Rational Quartic Spectrahedra

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Abstract

Rational quartic spectrahedra in 3-space are semialgebraic convex subsets in \( \mathbb{R}^3 \) of semidefinite, real symmetric \((4 \times 4)\)-matrices, whose boundary admit a rational parameterization. They are identified by the rank-2-locus and the rank-3 singular locus of the corresponding complex symmetroid, the Zariski closure of the boundary of the spectahedron in \( \mathbb{C}P^3 \). The symmetroid has a line or a smooth conic section in its rank-2-locus, or it has a triple point—in particular a rank-1-point—or a tacnode.

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1 Introduction

Spectrahedra are important basic objects in polynomial optimization and in convex algebraic geometry [BPT13]. They are intersections of the cone of positive-semidefinite matrices in the space of real symmetric \((n \times n)\)-matrices by an affine subspace. Quartic spectrahedra are the case of \((4 \times 4)\)-matrices intersected with a 3-dimensional affine space that contains a positive definite matrix. We identify the affine space with \( \mathbb{R}^3 \). The boundary of a quartic spectrahedron has a Zariski closure \( V(f_A) \subset \mathbb{R}P^3 \) defined by the determinant \( f_A(x) = f_A(x_0, x_1, x_2, x_3) \) of a symmetric matrix \( A(x) \), where explicitly

\[
A(x) = A_0 x_0 + A_1 x_1 + A_2 x_2 + A_3 x_3, \quad \text{(1.1)}
\]
and each $A_i$ is a real symmetric $(4 \times 4)$-matrix. Since the matrix $A(x)$ is symmetric, the surface $V(f_A)$ is called a (real) symmetroid. Similarly, the complex algebraic boundary $V_c(f_A) \subset \mathbb{CP}^3$ defined by $f_A$ is called a complex symmetroid, to distinguish it from its real points $V(f_A)$. If the complex algebraic boundary of a quartic spectrahedron is a rational complex symmetroid, we say that the spectrahedron is rational. In this paper we identify maximal families of rational quartic spectrahedra.

We show:

**Theorem 1.1.** The symmetroid of a rational quartic spectrahedron is singular along a line or a plane conic section, or it has a tacnode or a triple point. The general rational spectrahedron belongs to a maximal family whose general member has one of these singular loci and in addition $a \geq 0$ real nodes, of which $b \geq 0$ nodes lie on the boundary of the spectrahedron.

More precisely, the general member of each maximal family is one of the following:

1. A symmetroid singular along one line and $0 \leq b \leq a \leq b + 2 \leq 6$, both even.
2. A symmetroid singular along a smooth conic section with a real point that is disjoint from the spectrahedron and $a = b \leq 4$ even.
3. A symmetroid singular along a smooth conic section with a real point that lies on the boundary of the spectrahedron and $b \leq a \leq b + 2 \leq 4$, both even.
4. A symmetroid singular along a smooth conic section with no real points and $a = b = 2$.
5. A symmetroid with a triple point—a real rank-1-point—and $0 \leq b \leq a \leq 6$, with $a$ even.
6. A symmetroid with a tacnode not on the boundary of the spectrahedron and $b \leq a \leq b + 2 \leq 6$, both even.
7. A symmetroid with a tacnode on the boundary of the spectrahedron and $b \leq a \leq b + 2 \leq 6$, both even.

Appendix A contains examples of rational spectrahedral symmetroids realizing all solutions of the bounds given by Theorem 1.1, except two cases. The existence of spectrahedral symmetroids singular along a line, or along a conic section with a real point on the boundary of the spectrahedron, and $(a, b) = (0, 0)$ are open questions.

## 2 Real singularities of rational spectrahedral symmetroids

A general complex quartic symmetroid is singular, and a rational quartic surface is singular. The type of singularities that occur in the two cases are different, so we start by recalling from [Hel17] singularities of rational quartic symmetroids. We then go on to consider the cases of nonisolated singularities when the symmetroid is real. Lastly, we consider isolated singularities that are worse than nodes.
For a general matrix $A(x)$, the singular points of $V(f_A)$ is a finite set of double points, quadratic singularities called nodes. The nodal quartic spectrahedral symmetricoids form twenty maximal families identified by Degtyarev and Itenberg [DI11] and further investigated by Ottem et al. in [Ott+14]. A quartic surface with a finite set of nodes, or more generally rational double points, is irrational, in fact birational to a K3-surface.

A quartic surface is rational, i.e., can be rationally parameterized, only if it has an elliptic double point, a triple point or is singular along a curve [cf. Jes16; Noe89]. The first author identified maximal families of rational quartic symmetricoids in [Hel17].

At every point $x_p \in V(f_A) \subset \mathbb{C}P^3$, the matrix $A(x_p)$ has rank at most 3.

We say that $x_p \in V(f_A)$ is a rank-k-point, if $A(x_p)$ has rank $k$. The symmetric $V(f_A)$ has a double point at each rank-2-point, and a triple point at each rank-1-point. It may, however, also be singular at rank-3-points. This phenomenon is characterized by properties of the web of quadrics associated to the symmetric:

If $y := (y_0, y_1, y_2, y_3)$, then $q(x_p) := y \cdot A(x_p) \cdot y^T$ is a quadratic form, and its vanishing $Q(x_p) := V(q(x_p)) \subset \mathbb{C}P^3_y$ is a quadric surface. The set $Q(x) := \{Q(x_p) \mid x_p \in \mathbb{C}P^3\}$ is called a web of quadrics.

Lemma 2.1 ([Ili+17, Lemma 2.13; Wal81, Lemma 1.1]). The symmetric $V(f_A)$ has a singularity at a rank-3-point $x_p$ if and only if the web of quadrics $Q(x)$ has a basepoint at the singular point of $Q(x_p)$.

