ON THE BOUNDARY AND INTERSECTION MOTIVES OF GENUS 2
HILBERT-SIEGEL VARIETIES

MATTIA CAVICCHI

Abstract. We study genus 2 Hilbert-Siegel varieties, i.e. Shimura varieties $S_K$ corresponding to the group $\GSp_4$, along with the relative Chow motives $\lambda V$ of abelian type over $S_K$ obtained from irreducible representations $V_\lambda$ of $\GSp_4$. We analyse the weight filtration on the degeneration of such motives at the boundary of the Baily-Borel compactification and we find a criterion on the highest weight $\lambda$ which characterises the absence of the middle weights 0 and 1 in the corresponding degeneration. Thanks to Wildeshaus’ theory, the absence of these weights allows us to construct Hecke-equivariant Chow motives over $\Q$, whose realizations equal interior (or intersection) cohomology of $S_K$ with $V_\lambda$-coefficients. We give applications to the construction of motives associated to automorphic representations.

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Introduction

Background: motives for automorphic representations. Let $S_K$ be a Shimura variety associated to a reductive $\Q$-group $G$ and a neat open compact subgroup $K$ of $G(\A_f)$. The variety $S_K$ is then smooth and quasi-projective, and defined over its reflex field $E$ (a number field). Every algebraic representation $V$ of $G$ defines a local system $\mu(V)$ on $S_K(\C)$, whose interior cohomology $H^*_c$, i.e. the image of cohomology with compact supports into ordinary cohomology, contains very rich analytical and arithmetic information: in particular, the $K$-invariants of cohomological cuspidal automorphic representations of $G(\A)$ appear exactly inside the spaces of the form $H^*_c(S_K(\C), \mu(V))$. On the other hand, for all prime $\ell$, $V$ also defines an $\ell$-adic sheaf $\mu_\ell(V)$ on $S_K$, whose étale cohomology spaces are equipped with a Galois action and the action of Hecke operators. The study of the interaction of these different structures plays a pivotal role in the Langlands program.

Hence, following the general philosophy explained for example in [Cz], it is desirable to construct a Chow motive whose realizations equal interior cohomology - a subspace of the cohomology which is of pure weight, in the Galois or Hodge-theoretic sense. Moreover, one would like this construction to be functorial, in order to further decompose such a motive according to the Hecke action (maybe switching to homological motives).
The first successful example of such a construction was given by Scholl ([Sc]), who defined motives realizing to the Galois representations associated by Deligne to weight \( k \geq 2 \) modular cusp forms ([De1]).

The results of this paper imply that analogous motives exist for most coefficient systems in the case of genus 2 Hilbert-Siegel varieties, which are Shimura varieties \( S_K \) associated to (a subgroup of) the group \( G = \text{Res}_{F|Q} \text{GSp}_{4,F} \) over a totally real field \( F \) of degree \( d \) over \( Q \). More precisely, let \( V_\lambda \) be an irreducible representation of \( G \) of highest weight \( \lambda \); such a weight is in particular specified by a couple of vectors of non-negative integers \((k_i), (k'_i)\) such that \( k_i \geq k'_i \geq 0 \) for all \( i \) (i.e., the weight is dominant). It is called regular at \( i \) if \( k_i > k'_i > 0 \), and regular if it is regular at \( i \) for all \( i \). Denote moreover by \( \text{CHM}(Q) \) the category of Chow motives over \( Q \). Then, the main consequence of our results is the following:

**Theorem 1.** (Corollaries 3.1.0.2 and 3.1.0.5.(2) and Rmk. 3.1.0.6.(2)) If \( \lambda \) is regular, there exists an object \( s_\lambda^* \mathcal{V} \) of \( \text{CHM}(Q) \) whose Hodge-theoretical, resp. \( \ell \)-adic realization, equals \( H^{3d}(S_K(C), \mu_i(V_\lambda)) \), resp. \( H^{3d}(S_K \times_Q Q, \mu_\ell(V_\lambda)) \), and such that every element of the Hecke algebra \( \mathcal{S}(K, G(\mathbb{A}_f)) \) acts naturally on it.

Actually, we show that a functorial Chow motive realizing to interior cohomology exists under a less restrictive hypothesis on the weight \( \lambda \). To see how this is achieved, let us sketch the actual contents of this work, and explain the interest of the particular family of Shimura varieties which we consider.

**The role of the boundary motive.** Let us first come back to the general case of a Shimura variety \( S_K \) over \( E \), and denote by \( j: S_K \rightarrow S_K^0 \) the open immersion into the Baily-Borel compactification (a projective variety, still defined over \( E \)) and by \( i: \partial S_K^0 \hookrightarrow S_K^0 \) the closed immersion of the boundary \( \partial S_K^0 := S_K^0 \setminus S_K^0 \). The latter is itself stratified by (quotients by the action of a finite group of) Shimura varieties, still defined over \( E \).

Assume \( S_K \) to be of PEL type (loosely speaking, a moduli space of abelian varieties equipped with polarizations, endomorphisms and level structure). Then, according to [A], every irreducible representation \( V_\lambda \) of \( G \) of highest weight \( \lambda \) gives rise, in a functorial way, to an object \( \mathcal{V} \) of the category \( \text{CHM}(S_K) \) of Chow motives over \( S_K \), whose \( \ell \)-adic realization is \( \mu_\ell(V_\lambda) \). In this context, the theory developed by Wildeshaus (especially the works [W2], [W4], [W7], [W8]) implies that there exists a cohomological condition on the degeneration at the boundary \( i^* j_* \mu_\ell(V_\lambda) \) of the \( \ell \)-adic sheaves \( \mu_\ell(V_\lambda) \), which, once satisfied, allows for the construction of a Chow motive with the properties stated in Theorem 1: the condition consists in the absence of weights 0 and 1 in the latter complex of \( \ell \)-adic sheaves, in the sense of the weight filtration given by the Galois action. More precisely, this hypothesis allows to construct a functorial Chow motive which realizes to intersection cohomology with values in \( \mu_\ell(V_\lambda) \) (hence the name intersection motive) and to identify the latter with interior cohomology (cfr. Cor. 3.1.0.5). Moreover, the criterion contained in [W7] (recorded here as Theorem 2.1.0.7) implies that, in order to detect this weight avoidance, it suffices to analyse the weight filtration on the perverse cohomology objects of such complexes, stratum by stratum on the boundary: this is what we actually do in this paper.

Notice that the complex \( i^* j_* \mu_\ell(V_\lambda) \) is actually the \( \ell \)-adic realization of the (relative) boundary motive \( i^* j_*^! \mathcal{V} \), an object of the category \( \text{DM}_{B,C}(S_K) \) of Beilinson motives over \( \partial S_K^0 \), constructed over general bases in [CD1], along with its six functors formalism.

The analysis of the weight filtration brings into consideration contributions coming from different varieties of group cohomology: abstract cohomology of arithmetic and finitely generated free abelian groups, and Hochschild cohomology of algebraic groups. In previous work ([W3] for Hilbert-Blumenthal varieties, [W5] for Picard surfaces, [Cl] for Picard varieties of arbitrary dimension and [W8] for Siegel threefolds), these facts have been employed in order to show that regularity of the coefficient systems implies the avoidance of the weights 0 and 1 at the boundary. Two natural questions then arise. First: does this hold for other families of Shimura varieties? Moreover, since, in the first and third case above, it can be seen that there exist non-regular representations, which nonetheless satisfy the weight avoidance, one is lead to ask: does there exist a general condition on the highest weights of irreducible representations of \( G \), which is equivalent to the absence of the weights 0 and 1 in the degeneration at the boundary? In this paper we answer both questions in the case of genus 2 Hilbert-Siegel varieties.

Before explaining what the response is, let us just discuss the role of this special case. In the first three of the examples studied before, the strata of the boundary of the Baily-Borel compactification are simply of dimension 0, while one-dimensional strata appear in the boundary of Siegel threefolds: this makes the analysis sensibly more difficult and forces one to use the relative formalism of Beilinson motives. Genus 2 Hilbert-Siegel varieties represent then a natural following step: the boundary still presents only two types of strata, but the higher-dimensional ones can be of arbitrary dimension - equal, in fact, to the degree of the field \( F \) over \( Q \). In addition, for the first time with respect to the preceding cases, the phenomena, which are caused by the three types of cohomology listed before, make their appearance all together (especially because of the arithmetic of...
the field $F$, a point which we will comment better later).

Let us now define the corank\(^1\) of a weight $\lambda = ((k_i)_i$, $(k'_i)_i)$ as 0 if $k'_i$ is not the same integer for all $i$, 1 if $k'_i$ is constant but there exists an $i$ such that $k_i \neq k'_i$, and 2 if there exists an integer $\kappa$ such that $k_i = k'_i = \kappa$ for all $i$. Moreover, let us call the weight completely irregular if, for all $i$, it is not regular at $i$. Then, for genus 2 Hilbert-Siegel varieties, as a consequence of our main technical result (Thm. 2.1.0.3), we are able to exhibit the sought-for characterization of the absence of weights, by showing that it is precisely the notion of corank which allows for its formulation:

**Theorem 2.** (Cfr. Cor. 2.1.0.4) The weights 0 and 1 appear in the complex $i^\ast j_\ast \mu_\ell(V_\lambda)$ if and only if $\lambda$ is completely irregular of corank $\geq 1$.

By the theory cited above, and explained in more detail in subsection 2.1 and Section 3, Theorem 2 implies the validity of Theorem 1, and more generally of the following fact: a Chow motive which realizes to interior cohomology, equipped with a Hecke action, exists as soon as the weight $\lambda$ is either not completely irregular or of corank 0.

The characterization given by Theorem 2 subsumes all cases previously treated, and lends itself to further generalization. In order to conclude this introduction, let us put this result into perspective, by explaining why we think of the corank as an important property for the study of the cohomology of Shimura varieties.

**Corank and weight filtration in the cohomology of Shimura varieties.** For a general (smooth) Shimura variety $S_K$, it is a very important problem to understand the weight filtration (say, in the Hodge-theoretical sense) on the spaces $H^n(S_K(\mathbb{C}), \mu(V))$. By Hodge theory and consideration of the long exact sequence associated to the complementary, closed-open immersions $i$ and $j$ given by any (topological) compactification $\bar{S}$ of $S_K(\mathbb{C})$ and by its boundary $\partial\bar{S}$, one sees that, for each $n$, interior cohomology is contained in the weight $n$ subspace of $H^n(S_K(\mathbb{C}), \mu(V))$ (the pure part), while the rest of the weight filtration is determined by a subspace of boundary cohomology $\partial H^n(S_K(\mathbb{C}), \mu(V)) := H^n(\partial\bar{S}, i^\ast j_\ast \mu_\ell(V))$. The latter is precisely the hypercohomology of (the Hodge-theoretical analogues of) the complexes studied in this paper.

In this context, one approach to cohomology of $i^\ast j_\ast \mu_\ell(V)$ consists in taking a closed cover of $\partial\bar{S}$ and in studying the associated spectral sequence, abutting to $\partial H^\ast(S_K(\mathbb{C}), \mu(V))$. For $\bar{S}$ equal to the analytification of a smooth toroidal compactification\(^2\), the study of this spectral sequence (of mixed Hodge structures) is the subject of [HZ1], [HZ2]. In these works, the authors remark (in a slightly different language) that the interaction between the two filtrations of $i^\ast j_\ast \mu_\ell(V)$, the one coming from the closed cover of the boundary and the weight filtration, is far from being understood. The same is then true for the two corresponding filtrations on the spaces $H^n(S_K(\mathbb{C}), \mu(V))$.

On the other hand, after [Na], these same spaces are endowed with a third interesting filtration, whose graded objects are equipped with semisimple mixed Hodge structures: in loc. cit., the author uses the work of Franke ([Fra]) to show in particular that this filtration has an automorphic origin. Once again, the precise relationship with the weight filtration remains mysterious.

In order to read our main result in the light of the above considerations, let us come back to the situation of a genus 2 Hilbert-Siegel variety $S_K$. One knows (cfr. subsections 1.3.1-1.3.3) that the boundary $\partial S_K^\ast$ of its Baily-Borel compactification admits a stratification of the form

$$\partial S_K^\ast = \bigcup_{i=1}^2 \bigcup_{g \in C_i} S_{i,g},$$

where the $C_i$’s are some finite subsets of $G(\mathbb{A}_f)$ and the open, resp. closed subschemes $S_{1,g}$, resp. $S_{2,g}$ of $\partial S_K^\ast$ are (quotients by the action of a finite group of) Shimura varieties of dimension $d$, resp. 0, for $d$ the degree of $F$ over $\mathbb{Q}$. Denote, for $i = 1, 2$, $S_i := \bigcup_{g \in G} S_{i,g}$.

For an irreducible representation $V_\lambda$ of $G$ determined by a dominant weight $\lambda$, let us recall without definition, just for the needs of this introduction, the associated automorphic (coherent) sheaf $\omega(\lambda)$ over a fixed toroidal compactification $S_{K,\Sigma}$ of $S_K$, which, by pushforward along the projection $\pi : S_{K,\Sigma} \to S_K$, gives rise to a

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\(^1\)See Def. 2.1.0.2 for an equivalent, more transparent and more easily generalizable definition.

\(^2\)But also, in the - a priori - purely topological case, equal to the Borel-Serre compactification of $S_K$. The fact that this non-algebraic compactification still gives rise to spectral sequences of Hodge structures represents one of the most important discoveries in [HZ1].
Theorem. (cfr. [BR, Def. 1.10]) Let \( f \in M_{\lambda,K} \) be a non-zero automorphic form. The corank of \( f \) is the minimal integer \( \text{cor}(f) := q \) such that \( \oplus_{g \in C_{q+1}} f \big|_{S_{q+1,g}} = 0 \) (with \( C_3 = \emptyset \) by convention).

Notice that this notion gives a measure, in some sense, of the degree of cuspidality of \( f \): for example, \( \text{cor}(f) = 0 \iff f \) is cuspidal, and we could define completely non-cuspidal forms as the \( f \)'s such that \( \text{cor}(f) = 2 \).

Recall then the following theorem from [BR] (where the role of \( \lambda \), resp. \( M_{\lambda,K} \), is played by \( k \), resp. \( M_k(\mathcal{H},R) \), in which the notion of corank of \( \lambda \) as defined before arises in an essential way:

Theorem. ([BR, Thm. 1.12]) If \( f \in M_{\lambda,K} \) is non-zero, then \( \text{cor}(\lambda) \geq \text{cor}(f) \).

This implies for example that, in order to have non-zero non-cuspidal forms, it is necessary for \( \lambda \) to be of corank \( \geq 1 \); moreover, in order to have non-zero completely non-cuspidal forms, it is necessary for \( \lambda \) to be of corank \( 2 \), and in particular completely irregular.

Observe now that, reasoning along the lines of the lines of Corollary 3.1.0.1.(1), we can immediately deduce the following\(^3\) from Theorem 2:

Theorem 3. An irreducible representation \( V_\lambda \) has the property that, for some \( n \), weight \( n+1 \) appears in \( H^n(S_K(\mathcal{C}),\mu(V_\lambda)) \), if and only if \( \lambda \) is completely irregular and of corank \( \geq 1 \).

Even if the proofs of Theorem 2 and of the theorem from [BR] are completely independent, the analogy between the two is striking. In fact, thanks to the notion of corank, we see that weight \( 1 \) appears in the complex computing \( V_\lambda \)-valued cohomology of \( S_K \) if and only if \( \lambda \) is completely irregular and satisfies the necessary conditions for the presence of a non-zero non-cuspidal form on \( S_K \). Here we have in mind a non-zero non-cuspidal form as giving a class in the quotient of \( H^n(S_K(\mathcal{C}),\mu(V_\lambda)) \) of weight strictly bigger than \( n \), for some \( n \); since the corank of an automorphic form is in turn defined in terms of its behaviour along the different types of strata in \( \partial S_K \), Theorem 3 pushes us to think to our result as a (very little!) hint towards the understanding of the links between weight filtration, "simplicial" filtration coming from the stratification of the boundary and "automorphic" filtration in the cohomology of Shimura varieties. One could ask if investigating the relationship with the filtration by holomorphic rank considered in [HZ2], Sec. 4.4-4.5 could be a good starting point for studying such questions.

Notice that, in order to prove the presence of the weights \( 0 \) and \( 1 \), we need exactly the existence of a non-zero automorphic form of a certain type: however, it is a cuspidal Hilbert modular form over a "virtual" Hilbert-Blumenthal variety, which does not appear in \( \partial S_K \) (cfr. Prop. 2.3.3.2 and Rmk. 2.6.1.6).

As a last remark, let us stress the fact that the main phenomenon, which leads us to the notion of corank for \( \lambda \), appears precisely as a manifestation of the interaction between different group cohomologies alluded to before. The non-triviality of the totally real extension \( F \) gives in fact rise to an action of some subgroups of units of the integers of \( F \), which coincides with the action of the local Hecke operator from [LR] and puts essential restrictions on the possible weights. We refer to Remarks 2.3.2.8 and 2.4.2.4 for the precise formulation of these facts; let us just recall here that it is this same operator who plays a crucial role for the results of [Na] cited before.

Organisation of the paper. In the preliminary Section 1 we first introduce the Shimura datum \((G,X)\) which allows one to define the Hilbert-Siegel varieties \( S_K \). We recall then the structure of the Baily-Borel compactification \( S^*_K \), and we introduce the canonical construction functors, which produce a variation of Hodge structure \( \mu_H(V) \) or an \( \ell \)-adic sheaf \( \mu_\ell(V) \) over \( S_K \) from a representation \( V \) of \( G \), along with the relative Chow motives \( ^oV \) over \( S_K \) associated to irreducible representations \( V_\lambda \).

In Section 2, the heart of the paper, we begin by stating the main result (the description of the limit weights of \( ^oV \) in terms of the corank of \( \lambda \), Thm. 2.1.0.3) and the strategy of its proof, which makes use of the criterion 2.1.0.7. After recalling the theorems of Pink and Kostant, which are essential for describing the

\(^3\)The results stated so far concern a priori the weight filtration on the complexes of \( \ell \)-adic sheaves obtained from the \( \ell \)-adic realization of \( ^oV \), but they actually give information on the weight filtration in the sense of Bondarko ([B]) on the boundary motive itself. This doesn’t imply directly anything on the Hodge side, because the Hodge realization functor on Beilinson motives over singular bases (as \( \delta S^o \)) has not been constructed yet. However, thanks to the complete formal analogy between the results of [P2] (\( \ell \)-adic context) and [BuW] (Hodge-theoretic context), and between the formalisms of six functors in the respective derived categories, our computations are also valid in the case of mixed Hodge modules; it is in fact the Hodge theoretic picture which guides these computations (cfr. Rmk. 1.4.2.2.(1)).
degeneration of the ℓ-adic sheaves μℓ(Vλ), we study such degeneration along the dimension 0 strata (subsection 2.3) and along the strata of dimension d = [F : Q] (subsection 2.4), before passing to the study of the double degeneration along the cusps of the d-dimensional strata (subsection 2.5). In subsection 2.6 we glue together all this information by using the perverse t-structure, and we complete the proof of the main theorem.

In Section 3 we give the applications of our result: the construction of a Hecke-equivariant Chow motive which realises to interior cohomology of SK with coefficients which are not completely irregular of corank ≥ 1.

In the case of regular coefficients, we describe the consequences of the above for the construction of motives associated to automorphic representations, obtaining in particular (homological) motives corresponding to the representations studied in [F] (cfr. Rmk. 3.2.0.8).

**Notations.** The symbols A, resp. A_f will denote the ring of adèles, resp. finite adèles. Throughout the whole paper, F will be a fixed totally real field of degree d over Q, IF its set of real embeddings (thus, of cardinal d), and L a fixed Galois closure of F.

An empty entry in a ring-valued matrix will mean that the corresponding coefficient is zero.

A variety over a field k will be a separated scheme of finite type over k, and an algebraic group over k will mean a scheme of finite type over k with a group scheme structure. If f : T → S is a morphism of schemes and X is a S-scheme, the notation X ×S,t T will be used to specify that the fiber product has been taken along f: this can be shortened to (X_T)l and furthermore to X_T (if the choice of f is evident from the context).

We will denote by Res_{k/k}G the Weil restriction of an algebraic group G from a field k′ to a subfield k, and by S the algebraic group Res_{C/Q}G_{m,C} (Deligne’s torus).

We will use the symbol π_0(X) for the set of connected components of a topological space X.

If C is a category, then Gr_C will denote its category of graded objects. A sub-category B of an additive category A is dense if for any object B of B, any direct factor of B formed in A belongs already to B. The pseudo-Abelian completion of an additive category A is denoted by A^a. If A is an abelian category, D^b(A) will denote its bounded derived category.

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1. The Shimura datum, the Baily-Borel compactification and the canonical construction

In this preliminary section we introduce the basic objects of interest: the Shimura datum (G, X) which defines the genus 2 Hilbert-Siegel varieties SK, their Baily-Borel compactifications SK and the relative Chow motives V over SK associated to irreducible representations Vλ of G.

1.1. The group for the Shimura datum.

1.1.1. The group of symplectic similitudes and the group G. Define, for all positive integer n, the 2n × 2n matrix

\[ J_n := \left( \begin{array}{cc} -I_n & I_n \\ \end{array} \right) \in \text{GL}_{2n}(\mathbb{Q}). \]

The group GSp2n of symplectic similitudes of dimension 2n is the algebraic group over Q defined, for all Q-algebra R, by posing

\[ \text{GSp}_{2n}(R) := \{ g \in \text{GL}_{2n}(R) | g J_n g = \nu(g) J_n, \nu(g) \in \mathbb{G}_m(R) \} . \]

It is a reductive group, whose center Z is isomorphic to \( \mathbb{G}_m \) and whose derived subgroup is isomorphic to Sp2n, the usual symplectic group over Q of dimension 2n.

Letting π denote the projection on the adjoint group GSp2n := GSp2n/Z ≃ PGSp2n and ν : GSp2n → \( \mathbb{G}_m \) the multiplier (or similitude factor), one has exact sequences

\[ 1 \to \mathbb{G}_m \to \text{GSp}_{2n} \xrightarrow{\pi} \text{PGSp}_{2n} \to 1, \]

\[ 1 \to \text{Sp}_{2n} \to \text{GSp}_{2n} \xrightarrow{\nu} \mathbb{G}_m \to 1. \]
For the rest of this paper, fix \( n = 2 \). Recall that we have fixed a totally real field \( F \) of degree \( d \) over \( \mathbb{Q} \) and that we denote by \( I_F \) the set of real embeddings of \( F \). Define then the \( \mathbb{Q} \)-algebraic group \( \tilde{G} \) by posing

\[
\tilde{G} := \text{Res}_{F|\mathbb{Q}} \text{GSp}_{4,F}.
\]

The application of the composition of base change to \( F \) and restriction of scalars to \( \mathbb{Q} \) to every object of the exact sequences (1.1) and (1.2) yields sequences which are again exact.

**Remark 1.1.1.** For all subfield \( k \) of \( \mathbb{C} \) containing \( \sigma(F) \) for all \( \sigma \in I_F \), one has, for all \( k \)-algebra \( R \), an isomorphism

\[
(1.3) \quad F \otimes_{\mathbb{Q}} R \rightarrow \prod_{\sigma \in I_F} R, \ f \otimes r \mapsto (\sigma(f) \cdot r)_{\sigma}
\]

which induces a canonical isomorphism

\[
\tilde{G}_k \cong \prod_{\sigma \in I_F} (\text{GSp}_{4,k})_{\sigma},
\]

Consider now the canonical adjunction morphism \( \mathcal{G}_m \rightarrow \text{Res}_{F|\mathbb{Q}} \mathcal{G}_m,F \), induced by the fact that Weil restriction is right adjoint to base change, and the morphism \( \tilde{G} \rightarrow \text{Res}_{F|\mathbb{Q}} \mathcal{G}_m,F \), induced by functoriality by the multiplier \( \nu : \text{GSp}_{2n} \rightarrow \mathcal{G}_m \).

**Definition 1.1.1.2.** The reductive \( \mathbb{Q} \)-group \( G \) is defined by

\[
(1.4) \quad G := \mathcal{G}_m \times_{\text{Res}_{F|\mathbb{Q}} \mathcal{G}_m,F} \tilde{G},
\]

where the fibred product has been taken with respect to the preceding morphisms.

**Remark 1.1.1.3.** (1) The isomorphism (1.3) induces, for all subfield \( k \) of \( \mathbb{C} \) containing \( \sigma(F) \) for all \( \sigma \in I_F \), an isomorphism

\[
(1.5) \quad G_k \cong \mathcal{G}_m \times_{\sigma \in I_F} (\mathcal{G}_m,k)_{\sigma} \prod_{\nu} \prod_{\sigma \in I_F} (\text{GSp}_{4,k})_{\sigma}.
\]

(2) The center of \( G \) is such that \( Z(G) \cong \mathcal{G}_m \times_{\text{Res}_{F|\mathbb{Q}} \mathcal{G}_m,F,x \mapsto x^2} \text{Res}_{F|\mathbb{Q}} \mathcal{G}_m,F \). Its neutral component is then isogenous to \( \mathcal{G}_m \).

1.1.2. **The structure of parabolic subgroups of \( G \).** The group \( \text{GSp}_{4,F}(F) \) acts on \( F^\oplus 4 \) through the natural action induced by its inclusion into \( \text{GL}_{4,F}(F) \). The standard \( F \)-basis \( \{e_1,e_2,e_3,e_4\} \) gives then a symplectic basis for the non-degenerated, \( F \)-bilinear alternated form defined by \( J_2 \in \text{GSp}_{4,F}(F) \), which we will also denote \( J_2 \). Fix as a maximal torus of \( \text{GSp}_{4,F} \) the standard diagonal torus \( T \) defined on \( F \)-points by

\[
(1.6) \quad \check{T}(F) := \{\text{diag}(\alpha_1,\alpha_2,\alpha_1^{-1}\nu,\alpha_2^{-1}\nu)|\alpha_1,\alpha_2,\nu \in \mathcal{G}_m(F)\},
\]

along with the standard Borel \( \check{B} \) containing it, defined on \( F \)-points as the subgroup of matrices in \( \text{GSp}_{4,F}(F) \) of the form

\[
\begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}
\]

One knows that the parabolic subgroups of \( \text{GSp}_{4,F}(F) \) correspond bijectively to subgroups of the form \( \text{Stab}(V) \), for \( V \) a sub-\( F \)-vector space of \( F^\oplus 4 \) which is totally isotropic for the form \( J_2 \). The case \( V = \{0\} \) corresponds to the whole group \( \text{GSp}_{4,F}(F) \), while \( V = \langle e_1 \rangle \) gives the Klingen parabolic

\[
\check{Q}_1(F) := \left\{ \begin{pmatrix} \alpha & * & * & * \\
a & b & * & * \\
c & * & d & * \\
\beta & & & 
\end{pmatrix} \big| ad-bc = \alpha \beta \in \mathcal{G}_m,F(F) \right\} \cap \text{GSp}_{4,F}(F)
\]

and \( V = \langle e_1,e_2 \rangle \) gives the Siegel parabolic

\[
\check{Q}_0(F) := \left\{ \begin{pmatrix} \alpha & AM \\
A^{-1} & \end{pmatrix} \big| \alpha \in \mathcal{G}_m,F(F), A \in \text{GL}_{2,F}(F), ^t M = M \right\};
\]
every other parabolic subgroup is conjugated to one of the above.

