The Bloch vectors formalism for a finite-dimensional quantum system

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Abstract

In the present article, we consistently develop the main issues of the Bloch vectors formalism for an arbitrary finite-dimensional quantum system. In the frame of this formalism, qudit states and their evolution in time, qudit observables and their expectations, entanglement and nonlocality, etc. are expressed in terms of the Bloch vectors – the vectors in the Euclidean space $\mathbb{R}^{d^2-1}$, arising under decompositions of observables and states in different operator bases. Within this formalism, we specify for all $d \geq 2$ the set of Bloch vectors of traceless qudit observables and describe its properties; also, find for the sets of the Bloch vectors of qudit states, pure and mixed, the new compact expressions in terms of the operator norms that explicitly reveal the general properties of these sets and have the unified form for all $d \geq 2$. For the sets of the Bloch vectors of qudit states under the generalized Gell-Mann representation, these general properties cannot be analytically extracted from the known equivalent specifications of these sets via the system of algebraic equations. We derive the general equations describing the time evolution of the Bloch vector of a qudit state if a qudit system is isolated and if it is open and find for both cases the main properties of the Bloch vector evolution in time. For a pure bipartite state of a dimension $d_1 \times d_2$, we quantify its entanglement via the characteristics of the Bloch vectors for its reduced states. The introduced general formalism is important both for the theoretical analysis of quantum system properties and for quantum applications, in particular, for optimal quantum control, since, for systems where states are described by vectors in the Euclidean space, the methods of optimal control, analytical and numerical, are well developed.

1 Introduction

For qubit states and qubit observables, the formalism of Bloch vectors (coherence vectors) is well developed \cite{1, 2, 3, 4} and is widely used in many quantum information fields, for example, in quantum computation \cite{5}. This is not, however, the case for a qudit system of an arbitrary dimension $d \geq 2$. 
For \( d \geq 3 \), in the literature, mostly the properties of the Bloch vectors of qudit states under the generalized Gell-Mann representation \([5, 6, 7, 8, 9, 10, 11]\) and the problems of their visualization \([12, 13, 14, 15]\) have been analyzed. It is, however, important to develop a general formalism, where, for a finite-dimensional quantum system of an arbitrary dimension \( d \geq 2 \), not only its states but also its observables, its evolution in time, the entanglement and nonlocality, etc. would be described in terms of the Bloch vectors – the vectors in the Euclidean space \( \mathbb{R}^{d^2 - 1} \), arising under decompositions of observables and states in different operator bases. The new results in this direction have been recently presented in \([16, 17, 18]\). The development of this general formalism is important both for the theoretical analysis of quantum system properties and for quantum applications, in particular, for optimal quantum control, since, for systems, which states are described by vectors in the Euclidean space, the methods of optimal control, analytical and numerical, are well known.

In the present article, we consistently formalize (Sections 2, 3) and classify the main properties of Bloch-like representations for linear operators on a finite-dimensional complex Hilbert space. This allows us: (a) to specify (Section 4) the geometry properties of the set of Bloch vectors for all traceless qudit observables; (b) to find (Section 5) for the sets of the Bloch vectors of all qudit states, pure and mixed, the new compact expressions in terms of the operator norms, which explicitly reveal the general geometry properties of these sets and have the unified form for all \( d \geq 2 \); (c) to derive (Sections 6) the images in the Euclidean space of the Liouville–von Neumann equation and the Lindblad master equation and to find for the Bloch vector of a qudit state the main properties of its time evolution if a qudit system is isolated and if it is open; (d) to quantify (Section 7) the entanglement of a pure bipartite state in terms of the Bloch vectors for its reduced states.

2 Operator bases

Let \( \mathcal{H}_d \) be a complex Hilbert space of a finite dimension \( d \geq 2 \) and \( \mathcal{L}_d \) denote the vector space of all linear operators \( X \) on \( \mathcal{H}_d \) equipped with the scalar product

\[
\langle X_i, X_j \rangle_{\mathcal{L}_d} := \text{tr} \left( X_i^\dagger X_j \right).
\]

Denote by

\[
\mathfrak{B}_d := \{ \mathbb{I}_d, \ U^{(k)}_d \in \mathcal{L}_d, \ k = 1, \ldots, (d^2 - 1) \},
\]

\[
U^{(k)}_d = \left( U^{(k)}_d \right)^\dagger \neq 0, \quad \text{tr} \left( U^{(k)}_d \right) = 0, \quad \text{tr} \left( U^{(k)}_d U^{(m)}_d \right) = 2\delta_{km},
\]

a basis of \( \mathcal{L}_d \) consisting of the identity operator \( \mathbb{I}_d \) on \( \mathcal{H}_d \) and a tuple

\[
\Upsilon_d := \left( \Upsilon^{(1)}_d, \ldots, \Upsilon^{(d^2 - 1)}_d \right)
\]
of mutually orthogonal traceless Hermitian operators in $L_d$. Examples of operator bases
\{I_d, \Upsilon_d\} where elements are non Hermitian were introduced in \cite{11}.

For every qudit observable $W \in L_d$, $W = W^\dagger$, the decomposition in a basis $\mathfrak{B}_\Upsilon_d$ is
given by

$$W = \text{tr} (W) \frac{I_d}{d} + p_{\Upsilon_d} \cdot \Upsilon_d, \quad p_{\Upsilon_d} := \sum_{j=1}^{d^2-1} p_{\Upsilon_d}^{(j)} \Upsilon_d^{(j)},$$

(4)

and has the form of the representation via vector $p_{\Upsilon_d} \in \mathbb{R}^{d^2-1}$, satisfying the relation

$$\text{tr} (W^2) = \frac{1}{d} (\text{tr} W)^2 + 2 \|p_{\Upsilon_d}\|_2^2. \quad (5)$$

This implies that, for an observable $W$, the norm of the vector $p_{\Upsilon_d}$ in decomposition (4)
does not depend on which operator basis $\mathfrak{B}_\Upsilon_d$ of type (2) is used in this decomposition:

$$\|p_{\Upsilon_d}\|_2^2 = \|p_{\Upsilon_d}'\|_2^2, \quad \forall \Upsilon_d, \Upsilon_d'.$$

(6)

Notation 1 For the vector in $\mathbb{R}^{d^2-1}$ with components $\text{tr}(\Upsilon_d W)$, $j = 1, ..., (d^2 - 1)$, we
further use notation $\text{tr}(\Upsilon_d W)$, for short.

The most known decomposition via a basis of type (2) is the generalized Gell-Mann
representation \cite{6, 7, 9, 10, 15, 16, 17} specified in \cite{5} by the tuple

$$\Lambda_d := (\Lambda_d^{(1)}, ..., \Lambda_d^{(d^2-1)})$$

(7)

of the generalized Gell-Mann operators $\Lambda_d^{(k)}$ on $\mathbb{C}^d$ which are the higher-dimensional
extensions of the Pauli operators $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ on $\mathbb{C}^2$ and the Gell-Mann operators
on $\mathbb{C}^3$. For the product of the generalized Gell-Mann operators, the decomposition in
the basis $\mathfrak{B}_{\Lambda_d}$ is given by

$$\Lambda_d^{(k)} \Lambda_d^{(m)} = 2\delta_{km} \frac{I_d}{d} + \sum_l \left( g_{kml}^{(\Lambda_d)} + if_{kml}^{(\Lambda_d)} \right) \Lambda_d^{(l)}, \quad \forall k, m, \quad (8)$$

and implies

$$[\Lambda_d^{(k)}, \Lambda_d^{(m)}] = 2i \sum_l f_{kml}^{(\Lambda_d)} \Lambda_d^{(l)}, \quad (9)$$

$$\Lambda_d^{(k)} \circ \Lambda_d^{(m)} := \Lambda_d^{(k)} \circ \Lambda_d^{(m)} + \Lambda_d^{(m)} \circ \Lambda_d^{(k)} = 4\delta_{km} \frac{I_d}{d} + 2 \sum_l g_{kml}^{(\Lambda_d)} \Lambda_d^{(l)},$$

where the real constants

$$g_{kml}^{(\Lambda_d)} = \frac{1}{4} \text{tr} \left\{ \left( \Lambda_d^{(k)} \circ \Lambda_d^{(m)} \right) \Lambda_d^{(l)} \right\}, \quad f_{kml}^{(\Lambda_d)} = \frac{1}{4i} \text{tr} \left\{ [\Lambda_d^{(k)}, \Lambda_d^{(m)}] \Lambda_d^{(l)} \right\}, \quad (10)$$
are symmetric and antisymmetric, respectively, under the permutation of indices and constitute the structure constants of group SU(d).

For \( d = 2 \) and the tuple \( \Lambda_2 = \sigma = (\sigma_1, \sigma_2, \sigma_3) \) of the Pauli qubit operators, all symmetric constants \( g^{(\sigma)}_{kml} = 0 \) while the antisymmetric constants have the form \( f^{(\sigma)}_{kml} = \varepsilon_{kml} \) where \( \varepsilon_{kml} := (\epsilon_k, \epsilon_m, \epsilon_l) \) are the components of the Levi-Civita symbol, defined via the mixed product of the corresponding elements of the standard basis of \( \mathbb{R}^3 \).

Except for the tuple \( \Lambda_d \) of the generalized Gell-Mann operators on \( \mathbb{C}^d, d \geq 2, \) possible tuples of operators specified in (2), for example, include: (i) for \( \mathcal{H}_2 = \mathbb{C}^2, \) the operator tuple \( (\sigma_+, \sigma_-, \sigma_3) \) where \( \sigma \pm = \frac{\sigma_1 \pm \sigma_2}{\sqrt{2}} \); (ii) for \( \mathcal{H}_{d_1 \times d_2} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}, d_1, d_2 \geq 2, \) the tuple \( \Gamma_{\Lambda_{d_1} \otimes \Lambda_{d_2}} \) of operators

\[
\Lambda^{(j)}_{d_1} \otimes \frac{\mathbb{I}_{d_2}}{\sqrt{d_2}}, \frac{\mathbb{I}_{d_1}}{\sqrt{d_1}} \otimes \Lambda^{(k)}_{d_2}, \quad \frac{1}{\sqrt{2}} \Lambda^{(j)}_{d_1} \otimes \Lambda^{(k)}_{d_2}, \quad j = 1, \ldots, d_1, \; k = 1, \ldots, d_2, \tag{11}
\]

for which the renormalized version of decomposition (4), namely:

\[
W = \text{tr}(W) \frac{\mathbb{I}_{d_1} \otimes \mathbb{I}_{d_2}}{d_1 d_2} + \sqrt{\frac{d_1 - 1}{2d_1 d_2}} (r_1 \cdot \Lambda_{d_1}) \otimes \mathbb{I}_{d_2} + \sqrt{\frac{d_2 - 1}{2d_1 d_2}} \mathbb{I}_{d_1} \otimes (r_2 \cdot \Lambda_{d_2}) \tag{12}
\]

+ \[
\frac{(d_1 - 1)(d_2 - 1)}{4d_1 d_2} \sum_{i,j} T_{ij} \Lambda^{(i)}_{d_1} \otimes \Lambda^{(j)}_{d_2},
\]

constitutes a generalization to higher dimensions of the Pauli representation in the two-qubit case.

