CONVERGENCE OF THE COMPLETE ELECTROMAGNETIC FLUID SYSTEM TO THE FULL COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. The full compressible magnetohydrodynamic equations can be derived from the complete electromagnetic fluid system as the dielectric constant tends to zero. In this paper we justify this singular limit rigorously in the framework of smooth solutions for well-prepared initial data.

1. Introduction and Main Results

The electromagnetic dynamics studies the flow of an electrically conducting fluid in the presence of an electromagnetic field. In the electromagnetic dynamics the flow and the electromagnetic field are closely connected with each other, hence the fundamental system of the electromagnetic dynamics usually contains the hydrodynamical equations and the electromagnetic ones. The complete electromagnetic fluid system includes the conservation of mass, momentum, and energy to the fluid, the Maxwell system, and the conservation of electric charge, which take the form ([5, 14, 20])

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{u}) &= 0, \\
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla P &= \text{div} \Psi(\mathbf{u}) + \rho \mathbf{E} + \mu_0 \mathbf{J} \times \mathbf{H}, \\
\frac{\rho \mathbf{E}}{\varepsilon} (\partial_t \mathbf{E} + \mathbf{u} \cdot \nabla \mathbf{E}) + \mathbf{u} \cdot \nabla \mathbf{u} &= \text{div} (\kappa \nabla \theta) + \Psi(\mathbf{u}) : \nabla \mathbf{u} + (\mathbf{J} - \rho \mathbf{E}) \cdot (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}), \\
\epsilon \partial_t \mathbf{E} - \text{curl} \mathbf{H} + \mathbf{J} &= 0, \\
\partial_t \mathbf{H} + \frac{1}{\mu_0} \text{curl} \mathbf{E} &= 0,
\end{align*}
\]

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\[ \hat{\partial}_t (\rho_e) + \nabla \cdot \mathbf{J} = 0, \]  
\[ \epsilon \nabla \cdot \mathbf{E} = \rho_e, \quad \nabla \cdot \mathbf{H} = 0. \]  
\[ (1.6) \]
\[ (1.7) \]

Here the unknowns \( \rho, \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3, \theta, \mathbf{E} = (E_1, E_2, E_3) \in \mathbb{R}^3, \mathbf{H} = (H_1, H_2, H_3) \in \mathbb{R}^3, \) and \( \rho_e \) denote the density, velocity, and temperature of the fluid, the electric field, the magnetic field, and the electric charge density, respectively; \( \Psi(\mathbf{u}) \) is the viscous stress tensor given by

\[ \Psi(\mathbf{u}) = 2\mu \mathbb{D}(\mathbf{u}) + \lambda \nabla \cdot \mathbf{u} \mathbf{I}_3, \quad \mathbb{D}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2, \]  
\[ (1.8) \]

where \( \mathbf{I}_3 \) denotes the \( 3 \times 3 \) identity matrix, and \( \nabla \mathbf{u}^T \) the transpose of the matrix \( \nabla \mathbf{u} \). The pressure \( P = P(\rho, \theta) \) and the internal energy \( e = e(\rho, \theta) \) are smooth functions of \( \rho \) and \( \theta \) of the flow, and satisfy the Gibbs relation

\[ \theta dS = de + P d\left( \frac{1}{\rho} \right), \]  
\[ (1.9) \]

for some smooth function (entropy) \( S = S(\rho, \theta) \) which expresses the first law of the thermodynamics. The current density \( \mathbf{J} \) is expressed by Ohm’s law, i.e.,

\[ \mathbf{J} - \rho_e \mathbf{u} = \sigma (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}). \]  
\[ (1.10) \]

The symbol \( \Psi(\mathbf{u}) : \nabla \mathbf{u} \) denotes the scalar product of two matrices:

\[ \Psi(\mathbf{u}) : \nabla \mathbf{u} = \sum_{i, j=1}^{3} \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 + \lambda |\nabla \mathbf{u}|^2 = 2\mu |\mathbb{D}(\mathbf{u})|^2 + \lambda |\mathrm{tr} \mathbb{D}(\mathbf{u})|^2. \]  
\[ (1.11) \]

The viscosity coefficients \( \mu \) and \( \lambda \) of the fluid satisfy \( \mu > 0 \) and \( 2\mu + 3\lambda > 0 \). The parameters \( \epsilon > 0 \) is the dielectric constant, \( \mu_0 > 0 \) the magnetic permeability, \( \kappa > 0 \) the heat conductivity, and \( \sigma > 0 \) the electric conductivity coefficient. For simplicity we assume that \( \mu, \lambda, \epsilon, \mu_0, \kappa, \) and \( \sigma \) are constants.

Mathematically, it is difficult to study the properties of solutions to the electromagnetic fluid system (1.1)–(1.7). The reason is that, as pointed out by Kawashima [20], the system of the electromagnetic quantities \( (\mathbf{E}, \mathbf{H}, \rho_e) \) in the system (1.1)–(1.7), which are regarded as a first order hyperbolic system, is neither symmetric hyperbolic nor strictly hyperbolic in the three-dimensional case. The same difficulty also occurs in the first order hyperbolic system of \( (\mathbf{E}, \mathbf{H}) \) which is obtained from the above system by eliminating \( \rho_e \) with the aid of the first equation of (1.7). Therefore, the classic hyperbolic-parabolic theory (for example [36]) can not be applied here. There are only a few mathematical results on the electromagnetic
fluid system (1.1)–(1.7) in some special cases. Kawashima [20] obtained the global existence of smooth solutions in the two-dimensional case when the initial data are a small perturbation of some given constant state. Umeda, Kawashima and Shizuta [34] obtained the global existence and time decay of smooth solutions to the linearized equations of the system (1.1)–(1.7) in the three-dimensional case near some given constant equilibrium.

On the other hand, as it was pointed out in [14], the assumption that the electric charge density \( \rho_e \approx 0 \) is physically very reasonable for the study of plasmas. In this situation, we can eliminate the terms involving \( \rho_e \) in the electromagnetic fluid system (1.1)–(1.7) and obtain the following simplified system:

\[
\begin{align*}
\dot{\rho} + \text{div} (\rho u) &= 0, \\
\rho (\dot{u} + u \cdot \nabla u) + \nabla P &= \text{div} \Psi(u) + \mu_0 J \times H, \\
\rho \dot{\theta} + u \cdot \nabla \theta + \theta \frac{\partial P}{\partial \theta} \text{div} u &= \kappa \Delta \theta + \Psi(u) : \nabla u + (E + \mu_0 u \times H), \\
\varepsilon \dot{\epsilon} E - \text{curl} H + J &= 0, \\
\dot{\theta} H + \frac{1}{\mu_0} \text{curl} E &= 0, \\
\text{div} H &= 0,
\end{align*}
\]

with

\[ J = \sigma (E + \mu_0 u \times H). \]

Formally, if we take the dielectric constant \( \varepsilon = 0 \) in (1.15), i.e. the displacement current is negligible, then we obtain that \( J = \text{curl} H \). Thanks to (1.17), we can eliminate the electric field \( E \) in (1.13), (1.14), and (1.16) and finally obtain the system

\[
\begin{align*}
\dot{\rho} + \text{div} (\rho u) &= 0, \\
\rho (\dot{u} + u \cdot \nabla u) + \nabla P &= \text{div} \Psi(u) + \mu_0 \text{curl} H \times H, \\
\rho \dot{\theta} + u \cdot \nabla \theta + \theta \frac{\partial P}{\partial \theta} \text{div} u &= \kappa \Delta \theta + \Psi(u) : \nabla u + \frac{1}{\sigma} |\text{curl} H|^2, \\
\dot{\theta} H - \text{curl} (u \times H) &= -\frac{1}{\sigma \mu_0} \text{curl} (\text{curl} H), \\
\text{div} H &= 0.
\end{align*}
\]

The equations (1.18)–(1.21) are the so-called full compressible magnetohydrodynamic equations, see [5, 27, 29].
The above formal derivation is usually referred as magnetohydrodynamic approximation, see [5,14]. In [22], Kawashima and Shizuta justified this limit process rigorously in the two-dimensional case (i.e., \( u = (u_1, u_2, 0) \), \( E = (0, 0, E_3) \), and \( H = (H_1, H_2, 0) \) with spatial variable \( x = (x_1, x_2) \in \mathbb{R}^2 \)) for local smooth solutions. Later, in [23], they also obtained the global convergence of the limit in the two-dimensional case under the assumption that both the initial data of the electromagnetic fluid equations and those of the compressible magnetohydrodynamic equations are a small perturbation of some given constant state in some Sobolev spaces in which the global smooth solution can be obtained. Recently, we studied the magnetohydrodynamic approximation for the isentropic electromagnetic fluid system in a three-dimensional period domain and deduced the isentropic compressible magnetohydrodynamic equations [18].

The purpose of this paper is to give a rigorous derivation of the full compressible magnetohydrodynamic equations (1.18)–(1.21) from the electromagnetic fluid system (1.12)–(1.17) as the dielectric constant \( \epsilon \) tends to zero. For the sake of simplicity and clarity of presentation, we shall focus on the ionized fluids obeying the perfect gas relations

\[
P = \mathfrak{R}\rho \theta, \quad e = c_V \theta,
\]

where the parameters \( \mathfrak{R} > 0 \) and \( c_V > 0 \) are the gas constant and the heat capacity at constant volume. We consider the system (1.12)–(1.17) in a periodic domain of \( \mathbb{R}^3 \), i.e., the torus \( \mathbb{T}^3 = (\mathbb{R}/(2\pi \mathbb{Z}))^3 \).

