The Two Roads to ‘Intrinsic Charm’ in B Decays

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Abstract

We describe two complementary ways to show the presence of higher order effects in the $1/m_Q$ expansion for inclusive $B$ decays that have been dubbed ‘Intrinsic Charm’. Apart from the lessons they can teach us about QCD’s nonperturbative dynamics their consideration is relevant for precise extractions of $|V_{cb}|$: for they complement the estimate of the potential impact of $1/m_Q^4$ contributions. We draw semiquantitative conclusions for the expected scale of Weak Annihilation in semileptonic $B$ decays, both for its valence and non-valence components.

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1 Introduction

The Operator Product Expansion (OPE) has turned out to be a highly powerful tool for describing inclusive decays of beauty hadrons. Among the major results and successes are the precision determinations of CKM elements using inclusive semileptonic decays: e.g., for $|V_{cb}|$ a relative uncertainty of less than 2% has been achieved based on OPE methods.

To validate and even improve on such accuracy one has to analyze all potential sources of uncertainties. The OPE is widely viewed as yielding an expansion in inverse powers of the $b$ quark mass $m_b$ for sufficiently inclusive quantities. However, in $b \rightarrow c$ transitions a second scale is present that has a reasonable claim to be considered heavy, namely the charm quark mass with $m_c \gg \Lambda_{QCD}$. It is then natural to integrate out the charm quark as well. This is usually done for both quarks at the same scale, which means that the ratio $m_c/m_b$ is treated as a number of order unity.

On the other hand, numerically we have $m_c^2 \sim m_b \Lambda_{QCD}$ and hence $m_b \gg m_c \gg \Lambda_{QCD}$. This suggests an alternative where one integrates out the beauty quark in a first step, while leaving the charm quark still a dynamical quark; the latter is subsequently removed in a second step. In this procedure one has an intermediate theory still containing charm quarks as dynamical entities. Such a description contains dimension-six (or higher) operators of the schematic form

$$O_{IC} = (\bar{b}_v \Gamma^c c) (\bar{c} \Gamma b_v) \quad (1.1)$$

where $b_v$ is the (now) nonrelativistic $b$ quark field and $\Gamma$ a Dirac matrix depending on the process under consideration; it may contain derivatives yielding higher-dimension operators. It has been discussed in some detail in [1, 2] that these operators match onto a contribution proportional to the Darwin term $\rho_D^3$ upon integrating out the charm quark. Yet new subtleties emerge at this point. This contribution contains an infrared (IR) sensitive piece proportional to $\ln m_b/m_c$, which diverges for $m_c \rightarrow 0$. It is generated in the two-step procedure by the RG running between $m_b$ and $m_c$, while it appears simply in the coefficient function when integrating out bottom and charm together. In fact, it has been shown in [1] that at higher orders in the OPE even inverse powers of $m_c$ do appear which have a stronger infrared sensitivity.

Based on a superficial similarity with various non-valence nonperturbative charm effects previously discussed in the literature [3], these infrared sensitive terms have been dubbed “Intrinsic Charm” (IC) in inclusive B decays. They were largely treated as a factor contributing to the theoretical uncertainty from higher-order effects [1]. In this paper we show how the seemingly different reasonings given in [1, 2] are equivalent qualitatively as well as quantitatively and perform a systematic and more complete treatment of these terms. It turns out that the conventional OPE yields a combined expansion in powers of $1/m_b^k \times 1/m_c^l$ with $l > 0$ (in fact, starting with $l=2$ at tree level, without extra gluon loop) appearing first for $k=3$. This gives rise to a quantitative subtlety as well: Since as mentioned above $m_c^2 \sim m_b \Lambda_{QCD}$, one has to account for the terms $1/(m_c^2 m_b^2)$ in order to complement the full set of $1/m_b^k$ corrections.

The remainder of the paper is organized as follows. In the next section we perform the standard procedure and integrate out bottom and charm together in a similar way at a scale below $m_c$ and trace the origin of the intrinsic charm contributions in this approach. In Sect. 3 we perform the two-step procedure; i.e., first we integrate out just the bottom quark initially leaving charm as a dynamical quark in the theory before integrating it out as well at a lower scale. As expected these two ways yield the same results equivalent to what has been discussed in [1]. We provide a more consistent and concise derivation of the relation between (generalized) Weak Annihilation and the four-quark operators than it was done in Ref. [4]. In Sect. 4 we discuss
explicitly the dimension-eight operators that yield $1/m_b^2 \times 1/m_c^2$ terms at tree level qualitatively as well as quantitatively, confirming the results of [1]. We summarize the lessons learnt on QCD’s non-perturbative dynamics, the implementation of the OPE and on the uncertainties in the extraction of $V_{cb}$ in Sect. [6]. An estimate of the potential scale of Weak Annihilation (WA) for $b \rightarrow u$ is given there as well.

2 The standard OPE for $B \rightarrow X_c \ell \bar{\nu}_\ell$ revisited

One starts by considering the doubly differential rate for $B \rightarrow X_c \ell \bar{\nu}_\ell$

$$\frac{d^2 \Gamma}{d(v \cdot p) dp^2} = \frac{G_F^2 |V_{cb}|^2}{24 \pi^3 \sqrt{(v \cdot p)^2 - p^2}} \theta((v \cdot p)^2 - p^2) \theta(m_b - v \cdot p) \theta(m_b^2 + p^2 - 2m_b v \cdot p) \ L_{\mu \nu} W^{\mu \nu}(p)$$

(2.1)

where $p = m_b v - q$ with $q$ denoting momentum transfer to the lepton pair and $v$ the $B$ meson velocity. $L_{\mu \nu}$ is the leptonic tensor with

$$L_{\mu \nu}(p) = -m_b^2 [g_{\mu \nu} - v_\mu v_\nu] + m_b [2g_{\mu \nu}(v \cdot p) - v_\mu p_\nu - p_\mu v_\nu] - [g_{\mu \nu} p^2 - p_\mu p_\nu]$$

(2.2)

and $W^{\mu \nu}$ the hadronic counterpart given by

$$2M_B W_{\mu \nu}(p) = \int d^4 x \ \exp(i p \cdot x) \langle B(v) | \bar{b}_v(x) \Gamma_\nu c(x) \bar{c}(0) \Gamma_\mu b_v(0) | B(v) \rangle$$

(2.3)

with $b_v$ the ‘rephased’ (nonrelativistic) $b$ quark field, $b_v(x) = \exp(i m_b(v \cdot x)) b(x)$ and $\Gamma_\mu = \gamma_\mu (1 - \gamma_5)$.

2.1 General argument

Here we scrutinize the standard way of setting up the OPE for this inclusive process treating both the bottom and the charm quarks as heavy and neglecting a hierarchy of scales between the bottom and the charm masses. We remove both from the dynamical degrees of freedom in one step at a scale $\mu$ below the charm mass. No operators with charm fields can then emerge in the OPE, and only operators with nonrelativistic $b$ quarks and their (covariant) derivatives appear [1].

