ON THE GENERAL FORM OF LINEAR FUNCTIONAL ON
THE HARDY SPACES $H^1$ OVER COMPACT ABELIAN GROUPS
AND SOME OF ITS APPLICATIONS

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1 Introduction

It is well known that the dual spaces $H^1$ and $BMO$ play a fundamental role in PDEs
and some other branches of analysis because they are often the natural replacements
for $L^1$ and $L^\infty$.

Hardy spaces $H^p$ over compact Abelian groups with totally ordered duals were
introduced by Helson and Lowdenslager [1], see also [2, Chapter 8]. In this note we
generalize the celebrated Fefferman’s theorems on the dual of the complex and real
Hardy spaces $H^1$ on the circle group (see [3], [4]) to the case of arbitrary compact
and connected Abelian group and give some applications of these results to lacunary
multiple Fourier series, multidimensional Hankel operators, and atomic theory on the
two dimensional torus. Our main tool in this study is the theory of Hilbert transform
on (locally-)compact Abelian groups [2], [5].

In the following $G$ stands for compact Abelian group with the normalized Haar
measure $m$ and totally ordered dual $X$, $X_+ := \{ \chi \in X : \chi \geq 1 \}$ the positive cone
in $X$ (1 denotes the unit character). As is well known, a (discrete) Abelian group $X$
can be totally ordered if and only if it is torsion-free (see, for example, [2]), which
in turn is equivalent to the condition that its character group $G$ is connected [6];
the total order on $X$ here is not, in general, unique. In applications, often $X$ is
a dense subgroup of $\mathbb{R}^n$ endowed with the discrete topology so that $G$ is its Bohr
compactification, or $X = \mathbb{Z}^n$ so that $G = \mathbb{T}^n$ is the $n$-torus ($\mathbb{T}$ is the circle group and
$\mathbb{Z}$ is the group of integers). For other examples we refer to [7].

We denote by $\hat{\varphi}$ the Fourier transform of $\varphi \in L^1(G)$, and by $\| \cdot \|_\infty$ the norm in
$L^\infty(G)$. We put also

$$
\| f \|_p = \left( \int_G |f|^p dm \right)^{1/p}
$$

for $f \in L^p(G)$ $(0 < p < \infty)$.

The Hardy space $H^p(G)$ $(1 \leq p \leq \infty)$ over $G$ (with respect to the distinguished
order on $X$) is the subspace of $L^p(G)$ defined as follows

$$
H^p(G) = \{ f \in L^p(G) : \hat{f}(\chi) = 0 \ \forall \chi \notin X_+ \}.
$$
In particular $H^2(G)$ is the subspace of $L^2(G)$ with Hilbert basis $X_+$. Let $P_+: L^2(G) \to H^2(G)$ be the orthogonal projection, $P_- = I - P_+$.

For every $u \in L^2(G, \mathbb{R})$ there is a unique $\tilde{u} \in L^2(G, \mathbb{R})$ such that $\tilde{u}(1) = 0$ and $u + i\tilde{u} \in H^2(G)$. The linear continuation of the mapping $u \mapsto \tilde{u}$ to the complex $L^2(G)$ is called a \textit{Hilbert transform} on $G$. This operator extends to a bounded operator $H : \varphi \mapsto \tilde{\varphi}$ on $L^p(G)$ for $1 < p < \infty$ (generalized Marcel Riesz’s inequality), in particular $\|H\varphi\|_2 \leq \|\varphi\|_2$ for every $\varphi \in L^2(G)$ \cite{2, 8.7}; \cite[Theorem 8, Corollary 20]{3}. Note also that the Hilbert transform is a continuous map from $L^1(G)$ to $L^p(G)$ for $0 < p < 1$ (see, e.g.,\cite[Theorem 8.7.6]{2}).

\textbf{Definition 1} \cite{8} (cf. \cite[p. 189]{9}). We define the space $\text{BMO}(G)$ of functions of bounded mean oscillation on $G$ and its subspace $\text{BMOA}(G)$, as follows

$$\text{BMO}(G) := \{ f + \tilde{g} : f, g \in L^\infty(G) \}, \quad \text{BMOA}(G) := \text{BMO}(G) \cap H^1(G),$$

$$\|\varphi\|_{\text{BMO}} := \inf \{ \|f\|_\infty + \|g\|_\infty : \varphi = f + \tilde{g}, f, g \in L^\infty(G) \} \quad (\varphi \in \text{BMO}(G)).$$

\textbf{Lemma 1.} The following equalities hold:

1. $\text{BMO}(G) = P_- L^\infty(G) + P_+ L^\infty(G)$, with an equivalent norm

$$\|\varphi\|_* := \inf \{ \max(\|f_1\|_\infty, \|g_1\|_\infty) : \varphi = P_- f_1 + P_+ g_1, f_1, g_1 \in L^\infty(G) \};$$

2. $\text{BMOA}(G) = P_+ L^\infty(G)$. Moreover, for the norm

$$\|\varphi\|_* := \inf \{ \|h\|_\infty : \varphi = P_+ h, h \in L^\infty(G) \}$$

in this space the following inequalities take place:

$$\frac{2}{3}\|\varphi\|_{\text{BMO}} \leq \|\varphi\|_* \leq 2\|\varphi\|_{\text{BMO}}.$$  

\textbf{Proof.} The statement of the Lemma, except for the left hand side of the last inequality is contained in \cite[Proposition 3]{8}. But the equality $\varphi = P_+ h, h \in L^\infty(G)$ implies $\varphi = f + \tilde{g}$, where $f = 1/2(h + \tilde{h}(1)), g = i/2h \in L^\infty(G)$ \cite[Lemma 2]{8}. Thus $\|\varphi\|_{\text{BMO}} \leq \|f\|_\infty + \|g\|_\infty \leq 3/2\|h\|_\infty$, and it left to go to the infimum when $h$ runs through the space $L^\infty(G)$. \hfill $\Box$

\section{Duality theorems}

In the following we denote by $Y^*$ the dual of the Banach space $Y$.

