TOWARDS A PSEUDEQUATIONAL PROOF THEORY

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Abstract. A new scheme for proving pseudoidentities from a given set $\Sigma$ of pseudoidentities, which is clearly sound, is also shown to be complete in many instances, such as when $\Sigma$ defines a locally finite variety, a pseudovariety of groups, more generally, of completely simple semigroups, or of commutative monoids. Many further examples when the scheme is complete are given when $\Sigma$ defines a pseudovariety $V$ which is $\sigma$-reducible for the equation $x = y$, provided $\Sigma$ is enough to prove a basis of identities for the variety of $\sigma$-algebras generated by $V$. This gives ample evidence in support of the conjecture that the proof scheme is complete in general.

1. Introduction

Pseudovarieties are classes of finite algebras closed under taking homomorphic images, subalgebras and finite direct products. They have been studied mostly in the context of finite semigroup theory due to the connections with automata and formal languages. In the framework of Eilenberg’s correspondence [26], determining whether a regular language enjoys a suitable property of a certain kind is converted to the membership problem of its syntactic semigroup in the corresponding pseudovariety of semigroups.

On the other hand, pseudovarieties are in many respects like the varieties of classical Universal Algebra, admitting relatively free algebras, albeit in general not finite, but rather profinite, and thus being defined by formal equations, where pseudoidentities play the role of identities [2]. While there is a natural proof scheme for identities that is sound and complete, which is sometimes referred as the completeness theorem for equational logic, the situation in the theory of pseudovarieties is not so simple.

Indeed, the first author has shown that there is no complete finite deduction system that is sufficient to prove a given pseudoidentity from a set of hypotheses (or basis) assuming that all models of the basis are also models of the pseudoidentity [2, Section 3.8]. Some sort of topological closure operator seems to be required. While such an operator was also proposed by the first author (see [2, Section 3.8]), it is very hard to handle and only one instance of its application has been found so far [12].

The main contribution of this paper is a new approach which consists in starting with all evaluation consequences of the basis, and completing them in the same term; then, transfinitely alternating transitive closure and topological closure. This proof scheme is clearly sound and, by definition, it is suitable for transfinite induction proofs. We show that it is complete.
in many familiar instances of concrete bases: whenever they define locally
finite varieties (in their algebraic language), pseudovarieties of groups or,
more generally, of completely simple semigroups, or pseudovarieties of com-
mutative monoids. For the proof of some of these results, we need to show
that concrete pseudoidentities that valid in a pseudovariety with a given ba-
sis are provable from the basis. The main technique in proving such results
is invoking excluded structures.

We also show that the proof scheme is complete when the pseudo variety
defined by the basis $\Sigma$ is $\sigma$-reducible with respect to the equation $x = y$,
where $\sigma$ is an implicit signature such that the variety of $\sigma$-algebras generated
by the pseudovariety defined by $\Sigma$ admits a basis whose elements can be
obtained from $\Sigma$ by our proof scheme. Combining with several reducibility
results that can be found in the literature, an exercise that by no means we
carry out exhaustively, this provides ample evidence in favor of the conjecture
that our proof scheme is complete in general.

2. Preliminaries

We recall quickly in this section some basic notions from general algebra,
which also serves to fix some notation. The reader is referred to [19] for
basic notions on Universal Algebra and to [2, 5, 36] for an intro-
duction to pseudovarieties and profinite structures.

By an algebraic type $\tau$ we mean a set of operation symbols, each of which
has an associated finite arity. We assume from hereon that $\tau$ is finite. An
algebra of type $\tau$ or a $\tau$-algebra is a nonempty set endowed with an inter-
pretation of each operation symbol of the type in question as an op-
eration of the corresponding arity. In general, we fix an algebraic type and consider only
algebras of that type. The (symbols of) operations of the type are sometimes
called the basic operations.

By a topological algebra we mean an algebra endowed with a topology such
that the interpretations of the basic operations are continuous functions.
Such an algebra is compact if so is its topology. We endow finite algebras
with the discrete topology. A topological algebra $S$ is residually $C$ for a
class $C$ of algebras if distinct points in $S$ may be separated by continuous
homomorphisms into members of $C$.

Given a family $(S_i)_{i \in I}$ of topological algebras, where the index set is di-
rected, and for each pair $(i, j)$ of indices with $i \geq j$, a continuous homomor-
phism $\varphi_{i,j} : S_i \to S_j$ such that $\varphi_{i,i}$ is the identity mapping on $S_i$ and, for
$i \geq j \geq k$, $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$, we may consider the inverse limit $\lim_{\leftarrow \ i \in I} S_i$,
which may be described as the subset of the product $\prod_{i \in I} S_i$ consisting of all families $(s_i)_{i \in I}$ such that each $s_i \in S_i$ and, for $i \geq j$, $\varphi_{i,j}(s_i) = s_j$. In case each $S_i$ is compact and the functions $\varphi_{i,j}$ are onto, $\lim_{\leftarrow \ i \in I} S_i$ is nonempty
and a closed subalgebra of $\prod_{i \in I} S_i$.

Recall that a pseudovariety is a (nonempty) class of finite algebras of a
given type that is closed under taking homomorphic images, subalgebras
and finite direct products. Let $U$ be a pseudovariety. A pro-$U$ algebra is an
inverse limit of algebras from $U$. In other words, a pro-$U$ algebra is a compact
algebra that is residually $U$. A profinite algebra is a pro-$U$ algebra for the
class $U$ of all finite algebras of the given type. In case a profinite algebra $S$ is
finitely generated as a topological algebra, meaning that a finitely generated subalgebra is dense, since the signature is assumed to be finite, there are, up to isomorphism, only countably many finite homomorphic images of $S$. Hence, $S$ embeds in a countable product of finite algebras, which implies that its topology is metrizable; in particular, the topology of $S$ is characterized by the convergence of sequences.

Given an arbitrary algebra $S$, one may consider all homomorphisms $S \rightarrow F$ onto algebras from a set of representatives of isomorphism classes of algebras from a given pseudovariety $U$. These homomorphisms form a directed set and so we may consider the inverse limit $\lim_{\leftarrow S \rightarrow F}$, which is called the pro-$U$ completion of $S$ and is denoted $\hat{S}_U$. Note that the natural mapping $S \rightarrow \hat{S}_U$ is injective if and only if $S$ is residually $U$. In case $U$ consists of all finite algebras of the given type, we drop the index $U$ and talk about the profinite completion of $S$.

By an alphabet we simply mean a finite set. Alphabets will appear as free generating sets of various structures.

By a term (of a given algebraic type) we mean a formal expression in a fixed alphabet constructed using the basic operations according to their arities. In other words, it is an element of the free algebra in the (Birkhoff) variety of all algebras of the given type.

A pro-$U$ algebra $S$ is said to be freely generated by $A$ if it comes endowed with a function $\iota : A \rightarrow S$ such that, for every function $\varphi : A \rightarrow T$ into a pro-$U$ algebra $T$, there is a unique continuous homomorphism $\hat{\varphi} : S \rightarrow T$ such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & S \\
\downarrow & & \downarrow \hat{\varphi} \\
\varphi & & T
\end{array}
\]

Such a structure $S$ is unique up to isomorphism of topological algebras and can be easily shown to be precisely the inverse limit of all $A$-generated members of $U$. It is denoted $\Omega_A U$. If $n = |A|$, then we may sometimes write $\Omega_n U$ instead of $\Omega_A U$ as it is easy to see that, up to isomorphism of topological algebras, $\Omega_A U$ only depends on the cardinality of the set $A$ and not on the set itself. The subalgebra of $\Omega_A U$ generated by $\iota(A)$ is denoted $\Omega_A U$; it is the algebra in the variety generated by $U$ that is freely generated by the alphabet $A$. Note that $\Omega_A U$ may also be obtained as the profinite completion $\hat{\Omega_A U}$.

An element of $\Omega_A U$ is sometimes called a $U$-pseudoword; those that lie in $\Omega_A U$ are said to be finite whereas the remaining $U$-pseudowords are said to be infinite. Every pseudoword $w \in \Omega_A U$ has a natural interpretation as an operation of arity $|A|$ on a pro-$U$ algebra $T$: $|A|$-tuples of elements of $T$ may be identified with functions $\varphi : A \rightarrow T$ and so the interpretation of $w$ becomes a function $w_T : T^A \rightarrow T$; the image $w_T(\varphi)$ is defined to be $\hat{\varphi}(w)$, where $\hat{\varphi}$ is given by the above commutative diagram. Viewed as operations, pseudowords are sometimes called implicit operations because their natural interpretations commute with continuous homomorphisms between pro-$U$ algebras.
By a U-pseudoidentity we mean a formal equality \( u = v \) with \( u, v \in \overline{\Omega}_A U \) for some alphabet \( A \). A pseudoidentity \( u = v \) holds in an algebra \( T \in U \) if \( u_T = v_T \). For a set \( \Sigma \) of U-pseudoidentities the class \( \llbracket \Sigma \rrbracket_U \) consists of all algebras \( T \in U \) in which all pseudoidentities from \( \Sigma \) hold. When the ambient pseudovariety \( U \) is understood from the context, we may also write \( \llbracket \Sigma \rrbracket \). This defines a pseudovariety contained in \( U \), that is, a subpseudovariety of \( U \), and Reiterman’s theorem [35] states that every subpseudovariety of \( U \) is of this form.

By an implicit signature (over \( U \)) we mean a set \( \sigma \) of \( U \)- pseudowords including those of the form \( s(a_1, \ldots, a_n) \in \Omega_A U \), where \( A = \{a_1, \ldots, a_n\} \) is an \( n \)-letter alphabet and \( n \) is the arity of the basic operation \( s \). By the above, every pro-\( U \) algebra has a natural structure of \( \sigma \)-algebra. In particular, this is the case for the algebras from \( U \), which, as \( \sigma \)-algebras, generate a variety of \( \sigma \)-algebras denoted \( U^\sigma \). It is also the case of \( \overline{\Omega}_A U \); the \( \sigma \)-subalgebra generated by \( \nu(A) \) is denoted \( \overline{\Omega}_A^\sigma U \) and it is easily shown to be the algebra of the variety \( U^\sigma \) freely generated by \( A \).

Two pseudovarieties that have received a lot of attention are the pseudovarieties \( S \), of all finite semigroups, and \( M \), of all finite monoids. For an element \( s \) of a finite semigroup \( S \), there is a unique power \( s^n (n \geq 1) \) of \( s \) which is an idempotent and is denoted \( s^\omega \). The element \( s^{2n-1} \) is then also denoted \( s^{\omega-1} \). In terms of the discrete topology, \( s^\omega \) is the limit of the sequence \( (s^{n!})_n \) while \( s^{\omega-1} \) is the limit of the sequence \( (s^{n!-1})_n \). It follows that if, instead of taking \( S \) finite we take \( S \) to be profinite, the sequences in question still converge and the notation for the limits is retained.

Consider the semiring \( \mathbb{N} \) of all natural numbers (including zero) and the ring \( \mathbb{Z} \) of integers, both under the usual addition and multiplication. These may be viewed as algebras of type consisting of two binary operation symbols, of addition and multiplication. Since both underlying additive monoids are monogenic, so are there finite homomorphic images. Note also that a finite monogenic additive monoid \( M \) carries a natural structure of semiring via the natural additive homomorphism \( \mathbb{N} \to M \). Moreover, in case \( M \) is a group, such a homomorphism factors through the embedding \( \mathbb{N} \to \mathbb{Z} \) as a ring homomorphism \( \mathbb{Z} \to M \). It follows that the profinite completion of each of \( \mathbb{N} \) and \( \mathbb{Z} \) as additive structures carries a multiplication which makes it isomorphic (as topological algebra) with the semiring \( \hat{\mathbb{N}} \), respectively with the ring \( \hat{\mathbb{Z}} \).

Let \( G \) and \( G_p \) denote, respectively, the pseudovarieties of all finite groups and of all finite \( p \)-groups. The above allows to describe the structure of the monoid \( \overline{\Omega}_1 M \) and of the groups \( \overline{\Omega}_1 G \) and \( \overline{\Omega}_1 G_p \). Indeed we know that \( \overline{\Omega}_1 M \) and \( \overline{\Omega}_1 G \) are respectively the inverse limits of the finite monogenic monoids and of the finite cyclic groups, which we know that, as additive algebras, carry multiplicative structures which make them isomorphic to \( \hat{\mathbb{N}} \) and \( \hat{\mathbb{Z}} \) respectively. The isomorphisms \( \hat{\mathbb{N}} \to \overline{\Omega}_1 M \) and \( \hat{\mathbb{Z}} \to \overline{\Omega}_1 G \) are easy to describe: if the free generator is denoted \( x \), they send each natural \( n \) in \( \mathbb{N} \) or in \( \mathbb{Z} \) to \( x^n \), respectively in \( \overline{\Omega}_1 M \) and in \( \overline{\Omega}_1 G \). For this reason, we also denote, for each \( \alpha \) in \( \hat{\mathbb{N}} \) or in \( \hat{\mathbb{Z}} \) the image in \( \overline{\Omega}_1 M \) or in \( \overline{\Omega}_1 G \), respectively, by \( x^\alpha \). Thus the usual laws of exponents hold and \( (x^\alpha)^\beta = x^{\alpha \beta} \) can be viewed as a composition of implicit operations.
Similarly, $\prod_p G_p$ may be identified with the completion $\hat{\mathbb{Z}}_{G_p} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, which is frequently denoted $\mathbb{Z}$. Moreover, thanks to the Chinese Remainder Theorem, it is easy to see that $\hat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$, where the index $p$ runs over all primes. The structure of the ring $\mathbb{Z}_p$ is quite transparent: it is an integral domain and every ideal is both principal and closed. It follows that the principal ideals of the ring $\hat{\mathbb{Z}}$ are the closed ideals, which in turn are the finitely generated ideals. In particular, every subset of $\mathbb{Z}$ has a greatest common divisor, which is a generator of the closed ideal generated by the given set.

Since each of the (semi)rings $\mathbb{N}$ and $\mathbb{Z}$ is residually finite, and the later is even residually $G_p$, there are natural embeddings $\mathbb{N} \to \hat{\mathbb{N}}$, $\mathbb{Z} \to \hat{\mathbb{Z}}$, and $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}_p$, which we view as inclusion mappings. It is easy to see that the invertible elements of $\mathbb{Z}_p$ are those that are not divisible by $p$. Hence, the invertible elements of $\hat{\mathbb{Z}}$ are those that are not divisible by any prime $p$.

