FEET IN ORTHOGONAL-BUEKENHOUT-METZ UNITALS

N. ABARZÚA, R. POMAREDA, AND O. VEGA

ABSTRACT. Given an Orthogonal-Buekenhout-Metz unital $U_{\alpha,\beta}$, embedded in $PG(2,q^2)$, and a point $P \notin U_{\alpha,\beta}$, we study the set of feet, $\tau_P(U_{\alpha,\beta})$, of $P$ in $U_{\alpha,\beta}$. We characterize geometrically each of these sets as either $q+1$ collinear points or as $q+1$ points partitioned into two arcs. Other results about the geometry of these sets are also given.

1. Preliminares

Most definitions and theorems in this section may be found in [5]. We direct the reader to this source for more information, and details, about projective planes.

Let $GF(q^2)$ be the field with $q^2$ elements, where $q = p^n$ with $p$ prime (we will always consider $p$ to be odd in this article), and $n \in \mathbb{N}$. Throughout this article we will use

$$GF(q^2) = \{a + cb; \ a, b \in GF(q), \ and \ c^2 = w \in GF(q)\},$$

and we will write $\overline{x} = x^q$, $T(x) = x + \overline{x}$ and $N(x) = x\overline{x}$, for all $x \in GF(q^2)$.

We let $V$ be a 3-dimensional vector space over $GF(q^2)$ and we consider the projective plane $\Pi = PG(2,q^2)$, defined by letting its points to be the 1-dimensional subspaces of $V$ and its lines be the 2-dimensional subspaces of $V$. A point $P$ of $\Pi$ will be denoted by

$$P = [a, b, c],$$

where $(a, b, c)$ is a vector generating the subspace defining $P$. If $l$ is a line of $\Pi$ then it will be denoted by

$$l = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x, y, z]^t,$$

where $(x, y, z)$ is a vector that is orthogonal (using the standard dot product) to the 2-dimensional subspace defining $l$. The incidence in $\Pi$ is given by natural set-theoretic containment. Thus,

$$P \in l \iff \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \iff ax + by + cz = 0.$$

It is known that $\Pi$ is a projective plane of order $q^2$. Hence, the following properties hold in $\Pi$:

1. Every line of $\Pi$ contains exactly $q^2 + 1$ points.
(2) Every point of $\Pi$ is on exactly $q^2 + 1$ lines.
(3) The number of points, and the number of lines, in $\Pi$ is $q^4 + q^2 + 1$.

In order to motivate the concept of unital, we define another important object.

**Definition 1.** A *blocking set* $\beta$ is a subset of points of $\Pi$ such that every line of $\Pi$ contains at least one point in $\beta$. A minimal blocking set is a blocking set in which removing any of its points never yields a blocking set.

**Remark 1.** The collection of points on a line $l$ of $\Pi$ is a blocking set. We will say that a blocking set containing all the points on a line is a trivial blocking set.

A good summary of the basics on blocking sets may be found in Chapter 13 of [7]. The following result may be found there.

**Theorem 1.** Blocking sets exist in $\Pi$. Moreover, if $\beta$ is a minimal blocking set of $\Pi$ then $|\beta| \leq q^3 + 1$.

A very special kind of a largest possible minimal blocking set of $\Pi$ is the object we want to focus our attention from now on.

**Definition 2.** A *unital* in $\Pi$ is a set $U$ of $q^3 + 1$ points of $\Pi$ such that every line of $\Pi$ intersects $U$ in exactly 1 or $q+1$ points. Lines of $\Pi$ will be called tangent or secant to $U$ depending on whether they intersect $U$ in 1 or $q+1$ points, respectively.

**Remark 2.** Unitals may be defined in a much more general way but in this article we will focus only on unitals embedded in $\Pi$. So, our definition has been written with this purpose in mind. We refer the reader to [3] for a detailed exposition about unitals and for the concepts we use in this article that we may fail to explain in detail.

Two standard examples of unitals are

1. The set of absolute points of a non-degenerate unitary polarity of $\Pi$: $H = \{(x, y, z) \in \Pi; N(x) + N(y) + N(z) = 0\}$ is a unital in $\Pi$, called *classical*.
2. Buekenhout [4] proved that, for $\alpha, \beta \in GF(q^2)$ such that $4N(\alpha) + (\beta - \beta)^2$ is a non-square in $GF(q)$, the set $U_{\alpha,\beta} = \{x, \alpha x^2 + \beta N(x) + r, 1; x \in GF(q^2), r \in GF(q)\} \cup \{P_\infty\}$ is a unital (said to be an *orthogonal-Buekenhout-Metz unital*) in $\Pi$, where $P_\infty = [0, 1, 0]$. Moreover, $\alpha = 0$ if and only if the unital $U_{\alpha,\beta}$ is classical, and $\beta = \overline{\beta}$ if and only if the unital $U_{\alpha,\beta}$ is a union of conics (see [2] or [5], and [6]).