\textbf{Theorem 1.} For every $\varphi \in \text{BMOA}(G)$ the formula

$$F(f) = \int_G f \varphi dm$$

\footnote{Here we correct a typo made in \cite[p. 139]{8}.}
defines a linear functional on $H^\infty(G)$, and this functional extends uniquely to a continuous linear functional $F$ on $H^1(G)$. Moreover, the correspondence $\varphi \mapsto F$ is an isometrical isomorphism of $(BMOA(G), \| \cdot \|)$ and $H^1(G)^*$, and a topological isomorphism of $(BMOA(G), \| \cdot \|_{BMO})$ and $H^1(G)^*$.

**Proof.** Let $\varphi \in BMOA(G)$ and the functional $F$ on $H^\infty(G)$ is defined by the formula (1). By Lemma 1, $\varphi = P_+ h$, where $h \in L^\infty(G)$. Moreover, $F(f) = \int_G f P_+ h dm = \int_G P_+ fh dm = \int_G f h dm$, which implies that $|F(f)| \leq \|h\|_\infty \|f\|_1$ ($f \in H^\infty(G)$). Passing to the infimum over $h$, we get $|F(f)| \leq \|\varphi\|_\ast \|f\|_1$ for every $f \in H^\infty(G)$. Since $H^\infty(G)$ is dense in $H^1(G)$ [Lemma 1], $F$ extends uniquely to a continuous linear functional $F$ on $H^1(G)$, and $\|F\| \leq \|\varphi\|_\ast$.

Conversely, for every linear functional $F \in H^1(G)^*$ there is a norm preserving extension $F$ to $L^1(G)$. Therefore there is such $g \in L^\infty(G)$, that

$$F(f) = \int_G f \overline{g} dm \ (f \in L^1(G)), \text{ and } \|F\| = \|g\|_\infty.$$ 

So for every $f \in H^\infty(G)$ we have

$$F(f) = \int_G P_+ f \overline{g} dm = \int_G f \overline{P_+ g} dm.$$ 

Thus $F$ has the representation (1) with $\varphi = P_+ g \in BMOA(G)$ (Lemma 1). Moreover, $\|F\| = \|g\|_\infty \geq \|\varphi\|_\ast$. It follows that $\|F\| = \|\varphi\|_\ast$. We conclude that the linear map $(BMOA(G), \| \cdot \|_\ast) \to H^1(G)^*$, $\varphi \mapsto F$ is surjective and isometric and as a result it is bijective. Application of Lemma 1 completes the proof. □

Note that by *trigonometric polynomial on $G$* we as usual mean the linear combination of characters (with complex coefficients in general).

**Definition 2.** We define the space $H^1_\mathbb{R}(G)$ (the real $H^1$ space on $G$) as the completion of the space $\text{Pol}(G, \mathbb{R})$ of real-valued trigonometric polynomials on $G$ with respect to the norm

$$\|q\|_{1\ast} := \|P_- q\|_1 + \|P_+ q\|_1.$$ 

We denote the norm in $H^1_\mathbb{R}(G)$ by $\| \cdot \|_{1\ast}$, too.

In the next proposition we list several impotent properties of $H^1_\mathbb{R}(G)$.

**Proposition 1.** (i) Projectors $P_\pm$, and the Hilbert transform $\mathcal{H}$ are bounded operators on $H^1_\mathbb{R}(G)$;
(ii) restrictions $P_{x}\mid Pol(G, \mathbb{R})$ extend to bounded operators $P_{x}^{1}$ from $H_{x}^{1}(G)$ to $L^{1}(G)$ and
\[ \|f\|_{x} = \|P_{-}f\|_{x} + \|P_{+}f\|_{x} = \|P_{-}^{1}f\|_{1} + \|P_{+}^{1}f\|_{1} (f \in H_{x}^{1}(G)) ; \]

(iii) $H_{x}^{1}(G) = \text{Im}P_{-} + \text{Im}P_{+}$ (the direct sum of closed subspaces);

(iv) $\bigcup_{n > 1}L^{p}(G, \mathbb{R}) \subset H_{x}^{1}(G) \subset L^{1}(G, \mathbb{R})$;

(v) $\|f\|_{H} := \|f\|_{1} + \|Hf\|_{1}$ is an equivalent norm in $H_{x}^{1}(G)$;

(vi) $H_{x}^{1}(G) = \text{Re}H^{1}(G)$.

**Proof.** (i) The boundedness of $P_{x}$ follows from the inequalities $\|P_{x}q\|_{x} = \|P_{x}q\|_{1} \leq \|q\|_{1}$, and the boundedness of the Hilbert transform is the consequence of the equality $iHq = 2P_{+}q - q - 2\hat{q}(1)$ [5, Lemma 22], since $\|\hat{q}(1)\|_{1} = \|q\|_{1} \leq \|q\|_{1}$ ($q \in Pol(G, \mathbb{R})$).

(ii) Inequalities $\|P_{x}q\|_{1} \leq \|q\|_{1}$ implies that $P_{x}$ extend to bounded operators $P_{x}^{1}$ from $H_{x}^{1}(G)$ to $L^{1}(G)$. The first equality follows from (i) and the equality $\|\hat{q}\|_{1} = \|P_{+}q\|_{1} + \|P_{-}q\|_{1}$ ($q \in Pol(G, \mathbb{R})$).