The OPE is constructed by contracting the two charm quarks into (the imaginary part of) a propagator, while the beauty quark field is replaced by its expansion in inverse powers of $m_b$. Formally expanding the bi-local operator in (2.3) in local operators yields, upon integration, the well known Heavy Quark Expansion (HQE). The expansion can be done by external field methods particularly economical at tree level, see e.g. [5]. This yields ($b_v \equiv b_v(0)$)

$$2M_B W_{\mu \nu}(p) = \langle B(v) | \bar{b}_v \Gamma_\nu (\phi + m_c) \Gamma_\mu b_v | B(v) \rangle \ \delta_+ (p^2 - m_c^2)$$

$$+ \langle B(v) | \bar{b}_v \Gamma_\nu (\phi + m_c) (i \nabla) (\phi + m_c) \Gamma_\mu b_v | B(v) \rangle \ \delta_+^\prime (p^2 - m_c^2)$$

$$+ \frac{1}{2} \langle B(v) | \bar{b}_v \Gamma_\nu (\phi + m_c) (i \nabla)(\phi + m_c) (i \nabla) (\phi + m_c) \Gamma_\mu b_v | B(v) \rangle \ \delta_+^\prime (p^2 - m_c^2)$$

$$+ \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle B(v) | \bar{b}_v \Gamma_\nu (\phi + m_c) \left[ (i \nabla)(\phi + m_c) \right]^n \Gamma_\mu b_v | B(v) \rangle \ \delta_+ (p^2 - m_c^2).$$

(2.4)

\footnotetext{1}{We do not discuss here operators involving light quarks appearing additionally at $O(\alpha_s)$ level, their contribution is small.}
The main challenge in evaluating higher order contributions lies in identifying the independent hadronic parameters controlling the ‘string’ of matrix elements

\[ \langle B(v) | \bar{b} \gamma_{\mu} (iD_{\mu_1}) \cdots (iD_{\mu_n}) b | B(v) \rangle \]

entering at order \( n + 1 \). It should be noted that these quantities do not depend on \( p \).

Contracting the hadronic and leptonic tensors for the pseudoscalar \( B \) mesons yields a function of \((v \cdot p)\) and \( p^2 \) of the general form

\[ L_{\mu \nu} W^{\mu \nu} = \sum_{n=0}^{\infty} \sqrt{(v \cdot p)^2 - p^2} P_n((v \cdot p), p^2) \delta_c^{(n)} (p^2 - m_c^2) \]

with \( P_n(v \cdot p) \) denoting a polynomial in \( v \cdot p \). The analysis so far referred to a fully differential distribution in general kinematics. To proceed to the inverse mass expansions we need to consider partially integrated probabilities.

The integration over the variable \( v \cdot p \) has the limits \( \sqrt{p^2} \leq v \cdot p \leq (m_b^2 + p^2)/(2m_b) \), which yields terms logarithmic in \( p^2 \) from the lower end of integration. Focusing on these logarithms we get for \( l=0,1, \ldots \)

\[ \int \frac{d(v \cdot p)}{\sqrt{v \cdot p}} \sqrt{(v \cdot p)^2 - p^2} (v \cdot p)^{2l} = C_l (p^2)^{l+1} \ln \left( \frac{p^2}{m_b^2} \right) + \cdots, \quad C_l = \frac{\Gamma(l+\frac{1}{2})}{4\sqrt{\pi} \Gamma(l+2)} \]

where the ellipses denote polynomial terms in \( p^2 \), and the coefficients \( C_l \) are simple fractions: \( C_0 = 1/4, \ C_1 = 1/16, \ C_2 = 1/32, \ C_3 = 3/256 \ldots \)

These logarithms are the source of the IR sensitivity of the coefficient functions to the charm mass in this approach. This becomes manifest when they are combined with the derivatives of the \( \delta \)-function in Eq. (2.6):

\[ (p^2)^k \ln p^2 \delta_c^{(k)} (p^2 - m_c^2) = (-1)^k k! \ln m_c^2 \delta_c^{(p^2 - m_c^2)} + \cdots \]

\[ (p^2)^k \ln p^2 \delta_c^{(n)} (p^2 - m_c^2) = (-1)^{n-k} (n-k)! \left( \frac{1}{m_c^2} \right)^{n-k} \delta_c^{(p^2 - m_c^2)} \text{ for } n > k \]

where \( k \) is some integer power and the ellipses point to less singular terms as \( m_c \to 0 \). Note that at the lower limit of integration in Eq. (2.7) we have \( p_\| = p^2 \), i.e. \( p^\| \to 0 \). This is the infrared regime for the charm quark in the final state.

The leading IR sensitive terms in the integrated rates arise from only the leading term in the leptonic tensor

\[ L_{\mu \nu}^{\text{lead}} = -m_b^2 [g_{\mu \nu} - v_{\mu} v_{\nu}] \]

already in the expression for the differential rate. For the terms containing the vector \( p \) would lead to powers of \( v \cdot p \) and \( p^2 \). To obtain non-leading in \( 1/m_b \) terms, we consider the subleading terms in the leptonic tensor

\[ L_{\mu \nu}^{\text{sub}} = m_b [2g_{\mu \nu} (v \cdot p) - v_{\mu} p_{\nu} - p_{\mu} v_{\nu}] \]

\[ L_{\mu \nu}^{\text{subsub}} = - [g_{\mu \nu} p^2 - p_{\mu} p_{\nu}] \]
On the other hand, the $n^{th}$ term in the sum (2.4) for the hadronic tensor contains
\[ P_n \propto \Gamma_{\mu}(\p + m_c)\gamma_{\mu_1}(\p + m_c)\gamma_{\mu_2} \cdots (\p + m_c)\gamma_{\mu_n}(\p + m_c)\Gamma_{\mu} \] (2.13)
which yields upon contraction with the leptonic tensor (2.10) and with the nonperturbative matrix elements (2.5) a contribution of the form (see (2.6))
\[ P_n = \sum_{ijl} a_{ijl} (v \cdot p)^i (p^2)^j (m_c^2)^l \quad \text{with} \quad i + 2j + 2l = n + 1. \] (2.14)

Note that due to the purely left-handed structure of the current $\Gamma$, we can have only even powers of $m_c$ here. Furthermore, if $n$ is even, $i$ necessarily is odd (and vice versa). Hence from (2.7) we conclude that IR sensitive terms can appear only if $n$ is odd, which means that the number of covariant derivatives in the hadronic matrix element has to be odd as well. Therefore all operators which contribute to intrinsic charm have to match onto partonic matrix elements with at least one gluon. In turn, it implies that there is no intrinsic charm contribution to operators of the form $(k^2)^n$ where $\vec{k}$ are the spatial components of the residual $b$ quark momentum.

Now we can trace how singular terms actually emerge. Putting everything together, we get for $i = 2m$
\[ \int d(v \cdot p)\sqrt{(v \cdot p)^2 - p^2} P_n((v \cdot p), p^2)\delta^{(n)}(p^2 - m_c^2) \] (2.15)
\[ = \sum_{ijl} a_{ijl} \int d(v \cdot p)\sqrt{(v \cdot p)^2 - p^2} [(v \cdot p)^i (p^2)^j (m_c^2)^l] \delta^{(n)}(p^2 - m_c^2) \] (2.16)
\[ \Rightarrow \sum_{ijl} C_l a_{ijl} (p^2)^{m+j+1} (m_c^2)^l \ln \left( \frac{p^2}{m_b^2} \right) \delta^{(n)}(p^2 - m_c^2). \] (2.17)

Terms with odd $i$, on the other hand, do not contain a logarithm. Note that we have $2m + 2j + 2l = n + 1$, which can be satisfied at any odd $n$. Thus we arrive at
\[ \int d(v \cdot p)\sqrt{(v \cdot p)^2 - p^2} P_n((v \cdot p), p^2)\delta^{(n)}(p^2 - m_c^2) \] (2.18)
\[ \Rightarrow \sum_{ijl} C_l a_{ijl} (p^2)^{(n+3-2l)/2} (m_c^2)^l \ln \left( \frac{p^2}{m_b^2} \right) \delta^{(n)}(p^2 - m_c^2). \] (2.19)

Hence IR sensitive terms can appear starting at $n = 3$ with the logarithmic dependence on $m_c$ of the Darwin term. For $n > 3$ we get from these terms a tree level contribution to the total rate of the form
\[ \Gamma_n \propto \frac{1}{m_b^3} \left( \frac{1}{m_c^2} \right)^{(n-3)/2} \quad \text{with} \quad n = 5, 7, 9\ldots, \] (2.20)
i.e., even a powerlike singularity for $m_c \to 0$.

