Global Existence of Solutions for the Einstein-Boltzmann System with Cosmological Constant in the Robertson-Walker space-time for arbitrarily large initial data

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Abstract
We prove a global in time existence theorem for the initial value problem for the Einstein-Boltzmann system, with positive cosmological constant and arbitrarily large initial data, in the spatially homogeneous case, in a Robertson-Walker space-time.

1 Introduction
In the mathematical study of General Relativity, one of the main problems is to establish the existence and to give the properties of global solutions of the Einstein equations coupled to various field equations. The knowledge of the global dynamics of the relativistic kinetic matter is based on such results. In the case of Collisionless matter, the phenomena are governed by the Einstein-Vlasov system in the pure gravitational case, and by this system coupled to other fields equations, if other fields than the gravitational field are involved. In the collisionless case, several authors proved global results, see [10], [18] for reviews, [17], [22] and [5] for scalar matter fields, also see [20], [21] for the Einstein-Vlasov system with a cosmological constant. Now in the case of Collisional matter, the Einstein-Vlasov system is replaced by the Einstein-Boltzmann system, that seems to be the best approximation available and that describes the case of instantaneous, binary and elastic collisions. In contrast with the abundance of works in the collisionless case, the literature is very poor in the collisional case. If, due to its importance in collisional kinetic theory, several authors studied and proved global results for the single Boltzmann equation, see [5], [11], [11] for the non-relativistic case, and [7], for the full relativistic case, very few authors studied the Einstein-Boltzmann system, see [8] for a local existence theorem. It then seems interesting for us, to extend to the collisional case, some global
results obtained in the collisionless case. This was certainly the objective of the author in [13] and [14], in which he studied the existence of global solutions of the Einstein-Boltzmann system. Unfortunately, several points of the work are far from clear; such as, the use of a formulation which is valid only for the non-relativistic Boltzmann equation, or, concerning the Einstein equations, to abandon the evolution equations which are really relevant, and to concentrate only on the constraint equations, which, in the homogeneous case studied, reduce as we will see to a question of choice for the initial data.

In this paper, we study the collisional evolution of a kind of uncharged massive particles, under the only influence of their own gravitational field, which is a function of the position of the particles.

The phenomenon is governed, as we said above, by the coupled Einstein-Boltzmann system we now introduce. The Einstein equations are the basic equations of the General Relativity. These equations express the fact that, the gravitational field is generated by the matter contents, acting as its sources. The gravitational field is represented, by a second order symmetric 2-tensor of Lorentzian type, called the metric tensor, we denote by g, whose components $g_{\alpha\beta}$, sometimes called "gravitational potentials", are subject to the Einstein equations, with sources represented by a second order symmetric 2-tensor we denote $T_{\alpha\beta}$, that summarizes all the matter contents and which is called the stress-matter tensor. Let us observe that, solving the Einstein equations is determining both the gravitational field and its sources. In our case, the only matter contents are the massive particles statistically described in terms of their distribution function, denoted f, and which is a non-negative real valued function, of both the position and the momentum of the particles, and that generates the gravitational field g, through the stress-matter tensor $T_{\alpha\beta}$. The scalar function f is physically interpreted as "probability of the presence density" of the particles during their collisional evolution, and is subject to the Boltzmann equation, defined by a non linear operator Q called the "Collision Operator". In the binary and elastic scheme due to Lichnerowicz and Chernikov (1940), we adopt, at a given position, only 2 particles collide, in an instantaneous shock, without destroying each other, the collision affecting only their momenta which are not the same, before, and after the shock, only the sum of the 2 momenta being preserved.

We then study the coupled Einstein-Boltzmann system in (g,f). The system is coupled in the sense that f, which is subject to the Boltzmann equation generates the sources $T_{\alpha\beta}$ of the Einstein Equations, whereas the metric g, which is subject to the Einstein equations, is in both the Collision operator, which is the r.h.s of the Boltzmann equation, and in the Lie derivative of f with respect to the vectors field tangent to the trajectories of the particles, and which is the l.h.s of the Boltzmann equation.

We now specify the geometric frame, i.e. the kind of space-time we are looking for. An important part of general relativity is the Cosmology, which is the study of the Universe on a large scale. A. Einstein and W. de Sitter introduced the cosmological models in 1917; A. Friedman and G. Lemaitre introduced the concept of expanding Universe in 1920. Let us point out the fact that the
Einstein equations are **overdetermined**, and physically meaning symmetry assumptions reduce the number of unknowns. A usual assumption is that the spatial geometry has constant curvature which is positive, zero or negative, respectively. Robertson and Walker showed in 1944 that "exact spherical symmetry about every point would imply that the universe is spatially homogeneous", see [10], p. 135. We look for a spatially homogeneous Friedman-Lemaître-Robertson-Walker space-time, we will call a "Robertson-Walker space-time", which is, in Cosmology, the basic model for the study of the expanding Universe. The metric tensor g has only one unknown component we denote a, which is a **strictly positive** function called Cosmological expansion factor; the spatial homogeneity means that a depends only on the time t and the distribution function f depends only on the time t and the 4-momentum p of the particles. The study of the Einstein-Boltzmann system then turns out to the determination of the couple of scalar function (a, f).

In the present work, we consider the Einstein Equation with cosmological constant \( \Lambda \). Our motivation is of a physical point of view. Recent measurements show that the case \( \Lambda > 0 \) is physically very interesting in the sense that one can prove, as we will see, that the expansion of the Universe is accelerating; in mathematical terms, this means that the mean curvature of the space-time tends to a constant at late times. For more details on the cosmological constant, see [19].

Let us now sketch the strategy we adopted to prove the global in time existence of a solution \((a, f)\) of the initial values problem for the Einstein-Boltzmann system for arbitrarily large initial data \((a_0, f_0)\) at \( t = 0 \). In the homogeneous case we consider, the Einstein Equations are a system of 2 non-linear o.d.e for the cosmological expansion factor a; the Boltzmann equation is a non-linear first order p.d.e for the distribution function \( f \).

In a first step, we suppose \( a \) given, with the only assumption to be bounded away from zero, and we give, following Glassey, R.T., [7], the correct formulation of the relativistic Boltzmann equation in \( f \), on a Robertson-Walker space-time. We then prove that on any bounded interval \( I = [t_0, t_0 + T] \) with \( t_0 \in \mathbb{R}_+ \), \( T \in \mathbb{R}_+^* \), the Cauchy problem for the Boltzmann equation has a unique solution \( f \in C[I; L^2_2(\mathbb{R}^3)] \), where \( L^2_2(\mathbb{R}^3) \) is a weighted subspace of \( L^1(\mathbb{R}^3) \), whose weight is imposed by the expression of the sources terms \( T_{\alpha\beta} \) of the Einstein equations. We follow the method developed in [16] that prove a global existence of the solution \( f \in C[0, +\infty[; L^1(\mathbb{R}^3)] \) for the Cauchy problem for the Boltzmann equation, but here, the norm of \( L^2_2(\mathbb{R}^3) \) could allow us to prove the existence theorem only on bounded intervals \([t_0, t_0 + T] \), and this was enough for the coupling with the Einstein Equations.

In a second step, we suppose \( f \) given in \( C[I; L^2_2(\mathbb{R}^3)] \), and we consider the Einstein equations in \( a \), that split into the constraints equations and the evolution equation. The constraint equations contain the momentum constraint which is automatically satisfied in the homogeneous case we consider, and the Hamiltonian constraint that reduces to a question of choice for the initial data. The main problem is then to solve the evolution equation, which is a non-linear second order o.d.e in \( a \). We set: \( e = \frac{1}{a}, \theta = \frac{3\dot{a}}{a} \) (where \( \dot{a} = \frac{da}{dt} \)), and we
prove that, the Einstein evolution equation in $a$ is equivalent to the non-linear first order system in $(e, \theta)$ defined by $\dot{\theta} = -\frac{\theta}{3}$ and the non-linear first order Raychaudhuri equation in $\theta$, $(\theta$ is called the "Hubble variable"), and for which we deduce that there could exist no global solution in the case $\Lambda < 0$. In the case $\Lambda > 0$, we deduce by applying standard theorems to the first order system quoted above, the existence of the solution $a$ of the evolution Einstein equation, in the space of increasing and continuous functions on $I = [t_0, t_0 + T]$. We show that $a$ has an exponential growth and the assumption of the first step on $a$ is then satisfied.

In a third step, we consider the coupled Einstein-Boltzmann system, and, relying on the above results, we prove a global in time existence theorem of the solution $(a, f)$ of the Cauchy problem, by applying the fixed point theorem in an appropriate function space.

Our method preserves the physical nature of the problem that imposes to the distribution function $f$ to be a non-negative function and nowhere, we had to require smallness assumption on the initial data, which can, consequently, be taken arbitrarily large.

The paper is organized as follows:
In section 2, we introduce the Einstein and Boltzmann equations on a Robertson-Walker space-time.
In section 3, we study the Boltzmann equation in $f$.
In section 4 we study the Einstein equation in $a$.
In section 5, we prove the local existence theorem for the coupled Einstein-Boltzmann system.
In section 6, we prove the global existence theorem for the coupled Einstein-Boltzmann system.

2 The Boltzmann equation and the Einstein equations on a Robertson-Walker space-time.

2.1 Notations and function spaces

A greek index varies from 0 to 3 and a Latin index from 1 to 3, unless otherwise specified. We adopt the Einstein summation convention $a_\alpha b^\alpha = \sum_{\alpha=0}^{3} a_\alpha b^\alpha$. We consider the flat Robertson-Walker space-time denoted $(\mathbb{R}^4, g)$ where, for $x = (x^\alpha) = (x^0, x^i) \in \mathbb{R}^4$, $x^0 = t$ denotes the time and $\bar{x} = (x^i)$ the space. $g$ stands for the metric tensor with signature $(-, +, +, +)$ that can be written:

$$g = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

in which $a$ is a strictly positive function of $t$, called the cosmological expansion factor. We consider the collisional evolution of a kind of uncharged massive relativistic particles in the time oriented space-time $(\mathbb{R}^4, g)$. The particles are statistically described by their distribution function we denote $f$, which is a
non-negative real-valued function of both the position \((x^\alpha)\) and the 4-momentum \(p = (p^\alpha)\) of the particles, and that coordinatize the tangent bundle \(T(\mathbb{R}^4)\) i.e:

\[ f : T(\mathbb{R}^4) \simeq \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^+ \,, \quad (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha) \in \mathbb{R}^+ \]

For \(\bar{p} = (p^i), \bar{q} = (q^i) \in \mathbb{R}^3\), we set, as usual:

\[\bar{p} \cdot \bar{q} = \sum_{i=1}^{3} p^i q^i \quad |\bar{p}| = \left(\sum_{i=1}^{3} (p^i)^2\right)^{1/2} \quad (2.2)\]

We suppose the rest mass \(m > 0\) of the particles normalized to unity, i.e we take \(m = 1\). The relativistic particles are then required to move on the future sheet of the mass-shell whose equation is \(g(p, p) = -1\).

From this, we deduce, using (2.1) and (2.2):

\[p^0 = \sqrt{1 + a^2 |\bar{p}|^2} \quad (2.3)\]

where the choice of \(p^0 > 0\) symbolizes the fact that the particles eject towards the future . (2.2) shows that in fact, \(f\) is defined on the 7-dimensional subbundle of \(T(\mathbb{R}^4)\) coordinalized by \((x^\alpha), (p^\alpha)\). Now, we consider the spatially homogeneous case which means that \(f\) depends only on \(t\) and \(\bar{p} = (p^i)\). The framework we will refer to will be the subspace of \(L^1_2(\mathbb{R}^3)\), denote \(L^1_2(\mathbb{R}^3)\) and defined by :

\[L^1_2(\mathbb{R}^3) = \{ f \in L^1(\mathbb{R}^3), \| f \| := \int_{\mathbb{R}^3} \sqrt{1 + |\bar{p}|^2}|f(\bar{p})|d\bar{p} < +\infty \} \quad (2.4)\]

where \(|\bar{p}|\) is given by (2.2); \(\| \cdot \|\) is a norm on \(L^1_2(\mathbb{R}^3)\) and \((L^1_2(\mathbb{R}^3), \| \cdot \|)\) is a Banach space.

Let \(r\) be an arbitrary strictly positive real number. We set:

\[X_r = \{ f \in L^1_2(\mathbb{R}^3), f \geq 0 \text{ a.e, } \| f \| \leq r \} \quad (2.5)\]

Endowed with the metric induced by the norm \(\| \cdot \|\), \(X_r\) is a complete and connected metric subspace of \((L^1_2(\mathbb{R}^3), \| \cdot \|)\). Let \(I\) be a real interval. Set:

\[C[I; L^1_2(\mathbb{R}^3)] = \{ f : I \to L^1_2(\mathbb{R}^3), f \text{ continuous and bounded } \} \]

endowed with the norm:

\[\| f \| = \sup_{t \in I} \| f(t) \| \quad (2.6)\]

\(C[I; L^1_2(\mathbb{R}^3)]\) is a Banach space. \(X_r\) being defined by (2.4). We set:

\[C[I; X_r] = \{ f \in C[I; L^1_2(\mathbb{R}^3)], f(t) \in X_r, \forall t \in I \} \quad (2.7)\]

Endowed with the metric induced by the norm \(\| \cdot \|\) defined by (2.4), \(C[I; X_r]\) is a complete metric subspace of \((C[I; L^1_2(\mathbb{R}^3)], \| \cdot \|)\).
2.2 The Boltzmann equation in \((\mathbb{R}^4, g)\)

In its general form, the Boltzmann equation on the curved space-time \((\mathbb{R}^4, g)\) can be written:

\[ L_X f = Q(f, f) \quad (2.8) \]

where

\[ L_X f := p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^\alpha} \quad (2.9) \]

denotes the Lie derivative of \(f\) in the direction of the vector field \(X\) tangent to the trajectories of the particles in \(T\mathbb{R}^4\), and whose local coordinates are \((p^\alpha, -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu)\) in which \(\Gamma^\alpha_{\lambda\mu}\) are the Christoffel symbols of \(g\); \(Q\) is a non-linear integral operator called the "Collision Operator". We specify this operator in detail in next section. Now, since \(f\) depends only on \(t\) and \((p^i)\), \(2.8\) can be written using \(2.9\):

\[ p^0 \frac{\partial f}{\partial t} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = Q(f, f) \quad (2.10) \]

We now express the Christoffel symbols defined by:

\[ \Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} [\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}] \quad (2.11) \]

in which, the metric \(g\) is defined by \(2.1\) and \(g^{\lambda\mu}\) denotes the inverse matrix of \(g_{\lambda\mu}\); \(2.1\) gives:

\[

g^{00} = g_{00} = -1; \quad g^{ii} = a^2; \quad g^{i0} = a^{-2}; \quad g_{0i} = g^{0i} = 0; \quad g_{ij} = g^{ij} = 0 \quad \text{for} \quad i \neq j
\]

A direct computation, using \(2.11\) then gives, with \(\dot{a} = \frac{da}{dt}\):

\[

\Gamma^0_{i0} = \dot{a} a; \quad \Gamma^0_{0i} = \frac{\dot{a}}{a}; \quad \Gamma^0_{00} = 0; \quad \Gamma^0_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta; \quad \Gamma^k_{ij} = 0
\]

\(2.12\)

The Boltzmann equation \(2.10\) then writes, using \(2.12\):

\[ \frac{\partial f}{\partial t} - 2 \dot{a} \sum_{i=1}^3 p^i \frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(f, f) \quad (2.13) \]

in which \(p^0\) is given by \(2.3\). \(2.13\) is a non-linear p.d.e in \(f\) we study in next section.

