Isogeny in superstable groups

James Freitag

Abstract We study and develop a notion of isogeny for superstable groups inspired by the notion in algebraic groups and differential algebraic notions developed by Cassidy and Singer. We prove several fundamental properties of the notion. Then we use it to formulate and prove a uniqueness results for a decomposition theorem about superstable groups similar to one proved by Baudisch. Connections to existing model theoretic notions and existing differential algebraic notions are explained.

Keywords Superstable groups · Isogeny

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1 Introduction

The main goal of this note is to develop notions of strong connectedness, almost simplicity and isogeny for the class of superstable groups, in analogy to the related notions for algebraic groups. In this paper, notions like simple, quasi-simple, and almost simple are group theoretic notions and have nothing to do with the similarly named model theoretic property of first order theories. We will then use the notion of isogeny to prove a result of the form “the construction is unique up to isogeny”. For a familiar example from algebraic groups, see the Jordan-Hölder theorem. The other guiding example will be the Cassidy-Singer analysis of differential algebraic groups.

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J. Freitag (E) Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720-3840, USA e-mail: freitagj@gmail.com
The results here specialize to long known results in algebraic groups. The algebraic groups case also inspired the work of Cassidy and Singer in the differential setting. Many of the proofs in this paper are translations of proofs from these cases, generalized and modified appropriately. One interesting note is that while $U$-rank specializes to (Krull) dimension in algebraic groups, the notion of dimension that Cassidy and Singer use in their analysis is known to not be $U$-rank in differentially closed fields. Cassidy and Singer use the gauge of the differential algebraic group, that is the pair $(a_\tau, \tau)$, where $a_\tau$ is the typical differential dimension and $\tau$ is the differential type. From these differential birational invariants, one can formulate an upper bound for Lascar rank in differential fields. There is no known lower bound for Lascar rank in terms of these invariants [10]. We will define a similar notion of gauge in the superstable setting.

Strong results on the structure of infinite rank superstable groups were first established in [2]. Further model theoretic analysis continued over the next several years and is recalled in [9]. Our purpose here is somewhat different from the existing model theoretic analysis. The basic notion we consider is isogeny. The notion is interesting in its own right, and we prove several results about the properties of isogeny. We hope to illustrate how to import techniques from difference-differential algebraic groups into superstable groups, even when (as in this case) the results are not necessarily generalizations. This translation goes via thinking about Lascar rank in the way that differential algebraists think about the gauge of a group. Further, we hope this will lead to future work in model theory of fields with more general operators in which Lascar rank is either difficult to understand and calculate or is simply not available.

The decomposition theorem proved here is close to the one proved by Baudisch [1]. The quotients in our decomposition are almost simple and might have infinite centers; as stated, our decomposition is coarser than Baudisch’s decomposition, but the general techniques here can be easily adapted to prove his decomposition result. Baudisch’s paper does not mention the issue of uniqueness of the decomposition, which is one of the main issues in this paper. The style and techniques for proving the decomposition theorem in this paper follow developments from algebraic groups and differential algebraic groups much more closely than then development contained in [1].

2 Notation and preliminaries

Throughout this note, $G$ is a group definable or even type-definable in a monster model of a superstable theory $T$. Definable always means perhaps with parameters and perhaps in $T^{eq}$. We will heavily use the notion of Lascar rank on types, denoted $RU(p)$. Though this is a rank on types, one can abuse notation and denote, by $RU(G) = RU(p_G)$, where $p_G$ is a generic type of $G$. For certain technical reasons, this might be somewhat problematic when dealing with arbitrary definable sets, but not when dealing with (type-)definable groups. So, when we write $RU(X)$, where $X$ is a (type-)definable set (usually a group). In this case, the Lascar rank of the set is the supremum of the Lascar ranks of the complete types which include the formula “$x \in X$.”

A group is called type-definable if it is an intersection of definable subgroups; the groups which appear will be (implicitly) assumed to be type-definable unless otherwise
noted. We will be assuming standard notation from superstable group theory except where we define new notation. Poizat’s Stable Groups [9] is suggested as a reference for the notation which is not explicitly defined. The reader is advised that we will make frequent use of the Lascar inequality in particular. We emphasize that we are working in some fixed superstable theory $T$, and are calculating Lascar rank within that theory.

**Definition 2.1** Define $\tau_U : \{\text{Def}(G)\} \to \mathbb{O}_n$ to be the highest power $\alpha$, such that $\omega^\alpha$ appears in the Cantor normal form of the Lascar rank of definable set in question.

