One a way for Constructing Hybrid Methods with the Constant Coefficients and their Applied

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Abstract. As is known, the model problem for some scientific and technical problems is formulated with the help of the integro-differential equations of Volterra type of the second order. And for solving of such equations, numerical methods are mainly used. Numerical solution of nonlinear integro-differential equations of Volterra type has been studied relatively little. Therefore, here we consider the construction and application of numerical methods to solving Volterra's nonlinear integro-differential equations. Here, are constructed specific methods which are applied to solving of the model problem and are proved the advantage of the proposed method.

1. Introduction

It is known that the solution of higher-order integro-differential equations can be reduced to solving a system of integro-differential equations of Volterra type of the first order. Thus, the question of the numerical solution of higher-order integro-differential equations can be considered exhausted. But, we will show here, that specially constructed methods for solving integro-differential Volterra equations are more accurate than the known ones.

Let us consider the following initial-value problem for the nonlinear Volterra integro-differential equation, which consists in finding the solution of the next integro-differential equation:

\[ y' = f(x, y, y') + \int_{s_0}^{x} K(x, s, y(s)) ds, \quad x_0 \leq s \leq x \leq X, \quad (1) \]

satisfying the following initial conditions:

\[ y(x_0) = y_0, \quad y'(x_0) = y'_0. \quad (2) \]

We assume that the problem (1) - (2) has a unique solution defined on an interval \([x_0, X]\). The aim of this paper is to construct and apply the numerical methods to finding the values of the solution of problem (1) - (2). Therefore, the segment \([x_0, X]\) is divided into \(N\) equal parts with the mesh points \(x_i = x_0 + ih \quad (i = 0, 1, 2, ..., N)\). Here the positive parameter \(h\) is the step size.
We denote by the $y_i$ approximately, and the $y_i(x, (i = 0, 1, ..., N)$ exact value of the solution of the problem (1) - (2) at the mesh points $x_i (i = 0, 1, ..., N)$. Usually, depending from the accuracy of the constructed method set some conditions on the functions $f(x, y, z)$ and $K(x, y, z)$. In this connection, we assume that functions $f(x, y, z)$ and $K(x, y, z)$ continuous on the totality of arguments and defined in some closed set, where they have continuous partial derivatives up to order $P$, inclusively (here the integer value of $P$ is the order of accuracy for the used method).

As is known, the first approximate method for solving of the problem (1) - (2) was constructed by Vito Volterra (see [13]), which consisted in replacing the integral participating in equation (1) by the quadrature formula. There are numerous scientific papers devoted to the investigation and application of quadrature formulas to solving of problem (1) - (2) (see e.g. [2] - [4].) However, there are papers in which to solving of the problem (1) - (2) other methods are used, such as the spline function, the collocation method, etc. (see e.g. [1], [13], [15], [16].) Here, for solving problem (1) - (2) are proposed the hybrid methods constructed by using the multi-step methods with the constant coefficients and forward-jumping methods. As is known, hybrid methods are constructed using one-step and multi-step methods (see e.g. [17] - [23]).

Note that if the function $f(x, y, z)$ is independent from the variable $z$, then it is possible to construct effective methods, such as the methods of Stoermer or Numerov. But in this case the order of accuracy for these methods will be low. Obviously, in this case it is possible to use the methods proposed for solving of the problem (1) - (2), but in this case there arises the need to calculate the values $y'_m (m = 1, 2, ..., )$. We now consider the construction of numerical methods for solving of problem (1) - (2).

2. The construction of methods at the junction of implicit and hybrid methods.

It is obvious that equation (1) can be rewritten in the form:

$$y' = f(x, y, y') + v(x),$$

where $v(x) = \int K(x, s, y(x))ds$.

If the function $v(x)$ is known, then the equation (3) is an ODE of the second order, resolved to the highest derivative.

