The Bipartition Polynomial of a Graph: Reconstruction, Decomposition, and Applications

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Abstract

The bipartition polynomial of a graph, introduced in [Dod+15], is a generalization of many other graph polynomials, including the domination, Ising, matching, independence, cut, and Euler polynomial. We show in this paper that it is also a powerful tool for proving graph properties. In addition, we can show that the bipartition polynomial is polynomially reconstructible, which means that we can recover it from the multiset of bipartition polynomials of one-edge-deleted subgraphs.

1 Introduction

The bipartition polynomial $B(G; x, y, z)$ of a simple graph $G$ has been introduced in [Dod+15]. The bipartition polynomial is related to the set of bipartite subgraphs of $G$: it generalizes the Ising polynomial [AM09], the matching polynomial [Far79], the independence polynomial (in case of regular graphs) [LM05; GH83], the domination polynomial [AL00], the Eulerian subgraph polynomial [Aig07], and the cut polynomial of a graph. In this paper, we consider the natural generalization of the bipartition polynomial to graphs with parallel edges.

Let $G = (V, E)$ be a simple undirected graph with vertex set $V$ and edge set $E$. The open neighborhood of a vertex $v$ of $G$ is denoted by $N(v)$ or $N_G(v)$. It is the set of all vertices of $G$ that are adjacent to $v$. The closed neighborhood of $v$ is defined by $N_G[v] = N_G(v) \cup \{v\}$. The neighborhood of a vertex subset $W \subseteq V$ is:

$$N_G(W) = \bigcup_{w \in W} N_G(w) \setminus W,$$

$$N_G[W] = N_G(W) \cup W.$$

The edge boundary $\partial W$ of a vertex subset $W$ of $G$ is

$$\partial W = \{\{u, v\} \mid u \in W \text{ and } v \in V \setminus W\},$$

i.e., the set of all edges of $G$ with exactly one end vertex in $W$. Throughout this paper, we denote by $n$ the order, $n = |V|$, by $m$ the size, $m = |E|$, and by $k(G)$ the number of components of $G$. 

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The bipartition polynomial of a graph $G$ is defined by

$$B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \sum_{F \subseteq \partial W} y^{|N(v, x)(W)|} z^{|F|}.$$  \hspace{1cm} (1)

Note that the definitions of neighborhood, edge boundary and Equation (1) can be easily extended to graphs with parallel edges. From now on, unless otherwise stated, we allow graphs to have parallel edges. Note that adding loops does not change the bipartition polynomial.

Figure 1 provides an illustration of the definition given in Equation (1). First we select a vertex subset $W$, which is located within the left gray-shaded bubble in Figure 1. The cardinality of the set $W$ is counted in the exponent of the variable $x$. The edge boundary $\partial W$ consists of all edges that stick out from that bubble. Assume we select the edges shown in bold as subset $F \subseteq \partial W$. The end vertices of these edges outside $W$ are presented within the next bubble that is labeled with $Y$. The cardinality of the set $Y$ is counted in the exponent of variable $y$ of the bipartition polynomial. The third variable, $z$, counts the edges in $F$. We see that by the definition of $F$ always a bipartite subgraph of $G$ is defined, which is the reason for the naming ‘bipartition polynomial’. If $H = (S \cup T, F)$ is a connected bipartite graph, then the partition sets $S$ and $T$ are uniquely defined (up to order).

Equation (1) implies that we can derive the order and size of a graph from its bipartition polynomial:

$$n = \deg B(G; x, 1, 1)$$  \hspace{1cm} (2)

$$m = \frac{1}{2}[xyz]B(G; x, y, z),$$  \hspace{1cm} (3)

where $[xyz]B(G; x, y, z)$ denotes the coefficient of $xyz$ in $B$.

**Proposition 1** A loopless graph $G$ is bipartite if and only if

$$\frac{1}{2}[xyz]B(G; x, y, z) = \deg B(G; 1, 1, z).$$

**Proof.** The left-hand side is, according to Equation (3), the number of edges of $G$. A graph $G = (V, E)$ is bipartite if and only if there is a vertex subset $W \subseteq V$ with $\partial W = E$. Equation (1) shows that in this case the degree of $z$ in $B(G; x, y, z)$ is equal to $m$. \[ Q.E.D. \]
In the remaining part of this paper, we present different representations and decompositions of the bipartition polynomial (Section 2), derive relations to other graph polynomials (Section 3), prove its polynomial reconstructibility (Section 4), and provide some applications for proving graph properties (Section 5).

2 Representations and Decomposition

In this section, we provide some different representations of the bipartition polynomial and decomposition formulae with respect to vertex and edge deletions.

2.1 Representations of the Bipartition Polynomial

Theorem 2 (product representation, \cite{Dod+15}) The bipartition polynomial of a graph \(G\) can be represented as

\[
B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \prod_{v \in N_G(W)} \left[ y \left(1 + z\right)^{|\partial v \cap \partial W|} - 1 \right] + 1.
\] (4)

The bipartition polynomial of a simple graph \(G = (V, E)\) satisfies

\[
B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \prod_{v \in N_G(W)} \left[ y \left(1 + z\right)^{|N_G(v) \cap W|} - 1 \right] + 1.
\] (5)

Corollary 3 The number of components of a graph \(G\) is \(\log_2 B(G; 1, 1, -1)\).

Proof. From Equation (4), we obtain

\[B(G; 1, 1, -1) = \sum_{W \subseteq V} \prod_{v \in N_G(W)} 0^{|\partial v \cap \partial W|}.
\]

The product vanishes for all \(W \subseteq V\) with \(N_G(W) \neq \emptyset\), since \(\partial v \cap \partial W \neq \emptyset\) for all \(v \in N_G(W)\). The product equals 1 if \(N_G(W) = \emptyset\) if and only if \(W\) is the (possibly empty) union of vertex sets of components of \(G\). For a graph with \(k\) components, there are \(2^k\) ways to form a union of the vertex sets of the components. Hence we obtain \(B(G; 1, 1, -1) = 2^k\). \(\blacksquare\)

The proof of the last proposition also yields the following statement.

Corollary 4 If \(G\) consists of \(k\) components \(G_1, \ldots, G_k\) such that the order of \(G_i\) is \(k_i\), then

\[B(G; x, 1, -1) = \prod_{i=1}^{k} (1 + x^{k_i}).\]

Consequently, we can derive the order of all components of \(G\) by the following simple procedure. The order of the first component is the smallest positive power, say \(k_1\), of \(x\) in \(B(G; x, 1, -1)\). Now divide \(B(G; x, 1, -1)\) by \((1 + x^{k_1})\) and proceed step by step with the resulting polynomial in the same manner until you obtain the constant polynomial 1.