Let
\[ S = \{ H \subset G \mid RU(G/H) < \omega^\alpha \} = \{ H \subset G \mid \tau_U(G/H) < \alpha \}. \]

It can be shown that $S$ is closed under intersection (for more details and proofs, see [2, Section IV]). This justifies the following definition.

**Definition 2.2** Define the $\alpha$-connected component of the identity, $G_\alpha$, to be the minimal subgroup of $S$. $G$ is $\alpha$-connected if $G = G_\alpha$; equivalently, for every proper type-definable normal subgroup $H$ of $G$, $\tau_U(G/H) \geq \alpha$.

If $G$ is $\alpha$-connected, this implies that $RU(G) \geq \omega^\alpha$, and for the remainder of this section, we assume $RU(G) \geq \omega^\alpha$. We will be particularly interested in the following special case in which we let $\alpha = \tau_U(G)$:

**Definition 2.3** The strongly connected component of the identity of $G$ is defined to be $G_\alpha$ with $\alpha = \tau_U(G)$; that is, for every proper type-definable normal subgroup $H$ of $G$, $\tau_U(G/H) = \tau_U(G)$.

We will also be interested in the following strengthening of strongly connected:

**Definition 2.4** $G$ is almost simple if there is no proper type-definable normal subgroup $H$ of $G$, with $\tau_U(H) = \tau_U(G)$.

**Remark 2.5** Much of the above notation is not standard, but it is convenient for the purposes here. It is inspired by the notation of [3]. The definition of $\alpha$-connected agrees with that of [2]. The following open question depends on the relationship between Lascar rank and gauge in differential algebraic groups:

**Question 2.6** Is a strongly connected differential algebraic group (in the sense of differential gauge, [3]) actually strongly connected in the sense of Lascar rank?

It is known, by results of Berline and Lascar [2, Section IV], that $G$ is $\alpha$-connected if and only if $RU(G) = \omega^\alpha \cdot n_1 + \cdots + \omega^\alpha \cdot n_k + \omega^\alpha \cdot n$ and $G$ is connected (in the traditional sense that there is no type-definable subgroup of finite index). So, when we consider strongly connected groups, we are limited to groups of monomial valued $U$-rank. In that case, being strongly connected is equivalent to being connected.

**Proposition 2.7** Suppose that $G$ is strongly connected. Every type-definable normal subgroup, $N$, with $\tau_U(N) < \tau_U(G)$ is central.
Proof Consider the map $\alpha : G \times N \to N$ given by $(g, a) \mapsto gag^{-1}$. For any fixed $a \in N$, $\alpha_a(g) := gag^{-1}$ is a definable map from $G$ to $N$, such that $\alpha_a$ is constant on left cosets of the centralizer of $a$, $Z_G(a)$. So, there is a definable map $\beta$, such that the diagram commutes,

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha_a} & N \\
\pi \downarrow & & \downarrow \beta \\
G / Z_G(a) & \nearrow & \\
\end{array}
\]

We note that $\alpha_a(g) = \alpha_a(h)$ implies that $h^{-1}g \in Z_G(a)$. Thus, $\beta$ is injective. But, then $\tau_U(G / Z_G(a)) \leq \tau_U(N) < \tau(G)$, so $Z_G(a)$ must be all of $G$, since otherwise we have found a subgroup such that the $U$-rank of the coset space has leading monomial in its Cantor normal form less than $\tau_U(G)$. This means that the rank of $Z_G(a)$ is at least equal to the leading monomial. On the face of things, this should not force $Z_G(a)$ to be all of $G$, since we do not know that $Z_G(a)$ is a normal subgroup of $G$. But, in general, one now knows that the set $S$ of subgroups $H$ of $G$ such that the coset space has rank less than $\omega^\theta$ is nonempty. The set $S$ is closed under intersections and the minimal element will be a type-definable characteristic (so normal) subgroup of $G$ which shows that $G$ is not strongly connected. So, it must be that $G = Z_G(a)$.

**Proposition 2.8** The image of a strongly connected group under a definable homomorphism is strongly connected or trivial.

*Proof* Suppose that the image is nontrivial and not strongly connected. Then taking the inverse image of the definable subgroup of the image which shows non-strong connectedness would show the non-strong connectedness of $G$ itself.

### 3 Isogeny

We remind the reader that we are working within a monster model of a superstable theory. As usual, all of the groups and maps between them are (type-)definable (perhaps with parameters) in the monster model. The notion of strongly connected plays the role that connected plays in algebraic groups. Almost simple plays the role of quasi simple. Now we define isogeny in this setting.

