Research Article

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**Cesàro and Abel ergodic theorems for integrated semigroups**

https://doi.org/10.1515/conop-2020-0119
Received February 15, 2021; accepted September 7, 2021

Abstract: Let \( \{S(t)\}_{t \geq 0} \) be an integrated semigroup of bounded linear operators on the Banach space \( X \) into itself and let \( A \) be their generator. In this paper, we consider some necessary and sufficient conditions for the Cesàro mean and the Abel average of \( S(t) \) converge uniformly on \( B(X) \). More precisely, we show that the Abel average of \( S(t) \) converges uniformly if and only if \( X = R(A) \oplus N(A) \), if and only if \( R(A^k) \) is closed for some integer \( k \) and \( \|\lambda^2 R(A, A)\| \to 0 \) as \( \lambda \to 0^+ \), where \( R(A), N(A) \) and \( R(A, A) \), be the range, the kernel, the resolvent function of \( A \), respectively. Furthermore, we prove that if \( S(t)/t^2 \to 0 \) as \( t \to \infty \), then the Cesàro mean of \( S(t) \) converges uniformly if and only if the Abel average of \( S(t) \) is also converges uniformly.

Keywords: Cesàro means, Abel averages, Integrated semigroups, Uniform Abel ergodic, Uniform Cesàro ergodic.

MSC: 47D62

1 Introduction

Throughout this paper \( B(X) \) denotes the Banach algebra of all bounded linear operators on Banach space \( X \) into itself. Let \( A \) be a closed linear operator on \( X \) with domain \( D(A) \subseteq X \), we denote by \( N(A), R(A), \sigma(A), \rho(A) \) and \( R(\cdot, A) \), the kernel, the range, the spectrum, the resolvent set and the resolvent operator of \( A \), respectively.

The family \( \{T(t)\}_{t \geq 0} \) of bounded linear operator on \( X \) is called a strongly continuous semigroup (\( C_0 \)-semigroup in short [2]) if it has the following properties:

1. \( T(0) = I \),
2. \( T(t)T(s) = T(t+s) \),
3. The map \( t \to T(t)x \) from \([0, +\infty[\) into \( X \) is continuous for all \( x \in X \).

Their infinitesimal generator \( A \) is defined by:

\[
Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \quad \text{for all} \ x \in D(A),
\]

where \( D(A) = \{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \} \).

The Laplace transformation \( R(\lambda) \) of a \( C_0 \)-semigroup \( T(t) \) on \( B(X) \), defined as

\[
R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt,
\]

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which is exactly the resolvent function of $A$. Moreover, the infinitesimal generator of a $C_0$-semigroup is a linear closed densely defined operator on a Banach space $X$, see for instance [12] and [3].

Integrated semigroups and $n$-time Integrated semigroups, $n \in \mathbb{N}$, of operators in Banach space were introduced by Arendt [1] and studied by Arendt, Kellermann, Hieber [7], Thieme [17] and many others. A relevant example is obtained if we assume that $\{T(t)\}_{t \geq 0}$ is a $C_0$-semigroup of bounded linear operator on $X$, then $S(t) = \int_0^t T(r)dr$ defines an integrated semigroup $\{S(t)\}_{t \geq 0}$ having the following three properties:

1. $S(0) = 0$,
2. $S(s)S(t) = \int_0^s S(r + t) - S(r)dr$ for $t, s \geq 0$,
3. The map $t \to S(t)$ from $[0, +\infty]$ into $X$ is strongly continuous.

A relevant result obtained by S.Y. Shaw in [13] for a locally integrated semigroup, under an assumption weaker than $\omega_0 \leq 0$, that means $T(t)$ is uniformly Cesàro ergodic if and only if it satisfies the following conditions:

(i) The Laplace transformation $\mathcal{R}A$ exists for every $\lambda > 0$,
(ii) $\| T(t) \mathcal{R}A \|/t \to 0$ as $t \to \infty$, for some $\lambda > 0$, and
(iii) $T(t)$ is uniformly Abel ergodic.

The condition (i) holds whenever $\omega_0 \leq 0$, let us also mention that somewhat different necessary and sufficient
conditions are obtained in [14]. Clearly, if \( T(t) \) is uniformly Cesàro ergodic then is uniformly Abel ergodic, but the reverse is not true, for more information see [9, Chapter 2]. It is useful to mention that the limit of Cesàro averages and of Abel averages of \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) is the same, is the projection \( P \) of \( X \) onto \( N(A) \) parallel to \( \Re(A) \), corresponding to the ergodic decomposition

\[
\Re = \Re(A) \oplus N(A).
\]

The classical uniform ergodic theorem for \( C_0 \)-semigroups \( \{ T(t) \}_{t \geq 0} \) of bounded linear operators on \( X \), goes back to M. Lin in [10], he treats the Cesàro ergodicity of a \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) under the assumption \( \lim_{t \to 0^+} \| T(t) \|/t = 0 \), it showed that \( T(t) \) is uniformly Cesàro ergodic if and only if its infinitesimal generator \( A \) has a closed range if and only if \( T(t) \) is uniformly Abel ergodic. In this case, and under this latter assumption, we can easy checked that \( T(t) \) is uniformly Abel ergodic if and only if \( X = \Re(A) \oplus N(A) \). Moreover, this theory also plays an important role in the study of power convergence of linear operators. Recall that, an operator \( T \in \mathcal{B}(X) \) is called uniformly power convergent if there exists an operator \( P \in \mathcal{B}(X) \) such that \( \lim_{n \to \infty} \| T^n - P \| = 0 \). Lin, Shoikhet and Suciu in [11], showed that for a \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) on \( \mathcal{B}(X) \) satisfying \( \lim_{t \to 0^+} \| T(t) \|/t = 0 \), \( \{ T(t) \}_{t \geq 0} \) is uniformly ergodic if and only if there exists some \( \lambda > 0 \) such that the Abel average \( \lambda R(A, A) \) is uniformly power convergent. Kozitsky, Shoikhet and Zemànek in [8], obtained necessary and sufficient conditions for which the Abel averages of a \( C_0 \)-semigroup \( T(t) \) can be uniformly power convergent. Further condition have been obtained more recently by several authors [8, 16].

