Asymptotic approximation of the probability of ruin for large values of the Poisson rate

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Abstract

We analyze the probability of ruin for the scaled classical Cramér-Lundberg (CL) risk process and the corresponding diffusion approximation. The scaling, introduced by Iglehart [10] to the actuarial literature, amounts to multiplying the Poisson rate $\lambda$ by $n$, dividing the claim severity by $\sqrt{n}$, and adjusting the premium rate so that net premium income remains constant. Therefore, we think of the associated diffusion approximation as being “asymptotic for large values of $\lambda$.”

We are the first to use a comparison method to prove convergence of the probability of ruin for the scaled CL process and to derive the rate of convergence. Specifically, we prove a comparison lemma for the corresponding integro-differential equation and use this comparison lemma to prove that the probability of ruin for the scaled CL process converges to the probability of ruin for the limiting diffusion process. Moreover, we show that the rate of convergence for the ruin probability is of order $O(n^{-1/2})$, and we show that the convergence is uniform with respect to the surplus. To the best of our knowledge, this is the first rate of convergence achieved for these ruin probabilities, and we show that it is the tightest one. For the case of exponentially-distributed claims, we are able to improve the approximation arising from the diffusion, attaining a uniform $O(n^{-1})$ rate of convergence. We also include two examples that illustrate our results.

Keywords: Probability of ruin, Cramér-Lundberg risk process, diffusion approximation, approximation error.

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1 Introduction

Approximating and bounding the probability of ruin has a long-standing history in risk theory. Arguably, the earliest approximation and bound are the well known Cramér-Lundberg approximation and related Lundberg bound; see Lundberg [15] and Cramér [6].

The Cramér-Lundberg approximation is asymptotic for large values of the surplus process, and in most of the literature in ruin theory, this is what asymptotic refers to. In this paper, asymptotic refers to the large values of the Poisson rate, together with small claim severity; in that case, the classical Cramér-Lundberg (CL) risk process approaches a diffusion. We combine two areas of research.

The first area is that of using a diffusion process to approximate the discrete risk process, with many small jumps. This method takes advantage of the mathematical tractability of diffusion processes to deduce properties of the process it approximates. This technique was introduced by Kingman [13] in the analysis of a single-server queue and by Iglehart and Whitt [11, 12] in the context of multiple channel queues. Since then, it has gained popularity in the stochastic networks community, where it is referred to as the heavy-traffic approximation. In this field, the length of the queues are scaled (divided) by $n^{1/2}$ and the rates are scaled (multiplied) by $n$ in such a way that the system is critically loaded in the sense that the traffic intensity (utilization) converges to 1 from below. The martingale/functional central limit theorem, then, implies that, in the limit, one attains a diffusion process. The approximation helps in finding asymptotic optimal controls and behavior of complicated systems. For a basic introduction to the heavy-traffic approximation, please see Chen and Yao [5], Kushner [14], and the references therein.

Iglehart [10] introduced the diffusion approximation to the actuarial literature. He used probabilistic techniques (weak convergence) to prove that the probability of ruin for the scaled model approaches the probability of ruin for the limiting diffusion process. Grandell [9] and Asmussen [11] further used the approximating diffusion process to approximate the probability of ruin in finite time; Asmussen’s work was inspired by Siegmund [18]. In these works, the limits hold pointwise and no rate of convergence is provided. More recently, Bäuerle [3] used probabilistic techniques to prove limiting results under optimal control of the surplus process.

Instead of probabilistic techniques, we rely on comparison analysis of the integro-differential
equation that the probability of ruin solves, and this is the second area of research. The key element of this technique is an “increasing” functional that vanishes when evaluated at the probability of ruin (in the \( n \)-scaled problem). By perturbing the probability of ruin by \( O(n^{-1/2}) \) in both directions and by using the monotonicity of the functional, we get the required bounds (Propositions 4.1 and 4.2). This in turn implies a rate of convergence of order \( O(n^{-1/2}) \), uniformly in the initial surplus (Theorem 4.1); hence, we improve the estimates given by Gerber, Shiu, and Smith [8], which hold asymptotically with the initial surplus. In fact, due to the \( O(n^{-1/2}) \) jump sizes, it is the best rate that can be achieved for the general case (Remark 4.3). Moreover, this is the first time that a comparison principle is used to obtain the rate of convergence of these ruin probabilities. We believe that this technique can be applied in other actuarial and queueing applications, which we detail in Section 5.

Several actuarial researchers used comparison to bound the probability of ruin; however, they worked in the primary problem \( (n = 1) \) and did not apply comparison to the \( n \)-scaled problem. Specifically, Taylor [20] bounded the probability of ruin by using the integral version of this equation and comparison results for Volterra integral operators; see Walter [21] for these comparison results. De Vylder and Goovaerts [7] and Broeckx, De Vylder, and Goovaerts [4] continued the work of Taylor [20], using a simpler comparison lemma.

The remainder of the paper is organized as follows. In Section 2 we present the Cramér-Lundberg model and prove a comparison lemma for the integro-differential equation that determines the probability of ruin in that model. In Section 3 we scale the model by \( n \) and formally expand the probability of ruin as a sum of functions with coefficients equal to powers of \( n^{-1/2} \), which is related in flavor to the asymptotic analysis introduced by Sircar and Papanicolaou [19] in the mathematical finance literature. In that section, we observe that the first term in the expansion, \( \psi(0) \), the term with coefficient \( n^0 \) is identical to the probability of ruin under the diffusion approximation, and we end that section with two examples. In Section 4 we prove that that the probability of ruin in the scaled model approaches \( \psi(0) \) at a rate of convergence of order \( O(n^{-1/2}) \), uniformly in the initial surplus. We also strengthen this result for the special case of exponentially distributed claims. Section 5 concludes our paper.
2 Classical risk model and comparison lemma

2.1 Cramér-Lundberg model

Consider an insurer whose surplus process $X = \{X_t\}_{t \geq 0}$ is described by the classical Cramér-Lundberg model, that is, the insurer receives premium income at a constant rate $c$ and pays claims according to a compound Poisson process. Specifically,

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$

in which $X_0 = x \geq 0$ is the initial surplus, $N = \{N_t\}_{t \geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and the claim sizes $Y_1, Y_2, \ldots$ are independent and identically distributed, positive random variables, independent of $N$. Let $F_Y$ denote the common cumulative distribution function of $\{Y_i\}_{i \in \mathbb{N}}$. Assume that $Y$ has finite moment generating function $M_Y(u) = \mathbb{E}(e^{Yu})$ for $u$ in a neighborhood of 0, say, for $u \in (-u_0, u_0)$ for some $u_0 > 0$; thus, $\mathbb{E}(Y^k) < \infty$ for $k = 1, 2, \ldots$. Finally, assume that the premium rate $c$ satisfies $c > \lambda \mathbb{E}Y$ (otherwise, eventual ruin is certain), and write $c = (1 + \theta)\lambda \mathbb{E}Y$, with positive risk loading $\theta > 0$.

Define the time of ruin $\tau$ by

$$\tau = \inf\{t \geq 0 : X_t < 0\},$$

and define the probability of ultimate ruin by

$$\psi(x) = \mathbb{P}(\tau < \infty \mid X_0 = x).$$

Recall that $\psi(0) = 1/(1 + \theta)$.

By standard stochastic control theory and verification results, if we find a classical solution $v$ of the following integro-differential equation on $\mathbb{R}^+$, then $v$ equals the probability of ruin $\psi$.

$$\begin{cases}
\lambda v(x) = cv_x(x+) + \lambda \int_0^x v(x-y)dF_Y(y) + \lambda S_Y(x), & x > 0, \\
\lim_{x \to \infty} v(x) = 0,
\end{cases}$$

in which $S_Y = 1 - F_Y$ is $Y$'s survival function.

Remark 2.1. In a recent risk theory text, Schmidli [17] showed that one can rewrite the diff-
ferential equation in (2.4) as an integral equation, as follows:

\[ cv(x) = \lambda \int_0^x v(x - y)S_Y(y)dy + \lambda \int_x^\infty S_Y(y)dy. \quad (2.5) \]

It is this form of the equation that Taylor [20], De Vylder and Goovaerts [7], and Broeckx, De Vylder, and Goovaerts [4] used to find bounds for the probability of ruin. Theorem 1.2.1 in De Vylder and Goovaerts [7] proves that (2.5) has a unique solution; thus, (2.4) has a unique solution, and it equals the probability of ruin.

In future work, we will control the surplus process \( X \), and an integral equation of the form (2.5) will not readily apply in that case. Thus, anticipating that future work, we continue with the integro-differential equation in (2.4).

\[ \square \]

### 2.2 Comparison lemma

We look for bounds for the probability of ruin \( \psi \) as sub- and super-solutions of (2.4). Thus, we begin by proving a comparison lemma, which we use to determine whether a given function is a lower or upper bound for \( \psi \).

Define the operator \( F \) by

\[
F(x, u(x), u_x(x+), u(\cdot)) = -cu_x(x+) - \lambda \left( \int_0^x u(x - y)dF_Y(y) + S_Y(x) - u(x) \right). \quad (2.6)
\]

**Lemma 2.1.** (Comparison lemma). Let \( 0 \leq a < b \leq \infty \), and consider functions \( u, v \in C([0, b]) \) with continuous first derivatives, except possible at points of discontinuity of \( F_Y \), where \( u \) and \( v \) have left- and right-derivatives. Suppose \( u \) and \( v \) are such that \( u(x) \leq v(x) \) for all \( 0 \leq x \leq a \) and \( u(b) \leq v(b) \). Furthermore, suppose

\[
F(x, u(x), u_x(x+), u(\cdot)) < F(x, v(x), v_x(x+), v(\cdot)),
\]

for all \( x \in (a, b) \), then \( u(x) < v(x) \) for all \( x \in (a, b) \).

**Proof.** First, if the maximum of \( u - v \) on \([a, b]\) occurs at \( x = a \) or \( x = b \), but not in the interior of \((a, b)\), then \( u < v \) in the interior because \( u - v \leq 0 \) on the boundary, by assumption.