Now we claim that there is a continuous embedding $H_{x}^{1}(G) \subset L^{1}(G, \mathbb{R})$. Indeed, the norms $\cdot \parallel_{1}$ and $\cdot \parallel_{1}$ in $Pol(G, \mathbb{R})$ are comparable, since $\|q\|_{1} \leq \|q\|_{1}$ for $q \in Pol(G, \mathbb{R})$. Moreover, they are compatible in the sense that every sequence $(q_{n})$, which is fundamental with respect to both norms and converges to the zero element with respect to $\cdot \parallel_{1}$, also converges to the zero element with respect to $\cdot \parallel_{1}$. [10, p. 13]. For the proof of this statement first note that $\hat{q}_{n} \to 0$ (the uniform convergence on $X$). Since
\[ \|P_{-}(q_{n} - q_{m})\|_{1} + \|P_{+}(q_{n} - q_{m})\|_{1} = \|q_{n} - q_{m}\|_{1} \to 0 (m, n \to \infty), \]
\[ \|P_{x}q_{n} - h_{x}\|_{1} \to 0 \text{ for some } h_{x} \in L^{1}(G, \mathbb{R}), \text{ and therefore } \hat{P}_{x}q_{n} \to \hat{h}_{x} \text{ (the uniform convergence on } X). \]

On the other hand $2P_{+}q_{n} = iHq_{n} + q_{n} + 2\hat{q}_{n}(1)$ and therefore
\[ 2\hat{P}_{x}q_{n} = iH\hat{q}_{n} + \hat{q}_{n} + 2\hat{q}_{n}(1) = (i\text{sgn} X_{x} + 1)\hat{q}_{n} + 2\hat{q}_{n}(1) \to 0 (n \to \infty) \]
(see [5, Lemma 5]). So $h_{x} = \hat{h}_{x}$, $\|q_{n}\|_{1} \to 0$ and the continuous embedding $H_{x}^{1}(G) \subset L^{1}(G, \mathbb{R})$ follows [10, p. 14].

Hence for $f \in H_{x}^{1}(G)$ and a sequence $q_{n} \in Pol(G, \mathbb{R})$ such that $\|q_{n} - f\|_{1} \to 0$ we have $\|q_{n} - f\|_{1} \to 0$ and thus
\[ \|f\|_{1} = \lim_{n} \|q_{n}\|_{1} = \lim_{n}(\|P_{-}q_{n}\|_{1} + \|P_{+}q_{n}\|_{1}) = \|P_{-}^{1}f\|_{1} + \|P_{+}^{1}f\|_{1}. \]

(iii) Since $P_{-}, P_{+}$ are bounded projectors on $H_{x}^{1}(G)$, their images $\text{Im}P_{-}, \text{Im}P_{+}$ are closed subspaces of $H_{x}^{1}(G)$. The equality $H_{x}^{1}(G) = \text{Im}P_{-} + \text{Im}P_{+}$ follows from the boundedness of $P_{x}$. Now if $f \in \text{Im}P_{-} \cap \text{Im}P_{+}$, the property (ii) implies that $\|f\|_{1} = 0$. 

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L. Then map the Hilbert transform, and therefore $\|\hat{f}\|_{1} = \|\hat{f}\|_{1}$ for some $\xi \in \mathcal{F}(G)$. By the definition of the Hilbert transform

\[
\mathcal{H} f = \int_{G} f(x) \frac{1}{\pi} \frac{e^{ix\xi}}{x} \, dx,
\]

and therefore $\|\mathcal{H} f\|_{1} = \|f\|_{1}$ for some $\xi \in \mathcal{F}(G)$. Thus $P_{+} q_{n} \to \Phi f$ and $P_{-} q_{n} \to (I - \Phi) f$ in $L^{1}(G)$. This implies that

\[
\|q_{n} - q_{m}\|_{1} := \|P_{-}(q_{n} - q_{m})\|_{1} + \|P_{+}(q_{n} - q_{m})\|_{1} \to 0 \quad (n, m \to \infty),
\]

and therefore $\|q_{n} - g\|_{1} \to 0$ for some $g \in H_{R}^{1}(G)$. But from (ii) it follows that $P_{\pm} q_{n} \to P_{\pm} g$ in $L^{1}$ norm. So $P_{1} g = \Phi f, P_{2} g = (I - \Phi) f$, and we conclude that $f = P_{1} g + P_{2} g = g \in H_{R}^{1}(G)$.

(vii) Let $u \in H_{R}^{1}(G)$ and $\|q_{n} - u\|_{1} \to 0$ for some $q_{n} \in \mathcal{P}(G)$ $(n \to \infty)$. In view of (v) $\|q_{n} - u\|_{1} \to 0$ and $\|\mathcal{H} q_{n} - \mathcal{H} u\|_{1} \to 0$ $(n \to \infty)$. Since, by the definition of the Hilbert transform, $q_{n} + \mathcal{H} q_{n} \to H^{1}(G)$, it follows that $u + i \mathcal{H} u \in H^{1}(G)$ and thus $u \in \mathcal{R} H^{1}(G)$.

Conversely, let $u \in \mathcal{R} H^{1}(G)$ and $f = u + iv \in H^{1}(G)$. Then $\|p_{n} - f\|_{1} \to 0$ $(n \to \infty)$ for some $p_{n} \in \mathcal{P}(G)$ [Lemma 1]. Adding, if necessary, to $f$ a pure imaginary constant, we can assume that $\hat{v}(1) = 0$. Let $p_{n} = q_{n} + ih_{n}$, where $q_{n}, h_{n} \in \mathcal{P}(G)$. Then $\|q_{n} - u\|_{1} \to 0$, $\|h_{n} \to v\|_{1} \to 0$, and $\hat{h}_{n}(1) \to \hat{v}(1) = 0$ $(n \to \infty)$. Replacing, if necessary, $p_{n}$ with $p_{n} - ih_{n}(1)$ we can assume that $\hat{h}_{n}(1) = 0$. Fix $p \in (0, 1)$. Since the Hilbert transform continuously maps $L^{1}(G)$ into $L^{p}(G)$ [Chapter 8], we have $\|\mathcal{H} q_{n} - \mathcal{H} u\|_{p} \to 0$. By the definition of the Hilbert transform $\mathcal{H} q_{n} = h_{n}$, which implies that $\|h_{n} - \mathcal{H} u\|_{p} \to 0$. On the other hand, $\|h_{n} - v\|_{p} \leq \|h_{n} - v\|_{1} \to 0$ and therefore $v = \mathcal{H} u$. It follows that

\[
\|q_{n} - u\|_{H} = \|q_{n} - u\|_{1} + \|\mathcal{H} q_{n} - \mathcal{H} u\|_{1} = \|q_{n} - u\|_{1} + \|h_{n} - v\|_{1} \to 0 \quad (n \to \infty),
\]

and the application of the statement (v) finishes the proof. \hfill \Box

Now we are in position to prove the real version of Feffermans’ duality theorem. By $BMO(G, \mathbb{R})$ we denote the subspace of real-valued functions from $BMO(G)$.

