Veronese subspace codes

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Proposed Running Head: Veronese subspace codes

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Abstract

Using the geometry of quadrics of a projective plane $\text{PG}(2, q)$ a
family of $(6, q^3(q^2 - 1)(q - 1)/3 + (q^2 + 1)(q^2 + q + 1), 4; 3)_q$ constant
dimension subspace codes is constructed.

KEYWORDS: projective bundle; constant dimension subspace code; Singer
cyclic group; Veronese map;
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1 Introduction

Let $V$ be an $n$–dimensional vector space over $\text{GF}(q)$, $q$ any prime power.
The set $S(V)$ of all subspaces of $V$, or subspaces of the projective space
$\text{PG}(V)$, forms a metric space with respect to the subspace distance defined
by $d_s(U, U') = \dim(U + U') - \dim(U \cap U')$. In the context of subspace
codes, the main problem is to determine the largest possible size of codes
in the space $(S(V), d_s)$ with a given minimum distance, and to classify the
Corresponding optimal codes. The interest in these codes is a consequence of
the fact that codes in the projective space and codes in the Grassmannian
over a finite field referred to as subspace codes and constant–dimension
codes, respectively, have been proposed for error control in random linear
network coding. An $(n, M; d; k)_q$ constant–dimension subspace code (CDC)
is a set $C$ of $k$–subspaces of $V$ with $|C| = M$ and minimum subspace distance
d_s(C) = \min\{d_s(U, U') \mid U, U' \in C, U \neq U'\} = d$. The smallest open constant–
dimension case occurs when $n = 6$ and $k = 3$. From a projective geometry
point of view it translates in the determination of the maximum number of
planes in $\text{PG}(5, q)$ mutually intersecting in at most one point. In [10], the
authors show that the maximum size of a binary subspace code of packet
length $n = 6$, minimum subspace distance $d = 4$ and constant dimension $k = 3$ is $M = 77$. Therefore the maximum number of planes in $\text{PG}(5, 2)$ mutually
intersecting in at most one point is 77. In the same paper, the authors,
with the aid of a computer, classify all $(6, 77, 4; 3)_2$ subspace codes into 5
isomorphism types [10, Table 6] and a computer–free construction of one
isomorphism type [10, Table 6, A] is provided. This last isomorphism type
is then generalized to any $q$ providing a family of $(6, q^6 + 2q^2 + 2q + 1, 4; 3)_q$
subspace codes [10, Lemma 12]. In [6] the authors provided a construction of families of $(6, q^6 + 2q^2 + 2q + 1, 4; 3)_q$ subspace codes potentially including
the infinite family constructed in [10].

In this paper we construct a family of $(6, q^3(q^2 - 1)(q - 1)/3 + (q^2 + 1)(q^2 + q + 1), 4; 3)_q$ CDC. Our approach is purely geometric and the con-
struction relies on the geometry of quadrics of a projective plane $\text{PG}(2, q)$. More precisely, we use the correspondence between quadrics of $\text{PG}(2, q)$ and points of $\text{PG}(5, q)$. In this setting, we show that a special net of conics (circumscribed bundle) yields a $(6, q^3(q^2 - 1)(q - 1)/3, 4; 3)_q$ CDC admitting the linear group $\text{PGL}(3, q)$ as an automorphism group. Although the size of such a code asymptotically reaches the theoretical upper bound of a $(6, M, 4; 3)_q$ CDC [10], it turns out that it can be enlarged. This is done in the second part of the paper where we are able to find a set of further $(q^2 + 1)(q^2 + q + 1)$ planes of $\text{PG}(5, q)$ mutually intersecting in at most one point and extending the previous code. The $(6, q^3(q^2 - 1)(q - 1)/3 + (q^2 + 1)(q^2 + q + 1), 4; 3)_q$ CDC so obtained admits the normalizer of a Singer cyclic group of $\text{PGL}(3, q)$ as an automorphism group.

2 The Veronese embedding

Let $\text{PG}(2, q)$ the Desarguesian projective plane of order $q$. A quadric of $\text{PG}(2, q)$ is the locus of zeros of a quadratic polynomial, say $a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 + a_{12}X_1X_2 + a_{13}X_1X_3 + a_{23}X_2X_3$. There are six parameters associated to such a curve and hence the set of quadrics of $\text{PG}(2, q)$ forms a 5–dimensional projective space. There exist four kinds of quadrics in $\text{PG}(2, q)$, three of which are degenerate (splitting into lines, which could be in the plane $\text{PG}(2, q^2)$) and one of which is non–degenerate [11].

The Veronese map $v$ defined by

$$a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 + a_{12}X_1X_2 + a_{13}X_1X_3 + a_{23}X_2X_3 \mapsto (a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}),$$

is the correspondence between plane quadrics and the points of $\text{PG}(5, q)$.

The quadrics in $\text{PG}(2, q)$ are:

1. $q^2 + q + 1$ repeated lines;

2. $(q^2 + q + 1)(q + 1)q/2$ quadrics consisting of two distinct lines of $\text{PG}(2, q)$ (bi–lines);

3. $(q^2 + q + 1)(q - 1)q/2$ quadrics consisting of two distinct conjugate lines of $\text{PG}(2, q^2)$ (imaginary bi–lines).

4. $q^5 - q^2$ non–degenerate quadrics (conics).

We will say that a bi–line or an imaginary bi–line is centered at $A$ if its lines meet in the point $A$. 
When \( q \) is even, all tangent lines to a conic \( C \) pass through a point of \( \text{PG}(2,q) \) called the \textit{nucleus} of \( C \).

It is not difficult to see that a quadric of \( \text{PG}(2,q) \) is degenerate if and only if its parameters satisfy the polynomial

\[
P_1 := X_4X_5X_6 + X_1X_6^2 + X_2X_5^2 + X_3X_4^2,
\]

when \( q \) is even and

\[
P_2 := X_1X_2X_3 + 2X_4X_5X_6 + X_1X_6^2 + X_2X_5^2 + X_3X_4^2,
\]

when \( q \) is odd.

Notice that from \([13, \text{Theorem 25.1.3}]\) the image of the Veronese map \( v \) is the dual of the image of the map \( \zeta \) defined in \([13, \text{p. 146}]\).

The group \( G := \text{PGL}(3,q) \) acts on \( \text{PG}(2,q) \) and so it also acts naturally on the plane quadrics, and hence also on \( \text{PG}(5,q) \). The four sets of quadrics described above are \( G \)–orbits. With a slight abuse of notation we will denote by \( G \) the group \( \text{PGL}(3,q) \) acting on \( \text{PG}(5,q) \). We will denote by \( O_i, i = 1,2,3,4 \) the images under \( v \) in \( \text{PG}(5,q) \) of the four types of quadrics, respectively. It turns out that \( O_1 \) is the \textit{Veronese surface} when \( q \) is odd and a plane (called \textit{degenerate Veronese surface}) when \( q \) is even. The orbits \( O_i, i = 1,2,3 \), partition the cubic hypersurface \( S \) of \( \text{PG}(5,q) \) with equation \( P_1 = 0 \) when \( q \) is even and with equation \( P_2 = 0 \) when \( q \) is odd.

