A DESCRIPTION OF QUANTUM CHAOS

Kei Inoue†, Andrzej Kossakowski‡ and Masanori Ohya†
†Department of Information Sciences,
Science University of Tokyo,
Noda City, Chiba 278-8510, JAPAN.
‡Institute of Physics,
N. Copernicus University,
Grudziadzka 5, 87-100 Torun, POLAND

Abstract

A measure describing the chaos of a dynamics was introduced by two complexities in information dynamics, and it is called the chaos degree. In particular, the entropic chaos degree has been used to characterized several dynamical maps such that logistis, Baker’s, Tinkerbell’s in classical or quantum systems. In this paper, we give a new treatment of quantum chaos by defining the entropic chaos degree for quantum transition dynamics, and we prove that every non-chaotic quantum dynamics, e.g., dissipative dynamics, has zero chaos degree. A quantum spin 1/2 system is studied by our chaos degree, and it is shown that this degree well describes the chaotic behavior of the spin system.

1 Introduction

There exist several approaches in the study of chaotic behavior of dynamical systems using the concepts such as (1) entropy and dynamical entropy, (2) Chaitin’s complexity, (3) Lyapunov exponent (4) fractal dimension (5) bifurcation (6) ergodicity [13 5 6 7 13 13]. But these concepts are rather
independently used in each field. In 1991, one of the authors proposed Information Dynamics (ID for short) [15, 17, 8] to try to treat such chaotic behavior of systems from a common standing point. Then a chaos degree to measure the chaos in dynamical systems is defined by means of two complexities in ID [16, 17]. In particular, among several chaos degrees, the entropic chaos degree was introduced in [18] and it is applied to some dynamical systems [18, 9, 10], recently, semiclassical properties and chaos degree for quantum Baker’s map has been considered in [10, 11].

In this paper, we give a new treatment of quantum chaos by introducing the chaos degree for quantum transition dynamics, and we prove some fundamental properties for non-chaotic maps. Moreover we show, as an example, that our chaos degree well describes chaotic behavior of spin systems.

2 Entropic chaos degree

In order to contain more general dynamics such as one in continuous systems, we define the entropic chaos degree in C*-algebraic terminology. This setting will be too general in the sequel discussions, but for mathematical completeness including both classical and quantum systems, we start from the C*-algebraic setting.

Let \((\mathcal{A}, \mathcal{S})\) be an input C* system and \((\overline{\mathcal{A}}, \overline{\mathcal{S}})\) be an output C* system; namely, \(\mathcal{A}\) is a C* algebra with unit \(I\) and \(\mathcal{S}\) is the set of all states on \(\mathcal{A}\). We assume \(\overline{\mathcal{A}} = \mathcal{A}\) in the sequel for simplicity. For a weak* compact convex subset \(\mathcal{S}\) (called the reference space) of \(\overline{\mathcal{S}}\), take a state \(\varphi\) from the set \(\mathcal{S}\) and let

\[ \varphi = \int_{\mathcal{S}} \omega d\mu_{\varphi} \]

be an extremal orthogonal decomposition of \(\varphi\) in \(\mathcal{S}\), whose measure \(\mu_{\varphi}\) describes a certain degree of mixture of \(\varphi\) in the reference space \(\overline{\mathcal{S}}\). The measure \(\mu_{\varphi}\) is not uniquely determined unless \(\mathcal{S}\) is the Choquet simplex, so that the set of all such measures is denoted by \(M_{\varphi}(\mathcal{S})\).

Definition 1 The entropic chaos degree with respect to \(\varphi \in \mathcal{S}\) and a channel \(\Lambda^*\), a map from \(\mathcal{S}\) to \(\overline{\mathcal{S}}\), is defined by
\[ DS(\varphi; \Lambda^*) \equiv \inf \left\{ \int S^S(\Lambda^* \varphi) \, d\mu; \mu \in M_\varphi(S) \right\} \] (1)

where \( S^S(\Lambda^* \varphi) \) is the mixing entropy of a state \( \Lambda^* \varphi \) in the reference space \( S \), hence it becomes von Neumann entropy when \( S \) is the set of all density operators, or it does Shannon entropy when \( S \) is the set of all probability distributions. This \( DS(\varphi; \Lambda^*) \) contains the classical chaos degree and the quantum one. Now in the case of \( S = \mathcal{S} \), we simply denote \( DS(\varphi; \Lambda^*) \) by \( D(\varphi; \Lambda^*) \).