The semiring $\mathbb{N}$ has two idempotents, namely $0$ and $\omega = \lim_n n!$. The maximal additive group $H_\omega$ containing $\omega$ is a closed ideal of $\mathbb{N}$ which is generated by $\omega + 1$. The natural continuous homomorphism $\pi : \mathbb{N} \to \hat{\mathbb{Z}}$, mapping $1$ to $1$, also maps $\omega+1$ to $1$ and therefore restricts to an isomorphism $H_\omega \to \hat{\mathbb{Z}}$. Thus, $\hat{\mathbb{Z}}$ may be identified with $H_\omega$, which we do from hereon; it is a retract of $\mathbb{N}$ under the mapping $\alpha \mapsto \omega + \alpha$. Also note that $\mathbb{N}$ is the disjoint union of $\mathbb{N}$ with $\hat{\mathbb{Z}}$.

3. A Proof Scheme

Let $U$ be a pseudovariety of a certain finite type of algebras involving only finitary operations. Let $\Sigma$ be a set of pseudoidentities $u = v$ with $u$ and $v$ elements of the free pro-$U$ algebra over some arbitrary finite alphabet. We seek a complete proof scheme for pseudoidentities valid in the pseudovariety $[\Sigma]$, that is, a deduction system that is capable of deducing from $\Sigma$ exactly the pseudoidentities valid in $[\Sigma]$.

By transfinite recursion, we define, for each ordinal $\alpha$, a set $\Sigma_\alpha$ of pseudoidentities over $A$ as follows:

- $\Sigma_0$ consists of all pairs of the form $\langle t(\varphi(u), w_1, \ldots, w_n), t(\varphi(v), w_1, \ldots, w_n) \rangle$

such that either $u = v$ or $v = u$ is a pseudoidentity from $\Sigma$, say with $u, v \in \prod_B U$, $\varphi : \prod_B U \to \prod_A U$ is a continuous homomorphism, $t$ is a term (in the algebraic language of $U$), and $w_i \in \prod_A U$ ($i = 1, \ldots, n$);
- $\Sigma_{2\alpha+1}$ is the transitive closure of the binary relation $\Sigma_{2\alpha}$;
- $\Sigma_{2\alpha+2}$ is the topological closure of the relation $\Sigma_{2\alpha+1}$ in the space $\prod_A U \times \prod_A U$;
- if $\alpha$ is a limit ordinal, then $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$.

Note that, if $\Sigma_{\alpha+2} = \Sigma_\alpha$, then $\Sigma_\alpha$ is both transitive and topologically closed, so that $\Sigma_\beta = \Sigma_\alpha$ for every ordinal $\beta$ with $\beta \geq \alpha$. Such a condition must hold for $\alpha$ at most $\omega_1$, the least uncountable ordinal. Hence, the union $\Sigma = \bigcup_\alpha \Sigma_\alpha$ defines a transitive closed binary relation on $\prod_A U$.
Consider a binary relation $\theta$ on $\overline{\mathcal{A}}^\delta U$. We say that $\theta$ is stable if $(u, v) \in \theta$ implies 

$$(t(u, w_1, \ldots, w_n), t(v, w_1, \ldots, w_n)) \in \theta$$

for every term $t$ and $w_1, \ldots, w_n \in \overline{\mathcal{A}}^\delta U$. We also say that $\theta$ is fully invariant if, for every continuous endomorphism $\varphi$ of $\overline{\mathcal{A}}^\delta U$ and $(u, v) \in \theta$, we have $(\varphi(u), \varphi(v)) \in \theta$. A stable equivalence relation is also called a congruence.

The following result establishes the soundness of the above proof scheme.

**Proposition 3.1.** The relation $\tilde{\Sigma}$ is a fully invariant closed congruence on $\overline{\mathcal{A}}^\delta U$. For every $(u, v) \in \tilde{\Sigma}$, the pseudoidentity $u = v$ is valid in $\llbracket \Sigma \rrbracket$.

**Proof.** Note that $\Sigma_0$ is a reflexive and symmetric binary relation on $\overline{\mathcal{A}}^\delta U$, whence so are all $\Sigma_\alpha$ as well as $\tilde{\Sigma}$. We prove by transfinite induction on $\alpha$ that each $\Sigma_\alpha$ is a stable fully invariant binary relation on $\overline{\mathcal{A}}^\delta U$ whose elements, viewed as pseudoidentities, are valid in the pseudovariety $\llbracket \Sigma \rrbracket$.

For $\alpha = 0$, the claimed properties are immediate from the definition of $\Sigma_0$. Assuming the claim holds for all $\alpha < \beta$, then it clearly also holds for $\Sigma_\beta$ in case $\beta$ is a limit ordinal. Otherwise, $\beta$ is a successor ordinal, and we distinguish the cases where $\beta$ is odd or even.

In case $\beta$ is odd, that is, it is of the form $\beta = 2\gamma + 1$, $\Sigma_\beta$ is the transitive closure of $\Sigma_{2\gamma}$, which, by the induction hypothesis, is a stable fully invariant binary relation on $\overline{\mathcal{A}}^\delta U$ whose elements define pseudoidentities valid in the pseudovariety $\llbracket \Sigma \rrbracket$. The elements of $\Sigma_\beta$ are pairs $(u_0, u_n)$ such that there exist $u_1, \ldots, u_{n-1}$ with each $(u_i, u_{i+1}) \in \Sigma_{2\gamma}$ $(i = 0, \ldots, n - 1)$. In particular, each pseudoidentity $u_i = u_{i+1}$ is valid in $\llbracket \Sigma \rrbracket$, whence so is $u_0 = u_n$. If $t$ is a term and $w_1, \ldots, w_m$ are elements from $\overline{\mathcal{A}}^\delta U$ then, by the induction hypothesis, each pair $(t(u_i, w_1, \ldots, w_m), t(u_{i+1}, w_1, \ldots, w_m))$ belongs to $\Sigma_{2\gamma}$, and so the pair $(t(u_0, w_1, \ldots, w_m), t(u_n, w_1, \ldots, w_m))$ also belongs to $\Sigma_\beta$. Finally, if $\varphi$ is a continuous endomorphism of $\overline{\mathcal{A}}^\delta U$, by the induction hypothesis each of the pairs $(\varphi(u_i), \varphi(u_{i+1}))$ belongs to $\Sigma_{2\gamma}$, which entails that $(\varphi(u_0), \varphi(u_n))$ belongs to $\Sigma_\beta$.

Consider next the case where $\beta = 2\gamma + 2$, that is, a nonzero and non-limit even ordinal. Then, since $\overline{\mathcal{A}}^\delta U$ is a metric space, every element from $\Sigma_\beta$ is the limit $(u, v)$ of a sequence $(u_n, v_n)_n$ of elements from $\Sigma_{2\gamma+1}$. By the induction hypothesis, as a pseudoidentity, every element of the sequence is valid in $\llbracket \Sigma \rrbracket$, whence so is $u = v$ since, under an evaluation of the elements of $\mathcal{A}$ in a finite algebra, the sequences $(u_n)_n$ and $(v_n)_n$ eventually stabilize, precisely at the values of $u$ and $v$, respectively. Since $\Sigma_{2\gamma+1}$ is assumed to be stable, for a term $t$ and $w_1, \ldots, w_m \in \overline{\mathcal{A}}^\delta U$, the sequence of pairs $(t(u_n, w_1, \ldots, w_m), t(v_n, w_1, \ldots, w_m))_n$ consists of elements of $\Sigma_{2\gamma+1}$, whence its limit $(t(u, w_1, \ldots, w_m), t(v, w_1, \ldots, w_m))$ belongs to $\Sigma_\beta = \Sigma_{2\gamma+1}$. Finally, if $\varphi$ is a continuous endomorphism of $\overline{\mathcal{A}}^\delta U$, then the elements of the sequence $(\varphi(u_n), \varphi(v_n))_n$ belong to $\Sigma_{2\gamma+1}$ and, therefore, its limit $(\varphi(u), \varphi(v))$ belongs to $\Sigma_\beta$. This completes the transfinite induction.

In view of the already established initial claim, we know that $\tilde{\Sigma}$ is a reflexive symmetric stable fully invariant binary relation consisting of pairs $(u, v)$ such that the pseudoidentity $u = v$ is valid in $\llbracket \Sigma \rrbracket$. To complete the proof, it remains to recall that we already observed that $\tilde{\Sigma}$ is a closed transitive binary relation. \[\square\]
We call the elements of $\tilde{\Sigma}$ the pseudoidentities provable from $\Sigma$. A proof consists in a transfinite sequence of steps in which in step 0 we invoke pseudoidentities from $\Sigma$, suitably evaluated in $\mathfrak{P}_A\mathcal{U}$, in which both sides are plugged in the same place of an arbitrary term, and in later steps we either use transitivity of equality or take limits, in the latter two cases from already proved steps, or simply collect together all pseudoidentities in previous steps. In the last step in the proof we should have a set of pseudoidentities containing the one to be proved.

An alternative but equivalent, in the sense of capturing the same proved pseudoidentities, definition of proof would be to take a transfinite sequence of pseudoidentities in which in each step we allow one of the pseudoidentities of the above step 0, we take $u = w$ if there are two previous steps of the form $u = v$ and $v = w$, or we take $u = v$ provided there is a sequence of earlier steps $(u_n = v_n)_n$ with $u = \lim u_n$ and $v = \lim v_n$. The latter steps are called limiting steps. The last step in such a proof should be the pseudoidentity to be proved.

If such a proof only involves a finite number of non-limiting steps and no limiting steps, then we say the proof is algebraic. Note that such a proof exists precisely for the pseudoidentities of $\Sigma_1$.

Several examples of proofs in the above general sense can be found in the literature. In fact, all proofs that a pseudovariety defined by certain pseudoidentities satisfies a given pseudoidentity that we have been able to find in the literature seem to be expressible in this form. This suggests the following general conjecture which amounts to completeness of our proof scheme.

**Conjecture 3.2.** A $\mathcal{U}$-pseudoidentity $u = v$ is provable from a set $\Sigma$ of $\mathcal{U}$-pseudoidentities whenever $[\Sigma]$ satisfies $u = v$.

The above statement is to be interpreted as a logical clause depending on three parameters: $\mathcal{U}$, $\Sigma$, and $u = v$, where the latter two determine the first one. Since we have been unable to establish the conjecture in full generality, particular instances of the conjecture, where one of the parameters is fixed may be of interest as they provide evidence towards the conjecture. If a parameter is fixed then those that are not determined by it are interpreted as being universally quantified.

We say that a set $\Sigma$ of $\mathcal{U}$-pseudoidentities is $h$-strong (within $\mathcal{U}$, or for $\mathcal{U}$) if the statement in the conjecture holds for the given fixed choice of $\Sigma$ and an arbitrary choice of the $\mathcal{U}$-pseudoidentity $u = v$. In case $\Sigma = \{u = v\}$ consists of a single pseudoidentity, we also say that $u = v$ is $h$-strong if so is $\Sigma$.

A $\mathcal{U}$-pseudoidentity $u = v$ is said to be $t$-strong (within $\mathcal{U}$, or for $\mathcal{U}$) if the statement in the conjecture holds for the given fixed choice of $u = v$ and an arbitrary choice of $\Sigma$.

The conjecture also involves an ambient pseudovariety $\mathcal{U}$ with respect to which pseudoidentities are taken. We say that the pseudovariety $\mathcal{U}$ is strong if

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1The letters $h$ and $t$ in “$h/t$-strong” are meant to refer to whether the pseudoidentities appear as hypotheses or thesis in the proofs.
the statement in the conjecture holds that is it holds for every set \( \Sigma \cup \{ u = v \} \) of \( U \)-pseudoidentities.

Taking into account the results of [2, Section 3.8], showing that \( \Sigma \) is h-strong is equivalent to showing that \( \tilde{\Sigma} \) is a profinite congruence in the sense of [36, Page 139], that is, for every finite alphabet \( A \), that \( \tilde{\Sigma} \) is a closed congruence on \( \Omega_A U \) such that the quotient topological algebra \( \Omega_A U / \tilde{\Sigma} \) is a profinite algebra. A further equivalent formulation of this property is that, given any two distinct \( \tilde{\Sigma} \)-classes, there is a clopen union of \( \tilde{\Sigma} \)-classes separating them. The difficulty in establishing this property in general is to obtain \( \tilde{\Sigma} \)-saturation of such clopen sets. In case all congruence classes are determined by a single class, as in the group case, this program is much easier to achieve (cf. Section 8).

We are able to prove the conjecture in several cases of interest. The remainder of the paper is concerned with gathering evidence for the conjecture.

### 4. A Transfer Result

The purpose of this section is to show that it is possible to extend the validity of the conjecture within a certain ambient pseudovariety \( V \) to a larger pseudovariety \( U \) provided that \( V \) is defined within \( U \) by a basis satisfying the conjecture. We first note that the converse is also true without the additional assumption on a basis of \( V \).

**Proposition 4.1.** Let \( U \) be a pseudovariety and let \( V \) be a subpseudovariety of \( U \). Suppose that the set \( \Gamma \) of \( U \)-pseudoidentities defines a subpseudovariety of \( V \) and \( \Gamma \) is h-strong within \( U \). Then the set \( \Gamma' = \{ \pi(u) = \pi(v) : (u = v) \in \Gamma \} \) is h-strong within \( V \), where \( \pi \) stands for the natural continuous homomorphism \( \Omega_A U \to \Omega_A V \).

*Proof.* Let \( u, v \in \Omega_A U \) be such that the \( V \)-pseudoidentity \( \pi(u) = \pi(v) \) holds in the pseudovariety \( \llbracket \Gamma' \rrbracket_V = \llbracket \Gamma \rrbracket_U \). Then, the \( U \)-pseudoidentity \( u = v \) holds in the pseudovariety \( \llbracket \Gamma \rrbracket_U \). Since we assume that \( \Gamma \) is h-strong within \( U \), it follows that there is a proof of the pseudoidentity \( u = v \) from \( \Gamma \). Projecting by \( \pi \) into \( \Omega_A V \) all steps in such a proof, we obtain a proof of \( \pi(u) = \pi(v) \) from \( \Gamma' \); more precisely, one may easily prove by induction on the ordinal \( \alpha \) that \( (\pi \times \pi)(\Gamma_\alpha) \subseteq \Gamma'_\alpha \). Hence, \( \Gamma' \) is h-strong within \( V \). \( \square \)

Going in the opposite direction is more interesting, but requires an additional assumption.

**Proposition 4.2.** Let \( U \) be a pseudovariety and let \( V \) be a subpseudovariety of \( U \). Suppose that \( V \) admits a basis \( \Sigma \) of \( U \)-pseudoidentities that is h-strong within \( U \) and consists of t-strong pseudoidentities. Let \( \Gamma \) be a set of \( U \)-pseudoidentities such that \( \llbracket \Gamma \rrbracket_U \subseteq V \). If the set \( \Gamma' = \{ \pi(u) = \pi(v) : (u = v) \in \Gamma \} \) is h-strong within \( V \), where \( \pi \) is as above, then the set \( \Gamma \) is h-strong within \( U \).