It should be noted that under the map \( v \) a \( k \)–dimensional linear system of quadrics of \( \text{PG}(2,q) \) corresponds to a \( (k-1) \)–dimensional projective subspace of \( \text{PG}(5,q) \). This means that pencils, nets and webs of quadrics, are represented by lines, planes and solids of \( \text{PG}(5,q) \), respectively.

Let us fix a point \( A \) of \( \text{PG}(2,q) \). The \( q+1 \) lines passing through \( A \) considered as repeated lines, the \( q(q+1)/2 \) bi–lines centered at \( A \) and the \( q(q-1)/2 \) imaginary bi–lines centered at \( A \) form a net that under the Veronese map \( v \) corresponds to a plane \( \pi_A \) contained in \( S \) and meeting \( O_1 \) at \( q + 1 \) points forming either a conic (\( q \) odd) or a line (\( q \) even). Hence, there is a set \( N \) of \( q^2 + q + 1 \) such planes. Also, through a point \( P \in S \setminus O_1 \) there passes exactly one plane of \( N \) whereas through a point of \( O_1 \) there pass \( q + 1 \) planes of \( N \). It follows that two distinct planes in \( N \) meet in a point of \( O_1 \).

Let us fix two distinct points of \( \text{PG}(2,q) \), say \( A \) and \( B \). Let \( \ell \) be the line joining \( A \) and \( B \). There are \( q^2 + q \) bi–lines of \( \text{PG}(2,q) \) containing \( \ell \). These bi–lines together \( \ell \) (considered as a repeated line) form a net that under the Veronese map \( v \) corresponds to a plane \( \pi_{\ell} \) contained in \( S \) and tangent to \( O_1 \). Hence, there is a set \( T \) of \( q^2 + q + 1 \) such planes. Also, through a point \( P \in O_2 \) there pass exactly two planes of \( T \) whereas through a point \( P \in O_1 \)
there passes exactly one plane of $\mathcal{T}$. It follows that two distinct planes in $\mathcal{T}$ meet in a point of $\mathcal{O}_2$.

3 Circumscribed bundles

There exists a collection of $q^2+q+1$ conics in $\text{PG}(2,q)$ that mutually intersect in exactly one point, and hence serve as the lines of another projective plane on the points of $\text{PG}(2,q)$. Such a collection of conics is called a projective bundle of $\text{PG}(2,q)$. For more details on projective bundles, see [1].

Remark 3.1. Let us embed $\text{PG}(2,q)$ into $\text{PG}(2,q^3)$, and let $\sigma$ be the period 3 collineation of $\text{PG}(2,q^3)$ fixing $\text{PG}(2,q)$. Let us fix a triangle $T$ of vertices $P, P', P'^2$ in $\text{PG}(2,q^3)$. Up to date, the known types of projective bundles are as follows [5], [3]:

1. circumscribed bundle consisting of all conics of $\text{PG}(2,q)$ containing the vertices of $T$. This exists for all $q$;

2. inscribed bundle consisting of all conics of $\text{PG}(2,q)$ that are tangent to the three sides of $T$. This exists for all odd $q$;

3. self–polar bundle consisting of all conics of $\text{PG}(2,q)$ with respect to which $T$ is self–polar. This exists for all odd $q$.

From [1] the conics of a circumscribed bundle form a net.

Remark 3.2. A cyclic group of $G$ permuting points (lines) of $\text{PG}(2,q)$ in a single orbit is called a Singer cyclic group of $G$. A generator of a Singer cyclic group is called a Singer cycle.

A Singer cyclic group of $G$ has order $q^2 + q + 1$ and its normalizer in $G$ turns out to be a metacyclic group of order $3(q^2 + q + 1)$. For more details, see [14].

Remark 3.3. All these projective bundles are invariant under the normalizer of a Singer cyclic group of $G$.

Let $B$ be a circumscribed bundle of $\text{PG}(2,q)$. We will need the following result, which extends [6] Lemma 3.2].

Lemma 3.4. Consider two distinct conics $C_0, C_\infty$ of a circumscribed bundle $B$. If $q$ is even, their nuclei are distinct. If $q$ is odd, for a point $P \in \text{PG}(2,q)$ the polar lines of $P$ with respect to $C_0$ and $C_\infty$ are distinct.
Proof. If \( q \) is even or if \( q \) is odd and \( P = C_0 \cap C_\infty \) then the result follows from \([6, \text{Lemma 3.2}]\). Assume that \( q \) is odd and \( P \neq C_0 \cap C_\infty \). Let \( r_0 \) and \( r_\infty \) be the polar lines of \( P \) with respect to \( C_0 \) and \( C_\infty \), respectively. By way of contradiction let \( r_0 = r_\infty \). If \( P \in C_0 \) then \( P \in r_0 \) but \( P \notin r_\infty \), a contradiction.

Let \( A_0, A_\infty \), be the symmetric \( 3 \times 3 \) matrices associated to \( C_0 \) and \( C_\infty \), respectively. Let \( \mathcal{F} = \{ C_\lambda, \lambda \in \mathbb{GF}(q) \cup \{ \infty \} \} \) be the pencil generated by \( C_0 \) and \( C_\infty \). We have that the quadrics of \( \mathcal{F} \) are the conics (non degenerate quadrics) of \( \mathcal{B} \) through \( C_0 \cap C_\infty \) and they cover all points of \( \text{PG}(2,q) \). Let \( \perp_\lambda \) denote the polarity associated with the conic in \( C_\lambda \in \mathcal{F} \). The product \( \perp_0 \perp_\infty \) is then a projectivity of \( \text{PG}(2,q) \) fixing \( P \) whose associated matrix is \( A_0^{-1}A_\infty \), where \( t \) denotes transposition. In other terms \( (A_0^{-1}A_\infty)(P^t) = \rho P^t \), for some \( \rho \in \mathbb{GF}(q) \setminus \{0\} \). Analogously, \( \perp_0 \perp_\lambda \) is a projectivity of \( \text{PG}(2,q) \) whose associated matrix is \( A_0^{-1}(A_0 + \lambda A_\infty) \) and fixing \( P \). Indeed, \( (A_0^{-1}(A_0 + \lambda A_\infty))(P^t) = (I + \lambda A_0^{-1}A_\infty)(P^t) = P^t + \lambda \rho P^t = (1 + \lambda \rho)(P^t) \). It turns out that \( (P^t)^{\perp_\lambda \perp_0} = P \) if and only if \( P^{\perp_\lambda \perp_0} = r_0 \) if and only if \( r_0^{\perp_\lambda} = P \) for every \( \lambda \in \mathbb{GF}(q) \setminus \{0\} \). Let \( \lambda_0 \in \mathbb{GF}(q) \setminus \{0\} \) such that \( P \in C_{\lambda_0} \). Then \( P \in P^{\perp_\lambda \perp_0} = r_0 \) and hence \( P \in r_0 \), a contradiction. \( \square \)

Remark 3.5. Notice that if \( q \) is odd then the projectivity obtained as the product of two polarities associated to distinct conics of a circumscribed bundle is fixed point free.