Theorem 2.2 ([Hel17]). The rational complex quartic symmetricoids in $\mathbb{C}P^3$ form irreducible families whose general member has the following singularities:

1. $V_C(f_A)$ is singular of rank 3 along a line and has four additional nodes, and the web $Q(x)$ has one basepoint.

2. $V_C(f_A)$ has rank 2 along a line and has six additional nodes, and the web $Q(x)$ has four coplanar basepoints.

3. $V_C(f_A)$ is singular along a conic and has four additional nodes, and the web $Q(x)$ has four linearly independent basepoints.

4. A triple point and six additional nodes.

5. A tacnode and six additional nodes, and the web $Q(x)$ has two basepoints.

We now restrict the attention to real rational quartic symmetricoids $V(f_A)$ with a nonempty spectrahedron, i.e., with $A(x_p)$ definite for some $x_p \in \mathbb{R}^3$. First note that if $A(x_p)$ is definite, then $Q(x_p)$ has no real points. So if the quartic spectrahedron of $A(x)$ is nonempty, then the web of quadric surfaces $Q(x)$ has no common real points, i.e., no real basepoints, so they have an even number of complex conjugate basepoints. We therefore consider real singularities of the symmetric $V(f_A)$ that represent real rank-2-quadrics in a web of quadrics $Q(x)$ with complex basepoints. We say that a real quadric is semidefinite resp. indefinite, when the associated symmetric matrix is.
2.1 Rational spectrahedral symmetroids with nonisolated singularities

Lemma 2.3. Let \( p_1, p_1, p_2, p_2 \) be two pairs of complex conjugate points in \( \mathbb{C}P^3 \) that do not all lie in a line. Then a real rank-2-quadric that contains both pairs of points is indefinite if and only if it contains the real lines \( \overline{p_1p_1} \) and \( \overline{p_2p_2} \).

Proof. Assume that \( Q = M \cup N \) is a real rank-2-quadric, the union of two planes \( M \) and \( N \). If \( M \) and \( N \) are both real, then \( Q \) is indefinite, while if \( M \) and \( N \) are complex conjugates, then \( Q \) is semidefinite.

If \( M \) and \( N \) each contains only two of the four points \( p_1, p_1, p_2, p_2 \), the lemma follows. If \( M \) contains exactly three of the points, say \( p_1, p_1, p_2, \) and is not real, then \( N \) must contain \( p_1, p_1, p_2, \) so \( Q \) is semidefinite. If \( M \) contains all four basepoints, \( M \) is real, so \( Q \) is indefinite. ■

Lemma 2.4. Let \( p_1, p_1, p_2, p_2 \) be two pairs of complex conjugate points in \( \mathbb{C}P^3 \) that do not all lie in a line, and let \( Q(x) \) be the 5-dimensional linear system of all quadratic surfaces with basepoints at these four points.

If the basepoints are not coplanar, then the rank-2-quadrics in \( Q(x) \) form three quadratic surfaces, \( Q_i, Q_{s1}, Q_{s2} \), where the real quadrics in \( Q_i \) are indefinite and the real quadrics in \( Q_{s1} \) and \( Q_{s2} \) are semidefinite.

If the basepoints are coplanar, then the rank-2-quadrics in \( Q(x) \) form three quadratic surfaces, as in the nonplanar case, and in addition a web \( W \), whose real quadrics are indefinite. In this case, the double plane containing the basepoints is a rank-1-quadric that lies in the closure of each component of rank-2-quadrics in \( Q(x) \).

Proof. First, note that the lines \( \overline{p_1p_1} \) and \( \overline{p_2p_2} \) are real and distinct, so if they intersect they do so in a real point.

The quadrics in \( Q_i \) contain the two lines \( \overline{p_1p_1} \) and \( \overline{p_2p_2} \). Likewise, the quadrics in \( Q_{s1} \) contain the line \( \overline{p_1p_2} \) and the quadrics in \( Q_{s2} \) contain the line \( \overline{p_1p_2} \). The first part of the claim follows from Lemma 2.3.

Assume now that the basepoints span a plane \( M \), which is real. Then \( W \) consists of all quadrics \( Q = M \cup N \), where \( N \) is any plane. If \( Q \) is real, then \( N \) is also real, so \( Q \) is indefinite when \( N \) is distinct from \( M \). On the other hand, the semidefinite double plane \( 2M \) is contained in \( W, Q_i, Q_{s1} \) and \( Q_{s2} \). ■

We now give a preliminary analysis of real singularities for spectrahedral symmetroids with nonisolated singularities.

Lemma 2.5. Let \( S = V(f_A) \) be a rational quartic spectrahedral symmetroid with nonisolated singularities. Then \( V_C(f_A) \) has rank 2 along a real line or a real conic, or it is singular and has rank 3 along two intersecting complex conjugate lines. Furthermore:

1. A line of rank-2-points on \( S \) is disjoint from the spectrahedron.
2. A real conic of rank-2-points on \( S \) may have no real points, or have a real point and be disjoint from the spectrahedron, or lie on the boundary of the spectrahedron.
Proof. If the complex symmetroid $V_C(f_A)$ is singular along a curve, then, by Theorem 2.2, this curve contains a line or a smooth conic section. Furthermore, when $V_C(f_A)$ is singular along a line, the matrix $A(x)$ may have rank 2 or 3 along the line.

In the first case, when $A(x)$ has rank 2 along the line, the quadrics $Q(x)$ have four coplanar basepoints and the line is real. By Lemma 2.4, the matrix $A(x)$ is indefinite along the line, so on the real spectrahedral symmetroid $V(f_A)$, the singular line must be disjoint from the spectrahedron.