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2Here, we should write \( \pi_A \) instead of \( \pi \), since all finite sets of variables may need to be considered. Nevertheless, since the set \( A \) will usually be clear from the context, most often we abuse notation by writing simply \( \pi \) independently of the generating set \( A \).
Proof. Let \( u = v \) be a \( U \)-pseudoidentity valid in the pseudovariety \( \llbracket \Gamma \rrbracket_U \).
This means that the \( V \)-pseudoidentity \( \pi(u) = \pi(v) \) holds in \( \llbracket \Gamma' \rrbracket_V \) as \( \llbracket \Gamma \rrbracket_U \subseteq V = \llbracket \Sigma \rrbracket_U \). Since \( \Gamma' \) is \( \alpha \)-strong within \( V \), \( \pi(u) = \pi(v) \) is provable from \( \Gamma' \).
We need to show that \( u = v \) is also provable from \( \Gamma \). We first claim that there are \( u' \) and \( v' \) such that \( u' = v' \) is provable from \( \Gamma \), \( \pi(u') = \pi(u) \), and \( \pi(v') = \pi(v) \). Before proving the claim, we show how it allows us to conclude the proof of the proposition. Since \( \Sigma \) is \( \beta \)-strong within \( U \), it follows that \( \Sigma \) proves \( u' = u \) and \( v' = v \). On the other hand, since the pseudovariety \( \llbracket \Gamma \rrbracket \) is contained in \( V \), it satisfies all pseudoidentities from \( \Sigma \) and, as these are assumed to be \( \tau \)-strong, they are provable from \( \Gamma \). Hence, \( \Gamma \) proves \( u' = u \), \( v' = v \) and, assuming the claim, also \( u' = v' \), which entails that \( \Gamma \) proves \( u = v \). Thus, it remains to establish the claim.

Consider the sets \( \Gamma_\alpha' \) defined in Section 3. The proof will be complete once we establish the above claim that every pseudoidentity in \( \Gamma_\alpha' \) is of the form \( \pi(w) = \pi(z) \) for some pseudoidentity \( w = z \) provable from \( \Gamma \). To prove the claim, we proceed by transfinite induction on \( \alpha \).

In case \( \alpha = 0 \), we have a pseudoidentity of the form
\[
(1) \quad t(\varphi(\pi(w)), \pi(s_1), \ldots, \pi(s_n)) = t(\varphi(\pi(z)), \pi(s_1), \ldots, \pi(s_n)),
\]
where \( t \) is a term, \( w = z \) is a pseudoidentity from \( \Gamma \), say over the set of variables \( B \), \( \varphi : \prod_B V \to \prod_A V \) is a continuous homomorphism, and \( s_1, \ldots, s_n \in \prod_A U \). By the universal property of relatively free profinite algebras, there is a continuous homomorphism \( \psi : \prod_B U \to \prod_A U \) such that the following diagram commutes:
\[
\begin{array}{ccc}
\prod_B U & \xrightarrow{\psi} & \prod_A U \\
\downarrow{\pi_B} & & \downarrow{\pi_A} \\
\prod_B V & \xrightarrow{\varphi} & \prod_A V
\end{array}
\]

It follows that the pseudoidentity (1) is obtained from
\[
t(\psi(w), s_1, \ldots, s_n) = t(\psi(z), s_1, \ldots, s_n)
\]
by applying \( \pi \) to each member, which completes the basic step \( \alpha = 0 \) of the induction.

Suppose next that \( \alpha = 2\beta + 1 \) and that the pseudoidentities \( w_i = w_{i+1} \) \((i = 0, \ldots, k - 1)\) belong to \( \Gamma_{2\beta}^\prime \). By induction hypothesis, there exist \( w''_i, w'_{i+1} \in \prod_A U \) such that the pseudoidentity \( w''_i = w'_{i+1} \) is provable from \( \Gamma \), \( \pi(w''_i) = w_i \), and \( \pi(w''_{i+1}) = w_{i+1} \). In particular, we have \( \pi(w'_i) = w_i = \pi(w''_i) \) for \( i = 1, \ldots, k - 1 \), and so each pseudoidentity \( w'_i = w''_i \) is also provable from \( \Gamma \). Hence, \( w_0 = w_k \) is provable from \( \Gamma \).

For the case \( \alpha = 2\beta + 2 \), consider a sequence \( (w_n = z_n)_n \) of \( U \)-pseudoidentities such that each \( \pi(w_n) = \pi(z_n) \) belongs to \( \Gamma_{2\beta+1}^\prime \) and suppose that \( \pi(w_n) = \pi(z_n) \) converges to \( \pi(w) = \pi(z) \). By compactness and continuity of \( \pi \), we may as well assume that \( (w_n = z_n)_n \) converges to \( w = z \). Since each pseudoidentity \( w_n = z_n \) is provable from \( \Gamma \) by induction hypothesis, \( w = z \) is also provable from \( \Gamma \).

Since the case of limit ordinals is trivial, the transfinite induction is complete and the claim is established. 
\[\square\]
With essentially the same arguments, one may replace the t-strongness hypothesis by provability of $\Sigma$ from $\Gamma$, which yields the following corollary.

**Corollary 4.3.** Let $U$ be a pseudovariety and let $V$ be a subpseudovariety of $U$. Suppose that $V$ admits a basis of $U$-pseudoidentities $\Sigma$ that is h-strong within $U$ and that $V$ is strong. Then every set of $U$-pseudoidentities from which $\Sigma$ is provable is h-strong within $U$. □

5. **Locally finite sets of identities**

Recall that a variety is **locally finite** if all its finitely generated algebras are finite. We also say that a set $\Sigma$ of identities is **locally finite** if it defines a locally finite variety.

**Theorem 5.1.** Every locally finite set of identities is h-strong.

*Proof.* Consider a set $\Sigma$ of identities that is locally finite. Let $A$ be a finite set and let $\pi : \bar{\Omega}_A U \to \bar{\Omega}_A[\Sigma]$ be the natural continuous homomorphism. For each of the finitely many elements $s$ of $\bar{\Omega}_A[\Sigma]$ choose an element $f(s)$ of $\Omega_A U$ such that $\pi(f(s)) = s$. Let $\bar{f} = f \circ \pi$ and note that, since $\pi$ is continuous, so is $\bar{f}$.

For $u \in \Omega_A U$, consider the identity $u = \bar{f}(u)$. Since $\pi(u) = \pi \circ f \circ \pi(u) = \pi(\bar{f}(u))$, it is valid in the pseudovariety $[\Sigma]$. As the basis $\Sigma$ is locally finite, it is also valid in the variety $[\Sigma]$. By the completeness theorem for equational logic, it follows that the identity $u = \bar{f}(u)$ is algebraically provable from $\Sigma$.

Finally, consider an arbitrary pseudoidentity $u = v$ with $u, v \in \bar{\Omega}_A U$ and suppose that it is valid in $[\Sigma]$, that is, $\pi(u) = \pi(v)$. Let $(u_n)_n$ and $(v_n)_n$ be sequences in $\Omega_A U$ converging respectively to $u$ and $v$. Since $\pi$ is continuous and $\bar{\Omega}_A[\Sigma]$ is a discrete space, we may as well assume that the sequences $(\pi(u_n))_n$ and $(\pi(v_n))_n$ are constant. It follows that each of the identities $u_n = v_n$ is valid in $[\Sigma]$, whence the equality $\bar{f}(u_n) = \bar{f}(v_n)$ holds. By the preceding paragraph, we deduce that the identities $u_n = \bar{f}(u_n) = v_n$ are algebraically provable from $\Sigma$. Hence, the pseudoidentity $u = v$ is, sidewise, the limit of the sequence $(u_n = v_n)_n$ of algebraically provable pseudoidentities, which shows that $u = v$ is provable from $\Sigma$, thereby establishing that $\Sigma$ is h-strong. □

For a locally finite set of identities, we may take any basis of a variety generated by a single finite algebra. A classical example of locally finite identity for semigroups which is not of this type is $x^2 = x$ (see, for instance, [29]).

Along the same lines of the proof of Theorem 5.1, we may prove the following result.

**Theorem 5.2.** Every locally finite set of identities defines a strong pseudovariety.

*Proof.* Let $V$ be the pseudovariety defined by a locally finite set $\Sigma$ of identities and let $\mathcal{V}$ be the variety defined by $\Sigma$. Let $\Gamma$ be a set of $\mathcal{V}$-(pseudo)identities and let $u, v \in \Omega_A V = \bar{\Omega}_A V$ be such that the identity $u = v$ is valid in $[\Gamma]$. Since $\mathcal{V}$ is generated by $\mathcal{V}$, whence also $[\Gamma]$ is generated by $[\Gamma]$, it follows
that the variety \( \Gamma \) also satisfies the identity \( u = v \). By the completeness theorem for equational logic, we deduce that \( u = v \) is provable from \( \Gamma \). □

Combining Theorem 5.1 and 5.2 with Corollary 4.3, we obtain the following result.

**Corollary 5.3.** Every set of pseudoidentities which proves a locally finite set of identities is \( \text{h-strong} \). □

It would be nice to replace the provability assumption in Corollary 5.3 by the hypothesis that the given set of pseudoidentity is locally finite. This would follow from Corollary 4.3 if we could show that every identity is \( t\)-strong, which is a particular case of the conjecture which have not established in general.

6. **T-strongness**

The main proposition of this section gathers the statement of \( t\)-strongness of several pseudoidentities. Some of these results play a role in the application of Proposition 4.2 in later sections.

We start with a simple lemma which is used in several points in the sequel.

**Lemma 6.1.** If \( x^\alpha = x^\beta \) is a nontrivial one-variable \( M \)-pseudoidentity, then it proves the pseudoidentity \( x^\alpha = x^{\alpha + \omega} \).

_Proof._ Suppose first that \( \beta \) is either infinite or finite and greater than \( \alpha \). In this case, we can write the given pseudoidentity in the form \( x^\alpha = x^{\alpha (\beta - \alpha)} \) which proves algebraically \( x^\alpha = x^{\alpha \cdot (\beta - \alpha) n!} \) for every positive integer \( n \). Taking the limit as \( n \) goes to infinite, we deduce that \( x^\alpha = x^\beta \) proves \( x^\alpha = x^{\alpha (\beta - \alpha) \omega} = x^{\alpha + \omega} \).

If the initial assumption on \( \beta \) fails, then it holds when we interchange the roles of \( \alpha \) and \( \beta \). By the above, we deduce that, in that case, \( x^\alpha = x^\beta \) proves \( x^\beta = x^{\beta + \omega} \) and, therefore, also \( x^\alpha = x^\beta = x^\beta x^\omega = x^{\alpha + \omega} \). □

The following finite semigroups play a role below:

- \( SL_2 \) stands for the two-element semilattice;
- for positive integers \( m \) and \( n \), let \( B(m, n) \) denote the rectangular band \( m \times n \), consisting of the pairs \( (i,j) \) with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), where multiplication is described by \( (i,j)(k,\ell) = (i,\ell) \);
- for positive integers \( m \) and \( n \), let \( C_{m,n} = \langle a : a^m = a^{m+n} \rangle \) be the monogenic semigroup with \( m+n-1 \) elements and maximal subgroup with \( n \) elements;
- for a positive integer \( n \), let \( C_n \) be the cyclic group of order \( n \);
- \( B_2 \) is the five-element aperiodic Brandt semigroup, which is given by the presentation

\[
\langle a, b; aba = a, bab = b, a^2 = b^2 = 0 \rangle
\]

as a semigroup with zero;
- \( N \) is the semigroup with zero given by the presentation

\[
\langle a, b; a^2 = b^2 = ba = 0 \rangle;
\]
- \( T \) is the semigroup with zero given by the presentation

\[
\langle e, a; e^2 = e, ea = a, ae = 0 \rangle.
\]
We say that two sets of $U$-pseudoidentities are equivalent if every pseudoidentity from each of them is provable from the other set. Another simple result that is useful below is the following lemma.

**Lemma 6.2.** Let $\Gamma$ be a set of pseudoidentities and suppose that $\Gamma$ is equivalent to a single pseudoidentity $\varepsilon$. If each pseudoidentity in $\Gamma$ is $t$-strong, then so is $\varepsilon$.

**Proof.** Let $\Sigma$ be a set of pseudoidentities and suppose that $[\Sigma]$ satisfies $\varepsilon$. From the equivalence hypothesis, it follows that $[\Sigma]$ satisfies each pseudoidentity $\gamma$ from $\Gamma$. Since $\gamma$ is $t$-strong, we deduce that $\Sigma$ proves $\gamma$. By the equivalence hypothesis again, we conclude that $\Sigma$ proves $\varepsilon$. $\square$

We are now ready for the announced proposition.

**Proposition 6.3.** Each of the following pseudoidentities is $t$-strong:

1. $x^\omega = 1$;
2. $x^{\omega+1} = x$;
3. $x^{\omega+1} = x^\omega$;
4. $(x^\omega y)^\omega x^\omega = (x^\omega y)^\omega$;
5. $(x^\omega y)^\omega = (yx^\omega)^\omega$;
6. $(xy)^\omega x(xy)^\omega = (xy)^\omega$;
7. $(xy)^\omega x(xy)^\omega = (xy)^\omega$;
8. $((xy)^\omega x)^\omega = (xy)^\omega$;
9. $(xy)^\omega = (yx)^\omega$;
10. $(xy)^\omega x = (xy)^\omega$;
11. $x^{\omega} y = yx^\omega$;
12. $(xy)^\omega x = x$.

**Proof.** For simplicity, we may refer to one of (i)–(xii) as meaning either a part of the proposition or the pseudoidentity in it; which is the case should be clear from the context.

Let $\Sigma$ be a set of pseudoidentities. Except in the case of (xii), where we should take $S$-pseudoidentities to get a pseudoidentity that is not equivalent to $x^\omega = 1$, we assume that the elements of $\Sigma$ are $M$-pseudoidentities. For each of the pseudoidentities $\varepsilon$ in the statement of the proposition, we assume that the pseudovariety $[\Sigma]$ satisfies $\varepsilon$ and show that $\Sigma$ proves $\varepsilon$.

1. Since all finite semigroups satisfying $\Sigma$ are groups, in particular $SL_2$ fails some pseudoidentity $u = v$ from $\Sigma$, which means that there is some variable that occurs in one of the sides but not in the other. Substituting $x^\omega$ for that variable and 1 for all others, we conclude that $x^\omega = 1$ is algebraically provable from $\Sigma$.