This paper is organized as follows. In section 2, we give some definitions and fundamental properties for an integrated semigroup of bounded linear operators on \( X \). In section 3, we are motivated by application to the ergodic theory for an integrated semigroup \( \{ S(t) \}_{t \geq 0} \) on \( \mathcal{B}(X) \). More precisely, We show that \( S(t) \) is uniformly Abel ergodic if and only if \( \Re = \Re(A) \oplus N(A) \), if and only if \( \Re(A^k) \) is closed for some integer \( k \) and \( \| \lambda^2 R(A, A) \| \to 0 \) as \( \lambda \to 0^+ \). Also, we show that if \( S(t) \) satisfying \( S(t)/t^2 \to 0 \) as \( t \to \infty \), then we have the following equivalent

(i) \( S(t) \) is uniformly Cesàro ergodic,
(ii) \( S(t) \) is uniformly Abel ergodic,
(iii) \( \Re(A^k) \) is closed for some integer \( k \geq 1 \),
(iv) The descent \( \text{des}(A) \) of \( A \) is finite.

## 2 Preliminaries

We start this present section by recalling an interesting concept in operator theory that we need in the sequel. Let \( A \) be a closed linear operator on a Banach space \( X \), with domain \( D(A) \subset X \). The smallest non-negative integer \( p \) such that \( N(A^p) = N(A^{p+1}) \) is called the ascent of \( A \) and denoted by \( \text{asc}(A) \). If such an integer does not exist, we set \( \text{asc}(A) = \infty \). Similarly, the smallest non-negative integer \( q \) such that \( \Re(A^q) = \Re(A^{q+1}) \) is called the descent of \( A \) and denoted by \( \text{des}(A) \). If such an integer does not exist, we set \( \text{des}(A) = \infty \).

Let \( A \) be a closed linear operator with \( D(A) \subset X \), if \( \text{asc}(A) \) and \( \text{des}(A) \) are both finite, then \( \text{asc}(A) \leq \text{des}(A) \), the equality holds when \( A \in \mathcal{B}(X) \), see [18, Theorem 6.2]).

For \( A \in \mathcal{B}(X) \), we have the following equivalences, see [4, Lemma 1.1]:

\[
\text{asc}(A) \leq p \iff \Re(A^p) \cap N(A^j) = \{ 0 \}; \quad j = 1, 2, \ldots
\]

\[
\text{des}(A) \leq q \iff X = \Re(A^j) + N(A^q); \quad j = 1, 2, \ldots
\]

The family \( \{ S(t) \}_{t \geq 0} \) of bounded linear operator on \( \mathcal{B}(X) \) is called integrated semigroup [7, Definition 1.1] if it has the following properties:

1. \( S(0) = 0 \),
2. \( S(s)S(t) = \int_0^s S(r + t) - S(r)dr \) for \( t, s \geq 0 \),

3. The map \( t \rightarrow S(t) \) from \([0, +\infty[\) into \( \mathcal{X} \) is strongly continuous.

The differentiation spaces \( C^n, n \geq 0 \), are defined by \( C^0 = \mathcal{X} \) and

\[
C^n = \{x \in \mathcal{X} : S(.)x \in C^n(\mathbb{R}^+; \mathcal{X})\}.
\]

Using this notion (2) can equivalently be formulated by \( S(t)x \in C^1 \) for all \( x \in \mathcal{X} \), and

\[
S'(r)S(t) = S(r + t) - S(r).
\]

The set \( N = \{x \in \mathcal{X} ; S(t)x = 0, \forall t \geq 0\} \) is called the degeneration space of the integrated semigroup \( \{S(t)\}_{t \geq 0} \).

In this case, \( \{S(t)\}_{t \geq 0} \) is called non-degenerate if \( N = \{0\} \) and degenerate otherwise.

The generator \( A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X} \) of a non-degenerate integrated semigroup \( \{S(t)\}_{t \geq 0} \) is defined as follows:

\[
x \in D(A) \text{ and } Ax = y \text{ if and only if } x \in C^1 \text{ and } S'(t)x - x = S(t)y \text{ for } t \geq 0.
\]

Usually, a non-degenerate integrated semigroup is uniquely determined by its generator. Motivated from the Laplace transform theory we can define the generator of an integrated semigroup as

\[
Ax = (\lambda - R_A^{-1}) \text{ for all } x \in D(A),
\]

where \( R_A = \lambda \int_0^{+\infty} \mu^{\lambda}S(t)dt \) is the The Laplace Transformation of \( \{S(t)\}_{t \geq 0} \), this integral does not exist in general, for more details see [7, Exemple 1.2].

To ensure the existence of this integral, we recall the following definition. An integrated semigroup \( \{S(t)\}_{t \geq 0} \) is called exponentially bounded, if there exist constants \( M > 0 \) and \( \omega \in \mathbb{R} \) such that

\[
\|S(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.
\]

In this case, the Laplace Transformation \( R_A \) exists for all \( \lambda \) with \( Re\lambda > \omega \). In General, an operator \( A \) is called generator of an integrated semigroup, if there exists \( \omega \in \mathbb{R} \) such that \( (\omega, +\infty) \subset \rho(A) \), and there exists a strongly continuous exponentially bounded family \( \{S(t)\}_{t \geq 0} \) of bounded linear operators on \( \mathcal{X} \) such that

\[
S(0) = 0, \text{ and }
(\lambda - A)^{-1} = \lambda \int_0^{+\infty} e^{\mu t}S(t)dt, \text{ for all } \lambda \text{ with } Re\lambda > \omega.
\]

In this case, the Laplace Transformation \( R_A \) of \( \{S(t)\}_{t \geq 0} \) is exactly the resolvent function of \( A \) and satisfies the pseudo-resolvent:

\[
R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu).
\]

For more details, we refer the interested reader to [1, 5, 7].

**Example 1.** 1. Let \( \mathcal{X} = C[0, 1] \) with \( AF = f' \) on \( D(A) = \{f \in C^1([0, 1]) : f(1) = 0\} \). The integrated semigroup generated by \( A \) is given by

\[
S(t)f(x) = \int_x^m f(s)ds, \text{ where } m = \min\{1, x + t\}.
\]

2. We consider \( \mathcal{X} = l^2 \) and the family \( \{S(t)\}_{t \geq 0} \) of bounded linear operators on \( \mathcal{X} \) defined by:

\[
S(t)(x_n)_{n \in \mathbb{N}^*} = \left(\int_0^t e^{\theta s}dxs_n\right)_{n \in \mathbb{N}^*},
\]
where \( a_n = n + 2^n \pi i \). Then \( \{ S(t) \}_{t \geq 0} \) is an integrated semigroup on \( X \).

