FRW Cosmological model with Modified Chaplygin Gas and Dynamical System

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Abstract

The Friedmann-Robertson-Walker (FRW) model with dynamical Dark Energy (DE) in the form of modified Chaplygin gas (MCG) has been investigated. The evolution equations are reduced to an autonomous system on the two dimensional phase plane and it can be interpreted as the motion of the particle in an one dimensional potential.

Keywords : Dynamical System, Phase plane, FRW Cosmology, Modified Chaplygin Gas.

1 Introduction

The exciting observational evidences [1, 2] in the last decade raise a challenge to the standard cosmology. To incorporate the present accelerating phase of the universe within the framework of Einstein gravity, one has to introduce a non-gravitating type of matter with a hugely negative pressure (of the order of its energy density) called dark energy (DE hereafter). For the mysterious DE, there are only very weak constrains on its form of an equation of state [2, 3]. The most common candidate for DE is the cosmological constant $\Lambda$, which can be considered as a perfect fluid with equation of state (EoS) $p = -\rho$, $\rho = \Lambda$. But it is discarded due to inconsistency in the value of $\Lambda$ from type Ia supernovae (SNIa) observations in comparison with the value of $\Lambda$ interpreted as vacuum energy (i.e., the Planck mass scale). So usually, investigations in DE are modelled as quintessence scalar field [4] or field with barotropic equation of state [5]. But the transition from a universe filled with matter to an exponentially expanding universe does not necessarily require the presence of a scalar field as the only alternative. Subsequently, attempts were made to use an exotic type of fluid—the so-called Chaplygin gas (CG) having the EoS $p = -\frac{B}{\rho}$, and then it is extended as generalised CG (GCG) $p = -\frac{B}{\rho^\alpha}$, $B > 0$, $0 \leq \alpha \leq 1$. It is further generalised to modified CG (MCG) with EoS [6, 7]

$$p = \gamma \rho - \frac{B}{\rho^\beta}$$

with $\gamma, \beta > 0$ and $0 < \alpha \leq 1$.

This EoS shows a radiation era ($\gamma = \frac{1}{3}$) at one extreme (when scale factor $a(t)$ is vanishingly small) and a $\Lambda$CDM model at the other extreme (when $a'$ is infinitely large). So at all stages it shows a mixture. Also at an intermediate stage the pressure vanishes and the matter content is equivalent to pure dust.

In the present work, we formulate the dynamics of FRW cosmology with MCG as the matter in it and the evolution equations are shown to be represented in two dimensional autonomous system by suitable transformation of variables. The nature of the critical points are analysed by evaluating the eigen values of the linearized Jacobi matrix. The paper is organized as follows: Basic equations for FRW cosmology with MCG are presented and dynamical system is formulated in section 2. Both finite and asymptotic critical points are analysed in section 3. The paper ends with a short discussion at the end, in the section 4.
2 Basic Equations for FRW cosmology with MCG

The Friedmann equations which govern the dynamics are given by (assuming $8\pi G = 1 = c$)

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\rho}{3}$$

(2)

and

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -p$$

(3)

and the energy conservation relation reads

$$\dot{\rho} + 3\frac{\dot{a}}{a} (\rho + p) = 0$$

(4)

From the above field equations the acceleration has the expression

$$\ddot{a} a = -\frac{1}{6} (\rho + 3p)$$

(5)

Using the MCG EoS in the above conservation equation and integrating we have

$$\rho(a) = \left[ \frac{1}{(\gamma + 1)} \left( B + \frac{c}{a^\mu} \right) \right]^{\frac{1}{\gamma + 1}}$$

(6)

where $\mu = 3(\gamma + 1)(\alpha + 1)$ and $c$ is an integration constant.

So the EoS parameter can be written as

$$\omega(a) = \frac{p}{\rho} = \frac{\gamma c - Ba^\mu}{c + Ba^\mu}$$

(7)

To form an autonomous system from the dynamical equations let us introduce a new variable

$$x = \dot{a}$$

then we have

$$\begin{align*}
\dot{a} &= x \\
\dot{x} &= -\frac{1}{2} \left( 1 + 3\omega(a) \right) \rho(a)
\end{align*}$$

(8)

Thus set of equations (8) form an autonomous system in the phase plane $(a, x)$. Further using the phase space variables the field equation (2) can be written as

$$\frac{1}{2} x^2 + V(a) = -\frac{k}{2}$$

(9)

with

$$V(a) = -\frac{a^2 \rho(a)}{6}.$$  

(10)

Equation (9) is termed as the first integral of the solution of the autonomous system. Also geometrically solutions of the dynamical system (8) should lie on the curve given by equation (9) on the phase plane.

