Report on the trimestre “Heat Kernels, Random Walks, and Analysis on Manifolds and Graphs” at the Centre Émile Borel (Institut Henri Poincaré, Spring, 2002)

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“If it’s Tuesday, this must be Belgium.”

This is the name of a film about a group of tourists who were going from city to city a little too fast. Fortunately in this trimestre there was more time, and while the activities were numerous and extensive, one had the opportunity to delve into various topics in some detail.

To give a part of the mathematical setting, let us review a few classical matters related to calculus and partial differential equations. Fix a positive integer \( n \), and let \( \mathbb{R}^n \) be the usual \( n \)-dimensional Euclidean space, consisting of \( n \)-tuples of real numbers. If \( f(x) \) is a real-valued function on \( \mathbb{R}^n \) which is

Some historical notes, mentioned by a colleague: Émile Borel spoke at the opening of the author’s home institution, Rice University (originally the Rice Institute) in Houston, Texas, in 1912. Borel published “Molecular theories and mathematics” in connection with his lectures in the Rice Institute Pamphlet, Volume I (1915), 163–193. Henri Poincaré was also invited by President Edgar Odell Lovett and accepted, conditioned on the state of his health, but eventually declined the invitation and subsequently passed away. Borel’s paper begins with a tribute to Poincaré, and relates a discussion they had about the trip. Borel indicates that he would have changed his subject to an appreciation of Poincaré’s work, except that Vito Volterra was doing exactly that. Volterra’s paper appears in the same issue of the Rice Institute Pamphlet, “Henri Poincaré”, pp. 133–162. Jacques Hadamard contributed “The early scientific work of Henri Poincaré” and “The later scientific work of Henri Poincaré” to the Rice Institute Pamphlet, Volume IX (1922), 111-183 and Volume XX (1933), 1–86. Hadamard makes the point in the introduction to the first paper that uses for Poincaré’s work seemed to take 25 years to be found.
twice-continuously differentiable, say, then the Laplacian of $f$ is denoted $\Delta f$ and defined by

$$\Delta f = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f. \tag{1}$$

Let $f_1(x), f_2(x)$ be two real-valued functions on $\mathbb{R}^n$ which are continuous and have compact support, so that they are both equal to 0 outside of a bounded set. More generally, one can assume that $f_1, f_2$ satisfy suitable decay conditions, etc. The standard inner product of such functions is defined by

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^n} f_1(x) f_2(x) \, dx. \tag{2}$$

There is another symmetric bilinear form which is closely related to the Laplacian, given by

$$\mathcal{E}(f_1, f_2) = \frac{1}{2} \int_{\mathbb{R}^n} \nabla f_1(x) \cdot \nabla f_2(x) \, dx \tag{3}$$

when $f_1, f_2$ are continuously differentiable, or satisfy other appropriate regularity conditions. Here $\nabla f(x)$ denotes the gradient of $f$ at $x$, i.e., the vector with components $(\partial / \partial x_j)f(x)$, and $v \cdot w$ is the usual inner product on $\mathbb{R}^n$, so that $v \cdot w = \sum_{j=1}^{n} v_j w_j$. If in addition $f_1$ is twice continuously-differentiable, then

$$\mathcal{E}(f_1, f_2) = -\frac{1}{2} \int_{\mathbb{R}^n} \Delta f_1(x) f_2(x) \, dx. \tag{4}$$

This follows from integration by parts.