In this case, there are wide classes of methods for solving problem (3) - (2). One of these methods can be presented in the following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + \frac{h^2}{2} \sum_{i=0}^{k} \gamma_i y''_{n+i}$$

(4)

Here the coefficients, $\alpha_i, \beta_i, \gamma_i (i = 0, 1, ..., k)$ are some real numbers that satisfy conditions A, B, C from [24]. Note that the method (4) is impossible to solve problems (3) - (2) which the fundamentally investigated by Dahlquist (see [24]). It is easy to understand that for application of the method (4), to solving problem (3) - (2) are known the values $y'_m (m = n, n+1, ..., n+k)$. To calculate these values, there are different methods. One of them has the following form:

$$\sum_{i=0}^{k} \alpha'_i y'_{n+i} = h \sum_{i=0}^{k} \beta'_i y''_{n+i} (i = 0, 1, ..., k)$$

(5)

The coefficients $\alpha'_i, \beta'_i (i = 0, 1, ..., k)$ satisfy the conditions A, B, C from [25].

One of the basic characteristic of numerical methods for applying that to solving a particular problem is its stability. The stability for the methods (4) and (5) can be determined in the following form:

Definition 1. The method (4) is stable if the roots of the characteristic polynomial

$$\rho(\lambda) = \alpha_k \lambda^k + \alpha_{k-1} \lambda^{k-1} + ... + \alpha_1 \lambda + \alpha_0$$

lie inside the unit circle, on the boundary of which there are no multiple roots.
Obviously, method (5) is a particular case of method (4). Therefore, the stability for the method (5) can be established by definition 1.

As is known, when comparing approximate methods, their order of accuracy is mainly used. Since \( k \) is the order for the methods (4) and (5), Dahlquist used the notion of the degree for methods to determine the order of accuracy of the methods (4) and (5) and defined it in the following form:

Definition 2. An integer \( p \) is the degree of the method (5) if the following is holds:

\[
\sum_{i=0}^{k} \alpha_i y(x + i h) - h \sum_{i=0}^{k} (\beta_i y'(x + i h) + h \gamma_i y''(x + i h)) = O(h^{p+1}), \quad h \to 0.
\]

It is known that if the method (4) is stable, then \( p \leq 2k + 2 \). And if the method (5) is stable, then \( p \leq 2(k/2) + 2 \) (see [24], [25]). In [26] it is proved, that if the method (5) is stable and has the degree \( p_i \), and method (4) is stable and has a degree \( p \), then the method constructed with the help of these methods converges and the degree is not greater than \( p_i + 1 \) (we assume that \( p > p_i + 1 \)). It follows that for the solution of problem (3) and (2), the main question is the selection of a method for calculating the values of a quantity \( y_m' \) \((m \geq 0)\). To increase the degree of the method applied to the calculation of the values of the quantities \( y_m' \), one can use the forward-jumping method, which in a more general form can be written as:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i} + h^2 \sum_{i=0}^{m} \gamma_i y_{n+i}, \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (7)
\]

In [27], the maximum value for the degree of the method (6) was determined in the cases \( \phi = 0 \) and \( \phi \neq 0 \), where \( \phi = |y_0| + |y_1| + \ldots + |y_m| \). Here, to find the values of the quantity \( y_m' \), the following method is supposed to be used:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i} + h^2 \sum_{i=0}^{m} \gamma_i y_{n+i}^*, \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (8)
\]

It can be shown that in the class of methods (7) there are stable methods with degree \( p > 2k + 2 \). It is obvious that the method (8) is hybrid, if \( |y_0| + |y_1| + \ldots + |y_j| \neq 0 \). Therefore, if \( \beta_i = 0 \) \((i = 0, 1, \ldots, k)\), then the method (7) is hybrid. In the use of hybrid methods, the main difficulties lie in the construction of the methods for calculating the values of a quantity \( y_{n+i} \) \((0 \leq j \leq k)\). A numerous papers of different authors have been devoted to the investigation of hybrid methods (see, for example, [17] - [23], [28] - [30]). Note that to increase the order of accuracy for the method (4), it can be modified using of the values of quantities \( y_{n+i} \), \((\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \). For example, in the following form:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i} + h^2 \sum_{i=0}^{m} \gamma_i y_{n+i}^*, \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (9)
\]

However, the use methods of type (8) are more complicated than using methods of the type (4). In order to construct a simpler algorithm, to solving of the problem (3) and (2) can be used the following method:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i} + h^2 \sum_{i=0}^{m} \gamma_i y_{n+i}^*, \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (\gamma_i)_{i=0}^{\infty} \in \mathbb{R}^n \quad (10)
\]

by the addition of method (7). It is easy to show that the class of methods of type (7) and type (9) coincide. Therefore, when compiling an algorithm for solving the problem (3) and (2), one can use the same method to find the values of the quantities \( y_{n+i} \) and \( y_{n+i}' \) \((n = 0, 1, \ldots, N - k)\).