We use this degree to judge whether the dynamics \( \Lambda^* \) causes a chaos or not as follows:

**Definition 2** For a given state \( \varphi \), a dynamics \( \Lambda^* \) causes chaos iff \( D > 0 \), and it does cause chaos (i.e., may be called stable) iff \( D = 0 \).

In usual quantum system including classical discrete system, \( A \) is the set \( B(H) \) of all bounded operators on a Hilbert space \( H \) and \( \mathcal{S} \) is the set of all density operators, in which an extremal decomposition of \( \rho \in \mathcal{S} \) is a Schatten decomposition \( \rho = \sum_k p_k E_k \) (i.e., \( \{E_k\} \) are one dimensional orthogonal projections with \( \sum E_k = I \)), so that the entropic chaos degree is written as

\[ D(\rho; \Lambda^*) \equiv \inf \left\{ \sum_k p_k S(\Lambda^* E_k); \{E_k\} \right\}, \] (2)

where the infimum is taken over all possible Schatten decompositions and \( S \) is von Neumann entropy. Note that in classical discrete case, the Schatten decomposition is unique \( \rho = \sum_k p_k \delta_k \) with the delta measure \( \delta_k (j) \equiv \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases} \), and the entropic chaos degree is written by

\[ D(\varphi; \Lambda^*) = \sum_k p_k S(\Lambda^* \delta_k), \] (3)

where \( \rho \) is the probability distribution of the orbit obtained from a dynamics of a system, and that dynamics generates the channel \( \Lambda^* \), whose details are discussed in Section 2. Before closing this section we remark that in the
case when a certain decomposition of the state $\rho$ is fixed, say $\rho = \sum_k p_k \rho_k$, $\rho_k \in \mathcal{G}$, the entropic chaos degree (ECD in the sequel) becomes

$$D(\varphi; \Lambda^*) = \sum_k p_k S(\Lambda^* \rho_k)$$

(4)

without the infimum.

3 Entropic chaos degree for classical dynamics

Let us consider a map $f$ on $I \equiv [a, b] \subset \mathbb{R}^N$ with $x_{n+1} = f(x_n)$ (a difference equation), $x_0 \equiv x$. Take a finite partition of $I$: $I \equiv \bigcup_k B_k$ with $B_i \cap B_j = \emptyset$ ($i \neq j$). The state $\rho^{(n)}$ at time $n$ determined by the difference equation is the probability distribution $p^{(n)} \equiv (p_i^{(n)})$ of the orbit $\{f^n(x); n = 0, 1, \cdots\}$, that is,

$$p_i^{(n)} = \frac{1}{m+1} \sum_{k=n}^{m+n} 1_{B_i}(f^k x),$$

(5)

where $1_A$ is the characteristic function and $B \equiv \{B_i\}$. When the initial value $x$ is distributed due to a measure $\nu$ on $I$, the above $p_i^{(n)}$ is given as

$$p_i^{(n)} = \frac{1}{m+1} \int_I \sum_{k=n}^{m+n} 1_{B_i}(f^k x) \, d\nu.$$  

(6)

The joint probability distribution $(p_{ij}^{(n,n+1)})$ between the time $n$ and $n+1$ is defined by

$$p_{ij}^{(n,n+1)} = \frac{1}{m+1} \sum_{k=n}^{m+n} 1_{B_i}(f^k x) 1_{B_j}(f^{k+1} x)$$

(7)

or

$$p_{ij}^{(n,n+1)} = \frac{1}{m+1} \int_I \sum_{k=n}^{m+n} 1_{B_i}(f^k x) 1_{B_j}(f^{k+1} x) \, d\nu.$$  

(8)
Then the channel $\Lambda^*_n$ at $n$ is defined by

$$
\Lambda^*_n,B \equiv \left( \frac{p^{(n,n+1)}_{ij,B}}{p^{(n)}_{i,B}} \right) \Rightarrow p^{(n+1)}_B = \Lambda^*_n,B p^{(n)}_B ,
$$