### 2.2 Explicit expressions for $n = 0$ through 4 in the total width

For $n = 0$ the leading term reads
\[ P_0^{\text{lead}} = \frac{3}{2} m_b^2 (v \cdot p) \] (2.21)
which is an odd power of \(v \cdot p\) and hence does not lead to IR sensitive terms as just explained. One should note that the subleading contribution to the leptonic tensor \(L_{\mu\nu}\) yields upon contraction with the partonic hadronic tensor – a (subleading) term of the form

\[
P_0^{\text{sub}} = -2m_b(v \cdot p)^2 - p^2 m_b;
\]  

(2.22)

it leads to a contribution of the form \(m_c^6 \ln(m_c^2)\) in the phase space factor of the partonic total rate. We will focus eventually on those novel terms \(\propto 1/m_c^k\) that arise in leading order in \(1/m_b\); to low orders in \(\Lambda_{\text{QCD}}\) it is easy to keep as well the terms stemming from the subleading pieces in \(L_{\mu\nu}\).

The next term with \(n = 1\) is given by

\[
P_1 = \frac{\mu_r^2 - \mu_f^2}{12m_b} (5p^4 + 7m_c^2p^2 - 20(v \cdot p)^2p^2 - 10m_c^2(v \cdot p)^2),
\]  

(2.23)

where the hadronic tensor is contracted with the subsubleading part of the leptonic tensor \(L_{\mu\nu}^{\text{subsub}}\). This yields again an \(m_c^4 \ln(m_c^2)\) terms upon integration over the phase space.

For \(n = 2\) the leading term of the leptonic tensor again contains only odd powers of \((v \cdot p)\) which do not generate any logarithms.

At \(n = 3\) the IR sensitive contribution is the Darwin term. Explicitly, we have

\[
\langle B(v|\bar{b}_v(iD_\alpha)(iD_\gamma)(iD_\beta)b_v | B(v)\rangle = \frac{1}{6} M_B \rho_D^3 (g_{\alpha\beta} - v_\alpha v_\beta) v_\gamma (\not{p} + 1) + \cdots
\]  

(2.24)

from which we obtain

\[
P_3^{\text{Dar}} = -\frac{\rho_D^3}{12} m_b^2 (3p^2 - m_c^2)^2 + 8(v \cdot p)^4 - 8p^2(v \cdot p)^2).
\]  

(2.25)

Upon integration over \((v \cdot p)\) we arrive at terms with three types of prefactors:

\[
(p^2)^3 \ln\left(\frac{p^2}{m_b^2}\right), \quad m_c^2(p^2)^2 \ln\left(\frac{p^2}{m_b^2}\right), \quad m_c^4 p^2 \ln\left(\frac{p^2}{m_b^2}\right).
\]  

(2.26)

They have three derivatives with respect to \(p^2\) from the \(\delta^{(3)}\)-function, and hence the first term yields a \(\ln(m_c^2)\) factor which is the first infrared sensitive contribution. In the other two terms explicit factors of \(m_c^2\) kill the infrared singularity and they thus remain finite for \(m_c \to 0\). It is straightforward to check that in this way we end up with the correct prefactor for the infrared log in the Darwin contribution.

The terms with \(n = 4\) create again only odd powers of \((v \cdot p)\) and would yield infrared singularity for the \(1/m_b\)-subleading piece. Finally at \(n = 5\) the following nine structures arise

\[
P_5 \propto (v \cdot p)^6, \quad (v \cdot p)^4 p^2, \quad (v \cdot p)^4 m_c^2, \quad (v \cdot p)^2(p^2)^2, \quad (v \cdot p)^4 m_c^4;
\]  

(2.27)

\[
(v \cdot p)^2 p^2 m_c^2, \quad (p^2)^3, \quad (p^2)^2 m_c^2, \quad p^2 m_c^4
\]

Upon integration over \((v \cdot p)\) we obtain terms of the form

\[
(p^2)^4 \ln\left(\frac{p^2}{m_b^2}\right), \quad (p^2)^3 m_c^2 \ln\left(\frac{p^2}{m_b^2}\right), \quad (p^2)^2 m_c^4 \ln\left(\frac{p^2}{m_b^2}\right), \quad p^2 m_c^6 \ln\left(\frac{p^2}{m_b^2}\right)
\]  

(2.28)

coming with five derivatives of the \(\delta\)-function. All thus yield contributions of order \(1/m_c^2\) in the total rate. They will be addressed in detail in the next section. A similar consideration
extends in a straightforward way to higher orders where $n > 5$ emerge. These would generate terms inversely proportional to even larger powers of inverse charm mass.

As a first resumé we state that IR sensitive contributions – i.e., those singular for $m_c \to 0$ – unequivocally arise from the lower end of the integration over $v \cdot p$ (i.e. $\vec{p} \to 0$) due to the presence of the non-analytic factor $\sqrt{(v \cdot p)^2 - p^2} = |\vec{p}|$ in the integrand. For the dimension-six Darwin term they are of the form $\ln (m_b^2/m_c^2)$. Higher-dimension contributions exhibit an even stronger singularity, viz. powers of $1/m_c$. Without extra gluon loops this happens first for dimension eight.

The discussion above makes it clear that such IC effects emerge for the fully integrated width as well as for higher moments of the distributions. However the strength of the infrared singularity and the order in the heavy quark expansion it first emerges generally depends on the kinematic observable.

3 $m_b \gg m_c \gg \Lambda_{\text{QCD}}$: charm as a dynamical quark

In this section we shall consider an alternative way to describe IC effects. We now choose to integrate out the ‘heavy’ degrees of freedom only above the scale $m_c$. This leaves the charm quark as a dynamical entity, much in the same way as would be required for light quarks in QCD, e.g. in $b \to u \ell \nu$. The main difference that emerges is that, for inclusive decays, we have now to include four-quark operators explicitly containing charm quark fields.

The charm quarks in $b$ decay can act both as a hard and a soft degree of freedom. The hard component is treated the same way as described in the previous section. The matrix elements of the four-quark operators contain the “soft” part of the still dynamical quarks, which for now we treat nonperturbatively. Accordingly, we write the original QCD product of currents as

$$
\langle B \bar{b}(x) \Gamma_\nu c(x) \bar{c}(0) \Gamma_\rho b(0) |B \rangle
= \langle B \bar{b}(x) \Gamma_\nu \langle c(x) \bar{c}(0) \rangle \Gamma_\rho b(0) |B \rangle_{>\mu} + \langle B \bar{b}(x) \Gamma_\nu c(x) \bar{c}(0) \Gamma_\rho b(0) |B \rangle_{<\mu}.
$$

The product $c(x)\bar{c}(0)$ can be viewed as the Green function of the charm quark inside the external gluon field in the $B$ meson, averaged over the field configurations present in the meson. This is an exact relation as long as the beauty meson has no charm flavor: it is a consequence of the Gaussian integration over the quark fields. This applies to the l.h.s. as well as to each of the two terms on the r.h.s. The decomposition on the r.h.s. merely reflects the different treatment used to describe these terms.

The first term corresponds to the ‘perturbative’ (in $1/m_c$) calculation of the previous section. The second term has to be added now, since the charm quark is still a dynamic quark controlled by nonperturbative dynamics. The role of the intermediate scale $\mu$ is to draw the demarcation between the two dynamical regimes.