The energy $\mathcal{E}(f)$ of a function $f$ is defined by

$$\mathcal{E}(f) = \mathcal{E}(f, f) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx, \tag{5}$$

where $|v|$ denotes the standard Euclidean length of $v$, which is the same as saying that $|v|^2 = v \cdot v$. If $\eta(x)$ is another function on $\mathbb{R}^n$, then

$$\frac{d}{ds} \mathcal{E}(f + s \eta) \bigg|_{s=0} = -\int_{\mathbb{R}^n} \Delta f(x) \eta(x) \, dx. \tag{6}$$

under suitable conditions on $f$ and $\eta$. This is commonly rephrased as saying that the gradient of the energy functional $\mathcal{E}(f)$ is given by $-\Delta f$, where this statement implicitly uses the inner product $\langle \rangle$ on functions on $\mathbb{R}^n$. 
A function \( u(x,t) \) on \( \mathbb{R}^n \times (0, \infty) \) which is twice-continuously differentiable in \( x \) and continuously differentiable in \( x \) and \( t \) is said to satisfy the \textit{heat equation} if
\[
\frac{\partial}{\partial t} u = \Delta u.
\] (7)

Under modest growth conditions on a function \( f(x) \) on \( \mathbb{R}^n \), there is a unique continuous function \( u(x,t) \) on \( \mathbb{R}^n \times [0, \infty) \) such that \( u(x,0) = f(x) \), \( u(x,t) \) is infinitely differentiable in \( x \) and \( t \) when \( t > 0 \), \( u(x,t) \) satisfies the heat equation on \( \mathbb{R}^n \times (0, \infty) \), and \( u(x,t) \) also satisfies modest growth conditions (which can be related to those of \( f \)).

One way to look at the heat equation is as an ordinary differential equation in \( t \), acting in vector spaces of functions of \( x \). To find \( u(x,t) \) given \( f(x) \) as in the preceding paragraph, one might write
\[
u(x,t) = (\exp(t\Delta)f)(x).
\] (8)

In fact the Fourier transform gives a useful way to make sense of this.

**Aspects of symmetry**

Versions of these notions come up in a variety of situations, and a number of these were discussed in the trimestre. In the spirit of the book “Introduction to Fourier Analysis on Euclidean Spaces” by E. Stein and G. Weiss, which also provides a lot of helpful background information for these topics, one might start by considering the symmetries of the objects just described. They are all invariant under translations, and under rotations on \( \mathbb{R}^n \). They also behave nicely with respect to dilations on \( \mathbb{R}^n \), which is to say under transformations of the form \( x \mapsto ax \), where \( a \) is a positive real number. In the case of the heat equation, one should use the dilations \( (x,t) \mapsto (ax, a^2t) \), to adjust for the fact that there is one derivative in \( t \) and derivatives of order 2 in \( x \).

Instead of Euclidean spaces a basic setting is that of irreducible symmetric spaces of noncompact type, which was discussed in the course of J.-P. Anker. For these one again has translation invariance and forms of rotation invariance, but no dilation invariance. There are counterparts of Fourier analysis here too, for analyzing solutions to the heat equation, but this has some weaknesses differing from the Euclidean case.

In the Euclidean case the solution \( u(x,t) \) to the heat equation with initial data \( f(x) \) can be expressed in the form
\[
u(x,t) = \int_{\mathbb{R}^n} k_t(x-y) f(y) \, dy
\] (9)
for a function $k_t(x)$ called the heat kernel. The fact that the solution can be written in this manner, instead of

$$u(x, t) = \int_{\mathbb{R}^n} k_t(x, y) f(y) \, dy, \tag{10}$$

reflects the translation-invariance of the problem in $x$. The rotation-invariance of the problem implies in turn that $k_t(x)$ is a radial function of $x$, so that $k_t(x)$ can be written as $h_t(|x|)$ for a function $h_t(r)$ with $t \in (0, \infty)$ and $r \in [0, \infty)$. One can go further and use dilation-invariance to obtain that $k_t(x)$ is of the form $t^{-n/2}h(|x|/\sqrt{t})$ for a function $h(r)$, $r \in [0, \infty)$. It is a classical result, which is a good exercise to derive, that $k_t(x)$ is in fact a Gaussian function of $x$. This can be viewed in terms of the Fourier transform, or by working out an ordinary differential equation for the function $h(r)$.