Since \( \mathcal{B} \) is stabilized by the normalizer \( N \) of a Singer cyclic group \( S \) of \( G \) that is maximal in \( G \) \([4]\) we get that \( \mathcal{B}^G \) has size \( \frac{q^3(q^2-1)(q-1)}{3} \).

From \([7]\) the group \( G \) has three orbits on points and lines of \( \text{PG}(2,q^3) \). The orbits on points are the \( q^2+q+1 \) points of \( \text{PG}(2,q) \), the \( (q^3-q)(q^2+q+1) \) points on lines of \( \text{PG}(2,q^3) \) that are not in \( \text{PG}(2,q) \) and the remaining set \( E \) of \( q^3(q^2 - 1)(q-1) \) points of \( \text{PG}(2,q^3) \). The \( G \)-orbits on lines are the \( q^2 + q + 1 \) lines of \( \text{PG}(2,q) \), the \( (q^3-q)(q^2+q+1) \) lines meeting \( \text{PG}(2,q) \) in a point and the set \( L \) of \( q^3(q^2 - 1)(q-1) \) lines external to \( \text{PG}(2,q) \). The group \( N \) fixes a triangle \( T \) whose vertices are points of \( E \) and whose edges are lines of \( L \).

We will need the following lemma.

Lemma 3.6. Let \( T^G \) be the orbit of \( T \) under \( G \). If \( T_1 \) and \( T_2 \) are distinct elements of \( T^G \) then the union of their vertices always contains a 5–arc, i.e., 5 points of \( \pi \) no three of which are collinear.

Proof. The stabilizer of \( T \) in \( G \) contains \( N \) that is maximal in \( G \) and hence \( |T^G| = \frac{q^3(q^2-1)(q-1)}{3} \). Let us consider the incidence structure whose
points are the points of $E$ and whose blocks are the vertex sets of the triangles of $T^G$. The incidence relation is containment. It turns out that through a point of $E$ there pass exactly one triangle of $T^G$. Analogously, let us consider the incidence structure whose points are the lines of $L$ and whose blocks are the edge sets of the triangles of $T^G$. The incidence relation is containment. It turns out that through a line of $L$ there exists exactly one triangle of $T^G$ having that line as an edge. As a consequence, the union of two triangles of $T^G$ always contains a 5–arc of $\pi$.

\[ \text{Corollary 3.7.} \] Two distinct circumscribed bundles of $B^G$ share at most one conic.

\[ \text{Proof.} \] Let $B_1$ and $B_2$ be two distinct circumscribed bundles in $B^G$. Let $T_i$ be the triangle associated to $B_i$, $i = 1, 2$. By way of contradiction assume that $C_1$ and $C_2$ are distinct conics in $B_1 \cap B_2$. Then $C_1$ and $C_2$, considered as conics of $\pi$, contain the vertices of both triangles $T_1$ and $T_2$. From Lemma 3.6 and from \cite[Corollary 7.5]{11} we get a contradiction. \[ \square \]

Under the Veronese map $v$ the circumscribed bundles in $B^G$ correspond to a set $C$ of $q^3(q^2 - 1)(q - 1)/3$ planes of $PG(5, q)$ mutually intersecting in at most one point. Since no quadric in a bundle is degenerate, a plane of $C$ is always disjoint from $S$.

4 Two special webs of quadrics

Firstly, we recall some basic properties of three–dimensional non–degenerate quadrics.

A \textit{hyperbolic quadric} $Q^+(3, q)$ of $PG(3, q)$ consists of $(q + 1)^2$ points of $PG(3, q)$ and $2(q + 1)$ lines that are the union of two reguli. A \textit{regulus} is the set of lines intersecting three skew lines and has size $q + 1$. Through a point of $Q^+(3, q)$ there pass two lines belonging to different reguli. A plane of $PG(3, q)$ is either secant to $Q^+(3, q)$ and meets $Q^+(3, q)$ in a conic or it is tangent to $Q^+(3, q)$ and meets $Q^+(3, q)$ in a bi–line.

An \textit{elliptic quadric} $Q^-(3, q)$ of $PG(3, q)$ consists of $q^2 + 1$ points of $PG(3, q)$ such that no three of them are collinear. A plane of $PG(3, q)$ is either secant to $Q^-(3, q)$ and meets $Q^-(3, q)$ in a conic or it is tangent to $Q^-(3, q)$ and meets $Q^-(3, q)$ in a point. For more details on hyperbolic and elliptic quadrics in a three–dimensional projective space we refer to \cite{12}.

Let $P_1, P_2$ be two distinct points of $PG(2, q)$. Since $G$ is 2–transitive on points of $PG(2, q)$ we can always assume that $P_1 = (1, 0, 0)$ and $P_2 = (0, 1, 0)$.
The set of quadrics of $\text{PG}(2,q)$ passing through $P_1$ and $P_2$ are those having the coefficients $a_{11} = a_{22} = 0$ and forms a web $W$. Under the Veronese map $\nu$, $W$ corresponds to the solid $\nu(W)$ with equations $X_1 = X_2 = 0$. The solid $\nu(W)$ intersects $\mathcal{S}$ into the set of points satisfying the equations $X_4(2X_5X_6 - X_3X_4) = 0$ and $X_4(X_5X_6 - X_3X_4) = 0$ accordingly as $q$ is odd or even, respectively. In both cases, this set consists of a hyperbolic quadric $Q$ and a plane tangent $\pi$ to $Q$ at the point $R = (0,0,1,0,0,0)$. In particular, $\pi$ meets $Q$ at $2q + 1$ points forming a bi–line centered at $R$. The point $R$ corresponds to the repeated line $P_1P_2$ and the remaining $2q$ points correspond to the bi–lines of $W$ centered at $P_1$ and $P_2$. It is easily seen that the number of such solids (hyperbolic solids) is $q(q + 1)(q^2 + q + 1)/2$.

