Considerations Concerning the Contributions of Fundamental Particles to the Vacuum Energy Density

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\textbf{Abstract}

The covariant regularization of the contributions of fundamental particles to the vacuum energy density is implemented in the Pauli-Villars, dimensional regularization, and Feynman regulator frameworks. Rules of correspondence between dimensional regularization and cutoff calculations are discussed. Invoking the scale invariance of free field theories in the massless limit, as well as consistency with the rules of correspondence, it is argued that quartic divergencies are absent in the case of free fields, while it is shown that they arise when interactions are present.

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1 Introduction

It has been pointed out by several authors that one of the most glaring contradictions in physics is the enormous mismatch between the observed value of the cosmological constant and estimates of the contributions of fundamental particles to the vacuum energy density \[1\]. Specifically, the observed vacuum energy density in the universe is approximately \(0.73\rho_c\), where \(\rho_c = 3H^2/8\pi G_N \approx 4 \times 10^{-47}\text{GeV}^4\) is the critical density, while estimates of the contributions of fundamental particles range roughly from \((\text{TeV})^4\) in broken supersymmetry scenarios to \((10^{19}\text{ GeV})^4 = 10^{76}\) (GeV)^4 if the cutoff is chosen to coincide with the Planck scale. Thus, there is a mismatch of roughly 59 to 123 orders of magnitude!

The aim of this paper is to discuss the nature of these contributions by means of elementary arguments.

In the case of free particles, it is easy to see that a covariant regularization is needed, which we implement in the Pauli-Villars (PV) \([2, 3]\), dimensional regularization (DR) \([4]\), and Feynman regulator (FR) \([3]\) frameworks.

We recall that the vacuum energy density is given by
\[
\rho = <0|T_{00}|0>,
\]
where \(T_{\mu\nu}\) is the energy-momentum tensor.

Defining \(t_{\mu\nu} \equiv <0|T_{\mu\nu}|0>\) and assuming the validity of Lorentz invariance, we have
\[
t_{\mu\nu} = \rho g_{\mu\nu},
\]
or, equivalently,
\[
t_{\mu\nu} = \frac{g_{\mu\nu}}{4} t^\lambda_{\lambda},
\]
where we employ the metric \(g_{00} = -g_{11} = -g_{22} = -g_{33} = 1\).

We first consider the case of a free scalar field. Expanding the fields in plane waves with coefficients expressed in terms of creation and annihilation operators, and using their commutation relations, one readily finds the familiar expression
\[
\rho = t_{00} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k^2}{\omega_k},
\]
where \(\omega_k = [(\vec{k})^2 + m^2]^{1/2}\), as well as
\[
p = t_{ii} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{(k_i)^2}{\omega_k}.
\]
In Eq. (5), \( i = 1, 2, 3 \) and there is no summation over \( i \).

Both Eq. (4) and Eq. (5) are highly divergent and therefore mathematically undefined. Moreover, as recently emphasized by E. Kh. Akhmedov [5], the usual procedure of introducing a three-dimensional cutoff leads to an obvious contradiction: since the integrands in Eqs. (4, 5) are positive, one would reach the conclusion that \( t_{00} \) and \( t_{ii} \) have the same sign, in contradiction with Eq. (2)! This reflects the fact that a three-dimensional cutoff breaks Lorentz invariance. Clearly, covariant regularization procedures are required!

This is an important issue, since expressions that are not properly regularized are often deceptive. A classical example is provided by the calculation of vacuum polarization in QED, in which a quadratically divergent contribution turns out to be zero upon the imposition of electromagnetic current conservation [6]. Similarly, the same requirement transforms linearly divergent contributions to the triangle diagrams into convergent ones [7]. In fact, it is important that the regularization procedure respects the symmetries and partial symmetries of the underlying theory.

The plan of the paper is the following. In Section 2, we discuss rules of correspondence between the position of the poles in DR and cutoff calculations. In Section 3 we implement the covariant regularization of \( t_{\mu\nu} \) in the PV, DR, and FR frameworks, starting from Eqs. (2, 4, 5). In Section 4 we consider the evaluation of \( t_{\lambda\lambda} \) on the basis of well-known expressions for the vacuum expectation value of products of free-field operators, as well as Feynman diagrams. Throughout the paper the role played by the scale invariance of free field theories in the massless limit is emphasized. Section 5 illustrates the important effects arising from interactions by means of two specific examples. Section 6 presents the conclusions. Appendix A proves a general theorem concerning the signs of the vacuum energy density contributions of a free scalar field when it is regularized in the PV framework with the minimum number of regulator fields, while Appendix B illustrates the rules of correspondence in the evaluation of the one-loop effective potential. Section 4 contains a phenomenological update of the Veltman-Nambu sum rule for \( m_H \) [8, 9] and of an alternative relation discussed in Ref. [10].
2 Rules of Correspondence

Since DR does not involve cutoffs explicitly, this approach is seldom employed in discussions concerning the cosmological constant and hierarchy problems. However, as it will be shown, it does give valuable information about the nature of the ultraviolet singularities. Furthermore, it has other important virtues for the problems under consideration: it respects the scale invariance of free-field theories in the massless limit, does not involve unphysical regulator fields, and it is relatively easy to use in the two-loop calculation carried out in Section 5.

In order to discuss the position of the poles corresponding to specific ultraviolet divergencies in multi-loop calculations, it is convenient to multiply each \( n \)-dimensional integration \( \int d^n k \) by \( \mu^{4-n} \), where \( \mu \) is the 't Hooft scale. This ensures that the combination of the prefactor and the integration has the canonical dimension 4.

Let us first consider quadratic ultraviolet divergencies. In cutoff calculations, aside from physical masses and momenta, such contributions are proportional to \( \Lambda^2 \), where \( \Lambda \) is the ultraviolet cutoff. In DR they must be proportional to suitable poles multiplied by \( \mu^2 \), since this is the only available mass independent of the physical masses and momenta. If \( L \) is the number of loops, we have the condition \( (\mu^{4-n})^L = \mu^2 \), or \( n = 4 - 2/L \). This means that quadratic divergencies exhibit poles at \( n = 2, 3, 10/3, \ldots \) for \( L = 1, 2, 3, \ldots \) loop integrals. The same conclusion has been stated long ago by M. Veltman [8].

It should be pointed out that this is a useful criterion for scalar integrals of the form

\[
I_{l,m} = i \int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^l}{(k^2 - M^2)^m},
\]

where \( l, m \) are integers \( \geq 0 \) and for brevity we have not included the \( i\epsilon \) instruction. When \( l \) is a negative integer, there may appear poles at \( n = 2 \) which correspond to infrared, rather than ultraviolet singularities. A useful example is provided by the relation [10]:

\[
\int \frac{d^n k}{k^2} = \int \frac{d^n k}{k^2 - M^2} - M^2 \int \frac{d^n k}{k^2(k^2 - M^2)}. \tag{7}
\]

As is well known, the l.h.s. is zero in DR. The first integral in the r.h.s. is quadratically divergent in four dimensions and consequently exhibits an ultraviolet pole at \( n = 2 \), while the second one involves a pole at \( n = 2 \) arising
from the Feynman parameter integration. This last singularity is related to
the fact that the second integral contains a logarithmic infrared divergence at
\( n = 2 \). Thus, in Eq. (7) we witness a cancellation between ultraviolet and
infrared poles. As pointed out in Ref. [10], in discussing ultraviolet singularities
one should include in that case the contribution from the first integral.

The above discussion can be extended to quartic divergencies. Since,
by an analogous argument, these must be proportional to \( \mu^4 \), we have the
relation \( (\mu^{4-n})^L = \mu^4 \), or \( n = 4 - 4/L \). Thus, quartic divergencies exhibit
poles at \( n = 0, 2, 8/3, \ldots \) for \( L = 1, 2, 3, \ldots \) loop scalar integrals. Of course,
as we will see in a specific example, a quartic divergence may also arise from
the product of two one-loop quadratically divergent integrals, each of which
has a pole at \( n = 2 \).