3. Consider the Schrödinger operator \( A = i \Delta \) on \( L^p(\mathbb{R}) \) for \( p \geq 1 \). Then \( A \) generates an integrated semigroup \( \{ S(t) \}_{t \geq 0} \) given by

\[
f \mapsto \mathcal{F}^{-1}(u_t f) \text{ with } u_t(\xi) = \int_0^t e^{-i\xi s} ds.
\]

**Definition 2.1.** Let \( \{ S(t) \}_{t \geq 0} \) be an integrated semigroup on \( \mathcal{B}(X) \).

(i) We say that \( \{ S(t) \}_{t \geq 0} \) is uniformly Cesàro ergodic if the Cesàro averages of \( \{ S(t) \}_{t \geq 0} \) defined by

\[
\mathcal{C}(t) = \frac{1}{t^2} \int_0^t S(r)dr; \text{ for } t \geq 0,
\]

converges in the norm operator topology as \( t \) tend to infinity. Moreover, \( \{ S(t) \}_{t \geq 0} \) is said to be uniformly Cesàro bounded if there exists \( M > 0 \) such that

\[
\sup_{t \geq 0} \| \mathcal{C}(t) \| \leq M.
\]

(ii) We say that \( \{ S(t) \}_{t \geq 0} \) is uniformly Abel ergodic if the Abel averages of \( \{ S(t) \}_{t \geq 0} \) defined by

\[
\mathcal{A}(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda t} S(t)dt; \text{ for } t \geq 0,
\]

converges in the norm operator topology as \( \lambda \) tend to zero.

Now, we will need the following relations between integrated semigroups and their generators. The first result was investigated by W. Arendt [1] in the case of \( n \)-times integrated semigroup on \( \mathcal{B}(X) \), where \( n \in \mathbb{N} \).

**Proposition 2.1.** [1, Proposition 3.3] Let \( A \) be the generator of an integrated semigroup \( \{ S(t) \}_{t \geq 0} \) on \( \mathcal{B}(X) \). Then

1. For all \( x \in D(A) \) and all \( t \geq 0 \), we have: \( S(t)x \in D(A) \) and \( AS(t)x = S(t)Ax \).

   Moreover, \( S(t)x = \int_0^t S(s)Ax ds + tx \).

2. For all \( x \in X \), \( \int_0^t S(s)x ds \in D(A) \), and

   \[
   A \int_0^t S(s)x ds = S(t)x - tx.
   \]

**Lemma 2.1.** [15, Lemma 2.3] Let \( A \) be the generator of an integrated semigroup \( \{ S(t) \}_{t \geq 0} \) on \( \mathcal{B}(X) \). Then we have the following assertions:

1. \( \mathcal{R}(A) = (A R(\lambda, A) - I)X \).
2. \( \mathcal{N}(A) = N(AR(\lambda, A) - I) = \{ x \in X : S(t)x = tx; \text{ for all } t \geq 0 \} \).

We recall the following result which was recently proved in [16] for an \( a \)-times integrated semigroup \( \{ S(t) \}_{t \geq 0} \) on \( \mathcal{B}(X) \), where \( a \geq 0 \).

**Theorem 2.1.** [16, Theorem 2.2] Let \( A \) be the generator of an integrated semigroup \( \{ S(t) \}_{t \geq 0} \) on \( \mathcal{B}(X) \) such that

\[
\lim_{t \to \infty} \frac{\| S(t) \|}{t} = 0.
\]

If \( \mathcal{R}(A) \) is closed, then \( \{ S(t) \}_{t \geq 0} \) is uniformly Cesàro ergodic.
3 Main results

We began this section by the following two lemmas which will be widely used in the sequel.

**Lemma 3.1.** Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(X)$. If $\|\lambda^2 R(\lambda, A)\| \to 0$ as $\lambda \to 0^+$, then $\mathcal{R}(A) \cap N(A) = \{0\}$, which yields $\text{asc}(A) \leq 1$.

**Proof.** Let $\{S(t)\}_{t \geq 0}$ be an integrated semigroup on $\mathcal{B}(X)$ with $A$ be their generator and $R(\lambda, A)$ the resolvent function of $A$. We assume that $\|\lambda^2 R(\lambda, A)\| \to 0$ as $\lambda \to 0^+$ and let $y \in \mathcal{R}(A) \cap N(A)$, then by the second assertion of Lemma 2.1, we get $\lambda R(\lambda, A)y = y$, for all $\lambda \in \rho(A)$.

Since $\mathcal{R}(A) = R(\mathcal{R}(A) - I)$, then there exists $x \in X$ and $M > 0$, such that $y = (\lambda R(\lambda, A) - I)x$ and $\|x\| \leq M\|y\|$.

Next, from the resolvent equation:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \quad \text{for all } \lambda \neq \mu \in \rho(A).$$

We get the following inequality, for all $\lambda \neq \mu$,

$$\|\lambda R(\lambda, A)y\| \leq |\mu - \lambda|^{-1}\left[\|\lambda^2 R(\lambda, A)\| + |\lambda||\mu R(\mu, A)\|\right]\|x\| \leq M|\mu - \lambda|^{-1}\left[\|\lambda^2 R(\lambda, A)\| + |\lambda||\mu R(\mu, A)\|\right]\|y\|.$$

It follows that $\lambda R(\lambda, A)y \to 0$ as $\lambda \to 0^+$, which yields that $y = 0$. Therefore $\mathcal{R}(A) \cap N(A) = \{0\}$. 

**Lemma 3.2.** Let $A$ be a closed linear operator with domain $D(A) \subset X$, such that $\text{asc}(A) = d < \infty$. If either of the following hold:

(i) $\mathcal{R}(A^n)$ is closed for some $n > d$, or
(ii) $\mathcal{R}(A^j) + N(A^k)$ is closed for some positive integers with $j + k = n > d$,

then $\mathcal{R}(A^n)$ is closed for all $n \geq d$, and $\mathcal{R}(A^j) + N(A^k)$ is closed for all $j + k > d$.