One then knows that a maximal torus, resp. a Borel, of \( G \) are given by \( \text{Res}_{F|Q}(\hat{T}) \), resp. \( \text{Res}_{F|Q}(\hat{B}) \), which we will still denote by \( \hat{T}, \hat{B} \) in the following; note that \( \hat{T} \) is not split over \( Q \). A maximal torus and a Borel containing it in \( G \) are then respectively defined by \( T := \mathbb{G}_m \times \text{Res}_{F|Q}\mathbb{G}_m,F \hat{T} \) and \( B := \mathbb{G}_m \times \text{Res}_{F|Q}\mathbb{G}_m,F \hat{B} \).

In the same way, the standard maximal parabolics of \( \hat{G} \) corresponding to the choice \( (\hat{T}, \hat{B}) \) are exactly given, up to conjugation, by \( \text{Res}_{F|Q}\hat{Q}_0, \text{Res}_{F|Q}\hat{Q}_1 \), which we will still denote by \( \hat{Q}_0, \hat{Q}_1 \). Then, \( Q_0 := \mathbb{G}_m \times \text{Res}_{F|Q}\mathbb{G}_m,F \hat{Q}_0 \), \( Q_1 := \mathbb{G}_m \times \text{Res}_{F|Q}\mathbb{G}_m,F \hat{Q}_1 \) are the standard maximal parabolics of \( G \) with respect to \( (T, B) \), still called the Siegel and the Klingen one.

1.1.3. The Levi components of parabolic subgroups. Let \( W_0 \) and \( W_1 \) be the unipotent radicals of the groups \( Q_0 \) and \( Q_1 \) defined above. The quotients \( Q_i/W_i \) will be canonically identified with subgroups of the \( Q_i \)'s, thanks to the Levi decomposition of the latter.

Fix now a subfield \( k \) of \( \mathbb{C} \) which contains \( \sigma(F) \) for all \( \sigma \in I_F \). One has the following explicit description of the diagonal embedding of \( Q_0/W_0(\mathbb{Q}) \) into \( Q_0/W_0(k) \):

\[
Q_0/W_0(\mathbb{Q}) \simeq \{(\left(\begin{array}{c}\alpha \sigma(A) \\
(\sigma(A)^{-1})^t\end{array}\right))_{\sigma \in I_F} | \alpha \in Q^\times, A \in \text{GL}_2(F)\} \hookrightarrow Q_0/W_0(k) = \{(\left(\begin{array}{c}\alpha A_{\sigma} \\
(A_{\sigma}^{-1})^t\end{array}\right))_{\sigma \in I_F} | \alpha \in k^\times, A_{\sigma} \in \text{GL}_2(k) \text{ for all } \sigma\}
\]

and of the diagonal embedding of \( (Q_1/W_1)(\mathbb{Q}) \) into \( (Q_1/W_1)(k) \):

\[
Q_1/W_1(\mathbb{Q}) \simeq \{(\left(\begin{array}{ccc}\sigma(t) \cdot (ad - bc) & \sigma(a) & \sigma(b) \\
\sigma(c) & \sigma(t^{-1}) & \sigma(d)\end{array}\right))_{\sigma \in I_F} | t \in F^\times, a, b, c, d \in F \text{ such that } ad - bc \in Q^\times\} \hookrightarrow Q_1/W_1(k) = \{(\left(\begin{array}{ccc}t_{\sigma} & a_{\sigma} d_{\sigma} - b_{\sigma} c_{\sigma} \\
a_{\sigma} & b_{\sigma} & t_{\sigma}^{-1} \\
c_{\sigma} & t_{\sigma}^{-1} & d_{\sigma}\end{array}\right))_{\sigma \in I_F} | t_{\sigma} \in k^\times \text{ for all } \sigma, a_{\sigma}, b_{\sigma}, c_{\sigma}, d_{\sigma} \in k \text{ such that } a_{\sigma} d_{\sigma} - b_{\sigma} c_{\sigma} = a_{\hat{\sigma}} d_{\hat{\sigma}} - b_{\hat{\sigma}} c_{\hat{\sigma}} \in k^\times \text{ for all } \sigma, \hat{\sigma} \in I_F\}.
\]

Thus, there is an isomorphism

\[
(1.7) \quad Q_0/W_0 \simeq \mathbb{G}_m \times \text{Res}_{F|Q}\text{GL}_2,F,
\]

given on \( k \)-points by

\[
(Q_0/W_0)(k) = \mathbb{G}_m(k) \times \prod_{\sigma \in I_F} (\text{GL}_2(k)_{\sigma}) \quad \left(\left(\begin{array}{c}\alpha A_{\sigma} \\
(A_{\sigma}^{-1})^t\end{array}\right)_{\sigma \in I_F} \mapsto (\alpha, (A_{\sigma})_{\sigma \in I_F})\right),
\]

and an isomorphism

\[
(1.8) \quad Q_1/W_1 \simeq (\text{Res}_{F|Q}\text{GL}_2,F \times \text{Res}_{F|Q}\mathbb{G}_m,F,\text{det } \mathbb{G}_m) \times \text{Res}_{F|Q}\mathbb{G}_m,F
\]

given on \( k \)-points by
\[(Q_1/W_1)(k) \simeq ((\prod_{\sigma \in I_F} (GL_{2,k})_\sigma) \times \prod_{\sigma \in I_F} (\mathbb{G}_{m,k})_\sigma) \times \prod_{\sigma \in I_F} (\mathbb{G}_m(k))_\sigma \]

\[
\left( \begin{array}{ccc}
t_\sigma & (a_\sigma d_\sigma - b_\sigma c_\sigma) \\
0 & t_\sigma^{-1} & b_\sigma \\
0 & c_\sigma & d_\sigma
\end{array} \right)_{\sigma \in I_F} \mapsto \left( \begin{array}{ccc}
a_\sigma & b_\sigma \\
c_\sigma & d_\sigma \\
t_\sigma
\end{array} \right)_{\sigma \in I_F}, \left( t_\sigma \right)_{\sigma \in I_F}.
\]

1.1.4. **Characters and dominant weights.** Fix once and for all a Galois closure $L$ of $F$. Using the isomorphism (1.5) (for $k = L$) and Definition 1.6, we get the following description for the points of the maximal torus $T_L$ of $G_L$:

\[T_L(L) = \{(\text{diag}(\alpha_{1,\sigma}, \alpha_{2,\sigma}, \alpha_{1,\sigma}^{-1} \nu, \alpha_{2,\sigma}^{-1} \nu))_{\sigma \in I_F} | \alpha_{1,\sigma}, \alpha_{2,\sigma} \in L^*, \nu \in \mathbb{Q}^x \} \simeq \prod_{\sigma \in I_F} T_\sigma(L),\]

where each $T_\sigma$ is a copy of the diagonal maximal torus of $\text{GSp}_{4,L}$.

The elements $\lambda$ of the group $X^*(T_L) \simeq \bigoplus_{\sigma \in I_F} X^*(T_\sigma)$ of characters (or "weights") of $T_L$ are then parametrized by the $(2d + 1)$-tuples of integers of the form

\[(k_\sigma, k'_\sigma)_{\sigma \in I_F}, c) \text{ such that } \sum_{\sigma \in I_F} (k_\sigma + k'_\sigma) \equiv c \pmod{2};\]

the character $\lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$ corresponding to $((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$ is defined by

\[\lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c) : (\text{diag}(\alpha_{1,\sigma}, \alpha_{2,\sigma}, \alpha_{1,\sigma}^{-1} \nu, \alpha_{2,\sigma}^{-1} \nu))_{\sigma \in I_F} \mapsto \prod_{\sigma \in I_F} \alpha_{1,\sigma}^{k_\sigma} \cdot \prod_{\sigma \in I_F} \alpha_{2,\sigma}^{k'_\sigma} \cdot \nu^{\frac{1}{2} [c - \sum_{\sigma \in I_F} (k_\sigma + k'_\sigma)]}.\]

The dominant weights are the characters such that $k_\sigma \geq k'_\sigma \geq 0 \forall \sigma$. A weight is called regular at $\sigma$ if $k_\sigma > k'_\sigma > 0$ and regular if it is regular at $\sigma$ for all $\sigma$.

1.1.5. **Root system and Weyl group.** The choice of $(T_L, B_1)$ (obtained from the couple $(T, B)$ fixed at the end of 1.1.2, by base change to our fixed Galois closure $L$ of $F$) allows one to identify the set of roots $\mathfrak{r}$ of $G_L$ with $\bigsqcup_{\sigma \in I_F} \mathfrak{r}_\sigma$, where each $\mathfrak{r}_\sigma$ is a copy of the set of roots of $\text{GSp}_{4,L}$ corresponding to the diagonal torus and the standard Borel. For all fixed $\hat{\sigma} \in I_F$, $\mathfrak{r}_\sigma$ contains two simple roots $\rho_{1,\hat{\sigma}}$ and $\rho_{2,\hat{\sigma}}$, which, through the inclusion of $\mathfrak{r}_\sigma$ into $\mathfrak{r}$, can respectively be written $\rho_{1,\hat{\sigma}} = \rho_{1,\hat{\sigma}}((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$, with

\[k_\sigma = \begin{cases} 1 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \quad k'_\sigma = \begin{cases} -1 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \quad c = 0,
\]

and $\rho_{2,\hat{\sigma}} = \rho_{2,\hat{\sigma}}((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$, with

\[k_\sigma = 0 \forall \sigma, \quad k'_\sigma = \begin{cases} 2 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \quad c = 0.
\]

The Weyl group $W$ of $G_L$ is in turn isomorphic to the product $\prod_{\sigma \in I_F} W_\sigma$, where, for every fixed $\hat{\sigma} \in I_F$, $W_\hat{\sigma}$ is a copy of the Weyl group of $\text{GSp}_{4,L}$. The latter is a finite group of order 8 acting on $X^*(T_\sigma)$, generated by two elements $s_1$ and $s_2$, whose images $s_{\rho_{1,\hat{\sigma}}}$ and $s_{\rho_{2,\hat{\sigma}}}$ through the inclusion $W_\hat{\sigma}$ into $W$ are characterised by their action on the elements of $X^*(T_L)$: if $\lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$, then $s_{\rho_{1,\hat{\sigma}}} \lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$, with

\[h_\sigma = \begin{cases} k'_\sigma & \text{if } \sigma = \hat{\sigma} \\ k_\sigma & \text{otherwise} \end{cases}, \quad h'_\sigma = \begin{cases} k_\sigma & \text{if } \sigma = \hat{\sigma} \\ k'_\sigma & \text{otherwise} \end{cases}
\]

and $s_{\rho_{2,\hat{\sigma}}} \lambda = \lambda((h_\sigma, h'_\sigma)_{\sigma \in I_F}, c)$, with

\[h_\sigma = k_\sigma \forall \sigma \in I_F, \quad h'_\sigma = \begin{cases} -k'_\sigma & \text{if } \sigma = \hat{\sigma} \\ k_\sigma & \text{otherwise} \end{cases}.
\]

These descriptions mean that $s_{\rho_{1,\hat{\sigma}}}$ corresponds to the reflection associated to $\rho_{1,\hat{\sigma}}$ and that $s_{\rho_{2,\hat{\sigma}}}$ corresponds to the reflection associated to $\rho_{2,\hat{\sigma}}$.  

1.1.6. **Irreducible representations.** Irreducible representations of a split reductive group over a characteristic 0 field are parametrized by its dominant weights. By the description of the dominant weights of \( G_L \) given in (1.9), we see that isomorphism classes of irreducible \( L \)-representations of \( G_L \) are in bijection with the set

\[
\Lambda := \{ \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c) | |k_\sigma, k'_\sigma, c \in \mathbb{Z} et k_\sigma \geq k'_\sigma \text{ for all } \sigma, \sum_{\sigma \in I_F} (k_\sigma + k'_\sigma) \equiv c \pmod{2}\}.
\]

1.2. **Hilbert-Siegel varieties.** In this subsection, let \( G \) denote the group defined in 1.1.1.2.

1.2.1. *The Shimura datum.* We are going to define a *pure Shimura datum* \((G, X)\) in the sense of [P1, Def. 2.1].

**Definition 1.2.1.1.** The complex analytic space \( \mathcal{H}_2 \) is the subspace of \( M_2(\mathbb{C}) \) formed by complex \( 2 \times 2 \) matrices which are symmetric and whose imaginary part is definite (positive or negative).

Recall that \( G(\mathbb{R}) = \{(A_\sigma)_{\sigma \in I_F} \in \prod_{\sigma \in I_F} \text{GSp}_4(\mathbb{R}) \mid \nu(A_\sigma) = \nu(A_{\bar{\sigma}}) \forall \sigma, \bar{\sigma} \in I_F\} \) and that \(|I_F| = d\). Then, \( G(\mathbb{R}) \) acts on \( \mathcal{H}_2^d \) by analytical isomorphisms. Let \( J_2 \) be the element of \( \text{GSp}_4(\mathbb{R}) \) introduced in 1.1.2.

**Proposition 1.2.1.2.** Let \( h : \mathcal{S} \to G(\mathbb{R}) \) be the morphism defined on real points by

\[
\mathcal{S}(\mathbb{R}) \to G(\mathbb{R})
\]

\[
x + iy \mapsto ((xI_2 + yJ_2)_{\sigma \in I_F})
\]

The \( G(\mathbb{R}) \)-conjugacy class \( X \) of \( h \) has a canonical structure of complex analytic space (of dimension \( 3d \)), such that there exists a \( G(\mathbb{R}) \)-equivariant isomorphism \( X \simeq \mathcal{H}_2^d \) as complex manifolds. Moreover, \((G, X)\) is a pure Shimura datum.

**Remark 1.2.1.3.** Thanks to Remark 1.1.1.3.(2), the Shimura datum \((G, X)\) satisfies condition (+) from [BuW, page 7].

1.2.2. **Hilbert-Siegel varieties.** Fix now a compact open subgroup \( K \) of \( G(\mathbb{A}_f) \) which is moreover neat ([P1, Section 0.6]). Then, the double quotient \( G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \) has the structure of a smooth complex analytic variety, which is the analytification of a canonical smooth quasi-projective variety \( S(G, X)_K \) (the *Shimura variety* corresponding to the datum \((G, X)\) and to the subgroup \( K \)), defined over a number field \( E(G, X) \), the *reflex field* of \((G, X)\) (independent of \( K \)). In our case, the reflex field is just \( \mathbb{Q} \).

**Definition 1.2.2.1.** The Hilbert-Siegel variety (of level \( K \), genus 2, associated to \( F \)) is the smooth quasi-projective \( \mathbb{Q} \)-variety \( S_K := S(G, X)_K \).

**Remark 1.2.2.2.**

1. \( S_K \) is of dimension \( 3d \).
2. The datum \((G, X)\) is a PEL datum. In particular, according to [De2, 4.12], \( S_K \) admits an interpretation as moduli space of abelian varieties of dimension \( 2d \) with additional structures, among which real multiplication by a sub-algebra \( \mathcal{O} \) of \( F \) of rank \( 2d \) over \( \mathbb{Z} \), which depends on \( K \). Thus, there exists a universal family \( p : \mathcal{A}_K \to S_K \) of abelian varieties over \( S_K \).

1.3. **The Baily-Borel compactification.** Let \( S_K \) be a Shimura variety corresponding to a general pure Shimura datum \((G, \mathcal{X})\) and to a neat compact open subgroup \( K \subset G(\mathbb{A}_f) \). Recall that \( S_K \) has a canonical compactification \( S^*_K \), called Baily-Borel compactification, a projective variety, in general singular, defined over the reflex field \( E(G, X) \) ([P1, Main Theorem 12.3, (a), (b)]), in which \( S_K \) embeds as an open dense sub-scheme. In the case of Hilbert-Siegel varieties, \( S^*_K \) is then defined over \( \mathbb{Q} \).

1.3.1. **Structure of the stratification of \( S^*_K \).** The variety \( S^*_K \) admits a stratification by locally closed strata, amongst which \( S_K \) is the only open stratum. The other ones form a stratification \( \Phi' \) of the boundary \( \partial S^*_K = S^*_K \backslash S_K \), described as follows (cfr. [W4, Sect. 8]).

Fix an admissible parabolic subgroup \( Q \) of \( G \) ([P1, Def. 4.5]). Then, there exist a canonical normal subgroup \( P \) of \( Q \) ([P1, 4.7]) and a finite collection of mixed Shimura data ([P1, Def. 2.1]), denoted by \((P, \mathcal{X})\) and called rational boundary components, indexed by the \( P(\mathbb{R}) \)-orbits inside \( \pi_0(\mathcal{X}) \) ([P1, 4.11]).

Fix now such a rational boundary component \((P, \mathcal{X})\) and an element \( g \in G(\mathbb{A}_f) \), write \( K^g := g \cdot K \cdot g^{-1} \), and define
\(K' := P(A_f) \cap K^g.\)

Denoting by \(W\) the unipotent radical of \(P\) (which in our case, since \((G, X)\) is a pure datum, equals the unipotent radical of \(Q\) - cfr. [P1, proof of Lemma 4.8]), and by

\[
\pi : (P, X') \to (G', \delta') := (P, X')/W
\]

the quotient of \((P, X')\) by \(W\), there exists then a canonical morphism

\[
(1.11) \quad i_g : S_{\pi(K')} := S_{\delta(K')} \to S_K^*
\]

whose image is locally closed, and identical to the stratum associated to \((P, X')\) and \(g\) ([P1, 7.6, Main Theorem 12.3 (c)]).

**Remark 1.3.1.1.** (1) (cfr. [P2, Section 3.6]) The group \(G(\mathbb{Q})\) acts on the collection of the data \((G', \delta')\) obtained from the rational boundary components. This action induces isomorphisms among such data; in particular, the isomorphisms between the \(G'\)'s are induced by conjugation. Two data \((G'_1, \delta_1), (G'_2, \delta_2)\) are conjugated if and only if the corresponding parabolics are conjugated.

(2) (cfr. [P2, Section 3.7]) The description in (1.11) shows that the strata of \(\partial S_K^*\) are indexed by (a quotient of the set \((\{(P, X'), g)\} | (P, X')\) rational boundary component, \(g \in G(A_f)\)). This quotient is in fact finite, and a set of representatives of the strata can be obtained as follows: let first \((G', \delta')\) vary amongst the \(G(\mathbb{Q})\)-conjugacy classes of data obtained from the rational boundary components, and then, for each fixed \((G', \delta')\), let \(g\) vary amongst a set of representatives of the (finite) double quotient

\[
Stab_{Q(\mathbb{Q})}(\delta)P(A_f)\backslash G(\mathbb{A})/K.
\]

**1.3.2. Strata as quotients of Shimura varieties.** Suppose that the pure Shimura datum \((G, X)\) satisfies condition (+) from [BuW, page 7] (this is in particular true for the Hilbert-Siegel datum of 1.2.1.2 by Remark 1.2.1.3). We can describe more explicitly the stratum \(i_g(S_{\pi(K')})\) of (1.11). In order to do so, define

\[
H_Q := \text{Stab}_{Q(\mathbb{Q})}(\delta) \cap P(A_f) \cdot K^g,
\]

\[
(1.12)
\]

\[
H_C := \text{Cent}_{Q(\mathbb{Q})}(\delta) \cap W(A_f) \cdot K^g,
\]

\[
(1.13)
\]

\[
\Delta := H_Q/P(\mathbb{Q})H_C,
\]

where we denoted by \(\text{Cent}_{Q(\mathbb{Q})}(\delta)\) the group of elements of \(\text{Stab}_{Q(\mathbb{Q})}(\delta)\) which induce the identity on \(\delta\). The results we are interested in can be then resumed as follows:

**Proposition 1.3.2.1.** a) The group \(\Delta\) is finite and acts naturally and freely on \(S_{\pi(K')}\).

b) The stratum \(i_g(S_{\pi(K')})\) equals the quotient of \(S_{\pi(K')}\) by the action of \(\Delta\) and is smooth over \(E(G, X)\).

**Proof.** Everything follows from [W4, Lemma 8.2] and its proof, remembering that \(K\) is neat by hypothesis. \(\square\)

**1.3.3. Explicit description of strata.** Let us now describe in detail the (pure) Shimura data underlying the strata \(\partial S_K^*\) in the case of the Hilbert-Siegel datum \((G, X)\) of Proposition 1.2.1.2.

Each admissible parabolic subgroup \(Q\) of \(G\) is conjugated to exactly one of the subgroups \(Q_0\) (Siegel parabolic) or \(Q_1\) (Klingen parabolic) defined in 1.1.2.

Denote respectively by \(F_0\) and \(F_1\) the canonical normal subgroups \(Q_0\) and \(Q_1\) considered in 1.3.1. Denote also by \(G_0\), resp. \(G_1\) their quotients by the respective unipotent radicals, and by \((G_0, \delta_0)\), resp. \((G_1, \delta_1)\) the associated Shimura data. An immediate generalisation to \(\text{Res}_{F_0|Q}GSp_{4,F}\) (and then to \(G\)) of [P1, 4.25] (which treats the case of \(GSp_4\)) gives us the following:

- The group \(G_0\) is identified with the factor \(G_m\) inside \(Q_0/W_0 \simeq G_m \times \text{Res}_{F_0|Q}GL_{2,F}\) (remember (1.7)). Moreover, let \(k\) be the morphism \(S \to G_{0,R}\) which induces on real points, via the above identification,

\[
\mathbb{S}(\mathbb{R}) \to G_{0,R}
\]

\[
(1.15)
\]

\[
z \mapsto \left(\begin{array}{cc}
z \overline{z} & I_2 \\
I_2 & I_2\end{array}\right)_{\sigma \in \mathcal{I}_F}
\]
and let \( \mathcal{S}_0 \) be the set of isomorphisms between \( \mathbb{Z} \) and \( \mathbb{Z}(1) \). Consider the unique transitive action of \( \pi_0(\mathbb{G}_m(\mathbb{R})) \) on \( \mathcal{S}_0 \) and denote by \( h_0 \) the constant map \( \mathcal{S}_0 \to \{k\} \subset \text{Hom}(\mathbb{S}, \mathbb{G}_0(\mathbb{R})) \). Then, the Shimura datum corresponding to \( G_0 \) is given by \((G_0, \mathcal{S}_0)\). Thus, \( G_0 \) contributes with 0-dimensional strata to \( \partial S_K^0 \).

- The group \( G_1 \) is identified with the factor \( \text{Res}_{F|Q} \mathbb{G}_m(\mathbb{R}) \times \text{Res}_{F|Q} \mathbb{G}_m(\mathbb{R}) \mathbb{G}_m \) inside 

  \[
  Q_1/W_1 \simeq (\text{Res}_{F|Q} \mathbb{G}_m(\mathbb{R}) \times \text{Res}_{F|Q} \mathbb{G}_m(\mathbb{R})) \mathbb{G}_m
  \]

  (remember (1.8)). Denoting by \( \mathcal{S}_1 \) the \( G_1(\mathbb{R}) \)-conjugacy class of the morphism

  \[
  h_1 : S(\mathbb{R}) \to G_1(\mathbb{R})
  \]

  \[
  x + iy \mapsto \left( \begin{array}{c}
  x^2 + y^2 & x \\
  x & y \\
  -y & x 
  \end{array} \right)_{\sigma \in I_F}
  \]

  the Shimura datum corresponding to \( G_1 \) is then given by \((G_1, \mathcal{S}_1)\). Thus, \( G_1 \) contributes with \( d \)-dimensional strata to \( \partial S_K^d \). The description of the Shimura datum shows that these strata are in particular isomorphic to (quotients by the action of a finite group of) Hilbert-Blumenthal varieties.

By Remark 1.3.1.1, each stratum of \( \partial S_K^d \) corresponds to a Shimura datum of one of the above types. In particular, it is either of dimension 0 (and it will be then called a Siegel stratum) or of dimension \( d \) (and it will be then called a Klingen stratum).

### 1.4. The canonical construction functor and its motivic version.

#### 1.4.1. Conventions for Hodge structures.

Let \( w : \mathbb{G}_{m,R} \to \mathbb{S} \) be the cocharacter which induces the inclusion \( \mathbb{R}^+ \hookrightarrow \mathbb{C}^\times \) on real points. A representation \( (\rho, V) \) of \( \mathbb{S} \) induces a (semisimple) mixed Hodge structure on the real vector space \( V \); coherently with the convention of [P1] 1.3, we will say that the subspace of \( V \) where \( \rho \circ w \) acts as multiplication by \( t^{-k} \) is the quotient of \( V \) of weight \( k \).

Moreover, if \((\mathcal{G}, \mathfrak{X})\) is a Shimura datum, defined by \( h : \mathfrak{X} \to \text{Hom}(\mathbb{S}, \mathbb{G}_b) \), then every representation \( \rho : \mathcal{G} \to \text{GL}(V) \) gives rise, for all \( x \in \mathfrak{X} \), to a Hodge structure on \( V \), by applying the above observation to \( \rho \circ h(x) \circ w \).

#### 1.4.2. The canonical construction functor.

The axioms which define Shimura data are essentially designed in order to get a sheaf-theoretic version of the construction in 1.4.1:

**Proposition 1.4.2.1.** (cfr. [WIII, Chap. 2 and 4])

Let \((\mathcal{G}, \mathfrak{X})\) be a Shimura datum satisfying condition (+) from [BuW, page 7], \( K \) a neat compact open subgroup of \( \mathcal{G}(\mathbb{A}_f) \), and \( S_K \) the corresponding Shimura variety, defined over its reflex field \( E(\mathcal{G}, \mathfrak{X}) \). Let \( R \) be a subfield of \( \mathbb{R} \), and denote by \( \text{Rep}(\mathcal{G}_R) \) the Tannakian category of algebraic representations of \( \mathcal{G} \) in finite dimensional \( R \)-vector spaces.

