On Explicit Point Multi-Monopoles in SU(2) Gauge Theory

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Abstract

It is well known that the Dirac monopole solution with the U(1) gauge group embedded into the group SU(2) is equivalent to the SU(2) Wu-Yang point monopole solution having no Dirac string singularity. We consider a multi-center configuration of $m$ Dirac monopoles and $n$ anti-monopoles and its embedding into SU(2) gauge theory. Using geometric methods, we construct an explicit solution of the SU(2) Yang-Mills equations which generalizes the Wu-Yang solution to the case of $m$ monopoles and $n$ anti-monopoles located at arbitrary points in $\mathbb{R}^3$.

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1 Introduction

Abelian magnetic monopoles play a key role in the dual superconductor mechanism of confinement [1] which has been confirmed by many numerical simulations of the lattice gluodynamics (see e.g. [2, 3] and references therein). Due to a dominant role of abelian monopoles in the confinement phenomena, it is important to understand better how do they arise in nonabelian pure gauge theories.

A spherically-symmetric monopole solution of the SU(2) pure gauge field equations was obtained by Wu and Yang in 1969 [4]. This solution is singular at the origin and smooth on $\mathbb{R}^3 - \{0\}$. Initially it was thought that it is genuinely nonabelian, yet later it was shown [5] that this solution is nothing but the abelian Dirac monopole [6] in disguise. Note that the gauge potential of the finite-energy spherically symmetric 't Hooft-Polyakov monopole [7] approaches just the Wu-Yang gauge potential for large $r^2 = x^a x^a$.

In this note, we generalize the Wu-Yang solution to a configuration describing $m$ monopoles and $n$ anti-monopoles with arbitrary locations in $\mathbb{R}^3$. This explicit solution to the Yang-Mills equations can also be used as a guide to the asymptotic $r \to \infty$ behaviour of unknown finite-energy solutions in Yang-Mills-Higgs theory, whose form for small $r$ is determined by multiplying the solution by arbitrary functions and minimizing the energy functional, as was proposed in [8].

2 Generic U(1) configurations

We consider the configuration of $m$ Dirac monopoles and $n$ anti-monopoles located at points $\vec{a}_i = \{a^1_i, a^2_i, a^3_i\}$ with $i = 1, \ldots, m$ and $i = m + 1, \ldots, m + n$, respectively. There are delta-function sources for the magnetic field at these points.

Let us introduce the following two regions in $\mathbb{R}^3$:

$$R^3_{N,m+n} := \mathbb{R}^3 - \bigcup_{i=1}^{m+n} \left\{ x^1 = a^1_i, x^2 = a^2_i, x^3 \leq a^3_i \right\},$$

$$R^3_{S,m+n} := \mathbb{R}^3 - \bigcup_{i=1}^{m+n} \left\{ x^1 = a^1_i, x^2 = a^2_i, x^3 \geq a^3_i \right\}. \quad (1)$$

For simplicity we restrict ourselves to the generic case

$$a_i^{1,2} \neq a_j^{1,2} \quad \text{for} \quad i \neq j, \quad (2)$$

when

$$R^3_{N,m+n} \cup R^3_{S,m+n} = \mathbb{R}^3 - \{ \vec{a}_1, \ldots, \vec{a}_{m+n} \}, \quad (3)$$

and the two open sets are enough for describing the above $(m, n)$-configuration. Namely, the generic configuration of $m$ Dirac monopoles and $n$ anti-monopoles is described by the gauge potentials

$$A^{N,m+n} = \sum_{j=1}^{m} A^{N,j} + \sum_{j=m+1}^{m+n} \bar{A}^{N,j} \quad \text{and} \quad A^{S,m+n} = \sum_{j=1}^{m} A^{S,j} + \sum_{j=m+1}^{m+n} \bar{A}^{S,j}, \quad (4)$$

where $A^{N,m+n}$ and $A^{S,m+n}$ are well defined on $R^3_{N,m+n}$ and $R^3_{S,m+n}$, respectively. Here