Below for simplicity of presentation, we take the physical constants \( \mathfrak{R}, c_V, \sigma, \) and \( \mu_0 \) to be one. To emphasize the unknowns depending on the small parameter \( \epsilon \), we rewrite the electromagnetic fluid system (1.12)–(1.17) as

\[
\begin{align*}
\partial_t \rho + \nabla (\rho \mathbf{u}) &= 0, \\
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla (\rho \theta) &= \nabla \Psi (\mathbf{u}) + (\mathbf{E} + \mathbf{u} \times \mathbf{H}) \times \mathbf{H}, \\
\rho (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) + \rho \theta \partial_t \mathbf{u} &= \kappa \Delta \theta + \Psi (\mathbf{u}) : \nabla \mathbf{u} + |\mathbf{E} + \mathbf{u} \times \mathbf{H}|^2, \\
\epsilon \partial_t \mathbf{E} - \text{curl} \mathbf{H} &= \mathbf{E} + \mathbf{u} \times \mathbf{H}, \\
\partial_t \mathbf{H} + \text{curl} \mathbf{E} &= 0, \quad \text{div} \mathbf{H} = 0,
\end{align*}
\]
where \( \Psi(\mathbf{u}^\prime) \) and \( \Psi(\mathbf{u}^\prime) : \nabla \mathbf{u}^\prime \) are defined through (1.8) and (1.11) with \( \mathbf{u} \) replaced by \( \mathbf{u}^\prime \). The system (1.23)–(1.27) are supplemented with initial data

\[
(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{E}^0, \mathbf{H}^0)|_{t=0} = (\rho_0(x), \mathbf{u}_0(x), \theta_0(x), \mathbf{E}_0(x), \mathbf{H}_0(x)), \quad x \in \mathbb{T}^3. \tag{1.28}
\]

We also rewrite the limiting equations (1.18)–(1.21) (recall that \( \Re = c_V = \sigma = \mu_0 = 1 \)) as

\begin{align*}
\partial_t \rho^0 + \text{div} (\rho^0 \mathbf{u}^0) &= 0, \tag{1.29}
\rho^0 (\partial_t \mathbf{u}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0) + \nabla (\rho^0 \theta^0) &= \text{div} \Psi(\mathbf{u}^0) + \text{curl} \mathbf{H}^0 \times \mathbf{H}^0, \tag{1.30}
\rho^0 (\partial_t \theta^0 + \mathbf{u}^0 \cdot \nabla \theta^0) + \rho^0 \theta^0 \text{div} \mathbf{u}^0 &= \kappa \Delta \theta^0 + \Psi(\mathbf{u}^0) : \nabla \mathbf{u}^0 + |\text{curl} \mathbf{H}^0|^2, \tag{1.31}
\partial_t \mathbf{H}^0 - \text{curl}(\mathbf{u}^0 \times \mathbf{H}^0) &= -\text{curl}(\text{curl} \mathbf{H}^0), \quad \text{div} \mathbf{H}^0 = 0, \tag{1.32}
\end{align*}

where \( \Psi(\mathbf{u}^0) \) and \( \Psi(\mathbf{u}^0) : \nabla \mathbf{u}^0 \) are defined through (1.8) and (1.11) with \( \mathbf{u} \) replaced by \( \mathbf{u}^0 \). The system (1.29)–(1.32) are equipped with initial data

\[
(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)|_{t=0} = (\rho_0(x), \mathbf{u}_0(x), \theta_0(x), \mathbf{H}_0(x)), \quad x \in \mathbb{T}^3. \tag{1.33}
\]

Notice that the electric field \( \mathbf{E}^0 \) is induced according to the relation

\[
\mathbf{E}^0 = \text{curl} \mathbf{H}^0 - \mathbf{u}^0 \times \mathbf{H}^0 \tag{1.34}
\]

by moving the conductive flow in the magnetic field.

Before stating our main result, we recall the local existence of smooth solutions to the problem (1.29)–(1.33). Since the system (1.29)–(1.32) is parabolic-hyperbolic, the results in [36] imply that

**Proposition 1.1** ([36]). Let \( s > 7/2 \) be an integer and assume that the initial data \((\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0)\) satisfy

\[
\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0 \in H^{s+1}(\mathbb{T}^3), \quad \text{div} \mathbf{H}_0^0 = 0, \quad 0 < \bar{\rho} = \inf_{x \in \mathbb{T}^3} \rho_0^0(x) \leq \rho_0^0(x) \leq \sup_{x \in \mathbb{T}^3} \rho_0^0(x) < +\infty,
\]

\[
0 < \bar{\theta} = \inf_{x \in \mathbb{T}^3} \theta_0^0(x) \leq \theta_0^0(x) \leq \sup_{x \in \mathbb{T}^3} \theta_0^0(x) < +\infty
\]

for some positive constants \( \bar{\rho}, \bar{\rho}, \bar{\theta}, \) and \( \bar{\theta} \). Then there exist positive constants \( T_* \) (the maximal time interval, \( 0 < T_* \leq +\infty \)), and \( \bar{\rho}, \bar{\rho}, \bar{\theta}, \bar{\theta} \), such that the problem (1.29)–(1.33) has a unique classical solution \((\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)\) satisfying \( \text{div} \mathbf{H}^0 = 0 \) and \( \rho^0 \in C^l([0, T_*), H^{s+1-l}(\mathbb{T}^3)), \quad \mathbf{u}^0, \theta^0, \mathbf{H}^0 \in C^l([0, T_*), H^{s+1-2l}(\mathbb{T}^3)), \quad l = 0, 1; \)
The main result of this paper can be stated as follows.

**Theorem 1.2.** Let $s > 7/2$ be an integer and $(\rho^0, u^0, \theta^0, H^0)$ be the unique classical solution to the problem (1.29)–(1.33) given in Proposition 1.1. Suppose that the initial data $(\rho_0^0, u_0^0, \theta_0^0, E_0^0, H_0^0)$ satisfy

$$\rho_0^0, u_0^0, \theta_0^0, E_0^0, H_0^0 \in H^s(\mathbb{T}^3), \ \text{div} H_0^0 = 0, \ \inf_{x \in \mathbb{T}^3} \rho_0^0(x) > 0, \ \inf_{x \in \mathbb{T}^3} \theta_0^0(x) > 0,$$

and

$$\| (\rho_0^0 - \rho_0^0, u_0^0 - u_0^0, \theta_0^0 - \theta_0^0, H_0^0 - H_0^0) \|_s + \sqrt{\epsilon} \| E_0^0 - (\text{curl} H_0^0 - u_0^0 \times H_0^0) \|_s \leq L_0 \epsilon,$$  

(1.35)

for some constant $L_0 > 0$. Then, for any $T_0 \in (0, T_*]$, there exist a constant $L > 0$, and a sufficient small constant $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, the problem (1.23)–(1.28) has a unique smooth solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon, E^\epsilon, H^\epsilon)$ on $[0, T_0]$ enjoying

$$\| (\rho^\epsilon - \rho^0, u^\epsilon - u^0, \theta^\epsilon - \theta^0, H^\epsilon - H^0)(t) \|_s + \sqrt{\epsilon} \| \{ E^\epsilon - (\text{curl} H^0 - u^0 \times H^0) \} (t) \|_s \leq L \epsilon, \quad t \in [0, T_0].$$  

(1.36)

Here $\| \cdot \|_s$ denotes the norm of Sobolev space $H^s(\mathbb{T}^3)$.

**Remark 1.1.** The inequality (1.36) implies that the sequences $(\rho^\epsilon, u^\epsilon, \theta^\epsilon, H^\epsilon)$ converge strongly to $(\rho^0, u^0, \theta^0, H^0)$ in $L^\infty(0, T; H^s(\mathbb{T}^3))$ and $E^\epsilon$ converges strongly to $E^0$ in $L^\infty(0, T; H^s(\mathbb{T}^3))$ but with different convergence rates, where $E^0$ is defined by (1.34).

**Remark 1.2.** Theorem 1.2 still holds for the case with general state equations. Furthermore, our results also hold in the whole space $\mathbb{R}^3$. Indeed, neither the compactness of $\mathbb{T}^3$ nor Poincaré-type inequality is used in our arguments.

**Remark 1.3.** In the two-dimensional case, our result is similar to that of [22] (see Remark 5.1 of [22]). In addition, if we assume that the initial data are a small perturbation of some given constant state in the Sobolev norm $H^s(\mathbb{T}^3)$ for $s > 3/2 + 2$, we can extend the local convergence result stated in Theorem 1.2 to a global one.
Remark 1.4. For the local existence of solutions \((\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)\) to the problem (1.29)–(1.33), the assumption on the regularity of initial data \((\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0)\) belongs to \(H^s(\mathbb{T}^3)\), \(s > 7/2\), is enough. Here we have added more regularity assumption in Proposition 1.1 to obtain more regular solutions which are needed in the proof of Theorem 1.2.

Remark 1.5. Our approach used in the proof of Theorem 1.2 is different from that in [22]. We make use of the nonlinear energy method inspired by [18,31] instead of the linearized energy method as done in [22].

Remark 1.6. The viscosity and heat conductivity terms in the system (1.23)–(1.27) play a crucial role in our uniformly bounded estimates (in order to control some undesirable higher-order terms). In the case of \(\lambda = \mu = \kappa = 0\), the original system (1.23)–(1.27) are reduced to the so-call non-isentropic Euler-Maxwell system. Our arguments can not be applied to this case directly and we shall report the results in a forthcoming paper.

We give some comments on the proof of Theorem 1.2. The main difficulty in dealing with the zero dielectric constant limit problem is the oscillatory behavior of the electric field. As pointed out in [18], besides the singularity in the Maxwell equations, there exists an extra singularity caused by the strong coupling of the electromagnetic field (nonlinear source term) in the momentum equations. Moreover, compared to the isentropic case studied in [18], we have to circumvent additional difficulties in the derivation of uniform estimates induced by the higher order terms (such as \(\Psi(\mathbf{u}'') : \nabla \mathbf{u}'\)) and more nonlinear terms (such as \(|\mathbf{E}' + \mathbf{u}' \times \mathbf{H}'|^2\)) involving \(\mathbf{u}', \mathbf{E}', \text{ and } \mathbf{H}'\) in the temperature equation. In this paper, we shall control all these singularities and then derive rigorously the full compressible magnetohydrodynamic equations from the electromagnetic fluid equations by employing an elaborate nonlinear energy method inspired by [18,31]. The key step in our proof is to to obtain higher order estimates of the density, velocity, temperature, the magnetic field, and the electric field by applying the Sobolev imbedding, Moser-type inequalities, and the regularity of the limiting equations. Then, we combine these estimates and apply Gronwall’s type inequality to obtain desired results. We remark that in the
isentropic case in [18], the density is controlled by the pressure, while in our case the density is controlled through the viscosity terms in the momentum equations.

It should be pointed out that there are a lot of literatures on the studies of compressible magnetohydrodynamic equations by physicists and mathematicians due to its physical importance, complexity, rich phenomena, and mathematical challenges. Below we just mention some mathematical results on the full compressible magnetohydrodynamic equations (1.18)–(1.21), we refer the interested reader to [1, 27, 29, 32] for many discussions on physical aspects. For the one-dimensional planar compressible magnetohydrodynamic equations, the existence of global smooth solutions with small initial data was shown in [21]. In [11, 33], Hoff and Tsyganov obtained the global existence and uniqueness of weak solutions with small initial energy. Under some technical conditions on the heat conductivity coefficient, Chen and Wang [2, 3, 35] obtained the existence, uniqueness, and Lipschitz continuous dependence of global strong solutions with large initial data, see also [7, 8] on the global existence and uniqueness of global weak solutions, and [6] on the global existence and uniqueness of large strong solutions with large initial data and vacuum. For the full multi-dimensional compressible magnetohydrodynamic equations, the existence of variational solutions was established in [4, 9, 13], while a unique local strong solution was obtained in [10]. The low Mach number limit is a very interesting topic in magnetohydrodynamics, see [19, 26, 28, 30] in the framework of the so-called variational solutions, and [15–17] in the framework of the local smooth solutions with small density and temperature variations, or large density/entropy and temperature variations.

Before ending this introduction, we give some notations and recall some basic facts which will be frequently used throughout this paper.

1) We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(\mathbb{T}^3)$ with $\langle f, f \rangle = \| f \|^2$, by $H^k$ the standard Sobolev space $W^{k,2}$ with norm $\| \cdot \|_k$. The notation $\|(A_1, A_2, \ldots, A_l)\|_k$ means the summation of $\| A_i \|_k$ from $i = 1$ to $i = l$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$. For an integer $m$, the symbol $D_x^m$ denotes the summation of all terms $\partial_x^\alpha$ with the multi-index $\alpha$ satisfying $|\alpha| = m$. We use $C_i$, $\delta_i$, $K_i$, and $K$ to denote the constants.
which are independent of $\epsilon$ and may change from line to line. We also omit the
spatial domain $\mathbb{T}^3$ in integrals for convenience.