2. Since the monoid $C_{2,1}^1$ fails (i), it must fail some pseudoidentity $u = v$ from $\Sigma$ under some suitable evaluation. If such an evaluation gives the values 0 and 1 for $u$ and $v$, then $u$ and $v$ do not involve the same variables, so that, substituting $x^\omega$ for one variable and 1 for all others, one gets $x^\omega = 1$, from which (ii) follows. Otherwise, one of the sides, say $u$, is evaluated to $a$ and the other to either 0 or 1. Substituting 1 for all variables in $u$ that are not evaluated to $a$ and $x$ for every other variable, we obtain a pseudoidentity of the form $x = x^\alpha$, with $\alpha \in \mathbb{N} \setminus \{1\}$. By Lemma 6.1, each such pseudoidentity proves (ii).
Each of the cyclic groups of prime order fails the pseudoidentity (iii). For a pseudoword \( w \) and a variable \( x \), let \( w_x \) be the pseudoword that is obtained from \( w \) by substituting 1 for every variable except \( x \). Since the cyclic group \( C_p \) satisfies a pseudoidentity \( u = v \) if and only if, for every variable \( x \), it satisfies the pseudoidentity \( u_x = v_x \), it follows that, if \( C_p \) fails \( \Sigma \), then there is \( \alpha_p \in \mathbb{N} \setminus \mathbb{N} = \hat{\mathbb{Z}} \) such that \( p \) does not divide \( \alpha_p \) and \( \Sigma \) proves \( x^{\alpha_p} = x^\omega \).

Let \( \alpha \) be a greatest common divisor of the \( \alpha_p \), which is a limit of a sequence of linear combinations of the \( \alpha_p \). Since each prime \( p \) does not divide \( \alpha_p \), it cannot divide \( \alpha \). Hence, \( \alpha \) is invertible in the ring \( \hat{\mathbb{Z}} \) and there exists \( \beta \in \hat{\mathbb{Z}} \) such that \( \alpha \beta = \omega + 1 \). Now, if \( \gamma = \sum_{i=1}^n \gamma_i \alpha_{p_i} \), then \( \Sigma \) proves \( x^\gamma = x^\omega \) by raising both sides of each pseudoidentity \( x^{\alpha_{p_i}} = x^\omega \) to the \( \gamma_i \) power and multiplying the results side by side. The pseudoidentity \( x^\alpha = x^\omega \) is therefore a limit of pseudoidentities provable from \( \Sigma \), and whence it is provable from \( \Sigma \). Finally, raising both sides of \( x^\alpha = x^\omega \) to the \( \beta \) power, we deduce that the pseudoidentity (iii) is provable from \( \Sigma \).

As it is easy to see, and well known, \( B(1, 2)^1 \) fails a pseudoidentity \( u = v \) if and only if, from right to left, the order of the first occurrence of variables in \( u \) and \( v \) is not the same. In particular, the monoid \( B(1, 2)^1 \) fails (iv) and, therefore, it fails some pseudoidentity from \( \Sigma \). Hence, there is a substitution that sends all variables but two to 1 that yields from some pseudoidentity in \( \Sigma \) a two-variable pseudoidentity of the form \( ux = vy \), where \( x \) and \( y \) are distinct variables, or a nontrivial pseudoidentity \( u = 1 \) or \( 1 = u \).

In the latter case, by substituting all the variables by \( x^\omega \), we conclude that \( \Sigma \) proves \( x^\omega = 1 \) and, hence, every pseudoidentity in which both sides are products of \( \omega \) powers, as is the case of (iv). Thus, it remains to consider the former case. Applying the substitution \( x \mapsto x^\omega \) and \( y \mapsto x^\omega y \) and raising both sides to the \( \omega \) power, we conclude that \( \Sigma \) proves the pseudoidentity (iv).

Noting that the sets of pseudoidentities \( \{(x^\omega y)^\omega x^\omega = (x^\omega y)^\omega , x^\omega (y^\omega x^\omega )^\omega = (yx^\omega)^\omega \} \) and \( \{(x^\omega y)^\omega = (yx^\omega)^\omega \} \) are equivalent, in view of (iv) and its dual, it suffices to apply Lemma 6.2.

The evaluation of \( x \) to \( a \) and \( y \) to \( b \) shows that the monoid \( B_1^1 \) fails the pseudoidentity (vi). Hence, there is some pseudoidentity \( u = v \) from \( \Sigma \) and an evaluation of the variables in \( B_1^2 \) which yields different values for \( u \) and \( v \). A variable being assigned the value 1 corresponds to deleting that variable. For all other values in \( B_1^2 \), since \( B_2 \) is generated by \( \{a, b\} \) as a semigroup, we may first substitute the variable by a word in the variables \( x \) and \( y \) and then evaluate \( x \) to \( a \) and \( y \) to \( b \). Thus, under the assumption that \( B_2^2 \) fails \( u = v \), we conclude that there is a pseudoidentity \( u' = v' \) in \( x \) and \( y \) that can be proved from \( \Sigma \) and which fails in \( B_1^2 \) under the evaluation \( x \mapsto a \) and \( y \mapsto b \). Under such an evaluation, not both sides are evaluated to 0. If one of the sides is 1 then, substituting \( x \) for \( y \), we get a pseudoidentity of the form \( x^\alpha = 1 \), which proves \( x^\omega = 1 \) and, therefore, also the pseudoidentity (vi). Otherwise, one of the sides of the pseudoidentity \( u' = v' \), say \( u' \), must be a factor of \( (xy)^\omega \) while \( v' \) either admits \( x^2 \) or \( y^2 \) as a factor or does not start or end with the same letter as \( u' \). By multiplying both sides of \( u' = v' \) by suitable factors of \( (xy)^\omega \), we may prove from \( u' = v' \) a pseudoidentity \( u'' = v'' \) of the form \( (xy)^\omega = w \) where \( w \) is a pseudoword that admits at least one of
the words \(x^2\) and \(y^2\) as a factor, starts with \(x\) and ends with \(y\). Substituting \((xy)\omega x\) for \(x\) and \(y(xy)\omega\) for \(y\), we obtain from \(u'' = v''\) a pseudoidentity of the form

\[(xy)^\omega = (xy)^\alpha x(xy)^\beta w\]

or of the form

\[(xy)^\omega = (xy)^\alpha y(xy)^\beta w,\]

where \(\alpha\) and \(\beta\) are infinite exponents and \(w\) is some pseudoword. From the pseudoidentity (2), we may prove, algebraically,

\[(xy)^\omega = (xy)^\alpha x(xy)^\beta w = \cdots = ((xy)^\alpha x)^n(xy)^\omega ((xy)^\beta w)^n\]

and so, taking limits, also

\[(xy)^\omega = ((xy)^\alpha x(xy)^\omega)^\omega (xy)^\omega w\]

which entails \((xy)^\omega = ((xy)^\alpha x(xy)^\omega)^\omega\). Similarly, from

\[(xy)^\omega = ((xy)^\alpha x(xy)^\omega)^\omega-1 (xy)^\alpha \cdot (xy)^\omega \cdot x(xy)^\omega,\]

centering on the rightmost factor, we may deduce the desired pseudoidentity (vi). If we start from the pseudoidentity (3) instead of (2), we reach similarly the pseudoidentity \((xy)^\omega = ((xy)^\alpha x(xy)^\omega)^\omega\). Interchanging \(x\) and \(y\), we obtain \((yx)^\omega = ((yx)^\alpha x(yx)^\omega)^\omega\). Multiplying both sides on the left by \(x\) and on the right by \(y(xy)^\omega\), and rearranging the right hand side (using equalities valid in \(M\)) yields the pseudoidentity (vi).

(iii) First note that (vii) proves (vi) by simply raising both sides to the \(\omega\) power; it also proves (iii) by substituting \(y\) by \(x\) and using equalities that are valid in \(M\). Conversely, from (vii), using (iii), we deduce that

\[(xy)^\omega = ((xy)^\alpha x(xy)^\omega)^\omega-1 (xy)^\alpha \cdot (xy)^\omega \cdot x(xy)^\omega,\]

and, therefore, (ix) proves (vi). In view of the above and Lemma 6.2, it remains to establish that, together, the pseudoidentities (iv) and (vi) prove (viii). Indeed, the substitution \(x \mapsto (xy)^\omega x, y \mapsto (xy)^\omega\) in (iv) gives

\[((xy)^\omega x)^\omega = ((xy)^\omega x(xy)^\omega)^\omega.\]

Combining the latter pseudoidentity with (vi), by transitivity we obtain (viii).

Note first that (ix) proves the pseudoidentities

\[(xy)^\omega = (yx)^\omega = y(xy)^\omega-1 \cdot (xy)^\omega \cdot (xy)^\omega x.\]

Hence, (ix) proves \((xy)^\omega = (y(xy)^\omega-1)^n \cdot (xy)^\omega \cdot ((xy)^\omega x)^n\); by taking limits, we get \((xy)^\omega = (y(xy)^\omega-1)^\omega \cdot (xy)^\omega \cdot ((xy)^\omega x)^\omega\) and, therefore, (ix) proves (viii). Since (ix) is its own left right dual, (ix) also proves the dual of (viii), namely the pseudoidentity \((x(yx)^\omega)^\omega = (yx)^\omega\). Conversely, from (viii) and its dual, we may prove

\[(xy)^\omega = (y(xy)^\omega)^\omega = ((yx)^\omega y)^\omega = (yx)^\omega.\]

Taking into account previous parts of the proposition, it suffices to invoke Lemma 6.2.
(xiii) Substituting $y$ by $x$ in the pseudoidentity (xii) yields (iii) while, raising both sides to the $\omega$ power we obtain (viii). Once again, in view of Lemma 6.2, it suffices to show that, together, (iii) and (viii) also prove (x) which can be done as in proof of (vi):

$$(xy)^\omega = ((xy)^\omega x)^\omega = ((xy)^\omega x)(xy)^\omega x = (xy)^\omega x.$$  

(xiv) Raising both sides of (xiii) to the $\omega$ power, we obtain (vi) and so $\Sigma$ proves (vi). On the other hand, the monoid $T^1$ and its left right dual fails the pseudoidentity (xiii). Hence, each of them fails some pseudoidentity from $\Sigma$.

Let us consider first the fact that $T^1$ fails some pseudoidentity from $\Sigma$. As in earlier arguments, we deduce that $\Sigma$ proves either the pseudoidentity $x = x^{\omega+1}$ or a two-variable pseudoidentity of the form $x^\omega y = w$, where $w$ is a pseudoword that admits $yx$ as a factor. If $y$ occurs more than once in $w$ then, substituting $x$ by 1, we get a nontrivial pseudoidentity $y = y^\omega$, which entails $x = x^{\omega+1}$ by Lemma 6.1. Hence, $\Sigma$ proves either $x = x^{\omega+1}$ or $x^\omega y = x^\omega y x^\omega$.

Working instead with the dual of the monoid $T^1$, we deduce dually that $\Sigma$ proves either $x = x^{\omega+1}$ or $yx^\omega = x^\omega y x^\omega$.

Suppose first that $\Sigma$ proves $x = x^{\omega+1}$. Since $\Sigma$ also proves (vi), it proves the following pseudoidentities:

$$x^\omega y = (x^\omega y)^{\omega+1} = x^\omega (yx^\omega)^\omega y = x^\omega (x^\omega y)^\omega y = (x^\omega y)^\omega y = (yx^\omega)^\omega y.$$  

Dually, $\Sigma$ proves $yx^\omega = y(x^\omega y)^\omega$ and so also (xiii). Thus, we may assume that $\Sigma$ does not prove $x = x^{\omega+1}$. From the above, it follows that $\Sigma$ proves $x^\omega y = x^\omega y x^\omega = yx^\omega$, as required.

(xiii) Substituting $y$ by $x$ in (xiii), we obtain the pseudoidentity (iii). On the other hand, since $S_l^2$ fails (xiii), it also fails some pseudoidentity from $\Sigma$, and so there is some variable $z$ that occurs only on one side of that pseudoidentity. Substituting $x^\omega y x^\omega$ for $z$ and $x^\omega$ for every other variable, we conclude that $\Sigma$ proves a pseudoidentity of the form $(x^\omega y x^\omega)^\alpha = x^\omega$ and so also the special case where $\alpha = \omega$, which may be written in the form $(x^\omega y)^{\omega+1} = x^\omega$. Substituting $xy$ for $y$, multiplying both sides on the right by $x$, and using additionally the pseudoidentity $x^{\omega+1} = x$, we obtain the required pseudoidentity (xiii).

The choice of the monoids and semigroups considered in the proof of Proposition 6.3 was guided by several results in the literature, even though such results are not explicitly used in the proof. For a pseudoidentity $\varepsilon$, we essentially take a complete set of excluded monoids, that is, a set of finite monoids such that a pseudovariety $V$ satisfies $\varepsilon$ if and only if contains none of the monoids from the set. Such sets can be found in the literature for several pseudoidentities. See [2] for further details. It should be noted that the proof of Proposition 6.3 in fact implies that the sets in question are complete sets of excluded monoids.

7. H-strongness: the role of reducibility

In this section, we consider a method that allows us to give a class of examples of h-strong sets of pseudoidentities. In all of them, the key property of the pseudovariety $V = [\Sigma]$ is that every pseudoidentity $u = v$ valid in $V$ is the limit of a sequence of identities in a suitable implicit signature $\sigma$ which
are also valid in \( V \). In this case we say that the pseudovariety \( V \) is \( \sigma \)-reducible for the equation \( x = y \).

In terms of formal proofs, the above property reduces the conjecture for \( \Sigma \) to proving from \( \Sigma \) the \( \sigma \)-identities which are valid in \( V \). If, additionally, one knows a basis \( \Sigma' \) of identities for the variety \( V' \), all that is needed is to show that \( \Sigma' \) is provable from \( \Sigma \). In summary, we have the following simple observation which provides a method of proof that underlies all the examples in the remainder of the section.

**Proposition 7.1.** Let \( \sigma \) be an implicit signature and let \( \Sigma \) be a set of \( U \)-pseudoidentities defining a \( \sigma \)-reducible pseudovariety \( V \) for the equation \( x = y \). If the variety \( V' \) admits a basis whose identities are provable from \( \Sigma \), then \( \Sigma \) is \( h \)-strong. \( \square \)

We apply below Proposition 7.1 to the usual bases of pseudoidentities of several extensively studied pseudovarieties of semigroups. Before doing so, it is worth explaining why we use the above terminology introduced in [13] that has been widely adopted in the literature. This connection needs to be clarified in order to justify invoking several published results that are required in our application of Proposition 7.1.