$$A^{N,j} = A^{N,j}_{a} dx^a \quad \text{with} \quad A^{N,j}_{1} = \frac{i x_j^2}{2 r_j (r_j + x_j^3)}, \quad A^{N,j}_{2} = -\frac{i x_j^1}{2 r_j (r_j + x_j^3)}, \quad A^{N,j}_{3} = 0, \quad (5)$$
\[ A^S,j = A^S,j \, dx^a \quad \text{with} \quad A^1_j = -\frac{ix^2_j}{2r_j(r_j - x^2_j)} , \quad A^2_j = \frac{ix^1_j}{2r_j(r_j - x^2_j)} , \quad A^3_j = 0 , \quad (6) \]

where \[ x^c_j = x^c - a^c_j , \quad r^2_j = \delta_{ab} x^a_j x^b_j , \quad a, b, c = 1, 2, 3 , \quad (7) \]
and \[ A^{N,j} = -A^{N,j} , \quad A^{S,j} = -A^{S,j} . \] On the intersection \( R^3_{N,m+n} \cap R^3_{S,m+n} \) we have

\[ A^{N,m+n} = A^{S,m+n} + d \ln \left( \prod_{i=1}^{m+n} \frac{\bar{y}_i}{y_i} \right) , \quad (8) \]

where \( y_j = x^1_j + ix^2_j \) and bar denotes a complex conjugation.

**Remark.** Note that in the case when \( a^1_{i,2} = a^1_{j,2} \) for some \( i \neq j \), one has to introduce more than two open sets covering the space \( \mathbb{R}^3 - \{ \vec{a_1}, \ldots, \vec{a}_{m+n} \} \) and define gauge potentials on each of these sets as well as transition functions on their intersections. However, for the case \( \vec{a}_1 = \ldots = \vec{a}_{m+n} = \vec{a} \) the two sets (1) are again enough to cover \( \mathbb{R}^3 - \{ \vec{a} \} \) and the gauge potential (4)-(6) will describe \( m - n \) monopoles (if \( m > n \)) or \( n - m \) anti-monopoles (if \( m < n \)) sitting on top of each other.

One can simplify expressions (4)-(8) by introducing functions of coordinates

\[ w_j := \frac{y_j}{r_j - x^2_j} = e^{i\varphi_j} \cot \frac{\vartheta_j}{2} \quad \text{and} \quad v_j := \frac{1}{w_j} = \frac{\bar{y}_j}{r_j + x^2_j} = e^{-i\varphi_j} \tan \frac{\vartheta_j}{2} , \quad (9) \]

where

\[ x^1_j = r_j \sin \vartheta_j \cos \varphi_j , \quad x^2_j = r_j \sin \vartheta_j \sin \varphi_j \quad \text{and} \quad x^3_j = r_j \cos \vartheta_j . \quad (10) \]

Note that \( w_j \to \infty \) for \( x^1,2 \to a^1_{1,2}, x^3 \to a^3_1 \), and \( v_i \to \infty \) for \( x^1,2 \to a^1_{1,2}, x^3 \leq a^3_1 \). In terms of \( w_j \) and \( v_j \) the gauge potentials (4)-(6) have the form

\[ A^{N,m+n} = \sum_{i=1}^{m} \frac{1}{2(1 + w_i \bar{w}_i)} (\bar{w}_i dv_i - v_i d\bar{w}_i) + \sum_{i=m+1}^{m+n} \frac{1}{2(1 + v_i \bar{v}_i)} (v_i d\bar{w}_i - \bar{v}_i dv_i) , \quad (11) \]

\[ A^{S,m+n} = \sum_{i=1}^{m} \frac{1}{2(1 + w_i \bar{w}_i)} (\bar{w}_i dw_i - w_i d\bar{w}_i) + \sum_{i=m+1}^{m+n} \frac{1}{2(1 + v_i \bar{v}_i)} (w_i d\bar{w}_i - \bar{v}_i dw_i) . \quad (12) \]

On the intersection \( R^3_{N,m+n} \cap R^3_{S,m+n} \) of two domains (1) these configurations are related by the transformation

\[ A^{N,m+n} = A^{S,m+n} + d \ln \left( \prod_{i=1}^{m} \left( \frac{\bar{w}_i}{w_i} \right)^{\frac{1}{2}} \prod_{j=m+1}^{m+n} \left( \frac{\bar{w}_j}{\bar{v}_j} \right)^{\frac{1}{2}} \right) , \quad (13) \]

since \( \bar{y}_i/y_i = \bar{w}_i/w_i \). Note that the transition function in (13) can also be written in terms of \( v_i \) by using the relation \( v_i/\bar{v}_i = w_i/w_j \).