**Theorem 2.** For every $\varphi \in BMO(G, \mathbb{R})$ the linear functional

\[
F(q) = \int_{G} q \varphi \, dm
\]

(2)
on \( \text{Pol}(G, \mathbb{R}) \) extends uniquely to a continuous linear functional \( F \) on \( H^1_\mathbb{R}(G) \). Moreover, the correspondence \( \varphi \mapsto F \) is an isometrical isomorphism of \( (BMO(G, \mathbb{R}), \| \cdot \|_{BMO}) \) and \( H^1_\mathbb{R}(G)^* \), and a topological isomorphism of \( (BMO(G, \mathbb{R}), \| \cdot \|_{BMO}) \) and \( H^1_\mathbb{R}(G)^* \).

**Proof.** Let \( \varphi \in BMO(G, \mathbb{R}) \) and the functional \( F \) on \( \text{Pol}(G, \mathbb{R}) \) is defined by the formula (2). By Lemma 1, \( \varphi = P_-g + P_+h \), where \( g,h \in L^\infty(G) \). Then for every \( q \in \text{Pol}(G, \mathbb{R}) \) we have

\[
F(q) = \int_G P_-g \overline{q} dm + \int_G P_+h \overline{q} dm = \int_G g P_-q dm + \int_G h P_+q dm,
\]

which implies that \( |F(q)| \leq \max(\|g\|_\infty, \|h\|_\infty)(\|P_-q\|_1 + \|P_+q\|_1) \). So, \( \|F\| \leq \|\varphi\|_* \) and there is a unique extension of \( F \) to a continuous linear functional \( F \) on \( H^1_\mathbb{R}(G) \) with the same norm.

Conversely, for every linear functional \( F \in H^1_\mathbb{R}(G)^* \) let \( F_- \) and \( F_+ \) denote its restrictions to \( \text{Im}P_- \) and \( \text{Im}P_+ \) respectively. If \( f \in \text{Im}P_\pm, f = P_\pm g \), where \( g \in H^1_\mathbb{R}(G) \), then \( |F_\pm(f)| \leq \|F\|\|P_\pm g\|_1 = \|F\|_1 \). Therefore these functionals extend to linear bounded functionals \( F_- \) and \( F_+ \) on \( L^1(G, \mathbb{R}) \) with preservation of norms. Let \( g_\pm \in L^\infty(G) \) be such that

\[
F_\pm(f) = \int_G f \overline{g_\pm} dm \quad (f \in L^1(G, \mathbb{R})), \quad \|F_\pm\| = \|g_\pm\|_\infty.
\]

It follows that

\[
F(q_\pm) = \int_G P_\pm q_\pm \overline{g_\pm} dm = \int_G q_\pm P_\pm \overline{g_\pm} dm
\]

for \( q_\pm \in P_\pm \text{Pol}(G, \mathbb{R}) \). Since every \( q \in \text{Pol}(G, \mathbb{R}) \) has the form \( q = q_+ + q_- \) where \( q_\pm \in P_\pm \text{Pol}(G, \mathbb{R}) \) we have

\[
F(q) = F(q_+) + F(q_-) = \int_G q_+ P_+ \overline{g_+} dm + \int_G q_- P_- \overline{g_-} dm = \int_G q \overline{(P_+ g_+) + (P_- g_-)} dm.
\]

Thus if we put \( \varphi := \overline{P_+ g_+} + \overline{P_- g_-} \), the equality (2) holds. Since \( F(q) \) is real-valued for every \( q \in \text{Pol}(G, \mathbb{R}) \) so is \( \varphi \). In fact, putting \( q = \chi + \overline{\chi} \) and then \( q = i(\chi - \overline{\chi}) \), we deduce from the equality \( F(q) = \overline{F(q)} \) in view of (2) that for every \( \chi \in X \)

\[
\int_G (\varphi - \overline{\varphi}) \chi dm \pm \int_G (\varphi - \overline{\varphi}) \overline{\chi} dm = 0.
\]

The last two equalities imply that the Fourier transform of the imaginary part of \( \varphi \) equals to zero. Thus \( \varphi = \overline{P_+ g_+} + P_- g_- \in BMO(G, \mathbb{R}) \). Moreover, \( \|F\| \geq \|F_\pm\| = \|g_\pm\|_\infty \). Passing to the infimum over \( g_\pm \), we get \( \|F\| \geq \|\varphi\|_* \), and the rest of the proof is exactly the same as the rest of the proof of Theorem 1. \( \square \)
3 Applications. Lacunary series

Now we apply Theorem 1 in order to generalize some results on lacunary series in one variable (see [9, p. 191, 1.6.4 (a), (d)]). The following definition is implicitly contained in [2, 8.6].

**Definition 3.** We call a subset \( E \subset X_+ \) lacunary (in the sense of Rudin) if there is a constant \( K = K_E \) such that the number of terms of the set \( \{ \xi \in E : \chi \leq \xi \leq \chi^2 \} \) do not exceed \( K \) for every \( \chi \in X_+ \).