From now on we focus our study on non-classical orthogonal-Buekenhout-Metz unitals $U_{\alpha,\beta}$. So, for the rest of this article we assume $\alpha \neq 0$.

Elementary counting shows that if $U_{\alpha,\beta}$ is an orthogonal-Buekenhout-Metz unital in $\Pi$ and $P \in U_{\alpha,\beta}$ then there is exactly one tangent line to $U_{\alpha,\beta}$ through $P$ and there are exactly $q^2$ secant lines to $U_{\alpha,\beta}$ through $P$. Similarly, if $P \notin U_{\alpha,\beta}$ then there are exactly $q+1$ lines tangent to $U_{\alpha,\beta}$ through $P$ and there are exactly $q^2 - q$ secant lines to $U_{\alpha,\beta}$ through $P$.

**Definition 3.** Let $U_{\alpha,\beta}$ be an orthogonal-Buekenhout-Metz unital and $P$ a point not in $U_{\alpha,\beta}$. Each of the $q+1$ points of $U_{\alpha,\beta}$ that are on a tangent line to $U_{\alpha,\beta}$
through $P$ is said to be a foot of $P$. We will denote the set of feet of $P$ by $\tau_P(U_{\alpha,\beta})$ and we will call it the pedal of $P$.

It is known that $\tau_P(U_{\alpha,\beta})$ has the following properties:

1. $\tau_P(U_{\alpha,\beta})$ is contained in a line of $\Pi$, for all $P \in \Pi \setminus U_{\alpha,\beta}$ if and only if $U_{\alpha,\beta}$ is classical (see Thas [14]). The conditions for this result have been relaxed after Thas’s work, see [1] for a more recent result on this characterization.

2. For $U_{\alpha,\beta}$ non-classical, $\tau_P(U_{\alpha,\beta})$ is contained in a line of $\Pi$ if and only if $P \in \ell_{\infty}$. Note that $\ell_{\infty} \cap U_{\alpha,\beta} = P_{\infty}$ (see, e.g. [3]).

Note that this result implies that every line, different from $\ell_{\infty}$, through $P_{\infty}$ contains a pedal.

Finally, there is a group $G \leq PTL(3, q^2)$ leaving $U_{\alpha,\beta}$ invariant and fixing $P_{\infty}$ such that

1. $G$ is transitive on the set of points of $U_{\alpha,\beta} \setminus \ell_{\infty}$.

2. $G$ is transitive on the points of $\ell_{\infty} \setminus \{P_{\infty}\}$.

3. $G$ has either one or two orbits on the points of $\Pi \setminus (U_{\alpha,\beta} \cup \ell_{\infty})$. Moreover, these orbits are those of $P_{\lambda} = [0, \lambda \epsilon, 1]$, with $\lambda = 1$ or $\lambda = \omega = \epsilon^2$.

2. Intersections of Lines and Pedals

Our objective is to find geometric properties that can describe the pedals of points $P \notin \ell_{\infty}$ in unitals $U_{\alpha,\beta}$, where $\alpha \neq 0$. In particular, we care about how lines of $\Pi$ intersect these sets. Not much is known about pedals in non-classical unitals, albeit the work by Krčadinac and Smoljak is pertinent; in [9] they study all possible configurations for pedals in unitals that are embedded in (not-necessarily Desarguesian) projective planes of order 9 and 16.

Because of the results listed above about the group $G$ we will now only study $\tau_P(U_{\alpha,\beta})$ for $P = P_{\lambda} = [0, \lambda \epsilon, 1]$, with $\lambda = 1$ or $\lambda = \omega$; this decision is justified in the following lemma.

**Lemma 1.** If $\sigma \in G$, $A, C \in \tau_P(U_{\alpha,\beta})$, $\sigma(A) = B$, $\sigma(C) = D$, $Q = \sigma(P)$, then $\sigma(\tau_P(U_{\alpha,\beta})) = \tau_Q(U_{\alpha,\beta})$ and

$$|AC \cap \tau_P(U_{\alpha,\beta})| = |BD \cap \tau_Q(U_{\alpha,\beta})|.$$  

**Proof.** It is easy to see that $\sigma$ preserves the number of points of intersection between lines and $U_{\alpha,\beta}$, and so $\sigma$ maps tangent lines into tangent lines. It follows that $\sigma(\tau_P(U_{\alpha,\beta})) = \tau_Q(U_{\alpha,\beta})$.

Note that $\sigma(AC) = BD$ and that $B, D \in \tau_Q(U_{\alpha,\beta})$. So, if we repeat this argument with $A$ and any other point $E \in AC \cap \tau_P(U_{\alpha,\beta})$ we would get another point in $BD \cap \tau_Q(U_{\alpha,\beta})$. Hence, since $\sigma$ is injective we get one direction of the desired inequality. We obtain the other direction by repeating the argument using $\sigma^{-1}$ instead of $\sigma$. 

