Algorithm for the $k$-Position Tree Automaton Construction

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Abstract. The word position automaton was introduced by Glushkov and McNaughton in the early 1960. This automaton is homogeneous and has $\langle|E|\rangle + 1$ states for a word expression of alphabetic width $|E|$. This kind of automata is extended to regular tree expressions. In this paper, we give an efficient algorithm that computes the Follow sets, which are used in different algorithms of conversion of a regular expression into tree automata. In the following, we consider the $k$-position tree automaton construction. We prove that for a regular expression $E$ of a size $|E|$ and alphabetic width $|E|$, the Follow sets can be computed in $O(|E| \cdot |E|)$ time complexity.

1 Introduction

This paper is an extended version of [8].

Regular expressions, which are finite representatives of potentially infinite languages, are widely used in various application areas such as XML Schema Languages [13], logic and verification, etc. The concept of word regular expressions has been extended to tree regular expressions.

In the case of words, it is agreed that each regular expression can be transformed into a non-deterministic finite automaton. Computer scientists have been interested in designing efficient algorithms for the computation of the position automaton. Three well-known algorithms for the computation of this automaton exist. The first makes use of the notion of star normal form [2] of a regular expression. The second is based on a lazy computation technique [3]. The third is built on the so-called ZPC-structure [16]. The complexity of these three algorithms is quadratic with regard to the size of the regular expression.

This study is motivated by the development of a library of functions for handling rational kernels [5] in the case of trees. The first problem consists of the conversion of a regular expression into a tree automaton.

Recently Kuske and Meinecke [7] proposed an Algorithm to construct an equation automaton [14] from a regular tree expression $E$ with an $O(R \cdot |E|^2)$ time complexity where $|E|$ is the size of $E$ and $R$ is the maximal rank appearing in the ranked alphabet. This algorithm is an adaptation to trees of the one given

* D. Ziadi was supported by the MESRS - Algeria under Project 8/U03/7015.
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by Champarnaud and Ziadi in the case of words [10]. This generalization is interesting although the adaptation of the word algorithm to trees is not obvious at all. Indeed, the Champarnaud and Ziadi Algorithm, for the construction of the set of transitions, is based on the computation of some function called "Follow" which is not yet defined on trees. Notice that the star normal form of a regular tree expression $E$ can not be defined, this notion doesn’t make sense. For these reasons the definition of the Follow function in the case of trees is given in this paper, while an efficient algorithm for its computation (computation of the $k$-position tree automaton) is proposed.

The paper is organized as follows: Section 2 outlines finite tree automata over ranked alphabets, regular tree expressions, and linearized regular tree expressions. Next, in Section 3 the notions of First and Follow of regular expressions and the $k$-position automaton are recalled. Then, in Section 4 we present an efficient algorithm which builds the $k$-position tree automaton with an $O(|E|)$ time complexity. Finally, the different results described in this paper are given in the conclusion.

2 Preliminaries

Let $(\Sigma, r)$ be a ranked alphabet, where $\Sigma$ is a finite set and $r$ represents the rank of $\Sigma$ which is a mapping from $\Sigma$ into $\mathbb{N}$. The set of symbols of rank $n$ is denoted by $\Sigma_n$. The elements of rank 0 are called constants. A tree $t$ over $\Sigma$ is inductively defined as follows: $t = a, t = f(t_1, \ldots, t_k)$ where $a$ is any symbol in $\Sigma_0$, $k$ is any integer satisfying $k \geq 1$, $f$ is any symbol in $\Sigma_k$ and $t_1, \ldots, t_k$ are any $k$ trees over $\Sigma$. We denote by $T_\Sigma$ the set of trees over $\Sigma$. A tree language is a subset of $T_\Sigma$. Let $\Sigma_\Delta = \Sigma \setminus \Sigma_0$ denote the set of non-constant symbols of the ranked alphabet $\Sigma$. A Finite Tree Automaton (FTA) $A$ is a tuple $(Q, \Sigma, Q_T, \Delta)$ where $Q$ is a finite set of states, $Q_T \subset Q$ is the set of final states and $\Delta \subset \bigcup_{n \geq 0} (Q \times \Sigma_n \times Q^n)$ is the set of transition rules. This set is equivalent to the function $\Delta$ from $Q^* \times \Sigma_n$ to $2^Q$ defined by $(q, f, q_1, \ldots, q_n) \in \Delta \iff q \in \Delta(q_1, \ldots, q_n, f)$. The domain of this function can be extended to $(2^Q)^n \times \Sigma_n$ as follows: $\Delta(Q_1, \ldots, Q_n, f) = \bigcup_{(q_1, \ldots, q_n) \in Q_1 \times \cdots \times Q_n} \Delta(q_1, \ldots, q_n, f)$. Finally, we denote by $\Delta^*$ the function from $T_\Sigma \rightarrow 2^Q$ defined for any tree in $T_\Sigma$ as follows: $\Delta^*(t) = \Delta(a)$ if $t = a$ with $a \in \Sigma_0$, $\Delta^*(t) = \Delta(\Delta^*(t_1), \ldots, \Delta^*(t_n), f)$ if $t = f(t_1, \ldots, t_n)$ with $f \in \Sigma_n$ and $t_1, \ldots, t_n \in T_\Sigma$. A tree is accepted by $A$ if and only if $\Delta^*(t) \cap Q_T \neq \emptyset$.

The language $L(A)$ recognized by $A$ is the set of trees accepted by $A$, i.e. $L(A) = \{ t \in T_\Sigma \mid \Delta^*(t) \cap Q_T \neq \emptyset \}$.

For any integer $n \geq 0$, for any $n$ languages $L_1, \ldots, L_n \subset T_\Sigma$, and for any symbol $f \in \Sigma_n$, $f(L_1, \ldots, L_n)$ is the tree language $\{ f(t_1, \ldots, t_n) \mid t_i \in L_i \}$. The tree substitution of a constant $c$ in $\Sigma$ by a language $L \subset T_\Sigma$ in a tree $t \in T_\Sigma$, denoted by $t(c \leftarrow L)$, is the language inductively defined by: $L$ if $t = c$; $\{ d \}$ if $t = d$ where $d \in \Sigma_0 \setminus \{ c \}$; $f(t_1(c \leftarrow L), \ldots, t_n(c \leftarrow L))$ if $t = f(t_1, \ldots, t_n)$ with $f \in \Sigma_n$ and $t_1, \ldots, t_n$ any $n$ trees over $\Sigma$. Let $c$ be a symbol in $\Sigma_0$. The $c$-product $L_1 \cdot c \cdot L_2$ of two languages $L_1, L_2 \subset T_\Sigma$ is defined by $L_1 \cdot c \cdot L_2 = \bigcup_{t \in L_1} \{ t(c \leftarrow L_2) \}$. 

2
The \textit{iterated c-product} is inductively defined for \( L \subset T_\Sigma \) by: \( L^0 = \{e\} \) and \( L^{(n+1)} = L^n \cup L \cdot c \cdot L^n \). The \textit{c-closure} of \( L \) is defined by \( L^c = \bigcup_{n \geq 0} L^n \).