**Proof.** Let $A$ and $B$ are closed linear operators on a Banach space $X$, if $\mathcal{R}(B) \subset D(A)$ then we have the following identity:

$$A^{-1}\mathcal{R}(AB) = \mathcal{R}(B) + N(A).$$

Then, for a linear operator $A$ on a Banach space $X$ and for some integers $j$ and $k$, we get

$$A^{-k}\mathcal{R}(A^j A^k) = \mathcal{R}(A^j) + N(A^k).$$

Then, we obtain the following results:

(i) If $\mathcal{R}(A^n)$ is closed, so is $\mathcal{R}(A^j) + N(A^k)$ whenever $j + k = n$.

(ii) For $n \geq d$, $\mathcal{R}(A^n)$ is closed whenever $\mathcal{R}(A^m) + N(A^m)$ is closed for some $m \geq 1$.

Assume that $A$ be a closed linear operator with domain $D(A) \subset X$ such that $\text{asc}(A) = d < \infty$, then

$$\mathcal{R}(A^d) \cap N(A^m) = \{0\}, \text{ for all } m = 1, 2, \ldots.$$
Proof. Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:

1. $S(t)$ is uniform Abel ergodic,
2. $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$,
3. $\|\lambda^2 R(\lambda, A)\| \to 0$ as $\lambda \to 0^+$ and $\mathcal{B}(A)$ is closed,
4. $\|\lambda^2 R(\lambda, A)\| \to 0$ as $\lambda \to 0^+$ and the descent $\text{des}(A)$ is finite.

Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(\mathcal{X})$, then there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$ 

Therefore, the Laplace Transformation $R_1$ of $\{S(t)\}_{t \geq 0}$ is exactly the resolvent of $A$. It follows that the Abel averages $A(\lambda)$ of $\{S(t)\}_{t \geq 0}$ satisfying the following equality

$$\lim_{\lambda \to 0^+} A(\lambda) = \lim_{\lambda \to 0^+} \lambda^2 \int_0^\infty e^{-\lambda t} S(t) dt; \text{ for } t \geq 0,$$

$$= \lim_{\lambda \to 0^+} \lambda R(A, A).$$

(1) $\Rightarrow$ (2) It is known from the mean ergodic theorem [19, p. 217] that if there exists an operator $P \in \mathcal{B}(\mathcal{X})$ such that $\|\lambda R(A, A) - P\| \to 0$ as $\lambda \to 0^+$, then $P$ is the projection of $\mathcal{X}$ onto $\mathcal{N}(\lambda R(A, A) - I)$ parallel to $(\lambda R(A, A) - I)\mathcal{X}$, and by Lemma 2.1, we get

$$\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).$$

(2) $\Rightarrow$ (3) Assume that $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A)$ where $\mathcal{R}(A)$ is closed. It follows from the second assertion of Lemma 2.1 that for all $x \in \mathcal{N}(A)$ we have $\lambda R(A, A)x = x$. Then

$$\|\lambda^2 R(\lambda, A)|_{\mathcal{N}(A)}\| \to 0 \text{ as } \lambda \to 0^+.$$

So, to complete the proof we show that $\|\lambda^2 R(\lambda, A)|_{\mathcal{R}(A)}\| \to 0$ as $\lambda \to 0^+$. Indeed, let's denote $Y = \mathcal{R}(A)$ and $A_1$ be the generator of the restriction of $T(t)$ to $Y$, which is equal to the restriction of $A$ to $Y \cap \mathcal{D}(A)$. From the first assertion of Lemma 2.1 we have $Y = \mathcal{R}(\lambda R(A, A) - I)$. Since $\mathcal{R}(\lambda R(A, A) - I) \cap \mathcal{N}(\lambda R(A, A) - I) = \{0\}$, then the operator $(\lambda R(A, A) - I)$ is invertible on $Y$.

Now, let $y \in Y \cap \mathcal{D}(A)$ such that $A_1y = 0$, hence

$$y = R(A, A)(\lambda - A)y = \lambda R(A, A)y - R(A, A)y = AR(A, A)y.$$ 

Then $y \in \mathcal{N}(\lambda R(A, A) - I)$, which implies that $y = 0$. Thus $A_1$ is one to one. Clearly, we have $R(A, A)Y \subseteq Y$, hence we obtain that $(\lambda R(A, A) - I)Y \subseteq \mathcal{R}(A_1)$.

Then, we get the following

$$Y \supseteq \mathcal{R}(A_1) \supseteq (\lambda R(A, A) - I)Y = (\lambda R(A, A) - I)\mathcal{X} = \mathcal{R}(A) = Y.$$ 

Hence $Y = \mathcal{R}(A_1)$, so $A_1^{-1}$ is defined on all $Y$, since $A_1$ is closed, therefore $A_1^{-1}$ is closed, and by the closed graph theorem $A_1^{-1}$ is continuous.

Let $0 < \lambda < \delta < \frac{1}{\|A_1^{-1}\|}$ and $y \in Y$, we get

$$\|\lambda^2 R(\lambda, A)y\| = \|\lambda^2 R(\lambda, A)A_1^{-1}y\|.$$
Therefore
\[
\|A^2 R(\lambda, A) y\| \leq \lambda^2 \left( \| \lambda R(\lambda, A) - I \| + 1 \right) \| A_1^{-1} \| \| y \|.
\]

Also, we have
\[
\| \lambda R(\lambda, A) \| \leq \delta \left( \| \lambda R(\lambda, A) \| + 1 \right) \| A_1^{-1} \|.
\]

Then, we get
\[
\| \lambda R(\lambda, A) \| \leq \frac{\delta \| A_1^{-1} \|}{1 - \delta \| A_1^{-1} \|} = M.
\]

Therefore
\[
\| A^2 R(\lambda, A) y\| \leq \lambda^2 \| \lambda R(\lambda, A) - I \| \| A_1^{-1} \| \| y \|
\leq \lambda^2 \left( \| \lambda R(\lambda, A) \| + 1 \right) \| A_1^{-1} \| \| y \|
\leq \lambda^2 (M + 1) \| A_1^{-1} \| \| y \|,
\]
which implies that \( \| A^2 R(\lambda, A) y \| \to 0 \) as \( \lambda \to 0^+ \). Hence the assertion (3) holds.

(3) \(\Rightarrow\) (4) suppose that \( \| A^2 R(\lambda, A) \| \to 0 \) as \( \lambda \to 0^+ \), and \( \mathcal{R}(A) \) is closed.