2.3 The Einstein Equations and the coupled system

We consider the Einstein equations with a cosmological constant \(\Lambda\) and that can be written:

\[ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta} \quad (2.14) \]

in which:

\(R_{\alpha\beta}\) is the Ricci tensor of \(g\), contracted of the curvature tensor of \(g\);
\[ R = g^{\alpha\beta} R_{\alpha\beta} = R^\alpha_\alpha \] is the scalar curvature; 
\[ T_{\alpha\beta} \] is the stress-matter tensor that represents the matter contents, and that is generated by the distribution function \( f \) of the particles by:

\[
T^{\alpha\beta} = \int_{\mathbb{R}^3} \frac{p^\alpha p^\beta f(t, \bar{p}) |g|^\frac{3}{2}}{p^0} dp^1 dp^2 dp^3 \tag{2.15}
\]

in which \( |g| \) is the determinant of \( g \), we have, using (2.1), \( |g|^{\frac{3}{2}} = a^3 \). Recall that \( f \) is a function of \( t \) and \( \bar{p} = (p^i) \); then \( T^{\alpha\beta} \) is a function of \( t \).

\( G \) is the universal gravitational constant. We take \( G = 1 \). The contraction of the Bianchi identities gives the identities \( \nabla_\alpha S^{\alpha\beta} = 0 \), where \( S^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \) is the Einstein tensor. The Einstein equation \( (2.14) \) then implies, since \( \nabla g = 0 \), that the stress-matter tensor \( T^{\alpha\beta} \) must satisfy the four relations \( \nabla_\alpha T^{\alpha\beta} = 0 \) called the conservation laws. But it is proved in [6] that these laws are satisfied for all solutions \( f \) of the Boltzmann equation. Now if \( R^{\lambda}_{\quad \alpha,\mu\beta} \) are the components of the curvature tensor of \( g \), we have:

\[
\begin{align*}
R^{\alpha\beta} &= R^{\lambda}_{\quad \alpha,\lambda\beta} \\
R^{\lambda}_{\quad \mu,\alpha\beta} &= \partial_\alpha \Gamma^\lambda_{\mu\beta} - \partial_\beta \Gamma^\lambda_{\mu\alpha} + \Gamma^\lambda_{\nu\alpha} \Gamma^\nu_{\mu\beta} - \Gamma^\lambda_{\nu\beta} \Gamma^\nu_{\mu\alpha}
\end{align*}
\tag{2.16}
\]

in which \( \Gamma^\lambda_{\mu\beta} \) is defined by (2.11). This shows that the Einstein equations \( (2.14) \) are a system of non-linear second order p.d.e in \( g_{\alpha\beta} \). In order to have things fresh in mind, when we will study in details the Einstein equations, we leave the expression of \( (2.14) \) in term of \( a \), to section 4 which is devoted to this study. The Einstein-Boltzmann system is then the system \( (2.13)-(2.14) \) in \( (a, f) \).

3 Existence Theorem for the Boltzmann Equation

In this section, we suppose that the cosmological expansion factor \( a \) is given, and we prove an existence theorem for the initial value problem for the Boltzmann equation \( (2.13) \), on every bounded interval \( I = [t_0, t_0 + T] \) with \( t_0 \in \mathbb{R}_+, T \in \mathbb{R}_+ \).

We begin by specifying the collision operator \( Q \) in \( (2.10) \).

3.1 The Collision Operator

In the instantaneous, binary and elastic scheme due to Lichnerowicz and Chernikov, we consider, at a given position \((t, x)\), only 2 particles collide instantaneously without destroying each other, the collision affecting only the momenta of the 2 particles that change after the collision, only the sum of the 2 momenta being preserved, following the scheme:
The collision operator $Q$ is then defined, using functions $f$, $g$ on $\mathbb{R}^3$ by:

$$Q(f, g) = Q^+(f, g) - Q^-(f, g)$$ (3.1)

where

$$Q^+(f, g)(\bar{p}) = \int_{\mathbb{R}^3} \frac{a^3 d\bar{q}}{q^0} \int_{S^2} f(\bar{p}')g(\bar{q}')A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega$$ (3.2)

and

$$Q^-(f, g)(\bar{p}) = \int_{\mathbb{R}^3} \frac{a^3 d\bar{q}}{q^0} \int_{S^2} f(\bar{p})g(\bar{q})A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega$$ (3.3)

whose elements we now introduce step by step, specifying properties and hypotheses:

1) $S^2$ is the unit sphere of $\mathbb{R}^3$ whose volume element is denoted $dw$.

2) $A$ is a non-negative real-valued regular function of all its arguments, called the collision kernel or the cross-section of the collisions, on which we require the following boundedness, symmetry and Lipschitz continuity assumptions:

$$0 \leq A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') \leq C_1$$ (3.4)

$$A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') = A(a, \bar{q}, \bar{p}, \bar{q}', \bar{p}')$$ (3.5)

$$A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') = A(a, \bar{p}', \bar{q}', \bar{p}, \bar{q})$$ (3.6)

$$|A(a_1, \bar{p}, \bar{q}, \bar{p}', \bar{q}') - A(a_2, \bar{p}, \bar{q}, \bar{p}', \bar{q}')| \leq \gamma |a_1 - a_2|$$ (3.7)

where $C_1$ and $\gamma$ are strictly positive constants.

3) The conservation law $p + q = p' + q'$ splits into:

$$p_0 + q_0 = p'_0 + q'_0$$ (3.8)

$$\bar{p} + \bar{q} = \bar{p}' + \bar{q}'$$ (3.9)

and shows, using the conservation of the quantity:

$$e = \sqrt{1 + a^2} |\bar{p}|^2 + \sqrt{1 + a'^2} |\bar{q}|^2$$ (3.10)
called elementary energy of the unit rest mass particles; we can interpret (3.9) by setting, following Glassey, R., T., in [?

\[
\begin{align*}
\bar{p}' &= \bar{p} + b(\bar{p}, \bar{q}, \omega)\omega \\
\bar{q}' &= \bar{q} - b(\bar{p}, \bar{q}, \omega)\omega; \quad \omega \in S^2
\end{align*}
\]

in which \(b(\bar{p}, \bar{q}, \omega)\) is a real-valued function. We prove, by a direct calculation, using (2.3) to express \(\bar{p}'_0, \bar{q}'_0\) in term of \(\bar{p}, \bar{q}\), and now (3.11) to express \(\bar{p}', \bar{q}'\) in term of \(\bar{p}, \bar{q}\), that equation (3.8) leads to a quadratic equation in \(b\), that solves to give:

\[
b(\bar{p}, \bar{q}, \omega) = \frac{2p^oq^o e a^2 (\hat{q} - \hat{p})}{c^2 - a^4 (\bar{p} + \bar{q})^2}
\]

in which \(\hat{p} = \frac{\bar{p}}{p^o}, \hat{q} = \frac{\bar{q}}{q^o}\), and \(e\) is given by (3.10). Another direct computation shows, using the classical properties of the determinants, that the Jacobian of the change of variables \((\bar{p}, \bar{q}) \rightarrow (\bar{p}', \bar{q}')\) in \(\mathbb{R}^3 \times \mathbb{R}^3\), defined by (3.11) is given by:

\[
\frac{\partial (\bar{p}', \bar{q}')}{\partial (\bar{p}, \bar{q})} = -\frac{p^o q^o}{p^o q^o}
\]

(3.13) shows, using once more (2.3) and the implicit function theorem, that the change of variable (3.11) is invertible and also allows to compute \(\bar{p}, \bar{q}\) in term of \(\bar{p}', \bar{q}'\).

Finally, formulae (2.3) and (3.11) show that the functions to integrate in (3.2) and (3.3) completely express in terms of \(\bar{p}, \bar{q}, \omega\); the integration with respect to \(\bar{q}\) and \(\omega\) leave functions \(Q^+(f, g)\) and \(Q^-(f, g)\) of the single variable \(\bar{p}\). In practice, we will consider functions \(f\) on \(\mathbb{R} \times \mathbb{R}^3\), that induce, for \(t\) fixed in \(\mathbb{R}\), functions \(f(t)\) on \(\mathbb{R}^3\), defined by \(f(t)(\bar{p}) = f(t, \bar{p})\).

**Remark 3.1**

1) Formulae (3.12) and (3.13) are generalizations to the case of the Robertson-Walker space-time, of analogous formulae established by Glassey, R.T, in [7], in the case of the Minkowski space-time, to which the Robertson-Walker space-time reduces, when we take \(a(t) = 1\) in the metric (2.1).

2) The expression \(b(\bar{p}, \bar{q}, \omega) = \omega.(\hat{p} + \hat{q})\) used by the autor in [13] and [14] is valid only in the non-relativistic case.

3) \(A = ke^{-a^2-|\bar{p}|^2-|\bar{q}|^2-|\bar{p}'|^2-|\bar{q}'|^2}, k > 0\), is a simple example of functions satisfying assumptions (3.4), (3.5), (3.6) and (3.7).

### 3.2 Resolution of the Boltzmann equation

We consider the Boltzmann equation on \([t_0, t_0 + T]\) with \(t_0 \in \mathbb{R}_+, T \in \mathbb{R}^*_+\) and \(a\) is supposed to be given and defined on \([t_0, t_0 + T]\).
The Boltzmann equation (2.13) is a first order p.d.e and its resolution is equivalent to the resolution of the associated characteristic system, which can be written, taking \( t \) as parameter:

\[
\frac{dp}{dt} = -2 \frac{\dot{a}}{a} p^i; \quad \frac{df}{dt} = \frac{1}{p^0} Q(f, f)
\]  

(3.14)

We solve the initial value problem on \( I = [t_0, t_0 + T] \) with initial data:

\[
p^i(t_0) = y^i; \quad f(t_0) = f_{t_0}
\]  

(3.15)

The equation in \( \bar{p} = (p^i) \) solve directly to give, setting \( y = (y^i) \in \mathbb{R}^3 \):

\[
\bar{p}(t_0 + t, y) = a^2(t_0) \frac{a(t)}{a^2(t_0 + t)} y, \quad t \in [0, T]
\]  

(3.16)

The initial value problem for \( f \) is equivalent to the following integral equation in \( f \), in which \( \bar{p} \) stands this time for any independent variable in \( \mathbb{R}^3 \):

\[
f(t_0 + t, \bar{p}) = f_{t_0}(\bar{p}) + \int_{t_0}^{t_0 + t} \frac{1}{p^0} Q(f, f)(s, \bar{p}) ds \quad t \in [0, T]
\]  

(3.17)

Finally, solving the Boltzmann equation (2.13) is equivalent to solving the integral equation (3.17). We prove:

**Theorem 3.1** Let \( a \) be a strictly positive continuous function such that \( a(t) \geq \frac{3}{2} \) whenever \( a \) is defined .

Let \( f_{t_0} \in L^1_2(\mathbb{R}^3) \), \( f_{t_0} \geq 0 \), a.e, \( r \in \mathbb{R}^*_0 \) such that \( r > \| f_{t_0} \| \). Then, the initial value problem for the Boltzmann equation on \([t_0, t_0 + T]\), with initial data \( f_{t_0} \), has a unique solution \( f \in C([t_0, t_0 + T]; X_r) \). Moreover, \( f \) satisfies the estimation:

\[
\sup_{t \in [t_0, t_0 + T]} \| f(t) \| \leq \| f_{t_0} \|
\]  

(3.18)

Theorem 3.1 is a direct consequence of the following result:

**Proposition 3.1** Assume hypotheses of theorem 3.1 on: \( a, f_{t_0} \) and \( r \).

1) There exists an integer \( n_0(r) \) such that, for every integer \( n \geq n_0(r) \) and for every \( v \in X_r \), the equation

\[
\sqrt{n}u - \frac{1}{p^0} Q(u, u) = v
\]  

(3.19)

has a unique solution \( u_n \in X_r \)

2) Let \( n \in \mathbb{N}, n \geq n_0(r) \)
   i) For every \( u \in X_r \), define \( R(n, Q)u \) to be the unique element of \( X_r \) such that:

\[
\sqrt{n}R(n, Q)u - \frac{1}{p^0} Q[R(n, Q)u, R(n, Q)u] = u
\]  

(3.20)
ii) Define operator $Q_n$ on $X_r$ by:

$$Q_n(u,v) = \sqrt{n}R(n,Q)u - nu$$  \hspace{1cm} (3.21)

Then

a) The integral equation

$$f(t_0 + t,\bar{p}) = f_0(\bar{p}) + \int_{t_0}^{t_0 + t} Q_n(f,f)(s,\bar{p})ds \hspace{1cm} t \in [0,T]$$  \hspace{1cm} (3.22)

has a unique solution $f_n \in C[[t_0,t_0 + T]; X_r]$. Moreover, $f_n$ satisfies the estimation:

$$\sup_{t \in [t_0,t_0 + T]} \|f_n(t)\| \leq \|f_0\|$$  \hspace{1cm} (3.23)

b) The sequence $(f_n)$ converges in $C[[t_0,t_0 + T]; X_r]$ to an element $f \in C[[t_0,t_0 + T]; X_r]$, which is the unique solution of the integral equation (3.17). The solution $f$ satisfies the estimation (3.18).

**Proof of the proposition 3.1**

The proof follows the same lines as the proof of theorem 4.1 in [16]. We will emphasize only on points where differences arise with the present case and we show how we proceed in such cases.