Second, if \( u - v \) attains a strictly negative maximum in the interior of \((a, b)\), then we also have \( u < v \) in the interior.

\[ ^1 \text{If } b = \infty, \text{ then } u(b) \text{ denotes } \lim_{x \to \infty} u(x); \text{ similarly, for } v(b). \]
Third, if \( u - v \) attains a non-negative maximum at \( x_0 \in (a, b) \), then \( u_x(x_0+) - v_x(x_0+) \leq 0 \). It follows that

\[
0 < F(x_0, v(x_0), v_x(x_0+), v(\cdot)) - F(x_0, u(x_0), u_x(x_0+), u(\cdot)) \\
= -cv_x(x_0+) - \lambda \int_0^{x_0} v(x_0 - y)dF_{Y}(y) - \lambda S_Y(x_0) + \lambda v(x_0) \\
+ cu_x(x_0+) + \lambda \int_0^{x_0} u(x_0 - y)dF_{Y}(y) + \lambda S_Y(x_0) - \lambda u(x_0) \\
\leq \lambda \int_0^{x_0} (u(x_0 - y) - v(x_0 - y))dF_{Y}(y) - \lambda(u(x_0) - v(x_0)).
\]

The last line is non-positive. Indeed, because \( u - v \) reaches a non-negative maximum at \( x = x_0 \in (a, b) \), we have \( u(x_0) - v(x_0) \geq u(x) - v(x) \) for all \( x \in (a, b) \). Furthermore, because \( u(x) \leq v(x) \) for all \( 0 \leq x \leq a \), we have \( u(x_0) - v(x_0) \geq u(x) - v(x) \) for all \( x \in (0, b) \). Without loss of generality, we can extend \( u \) and \( v \) into \( \mathbb{R}^- \) by setting \( u(x) = v(x) = 1 \) for \( x < 0 \), from which it follows that \( u(x_0) - v(x_0) \geq u(x) - v(x) \) for all \( x < b \). We deduce that \( u(x) - u(x_0) \leq v(x) - v(x_0) \) for all \( x \leq x_0 \), which implies

\[
0 < \lambda \int_0^{x_0} (u(x_0 - y) - v(x_0 - y))dF_{Y}(y) - \lambda(u(x_0) - v(x_0)) \\
= \lambda \int_0^{\infty} \left((u(x_0 - y) - u(x_0)) - (v(x_0 - y) - v(x_0))\right)dF_{Y}(y) \leq 0,
\]

a contradiction. Thus, \( u < v \) in \( (a, b) \).

\( \square \)

**Remark 2.2.** If we only want non-strict comparison, that is, \( u \leq v \), then we can weaken the sub-(super-)solution property to \( F(x, u(x), u_x(x+), u(\cdot)) < F(x, v(x), v_x(x+), v(\cdot)) \), with \( F(x, u(x), u_x(x+), u(\cdot)) \) finite.

\( \square \)

In the next example, we use Lemma 2.1 and Remark 2.2 to re-prove the well known Lundberg bound.

**Example 2.1.** Suppose \( R > 0 \) solves

\[
cR = \lambda(M_Y(R) - 1),
\]

that is, \( R \) is the adjustment coefficient, and define \( v(x) = e^{-Rx} \). Let \( a = 0 \) and \( b = \infty \) in Lemma 2.1 then, \( \psi(0) = 1/(1 + \theta) \leq 1 = v(0) \). Also, for \( x > 0 \), \( F(x, \psi(x), \psi_x(x+), \psi(\cdot)) = 0 \),
and
\[ F(x, v(x), v_x(x), v(\cdot)) = cRe^{-Rx} - \lambda \left( \int_0^x e^{-R(x-y)} dF_Y(y) + S_Y(x) - e^{-Rx} \right) \]
\[ = \lambda e^{-Rx} \left[ (M_Y(R) - 1) - \left( \int_0^x e^{Ry} dF_Y(y) + e^{Rx} S_Y(x) - 1 \right) \right] \]
\[ = \lambda e^{-Rx} \int_x^\infty (e^{Ry} - e^{Rx}) dF_Y(y) \geq 0. \]

We deduce from the non-strict version of Lemma 2.1 that \( \psi(x) \leq e^{-Rx} \) for \( x > 0 \), the Lundberg bound. \( \square \)

3 Scaled model and asymptotic expansion

3.1 Scaled model

Next, we scale our model by \( n > 0 \). In the scaled system, define \( \lambda_n = n\lambda \), so \( n \) large is essentially equivalent to \( \lambda \) large. Scale the claim severity by defining \( Y_n = Y/\sqrt{n} \); thus, the variance of total claims during \([0, t]\) is invariant under the scaling, that is, \( \lambda_n \mathbb{E}(Y_n^2) = \lambda \mathbb{E}(Y^2) \) for all \( n > 0 \). Finally, define the premium rate by \( c_n = c + (\sqrt{n} - 1)\lambda \mathbb{E}Y \); thus, \( c_n - \lambda_n \mathbb{E}Y_n = c - \lambda \mathbb{E}Y \) is also invariant under the scaling. We can also write \( c_n = (\sqrt{n} + \theta)\lambda \mathbb{E}Y \), in which \( c = (1 + \theta)\lambda \mathbb{E}Y \); moreover, we can write \( c_n = (1 + \theta_n)\lambda_n \mathbb{E}Y_n \), in which \( \theta_n = \theta/\sqrt{n} \). The diffusion approximation of the scaled surplus process is, therefore,

\[ (c_n - \lambda_n \mathbb{E}Y_n) dt + \sqrt{\lambda_n \mathbb{E}(Y_n^2)} dB_t = (c - \lambda \mathbb{E}Y) dt + \sqrt{\lambda \mathbb{E}(Y^2)} dB_t, \]

for some standard Brownian motion \( B = \{B_t\}_{t \geq 0} \), independent of \( n \). See Iglehart [10], Baüerle [3], Gerber, Shiu, and Smith [8], and Schmidli [17] for more information about this scaling.

3.2 Formal expansion of \( \psi_n \) and examples

Let \( \psi_n \) denote the probability of ruin under the scaled CL model. We wish to derive an expansion of \( \psi_n \) of the form

\[ \psi^{(0)}(x) + \frac{1}{\sqrt{n}} \psi^{(1)}(x) + \frac{1}{n} \psi^{(2)}(x) + O(n^{-3/2}), \]

(3.2)
in which \( \psi^{(0)} \), \( \psi^{(1)} \), and \( \psi^{(2)} \) are independent of \( n \). To that end, consider the last two terms in (2.4) without the common factor of \( \lambda_n \):

\[
\int_0^x \psi_n(x - y) dF_Y(y) + S_Y(x) = \int_0^x \psi_n(x - y) dF_Y(\sqrt{n}y) + S_Y(\sqrt{n}x) \\
= \int_0^{\sqrt{n}x} \psi_n \left( x - \frac{t}{\sqrt{n}} \right) dF_Y(t) + S_Y(\sqrt{n}x).
\]

Expand \( \psi_n \left( x - \frac{t}{\sqrt{n}} \right) \) in a series about \( x \), so that the above expression becomes

\[
\int_0^{\sqrt{n}x} \left\{ \psi_n(x) - \frac{t\psi'_n(x)}{\sqrt{n}} + \frac{t^2\psi''_n(x)}{2n} - \frac{t^3\psi'''_n(x)}{6n^{3/2}} + \frac{t^4\psi''''_n(x)}{24n^2} + O(n^{-5/2}) \right\} dF_Y(t) + S_Y(\sqrt{n}x)
\]

\[
= \int_0^{\sqrt{n}x} \left\{ \psi_n(x) - \frac{t\psi'_n(x)}{\sqrt{n}} + \frac{t^2\psi''_n(x)}{2n} - \frac{t^3\psi'''_n(x)}{6n^{3/2}} + \frac{t^4\psi''''_n(x)}{24n^2} \right\} dF_Y(t)
\]

\[
+ \int_{\sqrt{n}x}^{\infty} \left\{ (1 - \psi_n(x)) + \frac{t\psi'_n(x)}{\sqrt{n}} - \frac{t^2\psi''_n(x)}{2n} + \frac{t^3\psi'''_n(x)}{6n^{3/2}} - \frac{t^4\psi''''_n(x)}{24n^2} \right\} dF_Y(t) + O(n^{-5/2}).
\]

From \( \mathbb{E}(Y^6) < \infty \), we deduce that, for every \( x > 0 \) and for every \( \varepsilon > 0 \), there exists \( N > 0 \) such that if \( n > N \), then

\[
0 \leq \int_{\sqrt{n}x}^{\infty} t^6 dF_Y(t) < \varepsilon,
\]

which implies

\[
0 \leq \int_{\sqrt{n}x}^{\infty} \frac{t^k}{k!n^{k/2}} dF_Y(t) < \frac{\varepsilon}{k!n^3x^{6-k}}, \quad (3.3)
\]

for \( k = 0, 1, \ldots, 6 \). Thus, to order \( 1/n^{3/2} \) (not uniformly in \( x \)), we can, therefore, replace the last two terms in (2.4) with

\[
n\lambda \left\{ \psi'_n(x)\mathbb{E}Y + \psi''_n(x)\mathbb{E}(Y^2) - \psi'''_n(x)\mathbb{E}(Y^3) + \psi''''_n(x)\mathbb{E}(Y^4) \right\} + O(n^{-3/2}),
\]

which gives us the following modified equation for \( \psi_n \):

\[
n\psi_n(x) = (\sqrt{n} + \theta)\mathbb{E}Y \psi'_n(x)
\]

\[
+ n \left\{ \psi'_n(x)\mathbb{E}Y + \psi''_n(x)\mathbb{E}(Y^2) - \psi'''_n(x)\mathbb{E}(Y^3) + \psi''''_n(x)\mathbb{E}(Y^4) \right\} + O(n^{-3/2}),
\]
or equivalently,

\[ 0 = \theta E Y'_n(x) + \frac{\psi''_n(x)E(Y^2)}{2} - \frac{\psi'''_n(x)E(Y^3)}{6\sqrt{n}} + \frac{\psi''''_n(x)E(Y^4)}{24n} + O(n^{-3/2}), \quad (3.4) \]

with boundary conditions \( \psi_n(0) = 1/(1 + \theta n) = \sqrt{n}/(\sqrt{n} + \theta) \) and \( \lim_{x \to \infty} \psi_n(x) = 0. \)

Insert the asymptotic expression for \( \psi_n \) from (3.2) into (3.4), and collect terms of the same order of \( 1/\sqrt{n} \). Also, expand the boundary condition at \( x = 0 \) in powers of \( 1/\sqrt{n} \):

\[ \psi_n(0) = 1 - \frac{\theta}{\sqrt{n}} + \frac{\theta^2}{n} + O(n^{-3/2}). \]

First, the terms of order \( 1/n^0 \) yield the following boundary-value problem (BVP):

\[
\begin{cases}
0 = \theta E Y'_n(x) + \frac{E(Y^2)}{2} \psi_n^{(0)}(x), & x > 0, \\
\psi_n^{(0)}(0) = 1, & \lim_{x \to \infty} \psi_n^{(0)}(x) = 0.
\end{cases} \quad (3.5)
\]

The solution of this equation is given by

\[ \psi_n^{(0)}(x) = e^{-\gamma x}, \quad (3.6) \]

for all \( x > 0 \), in which \( \gamma \) is given by

\[ \gamma = \frac{2\theta E Y}{E(Y^2)}. \quad (3.7) \]