A \textit{regular expression} over a ranked alphabet \( \Sigma \) is inductively defined by \( E = 0, E \in \Sigma_0 \), \( E = f(E_1, \ldots, E_n) \), \( E = (E_1 + E_2) \), \( E = (E_1 \cdot c) \), \( E = (E_1^* \cdot c) \), where \( c \in \Sigma_0 \), \( n \in \mathbb{N} \), \( f \in \Sigma \) and \( E_1, E_2, \ldots, E_n \) are any \( n \) regular expression \( s \) over \( \Sigma \). Parenthesis can be omitted when there is no ambiguity. We write \( E_1 = E_2 \) if \( E_1 \) and \( E_2 \) graphically coincide. We denote by \( \text{RegExp}(\Sigma) \) the set of all regular expression \( s \) over \( \Sigma \). Every regular expression \( E \) can be seen as a tree over the ranked alphabet \( \Sigma \cup \{+, \cdot, ^*, c | c \in \Sigma_0\} \) where + and \( \cdot \) can be seen as symbols of rank 2 and \( \cdot c \) has rank 1. This tree is the syntax-tree \( T_E \) of \( E \). We denote by \( |E|_f \) the number of occurrences of a symbol \( f \) in a regular expression \( E \). The \textit{alphabet width} \( ||E|| \) of \( E \) is the number of occurrences of symbols of \( \Sigma_> \) in \( E \) (\( ||E|| = (\sum_{f \in \Sigma_>} |E|_f) \)). The \textit{size} \( |E| \) of \( E \) is the size of its syntax tree \( T_E \). The \textit{language} \( [E] \) denoted by \( E \) is inductively defined by \( [0] = \emptyset, [c] = \{c\}, [f(E_1, E_2, \ldots, E_n)] = f([E_1], \ldots, [E_n]), [E_1 + E_2] = [E_1] \cup [E_2], [E_1 \cdot c] = [E_1] \cdot c, [E_1^*] = [E_1]^* \) where \( n \in \mathbb{N} \), \( E_1, E_2, \ldots, E_n \) are any \( n \) regular expression \( s \), \( f \in \Sigma \) and \( c \in \Sigma_0 \). It is well known that a tree language is accepted by some tree automaton if and only if it can be denoted by a regular expression [17]. A regular expression \( E \) defined over \( \Sigma \) is \textit{linear} if every symbol of rank greater than 1 appears at most once in \( E \). Note that any constant symbol may occur more than once. Let \( E \) be a regular expression over \( \Sigma \). The \textit{linearized regular expression} \( \widehat{E} \) in \( E \) of a regular expression \( E \) is obtained from \( E \) by marking differently all symbols of a rank greater than or equal to 1 (symbols of \( \Sigma_> \)). The marked symbols form together with the constants in \( \Sigma_0 \) a ranked alphabet \( \text{Pos}_E(E) \) the symbols of which we call \textit{positions}. The mapping \( h \) is defined from \( \text{Pos}_E(E) \) to \( \Sigma \) with \( h(\text{Pos}_E(E)_m) \subset \Sigma_m \) for every \( m \in \mathbb{N} \). It associates with a marked symbol \( f_j \in \text{Pos}_E(E)_m \) the symbol \( f \in \Sigma \) and for a symbol \( c \in \Sigma_0 \) the symbol \( h(c) = c \). We can extend the mapping \( h \) naturally to \( \text{RegExp}(\text{Pos}_E(E)) \rightarrow \text{RegExp}(\Sigma) \) by \( h(a) = a, h(E_1 + E_2) = h(E_1) + h(E_2), h(E_1 \cdot c, E_2) = h(E_1) \cdot h(E_2), h(E_1^*) = h(E_1)^*, h(f_j(E_1, \ldots, E_n)) = f(h(E_1), \ldots, h(E_n)), \) with \( n \in \mathbb{N}, a \in \Sigma_0, f \in \Sigma, f_j \in \text{Pos}_E(E)_m \) such that \( h(f_j) = f \) and \( E_1, \ldots, E_n \) any regular expression \( s \) over \( \text{Pos}_E(E) \).

\section{The \textit{k}-Position Tree Automaton}

The set of positions associated to \( E \) are straightforwardly deduced from the set of symbols associated to \( E \). In order to construct a non-deterministic finite automaton (position tree automaton) associated to the regular expression \( E \) that recognizes \([E]\), we need to define two sets, the set \( \text{First}(\widehat{E}) \) and the set \( \text{Follow}(\widehat{E}, f_j, k) \) for a position \( f_j \in \text{Pos}_E(E)_m \).

In the following of this section, \( E \) is a regular expression over a ranked alphabet \( \Sigma \). The set of symbols in \( \Sigma \) that appear in an expression \( F \) is denoted by \( \Sigma^F \).
In this section, we show how to compute the $k$-position tree automaton of a regular expression $E$, recognizing $\langle E \rangle$. This is an extension of the well-known position automaton $\mathbb{E}$ for word regular expression $s$ where the $k$ represents the fact that any $k$-ary symbol is no longer a state of the automaton, but is exploded into $k$ states. The same method was presented independently by McNaughton and Yamada [9]. Its computation is based on the computations of particular position functions, defined in the following.

In what follows, for any two trees $s$ and $t$, we denote by $s \preceq t$ the relation "$s$ is a subtree of $t$". Let $t = f(t_1, \ldots, t_n)$ be a tree. We denote by root($t$) the root of $t$, by $k$-child($t$) the $k^{th}$ child of $f$ in $t$, that is the root of $t_k$ if it exists, and by Leaves($t$) the set of the leaves of $t$, i.e. $\{s \in \Sigma_0 \mid s \preceq t\}$. We denote by root($t$) the root of $t$, by $k$-child($t$) the $k^{th}$ child of $f$ in $t$, that is the root of $t_k$ if it exists, and by Leaves($t$) the set of the leaves of $t$, i.e. $\{s \in \Sigma_0 \mid s \preceq t\}$.

Let $E$ be a regular expression and $\mathbb{E}$ its linearized form, $1 \leq k \leq m$ be two integers and $f_j$ be a position in $\text{Pos}_E(E)_m$ with $h(f_j) = f$.

The set $\text{First}(E)$ is the subset of $\text{Pos}_E(E)$ defined by $\{\text{root}(t) \in \text{Pos}_E(E) \mid t \in [\mathbb{E}]\}$: The set $\text{Follow}(E, f_j, k)$ is the subset of $\text{Pos}_E(E)$ defined by $\{g_i \in \text{Pos}_E(E) \mid \exists t \in [\mathbb{E}], \exists s \preceq t, \text{root}(s) = f, k$-child($s$) = $g_i\}$; The set $\text{Last}(E)$ is the subset of $\text{Pos}_E(E)_0$ defined by $\text{Last}(E) = \bigcup_{t \in [\mathbb{E}]} \text{Leaves}(t)$.

Example 1. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ be defined by $\Sigma_0 = \{a, b, c\}$, $\Sigma_1 = \{f, h\}$ and $\Sigma_2 = \{g\}$. Let us consider the regular expression $E$ and its linearized form defined by:

$$E = (f(a)^* a b + h(b))^* + g(c, a)^* c (f(a)^* a b + h(b))^*,$$

$$\mathbb{E} = (f_1(a)^* a b + h_2(b))^* + g_3(c, a)^* c (f_4(a)^* a b + h_5(b))^*.$$

The language denoted by $E$ is $[E] = \{b, f_1(b), f_1(f_1(b)), f_1(h_2(b)), h_2(b), h_3(f_1(b)), h_4(h_2(b)), \ldots, g_3(b, a), g_3(g_3(b, a), a), g_3(f_4(b), a), g_3(h_5(b), a), f_4(f_1(b)), f_4(h_5(b), h_5(f_1(b)), h_5(h_5(b))), \ldots\}$.

Consequently, $\text{First}(E) = \{b, f_1, h_2, g_3, f_4, h_5\}$ and $\text{Follow}(E, f_1, 1) = \{b, f_1, h_2\}$, $\text{Follow}(E, h_2, 1) = \{b, f_1, h_2\}$, $\text{Follow}(E, g_3, 1) = \{b, g_3, f_4, h_5\}$, $\text{Follow}(E, g_3, 2) = \{a\}$, $\text{Follow}(E, f_4, 1) = \{b, f_4, h_5\}$, $\text{Follow}(E, h_5, 1) = \{b, f_4, h_5\}$.