In the second case, when $A(x)$ has rank 3 along the singular line, the web of quadrics $Q(x)$ contains a pencil $L \subset Q(x)$ of quadrics that are all singular at a basepoint [cf. Hel17, Proof of Proposition 3.5]. Since this basepoint cannot be real, the complex conjugate is also a basepoint. But then, the complex conjugate pencil $\overline{L} \subset Q(x)$ must be distinct from $L$, and $V_C(f_A)$ must be singular of rank 3 along two complex conjugate lines. If these lines do not intersect, the symmetroid $V_C(f_A)$ is a scroll of lines. The lines of this scroll form a curve of bidegree $(2, 2)$ on a quadratic surface in the Grassmannian of lines in $\mathbb{CP}^3$, so the scroll is birational to an elliptic scroll, i.e., irrational. Therefore, the symmetroid $V_C(f_A)$ that is singular, but of rank 3 along two lines, is rational only if the two lines intersect. When the lines are complex conjugates, they of course intersect in a real point. Thus the real symmetroid $V(f_A)$ is singular at this point. If $V(f_A)$ is a rational spectrahedral symmetroid singular along a smooth conic section, then, by Theorem 2.2, $A(x)$ must have rank 2 along this curve and the web of quadrics $Q(x)$ has two pairs of complex conjugate basepoints that are linearly independent. Clearly the conic section is real and the a priori listed possibilities follow.

The existence of rational spectrahedral symmetroids of the kinds listed in Lemma 2.5 is shown in Appendices A.1 and A.2, with two exceptions.

Remark 2.6. A quartic surface with two double lines is a degeneration of tacnodal surfaces [cf. Hel17, Remark 9.2], so we do not treat this case any further.

2.1.1 Symmetroids with a double line

Proposition 2.7. Let $S = V(f_A)$ be a general quartic symmetroid with rank 2 along a line and a nonempty spectrahedron. The symmetroid has an even number $a \geq 0$ of additional real nodes, of which an even number $b \geq 0$ nodes lie on the boundary of the spectrahedron. Furthermore $b \leq a \leq b + 2 \leq 6$.

Proof. By Item 2 of Theorem 2.2, $S$ is the discriminant of a web $Q_A(x)$ of quadrics with four coplanar basepoints. Since $S$ has a nonempty spectrahedron, the basepoints occur in two complex conjugate pairs. In the notation of Lemma 2.4, $Q_A(x)$ intersects $W$ in a line and the surfaces $Q_1, Q_s, Q_s$ in two points each. These are the only singularities on $S$, so we get that

$$b \leq a \leq b + 2 \leq 6,$$

and $a, b$ are even from Lemma 2.4.
Remark 2.8. Consider the space $Q(x)$ of quadrics with coplanar basepoints $p_1$, $\bar{p}_1$, $p_2$, $\bar{p}_2$. After a change of coordinates, we may assume that $p_1 := [1 : i : 0 : 0]$ and $p_2 := [1 : 0 : i : 0]$. Then the quadrics in $Q(x)$ have matrices on the form

$$
\begin{bmatrix}
 x_{00} & 0 & 0 & x_{03} \\
 0 & x_{00} & x_{12} & x_{23} \\
 0 & x_{12} & x_{00} & x_{23} \\
 x_{03} & x_{13} & x_{23} & x_{33}
\end{bmatrix}.
$$

(2.1)

Hence if (1.1) is on the form (2.1), and $A_0$ is positive definite, then $A(x)$ defines a spectrahedral symmetroid with a double line.

2.1.2 Symmetroids with a double conic

Proposition 2.9. Let $S = V(f_A)$ be a general quartic symmetroid that is singular along a smooth conic section $C$, with a nonempty spectrahedron. The symmetroid has $a \geq 0$ additional real nodes, of which $b \geq 0$ nodes lie on the boundary of the spectrahedron. Assume that $C$ has real points. If $C$ is disjoint from the spectrahedron, then $a = b \leq 4$, otherwise $b \leq a \leq b + 2 \leq 4$.

Proof. By Item 3 of Theorem 2.2, $S$ is the discriminant of a web $Q_A(x)$ of quadrics with four linearly independent basepoints. In the notation of Lemma 2.4, $Q_A(x)$ intersects one of the surfaces $Q_1$, $Q_{s1}$ or $Q_{s2}$ in $C$ and the remaining two surfaces in two points each. Hence $a \leq 4$.

If $C$ is disjoint from the spectrahedron, then $C \subset Q_i$. The real quadrics in $Q_{s1}$ and $Q_{s2}$ are semidefinite, so $a = b$. If $C$ lies on the boundary of the spectrahedron, then $C$ is contained in either $Q_{s1}$ or $Q_{s2}$. It follows that $a \leq b + 2$, since the real quadrics in $Q_i$ are indefinite.

Remark 2.10. Consider the space $Q(x)$ of quadrics with linearly independent basepoints $p_1$, $\bar{p}_1$, $p_2$, $\bar{p}_2$. After a change of coordinates, we may assume that $p_1 := [1 : i : 0 : 0]$ and $p_2 := [0 : 0 : 1 : i]$. Then the quadrics in $Q(x)$ have matrices on the form

$$
\begin{bmatrix}
 x_{00} & 0 & x_{02} & x_{03} \\
 0 & x_{00} & x_{12} & x_{13} \\
 x_{02} & x_{12} & x_{22} & 0 \\
 x_{03} & x_{13} & 0 & x_{22}
\end{bmatrix}.
$$

(2.2)

Moreover, the components of the rank-2-locus of $Q(x)$ are

$$
Q_1 = V(x_{00}, x_{22}, x_{02}x_{13} - x_{03}x_{12}),
Q_{s1} = V(x_{02} - x_{13}, x_{03} + x_{12}, x_{00}x_{22} - x_{12}^2 - x_{13}^2),
Q_{s2} = V(x_{02} + x_{13}, x_{03} - x_{12}, x_{00}x_{22} - x_{12}^2 - x_{13}^2).
$$

If $S = V(f_A)$ is a spectrahedral symmetroid singular along a smooth conic section with real points disjoint from the spectrahedron, then the web of quadrics $Q_A(x)$ associated to $S$ intersects $Q_i = V(x_{00}, x_{22})$ in a plane. To achieve this, we may assume that all the matrices in (1.1) are on the form (2.2), that $A_0$ is definite and that $x_{00} = x_{22} = 0$ for $A_1$, $A_2$, $A_3$.