**Remark 3.1.** Note that \( \psi_n^{(0)} \) in (3.6) is identical to the basic diffusion approximation of the probability of ruin found by Iglehart [10]; see Theorem 8 in that paper. More precisely, \( \psi_n^{(0)} \) equals the probability of ruin for a surplus process that follows the diffusion given in (3.1). Because the diffusion in (3.1) approximates the CL risk process in (2.1) with \( \lambda, Y, \text{ and } c \) replaced by \( \lambda_n, Y_n, \text{ and } c_n \), respectively, researchers often say that \( \psi_n^{(0)} \) approximates \( \psi_n \). In Theorem 4.1 in the next section, we quantify the degree to which \( \psi_n^{(0)} \) approximates \( \psi_n \). \( \square \)

Second, the terms of order \( 1/\sqrt{n} \) yield the following BVP:

\[
\begin{cases}
0 = \theta E Y'_n(x) + \frac{E(Y^2)}{2} \psi_n^{(1)}(x) - \frac{E(Y^3)}{6} \psi_n^{(0)}(x), & x > 0, \\
\psi_n^{(1)}(0) = -\theta, & \lim_{x \to \infty} \psi_n^{(1)}(x) = 0.
\end{cases} \quad (3.8)
\]
After substituting for $\psi^{(0)}$ and solving the resulting differential equation, we obtain

$$
\psi^{(1)}(x) = \left[ \frac{\gamma^2}{3} \frac{\mathbb{E}(Y^3)}{\mathbb{E}(Y^2)} x - \theta \right] e^{-\gamma x}.
$$

(3.9)

Third, the terms of order $1/n$ yield the following BVP:

$$
\begin{cases}
0 = \theta \mathbb{E} \psi^{(2)}_x(x) + \frac{\mathbb{E}(Y^2)}{2} \psi^{(2)}_x(x) - \frac{\mathbb{E}(Y^3)}{6} \psi^{(1)}_{xxx}(x) + \frac{\mathbb{E}(Y^4)}{24} \psi^{(0)}_{xxxx}(x), & x > 0, \\
\psi^{(2)}(0) = \theta^2, & \lim_{x \to \infty} \psi^{(2)}(x) = 0.
\end{cases}
$$

(3.10)

After substituting for $\psi^{(0)}$ and $\psi^{(1)}$ and solving the resulting differential equation, we obtain

$$
\psi^{(2)}(x) = \left[ \left( \frac{\gamma^2}{3} \frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y^2)} \right)^2 \left( \frac{x^2}{2} - \frac{2x}{\gamma} \right) + \left( \frac{\gamma^3}{12} \frac{\mathbb{E}(Y^4)}{\mathbb{E}(Y^2)} - \frac{\theta \gamma^2}{3} \frac{\mathbb{E}(Y^3)}{\mathbb{E}(Y^2)} \right) x + \theta^2 \right] e^{-\gamma x}.
$$

(3.11)

By putting these three terms together, we obtain the following asymptotic expansion for $\psi_n$:

$$
\psi_n(x) = e^{-\gamma x} + \frac{1}{\sqrt{n}} \left[ \frac{\gamma^2}{3} \frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y^2)} x - \theta \right] e^{-\gamma x} \\
+ \frac{1}{n} \left[ \left( \frac{\gamma^2}{3} \frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y^2)} \right)^2 \left( \frac{x^2}{2} - \frac{2x}{\gamma} \right) + \left( \frac{\gamma^3}{12} \frac{\mathbb{E}(Y^4)}{\mathbb{E}(Y^2)} - \frac{\theta \gamma^2}{3} \frac{\mathbb{E}(Y^3)}{\mathbb{E}(Y^2)} \right) x + \theta^2 \right] e^{-\gamma x} + O(n^{-3/2}).
$$

(3.12)

In Section 4, we use Lemma 2.1 to compare the first term in this expansion with $\psi_n$. In Appendix B we discuss the difficulty in comparing the first two terms with $\psi_n$.

Gerber, Shiu, and Smith also obtained an expansion for $\psi$ from an asymptotic-in-$x$ expression for $\psi(x)$; their parameter $m$ corresponds to $1/\sqrt{n}$ in this paper. For comparison purposes, their equations (9.11) and (9.12) give the following asymptotic expansion; it is
asymptotic in both $x$ and $n$:

$$\psi(x) \sim e^{-\gamma x} + \frac{1}{\sqrt{n}} \frac{\gamma}{3\mathbb{E}(Y^2)} (\gamma x - 1) e^{-\gamma x}$$

$$+ \frac{1}{n} \left[ \left( \frac{\gamma^2}{3\mathbb{E}(Y^2)} \right)^2 \left( \frac{x^2}{2} - \frac{3x}{\gamma} + \frac{3}{\gamma^2} \right) + \frac{\gamma^3}{12\mathbb{E}(Y^4)} \left( x - \frac{2}{\gamma} \right) \right] e^{-\gamma x} + O(n^{-3/2}).$$

(3.13)

Note that the $x$-large-dominant terms in (3.12) and (3.13) of each order equal. Specifically, the $1/n^0$-order terms match; in the $1/\sqrt{n}$-order terms, the coefficients of $xe^{-\gamma x}$ match; and, in the $1/n$-order terms, the coefficients of $x^2e^{-\gamma x}$ match.

In the first example, we compute the asymptotic expansion when $Y$ is distributed exponentially.

\textbf{Example 3.1.} Suppose $Y \sim \text{Exp}(\beta)$ with mean $1/\beta$; then, $Y_n \sim \text{Exp}(\sqrt{n}\beta)$ and

$$\psi_n(x) = \frac{1}{1 + \theta_n} \exp \left( - \frac{\theta_n x}{(1 + \theta_n)EY_n} \right)$$

$$= \frac{1}{1 + \frac{\theta}{\sqrt{n}}} \exp \left( - \frac{\theta \beta x}{1 + \frac{\theta}{\sqrt{n}}} \right) \quad (3.14)$$

$$= e^{-\theta \beta x} \left\{ 1 + \frac{\theta}{\sqrt{n}} (\theta \beta x - 1) + \frac{\theta^2}{n} \left( \frac{1}{2} (\theta \beta)^2 x^2 - 2 \theta \beta x + 1 \right) \right\} + O(n^{-3/2}), \quad (3.15)$$

which one can show equals the expansion in (3.12). By comparison, the expansion in (3.13) yields the identical expression.

In Figure 1, we plot the graphs of $\psi_n$ in (3.14) and $\psi^{(0)} + \psi^{(1)}/\sqrt{n} + \psi^{(2)}/n$ in (3.15). The parameter values are $\beta = 1$, $\theta = 0.4$, and $n = 1$. From this figure, we observe that $\psi_n$ and its approximation are close, especially given that $\theta$ is not small. Recall that $\theta_n = \theta/\sqrt{n}$ will become small as $n$ increases.

In the second example, we compute the asymptotic expansion when $Y$ is distributed according to the Gamma with shape parameter 2.
Example 3.2. Suppose $Y \sim \text{Gamma}(2, \beta)$ with mean $2/\beta$; then, $Y_n \sim \text{Gamma}(2, \sqrt{n}\beta)$ and

$$
\psi_n(x) = \frac{\theta_n}{2(1 + \theta_n)} \left\{ \frac{2\sqrt{n}\beta - R_n}{2\theta_n \cdot \sqrt{n}\beta - \frac{3+4\theta_n}{2} R_n} e^{-R_n x} + \frac{2\sqrt{n}\beta - r_n}{2\theta_n \cdot \sqrt{n}\beta - \frac{3+4\theta_n}{2} r_n} e^{-r_n x} \right\}
$$

$$
= \frac{\theta}{2(1 + \theta \sqrt{n})} \left\{ \frac{2\beta - R_n}{2\theta \beta - \frac{3+4\theta / \sqrt{n}}{2} R_n} e^{-R_n x} + \frac{2\beta - r_n}{2\theta \beta - \frac{3+4\theta / \sqrt{n}}{2} r_n} e^{-r_n x} \right\},
$$

(3.16)
in which $R_n$ is the adjustment coefficient

$$
R_n = \frac{\sqrt{n}\beta}{4(1 + \theta / \sqrt{n})} \left[ \left( 3 + \frac{4\theta}{\sqrt{n}} \right) - \sqrt{9 + \frac{8\theta}{\sqrt{n}}} \right],
$$

and $r_n$ is

$$
r_n = \frac{\sqrt{n}\beta}{4(1 + \theta / \sqrt{n})} \left[ \left( 3 + \frac{4\theta}{\sqrt{n}} \right) + \sqrt{9 + \frac{8\theta}{\sqrt{n}}} \right].
$$

One can show that $\lim_{n \to \infty} R_n = \frac{2}{3} \theta \beta = \gamma$, and the asymptotic expansion in (3.12) equals

$$
e^{-\frac{2}{3} \theta \beta x} \left\{ 1 + \frac{\theta}{\sqrt{n}} \left( \frac{16}{27} \theta \beta x - 1 \right) + \frac{\theta^2}{n} \left( \frac{128}{729} (\theta \beta)^2 x^2 - \frac{232}{243} \theta \beta x + 1 \right) \right\} + \mathcal{O}(n^{-3/2}).
$$

(3.17)

By comparison, the asymptotic expansion in (3.13) equals

$$
e^{-\frac{2}{3} \theta \beta x} \left\{ 1 + \frac{\theta}{\sqrt{n}} \left( \frac{16}{27} \theta \beta x - \frac{8}{9} \right) + \frac{\theta^2}{n} \left( \frac{128}{729} (\theta \beta)^2 x^2 - \frac{88}{81} \theta \beta x + \frac{8}{9} \right) \right\} + \mathcal{O}(n^{-3/2}),
$$

(3.18)

which one can show equals the asymptotic-in-n expansion of the $e^{-R_n x}$ term of $\psi_n$ in (3.16).

There is not obvious corresponding expansion for the $e^{-r_n x}$ because $r_n$ is of order $\sqrt{n}$ and, therefore, goes to infinity as $n$ goes to infinity. Recall that the expression in (3.13) is asymptotic in both $x$ and $n$; thus, it makes sense that it corresponds to the asymptotic-in-n expansion of the $e^{-R_n x}$ term of $\psi_n$ because this term dominates as $x$ gets large. On the other hand, the expression in (3.17) better matches the boundary condition at $x = 0$.