For the abelian curvature \( F^{D,m+n} \) we have

\[ F^{D,m+n} = d A^{N,m+n} = -\sum_{i=1}^{m} \frac{dv_i \wedge d\bar{v}_i}{(1 + v_i \bar{v}_i)^2} + \sum_{i=m+1}^{m+n} \frac{dv_i \wedge d\bar{v}_i}{(1 + v_i \bar{v}_i)^2} = \]
and \( n \) are well defined on \( \mathbb{R} \).

It is not difficult to see that \( F^{D,m+n} \) is singular only at points \( \{ \vec{a}_1, \ldots, \vec{a}_{m+n} \} \), where monopoles and anti-monopoles are located.

3 \ Point SU(2) configurations

The generalization of the Wu-Yang SU(2) monopole [4] to a configuration describing \( m \) monopoles and \( n \) anti-monopoles can be obtained as follows. Let us multiply equation (13) by the Pauli matrix \( \sigma_3 \) and rewrite it as

\[
A^{N,m+n} \sigma_3 = f^{(m,n)}_{NS} A^{S,m+n} \sigma_3 (f^{(m,n)}_{NS})^{-1} + f^{(m,n)}_{NS} d (f^{(m,n)}_{NS})^{-1},
\]

where

\[
f^{(m,n)}_{NS} = \begin{pmatrix}
\prod_{i=1}^{m} \left( \frac{w_i}{\bar{w}_i} \right)^{\frac{1}{2}} \prod_{j=m+1}^{m+n} \left( \frac{w_j}{\bar{w}_j} \right)^{\frac{1}{2}} & 0 \\
0 & \prod_{i=1}^{m} \left( \frac{w_i}{\bar{w}_i} \right)^{\frac{1}{2}} \prod_{j=m+1}^{m+n} \left( \frac{w_j}{\bar{w}_j} \right)^{\frac{1}{2}}
\end{pmatrix}.
\]

It can be checked by direct calculation that the transition matrix (16) can be splitted as

\[
f^{(m,n)}_{NS} = (g^{(m,n)}_N)^{-1} g^{(m,n)}_S,
\]

where the 2 \times 2 unitary matrices

\[
g^{(m,n)}_N = \frac{1}{(1 + \prod_{i=1}^{m+n} v_i \bar{v}_i)^{\frac{1}{2}}} \begin{pmatrix}
\prod_{j=1}^{m} w_j \prod_{k=m+1}^{m+n} \bar{v}_k & 1 \\
-1 & \prod_{j=1}^{m} \bar{v}_j \prod_{k=m+1}^{m+n} v_k
\end{pmatrix}
\]

and

\[
g^{(m,n)}_S = \frac{1}{(1 + \prod_{i=1}^{m+n} w_i \bar{w}_i)^{\frac{1}{2}}} \begin{pmatrix}
1 & \prod_{j=1}^{m} \bar{w}_j \prod_{k=m+1}^{m+n} w_k \\
-\prod_{j=1}^{m} w_j \prod_{k=m+1}^{m+n} \bar{w}_k & 1
\end{pmatrix}
\]

are well defined on \( R^3_{N,m+n} \) and \( R^3_{S,m+n} \), respectively. Using formulae (9) and (10), one can rewrite these matrices in the coordinates \( x^i \) with explicit dependence on moduli \( \vec{a}_i \) for \( i = 1, \ldots, m+n \).