(9)

and the chaos degree at time $n$ is given by

$$
D (x; f) \equiv \sup_{\{B_i\}} D \left( p^{(n)}_B; \Lambda^*_n,B \right) = \sup_{\{B_i\}} \sum_i p^{(n)}_{i,B} S(\Lambda^*_n,B \delta_i) = \sum_{i,j} p^{(n,n+1)}_{ij,B} \log \frac{p^{(n)}_{i,B}}{p^{(n,n+1)}_{ij,B}},
$$

(10)

Therefore once we find a suitable partition $B$ such that $D \left( p^{(n)}_B; \Lambda^*_n,B \right)$ becomes positive, we conclude that the dynamics $f$ produces chaos.

This entropic chaos degree has been applied to several dynamical maps such logistic map, Baker’s transformation and Tinkerbel map, and it could explain their chaotic characters\[17, 9\]. This chaos degree has several merits to usual measures such as Lyapunov exponent.

## 4 Entropic chaos degree for quantum dynamics

Let us consider von Neumann-Liouville equation

$$
i \frac{d\rho_t}{dt} = [H(t), \rho_t]
$$

(11)

with the initial condition

$$
\rho_s = \rho
$$

(12)

The solution of (11) is given in the form

$$
\rho_{t,s} = \Lambda^*_{t,s} \rho = U_{t,s} \rho U^*_{t,s}
$$

(13)
where

\[ U_{t,s} = T \exp \left( -i \int_{s}^{t} H \left( t' \right) dt' \right) \]  \hspace{1cm} (14)

and it follows from (14) and (13) that the relations

\[ U_{t,s}U_{s,u} = U_{t,u}, \quad U_{t,t} = I, \quad t \geq s \geq u \]  \hspace{1cm} (15)

and

\[ \Lambda^{*}_{t,s} \Lambda^{*}_{s,u} = \Lambda^{*}_{t,u}, \quad \Lambda^{*}_{t,t} = id, \quad t \geq s \geq u \]  \hspace{1cm} (16)

hold.

From (15) and (16) one finds that

\[ U_{t+\Delta,s} = U_{t+\Delta,t}U_{t,s} \]  \hspace{1cm} (17)

and

\[ \Lambda^{*}_{t+\Delta,s} = \Lambda^{*}_{t+\Delta,t} \Lambda^{*}_{t,s}. \]  \hspace{1cm} (18)

That is, the relation between \( U_{t+\Delta,s} \left( \Lambda^{*}_{t+\Delta,s} \right) \) and \( U_{t,s} \left( \Lambda^{*}_{t,s} \right) \) is linear one.

Let us put

\[ s = 0, \quad t = n\tau \]
\[ U_{n} = U_{n\tau,0}, \quad \Lambda^{*}_{n} = \Lambda^{*}_{n\tau,0} \]  \hspace{1cm} (19)
\[ V_{n} = T \exp \left( -i \int_{(n-1)\tau}^{n\tau} H \left( t' \right) dt' \right). \]  \hspace{1cm} (20)

Then one finds

\[ U_{n} = V_{n}V_{n-1} \cdots \cdots V_{1} \]  \hspace{1cm} (21)
\[ \Lambda^{*}_{n}\rho = V_{n}V_{n-1} \cdots \cdots V_{1}\rho \left( V_{n}V_{n-1} \cdots \cdots V_{1} \right)^{*} = U_{n}\rho U^{*}_{n} \]  \hspace{1cm} (22)

The time dependence of \( H \left( t \right) \) is generally very complicated.
One can consider as an example the following case. Let $H_1, \ldots, H_n$ be selfadjoint operators such that $I, F_1, \ldots, F_n$ are linearly independent. Suppose that

$$H(t) = \sum_{k=1}^{N} c_k(t) F_k + H_0 \quad \text{for} \quad t \geq 0$$

where $c_1(t), \ldots, c_n(t)$ are solutions of the equations

$$\frac{d c_k(t)}{dt} = f_k(c_1(t), \ldots, c_N(t)) \quad k = 1, \ldots, N \quad (23)$$

with initial conditions

$$c_k(0) = c_k^0$$

and it is assumed that

$$H(t) = H_0 \quad \text{for} \quad t < 0.$$ 

The equations (23) can lead to chaos. However its discrete version is quite complicated.