**Proof of point 1) of prop 3.1**

We use:

**Lemma 3.1** Let $f, g \in L^1_2(\mathbb{R}^3)$. then $\frac{1}{p^0}Q^+(f,g), \frac{1}{p^0}Q^-(f,g) \in L^1_2(\mathbb{R}^3)$ and

$$\left\| \frac{1}{p^0}Q^+(f,g) \right\| \leq C(t) \| f \| \| g \|, \quad \left\| \frac{1}{p^0}Q^-(f,g) \right\| \leq C(t) \| f \| \| g \|$$  \hspace{1cm} (3.24)

$$\left\| \frac{1}{p^0}Q^+(f,f) - \frac{1}{p^0}Q^+(g,g) \right\| \leq C(t)(\| f \| + \| g \|) \| f - g \|$$  \hspace{1cm} (3.25)

$$\left\| \frac{1}{p^0}Q^-(f,f) - \frac{1}{p^0}Q^-(g,g) \right\| \leq C(t)(\| f \| + \| g \|) \| f - g \|$$  \hspace{1cm} (3.26)

where

$$C(t) = 32\pi C_1 a^3(t)$$  \hspace{1cm} (3.27)

**Proof of lemma 3.1**

We deduce from (3.8) and $a > 1$ that:

$$\sqrt{1 + \frac{1}{p^0}} \leq \sqrt{1 + \frac{1}{a^2 p^0}} \leq \sqrt{1 + \frac{1}{a^2 (p^0 + q^0)}} = \sqrt{1 + \frac{1}{a^2 (p^0 + q^0)}} \leq 2(p^0 + q^0)$$

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Expression (3.2) of $Q^+(f,g)$ then gives, using (3.3):

$$\left\| \frac{1}{p^0} Q^+(f,g) \right\| = \int_{\mathbb{R}^3} \left| \frac{\sqrt{1+p^2}}{p^0} Q^+(f,g) \right| \, dp \leq 2a^3(t)C_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{p_0^0 + q_0^0}{p^0 q^0} \left| f(\bar{p}') \right| \left| g(\bar{q}') \right| \, dp dq d\omega$$

We then deduce, using the change of variables and that gives $dp dq = \frac{q_0^0 p_0^0}{p^0 q^0} dp' dq'$ and the fact that, by (3.25) $\frac{p_0^0 + q_0^0}{q_0^0 p^0} = \frac{1}{q_0^0} + \frac{1}{q_0^0} \leq 2$

$$\left\| \frac{1}{p^0} Q^+(f,g) \right\| \leq 8\pi a^3(t)C_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| f(\bar{p}') \right| \left| g(\bar{q}') \right| \, dp' dq' \leq C(t) \|f\| \|g\|$$

The estimation of $\left\| \frac{1}{p^0} Q^-(f,g) \right\|$ follows the same way without change of variables and (3.24) follows. The inequalities (3.25) are consequences of (3.24) and the bilinearity of $Q^+$ and $Q^-$, that allows us to write, $P$ standing for $\frac{1}{p^0} Q^+$ or $\frac{1}{p^0} Q^-$:

$$P(f, f) - P(g,g) = P(f,-g) + P(f, g).$$

Finally, (3.26) is a consequence of (3.24) and $Q = Q^+ - Q^-$. This completes the proof of the lemma 3.1.

Now the continuous and strictly positive function $t \to a^3(t)$ is bounded from above, on the line segment $[t_0, t_0 + t]$, and $C(t)$ given by (3.27) is bounded from above by a constant $C(t_0, t) > 0$. Hence, if we replace C(t) by $C(t_0, t)$ in the inequalities in lemma 3.1, we obtain the same inequalities with an absolute constant than the inequalities in proposition 3.1 in [10]. The proof of the point 1) of prop 3.1 is then exactly the same as the proof of prop 3.2 in [10]. ■

**Proof of the point 2) of prop 3.1**

We use, $n_0(r)$ being the integer introduced in point 1:

**Lemma 3.2** We have, for every integer $n \geq n_0(r)$ and for every $u \in X_r$

$$\| \sqrt{n} R(n, Q) u \| = \| u \|$$

**Proof of lemma 3.2**

(3.28) is a consequence of:

$$\int_{\mathbb{R}^3} Q(f, f)(\bar{p}) dp = 0, \quad \forall f \in L^1_2(\mathbb{R}^3)$$

we now establish. It is here that assumption (3.5) on the collision kernel $A$ are required. Define operator $Q^*$, for $f, g \in L^1_2(\mathbb{R}^3)$ by:

$$Q^*(f, g)(\bar{p}) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') \left[ f(\bar{p}') g(\bar{q}') + f(\bar{q}') g(\bar{p}') \right] - f(\bar{p}) g(\bar{q}) - f(\bar{q}) g(\bar{p}) \, dq dp d\omega$$

Clearly, $Q^*(f, f) = Q(f, f)$, where $Q$ is the collision operator defined by (3.1), (3.2), (3.3). Now let $\Phi$ be a regular function on $\mathbb{R}^3$, such that:

$$\Phi(\bar{p}) + \Phi(\bar{q}) = \Phi(\bar{p'}) + \Phi(\bar{q'})$$

(3.30)
Multiplying (a) by \( \frac{1}{p^0} \Phi(\bar{p}) \) and integrating on \( \mathbb{R}^3 \) yields:

\[
\int_{\mathbb{R}^3} \frac{1}{p^0} Q^*(f, g)(\bar{p}) \Phi(\bar{p}) d\bar{p} = \frac{a^3}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \frac{A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')}{p^0 q^0} \left[ f(\bar{p}') g(\bar{q}') + f(\bar{q}') g(\bar{p}') \right] \\
- f(\bar{p}) g(\bar{q}) - f(\bar{q}) g(\bar{p}) \Phi(\bar{p}) d\bar{p} d\bar{q} d\omega
\]

(c)

Let us compute the integral of the r.h.s of (c) using the change of variables \((\bar{p}, \bar{q}) \mapsto (\bar{q}, \bar{p}); \) formulae \(3.12\) shows, since \( p^0 q^0 \) remains unchanged, that \( b(\bar{p}, \bar{q}, \omega) \) change to \(-b(\bar{q}, \bar{p}, \omega)\) and formulae \(3.11\) then show that \((\bar{p}', \bar{q}')\) change to \((\bar{q}', \bar{p}');\) next assumption \(3.5\) on \( A \) show that \( A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') \) remains unchanged. We then have, since the Jacobian of the change of variables \((\bar{p}, \bar{q}) \mapsto (\bar{q}, \bar{p})\) is 1

\[
\int_{\mathbb{R}^3} \frac{1}{p^0} Q^*(f, g) \Phi(\bar{p}) d\bar{p} = \frac{a^3}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \frac{A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')}{p^0 q^0} \left[ f(\bar{p}') g(\bar{q}') + f(\bar{q}') g(\bar{p}') \right] \\
- f(\bar{p}) g(\bar{q}) - f(\bar{q}) g(\bar{p}) \Phi(\bar{p}) d\bar{p} d\bar{q} d\omega
\]

(d)

The sum of (c) and (d) gives:

\[
\int_{\mathbb{R}^3} \frac{1}{p^0} Q^*(f, g) \Phi(\bar{p}) d\bar{p} = \frac{a^3}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \frac{S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')}{p^0 q^0} \left[ f(\bar{p}') g(\bar{q}') + f(\bar{q}') g(\bar{p}') \right] \\
- f(\bar{p}) g(\bar{q}) - f(\bar{q}) g(\bar{p}) \Phi(\bar{p}) d\bar{p} d\bar{q} d\omega
\]

where

\[
S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') = A(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')[\Phi(\bar{p}) + \Phi(\bar{q})]
\]

(f)

we can write using, (e)

\[
\int_{\mathbb{R}^3} \frac{1}{p^0} Q^*(f, g)(\bar{p}) \Phi(\bar{p}) d\bar{p} = I_0 - J_0
\]

(g)

where

\[
\begin{aligned}
I_0 &= \frac{a^3}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \frac{S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')}{p^0 q^0} \left[ f(\bar{p}') g(\bar{q}') + f(\bar{q}') g(\bar{p}') \right] d\bar{p} d\bar{q} d\omega \\
J_0 &= \frac{a^3}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \frac{S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')}{p^0 q^0} \left[ f(\bar{p}) g(\bar{q}) + f(\bar{q}) g(\bar{p}) \right] d\bar{p} d\bar{q} d\omega
\end{aligned}
\]

Let us make in \( I_0 \) the change of variable \((\bar{p}, \bar{q}) \mapsto (\bar{q}', \bar{q})\) defined by \(3.11\) and whose Jacobian \(3.13\) gives \( d\bar{p} d\bar{q} = \frac{p^0 q^0}{p^0 q^0} d\bar{q}' d\bar{q}'\); we have, since \((\bar{p}, \bar{q})\) is expressible in term of \((\bar{p}', \bar{q}')\) by virtue of \(3.14\):

\[
I_0 = \frac{a^3}{2} \int_{S^2} d\omega \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}')}{p^0 q^0} \left[ f(\bar{p}') g(\bar{q}') + f(\bar{q}') g(\bar{p}') \right] d\bar{p}' d\bar{q}'
\]

(h)

Now, assumption \(3.10\) on \( A \) and assumption (b) on \( \Phi \) imply that \( S \) defined by (f) satisfies the relation: \( S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') = S(a, \bar{p}', \bar{q}', \bar{p}, \bar{q}). \) Replacing in \( I_0 \)
\[ S(a, \bar{p}, \bar{q}, \bar{p}', \bar{q}') \) by \( S(a, \bar{p}', \bar{q}, \bar{p}, \bar{q}) \), we deduce, from the expression (h) of \( I_0 \) that \( I_0 = J_0 \). Then (g) implies:

\[
\int_{\mathbb{R}^3} \frac{1}{p^0} Q^*(f, g) \Phi(\bar{p}) d\bar{p} = 0 \tag{i}
\]

Now the function \( \Phi(\bar{p}) = p^0 = \sqrt{1 + a^2|\bar{p}|^2} \) satisfies hypothesis (b) as a consequence of the conservation law 3.8. The relation \( \mathcal{E} = J_0 \) then follows from the above choice of \( \Phi(\bar{p}) \), (i) and the relation \( Q^*(f, f) = Q(f, f) \).

Now let us prove (3.28).

We have, multiplying equation (3.20) by \( p^0 = \sqrt{1 + a^2|\bar{p}|^2} \), integrating over \( \mathbb{R}^3 \), and using (3.29):

\[
\sqrt{n} \int_{\mathbb{R}^3} \sqrt{1 + a^2|\bar{p}|^2} R(n, Q) u(\bar{p}) d\bar{p} = \int_{\mathbb{R}^3} \sqrt{1 + a^2|\bar{p}|^2} u(\bar{p}) d\bar{p} \tag{3.30}
\]

If we make in each side of (3.30) the change of variables \( \bar{q} = B\bar{p} \) where \( B = \text{Diag}(a, a, a) \), then \( |\bar{q}|^2 = a^2|\bar{p}|^2 \); \( d\bar{p} = \frac{1}{a^3} d\bar{q} \) and (3.30) gives, using definition 2.4 of \( \| \cdot \| \):

\[
\sqrt{n} \| R(n, Q) u \| B^{-1} = \| u \| \tag{3.31}
\]

But, if we compute \( \| u B \| \), using the above change of variable, we have:

\[
\| u B \| = \int_{\mathbb{R}^3} \sqrt{1 + |\bar{p}|^2} u(\bar{p}) d\bar{p} = \int_{\mathbb{R}^3} \sqrt{1 + a^2|\bar{q}|^2} u(\bar{q}) \frac{1}{a^3} d\bar{q}
\]

\[
\leq \frac{1}{a^3} \sqrt{1 + \frac{1}{a^2}} \int_{\mathbb{R}^3} \sqrt{1 + |\bar{q}|^2} u(\bar{q}) d\bar{q}
\]

i.e

\[
\| u B \| \leq \frac{1}{a^3} \left( 1 + \frac{1}{a} \right) \| u \|
\]

The assumption \( a \geq \frac{3}{2} \) implies \( \frac{1}{a^3} \left( 1 + \frac{1}{a} \right) \leq 1 \), so that \( \| u B \| \leq \| u \| \), this implies that \( u B \in X_r \) if \( u \in X_r \). Now since (3.31) holds for every \( u \in X_r \), we have, replacing in (3.31) \( u \) by \( u B \), \( \| \sqrt{n} R(n, Q) u \| = \| u \| \). We then have 3.28 and lemma 3.2 is proved.

Now (3.28) is exactly equality (3.10) in proposition 3.3 in [16], we then prove exactly as for prop. 3.3 in [16], that all the other relations of that proposition hold in the present case. Using this result, the proof of point 2)a) of proposition 3.1 is the same as the proof of prop 4.1 in [16] and the proof of point 2) b) of prop 3.1 is the same as the proof of theorem 4.1 in [16], just replacing, \( [0, +\infty[ \) by \( [t_0, t_0 + T] \). This completes the proof of prop 3.1 which give directly theorem 3.1. \( \blacksquare \)
4 Existence Theorem for the Einstein Equations

4.1 Expression and Reduction of the Einstein Equations

We express the Einstein equations (2.14) in terms of the cosmological expansion factor $a$ which is the only unknown. We have to compute the Ricci tensor $R_{\alpha\beta}$ given by (2.16).

The expression (2.12) of $\Gamma^\lambda_{\alpha\beta}$ shows that the only non-zero components of the Ricci tensor are $R_{\alpha\alpha}$ and that $R_{11} = R_{22} = R_{33}$. Then, it will be enough to compute $R_{00} = R^0_{\lambda\alpha0}$ and $R_{11} = R^1_{\lambda\alpha1}$. The expression (2.12) of $\Gamma^\lambda_{\alpha\beta}$ and formulae (2.16) give:

\[
R_{00} = 0; \quad R^1_{0,10} = R^2_{0,20} = R^3_{0,30} = -\frac{\ddot{a}}{a};
\]
\[
R^0_{1,01} = a\ddot{a}; \quad R^1_{1,11} = 0; \quad R^2_{1,21} = R^3_{1,31} = (\dot{a})^2.
\]

We then deduce that:

\[
R_{00} = -3\frac{\ddot{a}}{a} \text{ and } R_{11} = a\ddot{a} + 2(\dot{a})^2
\]

We can then compute the scalar curvature $R$ to be:

\[
R = R^\alpha_{\alpha} = g^{\alpha\beta} R_{\alpha\beta} = 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right]
\]

The Einstein equations (2.14) then take the reduced form, using expression (2.1) of $g$:

\[
\begin{cases}
3(\dddot{a})^2 - \Lambda = 8\pi T_{00} \\
-(\dot{a})^2 - 2a\ddot{a} + a^2\Lambda = 8\pi T_{11}
\end{cases}
\]

that can be written, using $T^{\alpha\beta} = g^{\alpha\lambda} g^{\beta\mu} T_{\lambda\mu}$:

\[
(\frac{\ddot{a}}{a})^2 = \frac{8\pi}{3} T^{00} + \frac{\Lambda}{3} \quad (4.1)
\]
\[
\frac{\ddot{a}}{a} = -\frac{4\pi}{3} (T^{00} + 3a^2 T_{11}) + \frac{\Lambda}{3} \quad (4.2)
\]

in which $T^{\alpha\beta}$ is defined in term of $f$ by (2.15). In this paragraph, we suppose $f$ fixed in $C[[0,T]; X_r]$, with $f(0) = f_0 \in L^2(\mathbb{R}^3)$, $f_0 \geq 0$ a.e. and $r > \|f_0\|$.