We first look at the lines through $P_{\infty}$. It was mentioned earlier that there is a bijection between the set of pedals containing $P_{\infty}$ and the set of lines, different from $\ell_{\infty}$, through this point. We now want to look at how these $q^2$ lines intersect $\tau_{P_{\lambda}}(U_{\alpha,\beta})$. The following remark gives enough information for us to address this issue in the subsequent lemma.
Remark 3. Two distinct pedals can intersect in at most one point, as for every point \( A \in \tau_P(U_{\alpha,\beta}) \cap \tau_Q(U_{\alpha,\beta}) \) we always get that \( A, P \) and \( Q \) are collinear. Moreover, two distinct pedals intersect if and only if they are the pedals of two points on a line tangent to \( U_{\alpha,\beta} \); their intersection is the tangency point.

The following lemma is immediate.

Lemma 2. Let \( \ell \neq \ell_{\infty} \) be a line such that \( P_{\infty} \in \ell \). Then, \( \ell \) is either tangent or exterior to all the pedals not contained in \( \ell \).

The generalization of this lemma to lines intersecting pedals of points not on \( \ell_{\infty} \) is not true (see Section 3). However, Lemma 2 implies that the line \( \ell \neq \ell_{\infty} \) can be partitioned into singletons, all of them in distinct pedals. We are able to prove that result for all other pedals as well.

Lemma 3. Let \( \ell \) be a line that is not tangent to \( U_{\alpha,\beta} \). Then, there are \( q+1 \) distinct pedals intersecting \( \ell \) in singletons, creating a partition of the points in \( \ell \cap U_{\alpha,\beta} \).

Proof. The case when \( P_{\infty} \in \ell \) follows immediately from Lemma 2.

Now, if \( P_{\infty} \notin \ell \) then we use that every point in \( U_{\alpha,\beta} \) is in \( q^2 \) pedals, and \( \ell \) contains \( q+1 \) points of \( U_{\alpha,\beta} \) then for each point in \( \ell \cap U_{\alpha,\beta} \) there are at least \( q^2-(q+1) \) pedals containing no other point of \( \ell \cap U_{\alpha,\beta} \). Hence, using that \( q \geq 3 \) implies that \( q^2-(q+1) \geq q+1 \), we can choose the pedals to create the desired partition.

□

Now our interest shifts to learn about the intersections of lines, not through \( P_{\infty} \), with pedals of points not on \( \ell_{\infty} \).

3. Lines Not Containing \( P_{\infty} \)

In this section we will study lines that do not go through \( P_{\infty} \). We consider the orthogonal-Buekenhout-Metz unital in \( \Pi \)

\[ U_{\alpha,\beta} = \{ [x, \alpha x^2 + \beta N(x) + r, 1] \mid x \in GF(q^2), \ r \in GF(q) \} \cup \{ P_{\infty} \} . \]

The tangent line to \( U_{\alpha,\beta} \) through \([x, \alpha x^2 + \beta N(x) + r, 1]\) is

\[ [-2\alpha x + (\beta - \beta)\overline{x}, 1, \alpha x^2 - \overline{\beta} N(x) - r]^T. \]

In order to study \( \tau_{P_{\lambda}}(U_{\alpha,\beta}) \) we need to determine all \( x \in GF(q^2) \) and \( r \in GF(q) \) such that

\[ [0, \lambda \epsilon, 1] \in [-2\alpha x + (\beta - \beta)\overline{x}, 1, \alpha x^2 - \overline{\beta} N(x) - r]^T \]

which means

\[ \begin{bmatrix} -2\alpha x + (\beta - \beta)\overline{x} \\ 1 \\ \alpha x^2 - \overline{\beta} N(x) - r \end{bmatrix} = 0 \iff \lambda \epsilon + \alpha x^2 - \overline{\beta} N(x) - r = 0 \]

\[ \iff r = \lambda \epsilon + \alpha x^2 - \overline{\beta} N(x). \]

Since \( r \in GF(q) \) we get \( \overline{x} = r \). Hence,

\[ \lambda \epsilon + \alpha x^2 - \overline{\beta} N(x) = \lambda \epsilon + \alpha x^2 - \overline{\beta} N(x) \]

and thus

\[ 2\lambda \epsilon + \alpha x^2 - \overline{x} x^2 + (\beta - \overline{\beta}) N(x) = 0. \]
We let
\[ M_{\alpha,\beta} = \begin{bmatrix} \frac{\alpha}{2} (\beta - \beta) & \frac{\alpha}{2} (\beta - \beta) \\ \frac{\beta - \beta}{\alpha} & \frac{\beta - \beta}{\alpha} \end{bmatrix}, \]
and notice that
\[ 2\lambda + \alpha x^2 - \alpha x^2 + (\beta - \beta) N(x) = 0 \iff 2\lambda + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0. \]