In the context of symmetric spaces one can start with a general form for $u(x, t)$ as in (10), and use translation-invariance to reduce to something more like (9). The counterpart of rotation-invariance permits one to reduce the number of variables further, but not in general to 2 variables. Fourier analysis leads to interesting representations for the heat kernel, but fundamental features concerning size and localization are not always so clear from this representation.

Now let us go in a different direction and suppose that we are working on $\mathbb{R}^n$ again, but with a differential operator $L$ with variable coefficients in place of the Laplacian. Specifically, we assume that $L$ is of the form

$$L = \sum_{j,m=1}^{n} \frac{\partial}{\partial x_j} a_{j,m}(x) \frac{\partial}{\partial x_m}, \tag{11}$$

where $a_{j,m}(x)$ are bounded real-valued functions which satisfy

$$a_{j,m}(x) = a_{m,j}(x) \tag{12}$$

and

$$|v|^2 \leq \sum_{j,m=1}^{n} a_{j,m}(x) v_j v_m \tag{13}$$

for all $v \in \mathbb{R}^n$. In other words, $(a_{j,m}(x))_{j,m}$ are positive-definite real symmetric matrices which are uniformly bounded in $x$ and bounded from below in the sense of matrices by the identity matrix. Because the coefficients are allowed to depend on $x$, we lose in general the invariance under translations,
rotations, or dilations, and the heat kernel should be written as \( k_t(x, y) \), with \( x, y \in \mathbb{R}^n \) and \( t > 0 \), as in (10). However, there are vestiges of these invariances, in that translations and rotations of \( L \) lead to operators of the same type, and similarly for dilations if one includes suitable scale-factors. While the precise form of the heat kernel may not be easy to describe, one can try to show that it has many properties in common with the Gaussian kernels in the case of the standard Laplacian.

One can go further and consider coefficients \( a_{j,m}(x) \) which are not symmetric in \( j \) or \( m \), and perhaps not even real-valued. For the latter one can adjust (13) by taking the real part of the right side, so that one still has “uniform ellipticity”. More generally one can allow operators of order larger than 2, and vector-valued functions and systems of differential equations. Questions related to these situations were discussed in the courses of P. Auscher and P. Tchamitchian, and of S. Hofmann and A. McIntosh.

Note that it still makes sense to talk about
\[
\exp(tL)
\]
in this type of situation, using spectral theory. This works more nicely when the coefficients \( a_{j,m}(x) \) are real and symmetric, so that the operator \( L \) is self-adjoint (with a suitable choice of domain). Even without these conditions, one can define (14), using resolvent integrals. For that matter, one can define more general functions of \( L \), and part of the interest of the heat kernels is that the exponentials (14) and related operators can make good building blocks for studying other functions of \( L \).

On a connected Lie group \( H \) one can again look at second-order elliptic differential operators \( L \) which are invariant under translations, but in general \( H \) can be noncommutative and one should be careful to specify whether \( L \) is invariant under left translations, right translations, or both. In the case of Lie groups which are nilpotent, such as the Heisenberg groups, dilations can be used in much the same manner as on Euclidean spaces to have an extra degree of symmetry. In the course of W. Hebisch, solvable Lie groups and operators on them were treated, for which there is a delicate interplay between exponential growth on the one hand and having a fair amount of commutativity around on the other hand.

S. Lang gave a series of lectures concerning deep questions of expansions for heat kernels on the locally symmetric spaces (of finite volume)
\[
SL(n, \mathbb{R})/SL(n, \mathbb{Z}), \quad SL(n, \mathbb{C})/SL(n, \mathbb{Z}[i]),
\]
where \( \mathbb{Z} \) denotes the set of integers, and \( \mathbb{Z}[i] \) is the set of complex numbers whose real and imaginary parts are integers.