The two functions $\text{First}$ and $\text{Follow}$ are sufficient to construct the $k$-position tree automaton from a regular expression $E$.

Definition 1. [12] Let $E$ be a regular expression, $f$ and $g$ be symbols in $\Sigma$ and $f_j$ and $g_i$ be positions in $\text{Pos}_E(E)$ with $h(f_j) = f$ and $h(g_i) = g$. The $k$-Position Tree Automaton $P_E$ is the automaton $(Q, \Sigma, Q_T, \Delta)$ defined by

$$Q = \{f_k^j \mid f_j \in \text{Pos}_E(E)_m \land 1 \leq k \leq m\} \cup \{\varepsilon^1\} \text{ with } \varepsilon^1 \text{ a new symbol not in } \Sigma, Q_T = \{\varepsilon^1\}$$

$$\Delta = \{(f_k^j, h(g_i), g_i^1, \ldots, g_i^n) \mid g_i \in \text{Follow}(E, f_j, k)\} \cup \{(\varepsilon^1, h(f_j), f_j^1, \ldots, f_j^m) \mid f_j \in \text{First}(E)\}$$

It has been shown in [12] that the $k$-position tree automaton of $E$ accepts $\langle E \rangle$, hence the following theorem:
Theorem 1. Let $E$ be a regular expression, then $\mathcal{L}(\mathcal{P}_E) = \llbracket E \rrbracket$.

Example 2. The $k$-Position Automaton $\mathcal{P}_E$ associated with $E$ of Example 1 is given in Figure 1. The set of states is $Q = \{ \varepsilon^1, f_1^1, h_2^1, g_3^2, f_4^1, h_5^1 \}$. The set of final states is $Q_T = \{ \varepsilon^1 \}$. The set of transition rules $\Delta$ is

- $f(f_1^1) \rightarrow \varepsilon^1$,
- $f(f_1^1) \rightarrow f_1^1$,
- $f(f_1^1) \rightarrow h_2^1$,
- $h(h_2^1) \rightarrow \varepsilon^1$,
- $h(h_2^1) \rightarrow f_1^1$,
- $h(h_2^1) \rightarrow h_2^1$,
- $g(g_3^2, g_3^2) \rightarrow g_3^2$,
- $g(g_3^2, g_3^2) \rightarrow \varepsilon^1$,
- $f(f_1^1) \rightarrow \varepsilon^1$,
- $f(f_1^1) \rightarrow g_3^2$,
- $f(f_1^1) \rightarrow f_1^1$,
- $f(f_1^1) \rightarrow h_3^1$,
- $h(h_3^1) \rightarrow \varepsilon^1$,
- $h(h_3^1) \rightarrow g_3^2$,
- $h(h_3^1) \rightarrow f_1^1$,
- $h(h_3^1) \rightarrow h_3^1$,
- $a \rightarrow g_3^2$,
- $b \rightarrow \varepsilon^1$,
- $b \rightarrow f_1^1$,
- $b \rightarrow h_2^1$,
- $b \rightarrow g_3^2$,
- $b \rightarrow f_1^1$,
- $b \rightarrow h_5^1$.

The $k$-Position Automaton $\mathcal{P}_E$ associated with $E$ is represented in Figure 1.

![Fig. 1. The $k$-Position Automaton $\mathcal{P}_E$ of $E = (f(a)^* \cdot a \cdot b + h(b))^* + g(c, a)^* \cdot (f(a)^* \cdot a \cdot b + h(b))^*$](image)

In the following sections, we will show how we can efficiently compute the function $\text{Follow}(E, f_j, k)$. This algorithm can be used in different constructions such as the equation automaton, $k$-$C$-continuation automaton, and Follow Automaton.
4 Efficient computation of the function Follow

In [1] Champanaund and Ziadi gave in the case of words an algorithm with an $O(|E| \cdot |E|^2)$ space and time complexity. They enhanced the algorithm to one with an $O(|E|^3)$ time and space complexity. In [7], Kuske and Meinecke extend the algorithm based on the notion of word partial derivatives [1] to tree partial derivatives in order to compute from a regular expression $E$ a tree automaton recognizing $[E]$. Lauerotte et al. proposed an algorithm for the computation of the position tree automaton and the reduced tree automaton in [8]. This is an extended version of [8]. In [10,11] Mignot et al. gave an efficient algorithm for the computation of the equation automaton using the $k$-c-continuations.

In this section we will describe an algorithm for the computation of the $k$-position tree automaton based on the computation of the Follow function.

In the following, we will inductively replace each regular subexpression $F^c$ of $E$ by the regular subexpression $(F + c)^*$. The regular expressions considered thereafter are already dealt by this transformation.

By misuse of language we will denote by First$(E)$ for First$(E)$ and by Follow$(E, f_j, k)$ for Follow$(E, f_j, k)$. Let us first show that the functions First and Follow can be inductively computed.

**Lemma 1.** [12] Let $E$ be a linear regular expression. The set First$(E)$ can be computed as follows:

$$
\begin{align*}
\text{First}(0) &= \emptyset, \\
\text{First}(a) &= \{a\}, \\
\text{First}(f_j(E_1, \ldots, E_m)) &= \{f_j\}, \\
\text{First}(E_1 + E_2) &= \text{First}(E_1) \cup \text{First}(E_2), \\
\text{First}(E_1^*) &= \text{First}(E_1), \\
\text{First}(E_1 \cdot c E_2) &= \begin{cases} \text{First}(E_1) \setminus \{c\} \cup \text{First}(E_2) & \text{if } c \in [E_1] \\
\text{First}(E_1) & \text{otherwise.} \end{cases}
\end{align*}
$$

**Lemma 2.** [12] Let $E$ be a linear regular expression, $1 \leq k \leq m$ be two integers and $f_j$ be a symbol in $\Sigma_m$.

The set of symbols Follow$(E, f_j, k)$ can be computed inductively as follows:

$$
\begin{align*}
\text{Follow}(0, f_j, k) &= \emptyset, \\
\text{Follow}(a, f_j, k) &= \emptyset, \\
\text{Follow}(f_j(E_1, \ldots, E_m), f_j, k) &= \begin{cases} \text{First}(E_k) & \text{if } g_i = f_j, \\
\text{Follow}(E_1, f_j, k) & \text{if } \exists l | f_j \in \Sigma_{E_l}, \\
\emptyset & \text{otherwise.} \end{cases} \\
\text{Follow}(E_1 + E_2, f_j, k) &= \begin{cases} \text{Follow}(E_1, f_j, k) & \text{if } f_j \in \Sigma_{E_1}, \\
\text{Follow}(E_2, f_j, k) & \text{if } f_j \in \Sigma_{E_2}, \\
\emptyset & \text{otherwise.} \end{cases} \\
\text{Follow}(E_1 \cdot c E_2, f_j, k) &= \begin{cases} (\text{Follow}(E_1, f_j, k) \setminus \{c\}) \cup \text{Follow}(E_2) & \text{if } f_j \in \Sigma_{E_1} \\
\text{Follow}(E_1, f_j, k) & \text{if } f_j \in \Sigma_{E_1} \\
\text{Follow}(E_2, f_j, k) & \text{if } f_j \in \Sigma_{E_2}, \\
\emptyset & \text{if } f_j \in \Sigma_{E_1} \land (c \notin \text{Follow}(E_1, f_j, k)), \\
\emptyset & \text{if } f_j \in \Sigma_{E_2} \land (c \notin \text{Follow}(E_1, f_j, k)), \\
\emptyset & \text{if } f_j \in \Sigma_{E_1} \land (c \notin \text{Last}(E_1)), \\
\emptyset & \text{otherwise.} \end{cases}
\end{align*}
$$
\[ \text{Follow}(E^c_{1}, f_j, k) = \begin{cases} \text{Follow}(E_1, f_j, k) \cup \text{First}(E_1) & \text{if } c \in \text{Follow}(E_1, f_j, k), \\ \text{Follow}(E_1, f_j, k) & \text{otherwise}. \end{cases} \]

The main idea of our algorithm consists of the separation of the computation of the function First (resp. Follow) to the computation of two subsets \( \text{Fr}_0 \) (resp. \( \text{Fl}_0 \)) and \( \text{Fr}_> \) (resp. \( \text{Fl}_> \)) that are respectively the projection of the set First (resp. Follow) to the positions associated with symbols of a rank 0 and a rank greater than 0.