In Section 3, we show how these rules permit to establish a correspon
dence between DR and cutoffs calculations in one-loop amplitudes.

3 Regularization of \( t_{\mu\nu} \)

In order to implement a covariant regularization of \( t_{\mu\nu} \), we first search for a
four dimensional representation of Eqs. (4, 5).

Inserting the well-known identity
\[
\frac{1}{2\omega_k} = \int_{-\infty}^{\infty} dk_0 \, \delta(k^2 - m^2) \, \theta(k_0),
\]

(8)
k\(^2 \equiv k_0^2 - (\vec{k})^2 \), in Eqs. (4, 5), we see that \( \rho \) and \( p \) are the zero-zero and \( i-i \)
components of the formal tensor
\[
t_{\mu\nu} = \int \frac{d^4k}{(2\pi)^3} k_\mu k_\nu \, \delta(k^2 - m^2) \, \theta(k_0).
\]

(9)

As we will discuss in detail later on, Eq. (9) can be regularized in the PV and
DR frameworks. Since \( t_{\mu\nu} \) is proportional to \( g_{\mu\nu} \), in analogy with Eqs. (2, 3)
it follows that
\[
t_{\mu\nu} = \frac{g_{\mu\nu}}{4} m^2 \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \, \theta(k_0),
\]

(10)

where we have employed \( k^2 \, \delta(k^2 - m^2) = m^2 \, \delta(k^2 - m^2) \). Using Eq. (8) in
reverse and the identity
\[
\frac{1}{\omega_k} = \frac{1}{\pi} \int_{-\infty}^{\infty} dk_0 \, \frac{i}{k^2 - m^2 + i\epsilon},
\]

(11)
which follows from contour integration, Eq. (10) can be cast in the form

\[
t_{\mu\nu} = \frac{g_{\mu\nu} m^2}{4} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon},
\]

(12)

which implies

\[
t^\lambda_\lambda = m^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}.
\]

(13)

We note that the integral in Eqs. (12, 13) is \(i\Delta_F(0)\), the Feynman propagator evaluated at \(x = 0\).

An alternative derivation of Eqs. (12, 13) can be obtained by using the starting Eqs. (4, 5) to evaluate the trace \(t^\lambda_\lambda\):

\[
t^\lambda_\lambda = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k^2 - (\vec{k})^2}{\omega_k} = \frac{m^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k}.
\]

(14)

Combining Eqs. (3, 11, 14), we immediately recover Eqs. (12, 13). It is important to note that these expressions are proportional to \(m^2\) and to the quadratically divergent integral \(i\Delta_F(0)\).

Returning to the issue of regularization, in the PV framework Eq. (9) is replaced by the regularized expression

\[
(t_{\mu\nu})_{PV} = \sum_{i=0}^{N} C_i \int \frac{d^4k}{(2\pi)^3} k_\mu k_\nu \delta(k^2 - M_i^2) \theta(k_0),
\]

(15)

where \(M_0 = m, C_0 = 1, N\) is the number of regulator fields, \(M_j (j = 1, 2, \ldots, N)\) denote their masses, and the \(C_i\) obey the constraints

\[
\sum_{i=0}^{N} C_i (M_i^2)^p = 0 \quad (p = 0, 1, 2).
\]

(16)

From Eq. (16) we see that in our case \(N = 3\) is the minimum number of regulator fields. It is worthwhile to note that if the limit \(M_j \to \infty\) is taken before the integration is carried out, Eq. (15) reduces to the original, unregularized expression of Eq. (9). In fact, \(\lim_{M_j \to \infty} \delta(k^2 - M_j^2) = 0\).

Following the steps leading from Eq. (9) to Eq. (12), Eq. (15) becomes

\[
(t_{\mu\nu})_{PV} = \frac{g_{\mu\nu}}{4} \sum_{i=0}^{N} C_i M_i^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M_i^2 + i\epsilon},
\]

(17)
which is the PV regularized version of Eq. (12).

The simplest way to evaluate Eq. (17) is to differentiate twice

\[ I(M_i^2) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M_i^2 + i\epsilon}, \tag{18} \]

with respect to \( M_i^2 \), so that it becomes convergent. This leads to \( I''(M_i^2) = 1/16\pi^2M_i^2 \). Integrating twice \( I''(M_i^2) \) with respect to \( M_i^2 \), we obtain

\[ I(M_i^2) = \frac{1}{16\pi^2} \left[ M_i^2 (\ln M_i^2 - 1) + K_1 M_i^2 + K_2 \right], \tag{19} \]

where \( K_1 \) and \( K_2 \) are arbitrary constants of integration. When inserted in Eq. (17), the terms involving \( K_1, K_2, \) and \(-M_i^2\) cancel on account of Eq. (16), and Eq. (17) becomes

\[ (t_{\mu\nu})_{PV} = \frac{g_{\mu\nu}}{64\pi^2} \sum_{i=0}^{N} C_i M_i^4 \ln \left( \frac{M_i^2}{\nu^2} \right). \tag{20} \]

Here \( \nu \) is a mass scale that can be chosen arbitrarily since its contribution vanishes on account of Eq. (16) with \( p = 2 \).

If we choose \( N = 3 \), the minimum number of regulator fields, the constants \( C_i \) can be expressed in terms of the \( M_i \) by solving Eq. (16) for \( p = 0, 1, 2 \) (see Appendix A). One finds that Eq. (20) contains three classes of contributions: i) terms quartic in the regulator masses \( M_j (j = 1, 2, 3) \); ii) terms of \( O(m^2) \) which are quadratic in \( M_j \); iii) terms of \( O(m^4) \). As shown in Eq. (20), all these contributions are accompanied by logarithms. If one rescales the regulator masses by a common factor \( \Lambda \), one finds that, modulo logarithms, the three classes become proportional to \( \Lambda^4 \), \( \Lambda^2 \), and \( \Lambda^0 \), respectively. Thus, the first class of terms exhibit the quartic divergence frequently invoked in discussions of the cosmological constant problem. However, as shown in the Appendix, the results show a curious and at first hand unexpected feature: for arbitrary values of \( M_j \), the sign of the quartic contribution to \( \rho \) is negative! Instead, the sign of the \( O(m^2\Lambda^2) \) term is positive, and that of \( O(m^4) \) contribution is negative.

The PV expression greatly simplifies in the limit \( M_3 \to M_2 \to M_1 = \Lambda \) and becomes

\[ (t_{\mu\nu})_{PV} = \frac{g_{\mu\nu}}{128\pi^2} \left[ -\Lambda^4 + 4m^2\Lambda^2 - m^4 \left( 3 + 2\ln\frac{\Lambda^2}{m^2} \right) \right]. \tag{21} \]
It is interesting to note that Eq. (17) also follows from the PV regularization of the formal, quartically divergent tensor

\[
J_{\mu\nu} = i \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^2 - m^2 + i\epsilon}.
\]

In fact

\[
(J_{\mu\nu})_{PV} = i \sum_{i=0}^{N} C_i \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^2 - M_i^2 + i\epsilon},
\]

which, upon the replacement \(k_\mu k_\nu \to g_{\mu\nu}k^2/4\) and the decomposition \(k^2 = k^2 - M_i^2 + M_i^2\), reduces to Eq. (14), since the contribution of \(k^2 - M_i^2\) vanishes on account of Eq. (16). Eq. (22) may be also regulated by means of a Feynman regulator which, for this application, we choose to be of the form \([\Lambda^2 - m^2]/(\Lambda^2 - k^2 + i\epsilon)^3\). Thus,

\[
(J_{\mu\nu})_{FR} = i \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{k^2 - m^2 + i\epsilon} \frac{(\Lambda^2 - m^2)^3}{(\Lambda^2 - k^2 + i\epsilon)^3}.
\]

Evaluating Eq. (23) we find the very curious result that it exactly coincides with Eq. (21)! An advantage of Eq. (23) is that one can discern immediately its sign by means of a Wick rotation of the \(k_0\) axis. Replacing \(k_\mu k_\nu \to g_{\mu\nu}k^2/4\), performing the rotation and introducing \(k_0 = iK_0\), we obtain the Euclidean representation

\[
(J_{\mu\nu})_{FR} = -g_{\mu\nu} \frac{4}{4} \int \frac{d^4K}{(2\pi)^4} \frac{K^2}{K^2 + m^2} \frac{(\Lambda^2 - m^2)^3}{(\Lambda^2 + K^2)^3},
\]

which shows that the cofactor of \(g_{\mu\nu}\) is manifestly negative, in conformity with Eq. (21).