Hence, \( \tau_{\alpha}(U_{\alpha,\beta}) \) is the set of all the points of the form
\[ [x, 2\alpha x^2 + (\beta - \beta) N(x) + \lambda, 1], \]
where \( x \in GF(q^2) \), and
\[ 2\lambda + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0. \]

We can now use Equation (1) to find a different way to represent points in \( \tau_{\alpha}(U_{\alpha,\beta}) \). Notice that
\[ T(\alpha x^2) - \lambda = (\alpha x^2 + (\beta - \beta) N(x) - \lambda \]
\[ = \alpha x^2 + (\beta - \beta) x^2 - \lambda \]
\[ = \alpha x^2 + (2\lambda + \alpha x^2 + (\beta - \beta) N(x)) - \lambda \]
\[ = 2\alpha x^2 + \lambda + (\beta - \beta N(x). \]

Hence, letting \( Q_x = [x, T(\alpha x^2) - \lambda, 1] \) we get
\[ \tau_{\alpha}(U_{\alpha,\beta}) = \left\{ Q_x; x \in GF(q^2), 2\lambda + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0 \right\}. \]

**Remark 4.** Equation (1) can be re-written as
\[ 2\lambda + 2\lambda Im(\alpha x^2) + (\beta - \beta) N(x) = 0 \]
It follows that for two points \( Q_x, Q_y \in \tau_{\alpha}(U_{\alpha,\beta}) \) we get \( N(x) = N(y) \) if and only if \( Im(\alpha x^2) = Im(\alpha y^2) \).

We now introduce some notation. Let
\[ T_{\lambda} = \left\{ x \in GF(q^2); Q_x \in \tau_{\alpha}(U_{\alpha,\beta}) \right\} \]
\[ = \left\{ x \in GF(q^2); 2\lambda + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0 \right\}. \]

It is easy to see that \( x \in T_{\lambda} \) if and only if \( -x \in T_{\lambda} \). Moreover, if \( x, -x \in T_{\lambda} \) then they have the same value of \( r \) associated to them (in the representation of \( Q_x \) and \( Q_{-x} \) as points in \( U_{\alpha,\beta} \)).

**Lemma 4.** Let \( l_{x, -x} \) be the line through \( Q_x \) and \( Q_{-x} \), where \( \pm x \in T_{\lambda} \). If \( Q_y \in l_{x, -x}, \) for some \( y \in T_{\lambda} \), then \( Q_{-y} \in l_{x, -x} \).

**Proof.** The line passing through \( Q_x \) and \( Q_{-x} \) is given by
\[ l_{x, -x} = \begin{bmatrix} 0 \\ -1 \\ T(\alpha x^2) - \lambda \end{bmatrix}. \]
If \( Q_y \in l_{x,-x} \) then

\[
[y, T(\alpha y^2) - \lambda \epsilon, 1] \begin{bmatrix} 0 \\ -1 \\ T(\alpha x^2) - \lambda \epsilon \end{bmatrix} = 0,
\]

which can be simplified to

\[ T(\alpha x^2) - T(\alpha y^2) = 0. \]

On the other hand,

\[
[y, T(\alpha y^2) - \lambda \epsilon, 1] \begin{bmatrix} 0 \\ -1 \\ T(\alpha x^2) - \lambda \epsilon \end{bmatrix} = T(\alpha x^2) - T(\alpha y^2),
\]

which is equal to zero. Hence, \( Q_{-y} \in l_{x,-x} \).

\[ \square \]

**Remark 5.** All lines of the form \( l_{x,-x} \) pass through \([1, 0, 0]\).

We want to learn about the conditions under which the line through \( Q_x \) and \( Q_y \), for \( x, y \in T_\lambda \), contains more points of \( U_{\alpha,\beta} \) besides \( Q_x \) and \( Q_y \).

**Lemma 5.** Let \( l_{x,y} \) be the line through \( Q_x \) and \( Q_y \), for \( x \neq y \) in \( T_\lambda \). If \( Q_z \in l_{x,y} \), \( z \in T_\lambda \), and \( T(\alpha x^2) = T(\alpha y^2) \), then \( Q_{-z} \in l_{x,y} \).