**Definition 3.1** Suppose that $G$ and $H$ are strongly connected. Then a definable group homomorphism $\phi : G \to H$ is an isogeny if $\phi$ is surjective and $\tau_U(Ker\phi) < \tau_U(G)$. We say that $H_1$ and $H_2$ are isogenous if there is some $G$ and there are $\phi_i : G \to H_i$ which are isogenies.

We are not generally dealing with definability problems in this paper, so even if we do not explicitly say so, groups and homomorphisms are assumed to be (type-) definable.

**Proposition 3.2** Let $G_1$ and $G_2$ be strongly connected groups. The following are equivalent:

\[\text{Springer}\]
1. There is a strongly connected group $H$ and isogenies $\phi_i : H \to G_i$:

\[
\begin{array}{ccc}
\phi_1 & \quad & \phi_2 \\
\downarrow & & \downarrow \\
G_1 & & G_2 \\
\end{array}
\]

2. There is a strongly connected group $K$ and isogenies $\psi_i : G_i \to K$:

\[
\begin{array}{ccc}
\psi_1 & \quad & \psi_2 \\
\downarrow & & \downarrow \\
G_1 & & G_2 \\
\end{array}
\]

Proof Let $H$ and $\phi_1$ be as in condition 1). Let $H_1 = \phi_1(\ker \phi_2)$ and $H_2 = \phi_2(\ker \phi_1)$. Then $H_1 = \phi_1(\ker \phi_1 \ker \phi_2)$, and $H_2 = \phi_2(\ker \phi_1 \ker \phi_2)$. Then

\[
\begin{align*}
G_1/H_1 &= \phi_1(H)/\phi_1(\ker \phi_2) = \phi_1(H)/\phi_1(\ker \phi_1 \ker \phi_2) = H/(\ker \phi_1 \ker \phi_2). \\
G_2/H_2 &= \phi_2(H)/\phi_2(\ker(\phi_2)) = \phi_2(H)/\phi_2(\ker \phi_1 \ker \phi_2) = H/(\ker \phi_1 \ker \phi_2).
\end{align*}
\]

So, let $K = H/(\ker \phi_1 \ker \phi_2)$. $K$, being the image of a strongly connected group $H$ is strongly connected. Further, $\tau_U(\ker \phi_1) < \alpha$, so $\tau_U(\ker \phi_1 \ker \phi_2) < \alpha$. But, then letting $\psi_i$ be the projection map $G_i \to G_i/H_i = K$, we have shown that $\psi_i$ is an isogeny.

Now, assume condition 2). We let $H = \{(g_1, g_2) \in G_1 \times G_2 \mid \psi_1(g_1) = \psi_2(g_2)\}$. Then there are natural surjective projections $\phi_i : H \to G_i$. But, then we see that $\tau_U(H) \geq \tau_U(G_i)$. As the kernel of the projection maps, $\phi_i$, are contained in $\ker \psi_1 \times \ker \psi_2$, the Lascar rank of the kernels of the maps is less than $\omega^\alpha$, since both of the groups in the product are (by virtue of $\psi_i$ being an isogeny). So, $\phi_i$ is an isogeny.

Let $H^*$ be the strongly connected component of $H$. It must be the case that $\phi_i(H^*) = G_i$, because if not, $\phi_i(H^*)$ is a proper subgroup of $G_i$ which witnesses that $G_i$ is not strongly connected (this follows from a standard application of the Lascar inequalities). Thus, $H^*$ satisfies the hypotheses of condition 1.

Proposition 3.3 Isogeny is an equivalence relation on the strongly connected type-definable groups. Let $G$ and $K$ be type-definable, strongly connected, isogenous groups. Then,

1. There is a bijection, $r$, between the type-definable strongly connected subgroups $G_1 \leq G$ with $\tau_U(G_1) = \tau_U(G)$ and those $K_1 \leq K$ with $\tau_U(K_1) = \tau_U(K)$.

2. Let $G_1, G_2 \leq G$ and $K_1, K_2 \leq K$ be strongly connected. Suppose that $r(G_1) = K_1$ and $r(G_2) = K_2$.

\[
\begin{align*}
G_1 &\leq G_2 \text{ if and only if } K_1 \leq K_2, \\
G_1 &\triangleleft G_2 \text{ if and only if } K_1 \triangleleft K_2.
\end{align*}
\]

3. Let $G_1, G_2, K_1, K_2$ be as in 2). If $G_1 \triangleleft G_2$, then $\tau_U(G_2/G_1) = \alpha$, $G_2/G_1$ is strongly connected, and $G_2/G_1$ is isogenous to $K_2/K_1$. 
4. Products of isogenous groups are isogenous.