For each tuple \( \Upsilon_d \neq \Lambda_d \) of mutually orthogonal traceless Hermitian operators on \( \mathbb{C}^d, \) all its elements \( \Upsilon_{d}^{(k)} \) admit the generalized Gell-Mann representation

\[
\Upsilon_{d}^{(k)} = v_{\Lambda_{d}}^{(k)} \cdot \Lambda_{d}, \quad v_{\Lambda_{d}}^{(k)} = \frac{1}{2} \text{tr}(\Lambda_{d} \Upsilon_{d}^{(k)}) \in \mathbb{R}^{d^2 - 1}, \tag{13}
\]

where \( \left\{ v_{\Lambda_{d}}^{(k)} \in \mathbb{R}^{d^2 - 1}, k = 1, \ldots, (d^2 - 1) \right\} \) is an orthonormal basis of \( \mathbb{R}^{d^2 - 1}, \) different from its standard basis. Similarly to (5) the decomposition in basis \( \mathfrak{B}_{\Upsilon_{d}} \) of the product \( \Upsilon_{d}^{(k)} \Upsilon_{d}^{(m)} \) reads

\[
\Upsilon_{d}^{(k)} \Upsilon_{d}^{(m)} = \frac{2}{d} \delta_{km} \mathbb{I}_{d} + \sum_{l} \left( g_{kml}^{(\Upsilon_{d})} + i f_{kml}^{(\Upsilon_{d})} \right) \Upsilon_{d}^{(l)}, \quad \forall k, m, \tag{14}
\]

\[
g_{kml}^{(\Upsilon_{d})} = \frac{1}{4} \text{tr} \left\{ \left( \Upsilon_{d}^{(k)} \circ \Upsilon_{d}^{(m)} \right) \Upsilon_{d}^{(l)} \right\}, \quad f_{kml}^{(\Upsilon_{d})} = \frac{1}{4} \text{tr} \left\{ \left[ \Upsilon_{d}^{(k)}, \Upsilon_{d}^{(m)} \right] \Upsilon_{d}^{(l)} \right\},
\]

where the real constants \( g_{kml}^{(\Upsilon_{d})} \cdot f_{kml}^{(\Upsilon_{d})} \) are symmetric and antisymmetric with respect to the permutation of indices and are expressed via the symmetric and antisymmetric structure constants \( g_{j_1 j_2 j_3}^{(\Lambda_{d})} \) and \( f_{j_1 j_2 j_3}^{(\Lambda_{d})} \) of group SU(d), given in (6), via the relations

\[
g_{kml}^{(\Upsilon_{d})} = \sum_{j_1, j_2, j_3} \left( v_{\Lambda_{d}}^{(k)} \right)_{j_1} \left( v_{\Lambda_{d}}^{(m)} \right)_{j_2} \left( v_{\Lambda_{d}}^{(l)} \right)_{j_3} g_{j_1 j_2 j_3}^{(\Lambda_{d})}, \tag{15}
\]

\[
f_{kml}^{(\Upsilon_{d})} = \sum_{j_1, j_2, j_3} \left( v_{\Lambda_{d}}^{(k)} \right)_{j_1} \left( v_{\Lambda_{d}}^{(m)} \right)_{j_2} \left( v_{\Lambda_{d}}^{(l)} \right)_{j_3} f_{j_1 j_2 j_3}^{(\Lambda_{d})},
\]

4
For representation (4) of qudit observables \(W\) and \(\tilde{W}\) on \(\mathcal{H}_d\), specified for operator tuples \(\Upsilon_d\) and \(\Upsilon'_d\) in (2):

\[
W = \text{tr} \left( W \right) \mathbb{I}_d + p_{\Upsilon_d} \cdot \Upsilon_d = \text{tr} \left( W \right) \mathbb{I}_d + p_{\Upsilon'_d} \cdot \Upsilon'_d,
\]

\[
\tilde{W} = \text{tr} \left( \tilde{W} \right) \mathbb{I}_d + \tilde{p}_{\Upsilon_d} \cdot \Upsilon_d = \text{tr} \left( \tilde{W} \right) \mathbb{I}_d + \tilde{p}_{\Upsilon'_d} \cdot \Upsilon'_d,
\]

relation (4) implies

\[
\|p_{\Upsilon_d}\|_{\mathbb{R}^{d^2-1}}^2 = \|p_{\Upsilon'_d}\|_{\mathbb{R}^{d^2-1}}^2, \quad \|\tilde{p}_{\Upsilon_d}\|_{\mathbb{R}^{d^2-1}}^2 = \|\tilde{p}_{\Upsilon'_d}\|_{\mathbb{R}^{d^2-1}}^2.
\]

Moreover, since, similarly to (13),

\[
\Upsilon_d^{(k)} = v_{\Upsilon_d}^{(k)} \cdot \Upsilon_d, \quad v_{\Upsilon_d}^{(k)} = \frac{1}{2} \text{tr} \left( \Upsilon_d' \Upsilon_d^{(k)} \right) \in \mathbb{R}^{d^2-1},
\]

where \(\{v_{\Upsilon_d}^{(k)} \in \mathbb{R}^{d^2-1}, \ k = 1, ..., (d^2 - 1)\}\) is an orthonormal basis of \(\mathbb{R}^{d^2-1}\), from (18) it follows

\[
p_{\Upsilon_d}^{(j)} = \sum_{k=1}^{d^2-1} \left( v_{\Upsilon_d}^{(k)} \right)_j p_{\Upsilon_d}^{(k)}, \quad \tilde{p}_{\Upsilon_d}^{(j)} = \sum_{k=1}^{d^2-1} \left( v_{\Upsilon_d}^{(k)} \right)_j \tilde{p}_{\Upsilon_d}^{(k)},
\]

where \([T_{jk}] := [\left( v_{\Upsilon_d}^{(k)} \right)_j]\) is an orthogonal matrix.

Denote by \(\mathcal{D}_\omega \subset \mathcal{L}_d\) the set of all qudit observables on \(\mathcal{H}_d\) with a fixed value \(\omega = \text{tr}(W) \in \mathbb{R}\) of trace. Since representation (4) is a decomposition via a basis \(\mathcal{B}_{\Upsilon_d}\), and, for all observables \(W \in \mathcal{D}_\omega\), the decomposition coefficient at element \(\mathbb{I}_d \in \mathcal{B}_{\Upsilon_d}\) is fixed, the mapping

\[
W \mapsto p_{\Upsilon_d} = \frac{1}{2} \text{tr} \left( \Upsilon_d W \right) \in \mathbb{R}^{d^2-1}, \quad W \in \mathcal{D}_\omega,
\]

due to (4) is injective and, for all \(d \geq 2\) and any tuple \(\Upsilon_d\) of operators satisfying relations in (2), establishes the one-to-one correspondence

\[
\mathcal{D}_\omega \leftrightarrow \mathcal{I}_{\mathcal{D}_\omega}^{\Upsilon_d}
\]

between set \(\mathcal{D}_\omega\) and its image \(\mathcal{I}_{\mathcal{D}_\omega}^{\Upsilon_d} \subset \mathbb{R}^{d^2-1}\) under the injective mapping (20).

Relations (16)–(21) imply.

**Proposition 1** Let \(\mathcal{D}_\omega \subset \mathcal{L}_d\) be the set of qudit observables with a fixed value \(\omega = \text{tr}(W)\) of trace. Under representations (4) specified for arbitrary tuples \(\Upsilon_d \neq \Upsilon'_d\) of operators, satisfying relations in (3), the images \(\mathcal{I}_{\mathcal{D}_\omega}^{\Upsilon_d} \subset \mathbb{R}^{d^2-1}\) and \(\mathcal{I}_{\mathcal{D}_\omega}^{\Upsilon'_d} \subset \mathbb{R}^{d^2-1}\) of set \(\mathcal{D}_\omega\) are isometrically isomorphic.
3 Bloch vectors

Let $\mathcal{X}_d \subset \mathcal{L}_d$ be the set of all traceless qudit observables $X$ on $\mathcal{H}_d$ with eigenvalues in $[-1, 1]$ and $\mathcal{S}_d \subset \mathcal{L}_d$ be the set of all qudit states (density operators) $\rho_d$ on $\mathcal{H}_d$, that is, positive Hermitian operators with the unit trace:

$$\rho_d = \rho_d^\dagger, \quad \rho_d \geq 0, \quad \text{tr} (\rho_d) = 1. \quad (22)$$

For qudit observables $X \in \mathcal{X}_d$, representation (4) reduces to

$$X = x \Upsilon_d \cdot \Upsilon_d, \quad x \Upsilon_d = 1 \frac{\text{tr} (\Upsilon_d X)}{2} \in \mathbb{R}^{d^2 - 1}. \quad (23)$$

Replacing in (23) $x \Upsilon_d \to \sqrt{\frac{d}{2}} n \Upsilon_d$, we rewrite this representation in the form [16]

$$X = \sqrt{\frac{d}{2}} (n \Upsilon_d \cdot \Upsilon_d), \quad n \Upsilon_d = \sqrt{\frac{1}{2d}} \text{tr} (\Upsilon_d X) \in \mathbb{R}^{d^2 - 1}, \quad (24)$$

which implies

$$\text{tr} (X^2) = d \|n \Upsilon_d\|_2^2. \quad (25)$$

For qudit states $\rho_d \in \mathcal{S}_d$, representation (4) takes the form

$$\rho_d = \frac{1}{d} I_d + p \Upsilon_d \cdot \Upsilon_d, \quad p \Upsilon_d = \frac{1}{2} \text{tr} (\Upsilon_d \rho_d) \in \mathbb{R}^{d^2 - 1}, \quad (26)$$

and the renormalization $p \Upsilon_d \to \sqrt{\frac{d-1}{2d}} r \Upsilon_d$ leads to the representation

$$\rho_d = \frac{1}{d} I_d + \sqrt{\frac{d-1}{2d}} (r \Upsilon_d \cdot \Upsilon_d),$$

$$r \Upsilon_d = \sqrt{\frac{d}{2(d-1)}} \text{tr} (\rho_d \Upsilon_d) \in \mathbb{R}^{d^2 - 1}, \quad (27)$$

for which

$$\text{tr} (\rho_d^2) = \frac{1}{d} + \frac{d-1}{d} \|r \Upsilon_d\|_2^2. \quad (28)$$