3 Critical points : Analysis of Dynamical System

The critical points on the phase plane have coordinates

$$(0, 0) \text{ and } a_c = \left[ \frac{c(1 + 3\gamma)}{2B} \right]^{\frac{1}{\mu}}, 0$$

(11)
and both lie on the $a$-axis. It is to be noted that the critical points correspond to static universe(s). Moreover, the first one corresponds to big-bang singularity and so we will concentrate mainly on the second critical point. As critical point lies on the curve (9) on the phase plane so we must have $V(a) = -\frac{k}{2}$ i.e., $k = +1$ is the only possibility. Hence we can say that the critical points lie in the portion of the phase plane represented by the trajectories of the closed model. Moreover, the first integral corresponding to flat model, i.e., $\frac{\dot{x}^2}{2} + V(a) = 0$ divides the phase plane into two regions namely $\frac{\dot{x}^2}{2} + V(a) \geq 0$ or $\leq 0$ which correspond to open and closed models respectively. Further the boundary of the strong energy condition is the straight line $a = \left[ \frac{c(1+3\gamma)}{2B} \right]^{\frac{1}{\alpha}}$ parallel to the x-axis and the second critical point lies on the line. The left of this line is the region in the phase space corresponds to decelerating phase (strong energy condition is satisfied) while the accelerating phase of evolution (strong energy condition is violated) is characterized by the right hand region of the above line \[8\].

Now we shall examine the character of the second critical point by evaluating the eigen values of the following linearization matrix $A$ of the autonomous system:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{\partial}{\partial a} \left[ -\frac{\partial}{\partial a} \rho(a) \{1 + 3\omega(a)\} \right] & 0 \end{bmatrix}_{at \ (a=\alpha_c, x=0)}.$$ 

So $Tr(A) = 0$ and $det(A) = \frac{1}{2}a_c\rho(a_c)\frac{\partial \omega}{\partial a}|_{a=a_c}$ with $\rho(a_c) = \left( \frac{3B}{1+3\gamma} \right)^{\frac{1}{\alpha}}$ and

$$\frac{\partial \omega}{\partial a} = -\frac{B\mu_c(1+\gamma)}{(C+Ba_c^\alpha)}.$$

Then the eigen value problem, namely $det [A - \lambda I] = 0$, i.e., $\lambda^2 - \lambda \times Tr(A) + det(A) = 0$ simplifies to $\lambda^2 + det(A) = 0$.

Thus we have

(i) real eigen values of opposite sign if $det(A) < 0$.

(ii) purely imaginary conjugate eigenvalues if $det(A) > 0$.

In the first case the critical point is a saddle point while it is a centre in the other case. As in the present case
$det(A) < 0$ so we have critical point which is saddle in nature and hence it is unstable in character \([9]\) as shown in figures 1(a),(b) for different choices of the parameters.

To complete the study of the dynamical system \([9]\) we now examine the critical points at infinity. To do this we make the following coordinate transformation in the phase plane to cover a circle $S'$ at infinity:

$$ (a, x) \to (p, q) : p = \frac{1}{a}, q = \frac{x}{a}; p = 0, -\infty < q < +\infty $$ (14)

Thus the autonomous system \([8]\) transforms to

$$ \begin{align*}
\dot{p} &= -pq \\
\dot{q} &= -\frac{c}{6} \{1 + 3\omega\} - q^2
\end{align*} $$ (15)

and the first integral becomes

$$ q^2 + 2p^2V \left( \frac{1}{p} \right) = -kp^2 $$ (16)

The critical points on the circle at infinity for the autonomous system \([15]\) are

$$ \begin{align*}
p_c &= 0 \\
q_c &= \pm \sqrt{-\frac{c}{6} \{1 + 3\omega\}}
\end{align*} $$ (17)

which shows that critical points at infinity exists only in the portion of the phase plane where strong energy condition is violated (i.e., in the accelerating domain). It should be noted that for the autonomous system \([15]\) there are other critical points in the finite domain which are already taken in to account.

The linearization matrix which characterizes the nature of the critical points \([17]\) is given by

$$ A = \begin{bmatrix}
-q_c & 0 \\
-\frac{1}{6} \frac{d}{dp} (\rho (1 + 3\omega)) & -2q_c
\end{bmatrix} $$ (18)

So clearly, $Tr(A) = -3q_c$ and let $det(A) = 2q_c^2$ and the eigen values are $-q_c$ and $-2q_c$, i.e., eigenvalues are real and of same sign. So both the critical points are node and they are asymptotically stable when $q_c > 0$ and are unstable for $q_c < 0$.

4 Discussion

In this work we study the FRW comology with MCG as the matter contained. The evolution equations are reduced into an autonomous dynamical system with a suitable change of variables. The dynamical system has a first integral and solutions to the system should lie on the curve represented by the first integral in the phase plane. Also the first integral can be interpreted as the energy conservation relation of a particle moving in an one-dimensional potential. Except the big bang singularity, there is only one critical point for the system which is saddle type and hence unstable in nature. Also there are two critical points on the circle at infinity and are node type asymptotically stable or unstable depending on the sign of the coordinate of the critical point. Finally, it should be noted that strong energy condition has a vital role for existence and type of critical points both at finite domain and at infinity.

Acknowledgement:
RB and NM are thankfull to State Govt. of West Bengal, India and CSIR, India respectively. All the authors are thankful to IUCAA, Pune as this work was done there during a visit.
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