Proof Reflexivity and symmetry of the isogeny relation are clear. Now, suppose that $H_1$ is isogenous to $H_2$ and $H_2$ is isogenous to $H_3$. Then, we have a diagram of isogenies with strongly connected $K_1$ and $K_2$:

$$
\begin{array}{c}
\phi_1 & \phi_2 & \psi_1 & \psi_2 \\
K_1 & H_2 & K_2 & H_3 \\
\end{array}
$$

But, by Proposition 3.2, we get the following diagram, with isogenies and strongly connected $L$:

$$
\begin{array}{c}
\pi_1 & \pi_2 \\
K_1 & K_2 \\
\phi_1 & \phi_2 & \psi_1 & \psi_2 \\
H_1 & H_2 & H_3 \\
\end{array}
$$

For $H_1$ to be isogenous to $H_3$, we would require that $\phi_1 \circ \pi_1$ and $\psi_2 \circ \pi_2$ are isogenies. Surjectivity is obvious. We check that the kernel of either of the compositions is of $U$-rank less than $\omega^\alpha$. The fiber of $\pi_1$ over any point of $K_1$ is a coset of the kernel of $\pi_1$. Therefore, by the Lascar inequality, the kernel of the map $\phi_1 \circ \pi_1$ is bounded above by $RU(ker(\phi_1)) \oplus RU(ker(\pi_1))$. Of course, this implies that $RU(ker(\phi_1)) < \omega^\alpha$. So, $\tau_U(ker(\phi_1 \circ \pi_1)) < \alpha$. Then, by a symmetric argument on $\psi_2 \circ \pi_2$, both maps are isogenies.

Now we prove item one of the proposition. Suppose that we have the following diagram:

$$
\begin{array}{c}
\phi_G & \phi_K \\
G & K \\
\end{array}
$$

Then we claim there is a bijection between the sets of strongly connected subgroups $G_1 \leq G$ and $K_1 \leq K$ with $\tau_U(G_1) = \tau_U(G) = \tau_U(K_1) = \tau_U(K)$. We will now set up a correspondence between these two types of subgroups. Let $r(G_1) = K_1$ if there is a type-definable strongly connected subgroup $H_1 \leq H$ with $\phi_G(H_1) = G_1$ and $\phi_K(H_1) = K_1$. To show that the map $r$ is well-defined and bijective, by the symmetry of the situation for $G$ and $K$, it suffices to show that there is a unique choice of strongly connected subgroup $H_1 \leq H$ with $\phi_G(H_1) = G_1$ and that $\phi_G(H_1)$ is a strongly connected subgroup of $G$. The second part follows from Proposition 2.8.
For the first part, there is one natural candidate, namely, the strongly connected component of the inverse image of $G_1$, which we will denote $\phi^{-1}_G(G_1)$. Since $\tau_U(G_1) = \tau_U(H) = \alpha$, we know that $\tau_U(\phi^{-1}_G(G_1)) = \alpha$. So, at least $\phi^{-1}_G(G_1)$ is a type-definable group which is strongly connected and of suitable rank. We claim that $\phi_G(\phi^{-1}_G(G_1)) = G_1$. Of course, the image is contained in $G_1$. Suppose that $RU(G_1) = \omega^\alpha \cdot n$. Then $RU(\phi^{-1}(G_1)) = \omega^\alpha \cdot n$. That the Lascar rank of the inverse image is at least this big is trivial. That it is at most this big follows from the Lascar inequality and the fact that $RU(ker \phi_G \cap \phi^{-1}_G(G_1)) \leq RU(ker \phi_G) < \omega^\alpha$.

So, the image of $\phi^{-1}_G(G_1)$ is a strongly connected subgroup of $G_1$ of the same leading monomial $U$-rank. This implies that the image is $G_1$. Now, we claim that there is no other choice of $H_1$. If there was, it would have to be a proper type-definable subgroup of $\phi^{-1}_G(G_1)$. But, we know that all such subgroups have leading monomial $U$-rank less than $\phi^{-1}_G(G_1)$ by virtue of strong connectedness. Of course, then the image of such a group can not be all of $G_1$, simply by virtue of rank.

Now we move on to item two. All of the subgroups in the following paragraph are strongly connected. Suppose that $r(G_1) = K_1$ and $r(G_2) = K_2$. Now suppose that $G_1 \leq G_2$. Then $\phi^{-1}_G(G_1) \leq \phi^{-1}_G(G_2)$, because strongly connected subgroups of $\phi^{-1}_G(G_2)$ must be contained in the strongly connected component.