For the qubit case ($d = 2$) and the operator basis $\{I_2, \sigma\}$ comprised of the Pauli operators on $\mathbb{C}^2$, representations (24) and (27) reduce to the well-known Bloch representations for qubit states and traceless qubit observables [1] [2] [3]:

$$\rho_2 = \frac{I_2 + p_\sigma \cdot \sigma}{2}, \quad p_\sigma = \text{tr} (\rho_2 \sigma) \in \mathbb{R}^3,$$

$$X = n_\sigma \cdot \sigma, \quad n_\sigma = \frac{1}{2} \text{tr} (\sigma X) \in \mathbb{R}^3. \quad (29)$$

For a unit vector $\|n\|_{\mathbb{R}^3} = 1$, the traceless qubit observable $n \cdot \sigma := \sigma_n$ has eigenvalue $\pm 1$ and is interpreted as a projection $\sigma_n$ of a qubit spin along a direction $n \in \mathbb{R}^3$. 

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Notation 2 In the Bloch representation (29), vector $p_\rho \in \mathbb{R}^3$ is called the Bloch vector (coherence vector) for a qubit state. For an arbitrary $d \geq 2$ and an arbitrary operator tuple $\Upsilon_d$, for definiteness, we also further refer to vectors $n_{\Upsilon_d^k}, r_{\Upsilon_d} \in \mathbb{R}^{d^2-1}$ in representations (24) and (27) as (generalized) Bloch vectors and, if it is clear from a context, we omit, for short, subscript $\Upsilon_d$ in their notation.

Due to this terminology, in Proposition 1 the image $\mathcal{Y}_{\Upsilon_d}(X_d)$ of the set $X_d$ under representation (24) constitutes the set of Bloch vectors for all qudit observables $X \in X_d$ under this representation while the image $\mathcal{Y}_{S_d}(\Upsilon_d)$ of the set $S_d$ under representation (27) – the set of the Bloch vectors for all qudit states $\rho_d \in S_d$.

Since $\text{tr} (X^2) \leq d$, $\forall X \in X_d$, and $\text{tr} (\rho_d^2) \leq 1$, $\forall \rho_d \in S_d$, relations (25), (28) imply.

Proposition 2 (Necessary conditions) Let $\Upsilon_d$ be a tuple of qudit operators satisfying relations in (3). For each traceless qudit observable $X$ with eigenvalues in $[-1, 1]$, the Bloch vector $n_{\Upsilon_d} \in \mathbb{R}^{d^2-1}$, $d \geq 2$, in representation (24) is necessarily

$$\|n\|_{\mathbb{R}^{d^2-1}} \leq 1,$$

(30)

where the equality holds only for the Bloch vectors corresponding by (24) to traceless qudit observables with all its eigenvalues equal to $\pm 1$. For any qudit state $\rho_d$, the Bloch vector $r_{\Upsilon_d} \in \mathbb{R}^{d^2-1}$ in representation (27) is necessarily

$$\|r\|_{\mathbb{R}^{d^2-1}} \leq 1,$$

(31)

where the equality holds only for the Bloch vectors, corresponding by (27) to pure qudit states.

The necessary conditions (30) and (31) mean that, for all $d \geq 2$, set $\mathcal{Y}_{\Upsilon_d}(X_d)$ of the Bloch vectors for all observables $X \in X_d$ under representation (24) and set $\mathcal{Y}_{S_d}(\Upsilon_d)$ of the Bloch vectors for all qudit states under representation (27) constitute subsets of the unit ball:

$$\mathcal{Y}_{\Upsilon_d}(X_d) \subseteq \{ n \in \mathbb{R}^{d^2-1} \mid \|n\|_{\mathbb{R}^{d^2-1}} \leq 1 \},$$

(32)

$$\mathcal{Y}_{S_d}(\Upsilon_d) \subseteq \{ r \in \mathbb{R}^{d^2-1} \mid \|r\|_{\mathbb{R}^{d^2-1}} \leq 1 \}.$$

Let $r, \tilde{r} \in \mathcal{Y}_{S_d}(\Upsilon_d)$ be the Bloch vectors of qudit states $\rho_d, \tilde{\rho}_d \in S_d$ under representation (27). Taking into account that $\text{tr} (\Upsilon_d^{(k)} \Upsilon_d^{(m)}) = 2 \delta_{km}$, we have $\text{tr} (\rho_d \tilde{\rho}_d) = \frac{1}{d} + \frac{d-1}{d} (r \cdot \tilde{r}) \geq 0$. Therefore, $(r \cdot \tilde{r}) \geq -\frac{1}{d-1}$.

Similarly, let observables $X, \tilde{X} \in X_d$ be mutually orthogonal in $L_d$ and $n, \tilde{n} \in \mathcal{Y}_{\Upsilon_d}(X_d)$ be their Bloch vectors under representation (24). Then by (24) and the mutual orthogonality $\text{tr} (X \tilde{X}) = d (n \cdot \tilde{n}) = 0$.

This implies the following properties, characterizing the Bloch vectors sets $\mathcal{Y}_{\Upsilon_d}(X_d)$ and $\mathcal{Y}_{\Upsilon_d}(S_d)$.

This condition is a generalization of the necessary condition [6, 7] for the Bloch vector of a qudit state under the generalized Gell-Mann representation.
Proposition 3 Let $\Upsilon_d$ be an arbitrary tuple of traceless Hermitian operators on $H_d$ satisfying relations in (2). If vectors $r, \tilde{r} \in J_{\Upsilon_d}^{(\Upsilon_d)} \subset \mathbb{R}^{d^2-1}$ are the Bloch vectors of qudit states $\rho_d, \tilde{\rho}_d \in \mathcal{S}_d$ in representation (27), then their scalar product

$$r \cdot \tilde{r} \geq -\frac{1}{d-1}. \quad (33)$$

If vectors $n, \tilde{n} \in J_{X_d}^{(\Upsilon_d)} \subset \mathbb{R}^{d^2-1}$ are the Bloch vectors in representation (24) of observables $X, \tilde{X} \in X_d$, mutually orthogonal in space $L_d$, then their scalar product

$$n \cdot \tilde{n} = 0. \quad (34)$$

For the Bloch vectors of qudit states under the generalized Gell-Mann representation, condition (33) was presented in [5, 6, 7].

The expectation of a quantum observable $X$ in a qudit state $\rho_d \in \mathcal{S}_d$ has the form

$$E_{\rho_d}(X) := \text{tr}(\rho_d X). \quad (35)$$

Substituting into the right hand-side of (35) representations (24) and (27) for an observable $X \in X_d$ and a state $\rho_d \in \mathcal{S}_d$, respectively, we come to the following representation

$$E_{\rho_d}(X) = \sqrt{d-1} (r \cdot n) \quad (36)$$

of the quantum expectation (35) via the scalar product of the Bloch vectors $n \in J_{X_d}^{(\Upsilon_d)}$, $r \in J_{\Upsilon_d}^{(\Upsilon_d)}$ bijectively corresponding to an observable $X \in X_d$ and a state $\rho_d \in \mathcal{S}_d$ under representations (24) and (27), respectively.

Remark 1 Expressed in terms of Bloch vectors, the quantum analogs of bipartite Bell inequalities for correlation functions constitute linear combinations of scalar products of the corresponding Bloch vectors, for details, see our results in [16, 18].

Note that, for an observable $X \in X_d$, expectation $|\text{tr}(\rho_d X)| \leq 1$. This and relation (36) imply.

Proposition 4 Let $n \in J_{X_d}^{(\Upsilon_d)}$, $r \in J_{\Upsilon_d}^{(\Upsilon_d)}$ be the Bloch vectors of an observable $X \in X_d$ and a state $\rho_d \in \mathcal{S}_d$ under representations (24) and (27), respectively. Then

$$\sqrt{d-1} (r \cdot n) \leq 1. \quad (37)$$

From relation (37) and Proposition 2 it follows that, for $d > 2$, an observable $X \in X_d$ with eigenvalues $\pm 1$ and a pure state $\rho_d \in \mathcal{S}_d$ cannot be described by the same unit vector in representations (24) and (27).

Proposition 5 Let $J_{X_d}^{(\Upsilon_d)}$ be the set of Bloch vectors for all qudit observables $X \in X_d$ under representation (24) and $J_{\Upsilon_d}^{(\Upsilon_d)}$ be the set of the Bloch vectors for all qudit states $\rho_d \in \mathcal{S}_d$ under representation (27). Then for all $d > 2$

$$\left\{ n \in J_{X_d}^{(\Upsilon_d)} \mid \|n\|_{\mathbb{R}^{d^2-1}} = 1 \right\} \cap \left\{ r \in J_{\Upsilon_d}^{(\Upsilon_d)} \mid \|r\|_{\mathbb{R}^{d^2-1}} = 1 \right\} = \emptyset. \quad (38)$$
For our further consideration, we need to generalize the statement of Lemma 1 in \cite{16}, formulated for the tuple $\Lambda_d$ of the generalized Gell-Mann operators on $\mathbb{C}^d$, to the case of an arbitrary operator tuple $\Upsilon_d$ in \cite{2}. As we stress in \cite{16}, the proofs of the main statements in Section 2 of this article do not involve the specific forms of the generalized Gell-Mann operators and hold for every operator tuple $\Upsilon_d$ with elements satisfying the relations:

\[
\Upsilon_d^{(k)} = \left( \Upsilon_d^{(k)} \right)^\dagger \neq 0, \quad \text{tr} \left( \Upsilon_d^{(k)} \right) = 0, \quad \text{tr}(\Upsilon_d^{(k)} \Upsilon_d^{(m)}) = 2\delta_{km}, \quad k, m = 1, \ldots, (d^2 - 1), \quad (39)
\]

that is, for any operator tuple $\Upsilon_d$ specified in \cite{2}.