If $S$ is singular along a smooth conic section with real points on the boundary of the spectrahedron, then $Q_A(x)$ intersects $Q_{s1}$ or $Q_{s2}$ in a plane. We may
therefore assume that $x_{02} \pm x_{13} = x_{03} \mp x_{12} = 0$ for $A_1$, $A_2$ and $A_3$ in (1.1).

A line through a definite matrix intersects $S$ only in real points. Therefore, if $Q_A(x)$ contains a definite point lying in $Q_{A_1} \cap Q_{A_2}$, then $S$ has two real, isolated nodes on the boundary of the spectrahedron. To construct examples with no real isolated nodes on the spectrahedron, we choose $A_0$ to be a definite matrix not in $Q_{A_1} \cap Q_{A_2}$. In particular, $A_0 \neq I_4$.

Note that all conics on $Q_1$ have real points. Hence if $S$ is singular along a conic with no real points, then the description in the previous paragraph applies. With an extra condition on the coefficients—for instance that $A_1$, $A_2$, $A_3$ satisfy $x_{00}x_{22} \leq 0$—we can ensure that the double conic has no real points.

Conic with no real points

A real quartic surface singular along a conic section with no real points in the plane at infinity, is known as a cyclide [Jes16, Chapter V]. The cyclides were first studied in a special case by Dupin [Dup22], and later in more generality by Darboux [Dar73]. An irreducible symmetroid in $\mathbb{RP}^3$ is therefore called a cyclide, if it is singular along a double conic with no real points.

Proposition 2.11 ([Jes16, Article 68]). If a cyclide has four additional nodes, then at most two of the isolated nodes are real.

Corollary 2.12. A general, real, quartic symmetroid singular along a smooth conic section with no real points has either two or no real nodes.

In a paper by Chandru, Dutta and Hoffmann, the authors summarize classical works by Cayley [Cay73] and Maxwell [Max68]. This is used to produce a classification of the various forms of the cyclides [CDH89, Table 1]. The only cyclides with a part that bounds a convex region are the spindle cyclides, see Figure 4. Hence these are the only cyclides that can occur as spectrahedral symmetroids. A spindle cyclide has two real nodes, or pinch points, where the “spindle” connects with the rest of the cyclide. We conclude:

Proposition 2.13. Let $S = V(f_A)$ be a real quartic symmetroid with a nonempty spectrahedron that is singular along a real conic section with no real points. Then $S$ is a spindle cyclide, it has two nodes, both on the boundary of the spectrahedron.

The 2-nodal spindle cyclide occurs as a spectrahedral symmetroid, as shown in Appendix A.2.3.

2.2 Rational spectrahedral symmetroids with only isolated singularities

2.2.1 Symmetroids with a triple point

By Item 4 of Theorem 2.2, a general complex symmetroid with a triple point has six nodes. Since a spectrahedral symmetroid is a real surface, the number $a$ of real nodes is even. There are no further restraints on $b$, the number of real semidefinite nodes, as the examples in Appendix A.3 show. Hence:

Proposition 2.14. Let $S$ be a general quartic symmetroid with a triple point and a nonempty spectrahedron. Let $a$ denote the number of additional real nodes
on $S$, and $b$ the number of those which lie on the boundary of the spectrahedron. Then $0 \leq b \leq a \leq 6$, with $a$ even. 

Remark 2.15. Triple points correspond precisely to rank-1-points. Thus (1.1) defines a spectrahedral symmetroid with a triple point if $A_0$ is a positive definite matrix, $A_1$ is a rank-1-matrix, and $A_2, A_3$ are any symmetric matrices. ♠

2.2.2 Symmetroids with a tacnode

Finally, we consider rational spectrahedral symmetroids with an elliptic double point. According to Noether’s classification [cf. Noe89], there are three families of rational quartic surfaces with an elliptic double point, i.e., a singular point so that there is a curve of arithmetic genus 1 supported on the exceptional curve of a minimal resolution of the singularity. Only the first kind in Noether’s classification—the tacnode—occurs for irreducible quartic symmetroids [Hel17, Propositions 7.9 and 7.10].

Proposition 2.16. Let $S = V(f_A)$ be a general quartic symmetroid with a tacnode and a nonempty spectrahedron. The symmetroid has $a \geq 0$ additional real nodes, of which $b \geq 0$ nodes lie on the boundary of the spectrahedron. Then $b \leq a \leq b + 2 \leq 6$.

Proof. By Item 5 of Theorem 2.2, the web $Q_A(x)$ has two basepoints. Since $S$ is spectrahedral, the basepoints are complex conjugates, $p$ and $\overline{p}$. Let $Q(x)$ be the 7-dimensional linear system of all quadratic surfaces with $p$ and $\overline{p}$ as basepoints. The rank-2-locus of $Q(x)$ consists of two fourfolds, $X_i$ and $X_s$. The quadrics in $X_i$ are pairs of planes, where one of the planes contains the line $p, \overline{p}$. In $X_s$, the quadrics consists of two planes, where the planes contains one basepoint each. The set $\text{Sing}(X_i) = \text{Sing}(X_s)$ consists of pairs of planes that both contain $p, \overline{p}$. The real quadrics in $X_i \setminus \text{Sing}(X_i)$ are indefinite and the real quadrics in $X_s \setminus \text{Sing}(X_s)$ are semidefinite. The real quadrics in $\text{Sing}(X_i) = \text{Sing}(X_s)$ are either semidefinite or indefinite.