In summary, according to Eq. (20) with the minimum number \(N = 3\) of regulator fields, or its limit in Eq. (21), the leading contribution for a bosonic field would be \(\rho = -O(\Lambda^4), p = O(\Lambda^4)!\) Such a result is theoretically unacceptable since for a scalar field

\[
< 0|T_{00}|0 > = \frac{1}{2} < 0|\partial_\mu \varphi \partial_\mu \varphi + \partial_i \varphi \partial_i \varphi + m^2 \varphi^2|0 >
\]

should be positive. We therefore interpret the sign problem as an artifact of the regularization procedure that arises in the case \(N = 3\) due to the fact that some of the regulator fields have negative metric.
The presence of quartic divergencies, of either sign, has another highly unsatisfactory consequence, namely it breaks down the scale invariance of free field theories in the massless limit! We recall that the divergence of the dilatation current for a scalar field has the form

$$\partial_\mu D^\mu = T^\lambda_\lambda + \frac{\Box \phi^2}{2} = \Theta^\lambda_\lambda,$$

(25)

where $\Theta_{\mu\nu}$ is the “improved” energy-momentum tensor [11]. This leads to

$$<0|\partial_\mu D^\mu|0> = t^\lambda_\lambda = m^2 <0|\phi^2|0>,$$

(26)

where we used $\Box <0|\phi^2|0> = 0$ and, in deriving the second equality in Eq. (26), we employed the equation of motion. Thus, for free fields, $t^\lambda_\lambda$ should vanish as $m \to 0$, a property that is violated by quartic divergencies of either sign.

In order to circumvent the dual problems of sign and breakdown of scale invariance of the free-field theory in the massless limit within the PV framework, there are two possibilities: one is to subtract the offending $O(\Lambda^4)$ term in Eq. (21); the other is to employ $N \geq 4$, in which case the $C_i$ are not determined by the $M_i$, and the sign of the $O(\Lambda^4)$ contributions is undefined. In the last approach one can in principle impose the cancellation of the quartic divergence as a symmetry requirement. However, the sign of the leading $O(m^2\Lambda^2)$ term remains undefined, which is not an attractive state of affairs.

A simpler and more satisfactory approach is to go back to Eq. (12) as a starting point to implement the regularization procedure. Regularization of Eq. (12) in the PV framework would lead us back to Eqs. (17, 21). Instead, we may regularize the integral in Eq. (12) with a Feynman regulator, which we choose to be of the form $[\Lambda^2/(\Lambda^2 - k^2 - i\epsilon)]^2$. Neglecting terms of $O(m^2/\Lambda^2)$, this leads to

$$(t_{\mu\nu})_R = \frac{g_{\mu\nu}}{64\pi^2} \left[ m^2 \Lambda^2 - m^4 \left( \ln \frac{\Lambda^2}{m^2} - 1 \right) \right],$$

(27)

which implies

$$(t^\lambda_\lambda)_R = \frac{1}{16\pi^2} \left[ m^2 \Lambda^2 - m^4 \left( \ln \frac{\Lambda^2}{m^2} - 1 \right) \right].$$

(28)

Eqs. (27, 28) have the correct sign and conform with scale invariance in the massless limit! A similar result is obtained if a Wick rotation is implemented in Eq. (12), and an invariant cutoff is employed to evaluate the integral.

9
We now turn to DR. Since the steps from Eq. (9) to Eq. (12) involve only the \( k_0 \) integration, in DR the regularized expressions of Eq. (9) and Eq. (12) are equivalent and we obtain

\[
(t_{\mu\nu})_{\text{DR}} = \frac{g_{\mu\nu}}{n} \frac{m^2}{(2\pi)^n} \frac{\mu^{(4-n)}}{k^2 - m^2 + i\epsilon},
\]

(29)

which leads to

\[
(t_{\lambda\lambda})_{\text{DR}} = m^2 \frac{\mu^{(4-n)}}{(2\pi)^n} \frac{\mu^{(4-n)}}{k^2 - m^2 + i\epsilon}.
\]

(30)

We note that the \( n = 0 \) pole in Eq. (29) arises from the replacement \( k_\mu k_\nu \rightarrow g_{\mu\nu} k^2 / n \). Since \( g^{\mu\nu} g_{\mu\nu} = n \), this pole is absent in the evaluation of the trace in Eq. (30). In order to use the rules of correspondence in an unambiguous manner, we apply them to the Lorentz scalar \( t_{\lambda\lambda} \) evaluated in the FR and DR frameworks. Carrying out the integration in Eq. (30), we find

\[
(t_{\lambda\lambda})_{\text{DR}} = \frac{4 m^4}{(2\sqrt{\pi})^n} \frac{\mu/m^{(4-n)}}{(2 - n)(4 - n)} \Gamma(3 - n/2). \]

(31)

This expression exhibits poles at \( n = 4 \) and \( n = 2 \) which, according to the rules of correspondence for one-loop integrals, indicate the presence of logarithmic and quadratic divergencies.

A heuristic way to establish a correspondence between the cutoff calculation in Eq. (28) and the DR expression in Eq. (31), is to carry out the expansion about \( n = 4 \) in the usual way, but at the same time separate out the \( n = 2 \) pole in such a manner that the overall result is only modified in \( \mathcal{O}(n - 4) \). This leads to

\[
(t_{\lambda\lambda})_{\text{DR}} = \frac{\mu^2 m^2}{2\pi} \left[ \frac{1}{2 - n} + \frac{1}{2} \right] - m^4 \frac{2}{16\pi^2} \left[ \frac{2}{4 - n} + \ln \frac{\mu^2}{m^2} - 2C + 1 \right] + \mathcal{O}(n - 4),
\]

(32)

where \( C = [\gamma - \ln 4\pi]/2 \). The contribution proportional to \( m^4 \) represents the usual result. The first term contains the pole at \( n = 2 \), and only modifies the expansion in \( \mathcal{O}(n - 4) \). A correspondence with Eq. (28) can be implemented by means of the identifications

\[
\left[ \frac{1}{4 - n} + \ln \frac{\mu}{m} - C \right]_{n=4} \rightarrow \ln \frac{\Lambda}{m} - 1, \quad (33)
\]

\[
\frac{\mu^2}{2\pi} \left[ \frac{1}{2 - n} + \frac{1}{2} \right]_{n=2} \rightarrow \frac{\Lambda^2}{16\pi^2}, \quad (34)
\]
where, for instance, \( n \approx 2 \) means that \( n \) is in the immediate neighborhood of 2. It is interesting to note that if one approaches the ultraviolet poles from below, as it seems natural in DR, the signs of the left and right sides of Eqs. (33, 34) coincide!

Another interesting information contained in the DR expression of Eq. (31) is that the \( \mathcal{O}(m^2\Lambda^2) \) contribution is not accompanied by a \( \ln (\Lambda^2/m^2) \) cofactor. This can be seen as follows: since Eq. (31) is proportional to \( (m^2)^{n/2} \), if we differentiate twice with respect to \( m^2 \) we see that the pole at \( n = 2 \) disappears. As a consequence, terms of \( \mathcal{O}(m^2\Lambda^2\ln (\Lambda^2/m^2)) \) cannot be present, since otherwise contributions of \( \mathcal{O}(\Lambda^2) \) would survive under the double differentiation. Indeed, this observation agrees with Eq. (28). Thus, in one-loop calculations depending on \( m^2, \Lambda^2 \), terms of \( \mathcal{O}(m^2\Lambda^2\ln (\Lambda^2/m^2)) \) would require a double pole at \( n = 2 \) in the DR expression.