A connected bipartite graph with at least one edge is called proper. For any given graph \(G\), we denote by \text{Comp}(G) the set of proper components of \(G\). As an abbreviation, we use \text{Comp}(V, E) instead of \text{Comp}((V, E)). The number of isolated vertices of a graph \(G = (V, E)\) is denoted by iso(G) or by iso(V, E).
Theorem 5 (bipartite representation, [Dod+15]) The bipartition polynomial of a graph $G = (V, E)$ satisfies
\[
B(G; x, y, z) = \sum_{F \subseteq E} (1 + x)^{\text{iso}(V,F)} \prod_{(S \cup T,A) \in \text{Comp}(V,F)} (x^{|S|} y^{|T|} + x^{|T|} y^{|S|}).
\] (6)

For another representation of the bipartition polynomial using so-called activity, we assume that the edge set $E = \{e_1, \ldots, e_m\}$ of the graph $G = (V, E)$ is linearly ordered, that is $e_1 < e_2 < \cdots < e_m$. Let $H$ be a spanning forest of $G$, which is a forest $H = (V, F)$ with the same vertex set as $G$ and $F \subseteq E$. An edge $e \in E \setminus F$ is externally active with respect to the forest $H$ if it is the largest edge in a cycle of even length of $H + e$. We denote by $\text{ext}(H)$ the number of externally active edges of $H$. Note that our definition of external activity is little different from that of Tutte [Tut54].

Theorem 6 (forest representation, [Dod+15]) The bipartition polynomial of a graph $G = (V, E)$ satisfies
\[
B(G; x, y, z) = \sum_{H \text{ is spanning forest of } G} (1 + x)^{n - k(H)} (1 + z)^{\text{ext}(H)} \prod_{(S \cup T,A) \in \text{Comp}(H)} (x^{|S|} y^{|T|} + x^{|T|} y^{|S|}).
\] (7)

Remark 7 In [Dod+15], the Theorems 2, 3, and 4 are proven for simple graphs only. However, the generalization to non-simple graphs is straightforward.

We need also the following result, which is proven in [Dod+15] too.

Theorem 8 Let $G$ be a graph consisting of $k$ components $G_1, \ldots, G_k$. Then
\[
B(G; x, y, z) = \prod_{i=1}^{k} B(G_i; x, y, z).
\]

2.2 Vertex and Edge Deletion

First we consider decompositions for the bipartition polynomial of a graph with respect to local vertex and edge operations.

Theorem 9 The bipartition polynomial of a graph $G = (V, E)$ satisfies for each vertex $v \in V$ the relation
\[
B(G; x, y, z) = (1 + x)B(G - v; x, y, z) + \sum_{(S \cup T,F) \text{ conn. bip.}} z^{|F|} (x^{|S|} y^{|T|} + x^{|T|} y^{|S|}) B(G - (S \cup T); x, y, z)
\]
\[
= (1 + x)B(G - v; x, y, z) + \sum_{(S \cup T,F) \text{ tree of } G} z^{|F| - 1}(1 + z)^{\text{ext}(S \cup T,F)}
\]
\[
\times (x^{|S|} y^{|T|} + x^{|T|} y^{|S|}) B(G - (S \cup T); x, y, z),
\]
where the first sum is taken over all proper subgraphs, and the second sum is over all nontrivial trees (having at least one edge) of $G$ that contain the vertex $v$. 

Proof. We show the proof for the first equality and the second one can be shown similarly. From Theorem 5, we obtain

\[ B(G; x, y, z) = \sum_{F \subseteq E, \text{ (V,F) is bipartite}} (1 + x)^{|F|} \prod_{(S \cup T, A) \in \text{Comp(V,F)}} (x^{\vert S \vert}y^{\vert T \vert} + x^{\vert T \vert}y^{\vert S \vert}) \]

\[ = \sum_{F \subseteq E \setminus \partial_v, \text{ (V,F) is bipartite}} (1 + x)^{|F|} \prod_{(S \cup T, A) \in \text{Comp(V,F)}} (x^{\vert S \vert}y^{\vert T \vert} + x^{\vert T \vert}y^{\vert S \vert}) \]

\[ + \sum_{F \subseteq E, F \cap \partial_v \neq \emptyset, \text{ (V,F) is bipartite}} (1 + x)^{|F|} \prod_{(S \cup T, A) \in \text{Comp(V,F)}} (x^{\vert S \vert}y^{\vert T \vert} + x^{\vert T \vert}y^{\vert S \vert}) B(G - (S \cup T); x, y, z) \]

The last equality results from factoring out the term of the product that corresponds to the component containing \(v\) and applying Theorem 5. 

The proof of the next statement can be performed in the same way.

Theorem 10 Let \(G = (V, E)\) be a graph and \(e \in E\); then

\[ B(G; x, y, z) = B(G - e; x, y, z) \]

\[ + \sum_{e \in F, \text{ conn. bip.}} z^{|F|}(x^{\vert S \vert}y^{\vert T \vert} + x^{\vert T \vert}y^{\vert S \vert}) B(G - (S \cup T); x, y, z). \]

3 Graph Polynomials that can be Derived from the Bipartition Polynomial

Several well-known graph polynomials can be obtained by substitution of the variables of the bipartition polynomial and (in some case) by multiplication with a certain factor that can easily be obtained from graph parameters like order, size, and the number of components. First we recall some results from [Dod+15].

**Domination polynomial** The domination polynomial of a graph \(G = (V, E)\), introduced in [AL00], is the ordinary generating function for the number of dominating sets of \(G\). Let \(d_k(G)\) be the number of dominating sets of size \(k\) of \(G\). We define the domination polynomial of \(G\) by

\[ D(G, x) = \sum_{k=0}^{n} d_k(G)x^k. \]

The domination polynomial satisfies, [Dod+15],

\[ D(G, x) = (1 + x)^n B\left(G; \frac{-1}{1 + x}, \frac{x}{1 + x}, -1\right). \]
A *generalized domination polynomial* is given by

\[ B(G; x, 1 - y, -1) = \sum_{W \subseteq V} x^{|W|} y^{|N_G(W)|}, \]

which follows directly from Theorem 2. The variable \( y \) counts here the number of vertices that are dominated by a given set \( W \). Consequently, the coefficient of \( x^i y^j \) in \( B(G; x, 1 - y, -1) \) gives the number of vertex subsets of cardinality \( i \) that dominate a vertex set of size \( j \).