In the proof of [Hel17, Proposition 7.4], it is shown that the tacnode corresponds to a point in $\text{Sing}(X_i) = \text{Sing}(X_s)$, and that $Q_A(x)$ intersects $X_i \setminus \text{Sing}(X_i)$ in two points and $X_s \setminus \text{Sing}(X_s)$ in four points. The claim follows. ■

Remark 2.17. Let $S$ be a spectrahedral symmetroid with a tacnode. Let $A(x)$ be the web of matrices defining $S$. Since its spectrahedron is nonempty, $A(x)$ contains a definite matrix. After a change of variables, we may assume that this matrix is the identity matrix, and that this matrix is $A(x_0)$.

Noether shows that for an elliptic double point on a surface, the tangent cone is a square. Moreover, a tacnode is rank-2-point [Hel17, Lemma 7.1]. We may assume that $A(x_1)$ corresponds to the tacnode, and that $A(x_1)$ is a diagonal rank-2-matrix with eigenvalues $\alpha$ and $\beta$. Since the tangent cone to $S$ at $A(x_1)$ is a square, we get that $A(x)$ has the form

$$A(x) = \begin{bmatrix} x_0 + l_{11} & 0 & l_{13} & l_{14} \\ 0 & x_0 + l_{11} & l_{23} & l_{24} \\ l_{13} & l_{23} & x_0 + \alpha x_1 + l_{33} & l_{34} \\ l_{14} & l_{24} & l_{34} & x_0 + \beta x_1 + l_{44} \end{bmatrix},$$
where each $l_{ij}$ is a linear form in $x_2, x_3$. Thus $S$ is defined by
\[
\det A(x) = \alpha \beta (x_0 + l_{11})^2 x_1^2 + (x_0 + l_{11}) f_2 x_1 + f_4,
\]
where the $f_i$ are forms of degree $i$ in $x_0, x_2, x_3$, and the tangent cone to $S$ is $V((x_0 + l_{11})^2)$.

Note that if $\alpha = \beta$, then the line spanned by $A(x_0)$ and $A(x_1)$ contains the matrix

\[
\begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus in order to construct examples of tacnodal symmetroids with no nodes on the boundary of the spectrahedron, we must choose $\alpha \neq \beta$.

2.3 Proof of Theorem 1.1

The theorem follows by combining Propositions 2.7, 2.9, 2.13, 2.14 and 2.16.

A Examples of existence

This appendix lists matrices that define spectrahedral symmetroids with the different values of $(a, b)$ described by Theorem 1.1. There are two missing instances, namely $(a, b) = (0, 0)$ both for symmetroids singular along a line and for symmetroids singular along a smooth conic section lying on the boundary of the spectrahedron.

A.1 Spectrahedral symmetroids with a double line

By Item 1 of Theorem 1.1, the possible number of real nodes $a$ and semidefinite real nodes $b$ satisfy
\[0 \leq b \leq a \leq b + 2 \leq 6,\]
with both $a$ and $b$ even. Below are examples of matrices that follow Remark 2.8 and produce all the different values of $(a, b)$, except $(a, b) = (0, 0)$. They were found by testing samples from a pseudorandom generator.

(6, 4):
\[
\begin{bmatrix}
x + w & 0 & 0 & 2z + 2w \\
0 & x + w & -2y & -y - 2z \\
0 & -2y & x + w & -2w \\
2z + 2w & -y - 2z & -2w & x
\end{bmatrix}
\]

(4, 4):
\[
\begin{bmatrix}
x - w & 0 & 0 & z - 2w \\
0 & x - w & 2y & -y - z \\
0 & 2y & x - w & -y - 2w \\
z - 2w & -y - z & -y - 2w & x
\end{bmatrix}
\]

(4, 2):
\[
\begin{bmatrix}
x & 0 & 0 & -2z - w \\
0 & x & y & -z \\
0 & y & x & -2y - w \\
-2z - w & -z & -2y - w & x
\end{bmatrix}
\]
Figure 1: Two spectrahedral symmetroids with a double line and two real nodes. The nodes lie on the spectrahedron in the left picture, but not in the right picture.

A.2 Spectrahedral symmetroids with a double conic

A.2.1 Conic disjoint from the spectrahedron

By Item 2 of Theorem 1.1, the possible number of real nodes $a$ and semidefinite real nodes $b$ satisfy

$$0 \leq a = b \leq 4,$$

with both $a$ and $b$ even. Below are examples of matrices that produce all the different values of $(a, b)$. They were constructed by following Remark 2.10.

(4, 4):

$$\begin{bmatrix} x & 0 & 2y & z \\ 0 & x & w & y \\ 2y & w & x & 0 \\ z & y & 0 & x \end{bmatrix}$$

(2, 2):

$$\begin{bmatrix} x & 0 & 2y - 2x & z \\ 0 & x & w & y \\ 2y - 2x & w & x & 0 \\ z & y & 0 & x \end{bmatrix}$$

(0, 0):

$$\begin{bmatrix} x & 0 & 2y - 4x & z \\ 0 & x & w & y \\ 2y - 4x & w & x & 0 \\ z & y & 0 & x \end{bmatrix}$$
Figure 2: A spectrahedral symmetroid with rank 2 along a conic that is disjoint from the spectrahedron. It has \((a, b) = (4, 4)\).

### A.2.2 Conic on the boundary of the spectrahedron

By Item 3 of Theorem 1.1, the possible number of real nodes \(a\) and semidefinite real nodes \(b\) satisfy

\[
0 \leq b \leq a \leq b + 2 \leq 4,
\]

with both \(a\) and \(b\) even. Following Remark 2.10, let \(A_0\) be the positive definite matrix

\[
\begin{bmatrix}
2 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Below are examples of matrices that produce all the different values of \((a, b)\), except \((a, b) = (0, 0)\). They were found by testing samples from a pseudorandom generator.