In Figure 2, we plot the graphs of $\psi_n$ in (3.16), $\psi^{(0)} + \psi^{(1)} / \sqrt{n} + \psi^{(2)} / n$ in (3.17), and the asymptotic-in-x-and-n expansion in (3.18) from Gerber, Shiu, and Smith [8]. We used the same parameter values as in Figure 1. From this figure, we see that all three graphs are close, with $\psi^{(0)} + \psi^{(1)} / \sqrt{n} + \psi^{(2)} / n$ approximating $\psi_n$ slightly better than the approximation of Gerber, Shiu, and Smith [8] for small values of $x$, which is not surprising given that $\psi^{(0)}(0) + \psi^{(1)} / \sqrt{n}(0) +$
\[ \psi^{(2)}/n(0) = 1 - \theta/\sqrt{n} + \theta^2/n, \text{ which equals } \psi_n(0) = 1/(1 + \theta/\sqrt{n}) \text{ to order } O(n^{-3/2}). \]

\section{Asymptotic analysis}

In this section, we analyze the first term in the asymptotic expansion in (3.12) using Lemma 2.1 as it applies to the scaled problem.

Throughout this section, let \( F_n \) denote the operator in (2.6) with \( c_n, \lambda_n, \text{ and } Y_n \) replaced by \( c_n, \lambda_n, \text{ and } Y_n \), respectively, and write \( F_n \) as follows:

\[
F_n(x, u(x), u_x(x+), u(\cdot)) = -\lambda \left[ (\sqrt{n} + \theta)E Y u_x(x+) + n \int_{\sqrt{n}x}^{\sqrt{n}x} u(x - t/\sqrt{n}) dF_Y(t) + S_Y(\sqrt{n}x) - u(x) \right]. \tag{4.1}
\]

In the next two propositions, we modify \( \psi^{(0)} \) by functions of order \( O(n^{-1/2}) \) to obtain lower and upper bounds for \( \psi_n \), respectively. In Appendix A we present background calculations that inspired these bounds. We begin by modifying \( \psi^{(0)} \) to obtain a lower bound for \( \psi_n \).

**Proposition 4.1.** Formally, define the random variable \( Z_d = (Y - d) \mid (Y > d) \) for \( d \geq 0 \), and let \( N \) be such that \( M_Y(\gamma/\sqrt{N}) < \infty \). Assume

\[
\sup_{d \geq 0} \mathbb{E} \left( Z_d^2 e^{\gamma \sqrt{N}Z_d} \right) < \infty, \tag{4.2}
\]

for all \( \gamma > 0 \). If we define \( \delta \) by

\[
\delta = \max \left[ \theta, \sup_{d \geq 0} \left( \gamma \mathbb{E} Z_d + \frac{\gamma^2}{\sqrt{N}} \int_{0}^{1} (1 - \omega) \mathbb{E} \left( Z_d^2 e^{\gamma \sqrt{N}Z_d} \right) d\omega \right) \right], \tag{4.3}
\]

then, for all \( n > N \),

\[
\left( 1 - \frac{\delta}{\sqrt{n}} \right) \psi^{(0)}(x) < \psi_n(x), \tag{4.4}
\]

for all \( x \geq 0 \).

**Proof.** Because \( \delta \geq \theta \), we have

\[
\left( 1 - \frac{\delta}{\sqrt{n}} \right) \psi^{(0)}(0) = 1 - \frac{\delta}{\sqrt{n}} \leq 1 - \frac{\theta}{\sqrt{n}} < \frac{1}{1 + \frac{\theta}{\sqrt{n}}} = \psi_n(0).
\]

Also, \( \lim_{x \to \infty} \left( 1 - \frac{\delta}{\sqrt{n}} \right) \psi^{(0)}(x) = 0 = \lim_{x \to \infty} \psi_n(x) \).
Next, consider $F_n$ evaluated at $\ell_n$, in which $\ell_n(x) = (1 - \delta/\sqrt{n})\psi^{(0)}(x)$, and assume without loss of generality that $n > \delta^2$:

$$F_n(x, \ell_n(x), \ell'_n(x), \ell_n(\cdot))$$

$$= \lambda \left(1 - \frac{\delta}{\sqrt{n}}\right) e^{-\gamma x} \left\{ \left(\sqrt{n} + \theta\right) EY \gamma - n \int_{0}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - 1\right) dF_Y(t) \right\}$$

$$+ n\lambda \int_{\sqrt{n}x}^{\infty} \left(1 - \frac{\delta}{\sqrt{n}}\right) e^{\gamma \left(\frac{\gamma}{\sqrt{n}} - x\right)} - 1 \right) dF_Y(t)$$

$$\propto \left(\sqrt{n} + \theta\right) EY \gamma - n \int_{0}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - 1\right) dF_Y(t) + n \int_{\sqrt{n}x}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - \frac{1}{1 - \frac{\delta}{\sqrt{n}}} e^{\gamma x}\right) dF_Y(t)$$

$$\propto - \int_{0}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - e^{\gamma t} - \gamma^2 t^2 \frac{\gamma^2 t^2}{2n}\right) dF_Y(t) + \int_{\sqrt{n}x}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - \frac{1}{1 - \frac{\delta}{\sqrt{n}}} e^{\gamma x}\right) dF_Y(t). \quad (4.5)$$

The first integral is automatically negative; thus, if we find values of $\delta$ and $N > \delta^2$ for which the second integral is non-positive for all $n > N$ and for all $x \geq 0$, then Lemma 2.1 implies that $\ell_n(x) < \psi_n(x)$ for all $x \geq 0$ and for all $n > N$. To that end, consider the following inequality:

$$\int_{\sqrt{n}x}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - \frac{1}{1 - \frac{\delta}{\sqrt{n}}} e^{\gamma x}\right) dF_Y(t) \leq 0.$$

If $S_Y(\sqrt{n}x) = 0$, then the left side is identically 0, so suppose that $S_Y(\sqrt{n}x) > 0$. After replacing $\sqrt{n}x$ by $d$ and dividing by $e^{\gamma x}S_Y(d)$, the above inequality becomes

$$\int_{d}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - \frac{1}{1 - \frac{\delta}{\sqrt{n}}} e^{\gamma x}\right) \frac{dF_Y(t)}{S_Y(d)} \leq 0,$$

for $d \geq 0$. Define $Z_d = (Y - d)|Y > d$; then, this inequality becomes

$$\int_{0}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - \frac{1}{1 - \frac{\delta}{\sqrt{n}}} e^{\gamma x}\right) dF_{Z_d}(z) \leq 0,$$

or equivalently,

$$\int_{0}^{\infty} \left(e^{\frac{\gamma t}{\sqrt{n}}} - 1\right) dF_{Z_d}(z) \leq \frac{\delta}{1 - \frac{\delta}{\sqrt{n}}}.$$

If we find $\delta$ that satisfies the following stronger inequality, then the above sequence of inegal-
Rewrite the integrand from the left side of inequality (4.6) as follows:

\[ e^{\frac{\gamma z}{\sqrt{n}}} - 1 = \frac{\gamma z}{\sqrt{n}} + \frac{\gamma^2 z^2}{n} \int_0^1 (1 - \omega)e^{\frac{\gamma z}{\sqrt{n}} \omega} \, d\omega. \]

Thus, inequality (4.6) is equivalent to

\[
\int_0^\infty \left( \frac{\gamma z}{\sqrt{n}} + \frac{\gamma^2 z^2}{n} \int_0^1 (1 - \omega)e^{\frac{\gamma z}{\sqrt{n}} \omega} \, d\omega \right) dF_{Z_d}(z) \leq \frac{\delta}{\sqrt{n}},
\]

or, after multiplying both side by \( \sqrt{n} \) and switching the order of integration,

\[
\gamma \mathbb{E}Z_d + \frac{\gamma^2}{\sqrt{n}} \int_0^1 (1 - \omega) \mathbb{E}\left( Z_d^2 e^{\frac{\gamma z}{\sqrt{n}} Z_d} \right) \, d\omega \leq \delta. \tag{4.7}
\]

Note that the left side decreases with increasing \( n \). It follows that if we define \( N \) so that \( Y^3 \)‘s moment generating function is finite at \( \gamma/\sqrt{N} \), and if we define \( \delta \) as in (4.3), then inequality (4.7) holds for all \( d \geq 0 \) and all \( n > N \), which implies that \( F_n \) evaluated at \( \ell_n \) is negative for all \( x \geq 0 \) and all \( n > N \). The conclusion in (4.4), then, follows from Lemma 2.1.

In the following proposition, we modify \( \psi^{(0)} \) to obtain an upper bound for \( \psi_n \).

**Proposition 4.2.** Define the function \( \upsilon_n \) by

\[ \upsilon_n(x) = e^{-\left(\frac{\gamma}{\sqrt{n}} - \alpha\right)x} = \psi^{(0)}(x) e^{\frac{\alpha}{\sqrt{n}} x}. \tag{4.8} \]

If

\[ \alpha > \frac{\gamma^2 \mathbb{E}(Y^3)}{3 \mathbb{E}(Y^2)}, \tag{4.9} \]

then there exists \( N > 0 \) such that, for all \( n > N \),

\[ \psi_n(x) < \upsilon_n(x), \tag{4.10} \]

for all \( x \geq 0 \).

**Proof.** \( \psi_n(0) < 1 = \upsilon_n(0) \). Also, \( \lim_{x \to \infty} \psi_n(x) = 0 = \lim_{x \to \infty} \upsilon_n(x) \), if \( n > (\alpha/\gamma)^2 \).
Next, consider $F_n$ evaluated at $v_n$:

\[
F_n(x, v_n(x), v'_n(x), v_n(\cdot)) = \lambda e^{-\left(\gamma - \frac{\alpha}{\sqrt{n}}\right)x} \left\{ (\sqrt{n} + \theta)\mathbb{E} (\gamma - \frac{\alpha}{\sqrt{n}}) - n \int_0^\infty \left( e^{\left(\gamma - \frac{\alpha}{\sqrt{n}}\right)t} - 1 \right) dF_Y(t) \right\} \\
+ n\lambda \int_0^\infty \left( e^{\left(\gamma - \frac{\alpha}{\sqrt{n}}\right)(\sqrt{n} - x)} - 1 \right) dF_Y(t).
\]

(4.11)

The last line of (4.11) is automatically non-negative if $n > (\alpha/\gamma)^2$. The expression in curly brackets is independent of $x$; denote it by $A_n$. If we find values of $\alpha$ and $N > (\alpha/\gamma)^2$ for which $A_n$ is positive for all $n > N$, then Lemma 2.1 implies that $\psi_n(x) < v_n(x)$ for all $x \geq 0$ and for all $n > N$. Expand the exponential in the integrand in $A_n$ to obtain

\[
A_n = (\sqrt{n} + \theta)\mathbb{E} (\gamma - \frac{\alpha}{\sqrt{n}})
- n \int_0^\infty \left( e^{\left(\gamma - \frac{\alpha}{\sqrt{n}}\right)x} - 1 - \left( \gamma - \frac{\alpha}{\sqrt{n}} \right) \frac{t}{\sqrt{n}} - \left( \gamma - \frac{\alpha}{\sqrt{n}} \right)^2 \frac{t^2}{2n} - \left( \gamma - \frac{\alpha}{\sqrt{n}} \right)^3 \frac{t^3}{6n^{3/2}} \right) dF_Y(t)
- n \left( \gamma - \frac{\alpha}{\sqrt{n}} \right) \mathbb{E} (\gamma - \frac{\alpha}{\sqrt{n}})^2 \frac{\mathbb{E}(Y^2)}{2n} + \left( \gamma - \frac{\alpha}{\sqrt{n}} \right)^3 \frac{\mathbb{E}(Y^3)}{6n^{3/2}}
- \frac{\gamma}{2\sqrt{n}} \left( \alpha \mathbb{E}(Y^2) - \frac{\gamma^2}{3} \mathbb{E}(Y^3) \right) + \mathcal{O}(n^{-1}).
\]

Choose $\alpha$ as in (4.9); then, the first term in the above expression is strictly positive. Next, choose $N > (\alpha/\gamma)^2$ so that the absolute value of the remainder term in $A_N$ (if it is negative) is less than the first term. It follows that $A_n > 0$ for that choice of $\alpha$ and for all $n > N$. The conclusion in (4.10), then, follows from Lemma 2.1.