1. There exists an exact tensor functor (called Hodge canonical construction)

   \[
   \mu^K_H : \text{Rep}(\mathcal{G}_R) \to \text{MVar}_R(S_K(\mathbb{C})),
   \]

   where \( \text{MVar}_R(S_K(\mathbb{C})) \) is the category of graded-polarizable admissible variations of mixed \( R \)-Hodge structure over \( S_K(\mathbb{C}) \).

2. Let \( \ell \) be a prime and \( R \) be finite over \( \mathbb{Q} \) and fix a prime \( l \) of \( R \) above \( \ell \). There exists an exact tensor functor (called \( \ell \)-adic canonical construction)

   \[
   \mu^K_{\ell} : \text{Rep}(\mathcal{G}_R) \to \text{Et}_{\ell,R}(S_K),
   \]

   where \( \text{Et}_{\ell,R}(S_K) \) is the \( \ell \)-linear version of the category of lisse \( \ell \)-adic sheaves over \( S_K \).

**Remark 1.4.2.2.**

1. If \((\mathcal{G}, \mathfrak{X})\) is a Shimura datum of abelian type, and \( S_K \) any of the corresponding Shimura varieties, then the functor \( \mu^K_{\ell} \) takes values in the full subcategory \( \text{Et}^M_{\ell,R}(S_K) \) of \( \text{Et}_{\ell,R}(S_K) \) formed by the mixed lisse sheaves with weight filtration, in the sense of [W11, Definition before Theorem 2.8]. Moreover, if \( V \in \text{Rep}(\mathcal{G}_R) \), then the weights of \( \mu^K_{\ell}(V) \) in the sense of the latter definition are identical to the weights of \( \mu^K_H(V) \) as a variation of mixed Hodge structure. These facts follow from [P2, Proposition (5.6.2)]. The hypothesis is in particular satisfied by PEL type Shimura data, and so by the data \((G, X), (G_0, \mathcal{S}_0)\) and \((G_1, \mathcal{S}_1)\) defined in subsections 1.2 and 1.3.
(2) If \( G \) is the group defined in 1.1.1.2 and \( V_\lambda \) is an irreducible representation of \( G \) of highest weight \( \lambda = \lambda(k_\sigma, k'_\sigma, c) \), then \( \mu^K_H(V_\lambda) \), resp. \( \mu^K_t(V_\lambda) \), is a variation of Hodge structure, resp. an \( \ell \)-adic sheaf, pure of weight \( w(\lambda) := -c \) (cfr. the convention fixed in 1.4.1, which is extended to variations of Hodge structure in the obvious way).

Remark 1.4.2.3.  
(1) The functor \( \mu^K_t \) factorises uniquely through the category \( \text{Gr}_2 \text{Var}_R(S_K(\mathbb{C})) \) (where \( \text{Var}_R(S_K(\mathbb{C})) \) is the category of pure polarizable variations of \( R \)-Hodge structure), in such a way that, still denoting by \( \mu^K_t \) such factorization, \( (\mu^K_t(V))_n \) is the graded object of weight \( n \) of \( \mu^K_t(V) \). If \( (\mathcal{G}, \mathcal{X}) \) is a Shimura datum of abelian type, then, by 1.4.2.2.(1), the functor \( \mu^K_t \) also factorises uniquely through the category \( \text{Gr}_2 \text{Et}_c(S_K) \), in such a way that, still denoting by \( \mu^K_t \) such factorization, \( (\mu^K_t(V))_n \) is the graded object of weight \( n \) of \( \mu^K_t(V) \).

(2) The exact functor \( \mu^K_t \) extends to a triangulated functor, denoted by the same symbol,

\[
\mu^K_t : D^b(\text{Rep}(\mathcal{G}_R)) \to D^b(S_K)_R,
\]

where \( D^b(S_K)_R \) is the \( R \)-linear version of the "derived" bounded category of \( \ell \)-adic constructible sheaves over \( S_K \) ([E, Section 6]).

1.4.3. **The motivic version of the canonical construction.** Let us adopt again the notation in the beginning of 1.3, applied to the Hilbert-Siegel Shimura datum \((G, X)\) of 1.2.1.2. Recall, for a base scheme \( X \) (say, for our purposes, a separated, finite type \( \mathbb{Q} \)-scheme), the triangulated \( R \)-linear category \( DM_{B,c}(X)_R \) of constructible Beilinson motives over \( X \) ([CD1]) and the \( \ell \)-adic realization functor \( R^\ell \) on it, with values in the category \( D^b(X)_R \) of (1.19) ([CD2, Sec. 7.2]). In our case, we will consider a base \( S \in \{S_K, S^K_\ast, \partial S^K_\ast\} \) or \( \in \{\text{strata of } \partial S^K_\ast\} \). Then, composition with the collection of cohomology functors, resp. perverse cohomology functors, \( R^\ast : D^b_c(S)_R \to \text{Gr}_2 \text{Et}_c(S), \) resp. \( H^\ast : D^b_c(S)_R \to \text{Gr}_2 \text{Perv}(\text{Et})_{\ell,R}(S) \) (where \( \text{Perv}(\text{Et})_{\ell,R}(S) \) is the \( R \)-linear version of the category of \( \ell \)-adic perverse sheaves over \( S \)), gives rise to the \( \ell \)-adic cohomological realization, resp. perverse cohomological realization functors.

Consider in particular the case \( S = S_K \). The \( \ell \)-linear, tensor pseudo-abelian category \( \text{CHM}(S_K)_R \) of Chow motives over \( S_K \) ([CH]) faithfully embeds into \( DM_{B,c}(S_K)_R \) (more on this at the beginning of subsection 2.1). If \( S = S_K \), the restriction of \( R^\ast \circ R^\ell \) to \( \text{CHM}(S_K)_R \) (still denoted by \( R^\ell \)) equals the usual \( \ell \)-adic realization on this category; over \( \text{CHM}(S_K)_R \), there also exists the \textit{Hodge cohomological realization}, with values in \( \text{Gr}_2 \text{Var}_R(S_K(\mathbb{C})) \), denoted by \( R^H \) (for these realization functors in the context of relative Chow motives, cfr. [A, Sec. 3]).

Recall now the universal family \( p : A_K \to S_K \) from Remark 1.2.2.2.(2). The following result, valid for every \( \text{PEL} \)-type Shimura variety, is crucial:

**Theorem 1.4.3.1.** ([A, Thm. 8.6], stated as in [W7, Thm. 1.4])

Let \( R \) be a subfield of \( \mathbb{R} \). There exists a \( R \)-linear tensor functor

\[
\tilde{\mu} : \text{Rep}(G_R) \to \text{CHM}(S_K)_R
\]

with the following properties:

(1) The composition of \( \tilde{\mu} \) with the cohomological Hodge realization is isomorphic to \( \mu^K_H \) (cfr. 1.4.2.3. (1)).

(2) For all prime \( \ell \), the composition of \( \tilde{\mu} \) with the \( \ell \)-adic cohomological Hodge realization is isomorphic to \( \mu^K_\ell \) (cfr. 1.4.2.3. (1)).

(3) \( \tilde{\mu} \) commutes with Tate twists.

(4) If \( V \) is the standard \( G_R \)-representation on \( \mathbb{C}^{2g+4} \), then \( \tilde{\mu} \) sends \( V \) to the dual of the Chow motive \( p^! \mathbb{1}_{A_K} \) over \( S_K \) (the first Chow-Künneth component of the Chow motive \( p^! \mathbb{1}_{A_K} \) sur \( S_K \), cfr. [DM, Thm. 3.1]).

**Remark 1.4.3.2.** For all positive integer \( n \), let \( p_n : A^K_n \to S_K \) be the \( n \)-fold fibred product of \( A_K \) with itself over \( S_K \). Then, since the tensor product \( V \otimes V^{\vee} \) of the standard representation \( V \) with its dual is a tensor generator for the Tannakian category \( \text{Rep}(G_R) \) ([DeM, Prop. 2.20]), Theorem 1.4.3.1 implies that every object in the essential image of \( \tilde{\mu} \) is isomorphic to a finite direct sum \( \bigoplus M_i \), where each \( M_i \) is a direct factor of a Tate twist of a Chow motive of the form \( p_{n_i} \mathbb{1}_{A^K_{n_i}} \), for suitable \( n_i \)'s.

Let now \( V_\lambda \) be an irreducible \( L \)-representation of \( G_L \) of highest weight \( \lambda \), where the latter is as in 1.1.6.
Remark 1.4.3.4. (1) Since Definition 1.4.3.3. the Chow motive \( R(1.21) \) the notion of (cfr. Definition 2.1.0.2. (cfr. Definition 2.1.0.1. to the homonymous category introduced in that subsection. category of heart \( \text{DM} \) such that \( M \) avoidance (1.22) Statement of the main result. 2.1. of the motive \( D \). Let \( k \) (1) \( B,c \) decomposition \( \ell \) resp. \( \mu \). Fix a subfield \( R \) of \( k(1) \) is equipped with a canonical \( D \) resp. \( V \). Then, after \([Fa]\), this category is equivalent \( R \) as in subsection 1.4.3, then, after \([Fa]\), this category is equivalent \( \text{CHM}(X)_R \) and called the \( (R\text{-linear version of the}) \) category of Chow motives over \( X \). If \( X = S_K \) as in subsection 1.4.3, then, after \([Fa]\), this category is equivalent to the homonymous category introduced in that subsection.

Definition 1.4.3.3. The Chow motive \( \lambda^V \) over \( S_K \) is defined by
\[
\lambda^V := \check{\mu}(V_\lambda).
\]

Remark 1.4.3.4. (1) Since \( \mu^H_\ell(V_\lambda) \) and \( \mu^F_\ell(V_\lambda) \) are pure of weight \( w(\lambda) \) (cfr. 1.4.2.2.(2)), the Hodge, resp. \( \ell \)-adic cohomological realizations of \( \lambda^V \) are zero in degree \( \neq w(\lambda) \), and identical to \( \mu^H_\ell(V_\lambda) \), resp. \( \mu^F_\ell(V_\lambda) \), in degree \( w(\lambda) \).

(2) Since \( S_K \) is a variety of dimension \( 3d \), (1) can be reformulated by saying that the perverse Hodge, resp. \( \ell \)-adic cohomological realizations are zero in perverse degree \( \neq w(\lambda)+3d \), and identical to \( \mu^H_\ell(V_\lambda) \), resp. \( \mu^F_\ell(V_\lambda) \), in perverse degree \( w(\lambda)+3d \).

(3) Let \( \mathbb{D}_{\ell,S_K} \) denote the \( \ell \)-adic local duality endofunctor over \( S_K \). Then
\[
R_{\ell,S_K}(\lambda^V) = \mu^K_\ell(V_\lambda)[-w(\lambda)]
\]
and \( \mathbb{D}_{\ell,S_K}(R_{\ell,S_K}(\lambda^V)) \cong R_{\ell,S_K}(\lambda^V)(w(\lambda)+3d)[2w(\lambda)+6d] \).

Proof. The first two assertions and (1.21) are immediate consequences of what we said above.

For the second part of the third assertion, notice first that, since \( S_K \) is smooth of dimension \( 3d \), there exists an isomorphism \( \mathbb{D}_{\ell,S_K}(\mu_\ell(V_\lambda)) \cong RHom(\mu_\ell(V_\lambda), L \otimes_{Q \ell}(3d)[6d]) \). On the other hand, following \([P1, \text{Summary 1.18 (d)}]\), there exists a perfect pairing
\[
V_\lambda \otimes_L V_\lambda \rightarrow L(-w(\lambda))
\]
in \( \text{Rep}(G_L) \), so that one has a perfect pairing
\[
\mu_\ell(V_\lambda) \otimes_{L \otimes_{Q \ell}} \mu_\ell(V_\lambda) \rightarrow L \otimes_{Q \ell}(-w(\lambda))
\]
of lisse \( \ell \)-adic sheaves over \( S_K \), which gives a second isomorphism
\[
\mu_\ell(V_\lambda) \cong RHom(\mu_\ell(V_\lambda), L \otimes_{Q \ell}(-w(\lambda))).
\]
These isomorphisms then imply that
\[
\mathbb{D}_{\ell,S_K}(\mu_\ell(V_\lambda)) \cong \mu_\ell(V_\lambda)(w(\lambda)+3d)[6d].
\]
So, the desired isomorphism comes from (1.21). \( \square \)

2. The degeneration at the boundary of the canonical construction

In this section we prove our main result (Thm. 2.1.0.3), i.e. the description of the interval of weight avoidance of the motive \( i^*j_!^!\lambda^V \in DM_{B,c}(\partial S_K^*)_L \) in terms of the corank of \( \lambda \).

2.1. Statement of the main result. In order to state our central theorem, we need the key notion of weight avoidance. Fix a subfield \( R \) of \( \mathbb{R} \) and a scheme \( X \) of finite type over \( \mathbb{Q} \): according to \([He]\), the category \( DM_{B,c}(X)_R \) (see 1.4.3) is equipped with a canonical weight structure, the motivic weight structure, whose heart (the subcategory of weight 0 objects) is denoted by \( \text{CHM}(X)_R \) and called the \( (R\text{-linear version of the}) \) category of Chow motives over \( X \). If \( X = S_K \) as in subsection 1.4.3, then, after \([Fa]\), this category is equivalent to the homonymous category introduced in that subsection.

Definition 2.1.0.1. (cfr. \([W2, \text{Defs. 1.6-1.10}]\)) Let \( M \in DM_{B,c}(X)_R \) and let \( \alpha, \beta \) be integers. We say that \( M \) avoids weights \( \alpha, \ldots, \beta \) if \( \alpha \leq \beta \) and there exists an exact triangle in \( DM_{B,c}(X)_R \)
\[
M_{\leq \alpha-1} \rightarrow M \rightarrow M_{\geq \beta+1} \rightarrow M_{\leq \alpha-1}[1]
\]
such that \( M_{\leq \alpha-1} \) is of weight at most \( \alpha - 1 \) and \( M_{\geq \beta+1} \) of weight at least \( \beta + 1 \).

Such a triangle is called a weight filtration of \( M \) avoiding weights \( \alpha, \ldots, \beta \).

Recall now the notations used and introduced in 1.4.3. We will need some more notions about \( \lambda \), especially the notion of corank (see the introduction for a motivation in the context of automorphic forms):

Definition 2.1.0.2. (cfr. \([BR, \text{Def. 1.9}]\)) Let \( \lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F, c}) \) (cfr. 1.1.6) be a weight of \( G_L \).

(1) \( k_1 := (k_\sigma)_{\sigma \in I_F} \) or \( k_2 := (k'_\sigma)_{\sigma \in I_F} \) is called parallel if \( k_{\sigma_1}^{\lambda} \) is constant on \( I_F \), equal to a positive integer \( \kappa \) (and we write \( k_1 = k_2 \)). The weight \( \lambda \) is called \( \kappa \)-Kostant parallel if there exist a \( \kappa \in \mathbb{Z} \) and a decomposition \( I_F = I'_F \sqcup I''_F \) such that
\[
\begin{cases} 
  k_\sigma = \kappa & \forall \sigma \in I'_F \\
  k'_\sigma = \kappa + 1 & \forall \sigma \in I''_F 
\end{cases}
\]
(2) We say that $\lambda$ has corank $q$ (and we write $\text{cor}(\lambda) = q$) if

$$q = |\{1 \leq i \leq 2|k_i = k_2\}|$$

and $k_2$ is parallel. If there is no such $q$, we say that the corank is 0.

(3) $\lambda$ is completely irregular if $(k_\sigma, k'_\sigma)$ is irregular for all $\sigma \in I_F$.

Assume $\lambda$ to be dominant. Notice then that if $\text{cor}(\lambda) = 2$, then $\lambda$ is completely irregular, and that if $\text{cor}(\lambda) \geq 1$, then $\lambda$ is $\kappa$-Kostant parallel with respect to the decomposition $I_F = I^1_F$, with $k_2 = k_2^\alpha$: this decomposition and this $\kappa$ are then the only ones with respect to which $\lambda$ is Kostant-parallel. Analogously, if $\text{cor}(\lambda) = 0$, there are at most a $\kappa$ and a decomposition $I_F = I^0_F \sqcup I^1_F$ with respect to which $\lambda$ is $\kappa$-Kostant parallel. For a motivation for the terminology Kostant parallel, see subsection 2.4.2.

We can now state our main result:

**Theorem 2.1.0.3.** Let $V_\lambda$ the irreducible representation of $G_L$ of highest weight $\lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, \epsilon$), $S_K$ the genus 2 Hilbert-Siegel variety of level $K$ corresponding to $(G, X)$ and $^{\lambda}V \in CHM(S_K)$ the Chow motive over $S_K$ introduced in Definition 1.4.3.3. Let moreover $j : S_K \to S^*_K$, resp. $i : \partial S^*_K \to S^*_K$ denote the open, resp. closed immersion in the Baily-Borel compactification $S^*_K$ of $S_K$. Then:

1. If $\lambda$ is not $\kappa$-Kostant parallel for any $\kappa \in \mathbb{Z}$, then the boundary motive $i^* j^*_k V$ is zero.

2. If $\text{cor}(\lambda) = 0$ and $\lambda$ is $\kappa$-Kostant parallel with respect to $(I^\alpha_F \neq \emptyset, I^1_F)$, then $i^* j^*_k V$ avoids weights $-d_1 = d_\kappa + 1, \ldots, d_1 + d_\kappa$, where $d_1 := |I^1_F|$. The weights $-d_1 = d_\kappa, d_1 + d_\kappa + 1$ do appear in $i^* j^*_k V$.

3. If $\text{cor}(\lambda) = 1$, with $k_2 = \kappa$, and $k_1$ is not constant on $I_F$, then $i^* j^*_k V$ avoids weights $d_\kappa + 1, \ldots, d_\kappa$. The weights $-d_\kappa + 1$ do appear in $i^* j^*_k V$.

4. If $\text{cor}(\lambda) \geq 1$ with $k_2 = \kappa_2$ and if moreover $k_1 = \kappa_1$, then, by posing $\kappa := \min\{\kappa_1 - \kappa_2, \kappa_2\}$, we have that $i^* j^*_k V$ avoids weights $-d_\kappa + 1, \ldots, d_\kappa$. The weights $-d_\kappa + 1$ do appear in $i^* j^*_k V$, and if $\kappa_1$, $\kappa_2$ have the same parity, then the weights $-d(\kappa_1 - \kappa_2), d(\kappa_1 - \kappa_2) + 1$ do appear in $i^* j^*_k V$.

Theorem 2.1.0.3 will be proved at the end of subsection 2.6.2.2. Admitting this theorem for the moment, we can prove its most important corollary for the applications to the intersection motive:

**Corollary 2.1.0.4.** The weights 0 and 1 appear in the boundary motive $i^* j^*_k V$ if and only if $\lambda$ is completely irregular of corank $\geq 1$.

**Proof.** Suppose $\lambda$ to be $\kappa$-Kostant parallel with respect to $(I^\alpha_F, I^1_F)$ (otherwise, by point (1) of the above theorem, there is nothing to do).

If $\text{cor}(\lambda) = 0$, then, by point (2) of the above theorem, the weights 0 and 1 appear if and only if $d_1 = 0 = \kappa$. But $d_1 = 0$ means that $I^\alpha_F = \emptyset$, i.e. $I^\alpha_F = I_F$, and by definition of Kostant-parallelism this implies $k_1 = \kappa$. Now, necessarily $\kappa > 0$, because otherwise $k_2 = 0$ (remember that $k_\sigma \geq k'_\sigma$ for all $\sigma \in I_F$) and $\text{cor}(\lambda) = 2$, a contradiction.

If $\text{cor}(\lambda) = 1$, with $k_2 = \kappa_2$, then, by point (3) and (4) of the above theorem, the weights 0 and 1 appear if and only if $k_2 = 0$: observe in fact that, even if $k_1 = \kappa_1$, we have $\kappa_1 - \kappa_2 > 0$ (otherwise $\text{cor}(\lambda) = 2$, a contradiction). But if $k_2 = 0$, $\lambda$ is completely irregular.

If $\text{cor}(\lambda) = 2$, then $k_1 = \kappa = k_2$; this means that $\lambda$ is completely irregular, and implies that, in point (4) of the above theorem, the parity condition is trivially satisfied and that $k_1 - k_2 = 0$, so that the weights 0 and 1 appear.

To conclude, we only have to observe that if $\text{cor}(\lambda) \geq 1$ and $\lambda$ is completely irregular, either $k_2 = 0$ or $k_1 = \kappa = k_2$. □

For a discussion of the definition of the intersection motive when the weights 0 and 1 are avoided, of its properties, and of the applications to the construction of motives associated to automorphic representations, we refer to Section 3. The rest of this section is occupied by the proof of Theorem 2.1.0.3.

The strategy of proof is based on two important points. The first one relies on the fundamental notion of motive of abelian type:

**Definition 2.1.0.5.** (Cfr. [W8, Def. 2.1]) An object $M \in DM_{B,c}(\partial S^*_K)_L$ is said to be a motive of abelian type over $\partial S^*_K$, and a stratification $\Phi$ of $\partial S^*_K$ is said to be adapted to $M$, if the following condition is verified:

---

4Cfr. Footnote 6 for this supplementary hypothesis.
the motive $M$ belongs to the strict, full, dense, L-linear triangulated subcategory $DM_{B,c}^a(\partial S_K^*)_L$ generated by the images via $\pi_*$ of the $\mathcal{S}$-constructible Tate motives ([W4, Def. 4.6 (a)]) over $S(\mathcal{S})$, where

$$\pi : S(\mathcal{S}) \to \partial S_K^*$$

runs over the morphisms of abelian type ([W4, page 579]) whose target is $\partial S_K^*$.

Consider now the structure of $\partial S_K^*$. By 1.3.1, $\partial S_K^*$ admits a natural stratification $\Phi^i$ by locally closed strata, which has been made explicit in 1.3.3, but let us replace it with the following coarser stratification $\partial S_K^* = Z_0 \sqcup Z_1$, indexed by $\Phi : \{0, 1\}$: $i_0 : Z_0 \to \partial S_K^*$ denotes the immersion of the disjoint union of strata corresponding to the conjugacy class of the parabolic $Q_0$ (i.e. the closed, 0-dimensional Siegel strata in $\Phi^i$), whereas $i_1 : Z_1 \to \partial S_K^*$ denotes the immersion of the disjoint union of strata corresponding to the conjugacy class of the parabolic $Q_1$ (i.e. the open, $d$-dimensional Klingen strata in $\Phi^i$).

The first result we are interested in is then the following:

**Proposition 2.1.0.6.** The motive $\pi^*j^\lambda \mathcal{V} \in DM_{B,c}(\partial S_K^*)_L$ is of abelian type, and $\Phi$ is adapted to $\pi^*j^\lambda \mathcal{V}$.

**Proof.** The proof of [W8, Thm. 2.2] can be translated word by word in our setting. We include a sketch for the convenience of the reader: by Remark 1.4.3.2, $\lambda \mathcal{V}$ is an object of the $\mathbb{Q}$-linear triangulated category $p_{n,*}DMT(\mathcal{A}^n_K)^{\lambda \mathcal{V}}$ generated by the images via $p_{n,*}$ of the objects of the category of Tate motives over the fibred product $p_n : \mathcal{A}^n_K \to S_K$ of the universal abelian variety over $S_K$. Moreover, $\mathcal{A}^n_K$ is identified with a mixed Shimura variety $M_{K_n}$, the morphism $p_n$ is identified with a morphism induced by a morphism of mixed Shimura data, and, by choosing a suitable toroidal compactification $M_{K_n}(\mathcal{S})$ of $M_{K_n}$, one has that $p_n$ extends to a surjective, proper morphism $M_{K_n}(\mathcal{S}) \to S_K^*$ which is of abelian type, so that the category $p_{n,*}DMT_{\mathcal{S}}(p_{n,1}(\partial S_K^*))^{\lambda \mathcal{V}}$ generated by the images via $p_{n,*}$ of $\mathcal{S}$-constructible Tate motives over $p_{n,1}(\partial S_K^*)$ is contained inside $DM_{B,c}^{ab}(\partial S_K^*)_L$. Now, one sees that $\pi^*j^\lambda \mathcal{V} \in p_{n,*}DMT_{\mathcal{S}}(p_{n,1}(\partial S_K^*))^{\lambda \mathcal{V}}$.

The key fact is that, the motive $\pi^*j^\lambda \mathcal{V}$ being of abelian type, the study of its weight filtration will be reduced to the study of the weight filtration on its $\ell$-adic realization, thanks to *weight conservativity* of the $\ell$-adic realization functor ([W6]).

The second important point for the proof of Theorem 2.1.0.3 is then the reduction to a *stratum-by-stratum* study of the weights (over $\partial S_K^*$). Consider in fact the intermediate extension functor $j_n$, from the category $\text{Perv}(\text{Et})_{\ell,L}(S_K)$ to the category $\text{Perv}(\text{Et})_{\ell,L}(S_K^*)$ (cfr. 1.4.3). Then, [W7] gives us a criterion to determine the weights avoided by $\pi^*j^\lambda \mathcal{V}$.

**Theorem 2.1.0.7.** (cfr. [W7, Corollary (3.6)(b)])

Let $\beta \geq 1$ be an integer and fix a prime number $\ell$. For any stratum $Z$ of $\partial S_K^*$, denote by $\mathcal{H}^n$ the $n$-th perverse cohomology functor on $D^b_c(Z)_L$ and write $j_{n,*}(R\ell(\lambda \mathcal{V}))$ for $(j_{n,*}(R\ell(\lambda \mathcal{V}))[w(\lambda) + 3d]) [w(\lambda) - 3d]$. The following assertions are then equivalent:

1. The motive $\pi^*j^\lambda \mathcal{V}$ avoids weights $-\beta + 1, -\beta + 2, \ldots, \beta$;
2. For all $n \in \mathbb{Z}$, $\mathcal{H}^n\pi^*i^\ast j_{n,*}(R\ell(\lambda \mathcal{V}))$ and $\mathcal{H}^n\pi^*i^\ast j_{n,*}(R\ell(\lambda \mathcal{V}))$ are of weights $\leq n - \beta$.

**Proof.** (Sketch) The $\ell$-adic realization functors are compatible with (exceptional) direct and inverse images. Moreover, by [W6], they are weight conservative on the category of motives of abelian type. Then, one uses in particular the auto-duality (up to a twist and a shift) of the motive $\lambda \mathcal{V}$ (cfr. Remark 1.4.3.4.(3)) and the auto-duality properties of the functor $j_n$, together with Remark 1.4.3.4.(2), to get the desired result. We refer to [W7] for details.