Assume that $P_1, P_2$ are points of $\text{PG}(2,q^2) \setminus \text{PG}(2,q)$ conjugate over $\text{GF}(q)$. Since $G$ is transitive on points of $\text{PG}(2,q^2) \setminus \text{PG}(2,q)$ we can assume that $P_1 = (1,\alpha,0)$ and so $P_2 = (1,\alpha^q,0)$, where $\alpha$ is a primitive element of $\text{GF}(q^2)$ over $\text{GF}(q)$. Again, the set of quadrics of $\text{PG}(2,q^2)$ passing through $P_1$ and $P_2$ are those whose coefficients satisfy $a_{11} = \alpha^{q+1}a_{22}$ and $a_{12} = -(\alpha + \alpha^q)a_{22}$ and forms a web $U$. Under the Veronese map $\nu$, $U$ corresponds to the solid $\nu(U)$ with equations $X_1 = \alpha^{q+1}X_2$ and $X_4 = -(\alpha + \alpha^q)X_2$. The solid $\nu(U)$ intersects $\mathcal{S}$ into the set of points satisfying the equations $X_2(X_6^2 + \alpha^{q+1}X_6^2 + (\alpha + \alpha^q)X_5X_6 + ((\alpha + \alpha^q)^2 - \alpha^{q+1})X_2X_3) = 0$ and $X_2((\alpha + \alpha^q)^2X_2X_3 + \alpha^{q+1}X_6^2 + (\alpha + \alpha^q)X_5X_6 + X_6^2) = 0$ accordingly as $q$ is odd or even, respectively. Notice that the polynomial $X^2 + (\alpha + \alpha^q)X + \alpha^{q+1}$ is irreducible over $\text{GF}(q)$ and that, if $q$ is odd, $\alpha^{q+1}$ is a nonsquare element of $\text{GF}(q)$. Therefore, in both cases, this set consists of an elliptic quadric $Q'$ and a plane $\pi$ tangent to $Q'$ at the point $R = (0,0,1,0,0,0)$. In this case, the number of such solids (elliptic solids) is $q(q - 1)(q^2 + q + 1)/2$.

The plane $\pi$ is contained in $\mathcal{S}$, belongs to $\mathcal{T}$ and meets $\mathcal{O}_1$ at the point $R$.

In the sequel a hyperbolic or elliptic solid will be denoted by $\Sigma = (\tau,Q)$ where $\tau \in \mathcal{T}$ is contained in $\Sigma$ and $Q$ is the three–dimensional hyperbolic or elliptic quadric contained in $\Sigma \cap \mathcal{S}$. We will denote by $\mathcal{H}$ and $\mathcal{E}$ the set of hyperbolic solids and elliptic solids, respectively.

Now, we do investigate how two solids (elliptic or hyperbolic) can intersect.

**Proposition 4.1.** Let $\Sigma_1 = (\pi_1,Q_1)$, $\Sigma_2 = (\pi_2,Q_2)$ be two distinct hyperbolic solids. Then, one of the following cases occur:

1. $\Sigma_1 \cap \Sigma_2$ is a plane, $\pi_1 = \pi_2$ and $|Q_1 \cap Q_2| = q + 1$;

2. $\Sigma_1 \cap \Sigma_2$ is a plane, $\pi_1 = \pi_2$ and $|Q_1 \cap Q_2| = 1$;
3. $\Sigma_1 \cap \Sigma_2$ is a plane, $|\pi_1 \cap \pi_2| = 1$ and $|Q_1 \cap Q_2| = q + 2$;

4. $\Sigma_1 \cap \Sigma_2$ is a line, $|\pi_1 \cap \pi_2| = 1$ and $|Q_1 \cap Q_2| = 2$;

Proof. Let us assume that $\Sigma_i$ corresponds to the web defined by the points $A_i, B_i$, $i = 1, 2$. Let $\ell_i$ be the line $A_iB_i$, $i = 1, 2$.

1. The pairs $A_1, B_1$ and $A_2, B_2$ share a point and $\ell_1 = \ell_2$. Then we can assume that $A_2 = B_1$. In this case it is clear that $\pi_1 = \pi_2$ and the $q + 1$ points of $Q_1 \cap Q_2$ correspond to the bi–lines centered at $A_2 = B_1$ of the relevant webs together with $\ell_1 = \ell_2$ considered as a repeated line.

2. The pairs $A_1, B_1$ and $A_2, B_2$ share no point and $\ell_1 = \ell_2$. In this case it is clear that $\pi_1 = \pi_2$ and the point of $Q_1 \cap Q_2$ corresponds to $\ell_1 = \ell_2$ considered as a repeated line.

3. The pairs $A_1, B_1$ and $A_2, B_2$ share the point $A_2 = B_1 = \ell_1 \cap \ell_2$. In this case the planes $\pi_1$ and $\pi_2$ share only the point corresponding to the bi–line $\ell_1 \ell_2$. On the other hand, $Q_1 \cap Q_2$ contains the $q + 1$ points corresponding to the bi–lines centered at a point of the line $A_1B_2$ and containing the line $A_1B_2$ and the line through $A_2$. Also, $Q_1 \cap Q_2$ contains the point corresponding to the bi–line $\ell_1 \ell_2$.

4. The pairs $A_1, B_1$ and $A_2, B_2$ share no point and $\ell_1 \neq \ell_2$.

4.1 $\ell_1 \cap \ell_2 = A_2$. In this case $\pi_1$ and $\pi_2$ share only the point corresponding to the bi–line $\ell_1 \ell_2$. Here, $Q_1 \cap Q_2$ consists of the two points corresponding to the bi–line $\ell_1 A_1 B_2$ centered in $A_1$ and the bi–line $\ell_1 B_1 B_2$ centered at $B_2$. The line joining the points of $Q_1 \cap Q_2$ lies on $\pi_1$.

4.2 The points $A_1, B_1, A_2, B_2$ form a 4–arc in PG($2, q$). In this case $\pi_1$ and $\pi_2$ share only the point corresponding to the bi–line $\ell_1 \ell_2$. Here, $Q_1 \cap Q_2$ consists of the two points corresponding to the bi–line containing the lines $A_1 A_2$ and $B_1 B_2$ and the bi–line containing $A_1 B_2$ and $A_2 B_1$.

4.3 $\ell_1 \cap \ell_2 = B_1$. In this case by switching $\ell_1$ and $\ell_2$ we are again in the case 4.1.

\[\square\]

**Proposition 4.2.** Let $\Sigma_1 = (\pi_1, Q_1)$, $\Sigma_2 = (\pi_2, Q_2)$ be two distinct elliptic solids. One of the following cases occur:
1. \( \Sigma_1 \cap \Sigma_2 \) is a plane, \( \pi_1 = \pi_2 \) and \( |Q_1 \cap Q_2| = 1 \);

2. \( \Sigma_1 \cap \Sigma_2 \) is a line, \( |\pi_1 \cap \pi_2| = 1 \) and \( |Q_1 \cap Q_2| = 2 \);

Proof. Let us assume that \( \Sigma_i \) corresponds to the web defined by the points \( A_i, A^i_q, i = 1, 2 \). Let \( \ell_i \) be the line \( A_i A^i_q, i = 1, 2 \).

1. Assume that \( \ell_1 = \ell_2 \). In this case it is clear that \( \pi_1 = \pi_2 \) and that the unique intersection point between \( Q_1 \) and \( Q_2 \) corresponds to the repeated line \( \ell_1 = \ell_2 \).