In the following theorem, we combine the results of Propositions 4.1 and 4.2.

**Theorem 4.1.** If (4.2) holds, then there exists $C > 0$ such that, for all $n > 0$ and $x \geq 0$,

\[
|\psi_n(x) - \psi^{(0)}(x)| \leq \frac{C}{\sqrt{n}}.
\]

(4.12)

Recall from (3.6) and (3.7) that $\psi^{(0)}(x) = e^{-\gamma x}$, with $\gamma = 2\theta \mathbb{E} Y / \mathbb{E}(Y^2)$.

**Proof.** From Propositions 4.1 and 4.2 it follows that

\[
\left( 1 - \frac{\delta}{\sqrt{n}} \right) e^{-\gamma x} < \psi_n(x) < e^{-\left( \gamma - \frac{\alpha}{\sqrt{n}} \right)x}.
\]
Subtracting $e^{-\gamma x}$ from each side yields,
\[-\frac{\delta}{\sqrt{n}} e^{-\gamma x} < \psi_n(x) - e^{-\gamma x} < e^{-(\gamma - \frac{\alpha}{\sqrt{n}})x} - e^{-\gamma x}.\]

Clearly, the left side is bounded below by $-\delta/\sqrt{n}$. Basic calculus implies that, for every $n > \left(\frac{\alpha}{\gamma}\right)^2$, the right side is bounded above by
\[
\left(1 - \frac{\alpha}{\gamma\sqrt{n}}\right)^{\frac{2\sqrt{n}}{\alpha}} \left(\frac{\alpha}{\gamma - \frac{\alpha}{\sqrt{n}}} \right) - 1.
\]

The first term converges to $e^{-1}$ and the second term is of order $O(n^{-1/2})$. Combining this upper bound with the lower bound, we deduce inequality (4.12). \(\square\)

**Remark 4.1.** Theorem 4.1 asserts that the rate of convergence of $\psi_n$ to $\psi^{(0)}$ is of order $O(n^{-1/2})$, and, moreover, that the convergences is uniform over $x \in [0, \infty)$. By using probabilistic techniques and relying on convergence in distribution of the underlying processes, others prove the pointwise convergence $\lim_{n \to \infty} \psi_n(x) = \psi^{(0)}(x)$ without estimating the rate. The first to do so in the actuarial literature is Iglehart [10]; for more recent work in this vein, see Baüerle [3]. \(\square\)

**Remark 4.2.** From the proof of Theorem 4.1 relative to the limit $e^{-\gamma x}$, we see that the relative error between $\psi_n$ and $e^{-\gamma x}$ is bounded as follows:
\[-\frac{\delta}{\sqrt{n}} < \frac{\psi_n(x) - e^{-\gamma x}}{e^{-\gamma x}} < e^{\frac{\alpha}{\sqrt{n}} x} - 1.
\]

Both lower and upper bounds are of order $O(n^{-1/2})$, but the upper bound is not uniform in $x$. \(\square\)

**Remark 4.3.** In this remark, we motivate the $O(n^{-1/2})$ rate of convergence. Note that the pre-limit process weakly converges to a Brownian motion (with drift), and by the Skorokhod representation theorem, we can think about the convergence as uniform over compact time intervals. Now, because the pre-limit process has jumps of order $O(n^{-1/2})$, it follows that at the first hitting time of 0 of the Brownian motion, the pre-limit process is in an $O(n^{-1/2})$-neighborhood of 0. Arguing by contradiction, assume for a moment that the rate in [11,12] can be improved to $o(n^{-1/2})$, then the probability of ever hitting 0, when starting at $c_1/\sqrt{n}$, is $e^{-(c_1/\sqrt{n})} + o(n^{-1/2}) \approx 1 + c_2/\sqrt{n} + o(n^{-1/2})$, for some scalar $c_2 \in \mathbb{R}$. The $c_2/\sqrt{n}$-term
yields that the difference between the hitting probabilities is of order $\mathcal{O}(n^{-1/2})$, contradicting the improvement we conjectured.

When $Y$ is exponentially distributed, we can strengthen the result of Theorem 4.1.

**Theorem 4.2.** If $Y$ is an exponential random variable, then there exists $C > 0$ such that, for all $n > 0$ and $x \geq 0$,

$$\left| \psi_n(x) - \left( \psi^{(0)}(x) + \frac{1}{\sqrt{n}} \psi^{(1)}(x) \right) \right| \leq \frac{C}{n}, \quad (4.13)$$

in which $\psi^{(1)}$ is given in (3.9).

**Proof.** From Propositions C.1 and C.2 in Appendix C, we get the following inequalities

$$-\frac{\kappa}{n} xe^{-\gamma x} < \psi_n(x) - \left( \psi^{(0)}(x) + \frac{1}{\sqrt{n}} \psi^{(1)}(x) \right) < \frac{1}{n} (\zeta + \delta x) e^{-\alpha x},$$

in which $\zeta > \theta^2$, $\delta$ satisfies (C.6), $\gamma / 2 < \alpha < \gamma$, and $\kappa$ is defined in (C.3). The bound in (4.13) follows since the functions $x \mapsto xe^{-\gamma x}$ and $x \mapsto (\zeta + \delta x) e^{-\alpha x}$ are bounded over all $x \geq 0$.

## 5 Summary and future research

We proved a comparison lemma (Lemma 2.1) for the integro-differential equation that determines the probability of ruin for the CL model. We also derived an asymptotic expansion for the probability of ruin $\psi_n$ in powers of $1/\sqrt{n}$. We showed that $\lim_{n \to \infty} \psi_n = \psi^{(0)}$, the 0-order term from that expansion. Moreover, in Theorem 4.1, we showed that the rate of convergence is of order $\mathcal{O}(n^{-1/2})$ and is uniform in $x \geq 0$. Generally, one cannot improve on this rate of convergence, which we detailed in Appendix B. That said, for exponentially distributed claims, in Theorem 4.2, we showed that $\psi^{(0)} + \psi^{(1)}/\sqrt{n}$ approximates $\psi_n$ up to order $\mathcal{O}(n^{-1})$, also uniformly in $x \geq 0$.

Many of the references in the bibliography also consider the finite-time ruin problem, which we did not address in this paper. So, in future work we will find expressions for the probability of ruin in finite time that are asymptotic in the Poisson rate of claims. More importantly, we will consider optimal control of the surplus process via reinsurance or optimal dividends. Diffusion approximations (DA) are commonly applied to the surplus process before applying controls because the problem becomes tractable. However, the optimal strategy in the CL case can be much different than the one obtained in the DA case. For example, when minimizing...
the probability of ruin under the CL model, the optimal per-claim retention strategy for small values of surplus is to retain all of one’s claims; by contrast, under the DA model, the optimal per-claim retention is strictly positive as surplus approaches 0. It would be interesting to see if the first-order term in an asymptotic-in-$n$ expansion of the optimal retention strategy were to reflect this difference.

Gerber, Shiu, and Smith [8] addressed approximations to the dividend problem. Also, Baüerle [3] considered the large-$\lambda$ approximation of the dividend problem and proved that, as the Poisson rate increases without bound with the corresponding scaling of claim severity as in this paper, the optimal value function converges to the one under the DA as $\lambda$ goes to infinity. It would be interesting to determine if the rate of convergence of the optimal barrier is of order $O\left(n^{-1/2}\right)$, as suggested by the work in this paper.

Another line of research we will pursue is to improve the existing estimates for busy periods and sojourn times in queueing systems. Recall, from the Introduction, that the diffusion approximation is often used in stochastic networks and is called the heavy-traffic approximation. Also, integro-differential equations are commonly used to estimate expectations and probabilities, see, for example, [20, 16, 2]. Hence, it would be natural to formulate the integro-differential for the scaled system and apply the method in this paper to attain convergence plus its rate for some magnitudes of interest.