As above, fix an ambient pseudovariety \( U \). We also consider a subpseudovariety \( V \) of \( U \). By a *system of equations* we mean a set \( S \) whose elements \( u = v \) are formal equalities (that is, pairs) of terms in the algebraic signature of \( U \). Let \( X \) be the set of variables that occur in \( S \), a set which, for simplicity, we assume to be finite. We are interested in \( V \)-solutions of \( S \) on a fixed but arbitrary finite set of generators \( A \), which consist of a mapping assigning to each variable \( x \in X \) an element \( \phi(x) \) of \( \Omega_A U \) whose natural extension \( \hat{\phi} \) to terms is such that \( V \) satisfies each pseudoidentity \( \hat{\phi}(u) = \hat{\phi}(v) \) with \( u = v \) a member from \( S \). Such systems are often constrained by assigning to each variable \( x \in X \) a clopen subset \( K_x \) of \( \Omega_A U \). The \( V \)-solution \( \phi \) is said to *satisfy* the constraints if \( \phi(x) \in K_x \) for every \( x \in X \).

The key property of the pseudovariety \( V \) introduced in [13] is the following. Let \( \sigma \) be an implicit signature, each of whose elements belongs to some \( \Omega_B U \), where \( B \) is a finite set. We say that \( V \) is \( \sigma \)-reducible for \( S \) if, for every choice of constraints for \( X \), if there is a \( V \)-solution of \( S \) satisfying the constraints, then there is such a solution taking its values in \( \Omega_A U \), which we call a *(\( V, \sigma \))*-solution. Since the clopen sets form a basis of the topology of \( \Omega_A U \), one immediately recognizes the following result which is nothing but a topological reformulation of the definition of reducibility. For that purpose, we view solutions of the system \( S \) as elements of the product space \((\Omega_A U)^X\).

**Proposition 7.2.** The pseudovariety \( V \) is \( \sigma \)-reducible for the system of equations \( S \) over the set of variables \( X \) if and only if the set of \( (V, \sigma) \)-solutions is dense in the set of all \( V \)-solutions. \( \square \)

In particular, the terminology adopted at the beginning of the section is consistent with that from [13].
The property of $\sigma$-reducibility of a pseudovariety $V$ was conceived as part of a strong form of decidability called $\sigma$-tameness. The remaining requirements for $\sigma$-tameness are computability assumptions, namely: the pseudovariety $V$ is assumed to be recursively enumerable, the signature $\sigma$ also recursively enumerable and to consist of operations that are computable in elements of $V$, and the word problem in $\Omega^\sigma_V$ is supposed to be decidable.

The computability assumptions on the pseudovariety $V$ and the implicit signatures considered in all our examples are immediately verified.

It should be pointed that the word problem in $\Omega^\sigma_V$ is equivalent to the variety $V^\sigma$ to admit a recursive basis of identities. Besides $\sigma$-reducibility for the identity $x = y$, it is the knowledge of such a basis that underlies all our applications of Proposition 7.1. Thus, our results rely more properly on $\sigma$-tameness of $V$ rather than just $\sigma$-reducibility for the identity $x = y$.

We are now ready to present our concrete examples of evidence for the conjecture obtained as applications of Proposition 7.1. In the following, the ambient pseudovariety $U$ will be either the pseudovariety $S$ of all finite semigroups or the pseudovariety $M$ of all finite monoids. The implicit signatures involved are often either $\kappa = \{\cdot, \omega^{-1}\}$ or $\{\cdot, \omega\}$ the latter of which, by abuse of notation, we also denote $\omega$.

7.1. Some simple examples. Our first example is given by the pseudoidentity $x^{\omega+1} = x^\omega$, which defines the pseudovariety $A$ of all finite aperiodic monoids. In view of Schützenberger’s characterization of star-free languages [37], this is a very important pseudovariety.

**Theorem 7.3.** For $U = M$, the pseudoidentity $x^{\omega+1} = x^\omega$ is $h$-strong.

**Proof.** The first key ingredient here is that the pseudovariety $A$ is $\omega$-reducible for the equation $x = y$. This is proved in [10, Corollary 3.2] based on Henckell’s algorithm for the computation of $A$-pointlike sets of finite monoids [27, 28].

In fact, the same holds for arbitrary systems of $\omega$-equations [8]. The following

$$
(xy)z = x(yz), \quad 1x = x = x1
$$

$$
(x^\omega)^\omega = (x^r)^\omega = x^\omega x^\omega = x^\omega
$$

$$
(xy)^\omega x = x(yx)^\omega
$$

$$
x^{\omega+1} = x^\omega
$$

Except for the identities in the last line, which are immediately provable from $x^{\omega+1} = x^\omega$, all the other identities are valid in all finite monoids and so they require no proof in our proof setup within the ambient pseudovariety $M$. Applying Proposition 7.1 we conclude that the pseudoidentity $x^{\omega+1} = x^\omega$ is $h$-strong. □

Our next example is the usual basis of the pseudovariety $R$ of all finite $R$-trivial monoids.

**Theorem 7.4.** For $U = M$, the pseudoidentity $(xy)^\omega x = (xy)^\omega$ is $h$-strong.

**Proof.** The $\omega$-reducibility of $R$ for the equation $x = y$ was first proved in [7]. In fact, the same holds for arbitrary systems of $\omega$-equations [8]. The following
basis of identities for the variety $R^\omega$ was obtained in [17]:

\[(x^r)^\omega = (x^\omega)^\omega = x^\omega \ (r \geq 2)\]

\[(xy)z = x(yz), \ x1 = 1x = x\]

\[(xy)^\omega = (xy)^\omega x = (xy)^\omega x^\omega = x(xy)^\omega.\]

Of all the above identities the only one that is not obviously provable from $(xy)^\omega x = (xy)^\omega$ is $(xy)^\omega x^\omega = (xy)^\omega$. Yet, iterating the hypothesis, one gets $(xy)^\omega x^n = (xy)^\omega$, whence also $(xy)^\omega x^\omega = (xy)^\omega$ by taking limits. □

While the two previous examples could be reformulated in the language of semigroups, as the pseudovariety of semigroups generated in each case has no additional monoids, for the next one the monoids in the pseudovariety constitute a much smaller class. Indeed, we now consider the pseudovariety $LSl$ of all finite semigroups which are locally semilattices, that is, the pseudovariety defined by the set in the following result.

**Theorem 7.5.** For $U = S$, the set of pseudoidentities

\[\Sigma = \{x^\omega yx^\omega z x^\omega = x^\omega z x^\omega y x^\omega, x^\omega y x^\omega y x^\omega = x^\omega y x^\omega\}\]

is h-strong.

**Proof.** The $\omega$-reducibility of $LSl$ for the equation $x = y$ was first proved in [21], where graph systems of equations are also considered. The following basis of identities for the variety $LSl$ may be found in [21]:

\[(x^r)^\omega = x^\omega x^\omega = x^\omega x = x^\omega \ (r \geq 2)\]

\[(xy)z = x(yz), \ (xy)^\omega x = x(xy)^\omega\]

\[x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega, x^\omega y x^\omega y x^\omega = x^\omega y x^\omega\]

\[(xy^\omega z)^\omega = (xy^\omega z)^2.\]

Of all the above identities, the only one that requires a proof from $\Sigma$ is the last one. We first note that the second pseudoidentity from $\Sigma$ immediately infers $(xy^\omega z)^3 = (xy^\omega z)^2$, which entails $(xy^\omega z)^{n+1} = (xy^\omega z)^n$ for every $n \geq 2$, whence also $(xy^\omega z)^n = (xy^\omega z)^2$. Taking limits, we may further prove $(xy^\omega z)^\omega = (xy^\omega z)^2$. In view of Proposition 7.1 this concludes the proof of the theorem. □

### 7.2. Monoids in which regular elements are idempotents.

The next example is given by two simple bases of pseudoidentities for the pseudovariety $DA$ of all finite semigroups whose regular $D$-classes are aperiodic subsemigroups, a property which is equivalent to all regular elements being idempotents.

**Theorem 7.6.** For $U = M$, the sets of pseudoidentities

\[\Sigma = \{x^\omega+1 = x^\omega, \ (xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega\}\]

\[\Gamma = \{(xy)^\omega x^2 = (xy)^\omega x\}\]

---

3Later, reducibility of $LSl$ was extended to arbitrary systems of $\kappa$-equations [22]. In view of the well-known decomposition $LSl = SI \ast D$ (see, for instance, [27 Section 10.8]), a generalization in a different direction has been obtained in [23] where, in particular, it is proved that if $V$ is $\kappa$-reducible for the equation $x = y$, then so is $V \ast D$. This was later extended to the case of graph systems of equations in [25].
are h-strong.

Proof. The \( \omega \)-reducibility of \( DA \) for the equation \( x = y \) has been proved in [10]. The following basis of identities for \( DA^\omega \) has been recently obtained [11]:

\[
\begin{align*}
(x^r)^\omega &= (x^\omega)^\omega = x^\omega \ (r \geq 2); \\
(xy)^\omega x &= x(yx)^\omega; \\
(xy)z &= x(yz), \ x1 = 1x = x \\

u^\omega vu^\omega &= u^\omega,
\end{align*}
\]

where \( u, v \in \mathbb{P}_B \mathbb{U} \) for an arbitrary finite alphabet \( B \) and every variable that occurs in \( v \) also occurs in \( u \). Only the family of identities in the last line requires a proof since the others are valid in every finite monoid.

We start by showing that \( \Sigma \) proves \( \Gamma \). We may prove algebraically:

\[
(xy)^\omega = (xy)^\omega (yx)\omega(xy)^\omega = (xy)^\omega (yx)^{\omega+1}(xy)^\omega
\]

\[
= (xy)^\omega y \cdot (xy)^\omega x(xy)^\omega = \cdots = ((xy)^\omega y)^{n_1}(xy)^\omega (x(xy)^\omega)^{n_1}.
\]

Taking limits, we get

\[
(xy)^\omega = ((xy)^\omega y)(xy)^\omega (x(xy)^\omega)^\omega
\]

which, upon multiplication of both sides on the right by \( x(xy)^\omega x \), yields

\[
((xy)^\omega x)^2 = ((xy)^\omega y)(xy)^\omega x(xy)^\omega)^{\omega+1}x
\]

\[
= ((xy)^\omega y)(xy)^\omega x(xy)^\omega)^{\omega}x = (xy)^\omega x.
\]

Conversely, we may algebraically prove \( \Sigma \) from \( \Gamma \) as follows. First, substituting \( x \) for \( y \) in the pseudoidentity \( ((xy)^\omega x)^2 = (xy)^\omega x \) and multiplying both sides by \( x^{\omega-1} \), we obtain

\[
x^{\omega+1} = x^{\omega-1}((xx)^\omega x)^2 = x^{\omega-1}(xx)^\omega x = x^\omega.
\]

Also, multiplying both sides of the pseudoidentity from \( \Gamma \) on the right by \( y(xy)^{\omega-1} \) yields

\[
(xy)^\omega x(xy)^\omega = (xy)^\omega.
\]

Next, we obtain

\[
(xy)^\omega = (xy)^{\omega-1}xy(xy)^\omega = (xy)^{\omega-1}x(yx)^\omega y = (xy)^{\omega-1}x(yx)^\omega y(yx)^\omega y
\]

\[
= (xy)^\omega (yx)^\omega y = (xy)^\omega y(xy)^\omega,
\]

and so

\[
(xy)^\omega y(xy)^\omega = (xy)^\omega,
\]

which entails

\[
(xy)^\omega = (xy)^\omega y(xy)^\omega = (xy)^\omega y(xy)^\omega x(xy)^\omega
\]

\[
= (xy)^\omega (yx)^{\omega+1}(xy)^\omega = (xy)^\omega (yx)^\omega (xy)^\omega.
\]

From hereon, we are thus allowed to use \( \Sigma \cup \Gamma \) as hypothesis in our proofs and we assume them without further mention. Recall that our objective is to prove that every \( \omega \)-identity of the last form \( u^\omega vu^\omega = u^\omega \) under the above assumptions on \( u \) and \( v \). Since \( u \) and \( v \) may expressed as limits of sequences
of words with constant content, we may as well assume that both $u$ and $v$ are words.

To simplify the notation we let $e = u \omega$. We claim that the following statements hold:

(i) if $e e v e = e$ may be proved, then $e w^2 e = e$ may also be proved;
(ii) if $e v e = e$ may be proved, then $e v w e = e$ may also be proved;
(iii) the pseudoidentity $(x y z)^\omega y (x y z)^\omega = (x y z)^\omega$ is provable.

To establish (i), assume that we have proved $e v e = e$. Then we may also prove $e w = e w e w = \cdots = (e w)^n$ and, taking limits, $e w = (e w)^\omega = (e w)^\omega$. Similarly, we can prove $w e = (w e)^\omega$. This yields the following equalities:

$$e = e v e = (e w)^\omega e = (e w)^\omega (w e)^\omega (e w)^\omega e = e w e w e = e w^2 e.$$ For (iii), suppose we have proved $e v e = e = e w e$. Then, in view of (i), we may prove

$$e = e e = e w e v e = e \cdot w e v \cdot e = e \cdot w e v \cdot w e v \cdot w e \cdot e = e w e v e.$$

To prove (ii), observe first that, from (iii) and (i) we deduce that the pseudoidentities

$$(y z x)^\omega = (y z x)^\omega y (y z x)^\omega = (y z x)^\omega x (y z x)^\omega$$

are provable, so that, by (i), we have $(y z x)^\omega = (y z x)^\omega y (y z x)^\omega$ and

$$(x y z)^\omega = (x y z)^\omega x (y z x)^\omega y (y z x)^\omega y z = (x y z)^\omega y (y z x)^\omega y (y z x)^\omega y.$$

Let $v = v_1 \cdots v_n$, where the $v_i$ are letters. By (iii) since every letter appearing in $v$ also appears in $u$, we may prove $e v_i e = e$ ($i = 1, \ldots, n$). Applying (i) $n - 1$ times, we deduce that we may also prove $e v e = e$, as required. □

7.3. $J$-trivial monoids. The next example consists of the two bases commonly used to describe the pseudovariety $J$ of all finite $J$-trivial semigroups.

**Theorem 7.7.** For $U = M$, the sets of pseudoidentities

$$\Sigma = \{x^{\omega+1} = x^\omega, (x y)^\omega = (y x)^\omega\} \text{ and } \Gamma = \{(x y)^\omega x = (x y)^\omega = y (x y)^\omega\}$$

are $h$-strong.

**Proof.** It is not difficult to show that $J$ is $\omega$-reducible for the equation $x = y$. In fact, $J$ is $\omega$-reducible for all finite systems of equations, and even of $\kappa$-equations [4, Theorem 12.3]. On the other hand, the following basis of identities for the variety $J^\omega$ is given in [2, Section 8.2]:

$$(x y) z = x (y z), \quad x 1 = x, \quad (x^\omega)^\omega = x^\omega \quad x^\omega x = x x^\omega = x^\omega \quad (x y)^\omega = (y x)^\omega = (x^\omega y^\omega)^\omega.$$ The identities in the first line are valid in all finite monoids and, therefore require no proof. In view of Proposition 7.1 to finish the proof it suffices to show that the remaining identities are provable from both $\Sigma$ and $\Gamma$.