**Discrete settings**

Let us consider \( \mathbb{Z}^n \) now instead of \( \mathbb{R}^n \). If \( x, y \) are elements of \( \mathbb{Z}^n \), let us say that \( x \) and \( y \) are adjacent if \( |x - y| = 1 \). Thus \( x \) and \( y \) are adjacent if they agree in all but one component, where they differ by \( \pm 1 \). If \( f(x) \) is a function on \( \mathbb{Z}^n \), define \( A(f) \) on \( \mathbb{Z}^n \) by

\[
A(f)(x) = \frac{1}{2n} \sum_{y \in \mathbb{Z}^n, |x-y| = 1} f(y),
\]

so that \( A(f)(x) \) is the average of \( f \) over the \( 2n \) elements of \( \mathbb{Z}^n \) adjacent to \( x \).

The linear operator \( A - I \) on functions on \( \mathbb{Z}^n \), where \( I \) denotes the identity operator, is a discrete version of the Laplacian. This makes more sense if one writes the classical Laplacian of a twice continuously-differentiable function \( h \) at a point \( x \) as

\[
\Delta(h)(x) = \lim_{r \to 0} \frac{1}{r^2} (\text{Av}(h)(x, r) - h(x)),
\]

with \( \text{Av}(h)(x, r) \) equal to the average of \( h \) over the sphere with center \( x \) and radius \( r \).

The analogue of the heat equation for a function \( u(x, t) \) with \( x \) in \( \mathbb{Z}^n \) and \( t \) ranging through nonnegative integers can be written as

\[
u(x, t + 1) = \frac{1}{2n} \sum_{y \in \mathbb{Z}^n, |x-y| = 1} u(x, t),
\]

which is the same as saying that \( u(x, t + 1) \) is given by applying the operator \( A \) to \( u(x, t) \) as a function of \( x \). To make this look more like the classical heat equation, one can reexpress this as saying that \( u(x, t + 1) - u(x, t) \), which is like the “derivative” of \( u \) in \( t \), is equal to \( A - I \) applied to \( u(x, t) \) as a function of \( x \). Clearly, for any function \( f(x) \) on \( \mathbb{Z}^n \), there is a unique function \( u(x, t) \) defined for \( x \) in \( \mathbb{Z}^n \) and \( t \) a nonnegative integer such that \( u(x, 0) = f(x) \) for all \( x \) in \( \mathbb{Z}^n \) and \( u(x, t) \) satisfies the heat equation above for all \( x \) and \( t \). In fact, \( u(x, t) \) can be written as

\[
u(x, t) = (A^t)(f)(x),
\]

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in analogy with (8).

In analogy with (9), we can write

\[ u(x, t) = \sum_{y \in \mathbb{Z}^n} p_t(x - y) f(y), \]

where the “heat kernel” \( p_t(w) \) is defined for \( t \) a nonnegative integer and \( w \) in \( \mathbb{Z}^n \). Specifically, \( p_0(w) \) is equal to 0 when \( w \neq 0 \) and to 1 when \( w = 0 \), \( p_1(w) \) is equal to 0 when \( w \) is not adjacent to 0 and to \( 1/(2^n) \) when \( w \) is adjacent to 0, and \( p_t(w) \) can easily be determined explicitly.

In fact, \( p_t(x - y) \) is the probability that the standard random walk on \( \mathbb{Z}^n \) goes from \( x \) to \( y \) in exactly \( n \) steps. In the continuous setting there are similar statements for Brownian motion and other processes associated to second-order differential operators.

That the heat kernel in (20) is of the form \( p_t(x - y) \), rather than \( p_t(x, y) \), reflects the translation-invariance here, just as in the classical case on \( \mathbb{R}^n \). Of course one can consider other graphs instead of \( \mathbb{Z}^n \), with similar objects as defined above, and with a formula of the type

\[ u(x, t) = \sum_{y \in \mathbb{Z}^n} p_t(x, y) f(y) \]

in place of (20).