Substituting (17) into (15), we obtain

\[
A^{N,m+n} g^{(m,n)}_N \sigma_3 (g^{(m,n)}_N)^\dagger + g^{(m,n)}_N d (g^{(m,n)}_N)^\dagger = A^{S,m+n} g^{(m,n)}_S \sigma_3 (g^{(m,n)}_S)^\dagger + g^{(m,n)}_S d (g^{(m,n)}_S)^\dagger =: A^{(m,n)}_{su(2)},
\]

where by construction \( A^{(m,n)}_{su(2)} \) is well defined on \( R^3_{N,m+n} \cup R^3_{S,m+n} = \mathbb{R}^3 - \{ \vec{a}_1, \ldots, \vec{a}_{m+n} \} \). Geometrically, the existence of splitting (17) means that Dirac’s nontrivial U(1) bundle over \( \mathbb{R}^3 - \{ \vec{a}_1, \ldots, \vec{a}_{m+n} \} \) trivializes when being embedded into an SU(2) bundle. The matrices (18) and (19) define this trivialization since \( f^{(m,n)}_{NS} \mapsto f^{(m,n)}_{NS} = g^{(m,n)}_N f^{(m,n)}_{NS} (g^{(m,n)}_S)^{-1} = 1_2 \).
Remark. Recall that we consider generic configurations with the conditions (2). In the case of $a_i^{1,2}$ coinciding for some $i \neq j$, one has $R_{N, m+n}^3 \cup R_{S, m+n}^3 \neq \mathbb{R}^3 - \{\tilde{a}_1, \ldots, \tilde{a}_{m+n}\}$ and the gauge potential (20) can have singularities outside $R_{N, m+n}^3 \cup R_{S, m+n}^3$. For example, in the case $m = 2, n = 0$, $a_1^{1,2} = a_2^{1,2} = 0$ and $a_3 = -a_3^2 = a$, the gauge potential describing two separated monopoles will be singular on the interval $-a < x^3 < a$. To have nonsingular $A_{su(2)}^{(2,0)}$ one should consider $a_1^{1,2} \neq a_2^{1,2}$ or to use three open sets covering $\mathbb{R}^3 - \{\tilde{a}_1, \tilde{a}_2\}$ instead of two ones.

The field strength for the configuration (20) is given by

$$F_{su(2)}^{(m,n)} = dA_{su(2)}^{(m,n)} + A_{su(2)}^{(m,n)} \wedge A_{su(2)}^{(m,n)} = iF_{D,m+n}Q_{(m,n)},$$

where the $su(2)$-valued matrix

$$Q_{(m,n)} := -ig^{(m,n)}_N \sigma_3 (g^N_{(m,n)})^\dagger = -ig^{(m,n)}_S \sigma_3 (g^S_{(m,n)})^\dagger$$

is well defined on $R_{N, m+n}^3 \cup R_{S, m+n}^3$. It is easy to see that $Q_{(m,n)}^2 = -1$ and $Q_{(m,n)}$ may be considered as the generator of the group $U(1)$ embedded into $SU(2)$. Then the abelian nature of the configuration (20)-(21) becomes obvious. Furthermore, for

$$A_{su(2)}^{(m,n)} = A_a^{(m,n)} dx^a \quad \text{and} \quad F_{su(2)}^{(m,n)} = \frac{1}{2} F^{(m,n)}_{ab} dx^a \wedge dx^b$$

one can easily show that

$$\partial_a F_{ab}^{(m,n)} + [A_a^{(m,n)}, F_{ab}^{(m,n)}] = i \left( \partial_a F_{D,m+n}^{(m,n)} \right) Q_{(m,n)}$$

and therefore on the space $\mathbb{R}^3 - \{\tilde{a}_1, \ldots, \tilde{a}_{m+n}\}$ we have

$$\partial_a F_{ab}^{(m,n)} + [A_a^{(m,n)}, F_{ab}^{(m,n)}] = 0,$$

which follows from the field equations describing $m$ Dirac monopoles and $n$ anti-monopoles. Note that the solution (20)-(23) of the $SU(2)$ gauge theory can be embedded in any larger gauge theory following e.g. [9].