Another example can be given in the form

$$H(t) = H_0 + H_n \quad \text{for} \quad (n-1)\tau \leq t < n\tau, \quad n = 1, 2, \ldots \quad (24)$$

$$H(t) = H_0 \quad \text{for} \quad t < 0$$

and suppose that $H_n$ are determined as follows

$$H_{n+1} = G(H_n) \quad (25)$$

or more explicitly

$$H_n = \sum_{k=1}^{N} c_n^{(k)}(t) F_k$$

where

$$c_{n+1}^{(k)}(t) = g_k(c_1^n, \ldots, c_N^n), \quad n = 1, 2, \ldots$$
and $F_k$ are as above.

Thus the channel describing a discrete dynamics of a quantum systems as in the above examples is written as

$$\Lambda_{n+1}^* = \Theta_n^* \Lambda_n^*.$$

and

$$\rho_{n+1} = \Lambda_{n+1}^* \rho = \Theta_n^* \rho_n.$$

It follows from the above examples that operators $V_n$ and consequently maps $\Theta_n^*$ may inherit some chaotic properties of the equations (23) or (25). However it seems that the only way to investigate the properties of $\Theta_n^*$ is to calculate expectation values of some observables. The simple choice is to consider one observable.

Let $X \in B(\mathcal{H})$ be an observable and $\rho \in \mathcal{S}(\mathcal{H})$ be an state of the system.

Let us consider the sequence

$$x_n = \text{tr} X \Theta_n^* \rho,$$

where

$$\Theta_n^* \rho = V_n \rho V_n^*$$

and

$$V_n = T \exp \left( -i \int_{(n-1)\tau}^{n\tau} H(t') \, dt' \right).$$

In the special case (24) one has

$$V_n = e^{-i\tau (H_0 + H_n)}, \quad n = 1, 2, \ldots$$

The sequence $\{x_n\}$ characterize the changes of $\Theta_n^*$. Let $X$ and $\rho$ be fixed and take a proper $N$. Let $r, R$ be

$$r = \inf_n \text{tr} X \Theta_n^* \rho = \inf_n x_n, \quad R = \sup_n \text{tr} X \Theta_n^* \rho = \sup_n x_n, \quad n = 1, 2, \ldots, N.$$
In this case, the interval $I$ in Section 2 can be written by

$$I = [r, R].$$

(30)

Using the same way as we mentioned in Section 3, one can calculate the entropic chaos degree $D$.

One can generalize the chaos degree taking the set $X_1, X_2, \ldots, X_L (L \in \mathbb{N})$ of observables.

The quantum entropic chaos degree is applied to analyze quantum spin systems and quantum baker’s map, and we could measure the chaos of these systems.

5 Properties of entropic chaos degree for quantum dynamics

In this section we explain some properties of the entropic chaos degree for quantum dynamics. We have the following theorem.

**Theorem 3** For any $\rho \in \mathcal{S}(\mathcal{H})$, $\Lambda_n^* : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$, $n = 1, 2, \cdots$, we have:

1. Let $U$ be a unitary operator. If $\Lambda_n^* \rho = U^n \rho U^{*n}$, then $D(\rho; \Lambda_n^*) = 0$,
2. If $\Lambda_n^* \rho = \rho$, then $D(\rho; \Lambda_n^*) = 0$,
3. Let $\rho_0$ be a fixed state on $\mathcal{H}$. If $\Lambda_n^* \rho = \rho_0$, then $D(\rho; \Lambda_n^*) = 0$,
4. Let $\{P_n\}$ be one dimensional projections such that $\sum_k P_k = I$. If $\Lambda_n^* \rho = \sum_{k_1, k_n} P_{k_1} P_{k_{n-1}} \cdots P_{k_1} \rho P_{k_1} \cdots P_{k_{n-1}} P_{k_n}$, then $D(\rho; \Lambda_n^*) = 0$.