A conclusion essentially identical to Eq. (27), namely that the divergence of the zero-point energy for free particles is quadratic rather than quartic, and that massless particles don’t contribute, has been recently advocated by E. Kh. Akhmedov [5], invoking arguments of relativistic invariance. The analysis of the present paper shows that this is not enough to single out Eq. (27), since the PV regularization leads to the covariant expressions of Eqs. (20, 21) that exhibit a quartic divergence. What singles out Eq. (27) are the combined requirements of relativistic covariance and scale invariance of free-field theories in the massless limit, as well as consistency with the rules of correspondence.

We conclude this Section by recalling that the contributions of all bosons (fermions) carry the same (opposite) sign as Eq. (27). Each contribution must be multiplied by a factor \( \eta \) that takes into account the color and helicity degrees of freedom, as well as the particle-antiparticle content.

4 Evaluation of \( t^\lambda_\lambda \) based on Feynman Diagrams

In Section 3 we have discussed the regularization of \( t_{\mu\nu} \) and its trace in the free-field theory case, starting from the familiar expressions for \( \rho \) and \( p \) given in Eqs. (4, 5). It is instructive to revisit the evaluation of \( t^\lambda_\lambda \) on the basis of well known expressions for the vacuum expectation value of products of free-field operators on the one hand, and Feynman diagrams on the other.
This will also pave the way to the discussion of the effect of interactions in Section 5.

We will consider three examples: an hermitian scalar field, a spinor field, and a vector boson, all endowed with mass $m$. We recall the free-field theory expressions for $T^\lambda_\lambda$ in the three cases:

$$T^\lambda_\lambda = -\partial_\lambda \varphi \partial^\lambda \varphi + 2m^2 \varphi^2$$  \hspace{1cm} (35)

$$T^\lambda_\lambda = -3\bar{\psi} \left\{ i\frac{\partial}{2} - m \right\} \psi + m\bar{\psi}\psi$$  \hspace{1cm} (36)

$$T^\lambda_\lambda = -m^2 A^\lambda_\lambda$$  \hspace{1cm} (37)

The above formulae are valid in four dimensions and, in deriving Eq. (37), we have employed the symmetric version of $T^\mu_\nu$ for the spin 1 field.

A direct way of evaluating $t^\lambda_\lambda$ is to consider the vacuum expectation value of two fields at $x$ and $y$, carry out the differentiations exhibited in Eqs. (35, 36) and take the limit $x \to y$. For instance, in the free-field scalar case we have the well-known representation:

$$<0|\varphi(x)\varphi(y)|0> = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \theta(k_0) e^{-ik(x-y)}.$$  \hspace{1cm} (38)

The r.h.s. of Eq. (38), the $i\Delta^+(x-y)$ function, is the contribution of the one-particle intermediate state in the Källen-Lehmann representation which, of course, is the only one that survives in the free-field theory case. From Eq. (38) we find

$$<0|-\partial_\lambda \varphi(x)\partial^\lambda \varphi(y) + 2m^2 \varphi(x)\varphi(y)|0> =$$

$$= \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \theta(k_0) (2m^2 - k^2) e^{-ik(x-y)},$$  \hspace{1cm} (39)

which is well defined. Taking the limit $x \to y$ and recalling Eq. (35), we obtain the formal expression

$$(t^\lambda_\lambda)_\varphi = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \theta(k_0) (2m^2 - k^2).$$  \hspace{1cm} (40)

If instead of $T^\mu_\mu$, the “improved” tensor $\Theta^\mu_\nu$ is employed for scalar fields, Eq. (35) is replaced by $\Theta^\lambda_\lambda = \varphi \Box \varphi + 2m^2 \varphi^2$, which again leads to Eqs. (39)
(40). If we replace \(k^2 \rightarrow m^2\) in these expressions on account of \(\delta(k^2 - m^2)\), we recover Eq. (10), the result of our previous analysis in Section 3. Parenthetically, we recall that, at the classical level, use of the equations of motion leads to \(\Theta^\lambda_\lambda = m^2 \varphi^2\) even in the presence of the \(\lambda \varphi^4\) interaction [11].

Eq. (40) admits another representation that can be linked with a Feynman vacuum diagram, to wit

\[
(t^\lambda_\lambda)_\varphi = \text{Re} \int \frac{d^4k}{(2\pi)^4} \frac{i [2m^2 - k^2]}{k^2 - m^2 + i\epsilon},
\]

where we have employed \(\pi \delta(k^2 - m^2) = \text{Re}(i/k^2 - m^2 + i\epsilon)\) and used the fact that the integrand is even in \(k_0\) to replace \(\theta(k_0) \rightarrow 1/2\).

Equivalently, we have

\[
(t^\lambda_\lambda)_\varphi = \text{Re} \left\{ m^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} - i \int \frac{d^4k}{(2\pi)^4} \right\}.
\]

Eq. (41) is depicted in Fig. 1, where the cross indicates the insertion of the operator \(T^\lambda_\lambda\) given in Eq. (35). We note that the “Re” instruction is important in the passage from Eq. (40) to Eq. (41) and ensures that the answer is real, as required for diagonal matrix elements of the hermitian operator \(T^\lambda_\lambda\).

Using Eq. (36), the corresponding expression in the fermion case is

\[
(t^\lambda_\lambda)_\psi = -\text{Re} \text{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{i [m - 3(\not{k} - m)]}{\not{k} - m + i\epsilon}.
\]

This can be cast in the form

\[
(t^\lambda_\lambda)_\psi = -\text{Re} \left\{ 4 m^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} - 12 i \int \frac{d^4k}{(2\pi)^4} \right\},
\]

where we have employed \(\int d^4k \not{k}/(k^2 - m^2 + i\epsilon) = 0\).

Finally, using Eq. (37), we have

\[
(t^\lambda_\lambda)_A = \text{Re} \left\{ 3 m^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} - i \int \frac{d^4k}{(2\pi)^4} \right\}.
\]

These expressions exhibit interesting features: the first terms in Eqs. (42, 44, 45) are quadratically divergent and real. The reality condition is easily checked by performing the \(k_0\) contour integration or by means of a Wick
rotation of the $k_0$ axis accompanied by the change of variable $k_0 = iK_0$. This rotation is mathematically allowed since the $k_0$ integrations in those contributions are convergent. The second terms in Eqs. (42, 44, 45) formally exhibit a quartic divergence. However, since the integrations are over the real axes, such terms are purely imaginary in Minkowski space and therefore do not contribute if the “Re” restriction is imposed. We note parenthetically that, unlike in the previous case, the Wick rotation cannot be applied to the unregularized $\int d^4k$ as it stands since, in performing the $k_0$ integration, the contributions of the large quarter circles in the complex plane are not negligible and, in fact, they are necessary to satisfy Cauchy’s theorem.

In the PV and DR approaches the regularized versions of the imaginary contributions in Eqs. (42, 44, 45) vanish automatically. In contrast, a Feynman regulator of the form $[\Lambda^2/(\Lambda^2 - k^2 - i\epsilon)]^3$ leads to

$$-i \int \frac{d^4k}{(2\pi)^4} \left( \frac{\Lambda^2}{\Lambda^2 - k^2 - i\epsilon} \right)^3 = \frac{\Lambda^4}{32\pi^2},$$

(46)

which is real and positive, and exhibits the frequently invoked quartic divergence. The reality property can also be checked by performing a Wick rotation in Eq. (46), which is now mathematically allowed. Thus, we see that the quartically divergent contributions in the one-loop vacuum diagrams have a very ambivalent and disturbing property: their contributions to $\rho$ are imaginary in Minkowski space and real, if regularized according to Eq. (46), in Euclidean space.

However, if Eq. (46) is applied to regularize the imaginary parts between curly brackets in Eqs. (42, 44, 45), serious inconsistencies emerge. In fact, their coefficients do not conform with the relations $(t^\lambda_\lambda)_{\psi} = -4 (t^\lambda_\lambda)_{\phi}$ and $(t^\lambda_\lambda)_A = 3 (t^\lambda_\lambda)_{\phi}$, which arise on account of the helicity and particle-antiparticle degrees of freedom of Dirac spinors and massive vector bosons in four dimensions.