Denote by

\[
\|X\|_0 := \sup_{\|\psi\|_{\mathcal{H}_d} = 1} \|X\psi\|_{\mathcal{H}_d} = \max_{\lambda_m(X)} |\lambda_m(X)| \quad (40)
\]

the operator norm of a qudit observable $X$ with eigenvalues $\lambda_m(X)$. The following statement is a generalization of Lemma 1 in \cite{16}.

**Proposition 6** For each tuple $\Upsilon_d = (\Upsilon_d^{(1)}, \ldots, \Upsilon_d^{(d^2 - 1)})$ of traceless Hermitian operators on $\mathcal{H}_d$ satisfying relations (39), the upper and the lower bounds on the operator norm

\[
\sqrt{\frac{2}{d}} \|p\|_{\mathbb{R}^{d^2 - 1}} \leq \|p \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{2(d - 1)}{d}} \|p\|_{\mathbb{R}^{d^2 - 1}} \quad (41)
\]

of a traceless Hermitian qudit operator $p \cdot \Upsilon_d$ hold for all vectors $p \in \mathbb{R}^{d^2 - 1}$ and all dimensions $d \geq 2$.

Bounds (41) imply.

**Corollary 1** For each tuple $\Upsilon_d = (\Upsilon_d^{(1)}, \ldots, \Upsilon_d^{(d^2 - 1)})$ of traceless Hermitian operators on $\mathcal{H}_d$ satisfying relations (39) and all $d \geq 2$:

(a) $\|p\|_{\mathbb{R}^{d^2 - 1}} \leq \sqrt{\frac{a}{d^2 - 1}} \Rightarrow \|p \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{a}{2}}$;

(b) $\|p\|_{\mathbb{R}^{d^2 - 1}} \leq \frac{1}{d - 1} \Rightarrow \|p \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{2}{d(d - 1)}}$;

(c) $\|p\|_{\mathbb{R}^{d^2 - 1}} \leq 1 \Rightarrow \|p \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{2(d - 1)}{d}}$;

(d) $\|p \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{2}{d}} \Rightarrow \|p\|_{\mathbb{R}^{d^2 - 1}} \leq 1$;

(e) $\|p \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{2}{d}} \|p\|_{\mathbb{R}^{d^2 - 1}} \Rightarrow \|p \cdot \Upsilon_d\|_0 = \sqrt{\frac{2}{d}} \|p\|_{\mathbb{R}^{d^2 - 1}}$.

4 **Bloch vectors of traceless qudit observables**

Under the generalized Gell-Mann representation $X = \sqrt{\frac{2}{d}} (n \cdot \Lambda_d)$, we specified the set $\mathcal{Y}_{\mathcal{X}_d}^{(\Lambda_d)}$ of Bloch vectors for all qudit observables $X \in \mathcal{X}_d$ in article \cite{16}. This representation is a particular case of representation \cite{21} with $\Upsilon_d \to \Lambda_d$. As we stressed above, the proofs
of our main statements in Section 2 of [16], formulated for the generalized Gell-Mann representation \( X = \sqrt{d/2} (n \cdot \Lambda_d) \), hold for representation (24) with every operator tuple \( \Upsilon_d = (\Upsilon_1, \ldots, \Upsilon_{d^2-1}) \) of operators on \( H_d \) satisfying relations (39).

For decomposition (24) via any basis \( \mathfrak{B}_{\Upsilon_d} \) of type (2), the generalization of Theorem 1 in [16] and the above Propositions 1–3 and Corollary 2 imply.

**Theorem 1** Let \( \Upsilon_d = (\Upsilon_{d}^{(1)}, \ldots, \Upsilon_{d}^{(d^2-1)}) \), \( d \geq 2 \), be a tuple of traceless Hermitian operators on \( H_d \) satisfying relations (39) and \( \mathcal{X}_d \) be the set of all traceless qudit observables with the operator norm \( \|X\|_0 \leq 1 \). The representation

\[
X = \sqrt{d/2} (n \cdot \Upsilon_d)
\]

establishes the one-to-one correspondence \( \mathcal{X}_d \leftrightarrow \mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \) between observables \( X \in \mathcal{X}_d \) and vectors \( n \in \mathbb{R}^{d^2-1} \) in the set

\[
\mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} = \left\{ n \in \mathbb{R}^{d^2-1} \mid \|n \cdot \Upsilon\|_0 \leq \sqrt{d/2} \right\},
\]

which is a subset of the unit ball:

\[
\mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \subseteq \left\{ n \in \mathbb{R}^{d^2-1} \mid \|n\|_{\mathbb{R}^{d^2-1}} \leq 1 \right\},
\]

and contains the ball of radius \( 1/\sqrt{d-1} \):

\[
\mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \supseteq \left\{ n \in \mathbb{R}^{d^2-1} \mid \|n\|_{\mathbb{R}^{d^2-1}} \leq \frac{1}{\sqrt{d-1}} \right\}.
\]

The boundary of \( \mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \) has the form

\[
\partial \mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} = \left\{ n \in \mathbb{R}^{d^2-1} \mid \|n\|_0 = \sqrt{\frac{2}{d}} \right\}.
\]

For \( d \geq 3 \), the geometry of the set

\[
\mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \cap \left\{ n \in \mathbb{R}^{d^2-1} \mid \frac{1}{\sqrt{d-1}} < \|n\|_{\mathbb{R}^{d^2-1}} \leq 1 \right\}
\]

is rather complicated. The maximal norm of a vector \( n \in \mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \) is equal to 1 if a dimension \( d \geq 2 \) is even and to \( \sqrt{\frac{d-1}{d}} \) if a dimension \( d > 2 \) is odd. Under the one-to-one correspondence \( \mathcal{X}_d \leftrightarrow \mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \), established by representation (42), the sets

\[
\{ X \in \mathcal{X}_d \mid \lambda_m(X) = \pm 1, \ m = 1, \ldots, d \} \leftrightarrow \left\{ n \in \mathfrak{J}_{\mathcal{X}_d}^{(\Upsilon_d)} \mid \|n\| = 1 \right\}
\]

and are not empty if and only if a dimension \( d \geq 2 \) is even.
From Theorem 1 it follows that the boundary of $\mathcal{J}^{(T_d)}_{X_d}$ contains unit vectors
\[
\partial \mathcal{J}^{(T_d)}_{X_d} \cap \left\{ n \in \mathbb{R}^{d^2-1} \mid \|n\|_{\mathbb{R}^{d^2-1}} = 1 \right\} \neq \emptyset
\] (49)
if and only if a qudit dimension $d \geq 2$ is even.

5 Bloch vectors of qudit states

Under representation (27) specified with the operator tuple $\Lambda_d$ of the generalized Gell-Mann operators (i.e., under the generalized Gell-Mann representation), the set $\mathcal{J}^{(\Lambda_d)}_{\mathcal{E}_d}$ of Bloch vectors set $\mathcal{J}^{(\Lambda_d)}_{\mathcal{E}_d}$ was specified in [6, 7] where it was proved that a Hermitian operator $\tau_d = \frac{1}{d} + \sqrt{\frac{d-1}{2d}} (r \Lambda_d \cdot \Lambda_d)$ with the unit trace $\text{tr} (\tau_d) = 1$ is positive $\tau_d \geq 0$, hence, constitutes a qudit state, if an only if
\[
a_j(\tau_d) \geq 0, \quad j = 2, \ldots, d, \tag{50}
\]
where coefficients $a_j(\tau_d)$ are derived in [6, 7] via the recursive relations and have the forms:
\[
2! a_2 = 1 - \text{tr} (\tau_d^2), \tag{51}
3! a_3 = 1 - 3 \text{tr} (\tau_d^2) + 2 \text{tr} (\tau_d^3),
4! a_4 = 1 - 6 \text{tr} (\tau_d^2) + 8 \text{tr} (\tau_d^3) + 3 (\text{tr} (\tau_d^2))^2 - 6 \text{tr} (\tau_d^4),
5! a_5 = \cdots.
\]
Moreover, the operator $\tau_d$ constitutes a pure qudit state if and only if $a_j(\tau_d) = 0$, for all $j = 2, \ldots, d$.

The proofs of these results in [6, 7] do not involve the specific forms of the generalized Gell-Mann operators but are only based on relations (39) and the application of Newton’s formulas for sums of the powers of roots $\lambda_j = 1, \ldots, d$ of the characteristic equation for the matrix representation of operator $\tau_d$. This means that relations (50), (51) are also true for the decomposition
\[
\tau_d = \frac{1}{d} + \sqrt{\frac{d-1}{2d}} (r \Upsilon_d \cdot \Upsilon_d), \quad r \in \mathbb{R}^{d^2-1}, \tag{52}
\]
where $\Upsilon_d$ is an arbitrary tuple of operators satisfying conditions (39). Substituting (52)
into (51) and taking into account relations (3) we derive:

\[
2!a_2(\Upsilon_d)(r) = \frac{d-1}{d} \left(1 - \|r\|_{\mathbb{R}^{d^2-1}}^2\right),
\]

\[
3!a_3(\Upsilon_d)(r) = \frac{(d-1)(d-2)}{d^2} \left(1 - 3\|r\|_{\mathbb{R}^{d^2-1}}^2\right) + 2\frac{d-1}{d} \sqrt{\frac{d-1}{2d}} \sum_{i,j,k} g_{ijk} r_i r_j r_k,
\]

\[
4!a_4(\Upsilon_d)(r) = \frac{(d-1)(d-2)(d-3)}{d^3} \left(1 - 6\|r\|_{\mathbb{R}^{d^2-1}}^2\right) + 3\frac{(d-1)^2(d-2)}{d^2} \|r\|_{\mathbb{R}^{d^2-1}}^4
\]

\[+ \frac{8(d-1)(d-3)}{d^2} \sqrt{\frac{d-1}{2d}} \sum_{i,j,k} g_{ijk} r_i r_j r_k - 3\frac{(d-1)^2}{d^2} \sum_{i,j,l,m,k} g_{kij} g_{kjm} r_i r_j r_k r_l,
\]

\[
5!a_5(\Upsilon_d)(r) = \cdots,
\]

where, for short of notations, we omit the lower index \(\Upsilon_d\) at \(r_{\Upsilon_d}\). Note that by (3) the norm \(\|r\|_{\mathbb{R}^{d^2-1}}^2\) of the Bloch vector of a state \(\tau_d\) is the same for representation (27) via different operator tuples \(\Upsilon_d\).