(2) We shall frequently use the following Moser-type calculus inequalities (see
[24]):

(i) For $f, g \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$ and $|\alpha| \leq s$, it holds that

$$\|\partial_x^\alpha (fg)\| \leq C_s(\|f\|_{L^\infty} \|D_x^\alpha g\| + \|g\|_{L^\infty} \|D_x^\alpha f\|). \quad (1.37)$$

(ii) For $f \in H^s(\mathbb{T}^3), D_x^1 f \in L^\infty(\mathbb{T}^3), g \in H^{s-1}(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$ and $|\alpha| \leq s$, it
holds that

$$\|\partial_x^\alpha (fg) - f^\partial_x^\alpha g\| \leq C_s(\|D_x^1 f\|_{L^\infty} \|D_x^{s-1} g\| + \|g\|_{L^\infty} \|D_x^s f\|). \quad (1.38)$$

(3) Let $s > 3/2, f \in C^s(\mathbb{T}^3)$, and $u \in H^s(\mathbb{T}^3)$, then for each multi-index $\alpha,$
$1 \leq |\alpha| \leq s$, we have ( [24, 25]):

$$\|\partial_x^\alpha (f(u))\| \leq C(1 + \|u\|_{L^\infty}^{|\alpha|-1})\|u\|_{|\alpha|}; \quad (1.39)$$

moreover, if $f(0) = 0$, then ( [12])

$$\|\partial_x^\alpha (f(u))\| \leq C(\|u\|_{L^s})\|u\|_{s}. \quad (1.40)$$

This paper is organized as follows. In Section 2, we utilize the primitive system
(1.23)–(1.27) and the target system (1.29)–(1.32) to derive the error system and
state the local existence of the solution. In Section 3 we give the a priori energy
estimates of the error system and present the proof of Theorem 1.2.

2. Derivation of the error system and local existence

In this section we first derive the error system from the original system (1.23)–
(1.27) and the limiting equations (1.29)–(1.32), then we state the local existence of
solution to this error system.

Setting $N^\epsilon = \rho^\epsilon - \rho^0, U^\epsilon = u^\epsilon - u^0, \Theta^\epsilon = \theta^\epsilon - \theta^0, F^\epsilon = E^\epsilon - E^0,$ and $G^\epsilon = H^\epsilon - H^0,$
and utilizing the system (1.23)–(1.27) and the system (1.29)–(1.32) with (1.34), we
obtain that

$$\partial_t N^\epsilon + (N^\epsilon + \rho^0)\text{div} U^\epsilon + (U^\epsilon + u^0) \cdot \nabla N^\epsilon = -N^\epsilon \text{div} u^0 - \nabla \rho^0 \cdot U^\epsilon, \quad (2.1)$$

$$\partial_t U^\epsilon + [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon + \nabla \Theta^\epsilon + \frac{1}{N^\epsilon + \rho^0} \nabla N^\epsilon - \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(U^\epsilon)$$
\[
\begin{align*}
&= -(U^\varepsilon \cdot \nabla)u^0 - \left[ \frac{1}{N^\varepsilon + \rho^0} - \frac{1}{\rho^0} \right] \nabla \rho^0 + \left[ \frac{1}{N^\varepsilon + \rho^0} - \frac{1}{\rho^0} \right] \text{div} \Psi(u^0) \\
&- \frac{1}{\rho^0} \text{curl} H^0 \times H^0 + \frac{1}{N^\varepsilon + \rho^0} \left[ F^\varepsilon + u^0 \times G^\varepsilon + U^\varepsilon \times H^0 \right] \times H^0 \\
&+ \frac{1}{N^\varepsilon + \rho^0} \left[ F^\varepsilon + u^0 \times G^\varepsilon + U^\varepsilon \times H^0 \right] \times G^\varepsilon \\
&+ \frac{1}{N^\varepsilon + \rho^0} (U^\varepsilon \times G^\varepsilon) \times (G^\varepsilon + H^0),
\end{align*}
\]
\begin{equation}
\tag{2.2}
\end{equation}
\begin{align*}
\partial_t \Theta^\varepsilon + [(U^\varepsilon + u^0) \cdot \nabla] \Theta^\varepsilon + (\Theta^\varepsilon + \theta^0) \text{div} U^\varepsilon - \frac{\kappa}{N^\varepsilon + \rho^0} \Delta \Theta^\varepsilon \\
&= -(U^\varepsilon \cdot \nabla)\theta^0 - \Theta^\varepsilon \text{div} u^0 + \left[ \frac{\kappa}{N^\varepsilon + \rho^0} - \frac{\kappa}{\rho^0} \right] \Delta \theta^0 \\
&+ \frac{2\mu}{N^\varepsilon + \rho^0} |\mathcal{D}(U^\varepsilon)|^2 + \frac{\lambda}{N^\varepsilon + \rho^0} |\text{tr}\mathcal{D}(U^\varepsilon)|^2 \\
&+ \frac{4\mu}{N^\varepsilon + \rho^0} \mathcal{D}(U^\varepsilon) : \mathcal{D}(u^0) + \frac{2\lambda}{N^\varepsilon + \rho^0} [\text{tr}\mathcal{D}(U^\varepsilon) \text{tr}\mathcal{D}(u^0)] \\
&+ \left[ \frac{2\mu}{N^\varepsilon + \rho^0} - \frac{2\mu}{\rho^0} \right] |\mathcal{D}(u^0)|^2 + \left[ \frac{\lambda}{N^\varepsilon + \rho^0} - \frac{\lambda}{\rho^0} \right] (\text{tr}\mathcal{D}(u^0))^2 \\
&+ \frac{1}{N^\varepsilon + \rho^0} |F^\varepsilon + U^\varepsilon \times G^\varepsilon|^2 + \frac{1}{N^\varepsilon + \rho^0} |u^0 \times G^\varepsilon + U^\varepsilon \times H^0|^2 \\
&+ \frac{2}{N^\varepsilon + \rho^0} (F^\varepsilon + U^\varepsilon \times G^\varepsilon) \cdot [\text{curl} H^0 + u^0 \times G^\varepsilon + U^\varepsilon \times H^0] \\
&+ \frac{2}{N^\varepsilon + \rho^0} \text{curl} H^0 \cdot (u^0 \times G^\varepsilon + U^\varepsilon \times H^0) \\
&+ \left[ \frac{1}{N^\varepsilon + \rho^0} - \frac{1}{\rho^0} \right] |\text{curl} H^0|^2, 
\end{align*}
\begin{equation}
\tag{2.3}
\end{equation}
\begin{align*}
\epsilon \partial_t F^\varepsilon - \text{curl} G^\varepsilon &= -[F^\varepsilon + U^\varepsilon \times H^0 + u^0 \times G^\varepsilon] - U^\varepsilon \times G^\varepsilon \\
&- \epsilon \partial_t \text{curl} H^0 + \epsilon \partial_t (u^0 \times H^0), \\
\partial_t G^\varepsilon + \text{curl} F^\varepsilon &= 0, \quad \text{div} G^\varepsilon = 0,
\end{align*}
\begin{equation}
\tag{2.4}
\end{equation}
\begin{equation}
\tag{2.5}
\end{equation}

with initial data
\[
(N^0, U^0, \Theta^0, F^0, G^0)|_{t=0} := (N^0_0, U^0_0, \Theta^0_0, F^0_0, G^0_0)
= (\rho^0_0 - \rho^0_0, u^0_0 - u^0_0, \theta^0_0 - \theta^0_0, E^0_0 - (\text{curl} H^0_0 - u^0_0 \times H^0_0), H^0_0 - H^0_0).
\]
\begin{equation}
\tag{2.6}
\end{equation}
Denote

\[
W^e = \begin{pmatrix}
N_e \\
U^e \\
\Theta^e \\
F^e \\
G^e
\end{pmatrix},
W_0^e = \begin{pmatrix}
N_0^e \\
U_0^e \\
\Theta_0^e \\
F_0^e \\
G_0^e
\end{pmatrix},
D^e = \begin{pmatrix}
I_5 & 0 \\
0 & eI_3 & 0
\end{pmatrix},
\]

\[
A_i^e = \begin{pmatrix}
(U_e + u^0)_i & (N_e + \rho^0)e_i^T & 0 \\
\frac{\Theta^e + \rho^0}{N_e + \rho^0}e_i & (U_e + u^0)_i & e_i \\
0 & (\Theta^e + \rho^0)e_i^T & (U_e + u^0)_i
\end{pmatrix},
\]

\[
A_{ij}^e = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{\mu^e}{N_e + \rho^0}e_i^T I_3 + e_i^T e_j + \frac{\Lambda_{ij}}{N^e + \rho^0}e_i^T e_i & 0 & 0 \\
0 & 0 & \frac{\kappa_{ij}}{N^e + \rho^0}e_i^T e_j & 0
\end{pmatrix},
\]

\[
S^e(W^e) = \begin{pmatrix}
-N^e \text{div } u^0 - \nabla \rho^0 \cdot U^e \\
R_1^e \\
R_2^e \\
R_3^e \\
0
\end{pmatrix},
\]

where \( R_1^e, R_2^e, \) and \( R_3^e \) denote the right-hand side of (2.2), (2.3), and (2.4), respectively. \((e_1, e_2, e_3)\) is the canonical basis of \( \mathbb{R}^3 \), \( I_d \ (d = 3, 5) \) is a \( d \times d \) unit matrix, \( y_i \) denotes the \( i \)-th component of \( y \in \mathbb{R}^3 \), and

\[
B_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix},
B_2 = \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
B_3 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Using these notations we can rewrite the problem (2.1)–(2.6) in the form

\[
\begin{aligned}
\mathbf{D}^\varepsilon \partial_t \mathbf{W}^\varepsilon + \sum_{i=1}^{3} \mathbf{A}^\varepsilon_i \mathbf{W}^\varepsilon_{x_i} + \sum_{i,j=1}^{3} \mathbf{A}^\varepsilon_{ij} \mathbf{W}^\varepsilon_{x_i x_j} = \mathbf{S}^\varepsilon(\mathbf{W}^\varepsilon), \\
\mathbf{W}^\varepsilon|_{t=0} = \mathbf{W}^\varepsilon_0.
\end{aligned}
\]

(2.7)

It is not difficult to see that the system for \( \mathbf{W}^\varepsilon \) in (2.7) can be reduced to a quasilinear symmetric hyperbolic-parabolic one. In fact, if we introduce

\[
\mathbf{A}^\varepsilon = \left( \begin{array}{ccc}
\Theta + \theta^0 \\
\Theta + \theta^0 \rho N^\varepsilon \\
\Theta + \theta^0 \\
0 & I_3 & 0 \\
0 & 0 & \rho \Theta + \theta^0 \\
0 & 0 & I_6
\end{array} \right),
\]

which is positively definite when \( |N^\varepsilon|_{L^\infty L^2} \leq \hat{\rho}/2 \) and \( \|\Theta^\varepsilon\|_{L^\infty L^2} \leq \hat{\Theta}/2 \), then \( \hat{\mathbf{A}}^\varepsilon = \mathbf{A}^\varepsilon \mathbf{D}^\varepsilon \) and \( \hat{\mathbf{A}}^\varepsilon = \mathbf{A}^\varepsilon \mathbf{A}^\varepsilon \) are positive symmetric on \([0,T]\) for all \( 1 \leq i \leq 3 \).

