Polyhedron under Linear Transformations

Zhang Zaikun†

May 6, 2008

Abstract

The image and the inverse image of a polyhedron under a linear transformation are polyhedrons.

Keywords: polyhedron, linear transformation, Sard quotient theorem.

1 Introduction

All the linear spaces discussed here are real.

Definition 1.1.
i.) Suppose that $X$ is a linear space, a subset $P$ of $X$ is said to be a polyhedron if it has the form

$$P = \{ x \in X; f_k(x) \leq \lambda_i \},$$

where $n$ is a positive integer, $\{f_k\}_{k=1}^n \subset X'$, and $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$.

If $\lambda_k = 0$ ($k = 1, 2, 3, ..., n$), then $P$ is said to be a polyhedral cone.

ii.) Suppose that $X$ is a TVS, a subset $P$ of $X$ is said to be a closed polyhedron if it has the form

$$P = \{ x \in X; f_k(x) \leq \lambda_i \},$$

where $n$ is a positive integer, $\{f_k\}_{k=1}^n \subset X^*$, and $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$.

If $\lambda_k = 0$ ($k = 1, 2, 3, ..., n$), then $P$ is said to be a closed polyhedral cone.

It is obvious that both $\emptyset$ and $X$ itself are (closed) polyhedral cones.

2 Main Results

Our main results are as follows.

Theorem 2.1. Suppose that $X$ and $Y$ are linear spaces, and $T : X \rightarrow Y$ is a linear operator.

i.) If $A \subset X$ is a polyhedron (polyhedral cone) and $T$ is surjective, then $T(A)$ is a

†Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing 100190, CHINA.
polyhedron (polyhedral cone).

ii.) If \( B \subset Y \) is a polyhedron (polyhedral cone), then \( T^{-1}(B) \) is a polyhedron (polyhedral cone).

**Theorem 2.2.** Suppose that \( X \) and \( Y \) are Fréchet spaces, and \( T : X \to Y \) is a bounded linear operator.

i.) If \( A \subset X \) is a closed polyhedron (closed polyhedral cone) and \( T \) is surjective, then \( T(A) \) is a closed polyhedron (closed polyhedral cone).

ii.) If \( B \subset Y \) is a closed polyhedron (closed polyhedral cone), then \( T^{-1}(B) \) is a polyhedron (closed polyhedral cone).

The conclusions above will be verified in section 4.

### 3 A Lemma

The following conclusion is significant in our proof.

**Lemma 3.1** (Sard Quotient Theorem).

i.) Suppose that \( X, Y \) and \( Z \) are linear spaces, and \( S : X \to Y \), \( T : X \to Z \) are linear operators with \( S \) surjective. If \( \ker S \subset \ker T \), then there exists a uniquely specified linear operator \( R : Y \to Z \), such that \( T = RS \).

ii.) Suppose that \( X, Y \) and \( Z \) are TVS’, and \( S : X \to Y \), \( T : X \to Z \) are bounded linear operators with \( S \) surjective. If \( X \) and \( Y \) are Fréchet spaces and \( \ker S \subset \ker T \), then there exists a uniquely specified bounded linear operator \( R : Y \to Z \), such that \( T = RS \).

**Proof.** We will prove only ii.).

Define

\[
\tilde{S} : X / \ker S \to Y \\
\tilde{S} [x] \mapsto Sx,
\]

and

\[
\tilde{T} : X / \ker S \to Z \\
\tilde{T} [x] \mapsto Tx.
\]

Then both \( \tilde{S} \) and \( \tilde{T} \) are well defined (note that \( \ker S \subset \ker T \)) and bounded. Besides, \( \tilde{S} \) is bijective and \( \tilde{S}^{-1} \) is bounded, since \( X / \ker S \) and \( Y \) are both Fréchet spaces. Now define

\[
R = \tilde{T} \tilde{S}^{-1},
\]

then it is easy to show that \( R \) satisfies the requirements.

The uniqueness of \( R \) is trivial. \( \blacksquare \)
4 Proofs of Main Results

We will prove only theorem 2.2, because the proof of theorem 2.1 is similar. Only the polyhedron case will be discussed.

Proof of Theorem 2.2.

i.) Suppose that

\[ A = \bigcap_{k=1}^{n} \{ x \in X; f_k(x) \leq \lambda_k \}, \]

where \( n \) is a positive integer, \( \{ f_k \}_{k=1}^{n} \subset X^* \), and \( \{ \lambda_k \}_{k=1}^{n} \subset \mathbb{R} \). The proof will be presented in four steps.