4.2 Compatibility

The relations $R_{0i} = 0$, $R_{ij} = 0$ if $i \neq j$, $R_{11} = R_{22} = R_{33}$, imply for the Einstein tensor $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$, that:

\[
T_{11} = T_{22} = T_{33}, \quad T_{0i} = 0, \quad T_{ij} = 0 \text{ for } i \neq j \quad (4.3)
\]
But the stress-matter tensor $T_{\alpha\beta}$ is defined by (2.15) in terms of the distribution function $f$. So, the relation (4.3) are in fact conditions to impose to $f$. We prove:

**Proposition 4.1** Let $f_{t_0}$ and $r > 0$ be defined as in theorem 3.1. Assume that, in addition, $f_{t_0}$ is invariant by $SO_3$ and that, the collision kernel $A$ satisfies

$$A(a(t), M\bar{p}, M\bar{q}, M\bar{p}', M\bar{q}') = A(a(t), \bar{p}, \bar{q}, \bar{p}', \bar{q}'), \quad \forall M \in SO_3 \tag{4.4}$$

then

1) The solution $f$ of the integral equation (4.7) satisfies:

$$f(t_0 + t, M\bar{p}) = f(t_0 + t, \bar{p}), \quad \forall t \in [0, T], \quad \forall \bar{p} \in \mathbb{R}^3, \quad \forall M \in SO_3 \tag{4.5}$$

2) The stress-matter tensor $T_{\alpha\beta}$ satisfies the conditions (4.4).

**Proof**

1) Let $M \in SO_3$; (4.7) gives, since $f_{t_0}(M\bar{p}) = f_{t_0}(\bar{p})$ and $p^0 = p^0(\bar{p})$:

$$f(t_0 + t, M\bar{p}) = f_{t_0}(\bar{p}) + \int_{t_0}^{t_0+t} \frac{1}{p^0oM}Q(f, f)(s, M\bar{p})ds, t \in [0, T] \tag{4.6}$$

Notice that $p^0 = p^0(\bar{p})$ is invariant by $SO_3$. Now, definition (3.1), (3.2), (3.3) of $Q$ gives:

$$Q(f, f)(s)(M\bar{p}) = \int_{\mathbb{R}^3} \frac{a^3(s)d\bar{q}}{q^0} \int_{\mathbb{S}^2} [f(s, \bar{p}')f(s, \bar{q}') - f(s, M\bar{p})f(s, \bar{q})]A(a, M\bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega \tag{4.7}$$

Let us set in (4.7) $\bar{q} = M\bar{q}_1$; $\omega = M\omega_1$. Then formulae (3.11) give using expression (3.12) of $b$, the invariance of the scalar product in $\mathbb{R}^3$ by $SO_3$:

$$\begin{cases}
\bar{p}' = M\bar{p} + b(M\bar{p}, \bar{q}, \omega)\omega = M\bar{p} + b(M\bar{p}, M\bar{q}_1, M\omega_1)M\omega_1 = M\bar{p} + b(\bar{p}, \bar{q}_1, \omega_1)M\omega_1 \\
\bar{q}' = \bar{q} - b(M\bar{p}, \bar{q}, \omega)\omega = M\bar{q}_1 - b(M\bar{p}, M\bar{q}_1, M\omega_1)M\omega_1 = M\bar{q}_1 + b(\bar{p}, \bar{q}_1, \omega_1)M\omega_1
\end{cases}$$

So that $\bar{p}' = M\bar{p}_1'$; $\bar{q}' = M\bar{q}_1'$ where:

$$\bar{p}_1' = \bar{p} + b(\bar{p}, \bar{q}_1, \omega_1)\omega_1; \quad \bar{q}_1' = \bar{q} - b(\bar{p}, \bar{q}_1, \omega_1)\omega_1$$

Then, (4.4) implies, using assumption (4.3) on $A$, $q^0 = q^0_1$ and the invariance of the volume elements $d\bar{q}, d\omega$ by $SO_3$:

$$Q(f, f)(s)(M\bar{p}) = Q[f(s)oM, f(s)oM](\bar{p}) \tag{4.8}$$

which shows, by setting $h(s) = f(s)oM$, that $\|h(s)\| = \|f(s)\|$ setting $\bar{q} = M\bar{p}$ in the integral in $\bar{p}$ defining $\|f(s)oM\|$ , and that, $h$ and $f$ are 2 solutions in
2) Let us consider the following element of $SO_3$ in which $\theta \in \mathbb{R}^3$:

$$M_1(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \quad M_3(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

Consider the expression (2.15) of $T_{\alpha\beta}$ in which $f$ satisfies (4.5) and observe that $p_0[M_k(\theta)\bar{q}] = \bar{q}_0$, $k = 1, 2, 3$, $\theta \in \mathbb{R}$.

(i) Set in (2.15) $\alpha = 0$, $\beta = i$ with $i = 1, 2$; now compute the integral using the change of variable $\bar{p} = M_1(\pi)\bar{q}$; the integral in $\bar{q}$ gives, using (4.5): $T_{0i} = -T_{0i}$, $i = 1, 2$; hence $T_{01} = T_{02} = 0$.

(ii) Set in (2.15) $\alpha = 0$, $\beta = 3$ and compute the integral in $\bar{q}$ using the change of variable $\bar{p} = M_3(\pi)\bar{q}$; the integral in $\bar{q}$ gives, using (4.5): $T_{03} = -T_{03}$; hence $T_{03} = 0$.

(iii) Set in (2.15) $\alpha = i$, $\beta = j$ if $i \neq j$ and compute the integral using the change of variable $\bar{p} = M_k(\pi)\bar{q}$ with $k = 1$ if $i = 1$, $j = 2$; $k = 2$ if $i = 1$, $j = 3$; $k = 3$ if $i = 2$, $j = 3$ the integrals in $\bar{q}$ give, using (4.5): $T_{12} = -T_{12}$, $T_{13} = -T_{13}$, $T_{23} = -T_{23}$ hence $T_{12} = T_{13} = T_{23} = 0$.

(iv) Set in (2.15) $\alpha = \beta = i$ with $i = 1, 2$; and compute the integral using the change of variable $\bar{p} = M_k(\pi)\bar{q}$; taking $k = 1$ if $i = 1$ and $k = 3$ if $i = 2$, to obtain using (4.5) $T_{11} = T_{22}$ and $T_{22} = T_{33}$.

This completes the proof of prop 4.1.

In all what follows, we assume that $f_{t_0}$ is invariant by $SO_3$ and that the collision kernel $A$ satisfies assumption (4.4). Notice that $A$ defined in Remark 3.1 is an example of such a kernel.

### 4.3 The Constraint Equation

We study the Cauchy problem for the system (4.1)-(4.2) on $[0, T]$ with initial data:

$$a(0) = a_0; \quad \dot{a}(0) = \dot{a}_0$$

(4.8)

The Einstein equations (2.14) with $G = 1$, give, using the Einstein tensor $S^\alpha_\beta$:

$$H^0_\alpha := S^\alpha_\alpha + \Lambda g^\alpha_\alpha - 8\pi T^\alpha_\alpha = 0$$

It is proved in [12], p. 39, that, in the general case, the four quantities $H^0_\alpha$ do not involve the second derivative of the metric tensor $g$ with respect to $t$, and, using the identities $\nabla_\alpha(S^\alpha_\beta - \Lambda g^\alpha_\beta - 8\pi T^\alpha_\beta) = 0$, that the four quantities $H^0_\alpha$ satisfy a linear homogeneous first order differential system. consequently:

1) For $t = 0$, the quantities $H^0_\alpha$ express uniquely in term of the initial data $a_0$, $\dot{a}_0$ and $f_{t_0}$.
2) If $H^0_\alpha(0) = 0$, then $H^0_\alpha(t) = 0$ in the whole existence domain of the solution $g$ of the Einstein equations.

But $H^0_\alpha(0) = 0$ are 4 constraints to impose to the initial data at $t = 0$ and the equation $H^0_\alpha = 0$ are called constraint equations. In our case, the relations $S^0_i = 0$, $T^0_i = 0$ (see prop 4.1) show that the constraints $H^0_i = 0$, called the momentum constraints are automatically satisfied. It then remains the constraint with $\alpha = 0$, which is equivalent to $S^{00} - \Lambda - 8\pi T^{00} = 0$, called the Hamiltonian constraint, and which is exactly (4.1).

So, following what we said above, the Hamiltonian constraint (4.1) is satisfied in the whole existence domain of the solution $a$ of the initial values problem on $[0, T]$, if the initial data $a_0$, $\dot{a}_0$, $f_0$ at $t = 0$, satisfy, using expression (2.15) of $T^{00}$, the constraint

\[
\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{8\pi a_0^3}{3} \int_{\mathbb{R}^3} \sqrt{1 + a_0^2 \bar{p}^2} f_0(\bar{p}) \, d\bar{p} + \frac{\Lambda}{3} \quad (4.9)
\]

(4.9) gives two possible choices of $\dot{a}_0$, when $a_0$ and $f_0$ are given. We will choose, taking also into account the hypothesis on $a(t)$ in theorem 3.1:

\[
a_0 \geq \frac{3}{2}; \quad f_0 \in L^1_2(\mathbb{R}^3); \quad f_0 \geq 0 \ a.e \quad \dot{a}_0 > 0. \quad (4.10)
\]

We now concentrate on (4.2) which is the evolution equation.

4.4 The Evolution Equation

We set $\theta = 3\frac{\dot{a}}{a}$, then $\dot{\theta} = 3\left[\frac{\frac{\dot{a}}{a} - (\frac{\dot{a}}{a})^2}{a}\right]$ and (4.2) gives:

\[
\dot{\theta} =\frac{\theta^2}{3} - 4\pi(T^{00} + 3a^2T^{11}) + \Lambda \quad (4.11)
\]

(4.11) is the Raychaudhuri equation in $\theta$. We prove:

Proposition 4.2 There can exist no global regular solution for the coupled Einstein-Boltzmann system in the case $\Lambda < 0$

For the proof, we use the following result, proved in [9]:

Lemma 4.1 Let $u$ and $\theta$ be 2 differentiable functions of $t$ satisfying:

\[
\dot{\theta} < -\frac{\theta^2}{3}; \quad \dot{u} = -\frac{u^2}{3}; \quad \theta(t_1) = u(t_1) \quad (4.12)
\]

for a given value $t_1$ of $t$. Then $\theta(t) \leq u(t)$ for $t \geq t_1$.

Proof of Proposition 4.2

Suppose $\Lambda < 0$. Then the Raychaudhuri equation (4.11) gives $\dot{\theta} < -\frac{\theta^2}{3}$. We have, integrating the equation in $u$ on $[t_1, t]$ when $u \neq 0$ and since $\theta(t_1) = u(t_1)$:

\[
u(t) = \frac{3\theta(t_1)}{3 + \theta(t_1)(t - t_1)}; \quad t \geq t_1 \quad (4.13)
\]
Let us show that we have necessary $\dot{a} < 0$.

The Hamiltonian constraint (4.1) writes, using (3.17) and $p^0 = \sqrt{1 + a^2(t)\bar{p}^2}$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi a^3}{3} \int_{\mathbb{R}^3} \sqrt{1 + a^2|\bar{p}|^2} f(t, \bar{p}) d\bar{p} + \frac{\Lambda}{3}$$  \hspace{1cm} (a)

The derivative of the l.h.s is: $A = 2\left(\frac{\dot{a}}{a}\right) \left[\frac{a}{a} - \left(\frac{\dot{a}}{a}\right)^2\right]$. The derivative of the r.h.s is

$$B(t) = \frac{8\pi}{3} \left[3a^2 \ddot{a} \int_{\mathbb{R}^3} p^0 f(t, \bar{p}) d\bar{p} + a^3 \int_{\mathbb{R}^3} \frac{a\dot{a}|\bar{p}|^2}{p^0} f(t, \bar{p}) d\bar{p} + \int_{\mathbb{R}^3} p \frac{\partial f}{\partial t}(t, \bar{p}) d\bar{p}\right]$$  \hspace{1cm} (b)

But we have by (3.21) $\frac{\partial f}{\partial t} = 2\dot{f}(\bar{p})$, then (3.22) shows that the last integral in $B(t)$ is zero. We have, naturally, $A(t) = B(t)$. Suppose $\dot{a}(t_1) > 0$; (b) implies $B(t_1) > 0$. Now $\dot{\theta} < -\frac{2}{3} \Rightarrow \dot{\theta} < 0$. But $\dot{\theta} = 3\left[\frac{2}{3} - \left(\frac{\dot{a}}{a}\right)^2\right] < 0$, so that $A(t_1) < 0$ and the equality $A(t_1) = B(t_1)$ is impossible. We then have $\dot{a}(t_1) \leq 0$. The hypothesis $\dot{a}(t_1) = 0$ would imply that the r.h.s of (a) be a constant, but it is not the case, since it depends on $f$ which changes with $f_{t_1}$. We then conclude that, necessarily $\dot{a} < 0$, which implies $\theta < 0$, so that in (4.12) we have $\dot{a}(t_1) < 0$. Now (1.14) shows that, since by (1.12) $u$ is a decreasing function:

$$u(t) \rightarrow -\infty \text{ when } 3 + \theta(t_1)(t - t_1) \rightarrow 0, \text{ and } t_1 \leq t < t_1 - \frac{3}{\theta(t_1)} := t^*$$  \hspace{1cm} (4.14)

By lemma 4.1, (4.14) implies that $\theta(t) = 3\frac{\dot{a}}{a}(t) \rightarrow -\infty$ when $t \searrow t^*$, then:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 \rightarrow +\infty \text{ when } t \searrow t^*$$  \hspace{1cm} (4.15)

Now, since $\dot{a} < 0$, $a$ is a decreasing function on $[t_1, t^*[$, and then, $a(t) \leq a(t_1)$, $\forall t \in [t_1, t^*[$; (a) and (4.1) then imply, since $\Lambda < 0$:

$$(\dot{a})^2 \leq \frac{8\pi a^5(t_1)}{3} \sqrt{1 + a^2(t_1)||f||} := C^2(t_1, f), \forall t \in [t_1, t^*]$$

so that:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 \leq \frac{C^2(t_1, f)}{a^2(t)} \forall t \in [t_1, t^*[$$

(4.15) then implies that $\frac{C^2(t_1, f)}{a^2(t)} \rightarrow +\infty$ when $t \searrow t^*$ and this can happen only if: $a(t) \rightarrow 0$ when $t \searrow t^*$. So the cosmological expansion factor $a$ tends to zero in a finite time and such a solution $(a, f)$ cannot be global towards the future. This complete the proof of proposition 4.2. ■