Moreover, the assumptions that \( \mu > 0, 2\mu + 3\lambda > 0 \), and \( \kappa > 0 \) imply that

\[
\mathbf{A}^\varepsilon = \sum_{i,j=1}^{3} \mathbf{A}^\varepsilon \mathbf{A}^\varepsilon_i \mathbf{W}^\varepsilon_{x_i x_j}
\]
is an elliptic operator. Thus, we can apply the result of Vol’pert and Hudiaev [36] to obtain the following local existence for the problem (2.7).

\textbf{Proposition 2.1.} Let \( s > 7/2 \) be an integer and \( (\rho_0^0, u_0^0, \theta_0^0, H_0^0) \) satisfy the conditions in Proposition 1.1. Assume that the initial data \( (N^0_0, U^0_0, \Theta^0_0, F^0_0, G^0_0) \) satisfy

\[
\inf_{x \in \mathbb{T}^3} N^0_0(x) > 0, \quad \inf_{x \in \mathbb{T}^3} \Theta^0_0(x) > 0, \quad |N^0_0|_s \leq \delta, \quad |\Theta^0_0|_s \leq \delta
\]

for some small constant \( \delta > 0 \). Then there exist positive constants \( T^* \) \( (0 < T^* \leq +\infty) \), \( \hat{N}, \hat{N}, \hat{\Theta}, \hat{\Theta}, \) and \( K \), such that the Cauchy problem (2.7) has a unique classical solution \( (N^*, U^*, \Theta^*, F^*, G^*) \) satisfying

\[
N^*, F^*, G^* \in C^l ([0, T^*), H^{s-l}), \quad U^*, \Theta^* \in C^l ([0, T^*), H^{s-2l}), \quad l = 0, 1; \quad \text{div} \mathbf{G}^* = 0;
\]

\[
0 < \hat{N} = \inf_{x \in \mathbb{T}^3} N^*(x, t) \leq \hat{N} = \sup_{x \in \mathbb{T}^3} N^*(x, t), \quad |N^*(t)|_s \leq K\delta, \quad t \in [0, T^*),
\]

\[
0 < \hat{\Theta} = \inf_{x \in \mathbb{T}^3} \Theta^*(x, t) \leq \hat{\Theta} = \sup_{x \in \mathbb{T}^3} \Theta^*(x, t), \quad |\Theta^*(t)|_s \leq K\delta, \quad t \in [0, T^*).
\]
Note that if $\|N^\epsilon\|_{L_t^\infty L_x^2} \leq \hat{\rho}/2$ and $\|\Theta^\epsilon\|_{L_t^\infty L_x^2} \leq \hat{\theta}/2$ then, for smooth solutions, the electromagnetic fluid system (1.23)–(1.27) with initial data (1.28) are equivalent to (2.1)–(2.6) or (2.7) on $[0, T], T < \min\{T^*, T_0\}$. Therefore, in order to obtain the convergence of electromagnetic fluid equations (1.23)–(1.27) to the full compressible magnetohydrodynamic equations (1.29)–(1.32), we only need to establish the uniform decay estimates with respect to the parameter $\epsilon$ of the solution to the error system (2.7). This will be achieved by the elaborate energy method presented in next section.

3. Uniform energy estimates and proof of Theorem 1.2

In this section we derive the uniform decay estimates with respect to the parameter $\epsilon$ of the solution to the problem (2.7) and justify rigorously the convergence of electromagnetic fluid system to the full compressible magnetohydrodynamic equations (1.29)–(1.32). Here we adapt and modify some techniques developed in [18,31] and put main efforts on the estimates of higher order nonlinear terms.

We first establish the convergence rate of the error equations by establishing the a priori estimates uniformly in $\epsilon$. For presentation conciseness, we define

$$
\|E^\epsilon(t)\|_s^2 := \|\text{proj}(N^\epsilon, U^\epsilon, \Theta^\epsilon, G^\epsilon)(t)\|_s^2,
$$

$$
\|E^\epsilon\|_\infty^2 := \|E^\epsilon(t)\|_s^2 + \epsilon\|F^\epsilon(t)\|_s^2,
$$

$$
\|E^\epsilon\|_{s,T} := \sup_{0 < t < T} \|E^\epsilon(t)\|_s.
$$

The crucial estimate of our paper is the following decay result on the error system (2.1)–(2.5).

**Proposition 3.1.** Let $s > 7/2$ be an integer and assume that the initial data $(N_0^\epsilon, U_0^\epsilon, \Theta_0^\epsilon, F_0^\epsilon, G_0^\epsilon)$ satisfy

$$
\|\text{proj}(N_0^\epsilon, U_0^\epsilon, \Theta_0^\epsilon, G_0^\epsilon)\|_s^2 + \epsilon\|F_0^\epsilon\|_s^2 = \|E^\epsilon(t = 0)\|_s^2 \leq M_0 \epsilon^2 \tag{3.1}
$$

for sufficiently small $\epsilon$ and some constant $M_0 > 0$ independent of $\epsilon$. Then, for any $T_0 \in (0, T_*)$, there exist two constants $M_1 > 0$ and $\epsilon_1 > 0$ depending only on $T_0$, such that for all $\epsilon \in (0, \epsilon_1]$, it holds that $T^* \geq T_0$ and the solution $(N^\epsilon, U^\epsilon, \Theta^\epsilon, F^\epsilon, G^\epsilon)$ of the problem (2.1)–(2.6), well-defined in $[0, T_0]$, enjoys that

$$
\|E^\epsilon\|_{s,T_0} \leq M_1 \epsilon. \tag{3.2}
$$
Once this proposition is established, the proof of Theorem 1.2 is a direct procedure. In fact, we have

**Proof of Theorem 1.2.** Suppose that Proposition 3.1 holds. According to the definition of the error functions \( \langle N^\varepsilon, U^\varepsilon, \Theta^\varepsilon, F^\varepsilon, G^\varepsilon \rangle \) and the regularity of \( (\rho^0, u^0, \theta^0, H^0) \), the error system (2.1)–(2.5) and the primitive system (1.23)–(1.27) are equivalent on \([0, T]\) for some \( T > 0 \). Therefore the assumption (1.35) in Theorem 1.2 imply the assumption (3.1) in Proposition 3.1, and hence (3.2) implies (1.36).

Therefore, our main goal next is to prove Proposition 3.1 which can be approached by the following a priori estimates. For some given \( \hat{T} < 1 \) and any \( \tilde{T} < \hat{T} \) independent of \( \varepsilon \), we denote \( T = T_\varepsilon = \min\{\tilde{T}, T^\varepsilon\} \).

**Lemma 3.2.** Let the assumptions in Proposition 3.1 hold. Then, for all \( 0 < t < T \), \( \varepsilon \) sufficiently small, there exist two positive constants \( \delta_1 \) and \( \delta_2 \), such that

\[
\|E^\varepsilon(t)\|_2^2 + \int_0^t \left\{ \delta_1 \|\nabla U^\varepsilon\|_2^2 + \delta_2 \|\nabla \Theta^\varepsilon\|_2^2 + \frac{1}{4} \|\nabla \Theta^\varepsilon\|_2^2 \right\} (\tau) d\tau 
\leq \|E^\varepsilon(t = 0)\|_2^2 + C \int_0^t \left\{ \left( \|E^\varepsilon\|_2^{2\varepsilon} + \|E^\varepsilon\|_2^2 + 1 \right) \|E^\varepsilon\|_2^2 \right\} (\tau) d\tau + C\varepsilon^2. \quad (3.3)
\]

**Proof.** Let \( 0 \leq |\alpha| \leq s \). In the following arguments the commutators will disappear in the case of \( |\alpha| = 0 \).

Applying the operator \( \partial_x^\alpha \) to (2.1), multiplying the resulting equation by \( \partial_x^\alpha N^\varepsilon \), and integrating over \( T^3 \), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \langle \partial_x^\alpha N^\varepsilon, \partial_x^\alpha N^\varepsilon \rangle = - \langle \partial_x^\alpha \left( (U^\varepsilon + u^0) \cdot \nabla \right) N^\varepsilon, \partial_x^\alpha N^\varepsilon \rangle 
- \langle \partial_x^\alpha ((N^\varepsilon + \rho^0) \text{div} U^\varepsilon), \partial_x^\alpha N^\varepsilon \rangle 
+ \langle \partial_x^\alpha ((-N^\varepsilon \text{div} u^0 - \nabla \rho^0 \cdot U^0), \partial_x^\alpha N^\varepsilon \rangle. \quad (3.4)
\]

Next we bound every term on the right-hand side of (3.4). By the regularity of \( u^0 \), Cauchy-Schwarz’s inequality, and Sobolev’s imbedding, we have

\[
\langle \partial_x^\alpha \left( (U^\varepsilon + u^0) \cdot \nabla \right) N^\varepsilon, \partial_x^\alpha N^\varepsilon \rangle 
= \langle (U^\varepsilon + u^0) \cdot \nabla \partial_x^\alpha N^\varepsilon, \partial_x^\alpha N^\varepsilon \rangle 
+ \langle \mathcal{H}^{(1)}, \partial_x^\alpha N^\varepsilon \rangle 
= - \frac{1}{2} \langle \text{div} (U^\varepsilon + u^0) \partial_x^\alpha N^\varepsilon, \partial_x^\alpha N^\varepsilon \rangle 
+ \langle \mathcal{H}^{(1)}, \partial_x^\alpha N^\varepsilon \rangle 
\leq C(\|E^\varepsilon(t)\|_s + 1) \|\partial_x^\alpha N^\varepsilon\|_2^2 + \|\mathcal{H}^{(1)}\|_2^2, \quad (3.5)
\]
where the commutator
\[ \mathcal{H}^{(1)} = \partial_x^\alpha \left( [(U^\tau + u^0) \cdot \nabla]N^\tau \right) - \left( [(U^\tau + u^0) \cdot \nabla]\partial_x^\alpha N^\tau \right) \]
can be bounded as follows, using the Moser-type and Cauchy-Schwarz’s inequalities, the regularity of \( u^0 \) and Sobolev’s imbedding.
\[
\| \mathcal{H}^{(1)} \| \leq C(\| D_x^1 (U^\tau + u^0) \|_{L^\infty} \| D_x^\alpha N^\tau \| + \| D_x^1 N^\tau \|_{L^\infty} \| D_x^{\alpha-1} (U^\tau + u^0) \|)
\]
\[
\leq C\| \mathcal{E}(t) \|_{s}^2 + C \epsilon^2. \tag{3.6}
\]