**Proof.** The line \( l_{x,y} \) is given by:

\[
l_{x,y} = \begin{bmatrix} T(\alpha x^2) - T(\alpha y^2) \\ y - x \\ xT(\alpha y^2) - yT(\alpha x^2) + (y - x)\lambda \epsilon \end{bmatrix}.
\]

But, \( T(\alpha x^2) = T(\alpha y^2) \) and \( x \neq y \), so \( l_{x,y} \) is represented by

\[
l_{x,y} = \begin{bmatrix} 0 \\ 1 \\ T(\alpha x^2) - \lambda \epsilon \end{bmatrix}.
\]

If \( Q_z \in l_{x,y} \) then, after routine simplifications, we get

\[ -T(\alpha z^2) + T(\alpha x^2) = 0. \]

It follows that \( l_{x,y} \) can be represented by

\[
l_{x,y} = \begin{bmatrix} 0 \\ 1 \\ T(\alpha z^2) - \lambda \epsilon \end{bmatrix},
\]

which is the line \( l_{z,-z} \). \[ \square \]

**Theorem 2.** Let \( U_{\alpha,\beta} \) be an orthogonal-Buekenhout-Metz unital with \( \alpha \neq 0 \). Let \( Q_x \) and \( Q_y \) be two distinct points in \( \tau_{P_\lambda}(U_{\alpha,\beta}) \), and let \( l_{x,y} \) be the line through them. If \( T(\alpha x^2) \neq T(\alpha y^2) \) then

\[ l_{x,y} \cap \tau_{P_\lambda}(U_{\alpha,\beta}) = \{Q_x, Q_y\}. \]

**Proof.** Suppose that \( l_{x,y} \) contains a point \( Q_z \in \tau_{P_\lambda}(U_{\alpha,\beta}) \), different from \( Q_x \) and \( Q_y \), then there is a \( \mu \in GF(q^2) \setminus \{0\} \) such that

\[ Q_z = Q_x + \mu Q_y. \]
Note that $1+\mu \neq 0$, otherwise $Q_x + \mu Q_y \notin U_{\alpha,\beta}$. Then,

$$[z, T(\alpha z^2) - \lambda x, 1] = Q_x + \mu Q_y = \left[ x + \frac{\mu y}{1+\mu}, T(\alpha x^2) + \mu T(\alpha y^2), \frac{1+\mu}{1+\mu} \right].$$

This expression implies

$$T \left( \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 \right) = \frac{T(\alpha x^2) + \mu T(\alpha y^2)}{1+\mu},$$

which we re-write as:

$$(1+\mu)T \left( \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 \right) = T(\alpha x^2) + \mu T(\alpha y^2).$$

It follows that

$$T \left( \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 - \alpha x^2 \right) = \mu T \left( \alpha y^2 - \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 \right).$$

If

$$T \left( \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 - \alpha x^2 \right) = T \left( \alpha y^2 - \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 \right) = 0$$

then

$$T \left( \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 \right) = T(\alpha x^2) \quad \text{and} \quad T(\alpha y^2) = T \left( \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 \right)$$

and thus $T(\alpha x^2) = T(\alpha y^2)$, which contradicts our hypothesis. It follows that Equation (2) implies $\mu \in GF(q)$, and thus $1+\mu \in GF(q)$.

Since $z \in T_{\lambda}$ and $z = \frac{x + \frac{\mu y}{1+\mu}}{1+\mu}$ we get

$$2\lambda x + \alpha \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 - \alpha x^2 \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^2 + (\beta - \overline{\beta}) \left( \frac{x + \frac{\mu y}{1+\mu}}{1+\mu} \right)^{q+1} = 0,$$

which is equivalent to

$$2\lambda x (1+\mu)^2 + \alpha(x + \frac{\mu y}{1+\mu})^2 - \alpha(x + \frac{\mu y}{1+\mu})^2 + (\beta - \overline{\beta})(x + \frac{\mu y}{1+\mu})^{q+1} = 0.$$

After some simplifications we get

$$\begin{align*}
(2\lambda x + \alpha x^2 - \alpha \overline{x}^2 + (\beta - \overline{\beta})N(x)) + \mu^2 (2\lambda x + \alpha y^2 - \alpha \overline{y}^2 + (\beta - \overline{\beta})N(y)) + 
\mu (4\lambda x + 2\alpha xy - 2\alpha \overline{x} \overline{y} + (\beta - \overline{\beta})(\overline{y} x + \overline{x} y)) = 0.
\end{align*}$$

Since $x, y \in T_{\lambda}$, we know

$$2\lambda x + \alpha x^2 - \alpha \overline{x}^2 + (\beta - \overline{\beta})N(x) = 2\lambda x + \alpha y^2 - \alpha \overline{y}^2 + (\beta - \overline{\beta})N(y) = 0$$

and thus, Equation (3) implies

$$4\lambda x + 2\alpha xy - 2\alpha \overline{x} \overline{y} + (\beta - \overline{\beta})(\overline{y} x + \overline{x} y) = 0,$$

as $\mu \neq 0$.