A direct way to see that these terms are inconsistent with Eq. (40) is to go back to that expression, replace $k^2 \to m^2$ on account to the $\delta$-function and then use $\delta(k^2 - m^2) = (1/\pi)\text{Re}(i/k^2 - m^2 + i\epsilon)$. This leads to the first term of Eq. (42), a result that is only consistent with Eq. (40) if the second contribution vanishes.

We conclude that, in order to avoid inconsistencies, the quartically divergent imaginary parts in Eqs. (42, 44, 45) must be subtracted either by imposing the reality condition in Minkowski space or by means of the regularization
procedure, as in the DR and PV cases. The surviving terms in Eqs. \(42, 44, 45\) are proportional to \(m^2 \Delta_F(0)\), satisfy the relations \((t^A_\lambda)\psi = -4 (t^A_\lambda)\phi\) and \((t^A_\lambda)A = 3 (t^A_\lambda)\phi\), and coincide with the result in Eq. \(40\) and its equivalent expression in Eq. \(10\). Furthermore, they conform with the scale invariance of free field theories in the massless limit.

The PV, DR, and FR regularizations and their correspondence was discussed in detail in Section 3 in the case of the free scalar field, starting with Eqs. \(12, 13\). In writing down the rules of correspondence between DR and four dimensional cutoff calculations in the case of spin 1 and spin 1/2 fields, there is a subtlety that should be pointed out. In the case of the spin 1 field, the DR version of \(t^A_\lambda\) is given by the expression for the scalar field (Eq. \(30\)) multiplied by \(n - 1\), the number of helicity degrees of freedom in \(n\) dimensions. In separating out the contribution of the \(n = 2\) pole (Cf. Eq. \(32\)), the residue carries then a factor 1, rather than 3. In order to maintain the proper relation with the four dimensional calculation, in the spin 1 case the l.h.s. of Eq. \(34\) corresponds to \(3\Lambda^2/16\pi^2\) rather than \(\Lambda^2/16\pi^2\), the factor 3 reflecting the number of helicity degrees of freedom in four dimensions. A similar rule holds for spin 1/2 fields: if in evaluating the \(n = 2\) residue one employs \(\text{Tr} \ 1 = 2\), as befits a spinor in two-dimensions, in the rule of correspondence with the four-dimensional cutoff calculation one includes an additional factor 2 to reflect the fact that \(\text{Tr} \ 1 = 4\) for four dimensional spinors.

The possible dichotomy in the treatment of the helicity degrees of freedom has had an interesting effect in the derivation of sum rules based on the speculative assumption that one-loop quadratic divergencies cancel in the Standard Model (SM). As explained in Ref. \[10\], in DR the condition of cancellation of quadratic divergencies in one-loop tadpole diagrams is given by

\[
\text{Tr} \ 1 \sum_f m_f^2 = 3 m_H^2 + (2 m_W^2 + m_Z^2)(n - 1), \quad (47)
\]

where the \(f\) summation is over fermion masses and includes the color degree of freedom. The factor \(n - 1\) reflects once more the helicity degrees of freedom of spin 1 bosons in \(n\) dimensions. Eq. \(47\) leads also to the cancellation of all quadratic divergencies in the one-loop contributions to the Higgs boson and fermion self-energies. Setting \(n = \text{Tr} \ 1 = 4\), and neglecting the contributions of the lighter fermions one obtains the Veltman-Nambu sum rule \[8, 9\]:

\[
m_H^2 = 4 m_t^2 - 2 m_W^2 - m_Z^2, \quad (48)
\]
On the other hand, it was pointed out in Ref. [10] that in DR Eq. (47) with $n = 4$ is not sufficient to cancel the remaining quadratic divergencies in the $W$ and $Z$ self-energies. Associating once more the one-loop quadratic divergencies with the $n = 2$ poles, the cancellation of the residues in all cases ($f, H, W, Z$) takes place when $n = 2$ is chosen. With $n = 2$ and $\text{Tr} \mathbb{1} = 2$ [12], this leads to the alternative sum rule [10]

$$m_H^2 = 2 m_t^2 - \frac{(2 m_W^2 + m_Z^2)}{3}.$$  \hspace{1cm} (49)

Inserting the current values, $m_t = 174.3$ GeV, $m_Z = 91.1875$ GeV, and $m_W = 80.426$ GeV [13], Eq. (48) and Eq. (49) lead to the predictions $m_H = 317$ GeV, and $m_H = 232$ GeV, respectively. The current 95% CL upper bound from the global fit to the SM is $m_H^{95} = 211$ GeV [13], so that the above values are somewhat larger than the range favored by the electroweak analysis. It will be interesting to see whether these predictions ultimately bear any relation to reality!

### 5 Effect of Interactions

An important issue is what happens when interactions are taken into account. The investigation of their effect on $t^4$ is an open and difficult one, since vacuum matrix elements are factored out and then cancelled in the usual treatment of Quantum Field Theory. As it is well-known, in the conventional framework, interactions break the scale invariance of free-field theories in the massless limit, a phenomenon referred to as the trace anomaly [14, 15, 16]. One naturally expects that a similar phenomenon takes place in vacuum amplitudes, an occurrence that would lead to the emergence of quartic divergencies. In this Section we limit our analysis to two instructive examples.

We first discuss the question in the scalar theory with $L_{\text{int}} = -(\lambda/4!)\phi^4$. One readily finds that in $\mathcal{O}(\lambda)$ this interaction contributes $-(\lambda/8)\Delta_F^2(0)$ to $\rho$, which is quartically divergent. This result is obtained by either using the familiar plane wave expansion involving annihilation and creation operators and their commutation relations, or by the calculation of the relevant Feynman diagram, which is a “figure 8” with the interaction at the intersection (see Fig. 2). We note that $1/8 = 3/4!$ is the symmetry number for this diagram. However, the mass in Eq. [12] is the unrenormalized mass

$$m_H^2 = 2 m_t^2 - \frac{(2 m_W^2 + m_Z^2)}{3}.$$  \hspace{1cm} (49)
\(m_0\), since this is the parameter that appears in the Lagrangian. Writing \(m_0^2 = m^2 - \Pi(m^2)\), where \(\Pi\) is the self-energy, Eq. (12) generates a counterterm \(-\Pi(m^2)i\Delta F(0)/4\). In \(\mathcal{O}(\lambda)\) the only contribution to \(\Pi(m^2)\) is the seagull diagram and equals \((\lambda/2)i\Delta F(0)\), where 1/2 is the symmetry number. Thus, the counterterm \((\lambda/8)\Delta F^2(0)\) exactly cancels the quartic divergence from the “figure 8” diagram! The same cancellation occurs when regulator fields are present, since Eq. (17) involves just a linear combination of terms analogous to Eq. (12)!

Next we consider the example of QED in \(\mathcal{O}(e^2)\). Choosing the symmetric and explicitly gauge invariant version of \(T^{\mu \nu}\), one finds in n-dimensions:

\[
T^A_{\lambda} = \frac{n - 4}{4} F_{\mu \nu} F^{\mu \nu} - (n - 1)\bar{\psi} \left[ i \frac{\not{D}}{2} - m_0 \right] \psi + m_0 \bar{\psi} \psi,
\]

where \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), \(D_\mu = \partial_\mu + ieA_\mu\) is the covariant derivative and \(\psi\) and \(A_\mu\) are the electron and photon fields. It is easy to see that the insertion of \(-(n - 1)\bar{\psi}(i \not{D}/2 - m_0)\psi\) in the electron-loop, corrected by the interaction in order \(e^2\) (Fig. 3), cancels against the contribution of \((n - 1)e\bar{\psi}A\psi\) corrected in \(\mathcal{O}(e)\) (see Fig. 4). This reflects the validity of the equation of motion \((iD_\mu - m_0)\psi = 0\).