\((4, 2)\):

\[
\begin{bmatrix}
2x + y & 0 & x - z & 4z + w \\
0 & 2x + y & -4z - w & -z \\
x - z & -4z - w & x - 2y + 4w & 0 \\
4z + w & -z & 0 & x - 2y + 4w
\end{bmatrix}
\]

\((2, 2)\):

\[
\begin{bmatrix}
2x + y & 0 & x - z & 3w \\
0 & 2x + y & -3w & -z \\
x - z & -3w & x - 2y - w & 0 \\
3w & -z & 0 & x - 2y - w
\end{bmatrix}
\]

\((2, 0)\):

\[
\begin{bmatrix}
2x + y & 0 & x + z & -2z + w \\
0 & 2x + y & 2z - w & z \\
x + z & 2z - w & x - 2y + 4w & 0 \\
-2z + w & z & 0 & x - 2y + 4w
\end{bmatrix}
\]
Figure 3: A spectrahedral symmetroid with rank 2 along a conic on the boundary of the spectrahedron. It has \((a, b) = (2, 2)\).

A.2.3 Conic with no real points

By Item 4 of Theorem 1.1, the possible number of real nodes \(a\) and semidefinite real nodes \(b\) are \(a = b = 2\). Below is an example of a matrix that produces \((a, b) = (2, 2)\). It was constructed by following Remark 2.10.

\[
\begin{bmatrix}
2x + y + z & 0 & x + w & z \\
0 & 2x + y + z & -z & w \\
x + w & -z & x - y & 0 \\
z & w & 0 & x - y
\end{bmatrix}
\]

Figure 4: A spectrahedral symmetroid with rank 2 along a smooth conic section with no real points. It has \((a, b) = (2, 2)\). The surface is known as a “spindle cyclide”.

A.3 Spectrahedral symmetroids with a triple point

By Item 5 of Theorem 1.1, the possible number of real nodes \(a\) and semidefinite real nodes \(b\) satisfy

\[0 \leq b \leq a \leq 6,\]

with \(a\) even. Following Remark 2.15, let \(A_0\) be the identity matrix \(I_4\) and

\[
A_1 := \\
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Below are examples of $A_2$ and $A_3$ such that (1.1) produce all the different values of $(a, b)$. They were found by testing samples from a pseudorandom generator.

| (6, 6): | $A_2 := \begin{bmatrix} -9 & -1 & 2 & -2 \\ -1 & -3 & 7 & 7 \\ 2 & 7 & 5 & 5 \\ -2 & 7 & 5 & -4 \end{bmatrix}$ | $A_3 := \begin{bmatrix} 3 & -4 & -7 & 3 \\ -4 & -1 & -2 & -9 \\ -7 & 2 & 4 & 8 \\ 3 & -9 & 8 & 6 \end{bmatrix}$ |
| (6, 5): | $A_2 := \begin{bmatrix} 3 & -7 & 5 & 4 \\ -7 & 3 & 2 & 9 \\ 5 & 2 & 5 & 7 \\ 4 & 9 & 7 & -8 \end{bmatrix}$ | $A_3 := \begin{bmatrix} -3 & 5 & 5 & -5 \\ 5 & -3 & 6 & 3 \\ 5 & 6 & -2 & 0 \\ -5 & 3 & 0 & -7 \end{bmatrix}$ |
| (6, 4): | $A_2 := \begin{bmatrix} 7 & -7 & 6 & -6 \\ -7 & -5 & -5 & -5 \\ 6 & -5 & 8 & 2 \\ -6 & -5 & 2 & -1 \end{bmatrix}$ | $A_3 := \begin{bmatrix} -3 & 8 & -2 & 1 \\ 8 & -4 & 1 & 5 \\ -2 & 1 & -8 & 8 \\ 1 & 5 & 8 & -7 \end{bmatrix}$ |
| (6, 3): | $A_2 := \begin{bmatrix} 0 & 0 & 3 & -3 \\ 0 & 0 & -2 & 8 \\ 3 & -2 & 6 & 7 \\ -3 & 8 & 7 & 1 \end{bmatrix}$ | $A_3 := \begin{bmatrix} 2 & -1 & -5 & -4 \\ -1 & -5 & -6 & 8 \\ -5 & -6 & -5 & -1 \\ -4 & 8 & -1 & -6 \end{bmatrix}$ |
| (6, 2): | $A_2 := \begin{bmatrix} 9 & 7 & 2 & 3 \\ 7 & -5 & -9 & -2 \\ 2 & -9 & -2 & 3 \\ 3 & -2 & 3 & -5 \end{bmatrix}$ | $A_3 := \begin{bmatrix} 1 & 3 & -5 & 6 \\ 3 & -3 & 6 & 5 \\ -5 & 6 & -8 & -7 \\ 6 & 5 & -7 & -1 \end{bmatrix}$ |
| (6, 1): | $A_2 := \begin{bmatrix} 8 & -5 & 2 & -9 \\ -5 & 1 & -1 & 2 \\ 2 & -1 & -5 & 9 \\ -9 & 2 & 9 & -8 \end{bmatrix}$ | $A_3 := \begin{bmatrix} -9 & 6 & -3 & 3 \\ 6 & -7 & 2 & -1 \\ -3 & 2 & 2 & -7 \\ 3 & -1 & -7 & 5 \end{bmatrix}$ |
| (6, 0): | $A_2 := \begin{bmatrix} -3 & 6 & -4 & 1 \\ 6 & 2 & 6 & 9 \\ -4 & 6 & 0 & -7 \\ 1 & 9 & -7 & 8 \end{bmatrix}$ | $A_3 := \begin{bmatrix} 8 & 6 & 3 & -4 \\ 6 & 5 & 9 & 7 \\ 3 & 9 & 3 & 7 \\ -4 & 7 & 7 & -9 \end{bmatrix}$ |
| (4, 4): | $A_2 := \begin{bmatrix} 1 & -6 & -6 & 4 \\ -6 & 6 & 2 & 5 \\ -6 & 2 & -5 & -1 \\ 4 & 5 & -1 & -8 \end{bmatrix}$ | $A_3 := \begin{bmatrix} -5 & 1 & -7 & 6 \\ 1 & 9 & 9 & 7 \\ -7 & 9 & -9 & -8 \\ 6 & 7 & -8 & -2 \end{bmatrix}$ |
| (4, 3): | $A_2 := \begin{bmatrix} 6 & 3 & 9 & 9 \\ 3 & -8 & 9 & 0 \\ 9 & 9 & -7 & 6 \\ 9 & 0 & 6 & -7 \end{bmatrix}$ | $A_3 := \begin{bmatrix} -1 & 3 & 3 & 3 \\ 3 & -9 & 5 & 6 \\ 3 & 5 & 5 & 4 \\ 3 & -6 & 4 & -9 \end{bmatrix}$ |
| (4, 2): | $A_2 := \begin{bmatrix} -6 & -8 & 3 & -9 \\ -8 & -2 & 2 & 9 \\ 3 & 2 & -4 & -6 \\ -9 & 9 & -6 & -7 \end{bmatrix}$ | $A_3 := \begin{bmatrix} -2 & 9 & -4 & -2 \\ 9 & 8 & -1 & 9 \\ -4 & -1 & 1 & -4 \\ -2 & 9 & -4 & 4 \end{bmatrix}$ |
\(A_2 := \begin{bmatrix} 2 & 9 & -1 & -8 \\ 9 & 1 & 0 & -1 \\ -1 & 0 & -8 & 6 \\ -8 & -1 & 6 & -2 \end{bmatrix} \) 
\(A_3 := \begin{bmatrix} 2 & -6 & 0 & -6 \\ -6 & -5 & 2 & -1 \\ 0 & 2 & 6 & -1 \\ -6 & -1 & -1 & 9 \end{bmatrix} \) 