Since the expression above is zero, independent of the value of $\mu$, every point $Q_x + \mu Q_y$, for $\mu \in GF(q) \setminus \{-1\}$, is in $\tau_{P_{\lambda}}(U_{\alpha,\beta})$. Hence, there are exactly $q$ feet of $P_{L}$ in $U_{\alpha,\beta}$ lying on the same line $\ell$. But since $x \in T_{\lambda}$ implies $-x \in T_{\lambda}$, there
is a \( y \in T_\lambda \) such that \( \ell = l_{y,-y} \). However, by Lemma 3 the number of points on \( \ell \cap \tau_{P_\lambda}(U_{\alpha,\beta}) \) must be even, which contradicts \( q \) being odd. \qed

As of now we have that a secant line cannot intersect \( \tau_{P_\lambda}(U_{\alpha,\beta}) \) in 3 points, and that if the line is not of the form \( l_{x,-x} \) then this intersection contains at most 2 points. Next we obtain a bound for the maximum number of collinear points on \( \tau_{P_\lambda}(U_{\alpha,\beta}) \).

**Theorem 3.** Let \( U_{\alpha,\beta} \) be an orthogonal-Buekenhout-Metz unital with \( \alpha \neq 0 \) and let \( \ell \) be the line through two distinct points in \( \tau_{P_\lambda}(U_{\alpha,\beta}) \). Then, \( \ell \) intersects \( \tau_{P_\lambda}(U_{\alpha,\beta}) \) in at most four points.

**Proof.** Because of the previous results, the only case to consider is when \( \ell \) is of the form \( l_{x,-x} \), for some \( x \in T_\lambda \). Hence, the conditions for \( Q_\ell \in \ell \cap \tau_{P_\lambda}(U_{\alpha,\beta}) \) are

\[
T(\alpha x^2) - T(\alpha z^2) = 0 \quad \text{and} \quad 2\lambda \epsilon + 2\epsilon \text{Im}(\alpha z^2) + (\beta - \overline{\beta}) N(z) = 0.
\]

We let \( z = z_1 + z_2 \epsilon \), \( \alpha = \alpha_1 + \alpha_2 \epsilon \), and \( \beta = \beta_1 + \beta_2 \epsilon \), where \( z_1, z_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in GF(q) \). Using these variables we can re-write Equations (4) as the system

\[
2^{-1}T(\alpha x^2) = \alpha_1 z_1^2 + \alpha_1 w z_2^2 + 2\alpha_2 w z_1 z_2 \\
-\lambda = (\alpha_2 + \beta_2) z_1^2 + (\alpha_2 - w \beta_2) z_2^2 + 2\alpha_1 z_1 z_2
\]

We define the following elements in \( GF(q) \).

\[
A = \alpha_1 \quad B = \alpha_1 w \quad C = 2\alpha_2 w \quad D = -2^{-1}T(\alpha x^2) \\
E = \alpha_2 + \beta_2 \quad F = \alpha_2 - w \beta_2 \quad G = 2\alpha_1 \quad H = \lambda.
\]

These elements allow us to re-write System (5) as the following system of equations with coefficients in \( GF(q) \):

\[
A z_1^2 + B z_2^2 + C z_1 z_2 + D = 0 \\
E z_1^2 + F z_2^2 + G z_1 z_2 + H = 0
\]

If these equations have a common linear factor then we get three linear equations equal to zero, which is three intersecting lines. This yields one solution or a triplet of coinciding lines, which would imply that each equation in System (6) is a multiple of the other. But we know that the equation \( T(\alpha x^2) - T(\alpha z^2) = 0 \) has exactly \( 2(q+1) \) solutions, implying that System (6) has \( 2(q+1) \) solutions, which is more than the maximum number of points on \( \ell \cap \tau_{P_\lambda}(U_{\alpha,\beta}) \), which is \( q+1 \).

In the case the equations in System (6) do not have common factors we can use Bézout’s Theorem for the curves given by

\[
p(z_1, z_2) = A z_1^2 + B z_2^2 + C z_1 z_2 + D \\
q(z_1, z_2) = E z_1^2 + F z_2^2 + G z_1 z_2 + H
\]

and since both are polynomials in two variables with coefficients in \( GF(q) \), and both have degree two, we get that System (6) has at most \( 4 = \deg(p) \cdot \deg(q) \) solutions. \qed

We summarize our results on the size of the intersections between lines and pedals in the following theorem.

**Theorem 4.** Let \( P \neq \ell_{\infty} \) and \( \alpha \neq 0 \). Then,

1. lines in \( \Pi \) intersect \( \tau_P(U_{\alpha,\beta}) \) in exactly 0, 1, 2, or 4 points.
2. the points of \( \tau_P(U_{\alpha,\beta}) \) may be partitioned into two arcs.
Proof. The first part of the theorem follows from Theorems \[2\] and \[3\] and Lemmas \[1\], \[4\] and \[5\].

For the second part, we use Lemma \[1\] to allow ourselves to consider the particular case \(P = P_{\lambda}\). Since we know that only the lines of the form \(l_{x, -x}\) can intersect \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\) in four points, we look at these lines first.