Even though the first term in Eq. (3.1) is evaluated in the ‘direct’ way detailed in the previous section, the result differs due to the introduction of the cutoff $\mu$. The precise form of how $\mu$ enters depends on the concrete way the (hard) separation is implemented. Let us mention that for tree-level calculations without extra perturbative loops it is sufficient and convenient to simply integrate the distribution with the constraint

$$
0 \leq q^2 < (m_b - \mu)^2.
$$

The upper bound above effectively introduces the separation scale in Eq. (3.1) when the correlator is integrated over the phase space to obtain the inclusive width or its moments.
Evaluating the first term in Eq. (3.1) in the previous section gave contributions IR sensitive to the charm mass. Taking the formal limit \( m_c \to 0 \) separately term by term in the expansion would have yielded divergent expressions. With a nonzero \( \mu \gg \Lambda_{\text{QCD}} \) this changes: all individual terms remain regular by themselves at \( m_c \to 0 \). For the role of the infrared regulator is now taken over by the Wilsonian cutoff.

This is evident on general grounds: the terms IR-singular for \( m_c \to 0 \) came only from the soft charm configuration with momentum \( p \lesssim m_c \) – the domain now excluded. Alternatively, this can be traced explicitly in the formalism of Sect. 2. For instance, with the constraint of Eq. (3.2) the lower limit of integration in \( (v \cdot p) \) rises from \( \sqrt{p^2} \) to \( p^2 + 2m_b \mu - m_c^2 \approx \mu \); therefore \( \log \frac{m_b^2}{p^2} \) in Eq. (2.8) turns into \( \log \frac{m_b^2}{\mu^2} \). All the integrals become analytic functions of \( m_c \) at \( m_c \ll \mu \); the logs and inverse powers of \( m_c \) are replaced by those of the cutoff mass \( \mu \).

The second term in the r.h.s. of Eq. (3.1) has a smooth \( m_c \to 0 \) limit. Although it may have soft “chiral” singularities when both \( m_c \) and one of the light quarks become massless, the expectation values should remain finite; only higher derivatives with respect to the charm mass may have singularities if \( m_u \) or \( m_d \) vanish. We note that this expectation value, being regularized in the ultraviolet, is well-defined and these conclusions hold with no reservations.

We may briefly discuss at this point the parametric dependence of the total semileptonic width \( \Gamma_d(b \to c) \) on \( m_c \) when the latter is small. As long as \( m_c \gg \Lambda_{\text{QCD}} \) holds, we have non-analytic nonperturbative terms at order \( 1/m_c^3 \) scaling like \( A_{\text{QCD}}^{2} \ln \frac{m_b^2}{m_c^2} \), \( A_{\text{QCD}}^{2} \left( \frac{A_{\text{QCD}}^{2}}{m_c^2} \right)^k \) with \( k > 0 \) (in general, odd powers of \( m_c \) also emerge). Each of these terms separately are singular at \( m_c \to 0 \). The leading term is driven by the Darwin expectation value; it represents an IR singularity in \( m_b/m_c \) which, in principle, is observable, at least at large \( m_b \) and sufficient accuracy.

The expansion in \( \Lambda_{\text{QCD}}/m_c \) makes, however, sense only as long as charm remains heavy on the scale of QCD dynamics. At lower \( m_c \) the successive terms with higher \( k \) would formally dominate. The whole function of \( m_c \) stabilizes at \( m_c \lesssim \Lambda_{\text{QCD}} \) and approaches a finite value at \( m_c \to 0 \). A model for such a behavior can be given, for instance, by

\[
\frac{\rho_D^3}{m_b^3} \ln \frac{m_b^2}{m_c^2 + \Lambda^2} \tag{3.3}
\]

(here \( \Lambda \) is a strong interaction mass scale parameter of the order of \( \Lambda_{\text{QCD}} \)) which, expanded in \( 1/m_c^3 \), would yield the whole series in \( 1/m_c^2 \). The actual coefficients for the \( 1/m_c^{2k} \) terms may, of course, be different, and they can be calculated in the OPE along either road.

Returning to the OPE analysis proper, we can bridge the two ways of accounting for the nonperturbative charm effects by looking at the \( \mu \)-dependence of both terms in Eq. (3.1) (more accurately, when it is integrated to obtain the inclusive probability). Their sum must be \( \mu \) independent which provides useful relations. The following note should be kept in mind.

The form of the \( \mu \)-dependence is determined by the regularization scheme. In the Wilsonian procedure with a hard cutoff the leading log dependence is accompanied by power terms. The resulting dependence is qualitatively different for \( \mu \ll m_c \) and for \( \mu \gg m_c \).

(i) When \( \mu \) is taken small compared to \( m_c \), it enters only as a small power correction, \( \sim (\mu/m_c)^l \); in the formal limit \( \mu \to 0 \) \( (m_c \gg \Lambda_{\text{QCD}} \) fixed) the last term would vanish, the whole correction to the width is then given by the first term. Raising \( \mu \) up to \( m_c \) and above moves the correction to the width from the first term to the second. The calculation of Sect. 2 represents the evaluation of the last term contribution in the \( \Lambda_{\text{QCD}}/m_c \)-expansion for \( \mu \gg m_c \). Such a calculation makes sense only as long as charm is sufficiently heavy.
(ii) At $\mu \gg m_c$ the situation is different. Beyond the leading term $O_D \ln \frac{\mu^2}{m_c^2}$ the $\mu$ dependence is suppressed by powers of $m^2/\mu^2$. In the second term in Eq. (3.1) these appear as a residual dependence of the high-dimension terms on the ultraviolet cutoff for integrals intrinsically convergent at the scale $m_c$. In the first term it shows up as now small coefficient functions of higher-dimension operators – which would normally be saturated at soft charm configurations $p \sim m_c$ – proportional to $1/m^3_c 1/\mu^{2k}$, instead of $1/m^3_c 1/m^{2k}_c$ without a cutoff. It is then convenient to assume $\mu \gg m_c$ and neglect these terms altogether.

A short comment is in order on how the separation would look like in the often adopted dimensional regularization scheme (DR) where no powerlike dependence on renormalization scale ever arises. For small $m_c$ the only $\mu$-dependence within DR enters through $\ln \mu$ in the Darwin operator. In such a scheme, however, charm quarks must be treated as massless; the dyonic four-quark operators have to be renormalized in the UV likewise in the way of DR. The requirement to set $m_c = 0$ then is rather clear. For keeping $m_c$ finite would destroy naive DR in a direct calculation: it yields a finite result for the first term around $D = 4$ because the potential IR singularity at $D = 4$ is regularized by a non-zero charm mass. As a result, with $m_c \neq 0$ calculating the first term in Eq. (3.1) with DR precisely reproduces the total contribution of Sect. 2. Consequently, in DR at $m_c \neq 0$ the last term in Eq. (3.1) should be considered as vanishing regardless of the concrete value of the charm mass; this is not a very physical result, at least for a relatively light quark.

This is a rather typical feature of dimensional regularization. DR may provide a convenient highly efficient technical tool to analyze the case of massless final state quark or when a non-zero charm mass can be neglected. However, it would require additional matching procedure, usually order by order in $m_c$ if the charm mass dependence is important.

In logarithmic terms – as for the Darwin operator in the integrated width – the charm mass plays the role of a renormalization point. Therefore, as discussed in Ref. [2], the infrared dependence on $m_c$ in the ‘conventional’ calculation along the ‘first’ road must match the UV dependence of the corresponding four-quark expectation value given by $\rho^2_3 \ln \frac{\Lambda^2_{QCD}}{m^2}$. A similarly dual description applies also for the terms scaling like inverse powers of $m_c$. They can be calculated conventionally assuming charm to be heavy following the route of Sect. 2. Alternatively, they can be obtained as the corresponding pieces of the four-quark expectation value (normalized at $\mu \gg m_c, \Lambda_{QCD}$). The results over either road must be identical whenever one is in the domain where the expansion can be applied. The first road (without implementing a cutoff) is, of course, justified only for $m_c \gg \Lambda_{QCD}$. The second route is formally valid for an arbitrary hierarchy between $m_c$ and $\Lambda_{QCD}$ – however, we do not have the means to calculate the expectation value through gluon operators without charm fields when charm becomes light.