\(A_2 := \begin{bmatrix} -8 & 0 & -9 & 6 \\ 0 & 3 & -6 & 3 \\ -9 & -6 & -7 & 6 \\ 6 & 3 & 6 & 6 \end{bmatrix} \) 
\(A_3 := \begin{bmatrix} 1 & -5 & -1 & 8 \\ -5 & 9 & 0 & 4 \\ -1 & 0 & -2 & 8 \\ 8 & 4 & 8 & -3 \end{bmatrix} \)

\(A_2 := \begin{bmatrix} -4 & 4 & -1 & -6 \\ 4 & -3 & -2 & 0 \\ -1 & -2 & 5 & -1 \\ -6 & 0 & -1 & 3 \end{bmatrix} \) 
\(A_3 := \begin{bmatrix} 4 & 3 & 9 & 5 \\ 3 & -6 & -3 & -5 \\ 9 & -3 & 2 & 0 \\ 5 & -5 & 0 & 4 \end{bmatrix} \)

\(A_2 := \begin{bmatrix} 5 & -8 & 1 & 6 \\ -8 & -7 & -8 & -9 \\ 1 & -8 & -5 & -9 \\ 6 & -9 & -9 & -6 \end{bmatrix} \) 
\(A_3 := \begin{bmatrix} 3 & 1 & 4 & 5 \\ 1 & -4 & -6 & -8 \\ 4 & -6 & -2 & 8 \\ 5 & -8 & 8 & 5 \end{bmatrix} \)

\(A_2 := \begin{bmatrix} -9 & -6 & 5 & -1 \\ -6 & 4 & 0 & 1 \\ 5 & 0 & 8 & 1 \\ -1 & 1 & 1 & 6 \end{bmatrix} \) 
\(A_3 := \begin{bmatrix} -7 & 5 & -7 & 4 \\ 5 & 5 & 0 & 3 \\ -7 & 0 & 6 & 4 \\ 4 & 3 & 4 & 7 \end{bmatrix} \)

\(A_2 := \begin{bmatrix} 5 & -9 & -5 & -1 \\ -9 & -1 & 2 & -9 \\ -5 & 2 & 2 & 0 \\ -1 & -9 & 0 & -2 \end{bmatrix} \) 
\(A_3 := \begin{bmatrix} -7 & 6 & -5 & -1 \\ 6 & 4 & -1 & -3 \\ -5 & -1 & 9 & -8 \\ -1 & -3 & 8 & 5 \end{bmatrix} \)

Figure 5: A spectrahedral symmetroid with a triple point and \((a, b) = (6, 4)\).

### A.4 Spectrahedral symmetroids with a tacnode

#### A.4.1 Tacnode disjoint from the spectrahedron

By Item 6 of Theorem 1.1, the possible number of real nodes \(a\) and semidefinite real nodes \(b\) satisfy

\[ 0 \leq b \leq a \leq b + 2 \leq 6, \]
with both $a$ and $b$ even. Following Remark 2.17, let $A_0$ be the identity matrix $I_4$ and

\[
A_1 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2
\end{bmatrix}.
\]

Below are examples of $A_2$ and $A_3$ such that \((1.1)\) produce all the different values of \((a, b)\). They were found by testing samples from a pseudorandom generator.