Thus the computation of the set First \((E)\) can be written as follows:

\[ \text{First}(E) = \text{Fr}_0(E) \cup \text{Fr}_>(E). \]

**Proposition 1.** Let \( E \) be a linear regular expression and \( H \) be a subexpression of \( E \). The set of symbols \( \text{Fr}_0(H) \) is defined as follows:

\[
\begin{align*}
\text{Fr}_0(f_j(E_1, \ldots, E_m)) &= \emptyset, \\
\text{Fr}_0(0) &= \emptyset, \quad \text{Fr}_0(a) = \{a\}, \\
\text{Fr}_0(E_1 + E_2) &= \text{Fr}_0(E_1) \cup \text{Fr}_0(E_2), \\
\text{Fr}_0(E_1 \cdot c E_2) &= \begin{cases} 
(\text{Fr}_0(E_1) \setminus \{c\}) \cup \text{Fr}_0(E_2) & \text{if } c \in \llbracket E_1 \rrbracket, \\
\text{Fr}_0(E_1) & \text{otherwise}.
\end{cases} \\
\text{Fr}_0(E_{1}^c) &= \text{Fr}_0(E_1).
\end{align*}
\]

**Proof.** Let \( E \) be a linear regular expression, \( 1 \leq k \leq m \) be two integers and \( f_j \) be a symbol in \( \Sigma_m^E \).

1. If \( E = 0 \) or if \( E = f_j(E_1, \ldots, E_m) \), then \( \text{Fr}_0(E) = \emptyset \) and for \( E = a \), \( \text{Fr}_0(E) = \{a\} \).
2. Let us prove this proposition for the case \( E = E_1 \cdot c E_2 \).

We have \( \text{Fr}_0(E_1 \cdot c E_2) = \text{First}(E_1 \cdot c E_2, f_j, k) \cap \Sigma_0 \)

\[
\begin{align*}
\text{Fr}_0(E_1 \cdot c E_2) &= \text{First}(E_1 \cdot c E_2) \cap \Sigma_0 \\
&= \begin{cases} 
(\text{First}(E_1) \setminus \{c\}) \cup \text{First}(E_2) \cap \Sigma_0 & \text{if } c \in \llbracket E_1 \rrbracket, \\
\text{First}(E_1) \cap \Sigma_0 & \text{otherwise}.
\end{cases} \\
&= \begin{cases} 
(\text{Fr}_>(E_1) \cup \text{Fr}_0(E_1) \setminus \{c\}) \cup (\text{Fr}_>(E_2) \cup \text{Fr}_0(E_2)) \cap \Sigma_0 & \text{if } c \in \llbracket E_1 \rrbracket, \\
(\text{Fr}_>(E_1) \cup \text{Fr}_0(E_1)) \cap \Sigma_0 & \text{otherwise}.
\end{cases} \\
&= \begin{cases} 
(\text{Fr}_0(E_1) \setminus \{c\}) \cup \text{Fr}_0(E_2) & \text{if } c \in \llbracket E_1 \rrbracket, \\
\text{Fr}_0(E_2) & \text{otherwise}.
\end{cases}
\end{align*}
\]

The following proposition shows that \( \text{Fr}_>(E) \) can be computed in a similar way to the case of words.
Proposition 2. Let $E$ be a linear regular expression and $H$ be a subexpression of $E$. The set of symbols $Fr > (H)$ is defined as:

\[
\begin{align*}
Fr > (a) &= Fr > (0) = \emptyset, \\
Fr > (f_j(E_1, \ldots, E_m)) &= \{f_j\}, \\
Fr > (E_1 + E_2) &= Fr > (E_1) \cup Fr > (E_2), \\
Fr > (E_1 \cdot E_2) &= \begin{cases} 
Fr > (E_1) \cup Fr > (E_2) & \text{if } c \in [E_1], \\
Fr > (E_1) & \text{otherwise.}
\end{cases} \\
Fr > (E_1^*) &= Fr > (E_1),
\end{align*}
\]

Proof. Let $E$ be a linear regular expression.

1. If $E = 0$ or if $E = f_j(E_1, \ldots, E_m)$, then $Fr_0 (E) = \emptyset$ and for $E = a$, $Fr_0 (E) = \{a\}$.
2. Let us prove this proposition for the cases $E = E_1 \cdot E_2$.

We have $Fr_0 (E_1 \cdot E_2) = First(E_1 \cdot E_2, f_j, k) \cap \Sigma >$

\[
\begin{align*}
Fr > (E_1 \cdot E_2) &= First(E_1 \cdot E_2) \cap \Sigma > \\
&= \begin{cases} 
(First(E_1) \setminus \{c\} \cup First(E_2)) \cap \Sigma > & \text{if } c \in [E_1], \\
First(E_1) \cap \Sigma > & \text{otherwise.}
\end{cases} \\
&= \begin{cases} 
((Fr > (E_1) \cup Fr_0 (E_1)) \setminus \{c\}) \cup (Fr > (E_2) \cup Fr_0 (E_2)) \cap \Sigma > & \text{if } c \in [E_1], \\
(Fr > (E_1) \cup Fr_0 (E_1)) \cap \Sigma > & \text{otherwise.}
\end{cases} \\
&= \begin{cases} 
((fr_0 (E_1) \setminus \{c\}) \cup Fr > (E_1)) \cup (Fr > (E_2) \cup Fr_0 (E_2)) \cap \Sigma > & \text{if } c \in [E_1], \\
(Fr > (E_1) \cup Fr_0 (E_1)) \cap \Sigma > & \text{otherwise.}
\end{cases} \\
&= \begin{cases} 
Fr > (E_1) \cup Fr > (E_2) & \text{if } c \in [E_1], \\
Fr > (E_2) & \text{otherwise.}
\end{cases}
\end{align*}
\]

Let us recall that $Fl_0 (E, f_j, k)$ and $Fl > (E, f_j, k)$ are, respectively, the projection of the set $Follow (E, f_j, k)$ to the symbols associated with symbols of a rank 0 and a rank greater than 0. We have:

\[
Follow (E, f_j, k) = Fl_0 (E, f_j, k) \cup Fl > (E, f_j, k)
\]