In order to make the above mentioned correspondence explicit, we will address the 1/m^3_c 1/m^n_c terms where at tree level only even $n$ emerge. To this end, we consider the contribution of the second term in Eq. (3.1) involving the explicit charm quark operators. Inserting it into the hadronic tensor we get

$$2M_B W_{\mu\nu}^{(IC)} = \int d^4x \exp(ip \cdot x) \langle B(v) | \bar{b}_v(x) \Gamma_{\nu} c(x) \bar{c}(0) \Gamma_\mu b_v(0) | B(v) \rangle |_\mu$$

$$= \int d^4x \exp(ip \cdot x) \langle B(v) | \bar{b}_v(0) \Gamma_{\nu} c(0) \bar{c}(0) \Gamma_\mu b_v(0) | B(v) \rangle |_\mu$$

$$+ \int d^4x \exp(ip \cdot x) \rho_\alpha B(v) [\partial^\alpha \bar{b}_v(0) \Gamma_{\nu} c(0)] \bar{c}(0) \Gamma_\mu b_v(0) | B(v) \rangle |_\mu + \cdots$$

where the ellipses denote operators of dimension eight and higher. Extra derivative for (renor-
malized) expectation values can bring in a factor of $m_c$ or $\mu$ at most; it can be traced that powers of $x_\alpha$ translate into powers of $1/m_b$. Therefore, the higher terms in this expansion yield higher powers in the $1/m_b$ expansion. We then can focus on the first term which is the conventional $D=6$ four-quark operator. The corresponding hadronic tensor reads

$$2M_B W^{(IC)}_{\mu\nu} = (2\pi)^4 \delta^4(p) \langle B| \bar{b}(0) \Gamma_\nu c(0) \bar{c}(0) \Gamma_\mu b(0) | B \rangle_\mu + \ldots$$

and we retain only the first term. Note that the $\delta$-function projects out the leading term of the leptonic tensor (2.10). Furthermore, this contribution is localized at $p = 0$ – hence at $p^2 = 0$ and $v \cdot p = 0$, in agreement with the findings of the last section. As a consequence, step-functions in Eq. (2.1) become superfluous. The last relation completing the arithmetic part is

$$\int d(v \cdot p) \, dp^{2} \sqrt{(v \cdot p)^{2} - p^{2}} \delta^4(p) = \frac{1}{2\pi}.$$  (3.6)

Omitting QCD corrections, the relevant diagrams for calculating $W^{(IC)}_{\mu\nu}$ in Eq. (3.5) are the one loop diagrams involving the charm-quark loop with an arbitrary number of external gluons. These diagrams have been considered already in [1]. It is advantageous to first perform a Fierz rearrangement of the four quark operator according to

$$\bar{b} \Gamma^\nu c \bar{c} \Gamma^\mu b = -\frac{1}{2} \bar{b}_\alpha \Gamma_\mu b_\beta \bar{c}_\beta \Gamma_\sigma c_\alpha [-i\epsilon^{\mu\nu\rho\sigma} + g^{\sigma\mu} \epsilon^{\rho\nu} + g^{\sigma\nu} \epsilon^{\rho\mu} - g^{\rho\sigma} \epsilon^{\mu\nu}],$$  (3.7)

where $\alpha, \beta$ are color indices, and the minus sign comes from anticommutativity of the quark fermion fields.

We may construct the charm mass expansion of this charm loop in the external gauge field and take the average over the $B$-meson state. There are a few subtleties related to this procedure, since the two charm quark operators are taken at coinciding space-time points. As in [1] we may start from the time-ordered product of two charm quark operators at displaced points, which amounts to consider the conventional charm propagator in an external field. This propagator is generally gauge dependent, yet in the end this is compensated by the same displacement in the $b$-quark fields. For constructing the short-distance expansion of the Green function the fixed-point gauge is convenient.

The limit of coinciding points in the charm Green function is formally divergent, yet gauge independent. Subtracting the free Green function (this piece is accounted for in the purely partonic width) we end up with a mild log divergence present in the vector current, proportional to $[D_\mu, G_{\mu\nu}]$:

$$\langle \bar{c}_\alpha \gamma^\nu c_\beta \rangle_A = \frac{2}{3} \frac{\Lambda_{UV}^2}{(4\pi)^2} \ln \left( \frac{\Lambda_{UV}^2}{m_c^2} \right) [D_\kappa, G^\kappa_{\mu\nu}]_{\beta\alpha} + \ldots$$  (3.8)

where $G_{\mu\nu}$ is the gauge field strength tensor; $\Lambda_{UV}$ should be identified with $\mu$ in this context, and the ellipses denote finite terms to be considered below. As discussed in [2], this ultraviolet-singular log matches onto the infrared piece of the conventionally calculated Darwin coefficient function. The contributions from the axial-vector $\bar{c}c$ current are convergent ab initio. Examples of the Feynman diagrams are shown in Fig. 1.
Figure 1: Diagrams illustrating calculation of the charm loop in the external field. The wavy lines generically reflect insertions of the external gluon field.

The lowest finite terms yield $1/m_b^3 1/m_c^2$ contributions given by

$$\langle \bar{c}_\alpha \gamma_\nu \gamma_5 c_\beta \rangle_A = \frac{1}{48\pi^2 m_c^2} \left( 2 \left\{ [D_\kappa, G^{\kappa\lambda}], \tilde{G}_{\nu\lambda} \right\} + \left\{ [D_\kappa, \tilde{G}_{\nu\lambda}], G^{\kappa\lambda} \right\} \right)_{\beta\alpha} + \cdots$$  \hspace{1cm} (3.9)

$$\langle \bar{c}_\alpha \gamma_\nu c_\beta \rangle_A = \frac{i}{240\pi^2 m_c^2} \left( 13 [D_\kappa, [G_{\lambda\nu}, G^{\lambda\kappa}]] + 8i [D^\kappa, [D^\lambda, [D_\lambda, G_{\kappa\nu}]]] \right)_{\beta\alpha} + \cdots$$  \hspace{1cm} (3.10)

This assumes $\mu \gg m_c$ and neglecting power terms $\sim (m_c/\mu)^k$. Inserting this into (3.7) we end up with dimension-eight $\bar{b}...b$ operators with gluon fields; their coefficient functions compared to the partonic $D=3$ operator $\bar{b}b$ are proportional to $1/m_b^3 1/m_c^2$.

A closer look into the calculations of the charm loop in the external field reveals that the resulting expressions exactly parallel those in Sect. 2 once the leading-order approximation Eq. (2.10) is adopted and only non-analytic terms according to Eq. (2.7) are retained. This occurs before the full integration is performed, when one takes the integral over the timelike component of the loop momentum by the residues at $p^2 = m_c^2$, for each power term in the expanded propagator.

In fact, the technique of the loop calculation in the external field itself allows to derive a number of relations which strongly constrain the form of the operators which can appear in such an expansion. It has been presented in detail in the first part of Ref. [6]. These relations ensure that the result has always the form of multiple commutators. This sharpens one observation made already in Sec. 2. There we noted that the partonic matrix elements of IC contributions necessarily have to be at least one-gluon matrix elements. From the arguments given in this section we conclude that intrinsic charm contributions involve only gluon fields and their derivatives; there will be no derivatives acting on the beauty quark fields that would generate a dependence on the ‘residual momentum’ of the decaying $b$-quark.

We have verified though explicit calculations of the $1/m_b^3 1/m_c^2$ terms that these two ways to calculate $1/m_c$-singular terms yield the same result for the operator expansion.