Of course, this implies that $K_1 = \phi_K(\phi^{-1}_G(G_1)) \leq \phi_K(\phi^{-1}_G(G_2)) = K_2$.

Now we assume that $G_1 \triangleleft G_2$. Then $\phi^{-1}_G(G_1) \triangleleft \phi^{-1}_G(G_2)$. Since the strongly connected component of a group is definably characteristic, $\phi^{-1}_G(G_1) \triangleleft \phi^{-1}_G(G_2)$. So, $\phi^{-1}_G(G_1) \triangleleft \phi^{-1}_G(G_2)$.

Now we prove the third item. The maps induced by $\phi_G$ and $\phi_K$ on the quotient $H_2/H_1$ are isogenies, since they are surjective onto their image, their kernels are quotients of the kernels of isogenies, and $\tau_U(H_2/H_1) = \tau_U(H)$ and $\tau_U(G_2/G_1) = \tau_U(G)$. For the final item, first note that products of strongly connected groups are strongly connected. Products of isogenous groups are isogenous, because taking a product of the isogeny maps gives an isogeny map (surjectivity is clear and the $U$-rank of the kernel is bounded by the Cantor sum of the $U$-rank of the kernels in the product).

Remark 3.4 For more details on the following brief remarks, see [9]. In superstable theories, all types are coordinatized by regular types. One often considers the equivalence relation of nonorthogonality of the regular types. The strongly connected groups considered here have generics which are a product of regular types, each nonorthogonal to a type of rank $\omega^\alpha$. The equivalence relation of nonorthogonality is much coarser than isogeny. The isogeny relation on almost simple groups is finer, and takes into account the group theoretic properties of the definable group in ways which nonorthogonality does not.

Let $G$ be a (non-commutative) quasi-simple algebraic group. In algebraically closed fields, the nonorthogonality relation is rather trivial, since any two positive rank types are nonorthogonal. The isogeny relation is nontrivial, and it matches the classical definition. Even in settings in which the nonorthogonality relation is highly nontrivial (for instance differentially closed fields), the isogeny relation is finer. Of course, almost
simplicity is not a sufficient condition for a connected group to have regular generic type. In the setting of differential algebraic groups, is it necessary?

**Lemma 3.5** Let $G$ be a strongly connected and non-commutative group. Then $\tau_U([G, G]) = \tau_U(G)$.

**Proof** Let $H := [G, G]$. Implicit in the lemma is the fact that $H$ is type-definable. This follows from $\alpha$-indecomposability theorem of [2]. In fact, if $G$ is actually definable, then so is the commutator subgroup. Suppose $\tau_U(H) < \tau_U(G)$, then $H \leq Z(G)$ by Proposition 2.7. Choose $a \in G \setminus Z(G)$, and define

$$c_a : G \to G \quad x \mapsto axa^{-1}x^{-1}.$$ 

Since $H \leq Z(G)$, $c_a$ is a definable homomorphism from $G$ to $H$. So, the kernel of the map is a subgroup of $G$ with the property that $RU(Ker(c_a)) \oplus RU(H) \geq RU(G)$ by the Lascar inequality. As $\tau_U(H) < \tau_U(G)$, this implies that $RU(Ker(c_a)) \geq \omega^\alpha n$. This is impossible since $G$ is strongly connected where $RU(G) = \omega^\alpha \cdot n$.

**Remark 3.6** Even in the case that $[G, G]$ (or another normal abstract subgroup) is not definable, one can consider the smallest type-definable subgroup, $H$, containing the $[G, G]$. One can still show $H$ is normal. It appears that Cassidy and Singer [3] need this fact for their Lemma 2.24, since they did not know until [4] that the commutator subgroup is definable. I will offer a proof. Take $A \triangleleft G$ where there are no definability conditions on $A$. Then, let $H$ be the smallest definable subgroup containing $A$ (differentially closed fields are $\omega$-stable, so we have the descending chain condition on definable groups). Now, consider a $G$-conjugate $H^g$ of $H$. Since $A$ is normal, $H^g$ is still a definable subgroup containing $A$. So, $H \cap H^g$ is a definable subgroup containing $A$. By the minimality of $H$, $H = H^g$. Thus, $H \triangleleft G$.

**Proposition 3.7** Let $G$ and $H$ be isogenous strongly connected groups. Both are almost simple or neither is. Both are commutative or neither is.

**Proof** The almost simple portion of the proposition is an easy consequence of items 1 and 2 of Proposition 3.3.

We have the following diagram, since $G$ and $H$ are isogenous, $G \xrightarrow{\phi_G} H \xleftarrow{\phi_H} K$

$G$ commutative implies $K$ is commutative.