A $F_n$ evaluated at $\psi^{(0)}$

In this appendix, we present the calculations that inspired Propositions 4.1 and 4.2.

\begin{align}
F_n(x, \psi^{(0)}(x), \psi_x^{(0)}(x), \psi^{(0)}(\cdot))
= \lambda e^{-\gamma x} \left\{ (\sqrt{n} + \theta) \mathbb{E}Y - n \int_0^\infty \left( e^{\frac{\gamma t}{\sqrt{n}}} - 1 \right) dF_Y(t) \right\} + n\lambda \int_\mathbb{R} \left( e^{\gamma (\frac{\gamma t}{\sqrt{n}} - 1)} - 1 \right) dF_Y(t)
= \lambda e^{-\gamma x} \left\{ (\sqrt{n} + \theta) \mathbb{E}Y - n \int_0^\infty \left( \frac{\gamma t}{\sqrt{n}} + \frac{\gamma^2 t^2}{2n} \right) dF_Y(t) \right\} + n\lambda \int_\mathbb{R} \left( e^{\gamma (\frac{\gamma t}{\sqrt{n}} - 1)} - 1 \right) dF_Y(t)
- n\lambda e^{-\gamma x} \int_0^\infty \left( e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} \right) dF_Y(t)
\quad \text{ (A.1)}
\end{align}
The terms in the curly brackets cancel, and we are left with

\[ F_n(x, \psi^{(0)}(x), \psi^{(0)}(\cdot)) \]

\[ = -n\lambda e^{-\gamma x} \int_0^\infty \left( e^{\frac{\gamma t}{\sqrt{n}}} - 1 \right) dF_Y(t) + n\lambda \int_0^\infty \left( e^{\frac{\gamma}{\sqrt{n} \gamma_x} - 1} \right) dF_Y(t) \]

\[ = -n\lambda e^{-\gamma x} \int_0^\infty \frac{\gamma^3 t^3}{2n^{3/2}} \int_0^1 (1 - \omega)^2 e^{\frac{\gamma t}{\sqrt{n}}} \omega d\omega dF_Y(t) + n\lambda \int_0^\infty \left( e^{\frac{\gamma}{\sqrt{n} \gamma_x} - 1} \right) dF_Y(t) \]

\[ = -\lambda e^{-\gamma x} \frac{\gamma^3}{2\sqrt{n}} \int_0^1 (1 - \omega)^2 \mathbb{E}\left(Y^3 e^{\frac{\gamma^2}{n} \omega Y}\right) d\omega + n\lambda \int_0^\infty \left( e^{\frac{\gamma}{\sqrt{n} \gamma_x} - 1} \right) dF_Y(t). \]

The first term is negative and of order \( \mathcal{O}(n^{-1/2}) \). If we (formally) define \( Z_d = (Y - d) | (Y > d) \) and set \( d = \sqrt{n}x \), then the second term becomes

\[ n\lambda \int_0^\infty \left( e^{\frac{\gamma}{\sqrt{n} \gamma_x} - 1} \right) dF_Y(t) = n\lambda S_Y(d) \int_d^\infty \left( e^{\frac{\gamma}{\sqrt{n} \gamma_x} (t-d)} - 1 \right) \frac{dF_Y(t)}{S_Y(d)} \]

\[ = n\lambda S_Y(d) \int_0^\infty \left( e^{\frac{\gamma^2}{n} \omega} - 1 \right) dF_{Z_d}(z) = n\lambda S_Y(d) \int_0^\infty \frac{\gamma z}{\sqrt{n}} \int_0^1 e^{\frac{\gamma^2}{n} \omega} d\omega dF_{Z_d}(z) \]

\[ = \sqrt{n} \gamma S_Y(d) \int_0^1 \mathbb{E}\left(Z_d e^{\frac{\gamma^2}{n} Z_d}\right) d\omega, \]

which is positive and of order \( \mathcal{O}(\sqrt{n}) \).

To obtain a lower bound for \( \psi_n \), we modify \( \psi^{(0)} \) so that the corresponding modified second term is negative, and that is the gist of Proposition 4.1. The scaling does not affect the negative sign of the first term, and it makes the second term negative. Also, note that the scaling effectively subtracts a term of order \( \mathcal{O}(n^{-1/2}) \) from \( \psi^{(0)} \).

To obtain an upper bound for \( \psi_n \), we modify \( \psi^{(0)} \) so that the corresponding modified first term is positive, and that is the gist of Proposition 4.2. The additional exponent of \( \alpha/\sqrt{n} \) does not affect the positive sign of the second term, and it makes the first term positive. Also, note that the modification of \( \psi^{(0)} \)'s exponent effectively adds a term of order \( \mathcal{O}(n^{-1/2}) \) to \( \psi^{(0)} \).

**B** \( F_n \) evaluated at \( \psi^{(0)} + \psi^{(1)} / \sqrt{n} \)

In this section, we analyze \( F_n \) evaluated at \( \psi^{(0)} + \psi^{(1)} / \sqrt{n} \) for the purpose of modifying it to obtain upper and lower bounds for \( \psi_n \), and we will discover the difficulty in doing so.
\[ F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi_x^{(0)}(x) + \psi_x^{(1)}(x)/\sqrt{n}, \psi(0)(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n}) \]

\[
= -\lambda e^{-\gamma x} (\sqrt{n} + \theta) \mathbb{E} \left\{ -\gamma + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \mathbb{E}(Y^3) (1 - \gamma x) + \gamma \theta \right) \right\} \\
- n\lambda e^{-\gamma x} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 \right\} dF_Y(t) \\
- \sqrt{n} \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} \mathbb{E}(Y^2) - \theta \right\} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 \right\} dF_Y(t) + \lambda e^{-\gamma x} \frac{\gamma^2}{3} \mathbb{E}(Y^3) \int_0^\infty t e^{\frac{\gamma t}{\sqrt{n}}} dF_Y(t) \\
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma (x - \frac{t}{\sqrt{n}})} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \mathbb{E}(Y^2) \left( x - \frac{t}{\sqrt{n}} \right) - \theta \right) e^{-\gamma (x - \frac{t}{\sqrt{n}})} - 1 \right\} dF_Y(t).
\]

Expand the first integral, as in (A.1), and cancel some of the terms to obtain

\[ F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi_x^{(0)}(x) + \psi_x^{(1)}(x)/\sqrt{n}, \psi(0)(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n}) \]

\[
= -\lambda e^{-\gamma x} \left( 1 + \frac{\theta}{\sqrt{n}} \right) \mathbb{E} \left\{ \frac{\gamma^2}{3} \mathbb{E}(Y^3) (1 - \gamma x) + \gamma \theta \right\} \\
- n\lambda e^{-\gamma x} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} \right\} dF_Y(t) \\
- \sqrt{n} \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} \mathbb{E}(Y^2) - \theta \right\} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 \right\} dF_Y(t) + \lambda e^{-\gamma x} \frac{\gamma^2}{3} \mathbb{E}(Y^3) \int_0^\infty t e^{\frac{\gamma t}{\sqrt{n}}} dF_Y(t) \\
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma (x - \frac{t}{\sqrt{n}})} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \mathbb{E}(Y^2) \left( x - \frac{t}{\sqrt{n}} \right) - \theta \right) e^{-\gamma (x - \frac{t}{\sqrt{n}})} - 1 \right\} dF_Y(t).
\]
Expand the integrands, as needed, to be able to work with the terms up to order \(O(n^{-1/2})\), except for the integral from \(\sqrt{n}x\) to infinity.

\[
F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n})
= -\lambda e^{-\gamma x} \left(1 + \frac{\theta}{\sqrt{n}} \right) EY \left\{ \frac{\gamma^2}{3} E(Y^3) (1 - \gamma x) + \gamma \theta \right\} \\
- n\lambda e^{-\gamma x} \int_0^\infty \left\{ e^{x/\sqrt{n}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} - \frac{\gamma^3 t^3}{6n^{3/2}} \right\} dF_Y(t) - \lambda e^{-\gamma x} \frac{\gamma^3}{6\sqrt{n}} E(Y^3) \\
- \sqrt{n} \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} E(Y^3) x - \theta \right\} \int_0^\infty \left\{ e^{x/\sqrt{n}} - 1 - \frac{\gamma t}{\sqrt{n}} \right\} dF_Y(t) \\
- \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} E(Y^3) x - \theta \right\} \left( \gamma EY + \frac{\gamma^2}{2\sqrt{n}} E(Y^2) \right) \\
+ \lambda e^{-\gamma x} \frac{\gamma^2}{3} E(Y^3) \int_0^\infty t \left\{ e^{x/\sqrt{n}} - 1 - \frac{\gamma t}{\sqrt{n}} \right\} dF_Y(t) + \lambda e^{-\gamma x} \frac{\gamma^2}{3} E(Y^3) \left( \gamma EY + \frac{\gamma}{\sqrt{n}} E(Y^2) \right) \\
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma (x - \frac{1}{\sqrt{n}})} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} E(Y^3) \left( x - \frac{t}{\sqrt{n}} \right) - \theta \right) e^{-\gamma (x - \frac{1}{\sqrt{n}})} - 1 \right\} dF_Y(t).
\]

Group the terms according to their order with respect to powers of \(n\).

\[
F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n})
= \lambda e^{-\gamma x} \left\{ -EY \left( \frac{\gamma^2}{3} E(Y^3) (1 - \gamma x) + \gamma \theta \right) - \left( \frac{\gamma^2}{3} E(Y^3) x - \theta \right) \gamma EY + \frac{\gamma^2}{3} E(Y^3) EY \right\} \\
+ \frac{\lambda}{\sqrt{n}} \left( e^{-\gamma x} \left( -\theta EY \left( \frac{\gamma^2}{3} E(Y^3) (1 - \gamma x) + \gamma \theta \right) - \frac{\gamma^3}{6} E(Y^3) \\
- \left( \frac{\gamma^2}{3} E(Y^3) x - \theta \right) \frac{\gamma^2}{2} E(Y^2) + \frac{\gamma^3}{3} E(Y^3) \right) \right) \\
-n\lambda e^{-\gamma x} \int_0^\infty \left\{ e^{x/\sqrt{n}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} - \frac{\gamma^3 t^3}{6n^{3/2}} \right\} dF_Y(t) \\
- \sqrt{n} \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} E(Y^3) x - \theta \right\} \int_0^\infty \left\{ e^{x/\sqrt{n}} - 1 - \frac{\gamma t}{\sqrt{n}} \right\} dF_Y(t) \\
+ \lambda e^{-\gamma x} \frac{\gamma^2}{3} E(Y^3) \int_0^\infty t \left\{ e^{x/\sqrt{n}} - 1 - \frac{\gamma t}{\sqrt{n}} \right\} dF_Y(t) \\
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma (x - \frac{1}{\sqrt{n}})} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} E(Y^3) \left( x - \frac{t}{\sqrt{n}} \right) - \theta \right) e^{-\gamma (x - \frac{1}{\sqrt{n}})} - 1 \right\} dF_Y(t).
\]
All the terms on the first three lines cancel, and we are left with

\[
F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}_x(x) + \psi^{(1)}_x(x)/\sqrt{n}, \psi^{(0)}(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n})
\]

\[
= -n\lambda e^{-\gamma x} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} - \frac{\gamma^3 t^3}{6n^{3/2}} \right\} dF_Y(t)
\]

\[\]

\[
- \sqrt{n} \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} x - \theta \right\} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} \right\} dF_Y(t)
\]

\[\]

\[
+ \lambda e^{-\gamma x} \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \int_0^\infty t \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} \right\} dF_Y(t)
\]

\[\]

\[
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma \left(x - \frac{t}{\sqrt{n}}\right)} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \left(x - \frac{t}{\sqrt{n}}\right) - \theta \right) e^{-\gamma \left(x - \frac{t}{\sqrt{n}}\right)} - 1 \right\} dF_Y(t).
\]

Next, continue expanding the integrands to pull out the terms of order \(O(n^{-1})\).