**Ising polynomial**

The *Ising polynomial* of a graph \( G \) is defined by

\[ Z(G; x, y) = x^n y^m \sum_{W \subseteq V} x^{-|W|} y^{-|\partial W|}. \]

The Ising polynomial has been differently introduced in \[AM09\] by

\[ \tilde{Z}(G; x, y) = \sum_{\sigma \in \Omega} x^{\sigma(v)} y^{M(\sigma)}, \]

where \( \sigma : V \to \{-1, 1\} \) is the state of \( G = (V, E) \), \( \sigma(v) \) the magnetization of \( v \in V \), \( \Omega \) the set of all states of \( G \). The sum

\[ M(\sigma) = \sum_{v \in V} \sigma(v) \]

is called magnetization of \( G \) with respect to \( \sigma \). The parameter \( \epsilon(\sigma, e) \) defines the energy of the edge \( e \in E \). The energy of \( G \) with respect to \( \sigma \) is

\[ \epsilon(\sigma) = \sum_{e \in E} \epsilon(\sigma, e). \]

The here given notions result from the interpretation of the Ising polynomial (Ising model) in statistical physics. The relation between the two abovementioned representations of the Ising polynomial is

\[ \tilde{Z}(G; x, y) = x^{-n} y^{-m} Z(G; x^2, y^2). \]

Generalizations and modifications of the Ising polynomial and their efficient computation in graphs of bounded clique-width are considered in \[KM15\].

The Ising polynomial can be obtained from the bipartition polynomial by,

\[ Z(G; x, y) = x^n y^m B\left(G; \frac{1}{x}, 1, \frac{1}{y} - 1\right). \]  

(9)

**Cut polynomial**

The *cut polynomial* of a graph \( G = (V, E) \) is the ordinary generating function for the number of cuts of \( G \),

\[ C(G, z) = \frac{1}{2k(G)} \sum_{W \subseteq V} z^{|\partial W|}. \]

The relation between cut polynomial and bipartition polynomial is given by, see \[Dod+15\],

\[ C(G, z) = \frac{1}{2k(G)} B(G; 1, 1, z - 1). \]  

(10)
Corollary 11 A graph $G$ of order $n$ with $k$ components is a forest if and only if $C(G, z) = (1 + z)^{n-k}$.

Proof. The statement follows from the fact that a forest is the only graph for which any edge subset is a cut. ■

The polynomial

$$B(G; x, 1, z - 1) = \sum_{W \subseteq V} x^{|W|} z^{|\partial W|}$$

(11)
can be considered as a generalized cut polynomial; it is equivalent to the Ising polynomial. Equation (11) implies also

$$\frac{1}{\partial x} B(G; x, 1, t - 1) \bigg|_{x=0} = \sum_{v \in V} t^{\deg v},$$

(12)

which is the degree generating function of $G$.

Euler polynomial An Eulerian subgraph of a graph $G = (V, E)$ is a spanning subgraph of $G$ in which all vertices have even degree. The Euler polynomial of $G$ is defined by

$$\mathcal{E}(G, z) = \sum_{F \subseteq E} z^{|F|},$$

(13)

In [Aig07] it is shown that the Euler polynomial is related to the Tutte polynomial via

$$\mathcal{E}(G, z) = (1 - z)^{m-n+k(G)} z^{n-k(G)} T\left(G; \frac{1}{z}, \frac{1}{1-z}\right).$$

There is also a nice direct relation between cut polynomial and Euler polynomial, which is also shown in [Aig07],

$$C(G, z) = \frac{(1 + z)^{|E|}}{2^{|E|} - |V| + k(G)} \mathcal{E}\left(G, \frac{1 - z}{1 + z}\right)$$

(13)

Solving Equation (13) for $\mathcal{E}(G, z)$ and substituting $C$ according to Equation (10) yields

$$\mathcal{E}(G, z) = \frac{(1 + z)^{|E|}}{2^{|V|}} B\left(G; 1, 1, \frac{-2z}{1+z}\right).$$

(14)

Let $G$ be a plane graph (a planar graph with a given embedding in the plane) and $G^*$ its geometric dual. The set of cycles of $G$ is in one-to-one correspondence with the set of cuts of $G^*$, which yields

$$\mathcal{E}(G, z) = C(G^*, z)$$

or, corresponding to Equations (10) and (13),

$$(1 + z)^n B\left(G; 1, 1, \frac{-2z}{1+z}\right) = 2^{n-1} B(G^*; 1, 1, z - 1).$$

(15)
Van der Waerden polynomial  The definition of this polynomial is presented in [AM09] and based on an idea given in [Wae41]. Let $G = (V, E)$ be a graph of order $n$ and size $m$. Let $w_{ij}(G)$ be the number of subgraphs of $G$ with exactly $j$ edges and $i$ vertices of odd degree. The van der Waerden polynomial of $G$ is defined by

$$W(G; x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} w_{ij}(G)x^iy^j.$$  

From [AM09] (Theorem 2.9), we obtain easily

$$W(G; x, y) = \left(\frac{1-\sqrt{x}}{2}\right)^n (1-y)^m Z\left(G; \frac{1+x}{1-y}, \frac{1+y}{1+y}\right),$$

where $Z$ is the Ising polynomial. The van der Waerden polynomial can be derived from the bipartition polynomial by

$$W(G; x, y) = \left(\frac{1+x}{2}\right)^n (1+y)^m B\left(G; \frac{1-x}{1+y}, 1, -\frac{2y}{1+y}\right), \quad (16)$$

where we use Equation (16).

Matching polynomial  The matching polynomial, see [Far79], of $G$ is defined by

$$M(G, x) = \sum_{F \subseteq E} x^{|F|}.$$  

Notice that the definition that is given here corresponds to matching generating polynomial from [LP09]. A subgraph of $G$ with exactly $k$ edges and exactly $2k$ vertices of odd degree is a matching in $G$, which yields

$$M(G, t) = \lim_{y \to 0} W(G; ty^{-\frac{1}{2}}, y).$$

Substituting Equation (16) for $W$, we obtain

$$M(G, t) = \lim_{y \to 0} \left(\frac{\sqrt{y} + t}{2\sqrt{y}}\right)^n (1+y)^m B\left(G; \frac{\sqrt{y} - t}{\sqrt{y} + t}, 1, -\frac{2y}{1+y}\right). \quad (17)$$

Independence polynomial  The independence polynomial of a graph $G = (V, E)$ is the ordinary generating function for the number of independent set of $G$,

$$I(G, x) = \sum_{W \subseteq V} x^{|W|}.$$  

If $G$ is a simple $r$-regular graph, then

$$I(G, t) = \lim_{x \to 0} B\left(G; tx^r, 1, \frac{1}{x} - 1\right).$$

The proof of this relation is given in [Dod+15]. We can easily rewrite the last equation in order to avoid the limit:

$$I(G, t) = \frac{1}{2\pi} \int_{0}^{2\pi} B(G; te^{ix}, 1, e^{-ix} - 1)dx.$$
The substitution \( x \mapsto e^{ix} \) transforms each power of \( x \) into a periodic function whose period divides \( 2\pi \) such that the integration over \([0, 2\pi]\) yields the constant term (with respect to \( x \)) multiplied by \( 2\pi \).

4 Polynomial Reconstruction

One of the important questions about graph polynomials is their distinguishing power, which can be stated as follows:

Let \( C \) be a graph class and let \( P \) be a polynomial-valued isomorphism invariant defined on \( C \). Are there nonisomorphic graphs \( G \) and \( H \) in \( C \) such that \( P(G) = P(H) \)?