Suppose for a contradiction that $H$ is not commutative. Let $H_1 := [H, H]$. We know that $\tau_U(H_1) = \tau_U(H)$ by Lemma 3.5. But, $\tau_U(ker\phi_H) < \tau_U(H_1)$, so the image is nontrivial. On the other hand, $\phi_H(H_1) \subseteq [K, K] = \{e\}$, a contradiction.

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The main reason for the notion of isogenous in this paper is to utilize it to prove uniqueness results of the form “up to isogeny” similar to the case of algebraic groups or differential algebraic groups. In particular, we will start, in the next section with a theorem similar to Baudisch’s Jordan-Hölder style decomposition based on Berline-Lascar analysis of superstable groups.

4 Jordan-Hölder Theorem

The proof of the following theorem follows the proof of the Jordan-Hölder theorem in the case of partial differential fields due to Cassidy and Singer. We should mention again that though Lascar rank does not specialize to the notions of dimension that Cassidy and Singer use, it shares enough of the same properties to make the proofs work similarly after the correct translation of the statements is known.

**Theorem 4.1** Let $G$ be a strongly connected superstable group. Then there exists a normal sequence
\[ 1 = G_r \triangleleft G_{r-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G \]
For each $i \in \{0, \ldots, r - 1\}$:

1. $G_i$ is strongly connected and $\tau_U(G_i) = \tau_U(G)$.
2. $RU(G_i) > RU(G_{i+1})$.
3. $G_i / G_{i+1}$ is almost simple and $\tau_U(G_i / G_{i+1}) = \tau_U(G_i)$, where $RU(G_i) = \omega^\alpha \cdot n$ and $RU(G_{i+1}) = \omega^\alpha \cdot m$.

If
\[ 1 = H_s \triangleleft H_{s-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G \]
is another sequence which satisfies the above properties, then the sequences must be of the same length ($r = s$). There is a permutation (call it $\sigma$) of the indices so that the quotients are isogenous. That is, $G_{\sigma(i)} / G_{\sigma(i)+1}$ is isogenous to $H_i / H_{i+1}$.

**Proof** If $G$ is already almost simple, then there is nothing to do. If this is not the case, then there is a nonempty collection of proper, type-definable, normal subgroups $H$ of $G$ with $\tau_U(G) = \tau_U(H)$. Pick any such $H$ so that if $RU(H) = \omega^\alpha \cdot n_0 + \beta$; then there is no other $H_1$ in the collection so that $RU(H) = \omega^\alpha \cdot n_1 + \beta_1$, where $n_1 > n_0$.

We let $G_1$ be the strongly connected component of $H$, $G_1 = H^{\langle \alpha \rangle}$. $G_1$ is a definably characteristic subgroup of $H \triangleleft G$, so $G_1 \triangleleft G$. Since the Cantor sum on ordinals is equal to the sum when the ordinals in question are monomials of the same “degree”,

\[ RU(G / G_1) = \omega^\alpha \cdot (n - m), \quad RU(G) = \omega^\alpha \cdot n \quad RU(G_1) = \omega^\alpha \cdot m. \]

Suppose that the quotient $G / G_1$ is not almost simple. Then, there is a proper, type-definable, normal subgroup $H_1 \triangleleft G / G_1$ with $\tau_U(G) = \tau_U(G / G_1) = \tau_U(H_1)$. But, then the preimage of $H_1$ under the quotient map is a subgroup of $G$ which violates the maximality condition with which $H$ was chosen, namely, the leading monomial of the Lascar rank of the preimage of $H_1$ is larger than that of $H$. So, the quotient is almost simple. Continuing in this way, we can find $G_2, G_3, \ldots$ having the properties prescribed in the statement of the theorem.

The proof of uniqueness proceeds in a similar manner to the proof of Cassidy-Singer decomposition in the differential field context. In turn, that proof follows the
one in [5, chapter 1, Section 3]. So, suppose we have two sequences as above \( \langle G_i \rangle_{i \leq r} \) and \( \langle H_j \rangle_{j \leq s} \). For each pair \((i, j)\) with \( i < r \) and \( j \leq s \), we define:

\[
G_{i,j} := G_{i+1}(H_j \cap G_i).
\]

Note that

\[
G_{i,s} := G_{i+1}.
\]

Then,

\[
\begin{align*}
1 & \lhd G_{r-1,s-1} \lhd G_{r-1,s-2} \lhd \cdots \lhd G_{r-1} \lhd G_{r-2,s-1} \lhd \\
& \cdots \lhd G_1 \lhd G_{0,s-1} \lhd \cdots \lhd G_{0,0} = G.
\end{align*}
\]

Of course, one can apply the definition in the opposite way as well, so get a refinement of \( \langle H_j \rangle \),

\[
H_{j,i} = H_{j+1}(G_i \cap H_j)
\]

for \( j < s \) and \( i \leq r \). By [5, chapter 1, Theorem 3.3], \( G_{i,j} / G_{i,j+1} \) is isomorphic to \( H_{j,i} / H_{j,i+1} \) for \( i < r \) and \( j < s \). Further, the isomorphism is definable.