From (50) and (53) it follows that, under representation (27), set \(\mathcal{J}(\Upsilon_d)\) of the Bloch vectors of all qudit states and its subset \(\mathcal{J}^{(\tau_d)}_{\mathcal{E}_d}\) of the Bloch vectors of all pure qudit states are given by

\[
\mathcal{J}(\Upsilon_d) = \left\{ r \in \mathbb{R}^{d^2-1} \mid \alpha_j(\Upsilon_d)(r) \geq 0, \text{ } j = 2, \ldots, d \right\},
\]

\[
\mathcal{J}^{(\tau_d)}_{\mathcal{E}_d} = \left\{ r \in \mathbb{R}^{d^2-1} \mid \alpha_j(\Upsilon_d)(r) = 0, \text{ } j = 2, \ldots, d \right\},
\]

respectively.

For the qubit case \((d = 2)\) and the operator type \(\Upsilon_2 = \Lambda_2 \equiv \sigma = (\sigma_1, \sigma_2, \sigma_3)\), the sets (54) and (55) reduce correspondingly, to the unit ball and the unit sphere in \(\mathbb{R}^3\) – the well-known results from the Bloch vectors formalism for qubit states. For higher dimensions, the geometrical properties of set \(\mathcal{J}(\Lambda_d)\) of the Bloch vectors of all qutrit states under the generalized Gell-Mann representation, also, the two-dimensional and three-dimensional sections of set \(\mathcal{J}(\Lambda_d)\) for \(d = 3, 4\) were analyzed in [6, 12, 13, 14, 15].

However, the specification of sets \(\mathcal{J}(\Upsilon_d)\) and \(\mathcal{J}^{(\tau_d)}_{\mathcal{E}_d}\) via the systems of algebraic equations in (51) and (55) does not allow to characterize these sets in a compact unified analytical form for all \(d \geq 2\), also, to find their general geometry properties.

In what follows, we introduce for set \(\mathcal{J}(\Upsilon_d)\) of the Bloch vectors for all qudit states and its subset \(\mathcal{J}^{(\tau_d)}_{\mathcal{E}_d}\) of the Bloch vectors for all pure qudit states, the new compact expressions in terms of operator norms.

These new expressions have the unified forms for all \(d \geq 2\) and reveal the general geometry properties of sets \(\mathcal{J}(\Upsilon_d)\) and \(\mathcal{J}^{(\tau_d)}_{\mathcal{E}_d}\) which for \(d \geq 3\) cannot be analytically extracted from the systems of algebraic equations specified in (54), (55).
Denote by $\lambda^{(+)}_m(r) > 0$ and $\lambda^{(-)}_m(r) \leq 0$ the positive and non-positive eigenvalues of a traceless Hermitian operator $(r \cdot \Upsilon_d)$ and by $k_{\lambda_m}$ – the multiplicity of an eigenvalue $\lambda_m$. The spectral decomposition of a Hermitian operator (52) reads

$$\tau_d = \sum_{\lambda_m^{(+)}} \left( \frac{1}{d} + \sqrt{\frac{d-1}{2d} \lambda^{(+)}_m(r)} \right) E_{\lambda_m^{(+)}} + \sum_{\lambda_m^{(-)}} \left( \frac{1}{d} - \sqrt{\frac{d-1}{2d} \lambda^{(-)}_m(r)} \right) E_{\lambda_m^{(-)}}, \quad (56)$$

where $E_{\lambda_m}$ is the spectral projection of a Hermitian operator $(r \cdot \Upsilon_d)$ corresponding to its eigenvalue $\lambda_m(r)$. Relation (56) implies that, for the Hermitian operator (52), all its eigenvalues are given by

$$\xi_m(r) = \left( \frac{1}{d} + \sqrt{\frac{d-1}{2d} \lambda^{(+)}_m(r)} \right) > 0, \quad (57)$$

$$\eta_j(r) = \left( \frac{1}{d} - \sqrt{\frac{d-1}{2d} \lambda^{(-)}_j(r)} \right),$$

and have multiplicities $k_{\lambda_m}$ of the corresponding eigenvalues $\lambda^{(\pm)}_m(r)$. Therefore, a Hermitian operator $\tau_d$ constitutes a qudit state if and only if all its eigenvalues $\eta_j(r)$ are non-negative. From (57) it follows that this is true if and only if

$$\max_{\lambda^{(-)}_j} \left| \lambda^{(-)}_j(r) \right| \leq \sqrt{\frac{2}{d(d-1)}}. \quad (58)$$

Recall [4] that any qudit observable $Z$ admits the decomposition

$$Z = Z^{(+)}, Z = Z^{(-)}, \quad Z^{(+)}, Z^{(-)} \geq 0, \quad (59)$$

via positive Hermitian operators $Z^{(\pm)} \geq 0$. This, in particular, refers to traceless qudit observables $(r \cdot \Upsilon)$ with the operator norm $\|r \cdot \Upsilon_d\|_0$ satisfying bounds (40).

In view of this and relation (40), the necessary and sufficient condition (58) is equivalent to $\| (r \cdot \Upsilon_d)^{-1} \|_0 \leq \sqrt{\frac{2}{d(d-1)}}$ in expression (52). The latter, in turn, implies $\text{tr}(\tau_d^2) \leq 1$, and, hence,

$$\text{tr}(\tau_d^2) = \frac{1}{d} + \frac{d-1}{d} \|r\|_{\mathbb{R}^{d^2-1}}^2 \leq 1 \quad \Leftrightarrow \quad \|r\|_{\mathbb{R}^{d^2-1}}^2 \leq 1, \quad (60)$$

so that by item (c) of Corollary 1

$$\|r \cdot \Upsilon_d\|_0 \leq \sqrt{\frac{2(d-1)}{d}}. \quad (61)$$

Furthermore, a Hermitian operator (52) is a pure qudit state if and only if it is positive and $\text{tr}(\tau_d^2) = 1 \Leftrightarrow \|r\|_{\mathbb{R}^{d^2-1}}^2 = 1$. Moreover, in this case $\tau_d$ has only the eigenvalue equal
to 1 with multiplicity 1 and the eigenvalue equal to 0 with multiplicity \((d - 1)\). In notation of (57) these are \(\xi = 1\) with multiplicity \(k_\xi = 1\) and \(\eta = 0\) with multiplicity \(k_\eta = d - 1\). But the latter is possible iff 
\[ \| (r \cdot \Upsilon) \|_0 \leq \sqrt{\frac{2}{d(d - 1)}}. \]

Relations (56)–(61) prove the following statement.

**Proposition 7** A Hermitian operator \((\Upsilon)\) constitutes a qudit state if and only if
\[ \| (r \cdot \Upsilon)(-) \|_0 \leq \sqrt{\frac{2}{d(d - 1)}} \]  
and this condition implies
\[ \| r \|_{\mathbb{R}^{d^2 - 1}} \leq 1. \]  
A Hermitian operator \((\Upsilon)\) constitutes a pure qudit state if and only if
\[ \| (r \cdot \Upsilon)(-) \|_0 = \sqrt{\frac{2}{d(d - 1)}}, \]
\[ \| r \|_{\mathbb{R}^{d^2 - 1}} = 1. \]

Since \[ \| (r \cdot \Upsilon)(-) \|_0 \leq \| r \cdot \Upsilon \|_0, \] from \((62)\) and item (b) of Corollary 1 it follows.

**Corollary 2** ((Sufficient condition)) For all
\[ \| r \|_{\mathbb{R}^{d^2 - 1}} \leq \frac{1}{d - 1}, \]  
condition \((65)\) is fulfilled.

Propositions 2, 3, 7 and Corollary 2 imply.

**Theorem 2** Let \(\Upsilon_d = (\Upsilon_d^{(1)}, \ldots, \Upsilon_d^{(d^2 - 1)})\), \(d \geq 2\), be a tuple of traceless Hermitian operators on \(\mathcal{H}_d\) satisfying relations \((66)\). The representation
\[ \rho_d = \mathbb{1}_d + \sqrt{\frac{d - 1}{2d}} (r \cdot \Upsilon_d) \]  
establishes the one-to-one correspondence \(\mathcal{S}_d \leftrightarrow \mathcal{J}_d^{(\Upsilon_d)}\) between qudit states \(\rho_d \in \mathcal{S}_d\) and vectors \(r \in \mathbb{R}^{d^2 - 1}\) in the set
\[ \mathcal{J}_d^{(\Upsilon_d)} = \left\{ r \in \mathbb{R}^{d^2 - 1} \mid \| (r \cdot \Upsilon_d)(-) \|_0 \leq \sqrt{\frac{2}{d(d - 1)}} \right\}, \]  
which is a subset
\[ \mathcal{J}_d^{(\Upsilon_d)} \subseteq \left\{ r \in \mathbb{R}^{d^2 - 1} \mid \| r \|_{\mathbb{R}^{d^2 - 1}} \leq 1 \right\}. \]
of the unit ball and contains the ball of radius \( \frac{1}{d-1} \):

\[
\mathcal{J}_{\mathbb{S}_d}^{(\Upsilon_d)} \supseteq \left\{ r \in \mathbb{R}^{d^2-1} \mid \| r \|_{\mathbb{R}^{d^2-1}} \leq \frac{1}{d-1} \right\}.
\]

(69)

For any two vectors \( r_1, r_2 \in \mathcal{J}_{\mathbb{S}_d}^{(\Upsilon_d)} \),

\[
r_1 \cdot r_2 \geq -\frac{1}{d-1}.
\]

(70)

The boundary of \( \mathcal{J}_{\mathbb{S}_d}^{(\Upsilon_d)} \) has the form

\[
\partial \mathcal{J}_{\mathbb{S}_d}^{(\Upsilon_d)} = \left\{ r \in \mathbb{R}^{d^2-1} \mid \left\| (r \cdot \Upsilon_d)^(-) \right\|_0 = \sqrt{\frac{2}{d(d-1)}} \right\}.
\]

(71)

For \( d \geq 3 \), the geometry of the set

\[
\mathcal{J}_{\mathbb{S}_d}^{(\Upsilon_d)} \cap \left\{ r \in \mathbb{R}^{d^2-1} \mid \frac{1}{d-1} < \| r \|_{\mathbb{R}^{d^2-1}} \leq 1 \right\}
\]

(72)

is rather complicated. Under the one-to-one correspondence \( \mathbb{S}_d \leftrightarrow \mathcal{J}_{\mathbb{S}_d}^{(\Upsilon)} \), established by representation \((66)\), pure qudit states are bijectively mapped to vectors \( r \in \mathbb{R}^{d^2-1} \) in the subset

\[
\mathcal{J}_{\mathbb{S}_d}^{(\Upsilon_d)_{\text{pure}}} = \left\{ r \in \mathbb{R}^{d^2-1} \mid \left\| (r \cdot \Upsilon)^(-) \right\|_0 = \sqrt{\frac{2}{d(d-1)}}, \ |r|^2_{\mathbb{R}^{d^2-1}} = 1 \right\}
\]

(73)

of the unit sphere in \( \mathbb{R}^{d^2-1} \).