At order $1/m_b^3 1/m_c^2$ the result can thus be expressed through five operators, which are determined by two contributions to the axial vector current and three contributions to the
vector current:

\[
2M_B \tilde{f}_1 = \langle B | \tilde{b}_v \left[ iD_\kappa, [iD_\lambda, [iD_\lambda^*, iG^{\kappa\alpha}]] \right] b_v | B \rangle \ v_\alpha \tag{3.11}
\]

\[
2M_B \tilde{f}_2 = \langle B | \tilde{b}_v \left[ iD_\kappa, [iD_\lambda, [iD_\lambda^*, iG^{\kappa\alpha}]] \right] b_v | B \rangle \ v_\alpha \tag{3.12}
\]

\[
2M_B \tilde{f}_3 = \langle B | \tilde{b}_v \left[ iD_\kappa, [iG^{\lambda\alpha}, iG^{\lambda\kappa}] \right] b_v | B \rangle \ v_\alpha \tag{3.13}
\]

\[
2M_B \tilde{f}_4 = \langle B | \tilde{b}_v \left\{ [iD^0, iG_{\rho\lambda}], iG_{\rho\gamma} \right\} (-i\sigma_{\alpha\beta}) b_v | B \rangle \times \frac{1}{2} \left( g^{\alpha\kappa} g^{\delta\beta} v^\gamma - g^{\alpha\lambda} g^{\gamma\beta} v^\delta + g^{\beta\kappa} g^{\gamma\lambda} v^\delta \right) \tag{3.14}
\]

\[
2M_B \tilde{f}_5 = \langle B | \tilde{b}_v \left\{ [iD^0, iG_{\rho\lambda}], iG_{\rho\gamma} \right\} (-i\sigma_{\alpha\beta}) b_v | B \rangle \times \frac{1}{2} \left( g^{\sigma\alpha} g^{\lambda\beta} v^\gamma - g^{\sigma\alpha} g^{\gamma\beta} v^\lambda + g^{\lambda\alpha} g^{\gamma\beta} v^\lambda \right) . \tag{3.15}
\]

The contributions originating from the axial current yield spin-triplet operators, while those of the vector current yield spin singlet operators.

## 4 Quantitative estimates of Intrinsic Charm

To estimate the IC contributions we follow the lines of [1] and apply the “ground-state factorization” approximation to the matrix elements.

The effect on the total rate has been considered already in [1], yielding a reasonably small contribution. In fact, the total rate, expressed in terms of the operators given in (3.11) reads as

\[
\frac{m_b^3 m_c^2}{\Gamma_0} \bigg| \frac{1}{m_c^2} \bigg| = -\frac{3}{2} \frac{2}{15} \left( -8 \tilde{f}_1 + 4 \tilde{f}_2 - 13 \tilde{f}_3 \right) + \frac{1}{2} \frac{2}{3} \left( -2 \tilde{f}_4 - \tilde{f}_5 \right) . \tag{4.1}
\]

Following the way to evaluate the expectation values suggested in Ref. [1] we obtain numerically\(^2\)

\[
\tilde{f}_1 \approx 0.31 \text{ GeV}^5, \quad \tilde{f}_2 \approx 0.25 \text{ GeV}^5, \quad \tilde{f}_3 \approx 0.14 \text{ GeV}^5, \quad \tilde{f}_4 \approx 0.34 \text{ GeV}^5, \quad \tilde{f}_5 \approx -0.40 \text{ GeV}^5 \tag{4.2}
\]

which lead to\(^3\)

\[
\delta \Gamma \bigg| \frac{1}{m_c^2} \bigg| \approx (0.7\%) \times \Gamma_{\text{Parton}} . \tag{4.3}
\]

This numerical estimate should not be considered bullet-proof. While the individual matrix elements are predicted with reasonable confidence in their signs and magnitudes, we are faced with a set of terms with different signs. Thus cancellations will in general occur among them. Their degree may depend on the numerical accuracy of the applied ground-state factorization, as well as on the precise values of the lower-dimension expectation values \(\mu_2, \rho_D^3\) and \(\rho_{LS}^3\). A more elaborate discussion of the nonfactorizable effects will be presented in a separate publication.

Finally we note that for the moments the situation is different. Contributions that introduce an IR sensitivity to the charm quark mass in the total rate may become regular for the moments. This becomes evident if we consider moments of the partonic invariant mass such as \(\langle (p^2 - m_c^2)^n \rangle\) which remain regular as \(m_c \to 0\) till higher orders in \(\Lambda_{\text{QCD}}\).

---

\(^2\)We have found a discrepancy with Ref. [1] in the overall factor for one of the expectation values. It does not produce a noticeable numerical change for the correction to the width, however.

\(^3\)Ref. [1] included an additional phase-space suppression factor for the IC kinematic of \((1-m_c/m_b)^2\). Based on the operator analysis we can show that actually it is absent in the case at hand.
Having at hand the numerical estimates of the higher-dimension expectation values allows us to refine the model (3.3) for the charm-mass dependence in the IR regime. Since the $1/(m_b^3 m_c^2)$ corrections calculated in this ansatz are fixed in terms of parameter $\Lambda$, we assign the latter the value which would reproduce these leading corrections. This yields

$$\Lambda^2 \equiv M_*^2 = \frac{\hat{f}_1}{5 \rho_D^3} - \frac{\hat{f}_2}{10 \rho_D^3} + \frac{13 \hat{f}_3}{40 \rho_D^3} - \frac{\hat{f}_4}{12 \rho_D^3} - \frac{\hat{f}_5}{24 \rho_D^3} \simeq (0.7 \text{GeV})^2,$$

and the model predicts

$$(-\delta^{\alpha\beta} + v^\alpha v^\beta) \frac{1}{2 M_B} \langle B | \bar{b} \gamma_\alpha (1-\gamma_5) c \bar{c} \gamma_\beta (1-\gamma_5) b | B \rangle_\mu \simeq -\frac{\rho_D^3}{4 \pi^2} \ln \frac{\mu^2}{m_c^2 + M_*^2},$$

see Fig. 2. In this case the correction to the width at $m_c^2 \ll m_b^2$ takes the form

$$\frac{\delta \Gamma_{\text{sl}}}{\Gamma_{\text{sl}}} \simeq -\frac{8 \rho_D^3}{m_b^2} \left( \ln \frac{m_b^2}{m_c^2 + M_*^2} - \frac{77}{48} \right),$$

where the constant term accounts for the explicit UV contribution in this limit. This model illustrates to which extent the charm quark may be considered heavy in this context.

![Figure 2: The WA expectation value $\frac{1}{2 M_B} \langle B | \bar{b} \gamma^k (1-\gamma_5) c \bar{c} \gamma^k (1-\gamma_5) b | B \rangle_\mu$ as a function of charm mass: the leading contribution logarithmic in $m_c$ (blue dashed), including additionally the $1/m_c^2$ power correction (orange dotted), and the complete behavior according to ansatz Eq. (4.5) (red solid). We assume the value $\rho_D^3 = 0.15 \text{GeV}^3$ and the normalization point $\mu$ is taken 4.6 GeV.](image)

It should be acknowledged that Eqs. (3.3), (4.4) and (4.6) are only a reasonable model for the effects at intermediate to small $m_c$. In fact, the size of the effective mass scale $M_*$ as determined by matching the leading charm power corrections may not be fully universal: it depends on the particular Lorentz structure of the weak vertices. It would likewise differ, say, in weak $B^*$ decays – although numerically such a variation in $M_*$ would be insignificant.

## 5 Facing up to Weak Annihilation

Our preceding discussion can suggest a dual description of IR charm effects that allows insights into a higher-order OPE calculation of the heavy-to-light case $B \to X_u \ell \bar{\nu}_\ell$. More specifically the limiting case $m_c \to 0$ is of relevance when treating the heavy-to-light case beyond order $1/m_b^2$. The inherent IR divergence of the standard calculation for $m_c/m_b \to 0$ underlies the importance
of the four quark operator matrix element in the heavy-to-light case, which is usually called WA contribution. More precisely, here it corresponds to its valence-quark insensitive piece which affects semileptonic $B^+$ and $B^0$ decays equally. We refer to it as non-valence WA.