In the case of $\Sigma$, only the identity $(x y)^\omega = (x^\omega y^\omega)^\omega$ needs to be considered. The following describes a proof from $\Sigma$. First, we do an algebraic proof:

$$(7) \quad (x y)^\omega = (x y)^{\omega+1} = x (x y)^\omega y = x (x y)^\omega y \cdots = x^m (x y)^\omega y^m.$$
Hence, we may also prove from $\Sigma$

$$(xy)^\omega = (yx)^\omega = y^m(xy)^\omega x^m = y^m(xy)^\omega x^m,$$

which, combined with (7), yields

(8)

$$(xy)^\omega = x^m y^m(xy)^\omega x^m y^m$$

Iterating (8), we get an algebraic proof of

$$(xy)^\omega = (x^m y^m)^m(xy)^\omega (x^m y^m)^m.$$\)

Letting $m = n!$ and taking limits, we obtain that $\Sigma$ proves

(9)

$$(xy)^\omega = (x^\omega y^\omega)^\omega (xy)^\omega (x^\omega y^\omega)^\omega.$$\)

Similarly, we may prove algebraically

$$(x^\omega y^\omega)^\omega = x^\omega y^\omega (x^\omega y^\omega)^\omega = xx^\omega y^\omega (x^\omega y^\omega)^\omega = x(x^\omega y^\omega)^\omega$$

and so also

$$(x^\omega y^\omega)^\omega = xy(x^\omega y^\omega)^\omega = \cdots = (xy)^{m!}(x^\omega y^\omega)^\omega.$$\)

Taking limits, we get

$$(x^\omega y^\omega)^\omega = (xy)^\omega (x^\omega y^\omega)^\omega.$$\)

Combining with (9) and taking into account that $(x^\omega y^\omega)^\omega$ is idempotent, we finally complete the proof of $(xy)^\omega = (x^\omega y^\omega)^\omega$ from $\Sigma$.

For $\Gamma$, we first note that, substituting $x$ for $y$ in $(xy)^\omega x = (xy)^\omega$, yields $x^{\omega + 1} = x^\omega$. Hence, it suffices to show that the pseudoidentity $(xy)^\omega = (yx)^\omega$ is provable from $\Gamma$, which can be established algebraically:

$$(xy)^\omega = y(xy)^\omega = (yx)^\omega y = (yx)^\omega.$$\)

Note that, in the proof of Theorem 7.7, we alternated several times topological and transitive closure. More precisely, we actually proved that $\bar{\Sigma} = \Sigma_4$. We do not know whether $\bar{\Sigma} = \Sigma_3$ but show below that $\bar{\Sigma} \neq \Sigma_2$.

We start with an auxiliary lemma involving equidivisibility. We say that a semigroup $S$ is equidivisible if any two factorizations of the same element admit a common refinement. We say that a pseudovariety (of semigroups or monoids) $V$ is equidivisible if, for each finite set $A$, the semigroup $\overline{\bigwedge}_A V$ is equidivisible. The equidivisible pseudovarieties of semigroups have been characterized in [G]. The characterization of equidivisible pseudovarieties of monoids can be derived from it by noting that, for a pseudovariety $V$ of monoids, $\overline{\bigwedge}_A V = (\overline{\bigwedge}_A W)^1$, where $W$ is the pseudovariety of semigroups generated by $V$, which amounts to a simple exercise, together with the obvious observation that a semigroup $S$ is equidivisible if and only if so is the monoid $S^1$. In particular, $M$ is equidivisible.

The following lemma can surely be generalized but is already sufficient for our purposes.

Lemma 7.8. Let $w \in A^*$ and $u, v \in \overline{\bigwedge}_A M$ be such that $w^\omega$ is a factor of $uv$. Then, $w^\omega$ is a factor of at least one of the factors $u$ and $v$.

Proof. By equidivisibility, from the two factorizations $uv = xw^\omega y$ (for some $x, y \in \overline{\bigwedge}_A M$), we know that there is a common refinement. Hence, if $w^\omega$ is a factor of neither $u$ nor $v$, then there is a factorization $w^\omega = zt$ with $u = xz$ and $v = ty$. We reach a contradiction by showing that $w^\omega$ must be a factor of at least one of $z$ and $t$.

First note that if $z$ and $t$ is a not a finite word for, otherwise, so would be $w^\omega$. By symmetry, we may as well assume that $z$ is not a finite
word. We claim that $w^n$ is a prefix of $z$ for every $n \geq 1$ and, therefore, so is $w^\omega$, thereby reaching the desired contradiction. To prove the claim, consider the monoid $M_k$ consisting of all words of $A^*$ of length at most $k = |w^n|$ where the product is defined by $r \cdot s = rs$ if $|rs| \leq k$, while $r \cdot s$ is taken to be the prefix of $rs$ of length $k$ otherwise. Consider also the unique continuous homomorphism $\varphi_k : \Omega_A M \to M_k$ which maps each letter $a$ from $A$ to $a$ as an element of $M_k$. Note that $\varphi_k$ maps each word of length at most $k$ to itself and every other finite word to its prefix of length $k$. It follows that $\varphi_k(s)$ is a prefix of $s$ for every pseudoword $s \in \Omega_A M$.

Since $z$ is not a finite word, there is a sequence of words $(z_m)_m$ converging to $z$ with $|z_m| \geq k$ for every $m$. Then, from the equalities $\varphi_k(z) = \varphi_k(z_t) = \varphi_k(u^\omega) = u^n$, we deduce that $w^n$ is a prefix of $z$, as was claimed. 

\begin{proposition}
For $U = M$ and $\Sigma = \{x^{\omega+1} = x^\omega, (xy)^\omega = (yx)^\omega\}$, we have $\Sigma \neq \Sigma_2$.
\end{proposition}

\begin{proof}
For the purpose of the present proof, we take $A = \{x, y\}$.

We have shown in the proof of Theorem 7.7 that the $\omega$-identity
\begin{equation}
(x^\omega y^\omega)^\omega = (xy)^\omega
\end{equation}

belongs to $\Sigma_3$. We prove that it does belong to $\Sigma_2$. For that purpose, we claim that, if the pseudoidentity $w = (xy)^\omega$ is in $\Sigma_2$, then $(xy)^\omega$ is a factor of $w$. Since not even the word $xyx$ is a factor of $(x^\omega y^\omega)^\omega$ (cf. [16, Lemma 8.2]), we conclude that the pseudoidentity (10) cannot belong to $\Sigma_2$.

The hypothesis of the claim implies the existence of a sequence of pseudoidentities $(w_n = v_n)_n$ in $\Sigma_1$ which converges to $w = (xy)^\omega$. Now, by taking subsequences, we may as well assume that either the sequence $(v_n)_n$ consists only of finite words or only of infinite pseudowords. In the first case, the only $\Sigma_1$-pseudoidentity of the form $u = v_n$ is the trivial pseudoidentity $v_n = v_n$. Since the pseudoidentity (10) is not trivial over $M$, the first case is excluded. In the second case, we note that $(xy)^\omega$ is a factor of $v_n$ for every sufficiently large $n$. Indeed, the language $(xy)^*$ has an open closure in $\Omega_A M$ [2, Theorem 3.6.1], which consists of all powers of $xy$.

Thus, to establish our claim, it suffices to show that, if the pseudoidentity $u = v$ belongs to $\Sigma_1$ and the pseudoword $(xy)^\omega$ is a factor of $v$ then it is also a factor of $u$. Since $\Sigma_1$ is the transitive closure of $\Sigma_0$, by a straightforward induction argument it suffices to treat the case where $u = v$ belongs to $\Sigma_0$. Hence, there are a nonempty word $t$, pseudowords $w_1, \ldots, w_n$, a pseudoidentity $v' = v'$ such that either $v'$ or $v' = v'$ belongs to $\Sigma$, and a continuous endomorphism $\varphi$ of $\Omega_A M$ such that $u = t(\varphi(w_1), w_1, \ldots, w_n)$ and $v = t(\varphi(v'), w_1, \ldots, w_n)$. Note that, since $v'$ and $v'$ are $J$-equivalent, so are $\varphi(v')$ and $\varphi(v')$, which means that these two pseudowords have the same factors. Without loss of generality, we may assume that, when $t = t(x_0, x_1, \ldots, x_n)$, all the letters $x_i$ ($i = 1, \ldots, n$) appear at least once in $t$. By Lemma 7.3, we deduce from the assumption that $(xy)^\omega$ is a factor of $v$ that it must also be a factor of either $\varphi(v')$ or one of the $w_i$. Hence, $(xy)^\omega$ is a factor of either $\varphi(v')$ or one of the $w_i$ and, therefore, also of $u$. \qed

8. The group case

Let \( G \) be the pseudovariety of all finite groups. As far as subpseudovarieties of \( G \) are concerned, whether we view groups in the natural signatures for semigroups, monoids, or groups is irrelevant, since the identity element is the only idempotent \( x^\omega \) and inversion is also captured by the semigroup pseudoword \( x^{\omega^{-1}} \). However, for the purpose of this section, we prefer to deal with the group signature, consisting of a binary multiplication, a constant symbol 1, for the identity element, and the unary operation of inversion.

Note that for \( u, v \in \Omega_A G \), each of the pseudoidentities \( u = v \) and \( u^{-1}v = 1 \) is provable from the other. Hence, in the language of groups, it suffices to deal with pseudoidentities of the form \( w = 1 \).

For a set \( \Upsilon \) of group pseudowords, consider the pseudovariety

\[
H_\Upsilon = \{ u = 1 : u \in \Upsilon \}.
\]

Denote by \( N_\Upsilon \) the subgroup of \( \Omega_A G \) generated by all conjugates of elements of the form \( \varphi(u) \) with \( B \) a finite alphabet, \( u \in \Upsilon \cap \Omega_B G \), and \( \varphi : \Omega_B G \rightarrow \Omega_A G \) a continuous homomorphism.

**Theorem 8.1.** Let \( \Upsilon \) be a set of group pseudowords. Then, for every \( w \in \Omega_A G \), the pseudovariety \( H_\Upsilon \) satisfies the pseudoidentity \( w = 1 \) if and only if \( w \) belongs to the closure of \( N_\Upsilon \) in \( \Omega_A G \).

**Proof.** Suppose that \( w \in \overline{N_\Upsilon} \), say \( w = \lim w_n \) for a sequence \((w_n)_n \) of elements of \( N_\Upsilon \). Now, for each \( u \in \Upsilon \cap \Omega_B G \) and each continuous homomorphism \( \varphi : \Omega_B G \rightarrow \Omega_A G \), the pseudoidentity \( \varphi(u) = 1 \) is valid in \( H_\Upsilon \), whence so is \( g\varphi(u)g^{-1} = 1 \) for every \( g \in \Omega_A G \). Hence, each pseudoidentity \( w_n = 1 \) is also valid in \( H_\Upsilon \), which entails that so is \( w = 1 \).

Conversely, suppose that \( w \in \Omega_A G \) is such that \( H_\Upsilon \) satisfies the pseudoidentity \( w = 1 \), but, arguing by contradiction, \( w \) does not belong to \( \overline{N_\Upsilon} \).

Since \( \Omega_A G \) is a profinite group, for each \( v \in \overline{N_\Upsilon} \) there is a clopen normal subgroup \( K_v \) of \( \Omega_A G \) such that \( wK_v \neq vK_v \). As \( N_\Upsilon \) is compact, we may extract a finite covering from \( \bigcup_{v \in \overline{N_\Upsilon}} vK_v \), say \( \bigcup_{i=1}^n v_iK_{v_i} \). Let \( K = \bigcap_{i=1}^n K_{v_i} \), which is a clopen normal subgroup of \( \Omega_A G \). Finally, let \( L = \overline{N_\Upsilon K} \), which is again a clopen normal subgroup of \( \Omega_A G \). Clearly, \( L \) contains \( N_\Upsilon \) and we claim that it does not contain \( w \). Indeed, if \( w \in L \), then there is \( v \in \overline{N_\Upsilon} \) such that \( w \in vK \). Since the open sets \( v_iK_{v_i} \) cover \( \overline{N_\Upsilon} \), we deduce that there is \( i \in \{1, \ldots, n\} \) such that \( v \in v_iK_{v_i} \). Hence, we have

\[
w \in vK \subseteq v_iK_{v_i}K \subseteq v_iK_{v_i},
\]

which contradicts the choice of \( K_{v_i} \).

Let \( G \) be the quotient group \( \Omega_A G / L \), which is a finite group. Since \( L \) contains \( \overline{N_\Upsilon} \), by the first part of the proof we infer that \( G \in H_\Upsilon \). Hence, by hypothesis, \( G \) satisfies the pseudoidentity \( w = 1 \), which, for the evaluation of each letter \( a \in A \) by the coset \( aL \), yields \( w \in L \), in contradiction with the choice of \( L \). It follows that \( w \) must belong to \( \overline{N_\Upsilon} \).

**Corollary 8.2.** The pseudovariety \( G \) is strong.

The same method may be applied to show that the pseudovariety of all finite rings (not necessarily with identity element) is strong.
Theorem 8.3. For $U = M$, the pseudoidentity $x^ω = 1$ is $h$-strong.

Proof. The key result upon which the proof is based is the $κ$-reducibility of $G = [x^ω = 1]$ for the equation $x = y$. More generally, based on a celebrated theorem of Ash [18], which contains all the essential hard work, it has been observed in [13, Theorem 4.9] that $G$ is $κ$-reducible for all finite graph systems of equations. In view of Proposition 7.1, it remains to show that every $κ$-identity valid in $G$ may be proved from $x^ω = 1$ in the sense of Section 3. In fact, we show that it may be algebraically proved.

The essential observation is that the $κ$-identity

\[(xy)^ω^{-1} = y^ω^{-1}x^ω^{-1}\]

may be algebraically proved from $x^ω = 1$. Indeed here is a description of such a proof:

\[
(xy)^ω^{-1} = y^ω(xy)^ω^{-1}x^ω = y^ω^{-1}y(xy)^ω^{-1}x^ω^{-1} = y^ω^{-1}(yx)^ω^{-1} = y^ω^{-1}x^ω^{-1}.
\]

Note also that $((x^ω)^{-1})^ω^{-1} = x^{ω+1} = x$, where the latter equality is provable from $x^ω = 1$. Applying repeatedly the identity (11), every $κ$-term may be rewritten in the form $x_{i1}^{ε_1} \cdots x_{in}^{ε_n}$, where each $x_i$ is a variable and each exponent $ε_i$ is either 1 or $ω - 1$. We say that a $κ$-term of this type is in standard form; a $κ$-identity whose sides are in standard form is also said to be in standard form. Thus, in the presence of the pseudoidentity $x^ω = 1$, every $κ$-identity valid in $G$ is algebraically provably equivalent to a $κ$-identity $u = v$ in standard form that is also valid in $G$. Since the free group is residually finite, $u = v$ may be viewed as a group identity (by removing the $ω$'s from the exponents) that is satisfied by all groups, which means that, by applying the reduction rules $aa^{-1} → 1$ and $a^{-1}a → 1$ a finite number of times, both sides may be transformed to the same reduced group word. Since, in the context of finite monoids, both rules follow from the pseudoidentity $x^ω = 1$, we conclude that $u = v$ is algebraically provable from $x^ω = 1$. □

Although the next result is superseded by Corollary 9.4, it seems worthwhile to include it at this stage.