4 Point monopoles via Riemann-Hilbert problems

Here we want to rederive the described configurations by solving a matrix Riemann-Hilbert problem. For simplicity, we restrict ourselves to the case of $m$ monopoles.

Let us consider the Bogomolny equations [10]

$$F_{ab} = \epsilon_{abc} D_c \chi,$$

where $D_c = \partial_c + [A_c, \cdot]$ and the fields $A_a, F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ and $\chi$ take values in the Lie algebra $u(q)$. Obviously, in the abelian case $D_c \chi = \partial_c \chi$. Note that for the gauge fields $F_{D,m}$ given by (14) we have

$$F_{D,m}^{(m)} = \epsilon_{abc} \partial_c \phi^{(m)} \quad \text{with} \quad \phi^{(m)} = \sum_{k=1}^{m} \frac{i}{2r_k}.$$

(27)

Analogously, for the field $F_{ab}^{(m)}$ from (23) we have

$$F_{ab}^{(m)} = \epsilon_{abc} D_c \Phi^{(m)} \quad \text{with} \quad \Phi^{(m)} = i \phi^{(m)} Q_{(m)},$$

(28)
where $\phi^{(m)}$ is given in (27) and $Q^{(m)}$ in (22). Thus, both U(1) and SU(2) multi-monopoles as well as $(m,n)$-configurations (11)-(14) and (20)-(22) can be considered as solutions of the Bogomolny equations (26). In fact, the second order pure Yang-Mills equations for $F^{D,m}_{ab}$ and $F^{ab(m)}_{ab}$ can be obtained by differentiating (27) and (28), respectively. Moreover, in pure SU(2) Yang-Mills theory in (3+1)-dimensional Minkowski space-time, one can choose the component $A_0$ of the gauge potential $A=A_0 dt+A_a dx^a$ to be nonzero and proportional to $\Phi^{(m)}$ (the abelian case is similar). Then the configuration $\{A_0^{(m)}, A_a^{(m)}\}$ will be a static multi-dyon solution of the Yang-Mills equations.

Recall that the Bogomolny equations (26) can be obtained as the compatibility conditions of the linear system

$$[D_y - \frac{\lambda}{2}(D_3 + i\chi)]\psi = 0 \quad \text{and} \quad \left[\frac{1}{2}(D_3 - i\chi) + \lambda D_y\right]\psi = 0,$$

(29)

where $D_y = \frac{1}{2}(D_1 + iD_2)$, $D_y = \frac{1}{2}(D_1 - iD_2)$ and the auxiliary $q \times q$ matrix $\psi(x^a, \lambda)$ depends holomorphically on a new variable $\lambda \in U \subset \mathbb{C}P^1$. Such matrices $\psi$ can be found via solving a parametric Riemann-Hilbert problem which is formulated in the monopole case as follows [11].

Suppose we are given a $q \times q$ matrix $f_{+\lambda}$ depending holomorphically on

$$\eta = y - 2\lambda x^3 - \lambda^2 \bar{y}$$

and $\lambda$ for $\lambda \in U_+ \cap U_-$, where $U_+ = \mathbb{C}P^1 - \{\infty\}$ and $U_- = \mathbb{C}P^1 - \{0\}$. Then for each fixed $(x^a) \in \mathbb{R}^3$ and $\lambda \in S^1 \subset U_+ \cap U_-$ one should factorize this matrix-valued function,

$$f_{+\lambda}(x, \lambda) = \psi_+^{-1}(x, \lambda)\psi_-(x, \lambda)$$

(30)

in such a way that $\psi_+$ and $\psi_-$ extend holomorphically in $\lambda$ onto subsets of $U_+$ and $U_-$, respectively. In order to insure that $A_0^a = -A_a$ and $\chi^\dagger = -\chi$ in (29) with $\psi = \psi_\pm$ one should also impose the (reality) conditions

$$f_{+\lambda}^\dagger(x, -\lambda^{-1}) = f_{-\lambda}(x, \lambda) \quad \text{and} \quad \psi_+^\dagger(x, -\lambda^{-1}) = \psi_-^{-1}(x, \lambda).$$