6 Evolution in time

For the Bloch vector \( r(t) \in \mathcal{J}_{\mathbb{S}_d}^{(\Upsilon)} \) of a qudit state \( \rho_d(t) \) under representation \((27)\), let us now consider its evolution in time if a qudit system is isolated and if it is open.

Recall that if a qudit system is isolated, then, under a qudit Hamiltonian \( H_d(t) = H_d^\dagger(t) \), the evolution of its state \( \rho_d(t), t > t_0 \), in time is described by the relation

\[
\rho_d(t) = U(t, t_0)\rho_d(t_0)U^\dagger(t, t_0),
\]

(74)

where \( \rho_d(t_0) \) is an initial state of a qudit system and \( U(t, t_0) \) is the unitary operator on \( \mathcal{H}_d \), satisfying the Cauchy problem for the Schrödinger equation:

\[
i\frac{d}{dt} U(t, t_0) = H_d(t)U(t, t_0), \quad t > t_0,
\]

(75)

\[
U(t_0, t_0) = \mathbb{I}_d.
\]

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Eqs. (75) and (74) imply that, for an isolated qudit system, the time evolution of its state $\rho_d(t)$ is described by the solution of the Liouville–von Neumann equation
\[
\frac{d}{dt}\rho_d(t) = -i \left[ H_d(t), \rho_d(t) \right], \quad t > t_0, \tag{76}
\]
satisfying the initial condition $\rho_d(t_0)$.

If, however, a qudit system is open, i.e., interacts with an environment, then, in the Markovian case, the evolution in time of its state $\rho_d(t)$ is described by the Lindblad master equation \[19, 4\] which we take in the following generalized form
\[
\frac{d}{dt}\rho_d(t) = -i \left[ \tilde{H}_d(t), \rho_d(t) \right] + \sum_k \gamma_k \left( L_k(t)\rho_d(t)L_k^\dagger(t) - \frac{1}{2} L_k^\dagger(t)L_k(t) \odot \rho_d(t) \right), \tag{77}
\]
\[
\gamma_k \geq 0.
\]

Here, in the right hand side: (i) the first term describes the time evolution under a qudit Hamiltonian $\tilde{H}_d(t) = \tilde{H}_d^\dagger(t)$, including, in general, a ”bare” Hamiltonian $H_d(t)$ of a qudit system and an additive due to its interaction with an environment; (ii) the second term describes the dissipative part with, in general, nonstationary operators $L_k(t) \in \mathcal{L}_d$; (iii) notation $\odot$ is determined in (8).

Taking into account that representation (27) of a qudit state holds for all moments of time:
\[
\rho_d(t) = \mathbb{1}_d + \sqrt{\frac{d-1}{2d}} (r(t) \cdot \Upsilon_d), \tag{78}
\]
\[
r(t) = \sqrt{\frac{d}{2(d-1)}} \text{tr} (\rho_d(t)\Upsilon_d) \in \mathcal{J}_{\Upsilon_d} \subset \mathbb{R}^{d^2-1}, \quad t \geq t_0,
\]
we have
\[
\frac{d}{dt} r(t) = \sqrt{\frac{d}{2(d-1)}} \text{tr} \left( \Upsilon_d \frac{d}{dt} \rho_d(t) \right). \tag{79}
\]

In what follows, based on relation (79) and Eqs. (76) and (77), we specify the general equations describing the time evolution of the Bloch vector $r(t)$ of a qudit system state $\rho_d(t)$ under representation (78) if a qudit system is isolated and if a qudit system is open.

### 6.1 Isolated qudit system

For an isolated qudit system, the evolution in time of its state $\rho_d(t)$ under a Hamiltonian $H_d(t)$ is described by the Liouville–von Neumann equation (76). This equation and relation (49) imply
\[
\frac{d}{dt} r(t) = -i \sqrt{\frac{d}{2(d-1)}} \text{tr} \left( [H_d(t), \rho_d(t)] \Upsilon_d \right). \tag{80}
\]
Taking into account representation (78) for state \( \rho(t) \), the decomposition (4) of a general qudit Hamiltonian \( H_d(t) \) in a basis \( \mathcal{B}_Y \):

\[
H_d(t) = h_0(t) \frac{I_d}{d} + h(t) \cdot Y_d, \quad (81)
\]

\[
h_0(t) = \text{tr} \left( H_d(t) \right) \in \mathbb{R}, \quad h(t) = \frac{1}{2} \text{tr} \left( Y_d H_d(t) \right) \in \mathbb{R}^{d^2-1},
\]

and relations (14), we have

\[
\sqrt{\frac{d}{2(d-1)}} [H_d(t), \rho_d(t)] = \frac{1}{2} \left[ \left( h(t) \cdot Y_d \right), \left( r(t) \cdot Y_d \right) \right] = \frac{1}{2} \sum_{k,m,l} h_k(t) r_m(t) \left[ Y_d^{(k)}, Y_d^{(m)} \right]
\]

\[
= i \sum_{k,m,l} f_{kml}^{(Y_d)} h_k(t) r_m(t) Y_d^{(l)}, \quad (82)
\]

where constants \( f_{kml}^{(Y_d)} \) are defined in (14).

The substitution of (82) into relation (80) proves the following statement.

**Theorem 3** Let \( Y_d = \left( Y_d^{(1)}, ..., Y_d^{(d^2-1)} \right) \), \( d \geq 2 \), be a tuple of traceless Hermitian operators on \( \mathcal{H}_d \) satisfying conditions (39) and Eq. (81) be the decomposition of a general nonstationary qudit Hamiltonian \( H_d(t) \) in basis \( \mathcal{B}_Y \). Under the time evolution of a qudit state \( \rho_d(t) \) due to the Liouville–von Neumann equation (76), the evolution in time of its Bloch vector \( r(t) \in J^{(Y_d)} \subset \mathbb{R}^{d^2-1} \) in representation (78) is described by

\[
\frac{d}{dt} r(t) = \mathbb{B}_{H_d}(t) r(t), \quad t > t_0, \quad (83)
\]

\[
r(t_0) = \sqrt{\frac{d}{2(d-1)}} \text{tr} \left( \rho_d(t_0) Y_d \right),
\]

where \( \mathbb{B}_{H_d} : \mathbb{R}^{d^2-1} \to \mathbb{R}^{d^2-1} \) is the skew-symmetric linear operator defined via its matrix representation in the standard basis of \( \mathbb{R}^{d^2-1} \):

\[
\mathbb{B}_{H_d}^{(lm)}(t) = -2 \sum_k f_{kml}^{(Y_d)} h_k(t) = -\mathbb{B}_{H_d}^{(ml)}(t), \quad (84)
\]

with constants \( f_{kml}^{(Y_d)} \) given in (14).

For the qubit case \( (d = 2) \) and the tuple \( Y_2 = \sigma \), i.e. in case of the Bloch representation (29), Eq. (83) reduces to

\[
\frac{d}{dt} r(t) = 2 h(t) \times r(t), \quad (85)
\]

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where \( h(t) \in \mathbb{R}^3 \) is the vector in the decomposition \( H_3(t) = h_0(t) \frac{\mathbf{I}}{2} + h(t) \cdot \sigma \) of a general qubit Hamiltonian \( H_3(t) \) in the operator basis \( \{ \mathbf{I}_2, \sigma \} \) and notation \((a \times c)\) means the vector product of vectors \( a, c \in \mathbb{R}^3 \).

Taking into account that by (85)
\[
\frac{1}{2} \frac{d}{dt} \|r(t)\|_{\mathbb{R}^{d^2-1}}^2 = r \cdot (\mathbb{B}_{H_d}(t)r(t)), \quad t \geq t_0,
\]
and that operator \( \mathbb{B}_{H_d}(t) \) is skew symmetric, we have \( \frac{d}{dt} \|r(t)\|_{\mathbb{R}^{d^2-1}}^2 = 0 \). This implies the following statement.

**Proposition 8** Under the unitary evolution (74), the norm of the Bloch vector \( r(t) \in \mathcal{V}^{(Y_d)}_{\mathcal{H}_d} \subset \mathbb{R}^{d^2-1} \) of a qudit state system \( \rho_d(t) \) in representation (78) is invariant in time:
\[
\|r(t)\|_{\mathbb{R}^{d^2-1}} = \|r(t_0)\|_{\mathbb{R}^{d^2-1}}, \quad t \geq t_0,
\]
for all \( d \geq 2 \).

Note that by (17) the norm \( \|r(t)\|_{\mathbb{R}^{d^2-1}} \) of the Bloch vector of a qudit state does not also depend on a choice of an operator tuple \( Y_d \) in representation (78).

For the unitary operator \( U(t, t_0) \) in (19), the differential equations describing the evolution in time of its decomposition coefficients under the generalized Gell-Mann representation were found by us recently in [16], see there Eq. (26). The derivation in [16] of these equations does not involve the specific forms of the generalized Gell-Mann operators and is only based on the validity of relations (39), which are, however, true for all tuples \( Y_d \) in operator bases of type (2).