Corollary 8.4. For $U = M$, every set $Σ$ of pseudoidentities defining a group pseudovariety is $h$-strong.

Proof. It suffices to apply Proposition 4.2 taking into account Theorem 8.3, Corollary 8.2, and Proposition 6.3(1). □
9. The Completely Simple Semigroup Case

Let $\text{CS}$ be the pseudovariety of all finite completely simple semigroups. It is defined, for instance, by the pseudoidentity $(xy)^2x = x$. By a well-known theorem of Rees, completely simple semigroups are precisely those that admit a Rees matrix representation $\mathcal{M}(I, G, \Lambda, P)$, where $I$ and $\Lambda$ are sets, $G$ is a group, and $P : \Lambda \times I \to G$ is a function (the sandwich matrix), the image $P(\lambda, i)$ being usually denoted $p_{\lambda, i}$; as a set, it is the Cartesian product $I \times G \times \Lambda$, and multiplication is given by the formula

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda, j}h, \mu).$$

Assuming that 1 is a common element of $I$ and $\Lambda$, the sandwich matrix may be supposed to be normalized in the sense that $p_{1,i} = p_{\lambda,1} = 1$ for all $i \in I$ and $\lambda \in \Lambda$.

The purpose of this section is to establish that $\text{CS}$ is strong. The key ingredient is the following characterization of the congruences on a Rees matrix semigroup which may be extracted from [20, Theorem 10.48].

**Theorem 9.1.** Let $S = \mathcal{M}(I, G, \Lambda, P)$ be a Rees matrix semigroup and let $\rho$ be a congruence on $S$. Consider the relations

$$\rho_1 = \{(i, j) \in I \times I : (i, 1, 1) \rho (j, 1, 1)\}$$

$$\rho_2 = \{((\lambda, \mu) \in \Lambda \times \Lambda : (1, 1, \lambda) \rho (1, 1, \mu)\}$$

Then $\rho_1$ (respectively $\rho_2$) is an equivalence relation on the set $I$ (resp. $\Lambda$) and $N$ is a normal subgroup of $G$ such that

$$(i, j) \in \rho_1 \implies p_{\lambda, i}N_\rho = p_{\lambda, j}N_\rho$$

$$(\lambda, \mu) \in \rho_2 \implies p_{\lambda, i}N_\rho = p_{\mu, i}N_\rho.$$

Conversely, for every triple $\tau = (\rho_1, \rho_2, N_\rho)$, where $\rho_1$ (resp. $\rho_2$) is an equivalence relation on the set $I$ (resp. $\Lambda$) and $N_\rho$ is a normal subgroup of $G$, satisfying properties (12) and (13), the relation

$$\rho_\tau = \{(i, g, \lambda), (j, h, \mu) \in S \times S : i \rho_1 j, \lambda \rho_2 \mu, gN_\rho = hN_\rho\}$$

is a congruence on $S$ and every congruence on $S$ is of this form.

We may now proceed as in the proof of Theorem 8.1 to obtain our next result. For a set $\Sigma$ of $\text{CS}$-pseudoidentities, consider the closed congruence $\rho = \Sigma$ generated by the pairs of the form $(\varphi(u), \varphi(v))$ with $u = v$ in $\Sigma$, $u, v \in \overline{\Pi}_B \text{CS}$, and $\varphi : \overline{\Pi}_B \text{CS} \to \overline{\Pi}_A \text{CS}$ a continuous homomorphism.

**Theorem 9.2.** The pseudovariety $\text{CS}$ is strong.

**Proof.** Suppose that $u, v \in \overline{\Pi}_A \text{CS}$ are such that the pseudoidentity $u = v$ is valid in $[\Sigma]$. We claim that $u = v$ is provable from $\Sigma$, which amounts to the condition $u \rho v$.

Arguing by contradiction, suppose that $(u, v) \notin \rho$. By [1], there is an isomorphism $\psi : \overline{\Pi}_A \text{CS} \to S = \mathcal{M}(A, \overline{\Pi}_X G, A, P)$ where, choosing an element $a_0$ from $A$ and letting $A' = (A \setminus \{a_0\})^2$ be the Cartesian square, we have $X = A \cup A'$, $p_{a_0, b} = p_{b, a_0} = 1$, and $p_{a, b} = (a, b)$ for each $(a, b) \in A'$. Note
that the letter \( a_0 \) plays the role of 1 in the normalization of the sandwich matrix.

Consider the normal subgroup \( N_\rho \) and the equivalence relations \( \rho_1 \) and \( \rho_2 \) as defined in Theorem 9.1. Note that \( N_\rho \) is a closed normal subgroup of \( \Omega X G \).

Let \( \psi(u) = (a, g, b) \) and \( \psi(v) = (c, h, d) \). Since we are assuming that \( (u, v) \notin \rho \), at least one of the following conditions must hold:

\[
(a, c) \notin \rho_1, \ (b, d) \notin \rho_2, \ gN_\rho \neq hN_\rho.
\]

In case one of the first two conditions holds, the mapping from \( S \) onto the rectangular band \( T = A/\rho_1 \times A/\rho_2 \) that maps each triple \( (x, w, y) \in S \) to \( (x/\rho_1, y/\rho_2) \) is a continuous homomorphism onto a semigroup from \( \mathcal{CS} \) that distinguishes \( u \) and \( v \). Moreover, its kernel congruence is contained in \( \rho \), which implies that \( T \in [\Sigma] \), contradicting the assumption that \( [\Sigma] \) satisfies \( u = v \). Hence, we may assume that \( (a, c) \in \rho_1 \) and \( (b, d) \in \rho_2 \), so that \( g^{-1}h \) does not belong to \( N_\rho \). As in the proof of Theorem 8.1, we deduce that there is a clopen normal subgroup \( K \) of \( \Omega X G \) such that \( N_\rho \subseteq K \) and \( g^{-1}h \notin K \).

We proceed with some observations on the variety \( \mathcal{CS} \) of completely simple semigroups, consisting of algebras with a binary multiplication and unary “inversion” \( _{-1} \) satisfying the following identities, where \( u^0 \) abbreviates \( uu^{-1} \):

\[
\begin{align*}
(14) \quad (xy)z &= x(yz), \ x^{-1}x = x^0, \ x^0x = x, \ (x^{-1})^{-1} = x \\
(15) \quad (xy)^0 &= x^0.
\end{align*}
\]

The identities (14) define the variety of completely regular semigroups \( \mathcal{CSR} \). In the presence of them, it is well known that the identity (15) is equivalent to

\[
(16) \quad (xy)^0x = x.
\]

Note that, when the inversion operation \( _{-1} \) is interpreted as \( _{-\omega} \) in a finite semigroup, \( _0 \) becomes \( _-\omega \) and the first two identities in (14) are verified while the last two identities in (14) are valid in every completely regular finite semigroup.

**Theorem 9.3.** For \( U = S \), each of the sets of pseudoidentities

\[
\Sigma = \{(xyx)^\omega = x^\omega, x^{\omega+1} = x\} \quad \text{and} \quad \Gamma = \{(xy)^\omega x = x\}
\]

is \( h \)-strong.

**Proof.** As implied by the entry for \( \mathcal{CS} \) in [8, Table 2], the methods of [2] show that the pseudovariety \( \mathcal{CS} \) is \( \kappa \)-reducible for graph systems of equations. In fact, in [3] vertices were allowed to be constrained by the clopen subset
\{1\} \subseteq (\bigwedge A S)^1$, which forces the corresponding variable to be evaluated by 1. It follows that $CS$ is $\kappa$-reducible for the equation $x = y$.

Consider next the variety $CS^\kappa$. Since free completely simple semigroups are residually finite [33, Proposition 2.5], in view of the above remarks a basis of identities for $CS^\kappa$ is given by

\[(xy)z = x(yz), \quad x^{\omega - 1}x = xx^{\omega - 1}, \quad xx^{\omega - 1}x = x, \quad (x^{\omega - 1})^{\omega - 1} = x\]

All the identities in the first line are obviously provable from $\Sigma$. On the other hand, substituting $x$ for $y$ gives a proof of $x^{\omega + 1} = x$ from $\Gamma$ and we already observed that in the presence of the identities in the above first line, the identities $(xy)^x = x^x$ and $(xy)^x = x$ are provable from each other. Thus, the result follows from Proposition 7.1.

Combining Theorems 9.2 and 9.3 with Proposition 6.3(xii), and applying Proposition 4.2, we obtain the following result.

**Corollary 9.4.** For $U = S$, every set $\Sigma$ of pseudoidentities such that $[\Sigma] \subseteq CS$ is h-strong.

Taking into account the $\kappa$-reducibility for the equation $x = y$ of the pseudovariety $CR = [x^{\omega + 1} = x]$, of all finite completely regular semigroups, (cf. [13]) and the residual finiteness of free completely regular semigroups (viewed as algebras in the signature \{., ., .\}), the following theorem can be proved in the same way as Theorem 9.3.

**Theorem 9.5.** For $U = S$, the pseudoidentity $x^{\omega + 1} = x$ is h-strong.

10. **The commutative case**

Our next example is the usual basis of the pseudovariety $Com$ of all finite commutative monoids.

**Theorem 10.1.** For $U = M$, the identity $xy = yx$ is h-strong.

**Proof.** Although [38] deals with the notion of hyperdecidability, which is weaker than $\sigma$-reducibility provided the implicit signature $\sigma$ has suitable algorithmic properties (cf. [13, 14]), it is observed in [13] that the same methods show that $Com$ is $\kappa$-reducible, in particular for the equation $x = y$. We claim that the following is a basis of identities for the variety $Com^\kappa$, as usual, we write $x^\omega$ for the product $x^{\omega - 1}x$:

\[x^{\omega - 1}x^{\omega} = x^{\omega - 1}, \quad (x^{\omega - 1})^{\omega - 1} = x^{\omega + 1}\]

\[(xy)z = x(yz), \quad x1 = 1x = x\]

\[xy = yx\]

\[(xy)^{\omega - 1} = x^{\omega - 1}y^{\omega - 1}.

Indeed all such identities are clearly valid in $Com$. On the other hand, using the above identities, one may reduce every $\kappa$-term to one of the form $u = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where each exponent $\varepsilon_i$ is either 1 or $\omega - 1$. Moreover, we may rearrange the factors so that the powers with the same base are not separated by other powers. On the other hand, powers with the same base can collected together to powers of one of the forms $x^n$ or $x^{\omega - 1}x^n$. 

where $n$ is a nonnegative integer, or $(x^\omega)^n$, where $n \geq 2$. For convenience, we write $x^{\omega-1}x^n$ as $x^{\omega-1+n}$ and $(x^\omega)^n$ as $x^{(\omega-1)n}$. In this form, we say that we have a $\kappa$-term in completely reduced form. Consider two $\kappa$-terms in completely reduced form $u = a_1^{\varepsilon} \cdots a_n^{\varepsilon}$ and $v = a_1^{\delta} \cdots a_n^{\delta}$ over the alphabet $\{a_1, \ldots, a_n\}$ and assume that the identity $u = v$ is valid in $\text{Com}$. Then, substituting 1 for every variable but one, we obtain an identity valid in $\text{Com}$ of the form $x^\varepsilon = x^\delta$ with $\varepsilon$ and $\delta$ exponents of one of the forms $n$, $\omega - 1 + n$ or $(\omega - 1)n$. By considering a suitable monogenic finite semigroup, one immediately verifies that $\varepsilon = \delta$. This proves the claim.

To conclude the proof, it remains to observe that the last identity in the above basis is provable from $xy = yx$. Indeed, each identity $(xy)^{n!-1} = x^{n!-1}y^{n!-1}$ can be easily proved and the result follows by taking limits. □

We next show that the pseudoidentity $xy = yx$ also defines a strong pseudovariety.

**Theorem 10.2.** The pseudovariety $\text{Com}$ is strong.

**Proof.** First of all, we recall that elements of the free profinite monoid $\Omega_A \text{Com}$ over the set $A = \{a_1, \ldots, a_n\}$ can be written in the form $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$, where $\varepsilon_1, \ldots, \varepsilon_n \in \hat{\mathbb{N}}$. Each pseudoidentity $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} = a_1^{\delta_1} \cdots a_n^{\delta_n}$ is provably equivalent to the set of pseudoidentities $\{a_1^{\varepsilon_1} = a_1^{\delta_1}, \ldots, a_n^{\varepsilon_n} = a_n^{\delta_n}\}$. Therefore, we may assume that all pseudoidentities which are under consideration are over a single variable.

Recall that $\Omega_A \text{Com}$ and $\Omega_A \text{M}$ are both isomorphic to $\hat{\mathbb{N}}$ via the mapping $\varepsilon \mapsto x^\varepsilon$ and that $\hat{\mathbb{Z}} = \hat{\mathbb{N}} \setminus \mathbb{N}$ is isomorphic to $\Omega_A \text{G}$. For the purpose of simplification of notation, we identify isomorphic structures, so, for example, we denote by $\pi$ the continuous homomorphisms $\pi : \Omega_A \text{Com} \to \Omega_A \text{G}$ given by the rule $\pi(x^\varepsilon) = x^{\omega+\varepsilon}$, which is formally a projection $\pi_G : \Omega_A \text{M} \to \Omega_A \text{G}$ after the identification $\Omega_A \text{Com} = \Omega_A \text{M}$. In this way, the group $\Omega_A \text{G}$ can be viewed as a retract of the monoid $\Omega_A \text{Com}$.

Let $\Sigma$ be a set of pseudoidentities in the variable $x$ and $x^\varepsilon = x^\delta$ be a nontrivial pseudoidentity satisfied by $\llbracket \Sigma \rrbracket$. We need to show that $x^\varepsilon = x^\delta$ belongs to $\Sigma$.

By the finite index $i(\Sigma)$ of $\Sigma$ is meant the minimal natural number $n$ such that $(x^n = x^\mu) \in \Sigma$ where $\mu \neq n$. If such $n \in \mathbb{N}$ does not exist, we say that $\Sigma$ has infinite index.