This theorem tells us that we have to analyse the weights of the perverse sheaves $\mathcal{H}^n\pi^*i^\ast j_{n,*}(R\ell(\lambda \mathcal{V}))$, for $m \in \{0, 1\}$. This is the aim of the rest of this section.

### 2.2. Pink’s and Kostant’s theorems.

#### 2.2.1. Pink’s theorem.

Let $j : S_K \to S_K^*$ be the open immersion of a Shimura variety $S_K$, associated to a datum $(X, G)$ and to a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, into its Baily-Borel compactification. Recall the finite stratification $\Phi$ of $\partial S_K^*$ introduced in 1.3.1 and fix a numbering $\Phi \subset \mathbb{N}$ of the conjugacy classes of admissible $\mathbb{Q}$-parabolics in $G$. For $m \in \Phi$, denote by $\iota_m : Z_m \to \partial S_K^*$ the locally closed immersion of the disjoint union of strata inside $\Phi^i$ who correspond to parabolics in the $m$-th class. This identifies $\Phi$ with a new stratification of $\partial S_K^*$, coarser than $\Phi^i$, that in the case of the Hilbert-Siegel Shimura datum of 1.2.1.2 gives back the situation of 2.1.
Let us now adopt the notations for the various groups, spaces and morphisms defined in 1.3.2.1, changing them in the following way: if an object corresponds to a stratum $Z$ of $\Phi'$ which contributes to $Z_m$, then suppress any superscript $'$ and add a subscript $m$, so that, in particular, $Z$ will be written as the quotient of a Shimura variety $S_{\pi_m(K_m)}$, corresponding to a datum $(G_m,\mathcal{H}_m)$, by the action of the finite group $\Delta_m = H_{Q_m/P_m(Q)}H_{C_m}$.

Denote again by $\pi_m$ the projection $Q_m \to Q_m/W_m$, and define

$$\Gamma_m := \pi_m(H_{C_m}).$$

It is an arithmetic subgroup $Q_m/W_m(Q)$, which is moreover torsion-free (because $K$ is neat).

**Remark 2.2.1.1.** $\Gamma_m$ is such that $\Gamma_m \cap G_m(Q) = \{1\}$ (cfr. [BuW, Sec. 2]). Since $Q_m/W_m$ is reductive, there exists a complement $M_m$ of $G_m$ inside $Q_m/W_m$, i.e. a normal subgroup $M_m$ of $Q_m/W_m$ which is connected and reductive and such that $Q_m/W_m \cong G_m \cdot M_m$, with $G_m \cap M_m$ finite. We will then see $\Gamma_m$ as a subgroup of $M_m(Q)$.

Denote now by $\mu^K_\ell, \mu_{\pi_m(K_m)}$ the extensions of the $\ell$-adic canonical construction functors introduced in Remark 1.4.2.3.(2), and by $R^n$ the $n$-th classical, i.e. non-perverse, cohomology functor on the category $D^b_c(Z)_L$ (see 1.4.3), for any stratum $Z$ of $\partial S^*_K$. Our first main tool for the analysis of the weights is the following theorem of Pink's:

**Theorem 2.2.1.2.** ([P2, Thms. (4.2.1)-(5.3.1)], re-stated in the shape of [BuW, Thms. 2.6-2.9])

Let $R$ be a subfield of $\mathbb{R}$, $V \in D^b(\text{Rep}_R(G))$ and $m \in \Phi$. Let $Z$ be a stratum of $\Phi'$ contributing to $Z_m$.

1. There exists a canonical isomorphism in $D^b_c(Z)_R$

$$i^*_m j_* ^*\mu^K_\ell(V)|_Z \simeq \bigoplus_n R^n i^*_m j_* ^*\mu^K_\ell(V)|_Z[-n].$$

2. For all $n$, there exists a canonical and functorial isomorphism in $\text{Et}_{\ell,R}(Z)$

$$R^n i^*_m j_* ^*\mu^K_\ell(V)|_Z \simeq \bigoplus_{p+q=n} \mu_{\pi_m(K_m)}(\text{Et}_{\ell,R}(\Gamma_m, H^q(W_{m,R}, V))).$$

3. Suppose that the datum $(G_m, \mathcal{H}_m)$ is of abelian type. Then, denoting by $\mathcal{W}$ both the weight filtration in the sense of Remark 1.4.2.2.(1) and the one introduced in 1.4.1 (applied to representations of $G_{m,R}$), the sheaf $R^n i^*_m j_* ^*\mu^K_\ell(V)|_Z$ is the direct sum of its weight-graded objects (in particular, it is a semisimple object) and there exist canonical and functorial isomorphisms in $\text{Et}_{\ell,R}(Z)$

$$\text{Gr}^\mathcal{W}_n R^n i^*_m j_* ^*\mu^K_\ell(V)|_Z \simeq \bigoplus_{p+q=n} \mu_{\pi_m(K_m)}(\text{Gr}^\mathcal{W}_n H^p(\Gamma_m, \text{Gr}^\mathcal{W}_n H^q(W_{m,R}, V))).$$

In order to explain the above statements, some remarks are in order:

**Remark 2.2.1.3.**

1. Reasoning as in [BuW], before Definition 2.2, we see that the functor $\mu_{\pi_m(K_m)}$, a priori with values in $\text{Et}_{\ell,R}(S_{\pi_m(K_m)})$, gives rise to a functor with values in $\text{Et}_{\ell,R}(Z)$, still denoted by the same symbol.

2. $(Q_m/W_m)_R$ (seen as a subgroup of $Q_{m,R}$) acts on $H^q(W_{m,R}, V)$ via its action on $W_m$ and on $V$, and so it acts on $H^p(\Gamma_m, H^q(W_{m,R}, V))$. Hence, the latter space is seen as a representation of $G_{m,R}$ via the inclusion $G_{m,R} \subset (Q_m/W_m)_R$.

3. The statement of point 2.2.1.2(3) contains in particular the fact that $\text{Gr}^\mathcal{W}_n H^p(\Gamma_m, H^q(W_{m,R}, V)) \simeq H^p(\Gamma_m, \text{Gr}^\mathcal{W}_n H^q(W_{m,R}, V)).$}

2.2.2. **Kostant’s theorem.** The second ingredient for the analysis of the weights is a theorem of Kostant which allows one to make explicit the $(Q_m/W_m)_R$-representations $H^q(W_{m,R}, V)$ appearing in Theorem 2.2.1.2.

Fix a split reductive group $G$ over a field of characteristic zero and with root system, resp. Weyl group, $\Phi$, resp. $W$. Denote by $\tau^+$ the set of positive roots and fix moreover a parabolic subgroup $Q$ with its unipotent radical $W$. Let $W$ be the Lie algebra of $W$ and $\mathfrak{r}_W$ the set of roots appearing inside $W$ (necessarily positive). For all $w \in W$, we define:

$$\tau^+(w) := \{ \alpha \in \mathfrak{r}_W^{-1} \alpha \notin \tau^+ \},$$

$$\text{Gr}^\mathcal{W}_n H^p(\Gamma_m, H^q(W_{m,R}, V)) \simeq H^p(\Gamma_m, \text{Gr}^\mathcal{W}_n H^q(W_{m,R}, V)).$$
at the same time.

\[ l(w) := |r^+(w)|, \]

\[ \mathcal{W}' := \{ w \in \mathcal{W} | r^+(w) \subset \tau_W \}. \]

We can now state Kostant’s theorem:

**Theorem 2.2.2.1.** ([V, Thm. 3.2.3])

Let \( \mathcal{V}_\lambda \) be an irreducible \( G \)-representation of highest weight \( \lambda \), and let \( \rho \) be the half-sum of the positive roots of \( G \). Then, as \((Q/W)\)-representations,

\[ H^q(W, \mathcal{V}_\lambda) \simeq \bigoplus_{w \in \mathcal{W}' | l(w) = q} U_{w, (\lambda + \rho) - \rho}, \]

where \( U_\mu \) denotes an irreducible \((Q/W)\)-representation of highest weight \( \mu \).

**Remark 2.2.2.2.**

1. Recall our fixed Galois closure \( L \) of the totally real field \( F \) and the group \( G \) underlying the Hilbert-Siegel Shimura datum of 1.2.1.2. We have seen in 1.1.5 that the root system of \( G_L \) is given by \( r = \bigcup_{\sigma \in I_F} r_\sigma \) and that every component \( r_\sigma \) contains two simple roots \( \rho_1, \rho_2 \); the other positive roots in such a component are then given by \( \rho_1 + \rho_2 \) and \( 2\rho_1 + \rho_2 \).

2. By 1.1.5, the Weyl group of \( G_L \) is given by \( \mathcal{W} = \prod_{\sigma \in I_F} \mathcal{W}_\sigma \). Denote by \( \mathcal{W}'_m \), for \( m \in \{0,1\} \), the sets defined in (2.4), corresponding to the choices \( G = G_L \) and \( Q = Q_{m,L} \) (cfr. 1.1.2). In both cases, we will see that, for all \( \sigma \in I_F \), there exist sets \( \mathcal{W}'_{m,\sigma} = \{ w_\sigma^i \}_{i=0,...,3} \subset \mathcal{W}_\sigma \) such that \( l(w_\sigma^i) = j \iff i = j \) and that \( \mathcal{W}'_m = \prod_{\sigma \in I_F} \mathcal{W}'_{m,\sigma} \). This means that every \( w = (w_\sigma)_{\sigma \in I_F} \in \mathcal{W}'_m \) determines a partition\(^5\)

\[ I_F = \bigsqcup_{i=0,...,3} I_F^i \]

where \( I_F^i := \{ \sigma \in I_F | w_\sigma = w_\sigma^i \} \), and that, since \( l((w_\sigma)_{\sigma \in I_F}) = \sum_{\sigma \in I_F} l(w_\sigma) \), one has \( 0 \leq l(w) \leq 3d \) for all \( w \in \mathcal{W}'_m \).

For all integer \( q \in \{0,\ldots,3d\} \), the above considerations imply that there exists a bijection between the set \( \{ w \in \mathcal{W}'_m | l(w) = q \} \) and the set of \( q \)-admissible partitions of \( I_F \)

\[ (2.5) \quad \mathcal{P}_q := \{ \text{partitions } I_F = \bigsqcup_{i=0,...,3} I_F^i | \text{there exists a 4-tuple of non-negative integers } (d_0, d_1, d_2, d_3) \text{ such that } |I_F^i| = d_i \text{ and } d_1 + 2d_2 + 3d_3 = q \}. \]

2.3. **The degeneration along the Siegel strata.** With notation as in the statement of Thm. 2.1.0.3, fix an irreducible \( G_L \)-representation \( \mathcal{V}_\lambda \) of highest weight \( \lambda = \lambda(k, \kappa'_c) \sigma_{\in I_F}, c \text{ (cfr. 1.3.3.3).} \) We want to employ Theorem 2.2.1.2 to study the degeneration of \( \mu^{|x|} L \lambda \) along the Siegel strata, whose underlying Shimura datum is \((G_0, \mathcal{H}_0)\), where \( G_0 \simeq G_m \), as explained in 1.3.3.

2.3.1. **Cohomology of the unipotent radical.** We begin by studying the cohomology spaces \( H^q(W_{0,L}, \mathcal{V}_\lambda) \), identified as \((Q_0/W_0)_{L\sigma}\)-representations by Theorem 2.2.1.2: we have \((Q_0/W_0)_{L\sigma} \simeq \mathbb{G}_m L \times \prod_{\sigma \in I_F} (GL_{2,L}\sigma)\), and

(recalling Definition (2.5)) an isomorphism

\[ (2.6) \quad H^q(W_{0,L}, \mathcal{V}_\lambda) \simeq \bigoplus_{\Psi \in \mathcal{P}_q} V_\Psi^{0,q}, \]

where each \( V_\Psi^{0,q} \) is an irreducible \((Q_0/W_0)_{L\sigma}\)-representation whose highest weight will now be computed (as a generalisation of the computation of [Le, Sec. 4.3]).

\[^5\text{We use the terminology } \text{partition} \text{ in the broad sense: in the following, the sets } I_F^i \text{ are allowed to be empty - although not all at the same time.} \]
Recall then the notation of Remark 2.2.2.2.1: by fixing a component $r_{\sigma}$ of the root system of $G_{L}$, one easily sees that the positive roots which are contained in such a component and which appear in the Lie algebra of $W_{0,L}$ are given by \{\rho_{1,\sigma} + \rho_{2,\sigma}, 2\rho_{1,\sigma} + \rho_{2,\sigma}, \rho_{2,\sigma}\}.

Coherently with the notation of 1.1.5, denote by $s_{\rho}$ the reflection, belonging to the Weyl group $W$, whose axis is orthogonal to the root $\rho$. By direct inspection of the action of the component $W_{\sigma}$ on $r_{\sigma}$, one sees that the elements of the sets $W_{0,\sigma}^{0} = \{w_{i,\sigma}^{0}\}_{i=0,\ldots,3}$ defined in Remark 2.2.2.2 (for $m = 0$) are given by

$$w_{0}^{0} = \text{id},$$

$$w_{1}^{0} = s_{\rho_{2,\sigma}},$$

$$w_{2}^{0} = s_{\rho_{1,\sigma}} s_{\rho_{2,\sigma}},$$

$$w_{3}^{0} = s_{\rho_{1,\sigma}} + s_{\rho_{2,\sigma}}$$

and that $W_{0}^{0} = \prod_{\sigma \in I_{F}} W_{0,\sigma}^{0}$.

Theorem 2.2.2.1 and the explicit computation of $w_{\lambda}^{0}(\lambda + \rho) - \rho$ for $w \in W_{0}^{0}$ now tell us that, in the isomorphism (2.6), the highest weight of the irreducible $(Q_{0}/W_{0})_{L}$-representation $V_{\Psi}^{0,q}$ is given by the restriction of the character

$$\lambda((\eta_{\sigma}, \eta_{\sigma}')_{\sigma \in I_{F}}, c),$$

where $(\Psi = (I_{F}^{0}, I_{F}^{1}, I_{F}^{2}, I_{F}^{3}))$

$$\eta_{\sigma} = \begin{cases} k_{\sigma} & \text{if } \sigma \in I_{F}^{0}, I_{F}^{1} \\ k_{\sigma}' - 1 & \text{if } \sigma \in I_{F}^{2} \\ -k_{\sigma}' - 3 & \text{if } \sigma \in I_{F}^{3} \end{cases}, \quad \eta_{\sigma}' = \begin{cases} k_{\sigma}' & \text{if } \sigma \in I_{F}^{0} \\ -k_{\sigma}' - 2 & \text{if } \sigma \in I_{F}^{1} \\ -k_{\sigma} - 3 & \text{if } \sigma \in I_{F}^{2} \\ -k_{\sigma} - 3 & \text{if } \sigma \in I_{F}^{3} \end{cases}$$

The $L$-points of the factor $G_{m,L}$ of $(Q_{0}/W_{0})_{L}$ are identified with the subgroup \{\left((\alpha I_{2}^{0}, I_{2}^{1}, I_{2}^{2}, I_{2}^{3})_{\sigma \in I_{F}} | \alpha \in L^{\times}\right)$ of $Q_{0}/W_{0}(L)$. Recalling (1.10), $G_{m,L}(L)$ then acts on $V_{\Psi}^{0,q}$ via the character

$$\alpha \mapsto \alpha^{|c| + \sum_{\sigma \in I_{F}} (\eta_{\sigma} + \eta_{\sigma}')},$$

and by the convention fixed in 1.4.1 and the definition in (1.15) of the Shimura datum $(G_{0}, \mathcal{S}_{0})$, this induces on $V_{\Psi}^{0,q}$ a pure Hodge structure of weight

$$w(\lambda) - \left[\sum_{\sigma \in I_{F}^{0}} (k_{\sigma} + k_{\sigma}') + \sum_{\sigma \in I_{F}^{1}} (k_{\sigma} - k_{\sigma}') - 2 - \sum_{\sigma \in I_{F}^{2}} (k_{\sigma} - k_{\sigma}') - 4 - \sum_{\sigma \in I_{F}^{3}} (k_{\sigma} + k_{\sigma}') + 6\right].$$

Notice for later use that if $V$ is the standard 2-dimensional $L$-representation of $GL_{2,L}$, the above computations imply that the representation obtained by restriction to the factor $\prod_{\sigma \in I_{F}} (GL_{2,L})_{\sigma}$ of $(Q_{0}/W_{0})_{L}$ is isomorphic to

$$\bigotimes_{\sigma \in I_{F}^{0}} \text{Sym}^{k_{\sigma} - k_{\sigma}'} V \otimes \text{det}^{k_{\sigma}'} \otimes \left(\bigotimes_{\sigma \in I_{F}^{1}} \text{Sym}^{k_{\sigma} + k_{\sigma}'} + 2V \otimes \text{det}^{-k_{\sigma}' - 2}\right) \otimes \left(\bigotimes_{\sigma \in I_{F}^{2}} \text{Sym}^{k_{\sigma} + k_{\sigma}'} + 2V \otimes \text{det}^{-k_{\sigma} - 3}\right) \otimes \left(\bigotimes_{\sigma \in I_{F}^{3}} \text{Sym}^{k_{\sigma} - k_{\sigma}'} V \otimes \text{det}^{-k_{\sigma}'}\right)$$

2.3.2. Cohomology of the arithmetic subgroup. Consider now the arithmetic group $\Gamma_{0}$ as defined in (2.1): we need to identify its cohomology spaces

$$H^{p}(\Gamma_{0}, H^{q}(W_{0,L}, V_{\lambda})) \simeq \bigoplus_{\Psi \in P_{q}} H^{p}(\Gamma_{0}, V_{\Psi}^{0,q})$$

and their weight-graded objects $G_{k}^{W} H^{p}(\Gamma_{0}, H^{q}(W_{0,L}, V_{\lambda})) \simeq \bigoplus_{\Psi \in P_{q}} H^{p}(\Gamma_{0}, G_{k}^{W} V_{\Psi}^{0,q})$ (cfr. Remark 2.2.1.3(3)). As the cohomological dimension of $W_{0,L}$ is $3d$, these spaces can be non-zero only for $q \in \{0, \ldots, 3d\}$. 
Remark 2.3.2.1. \( \Gamma_0 \) is identified with a neat (so, torsion-free) arithmetic subgroup \( \text{Res}_{F|Q} \text{GL}_2(F) = \text{GL}_2(F) \) (Remark 2.2.1.1). Let \( \pi \) be the projection \( \text{GL}_2(F) \to \text{GL}_2(F)/Z(\text{GL}_2(F)) \) and define \( \Gamma_{0,Z} := \Gamma_0 \cap Z(\text{GL}_2(F)) \) and \( \Gamma_0' := \pi(\Gamma_0) \) (non trivial, torsion-free arithmetic subgroups of \( Z(\text{GL}_2(F)) \) \( \simeq F^x \), resp. \( \text{PGL}_2(F) \)). Then, \( \Gamma_0 \) can be written as an extension

\[
(2.11) \quad 1 \to \Gamma_{0,Z} \to \Gamma_0 \xrightarrow{\pi} \Gamma_0' \to 1,
\]

and applying the Lyndon-Hochschild-Serre spectral sequence to this extension

\[
(2.12) \quad E_2 = H^r(\Gamma_0',H^s(\Gamma_{0,Z},V_\phi^{0,q})) \Rightarrow H^{r+s}(\Gamma_0,V_\phi^{0,q})
\]

we see that every subspace \( H^p(\Gamma_0,V_\phi^{0,q}) \) is (non-canonically) isomorphic to a direct sum

\[
(2.13) \quad \bigoplus_{r+s=\rho} U^{r,s}
\]

where every \( U^{r,s} \) is a subquotient of \( H^r(\Gamma'_0,H^s(\Gamma_{0,Z},V_\phi^{0,q})) \). Thus, if \( H^s(\Gamma_{0,Z},V_\phi^{0,q}) \) is zero for all \( s \), then \( H^p(\Gamma_0,V_\phi^{0,q}) \) is.

Now, \( \mathbb{G}_{m,F}(\mathbb{Z}) \simeq \mathbb{O}_F^\times \simeq \mathbb{Z}^{d-1} \times \mathbb{Z}/2\mathbb{Z} \) by Dirichlet’s unit theorem; on the other hand, the torsion-free group \( \Gamma_{0,Z} \) has a finite-index subgroup, which is again a finite-index subgroup of \( \mathbb{O}_F^\times \). The group \( \Gamma_{0,Z} \) is then isomorphic to \( \mathbb{Z}^{d-1} \), and by choosing generators \( \gamma_1, \ldots, \gamma_{d-1} \) it is identified with the subgroup

\[
(2.14) \quad \{ (t, \sigma(t), \sigma(t)^{-1}) \mid t = \gamma_1^{p_1} \cdots \gamma_{d-1}^{p_{d-1}}, p_1, \ldots, p_{d-1} \in \mathbb{Z} \} \hookrightarrow Q_0/W_0(L).
\]

Then, an element \( t = \gamma_1^{p_1} \cdots \gamma_{d-1}^{p_{d-1}} \in \Gamma_{0,Z} \) acts on \( V_\phi^{0,q} \) via multiplication by

\[
(2.15) \quad \prod_{\sigma \in \mathcal{I}_F} \sigma(t)^{\eta_1+\eta_2} = \prod_{\sigma \in \mathcal{I}_F} \sigma(t)^{k_1+k_2} \cdot \prod_{\sigma \in \mathcal{I}_E} \sigma(t)^{k_3+k_4} \cdot \prod_{\sigma \in \mathcal{I}_E} \sigma(t)^{-(k_1-k_2+4)} \cdot \prod_{\sigma \in \mathcal{I}_E} \sigma(t)^{-(k_3-k_4+6)}.
\]

Remark 2.3.2.2. The cohomological dimension of \( \Gamma_{0,Z} \simeq \mathbb{Z}^{d-1} \) is \( d-1 \). Since it acts by semisimple endomorphisms, by choosing free subgroups of decreasing rank of \( \Gamma_{0,Z} \) and by applying the Lyndon-Hochschild-Serre spectral sequence until reducing to cohomology of \( \mathbb{Z} \), we see that \( H^s(\Gamma_{0,Z},V_\phi^{0,q}) \) is non-zero for at least a \( s \in \{0, \ldots, d-1\} \) \( \iff \) it is non-zero for all \( s \in \{0, \ldots, d-1\} \) \( \iff \) the action of \( \Gamma_{0,Z} \) on \( V_\phi^{0,q} \) is trivial. More precisely, \( H^s(\Gamma_{0,Z},V_\phi^{0,q}) \simeq H^0(\Gamma_{0,Z},V_\phi^{0,q})^{(d-1)} \).

The following (standard) lemma holds, whose proof we include for the convenience of the reader:

Lemma 2.3.2.3. Let \( \mathcal{O}_F \) be the ring of integers of \( F \) and fix \( (0, \ldots, 0) \neq (n_1, \ldots, n_d) \in \mathbb{Z}^d \). Then,

\[
\prod_{i=1}^{d} |\sigma_i(t)|^{n_i} = 1 \quad \text{for all} \ t \in \mathcal{O}_F^\times.
\]

if and only if \( (n_1, \ldots, n_d) \in \mathbb{Z} \cdot (1, 1, \ldots, 1) \).

Proof. Choose a base of \( \{\gamma_1, \ldots, \gamma_{d-1}\} \) of \( \mathcal{O}_F^\times \) as \( \mathbb{Z} \)-module. By writing \( t = \prod_{0, \ldots, d-1} \gamma_1^{a_i} \) and choosing a \( d \)-tuple of integers \( (n_1, \ldots, n_d) \neq (0, \ldots, 0) \), we have

\[
\prod_{i=1}^{d} |\sigma_i(t)|^{n_i} = 1 \quad \forall \ t \iff \sum_{j=1}^{d-1} \left( \sum_{i=1}^{d} a_j n_i \log|\sigma_i(\gamma_j)| \right) = 0 \ \forall \ (a_1, \ldots, a_{d-1}) \neq (0, \ldots, 0) \in \mathbb{Z}^{d-1}
\]

\[
\iff \left( \begin{array}{c} a_1 \\ \vdots \\ a_{d-1} \end{array} \right) \cdot \Lambda \cdot \left( \begin{array}{c} n_1 \\ \vdots \\ n_d \end{array} \right) = 0
\]

\[
\iff \left( \begin{array}{c} \log|\sigma_1(\gamma_1)| & \ldots & \log|\sigma_d(\gamma_1)| \\ \vdots & \ddots & \vdots \\ \log|\sigma_1(\gamma_{d-1})| & \ldots & \log|\sigma_d(\gamma_{d-1})| \end{array} \right) \cdot \left( \begin{array}{c} n_1 \\ \vdots \\ n_d \end{array} \right) = 0,
\]

\[
\forall \ (a_1, \ldots, a_{d-1}) \neq (0, \ldots, 0) \in \mathbb{Z}^{d-1}
\]
(where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{d-1}$) $\iff (n_1, \ldots, n_d) \in \ker \Lambda$. But by Dirichlet’s unit theorem, $\ker \Lambda = \mathbb{R} \cdot (1, \ldots, 1)$.

**Remark 2.3.2.4.**

1. By choosing adapted bases, Lemma 2.3.2.3 generalises immediately to the case where $O_0^\times$ is replaced by a finite-index subgroup of $O_0^\times$, and furthermore to the case where it is replaced by an arithmetic subgroup of $F^\times$.

2. Consider the norm morphism $N : O_0^\times \to \{\pm 1\}$. As the image of a neat subgroup by a morphism is again neat, the elements of a neat subgroup of $O_0^\times$ are of norm 1. Thus, if the neat subgroup $\Gamma_{0,Z}$ is a finite-index subgroup of $O_0^\times$, we have that $\prod_{i=1, \ldots, d} |\sigma_i(t)|^{n_i} = 1 \ \forall \ t \in \Gamma_{0,Z} \iff \prod_{i=1, \ldots, d} |\sigma_i(t)|^{n_i} = 1 \ \forall \ t \in \Gamma_{0,Z}$, and, by (1), Lemma 2.3.2.3 tells us that the action of $\Gamma_{0,Z}$ on a vector space by multiplication by $\prod_{i=1, \ldots, d} |\sigma_i(t)|^{n_i}$ is trivial if and only if $(n_1, \ldots, n_d) \in \mathbb{Z} \cdot (1, \ldots, 1)$. This equivalence continues to hold in the general case where $\Gamma_{0,Z}$ is a (neat) arithmetic subgroup of $F^\times$, again via (1).