Although we know that the bipartition polynomial cannot distinguish all graphs up to isomorphism, see \[\text{Dod+15}\], we do not know yet whether there are two nonisomorphic trees with the same bipartition polynomial. Instead, as trees are well-known to be ‘reconstructible’ in various senses, we show that the bipartition polynomial of a graph is edge-reconstructible from its polynomial-deck, which shall be defined precisely below.

For a graph \( G \), its polynomial-deck is the multiset \( \{B(G - e)\}_{e \in E(G)} \). We show that the bipartition polynomial is ‘edge-reconstructible’ in most cases in the following sense:

A graph \( G \) is bp-reconstructible if whenever a graph \( H \) has the same polynomial-deck as \( G \) we have \( B(H) = B(G) \).

Unfortunately, there are some graphs with few edges that are not bp-reconstructible. To describe such examples, let \( P_s \) and \( C_s \) denote respectively the path and cycle on \( s \) vertices. We denote by \( C_s + tP_1 \) the disjoint union of \( C_s \) and \( t \) isolated vertices, and the graphs \( P_s + tP_1 \), \( sP_2 + tP_1 \) etc. are defined similarly. The following graphs in each line have the same polynomial-deck but have different bipartition polynomial.

\[\begin{align*}
\bullet & \quad C_2 + (t + 2)P_1, \ P_3 + (t + 1)P_1 \text{ and } 2P_2 + tP_1 \text{ for } t \geq 0. \\
\bullet & \quad C_3 + (t + 1)P_1 \text{ and } K_{1,3} + tP_1 \text{ for } t \geq 0.
\end{align*}\]

Note that the graphs on each line for fixed \( t \) not only have the same polynomial-deck but also have the same collection of one-edge-deleted subgraphs.

We prove the following in this section.

**Theorem 12** A graph \( G \) is bp-reconstructible unless \( G \) is one of the exceptions above. In particular, all graphs with at least four edges are bp-reconstructible.

4.1 Graphs with Isolated Vertices

We shall use the following information on graphs that are deducible from the bipartition polynomial. The statement combines the results given in Equations (2), (3), (12), Proposition H and Corollaries H, I, J.

**Theorem 13** Let \( G \) be a graph. The bipartition polynomial of \( G \) yields \( |V(G)| \), \( |E(G)| \), \( k(G) \), \( \text{iso}(G) \), the degree sequence, and the multiset of orders of all components of \( G \). We can also decide from \( B(G) \) whether \( G \) is bipartite, a forest, a path, or connected. (The last two properties follow from the other ones.)
We begin the proof of Theorem 12 with the case when two graphs $G$ and $H$ have different number of isolated vertices but the same polynomial-deck. Note that from Theorem 13, we know that two graphs with a different number of isolated vertices have a different bipartition polynomial.

**Lemma 14** Let $G$ and $H$ be two graphs having different number of isolated vertices. If $G$ and $H$ have the same polynomial-deck, then there exists $t \geq 0$ such that either

1. $\{G, H\} \subset \{C_2 + (t + 2)P_1, P_3 + (t + 1)P_1, 2P_2 + tP_1\}$ or
2. $\{G, H\} = \{C_3 + (t + 1)P_1, K_{1,3} + tP_1\}$.

**Proof.** Suppose $G$ and $H$ have the same polynomial-deck and $\text{iso}(G) = t$ while $\text{iso}(H) > t$. Since $\text{iso}(G - e) \leq t + 2$ for every edge $e \in E(G)$, we have $\text{iso}(H) = t + 1$ or $t + 2$. As $\text{iso}(H - f) > t$ and $f \in E(H)$, we have $\text{iso}(G - e) > \text{iso}(G)$ for all $e \in E(G)$ implying that every edge of $G$ is incident with a vertex of degree 1. That is, the components of $G$ are stars and isolated vertices. By Theorem 13, we deduce that $H - f$ is a forest for every $f \in E(H)$. Hence either $H$ itself is a forest or $H = C_s + t'P_1$ for some $s$ and $t + 1 \leq t' \leq t + 2$.

If $\text{iso}(H) = t + 2$ then every edge removal from $G$ produces two new isolated vertices, so that $G = s'P_2 + tP_1$ for some $s'$. Moreover, no edge of $H$ is incident with a vertex of degree 1, that is, $H = C_s + (t + 2)P_1$. Since $G$ and $H$ have the same number and an equal number of edges, we conclude $G = 2P_2 + P_1$ and $H = C_2 + (t + 2)P_1$.

Now we assume $\text{iso}(H) = t + 1$. If $H = C_s + (t + 1)P_1$ for some $s$, then for all $f \in E(H)$, $\text{iso}(H - f) = t + 1$ and $H - f$ has maximum degree at most two. As $G$ and $H$ have the same polynomial-deck, the same holds for $G - e$ for all $e \in E(G)$. Since $G$ is a disjoint union of stars with $t$ isolated vertices, the only possibility for this case is $G = K_{1,3} + tP_1$ and $H = C_3 + (t + 1)P_1$.

Now we also assume that $H$ is a forest. Theorem 13 states that we can decide the orders of the components from the bipartition polynomial. If $G - e$ for some $e \in E(G)$ has three $P_2$-components, then $H - f$ has it too for some $f \in E(H)$ and $H$ has a $P_2$-component. Removing its edge produces a subgraph with $t + 3$ isolated vertices, which cannot be obtained from $G$ by removing only one edge. Thus for all $e \in E(G)$, $G - e$ can have at most two $P_2$-components and $G$ may have at most three $P_2$-components. If $G$ has three $P_2$-components, then they are the only nontrivial components of $G$.

On the other hand, as $H$ is a forest, each nontrivial component of $H$ has at least two leaves which leave $t + 2$ isolated vertices each when removed. The number of leaves of $H$ must be equal to the number of $P_2$-components of $G$, so that either $G = 3P_2 + tP_1$ or $H = P_s + (t + 1)P_1$ for some $s$. It is easy to check that for this case the only possibility of non-isomorphic pair $G$ and $H$ with same polynomial-deck is $G = 2P_2 + tP_1$ and $H = P_3 + (t + 1)P_1$. ■

### 4.2 Cyclic Graphs

Because of Lemma 14 we only need to compare those graphs without isolated vertices. The remaining part of our proof of Theorem 12 is presented in the following order.

1. Every non-bipartite graph except $C_3 + tP_1$ for $t \geq 1$ is bp-reconstructible.
2. Every bipartite graph with a cycle except $C_2 + tP_1$ for $t \geq 2$ is bp-reconstructible.

3. Every forest except $P_3 + (t + 1)P_1$, $2P_2 + tP_1$, $K_{1,3} + tP_1$ for $t \geq 0$ is bp-reconstructible.

The first two are simple but the proof for the third case is a bit longer so we defer it to Section 4.3.

