u^{\delta^2} - u\|^2_{L^2} + 2\delta(1 + \gamma)^2(\delta + 1)^2 \|u - W\|^2_{L^2}.
\]

Using the Cauchy-Schwarz inequality and Young’s inequality, we estimate \( J_6 \) as

\[
J_6 \leq 2\beta\gamma \|u^h - u\|_{L^2} \|u - W\|_{L^2} \leq 2\beta\gamma \|u^h - u\|^2_{L^2} + \frac{\beta\gamma}{2} \|u - W\|^2_{L^2}.
\]

Making use of Taylor’s formula, Hölder’s and Young’s inequalities, we estimate \( J_7 \) as

\[
J_7 = -2(2\delta + 1) \beta(\|\theta u^h + (1 - \theta)u\|^{2\delta}(u^h - u), u - W)
\]

\[
\leq 2\delta(2\delta + 1) \beta \|\theta u^h\|_{L^2}^{2\delta} + \|u\|_{L^4}^{2\delta}(u^h - u\|_{L^4}^2 + \|u\|_{L^4}^{2\delta}) \|u^h - u\|_{L^2}.
\]
where we used the fact that $H^1_0(\Omega) \subset L^\infty(\Omega)$. Note also that
\[
\|u^h - u\|_{L^2}^{\delta+2} = \int_\Omega |u^h(x) - u(x)|^{\delta+2} |u^h(x) - u(x)|^2 \, dx \\
\leq 2^{\delta-1} \int_\Omega (|u^h(x)|^{\delta+2} + |u(x)|^{\delta}) |u^h(x) - u(x)|^2 \, dx \\
= 2^{\delta-1} \|u^h - u\|_{L^2}^2 + 2^{\delta-1} \|u^h - u\|_{L^2}^2.
\]
Combining (190)-(197) and then substituting it in (189), we deduce that
\[
\frac{d}{dt} \|u^h(t) - u(t)\|_{L^2}^2 + \nu \|\partial_x (u^h(t) - u(t))\|_{L^2}^2 + \frac{\beta}{2^{\delta+1}} \|u^h(t) - u(t)\|_{L^2}^{2\delta+2} \\
\leq -2\partial_t (u^h(t) - u(t), u(t) - W(t)) + \|\partial_t (u(t) - W(t))\|_{L^2}^2 \\
+ \left(1 + \frac{2}{\nu} + \frac{2^{2(\delta+1)}\alpha^2}{\beta}\right) \|\partial_x (u(t) - W(t))\|_{L^2}^2 \\
+ 2^{\delta+1} \beta(1 + \gamma)^2 (\|u^h(t)\|_{L^2}^4 + \|u(t)\|_{L^2}^4) \|u^h(t) - u(t)\|_{L^2}^2 \\
+ C(\nu, \alpha, \delta) \left(\|u^h(t)\|_{L^2}^{4\delta+4} + \|u(t)\|_{L^2}^{4\delta+4}\right) \|u^h(t) - u(t)\|_{L^2}^2.
\]
Using the above estimates in (199) and then applying Gronwall’s inequality, we get
\[
\sup_{t \in [0, T]} \| u^h(t) - u(t) \|_{L^2}^2 + \nu \int_0^T \| \partial_x(u^h(t) - u(t)) \|_{L^2}^2 dt \\
+ \frac{\beta}{2^{2\delta}} \int_0^T \| u^h(t) - u(t) \|_{L^{2\delta+2}}^{2\delta+2} dt \\
\leq 2 \left\{ 2\| u_0^h - u_0 \|_{L^2}^2 + \| u_0 - W(0) \|_{L^2}^2 \\
+ 2^{2(\delta+1)}(1 + \gamma)^2(\delta + 1)^2 \int_0^T \| u(t) - W(t) \|_{L^2}^2 dt + \int_0^T \| \partial_x(u(t) - W(t)) \|_{L^2}^2 dt \\
+ \left( 1 + \frac{2}{\nu} + \frac{2^{2(\delta+1)}\beta^2}{\nu} \right) \int_0^T \| \partial_x(u(t) - W(t)) \|_{L^2}^2 dt \right\} \\
\times e^{(1 + \frac{2^{2\delta+1}}{\nu} + C(\beta, \alpha, \delta)) T} \exp \left\{ C^2(\delta+1)(2\delta + 1)^2 \int_0^T \| u^h(t) \|_{L^{2\delta+2}}^{2\delta+2} + \| u(t) \|_{L^{2\delta+2}}^{2\delta+2} dt \right\}.
\]
But we know that \( u, u^h \in L^\infty(0, T; H^3_0(\Omega)) \), so that the term appearing in the exponential is bounded uniformly and is independent of \( h \) (see 58). Using (184), we obtain
\[
\| u_0 - W(0) \|_{L^2} = \| u_0 - R^h u_0 \|_{L^2} \leq Ch^2 \| u_0 \|_{H^2},
\]
\[
\int_0^T \| u(t) - W(t) \|_{L^2}^2 dt = \int_0^T \| u(t) - R^h u(t) \|_{L^2}^2 dt \leq Ch^4 \int_0^T \| u(t) \|_{H^2}^2 dt,
\]
\[
\int_0^T \| \partial_x(u(t) - W(t)) \|_{L^2}^2 dt = \int_0^T \| \partial_x(u(t) - R^h u(t)) \|_{L^2}^2 dt \leq Ch^4 \int_0^T \| u(t) \|_{H^2}^2 dt,
\]
\[
\int_0^T \| \partial_x(u(t) - W(t)) \|_{L^2}^2 dt = \int_0^T \| \partial_x(u(t) - R^h u(t)) \|_{L^2}^2 dt \leq Ch^4 \int_0^T \| \partial_x u(t) \|_{H^2}^2 dt.
\]
From (81), we infer that if \( u_0 \in H^3(\Omega) \cap H^1_0(\Omega), f \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)), \) then \( \partial_t u \in L^2(0, T; H^2(\Omega)) \). Thus, from (200), it is immediate that
\[
\sup_{t \in [0, T]} \| u^h(t) - u(t) \|_{L^2}^2 + \nu \int_0^T \| \partial_x(u^h(t) - u(t)) \|_{L^2}^2 dt \\
+ \frac{\beta}{2^{2\delta}} \int_0^T \| u^h(t) - u(t) \|_{L^{2\delta+2}}^{2\delta+2} dt \\
\leq C(\| u_0 \|_{H^4}, \nu, \alpha, \beta, \gamma, \delta, T, \| f \|_{L^2(0, T; L^2(\Omega)))} \left\{ \| u_0^h - u_0 \|_{L^2}^2 + h^4 \| u_0 \|_{H^2}^2 \\
+ (h^2 + h^4) \int_0^T \| u(t) \|_{H^2}^2 dt + h^4 \int_0^T \| \partial_t u(t) \|_{L^2}^2 dt \right\} \\
\leq C(\| u_0 \|_{H^4}, \nu, \alpha, \beta, \gamma, \delta, T, \| f \|_{L^2(0, T; H^2(\Omega))}, \| \partial_t f \|_{L^2(0, T; L^2(\Omega)))} \left\{ \| u_0^h - u_0 \|_{L^2}^2 + h^2 \right\},
\]
and hence the estimate (185) follows.
we use the default sparse direct solver (Intel Core i5 processor with 16GB (RAM) memory). To solve the algebraic system, we omit it here. For non-linearity, the Picard fixed point iteration method is used.