Proposition 3. Let $E$ be a linear regular expression, $1 \leq k \leq m$ be two integers and $f_j$ be a symbol in $\Sigma _m ^E$. The function $Fl_0 (E, f_j, k)$ can be computed inductively
as follows:

\[
\begin{align*}
\text{Fl}_0(a, f_j, k) &= \text{Fl}_0(0, f_j, k) = \emptyset, \\
\text{Fl}_0(g_i(E_1, \cdots, E_m), f_j, k) &= \begin{cases} \\
\text{Fr}_0(E_k) & \text{if } g_i = f_j, \\
\text{Fl}_0(E_i, f_j, k) & \text{if } f_j \in \Sigma^{E_i}, \\
\end{cases} \\
\text{Fl}_0(E_1 + E_1, f_j, k) &= \begin{cases} \\
\text{Fl}_0(E_1, f_j, k) & \text{if } f_j \in \Sigma^{E_1}, \\
\text{Fl}_0(E_2, f_j, k) & \text{if } f_j \in \Sigma^{E_2}, \\
\end{cases} \\
\text{Fl}_0(E_1 \cdot c E_1, f_j, k) &= \begin{cases} \\
\text{Fl}_0(E_1, f_j, k) & \text{if } f_j \in \Sigma^{E_1}, \\
\text{Fl}_0(E_2, f_j, k) & \text{if } f_j \in \Sigma^{E_2}, \\
\emptyset & \text{if } f_j = \emptyset, \\
\end{cases} \\
\text{Fl}_0(E_1^*, f_j, k) &= \begin{cases} \\
\text{Fl}_0(E_1, f_j, k) \cap \Sigma_0 & \text{if } c \in \text{Fl}_0(E_1, f_j, k), \\
\text{Fl}_0(E_1, f_j, k) \cap \Sigma_0 & \text{otherwise}. \\
\end{cases}
\end{align*}
\]

**Proof.** Let E be a linear regular expression, 1 ≤ k ≤ m be two integers and f_j be a symbol in Σ_{E_k}.

1. If E = 0 or E = a, then Fl_0(E, f_j, k) = ∅.

Let us prove this proposition for the cases E = E_1 \cdot E_2 and E = E_1^*.

2. Let us consider that E = E_1 \cdot E_2.

We have Fl_0(E_1 \cdot E_2, f_j, k) = Follow(E_1 \cdot E_2, f_j, k) ∩ Σ_0

\[
\begin{align*}
\text{Fl}_0(E_1 \cdot E_2, f_j, k) &= \text{Follow}(E_1 \cdot E_2, f_j, k) \cap \Sigma_0 \\
&= \begin{cases} \\
((\text{Follow}(E_1, f_j, k) \setminus \{c\}) \cup \text{First}(E_2)) \cap \Sigma_0 & \text{if } f_j \in \Sigma^{E_1} \land c \in \text{Fl}_0(E_1, f_j, k), \\
\text{Follow}(E_1, f_j, k) \cap \Sigma_0 & \text{if } f_j \in \Sigma^{E_1} \land c \notin \text{Fl}_0(E_1, f_j, k), \\
\emptyset & \text{otherwise}. \\
\end{cases} \\
&= \begin{cases} \\
((\text{Fl}_0(E_1, f_j, k) \cup \text{Fr}_0(E_2)) \setminus \{c\}) \cup \Sigma_0 & \text{if } f_j \in \Sigma^{E_1} \land c \in \text{Fl}_0(E_1, f_j, k), \\
\text{Fl}_0(E_1, f_j, k) \cap \Sigma_0 & \text{if } f_j \in \Sigma^{E_1} \land c \notin \text{Fl}_0(E_1, f_j, k), \\
\emptyset & \text{otherwise.} \\
\end{cases} \\
&= \begin{cases} \\
\text{Fl}_0(E_1, f_j, k) \cup \text{Fr}_0(E_2) & \text{if } f_j \in \Sigma^{E_1} \land c \in \text{Fl}_0(E_1, f_j, k), \\
\text{Fl}_0(E_1, f_j, k) & \text{if } f_j \in \Sigma^{E_1} \land c \notin \text{Fl}_0(E_1, f_j, k), \\
\emptyset & \text{otherwise.} \\
\end{cases}
\end{align*}
\]
3. Let us consider that \( E = E_1^e \). By definition we have \( F_{l_0} (E_1^e, f_j, k) = \) Follow\((E_1^e, f_j, k) \cap \Sigma_0 \). Then:

\[
F_{l_0} (E_1^e, f_j, k) = \text{Follow}(E_1^e, f_j, k) \cap \Sigma_0
\]

\[
= \begin{cases} 
\{ (\text{Follow}(E_1, f_j, k) \setminus \{c\}) \cup \text{First}(E_1) \} \cap \Sigma_0 & \text{if } c \in F_{l_0} (E_1, f_j, k), \\
\{ \text{Follow}(E_1, f_j, k) \cap \Sigma_0 \} & \text{otherwise.}
\end{cases}
\]

\[
= \begin{cases} 
\{ ( \text{Fr}_0 (E_1) \cup \text{Fr}_r (E_1)) \} \cap \Sigma_0 & \text{if } c \in F_{l_0} (E_1, f_j, k), \\
\{ \text{Follow}(E_1, f_j, k) \cap \Sigma_0 \} & \text{otherwise.}
\end{cases}
\]

\[
= \begin{cases} 
F_{l_0} (E_1, f_j, k) \cup F_{r_0} (E_1) & \text{if } c \in F_{l_0} (E_1, f_j, k), \\
F_{l_0} (E_1, f_j, k) & \text{otherwise.}
\end{cases}
\]

\( \Box \)

\textbf{Proposition 4}. Let \( E \) be a linear regular expression, \( 1 \leq k \leq m \) be two integers and \( f_j \) be a symbol in \( \Sigma_m^E \). We define inductively the set \( F_{l_0} (E, f_j, k) \) as follows:

\[
F_{l_0} (a, f_j, k) = F_{l_0} (0, f_j, k) = \emptyset,
\]

\[
F_{l_0} (g(E_1, \ldots, E_m), f_j, k) = \begin{cases} 
F_{r_0} (E_k) & \text{if } g_i = f_j, \\
F_{l_0} (E_i, f_j, k) & \text{if } f_j \in \Sigma_{E_i}^E.
\end{cases}
\]

\[
F_{l_0} (F + G, f_j, k) = \begin{cases} 
F_{l_0} (F, f_j, k) & \text{if } f_j \in \Sigma_F, \\
F_{l_0} (G, f_j, k) & \text{if } f_j \in \Sigma_G,
\end{cases}
\]

\[
F_{l_0} (F \cdot c G, f_j, k) = \begin{cases} 
F_{l_0} (F, f_j, k) \cup F_{r_0} (G) & \text{if } c \in F_{l_0} (F, f_j, k), \\
F_{l_0} (F, f_j, k) & \text{if } f_j \in \Sigma_F, \\
F_{l_0} (G, f_j, k) & \text{if } f_j \in \Sigma_G, \\
\emptyset & \text{and } c \notin F_{l_0} (F, f_j, k), \\
\emptyset & \text{if } f_j \in \Sigma_G, \\
\emptyset & \text{and } c \in \text{Last}(F), \\
\emptyset & \text{otherwise.}
\end{cases}
\]

\[
F_{l_0} (F^c, f_j, k) = \begin{cases} 
F_{l_0} (F, f_j, k) \cup F_{r_0} (F) & \text{if } c \in F_{l_0} (F, f_j, k), \\
F_{l_0} (F, f_j, k) & \text{otherwise.}
\end{cases}
\]

\textit{Proof}. Let \( E \) be a linear regular expression, \( 1 \leq k \leq m \) be two integers and \( f_j \) be a symbol in \( \Sigma_m^E \).

1. If \( E = 0 \) or if \( E = a \), then \( F_{l_0} (E, f_j, k) = \emptyset \).

Let us prove this proposition for the cases \( E = E_1 \cdot c E_2 \) and \( E = E_1^e \).