We now study the case $\Lambda > 0$. Notice that since $T^{00} \geq 0$, $T^{11} \geq 0$ (4.1)-(4.2) gives by subtraction $\frac{2}{3} - \left(\frac{\dot{a}}{a}\right)^2 < 0$; this implies $\frac{2}{3} - \left(\frac{\dot{a}}{a}\right)^2 < 0$, which shows that, in all the cases, $\theta = 3\frac{\dot{a}}{a}$ is a decreasing function. In the case $\Lambda > 0$, the Hamiltonian
constraint (4.1) gives:

\[
\left( \frac{\dot{a}(t)}{a(t)} \right)^2 \geq \frac{\Lambda}{3} \text{ i.e. } \left[ \frac{\dot{a}(t)}{a(t)} - \sqrt{\frac{\Lambda}{3}} \right] \left[ \frac{\dot{a}(t)}{a(t)} + \sqrt{\frac{\Lambda}{3}} \right] \geq 0,
\]

which is equivalent to:

\[
\frac{\dot{a}(t)}{a(t)} \geq \sqrt{\frac{\Lambda}{3}} \quad (4.16)
\]

or

\[
\frac{\dot{a}(t)}{a(t)} \leq -\sqrt{\frac{\Lambda}{3}} \quad (4.17)
\]

The continuity of \( t \rightarrow \frac{\dot{a}(t)}{a(t)} \) implies that we have to choose between (4.16) and (4.17). Since \( a > 0 \), (4.17) implies \( \dot{a} < 0 \) and \( a \) is decreasing; (4.16) implies that \( \dot{a} > 0 \), then \( a \) is increasing and since \( \frac{\dot{a}}{a} \) is decreasing, this gives on \([t_0, t]\):

\[
a(t) \geq a(t_0); \quad \sqrt{\frac{\Lambda}{3}} \leq \frac{\dot{a}(t)}{a(t)} \leq \frac{\dot{a}(t_0)}{a(t_0)}, \quad t \geq t_0 \quad (4.18)
\]

Recall that our aim is to study the coupled Einstein-Boltzmann system and we had to require, for the study of the Boltzmann equation, that \( a \), which is positive be bounded away from zero. This problem is solved by choosing (4.16). Another important consequence of (4.16) is that it implies on \([t_0, t]\)

\[
a(t) \geq a(t_0)e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}, \quad t \geq t_0
\]

Which shows that, the cosmological expansion factor has an exponential growth; but, it also shows that, an eventual global solution \( a \) will be unbounded, and this is why, in order to use standard results, we make the change of variable:

\[
c = \frac{1}{a} \quad (4.19)
\]

which gives:

\[
\dot{c} = -\frac{\dot{a}}{a^2} \quad (4.20)
\]

Then, using (4.16) \( T^{00} \) and \( T^{11} \) in the r.h.s of (4.2) express in, terms of \( e \) and \( f \), and we set:

\[
\rho = T^{00} = \frac{1}{e^3} \int_{\mathbb{R}^3} \sqrt{1 + \frac{1}{e^2} |\vec{p}|^2 f(t, \vec{p}) d\vec{p}} \quad (4.21)
\]

\[
P = a^2 T^{11} = \frac{1}{e^3} \int_{\mathbb{R}^3} (p^1)^2 f(t, \vec{p}) \sqrt{1 + \frac{1}{e^2} |\vec{p}|^2} d\vec{p} \quad (4.22)
\]

\( \rho \) stands for the density and \( P \) for the pressure. One verifies that:

\[
P \leq \rho \quad (4.23)
\]

Recall that \( r > 0 \) is such that \( r > \|f_0\| \). If we set:

\[
d_0 = 3 \sqrt{\frac{\Lambda}{3} + \frac{16\pi}{3} ra_0^4} \quad (4.24)
\]
One checks easily, using $a_0 \geq \frac{3}{2}$, that
\[ e = \frac{1}{a} \in [0, \frac{2}{3}]; \quad \text{et} \quad \theta = 3 \frac{\dot{a}}{a} \in [\sqrt{3}X, d_0] \] (4.25)
A direct calculation shows, using (4.19), (4.20), (4.21), (4.22), and (4.11), that the Einstein evolution equation (4.2) is equivalent to the following first order system in $(e, \theta)$:

\[
\dot{e} = -\frac{\theta}{3} \times e \quad (4.26)
\]

\[
\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(\rho + 3P) + \Lambda \quad (4.27)
\]
with $\rho = \rho(e, f)$ and $P = P(e, f)$ given by (4.21) and (4.22).

we will study the initial values problem for the system (4.26)-(4.27) with initial data $(e_0, \theta_0)$ and $t = 0$. By virtue of the change of variables (4.25), we will deduce solution for the Einstein evolution equation (4.2) by setting:

\[ e(0) = \frac{1}{a_0}, \quad \theta(0) = 3 \frac{\dot{a}_0}{a_0} \quad (4.28) \]

in which $a_0, \dot{a}_0$ satisfy the constraint (4.10)-(4.9) with $f_0$ given. By virtue of (4.25) it will be enough for the study of the initial value problem for (4.26)- (4.27) to take $e_0, \theta_0$ such that

\[ 0 < e_0 \leq \frac{2}{3}, \quad 0 < \theta_0 \leq d_0 \quad (4.29) \]

**Remark 4.1** The equivalence of the evolution equation (4.2) and the system (4.20)-(4.27) requires that any solution $(e, \theta)$ of (4.20)-(4.27) satisfies $e > 0$ and $\theta > 0$.

In fact, (4.21), (4.22) require $e \neq 0$; by (4.20), $e(0) = e_0 > 0$. Since $e$ is continuous and cannot vanish, by virtue of the mean values problem, $e$ remains strictly positive. Now if we have $\theta(t_1) = 0$, then (4.20) implies $\dot{e}(t_1) = 0$ and (4.21) implies $\dot{a}(t_1) = 0$, which is impossible given (4.10). So $\theta$ never vanishes and by (4.25) $\theta(0) = \theta_0 > 0$; So $\theta > 0$.

For the global existence theorem, we will need the following a priori estimation

**Proposition 4.3** Let $\delta > 0$ and $t_0 \in \mathbb{R}_+$ be given; suppose that in (4.21) and (4.22), $f \in C[t_0, t_0 + \delta; L^1_2(\mathbb{R}^3)]$ is given. Let $(e, \theta)$ be any solution of the system (4.20)-(4.27) on $[t_0, t_0 + \delta]$. Then $\theta$ and $a = \frac{1}{e}$ satisfy the inequalities:

\[ \theta(t_0 + t) \leq \theta(t_0) + \Lambda, \quad t \in [0, \delta] \quad (4.30) \]

\[ a(t_0 + t) \leq a(t_0)e^{\left(\frac{\theta(t_0)}{3} + \frac{\Lambda}{3}\right)(t_0 + t + 1)^2}, \quad t \in [0, \delta] \quad (4.31) \]

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Proof.
Since \( \rho \geq 0 \), \( P \geq 0 \), Proposition 4.3 implies: \( \dot{\theta} \leq \Lambda \). Integrating this inequality on \([t_0, t_0 + \delta]\) where \( \delta \) yields (4.31).
Next, since \( \epsilon > 0 \), integrating (4.26) that writes \(-\frac{e}{3} = \frac{\dot{\epsilon}}{3}\) over \([t_0, t_0 + \delta]\), \( t \in [0, \delta] \) gives:
\[
a(t_0 + t) \leq a(t_0)e^{\frac{\epsilon t}{3}} \int_{t_0}^{t_0 + \delta} \frac{\dot{\epsilon}}{3} ds, \quad t \in [0, \delta]
\]
(4.32)
Now setting in (4.31) \( s = t_0 + t \in [t_0, t_0 + \delta]\), and integrating on \([t_0, t_0 + t]\) yields:
\[
\int_{t_0}^{t_0 + t} \frac{\dot{\theta}}{3} ds \leq \left( \frac{\theta(t_0)}{3} + \frac{\Lambda}{3} \right)(t_0 + t + 1)^2, \quad t \in [0, \delta]
\]
(4.33)
then follows from (a) and (b). We deduce:

**Proposition 4.4** Let \( T > 0 \) and \( f \in \mathbb{R} \) be given. Suppose that the initial value problem \((4.26)-(4.27)-(4.28)-(4.29)\) with initial data \((a, \theta(t_0))\) at \( t = 0 \) satisfying the constraints \((4.10)-(4.11)-(4.12)-(4.13)\) has a solution \((\Xi = \Xi, \Theta)\) on \([0, t_0]\) with \( 0 \leq t_0 < T \). Then, any solution \((\epsilon, \theta)\) of the initial value problem for the system \((4.26)-(4.27)-(4.28)-(4.29)\) on \([t_0, t_0 + \delta]\), \( \delta > 0 \), with initial data \((e, \theta(t_0)) = (\Xi, \Theta)(t_0)\) at \( t = t_0 \), satisfy the inequalities:
\[
a(t_0 + t) \leq C_2 e^{C_3 (t_0 + t + 1)^2}, \quad t \in [0, \delta]
\]
(4.34)
\[
\theta(t_0 + t) \leq 3\gamma_1 + \Lambda(T + t), \quad t \in [0, \delta]
\]
(4.35)
where:
\[
C_2 = a_0 e^{\gamma_1}; \quad C_3 = \gamma_1 + \frac{\Lambda}{3}; \quad \gamma_1 = \gamma_1(a_0, r, T) = \left( \frac{\Lambda}{3} + \sqrt{\frac{\Lambda}{3} + 3 a_0^2} \right)(T + 1)^2
\]
(4.36)

**Proof**
We apply proposition 4.3 to the solution \((e, \theta)\) of \((4.26)-(4.27)-(4.28)-(4.29)\) on \([t_0, t_0 + \delta]\); (4.31) gives, since \( a(t_0) = \Xi(t_0) \), \( \theta(t_0) = \Theta(t_0) \):
\[
a(t_0 + t) \leq \Xi(t_0)e^{\left( \frac{a_0}{a_0 + \frac{1}{3}} \right)(t_0 + t + 1)^2}, \quad t \in [0, \delta]
\]
(a)
Now apply (4.31) to the solution \((\Xi, \Theta)\) of \((4.26)-(4.27)-(4.28)-(4.29)\) on \([0, t_0]\), at the point \( t_0 \). We obtain by setting in (4.31), \( t_0 = 0 \), \( t = t_0 \) and since \( a(t_0) = \Xi(t_0) \), \( \Xi(0) = a_0 \), \( \Theta(0) = 3 a_0 \), \( 0 \leq t_0 < T \):
\[
\Xi(t_0) \leq a_0 e^{\left( \frac{a_0}{a_0 + \frac{1}{3}} \right)(t_0 + 1)^2} \leq a_0 e^{\left( \frac{a_0}{a_0 + \frac{1}{3}} \right)(T + 1)^2}, \quad t \in [0, \delta]
\]
(b)
Now since the initial data at \( t = 0 \) satisfy (4.10), (4.11), this imply, using \( a_0 > 0 \), \( a_0 > 1 \), \( \|f_0\| < r \):
\[
\dot{a_0} \leq \sqrt{\frac{\Lambda}{3} + \frac{8\pi}{3} r a_0^3}
\]
(c)
and (b) gives:
\[ \Xi(t_0) \leq a_0 e^{\gamma_1} \]  
\hspace{1cm} (d)

with
\[ \gamma_1 = \left( \frac{\Lambda}{3} + \sqrt{\frac{\Lambda}{3} + \frac{8\pi}{3} r a_0^4} \right) (T + 1)^2 \]  
\hspace{1cm} (e)

Now, we apply (4.30) to the solution \((\Xi, \Theta)\) of (4.26)-(4.27)-(4.28), on \([0, t_0]\) at point \(t_0\), \(t = t_0\) and since \(\Xi(0) = 3\frac{\dot{a}_0}{a_0}\), \(0 \leq t_0 < T\):
\[ \Xi(t_0) \leq \Xi(0) + \Lambda t_0 \leq 3 \frac{\dot{a}_0}{a_0} + \Lambda T \]  
\hspace{1cm} (f)

We obtain, using (c) and (f):
\[ \Xi(t_0) \leq 3 + \Lambda + \Lambda T \]  
\hspace{1cm} (g)

and (4.35) follows from (a), (d) and (g).

The inequality (4.35) follows from (4.30), \(\theta(t_0) = \Xi(t_0)\), (f), (c) and (e). This completes the proof of prop. 4.4. ■

We can now prove.

Proposition 4.5 Let \(T > 0\) and \(f \in C([0,T]; X_r)\) be given. Then the initial value problem for the system (4.26)-(4.27) with initial data \((e_0, \theta_0)\) at \(t = 0\) satisfying (4.29), has an unique solution \((e, \theta)\) on \([0, T]\)

The proof of proposition 4.5 will use the following result.

Lemma 4.2 Let \(e_1, e_2 \in [0, \frac{T}{2}]\); then we have, \(C\) being a constant:
\[ |\rho(e_1, f) - \rho(e_2, f)| \leq \frac{C}{e_1^2 e_2^2} \|f(t)\| |e_1 - e_2| \]  
\hspace{1cm} (4.37)

\[ |P(e_1, f) - P(e_2, f)| \leq \frac{C}{e_1^2 e_2^2} \|f(t)\| |e_1 - e_2| \]  
\hspace{1cm} (4.38)

Proof of the lemma 4.2
1) To prove, (4.37), we write (4.21) in \(e_1, e_2 \in [0, \frac{T}{2}]\) and we subtract. We will use:
\[ \frac{1}{e_1^2} - \frac{1}{e_2^2} = \frac{|e_1^2 - e_2^2|}{e_1^2 e_2^2} = \frac{|e_1 - e_2|}{e_1^2 e_2^2} |e_1^2 + e_1 e_2 + e_2^2| \leq 3 \frac{|e_1 - e_2|}{e_1^2 e_2^2} \]  
\hspace{1cm} (a)
We will also use, applying $|\sqrt{a} - \sqrt{b}| = |a - b|/\sqrt{|a + b|^{-1}}$, $0 < e_i < 1$ and (a):

$$\left| \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} - \sqrt{1 + \frac{1}{e_2^2} |\vec{p}|^2} \right| \leq \frac{1}{e_1^2} - \frac{1}{e_2^2} \frac{|\vec{p}|^2}{\sqrt{1 + |\vec{p}|^2}} \left| e_1^2 - e_2^2 \right| \leq 3\sqrt{1 + |\vec{p}|^2} \left| e_1 - e_2 \right| \left( \frac{1}{e_1^2} - \frac{1}{e_2^2} \right)$$

(b)

Now (4.21) gives:

$$\left| \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} - \sqrt{1 + \frac{1}{e_2^2} |\vec{p}|^2} \right| \leq \frac{1}{e_1^2} - \frac{1}{e_2^2} \frac{|\vec{p}|^2}{\sqrt{1 + |\vec{p}|^2}} \left| e_1^2 - e_2^2 \right| \leq 3\sqrt{1 + |\vec{p}|^2} \left| e_1 - e_2 \right| \left( \frac{1}{e_1^2} - \frac{1}{e_2^2} \right)$$

(c)

and (4.37) follows from (d), (e), (f), and $e_i \leq 1$.