Note that the above mentioned methods are easily applied to solving of problem (3) and (2) in the case of \( n(x) = 0 \).
Therefore, we consider the application of the above mentioned methods to solving of the problem (3) and (2) in the case, when \( v(x) \neq 0 \). If method (4) is applicable to solving of the equation (3), then we have:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma'_{i} y_{n+i}, (10)
\]

where the discrete functions \( f_n \) and \( v_n \) are defined in the form:

\[
f_m = f(x_m, y_m, y'_m), \quad v_m = \int_{x_0}^{x_2} K(x_m, s, y(s))ds \quad (m = 0, 1, 2, \ldots).
\]

Obviously, if the values of the quantities \( y_0, y_1, \ldots, y_{k-1}, y'_0, y'_1, \ldots, y'_{k-1}, v_0, v_1, \ldots, v_{k-1} \), are known, then by using the method (10), we can calculate the values of the quantities \( y_k \). In this case, depending by from the values of the coefficients \( \beta_i \) and \( \gamma_i \) there may be requirements for calculating the quantities \( y'_k \) and \( v_k \). For this purpose, one can use the following method:

\[
y'_{n+k} = \sum_{i=0}^{k} \alpha_i y'_{n+i} + h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i v_{n+i} + \sum_{i=0}^{k} \gamma'_{i} y_{n+i}, (11)
\]

\[
v_{n+k} = \sum_{i=0}^{k} \lambda_i v_{n+i} + h \sum_{i=0}^{k} \lambda_i \gamma_i (x_{n+i}, x_{n+i}, y_{n+i}) + \sum_{i=0}^{k} \lambda'_{i} y_{n+i}, (12)
\]

Thus, for solving of the problem (3) and (2), the system of difference methods consisting from the formula (10) - (12) was constructed. Note that in this system formula (10) can be replaced by method (9). Obviously, each of these methods has its advantages and disadvantages. We believe that to find numerical solutions of problem (1) and (2), it is desirable to use methods compiled by using (9), (11), and (12). It is easy to see that in this sequence of methods, the same method can be used to compute \( u \).

We consider a special case when the function \( f(x, y, z) \) is independent of the variable \( z \), i.e. \( f(x, y, z) = \phi(x, y) \). Then for solving of the problem (1) and (2) one can construct a sequence of methods consisting of two formulas. To this end, we use the following hybrid method:

\[
\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i y'_{n+i} + h^2 \sum_{j=0}^{k} \gamma_{j} y_{n+j}, \quad (|\gamma_i| < 1; \quad 0 \leq i \leq k). (13)
\]

Here the coefficients \( \alpha_i, \beta_i, \gamma_i (i = 0, 1, 2, \ldots, k) \) are some real numbers, and \( \alpha_k 
eq 0 \). We note that this concept of stability and degree for method (13) is different (see [24]).

3. Illustration of the results

As is known, for the methods often are compared use a model problem. Therefore, we consider to solving of the following problem:

\[
y'' = \lambda^2 (1 + a - ay(x)) + (1 + a) \lambda^3 \int_{0}^{x} y(s)ds, \quad y(0) = 1, \quad y'(0) = \lambda, \quad 0 \leq x \leq 1,
\]

the exact solution for which can be represented as: \( y(x) = \exp(\lambda x) \) since the right-hand side does not depend on \( y'(x) \), to find the solution of the example we use a method of the type (13). It is easy to obtain the following method:

\[
y'_{n+2} = 2 y_{n+1} - y_n + h^2 (4 y''_{n+1} + y'_{n+1} + 4 y_{n+1} + y_{n+2} + 4 y_{n+2}), \quad \beta = \sqrt[3]{3} / 4, \quad p = 4, (14)
\]

which was applied to solving of our example in the case, when \( a = 0 \). To compare the results obtained, we used the following method of Numerov:

\[
y'_{n+2} = 2 y_{n+1} - y_n + h^2 (7 y''_{n+1} - 10 y''_{n+1} + y_{n+1} + y_{n+2} + y_{n+2} + 12), \quad p = 4.
\]