By Lemma 2.1, we have \( \mathcal{R}(A) = \{ \lambda R(\lambda, A) - I \} \mathcal{X} \), which means that for all \( \lambda > 0 \), the operator \( \lambda R(\lambda, A) - I \) has a closed range. Fix \( \mu > 0 \) such that for each \( y \in \{ \mu R(\mu, A) - I \} \mathcal{X} \), there exists a \( M > 0 \) and \( x \in \mathcal{X} \) such that \( y = (\mu R(\mu, A) - I) x \) and \( \| x \| \leq M \| y \| \). So, we have
\[
\lambda R(\lambda, A) (\mu R(\mu, A) - I) = \lambda \mu R(\lambda, A) R(\mu, A) - \lambda R(\lambda, A).
\]

By the resolvent equation:
\[
R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A),
\]
we obtain
\[
\lambda R(\lambda, A) (\mu R(\mu, A) - I) = \lambda \mu R(\lambda, A) R(\mu, A) - \lambda (\mu - \lambda) R(\lambda, A) R(\mu, A) - \lambda R(\lambda, A)
\]
\[
= \lambda^2 R(\lambda, A) R(\mu, A) - \lambda R(\lambda, A)
\]
\[
= \lambda^2 (\mu - \lambda)^{-1} \left[ R(\lambda, A) - R(\mu, A) \right] - \lambda R(\lambda, A)
\]
\[
= (\mu - \lambda)^{-1} \left[ \lambda^2 (\lambda, A) - \lambda \mu R(\mu, A) \right].
\]

This gives
\[
|\lambda R(\lambda, A) y\| = |\lambda R(\lambda, A) (\mu R(\mu, A) - I) x|
\]
\[
= |(\mu - \lambda)^{-1} \left[ \lambda^2 R(\lambda, A) - \lambda \mu R(\mu, A) \right] x|
\]
\[
\leq |\mu - \lambda|^{-1} \left[ |\lambda^2 R(\lambda, A)\| + |\lambda| \| \mu R(\mu, A) \| \right] M \| y \|.
\]

It follows that
\[
|\lambda R(\lambda, A) y\|_{(AR(\lambda, A) - I) \mathcal{X}} \to 0 \text{ as } \lambda \to 0^+.
\]

Then for a small \( \lambda > 0 \), the operator \( \lambda R(\lambda, A) - I \) is invertible on \( (\lambda R(\lambda, A) - I) \mathcal{X} \). Therefore
\[
(\lambda R(\lambda, A) - I)^{-2} \mathcal{X} = (\lambda R(\lambda, A) - I) \mathcal{X},
\]
which yields \( \mathcal{X} = (\lambda R(\lambda, A) - I) \mathcal{X} + N(\lambda R(\lambda, A) - I) \). It follows from Lemma 2.1 that \( \mathcal{X} = \mathcal{R}(A) + N(A) \) where \( \mathcal{R}(A) \cap N(A) = \{0\} \), hence \( \mathcal{X} = \mathcal{R}(A) \oplus N(A) \). Therefore, it follows from Lemma 2.1
\[
\mathcal{X} = (\lambda R(\lambda, A) - I) \mathcal{X} \oplus N(\lambda R(\lambda, A) - I).
\]

Moreover, as shown above that \( |\lambda R(\lambda, A) y\|_{(AR(\lambda, A) - I) \mathcal{X}} \to 0 \) as \( \lambda \to 0^+ \). Since for all \( x \in N(\lambda R(\lambda, A) - I) \), we have \( \lambda R(\lambda, A) x = x \), then \( \lambda R(\lambda, A) y_{|N(AR(\lambda, A) - I)} \) converge to the identity operator \( I \) as \( \lambda \to 0^+ \). It follows from the decomposition (3.2) that \( \lambda R(\lambda, A) \) converges uniformly on \( \mathcal{X} \). Hence the assertion (1) holds. \( \square \)
The following corollary is an immediate consequence of the Theorem 3.2 and of the Lemma 3.2.

**Corollary 3.1.** Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(X)$. Then the following assertions are equivalent:

1. $S(t)$ is uniform Abel ergodic,
2. $\|\lambda^2 R(\lambda, A)\| \rightarrow 0$ as $\lambda \rightarrow 0^+$ and $\mathcal{B}(A^k)$ is closed for some integer $k$,
3. $\|\lambda^2 R(\lambda, A)\| \rightarrow 0$ as $\lambda \rightarrow 0^+$ and $\mathcal{B}(A^k) + \mathcal{N}(A^l)$ is closed for some $k, j > 0$.

**Proof.** Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(X)$ and let $R(\lambda, A)|_{\mathcal{R}(A)}$ be the restriction of $R(\lambda, A)$ onto $\mathcal{R}(A)$. As shown above in the proof of Theorem 3.2 that if $S(t)$ is uniformly Abel ergodic, then $\mathcal{R}(A)$ is closed and $\|AR(\lambda, A)|_{\mathcal{R}(A)}\| \rightarrow 0$ as $\lambda \rightarrow 0^+$ by (3.1).

Conversely, we assume that $\mathcal{R}(A)$ is closed and $\|AR(\lambda, A)|_{\mathcal{R}(A)}\| \rightarrow 0$ as $\lambda \rightarrow 0^+$.

Since $\mathcal{R}(A) = (AR(\lambda, A) - I)X$, then $\|AR(\lambda, A)|_{(AR(\lambda, A) - I)X}\| \rightarrow 0$ as $\lambda \rightarrow 0^+$, then for a small $\lambda$, the operator $(AR(\lambda, A) - I)|_{(AR(\lambda, A) - I)X}$ is invertible and we have

$$\mathcal{R}(AR(\lambda, A) - I) = \mathcal{R}((AR(\lambda, A) - I)|_{(AR(\lambda, A) - I)X}) = \mathcal{R}((AR(\lambda, A) - I)^2).$$

Therefore

$$X = (AR(\lambda, A) - I)X + N(AR(\lambda, A) - I). \quad (3.3)$$

Now, let $y \in \mathcal{R}(AR(\lambda, A) - I) \cap N(AR(\lambda, A) - I)$, so $AR(\lambda, A)y = y$ for all $\lambda > 0$, and by assumption $AR(\lambda, A)y \rightarrow 0$ as $\lambda \rightarrow 0^+$, hence $y = 0$ which yields that

$$(AR(\lambda, A) - I)X \cap N(AR(\lambda, A) - I) = \{0\}. $$

Then, the summation in (3.3) is direct and from Lemma 2.1 $X = \mathcal{R}(A) \oplus N(A)$.