The important question here is the potential scale of the non-valence WA contributions, once the effect of the Darwin operator has been separated out. A direct computation based on the method discussed above is not possible. Yet one may try to estimate the natural scale expected for WA by approaching the massless case from the heavy quark side. For this purpose we use the model suggested in the previous sections. In spite of its admitted oversimplification, we can be sure that the difference between this ansatz and the actual QCD contribution remains finite at any $m_c$ including the limit $m_c \to 0$. The model has the advantage of being sufficiently accurate when extrapolating down from the side of intermediately heavy charm quarks.

It is therefore plausible that setting $m_c=0$ and applying this model we do not stray far away from the leading effect of the non-valence component of WA in $b \to u \ell \nu$ decays (which is its only contribution in $B_d$ decays). This guesstimate, though, cannot be validated in the context of the $1/m_c$ expansion examined here. In some sense such an assumption implies that no ‘phase transition’ occurs when going down in mass from heavy to light quarks. We know such a phase transition takes place in the QCD vacuum, yet it may not necessarily be important for the expectation values over $B$ meson states. One may a priori expect larger WA effects in $b \to u \ell \nu$ coming from its flavor-specific piece manifesting itself in the decays of charged $B$-mesons.

Using the model of the previous sections to interpolate between the regimes of heavy and light charm we get an estimate

$$\frac{1}{2M_B} \langle B_d | \bar{b} \gamma(1-\gamma_5) u \bar{u} \gamma(1-\gamma_5) b | B_d \rangle \simeq -\frac{\rho_D^3}{4\pi^2} \ln \left( \frac{\mu^2}{M^2} + 1 \right) \approx -0.005 \text{GeV}^3,$$  

(5.1)

with $\mu$ denoting the normalization scale. This is an educated guess and cannot guarantee to yield even the correct sign of the effect at $\mu \lesssim 1 \text{GeV}$. The negative sign physically means that the propagation of the soft $u$ quark (projected onto the spin state specified by the Lorentz structure in question) is suppressed compared to free propagation. Taken at face value, the WA expectation value in Eq. (5.1) would yield the isoscalar shift in the semileptonic $B$ decay width

$$\frac{\delta \Gamma_{sl}^{WA}(b \to u)}{\Gamma_{sl}(b \to u)} \simeq -\frac{8\rho_D^3}{m_b^3} \ln \left( \frac{\mu^2}{M^2} + 1 \right) \approx -0.015.$$  

(5.2)

A more refined way to assess non-valence WA for $b \to u \ell \nu$ would be to consider the real massless case for the final-state quark, $m_u=0$, yet to introduce an IR cutoff via the kinematic restriction Eq. (3.2). The first term in Eq. (3.1) representing the ‘UV’ piece of the $\bar{b} u \bar{u} b$ expectation value from the domain of quark momenta above $\mu$ is then calculated in the direct way of Sect. 2 (with $m_c \to 0$). The result is expressed in terms of the same five expectation values, with the coefficients scaling as $1/\mu^2$; yet they would combine to yield in general a

4The difference due to the valence part of WA in $b \to u \ell \nu$ can be experimentally probed by analyzing the difference in the semileptonic spectra spectra of charged and neutral $B$ decays.

5Paper [4] which first analyzed the effects of generalized WA in semileptonic decays, focussed on the differences between mesons with different spectators and therefore explicitly subtracted the $\bar{b} u \bar{u} b$ expectation value in $B_f$ from that in $B^-$. Following an earlier classification of the preasymptotic power corrections to the inclusive widths it referred to this as the spectator-dependent correction, a terminology continued to a number of later publications. The valence and non-valence effects separately were considered in Ref. [7] where valence effects were sometimes also called ‘spectator’ contributions. The importance of the non-valence WA for light quarks, although conjectured already back in 1994, was emphasized in Ref. [8]. There the ‘singlet’ and the WA proper pieces referred to the average and the difference of the valence and non-valence components.
different combination of the operators and, consequently, a different number. Evaluating the result with $\mu \approx 0.6$ GeV would provide an estimate of the minimal natural scale of non-valence WA.

Even such an estimate would be admittedly incomplete; beyond its lack of precision in evolving the $\mu$ dependence to as low a value as 0.6 GeV, the total WA should also include the $\langle B \mid b u \bar{u} b \mid B \rangle_{<0.6 \text{ GeV}}$, the last term in Eq. (3.1). The contribution from the low momenta plausibly exceeds the numerical estimate above, and may even change the overall sign. However, it would be unnatural to allow the contributions from physically distinct domains of low and high momenta to show significant cancellations. Therefore, we would view the thus obtained estimate as a firmer lower bound on the scale of non-valence WA in $b \to u \ell \nu$.

One may anticipate a potentially more significant effect for the ‘valence’ part of WA which describes this effect in $\Gamma(B^+ \to X_u \ell \nu)$. On physical grounds we expect this contribution to contain a piece independent of the non-valence WA and not related to something that can be traced from the $b \to c \ell \nu$ decays in the limit of small charm mass. In the formal derivation of relating $c(x) \bar{c}(0)$ in Eq. (3.1) to charm Green functions it would be associated with an additional term. That term appears through the contraction of the quark fields with those of the same flavor whose presence is required in the interpolating currents to produce the initial and to annihilate the final $B$ meson state.

On the other hand, as pointed out in Refs. [4, 9] and exploited in later papers [8], [7], one may infer certain information about WA, and in particular about valence WA from the $D$ meson decays. The charm quark is marginally heavy to apply precision heavy quark expansions to its decays. Extrapolation from charm to beauty may thus be semi-quantitative at best, yet it should provide some constraint on the expected scale of WA’s physical implementation. In particular, we want to point out that the preliminary CLEO-c [10] data on the semileptonic branching fraction of $D_s$ indicate the presence of a destructive spectator-related WA contribution of around 20%; it must be the result of non-factorizable four-quark expectation values. In interpreting this observation one has to address the question of $SU(3)$-breaking in the leading nonperturbative corrections described by the kinetic and chromomagnetic operators (and, possibly, in the Darwin expectation value).

While semileptonic ‘valence’ WA cannot occur at all in $D^0$ decays, it can contribute in $D^+$ and $D_s$ decays on the Cabibbo suppressed and Cabibbo allowed levels, respectively. Assuming possible $SU(3)$ breaking to be under control in WA proper, we conclude that the observed difference of the total semileptonic widths for $D_s$ and $D^0$

$$\frac{\Gamma_{sl}(D_s)}{\Gamma_{sl}(D^0)} = \frac{\text{Br}_{sl}(D_s)}{\text{Br}_{sl}(D^0)} \frac{\tau_{D^0}}{\tau_{D_s}} \simeq 0.81$$

must be dominated by valence WA in $D_s$. To describe the pattern on the WA effects in $D$ mesons we may adopt the nomenclature for the generalized ‘annihilation’ correction following Ref. [4]: The valence WA for a particular transition $c \to q \ell \nu$ ($q = s$ or $d$), $\text{WA}^{\text{val}}_q$ refers to the difference in the matrix elements between $D_q$ and $D^0$. The nonvalence $\text{WA}^{\text{val}}_d$ is directly the expectation value in the (non-strange) state $D^0$ containing no valence $d$ quark. To allow for $SU(3)$ asymmetry we also have to distinguish $\text{WA}^{\text{val}}_d$ from $\text{WA}^{\text{val}}_d$ where it is considered in the strange $D_s$ state. The WA operators above may be general products like $\bar{c} q \bar{q} c$, either local or nonlocal. (For the decaying quark being heavy enough, like in $B$-mesons, it would be sufficient to include only the leading local four-quark operators.)
With this convention, we in general have in $D$ mesons

$$
\Gamma_{sl}(D^+) - \Gamma_{sl}(D^0) = \sin^2 \theta_c \cdot W_A^{\text{val}}
$$

$$
\Gamma_{sl}(D_s) - \Gamma_{sl}(D^0) = \cos^2 \theta_c \cdot W_A^{\text{val}} - \sin^2 \theta_c \left[ W_A^{n\text{val}} - W_A^{n\text{val}(s)} \right] + \Delta_{SU(3)}. \quad (5.4)
$$