Similarly, the second term on the right-hand side of (3.4) can bounded as follows.
\[
\langle \partial_x^\alpha ((N^\tau + \rho^0)\text{div} U^\tau), \partial_x^\alpha N^\tau \rangle
\]
\[
= \langle [(N^\tau + \rho^0)\partial_x^\alpha \text{div} U^\tau, \partial_x^\alpha N^\tau] + \langle \mathcal{H}^{(2)}, \partial_x^\alpha N^\tau \rangle \rangle
\]
\[
\leq \eta_1 \| \nabla \partial_x^\alpha U^\tau \|^2 + C\eta_1 \| \partial_x^\alpha N^\tau \|^2 + \| \mathcal{H}^{(2)} \|^2 \tag{3.7}
\]
for any \( \eta_1 > 0 \), where the commutator
\[ \mathcal{H}^{(2)} = \partial_x^\alpha ((N^\tau + \rho^0)\text{div} U^\tau) - (N^\tau + \rho^0)\partial_x^\alpha \text{div} U^\tau \]
can be estimated by
\[
\| \mathcal{H}^{(2)} \| \leq C(\| D_x^1 (N^\tau + \rho^0) \|_{L^\infty} \| D_x^\alpha U^\tau \| + \| D_x^1 U^\tau \|_{L^\infty} \| D_x^{\alpha-1} (N^\tau + \rho^0) \|)
\]
\[
\leq C\| \mathcal{E}(t) \|_{s}^2 + C \epsilon^2. \tag{3.8}
\]

By the Moser-type and Cauchy-Schwarz’s inequalities, and the regularity of \( u^0 \) and \( \rho^0 \), we can control the third term on the right-hand side of (3.4) by
\[
\left| \left\langle \partial_x^\alpha \left( -N^\tau \text{div} u^0 - \nabla \rho^0 \cdot U^\tau \right), \partial_x^\alpha N^\tau \right\rangle \right| \leq C(\| \partial_x^\alpha N^\tau \|^2 + \| \partial_x^\alpha U^\tau \|^2). \tag{3.9}
\]
Substituting (3.5)–(3.9) into (3.4), we conclude that
\[
\frac{1}{2} \frac{d}{dt} \left\langle \partial_x^\alpha N^\tau, \partial_x^\alpha N^\tau \right\rangle \leq \eta_1 \| \nabla \partial_x^\alpha U^\tau \|^2 + C\eta_1 \| \partial_x^\alpha N^\tau \|^2
\]
\[
+ C\left[ (\| \mathcal{E}(t) \|_{\tau} + 1) \| \partial_x^\alpha N^\tau \|^2 + \| \mathcal{E}(t) \|_{s}^2 + \epsilon^4 \right]. \tag{3.10}
\]
Applying the operator \( \partial_x^\alpha \) to (2.2), multiplying the resulting equation by \( \partial_x^\alpha U^\tau \), and integrating over \( T^3 \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\langle \partial_x^\alpha U^\tau, \partial_x^\alpha U^\tau \right\rangle + \left\langle \partial_x^\alpha \left( [(U^\tau + u^0) \cdot \nabla]U^\tau \right), \partial_x^\alpha U^\tau \right\rangle
\]
\[
+ \left\langle \partial_x^\alpha \nabla \Theta^\tau, \partial_x^\alpha U^\tau \right\rangle + \left\langle \partial_x^\alpha \left( \frac{1}{N^\tau + \rho^0} \nabla N^\tau \right), \partial_x^\alpha U^\tau \right\rangle
\]
\[ - \left\langle \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div } \Psi(U^\epsilon) \right), \partial_x^\alpha U^\epsilon \right\rangle \\
= - \left\langle \partial_x^\alpha \left( (U^\epsilon \cdot \nabla) u^0 \right), \partial_x^\alpha U^\epsilon \right\rangle - \left\langle \partial_x^\alpha \left\{ \frac{1}{\rho^0} \text{curl } H^0 \times H^0 \right\}, \partial_x^\alpha U^\epsilon \right\rangle \\
+ \left\langle \partial_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} - \frac{1}{\rho^0} \right\} \nabla \rho^0 \right\}, \partial_x^\alpha U^\epsilon \right\rangle \\
+ \left\langle \partial_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} - \frac{1}{\rho^0} \right\} \text{div } \Psi(u^0), \partial_x^\alpha U^\epsilon \right\rangle \\
+ \left\langle \partial_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} [F^\epsilon + u^0 \times G^\epsilon + U^\epsilon \times H^0] \times H^0 \right\}, \partial_x^\alpha U^\epsilon \right\rangle \\
+ \left\langle \partial_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} [F^\epsilon + u^0 \times G^\epsilon + U^\epsilon \times H^0] \times G^\epsilon \right\}, \partial_x^\alpha U^\epsilon \right\rangle \\
+ \left\langle \partial_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} (U^\epsilon \times G^\epsilon) \times (G^\epsilon + H^0) \right\}, \partial_x^\alpha U^\epsilon \right\rangle \\
:= \sum_{i=1}^{7} R_i^{(1)}. \tag{3.11} \]

We first bound the terms on the left-hand side of (3.11). Similar to (3.5) we infer that

\[
\left\langle \partial_x^\alpha \left( [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon \right), \partial_x^\alpha U^\epsilon \right\rangle \\
= \left\langle [(U^\epsilon + u^0) \cdot \nabla] \partial_x^\alpha U^\epsilon, \partial_x^\alpha U^\epsilon \right\rangle + \left\langle \mathcal{H}^{(3)}, \partial_x^\alpha U^\epsilon \right\rangle \\
= -\frac{1}{2} \left\langle \text{div } (U^\epsilon + u^0) \partial_x^\alpha U^\epsilon, \partial_x^\alpha U^\epsilon \right\rangle + \left\langle \mathcal{H}^{(3)}, \partial_x^\alpha U^\epsilon \right\rangle \\
\leq C(\|E^\epsilon(t)\|_s + 1)\|\partial_x^\alpha U^\epsilon\|^2 + \|\mathcal{H}^{(3)}\|^2, \tag{3.12} \]

where the commutator

\[
\mathcal{H}^{(3)} = \partial_x^\alpha \left( [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon \right) - [(U^\epsilon + u^0) \cdot \nabla] \partial_x^\alpha U^\epsilon
\]

can be bounded by

\[
\|\mathcal{H}^{(3)}\| \leq C(\|D_x^1(U^\epsilon + u^0)\|_{L^\infty} \|D_x^2 U^\epsilon\| + \|D_x^1 U^\epsilon\|_{L^\infty} \|D_x^{s-1}(U^\epsilon + u^0)\|) \\
\leq C\|E^\epsilon(t)\|^2 + C\epsilon^2. \tag{3.13} \]

By Holder's inequality, we have

\[
\left\langle \partial_x^\alpha \nabla \Theta^\epsilon, \partial_x^\alpha U^\epsilon \right\rangle \leq \eta_2 \|\partial_x^\alpha \nabla \Theta^\epsilon\|^2 + C_{\eta_2} \|\partial_x^\alpha U^\epsilon\|^2 \tag{3.14} \]
for any \( \eta_2 > 0 \). For the fourth term on the left-hand side of (3.11), similar to (3.7), we integrate by parts to deduce that
\[
\left\langle \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \nabla N^\epsilon \right), \partial_x^\alpha U^\epsilon \rightangle
\]
\[
= \left\langle \frac{1}{N^\epsilon + \rho^0} \partial_x^\alpha \nabla N^\epsilon, \partial_x^\alpha U^\epsilon \rightangle + \left\langle \mathcal{H}^{(4)}, \partial_x^\alpha U^\epsilon \rightangle
\]
\[
= - \left\langle \partial_x^\alpha N^\epsilon, \text{div} \left( \frac{1}{N^\epsilon + \rho^0} \partial_x^\alpha U^\epsilon \right) \right\rangle + \left\langle \mathcal{H}^{(4)}, \partial_x^\alpha U^\epsilon \rightangle
\]
\[
\leq \eta_3 \| \nabla \partial_x^\alpha U^\epsilon \|_2^2 + C_{\eta_3} \| \partial_x^\alpha N^\epsilon \|_2^2 + C \| \mathcal{E}^\epsilon(t) \|_s^4 + \| \mathcal{H}^{(4)} \|_s^2
\]
(3.15)
for any \( \eta_3 > 0 \), where the commutator
\[
\mathcal{H}^{(4)} = \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \nabla N^\epsilon \right) - \frac{1}{N^\epsilon + \rho^0} \partial_x^\alpha \nabla N^\epsilon
\]
can be bounded as follows, using (1.38) and (1.39).
\[
\| \mathcal{H}^{(4)} \| \leq C \left( \| D_x^1 \left( \frac{1}{N^\epsilon + \rho^0} \right) \|_{L^\infty} \| D_x^s \nabla N^\epsilon \| + \| D_x^1 N^\epsilon \|_{L^\infty} \| D_x^{s-1} \left( \frac{1}{N^\epsilon + \rho^0} \right) \|_s \right)
\]
\[
\leq C \| \mathcal{E}^\epsilon(t) \|_s^2 + C \| \mathcal{E}^\epsilon(t) \|_s^4 + C \epsilon^2.
\]
(3.16)

For the fifth term on the left-hand side of (3.11), we integrate by parts to deduce
\[
- \left\langle \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(U^\epsilon) \right), \partial_x^\alpha U^\epsilon \rightangle
\]
\[
= - \left\langle \frac{1}{N^\epsilon + \rho^0} \partial_x^\alpha \text{div} \Psi(U^\epsilon), \partial_x^\alpha U^\epsilon \rightangle - \left\langle \mathcal{H}^{(5)}, \partial_x^\alpha U^\epsilon \rightangle
\]
(3.17)
where the commutator
\[
\mathcal{H}^{(5)} = \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(U^\epsilon) \right) - \frac{1}{N^\epsilon + \rho^0} \partial_x^\alpha \text{div} \Psi(U^\epsilon).
\]

By the Moser-type and Cauchy-Schwarz inequalities, the regularity of \( \rho^0 \) and the positivity of \( N^\epsilon + \rho_0 \), the definition of \( \Psi(U^\epsilon) \) and Sobolev’s imbedding, we find that
\[
\| \left\langle \mathcal{H}^{(5)}, \partial_x^\alpha U^\epsilon \right\rangle \| \leq \| \mathcal{H}^{(5)} \| \cdot \| \partial_x^\alpha U^\epsilon \|
\]
\[
\leq C \left( \| D_x^1 \left( \frac{1}{N^\epsilon + \rho^0} \right) \|_{L^\infty} \| \text{div} \Psi(U^\epsilon) \|_{s-1} + \| \text{div} \Psi(U^\epsilon) \|_{L^\infty} \| \frac{1}{N^\epsilon + \rho^0} \|_s \right) \| \partial_x^\alpha U^\epsilon \|
\]
\[
\leq \eta_4 \| \nabla U^\epsilon \|_s^2 + C_{\eta_4} (\| \mathcal{E}^\epsilon(t) \|_s^4 + 1) (\| \partial_x^\alpha U^\epsilon \|_s^2 + \| \partial_x^\alpha N^\epsilon \|_s^2 + \| \mathcal{E}^\epsilon(t) \|_s^4)
\]
(3.18)
for any \( \eta_4 > 0 \), where we have used the assumption \( s > 3/2 + 2 \) and the imbedding \( H^l(T^3) \to L^\infty(\mathbb{R}^3) \) for \( l > 3/2 \). By virtue of the definition of \( \Psi(U^\epsilon) \) and partial
integrations, the first term on the right-hand side of (3.17) can be rewritten as

$$-\left\langle \frac{1}{N^\varepsilon + \rho^0} \partial_x^a \text{div} \Psi (\mathbf{u}^\varepsilon), \partial_x^a \mathbf{u}^\varepsilon \right\rangle = 2\mu \left\langle \frac{1}{N^\varepsilon + \rho^0} \partial_x^a \mathbf{D} (\mathbf{u}^\varepsilon), \partial_x^a \mathbf{D} (\mathbf{u}^\varepsilon) \right\rangle + \lambda \left\langle \frac{1}{N^\varepsilon + \rho^0} \partial_x^a \text{div} \mathbf{u}^\varepsilon, \partial_x^a \text{div} \mathbf{u}^\varepsilon \right\rangle + 2\mu \left\langle \nabla \left( \frac{1}{N^\varepsilon + \rho^0} \right) \otimes \partial_x^a \mathbf{u}^\varepsilon, \partial_x^a \mathbf{D} (\mathbf{u}^\varepsilon) \right\rangle + \lambda \left\langle \nabla \left( \frac{1}{N^\varepsilon + \rho^0} \right) \cdot \partial_x^a \mathbf{u}^\varepsilon, \partial_x^a \text{div} \mathbf{u}^\varepsilon \right\rangle$$

$$:= \sum_{i=1}^4 \mathcal{I}^{(i)}. \quad (3.19)$$

Recalling the facts that $\mu > 0$ and $2\mu + 3\lambda > 0$, and the positivity of $N^\varepsilon + \rho_0$, the first two terms $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ can be bounded as follows.