**Theorem 3.** Let \( \varphi \in L^1(G) \), and \( \widehat{\varphi} \) vanishes outside some lacunary set \( E \). Then \( \varphi \in BMOA(G) \) if and only if \( \varphi \in H^2(G) \); moreover,

\[
\|\varphi\|_2 \leq \|\varphi\|_{BMO} \leq 3\sqrt{K_E}\|\varphi\|_2.
\]

**Proof.** By the Lemma 1 \( BMOA(G) \subset H^2(G) \). Now let \( \varphi \in H^2(G) \) and \( \widehat{\varphi} \) vanishes outside \( E \subset X_+ \). Then \( \widehat{\varphi} \in l^2(E) \). We claim that the functional

\[
\Lambda(f) := \sum_{\chi \in E} \widehat{f}(\chi)\overline{\widehat{\varphi}(\chi)}
\]

is defined and bounded on \( H^1(G) \). Indeed, for every \( f \in H^1(G) \) the restriction of \( \widehat{f} \) to \( E \) belongs to \( l^2(E) \) by [2, Theorem 8.6]. Moreover, \( \|\widehat{f}\|_{l^2(E)} \leq 2\sqrt{K_E}\|f\|_{1} \) by [2, p. 214, (5)] and therefore

\[
|\Lambda(f)| \leq \|\widehat{f}\|_{l^2(E)}\|\widehat{\varphi}\|_{l^2(E)} \leq 2\sqrt{K_E}\|\widehat{\varphi}\|_{l^2(E)}\|f\|_1.
\]

Hence accordingly to Theorem 1 there exists such \( \varphi_1 \in BMOA(G) \) that

\[
\Lambda(f) = \int_G f\varphi_1 dm \quad (f \in H^\infty(G)).
\]

On the other hand it follows from the Plancherel Theorem that

\[
\Lambda(f) = \sum_{\chi \in X} \widehat{f}(\chi)\overline{\widehat{\varphi}(\chi)} = \int_G f\varphi dm \quad (f \in H^\infty(G)).
\]

for every \( f \in H^\infty(G) \). Now for \( f \in X_+ \) formulas (3) and (4) imply that the function \( \varphi_1 - \varphi_1 \) vanishes on \( X_+ \). Since this function is concentrated on \( X_+ \), we get \( \varphi = \varphi_1 \), which completes the proof of the first statement.

Next, let \( \varphi = f + \tilde{g} \) where \( f, g \in L^\infty(G) \). Since \( \|\tilde{g}\|_2 \leq \|g\|_2 \) [3, Theorem 8], we have

\[
\|\varphi\|_2 \leq \|f\|_2 + \|\tilde{g}\|_2 \leq \|f\|_\infty + \|g\|_\infty.
\]

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and the first inequality follows. On the other hand if the functional $F$ is defined by the formula (1), then $\|F\| = \| \varphi \|_*$ (see the proof of Theorem 1). But (4) implies that $F = \Lambda$, and we already have seen that

$$\|\Lambda\| \leq 2 \sqrt{K_E} \| \hat{\varphi} \|_{l_2(E)} = 2 \sqrt{K_E} \| \varphi \|_2.$$ 

The application of Lemma 1 completes the proof. □

For the next corollary of Theorem 1 recall that Hankel operator $H_{\varphi} : H^2(G) \to H^2_-(G)$ ($H^2_-(G) = L^2(G) \ominus H^2(G)$) with symbol $\varphi \in L^2(G)$ is initially defined on the subspace of trigonometric polynomials of analytic type (the linear span of $X_+$) by the equality

$$H_{\varphi}f = P_-(\varphi f),$$

(see, e.g., [8]).

For compact Abelian groups the fundamental Nehari Theorem for Hankel forms was proved by J. Wang [11]. Its version for Hankel operators looks as follows.

**Theorem (Z. Nehari).** A bounded operator $H : H^2(G) \to H^2_-(G)$ is of the form $H_{\varphi}$ for some $\varphi \in L^\infty(G)$ if and only if

$$HS_\chi = P_- S_\chi H \quad \forall \chi \in X_+, \quad (5)$$

where $S_\chi f := \chi f$ ($f \in L^2(G)$). Moreover, $\|H\| = \|\varphi\|_\infty$ for some $\varphi \in L^\infty(G)$ such that $H = H_{\varphi}$.

For the proof one can consider the bilinear form $(f,g) \mapsto \langle Hf, \overline{g} \rangle$ on $H^2(G) \times H^2(G)$ and apply the result from [11] to the corresponding bilinear Hankel form on $l_2(X_+) \times l_2(X_+)$ ($l_2(X_+)$ is isomorphic to $H^2(G)$ via the Fourier transform; see [12] for details). □

**Theorem 4.** Let $E$ be a lacunary set and $1 \notin E$. Then for every $\varphi \in H^2(G)$ such that $\hat{\varphi}$ vanishes outside $E$ the operator $H_{\varphi}$ is bounded and

$$A_E \|\varphi\|_2 \leq \|H_{\varphi}\| \leq 6 \sqrt{K_E} \|\varphi\|_2 \quad (6)$$

where $A_E$ is independent of $\varphi$.

**Proof.** By virtue of Theorem 3 and Lemma 1 $\varphi = P_+ h, h \in L^\infty(G)$. So $P_-(\varphi - \bar{h}) = 0$, and therefore $H_{\varphi} = H_\varphi$. Consequently $H_{\varphi}$ is bounded and $\|H_{\varphi}\| \leq \|h\|_\infty.$

It follows that $\|H_{\varphi}\| \leq \|\varphi\|_*$, and Lemma 1 and Theorem 3 entail that $\|\varphi\|_* \leq 6 \sqrt{K_E} \|\varphi\|_2$. This proves the second inequality (without assumption that $1 \notin E$).

To prove the first one consider the following subspace of $H^2(G)$:

$$H^2_0(G) := \{ \varphi \in H^2(G) : \hat{\varphi} \text{ vanishes outside } E \}.$$ 

Being isomorphic to $l_2(E)$ via the Fourier transform, this space is complete with respect to the $L^2$ norms. It is easy to verify that the seminorm $\|\varphi\|_H := \|H_{\varphi}\|$ is a
norm in $H^2_{E}(G)$ if $1 \notin E$. We claim that $H^2_{E}(G)$ is complete with respect to this norm as well. Indeed, let the sequence $(\varphi_n) \subset H^2_{E}(G)$ be fundamental with respect to $\| \cdot \|_H$. Then $\|H_{\varphi_n} - H\| \to 0$ ($n \to \infty$) for some bounded operator $H, H : H^2(G) \to H^2_{E}(G)$.