(31)

After finding such $\psi_\pm$ for an educated guess of $f_{+\lambda}$, one can get $A_a$ and $\chi$ from the linear system (29) with the matrix function $\psi_+$ or $\psi_-$ instead of $\psi$. Namely, from (29) we get

$$A_\gamma := \frac{1}{2}(A_1 + iA_2) = \psi_+\partial_\gamma\psi_+^{-1}|_{\lambda=0}, \quad A_3 - i\chi = \psi_+\partial_3\psi_+^{-1}|_{\lambda=0},$$

(32)

$$A_\gamma := \frac{1}{2}(A_1 - iA_2) = \psi_-\partial_\gamma\psi_-^{-1}|_{\lambda=\infty}, \quad A_3 + i\chi = \psi_-\partial_3\psi_-^{-1}|_{\lambda=\infty}. $$

(33)

For more details see [11, 12] and references therein.

The construction of U(1) multi-monopole solutions via solving the Riemann-Hilbert problem for the function

$$f_{+\lambda}^{D,m} = \frac{\lambda^m}{\prod_{k=1}^m \eta_k} =: \rho_m \quad \text{with} \quad \eta_k = \eta - h(a^1_k, a^2_k, a^3_k, \lambda) = (1-\lambda^2)x^1_k + i(1+\lambda^2)x^2_k - 2\lambda x^3_k$$

(35)

was discussed in [12] and here we describe only the SU(2) case. The ansatz for $f_{+\lambda}^{(m)}$ which satisfies (32) only for odd $m$ was written down in the appendix C of [12]. Here we introduce the ansatz

$$f_{+\lambda}^{(m)} = \left(\frac{\rho_m}{(-1)^m\lambda^m} \rho_m^{-1} + (-1)^m \rho_m^{-1}\right)$$

(36)
satisfying the reality condition (32) for any \( m \). It is not difficult to see that

\[
J^{(m)}_{+-} = \begin{pmatrix}
1 & 0 \\
(-1)^m \lambda^m \rho_m^1 & 1
\end{pmatrix} \begin{pmatrix}
J^{D,m}_{+-} \\
(J^{D,m}_{+-})\dagger
\end{pmatrix} \begin{pmatrix}
1 & \lambda^{-m} \rho_m^{-1} \\
0 & 1
\end{pmatrix} \sim \begin{pmatrix}
J^{D,m}_{+-} \\
(J^{D,m}_{+-})\dagger
\end{pmatrix},
\]

where the diagonal matrix in (37) describes the Dirac line bundle \( L \) (the U(1) gauge group) embedded into the rank 2 complex vector bundle (the SU(2) gauge group) as \( L \oplus L^{-1} \). This gives another proof of the equivalence of U(1) and SU(2) point monopole configurations (see [12] for more details). Furthermore, the matrix (36) can be splitted as follows:

\[
J^{(m)}_{+-} = (\psi^{(m)}_+)^{-1} \psi^{(m)}_-, \quad \text{(38)}
\]

where

\[
\psi^{(m)}_+ = \psi^{(m)}_+ \begin{pmatrix}
1 & 0 \\
(-1)^{m+1} \lambda^m \rho_m^{-1} & 1
\end{pmatrix}, \quad \psi^{(m)}_- = \psi^{(m)}_- \begin{pmatrix}
1 & -\lambda^{-m} \rho_m^{-1} \\
0 & 1
\end{pmatrix}, \quad \text{(39)}
\]

\[
\hat{\psi}^{(m)}_+ = g^{(m)}_S \begin{pmatrix}
\psi^{S,m}_+ \\
(\psi^{S,m}_+)^\dagger
\end{pmatrix}, \quad \hat{\psi}^{(m)}_- = g^{(m)}_N \begin{pmatrix}
\psi^{N,m}_- \\
(\psi^{N,m}_-)^\dagger
\end{pmatrix}, \quad \text{(40)}
\]