Therefore, according to our results in [16], for an arbitrary operator tuple \( Y_d \), the decomposition of \( U(t, t_0) \) in a basis \( \mathcal{B}_{Y_d} \) takes the form
\[
U(t, t_0) = \exp \left\{ -i \int_{t_0}^t h_0(\tau)d\tau \right\} \left( u_0(t, t_0)\mathbf{I}_d - i \sqrt{\frac{d}{2}} u(t, t_0) \cdot Y_d \right),
\]
where\(^2\)
\[
|u_0(t, t_0)|^2 + \|u'(t, t_0)\|_{\mathbb{C}^{d^2-1}}^2 = 1,
\]
\[
u_0(t_0)\mathbf{v}^{(j)}(t, t_0) - \mathbf{v}_0(t_0)u^{(j)}(t, t_0) = \sqrt{\frac{d}{2}} \sum_{k,m} \left( f_{kmj} - ig_{kmj}^{(Y_d)} \right) u^{(k)}(t, t_0)\mathbf{v}^{(m)}(t, t_0),
\]
and, under the time evolution of the unitary operator \( U(t, t_0) \) due to the Schrödinger equation (74), the evolution in time of its decomposition coefficients in representation

\(^2\)Here, notation \( u' \) means the vector-column in \( \mathbb{C}^{d^2-1} \) comprised of components of vector \( u = (u_1, ..., u_{d^2-1}) \) and \( \mathbf{v}^{(j)} \) – the complex conjugate of \( u_j \in \mathbb{C} \).
is described by the system of linear ordinary differential equations:

\[
\begin{align*}
\dot{\gamma}_t(t) & = h(t) \cdot u(t, t_0), \\
\frac{d}{dt} u_j(t, t_0) & = -u_0(t, t_0) h_j(t) + \sqrt{\frac{d}{2}} \sum_{m,k} (f^{(Y_d)}_{jkm} - ig^{(Y_d)}_{jkm}) h_k(t) u_m(t, t_0), \\
\end{align*}
\]

where \( g^{(Y_d)}_{jkm} \) and \( f^{(Y_d)}_{jkm} \) are symmetric and antisymmetric constants defined in \( \text{[14]} \).

### 6.2 Open qudit system

Let a qudit system be open and the evolution in time of its state \( \rho_d(t) \) be described by the Lindblad master equation \( \text{[77]} \). From \( \text{[79]} \) and \( \text{[77]} \) it follows

\[
\frac{d}{dt} r(t) = \sqrt{\frac{d}{2(d-1)}} \text{tr} \left( Z(t) \right),
\]

\[
Z(t) = Y_d^{(l)} \left( -i \left[ \hat{H}_d(t), \rho_d(t) \right] + \sum_k \gamma_k \left( L_k(t) \rho_d(t) L_k^\dagger(t) - \frac{1}{2} L_k^\dagger(t) L_k(t) \rho_d(t) \right) \right).
\]

Let \( \hat{H}_d(t) = \hat{h}_0(t) \frac{\hat{\tau}}{2} + \hat{h}(t) \cdot Y_d \) be the decomposition a general Hamiltonian \( \hat{H}_d(t) \) in a basis \( \mathfrak{B}_Y_d \). Then, similarly to \( \text{[82]} \), the commutator

\[
\sqrt{\frac{d}{2(d-1)}} \left[ \hat{H}_d(t), \rho_d(t) \right] = i \sum_{k,m,l} f^{(Y_d)}_{kml} \hat{h}_k(t) r_m(t) Y_d^{(l)}
\]

and substituting this relation and decomposition \( \text{[82]} \) into Eq. \( \text{[91]} \), we derive

\[
\sqrt{\frac{d}{2(d-1)}} \text{tr} \left( Z(t) \right) = \sum_m \mathbb{B}^{(lm)}_{\hat{H}_d}(t) r_m(t) + \frac{1}{2d(d-1)} \sum_k \gamma_k \text{tr} \left( Y_d^{(l)} \left[ L_k(t), L_k^\dagger(t) \right] \right) + \frac{1}{2} \sum_{k,m} \gamma_k r_m(t) \text{tr} \left( Y_d^{(l)} \left( L_k(t) Y_d^{(m)} L_k^\dagger(t) - \frac{1}{2} L_k^\dagger(t) L_k(t) \rho_d(t) \right) \right),
\]

where matrix \( \mathbb{B}^{(lm)}_{\hat{H}_d}(t) = -2 \sum_k f^{(Y_d)}_{kml} \hat{h}_k(t) \) is skew-symmetric. Noting further that

\[
\text{tr} \left( L_k^\dagger(t) Y_d^{(l)} L_k(t) Y_d^{(m)} \right) = \frac{1}{2} \text{tr} \left( \left[ Y_d^{(l)}, L_k(t) \right] Y_d^{(m)} L_k^\dagger(t) + \left[ L_k^\dagger(t), Y_d^{(l)} \right] L_k(t) Y_d^{(m)} \right)
\]

we come to the following general statement.
Theorem 4 Let $\Upsilon_d = (\Upsilon_d^{(1)}, ..., \Upsilon_d^{(d^2-1)})$, $d \geq 2$, be a tuple of traceless Hermitian operators on $H_d$, satisfying conditions (74) and $\tilde{H}_d(t) = \tilde{h}_0(t) \frac{1}{2} + \tilde{h}(t) \cdot \Upsilon_d$ be the decomposition of a general qudit Hamiltonian $\tilde{H}_d(t)$ in basis $\mathcal{B}$ of qudit. Under the time evolution of a qudit state $\rho_d(t)$ due to the Lindblad master equation (77), the evolution in time of its Bloch vector $r(t) \in \mathbb{J}^{(\Upsilon_d)} \subset \mathbb{R}^{d^2-1}$ in representation (78) is described by

\[
\frac{d}{dt} r(t) = \left( \mathbb{B}_{\tilde{H}_d}(t) + \mathbb{B}_{\text{dis}}(t) \right) r(t) + \sqrt{\frac{1}{2d(d-1)}} \sum_k \gamma_k \text{tr} \left( \Upsilon_d \left[ L_k(t), L_k^\dagger(t) \right] \right),
\]

where $\mathbb{B}_{\tilde{H}_d}(t) : \mathbb{R}^{d^2-1} \rightarrow \mathbb{R}^{d^2-1}$ and $\mathbb{B}_{\text{dis}}(t) : \mathbb{R}^{d^2-1} \rightarrow \mathbb{R}^{d^2-1}$ are linear operators defined via their matrix representations in the standard basis of $\mathbb{R}^{d^2-1}$:

\[
\mathbb{B}_{\tilde{H}_d}^{(lm)}(t) := -2 \sum_k f_{kmk}^l \tilde{h}_0(t) = -\mathbb{B}_{\tilde{H}_d}^{(ml)}(t),
\]

\[
\mathbb{B}_{\text{dis}}^{(lm)}(t) = \frac{1}{2} \sum_k \gamma_k \text{Re} \left( \text{tr} \left( \left[ \Upsilon_d^{(l)}(t), L_k(t) \right] \Upsilon_d^{(m)} L_k^\dagger(t) \right) \right), \quad \gamma_k > 0.
\]

Operator $\mathbb{B}_{\tilde{H}_d}(t)$ is skew symmetric, constants $f_{kmk}^l$ are given in (14).

Let us analyze the evolution in time of the norm $\|r(t)\|_{\mathbb{R}^{d^2-1}}$ of the Bloch vector of a qudit state $\rho_d(t)$. We multiply the left-hand and the right-hand sides of Eq. (95) by component $r_j(t)$, consider further the sum of the resulting expressions over $l = 1, ..., (d^2 - 1)$, take into account $r(t) \cdot \left( \mathbb{B}_{\tilde{H}_d}(t)r(t) \right) = 0$ since operator $\mathbb{B}_{\tilde{H}_d}$ is skew symmetric and finally derive:

\[
\frac{1}{2} \frac{d}{dt} \|r(t)\|_{\mathbb{R}^{d^2-1}}^2 = \sqrt{\frac{1}{2d(d-1)}} \sum_k \gamma_k \text{tr} \left( X_d(t) \left[ L_k(t), L_k^\dagger(t) \right] \right) + \frac{1}{2} \sum_k \gamma_k \text{Re} \left( \text{tr} \left( \left[ X_d(t), L_k(t) \right] X_d(t)L_k^\dagger(t) \right) \right),
\]

where $X_d(t) := r(t) \cdot \Upsilon_d$.

Since the qudit operator $X_d(t)$ is Hermitian at each moment time, its spectral decomposition reads

\[
X_d = \sum_{j=1,...,d} x_j |x_j\rangle\langle x_j|,
\]

where $\{|x_j\rangle \in H_d, j = 1, ..., d\}$ is the orthonormal basis comprised of the eigenvectors of operator $X_d$ corresponding to eigenvalues $x_1 \leq x_2 \leq \cdots \leq x_d$. Here and further, for short, we omit in Eqs. (100)–(101) the dependence of time in notations for operators $X_d$, $L_k$, eigenvalues $x_j$ and eigenvectors $|x_j\rangle$.

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Substituting (99) into the right-hand side of (98) we have

\[
[X_d, L_k] = \sum_{j=1, \ldots, d} x_j \{ |x_j\rangle\langle x_j| L_k - L_k |x_j\rangle\langle x_j| \},
\]

(100)

\[
X_d L_k^\dagger = \sum_{i=1, \ldots, d} x_i |x_i\rangle\langle x_i| L_k^\dagger,
\]

\[
\text{tr}\left( [X_d, L_k] X_d L_k^\dagger \right) = \sum_{j=1, \ldots, d} \langle x_j| [X_d, L_k] X_d L_k^\dagger |x_j\rangle
\]

\[
= \sum_{i,j} x_j x_i \langle x_j| L_k |x_i\rangle \langle x_i| L_k^\dagger |x_j\rangle - \sum_{j,i} x_i^2 \langle x_j| L_k |x_i\rangle \langle x_i| L_k^\dagger |x_j\rangle.
\]

(101)

Taking into account that, in (101),

\[
x_j x_i \leq \frac{1}{2} (x_j^2 + x_i^2), \quad \sum_{n=1, \ldots, d} |x_n\rangle\langle x_n| = \mathbb{I}_d,
\]

(102)

we derive

\[
\text{Re}\left( \text{tr}\left( [X_d, L_k] X_d L_k^\dagger \right) \right) \leq \frac{1}{2} \sum_{i,j} (x_j^2 + x_i^2) \langle x_j| L_k |x_i\rangle \langle x_i| L_k^\dagger |x_j\rangle - \sum_{j,i} x_i^2 \langle x_j| L_k |x_i\rangle \langle x_i| L_k^\dagger |x_j\rangle
\]

\[
= \frac{1}{2} \sum_{j} x_j^2 \langle x_j| L_k L_k^\dagger |x_j\rangle - \frac{1}{2} \sum_{i} x_i^2 \langle x_i| L_k^\dagger L_k |x_i\rangle
\]

\[
= \frac{1}{2} \sum_{m} x_m^2 \langle x_m| [L_k, L_k^\dagger] |x_m\rangle.
\]

(103)

Relations (105)–(108) imply.