2. Assume that \( \ell_1 \neq \ell_2 \). In this case \( \pi_1 \) and \( \pi_2 \) share a unique point corresponding to the bi–line \( \ell_1 \ell_2 \). Here \( Q_1 \cap Q_2 \) consists of the two points corresponding to the two imaginary bi–lines containing the lines \( A_1 A_2, A^1_q A^2_q \) and \( A_1 A^2_q, A^1_q A_2 \), respectively.

\[ \square \]

5 Two special nets of quadrics

As already observed, the Singer cyclic group \( S \) permutes the points (lines) of \( \text{PG}(2, q) \) in a single orbit. Under the action of \( S \), the set of \( q(q + 1)(q^2 + q + 1)/2 \) bi–lines of \( \text{PG}(2, q) \) is partitioned into \( q(q + 1)/2 \) orbits of size \( q^2 + q + 1 \). Let us fix one of the \( q(q + 1)/2 \) orbits of bi–lines, say \( b \), and let us consider the incidence structure whose points are the lines of \( \text{PG}(2, q) \) and whose blocks are the bi–lines of \( b \). It turns out that a line \( \ell \) is contained in exactly two bi–lines, say \( b_1 \) and \( b_2 \), of \( b \) centered at two distinguished points of \( \ell \), say \( A_1 \) and \( A_2 \), respectively. Let \( s \) be the unique element of \( S \) such that \( A^1_q = A_2 \). Then \( b^1_q = b_2 \).

Let \( \mathcal{P}_{A_1} \) and \( \mathcal{P}_{A_2} \) be the pencils of lines with vertices \( A_1 \) and \( A_2 \). Clearly, \( s \) is a projectivity sending \( \mathcal{P}_{A_1} \) to \( \mathcal{P}_{A_2} \) that does not map the line \( \ell \) onto itself. In [15] it is proved that the set of points of intersection of corresponding lines under \( s \) is a conic \( C \) passing through \( A_1 \) and \( A_2 \) (Steiner’s argument). The projectivity \( s \) maps the tangent line to \( C \) at \( A_1 \) onto the line \( \ell \) and the line \( \ell \) onto the tangent line to \( C \) at \( A_2 \). Moreover, for any two distinct points \( A \) and \( B \) of a conic there exists a projectivity \( \psi \in S \) sending \( A \) to \( B \) and such that \( C \) is the set of points of intersection of corresponding lines under \( \psi \). Assume that \( b_i = \ell \ell_i, i = 1, 2 \). Since \( s \) sends \( \ell_1 \) to \( \ell \) and \( \ell \) to \( \ell_2 \), it follows that \( \ell_i \) is tangent to \( C \) at \( A_i, i = 1, 2 \). Embed \( \text{PG}(2, q) \) into \( \text{PG}(2, q^3) \). We have denoted by \( T \) be the unique triangle of \( \text{PG}(2, q^3) \) fixed by \( S \). Considering \( \mathcal{P}_{A_1} \) and \( \mathcal{P}_{A_2} \) as pencils in \( \text{PG}(2, q^3) \) and repeating the
previous argument, a conic $C$ of $\text{PG}(2, q^3)$ passing through the vertices of $T$ and containing $C$ arises. It follows that $C$ is a member of the circumscribed bundle $B$ of $\text{PG}(2, q)$ left invariant by $S$. Let $A_3 \in C \setminus \{A_1, A_2\}$ and let $b_3$ the bi–line of $b$ centered at $A_3$. Let $s'$ be the unique element of $S$ sending $A_1$ to $A_3$. Then $b_1' = b_3$. Steiner’s argument with $s$ replaced by $s'$, applied to the pencils $P_{A_1}$ and $P_{A_3}$, gives rise to a conic $C'$ that necessarily belongs to $B$. Furthermore, being unique the conic of $B$ through two distinct points of $\text{PG}(2, q)$ it follows that $C = C'$. Since $A_1' = A_3$ and $A_1 \in \ell_1$ then $A_1' = A_3 \in \ell_1'. \ell_1$ On the other hand, the point $\ell_1' \cap \ell_1$ lies on $C$ and of course it lies on $\ell_1$. Since the line $\ell_1$ is tangent to $C$ at $A_1$ we have that $\ell_1' \cap \ell_1$ is the point $A_1$ or, in other words, $\ell_1'$ is the line $A_1A_3$. Analogously, the point $\ell_1 \cap \ell$ lies on $C$ and of course it lies on $\ell$. Therefore $\ell_1'$ is either the line $A_1A_2$ or the line $A_1A_3$. Since $\ell_1 \neq \ell$ it follows that $\ell_1' = A_2A_3$. We have showed that $b_3$ is the bi–line containing the lines $A_1A_3$ and $A_2A_3$.

We have proved the following Proposition.

**Proposition 5.1.** For any conic $C$ of $B$ there exists two distinguished points $P_1$ and $P_2$ of $C$ such that the elements of $b$ centered at a point of $C$ are as follows: $t_{P_i}r$, $t_{P_2}r$, $r_1r_2$, where $t_{P_i}$ is the tangent line to $C$ at $P_i$, $i = 1, 2$, $r$ is the line $P_1P_2$, $v_i$ is the line $PP_i$, $i = 1, 2$, and $P$ ranges over $C \setminus \{P_1, P_2\}$.

**Remark 5.2.** Notice that there exists a one to one correspondence between the orbits of $S$ on bi–lines and secant lines to $C$.

**Remark 5.3.** With the notation introduced in Proposition 5.1 notice that, from [11, Table 3.7] the pencil generated by the bi–lines $r_1r_2$ and $ru$, where $t_{P_1} \cap t_{P_2} \in u$ and $P_1, P_2, P_3 \not\in u$, contains exactly a further bi–line. Moreover, this bi–line is centered at a point of $C$. Indeed, let $C$ be the conic with equation $X_1X_3 - X_2^2 = 0$. The stabilizer of $C$ in $G$ is isomorphic to $\text{PGL}(2, q)$ and acts 3–transitively on points of $C$. Hence, without loss of generality, we can assume that $P_1 = (1, 0, 0)$, $P_2 = (0, 0, 1)$ and $P_3 = (1, 1, 1)$. Let $v_i$ be the line joining the point $P_3$ and the point $P_i$, $i = 1, 2$. Then $b_3 = v_1v_2$. Notice that $U = t_{P_1} \cap t_{P_2} = (0, 1, 0)$. Let $u$ be a line passing through $U$ and containing none of the points $P_i$, $i = 1, 2, 3$. Let $b_4$ be the bi–line $uP_1P_2$. Then $b_3 \cap b_4$ consists of the four points $P_1, P_2, (1, 1, t), (1, t, t)$, with $t \neq 0, 1$. It turns out that the bi–line consisting of the lines $(1, t, t)P_2$ and $(1, 1, t)P_1$ is centered at the point $(1, t, t^2) \in C$.

The following Proposition could be of some interest.