**Step 1.** We will prove that the conclusion holds if \( \ker T \subset \bigcap_{k=1}^{n} \ker f_k \).

In this case, for any \( k \in \{ 1, 2, 3, ..., n \} \), we can choose a functional \( g_k \in Y^* \) such that \( f_k = g_k T \) (Sard quotient theorem). It can be shown without difficulty that

\[ T(A) = \bigcap_{k=1}^{n} \{ y \in Y; g_k(y) \leq \lambda_k \}. \]

**Step 2.** We will prove that the conclusion holds if \( \dim(\ker T) = 1 \). This is the most critical part of the proof.

Suppose that \( \xi \) is a point in \( \ker T \setminus \{ 0 \} \). Let

\[ K_+ = \{ k; 1 \leq k \leq n \text{ and } f_k(\xi) > 0 \}, \]
\[ K_- = \{ k; 1 \leq k \leq n \text{ and } f_k(\xi) < 0 \}, \]
\[ K_0 = \{ k; 1 \leq k \leq n \text{ and } f_k(\xi) = 0 \}. \]

For any \( i \in K_+ \) and \( j \in K_- \), define

\[ h_{ij} = f_i - \frac{f_i(\xi)}{f_j(\xi)} f_j. \]

Then define

\[ A_1 = \bigcap_{i \in K_+, \ j \in K_-} \{ x \in X; h_{ij}(x) \leq \lambda_i - \frac{f_i(\xi)}{f_j(\xi)} \lambda_j \}, \]
\[ A_2 = \bigcap_{k \in K_0} \{ x \in X; f_k(x) \leq \lambda_k \}. \]

If \( K_+ = \emptyset \) or \( K_- = \emptyset \), we take \( A_1 \) as \( X \). Similarly, if \( K_0 = \emptyset \), we take \( A_2 \) as \( X \). We will prove that \( T(A) = T(A_1 \cap A_2) \). It suffices to show that \( T(A_1 \cap A_2) \subset T(A) \).
• If $K_+ = \emptyset = K_-$, nothing needs considering.

• If $K_+ \neq \emptyset = K_-$, fix a point $x \in A_1 \cap A_2$, define

$$s = \min_{i \in K_+} \frac{\lambda_i - f_j(x)}{f_i(\xi)},$$

then it is easy to show that $x + s\xi \in A$ and $T(x + s\xi) = Tx$. The case with $K_+ \neq \emptyset = K_-$ is similar.

• If $K_+ \neq \emptyset \neq K_-$, fix a point $x \in A_1 \cap A_2$, define

$$t = \max_{j \in K_-} \frac{\lambda_j - f_j(x)}{f_j(\xi)}$$

and consider $x + t\xi$. It is obvious that

$$T(x + t\xi) = y$$

and that

$$f_j(x + t\xi) \leq \lambda_j, \quad \forall j \in K_- \cup K_0.$$ 

Suppose

$$t = \frac{\lambda_{j_0} - f_{j_0}(x)}{f_{j_0}(\xi)} \quad (j_0 \in K_-),$$

then for any $i \in K_+$,

$$f_i(x + t\xi) = h_{ij_0}(x + t\xi) + \frac{f_i(\xi)}{f_{j_0}(\xi)} f_{j_0}(x + t\xi) \leq \lambda_i \frac{f_i(\xi)}{f_{j_0}(\xi)} \lambda_{j_0} + \frac{f_i(\xi)}{f_{j_0}(\xi)} \lambda_{j_0} = \lambda_i.$$

Thus $x + t\xi \in A$.

It has been shown that $T(A_1 \cap A_2) \subset T(A)$, and consequently $T(A_1 \cap A_2) = T(A)$. According to Step 1, the conclusion holds under the assumption $\dim(\ker T) = 1$.