By Remark 2.3.2.4 and (2.15), the action of $\Gamma_{0,Z}$ on $V_0^0,q$, for $\Psi = (I_{F}^0, I_{F}^1, I_{F}^2, I_{F}^3) \in \mathcal{P}_q$, is then trivial if and only if there exists an integer $\kappa$ such that

\begin{equation}
\begin{cases}
  k_\sigma + k'_\sigma = \kappa & \forall \sigma \in I_{F}^0 \\
  k_\sigma - k'_\sigma - 2 = \kappa & \forall \sigma \in I_{F}^1 \\
  -(k_\sigma - k'_\sigma + 4) = \kappa & \forall \sigma \in I_{F}^2 \\
  -(k_\sigma + k'_\sigma + 6) = \kappa & \forall \sigma \in I_{F}^3
\end{cases}
\end{equation}

**Definition 2.3.2.5.** If $\lambda$ satisfies the above condition with respect to a $q$-admissible partition $\Psi$ and to $\kappa \in \mathbb{Z}$, we say that $\lambda$ is $(\kappa, 0)$-Kostant parallel with respect to $\Psi$.

**Definition 2.3.2.6.** A $q$-admissible partition $\Psi$ is said to be $(\lambda, 0)$-admissible if there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 0)$-Kostant parallel with respect to $\Psi$. The set of $q$-admissible partitions which are moreover $(\lambda, 0)$-admissible will be denoted by $\mathcal{P}_q^{(\lambda, 0)}$.

With these definitions in hand, we can prove:

**Lemma 2.3.2.7.** There exist integers $0 \leq s \leq d - 1$ and $q \in \{0, \ldots, 3d\}$ and a $q$-admissible partition $\Psi$ such that $H^s(\Gamma_{0,Z}, V_0^{0,q})$ is non-zero if and only if $\lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_{F}, c})$ satisfies one of the following conditions:

1. there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 0)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = (I_{F}^0 \neq \emptyset, I_{F}^1, \emptyset, \emptyset)$. In this case, necessarily $q \in \{0, \ldots, d - 1\}$; more precisely, $q = d_1 = d - d_0$ with $d_0 = |I_{F}^0| \in \{1, \ldots, d\}$. We have then $G_{w(\lambda)-d_0}^{W} V_0^{0,d-d_0} \neq \{0\}$;

2. there exists $\kappa \in \mathbb{Z}$ such that $k_\sigma - k'_\sigma$ is equal to $\kappa$ for all $\sigma \in I_{F}$. In this case, necessarily $q = d$, and $\lambda$ is $(\kappa - 2, 0)$-Kostant parallel with respect to $\Psi = (\emptyset, I_{F}^1, \emptyset, \emptyset) = (\emptyset, I_{F}^1, \emptyset, \emptyset)$. We have then $G_{w(\lambda)-d_0}^{W} V_0^{0,d-d_0} \neq \{0\}$;

3. there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 0)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = (\emptyset, \emptyset, I_{F}^1 \neq \emptyset, I_{F}^3)$. In this case, necessarily $q \in \{2d, \ldots, 3d-1\}$; more precisely, $q = 2d_2 + 3d_3 = 2d + d_3$, for $d_3 = |I_{F}^1| \in \{0, \ldots, d - 1\}$. We have then $G_{w(\lambda)-d_0}^{W} V_0^{0,2d+d_2} \neq \{0\}$;

4. there exists $\kappa \in \mathbb{Z}$ such that $k_\sigma + k'_\sigma$ is equal to $\kappa$ for all $\sigma \in I_{F}$. In this case, necessarily $q = 3d$, and $\lambda$ is $\lambda - \kappa - 6, 0$-Kostant parallel with respect to $\Psi = (\emptyset, \emptyset, \emptyset, I_{F}^1) = (\emptyset, \emptyset, \emptyset, I_{F}^1)$. We have then $G_{w(\lambda)+d_0}^{W} V_0^{0,3d} \neq \{0\}$.

**Proof.** Suppose first that there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 0)$-Kostant parallel with respect to a partition $\Psi$ with $I_{F} \neq \emptyset$. If $I_{F}^0$ was also $\neq \emptyset$, we would have $k_\sigma + k'_\sigma = \kappa = -(k_\rho - k'_\rho + 4)$ for all $\sigma \in I_{F}^0, \rho \in I_{F}^2$, which is impossible since $k_\sigma \geq k'_\sigma \geq 0$ for all $\sigma \in I_{F}^0$, while $k_\sigma \geq k'_\rho \geq 0$ for all $\rho \in I_{F}^3$ (remember that $\lambda$ is dominant). This implies $I_{F}^2 = \emptyset$, and we obtain $I_{F}^3 = \emptyset$ in the same way. Consequently, $d_0 \neq 0, d_2 = d_3 = 0$; by putting together Remark 2.3.2.2, Definition 2.3.2.5 and (2.9), the relations $d_0 + d_1 = d, d_1 = q$ then give the conclusions of (1).

The other cases follow by supposing that $I_{F}^0 = \emptyset$ and arguing in an analogous way.

**Remark 2.3.2.8.** The above lemma, which is an essential step towards Theorem 2.1.0.3, implicitly makes use of the following phenomena, made evident by the "coincidences" in the computations in (2.9) and in (2.16).

Let $\iota : \mathbb{G}_{m,L} \to Z(M_0)L$ be the composition of the adjunction embedding $\mathbb{G}_{m,L} \to \mathbb{G}_{d_{\mathbb{S}},L}$ and of the isomorphism...
and for all $H$
the following lemma will allow us to identify the cohomology spaces corresponding to this action:

\[ \lambda|_{G_{0,\mathbb{R}}} \circ k \circ w = \lambda|_{Z(M_0)_{\mathbb{R}}} \circ L. \]

In other words, the "Hodge weight", determined by the restriction of $\lambda$ to the center of the $G_0$-component of $Q_0/W_0$, equals the (a priori different) character obtained by restriction to the center of the $M_0$-component.

Denote now by $A$ the maximal $\mathbb{Q}$-split torus in the center of $(Q_0/W_0) \cap G^{\text{ss}}$, which is a subgroup of $Z(M_0)$ isomorphic to $\mathbb{G}_m$. If $i_A$ is the isomorphism $A \simeq \mathbb{G}_m$ obtained in the same way as $\ell$, then $\lambda|_{Z(M_0)_{\mathbb{R}}} \circ \ell = \lambda|_{A_{\mathbb{R}}} \circ i_A$. Hence, we see that (2.17) is a consequence of [LR, Prop. 6.4]: the proof in loc. cit. is based on the description of the action of $A$ (through $\lambda$) via local Hecke operators.

Remembering Definition 2.3.2.6, the isomorphism from Theorem 2.2.1.2.(2) for a stratum $Z'$ of $\Phi'$ contributing to $Z_0$ now becomes

\[ R^n i_0^* j_* \mu_{\ell}^K (V_\lambda) \big|_{Z'} \simeq \bigoplus_{p+q=n} \mu_{\ell} (\prod_{\psi \in P_q^{(\lambda, 0)}} H^p(G_0, V_\psi^{q, 0})). \]

We are interested in the weight-graded objects

\[ \text{Gr}^W_k R^n i_0^* j_* \mu_{\ell}^K (V_\lambda) \big|_{Z'} \simeq \bigoplus_{p+q=n} \mu_{\ell} (\prod_{\psi \in P_q^{(\lambda, 0)}} H^p(G_0, \text{Gr}^W_k V_\psi^{q, 0})). \]

In order to determine whether they are trivial or not, we will need a dévissage which is "orthogonal" to the one described in Remark 2.3.2.1.

Remark 2.3.2.9. The groups $\Gamma_{0,\text{ss}} := \Gamma_0 \cap \text{SL}_2(F)$, resp. $\Gamma_0$ are non-trivial subgroups of $\text{SL}_2(F)$, resp. $F^\times$, which are again arithmetic and torsion-free. In particular, $\det \Gamma_0 \simeq \mathbb{Z}^{d-1}$, so that its cohomological dimension is $d-1$ (cfr. Remark 2.3.2.2 and the preceding discussion). Moreover, $\Gamma_0$ can be written as an extension

\[ 1 \rightarrow \Gamma_{0,\text{ss}} \rightarrow \Gamma_0 \xrightarrow{\det} \det \Gamma_0 \rightarrow 1, \]

so that the Lyndon-Hochschild-Serre spectral sequence applied to this extension

\[ E_2 = H^r(\det \Gamma_0, H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0})) \Rightarrow H^{r+s}(\Gamma_0, V_\psi^{q, 0}) \]

tells us that each space $H^p(\Gamma_0, V_\psi^{q, 0})$ is (non-canonically) isomorphic to a direct sum

\[ \bigoplus_{r+s=p} N^{r,s} \]

where each $N^{r,s}$ is a subquotient of $H^r(\det \Gamma_0, H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0}))$. If $H^r(\det \Gamma_0, H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0}))$ is zero for all $r$ or $H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0})$ is zero for all $s$, then $H^p(\Gamma_0, V_\psi^{q, 0})$ is.

For all integer $q \in \{0,\ldots, 3d\}$, we know by (2.10) that $\det \Gamma_0$ acts on $V_\psi^{q, 0}$, and a fortiori on its subspace $H^0(\Gamma_{0,\text{ss}}, V_\psi^{q, 0})$, via multiplication by the character $\chi$ defined by

\[ t \mapsto \prod_{\sigma \in \Gamma_{0,\text{ss}}} \sigma(t)^{k_\sigma} \cdot \prod_{\sigma \in \Gamma_{0,\text{ss}}} \sigma(t)^{-k_\sigma - 2} \cdot \prod_{\sigma \in \Gamma_{0,\text{ss}}} \sigma(t)^{-k_\sigma - 3} \cdot \prod_{\sigma \in \Gamma_{0,\text{ss}}} \sigma(t)^{-k_\sigma - 3} \]

and the following lemma will allow us to identify the cohomology spaces corresponding to this action:

Lemma 2.3.2.10. Let $r \in \{0,\ldots, d-1\}$ be an integer. Then, for all integers $s \in \{0,\ldots, 3d\}$ and $q \in \{0,\ldots, 3d\}$ and for all $q$-admissible partition $\Psi = (I_0^0, I_1^0, I_2^0, I_3^0)$ such that $H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0})$ is non-zero, we have that

\[ H^r(\det \Gamma_0, H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0})) \neq \{0\} \iff H^0(\det \Gamma_0, H^s(\Gamma_{0,\text{ss}}, V_\psi^{q, 0})) \neq \{0\} \]

\[ \iff \text{one of the following conditions is satisfied:} \]

1. $q = 0$, $\Psi = (I_0^0, \emptyset, \emptyset, \emptyset) = (I_F, \emptyset, \emptyset, \emptyset)$ and $k_{\emptyset}$ is constant on $I_F$;
2. $q = d$, $\Psi = (\emptyset, I_F, \emptyset, \emptyset) = (\emptyset, I_F, \emptyset, \emptyset)$ and $k_{\emptyset}$ is constant on $I_F$;
3. $q \in \{2d,\ldots, 3d\}$, $\Psi$ is of the form $(\emptyset, \emptyset, I_F^0, I_F^1)$ and $k_{\emptyset}$ is constant on $I_F$. 

Proof. The point is to reduce oneself to the case where the action of det $\Gamma_0$ on the spaces $H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})$, for $s > 0$, remains semisimple. Now, if $[\Phi] \in H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})$ is the class of a $s$-cocycle

$$\Phi \in \text{Hom}_L(L[\Gamma_{0,ss}]^{s+1},V_{\Psi}^{0,q}),$$

then, for $t \in \text{det} \Gamma_0$, the element $t.[\Phi] \in H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})$ equals the class of the morphism

$$t.\Phi \in \text{Hom}_L(L[\Gamma_{0,ss}]^{s+1},V_{\Psi}^{0,q})$$

that to every $(t_0,\ldots,t_s)$ associates $\chi(t)\Phi(\tilde{t}^{-1}(t_0,\ldots,t_s)\tilde{t})$ (where $\tilde{t}$ is any lifting of $t$ in $\Gamma_0$, and $\chi$ is as in (2.23)).

Consider now the subgroup of $\Gamma_0$ defined in (2.14), which is free abelian, generated by $\{\gamma_1,\ldots,\gamma_{d-1}\}$. The elements $\{(\gamma_1)^2,\ldots,(\gamma_{d-1})^2\}$ generate a free abelian subgroup $\tilde{\Gamma}$ of $\text{det} \Gamma_0$, of rank $d-1$, each of whose elements has a central lifting in $\Gamma_0$. Then, for all $s$, $\tilde{\Gamma}$ acts via the same character $\chi$ on $H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})$. We can now apply Remark 2.3.2.2 to $\tilde{\Gamma}$ and conclude that if $H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})$ is non-zero, then

$$H^s(\tilde{\Gamma},H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})) \simeq H^0(\tilde{\Gamma},H^s(\Gamma_{0,ss},V_{\Psi}^{0,q}))(s-1) \neq \{0\}$$

if and only if there exists an integer $\theta$ such that

$$\left\{ \begin{array}{l}
    k'_\sigma = \theta \quad \forall \sigma \in I^F_p \\
    -k'_\sigma - 2 = \theta \quad \forall \sigma \in I^F_1 \\
    -k'_\sigma - 3 = \theta \quad \forall \sigma \in I^F_2 \\
    -k'_\sigma - 3 = \theta \quad \forall \sigma \in I^F_3
\end{array} \right.$$ 

(also using Remark 2.3.2.4, and remembering the definition of $\chi$). Now recall that $k_\sigma \geq k'_\sigma \geq 0$: the above condition is then equivalent to the one in the statement.

In order to finish the proof, put $F := \text{det} \Gamma_0/\tilde{\Gamma}$: it is a finite group, that we can assume non-trivial, of a certain order $f$ (otherwise, there is nothing else to do). By considering the Lyndon-Hochschild-Serre spectral sequence associated to this quotient and by applying [We, Prop. 6.1.10], we see that

$$H^s(\text{det} \Gamma_0,H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})) \simeq H^0(F,H^s(\tilde{\Gamma},H^s(\Gamma_{0,ss},V_{\Psi}^{0,q}))) \simeq e \cdot H^s(\tilde{\Gamma},H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})), $$

where, if $\{\phi_1,\ldots,\phi_f\}$ is a system of representatives of $F$ inside det $\Gamma_0$, $e$ denotes the endomorphism of multiplication by $\frac{1}{f} \sum_{i=1}^f \chi(\phi_i)$. If $e$ is not the zero endomorphism, then $e \cdot H^s(\tilde{\Gamma},H^s(\Gamma_{0,ss},V_{\Psi}^{0,q})) \simeq H^s(\tilde{\Gamma},H^s(\Gamma_{0,ss},V_{\Psi}^{0,q}))$, and the lemma is demonstrated.

But the fact that $\Gamma_0$ is a neat subgroup of $M_0(F) \simeq \prod_{\sigma \in E_F}(GL_2(F))_\sigma$, along with the fact that the group of $F$-points of the quotient $M_0/(M_0^{der}.Z(M_0))$ is isomorphic to the torsion group $F^*/F^{*2}$, and that the image of a neat subgroup by a morphism of algebraic groups remains neat (in particular, torsion-free) imply that the image of $\Gamma_0$ in $M_0/(M_0^{der}.Z(M_0))(F)$ is trivial. Hence, for all $t \in \text{det} \Gamma_0$, for all embedding $\sigma : F \to \mathbb{R}$, the elements $\sigma(t)$ are squares inside $\mathbb{R}^\times$, and the endomorphism $e$ cannot be zero. \qed

Remark 2.3.2.11. Denote by $D := G^0(\mathbb{R})/K_\infty A_{G^0} \simeq \mathcal{H}^d \times \mathbb{R}^{d-1}$ the symmetric space associated to $M_0 \simeq \text{Res}_{F|Q}GL_{2,F}$, where $K_\infty \simeq \prod_{\sigma \in E_F}SO_2(\mathbb{R})$ is a maximal compact subgroup of $M_0(\mathbb{R})$, $A_{G^0} := S(\mathbb{R})^0$ with $S$ the maximal $\mathbb{Q}$-split torus inside $Z(M_0)$ and $\mathcal{H}$ is the complex upper half space. Then, every $V_{\Psi}^{0,q}$ (as a $M_{0,LL}$-representation) defines a local system $\mathcal{V}_{\Psi}^{0,q}$ on $X := D/\Gamma_0$ such that $H^p(\Gamma_0,V_{\Psi}^{0,q}) \simeq H^p(\mathcal{X},\mathcal{V}_{\Psi}^{0,q})$ for all $p$. On the other hand, if $\Gamma_{0,ss}$ is the group defined in Remark 2.3.2.9, then, for all $p$, we have $H^p(\Gamma_{0,ss},V_{\Psi}^{0,q}) \simeq H^p(X,\mathcal{V}_{\Psi}^{0,q})$, where $X$ is the (complex analytic, connected) Hilbert-Blumenthal variety $\mathcal{H}^d/\Gamma_{0,ss}$ and $\mathcal{V}_{\Psi}^{0,q}$ also abusively denotes the local system on $X$ induced by the the restriction of the representation $V_{\Psi}^{0,q}$ to $M_{0,LL}$.

In particular:

(1) $H^p(\Gamma_0,V_{\Psi}^{0,q}) = \{0\}$ for all $p < 0$ and all $p > \text{dim}(D) - r_Q(M_0/R(M_0)) = 3d - 2$, where $r_Q$ denotes the $\mathbb{Q}$-rank of a $\mathbb{Q}$-algebraic group [BoS, Thm. 11.4.4]).

(2) If the highest weight of $V_{\Psi}^{0,q}$ is regular, then $H^p(\Gamma_0,V_{\Psi}^{0,q}) = \{0\}$ for all $0 \leq p \leq \frac{1}{2}(\text{dim}(D) - l_0) = d - \frac{1}{2}$, where $l_0 := \text{rk}(M_0(\mathbb{R})) - \text{rk}(K)$ and $\text{rk}$ denotes the absolute rank of a Lie group [LS, Cor. 5.6]). But since $H^p(X,\mathcal{V}_{\Psi}^{0,q}) = \{0\}$ for all $0 \leq p < d$ if $V_{\Psi}^{0,q}$ is non-trivial ([M-SSYZ, Thm. 1.1(i)]), by employing Remark 2.3.2.9 we see that, for every non-trivial irreducible representation $V_{\Psi}^{0,q}$, $H^p(\Gamma_0,V_{\Psi}^{0,q}) = \{0\}$ for all $0 \leq p < d$ (i.e., regularity of the representation is not necessary for the vanishing statement).
2.3.3. **Computation of weights along the Siegel strata.** We can finally describe the weights appearing in the degeneration of the canonical construction along the Siegel strata, in the cohomological degrees which we will need in the sequel:

**Proposition 2.3.3.1.** Let $V_\lambda$ be the irreducible $L$-representation of $G_L$ of highest weight $\lambda = \lambda((k_\sigma, k'_\sigma), c)$ and $Z'$ a stratum of the stratification $\Psi'$ of $\partial S^*_K$ which contributes to $Z_0$.

1. Let $n < 0$. Then $R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'}$ is zero.
2. Let $0 \leq n < d$. Then $R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'}$ can be non-zero only if $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, equal to the same integer $\kappa$. In this case,

$$R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'} \cong \mu^\sigma_\ell(K_0)(H^n(\Gamma_0, V_0^{0,0}|_{I_F^0 = I_F}))$$

is pure of weight $w(\lambda) = d(\kappa_1 + \kappa_2)$, and non-zero for $n = 0$.
3. Let $n \in \{d, \ldots, 2d - 1\}$. Then $R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'}$ can be non-zero only if $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, respectively equal to integers $\kappa_1$ and $\kappa_2$. In this case,

$$(2.24) R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'} \cong \mu^\sigma_\ell(K_0)(H^n(\Gamma_0, V_0^{0,0}|_{I_F^0 = I_F}))$$

is pure of weight $w(\lambda) = d(\kappa_1 + \kappa_2)$, and non-zero for $n = 0$.
4. Let $n \in \{2d, \ldots, 3d - 1\}$. Then $R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'}$ can be non-zero only if $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, respectively equal to integers $\kappa_1$ and $\kappa_2$. In this case, it is isomorphic to

$$\mu^\sigma_\ell(K_0)(H^n(\Gamma_0, V_0^{0,0}|_{I_F^0 = I_F})) \oplus \mu^\sigma_\ell(K_0)(H^{n-d}(\Gamma_0, V_0^{0,d}|_{I_F^0 = I_F})).$$

where the first factor is isomorphic to

$$\text{Gr}_{w(\lambda) - d(\kappa_1 + \kappa_2)} W^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'},$$

and the second one to

$$\text{Gr}_{w(\lambda) + 2d - d(\kappa_1 - \kappa_2)} W^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'}.$$ If $n = 2d$, then the second factor is non-zero.

**Proof.** Point (1) is clear from Remark 2.3.2.11.(1). The isomorphisms (2.18) and Lemma 2.3.2.10 (taking into account Remark 2.3.2.9) imply that, in order to have non-zero cohomology objects in the degrees considered in points (2) and (3), $k'_\sigma$ has to be constant, and that $q$ can only take the values 0 or $d$. But the Kostant-paral- lelism conditions imposed by Lemma 2.3.2.7 then imply constancy of $k_\sigma$. As far as point (4) is concerned, an analogous reasoning is valid if $q < 2d$. If $2d \leq q \leq 3d - 1$, then the fact that $p \in \{0, \ldots, d - 1\}$ and Remark 2.3.2.11.(2) imply that the spaces $H^p(\Gamma_0, V_0^{0,0})$ can give non-trivial contributions to the cohomology objects if and only if $V_0^{0,q}$ is the trivial $M^*_{0,L}$-representation, which is never the case, by the description in (2.10) (remember that in this case, $\Psi$ is of the form $(\varnothing, \varnothing, I_F^2 \neq \varnothing, I_F^2)$).

The first half of (2) then follows from Remark 2.3.2.11.(2) and from the fact that, again by (2.10), $V_0^{0,0}|_{I_F^0 = I_F}$ is the trivial $M^*_{0,L}$-representation if and only if $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, equal to the same integer. Also notice that in point (3), the $\ell$-adic sheaf $R^n i_0^!* j_* \mu^K_\ell(V_\lambda)|_{Z'}$ could have a direct factor which is isomorphic to $\mu^\sigma_\ell(K_0)(H^{n-d}(\Gamma_0, V_0^{0,d}|_{I_F^0 = I_F}))$, which is however zero, again by Remark 2.3.2.11.(2), since $n - d < d$ and $V_0^{0,d}|_{I_F^0 = I_F}$ is non-trivial as a $M^*_{0,L}$-representation (cfr. (2.10)).

The statements about weight-graded objects then follow from Remark 2.3.2.7 and from the isomorphisms (2.19), while the non-triviality statements are consequences of the following proposition.

**Proposition 2.3.3.2.** Let $\lambda$, $V_\lambda$ and $Z'$ be as in Proposition 2.3.3.1. If $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, respectively equal to integers $\kappa_1$ and $\kappa_2$, then:

1. if $\kappa_1 = \kappa_2$, then the lisse $\ell$-adic sheaf $\mu^\sigma_\ell(K_0)(H^0(\Gamma_0, V_0^{0,0}|_{I_F^0 = I_F}))$ on $Z'$ is non-zero;
2. if $\kappa_1 \neq \kappa_2$, then the lisse $\ell$-adic sheaf $\mu^\sigma_\ell(K_0)(H^d(\Gamma_0, V_0^{0,d}|_{I_F^0 = I_F}))$ on $Z'$ is non-zero;
(3) the lisse ℓ-adic sheaf $\mu_{\ell}^{\pi_0(K)}(H^d(\Gamma_0, V^0_\ell))$ on $Z'$ is non-zero. If moreover $\kappa_1$ and $\kappa_2$ have the same parity\footnote{This restriction on parity is necessary in order to apply the results from [Fre], which in turn depend on the formulae for the dimension of certain spaces of cusp forms proved in [Sh]. By [Sh, Note 11, pag. 63], it is possible that these formulae could admit a suitable generalisation, such that the hypothesis on parity could be removed.}, it is locally of dimension $h > 0$, where $h := |\mathbb{P}^1(F)/\Gamma_{0,ss}|$ is the (strictly positive) number of cusps of the (complex analytic, connected) Hilbert-Blumenthal variety $H^d/\Gamma_{0,ss}$.

Proof. If $\kappa_1 = \kappa_2$, then the spectral sequence considered in Remark 2.3.2.9 shows that the space $H^0(\Gamma_0, V)$ is isomorphic to $H^0(\det \Gamma, H^0(\Gamma_{0,ss}, V))$, which, by the proof of Lemma 2.3.2.10, is in turn isomorphic to $H^0(\Gamma_{0,ss}, V)$, where $V$ is the trivial representation (by (2.10)). Thus, it is a 1-dimensional $L$-vector space. This shows point (1).

Assume then to be in one of the two following cases: either $\kappa_1 \neq \kappa_2$ and $V$ is the irreducible $L$-representation of $M_{0,\ell} \simeq (\text{Res}_{\overline{\mathbb{Q}}}[\mathbb{Q} \mathbb{SL}_2, F])_L$ given by $V_0^0 = \overline{I}_F$ (which in this case is non-trivial), or $V := V_0^0 = \overline{I}_F$ (which by (2.10) is isomorphic to $\bigotimes_{\sigma \in I_F} \text{Sym}^\kappa V$, where $V$ is the standard 2-dimensional $L$-representation of $\mathbb{SL}_2, L$, so that $V_0^0 = \overline{I}_F$ is never trivial).

In both cases, the same Remark 2.3.2.9 and Remark 2.3.2.11 show that the space $H^d(\Gamma_0, V)$ is isomorphic to $H^0(\det \Gamma, H^d(\Gamma_{0,ss}, V))$, which, by the hypothesis on $\kappa_1$ and $\kappa_2$, and by the proof of Lemma 2.3.2.10 in turn isomorphic to $H^d(\Gamma_{0,ss}, V)$. Now, for all integer $\tilde{\kappa} > 0$, [M-SSYZ, Thm. 1.1(iv)] shows that $\dim H^d(\Gamma_{0,ss}, \bigotimes_{\sigma \in I_F} \text{Sym}^\kappa V) = h + \delta(\Gamma_{0,ss}, \tilde{\kappa})$, where $\delta(\Gamma_{0,ss}, \tilde{\kappa})$ is a non-negative integer which depends on $\Gamma_{0,ss}$ and on $\tilde{\kappa}$. This is enough to show (2) and the first half of (3).