2) To prove (4.37), we write (4.21) in $e_1, e_2 \in [0, \frac{2}{4}]$ and we subtract. We will use:

$$\left| \frac{1}{e_1^2} - \frac{1}{e_2^2} \right| = \frac{e_2^2 - e_1^2}{e_1^2 e_2^2} = \frac{|e_1 - e_2|}{e_1^2 e_2^2} |e_1^2 + e_1^2 e_2 + e_2^2 + e_1 e_2| \leq \frac{|e_1 - e_2|}{e_1^2 e_2^2} \left| e_1 - e_2 \right| \leq \frac{|e_1 - e_2|}{e_1 e_2}$$

(d)

and we also use $0 < e_i < 1$ and $(\frac{p^1}{|\vec{p}|})^2 \leq 1$ to obtain:

$$\left| \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} - \sqrt{1 + \frac{1}{e_2^2} |\vec{p}|^2} \right| \leq \frac{1}{e_1^2} - \frac{1}{e_2^2} \frac{(\vec{p})^2}{\sqrt{1 + |\vec{p}|^2}} \left| e_1^2 - e_2^2 \right| \leq \frac{2}{e_2^2} \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} |e_1 - e_2| \leq \frac{2}{e_1 e_2} \sqrt{1 + |\vec{p}|^2} |e_1 - e_2|$$

(e)

Now (4.21) gives:

$$\left| P(e_1, f) - P(e_2, f) \right| \leq \frac{1}{e_1^2} - \frac{1}{e_2^2} \int_{\mathbb{R}^3} \left| \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} f(t, \vec{p}) \right| d\vec{p} \leq \frac{1}{e_2^2} \int_{\mathbb{R}^3} \left| \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} \right| d\vec{p} \left| \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} \right| f(t, \vec{p}) d\vec{p}$$

(f)

(4.37) then follows from (d), (e), (f), $0 < e_i < 1$, and using in the first integral:

$$\frac{(p^1)^2}{\sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2}} \leq e_1^2 \sqrt{1 + \frac{1}{e_1^2} |\vec{p}|^2} \leq e_1 \sqrt{1 + |\vec{p}|^2}$$

(4.39)
This completes the proof of the lemma 4.2. ■

Proof of proposition 4.5

Define using (4.21), (4.22), the function \( F \) on \([0, \delta] \times [0, d_0] \) by:

\[
F(e, \theta) = \left[ \frac{\theta e}{3} - \frac{\theta^2}{3} \right] - 4\pi(\rho + 3P) + \Lambda
\]

To prove the proposition 4.5, we first prove the existence of a local solution. To have that result, we have to show that \( F \) defined by (4.40) is continuous with respect to \( t \), and locally Lipschitzian in \((e, \theta)\) with respect to the norm of \( \mathbb{R}^2 \).

The dependence of \( F \) on \( t \) is through \( f \) that appears in \( \rho \) and \( P \) (see formulae (4.21) and (4.22)). But, since \( f \in C[t_0, t_0 + T, X_r] \), the definition (4.20) of this space shows that \( F \) is a continuous function of \( t \). Now we have, with \((e_1, \theta_1), (e_2, \theta_2) \in [0, \frac{\delta}{2}] \times [0, d_0] \), and by usual factorization:

\[
\left| \frac{\theta_1 e_1}{3} - \frac{\theta_2 e_2}{3} \right| \leq \frac{1 + d_0}{3} (|\theta_1 - \theta_2| + |e_1 - e_2|) \quad (4.41)
\]

\[
\left| \frac{1}{3}(\theta_1^2 - \theta_2^2) \right| \leq \frac{2d_0}{3}|\theta_1 - \theta_2| \quad (4.42)
\]

Let us fix \( e^1 \in [0, \frac{\delta}{2}] \) and we take \( e_1, e_2 \in \left[ \frac{e^1}{2}, \frac{e^1 + \frac{\delta}{2}}{2} \right] \) then:

\[
|\rho(e_1, f) - \rho(e_2, f)| + 3|P(e_1, f) - P(e_2, f)| \leq N(e^1) r|e_1 - e_2| \quad (4.43)
\]

Where \( N \) is a constant depending only on \( e^1 \).

Then, if \( \mathbb{R}^2 \) is endowed with the norm \( \|(x, y)\|_{\mathbb{R}^2} = |x| + |y| \), we have for \( F \) defined by (4.40), using (4.41) + (4.42) + (4.43):

\[
\|F(e_1, \theta_1) - F(e_2, \theta_2)\|_{\mathbb{R}^2} \leq M(|\theta_1 - \theta_2| + |e_1 - e_2|) = M\|(e_1, \theta_1) - (e_1, \theta_1)\|_{\mathbb{R}^2} \quad (4.44)
\]

Where \( M \) is a constant depending only on \( e^1 \), and \( d_0 \).

(4.43) shows that \( F \) is locally Lipschitzian in \((e, \theta)\) with respect to the norm of \( \mathbb{R}^2 \). The standard theorem of the first order differential system implies that the initial value problem (4.20), (4.21), (4.22) has a unique local solution \((e, \theta)\) on \([0, \delta], \delta > 0 \). Notice that since \( f \in C[[0, T]; X_r] \), the system (4.20) is defined on \([0, T] \). Hence the maximum value of \( \delta \) is \( T \). Now if \( 0 < \delta < T \), prop. 4.5 in which we set \( t_0 = 0 \), \( 0 \leq t \leq \delta < T \), shows, applying (4.31) and (4.32), that for every solution \( e \), the function \( \frac{1}{e} \) is uniformly bounded, and, \( \rho \) and \( P \) defined by (4.21) and (4.22) are uniformly bounded. Consequently, the function \( F \) defined by (4.40) is uniformly bounded. The standard theorem on the first order system then implies that the solution \((e, \theta)\) is global on \([0, T] \).

This complete the proof of prop. 4.5. ■

We deduce a result that will be useful to prove the global existence for the coupled Einstein-Boltzmann system. We will use the number \( D_0 \) defined by:

\[
D_0 = 3\gamma_1 + \Lambda(T + 1) \quad (4.45)
\]

where \( \gamma_1 \) is defined by (4.33). We prove:
Proposition 4.6 Let $T > 0$ and $f \in C[[0,T];X_r]$ be given. Let $(\Xi = \frac{1}{r}, \Theta)$ be the solution of the initial value problem for the system (4.20)-(4.27) with initial data $(e_0, \theta_0)$ at $t = 0$ given by (4.28) with $a_0, \dot{a}_0$ satisfying the constraints (4.10)–(4.30). Let $t_0 \in [0,T]$. Then the initial values problem for the system (4.20)-(4.27), with the initial data $(\Xi = \frac{1}{r}, \Theta)(t_0)$ at $t = t_0$ has a unique solution $(e = \frac{1}{r}, \theta)$ on $[t_0, t_0 + \delta]$, where $\delta > 0$ is independent of $t_0$. The solution $(e = \frac{1}{r}, \theta)$ satisfies the inequalities:

\[ \frac{3}{2} \leq a(t_0 + t) \leq C_2 e^{C_3(t_0 + t + 1)^2}, \quad t \in [0, \delta] \quad (4.46) \]

\[ \sqrt{3\Lambda} \leq \theta(t_0 + t) \leq 3\gamma_1 + \Lambda(T + t), \quad t \in [0, \delta] \quad (4.47) \]

where $C_2$, $C_3$ and $\gamma_1$ are defined by (4.30)

Proof

We have $\theta_0 = \frac{3\delta}{a_0}$ and (4.10) implies $\theta_0 \geq \sqrt{3\Lambda}$, so, the proof given for prop. 4.5 with the function $F$ given by (4.40) and defined this time on $[0, \frac{\delta}{3}] \times [\sqrt{3\Lambda}, D_0]$ leads to a solution $(\Xi, \Theta)$ satisfying $\Theta \geq \sqrt{3\Lambda}$, hence, $\Theta(t_0) \geq \sqrt{3\Lambda}$. Now suppose that we look for solutions $(e, \theta)$ on $[t_0, t_0 + \delta]$ with $0 < \delta < 1$; then (4.30) shows that every solution $(e, \theta)$ satisfies $\theta(t_0 + t) \leq D_0$, $0 < t < \delta$, where $D_0$ is defined by (4.45). We prove the existence of a local solution $(e, \theta)$ of (4.20)-(4.27) on $[t_0, t_0 + \delta]$, with initial data at $t = t_0$: $e(t_0) = \Xi(t_0)$, $\theta(t_0) = \Theta(t_0) \geq \sqrt{3\Lambda}$, following the same lines as in the proof of prop. 4.5, with the function $F$ given by (4.40), defined this time on $[0, \frac{\delta}{3}] \times [\sqrt{3\Lambda}, D_0]$. This leads to the existence of a local solution $(e, \theta)$ on some interval $[t_0, t_0 + \delta]$, $0 < \delta < 1$. On the other hand, (4.33) gives, using $0 \leq t_0 < T$, $0 < \delta < 1$: $a(t_0 + t) = \frac{1}{\sqrt{a(t_0 + t)}} \leq C_2 e^{C_3(T + 2)^2}$, which shows that, for every solution $(e, \theta)$, $\frac{1}{e}$ is uniformly bounded. From there, the existence of a number $0 < \delta < 1$, independent of $t_0$. Finally, (4.40) follows from (4.33) and $a(t_0 + t) = e(t_0 + t) \leq \frac{3}{2}$; (4.41) follows from (4.33) and what we said above. This ends the proof of prop. 4.6. \[ \square \]

Theorem 4.1 Let $T > 0$ and $f \in C[[0,T];X_r]$ be given. Then the initial value problem for the Einstein equation (4.1)-(4.2)-(4.8) with initial data $a_0, \dot{a}_0$ satisfying the constraints (4.10)-(4.50) has a unique solution $a$ on $[0,T]$.

Proof

Apply proposition 4.5, choosing the initial data $(e_0, \theta_0)$ at $t = 0$ by (4.28). Prop. 4.5 then proves the existence of the unique solution $(e, \theta)$ of (4.20)-(4.27)-(4.28) on $[0,T]$. Now the equivalence of the evolution equation (4.12) with the system (4.20)-(4.27) show that $a = \frac{1}{r}$ is the unique solution of (4.12) with the initial data $a_0, \dot{a}_0$. Now since $a_0, \dot{a}_0$ satisfy the initial constraint (4.10), The Hamiltonian constraint (10) is satisfied on the whole existence domain $[0,T]$ of $a$. This ends the proof of theorem 4.1. \[ \square \]
5 Local Existence for the Coupled Einstein-Boltzmann System

5.1 Equations and Functional Framework

Given the equivalence of the initial value problems (4.1)-(4.2)-(4.8) with constraint (4.10)-(4.9) and (4.26)-(4.27)-(4.28), and following the study of the Boltzmann equation, paragraph 3, by the characteristic method that leads to (3.14), we study the initial value problem for the first order system:

\[
\frac{df}{dt} = \frac{1}{\rho_0}Q(f, f) \quad (5.1)
\]

\[
\dot{e} = -\frac{\theta}{3}e \quad (5.2)
\]

\[
\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(\rho + 3P) \quad (5.3)
\]

with, in (5.3) \( \rho = \rho(e, f) \) and \( P = P(e, f) \) defined by (4.21) and (4.22). The initial data at \( t = 0 \) are denoted \( f_0, e_0, \theta_0 \) i.e.

\[
f(0, \vec{p}) = f_0(\vec{p}); \quad e(0) = e_0; \quad \theta(0) = \theta_0 \quad (5.4)
\]

where we take

\[
f_0 \in L^2_1(\mathbb{R}^3); \quad f_0 \geq 0 \quad a.e; \quad e_0 = \frac{1}{a_0}; \quad \theta_0 = 3\frac{\dot{a}_0}{a_0} \quad (5.5)
\]

With, following (4.10)-(4.9), \( a_0, \dot{a}_0, f_0 \) subject to the constraints:

\[
a_0 \geq \frac{3}{2}, \quad \frac{\dot{a}_0}{a_0} = \sqrt{\frac{8\pi}{3} \int_{\mathbb{R}^3} a_0^3 \sqrt{1 + a_0^2|\vec{p}|^2f_0(\vec{p})d\vec{p}} + \frac{\Lambda}{3}} \quad (5.6)
\]

We are going to prove the existence of the solution \((f, e, \theta)\) of the above initial value problem, on an interval \([0, l] \), \( l > 0 \). Given the study in paragraphs 3 and 4, the functional framework will be the Banach space \( E = L^2_1(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R} \), endowed with the norm

\[
\|(f, e, \theta)\|_E = \|f\| + |e| + |\theta|
\]

where \( \|\cdot\| \) is defined by (2.4) and \( |\cdot| \) is the absolute value in \( \mathbb{R} \). (5.1)-(5.2)-(5.3)-(5.4) can be written in the following standard form of first order differential systems:

\[
\begin{aligned}
\frac{dX}{dt} &= F(t, X) \\
X(0) &= X_0
\end{aligned} \quad \text{with} \quad \begin{aligned}
X &= (f, e, \theta) \in E \\
X(0) &= (f_0, e_0, \theta_0)
\end{aligned} \quad (5.7)
\]

We will prove the local existence theorem by applying the standard theorem for such systems in a Banach space, and that requires that \( F \) be continuous with
We will deal with the quantities:  

\[ F_0, \quad F_1, \quad F_2, \quad F_3, \quad Q, \quad \rho, \quad P \]

and the collision kernel \( A \) given by (3.4) depends on \( e \), and this implies that in \( \mathbb{R}^3 \), \( F \) does not depend explicitly on \( t \), but only on \( f, e, \theta \).

5.2 The Local Existence Theorem

Proposition 5.1 There exist an interval \([0, l]\), \( l > 0 \) such that, the initial value problem for the system (5.4), (5.5), (5.6) with initial data

\( (f_0, e_0, \theta_0) \in L^1_2(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R} \) has a unique solution \( (f, e, \theta) \) on \([0, l] \).