2. Let us consider that \( E = E_1 \cdot c E_2 \).
4.1 ZPC-Structure for Follow Computation

We have $\text{Fl}_> (E_1 \cdot E_2, f_j, k) = \text{Follow}(E_1 \cdot E_2, f_j, k) \cap \Sigma_>$

$$\text{Fl}_> (E_1 \cdot E_2, f_j, k) = \text{Follow}(E_1 \cdot E_2, f_j, k) \cap \Sigma_>
\begin{cases}
\left( (\text{Follow}(E_1, f_j, k) \setminus \{c\}) \cup \text{First}(E_2) \right) \cap \Sigma_> & \text{if } f_j \in \Sigma^{E_1} \land c \in \text{Fl}_0 (E_1, f_j, k), \\
\text{Follow}(E_1, f_j, k) \cap \Sigma_> & \text{if } f_j \in \Sigma^{E_1} \land c \notin \text{Fl}_0 (E_1, f_j, k), \\
\text{Follow}(E_2, f_j, k) \cap \Sigma_> & \text{if } f_j \in \Sigma^{E_2} \land c \in \text{Last}(E_1), \\
\emptyset & \text{otherwise}.
\end{cases}$$

$\text{Fl}_> (E_1, f_j, k) \cup \text{Fl}_> (E_2, f_j, k) = \text{Follow}(E_1 \cdot E_2, f_j, k) \cap \Sigma_>$

3. Let us consider that $E = E_1^*$. By definition we have $\text{Fl}_> (E_1^*, f_j, k) = \text{Follow}(E_1^*, f_j, k) \cap \Sigma_>$.

$$\text{Fl}_> (E_1^*, f_j, k) = \text{Follow}(E_1^*, f_j, k) \cap \Sigma_>
\begin{cases}
\left( (\text{Follow}(E_1, f_j, k) \setminus \{c\}) \cup \text{First}(E_1) \right) \cap \Sigma_> & \text{if } f_j \in \Sigma^{E_1} \land c \in \text{Fl}_0 (E_1, f_j, k), \\
\text{Follow}(E_1, f_j, k) \cap \Sigma_> & \text{if } f_j \in \Sigma^{E_1} \land c \notin \text{Fl}_0 (E_1, f_j, k), \\
\emptyset & \text{otherwise}.
\end{cases}$$

Remark 1. The definition of the set $\text{Fl}_> (E, f_j, k)$ is identical to the function Follow in the case of words [16]. We have the same formulas.

The construction of the $k$-position tree automaton $P_E$ from the regular expression as it has been presented in this article complies with the properties of the position automaton proposed by Glushkov. This is the generalization of the position automaton from words to trees.

4.1 ZPC-Structure for Follow Computation

In the word case, the construction of the position automaton, has been developed in [15, 16]. This construction will be extended to trees in the following.
Let $T_E$ be the syntax tree associated with the regular expression $E$.

The set of nodes of $T_E$ is written as $\text{Nodes}(E)$. For a node $\nu$ in $\text{Nodes}(E)$, $\text{sym}(\nu)$, $\text{father}(\nu)$, $\text{son}(\nu)$, $\text{right}(\nu)$ and $\text{left}(\nu)$ denote respectively the symbol, the father, the son, the right son and the left son of the node $\nu$ if they exist.

We denote by $E_\nu$ the subexpression rooted at $\nu$; in this case we write $\nu E$ to denote the node associated to $E_\nu$. Let $\gamma : \text{Nodes}(E) \cup \{\bot\} \to \text{Nodes}(E) \cup \{\bot\}$ be the function defined by:

$$\gamma(\nu) = \begin{cases} 
\text{father}(\nu) & \text{if } \text{sym}(\text{father}(\nu)) = \ast_c \text{ and } \nu \neq \nu_E \\
\bot & \text{otherwise}
\end{cases}$$

where $\bot$ is an artificial node such that $\gamma(\bot) = \bot$. The ZPC-Structure is the syntax tree equipped with $\gamma(\nu)$ links.

We extend the relation $\preceq$ to the set of nodes of $T_E$: For two nodes $\nu$ and $\mu$ we write $\nu \preceq \mu$ if $T_\nu \succeq T_\mu$. We define the set $\Gamma_\nu(E) = \{\mu \in \text{Nodes}(E) | \nu \preceq \mu \land \gamma(\mu) \neq \bot\}$ which is totally ordered by $\preceq$.

**Proposition 5.** Let $E$ be linear regular expression, $1 \leq k \leq n$ be two integers and $f$ be in $\Sigma^k \cap \Sigma_n$. Then $\text{Follow}(E, f, k) = \{(\text{First}(E_{\nu_0}), \text{op}(\nu_1), \text{First}(E_{\gamma(\nu_1)})), \text{op}(\nu_2), \ldots, \text{op}(\nu_m), \text{First}(E_{\gamma(\nu_m)})\}$ where $\nu_f$ is the node of $T_E$ labelled by $f$, $\nu_0$ is the $k$-child($\nu_f$), $\Gamma_{\nu_f}(E) = \{\nu_1, \ldots, \nu_m\}$ and for $1 \leq i \leq m$, $\text{op}(\nu_i) = c$ such that $\text{sym}(\text{father}(\nu_i)) \in \{\ast_c, \ast_e\}$.

**Proof.** By induction over the structure of $E$.

1. Let us suppose that $E = f(E_1, \ldots, E_n)$. Then $\text{Follow}(E, f, k) = \text{First}(E_k)$.

   Since by definition $\nu_f$ is the root of $T_E$, the $k$-child($\nu_f$) is the root of $E_k$.

   Hence $\text{First}(E_{\nu_0}) = \text{First}(E_k) = \text{Follow}(E, f, k)$.

2. Let us suppose that $E = g(E_1, \ldots, E_m) \cdot E_2$ with $g \neq f$, or $E = E_1 + E_2$, or $E = E_1 \cdot E_2$ with $f \in \Sigma^E_\ast$. Then $\text{Follow}(E, f, k) = \text{Follow}(E_j, f, k)$ with $f \in \Sigma^{E_j}$. By induction hypothesis, $\text{Follow}(E_j, f, k) = \{(\text{First}(E_{\nu_0}), \text{op}(\nu_1), \text{First}(E_{\gamma(\nu_1)})), \text{op}(\nu_2), \ldots, \text{op}(\nu_m), \text{First}(E_{\gamma(\nu_m)})\}$ where $\nu_f$ is the node of $T_E$ labelled by $f$, $\nu_0$ is the $k$-child($\nu_f$), $\Gamma_{\nu_f}(E_j) = \{\nu_1, \ldots, \nu_m\}$ and for $1 \leq i \leq m$, $\text{op}(\nu_i) = c$ such that $\text{sym}(\text{father}(\nu_i)) \in \{\ast_c, \ast_e\}$.

   Since $T_{E_1} \preceq T_{E_2}$, $\text{Follow}(E_j, f, k) = \{(\text{First}(E_{\nu_0}), \text{op}(\nu_1), \text{First}(E_{\gamma(\nu_1)})), \text{op}(\nu_2), \ldots, \text{op}(\nu_m), \text{First}(E_{\gamma(\nu_m)})\}$ where $\nu_f$ is the node of $T_E$ labelled by $f$, $\nu_0$ is the $k$-child($\nu_f$), $\Gamma_{\nu_f}(E_j) = \{\nu_1, \ldots, \nu_m\}$ and for $1 \leq i \leq m$, $\text{op}(\nu_i) = c$ such that $\text{sym}(\text{father}(\nu_i)) \in \{\ast_c, \ast_e\}$.

3. Let us suppose that $E = E_1 \cdot E_2$ with $f \in \Sigma^{E_1}$ (resp. $E = E_1^\ast$). Then $\text{Follow}(E, f, k) = \text{Follow}(E_1, f, k) \cdot \text{First}(G)$ with $G \in \{E_1^\ast, E_2\}$. By induction hypothesis, $\text{Follow}(E_1, f, k) = \{(\text{First}(E_{\nu_0}), \text{op}(\nu_1), \text{First}(E_{\gamma(\nu_1)})), \text{op}(\nu_2), \ldots, \text{op}(\nu_m), \text{First}(E_{\gamma(\nu_m)})\}$ where $\nu_f$ is the node of $T_{E_1}$ labelled by $f$, $\nu_0$ is the $k$-child($\nu_f$), $\Gamma_{\nu_f}(E_1) = \{\nu_1, \ldots, \nu_m\}$ and for $1 \leq i \leq m$, $\text{op}(\nu_i) = c$ such that $\text{sym}(\text{father}(\nu_i)) \in \{\ast_c, \ast_e\}$.