Next, we focus on the insertion of \([(n - 4)/4]F_{\mu \nu}F^{\mu \nu}\) in the photon line (Fig. 5). The fermion-loop integral with two external off-shell photons of momentum \(q\) is given by

\[
\frac{-i8e^2}{(2\sqrt{\pi})^n} \mu^{(4-n)} \Gamma(2 - n/2) (g_{\mu \nu}q^2 - q_{\mu}q_{\nu}) \int_0^1 dx \ x \ (1 - x) \ [m^2 - q^2 x(1 - x)]^{n/2 - 2},
\]

where \(m\) is the fermion mass. Closing the photon line and inserting the vertex \([(n - 4)/4]F_{\mu \nu}F^{\mu \nu}\) multiplies Eq. (51) by

\[
\mu^{(4-n)} \int \frac{d^n q}{(2\pi)^n} \frac{n - 4}{2} q^2 g^{\mu \nu} \left( \frac{-i}{q^2} \right)^2
\]

and we obtain for the diagram of Fig. 5:

\[
A_{DR} = \frac{4ie^2}{(2\sqrt{\pi})^n} \ (\mu^{4-n})^2 \ (n - 4)(n - 1)\Gamma(2 - n/2)
\times \int_0^1 dx \ x \ (1 - x) \int \frac{d^n q}{(2\pi)^n} [m^2 - q^2 x(1 - x)]^{n/2 - 2}.
\]
We note that $(1/q^2)^2$, arising from the two photon propagators in Fig. 5, has been cancelled by two $q^2$ factors, one from Eq. (51), the other from the $[(n - 4)/4]F_{\mu\nu}F^{\mu\nu}$ vertex.

Performing a Wick rotation of the $q_0$ axis, and introducing

$$q_0 = iQ_0, \quad \bar{q} = \bar{Q}, \quad d^nQ = \frac{\pi^{n/2}}{\Gamma(n/2)} Q^{2(\frac{n-2}{2})} dQ^2, \quad u = Q^2,$$

we have

$$A_{DR} = -e^2 C(n) \int_0^1 dx [x(1-x)]^{\frac{n}{2}-1} \int_0^\infty du \, u^{\frac{n}{2}-1} \left[ u + \frac{m^2}{x(1-x)} \right]^{\frac{n}{2}-2} ,$$

where

$$C(n) = \frac{4}{(4\pi)^n} \frac{\Gamma(2-n/2)}{\Gamma(n/2)} (n-4)(n-1)(\mu^{4-n})^2.$$

The $u$-integral in Eq. (53) equals

$$[\Gamma(n/2)\Gamma(2-n)/\Gamma(2-n/2)] \, m^{2n-4} [x(1-x)]^{2-n},$$

and Eq. (53) becomes

$$A_{DR} = -\frac{4}{(4\pi)^n} (n-4)(n-1)(\mu^{4-n})^2 \, \Gamma(2-n)m^{2n-4} \int_0^1 dx [x(1-x)]^{1-n/2}.$$

The remaining integral equals $B(2-n/2, 2-n/2) = \Gamma^2(2-n/2)/\Gamma(4-n)$ and we find

$$A_{DR} = \frac{16 \, e^2 \, m^4 \, (\mu/m)^{2(4-n)}(n-1) \, \Gamma^2(3-n/2)}{(4\pi)^n(2-n)(3-n)(4-n)}.$$

Eq. (55) exhibits simple poles at $n = 2, 3, 4$. According to the rules of correspondence for two-loop diagrams, this indicates the presence of quartic, quadratic, and logarithmic divergencies.

For clarity, we point out that potentially there is a counterterm diagram associated with Fig. 5, in which the fermion loop is replaced by the insertion of the field renormalization vertex $-i\deltaZF_{\mu\nu}F^{\mu\nu}/4$ in the closed photon loop. However, such diagram, involving two vertices proportional to $F_{\mu\nu}F^{\mu\nu}$ and two photon propagators, leads to a result proportional to $i \int d^nq$, which is imaginary in Minkowski space and furthermore vanishes in DR. In particular, this also means that if the result for the $k$-subintegration given in
Eq. (51) were expanded about $n = 4$, the pole contribution would cancel when the $q$ integration is performed. Since we need the full dependence on $n$ to determine the possible pole positions and the pole contribution from the $k$-subintegration vanishes, in Eq. (55) we have evaluated the full two-loop integral, without expanding the $k$-subintegration about $n = 4$.

There remains the contribution of $m_0\bar{\psi}\psi$, the last term in Eq. (50), corrected by the interaction in order $\epsilon^2$. The superficial degree of divergence in four dimensions of the corresponding two-loop diagram is trilinear, so that one expects a quadratic divergence. Indeed, the DR calculation of the diagram shows a pole at $n = 3$ and a double pole at $n = 4$ which, according to the rules of correspondence for two-loop diagrams, indicate a quadratic divergence and logarithmic singularities proportional to $\mathcal{O}(m^4 \ln (\Lambda^2/m^2))$ and $\mathcal{O}(m^4 \ln^2 (\Lambda^2/m^2))$.

6 Conclusions

In this paper we discuss a number of issues related to the nature of the contributions of fundamental particles to the vacuum energy density. This problem is of considerable conceptual interest since what may be called the physics of the vacuum is not addressed in the usual treatment of Quantum Field Theory. On the other hand, it also represents a major unsolved problem since estimates of these contributions show an enormous mismatch with the observed cosmological constant.

As a preamble to our analysis, in Section 2 we use an elementary argument to derive rules of correspondence between the poles’ positions in DR and ultraviolet cutoffs in four dimensional calculations. In the case of quadratic divergencies, they coincide with Veltman’s dictum [8], while they are extended here to quartic singularities. A specific example of this correspondence is given at the one-loop level in Section 3.

In Section 3 we address the Lorentz-covariant regularization of $t_{\mu\nu}$ in free-field theories, starting from the elementary expressions for $\rho$ and $p$. Making use of a mathematical identity, we are led to a covariant expression, which is immediately confirmed by the direct evaluation of $t^{\lambda\lambda}$. The regularization of this result is then implemented in the PV, DR, and FR frameworks. In Section 4, we re-examine $t_{\mu\nu}$ on the basis of the well-known expression for the vacuum expectation value of products of free fields, as well as one-loop Feynman vacuum diagrams, with results that are consistent with those of
Section 3. In Section 5, we consider two cases involving interactions: \(\lambda \phi^4\) theory in \(\mathcal{O}(\lambda)\) and QED in \(\mathcal{O}(e^2)\), which require the examination of two-loop vacuum diagrams.

Our general conclusion, based on Lorentz covariance and the scale invariance of free-field theories in the massless limit, as well as consistency with the rules of correspondence applied to \(t^\lambda \chi\), is that quartically divergent contributions to \(\rho\) are absent in the case of free fields.

At first hand, the notion that free photons do not contribute to \(\rho\) may seem strange. However, we point out that this immediately follows from Eq. (50), which tells us that for free photons \((T^\lambda \chi)^\gamma = 0\) in four dimensions. This implies \((t^\lambda \chi)^\gamma = 0\) and, using Eqs. (2, 3), \(\rho_\gamma = 0!\) In more pictorial language: \((T^\lambda \chi)^\gamma = 0\) implies \(\rho_\gamma = 3p_\gamma\), the equation of state of a photon gas but, in the vacuum case, Eq. (2) tells us that \(\rho_{\text{vac}} = -p_{\text{vac}}\) for any field. The only way of satisfying the two constraints in the vacuum case is \(\rho_{\text{vac}}^\gamma = p_{\text{vac}}^\gamma = 0\).

As pointed out in Section 3, in the case of free fields the same conclusion was recently advocated in the interesting work of E. Kh. Akhmedov \[5\] on the basis of a less complete line of argumentation, and without examining the effect of interactions.