The results for the Numerov method were placed in Table 1 and for the method (14) in Table 2.
Table 1.

\[a = 1; h = 0.01; \lambda = \pm 1; \pm 5 \pm 10\]

| Variable | \(\lambda = 1\) | \(\lambda = -1\) | \(\lambda = 5\) | \(\lambda = -5\) | \(\lambda = 10\) | \(\lambda = -10\) |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.20     | 2.04E-12        | 2.16E-12        | 2.53E-08        | 3.26E-08        | 8.86E-07        | 1.34E-06        |
| 0.60     | 1.7E-11         | 2.05E-11        | 3.17E-08        | 1.17E-08        | 1.79E-04        | 6.22E-06        |
| 1.00     | 4.1E-11         | 5.53E-11        | 2.86E-06        | 1.97E-07        | 2.2E-02         | 1.84E-05        |

Table 2.

\[a = -1; h = 0.01; \lambda = \pm 1; \pm 5 \pm 10\]

| Variable | \(\lambda = 1\) | \(\lambda = -1\) | \(\lambda = 5\) | \(\lambda = -5\) | \(\lambda = 10\) | \(\lambda = -10\) |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.20     | 5.66E-13        | 4.92E-13        | 1.28E-08        | 6.5E-09         | 1.49E-06        | 4.16E-07        |
| 0.60     | 6.23E-12        | 4.17E-12        | 4.31E-07        | 8.13E-08        | 3.04E-04        | 2.52E-05        |
| 1.00     | 2.12E-11        | 1.10E-11        | 5.76E-06        | 6.11E-07        | 2.88E-02        | 1.37E-03        |

4. Conclusions.
We have here proposed several methods for solving problem (1) - (2). Obviously, each of them has its own shortcomings and advantages. With the help of concrete methods it was proved that stable methods of hybrid type are more accurate than the known ones. It is known that more accurate methods do not always give the best result, which is related to their area of stability. Therefore, for comparison, the results obtained by well-known Numerov methods were used. Because the methods (14) and (15) have the same degree and are stable. Comparing the above results, we find that the methods proposed here are promising.

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6. References
[1] Volterra V. Theory of functional and of integral and integro-differential equations, Moskow, Nauka, 1982.
[2] A.V. Manzhirov, A.D. Polyanin Handbook of Integral Equations: Methods of solutions. Moscow: Publishing House of the "Factorial Press", 2000, 384 p.
[3] Verlan A.F, Sizikov V.S. Integral equations: methods, algorithms, programs. Kiev, Naukova Dumka, 1986.
[4] Brunner H. The Solution of Volterra integral equations of the first kind by piecewise polynomials. - J.Inst.Math. and Appl., 1973,12,No3, p.295-302.
[5] Baker C.T.H. The numerical treatment of integral equations.- Oxford: Claerdon, 1977. - 1034p.
[6] Lubich Ch. Runge-Kutta theory for Volterra and Abel Integral Equations of the Second Kind. Mathematics of computation, volume 41, number 163, July 1983, p. 87-102.
[7] Linz P. Linear Multistep methods for VolterraIntegro-Differential equations. Journal of the Association for Computing Machinery, Vol.16, No.2, April, 1969, 295-301.
[8] Feldstein, A., &Sopka, J.R. Numerical methods for nonlinear Volterra integro differential equations. SIAM J. Numer. Anal. V. 11, 1974, 826-846.
[9] MehdiyevaG.Yu., Imanova M.N., Ibrahimov V.R. On one application of forward jumping methods. Applied Numerical Mathematics, Volume 72, October, 2013, 234-245.
[10] Mehdiyeva G., Imanova M., Ibrahimov V. Solving VolterraIntegro-Differential Equation by the Second Derivative Methods, Applied Mathematics and Information Sciences, Volume 9, No. 5, Sep. 2015, pp. 2521-2527.