**Proof:** Let $X$ be an observable, $r, R$ be real numbers given by (29), $n, m$, $M$ be large natural numbers and

$$B_i = \left[\frac{i}{M} (R - r) + r, \frac{i + 1}{M} (R - r) + r\right].$$

9
for $i = 0, \cdots, M - 2$,

$$B_{M-1} = \left[ \frac{M - 1}{M} (R - r) + r, R \right].$$

(1) By a direct calculation, we obtain

$$x_k = \text{tr} X \Theta_k^* \rho = \text{tr} X U \rho U^*, \quad k = 1, 2, \cdots.$$  

Let $B_{n_0}$ be a domain including the point $\text{tr} X U \rho U^*$. Then we have

$$x_k \in B_{n_0}$$

because $x_k$ is independent of $k$. One finds that

$$p^{(n)}_{n_0, B} = \frac{1}{m + 1} \sum_{k=n}^{m+n} 1_{B_{n_0}} (x_k) = 1 \quad (31)$$

and

$$p^{(n,n+1)}_{n_0,n_0,B} \equiv \frac{1}{m + 1} \sum_{k=n}^{m+n} 1_{B_{n_0}} (x_k) 1_{B_{n_0}} (x_{k+1}) \quad (32)$$

(31) and (32) imply that

$$D \left( p^{(n)}_{B}; \Lambda_n^{*} \right) = 0$$

for any partition $B$, hence

$$D (\rho; \Lambda_n^{*}) = 0.$$

(2) Note that

$$x_k = \text{tr} X \Theta_k^* \rho = \text{tr} A \rho, \quad k = 1, 2, \cdots.$$  

for $k \in \mathbb{N}$.

Let $B_{n_1}$ be a domain including the point $tr (X \rho)$. Then we have

$$x_k \in B_{n_1}$$
because $x_k$ is independent of $k$. One finds that

One can show that

$$p^{(n)}_{n_1B} = p^{(n,n+1)}_{n_1n_1B} = 1$$  \hspace{1cm} (33)

This equation (33) implies

$$D(\rho; \Lambda^*_n) = 0.$$  \hspace{1cm} (3)

A direct calculation yields

$$x_k = \text{tr} X \Theta_k^* \rho = \text{tr} X \rho_0, \quad k = 1, 2, \cdots.$$  

for $k \in \mathbb{N}$.

Let $B_{n_2}$ be a domain including the point $\text{tr} A \rho_0$. Then we have

$$x_k \in B_{n_2}$$

because $x_k$ is independent of $k$. It follows that

$$p^{(n)}_{n_2B} = p^{(n,n+1)}_{n_2n_2B} = 1$$

Similarly (2), we obtain

$$D(\rho; \Lambda^*_n) = 0.$$  \hspace{1cm} (4)

By a direct calculation, we have

$$x_k = \text{tr} X \Theta_k^* \rho = \text{tr} X \left( \sum_j P_j \rho P_j \right), \quad k = 1, 2, \cdots.$$  

for $k \in \mathbb{N}$.

Let $B_{n_3}$ be a domain including the point $\text{tr} X \left( \sum_j P_j \rho P_j \right)$. Then we have

$$x_k \in B_{n_3}$$
because \( x_k \) is independent of \( k \). One can show that
\[
      p_{n_3, B}^{(n)} = p_{n_3 n_3, B}^{(n, n+1)} = 1
\]
(34)

This equation (34) implies
\[
      D (\rho; \Lambda^*_n) = 0. \blacksquare
\]

6 Entropic chaos degree for quantum dynamics in spin 1/2 system

Let us study a spin 1/2 system in external magnetic field. The Hamiltonian of the system has the form
\[
      H (t) = \frac{e \hbar}{2mc} \vec{\sigma} \vec{B} (t) = \frac{1}{2} \vec{\sigma} \vec{y}_t
\]
(35)

where
\[
      \vec{y}_t = \frac{e \hbar}{mc} \vec{B} (t) \in \mathbb{R}^3.
\]
(36)