**Proposition 5.4.** The incidence structure whose points are the elements of $b$ and whose lines are the conics of the circumscribed bundle $B$, where a
bi–line is incident with a conic if it is centered at one of its points, forms a projective plane.

Proof. We have that \(|b| = |\mathcal{B}| = q^2 + q + 1\). Since through a point of \(PG(2, q)\) there pass \(q + 1\) conics of \(\mathcal{B}\), we have that a bi–line of \(b\) is incident with \(q + 1\) conics of \(\mathcal{B}\). On the other hand a conic is incident with \(q + 1\) bi–lines of \(b\). In particular we have seen that to a conic \(C\) of \(\mathcal{B}\) are associated two distinguished points \(P, P^s \in C\), where \(s \in S\) and all the bi–lines of \(b\) incident with \(C\) contain both \(P, P^s\). Let us consider now a conic of \(\mathcal{B} \setminus \{C\}\). Then it is necessarily of the form \(C^\mu\), for some non–trivial element \(\mu \in S\). Then two possibilities occur according as one of the points \(P^\mu, P^{s\mu}\) does belong to \(C\) or does not. If the first case occurs then, assuming that \(P^\mu\) is the point belonging to \(C\), we have that \(C \cap C^\mu = P^\mu\). If \(t_P\) denotes the tangent line to \(C\) at the point \(P\), it turns out that \(t_P^\mu\) is the tangent line to \(C^\mu\) at the point \(P^\mu\) and \(t_P^\mu = PP^\mu\). Therefore the unique bi–line of \(b\) incident with both \(C\) and \(C^\mu\) is centered at \(P^\mu\). If the latter case occurs, then, by construction (a la Steiner), \(PP^\mu \cap P^s P^{s\mu} = PP^\mu \cap P^s P^{s\mu} = PP^\mu \cap (PP^\mu)^s\) is the unique point in common between \(C\) and \(C^\mu\). Therefore the unique bi–line of \(b\) incident with both \(C\) and \(C^\mu\) is centered at \(C \cap C^\mu\).

With the notation introduced in Proposition 5.1 let us consider the three bi–lines \(t_{P_1} r, t_{P_2} r\) and \(r_1 r_2\), where \(r_1 \cap r_2 = P \in C \setminus \{P_1, P_2\}\). Under the map \(v\) they correspond to three points \(R_1, R_2, R_3\) of \(\mathcal{O}_2\), respectively. From the classification of pencils of quadrics of \(PG(2, q)\) in [11, Table 7.7] the line joining \(R_1\) and \(R_2\) corresponds to the unique pencil \(\mathcal{P}\) whose members are all bi–lines and having a base consisting of \(q + 2\) points. Hence the line \(R_1 R_2\) is completely contained in \(\mathcal{O}_2\). In particular, the bi–lines of \(\mathcal{P}\) are those containing the line \(r\) and the line \(t_{P_1} \cap t_{P_2} A\), where \(A\) ranges over \(r\). It follows that the bi–line corresponding to \(R_3\) cannot belong to \(\mathcal{P}\). Let \(v(b)\) be the image of \(b\) under \(v\). Of course \(v(b)\) contains \(R_i, i = 1, 2, 3\). Let \(\pi_e\) be the plane of \(PG(5, q)\) generated by \(R_1, R_2, R_3\) and let \(\Pi_e\) denote the set of planes obtained in this way.

The plane \(\pi_e\) meets \(\mathcal{O}_2\) at \(2q\) points consisting of the line \(R_1 R_2\) and of further \(q – 1\) points. Also, the plane \(\pi_e\) meets \(v(b)\) in \(q + 1\) points containing \(R_1, R_2, R_3\). Indeed, from Remark 5.3 through the point \(R_3\) there are \(q – 2\) lines intersecting \(\mathcal{O}_2\) in three points and \(v(b)\) in two points.

It follows that the points of \(\pi_e \cap \mathcal{O}_2\) correspond to the bi–lines of \(b\) centered at points of \(C\). We have that \(|\Pi_e| = q(q + 1)(q^2 + q + 1)/2\)

On the other hand, the plane \(\pi_e\) is contained in the hyperbolic solid defined by the points \(P_1, P_2\) and then \(\pi_e \cap \mathcal{O}_2\) consists of a conic and a line secant to it.
Since the number of hyperbolic solids equals $|\Pi_e|$ and each plane of $\Pi_e$ is contained in at least a hyperbolic solid, it follows that there exists a one-to-one correspondence between planes of $\Pi_e$ and hyperbolic solids.

Now, let $S'$ be the unique Singer cyclic group of $\text{PGL}(3, q^2)$ containing $S$. It is clear that the circumscribed bundle $\mathcal{B}'$ of $\text{PG}(2, q^2)$ fixed by $S'$ induces the circumscribed bundle $\mathcal{B}$ of $\text{PG}(2, q)$ fixed by $S$.

Let $b_1'$ be the imaginary bi-line containing the lines $r, r^q$ and centered at the point $P \in \text{PG}(2, q)$. Let $C$ be a conic of $\mathcal{B}$ through $P$ and let $C'$ be the unique conic of $\mathcal{B}'$ containing $C$.

Let $b'$ be the orbit of $b_1'$ under $S'$. As already observed above there exist two points, say $P_1, P_2$ on $C$ such that all elements of $b'$ centered at a point of $C$ pass through $P_1$ and $P_2$. Also, since $P_1 \cup P_2 \in r \cup r^q$ and the tangent line to $C$ at $P$ is a line of $\text{PG}(2, q)$, it follows that $P_1, P_2 \notin \text{PG}(2, q)$. Under the action of $S$, $b'$ is partitioned into $q^2 - q + 1$ orbits of size $q^2 + q + 1$. Among these, we denote by $b'$ the unique $S$–orbit consisting of imaginary bi–lines. It turns out that a member of $b'$ consists of the lines $z = RP_1$ and $z^q = RP_2$ for some $R \in C$. Let $R_1, R_2 \in C$, $R_1 \neq R_2$. Let $r_i = R_iP_1$ and $r_i^q = R_iP_2$, $i = 1, 2$. Since $r_1 \cap r_2 = P_1$ it follows that $r_1^q \cap r_2^q = P_1^q$ and then $P_2 = P_1^q$.

Notice that the line $P_1P_2$ arises from a line $a$ of $\text{PG}(2, q)$ that is external to $C$. Let $A = t_{P_1} \cap t_{P_2}$, where $t_{P_1}$ and $t_{P_2}$ are the tangent lines to $C$ at $P_1$ and $P_2$, respectively. Then $A \in \text{PG}(2, q)$. Indeed, when $q$ is odd, $A$ is the conjugate of $a$ with respect to $C$. When $q$ is even, $A$ is the nucleus of both $C$ and $C'$.