Assume that the lines of the form \(l_{x, -x}\) intersecting \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\) are partitioned as follows: \(\ell_1, \ell_2, \ldots, \ell_n\) intersect \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\) in exactly two points and \(\ell_{n+1}, \ell_{n+2}, \ldots, \ell_t\) intersect \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\) in four. We label the points of \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\) by \(Q_{x_{ij}}\), where \(x_{ij} \in \ell_i\), and \(x_{ij} = -x_{ij}\), for all \(i\). Then, the points of \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\) can be partitioned into the following two arcs

\[
A_1 = \{Q_{x}; x = x_{ij}, \ i = 1, \ldots, t \text{ and } j = 1, 2\} \cup \{P_{\lambda}\}
\]

and

\[
A_2 = \{Q_{x}; x = x_{ij}, \ i = n + 1, \ldots, t \text{ and } j = 3, 4\}.
\]

Note that the partition given is just one of the many possible ones. \(\square\)

In the particular case when \(\beta = \beta\) we can get an even stronger result.

**Corollary 1.** If \(\alpha \neq 0\) and \(\beta = \beta\), then the points of \(\tau_{P}(U_{\alpha, \beta})\) are contained in lines or arcs.

Proof. We already know that \(\tau_{P}(U_{\alpha, \beta})\) is contained in a line when \(P \in \ell_{\infty}\). For when \(P \notin \ell_{\infty}\) we use Lemma \[1\] to restrict ourselves to study the structure of \(\tau_{P_{\lambda}}(U_{\alpha, \beta})\).

Let \(x \neq y\) and let \(l_{x, y}\) be the line through \(Q_x, Q_y \in \tau_{P_{\lambda}}(U_{\alpha, \beta})\). Since \(\beta = \beta\) we get that \(T(ax^2) = 2\alpha x^2 + 2\lambda\) for all \(x \in T_{\lambda}\) (this follows from the argument before Remark \[3\]). Hence, a point \(Q_z \in \tau_{P_{\lambda}}(U_{\alpha, \beta})\) now looks like \(Q_z = [z, 2\alpha z^2 + \lambda \epsilon, 1]\), and the line \(l_{x, y}\) is given by

\[
\begin{bmatrix}
-2\alpha(x + y) \\
1 \\
2\alpha xy - \lambda \epsilon
\end{bmatrix}.
\]

Thus, \(Q_z \in l_{x, y}\) if and only if

\[
2\alpha[(x^2 - y^2)z - (x - y)z^2 - xy(x - y)] = 0.
\]

Since \(\alpha \neq 0\) and \(x \neq y\), this equation reduces to

\[
z^2 - (x + y)z + xy = 0,
\]

which can be re-written as

\[(z - x)(z - y) = 0.\]

The result follows. \(\square\)
4. The Elation Group of $U_{\alpha,\beta}$

Let us consider the collineation group of $U_{\alpha,\beta}$ given by

$$E = \{ E_t : (x, y, z) \mapsto (x, y + tz, z) : t \in GF(q) \}.$$ 

Note that $E$ is an elation group with center $P_\infty$ and axis $\ell_\infty$. It is easy to show that lines of the form $AE_t(A)$ must pass through $P_\infty$, for all $A \notin \ell_\infty$ and $E_t \in E$. Also, since $E$ stabilizes a $\tau_P(U_{\alpha,\beta})$, and if $Q = E_t(P)$, for some $E_t \in E$, we get that $\tau_P(U_{\alpha,\beta})$ and $\tau_Q(U_{\alpha,\beta})$ are disjoint.

It has been mentioned before that every line through $P_\infty$, except from $\ell_\infty$, contains a pedal (of a point on $\ell$). Since it is enough to look at how lines intersect these sets. We take this observation as a ‘suggestion’ to take a closer look at the orbits of pedals under $E$ and to study how lines intersect these sets.

From now on, we will use $O(X)$ to denote the orbit of a set $X$ under the group $E$.

**Lemma 6.** Given a pedal $\tau_P(U_{\alpha,\beta})$, there is a point $Q \in \ell_\infty$ and $q$ lines through $Q$ that partition $O(\tau_P(U_{\alpha,\beta}))$. That is, the intersection of each of these lines with $U_{\alpha,\beta}$ is completely contained in $O(\tau_P(U_{\alpha,\beta}))$.