Following the scheme presented in Section 4, one can choose operators \( F_1, F_2, F_3 \) in (23) as \( \sigma_1, \sigma_2, \sigma_3 \), then the equations (23) will take the form
\[
      \frac{d y^{(k)}_t}{dt} = Y_k (\vec{y}_t), \quad k = 1, 2, 3, \quad \vec{y}_{t_0} = \vec{y}_0
\]
(37)

and
\[
      \vec{y}_t = \vec{c} \quad for \quad t < 0.
\]

In what follows we will proceed according to the second example (23) one has
\[
      \vec{y}_{n+1} = \vec{Z} (\vec{y}_n), \quad n = 0, 1, 2, \ldots
\]
(38)

Denoting
\[
      \omega_n = |\vec{y}_n|, \quad \vec{c}_n = \frac{\vec{y}_n}{|y_n|},
\]
one has
\[ \vec{y}_n = \omega_n \vec{e}_n. \]
In this case
\[ V_n = \exp \left( -i \frac{\omega_n \tau}{2} (\vec{e}_n, \vec{\sigma}) \right). \]
Instead of (38) we can rewrite
\[ \vec{e}_{n+1} = \vec{e} [\vec{e}_n], \quad \vec{e}_n = (e_n^{(1)}, e_n^{(2)}, e_n^{(3)}). \tag{39} \]
Any observable can be written in the form \( X = \vec{a} \vec{\sigma}, \vec{a} \in \mathbb{R}^3 \), then one finds
\[ V_n^* \vec{a} \vec{\sigma} V_n = \Theta_n \vec{a} \vec{\sigma} = \vec{\sigma} R(\omega \tau, \vec{e}_n) \vec{\sigma} \]
where
\[ R(\omega \tau, \vec{e}_n) \vec{\sigma} = [\vec{\sigma} - \vec{e}_n (\vec{e}_n \vec{a})] \cos \omega \tau + \vec{e}_n (\vec{e}_n \vec{a}) - (\vec{e}_n \wedge \vec{a}) \sin \omega \tau. \]
Using (27), (28), (39) and the standard form of state in \( \mathbb{C}^2 \)
\[ \rho = \frac{1}{2} (I + \vec{\sigma} \vec{\rho}), \quad \vec{\rho} \in \mathbb{R}^3, \quad \| \vec{\rho} \| \leq 1 \tag{40} \]
One finds that
\[ x_n (X, \rho) = (\vec{\rho}, R(\omega \tau, \vec{e}_n) \vec{a}) = x_n (\vec{a}, \vec{\rho}) \tag{41} \]
\[ = \vec{\rho} [\vec{a} - \vec{e}_n (\vec{e}_n \vec{a})] \cos \omega \tau + \vec{\rho} \vec{e}_n (\vec{e}_n \vec{a}) - \vec{\rho} (\vec{e}_n \wedge \vec{a}) \sin \omega \tau \tag{42} \]
The special examples of (39) are the followings:

**Example 4**
\[ e_{n+1}^{(1)} = (1 - \cos \theta) e_n^{(3)} - (\sin \theta) e_n^{(2)} \]
\[ e_{n+1}^{(2)} = (1 - \cos \theta) e_n^{(3)} + (\sin \theta) e_n^{(1)} \]
\[ e_{n+1}^{(3)} = \cos \theta + (1 - \cos \theta) e_n^{(3)} \]

The last relation can be written in the form
\[ z_{n+1} = 4b z_n (1 - z_n) \]
where
\[ z_n = \frac{1}{2} (1 - e_n^{(3)}), \quad b = \left( \frac{\sin \theta}{2} \right)^2. \]
Example 5

\begin{align*}
e^{(1)}_{n+1} &= \left[ -1 + 2 \left( 1 - \cos \theta \right) \left( e^{(3)}_n \right)^2 \right] e^{(1)}_n - 2 \left( \sin \theta \right) e^{(3)}_n e^{(2)}_n \\
e^{(2)}_{n+1} &= \left[ -1 + 2 \left( 1 - \cos \theta \right) \left( e^{(3)}_n \right)^2 \right] e^{(2)}_n - 2 \left( \sin \theta \right) e^{(3)}_n e^{(1)}_n \\
e^{(3)}_{n+1} &= \left( 1 - a \right) e^{(3)}_n + a \left( e^{(3)}_n \right)^2
\end{align*}

with

\begin{align*}
a &= 2 \left( 1 - \cos \theta \right), \quad 0 \leq a \leq 4
\end{align*}