\[
F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}_x(x) + \psi^{(1)}_x(x)/\sqrt{n}, \psi^{(0)}(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n})
\]

\[
= \frac{\lambda}{n} e^{-\gamma x} \left\{ - \frac{\gamma^4}{24} \frac{E(Y^4)}{E(Y^3)} - \left( \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} x - \theta \right) \frac{\gamma^3}{6} \frac{E(Y^3)}{E(Y^2)} + \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \cdot \frac{\gamma^2}{2} \frac{E(Y^3)}{E(Y^2)} \right\}
\]

\[\]

\[
- n\lambda e^{-\gamma x} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} - \frac{\gamma^3 t^3}{6n^{3/2}} - \frac{\gamma^4 t^4}{24n^2} \right\} dF_Y(t)
\]

\[\]

\[
- \sqrt{n} \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} x - \theta \right\} \int_0^\infty \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} - \frac{\gamma^3 t^3}{6n^{3/2}} \right\} dF_Y(t)
\]

\[\]

\[
+ \lambda e^{-\gamma x} \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \int_0^\infty t \left\{ e^{\frac{\gamma t}{\sqrt{n}}} - 1 - \frac{\gamma t}{\sqrt{n}} - \frac{\gamma^2 t^2}{2n} \right\} dF_Y(t)
\]

\[\]

\[
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma \left(x - \frac{t}{\sqrt{n}}\right)} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \left(x - \frac{t}{\sqrt{n}}\right) - \theta \right) e^{-\gamma \left(x - \frac{t}{\sqrt{n}}\right)} - 1 \right\} dF_Y(t).
\]

Use the identity

\[
e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \int_0^1 (1 - \omega)^n e^{\omega x} d\omega
\]
to rewrite the three integrals from 0 to infinity.

\[
F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}_x(x) + \psi^{(1)}_x(x)/\sqrt{n}, \psi^{(0)}(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n})
\]

\[
= \frac{\gamma^3}{6n} \lambda e^{-\gamma x} \left\{ -\frac{\gamma}{4} E(Y^4) - \frac{\gamma^2}{3} \left( \frac{E(Y^3)^2}{E(Y^2)} \right) x + \left( \theta + \gamma \frac{E(Y^3)}{E(Y^2)} \right) E(Y^3) \right\}
\]

\[
- \lambda e^{-\gamma x} \int_0^\infty \frac{\gamma_5 t^5}{24n^{3/2}} \int_0^1 (1 - \omega)^4 e^{\frac{\gamma t}{\sqrt{n}}} \omega dF_Y(t)
\]

\[
- \lambda e^{-\gamma x} \left\{ \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} x - \theta \right\} \int_0^\infty \frac{\gamma^4 t^4}{6n^{3/2}} \int_0^1 (1 - \omega)^3 e^{\frac{\gamma t}{\sqrt{n}}} \omega dF_Y(t)
\]

\[
+ \lambda e^{-\gamma x} \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \int_0^\infty \frac{\gamma^2 t^2}{2n^{3/2}} \int_0^1 (1 - \omega)^2 e^{\frac{\gamma t}{\sqrt{n}}} \omega dF_Y(t)
\]

\[
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma(x-\frac{1}{\sqrt{n}})} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \left( x - \frac{t}{\sqrt{n}} \right) - \theta \right) e^{-\gamma(x-\frac{1}{\sqrt{n}})} - 1 \right\} dF_Y(t).
\]

Switch the order of integration to obtain

\[
F_n(x, \psi^{(0)}(x) + \psi^{(1)}(x)/\sqrt{n}, \psi^{(0)}_x(x) + \psi^{(1)}_x(x)/\sqrt{n}, \psi^{(0)}(\cdot) + \psi^{(1)}(\cdot)/\sqrt{n})
\]

\[
= \frac{\gamma^3}{6n} \lambda e^{-\gamma x} \left\{ -\frac{\gamma}{4} E(Y^4) - \frac{\gamma^2}{3} \left( \frac{E(Y^3)^2}{E(Y^2)} \right) x + \left( \theta + \gamma \frac{E(Y^3)}{E(Y^2)} \right) E(Y^3) \right\}
\]

\[
- \lambda e^{-\gamma x} \frac{\gamma_5 t^5}{24n^{3/2}} \int_0^1 (1 - \omega)^4 \frac{e^{\frac{\gamma t}{\sqrt{n}}}{Y^5}}{E(Y^2)} \omega d\omega
\]

\[
- \lambda e^{-\gamma x} \frac{\gamma^4}{6n^{3/2}} \left\{ \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} x - \theta \right\} \int_0^1 (1 - \omega)^3 \frac{e^{\frac{\gamma t}{\sqrt{n}}}{Y^4}}{E(Y^2)} \omega d\omega
\]

\[
+ \lambda e^{-\gamma x} \frac{\gamma_5}{6n^{3/2}} \frac{E(Y^3)}{E(Y^2)} \int_0^1 (1 - \omega)^2 \frac{e^{\frac{\gamma t}{\sqrt{n}}}{Y^4}}{E(Y^2)} \omega d\omega
\]

\[
+ n\lambda \int_{\sqrt{n}x}^\infty \left\{ e^{-\gamma(x-\frac{1}{\sqrt{n}})} + \frac{1}{\sqrt{n}} \left( \frac{\gamma^2}{3} \frac{E(Y^3)}{E(Y^2)} \left( x - \frac{t}{\sqrt{n}} \right) - \theta \right) e^{-\gamma(x-\frac{1}{\sqrt{n}})} - 1 \right\} dF_Y(t). \quad (B.1)
\]

Similar to the work in Propositions 4.1 and 4.2 we can modify \( \psi^{(0)} + \psi^{(1)}/\sqrt{n} \) by functions of order \( O(n^{-1}) \) to change \( F_n \) in (B.1) so that the result is either negative (for a lower bound) or positive (for an upper bound), except for the last integral, the one from \( \sqrt{n}x \) to infinity. The dominant terms in the integrand of that integral are of order \( O(n^{-1/2}) \); therefore, changing \( \psi^{(0)} + \psi^{(1)}/\sqrt{n} \) by functions of order \( O(n^{-1}) \) will not affect those terms.

In the rest of this appendix, we further analyze the integral from \( \sqrt{n}x \) to infinity. If we (formally) define \( Z_d = (Y - d)|(Y > d) \) and set \( d = \sqrt{n}x \), and if we denote that integral by \( J_n \)
(including the factor of $n\lambda S_Y(d)$), then

$$J_n = n\lambda S_Y(d) \int_0^\infty \left\{ \left( e^{\gamma z} - 1 \right) - \frac{1}{\sqrt{n}} \left( \gamma^2 \mathbb{E}(Y^3) \frac{z}{\sqrt{n}} + \theta \right) e^{\gamma z} \right\} dF_Z(d).$$

Expand the integrand to expose the terms up to order $\mathcal{O}(n^{-1})$.

$$J_n = n\lambda S_Y(d) \left[ \int_0^\infty \left\{ \frac{1}{\sqrt{n}} \left( \gamma z \mathbb{E}(Z^d) - \gamma \mathbb{E}(Z^d) \right) + \frac{1}{n} \left( \mathbb{E}(Y) z^2 \mathbb{E}(Z^d) - \mathbb{E}(Y) \mathbb{E}(Z^d) \right) + \frac{\gamma^2}{2n^{3/2}} \int_0^1 (1 - \omega)^2 e^{\gamma z} d\omega \right\} dF_Z(z) \right.$$

$$- \frac{\gamma^2}{3n^{3/2}} \int_0^\infty \left\{ z + \frac{\gamma z^2}{\sqrt{n}} \right\} e^{\gamma z} d\omega \left. \right\} dF_Z(z)$$

$$- \frac{\theta}{\sqrt{n}} \int_0^\infty \left\{ 1 + \frac{\gamma z}{\sqrt{n}} \right\} e^{\gamma z} d\omega \left. \right\} dF_Z(z) \right].$$

Next, group the terms according to order.

$$J_n = n\lambda S_Y(d) \left[ \frac{1}{\sqrt{n}} \left( \gamma \mathbb{E}(Z^d) - \theta \mathbb{E}(Z^d) \right) + \frac{1}{n} \left( \frac{\gamma^2}{2} \mathbb{E}(Z^d) - \frac{\gamma}{3} \mathbb{E}(Y) \mathbb{E}(Z^d) - \gamma \mathbb{E}(Z^d) \right) \right.$$

$$+ \frac{\gamma^2}{n^{3/2}} \int_0^1 \left\{ \frac{\gamma}{2} (1 - \omega)^2 \mathbb{E}(Z^2_d e^{\gamma z}) - \left( \frac{\gamma}{3} \mathbb{E}(Y^3) + \theta (1 - \omega) \right) \mathbb{E}(Z^2_d e^{\gamma z}) \right\} d\omega \right].$$

(B.2)

Thus, we see that the dominant terms in the integrand are of order $\mathcal{O}(n^{-1/2})$, which would not be affected by adding or subtracting a function of order $\mathcal{O}(n^{-1})$ to or from $\psi(0) + \psi(1)/\sqrt{n}$. For example, if $Y \sim \text{Gamma}(2, \beta)$, as in Example 3.2 then $S_{Z_d}(z) = \mathbb{E}(Z^2_d e^{\gamma z})$ and the terms of order $\mathcal{O}(n^{-1/2})$ equal

$$\gamma \mathbb{E}(Z^d) - \theta \propto 2 \mathbb{E}(Y) \mathbb{E}(Z^d) - \mathbb{E}(Y^2) = \frac{4}{\beta} \frac{2 + \beta d}{\beta (1 + \beta d)} - \frac{6}{\beta^2} \propto 1 - \beta d,$$

which decreases from positive to negative as $d$ increases from 0 to infinity.

However, if $Y \sim \text{Exp}(\beta)$, as in Example 3.1 then $Z_d \sim Y$ for all $d \geq 0$, and the terms of order $\mathcal{O}(n^{-1/2})$ equal

$$\gamma \mathbb{E}(Z^d) - \theta \propto 2 \mathbb{E}(Y) \mathbb{E}(Z^d) - \mathbb{E}(Y^2) = 2 (\mathbb{E}(Y)^2 - \mathbb{E}(Y)^2) = \frac{2}{\beta^2} - \frac{2}{\beta^2} = 0;$$

thus, if we were to change $\psi(0) + \psi(1)/\sqrt{n}$ by a function of order $\mathcal{O}(n^{-1})$, then we could perhaps find bounds for $\psi_n$ of order $\mathcal{O}(n^{-1})$. This is the topic of the next appendix.
C Comparing $\psi(0) + \psi(1)/\sqrt{n}$ with $\psi_n$ when $Y \sim Exp$

Throughout this appendix, we assume that $Y \sim Exp(\beta)$ with mean $1/\beta$; then, $J_n$ in (B.2) equals

$$J_n = n\lambda S_Y(d) \left[ \frac{1}{n} \left( \frac{\gamma^2}{2} \mathbb{E} (Y^2) - \frac{\gamma^2}{3} \mathbb{E} (Y^3) \mathbb{E} Y - \theta \gamma \mathbb{E} Y \right) \right. $$

$$+ \left. \frac{\gamma^2}{n^{3/2}} \int_0^1 \left\{ \frac{\gamma}{2} (1 - \omega)^2 \mathbb{E} (Y^3 e^{\gamma \omega \sqrt{n} Y}) - \left( \frac{\gamma}{3} \frac{\mathbb{E} (Y^3)}{\mathbb{E} (Y^2)} + \theta (1 - \omega) \right) \mathbb{E} (Y^2 e^{\gamma \omega \sqrt{n} Y}) \right\} d\omega \right]$$

$$= n\lambda \theta e^{-\sqrt{n x}/\beta} \left[ -1/n + \frac{\beta^2}{n^{3/2}} J_n' \right],$$

which is finite if $n > (\theta \beta^2)^2$ and is of order $O(1)$. Thus, $J_n$ is negative for $n$ large enough, and the expression in square brackets is of order $O(n^{-1})$.