[11] Makroglou A.A. Block - by-block method for the numerical solution of Volterra delay integro-differential equations, Computing 3, 1983, 30, №1, p.49-62.

[12] Mehdiyeva G., Ibrahimov V., Imanova M. Some refinement of the notion of symmetry for the Volterra integral equations and the construction of symmetrical methods to solve them, Journal of Computational and Applied Mathematics, 306 (2016), 1–9.

[13] Makroglou A. Hybrid methods in the numerical solution of Volterraintegro-differential equations. Journal of Numerical Analysis 2, 1982, 21-35

[14] Mehdiyeva G., Imanova M., Ibrahimov V. An Application of Mathematical Methods for Solving of Scientific Problems, British Journal of Applied Science & Technology - Volume 14, issue 2, 2016, 1-15.

[15] Brunner H. ImplicitRunge-Kutta Methods of Optimal order for Volterraintegro-differential equation. Mathematics of computation, Volume 42, Number 165, January 1984, 95-109

[16] Bulatov M.V, Chistyakov E.V Numerical solution of integro-differential systems with a degenerate matrix before the derivative by multi-step methods. Differents. Equations, 2006, Vol. 42, No. 9, p. 1248-1255.

[17] Butcher J.C. A modified multistep method for the numerical integration of ordinary differential equations. J. Assoc. Comput. Math., v.12, 1965, 124-135.

[18] Gear C.S. Hybrid methods for initial value problems in ordinary differential equations. SIAM, J. Numer. Anal. v. 2, 1965, 69-86.

[19] Mehdiyeva G., Imanova M., Ibrahimov V. A way to construct an algorithm that uses hybrid methods. Applied Mathematical Sciences, HIKARI Ltd, Vol. 7, 2013, no. 98, p.4875-4890.

[20] Mehdiyeva G., Imanova M., Ibrahimov V. A way for constructing hybrid methods with high order of accuracy and their application to solving odes of first order, International Journal of Current Research, Volume 7, Issue 08, August, 2015, p.19379-19382.

[21] Mehdiyeva G., Imanova M., Ibrahimov V. Application of hybrid methods to solving Volterra integral equation with the fixed boundaries, International journal of advance research and innovative ideas in education, volume-1, issue-5, 2015, 491-502.

[22] Mehdiyeva G., Ibrahimov V., Imanova M. On One Application of Hybrid Methods for Solving Volterra Integral Equations World Academy of Science, engineering and Technology, Dubai, 2012, 809-813.

[23] Skvortsov L.M. Explicit two-step Runge-Kutta methods Math. modeling, 21, 9 (2009), P. 54-65.

[24] Dahlquist G. Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations. Trans. Of the Royal Inst. Of Techn. Stockholm, Sweden, 1959, №130, p.3-87.

[25] Dahlquist G. Convergence and stability in the numerical integration of ordinary differential equations. Math. Scand. 1956, №4, p.33-53.

[26] Ibrahimov V. On the maximal degree of the k-step Obrechkoff’s method. Bulletin of Iranian Mathematical Society, Vol.28, 2002, №1, p. 1-28.

[27] V.R. Ibrahimov On a relation between order and degree for stable forward jumping formula. Zh. Vychis. Mat. № 7, 1990, p.1045-1056.

[28] Gupta G.K. A polynomial representation of hybrid methods for solving ordinary differential equations, Mathematics of comp., volume 33, number 148, 1979, 1251-1256.

[29] Aree E.A., Ademiluyi R.A., Babatola P.O. Accurate collocation multistep method for integration of first order ordinary differential equations. J.of Modern Math.and Statistics, 2(1), 1-6, 2008, p.1-6.

[30] Mehdiyeva G., Imanova M., Ibrahimov V. (2011). Application of the hybrid methods to solving Volterraintegro-differential equations. World Academy of Science, engineering and Technology, Paris, 1197-1201.