But it is easy to verify that $H_{\varphi_n} \sigma G \xi = P_{-} S_{\chi} H_{\varphi_n} \xi, \xi \in X$+. Since the operator $H_{\varphi_n}$ is bounded, it follows that it satisfies the condition (5) and therefore $H$ satisfies (5), too. So by Nehari's Theorem $H = H_\tau$ for some $g \in L^\infty(G)$. Let $f := P_\tau g$. Then $\widehat{f}$ vanishes outside $X_-$ and $\|H_{\varphi_n - \hat{f}}\| \to 0$ ($n \to \infty$). Again by Nehari's Theorem there is such $\varepsilon_n \in L^\infty(G)$ that $H_{\varphi_n - \hat{f}} = H_{\varphi_n}$ and $\|H_{\varphi_n - \hat{f}}\| = \|\varepsilon_n\|_\infty$. Put $\psi_n := \varphi_n - \hat{f} - \varepsilon_n$. Then $\psi_n \in H^2(G)$, since $H_{\psi_n} = 0$. Thus for every $\chi \in X_+ \setminus E, \chi \neq 1$ we have

$$\widehat{f}(\chi) = \varphi_n(\chi) - \psi_n(\chi) - \varepsilon_n(\chi) = -\varepsilon_n(\chi),$$

and therefore $|\widehat{f}(\chi)| \leq \|\varepsilon_n\|_\infty$. It follows that $f \in H^2_{E}(G)$ and since $\|\varphi_n - f\|_H \to 0$ ($n \to \infty$) the space $(H^2_{E}(G), \| \cdot \|_H)$ is complete. In view of the second inequality in (6) and the well known Banach Theorem, the norms $\| \cdot \|_2$ and $\| \cdot \|_H$ are equivalent in $H^2_{E}(G)$. □

**Remark 1.** If we take $G = \mathbb{T}^n$, the $n$-dimensional torus, and choose some linear order on its dual group $\mathbb{Z}^n$ Theorems 3 and 4 turn into results on lacunary multiple Fourier series and multidimensional Hankel operators. A description of all linear orders on $\mathbb{Z}^n$ one can find in [13, 14]. The case of infinite dimensional torus $\mathbb{T}^\infty$ (see, e.g., [7, Examples 2, 3]) is also of interest.

### 4 Some results related to atomic theory on $H^1_{\mathbb{R}}(\mathbb{T}^2)$

The "atomic" theory for functions from $H^1_{\mathbb{R}}(\mathbb{T})$ was developed in [15]. A general approach to atomic decompositions was proposed by Coifman and Weiss [16], but their notion of an atom [16, p. 591] differs from ours (see Definition 4 below; a remarkable feature of our atoms is that these atoms have only partial cancellation conditions), and the definition of Hardy spaces in [16, p. 592] differs from ours, too. For more recent results in this area see, e.g., [17 — 21].

The problem of developing an atomic theory for Hardy spaces on the polydisc was posed in [16, p. 642]. In this section, we get some results related to an atomic theory for $H^1_{\mathbb{R}}(\mathbb{T}^2)$. It should be noted that we consider the last space with respect to the lexicographic order on the dual group $\mathbb{Z}^2$ of $\mathbb{T}^2$.

**Definition 4.** By a $\mathbb{T}^2$-atom we mean either the function $1$ or a real-valued function $a(\theta_1, \theta_2)$ supported on a rectangle $J_1 \times J_2 \subset \mathbb{T}^2$ having the property

(i) $|a(\theta_1, \theta_2)| \leq \min\{1/|J_1|, 1/|J_2|\}$,

(ii) $\int_{J_1} a(\theta, \theta_2) d\theta_1 = 0 = \int_{J_2} a(\theta_1, \theta) d\theta_2$ for every $(\theta_1, \theta_2) \in J_1 \times J_2$.

By a $\mathbb{T}^1$-atom we mean either the function $1$ or a real-valued function $a(\theta)$ on $\mathbb{T}$ supported on a rectangle $J_1 \times \mathbb{T}$ (respectively $\mathbb{T} \times J_2$) having the property
\[(i') \quad |a(\theta_i)| \leq 1/|J_i|,
(ii') \quad \int a(\theta_i) d\theta_i = 0.\]

Above \(J_i\) denotes an arc in \(T\) with normalized Lebesgue measure \(|J_i| (|T| = 1), i = 1, 2.\)

**Proposition 2.** Atoms form a bounded subset of \(H^1_T(T^2)\) (and generate a dense subspace of this space).

We need two preliminary results to prove Proposition 2.

**Proposition 3.** The Hilbert transform in \(L^2(T^2)\) (with respect to the lexicographic order on \(\mathbb{Z}^2\)) has the form

\[
\mathcal{H}f(t_1, t_2) = P.V. \int_{-\pi}^{\pi} f(\theta_1, t_2) \cot \frac{t_1 - \theta_1}{2} \frac{d\theta_1}{2\pi} \\
+ \int_{-\pi}^{\pi} P.V. \int_{-\pi}^{\pi} f(\theta_1, \theta_2) \cot \frac{t_2 - \theta_2}{2} \frac{d\theta_2}{2\pi} \frac{d\theta_1}{2\pi} \\
= \mathcal{H}_{\theta_1\rightarrow t_1} f(\theta_1, t_2) + \int_{-\pi}^{\pi} (\mathcal{H}_{\theta_2\rightarrow t_2} f(\theta_1, \theta_2)) \frac{d\theta_1}{2\pi} \tag{7}
\]

where \(\mathcal{H}_1\) and \(\mathcal{H}_2\) stand for the Hilbert transform in \(L^2(T)\) in the first and the second variable independently.