When interactions are turned on, as illustrated in the QED case in Section 5, our conclusion is that quartic divergencies generally emerge. Thus, in some sense there is a parallelism between the analysis of vacuum amplitudes and conventional Quantum Field Theory. Free-field theories are scale invariant in the massless limit and, according to our interpretation, this partial symmetry protects the theory from the emergence of quartic divergencies. However, in the presence of interactions, the symmetry is broken even in the massless limit and consequently such singularities generally arise. On the other hand, it is worthwhile to recall that \(T^\lambda \chi\) becomes a soft operator even in the presence of interactions under the speculative assumption that the coupling constants are zeros of the relevant \(\beta\)-functions \[15\, 16\].

From the point of view of formal renormalization theory, the presence of the highly divergent expressions encountered in the study of vacuum amplitudes does not present an insurmountable difficulty. For instance, in the PV approach discussed in Section 3 with a sufficiently large number of regulator fields, the coefficients of the free-field divergencies are undefined and in principle can be chosen to cancel the corresponding singularities emerging from interactions. More generally, the \(\lambda\) constant that appears in Einstein’s equation can be adjusted to cancel such singularities. As emphasized by several
the crisis resides in the extraordinarily unnatural fine-tuning that these cancellations entail.

From a practical point of view, the conclusions in the present paper hardly affect the cosmological constant problem: clearly, it makes very little difference phenomenologically whether the mismatch is 123 or 120 orders of magnitude! On the other hand, they place the origin of the problem on a different conceptual basis.

The simplest framework in which quartic divergencies cancel remains supersymmetry since it implies an equal number of fermionic and bosonic degrees of freedom. In some effective supergravity theories derived from four dimensional superstrings, with broken supersymmetry, it is possible to ensure also the cancellation of the $\mathcal{O}(m^2\Lambda^3)$ terms in the one-loop effective potential $^{[17]}$. In such scenarios, the $\mathcal{O}(m^4)$ terms become $\mathcal{O}(m_{3/2}^4)$, where $m_{3/2}$, the gravitino mass, is associated with the scale of supersymmetry breaking. Assuming $m_{3/2} = \mathcal{O}(1 \text{ TeV})$ this leads to the rough estimate $\rho = \mathcal{O}(\text{TeV}^4)$ mentioned in the Introduction.

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Appendix A

In this Appendix we analyze Eq. $^{(20)}$, which is the PV-regularized version of Eq. $^{(12)}$. Choosing the minimum number $N = 3$ of regulator fields, the constants $C_j$ ($j = 1, 2, 3$) can be expressed in terms of the $m^2$ and the $M_j^2$ ($j = 1, 2, 3$) by solving Eq. $^{(16)}$ for $p = 0, 1, 2$. This leads to

$$C_1 = \frac{-M_2^2M_3^2 + m^2(M_2^2 + M_3^2) - m^4}{(M_2^2 - M_1^2)(M_3^2 - M_1^2)}.$$

(A1)
\( C_2 \) is obtained from \( C_1 \) by applying the cyclic permutation (1 2 3), while \( C_3 \) is obtained from \( C_2 \) by means of the same permutation. Focusing on the quartic divergencies, we neglect the terms proportional to \( m^2 \) and \( m^4 \) and, since \( \nu^2 \) in Eq. (20) is arbitrary, we choose \( \nu^2 = M_3^2 \). Then the quartically divergent terms in Eq. (20) can be cast in the form

\[
(t_{\mu\nu})_{PV}(m^2 = 0) = \frac{g_{\mu\nu}}{64\pi^2} M_1^2 M_2^2 M_3^2 \left[ f\left(\frac{M_3^2}{M_1^2}\right) - f\left(\frac{M_3^2}{M_2^2}\right)\right], \tag{A2}
\]

where

\[
f(x) = \frac{\ln x}{x - 1}. \tag{A3}
\]

We note that \( f(x) \) is positive definite for all \( x \geq 0 \) while its derivative

\[
f'(x) = \frac{1}{x - 1} \left[ \frac{1}{x} - \frac{\ln x}{x - 1} \right], \tag{A4}
\]

is negative definite. Thus, \( f(x) \) is a positive definite and decreasing function of its argument. Consider now the case \( M_2^2 > M_1^2 \). Then \( M_3^2/M_1^2 > M_3^2/M_2^2 \) and the expression between square brackets is negative. Since \( (M_2^2 - M_1^2) > 0 \), we conclude that the coefficient of \( g_{\mu\nu} \) in Eq. (A2) is negative for any value of \( M_3^2 \). If \( M_1^2 > M_2^2 \), \( M_3^2/M_1^2 < M_3^2/M_2^2 \), the square bracket is positive but \( (M_2^2 - M_1^2) < 0 \), so that we reach the same conclusion. Thus, for all possible values of the regulator masses \( M_j^2 \), the coefficient of \( g_{\mu\nu} \) in Eq. (20) is negative definite which, as explained in Section 3, is physically unacceptable. Analogous arguments show that for all \( M_j^2 \), the contributions of \( O(m^2M_j^2) \) and \( O(m^4) \) in the cofactor of \( g_{\mu\nu} \) are positive and negative definite, respectively. In the limit \( M_3 \to M_2 \to M_1 = \Lambda \), Eq. (A2) greatly simplifies and reduces to the first term in Eq. (21).

As mentioned in Section 3, one possible solution of the sign problem is to subtract the offending \( O(M^4) \) contributions. Another possibility is to consider \( N \geq 4 \) regulator fields. In that case Eq. (16) for \( p = 0, 1, 2 \) are not sufficient to determine the \( C_j \ (j = 1, \ldots, N) \) in terms of the masses. As expected, we have checked that the coefficient of the \( O(M^4) \) term in Eq. (20) becomes undetermined while still satisfying the three relations of Eq. (16), so that it may be chosen to be positive or, for that matter, zero. The last possibility would naturally follow by invoking the scale invariance of free field theories in the massless limit and would conform with the analysis based on DR. As mentioned in Section 3, in the \( N \geq 4 \) solution of the
problem, the coefficient of the leading $O(m^2\Lambda^2)$ is also undefined, which is not a satisfactory state of affairs!

**Appendix B**

In this Appendix we apply DR and the rules of correspondence to integrals that occur in the evaluation of the one-loop effective potential in the $\lambda\phi^4$ theory [18]:

$$V(\phi_c) = -\frac{i}{2} \mu^{4-n} \int \frac{d^nk}{(2\pi)^n} \ln \left( \frac{k^2 - m^2 - \frac{1}{2} \lambda\phi_c^2 + i\epsilon}{k^2 - m^2 + i\epsilon} \right). \tag{B1}$$

We recall that in the path integral formalism, the denominator in the argument of the logarithm arises from the normalization of the generating functional $W[J]$, namely $W[0] = 1$.

We consider the integral:

$$K = -\frac{i}{2} \mu^{4-n} \int \frac{d^nk}{(2\pi)^n} \ln (k^2 - c + i\epsilon), \tag{B2}$$

which can be obtained from

$$L = -\frac{i}{2} \mu^{4-n} \int \frac{d^nk}{(2\pi)^n} (k^2 - c + i\epsilon)^\alpha \tag{B3}$$

by differentiating with respect to $\alpha$ and setting $\alpha = 0$. The last integral is given by

$$L = \frac{\mu^{4-n}}{2(2\sqrt{\pi})^n} (-1)^\alpha c^{\frac{n}{2}+\alpha} \frac{\Gamma(-\alpha - \frac{n}{2})}{\Gamma(-\alpha)}. \tag{B4}$$

Since $1/\Gamma(0) = 0$, the only non-vanishing contribution to $K$ involves the differentiation of $\Gamma(-\alpha)$. Thus

$$K = -\frac{\mu^{4-n}}{2(2\sqrt{\pi})^n} c^{\frac{n}{2}} \Gamma\left(-\frac{n}{2}\right) \tag{B5}$$

where we have employed $\lim_{\alpha \to 0} \psi(-\alpha)/\Gamma(-\alpha) = -1$. We see that $K$ contains poles at $n = 0, 2, 4$ which, according to the rules of correspondence,
indicate quartic, quadratic and logarithmic ultraviolet singularities. However, Eq. (B1) involves the difference of two integrals of the \( K \) type and we find

\[
V(\phi_c) = -\frac{\mu^{4-n}}{2(2\sqrt{\pi})^n} \left[ \left( m^2 + \frac{\lambda \phi_c^2}{2} \right)^{n/2} - (m^2)^{n/2} \right] \Gamma\left(-\frac{n}{2}\right).
\]  

(B6)

Clearly, the residue of the \( n = 0 \) pole cancels in Eq. (B6) and the leading singularity is given by the \( n = 2 \) pole:

\[
V(\phi_c) = \frac{\mu^2}{8\pi} \frac{\lambda \phi_c^2}{2-n} + \cdots ,
\]  

(B7)

which corresponds to a quadratic divergence. This conforms with the result for the leading singularity obtained by expanding Eq. (B1) in powers of \( \lambda \phi_c^2 \). Moreover, if one approaches the pole from below, as discussed after Eq. (34), the signs also coincide! As it is well-known, the quadratic and logarithmic singularities in Eq. (B6) are cancelled by the \( \delta m^2 \) and \( \delta \lambda \) counterterms \[18\].