The course of T. Sunada dealt with crystal lattices, which are characterized in terms of a large abelian group of symmetries. The graphs \( \mathbb{Z}^n \) are a very special case of this, and numerous other configurations are possible. In W. Woess’ course, techniques of generating functions were discussed, which can lead to remarkable formulas and information about random walks. Part of M. Barlow’s course was concerned with random walks on graphs with self-similarity, and the effect of self-similarity on the heat kernel.

In analogy with second-order differential operators on \( \mathbb{R}^n \) with variable coefficients, one can consider random walks and discrete Laplacians on \( \mathbb{Z}^n \) in which the weighting factors vary from point to point. One does not need to stick to \( \mathbb{R}^n \) or \( \mathbb{Z}^n \) here; one can work on manifolds or graphs, or more generally metric spaces equipped with a measure. Several of the courses dealt with different facets of this, including Sobolev spaces and Sobolev or Poincaré inequalities.

R. Brooks discussed in his course Riemann surfaces, graphs, correspondences between them, and lower bounds for positive eigenvalues for the Laplacian for both.
Additional topics

Let $p$ be a real number, $p > 1$. For suitable functions $f(x)$ on $\mathbb{R}^n$, consider the $p$-energy functional

$$E_p(f) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla f(x)|^p. \quad (22)$$

This is the same as $E(f)$ in (5) when $p = 2$, but there is not a bilinear version as in (3) when $p \neq 2$. However, one can again consider the derivative of $E_p(f)$ in $f$ for all $p$, and this leads to a nonlinear (when $p \neq 2$) second-order differential operator known as the $p$-Laplacian.

The $p$-energy is invariant under translations and rotations, and scales under dilations in a simple way, just as when $p = 2$. For $p = n$ there is additional symmetry, known as conformal invariance.

One can consider more complicated functionals which behave in roughly the same manner in terms of size, but which incorporate “variable coefficients” into the picture. When $p = n$ there is a “quasi-invariance” of the energy under quasiregular mappings, which are defined in terms of a pointwise quasiconformality property (where the $n$th power of the norm of the differential of the mapping is bounded by a constant times the Jacobian, i.e., the determinant of the differential of the mapping). Quasiregular mappings, unlike quasiconformal mappings, are allowed to have branching, analogous to holomorphic mappings in the complex plane which are not one-to-one. The quasi-invariance of the $p$-energy when $p = n$ states that the energy functional is transformed by a quasiregular change of variables into an energy functional of roughly the same type, but with variable coefficients which satisfy bounds in terms of the quasiregularity constant. As a result, a solution of the $n$-Laplace equation is transformed, after composition with a quasiregular mapping, into a solution of an analogous equation with variable coefficients, still with suitable boundedness and ellipticity conditions. This is an important tool in the study of quasiregular mappings, as discussed in the course of I. Holopainen.

Even with the extra nonlinearity, there are similar issues concerning the relationship between the geometry of a space and the behavior of solutions of differential equations or inequalities as before.

A different kind of nonlinearity was treated in the course of K.-T. Sturm, with averages, heat flows, and random processes taking values in a metric space, under general conditions of nonpositive curvature. It can be clear how
to take a weighted average of two points in a metric space, using a point along a geodesic arc that joins them, but for more than two points not lying on the same geodesic the situation becomes more complicated. A fascinating feature of the probabilistic point of view is that in a sequence of independent samples one can use the ordering of the sequence to apply the two-point case step-by-step; it turns out that there are results to the effect that the limit of this exists and is the same almost surely, and that the common answer is the same as one produced from another procedure which deals with all points in the average at the same time.

The courses of B. Driver and L. Saloff-Coste were concerned with analysis on infinite-dimensional spaces. Specifically, Driver’s course dealt with Weiner space, spaces of paths in manifolds, and loop groups, while Saloff-Coste’s course addressed locally-compact and connected topological groups, such as infinite products of finite-dimensional compact connected Lie groups.

Of course the brief overview given here is not at all intended to be exhaustive. Fortunately, a volume is in preparation containing surveys and other material from the trimestre, in which much more information can be found.