Given a proper bipartite graph $K$ with bipartition $(U_1, U_2)$, let $m(K) = \min(|U_1|, |U_2|)$ and $M(K) = \max(|U_1|, |U_2|)$. Theorem 3 states

$$B(G; x, y, z) = \sum_{F \subseteq E} z^{|F|}(1 + x)^{\text{iso}(V,F)} \prod_{K \in \text{Comp}(V,F)} [x^{M(K)}y^m(K) + x^m(K)y^{M(K)}].$$

(18)

Let us define for each $F \subseteq E$, a polynomial $\phi_G(F; x, y)$ or simply $\phi_G(F)$ as

$$\phi_G(F) := \begin{cases} (1 + x)^{\text{iso}(V,F)} \prod_{K \in \text{Comp}(V,F)} [x^{M(K)}y^m(K) + x^m(K)y^{M(K)}] & \text{if } (V,F) \text{ is bipartite,} \\ 0 & \text{otherwise.} \end{cases}$$

We also write $B(G)$ instead of $B(G; x, y, z)$ for convenience. With this definition, Equation (18) simplifies to

$$B(G) = \sum_{F \subseteq E} z^{|F|}\phi_G(F).$$

(19)

We consider the sum $B'(G) = \sum_{e \in E} B(G - e)$ for a given multiset $\{B(G - e)\}_{e \in E}$.

For each $F \subseteq E$, the term $z^{|F|}\phi_G(F)$ appears precisely $|E| - |F|$ times on the right-hand-side so that

$$B'(G) = \sum_{F \subseteq E} (|E| - |F|)z^{|F|}\phi_G(F).$$

Thus for each $k = 0, 1, \ldots, |E| - 1$, the coefficient of $z^k$ in $B'(G)$ is the coefficient of $z^k$ in $B'(G)$ divided by $|E| - k$. The only remaining term to decide $B(G)$ is $z^{|E|}\phi_G(E)$. Therefore, to show $G$ is bp-reconstructible it is enough to show that if $H = (V', E')$ is another graph with the same polynomial-deck as $G$, then $\phi_H(E') = \phi_G(E)$.

Now we show that nonbipartite graphs are bp-reconstructible except $C_3 + tP_1$ for $t \geq 1$.

**Lemma 15** Every nonbipartite graph except $C_3 + tP_1$ for $t \geq 1$ is bp-reconstructible.

**Proof.** By Lemma 14 it is enough to show that if $G$ is not bipartite, $\text{iso}(G) = \text{iso}(H)$ and $H$ has the same polynomial-deck as $G$ then $B(G) = B(H)$. We may additionally assume that $G$ and $H$ have no isolated vertices.

Suppose $G = (V, E)$ is not bipartite, $\text{iso}(G) = 0$ and let $D = \{B(G - e)\}_{e \in E}$. Let $H = (V', E')$ be a graph with $\text{iso}(H) = 0$ whose polynomial-deck is equal to $D$ as a multiset. If $G - e$ is not bipartite for some $e \in E$ then from the corresponding bipartition polynomial, we infer that $H - e'$ is nonbipartite for
some $e' \in E'$ and $\phi_H(E') = 0$, that is, $B(G) = B(H)$. If there is no such $e$, then $G$ is an odd cycle. Applying Theorem 13 to $D$, we deduce that every one-edge-deleted subgraph of $H$ is a path consisting of odd number of vertices and the only graph with such property is an odd cycle, and hence $\phi_H(E') = 0$.

We now suppose that $G = (V, E)$ is bipartite. Note that the degree of $\phi_G(F)$ is precisely $|V|$. Since

$$x^{M(K)}y^{m(K)} + x^{m(K)}y^{M(K)} = (xy)^{m(K)} \left(x^{M(K) - m(K)} + y^{M(K) - m(K)}\right),$$

to decide $\phi_G(E)$ we only need $M(K) - m(K)$ for each nontrivial component $K$ in $G$. If $G$ has $k$ nontrivial components with bipartitions $(U_i, V_i)$ for $i = 1, 2, \ldots, k$ and $t$ isolated vertices, then we say $G$ has type

$$(a_1, a_2, \ldots, a_k, *, *, \ldots, *)$$

where $a_i = ||U_i| - |V_i||$ and the number of $*$’s are $t$. We will ignore the order of entries in types. From now on we consider the type instead of $\phi_G(E)$.

**Lemma 16** Let $G$ be a bipartite graph. If $G$ has a cycle then $G$ is bp-reconstructible unless $G = C_2 + tP_1$ for some $t \geq 2$.

**Proof.** By Lemmas 14 and 15 it is enough to show that if $H = (V', E')$ is another bipartite graph with the same polynomial-deck as $G$ and $\text{iso}(G) = \text{iso}(H) = 0$, then the type of $H$ is uniquely determined. Note that the exceptions $C_2 + tP_1$ are automatically excluded since $t \geq 2$.

If $G$ is connected, then $G - e$ is connected for some $e$, since $G$ has a cycle. Let $e' \in E'$ be an edge such that $B(H - e') = B(G - e)$. We know that $H$ is bipartite and, by Theorem 13, $H - e'$ is connected. The coefficient of $z^{|E'| - 1}$ in $B(H - e') = B(G - e) \mid E'(G) = \text{iso}(E')$ tells us the type of $H'$, which must be the same as the type of $H$ and hence $\phi_H(E') = \phi_G(E)$.

Suppose $G$ is not connected. Then $G - e$ contains a cycle for some $e \in E$, and by Theorem 13 $H$ also has an edge $e'$ such that $H - e'$ contains a cycle. Thus $H$ has a cycle, and we choose $e' \in E'$ such that $H - e''$ has minimum number of components among the one-edge-deleted subgraphs of $H$. The components of $H$ are vertex-wise same as the components of $H - e''$ and have precisely the same bipartitions. That is, the type of $H$ is the type of $H - e''$ which is again equal to the type of $G$. Hence $\phi_H(E') = \phi_G(E)$ and $G$ is bp-reconstructible. ■

### 4.3 Bipartition Polynomials of Forests

In this section we prove the following lemma, thereby completing the proof of Theorem 12.

**Lemma 17** Every forest except $2P_2 + tP_1$, $P_3 + (t + 1)P_1$ and $K_{1,3} + tP_1$ for $t \geq 0$ is bp-reconstructible.

To prove Lemma 17 for forests with at least four edges we show the following:

**Lemma 18** Let $F$ be a forest. The type of $F$ is uniquely determined from the degree sequence of $F$ and the multiset consisting of types of $F - e$ for all $e \in E(F)$.
In [DFR02], it was shown that if \( G \) has at least four edges, then the degree sequence of \( G \) is completely determined from the degree sequences of one-edge-deleted subgraphs. Theorem 13 states that the degree sequence is obtainable from the bipartition polynomial, so that Lemma 14 follows for forests with at least four edges. The missing cases for Lemma 17 without isolated vertices, \( P_2, 3P_2, P_3 + P_2 \) and \( P_4 \) as simple graphs and also the non-simple ones are easy to check.