$$\mathcal{I}^{(1)} + \mathcal{I}^{(2)} = \int \frac{1}{N^\varepsilon + \rho^0} \{2\mu |\partial_x^a \mathbf{D} (\mathbf{u}^\varepsilon)|^2 + \lambda |\partial_x^a \text{tr} \mathbf{D} (\mathbf{u}^\varepsilon)|^2 \} dx \geq 2\mu \int \frac{1}{N^\varepsilon + \rho^0} \left( |\partial_x^a \mathbf{D} (\mathbf{u}^\varepsilon)|^2 - \frac{1}{3} |\partial_x^a \text{tr} \mathbf{D} (\mathbf{u}^\varepsilon)|^2 \right) dx = \mu \int \frac{1}{N^\varepsilon + \rho^0} \left( |\partial_x^a \nabla \mathbf{u}^\varepsilon|^2 + \frac{1}{3} |\partial_x^a \text{div} \mathbf{u}^\varepsilon|^2 \right) dx \geq \mu \int \frac{1}{N^\varepsilon + \rho^0} |\partial_x^a \nabla \mathbf{u}^\varepsilon|^2 dx. \quad (3.20)$$

By virtue of Cauchy-Schwarz’s inequality, the regularity of $\rho^0$ and the positivity of $N^\varepsilon + \rho_0$, the terms $\mathcal{I}^{(3)}$ and $\mathcal{I}^{(4)}$ can be bounded by

$$|\mathcal{I}^{(3)}| + |\mathcal{I}^{(4)}| \leq \eta_5 \| \nabla \partial_x^a \mathbf{u}^\varepsilon \|^2 + C_{\eta_5} (\| \mathbf{E}^\varepsilon (t) \|^2 + 1) (\| \partial_x^a \mathbf{u}^\varepsilon \|^2 + \| \partial_x^a N^\varepsilon \|^2) \quad (3.21)$$

for any $\eta_5 > 0$, where the assumption $s > 3/2 + 2$ has been used.

Substituting (3.12)–(3.21) into (3.11), we conclude that

$$\frac{1}{2} \frac{d}{dt} \langle \partial_x^a \mathbf{u}^\varepsilon, \partial_x^a \mathbf{u}^\varepsilon \rangle + \int \frac{\mu}{N^\varepsilon + \rho^0} |\nabla \partial_x^a \mathbf{u}^\varepsilon|^2 dx - (\eta_1 + \eta_3 + \eta_4 + \eta_5) \| \nabla \partial_x^a \mathbf{u}^\varepsilon \|^2 \leq C_{\eta} \left( \langle \mathbf{E}^\varepsilon (t) \rangle^2 + 1 \right) (\| \partial_x^a \mathbf{u}^\varepsilon \|^2 + \| \partial_x^a N^\varepsilon \|^2 + \| \mathbf{E}^\varepsilon (t) \|^2) + \eta_2 \| \partial_x^a \nabla \mathbf{u}^\varepsilon \|^2 + \sum_{i=1}^7 \mathcal{R}^{(i)} \quad (3.22)$$

for some constant $C_{\eta} > 0$ depending on $\eta_i$ ($i = 1, \ldots, 5$).

We have to estimate the terms on the right-hand side of (3.22), and the estimate process is similar to that in [18], we present it here for completeness. In view of the
regularity of $(\rho^0, \mathbf{u}^0, \mathbf{H}^0)$, the positivity of $N^\varepsilon + \rho^0$ and Cauchy-Schwarz’s inequality, the first two terms $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ can be controlled by

$$\mathcal{R}^{(1)} + \mathcal{R}^{(2)} \leq C(\|\mathcal{E}^\varepsilon(t)\|_s^2 + 1)(\|\partial_x^\alpha N^\varepsilon\|^2 + \|\partial_x^\alpha \mathbf{U}^\varepsilon\|^2).$$  \hspace{1cm} (3.23)

For the terms $\mathcal{R}^{(3)}$ and $\mathcal{R}^{(4)}$, by the regularity of $\rho^0$ and $\mathbf{u}^0$, the positivity of $N^\varepsilon + \rho^0$, Cauchy-Schwarz’s inequality and (1.40), we see that

$$\mathcal{R}^{(3)} + \mathcal{R}^{(4)} \leq C(\|\mathcal{E}^\varepsilon(t)\|_s^2 + \epsilon^2) + C\|\partial_x^\alpha \mathbf{U}^\varepsilon\|^2.$$ \hspace{1cm} (3.24)

For the fifth term $\mathcal{R}^{(5)}$, we utilize the positivity of $N^\varepsilon + \rho^0$ to deduce that

$$\mathcal{R}^{(5)} = \left\langle \partial_x^\alpha \mathbf{F}^\varepsilon \times \frac{\mathbf{H}^0}{N^\varepsilon + \rho^0}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle + \left\langle \mathcal{H}^{(6)}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle + \sigma \mathcal{R}^{(51)}$$

$$\leq \frac{1}{16} \|\partial_x^\alpha \mathbf{F}^\varepsilon\|^2 + C\|\partial_x^\alpha \mathbf{U}^\varepsilon\|^2 + \left\langle \mathcal{H}^{(6)}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle + \mathcal{R}^{(51)},$$ \hspace{1cm} (3.25)

where

$$\mathcal{H}^{(6)} = \partial_x^\alpha \left\{ \frac{\mathbf{F}^\varepsilon}{N^\varepsilon + \rho^0} \times \mathbf{H}^0 \right\} - \partial_x^\alpha \mathbf{F}^\varepsilon \times \frac{\mathbf{H}^0}{N^\varepsilon + \rho^0}$$

and

$$\mathcal{R}^{(51)} = \left\langle \partial_x^\alpha \left\{ \frac{\sigma}{N^\varepsilon + \rho^0}[\mathbf{u}^0 \times \mathbf{G}^\varepsilon + \mathbf{U}^\varepsilon \times \mathbf{H}^0] \times \mathbf{H}^0 \right\}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle.$$

If we make use of the Moser-type inequality, (1.39) and the regularity of $\rho^0$ and $\mathbf{H}^0$, we obtain that

$$\left| \left\langle \mathcal{H}^{(6)}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle \right| \leq \|\mathcal{H}^{(6)}\| \cdot \|\partial_x^\alpha \mathbf{U}^\varepsilon\|$$

$$\leq C\left[ \left\| D_{x}^\frac{1}{2} \left( \frac{\mathbf{H}^0}{N^\varepsilon + \rho^0} \right) \right\|_{L^\infty} \|\mathbf{F}^\varepsilon\|_{s-1} + \left\| \mathbf{F}^\varepsilon \right\|_{L^\infty} \left\| \frac{\mathbf{H}^0}{N^\varepsilon + \rho^0} \right\|_s \|\partial_x^\alpha \mathbf{U}^\varepsilon\| \right]$$

$$\leq \eta_6 \|\mathbf{F}^\varepsilon\|_{s-1}^2 + C\eta_6(\|\mathcal{E}^\varepsilon(t)\|_s^{2(s+1)} + 1)\|\partial_x^\alpha \mathbf{U}^\varepsilon\|^2$$ \hspace{1cm} (3.26)

for any $\eta_6 > 0$. Recalling the regularity of $\mathbf{u}^0$ and $\mathbf{H}^0$, (1.37) and (1.39) and Hölder’s inequality, we find that

$$|\mathcal{R}^{(51)}| \leq C(\|\mathcal{E}^\varepsilon(t)\|_s^2 + 1)(\|\partial_x^\alpha N^\varepsilon\|^2 + \|\partial_x^\alpha \mathbf{U}^\varepsilon\|^2 + \|\partial_x^\alpha \mathbf{G}^\varepsilon\|^2).$$ \hspace{1cm} (3.27)

For the sixth term $\mathcal{R}^{(6)}$ we again make use of the positivity of $N^\varepsilon + \rho^0$ and Sobolev’s inbedding to infer that

$$\mathcal{R}^{(6)} = \left\langle \partial_x^\alpha \mathbf{F}^\varepsilon \times \frac{\mathbf{G}^\varepsilon}{N^\varepsilon + \rho^0}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle + \left\langle \mathcal{H}^{(7)}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle + \mathcal{R}^{(61)}$$