To finish the proof of (3), suppose that $\kappa_1$ and $\kappa_2$ have the same parity and put $\kappa_1 + \kappa_2 =: 2\kappa$: we will show that, in this case, $\delta := \delta(\Gamma_{0,ss}, 2\kappa + 2) > 0$. Actually, [M-SSYZ, Thm. 1.1(iv)] shows that, more precisely, $\delta = h_{I_F} + \delta'$, where $\delta'$ is a certain positive integer and $h_{I_F}$ is the dimension of the space of cusp forms of weight $2\kappa + 4$ with respect to the group $\Gamma_{0,ss}$. Thus, in order to conclude, it is enough to show that this dimension is strictly positive.

Let $X$ be the complex analytic Hilbert-Blumenthal variety $H^d/\Gamma_{0,ss}$. According to [Fre, Chap. II, Thm. 3.5], we have

$$h_{I_F} = \text{vol}(X)(2\kappa + 3)^d + L_{\text{cusp}},$$

where $L_{\text{cusp}}$ is a (not necessarily positive) integer which does not depend on $\kappa$ (recall that $\Gamma_{0,ss}$ is neat). Now, if $d$ is odd, then the discussion in [Fre, page 111] implies that $L_{\text{cusp}} = 0$, so that we obtain $h_{I_F} > 0$, as desired.

If instead $d$ is even, let us consider a smooth toroidal compactification $\overline{X}$ of $X$. Then, by applying the Hirzebruch-Riemann-Roch theorem to certain locally free (automorphic) coherent sheaves on $\overline{X}$, the authors show in [M-SSYZ, Prop. 7.10] that

$$h_{I_F} = \chi(\overline{X}, \mathcal{O}_{\overline{X}}) + \epsilon$$

for a certain integer $\epsilon$. Now, [Fre, Chap. II, Thm. 4.8], implies that, if $d$ is even, $\chi(\overline{X}, \mathcal{O}_{\overline{X}}) > 0$ (this quantity is in particular equal to 1 plus the dimension of the space of cusp forms of weight 2 with respect to $\Gamma_{0,ss}$), and that

$$\chi(\overline{X}, \mathcal{O}_{\overline{X}}) = \text{vol}(X) + L_{\text{cusp}}$$

(see us stress the fact that $L_{\text{cusp}}$ is the same integer of equation (2.25)). By replacing the expression for $\chi(\overline{X}, \mathcal{O}_{\overline{X}})$ into equation (2.26), the equality between the two expressions (2.25) and (2.26) for $h_{I_F}$ tells us that $\epsilon = \text{vol}(X)(2\kappa + 3)^d - \text{vol}(X) > 0$. The equation (2.26) then implies that $h_{I_F} > 0$ in this case too. \qed

2.4. The degeneration along the Klingen strata. Let $\lambda, V_\lambda$ be as in subsection 2.3 and let us now study, by using Theorem 2.2.1.2, the degeneration of $\mu^{\overline{\kappa}}_\ell(V_\lambda)$ along the Klingen strata. The group $G_1$ in their underlying Shimura datum is isomorphic to $\text{Res}_{\overline{\mathbb{Q}}}[\mathbb{GL}_2, F] \times \text{Res}_{\overline{\mathbb{Q}}}[\mathbb{G}_m, F, \det \mathbb{G}_m]$ (cfr. 1.3.3).
2.4.1. **Cohomology of the unipotent radical.** In this case, \((Q_1/W_1)_L \simeq \left( \prod_{\sigma \in I_F} (GL_{2,L})_{\sigma} \times \prod_{\sigma \in I_F} (G_{m,L})_{\sigma} \right) \times \prod_{\sigma \in I_F} (G_{m,L})_{\sigma}, \) and Theorem 2.2.2.1 identifies the cohomology spaces \(H^q(W_{1,L}, V_{\lambda})\) as \((Q_1/W_1)_L\)-representations: by employing Definition (2.5), we have the isomorphism

\[
H^q(W_{1,L}, V_{\lambda}) \simeq \bigoplus_{\Psi \in \mathcal{P}_q} V^{1,q}_{\Psi},
\]

where each \(V^{1,q}_{\Psi}\) is an irreducible \((Q_1/W_1)_L\)-representation. The following computation of its highest weight is again a generalisation of [Le, Sec. 4.3].

Recall the notations of 2.2.2.2 and fix a component \(\tau_{\sigma}\) of the root system of \(G_L\): the positive roots contained in this component which appear inside the Lie algebra of \(W_{1,L}\) are given this time by \(\{\rho_1, \rho_1 + \rho_2, 2\rho_1, 3\rho_1 + \rho_2\}.\)

With notations as in the analogous discussion in 2.3.1, we see that the elements in the sets \(\mathcal{W}_{1,\sigma} = \{w_{\sigma}^i\}_{i=0,\ldots,3} \subset \mathcal{W}_{\sigma}\) of Remark 2.2.2.2.(2) (for \(m=1\)) are given by

\[
\begin{align*}
\rho_0^0 &= \text{id}, \\
\rho_0^1 &= s_{\rho_1,\sigma}, \\
\rho_0^2 &= s_{\rho_1,\sigma} + s_{\rho_2,\sigma}, \\
\rho_0^3 &= 2s_{\rho_1,\sigma} + s_{\rho_2,\sigma},
\end{align*}
\]

and that \(\mathcal{W}_1' = \bigcap_{\sigma \in I_F} \mathcal{W}_{1,\sigma}'.\)

Theorem 2.2.2.1 and the explicit computation of \(w_1(\lambda + \rho) - \rho\) for \(w_1 \in \mathcal{W}_1'\) now tell us that, in the isomorphism (2.28), the highest weight of the irreducible \((Q_1/W_1)_L\)-representation \(V^{1,q}_{\Psi}\) is the restriction of the character

\[
\lambda((\epsilon_{\sigma}, \epsilon'_{\sigma})_{\sigma \in I_F}, c)
\]

where (if \(\Psi = (I_F^0, I_F^1, I_F^2, I_F^3)\))

\[
\epsilon_{\sigma} = \begin{cases} k_{\sigma} & \text{if } \sigma \in I_F^0 \\
 k_{\sigma}' - 1 & \text{if } \sigma \in I_F^1 \\
 -k_{\sigma}' - 3 & \text{if } \sigma \in I_F^2 \\
 -k_{\sigma}' - 4 & \text{if } \sigma \in I_F^3
\end{cases},
\]

\[
\epsilon'_{\sigma} = \begin{cases} k_{\sigma}' & \text{if } \sigma \in I_F^0 \\
 k_{\sigma}' + 1 & \text{if } \sigma \in I_F^1 \\
 k_{\sigma}' + 1 & \text{if } \sigma \in I_F^2 \\
 k_{\sigma}' & \text{if } \sigma \in I_F^3
\end{cases}
\]

The \(L\)-points of the factor \((\prod_{\sigma \in I_F} (GL_{2,L})_{\sigma}) \times \prod_{\sigma \in I_F} (G_{m,L})_{\sigma}, \det G_{m,L}\) of \((Q_1/W_1)_L\) are identified with the subgroup

\[
\left\{ \begin{pmatrix} \rho & \tau_{\sigma} \\ 1 & \tau_{\sigma}^{-1} \rho \end{pmatrix} \right\}_{\sigma \in I_F} | \rho \in L^\times, \tau_{\sigma} \in L^\times \text{ for all } \sigma \in I_F
\]

of \(Q_1/W_1(L)\). Remembering Definition (1.10), the highest weight of the action of \((\prod_{\sigma \in I_F} GL_{2}(L)_{\sigma}) \times (L^\times)_{\sigma}, L^\times\) is then the character

\[
((\tau_{\sigma})_{\sigma \in I_F}, \rho) \mapsto \prod_{\sigma \in I_F} \tau_{\sigma}^\epsilon_{\sigma} \cdot \rho \cdot \text{exp} \left( \frac{1}{2} \left| c + \sum_{\sigma \in I_F} (\epsilon_{\sigma} - \epsilon'_{\sigma}) \right| \right),
\]

so that, by the convention in 1.4.1 and the definition 1.16 of the Shimura datum \((G_1, \mathcal{M}_1)\), this induces on \(V^{1,q}_{\Psi}\) a pure Hodge structure of weight

\[
w(\lambda) - \left[ \sum_{\sigma \in I_F^0} k_{\sigma} + \sum_{\sigma \in I_F^1} (k_{\sigma}' - 1) - \sum_{\sigma \in I_F^2} (k_{\sigma}' + 3) - \sum_{\sigma \in I_F^3} (k_{\sigma} + 4) \right].
\]
2.4.2. **Cohomology of the arithmetic subgroup.** Consider now the arithmetic group $\Gamma_1$ defined in (2.1). We need to identify the cohomology spaces

$$H^p(\Gamma_1, H^q(W_{1,L}, V_\lambda)) \simeq \bigoplus_{\Psi \in P_q} H^p(\Gamma_1, V^1_{\Psi}).$$

Since the cohomological dimension of $W_{1,L}$ is $3d$, these cohomology spaces can be non-zero only for $q \in \{0, \ldots, 3d\}$.

The subgroup $\Gamma_1$ is identified with a torsion-free arithmetic subgroup of $\text{Res}_{F|Q} \mathbb{G}_m, F(\mathbb{Q}) = F^\times$ (cfr. Remark 2.2.1.1), and by reasoning as before Remark 2.3.2.2 we see that $\Gamma_1 \simeq \mathbb{Z}^{d-1}$. In particular, it has cohomological dimension $d - 1$.

By choosing generators $\delta_1, \ldots, \delta_{d-1}$, $\Gamma_1$ is then identified with the subgroup

$$(2.32) \quad \{ (\sigma(t) \begin{pmatrix} 1 & \sigma(t^{-1}) \\ 1 & 1 \end{pmatrix})_{t \sigma} \in I_{F} \mid t = \delta_1^{p_1} \ldots \delta_{d-1}^{p_{d-1}}, p_1, \ldots, p_{d-1} \in \mathbb{Z} \} \hookrightarrow Q_1/W_1(L).$$

Thus, an element $t = \delta_1^{p_1} \ldots \delta_{d-1}^{p_{d-1}} \in \Gamma_1$ acts on $V^1_{\psi}$ via multiplication by

$$(2.33) \quad \prod_{\sigma \in I_{F}} \sigma(t)^{f_{\sigma}} = \prod_{\sigma \in I_1} \sigma(t)^{k_{\sigma}} \cdot \prod_{\sigma \in I_2} \sigma(t)^{k_{\sigma}^* - 1} \cdot \prod_{\sigma \in I_3} \sigma(t)^{-(k_{\sigma}^* + 3)} \cdot \prod_{\sigma \in I_4} \sigma(t)^{-(k_{\sigma} + 4)}.$$

We can now apply Remark 2.3.2.4 to $\Gamma_1$. Then, taking into account (2.33), the action of $\Gamma_1$ on $V^1_{\psi}$, with $\Psi$ equal to $(I_{F}, I_1, I_2, I_3)$ inside $P_q$, is trivial if and only if there exists an integer $\kappa$ such that

$$(2.34) \quad \begin{cases} k_{\sigma} = \kappa & \forall \sigma \in I_{F}^0, \\ k_{\sigma}^* - 1 = \kappa & \forall \sigma \in I_{F}^1, \\ -(k_{\sigma}^* + 3) = \kappa & \forall \sigma \in I_{F}^2, \\ -(k_{\sigma} + 4) = \kappa & \forall \sigma \in I_{F}^3. \end{cases}$$

As in subsection 2.3.2, we pose the following:

**Definition 2.4.2.1.** If $\lambda$ satisfies the above condition with respect to a $q$-admissible partition $\Psi$ and to $\kappa \in \mathbb{Z}$, we say that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to $\Psi$.

**Definition 2.4.2.2.** A $q$-admissible partition $\Psi$ is said to be $(\lambda, 1)$-admissible if there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to $\Psi$. The set of $q$-admissible partitions which are moreover $(\lambda, 1)$-admissible will be denoted by $P_q^{(\lambda, 1)}$.

We can again apply Remark 2.3.2.2 to $\Gamma_1$. By employing Definition 2.4.2.1 and by (2.31) we obtain:

**Lemma 2.4.2.3.** There exist integers $0 \leq p \leq d - 1$ and $q \in \{0, \ldots, 3d\}$ and a $q$-admissible partition $\Psi$ such that $H^p(\Gamma_1, V_{\Psi}^{d,q})$ is non-zero if and only if $\lambda = \lambda(k_{\sigma}, k_{\sigma}^*, \sigma_{\in I_F}, c)$ satisfies one of the following conditions:

1. there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = \{ (I^0_F \neq \varnothing, I^1_F, \varnothing, \varnothing) \}$. In this case, necessarily $q \in \{0, \ldots, d - 1\}$. More precisely, $q = d_1 = |I^0_F| \in \{0, \ldots, d - 1\}$. We have then $\text{Gr}_{\lambda}^{d-1-q} V^{1,d-q}_{\psi} \neq \{0\}$;

2. there exists $\kappa \in \mathbb{Z}$ such that $k_{\sigma}^*$ is equal to $\kappa$ for all $\sigma \in I_F$. In this case, necessarily $q = d$, and $\lambda$ is $(\kappa - 1, 0)$-Kostant parallel with respect to $\Psi = \{ (\varnothing, I^1_F, \varnothing, \varnothing) \}$. We have then $\text{Gr}_{\lambda}^{d-1-q} V^{1,d-q}_{\psi} \neq \{0\}$;

3. there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = \{ (\varnothing, \varnothing, I^2_F \neq \varnothing, I^3_F) \}$. In this case, necessarily $q \in \{2d, \ldots, 3d-1\}$; more precisely, $q = 2d_2 + 3d_3 = 2d_3 + d_4$, with $d_3 = |I^3_F| \in \{0, \ldots, d - 1\}$. We have then $\text{Gr}_{\lambda}^{d-1-q} V^{1,2d+3d-q}_{\psi} \neq \{0\}$;

4. there exists $\kappa \in \mathbb{Z}$ such that $\lambda$ is $(\kappa - 4, 0)$-Kostant parallel with respect to $\Psi = \{ (\varnothing, \varnothing, \varnothing, I^1_F) \}$. We have then $\text{Gr}_{\lambda}^{d-1-q} V^{1,d-q}_{\psi} \neq \{0\}$.

**Proof.** Completely analogous to the proof of Lemma 2.3.2.7. $\Box$
Remark 2.4.2.4. The computations in (2.31) and in (2.34) show the same phenomenon of 2.3.2.8, which is again an essential ingredient in order for the above proof to work.

By employing Definition 2.4.2.2, the isomorphism in 2.2.1.2.(2) for a stratum $Z''$ of $\Phi'$ contributing to $Z_1$ becomes now
\begin{equation}
R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''} \simeq \bigoplus_{p+q=n} \mu^{\pi_1(K)}_{\ell^*}(H^p(\Gamma_1, V^{1,q}_\Psi))
\end{equation}
and we want to study the weight-graded objects
\begin{equation}
\text{Gr}^W_k R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''} \simeq \bigoplus_{p+q=n} \mu^{\pi_1(K)}_{\ell^*}(H^p(\Gamma_1, \text{Gr}^W_k V^{1,q}_k)).
\end{equation}

2.4.3. Computation of weights along the Klingen strata. We can now describe the weights appearing in the degeneration of the canonical construction along the Klingen strata (in the cohomological degrees which will be needed in the sequel). We just need a last preliminary remark:

Remark 2.4.3.1. Suppose the dominant weight $\lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$ of $G_L$ to be $(\kappa, 1)$-Kostant parallel with respect to a partition $\Psi$ of the form $(I^0_F, I^1_F, 2, 2)$. Then, by the definition of Kostant-parallelism and the hypothesis on $\lambda$, we see that such $\kappa$ and $\Psi$ are necessarily unique, except if $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, equal respectively to $k_1$ and $k_2$. In this last case, there exist exactly two pairs $(\kappa, (I^0_F, I^1_F))$ such that $\lambda$ is $\kappa$-Kostant parallel with respect to $(I^0_F, I^1_F)$, i.e. $(k_1, I^0_F = I^1_F, \emptyset)$ and $(k_2 - 1, (\emptyset, I^0_F = I^1_F))$.

Proposition 2.4.3.2. Let $V_L$ be the irreducible $L$-representation of $G_L$ of highest weight $\lambda = \lambda((k_\sigma, k'_\sigma)_{\sigma \in I_F}, c)$ and $Z''$ a stratum of the stratification $\Phi'$ of $\partial S_K^*$ contributing to $Z_1$.

1. Let $n < 0$ or $n > 4d - 1$. Then, $R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''}$ is zero.

2. Let $n \in \{0, \ldots, d - 1\}$. Then the $\ell$-adic sheaf $R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''}$ on $Z''$ is non-zero if and only if there exists $\kappa \in Z$ such that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = (I^0_F, I^1_F, 2, 2)$ and if $n \geq d_1$, where $d_1 := |I^0_F| \in \{0, \ldots, d - 1\}$.

In this case, $\kappa$ and $\Psi$ are necessarily unique, and we have the isomorphism
\begin{equation}
R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''} \simeq \mu^{\pi_1(K)}_{\ell^*}(H^{n-d_1}(\Gamma_1, V^{1,d_1}_\Psi)).
\end{equation}

3. If $n = 0$, then the $\ell$-adic sheaf $R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''}$ on $Z''$ is non-zero if and only if there exists $\kappa \in Z$ such that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = (I^0_F, I^1_F, 2, 2)$.

In this case, $\kappa$ and $\Psi$ are necessarily unique, and we have the isomorphism
\begin{equation}
R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''} \simeq \mu^{\pi_1(K)}_{\ell^*}(H^{n-d_1}(\Gamma_1, V^{1,d_1}_\Psi)).
\end{equation}

4. Let $n \in \{2d, \ldots, 3d - 1\}$. Then the $\ell$-adic sheaf $R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''}$ on $Z''$ is non-zero if and only if there exists $\kappa \in Z$ such that $\lambda$ is $(\kappa, 1)$-Kostant parallel with respect to a partition $\Psi$ of the form $\Psi = (2, 2, I^0_F, I^1_F)$.

In this case, $\kappa$ and $\Psi$ are necessarily unique, and we have the isomorphism
\begin{equation}
R^{n_i^*j^*}_{\ell^*}\mu^K_{p(q)}(V_{\lambda})|_{Z''} \simeq \mu^{\pi_1(K)}_{\ell^*}(H^{n-2d}(\Gamma_1, V^{1,2d+d}_\Psi)).
\end{equation}

Proof. The description of the cohomological objects follows from (2.35), taking into account the cohomological dimensions of the groups $\Gamma_1, W_{1, L}$ and the Kostant-parallelism restrictions imposed by Lemma 2.4.2.3, along with Remark 2.4.3.1. Notice in particular that in point (2), we have necessarily $q \in \{0, \ldots, d - 1\}$, and that necessarily $q \in \{0, \ldots, d\}$ in point (3). On the other hand, necessarily $q \in \{2d, \ldots, 3d - 1\}$ in point (4). The description of the weight-graded objects follows from the same Lemma 2.4.2.3 and from the isomorphisms (2.36).
2.5. The double degeneration along the cusps of the Klingen strata. Keep the notation of Thm. 2.1.0.3. In order to study the weights of the motive \( * j^! \mathcal{V} \), the study of the degeneration of the canonical construction to each stratum of \( \partial S^*_K \) will not be enough: we will also need to consider a double degeneration, the one of mixed sheaves on the Klingen strata, already obtained by degeneration, along the boundary of the closure in \( \partial S^*_K \) of the Klingen strata themselves.

By subsection 1.3.3, every stratum \( Z'' \) of \( \Phi' \) contributing to \( Z_1 \) (as defined in 2.2.1) is (a smooth quotient by the action of a finite group of) a Hilbert-Blumenthal variety \( S_{\pi_1(K_1)} \) of dimension \( d \). Remember from the same subsection 1.3.3 that the Shimura datum underlying \( S_{\pi_1(K_1)} \) corresponds to the algebraic group \( G_1 \cong \text{Res}_{F/\mathbb{Q}} \text{GL}_2,F \times \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m, \text{det} \mathbb{G}_m \), whose \( L \)-points are identified, up to conjugation, with

\[
G_1(L) = \left\{ \begin{pmatrix} \rho & a_\sigma & b_\sigma \\ 0 & 1 & c_\sigma \\ 0 & d_\sigma & 1 \end{pmatrix} \big| a_\sigma, b_\sigma, c_\sigma, d_\sigma \in L, \rho \in L^\times, \right. \]

such that \( \rho = a_\sigma d_\sigma - b_\sigma c_\sigma \) for all \( \sigma \in I_F \} = \{(A_\sigma)_{\sigma \in I_F} \in \prod_{\sigma \in I_F} \text{GL}_2,L(L) \text{ such that } \det(A_\sigma) = \det(A_\sigma) \forall \sigma, \sigma \in I_F \}.

The boundary \( \partial S^*_\pi_1(K_1) \) of the Baily-Borel compactification \( S^*_\pi_1(K_1) \) of \( S_{\pi_1(K_1)} \) is 0-dimensional: it is in fact a finite disjoint union of strata (called cusps), obtained as Shimura varieties coming from the group \( \mathbb{G}_m \). Fix such a stratum \( Z'' \), corresponding up to conjugation, in the formalism of 1.3.1, to the standard Borel subgroup of \( G_1 \) (denoted by \( Q_2 \) for the sake of coherence with the notations in the sequel): it is a representative of the unique \( G_1(\mathbb{Q}) \)-conjugacy class of standard maximal parabolics of \( G_1 \). The Levi component of \( Q_2 \) is a torus \( T_1 \) isomorphic to

\[
\mathbb{G}_m \times \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m,F
\]

via the isomorphism defined on \( L \)-points by

\[
T_1(L) \cong \mathbb{G}_m(L) \times \prod_{\sigma \in I_F} \mathbb{G}_m(L) \sigma
\]

\[
\begin{pmatrix}
\beta \\
\beta u_\sigma \\
1
\end{pmatrix}
\begin{pmatrix}
\beta \\
1
\end{pmatrix}
\rightarrow \left( \begin{pmatrix}
\beta \\
\beta u_\sigma \\
1
\end{pmatrix} \right)_{\sigma \in I_F} \rightarrow \left( \begin{pmatrix}
\beta, \sigma \end{pmatrix} \right)_{\sigma \in I_F}.
\]

Then, the Shimura datum \((G_2, \mathcal{H}_2)\) underlying \( Z'' \) is such that the \( L \)-points of the group \( G_2 \cong \mathbb{G}_m \) are identified with

\[
G_2(L) = \left\{ \left( \begin{pmatrix}
\beta & I_2 \\
I_2
\end{pmatrix} \right)_{\sigma \in I_F} \big| \beta \in L^\times \right\} \hookrightarrow T_1(L)
\]

and \( \mathcal{H}_2 \) is defined exactly as \( \mathcal{H}_0 \) in (1.15) (cfr. [P1, Example 12.21]).

2.5.1. The degeneration of the canonical construction along the cusps of Hilbert-Blumenthal varieties. Let \( j' \) be the open immersion of \( S_{\pi_1(K_1)} \) in \( S^*_\pi_1(K_1) \) and adopt the notations of subsection 2.2.1, by replacing \( j \) with \( j' \) and \( K \) with \( \pi_1(K_1) \): the stratification \( \Phi \) of \( \partial S^*_\pi_1(K_1) \) is formed by only one element, called \( Z_2 \). Denote then by \( i_2 : Z_2 \hookrightarrow S^*_\pi_1(K_1) \) the closed immersion complementary to \( j' \). Let us consider a stratum \( Z'' \) contributing to \( Z_2 \) and let us make explicit, thanks to Theorem 2.2.2.1, the conclusions of Theorem 2.2.1.2, applied this time to \( \mu_{\pi_1(K_1)}^T(U_{k_1}) \), where \( U_{k_1} \) is an irreducible \( L \)-representation of \( G_{1,L} \).

Such a representation is determined by its highest weight \( \chi = \chi((h_\sigma)_{\sigma \in I_F}, g) \), where \( h_\sigma \in \mathbb{Z}, h_\sigma \geq 0 \forall \sigma \in I_F, g \in \mathbb{Z} \). This character is defined on the points of the maximal torus \( T_{1,L} \) of \( G_{1,L} \) by

\[
\left( \begin{pmatrix}
\beta \\
\beta u_\sigma \\
1
\end{pmatrix} \right)_{\sigma \in I_F} \rightarrow \prod_{\sigma \in I_F} u_{\sigma}^{h_\sigma} \cdot \beta^g.
\]
Remark 2.5.1.1. Notice that, with these conventions, the restriction to $T_{1,L}$ of the character $\lambda((k_\sigma,k'_\sigma),c)$ defined in 1.10 is given by

\begin{equation}
(2.41)
\chi((k'_\sigma),\frac{1}{2} \cdot [c + \sum_\sigma (k_\sigma + k'_\sigma)]).
\end{equation}

We have an identification

\begin{equation}
(Q_2/W_2)_L \simeq T_{1,L},
\end{equation}

so that, by Theorem 2.2.2.1, the cohomology spaces $H^q(W_{2,L},U_\chi)$ are identified with representations of the group $T_{1,L} \simeq \mathbb{G}_{m,L} \times \prod_{\sigma \in I_F} \mathbb{G}_{m,L}$. Let us determine the characters through which this torus acts on the irreducible factors.

By choosing $(T_{1,L},Q_{2,L})$ as a maximal torus and a Borel of $G_{1,L}$, we can identify the set of roots $\varpi$ of $G_{1,L}$ with $\bigcup \tau_\sigma$, where each $\tau_\sigma$ is a copy of the set of roots of $GL_{2,L}$ corresponding to the obvious choice of maximal torus and Borel. For each fixed $\hat{\sigma} \in I_F$, $\tau_\sigma$ contains only one simple root $\rho_\sigma$, which, through the inclusion of $\tau_\sigma$ inside $\tau$, acquires the expression $\rho_\sigma = \rho_\sigma((h_\sigma)_{\sigma \in I_F},g)$, where

\begin{equation}
h_\sigma = \begin{cases}
2 & \text{if } \sigma = \hat{\sigma} \\
0 & \text{otherwise}
\end{cases}, \quad g = 1.
\end{equation}

The Weyl group $W$ of $G_{1,L}$ is in turn isomorphic to the product $\prod_{\sigma \in I_F} W_\sigma$, where, for each fixed $\hat{\sigma} \in I_F$, $W_\sigma$ is a copy of the Weyl group of $GL_{2,L}$. The latter is a finite group of order 2, the image of whose only non-trivial element through the inclusion of $W_\sigma$ in $W$ is given by the element of $\tau_\sigma$ which acts on $X^*(T_{1,L})$ in the following way: if $\chi = \chi((h_\sigma)_{\sigma \in I_F},g)$, then $\tau_\sigma.\chi = \chi((\ell_\sigma)_{\sigma \in I_F},f)$, where

\begin{equation}
\ell_\sigma = \begin{cases}
-h_\sigma & \text{if } \sigma = \hat{\sigma} \\
h_\sigma & \text{otherwise}
\end{cases}, \quad f = g - h_\sigma.
\end{equation}

By employing the notations of 2.2.2, it is now clear that, with respect to the only parabolic of $G_{1,L}$ (up to conjugation), i.e. $Q_{2,L}$, we have $W' = W$, and that, if $w = (w_\sigma)_{\sigma \in I_F} \in W' \simeq \prod_{\sigma \in I_F} W_\sigma$, we have $\ell(w) = \sharp\{\sigma \in I_F\mid w_\sigma = \tau_\sigma\}$.