We shall use some lemmas about trees. For the definition of the type of a bipartite graph see the discussion preceding Lemma 16. In a tree, a vertex of degree 1 is a leaf and an edge incident with a leaf is a leaf-edge. An edge is internal if it is not a leaf-edge.

**Lemma 19** Let \( T \) be a tree with at least one edge. Let \( (U, V) \) be the bipartition of \( T \).

(i) If \( U \) has all the leaves, then \( |U| > |V| \).

(ii) If \( V \) has only one leaf, then \( |U| \geq |V| \).

(iii) If \( T \) has type \((a)\) for \( a \geq 1 \), then \( T \) has two edges \( e_1, e_2 \) such that both \( T - e_1 \) and \( T - e_2 \) have type \((a - 1, *)\).

(iv) Suppose \( T \) has type \((0)\). If \( T - e \) has type \((1, 1)\) for every internal edge \( e \), then the degrees of vertices of \( T \) are all odd.

(v) Suppose \( T \) has type \((2)\). If the types of \( T - e \) with * are all \((1, *)\), then either \( T \) is \( K_{1,3} \) or \( T - f \) has type \((0, 2)\) for some edge \( f \).

**Proof.** Let \( u \) be a vertex in \( U \). Consider \( u \) as a root and direct every edge of \( T \) away from \( u \). Then

\[
|U| = 1 + \sum_{v \in V(T)} d^+(v) = 1 + \sum_{v \in V} (d(v) - 1) = 1 + \sum_{v \in V} (d(v) - 2) + |V|,
\]

so that

\[
|U| - |V| = 1 + \sum_{v \in V} (d(v) - 2)
\]

and \((*)\) follows immediately.

Suppose \( T \) has type \((a)\) for some \( a \geq 1 \). We may assume \( |U| - |V| = a \). By \( (i) \) and \( (ii) \), \( U \) contains at least two leaves and removing their incident edges produce forests, each of type \((a - 1, *)\). Thus \( (iii) \) holds.

Now we consider \((iv)\). Suppose \( T \) has type \((0)\) and \( T - e \) has type \((1, 1)\) for every internal edge \( e \) of \( T \). If \( T \) consists of only one edge then the conclusion holds. Thus we assume that \( T \) has at least one internal edge. Suppose that \( T \) has a vertex \( v \) of even degree, for contradiction. Since \( T \) has type \((0)\) \( v \) is incident to at least one internal edge. Let us say \( v \) is adjacent to \( s \) leaves and is incident to \( t \) internal edges \( e_1, e_2, \ldots, e_t \) where \( e_i = vu_i \) for \( i = 1, 2, \ldots, t \). Let us denote by \((U_i, V_i)\) the bipartition of the component of \( T - e_1 \) containing \( u_i \) such that \( u_i \in U_i \). By the assumptions on \( T \), we know \( |U_i| - |V_i| \) is odd for all \( i \). We consider the component of \( T - e_1 \) containing \( v \). Its type is given by

\[
\left| \sum_{i=2}^{t} |U_i| + s - \sum_{i=2}^{t} |V_i| - 1 \right|,
\]
which is an even number contradicting the assumption that $T - e_1$ has type (1, 1). Hence (iv) follows.

Lastly we show (v). Let $T$ be a tree with bipartition $(U, V)$ such that $|U| - |V| = 2$. Suppose that for every leaf-edge $e$ of $T$, the forest $T - e$ has type (1, *).

That is, $U$ contains all the leaves. From Equation ($\ast$), we deduce that $V$ has a unique vertex of degree 3 and all other vertices in $V$ have degree 2. If $V$ has no vertex of degree 2 then $|V| = 1$ and $T$ is $K_{1,3}$. Suppose $V$ has a vertex, say $v$, of degree 2. Let $e, f$ be the edges incident with $v$. If $e$ is a leaf-edge then $T - f$ has type $(0,2)$. We assume that both $e$ and $f$ are internal edges. The graph $T - e$ has two components, say $T_1 = (W_1, E_1)$ and $T_2 = (W_2, E_2)$ where $f \in E_2$.

Note that all the leaves of $T_1$ are in $U$ and all but one leaves of $T_2$ are in $U$. By (i) and (ii), we have

$$|U \cap W_1| > |V \cap W_1| \quad \text{and} \quad |U \cap W_2| \geq |V \cap W_2|.$$ 

Since $2 = |U| - |V| = (|U \cap W_1| - |V \cap W_1|) + (|U \cap W_2| - |V \cap W_2|)$, we have either

$$|U \cap W_1| - |V \cap W_1| = 2 \quad \text{and} \quad |U \cap W_2| - |V \cap W_2| = 0$$

or

$$|U \cap W_1| - |V \cap W_1| = 1 \quad \text{and} \quad |U \cap W_2| - |V \cap W_2| = 1.$$ 

For the former $T - e$ has type $(0,2)$. For the latter $T - f$ has type $(0,2)$. Thus (v) holds. ■

**Proof of Lemma** 18

We shall divide the proof of Lemma 18 into three cases, depending on whether the forest $F$ has one component, two components or more than two components. In each case we show how to retrieve the type of $F$. We call the multiset of the types of $F - e$ for all edges $e$ as the type-deck of $F$. The sub-multiset consisting of those types with * is called *-deck. We can assume the following in all three cases. The reasoning is given below.

- The types in the *-deck contains precisely one *.
- If $F$ has more than one component, then at least one type in the *-deck has a zero as entry.
- No type in the *-deck has more than one zero.

As the degree sequence of $F$ is given by the assumption, we may assume that $F$ has no isolated vertices. If the the *-deck of $F$ contains a type with two *’s, then the two isolated vertices came from deleting an edge in a $P_2$-component of $F$, so that we can recover the type of $F$ by replacing the two *’s with a zero. Thus we may assume that the types in the *-deck contains precisely one *.

Let $m$ be the minimum of integral entries in the types in the *-deck. If $m > 0$, then $F$ cannot induce a component of type $(m)$ by Lemma 19 (iii). If $m \neq 1$, the entry $m$ is obtained from a component of type $(m + 1)$, and the type of $F$ is obtained by replacing $(m, *)$ with a $m + 1$. If $m = 1$ and $F$ has more than one component, then $F$ cannot have a component of type $(0)$ so that again, by replacing $(m, *)$ with $m + 1$ we retrieve the type of $F$. Hence we may assume that some types in the *-deck have a zero.

Suppose that a type in the *-deck has at least two zeros. Then $F$ has a component of type $(0)$. Among the types in the *-deck, we choose one with a minimum number of zeros, denote this type by $X$. The zeros in $X$ came directly
from $F$ and there must be a 1 and a * which was obtained by removing a leaf-edge of a component with type (0). Thus we replace (1, *) by (0) to retrieve the type of $F$. Now we may assume that no type in the *-deck has more than one zero.