**Proof.** First note that both summands in the right-hand side of formula (7) are continuous linear operators on \(L^2(T^2)\). And as was mentioned in the introduction of this paper the left-hand side of (7) is continuous, too. So it remains to verify (7) for functions of the form \(f = u_i \otimes u_j\) where \(u_i\) and \(u_j\) run over some orthogonal base of \(L^2(T)\). To this end note that the equality \(g = \mathcal{H}f\) is equivalent to

\[
\hat{g} = -i \text{sgn}_{X_+} \hat{f},
\]

where \(X_+\) is the positive cone in \(\mathbb{Z}^2\) with respect to the lexicographic order, and \(\text{sgn}_{X_+}(n_1, n_2) := 1(-1)\) for \((n_1, n_2) \in X_+ \setminus \{0\}\) (respectively \((n_1, n_2) \notin X_+\)), \(\text{sgn}_{X_+}(0, 0) := 0\). It is easy to verify that

\[
\text{sgn}_{X_+}(n_1, n_2) = \text{sgn}_{Z_+}(n_1) + \text{sgn}_{\{0\} \times Z_+}(n_1, n_2). \tag{8}
\]

Since \(\hat{f}(n_1, n_2) = \hat{u}_i(n_1) \hat{u}_j(n_2)\) for all \((n_1, n_2) \in \mathbb{Z}^2\), we have using (8)

\[
\hat{g}(n_1, n_2) = -i \text{sgn}_{X_+}(n_1, n_2) \hat{f}(n_1, n_2) \\
= (-i \text{sgn}_{Z_+}(n_1) \hat{u}_i(n_1)) \hat{u}_j(n_2) + \hat{u}_i(0)(-i \text{sgn}_{Z_+}(n_2) \hat{u}_j(n_2)).
\]

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This implies that
\[ g(t_1, t_2) = (H u_i(t_1)) u_j(t_2) + \hat{v}_i(0) H u_j(t_2) \]
\[ = \text{P.V.} \int_{-\pi}^{\pi} f(\theta_1, t_2) \cot \frac{t_1 - \theta_1}{2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta_1, \theta_2) \cot \frac{t_2 - \theta_2}{2} \frac{d\theta_1}{2\pi}, \]
concluding the proposition. □

Lemma 2. There is a universal constant \( C > 0 \) such that \( \|a\|_{1*} \leq C \) for every \( \mathbb{T}^2 \)- or \( \mathbb{T}^1 \)-atom \( a \).

Proof. We shall use the statement (v) of Proposition 1. It is evident that \( \|a\|_1 \leq 1 \) for every \( \mathbb{T}^2 \)- or \( \mathbb{T}^1 \)-atom \( a \).

By Proposition 3 \( Ha(t_1, t_2) = A_1(t_1, t_2) + A_2(t_2) \), where
\[ A_1(t_1, t_2) = \mathcal{H}_1 a(t_1, t_2) = \text{P.V.} \int_{-\pi}^{\pi} a(\theta_1, t_2) \cot \frac{t_1 - \theta_1}{2} \frac{d\theta_1}{2\pi}, \]
and
\[ A_2(t_2) = \int_{-\pi}^{\pi} \mathcal{H}_2 a(\theta_1, t_2) \frac{d\theta_1}{2\pi}. \]

Let \( a \) be a \( \mathbb{T}^1 \)-atom, \( a \neq 1 \). If \( a \) depends of \( \theta_1 \) only, then \( \|A_1\|_1 = \|\mathcal{H}_1 a\|_{L^1(dt_1/2\pi)} \leq \text{const} \) by the classical result on \( \mathbb{T}^1 \)-atoms (see, e. g. [22, p. 27]), and if \( a \) depends of \( \theta_2 \) only, we get \( A_1 = 0 \).

Now let \( a \) be a \( \mathbb{T}^2 \)-atom, \( a \neq 1 \). If \( J_1 \) is the arc appearing in (i) and (ii), we can assume, without loss of generality, that \( J_1 = (-\delta; \delta) \). As in the one dimensional case we can assume also, that \( \delta < 1/2 \), since \( |a| \leq 1 \) for \( \delta \geq 1/2 \). Then we have
\[ \|A_1\|_1 = \int_{J_2} \left( \int_{-2\delta}^{2\delta} |A_1(t_1, t_2)| \frac{dt_1}{2\pi} + \int_{2\delta < |t_1| < \pi} |A_1(t_1, t_2)| \frac{dt_1}{2\pi} \right) \frac{dt_2}{2\pi} \]
\[ = \int_{J_2} (I_1(t_2) + I_2(t_2)) \frac{dt_2}{2\pi}. \]

Using the Hölder’s and generalized Marcel Riesz’s inequality, and property (i) of atom we obtain (below we denote by \( C \) or const any universal constant)
\[ I_1(t_2) = \int_{-2\delta}^{2\delta} |A_1(t_1, t_2)| \frac{dt_1}{2\pi} \leq C \sqrt{\delta} \|A_1(\cdot, t_2)\|_{L^2(dt_1/2\pi)}. \]
Therefore, \( \int_{J_2} I_1(t_2) dt_2 / 2\pi \leq \text{const.} \)

To estimate \( I_2 \) we use the following classical estimate for the one-dimensional case (see, e.g., [22, p. 28])

\[
|A_1(t_1, t_2)| = |\mathcal{H}_1 a(t_1, t_2)| \leq C\delta \|a(\cdot, t_2)\|_{L_1(\frac{dt_1}{2\pi})} t_1^{-2} (|t_1| \geq 2\delta).
\]

(9)

It follows that

\[
I_2(t_2) = \int_{2\delta < |t_1| < \pi} |A_1(t_1, t_2)| \frac{dt_1}{2\pi} \leq C\delta \int_{J_1} |a(\theta_1, t_2)| \frac{d\theta_1}{2\pi} \int_{2\delta < |t_1| < \pi} t_1^{-2} \frac{dt_1}{2\pi} \leq \text{const},
\]

and therefore \( \int_{J_2} I_2(t_2) dt_2 / 2\pi \leq \text{const.} \), as well. We conclude that

\[
\|A_1\|_1 = \int_{-\pi}^{\pi} |\mathcal{H}_1 a(t_1, t_2)| \frac{dt_1}{2\pi} \frac{dt_2}{2\pi} \leq \text{const.}
\]