$$\leq \frac{1}{16} \|\partial_x^\alpha \mathbf{F}^\varepsilon\|^2 + C(\|\mathcal{E}^\varepsilon(t)\|_s^2 \|\partial_x^\alpha \mathbf{U}^\varepsilon\|^2 + \left\langle \mathcal{H}^{(7)}, \partial_x^\alpha \mathbf{U}^\varepsilon \right\rangle + \mathcal{R}^{(61)},$$ \hspace{1cm} (3.28)
where
\[
\mathcal{H}^{(7)} = \partial_x^\alpha \left\{ \frac{F^c}{\sigma + \rho} \times G^c \right\} - \partial_x^\alpha F^c \times \frac{G^c}{\sigma + \rho}
\]
and
\[
\mathcal{R}^{(6)}_1 = \left\langle \partial_x^\alpha \left\{ \frac{\sigma}{\sigma + \rho} [u^0 \times G^c + U^c \times H^0] \times G^c \right\}, \partial_x^\alpha U^c \right\rangle.
\]
From the Hőlder and Moser-type inequalities we get
\[
\left| \left\langle \mathcal{H}^{(7)}, \partial_x^\alpha U^c \right\rangle \right| \leq \left\| \mathcal{H}^{(7)} \right\| \cdot \left\| \partial_x^\alpha U^c \right\|
\leq C \left[ D^1 \left( \frac{G^c}{\sigma + \rho} \right) \right]_{L^\infty} \left\| F^c \right\|_{L^1} + \left\| F^c \right\|_{L^\infty} \left\| \frac{G^c}{\sigma + \rho} \right\|_{L^1} \left\| \partial_x^\alpha U^c \right\|
\leq \eta \left\| F^c \right\|^2_{L^1} + C_\eta \left( \left\| \mathcal{E}^c(t) \right\|^2 + 1 \right) \left\| \partial_x^\alpha U^c \right\|^2
\] 
(3.29)
for any \( \eta > 0 \), while for the term \( \mathcal{R}^{(6)}_1 \) one has the following estimate
\[
\left| \mathcal{R}^{(6)}_1 \right| \leq C \left( \left\| \mathcal{E}^c(t) \right\|^2_{L^\infty} + \left\| \mathcal{E}^c(t) \right\|_{L^1} \right) \left\| \mathcal{E}^c(t) \right\|^2_{L^3}.
\] 
(3.30)
For the last term \( \mathcal{R}^{(7)} \), recalling the formula \( (a \times b) \times c = (a \cdot c)b - (b \cdot c)a \) and applying (1.37), (1.39), and Hőlder’s inequality, we easily deduce that
\[
\left| \mathcal{R}^{(7)} \right| = \left| \left\langle \partial_x^\alpha \left\{ \frac{1}{\sigma + \rho} \left[ [U^c \cdot (G^c + H^0)]G^c - [G^c \cdot (G^c + H^0)]U^c \right] \right\}, \partial_x^\alpha U^c \right\rangle \right|
\leq C \left( \left\| \mathcal{E}^c(t) \right\|^2_{L^\infty} + \left\| \mathcal{E}^c(t) \right\|^3_{L^3} \right) \left\| \mathcal{E}^c(t) \right\|^2_{L^3}.
\] 
(3.31)
Substituting (3.23)–(3.31) into (3.22), we conclude that
\[
\frac{1}{2} \frac{d}{dt} \left\| \partial_x^\alpha U^c \right\|^2 + \int \frac{\mu}{\sigma + \rho} \left| \nabla \partial_x^\alpha U^c \right|^2 dx - (\eta_1 + \eta_3 + \eta_4 + \eta_5) \left\| \nabla \partial_x^\alpha U^c \right\|^2
\leq \tilde{C}_\eta \left( \left\| \mathcal{E}^c(t) \right\|^2_{L^\infty} + \left\| \mathcal{E}^c(t) \right\|^3_{L^3} \right) \left\| \mathcal{E}^c(t) \right\|^2_{L^3}
\]
\[
+ \eta_2 \left\| \partial_x^\alpha \Delta \Theta \right\|^2 + \left( \eta_6 + \eta_7 + \frac{1}{8} \right) \left\| F^c \right\|^2_{L^2} + C\epsilon^4.
\] 
(3.32)
for some constant \( \tilde{C}_\eta > 0 \) depending on \( \eta_i \ (i = 1, \ldots, 7) \).

Applying the operator \( \partial_x^\alpha \) to (2.3), multiplying the resulting equation by \( \partial_x^\alpha \Theta^c \), and integrating over \( T^3 \), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \left\| \partial_x^\alpha \Theta^c \right\|^2 + \left\langle \partial_x^\alpha \left\{ \left[ (U^c \cdot \nabla)^{\Theta^c} \right] \Theta^c + \left[ (U^c \cdot \Theta^c) \right] \Theta^c \right\}, \partial_x^\alpha \Theta^c \right\rangle
\]
\[
+ \left\langle \partial_x^\alpha \left\{ (\Theta^c + \Theta^3) \text{div} U^c \right\}, \partial_x^\alpha \Theta^c \right\rangle - \left\langle \partial_x^\alpha \left\{ \frac{\kappa}{\sigma + \rho} \Delta \Theta^c \right\}, \partial_x^\alpha \Theta^c \right\rangle
\]
\[
= - \left\langle \partial_x^\alpha \left\{ (U^c \cdot \nabla) \Theta^3 - \Theta^c \text{div} U^c \right\}, \partial_x^\alpha \Theta^c \right\rangle
\]
We first bound the terms on the left-hand side of (3.33). Similar to (3.5), we have

\[
\langle \hat{\partial}_x^\alpha \left\{ \left[ \frac{\kappa}{N^\epsilon + \rho^0} - \frac{\kappa}{\rho^0} \right] \Delta \theta^0 \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \left[ \frac{2\mu}{N^\epsilon + \rho^0} - \frac{2\mu}{\rho^0} \right] |\mathbb{D}(u^0)|^2 \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \left[ \frac{\lambda}{N^\epsilon + \rho^0} - \frac{\lambda}{\rho^0} \right] (\text{tr} \mathbb{D}(u^0))^2 \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \left[ \frac{1}{N^\epsilon + \rho^0} \right] |\text{curl} \mathbf{H}^0|^2 \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{2\mu}{N^\epsilon + \rho^0} |\mathbb{D}(u^\epsilon)|^2 + \frac{\lambda}{(N^\epsilon + \rho^0) |\text{tr} \mathbb{D}(u^\epsilon)|^2} \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{4\mu}{N^\epsilon + \rho^0} \mathbb{D}(u^\epsilon) : \mathbb{D}(u^\epsilon) \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{2\lambda}{N^\epsilon + \rho^0} |\text{tr} \mathbb{D}(u^\epsilon) \text{tr} \mathbb{D}(u^0)| \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} |\mathbf{F}^\epsilon + \mathbf{U}^\epsilon \times \mathbf{G}^\epsilon|^2 \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{1}{N^\epsilon + \rho^0} |\mathbf{u}^0 \times \mathbf{G}^\epsilon + \mathbf{U}^\epsilon \times \mathbf{H}^0|^2 \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{2\mathbf{F}^\epsilon}{N^\epsilon + \rho^0} \cdot [\text{curl} \mathbf{H}^0 + \mathbf{u}^0 \times \mathbf{G}^\epsilon + \mathbf{U}^\epsilon \times \mathbf{H}^0] \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{2(\mathbf{U}^\epsilon \times \mathbf{G}^\epsilon)}{N^\epsilon + \rho^0} \cdot [\text{curl} \mathbf{H}^0 + \mathbf{u}^0 \times \mathbf{G}^\epsilon + \mathbf{U}^\epsilon \times \mathbf{H}^0] \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
+ \langle \hat{\partial}_x^\alpha \left\{ \frac{2}{N^\epsilon + \rho^0} \text{curl} \mathbf{H}^0 \cdot (\mathbf{u}^0 \times \mathbf{G}^\epsilon + \mathbf{U}^\epsilon \times \mathbf{H}^0) \right\}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
:= \sum_{i=1}^{13} S^{(i)}.
\]

(3.33)

We first bound the terms on the left-hand side of (3.33). Similar to (3.5), we have

\[
\langle \hat{\partial}_x^\alpha \left( [\mathbf{U}^\epsilon + \mathbf{u}^0] \cdot \nabla \right) \Theta^\epsilon, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
= \langle [(\mathbf{U}^\epsilon + \mathbf{u}^0) \cdot \nabla] \hat{\partial}_x^\alpha \Theta^\epsilon, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle + \langle \mathcal{H}^{(8)}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
= -\frac{1}{2} \langle \text{div} (\mathbf{U}^\epsilon + \mathbf{u}^0) \hat{\partial}_x^\alpha \Theta^\epsilon, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle + \langle \mathcal{H}^{(8)}, \hat{\partial}_x^\alpha \Theta^\epsilon \rangle \\
\leq C (\|\mathcal{E}^\epsilon(t)\|_s + 1) \|\hat{\partial}_x^\alpha \Theta^\epsilon\|^2 + \|\mathcal{H}^{(8)}\|^2,
\]

(3.34)

where the commutator

\[
\mathcal{H}^{(8)} = \hat{\partial}_x^\alpha \left( [\mathbf{U}^\epsilon + \mathbf{u}^0] \cdot \nabla \right) \Theta^\epsilon - [(\mathbf{U}^\epsilon + \mathbf{u}^0) \cdot \nabla] \hat{\partial}_x^\alpha \Theta^\epsilon
\]
can be bounded by
\[
\|\mathcal{H}^{(8)}\| \leq C\left(\|D_x^1(U^\varepsilon + u^0)\|_{L^\infty} + \|D_x^2 U^\varepsilon\|_{L^\infty} \right) + \|D_x^1 U^\varepsilon\|_{L^\infty} \|D_x^{s,1}(U^\varepsilon + u^0)\|
\]
\[
\leq C\|\mathcal{E}^\varepsilon(t)\|_8^2 + C\varepsilon^2.
\] (3.35)

The second term on the left-hand side of (3.33) can bounded, similar to (3.7), as follows.
\[
\langle \partial_2^\alpha (\Theta^\varepsilon + \theta^0) \text{div} U^\varepsilon, \partial_2^\alpha \Theta^\varepsilon \rangle
\]
\[
= \langle (\Theta^\varepsilon + \rho^0) \partial_2^\alpha \text{div} U^\varepsilon, \partial_2^\alpha \Theta^\varepsilon \rangle + \langle \mathcal{H}^{(9)}, \partial_2^\alpha \Theta^\varepsilon \rangle
\]
\[
\leq \eta_8 \|\nabla \partial_2^\alpha U^\varepsilon\|^2 + C_{\eta_8} \|\partial_2^\alpha \mathcal{N}^\varepsilon\|^2 + \|\mathcal{H}^{(9)}\|^2
\] (3.36)
for any \(\eta_8 > 0\), where the commutator
\[
\mathcal{H}^{(9)} = \partial_2^\alpha (\Theta^\varepsilon + \rho^0) \text{div} U^\varepsilon - (\Theta^\varepsilon + \theta^0) \partial_2^\alpha \text{div} U^\varepsilon
\]

can be controlled as
\[
\|\mathcal{H}^{(9)}\| \leq C\left(\|D_x^1 (\Theta^\varepsilon + \theta^0)\|_{L^\infty} + \|D_x^2 U^\varepsilon\|_{L^\infty} \right) + \|D_x^1 U^\varepsilon\|_{L^\infty} \|D_x^{s,1}(\Theta^\varepsilon + \theta^0)\|
\]
\[
\leq C\|\mathcal{E}^\varepsilon(t)\|_8^2 + C\varepsilon^2.
\] (3.37)

For the fourth term on the left-hand side of (3.33), we integrate by parts to deduce that
\[
-k \langle \partial_2^\alpha \left( \frac{1}{N^\varepsilon + \rho^0} \Delta U^\varepsilon \right), \partial_2^\alpha \Theta^\varepsilon \rangle
\]
\[
= -k \langle \frac{1}{N^\varepsilon + \rho^0} \Delta \partial_2^\alpha \Theta^\varepsilon, \partial_2^\alpha \Theta^\varepsilon \rangle - k \langle \mathcal{H}^{(10)}, \partial_2^\alpha \Theta^\varepsilon \rangle
\]
\[
= k \langle \frac{1}{N^\varepsilon + \rho^0} \nabla \partial_2^\alpha \Theta^\varepsilon, \nabla \partial_2^\alpha \Theta^\varepsilon \rangle
\]
\[
+ k \langle \nabla \left( \frac{1}{N^\varepsilon + \rho^0} \right) \nabla \partial_2^\alpha \Theta^\varepsilon, \partial_2^\alpha \Theta^\varepsilon \rangle - k \langle \mathcal{H}^{(10)}, \partial_2^\alpha \Theta^\varepsilon \rangle,
\] (3.38)

where
\[
\mathcal{H}^{(10)} = \partial_2^\alpha \left( \frac{1}{N^\varepsilon + \rho^0} \Delta \Theta^\varepsilon \right) - \frac{1}{N^\varepsilon + \rho^0} \Delta \partial_2^\alpha \Theta^\varepsilon.
\]