The explicit computation of $w.(\chi + \rho) - \rho$ (for $w \in W'$) and Theorem 2.2.2.1 now give the isomorphisms

\begin{equation}
(2.42)
H^q(W_{2,L},U_\chi) \simeq \bigoplus_{I \subset I_F} q_{|I|=q} \bigoplus_{I \subset I_F} U_{\ell}^q,
\end{equation}

where the $U_{\ell}^q$'s are 1-dimensional $L$-vector spaces on which $\mathbb{G}_{m,L} \times \prod_{\sigma \in I_F} \mathbb{G}_{m,L}$ acts via the character $\chi'(\ell_\sigma)_{\sigma \in I_F},g'$ defined by

\begin{equation}
l_\sigma = \begin{cases}
h_\sigma & \text{if } \sigma \notin I \\
-h_\sigma & \text{if } \sigma \in I
\end{cases}, \quad g' = g - \sum_{\sigma \in I} (h_\sigma + 1).
\end{equation}

By the definition of the Shimura datum $(G_2,\mathfrak{H}_2)$ in the beginning of 2.5 and by the convention fixed in 1.4.1, the action of the real points of the factor $\mathbb{G}_m$ corresponding to the Shimura datum induces a pure Hodge structure of weight

\begin{equation}
(2.43)
-2g + 2\sum_{\sigma \in I} (h_\sigma + 1)
\end{equation}

on each $U_{\ell}^q$.

Consider now the group $\Gamma_2$ as defined in (2.1), which is a torsion-free arithmetic subgroup of $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m(\mathbb{Q})$, i.e. of $F^\times$ (cfr. the isomorphism (2.40) and Remark 2.2.1.1), thus isomorphic, by the same reasoning as before Remark 2.3.2.2, to $\mathbb{Z}^d$. We need to study the cohomology spaces

\begin{equation}
H^p(\Gamma_2, H^q(W_{2,L},U_\chi)) \simeq \bigoplus_{I \subset I_F} q_{|I|=q} H^p(\Gamma_2, U_{\ell}^q).
\end{equation}

By choosing generators $\omega_1,\ldots,\omega_{d-1}$, $\Gamma_2$ is identified with the subgroup
and an element \( t \in \Gamma_2 \) acts on \( U^I \) by multiplication by \( \prod_{\sigma \in I} \sigma(\omega)^{h_{\sigma}} \cdot \prod_{\sigma \in I} \sigma(\omega)^{-h_{\sigma} - 2} \). By reasoning as in subsection 2.4.2, we get:

**Lemma 2.5.1.2.** There exist integers \( 0 \leq p \leq d - 1 \) and \( q \in \{0, \ldots, d\} \) such that \( H^p(\Gamma_2, U^I_q) \) is non-zero if and only if \( h_{\sigma} \) is constant on \( I_F \) (equal to an integer \( h \)) and one of the following conditions is satisfied:

1. either \( I = \emptyset \), in which case necessarily \( q = 0 \) and the Hodge structure on \( U^I_0 = H^0(W_{2, \lambda}, U^I_\lambda) \) is pure of weight \(-2g\);
2. or \( I = I_F \), in which case necessarily \( q = d \) and the Hodge structure on \( U^I_d = H^d(W_{2, \lambda}, U^I_\lambda) \) is pure of weight \(-2g + 2 \sum (h_{\sigma} + 1) = -2g + 2d + 2dh \).

The isomorphism of Theorem 2.2.1.2.2 for a stratum \( Z'' \) contributing to \( \partial S_{\pi_1(K_1)}^* \) now becomes

\[
R^n(i_2^* j_* \mu^\pi_{1, \lambda}(U^I_\lambda)) \mid_{Z''} \cong \bigoplus_{p+q=n} \bigoplus_{I \subset I_F} \bigoplus_{tq, |I| = q} \mu^\pi_{1, \lambda}(H^p(\Gamma_2, U^I_\lambda)).
\]

The computation of the weights of these cohomology objects is then a direct consequence of 2.5.1.2:

**Proposition 2.5.1.3.** Let \( U^I_\lambda \) be the irreducible representation of \( G_{1, \lambda} \) of highest weight \( \chi((h_{\sigma})_{\sigma \in I_F}, g) \) and \( Z'' \) a stratum contributing to \( \partial S_{\pi_1(K_1)}^* \). Then:

1. Let \( n < 0 \) or \( n > 2d - 1 \). Then \( R^n(i_2^* j_* \mu^\pi_{1, \lambda}(U^I_\lambda)) \mid_{Z''} \) is zero.
2. Let \( n \in \{0, \ldots, d - 1\} \). Then \( R^n(i_2^* j_* \mu^\pi_{1, \lambda}(U^I_\lambda)) \mid_{Z''} \) is non-zero if and only if \( h_{\sigma} \) is constant on \( I_F \).

In this case, it is isomorphic to \( \mu^\pi_{1, \lambda}(H^n(\Gamma_2, U^I_\lambda)) \), and pure of weight \(-2g\).
3. Let \( n \in \{d, \ldots, 2d - 1\} \). Then \( R^n(i_2^* j_* \mu^\pi_{1, \lambda}(U^I_\lambda)) \mid_{Z''} \) is non-zero if and only if \( h_{\sigma} \) is constant on \( I_F \) (equal to an integer \( h \)). In this case, it is isomorphic to \( \mu^\pi_{1, \lambda}(H^{n-d}(\Gamma_2, U^I_\lambda)) \), and pure of weight \(-2g + 2d + 2dh\).

**Remark 2.5.1.4.** The results of Proposition 2.5.1.3 had been obtained in [W3, Thm. 3.5], by slightly different considerations, for representations \( U^I_\lambda \) such that \( g = 0 \).

2.5.2. **The double degeneration.** Let \( \lambda = \lambda((k_{\sigma}, k'_{\sigma}, \sigma), \epsilon), V_\lambda \) and \( S_K \) be as in subsection 2.4, and let \( Z'' \) and \( S_{\pi_1(K_1)}^* \) be as in subsection 2.5.1.

By (2.35) and Theorem (2.2.1.2.1)-2, we have the following isomorphism in the derived category:

\[
i_1^* i_*^* j_* \mu^\pi_{1, \lambda}(V_\lambda) \mid_{Z''} \cong \bigoplus_{m \geq 0} \bigoplus_{p+q=m} \mu^\pi_{1, \lambda}(V^{p,q}) \mid[m],
\]

where

\[
V^{p,q} := \bigoplus_{\Psi \in P(\lambda, 1)} H^p(\Gamma_1, V^{1, \Psi}_{\lambda}).
\]

In this latter direct sum, every factor is, by restriction, a certain power of an irreducible representation of \( G_{1, \lambda} \), whose dominant weight is the one prescribed by Remark (2.5.1.1) applied to the character \( \lambda((\epsilon_{\sigma}, \epsilon'_{\sigma})_{\sigma \in I_F}, \epsilon) \) defined in (2.29).

Recall that the functor \( \mu^\pi_{1, \lambda} \) used in the isomorphism (2.46), with values in \( E_{\ell, R}(Z'') \), is deduced from the canonical construction functor, which takes values in \( E_{\ell, R}(S_{\pi_1(K_1)}) \) (Remark 2.2.1.3.1). In order to study the degeneration of \( \mu^\pi_{1, \lambda}(V^{p,q}) \) along the points in the closure of \( Z'' \) in \( \partial S_K^* \), we will rather consider the sheaves on \( S_{\pi_1(K_1)} \), denoted by the same symbol, which are obtained by interpreting this time \( \mu^\pi_{1, \lambda}(K_1) \) as the canonical construction functor.
Let us now apply Theorem 2.2.1.2.(2) to $\mu^\pi_1(K_1)(V^{p,q})[m]$ and to $S_{\pi_1(K_1)}$, by posing $p+q=m$ and by adopting the notations of 2.5.1, for a stratum $\partial S_{\pi_1(K_1)}$, in order to study the weights of the objects $R^{n-m}(i_2)^*j_{\ast}^!\mu^\pi_1(K_1)(V^{p,q})|_{Z'''}$. We will only need this for $m \in \{2d,\ldots,3d-1\}$.

**Proposition 2.5.2.1.** Fix two positive integers $p$ and $q$ such that $p+q \in \{2d,\ldots,3d-1\}$ and let $V^{p,q}$ be the $L$-representation of $G_{1,L}$ defined in (2.47), deduced from the irreducible $L$-representation $V_\lambda$ of $G_L$ of highest weight $\lambda = \lambda(k',k'')_{m \in I,F,c}$ (in particular, $q \in \{d+1,\ldots,3d-1 \}$). Let $Z'''$ be a stratum contributing to $\partial S_{\pi_1(K_1)}$. Then:

(i) if $m' \in \{0,\ldots,d-1\}$, the $\ell$-adic sheaf $R^{m'}(i_2)^*j_{\ast}^!\mu^\pi_1(K_1)(V^{p,q})|_{Z'''}$ on $Z'''$ is non-zero if and only if $k_\sigma$ are $k'_\sigma$ constant on $I_F$, respectively equal to integers $k_1$ and $k_2$. In this case, it is pure of weight $w(\lambda)+2d-d(k_1-k_2)$;

(ii) if $m' \in \{d,\ldots,2d-1\}$, the $\ell$-adic sheaf $R^{m'}(i_2)^*j_{\ast}^!\mu^\pi_1(K_1)(V^{p,q})|_{Z'''}$ on $Z'''$ is non-zero if and only if $k_\sigma$ and $k'_\sigma$ are constant on $I_F$, respectively equal to integers $k_1$ and $k_2$. In this case, it is pure of weight $w(\lambda)+6d-d(k_1-k_2)$;

(iii) if $m' \not\in \{0,\ldots,2d-1\}$, then the $\ell$-adic sheaf $R^{m'}(i_2)^*j_{\ast}^!\mu^\pi_1(K_1)(V^{p,q})|_{Z'''}$ on $Z'''$ is zero.

**Proof.** Lemma 2.4.2.3 and the fact that $q \in \{d+1,\ldots,3d-1\}$ imply that $V^{p,q}$ is non-zero if and only if there exists a partition $\Psi = (P^2 \neq \emptyset, P^3)$ of $I_F$ and an integer $\iota_1$ such that

$$\begin{cases} k'_\sigma = \iota_1 & \forall \sigma \in P^2 \\ k_\sigma = \iota_1-1 & \forall \sigma \in P^3 \end{cases}$$

In this case, if $P^{\mathcal{P},(\lambda,d)}$ is the set of such partitions, we have $V^{p,q} = \bigoplus_{\Psi \in P^{\mathcal{P},(\lambda,d)}} H^p(\Gamma_1, V_\Psi^{q,1})$; the highest weight of the action of $G_{1,L}$ on $H^p(\Gamma_1, V_\Psi^{q,1})$ is then, by (2.30), the restriction of the character $\lambda((\epsilon'_\sigma,\epsilon_\sigma)_{\sigma \in I_F,c})$ defined in (2.29), where

$$\epsilon_\sigma = \begin{cases} -k'_\sigma - 3 & \text{if } \sigma \in P^2 \\ -k_\sigma - 4 & \text{if } \sigma \in P^3, \end{cases} \quad \epsilon'_\sigma = \begin{cases} k_\sigma + 1 & \text{if } \sigma \in P^3 \\ k'_\sigma & \text{if } \sigma \in P^2 \end{cases}$$

By Remark 2.5.1.1, this restriction, as a character of the maximal torus $T_{1,L}$ of $G_{1,L}$, has the form $\chi((\epsilon'_\sigma,\epsilon_\sigma),\frac{1}{2},[c+\sum_{\sigma \in I_F} (\epsilon'_\sigma + \epsilon_\sigma)])$.

Now, by Proposition 2.5.1.3, $R^{m'}(i_2)^*j_{\ast}^!\mu^\pi_1(K_1)(V^{p,q})|_{Z'''}$ is non-zero if and only if $m' \in \{0,\ldots,2d-1\}$ and $\epsilon'_\sigma$ is constant on $I_F$, say equal to an integer $\iota_2$. This means that

$$\begin{cases} k_\sigma = \iota_2 - 1 & \forall \sigma \in P^2 \\ k'_\sigma = \iota_2 & \forall \sigma \in P^3 \end{cases}$$

Thus, the fact that $k_\sigma \geq k'_\sigma$ for all $\sigma \in I_F$ and that $P^2 \neq \emptyset$ imply that the sheaves we are interested in are non-zero if and only if $I_F = P^2$ and $k_\sigma$, $k'_\sigma$ are constant on $I_F$, respectively equal to $k_1 := \iota_2 - 1$ and $k_2 := \iota_1$. In order to conclude, it is now enough to apply again Proposition 2.5.1.3, by observing that if $m' \in \{0,\ldots,d-1\}$ then the highest weight of the action of $G_{1,L}$ on $H^p(\Gamma_1, V_\Psi^{q,1})$ is the character

$$(\iota_2,\beta) \mapsto \prod_{\sigma \in I_F} u_\sigma^{\iota_2+1} \cdot \beta^{\frac{1}{2}[c-2d+d(k_1-k_2)]}$$

(and analogously for the case $m' \in \{d,\ldots,2d-1\}$).

2.6. **Weight avoidance.** In this section, we employ the notations of 2.1 and 2.2.1. Our aim is to use the results of the preceding subsections in order to prove Theorem 2.1.0.3, thanks to the criterion given by Theorem 2.1.0.7. Thus, we have to rely, for $m \in \{0,1\}$, the weights of the objects $H^p(i_n^*,i_\ast^!j_{\ast})(R_c(\lambda V))$ to the weights of the objects $R^n(i_n^*,i_\ast^!j_{\ast})(R_c(\lambda V))$, which are now known.

In the following, the symbols $\tau_{\mathcal{Z}}$ will denote the truncation functors with respect to the perverse $t$-structure on $Z$. 


2.6.1. Weight avoidance on the Siegel strata. Let us begin by studying the weight avoidance on the Siegel strata, by employing Propositions 2.3.3.1 and 2.5.2.1.

Remark 2.6.1.1. Reasoning as in [W8, Rmk. 2.7 (a)-(b)-(c)-(d)], we have exact sequences

\[
(2.49)\quad \mathcal{H}^{n-1}(i_0^*i_1^*\tau_{Z_1}^\geq w(\lambda)+3d)_{\ell}(\lambda V) \rightarrow \mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V)) \rightarrow \mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V))
\]

for \(n \leq w(\lambda) + 3d - 1\), whereas \(\mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V))\) is zero for \(n \geq w(\lambda) + 3d\); one also sees, again reasoning as in (loc. cit.), that \(\mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V))\) is zero for \(n < w(\lambda) + 2d\), so that \(\mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V)) \simeq \mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V))\) for \(n < w(\lambda) + 2d\).

Thus, if \(n \leq w(\lambda) + 2d - 1\), the weights of the perverse sheaf \(\mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V))\) are the same as the weights of \(\mathcal{H}^n(i_0^*i_1^*j_*R(\lambda V))\), which have already been computed, while there is nothing to do for \(n \geq w(\lambda) + 3d\). It remains to study the interval \([w(\lambda) + 2d, w(\lambda) + 3d - 1]\).

Remark 2.6.1.2. Each stratum \(Z''\) of \(\Phi'\) contributing to \(Z_1\) is the quotient of a Hilbert-Blumenthal variety \(S_{K,Z''}\) by the action of a finite group; let \(S_{K,Z''}^*\) be its Baily-Borel compactification. If \(Z_1\) is the closure of \(Z_1\) in \(\partial S_{K,Z''}^*\) and

\[
(2.50)\quad Z_1^* := \bigcup_{\text{stratum of } \Phi' \text{ contributing to } Z_1} S_{K,Z''}^*,
\]

then there exists a surjective, finite morphism

\[
(2.51)\quad \varrho : Z_1^* \rightarrow \bar{Z}_1
\]

whose restriction to each \(S_{K,Z''}^*\) is the quotient morphism from \(S_{K,Z''}\) to \(Z''\) (cfr. [P1, Main Thm. 12.3 (c), Sec. 7.6]).

Thanks to the above Remark, we can now compute the weights of \(\mathcal{H}^n(i_0^*i_1^*\tau_{Z_1}^\geq w(\lambda)+3d)_{\ell}(\lambda V)\) in the degrees we are interested in.

Lemma 2.6.1.3. If \(n \in [w(\lambda) + 2d, w(\lambda) + 3d - 1]\), then \(\mathcal{H}^n(i_0^*i_1^*\tau_{Z_1}^\geq w(\lambda)+3d)_{\ell}(\lambda V)\) is non-zero if and only if \(k_\sigma\) and \(k'_\sigma\) are constant on \(IF\), equal respectively to \(\kappa_1, \kappa_2\). In this case, it is pure of weight \(w(\lambda) + 2d - d(\kappa_1 - \kappa_2)\).

Proof. Recall that \(R_{\ell}(\lambda V) = \mu^K_{\ell}(V_{\lambda})[-w(\lambda)]\). Then, for each stratum \(Z''\) contributing to \(Z_1\), we have, by Theorem 2.2.1.2,(1),

\[
i_1^*i_1^*j_*R_{\ell}(\lambda V)|_{Z''} \simeq \bigoplus_k R_k^{i_1^*i_1^*j_\mu^K}(V_{\lambda})[-w(\lambda) - k]|_{Z''}.
\]

Moreover, by Theorem 2.2.1.2,(2), the objects \(R^{w(\lambda) - w(\lambda) - k_\sigma}j_*\mu^K_{\ell}(V_{\lambda})\) are all isse: since \(Z_1\) is of dimension \(d\), perverse truncation above degree \(w(\lambda) + 3d\) equals classical truncation above degree \(w(\lambda) + 2d\). Thus, by fixing a stratum \(Z'\) contributing to \(Z_0\), we obtain

\[
(2.52)\quad \mathcal{H}^n(i_0^*i_1^*\tau_{Z_1}^\geq w(\lambda)+3d)_{\ell}(\lambda V)|_{Z_1} \simeq \bigoplus_{k \geq 2d} \mathcal{H}^n(i_0^*i_1^*(\bigoplus_{k \geq 2d} R_k^{i_1^*i_1^*j_*\mu^K_{\ell}}(V_{\lambda})[-w(\lambda) - k]|_{Z''}))|_{Z_1},
\]

where the direct sum runs over all strata \(Z''\) contributing to \(Z_1\) and containing \(Z'\) in their closure. Fix now such a stratum: as in (2.46) and (2.47), we get

\[
(2.53)\quad \bigoplus_{k \geq 2d} R_k^{i_1^*j_*\mu^K_{\ell}}(V_{\lambda})[-w(\lambda) - k]|_{Z''} \simeq \bigoplus_{k \geq 2d} \left( \bigoplus_{p+q=k} \mu^K_{\ell}(V_p)[-w(\lambda) - k] \right),
\]

and as a consequence, by taking into account the fact that \(Z'\) is of dimension 0,

\[
(2.54)\quad \left( \mathcal{H}^n(i_0^*i_1^*(\bigoplus_{k \geq 2d} R_k^{i_1^*j_*\mu^K_{\ell}}(V_{\lambda})[-w(\lambda) - k]|_{Z''})) \right)|_{Z_1} \simeq \bigoplus_{k \geq 2d} \left( \bigoplus_{p+q=k} R^{w(\lambda)-k}i_0^*i_1^*\mu^K_{\ell}(V_p)[-w(\lambda) - k] \right)|_{Z_1}.
\]
Now, let us adopt the notations of Remark 2.6.1.2, and extend the notations of 2.5.1 in the following way: \( j' \) will denote the open immersion of the union of the \( S_{K,S''}^* \)'s in the union of the \( S_{K,S''}^* \)'s, while \( i_2 \) will denote the complementary closed immersion of the union of the \( \partial S_{K,S''}^* \)'s in the union of the \( S_{K,S''}^* \)'s. By restriction to \( Z'' \), we get, by proper base change, the relation
\[
(2.55) \quad i_{0!}^{*}i_{1*}^{*}\mu_{*}^{s}(K_{1}) \simeq q^{*}i_{2!}^{*}j'^{*}\mu_{*}^{s}(K_{1}),
\]
where on the left, resp. right hand side, we have interpreted the functor \( \mu_{*}^{s}(K_{1}) \) as a functor with values in \( \text{Et}_{L}(Z'') \), resp. in \( \text{Et}_{L}(S_{\pi_{1}(K_{1})}) \). Since the morphism \( q \) is finite, we deduce that, for all \( k, p, q \),
\[
(2.56) \quad R^{n{-}w(\lambda{-}k^{*}i_{0!}^{*}i_{1*}^{*}\mu_{*}^{s}(K_{1}))}(V_{p,q}) \mid_{Z''} \simeq q_{*}(R^{n{-}w(\lambda{-}k^{*}i_{2!}^{*}j'^{*}\mu_{*}^{s}(K_{1}))}(V_{p,q}) \mid_{\partial Z''} ),
\]
where \( \partial Z'' \) is the stratum of \( S_{K,S''}^* \) such that \( q(\partial Z'') = Z' \) (such a stratum is unique, because two rational boundary components (cfr. 1.3.1) are conjugated by \( G_{1}(Q) \) if and only if they are conjugated by \( G(Q) \), by [P1, Rmk. at page 91, (iii)]).

Now, the functor \( q_* \) preserves weights, because the morphism \( q \) is finite. Thus, the isomorphisms (2.52)-(2.56) allow us to deduce the weights of \( H^{n}(i_{0!}^{*}i_{1*}^{*}Z^{2w(\lambda)+3d}i_{1*}^{*}j_{*}R_{d}(\lambda\mathcal{V})) \) from the weights of the sheaf \( R^{n{-}w(\lambda{-}k^{*}i_{2!}^{*}j'^{*}\mu_{*}^{s}(K_{1}))}(V_{p,q}) \mid_{Z''} \). Now, if \( n \in \{w(\lambda)+2d, w(\lambda)+3d-1\} \) and \( k \geq 2d \), then, by Proposition 2.5.2.1, the objects which appear as summands in the right hand side of (2.54) are non-zero only for indices \( n{-}w(\lambda)+k \in \{0, \ldots, d-1\} \). We can then conclude by Proposition 2.5.2.1(i).

We now dispose of all the necessary information in order to determine an interval of weight avoidance on the Siegel strata:

**Proposition 2.6.1.4.** There exists a \( n \in \mathbb{Z} \) such that \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is non-zero if and only if \( k_{\sigma} \) and \( k'_{\sigma} \) are constant on \( I_{F} \), respectively equal to integers \( \kappa_{1}, \kappa_{2} \). In this case, \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is of weight \( \leq n-d(\kappa_{1}+\kappa_{2}) \) for each \( n \in \mathbb{Z} \).

If \( \kappa_{1} \) and \( \kappa_{2} \) have the same parity, then the weight-graded object of weight \( w(\lambda)+2d-d(\kappa_{1}+\kappa_{2}) \) of the perverse sheaf \( H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is non-zero.

**Proof.** If \( n < w(\lambda)+2d \), then \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \simeq H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \), by Remark 2.6.1.1. Now, \( Z_{0} \) is of dimension 0; hence,
\[
(2.57) \quad H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \simeq R^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) = R^{n{-}w(\lambda)}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})).
\]

Thus, the perverse sheaf \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is zero for all \( n < w(\lambda) \). If \( k_{\sigma} \) and \( k'_{\sigma} \) are not constant on \( I_{F} \), then Proposition 2.3.3.1 tells us that \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is zero for all \( w(\lambda) \leq n < w(\lambda)+2d \). If instead \( k_{\sigma} = \kappa_{1}, \ k'_{\sigma} = \kappa_{2} \) for all \( \sigma \in I_{F} \), then the same Proposition tells us that for all \( w(\lambda) \leq n < w(\lambda)+2d, \ H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is of weight \( \leq n-d(\kappa_{1}+\kappa_{2}) \).

Let now \( n \geq w(\lambda)+2d \). If \( n \geq w(\lambda)+3d \), Remark 2.6.1.1 implies that \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is zero. Assume then \( n \in \{w(\lambda)+2d, \ldots, w(\lambda)+3d-1\} \). In this case, by reasoning as in (2.57) and by applying again Proposition 2.3.3.1, along with Lemma 2.6.1.3, we also see, by the exact sequence (2.49), that if \( k_{\sigma} \) and \( k'_{\sigma} \) are not constant on \( I_{F} \), then \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is zero for all \( n \in \{w(\lambda)+2d, \ldots, w(\lambda)+3d-1\} \). If instead \( k_{\sigma} = \kappa_{1}, \ k'_{\sigma} = \kappa_{2} \) for all \( \sigma \in I_{F} \), then we see in a similar way that the weights that can appear in \( H^{n}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) are of the form \( w(\lambda) \leq n-d(\kappa_{1}+\kappa_{2}) \).