\[
(6, 4): \quad A_2 := \begin{bmatrix}
8 & 0 & -6 & -8 \\
0 & 8 & 7 & 1 \\
-6 & 7 & -7 & 4 \\
-8 & 1 & 4 & 7
\end{bmatrix} \quad A_3 := \begin{bmatrix}
-6 & 0 & -2 & 2 \\
0 & -6 & 7 & -6 \\
-2 & 7 & 1 & -9 \\
2 & -6 & -9 & -7
\end{bmatrix}
\]

\[
(4, 4): \quad A_2 := \begin{bmatrix}
4 & 0 & -8 & -2 \\
0 & 4 & 5 & -3 \\
-8 & 5 & 5 & -8 \\
-2 & -3 & -8 & 9
\end{bmatrix} \quad A_3 := \begin{bmatrix}
3 & 0 & 2 & -4 \\
0 & 3 & 0 & 4 \\
2 & 0 & 0 & 2 \\
-4 & 4 & 2 & 6
\end{bmatrix}
\]

\[
(4, 2): \quad A_2 := \begin{bmatrix}
-1 & 0 & -5 & 7 \\
0 & -1 & -6 & -4 \\
-5 & -6 & 6 & 3 \\
7 & -4 & 3 & -6
\end{bmatrix} \quad A_3 := \begin{bmatrix}
6 & 0 & 2 & -8 \\
0 & 6 & 1 & -3 \\
2 & 1 & 5 & 4 \\
-8 & -3 & 4 & 8
\end{bmatrix}
\]

\[
(2, 2): \quad A_2 := \begin{bmatrix}
7 & 0 & 6 & 5 \\
0 & 7 & 2 & -4 \\
6 & 2 & 0 & -7 \\
5 & -4 & -7 & 2
\end{bmatrix} \quad A_3 := \begin{bmatrix}
0 & 0 & -8 & 2 \\
0 & 0 & 3 & 7 \\
-8 & 3 & 8 & -3 \\
2 & 7 & -3 & 7
\end{bmatrix}
\]

\[
(2, 0): \quad A_2 := \begin{bmatrix}
5 & 0 & 6 & -6 \\
0 & 5 & -6 & 8 \\
6 & -6 & 8 & 7 \\
-6 & 8 & 7 & 6
\end{bmatrix} \quad A_3 := \begin{bmatrix}
4 & 0 & 2 & -1 \\
0 & 4 & 6 & -6 \\
2 & 6 & -2 & 3 \\
-1 & -6 & 3 & 5
\end{bmatrix}
\]

\[
(0, 0): \quad A_2 := \begin{bmatrix}
-9 & 0 & -3 & 9 \\
0 & -9 & -6 & 7 \\
-3 & -6 & 4 & -1 \\
9 & 7 & -1 & -1
\end{bmatrix} \quad A_3 := \begin{bmatrix}
4 & 0 & 5 & -9 \\
0 & 4 & -2 & 5 \\
5 & -2 & 3 & 1 \\
-9 & 5 & 1 & -4
\end{bmatrix}
\]

### A.4.2 Tacnode on the boundary of the spectrahedron

By Item 7 of Theorem 1.1, the possible number of real nodes $a$ and semidefinite real nodes $b$ satisfy

\[0 \leq b \leq a \leq b + 2 \leq 6,\]

with both $a$ and $b$ even. Following Remark 2.17, let $A_0$ be the identity matrix $I_4$ and

\[
A_1 := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}.
\]
Below are examples of $A_2$ and $A_3$ such that (1.1) produce all the different values of $(a, b)$. They were found by testing samples from a pseudorandom generator.

(6, 4): $A_2 := \begin{bmatrix} 6 & 0 & -3 & 6 \\ 0 & 6 & -6 & -4 \\ -3 & -6 & 0 & 3 \\ 6 & -4 & 3 & 6 \end{bmatrix}$ $A_3 := \begin{bmatrix} 3 & 0 & 5 & -8 \\ 0 & 3 & -3 & -4 \\ 5 & -3 & 2 & -5 \\ -8 & -4 & -5 & 8 \end{bmatrix}$

(4, 4): $A_2 := \begin{bmatrix} 0 & 0 & -4 & 4 \\ -4 & -3 & -1 & 8 \\ 4 & -3 & 8 & -4 \end{bmatrix}$ $A_3 := \begin{bmatrix} 5 & 0 & -2 & 4 \\ 0 & 5 & -5 & 2 \\ -2 & -5 & 1 & -5 \\ 4 & 2 & -5 & -8 \end{bmatrix}$

(4, 2): $A_2 := \begin{bmatrix} 8 & 0 & -5 & 8 \\ 0 & 8 & 8 & -3 \\ -5 & 8 & 5 & -3 \\ 8 & -3 & -3 & 2 \end{bmatrix}$ $A_3 := \begin{bmatrix} -8 & 0 & 1 & 9 \\ 0 & -8 & -9 & 7 \\ 1 & -9 & 8 & 2 \\ 9 & 7 & 8 & 9 \end{bmatrix}$

(2, 2): $A_2 := \begin{bmatrix} -6 & 0 & 8 & 4 \\ 0 & -6 & -1 & -9 \\ 8 & -1 & 8 & 0 \\ 4 & -9 & 0 & -3 \end{bmatrix}$ $A_3 := \begin{bmatrix} -3 & 0 & 2 & -7 \\ 0 & -3 & -7 & -6 \\ 2 & -7 & 6 & -4 \\ -7 & -6 & -4 & -1 \end{bmatrix}$

(2, 0): $A_2 := \begin{bmatrix} 6 & 0 & 6 & -5 \\ 0 & 6 & 6 & -1 \\ 6 & 6 & 1 & -1 \\ -5 & -1 & -1 & 7 \end{bmatrix}$ $A_3 := \begin{bmatrix} -8 & 0 & -5 & 8 \\ 0 & -8 & 6 & -1 \\ -5 & 6 & -2 & 2 \\ 8 & -1 & 2 & 9 \end{bmatrix}$

(0, 0): $A_2 := \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & -1 \\ -2 & 0 & -7 & -5 \\ -4 & -1 & -5 & -4 \end{bmatrix}$ $A_3 := \begin{bmatrix} 7 & 0 & 4 & -6 \\ 0 & 7 & 5 & 8 \\ 4 & 5 & -2 & 0 \\ -6 & 8 & 0 & -4 \end{bmatrix}$

Figure 6: A spectrahedral symmetroid with a tacnode on the spectrahedron and $(a, b) = (6, 4)$. 

16
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