If $n$ is the index of $\Sigma$, then in a nontrivial pseudoidentity $(x^n = x^\mu) \in \Sigma$ we have $\mu$ infinite or $n < \mu \in \mathbb{N}$. In both cases, in view of Lemma 6.1, the pseudoidentity $x^n = x^{\omega+n}$ is provable from $\Sigma$. On the other hand, the monogenic monoid $C_{n,1}^{\omega}$ belongs to $\llbracket \Sigma \rrbracket$, whence it satisfies $x^\varepsilon = x^\delta$. Hence, both $\varepsilon$ and $\delta$ are infinite or greater than or equal to $i(\Sigma) = n$. From Lemma 6.1 it follows that $x^\varepsilon = x^{\omega+\varepsilon}$ and $x^\delta = x^{\omega+\delta}$ are provable from $\Sigma$. Therefore, $x^\varepsilon = x^\delta$ is provable from $\Sigma$ if and only if so is $x^{\omega+\varepsilon} = x^{\omega+\delta}$. On the other hand, if $\Sigma$ has infinite index, then the monoid $C_{n,1}^{\omega}$ belongs to $\llbracket \Sigma \rrbracket$ for every $n \in \mathbb{N}$ and we see that $\varepsilon, \delta \notin \mathbb{N}$. Altogether, we can deal just with the case $\varepsilon, \delta \in \hat{\mathbb{Z}}$.

Before finishing the proof, we make one technical observation. Let $\rho$ be the restriction of the relation $\Sigma$ to the set $\{x^\lambda : \lambda \in \hat{\mathbb{Z}}\}$ and $\tau$ denote the
relation \( \overline{\pi(\Sigma)} \) on \( \overline{\Omega_1 G} \). We claim that these relations are equal, under our identification of underlying sets with \( \hat{\mathbb{Z}} \). Since \( \hat{\mathbb{Z}} \) is a closed ideal in \( \hat{\mathbb{N}} \) and \( \Sigma \) is a closed congruence on \( \hat{\mathbb{N}} = \overline{\Omega_1 \text{Com}} \), the relation \( \rho \) is a closed congruence on \( \hat{\mathbb{Z}} \). We have \( \pi(\Sigma) \subseteq \Sigma_0 \), because each pseudoidentity \( x^{\omega+\lambda} = x^{\omega+\mu} \) is provable from \( x^{\lambda} = x^{\mu} \). Hence, \( \pi(\Sigma) \subseteq \Sigma \), where \( \pi(\Sigma) \) is a relation on \( \hat{\mathbb{Z}} \). Therefore, \( \pi(\Sigma) \subseteq \rho \) and the transitive-topological closure \( \tau = \pi(\Sigma) \) is also a subset of the closed congruence \( \rho \). The reverse inclusion \( \rho \subseteq \tau \) will be proved if we establish (by the induction), for each \( \alpha \), the inclusion \( \Sigma_0 |_{\hat{\mathbb{Z}}^x} \subseteq (\pi(\Sigma))_0 \).

Let \( \alpha = 0 \) and \( (x^\lambda, x^\mu) \in \Sigma_0 \), where \( \lambda, \mu \in \hat{\mathbb{Z}} \). Taking into account commutativity, we may assume that \( x^\lambda = x^{a+bk} \), \( x^\mu = x^{a+bl} \), where \( a, b, k, \ell \in \hat{\mathbb{N}} \) and \( (x^k = x^l) \in \Sigma \). Then, \( (x^{a+b(\omega+k)} = x^{a+b(\omega+l)}) \in \pi(\Sigma) \) and we have \( (x^{a+b(\omega+k)}) \) is a relation on \( \hat{\mathbb{Z}} \). Therefore, \( \pi(\Sigma) \subseteq \Sigma \), where \( \pi(\Sigma) \) is a relation on \( \hat{\mathbb{Z}} \). Hence, \( \pi(\Sigma) \subseteq \rho \) and the transitive-topological closure \( \tau = \pi(\Sigma) \) is also a subset of the closed congruence \( \rho \). The reverse inclusion \( \rho \subseteq \tau \) will be proved if we establish (by the induction), for each \( \alpha \), the inclusion \( \Sigma_0 |_{\hat{\mathbb{Z}}^x} \subseteq (\pi(\Sigma))_0 \).

Now assume that \( \Sigma_{2\gamma} |_{\hat{\mathbb{Z}}^x} \subseteq (\pi(\Sigma))_{2\gamma} \) and let \( (x^\lambda = x^\mu) \in \Sigma_{2\gamma+1} \), where \( \lambda, \mu \in \hat{\mathbb{Z}} \). Then, there is a finite sequence \( \lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = \mu \in \hat{\mathbb{N}} \) such that \( (x^{\lambda_{i-1}} = x^{\lambda_i}) \in \Sigma_{2\gamma_i} \), for each \( i = 1, \ldots, m \). Since \( \Sigma_{2\gamma_i} \) is a closed congruence on \( \hat{\mathbb{Z}} \) with the limit \( \tau = \pi(\Sigma) \), we obtain the sequence of pseudoidentities from \( \Sigma_{2\gamma+1} \) converging to \( x^\lambda = x^\mu \). Multiplying the pseudoidentities in the sequence by \( x^\omega \) and using the fact that \( \Sigma_{2\gamma+1} \) is a congruence, we obtain the sequence of pseudoidentities from \( \Sigma_{2\gamma+1} |_{\hat{\mathbb{Z}}^x} \) with the limit \( x^\lambda = x^\mu \). Now, using the induction hypothesis we obtain, that \( (x^\lambda = x^\mu) \in (\pi(\Sigma))_{2\gamma+2} \).

To accomplish the proof of the claim by transfinite induction we need to consider a limit ordinal \( \alpha \), but the inclusion follows immediately from the inclusions for smaller ordinals.

Now, we are ready to finish the proof of the theorem. We assumed that \( \lfloor \Sigma \rfloor \) satisfies the pseudoidentity \( x^\varepsilon = x^\delta \), where \( \varepsilon, \delta \in \hat{\mathbb{Z}} \). We need to show that \( (x^\varepsilon = x^\delta) \in \Sigma \), or equivalently written \( (\varepsilon, \delta) \in \rho \). Assume on the contrary that \( (\varepsilon, \delta) \notin \rho \). We have proved the equality of two closed congruences \( \rho \) and \( \tau \). So, we can consider the monoid \( \hat{\mathbb{Z}}/\rho = \hat{\mathbb{Z}}/\tau \), which is a profinite monoid, because it is a quotient of \( \overline{\Omega_1 G} \) by the relation \( \tau = \pi(\Sigma) \) and \( G \) is strong by Corollary 8.2. Thus, there is a continuous homomorphism \( f : \hat{\mathbb{Z}} \to G \) onto a finite (cyclic) group \( G \in \lfloor \pi(\Sigma) \rfloor \subseteq \lfloor \Sigma \rfloor \), such that \( f(\varepsilon) \neq f(\delta) \). Now, we can consider the composition of \( f \) with \( \pi \) and we obtain the continuous homomorphism \( g : \overline{\Omega_1 \text{Com}} \to G \), \( g(x^\lambda) = f(\lambda + \omega) \), such that \( g(x^\varepsilon) \neq g(x^\delta) \). This is a contradiction because the group \( G \in \lfloor \Sigma \rfloor \) must satisfy the pseudoidentity \( x^\varepsilon = x^\delta \). Therefore, we have \( (x^\varepsilon = x^\delta) \in \Sigma \) and the proof is finished. \( \square \)

To complete the program already followed in Sections 8 and 9 we establish missing strongness property for the pseudoidentity \( xy = yx \). It was not
established earlier, namely as part of Proposition 6.3 because our argument depends on Corollary 8.4.

**Proposition 10.3.** The pseudoidentity of monoids $xy = yx$ is $t$-strong.

**Proof.** Let $\Sigma$ be a set of M-pseudoidentities such that all monoids in $[\Sigma]$ are commutative. We need to show that $\Sigma$ proves $xy = yx$.

In the terminology introduced at the end of Section 6 by a theorem of Margolis and Pin [30], a complete set of excluded monoids for the pseudoidentity $xy = yx$ is given by all non-Abelian groups together with $N^1$, $B(1, 2)^1$, and $B(2, 1)^1$. Hence, each such monoid fails some pseudoidentity from $\Sigma$. Actually, since $xy = yx$ entails $x^\omega y = yx^\omega$ and the latter pseudoidentity is $t$-strong by Proposition 6.3(xi), we already know that $\Sigma$ proves $x^\omega y = yx^\omega$.

On the other hand, as $N^1$ fails some pseudoidentity $u = v$ from $\Sigma$, either $\Sigma$ proves $x^\omega = 1$ or $N^1$ fails a two-variable pseudoidentity provable from $\Sigma$ by evaluating $x$ as $a$ and $y$ as $b$. Taking into account that the only nonzero product involving at least two factors using either $a$, $b$, or both is $ab$, we deduce that $\Sigma$ proves either $x = x^{\omega+1}$ or a nontrivial pseudoidentity of the form $xy = w$. If $u$ is not $yx$, then one may substitute one of the variables by $1$ to get a nontrivial pseudoidentity of the form $x = x^\omega$, which again entails $x = x^{\omega+1}$ by Lemma 6.1. Hence, $\Sigma$ deduces either $x = x^{\omega+1}$ or $xy = yx$, which means that we may as well assume that $\Sigma$ proves the former pseudoidentity. Thus, $\Sigma$ proves the pseudoidentities

\[
\begin{align*}
(17) & \quad x^{\omega+1} = x \\
(18) & \quad x^\omega y = yx^\omega,
\end{align*}
\]

which are known to define the join $Sl \lor G$, that is, the pseudovariety of all finite monoids that are semilattices of groups [2, Exercise 9.1.4]. Since it is not so easy to deduce pseudoidentities from $\Sigma$ from the knowledge that non-Abelian groups fail some pseudoidentity in $\Sigma$, we proceed instead to reduce our problem to the group case and apply Corollary 8.4 to draw the conclusion that $\Sigma$ proves $xy = yx$.

First, we exhibit other provable consequences of $\Sigma$. Since we deal with the pseudoidentity $xy = yx$, we work over the alphabet $A = \{x, y\}$ only, even though more general pseudoidentities may be handled similarly. In fact we list some useful pseudoidentities that are provable from (17) and (18).

We have $(xy)^\omega = x^\omega (xy)^\omega y^\omega = y^\omega (xy)^\omega x^\omega$, where we used (17) first and (18) in the second step. Then we get $y^\omega (xy)^\omega x^\omega = y^\omega-1 (yx)^{\omega+1} x^{\omega-1} = y^\omega-1 y x x^{\omega-1} = y^\omega x^{\omega-1} = x^\omega y^{\omega-1}$, where the first and third equality hold in $\overline{\Omega}_A\mathcal{M}$ and the second and fourth equality follow respectively from (17) and (18).

Now, for each word $w \in A^*$ containing both variables $x$ and $y$, if we use the pseudoidentities $(xy)^\omega = x^\omega y^\omega$ and (18) repeatedly, we may prove $w^\omega = x^\omega y^\omega$. Hence, from (17) and (18) we may prove $w^\omega x^\omega y^\omega = x^\omega y^\omega$ for every pseudoword $w$ over the alphabet $\{x, y\}$.

Recall that $\Sigma$ is a relation on $\overline{\Omega}_A\mathcal{M}$. Let $\Gamma = \Sigma \cup \{x^\omega = 1\}$ and consider $\bar{\Gamma}$ on the same monoid $\overline{\Omega}_A\mathcal{M}$. Then $[\Gamma] = [\Sigma] \cap G$. Applying Corollary 8.4 we obtain that $(xy = yx) \in \bar{\Gamma}$. We claim, for any pseudoidentity $u = v$, that $(u = v) \in \bar{\Gamma}$ implies $(ux^\omega y^\omega = vx^\omega y^\omega) \in \Sigma$. This gives the proof of the
statement because the pseudoidentities $xy = yx^ωy^ω$ and $yx = yxx^ωy^ω$ are provable from (17) and (18).

The claim will be proved if we show, for every ordinal $\alpha$, the following implication:

$$(u = v) \in \Gamma_\alpha \implies (ux^\omega y^\omega = vx^\omega y^\omega) \in \tilde{\Sigma}.$$  

Let $\alpha = 0$ and $(u = v) \in \Gamma_0$. If we use in the proof of $u = v$ a pseudoidentity from $\Sigma$, then $(u = v) \in \Sigma_0$ and therefore also $(ux^\omega y^\omega = vx^\omega y^\omega) \in \Sigma_0$. So, we may assume, without loss of generality, that $u = t(\varphi(x^\omega), w_1, \ldots, w_n), v = t(\varphi(1), w_1, \ldots, w_n)$, where $\varphi : \Omega_A \to \Omega_A$ is a continuous homomorphism, $t$ is a word and $w_i \in \Omega_A$ ($i = 1, \ldots, n$). Using (18) and the fact that $(\varphi(x))^\omega x^\omega y^\omega = x^\omega y^\omega$ is $\Sigma$-provable, we get that $(ux^\omega y^\omega = vx^\omega y^\omega) \in \tilde{\Sigma}$.

The other steps of the induction proof are easy to see as $\tilde{\Sigma}$ is a closed congruence.

As in Sections 8 and 9, we may now apply Proposition 4.2 to obtain the following result.

**Corollary 10.4.** For $U = M$, every set $\Sigma$ of pseudoidentities defining a pseudovariety of commutative monoids is $h$-strong.

11. Conclusion

We have introduced a natural and sound proof scheme for pseudoidentities. We have given ample evidence for the conjecture that our proof scheme is complete. There is a nice connection with the much studied notions of reducibility and tameness. In fact, in all our examples built on reducible pseudovarieties, we have shown that $\Sigma_n = \tilde{\Sigma}$ for some integer $n$. But, in Proposition 7.9, we also established that $n = 2$ is not enough. It is expectable that in general there may be no $n \geq 1$ such that $\Sigma_n = \tilde{\Sigma}$, or perhaps even $\Sigma_\omega \neq \tilde{\Sigma}$, but we have found no example in which that is the case.

We have not tried to treat exhaustively all cases of pseudovarieties which are known to be tame. Those that we have considered are perhaps those that appear more frequently in the literature. Of course, since we believe in our conjecture, more such results would only be adding evidence to our claim that we have identified a completely general phenomenon.

It is the knowledge of congruences on suitable relatively profinite monoids or semigroups that allowed us to prove that the pseudovarieties $G$, $CS$, and $Com$ are strong. Much is also known about the congruences on $\Omega_A CR$ but we have not taken the natural path of trying to prove that the pseudovariety $CR$ is strong.

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