Numerical results. In this section, we present the numerical study of the problem (5) with $\Omega = (0,1)$ and $T = 1$. To discretize the problem (5), we use the standard finite element method ($P_1$ polynomials) in space and the backward Euler’s method in time. For non-linearity, the Picard fixed point iteration method is used. All results are computed running MATLAB on a MacBook Pro laptop (2.3GHz Intel Core i5 processor with 16GB (RAM) memory). To solve the algebraic system, we use the default sparse direct solver (\texttt{dlsymm}) given in MATLAB.

In this section, we present two numerical results based on the following exact solutions:

- **Case I**: $u(x,t) = t^2(x-x^2)$ (Polynomial based solution).
- **Case II**: $u(x,t) = \frac{1}{8} \exp(t) \sin^2(\pi x)$ (Exponential function and trigonometric function based solution).

In both examples, we choose the values of parameters as follows: $\delta = 1$, $\alpha = 1$, $\beta = 1$ and $\gamma = 0.5$. We substitute the exact solution and the values of parameters in the system (5) to compute the right hand side function $f(\cdot,\cdot)$ and the initial data $u_0(\cdot)$. In Fig. 1 and Fig 3, graphs of the exact solutions and the numerical solutions for **Case I** and **Case II** at the final time $t = 1$ are presented for the different values of the mesh sizes $h = 1/2, 1/4, 1/8, 1/16$ with the fixed time step size $k = 1/10$. It is clear from the Fig 1 and Fig 3 that the numerical solution becomes more accurate when the spatial mesh size ($h$) decreases. It ensures that the numerical solutions for **Case I** and **Case II** converge to the exact solutions. For the second possibility, we fix the spatial mesh size ($h$) and reduce the time step size ($k$). The second possibility is discussed in Fig 2 and Fig 4. Similarly, we see that the numerical solutions for **Case I** and **Case II** in the second possibility approach the exact solutions. Hence it confirms that the proposed algorithm converges numerically.

**Possible future extensions.** In this work, we have discussed the global solvability results of the generalized Burgers-Huxley equations (5) in one dimensional bounded domains only. In future, we will try to establish the global solvability of results for the generalized Burgers-Huxley equations in higher dimensions also (cf. [19]).
specific parameter regimes and other generalizations of the Burgers equation (cf. [1, 14, 17], etc) in the context of Burgers-Huxley equations. In section 8, we have used a standard approach to solve the system (5) numerically. In future, our plan is to develop a-priori analysis and a-posteriori analysis for the proposed approach.

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Figure 2. Plots of solution (Case I) for the fixed spatial mesh size $h = 1/32$ and the different values of time step size (a) $k = 1/2$; (b) $k = 1/6$; (c) $k = 1/10$ at $t = 1$.
Figure 3. Plots of solution (Case II) for the fixed time step size $k = 1/100$ and the different values of the spatial mesh size (a) $h = 1/2$; (b) $h = 1/4$; (c) $h = 1/8$; (d) $h = 1/16$ at $t = 1$.

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Figure 4. Plots of solution (Case II) for the fixed spatial mesh size $h = 1/32$ and the different values of time step size (a) $k = 1/2$; (b) $k = 1/6$; (c) $k = 1/10$ at $t = 1$.

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