**Proposition 5.5.** For any conic $C$ of $\mathcal{B}$ there exists two distinguished points $P$ and $P^q$ of $\bar{C}$ not on $C$ such that the elements of $b'$ centered at a point of $C$ are of the form $XP, XP^q$, where $X$ ranges over $C$.

**Remark 5.6.** Notice that there exists a one to one correspondence between the orbits of $S$ on imaginary bi–lines and lines external to $C$.

Similar arguments used in Proposition 5.4 give the following result.

**Proposition 5.7.** The incidence structure whose points are the elements of $b'$ and whose lines are the conics of the circumscribed bundle $\mathcal{B}$, where an imaginary bi–line is incident with a conic if it is centered at one of its points, forms a projective plane.

Let $d_i$ be the bi–line consisting of the lines $a$ and $D_iA$, $i = 1, 2$, where $D_1, D_2$ are distinct points of $a$.

With the notation introduced in Proposition 5.5, let us consider two bi–lines of the form $a, D_iA$, $i = 1, 2$ and the imaginary bi–line $XP, XP^q$, for
some $X \in C$. Under the map $v$ they correspond to three points $R_1, R_2, R_3$, respectively. The points $R_1$ and $R_2$ are in $O_2$, whereas $R_3 \in O_3$. From the classification of pencils of quadrics of $PG(2, q)$ in [11, Table 7.7] the line joining $R_1$ and $R_2$ corresponds to the unique pencil $\mathcal{P}$ whose members are all bi–lines and having a base consisting of $q + 2$ points. Hence the line $R_1R_2$ is completely contained in $O_2$. In particular, the bi–lines of $\mathcal{P}$ are those containing the line $a$ and the line $DA$, where $D$ ranges over $a$. Of course the imaginary bi–line corresponding to $R_3$ cannot belong to $\mathcal{P}$. Let $v(b')$ be the image of $b'$ under $v$. Of course $v(b')$ contains $R_i$, $i = 1, 2, 3$. Let $\pi_i$ be the plane of $PG(5, q)$ generated by $R_1, R_2, R_3$ and let $\Pi_i$ denote the set of planes obtained in this way.

The plane $\pi_i$ meets $O_2$ in the line $R_1R_2$ and $O_3$ in further $q + 1$ points. Indeed, from the classification of pencils of quadrics of $PG(2, q)$ in [11, Table 7.7], through the point $R_3$ there are $q$ lines intersecting $O_3$ in two points and $O_2$ in one point. Each of these lines corresponds to the unique pencil consisting of $q - 2$ conics, a bi–line and two imaginary bi–lines. Also, there exists a unique line through the point $R_3$ intersecting both $O_2$, $O_3$ in one point. Such a line corresponds to the unique pencil consisting of $q - 1$ conics one bi–line and one imaginary bi–line. It follows that the points of $\pi_i \cap O_3$ correspond to the imaginary bi–lines of $b'$ centered at points of $C$. We have that $|\Pi_i| = q(q - 1)(q^2 + q + 1)/2$

On the other hand, the plane $\pi_i$ is contained in the elliptic solid defined by the points $P,P^4$ and then $\pi_i \cap O_3$ consists of a conic and $\pi_i \cap O_2$ of a line external to it.

Since the number of elliptic solids equals $|\Pi_i|$ and each plane of $\Pi_i$ is contained in at least an elliptic solid, it follows that there exists a one to one correspondence between planes of $\Pi_i$ and elliptic solids.

6 Lifting Singer cycles

Here, we assume that $q$ is odd. From [14], we may assume that $S$ is given by

$$
\begin{pmatrix}
\omega & 0 & 0 \\
0 & \omega^q & 0 \\
0 & 0 & \omega^{q^2}
\end{pmatrix},
$$

where $\omega$ is a primitive element of $GF(q^3)$ over $GF(q)$. It follows that the lifting of $S$ to a collineation of $PG(5, q)$ fixing the Veronese surface $O_1$ has the following canonical form $A = diag(S^2, S^{q+1})$ [2].
The group $\langle A \rangle$ has order $q^2 + q + 1$. Geometrically, $\langle A \rangle$ fixes two planes of $\text{PG}(5,q)$, say $\rho_1, \rho_2$, and partition the remaining points of $\text{PG}(5,q)$ into Veronese surfaces. [2] Corollary 5. In particular, the planes $\rho_1$ and $\rho_2$ are both full orbits of $\langle A \rangle$ and disjoint from the cubic hypersurface $S$. 

From [2] the cubic hypersurface $S$ is partitioned under $\langle A \rangle$ into Veronese surfaces. The hypersurface $S$ has $(q^2 + 1)(q^2 + q + 1)$ points and hence it consists of $q^2 + 1$ Veronese surfaces.

7 The construction of subspace codes

In this Section we prove our main result.

Lemma 7.1. Two distinct planes of $\Pi_e$ can meet in at most one point.

Proof. Let $\sigma_1, \sigma_2$ be two distinct planes of $\Pi_e$. From Section 5 there exist uniquely determined hyperbolic solids $\Sigma_1 = (\pi_1, Q_1)$ and $\Sigma_2 = (\pi_2, Q_2)$ of $H$ containing $\sigma_1$ and $\sigma_2$, respectively. Let $c_i = \sigma_i \cap Q_i$ be the conic in $\Sigma_i$, $i = 1, 2$.

Assume first that $c_1, c_2$ belong to the same $S$–orbit. Then, from Proposition 5.4, $c_1$ and $c_2$ share exactly one point. Since $S$ permutes the planes of $T$ in a single orbit we have that $\pi_1 \neq \pi_2$. Therefore, from Proposition 4.1, $Q_1 \cap Q_2$ consists of either 2 or $q + 2$ points ($q + 1$ points on a line together with a further point $Y$). Assume that $y = \sigma_1 \cap \sigma_2$ is a line. If $|Q_1 \cap Q_2| = 2$, then the conics $c_1$ and $c_2$ should share two points, a contradiction. If $|Q_1 \cap Q_2| = q + 2$, then it turns out that $c_1 \cap c_2 = \{Y\}$. On the other hand, since $y \subseteq \Sigma_1 \cap \Sigma_2$, the line $y$ contains $Y$ and must be secant to both $Q_1$ and $Q_2$. Hence, again, the conics $c_1$ and $c_2$ should share two points, a contradiction.