Finally, Theorem 3.2 implies that $S(t)$ is uniformly Abel ergodic. $\square$

Now, we present our second main result as follows.

**Theorem 3.2.** Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(X)$. Assume that $\lim_{t \rightarrow \infty} \|S(t)\|/t^2 = 0$, then $S(t)$ is uniformly Abel ergodic if and only if $S(t)$ is uniformly Cesàro ergodic.

**Proof.** Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(X)$ such that $\lim_{t \rightarrow \infty} \|S(t)\|/t^2 = 0$. Let $S(t)$ be uniformly Abel ergodic, then by Theorem 3.1, we obtain the decomposition $X = \mathcal{R}(A) \oplus N(A)$, with $\mathcal{R}(A)$ is closed.

From the second assertion of Lemma 2.1, we can easily check that for all $x \in N(A)$,

$$\lim_{t \rightarrow \infty} \frac{1}{t^2} \int_0^t S(r)xdr - \frac{tx}{2} = 0.$$

So, to complete the proof we show that $\frac{1}{t^2} \int_0^t S(t)ydr \rightarrow 0$ as $t \rightarrow \infty$, for all $y \in \mathcal{R}(A)$. Since $\mathcal{R}(A)$ is closed, let’s denote $A_1$ be the generator of the restriction of $S(t)$ to $\mathcal{R}(A)$, which is equal to the restriction of $A$ to $\mathcal{R}(A) \cap D(A)$. As shown in the proof of Theorem 3.1 that $A_1^{-1}$ is defined on all $\mathcal{R}(A)$ and continuous, then for all
\( y \in \mathcal{R}(A) \), there exists \( x \in D(A) \) such that \( y = A_1 x \) and \( \|x\| \leq \|A_1^{-1}\| \|y\| \). the second assertion of Proposition 2.1 implies that for all \( x \in D(A) \), we have

\[
\int_0^t S(s)Axds = S(t)x - tx; \quad \text{for all } t \geq 0.
\]

It follows that, we get

\[
\left\| \frac{1}{t^2} \int_0^t S(r)yd\!r \right\| = \left\| \frac{1}{t^2} (S(t)x - tx) \right\|
\leq \|A_1^{-1}\| \left( \left\| S(t) \right\| + \left\| \frac{1}{t} \right\| \right) \|y\|.
\]

Since \( \lim_{t \to \infty} \left\| S(t) \right\| /t^2 = 0 \), then \( \left\| \frac{1}{t^2} \int_0^t S(r)yd\!r \right\| \to 0 \) as \( t \to \infty \) for all \( y \in \mathcal{R}(A) \). Therefore \( S(t) \) is uniformly Cesàro ergodic.

Conversely, let \( S(t) \) be uniformly Cesàro ergodic, hence there exists an operator \( P \in \mathcal{B}(X) \) such that \( \| \mathcal{C}(t) - P \| \to 0 \) as \( t \to \infty \), where \( \mathcal{C}(t) = \frac{1}{t^2} \int_0^t S(r)dr \) for \( t \geq 0 \). So, there exists \( \varepsilon > 0 \) and \( a > 0 \) such that \( \| \mathcal{C}(t) - P \| \leq \varepsilon \) for all \( t > a \).

Now, let’s denote \( \mathcal{W}(t) = \int_0^t S(r)dr \) for all \( t \geq 0 \). Then, using the integration by parts, we get the following identity:

\[
R(\lambda, A) = \lambda^2 \int_0^\infty e^{-\lambda t} \mathcal{W}(t)dt, \quad \text{for all } t \geq 0. \tag{3.4}
\]

Moreover, we have the following identity for any bounded linear operator \( P \in \mathcal{B}(X) \)

\[
\lambda^{a+1} \int_0^\infty e^{-\lambda t}t^ap\ dt = (a!)P, \quad \text{for all } \lambda \in \mathbb{C} \text{ and } a, t \geq 0. \tag{3.5}
\]

It follows from (3.4) and (3.5), we have for every \( x \in X \)

\[
\left\| \lambda R(\lambda, A)x - 2Px \right\| = \left\| \lambda R(\lambda, A)x - \lambda^3 \int_0^\infty e^{-\lambda t}t^2Px\ dt \right\|
= \left\| \lambda^3 \int_0^\infty e^{-\lambda t}t^2xdt - \lambda^3 \int_0^\infty e^{-\lambda t}t^2Px\ dt \right\|
\leq \left\| \lambda^3 \int_0^\infty e^{-\lambda t}(\mathcal{W}(t) - (t^2)P)dt \right\| \|x\|
\leq \left[ \lambda^3 \int_0^a e^{-\lambda t} \left( \|\mathcal{W}(t)\| + t^2\|P\| \right) dt \right] \|x\|
+ \left[ \lambda^3 \int_a^\infty e^{-\lambda t} \mathcal{W}(t) - t^2Pdt \right] \|x\|
\leq \left[ \lambda^3 \int_0^a e^{-\lambda t} \left( \|\mathcal{W}(t)\| + t^2\|P\| \right) dt \right]
+ \left[ \lambda^3 \int_a^\infty e^{-\lambda t} \mathcal{W}(t) - t^2Pdt \right] \|x\|
Proof. Let \(A\) be the generator of an integrated semigroup \(\{S(t)\}_{t \geq 0}\) on \(\mathcal{B}(X)\). Assume that 
\[
\lim_{t \to \infty} \|S(t)\|/t^2 = 0,
\]
then the following assertions are equivalent:

1. \(S(t)\) is uniformly Cesàro ergodic,
2. \(\mathcal{R}(A^k)\) is closed for some integer \(k \geq 1\),
3. \(\mathcal{R}(A^h) + \mathcal{N}(A^l)\) is closed for some integers \(h, l \geq 1\),
4. The descent \(\text{des}(A)\) of \(A\) is finite.