The \( K \) integral with \( c = m^2 \) is also interesting because in some formulations it is directly linked to the vacuum energy density contribution from free scalar fields \[19\]. In order to obtain a four-dimensional representation of Eq. (B2), we introduce a Feynman regulator \[\Lambda^2/(\Lambda^2 - k^2 - i\epsilon)\]^3 and perform a Wick rotation, which leads to the Euclidean-space expression

\[
K = \frac{1}{32\pi^2} \int_0^\infty du \ u \ln \left( \frac{u + m^2}{\sigma^2} \right) \left( \frac{\Lambda^2}{\Lambda^2 + u} \right)^3 .
\]  

(B8)

In order to give mathematical meaning to the logarithm, in Eq. (B8) we have introduced a squared-mass scale \( \sigma^2 \) which, for the moment, is unspecified. Evaluating Eq. (B8), we have

\[
K = \frac{1}{64\pi^2} \left\{ \Lambda^4 \left[ \ln \left( \frac{\Lambda^2}{\sigma^2} \right) + 1 \right] + m^2\Lambda^2 - m^4 \left[ \ln \left( \frac{\Lambda^2}{m^2} \right) - 1 \right] \right\} ,
\]  

(B9)

where we have neglected terms of \( \mathcal{O}(m^2/\Lambda^2) \). In a free-field theory calculation, one expects the answer to depend on \( m^2 \) and \( \Lambda^2 \), as we found, for instance, in Eq. (21) and Eq. (27). By arguments analogous to those explained at the end of Section 3, one finds that \( \sigma^2 \) cannot be identified with \( m^2 \). In fact, with \( c = m^2 \), Eq. (B5) is proportional to \( (m^2)^{n/2} \). Differentiating with respect to \( m^2 \), the \( n = 0 \) pole in Eq. (B5) cancels. This implies
that terms of $\mathcal{O}(\Lambda^4 \ln(\Lambda^2/m^2))$ cannot be present, since otherwise contributions of $\mathcal{O}(\Lambda^4)$ would survive the differentiation with respect to $m^2$. An attractive idea to fix $\sigma^2$ is to invoke symmetry considerations. In particular, since according to the arguments of this paper the terms of $\mathcal{O}(\Lambda^4)$ violate the scale invariance of free field theories in the massless limit, we may choose $\sigma = \sqrt{e} \Lambda$ to eliminate such contributions. In that case, Eq. (B9) reduces to our previous result in Eq. (27), obtained by more elementary and transparent means! Correspondingly, in the DR version of $K$, the $n = 0$ pole $\mu^4(4-n)/4n$ may be removed in order to conform with the scale invariance of free field theories in the massless limit. This can be achieved by appending an $n/4$ normalization factor to the r.h.s. of Eq. (B5), in which case the rules of correspondence between the DR and four-dimensional calculations reduce precisely to Eqs. (33, 34).

In summary, aside from the fact that the derivation of Eq. (12) and the calculation of Eq. (27) are particularly simple, they offer the additional advantage that they explicitly exhibit the partial scale invariance of free-field theories.

References

[1] See, for example, S. Weinberg, Rev. Mod. Phys. 61, 1 (1989); A. D. Dolgov, M. V. Sazhin, Ya. B. Zeldovich, “Basics of Modern Cosmology” (Editions Frontières, Gif-sur-Yvette Cedex-France, 1990), Section 5.4.

[2] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949); N. N. Bogoliubov and D. V. Shirkov, “Introduction to the theory of quantized fields” (Interscience Publishers Inc, New York, N.Y. 1959).

[3] R. P. Feynman, Phys. Rev. 76, 769 (1949).

[4] G. ’t Hooft and M. J. Veltman, Nucl. Phys. B 44, 189 (1972); C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B 12, 20 (1972); J. F. Ashmore, Lett. Nuovo Cim. 4, 289 (1972).

[5] E. Kh. Akhmedov, arXiv:hep-th/0204048

[6] See, for example, J. J. Sakurai, “Advanced Quantum Mechanics” (Addison-Wesley Publishing Co., Reading, MA, 1967), p. 273 et seq.
[7] L. Rosenberg, Phys. Rev. 129, 2786 (1963); S. L. Adler, “Perturbation Theory Anomalies”, (1970 Brandeis University Summer Institute in Theoretical Physics, Vol.1, edited by S. Deser, M. Grisaru, and Hugh Pendleton, M.I.T. Press, Cambridge, Mass., 1970).

[8] M. J. Veltman, Acta Phys. Polon. B 12, 437 (1981).

[9] Y. Nambu, Enrico Fermi Institute reports, EFI-89-08, 90-46.

[10] G. Degrassi and A. Sirlin, Nucl. Phys. B 383, 73 (1992).

[11] C. G. Callan, S. R. Coleman and R. Jackiw, Annals Phys. 59, 42 (1970).

[12] See also M. Capdequi Peyranere, J. C. Montero and G. Moultaka, Phys. Lett. B 260, 138 (1991).

[13] The LEP Electroweak Working Group, EP Preprint Winter 2003 – in preparation (http://lepewwg.web.cern.ch/LEPEWWG/).

[14] R. J. Crewther, Phys. Rev. Lett. 28, 1421 (1972); M. S. Chanowitz and J. R. Ellis, Phys. Lett. B 40, 397 (1972); M. S. Chanowitz and J. R. Ellis, Phys. Rev. D 7, 2490 (1973).

[15] S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. D 15, 1712 (1977).

[16] J. C. Collins, A. Duncan and S. D. Joglekar, Phys. Rev. D 16, 438 (1977); N. K. Nielsen, Nucl. Phys. B 120, 212 (1977).

[17] S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. B 429, 589 (1994).

[18] See, for example, C. Itzykson, J. B. Zuber, “Quantum Field Theory” (McGraw-Hill Book Co, Singapore, 1980), p.448 et seq.

[19] See, for example, M. E. Peskin, D. V. Schroeder, “An Introduction to Quantum Field Theory” (Addison-Wesley Publishing Co., Reading, MA, 1995), p.364 et seq.
Figure 1: One-loop vacuum amplitudes. The cross indicates the insertion of the trace $T^\lambda_\lambda$ of the energy-momentum tensor. The dashed line represents a scalar, spinor or massive vector particle (see Section 4).

Figure 2: Two-loop vacuum amplitude in $\lambda\phi^4$ theory. (See Section 5).
Figure 3: Two-loop vacuum amplitude in QED. The cross represents the insertion of $-(n-1)\bar{\psi}(i\not{\partial}/2-m)\psi$ (Section 5).

Figure 4: Two-loop vacuum amplitude in QED. The cross represents the insertion of $(n-1)e\bar{\psi}A\psi$ (Section 5).

Figure 5: Two-loop vacuum amplitude in QED. The cross represents the insertion of $[(n-4)/4]F_{\mu\nu}F^{\mu\nu}$ (Section 5).