**Proof.** Because of Lemma 1, it is enough to look at how lines intersect $\tau_P(U_{\alpha,\beta})$. We consider the point $[1, 0, 0]$ and the lines through it. We know that lines, different from $\ell_\infty$, through $[1, 0, 0]$ look like

$$l_\gamma = \begin{bmatrix} 0 \\ -1 \\ \gamma \end{bmatrix}.$$ 

It is easy to see that the orbit of $P_\lambda$ is contained on a line through $P_\infty$. So, we let $O(P_\lambda) = \{ P_1, P_2, \ldots, P_q \}$, where $P_t = E_t(P_\lambda)$, for all $t \in GF(q)$. Moreover, since

$$E_t(x, y, z) = (x, y + tz, z)$$

for all $t \in GF(q)$, and $P_\lambda = [0, \lambda e, 1]$, with $\lambda = 1$ or $\lambda = w$, we obtain

$$P_t = [0, \lambda e + t, 1]$$

Using the arguments at the beginning of Section 3, we get that

$$\tau_P(U_{\alpha,\beta}) = \left\{ R_y : y \in GF(q^2), \ 2\lambda e + \begin{bmatrix} y & \overline{y} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} y \\ \overline{y} \end{bmatrix} = 0 \right\}$$

where $R_y = [y, T(\alpha y^2) - \lambda e + t, 1]$.

Now, the points of intersection (if any) of $\tau_P(U_{\alpha,\beta})$ with $l_\gamma$ are given by

$$0 = [y, T(\alpha y^2) - \lambda e + t, 1] \begin{bmatrix} 0 \\ -1 \\ \gamma \end{bmatrix} = -(T(\alpha y^2) - \lambda e + t) + \gamma$$

which means

$$t = \gamma + \lambda e - T(\alpha y^2).$$

It follows that, if $y$ and $\gamma$ were given, and $\gamma = s - \lambda e$ for some $s \in GF(q)$, then we can always find a $t \in GF(q)$ that satisfies Equation 7. In this case, given a line
for every $y \in GF(q^2)$ such that $R_y \in \tau_{P_1}(U_{\alpha,\beta})$ there is a point of intersection between $l_{s - \lambda \epsilon}$ and $O(\tau_{P_1}(U_{\alpha,\beta}))$.

Hence, for $\gamma \neq s - \lambda \epsilon$, for all $s \in GF(q)$ the intersection is empty and for when $\gamma = ks - \lambda \epsilon$ then the intersection contains $q + 1$ points. Note that for every $s \in GF(q)$ we are able to choose such a $\gamma$, thus we get $q$ lines through $[1, 0, 0]$ intersecting $O(\tau_{P_1}(U_{\alpha,\beta}))$ in $q + 1$ points each. □

We would like to close this paper stating a few open problems.

(1) Do lines intersecting pedals in at least four points exist if and only if $\beta \neq \overline{\beta}$? Corollary \ref{cor:open_prob_1} gives us one direction of this conjecture.

(2) What geometric properties determine when a line of the form $l_{x, -x}$ intersects a given $\tau_P(U_{\alpha,\beta})$ in four points?

(3) When a $\tau_P(U_{\alpha,\beta})$ is partitioned into two arcs (or contained in one arc for the case $\beta = \overline{\beta}$), is any of these arcs contained in a conic?

(4) In how many points does a line intersect the set of points in the orbit of any given $\tau_P(U_{\alpha,\beta})$ under $E$? Lemma \ref{lem:open_prob_4} gives us a partial answer to this, but there are several other lines that are not considered in this result.

(5) Is there a combinatorial characterization for the structure formed by the lines of $\Pi$ and the points on $O(\tau_{P_1}(U_{\alpha,\beta}))$?

REFERENCES

[1] Aguglia, A.; Ebert, G. L. A combinatorial characterization of classical unitals. Arch. Math. (Basel) 78 (2002), no. 2, 166–172.

[2] Baker, R. D.; Ebert, G. L. Intersection of unitals in the Desarguesian plane. Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). Congr. Numer. 70 (1990), 87–94.

[3] Barwick, Susan; Ebert, Gary. Unitals in Projective Planes Springer Monographs in Mathematics, 2008.

[4] F. Buekenhout. Existence of Unitals in Finite Translation Planes of Order $q^2$ with a Kernel of Order $q$. Geom. Dedicata, 5 (1976) 189–194.

[5] Dembowski, Peter. Finite Geometries, Springer-Verlag, Berlin-New York, 1968.

[6] Durante, Nicola; Siciliano, Alessandro. Unitals of $PG(2,q^2)$ containing conics. J. Combin. Des. 21 (2013), no. 3, 101–111.

[7] Hirschfeld, J.W.P. Projective Geometries Over Finite Fields, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1979.

[8] Hirschfeld, J. W. P.; Szőnyi, T. Sets in a finite plane with few intersection numbers and a distinguished point. Discrete Math. 97 (1991), no. 1-3, 229–242.

[9] Krajadinac, Vedran; Smoljak, Ksenija. Pedal sets of unitals in projective planes of order 9 and 16. Sarajevo J. Math. 7(20) (2011), no. 2, 255–264.

[10] Thas, J. A. A combinatorial characterization of Hermitian curves. J. Algebraic Combin. 1 (1992), no. 1, 97–102.