**Step 3.** We will prove by induction that the conclusion holds if $\dim(\ker T)$ is finite. If $\dim(\ker T) = 0$, then $T$ is an isomorphism as well as a homeomorphism (inverse mapping theorem), thus nothing needs proving. Now suppose that the conclusion holds when $\dim(\ker T) \leq n$ ($n \geq 0$). To prove the case with $\dim(\ker T) = n + 1$, choose a
point η in ker \( T \setminus \{0\} \), find a functional \( F \in X^* \) such that \( F(\eta) = 1 \) (Hahn-Banach theorem), and define

\[
\tilde{T} : X \to Y \times \mathbb{R} \\
x \mapsto (Tx, F(x)), \\
\pi : Y \times \mathbb{R} \to Y \\
(y, \lambda) \mapsto y.
\]

Then we have

• \( T = \pi \tilde{T} \);
• \( \dim(\ker \tilde{T}) = n \);
• \( \dim(\ker \pi) = 1 \);
• both \( \tilde{T} \) and \( \pi \) are surjective bounded linear operators.

Thus by the induction hypothesis and the conclusion of Step 2, \( T(A) \) is a closed polyhedron.

**Step 4.** Now consider the general case.

Let

\[
M = \bigcap_{k=1}^{n} \ker f_k \cap (\ker T),
\]
then \( M \) is a closed linear subspace of \( M \), and therefore \( X/M \) is a Fréchet space. Define

\[
\tilde{T} : X/M \to Y \\
[x] \mapsto Tx, \\
\tilde{f}_k : X/M \to \mathbb{R} \\
[x] \mapsto \tilde{f}_k(x)
\]

where \( k = 1, 2, 3, ..., n \). Then \( \tilde{T} \) and \( \tilde{f}_k \) are well defined, \( T \) is a bounded linear operator from \( X/M \) onto \( Y \), and \( \{\tilde{f}_k\}_{k=1}^{n} \subset (X/M)^* \). Besides, we have

\[
\bigcap_{k=1}^{n} \ker \tilde{f}_k \cap (\ker \tilde{T}) = \{0\},
\]
which implies that

\[
\dim(\ker \tilde{T}) \leq n.
\]

Now let

\[
\tilde{A} = \bigcap_{k=1}^{n} \{ [x] \in X/M; \tilde{f}_k([x]) \leq \lambda_k \},
\]

5
then
\[ T(A) = \tilde{T}(\tilde{A}). \]
From what has been proved, it is easy to show that \( T(A) \) is a closed polyhedron. Proof of part i.) has been completed.

ii.) This part is much easier. Suppose that
\[
B = \bigcap_{k=1}^{m} \{ y \in Y; g_k(y) \leq \mu_k \},
\]
where \( m \) is a positive integer, \( \{ g_k \}_{k=1}^{m} \subset Y^* \), and \( \{ \mu_k \}_{k=1}^{m} \subset \mathbb{R} \). One can show without difficulty that
\[
T^{-1}(B) = \bigcap_{k=1}^{m} \{ x \in X; g_k(Tx) \leq \mu_k \},
\]
which is a closed polyhedron in \( X \).

5 Remarks
For part i) of theorem 2.2, the completeness conditions are essential. This can be seen from the following examples.

Example 5.1. Suppose that \((Y, \| \cdot \|_Y)\) is an infinite dimensional Banach space, and \( f \) is an unbounded linear functional on it\(^1\). Let \( X \) has the same elements and linear structure as \( Y \), but the norm on \( X \) is defined by
\[
\|x\|_X = \|x\|_Y + |f(x)|.
\]
It is clear that the identify mapping \( I : X \to Y \) is linear, bounded and bijective. Now consider \( \ker f \). It is a closed polyhedral cone in \( X \), while its image under \( I \) is not closed in \( Y \).

Example 5.2. Suppose that \( X \) is \( \ell^1 \). Let \( Y \) has the same elements and linear structure as \( X \), but the norm on \( Y \) is defined by
\[
\|(x_k)\| = \sup_{k \geq 1} |x_k|.
\]
Then \( f : (x_k) \mapsto \sum x_k \) is a bounded linear functional on \( X \), while it is unbounded on \( Y \). Now consider the identify mapping again.

The preceding examples also imply that inverse mapping theorem and Sard quotient theorem do not hold without completeness conditions.

\(^1\)For a locally bounded TVS \( Y \), there exist unbounded linear functionals on \( Y \) provided \( \dim Y = \infty \). One of them can be constructed as follows: Let \( U \) be a bounded neighborhood of 0, and \( \{ e_k; k \geq 1 \} \subset U \) be a sequence of linearly independent elements in \( Y \). Let \( M = \text{Span} \{ e_k; k \geq 1 \} \), and define \( g : M \to \mathbb{R}, \sum \alpha_k e_k \mapsto \sum k \alpha_k \). Then extend \( g \) to \( Y \).