By introducing the subscript marking the $d$ or $s$ flavor we have explicitly allowed for the $SU(3)$ breaking in the expectation values due to the different spectator in a meson or in the light quark field flavor in the corresponding operator. We have still neglected the explicit short-distance $SU(3)$-breaking $\propto m_s^2$ in the coefficient functions emerging due to the larger $m_s$ in the hard quark Green functions; it is expected to be strongly suppressed. $\Delta_{SU(3)}$ in Eq. (5.4) therefore denotes only the shift related to the $SU(3)$ violation in the (flavor-singlet) nonperturbative expectation values between the strange and non-strange heavy meson states. The analysis suggests that these effects are numerically suppressed for the kinetic and the chromomagnetic operators and should not exceed 5% level in the widths. Then the bulk of the difference in Eq. (5.3) should be equated with the valence component of WA, at least if $SU(3)$ violation in it is not too strong.

Translating these relations for WA from charm to beauty is associated with significant uncertainties due to a potentially poor representation of the contributions to the inclusive width for charm by the (truncated) OPE; for $m_c$ is manifestly not large enough for a precision treatment. Relating WA for $B$ decays to the expectation values of the $D=6$ operators can be done with acceptable theoretical accuracy. Yet expressing the WA contributions in Eqs. (5.4) through the analogous local expectation values in $D$ mesons is subjected to large corrections – first of all from the corresponding higher-dimension operators with additional derivatives. Related to this is the short-distance ‘hybrid’ renormalization of the operators in question from the scale of charm to beauty; in this case it may be even not fully perturbative.

Bearing in mind these potential caveats we nevertheless use this line of reasoning to estimate the expected size of the valence WA in the semileptonic $b \to u$ width of charged $B$ meson:

$$
\frac{\Gamma(B^+ \to X_{\text{light}} \ell \nu) - \Gamma(B^0 \to X_{\text{light}} \ell \nu)}{\Gamma(B \to X_{\text{light}} \ell \nu)} \approx -(0.005 \text{ to } 0.01)
$$

(5.5)

which is similar in magnitude, yet still below our estimates for the minimal scale of the non-valence WA.

### 6 Summary and Outlook

The main result of this study is that the OPE for inclusive $B \to X_c \ell \bar{\nu}_\ell$ contains terms with an infrared sensitivity to the charm-quark mass. Although this has been known, a complete discussion of these so-called “intrinsic charm” contributions had not been presented. We have given such a discussion here in the context of two theoretical frameworks or ‘roads’ for removing $c$ quarks from the dynamical degrees of freedom that a priori appear different, yet in the end yield identical results.

We have shown that starting at $1/m_b^3$ the standard OPE for $B \to X_c \ell \bar{\nu}_\ell$ exhibits terms of the form $1/m_b^m \times 1/m_c^n$ where at tree level only even $n$ and odd $m$ appear. The matrix elements of local operators parametrizing their nonperturbative input always contain gluon-field-strength operators and their covariant derivatives; in turn the residual momentum of the $b$ quark does not enter here.
We have performed a detailed analysis of the contributions of the form $1/m_b^3 \times 1/m_c^2$ at tree level, which is needed to complete the OPE calculation of $B \rightarrow X_c \ell \bar{\nu}_\ell$ up to order $1/m_b^4$, since parametrically $1/m_b^3 \times \Lambda_{QCD}/m_c^2$ is of the same order. The numerical estimates confirm the results presented in Ref. [1].

The conclusion for $B \rightarrow X_c \ell \bar{\nu}_\ell$ is that a calculation to order $1/m_b^3$ has to include also the terms of order $1/m_b^{n-k} \times 1/(m_c^2)^k$; furthermore, including radiative corrections one obtains also contributions of the order $1/m_b^n \times \alpha_s(m_c)/m_c^k$ where $k$ can also be odd. The lowest terms of this kind are of order $1/m_b^3 \times \alpha_s(m_c)/m_c$ and have been considered in Ref. [1].

These effects are of considerable theoretical interest with respect to subtleties that can arise in nonperturbative dynamics, yet they go beyond it towards more pragmatic goals: they help to validate the goal of reducing the theoretical uncertainty in extracting $|V_{cb}|$ from $B \rightarrow X_c \ell \nu$ to the 1% level; achieving such a goal is of obvious interest for the theoretical treatment of $B$ decays – yet also for a proper interpretation of the ultra-rare decays $K \rightarrow \pi \nu \bar{\nu}$. Their amplitudes have been calculated with high accuracy in terms of $m_c$ and $V_{ts}^*V_{td}$ [13]. Their widths thus scale with $V_{cb}^*$, and the error on the latter is at least a large component in the stated overall 2% uncertainty.

In $b \rightarrow u \ell \nu$ decays the straightforward calculation of the higher-order power corrections to the total width beyond order $1/m_b^3$ would yield terms which diverge power-like in the infrared. Yet this does not mean that one cannot go beyond $1/m_b^3$ order here. The analysis shows that to calculate $\Gamma_{sl}(b \rightarrow \ell \nu)$ without extra $\alpha_s$-corrections it is sufficient to introduce the corresponding WA four-quark operators, and then one should simply discard all the terms formally having inverse powers of $m_u$.

We have found in the process that analysis of IC effects can inform and focus our thinking about the possible impact of WA in the heavy-to-light transitions $B \rightarrow X_u \ell \nu$ – and the extraction of $|V_{ub}|$ there – and on its relation to charm decays. The IC effects for the inclusive distributions are conceptually similar to (generalized) WA corrections extensively discussed in connection to the lifetimes of heavy flavors since the late 1970s. Since the usual beauty hadrons do not contain valence charm quarks, we deal here with the case of non-valence WA contributions first noted in Ref. [4]. A profound difference with the conventional WA for light quarks is that charm quarks, even soft ones, to the leading approximation can be viewed perturbatively, and the nontrivial strong dynamics affect its propagation at the level of power corrections $(\Lambda_{QCD}/m_c^3)^k$, while for light quarks this would not represent a parametric suppression. Nevertheless, approaching the case of conventional WA with light flavors from the heavy-mass side and using a model which naturally interpolates between the regimes of heavy and light quark we get an estimate

$$\frac{\delta \Gamma_{sl}^{val}(b \rightarrow u)}{\Gamma_{sl}(b \rightarrow u)} \approx -0.015 . \quad (6.1)$$

This result should be viewed as an educated guess rather than a real evaluation; one cannot count even on the firm prediction of the sign. It also leaves out the more intuitive valence WA which, as a matter of fact, historically gave the phenomenon its name. (Yet, as clarified in Ref. [12], its interpretation in the presence of strong interactions is more subtle and may include interference-type contributions which allow for the net correction to the width even to become negative.)

We have used the recently reported [10] measurements of the $D_s$ semileptonic fraction to estimate the significance of spectator-related WA in the KM-suppressed semileptonic width of $B^+$. Taken literally, the correction turns out to be close to the non-valence case Eq. (6.1) or
even somewhat smaller, but still destructive:

\[ \frac{\delta \Gamma_{\text{al}}(b \to u)}{\Gamma_{\text{al}}(b \to u)} \approx -(0.005 \text{ to } 0.01). \] (6.2)

Since these contributions populate the kinematic domain of small hadronic invariant mass and energy of the final hadron state, they could have an amplified impact on the existing determinations of \( V_{ub} \) from \( B \to X_u \ell \bar{\nu}_\ell \).

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