Remark 5.1 For the proof of proposition 5.1, we will set

\[ F_1(e, f, \theta) = \frac{1}{p^0}Q(f, f), \quad F_2(e, f, \theta) = -\frac{\theta}{3}e, \quad F_3(e, f, \theta) = -\frac{\theta^2}{3} - 4\pi(\rho + 3P) + \Lambda \]

(5.8)
in \( F_2 \) and \( F_3 \), \( (f, e, \theta) \mapsto -\frac{\theta}{3}e \) and \( (f, e, \theta) \mapsto -\frac{\theta^2}{3} - 4\pi(\rho + 3P) + \Lambda \) are \( C^\infty \) functions, which are then, locally Lipschitzian with respect to \((f, e, \theta)\). The problem will be to prove that, in \( F_1 \) and \( F_3 \), \((f, e) \mapsto p^0Q(f, f)\) and \((f, e) \mapsto -4\pi(\rho + 3P)(f, e)\) are locally Lipschitzian with respect to \((f, e)\), in the \( L^1_2(\mathbb{R}^3) \times \mathbb{R} \) norm.

We will deal with the quantities:

\[ \frac{1}{p^0(e_1)}Q(f_1, f_1)(e_1) - \frac{1}{p^0(e_2)}Q(f_2, f_2)(e_2); \quad \rho(e_1, f_1) - \rho(e_2, f_2); \quad P(e_1, f_1) - P(e_2, f_2) \]

and this will lead us to look in detail inside the collision operator \( Q \). It then shows useful to begin by giving some preliminary results.

Lemma 5.1 We have for \( f, f_1, f_2 \in L^1_2(\mathbb{R}^3) \), \( e_1, e_2 \in [0, \frac{3}{2}] \), and \( C \) being a constant:

\[
\left\| \left( \frac{1}{p^0(e_1)} - \frac{1}{p^0(e_2)} \right) Q(f, f)(e_2) \right\| \leq \frac{C}{e_1 e_2} \| f \| \| e_1 - e_2 \| \tag{5.9}
\]

\[
\left\| \frac{Q(f_1, f_1)(e_2) - Q(f_2, f_2)(e_2)}{p^0(e_2)} \right\| \leq \frac{C}{e_2^2} (\| f_1 \| + \| f_2 \|) \| f_2 - f_1 \| \tag{5.10}
\]

Proof of the lemma 5.1

Recall that \( p^0 = \sqrt{1 + \frac{1}{e_2^2} |\bar{p}|^2} \):

1) we have:

\[
\left| \left( \frac{1}{p^0(e_1)} - \frac{1}{p^0(e_2)} \right) Q(f, f)(e_2) \right| = \left| \frac{p^0(e_2) - p^0(e_1)}{p^0(e_1)} \times Q(f, f)(e_2) \right|
\]

\[
= \frac{1}{e_2^2} - \frac{1}{e_2^2} \times \frac{|\bar{p}|^2}{(p^0(e_1) + p^0(e_2))p^0(e_1)} \left| Q(f, f)(e_2) \right|
\]

But we have:

\[
\frac{|\bar{p}|^2}{(p^0(e_1) + p^0(e_2))p^0(e_1)} \leq \frac{|\bar{p}|^2}{|\bar{p}|^2} e_2^2 = e_2^2 < 1
\]
So we have, since $0 < e_i < 1$ and using the expression (2.4) of $|.|$:

$$\left\| \left( \frac{1}{p^0(e_1)} - \frac{1}{p^0(e_2)} \right) Q(f, f)(e_2) \right\| \leq \frac{2}{e_1^2 e_2} \left\| Q(f, f)(e_2) \right\| |e_1 - e_2| \quad (a)$$

Then follows from (a), the inequality (3.26) on $Q$, in which we set $g = 0$, and with $a = \frac{1}{e_2}$ in (3.27).

2) (5.10) follows from (3.26) in which we set: $f = f_1$, $g = f_2$ and $a = \frac{1}{e_2}$ in (3.27). ■

**Lemma 5.2** We have for $f \in L_2^1(\mathbb{R}^3)$, $e_1, e_2 \in ]0,\frac{3}{2}]$ and $C$ being a constant:

$$\left\| \frac{1}{p^0(e_1)} [Q(f, f)(e_1) - Q(f, f)(e_2)] \right\| \leq \frac{C}{e_1^2 e_2}|e_1 - e_2| \quad (5.11)$$

**Proof of Lemma 5.2**

Here we use $Q = Q^+ - Q^-$, where $Q^+, Q^-$ are given by (3.2) and (3.3), in which $a = \frac{1}{2}$. We write:

$$Q(f, f)(e_1) - Q(f, f)(e_2) = [Q^+(f, f)(e_1) - Q^+(f, f)(e_2)] + [Q^-(f, f)(e_1) - Q^-(f, f)(e_2)] \quad (a)$$

We can write, using the expression (3.2) of $Q^+$

$$[Q^+(f, f)(e_1) - Q^+(f, f)(e_2)] = \int_{\mathbb{R}^3} \int_{S^2} B(e_1, e_2, \bar{p}, \bar{q}, \vec{p}, \vec{q}) f(\vec{p}) f(\vec{q}) d\bar{q} d\varpi \quad (b)$$

where

$$B = B(e_1, e_2, \bar{p}, \bar{q}, \vec{p}, \vec{q}) = \frac{1}{e_1^2 q^0(e_1)} [A(e_1) - A(e_2)] + \left( \frac{1}{e_1^2 q^0(e_1)} - \frac{1}{e_2^2 q^0(e_2)} \right) A(e_2) \quad (c)$$

in which $A(e_i)$ stands in fact for $A(e_i, \bar{p}, \bar{q}, \vec{p}, \vec{q})$, $i = 1, 2$. Assumption (3.4) on the collision kernel $A$, gives:

$$\left| \frac{1}{e_1^2 q^0(e_1)} (A(e_1) - A(e_2)) \right| \leq \frac{\gamma}{e_1^2 q^0(e_1)} |e_1 - e_2| \quad (d)$$

Now we can write, using $q^0(e_i) = \sqrt{1 + \frac{1}{e_i^2} |\vec{q}|^2}$ and $0 < e_i < 1$, $i = 1, 2$:

$$\left| \frac{1}{e_1^2 q^0(e_1)} - \frac{1}{e_2^2 q^0(e_2)} \right| = \left| \frac{(e_2^2 - e_1^2) q^0(e_2) + e_1 [q^0(e_2) - q^0(e_1)]}{e_1^2 e_2^3 q^0(e_1) q^0(e_2)} \right| \leq \frac{1}{q^0(e_1)} \left[ \frac{3}{(e_1 e_2)^3} + \frac{e_1^2 - e_2^2}{e_1^2 e_2^2 (q^0(e_2))^2} \right]$$

Hence:

$$\left| \frac{1}{e_1^2 q^0(e_1)} - \frac{1}{e_2^2 q^0(e_2)} \right| \leq \frac{1}{e_1^2 q^0(e_1)} \left( \frac{3}{e_2^2} + \frac{2}{e_1^2 e_2^2} \right) |e_1 - e_2| \quad (e)$$
Then apply (5.11) with $f$.

We can write:

**Proof of the lemma 5.3**

using the change of variable (\( \bar{\rho}, \bar{q} \)) in lemma 3.1, using the change of variable \((\bar{p}, \bar{q}) \mapsto (\bar{p'}, \bar{q'})\) defined by \(3.11\) and this leads, using \(3.27\) to

\[
\left\| \frac{Q^+(f_1, f_2)}{p^0(e_1)} (e_1) - \frac{Q^+(f_1, f_2)}{p^0(e_2)} (e_2) \right\| \leq \frac{C \| f \|^2 |e_1 - e_2|}{e_1^2 e_2^3} \]

Next, we proceed the same way for the second term \(Q^-(f, f)(e_2) - Q^-(f, f)(e_1)\) of (a), using this time the expression \(3.13\) of \(Q^-\). The only difference is that, in integrals (b) and (g) \(f(p')f(q')\) is replaced by \(f(\bar{p})f(\bar{q})\). A direct computation without change of variable, then leads to the same estimation (h) where \(Q^+\) is replaced by \(Q^-\), and lemma 5.2 follows.

**Lemma 5.3** We have for \(e_1, e_2 \in [0, \frac{2}{3}]\), \(f_1, f_2 \in L^1_2(\mathbb{R}^3)\) and \(C > 0\) being a constant.

\[
\left\| \frac{Q(f_1, f_1)}{p^0(e_1)} (e_1) - \frac{Q(f_2, f_2)}{p^0(e_2)} (e_2) \right\| \leq \frac{C(\| f_1 \|^2 + \| f_2 \|^2)(|e_1 - e_2| + \| f_1 - f_2 \|)}{e_1^2 e_2^3}
\]

(5.12)

**Proof of the lemma 5.3**

We can write:

\[
\frac{Q(f_1, f_1)}{p^0(e_1)} (e_1) - \frac{Q(f_2, f_2)}{p^0(e_2)} (e_2) = \frac{Q(f_1, f_1)}{p^0(e_1)} (e_1) - \frac{Q(f_1, f_1)}{p^0(e_2)} (e_2) + \left( \frac{1}{p^0(e_1)} - \frac{1}{p^0(e_2)} \right) Q(f_1, f_1)(e_2)
\]

\[
+ \frac{1}{p^0(e_2)} [Q(f_1, f_1)(e_2) - Q(f_2, f_2)(e_2)]
\]

Then apply \(5.11\) with \(f = f_1\) to the first term, \(5.12\) with \(f = f_1\) to the second term, and \(5.11\) to the third term to obtain \(5.12\), by addition of the inequalities and using \(0 < e_i < 1\):

We now consider \(F_3\) in \(5.8\). We can write since, by \(3.14\) and \(3.15\), \(\rho\) and \(P\) are linear in \(f\):

\[
\rho(e_1, f_1) - \rho(e_2, f_2) = \rho(e_1, f_1 - f_2) + [\rho(e_1, f_2) - \rho(e_2, f_2)]
\]

(5.13)

\[
P(e_1, f_1) - P(e_2, f_2) = P(e_1, f_1 - f_2) + [P(e_1, f_2) - P(e_2, f_2)]
\]

(5.14)
We then deduce from (5.12), (5.17) and (5.18) that:

\[ f \text{ are continuous and} \]

\[ 1 \]

Then we have \( P(e_1, f_1 - f_2) \leq \frac{1}{e_2^2} \| f_1 - f_2 \| \) (5.16)

From proposition 5.1, we deduce the following theorem:

Let \( f \), \( e \), \( \theta \) defined by (5.9) is locally Lipschitzian in \( (f, e, \theta) \) has the following properties:

(\( i \)) \( a \) is an increasing function.

\[ \text{Theorem 5.1} \]

Now let \( f^1 \in L^1_2(\mathbb{R}^3) \) and \( e^1 \in [0, \frac{3}{2}] \); take \( e_1, e_2 \in [\frac{e^1}{2}, \frac{e^1 + 2}{2}] \) and \( f^1, f_2 \in B(f^1, 1) := \{ f \in L^1_2(\mathbb{R}^3) \| f - f^1 \| < 1 \} \). Then we have \( \frac{1}{e_1}, \frac{1}{e_2} < \frac{2}{e^1} \) and \( \| f^1 \|, \| f_2 \| \leq \| f^1 \| + 1 \).

We then deduce from (5.12), (5.17) and (5.18) that:

\[ \left\| Q(f_1, f_1)(e_1) - Q(f_2, f_2)(e_2) \right\| \leq K(\| f_1 - f_2 \| + |e_1 - e_2|) \]

\[ |\rho(e_1, f_1) - \rho(e_2, f_2)| \leq K(\| f_1 - f_2 \| + |e_1 - e_2|) \]

\[ |P(e_1, f_1) - P(e_2, f_2)| \leq K(\| f_1 - f_2 \| + |e_1 - e_2|) \]

where \( K = K(e^1, f^1) \) is a constant depending only on \( e^1 \) and \( f^1 \). If we add this to the fact that \( (f, e, \theta) \mapsto -\frac{\theta}{2} f \) and \( (f, e, \theta) \mapsto -\frac{\theta^2}{4} + \Lambda \) are Lipschitzian with respect to the norm of \( E = L^1_2(\mathbb{R}^3)^3 \times \mathbb{R} \times \mathbb{R} \), we can conclude that \( F = (F_1, F_2, F_3) \) defined by (5.9) is locally Lipschitzian in \( (f, e, \theta) \) with respect to the norm of \( E \). Proposition 5.1 then follows from the standard theorem on first order differential system for functions with values in a Banach space. Notice the \( e, \theta \) are continuous and \( f \in C[0, l; L^1_2(\mathbb{R}^3)] \).

From proposition 5.1, we deduce the following theorem:

\[ \text{Theorem 5.1} \]

Let \( f_0 \in L^1_2(\mathbb{R}^3) \), \( f_0 \geq 0 \ a.e., \ a_0 \geq \frac{2}{3} \) and let a strictly positive cosmological constant \( \Lambda \), be given. Define \( a_0 \) by the relation (5.6).

Let \( r > \| f_0 \| \). Then, there exist a number \( l > 0 \) such that the initial value problem for the coupled Einstein-Boltzmann system (4.13), (4.14), (4.15) with the the initial data \( (f_0, a_0, a_0) \), has an unique solution \( (f, a) \) on \([0, l]\). The solution \((a, f)\) has the following properties:

(i) \( a \) is an increasing function.
\[ f \in C[[0, l], X_r] \quad (5.19) \]

\[ ||f|| \leq ||f_0|| \quad (5.20) \]

**Proof of theorem 5.1**

Choose in proposition 5.1, \( f_0, e_0 \) and \( \theta_0 \) as in (5.5). Let \((e, f, \theta)\) be the unique solution of that initial value problem defined on the interval \([0, l], l > 0\), whose existence is proved by that proposition. Set \( a = \frac{1}{e} \). \( (5.2) \) then implies \( \theta = 3 \dot{a} a \).

We know, by the characteristic method that the Boltzmann equation (2.13) is equivalent to (5.1). We also know that, in the framework described in §4.4, and in which we solve the Einstein equation in the case \( \Lambda > 0 \), the system (4.1)-(4.2) in \( \Lambda \) and the system (5.2)-(5.3) in \( (e, \theta) \) are equivalent. Since the relation (5.6) which implies (4.10), is satisfied, we know that the Hamiltonian constraint (4.1) is satisfied on \([0, l] \). We can conclude that, in the framework fixed by choosing (4.16) in the case \( \Lambda > 0 \), the unique solution \((f, e, \theta)\) on \([0, l] \), of the initial value problem (5.1)-(5.2)-(5.3)-(5.4) with \( f_0, e_0, \theta_0 \) given by (5.5) and satisfying (5.6), give the unique solution \((f, a = \frac{1}{e})\) on \([0, l] \), of the initial value problem (2.13)-(4.1)-(4.2) with the initial data \((f_0, a_0, \dot{a}_0)\) satisfying (5.6).