Assume that $c_1, c_2$ do not belong to the same $S$–orbit. Then $c_1$ and $c_2$ have no point in common. Assume that $y = \sigma_1 \cap \sigma_2$ is a line. If $Q_1 \cap Q_2$ consists of either 2 or $q + 1$ or $q + 2$ points, then since $y \subseteq \Sigma_1 \cap \Sigma_2$, from Proposition 4.1, the conics $c_1$ and $c_2$ should share at least one point, a contradiction. If $|Q_1 \cap Q_2| = 1$, since $y \subseteq \Sigma_1 \cap \Sigma_2$, then the line $y$ either contains the point $Q_1 \cap Q_2$ and, again, the conics $c_1$ and $c_2$ should share one point, a contradiction, or the line $y$ is secant to both $c_1, c_2$ and $y \cap (c_1 \cup c_2)$ consists of four distinct points. If this last case occurs, then, under the inverse of the map $v$, these four points correspond to four distinct bi–lines having in common $q + 1$ points of a line $z$ and a further point $Z \notin z$. In particular, let $c'_i$ denote the conic of the circumscribed bundle $B$ locus of centers of the bi–lines corresponding to points of $c_i$, $i = 1, 2$. It turns out
that $c'_1 \neq c'_2$ (see Remark 5.2) and $z$ is the polar line of the point $Z$ with respect to both $c'_1$ and $c'_2$, when $q$ is odd or $Z$ is the nucleus of both $c'_1$ and $c'_2$, when $q$ is even. But this contradicts Lemma 3.4.

**Lemma 7.2.** Two distinct planes of $\Pi_i$ can meet in at most one point.

**Proof.** Let $\sigma_1, \sigma_2$ be two planes of $\Pi_i$. From Section 5 there exist uniquely determined elliptic solids $\Sigma_1 = (\pi_1, Q_1)$ and $\Sigma_2 = (\pi_2, Q_2)$ of $E$ containing $\sigma_1$ and $\sigma_2$, respectively. Let $c_i = \sigma_i \cap Q_i$ be the conic in $\Sigma_i, i = 1, 2$.

Assume first that $c_1, c_2$ belong to the same $S$–orbit. Then, from Proposition 5.7 $c_1$ and $c_2$ share exactly one point. Since $S$ permutes the planes of $T$ in a single orbit we have that $\pi_1 \neq \pi_2$. Therefore, from Proposition 4.2 $Q_1 \cap Q_2$ consists of 2 points. Assume that $y = \sigma_1 \cap \sigma_2$ is a line, then the conics $c_1$ and $c_2$ should share two points, a contradiction.

Assume that $c_1, c_2$ does not belong to the same $S$–orbit. Then $c_1$ and $c_2$ have no point in common. Assume that $y = \sigma_1 \cap \sigma_2$ is a line. If $Q_1 \cap Q_2$ consists of 2 points, then, since $y \subseteq \Sigma_1 \cap \Sigma_2$, from Proposition 4.2 the conics $c_1$ and $c_2$ should share at least one point, a contradiction. If $|Q_1 \cap Q_2| = 1$, since $y \subseteq \Sigma_1 \cap \Sigma_2$, then the line $y$ either contains the point $Q_1 \cap Q_2$ and, again, the conics $c_1$ and $c_2$ should share one point, a contradiction, or the line $y \subset O_2$ is external to both $c_1, c_2$. If this last case occurs, then, under the inverse of the map $v$, the points of $y$ correspond to bi–lines having in common $q + 1$ points of a line $z$ and a further point $Z \notin z$. In particular, let $c'_i$ denote the conic of the circumscribed bundle $B$ for the loci of centers of the imaginary bi–lines corresponding to points of $c_i, i = 1, 2$. It turns out that $c'_1 \neq c'_2$ (see Remark 5.6) and $z$ is the polar line of the point $Z$ with respect to both $c'_1$ and $c'_2$, when $q$ is odd or $Z$ is the nucleus of both conics $c'_1$ and $c'_2$, when $q$ is even. But this, again, contradicts Lemma 3.4.

**Theorem 7.3.** The set $\mathcal{C} \cup \Pi_i \cup \Pi_e \cup \mathcal{N}$ consists of $q^3(q^2 - 1)(q - 1)/3 + (q^2 + 1)(q^2 + q + 1)$ planes mutually intersecting in at most one point.

**Proof.**

1. Assume that $\sigma_1 \in \mathcal{C}$ and $\sigma_2 \in \Pi_i \cup \Pi_e \cup \mathcal{N}$. In this case $\sigma_1 \subset O_4$ and $\sigma_2$ always contains a line in $O_2 \cup O_3$ and hence if $\sigma_1 \cap \sigma_2$ was a line then $\sigma_1$ should contain a point of $O_2 \cup O_3$.

2. Assume that $\sigma_1 \in \Pi_i \cup \Pi_e$ and $\sigma_2 \in \mathcal{N}$. In this case $\sigma_2 \subset S$ whereas $\sigma_1$ meets $S$ in the union of a conic and a line $r$. Hence if $\sigma_1 \cap \sigma_2$ was a line, such a line should be $r$. From Table 7.7 the line $r$ corresponds to the unique pencil of quadrics containing only bi–lines. On the other hand, a line of $\sigma_2$ corresponds to a pencil of quadrics always containing imaginary bi–lines or at most $q$ bi–lines.
3. Assume that $\sigma_1 \in \Pi_e$ and $\sigma_2 \in \Pi_i$.

Let $\Sigma_1 = (\pi_1, Q_1)$ the unique hyperbolic solid of $H$ containing $\sigma_1$ and let $\Sigma_2 = (\pi_2, Q_2)$ the unique elliptic solid of $E$ containing $\sigma_2$. Notice that $Q_1 \setminus \pi_1$ is always disjoint from $Q_2 \setminus \pi_2$. Let $c_i = \sigma_i \cap Q_i$ be the conic in $\Sigma_i$, $i = 1, 2$. Assume that $\sigma_1 \cap \sigma_2$ is a line $r$. Since $r \subset \Sigma_1 \cap \Sigma_2$ it follows that $r \subset \pi_1 \cap \pi_2$. Under the inverse of the map $v$, the points of $r$ correspond to bi–lines having in common $q + 1$ points of a line $z$ and a further point $Z \notin z$. In particular, let $c'_i$ denote the conic of the circumscribed bundle $B$ locus of centers of the (imaginary) bi–lines corresponding to points of $c_1$ ($c_2$). It turns out that $z$ is secant to $c'_1$ and external to $c'_2$. In particular, $z$ is the polar line of the point $Z$ with respect to both $c'_1$ and $c'_2$, when $q$ is odd or $Z$ is the nucleus of both conics $c'_1$ and $c'_2$, when $q$ is even. But this, again, contradicts Lemma 3.4.

\[ \square \]

**Corollary 7.4.** There exists a constant dimension subspace code $K$ with parameters $(6, q^2(q^2 - 1)(q - 1)/3 + (q^2 + 1)(q^2 + q + 1), 4; 3)_{q}$.

**Corollary 7.5.** The code $K$ admits a group of order $3(q^2 + q + 1)$ as an automorphism group. It is the normalizer of a Singer cyclic group of $\text{PGL}(3, q)$.

**Remark 7.6.** We say that a constant dimension subspace code is complete if it is maximal with respect to set–theoretic inclusion. Some computer tests performed with MAGMA \[5\] yield that our code is not complete when $q = 3$. Indeed there exist other 39 planes that can be added to our set. However, when $q = 4, 5$ our code is complete. We conjecture that our code is complete whenever $q \geq 4$.

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