Now we prove Lemma 19

**Case 1.** $F$ is a tree.

If the the *-deck has $(a, *)$ and $(a + 2, *)$ for some $a$ then the only possible type of $F$ is $(a + 1)$. Suppose $(a, *)$ is the unique type in the *-deck. If $a \neq 1$ then by Lemma 19 (iii) the type of $F$ is $(a + 1)$. Suppose $(1, *)$ is the only type in the *-deck. $F$ has two possibilities: (0) and (2). If the type-deck has (0, 2) then clearly (2) is the case. If the degree sequence of $F$ is $(3, 1, 1, 1)$ then $F$ is $K_{1,3}$. If the type-deck does not have (0, 2) and $F$ is not $K_{1,3}$ then by Lemma 19 (v) the type of $F$ is (0). This completes Case 1.

**Case 2.** $F$ has precisely two components. By the assumptions the *-deck has $(0, a, *)$ for some $a \geq 1$. If the *-deck has $(0, a + 2, *)$ also then $F$ has type $(0, a + 1)$. Thus we assume $(0, a, *)$ is the unique type in the *-deck with a 0. First we assume $a \geq 2$ and then consider the case $a = 1$.

Suppose $a \geq 2$. If $F$ had type $(0, a - 1)$, then by Lemma 19 (iii) its *-deck must have $(0, a - 2, *)$ which is a contradiction. Thus $F$ has type $(0, a + 1)$ or $(1, a)$. If the *-deck has $(1, a - 1, *)$ then the type of $F$ cannot be $(0, a + 1)$ and hence it is $(1, a)$. If $F$ has type $(1, a)$ by Lemma 19 (iii) the *-deck has $(1, a - 1, *)$. That is, $F$ has type $(1, a)$ if and only if the *-deck has $(1, a - 1, *)$, implying that we can decide the type of $F$ from its *-deck.

Now we assume $a = 1$ and the types in the *-deck with a 0 are $(0, 1, *)$. $F$ has one of three types: $(0, 0)$, $(1, 1)$ and $(0, 2)$. If the type-deck has $(0, 0, 2)$ then $F$ has type $(0, 2)$. Suppose the type-deck does not have $(0, 0, 2)$. If $F$ has type $(0, 2)$, then every leaf-edge of a component of type (2) produces $(1, *)$ when deleted. By Lemma 19 (v), the component is $K_{1,3}$ and the *-deck of $F$ has precisely three $(0, 1, *)$. On the other hand, if $F$ had type $(1, 1)$ then by Lemma 19 (iii) there are at least four $(0, 1, *)$ in the *-deck of $F$. If $F$ had type $(0, 0)$ then its *-deck also have at least four $(0, 1, *)$ since we assumed none of the components are $K_2$ and every tree has at least two leaves. Thus $(0, 1, *)$ appears precisely three times in the *-deck if and only if $F$ has type $(0, 2)$.

Now we assume the *-deck has at least four times $(0, 1, *)$. $F$ has type $(0, 0)$ or $(1, 1)$. We may assume:

(i) the *-deck has only $(0, 1, *)$.

(ii) the type-deck consists of precisely $(0, 1, *)$ and $(0, 1, 1)$.

The first assumption is because the only other possible type in the *-deck is $(1, 2, *)$, in which case $F$ has type $(1, 1)$. For the second assertion, recall that we assumed that no component is $P_2$. Thus a component of type $(0)$ has an internal edge and by removing an internal edge from $(0, 0)$ we get $(0, a, a)$. If $a \neq 1$ then $F$ has type $(0, 0)$.

If $F$ has type $(1, 1)$, then assumption (i) above implies for both components, one part of the bipartition contains all the leaves. From the proof of Lemma 19 (i), all vertices in the other part have degree 2.

On the other hand, if $F$ has type $(0, 0)$, then assumption (ii) above implies that for each component of $F$, every internal edge produces a forest of type $(1, 1)$ when removed. Lemma 19 (iv) asserts that all vertices of $F$ have odd degree.
Therefore, $F$ has type $(1,1)$ if $F$ has a vertex of degree 2, and $F$ has type $(0,0)$ otherwise. That is, we can determine the type of $F$ given all the assumptions so far. This completes Case 2.

**Case 3.** $F$ has more than two components. By the assumptions after Lemma 19 the $*$-deck has a type $(0,a_1,a_2,\ldots,a_k,* )$ where $a_i > 0$ for all $i$. The question is to decide whether $F$ has a component of type $(0)$ or not. If it has, then the $(0)$ component is unique and the $*$-deck has a type without zero. We replace $(1,*)$ with a $(0)$ to recover the type of $F$. If $F$ does not have a component of type $(0)$, then $F$ has type $(1,a_1,a_2,\ldots,a_k)$.

Suppose the $*$-deck of $F$ has another type $(0,b_1,b_2,\ldots,b_k,* )$ where $\{a_i\} \neq \{b_i\}$ as multisets. Then $F$ has a component of type $(0)$, since otherwise the zeros in the $*$-deck types come from a $(1)$ of $F$ and all such types must be the same up to order of entries.

Thus we assume $(0,a_1,a_2,\ldots,a_k,*)$ is the only type in the $*$-deck with a 0. If the type of $F$ had a 0 and two distinct nonzero numbers then the $*$-deck contains two distinct types with a 0. Hence we may assume that

$$(0,a_1,a_2,a_3,\ldots,a_k,*) = (0,a-1,a,a,\ldots,a,*)$$

for some $a \geq 2$.

The type of $F$ is either $(0,a,a,\ldots,a)$ or $(1,a-1,a,\ldots,a)$. Lemma 10 (iii) implies that in the latter case the $*$-deck has $(1,a-1,a-1,a,\ldots,*,*)$, whereas the former case cannot have the same type. Thus we can decide the type of $F$ from the $*$-deck in Case 3.

In all cases we can decide the type of $F$ from the degree sequence and the type-deck of $F$. Therefore Lemma 18 holds and Lemma 17 follows.

## 5 Applications

In this section we prove some facts about Euler polynomials, the number of dominating sets, and sums over spanning forests of a graph. The common theme of these results is a very simple way of proving by just using different representations of the bipartition polynomial.

We denote by $F(G)$ the set of spanning forests of the graph $G$.