   Since $T_{E_1} \preceq T_{E_2}$, by setting $H = E_{\nu_0+1}$ and $\text{op}(\nu_{m+1}) = c$, $\text{Follow}(E_1, f, k) \cdot \text{First}(G) = \{(\text{First}(E_{\nu_0}), \text{op}(\nu_1), \text{First}(E_{\gamma(\nu_1)})), \text{op}(\nu_2), \ldots, \text{op}(\nu_m), \text{First}(E_{\gamma(\nu_m)})\}$
First(E₁(...Eₙ))) · \text{op}(ν_{m+1}) \text{First}(E₁(ν_{m+1})) where νᵢ is the node of T_E labelled by f, ν₀ is the k-child(νᵢ), Γ_{νᵢ}(E) = {ν₁, ..., νₘ, ν_{m+1}} and for 1 ≤ i ≤ m + 1, \text{op}(νᵢ) = c such that \text{sym}(\text{father}(νᵢ)) ∈ \{·c, *c\}.

4.2 Description of the algorithm and complexity

An implicit construction of the word position automaton, the so-called ZPC-structure, has been developed by Ziadi et al. [15,16]. Algorithm 1 extends this construction to the regular tree expressions. It constructs a forest of trees where every tree rooted at a node ν_F represents the set Fr₁(F) according to Proposition 2.

Algorithm 1: ZPC-Structure Construction

Input: Regular Expression E.
Output: ZPC-Structure

Construct the syntax tree T_E of E;

# for each node ν_F on T_E do
  Compute Fr₀(F);
end for

# The construction of a First Forest
for each node ν_F · c in T_E do
  if c ∉ Fr₀(F) then
    Remove the link (ν_F · c, ν_G);
  end if
end for

# We have First (f_j(E₁, ..., Eₙ)) = \{f_j\}
for each node ν_f_j(E₁, ..., Eₙ) in T_E do
  for i = 1 to n do
    Remove the link (ν_f_j(E₁, ..., Eₙ), ν_E_i);
  end for
end for

for each node ν_F ∈ Σ₀ do
  Delete the node ν_F;
end for

# The construction of Follow links (γ_F links)
for each node ν_F · G in T_E do
  create a follow link from ν_F to ν_G;
end for

for each node ν_F *c in T_E do
  create a link from ν_F to ν_F *c;
end for

return ZPC-Structure
Example 3. The syntax tree $T_E$ associated with the regular expression $E = (f_1(a)^* \cdot a \cdot b + h_2(b))^* + g_3(c, a)^* \cdot c \cdot (f_4(a)^* \cdot a \cdot b + h_5(b))^*$ is given in Figure 2.

Fig. 2. The syntax tree $T_E$ of $E$

The ZPC-Structure associated with $E = (f_1(a)^* \cdot a \cdot b + h_2(b))^* + g_3(c, a)^* \cdot c \cdot (f_4(a)^* \cdot a \cdot b + h_5(b))^*$ is given in Figure 3.

Fig. 3. The ZPC-Structure of $E$
Theorem 2. The ZPC-Structure associated with $E$ can be computed in $O(|E|)$ time and space complexity.

Proof. The first step of our Algorithm 1 consists of computing the sets $F_{r0}(F)$ for all subexpressions $F$ of $E$. The set $F_{r0}(F)$ is represented by an array where the entries are indexed by symbols of $\mathcal{S}_0$. The computation of all sets $F_{r0}(F)$ requires $O(|E|)$ time and space complexity.

Now that we have computed the sets $F_{r0}()$, the second step consists of the construction of the First Forest. Recall that this First Forest encodes the $F_{r>}(F)$ sets for all subexpressions $F$ of $E$. Therefore, the set $F_{r>}(F)$ can be obtained by a prefix traversal of the syntax tree of $E$ in $O(|E|)$ time and space complexity. $\Box$

As each node $\nu_F$ encodes $F_{r>}(E_{\nu_F})$ we can state the following lemma.

Lemma 3. For a subexpression $F$ of $E$ the set $F_{r>}(F)$ can be computed in $O(|F|)$ time and space complexity.

For a regular expression $E$, the following algorithm allows to compute the set $\text{Follow}(E, f_j, k)$ for a symbol $f_j \in \mathcal{S}_E^m$ and integers $1 \leq k \leq m$.

Algorithm 2: Algorithm for the function Follow for $f_j$ and $k$

**Input**: Regular Expression $E$.

**Output**: $\{\text{Follow}(E, f_j, k) \mid f_j \in \mathcal{S}_E^m, 1 \leq k \leq m\}$.

1. Calculate $\text{Follow}(E, f_j, k) = F_{l0}(E, f_j, k) \cup F_{l>}(E, f_j, k)$

   (1.1) for $\nu = \nu_{f_j}$ to $\nu_E$ do
   
   Compute $F_{l0}(E_{\nu}, f_j, k)$;
   
   end for

   (1.2) Compute $F_{l>}(E, f_j, k)$;

   return $(F_{l0}(E, f_j, k) \cup F_{l>}(E, f_j, k))$

For each step of the Algorithm 2 we will evaluate the complexity in time and in space.

We denote by $\sum_{f_j \in \mathcal{S}_E^m} r(f_j)$ the sum of all ranks of symbols $f_j \in \mathcal{S}_>$. 

Step 1: Computation of Follow $(E, f_j, k) = F_{l0}(E, f_j, k) \cup F_{l>}(E, f_j, k)$

We are interested about the computation of the sets $F_{l0}(E, f_j, k)$ and $F_{l>}(E, f_j, k)$.

Step 1.1: Computation of sets $F_{l0}(E_{\nu}, f_j, k)$

At each node $\nu$ of the syntax tree $T_E$ of $E$, the set $F_{l0}(E_{\nu}, f_j, k)$ is represented by an array where the entries are indexed by symbols of $\mathcal{S}_0$. The computation of the set $F_{l0}(E_{\nu}, f_j, k)$ requires an $O(|E|)$ time and space complexity.

Step 1.2: Computation of $F_{l>}(E, f_j, k)$
Now that $\text{Fl}_0(E_\nu, f_j, k)$ for all node $\nu$, such that $\nu f_j \preceq \nu$, are computed, we can use the techniques outlined in the case of words to calculate the set $\text{Fl}_>^E(E, f_j, k)$. Indeed, our formulas given in the Proposition 4 for the computation of $\text{Fl}_>^E(E, f_j, k)$ are similar to that defined in the case of words [2][16]. We have the same formulas so we can use the same algorithms used in the paper [16] for the computation of the sets $\text{Follow}$. Therefore, the computation of $\text{Fl}_>^E(E, f_j, k)$ can be done in $O(|E|)$ time complexity.

We denote by $R$ the maximal rank of symbols of $\Sigma$ appearing in $E$. Recall that the alphabetic width $||E||$, of a regular expression $E$ is the sum of occurrences of symbols of a rank greater than 0 appearing in $E$ that is $||E|| = \sum_{f \in \Sigma >} |E|_f$.

The size of the ranked alphabet $\Sigma$ is considered as constant.

**Lemma 4.** Let $E$ be a regular expression, $f_j$ be a symbol in $\Sigma^E_m$ and $1 \leq k \leq m$ be two integers. The sets $\text{Follow}(E, f_j, k)$ for $1 \leq k \leq m$ can be computed in time $O(r(f_j) \cdot |E|)$.