**Claim 4.2** For \( i < r \), there is precisely one \( j < s \) so that \( \tau_U(G_{i,j} / G_{i,j+1}) = \tau_U(G) \). Further, for this specific value of \( j \), we have that \( G_{i,j} / G_{i,j+1} \) is isogenous to \( G_i / G_{i+1} \).

Assume that we have established the claim. Then, since the symmetric statement holds for the \( H_{j,i} \), we know that \( r = s \) and the theorem follows.

Now we prove the claim. By the Lascar inequality,

\[
RU(G_{i+1}) + \sum_{j=s-1}^{0} RU(G_{i,j} / G_{i,j+1})
\]

\[
\leq RU(G_i) \leq RU(G_{i+1}) \oplus \bigoplus_{j=s-1}^{0} RU(G_{i,j} / G_{i,j+1})
\]

So, for some \( j \),

\[
\tau_U(G_{i,j} / G_{i,j+1}) = \alpha.
\]

So choose a maximal such \( j < s \). Then

\[
\tau_U(G) = \tau_U(G_{i,j} / G_{i,j+1}) \leq \tau_U(G_{i,j} / G_{i+1}) \leq \tau_U(G_i / G_{i+1}) = \tau_U(G).
\]

Then note that for each \( k \leq j \),

\[
\tau_U(G_{i,j} / G_{i+1}) \leq \tau_U(G_{i,k} / G_{i+1}) \leq \tau_U(G_i / G_{i+1}).
\]

Thus, for all \( k \leq j \), \( \tau_U(G_{i,k} / G_{i+1}) = \tau_U(G) \).
Since \( G_i/G_{i+1} \) is almost simple, we get \( G_{i,0} = G_i = G_{i,1} \). Continuing in the same way, we can see

\[
G_i = G_{i,0} = \cdots = G_{i,j}.
\]

We have the canonical projection map

\[
G_i/G_{i+1} \rightarrow G_{i,j}/G_{i,j+1}.
\]

The kernel is a proper normal subgroup of an almost simple group, so the map is an isogeny. These equalities, along with the maximality of \( j \), imply that the \( j \) with the property \( \tau_u(G_{i,j}/G_{i,j+1}) = \alpha \) is unique.

5 Final remarks

In this section, we take \( \mathcal{U} \models DC F_{0,m} \) to be a differentially closed field in \( m \) commuting derivations, \( \Delta = \{ \delta_1, \ldots, \delta_m \} \), which we assume to be sufficiently saturated. In differential fields, there are examples, due to Cartan, Cassidy and Singer, which show that some sort of weaker notion of correspondence than isomorphism is necessary for the sort of theorem of the previous section to be true. We discuss the model theoretic aspects of an example of Cassidy and Singer here. For the next portion of the discussion, we assume \( m = 2 \), that is \( \Delta = \{ \delta_1, \delta_2 \} \). The purpose of this assumption is mainly to simplify the discussion, but the following examples could be developed in much greater generality. Let \( a \in \mathcal{U} \) be such that \( \delta_1(a) = 1 \) and \( \delta_2(a) = 0 \). Let \( G_1 \) be the zero set of \( \delta_1^2 z - \delta_2 z \). Let \( G_2 \) be the zero set of \( \delta_1 z - a \delta_2 z \). We consider these groups as subgroups of the additive group, \( \mathbb{C}_a(\mathcal{U}) \). Both of these differential algebraic groups have Lascar rank \( \omega \).

One can quickly see that \( \omega \) is a lower bound for the Lascar rank as follows. The solution sets to the equations are infinite dimensional vector spaces over the field \( \{ c \in \mathcal{U} \mid \delta_1(c) = \delta_2(c) = 0 \} \). To see this, consider, for \( j \in \mathbb{N} \), the definable subspaces of \( G_1 \) given by \( \delta_1^j z = c \). Then, for all \( j \),

\[
\{ z \in G_1 \mid \delta_1^j z = c \} \subsetneq \{ z \in G_1 \mid \delta_1^{j+1} z = 0 \} \subsetneq G_1
\]

The strictness of the containments follows easily from the axioms for differential fields. Further, consider the generic type of the zero set of \( G_1 \) over some differential field \( K \). Take \( c \in \mathcal{U} \) in a differential field extension of \( K \) such that \( c \) is transcendental over \( K \). Then the generic type of \( \{ z \in G_1 \mid \delta_1^j z = c \} \) is a forking extension of the generic type of \( G_1 \). Further, this type has Lascar rank \( j \) as it is the generic of a vector space of dimension \( j \) over the definable field \( \{ c \in \mathcal{U} \mid \delta_1(c) = \delta_2(c) = 0 \} \).