\[
\psi^{S,m}_+ = \prod_{i=1}^m \psi^{S}_+(x_i^a, \lambda), \quad \psi^{S}_+(x_i^a, \lambda) = \xi^+(x_i^a) - \lambda \xi^{-1}(x_i^a) \bar{y}_i, \quad \xi^+(x_i^a) = (r_i - x_i^3)^{\frac{1}{2}}, \quad \text{(41)}
\]

\[
\psi^{N,m}_- = \prod_{i=1}^m \psi^{N}_-(x_i^a, \lambda), \quad (\psi^{N}_-(x_i^a, \lambda))^{-1} = \xi^-(x_i^a) \bar{y}_i + \lambda^{-1} \xi^{-1}(x_i^a), \quad \xi^-(x_i^a) = (r_i + x_i^3)^{-\frac{1}{2}}. \quad \text{(42)}
\]

The explicit form of \( g^{(m)}_N \) and \( g^{(m)}_S \) is given in (18) and (19). Note that both \( \psi^{(m)}_\pm \) and \( \hat{\psi}^{(m)}_\pm \) satisfy the reality conditions (32).

Formulae (38)-(42) solve the parametric Riemann-Hilbert problem for our \( J^{(m)}_{+-} \) restricted to a contour on \( \mathbb{C}P^1 \) which avoids all zeros of the function \( \prod_{k=1}^{m} \eta_k \). Substituting (39)-(42) into formulæ (33)-(34), we get

\[
A^{(m)}_y = g^{(m)}_S \partial_y (g^{(m)}_S)^\dagger, \quad A^{(m)}_y = g^{(m)}_N \partial_y (g^{(m)}_N)^\dagger, \quad A^{(m)}_3 = g^{(m)}_S \partial_3 (g^{(m)}_S)^\dagger = g^{(m)}_N \partial_3 (g^{(m)}_N)^\dagger, \quad \text{(43)}
\]

\[
\chi^{(m)} = \frac{i}{2} (g^{(m)}_S \partial_3 (g^{(m)}_S)^\dagger - g^{(m)}_N \partial_3 (g^{(m)}_N)^\dagger), \quad \text{(44)}
\]

where

\[
g^{(m)}_S = g^{(m)}_S \begin{pmatrix}
\xi^+ & 0 \\
0 & \xi^+_1
\end{pmatrix} \quad \text{with} \quad g^{(m)}_S = \frac{1}{\prod_{i=1}^{m} (r_i - x_i^3)^2 + \prod_{i=1}^{m} y_i \bar{y}_i} \begin{pmatrix}
\prod_{j=1}^{m} (r_j - x_j^3) & \prod_{j=1}^{m} \bar{y}_j \\
- \prod_{j=1}^{m} y_j & \prod_{j=1}^{m} (r_j - x_j^3)
\end{pmatrix}, \quad \text{(45)}
\]

\[
g^{(m)}_N = g^{(m)}_N \begin{pmatrix}
\xi^- & 0 \\
0 & \xi^+_1
\end{pmatrix} \quad \text{with} \quad g^{(m)}_N = \frac{1}{\prod_{i=1}^{m} (r_i + x_i^3)^2 + \prod_{i=1}^{m} y_i \bar{y}_i} \begin{pmatrix}
\prod_{j=1}^{m} y_j & \prod_{j=1}^{m} (r_j + x_j^3) \\
- \prod_{j=1}^{m} (r_j + x_j^3) & \prod_{j=1}^{m} y_j
\end{pmatrix} \quad \text{(46)}
\]
and
\[ \xi_+ = \prod_{k=1}^{m} \xi_+(x_k^a) = \prod_{k=1}^{m} (r_k - x_k^3)^\frac{1}{2}, \quad \xi_- = \prod_{k=1}^{m} \xi_-(x_k^a) = \prod_{k=1}^{m} (r_k + x_k^3)^{-\frac{1}{2}}. \] (47)

It is not difficult to see that the configuration (43) coincides with (20) and \( \chi^{(m)} \) from (44) with \( \Phi^{(m)} \) from (28). Thus, we have derived SU(2) multi-monopole point-like solutions via a parametric Riemann-Hilbert problem.

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