As $(\sum_{f_j \in \Sigma >} (r(f_j)))$ is bounded by $(R \cdot ||E||)$ we can state the following theorem.

**Theorem 3.** The sets $\text{Follow}(E, f_j, k)$ for all symbols $f_j$ in $\Sigma^E_>$ and for all $1 \leq k \leq r(f_j)$ can be computed with an $O(R \cdot ||E|| \cdot |E|)$ time complexity.

### 4.3 Improving the computation of the function Follow

In this section we present a simple transformation of the regular expression $E$ which allows us to efficiently compute the sets $\text{Follow}$. For a subexpression $f_j(E_1, \ldots, E_m)$ of $E$ and a symbol $a$ in $\bigcup_{i=1}^m \text{Fr}_0(E_i)$ we associate an expression $E_{f_j}^a$, obtained from $E$ by replacing the subexpression $f_j(E_1, \ldots, E_m)$ by the expression $f_j(a)$.

**Example 4.** For the regular expression $E = f(a + g(b), a + b + h(a))^*a \cdot b l(b)$. We get $E_f^a = f(a)^*a \cdot b l(b)$ and $E_f^b = f(b)^*a \cdot b l(b)$.

For all subexpressions $f_j(E_1, \ldots, E_m)$ of $E$ and for a symbol $a \in \bigcup_{i=1}^m \text{Fr}_0(E_i)$, the following proposition gives the link between $\text{Follow}(E, f_j, k)$ and $\text{Follow}(E_{f_j}^a, f_j, 1)$.

**Proposition 6.** Let $E$ be a regular expression, $f_j(E_1, \ldots, E_m)$ be a subexpression of $E$ and $1 \leq k \leq m$ be two integers.

The set $\text{Follow}(E, f_j, k)$ can be computed as follows:

$$\text{Follow}(E, f_j, k) = \text{Fr}_>^E(E_k) \cup \bigcup_{a \in \text{Fr}_0(E_k)} \text{Follow}(E_{f_j}^a, f_j, 1)$$
Proof. For a subexpression \( f_j(E_1, \ldots, E_n) \) of \( E \) and from Proposition 5, the set \( \text{Follow}(E, f_j, k) \) is of the form: \( \text{Follow}(E, f_j, k) = (\text{First}(E_{\nu_0}) \cdot \text{op}(\nu_1) \text{First}(E_{\gamma_1})) \cdot \text{op}(\nu_2) \text{First}(E_{\gamma_2})) \cdot \cdots \cdot \text{op}(\nu_m) \text{First}(E_{\gamma_m})) \) where \( \nu, \nu_0 \) is the node of \( T_E \) labelled by \( f_j \), \( \nu_0 \) is the \( k \)-child of \( (\nu_j) = \{\nu_1, \ldots, \nu_m\} \) and for \( 1 \leq i \leq m, \text{op}(\nu_i) = c \) such that \text{sym(father}(\nu_i)) \in \{c, *c\}

\[
\text{Follow}(E, f_j, k) = ((((\text{First}(E_{\nu_0}) \cdot \text{op}(\nu_1) \text{First}(E_{\gamma_1}))) \cdot \text{op}(\nu_2) \text{First}(E_{\gamma_2}))) \cdot \cdots \cdot \text{op}(\nu_m) \text{First}(E_{\gamma_m}))
\]

By using this last formula and the modifications: for all symbols \( a \in \bigcup \text{Fr}_0(E_{\nu_0}) \) we associate an expression \( E_{f_j}^a \) obtained from \( E \) by replacing the subexpression \( f_j(E_1, \ldots, E_n) \) by the expression \( f_j(a) \), then we have for \( a \in \text{Fr}_0(E_{\nu_0}) \):

\[
\text{Follow}(E_{f_j}^a, f_j, 1) = (((\text{op}(\nu_1) \text{First}(E_{\gamma_1}))) \cdot \text{op}(\nu_2) \cdots \text{op}(\nu_m) \text{First}(E_{\gamma_m})))
\]

Therefore, for all symbols \( a \in \bigcup \text{Fr}_0(E_{\nu_0}) \):

\[
\text{Follow}(E, f_j, k) = \text{Fr}_>(E_{\nu_0}) \cup \bigcup_{a \in \text{Fr}_0(E_{\nu_0})} \text{Follow}(E_{f_j}^a, f_j, 1)
\]

\( \square \)

As the rank of the symbol \( f_j \) in \( E_{f_j}^a \) is 1 and by Lemma 4, the set \( \text{Follow}(E_{f_j}^a, f_j, 1) \) can be computed in time \( O(|E|) \). This step is considered as a preprocessing and is common to each symbol \( a \) such that \( a \) is in \( \bigcap_{k=1}^n \text{Fr}_0(E_k) \) for all \( 1 \leq k \leq n \).

So, one can compute in first time the sets \( \text{Follow}(E_{f_j}^a, f_j, 1) \) for all \( a \) in \( \bigcup_{k=1}^n \text{Fr}_0(E_k) \) in \( O(|E|) \) time complexity. In the second time, from these sets and the set \( \text{Fr}_>(E_k) \) we construct the set \( \text{Follow}(E, f_j, k) \) using formula of Proposition 6. This second step can be performed in \( O(|E_k| + |E|) \) time complexity. Indeed from Lemma 4, \( \text{Fr}_0(E_k) \) can be computed in time \( O(|E_k|) \) and the set \( \bigcup_{a \in \text{Fr}_0(E_k)} \text{Follow}(E_{f_j}^a, f_j, 1) \) can be constructed from the sets computed in the first step with an \( O(|E|) \) time complexity.
As \(\left(\sum_{k=1}^{n} |E_k|\right) < |E| \) and \((r(f_j) \cdot ||E||) < |E|\) and as the first step is performed once for all \(k, 1 \leq k \leq n\) and for all \(a \in \bigcap_{k=1}^{n} Fr_0(E_k)\), then, we can state the following proposition.

**Proposition 7.** Let \(E\) be a regular expression and \(f_j\) be a symbol in \(\Sigma_E^>\). The set \(\text{Follow}(E, f_j, k)\) for all \(1 \leq k \leq r(f_j)\) can be computed with an \(O(|E|)\) time complexity.

Finally we can state the following theorem.

**Theorem 4.** Let \(E\) be a regular expression. The computation of the Follow sets for all symbol \(f_j \in \Sigma_E^>\) and \(1 \leq k \leq r(f_j)\) can be done with an \(O(|E| \cdot ||E||)\) time complexity.

Our algorithm for the computation of the Follow sets can be used for the computation of the set of transition rules of the \(k\)-position automaton, the equation automaton \([7,10]\), the \(k\)-c-continuation automaton \([10,12]\) and the Follow automaton \([12]\).

**Remark 2.** By analogy to the word case, we have chosen to don’t consider the constant symbols \((\Sigma_0)\) in the alphabetic width of \(E\). For example for the regular expression \(E = f(a, \ldots, a)\), \(||E|| = 1\). However, in \([7]\), the alphabetic width is \(a\) \(n\)-times the number of occurrences of symbols of \(\Sigma\) in \(E\), that is \(||E|| = n + 1\).

**5 Conclusion**

In this paper the notion of \(k\)-position tree automaton associated with the regular tree expression has been recalled. This automaton is the generalization from words to trees of the position automaton introduced by Glushkov. We give an efficient algorithm that computes the Follow function from a regular expression \(E\) in \(O(||E|| \cdot |E|)\) time complexity.

This algorithm for the computation of the Follow sets can be used for the computation of the set of transitions of the \(k\)-position, equation, \(k\)-c-continuation and Follow automata.

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