Proof. Let \(A\) be the generator of an integrated semigroup \(\{S(t)\}_{t \geq 0}\) on \(\mathcal{B}(X)\). First we must show that 
\[
\lim_{t \to \infty} \|S(t)\|/t^2 = 0
\]
implies that \(\|\lambda^2 R(\lambda, A)\| \to 0\) as \(\lambda \to 0^+\). Indeed, if 
\[
\lim_{t \to \infty} \|S(t)\|/t^2 = 0,
\]
then there exists \(\varepsilon > 0\) and \(a > 0\) such that 
\[
\|S(t)\| \leq \varepsilon t^2, \quad \text{for all } t > a.
\]
Using the resolvent equation, we obtain for all \(x \in X\) and \(\mu \neq \lambda\):
\[
\|\lambda^2 R(\lambda, A)x\| = \|\lambda^2 [R(\mu, A) + (\mu - \lambda)R(\lambda, A)]x\| 
\leq \|\lambda^2 R(\mu, A)x\| + |\mu - \lambda| \|\lambda^2 R(\lambda, A)x\| 
\leq \|\lambda^2 R(\mu, A)x\| + |\mu - \lambda| \lambda^3 \int_0^a e^{-\lambda t} \|S(t)R(\mu, A)x\| dt 
\leq \|\lambda^2 R(\mu, A)x\| + |\mu - \lambda| \lambda^3 \int_0^a e^{-\lambda t} \|S(t)R(\mu, A)x\| dt 
+ \varepsilon \lambda^3 \int_a^\infty e^{-\lambda t^2} \|R(\mu, A)x\| dt.
\]
From the identity (3.5), we obtain 
\[
\|\lambda^2 R(\lambda, A)x\| \leq \|\lambda^2 R(\mu, A)x\| + |\mu - \lambda| \lambda^3 \left( \sup_{t \geq a} \|S(t)\| \|R(\mu, A)x\| \right) 
+ 2\varepsilon \|R(\mu, A)x\| \|x\|.
\]
It follows from the above estimate that \(\|\lambda^2 R(\lambda, A)\| \to 0\) as \(\lambda \to 0^+\). So, according to Corollary 3.1, Theorem 3.1 and Theorem 3.2, the equivalents hold.

Corollary 3.3. Let \(A\) be the generator of an integrated semigroup \(\{S(t)\}_{t \geq 0}\) on \(\mathcal{B}(X)\). Assume that 
\[
\lim_{t \to \infty} \|S(t)\|/t^2 = 0.
\]
Then the following assertions are equivalent:

1. \(S(t)\) is uniformly Cesàro ergodic,
2. \(\mathcal{R}(A^k)\) is closed for some integer \(k \geq 1\),
3. \(\mathcal{R}(A^h) + \mathcal{N}(A^l)\) is closed for some integers \(h, l \geq 1\),
4. The descent \(\text{des}(A)\) of \(A\) is finite.

Proof. Assume that \(\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)\) is uniformly Cesàro ergodic, then there exists \(M > 0\) such that 
\[
\sup_{t \geq 0} \left\| \frac{1}{t^2} \int_0^t S(s) ds \right\| \leq M.
\]
Let’s denote \( \mathcal{W}(t) = \int_0^t S(s) ds \) for all \( t \geq 0 \), and let \( \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > 0 \). Then, from the integration by parts, we have for all \( 0 < u < v \) and \( x \in \mathcal{X} \)

\[
\| \lambda \int_u^v e^{-\lambda t} S(t)x dt \| = \left\| \lambda \left[ (e^{-\lambda t} S(t)x)_v^\nu + \lambda^2 \int_u^v e^{-\lambda t} S(t)x dt \right] \right\|
\]

\[
\leq M \left\| \lambda \left[ (e^{-\lambda t} t^2)_v^\nu + \lambda^2 \int_u^v e^{-\lambda t} t^2 dt \right] \right\| \| x \|
\]

\[
\leq M \left[ \| \lambda \| (e^{-\lambda u} v^2 + e^{-\lambda u} u^2) + \| \lambda \|^2 \int_u^v e^{-\Re(\lambda) t^2} dt \right] \| x \|.
\]

It follows that \( \| \lambda \int_u^v e^{-\lambda t} S(t)dt \| \to 0 \) as \( u \to \infty \). Finally, we deduce that the Laplace Transformation \( R_A \) of \( \{S(t)\}_{t \geq 0} \) exists for all \( \lambda \) with \( \Re \lambda > 0 \).

\[\square\]

**Proposition 3.2.** Let \( \{S(t)\}_{t \geq 0} \) be an integrated semigroup on \( \mathcal{B}(\mathcal{X}) \). Then, \( S(t) \) is uniformly Cesàro ergodic if and only if the restriction of \( S(t) \) to \( \mathcal{R}(A) \) is uniformly Cesàro ergodic.

**Proof.** Let \( A \) be the generator of an integrated semigroup \( \{S(t)\}_{t \geq 0} \) on \( \mathcal{B}(\mathcal{X}) \). Let’s denote \( \{\Omega(t)\}_{t \geq 0} \) be the restriction of \( \{S(t)\}_{t \geq 0} \) to \( \mathcal{R}(A) \), and \( A_1 \) be their generator, which is exactly the restriction of \( A \) to \( \mathbb{R}(A) \cap D(A) \).

As shown above that if \( S(t) \) is uniformly Cesàro ergodic, then the Cesàro averages of \( S(t) \) converges uniformly to the projection \( \pi \) onto \( \mathcal{N}(A) \) along \( \mathcal{R}(A) \), corresponding to the ergodic decomposition \( \mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A) \) where \( \mathcal{R}(P) = \mathcal{N}(A) \) and \( \mathcal{N}(P) = \mathcal{R}(A) \). Since \( \mathcal{R}(A) \) is closed and \( S(t) \)-invariant for all \( t \geq 0 \), then we easily check that \( \Omega(t) \) is uniformly Cesàro ergodic to 0, which means that the Cesàro averages of \( \Omega(t) \) converge to 0 as \( t \to \infty \).