Concerning the properties of the solution \((f, a)\) on \([0, l] \): (4.10) shows that \( \dot{a} > 0; t \mapsto a(t) \) then satisfies the point i) and a is bounded from below, since \( a(t) \geq a_0 \geq \frac{3}{2} \). a then satisfies the hypotheses of theorem 3.1 in which we set \( t_0 = 0, T = l \). Note that \( C[[0, l], X_r] \subset C[[0, l]; L^2_1(\mathbb{R}^3)] \); (5.19) follows from the uniqueness and (5.20) from (3.18). ■

### 6 Global Existence Theorem for the Coupled Einstein-Boltzmann System

#### 6.1 The Method

Here, we prove that the local solution obtained in §5 is, in fact, a global solution. Let us sketch the method we adopt: Denote \([0, T]\), where \( T > 0 \), the maximal existence domain of the solution of (5.1)-(5.2)-(5.3), with initial data \((f_0, e_0, \theta_0)\) defined by (5.6) - (5.6); here we denote this solution \((\tilde{f}, \tilde{e}, \tilde{\theta})\), in order words we have, on \([0, T]\)

\[ \dot{\tilde{f}} = \frac{1}{p_0(\tilde{e})} Q(\tilde{f}, \dot{\tilde{f}}) \quad (6.1) \]

\[ \dot{\tilde{e}} = - \frac{\tilde{\theta}}{3} \tilde{e} \quad (6.2) \]

\[ \dot{\tilde{\theta}} = - \frac{\tilde{\theta}^2}{3} - 4\pi(\tilde{\rho} + 3\tilde{\Pi}) + \Lambda \quad (6.3) \]

\[ \tilde{f}(0) = f_0 \in X_r, \quad \tilde{e}(0) = e_0 = \frac{1}{a_0}, \quad \tilde{\theta}(0) = \frac{\dot{a}_0}{a_0} \quad (6.4) \]
with in \(6.3\) \(\tilde{\rho} = \rho(\tilde{f}, \tilde{e})\): \(\tilde{P} = P(\tilde{f}, \tilde{e})\) and in \(6.4\) \(f_0, a_0, \theta_0\) subject to the constraints \(5.4\), \(r > 0\) is given such that \(r > \|f_0\|\).

If \(T = +\infty\), the problem is solved. We are going to show that, if we suppose that \(T < +\infty\), then the solution \((\tilde{f}, \tilde{e}, \tilde{\theta})\) can be extended beyond \(T\), which contradicts the maximality of \(T\). Suppose \(0 < T < +\infty\) and let \(t_0 \in [0, T]\). We will show that, there exists a strictly positive number \(\delta > 0\), independent of \(t_0\), such that the following system in \((e, \theta)\) on \([t_0, t_0 + \delta]\), in which \(\tilde{a} = \frac{1}{e}\):

\[
\dot{f} = \frac{1}{p^0(e)} Q(f, f) \tag{6.5}
\]

\[
\dot{e} = -\frac{\theta}{3} e \tag{6.6}
\]

\[
\dot{\theta} = -\frac{\theta^2}{3} - 4\pi(\rho + 3P) + \Lambda \tag{6.7}
\]

has a solution \((f, e, \theta)\) on \([t_0, t_0 + \delta]\). Then, by taking \(t_0\) sufficiently close to \(T\), for example, to such that \(0 < T - t_0 < \frac{\delta}{2}\), hence \(T < t_0 + \frac{\delta}{2}\), we can extend the solution \((\tilde{f}, \tilde{e}, \tilde{\theta})\) to \([0, t_0 + \frac{\delta}{2}]\) that contains strictly \([0, T]\), and this will contradict the maximality of \(T\). We need some preliminaries results.

In what follows, we suppose \(0 < T < +\infty\) and \(t_0 \in [0, T]\).

### 6.2 The Functional Framework

In all what follows, \(C_2, C_3, D_0\) are the absolute constant defined by \((4.36)\) and \((4.49)\). We set, for \(\delta > 0\):

\[
E_{t_0}^{\delta} = \left\{ e \in C[t_0, t_0 + \delta], \frac{1}{C_2} e^{-C_3(t_0 + t + 1)^2} \leq e(t_0 + t) \leq \frac{2}{3}, \forall t \in [0, \delta] \right\}
\]

\[
F_{t_0}^{\delta} = \left\{ \theta \in C[t_0, t_0 + \delta], \theta, \sqrt{3}\Lambda \leq \theta(t_0 + t) \leq D_0, \forall t \in [0, \delta] \right\}
\]

where \(C[t_0, t_0 + \delta]\) is the space of continuous (and hence bounded) functions on \([t_0, t_0 + \delta]\). One verifies easily that \(E_{t_0}^{\delta}\) and \(F_{t_0}^{\delta}\) are complete metric subspaces of the Banach space \(C[t_0, t_0 + \delta], ||.||_\infty\) where \(\|u\|_\infty = \sup_{t \in [t_0, t_0 + \delta]} |u(t)|\)

### 6.3 The Global Existence Theorem

**Proposition 6.1** There exists a strictly positive real number \(\delta > 0\) depending only on the absolute constants \(a_0, \Lambda, r\) and \(T\) such that the initial value problem \((6.3)-(6.6)-(6.7)-(6.8)\) has a solution \((f, e = \frac{1}{a}, \theta)\) in \(C[[t_0, t_0 + \delta]; X, X] \times E_{t_0}^{\delta} \times F_{t_0}^{\delta}\)
Proof

It will be enough, if we look for \( \delta \) such that \( 0 < \delta < 1 \). By theorem 3.1, we know that if we fix \( \bar{e} \in E_t^\delta \) and if we set \( \bar{a} = \frac{1}{\alpha} \), then \( \Theta = \bar{\theta} \) has an unique fixed point \( f \in C[[t_0, t_0 + \delta]; X_r] \), such that, \( f(t_0) = \bar{f}(t_0) \), and, by (3.18) and (3.20):

\[
\|f(t)\| \leq \|\bar{f}(t_0)\| \leq f_0 \leq r \quad (6.9)
\]

Next, by proposition 4.6 in which we set \( \Xi = \tilde{\Theta} = \bar{\hat{\theta}} \), we know that if \( \bar{f} \) is given in \( C[[t_0, t_0 + \delta]; X_r] \), then (6.6)-(6.7) has an unique solution \( (e, \theta) \) on \([t_0, t_0 + \delta]\) such that \( e(t_0) = \frac{1}{a(t_0)} \), \( \theta(t_0) = 3\tilde{\alpha}(t_0) \). Now (3.16) and (4.17) show that \( (e = \frac{1}{a}, \theta) \in E_t^\delta \times F_t^\delta \). This allows us to define the application:

\[
G : C[[t_0, t_0 + \delta]; X_r] \times E_t^\delta \to C[[t_0, t_0 + \delta]; X_r] \times (E_t^\delta \times F_t^\delta) \quad (6.10)
\]

\[
(f, \tilde{e}) \mapsto G(f, \tilde{e}) = [f, (e, \theta)] \quad (6.11)
\]

We are going to show that we can find \( \delta > 0 \) such that \( G \) defined by (6.10) induces a contracting map of the complete metric space \( C[[t_0, t_0 + \delta]; X_r] \times E_t^\delta \) into itself, that will hence, have an unique fixed point \( (f, e) \); this will allow us to find \( \theta \) such that \( (f, e, \theta) \) be the unique solution of (6.5), (6.6), (6.7) in \( C[[t_0, t_0 + \delta]; X_r] \times (E_t^\delta \times F_t^\delta) \). So if we set in (6.5) \( e = \bar{e} \in E_t^\delta \), in (6.7) \( \rho = \bar{\rho} = \rho(e, \bar{f}) \), \( P = \bar{P} = P(e, \bar{f}) \) where \( \bar{f} \in C[[t_0, t_0 + \delta]; X_r] \), we have a solution \( (f, e, \theta) \) of that system, or, equivalently, a solution \( (f, e, \theta) \) of the following integral system:

\[
f(t_0 + t) = \bar{f}(t_0) + \int_{t_0}^{t_0 + t} \frac{1}{p(\bar{e})} Q(f, f)(\bar{e})(s) ds \quad (6.12)
\]

\[
e(t_0 + t) = \bar{e}(t_0) + \int_{t_0}^{t_0 + t} \frac{\theta(s)e(s)}{3} ds \quad (6.13)
\]

\[
\theta(t_0 + t) = \bar{\theta}(t_0) + \int_{t_0}^{t_0 + t} \left[ -\frac{\theta^2}{3} - 4\pi(\bar{\rho} + 3\bar{P}) + \Lambda \right](s) ds \quad (6.14)
\]

\( t \in [0, \delta] \)

Let \( \bar{e}_1, \bar{e}_2 \in E_t^\delta \); \( \bar{f}_1, \bar{f}_2 \in C[[t_0, t_0 + \delta]; X_r] \). To \( \bar{e}_i \) (resp \( \bar{f}_i \)), \( i = 1, 2 \), corresponds by \( G \), the solution \( f_i \) (resp \( (e_i, \theta_i) \)) of (6.12), (6.13) of (6.14). Writing each equation for \( i = 1, 2 \) and subtracting yields:

\[
\|f_1 - f_2\| \leq \int_{t_0}^{t_0 + t} \left\| \frac{1}{p(\bar{e}_1)} Q(f_1, f_1)(\bar{e}_1) - \frac{1}{p(\bar{e}_2)} Q(f_2, f_2)(\bar{e}_2) \right\| ds \quad (6.15)
\]

\[
|e_1 - e_2| \leq \int_{t_0}^{t_0 + t} \left| \frac{\theta_1 e_1}{3} - \frac{\theta_2 e_2}{3} \right| ds \quad (6.16)
\]

\[
|\theta_1 - \theta_2| \leq \int_{t_0}^{t_0 + t} \left| \frac{1}{3} |\theta_1^2 - \theta_2^2| + 12\pi(\bar{\rho}_1 - \bar{\rho}_2) + |\bar{P}_1 - \bar{P}_2| \right| ds \quad (6.17)
\]
We are going to bound the r.h.s of (6.15) and (6.17), using: (5.12) in which we set \( e_1 = \bar{e}_1, e_2 = \bar{e}_2 \) and (6.17)-(6.18), in which we set \( f_1 = \bar{f}_1, f_2 = \bar{f}_2 \). Since \( \bar{e}_1, \bar{e}_2, e_1, e_2 \in E^\delta_{E_0} \), (6.16), shows that: \( \frac{1}{|f|} \leq C_2 e e^{C_3|t+1|^2} \), since \( t_0 < T, t \leq \delta < 1 \). So we deduce from (6.12), (6.13) and (6.14), using: (5.17), (4.17), \( ||\bar{f}_i|| < r, 0 \leq t \leq \bar{\delta} \), the definition of \( E^\delta_{E_0}, F^\delta_{E_0} (|\theta_i| \leq D_0(a_0, r, \Lambda, T)) \), see (4.15), the definition (2.10) of the norm \( \|\cdot\| \) of function over \([t_0, t_0 + \bar{\delta}]\), that:

\[
\|f_1 - f_2\| \leq \delta M_1 (\|\bar{e}_1 - \bar{e}_2\| + \|f_1 - f_2\|) \tag{6.18}
\]

\[
\|e_1 - e_2\| \leq \delta M_2 (\|e_1 - e_2\| + \|\theta_1 - \theta_2\|) \tag{6.19}
\]

\[
\|\theta_1 - \theta_2\| \leq \delta M_3 (\|e_1 - e_2\| + \|\theta_1 - \theta_2\| + ||\bar{f}_1 - \bar{f}_2||) \tag{6.20}
\]

where \( M_1, M_2, M_3 \) are constants depending only on \( a_0, \Lambda, r \) and \( T \). We have by addition of (6.19) and (6.20):

\[
\|e_1 - e_2\| + \|\theta_1 - \theta_2\| \leq 2\delta(M_2 + M_3)(\|e_1 - e_2\| + \|\theta_1 - \theta_2\| + ||\bar{f}_1 - \bar{f}_2||) \tag{6.21}
\]

Then, if we choose \( \delta \) such that:

\[
\delta = \inf \left[ 1, \frac{1}{8(M_1 + M_2 + M_3)} \right] \tag{6.22}
\]

(6.22) implies that: \( \delta M_1 < \frac{1}{4} \); \( 2\delta(M_2 + M_3) < \frac{1}{4} \) and (6.18), (??) give:

\[
\begin{align*}
\|f_1 - f_2\| & \leq \frac{1}{2}\|\bar{e}_1 - \bar{e}_2\| \\
\|e_1 - e_2\| + \|\theta_1 - \theta_2\| & \leq \frac{1}{2}||\bar{f}_1 - \bar{f}_2||
\end{align*}
\]

and by addition:

\[
\|f_1 - f_2\| + \|e_1 - e_2\| + \|\theta_1 - \theta_2\| \leq \frac{1}{3} (\|\bar{f}_1 - \bar{f}_2\| + \|\bar{e}_1 - \bar{e}_2\|)
\]

from which, we deduce:

\[
\|f_1 - f_2\| + \|e_1 - e_2\| \leq \frac{1}{3} (\|\bar{f}_1 - \bar{f}_2\| + \|\bar{e}_1 - \bar{e}_2\|) \tag{6.23}
\]

(6.23) shows that the map \((\bar{f}, \bar{e}) \mapsto (f, e)\) is a contracting map from the complete metric space \( C[t_0, t_0 + \delta]; X_r \times E^\delta_{E_0} \) into itself, for every \( \delta \) satisfying (6.22), which shows that such a \( \delta \) depends only on \( a_0, r, \Lambda \) and \( T \). This map has an unique fixed point \((f, e)\); since \( e \) is known, (6.4) determines \( \theta \) by \( \theta = -3\bar{e} \), and \((f, e, \theta)\) is a solution of (6.3)-(6.4) in \( C[t_0, t_0 + \delta]; X_r \times E^\delta_{E_0} \times E^\delta_{E_0} \). This completes the proof of proposition 6.4.

We can then state:
Theorem 6.1 The initial value problem for the Einstein-Boltzmann system with a strictly positive cosmological constant $\Lambda$ on a Robertson-Walker spacetime has a global solution $(a, f)$ on $[0, +\infty]$, for arbitrarily large initial data $a_0$ and $f_0 \in L^1_2(\mathbb{R}^3)$, $f_0 \geq 0$ a.e.

Remark 6.1 1) Nowhere in the proof we had to restrict the size of the initial data $a_0$, $f_0$, which can then be taken arbitrarily large.

2) In [13], the author considered only the Hamiltonian constraint (4.1), in the case $\Lambda = 0$, without studying the evolution equation (4.2) that is, as we saw, the main problem to solve, since the Hamiltonian constraint is satisfied once it is the case for the initial data.

3) We will prove in a future paper, that theorem 6.1 extends to the case $\Lambda = 0$.

4) In the future, we will try to relax hypotheses on the collision kernel $A$.

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