(10)

As regards \( A_2 \), the case when \( a \) is a \( T^1 \)-atom can be considered as above. Let \( a \) be a \( T^2 \)-atom, \( a \neq 1 \). Then by the Fubini’s theorem we get in view of (10) that

\[
\|A_2\|_1 = \|A_2\|_{L^1(\frac{dt_1}{2\pi})} \leq \int_{-\pi}^{\pi} |\mathcal{H}_2 a(\theta_1, t_2)| \frac{dt_2}{2\pi} \frac{d\theta_1}{2\pi} \leq \text{const.}
\]

So \( \|\mathcal{H} a\|_1 \leq \text{const} \) for every \( T^2 \)- or \( T^1 \)-atom \( a \), as required. \( \square \)

**Proof of Proposition 2.** The first statement of the proposition follows from Lemma 2. To prove the second one consider a linear functional \( F \in H_2^1(T^2)^* \) such that the restriction of \( F \) to the subspace generated by atoms be zero. By Theorem 2 there is a function \( \varphi \in BMO(T^2) \) such that (2) holds. Using in formula (2) the atoms \( q_1 = (x + \bar{x})/2 \) and \( q_2 = (x - \bar{x})/(2i) \) \( (x \in X) \), one deduces that \( \tilde{\varphi} = 0 \) and therefore \( F = 0 \). The application of Hahn-Banach Theorem completes the proof. \( \square \)

**Remark 2.** Let \( H^1_{at} \) denotes the vector space of all function of the form \( f = \sum_{j=1}^{\infty} \lambda_j a_j \) where \( a_j \) are \( T^2 \)- or \( T^1 \)-atoms, \( \lambda_j \in \mathbb{R} \), and \( \sum_{j=1}^{\infty} |\lambda_j| < \infty \) endowed with the "atomic" norm

\[
\|f\|_{at} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.
\]

By Proposition 2 there is a continuous (and dense) embedding of \( H^1_{at} \) in \( H^1_{R}(T^2) \). By Theorem 2 this implies a continuous embedding of \( BMO(T^2) \) into \( (H^1_{at})^* \). In view of Hahn-Banach Theorem to prove the equality \( H^1_{R}(T^2) = H^1_{at} \) it remains to prove a continuous embedding of \( (H^1_{at})^* \) into \( BMO(T^2) \). The problem of the existence of such embedding (and therefore of an atomic decompositions for functions from \( H^1_{R}(T^2) \)) seems to be open.
References

[1] H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, Acta Math. 99 (1958), 165 – 202.

[2] W. Rudin, Fourier analysis on groups, Interscience Publishers, New York and London, 1962.

[3] C. Fefferman, Characterization of bounded mean oscillation, Bull. Amer. Math. Soc., 77 (1971), 587 – 588.

[4] J. B. Garnett, Bounded analytic functions, Academic Press, New York–London-Toronto–Sydney–San Francisco, 1981.

[5] A. R. Mirotin, On Hilbert Transform in Context of Locally Compact Abelian Groups, Int. J. Pure Appl. Math., 51:4 (2009), 463 – 474.

[6] L. S. Pontryagin, Topological groups, 2nd ed., GITTL, Moscow 1954; English transl., Gordon and Breach, New York–London–Paris, 1966.

[7] A. R. Mirotin, Fredholm and spectral properties of Toeplitz operators on the spaces $H_p$ over ordered groups, Sbornik Math., 202:5 (2011), 101 – 116, [http://dx.doi.org/10.1070/SM2011v202n05ABEH004163](http://dx.doi.org/10.1070/SM2011v202n05ABEH004163).

[8] R. V. Dyba, A. R. Mirotin, Functions of Bounded Mean Oscillation and Hankel Operators on Compact Abelian Groups, Trudy Institut matematiki i mehaniki UrO RAN, 20:2 (2014), 135 – 144. (In Russian).

[9] N. K. Nikolski, Operators, Functions, and Systems: An Easy Reading: in 2 vol., Vol. I. Amer. Math. Soc., 2002.

[10] I. M. Gelfand, G. E. Shilov, Generalized Functions. Vol. 2. Spaces of Fundamental and Generalized Functions, Academic Press, New York and London, 1968.

[11] J. Wang, Note on Theorem of Nehari on Hankel forms, Proc. Amer. Math. Soc., 24:1 (1970), 103 – 105.

[12] R. V. Dyba, Nehari Theorem on compact Abelian groups with totally ordered dual, Problemy fiziki, matematiki i tehnik, no 3 (8) (2010), 57 – 60 (Russian).

[13] H. H. Teh, Construction of orders in Abelian groups, Proc. Cambridge Phil. Soc. 57:3 (1961), 476 – 482.

[14] M. I. Zajtceva, On the set of orderings of Abelian groups, Uspehi Mat. Nauk, 8:1 (1953), 135 – 137 (Russian).
[15] R. P. Coifman, A real variable characterization of $H^p$, Stud. math. (PRL), 51:3 (1974), 269 – 274.

[16] R. P. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83:4 (1977), 569 — 645.

[17] M. Bownik, Boundedness of operators on Hardy spaces via atomic decompositions, Proc. Amer. Math. Soc., 133 (2005), 3535 – 3542.

[18] K.-P. Ho, Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces, Analysis Mathematica, 38 (2012), 173 – 185, DOI: 10.1007/s10476-012-0302-5.

[19] D. Yang and Y. Zhou, A boundedness criterion via atoms for linear operators in Hardy spaces, Constr. Approx. 29 (2009), no. 2, 207–218.

[20] D. Yang and Y. Zhou, Boundedness of sublinear operators in Hardy spaces on $RD$-spaces via atoms, J. Math. Anal. Appl. 339 (2008), 622—635.

[21] S. Dekel et. al. A new proof of the atomic decomposition of Hardy spaces, Sep. 1, 2014. Preprint, arXiv: 1409.0419 v. 1.

[22] G. Hoepfner, Hardy spaces, its variants and applications, Sierra Negra-SP, Brasil, 2007.