Choosing \( \vec{a} = (0, 0, 1) = \vec{a}_0 \) and \( \vec{\rho} = (0, 0, 1) = \vec{\rho}_0 \), we have

\begin{align*}
x_n (\vec{a}_0, \vec{\rho}_0) &= \left[ 1 - \left( e^{(3)}_n \right)^2 \right] \cos \omega \tau + \left( e^{(3)}_n \right)^2 \\
&= \cos \omega \tau + \left( e^{(3)}_n \right)^2 \left( 1 - \cos \omega \tau \right)
\end{align*}

It is clear that in order to investigate the properties of the sequence \( x_n (\vec{a}_0, \vec{\rho}_0) \), it is enough to consider the sequence

\begin{align*}
x'_n &= \left( e^{(3)}_n \right)^2
\end{align*}

which will be chaotic.

The entropic chaos degree \( D \) for the above two examples are shown in Fig. 6.1 and 6.2 under initial value \( e^{(3)}_0 = 1/\sqrt{3} (i = 1, 2, 3) \).
References

[1] S. Akashi, The asymptotic behavior of $\varepsilon$-entropy of a compact positive operator, J.Math.Anal.Appl., 153, 250-257 (1990).

[2] K.T. Alligood, T.D. Sauer and J.A. Yorke, Chaos-An Introduction to Dynamical Systems-, Textbooks in Mathematical Sciences, Springer (1996).

[3] R. Alicki, Quantum geometry of noncommutative Bernoulli shifts, Banach Center Publications, Mathematics Subject Classification 46L87 (1991).

[4] F. Benatti, Deterministic Chaos in Infinite Quantum Systems, Springer (1993).

[5] R.L. Devaney, An Introduction to Chaotic dynamical Systems, Benjamin (1986).

[6] G.G.Emch, H.Narnhofer, W.Thirring and G.L.Sewell, Anosov actions on noncommutative algebras, J.Math.Phys., 35, No.11, 5582-5599 (1994).

[7] H. Hasegawa, Dynamical formulation of quantum level statistics, Open Systems and Information dynamics, 4, 359-377 (1997).

[8] R.S. Ingarden, A. Kossakowski and M. Ohya, Information Dynamics and Open Systems, Kluwer Academic Publishers (1997).

[9] K. Inoue, M. Ohya and K. Sato, Application of chaos degree to some dynamical systems, Chaos, Solitons & Fractals, 11, 1377-1385 (2000).

[10] K.Inoue, M.Ohya and I.V.Volovich, Semiclassical properties and chaos degree for the quantum baker’s map

[11] K.Inoue, M.Ohya and I.V.Volovich, On correspondence between quantum and classical expectation value for baker’s Map

[12] A. Kossakowski, M. Ohya and N. Watanabe, Quantum dynamical entropy for completely positive maps, Infinite Dimensional Analysis, Quantum Probability and Related Topics, 2, No.2, 267-282 (1999).
[13] W.A. Majewski, Does quantum chaos exist? a quantum Lyapunov exponents approach, Preprint.

[14] M. Ohya, Some aspects of quantum information theory and their applications to irreversible processes, Reports on Mathematical Physics, 27, No.2, 19-47 (1989).

[15] M. Ohya, Information dynamics and its applications to optical communication processes, Lecture Note in Physics, 378, 81-92 (1991).

[16] M. Ohya, State change, complexity and fractal in quantum systems, Quantum Communications and Measurement, Plenum Press, New York, 309-320 (1995).

[17] M. Ohya, Complexity and fractal dimensions for quantum states, Open Systems and Information Dynamics, 4, 141-157 (1997).

[18] M. Ohya, Complexities and their applications to characterization of chaos, International Journal of Theoretical Physics, 37, No.1, 495-505 (1998).

[19] M. Ohya and D. Petz, Quantum Entropy and Its Use, Springer-Verlag, TMP (1993).
Fig. 6.1. The entropic chaos degree $D$ for example 1.
Fig. 6.2. The entropic chaos degree $D$ for example 2.