**Theorem 20** The Euler polynomial of a graph $G$ of order $n$ and size $m$ satisfies

$$E(G,z) = (1+z)^{m-n}(-z)^n \sum_{H \in F(G)} \left( -\frac{1+z}{z} \right)^{k(H)} \left( \frac{1-z}{1+z} \right)^{\text{ext}(H)}.$$ 

**Proof.** We use the forest representation of the bipartition polynomial that is given in Theorem 5 and Equation 14; we obtain

$$E(G,z) = \frac{(1+z)^m}{2^n} B(G;1,1,-\frac{2z}{1+z})$$

$$= \frac{(1+z)^m}{2^n} \sum_{H \in F(G)} 2^{\text{iso}(H)} \left( -\frac{2z}{1+z} \right)^{n-k(H)} \left( \frac{1-z}{1+z} \right)^{\text{ext}(H)} 2^{k(H)-\text{iso}(H)}$$

$$= (1+z)^{m-n}(-z)^n \sum_{H \in F(G)} \left( -\frac{1+z}{z} \right)^{k(H)} \left( \frac{1-z}{1+z} \right)^{\text{ext}(H)}.$$  

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For the second equality, we used the simple relation $|\text{Comp}(H)| + \text{iso}(H) = k(H)$.

The next theorem provides a representation of the Euler polynomial as a sum ranging over subsets of the vertex set.

Theorem 21 The Euler polynomial of a graph $G = (V, E)$ satisfies

$$E(G, z) = \frac{(1 + z)^{|E|}}{2^{|V|}} \sum_{W \subseteq V} \left( \frac{1 - z}{1 + z} \right)^{|\partial W|}.$$  

Proof. The result can be obtained via the multiplicative representation of the bipartition polynomial according to Theorem 2. The substitution of this representation for the bipartition polynomial in Equation (14) yields

$$E(G, z) = \frac{(1 + z)^n}{2^n} B \left( G; 1, 1, -\frac{2z}{1 + z} \right)$$

$$= \frac{(1 + z)^n}{2^n} \sum_{W \subseteq V} \prod_{v \in N_G(W)} \left( \left( 1 - \frac{2z}{1 + z} \right)^{|\partial v \cap \partial W|} \right)$$

$$= \frac{(1 + z)^n}{2^n} \sum_{W \subseteq V} \left( \frac{1 - z}{1 + z} \right)^{|\partial W|}.$$  

The last equality follows from the fact that $|N_G(v) \cap W|$ is the number of edges that connect $v$ to a vertex of $W$. Hence when we take the product over all vertices in $N_G(W)$, then we count each edge in $\partial W$ exactly once.

The following statement can be proven also via the principle of inclusion–exclusion. However, our knowledge about the bipartition polynomial offers an even faster way of proof.

Theorem 22 The number $d(G)$ of dominating sets of a graph $G$ satisfies

$$d(G) = 2^n \sum_{W \subseteq V} (-1)^{|W|} \left( \frac{1}{2} \right)^{|N_G(W)|}.$$  

Proof. Here we use the product representation of the bipartition polynomial of a simple graph. The restriction to simple graphs does not change the domination polynomial. According to Equation (8), we obtain

$$d(G) = D(G, 1)$$

$$= 2^n B \left( G; -\frac{1}{2}, \frac{1}{2}, -1 \right)$$

$$= 2^n \sum_{W \subseteq V} \left( -\frac{1}{2} \right)^{|W|} \prod_{v \in N_G(W)} \left[ \left( \frac{1}{2} \right)^{|N_G(v) \cap W|} - 1 \right] + 1$$

$$= 2^n \sum_{W \subseteq V} \left( -\frac{1}{2} \right)^{|W|} \prod_{v \in N_G(W)} \left( \frac{1}{2} \right)$$

$$= 2^n \sum_{W \subseteq V} \left( -\frac{1}{2} \right)^{|W|} \left( \frac{1}{2} \right)^{|N_G(W)|},$$

which is equivalent to the statement of the theorem. ■
Theorem 23 Let $G = (V, E)$ be a graph with a linearly ordered edge set and $\mathcal{F}_0(G)$ the set of all spanning forests of $G$ with external activity 0. Then

$$\sum_{H \in \mathcal{F}_0(G)} (-2)^{k(H)} = (-1)^n 2^{k(G)}.$$  

Proof. The statement follows immediately from the forest representation of the bipartition polynomial that is given in Theorem 6 by substituting $x = 1, y = 1, z = -1$ in $B(G; x, y, z)$. 

Theorem 24 Let $G = (V, E)$ be a simple undirected graph with $n$ vertices. The number of bicolored subgraphs of $G$ with exactly $i$ isolated vertices and exactly $j$ edges is given by the coefficient of $x^i z^j$ in the polynomial

$$(2x - 1)^n B\left(G; \frac{1}{2x-1}, \frac{1}{2x-1}, z\right).$$

Proof. An edge subset $F \subseteq E$ of $G$ induces a subgraph that can be properly colored with two colors if and only if $(V, F)$ is a bipartite graph. The number of bicolored graphs with edge set $F$ is then given by $2^{k(V,F)}$. Substituting $x$ and $y$ by $1/(2x - 1)$ in $B(G; x, y, z)$ and multiplying the resulting expression with $(2x - 1)^n$ yield

$$(2x - 1)^n \sum_{F \subseteq E \atop (V,F) \text{ is bipartite}} z^{|F|} \left(\frac{2x}{2x - 1}\right)^{iso(V,F)} \prod_{(U,A) \in Comp(V,F)} \left(\frac{2}{2x - 1}\right)^{|U|}. $$

Using the equations $iso(V,F) + |Comp(V,F)| = k(V,F)$ and $iso(V,F) + \sum_{(U,A) \in Comp(V,F)} |U| = n$, we obtain

$$(2x - 1)^n B\left(G; \frac{1}{2x-1}, \frac{1}{2x-1}, z\right) = \sum_{F \subseteq E \atop (V,F) \text{ is bipartite}} 2^{k(V,F)} x^{iso(V,F)} z^{|F|},$$

which proves the statement. 

Corollary 25 Let $G$ be a simple graph of order $n$. The number of bicolored subgraphs of $G$ without any isolated vertices is

$$(-1)^n B(G; -1, -1, 1).$$

Proof. This follows immediately from the last line of the proof of Theorem 24 by substituting $x = 0$ and $z = 1$. 

6 Conclusions and Open Problems

The bipartition polynomial emerges as a powerful tool for proving equations in graphical enumeration. It shows nice relations to other graph polynomials, offers a couple of useful representations, and is polynomially reconstructible. However, there are still many open questions for the bipartition polynomial; we consider the following ones most interesting:
• Equation (15) gives a relation between the bipartition polynomial of a planar graph and its dual. However, this equation is restricted to the evaluation of $B(G; x, y, z)$ at $x = y = 1$. Is there a way to generalize this result?

• The Ising and matching polynomial of a graph $G$ can be derived from the corresponding polynomials of the complement $\bar{G}$. Can we calculate the bipartition polynomial of a graph from the bipartition polynomial of its complement?

• There are known pairs of nonisomorphic graphs with the same bipartition polynomial. However, despite all efforts by extensive computer search, we could not find a pair of nonisomorphic trees with coinciding bipartition polynomial. We know that no such pair for trees with order less than 19 exists. Is the bipartition polynomial able to distinguish all nonisomorphic trees?

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