To see that if \( \kappa_{1} \) and \( \kappa_{2} \) have the same parity, then weight \( w(\lambda)+2d-d(\kappa_{1}+\kappa_{2}) \) does appear in the perverse sheaf \( H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \), notice that the long exact sequence (2.49) gives a short exact sequence
\[
(2.58) \quad 0 \to H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \to H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \to H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}Z_{1}^{2w(\lambda)+3d}i_{1*}^{*}j_{*}R_{d}(\lambda\mathcal{V})),
\]
so that \( H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) is identified with the kernel of the arrow \( ad \). Proposition 2.3.3.2 shows that if \( \kappa_{1} \) and \( \kappa_{2} \) have the same parity, then \( H^{n(\lambda)+2d}(i_{0!}^{*}i_{1*}^{*}R_{d}(\lambda\mathcal{V})) \) contains a direct factor, which is pure of weight \( w(\lambda)+2d-d(\kappa_{1}+\kappa_{2}) \) and locally of dimension \( h \), where \( h := \left[ P^{1}(F)/G_{0,ss} \right] \) is the (strictly positive) number of cusps of the Hilbert-Blumenthal variety \( \mathcal{H}/G_{0,ss} \) (cfr. the notations of Remark 2.3.2.9). In order to conclude,
it is then enough to show that locally, the kernel of \( ad \) has non-trivial intersection with this sub-object. Recall that above a stratum \( Z' \) of \( \partial S_K^* \) contributing to \( Z_0 \), the arrow \( ad \) has the form
\[
(2.59) \quad R^{2d}i_1^* \mu_1(K_1(V_\lambda)) \mid_{Z'} \to q_+ \bigoplus (R^{0}i_2^*j_{\pi_1(K_1)}(V^{p,q}) \big|_{\partial Z''})
\]
where notations have been established during the proof of Lemma 2.6.1.3; in particular, the index \( Z'' \) runs over all strata \( Z'' \) contributing to \( Z_1 \) and containing \( Z' \) in their closure, the morphism \( q : Z_1^* \to Z \) (cfr. Remark 2.6.1.2) is finite, and, for each fixed \( Z'' \), we have \( R^{0}i_2^*j_{\pi_1(K_1)}(V^{p,q}) \mid_{\partial Z''} \simeq \mu_1(K_2)(H^0(\Gamma_2, U^p_0)) \), where \( H^0(\Gamma_2, U^p_0) \) is a 1-dimensional \( L \)-vector space (cfr. subsection 2.5.1).

We are then reduced to show that, locally, the dimension of the target of \( ad \) is strictly smaller than the dimension of the source (remember that by Lemma 2.6.1.3, the target is pure of weight \( w(\lambda) + 2d - d(\kappa_2 - \kappa_2) \)). But this is true, thanks to the following proposition.

**Proposition 2.6.1.5.** Let \( Z', q \) and \( h \) be as in the proof of Proposition 2.6.1.4.

Then, above \( Z' \subset Z_0 \), the number of points in the geometrical fibers of \( q|_{\partial Z_1^*} : \partial Z_1^* \to Z_0 \) is \( \leq h \).

**Proof.** The statement can be proven on \( \mathbb{C} \)-points. Thus, let us consider the rational boundary components \((Q_0, x_0)\) and \((Q_1, x_1)\) corresponding to the standard maximal parabolics \( Q_0 \) and \( Q_1 \) of \( G \), along with the canonical normal subgroup \( P_0 \) of \( Q_0 \), as defined in subsection 1.3.1; put \( Q_0(\mathbb{Q}) \to Q_0/P_0(\mathbb{Q}) \), denote by \( q_0 \to q_0 \) the quotient morphism, and define \( Q_01 := Q_0 \cap Q_1 \).

Then, the proof of [W8, pp. 25-26] can be translated word by word in this context: first, use the adelic description of the morphism obtained from \( q \) by analytification, in order to identify the fibre \( q^{-1}(z) \) above \( z \in Z'(\mathbb{C}) \) (where \( Z' \) corresponds to the image of the morphism \( i_\gamma \) as in subsection 1.3.1, for a certain \( g \in G(\mathbb{A}_f) \)) with the set \( Q_01(\mathbb{Q}) \backslash Q_0(\mathbb{Q})/H'_C \) (where \( H'_C := \text{Cent}_{Q_0(\mathbb{Q})}(x_0) \mid Q_0(\mathbb{Q}) \cap p_0gKg^{-1}p_0^{-1} \), for a suitable \( p_0 \in P_0(\mathbb{A}_f) \) determined by \( z \)). Second, further identify the latter with \( Q_01(\mathbb{Q}) \backslash Q_0(\mathbb{Q})/H'_C = B_2(\mathbb{F}) / GL_2(\mathbb{F})/\Gamma_0 \), where \( B_2 \) is the standard Borel subgroup of \( GL_2 \). Now, this set is exactly the set of cusps of \( \Gamma_0 \), whose cardinality is \( \leq \) the cardinal of the set of cusps of \( \Gamma_0,ss \). \( \square \)

**Remark 2.6.1.6.** By the preceding proof, the number of \( d \)-dimensional strata in \( \partial S_K^* \) which contain a fixed cusp is related to the number of cusps in the Baily-Borel compactification of a (complex analytic, connected) virtual Hilbert-Blumenthal variety, which does not appear in \( \partial S_K^* \), i.e. \( \mathcal{H}^d \backslash \Gamma_0,ss \). Cfr. [W8, Rmk. 2.10 (c)] for an analogous remark.

### 2.6.2. Weight avoidance on the Klingen strata and proof of the main theorem.

Let us now study the weight avoidance on the Klingen strata, by means of Proposition 2.4.3.2.

**Remark 2.6.2.1.** By reasoning as in [W8, Rmk. 2.7 (a)-(b)-(c)], we see that
\[
(2.60) \quad i_1^*i_2^*j_{\lambda}(R_{\ell}(\mathbb{V})) \simeq \tau_{2d}^{\leq w(\lambda) + 3d - 1}i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V}),
\]

Thanks to the latter Remark, we are ready to determine the interval of weight avoidance on the Klingen strata.

**Proposition 2.6.2.2.** There exists a \( n \) such that \( \mathcal{H}^n(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V})) \) is non-zero if and only if, for some \( k \in \mathbb{Z} \), \( \lambda \) is \((\kappa_1,1)\)-Kostant parallel with respect to a partition \( \Psi \) of the form \((I^p_{\Gamma}, I^p_{\mathbb{A}}, \varnothing, \varnothing)\),

In this case, let \( k_1 := k \) et \( k_2 := k + 1 \). Then:

1. if there exists a \( \Psi \) for which \( \lambda \) is \((\kappa_1,1)\)-Kostant parallel, and such that \( I^p_{\mathbb{A}} \neq \varnothing \), then the perverse sheaf \( \mathcal{H}^n(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V})) \) is of weight \( \leq n - d_1 - d_1 \binom{n}{d_1} \), where \( d_1 := \mid I^p_{\mathbb{A}} \mid \in \{0, \ldots, d_1 - 1\} \), and the object \( \mathcal{H}^{w(\lambda)+d_1}(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V})) \) is non-zero and pure of weight \( w(\lambda) + d_1 \).

2. if there exists \( \Psi \) for which \( \lambda \) is \((\kappa_1,1)\)-Kostant parallel, and such that \( I^p_{\mathbb{A}} \neq \varnothing \), then the perverse sheaf \( \mathcal{H}^n(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V})) \) is of weight \( \leq n - d_2 \binom{n}{d_2} \), where \( d_2 := \mid I^p_{\mathbb{A}} \mid \in \{0, \ldots, d_2 - 1\} \), and the object \( \mathcal{H}^{w(\lambda)+2d_2}(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V})) \) is non-zero and pure of weight \( w(\lambda) + 2d_2 \).

**Proof.** By Remark 2.6.2.1,
\[
\mathcal{H}^n(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V})) \simeq \mathcal{H}^n(i_1^*i_2^*j_{\lambda}R_{\ell}(\mathbb{V}))
\]
for all \( n \leq w(\lambda) + 3d - 1 \), and \( \mathcal{H}^n(i_1^*j_*R_*^!(\lambda V)) \) is zero for all \( n \geq w(\lambda) + 3d \). Moreover,

\[
\mathcal{H}^n(i_1^*j_*R_*^!(\lambda V)) = (R^{n-w(\lambda)}-d_1^*j_*\mu_!(V_\lambda))[d].
\]

Then, by applying Proposition 2.4.3.2, we see the following facts.

If \( n < w(\lambda) + d \), then \( \mathcal{H}^n(i_1^*j_*R_*^!(\lambda V)) \) is zero.

If \( n \in \{w(\lambda) + d, \ldots, w(\lambda) + 2d - 1\} \), then \( \mathcal{H}^n(i_1^*j_*R_*^!(\lambda V)) \) is non-zero if and only if, for some \( k \in \mathbb{Z} \), \( \lambda \) is (\( k, 1 \))-Kostant parallel with respect to a partition \( \Psi \) of the form \( (I^0_\Psi \neq \emptyset, I^1_\Psi, \emptyset, \emptyset) \) and \( n \geq w(\lambda) + d + d_1 \), where \( d_1 = |I^1_\Psi| \in \{0, \ldots, d - 1\} \). In this case, it is pure of weight \( w(\lambda) + d - d_1 \); it is then of weight \( \leq n - d_1 - d_1 \).

If \( n \in \{w(\lambda) + 2d, \ldots, w(\lambda) + 3d - 1\} \), then \( \mathcal{H}^n(i_1^*j_*R_*^!(\lambda V)) \) is non-zero if and only if, for some \( k \in \mathbb{Z} \), \( \lambda \) is (\( k, 1 \))-Kostant parallel with respect to a partition \( \Psi \) of the form \( (I^0_\Psi \neq \emptyset, I^1_\Psi, \emptyset, \emptyset) \) and \( n \leq w(\lambda) + 2d + d_1 - 1 \), where \( d_1 = |I^1_\Psi| \in \{1, \ldots, d\} \). In this case, it is pure of weight \( w(\lambda) + 2d - d_1 \); it is then of weight \( \leq n - d_1 \).

We now have all the necessary ingredients for the proof of Theorem 2.1.0.3.

**Proof.** (of Theorem (2.1.0.3))

We only have to apply the criterion 2.1.0.7.(2) and use Propositions 2.6.1.4 and 2.6.2.2. □

### 3. THE INTERSECTION MOTIVE OF GENUS TWO HILBERT-SIEGEL VARIETIES

In this section we study the properties of the intersection motive of genus 2 Hilbert-Siegel varieties (with coefficients in suitable irreducible representations \( V_\lambda \)), whose existence follows from Thm. 2.1.0.3 and Wildeshaus’ theory, and the implications for the construction of motives associated to automorphic representations.

#### 3.1. Properties of the intersection motive

Adopt the notation of 2.1 and assume from now on that \( \lambda \) is either not completely irregular or of corank 0. Then, the weight avoidance proved in Corollary 2.1.0.4 allows us to apply to general theory developed in [W7]. In fact, absence of the weights 0 and 1 implies that there exist an intermediate extension functor \( j_* \) from the category \( CHM(S_K) \) to the category \( CHM(S_{K}^\circ) \). Denoting by \( s: S_K \to \text{Spec Q} \) the structural morphism, we can define, by applying [W7, Def. 2.7], the intersection motive of \( S_K \) with respect to \( S_{K}^\circ \) with coefficients in \( \lambda V \) as the object \( s_!j_!^!V \) of the category \( CHM(Q)L \). In the following, this object will be simply called intersection motive.

Let us spell out its main properties. For doing so, if \( \lambda \) verifies in addition the hypotheses of point (1) or (2) or (3), resp. (4), of Theorem 2.1.0.3, put \( \beta := d_\lambda \), resp. \( \beta := \min\{d_\lambda 1, d(k_1 - \nu_2)\} \) (with notations as in the Theorem). The general theory then implies the following:

**Corollary 3.1.0.1.** Let \( s \) and \( \beta \) be as above, and let \( s \) be the structural morphism of \( S_K \).

1. The motive \( s_!^!V \in DM_{B,c}(Q)L \) avoids weights -2, -\( \beta + 1, \ldots, 1 \), and the motive \( s_!^!V \in DM_{B,c}(Q)L \) avoids weights 2, \( \ldots, \beta \).

More precisely, the exact triangles

\[
s_*s_!^!j_!V[-1] \to s_!^!V \to s_*s_!^!V \to s_*s_!^!j_!V
\]

are weight filtrations of \( s_!^!V \), resp. of \( s_!^!V \), which avoid weights -2, -\( \beta + 1, \ldots, 1 \), resp. 2, \( \ldots, \beta \).

2. The intersection motive \( s_*j_!^!V \) is functorial with respect to \( \lambda V \) and to \( \lambda V \). In particular, all endomorphism of \( s_!^!V \) or \( s_!^!V \) induces an endomorphism of \( s_*j_!^!V \).

3. If \( s_!^!V \to N \to s_!^!V \) is a factorisation of \( s_!^!V \) through a Chow motive \( N \in CHM(Q)L \), then the intersection motive \( s_*j_!^!V \) is canonically identified with a direct factor of \( N \), with a canonical direct complement.

**Proof.** We only have to apply [W7, Thm. 2.4], resp. [W7, Thm. 2.5], resp. [W7, Thm. 2.6], in order to obtain the point (1), resp. (2), resp. (3) (thanks to Theorem 2.1.0.3).

Fix now an integer \( N \) such that, as in Remark 1.4.3.2, \( \lambda V \) is a direct factor of a Tate twist of \( p_{N,s}^!A^N_K \), where \( p_{N}: A^N_K \to S_K \) denotes the N-fold fibred product of the universal abelian variety \( A_K \) with itself over \( S_K \). The property stated in Corollary 3.1.0.1.(2) has important consequences for the Hecke algebra \( \mathcal{H}(\mathfrak{f}(K, G(\mathfrak{f})) \) associated to the open compact subgroup \( K \), as defined in [W4, pp. 591-592], along with the action of each of its elements on \( s_!p_{N,s}^!A^N_K \). By loc. cit., each element of \( \mathcal{H}(\mathfrak{f}(K, G(\mathfrak{f})) \) also acts on \( s_!^!V \). Then, corollary 3.1.0.1.(2) gives us immediately the following consequence:

**Corollary 3.1.0.2.** Each element of \( \mathcal{H}(\mathfrak{f}(K, G(\mathfrak{f})) \) acts naturally on the intersection motive \( s_*j_!^!V \).
It is also useful to explicitly formulate the property stated in Corollary 3.1.0.1.(3) in a specific context:

**Corollary 3.1.0.3.** Let $\mathcal{A}_K$ be a smooth compactification $\mathcal{A}_K$. Then, the intersection motive $s_j^\lambda \mathcal{V}$ is canonically identified with a direct factor of a Tate twist of $\mathcal{A}_K$ (where $\lambda$ is the structural morphism of $\mathcal{A}_K$ towards $\text{Spec } \mathbb{Q}$), with a canonical direct complement.

This corollary has important consequences for the realizations of $s_j^\lambda \mathcal{V}$.

**Corollary 3.1.0.4.** Let $\mathcal{O}$ be the order of $F$ prescribed by the PEL datum corresponding to $S_K$ (cfr. Remark 1.2.2.2.(2)), $D$ the discriminant of $\mathcal{O}$ as defined in [L1, Def. 1.1.1.6], and $N$ the level of $K$. Let $p$ be a prime which does not divide neither $D$, nor $N$. Then:

1. the $p$-adic realization of $s_j^\lambda \mathcal{V}$ is crystalline, and if $\ell$ is a prime different from $p$, the $\ell$-adic realization of $s_j^\lambda \mathcal{V}$ is unramified at $p$;
2. consider on the one hand the action of Frobenius $\phi$ on the $\phi$-filtered module associated to the (crystalline) $p$-adic realization of $s_j^\lambda \mathcal{V}$, and on the other hand the action of a geometrical Frobenius at $p$ on the $\ell$-adic realization of $s_j^\lambda \mathcal{V}$ (unramified at $p$). Then, the characteristic polynomials of the two actions coincide.

**Proof.** (1) By [W2, Thm. 4.14], and with the notations of the preceding Corollary, the existence of a smooth compactification of $\mathcal{A}_K$ with good reduction properties is enough to get the conclusion. Now, we have at our disposal the very general results of [L2] on the existence of smooth projective integral models of smooth compactifications of $\mathcal{O}$-type Kuga-Sato families: namely, Thm. 2.15 of loc. cit. (by taking into account Definition 1.6 of loc. cit. and [L1, Prop. 1.4.4.3]) implies that there exists a smooth compactification of $\mathcal{A}_K$ with good reduction at all prime $p$ which does not divide neither $D$, nor $N$. Thus, we can invoke [W2, Thm. 4.14] to conclude.

(2) We argue exactly as in [W8, Cor. 1.13], in order to use [KM, Thm. 2.2] and conclude.

In order to end this list of properties of $s_j^\lambda \mathcal{V}$, we recall that the reason for the name of the intersection motive is the behaviour of its realizations (recall that we are supposing that $\lambda$ is either not completely irregular or of corank 0):

**Corollary 3.1.0.5.** (1) For all $n \in \mathbb{Z}$, the natural maps

$$H^n(S_K^\lambda(\mathbb{C}), j_*\mu^K_L(V)) \to H^n(S_K(\mathbb{C}), \mu^K_L(V))$$

(between cohomology spaces of Hodge modules) and

$$H^n(S_K^\lambda(\mathbb{C}) \times \bar{\mathbb{Q}}, j_*\mu^K_L(V)) \to H^n(S_K(\mathbb{C}) \times \bar{\mathbb{Q}}, \mu^K_L(V))$$

(between cohomology spaces of $\ell$-adic perverse sheaves) are injective, and dually, the natural maps

$$H^n_c(S_K(\mathbb{C}), \mu^K_L(V)) \to H^n_c(S_K^\lambda(\mathbb{C}), j_*\mu^K_L(V))$$

and

$$H^n_c((S_K \times \bar{\mathbb{Q}}, \mu^K_L(V)) \to H^n_c(S_K^\lambda(\mathbb{C}) \times \bar{\mathbb{Q}}, j_*\mu^K_L(V))$$

are surjective. Consequently, the natural maps from intersection cohomology of $S_K$ towards intersection cohomology (with coefficients in $\mu^K_L(V)$, resp. $\mu^K(V)$) are isomorphisms.

(2) The Hodge realization, resp. $\ell$-adic realization of the intersection motive $s_j^\lambda \mathcal{V} \in CHM(\mathbb{Q})_L$ is identified with interior cohomology $H^*_c(S_K(\mathbb{C}), R_H^{\lambda}(\mathcal{V}))$, resp. $H^*_c((S_K \times \bar{\mathbb{Q}}, R_{\ell}^{\lambda}(\mathcal{V}))$.

**Proof.** Theorem 2.1.0.3 tells us that the motive $\iota^* j^\lambda \mathcal{V}$ avoids weights $-\beta, \ldots, \beta + 1$. Then, it is enough to apply [W7, Rmk. 2.13 (c)] to the complexes $\iota^* j_* R_{\ell}^{\lambda}(\mathcal{V})$, where $R$ is the Hodge or $\ell$-adic realization functor on $CHM(S_K)_L$.

Point (2) follows from (1) and from the fact that realizations of the intersection motive are identified with intersection cohomology ([W7, page 15], before Proposition 2.8; cfr. [W4, Thm. 7.2] for details).

**Remark 3.1.0.6.** (1) The vanishing theorems for cohomology of locally symmetric spaces with coefficients in regular representations of the underlying algebraic group ([LS, Cor. 5.6] ou [S, Thm. 5]) imply that, if $\lambda$ is regular, the spaces $H^n(S_K(\mathbb{C}), \mu^K_L(V))$, and so (by comparison) $H^n(S_K \times \bar{\mathbb{Q}}, \mu^K_L(V))$ are zero for $n < 3d = \dim S_K$. Dually, we get $H^n_c(S_K(\mathbb{C}), \mu^K_L(V)) = 0$ et $H^n_c(S_K \times \bar{\mathbb{Q}}, \mu^K(V)) = 0$ for
n > 3d. As a consequence, if λ is regular, then the interior cohomology spaces $H^p_n(S_K(\mathbb{C}), \mu^K_S(V_\lambda))$ et $H^p_n(S_K \times \mathbb{Q}, \mu^S(V_\lambda))$ are zero in degrees different from $n = 3d$.

(2) Corollary 3.1.0.5.(2) and the preceding point imply that, if λ is regular, the Hodge realization of the intersection motive $s_j^\lambda V \in CHM(\mathbb{Q})_L$ is given by $H^{3d}_i(S_K(\mathbb{C}), \mu^K_S(V_\lambda))[-(w(\lambda) + 3d)]$, and that its $\ell$-adic realization is given by $H^{3d}_i(S_K \times \mathbb{Q}, \mu^S(V_\lambda))[-(w(\lambda) + 3d)]$.

3.2. Homological motives for automorphic representations. Keep the notations of the preceding subsection and assume moreover that λ is regular. In this last part, following [W8, Sec. 3], we would like to exploit the action of the algebra $\mathcal{H}(K, M_{\lambda})$ on the intersection motive $s_j^\lambda V \in CHM(\mathbb{Q})_L$ (cfr. Corollary 3.1.0.2) to cut out certain homological sub-motives thereof. Recall that the algebra $\mathcal{H}(K, M_{\lambda})$ acts on $H^{3d}_i(S_K(\mathbb{C}), \mu^K_S(V_\lambda))$ and on $H^{3d}_i(S_K \times \mathbb{Q}, \mu^S(V_\lambda))$.

Theorem 3.2.0.1. ([H, Chap. 3, Sec. 4.3.5, page 75])

For all extension $E$ of $L$, the $\mathcal{H}(K, M_{\lambda}) \otimes E$-module $H^{3d}_i(S_K(\mathbb{C}), \mu^S(V_\lambda))$ is semisimple.

Corollary 3.2.0.2. Let $R(\pi) := R(\pi)(K, M_{\lambda})$ be the image of $\mathcal{H}(K, M_{\lambda})$ in the endomorphism algebra of $H^{3d}_i(S_K(\mathbb{C}), \mu^S(V_\lambda))$. Then, for all extension $E$ of $L$, the algebra $R(\pi) \otimes E$ is semisimple.

In particular, isomorphism classes of simple right $R(\pi) \otimes E$-modules are in bijection with isomorphism classes of minimal right ideals. Now, by fixing $E$, and one of these minimal right ideals $Y_{\pi_f}$ of $R(\pi) \otimes E$, there exists an idempotent $e_{\pi_f} \in R(\pi) \otimes E$ which generates $Y_{\pi_f}$.

Definition 3.2.0.3. (1) The Hodge structure $W(\pi_f)$ associated to $Y_{\pi_f}$ is defined by

$W(\pi_f) := \text{Hom}_{R(\pi) \otimes E}(Y_{\pi_f}, H^{3d}_i(S_K(\mathbb{C}), \mu^S(V_\lambda)))$.

(2) For all prime number $\ell$, the Galois module $W(\pi_f)_\ell$ associated to $Y_{\pi_f}$ is defined by

$W(\pi_f)_\ell := \text{Hom}_{R(\pi) \otimes E}(Y_{\pi_f}, H^{3d}_i(S_K \times \mathbb{Q}, \mu^S(V_\lambda)))$.

The proof of the following is immediate (see for example [W8, Prop. 3.4]):

Proposition 3.2.0.4. There are canonical isomorphisms of Hodge structures, resp. of Galois modules

$W(\pi_f) \simeq H^{3d}_i(S_K(\mathbb{C}), \mu^S(V_\lambda)) \cdot e_{\pi_f}$,

resp.

$W(\pi_f)_\ell \simeq H^{3d}_i(S_K \times \mathbb{Q}, \mu^S(V_\lambda)) \cdot e_{\pi_f}$.

Since we do not know if $e_{\pi_f}$ lifts to an idempotent element of $\mathcal{H}(K, M_{\lambda}) \otimes E$, we are forced to consider its action on the homological (or Grothendieck) motive which underlies the intersection motive $s_j^\lambda V \in CHM(\mathbb{Q})_L$. Denote then by $s_j^\lambda V'$ this homological motive, and define, thanks to Corollary 3.1.0.2:

Definition 3.2.0.5. The (homological) motive corresponding to $Y_{\pi_f}$ is defined by $W(\pi_f) := s_j^\lambda V' \cdot e_{\pi_f}$.

We finish by making explicit the properties of the latter motive which follow from the preceding constructions:

Theorem 3.2.0.6. The realizations of the motive $W(\pi_f)$ are concentrated in cohomological degree $w(\lambda) + 3d$, where in particular the Hodge realization equals $W(\pi_f)$, and the $\ell$-adic realizations equal $W(\pi_f)_\ell$, for all prime $\ell$.

Proof. Follows from the construction of $W(\pi_f)$ and Remark 3.1.0.6.(2) (remember that we are supposing $\lambda$ to be regular).

Corollary 3.2.0.7. Let $p$ be a prime number which does not divide neither the integer $D$ nor the integer $N$ from Corollary 3.1.0.4, and $\ell$ a prime different from $p$. Then:

(1) the $p$-adic realization of $W(\pi_f)$ is crystalline, and the $\ell$-adic realization of $W(\pi_f)$ is unramified at $p$;

(2) consider on the one hand the action of the Frobenius $\phi$ on the $\phi$-filtered module associated to the $p$-adic (crystalline) realization of $W(\pi_f)$, and on the other hand the action of a geometrical Frobenius in $p$ on the $\ell$-adic realization of $W(\pi_f)$ (unramified at $p$). Then, the characteristic polynomials of the two actions coincide.

Proof. (1) Follows from Corollary 3.1.0.4.(1), by taking into account the fact that $W(\pi_f)$ is a direct factor of $s_j^\lambda V'$.
Remark 3.2.0.8. (1) If $\lambda$ is a regular weight of $G$ whose restriction to the center is trivial, the $\ell$-adic realizations $W(\pi_f)_\ell$ of the motive $W(\pi_f)$ coincide with the Galois modules $H^*_c(\pi_{Hf})$ associated to suitable automorphic representations of $G$ in [Fl, Part 2, Chap. I.2, Thm. 2] (notice that, under the regularity assumption, cuspidal cohomology equals interior cohomology, which in turn equals intersection cohomology, as can be seen for example from [H, 3.2.4]). The existence of the motive $W(\pi_f)$ then adds to the description in [Fl] the information about the behaviour at $p$ of the Galois module $W(\pi_f)_p$, which has been obtained in Corollary 3.2.0.7.

(2) Let $l$ be a place of $E$ above the prime $\ell$. The Galois modules $W(\pi_f)_l$ are then of dimension $4^d$ or $\frac{1}{2} \cdot 4^d$ over $E_l$ ([Fl, Part 2, Chap. I.2, Thm. 2.(1),(4)]). One can expect that, in the case of a (Hilbert-Siegel) eigenform $f$, the motives $W(\pi_f)$ over $\mathbb{Q}$ can be written as tensor products over $E$ of rank-4 motives over $F$, whose $L$-function has the correct relation with the $L$-function of $f$. However, there are no known methods for constructing motives with such properties. It is the same problem which arises for motives corresponding to Hilbert modular forms, when cut out inside Kuga-Sato varieties over Hilbert modular varieties, cfr. for example [H, 5.2].

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