Conversely, we assume that \( \Omega(t) \) is uniformly Cesàro ergodic, so it follows from Theorem 3.2 that \( \Omega(t) \) is also uniformly Abel ergodic and \( \mathcal{R}(A) = \mathcal{R}(A_1) \oplus \mathcal{N}(A_1) \) where \( \mathcal{R}(A_1) \) is closed. Moreover, we have

\[
\left\| \frac{1}{t^2} \int_0^t S(r)x dr \right\| \to 0 \text{ as } t \to \infty, \text{ for all } x \in \mathcal{R}(A).
\]

Let \( x \in \mathcal{R}(A) \), hence from the above decomposition, we have \( x = y + z \) where \( y \in \mathcal{R}(A_1) \) and \( z \in \mathcal{N}(A_1) \). Since \( \mathcal{N}(A_1) \subset \mathcal{N}(A) \), then

\[
z = \Omega(t)z = \frac{1}{t^2} \int_0^t \Omega(r)z dr, \text{ for all } t \geq 0.
\]

Therefore, we obtain

\[
\left\| \frac{1}{t^2} \int_0^t \Omega(r)z dr \right\| = \left\| \frac{1}{t^2} \int_0^t \Omega(r)(x - y) dr \right\|
\]

\[
\leq \left\| \frac{1}{t^2} \int_0^t \Omega(r)x dr \right\| + \left\| \frac{1}{t^2} \int_0^t \Omega(r)y dr \right\|.
\]

It follows that \( \left\| \frac{1}{t^2} \int_0^t \Omega(r)z dr \right\| \to 0 \) as \( t \to \infty \), hence \( z = 0 \) which gives that \( x = y \). Therefore \( \mathcal{R}(A) = \mathcal{R}(A_1) \), since \( \mathcal{R}(A_1) \subset \mathcal{R}(A) \), then \( \mathcal{R}(A) \) is closed.
Furthermore, since $Q(t)$ is uniformly Abel ergodic and their Cesáro averages converge to 0, then
\[
\|AR(\lambda, A)\|_{\mathcal{B}(\mathcal{A})} \to 0 \text{ as } \lambda \to 0^+.
\]
Consequently, $T(t)$ is uniformly Abel ergodic by Corollary 3.2, and we obtain
\[
\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{N}(A).
\]
The projection $P$ of $\mathcal{X}$ onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$ corresponding to this decomposition is bounded, so we have $\mathcal{N}(I - P) = \mathcal{R}(P) = \mathcal{N}(A)$, and $\mathcal{R}(I - P) = \mathcal{N}(P) = \mathcal{R}(A)$. Then it follows from Lemma 2.1 that for all $x \in \mathcal{X}$
\[
\frac{1}{t^2} \int_0^t S(r) P x \, dr = \frac{1}{2} P x.
\]
So, we obtain
\[
\left\| C(t)x - \frac{1}{2} P x \right\| = \left\| \frac{1}{t^2} \int_0^t S(r) x \, dr - \frac{1}{t^2} \int_0^t S(r) P x \, dr \right\|
\]
\[
= \left\| \frac{1}{t^2} \int_0^t S(r) (x - P x) \, dr \right\|
\]
\[
= \left\| \frac{1}{t^2} \int_0^t Q(r) (x - P x) \, dr \right\|
\]
\[
\leq \left\| \frac{1}{t^2} \int_0^t Q(r) \, dr \right\| \left\| x - P x \right\|
\]
\[
\leq \left\| \frac{1}{t^2} \int_0^t Q(r) \, dr \right\| (1 + \|P\|) \|x\|.
\]
Then $\left\| C(t)x - \frac{1}{2} P x \right\| \to 0$ as $t \to \infty$. Finally, $S(t)$ is uniformly Cesáro ergodic. □

**Remark 3.1.** Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{B}(\mathcal{X})$. If $\lim_{t \to \infty} \|S(t)\|/t = 0$, then $A$ is one to one by [15, Corollary 2.4]. In this case, if the Cesáro averages and the Abel averages converge, they will converge to zero and we obtain $\mathcal{X} = \mathcal{R}(A)$. Moreover, the strong limit of $\frac{1}{t} \int_0^t S(r) \, dr$ may be divergent when $t \to \infty$, as the following example shows.

**Example:** Let $X = C([0, \infty])$, we consider the derivation operator $Af = -f'$ for all $f \in D(A)$, with $D(A) = \{f \in C^1([0, 1]) : f(0) = 0\}$. Since the domain $D(A)$ is not dense in $X$, then $A$ cannot be an infinitesimal generator of a $C_0$-semigroup. Furthermore, the semigroup $S(t)$ generated by $A$, is given by:
\[
(S(t)f)(x) = \begin{cases} 
- \int_0^{x-t} f(s) \, ds, & \text{si } x > t, \\
\int_x^t f(s) \, ds, & \text{si } 0 \leq x \leq t,
\end{cases}
\]
Note that $S(t)$ is an integrated semigroup of type $\omega_0 = 0$, where
\[
\omega_0 = \inf \{w \in \mathbb{R} : \text{ there exists } M \text{ such that } \|S(t)\| \leq Me^{wt}, \ t \geq 0\}.
\]
It follows that $\|S(t)\|/t \to 0$ as $t \to \infty$.

On the other hand, since the generator $A$ has an empty spectrum, then $X = R(A)$, and by Theorem 3.1 and Corollary 3.3, we deduce that $S(t)$ is both uniformly Cesàro ergodic and uniformly Abel ergodic. More precisely, we have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S(r)dr = \lim_{\lambda \to 0^+} \lambda^2 \int_0^\infty e^{-\lambda t}S(t)dt = 0.$$

Now, we suppose that then there exists an operator $P$ such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S(r)dr - P = 0.$$

Hence $P$ is a projection operator of $X$ onto $N(A)$ parallel to $R(A)$, corresponding to ergodic decomposition $X = R(A) \oplus N(P)$. As shown above that $X = R(A)$, then we deduce that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S(r)dr = 0.$$

Next, let $f$ be a non-zero function defined on $X$, hence there exists $g \in D(A)$ such that $f = Ag = -g'$. Therefore, we obtain for $0 < x \leq s$

$$\int_0^t (S(s)f)(x)ds = \int_0^t (S(s)Ag)(x)ds = -\int_0^t (S(s)g')(x)ds = -\int_0^t \int_0^x g'(r)dr = -\int_0^t (g(0) - g(x))ds = g(x)t.$$

By assumption $\lim_{t \to \infty} \frac{1}{t} \int_0^t S(r)dr = 0$, hence $g(x) = 0$ for all $0 < x \leq s$, absurd.

Finally, $\frac{1}{t} \int_0^t S(r)dr$ is divergent as $t$ tends to infinity.

**Acknowledgement:** The author wishes to express their indebtedness to the referee, for his suggestions and valuable comments on this paper.

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