By the Moser-type and Hölder’s inequalities, the regularity of \(\rho^0\), the positivity of \(N^\varepsilon + \rho_0\) and (1.39), we find that
\[
\langle \mathcal{H}^{(10)}, \partial_2^\alpha \Theta^\varepsilon \rangle \leq \|\mathcal{H}^{(10)}\| \cdot \|\partial_2^\alpha \Theta^\varepsilon\|
\]
while for the sixth term 
\( S^{(6)} \), we integrate by parts, and use Cauchy-Schwarz’s inequality and the positivity of \( \Theta^\varepsilon + \rho^0 \) to obtain that
\[
S^{(6)} = - \left\langle \partial_x^{\alpha-1} \left( \frac{2 \mu}{N^\varepsilon + \rho^0} |\mathbf{D}(U^\varepsilon)|^2 + \frac{\lambda}{N^\varepsilon + \rho^0} |\text{tr} \mathbf{D}(U^\varepsilon)|^2 \right) , \partial_x^{\alpha} \Theta^\varepsilon \right\rangle 
\leq \eta_{11} \| \nabla \partial_x^{\alpha} \Theta^\varepsilon \|^2 + C_{\eta_{11}} (\| \mathcal{E}^\varepsilon (t) \|_{s}^4 + \| \mathcal{E}^\varepsilon (t) \|_{s-1}^2) 
\tag{3.43}
\]
for any \( \eta_{11} > 0 \), where \( \alpha_1 = (1,0,0) \) or \( (0,1,0) \) or \( (0,0,1) \). Similarly, we have
\[
S^{(7)} + S^{(8)} \leq \eta_{12} \| \nabla \partial_x^{\alpha} \Theta^\varepsilon \|^2 + C_{\eta_{12}} (\| \mathcal{E}^\varepsilon (t) \|_{s}^4 + \| \mathcal{E}^\varepsilon (t) \|_{s-1}^2) 
\tag{3.44}
\]
for any \( \eta_{12} > 0 \).

For the ninth term \( S^{(9)} \), we rewrite it as
\[
S^{(9)} = \left\langle \partial_x^{\alpha} \left( \frac{1}{N^\varepsilon + \rho^0} |\mathbf{F}^\varepsilon + \mathbf{U}^\varepsilon \times \mathbf{G}^\varepsilon|^2 \right) , \partial_x^{\alpha} \Theta^\varepsilon \right\rangle 
= \left\langle \partial_x^{\alpha} \left( \frac{1}{N^\varepsilon + \rho^0} |\mathbf{F}^\varepsilon|^2 \right) , \partial_x^{\alpha} \Theta^\varepsilon \right\rangle 
\]
By the Cauchy-Schwarz and Moser-type inequalities, we obtain that

\[ S = S^{(9_1)} + S^{(9_2)} + S^{(9_3)}. \]

For any \( \gamma > 0 \), for the term \( S^{(9_1)} \), similar to \( R^{(6)} \), we have

\[
S^{(9_1)} = \frac{1}{N^\varepsilon + \rho^0} \langle \tilde{\Theta}^\alpha, \tilde{\Theta}^\alpha \rangle \\
+ \sum_{\beta < |\alpha|} (\tilde{\Theta}^{\alpha - \beta}, \tilde{\Theta}^{\alpha - \beta} \tilde{\Theta}^\alpha) \\
\leq \gamma_1 \| \tilde{\Theta}^\alpha \|_s^4 + C_{\gamma_1} \| \tilde{\Theta}^\alpha \|_s^2 (1 + \| \mathcal{E}(t) \|_s^{2(s+1)})
\]

(3.45)

for any \( \gamma_1 > 0 \). For the term \( S^{(9_2)} \), by the regularity of \( \theta^0, \varphi^0 \) and \( \mathbf{H}^0 \), the positivity of \( \Theta^\alpha + \rho^0 \), and Cauchy-Schwarz’s inequality, the first terms \( S^{(10)} \) and \( S^{(13)} \) can be bounded as follows

\[
S^{(10)} + S^{(13)} \leq C (\| \mathcal{E}(t) \|_s^2 + 1) (\| \tilde{\Theta}^\alpha \|_s^2 + \| \tilde{\Theta}^\alpha \|_s^2 + \| \tilde{\Theta}^\alpha \|_s^2 + \| \tilde{\Theta}^\alpha \|_s^2).
\]

(3.49)
In a manner similar to $S^{(11)}$, we can control the term $S^{(12)}$ by

$$
S^{(11)} \leq \gamma_3 ||\mathbf{F}^\alpha||^2_{s-1} + C_{\gamma_3} (||\mathbf{E}'(t)||^{2s} + 1)(||\mathbf{\theta}_x^\alpha \mathbf{G}'||^2 + ||\mathbf{\theta}_x^\alpha \mathbf{U}'||^2 + ||\mathbf{\theta}_x^\alpha \mathbf{G}^\alpha||^2).
$$

(3.50)

for any $\gamma_3 > 0$. Finally, similarly to $S^{(9s)}$, the term $S^{(12)}$ can be bounded by

$$
S^{(12)} \leq C ||\mathbf{\theta}_x^\alpha \mathbf{G}'||^2 (1 + ||\mathbf{E}(t)||^{2(s+1)}).
$$

(3.51)

Substituting (3.34)–(3.51) into (3.33), we conclude that

$$
\frac{1}{2} \frac{d}{dt} \left( \mathbf{\theta}_x^\alpha \mathbf{F}' \right) + \kappa \left( \frac{1}{N^\epsilon + \rho^0} \nabla \mathbf{\theta}_x^\alpha \mathbf{G}' - \nabla \mathbf{\theta}_x^\alpha \mathbf{F}' \right) \\
- (\eta \eta_1 + \eta_1 \eta_1 + \eta_2) ||\nabla \mathbf{\theta}_x^\alpha \mathbf{G}'||^2 \\
\leq C_{\eta_1, \gamma} \left( ||\mathbf{E}'(t)||^{2s+1} + ||\mathbf{E}'(t)||^2 + ||\mathbf{E}'(t)|| + 1 \right) ||\mathbf{E}'(t)||^2 \\
+ \gamma_1 ||\mathbf{F}'||^4 + \left( \gamma_2 + \gamma_3 + \frac{1}{16} \right) ||\mathbf{F}'||^2 + C \epsilon^4
$$

(3.52)

for some constant $C_{\eta, \gamma} > 0$ depending on $\eta_i$ $(i = 9, 10, 11, 12)$ and $\gamma_j$ $(j = 1, 2, 3)$. Applying the operator $\mathbf{\theta}_x^\alpha$ to (2.4) and (2.5), multiplying the results by $\mathbf{\theta}_x^\alpha \mathbf{F}'$ and $\mathbf{\theta}_x^\alpha \mathbf{G}'$ respectively, and integrating then over $T^3$, one obtains that

$$
\frac{1}{2} \frac{d}{dt} (\sqrt{\epsilon} \mathbf{\theta}_x^\alpha \mathbf{F}'^2) + ||\mathbf{\theta}_x^\alpha \mathbf{G}'||^2 \\
+ \int (\text{curl} \mathbf{\theta}_x^\alpha \mathbf{F}' \cdot \mathbf{\theta}_x^\alpha \mathbf{G}' - \text{curl} \mathbf{\theta}_x^\alpha \mathbf{G}' \cdot \mathbf{\theta}_x^\alpha \mathbf{F}') dx \\
= \langle \mathbf{\theta}_x^\alpha \mathbf{U}' \times \mathbf{H}^0 + \mathbf{\theta}_x^\alpha \mathbf{u}^0 \times \mathbf{G}' \rangle - \mathbf{\theta}_x^\alpha \mathbf{U}' \times \mathbf{G}' \\
- \langle \epsilon \mathbf{\theta}_x^\alpha \mathbf{\theta}_x \text{curl} \mathbf{H}^0 + \epsilon \mathbf{\theta}_x^\alpha \mathbf{\theta}_x (\mathbf{u}^0 \times \mathbf{H}^0), \mathbf{\theta}_x^\alpha \mathbf{F}' \rangle.
$$

(3.53)

Following a process similar to that in [18] and applying (3.53), we finally obtain that

$$
\frac{1}{2} \frac{d}{dt} (\sqrt{\epsilon} \mathbf{\theta}_x^\alpha \mathbf{F}'^2) + ||\mathbf{\theta}_x^\alpha \mathbf{G}'||^2 + \frac{3}{4} ||\mathbf{\theta}_x^\alpha \mathbf{F}'||^2 \\
\leq C (||\mathbf{E}'(t)||^{2s+1} + 1) ||\mathbf{\theta}_x^\alpha \mathbf{U}', \mathbf{\theta}_x^\alpha \mathbf{G}'||^2 + C \epsilon^2.
$$

(3.54)

Combining (3.10), (3.32), and (3.52) with (3.54), summing up $\alpha$ with $0 \leq |\alpha| \leq s$, using the fact that $N^\epsilon + \rho^0 \geq N + \dot{\rho} > 0$, $\mathbf{F}' \in C([0, T], H^{s-2})$ $(l = 0, 1)$, and choosing $\eta_i$ $(i = 1, \ldots, 12)$ and $\gamma_1, \gamma_2, \gamma_3$ sufficiently small, we obtain (3.3). This completes the proof of Lemma 3.2.

With the estimate (3.3) in hand, we can now prove Proposition 3.1.
Proof of Proposition 3.1. As in [18, 31], we introduce an \( \epsilon \)-weighted energy functional

\[
\Gamma^\epsilon(t) = \|E^\epsilon(t)\|_s^2.
\]

Then, it follows from (3.3) that there exists a constant \( \epsilon > 0 \) depending only on \( T \), such that for any \( \epsilon \in (0, \epsilon] \) and any \( t \in (0, T] \),

\[
\Gamma^\epsilon(t) \leq C\Gamma^\epsilon(t = 0) + C \int_0^t \left\{ \left( (\Gamma^\epsilon)^s + \Gamma^\epsilon + 1 \right) \Gamma^\epsilon \right\}(s)ds + C\epsilon^2.
\]

(3.55)

Thus, applying the Gronwall lemma to (3.55), and keeping in mind that \( \Gamma^\epsilon(t = 0) \leq C\epsilon^2 \) and Proposition 3.1, we find that there exist a \( 0 < T_1 < 1 \) and an \( \epsilon > 0 \), such that \( T^\epsilon \geq T_1 \) for all \( \epsilon \in (0, \epsilon] \) and \( \Gamma^\epsilon(t) \leq C\epsilon^2 \) for all \( t \in (0, T_1] \). Therefore, the desired a priori estimate (3.2) holds. Moreover, by the standard continuation induction argument, we can extend \( T^\epsilon \geq T_0 \) to any \( T_0 < T_\ast \). \( \square \)

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