Because $J_n$ is negative for $n$ large enough, we begin by modifying $\psi(0) + \psi(1)/\sqrt{n}$ to obtain a lower bound for $\psi_n$.

**Proposition C.1.** If $Y$ is an exponential random variable, then there exists $N > 0$ such that, for all $n > N$,

$$\psi(0)(x) + \frac{1}{\sqrt{n}} \psi(1)(x) - \frac{\kappa}{n} xe^{-\gamma x} < \psi_n(x),$$

for all $x \geq 0$, in which $\kappa$ is defined by

$$\kappa = \frac{\gamma^3}{6} \frac{\mathbb{E} (Y^3)}{\mathbb{E} Y} \left( 1 + \frac{2 \mathbb{E} Y}{\mathbb{E} (Y^2)} \frac{\mathbb{E} (Y^3)}{\mathbb{E} (Y^2)} \right).$$

**Proof.** First,

$$\psi(0)(0) + \frac{1}{\sqrt{n}} \psi(1)(0) - \frac{\kappa}{n} \cdot 0 \cdot e^{-\gamma \cdot 0} = 1 - \frac{\theta}{\sqrt{n}} < \frac{1}{1 + \theta/\sqrt{n}} = \psi_n(0).$$
If we subtract $\kappa x e^{-\gamma x}/n$ from $\psi^{(0)} + \psi^{(1)}/\sqrt{n}$, then a calculation similar to the one in Appendix \[B\] shows that $F_n$ changes by adding the following terms:

$$\begin{align*}
\kappa \lambda e^{-\gamma x} & \left\{ -\frac{\theta}{n} E Y + \frac{\gamma^3}{2 n^{3/2}} x \int_0^1 (1 - \omega)^2 E \left( Y^3 e^{\frac{\gamma Y}{\sqrt{n}}} \right) d\omega - \frac{\gamma^2}{n^{3/2}} \int_0^1 (1 - \omega) E \left( Y^3 e^{\frac{\gamma Y}{\sqrt{n}}} \right) d\omega \right\} \\
+ n \kappa \lambda e^{-\sqrt{n} x/\beta} & \int_{0}^{\infty} \frac{z}{n^{3/2}} e^{\gamma z} dF_Z(z) .
\end{align*}$$

If we ignore the integrals with respect to $F_Z$, the above expression and the one in (B.1) show that the terms of order $O(n^{-1})$ in $F_n$ evaluated at $\psi^{(0)} + \psi^{(1)}/\sqrt{n} - \kappa x e^{-\gamma x}/n$ equal

$$\begin{align*}
\frac{\gamma^3}{6} \lambda e^{-\gamma x} & \left\{ -\frac{\gamma}{4} E(Y^4) - \frac{\gamma^2}{3} \left( \frac{E(Y^3)}{E(Y^2)} \right)^2 x + \left( \theta + \gamma \frac{E(Y^3)}{E(Y^2)} \right) E(Y^3) \right\} - \kappa \theta E Y \lambda e^{-\gamma x} \\
= \frac{\gamma^3}{6} \lambda e^{-\gamma x} & \left\{ -\frac{\gamma}{4} E(Y^4) - \frac{\gamma^2}{3} \left( \frac{E(Y^3)}{E(Y^2)} \right)^2 x \right\} < 0,
\end{align*}$$

in which the equality follows from the choice of $\kappa$ in (C.3). Furthermore, the integrand of $J_n$ is negative for $n$ large enough because the integrand of the corresponding $\kappa$-integral is of order $O(n^{-3/2})$. Thus, the terms of order $O(n^{-1})$ are negative for all $x \geq 0$, and those terms will dominate those of order $O(n^{-3/2})$ for $n > N$ and for all $x > 0$, with $N$ large enough. The conclusion in (C.2), then, follows from Lemma 2.1.

Next, we modify $\psi^{(0)} + \psi^{(1)}/\sqrt{n}$ to obtain an upper bound for $\psi_n$.

**Proposition C.2.** If $Y$ is an exponential random variable with mean $1/\beta$, then there exists $N > 0$ such that, for all $n > N$,

$$\psi_n(x) < \psi^{(0)}(x) + \frac{1}{\sqrt{n}} \psi^{(1)}(x) + \frac{1}{n} \left( \zeta + \delta x \right) e^{-\alpha x} , \quad (C.4)$$

for all $x \geq 0$, in which $\alpha$ is any number strictly between $\theta \beta/2$ and $\theta \beta$, and $\zeta$ and $\delta$ are such that

$$\zeta > \theta^2 , \quad (C.5)$$

and

$$\delta > \frac{\theta^5 \beta^3}{\alpha (\beta \theta - \alpha)} . \quad (C.6)$$

**Proof.** First,

$$\psi_n(0) = \frac{1}{1 + \theta/\sqrt{n}} < 1 - \frac{\theta}{\sqrt{n}} + \frac{\zeta}{n} = \psi^{(0)}(0) + \frac{1}{\sqrt{n}} \psi^{(1)}(0) + \frac{1}{n} \left( \zeta + \delta \cdot 0 \right) e^{-\alpha \cdot 0} .$$

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Also,
\[
\lim_{x \to \infty} \psi_n(x) = 0 = \lim_{x \to \infty} \psi^{(0)}(x) + \frac{1}{\sqrt{n}} \psi^{(1)}(x) + \frac{1}{n} (\zeta + \delta x)e^{-\alpha x}.
\]

If we add \((\zeta + \delta x)e^{-\alpha x}/n\) to \(\psi^{(0)}/\sqrt{n}\), then a calculation similar to the one in Appendix B shows that \(F_n\) changes by adding

\[
\delta \lambda e^{-\alpha x} \left\{ \frac{1}{n} \left( \frac{\alpha}{2} E(Y^2) \right) - \frac{\alpha^3}{2n^{3/2}} \int_0^1 (1 - \omega)^2 E\left(Y^3 e^{\alpha \sqrt{n} Y}\right) d\omega \right\}
+ \frac{\alpha^2}{n^{3/2}} \int_0^1 (1 - \omega) E\left(Y^3 e^{\alpha \sqrt{n} Y}\right) d\omega
+ \zeta \lambda e^{-\alpha x} \left\{ \frac{\alpha}{n} \left( \frac{\alpha}{2} E(Y^2) \right) - \frac{\alpha^3}{2n^{3/2}} \int_0^1 (1 - \omega)^2 E\left(Y^3 e^{\alpha \sqrt{n} Y}\right) d\omega \right\}
+ n \lambda e^{-\alpha x / \beta} \int_0^\infty \frac{1}{n} \left( \frac{\zeta}{\sqrt{n}} - \frac{\delta z}{\sqrt{n}} \right) e^{\alpha z / \sqrt{n}} dF_{Z_d}(z).
\]

(C.7)

If we ignore the integrals with respect to \(F_{Z_d}\), then the expressions in (B.1) and (C.7) show that the terms of order \(O(n^{-1})\) in \(F_n\) evaluated at \(\psi^{(0)}/\sqrt{n} + (\zeta + \delta x)e^{-\alpha x}/n\) equal

\[
\frac{\gamma^3}{6} \lambda e^{-\gamma x} \left\{ -\gamma \frac{E(Y^4)}{4} - \gamma^2 \left( \frac{E(Y^3)}{E(Y^2)} \right)^2 x + \left( \theta + \gamma \frac{E(Y^3)}{E(Y^2)} \right) E(Y^3) \right\}
+ \delta \lambda e^{-\alpha x} \left\{ \theta E(Y^3) - \frac{\alpha}{2} E(Y^2) \right\} + \zeta \lambda e^{-\alpha x} \left\{ \theta E(Y^3) - \frac{\alpha}{2} E(Y^2) \right\},
\]

The choices of \(\alpha \in (\theta \beta / 2, \theta \beta)\) and \(\delta\) in (C.6) guarantee that the above expression is positive for all \(x \geq 0\), and that it will dominate the terms of order \(O(n^{-3/2})\) for \(n > N\) and for all \(x > 0\), with \(N\) large enough.

Next, from (C.1) and (C.7), we see that the integrals with respect to \(F_{Z_d}\) in \(F_n\) evaluated at \(\psi^{(0)}/\sqrt{n} + (\zeta + \delta x)e^{-\alpha x}/n\) equal

\[
n \lambda e^{-\alpha x / \beta} \left\{ \theta^2 \left[ -\frac{1}{n} + \frac{\beta^2}{n^{3/2}} J'_n \right] + \int_0^\infty \frac{1}{n} \left( \frac{\zeta}{\sqrt{n}} - \frac{\delta z}{\sqrt{n}} \right) e^{\alpha z / \sqrt{n}} dF_{Z_d}(z) \right\}.
\]

Because \(\zeta > \theta^2\), the terms in the integrand of order \(O(n^{-1})\) are positive and will dominate the terms of order \(O(n^{-3/2})\) for \(n > N\) and for all \(x > 0\), with \(N\) large enough. The conclusion in (C.4), then, follows from Lemma 2.1.

□

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Figure 1: Graph of $\psi_n$ (solid line) and $\psi^{(0)} + \psi^{(1)}/\sqrt{n} + \psi^{(2)}/n$ (dashed line) for $Y \sim Exp(\beta)$; see Example 3.1. For this figure, $\beta = 1$, $\theta = 0.4$, and $n = 1$. 
Figure 2: Graph of $\psi_n$ (solid line), $\psi^{(0)} + \psi^{(1)}/\sqrt{n} + \psi^{(2)}/n$ in (3.17) (dashed line), and the approximation in (3.18) (dotted line) for $Y \sim Gamma(2, \beta)$; see Example 3.2. For this figure, $\beta = 1$, $\theta = 0.4$, and $n = 1$. 