Seeing that \( \omega \) is an upper bound takes slightly more work. We will work with a slightly more general class of examples. Let us show \( Z(\delta_1 y - f(y)) \) where \( f \in K[\delta_2] \) is a linear \( \delta_2 \)-polynomial, is almost simple. A proper subgroup of the additive group, \( H \), is a kernel of a linear operator \( g \in K[\Delta] \) such that \( g(y) \notin \{ \delta_1 y - f(y) \} \) in \( K \{ y \} \) [10, Proposition 3.45]. We may assume that \( g \in K[\delta_2] \) (and is linear), since we are
only interested in working modulo the relation $\delta_1 y = f(y)$, where $f$ is linear as a $\delta_2$-polynomial. If $g$ has order $d$, then for any $y \in H$, we have $k(y, \delta_2 y, \delta_2^2 y, \ldots) = k(y, \delta_2 y, \ldots, \delta_2^{d-1} y)$. So, $H$ has $\Delta$-type 0, which implies finite Lascar rank. In partial differential fields, the $\Delta$-type of differential algebraic group is the degree of the Kolchin polynomial of a generic point on the group; $\Delta$-type zero means that the Kolchin polynomial is a constant, which is equivalent to saying that if $a$ is a generic point on the group over some differential field $K$, then $\{\delta_1^{j_1} \delta_2^{j_2} (a) \mid j_i \in \mathbb{N}\}$ has finite transcendence degree over $K$. In differentially closed fields, forking extensions correspond to proper differential algebraic subvarieties. So, we have proved that every forking extension has $\Delta$-type zero (that is, the differential field generated by a realization of the type is a finite transcendence degree field extension of the base), which implies finite Lascar rank; this implication comes from the fact that the transcendence degree of the differential function field generated by a generic point on the variety is an upper bound for the Krull dimension of the differential ideal [6, Theorem 5.2.2 gives this upper bound; however, note that the lower bound in that theorem is incorrect].

$G = G_1 + G_2$ is strongly connected and the series decomposition as above may be given $1 \triangleleft G_1 \triangleleft G$ or $1 \triangleleft G_2 \triangleleft G$. Cassidy and Singer [3] show that $G_1$ is not isomorphic to either $G_2$ or $G/G_2$. However, $G_1$ is isogenous to $G/G_2$.

All currently known non-commutative almost simple differential algebraic groups actually have finite center. Such groups are, by the results of [4], perfect central extensions of the $C'$ points of an algebraic group, where $C'$ is some definable subfield. Let $H$ denote the $C'$ points of the algebraic group. So, any almost simple $G$ has the following exact sequence:

$$1 \to Z(G) \to G \to H \to 1$$

Now, in an arbitrary superstable theory $T$ work with an arbitrary definable perfect central extension of an algebraic group $H$, perhaps restricted to a definable subfield, which is almost simple. Is $G$ a finite extension of $H$? The assumptions are weak enough so that one should guess that the answer is no. However, examples which show the negative conclusion would be of great interest if they could be translated to the setting of differential algebraic groups.

There are suitable theories of numerical polynomials in other algebraic settings from which a theory similar to that of Cassidy and Singer might be developed. An example of model theoretic interest is the setting of difference-differential fields [7]. In that setting, there is no nontrivial lower bound Lascar rank in terms of the appropriate generalization of differential gauge (there are definable sets of Morley rank one with infinite difference-differential transcendence degree). The results in this paper would have to be generalized to the supersimple setting in order to compare potential model theoretic and algebraic notions of strong connectedness.

In the setting of differential fields, the decomposition series of Cassidy-Singer theorem are connected to the factorization of differential operators and with differential galois theory. Subsequent work [8, for instance] related to that of Cassidy and Singer has used this connection. Of course, there is no analogue of this connection in the level of generality which we currently work; however such connections are plausible in various other model theory of fields settings. We hope that this paper and future
work on generalizing the setting might set the stage for the transfer of the ideas of the Cassidy–Singer theorem to the groups definable in other settings.

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