**Proposition 9** Let, in the Lindblad equation (77), each operator \(L_k(t)\) be normal \(L_k(t)L_k^\dagger(t) = L_k^\dagger(t)L_k(t)\) at all moments \(t > t_0\) of time. Then, under the time evolution of a qudit state \(\rho_d(t)\) due to the Lindblad equation (77), the norm \(\|r(t)\|_{\mathbb{R}^d-1}\) of its Bloch vector in representation (78) is a non-increasing function of time:

\[
\frac{d}{dt} \|r(t)\|_{\mathbb{R}^d-1} \leq 0, \quad t > t_0.
\]

(104)

7 Entanglement of a pure bipartite state

Let \(\rho_{d_1 \times d_2}\) be a pure bipartite state on a Hilbert space \(\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}\) with arbitrary dimensions \(\dim \mathcal{H}_j = d_j \geq 2\) and

\[
\rho_{d_1} := \text{tr}_{\mathcal{H}_{d_2}} (\rho_{d_1 \times d_2}), \quad \rho_{d_2} := \text{tr}_{\mathcal{H}_1} (\rho_{d_1 \times d_2})
\]

(105)
be the states on \( H_{d_j} \), \( j = 1, 2 \), reduced from a state \( \rho_{d_1 \times d_2} \).

By the Schmidt decomposition \([2][3][4]\), the non-zero eigenvalues of the reduced states \( \rho_{d_1} \) and \( \rho_{d_2} \) coincide, therefore,

\[
\text{tr} \left( \rho_{d_1}^2 \right) = \text{tr} \left( \rho_{d_2}^2 \right).
\] (106)

For a pure bipartite state \( \rho_{d_1 \times d_2} \), the parameter

\[
C_{\rho_{d_1 \times d_2}} := \sqrt{\alpha_{d_1 \times d_2} \left( 1 - \text{tr}(\rho_{d_j}^2) \right)}, \quad j = 1, 2, \quad \alpha_{d_1 \times d_2} > 0,
\] (107)

where \( \alpha_{d_1 \times d_2} > 0 \) is some positive constant, is monotone increasing in the entanglement of this state and constitutes an entanglement measure called the concurrence \([20][21]\). A pure bipartite state \( \rho_{d_1 \times d_2} \) is separable iff \( C_{\rho_{d_1 \times d_2}} = 0 \) and entangled iff \( C_{\rho_{d_1 \times d_2}} > 0 \).

In Eq. (107), a choice of coefficient \( \alpha_{d_1 \times d_2} > 0 \) depends on a normalization of the concurrence \( C_{\rho_{d_1 \times d_2}} \). In \([20]\), coefficient \( \alpha_{d_1 \times d_2} \) is taken to be equal to 2 for all \( d_1, d_2 \geq 2 \) — similarly as it is for a two-qubit pure state \( \rho_{2 \times 2} \). However, below we choose the normalization in (107) such that, for a maximally entangled two-qudit state \( \rho_{d_1 \times d_2}^{m-e} \), the concurrence \( C_{\rho_{d_1 \times d_2}^{m-e}} = 1 \), for all \( d_1, d_2 \geq 2 \). As we prove below, the latter results in the coefficient \( \alpha_{d_1 \times d_2} \) different from the value 2 taken in \([20]\).

For the reduced states \( \rho_{d_j} \), consider their decompositions \([27]\)

\[
\rho_{d_j} = \frac{I_{d_j}}{d_j} + \sqrt{\frac{d_j - 1}{2d_j}} \left( r_{\rho_{d_j}} \cdot \Upsilon_{d_j} \right), \quad j = 1, 2,
\] (108)

via arbitrary tuples \( \Upsilon_{d_j}, j = 1, 2 \), of traceless Hermitian operators satisfying relations \([39]\) and acting on \( H_{d_1} \) and \( H_{d_2} \), respectively. In decompositions (108), the Bloch vectors of the reduced state \( \rho_{d_j} \) are given by

\[
r_{\rho_{d_j}} = \sqrt{\frac{d_j}{2(d_j - 1)}} \text{tr} \left( \rho_{d_j} \Upsilon_{d_j} \right) \in \mathbb{R}^{d_j^2 - 1}
\]

and by Theorem 2

\[
||r_{\rho_{d_j}}||_{\mathbb{R}^{d_j^2 - 1}} \leq 1.
\] (109)

The relation

\[
\text{tr}(\rho_{d_j}^2) = \frac{1}{d_j} + \frac{d_j - 1}{d_j} ||r_{\rho_{d_j}}||_{\mathbb{R}^{d_j^2 - 1}}^2 \quad j = 1, 2,
\] (110)

implies that the norms \( ||r_{\rho_{d_j}}||_{\mathbb{R}^{d_j^2 - 1}}, j = 1, 2 \), of these Bloch vectors do not depend on what operator tuples \( \Upsilon_{d_j} \) are used in decomposition (108). Moreover, from relations (106) and (110) it follows that, for a pure bipartite state \( \rho_{d_1 \times d_2} \), the norms of the Bloch vectors of the reduced states under decompositions (108) satisfy the relation

\[
\frac{1}{d_1} + \frac{d_1 - 1}{d_1} ||r_{\rho_{d_1}}||_{\mathbb{R}^{d_1^2 - 1}}^2 = \frac{1}{d_2} + \frac{d_2 - 1}{d_2} ||r_{\rho_{d_2}}||_{\mathbb{R}^{d_2^2 - 1}}^2.
\] (111)
and

\[ 1 - \text{tr}(\rho_d^2) = \frac{d_j - 1}{d_j} \left( 1 - ||r_{\rho_d_j}||^2_{\mathbb{R}^{d_j^2-1}} \right). \quad (112) \]

Let \( d_k = \min\{d_1, d_2\} \). Substituting relations (112) into formula (107) specified for \( j = k \), we have

\[
C_{\rho_{d_1 \times d_2}} = \sqrt{\alpha_{d_1 \times d_2} \frac{d_k - 1}{d_k} \left( 1 - ||r_{\rho_{d_k}}||^2_{\mathbb{R}^{d_k^2-1}} \right)}, \quad (113)
\]

\[
d_k = \min\{d_1, d_2\}.
\]

Therefore, the normalization of the concurrence \( C_{\rho_{d_1 \times d_2}} \) for all \( d_1, d_2 \geq 2 \) to the maximal value 1, attained on a maximally entangled state, i.e. \( ||r_{\rho_{d_1 \times d_2}}||_{\mathbb{R}^{d_1^2-1}} = 0 \), implies

\[
\alpha_{d_1 \times d_2} := \frac{d_k}{d_k - 1} = \frac{\min\{d_1, d_2\}}{\min\{d_1, d_2\} - 1}. \quad (114)
\]

Relations (105) – (114) prove the following general statement.

**Theorem 5** Let \( \rho_{d_1 \times d_2} \), where \( d_1, d_2 \geq 2 \), be a pure bipartite state on \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \) and \( \rho_{d_j} \), \( j = 1, 2 \), be the states on \( \mathcal{H}_{d_j} \), \( j = 1, 2 \), reduced from \( \rho_{d_1 \times d_2} \) and admitting representations (108) with the Bloch vectors

\[
r_{\rho_{d_j}} = \sqrt{\frac{d_j}{2(d_j - 1)}} \text{tr} (\rho_{d_j} \Upsilon_{d_j}) \in \mathbb{R}^{d_j^2-1}, \quad j = 1, 2. \quad (115)
\]

The norms \( ||r_{\rho_{d_j}}||^2_{\mathbb{R}^{d_j^2-1}}, j = 1, 2 \), of these Bloch vectors do not depend on a choice of operator tuples \( \Upsilon_{d_j} \) in representation (108) and satisfy the relation (111). The concurrence \( C_{\rho_{d_1 \times d_2}} \) of a pure state \( \rho_{d_1 \times d_2} \) normalized to the maximal value 1, attained on a maximally entangled state, is given by

\[
C_{\rho_{d_1 \times d_2}} = \sqrt{1 - ||r_{\rho_{d_k}}||^2_{\mathbb{R}^{d_k^2-1}}}, \quad (116)
\]

\[
d_k := \min\{d_1, d_2\}.
\]

8 Conclusions

In the present article, we consistently develop the main issues of the Bloch vectors formalism for a finite-dimensional quantum system. Within this formalism, qudit states and their evolution in time, qudit observables and their expectations, entanglement and nonlocality, etc. are expressed in terms of vectors in the Euclidean space \( \mathbb{R}^{d^2-1} \). Our developments allow us:
• to formalize the main issues (Propositions 1–6, Corollary 1) of decompositions of linear operators on a finite-dimensional complex Hilbert space via different operator bases;

• to find (Theorem 1) for all $d \geq 2$ the new general expression for the set of Bloch vectors of all traceless qudit observables and to describe the properties of this set;

• to find (Proposition 7, Corollary 2, Theorem 2) for the sets of Bloch vectors of all qudit states, pure and mixed, the new compact general expressions in terms of the operator norms, which have the unified form for all $d \geq 2$ and explicitly reveal the geometry properties of these sets. For the sets of Bloch vectors under the generalized Gell-Mann representation these properties cannot be analytically extracted from the known [6, 7] equivalent specifications via systems (51) of algebraic equations;

• to derive (Theorems 3, 4) for all $d \geq 2$ the new general equations for the time evolution of the Bloch vector of a qudit state if a qudit system is isolated and if it is open and to characterize (Propositions 8, 9) in both cases the main properties of the Bloch vector evolution in time;

• to express (Theorem 5) the concurrence of a pure bipartite state on $H_{d_1} \otimes H_{d_2}$ of an arbitrary dimension $d_1 \times d_2$ via the norm of the Bloch vector of its reduced state $\rho_{d_k}$ on the Hilbert space $H_{d_k}$ of dimension $d_k := \min\{d_1, d_2\}$.

The introduced general formalism is important both for the theoretical analysis of quantum system properties and for quantum applications, in particular, for optimal quantum control, since, for systems where states are described by vectors in the Euclidean space, the methods of optimal control, analytical and numerical, are well developed.

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